Structural stability for a thermal convection model with temperature-dependent solubility

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Abstract

We study a problem involving thermosolutal convection in a fluid when the solute concentration is subject to a chemical reaction in which the solubility of the dissolved component is a function of temperature. When the spatial domain is a bounded one in $\mathbb{R}^2$ we show that the solution depends continuously on the reaction rate using true a priori bounds for the solution when the chemical equilibrium function is an arbitrary function of temperature.

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1 Introduction

The problem of double diffusion convection in a horizontal layer of fluid with simultaneous chemical reaction is well studied, cf. the account in Straughan [1], pp. 225-237. Indeed the governing equations for such a chemical reaction problem are derived by Morro and Straughan [2], employing ideas of continuum thermodynamics. It is also worth drawing attention to the fact that similar equations arise in fluid phase change problems where continuum thermodynamic theories are again employed and analysed by Berti et al. [3], Berti et al. [4], Bonetti et al. [5], Bonetti et al. [6], Fabrizio [7] and Fabrizio et al. [8, 9, 10]. A particular reaction where the solute concentration is subject to a chemical reaction in which the solubility of the dissolved component is a linear function of the temperature has been analysed recently in a porous medium context by Pritchard and Richardson [11], by Wang and Tan [12], and by Malashetty and Biradar [13]. The object of this paper is to analyse the effect of such a class of reaction terms in a fluid. The dependence on temperature is here taken to be arbitrary and not only linear. We allow the fluid to occupy a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma$ sufficiently smooth to allow application of the divergence theorem. We are particularly interested to investigate the continuous dependence of the solution on the reaction rate. It is very reasonable to expect that in such a mathematical model one can achieve an appropriate type of continuous dependence and this kind of stability problem belongs to the important class of structural stability questions.

Structural stability, or continuous dependence on the model itself, is a concept at least as important as the classical idea of stability which involves continuous dependence on the initial
data, as explained in some detail by Hirsch and Smale [14]. Structural stability was studied in elasticity by Knops and Payne [15], then further advanced in a variety of continuum mechanical contexts by Payne [16, 17, 18] and by Knops and Payne [19]. Since then structural stability studies in continuum mechanics have proved very popular as witnessed by the works of Aulisa et al. [20, 21], Celebi et al. [22], Chirita et al. [23], Ciarletta et al. [24, 25], Hoang and Ibragimov [26, 27], Hoang et al. [28], Kalantarov and Zelik [29], Kandem [30], Kang and Park [31], Kelliher et al. [32], Li et al. [33], Liu [34, 35], Liu et al. [36, 37], Ouyang and Yang [38], Passarella et al. [39], Ugurlu [40], You et al. [41].

2 Fundamental equations

The fundamental model we study is based upon the equations of balance of momentum, balance of mass, conservation of energy, and conservation of salt concentration, adopting a Boussinesq approximation in the body force term in the momentum equation. Thus, let $v_i(x, t), p(x, t), T(x, t)$ and $C(x, t)$ denote velocity, pressure, temperature and salt concentration, where $x \in \Omega$, $t$ denote time, with $0 < t < T$, for some $T < \infty$. Then, the equations of momentum, mass, energy, and salt concentration are taken to be

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \Delta v_i + g_i T - h_i C,$$
$$\frac{\partial v_i}{\partial x_i} = 0,$$
$$\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = \Delta T,$$
$$a \frac{\partial C}{\partial t} + b v_i \frac{\partial C}{\partial x_i} = \Delta C + L f(T) - KC,$$

where $g_i, h_i$ represent gravity vectors and without loss of generality we assume that $|g|, |h| \leq 1$.

Standard indicial notation is used throughout with a repeated index denoting summation over 1 and 2, and $a, b, L$ and $K$ are positive constants. Equations (1) follow in practice by employing a Boussinesq approximation which accounts for variable $C$ allowing the incompressibility condition to hold, cf. Fife [42], pp. 72-74.

The function $f$ is at least $C^1$ and the term $L f(T)$ is analogous to the chemical equilibrium term, $C_{eq}$, in Pritchard and Richardson [11], Wang and Tan [12], and Malashetty and Biradar [13], although all of these writers assume $C_{eq}(T)$ is a linear function of $T$. The terms $L f(T) - KC$ in equation (1) correspond to the mass supply term $m_\alpha$ in Morro and Straughan [2]. The justification for this is as given in Pritchard and Richardson [11] who write $L f(T) - KC = K(C_{eq} - C)$, $C_{eq}$ being a chemical equilibrium term. The logic is that in chemical equilibrium the chemical reaction arises solely due to the term $K(C_{eq}(T) - C)$.

In general (1) holds with $a = b = 1$. However, we allow for different $a$ and $b$ since these coefficients will change under a rescaling, i.e. under a different non-dimensionalization.

Equations (1) hold in the domain $\Omega \times (0, T)$, together with the initial conditions

$$v_i(x, 0) = v_i^0(x),$$
$$T(x, 0) = T_0(x),$$
$$C(x, 0) = C_0(x),$$

(2)
for \( x \in \Omega \), and the boundary conditions

\[
\begin{align*}
    v_i(x, t) &= 0, \\
    T(x, t) &= g(x, t), \\
    C(x, t) &= h(x, t),
\end{align*}
\]

\( x \in \Gamma, \ t \in [0, T] \).

Since we are interested in studying continuous dependence on the reaction rates \( L \) and \( K \), we let \((u_i, p_1, T_1, C_1)\) and \((v_i, p_2, T_2, C_2)\) be two solutions to (1)-(3) for the same initial and boundary data, but for different reaction coefficients \((L_1, K_1)\) and \((L_2, K_2)\). To progress we now introduce the difference variables \((w_i, \pi, \theta, \phi)\) and \( l \) and \( k \) by

\[
\begin{align*}
    w_i &= u_i - v_i, \quad \pi = p_1 - p_2, \quad \theta = T_1 - T_2, \\
    \phi &= C_1 - C_2, \quad l = L_1 - L_2, \quad k = K_1 - K_2.
\end{align*}
\]

Thus, from (1)-(4) we may determine the boundary-initial value problem for the difference variables as

\[
\begin{align*}
    \frac{\partial w_i}{\partial t} + w_j \frac{\partial u_i}{\partial x_j} + v_j \frac{\partial w_i}{\partial x_j} &= -\frac{\partial \pi}{\partial x_i} + \Delta w_i + g_i \theta - h_i \phi, \\
    \frac{\partial \theta}{\partial t} + w_i \frac{\partial T_1}{\partial x_i} + v_i \frac{\partial \theta}{\partial x_i} &= \Delta \theta, \\
    a \frac{\partial \phi}{\partial t} + b \left( w_i \frac{\partial C_1}{\partial x_i} + v_i \frac{\partial \phi}{\partial x_i} \right) &= \Delta \phi + L_1[f(T_1) - f(T_2)] + l f(T_2) - K_1 \phi - k C_2,
\end{align*}
\]

in \( \Omega \times (0, T) \), with

\[
\begin{align*}
    w_i(x, 0) &= 0, \quad \theta(x, 0) = 0, \quad \phi(x, 0) = 0,
\end{align*}
\]

\( x \in \Omega \), together with

\[
\begin{align*}
    w_i(x, t) &= 0, \quad \theta(x, t) = 0, \quad \phi(x, t) = 0,
\end{align*}
\]

\( x \in \Gamma, \ t \in [0, T] \). We write \( T_1 \equiv T^1 \) when the occasion needs, with similar notation for \( T_2, C_1 \) and \( C_2 \).

We wish to derive a continuous dependence estimate for a suitable measure of \( w_i, \theta, \phi \) in terms of \( l \) and \( k \). However, we require this estimate to be truly \textit{a priori} in the sense that the coefficients which appear associated to \( l \) and \( k \) involve only data. Thus, before we can achieve our continuous dependence result we need to derive some \textit{a priori} bounds for the solution to (1)-(3).

### 3 A priori estimates

We denote by \((\cdot, \cdot)\) and \(\|\cdot\|\) the inner product and norm on \( L^2(\Omega) \) and further let \(\|\cdot\|_p\) be the norm on \( L^p(\Omega) \) with \(\|\cdot\|_\infty\) being the \(L^\infty(\Omega)\) norm.

Define the quantity \( T_m \) by

\[
T_m = \max\{\|T_0\|_\infty, \sup_{[0, T]} \|g\|_\infty\}.
\]

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Payne et al. [43] show how we may employ the function
\[ \tilde{\psi} = [T - T_m]^+ = \sup(T - T_m, 0) \]
to show from (1)-(3) that
\[ \sup_{\Omega \times [0, T]} |T(x, t)| \leq T_m. \] (9)
This \textit{a priori} bound for the temperature is very important in what follows.

We need to establish \textit{a priori} bounds for certain norms of \( v \) and \( C \) and to do this we recall two lemmas. The first arises from a Rellich identity as used by Payne and Weinberger [44], and is given explicitly in Payne and Straughan [45], inequality (A10).

**Lemma 1.** Let \( \Phi \) be a harmonic function in \( \Omega \) with boundary values \( Q \), i.e. \( \Phi \) satisfies
\[ \Delta \Phi = 0 \quad \text{in} \quad \Omega, \]
\[ \Phi = Q \quad \text{on} \quad \Gamma. \] (10)
Then one may derive explicit constants \( c_1 \) and \( c_2 \) such that
\[ \| \nabla \Phi \|^2 + c_1 \int_\Gamma \left( \frac{\partial \Phi}{\partial n} \right)^2 dA \leq c_2 \int_\Gamma |\nabla_s Q|^2 dA, \] (11)
where \( \nabla_s \) denotes the tangential derivative.

Lemma 1 holds for a general domain \( \Omega \) not just one in \( \mathbb{R}^2 \). In the present case since we are in a two-dimensional domain it is to be understood that the integral element \( dA \) stands for an integral along a curve. We use this notation consistently. The second lemma is given as inequality (A12) in Payne and Straughan [45].

**Lemma 2.** Let \( \psi \) be the torsion function which satisfies the boundary value problem
\[ \Delta \psi = -1 \quad \text{in} \quad \Omega, \]
\[ \psi = 0 \quad \text{on} \quad \Gamma. \] (12)
Then by the maximum principle \( \psi > 0 \) in \( \Omega \), and for a function \( \Phi \) satisfying equations (10) we have the inequality
\[ 2(\psi \nabla \Phi, \nabla \Phi) + \|\Phi\|^2 \leq \psi_1 \int_\Gamma Q^2 dA, \] (13)
where
\[ \psi_1 = \max_{\Gamma} \left| \frac{\partial \psi}{\partial n} \right|. \]

We next take the inner product on \( L^2(\Omega) \) of equation (11) with \( v_i \) and using the boundary conditions (3), integration by parts, and the Cauchy-Schwarz, arithmetic-geometric mean and Poincaré inequalities, we obtain
\[ \frac{d}{dt} \frac{1}{2} \|v\|^2 + \|\nabla v\|^2 \leq \|T\| \|v\| + \|C\| \|v\| \]
\[ \leq \left(\|T\|^2 + \|C\|^2\right) \frac{\alpha}{2} + \frac{1}{\alpha} \|v\|^2 \]
\[ \leq \left(\|T\|^2 + \|C\|^2\right) \frac{\alpha}{2} + \frac{1}{\alpha \lambda_1} \|\nabla v\|^2, \] (14)
where $\alpha > 0$ is to be chosen and $\lambda_1$ is the first eigenvalue in the membrane problem for $\Omega$. Pick now $\alpha = 2/\lambda_1$ and integrate (13) to find
\[
\|v\|^2 + \int_0^t \|
abla v\|^2 ds \leq \|v_0\|^2 + \frac{2}{\lambda_1} \int_0^t \|T\|^2 ds + \frac{2}{\lambda_1} \int_0^t \|C\|^2 ds.
\] (15)

Thanks to the estimate (9) we may replace the first two terms on the right of (15) by the data term
\[
d_5 = \|v_0\|^2 + \frac{2mT^2}{\lambda_1},
\] (16)

where $m = m(\Omega)$ is the Lebesgue measure of $\Omega$.

Thus, from (15), (16) we may establish
\[
\|v\|^2 + \lambda_1 \int_0^t \|v\|^2 ds \leq d_5 + \frac{2}{\lambda_1} \int_0^t \|C\|^2 ds,
\] (17)

and from this with the aid of Poincaré’s inequality we see that
\[
\|v\|^2 + \lambda_1 \int_0^t \|v\|^2 ds \leq d_5 + \frac{2}{\lambda_1} \int_0^t \|C\|^2 ds.
\] (18)

From this point we introduce the function $H(x, t)$ by
\[
\Delta H = 0 \quad \text{in } \Omega,
H = h \quad \text{on } \Gamma,
\] (19)

where $h(x, t)$ is the boundary data function for $C$ as given in equations (3).

Now, multiply equation (14) by $C - H$ and integrate to find
\[
\begin{align*}
I_1 & = \int_0^t \int_\Omega C_s (C - H) dx ds = -b \int_0^t \int_\Omega v_i C_s (C - H) dx ds \\
& \quad + \int_0^t \int_\Omega \Delta C (C - H) dx ds + L \int_0^t \int_\Omega f(T) (C - H) dx ds \\
& \quad - K \int_0^t \int_\Omega C (C - H) dx ds.
\end{align*}
\] (20)

Denote the five terms in equation (20) by $I_1$–$I_5$ and we now develop these. By integration
\[
I_1 = \frac{a}{2} (\|C\|^2 - \|C_0\|^2) - a \int_\Omega HC dx + a \int_\Omega H_0 C_0 dx \\
+ a \int_0^t \int_\Omega CH_s dx ds.
\] (21)

Using the maximum principle $H$ may be bounded by $h_m$ where
\[
h_m = \max_{\Gamma \times [0, T]} |h|.
\]
Then

\[ I_2 = b \int_0^t \int_0^t v_i C_i H d \nu ds \leq bh_m \sqrt{\int_0^t \| v \|^2 ds \int_0^t \| \nabla C \|^2 ds} \]

\[ \leq bh_m \sqrt{d_5 \frac{d_5}{\lambda_1} + \frac{2}{\lambda_2^2} \int_0^t \| C \|^2 ds \sqrt{\int_0^t \| \nabla C \|^2 ds}}, \tag{22} \]

where in deriving (22) we have employed the Cauchy-Schwarz inequality and estimate (18).

For \( I_3 \) we integrate by parts and use (19) to obtain

\[ I_3 = -\int_0^t \| \nabla C \|^2 ds + \int_0^t (\nabla C, \nabla H) ds \]

\[ = -\int_0^t \| \nabla C \|^2 ds + \int_0^t \oint_{\Gamma} h \frac{\partial H}{\partial n} d Ads. \]

We further employ the Cauchy-Schwarz inequality, and then Lemma 1 to find

\[ I_3 \leq -\int_0^t \| \nabla C \|^2 ds + \sqrt{\frac{c_2}{c_1} \int_0^t \oint_{\Gamma} h^2 d Ads \int_0^t \oint_{\Gamma} |\nabla h|^2 d Ads}. \tag{23} \]

To estimate \( I_4 \) we use the arithmetic-geometric mean inequality with positive constants \( \gamma_1 \) and \( \gamma_2 \) to find

\[ I_4 \leq \frac{L}{2} (\gamma_1^{-1} + \gamma_2^{-1}) \int_0^t \| f \|^2 ds + \frac{L \gamma_1}{2} \int_0^t \| C \|^2 ds + \frac{L \gamma_2}{2} \int_0^t \| H \|^2 ds. \]

Now, \( f \) is known and \( f = f(T) \) and so using bound (9) we may bound \( \int_0^t \| f \|^2 ds \) by data, say \( d_4 \). Further, employ Lemma 2 on the \( \| H \| \) term and we then find

\[ I_4 \leq \frac{L}{2} (\gamma_1^{-1} + \gamma_2^{-1})d_4 + \frac{L \gamma_1}{2} \int_0^t \| C \|^2 ds + \frac{L \gamma_2 \psi_1}{2} \int_0^t \oint_{\Gamma} h^2 d Ads. \tag{24} \]

Finally employing the arithmetic-geometric mean inequality with a constant \( \zeta_3 > 0 \) together with Lemma 2 we obtain

\[ I_5 \leq -\left( K - \frac{K}{2 \zeta_3} \right) \int_0^t \| C \|^2 ds + \frac{\psi_1 K \zeta_3}{2} \int_0^t \oint_{\Gamma} h^2 d Ads. \tag{25} \]

We now group (21)-(25) together in equation (20) and with further use of the arithmetic-
geometric mean inequality we may obtain for positive constants $\lambda$, $\omega_1$ and $\omega_2$ at our disposal,

\[
\frac{a}{2} \|C\|^2 + \int_0^t \|\nabla C\|^2 ds \\
\leq a \|C_0\|^2 + \frac{a}{2} \|H_0\|^2 + \frac{a}{2\lambda} \|H\|^2 + \frac{\omega_1 a \psi_1}{2} \int_0^t \int_{\Gamma} h^2_d dA ds \\
+ \frac{bh_m d_5}{\omega_2 \lambda_1} + \frac{Ld_4}{2}(\gamma_1^{-1} + \gamma_2^{-1}) + \left( \frac{L \gamma_2 \psi_1}{2} + \frac{\psi_1 K \zeta_3}{2} \right) \int_0^t \int_{\Gamma} h^2_d dA ds \\
+ \sqrt{\frac{c_2}{c_1}} \int_0^t \int_{\Gamma} h^2_d dA ds \int_0^t \int_{\Gamma} |\nabla_s h|^2 dA ds \\
+ \frac{a \lambda}{2} \|C\|^2 + \frac{h_m b \omega_2}{2} \int_0^t \|\nabla C\|^2 ds \\
+ \int_0^t \|C\|^2 ds \left( -K + \frac{a}{2\omega_1} + \frac{h_m b}{\omega_2 \lambda_1} + \frac{L \gamma_1}{2} + \frac{K}{2 \zeta_3} \right).
\]

Next, we estimate the $\|H\|$ terms using (19) and Lemma 2, and then pick $\lambda = 1/2$ and $\omega_2 = 1/bh_m$. Define the constant $N$ and the data term $d_6$ by

\[
\frac{Na}{4} = -K + \frac{a}{2\omega_1} + \left( \frac{h_m b}{\lambda_1} \right)^2 + \frac{L \gamma_1}{2} + \frac{K}{2 \zeta_3}
\]

and

\[
d_6 = a \|C_0\|^2 + \frac{3}{2} \psi_1 \int_{\Gamma} h^2_d dA + \frac{bh_m d_5}{\omega_2 \lambda_1} + \frac{Ld_4}{2} \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) \\
+ \frac{\omega_1 a \psi_1}{2} \int_0^t \int_{\Gamma} h^2_d dA ds + \sqrt{\frac{c_2}{c_1}} \int_0^t \int_{\Gamma} h^2_d dA ds \int_0^t \int_{\Gamma} |\nabla_s h|^2 dA ds \\
+ \left( \frac{L \gamma_2 \psi_1}{2} + \frac{\psi_1 K \zeta_3}{2} \right) \int_0^t \int_{\Gamma} h^2_d dA ds.
\]

Then, from (26) one may derive

\[
\frac{a}{4} \|C\|^2 + \frac{1}{2} \int_0^t \|\nabla C\|^2 ds \leq d_6 + \frac{Na}{4} \int_0^t \|C\|^2 ds.
\]

Upon integration of (27) we may then obtain

\[
\int_0^t \|C\|^2 ds \leq d_8(t)
\]

where $d_8$ is the data term

\[
d_8 = \int_0^t e^{N(t-s)} d_6(s) ds.
\]

Employing (28) in (27) one may then derive the a priori bounds

\[
\|C\|^2 \leq Nd_8
\]
and
\[ \int_0^t \| \nabla C \|^2 \, ds \leq \frac{N a d_s}{2}. \] (30)

Inequalities (28) and (29) furnish the necessary \textit{a priori} bounds for \( \| C \| \) and we now proceed to derive a similar estimate involving \( \| C \|_4 \).

Introduce the function \( I(x, t) \) as the solution to
\[ \nabla I = 0 \quad \text{in } \Omega, \]
\[ I = h^3(x, t) \quad \text{on } \Gamma. \] (31)

Now form the identity
\[ a \int_0^t \int \nabla s(C^3 - I) \, dx \, ds = -b \int_0^t \int v_i C_{,i}(C^3 - I) \, dx \, ds \]
\[ + \int_0^t \int \nabla C(C^3 - I) \, dx \, ds + L \int_0^t \int f(T)(C^3 - I) \, dx \, ds \]
\[ - K \int_0^t \int C(C^3 - I) \, dx \, ds. \] (32)

Next, denote the five terms in equation (32) by \( J_1, \ldots, J_5 \). We now proceed in a similar manner to that involving the \( H \) terms, employing a weighted arithmetic-geometric mean inequality, Lemmas 1 and 2, although we now additionally use Young’s inequality on the term involving \( f(T) \). In this way we have
\[ J_1 = \frac{a}{4}(\| C \|^4 - \| C_0 \|^4) - a(I, C) + a(I_0, C_0) + \int_0^t (I, s, C) \, ds. \] (33)

Then,
\[ J_2 = + b \int_0^t \int v_i C_{,i} I \, dx \, ds \leq h^3 b \sqrt{\int_0^t \| v \|^2 \, ds} \int_0^t \| \nabla C \|^2 \, ds. \] (34)

Furthermore,
\[ J_3 = - \frac{3}{4} \int \| \nabla C^2 \|^2 \, ds + \int_0^t \int_{\Gamma} h \frac{\partial I}{\partial n} \, dA \, ds \]
\[ \leq - \frac{3}{4} \int \| \nabla C^2 \|^2 \, ds + \int_0^t \int_{\Gamma} h^2 dA \, \frac{c_2}{c_1} \int_{\Gamma} | \nabla s |^2 \, dA \, ds. \] (35)

For \( J_4 \) and \( J_5 \),
\[ J_4 + J_5 \leq \frac{L}{4 \varepsilon_1^3} \int \| f \|^4 \, ds + \frac{3 L \varepsilon_1^3}{4} \int \| C \|^4 \, ds \]
\[ + \frac{L}{2 \varepsilon_1} \int \| f \|^2 \, ds + \left( \frac{\varepsilon_1 L}{2} + \frac{K}{2 \varepsilon_1} \right) \int \| I \|^2 \, ds \]
\[ + \frac{K \varepsilon_2}{2} \int \| C \|^2 \, ds - K \int \| C \|^4 \, ds. \] (36)
We group (33)-(36) together in (32) to arrive at, after further use of the arithmetic-geometric mean inequality and Lemma 2

\[
\frac{a}{4} \|C\|_4^4 + \frac{3}{4} \int_0^t \|\nabla C\|^2 ds + K \int_0^t \|C\|_4^4 ds
\]

\[
\leq \frac{3a}{4} \|C\|^2 + \frac{a \delta_1}{2} \|C\|^2 + \left( \frac{a}{2 \delta_1} + \frac{a}{2} \right) \psi_1 \int \lambda h^6 dA
\]

\[
+ \sqrt{\psi_1} \int_0^t \|C\| \sqrt{\int \lambda \lambda h^2 dA} ds
\]

\[
+ \sqrt{\frac{c_2}{c_1}} \int_0^t \sqrt{\int \lambda h^2 dA} \int \lambda |\nabla h^3|^2 dA ds
\]

\[
+ \frac{L}{4 \varepsilon_1} \int_0^t \|f\|^4 ds
\]

\[
+ \frac{L}{2 \varepsilon_2} \int_0^t \|f\|^2 ds + \left( \frac{\varepsilon_1 L}{2} + \frac{K}{2 \varepsilon_2} \right) \psi_1 \int \lambda h^6 dAds
\]

\[
+ \frac{K \varepsilon_2}{2} \int_0^t \|C\|^2 ds + h^3 \alpha \sqrt{\int \lambda \lambda \lambda C\|^2 ds} \sqrt{\frac{d_5}{\lambda_1} + \frac{2}{\lambda_2}} \int_0^t \|C\|^2 ds
\]

\[
\frac{3L \varepsilon^{4/3}}{4} \int_0^t \|C\|_4^4 ds.
\]

We now select \( \varepsilon = (2K/3L)^{3/4} \). This removes the last term on the right of (37). Now, using the a priori bound on \( T \), and the a priori estimates (28)-(30), we see that what remains on the right hand side of inequality (37) may be bounded by data.

Let us denote this term by \( \frac{a}{2} d_9 \). Then

\[
\frac{a}{4} \|C\|_4^4 + \frac{3}{4} \int_0^t \|\nabla C\|^2 ds + K \int_0^t \|C\|_4^4 ds \leq \frac{a}{4} d_9(t)
\]

Inequality (39) thus yields an a priori bound for \( \|C\|_4 \), for \( \int_0^t \|C\|_4^4 ds \), and for the term \( \int_0^t \|\nabla C\|^2 ds \). Let us also recollect that using (17) and (28) we have the following a priori bound

\[
\|v\|^2 + \int_0^t \|\nabla v\|^2 ds \leq d_{10}.
\]

where

\[ d_{10} = d_5 + \frac{2}{\lambda_1} d_8. \]
4 Continuous dependence on the reaction coefficients

We now return to the boundary-initial value problem for the difference equations, (5)-(7). In the interests of clarity we recall the specific \textit{a priori} bounds we require in this section, namely,

\[
\int_0^t \|\nabla v\|^2 ds \leq d_{10}
\]

\[
\|C\|^4 \leq d_9
\]

\[
\|C\|^2 \leq N d_8
\]

(40)

We also require a Sobolev inequality in two-dimensions. This may be written in the form, see Payne [46],

\[
\int_\Omega |w|^4 dx \leq \Omega_1 \int_\Omega |w|^2 dx \int_\Omega |\nabla w|^2 dx.
\]

(41)

where \(\Omega_1\) is a positive constant. An estimate for \(\Omega_1\) is given by Payne [46], in Lemma 1, p. 132, as \(\Omega_1 = \frac{1}{2}\).

\textbf{Theorem.} Let \(\chi \equiv (v_i, p, T, C)\) be a solution to the boundary-initial value problem \((\text{1}) - (\text{3})\) in \(\Omega \times (0, \mathcal{T})\) for some \(\mathcal{T} < \infty\). Then the solution \(\chi\) depends continuously on the reaction coefficients \(L\) and \(K\) explicitly in \(L^2(\Omega)\) in the sense that the difference solution \((w_i, \pi, \theta, \phi)\) given in \((4)\) satisfies the inequality

\[
\|w(t)\|^2 + \|\theta(t)\|^2 + \|\phi(t)\|^2 \leq f_1(t)l^2 + f_2(t)k^2, \quad t \in (0, \mathcal{T}),
\]

where \(l, k\) are defined in \((4)\) and \(f_1\) and \(f_2\) are coefficients which depend only on \(\Omega, \mathcal{T}\) and the data functions \(v_0\), \(T_0\), \(C_0\), \(g\) and \(h\).

\textbf{Proof.} Multiply equation \((5)_1\) by \(w_i\) and integrate over \(\Omega\) to see that

\[
\frac{d}{dt} \frac{1}{2} \|w\|^2 = -\int_\Omega u_{i,j}w_iw_j dx - \|\nabla w\|^2 + (g, \theta, w_i) - (h, \phi, w_i)
\]

\[
\leq \|\nabla u\| \|w\|^2 - \|\nabla w\|^2 + \|\theta\| \|w\| + \|\phi\| \|w\|
\]

\[
\leq \sqrt{\Omega_1} \|\nabla u\| \|w\| - \|\nabla w\|^2 + \|\theta\| \|w\| + \|\phi\| \|w\|,
\]

where the Cauchy-Schwarz and Sobolev inequalities have been employed. Now employ the arithmetic-geometric mean inequality for \(\beta > 0\) to find

\[
\frac{d}{dt} \frac{1}{2} \|w\|^2 \leq \frac{\beta}{2} \|w\|^2 \|\nabla u\|^2 + \frac{\Omega_1}{2\beta} \|\nabla w\|^2 + \frac{\alpha}{2} \|\theta\|^2 + \frac{\gamma}{2} \|\phi\|^2 + \|w\|^2 \left( \frac{1}{2\alpha} + \frac{1}{2\gamma} \right).
\]

Pick now \(\beta = \Omega_1\) and employ the Poincaré inequality on the last term with \(\gamma = \alpha\) to obtain

\[
\frac{d}{dt} \frac{1}{2} \|w\|^2 + \frac{1}{2} \|\nabla w\|^2 \leq \frac{\Omega_1}{2} \|w\|^2 \|\nabla u\|^2 + \frac{\alpha}{2} (\|\theta\|^2 + \|\phi\|^2) + \frac{1}{\alpha \lambda_1} \|\nabla w\|^2.
\]
Now select $\alpha = 4/\lambda_1$ and then we find
\[
\frac{d}{dt} \|w\|^2 + 2 \|\nabla w\|^2 \leq \Omega_1 \|\nabla u\|^2 \|w\|^2 + \frac{4}{\lambda_1} (\|\theta\|^2 + \|\phi\|^2). \tag{42}
\]

Upon multiplying equation (53) by $\theta$ and integrating over $\Omega$ we derive
\[
\frac{d}{dt} \frac{1}{2} \|\theta\|^2 = - \|\nabla \theta\|^2 + \int_{\Omega} w_i T_i \theta_j \, dx \\
\leq - \|\nabla \theta\|^2 + T_m \|w\| \|\nabla \theta\| \\
\leq \frac{T_m^2}{4} \|w\|^2. \tag{43}
\]

The next stage requires us to multiply equation (54) by $\phi$ and integrate over $\Omega$ to find
\[
\frac{a}{2} \frac{d}{dt} \|\phi\|^2 = b \int_{\Omega} w_i C_1 \phi_j \, dx - \|\nabla \phi\|^2 - K_1 \|\phi\|^2 - k(C_2, \phi) \\
+ L_1 (f(T^1) - f(T^2), \phi) + l(f(T^2), \phi). \tag{44}
\]

Using Lagrange’s theorem we know $f(T^1) - f(T^2) = \theta f'(\xi)$ for some $\xi \in (T_1, T_2)$. Then since $T_m$ is a bound for $|T|$ and $f \in C^1$, we know $|f'(\xi)| \leq a_1$, $|f(T^2)| \leq d_2$ for data terms $d_1$ and $d_2$. Thus, the last two terms of (44) may be bounded by
\[
L_1 d_1 \left( \frac{\|\theta\|^2}{2\alpha} + \frac{\alpha}{2} \|\phi\|^2 \right) + L \left( \frac{2}{2\beta} \|f(T^2)\|^2 + \frac{\beta}{2} \|\phi\|^2 \right), \tag{45}
\]
for $\alpha, \beta > 0$ to be selected. Likewise for $\gamma > 0,$
\[
- k(C_2, \phi) \leq k^2 \frac{\|C_2\|^2}{2\gamma} + \frac{\gamma}{2} \|\phi\|^2. \tag{46}
\]

For the cubic term we have
\[
b \int_{\Omega} w_i C_1 \phi_j \, dx \leq \frac{b}{2\zeta} \int_{\Omega} |w|^2 C^2 \, dx + \frac{\zeta b}{2} \|\nabla \phi\|^2 \\
\leq \frac{b}{2\zeta} \|w\|^2 \|C_1\|^4 + \frac{\zeta b}{2} \|\nabla \phi\|^2 \\
\leq \frac{b\Omega_1}{4\zeta \mu} \|C_1\|^4 \|w\|^2 + \frac{b\mu}{4\zeta} \|\nabla w\|^2 + \frac{\zeta b}{2} \|\nabla \phi\|^2, \tag{47}
\]
for $\zeta, \mu > 0$ to be chosen.

We employ (45)-(47) in (44), pick $\beta = \gamma = K_1/2$, $\alpha = K_1/L_1 d_1$ and pick $\zeta = 2/b$. Then we obtain
\[
\frac{a}{2} \frac{d}{dt} \|\phi\|^2 \leq \frac{L_1^2 d_1^2}{2K_1} \|\theta\|^2 + \frac{\|f(T^2)\|^2}{K_1} - \frac{\Omega_1 b^2}{8} \|\nabla \phi\|^2 + \frac{b^2 \mu}{8} \|\nabla w\|^2. \tag{48}
\]

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Thus, from (42), (43) and (48) we may obtain
\[
\frac{d}{dt}(\|w(t)\|^2 + \|\theta(t)\|^2 + \|\phi(t)\|^2) + \frac{1}{2} \|\nabla w\|^2 \\
\leq \left(\frac{T_m^2}{2} + \frac{\Omega_1 b^2}{4a\mu} |C_1|^4 + \Omega_1 \|\nabla u\|^2\right) \\
+ \frac{b^2\mu}{4a} \|\nabla w\|^2 + \left(\frac{4}{\lambda_1} + \frac{L_1^2d_1^2}{aK_1}\right) \|\theta\|^2 + \frac{4}{\lambda_1} \|\phi\|^2, \\
+ \frac{2}{aK_1} \|f(T^2)\|^2 l^2 + \frac{2\|C_2\|^2}{aK_1} k^2.
\] (49)

Use the bounds (40) on \(|C_1|^4\) and \(|C_2|\), choose \(\mu = \frac{2a}{b^2}\), and set
\[
M = \max \left\{ \frac{\Omega_1 b^4d_0}{8a^2} + \frac{T_m^2}{2}, \frac{4}{\lambda_1} + \frac{L_1^2d_1^2}{aK_1} \right\}.
\]

The term \(\|f(T^2)\|\) is \textit{a priori} bounded and so we set
\[
\alpha_1 = \frac{2}{aK_1} \|f(T_2)\|^2, \quad \alpha_2 = \frac{2}{aK_1} \|C_2\|^2.
\]

Then, with \(F(t) = \|w(t)\|^2 + \|\theta(t)\|^2 + \|\phi(t)\|^2\) we obtain from (49)
\[
F' - (\Omega_1 \|\nabla u\|^2 + M) F \leq \alpha_1 l^2 + \alpha_2 k^2.
\] (50)

We integrate (51) via an integrating factor and we use (40) to now see that
\[
\|w(t)\|^2 + \|\theta(t)\|^2 + \|\phi(t)\|^2 \leq R(t)(\alpha_1 l^2 + \alpha_2 k^2),
\] (51)

where \(R(t)\) is the data term
\[
R(t) = \int_0^t \exp \left[ M(t-s) + \Omega_1 \int_s^t d_{10}(y) dy \right] ds.
\] (52)

Inequality (51) is the sought after result and establishes continuous dependence on the reaction coefficients.

\textbf{Remark.} We have established continuous dependence when \(\Omega\) is a bounded domain in \(\mathbb{R}^2\). The proof relies on the Sobolev inequality (41). This inequality does not hold when \(\Omega \subset \mathbb{R}^3\) and instead one must employ an alternative Sobolev inequality of form
\[
\|w\|^4 \leq \tilde{\Omega}_1 \|w\| \|\nabla w\|^3.
\]

In this case we cannot establish continuous dependence by methods analogous to those employed here. Evidently we may proceed along similar lines as Payne and Straughan [47] for the three-dimensional problem but only for \(t\) restricted. We do not give details of such a local result since we are interested in a truly global \textit{a priori} result.
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