The most general geometrical scenario in which the brane-world program can be implemented is investigated. The basic requirement is that it should be consistent with the confinement of gauge interaction, the existence of quantum states and the embedding in a bulk with arbitrary dimensions, signature and topology. It is found that the embedding equations are compatible with a wide class of Lagrangians, starting with a modified Einstein-Hilbert Lagrangian as the simplest one, provided minimal boundaries are added to the bulk. A non-trivial canonical structure is derived, suggesting a canonical quantization of the brane-world geometry relative to the extra dimensions, where the quantum states are set in correspondence with high frequency gravitational waves. It is shown that in the cases of at least six dimensions, there exists a confined gauge field included in the embedding structure. The size of extra dimensions compatible with the embedding is calculated and found to be different from the one derived with product topology.

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I. INTRODUCTION

The brane-worlds program proposes a solution of the hierarchy problem at the TeV scale, assuming that the usual matter and standard gauge interactions remain confined to a four-dimensional space-time embedded in a higher dimensional bulk, while the extra dimensions are probed by gravitons. The size of the extra dimensions is of the order of tenths of millimeter, as derived from the effective Planck scale in four dimensions and a fundamental scale in the bulk set at the TeV [1,2].

Brane-worlds inherits its name and some basic ideas from Horava-Witten's M-theory, where the standard model of interactions contained in the $E_8 \times E_8$ heterotic string theory is also confined to a 3-brane, but gravitons propagate in the 11-dimensional bulk [3]. However, the use of large extra dimensions and confined gauge interactions in higher dimensional models has been considered earlier, under distinct motivations [4]. Also, the idea of a space-time embedded and evolving in a higher dimensional space has been proposed in various related applications, such as the generation of internal symmetries, quantum gravity, alternative Kaluza-Klein theories and cosmology [5]. Recent problems and reviews can be found in this regard.

In this note we attempt to answer some basic questions which remain open, due in part to the fact that most of the recent developments are specific to particular models. For example this has given the wrong impression that the brane-world program is necessarily a five-dimensional theory based on the $AdS_5$ bulk, or that it is a higher dimensional theory defined on a bulk with a product topology. Thus we ask, what is the most general geometrical scenario in which such program can be developed? What are its essential, model independent, postulates? How are TeV gravitons to be defined and how do we confine the gauge interactions? Above all, what is the brane-world action principle?

As the program stands today and leaving aside model dependent properties such as the $AdS_5$ bulk, warp factors and junction conditions, we may identify four basic principles. One of phenomenological nature sets the fundamental scale of interactions at the TeV. The other three are of theoretical nature, asserting that the extra dimensions are probed by TeV gravitons, that the standard gauge interactions remain confined to the four dimensional space-time and that this space-time is embedded in a higher dimensional bulk space.

Our purpose is to avoid the limitations imposed by the hypersurface condition or by the use of specific topologies, studying the compatibility between those principles in the most general situation, assuming that the bulk has an arbitrary number of dimensions, arbitrary signature and topology. Thus, brane-worlds are considered here in the broader sense, characterized only by the above basic principles. That is, as dynamically embedded submanifolds, such that they retain the gauge interactions confined within and that they exhibit some sort of quantum fluctuations.

One of our results shows that under those very general conditions the Einstein-Hilbert action arises naturally as the simplest action derived from the embedding equations. We will see that the total divergence term...
can be removed before the application of the variational principle, resulting in a considerable simplification of the dynamics.

Although very little has been said about the symmetries of the extra dimensions, we have not found any motivation to mix this group with the space-time diffeomorphisms. Assuming that these are separate symmetries, we are able to derive a canonical formulation of brane-worlds and sketch a model of quantum theory.

Another result shows that when the number of extra dimensions is greater than one and that they admit an isometry group, the embedding equations contain a confined gauge-like potential, whose gauge group is defined by that symmetry.

Finally, we find that the size of the extra dimensions that can be probed by gravitons, compatible with the embedding, requires the existence of minimal boundaries. Using these boundaries we find that the size differs slightly from that estimated in [4]. However, for small incursions in a region where the embedding is smooth the difference is negligible.

The paper is organized as follows: In section II brane-worlds are described from the point of view of geometric perturbations, where each perturbation remains an embedded submanifold. The Lagrangian for the higher dimensional space geometry is derived from Gauss’ equation, without appeal to any particular symmetry in section III. A non-constrained canonical structure is also derived. In section IV we discuss the corresponding quantum description of a brane-world and the induced topological changes. Section V shows the confined gauge field included in the embedding and its implications to the number of the extra dimensions. The size of these dimensions compatible with the embedding is discussed at the end.

II. GEOMETRIC PERTURBATIONS AND STABILITY

The electromagnetic, weak and strong interactions together with the confined matter produce tensions, pressures and energy in the brane-world, which in turn cause deformations on its geometry. Therefore, a natural approach to brane-world perturbative analysis is to start with the perturbations of the source, and then find the consequent perturbation of the geometry. To do this, it is common practice to rely on junction conditions relating the energy-momentum tensor of the source to the extrinsic curvature. Besides being non-unique, it has been noted that these conditions are difficult to solve together with the other equations [3]. A more general procedure is to follow on the opposite direction, starting with the perturbations of the geometry and if desired, latter on find the perturbations of the confined source [5]. This has the advantage of requiring a simpler dynamics and it can be applied to any number of dimensions. Actually, by use of Nash’s perturbative embedding procedure, we shall see that the brane-worlds may be described as a family of stable perturbations of a given locally embedded background space-time.

The local embedding is constructed in a neighborhood of each point of the brane-world, defining an embedding bundle whose total space consists of all embedding spaces. Then, the embedding equations are derived from the curvature tensor of each local embedding space, written in the Gaussian frame defined by the embedded submanifold and the normal vectors [10]. From the point of view of brane-worlds, this amounts to have a dynamic bulk whose geometry depends on that of the brane-world, as opposed to static or rigid embedding.

Perturbations of embedded submanifolds with respect to a transverse direction has been used as a way to generate embedding theorems along the following lines [11]: Consider background $V_n$ with metric $\bar{g}_{ij}$ isometrically embedded in $V_D$, by a map $\bar{X} : V_n \rightarrow V_D$ such that

\[
\bar{X}_j^\mu \bar{X}_j^\nu G_{\mu\nu} = \bar{g}_{ij}, \quad \bar{X}_i^\mu \bar{X}_j^\nu G_{\mu\nu} = 0, \quad \bar{g}_{AB} G_{\mu\nu} = g_{AB} \tag{1}
\]

where we have denoted by $\bar{g}_{ij}$ the metric of $V_D$ in arbitrary coordinates and $g_{AB}$ denotes the components of the metric of the complementary space orthogonal to $V_n$, in the basis $\{\bar{g}_{ij}\}$. The perturbations of $V_n$ with respect to a small parameter $s$ along an arbitrary transverse direction $\zeta$ is given by

\[
Z^\mu(x^i, s) = \bar{X}_i^\mu + s \bar{X}_i^\nu \bar{X}_j^\mu = \bar{X}_i^\mu + s(\zeta, \bar{X})^\mu \tag{2}
\]

The presence of components of $\zeta$ tangent to $V_n$ is a cause for concern because it can induce undesirable coordinate gauges. In geometric perturbations it is possible to obtain coordinate gauge independency simply by selecting the $\zeta^\mu$ to be orthogonal to the background. In this case, we obtain the perturbations of the embedding map along a single orthogonal extra direction $\bar{\eta}_A$ as

\[
Z^\mu(x, s^A) = \bar{X}_i^\mu(x) + s^A \bar{\eta}_A(x). \tag{3}
\]

Since the vectors $\bar{\eta}_A$ are independent and they depend only of $x^i$, it also follows that

\[
\eta^\mu_A(x^i) = \bar{\eta}_A^\mu + s^B [\bar{\eta}_B, \bar{\eta}_A]^\mu = \bar{\eta}_A^\mu \tag{4}
\]

However, it is not obvious that this perturbation represents a new submanifold or even that it is embedded in

* All Greek indices run from 1 to $D$. Small case Latin indices run from 1 to $n$ and capital Latin indices run from $n+1$ to $D$. The covariant derivative with respect to the metric of the higher dimensional manifold is denoted by a semicolon and $\xi_i^\mu = \xi_i^\mu \bar{X}_i^\mu$ denotes its projection over $V_n$. The curvatures of $V_D$ are distinguished from that of $V_n$ by a calligraphic $\mathcal{R}$. Since we have not fixed the signature of $V_D$ the notation $\mathcal{G} = |\det(\mathcal{G}_{\alpha\beta})|$ is used throughout.
the same $V_D$. For example, the Schwarzschild space-time is known to be isometrically embedded in a six dimensional flat space with metric signature $(4,2)$. Its maximal analytic extension, the Kruskal space-time is also embedded in a six dimensional space, but with metric signature $(5,1)$ [10]. Now, the Kruskal space-time may be seen as a perturbation of the Schwarzschild space-time such that it becomes geodesically complete. Although the latter is a subset of the former, they do not fit into the same flat bulk, unless the signature of the six dimensional space is allowed to change. Therefore, in the general case the geometry and topology of the bulk should not be fixed.

The integrability conditions for the perturbed geometry are the Gauss, Codazzi and Ricci equations, respectively

$$ \begin{align*}
R_{ijkl} &= 2g^{MN}k_{[kM}k_{l]N} + R_{\mu
u\rho\sigma}Z^\mu_iZ^\nu_jZ^\rho_kZ^\sigma_l, \\
k_{ij|\tilde{A}|k} &= g^{MN}A_{[jM\tilde{A}k]iN} + R_{\mu
u\rho\sigma}Z^\rho_i\eta^\nu_jZ^\sigma_k\eta^\mu_l, \\
2A_{[iAB;k]} &= -2g^{MN}A_{[iMANkB]} - R_{\mu
u\rho\sigma}Z^\rho_j\eta^\nu_iZ^\sigma_k\eta^\mu_l
\end{align*} $$

The first two equations have been extensively applied to the analysis of brane-worlds in five dimensions [8], but as a whole they have not been appreciated in the case $D \geq 6$. Assuming that (6) hold true for all perturbations, the result is an $N$-parameter family of embedded submanifolds characterized by the parameters $s^A$, suitable for a perturbative description of the brane-worlds, after implementing the confinement and the quantization.

The perturbation $\tilde{\eta}_A$ and $\tilde{\eta}_B$ induce a perturbation of the metric $g_{ij}$ along those dimensions which can be written in general coordinates as

$$ g_{ij} = \bar{g}_{ij} + \chi_{ij}(x^i, s^A) $$

In particular, the linear perturbation obtained from the expansion in $s^A$ are

$$ g_{ij} = \bar{g}_{ij} + \epsilon^A\gamma_{ijA}(x^i) $$

where $\epsilon^A$ is a small expansion parameter. Applying this to Einstein’s equations under the de Donder gauge, we obtain the linear wave equation relative to the extra dimensions, where the back reaction of the background geometry must be taken into consideration. The wave equation is written in the most general form as

$$ \square_{kl}^{ij}\Psi_{ijA}(x,s) = 8\pi GT_{klA} $$

where $\Psi_{ijA} = \gamma_{ijA} - 1/2\gamma_{Aij}\bar{g}_{ij}$, $\gamma_A = \bar{g}^{mn}\gamma_{mn}$ and where (denoting $\nabla_k\xi = \xi_k$ for clarity)

$$ \square_{kl}^{ij} = \bar{g}^{ij}\nabla_k\nabla_l + 2\bar{R}_{kl}^{ij} + 2\bar{R}_{(k}^{ij}(\xi_{l)} $$

is the generalized (de Rahm) wave operator, containing curvature terms of the background geometry.

Assuming that the wave solutions of (6) correspond to the quantum modes of the brane-world geometry, they must represent gravitational waves of high frequency. That is, with a small wavelength $\lambda$ as compared with a local invariant characteristic length $\ell$ of the brane-world geometry, the curvature radius, which plays a relevant role on the determination of the classical modes. This radius has been characterized as $\text{inf} |R_{ijkl}|$ [2], but in brane-worlds it must be expressed in terms of a distance in the extra dimensions. To find this we follow the same definitions as in the geometry of surfaces. Consider the embedding equations of the perturbed geometry written in the particular Gaussian frame defined by the embedded geometry and the $\eta^A$'s

$$ Z^\mu_iZ^\nu_j\bar{g}_{\mu\nu} = \bar{g}_{ij}, \quad Z^\mu_i\eta^\nu_A\bar{g}_{\mu\nu} = \bar{g}_{ijA}, \quad \eta^A_{ij}\bar{g}_{\mu\nu} = g_{AB} $$

where $g_{ij} = s^MA_{iMA}$ and

$$ A_{iAB} = \eta^B_{ij}\eta^A_{ij}\bar{g}_{\mu\nu} = \eta^A_{ij}\eta^B_{ij}\bar{g}_{\mu\nu} = \bar{A}_{iAB} $$

Replacing (3) in (8), we may express the perturbed metric in the Gaussian frame defined by the embedding as

$$ g_{ij} = \bar{g}_{ij} - 2s^A\bar{k}_{ijA} + s^A\bar{B}(\bar{g}^{mn}\bar{k}_{imA}\bar{k}_{jnB} + g^{MN}A_{iMA}A_{jNB}) $$

and the perturbed extrinsic curvature

$$ k_{ijA} = -2\bar{Z}^\nu_i\eta^\nu_A\bar{g}_{\mu\nu} = \bar{k}_{ijA} + s^B(\bar{g}^{mn}\bar{k}_{imA}\bar{k}_{jnB} + g^{MN}A_{iMA}A_{jNB}) $$

The curvature radii of the background $\bar{V}_n$ are the $n \times N$ values $\ell_n^A$ of $s^A$, one for each principal direction $dx^i$ and for each normal $\eta_A$, satisfying the homogeneous equation

$$ (\bar{g}_{ij} - s^A\bar{k}_{ijA})dx^j = 0, \quad A \text{ fixed}. $$

The single curvature radius $\ell$ is the smallest of these solutions, corresponding to the direction in which the brane-world deviates more sharply from the tangent plane. Considering all contributions of $\ell_n^A$, in such a way that the smaller solution of (13) prevails, the curvature radius may be expressed as

$$ \frac{1}{\ell} = \sqrt{\bar{g}_{ij}g_{AB}} \frac{1}{\ell_A^i} $$

Since (13) can also be written as

$$ g_{ij} = \bar{g}^{mn}(\bar{g}_{im} - s^A\bar{k}_{imA})(\bar{g}_{jn} - s^B\bar{k}_{jnB}) + s^A\bar{B}\bar{g}^{MN}A_{iMA}A_{jNB} $$

it follows that the components

$$ \bar{g}_{ij} = \bar{g}^{mn}(\bar{g}_{im} - s^A\bar{k}_{imA})(\bar{g}_{jn} - s^B\bar{k}_{jnB}) $$

become singular at the solutions of (12). Therefore, $g_{ij}$ and consequently, the metric of the bulk written in matrix form

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3
\[ G_{\alpha\beta} = \left( \tilde{g}_{ij} + s^A s^B g^{MN} A_{iMA} A_{jNB} g_{AB} \right) \]  
(15)

becomes also singular at the points determined by those solutions. Of course, this is not real singularity of \( V_D \) but a property of the Gaussian system defined by the brane-world \( V_n \). However, this singularity breaks the continuity and regularity of the integrability equations \( \tilde{r} \) which are constructed with this system. Therefore, it represents also a singularity for the wave equation \( \tilde{r} \) which depends on the background geometry. In short, the curvature radius \( \ell \) sets a local limit for the region in the bulk accessed by the gravitons associated with those high frequency waves.

### III. FIELD EQUATIONS FOR BRANE-WORLDS

Among the three independent variables \( g_{ij} \), \( k_{ijA} \) and \( A_{iAB} \) in \( \tilde{r} \), only \( g_{ij} \) is normally assumed to propagate along the extra dimensions. However, comparing \( \tilde{r} \) with the derivative of \( \tilde{r} \) we obtain the generalized York’s relation

\[ \frac{\partial g_{ij}}{\partial s^A} = -2k_{ijA} \]  
(16)

which shows that the extrinsic curvature also propagates in the bulk, as a consequence of the metric propagation. Finally, from \( \tilde{r} \) it follows that the third variable \( A_{iAB} \) does not propagate.

Since we are not using any particular metric ansatz, we must follow a general procedure to determine the variational principle compatible with \( \tilde{r} \). For that purpose we note that

\[ g^{ij} Z^i_i Z^j_j = G^{\mu\nu} - g^{AB} \eta_A^{\mu} \eta_B^{\nu} \]  
(17)

Using this, the contractions of the first equation \( \tilde{r} \) gives the Ricci scalar of the perturbed geometry

\[ R = (K^2 - h^2) - \frac{1}{2} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) Z^\mu_i Z^\nu_j + \frac{1}{2} \left( K^2 - h^2 \right) g_{ij} \]  
(18)

where \( K^2 = k_{ijA} k^{ijA} \) corresponds to the Gaussian curvature, \( h_A = g^{ij} k_{ijA} \) is the mean curvature for \( \eta_A \) and \( h^2 = g^{AB} h_A h_B \). In the Gaussian frame of the embedding we may set \( g_{AB} = \epsilon A \delta_{AB}, \epsilon = \pm 1 \), so that the last term in \( \tilde{r} \) vanishes and

\[ g^{AB} R_{\mu\nu} \eta_A^\mu \eta_B^\nu = -g^{AB} \frac{\partial h_A}{\partial s^B} + K^2 \]  

Since this is a tensor equation, it holds in any frame and \( \tilde{r} \) reduces to

\[ R = R - \left( K^2 + h^2 \right) - 2h^A \]  

where the divergence can be discarded under a volume integration on \( s^A \), provided the mean curvatures \( h_A \) vanish at given boundaries. This is automatically satisfied when we assume that these boundaries are minimal submanifolds. These fixed boundaries replace the dynamical boundaries used in \( \tilde{r} \). With this, after discarding this divergence we obtain the Lagrangian for the brane-world geometry

\[ L(g) = R \sqrt{g} = R \sqrt{g} + (K^2 + h^2) \sqrt{g} \]  
(19)

Consequently, the dynamics of the gravitational field in brane-worlds follows from the Einstein-Hilbert dynamics of the bulk, modified by the presence of the extrinsic curvature term.

We may also construct other scalar invariants with contractions of various curvature terms and their powers to obtain higher derivative Lagrangians, or even an infinite series leading to the Nambu-Goto action. The modified Einstein-Hilbert Lagrangian \( \tilde{r} \) is just the simplest one that can be derived from the embedding equations \( \tilde{r} \), without further assumptions.

The use of a variational principle permits us to introduce an independent source in the brane-world. Thus we add to \( \tilde{r} \) the Lagrangian of the confined matter \( L_m \). Then, the field equations with respect to the metric \( g_{ij} \), with the confined matter represented by \( T^m_{ij} \) are

\[ R_{ij} - \frac{1}{2} R g_{ij} = 8 \pi G T^m_{ij} + (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) Z^\mu_i Z^\nu_j + Q_{ij} + S_{ij} \]  
(20)

where we have denoted

\[ Q_{ij} = g^{AB} (h^m_A b^m_{jB} - h_A b_{ijB}) - \frac{1}{2} (K^2 - h^2) g_{ij} \]  
(21)

and

\[ S_{ij} = g^{AB} R_{\mu\nu} \eta_A^\mu \eta_B^\nu g_{ij} - \frac{1}{2} g^{AB} R_{\mu\nu\rho\sigma} \eta_A^\mu \eta_B^\nu Z^\rho_i Z^\sigma_j \]  
(22)

The value of \( (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) \) depends on the definition of the geometry of the bulk, generally taken as a solution of the higher dimensional Einstein’s equations, with a bulk source represented by the energy momentum tensor \( T^B_{\mu\nu} \). In this case, due to presence of the factor \( Z^\mu_i Z^\nu_j \), at the end this bulk matter is projected onto the brane-world in accordance with the confinement hypothesis.

In some models with a particularly chosen bulk geometry, the last term \( S_{ij} \) vanishes. On the other hand the term \( Q_{ij} \) depends essentially on the extrinsic curvature and it does not necessarily vanish, even if the bulk is flat. Therefore, this term may effectively modify the usual Einstein dynamics. We will discuss its meaning in a cosmological application in a subsequent paper.

The solutions of \( \tilde{r} \) describe the gravitational field in the brane-world, showing the additional terms resulting from the embedding. This must be complemented by the description of the evolution of the brane-world geometry along the extra variables. For this purpose, we need a canonical structure compatible with \( \tilde{r} \) and compatible with the perturbative analysis of section II.
Unlike general relativity, the coordinates on the extra dimensions behave as the propagation parameters for the embedded geometry, so that the phase space has to be defined with respect to these parameters. Thus, the momentum conjugated to \( \mathcal{G}_{ij} \), relative to the extra dimensions \( \eta^A \), is given by

\[
p^{ij}(A) = \frac{\partial L}{\partial \dot{g}^{ij}}
\]

and in particular, using (16) we obtain the components

\[
p^{ij}(A) = -(k^{ij} + h_{AB}g^{ij})\sqrt{\mathcal{G}} (23)
\]

which corresponds to the propagation of \( g^{ij} \) along \( \eta^A \).

The confinement hypothesis implies that any gauge fields and matter sources which could eventually be contained in the extra components of the bulk metric should not propagate. Therefore, consistently with this we add the momentum constraints

\[
p^{iA}(B) = -2\frac{\partial R_{ij}^A \eta^{jA}_{ij}}{\partial \dot{p}^{iA}} \sqrt{\mathcal{G}} = 0, (24)
\]

\[
p^{AB}(C) = -2\frac{\partial R_{ij}^A \eta^{jA}_{ij}}{\partial \dot{p}^{AB}} \sqrt{\mathcal{G}} = 0 (25)
\]

These constraints are also consistent with our previous choice of orthogonal perturbations given by (3) and (4).

Using (24) and (25), the Hamiltonian corresponding to the displacement along a single direction \( \eta^A \), follows from the Legendre transformation

\[
\mathcal{H}_A(g,p) = p^{ij}(A)g_{ij,A} - \mathcal{L} = -R\sqrt{\mathcal{G}} - \frac{1}{\mathcal{G}} \left( p^2_A - p_{ij}(A)p^{ij} \right) (26)
\]

where we have denoted \( p_A = g_{ij}\dot{p}^{ij}(A) \). Hamilton’s equations relative to the extra coordinate \( \delta^A \) are

\[
\frac{dg_{ij}}{d\delta^A} = \delta H_A \delta p^{ij}(A) = -\frac{2}{\sqrt{\mathcal{G}}} \left( g_{ij} \dot{p}_{ij} - \frac{p_A}{n+1} \right), (27)
\]

\[
\frac{dp^{ij}(A)}{d\delta^A} = -\frac{\delta H_A}{\delta g_{ij}} = (R^{ij} - \frac{1}{2} \mathcal{G}^{ij})\sqrt{\mathcal{G}} + \frac{2p_{ij}p^{ij}(A)}{n+1} + \frac{1}{\sqrt{\mathcal{G}}} \left( \frac{p_A}{n+1} + \frac{1}{2} p_{mn}(A)p^{mn}(A) g^{ij} \right) (28)
\]

The first of these equations is the same as York’s relation expressed in terms of \( \dot{p}^{ij}(A) \), giving the propagation of the metric in terms of the extrinsic curvature. The second equation expresses the propagation of the extrinsic curvature expressed in terms of \( \dot{p}^{ij}(A) \).

We conclude that the Hamiltonian dynamics expressed by (27) through (28) describe the same a motion which equivalent to the one given by the perturbative analysis in section II.

IV. QUANTUM STATES

The compactification of the extra dimensions down to Planck’s length was introduced to make Kaluza-Klein theory compatible with quantum mechanics, where the normal modes of the harmonic expansion with respect to the internal parameters were set in correspondence with quantum modes \([13]\). As we know, in that theory the strong curvature of the internal space contributes to large mass fermion states, which are not observed at the electroweak scale. If the extra dimensions were large or non compact, then we would obtain massless or light Kaluza-Klein modes, which could be observed at that energy scale. However, it is not clear that these modes would still keep a correspondence with quantum states.

Contrasting with the Kaluza-Klein program, in brane-worlds only the gravitational field is expanded along the extra dimensions, with modes associated with gravitational waves \([4]\). Then the fermion chirality problem would not arise but the metric expansion should hold independently of the fact that these dimensions are large, compact or not. In other words, the quantum correspondence must be independent of the bulk topology but it must be compatible with the embedding. As remarked before, the gravitational waves associated with the quantum fluctuations of the geometry make sense only in the high frequency limit, which depend on the local geometry of the background, and not on the topology of the bulk.

In the previous sections we have seen that the same perturbations that lead to the wave equation also lead to a canonical formulation derived from the Hamiltonians [23]. Consequently, the quantum states associated with the high frequency waves can be, at least in principle, defined by the canonical quantization defined by those Hamiltonians with respect to the extra dimensions. The procedure would be similar to that of the ADM formulation of general relativity, with an important difference: Since the extra dimensions do not transform under the same diffeomorphism group of the brane-world, the Poisson bracket structure does not suffer the same propagation problem. Instead, it behaves differently under the brane-world diffeomorphisms and under the transformations of the extra coordinates. Therefore, the evolution of a functional \( \mathcal{F} \) in phase space relative to a single extra dimension \( \eta_A \), given in terms of Poisson brackets as

\[
[\mathcal{F}, \mathcal{H}_A] = \delta \mathcal{F} \delta \mathcal{H}_A \delta g_{ij} - \frac{\delta \mathcal{F}}{\delta \mathcal{H}_A} \delta g_{ij} = \frac{\delta \mathcal{F}}{\delta \eta^A}
\]

propagates covariantly along the evolution of the system. Thus, a canonical quantization may be defined for each separate \( \mathcal{H}_A \) associated with an operator \( \mathcal{H}_A \) acting on a Hilbert space, where the quantum state of the embedded brane-world is given by the wave function \( \Psi_{ij}(A) \) and the final state is given by a superposition

\[
\Psi_{ij} = \sum_A \alpha^A \Psi_{ij}(A).
\]

The wave functions \( \Psi_{ij}(A) \) describe spin-2 fields in the brane-world as solution of the Klein-Gordon-like equa-
tion associated with the de Rahm operator \( \mathcal{H}_A \). However, its evolution along the extra dimensions requires the explicit variation of these functions with respect to \( s^A \). Since they do not mix with the brane-world coordinates, they may be used as time parameters. As a naive example consider that the quantum states are described by Schrödinger’s equations with respect to \( s^A \)

\[
- \frac{i\hbar}{ds^A} \frac{d\Psi_{ij}(A)}{ds^A} = \mathcal{H}_A \Psi_{ij}(A)
\]

(29)

Then the probability of a brane-world to be in an embedding state \( \Psi_{ij(A)} \) is given by

\[
||\Psi_{ij(A)}||^2 = \int \Psi^\dagger_{ij(A)} \Psi_{ij(A)} dV
\]

where the integral extends over a volume in \( V_D \) with a base on a compact region of \( V_n \) and a finite extension of the extra coordinates, such that it does not break the limit \( \ell \) of regularity of the embedding functions.

Topological changes such as the emergence of handlers, black holes and wormholes, induced by the probability transitions, are expected to occur from high energy oscillations \[15\]. Thus, for example, if \( \eta_A \) and \( \eta_B \) are both space-like extra dimensions, then, the classical limit of the probability transition \( \langle \Psi_{ij(A)}, \Psi_{kl(B)} \rangle \) corresponds to a transition from a perturbation of \( V_n \) along \( \eta_A \) to a perturbation along \( \eta_B \). An observer in \( V_n \) may interpret the result as the emergence of a space-like handle. On the other hand, if \( \eta_A \) and \( \eta_B \) have both time-like signatures, then the classical limit would correspond to a closed loop involving two internal time-like parameters.

When \( \eta_A \) and \( \eta_B \) have different signatures, the transition probability must also take into account possible changes of signature. Considering again the Kruskal brane-world example, regarded as a geodesically complete perturbation of the Schwarzschild space-time, we may fit both space-times in the same dynamical six dimensional flat space, provided a quantum signature transition at the horizon is considered.

V. CONFINEMENT OF GAUGE INTERACTIONS

Since most of current discussion on brane-worlds is concentrated on models with just one extra dimension, not much has been said about the symmetries of the extra dimensions. In strings or M-theory all internal symmetries derive from the string group (e.g. \( E_8 \times E_8 \) or \( SO(32) \)) so that additional symmetries on the extra dimensions are not required or even wanted. Quite on the opposite direction, Kaluza-Klein theory with a ground state like \( M_4 \times B_N \) requires a maximal symmetry for the space \( B_N \) generated by the extra dimensions. In brane-worlds the gauge interactions remain confined independently of the state of the embedded geometry, suggesting that the gauge group should also be independent of the embedding state.

The bending of brane-worlds is described by the variation of the normal vector \( \eta_A \) when its foot is displaced along the brane-world. In general it has tangent and normal components with coefficients \( k_{ijA} \) and \( A_{ijAB} \) respectively. The variation of \( k_{ijA} \) produces a tension on the brane-world and consequently a change of the energy-momentum tensor of the confined source \[3\]. On the other hand, as evidenced by \[10\] \( A_{AB} \) does not propagate with \( s^A \). In order to understand its meaning, consider the cases \( D \geq 6 \) and that the space generated by the extra dimensions has a certain number of Killing vector fields. Then we may apply the relevant, but little explored fact that \( A_{iAB} \) transform as the components of a gauge potential under that group of isometries. This can be seen from the transformation of the mixed component of the metric tensor, of \( V_D \) under a local infinitesimal coordinate transformation of the extra coordinates but leaving fixed the coordinates of \( V_n \):

\[
s'^A = s^A + \xi^A \quad \text{with} \quad \xi^i = 0, \quad \text{and} \quad \xi^A = \theta^A_M(x^i)s^M
\]

where \( \theta^A_M \) are infinitesimal parameters. Denoting generic coordinates in \( V_D \) by \( \{ x^\mu \} = \{ x^i, s^A \} \), it follows that \[17\]

\[
\dot{g}_{iA} = g_{iA} + g_{i\mu} \xi^\mu_A + g_{AB} \xi^i_B + \xi^i_B \frac{\partial g_{iA}}{\partial x^\mu} + O(\xi^2)
\]

Therefore the transformation of \( A_{iAB} \) follows from

\[
A'_{iAB} = \frac{\partial g'_{iA}}{\partial x^B} - \frac{\partial g'_{iB}}{\partial x^A} - \xi^i_B \frac{\partial g_{iA}}{\partial x^\mu}
\]

Using \( \xi^A_B = \theta^A_B(x^i) \) and \( \xi^i_A = \theta^i_A s^B \) we obtain

\[
A'_{iAB} = A_{iAB} - 2g_{MN} A_{iM[A} \theta_{B]N} + g_{MB} \theta^M_{A,i} \quad (30)
\]

showing that in fact \( A_{iAB} \) transform as the components of a non-Abelian gauge potential, where the gauge group is the group of isometries of the extra dimensions. This property strongly suggests that \( A_{AB} \) should be considered as a confined gauge potential when \( D \geq 6 \) and the extra dimensions have an isometry group.

The simplest embedding theorem concerns the analytic embeddings in flat spaces \( M_D \) \[18\]. The analytic assumption greatly simplifies the embedding and it implies that 10 dimensions are sufficient and it can be done in even less dimensions. However, in brane-worlds the oscillations of the embedded geometry are taken as solutions of the differential equations \[3\], and the analyticity implies that these oscillations are represented by convergent positive power series. This represents a limitation of the spectrum of solutions, including the probing near singularities where the power series may become divergent. To avoid these limitations we assume that these oscillations correspond to differentiable solutions of \[3\]. In this case, a more powerful embedding theorem shows that the limiting dimension for a flat embeddings rises to 14, or, more generally for an n-dimensional submanifold \( n(n + 3)/2 \), with a wide range of compatible signatures \[19\].
Consequently, with the exception of five-dimensional bulks, we may use the gauge degree of freedom to determine the number of extra dimensions, such that $A_{iAB}$ is the confined gauge potential. Taking the standard model $SU(3) \times SU(2) \times U(1)$ acting on a seven-dimensional projective space, identified with the space generated by the extra dimensions, we obtain as in Kaluza-Klein and supergravity theories an 11 dimensional space, which may be realized in a flat bulk. On the other hand, it has been suggested that the new physics occurring at the TeV may require a larger gauge group $\mathbf{20}$. If we take this into account along with the motivations for $SO(10)$ GUT, the differentiable embedding gives a fourteen-dimensional flat bulk with signature $(11,3)$ where $A_{4AB}$ acts as a self-contained and confined $SO(10)$ gauge field.

Regardless of the topology of the extra dimensions we need to know how far these dimensions can be probed by gravitons. Currently there are two approaches to this problem: In $\mathbf{2}$, the volume of the space probed by gravitons is determined by the addition of two boundary terms to the Einstein-Hilbert Lagrangian. A radion field included in the metric takes care of the separation between the boundaries. On the other hand, the derivation of the size in $\mathbf{1}$ assumes that the bulk has a fixed product topology, where the volume of the extra dimensions is finite. The case of a single extra dimension is excluded because it leads to a very large extra dimension.

Here, for generality we have not specified a metric ansatz and for compatibility with the embedding we have not imposed any topological condition on the bulk. To find the size of the extra dimensions under these general conditions, take a compact region in $\mathcal{V}$ and a finite volume $\mathcal{V}$ of the space generated by the extra dimensions limited by two minimal boundaries for the variables $s^A$, such that this region is effectively probed by gravitons. From our previous discussion, to keep the regularity of embedding and wave equations we require that the length of the extra dimensions should not exceed $\ell$. Thus, the action integral for the brane-world in this region using $\mathbf{3}$ is (for $n = 4$)

$$\int \int R \sqrt{\bar{g}} d^4 x d^N s = \int \mathcal{R} \sqrt{\bar{g}} d^D x - \int (K^2 + h^2) \sqrt{\bar{g}} d^4 x d^N s$$

where we notice that all integrands depend on $x^i$ and $s^A$, so that the integrated integrals cannot be separated. However, for small oscillations of the brane-world such that $(s^A)^2 << s^A < \ell$, and using $\mathbf{13}$ in an appropriate frame we obtain $\mathcal{G} = g$, so that the Einstein-Hilbert action for the bulk is

$$\int \mathcal{R} \sqrt{\bar{g}} d^D x \approx \int \int R \sqrt{\bar{g}} d^4 x d^N s + \int (K^2 + h^2) \sqrt{\bar{g}} d^4 x d^N s$$

However, from $\mathbf{16}$ we see that $\beta_{ij}$ still has a linear dependence on $s^A$. This can be eliminated without further impositions on the bulk by selecting a sufficiently smooth background at the embedding neighborhood, so that $\bar{k}_{ij} \approx 0$. With this choice, using the same arguments as in $\mathbf{3}$ we may write similarly to $\mathbf{1}$

$$\frac{1}{M_*^{2+N}} \approx \left( \frac{1}{M_{Pl}^2} + \frac{1}{M_*^2} \right)^N$$

where $M_*$ and $M_{Pl}$ are respectively the fundamental and effective scales, and where we have introduced an extrinsic (or bending) energy scale given by

$$\frac{1}{M_e^2} = \int (K^2 + h^2) \sqrt{\bar{g}} d^d x$$

This is not necessarily zero, even when the brane-world is flat. As such it may represent an observable effect on the brane-world dynamics in the form of the extrinsic tensor $Q_{ij}$ in $\mathbf{2}$.

Denoting by $d$ the typical length of the extra dimensional space probed by the gravitons, we may set $\mathcal{V} \approx d^N$ and under the specified conditions we obtain

$$d \approx \frac{M_{Pl}^{2/N}}{M_*^{2+2/N}} \left( 1 + \frac{1}{M_*^{1/N}} \right)$$

The size predicted in $\mathbf{1}$ and $\mathbf{3}$ is recovered when $M_{Pl}^2 < < M_*^2$ which occurs with the suggested approximations. We cannot make such approximation in general without imposing limitations on the brane-world oscillations.

When we have several extra dimensions the bending is determined by $k_{ij}$ and the gauge field $A_{iAB}$. This eliminates the need to introduce a radion field in the brane-world metric which appear in the hypersurface cases.

VI. SUMMARY

We have investigated the most general geometrical scenarios in which a brane-world program compatible with the hypotheses of embedding, confinement and the existence of quantum states can be implemented. Our analysis is independent of any previous choice of geometry, topology, number of dimensions and signature for the bulk. Instead, we have used the natural assumption that the brane-world geometry must remain a local differentiable embedded submanifold oscillating between minimal boundaries. We have found that the four basic postulates are sufficient to go a long way towards the formulation of a brane-world theory, but some conclusions apply only when the bulk has at least six dimensions.

Our first result consists in the derivation of a general dynamical principle for brane-worlds. We have shown that the Lagrangian for the brane-world geometry differ from the Einstein-Hilbert Lagrangian by a term, which depends essentially of the extrinsic curvature. The implication of this is that in general the bulk responds to the dynamics of the brane-world and consequently it should be allowed to have a variable geometry.
It is also possible to add to the modified Einstein-Hilbert basic Lagrangian \( [19] \) a constant term and powers of scalar functions constructed with the curvature tensors derived from \([3]\), to obtain higher derivative Lagrangians provided we also take in account the corresponding powers of the extrinsic curvature term.

Using the fact that the extra dimensions do not obey the same symmetry of the four-dimensional brane-world, we have managed to derive a non-trivial canonical structure and suggested a canonical quantization of the brane-world relative to the extra dimensions based on the Hamiltonians \( \mathcal{H}_A \).

When the bulk has at least six dimensions, a confined gauge field is contained in the embedding structure. This novel confinement mechanism appear in the form of one of the basic embedding variables \( A_{iAB} \). When we identify this field with the physical gauge field, a simple arithmetic fixes also the number of extra dimensions: For the standard model it was found that the self-contained gauge structure requires 11 dimensions. On the other hand, the \( SO(10) \) gauge group implies in 14 dimensions, which can be realized by a flat bulk.

Five dimensional, or more generally hypersurface models are not excluded from our analysis but since they do not contain the field \( A_{iAB} \), the confined gauge fields need to be introduced by other mechanisms. In this case, the equations can be derived from the general case by setting \( D = n + 1, \ A, B \cdots = n + 1, \ g_{AB} = g_{n+1,n+1} = \pm 1 \) and \( b_{n+1} = 0 \). Only the first two equations in \([4]\) remain and are required to obtain a Lagrangian similar to \([19]\), suitable to describe the evolution of the brane-world with respects to the single extra dimension.

One difficulty associated with perturbations of hypersurface brane-worlds in a constant curvature bulk is due to a general result in geometry, stating that if a hypersurface has more than two finite curvature radii \( \ell_i \), then it becomes indelible \([10]\). This means that there is a certain degree of stiffness associated with perturbations, preventing the generation of more complicated configurations of the embedded geometry.

The typical size of the extra dimensions compatible with the embedding was found to be close to the one predicted with product topology, as long as we remain in the linear regime of perturbations in a very smooth background.

We have not included some relevant questions such as the emergence of a cosmological constant and the observable implications of the extrinsic terms in the dynamical equations \([20]\). Problems related to brane-world cosmology become extremely interesting under the Lagrangian \([14]\), where the extrinsic curvature contributes to the modification of Friedmann’s equation, as will be discussed in a subsequent paper.

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