Learning high dimensional stochastic partial differential equations in a predictor-corrector framework

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Abstract

In this paper, we propose a deep learning-based numerical method for approximating high dimensional stochastic partial differential equations in a predictor-corrector framework. At each time step, the original equation is decomposed into a degenerate stochastic partial differential equation to serve as the prediction step, and a second-order deterministic partial differential equation to serve as the correction step. The solution of the degenerate stochastic partial differential equation is approximated by the Euler method, and the solution of the partial differential equation is approximated by neural networks via the equivalent backward stochastic differential equation. Under standard assumptions, the rate of convergence and the error estimates in terms of the approximation error of neural networks are provided. The efficiency and accuracy of the proposed algorithm are illustrated by the numerical results.

Keywords: Stochastic partial differential equations in high dimension, deep learning based numerical algorithm, predictor–corrector methods

AMS subject classifications: 60H15, 60H35, 65M12

1 Introduction

In this paper, we propose and analyze a deep learning-based numerical scheme for the following stochastic partial differential equation (SPDE):

\[ u(t, x) = h(x) + \int_t^T \left( Lu(s, x) + f(s, x, u(s, x), (\nabla u \sigma)(s, x)) \right) \, ds \]
\[ + \int_t^T g(s, x, u(s, x)) \, dB_s, \]  

(1.1)
where $u \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^k$, $L u = [L u_1, \ldots, L u_k]^\top$, with

$$L = \frac{1}{2} \sum_{i,j=1}^d \left( \sigma \sigma^\ast \right)_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t,x) \frac{\partial}{\partial x_i},$$

$\nabla$ denotes the space gradient, $B$ is a Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, P)$, and the $dB$ integral is a backward Itô integral. The dimension $d$ is assumed to be high. SPDEs of this type appear in many real-world applications, including the filtering problems, population genetics, statistical hydromechanics (see [39] and the references therein).

Numerical approximations of SPDEs have been studied extensively. A comprehensive review of the literature can be found in Jentzen and Kloeden [34]. Traditional numerical methods such as the finite difference methods [26, 27, 53], finite element methods [20, 51], spectral methods [24], and Wiener chaos decomposition [41], are based on space-time discretization of the SPDE under consideration. In low dimensional cases, the classical numerical methods are easily implemented and perform quite well with reasonable accuracy. However, for higher space dimensions, these numerical approaches are infeasible due to their prohibitive computational costs and storage requirements. Several probabilistic methods have been developed in recent years to mitigate the curse of dimensionality challenge. Milstein and Tretyakov [44] proposed a numerical scheme for linear SPDEs based on the method of characteristics and Monte Carlo techniques. In [47], Pardoux and Peng introduced a probabilistic representation of the solution of (1.1) through the solution of the following forward backward doubly stochastic differential equation (FBDSDE)

$$X^{t,x}_s = x + \int_t^s \mu(r, X^{t,x}_r) \, dr + \int_t^s \sigma(r, X^{t,x}_r) \, dW_r,$$  \hspace{1cm} (1.2)

$$Y^{t,x}_s = h(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r) \, dr$$

$$- \int_s^T Z^{t,x}_r \, dW_r + \int_s^T g(r, X^{t,x}_r, Y^{t,x}_r) \, dB_r,$$  \hspace{1cm} (1.3)

where $\{X^{t,x}_s, t \leq s \leq T\}$ denotes the forward process starting from $x$ at time $t$, $W$ is a $d$-dimensional standard Brownian motion independent of $B$ in (1.1) and (1.3). Under smoothness assumptions, they proved that if $u$ is a classical solution of (1.1), then $\{Y^{t,x}_s = u(s, X^{t,x}_s), Z^{t,x}_s = (\nabla u)\sigma)(s, X^{t,x}_s), t \leq s \leq T\}$ is the unique solution of (1.3). Several generalizations have been proposed to derive similar probabilistic representations of weak solutions of (1.1) via FBDSDEs (see, e.g. [4]). Since then, various numerical methods for SPDEs based on the associated FBDSDEs have been investigated (see for instance [2, 3, 5, 30]). In particular, when $g$ vanishes, there is a vast literature on numerical methods for backward stochastic differential equations (BSDEs). In [54], Zhang introduced the $L^p$ regularity of $Z$, and proposed a Monte Carlo method for BSDEs. Bouchard and Touzi [11] approximated the conditional expectations using esti-
mators obtained from the kernel regression method and Malliavin calculus technique. Gobet et al. \[21\] presented a least squares regression method based on the approximation of conditional expectations by the $L^2$ projection on the space spanned by a finite basis of functions. This method has been extended to BSDEs with locally Lipschitz driver in Gobet and Turkedjiev \[22, 23\]. Gauss-Hermite quadrature rule and polynomial interpolations were applied to approximate the conditional expectations in \[52, 55\]. Using the sparse grid Gaussian-Hermite quadrature rule and the sparse grid approximations, \[18\] proposed an algorithm manage to solve FBSDEs up to dimension 6.

In recent years, there has been a significant growth in interest in the deep learning-based numerical methods for partial differential equations (PDEs). In the groundbreaking work of E et al. \[14, 28\], the problem of solving high-dimensional PDEs was reformulated as a deep learning problem using the nonlinear Feyman-Kac formula introduced in \[46\]. This approach was extended to fully nonlinear second-order PDEs in \[7\]. Backward induction schemes for high dimensional PDEs and variational inequalities were developed in \[51\]. A multilevel Picard iteration scheme was derived in \[17\] by combining the Feynman-Kac and Bismut-Elworthy-Li formulae with a multilevel Picard approximation. \[19, 31, 48\] proposed the deep learning version of regression methods and backward induction schemes. Although there has been extensive research on deep learning-based numerical approximations for PDEs, only few studies have attempted to investigate the deep learning-based algorithms for SPDEs, and there has been little discussion about the convergence analysis of the algorithms. We refer to \[6\] for a deep learning-based numerical methods for SPDEs with initial conditions, and to \[50\] for a deep learning-based approach for BDSDEs.

In this work, we present a deep learning-based numerical algorithm for the weak solution of \[1.1\] in a predictor-corrector framework as follows. Let $\pi^N: 0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of the interval $[0, T]$. The approach generates an approximating sequence $\{u_i(x)\}_{i=0}^N$ defined by $u_N = h(x)$ and

\[
u_i = \Phi_{\Delta t_i} \Psi_{t_i, t_{i+1}} u_{i+1}, \quad i = N-1, \cdots, 0, \tag{1.4}\]

where $\Delta t_i = t_{i+1} - t_i$, $\{\Phi_t, \ t \geq 0\}$ and $\{\Psi_{st}, \ 0 \leq s \leq t\}$ denote the solution operators associated with the equations

\[d\psi + g(t, x, \psi(t, x)) dB_t = 0, \quad \lim_{t \to t_i+1} \psi_t = u_{i+1} = \psi_{t_{i+1}}, \tag{1.5}\]

and

\[\partial_t \varphi + \mathcal{L} \varphi(t, x) + f(t, x, \varphi(t, x), (\nabla \varphi) (t, x)) = 0, \quad \lim_{t \to t_i+1} \varphi_t = \psi_{t_i}, \tag{1.6}\]

respectively. At each time step, the SDE (1.5) serves as a predictor by providing an a priori estimate of the solution at the next time step. Then we use its solution at time $t_i$ as the terminal condition at time $t_{i+1}$ of the PDE (1.6), and refine the initial estimate. $u_i(x) = \varphi_i(x)$ is therefore an approximation of the solution $u(t_i, X_{t_i}^x)$, for $i = N-1, \cdots, 0$. The numerical approximations of SDE
are well-understood (see, e.g., [38]), and the numerical solution of the PDE (1.6) is of primary interest. Inspired by [14, 28], we propose to approximate (1.6) by a neural network at each time step via its equivalent BSDE.

Our approximation strategy is to decompose (1.1) into two parts which are simpler for numerical computations, compute the solution to each component individually, and then integrate the two solutions to generate a solution to the original equation. One key motivation for employing splitting strategy is that it improves efficiency and provides remarkable flexibility in choosing different numerical methods or different discretizations, depending on the characteristics of each sub-problem. Applications of splitting methods for linear SPDEs such as the Zakai equation were formulated and studied by Bensoussan et al [8, 9, 11] and Legland [40], and further developed by Nagase [45] and Gyöngy and Krylov [25]. Our theoretical contribution is the convergence analysis of the proposed algorithm, with the error estimates depending on the approximation errors of the neural networks. We also emphasize the connection between this approximation strategy and the uncertainty quantification problems in deep neural network [1].

With properly chosen $f$ and $g$, the predictor (1.5) describes the dynamics of hidden units in continuous-depth residual networks with uncertainty, while the corrector (1.6) is the corresponding adjoint equation.

The paper is organized as follows. In section 2, we introduce the notations and the assumptions that will be used. Then, we describe the discrete time approximation scheme for (1.1) and analyze the time discretization error. In section 3, we present the deep learning algorithm based on the discrete time approximations in section 2. We provide the convergence analysis of the proposed algorithm and the error estimates in terms of the approximation errors of the neural networks as well. Section 4 is dedicated to some numerical tests.

2 Preliminaries

2.1 Notations and assumptions

Throughout this paper, we denote by $(\Omega, \mathcal{F}, P)$ a probability space on which two mutually independent standard Brownian motions $W$ and $B$ (with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively) are defined. Let $\mathcal{F}_{s,t}^W$ and $\mathcal{F}_{s,t}^B$ be the the usual P-augmentation of $\sigma\{W_r - W_s; s \leq r \leq t\}$ and $\sigma\{B_r - B_s; s \leq r \leq t\}$, respectively. Let $T > 0$ be a fixed time. For each $t \in [0, T]$, define two collections $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ and $\{\mathcal{G}_t\}_{0 \leq t \leq T}$ by

$$\mathcal{F}_t := \mathcal{F}_{0,t}^W \lor \mathcal{F}_{t,T}^B,$$

and $\mathcal{G}_t := \mathcal{F}_{0,t}^W \lor \mathcal{F}_{0,T}^B$.

Note that $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is neither increasing nor decreasing, while $\{\mathcal{G}_t\}_{0 \leq t \leq T}$ is an increasing filtration. The transpose of any vector or matrix $A$ is denoted by $A^T$. $|x| = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$ is the Euclidean norm of any vector $x \in \mathbb{R}^n$. $|A| = (\sum_{i,j}|a_{ij}|^2)^{1/2}$ is the Frobenius norm for any matrix $A \in \mathbb{R}^{m \times n}$.

The following spaces will be frequently used. $C^k(\mathbb{R}^m; \mathbb{R}^n)$ denotes the space of $C^k$-functions from $\mathbb{R}^m$ to $\mathbb{R}^n$; $C^k_c(\mathbb{R}^m; \mathbb{R}^n)$ denotes the space of compactly
supported $C^k$-functions; $C^k_c(R^m; R^n)$ is the space of all $C^k$-functions whose partial derivatives up to order $k$ are bounded. For $1 \leq p < \infty$, $L^p_c(0, T; R^n)$ is the space of all $\mathcal{F}_t$-measurable $R^n$-valued processes $\varphi(\cdot)$ such that $\mathbb{E}[\int_0^T |\varphi_t|^p \, dt] < \infty$. Denote similarly by $L^p_r(\Omega, C([0, T]; R^n))$ the space of all $\mathcal{F}_t$-measurable $R^n$-valued processes $\varphi(\cdot)$ such that $\mathbb{E}[\sup_{0 \leq t \leq T} |\varphi_t|^p] < \infty$. Let $L^2_c(R^n)$, where $\omega(x)$ is a continuous positive function such that $\int_{R^n} (1 + |x|^2) \omega(x) \, dx < \infty$, be a weighted Hilbert space with scalar product $(u, v)_\omega := \int_{R^n} u(x) v(x) \omega(x) \, dx$, which induced the norm $|u|_\omega := (u, u)_\omega$. For all $k \geq 0$, the weighted Sobolev space $H^k_c(R^n)$ be the completion of $C_c^\infty(R^n)$ with respect to the norm $|u|_{k, \omega}^2 = \sum_{0 \leq |\alpha| \leq k} |D^\alpha u|^2$. Denote by $\mathcal{H}$ the space of $\mathcal{F}^B_{t,x}$-measurable process $\varphi_t(\cdot)$ with values in $H^1_c(R^n)$ such that $|\varphi|_{H^1}^2 := \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\varphi_t|^2 \right] + \mathbb{E} \left[ \int_0^T |\nabla \varphi_t|^2 \, dt \right] < \infty$.

We will make the following assumptions.

1. The functions $f \in C^2_c([0, T] \times R^d \times R^k \times R^{k \times d}, R^k)$ and $g \in C^2_c([0, T] \times R^d \times R^k \times R^{k \times d}; R^k)$ are jointly measurable in $(t, x, y, z)$ such that $f(\cdot, \cdot, y, z) \in L^2_c(0, T; R^k)$ and $g(\cdot, \cdot, y, z) \in L^2_c(0, T; R^{k \times d})$, for any $(y, z) \in R^k \times R^{k \times d}$.

2. We assume that there exists a constant $K > 0$ such that

$$\left| f(t, x, y, z) - f(s, \bar{x}, \bar{y}, \bar{z}) \right|^2 + \left| g(t, x, y) - g(s, \bar{x}, \bar{y}) \right|^2 \leq K \left( |t - s| + |x - \bar{x}|^2 + |y - \bar{y}|^2 + |z - \bar{z}|^2 \right),$$

for any $t, s \in [0, T], x, \bar{x} \in R^d, y, \bar{y} \in R^k, z, \bar{z} \in R^{k \times d}$.

3. The terminal condition $h \in C^2_c(R^d; R^k)$ satisfies the Lipschitz condition

$$|h(x) - h(\bar{x})|^2 \leq K |x - \bar{x}|^2, \forall x, \bar{x} \in R^d.$$

4. The functions $\mu \in C^2_c(R^d; R^d)$ and $\sigma \in C^2_c(R^d; R^{d \times d})$ in $[1, 2]$ are uniformly $\frac{1}{2}$-Hölder continuous in $t \in [0, T]$ and uniformly Lipschitz continuous in $x \in R^d$, i.e., there exists a $K > 0$ such that

$$|\mu(t, x) - \mu(s, \bar{x})|^2 + |\sigma(t, x) - \sigma(s, \bar{x})|^2 \leq K \left( |t - s| + |x - \bar{x}|^2 \right), \forall x, \bar{x} \in R^d.$$

Furthermore we assume that there exists a constant $c > 0$, such that

$$\xi^T \sigma(t, x) \sigma^T(t, x) \xi \geq c |\xi|^2, \forall x, \xi \in R^d, t \in [0, T].$$

Under the above assumptions, Pardoux and Peng [47] prove that BDSDE [1,33] has a unique solution

$$\left( Y, Z \right) \in L^2_c(\Omega, C([0, T]; R^k)) \times L^2_c(0, T; R^{k \times d}).$$
Definition 2.1 ([4]). A function $u \in \mathcal{H}$ is called a weak solution of (1.1) if

$$
\int_t^T \int_{\mathbb{R}^d} u(s, x) \partial_s \phi(s, x) \, dx \, ds - \int_t^T \int_{\mathbb{R}^d} u(s, x) \mathcal{L}^* \phi(s, x) \, dx \, ds \\
+ \int_t^T \int_{\mathbb{R}^d} u(t, x) \phi(t, x) \, dx - \int_t^T h(x) \phi(T, x) \, dx \\
= \int_t^T \int_{\mathbb{R}^d} f(s, x, u(s, x), (\nabla u)\sigma(s, x)) \phi(s, x) \, dx \, ds \\
+ \int_t^T \int_{\mathbb{R}^d} g(s, x, u(s, x)) \phi(s, x) \, dx \, dB_s,
$$

for every $\phi \in C^\infty_c([0, T] \times \mathbb{R}^d)$, where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$.

At the end of this section we recall the following wellposedness and regularity results on BDSDEs to be used in this paper.

Proposition 2.2 ([2] [4]). Under the assumptions (H1)-(H3), there exists a unique weak solution $u \in \mathcal{H}$ of the SPDE (1.1). Moreover, $u(t, x) = Y^{t,x}_s$, where $(Y^{t,x}_s, Z^{t,x}_s)_{t \leq s \leq T}$ is the solution of the BDSDE (1.3). Furthermore, we have for all $s \in [t, T]$,

$$
Y^{t,x}_s = u(s, X^{t,x}_s), \quad Z^{t,x}_s = (\nabla u)\sigma(s, X^{t,x}_s).
$$

Proposition 2.3 ([2] [5] [17]). Let (H1)-(H3) hold. Then, for $0 \leq r \leq s \leq T$,

$$
E|Y^{t,x}_s - Y^{t,x}_r|^2 \leq C(1 + |x|^2)(s - r), \\
E|Z^{t,x}_s - Z^{t,x}_r|^2 \leq C(1 + |x|^2)(s - r).
$$

2.2 Time discretization approximations

In this section, we construct the discrete time approximations of the solution $\psi$ of (1.5) and the solution $\varphi$ of (1.6), to obtain the approximating sequence $(u_i(x))_{i \geq 0}$ defined in (1.4). We start with an approximation of SDE (1.5). Let $\pi: 0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$. Define the partition size by $|\pi| := \max \Delta t_i$. The Euler scheme associated with (1.5) and $\pi$ is defined by

$$\psi_{t_i}^{n} = \varphi_{t_{i+1}}^{n} + g(t_{i+1}, X_{t_{i+1}}^{n}, \varphi_{t_{i+1}}^{n}) \Delta B_i, \quad (2.1)$$

where $\Delta t_i = t_{i+1} - t_i$, $\Delta B_i = B_{t_{i+1}} - B_{t_i}$, $i = 0, 1, \cdots, N - 1$, $\varphi_{t_{i+1}}^{n}$ and $X_{t_{i+1}}^{n}$ are approximations of $\varphi(t, x)$ and $X_t$ at $t = t_{i+1}$ which will be clarified later.

On each interval $[t_i, t_{i+1})$, PDE (1.6) is solved through its equivalent BSDE

$$y^{t,x}_s = \psi_{t_i}(x) + \int_{t_i}^{t_{i+1}} f(r, X^{t,x}_r, y^{t,x}_r, z^{t,x}_r) \, dr - \int_{t_i}^{t_{i+1}} z^{t,x}_r \, dW_r, \quad t_i \leq s < t_{i+1}. \quad (2.2)$$
It should be noted that the family \{(y,z)\} differs from \{(Y,Z)\} in \text{(1.3)}, and \(y\) is discontinuous at points \(t_i, \ i = 1, \cdots, N\). We recall from Theorem 3.2 in \cite{34} that
\[
\varphi(t, x) = y^t_{\pi} \quad \text{and} \quad \nabla \varphi(t, x) \sigma(t, x) = z^t_{\pi}.
\] (2.3)
The forward process \(X_t\), as defined in \text{(1.2)}, is approximated by the Euler scheme
\[
X_{t_{i+1}}^{t,x,\pi} = X_{t_i}^{t,x,\pi} + \mu(t_i, X_{t_i}^{t,x,\pi}) \Delta t_i + \sigma(t_i, X_{t_i}^{t,x,\pi}) \Delta W_i,
\] (2.4)
where \(\Delta t_i = t_{i+1} - t_i\), \(\Delta W_i = W_{t_{i+1}} - W_{t_i}\), \(i = 0, 1, \cdots, N - 1\). Hereafter, the superscript \(t\), \(x\) will be dropped unless needed for clarity. The following estimate is well-known (see, e.g., \cite{34}).

**Lemma 2.4.** Assume that \(\mu\) and \(\sigma\) satisfy \((H3)\). Then there exists a constant \(C\), independent of \(\pi\), such that
\[
\max_{0 \leq i \leq N-1} \sup_{s \in [t_i, t_{i+1})} \mathbb{E} \left[ |X_s - X^\pi_{t_i}|^2 + |X_s - X^\pi_{t_{i+1}}|^2 \right] < C(1 + |x|^2)|\pi|.
\]
The solution \((y^\pi_t, z^\pi_t)\) of the following equation:
\[
y^\pi_{t_i} = \psi^\pi_{t_i} + f(t_i, X^\pi_{t_i}, y^\pi_{t_i}, z^\pi_{t_i}) \Delta t_i - z^\pi_{t_i} \Delta W_i,
\] (2.5)
The scheme in \text{(2.5)} is implicit in \(y\) and \(z\). Thus, by \text{(2.3)}, \(\varphi(t_i, \cdot)\) is approximated by
\[
\varphi^\pi_{t_i} = y^\pi_{t_i} = \psi^\pi_{t_i} + f(t_i, X^\pi_{t_i}, \varphi^\pi_{t_i}, \nabla \varphi^\pi_{t_i} \sigma(t_i, X^\pi_{t_i})) \Delta t_i - \nabla \varphi^\pi_{t_i} \sigma(t_i, X^\pi_{t_i}) \Delta W_i.
\]
To sum up, given time grid \(\pi\) and \(\{(X^\pi_{t_i}, y^\pi_{t_i}, z^\pi_{t_i})\}_{i=0}^N\), the solution \(\psi\) and \(\varphi\) of \text{(1.5)} and \text{(1.6)} is approximated by \(\psi^\pi_{t_i} \}_{i=0}^N\) and \(\varphi^\pi_{t_i} \}_{i=0}^N\), respectively, defined in a backward manner by \(\varphi^\pi_{t_N} = h(X^\pi_{t_N})\), and
\[
\psi^\pi_{t_i} = \psi^\pi_{t_{i+1}} + g(t_{i+1}, X^\pi_{t_{i+1}}, \varphi^\pi_{t_{i+1}}) \Delta B_i,
\]
\[
\varphi^\pi_{t_i} = \psi^\pi_{t_i} + f(t_i, X^\pi_{t_i}, \varphi^\pi_{t_i}, \nabla \varphi^\pi_{t_i} \sigma(t_i, X^\pi_{t_i})) \Delta t_i - \nabla \varphi^\pi_{t_i} \sigma(t_i, X^\pi_{t_i}) \Delta W_i,
\] (2.6)
where \(\Delta t_i = t_{i+1} - t_i\), \(\Delta W_i = W_{t_{i+1}} - W_{t_i}\), \(\Delta B_i = B_{t_{i+1}} - B_{t_i}\), for \(i = N - 1, \cdots, 0\). As we mentioned, the approximate sequence \(\{u^\pi_i\}_{i=0}^N\) in \text{(1.4)} is defined by
\[
u_i(x) = \varphi^\pi_{t_i}(x),
\] (2.7)
for \(i = N - 1, \cdots, 0\).

### 2.3 Time discretization error

In this section, we analyze the approximation error of the discrete time scheme described in Section \text{(2.2)}. We first investigate the square error between the solution of BDSDE \text{(1.3)} and the discrete time scheme \text{(2.5)}.
**Proposition 2.5.** Let \((H1)-(H3)\) hold. Then there exists a constant \(C\), independent of \(k, l\), the time partition \(\pi\) and the space dimension \(d\), such that

\[
\max_{0 \leq i \leq N-1} \mathbb{E}|Y_i^{t_i} - y_i^{t_i}|^2 \leq C(1 + |x|^2)|\pi|.
\]  

(2.8)

**Proof.** Let \(\mathbb{E}_i[\cdot]\) denote the conditional expectation with respect to the discrete filtration \(G^\pi_i\) defined by

\[ G^\pi_i := \sigma\{\Delta W_j; 0 \leq j \leq i - 1\} \cup \sigma\{\Delta B_j; 0 \leq j \leq N - 1\}. \]

Setting \(s = t_i\), taking the conditional expectation \(\mathbb{E}_i[\cdot]\) in (1.3) and using the tower property, we have

\[
Y_i = \mathbb{E}_i \left[ h(X_T) + \int_{t_i}^T f(r, X_r, Y_r, Z_r) \, dr + \int_{t_i}^T g(r, X_r, Y_r) \, dB_r \right]
\]

\[
= \mathbb{E}_i \left[ Y_{i+1} + \int_{t_i}^{t_{i+1}} f(r, X_r, Y_r, Z_r) \, dr + \int_{t_i}^{t_{i+1}} g(r, X_r, Y_r) \, dB_r \right].
\]

(2.9)

Define \(\Delta y_i := Y_i - y_i^{t_i}\) and \(\Delta z_i := Z_t - z_i^{t_i}\) for \(i = 0, \ldots, N - 1\). From \((2.9), (2.5)\) and \((2.1)\), we have

\[
\Delta y_i = \mathbb{E}_i[\Delta y_{i+1}] + \mathbb{E}_i \left[ \int_{t_i}^{t_{i+1}} \delta f^i(r) \, dr \right] + \mathbb{E}_i \left[ \int_{t_i}^{t_{i+1}} \delta g^i(r) \, dB_r \right],
\]

where

\[
\delta f^i(r) := f(r, X_r, Y_r, Z_r) - f(t_i, X_i^t, y_i^t, z_i^t),
\]

\[
\delta g^i(r) := g(r, X_r, Y_r) - g(t_{i+1}, X_{i+1}^t, y_{i+1}^t).\]

Square and then take the expectation on both sides of the above equation to obtain

\[
\mathbb{E} |\Delta y_i|^2 = \mathbb{E} [\mathbb{E}_i |\Delta y_{i+1}|^2] + \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \mathbb{E}_i [\delta f^i(r)] \, dr \right]^2 + \mathbb{E} \left[ \mathbb{E}_i \left[ \int_{t_i}^{t_{i+1}} \delta f^i(r) \, dr \right] \right]^2
\]

\[
+ 2 \mathbb{E} \left[ \mathbb{E}_i |\Delta y_{i+1}| \int_{t_i}^{t_{i+1}} \mathbb{E}_i [\delta f^i(r)] \, dr \right] + 2 \mathbb{E} \left[ \mathbb{E}_i |\Delta y_{i+1}| \mathbb{E}_i \left[ \int_{t_i}^{t_{i+1}} \delta g^i(r) \, dB_r \right] \right]
\]

\[
+ 2 \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \mathbb{E}_i [\delta f^i(r)] \, dr \right] \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \delta g^i(r) \, dB_r \right] := I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]

By Cauchy’s inequality, Jensen’s inequality, and \((H1)\), we have

\[
I_2 \leq \Delta t_i \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} (\mathbb{E}_i |\delta f^i(r)|)^2 \, dr \right]
\]

\[
\leq K \Delta t_i \int_{t_i}^{t_{i+1}} \left( |r - t_i| + \mathbb{E}|X_r - X_i^t|^2 + \mathbb{E}|Y_r - y_i^t|^2 + \mathbb{E}|Z_r - z_i^t|^2 \right) \, dr.
\]

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Next, by Itô’s isometry, Jensen’s inequality, and (H1), we obtain

\[
I_3 = E \left[ \int_{t_i}^{t_{i+1}} (E_i[\delta g^i(r)])^2 \, dr \right] \\
\leq K \int_{t_i}^{t_{i+1}} \left( |r - t_i| + E[X_r - X_{t_{i+1}}^\pi]^2 + E[Y_r - y_{t_{i+1}}^\pi]^2 \right) \, dr.
\]

Note that \( Y_r \) is \( F_{T_{i+1}} \)-measurable, then

\[
\mathbb{E} \left[ \mathbb{E}_i[\Delta y_{t+1}] \mathbb{E}_i[\delta g^{i+1} \Delta B_t] \right] = 0,
\]

where \( \delta g^{i+1} = g(t_{i+1}, X_{t_{i+1}}, Y_{t_{i+1}}) - g(t_{i+1}, X_{t_{i+1}}^\pi, Y_{t_{i+1}}^\pi) \). We now estimate \( I_4 \) and \( I_5 \). It follows from Young’s inequality, Jensen’s inequality, and the estimates of \( I_2 \) and \( I_3 \) that

\[
I_4 + I_5 \leq \epsilon_1 \Delta t_i \mathbb{E}[\Delta y_{t+1}]^2 + \frac{1}{\epsilon_1 \Delta t_i} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \mathbb{E}_i[\delta f^i(r)] \, dr \right]^2 \\
+ \frac{1}{\epsilon_2} \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \mathbb{E}_i[\delta f^i(r)] \, dr \right]^2 + \epsilon_2 \mathbb{E} \left[ \int_{t_i}^{t_{i+1}} \mathbb{E}_i[\delta g^i(r)] \, dB_r \right]^2 \\
\leq \epsilon_1 \Delta t_i E[\Delta y_{t+1}]^2 + \left( \frac{1}{\epsilon_1} + \frac{\Delta t_i}{\epsilon_2} \right) K \int_{t_i}^{t_{i+1}} \left( |r - t_i| + E[X_r - X_{t_{i+1}}^\pi]^2 + E[Y_r - y_{t_{i+1}}^\pi]^2 \right) \, dr \\
+ \epsilon_2 K \int_{t_i}^{t_{i+1}} \left( |r - t_i| + E[X_r - X_{t_{i+1}}^\pi]^2 + E[Y_r - y_{t_{i+1}}^\pi]^2 \right) \, dr.
\]

By Jensen’s inequality, we have

\[
E[Y_r - y_{t_{i+1}}^\pi]^2 \leq 2E[\Delta y_i]^2 + 2E[Y_r - Y_{t_i}]^2, \\
E[Y_r - y_{t_{i+1}}^\pi]^2 \leq 2E[\Delta y_{t+1}]^2 + 2E[Y_r - Y_{t_{i+1}}]^2, \\
E[Z_r - z_{t_{i+1}}^\pi]^2 \leq 2E[\Delta z_i]^2 + 2E[Z_r - Z_{t_i}]^2.
\]

Combining Proposition 2.3, Lemma 2.4, and (H1) with the above, we obtain

\[
E[\Delta y_i]^2 \leq E[E_i[\Delta y_{t+1}]^2 + \epsilon_1 \Delta t_i E[\Delta y_{t+1}]^2 + 2K(1 + \epsilon_2) \Delta t_i E[\Delta y_{t+1}]^2 \\
+ 2K \left( \frac{1}{\epsilon_1} + \Delta t_i + \frac{\Delta t_i}{\epsilon_2} \right) (\Delta t_i) E[\Delta y_i]^2 \\
+ 2K \left( \frac{1}{\epsilon_1} + \Delta t_i + \frac{\Delta t_i}{\epsilon_2} \right) (\Delta t_i) E[\Delta z_i]^2 + C(1 + |x|)(\Delta t_i)^2.
\] (2.10)
Now we turn to the estimation of $\Delta z_i$. Setting $s = t_i$, multiplying (1.3) by $\Delta W_t$, and taking the conditional expectation $E_i[\cdot]$, we obtain by the integration by parts formula that

$$
\Delta t_i Z_{t_i} = E_i[Y_{t_{i+1}} \Delta W_t] + E_i \left[ \int_{t_i}^{t_{i+1}} f(r, X_r, Y_r, Z_r) \, dr \Delta W_t \right] + 2 E_i \left[ \int_{t_i}^{t_{i+1}} g(r, X_r, Y_r) \, dB_t \Delta W_t \right] + \int_{t_i}^{t_{i+1}} E_i [Z_{t_i} - Z_r] \, dr.
$$

Similar to the discussions for $\Delta y_i$, we multiply (2.5) by $\Delta W_t$, taking the conditional expectation $E_i[\cdot]$, and combining with (2.1), we get

$$
\Delta t_i z_{t_i}^r = E_i \left[ \psi_i^r \Delta W_t \right] = E_i \left[ \left( \varphi_i^{t_{i+1}} + g(t_{i+1}, X_{t_{i+1}}, \varphi_i^{t_{i+1}}) \Delta B_t \right) \Delta W_t \right].
$$

Subtracting (2.12) from (2.11) gives

$$
\Delta t_i \Delta z_i = E_i [\Delta y_{t+1} \Delta W_t] + E_i \left[ \int_{t_i}^{t_{i+1}} f(r, X_r, Y_r, Z_r) \, dr \Delta W_t \right] + 2 E_i \left[ \int_{t_i}^{t_{i+1}} \delta g^i(r) \, dB_t \Delta W_t \right] + \int_{t_i}^{t_{i+1}} E_i [Z_{t_i} - Z_r] \, dr.
$$

Similar to the discussions for $\Delta y_i$, we square the above equation, and take expectation to get

$$
E |\Delta t_i \Delta z_i|^2 = E |E_i [\Delta y_{t+1} \Delta W_t]|^2
$$

$$
+ E \left[ E_i \left[ \int_{t_i}^{t_{i+1}} f(r, X_r, Y_r, Z_r) \, dr \Delta W_t \right] + E_i \left[ \int_{t_i}^{t_{i+1}} \delta g^i(r) \, dB_t \Delta W_t \right] + \int_{t_i}^{t_{i+1}} E_i [Z_{t_i} - Z_r] \, dr \right]^2
$$

$$
+ 2 E \left[ E_i [\Delta y_{t+1} \Delta W_t] \left[ E_i \left[ \int_{t_i}^{t_{i+1}} f(r, X_r, Y_r, Z_r) \, dr \Delta W_t \right] \right] + 2 E \left[ E_i [\Delta y_{t+1} \Delta W_t] \left[ \int_{t_i}^{t_{i+1}} \delta g^i(r) \, dB_t \Delta W_t \right] \right]
$$

$$
+ 2 E \left[ E_i [\Delta y_{t+1} \Delta W_t] \int_{t_i}^{t_{i+1}} E_i [Z_r - Z_{t_i}] \, dr \right]
$$

$$
= J_1 + J_2 + J_3 + J_4 + J_5.
$$

We now estimate term by term. By Cauchy’s inequality, Jensen’s inequality, Itô’s isometry, (H1) and the estimate of $I_3$, we have

$$
J_2 \leq 3 (\Delta t_i)^2 \int_{t_i}^{t_{i+1}} E|f(r, X_r, Y_r, Z_r)|^2 \, dr
$$

$$
+ 3 \Delta t_i \int_{t_i}^{t_{i+1}} E|\delta g^i(r)|^2 \, dr + 3 \Delta t_i \int_{t_i}^{t_{i+1}} E|Z_{t_i} - Z_r|^2 \, dr
$$

$$
\leq 6 K (\Delta t_i)^2 E |\Delta y_{t+1}|^2 + C(1 + |x|^2)(\Delta t_i)^3.
$$
Combining the above result with Young’s inequality, we have

\[ J_3 \leq \varepsilon_3 \mathbb{E}[|\Delta y_{i+1}|^2 + C(1 + |x|^2)(\Delta t_i)^3], \]
\[ J_4 \leq \varepsilon_4 \mathbb{E}[\Delta t_i \mathbb{E}|\Delta y_{i+1}|^2] + \frac{1}{\varepsilon_4} [2K(\Delta t_i)^2 \mathbb{E}|\Delta y_{i+1}|^2 + C(1 + |x|^2)(\Delta t_i)^3], \]
\[ J_5 \leq \varepsilon_5 \mathbb{E}[\Delta y_{i+1}]^2 + C(1 + |x|^2)(\Delta t_i)^3. \]

Thus

\[ (\Delta t_i)^2 \mathbb{E}|\Delta z_i|^2 \leq (1 + \varepsilon) \mathbb{E}[\Delta y_{i+1}]^2 + C_z(\Delta t_i)^2 \mathbb{E}|\Delta y_{i+1}|^2 + C(1 + |x|^2)(\Delta t_i)^3, \]  
(2.13)

where \( \varepsilon := \varepsilon_3 + \varepsilon_4 + \varepsilon_5 \), and \( C_z = 6K + \frac{2K}{\varepsilon_4} \). Thanks to Cauchy’s inequality, we have

\[ \mathbb{E}[\Delta y_{i+1}]^2 \leq \Delta t_i (\mathbb{E}[\Delta y_{i+1}]^2) - \mathbb{E}[\Delta y_{i+1}]^2. \]  
(2.14)

By plugging (2.14) and (2.13) into (2.10), we obtain

\[ C_1^2 \mathbb{E}|\Delta y_i|^2 \leq C_2^2 \mathbb{E}|\Delta y_{i+1}|^2 + C_3^2 \Delta t_i \mathbb{E}|\Delta y_{i+1}|^2 + C(1 + |x|^2)(\Delta t_i)^2, \]  
(2.15)

where \( C_1 = 1 - 2K \left( \frac{1}{\varepsilon_1} + \Delta t_i + \frac{\Delta t_i}{\varepsilon_2} \right)(\Delta t_i) \), \( C_2^2 = 1 + 2K \frac{1+\varepsilon}{\varepsilon_2} \), and \( C_3 = \varepsilon_1 + 2K(1 + \varepsilon_2) + 2K(1 + \frac{1}{\varepsilon_2})(1 + \varepsilon) + 2K(\frac{1}{\varepsilon_1} + \Delta t_i + \frac{\Delta t_i}{\varepsilon_2})C_z \). Combining (2.13), (2.14), and (2.10), we have

\[ \mathbb{E}|\Delta y_i|^2 \leq \mathbb{E}[\mathbb{E}[\Delta y_{i+1}]^2] + C_4^2(\mathbb{E}[\Delta y_{i+1}]^2 - \mathbb{E}[\mathbb{E}[\Delta y_{i+1}]^2]) \]
\[ + C \Delta t_i \mathbb{E}|\Delta y_{i+1}|^2 + C(1 + |x|^2)(\Delta t_i)^2, \]

where \( C_4^2 = 2K \frac{1+\varepsilon}{\varepsilon_1} \). We see that \( C_4^2 \leq 1 \), with appropriately chosen \( \varepsilon_1, \varepsilon_3, \varepsilon_4 \), and \( \varepsilon_5 \). Thus,

\[ \mathbb{E}|\Delta y_i|^2 \leq (1 + C \Delta t_i) \mathbb{E}|\Delta y_{i+1}|^2 + C(1 + |x|^2)(\Delta t_i)^2. \]  
(2.16)

Applying the discrete Gronwall’s inequality to (2.16) gives

\[ \max_{0 \leq i \leq N} \mathbb{E}|\Delta y_i|^2 \leq C(1 + |x|^2)|\pi|. \]

\[ \square \]

The following theorem shows the convergence of the time-discretization scheme (2.7).

**Theorem 2.6.** Let \( (H1)-(H3) \) hold. Then there exists a constant \( C \), independent of \( k, l \), the time partition \( \pi \) and the space dimension \( d \), such that

\[ \max_{0 \leq i \leq N-1} \mathbb{E}[u(t_i, X^{t_i}_i, x) - u_i]^2 \leq C|\pi|. \]
Proof. It follows from Proposition 2.5 that
\[
E \left[ \int_{\mathbb{R}^d} \left( \left| u\left( t_i, X_{t_i}^{x_i} \right) - u_i(x) \right|^2 \omega(x) \right) dx \right] \\
= E \left[ \int_{\mathbb{R}^d} \left( \left| Y_{t_i}^{x_i} X_{t_i}^{x_i} - y_i^* \right|^2 \omega(x) \right) dx \right] \\
\leq \int_{\mathbb{R}^d} \max_{0 \leq i \leq N-1} E \left| Y_{t_i}^{x_i} X_{t_i}^{x_i} - y_i^* \right|^2 \omega(x) dx \\\n\leq C|\pi| \int_{\mathbb{R}^d} (1 + |x|^2) \omega(x) dx \leq C|\pi|.
\]

\hfill \square

3 Deep learning-based algorithm

In this section, we propose a deep learning-based numerical scheme for the weak solution of (1.1) based on the discrete time approximations (2.6) and (2.7) in Section 2. We also analyze its convergence.

3.1 Notation and algorithm

Let us begin with a general review of how deep neural networks approximate unknown functions. Consider an \( L + 1 \)-layered \((L > 1)\) feedforward neural network \( N_{L+1} \). The input layer will be referred to as layer 0 and the output layer as layer \( L \). The layers in between are called hidden layers. The basic building block of a neural network is an artificial neuron or node. Each input \( i_k \) has an associated weight \( w_k \). For the sake of simplicity, the bias terms are not treated specially, as they correspond to a weight with a fixed input of 1. The sum of all weighted inputs, \( S = \sum_k i_k w_k \), is passed through a nonlinear activation function \( \sigma \), to transform the preactivation level of the neuron to an output \( o = \sigma(\sum_k i_k w_k) \). The output \( o \) is used as an input by nodes in the next layer. Let \( n_l \) denote the number of neurons on the \( l \)-th layer. For simplicity, we set \( n_l = \bar{n} \), for \( l = 1, \cdots, L-1 \). Thus a feedforward neural network is a function defined as the composition
\[
N_{L+1}^{n_0, \bar{n}, n_L}(x_0, W, \sigma_0, \cdots, \sigma_L) := \sigma_L \circ S_L \circ \sigma_{L-1} \circ S_{L-1} \circ \cdots \circ \sigma_0 \circ S_0(x_0),
\]
where \( x_0 \in \mathbb{R}^{n_0} \) is the network input, \( S_l \) and \( \sigma_l \) are the weighted sum and the activation function of the \( l \)-th \((l = 0, \cdots, L)\) layer, respectively. \( W \) contains all the parameters (weights and biases) of the network. The dimension of \( W \) is \( n_W = \sum_{l=1}^{L} n_l \times n_{l-1} + n_{l-1} = (n_0 + 1)\bar{n} + (\bar{n} + 1)\bar{n}_L \).

Let \( \pi : 0 = t_0 < t_1 < \cdots < t_N = T \) be a partition of \([0, T]\). The forward SDE (1.2) is solved numerically using the Euler method (2.4). For every fixed path \( \{ \Delta B_i, 0 \leq i \leq N-1 \} \) of the Brownian motion \( B \), the solution \( \psi \) of (1.5) at \( t = t_i \) is approximated by
\[
\psi(t_i, \cdot) \approx \Psi^*_i(X_{t_{i+1}}^\pi, \Delta B_i),
\]
with
\[
\Psi_i^*(X_{t_{i+1}}^*, \Delta B_i) = \mathcal{U}_i^*(X_{t_{i+1}}^*, \Delta B_i) + g(t_{i+1}, X_{t_{i+1}}^*, \mathcal{U}_i^*(X_{t_{i+1}}^*)) \Delta B_i,
\]
where the notation \( \mathcal{U}_i^*(X_{t_{i+1}}^*) \) will be explained later. At each time step \( t_i \), \( i = N - 1, \ldots, 0 \), the solution \( \varphi \) of (1.6) is approximated by
\[
\varphi(t_i, \cdot) \approx \Psi_i^*(X_{t_{i+1}}^*, \Delta B_i) - \mathcal{V}_i(X_{t_{i+1}}^*; \theta_i)\sigma(t_i, X_{t_{i+1}}^*) \Delta W_i + f(t_i, X_{t_{i+1}}^*, \mathcal{U}_i^*(X_{t_{i+1}}^*; \theta_i), \mathcal{V}_i(X_{t_{i+1}}^*; \theta_i)\sigma(t_i, X_{t_{i+1}}^*)) \Delta t_i,
\]
(3.2)
where \( \mathcal{U}_i(\cdot; \theta_i) \in \mathcal{N}_{d, \tilde{n}}^{d, \tilde{n}} \) and \( \mathcal{V}_i(\cdot; \theta_i) \in \mathcal{N}_{d, \tilde{n}}^{d, \tilde{n}} \) are a pair of neural networks with parameter \( \theta_i \). The optimal parameters \( \theta_i^* \) are determined by minimizing the loss function
\[
L_i(\theta_i) = \mathbb{E}[\Psi_i^*(X_{t_{i+1}}^*, \Delta B_i) - \mathcal{V}_i(X_{t_{i+1}}^*; \theta_i)\sigma(t_i, X_{t_{i+1}}^*) \Delta W_i + f(t_i, X_{t_{i+1}}^*, \mathcal{U}_i^*(X_{t_{i+1}}^*; \theta_i), \mathcal{V}_i(X_{t_{i+1}}^*; \theta_i)\sigma(t_i, X_{t_{i+1}}^*)) \Delta t_i - \mathcal{U}_i(X_{t_{i+1}}^*; \theta_i)]^2.
\]
Set \( \mathcal{U}_i^*(X_{t_{i+1}}^*) = \mathcal{U}_i(X_{t_{i+1}}^*; \theta_i^*) \) and \( \mathcal{V}_i^*(X_{t_{i+1}}^*) = \mathcal{V}_i(X_{t_{i+1}}^*; \theta_i^*) \). Then the value of the solution \( u \) of (1.1) at time \( t_i \) is approximated by \( \mathcal{U}_i^*(X_{t_{i+1}}^*) \). We should point out here that \( \mathcal{V}_i^*(X_{t_{i+1}}^*) \) is not an approximation of \( \nabla u \) at time \( t_i \). In comparison to the global optimization problem formulated in [47], which requires to keep in memory all the computed network approximations of \( u \), the memory requirements for our proposed algorithm are significantly lower. The complete computational pseudocode can be found in Algorithm 1.

**Algorithm 1:**

**Data:** Sample paths of \( \{X_{t_i}^N\}_{i=0}^{N} \), \( \{\Delta W_i\}_{i=0}^{N-1} \) and \( \{\Delta B_i\}_{i=0}^{N-1} \)

1. \( \mathcal{U}_N^* = h(X_{t_0}^N) \).
2. for \( i = N - 1, \ldots, 0 \) do
3. \( \Psi_i^*(X_{t_{i+1}}^*, \Delta B_i) = \mathcal{U}_i^*(X_{t_{i+1}}^*, \Delta B_i) + g(t_{i+1}, X_{t_{i+1}}^*, \mathcal{U}_i^*(X_{t_{i+1}}^*)) \Delta B_i \)
4. Train a pair of deep neural networks \( \mathcal{U}_i^*(X_{t_{i+1}}^*; \theta_i) \) and \( \mathcal{V}_i^*(X_{t_{i+1}}^*; \theta_i) \) for the approximation of \( (\varphi_{t_{i+1}}^*, \nabla \varphi_{t_{i+1}}^*) \) and minimize \( L_i(\theta_i) \). Set
5. \( \theta_i^* = \text{arg min} L_i(\theta_i) \).
6. \( u(0, X_0) = Y_0^0, \bar{X}_0 \approx \mathcal{U}_0^*(X_{t_0}^N) \).

The following theorem presents the convergence result for the algorithm.

**Theorem 3.1.** Let (H1)-(H3) hold. Suppose that the partition \( \pi \) satisfies the regularity constraint \( \frac{|\pi|}{\min \Delta t_i} \leq c_0 \). Then there exists a constant \( C \), independent of \( k, \bar{l} \), the time partition \( \pi \) and the space dimension \( d \), such that
\[
\max_{0 \leq t \leq N-1} \mathbb{E}[u(t_i, X_{t_i}^{i,x}) - \mathcal{U}_i^*(X_{t_i}^*)]^2 \leq C \left( |\pi| + \frac{c_0}{|\pi|} \sum_{i=0}^{N-1} \left( \inf_{\theta_i} \mathbb{E}[\mathcal{U}_i^*(X_{t_i}^*; \theta_i) - \mathcal{U}_i^*(X_{t_i}^*)]^2 \right) \right).
\]

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Proof. From (2.9), (3.1) and (3.2), we have

\[ Y_{t_0} - U_t(X^\pi_{t_0}; \theta_i) = E_i[Y_{t_0} - \Psi(t_0, X^\pi_{t_0}) + \int_{t_0}^{t_{i+1}} g(r, X, Y) \, dB_i + E_i \left[ \int_{t_0}^{t_{i+1}} g(r, X, Y) \, dB_r \right] \]

\[ + E_i \left[ \int_{t_0}^{t_{i+1}} f(r, X, Y, Z) \, dr \right] f(t_i, X^\pi_{t_0}, U(t_i, X^\pi_{t_0}; \theta_i), Y_i(X^\pi_{t_0}; \theta_i) \sigma(t_i, X^\pi_{t_0}) \Delta t_i \right] \]

\[ = E_i[Y_{t_{i+1}} - U_{t_{i+1}}(X^\pi_{t_{i+1}})] + E_i \left[ \int_{t_{i+1}}^{t_{i+1}} g(r, X, Y) \, dB_r \right] - g(t_{i+1}, X^\pi_{t_{i+1}}, U_{t_{i+1}}(X^\pi_{t_{i+1}})) \Delta t_i \]

Using an argument similar to the proof of Proposition 2.5, we obtain

\[ E[Y_{t_{i+1}} - U_{t_{i+1}}(X^\pi_{t_{i+1}})]^2 \leq (1 + C \Delta t_i) E[Y_{t_{i+1}} - U_{t_{i+1}}(X^\pi_{t_{i+1}})]^2 + C(1 + |x|^2)(\Delta t_i)^2. \]

It follows by the above and the Young's inequality that

\[ (1 - \Delta t_i) E[Y_{t_i} - U^*_t(X^\pi_{t_i})]^2 + \left(1 - \frac{1}{\Delta t_i} \right) E[U^*_t(X^\pi_{t_i}) - U_t(X^\pi_{t_i}; \theta_i)] \]

\[ \leq E[Y_{t_i} - U^*_t(X^\pi_{t_i})]^2 + \left(1 - \frac{1}{\Delta t_i} \right) E[U^*_t(X^\pi_{t_i}) - U_t(X^\pi_{t_i}; \theta_i)] \]

\[ \leq (1 + C \Delta t_i) E[Y_{t_{i+1}} - U_{t_{i+1}}(X^\pi_{t_{i+1}})]^2 + C(1 + |x|^2)(\Delta t_i)^2. \]

Thus

\[ E[Y_{t_i} - U^*_t(X^\pi_{t_i})]^2 \leq (1 + C \Delta t_i) E[Y_{t_{i+1}} - U_{t_{i+1}}(X^\pi_{t_{i+1}})]^2 + C \frac{1}{\Delta t_i} E[U^*_t(X^\pi_{t_i}) - U_t(X^\pi_{t_i}; \theta_i)] \]

\[ \leq (1 + C \Delta t_i) E[Y_{t_{i+1}} - U_{t_{i+1}}(X^\pi_{t_{i+1}})]^2 + C \frac{C_{\theta_i}}{|\sigma|} E[U^*_t(X^\pi_{t_i}) - U_t(X^\pi_{t_i}; \theta_i)] \]

\[ \leq (1 + C \Delta t_i) E[Y_{t_{i+1}} - U_{t_{i+1}}(X^\pi_{t_{i+1}})]^2 + C(1 + |x|^2)(\Delta t_i)^2. \]

Taking the infimum over network parameters \( \theta_i \) (the optimization/training process), we have, by the discrete Gronwall’s inequality that

\[ E[Y_{t_i} - U^*_t(X^\pi_{t_i})]^2 \leq C \sum_{i=0}^{N-1} \left( \frac{C_{\theta_i}}{|\sigma|} \inf_{\theta_i} E[U^*_t(X^\pi_{t_i}) - U_t(X^\pi_{t_i}; \theta_i)] \right) + C(1 + |x|^2)|\pi|. \]

By using an argument similar to the proof of Theorem 2.6, we obtain the desired result.

Remark 3.2. Theorem 3.1 indicates that the approximation error of the proposed algorithm is small if the neural network approximation errors \( E[U^*_t(X^\pi_{t_i}) - U_t(X^\pi_{t_i}; \theta_i)] \), \( i = 0, \cdots, N - 1 \), are small. In addition, the number of parameters \( n_W \) used in the neural network grows at most polynomially in both the space dimension \( d \) and the reciprocal of the prescribed accuracy of the neural network approximation errors, as proved in [32] for semilinear heat equations with gradient-independent nonlinearities.
4 Numerical results

In this section, we test the algorithm proposed in Section 3 on a concrete SPDE for which we know the exact solution. The numerical experiments are performed in Python using TensorFlow 2.7 on a laptop equipped with an Intel Core i5 Processor with 1.8GHz. We use fully connected layers and batch normalization \[33\] after each matrix multiplication and before activation. We employ the Leaky Rectified Linear Unit function as the activation function for the hidden layers, and the identity function as the activation function for the output layer. For optimization, we use Adam optimizer \[37\] with the exponential decay rate of 0.9 for the first moment estimates, and the exponential decay rate of 0.999 for the second moment estimates. The training was on mini-batches with 64 trajectories of \(X\) per batch for 100 epochs. All the weights in the network are initialized using a uniform distribution. We use adaptive learning rate for the training process with a starting value of 0.01 and drop it by half if the loss doesn’t decrease for 10 consecutive epochs.

The following example is adapted from Section 4 in \[50\], the original equation is a backward doubly stochastic differential equation. We test the algorithm in four cases, a one-dimensional case, a moderate dimensional case (5-dimensional), and two high-dimensional cases (50 and 100-dimensional).

Example 1. Let us consider the SPDE (1.1) with the following parameters:

\[
\begin{align*}
&k = 1, l = 1, \\
&\sigma = 0.25, T = 1, \\
&\mu(t,x) = \sigma \sqrt{d} \sin(x), \\
&\sigma(t,x) = \sigma \sqrt{d} I_d, \\
&h(x) = \sqrt{d} \arctan\left(\frac{1}{d} \sum_{j=1}^{d} x_j\right) + \frac{\pi}{2} \sqrt{d},
\end{align*}
\]

\[
\begin{align*}
f &= -\sigma \sin(x) \nabla u, \\
g &= -\sigma \sqrt{d} \sin^2\left(\frac{1}{\sqrt{d}} u\right).
\end{align*}
\]

Set \(X_0 = 0\) (To avoid overfitting, we set \(X_0 \sim U(-0.2, 0.2)\)). We explicitly know the solution:

\[
u(t, X_t) = \sqrt{d} \arctan\left(\frac{1}{d} \sum_{j=1}^{d} (X_t)_j - \sigma(B_T - B_t)\right) + \frac{\pi}{2} \sqrt{d}.
\]

In dimension one, we discretize the equation using \(N = 32\) time steps. Each of the neural networks consists of 2 hidden layers with 11 nodes. The number of iterations per epoch is set to 100. We present the simulation results in Table 1. For each fixed sample path of \(B\), we provide the average \(\bar{U}_{0}^*(X_{\pi 0}^*)\) and the standard deviation of \(\bar{U}_{0}^*(X_{\pi 0}^*)\) by performing 5 independent runs of the algorithm. The relative error is

\[
\frac{|\bar{U}_{0}^*(X_{\pi 0}^*) - u(0, X_0)|}{u(0, X_0)}.
\]

Figures 1-5 illustrate a comparison between the exact solution and the approximate solutions at time \(t = 0.03125\) for each realization of \(B\) in Table 1, respectively. The black solid line is the exact solution, and the dashed lines are the approximate solutions. We also report the relative \(L^2\) error, which is computed based on five realizations of \(B\).
| Averaged Approx. | Exact solution | Standard deviation | Relative error |
|-----------------|----------------|-------------------|---------------|
| 1.5998764       | 1.596926235    | 0.004728534       | 0.001847402   |
| 1.9575781       | 1.967079301    | 0.002868398       | 0.004830106   |
| 1.8510513       | 1.847140899    | 0.001399888       | 0.002117002   |
| 1.8063872       | 1.808914966    | 0.004120083       | 0.001397394   |
| 1.4118652       | 1.403613807    | 0.00232844        | 0.005878677   |

Relative $L^2$ error: 0.00368065398642483

Table 1: Numerical results for the one-dimensional case

Figure 1: Comparison between the exact solution and the approximate solutions for the 1st sample path of $B$ ($d = 1$)
Figure 2: Comparison between the exact solution and the approximate solutions for the 2nd sample path of $B (d = 1)$

Figure 3: Comparison between the exact solution and the approximate solutions for the 3rd sample path of $B (d = 1)$
Figure 4: Comparison between the exact solution and the approximate solutions for the 4th sample path of $B (d = 1)$

Figure 5: Comparison between the exact solution and the approximate solutions for the 5th sample path of $B (d = 1)$
Results in dimension 5, 50 and 100 are given in Tables 2-4. The number of
time steps is set to 16. Each of the neural networks consists of 2 hidden layers,
the number of nodes for each hidden layer is set to \(d+10\) for 5-dimensional case,
and \(d+50\) for 50 and 100-dimensional cases. To illustrate the accuracy of the
proposed algorithm, we plot in Figures 6-10 the exact solution and the approx-
imation solution of the 100-dimensional case as a function of \(\frac{1}{100} \sum_{j=1}^{100} (X_t)_j\) at
t = 0.0625 for each realization of \(B\) in Table 4.

| Averaged Approx. | Exact solution | Standard deviation | Relative error |
|------------------|----------------|--------------------|----------------|
| 3.7816436        | 3.783391082    | 0.01756304         | 0.000461883    |
| 4.0550942        | 4.092778688    | 0.01245616         | 0.009207556    |
| 3.5838552        | 3.589354974    | 0.025385832        | 0.01532246     |
| 3.9931598        | 4.034977758    | 0.03498262         | 0.010363863    |
| 3.665762         | 3.675363075    | 0.006242963        | 0.002662832    |

Relative L^2 error: 0.00635359012791855

Table 2: Numerical results for the 5-dimensional case.

| Averaged Approx. | Exact solution | Standard deviation | Relative error |
|------------------|----------------|--------------------|----------------|
| 13.869227        | 13.80994108    | 0.0389653          | 0.004292989    |
| 11.789759        | 11.7779362     | 0.026603295        | 0.001015927    |
| 11.142225        | 11.12731684    | 0.02187582         | 0.00133978     |
| 8.79433          | 8.691944684    | 0.0351124          | 0.011779334    |
| 8.846188         | 8.819915756    | 0.02828248         | 0.002978741    |

Relative L^2 error: 0.0058117548123784

Table 3: Numerical results for the 50-dimensional case.

| Averaged Approx. | Exact solution | Standard deviation | Relative error |
|------------------|----------------|--------------------|----------------|
| 13.589519        | 13.50290922    | 0.06967104         | 0.006414157    |
| 10.588926        | 10.61844227    | 0.061020687        | 0.002779717    |
| 19.077723        | 18.69746132    | 0.048847217        | 0.02033761     |
| 14.551335        | 14.54826577    | 0.023958674        | 0.000210969    |
| 17.445202        | 17.53231125    | 0.022984497        | 0.004968498    |

Relative L^2 error: 0.00987134339284379

Table 4: Numerical results for the 100-dimensional case.
Figure 6: Comparison between the exact solution and the approximate solutions for the 1st sample path of $B$ ($d = 100$)

Figure 7: Comparison between the exact solution and the approximate solutions for the 2nd sample path of $B$ ($d = 100$)
Figure 8: Comparison between the exact solution and the approximate solutions for the 3rd sample path of $B (d = 100)$

Figure 9: Comparison between the exact solution and the approximate solutions for the 4th sample path of $B (d = 100)$
Conclusions

In this work, we have studied a deep learning-based numerical method for solving high dimensional stochastic partial differential equations in a predictor-corrector framework. At each time step, the original SPDE is decomposed into a degenerate SPDE to serve as the prediction step, and a second-order deterministic PDE to serve as the correction step. The solution of the degenerate SPDE is approximated by the Euler method, and the solution of the PDE is approximated by neural networks via the equivalent backward stochastic differential equation. We have analyzed the convergence of the proposed algorithm and provide error estimates in terms of the approximation error of neural networks. The efficiency and accuracy of the proposed algorithm are illustrated by the numerical results.

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