An important area of study in arithmetic combinatorics is the $h$-fold sumset. For a set $A \subset \mathbb{Z}^d$, the $h$-fold sumset is

$$hA = \{a_1 + \cdots + a_h : a_i \in A\}.$$

An important contribution to this area of study is the following theorem due to Khovanskii:

**Theorem 1.1.** [K] Given a finite set $A \subset \mathbb{Z}^d$, there exists a polynomial $p \in \mathbb{Q}[x]$ of degree at most $d$ such that $|hA| = p(h)$ for all sufficiently large $h$. Further, if $A - A$ generates all of $\mathbb{Z}^d$ additively, then $\deg p = d$ and the leading coefficient of $p$ is the volume of the convex hull of $A$.

The proof of this theorem was, however, ineffective, as it yielded no information about the polynomial past its degree and leading term. There have been successes in bounding the integer $h_0$ such that $|hA| = p(h)$ for $h \geq h_0$ ([GSW], for instance).

However, in a recent paper ([CG]), the cardinality of $|hA|$ has been completely described for all positive integers $h$, where $A \subset \mathbb{Z}^d$ is a set with $d + 2$ elements, such that $A - A$ additively generates $\mathbb{Z}^d$.

**Definition 1.2.** For $A \subset \mathbb{Z}^d$, we will denote the convex hull of $A$ with $\Delta_A$.

The main result of the paper [CG] is

**Theorem 1.3.** [CG, Theorem 1.2] Suppose $A \subset \mathbb{Z}^d$ consists of $d + 2$ elements, and further that $A - A$ generates $\mathbb{Z}^d$ additively. Then
\[ |hA| = \left( \frac{h + d + 1}{d + 1} \right) \] whenever \( 0 \leq h < \text{vol}(\Delta_A) \cdot d! \)

and

\[ |hA| = \left( \frac{h + d + 1}{d + 1} \right) - \left( \frac{h - \text{vol}(\Delta_A) \cdot d! + d + 1}{d + 1} \right) \] whenever \( h \geq \text{vol}(\Delta_A) \cdot d! \).

The proof in [CG] treats two cases in two different ways. The first case is when \( \Delta_A \) is a simplex with \( d + 1 \) vertices and \((d + 2)\)-nd element of \( A \) is in \( \Delta_A \), and the second case is when \( \Delta_A \) is a polytope with \( d + 2 \) vertices. However, the proof of the second case contained a misstep as noticed in the first version of this paper, (see section 3). In the second version of the paper [CG] this proof is corrected using the same idea as in the original proof.

In this paper we provide a different approach to the proof of [CG, Theorem 1.2] and establish the more general result treating also the sets \( A \) for which the set \( A - A \) does not necessarily generate \( \mathbb{Z}^d \) additively. This is our main result and we show that the theorem from [CG] is its direct corollary. An additional advantage of our approach is that we obtain a shorter and simpler proof which does not treat two cases in different ways but provides a unified proof.

In the last section we briefly discuss the sets \( A \subset \mathbb{Z}^d \) of \( d + 3 \) elements. We want to show that this situation is considerably more complicated, as two such sets with the same convex hull and the same \( d + 2 \) elements could produce different polynomials. So, it might be of some interest to obtain an upper bound for \( hA \) in this case.

2. Lemmas

For \( v = (v_1, \ldots, v_d) \in \mathbb{Z}^d \), we define its lift to be \( \tilde{v} = (v_1, \ldots, v_d, 1) \in \mathbb{Z}^{d+1} \). If \( v = (v_1, \ldots, v_d) \in \mathbb{Z}^d \) and \( h \in \mathbb{N} \), then we write \((v, h)\) instead of \((v_1, \ldots, v_d, h)\), and refer to \( h \) as the height of this point.

**Definition 2.1.** For a set \( A = \{v_1, \ldots, v_k\} \subset \mathbb{Z}^d \), the cone of \( A \) is

\[ C_A := \text{span}_\mathbb{N}(\tilde{v}_1, \ldots, \tilde{v}_k) = \{n_1\tilde{v}_1 + \cdots + n_k\tilde{v}_k : n_1, \ldots, n_k \in \mathbb{N}\}. \]

To the cone \( C_A \), we associate the generating series

\[ C_A(t) := \sum_{a \in C_A} t^{\text{height}(a)}. \]

Since the points at height \( h \) form a copy of \( |hA| \) embedded in \( \mathbb{Z}^{d+1} \), we have that

\[ C(t) = \sum_{h \geq 0} |hA| t^h. \]
Let $A$ be a $d + 3$ element set in $\mathbb{Z}^d$ such that $A - A$ generates $\mathbb{Z}^d$ additively and $\Delta_A$ is a $d$-simplex. Denote the $d + 1$ vertices of $\Delta_A$ by $v_1, \ldots, v_{d+1}$. These span a lattice $\Lambda = \text{span}_\mathbb{Z}(\tilde{v}_1, \ldots, \tilde{v}_{d+1})$ in $\mathbb{Z}^{d+1}$. For such a lattice, we denote the set

$$\Pi := \left\{ \sum_{i=1}^{d+1} \lambda_i \tilde{v}_i : 0 \leq \lambda_i < 1 \right\} \cap \mathbb{Z}^{d+1}.$$

In the paper, we will also encounter $\Lambda^+ := \text{span}_\mathbb{N}(\tilde{v}_1, \ldots, \tilde{v}_{d+1})$, and define $N_\Lambda$ as the cardinality of $\Pi$.

We partition $C_A$ into residue classes $\pi$ (mod $\Lambda$), each of which can be represented by an element of $\Pi$. For $\pi \in \Pi$, we denote its residue class by $S_\pi$. An element $(g, N) \in S_\pi$ is said to be minimal if $(g, N) - \tilde{v}_i$ does not lie in $C_A$ for any $i$.

From the geometry of numbers, we know that if we have a lattice $\Lambda = \text{span}(\tilde{v}_1, \ldots, \tilde{v}_{d+1})$ in $\mathbb{Z}^{d+1}$ with a fundamental domain of nonzero volume, then $\mathbb{Z}^{d+1}/\Lambda$ can be identified with the set of lattice points in the fundamental domain of $\Lambda$, and that this number is equal to the determinant of the matrix whose columns are the generating vectors $\tilde{v}_i$. Therefore,

$$N_\Lambda = |\mathbb{Z}^{d+1}/\Lambda| = \text{vol}(\Delta_A)d!.$$

(For this claim, we refer the reader for example to [N, Ch. 6, Sec. 1].)

**Lemma 2.2.** [CG, Lemma 3.1] If $(\alpha, M)$ is a minimal element of $S_\pi$, then

$$M \leq N_\Lambda - 1.$$

**Lemma 2.3.** Let $v_1, \ldots, v_{d+1} \in \mathbb{Z}^d$ be vectors that generate $\mathbb{Z}^d$ and $\det(\tilde{v}_1, \ldots, \tilde{v}_{d+1}) \neq 0$, and let $w \in \mathbb{Z}^d$ be an integer vector such that $\tilde{w} = a_1 \tilde{v}_1 + \cdots + a_{d+1} \tilde{v}_{d+1}$, where $a_i$ are non-negative coefficients such that $a_1 + \cdots + a_{d+1} = 1$. Then $a_i$ are rational numbers.

**Proof.** Since $\tilde{v}_1, \ldots, \tilde{v}_{d+1}, \tilde{w}$ are all integer vectors and $\det(\tilde{v}_1, \ldots, \tilde{v}_{d+1}) \neq 0$, by Cramer’s rule coefficients $a_i$ will be rational numbers. $\square$

We will also need a well known result from Combinatorial geometry, Radon theorem and its extension which we prove here.

**Theorem 2.4.** Every set $S$ of $d + 2$ points in $\mathbb{R}^d$, could be split in two disjoint subsets $S = S_1 \cup S_2$ such that the convex hulls of $S_1$ and $S_2$ intersect, i.e. $\text{conv}S_1 \cap \text{conv}S_2 \neq \emptyset$.

Actually we need the following extension of this theorem, saying that in generic case (when no $d + 1$ points belong to the same hyperplane), this splitting is unique.

**Theorem 2.5.** Let $S = \{x_1, x_2, ..., x_{d+2}\}$ be the set of points in $\mathbb{R}^d$ (no $d + 1$ of which belong to the same hyperplane) and let $S = S_1 \cup S_2$ be the splitting satisfying $\text{conv}S_1 \cap \text{conv}S_2 \neq \emptyset$. Then two points $x_i, x_j \in S$ belong to the same of two sets $S_1$ and $S_2$ if and only if they belong to different halfspaces determined by the hyperplane spanned by the remaining $d$ points of the set $S$. 
Proof. Let the points \( x_i, x_j \in S \) belong to the same halfspace \( H_+ \) determined by the hyperplane \( H \) spanned by the remaining \( d \) points of \( S \). Then points \( x_i \) and \( x_j \) could not belong to the same of two sets \( S_1 \) and \( S_2 \). Namely, if \( x_i, x_j \in S_1 \), then \( S_2 \subseteq H \) and \( S_1 \subseteq H_+ \). Then \( S_1 \cap S_2 \subseteq H \) and so \( \operatorname{conv}(S_1 \setminus \{x_i, x_j\}) \cap \operatorname{conv}S_2 \neq \emptyset \). This is impossible since the set \( S \setminus \{x_i, x_j\} \) consists of \( d \) points in generic position.

Let the points \( x_i, x_j \in S \) belong to different halfspaces determined by the hyperplane \( H \) spanned by the remaining \( d \) points of \( S \). (For example, let \( x_i \in H_+ \) and \( x_j \in H_- \).) Then points \( x_i \) and \( x_j \) could not belong to different of two sets \( S_1 \) and \( S_2 \). Namely, if \( x_i \in S_1 \) and \( x_j \in S_2 \), then \( S_1 \subseteq H_+ \) and \( S_2 \subseteq H_- \). Then \( S_1 \cap S_2 \subseteq H \) and so \( \operatorname{conv}(S_1 \setminus \{x_i\}) \cap \operatorname{conv}(S_2 \setminus \{x_j\}) \neq \emptyset \). This is impossible since the set \( S \setminus \{x_i, x_j\} \) consists of \( d \) points in generic position. \( \square \)

3. Addressing a misstep in 1.3, non-simplicial case

The approach in [CG] in the non-simplicial case is based on presenting \( \Delta_A \) as the union of two simplices with a common \( d-1 \) face. But, some convex polytopes with \( d+2 \) vertices cannot be split into two simplices. To see this, we will need the extension of Radon’s theorem (see Theorem 2.5): a set \( X \) of \( d+2 \) elements in \( \mathbb{R}^d \) can be split into two disjoint subsets \( X = X_1 \cup X_2 \) such that the convex hulls of \( X_1 \) and \( X_2 \) intersect and this splitting is unique (in a generic case). Namely, we proved that two vertices are in the same set \((X_1 \text{ or } X_2)\) if and only if they belong to different half-spaces determined by the hyperplane spanned by the remaining \( d \) vertices.

Let \( A \) be a set with \( d+2 \) elements that has a convex hull that is split into two \( d \)-simplices with a common \( d-1 \) face. We will denote the set of vertices determined by the common face of these two simplices by \( X_1 \), and set \( X_2 \) will be comprised of the remaining two vertices. Then, the convex hulls of \( X_1 \) and \( X_2 \) will intersect. Therefore, in this situation, one set will always have \( d \) elements, and one will have 2 elements. If a set may be split into two sets which both have at least three elements, and their convex hulls intersect, then the convex hull \( \Delta_A \) cannot be split into two simplices.

An easy example of such a set is \( A = \{P_1(1, 0, 0, 0), P_2(0, 1, 0, 0), P_3(0, 0, 1, 0), Q_1(0, 0, 0, 1), Q_2(0, 0, 0, 0), Q_3(1, 1, 1, -1)\} \). We see that convex hulls of \( X_1 = \{P_1, P_2, P_3\} \) and \( X_2 = \{Q_1, Q_2, Q_3\} \) intersect at common barycenter \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)\). It is easy to see that \( A - A \) generates \( \mathbb{Z}^d \). If \( A \) did have a splitting into two simplices, one set would have 2, and one would have 4 elements, which is not the case here. Since these splittings are unique, we conclude that \( \Delta_A \) does not split into two simplices.

4. The main theorem

We start by an example.

Example 4.1. Let \( d = 2 \) and consider the set of four points \( A = \{(1, 0), (0, 1), (-1, 0), (0, -1)\} \). For any \( h \in \mathbb{N} \) the set \( hA \) contains one point with the first coordinate equal to \( h \) or \(-h\), two points with the first coordinate equal to \( h - 1 \)
or \(-(h - 1)\), three points with the first coordinate equal to \(h - 2\) or \(-(h - 2)\), etc. Finally, the set \(hA\) contains \(h + 1\) points with the first coordinate equal to 0.

Therefore the cardinality of the set \(hA\) is \(|hA| = 2(1 + 2 + \cdots + h) + h + 1 = (h + 1)^2\).

Applying Theorem 1.3 to the set \(A\) would imply that the cardinality of \(hA\) is \(\left(\frac{h + 3}{3} - \frac{h - 1}{3}\right) = 2h^2 + 2\), which is incorrect. This happens since the set \(A\) does not satisfy the assumption of Theorem 1.3 that the set \(A - A\) generates \(\mathbb{Z}^d\) additively.

Here we provide a little bit more general result which contains Theorem 1.3 as a special case, and also treats the sets of points not satisfying the assumption that \(A - A\) generates \(\mathbb{Z}^d\) additively, like in the example above.

Let us consider the set \(A = \{v_1, \ldots, v_{d+2}\}\) of \(d + 2\) points in \(\mathbb{Z}^d\), no \(d + 1\) of which are contained in the same hyperplane. By \(\tilde{v}_i = (v_i, 1)\) we denoted the lifts of these points in \(\mathbb{Z}^{d+1}\), for \(i \in \{1, 2, \ldots, d + 2\}\).

Let us now denote, for \(i \in \{1, 2, \ldots, d + 2\}\), \(D_i = \det(\tilde{v}_1, \ldots, \tilde{v}_{i-1}, \tilde{v}_{i+1}, \ldots, \tilde{v}_{d+2})\), and \(D = \text{GCD}(D_1, \ldots, D_{d+2})\). We state now our main theorem.

**Theorem 4.2.** Let \(A = \{v_1, \ldots, v_{d+2}\} \subset \mathbb{Z}^d\) be a set of \(d + 2\) points, no \(d + 1\) of which are contained in the same hyperplane. Then

\[
|hA| = \left(\frac{h + d + 1}{d + 1}\right), \quad \text{for } 1 \leq h < \text{Vol}(\Delta_A)d! / D
\]

and

\[
|hA| = \left(\frac{h + d + 1}{d + 1}\right) - \left(\frac{h - \text{Vol}(\Delta_A)d! / D + d + 1}{d + 1}\right), \quad \text{for } h \geq \text{Vol}(\Delta_A)d! / D.
\]

**Proof:**

Let \(A = \{v_1, \ldots, v_{d+2}\} \subset \mathbb{Z}^d\), and let \(h\) be a positive integer. First we will look for non-trivial solutions of the system of equations in variables \(\alpha_1, \ldots, \alpha_{d+2}\)

\[
\alpha_1 v_1 + \cdots + \alpha_{d+2} v_{d+2} = 0,
\]

\[
\alpha_1 + \cdots + \alpha_{d+2} = 0.
\]

This is equivalent to

\[
\alpha_1 \tilde{v}_1 + \cdots + \alpha_{d+2} \tilde{v}_{d+2} = (0, 0).
\]

Points \(v_1, \ldots, v_{d+2}\) are affine-dependent, so there exists a non-trivial solution \(\mu_1, \ldots, \mu_{d+2}\) of the above system of equations.

By the generic position, \(D_{d+2} = \det(\tilde{v}_1, \ldots, \tilde{v}_{d+1}) \neq 0\), and also \(\mu_{d+2} \neq 0\), since otherwise points \(v_1, \ldots, v_{d+1}\) would be affine-dependent.

Multiplying the equality \(\mu_1 \tilde{v}_1 + \cdots + \mu_{d+2} \tilde{v}_{d+2} = (0, 0)\) by \(1/\mu_{d+2}\), we get the identity

\[
\frac{\mu_1}{\mu_{d+2}} \tilde{v}_1 + \cdots + \frac{\mu_{d+1}}{\mu_{d+2}} \tilde{v}_{d+1} = -\tilde{v}_{d+2}. \tag{4.1}
\]
By Cramer’s rule, we have that

\[
\frac{\mu_i}{\mu_{d+2}} = \frac{\det(\tilde{v}_1, \ldots, \tilde{v}_i, -\tilde{v}_{d+2}, \tilde{v}_{i+1}, \ldots, \tilde{v}_{d+1})}{\det(\tilde{v}_1, \ldots, \tilde{v}_{d+1})}.
\]

Let \( \lambda_i := \det(\tilde{v}_1, \ldots, \tilde{v}_{d+1}) \cdot \frac{\mu_i}{\mu_{d+2}} = \det(\tilde{v}_1, \ldots, \tilde{v}_{i-1}, -\tilde{v}_{d+2}, \tilde{v}_{i+1}, \ldots, \tilde{v}_{d+1}) \in \mathbb{Z} \) for \( 1 \leq i \leq d+2 \). Notice that \( \lambda_i = \pm D_i \), i.e. these numbers are equal up to the sign. Multiplying identity (4.1) by \( \det(\tilde{v}_1, \ldots, \tilde{v}_{d+1}) \), we get

\[
\lambda_1 \tilde{v}_1 + \cdots + \lambda_k \tilde{v}_k + \lambda_{k+1} \tilde{v}_{k+1} + \cdots + \lambda_{d+2} \tilde{v}_{d+2} = (0, 0). \tag{4.2}
\]

Now, since the coefficients \( \lambda_i \) are all divisible by \( D \), we could divide this identity by \( D \) and obtain

\[
\frac{\lambda_1}{D} \tilde{v}_1 + \cdots + \frac{\lambda_k}{D} \tilde{v}_k + \frac{\lambda_{k+1}}{D} \tilde{v}_{k+1} + \cdots + \frac{\lambda_{d+2}}{D} \tilde{v}_{d+2} = (0, 0). \tag{4.3}
\]

Notice that all coefficients are integers and that they have no common divisor. Without loss of generality, we may assume that \( \lambda_1, \ldots, \lambda_k \geq 0 \) and \( \lambda_{k+1}, \ldots, \lambda_{d+2} < 0 \).

Let us now denote \( r = \lambda_1 + \cdots + \lambda_k \). On a side note, we can deduce from this equation that the convex hull of \( X_1 = \{ v_1, \ldots, v_k \} \) intersects with the convex hull of \( X_2 = \{ v_{k+1}, \ldots, v_{d+2} \} \). In particular, if \( k = 1 \), then one set of vertices has a \( d \)-simplex as a convex hull, and the other is a vertex contained in the mentioned simplex.

Now, let \( w \in hA \) have two representations (with non-negative coefficients):

\[
\alpha_1 v_1 + \cdots + \alpha_{d+2} v_{d+2} = \beta_1 v_1 + \cdots + \beta_{d+2} v_{d+2}.
\]

Then their difference is 0. Furthermore, the sum of coefficients is \( \sum_{i=1}^{d+2} (\alpha_i - \beta_i) = 0 \). Therefore, the difference has to be a multiple of the left-hand side of (4.3). To each element \( w \in hA \) corresponds exactly one non-negative representation \( \alpha_1 v_1 + \cdots + \alpha_{d+2} v_{d+2} \) for which \( \alpha_i < \lambda_i / D \) for at least one \( 1 \leq i \leq k \). Namely, if \( \alpha_i \geq \lambda_i / D \) for \( 1 \leq i \leq k \), we can reduce this representation to the also non-negative representation \( (\alpha_1 - \lambda_1 / D) \tilde{v}_1 + \cdots + (\alpha_{d+2} - \lambda_{d+2} / D) \tilde{v}_{d+2} \). To obtain other non-negative representations, we can only add a multiple of (4.3). Therefore, to obtain the number of elements in \( hA \), we need to take the number of all non-negative representations for which \( \sum_{i=0}^{d+2} \alpha_i = h \), and reduce it by the number of non-negative representations for which \( \sum_{i=0}^{d+2} \alpha_i = h \) and \( \alpha_i \geq \lambda_i / D \), for all \( 1 \leq i \leq k \). Therefore, if \( r / D \leq h \), we have that

\[
|hA| = \binom{d + h + 1}{h} - \binom{d + 1 + h - r / D}{h - r / D} = \binom{d + h + 1}{d + 1} - \binom{d + 1 + h - r / D}{d + 1}.
\]

Otherwise, we have that

\[
|hA| = \binom{d + h + 1}{d + 1}.
\]
Let us now determine \( r \). We will denote the \( d \)-simplex determined by the vertices \( v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{d+2} \) by \( \Delta_i \). Note that \( \lambda_i = \pm \text{vol}(\Delta_i) \cdot d! \).

By the extension of Radon’s theorem (Theorem 2.5), every generic point in \( \Delta_A \) (that is not contained in any \( d - 1 \) dimensional face of these simplices) is contained in exactly two simplices \( \Delta_i \) and \( \Delta_j \), and they are such that the vertices \( v_i \) and \( v_j \) belong to different sets \( X_1 \) and \( X_2 \).

Namely, if \( x \in \Delta_i \), let \( l \) be a half-line starting from \( v_i \) passing through \( x \), and let the final point of intersection of this half-line with the boundary of \( \Delta_i \) belong to the face \( (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{d+2}) \). Then \( \Delta_j \) is the unique other simplex containing the point \( x \). Since the interiors of \( \Delta_i \) and \( \Delta_j \) intersect, vertices \( v_i \) and \( v_j \) belong to the same half-space determined by the hyperplane spanned by the remaining \( d \) vertices. By Radon’s theorem, this means that one of \( v_i \) and \( v_j \) belongs to \( X_1 \), and the other belongs to \( X_2 \).

From this we see that \( \Delta_A \) has a covering:

\[
\Delta_A = \Delta_1 \cup \cdots \cup \Delta_k = \Delta_{k+1} \cup \cdots \cup \Delta_{d+2}.
\]

Since intersections of any two of the simplices \( \Delta_1, \ldots, \Delta_k \) and any two of the simplices \( \Delta_{k+1}, \ldots, \Delta_{d+2} \) have volume 0, and since \( \lambda_1, \ldots, \lambda_k \geq 0 \), we have that

\[
r = \lambda_1 + \cdots + \lambda_k = \text{vol}(\Delta_1) \cdot d! + \cdots + \text{vol}(\Delta_k) \cdot d! = \text{vol}(\Delta_A) \cdot d!.
\]

Now, we want to show that Theorem 1.3 is a direct corollary of Theorem 4.2. First, we prove the following

**Proposition 4.3.** For a set \( A = \{v_1, \ldots, v_{d+2}\} \) for which the set \( A - A \) generates \( \mathbb{Z}^d \) additively, the determinants \( D_1, \ldots, D_{d+2} \) have no common divisor.

**Proof:** Suppose, to the contrary, that the determinants \( D_1, \ldots, D_{d+2} \) have common divisor \( m \geq 2 \). Notice that for every \( i \in \{1, 2, \ldots, d + 1\} \), by subtracting the last column from other columns we have

\[
D_i = \begin{vmatrix}
v_1 - v_{d+2} & v_2 - v_{d+2} & \cdots & v_i - v_{d+2} & v_{i+1} - v_{d+2} & \cdots & v_{d+1} - v_{d+2} & v_{d+2} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
v_1 - v_{d+2} & v_2 - v_{d+2} & \cdots & v_i - v_{d+2} & v_{i+1} - v_{d+2} & \cdots & v_{d+1} - v_{d+2} & v_{d+2} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\
\end{vmatrix}
\]

Since the vectors \( v_1 - v_{d+2}, \ldots, v_{d+1} - v_{d+2} \) generate \( \mathbb{Z}^d \) additively, then the unit vectors \( e_j \) of standard basis could be represented as the linear combinations with integer coefficients of these vectors \( v_i - v_{d+2} \). This implies that the determinant of the identity matrix equals (by linearity) the combination with integer coefficients of determinants, some of which are 0 (if they have two columns equal) and the remaining are divisible by \( m \). This contradiction proves the proposition.
It is easy to see now that Theorem 1.3 is a direct corollary of Theorem 4.2. Namely, if we suppose that $A-A$ generates $\mathbb{Z}^d$ additively, by the above proposition, $D = \gcd(D_1, \ldots, D_{d+2}) = 1$, and the Theorem 1.3 follows.

Let us now turn back to Example 4.1. If we apply Theorem 4.2, we see that $D_1, D_2, D_3, D_4 = 2$ and so $D = 2$. So theorem says

$$hA = \left(\frac{h+3}{3}\right) - \left(\frac{h+3-2\cdot2}{3}\right) = \left(\frac{h+1}{3}\right) = (h+1)^2,$$

as we showed in Example 4.1.

5. Sumsets of a set with $d+3$ elements

In this section we treat the case of the set $A$ of $d+3$ points in $\mathbb{Z}^d$, especially the case when $d+1$ of them are the vertices of a simplex containing the remaining two points.

Let us start with some examples illustrating the fact that this case is more complicated. In particular, we will see that the value $|hA|$ does not depend only on the convex hull $\Delta_A$ of the set $A$ as in the previous case of the sets of $d$ points, but also on the position of two remaining points inside $\Delta_A$. Consequently, it is not a surprise that we do not determine the exact value of $|hA|$, but provide the upper bound for this value.

Example 5.1. Let $d = 1$. We will consider several sets of 4 integers, all of them containing integers 0, 1 and 8 and the fourth integer being one of 2, 3, 4, 5, 6, 7.

Let $A = \{0, 1, 2, 8\}$. It is easy to see that for $h$ large enough, the set $hA$ consists of all integers from 0 to $8h$ (so, $8h+1$ of them) except for the integers $8h-1 = 8(h-1) + 7, 8h-2 = 8(h-1) + 6, 8h-3 = 8(h-1) + 5, 8h-4 = 8(h-1)+4, 8h-5 = 8(h-1)+3, 8h-9 = 8(h-2)+7, 8h-10 = 8(h-2)+6, 8h-11 = h(h-2) + 5, 8h-17 = 8(h-3) + 7$. Therefore, $|hA| = 8h + 1 - 9 = 8h - 8$ in this case.

Similarly, if $h$ is large enough, for the set $A = \{0, 1, 3, 8\}$ we have $|hA| = 8h + 1 - 7 = 8h - 6$; for the set $A = \{0, 1, 4, 8\}$ we have $|hA| = 8h + 1 - 9 = 8h - 8$; for the set $A = \{0, 1, 5, 8\}$ we have $|hA| = 8h + 1 - 5 = 8h - 4$; for the set $A = \{0, 1, 6, 8\}$ we have $|hA| = 8h + 1 - 3 = 8h - 2$; and for the set $A = \{0, 1, 7, 8\}$ we have $|hA| = 8h + 1$.

Just for illustration, for $A = \{0, 1, 6, 8\}$, the set $hA$ consists of all integers from 0 to $8h$ except for the integers $8(h-1) + 3, 8(h-1) + 5, 8(h-1) + 7$. Notice that for the set $A = \{0, 1, 7, 8\}$, the set $hA$ consists of all integers from 0 to $8h$. The convex hull of all these sets is the same, the interval $[0, 8]$, and all of them contain the same integer 1. However, the values of $|hA|$ differ.

Denote the vertices of $\Delta_A$ by $v_1, \ldots, v_{d+1}$, let $w$ be the $(d+2)^{\text{nd}}$ element of $A$, and suppose that the $(d+3)^{\text{rd}}$ is 0. Set
Λ := span⁡(\tilde{v}_1, \ldots, \tilde{v}_{d+1}),
Λ^+ := span₂(\tilde{v}_1, \ldots, \tilde{v}_{d+1}),
Λ^+_{(0,1)} := span₂((0, 1), \tilde{v}_1, \ldots, \tilde{v}_{d+1}),
N_A := The number of integer points in the fundamental domain of Λ.

Let C_A be the cone over A. It is equal to

\[ \bigcup_{m=0}^{\infty} \left( (mw, m) + \Lambda^+_{(0,1)} \right). \]

However, vector (w, 1) has finite order in the group \( \mathbb{Z}^{d+1}/\Lambda \), which will be denoted by \( o_w \). It can be seen that \( o_w(w, 1) \in \Lambda^+ \).

To prove this, first note that w belongs in the interior of simplex \( \Delta_A \), which is why \( (w, 1) \) belongs to the boundary of the simplex determined by vertices \((0, 0), \tilde{v}_1, \ldots, \tilde{v}_{d+1}\). Therefore, vector \((w, 1)\) has barycentric coordinates \( 0 \leq \mu_1, \ldots, \mu_{d+1} \leq 1 \) such that

\[ \sum_{i=1}^{d+1} \mu_i \tilde{v}_i = (w, 1). \]

By Lemma 2.3, \( \mu_i \) must be rational, and therefore \( \mu_i = \frac{a_i}{q_i} \) for \( 1 \leq i \leq d+1 \), where \( 0 \leq a_i \leq q_i \) and \( (a_i, q_i) = 1 \). The order \( o_w \) of \((w, 1)\) is the least common container of \( q_1, \ldots, q_{d+1}, lcc(q_1, \ldots, q_{d+1}) \). Since \( o_w \frac{a_i}{q_i} \) are all non-negative integers, \( o_w(w, 1) \in \Lambda^+ \). Therefore,

\[ C_A = \bigcup_{m=0}^{o_w-1} \left( (mw, m) + \Lambda^+_{(0,1)} \right). \]

This union need not be disjoint. From [CG, Theorem 1.2, simplicial case], we have that \( \Lambda^+_{(0,1)}(t) = \frac{1-t^{\text{Vol}(\Delta_A)d}}{1-t} \). If \( B_A(t) \) is the generating series

\[ B_A(t) = \sum_{m=0}^{o_w-1} \sum_{h \geq 0} \binom{h + d + 1}{h} t^h, \]

the generating series \( C_A(t) = \sum_{h \geq 0} |hA| t^h \) will have coefficients \( |hA| \leq b_h \) (If the union in the cone had been disjoint, there would have been an equality instead).

From this we see an upper bound:

\[ |hA| \leq \]
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\sum_{m=0}^{h} \binom{m+d+1}{m}, & h \leq o_w - 1 \\
\sum_{m=0}^{o_w-1} \binom{h+d+1-m}{h-m}, & o_w \leq h \leq N_{\Lambda} - 1 \\
\sum_{m=0}^{o_w-1} \binom{h+d+1-m}{h-m} - \sum_{m=0}^{h-N_{\Lambda}} \binom{m+d+1}{m}, & N_{\Lambda} \leq h \leq N_{\Lambda} + o_w - 1 \\
\sum_{m=0}^{o_w-1} \binom{h+d+1-m}{h-m} - \sum_{m=0}^{h-N_{\Lambda}+d+1-m} \binom{h-N_{\Lambda}+d+1-m}{h-N_{\Lambda}-m}, & h \geq N_{\Lambda} + o_w.
\end{array} \right.
\]

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