Numerical Solution of the Ekpyrotic Scenario in the Moduli Space Approximation

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A numerical solution to the equations of motion for the ekpyrotic bulk brane scenario in the moduli space approximation is presented. The visible universe brane has positive tension, and we use a potential that goes to zero exponentially at large distance, and also goes to zero at small distance. In the case considered, no bulk brane, visible brane collision occurs in the solution. This property and the general behavior of the solution is qualitatively the same when the visible brane tension is negative, and for many different parameter choices.

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I. INTRODUCTION

The ekpyrotic scenario was first presented in [1]. Based on string theory [2, 3], it is a cosmological theory containing three-branes in a bulk space with four spatial dimensions. One of these branes is the observable three-dimensional universe. The theory was proposed as an alternative to the big bang/inflationary theory, and avoids the space-time singularity of the big bang. What is perceived by the inhabitants of the visible three-brane as a big bang, is explained five-dimensionally as a collision between a bulk brane and the visible brane. This collision is called ekpyrosis. As the collision is totally inelastic, this sends the universe into a hot, expanding phase.

This theory has later been modified into a cyclic universe model, without a bulk brane [4, 5]. In the cyclic model, the big bang is realized as a collision between the visible and the hidden brane. In this scenario, the boundary branes collide, which means that the collision is accompanied by a strong space-time singularity, whereby an entire dimension collapses to zero physical length.

In section II we recall the mathematical basis for the ekpyrotic scenario. Section III contains the static solutions that is the basis for the moduli space approximation. We describe the moduli space approximation in section IV. The numerical results from solving the moduli space approximation equations are given in section V. In section VI we give our conclusions.

II. MODEL

The ekpyrotic scenario is based on Simplified Heterotic M theory, which is Heterotic M theory with the fields whose equation of motion allow it, set to zero [6, 7].

The space-time of the theory is the five-dimensional manifold $M_5 \equiv \mathbb{R}^4 \times S_1/\mathbb{Z}_2$, which we call the bulk space and is given by the following construction. The $\mathbb{R}^4$ part is the usual four-dimensional infinite space-time, with coordinates $(t, x^1, x^2, x^3)$. The extra dimension $S_1/\mathbb{Z}_2$, with coordinate $y$, is obtained from the circle $S_1$ as follows. Let $y \in [-R, R]$ with identification $R \sim -R$ be a coordinate on $S_1$. Let $\mathbb{Z}_2$ be the group of reflections of $S_1$ about $y = 0$. We use these actions of $\mathbb{Z}_2$ on $S_1$ to define the new manifold $S_1/\mathbb{Z}_2$ by identifying points on $S_1$ that are related by an action of $\mathbb{Z}_2$. Since we then have the identification $y \sim -y$, the coordinate on $S_1/\mathbb{Z}_2$ is $y \in [0, R]$, so this dimension is a closed line element $[0, R]$. $S_1/\mathbb{Z}_2$ is an orbifold, since it consists of the orbits of the group action of $\mathbb{Z}_2$ on $S_1$, and will be referred to as the orbifold dimension.

Since the extra dimension is a line element, this space-time has two boundaries, at $y \in \{0, R\}$. Differentiation is not well-defined on the boundary of a manifold, so we define the action $\mathcal{M}$ of the theory on the boundary-less manifold $\mathbb{R}^4 \times S_1$, making differentiation well-defined everywhere. We must then demand $\mathbb{Z}_2$ reflection symmetry $y \sim -y$ from all the fields in the Lagrangian.

The action of the Simplified Heterotic M-theory is

$$S = \frac{M_5^3}{2} \int_{M_5} d^5x \sqrt{-g} \left( R - \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{3}{2} \frac{1}{5!} \epsilon^{i\beta_1\beta_2\gamma} A_{i\beta_1\beta_2\gamma} \right) + \frac{1}{4!} \epsilon^{\alpha\beta_1\beta_2\gamma} A_{\alpha\beta_1\beta_2\gamma} X_{i\beta_1}(\xi_{i\beta_1}) X_{i\beta_2}(\xi_{i\beta_2}) X_{i\gamma}(\xi_{i\gamma}) \partial_i X_{i\alpha} - \frac{1}{4!} \epsilon^{\alpha\beta_1\beta_2\gamma} A_{\alpha\beta_1\beta_2\gamma} \partial_i X_{i\beta_1}(\xi_{i\beta_1}) X_{i\beta_2}(\xi_{i\beta_2}) X_{i\gamma}(\xi_{i\gamma}) \partial_i X_{i\alpha},$$

where $M_5$ is the 5D Planck mass, $g_{AB}$ is the metric on $M_5$, and $x$ is its determinant. Uppercase Latin indices refer to coordinates in the bulk space, and small Greek indices refer to coordinates on a brane. $R$ is the 5D Ricci scalar, and $\epsilon^\phi$ is the size of the Calabi-Yau 6D compact manifold. $\mathcal{A}$ is a bulk four-form gauge field, with field strength $\mathcal{F} \equiv d\mathcal{A}$. Furthermore, we have contributions from the three-branes, which couple to the bulk fields $\phi$ and $A$. The brane tensions are $3\alpha_i M_5^3/2$, and their actions are given by the four dimensional integrals over their world-volumes $M_4^{(i)}$. $\xi_{i\alpha}$ are coordinates on the branes. The corresponding integrands contain the induced metric on their world-volumes, $h^{(i)}_{\alpha\beta}$, which is the pull-back of the bulk metric onto $M_4^{(i)}$. $X^\alpha_{(i)}(\xi_{(i)})$ are the coordinates in $M_5$ of a point whose coordinates in $M_4^{(i)}$ are $\xi_{(i)}$. $A_{\alpha\beta_1\beta_2\gamma} \partial_i X_{(i)}^{\alpha} \partial_{\beta_1} X_{(i)}^{\beta_2} \partial_{\gamma} X_{(i)}^{\gamma} \partial_i X_{(i)}^{\alpha}$ is the induced
four-form field on the brane world-volume, from the bulk four-form field $\mathcal{A}$. Anomaly cancellation, which must be satisfied to enable a consistent quantization of the theory, demands that the brane tensions sum up to zero. Therefore we parametrize them as $\alpha_1 = -\alpha$, $\alpha_2 = \alpha - \beta$ and $\alpha_3 = \beta$.

### III. STATIC SOLUTION

A vacuum solution of this theory contains flat, static, parallel branes. This means that we can express their embeddings as

$$X^A_{(i)} = (t, x^1, x^2, x^3, y(i)).$$

We choose brane 1 and 2 as the boundary branes at $y(1) = 0$ and $y(2) = R$. Brane 3 is the bulk brane at $y(3) \equiv Y$. A static solution to the equations of motion is given by the expression

$$\alpha \equiv \sqrt{CA}.$$

From (2) we note that the physical length of the orbifold dimension is given by the expression

$$L = B \int_0^R dy D(y)^2 = \frac{1}{3} B \left[ \frac{1}{\alpha} ((\alpha Y + C)^3 - C^3) + \frac{1}{\alpha - \beta} \right. \times \left. \left( ((\alpha - \beta)R + \beta Y)^3 - (\alpha Y + C)^3 \right) \right].$$

### IV. MODULI SPACE APPROXIMATION

The moduli space approximation involves making a static solution to the time-dependent functions, then substituting this into the action of the brane potential $V(Y)$ to obtain an action for the new time-dependent quantities. The result is only valid for slowly moving systems, and with insignificant matter production on the branes.

After integrating over the $y$-direction, the resulting moduli space approximation Lagrangian is

$$\mathcal{L} = -\frac{3M_s^2 A^3 B}{N} \left[ I_3 \left( \frac{A}{A} \right)^2 + \frac{A}{B} \left( \frac{I_3 B}{B} + 3I_2 C \right)^2 \right] + \frac{3}{12} \beta I_2 b \dot{Y} - \frac{1}{12} \beta I_3 b \dot{B}^2 + \frac{1}{2} \beta I_1 b \dot{C}^2 + \beta I_1 b Y \left( \dot{C} + \frac{3}{2} \dot{Y} \right) - \beta \left( \frac{1}{2} B^2 \dot{Y}^2 - N^2 V(Y) \right),$$

where we have added a dimensionless bulk brane potential $V(Y)$ that was not in the original action (1). The $I_m$ functions are integrals of powers of the function (1) over the orbifold dimension.

$$I_{m,a} \equiv 2 \int_0^Y D^m dy = \frac{2}{\alpha(m + 1)} \left[ (\alpha Y + C)^{m+1} - C^{m+1} \right]$$

$$I_{m,b} \equiv 2 \int_0^Y D^m dy = \frac{2}{(\alpha - \beta)(m + 1)} \times \left[ ((\alpha - \beta)R + \beta Y + C)^{m+1} - (\alpha Y + C)^{m+1} \right]$$

Since the function $D$ is different on each side of the bulk brane at $y = Y$, the limit $Y \rightarrow R$ of equation (4) is different from the limit $Y \rightarrow 0$ of equation (10). The equations of motion are the Euler-Lagrange equations obtained from the Lagrangian in equation (8). We use the gauge choice $N(t) = 1$ to obtain simpler equations. Note that since $g_{t\mu} = -D(y)N^2$, cosmic time gauge on the visible brane at $y = 0$ would correspond to the gauge choice $N(t) = 1/\sqrt{C(t)}$.

We will solve Euler-Lagrange equations from this action numerically, and the result will be different from the results in the original article (1), in which the authors analyzed the equations after choosing $B(t)$ and $C(t)$ to be constant, because that choice is not in accordance with the equations of motion.

### V. BULK BRANE POTENTIAL

In (1), it is argued that the bulk brane potential must satisfy the following conditions:

- It must depend only on the $y$ coordinate, since the bulk brane movement in the orbifold direction must be independent of all the other coordinates except time. This is so because the brane must remain approximately flat during its journey across the bulk. Otherwise it would hit the visible brane at significantly different times on different points in the visible universe, causing big inhomogeneities in the universe.
V′(y) must be small near y = R because otherwise the bulk brane will not be flat when it is emitted from the hidden brane. The bulk brane nucleation process starts at one point on the hidden brane, and grows outwards in all three spatial directions tangent to the brane. For the bulk brane to become flat, the nucleation growth rate must be much faster than the movement of the bulk brane in the orbifold direction.

We must have V(R) = 0 to prevent inflationary behavior on the hidden brane.

We need V(0) = 0 since otherwise it would contribute to a cosmological constant in the visible universe after the bulk brane collides.

As our dimensionless potential that satisfies the above conditions, we choose the following.

\[ V(Y) = -\frac{1}{10} (e^{-5Y/R} - e^{-5}) (1 - \frac{1}{10Y/R + 1}) \] (12)

VI. NUMERICAL RESULTS AND DISCUSSION

We solve the equations of motion derived from the Lagrangian \[ \mathcal{L} \], numerically from \( t = 0 \) to \( t = 4300/M_5 \) in Mathematica\textsuperscript{TM}, with parameter values \( R = 1/M_5 \), \( \alpha = -100M_5 \), \( \beta = M_5 \). The initial conditions are \( A(0) = B(0) = 1, C(0) = 200 \) and \( A'(0) = B'(0) = C'(0) = Y'(0) = 0 \). We use the potential in equation (12). The solutions are given graphically in figures 1 - 5.

In the course of the evolution, we see from figures 1 and 2 that the scale factors \( A(t) \) and \( B(t) \) increase monotonically. \( C(t) \) in figure 3, however, decreases monotonically. These three functions show no oscillatory tendencies, as opposed to \( Y(t) \) in figure 4 which moves back and forth in the bulk space. Analysis of the numerical solution shows that on its first encounter towards the visible brane, the time at which the separation is the smallest is \( t_c \approx 2080/M_5 \), with position \( Y(t_c) \approx 0.0017 \times R \), as seen in figure 5. It does not bump into the hidden brane either. The graph of \( Y(t) \) in figure 4 is resembles a damped cycloidal motion with decreasing wavelength. The amplitude and wavelength of the oscillations decrease until...
Figure 5: The bulk brane doesn’t collide with the visible universe.

Figure 6: The four-dimensional effective scale factor decreases as the system evolves.

The solution breaks down at around zero wavelength (not shown in the plot).

The effective scale factor $a_1(t)$ of the visible universe, given by equation (6), is plotted in figure 6. It decreases as the system evolves.

A plot of the physical length of the orbifold dimension, given by the expression (7), in figure 7. The physical length increases on the whole during the evolution, but displays small dips when the bulk brane is close to the visible brane.

So with these initial values for the unknown functions, there is no ekpyrosis. Simulations with many other values of the parameters and initial conditions show similar behavior with no ekpyrosis, so this is no special case. Using a parabolic potential also gives qualitatively the same type of solution. A collision can be induced by giving $Y(t)$ a sufficient, small initial velocity, so there no inherent barrier in the equations at $y = 0$ for the bulk brane movement. It is simply energy conservation that prohibits it from reaching $y = 0$.

This suggests that the action (11) in the moduli space approximation, is not capable of producing a big bang effect, which is necessary for the ekpyrotic scenario to send the visible universe into a hot, expanding phase. If there is significant matter production on the branes during evolution, the moduli space approximation is inaccurate, but this would probably lead to further energy dissipation in the bulk brane movement, and therefore not lead to ekpyrosis. For parameter choices where the branes reach each other within a length of the order of the Planck length, stringy effect could lead to ekpyrosis through modifications of the potential or brane capture, even though they don’t meet exactly in the classical solution.

VII. CONCLUSION

No collision is seen between the bulk brane and the visible brane in the case considered. The behavior of the solution is qualitatively the same when the visible brane tension is negative, and also when the bulk brane potential is parabolic, and also for many other values of the different parameters. The bulk brane seems to bounce back from the visible brane, without hitting it, in all cases.

Ways of modifying the model to provoke a collision are e.g. allowing $\lim_{Y \to 0} V(Y) < 0$, but this conflicts with the conditions on the potential put forth in [1]. A different method is to introduce extra kinetic energy into the initial condition for the bulk brane movement. This would introduce an extra fine-tuning into the theory, which is esthetically undesirable. Ideally, since the ekpyrotic scenario is built on M theory, the potential should be calculated from first principles before consequences can be deduced.
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