EXTENSIONS OF S-LEMMA FOR NONCOMMUTATIVE POLYNOMIALS

FENG GUO AND SIZHUO YAN AND LIHONG ZHI

Abstract. We consider the problem of extending the classical S-lemma from commutative case to noncommutative cases. We show that a symmetric quadratic homogeneous matrix-valued polynomial is positive semidefinite if and only if its coefficient matrix is positive semidefinite. Then we extend the S-lemma to three kinds of noncommutative polynomials: noncommutative polynomials whose coefficients are real numbers, matrix-valued noncommutative polynomials and hereditary polynomials. Some examples are given to demonstrate the relations between these new derived conditions.

1. Introduction

The classical S-lemma for commutative polynomials answers the question that when one quadratic inequality is a consequence of some other quadratic inequalities [16]. Thus, it is a special form of Positivstellensatz from real algebraic geometry which characterizes polynomials that are positive (nonnegative) on a semialgebraic set [3]. There are many important results of Positivstellensatz for noncommutative cases. Helton proved a remarkable result that positive noncommutative polynomials are sums of squares [6]. Helton and McCullough presented a noncommutative Positivstellensatz [11]. Helton, Klep, and McCullough gave a linear Positivstellensatz for characterizing the matricial linear matrix inequality (LMI) domination problems [7]. Their result was generalized by Zalar to solve the linear operator inequality (LOI) domination problems [20]. When the domain is convex [10], Helton, Klep, and McCullough established a perfect noncommutative Nichtnegativstellensatz in [5]. Furthermore, they studied the matrix convex hulls of free semialgebraic set in [9]. Our goal in this paper is to investigate how to extend S-lemma to noncommutative cases.

To state the main contributions of this paper, we need the following notations. The symbol $\mathbb{R}$ (resp. $\mathbb{N}, \mathbb{N}^+$) denotes the set of real (resp. natural, positive natural) numbers. For $n \in \mathbb{N}^+$, $\mathbb{R}^{n \times n}$ (resp. $\mathbb{S} \mathbb{R}^{n}$) stands for set of $n \times n$ real matrices (resp. symmetric matrices). For $m, n \in \mathbb{N}^+$, the symbol $(\mathbb{R}^{n \times n})^m$ (resp. $(\mathbb{S} \mathbb{R}^{n})^m$) denotes the vector space consisting of $m$-dimensional vectors of $n \times n$ real matrices (resp. symmetric matrices). The symbols $\phi, \psi$ are used to represent the linear maps between finite dimensional Euclidean spaces.

The main results of the paper are stated below. We start with the simplest case where the coefficients of noncommutative polynomials are real numbers.

---

2020 Mathematics Subject Classification. 90C20, 47A56, 46L07, 90C22, 14P10, 47A68.

Key words and phrases. S-lemma, noncommutative polynomials, positive semidefinite matrix, completely positive linear map.

This research is supported by the National Key Research Project of China 2018YFA0306702 (Zhi) and the National Natural Science Foundation of China 12071467 (Zhi).
Theorem 1.1. Let

\[ f = \sum_{i=1, j=1}^{m} a_{ij}x_i x_j, \quad g = \sum_{i=1, j=1}^{m} b_{ij}x_i x_j, \]

be homogeneous quadratic symmetric noncommutative polynomials, where \( a_{ij}, b_{ij} \in \mathbb{R} \) and \( a_{ij} = a_{ji}, b_{ij} = b_{ji} \) for all \( i, j \). Suppose that there is an \( \hat{X} \in (\mathbb{S} \mathbb{R}^{\hat{n}})^m \) for some \( \hat{n} \in \mathbb{N}^+ \), such that \( g(\hat{X}) > 0 \). Then the following three statements are equivalent:

1. For all \( X \in \mathbb{R}^m \), if \( g(X) \geq 0 \), then \( f(X) \geq 0 \).
2. For all \( X \in (\mathbb{S} \mathbb{R}^n)^m, n \in \mathbb{N}^+ \), if \( g(X) \geq 0 \), then \( f(X) \geq 0 \).
3. There is a nonnegative real number \( \lambda \) such that \( f(X) - \lambda g(X) \geq 0 \) for all \( X \in (\mathbb{S} \mathbb{R}^n)^m, n \in \mathbb{N}^+ \).

The main part of the paper is devoted to extend the S-lemma for noncommutative polynomials with matrix coefficients, i.e. matrix-valued polynomials. Let \( f(x) = \sum_{i=1, j=1}^{m} A_{ij} x_i x_j \) be a homogeneous quadratic symmetric matrix-valued polynomial, where \( A_{ij} = A_{ji}^T, A_{ij} \in \mathbb{R}^{q \times q} \) for all \( i, j \). We show first in Theorem 4.1 that \( f(X) = \sum_{i, j=1}^{m} A_{ij} \otimes X_i X_j \geq 0 \) for all \( X \in (\mathbb{S} \mathbb{R}^n)^m \) and \( n \in \mathbb{N}^+ \) if and only if its coefficient matrix \( A = (A_{ij}) \in \mathbb{S} \mathbb{R}^{mq} \) is positive semidefinite.

For \( n \in \mathbb{N}^+ \), let \( I_n \) represent the identity map from \( \mathbb{R}^{n \times n} \) to \( \mathbb{R}^{n \times n} \). Inspired by Choi’s characterization of a completely positive map via a positive semidefinite Choi matrix (Theorem 2.2), we generalize the condition of existing a nonnegative number \( \lambda \) such that \( f(X) - \lambda g(X) \geq 0 \) for all \( X \in \mathbb{R}^m \) to the existence of a completely positive linear mapping \( \phi : \mathbb{R}^{q \times q} \to \mathbb{R}^{q \times q} \) such that \( f(X) - (\phi \otimes I_n) g(X) \geq 0 \) for all \( X \in (\mathbb{S} \mathbb{R}^n)^m, n \in \mathbb{N}^+ \).

Theorem 1.2. Let

\[ f(x) = \sum_{i=1, j=1}^{m} A_{ij} x_i x_j, \quad g(x) = \sum_{i=1, j=1}^{m} B_{ij} x_i x_j, \]

be homogeneous quadratic symmetric matrix-valued polynomials, where \( A_{ij}, B_{ij} \in \mathbb{R}^{q \times q} \) and \( A_{ij} = A_{ji}^T, B_{ij} = B_{ji}^T \) for all \( i, j \). Suppose that there is an \( \hat{X} \in (\mathbb{S} \mathbb{R}^{\hat{n}})^m \) for some \( \hat{n} \in \mathbb{N}^+ \), such that \( g(\hat{X}) > 0 \). Then the following two statements are equivalent:

1. For all \( X \in (\mathbb{S} \mathbb{R}^n)^m, n > q \), if \( (I_q \otimes P) g(X)(I_q \otimes P) \geq 0 \), then \( (I_q \otimes P) f(X)(I_q \otimes P) \geq 0 \), where \( P : \mathbb{R}^n \to \mathbb{R}^q \) is the projection to the last \( q \) coordinates.
2. There is a nonzero completely positive linear mapping \( \phi : \mathbb{R}^{q \times q} \to \mathbb{R}^{q \times q} \) such that \( f(X) - (\phi \otimes I_n) g(X) \geq 0 \) for all \( X \in (\mathbb{S} \mathbb{R}^n)^m, n \in \mathbb{N}^+ \).

The following theorem is for a special case of matrix-valued hereditary polynomials. Its proof can be adjusted from the proof of Theorem 1.2.

Theorem 1.3. Let

\[ f(x) = \sum_{i=1, j=1}^{m} A_{ij} x_i x_j^T, \quad g(x) = \sum_{i=1, j=1}^{m} B_{ij} x_i x_j^T, \]

be homogeneous matrix-valued hereditary polynomials, where \( A_{ij}, B_{ij} \in \mathbb{R}^{q \times q} \) and \( A_{ij} = A_{ji}^T, B_{ij} = B_{ji}^T \) for all \( i, j \). Suppose that there is an \( \hat{X} \in (\mathbb{R}^{\hat{n} \times \hat{n}})^m \) for some \( \hat{n} \in \mathbb{N}^+ \), such that \( g(\hat{X}) > 0 \). Then the following two statements are equivalent:
For all $X \in (\mathbb{R}^{n \times n})^m$, $n \in \mathbb{N}^+$, if $g(X) \succeq 0$, then $f(X) \succeq 0$.

(2) There is a nonzero completely positive linear mapping $\phi : \mathbb{R}^{q \times q} \to \mathbb{R}^{q \times q}$, such that $f(X) - (\phi \otimes 1_n)g(X) \succeq 0$ for all $X \in (\mathbb{R}^{n \times n})^m$, $n \in \mathbb{N}^+$.

2. Preliminaries

2.1. Matrix-valued polynomials in symmetric entries. In this paper, we deal with matrix-valued noncommutative polynomials. Different from the commutative polynomials, the variables and coefficients are all matrices. The polynomial $p$ we considered in this paper has the following form:

$$p = \sum_{\omega \in \mathcal{W}_m} p_{\omega} \omega,$$

where $p_{\omega} \in \mathbb{R}^{q \times q}$, $q \in \mathbb{N}^+$ and $\mathcal{W}_m$ is a set of words generated by the entries of $x = [x_1, x_2, \ldots, x_m]^T$, and

$$p^T = \sum_{\omega \in \mathcal{W}_m} p_{\omega}^T \omega^T.$$

If $p = p^T$, we say $p$ is symmetric. When we evaluate a polynomial $p$ at $X \in (\mathbb{S} \mathbb{R}^n)^m$, we define the empty word as $\text{Id}_n$, where $\text{Id}_n$ denote the identity matrix in $\mathbb{R}^{n \times n}$ for $n \in \mathbb{N}^+$.

For symmetric quadratic homogeneous matrix-valued polynomials

$$f(x) = \sum_{i=1,j=1}^m A_{ij} x_i x_j \quad \text{and} \quad g(x) = \sum_{i=1,j=1}^m B_{ij} x_i x_j,$$

where $A_{ij} = A_{ji}^T, B_{ij} = B_{ji}^T, A_{ij} B_{ij} \in \mathbb{R}^{q \times q}$, the evaluations of $f$ and $g$ at $X \in (\mathbb{S} \mathbb{R}^n)^m$ are

$$f(X) = \sum_{i=1,j=1}^m A_{ij} \otimes X_i X_j \quad \text{and} \quad g(X) = \sum_{i=1,j=1}^m B_{ij} \otimes X_i X_j.$$

If we restrict the coefficients being real numbers, i.e., $q = 1$, then we have noncommutative polynomial

$$p = \sum_{\omega \in \mathcal{W}_m} p_{\omega} \omega, \quad p_{\omega} \in \mathbb{R}.$$

2.2. The classical S-lemma. When $f$ and $g$ are homogeneous quadratic polynomials, there are many different approaches for proving the S-lemma in commutative case. In [18], Yakubovich used the convexity result in [5] to prove the S-lemma. A modern proof can be found in the book by Ben-Tal and Nemirovski [2]. An elementary proof of the S-lemma could be derived based on a lemma given by Yuan [19]. See excellent survey on S-lemma by Pólik and Terlaky [16]. We introduce below one of them which is suitable for being extended to the noncommutative cases.

**Theorem 2.1.** Given $f, g : \mathbb{R}^m \to \mathbb{R}$ are homogeneous quadratic polynomials, and suppose there is an $X \in \mathbb{R}^m$ such that $g(X) > 0$. Then the following two statements are equivalent.

(1) For all $X \in \mathbb{R}^m$, if $g(X) \geq 0$, then $f(X) \geq 0$.

(2) There is a nonnegative real number $\lambda$ such that $f(X) - \lambda g(X) \geq 0$ for all $X \in \mathbb{R}^m$. 


Proof. Let \( f(x), g(x) \) be homogeneous quadratic polynomials. There are symmetric matrices \( A, B \in \mathbb{R}^{m \times m} \) such that
\[
f(x) = x^T Ax, \quad g(x) = x^T Bx.
\]
It is well known that \( h(X) = X^T H X \geq 0 \) for all \( X \in \mathbb{R}^m \) if and only if \( H \succeq 0 \).

The implication \((2) \implies (1)\) is obvious. Now, assume that the condition \((2)\) is false, we show the condition \((1)\) is false too. Consider two convex closed sets
\[
C = \{ M \succeq 0 \mid M \in \mathbb{R}^{m \times m} \},
\]
and
\[
D = \{ A - \lambda B \mid \lambda \geq 0 \}.
\]
As the condition \((2)\) is false, \( C \cap D = \emptyset \), i.e., there is no nonnegative real number \( \lambda \) such that \( A - \lambda B \succeq 0 \). Since there is an \( \hat{X} \) such that \( g(\hat{X}) > 0 \), we have \( g(\hat{X}) = \hat{X}^T B \hat{X} > 0 \), which means \( B \) must have a positive eigenvalue. Therefore, there must exist a large enough positive real number \( \lambda_0 \) such that \( A - \lambda_0 B \) has a negative eigenvalue. Therefore, for \( \lambda > \lambda_0 \), the distance between \( A - \lambda B \) and \( C \) will get larger when \( \lambda \to +\infty \). The topology and distance we used here are the general topology and distance of finite dimensional real Euclidean space.

It is clear that
\[
\inf\{\|M_1 - M_2\| \mid M_1 \in C, M_2 \in D\}
= \inf\{\|M_1 - M_2\| \mid M_1 \in C, M_2 \in \{ A - \lambda B \mid \lambda \leq \lambda_0 \}\} = d > 0.
\]

By the separation theorem \([17, \text{Theorem 11.4}]\), there exists an \( S \in \mathbb{R}^{m \times m} \), \( S \neq 0 \) such that
\[
\langle S, M_1 \rangle \geq a > \langle S, M_2 \rangle, \quad \text{for all} \quad M_1 \in C, M_2 \in D.
\]
As \( C \) is a positive semidefinite cone, \( S \succeq 0 \) and \( a = 0 \).

Since \( \langle S, A - \lambda B \rangle < 0 \) for all \( \lambda \geq 0 \). Let us assume \( \lambda = 0 \), then we have \( \langle S, A \rangle < 0 \). Let \( \lambda \to +\infty \), we have \( \langle S, B \rangle \geq 0 \). Since \( S \) is positive semidefinite, if \( \langle S, A \rangle < 0 \) and \( \langle S, B \rangle \geq 0 \), according to Corollary 6.1.4 in \([12]\), there exists an \( X \in \mathbb{R}^m \) such that \( X^T A X < 0 \) and \( X^T B X \geq 0 \). Hence we have found an \( X \in \mathbb{R}^m \) such that \( g(X) < 0 \) and \( f(X) < 0 \), which contradicts the condition \((1)\). \( \square \)

2.3. Completely positive linear map. A real number can be seen as a linear map form \( \mathbb{R} \) to \( \mathbb{R} \), and if the number is positive, the linear map translates a positive real number to a positive real number. Similarly, we can define positive linear maps and completely positive linear maps between real vector spaces of higher dimensions.

A linear map \( \phi : \mathbb{R}^{s \times s} \to \mathbb{R}^{t \times t} \), where \( s, t \in \mathbb{N}^+ \) can be represented by a matrix in \( \mathbb{R}^{(s \times t) \times (s \times t)} \)
\[
(2.1) \quad J(\phi) = \sum_{a,b=1}^s \phi(E_{ab}) \otimes E_{ab} = \begin{pmatrix} J_{11} & \cdots & J_{1t} \\ \vdots & \ddots & \vdots \\ J_{t1} & \cdots & J_{tt} \end{pmatrix},
\]
where \( J_{ij} \in \mathbb{R}^{s \times s} \) and \( E_{ab} \in \mathbb{R}^{s \times s} \) are matrices whose \( (a, b) \)-th entry is 1 and all others are 0. The matrix \( J(\phi) \) is called the Choi matrix of \( \phi \) \([4]\). It is easy to verify that for any \( M \in \mathbb{R}^{s \times s} \),
\[
\phi(M) = \begin{pmatrix} \langle J_{11}, M \rangle & \cdots & \langle J_{1t}, M \rangle \\ \vdots & \ddots & \vdots \\ \langle J_{t1}, M \rangle & \cdots & \langle J_{tt}, M \rangle \end{pmatrix}.
\]
We say that the linear map $\phi$ is positive, if for every positive semidefinite matrix $M \in \mathbb{R}^{s \times s}, M \succeq 0$, its image under the map $\phi$ is also positive semidefinite, i.e., $\phi(M) \succeq 0$. Recall that $\mathbb{1}_n$ represents the identity map from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n \times n}$. We say $\phi$ is completely positive, if for all $n \in \mathbb{N}^+$, the linear map $\phi \otimes \mathbb{1}_n$ is a positive linear map from $\mathbb{R}^{(sn) \times (sn)}$ to $\mathbb{R}^{(sn) \times (sn)}$.

**Theorem 2.2.** [4] The linear map $\phi : \mathbb{R}^{s \times s} \to \mathbb{R}^{t \times t}$ where $s, t \in \mathbb{N}^+$ is completely positive, if and only if the Choi matrix $J(\phi) \succeq 0$.

There is a one-to-one correspondence between the set of all completely positive maps from $\mathbb{R}^{s \times s}$ to $\mathbb{R}^{t \times t}$ and the set of positive semidefinite matrices in $\mathbb{R}^{(st) \times (st)}$.

3. **S-lemma of noncommutative polynomials**

In this section, we prove the S-lemma for noncommutative polynomials (Theorem 1.1). Suppose that we are given polynomials $f(x) = \sum_{i=1,j=1}^{m} a_{ij}x_i x_j$ and $g(x) = \sum_{i=1,j=1}^{m} b_{ij}x_i x_j$, where $a_{i,j}, b_{i,j} \in \mathbb{R}$ and $a_{i,j} = a_{j,i}, b_{i,j} = b_{j,i}$ for all $i,j$. Define

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mm} \end{pmatrix}.$$  

**Proof of Theorem 1.1** (3) $\Rightarrow$ (2) $\Rightarrow$ (1): The implications are obvious.

(1) $\Rightarrow$ (3): Assume that for all $X \in \mathbb{R}^{m}, g(X) = X^T B X \leq 0$. Then we know $B \preceq 0$ and hence

$$g(X) = X^T (B \otimes \mathbb{1}_n) X \leq 0,$$

for all $X \in (\mathbb{S}^n)^m, n \in \mathbb{N}^+$, which contradicts the condition that there is an $\tilde{X} \in (\mathbb{S}^n)^m$ for some $\hat{n} \in \mathbb{N}^+$, such that $g(\tilde{X}) \succ 0$. Hence, there always exists an $\tilde{X} \in \mathbb{R}^m$ such that $g(\tilde{X}) > 0$. According to Theorem 2.1, we can derive that there exists a positive real number $\lambda$ such that $f(X) - \lambda g(X) \succeq 0$ for all $X \in \mathbb{R}^m$, especially $A - \lambda B \succeq 0$. Then we know

$$f(X) - \lambda g(X) = X^T ((A - \lambda B) \otimes \mathbb{1}_n) X \succeq 0,$$

for all $X \in (\mathbb{S}^n)^m, n \in \mathbb{N}^+$.

4. **Positivity of symmetric quadratic homogeneous matrix-valued polynomials**

For a commutative polynomial $h(X) = X^T H X$ with $H \in \mathbb{S}^m$, we know $h(X) \succeq 0$ for all $X \in \mathbb{R}^m$ if and only if $H \succeq 0$. It is very interesting to see that this property can be extended to noncommutative polynomials.

**Theorem 4.1.** Let $f(x) = \sum_{i=1,j=1}^{m} A_{ij} x_i x_j$ be a symmetric quadratic homogeneous matrix-valued polynomial, where the matrices $A_{ij} = A^T_{ji} \in \mathbb{R}^{r \times q}$ for all $i,j$. Define the coefficient matrix

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}.$$
Then $f(X)$ is positive semidefinite for all $X \in (\mathbb{S}\mathbb{R}^n)^m$, $n \in \mathbb{N}^+$, if and only if $\mathcal{A}$ is positive semidefinite.

Proof. Let us rearrange the matrix $\mathcal{A}$ to define a matrix in $\mathbb{R}^{q \times q} \otimes \mathbb{R}^{m \times m}$.

\[
\mathcal{A}' = \begin{pmatrix}
A_{11}' & \cdots & A_{1q}' \\
\vdots & \ddots & \vdots \\
A_{q1}' & \cdots & A_{qq}'
\end{pmatrix}
\]

Using the matrix $\mathcal{A}'$ as the Choi matrix, define a linear map

\[
\psi_f : \mathbb{R}^{m \times m} \to \mathbb{R}^{q \times q}
M \mapsto \begin{pmatrix}
\langle A_{11}, M \rangle & \cdots & \langle A_{1q}, M \rangle \\
\vdots & \ddots & \vdots \\
\langle A_{q1}, M \rangle & \cdots & \langle A_{qq}, M \rangle
\end{pmatrix}.
\]

It is essential to notice that

\[
f(X) = \psi_f \otimes I_n \begin{pmatrix} X_1X_1 & \cdots & X_1X_m \\
\vdots & \ddots & \vdots \\
X_mX_1 & \cdots & X_mX_m
\end{pmatrix}
\]

for all $X \in (\mathbb{S}\mathbb{R}^n)^m$, $n \in \mathbb{N}^+$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ be the standard orthogonal basis of $\mathbb{R}^q$, $\{\beta_1, \beta_2, \ldots, \beta_m\}$ be the standard orthogonal basis of $\mathbb{R}^m$, and

\[
u = \sum_{j=1}^q \sum_{i=1}^m (\alpha_j \otimes \beta_i)(\beta_i \otimes \alpha_j)^T.
\]

Now let us assume that $\mathcal{A}$ is positive semidefinite. As the matrix $\mathcal{A}'$ is obtained after applying unitary transformation by $u$ to the matrix $\mathcal{A}$, $\mathcal{A}'$ is still positive semidefinite. According to Theorem 2.2, the linear map $\psi_f$ is completely positive. Hence, $\psi_f \otimes I_n$ is a positive linear map for all $n \in \mathbb{N}^+$. Since

\[
\begin{pmatrix} X_1X_1 & \cdots & X_1X_m \\
\vdots & \ddots & \vdots \\
X_mX_1 & \cdots & X_mX_m
\end{pmatrix} = \begin{pmatrix} X_1 \\
\vdots \\
X_m
\end{pmatrix} \cdot \begin{pmatrix} X_1, \ldots, X_m
\end{pmatrix}
\]

is positive semidefinite, we know that $f(X)$ is positive semidefinite for all $X \in (\mathbb{S}\mathbb{R}^n)^m$, $n \in \mathbb{N}^+$. 

On the other hand, we define $f'(X) = \sum_{i=1,j=1}^m X_iX_j \otimes A_{ij}$. It is obvious that for any $X \in (\mathbb{S}^n)^m$, $n \in \mathbb{N}^+$,

$$f(X) \succeq 0 \iff f'(X) \succeq 0.$$ 

Let $X^0 := (X^0_1, \ldots, X^0_m) \in (\mathbb{S}^{(m+1)}(\mathbb{R})^m$, where each $X^0_i$ is the matrix whose $(1, i+1)$-th entry and $(i+1, 1)$-th entry are 1 and all others are 0, i.e.,

$$X^0 := \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}.$$

It is easy to check that

$$f'(X^0) = \begin{pmatrix} \sum_{i=1}^m A_{i1} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}.$$ 

By assumption $f'(X^0) \succeq 0$, and hence we have $A \succeq 0$. \hfill \Box

From this theorem, using the spectral decomposition of $A$, we can factorize the symmetric quadratic homogeneous matrix-valued polynomials $f(x)$ which is positive semidefinite on $(\mathbb{S}^n)^m$ for all $n \in \mathbb{N}^+$ into the product of a linear homogeneous matrix-valued polynomial $U(x)$ and its transpose $U(x)^T$

$$f(x) = U(x)U(x)^T.$$ 

The dimension of coefficients of the polynomial $U$ is at most $(qm) \times (qm)$.

Let $h(x)$ be a matrix-valued polynomial having $m$ variables, its degree is at most $2l$ and coefficients are matrices belonging to $\mathbb{R}^{q \times q}$. If $h(X)$ is positive semidefinite for all $X \in (\mathbb{S}^n)^m$, $n \in \mathbb{N}^+$, McCullough has already shown there exists a matrix-valued polynomial $U(x)$, whose coefficients belong to $\mathbb{R}^{q(\sum_{i=0}^m l_i) \times (q \sum_{i=0}^m l_i)}$, such that $h(x) = U(x)^TU(x)$ \cite{13} Theorem 0.2. However, as the proofs use Arveson’s extension theorem \cite{11 \cite{14 \cite{15}}, it is unclear how to construct the factorization.

5. S-lemma of matrix-valued polynomials

Suppose that we are given polynomials

$$f(x) = \sum_{i=1,j=1}^m A_{ij}x_ix_j$$

and

$$g(x) = \sum_{i=1,j=1}^m B_{ij}x_ix_j,$$

where $A_{ij}, B_{ij} \in \mathbb{R}^{q \times q}$ and $A_{ij}^T = A_{ji}$, $B_{ij}^T = B_{ji}$ for all $i, j$. In this section, we prove the S-lemma of matrix-valued polynomials (Theorem \cite{12}). Define

$$A := \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mm} \end{pmatrix},$$ 

where $A_{ij}, B_{ij} \in \mathbb{R}^{q \times q}$.
Proof of Theorem 1.2. Assume that the condition (2) is satisfied. Let \( P : \mathbb{R}^n \rightarrow \mathbb{R}^q \) be the projection to the last \( q \) coordinates (actually \( P \) could be any orthogonal projection matrix in \( \mathbb{R}^{n \times n} \), or any matrix \( Q \in \mathbb{R}^{n \times \ell} \), \( \ell, n \in \mathbb{N}^+ \), see Corollary 5.1.

For all \( X \in (\mathbb{S} \mathbb{R}^n)^m \), \( n, m \in \mathbb{N}^+ \), we have

\[
\sum_{i=1,j=1}^m A_{ij} \otimes X_i X_j - \sum_{i=1,j=1}^m \phi(B_{ij}) \otimes X_i X_j \geq 0
\]

\[
\Rightarrow \sum_{i=1,j=1}^m A_{ij} \otimes X_i X_j - \sum_{i=1,j=1}^m (\phi \otimes I_n)(B_{ij} \otimes X_i X_j) \geq 0
\]

\[
\Rightarrow (\text{Id}_q \otimes P) \left( \sum_{i=1,j=1}^m A_{ij} \otimes X_i X_j \right) (\text{Id}_q \otimes P) - (\text{Id}_q \otimes P) \left( \sum_{i=1,j=1}^m (\phi \otimes I_n)(B_{ij} \otimes X_i X_j) \right) (\text{Id}_q \otimes P) \geq 0
\]

\[
\Rightarrow (\text{Id}_q \otimes P) \left( \sum_{i=1,j=1}^m A_{ij} \otimes X_i X_j \right) (\text{Id}_q \otimes P) - (\phi \otimes I_n)(\text{Id}_q \otimes P) \left( \sum_{i=1,j=1}^m B_{ij} \otimes X_i X_j \right) (\text{Id}_q \otimes P) \geq 0.
\]

As \( \phi \) is a completely positive linear map, if \((\text{Id}_q \otimes P)g(X)(\text{Id}_q \otimes P) \geq 0 \), then \((\phi \otimes I_n)(\text{Id}_q \otimes P)g(X)(\text{Id}_q \otimes P) \geq 0 \) and hence \((\text{Id}_q \otimes P)f(X)(\text{Id}_q \otimes P) \geq 0 \). The condition (1) is established.

Now we assume that the condition (2) is false, our aim is to show that the condition (1) is also false. For a fixed linear map \( \phi : \mathbb{R}^{q \times q} \rightarrow \mathbb{R}^{q \times q} \), let

\[
(5.1) \quad \phi g(x) = \sum_{i=1,j=1}^m \phi(B_{ij}) x_i x_j.
\]

Consider the set

\[
\{ f(x) - \phi g(x) \mid \phi : \mathbb{R}^{q \times q} \rightarrow \mathbb{R}^{q \times q} \text{ is a completely positive linear map} \}.
\]

\( f(x) - \phi g(x) \) is a homogeneous quadratic polynomial and its coefficient matrix has the following form

\[
\begin{pmatrix}
A_{11} - \phi(B_{11}) & \cdots & A_{1m} - \phi(B_{1m}) \\
\vdots & \ddots & \vdots \\
A_{m1} - \phi(B_{m1}) & \cdots & A_{mm} - \phi(B_{mm})
\end{pmatrix} = A - (I_m \otimes \phi)B.
\]

The set \( D \) which contains all such matrices is a closed convex cone in \( \mathbb{S} \mathbb{R}^{mq} \). Let \( \mathcal{C} \) denote the positive semidefinite cone in \( \mathbb{S} \mathbb{R}^{mq} \). Since the condition (2) is false, according to Theorem 11, the coefficient matrix of \( f(x) - \phi g(x) \) can not be positive semidefinite. Hence, we have \( \mathcal{C} \cap D = \emptyset \).

Let us define

\[
K = \{ J(\phi) \mid \phi : \mathbb{R}^{q \times q} \rightarrow \mathbb{R}^{q \times q} \text{ is a completely positive linear map}, ||J(\phi)|| = 1 \},
\]
where $J(\phi)$ is defined by (2.1). The set $K$ is compact. For any completely positive linear map $\phi: \mathbb{R}^{q \times q} \to \mathbb{R}^{q \times q}$ with $||J(\phi)|| = 1$, define

$$D_{J(\phi)} = \{A - \lambda(I_m \otimes \phi)B \mid \lambda \geq 0\},$$

and

$$k(J(\phi)) = \inf\{||\mathcal{M}_1 - \mathcal{M}_2|| \mid \mathcal{M}_1 \in \mathcal{C}, \mathcal{M}_2 \in D_{J(\phi)}\}.$$ 

Then, $k(J(\phi))$ can be seen as a continuous function on $K$. Since $K$ is compact, there is a completely positive linear map $\phi^0$ and $J(\phi^0) \in K$, such that $k(J(\phi^0)) = \min_{J(\phi) \in K} k(J(\phi))$.

For the completely positive linear map $\phi^0$, we have

$$g(\hat{X}) = \sum_{i=1, j=1}^{m} \phi^0(B_{ij}) \otimes \hat{X}_i \hat{X}_j \leq 0,$$

$$g(\hat{X}) = \sum_{i=1, j=1}^{m} \phi^0(B_{ij}) \otimes \hat{X}_i \hat{X}_j \neq 0.$$ (5.2)

Now we show that $(I_m \otimes \phi^0)B$ has a positive eigenvalue. If not, we have

$$(I_m \otimes \phi^0)B \preceq 0.$$ 

Then $(\phi^0 \otimes 1_n)g(X) \preceq 0$ for all $X \in (\mathbb{S}^n)^m$, which contradicts (5.2). The condition that $(I_m \otimes \phi^0)B$ has a positive eigenvalue ensures that

$$k(J(\phi^0)) = d > 0.$$ 

Therefore, we have

$$\inf\{||M_1 - M_2|| \mid M_1 \in \mathcal{C}, M_2 \in D\} = d > 0.$$ 

By the separation theorem [17] Theorem 11.4, there is a matrix $M^s \in \mathbb{R}^{(mq) \times (mq)}$, such that

$$\langle M_1, M^s \rangle \geq a_0 > \langle M_2, M^s \rangle, \quad \forall M_1 \in \mathcal{C}, M_2 \in D.$$ 

It is clear that $M^s \succeq 0$ and $a_0 > 0$. Then we have

$$\langle A, M^s \rangle < 0 \quad \text{and} \quad \langle (I_m \otimes \phi)B, M^s \rangle \geq 0,$$ (5.3)

for every completely positive linear map $\phi: \mathbb{R}^{q \times q} \to \mathbb{R}^{q \times q}$.

Let $\{e_1, e_2, \ldots, e_q\}$ be the standard orthogonal basis of $\mathbb{R}^q$, and $E = \sum_{i=1}^{q} e_i \otimes e_i$. The matrix $M^s$ can be written in the following form

$$M^s = \begin{pmatrix} M^s_{11} & \cdots & M^s_{1m} \\ \vdots & \ddots & \vdots \\ M^s_{m1} & \cdots & M^s_{mm} \end{pmatrix},$$ (5.4)

where each $M^s_{ij} \in \mathbb{R}^{q \times q}$. The condition (5.3) can be written in the following form:

$$\langle A, M^s \rangle = \sum_{i=1, j=1}^{m} \langle A_{ij}, M^s_{ij} \rangle = E^T \left( \sum_{i=1, j=1}^{m} A_{ij} \otimes M^s_{ij} \right) E < 0.$$ (5.5)
Moreover, we have

\[
\langle (1_m \otimes \phi)B, M^s \rangle = \sum_{i=1,j=1}^{m} \langle \phi(B_{ij}), M^s_{ij} \rangle
\]

\[
= \left\langle \sum_{i=1,j=1}^{m} \phi(B_{ij}) \otimes M^s_{ij}, \sum_{a=1,b=1}^{q} E_{ab} \otimes E_{ab} \right\rangle
\]

\[
= \left\langle \sum_{i=1,j=1}^{m} B_{ij} \otimes M^s_{ij}, \sum_{a=1,b=1}^{q} \phi(E_{ab}) \otimes E_{ab} \right\rangle
\]

\[
= \left\langle \sum_{i=1,j=1}^{m} B_{ij} \otimes M^s_{ij}, J(\phi) \right\rangle \geq 0,
\]

where \( E_{ab} \in \mathbb{R}^{q \times q} \) are matrices whose \((a, b)\)-th entry is 1 and all others are 0. According to Theorem 2.2, the set

\[
\{ J(\phi) \mid \phi : \mathbb{R}^{q \times q} \to \mathbb{R}^{q \times q} \text{ is completely positive} \}
\]

is equivalent to the positive semidefinite cone in \( \mathbb{S} \mathbb{R}^{q^2} \). We have

\[
(5.6) \quad \sum_{i=1,j=1}^{m} B_{ij} \otimes M^s_{ij} \succeq 0.
\]

In order to show that the condition (1) in Theorem 1.2 is not satisfied, we need to translate the inequality conditions (5.5) and (5.6) into the evaluations of \( f \) and \( g \) at some matrix vector \( X \in (\mathbb{S} \mathbb{R}^{q})^m \). Since the positive semidefinite matrix \( M^s = (M^s_{ij}) \in (\mathbb{S} \mathbb{R}^{q})^m \) may not belong to the set

\[
X = \{ YY^T \mid Y \in (\mathbb{S} \mathbb{R}^{q})^m \},
\]

which is a strict subset of the positive semidefinite cone \( C \subset \mathbb{S} \mathbb{R}^{mq} \). Hence, we can not ensure that there always exists an \( X \in (\mathbb{S} \mathbb{R}^{q})^m \) such that

\[
f(X) = \sum_{i=1,j=1}^{m} A_{ij} \otimes M^s_{ij} \quad \text{and} \quad g(X) = \sum_{i=1,j=1}^{m} B_{ij} \otimes M^s_{ij}.
\]

This is the main reason why we introduce a projection (5.8) to construct an evaluation point.

Since \( M^s \) defined in (5.4) is a positive semidefinite matrix, it has the decomposition

\[
M^s = \sum_{k=1}^{r} v_k v_k^T, \quad v_k \in \mathbb{R}^{mq}, \quad r = \text{rank}(M^s),
\]

\[
v_k = \begin{pmatrix} v^1_k \\ \vdots \\ v^m_k \end{pmatrix}, \quad v^l_k \in \mathbb{R}^{q}, \quad 1 \leq l \leq m, \quad k = 1, \ldots, r.
\]
We define $X^M := (X_1^M, \ldots, X_m^M) \in (\mathbb{R}^{(r+q) \times (r+q)})^m$, where for each $i = 1, \ldots, m$,

\[
X_i^M = \begin{pmatrix}
0 & (v_i^1)^T \\
\vdots & \vdots \\
0 & (v_i^n)^T
\end{pmatrix},
\]

and the projection $P^M : \mathbb{R}^{(r+q)} \to \mathbb{R}^q$ to the last $q$ coordinates

\[
P^M = \begin{pmatrix}
0 \\
\text{Id}_q
\end{pmatrix}.
\]

Then the condition (5.6) can be used to show

\[
(Id_q \otimes P^M)g(X^M)(Id_q \otimes P^M) = \sum_{i=1,j=1}^m B_{ij} \otimes P^M X_i^M X_j^M P^M = \sum_{i=1,j=1}^m B_{ij} \otimes M_{ij}^R \succeq 0.
\]

On the other hand, the condition (5.5) can be used to show

\[
E^T(Id_q \otimes P^M)f(X^M)(Id_q \otimes P^M)E = E^T \left( \sum_{i=1,j=1}^m A_{ij} \otimes P^M X_i^M X_j^M P^M \right) E
\]

\[
= E^T \left( \sum_{i=1,j=1}^m A_{ij} \otimes M_{ij}^R \right) E < 0.
\]

Therefore, we have

\[
(Id_q \otimes P^M)f(X^M)(Id_q \otimes P^M) \not\succeq 0.
\]

Hence, the condition (1) in Theorem 1.2 is false.

**Corollary 5.1.** Under the same assumption in Theorem 1.2, the statements (1) and (2) in Theorem 1.2 are also equivalent to the following two conditions:

(3) For all $X \in (\mathbb{S}\mathbb{R}^n)^m$, orthogonal projection matrices $P \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}^+$, if $(Id_q \otimes P)g(X)(Id_q \otimes P) \succeq 0$, then $(Id_q \otimes P)f(X)(Id_q \otimes P) \succeq 0$.

(4) For all $X \in (\mathbb{S}\mathbb{R}^n)^m$, $Q \in \mathbb{R}^{n \times \ell}$, $\ell, n \in \mathbb{N}^+$, if $(Id_q \otimes Q^T)g(X)(Id_q \otimes Q) \succeq 0$, then $(Id_q \otimes Q^T)f(X)(Id_q \otimes Q) \succeq 0$.

**Proof.** From the proof of (2) $\Rightarrow$ (1), we can see that (2) $\Rightarrow$ (4) also holds. The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious. 

**Remark 5.2.** Theorem 1.2 is still true when the dimension $q_f$ of the coefficients of the polynomial $f$ is smaller than the dimension $q_g$ of the coefficients of the polynomial $g$. In fact, we can always add zeros to the coefficients of $f$ to make $q_f = q_g$. Consider the case when $q_f > q_g$. Suppose that $k$ is the smallest positive integer satisfying $q_f \leq kq_g$. Define a new polynomial $\tilde{g} = \oplus^k g$. Then Theorem 1.2 is still valid after replacing $g$ by $\tilde{g}$. 


6. Other Variants of S-Lemma in Noncommutative Cases

6.1. Other Variant Conditions of S-Lemma. Different from commutative polynomials, there are many ways to extend the S-lemma for (matrix-valued) polynomials with matrix evaluations. Comparing with the condition (1) in Theorem 1.2, we consider the following condition which is a more direct extension of the classical S-lemma:

\[(1') \text{ For all } X \in (\mathbb{S} \mathbb{R}^n)^m, n \in \mathbb{N}^+, \text{ if } g(X) \succeq 0, \text{ then } f(X) \succeq 0.\]

Remark 6.1. It is straightforward to verify that the condition (2) in Theorem 1.2 implies (1’). Therefore, under the assumption that there is an $\hat{X} \in (\mathbb{R}^{\hat{n} \times \hat{n}})^m$ for some $\hat{n} \in \mathbb{N}^+$, such that $g(\hat{X}) \succ 0$, the condition (1) in Theorem 1.2 implies the condition (1’), but it is unknown if it is true the other way around.

As illustrated by the following example, without the assumption of existing an $\hat{X}$ such that $g(\hat{X}) \succ 0$, (1’) can not imply the condition (1) in Theorem 1.2.

Example 6.2. We construct two matrix-valued polynomials

\[f = \begin{pmatrix} x_1x_2 + x_2x_1 - x_2x_2 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\]

\[g = \begin{pmatrix} x_1x_1 - x_2x_2 & 0 \\ 0 & x_1x_2 + x_2x_1 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_1x_2 - x_2x_1 \\ x_2x_1 - x_1x_2 & 0 \end{pmatrix}.\]

For any $X \in (\mathbb{S} \mathbb{R}^n)^2$, if $g(X) \succeq 0$, we have $X_1X_2 - X_2X_1 = 0$. So $X_1, X_2$ have the same eigenspaces. Let $X = (X_1, X_2)$, where

\[X_1 = \sum_{i=1}^r \lambda_i v_i v_i^T, \quad X_2 = \sum_{i=1}^r \mu_i v_i v_i^T.\]

Assume that $g(X) \succeq 0$, then we have

\[(\lambda_i)^2 - (\mu_i)^2 \geq 0 \text{ and } \lambda_i \mu_i \geq 0.\]

It is easy to check that $f(X) \succeq 0$. Hence $f$ and $g$ satisfy the condition (1’).

On the other hand, let

\[X_1^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\]

\[P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \text{Id}_4.\]

It is straightforward to verify that

\[(\text{Id}_4 \otimes P)g(X^0)(\text{Id}_4 \otimes P) = \begin{pmatrix} P(X_1^0X_1^0 - X_2^0X_2^0)P & 0 \\ 0 & P(X_1^0X_2^0 + X_2^0X_1^0)P \end{pmatrix} \oplus \begin{pmatrix} 0 & P(X_1^0X_2^0 - X_2^0X_1^0)P \\ P(X_1^0X_1^0 - X_2^0X_2^0)P & 0 \end{pmatrix}.\]
The top left corner matrix is positive semidefinite

\[
P(X_1^0X_1^0 - X_2^0X_2^0)P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \succeq 0.
\]

The other submatrices are all zero matrices

\[
P(X_1^0X_2^0 + X_2^0X_1^0)P = \pm P(X_1^0X_2^0 - X_2^0X_1^0)P = 0.
\]

Therefore, we have

\[
(Id_4 \otimes P)g(X^0)(Id_4 \otimes P) \succeq 0.
\]

However, we have

\[
(Id_4 \otimes P)f(X^0)(Id_4 \otimes P) = \left( P(X_1^0X_2^0 + X_2^0X_1^0 - X_2^0X_2^0)P 0 \right) \oplus \left( 0 0 \right).
\]

The top left corner matrix is negative semidefinite

\[
P(X_1^0X_2^0 + X_2^0X_1^0 - X_2^0X_2^0)P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \preceq 0.
\]

Therefore, we have

\[
(Id_4 \otimes P)f(X^0)(Id_4 \otimes P) \preceq 0.
\]

Therefore, the condition (1) in Theorem 1.2 is false for the given \( f \) and \( g \).

With the assumption that there is an \( \tilde{X} \in (\mathbb{S}\mathbb{R}^n)^m \) for some \( \tilde{n} \in \mathbb{N}^+ \), such that \( g(\tilde{X}) \succ 0 \), whether or not (1') can imply the condition (1) in Theorem 1.2 is an interesting problem and we wish to investigate it in future.

Furthermore, one can also consider the following condition:

(1'') For all \( X \in (\mathbb{S}\mathbb{R}^n)^m \), \( n \in \mathbb{N}^+ \), given a vector \( v \in \mathbb{R}^m \), if \( v^Tg(X)v \geq 0 \) then

\[
v^Tf(X)v \geq 0.
\]

The following example shows that the condition (1'') is strictly stronger than the condition (1) in Theorem 1.2 and the condition (1').

**Example 6.3.** We are given the matrix-valued polynomials

\[
f = \begin{pmatrix}
x_1x_1 & 0 \\
0 & x_1x_1 - x_2x_2
\end{pmatrix},
\]

and

\[
g = \begin{pmatrix}
x_1x_1 - x_2x_2 & 0 \\
0 & x_1x_1
\end{pmatrix}.
\]

Let us define a linear map \( \phi_2 \) from \( \mathbb{R}^{2 \times 2} \) to \( \mathbb{R}^{2 \times 2} \)

\[
\phi_2 : \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \to \begin{pmatrix}
d & 0 \\
0 & a
\end{pmatrix}.
\]

It is easy to verify that \( \phi_2 \) is a completely positive linear map. We have

\[
f(X) - (\phi_2g)(X) = 0 \text{ for all } X \in \mathbb{S}\mathbb{R}^n, \ n \in \mathbb{N}^+.
\]
The condition (2) in Theorem 1.2 is satisfied. Therefore, the condition (1) in Theorem 1.2 and the condition (1') are satisfied too. However, let $X_0 = [1, 2]^T$, $v = [0, 1]^T$, we have

$$v^T g(X_0) v = 1 > 0, \quad \text{but} \quad v^T f(X_0) v = -3 < 0.$$ 

Therefore the condition (1'') above is not satisfied. □

6.2. Proof of Theorem 1.3. We assume that $f(x)$ and $g(x)$ are homogeneous matrix-valued polynomials with following form

$$f(x) = \sum_{i=1,j=1}^{m} A_{ij} x_i x_j^T, \quad g(x) = \sum_{i=1,j=1}^{m} B_{ij} x_i x_j^T,$$

where $A_{ij}, B_{ij} \in \mathbb{R}^{q \times q}$ and $A_{ij} = A_{ji}^T$, $B_{ij} = B_{ji}^T$ for all $i, j$. Now we prove Theorem 1.3 which implies that the condition (1) in Theorem 1.2 can be simplified to (1'), i.e., we do not need projection for the matrix-valued hereditary polynomials.

Define that $A' = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}$ and $B' = \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mm} \end{pmatrix}$.

Let $A'$ be defined as in (4.1) and $\psi_f : \mathbb{R}^{m \times m} \to \mathbb{R}^{q \times q}$ be the linear map defined by $A'$ (1.2). Then, it holds that

$$f(X) = \psi_f \otimes I_n \begin{pmatrix} X_1X_1^T & \cdots & X_1X_m^T \\ \vdots & \ddots & \vdots \\ X_mX_1^T & \cdots & X_mX_m^T \end{pmatrix} = \psi_f \otimes I_n(XX^T),$$

for all $X \in (\mathbb{R}^{n \times n})^m$, $n \in \mathbb{N}^+$. Similarly, let $B'$ be the rearrangement of $B$ and $\psi_g : \mathbb{R}^{m \times m} \to \mathbb{R}^{q \times q}$ be the linear map defined by $B'$ such that

$$g(X) = \psi_g \otimes I_n(XX^T),$$

for all $X \in (\mathbb{R}^{n \times n})^m$, $n \in \mathbb{N}^+$.

Proof of Theorem 1.3. The implication (2) ⇒ (1) is obvious.

Assume that the condition (2) in Theorem 1.3 is false. Similar to the discussion in the proof of Theorem 1.2, we can find a separation matrix $M^* \succeq 0$ which satisfies the condition (5.3) and has the following decomposition:

$$M^* = \sum_{k=1}^{r} v_k v_k^T, \quad v_k \in \mathbb{R}^{mq}, \quad r = \text{rank}(M^*),$$

$$v_k = \begin{pmatrix} v_1^k \\ \vdots \\ v_m^k \end{pmatrix}, \quad v_l^k \in \mathbb{R}^{q}, \quad 1 \leq l \leq m, \quad k = 1, \ldots, r.$$ 

Since we do not require the variable $X_i^M$ to be symmetric, instead of constructing $X_i^M$ as in (5.7), we let

$$X_i^M = \begin{pmatrix} v_1^i & \cdots & v_r^i \end{pmatrix}.$$
Letting \( n = \max\{r, q\} \), we add zero rows or columns into \( X_i^M \in \mathbb{R}^{q \times r} \) to make it a square matrix in \( \mathbb{R}^{n \times n} \). Without loss of generality, we assume that \( r > q \), and define new matrices \( \tilde{X}_i^M \in \mathbb{R}^{n \times n} \) for \( i = 1, \ldots, m \),

\[
\tilde{X}_i^M = \begin{pmatrix}
v_1^i & \cdots & v_r^i \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}.
\]

Let \( \tilde{X}^M := (\tilde{X}_1^M, \ldots, \tilde{X}_m^M) \in (\mathbb{R}^{n \times n})^m \). We can translate the inequality conditions (5.5) and (5.6) into the evaluations of \( f \) and \( g \) at \( \tilde{X}^M \in (\mathbb{R}^{n \times n})^m \). In particular, we have

\[
g(\tilde{X}^M) = \sum_{i=1, j=1}^{m} B_{ij} \otimes \begin{pmatrix} M_{ij}^s & 0 \\ 0 & 0 \end{pmatrix} \succeq 0.
\]

Let \( \{e_1, e_2, \ldots, e_q\} \) be the standard orthogonal basis of \( \mathbb{R}^q \), \( \{f_1, f_2, \ldots, f_n\} \) be the standard orthogonal basis of \( \mathbb{R}^n \), and \( E' = \sum_{i=1}^q e_i \otimes f_i \), we have

\[
E'^T f(\tilde{X}^M) E' = E'^T \left( \sum_{i=1, j=1}^{m} A_{ij} \otimes \begin{pmatrix} M_{ij}^s & 0 \\ 0 & 0 \end{pmatrix} \right) E' < 0.
\]

Therefore, the condition (1) in Theorem 1.3 is false.

**Some discussions.** In this paper, we show several variants of the S-lemma in noncommutative cases for quadratic homogeneous polynomials. Unlike the commutative case, the S-lemma for general quadratic nonhomogeneous polynomials in noncommutative case is still unknown.

In the commutative case, it is straightforward to convert a nonhomogeneous polynomial to a homogeneous one by introducing a new variable. For example, let

\[
f(x) = \sum_{i=1, j=1}^{m} a_{ij} x_i x_j + \sum_{i=1}^{m} a_i x_i + a_0,
\]

where \( a_{ij} = a_{ji}, a_i, a_0 \in \mathbb{R} \) for all \( i, j \). By introducing a new variable \( x_0 \), the homogenization of \( f(x) \) can be written in the following form:

\[
\tilde{f}(x_0, x) = \sum_{i=1, j=1}^{m} a_{ij} x_i x_j + \sum_{i=1}^{m} a_i x_i x_0 + a_0 x_0^2.
\]

Then we have

\[
\tilde{f}(X_0, X) = X_0^2 f(X/X_0), \quad \text{for all} \quad X \in \mathbb{R}^m, \quad X_0 \neq 0 \in \mathbb{R}.
\]

Using this fact, the proof of the classical S-lemma for commutative nonhomogeneous polynomials can be reduced to homogeneous ones (see [18]).

However, this process becomes more complicated in the noncommutative cases. First of all, due to the noncommutativity of variables, the homogenization of a noncommutative polynomial is not unique. For example, consider a nonhomogeneous quadratic matrix-valued polynomial

\[
f(x) = \sum_{i=1, j=1}^{m} A_{ij} x_i x_j + \sum_{i=1}^{m} A_i x_i + A_0,
\]
where \( A_{ij} = A^T_{ji}, A_i, A_0 \in \mathbb{R}^{q \times q} \) for all \( i, j \). By introducing a new variable \( x_0 \), we homogenize \( f \) to

\[
h(x_0, x) = \sum_{i=1}^{m} A_{ij} x_i x_j + \sum_{i=1}^{m} H_{i0} x_i x_0 + \sum_{i=1}^{m} H_{0i} x_0 x_i + A_0 x_0 x_0,
\]

where

\[
(6.1) \quad H_{i0} + H_{0i} = A_i, \quad \text{for all} \quad i = 1, \ldots, m.
\]

There exist different choices of \( H_{i0} \) and \( H_{0i} \) satisfying (6.1) for \( 1 \leq i \leq m \). Therefore, the homogenization of a quadratic nonhomogeneous noncommutative polynomial is not unique.

**Example 6.4.** For the quadratic nonhomogeneous noncommutative polynomial

\[
f(x) = \begin{pmatrix} x^2 & x \\ x & 1 \end{pmatrix},
\]

we have two different choices of homogenization:

\[
h_1(x_0, x) = \begin{pmatrix} x^2 & x x_0 \\ x x_0 & x_0^2 \end{pmatrix} \quad \text{and} \quad h_2(x_0, x) = \begin{pmatrix} x^2 & x_0 x \\ x x_0 & x_0^2 \end{pmatrix}.
\]

For all \( X \in \mathbb{S}^{\mathbb{R}^n} \), \( n \in \mathbb{N}^+ \), it holds that

\[
f(X) = h_1(\text{Id}_n, X) = h_2(\text{Id}_n, X).
\]

The coefficient matrices of \( h_1(X_0, X) \) and \( h_2(X_0, X) \) satisfy the following conditions:

\[
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \preceq 0.
\]

By Theorem 4.1, we know that \( h_1(X_0, X) \) is positive semidefinite for all \( X_0, X \in \mathbb{S}^{\mathbb{R}^n} \), \( n \in \mathbb{N}^+ \), while \( h_2(X_0, X) \) is not positive semidefinite for all \( X_0, X \in \mathbb{S}^{\mathbb{R}^n} \), \( n \in \mathbb{N}^+ \).

\( \square \)

Nevertheless, given a positive semidefinite polynomial \( f(x) \), there always exists a choice of \( H_{i0} \) and \( H_{0i} \) satisfying (6.1), such that the homogenization \( h(x) \) is positive semidefinite. In fact, according to [13, Theorem 0.2], if a quadratic polynomial \( f(X) \) is positive semidefinite for all \( X \in (\mathbb{S}^{\mathbb{R}^n})^m \), \( n \in \mathbb{N}^+ \), then there exists a matrix-valued linear polynomial \( U(x) \), whose coefficients belong to \( \mathbb{R}^{(q(m+1)) \times (q(m+1))} \), such that \( f(x) = U(x)^T U(x) \). Hence, we can let \( h(x_0, x) = \tilde{U}(x_0, x)^T \tilde{U}(x_0, x) \) where \( \tilde{U}(x_0, x) \) is obtained by homogenizing \( U(x) \). It is clear that \( h(X) \) is positive semidefinite for all \( X \in (\mathbb{S}^{\mathbb{R}^n})^m \), \( n \in \mathbb{N}^+ \). Thanks to Theorem 1.1, one can find such a homogenization \( h(x) \) by solving a semidefinite program with the positive semidefinite constraint of the coefficient matrix of \( h \) and the equality constraint (6.1).

However, unlike the commutative case proved in [13], it is unclear how to derive S-lemma for nonhomogeneous quadratic polynomials from homogeneous ones. In
particular, for a general nonhomogeneous quadratic polynomial $f$ and its homogenization $h$, we have

$$h(X_0, X) \neq X_0 f(X_0^{-\frac{1}{2}} XX_0^{-\frac{1}{2}}) X_0, \quad X \in (\mathbb{S}\mathbb{R}^n)^m, \quad X_0 \in \mathbb{S}\mathbb{R}^n \text{ is invertible.}$$

Thus, the S-lemma for general quadratic nonhomogeneous polynomials in noncommutative cases is still unknown and left for future research.

Acknowledgments: We would also like to acknowledge many valuable comments and suggestions from Ke Ye and Jianting Yang.

References
1. William B Arveson, Subalgebras of $C^*$-algebras, Acta Mathematica 123 (1969), 141–224.
2. Aharon Ben-Tal and Arkadi Nemirovski, Lectures on modern convex optimization - analysis, algorithms, and engineering applications, Society for Industrial and Applied Mathematics, 2001.
3. Jacek Bochnak, Michel Coste, and Marie-Françoise Roy, Real algebraic geometry, vol. 36, Springer Science & Business Media, 2013.
4. Man-Duen Choi, Completely positive linear maps on complex matrices, Linear Algebra and its Applications 10 (1975), no. 3, 285–290.
5. Lloyd L. Dines, On the mapping of quadratic forms, Bulletin of the American Mathematical Society 47 (1941), 494–498.
6. J. William Helton, “Positive” noncommutative polynomials are sums of squares, Annals of Mathematics. Second Series 2 (2002), 675–694.
7. J. William Helton, Igor Klep, and Scott McCullough, The matricial relaxation of a linear matrix inequality, Mathematical Programming 138 (2010), 401–445.
8. J. William Helton, Igor Klep, and Scott McCullough, The convex Positivstellensatz in a free algebra, Advances in Mathematics 231 (2012), no. 1, 516–534.
9. , Matrix convex hulls of free semialgebraic sets, Transactions of the American Mathematical Society 368 (2016), no. 5, 3105–3139.
10. J. William Helton and Scott McCullough, Convex noncommutative polynomials have degree two or less, SIAM Journal on Matrix Analysis and Applications 25 (2004), no. 4, 1124–1139.
11. J. William Helton and Scott Mccullough, A Positivstellensatz for non-commutative, Transactions of the American Mathematical Society 356 (2004), no. 9, 3721–3737.
12. Monique Laurent and Frank Vallentin, Semidefinite optimization, Lecture Notes, 2014.
13. Scott McCullough, Factorization of operator-valued polynomials in several non-commuting variables, Linear Algebra and Its Applications 326 (2001), 193–203.
14. Vern Paulsen, Completely bounded maps and operator algebras, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2003.
15. Vern I. Paulsen, Completely bounded maps and dilations, John Wiley & Sons, Inc., USA, 1987.
16. Imre Pólik and Tamás Terlaky, A survey of the S-lemma, SIAM Review 49 (2007), no. 3, 371–418.
17. R. Tyrrell Rockafellar, Convex analysis, Princeton University Press, 1970.
18. V. A. Yakubovic, S-procedure in nonlinear control theory, Vestnik Leningrad Univ 1 (1971), 62–77.
19. Y. Yuan, On a subproblem of trust region algorithms for constrained optimization, Mathematical Programming 47 (1990), 53–63.
20. Aljaž Zalar, Operator Positivstellensätze for noncommutative polynomials positive on matrix convex sets, Journal of Mathematical Analysis and Applications 445 (2017), no. 1, 32–80.
SCHOOL OF MATHEMATICAL SCIENCES, DALIAN UNIVERSITY OF TECHNOLOGY, DALIAN, 116024, CHINA  
Email address: fguo@dlut.edu.cn

KEY LAB OF MATHEMATICS MECHANIZATION, AMSS, UNIVERSITY OF CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA  
Email address: yansizhuo@amss.ac.cn, lzhi@mmrc.iss.ac.cn