Equivariant eta forms and equivariant differential K-theory

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Abstract In this paper, for a compact Lie group action, we prove the anomaly formula and the functoriality of the equivariant Bismut-Cheeger eta forms with perturbation operators when the equivariant family index vanishes. In order to prove them, we extend the Melrose-Piazza spectral section and its main properties to the equivariant case and introduce the equivariant version of the Dai-Zhang higher spectral flow for arbitrary-dimensional fibers. Using these results, we construct a new analytic model of the equivariant differential K-theory for compact manifolds when the group action has finite stabilizers only, which modifies the Bunke-Schick model of the differential K-theory. This model could also be regarded as an analytic model of the differential K-theory for compact orbifolds. Especially, we answer a question proposed by Bunke and Schick (2009) about the well-definedness of the push-forward map.

Keywords equivariant eta form, equivariant differential K-theory, equivariant spectral section, equivariant higher spectral flow, orbifold

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1 Introduction

The differential K-theory is the differential extension of the topological K-theory, whose basic idea is to combine the topological K-theory with the differential form information. It is partly motivated by the study of D-branes in theoretical physics (see, e.g., [25,52]). Various models of differential K-theory have been given by Hopkins and Singer [31], Bunke and Schick [16], Freed and Lott [27], Simons and Sullivan [49], Tradler et al. [51] and Gorokhovsky and Lott [29], etc. For a detailed survey, see [17].

Until now, the equivariant version of the differential K-theory is not well understood yet. When the group is finite, the equivariant differential K-theory was studied by Szabo and Valentino [50] and Ortiz [46]. In [18], Bunke and Schick extended their model to the orbifold case using the language of stacks, which could be regarded as a model of the equivariant differential K-theory when the action has finite stabilizers only.

Inspired by the model of Bunke and Schick [18], as a parallel version, in this paper, we will construct a purely analytic model of the equivariant differential K-theory for compact manifolds when the action...
has finite stabilizers only, using the local index technique developed in [12]. Moreover, a detailed proof of the well-definedness of the push-forward map is given here, which is a question proposed in [16, 18] and is the main motivation for this new construction. This model is a direct generalization of [16] without using the language of stacks and could also be regarded as an analytic model of the differential K-theory for compact orbifolds.

The study of the differential K-theory is always related to the Bismut-Cheeger eta form (see [9]), which is defined for a family of Dirac operators and is the family extension of the famous Atiyah-Patodi-Singer eta invariant. Usually, the well-definedness of the eta form needs one of the following additional conditions:

1. The kernels of the family of Dirac operators form a vector bundle over the base manifold (see [9,19]).
2. The family index of the family of Dirac operators vanishes as an element of the K-group of the base manifold (see [14,43,44]).

In the model of Freed and Lott [27], the eta form with the first condition is used to define the push-forward map. In the model of Bunke and Schick [16], the eta form under the second condition, which is defined by Bunke [14] using the taming, is used to define the differential K-group. From this point of view, in order to extend the differential K-theory to the equivariant case, we firstly need to extend the Bismut-Cheeger eta form to the equivariant case.

In [34], Liu systematically studied the equivariant eta forms under the first condition and proved the anomaly formula and the functoriality of them, which should be used to establish an equivariant version of the Freed-Lott model. In this paper, we will establish the properties of the eta form under the second condition, extend them to the equivariant case and use them to construct our model. In order to do finer spectral analysis, we use the notion of the spectral section developed by Melrose and Piazza [43,44] and the Dai-Zhang higher spectral flow (see [20]) instead of the taming and the Kasparov KK-theory in [14,16].

In [34,44], in order to prove the family index theorem for manifolds with boundary, Melrose and Piazza defined the spectral section and the eta form under the second condition. In [20], using the spectral section, Dai and Zhang introduced the higher spectral flow for a family of Dirac type operators on a family of odd-dimensional manifolds. In this paper, we will extend the spectral section, the higher spectral flow and the eta form to the equivariant case. Especially, we will define the higher spectral flow for a family of even-dimensional manifolds. Furthermore, we will prove the anomaly formula and the functoriality of equivariant eta forms using the language of the equivariant higher spectral flow and the techniques in [19,39–41], which are analogues of the results in [7,15,34]. Note that our proof of the anomaly formula of the eta forms for a family of even-dimensional manifolds relies on the functoriality of equivariant eta forms (see Theorem 3.18), which is highly nontrivial and is the main technical difficulty of this paper. Since the second condition is a topological condition, there is no additional rigidity assumption in the formulas here.

Let $\pi : W \to B$ be a proper smooth submersion of compact manifolds with orientable fibers $Z$. Let $TZ = \ker(d\pi)$ be the relative tangent bundle to the fibers $Z$ with the Riemannian metric $g^{TZ}$ and $T^HW$ be a horizontal subbundle of $TW$ such that $TW = T^HW \oplus TZ$. Let $o$ be an orientation of $TZ$. Let $\nabla^{TZ}$ be the Euclidean connection on $TZ$ defined in (2.14). We assume that $TW$ is a Spin$^c$ structure. Let $L_Z$ be the complex line bundle associated with the Spin$^c$ structure of $TZ$ with a Hermitian metric $h^{L_Z}$ and a Hermitian connection $\nabla^{L_Z}$. Let $(E,h^E)$ be a $\mathbb{Z}_2$-graded Hermitian vector bundle with a Hermitian connection $\nabla^E$. Let $G$ be a compact Lie group which acts on $W$ and $B$ such that $\pi \circ g = g \circ \pi$ for any $g \in G$. We assume that the $G$-action preserves everything. The family of $G$-equivariant geometric data $F = (W,L_Z,E,o,T^HW,g^{TZ},h^{L_Z},\nabla^{L_Z},h^E,\nabla^E)$ is enough to define the equivariant Bismut superconnection. We call $F$ an equivariant geometric family over $B$ for short. Let $D(F)$ be the fiberwise Dirac operators of $F$ defined in (2.23). Let $K_G^*(B)$ (i.e., 0,1) be the equivariant K-group of $B$. Then the family index map $\text{Ind}(D(F)) \in K_G^*(B)$, where $* = 0$ or $* = 1$ corresponds to the even or odd dimensions of fibers $Z$.

Let $F^0_G(B)$ (resp. $F^1_G(B)$) be the set of equivalence classes of isomorphic equivariant geometric families such that the dimensions of all the fibers are even (resp. odd). We write $F \sim F'$ if $\text{Ind}(D(F)) = \text{Ind}(D(F')$. The study of the differential K-theory is always related to the Bismut-Cheeger eta form (see [9]), which is defined for a family of Dirac operators and is the family extension of the famous Atiyah-Patodi-Singer eta invariant. Usually, the well-definedness of the eta form needs one of the following additional conditions:

1. The kernels of the family of Dirac operators form a vector bundle over the base manifold (see [9,19]).
2. The family index of the family of Dirac operators vanishes as an element of the K-group of the base manifold (see [14,43,44]).
\textbf{Proposition 1.1.} \hspace{0.02em} \textit{There is a ring isomorphism}

\begin{equation}
F^*_G(B)/ \sim \simeq K^*_G(B).
\end{equation}

Let \( D \) be a family of first-order pseudodifferential operators on the fibers of \( F \), which is self-adjoint, fiberwise elliptic and commutes with the \( G \)-action. Furthermore, if \( F \in F^*_G(B) \), we assume that \( D \) preserves the \( \mathbb{Z}_2 \)-grading of \( E \); if \( F \in F^0_G(B) \), we assume that \( D \) anti-commutes with the \( \mathbb{Z}_2 \)-grading of \( S(TZ, L_\mathbb{Z}) \otimes E \), where \( S(TZ, L_\mathbb{Z}) \) is the spinor with respect to the Spin\( ^c \) structure of \( TZ \). As in [20], we call such \( D \) an equivariant \( B \)-family on \( F \) (see Definition 3.1). If \( \text{Ind}(D) = 0 \in K^*_G(B) \) and at least one component of the fiber has the nonzero dimension, there exist an equivariant spectral section \( P \) (see Definition 3.2) and a family of smoothing operators \( A_P \) associated with \( P \) such that \( D + A_P \) is an invertible equivariant \( B \)-family (see Proposition 3.3). Let \( P \) and \( Q \) be equivariant spectral sections. We could define the difference \( [P - Q] \in K^*_G(B) \) (see (3.13) and (3.18)).

Let \( F, F' \in F^*_G(B) \) (resp. \( F^0_G(B) \)) which have the same topological structure, i.e., the only differences between them are horizontal subbundles, metrics and connections. Let \( D_0 \) and \( D_1 \) be two equivariant \( B \)-families on \( F \) and \( F' \), respectively. Let \( Q_0 \) and \( Q_1 \) be equivariant spectral sections of \( D_0 \) and \( D_1 \), respectively. We define the equivariant higher spectral flow \( \text{sf}_G\{ (D_0, Q_0), (D_1, Q_1) \} \) between the pairs \( (D_0, Q_0) \) and \( (D_1, Q_1) \) to be an element in \( K^*_G(B) \) (resp. \( K^0_G(B) \)) in Definitions 3.7 and 3.8. Note that when \( F \) is odd, it is the direct extension of the Dai-Zhang higher spectral flow in [20]; when \( F \) is even, it is defined by adding an additional dimension.

Moreover, besides the equivariant geometric family, we could also represent the elements of the equivariant \( K \)-group as equivariant higher spectral flows (see Proposition 3.9). From this point of view, the equivariant higher spectral flow here is the same as the term \( \text{Ind}((\mathcal{E} \times I)_B) \) in [18, Subsection 2.5.8], which is studied by using the KK-theory there. This enables us to replace the techniques of KK-theory in [18] by those of the equivariant higher spectral flow, which are purely analytic.

Let \( D \) be an equivariant \( B \)-family on \( F \). A perturbation operator with respect to \( D \) is a family of bounded pseudodifferential operators \( A \) such that \( D + A \) is an invertible equivariant \( B \)-family on \( F \), which is a generalization of \( A_P \). Note that if at least one component of the fibers of \( F \) has the nonzero dimension, a perturbation operator exists with respect to \( D \) if and only if \( \text{Ind} D = 0 \in K^*_G(B) \).

If the \( G \)-action on \( B \) is trivial, for any \( g \in G \), we define the equivariant eta form \( \bar{\tilde{\eta}}_g(F, A) \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}) \) with respect to a perturbation operator \( A \) in Definition 3.12. If the equivariant geometric families \( F \) and \( F' \) have the same topological structure, we prove the anomaly formula as follows.

\textbf{Theorem 1.2.} \hspace{0.02em} \textit{Assume that the \( G \)-action on \( B \) is trivial. Let \( F, F' \in F^*_G(B) \) which have the same topological structure. Let \( A \) and \( A' \) be perturbation operators with respect to \( D(F) \) and \( D(F') \) and \( P \) and \( P' \) be the Atiyah-Patodi-Singer (APS) projections onto the eigenspaces of the positive spectra of \( D(F) + A \) and \( D(F') + A' \), respectively. For any \( g \in G \), modulo exact forms on \( B \), we have}

\begin{equation}
\bar{\tilde{\eta}}_g(F', A') - \bar{\tilde{\eta}}_g(F, A) = \int_{Z^g} \overline{\text{Td}_g}(\nabla^{TZ}, \nabla^{L_Z}, \nabla^{T_Z}, \nabla^{L_Z}) \text{ch}_g(E, \nabla^E) + \int_{Z^g} \text{Td}_g(\nabla^{TZ}, \nabla^{L_Z}) \text{ch}_g(\nabla^E, \nabla^{\text{T}E}) + \text{ch}_g(\text{sf}_G\{ (D(F) + A, P), (D(F') + A', P') \}),
\end{equation}

\textit{where \( Z^g \) is the fixed point set of \( g \) on the fibers \( Z \), and the characteristic forms \( \text{ch}_g(\cdot) \) and \( \text{Td}_g(\cdot) \) and the Chern-Simons forms \( \text{ch}_g(\cdot) \) and \( \text{Td}_g(\cdot) \) are defined in Section 2.}

Note that when \( F, F' \in F^*_G(B) \), the proof of the anomaly formula relies on a special case of functoriality of equivariant eta forms.

If \( B = \text{pt} \) and \( F \in F^*_G(\text{pt}) \), by taking \( A = P_{\ker D} \), the orthogonal projection onto the kernel of \( D(F) \), the equivariant eta form here is just the classical reduced equivariant APS eta invariant. Using Theorem 1.2, we could write the equivariant spectral flow term in the anomaly formula of eta invariants explicitly.
Let $\pi : V \to B$ be an equivariant surjective proper submersion with compact orientable fibers $Y$. We assume that $B$ is compact, $G$ acts trivially on $B$ and $TY$ is equivariant Spin$^c$. Let $\mathcal{F}_X = (W, L_X, E, \sigma_X, T^*_Z W, g^T X, h^L X, \nabla^L X, h^E, \nabla^E)$ be an equivariant geometric family over $V$ for an equivariant surjective proper submersion $\pi_X : W \to V$ with compact orientable fibers $X$ (see (2.33)). Then $\pi_Z := \pi_Y \circ \pi_X : W \to B$ is an equivariant proper submersion with compact orientable fibers $Z$. We could obtain a new equivariant geometric family $\mathcal{F}_Z$ over $B$ in (2.37). For any $g \in G$, let $Y^g$ and $Z^g$ be the fixed point sets of $g$ on the fibers $Y$ and $Z$, respectively. We obtain the functoriality of equivariant eta forms.

**Theorem 1.3.** Let $A_Z$ and $A_X$ be perturbation operators with respect to $D(\mathcal{F}_Z)$ and $D(\mathcal{F}_X)$. Then there exists $T_0 \geq 1$ such that for any $T \geq T_0$ and any $g \in G$, modulo exact forms on $B$, we have

$$
\tilde{\eta}_g(\mathcal{F}_Z, A_Z) = \int_{Y^g} \text{Td}\left(\nabla^TY, \nabla^L Y\right) \tilde{\eta}_g(\mathcal{F}_X, A_X) + \int_{Z^g} \text{Td}\left(\nabla^TY, X, \nabla^L X, \nabla^E\right) \chi_g(E, \nabla^E) + \chi_g(\text{sf} G(\mathcal{D}(\mathcal{F}_Z, T) + 1 \otimes T \mathcal{A}_X, F), (\mathcal{D}(\mathcal{F}_Z) + A_Z, P'_t)),
$$

(1.3)

where $\mathcal{F}_{Z,T}$ is the equivariant geometric family defined in (3.9), $\nabla^TY, X$ is defined in (3.91), and $P$ and $P'$ are the associated APS projections, respectively.

In the last section, inspired by [16,18,46], we use the results above to define the equivariant differential K-theory for the compact manifolds when $G$ acts with finite stabilizers and study the properties of it.

Essential to our definition is that when the group action has finite stabilizers, $K^*_G(B) \otimes \mathbb{R}$ is isomorphic to the delocalized cohomology $H^*_\text{deloc}(G(B, \mathbb{R}))$ defined in (4.8), which is the cohomology of complex $(\Omega^*_G B, \mathbb{R}, d)$ of differential forms on the disjoint union of the fixed point sets of a representative element in the conjugacy classes. Furthermore, we could define $\delta_G(\mathcal{F}, A) \in \Omega^*_\text{deloc}(G(B, \mathbb{R}))/\text{Ind}$ (see Definition 4.2) when $G$ acts with finite stabilizers on $B$.

A cycle for an equivariant differential K-theory class over $B$ is a pair $(\mathcal{F}, \rho)$, where $\mathcal{F} \in F^*_G(B)$ and $\rho \in \Omega^*_\text{deloc}(G(B, \mathbb{R}))/\text{Ind}$. The cycle $(\mathcal{F}, \rho)$ is called even (resp. odd) if $\mathcal{F}$ is even (resp. odd) and $\rho \in \Omega^*_\text{even}(G(B, \mathbb{R}))/\text{Ind}$. Two cycles $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ are called isomorphic if $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic and $\rho = \rho'$. Let $\mathcal{I}_G^0(B)$ (resp. $\mathcal{I}_G^1(B)$) denote the set of isomorphism classes of even (resp. odd) cycles over $B$. Let $\mathcal{F}^{\text{top}}$ be the equivariant geometric family reversing the $\mathbb{Z}_2$-grading of $E$ in $\mathcal{F}$, which implies that $\text{Ind}(\mathcal{D}(\mathcal{F}^{\text{top}})) = -\text{Ind}(\mathcal{D}(\mathcal{F}))$. We call two cycles $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ paired if

$$
\text{Ind}(\mathcal{D}(\mathcal{F})) = \text{Ind}(\mathcal{D}(\mathcal{F}')), \quad (1.4)
$$

and there exists a perturbation operator $A$ with respect to $D(\mathcal{F} + \mathcal{F}^{\text{top}})$ such that

$$
\rho - \rho' = \tilde{\eta}_G(\mathcal{F} + \mathcal{F}^{\text{top}}, A). \quad (1.5)
$$

Note that from (1.4), $\text{Ind}(\mathcal{F} + \mathcal{F}^{\text{top}}) = 0$ in $K^*_G(B)$. So $\tilde{\eta}_G(\mathcal{F} + \mathcal{F}^{\text{top}}, A)$ is well defined. Let $\sim$ denote the equivalence relation generated by the relation “paired”.

**Definition 1.4** (Compare with [16, Definition 2.14]). The equivariant differential K-theory $\tilde{K}_G^0(B)$ (resp. $\tilde{K}_G^1(B)$) is the group completion of the abelian semigroup $\mathcal{I}_G^0(B)/\sim$ (resp. $\mathcal{I}_G^1(B)/\sim$).

Let $\pi_Y : V \to B$ be an equivariant proper submersion of compact smooth $G$-manifolds with compact fibers $Y$ such that $TY$ is oriented and equivariant Spin$^c$. We assume that the $G$-action on $B$ has only finite stabilizers. Thus, so is the action on $V$. As in [16], we define the equivariant differential K-orientation with respect to $\pi_Y$ in Definition 4.6 and the map $\tilde{\eta}_Y : \mathcal{I}_G^0(V) \to \mathcal{I}_G^1(B)$ in (4.24). Then using Theorems 1.2 and 1.3, we prove the following theorem.

**Theorem 1.5.** The map $\tilde{\eta}_Y : \tilde{K}_G^0(V) \to \tilde{K}_G^1(B)$ is well defined.
By Theorems 1.2 and 1.3, in Section 3, we also prove that our model is a ring-valued functor with the usual properties of the differential extension of a generalized cohomology. Finally, we explain that this model could be naturally regarded as a model of differential K-theory for orbifolds.

Note that there is no adiabatic limit in Theorem 1.3. So in the non-equivariant case, our proofs in the construction of the differential K-theory simplify those in [16] a little.

The rest of this paper is organized as follows. In Section 2, we give a geometric description of the equivariant K-theory. In Section 3, we extend the spectral section to the equivariant case, introduce the equivariant higher spectral flow for arbitrary-dimensional fibers and use them to obtain the anomaly formula and the functoriality of the equivariant eta forms. In Section 4, we construct an analytic model for the equivariant differential K-theory and prove some properties.

**Notation.** All the manifolds in this paper are smooth and without boundary. We denote by for the equivariant differential K-theory and prove some properties.

For the fiber bundle \(\pi: W \to B\), we will often use the integration of the differential forms along the oriented fibers \(Z\) in this paper. Since the fibers may be odd-dimensional, we must make precisely our sign convention: for \(\alpha \in \Omega^r(B)\) and \(\beta \in \Omega^s(W)\), we have

\[
\int_Z (\pi^* \alpha) \wedge \beta = \alpha \wedge \int_Z \beta.
\]

2 The equivariant K-theory

In this section, we explain a geometric description of the equivariant K-theory in [16,18] for any compact Lie group action and define the push-forward map of equivariant K-groups in this point of view. In Subsection 2.1, we recall some elementary results of the Clifford algebra. In Subsection 2.2, we introduce the equivariant geometric family. In Subsection 2.3, we give a geometric description of the equivariant K-theory. In Subsection 2.4, we study the push-forward map in equivariant K-theory using equivariant geometric families.

2.1 The Clifford algebra

Let \(E\) be an oriented Euclidean space of dimension \(n\). Let \(C(E)\) denote the complex Clifford algebra of \(E\). Relative to an orthonormal basis of \(E\), \(\{e_i\}_{1 \leq i \leq n}\), \(C(E)\) is defined by the relations

\[
e_i e_j + e_j e_i = -2\delta_{ij}.
\]

To avoid ambiguity, we denote by \(c(e_i)\) the element of \(C(E)\) corresponding to \(e_i\). The Clifford algebra \(C(E)\) is naturally \(\mathbb{Z}_2\)-graded from the \(\mathbb{N}\)-grading of the tensor algebra after reduction mod 2. We define \(C(E) = C_0(E) \oplus C_1(E)\). Let \(\text{Spin}^c_n\) be the Spin\(^c\) group associated with \(C(E)\) (see [32, Appendix D]).

If \(n = 2k\) is even, up to isomorphism, \(C(E)\) has a unique irreducible representation, the spinor \(S(E)\), which has a \(\mathbb{Z}_2\)-grading obtained from the chirality operator

\[
\tau_E = (\sqrt{-1})^k c(e_1) \cdots c(e_{2k}).
\]
We write $S(E) = S_+(E) \oplus S_-(E)$ with respect to this $\mathbb{Z}_2$-grading. In fact, there are isomorphisms of $\mathbb{Z}_2$-graded algebras

$$C(E) \simeq \text{End}(S(E)) \simeq S(E) \otimes S(E)^*.$$  \hfill (2.2)

For any $A \in C(E)$, we write $\text{Tr}_A[A] := \text{Tr}[\tau_E A]$ as the supertrace on $S(E)$. Note that $S(E)$ is also a representation of Spin$_n^\epsilon$ induced by the Clifford action.

If $n = 2k - 1$ is odd, $C(E)$ has only two inequivalent irreducible representations. For arbitrary $n$, $c(e_j) \mapsto c(e_j)c(e_{n+1})$, $1 \leq j \leq n$, defines an isomorphism $C(E) \simeq C_0(E \oplus \mathbb{R})$ of algebras. Since $n$ is odd, we can regard $S_\pm(E \oplus \mathbb{R})$ as the two inequivalent irreducible representations of $C(E)$. Their restrictions to Spin$_n^\epsilon$ are equivalent. In the following, we may and will take $S_+(E \oplus \mathbb{R})$ as the spinor for $C(E)$, also denoted by $S(E)$ for the convenience. In particular, the notation $\text{Tr}_{(\epsilon)}$ on the spinor refers to the representation $S_+(E \oplus \mathbb{R})$.

Let $F$ be another oriented Euclidean space. Let $S(E) \otimes S(F)$ be the $\mathbb{Z}_2$-graded tensor product of $S(E)$ and $S(F)$. Then it is a $\mathbb{Z}_2$-graded representation of $C(E) \otimes C(F)$ defined by

$$(a_1 \widehat{\otimes} a_2)(s_1 \widehat{\otimes} s_2) = (-1)^{|a_2||s_1|}(a_1 s_1) \widehat{\otimes} (a_2 s_2),$$  \hfill (2.3)

where $a_1 \in C(E)$, $a_2 \in C(F)$, $s_1 \in S(E)$, $s_2 \in S(F)$ and $|a_2|$ and $|s_1|$ are degrees of $a_2$ and $s_1$ associated with the $\mathbb{Z}_2$-gradings of $C(F)$ and $S(E)$, respectively. We express $S(E) \widehat{\otimes} S(F)$ by $S(E)$ and $S(F)$ using the ungraded tensor product as follows (see \cite[(1.10) and (1.11)]).

If both $\dim E$ and $\dim F$ are odd, let $C^2 = \mathbb{C} \oplus \mathbb{C}$ define the grading on $C^2$ and let $J, K \in \text{End}(C^2)$ denote the involutions

$$J = \begin{pmatrix} 0 & -1 \sqrt{-1} \\ -1 \sqrt{-1} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hfill (2.4)

Note that $J^2 = K^2 = 1$ and $JK = -KJ$. Then $S(E) \otimes S(F) \otimes C^2$ with involution $1 \otimes 1 \otimes \sqrt{-1}JK$ is the unique irreducible $\mathbb{Z}_2$-graded representation of $C(E) \widehat{\otimes} C(F)$ defined by

$$a_i \widehat{\otimes} b_j \mapsto a_i \otimes b_j \otimes J^i K^j, \quad a_i \in C_i(E), \quad b_j \in C_j(F).$$  \hfill (2.5)

It is isomorphic to $S(E) \widehat{\otimes} S(F)$ as $\mathbb{Z}_2$-graded $C(E) \widehat{\otimes} C(F)$-representations.

If $\dim E$ is even and $\dim F$ is odd, then as representations, $S(E) \widehat{\otimes} S(F)$ is isomorphic to $S(E) \otimes S(F)$ with $C(E) \widehat{\otimes} C(F)$-action defined by

$$a \widehat{\otimes} b_i \mapsto a \tau_F^i \otimes b_i, \quad a \in C(E), \quad b_i \in C_i(F).$$  \hfill (2.6)

If $\dim E$ is odd and $\dim F$ is even, then as representations, $S(E) \widehat{\otimes} S(F)$ is isomorphic to $S(E) \otimes S(F)$ with $C(E) \widehat{\otimes} C(F)$-action defined by

$$a_i \widehat{\otimes} b \mapsto a_i \otimes \tau_F^i b, \quad a_i \in C_i(E), \quad b \in C(F).$$  \hfill (2.7)

If both $\dim E$ and $\dim F$ are even, the representation $S(E) \otimes S(F)$ with $C(E) \widehat{\otimes} C(F)$-action defined by (2.7) is the unique irreducible one and $\mathbb{Z}_2$-graded for the tensor product grading on $S(E) \otimes S(F)$. It is isomorphic to $S(E) \widehat{\otimes} S(F)$ as representations. Since

$$C(E \oplus F) \simeq C(E) \widehat{\otimes} C(F),$$  \hfill (2.8)

$S(E) \widehat{\otimes} S(F)$ is also a $\mathbb{Z}_2$-graded representation of $C(E \oplus F)$. By (2.8), we have the isomorphism of representations

$$S(E \oplus F) \simeq S(E) \widehat{\otimes} S(F).$$  \hfill (2.9)
2.2 The equivariant geometric family

In this subsection, we introduce the equivariant geometric family (see [16,18]).

Let \( \pi : W \rightarrow B \) be a smooth surjective proper submersion of compact manifolds with compact fibers \( Z \) (possibly non-connected). Let \( B = \bigsqcup B_i \) be a finite disjoint union of compact connected manifolds. Let \( W_i \) be the restriction of \( W \) on \( B_i \). Let \( W_i = \bigsqcup W_{ij} \) be a finite disjoint union of compact connected manifolds. Let \( Z_{ij} \) be the fibers of the submersions restricted on \( W_{ij} \). We note here that the dimension of \( Z_{ij} \) might be zero. In the sequel, we will often omit the subscripts \( i \) and \( j \).

Let \( TZ = \ker(d\pi) \) be the relative tangent bundle to the fibers \( Z \) over \( W \), which is a subbundle of \( TW \). We assume that \( TZ \) is orientable and carries an orientation \( o \in H^1(W,\mathbb{Z}_2) \). Let \( T^HZW \) be a horizontal subbundle of \( TW \) such that

\[
TW = T^HZW \oplus TZ. \tag{2.10}
\]

The splitting (2.10) gives an identification

\[
T^HZW \cong \pi^*TB. \tag{2.11}
\]

If there is no ambiguity, we will omit the subscript \( \pi \) in \( T^HZW \). Let \( P^{TZ} \) be the projection

\[
P^{TZ}: TW = T^HZW \oplus TZ \rightarrow TZ. \tag{2.12}
\]

Let \( g^{TZ} \) and \( g^{TB} \) be Riemannian metrics on \( TZ \) and \( TB \), respectively. We equip \( TW = T^HZW \oplus TZ \) with the Riemannian metric

\[
g^{TW} = \pi^*g^{TB} \oplus g^{TZ}. \tag{2.13}
\]

Let \( \nabla^{TW} \) be the Levi-Civita connection on \((W,g^{TW})\). Set

\[
\nabla^{TZ} = P^{TZ}\nabla^{TW}P^{TZ}. \tag{2.14}
\]

Then \( \nabla^{TZ} \) is a Euclidean connection on \( TZ \). By [8, Theorem 1.9], we know that \( \nabla^{TZ} \) only depends on \((T^HZW,g^{TZ})\).

Let \( C(TZ) \) be the Clifford algebra bundle of \((TZ,g^{TZ})\), whose fiber at \( x \in W \) is the Clifford algebra \( C(T_xZ) \) of the Euclidean space \((T_xZ,g^{TZ})\).

We make the assumption that the oriented vector bundle \((TZ,o)\) has a Spin\(^c\) structure. Then there exists a complex line bundle \( L_Z \) over \( W \) such that \( \omega_2(TZ) = c_1(L_Z) \bmod 2 \), where \( \omega_2 \) denotes the second Stiefel-Whitney class and \( c_1 \) denotes the first Chern class. Let \( S(TZ,L_Z) \) be the fundamental complex spinor bundle for \((TZ,L_Z)\), which has a smooth action of \( C(TZ) \) (see [32, Appendix D.9]). Locally, the spinor bundle \( S(TZ,L_Z) \) may be written as

\[
S(TZ,L_Z) = S(TZ) \otimes L^{1/2}_Z, \tag{2.15}
\]

where \( S(TZ) \) is the fundamental spinor bundle for the (possibly non-existent) Spin structure on \( TZ \), and \( L^{1/2}_Z \) is the (possibly non-existent) square root of \( L_Z \). Let \( h^{L_Z} \) be a Hermitian metric on \( L_Z \). Then from (2.15), the metrics \( g^{TZ} \) and \( h^{L_Z} \) induce a Hermitian metric on \( S(TZ,L_Z) \), which we denote by \( h^{Sz} \) for simplicity. Let \( \nabla^{L_Z} \) be a Hermitian connection on \((L_Z,h^{L_Z})\). Similarly, we denote by \( \nabla^{Sz} \) the connection on \( S(TZ,L_Z) \) induced by \( \nabla^{TZ} \) and \( \nabla^{L_Z} \) from (2.15). Then \( \nabla^{Sz} \) is a Hermitian connection on \((S(TZ,L_Z),h^{Sz})\). Moreover, it is a Clifford connection associated with \( \nabla^{TZ} \), i.e., for any \( U \in TW \) and \( V \in \mathcal{C}^\infty(W,TZ) \),

\[
[\nabla^{Sz}_U,c(V)] = c(\nabla^{TZ}_U V). \tag{2.16}
\]

In the following, we often simply denote by \( S_Z \) the spinor bundle \( S(TZ,L_Z) \). If \( n = \dim Z \) is even, \( S_Z \) is \( \mathbb{Z}_2 \)-graded and the action of \( TZ \) exchanges the \( \mathbb{Z}_2 \)-grading.
Let $E = E_+ \oplus E_-$ be a $\mathbb{Z}_2$-graded smooth complex vector bundle over $W$ with the Hermitian metric $h^E$, for which $E_+$ and $E_-$ are orthogonal, and let $\nabla^E$ be a Hermitian connection on $(E, h^E)$ preserving the $\mathbb{Z}_2$-grading. Set

$$\nabla^{S_Z} \hat{\otimes} E := \nabla^{S_Z} \otimes 1 + 1 \otimes \nabla^E.$$  \hfill (2.17)

Then $\nabla^{S_Z} \hat{\otimes} E$ is a Hermitian connection on $(S_Z \hat{\otimes} E, h^{S_Z} \hat{\otimes} h^E)$.

Let $G$ be a compact Lie group which acts on $W$ and $B$ such that for any $g \in G$, $\pi \circ g = g \circ \pi$. We assume that the $G$-action preserves the splitting (2.10) and the Spin$^c$ structure of $TZ$. Thus, $TZ$, $L_Z$ and $S_Z$ are $G$-equivariant vector bundles. We assume that $g^{TZ}$, $h^{L_Z}$ and $\nabla^{L_Z}$ are $G$-invariant. We further assume that $E$ is a $G$-equivariant $\mathbb{Z}_2$-graded complex vector bundle and $h^E$ and $\nabla^E$ are $G$-invariant. Note that the $G$-action here may be nontrivial on $B$.

**Definition 2.1** (Compare with [16, Definition 2.2]). An equivariant geometric family $\mathcal{F}$ over $B$ is a family of $G$-equivariant geometric data

$$\mathcal{F} = (W, L_Z, E, o, T^H W, g^{TZ}, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E)$$  \hfill (2.18)
described above. We call that the equivariant geometric family $\mathcal{F}$ is even (resp. odd) if for any connected component of fibers, the dimension is even (resp. odd).

**Definition 2.2** (Compare with [16, Subsection 2.1.7]). Let $\mathcal{F}$ and $\mathcal{F}'$ be two equivariant geometric families over $B$. An isomorphism $\mathcal{F} \sim \mathcal{F}'$ consists of the following data:

$$\begin{array}{ccc}
E & \xrightarrow{f_E} & E' \\
\downarrow & & \downarrow \\
W & \xrightarrow{f} & W' \\
\downarrow & & \downarrow \\
B & \xrightarrow{\pi'} & B
\end{array}$$

where

1. $f$ is a diffeomorphism commuting with the $G$-action such that $\pi' \circ f = \pi$, which implies that $f$ preserves the relative tangent bundle;
2. $f$ preserves the orientation and the Spin$^c$ structure of the relative tangent bundle, which implies that there exists an equivariant complex line bundle isomorphism $f_L : L_Z \rightarrow L'_Z$;
3. $f_E : E \rightarrow E'$ is an equivariant vector bundle isomorphism over $f$, which preserves the $\mathbb{Z}_2$-grading;
4. $f$ preserves the horizontal subbundle and the vertical metric;
5. $f_L$ and $f_E$ preserve the metrics and the connections on the vector bundles.

If only the first three conditions hold, we say that $\mathcal{F}$ and $\mathcal{F}'$ have the same topological structure.

Let $\mathcal{F}^0_G(B)$ (resp. $\mathcal{F}^1_G(B)$) be the set of equivalence classes of even (resp. odd) equivariant geometric families over $B$.

For two equivariant geometric families $\mathcal{F}$ and $\mathcal{F}'$ over $B$, we can form their sum $\mathcal{F} + \mathcal{F}'$ over $B$ as a new equivariant geometric family: the underlying fibration of the sum is $\pi \sqcup \pi' : W \sqcup W' \rightarrow B$, where $\sqcup$ is the disjoint union and the remaining structures of $\mathcal{F} + \mathcal{F}'$ are induced in the obvious way. Let $\mathcal{F}^0_G(B) = \mathcal{F}^0_G(B) \oplus \mathcal{F}^1_G(B)$. It is an additive abelian semigroup.

For $\mathcal{F}, \mathcal{F}' \in \mathcal{F}^0_G(B)$, we can also form their product $\mathcal{F} \times_B \mathcal{F}'$ over $B$. The total space of the underlying fibration of $\mathcal{F} \times_B \mathcal{F}'$ is $W \times_B W' := \{(w, w') \in W \times W' : \pi(w) = \pi'(w')\}$ and the fiber is $Z \times Z'$. Let $\text{pr}_W : W \times_B W' \rightarrow W$ and $\text{pr}_{W'} : W \times_B W' \rightarrow W'$ be the obvious projections. The complex vector bundle now is $\text{pr}_W^* E \hat{\otimes} \text{pr}_{W'}^* E'$. The remaining structures of $\mathcal{F} \times_B \mathcal{F}'$ are induced in the obvious way.

Let $B$ and $B'$ be two compact manifolds with smooth $G$-actions. Let $f : B \times B' \rightarrow B$ be the projection onto the first part. For any $\mathcal{F} \in \mathcal{F}^0_G(B)$, we could construct the pullback $f^* \mathcal{F} \in \mathcal{F}^0_G(B \times B')$ in a natural way. Remark that in a general case, for a $G$-equivariant map $f : B' \rightarrow B$ and $\mathcal{F} \in \mathcal{F}^0_G(B)$, $f^* \mathcal{F}$ is hard
to define canonically because we cannot choose a canonical horizontal subbundle in $f^*\mathcal{F}$. We will show more details in Subsection 4.3 later.

**Definition 2.3.** The opposite family $\mathcal{F}^{op}$ of an equivariant geometric family $\mathcal{F}$ is obtained by reversing the $\mathbb{Z}_2$-grading of $E$.

### 2.3 The equivariant K-theory

In this subsection, we give some examples of the equivariant geometric families and a geometric description of the equivariant K-theory.

Let $K^0_G(B)$ be the $G$-equivariant $K^0$-group of $B$, which is the Grothendieck group of the equivalence classes of $G$-equivariant topological complex vector bundles over $B$ (see [48]). Since $G$ is compact, by Proposition A.4, it is also the Grothendieck group of the equivalence classes of $G$-equivariant smooth complex vector bundles. Note that the ring structure of the $K^0$-group is induced by the tensor product of the complex vector bundles.

We lift the $G$-action on $B \times S^1$ such that the $G$-action on $S^1$ is trivial. Take $s \in S^1$ fixed. Let $i : B \ni b \mapsto (b, s) \in B \times S^1$ be the $G$-equivariant inclusion map. Let $i^* : K^0_G(B \times S^1) \to K^0_G(B)$ be the induced homomorphism. Let $K^1_G(B)$ be the $G$-equivariant $K^1$-group of $B$. By [48, Definitions 2.7 and 2.8], we have the split short exact sequence

$$0 \to K^1_G(B) \xrightarrow{j} K^0_G(B \times S^1) \xrightarrow{i^*} K^0_G(B) \to 0,$$

(2.19)

where $j$ is induced by the suspension isomorphism $K^1_G(B) \simeq \tilde{K}^0_G(B \wedge S^1) \simeq \ker(i^*)$ (see [48, p. 136]). Here, $B \wedge S^1$ is the smash product of $B$ and $S^1$ and $\tilde{K}^0_G(B \wedge S^1)$ is the $G$-equivariant reduced $K^0$-group of $B \wedge S^1$.

Now we introduce another explanation of $K^2_G(B)$. Let $V$ be a finite-dimensional complex unitary representation of $G$. If $F \in \mathcal{E}^\infty(B, \text{End}(V))$ such that for any $b \in B$, $F(b) \in \text{End}(V)$ is unitary and for any $g \in G$ and $v \in V$,

$$g(F(b)v) = F(\sigma b)(g)v,$$

(2.20)

we say that $F$ is a $G$-invariant unitary element of $\mathcal{E}^\infty(B, \text{End}(V))$. In this case, for $(b, t, v) \in B \times [0, 1] \times V$, the relation $(b, 0, v) \sim (b, 1, F(b)v)$ forms a $G$-equivariant smooth Hermitian vector bundle $W$ over $B \times S^1$.

Let $U = B \times S^1 \times V$ be the $G$-equivariant trivial bundle over $B \times S^1$ as in (A.2). Then from (2.19), $[W] - [U] \in \ker(i^*)$ corresponds to an element $[F] \in K^2_G(B)$.

**Lemma 2.4.** For any $y \in K^1_G(B)$, there exists a finite-dimensional complex unitary representation $V$ of $G$ such that $y$ can be represented as a $G$-invariant unitary element of $\mathcal{E}^\infty(B, \text{End}(V))$.

**Proof.** By (2.19), an element $y \in K^1_G(B)$ can be represented as an element

$$x = j(y) \in K^0_G(B \times S^1)$$

such that $i^*x = 0 \in K^0_G(B)$. We write $x = W - U$, where $W$ and $U$ are equivariant smooth complex vector bundles over $B \times S^1$. By Proposition A.2, we may and we will assume that $U$ is an equivariant trivial complex vector bundle over $B \times S^1$ associated with a finite-dimensional complex $G$-representation $V$ as in (A.2). Note that $B \times S^1 \simeq B \times \mathbb{R}/\mathbb{Z}$. We assume that $i(B) = B \times \{1/2\}$. Since

$$i^*x = W|_{B \times \{1/2\}} - U|_{B \times \{1/2\}} = 0 \in K^0_G(B),$$

by Proposition A.2, we may and we will assume that $W|_{B \times \{1/2\}}$ is isomorphic to the equivariant trivial bundle $(B \times \{1/2\}) \times V$ over $B \times \{1/2\}$ as equivariant smooth complex vector bundles. Since $(0, 1)$ is contractible, as equivariant smooth complex vector bundles over $B \times (0, 1)$,

$$W|_{B \times (0,1)} \simeq (\text{Id}_B \times p_{1/2})^*(W|_{B \times \{1/2\}}),$$
where \( p_{1/2} : (0, 1) \to 1/2 \) is the constant map. Since \( B \times (0, 1) \times V = (\text{Id}_B \times p_{1/2})^* ((B \times \{1/2\}) \times V) \) as complex vector bundles over \( B \times (0, 1) \), there exists a \( G \)-equivariant smooth complex vector bundle isomorphism

\[
 f : W \mid_{B \times (0, 1)} \to B \times (0, 1) \times V.
\]

Let \( h : B \times (0, 1) \times V \to V \) be the obvious projection. For any \( b \in B \) and \( v \in V \), we could choose a section \( s \in \mathcal{C}^\infty (B \times S^1, W) \) such that \( \lim_{t \to 0^+} h \circ f (s(b, t)) = v. \) Then we define

\[
 F(b)v := \lim_{t \to 1} h \circ f (s(b, t)) \in V.
\]

Note that the definition of \( F(b) \in \text{End}(V) \) does not depend on the choices of the isomorphic map \( f \) and the section \( s. \) Take a \( G \)-invariant Hermitian metric on \( W \) which induces a \( G \)-equivariant Hermitian inner product on \( V. \) It is obvious that \( F \) is a \( G \)-invariant unitary element of \( \mathcal{C}^\infty (B, \text{End}(V)) \) and \( \{ F \} = y \in K^0_G(B). \)

The proof of Lemma 2.4 is completed. \( \square \)

For an equivariant geometric family \( F, \) the fiberwise Dirac operator \( D(F) \) associated with \( F \) is defined by

\[
 D(F) := \sum_i c(e_i) \nabla_{e_i} \overset{\cdot}{\otimes} E,
\]

where \( \{ e_i \} \) is a local orthonormal frame of \( TZ. \) Note that the definition of the fiberwise Dirac operator is independent of the choice of the local orthonormal frame. From (2.23), the \( G \)-action commutes with \( D(F). \)

If \( F \) is isomorphic to \( F', \) from Definition 2.2 and (2.23), the isomorphism preserves the fiberwise Dirac operator. So the fiberwise Dirac operator can be defined on an element of \( F_G^* (B). \) For an even (resp. odd) equivariant geometric family of \( F, \) the classical construction of Atiyah-Singer assigns to this family its equivariant (analytic) index \( \text{Ind}(D(F)) \in K^0_G(B) \) (resp. \( K^1_G(B) \)) (see [4, 5]). Remark that \( \text{Ind}(D(F)) \) depends only on the topological structure of \( F. \) It induces a map

\[
 \text{Ind} : F^*_G(B) \to K^*_G(B), \quad F \mapsto \text{Ind}(D(F)).
\]

Let \( K^*_G(B) = K^0_G(B) \oplus K^1_G(B). \) Since

\[
 \text{Ind}(D(F + F')) = \text{Ind}(D(F)) + \text{Ind}(D(F')) \in K^*_G(B),
\]

the equivariant index map in (2.24) is a semigroup homomorphism. It is well known that if \( F \) and \( F' \) are even,

\[
 \text{Ind}(D(F \times_B F')) = \text{Ind}(D(F)) \cdot \text{Ind}(D(F')) \in K^0_G(B).
\]

**Example 2.5.** (a) Let \((E, h^E)\) be an equivariant \(\mathbb{Z}_2\)-graded smooth Hermitian vector bundle over \( B \) with a \( G \)-invariant Hermitian connection \( \nabla^E. \) Then \((E, h^E, \nabla^E)\) can be regarded as an even equivariant geometric family \( F \) for \( Z = pt. \) In this case, \( D(F) = 0 \) and \( \text{Ind}(D(F)) = [E_+] - [E_-] \in K^0_G(B). \)

(b) Let \( W = B \times \mathbb{C}P^1 \) with the \( G \)-action which acts trivially on \( \mathbb{C}P^1. \) Then the complex line bundle \( \mathcal{O}(1) \) over \( \mathbb{C}P^1 \) can be naturally extended on \( W. \) Thus \((W, \mathcal{O}(1))\) with canonical metrics, connections, the standard orientation of \( \mathbb{C}P^1 \) and the Spin structure on \( \mathbb{C}P^1 \) form an even equivariant geometric family \( F_S \) over \( B. \) Let \( D_{\mathbb{C}P^1}^O \) be the Dirac operator on \( \mathbb{C}P^1 \) associated with \( \mathcal{O}(1). \) Since \( \text{Ind}(D_{\mathbb{C}P^1}^O) = \langle c_1(\mathcal{O}(1)), [\mathbb{C}P^1] \rangle = 1, \) from (2.26), for even equivariant geometric family \( F \) in (a), we have \( \text{Ind}(D(F \times_B F_S)) = \text{Ind}(D(F)) \in K^0_G(B). \)

(c) (Compare with [44, Section 5] and [14, Subsection 2.2.3.8]) Let \( B = S^1_\theta = \mathbb{R}/\mathbb{Z}, \) \( W = S^1_\theta \times S^1_t \) and \( \pi : W \to B \) be the projection onto the first part. We consider the Hermitian line bundle \((L, h^L)\) which is obtained by identifying

\[
 (\theta = 0, t, v), \quad (\theta = 1, t, \exp(-2\pi t\sqrt{-1})v) \in [0, 1] \times S^1 \times \mathbb{C}.
\]
Then
\[ \nabla^L = d + 2\pi(\theta - 1/2)\sqrt{-1}dt \] (2.28)
is a Hermitian connection on \((L, h^L)\) (see [11, p. 124]). We choose the \(\mathbb{Z}_2\)-grading of \(L\) such that \(L_+ = L\) and \(L_- = 0\). We consider the Spin structure on \(S^1\) as the desired Spin’ structure. Then we get an odd geometric family \(\mathcal{F}^L\) after choosing the natural geometric data. In fact, since \(c_1(L) = dt d\theta\), \(\text{Ind}(D(\mathcal{F}^L))\) is a generator of \(K^1(S^1) \simeq \mathbb{Z}\) by the family index theorem.

(d) Let \(\mathcal{F} \in \mathcal{F}^*_G(B)\). Let \(p_1\) and \(p_2\) be the projections onto the first and second parts of \(B \times S^1\), respectively. We take \(\mathcal{F}^L\) as in (c). Then \(p_1^*\mathcal{F} \times_{B \times S^1} p_2^*\mathcal{F}^L\) is an equivariant geometric family over \(B \times S^1\). From Proposition B.1 (see also the proof of [11, Theorem 2.10]), for \(\mathcal{F} \in \mathcal{F}^*_G(B)\), there exists an inclusion \(i: B \rightarrow B \times S^1\) such that \(i^*\text{Ind}(D(p_1^*\mathcal{F} \times_{B \times S^1} p_2^*\mathcal{F}^L)) = 0\). Moreover, as an element of \(K^*_G(B)\) in the sense of (2.19), by an equivariant version of [44, Proposition 6], we have
\[ j(\text{Ind}(D(\mathcal{F}))) = \text{Ind}(D(p_1^*\mathcal{F} \times_{B \times S^1} p_2^*\mathcal{F}^L)), \] (2.29)
where \(j\) is the map in (2.19). This example is essential in our construction of the higher spectral flow for the even case. For the reader’s convenience, we will show more details in Appendix B.

We write \(\mathcal{F} \sim \mathcal{F}'\) if \(\text{Ind}(D(\mathcal{F})) = \text{Ind}(D(\mathcal{F}'))\). It is an equivalence relation and compatible with the semigroup structure. So \(\mathcal{F}^*_G(B)/ \sim\) is a semigroup and the map
\[ \text{Ind}: \mathcal{F}^*_G(B)/ \sim \rightarrow K^*_G(B) \] (2.30)
is an injective semigroup homomorphism.

By Definition 2.3, we have
\[ \text{Ind}(D(\mathcal{F}^{\text{op}})) = - \text{Ind}(D(\mathcal{F})). \] (2.31)
After one defines \(-\mathcal{F} := \mathcal{F}^{\text{op}},\) the semigroup \(\mathcal{F}^*_G(B)/ \sim\) can be regarded as an abelian group. So by (2.31), the equivariant index map in (2.30) is a group homomorphism. Note that \(K^*_G(B)\) has a ring structure (see [2]), and by (2.29), (2.26) holds for any \(\mathcal{F}, \mathcal{F}' \in \mathcal{F}^*_G(B)\). Thus the equivariant index map in (2.30) is also a ring homomorphism. In fact, it is even a ring isomorphism [18, Subsection 2.5.5]. We rewrite the proof in our notation here for the completeness.

**Proposition 2.6.** The equivariant index map \(\text{Ind}\) in (2.30) is surjective. In other words, we have the \(\mathbb{Z}_2\)-graded ring isomorphism
\[ \mathcal{F}^*_G(B)/ \sim \simeq K^*_G(B). \] (2.32)

**Proof.** When \(* = 0\), we can get the proposition directly from Example 2.5(a) or Example 2.5(b).

When \(* = 1\), from the proof of Lemma 2.4, for any \([F] \in K^*_G(B)\), there exist equivariant complex vector bundles \(W\) and \(U\) such that \([W] - [U] \in \mathcal{K}^*_G(B \times S^1)\) corresponds to \([F] \in K^*_G(B)\). Moreover, after taking the natural geometric data, we get an odd equivariant geometric family \(\mathcal{F}\) over \(B\) with fibers \(S^1\) and the \(\mathbb{Z}_2\)-graded equivariant complex vector bundle \(W \oplus U\). As in [14, Subsection 2.2.2.3], we have \(\text{Ind}(D(\mathcal{F})) = [F] \in K^*_G(B)\).

The proof of Proposition 2.6 is completed. \(\square\)

**Remark 2.7.** Note that if we replace the Spin’ condition of the geometric family by the general Clifford module condition (which is the setting in [16,18]) or the Spin condition, Proposition 2.6 also holds. Since we do not use the language of Clifford modules here, our definition of \(\mathcal{F}^{\text{op}}\) in Definition 2.3 is simpler than that in [16,18]. In fact, in the sense of (2.32), they are the same.

### 2.4 The push-forward map

In this subsection, we define the push-forward map in the equivariant K-theory using the equivariant geometric families.
Let $\pi_Y : V \to B$ be a $G$-equivariant smooth surjective proper submersion of compact manifolds with compact orientable fibers $Y$. We simply assume that the dimensions of all the connected components of $Y$ have the same parity. Let $o_Y \in H^0(V, \mathbb{Z}_2)$ be an orientation of the relative tangent bundle $TY$.

**Definition 2.8** (Compare with [18, Definition 3.1]). An equivariant K-orientation of $\pi_Y$ is an equivariant Spin$^c$ structure of $TY$. Let $\mathcal{O}_G(\pi_Y)$ be the set of equivariant K-orientations.

Suppose that $\pi_Y$ has an equivariant K-orientation $O_Y \in \mathcal{O}_G(\pi_Y)$. For $j = 0, 1$, let $N(j) := j \pmod{2}$ if $\dim Y$ is even (resp. odd). We will use Proposition 2.6 to define the push-forward map of equivariant K-groups $\pi_Y! : K^j_G(V) \to K^N_G(B)$ as follows.

Let $\pi_X : W \to V$ be the submersion with compact orientable fibers $X$. Let $F_X = (W, L_X, E, o_X, T^H_{\pi_X} W, g^{TX}, h^{L_X}, \nabla^{L_X}, h^E, \nabla^E)$ be a $G$-equivariant geometric family in $F^0_G(V)$. Then $\pi_Z := \pi_Y \circ \pi_X : W \to B$ is a smooth submersion with compact orientable fibers $Z$. We have the diagram of smooth fibrations

$$
\begin{array}{ccc}
X & \rightarrow & Z \\
\downarrow \pi_X & & \downarrow \pi_Z \\
Y & \rightarrow & V & \rightarrow & B.
\end{array}
$$

Set $T^H_{\pi_X} Z := T^H_{\pi_X} W \cap TZ$. Then we have the splitting of smooth vector bundles over $W$:

$$TZ = T^H_{\pi_X} Z \oplus TX$$

and

$$T^H_{\pi_X} Z \cong \pi_X^* TY.$$  

Let $o_Z := \pi_X^* o_Y \cup o_X \in H^0(W, \mathbb{Z}_2)$. Since $TY$ and $TX$ have equivariant Spin$^c$ structures, so is $TZ$. Let $L_Y$ be the equivariant complex line bundle associated with the equivariant Spin$^c$ structure of $TY$. Set

$$L_Z := \pi_X^* L_Y \otimes L_X.$$  

Let $g^{TY}$ be a $G$-invariant Riemannian metric on $TY$. Let $h^{L_Y}$ be a $G$-invariant Hermitian metric on $L_Y$ and $\nabla^{L_Y}$ be a $G$-invariant Hermitian connection on $(L_Y, h^{L_Y})$. Take geometric data $(T^H_{\pi_X} W, g^{TX}, h^{L_X}, \nabla^{L_X})$ of $\pi_Z$ such that $T^H_{\pi_X} W \subset T^H_{\pi_X} W$, $g^{TX} = \pi_X^* g^{TY} \oplus g^{TX}$, $h^{L_X} = \pi_X^* h^{L_Y} \otimes h^{L_X}$ and $\nabla^{L_X} = \pi_X^* \nabla^{L_Y} \oplus 1 + 1 \otimes \nabla^{L_X}$. We get a new equivariant geometric family over $B$:

$$F_Z := (W, L_Z, E, o_Z, T^H_{\pi_X} W, g^{TZ}, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E).$$

We write $F_Z = \pi_Y!(F_X)$.

**Theorem 2.9.** For the equivariant K-orientation $O_Y \in \mathcal{O}_G(\pi_Y)$ fixed, the push-forward map

$$\pi_Y! : K^j_G(V) \to K^N_G(B),$$

$$[F_X] \mapsto [F_Z]$$

is a well-defined group homomorphism and independent of the geometric data $(T^H_{\pi_X} W, g^{TY}, h^{L_Y}, \nabla^{L_Y})$.

**Proof.** We firstly assume that the map $\pi_Y!$ is well defined. Then the remaining results follow from the definition of the equivariant family index.

The well-defined property of $\pi_Y!$ will be proved in Subsection 3.2 later.

Let $\pi_U : B \to S$ be a $G$-equivariant smooth surjective proper submersion of compact manifolds with compact oriented fibers $U$ and an equivariant K-orientation $O_U$. Then $\pi_A := \pi_U \circ \pi_Y : V \to S$ is a $G$-equivariant smooth submersion with an equivariant K-orientation constructed by $O_Y$ and $O_U$. From the construction of the push-forward map and Theorem 2.9, the following theorem is obvious.
Theorem 2.10. We have the equality of homomorphisms
\[ \pi_A! = \pi_U! \circ \pi_Y! : K_G^*(V) \to K_G^*(S). \] (2.39)

3 The equivariant higher spectral flow and the equivariant eta form

In this section, we extend the Melrose-Piazza spectral section to the equivariant case, introduce the equivariant version of the Dai-Zhang higher spectral flow for arbitrary-dimensional fibers and use them to prove the anomaly formula and the functoriality of the equivariant Bismut-Cheeger eta forms. In this section, we use the notation in Section 2.

In Subsection 3.1, we introduce the equivariant version of the spectral section and prove the main properties. In Subsection 3.2, we complete the proof of the well-definedness of the push-forward map in Theorem 2.9. In Subsection 3.3, we define the higher spectral flow for fibrations with even-dimensional fibers and extend the higher spectral flow to the equivariant case. Moreover, we prove that the equivariant K-group could be generated by the equivariant higher spectral flows. In Subsection 3.4, we explain the family local index theorem. In Subsection 3.5, we define the equivariant eta form associated with a perturbation operator. In Subsections 3.6–3.9, we prove the anomaly formula of equivariant eta forms in the odd case. In Subsections 3.7–3.9, we prove the functoriality of equivariant eta forms and use it to prove the anomaly formula in the even case.

3.1 The equivariant spectral section

In this subsection, we extend the spectral section of Melrose and Piazza [43, 44] and the main properties of them to the equivariant case.

Definition 3.1. Let \( F \in F_G^*(B) \) and at least one component of the fiber has the nonzero dimension. An equivariant \( B \)-family on \( F \) is a smooth family of self-adjoint pseudodifferential operators \( D = \{D_b\}_{b \in B} \) on the fibers of \( F \), which commutes with the \( G \)-action and is first-order on nonzero-dimensional fibers such that

(a) it preserves the \( \mathbb{Z}_2 \)-grading of \( E \) when the fiber is odd-dimensional;
(b) it anti-commutes with the \( \mathbb{Z}_2 \)-grading of \( \mathcal{S}_Z \hat{\otimes} E \) when the fiber is even-dimensional.

If the dimension of the fiber is zero, an equivariant \( B \)-family is a self-adjoint endomorphism of \( E \) which commutes with the \( G \)-action and anti-commutes with the \( \mathbb{Z}_2 \)-grading of \( E \).

If the principal symbol of \( D_b \) is the same as that of the fiberwise Dirac operator \( D(F)|_{Z_b} \) for any \( b \in B \), we call this equivariant \( B \)-family a \( B \)-family of the equivariant Dirac type operator. In this case, we have \( \text{Ind}(D) = \text{Ind}(D(F)) \in K_G^*(B) \). Recall that if the fiber is a point, the fiberwise Dirac operator is zero.

Definition 3.2. An equivariant Melrose-Piazza spectral section of an equivariant \( B \)-family \( D = \{D_b\}_{b \in B} \) is a continuous family of self-adjoint pseudodifferential projections \( P_b \) on the \( L^2 \)-completion of the domain of \( D_b \), which commutes with the \( G \)-action and anti-commutes with the \( \mathbb{Z}_2 \)-grading of \( E \) such that

(a) for some smooth function \( f : B \to \mathbb{R} \) (depending on \( P \)) and every \( b \in B \),
\[ D_b u = \lambda u \Rightarrow \begin{cases} P_b u = u, & \text{if } \lambda > f(b), \\ P_b u = 0, & \text{if } \lambda < -f(b); \end{cases} \] (3.1)
(b) when the fiber is odd-dimensional, \( P \) commutes with the \( \mathbb{Z}_2 \)-grading of \( E \);
(c) when the fiber is even-dimensional,
\[ \tau_{S_Z \hat{\otimes} E} \circ P + P \circ \tau_{S_Z \hat{\otimes} E} = \tau_{S_Z \hat{\otimes} E}, \] (3.2)

where \( \tau_{S_Z \hat{\otimes} E} \) is the \( \mathbb{Z}_2 \)-grading of \( S_Z \hat{\otimes} E \).
The following proposition is the equivariant extension of the results in [43,44]. Remark that in our setting the dimension of the fiber might be zero. In the proof of this proposition, we will use the equivariant version of the Fredholm theory for fiberwise elliptic operators. For the references, see [3] and [26, Appendix A.5]. We will also show some details in Appendix B.

**Proposition 3.3.** Let \( F \in \mathcal{F}_G^0(B) \) and \( D \) be an equivariant \( B \)-family on \( F \).

(i) (Compare with [43, Proposition 1] and [44, Proposition 2]) If there exists an equivariant spectral section of \( D \) on \( F \in \mathcal{F}_G^0(B) \) (resp. \( \mathcal{F}_G^1(B) \)), then Ind\((D) = 0 \in K_G^0(B) \) (resp. \( K_G^1(B) \)). Conversely, if \( F \in \mathcal{F}_G^0(B) \) (resp. \( \mathcal{F}_G^1(B) \)), Ind\((D) = 0 \in K_G^0(B) \) (resp. \( K_G^1(B) \)) and at least one component of the fibers has the nonzero dimension, there exists an equivariant spectral section of \( D \).

(ii) (Compare with [43, Proposition 2]) For \( F \in \mathcal{F}_G^1(B) \), given equivariant spectral sections \( P \) and \( Q \) of \( D \), there exists an equivariant spectral section \( R \) of \( D \) such that \( PR = R \) and \( QR = R \). We say that \( R \) majorizes \( P \) and \( Q \).

(iii) (Compare with [43, Lemma 8] and [44, Lemma 1]) If there exists an equivariant spectral section \( P \) of \( D \), then there exists a family of self-adjoint equivariant smoothing operators \( A_P \) (when the dimensions of the fibers are zero, it descends to a self-adjoint equivariant endomorphism of the complex vector bundle) with range in a finite sum of eigenspaces of \( D \) such that \( D + A_P \) is an invertible equivariant \( B \)-family and \( P \) is the Atiyah-Patodi-Singer (APS) projection onto the positive spectrum of \( D + A_P \).

**Proof.** **Case 1.** Let \( F \in \mathcal{F}_G^0(B) \). We use the notation of Appendix B in this part of the proof.

We write \( F = (W, o, E) \), \( E = E_+ \oplus E_- \) and omit other data for simplicity. Set \( F_+ = (W, o, E_+ \oplus \{0\}) \) and \( F_- = (W, o, \{0\} \oplus E_-) \). Then Ind\((D_{F_+}) = Ind(D_{F_+} + F_-) = Ind(D_{F_+} - F_-) \). Set \( F_{E_+} := (W, -o, E_+ \oplus \{0\}) \), where \( -o \) is the reversion of the orientation \( o \) and \( E_- \) is equipped with the positive \( \mathbb{Z}_2 \)-grading. Then there is a natural bijective map \( F_- \rightarrow F_{E_+} \) by identification. Note that Ind\((D_{F_+}) = Ind(D_{F_{E_+}}) \). We have Ind\((D_{F_+}) = Ind(D_{F_+} + F_{E_+}) \). So using this bijective map, we only need to prove our proposition in the odd case where \( E_- = 0 \) in \( F \).

(i) Let \( T = D/(1 + D^2)^{1/2} \). Then \( T \) is bounded, \( G \)-equivariant and Ind\((T) = Ind(D) \in K_G^1(B) \). As in Appendix B, \( \sqrt{-TT} \) can be extended to an equivariant map from \( B \) to \( \text{Fred}^1(L^2(G) \otimes H) \), where \( H \) is a separable Hilbert space. Assume that there exists an equivariant spectral section \( P \) of \( D \). It could be extended on \( L^2(G) \otimes C(\mathbb{R}) \otimes H \) in the same way as \( T \), which is also denoted by \( P \). From the definition of the equivariant spectral section, \( PT(1 - P) \) and \((1 - PT)P \) are \( G \)-equivariant self-adjoint finite rank operators. Let \( K = (1 + D^2)^{-1/2} \) on \( E \). Then \( K \) is a \( G \)-equivariant self-adjoint compact operator on \( L^2(G) \otimes C(\mathbb{R}) \otimes H \) by taking zero on the complement of \( E \). Thus from the \( G \)-homotopy invariance, the equivariant family index of \( T \) in \( K_G^1(B) \) is the same as that of \( P(T + rK)P + (1 - PT)(1 - K)(1 - P) \) for \( r \geq 0 \). When \( r \) is large enough, for any \( b \in B \), \( P_b(T_b + rK)P_b \) is positive and \((1 - P_b)(T_b - rK_b)(1 - P_b) \) is negative. Therefore, we have Ind\((T) = 0 \in K_G^1(B) \).

If Ind\((D) = Ind(T) = 0 \in K_G^1(B) \), we modify the construction of the spectral section in the proof of [43, Proposition 1]. All the equivariant operators here are regarded as families of equivariant operators acting on \( L^2(G) \otimes H \). Since Ind\((T) = 0 \in K_G^1(B) \), from (B.1), \( \sqrt{-TT} \) is \( G \)-homotopic to an invertible element in \( \text{Fred}^1(L^2(G) \otimes H) \) through \( \sqrt{-TT} : \text{Fred}^1(L^2(G) \otimes H), t \in [0, 1] \). As in [43], all the operators in these families have discrete spectra in some fixed open interval \((-\varepsilon, \varepsilon) \), \( 0 < \varepsilon < 1 \). Choose \( \chi \in C_c^\infty(\mathbb{R}) \) with \( \chi(\lambda) = 0 \) if \( \lambda < 0 \) and \( \chi(\lambda) = 1 \) if \( \lambda > \varepsilon/2 \). Set \( J = \chi(T) \). Then \( J \) is \( G \)-equivariant and smooth on \( b \in B \). From the \( G \)-homotopy above, we could construct a smooth family of \( G \)-equivariant projections \( P_b \) on \( L^2(G) \otimes H \) in the same way as in the proof of [43, Proposition 1] such that \( J - P_b \) has finite rank and the range of \( J - P_b \) lies in \( E \). By taking spectral cuts as in the proof of [33, Theorem 3], we could obtain an equivariant projection \( P \) which differs from \( J \) by an equivariant operator whose range lies in the span of a finite number of eigenfunctions of \( T \) on \( E \) for each \( b \in B \) (see also the proof of [54, Proposition 3.7]). So \( P|_E \) is an equivariant spectral section.

(ii) We extend the equivariant spectral section \( P \) on the equivariant trivial Hilbert bundle as before and consider the family of operators \( PTP \) on the range of \( P \). These are equivariant self-adjoint operators and from (3.1), there exists \( N > 0 \) such that all but the first \( N \) eigenfunctions of \( PTP \) are eigenfunctions.
of $D$. Since $B$ is compact, we could take $0 < a_1 < 1$ such that the first $N$ eigenvalues of $PTP$ are all less than $a_1$ for any $b \in B$. Take $a_2 \in (a_1, 1)$ and choose $\chi_1 \in C^\infty_0(\mathbb{R})$ with $\chi_1(\lambda) = 0$ if $\lambda \leq a_1$ and $\chi_1(\lambda) = 1$ if $\lambda \geq a_2$. Then for $M$ large enough, the range of $P - \chi_1(T)$ is modified to an equivariant subbundle of the range of $P$ (see [6, Lemma 9.9]) which contains the first $N$ eigenfunctions and is contained in the span of the first $M$ eigenfunctions of $PTP$. Let $R$ be the orthogonal projection on the complement of this subbundle in $E$. Then $R$ is an equivariant spectral section such that $PR = R$. If the integer $N$ is chosen large enough, then the projection $R$ will have range contained in the intersection of the range of any two given equivariant spectral sections $P$ and $Q$. So $QR = R$.

(iii) Let $\mathcal{P}_{\lambda \in [a_1, a_2], b}(D_b)$ be the span of the eigenfunctions corresponding to the eigenvalues $\lambda \in [a_1, a_2]$ of $D_b$. Since $B$ is compact, we can choose $s > 0$ such that $P$ is an equivariant spectral section for $f(b) \equiv s$. By the proof of (ii), we can choose equivariant spectral sections $R'$ and $R''$ such that for any $b \in B$, $R'_b = 0$ on $\mathcal{P}_{\lambda \leq s, b}(D_b)$ and $R''_b = I$ on $\mathcal{P}_{\lambda > s, b}(D_b)$. Then the operator

$$\tilde{D} = R'DR' + sPR''(I - R') + (I - R'')DR(I - R'') - s(I - P)R''(I - R')$$

(3.3)
is an invertible equivariant $B$-family (see [43, (8.3)]). Then $AP = \tilde{D} - D$ satisfies all the conditions.

**Case 2.** Let $F \in \mathbb{P}_G^0(B)$ and at least one component of the fibers has the nonzero dimension.

Let $D_{\pm} := D|_{(S \otimes E)\pm}$. Let $S$ be a first-order positive equivariant elliptic pseudodifferential operator. Then in the sense of (B.1), $D$ is $G$-homotopic to $(S \otimes D_{\mp})$, which is invertible. Thus the equivariant $K'$-index of the whole self-adjoint family $D$ vanishes. By the same process in the proof of (i) in the odd case, there exists an equivariant spectral section $P'$ in the odd sense, which means that it is an equivariant spectral section without the condition (3.2).

(iii) By the proof of (ii) for the odd case, we could choose $P'$, which is an equivariant spectral section in the odd sense such that $P'DP'$ is positive on the range of $P$. We simply define $\tau = \tau_{S \otimes E}$. Then the operator

$$AP = P - P' - \tau(P - P')\tau + P'DP' + \tau P'\tau D\tau P'\tau - D$$

(3.4)
satisfies all the conditions (see [44, (2.11) and (2.12)]).

(i) Assume that $\text{Ind}(D) = 0 \in K_G^0(B)$. As in the proof of (ii) for the odd case, for $r > 0$ fixed, we can choose an equivariant spectral section $P'$ in the odd sense such that $P' = 0$ on $\mathcal{P}_{\lambda < r}(D)$, and $\chi_1(\lambda) = 0$ if $\lambda \leq a_1$ and $\chi_1(\lambda) = 1$ if $\lambda \geq a_2$. Then the operator

$$\tilde{D} = R'DR' + sPR''(I - R') + (I - R'')DR(I - R'') - s(I - P)R''(I - R')$$

(3.3)
is an invertible equivariant $B$-family (see [43, (8.3)]). Then $AP = \tilde{D} - D$ satisfies all the conditions.

**Case 3.** The dimensions of all the fibers are zero.

In this case, for any self-adjoint projection $P \in \text{End}(E)$ commuting with the $G$-action, (3.1) holds. Thus the only restriction for $P$ as an equivariant spectral section is (3.2). If there exists an equivariant
skeletal section $P$, we take $A_P = P - \tau P - D$. Thus $D + A_P = 2P - \text{Id}$ is invertible and $P$ is the projection onto the eigenspaces of the positive eigenvalues of $D + A_P$. Thus $\text{Ind}(D) = 0 \in K_G^0(B)$.

The proof of Proposition 3.3 is completed.

Remark 3.4. In the zero-dimensional case, we could also construct an equivariant spectral section of $D(F + F^{op})$ as in (3.5).

Definition 3.5. Let $D$ be an equivariant $B$-family on $F$. A perturbation operator with respect to $D$ is a family of bounded pseudodifferential operators $A$ such that $D + A$ is an invertible equivariant $B$-family on $F$.

Note that if there exists an equivariant spectral section of $D$, the smoothing operator associated with it is a perturbation operator.

Remark that the tamings in [14, 16, 18] are perturbation operators when the manifolds there are smooth, compact and without boundary.

3.2 The well-defined property for the push-forward map

In this subsection, we show that the push-forward map defined in Theorem 2.9 is well defined. We use the notation in Subsection 2.4.

Lemma 3.6. If $\text{Ind}(D(F_X)) = 0 \in K^*_G(V)$, then $\text{Ind}(D(F_Z)) = 0 \in K^N_G(B)$, $j = 0, 1$.

Proof. We only need to prove this lemma when the dimensions of the fibers are nonzero. Let

$$g^{TZ}_T = \pi_X^* g^{TY} \oplus \frac{1}{T^2} g^{TX}. \hspace{1cm} (3.6)$$

We denote by $C_T(TZ)$ the Clifford algebra bundle of $TZ$ with respect to $g^{TZ}_T$. If $U \in TV$, let $U^H \in T^H_{X,x} W$ be the horizontal lift of $U$ such that $\pi_X^*(U^H) = U$. Let $\{e_i\}$ and $\{f_p\}$ be local orthonormal frames of $(TX, g^{TX})$ and $(TY, g^{TY})$, respectively. Then $\{f_{p,1}^H \cup \{T e_i\}$ is a local orthonormal frame of $(TZ, g^{TZ})$. We define a Clifford algebra isomorphism

$$\mathcal{G}_T : C_T(TZ) \to C(TZ) \hspace{1cm} (3.7)$$

by

$$\mathcal{G}_T(c(f_{p,1}^H)) = c(f_p^H), \hspace{0.5cm} \mathcal{G}_T(c(T e_i)) = c(e_i). \hspace{1cm} (3.8)$$

Under this isomorphism, we can consider $((\pi_X^* S_Y \otimes S_X) \otimes E, h^{\pi_X^* S_Y \otimes S_X} \otimes h^E)$ as a self-adjoint Hermitian equivariant Clifford module of $C_T(TZ)$. So

$$F_{Z,T} = (W, L_Z, E, o_Z, T_{\pi_Z}^H W, g^{TZ}_T, h^{L_Z}, \nabla^{L_Z}, h^E, \nabla^E) \hspace{1cm} (3.9)$$

is an equivariant geometric family over $B$ and $F_{Z,1} = F_Z$ in (2.37).

If $\text{Ind}(D(F_X)) = 0 \in K^*_G(V)$, from Proposition 3.3(i), there exists a perturbation operator $A_X$ such that

$$\ker(D(F_X) + A_X) = 0.$$

We extend $A_X$ to a pseudodifferential operator acting on $C^\infty(W, (\pi_X^* S_Y \otimes S_X) \otimes E)$ in the same way as the extension of $c(e_i)$ in Subsection 2.1, denoted by $1 \otimes A_X$. From the proof of [34, Lemma 5.3], there exists $T' \geq 1$ such that when $T \geq T'$, $\ker(D(F_{Z,T}) + 1 \otimes TA_X) = 0$. So by the homotopy invariance of the equivariant family index, for any $T \geq 1$, we have $\text{Ind}(D(F_{Z,T})) = 0$.

The proof of Lemma 3.6 is completed. □
3.3 The equivariant higher spectral flow

In [20], Dai and Zhang introduced the higher spectral flow for odd-dimensional fibers. In this subsection, we extend the Dai-Zhang higher spectral flow to the equivariant case and define the equivariant higher spectral flow for even-dimensional fibers inspired by [44, Proposition 4].

Note that a horizontal subbundle on $W$ is simply a splitting of the exact sequence

$$0 \rightarrow TZ \rightarrow TW \rightarrow \pi^*TB \rightarrow 0.$$  \hspace{1cm} (3.10)

As the space of the splitting map is affine and the $G$-action preserves (2.10), it follows that any pair of equivariant horizontal subbundles can be connected by a smooth path of equivariant horizontal distributions.

Assume that $F, F' \in F_G^*(B)$ have the same topological structure, i.e., they satisfy the first three conditions in Definition 2.2. Let $r \in I, I = [0,1]$ parametrize a smooth path of equivariant horizontal subbundles $\{T^H_{\pi,r}W\}_{r \in [0,1]}$ such that $T^H_{\pi,0}W = T^H_\pi W$ and $T^H_{\pi,1}W = T^H_\pi W$. Let $g^T_{\pi}, h^L_z$ and $h^E$ be the $G$-invariant metrics on $TZ, L_Z$ and $E$, respectively, depending smoothly on $r \in I$, which coincide with $g^T_z, h^L_z$ and $h^E$ at $r = 0$ and with $g^T_{\pi}, h^L_z$ and $h^E$ at $r = 1$. By the same reason, we can choose $G$-invariant Hermitian connections $\nabla^T_z$ and $\nabla^E_z$ on $L_Z$ and $E$ such that $\nabla^E_0 = \nabla^E, \nabla^E_1 = \nabla^E, \nabla^L_0 = \nabla^L_z$ and $\nabla^L_z = \nabla^L_z$.

Let $\tilde{B} = B \times I$. We consider the bundle $\tilde{\pi} : \tilde{W} := W \times I \rightarrow \tilde{B}$ together with the natural projection $\text{Pr} : \tilde{W} \rightarrow W$. Then the fiberwise $G$-action can be naturally extended to $\tilde{\pi} : \tilde{W} \rightarrow \tilde{B}$ such that $G$ acts as identity on $I$. Thus $T^H_{\pi,r} \tilde{W}_{(.,r)} = R \times T^H_{\pi,r} W$ defines an equivariant horizontal subbundle of $T\tilde{W}$, and $T\tilde{Z} := \text{Pr}^*T\tilde{Z}, \tilde{L}_Z := \text{Pr}^*\tilde{L}_Z$ and $\tilde{E} := \text{Pr}^*E$ are naturally equipped with $G$-invariant metrics $g^{\tilde{T}_z}, h^{\tilde{L}_z}$ and $h^E$ and $G$-invariant Hermitian connections $\nabla^T_z$ and $\nabla^E_z$. Let $\tilde{\alpha} = \text{Pr}^*\alpha$. Then we obtain equivariant geometric families

$$F_r = (W, L_Z, E, o, T^H_{\pi,r} W, g^T_{\pi}, h^L_z, \nabla^T_z, h^E, \nabla^E)$$  \hspace{1cm} (3.11)

over $B$ and

$$\tilde{F} = (\tilde{W}, \tilde{L}_Z, \tilde{E}, \tilde{\alpha}, T^H_{\pi,r} \tilde{W}, \tilde{g}^{\tilde{T}}, h^{\tilde{L}}_z, \nabla^{\tilde{T}}_z, h^\tilde{E}, \nabla^\tilde{E})$$  \hspace{1cm} (3.12)

over $\tilde{B}$ such that $F_0 = F$ and $F_1 = F'$.

If $F \in F^*_G(B)$ and $P$ and $Q$ are two equivariant spectral sections of an equivariant $B$-family $D$ such that $PR = R$, then the cokernel of $P_bR_b : \text{Im}(R_b) \rightarrow \text{Im}(P_b)$ for $b \in B$ forms an equivariant complex vector bundle over $B$, denoted by $[P - R]$. Hence for any two equivariant spectral sections $P$ and $Q$, the difference element $[P - Q]$ can be defined as an element in $K^0_G(B)$ as follows:

$$[P - Q] := [P - R] - [Q - R] \in K^0_G(B),$$  \hspace{1cm} (3.13)

where $R$ is an equivariant spectral section which majorizes $P$ and $Q$ as in Proposition 3.3(ii). From (3.13), we can obtain that if $P_1, P_2$ and $P_3$ are equivariant spectral sections of $D$, then

$$[P_3 - P_1] = [P_3 - P_2] + [P_2 - P_1] \in K^0_G(B).$$  \hspace{1cm} (3.14)

Thus the class in (3.13) is independent of the choice of $R$.

Let $E, F' \in F^*_G(B)$, which have the same topological structure. Let $F_r$ and $\tilde{F}$ are equivariant geometric families in (3.11) and (3.12). Now we consider a continuous family of operators $D_r$ on $F_r$ for $r \in I$ such that $D_0$ is an equivariant $B$-family on $F_r$. Assume that $\text{Ind}(D_0) = 0 \in K^0_G(B)$. Then the homotopy invariance of the equivariant family index implies that the equivariant indices of $D_r$ vanish. Let $Q_0$ and $Q_1$ be equivariant spectral sections of $D_0$ and $D_1$, respectively. If we consider the total family $D = \{D_r\}$ parametrized by $B \times I$, then there exists a total equivariant spectral section $\tilde{P}$. Let $P_r$ be the restriction of $\tilde{P}$ over $B \times \{r\}$. Thus we have the natural equivariant extension of the higher spectral flow in [20, Definition 1.5].
Definition 3.7. The equivariant Dai-Zhang higher spectral flow $sf_G\{(D_0, Q_0), (D_1, Q_1)\}$ between the pairs $(D_0, Q_0)$ and $(D_1, Q_1)$ is an element in $K^0_G(B)$ defined by

$$sf_G\{(D_0, Q_0), (D_1, Q_1)\} = [Q_1 - P_1] - [Q_0 - P_0] \in K^0_G(B).$$  \hspace{1cm} (3.15)$$

From (3.14), we know that this definition is independent of the choice of the total equivariant spectral section $P$.

In the following, we define the equivariant higher spectral flow for the even case.

Let $F \in \mathcal{F}^2_G(B)$. Let $D$ be an equivariant $B$-family on $F$. **We assume that there exists an equivariant spectral section $P$ with respect to $D$.** Let $A_P$ be the family of self-adjoint equivariant smoothing operators associated with $P$ by Proposition 3.3(iii).

Now we use the notation in Example 2.5(d). Let $p_1^*F \times_{B \times S^1} p_2^*F^L$ be the odd equivariant geometric family in Example 2.5(d) with fibers $Z \times S^1$. Let $\tau$ be the $\mathbb{Z}_2$-grading of $S_Z \hat{\otimes} E$ in $F$. We consider the vector bundle part in $p_1^*F \times_{B \times S^1} p_2^*F^L$ as an ungraded one. Then from Definition 3.1,

$$D_P = (D + A_P) \otimes 1 + \tau \otimes D(F^L)$$

is an equivariant $B \times S^1$-family on the odd geometric family $p_1^*F \times_{B \times S^1} p_2^*F^L$ and commutes with the group action.

Since $D$ and $A_P$ anti-commute with $\tau$,

$$D_P^2 = ((D + A_P) \otimes 1 + \tau \otimes D(F^L))^2 = (D + A_P)^2 \otimes 1 + 1 \otimes D(F^L)^2 > 0.$$  \hspace{1cm} (3.17)$$

It implies that $D_P$ is invertible. Thus the APS projection $P'$ is an equivariant spectral section of $D_P$.

Similarly, let $Q$ be another equivariant spectral section of $D$, and we can construct the equivariant spectral section $Q'$ of $D_Q$ as above. Since $p_1^*F \times_{B \times S^1} p_2^*F^L \in \mathcal{F}^2_G(B)$, from Definition 3.7, we could define $sf_G\{(D_P, P'), (D_Q, Q')\} \in K^0_G(B \times S^1)$.

Now we consider Example 2.5(c) more explicitly. It is easy to calculate that for $\theta \in [0, 1]$ fixed, the eigenvalues of $D(F^L)$ are $\lambda_k(\theta) = 2\pi k + 2\pi(\theta - 1/2)$, $k \in \mathbb{Z}$. So for $\theta \in [0, 1], \theta \neq 1/2$, we have $D(F^L)^2 > 0$. Thus as in (3.17), for any $s \in [0, 1], \theta \neq 1/2, \text{restricted on } B \times \{\theta\}$, $(1-s)D_P + sD_Q$ is invertible. From Definition 3.7, it means that for $\theta \neq 1/2$, $sf_G\{(D_P, P'), (D_Q, Q')\}|_{B \times \{\theta\}} = 0 \in K^0_G(B \times \{\theta\})$.

(From 2.19), there exists an element in $K^1_G(B)$, which we denote by $[Q - P]$ such that

$$j([Q - P]) = sf_G\{(D_P, P'), (D_Q, Q')\} \in K^0_G(B \times S^1).$$  \hspace{1cm} (3.18)$$

The idea for this construction comes from [44, Proposition 4]. We note that when the group $G$ is trivial, this definition is equivalent to that there.

Similarly, if $P_1, P_2$ and $P_3$ are equivariant spectral sections of $D$, then

$$[P_3 - P_1] = [P_3 - P_2] + [P_2 - P_1] \in K^1_G(B).$$  \hspace{1cm} (3.19)$$

Now we extend the difference $[Q - P]$ to the equivariant higher spectral flow. Let $F, F' \in \mathcal{F}^2_G(B)$, which have the same topological structure, and $D_0$ and $D_1$ be two equivariant $B$-families on $F$ and $F'$, respectively. For $i = 0, 1$, let $Q_i$ be an equivariant spectral section of $D_i$ with the corresponding smoothing operator $A_{Q_i}$. Let $D(r), r \in [0, 1]$ be a continuous curve of equivariant $B$-families on $F_r$ such that $D(0) = D_0 + A_{Q_0}, i = 0, 1$. Let

$$D_i, Q_i = (D_i + A_{Q_i}) \otimes 1 + \tau \otimes D(F^L).$$  \hspace{1cm} (3.20)$$

By (3.17), they are invertible. Let $Q'_i$ be their APS projections. Let $\tilde{D} = \{D(r) \otimes 1 + \tau \otimes D(F^L)\}$ parametrized by $B \times S^1 \times I$. By (3.17), $D_0, Q_0$ is invertible. Thus $\text{Ind}(D_0, Q_0) = 0 \in K^1_G(B \times S^1)$. So $\text{Ind}(\tilde{D}) = 0 \in K^0_G(B \times S^1 \times I)$. Let $\tilde{P} = \{P(r)\}_{r \in [0, 1]}$ be an equivariant spectral section with respect to $\tilde{D}$ such that $P(i)$ majorizes $Q'_i$ for $i = 0, 1$. Then from Definition 3.7,

$$sf_G\{(D_0, Q_0), (D_1, Q_1)\} = [Q_1 - P(1)] - [Q_0 - P(0)] \in K^0_G(B \times S^1).$$  \hspace{1cm} (3.21)$$
Furthermore, we could obtain that this equivariant higher spectral flow lies in the image of $j$ in (2.19). In fact, when restricted on $B \times \{\emptyset\} \times I$ for $\theta \neq 1/2$, as in (3.17), $\tilde{D}_{B \times \{\emptyset\} \times I}$ is invertible. For $\theta \neq 1/2$, let $\{P'(r)\}_{r \in (0,1]}$ be the APS projection of $\tilde{D}_{B \times \{\emptyset\} \times I}$. Then $P'(0) = Q'_{1|B \times \{\emptyset\}}$ and $P'(1) = Q'_{1|B \times \{\emptyset\}}$. Since $P'(r)$ and $P(r)$ are two equivariant spectral sections of $\tilde{D}_{B \times \{\emptyset\} \times I}$ and $P(i)$ majorizes $Q'_{1|B \times \{\emptyset\}} = P'(i)$ for $i = 0, 1$, we see that $\{P'(r) - P(r)\}_{B \times \{\emptyset\} \times I}$ forms an equivariant complex vector bundle over $B \times \{\emptyset\} \times I$. Thus we have $\left([Q'_{1} - P(1)] - [Q'_{0} - P(0)]\right)_{B \times \{\emptyset\}} = 0 \in K^{0}_{B}(B \times \{\emptyset\})$. It implies that $s_{F,G}(\{D_{0},Q_{0}\},(D_{1},Q_{1})) \in \text{Im}(j)$.

**Definition 3.8.** If $F,F' \in F^{0}_{G}(B)$, the equivariant higher spectral flow $s_{F,G}(\{D_{0},Q_{0}\},(D_{1},Q_{1}))$ between the pairs $(D_{0},Q_{0})$ and $(D_{1},Q_{1})$ is an element in $K^{1}_{G}(B)$ defined by

$$j(s_{F,G}(\{D_{0},Q_{0}\},(D_{1},Q_{1}))) = s_{F,G}(\{D_{0},Q_{0}\},(D_{1},Q_{1})).$$

(3.22)

Note that when $F = F'$ and $D_{0} = D_{1} = D$, the equivariant higher spectral flow $s_{F,G}(\{D_{0},Q_{0}\},(D_{1},Q_{1})) = [Q_{1} - Q_{0}]$.

The following proposition says that any element of the equivariant $K$-group could be generated by equivariant higher spectral flows. Our proof is constructive.

**Proposition 3.9.** (i) For any $x \in K_{G}^{0}(B)$, there exist $F_{1},F_{2} \in F^{0}_{G}(B)$ and equivariant spectral sections $P_{1}$ and $Q_{1}$, with respect to $D(F_{i})$ for $i = 1, 2$ such that $x = \left[\left[P_{1} - Q_{1}\right]\right] - \left[\left[P_{2} - Q_{2}\right]\right]$.

(ii) For any $x \in K_{G}^{1}(B)$, there exist $F \in F_{G}^{0}(B)$ and equivariant spectral sections $P$ and $Q$ with respect to $D(F)$ such that $x = [P - Q]$.

**Proof.** Let $(E,h^{E})$ be a Hermitian vector bundle and $\nabla^{E}$ be a Hermitian connection on $(E,h^{E})$. Let $\pi : B \times S^{1} \to B$ be the projection onto the first part. Let

$$F = (B \times S^{1},\pi^{*}E,o,T^{H}(B \times S^{1}),\tilde{g}^{T}S^{1},\pi^{*}h^{E},\pi^{*}\nabla^{E}) \in F_{G}^{0}(B),$$

where $o$ and $\tilde{g}^{T}S^{1}$ are the canonical orientation and the metric on $S^{1}$ and $T^{H}(B \times S^{1}) = TB \times S^{1}$. Let $\partial_{t}$ be the generator of $TS^{1}$. Then $D(F) = -\sqrt{-1}\partial_{t} \oplus \text{Id}_{E}$. We could calculate that the eigenvalues of $D(F)$ are $\lambda_{k} = k$ for $k \in \mathbb{Z}$. We denote by $P_{\lambda \geq k}$ the orthogonal projection onto the union of the eigenspaces of $\lambda \geq k$. Then for any $k$, $P_{\lambda \geq k}$ is an equivariant spectral section of $D(F)$. In particular, we have $\left[P_{\lambda \geq k} - P_{\lambda \geq k+1}\right] = [E] \in K_{G}^{0}(B)$. Thus we obtain Proposition 3.9 in the even case.

For any $x \in K_{G}^{1}(B)$, from Lemma 2.4, there exists a finite-dimensional complex unitary representation $V$ of $G$ such that $x$ can be represented as a $G$-invariant unitary element $F \in C^{\infty}(B,\text{End}(V))$. Let $F_{1} = (E_{+},E_{-} = B \times V) \in F_{G}^{0}(B)$ with the fiber $Z = pt$ and the trivial metric and the connection on $E_{\pm}$. Let

$$A_{0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{1} = \begin{pmatrix} 0 & F(b)^{*} \\ F(b) & 0 \end{pmatrix}$$

be endomorphisms of $V \oplus V$. Let $P_{1}$ be the orthogonal projection onto the positive part of the spectrum of $A_{1}$ for $i = 0, 1$. It is easy to calculate that for $i = 0, 1$, $P_{i\tau} + \tau P_{1} = \tau$. From Definition 3.2, we know that $P_{0}$ and $P_{1}$ are equivariant spectral sections with respect to $D(F_{1}) = 0$ on $F_{1}$. Let $D_{i} = A_{i} \oplus 1 + \tau \otimes D(F^{L})$ on $p_{0}^{*}F_{1} \times_{B \times S^{1}} p_{1}^{*}F^{L}$ and $P_{i}$ be the APS projections of $D_{i}$. Let $D_{s} = (1 - s)D_{0} + sD_{1}$ for $s \in [0, 1]$. We claim that

$$s_{F,G}(\{D_{0},P_{0}',(D_{1},P_{1}')\}) = [W] - [U] \in K_{G}^{0}(B \times S^{1}),$$

(3.23)

where $W$ and $U$ are bundles constructed in Lemma 2.4. Then from (3.18) and (3.23), we obtain Proposition 3.9 in the odd case.

We prove the claim (3.23) constructively. Let $\lambda_{b,i}$ be the eigenvalues of $F(b)$ on $V$ with unitary eigenvectors $v_{b,i}$. Then $\tilde{\lambda}_{b,i}$ are the eigenvalues of $F^{*}(b)$ on $V$ with the same eigenvectors. Let $v_{b,i}^{S}$ be the corresponding vectors in $E_{\pm,b}$. Let $v_{b}$ be the eigenvector of $\lambda_{b}(\theta)$ with respect to $D(F^{L})$ (see Appendix B). From (3.17), it is easy to calculate that the nonnegative eigenvalues of $D_{s}$ are

$$\lambda_{s,b,i,k}(\theta) = \sqrt{\lambda_{k}(\theta)^{2} + (1 - 2s)^{2} + s(1 - s)(\lambda_{b,i} + \tilde{\lambda}_{b,i} + 2)}.$$

(3.24)
Since $F$ is unitary, $|\lambda_{b,i}| = 1$. So $\lambda_{b,i,k}(0) = 0$ if and only if $k = 0, \theta = \frac{1}{2}, s = \frac{1}{4}$ and $\lambda_{b,i} = -1$. From (3.16), we calculate that the eigenfunctions of $\lambda_{b,i,1}(\theta)$ with respect to $D_s$ are

$$
\begin{align*}
    u_{s,b,i}^{(1)}(\theta) &= ((s\lambda_{b,i} + 1 - s)\nu_{b,i}^+ + (\lambda_{b,i,1}(\theta) - \lambda_1(\theta))\nu_{b,i}^-) \otimes v_1, \\
    u_{s,b,i}^{(2)}(\theta) &= ((\lambda_{b,i,-1}(\theta) + \lambda_{-1}(\theta))\nu_{b,i}^+ + (s\lambda_{b,i} + 1 - s)\nu_{b,i}^-) \otimes v_{r-1}.
\end{align*}
$$

Let

$$
\begin{align*}
    u_{s,b,i}^{(3)}(\theta) &= ((s\lambda_{b,i} + 1 - s)\nu_{b,i}^+ + (\lambda_{b,i,0}(\theta) - \lambda_0(\theta))\nu_{b,i}^-) \otimes v_0, \quad 0 \leq \theta < 1, \\
    u_{s,b,i}^{(4)}(\theta) &= ((\lambda_{b,i,0}(\theta) + \lambda_0(\theta))\nu_{b,i}^+ + (s\lambda_{b,i} + 1 - s)\nu_{b,i}^-) \otimes v_0, \quad 1/2 \leq \theta < 1.
\end{align*}
$$

Then $u_{s,b,i}^{(3)}(\theta)$ and $u_{s,b,i}^{(4)}(\theta)$ are the eigenfunctions of $\lambda_{b,i,0}(\theta)$ with respect to $D_s$. Let

$$
\begin{align*}
    w_{s,b,i}^{(j)}(\theta) &= u_{s,b,i}^{(j)}(\theta)/\|u_{s,b,i}^{(j)}(\theta)\|, \quad j = 1, 2, 3, 4.
\end{align*}
$$

Remark that when $\theta = 1/2$, if $s = 1/2$ and $\lambda_{b,i} = -1$, then $u_{s,b,i}^{(3)}(1/2) = u_{s,b,i}^{(4)}(1/2) = 0$. In this case, we define

$$
\begin{align*}
    w_{s,b,i}^{(3)}(1/2) &= \lim_{\theta \to 1/2^-} u_{s,b,i}^{(3)}(\theta)/\|u_{s,b,i}^{(3)}(\theta)\|, \\
    w_{s,b,i}^{(4)}(1/2) &= \lim_{\theta \to 1/2^+} u_{s,b,i}^{(4)}(\theta)/\|u_{s,b,i}^{(4)}(\theta)\|.
\end{align*}
$$

Choose $\chi(\theta) \in C^\infty([0, 1/2])$ with $\chi(\theta) = 1/2$ near $\theta = 0$ and $\chi(\theta) = 0$ near $\theta = 1/2$. Let

$$
\begin{align*}
    w_{s,b,i}^{(5)}(\theta) &= \chi(\theta)w_{s,b,i}^{(1)}(\theta) + (1 - \chi(\theta))w_{s,b,i}^{(3)}(\theta), \quad 0 \leq \theta \leq 1, \\
    w_{s,b,i}^{(6)}(\theta) &= \chi(1 - \theta)w_{s,b,i}^{(2)}(\theta) + (1 - \chi(1 - \theta))w_{s,b,i}^{(4)}(\theta), \quad 1/2 \leq \theta \leq 1.
\end{align*}
$$

Since $v_1(\theta = 0) = v_0(\theta = 1), v_0(\theta = 0) = v_{r-1}(\theta = 1), \lambda_0(0) = \lambda_{-1}(1) = -\pi$ and $\lambda_0(1) = \lambda_1(0) = \pi$, from (3.25)–(3.27), we have $w_{s,b,i}^{(1)}(0) = w_{s,b,i}^{(3)}(1)$ and $w_{s,b,i}^{(2)}(1) = w_{s,b,i}^{(4)}(0)$. By (3.29), we have

$$
\begin{align*}
    w_{s,b,i}^{(5)}(0) &= w_{s,b,i}^{(6)}(1).
\end{align*}
$$

So $\bigoplus_i C\{w_{s,b,i}^{(5)}(\theta), 0 \leq \theta < 1/2\}$ and $\bigoplus_i C\{w_{s,b,i}^{(6)}(\theta), 1/2 < \theta \leq 1\}$ can be connected as a trivial equivariant complex vector bundle over $B \times (S^1_\theta \setminus \{1/2\}) \times [0, 1]_s$. Then we could glue $w_{s,b,i}^{(5)}(1/2)$ and $w_{s,b,i}^{(6)}(1/2)$ for any $i$ to get an equivariant complex vector bundle $\bar{W}$ over $B \times S^1_\theta \times [0, 1]$. Let $\bar{R}$ be the orthogonal projection onto the sum of $\bar{W}$ and the eigenspaces with non-positive eigenvalues of $\bar{D} = \{D_s\}$. Then $\bar{Q} = 1 - \bar{R}$ is an equivariant spectral section with respect to $\bar{D}$. Since $\text{Ker}D_s \neq \emptyset$ only when $s = 1/2$, from (3.21), we have

$$
\begin{align*}
    \text{sf}_G\{(D_0, P'_0), (D_1, P'_1)\} &= [P'_1 - \bar{Q}|_{s=1} - P'_0 - \bar{Q}|_{s=0}] = [\bar{W}|_{s=1}] - [\bar{W}|_{s=0}] = [W|_{s=1}] - [W|_{s=0}].
\end{align*}
$$

For $s = 1$, from (3.26), (3.27) and (3.29), we have $w_{0,b,i}^{(5)}(1/2) = w_{0,b,i}^{(3)}(1/2) = (\lambda_{b,i}\nu_{b,i}^+ + \nu_{b,i}^-)/\sqrt{2}$ and $w_{0,b,i}^{(6)}(1/2) = w_{0,b,i}^{(4)}(1/2) = (\nu_{b,i}^+ + \lambda_{b,i}\nu_{b,i}^-)/\sqrt{2}$. So $w_{0,b,i}^{(6)}(1/2) = \lambda_{b,i} \cdot w_{0,b,i}^{(4)}(1/2)$. From the construction before Lemma 2.4, we have $[\bar{W}|_{s=1}] = [W]$. For $s = 0$, in the same way, we calculate that

$$
\begin{align*}
    w_{1,b,i}^{(6)}(1/2) &= w_{1,b,i}^{(5)}(1/2) = (\nu_{b,i}^+ + \lambda_{b,i}\nu_{b,i}^-)/\sqrt{2}.
\end{align*}
$$

So $[\bar{W}|_{s=0}] = [U]$. Therefore, we obtain the claim (3.23) from (3.31).

The proof of Proposition 3.9 is completed.

Note that the proof of Proposition 3.9 in the odd case gives a nontrivial example of the equivariant higher spectral flow for even-dimensional fibers and an example of the equivariant spectral section without the spectral gap.

**Remark 3.10.** In the non-equivariant case, there is a stronger version of Proposition 3.9 (see [44, Proposition 12]).
3.4 The equivariant family local index theorem

In this subsection, we use the notation in Subsection 2.2 to describe the equivariant local index theorem for $\mathcal{F} \in \mathcal{F}_{c}^{*}(B)$ when the $G$-action on $B$ is trivial.

For $b \in B$, let $\mathcal{E}$ be the set of smooth sections over $Z_{b}$ of $S_{Z} \otimes E|_{Z_{b}}$. As in [8], we will regard $\mathcal{E}$ as an infinite-dimensional vector bundle over $B$.

Let $\nabla^{TB}$ be the Levi-Civita connection on $(B, g^{TB})$. Let $0\nabla^{TW}$ be the connection on $TW = T^{H}W \oplus TZ$ defined by

$$0\nabla^{TW} = \pi^{*}\nabla^{TB} \oplus \nabla^{TZ}. \quad (3.32)$$

Then $0\nabla^{TW}$ preserves the metric $g^{TW}$ in (2.13). Set

$$S = \nabla^{TW} - 0\nabla^{TW}. \quad (3.33)$$

If $V \in TB$, let $V^{H} \in T^{H}W$ be its horizontal lift in $T^{H}W$ so that $\pi_{*}V^{H} = V$. For any $V \in TB$ and $s \in \mathcal{E}^{\infty}(B, \mathcal{E}) = \mathcal{E}^{\infty}(W, S_{Z} \otimes E)$, by [10, Proposition 1.4], the connection

$$\nabla_{V^{H}}s := \nabla^{S_{Z} \otimes E} s - \frac{1}{2}(S(e_{i})e_{i}, V^{H})s \quad (3.34)$$

preserves the $L^{2}$-product on $\mathcal{E}$.

Let $\{f_{p}\}$ be a local orthonormal frame of $TB$ and $\{f^{p}\}$ be its dual. We define $\nabla^{\mathcal{E}} = f^{p} \wedge \nabla^{\mathcal{E}}_{p}$. Let $T$ be the torsion of $0\nabla^{TW}$. Then $T(f^{H}, f^{H}) \in TZ$. We define

$$c(T) = \frac{1}{2}c(T(f^{H}, f^{H}))f^{p} \wedge f^{q} \wedge. \quad (3.35)$$

By [8, (3.18)], the (rescaled) Bismut superconnection

$$\mathbb{B}_{u} : \mathcal{E}^{\infty}(B, \Lambda(T^{*}B) \otimes \mathcal{E}) \to \mathcal{E}^{\infty}(B, \Lambda(T^{*}B) \otimes \mathcal{E}) \quad (3.36)$$

is defined by

$$\mathbb{B}_{u} = \sqrt{u}D(f) + \nabla^{\mathcal{E}} - \frac{1}{4\sqrt{u}}c(T). \quad (3.37)$$

Obviously, the Bismut superconnection $\mathbb{B}_{u}$ commutes with the $G$-action. Moreover, $\mathbb{B}_{u}^{2}$ is a second-order elliptic differential operator along the fibers $Z$ (see [8, (3.4)]). Let $\exp(-\mathbb{B}_{u}^{2})$ be the family of heat operators associated with the fiberwise elliptic operator $\mathbb{B}_{u}^{2}$. From [6, Theorem 9.50], $\exp(-\mathbb{B}_{u}^{2})$ is a smooth family of smoothing operators.

If $P$ is a trace class operator acting on $\Lambda(T^{*}B) \otimes \text{End}(\mathcal{E})$ which takes values in $\Lambda(T^{*}B)$, we use the convention that if $\omega \in \Lambda(T^{*}B)$,

$$\text{Tr}_{s}[\omega P] = \omega \text{Tr}_{s}[P]. \quad (3.38)$$

We denote by $\text{Tr}_{s}^{\text{odd/even}}[P]$ the part of $\text{Tr}_{s}[P]$ which takes values in odd or even forms. Set

$$\tilde{\text{Tr}}[P] = \begin{cases} \text{Tr}_{s}[P], & \text{if dim } Z \text{ is even,} \\ \text{Tr}_{s}^{\text{odd}}[P], & \text{if dim } Z \text{ is odd.} \end{cases} \quad (3.39)$$

Recall that in this subsection we assume that $G$ acts trivially on $B$. Take $g \in G$. Let $W^{g}$ be the fixed point set of $g$ on $W$. Then $W^{g}$ is a submanifold of $W$ and $\pi : W^{g} \to B$ is a fiber bundle with compact fibers $Z^{g}$. Set

$$\text{ch}_{g}(E, \nabla^{E}) = \text{Tr}_{s} \left[ g \exp \left( \sqrt{-1} \frac{1}{2\pi} R^{E} \bigg|_{W^{g}} \right) \right] \quad (3.40)$$
Let \( \text{ch}_g(E) \in H^{\text{even}}(W^g, \mathbb{C}) \) denote the cohomology class of \( \text{ch}_g(E, \nabla^E) \). When the fiber \( Z \) is a point, it descends to the equivariant Chern character map

\[
\text{ch}_g : K^0_G(B) \to H^{\text{even}}(B, \mathbb{C}).
\]  

By (2.19), for \( x \in K^1_G(B) \), \( j(x) \in K^0_G(B \times S^1) \). The odd equivariant Chern character map

\[
\text{ch}_g : K^1_G(B) \to H^{\text{odd}}(B, \mathbb{C})
\]  

is defined by

\[
\text{ch}_g(x) := \int_{S^1} \text{ch}_g(j(x)).
\]  

We adopt the sign notation in the integral as in (1.7). This is just the equivariant version of the odd Chern character in [28] and [53, (1.50)] (see, e.g., [36, (3.10)]).

Let \( N \) be the normal bundle of \( W^g \) in \( W \). As \( G \) is compact, there is an orthonormal decomposition of real vector bundles over \( W^g \):

\[
TZ|_{W^g} = TZ^g \oplus N.
\]  

Let \( \nabla \) be a Euclidean connection on \((TZ, g^{TZ})\) commuting with the \( G \)-action. Then its restriction on \( W^g \) preserves the decomposition (3.44). Let \( \nabla^{TZ^g} \) and \( \nabla^{N} \) be the corresponding induced connections on \( TZ^g \) and \( N \) with curvatures \( R^{TZ^g} \) and \( R^{N} \), respectively. Set

\[
\tilde{\Lambda}_g(TZ, \nabla) := \det^{1/2} \left( \frac{\sqrt{-1}}{4\pi} \frac{R^{TZ^g}}{\sinh(\frac{\sqrt{-1}}{4\pi} R^{TZ^g})} \right)
\times \left( \sqrt{-1} \frac{1}{\dim N} \det^{1/2} \left| 1 - g \exp \left( \frac{\sqrt{-1}}{2\pi} R^{N} \right) \right| \right)^{-1}.
\]  

If \( g \) acts on \( L|_{W^g} \) by multiplying by \( e^{\sqrt{-1}v} \), we write

\[
\text{ch}_g(L^{1/2}_Z; \nabla^{1/2}) := \exp \left( \frac{\sqrt{-1}}{4\pi} R^I \bigg|_{W^g} + \frac{\sqrt{-1}}{2} v \right).
\]  

We define

\[
\text{Td}_g(\nabla, \nabla^{L^Z}) := \tilde{\Lambda}_g(TZ, \nabla) \text{ch}_g(L^{1/2}_Z; \nabla^{1/2}L^Z).
\]  

Let \( \text{Td}_g(TZ, L_Z) \in H^{\text{even}}(W^g, \mathbb{C}) \) denote the cohomology class of \( \text{Td}_g(\nabla, \nabla^{L^Z}) \).

For \( \alpha \in \Omega^*(B) \), set

\[
\psi_B(\alpha) = \begin{cases} 
\left( \frac{1}{2\pi \sqrt{-1}} \right)^{i} \cdot \alpha, & \text{if } i \text{ is even}, \\
\left( \frac{1}{2\pi \sqrt{-1}} \right)^{\frac{1}{2}} \cdot \alpha, & \text{if } i \text{ is odd}.
\end{cases}
\]  

We state the equivariant family local index theorem here (see, e.g., [8, Theorem 4.17], [11, Theorem 2.10], [34, Theorem 2.2], [35, Theorem 2.2] and [37, Theorem 1.3]). Note that from [38, Lemma 4.1], \( Z^g \) is naturally oriented.

**Theorem 3.11.** For any \( u > 0 \) and \( g \in G \), the differential form \( \psi_B \tilde{\text{Tr}}[g \exp(-B^2_u)] \in \Omega^*(B, \mathbb{C}) \) is closed and its cohomology class represents \( \text{ch}_g(\text{Ind}(D(F))) \in H^*(B, \mathbb{C}) \). As \( u \to 0 \), we have

\[
\lim_{u \to 0} \psi_B \tilde{\text{Tr}}[g \exp(-B^2_u)] = \int_{Z^g} \text{Td}_g(\nabla^{TZ}, \nabla^{L^Z}) \text{ch}_g(E, \nabla^E).
\]
To simplify the notation, we set
\[
\text{FLI}_g(F) = \int_{Z^s} \text{Td}_g(\nabla^{TZ}, \nabla^{LZ}) \, \text{ch}_g(E, \nabla^E) \in \Omega^*(B, \mathbb{C}).
\tag{3.50}
\]

So Theorem 3.11 says that for \( F \in F_G^{0/1}(B) \),
\[
[\text{FLI}_g(F)] = \text{ch}_g(\text{Ind}(D(F))) \in H^{\text{even/odd}}(B, \mathbb{C}).
\tag{3.51}
\]

When \( F \) is the equivariant geometric family in Example 2.5(a) and \( Z = \text{pt} \), the equivariant family local index theorem degenerates to the equivariant Chern-Weil theory
\[
\psi_B \text{Tr}[g \exp(-B^2)] = \psi_B \text{Tr}[g \exp(-\nabla^2)] = \text{ch}_g(E, \nabla^E).
\tag{3.52}
\]

In this case, \( \text{FLI}_g(F) = \text{ch}_g(E, \nabla^E) = \text{ch}_g(E_+, \nabla^{E_+}) - \text{ch}_g(E_-, \nabla^{E_-}). \)

If \( \alpha \in \Lambda(T^*(\mathbb{R}_+ \times B)) \),
\[
\alpha = \alpha_0 + ds \wedge \alpha_1, \quad \alpha_0, \alpha_1 \in \Lambda(T^*B).
\tag{3.53}
\]

Set
\[
[\alpha]^{ds} = \alpha_1.
\tag{3.54}
\]

Let \( F, F' \in F_G(B) \), which have the same topological structure. By (3.51), we have \([\text{FLI}_g(F)] = [\text{FLI}_g(F')] \in H^*(B, \mathbb{C}).\)

We use the notation in (3.11) and (3.12). By [42, Theorem B.5.4], modulo exact forms on \( W^g \), the equivariant Chern-Simons forms
\[
\tilde{\text{Td}}_g(\nabla^{TZ}, \nabla^{LZ}, \nabla^{TZ}, \nabla^{LZ}) := - \int_0^1 [\text{Td}_g(\nabla^{TZ}, \nabla^{LZ})]^{ds} ds,
\tag{3.55}
\]
\[
\tilde{\text{ch}}_g(\nabla^E, \nabla^E) := - \int_0^1 [\text{ch}_g(\tilde{E}, \nabla^E)]^{ds} ds
\]
depend only on the connections in \( F \) and \( F' \). Moreover,
\[
d^{W^g} \tilde{\text{Td}}_g(\nabla^{TZ}, \nabla^{LZ}, \nabla^{TZ}, \nabla^{LZ}) = \text{Td}_g(\nabla^{TZ}, \nabla^{LZ}) - \text{Td}_g(\nabla^{TZ}, \nabla^{LZ}),
\tag{3.56}
\]
\[
d^{W^g} \tilde{\text{ch}}_g(\nabla^E, \nabla^E) = \text{ch}_g(E, \nabla^E) - \text{ch}_g(E, \nabla^E).
\]

Set
\[
\text{FLI}_g(F, F') = \int_{Z^y} \tilde{\text{Td}}_g(\nabla^{TZ}, \nabla^{LZ}, \nabla^{TZ}, \nabla^{LZ}) \text{ch}_g(E, \nabla^E)
\]
\[
+ \int_{Z^s} \text{Td}_g(\nabla^{TZ}, \nabla^{LZ}) \tilde{\text{ch}}_g(\nabla^E, \nabla^E) \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}).
\tag{3.57}
\]

From [6, (1.7)], for \( \sigma \in \Omega^*(W^g) \), using the sign convention in (1.7), we have
\[
d^B \int_{Z^y} \sigma = \int_{Z^y} d^{W^g} \sigma.
\tag{3.58}
\]

From (3.56) and (3.58), we have
\[
d^B \text{FLI}_g(F, F') = \text{FLI}_g(F') - \text{FLI}_g(F).
\tag{3.59}
\]
3.5 The equivariant eta form

In this subsection, we also assume that $G$ acts trivially on $B$. We define the equivariant Bismut-Cheeger eta form with the perturbation operator in Definition 3.5.

In this subsection, we assume that there exists a perturbation operator with respect to $D(\mathcal{F})$ on $\mathcal{F}$. It implies that $\text{Ind}(D(\mathcal{F})) = 0 \in K^*_G(B)$.

Let $A$ be a perturbation operator with respect to $D(\mathcal{F})$. We extend $A$ to $1 \otimes A$ on $\mathcal{E}^\infty(B, \pi^* \Lambda(T^*B) \otimes \mathcal{E})$ as an element of the $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded algebras. In this case,

$$ (\alpha \otimes 1)(1 \otimes A) = (-1)^{\text{deg}\alpha}(1 \otimes A)(\alpha \otimes 1). $$

(3.60)

We usually abbreviate $1 \otimes A$ by $A$ when there is no confusion.

Let $\chi \in \mathcal{C}^\infty(R)$ be a cut-off function such that

$$
\chi(u) = \begin{cases} 
0, & \text{if } u < 1, \\
1, & \text{if } u > 2.
\end{cases}
$$

(3.61)

Set

$$ B'_u = B_u + \sqrt{u} \chi(\sqrt{u}) A. $$

(3.62)

Since $\chi(\sqrt{u}) = 0$ if $u \in (0, 1)$, by (3.49) and (3.50),

$$
\lim_{u \to 0} \psi_B \text{Tr} \left[ g \exp(- (B'_u)^2) \right] = \text{FLI}_g(\mathcal{F}) \in \Omega^*(B, \mathbb{C}).
$$

(3.63)

Since $\chi(\sqrt{u}) = 1$ if $u \in (2, +\infty)$, from [6, Theorem 9.19], we have

$$
\lim_{u \to +\infty} \psi_B \text{Tr} \left[ g \exp(- (B'_u)^2) \right] = 0.
$$

(3.64)

Definition 3.12. For any $g \in G$, modulo exact forms on $B$, the equivariant eta form with the perturbation operator $A$ is defined by

$$
\tilde{\eta}_g(\mathcal{F}, A) = - \int_0^\infty \left\{ \psi_B \text{Tr} \left[ g \exp \left( - \left( B'_u + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \right\} du \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}).
$$

(3.65)

The regularities of the integral on the right-hand side of (3.65) are proved in [34, Subsection 2.4]. As in [34, (2.81)], we have

$$
d\tilde{\eta}_g(\mathcal{F}, A) = \text{FLI}_g(\mathcal{F}).
$$

(3.66)

As in [34, (2.95)], the value of $\tilde{\eta}_g(\mathcal{F}, A)$ in $\Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C})$ is independent of the choice of the cut-off function. Similarly, if $A_P$ and $A'_P$ are two smoothing operators associated with the same equivariant spectral section $P$, we have $\tilde{\eta}_g(\mathcal{F}, A_P) = \tilde{\eta}_g(\mathcal{F}, A'_P) \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C})$. In this case, we often simply denote it by $\tilde{\eta}_g(\mathcal{F}, P)$.

If the fiber $Z$ is connected, we could calculate the equivariant eta form explicitly:

$$
\tilde{\eta}_g(\mathcal{F}, A) = \begin{cases} 
\int_0^\infty \frac{1}{\sqrt{\pi}} \psi_B \text{Tr} \left[ g \frac{\partial B'_u}{\partial u} \exp(- (B'_u)^2) \right] du \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}), & \text{if } \mathcal{F} \text{ is odd,} \\
\int_0^\infty \frac{1}{2\sqrt{\pi}} \psi_B \text{Tr} \left[ \frac{\partial B'_u}{\partial u} \exp(- (B'_u)^2) \right] du \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}), & \text{if } \mathcal{F} \text{ is even and } \dim Z > 0,
\end{cases}
$$

(3.67)

$$
\int_0^\infty \frac{\sqrt{\pi}}{2\pi} \text{Tr} \left[ \frac{\partial \nabla^E_u}{\partial u} \exp \left( - \frac{(\nabla^E_u)^2}{2\sqrt{\pi} - 1} \right) \right] du \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}), & \text{if } \dim Z = 0,
$$

where $\nabla^E_u$ is the equivariant connection on $\mathcal{E}^\infty(B, \pi^* \Lambda(T^*B) \otimes \mathcal{E})$. 


where $\nabla^E_u = \nabla^E + \sqrt{u} \chi(\sqrt{u}) A$.

When $\dim Z = 0$, the equivariant geometric family degenerates to the case of Example 2.5(a). Then there exists a complex vector bundle $E'$ such that $E_+ \oplus E' \simeq E_- \oplus E'$ as complex vector bundles. As in [42, Definition B.5.3], from (3.52) and (3.66), the equivariant eta form in this case is just the equivariant transgression between $\mathrm{ch}_y(E_+ \oplus E', \nabla^{E_+\oplus E'})$ and $\mathrm{ch}_y(E_- \oplus E', \nabla^{E_-\oplus E'})$.

Furthermore, by changing variables (see also [34, Remark 2.4]), we could get another form of the equivariant eta form

$$
\tilde{\eta}_y(F, A) = -\int_0^1 \left\{ \psi_{\mathbb{R} \times B} \operatorname{Tr} \left[ g \exp \left( - \left( \frac{\mathbb{B} \otimes 1 + 1 \otimes uD(F')}{-\mathbb{B}_u^2 + du \wedge \partial}{du} \right)^2 \right] \right\}^2 \right\} \frac{du}{du}.
$$

(3.68)

Let $(Z', g^T Z')$ be an even-dimensional Spin$^c$ manifold and $(E', h^{E'}, \nabla^{E'})$ be a $\mathbb{Z}_2$-graded Hermitian vector bundle over $Z'$ with a Hermitian connection $\nabla^{E'}$. Let $\operatorname{pr}_2 : B \times Z' \to Z'$ be the projection onto the second part. Then all the bundles and geometric data above could be pulled back on $B \times Z'$. Thus the fiber bundle $B \times Z' \to B$ and the structures pulled back by $\operatorname{pr}_2$ form a geometric family $F'$ with fibers $Z'$. In this case, $\operatorname{Ind}(D(F'))$ is a trivial virtual complex vector bundle over $B$. It could also be regarded as a locally constant function on $B$ with values in $\mathbb{Z}$. We assume that the group action on $F'$ is trivial. For $F \in \mathbb{F}_G(B)$, let $A$ be a perturbation operator with respect to $D(F)$ on $F$. Let $\tau'$ be the $\mathbb{Z}_2$-grading of $S_{Z'} \oplus^e E'$. As in (2.7), we define

$$A \otimes 1 := A \otimes \tau'$$

(3.69)
on $F \times_B F'$. By (2.7), we have

$$\left( D(F \times_B F') + A \otimes 1 \right)^2 = \left( D(F) + A \right)^2 \otimes 1 \otimes D(F')^2 > 0.$$

(3.70)

Thus $A \otimes 1$ is a perturbation operator with respect to $D(F \times_B F')$.

Lemma 3.13. For $g \in G$, we have

$$\tilde{\eta}_y(F \times_B F', A \otimes 1) = \tilde{\eta}_y(F, A) \cdot \operatorname{Ind}(D(F')) \in \Omega^*(B, \mathbb{C})/d\Omega^*(B, \mathbb{C}).$$

(3.71)

Here, we consider $\operatorname{Ind}(D(F'))$ as a locally constant function on $B$ with values in $\mathbb{Z}$.

Proof. We denote by $\operatorname{Tr} |_F$ the trace operator associated with $F$. Then from (3.68),

$$\tilde{\eta}_y(F \times_B F', A \otimes 1)$$

$$= -\int_0^1 \left\{ \psi_{\mathbb{R} \times B} \operatorname{Tr} \left[ F \times_B F' \left[ g \exp \left( - \left( \frac{\mathbb{B} \otimes 1 + 1 \otimes uD(F')}{-\mathbb{B}_u^2 + du \wedge \partial}{du} \right)^2 \right] \right\} \right\} \frac{du}{du}$$

$$= \int_0^1 \left\{ \psi_{\mathbb{R} \times B} \operatorname{Tr} \left[ F \times_B F' \left[ g(1 \otimes D(F')) \exp(-u^2D(F')) \right] \right\} \frac{du}{du}$$

$$- \int_0^1 \left\{ \psi_{\mathbb{R} \times B} \operatorname{Tr} \left[ F \times_B F' \left[ g \exp \left( - \left( \frac{\mathbb{B} \otimes 1 + 1 \otimes uD(F')}{-\mathbb{B}_u^2 + du \wedge \partial}{du} \right)^2 \right] \right\} \right\} \frac{du}{du}$$

$$= \int_0^1 \left\{ \psi_{\mathbb{R} \times B} \operatorname{Tr} \left[ F \left[ g \exp(-u^2D(F')) \right] \right] \frac{du}{du}$$

$$- \int_0^1 \left\{ \psi_{\mathbb{R} \times B} \operatorname{Tr} \left[ F \left[ g \exp(-u^2D(F')) \right] \right] \frac{du}{du}$$

(3.72)

From the definition of $F'$ and the local index theorem, as functions on $B$, we have

$$\operatorname{Tr}_s[F \left[ D(F') \exp(-u^2D(F')) \right]] = 0,$$

$$\operatorname{Tr}_s[F \left[ \exp(-u^2D(F')) \right]] = \operatorname{Ind}(D(F')).$$

(3.73)

So we get $\tilde{\eta}_y(F \times_B F', A \otimes 1) = \tilde{\eta}_y(F, A) \cdot \operatorname{Ind}(D(F'))$.

The proof of Lemma 3.13 is completed.
3.6 The anomaly formula for odd equivariant geometric families

In this subsection, we will study the anomaly formula of the equivariant eta forms for two odd equivariant geometric families $F$ and $F'$ with the same topological structure. In this subsection, we also assume that $G$ acts on $B$ trivially.

Assume that $F \in \mathcal{F}_G(B)$. Let $A$ be a family of bounded pseudodifferential operators on $F$ such that $D(F) + A$ is an equivariant $B$-family. Let $P$ and $Q$ be two equivariant spectral sections with respect to $D(F) + A$. Let $A_P$ and $A_Q$ be smoothing operators associated with $P$ and $Q$. Then $A + A_P$ and $A + A_Q$ are perturbation operators of $D(F)$. In this case, by (3.66), the difference of $\tilde{\eta}_g(F, A + A_P)$ and $\tilde{\eta}_g(F, A + A_Q)$ is closed. Furthermore, we have the following lemma.

**Lemma 3.14** (Compare with [43, Proposition 17]). For any $g \in G$, modulo exact forms on $B$, we have

$$\tilde{\eta}_g(F, A + A_P) - \tilde{\eta}_g(F, A + A_Q) = \text{ch}_g([P - Q]) \in H^{\text{even}}(B, \mathbb{C}).$$  \hspace{1cm} (3.74)

**Proof.** Note that $A$, $A_P$ and $A_Q$ preserve the $\mathbb{Z}_2$-grading of $E$ and if we reverse the orientation of the fibers, the eta form is changed to its minus. From (3.14) and the orientation reversing trick in the proof of Proposition 3.3(i), we only need to prove the lemma when $Q$ majorizes $P$ and $E_\pm = 0$ in $F$.

Let $\tilde{F}$ be the equivariant geometric family defined in (3.12) such that $F_r = F$ for any $r \in [0, 1]$. Let $\tilde{B}_u$ be the Bismut superconnection associated with $\tilde{F}$. We choose $s > 0$ large enough such that $P$ and $Q$ satisfy (3.1) for $f(b) \equiv s$. We choose equivariant spectral sections $R'$ and $R''$ as in (3.3). Since the eta form is independent of the smoothing operators with respect to the same equivariant spectral section, we may choose the smoothing operators $A_P$ and $A_Q$ as in (3.3). Set $A_r := A + rA_Q + (1 - r)A_P$. Let

$$\tilde{B}_u \big|_{(u, r)} := \tilde{B}_u \big|_{(u, r)} + \sqrt{u} \chi(\sqrt{u}) A_r$$  \hspace{1cm} (3.75)

as in (3.62). We simply define

$$D_r := D(F) + A_r, \quad \tilde{\nabla} := \nabla^\phi + dr \wedge \frac{\partial}{\partial r}.$$  \hspace{1cm} (3.76)

Then from (3.37), when $u > 2$, we have

$$(\tilde{\nabla}_u) - 2 = u^2 D_r^2 + u[D_r, \tilde{\nabla}] + \tilde{\nabla}^2 + \frac{1}{4} [D_r, c(T)] + \frac{1}{4u} [\tilde{\nabla}, c(T)] + \frac{1}{16u^2} c(T)^2.$$  \hspace{1cm} (3.77)

For a family of bounded operators $A_u, u \in \mathbb{R}_+$, we write $A_u = O(u^{-k})$ as $a \to +\infty$ if there exists $C > 0$ such that if $u$ is large enough, the norm of $A_u$ is dominated by $C/u^k$.

Let $\Pi$ be the orthogonal projection onto $[P - Q]$, an equivariant complex vector bundle over $\tilde{B}$. Let

$$E_u = \Pi \circ (\tilde{B}_u)^2 \circ \Pi, \quad F_u = \Pi \circ (\tilde{B}_u^\phi)^2 \circ \Pi^\perp,$$

$$G_u = \Pi^\perp \circ (\tilde{B}_u^\phi)^2 \circ \Pi, \quad H_u = \Pi^\perp \circ (\tilde{B}_u^\phi)^2 \circ \Pi^\perp.$$  \hspace{1cm} (3.78)

From (3.3), since $\Pi$ commutes with $P, Q, R', R''$, $\Pi Q = \Pi R' = 0$ and $\Pi R'' = \Pi P = \Pi$, we have

$$\Pi \circ D_r \circ \Pi = r \Pi \circ (R'(D(F) + A)R' + sQR''(I - R') + (I - R')(D(F) + A)(I - R'')$$

$$- s(I - Q)R''(I - R') \circ \Pi + (1 - r) \Pi \circ (R'(D(F) + A)R' + sPR''(I - R')$$

$$+ (I - R'')(D(F) + A)(I - R'')) - s(I - P)R''(I - R') \circ \Pi$$

$$= s(1 - 2r) \Pi.$$  \hspace{1cm} (3.79)

Since $D_r$ preserves the splitting $\text{Range}(\Pi) \oplus \text{Range}(\Pi^\perp)$,

$$\Pi \circ [D_r, c(T)] \circ \Pi = [\Pi \circ D_r \circ \Pi, \Pi \circ \nabla^\phi \circ \Pi] = dr \wedge \frac{\partial}{\partial r} (\Pi \circ D_r \circ \Pi) = -2sdr \wedge \Pi.$$  \hspace{1cm} (3.80)

Similarly, we have $\Pi \circ [D_r, c(T)] \circ \Pi = 0$ and

$$\Pi \circ \nabla^\phi \circ \Pi = \Pi \circ (\nabla^\phi)^2 \circ \Pi.$$  \hspace{1cm} (3.81)
Let
\[
E' = u^2 s^2 (1 - 2r)^2 \Pi - 2usdr \land \circ \Pi + \Pi \circ (\nabla^\epsilon)^2 \circ \Pi, \quad F' = \Pi^\perp \circ [D_r, \tilde{\nabla}] \circ \Pi, \quad G' = \Pi \circ [D_r, \tilde{\nabla}] \circ \Pi^\perp, \quad H' = \Pi^\perp \circ D_r^2 \circ \Pi^\perp.
\]
(3.82)
From (3.77)–(3.82), when \( u \to +\infty \),
\[
E_u = E' + O(u^{-1}), \quad F_u = uF' + F'' + O(1), \quad G_u = uG' + G'' + O(1), \quad H_u = uH' + uH'' + H''' + O(1),
\]
where \( F'', G'', H'' \) and \( H''' \) are first-order differential operators along the fiber. Let
\[
\nabla^\Pi := \Pi \circ \nabla^\epsilon \circ \Pi.
\]
We have
\[
E' - F'H'^{-1}G' = u^2 s^2 (1 - 2r)^2 \Pi - 2usdr \land \circ \Pi + (\nabla^\Pi)^2.
\]
(3.85)
Following the same way as in the proof of [34, Theorem 5.13], we can obtain
\[
\exp(-\tilde{E}'_u)^2) = \Pi \circ \exp(-(E' - F'H'^{-1}G')) \circ \Pi + O(u^{-1}).
\]
(3.86)
Thus we have
\[
\int \exp(-\tilde{E}'_u)^2) dr = 2use^{-u^2s^2(1-2r)^2} \nabla \circ \exp(-(\nabla^\Pi)^2) \circ \Pi + O(u^{-1}).
\]
(3.87)
Set
\[
r_1(u,r) = \{ \psi_B \text{ Tr}_{P} \text{odd} [g \exp(-\tilde{E}'_u)^2]) dr |_{(u,r)} \}.
\]
From [34, (2.95)], modulo exact forms on \( B \), we have
\[
\tilde{g}(\bar{\mathcal{F}}, A + Ap) - \tilde{g}(\bar{\mathcal{F}}, A + AQ) = \lim_{u \to +\infty} \frac{1}{\sqrt{\pi}} \lim_{u \to +\infty} \int_{-u}^{u} e^{-x^2} dx \cdot \text{ch}_d([P - Q]) = \text{ch}_d([P - Q]).
\]
(3.88)
The proof of Lemma 3.14 is completed.

Using Lemma 3.14, we obtain the anomaly formula in the odd case as follows.

**Proposition 3.15** (Compare with [20, Theorem 0.1]). Let \( \bar{\mathcal{F}}, \bar{\mathcal{F}}' \in F^1_{B}(B) \), which have the same topological structure. Let \( A' \) be a perturbation operator with respect to \( D(\bar{\mathcal{F}}) \) and \( D(\bar{\mathcal{F}}') \) and \( P \) and \( P' \) be APS projections with respect to \( D(\bar{\mathcal{F}}) + A \) and \( D(\bar{\mathcal{F}}') + A' \), respectively. For any \( g \in G \), modulo exact forms on \( B \), we have
\[
\tilde{g}(\bar{\mathcal{F}}, A') - \tilde{g}(\bar{\mathcal{F}}, A) = \text{Fli}_{g}(\bar{\mathcal{F}}, \bar{\mathcal{F}}) + \text{ch}_d(\text{sg}(\{ D(\bar{\mathcal{F}}) + A, P \}, D(\bar{\mathcal{F}}') + A', P'))).
\]
(3.89)

**Proof.** Let \( \tilde{F} \) be the equivariant geometric family defined in (3.12). Let \( D_r = D(\tilde{F}_r) + (1 - r)A + ra' \) and \( \tilde{D} = \{ D_r \}_{r \in [0,1]} \) on \( \tilde{F} \). Since the equivariant family index of \( D(\bar{\mathcal{F}}) \) vanishes, so do \( D_r \) and \( \tilde{D} \). If we consider the total family \( \tilde{F} \), from Proposition 3.3(i), there exists a total equivariant spectral section \( \tilde{P} \) of \( \tilde{D} \). Let \( P_r \) be the restriction of \( \tilde{P} \) over \( \{ r \} \times B \). Then it is an equivariant spectral section of \( D_r \). Let \( A_{P_r} \) be an equivariant smoothing operator associated with \( P_r \). Following the proof of [34, Theorem 2.7], we can get
\[
\tilde{g}(\bar{\mathcal{F}}, A' + A_{P_r}) - \tilde{g}(\bar{\mathcal{F}}, A + A_{P_r}) = \text{Fli}_{g}(\bar{\mathcal{F}}, \bar{\mathcal{F}}').
\]
(3.90)
Thus Proposition 3.15 follows from Lemma 3.14, (3.15) and (3.90).

The proof of Proposition 3.15 is completed.
3.7 Functoriality of equivariant eta forms

In this subsection, we will study the functoriality of the equivariant eta forms and use it to prove the anomaly formula of equivariant eta forms for even equivariant geometric families. In this subsection, we use the notation in Subsection 2.4 and assume that $G$ acts trivially on $B$.

Recall that in (2.34),

$$TZ = T^H_{\pi_X} Z \oplus TX.$$  

Let $\nabla^{TY,TX}$ be the connection on $TZ$ defined by

$$\nabla^{TY,TX} = \pi_X \nabla^Y \oplus \nabla^X$$

as in (3.32).

Let $\nabla$ and $\nabla'$ be Euclidean connections on $(TZ,g^TZ)$ and $\nabla^{LZ}$ and $\nabla^{{ILZ}}$ be Hermitian connections on $(L_Z,h^{LZ})$. Similar to (3.50), we define

$$\text{FLI}_g(\nabla,\nabla^{LZ}) := \int_{Z^g} \text{Td}_g(\nabla,\nabla^{LZ}) \text{ch}_g(E,\nabla^E).$$

As in (3.55) and (3.56), there exists a well-defined equivariant Chern-Simons form $\text{Td}_g(\nabla,\nabla^{LZ},\nabla',\nabla^{{ILZ}}) \in \Omega^*(W^g,\mathbb{C})/d\Omega^*(W^g,\mathbb{C})$ such that

$$\text{d}^{W^g}\text{Td}_g(\nabla,\nabla^{LZ},\nabla',\nabla^{{ILZ}}) = \text{Td}_g(\nabla',\nabla^{{ILZ}}) - \text{Td}_g(\nabla,\nabla^{LZ}).$$

Set

$$\text{FI}_{\text{LI},g}(\nabla,\nabla^{LZ},\nabla',\nabla^{{ILZ}}) := \int_{Z^g} \text{Td}_g(\nabla,\nabla^{LZ},\nabla',\nabla^{{ILZ}}) \text{ch}_g(E,\nabla^E).$$

From (3.58) and (3.93), we have

$$\text{d}^B \text{FI}_{\text{LI},g}(\nabla,\nabla^{LZ},\nabla',\nabla^{{ILZ}}) = \text{FI}_{\text{LI},g}(\nabla',\nabla^{{ILZ}}) - \text{FI}_{\text{LI},g}(\nabla,\nabla^{LZ}).$$

From the proof of Lemma 3.6, we obtain that if $A_X$ is a perturbation operator of $D(F_X)$, there exists $T'>0$ such that when $T \geq T'$, $1 \otimes T_A X$ is a perturbation operator of $D(F_{X,T})$.

Note that when $A_X$ is a family of smoothing operators along the fibers $X$, $1 \otimes A_X$ is only bounded, not a family of smoothing operators along the fibers $Z$. This is the reason for us to define the eta form for the bounded perturbation operator instead of the smoothing operator in [14,16,20,43].

The following technical lemma is a modification of the main result in [34]. The proof of it will be left to the next subsection.

**Lemma 3.16.** Modulo exact forms on $B$, for $T \geq T'$, we have

$$\bar{\eta}_g(F_{X,T}, 1 \otimes T_A X) = \int_{Y^g} \text{Td}_g(\nabla^{TY}, \nabla^{LY}) \bar{\eta}_g(F_X, A_X) + \text{FI}_{\text{LI}}(\nabla^{TY,TX}, \nabla^{LZ}, \nabla^T_{Z^T}, \nabla^{LZ}).$$

Here, $\nabla^T_{Z^T}$ is the connection associated with $(T^H_{\pi_Y} W, g^T_{Z^T})$ as in (2.14).

Using Lemma 3.16, we could extend the anomaly formula Proposition 3.15 to the general case.

**Theorem 3.17.** Let $F, F' \in \mathcal{F}_G(B)$, which have the same topological structure. Let $A$ and $A'$ be perturbation operators with respect to $D(F)$ and $D(F')$ and $P$ and $P'$ be the APS projections with respect to $D(F) + A$ and $D(F') + A'$, respectively. For any $g \in G$, modulo exact forms on $B$, we have

$$\bar{\eta}_g(F', A') - \bar{\eta}_g(F, A) = \text{FI}_{\text{LI}}(F, F') + \text{ch}_g(s f_G((D(F) + A, P), (D(F') + A', P'))) \text{ch}_g((D(F) + A, P), (D(F') + A', P')).$$

**Proof.** We only need to prove the even case.

Let $L \to S^1 \times S^1$ be the Hermitian line bundle in Example 2.5(c) with $\nabla^L$ constructed there. We use the notation in Example 2.5(c). Let $p : B \times S^1 \times S^1 \to S^1 \times S^1$ be the natural projection. Then
all the bundles and geometric data in $\mathcal{F}^L$ could be pulled back on $B \times S^1 \times S^1$. Thus the fiber bundle $B \times S^1 \times S^1 \to B$ and the structures pulled back by $p$ form an even geometric family $\mathcal{F}_0$ over $B$. In this case, $\text{Ind}(D(\mathcal{F}_0)) = 1$. Here, we consider $\text{Ind}(D(\mathcal{F}_0))$ as a locally constant function on $B$ as in Lemma 3.13. The key observation is

$$p^1(p^*_1\mathcal{F} \times_{B \times S^1} p^*_2\mathcal{F}^L) = \mathcal{F} \times_B \mathcal{F}_0. \quad (3.98)$$

Recall that $A \otimes 1_{\mathcal{F}_0}$ is defined in (3.69). Since $A \otimes 1_{\mathcal{F}_0}$ is a perturbation operator of $D(\mathcal{F} \times_B \mathcal{F}_0)$, we could choose $T' = 1$ in Lemma 3.16. By Lemmas 3.13 and 3.16, we have

$$\tilde{\eta}_g(\mathcal{F}, A) = \int_{S^1} \{ \tilde{\eta}_g(p^*_1\mathcal{F} \times_{B \times S^1} p^*_2\mathcal{F}^L, A' \otimes 1_{\mathcal{F}_0}) - \tilde{\eta}_g(p^*_1\mathcal{F} \times_{B \times S^1} p^*_2\mathcal{F}^L, A \otimes 1_{\mathcal{F}_0}) \}$$

$$+ \int_{S^1} \tilde{T}_d_g(\nabla^{T(S^1 \times Z)}, \nabla_{L^Z}, \nabla^{T_S^1 TT^z}, \nabla_{L^x}) \text{ch}_g(E', \nabla^{E'}) - \tilde{T}_d_g(\nabla^{T(S^1 \times Z)}, \nabla_{L^Z}, \nabla^{T_S^1 TT^z}, \nabla_{L^x}) \text{ch}_g(E, \nabla^{E'})$$

$$= \text{FLI}_g(\mathcal{F}, A') + \int_{S^1} \text{ch}_g(\text{sf}_G\{(D(p^*_1\mathcal{F} \times_{B \times S^1} p^*_2\mathcal{F}^L) + A \otimes 1, P_0)\})$$

$$= \text{FLI}_g(\mathcal{F}, \mathcal{F}') + \text{ch}_g(\text{sf}_G\{(D(\mathcal{F}) + A, P), (D(\mathcal{F}'), A')\}), \quad (3.100)$$

where $P_0$ and $P'_0$ are the associated APS projections, respectively. Note that in order to adopt the sign convention (1.7), the sign at the beginning of the fifth line of (3.100) is altered.

The proof of Theorem 3.17 is completed. \hfill \square

Using Theorem 3.17, we could write Lemma 3.16 in a more elegant form.

**Theorem 3.18.** Let $A_Z$ and $A_X$ be perturbation operators with respect to $D(\mathcal{F}_Z)$ and $D(\mathcal{F}_X)$. Then modulo exact forms on $B$, for $T \geq 1$ large enough, we have

$$\tilde{\eta}_g(\mathcal{F}_Z, A_Z) = \int_{Y^*} \tilde{T}_d_g(\nabla^{TY}, \nabla^{L^Y}) \tilde{\eta}_g(\mathcal{F}_X, A_X) + \text{FLI}_g(\nabla^{TY,TX}, \nabla_{L^X}, \nabla^{TZ}, \nabla_{L^Z})$$

$$+ \text{ch}_g(\text{sf}_G\{(D(\mathcal{F}_Z,T) + 1 \otimes TA_X, P), (D(\mathcal{F}_X) + A_Z, P')\}), \quad (3.101)$$

where $P$ and $P'$ are the associated APS projections, respectively.

From Theorems 3.17 and 3.18, we could extend Lemma 3.13 to the general case.

**Theorem 3.19** (Compare with [16, (24)]). Let $\mathcal{F}, \mathcal{F}' \in \mathcal{F}^*_\mathcal{G}(B)$. Let $A$ and $A'$ be the perturbation operators with respect to $D(\mathcal{F})$ and $D(\mathcal{F} \times_B \mathcal{F}')$. Then there exists $x \in \mathcal{K}^*_\mathcal{G}(B)$ such that

$$\tilde{\eta}_g(\mathcal{F} \times_B \mathcal{F}', A') = \tilde{\eta}_g(\mathcal{F}, A) \text{FLI}_g(\mathcal{F}') + \text{ch}_g(x). \quad (3.102)$$
Proof. Here, we use a trick in [16] similar to (3.98). Let \( \pi' : W' \to B \) be the submersion in \( F' \). We could obtain a pullback family \( \pi^* F \) by choosing a horizontal subbundle \( T^H(\pi^* W) \) such that
\[
d\pi'(T^H(\pi^* W)) \subset T^H W.
\]
Let \( \pi^* F \otimes E' \) be the equivariant geometric family which is obtained from \( \pi^* F \) by twisting with \( P_{W'}^*(S_{Z'} \otimes E') \), where \( P_{W'} : W \times_B W' \to W' \). Then we have
\[
F \times_B F' \simeq \pi'! (\pi^* F \otimes E').
\]
(3.103)
Since the fibers of \( \pi^* W \to B \) are \( Z' \times Z \), the fiberwise connection \( \nabla^{T(Z' \times Z)} = \nabla^{T Z', T Z} \). So Theorem 3.19 follows from Theorem 3.18.

The proof of Theorem 3.19 is completed. \( \square \)

Remark 3.20. When the parameter space \( B \) is a point and \( \dim Z \) is odd, letting \( A = P_{\ker D} \) be the orthogonal projection onto the kernel of \( D(F) \), which we simply denote by \( D \), we have
\[
\hat{\eta}_g(F, A) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \text{Tr}[g(D + (u\chi(u))'P_{\ker D}) \exp(-(uD + u\chi(u)P_{\ker D})^2)] du
\]
\[
= \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \text{Tr}[g(D + (u\chi(u))'P_{\ker D}) \exp(-u^2 D - u^2 \chi(u)^2 P_{\ker D})] du
\]
\[
= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} u^{-1/2} \text{Tr}[gD \exp(-u^2 D^2)] du + \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \text{Tr}[g\chi(u)'P_{\ker D} \exp(-u^2 \chi(u)^2 P_{\ker D})] du
\]
\[
= \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} u^{-1/2} \text{Tr}[gD \exp(-u^2 D^2)] du + \frac{1}{2} \text{Tr}[gP_{\ker D}],
\]
(3.104)
which is just the usual equivariant reduced eta invariant in [23]. So Theorem 3.18 naturally degenerates to the case of equivariant reduced eta invariants and the equivariant higher spectral flow degenerates to the canonical equivariant spectral flow [24].

3.8 Proof of Lemma 3.16

The proof of Lemma 3.16 is almost the same as the proof of [34, Theorem 3.4]. Observe that [34, Assumptions 3.1 and 3.3] naturally hold in our case.

Let \( T' \geq 1 \) be the constant taken in the proof of Lemma 3.6. For \( T \geq T' \), let \( B_{u,T} \) be the Bismut superconnection associated with the equivariant geometric family \( F_{Z,T} \). Let \( \hat{B}_{(T,u)} = B_{u^2,T} + uT\chi(uT)(1 \otimes A_X) + dT \land \frac{\partial}{\partial T} + du \land \frac{\partial}{\partial u} \).

(3.105)
We define
\[
\beta_g = du \land \beta_g^u + dT \land \beta_g^T
\]
to be the part of \( \psi_B \text{Tr}[g \exp(-\hat{B}^2)] \) of degree one with respect to the coordinates \( (T,u) \) with functions \( \beta_g^u : \mathbb{R}_{T \times \mathbb{R}_{+u}} \to \Omega^*(B, \mathbb{C}) \).

Compared with [34, Proposition 4.2], there exists a smooth family \( \alpha_g : \mathbb{R}_{+T} \times \mathbb{R}_{+u} \to \Omega^*(B, \mathbb{C}) \) such that
\[
\left( du \land \frac{\partial}{\partial u} + dT \land \frac{\partial}{\partial T} \right) \beta_g = dT \land du \land d^B \alpha_g.
\]
(3.106)
Take \( \varepsilon, A, T_0, 0 < \varepsilon \leq 1 \leq A < \infty, T' \leq T_0 < \infty \). Let \( \Gamma = \Gamma_{\varepsilon,A,T_0} \) be the oriented contour in \( \mathbb{R}_{+T} \times \mathbb{R}_{+u} \).
The contour $\Gamma$ is made of four oriented pieces $\Gamma_1, \ldots, \Gamma_4$ indicated in Figure 1. For $1 \leq k \leq 4$, set $I_k^0 = \int_{\Gamma_k} \beta_g$. Then by the Stokes formula and (3.106),

$$\sum_{k=1}^{4} I_k^0 = \int_{\partial U} \beta_g = \int_{U} \left( du \land \frac{\partial}{\partial u} + dT \land \frac{\partial}{\partial T} \right) \beta_g = dB \left( \int_{U} \alpha dT \land du \right).$$

The following theorems are the analogues of [34, Theorems 4.3–4.6]. Note that Theorem 3.22 is the analogue of [34, (6.8)]. We will sketch the proofs in the next subsection.

**Theorem 3.21.** (i) For any $u > 0$, we have

$$\lim_{T \to \infty} \beta_g^u(T, u) = 0.$$  (3.108)

(ii) For $0 < u_1 < u_2$ fixed, there exists $C > 0$ such that for $u \in [u_1, u_2]$ and $T \geq 1$, we have

$$|\beta_g^u(T, u)| \leq C.$$  (3.109)

(iii) We have the following identity:

$$\lim_{T \to +\infty} \int_{1}^{\infty} \beta_g^u(T, u) du = 0.$$  (3.110)

**Theorem 3.22.** For $u_0 > 0$ fixed, there exist $C, C' > 0$ and $T_0 \geq 1$ such that for $u \geq u_0$ and $T \geq T_0$,

$$|\beta_g^u(T, u)| \leq C \exp(-C'u^2).$$  (3.111)

We know that $\tilde{\Lambda}_g(TZ, \nabla)$ only depends on $g \in G$ and $R := \nabla^2$. So we also denote it by $\tilde{\Lambda}_g(R)$. Let $R_{TZ}^2 := (\nabla^2_{TZ})^2$. Set

$$\gamma_\Omega(T) = -\frac{\partial}{\partial b} \bigg|_{b=0} \tilde{\Lambda}_g \left( R_{TZ}^2 + b \frac{\partial \nabla_{TZ}}{\partial T} \right).$$

By a standard argument in Chern-Weil theory, we know that

$$\frac{\partial}{\partial T} \tilde{\Lambda}_g(TZ, \nabla_{TZ}^2, \nabla_{TZ}^2) = -\gamma_\Omega(T).$$

**Theorem 3.23.** When $T \to +\infty$, we have $\gamma_\Omega(T) = O(T^{-2})$. Moreover, modulo exact forms on $W^g$, we have

$$\tilde{\Lambda}_g(TZ, \nabla_{TZ}^2, \nabla_{TY,TX}) = -\int_{T'}^{+\infty} \gamma_\Omega(T) dT.$$  (3.114)

Let $\mathcal{B}_{X,T}$ be the Bismut superconnection associated with the equivariant geometric family $F_{X,T}$, which is the same as $F_X$ except for replacing $g^{TX}$ by $T^{-2}g^{TX}$. Set

$$\gamma_1(T) = \left\{ \psi_{V^*} T_{V^*} \left| V^* \right. \right\} \left[ g \exp \left( - \left( \mathcal{B}_{X,T} |_{V^*} + T_\chi(T) A_X |_{V^*} + dT \land \frac{\partial}{\partial T} \right)^2 \right) \right] \right\}^{dT}. $$  (3.115)
Then from (3.68),
\[
\tilde{h}_g(F_X, A_X) = -\int_0^\infty \gamma_1(T) dT. \tag{3.116}
\]

**Theorem 3.24.** (i) For any \( u > 0 \), there exist \( C > 0 \) and \( \delta > 0 \) such that for \( T \geq T' \), we have
\[
|\beta_g^T(T, u)| \leq \frac{C}{T^{1+\delta}}. \tag{3.117}
\]
(ii) For any \( T > 0 \), we have
\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} \beta_g^{T}(T \varepsilon^{-1}, \varepsilon) = \int_{Y^g} Td_g(\nabla^{TY}, \nabla^{LY}) \gamma_1(T). \tag{3.118}
\]
(iii) There exists \( C > 0 \) such that for \( \varepsilon \in (0, 1/T'] \), \( \varepsilon T' \leq T \leq 1 \),
\[
\varepsilon^{-1} |\beta_g^{T}(T \varepsilon^{-1}, \varepsilon) + \int_{\Omega} \gamma_0(T \varepsilon^{-1}) \mathrm{ch}_g(L_{Y/T}^{1/2}, \nabla^{LY}_{1/2}) \mathrm{ch}_g(E, \nabla^E)| \leq C. \tag{3.119}
\]
Note that as in (3.46), \( \mathrm{ch}_g(L_{Y/T}^{1/2}, \nabla^{LY}_{1/2}) \) is well defined even if \( L_{Y/T}^{1/2} \) does not exist.
(iv) There exist \( \delta \in (0, 1] \) and \( C > 0 \) such that for \( \varepsilon \in (0, 1] \) and \( T \geq 1 \),
\[
\varepsilon^{-1} |\beta_g^{T}(T \varepsilon^{-1}, \varepsilon)| \leq \frac{C}{T^{1+\delta}}. \tag{3.120}
\]

Now we prove Lemma 3.16 using the theorems above.

By (3.107), we know that
\[
\int_{\varepsilon}^{A} \beta_g^a(T_0, u) du - \int_{T}^{T_0} \beta_g^a(T, A) dT - \int_{\varepsilon}^{A} \beta_g^a(T', u) du + \int_{T}^{T_0} \beta_g^a(T, \varepsilon) dT = \sum_{k=0}^{4} I_j^k \tag{3.121}
\]
is an exact form. We take the limits \( A \to +\infty \), \( T_0 \to +\infty \) and then \( \varepsilon \to 0 \) in the indicated order. Let \( I_j^k \) \((j = 1, 2, 3, 4, k = 1, 2, 3)\) denote the value of the part \( I_j^k \) after the \( k \)-th limit.

Since the definition of the equivariant eta form does not depend on the cut-off function, from (3.65), we obtain that modulo exact forms on \( B \),
\[
I_3^3 = \tilde{h}_g(F_{Z,T'}, 1 \otimes T' A_X). \tag{3.122}
\]
Furthermore, by Theorem 3.22, we get
\[
I_3^3 = I_2^2 = 0. \tag{3.123}
\]
From Theorem 3.21, we have
\[
I_3^1 = 0. \tag{3.124}
\]
Finally, using Theorem 3.24, we get
\[
I_4^3 = -\int_{Y^g} Td_g(\nabla^{TY}, \nabla^{LY}) \tilde{h}_g(F_X, A_X) + F \mathrm{Li}_g(\nabla^{T_Z}, \nabla^{L_Z}, \nabla^{TY}, T_X, \nabla^{L_Z}) \tag{3.125}
\]
from the following arguments. Notice that
\[
\int_{T}^{+\infty} \beta_g^T(T, \varepsilon) dT = \int_{\varepsilon}^{+\infty} \varepsilon^{-1} \beta_g^T(T \varepsilon^{-1}, \varepsilon) dT. \tag{3.126}
\]
Convergence of the integrals above is guaranteed by (3.117). Using Theorem 3.23 and (3.118)–(3.120), we get
\[
\lim_{\varepsilon \to 0} \int_{1}^{+\infty} \varepsilon^{-1} \beta_g^{T}(T \varepsilon^{-1}, \varepsilon) dT = \int_{Y^g} Td_g(\nabla^{TY}, \nabla^{LY}) \int_{1}^{+\infty} \gamma_1(T) dT \tag{3.127}
\]
Since ker($D(F,T)$) = 0, the proofs of Theorems 3.21–3.24 in our case are much easier than those in [34]. We only need to replace $D^X$ and $D^T_\varepsilon$ somewhere in [34] by $D(F,T) + A_X$ and $D(F,T) + 1 \otimes TA_X$ and take care with the local index computation in the proof of Theorem 3.24(ii). In this subsection, we only sketch the local index part here.

Set (see [34, (7.1)])
\[
B_{\varepsilon,T/\varepsilon} = (\mathbb{B}_{\varepsilon^2,T/\varepsilon} + T\chi(T)A_X)^2 + \varepsilon^{-1}dT\wedge \frac{\partial(\mathbb{B}_{\varepsilon^2,T/\varepsilon} + \varepsilon T\chi(T)A_X)}{\partial T} \bigg|_{T'=T+1}.
\]

By the definition of $\beta^T_{\varepsilon}(T,\varepsilon)$, we have
\[
\varepsilon^{-1}\beta^T_{\varepsilon}(T/\varepsilon,\varepsilon) = \{\psi_{\varepsilon,T/\varepsilon}[g\exp(-B_{\varepsilon,T/\varepsilon})]\}dT.
\]

Let $S_X$ be the tensor in (3.33) with respect to $\pi_X$. Let $\{e_i\}$, $\{f_p\}$ and $\{g_{\alpha,\beta}\}$ be the local orthonormal frames of $TX$, $TY$ and $TB$ and $\{f_p^H\}$ and $\{g_{\alpha,\beta}^H\}$ be the corresponding horizontal lifts. Precisely, by (3.37), we have
\[
\varepsilon^{-1}\frac{\partial(\mathbb{B}_{\varepsilon^2,T/\varepsilon} + \varepsilon T\chi(T)A_X)}{\partial T} \bigg|_{T'=T+1} = D^X + \chi(T)A_X + T\chi'(T)A_X - \frac{1}{8T^2}((e^2|f_{p,1},f_{q,1}],e_i)c(e_i)c(f_{p,1})c(f_{q,1})
+ 4\varepsilon(S_X(g_{\alpha,\beta}^H)e_i,f_{p,1}^H)c(e_i)c(f_{p,1}^H)g_{\alpha}^\beta \wedge + (g_{\alpha,\beta}^H,e_i)c(e_i)g_{\alpha}^\beta \wedge g_{\beta}^\gamma).\]
Lemma 3.25. When \( \varepsilon \to 0 \), the limit \( L_{0,T}^3 |_{V_s} \) exists in the sense of \([34, (7.108)]\) and

\[
L_{0,T}^3 |_{V_s} = - \left( \partial_{p_0} + \frac{1}{4} (R^{TV} |_{V_s} U, f_{p_1}^{H}) \right)^2 + \frac{1}{2} R^{LV} |_{V_s} + B_{T^2}^\gamma |_{V_s}. \tag{13.43}
\]

So all the computations in our case are the same as in \([34, \text{Section 7}]\).
4 Equivariant differential K-theory

In this section, we assume that the $G$-action on $B$ has finite stabilizers only, i.e., for any $b \in B$, $G_b := \{g \in G : gb = b\}$ is finite. With this action, we construct an analytic model of equivariant differential K-theory and prove some properties using the results in Section 3.

4.1 The definition of equivariant differential K-theory

In this subsection, we construct an analytic model of equivariant differential K-theory. When $G = \{e\}$, this construction is similar to that in [16] except replacing the taming and KK-theory by the spectral section and higher spectral flow.

Let $E$ be a $G$-equivariant complex vector bundle over $B$. Then its restriction to $B^g$ is acted on fiberwisely by $g$ for $g \in G$. So it decomposes as a direct sum of subbundles $E_v$ for each eigenvalue $v$ of $g$. Set $\phi_g(E) := \sum vE_v$. Then it induces a homomorphism (for $K^*_B$, we replace $B$ by $B \times S^1$ and use (2.19))

$$\phi_g : K^*_G(B) \otimes \mathbb{C} \to [K^*(B^g) \otimes \mathbb{C}]^{C_G(g)}, \tag{4.1}$$

where $C_G(g)$ is the centralizer of $g$ in $G$. Let $(g)$ be the conjugacy class of $g \in G$. For $g, g' \in (g)$, there exists $h \in G$ such that $g' = h^{-1}gh$. Furthermore, the map

$$h : B^{g'}/C_G(g') \to B^g/C_G(g) \tag{4.2}$$

is a homeomorphism. So

$$[K^*(B^g) \otimes \mathbb{C}]^{C_G(g)} \simeq [K^*(B^{g'}) \otimes \mathbb{C}]^{C_G(g')}. \tag{4.3}$$

By [1, Corollary 3.13], we know that the additive decomposition

$$\phi = \bigoplus_{(g), g \in G} \phi_g : K^*_G(B) \otimes \mathbb{C} \to \bigoplus_{(g), g \in G} [K^*(B^g) \otimes \mathbb{C}]^{C_G(g)} \tag{4.4}$$

is an isomorphism, where $(g)$ ranges over the conjugacy classes of $G$.

If $B^g \neq \emptyset$, then there exists $b \in B^g$ such that $g \in G_b$. The conjugacy class of $G_b$ is the type of the orbit $G \cdot b$. Since $B$ is compact, there are only finitely many orbit types. Since all the stabilizers are finite groups, we see that the direct sum in (4.4) only has finite terms. From the isomorphism (4.3), the direct sum in (4.4) does not depend on the choice of the element in $(g)$ in the sense of (4.3).

From (4.2), we also know that the map $h^*$ induces an isomorphism

$$h^* : [\Omega^*(B^g, \mathbb{C})]^{C_G(g)} \to [\Omega^*(B^{g'}, \mathbb{C})]^{C_G(g')} \tag{4.5}$$

We denote by

$$\Omega^*_{\text{deloc},G}(B, \mathbb{C}) := \bigoplus_{(g), g \in G} \{[\Omega^*(B^g, \mathbb{C})]^{C_G(g)}\} \tag{4.6}$$

the set of delocalized differential forms, where $\{\cdot\}$ denotes the isomorphic class in the sense of (4.5). The definition above does not depend on the choice of $g \in (g)$. It is easy to see that the exterior differential operator $d$ preserves $\Omega^*_{\text{deloc},G}(B, \mathbb{C})$. We denote by the delocalized de Rham cohomology $H^*_{\text{deloc},G}(B, \mathbb{C})$ the cohomology of the differential complex $(\Omega^*_{\text{deloc},G}(B, \mathbb{C}), d)$. Then from (4.1) and (4.4), the equivariant Chern character isomorphism can be naturally defined by

$$\text{ch}_G : K^*_G(B) \otimes \mathbb{C} \xrightarrow{\sim} H^*_{\text{deloc},G}(B, \mathbb{C}), \quad \mathcal{K} \mapsto \bigoplus_{(g), g \in G} \{\text{ch}(\phi_g(\mathcal{K}))\}. \tag{4.7}$$

We note that $\text{ch}(\phi_g(\mathcal{K})) = \text{ch}_g(\mathcal{K})$ is $C_G(g)$-invariant by the definition.
Observe that the fixed point set for the $g$-action coincides with that for $g^{-1}$-action. Set

$$H_{\text{deloc},G}(B, \mathbb{R}) := \left\{ c = \bigoplus_{(g), g \in G} \{ c_g \in H^*_{\text{deloc},G}(B, \mathbb{C}) : \forall g \in G, c_{g^{-1}} = \overline{c_g} \} \right\}. \quad (4.8)$$

Let $\Omega^*_{\text{deloc},G}(B, \mathbb{R}) \subset \Omega^*_{\text{deloc},G}(B, \mathbb{C})$ be the ring of forms $\omega = \bigoplus_{(g), g \in G} \{ \omega_g \}$ such that $\forall g \in G, \omega_{g^{-1}} = \overline{\omega_g}$. Then $H^*_{\text{deloc},G}(B, \mathbb{R})$ is the cohomology of the differential complex $(\Omega^*_{\text{deloc},G}(B, \mathbb{R}), d)$. Since $\text{ch}(\phi_g^{-1}(K)) = \overline{\text{ch}(\phi_g(K))}$, from (4.7), for any $K \in K^*_G(B)$, $\text{ch}_G(K) \in H^*_{\text{deloc},G}(B, \mathbb{R})$. Thus $\text{ch}_G(K^*_G(B) \otimes \mathbb{R}) \subseteq H^*_{\text{deloc},G}(B, \mathbb{R})$. Since (4.7) is an isomorphism, we obtain a group isomorphism

$$\text{ch}_G : K^*_G(B) \otimes \mathbb{R} \xrightarrow{\sim} H^*_{\text{deloc},G}(B, \mathbb{R}). \quad (4.9)$$

**Definition 4.1** (Compare with [16, Definition 2.4]). A cycle for an equivariant differential K-theory class over $B$ is a pair $(\mathcal{F}, \rho)$, where $\mathcal{F} \in F^*_G(B)$ and $\rho \in \Omega^*_{\text{deloc},G}(B, \mathbb{R})/\text{Imd}$. The cycle $(\mathcal{F}, \rho)$ is called even (resp. odd) if $\mathcal{F}$ is even (resp. odd) and $\rho \in \Omega^\text{even}_{\text{deloc},G}(B, \mathbb{R})/\text{Imd}$ (resp. $\rho \in \Omega^\text{odd}_{\text{deloc},G}(B, \mathbb{R})/\text{Imd}$).

Two cycles $(\mathcal{F}, \rho)$ and $(\mathcal{F}', \rho')$ are called isomorphic if $\mathcal{F}$ and $\mathcal{F}'$ are isomorphic and $\rho = \rho'$. Let $\mathcal{IC}^\text{even}_G(B)$ (resp. $\mathcal{IC}^\text{odd}_G(B)$) denote the set of isomorphic classes of even (resp. odd) cycles over $B$ with a natural abelian semi-group structure by $(\mathcal{F}, \rho) + (\mathcal{F}', \rho') = (\mathcal{F} + \mathcal{F}', \rho + \rho')$.

For $\mathcal{F} \in F^*_G(B)$, we assume that there exists a perturbation operator $A$ with respect to $D(\mathcal{F})$. For any $g \in G$, by Definition 3.12, the equivariant eta form restricted on the fixed point set of $g$ is $C_G(g)$-invariant, i.e., $\tilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g}) \in [\Omega^*(B^g, \mathcal{C})]^{C_G(g)}$. Let $h^*$ be the map in (4.5). Since the perturbation operator $A$ is equivariant, from Definition 3.12, we have

$$\tilde{\eta}_g(h^*(\mathcal{F}|_{B^g}, A|_{B^g})) = h^*\tilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g}). \quad (4.10)$$

From Definition 3.12, $\tilde{\eta}_g^{-1}(\mathcal{F}|_{B^g}, A|_{B^g}) = \eta_g(\mathcal{F}|_{B^g}, A|_{B^g})$. So the following definition is well defined.

**Definition 4.2.** The delocalized eta form $\tilde{\eta}_G(\mathcal{F}, A)$ is defined by

$$\tilde{\eta}_G(\mathcal{F}, A) = \bigoplus_{(g), g \in G} \{ \tilde{\eta}_g(\mathcal{F}|_{B^g}, A|_{B^g}) \} \in \Omega^*_{\text{deloc},G}(B, \mathbb{R})/\text{Imd}. \quad (4.11)$$

By the same process, we can define

$$\text{FLI}_G(\mathcal{F}) = \bigoplus_{(g), g \in G} \{ \text{FLI}_g(\mathcal{F}) \} \in \Omega^*_{\text{deloc},G}(B, \mathbb{R}). \quad (4.12)$$

From (3.66), we have

$$d\tilde{\eta}_G(\mathcal{F}, A) = \text{FLI}_G(\mathcal{F}). \quad (4.13)$$

Let $\mathcal{F} \in F^*_G(B)$ and $A$ be a perturbation operator with respect to $D(\mathcal{F})$. Then by Definition 2.3, there exists a perturbation operator $A^{\text{op}}$ with respect to $D(\mathcal{F}^{\text{op}})$ such that

$$\tilde{\eta}_G(\mathcal{F}^{\text{op}}, A^{\text{op}}) = -\tilde{\eta}_G(\mathcal{F}, A). \quad (4.14)$$

Let $\mathcal{F}, \mathcal{F}' \in F^*_G(B)$, and $A$ and $A'$ be perturbation operators with respect to $D(\mathcal{F})$ and $D(\mathcal{F}')$, respectively. By Definition 3.12, we have

$$\tilde{\eta}_G(\mathcal{F} + \mathcal{F}', A \bigoplus_{B} A') = \tilde{\eta}_G(\mathcal{F}, A) + \tilde{\eta}_G(\mathcal{F}', A'). \quad (4.15)$$

From Remark 3.4, we know that for any $\mathcal{F} \in F^*_G(B)$, there exists a perturbation operator $A$ with respect to $D(\mathcal{F} + \mathcal{F}^{\text{op}})$ and $A = A^{\text{op}}$. From (4.14), we have

$$\tilde{\eta}_G(\mathcal{F} + \mathcal{F}^{\text{op}}, A) = 0. \quad (4.16)$$
Definition 4.3 (Compare with [16, Definition 2.10]). We call two cycles \((\mathcal{F}, \rho)\) and \((\mathcal{F}', \rho')\) paired if \(\text{Ind}(D(\mathcal{F})) = \text{Ind}(D(\mathcal{F}'))\), and there exists a perturbation operator \(A\) with respect to \(D(\mathcal{F} + \mathcal{F}^{\text{top}})\) such that

\[
\rho - \rho' = \tilde{\eta}_G(\mathcal{F} + \mathcal{F}^{\text{top}}, A).
\]

(4.17)

From (4.14)–(4.16), we have the following lemma.

Lemma 4.4 (Compare with [16, Lemmas 2.11 and 2.12]). The relation "paired" is symmetric, reflexive and compatible with the semigroup structure on \(\widehat{\mathcal{K}}_G^*(B)\).

Definition 4.5 (Compare with [16, Definition 2.14]). Let \(\sim\) denote the equivalence relation generated by the relation "paired". The equivariant differential K-group \(\widehat{K}_G^0(B)\) (resp. \(\widehat{K}_G^1(B)\)) is the group completion of the abelian semigroup \(\widehat{\mathcal{K}}_G^\text{even}(B)/\sim\) (resp. \(\widehat{\mathcal{K}}_G^\text{odd}(B)/\sim\)).

If \((\mathcal{F}, \rho) \in \widehat{\mathcal{K}}_G^*(B)\), we denote by \([\mathcal{F}, \rho] \in \widehat{K}_G^*(B)\) the corresponding class in the equivariant differential K-group. From (4.14)–(4.16), for any \([\mathcal{F}, \rho], [\mathcal{F}', \rho'] \in \widehat{K}_G^*(B)\), we have

\[
[\mathcal{F}, \rho] = [\mathcal{F} + \mathcal{F}^{\text{top}}, \rho - \rho'] + [\mathcal{F}', \rho'].
\]

(4.18)

So every element of \(\widehat{K}_G^*(B)\) can be represented in the form \([\mathcal{F}, \rho]\). Furthermore, we have \(-[\mathcal{F}, \rho] = [\mathcal{F}^{\text{top}}, -\rho]\).

4.2 The push-forward map

In this subsection, we construct a well-defined push-forward map in equivariant differential K-theory and prove the functoriality of it using the theorems in Section 3. This answers a question proposed in [16] when \(G = \{e\}\). We use the notation in Subsection 2.4.

Let \(\pi_Y : V \rightarrow B\) be an equivariant smooth surjective proper submersion of compact \(G\)-manifolds with compact orientable fibers \(Y\). We assume that the \(G\)-action on \(B\) has finite stabilizers only. Thus, so is the action on \(V\). We assume that \(TY\) is oriented and \(\pi_Y\) has an equivariant K-orientation in Definition 2.8.

For \(g \in G\), the fixed point set \(V^g\) is the total space of the fiber bundle \(\pi_Y|_{V^g} : V^g \rightarrow B^g\) with fibers \(Y^g\). Since the pullback isomorphism \(h^*\) in (4.5) commutes with the integral along the fiber, for \(\alpha = \bigoplus_{(g), g \in G} \{\alpha_g\} \in \Omega^*_{\text{deloc},G}(V, \mathbb{R})\), the integral

\[
\int_{Y,G} \alpha := \bigoplus_{(g), g \in G} \left\{ \int_{Y^g} \alpha_g \right\} \in \Omega^*_{\text{deloc},G}(B, \mathbb{R})
\]

(4.19)

does not depend on \(g \in (g)\). So it defines an integral map

\[
\int_{Y,G} : \Omega^*_{\text{deloc},G}(V, \mathbb{R}) \rightarrow \Omega^*_{\text{deloc},G}(B, \mathbb{R}).
\]

(4.20)

Consider the set \(\mathcal{O}_G(\pi_Y)\) of equivariant geometric data \(\tilde{\sigma}_Y = (T^H_Y V, g^{TY}, \nabla^{LV}, \sigma_Y)\), where \(\sigma_Y \in \Omega^*_{\text{deloc},G}(V, \mathbb{R})\).

Let

\[
\text{Td}_G(\nabla^{TY}, \nabla^{LV}) := \bigoplus_{(g), g \in G} \{\text{Td}_g(\nabla^{TY}, \nabla^{LV})\} \in \Omega^*_{\text{deloc},G}(V, \mathbb{R}).
\]

(4.21)

Let \(\tilde{\sigma}_Y = (T^H_Y V, g^{TY}, \nabla^{LV}, \sigma_Y) \in \mathcal{O}_G(\pi_Y)\) be another equivariant tuple with the same equivariant K-orientation in Definition 2.8. As in (3.93), from [42, Theorem B.5.4], we can construct the Chern-Simons form \(d\tilde{\text{Td}}_G(\nabla^{TY}, \nabla^{LV}, \nabla^iT^{TY}, \nabla^iT^{LY}) = \text{Td}_G(\nabla^{TY}, \nabla^{LV}) - \text{Td}_G(\nabla^{TY}, \nabla^{LV})\).

(4.22)
We introduce a relation \( \hat{\sigma} \sim \hat{\sigma}' \) as in [16]: two equivariant tuples \( \hat{\sigma} \) and \( \hat{\sigma}' \) are related if and only if

\[
\sigma' - \sigma = \widetilde{\text{Td}}_G(\nabla^{TY}, \nabla^{LY}, \nabla^{TY'}, \nabla^{LY'}),
\]

where we mark the objects associated with the second tuple by ‘.’

**Definition 4.6** (Compare with [16, Definition 3.5]). The set of equivariant differential K-orientations is the set of equivalence classes \( \hat{\Theta}_G^d(\pi_Y)/\sim \).

We now start with the construction of the push-forward map \( \hat{\pi}_Y! : \hat{K}_G^*(V) \to \hat{K}_G^*(B) \) for a given equivariant differential K-orientation which extends Theorem 2.9 to the differential case. For \( [\mathcal{F}_X, \rho] \in \hat{K}_G^*(V) \), let \( \mathcal{F}_Z \) be the equivariant geometric family defined in (2.37). We define (see [16, (17)])

\[
\hat{\pi}_Y!([\mathcal{F}_X, \rho]) = \left[ \mathcal{F}_Z, \int_{Y,G} \text{Td}_G(\nabla^{TY}, \nabla^{LY}) \wedge \rho + F\text{ILI}_G(\nabla^{TY,TX}, \nabla^{LZ}, \nabla^{TZ}, \nabla^{LZ}) + \int_{Y,G} \sigma_Y \wedge (F\text{ILI}_G(\mathcal{F}_X) - d\rho) \right] \in \hat{K}_G^*(B),
\]

where \( F\text{ILI}_G := \bigoplus_{g \in G} F\text{ILI}_g \in \Omega_{\text{deloc},G}(B, \mathbb{R})/\text{Ind} \).

**Theorem 4.7** (Compare with [16, Lemma 3.14]). The map \( \hat{\pi}_Y! : \hat{K}_G^*(V) \to \hat{K}_G^*(B) \) in (4.24) is well defined.

**Proof.** Let \( (\mathcal{F}_X, \rho) \) and \( (\mathcal{F}_X', \rho') \) be two cycles over \( V \). By (4.24), we have

\[
\hat{\pi}_Y!((\mathcal{F}_X, \rho) - (\mathcal{F}_X', \rho')) = \hat{\pi}_Y!((\mathcal{F}_X + \mathcal{F}_X^\text{op}, \rho - \rho')).
\]

If \( (\mathcal{F}_X, \rho) \) is paired with \( (\mathcal{F}_X', \rho') \), there exists a perturbation operator \( A \) such that

\[
\rho - \rho' = \hat{\eta}_G(\mathcal{F}_X + \mathcal{F}_X^\text{op}, A).
\]

So we only need to prove that if there exists a perturbation operator \( A_X \) with respect to \( D(\mathcal{F}_X) \),

\[
\hat{\pi}_Y!([\mathcal{F}_X, \hat{\eta}_G(\mathcal{F}_X, A_X)]) = 0 \in \hat{K}_G^*(B).
\]

From (4.24), we have

\[
\hat{\pi}_Y!([\mathcal{F}_X, \hat{\eta}_G(\mathcal{F}_X, A_X)]) = \left[ \mathcal{F}_Z, \int_{Y,G} \text{Td}_G(\nabla^{TY}, \nabla^{LY}) \hat{\eta}_G(\mathcal{F}_X, A_X) + F\text{ILI}_G(\nabla^{TY,TX}, \nabla^{LZ}, \nabla^{TZ}, \nabla^{LZ}) + \int_{Y,G} \sigma_Y \wedge (F\text{ILI}_G(\mathcal{F}_X) - d\hat{\eta}_G(\mathcal{F}_X, A_X)) \right].
\]

From Proposition 3.3(iii) and Lemma 3.6, there exists a perturbation operator \( A_Z \) with respect to \( D(\mathcal{F}_Z) \).

By Theorem 3.18, (4.13), (4.15) and (4.19), there exists \( x \in K_G^*(B) \) such that

\[
\hat{\pi}_Y!(\mathcal{F}_Z, \hat{\eta}_G(\mathcal{F}_Z, A_Z)) = \left[ \mathcal{F}_Z, \hat{\eta}_G(\mathcal{F}_Z, A_Z) - \chi_G(x) \right].
\]

From Proposition 3.9, if \( x \in K_G^*(B) \), there exist \( F \in F_G^1(B) \) and equivariant spectral sections \( P \) and \( Q \) with respect to \( D(\mathcal{F}) \) such that \( [P - Q] = x \). Let \( A_P \) and \( A_Q \) be perturbation operators associated with \( P \) and \( Q \), respectively. From Theorem 3.17, we have

\[
\chi_G(x) = \hat{\eta}_G(F, A_P) - \hat{\eta}_G(F, A_Q).
\]

If \( x \in K_G^0(B) \), by Proposition 3.9, there exist \( F_1, F_2 \in F_G^0(B) \) and equivariant spectral sections \( P_i \) and \( Q_i \) of \( D(\mathcal{F}_i) \) and \( P_2 \) and \( Q_2 \) of \( D(\mathcal{F}_2) \) such that \( x = [P_1 - Q_1] - [P_2 - Q_2] \). Let \( A_{P_i} \) and \( A_{Q_i} \) be perturbation operators associated with \( P_i \) and \( Q_i \) for \( i = 0, 1 \). From Theorem 3.17, letting \( F = F_1 + F_2, A_P = A_{P_1} \sqcup_B A_{Q_2} \) and \( A_Q = A_{P_2} \sqcup_B A_{Q_1} \), we also have

\[
\chi_G(x) = \hat{\eta}_G(F_1, A_{P_1}) - \hat{\eta}_G(F_1, A_{Q_1}) - (\hat{\eta}_G(F_2, A_{P_2}) - \hat{\eta}_G(F_2, A_{Q_2})).
\]
By (4.14), (4.28)–(4.30) and Definition 4.3, we have
\[
\hat{\pi}_Y!(\mathcal{F}_X, \eta_G(\mathcal{F}_X, A_X)) = [\mathcal{F}_Z, \eta_G(\mathcal{F}_Z, A_Z)] - \eta_G(\mathcal{F}, A_F) - \eta_G(\mathcal{F}^\text{op}, A_Q^\text{op})
\]
\[
= [\mathcal{F} + \mathcal{F}^\text{op}, 0] - [\mathcal{F}, 0] = 0 \in \hat{\mathcal{K}}^*_G(B). \quad (4.31)
\]

Then from Theorem 2.9, we complete the proof of Theorem 4.7.

Lemma 4.8 (Compare with [16, Lemma 3.17]). The homomorphism \( \hat{\pi}_Y! : \hat{\mathcal{K}}_G^*(V) \to \hat{\mathcal{K}}_G^*(B) \) only depends on the equivariant differential K-orientation.

Proof. Let \( \hat{\pi}_Y = (T^H_{\mathcal{V}}V, g^{TY}, \nabla^{TY}, \sigma_Y) \) and \( \hat{\pi}_Y = (T^H_{\mathcal{V}}V, g^{TY}, \nabla^{TY}, \sigma_Y) \) be two representatives of an equivariant differential K-orientation. We will mark the objects associated with the second representative by ‘. From (3.94), we could get
\[
\hat{\mathcal{L}}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) = \int_{\mathcal{Y}, G} \frac{\rho}{\mathcal{L}}(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) \hat{\mathcal{L}}_G(\mathcal{F}_X).
\]

Then from (4.22), (4.23) and (4.32), we have
\[
\hat{\pi}_Y!(\mathcal{F}_X, \rho) - \hat{\pi}_Y!(\mathcal{F}_X, \rho)
\]
\[
= [\mathcal{F}_Z + \mathcal{F}_Z^\text{op}, \int_{\mathcal{Y}, G} (d\mathcal{L}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) - \mathcal{L}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) \wedge \rho - \hat{\mathcal{L}}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) \wedge \rho) - \hat{\mathcal{L}}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) \wedge \rho - \hat{\mathcal{L}}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) \wedge \rho)
\]
\[
= [\mathcal{F}_Z + \mathcal{F}_Z^\text{op}, \int_{\mathcal{Y}, G} \mathcal{L}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) \wedge \rho - \int_{\mathcal{Y}, G} \mathcal{L}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) \wedge \rho - \hat{\mathcal{L}}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) \wedge \rho - \hat{\mathcal{L}}_G(\nabla^{TY}, \nabla^{TY}, \nabla^{TY}, \nabla^{TY}) \wedge \rho)
\]
\[
= [\mathcal{F}_Z + \mathcal{F}_Z^\text{op}, \hat{\mathcal{L}}_G(\mathcal{F}_X, \mathcal{F}_Z)].
\]

By Proposition 3.3(iii) and Lemma 3.6, there exists a perturbation operator \( A \) with respect to \( D(\mathcal{F}_Z + \mathcal{F}_Z^\text{op}) \). By Theorem 3.17 and (4.16), there exists \( x \in \hat{\mathcal{K}}_G^*(B) \) such that
\[
\hat{\mathcal{L}}_G(\mathcal{F}_Z, \mathcal{F}_Z) = \hat{\mathcal{L}}_G(\mathcal{F}_Z + \mathcal{F}_Z^\text{op}, \mathcal{F}_Z + \mathcal{F}_Z^\text{op}) = -\eta_G(\mathcal{F}_Z + \mathcal{F}_Z^\text{op}, A) + \text{ch}_G(x). \quad (4.34)
\]

Following the same process in (4.28)–(4.31), we have \( \hat{\pi}_Y!(\mathcal{F}_X, \rho) = \pi_Y!(\mathcal{F}_X, \rho) \).

The proof of Lemma 4.8 is completed.

We now discuss the functoriality of the push-forward maps with respect to the composition of fiber bundles. Let \( \pi_Y : V \to B \) with fibers \( Y \) be as in the above subsection together with a representative of an equivariant differential K-orientation \( \hat{\pi}_Y = (T^H_{\mathcal{V}}V, g^{TY}, \nabla^{TY}, \sigma_Y) \). Let \( \pi_U : B \to S \) be another equivariant smooth surjective proper submersion with compact oriented fibers \( U \) together with a representative of an equivariant differential K-orientation \( \hat{\pi}_U = (T^H_{\mathcal{V}}B, g^{TU}, \nabla^{TU}, \sigma_U) \).

Let \( \pi_A := \pi_U \circ \pi_Y : V \to S \) be the composition of two submersions with fibers \( A \). Let \( T^H_{\pi_A}V \) be a horizontal subbundle associated with \( \pi_A \). We assume that \( T^H_{\pi_A}V \subset T^H_{\pi_Y}V \). Set \( g^{TA} = \pi_Y^*g^{TU} \otimes g^{TY} \) and \( \nabla^{LA} = \pi_Y^*\nabla^{LU} \otimes \nabla^{LY} \).
By Proposition 3.15 and Definition 4.3, if $B$ unital graded commutative rings. The unit is simply given by horizontal subbundles, $(\hat{K}_{\sigma}^\ast(B), \rho)$.

Note that the choice of the new horizontal subbundle is not unique. If $\hat{K}_{\sigma}^\ast(B)$, we choose the new horizontal subbundle $(\hat{K}_{\sigma}^\ast(B), \rho)$ as follows. For $F \in G^\ast(B)$, in order to define $f^\ast F$, as remarked in Subsection 2.2, we need take care with the pullback of the horizontal subbundle. Let $F$ be the natural map from $f^\ast W$ to $W$. We choose the new horizontal subbundle $T_{f^\ast W}^H(f^\ast W)$ by the condition that $dF(T_{f^\ast W}^H(f^\ast W)) \subseteq T_{f^\ast W}^H(W)$.

Note that the choice of the new horizontal subbundle is not unique. If $A$ is a perturbation operator with respect to $D(F)$, then $f^\ast A$ is a perturbation operator with respect to $D(f^\ast F)$. Moreover, from Definition 3.12, we have

$$\hat{\eta}_G(f^\ast F, f^\ast A) = (f^\ast \hat{\eta}_G(F, A)).$$

By Proposition 3.15 and Definition 4.3, if $F_1, F_2 \in G^\ast(B)$ are the pullbacks of $F$ associated with distinct horizontal subbundles, $(F_1, 0) \sim (F_2, 0)$. So we obtain a well-defined pullback map

$$f^\ast : \hat{K}_G^\ast(B_2) \to \hat{K}_G^\ast(B_1).$$

Evidently, $\text{Id}_B^\ast = \text{Id}_{\hat{K}_G^\ast(B)}$. Let $f' : B_0 \to B_1$ be another equivariant smooth map. We could get

$$f'^\ast f^\ast = (f \circ f')^\ast : \hat{K}_G^\ast(B_2) \to \hat{K}_G^\ast(B_0).$$

Let $[F, \rho] \in \hat{K}_G^\ast(B)$ and $[F', \rho'] \in \hat{K}_G^\ast(B)$, where $i = 0, 1$. We define

$$[F, \rho] \cup [F', \rho'] := [F \times_B F', (-1)^i\text{FLG}(F) \wedge \rho + \rho \wedge \text{FLG}(F') - (-1)^i\rho \wedge \rho'],$$

which is similar to [16, Definition 4.1]. It is obvious that the product is natural with respect to pullbacks.

Proposition 4.11 (Compare with [16, Propositions 4.2 and 4.5]). (i) The product is well defined. It turns $B \mapsto \hat{K}_G^\ast(B)$ into a contravariant functor from compact smooth $G$-manifolds with finite stabilizers to unital graded commutative rings. The unit is simply given by $[F, 0]$, where $F$ is the equivariant geometric family in Example 2.5(a) such that $E_+$ is a $1$-dimensional trivial representation and $E_- = 0$.

(ii) The product is associative.

(iii) Let $\pi_U : B \to S$ be an equivariant smooth proper submersion with oriented fibers and an equivariant differential $K$-orientation. For $x \in \hat{K}_G^\ast(B)$ and $y \in \hat{K}_G^\ast(S)$, we have

$$\hat{\pi}_U!(\pi_U^\ast y \cup x) = y \cup \hat{\pi}_U!(x).$$

\section{The cup product}

In this subsection, we construct the cup product in equivariant differential $K$-theory in our model as in [16, 18] and prove the desired properties.

Let $f : B_1 \to B_2$ be a $G$-equivariant smooth map. We define the induced homomorphism $f^\ast : \hat{K}_G^\ast(B_2) \to \hat{K}_G^\ast(B_1)$ as follows. For $F \in G^\ast(B_2)$, in order to define $f^\ast F$, as remarked in Subsection 2.2, we need take care with the pullback of the horizontal subbundle. Let $F$ be the natural map from $f^\ast W$ to $W$. We choose the new horizontal subbundle $T_{f^\ast W}^H(f^\ast W)$ by the condition that $dF(T_{f^\ast W}^H(f^\ast W)) \subseteq T_{f^\ast W}^H(W)$.

Proof. The topological part of Theorem 4.10 is just Theorem 2.10 and the differential part follows from direct calculation by using (4.24) and (4.36).

\subsection{The cup product}

In this subsection, we construct the cup product in equivariant differential $K$-theory in our model as in [16, 18] and prove the desired properties.

Let $f : B_1 \to B_2$ be a $G$-equivariant smooth map. We define the induced homomorphism $f^\ast : \hat{K}_G^\ast(B_2) \to \hat{K}_G^\ast(B_1)$ as follows. For $F \in G^\ast(B_2)$, in order to define $f^\ast F$, as remarked in Subsection 2.2, we need take care with the pullback of the horizontal subbundle. Let $F$ be the natural map from $f^\ast W$ to $W$. We choose the new horizontal subbundle $T_{f^\ast W}^H(f^\ast W)$ by the condition that $dF(T_{f^\ast W}^H(f^\ast W)) \subseteq T_{f^\ast W}^H(W)$.

Note that the choice of the new horizontal subbundle is not unique. If $A$ is a perturbation operator with respect to $D(F)$, then $f^\ast A$ is a perturbation operator with respect to $D(f^\ast F)$. Moreover, from Definition 3.12, we have

$$\hat{\eta}_G(f^\ast F, f^\ast A) = f^\ast \hat{\eta}_G(F, A).$$

By Proposition 3.15 and Definition 4.3, if $F_1, F_2 \in G^\ast(B)$ are the pullbacks of $F$ associated with distinct horizontal subbundles, $(F_1, 0) \sim (F_2, 0)$. So we obtain a well-defined pullback map

$$f^\ast : \hat{K}_G^\ast(B_2) \to \hat{K}_G^\ast(B_1).$$

Evidently, $\text{Id}_B^\ast = \text{Id}_{\hat{K}_G^\ast(B)}$. Let $f' : B_0 \to B_1$ be another equivariant smooth map. We could get

$$f'^\ast f^\ast = (f \circ f')^\ast : \hat{K}_G^\ast(B_2) \to \hat{K}_G^\ast(B_0).$$

Let $[F, \rho] \in \hat{K}_G^\ast(B)$ and $[F', \rho'] \in \hat{K}_G^\ast(B)$, where $i = 0, 1$. We define

$$[F, \rho] \cup [F', \rho'] := [F \times_B F', (-1)^i\text{FLG}(F) \wedge \rho + \rho \wedge \text{FLG}(F') - (-1)^i\rho \wedge \rho'],$$

which is similar to [16, Definition 4.1]. It is obvious that the product is natural with respect to pullbacks.

Proposition 4.11 (Compare with [16, Propositions 4.2 and 4.5]). (i) The product is well defined. It turns $B \mapsto \hat{K}_G^\ast(B)$ into a contravariant functor from compact smooth $G$-manifolds with finite stabilizers to unital graded commutative rings. The unit is simply given by $[F, 0]$, where $F$ is the equivariant geometric family in Example 2.5(a) such that $E_+$ is a $1$-dimensional trivial representation and $E_- = 0$.

(ii) The product is associative.

(iii) Let $\pi_U : B \to S$ be an equivariant smooth proper submersion with oriented fibers and an equivariant differential $K$-orientation. For $x \in \hat{K}_G^\ast(B)$ and $y \in \hat{K}_G^\ast(S)$, we have

$$\hat{\pi}_U!(\pi_U^\ast y \cup x) = y \cup \hat{\pi}_U!(x).$$
Proof. The product is obviously biadditive. From Theorem 3.19 and direct calculation, we could get the product is compatible with the equivalence relation in differential K-theory. Other properties are the direct extension of the discussions in [16, pp. 47–50].

**Theorem 4.12** (Compare with [16, Sections 3 and 4]). The equivariant differential K-theory $\hat{K}_G$ is a contravariant functor $B \to \hat{K}_G(B)$ from the category of compact smooth $G$-manifolds with finite stabilizers to unital $\mathbb{Z}_2$-graded commutative rings together with the following well-defined transformations:

1. $R: \hat{K}_G^*(B) \to \Omega^*_\text{deloc,cl}(B, \mathbb{R})$ ($\text{curvature}$);
2. $I: \hat{K}_G^*(B) \to K_G^*(B)$ ($\text{underlying K}_G$-group);
3. $a: \Omega^*_\text{deloc,G}(B, \mathbb{R})/\text{Imd} \to \hat{K}_G(X)$ ($\text{action of forms}$),

where $\Omega^*_\text{deloc,G}(B, \mathbb{R})$ denotes the set of closed delocalized differential forms such that

- (i) the following diagram commutes:
- (ii) it holds that $R \circ a = d$;

(iii) $a$ is of degree 1;

(iv) for $x, y \in \hat{K}_G^*(B)$ and $\alpha \in \Omega^*_\text{deloc,G}(B, \mathbb{R})/\text{Imd}$, we have $R(x \cup y) = R(x) \wedge R(y)$, $I(x \cup y) = I(x) \cup I(y)$, $a(\alpha) \cup x = a(\alpha \wedge R(x))$;

(v) the following sequence is exact:

$$K_G^{*-1}(B) \xrightarrow{\text{ch}_G} \Omega^*_{\text{deloc,G}}(B, \mathbb{R})/\text{Imd} \xrightarrow{a} \hat{K}_G^*(B) \xrightarrow{I} K_G^*(B) \to 0.$$  

Proof. We define the natural transformation

$I: \hat{K}_G^*(B) \to K_G^*(B)$

by

$I([\mathcal{F}, \rho]) := \text{Ind}(D(\mathcal{F})).$

From Definition 4.3, the transformation $I$ is well defined.

Let $a$ be a parity-reversing natural transformation

$a: \Omega^*_\text{even/odd}(B, \mathbb{R})/\text{Imd} \to \hat{K}_G^{-1,0}(B)$

by

$a(\rho) := [\emptyset, -\rho],$

where $\emptyset$ is the empty geometric family.

We define a transformation

$R: \hat{K}_G^*(B) \to \Omega^*_\text{deloc,G,cl}(B, \mathbb{R})$

by

$R((\mathcal{F}, \rho)) := \text{FLL}_G(\mathcal{F}) - d\rho.$
If $(\mathcal{F}', \rho')$ is paired with $(\mathcal{F}, \rho)$, there exists a perturbation operator $A$ with respect to $D(\mathcal{F} + \mathcal{F}^{op})$ such that

$$\rho - \rho' = \tilde{\eta}_G(\mathcal{F} + \mathcal{F}^{op}, A).$$

From (3.66) and (4.17), we have

$$R((\mathcal{F}, \rho)) = \text{FLI}_G(\mathcal{F}) - d\rho = \text{FLI}_G(\mathcal{F}) - d\rho' - d\tilde{\eta}_G(\mathcal{F} + \mathcal{F}^{op}, A) = \text{FLI}_G(\mathcal{F}) - d\rho' - \text{FLI}_G(\mathcal{F}') + \text{FLI}_G(\mathcal{F}') = R((\mathcal{F}', \rho')).$$

(4.52)

Since $R$ is additive, it descends to $\hat{\text{IC}}_G^*(B)/\sim$ and finally to the map $R: \hat{K}^*_G(B) \rightarrow \Omega_{\text{deloc}, G, cl}^*(B, \mathbb{R})$. Let $f: B_1 \rightarrow B_2$ be a $G$-equivariant smooth map. It follows from $\text{FLI}_G(f^*\mathcal{F}) = f^*\text{FLI}_G(\mathcal{F})$ that $R$ is natural.

From (4.49) and (4.51), we have

$$R \circ a = d. \quad (4.53)$$

By (3.51), the diagram commutes.

The formulas in (4.44) follow from straight calculations by using the definitions.

At last, we prove the exactness of the sequence (4.45).

The surjectivity of $I$ follows from Proposition 2.6.

Next, we show the exactness at $\hat{K}^*_G(B)$. It is obvious that $I \circ a = 0$. For a cycle $(\mathcal{F}, \rho)$, if $I([\mathcal{F}, \rho]) = 0$, we have $\text{Ind}(D(\mathcal{F})) = 0$. By Example 2.5(b), we could take $\mathcal{F}$ such that at least one component of the fiber has the nonzero dimension. So there exists a perturbation operator $A$ with respect to $D(\mathcal{F})$ from Proposition 3.3. By (4.17) and (4.49), we have

$$[\mathcal{F}, \rho] = a(\tilde{\eta}_G(\mathcal{F}, A) - \rho). \quad (4.54)$$

Finally, we prove the exactness at $\Omega_{\text{deloc}, G, cl}^{*-1}(B, \mathbb{R})/\text{Ind}$. Following the same process in (4.28)–(4.31), for any $x \in K^*_G(B)$, by (4.49),

$$a \circ \text{ch}_G(x) = (\emptyset, \tilde{\eta}_G(\mathcal{F}, A_Q) - \tilde{\eta}_G(\mathcal{F}, A_P)) = [\mathcal{F}, 0] - [\mathcal{F}, 0] = 0. \quad (4.55)$$

If $a(\rho) = 0$, for any equivariant geometric family $\mathcal{F}$ with a perturbation operator $A$ with respect to $D(\mathcal{F})$, by Definition 4.3 and (4.54), we have

$$[\mathcal{F}, \tilde{\eta}_G(\mathcal{F}, A) - \rho] = a(\rho) = 0 = [\mathcal{F}, \tilde{\eta}_G(\mathcal{F}, A)]. \quad (4.56)$$

So by Definition 4.5, there exists another cycle $(\mathcal{F}', \rho')$ such that

$$(\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A) - \rho) \sim (\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A)).$$

Since $\sim$ is generated by "paired", we have the cycles $\{(\mathcal{F}, \rho_i)\}_{0 \leq i \leq r}$ such that for any $1 \leq i \leq r$, $(\mathcal{F}, \rho_i)$ is paired with $(\mathcal{F}_{i-1}, \rho_{i-1})$, $(\mathcal{F}_0, \rho_0) = (\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A) - \rho)$ and $(\mathcal{F}_r, \rho_r) = (\mathcal{F} + \mathcal{F}', \rho' + \tilde{\eta}_G(\mathcal{F}, A))$. By Definition 4.3, for any $1 \leq i \leq r$, there exists a perturbation operator $A_i$ with respect to $D(\mathcal{F}_{i-1} + \mathcal{F}_{i}^{op})$ such that

$$\rho_{i-1} - \rho_i = \tilde{\eta}_G(\mathcal{F}_{i-1} + \mathcal{F}_{i}^{op}, A_i).$$

Let $A_i' (0 \leq i \leq r)$ be the perturbation operator with respect to $D(\mathcal{F}_i + \mathcal{F}_i^{op})$ taken in (4.16). Therefore, by Theorem 3.17, (4.15) and (4.16), there exists $x \in K^*_G(B)$ such that

$$-\rho = \sum_{i=1}^{r} (\rho_{i-1} - \rho_i) = \tilde{\eta}_G \left( \mathcal{F}_0 + \mathcal{F}_1^{op} + \cdots + \mathcal{F}_{r-1} + \mathcal{F}_r^{op}, A_1 \bigcup_{B} \cdots \bigcup_{B} A_r \right)$$
\[ = \tilde{\eta}_G \left( F_0 + F^\text{op}_{r-1} + A_1 \bigsqcup B \cdots \bigsqcup B A_r \right) \]

\[ - \tilde{\eta}_G \left( F_0 + F^\text{op}_{r-1} + A'_1 \bigsqcup B \cdots \bigsqcup B A'_{r-1} \right) = \text{ch}_G(x). \] (4.57)

The proof of Theorem 4.12 is completed. \( \square \)

The direct extension of [16, Proposition 3.19 and Lemma 3.20] shows that the pullback map and the exact sequence (4.45) are compatible with the push-forward maps.

**Remark 4.13.** If the group \( G \) is trivial, all the models of differential K-theory are isomorphic (see, e.g., [17]). For the equivariant case, the uniqueness is an open question.

### 4.4 Differential K-theory for orbifolds

In [18], Bunke and Schick constructed the first model of the differential K-theory for orbifolds by using the language of stacks and proved the desired properties. It could be regarded as a model of the equivariant differential K-theory when the action has finite stabilizers. In the subsections above, inspired by the constructions in [16,46], we construct a model of the equivariant differential K-theory when the action has finite stabilizers. In this subsection, we will explain that this model could also be regarded as a model for orbifolds.

Let \( \mathcal{X} \) be a compact orbifold (effective orbifold in some literature). There exist a compact smooth manifold \( B \) and a compact Lie group \( G \) such that \( \mathcal{X} \) is diffeomorphic to a quotient for a smooth effective \( G \)-action on \( B \) with finite stabilizers (see [1, Theorem 1.23]).

Let \( K^0_{\text{orb}}(\mathcal{X}) \) be the orbifold \( K^0 \)-group of the compact orbifold \( \mathcal{X} \) defined as the Grothendieck ring of the equivalence classes of orbifold vector bundles over \( \mathcal{X} \). Since \( \mathcal{X} \) is an orbifold, \( \mathcal{X} \times S^1 \) is an orbifold. Moreover, \( i : \mathcal{X} \rightarrow \mathcal{X} \times S^1 \) is a morphism in the category of orbifolds. As in (2.19), we define the orbifold \( K^1 \)-group

\[ K^1_{\text{orb}}(\mathcal{X}) := \ker(i^* : K^0_{\text{orb}}(\mathcal{X} \times S^1) \rightarrow K^0_{\text{orb}}(\mathcal{X})). \]

Let \( p : B \rightarrow B/G = \mathcal{X} \) be the projection. Then from [1, Proposition 3.6], it induces an isomorphism \( p^* : K^*_{\text{orb}}(\mathcal{X}) \rightarrow K^*_{\text{orb}}(B) \). Note that if the orbifold \( \mathcal{X} \) can be presented in two different ways as a quotient, say, \( B'/G' \simeq \mathcal{X} \simeq B/G \), it shows that \( K^*_{\text{orb}}(B') \simeq K^*_{\text{orb}}(\mathcal{X}) \simeq K^*_{\text{orb}}(B) \). So we can consider the orbifold K-theory as a special case of the equivariant K-theory.

Furthermore, from the definition of the differential structure on orbifolds, we know that

\[ \Omega^*_\text{deloc},G(B, \mathbb{R})/\text{Ind} \simeq \Omega^*_\text{deloc},G(B', \mathbb{R})/\text{Ind}. \]

From the exact sequence in (4.45) and the five lemma, we have

\[ \tilde{K}^*_G(B') \simeq \tilde{K}^*_G(B). \] (4.58)

Therefore, this model of equivariant differential K-theory for the \( G \)-action with finite stabilizers could be regarded as a model of differential K-theory for orbifolds.

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Appendix A  Equivariant K-theory for smooth complex vector bundles

In this appendix, we show that for a compact Lie group $G$ and a smooth compact manifold $B$ with a smooth $G$-action, $K_0^G(B)$ in [48] defined by $G$-equivariant topological complex vector bundles could be studied by using the $G$-equivariant smooth complex vector bundles. Although this is certainly well known (see, e.g., [22, (2.1)]), we were unable to find an explicit proof in the literature. We state it here for the completeness following the suggestion of a referee. In this appendix, all the vector bundles are complex.

For a representation $V$ of $G$, for $v \in V$, if $Gv$ generates a finite-dimensional subspace of $V$, we say that $v$ is a $G$-finite vector in $V$.

Let $E$ be a $G$-equivariant smooth vector bundle over $B$. Take a Hermitian metric on $E$ and let $\| \cdot \|_{\mathcal{C}^0(B,E)}$ be the corresponding $\mathcal{C}^0$-norm. For $s \in \mathcal{C}^\infty(X,E)$, $gs \in \mathcal{C}^\infty(X,E)$. Thus for any $f \in \mathcal{C}^\infty(G)$,

$$s_f := \int_B f(g)gs \, dg \in \mathcal{C}^\infty(B,E),$$

(A.1)

where $dg$ is the Haar measure. Since $G$ acts continuously on $\mathcal{C}^\infty(B,E)$, for any $\varepsilon > 0$, there exists a neighborhood $U$ of the unity $e \in G$ such that for any $g \in U$, $\|gs - s\|_{\mathcal{C}^0(B,E)} < \varepsilon / 2$. Let $v \in \mathcal{C}^\infty(G)$ be a non-negative function vanishing outside $U$ with $\int_G v(g) \, dg = 1$. Then $\|s_v - s\|_{\mathcal{C}^0(B,E)} < \varepsilon / 2$. Let

$$M_s := \left\| \int_G gs \, dg \right\|_{\mathcal{C}^0(B,E)}.$$

From the Peter-Weyl theorem, there exists a $G$-finite vector $u \in \mathcal{C}^\infty(G)$ such that

$$\|v - u\|_{\mathcal{C}^0(G)} < \frac{\varepsilon}{2M_s}.$$

Thus, $\|s_v - s_u\| < \varepsilon / 2$. Observe that $s_u$ is a $G$-finite vector in $\mathcal{C}^\infty(B,E)$. We have the following lemma.

**Lemma A.1** (See [45, Subsection 2.16]). For $s \in \mathcal{C}^\infty(B,E)$ and any $\varepsilon > 0$, there exists a $G$-finite vector $s' \in \mathcal{C}^\infty(B,E)$ such that $\|s - s'\|_{\mathcal{C}^0(B,E)} < \varepsilon$.

For a finite-dimensional complex representation $M$ of $G$, we consider the $G$-action on $B \times M$ given by

$$g(b,u) = (gb, gu), \quad \forall g \in G, \quad b \in B, \quad u \in M.$$

(A.2)

Thus $B \times M \to B$ is an equivariant smooth vector bundle over $B$. In this case, a $G$-invariant Hermitian inner product on $M$ forms a $G$-invariant Hermitian smooth metric on this vector bundle. The following proposition extends [48, Proposition 2.4] to the category of $G$-equivariant smooth vector bundles.

**Proposition A.2.** Let $E$ be a $G$-equivariant smooth vector bundle over $B$. There exist a finite-dimensional complex representation $M$ of $G$ and a $G$-equivariant smooth vector bundle $F$ over $B$ such that $E \oplus F$ is isomorphic to $B \times M$ as $G$-equivariant smooth vector bundles.
Proof. It suffices to find an equivariant smooth surjection from some $B \times M$ to $E$. Then $F$ is the orthogonal complement of $E$ in $B \times M$.

For any $b \in B$, we can choose a finite set $\sigma_b \subset \mathcal{C}^\infty(B, E)$ such that $\{s(b)\}_{s \in \sigma_b}$ spans $E_b$. From Lemma A.1, we can choose $\sigma_b$ such that it consists of $G$-finite vectors in $\mathcal{C}^\infty(B, E)$. There exists a neighborhood of $b$, $U_b$, such that for any $x \in U_b$, $\{s(x)\}_{s \in \sigma_x}$ spans $E_x$. Suppose that $U_{b_1}, \ldots, U_{b_m}$ covers $B$. Let $\sigma = \bigcup \sigma_b$. Let $M$ be the finite-dimensional subspace of $\mathcal{C}^\infty(B, E)$ generated by $\sigma$. Then the evaluation map $B \times M \to E$ is the required surjection. □

Lemma A.3 (Compare with [30, Theorem 3.5]). For every $G$-equivariant topological vector bundle $E$ over $B$, there exists a $G$-equivariant smooth vector bundle $E^s$ over $B$, which is unique up to isomorphism of $G$-equivariant smooth vector bundles, such that $E^s$ is isomorphic to $E$ as $G$-equivariant topological vector bundles.

Proof. By [48, Proposition 2.4], the $\mathcal{C}^0$-version of Proposition A.2, there exist a finite-dimensional complex representation $M$ of $G$ and an equivariant embedding $i : E \to B \times M$. Let $Gr_{M,r}$ be the Grassmannian parameterizing all the complex linear subspaces of the finite-dimensional complex representation $M$ of given dimension $r$. Since $G$ acts linearly on $M$, there is an induced smooth $G$-action on the smooth manifold $Gr_{M,r}$. Let $r$ be the rank of $E$. Define the continuous map $h : B \to Gr_{M,r}$ by $h(b) := i(E_b) \in Gr_{M,r}, \forall b \in B$. Since $i$ is equivariant, $h$ is an equivariant map. Let $\gamma_{M,r}$ be the universal bundle over $Gr_{M,r}$, which is an equivariant smooth vector bundle over $Gr_{M,r}$. Then $\gamma_{M,r}^* \gamma_{M,r}$ is isomorphic to $E$ as $G$-equivariant topological vector bundles.

For the equivariant continuous map $h : B \to Gr_{M,r}$, there exists an equivariant smooth map

$$h_G : B \to Gr_{M,r},$$

which is $G$-homotopic to $h$ continuously (see, e.g., [13, Theorem VI.4.2]). So $E^s := h_G^* \gamma_{M,r}$ is a $G$-equivariant smooth vector bundle which is isomorphic to $h^* \gamma_{M,r} \simeq E$ as $G$-equivariant topological vector bundles.

For two $G$-equivariant smooth vector bundles $E_1^s$ and $E_2^s$, which are isomorphic as $G$-equivariant topological vector bundles, there exist $G$-equivariant smooth maps $h_i : B \to Gr_{M,r}, i = 1, 2$ such that $E_i^s = h_i^* \gamma_{M,r}$ and $h_1$ is $G$-homotopic to $h_2$ continuously. Since $G$ is compact, $h_1$ is $G$-homotopic to $h_2$ smoothly (see, e.g., [13, Corollary VI.4.3]). Thus $E_1^s$ is isomorphic to $E_2^s$ as $G$-equivariant smooth vector bundles.

The proof of Lemma A.3 is completed. □

Proposition A.4. Let $K_{G, sm}^0(B)$ be the Grothendieck group of the $G$-equivariant smooth vector bundles over $B$. We have

$$K_{G, sm}^0(B) \simeq K_G^0(B).$$

(3.3)

Proof. Forgetting the smooth structure, we obtain a well-defined map $A : K_{G, sm}^0(B) \to K_G^0(B)$.

Let $\text{Vect}_G(B)$ and $\text{Vect}_{G, sm}(B)$ be the equivalence classes of $G$-equivariant topological and smooth vector bundles over $B$. Then Lemma A.3 induces a well-defined map $\text{Vect}_G(B) \to \text{Vect}_{G, sm}(B)$. For $E_1$ and $E_2$ in $\text{Vect}_G(B)$, if $[E_1] = [E_2] \in K_G^0(B)$, there exists a topological vector bundle $F$ such that $E_i \oplus F$ is isomorphic to $E_2 \oplus F$ as $G$-equivariant topological vector bundles. Let $E_1^s$, $E_2^s$ and $F^s$ be the corresponding $G$-equivariant smooth vector bundles. Since $E_1^s \oplus F^s$ is isomorphic to $E_2^s \oplus F^s$ as $G$-equivariant topological vector bundles, from the uniqueness in Lemma A.3, they are isomorphic as $G$-equivariant smooth vector bundles. Thus we get a well-defined map $B : K_G^0(B) \to K_{G, sm}^0(B)$.

The proof of Proposition A.4 is completed. □

Appendix B The equivariant family index for odd-dimensional fibers

In this appendix, we summarize some results on the equivariant family index for odd-dimensional fibers and the explanations for $K_G^0(B)$ (see [4, 26, 44]).
We consider the equivariant \( \mathbb{Z}_2 \)-graded Hilbert bundle \( \mathcal{E} \) with the fiber \( L^2(Z_b, E) \) over \( b \in B \). From [26, Lemma A.32], there exists an equivariant embedding from \( \mathcal{E} \) to the equivariant trivial Hilbert bundle \( B \times L^2(G) \otimes C(\mathbb{R}) \otimes H \), where \( C(\mathbb{R}) \) is the complex Clifford algebra and \( H \) is a separable Hilbert space. As in [26, Definitions A.39 and A.40], for any equivariant \( \mathbb{Z}_2 \)-graded Hilbert space \( \mathcal{H} \), we denote by \( \text{Fred}^0(\mathcal{H}) \) the space of odd skew-adjoint equivariant Fredholm operators \( A \), for which \( A^2 + 1 \) is compact, topologized as a subspace of \( B(\mathcal{H}) \times K(\mathcal{H}) \), where \( B(\mathcal{H}) \) and \( K(\mathcal{H}) \) are the sets of bounded linear operators and compact operators on \( \mathcal{H} \) given the compact-open topology and the norm topology, respectively. Denote by \( \text{Fred}^1(\mathcal{H}) \) the subspace of \( \text{Fred}^0(C(\mathbb{R}) \otimes \mathcal{H}) \) consisting of odd operators \( A \), which supercommute with the action of \( C(\mathbb{R}) \) and for which the essential spectrum of \( -\sqrt{-1}c(e)A \) contains both positive and negative eigenvalues, where \( c(e) \) is the basis element of \( C(\mathbb{R}) \). By [26, Subsection 3.5.4], \( K^0_G(B) \) is realized as the space of \( G \)-homotopy classes of \( G \)-equivariant maps from \( B \) to \( \text{Fred}^1(L^2(G) \otimes H) \):

\[
K^0_G(B) \cong \{ B, \text{Fred}^1(L^2(G) \otimes H) \}_G.
\]  

(B.1)

Let \( T = D(\mathcal{F})/(1 + D(\mathcal{F})^2)^{1/2} \). Then \( T \) is bounded, \( G \)-equivariant and \( \text{Ind}(T) = \text{Ind}(D(\mathcal{F})) \in K^0_G(B) \). Moreover, \( \sqrt{-1}T \) can be extended to an equivariant map from \( B \) to \( \text{Fred}^1(L^2(G) \otimes H) \) by taking the identity map on the complement of \( \mathcal{E} \) in \( B \times L^2(G) \otimes C(\mathbb{R}) \otimes H \). By [26, Subsection 3.5.4 and Proposition A.41], \( \text{Ind}(D(\mathcal{F})) = \text{Ind}(T) \in K^0_G(B) \) corresponds to the element of

\[
K^0_G \left( B \times \left( 0, \frac{1}{2} \right) \right) \cong \left[ B \times \left( 0, \frac{1}{2} \right), \text{Fred}^0(L^2(G) \otimes H) \right]_G
\]
given by

\[
D(\theta) = \cos(2\pi \theta) + \sqrt{-1}T \sin(2\pi \theta), \quad \theta \in \left( 0, \frac{1}{2} \right).
\]  

(B.2)

Here, \( \text{Fred}^0(L^2(G) \otimes H) \) consists of the elements in \( \text{Fred}^0(L^2(G) \otimes H) \) which is invertible outside a compact set of the parameter space (see also [4, p.6] and [44, (3.1)]). By applying the natural inclusion \( K^0_G(B \times (0, \frac{1}{2})) \rightarrow K^0_G(B \times S^1) \), we obtain an element of \( K^0_G(B \times S^1) \) which lies in the image of \( j \) in (2.19), and thus an element of \( K^0_G(B) \).

The following proposition is the equivariant version of [44, Proposition 6]. We prove it here using the notation in Example 2.5(d) for the completeness.

**Proposition B.1.** For \( \mathcal{F} \in F^1_G(B) \), there exists an inclusion \( i : B \rightarrow B \times S^1 \) such that

\[
i^* \text{Ind}(D(p^*_b \mathcal{F} \times_{B \times S^1} p^*_2 \mathcal{F}^L)) = 0.
\]

Moreover, as an element of \( K^0_G(B) \), we have

\[
j(\text{Ind}(D(\mathcal{F}))) = \text{Ind}(D(p^*_b \mathcal{F} \times_{B \times S^1} p^*_2 \mathcal{F}^L)).
\]  

(B.3)

**Proof.** In order to compare our definition with that in [4], we replace the connection in (2.28) by

\[
\nabla^L = d + 2\pi(\theta - 1/4)\sqrt{-1}dt.
\]

Since from (2.4) and (2.5),

\[
D(p^*_b \mathcal{F} \times_{B \times S^1} p^*_2 \mathcal{F}^L) = D(\mathcal{F}) \otimes J + D(\mathcal{F}^L) \otimes K,
\]  

(B.4)

the index on the right-hand side of (B.3) does not vary in \( K^0_G(B \times S^1) \) after the replacement. Then we could calculate that for \( \theta \in [0, 1) \), the eigenvalues of \( D(\mathcal{F}^L) \) are

\[
\left\{ \lambda_k = 2\pi k + 2\pi \left( \theta - \frac{1}{4} \right) \right\}_{k \in \mathbb{Z}}
\]
and the eigenspace of $\lambda_k$ is one-dimensional for any $k \in \mathbb{Z}$. Let $s$ be a local frame of $L$. The eigenfunction of $\lambda_k$ is $v_k(t) = \exp(2\pi k \sqrt{-1} t) s$. From (B.4),

$$D(p_1^* F \times_{B \times S^1} p_2^* F^L)^2 = (D(F)^2 + D(F^L)^2) \otimes \text{Id}_{C^2}.$$ 

So it is invertible if and only if $\theta \neq 1/4$. Thus for the inclusion $i : B \to B \times S^1$ and $i(B) = B \times \{1/2\}$,

$$i^* \text{Ind}(D(p_1^* F \times_{B \times S^1} p_2^* F^L)) = 0.$$ 

Fix $b \in B$. Since $\mathcal{S}_{S^1 \times Z} = \mathcal{S}_{S^1} \otimes \mathcal{S}_Z \otimes \mathbb{C}^2$, we have

$$L^2(S^1_1 \times Z_b, \mathcal{S}_{S^1 \times Z} \hat{\otimes} L \hat{\otimes} E) = ' \bigoplus_{k \in \mathbb{Z}} \mathbb{C}v_k(t) \otimes L^2(Z_b, \mathcal{S}_Z \hat{\otimes} E) \otimes \mathbb{C}^2. \quad (B.5)$$

Here, $' \oplus$ stands for the direct sum in the category of Hilbert spaces. If $k \neq 0$,

$$D(p_1^* F \times_{B \times S^1} p_2^* F^L)^2 |_{\mathbb{C}v_k(t) \otimes L^2(Z_b, \mathcal{S}_Z \hat{\otimes} E) \otimes \mathbb{C}^2} > 0.$$ 

Let $H_+ = \mathbb{C}v_0(t) \otimes L^2(Z_b, \mathcal{S}_Z \hat{\otimes} E) \otimes (\mathbb{C} \oplus \{0\})$ and $H_- = \mathbb{C}v_0(t) \otimes L^2(Z_b, \mathcal{S}_Z \hat{\otimes} E) \otimes (\{0\} \oplus \mathbb{C})$. Then

$$\text{Ind}(D(p_1^* F \times_{B \times S^1} p_2^* F^L)) = \text{Ind}(D(p_1^* F \times_{B \times S^1} p_2^* F^L))_+ : H_+ \to H_- \quad (B.6)$$

We define the isomorphisms

$$\phi_+ : H_+ \to L^2(Z_b, \mathcal{S}_Z \hat{\otimes} E) \quad \text{and} \quad \phi_- : H_- \to L^2(Z_b, \mathcal{S}_Z \hat{\otimes} E)$$

by $\phi_+(v_0(t) \otimes l \otimes (1,0)) = l$ and $\phi_-(v_0(t) \otimes l \otimes (0,1)) = l$, respectively. Set

$$D_+ := \phi_- \circ D(p_1^* F \times_{B \times S^1} p_2^* F^L)_+ \circ \phi_+^{-1}.$$ 

Then on $L^2(Z_b, \mathcal{S}_Z \hat{\otimes} E)$, from (2.4) and (B.4), we have

$$D_+ = \sqrt{-1} D(F) + \lambda_0(\theta). \quad (B.7)$$

From (B.2) and (B.7), for $\theta \in (0, 1/2)$, we have

$$\text{Ind}(D(p_1^* F \times_{B \times S^1} p_2^* F^L)) = \text{Ind} \left( \frac{D(p_1^* F \times_{B \times S^1} p_2^* F^L)_+}{\sqrt{1 + D(p_1^* F \times_{B \times S^1} p_2^* F^L)^2}} : H_+ \to H_- \right)$$

$$= \text{Ind} \left( \frac{D_+}{\sqrt{1 + \lambda_0(\theta)^2 + D(F)^2}} \right) = \text{Ind} \left( \frac{\lambda_0(\theta) + \sqrt{-1} D(F)}{\sqrt{1 + \lambda_0(\theta)^2 + D(F)^2}} \right)$$

$$= \text{Ind}(D(\theta)). \quad (B.8)$$

Since $D(p_1^* F \times_{B \times S^1} p_2^* F^L)$ and $D(\theta)$ are invertible for $\theta \neq 1/4$, we obtain Proposition B.1. \qed