Influence of small-scale $E_M$ and $H_M$ on the growth of large-scale magnetic field

Kiwan Park*

Department of Physics and Astronomy, Ulsan National Institute of Science and Technology, Ulsan 689-798, South Korea

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ABSTRACT

We investigated the influence of small-scale magnetic energy ($E_M$) and magnetic helicity ($H_M$) on the growth rate ($\gamma$) of $B$ field (large-scale magnetic field). $H_M$ that plays a key role in magnetohydrodynamic (MHD) dynamo is a topological concept describing the structural properties of magnetic fields. Since $E_M$ is a prerequisite of $H_M$, it is not easy to differentiate the intrinsic properties of $H_M$ from the influence of $E_M$. However, to understand MHD dynamo, the features of helical and non-helical magnetic field should be made clear. For this, we made a detour: we gave each simulation set its own initial condition (IC, same $E_M(0)$ and specific $H_M(0)$ at the forced wavenumber $k_f = 5$), and then drove the system with positive helical kinetic energy ($k_f = 5$). According to the simulation results, $E_M(0)$, whether or not helical, increases the growth rate of $\mathbf{B}$. The positive $H_M(0)$ boosts the increased growth rate, but the negative $H_M(0)$ decreases it. To explain these results, two coupled equations of $H_M$ and $E_M$ were derived and solved using a simple approximate method. The equations imply that helical magnetic field evolves into the total (helical and non-helical) magnetic field but quenches itself. Non-helical magnetic field also evolves into the total magnetic field but quenches itself. The initially given $E_M(0)$ modifies the electromotive force (EMF, $\langle u \cdot b \rangle$) and generates new terms. The effects of these terms depend on the magnetic diffusivity $\eta$, position of initial conditions $k_f$, and magnetic diffusion time. But the influence disappears exponentially as time passes, so the saturated magnetic fields are eventually independent of the pre-existing initial conditions.

Key words: dynamo – MHD – plasmas – turbulence – methods: analytical – ISM: magnetic fields.

1 INTRODUCTION

The evolutions of magnetic fields such as generation, amplification (dynamo), and annihilation (reconnection) are commonly observed in most celestial phenomena which include the interactions between magnetic fields and conducting fluids. The kinetic energy in the plasma motion can be transferred into the magnetic energy (dynamo), and this energy cascades towards smaller scale eddies and grows (small-scale dynamo), or cascades towards larger scale ones and grows (large-scale dynamo). In magnetohydrodynamic (MHD) dynamo, the role of helical kinetic motion (kinetic helicity, ($u \cdot \omega$), $\omega = \nabla \times u$) is relatively clear: it generates the magnetic energy (helicity) and cascades the energy (helicity) to the larger-scale magnetic eddies. However, the physical role and meaning of helical magnetic field (magnetic helicity, $H_M \equiv \langle A \cdot B \rangle$, $B = \nabla \times A$) are not yet fully understood.

$H_M$ is defined as a topological measure of twist and linkage of magnetic field lines ($2\Phi_1 \Phi_2$, $\Phi = \int_A \mathbf{B} \cdot dS$; Moffatt 1978; Brown, Canfield & Pevtsov 1999) in the minimum state of energy equilibrium. Helical magnetic field is called “force-free field” because it makes Lorentz force ($J \times B$) zero (Biskamp 2003). So in principle, the motion of kinetic eddy in MHD system is not influenced by the maximally helical magnetic field. $H_M$ is also related to the particle resonant scattering in the interplanetary magnetic fields when the handedness of helical magnetic field is the same as that of helical motion of a particle (Brown et al. 1999). Like magnetic energy($E_M$), $H_M$ is conserved in ideal plasmas. Increasing large-scale magnetic helicity leads to the generation and cascade of oppositely signed magnetic helicity towards smaller scale in an MHD system. For example, quickly grown $H_M$ in the sun is ejected into the solar wind, and the equal amount of oppositely signed $H_M$ is simultaneously generated and stays there. In helical MHD turbulence dynamo model $H_M$ in the small scale, more exactly, current helicity ($j \cdot b$)($= k^2 H_M$), $j = \nabla \times A \rightarrow k^2 a_k$ in Fourier space) is thought to constrain the growth of $\mathbf{B}$ fields (Blackman & Field 2002).

The quenching effect of small-scale $H_M$ is from the definition of $\alpha$ coefficient ($\sim \langle j \cdot b \rangle - (u \cdot \omega)$) with the conservation and redistribution of $H_M$ in the system. However, strictly speaking,
(large or small) magnetic helicity and kinetic helicity are the solutions of MHD equations. This implies that an external driving force, viscosity(ν) or diffusivity(η) of the coupled MHD equations, decides the solutions ‘u’ and ‘b’. Namely, the change of these intrinsic features in the system leads to the change of solutions and the subordinate results.

We are interested in the unique roles of helical and non-helical magnetic energy and their relation in MHD dynamo. But, it is not easy to answer these questions because magnetic helicity assumes the existence of magnetic energy. In fact, \( E_M \) can have arbitrary \( |H_M| \) as long as the realizability relation \( 2E_M \geq |H_M| \); Frisch et al. (1975) is satisfied. So, we look for another indirect way to investigate \( E_M \) and \( H_M \) in MHD dynamo.

Before we go further, we need to make clear the statistical meaning of \( H_M \). The correlation \( \langle B_i(k)B_j(-k) \rangle \) can be represented by two invariants \( E_M \) and \( H_M \) like (Lesieur 1987; Yoshizawa 2011; Park 2013)

\[
\langle B_i(k)B_j(-k) \rangle = \frac{E_M(k)}{4\pi k^2} + \frac{i\epsilon_{ijk}k_j}{8\pi k^2} H_M(k).
\]

(1)

In a homogeneous and isotropic (reflectionally symmetric) system, only the trace \( \langle B_i(k)B_i(-k) \rangle \) (\( \sim E_M \)) survives. Off-diagonal term \( \langle B_i(k)B_j(-k) \rangle \) (\( i \neq j \)) which is related to \( H_M \) does not exist. This means that the second-order correlation independent of translation and rotation (including reflection symmetry) can be described by the invariant variable \( E_M \) (Robertson 1940). Actually most of the turbulence theories accept this assumption, and \( E_M \) is used to describe the correlation \( \langle B_i(k)B_j(-k) \rangle \). However, a system with such a strict condition is not common in nature. If there is a rotation, although the system is still isotropic, the reflection symmetry is broken so that \( \langle B_i(k)B_j(-k) \rangle \) cannot be ignored. This off-diagonal term can be described by another invariant quantity, helicity. This formula implies that helical fields are essentially related to the statistical correlations between ‘B’ and ‘B’ in an isotropic system. For example, current helicity \( (J \cdot B) = \int (J(k) \cdot B(-k)) \, dk \) cannot exist without the off-diagonal correlation

\[
(J \cdot B) = \int \varepsilon_{ijk} k_j \langle B_i(k)B_j(-k) \rangle \, dk.
\]

(2)

But strictly speaking, equation (1) is a description of the second correlation tensor rather than a conservation law. Although \( E_M \) is described as a trace in this formula, it can include helical magnetic energy \( (kH_M/2) \) and non-helical magnetic energy \( (E_M - kH_M/2) \).

2 PROBLEM TO BE SOLVED AND METHODS

The main aim of this paper is to figure out the effect of initial conditions \( (H_M(0) \) and \( E_M(0) \)) in small scale on the growth of large-scale MHD dynamo. Pouquet, Frisch & Leorat (1976) derived the equations of \( E_M, H_M, E_k \) (kinetic energy), and \( H_k \) (kinetic helicity) using Eddy-Damped Quasi-Normal Markovian (EDQNRM). The results show the features of the variables and explain how the inverse cascade of \( E_M \) and \( H_M \) with \( \alpha \) coefficient occurs. But the physical difference between \( E_M \) and \( H_M \) in MHD dynamo is not clearly shown. Driving the system with the mixed helical and non-helical kinetic energy, Maron & Blackman (2002) tried to see the effects of various helicity ratios. The results show the mixed effect of partially helical and non-helical kinetic energy, but the influence of \( H_M \) or \( E_M \) on MHD dynamo is not shown. In Park (2013) and Park, Blackman & Subramanian (2013), it was shown that \( H_M(0) \) and \( E_M(0) \) in the large scale boosted the generation of \( \mathbf{B} \) field. But the work was chiefly focused on the influence of \( E_M(0) \). So, we need more detailed analytic and experimental work which can show the effect of \( H_M \) and \( E_M \). For this purpose, we prepared some simulation sets. Magnetic energy \( E_M \) with a fractional viscosity (fkm) drove a system as a precursor simulation \( (k_i = 5, t < 0.005, \) one simulation step) to generate \( E_M(0) \) and \( H_M(0) \) in the system. Then fully helical kinetic energy \( (fkh = 1.0) \) was injected into the kinetic eddy at \( k_i = 5 \) (helical kinetic forcing, HKF) to drive the system as a main simulation.

All simulations were done with the high-order finite-difference \textsc{pencil} code (Brandenburg 2001) and the mesh passing interface in a periodic box of spatial volume \( (2\pi a)^3 \) with mesh size 2563. The basic equations solved in the code are

\[
\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u},
\]

(3)

\[
\frac{Du}{Dt} = -c_s^2 \nabla \ln \rho + \frac{J \times B}{\rho} + \nu \left( \nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right) + \mathbf{f},
\]

(4)

\[
\frac{dA}{dt} = \mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B} + \mathbf{f}_{pre}.
\]

(5)

\( \rho \): density; \( \mathbf{u} \): velocity; \( \mathbf{B} \): magnetic field; \( \mathbf{A} \): vector potential; \( J \): current density; \( D/ Dt = (\partial/ \partial t + \mathbf{u} \cdot \nabla) \): advective derivative; \( \eta \): magnetic diffusivity \( (=c_s^2/4\pi \sigma) \); \( \sigma \): conductivity; \( \nu \): kinematic viscosity \( (=\mu/\rho) \); \( \mu \): viscosity; \( c_s \): sound speed. Velocity is expressed in units of \( c_s \) and magnetic fields in units of \( (\rho_0 \mu_0)^{1/2} c_s \) \( |B| = \sqrt{\rho_0 |\mu_0|} |\mathbf{v}| \) from \( E_M \sim B^2/\mu_0 \) and \( E_{kin} \sim \rho_0 u^2 \). \( \mu_0 \) is magnetic permeability and \( \rho_0 \) is the initial density. Note that \( \rho_0 \sim \rho \) in the weakly compressible simulations. These constants \( c_s, \mu_0, \) and \( \rho_0 \) are set to be ‘1’. In the simulations, \( \eta \) and \( v \) are 0.006. To force the magnetic eddy, the forcing function \( f_{pre}(x, t) \) is placed at equation (5) first, and then \( \mathbf{f} \) is placed at equation (4) to drive the momentum equation (HKF). Magnetic forcing \( f_{pre}(x, t) \) used here is artificial to make IC. However, \( f_{pre}(x, t) \) physically represents a variable having the dimension of electric field in Ohm’s law \( J = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \mathbf{f} \). For example, a variable like electromagnetic (EM) wave that affects the current density in Ohm’s law \( \mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B} + \mathbf{f}) \). The amplitude of magnetic forcing function \( f_{pre}(x, t) \) was 0.01 with various magnetic helicity ratios modifying \( \mathbf{f}_{pre} \) in \( \text{HMF, f}_{0(0)} \) of HKF was 0.07. The variables in the \textsc{pencil} code are independent of a unit system. For example, if the length of cube box \( L \) is 2\pi and \( u_{\text{rms}} \sim 0.2 \) after \( t = 3 \), these can be interpreted as \( L = 2\pi \text{ m} \), \( u_{\text{rms}} \sim 0.2 \text{ m s}^{-1} \), \( t = 3 \text{ s} \), or \( L = 2\pi \text{ pc} \), \( u_{\text{rms}} \sim 0.2 \text{ pc Myr}^{-1} \), \( t = 3 \text{ Myr} \). And for the theoretical analysis, we use semi-analytic and statistical methods. The equations of \( E_M \) and \( H_M \) with the solutions are derived again using an approximation like first-order smoothing approximation (Moffatt 1978).
one. This positive $H_M$ is thought to be caused by the tendency of conserving $H_{M,0}$ against the injected negative $H_M$. However, if the magnitude of negative $H_{M}(0)$ is large ($\operatorname{fhm} = -1$) or $E_M$ is not so large (reference HKF), $H_{M,L}$ does not change its sign.

Fig. 2(d) shows the evolving profiles of $H_M$ at $k = 1, 2, 5$. $H_M$ at $k = 1, 2$ are negative. But $H_M$ of $k = 5$ turns into a positive value regardless of the sign of $H_{M}(0)$, which is due to the backreaction of the larger scale magnetic field. While $H_M(0)$ at $k = 5$ is positive (but $H_M$ at $k = 1, 2$ is negative), $H_M(0)$ decreases faster than the negative $H_{M}(0)$. And this fast decrease of $H_M$ boosts the growth of $|H_M|$ at $k = 1, 2$. In contrast, for the negative $H_{M}(0)$ at $k = 5$, the injected (negative) $H_M$ mitigates the decreasing speed of $|H_{M}|$ at $k = 5$ and growing speed of $|H_{M}|$ at $k = 1, 2$. Similarly for $\operatorname{fhm} = 0$, $H_M$ first drops. However, as the magnitude of $\overline{B}$ field grows, the diffusion of positive $H_M$ from large scale makes $H_M$ at $k = 5$ to grow to be positive.

Fig. 2(e) includes the evolving profiles of non-helical $E_M$ ($E_M = -k|H_M|/2$). Larger $H_M(0)$ at $k = 5$ leads to the larger growth rate of non-helical $E_M$ at $k = 1, 2$. And when the diffusion of energy from larger scale grows, the flat profile (in non-helical $E_M$) at $k = 5$ shows up ($-2 < t < \sim 7$). And then the profiles of $k = 5$ for each case evolve together independent of the different evolutions of large-scale $B$ fields for a while ($t < -20$). The profiles of $E_M$ at $k = 2$ also show the similar, but short pattern.

Fig. 2(f) includes the growth rates of large-scale $B$ field for $\operatorname{fhm} = 1.0, -1.0$, and the reference simulation. The positive $H_M(0)$ causes the highest $\gamma$ in the early-time regime. Also, the comparison of growth rate between $\operatorname{fhm} = -1$ and reference HKF implies that $E_M(0)$ is a more important factor than $H_M(0)$ in MHD dynamo.

4 Analytic solutions to $H_M$ and $E_M$

For the analytic approach, we use more simplified equations than equations (3)–(5). If we combine Faraday's law $\partial B/\partial t = -cV \times E$ and Ohm's law $J = \sigma(E + 1/\epsilon U \times B)$, we get the magnetic induction equation

$$ \frac{\partial B}{\partial t} = \nabla \times (U \times B) + \eta \nabla^2 B. $$

All variables can be split into the mean (or large scale) and fluctuating values like $U = \overline{U} + u$ (denoted $u$ ($\overline{U} = 0$, Galilean transformation) and $B = \overline{B} + b$. Then, the magnetic induction equation for $\overline{B}$ field becomes (Krause & Rädler 1980)

$$ \frac{\partial \overline{B}}{\partial t} = \nabla \times (u \times b) + \eta \nabla^2 \overline{B} $$

$$ \sim \nabla \times u \overline{B} + (\eta + \beta) \nabla^2 \overline{B}. $$

[Here, the electromotive force EMF $(u \times b)$ was replaced by $\alpha \overline{B} - \beta \nabla \times \overline{B}$ (ansatz). $\alpha = 1/3 \int ((j \cdot b) - (u \cdot \omega)) \, \mathrm{d}t$, $\beta = 1/3 \int (u^2) \, \mathrm{d}t$ (Moffatt 1978). The helicity terms in ‘$\alpha$’ indicate that the MHD system is isotropic without the reflection symmetry.] $E_M(t)$ or $H_M(t)$ can be derived using EDQNM (Pouquet et al. 1976), but the same equations can be derived using a mean field method (Park & Blackman 2012a,b). With equation (8), we get $\partial H_M/\partial t$ (Krause & Rädler 1980; Blackman & Field 2002):
Figure 2. (a) The plots of $E_{M,L}$ (large-scale magnetic energy $k = 1$) with various $H_M(0)$ at $k_t = 5$. (b) $E_M$ at $k = 1$, 2, and 5. Thin lines indicate $E_M$ with negative $H_M(0)$ ($fhm = -1$), thick lines are for $E_M$ with zero $H_M(0)$ ($fhm = 0$), and thicker lines are for $E_M$ with positive $H_M(0)$ ($fhm = +1$). The small box includes the magnified plots of $E_M$. (c) $|H_M, L|/2 (H_M, L$: large-scale magnetic helicity) with various $H_M(0)$. $H_M, L$ is negative when the system is driven by the positive $\langle u \cdot \omega \rangle$, but it is positive (thick lines) at $0.3 < t < 8$. (d) $H_M$ of $k = 1, 2$, and 5. (e) Non-helical $E_M (k=5) - HKF(k=5)$. Thin line is for $fhm = -1$, thick line is for $fhm = 0$, and thicker line is for $fhm = 1$. (f) Growth rate $\gamma (d \log E_m/dt)$.

Considering that helicity is a pseudo-scalar, this equation in Fourier space can be represented like

$$\frac{\partial}{\partial t} H_{M,L} = 4\alpha E_{M,L} - 2k^2(\beta + \eta)H_{M,L} \quad (k = 1). \quad (10)$$

(Here, $H_{M,L}$ and $E_{M,L}$ mean the large-scale ($k = 1$) magnetic helicity and energy in Fourier space.)

Also $\partial E_{M,L}$ can be derived from equation (8):

$$\frac{\partial}{\partial t} \left( \frac{1}{2} |\mathbf{B}|^2 \right) = \left( \mathbf{B} \cdot \nabla \times \mathbf{E} \right) - \frac{c}{\sigma} \left( \mathbf{B} \cdot \nabla \times \mathbf{J} \right)$$

$$= \left( \alpha \mathbf{B} \cdot \nabla \times \mathbf{B} \right) - \left( \beta \nabla \times \mathbf{B} \cdot \nabla \times \mathbf{B} \right) - \frac{c}{\sigma} \left( \mathbf{J} \cdot \nabla \times \mathbf{B} \right). \quad (11)$$

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In Fourier space,
\[
\frac{\partial}{\partial t} E_{M, l}(t) = \alpha k^2 (\mathbf{A} \cdot \mathbf{B})_k - k^2 (\beta + \eta)(\mathbf{B}^2)_k
\]
\[= \alpha k^2 H_{M, l}(t) - 2k^2 (\beta + \eta) E_{M, l}(t) \quad (k = 1). \tag{12}\]

\(\mathbf{B}\) or \(E_{M, l}\) itself includes the helical and non-helical part, but the non-helical one is excluded in \((\mathbf{A} \cdot \mathbf{B})\) or \((\mathbf{B} \cdot \nabla \times \mathbf{B})\).

Helical magnetic field in small scale constrains the growth of \(\mathbf{B}\) field, and non-helical magnetic field \((\sim E_{M, l} - k H_{M, l}/2)\) restricts the plasma motion through Lorentz force \((\mathbf{J} \times \mathbf{B}) \approx (\mathbf{B} \cdot \nabla \mathbf{B} - \nabla (\mathbf{B}^2)/2)\). Equations (9) and (11) show additional relations between \(H_{M, l}\) and \(E_{M, l}\) : the growing correlation \((\mathbf{B}, \mathbf{B})_l\) leads to the increase of \(E_{M, l}\), and growing \(E_{M, l}\) increases the correlation \((\mathbf{B}, \mathbf{B})_l\), but at the same time, the dissipation effect of \(E_{M, l}\) \((H_{M, l})\) grows with increasing \(E_{M, l}\) \((H_{M, l})\). Besides, the initially given magnetic energy at \(k = 5\) as IC interacts with small-scale magnetic and kinetic field to produce a couple of additional terms in the EMF. These terms modify the growth rate of \(\mathbf{B}\) field whether the field is helical or not.

\[
\frac{\partial E_{M, l}}{\partial t} \approx \frac{\partial}{\partial t} [H_{M, l}(t) - 2E_{M, l}(t)] e^{-i(\alpha + \beta + \eta)t} ,
\]
\[
\frac{\partial E_{M, l}}{\partial t} \approx \frac{\partial}{\partial t} [H_{M, l}(t) - 2E_{M, l}(t)] e^{-i(\alpha + \beta + \eta)t} .
\]

(As mentioned, \(E_{M, l}(t)\) and \(H_{M, l}(t)\) \((k = 1)\) indicate the initial values which are necessary in the differential equation. Also the results \(E_{M, l}(t)\) and \(H_{M, l}(t)\) become new initial values for the next simulation step.) \(E_{M, l}(t)\) and \(H_{M, l}(t)\) proportionally depend on \(E_{M, l}(0)\) and \(H_{M, l}(0)\), and their evolutions also depend on \(\int_0^t (\alpha - \beta - \eta) \, dt\) and \(\int_0^t (\alpha + \beta + \eta) \, dt\). Also initial small-scale fields influence the dynamo, but the effect is clear while a large-scale \(\mathbf{B}\) field is weak. The profile of small-scale eddies becomes subordinate to the large-scale magnetic field in a few eddy turnover times. While \(\beta\) and \(\eta\) are always positive, \(\alpha\) is negative when the system is driven by the positive helical velocity field. Thus, the second terms on the right-hand side in equations (13) and (14) dominantly decide the profile of \(E_{M, l}\) and \(H_{M, l}\). And negative \(\dot{H}_{M, l} = 2E_{M, l}\) indicates that the evolving \(E_{M, l}\) is positive but \(H_{M, l}\) is negative.

The initially given \(E_{M, l}(0) \sim h_0^2\) \((k = 5)\) changes EMF. Since the interaction between \(u\) and \(b_{ic}(0)\) can be ignored in the very early time regime, the magnetic induction equation is

\[
\frac{\partial b_{ic}}{\partial t} \approx \eta k^2 b_{ic} .
\]

In Fourier space,

\[
\frac{\partial b_{ic}}{\partial t} \approx -\eta k^2 b_{ic} \Rightarrow b_{ic}(t) = b_{ic}(0)e^{-\eta k^2 t} .
\]

Total magnetic field is composed of \(\mathbf{B}\) \((k = 1)\), \(b_{ic}(k = 5)\), and \(b(2 \leq k \leq k_{max})\). Strictly speaking, \(b_{ic}\) is in the small scale. However, since such large \(E_{M, l}(0)\) decreases quickly before \(u\) grows enough to interact with \(b_{ic}\), we can think that \(b_{ic}\) evolves independently (Fig. 3).

Ignoring the dissipation term for simplicity, the approximate small-scale magnetic field \(b\) is

\[
\frac{\partial b}{\partial t} = \nabla \times (u \times \mathbf{B}) + \nabla \times (u \times b_{ic}) .
\]

This equation indicates that \(EMF(\langle u \times b \rangle \equiv \xi)\) can be represented by a linear combination of \(\mathbf{B}\) and \(b_{ic}\) such as \(B, \nabla \times B, b_{ic}, \) and \(\nabla \times b_{ic}\). Thus, we assume that the basic structure of EMF is

\[
\xi = \xi_1 + \xi_2 = \alpha_1 B - \beta_1 \nabla \times B + \alpha_2 b_{ic} - \beta_2 \nabla \times b_{ic} .
\]

For \(\xi\), we calculate \(\partial / \partial t (u \times B) = (\partial u / \partial t \times B) + (u \times \partial b / \partial t)\) to use the known momentum and magnetic induction equation.

After the simulation begins with the large \(b_{ic}(0)\) or \(E_{M, l}(0)\), \(E_{M, l}(0)\) decreases very quickly as \(u\) grows. In a few time units \((t \sim 5)\), \(u\) gets almost saturated, but \(b\) is still growing. Thus, we start the calculation using \((u \times \partial b / \partial t)\) \((\sim \xi_2)\). From equation (17),

\[
b(t') = \int_0^{t'} (\nabla \times (u \times \mathbf{B}) + \nabla \times (u \times b_{ic})) \, dt' .
\]

Since the basic structures of \(\xi_{1, 2}(B)\) and \(\xi_{1, 2}(b_{ic})\) are the same, we calculate \(\xi_{1, 2}\) and then change the variables to get \(\xi_{1, 2}, \xi_{1, 2}(b_{ic})\) is

\[
\xi_{1, 2} = \int_0^{t'} u(x, t') \times [\nabla \times (u(x, t') \times b_{ic}(x, t'))] \, dt' .
\]

Using equation (18), we get \(\xi_{1, 2} = \xi_{1, 2} \times \xi_{1, 2} \times (\nabla \times b_{ic})\).

\[
\xi_{1, 2} = \alpha_{x, y} \frac{\partial u_{xx}}{\partial x} - \beta_{x, y} \frac{\partial b_{ic, x}}{\partial x} - \beta_{y, z} \frac{\partial b_{ic, y}}{\partial y} + \epsilon_{x, y} \frac{\partial b_{ic, y}}{\partial z} + \alpha_{y, x} \frac{\partial b_{ic, y}}{\partial z} - \beta_{x, z} (\nabla \times b_{ic}) .
\]
\[ \xi_{k,2} \] has the same structure but the variables rotate: \( x \rightarrow y, y \rightarrow z, z \rightarrow x \). And for \( \xi_{k,2}, x \rightarrow z, y \rightarrow x, z \rightarrow y \). Since we assume that the system is isotropic, the coefficients of \( b_{k,x}, b_{k,y}, b_{k,z} \) are the same:

\[
\begin{align*}
    u_x \frac{\partial u'_t}{\partial x} - u_y \frac{\partial u'_t}{\partial y} = & \quad u_x \frac{\partial u'_t}{\partial z} - u_z \frac{\partial u'_t}{\partial z} \\
    \Rightarrow \quad & \frac{1}{3} \left( u_x \frac{\partial u'_t}{\partial x} - u_y \frac{\partial u'_t}{\partial y} + u_z \frac{\partial u'_t}{\partial z} + u_x \frac{\partial u'_t}{\partial z} - u_y \frac{\partial u'_t}{\partial z} - u_z \frac{\partial u'_t}{\partial z} \right).
\end{align*}
\]

(22)

Then, \( \alpha_{k,2} \) is

\[
\alpha_{k,2} = \frac{1}{3} \int_{-\infty}^{t} \left( \nabla \times u(x,t) \right)dt'.
\]

(23)

Similarly,

\[
\alpha_{k,2} = \frac{1}{3} \int_{-\infty}^{t} \left( \nabla \times u(x,t) \right)dt'.
\]

(24)

The coefficients \( \alpha_{k,1} \) and \( \beta_{k,1} \) are the same mentioned.

While \( \mathbf{b} \) field is even larger than growing velocity field, or stationary \( \mathbf{b} \) \((2 \leq k \leq k_{\text{max}}) \) is large enough to affect the plasma motion, we calculate \( \langle \mathbf{u} \mathbf{\cdot} \frac{\partial \mathbf{u}}{\partial t} \mathbf{\cdot} \mathbf{b} \rangle \).

We assume that dissipation effect is ignorably small and Lorentz force is a dominant term in the momentum equation. Then,

\[
\frac{\partial \mathbf{u}}{\partial t} \sim \mathbf{j} \times \mathbf{b} = (\mathbf{j} + \mathbf{\omega} \times \mathbf{u} \times \mathbf{b} + \mathbf{b} \times \mathbf{u} + \mathbf{b} \times \mathbf{b}).
\]

(26)

Small-scale momentum equation is

\[
\frac{\partial \mathbf{u}}{\partial t} \sim \mathbf{\nabla} \times \mathbf{b} + \mathbf{\nabla} \times \mathbf{b}.
\]

(27)

Here, we assume that the averages of \( \mathbf{b}_{\text{ic}} \) and \( \mathbf{B} \) are not zero, and their spatial changes are ignorable small within the small-scale eddy turnover time. Then EMF \( \langle \xi_{\text{ic}} \rangle = \xi_{M,1} \langle \mathbf{B} \rangle + \xi_{M,2} \mathbf{b}_{\text{ic}} \rangle \) is

\[
\mathbf{u} \times \mathbf{b} = \int_{-\infty}^{t} \mathbf{[} \mathbf{B}(x,t) \times \mathbf{\nabla} \mathbf{b}(x,t) \mathbf{]}dt \times \mathbf{b}(x,t)
\]

\[
+ \int_{-\infty}^{t} \mathbf{[} \mathbf{b}_{\text{ic}}(x,t) \times \mathbf{\nabla} \mathbf{b}(x,t) \mathbf{]}dt \times \mathbf{b}(x,t).
\]

(28)

The integrands of \( \xi_{M,1} \) and \( \xi_{M,2} \) are of the same structure. So if we consider \( \xi_{M,1} \),

\[
\mathbf{\xi}_{M,1} \times \mathbf{b}_{\text{ic}} = \mathbf{b}_{\text{ic}} \frac{\partial \mathbf{b}}{\partial x}, \mathbf{b}_{\text{ic}} \frac{\partial \mathbf{b}}{\partial y}, \mathbf{b}_{\text{ic}} \mathbf{\nabla} \mathbf{b},
\]

(29)

\( \mathbf{\xi}_{M,1} \) and \( \mathbf{\xi}_{M,2} \) have the same results with the rotation of variables mentioned before. Also, the assumption of isotropy makes the results simple:

\[
\frac{\partial \mathbf{b}}{\partial x} = \frac{\partial \mathbf{b}}{\partial y} = \frac{\partial \mathbf{b}}{\partial z} = \frac{\partial \mathbf{b}}{\partial y} = \frac{\partial \mathbf{b}}{\partial z} = \frac{\partial \mathbf{b}}{\partial z} = \frac{\partial \mathbf{b}}{\partial z} = \frac{\partial \mathbf{b}}{\partial z} = \frac{\partial \mathbf{b}}{\partial z} = \frac{\partial \mathbf{b}}{\partial z} = \frac{\partial \mathbf{b}}{\partial z}.
\]

(30)

Thus, \( \alpha_{M,1} \) related to \( \mathbf{B} \) field is

\[
\alpha_{M,1} = \alpha_{M,2} = \frac{1}{3} \int_{-\infty}^{t} \mathbf{b}(x,t) \times \mathbf{j}(x,t)dt'.
\]

(31)

Finally, the complete EMF is \( \xi = \xi_{k,1} + \xi_{k,2} + \xi_{M,1} + \xi_{M,2} \):

\[
\xi = \frac{1}{3} \left[ \left( j \mathbf{\cdot} \mathbf{b} \right) - \left( \mathbf{\omega} \times \mathbf{u} \right) \right] \mathbf{\tau} \mathbf{B} + \frac{1}{3} \left[ \left( j \mathbf{\cdot} \mathbf{b} \right) - \left( \mathbf{\omega} \times \mathbf{u} \right) \right] \mathbf{b}_{\text{ic}}(0) e^{-\alpha k t}.
\]

(32)

(\( \tau \) is substituted for the integration. Only the magnitude is considered.)

\[
E_{\text{M.L}} \text{ in equation (12) is}
\]

\[
\frac{\partial}{\partial t} E_{\text{M.L}} = \alpha k^2 H_{\text{M.L}} - \tau^2 (\beta + \eta) E_{\text{M.L}}
\]

\[
+ \left< \mathbf{B} \cdot \mathbf{\nabla} \times \mathbf{b}_{\text{ic}}(0) \right> e^{-\alpha k t}.
\]

(33)

The first and third terms on the right-hand side are the sources of \( E_{\text{M.L}} \). These two terms describe the inverse cascade of energy in small scale to \( E_{\text{M.L}} \) with \( \alpha \). In fact, Fourier-transformed representation shows that the mean correlation \( \langle \mathbf{B} \cdot \mathbf{\nabla} \times \mathbf{b}_{\text{ic}}(0) \rangle \) is decomposed into helical and non-helical part. With the positive helical \( \mathbf{b}_{\text{ic}}(0) \), this term is

\[
k_1 \langle \mathbf{B} \cdot \mathbf{b}_{\text{ic}}(0) \rangle_{\text{hel}} + \langle \mathbf{B} \cdot \mathbf{\nabla} \times \mathbf{b}_{\text{ic}}(0) \rangle_{\text{non-hel}}
\]

(34)

and with the negative helicity,

\[
- k_1 \langle \mathbf{B} \cdot \mathbf{b}_{\text{ic}}(0) \rangle_{\text{hel}} + \langle \mathbf{B} \cdot \mathbf{\nabla} \times \mathbf{b}_{\text{ic}}(0) \rangle_{\text{non-hel}}.
\]

(35)

When non-helical \( E_{\text{M.L}} \) is given,

\[
\langle \mathbf{B} \cdot \mathbf{\nabla} \times \mathbf{b}_{\text{ic}}(0) \rangle_{\text{non-hel}}.
\]

(36)

These three equations split the spectrum of \( E_{\text{M.L}} \) in large scale. For the detailed analysis, we have to consider fourth-order correlation, but we will leave this for the future work with more detailed simulation.

Up to now, we have used the fact that the left-handed magnetic helicity \( \langle \mathbf{h}_{\text{L}} \rangle \) is generated when the system is driven by the right-handed kinetic helicity \( \langle \mathbf{h}_{\text{R}} \rangle \) without enough consideration. Mathematically, the growth of larger scale magnetic field \( \langle \mathbf{B} \rangle \) is described by a differential equation like equation (8) or (10). However, since the differential equation in itself cannot describe the change of sign of variables, more fundamental and physical approach is necessary. In the case of \( \alpha \omega \) dynamo, there was a trial to explain the handedness of twist and writhe in corona ejection using the concept of magnetic helicity conservation (Blackman & Field 2002). But, even when the effect of differential rotation cannot be expected \( \alpha \omega \) dynamo, the sign of generated \( H_{\text{M,L}} \) and injected \( \mathbf{u} \mathbf{\cdot} \mathbf{\omega} \) is opposite.

We assume that the magnetic field \( \mathbf{B} \) interacts with right-handed helical kinetic plasma motion. The velocity can be divided into toroidal component \( \mathbf{u}_t \) and poloidal component \( \mathbf{u}_p \). The interaction of this toroidal motion with \( \mathbf{B} \) produces \( j_2 \sim \mathbf{u}_t \times \mathbf{B} \). The induced current density \( j_{2,f} \) in the front is towards the positive \( \hat{z} \) direction, but the rear current density \( j_{2,f} \) is along with the negative \( \hat{z} \). These two current densities become the sources of magnetic field \( -\mathbf{b}_2 \). Again, this
induced magnetic field interacts with the poloidal kinetic velocity \( u_{2,p} \) and generates \( -J_{\omega y} \). Finally, this \( J_{\text{in}} \) produces \( B_{\text{in}} \), which forms a circle from the magnetic field \( B_1 \) (upper picture in Fig. 4b). If we go one step further from here, we see that \( B_{\text{in}} \) can be considered as a new toroidal magnetic field \( B_{\text{tor}} \), and \( B_1 \) as \( B_{\text{pol}} \). This new helical magnetic field structure has the left-handed polarity, i.e., \( \langle a \cdot b \rangle < 0 \) (lower plot of Fig. 4b). \( B_{\text{tor}} \) interacts with the positive \( \langle u_2 - \omega_z \rangle \) and induces the current density \( J \) which is antiparallel to \( B_{\text{tor}} \). Then \( B_{\text{pol}} \) is reinforced by this \( J \), which is the typical \( \alpha^2 \langle B_{\text{pol}} \rangle \) dynamo with the external forcing source.

5 CONCLUSION

We have seen how the initially given magnetic energy \( E_{M}(0) \) and magnetic helicity \( H_{M}(0) \) affect the growth rate of large-scale \( B \) field. To analyse the simulation results, we calculated EMF (equation 18) modified by the pre-existing magnetic field \( b_2(t) \) from \( E_{M}(0) \) at \( k = 5 \) generates a couple of terms in EMF. The interaction among \( b_{2c}, u, b, \) and \( \mathbf{B} \) field in the early-time regime leads to the change of growth rate of \( E_{M,1} \).

Equations (18) and (34)–(36) show that a non-trivial interaction between \( b_{2c} \) and \( \mathbf{B} \) occurs through the \( \alpha \) coefficient. Simulation results with these equations show that the growth rate is proportional to \( E_{M}(0) \). Positive \( H_{M}(0) \) boosts the enhanced growth rate, but negative \( H_{M}(0) \) decreases it. These results differ from the so-called quenching effect of the small-scale magnetic helicity. In fact, the quenching effect of current helicity in small scale is inferred from the representation of \( \alpha \sim (\mathbf{j} \cdot \mathbf{b}) - \langle \mathbf{u} \cdot \omega \rangle \). However, strictly speaking, current(magnetic) helicity in small scale is passively decided by \( \langle \mathbf{u} \cdot \omega \rangle \) with \( v \) and \( \eta \), not being able to constrain the large-scale magnetic field. Besides, the effect of \( b_{2c} \) depends on time. So the effect of IC eventually disappears as time passes, and the saturated value of magnetic field becomes independent of the initial conditions.

We have seen the physical meaning and role of magnetic helicity and non-helical magnetic fields. The magnetic helicity \( \langle \mathbf{A} \cdot \mathbf{B} \rangle \) is defined as the topological measure of twist and linkage of magnetic field lines, and the statistical description of \( H_{M} \) expands its intuitive meaning. With the realizability condition, these conceptual definitions give us some clue about how helical and non-helical field constrain each other.

Finally, we have seen the relation between the sign of driving kinetic helicity and that of the produced magnetic helicity. Because of the curl operator in the magnetic induction equation, velocity ‘\( \mathbf{u} \)’ and magnetic field ‘\( \mathbf{B} \)’ essentially have a phase difference. So when a system is driven by the (positive) kinetic helicity, negative sign of magnetic helicity(field) is generated at the injection scale. This magnetic helicity (field) grows and eventually becomes a (negative) large-scale magnetic helicity (field). Then (positive) magnetic helicity is diffused from the large scale and migrates into the small scale to conserve total \( H_{M} \) in the system. Equations (13) and (14) for \( E_{M,1} \) and \( H_{M,1} \) mathematically explain their evolving profiles. Also the conservation of total magnetic helicity in MHD system plays an important role in the growth \( \mathbf{B} \) field. But due to the scale dependence of the dissipation effect and two opposite signs of \( (\pm)H_{M} \), more detailed investigations are necessary to make clear the relation between helicity and growth of magnetic field.

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