VON NEUMANN TYPE OF TRACE INEQUALITIES FOR SCHATTEN-CLASS OPERATORS

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ABSTRACT. We generalize von Neumann’s well-known trace inequality, as well as related eigenvalue inequalities for hermitian matrices, to Schatten-class operators between complex Hilbert spaces of infinite dimension. To this end, we exploit some recent results on the C-numerical range of Schatten-class operators. For the readers’ convenience, we sketched the proof of these results in the Appendix.

KEYWORDS: C-numerical range; Schatten-class operators; trace inequality; von Neumann inequality

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1. INTRODUCTION

In the mid thirties of the last century, von Neumann [20, Thm. 1] derived the following beautiful and widely used trace inequality for complex $n \times n$ matrices:

Let $A, B \in \mathbb{C}^{n \times n}$ with singular values $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$ and $s_1(B) \geq s_2(B) \geq \ldots \geq s_n(B)$, respectively, be given. Then

$$
\max_{U,V \in \mathcal{U}_n} \left| \text{tr}(AUBV) \right| = \sum_{j=1}^{n} s_j(A)s_j(B),
$$

where $\mathcal{U}_n$ denotes the unitary group.

In fact, the above result can be reinterpreted as a characterization of the image of the unitary double-coset $\{ AUBV \mid U, V \in \mathbb{C}^{n \times n} \text{ unitary} \}$ under the trace-functional, i.e.

$$
\{ \text{tr}(AUBV) \mid U, V \in \mathcal{U}_n \} = \mathcal{K}_r(0)
$$

with $r := \sum_{j=1}^{n} s_j(A)s_j(B)$ and $\mathcal{K}_r(0) = \{ z \in \mathbb{C}, \ |z| \leq r \}$ being the closed disk of radius $r$ centred around the origin. This results from the elementary observation that the left-hand side of (1.2) is circular (simply replace $U$ by $e^{i\theta}U$). Another
well-known consequence of (1.1), a von Neumann inequality for hermitian matrices [10, Ch. 9.H.1], reads as follows.

Let $A, B \in \mathbb{C}^{n \times n}$ hermitian with respective eigenvalues $(\lambda_j(A))_{j=1}^n$ and $(\lambda_j(B))_{j=1}^n$ be given. Then

$$\sum_{j=1}^n \lambda_j^\downarrow(A) \lambda_j^\uparrow(B) \leq \text{tr}(AB) \leq \sum_{j=1}^n \lambda_j^\downarrow(A) \lambda_j^\downarrow(B),$$

where the superindices $\downarrow$ and $\uparrow$ denote the decreasing and increasing sorting of the eigenvalue vectors, respectively.

The area of applications of von Neumann’s inequalities and, more generally, singular value decompositions (SVD) is enormous. It ranges from operator theory [6, 17] and numerics [8] to more applied fields like control theory [9], neural networks [14] as well as quantum dynamics and quantum control [7, 18]. An overview can be found in [10, 12]. Now the goal of this short contribution is to generalize these inequalities to Schatten-class operators on infinite-dimensional Hilbert spaces. In doing so, some recent results on the $C$-numerical range of Schatten-class operators [3, 4] turn out to be quite helpful. For the readers’ convenience, we sketched the corresponding proofs in Appendix A.

This paper is organized as follows: Section 2 introduces the key notions and concepts of this work such as 2.1 Schatten classes, 2.2 convergence of compact sets via the Hausdorff metric as well as 2.3 the $C$-numerical range for Schatten-class operators. Section 3 then presents the main results as mentioned above. Appendix A outlines the outsourced proof of some crucial geometrical results regarding the $C$-numerical range.

2. NOTATION AND PRELIMINARIES

Unless stated otherwise, here and henceforth $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary infinite-dimensional complex Hilbert spaces while $\mathcal{H}$ and $\mathcal{G}$ are reserved for infinite-dimensional separable complex Hilbert spaces. Moreover, let $\mathcal{B}(\mathcal{X}, \mathcal{Y}), \mathcal{U}(\mathcal{X}, \mathcal{Y}), \mathcal{K}(\mathcal{X}, \mathcal{Y}), \mathcal{F}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{B}_p(\mathcal{X}, \mathcal{Y})$ denote the set of all bounded, unitary, compact, finite-rank and $p$-th Schatten-class operators between $\mathcal{X}$ and $\mathcal{Y}$, respectively. As usual, if $\mathcal{X}$ and $\mathcal{Y}$ coincide we simply write $\mathcal{B}(\mathcal{X}), \mathcal{U}(\mathcal{X}),$ etc.

Scalar products are conjugate linear in the first argument and linear in the second one. For an arbitrary subset $S \subset \mathbb{C}$, the notations $\overline{S}$ and $\text{conv}(S)$ stand for its closure and convex hull, respectively. Finally, given $p, q \in [1, \infty]$, we say $p$ and $q$ are conjugate if $\frac{1}{p} + \frac{1}{q} = 1$.

2.1. INFINITE-DIMENSIONAL HILBERT SPACES AND THE SCHATTEN CLASSES. For a comprehensive introduction to Hilbert spaces of infinite dimension as well as Schatten-class operators, we refer to, e.g., [11] and [5]. Here, we
recall only some basic results which will be used frequently throughout this paper.

**Lemma 2.1 (Schmidt decomposition).** For each \( C \in \mathcal{K}(\mathcal{X}, \mathcal{Y}) \), there exists a decreasing null sequence \( (s_n(C))_{n \in \mathbb{N}} \) in \([0, \infty)\) as well as orthonormal systems \( (f_n)_{n \in \mathbb{N}} \) in \( \mathcal{X} \) and \( (g_n)_{n \in \mathbb{N}} \) in \( \mathcal{Y} \) such that

\[
C = \sum_{n=1}^{\infty} s_n(C) \langle f_n, \cdot \rangle g_n,
\]

where the series converges in the operator norm.

As the singular numbers \( (s_n(C))_{n \in \mathbb{N}} \) in Lemma 2.1 are uniquely determined by \( C \), the \( p \)-th Schatten-class \( B^p(\mathcal{X}, \mathcal{Y}) \) is well-defined via

\[
B^p(\mathcal{X}, \mathcal{Y}) := \left\{ C \in \mathcal{K}(\mathcal{X}, \mathcal{Y}) \mid \sum_{n=1}^{\infty} s_n(C)^p < \infty \right\}
\]

for \( p \in [1, \infty) \). The Schatten-\( p \)-norm

\[
\|C\|_p := \left( \sum_{n=1}^{\infty} s_n(C)^p \right)^{1/p}
\]

turns \( B^p(\mathcal{X}, \mathcal{Y}) \) into a Banach space. Moreover, for \( p = \infty \), we identify \( B^\infty(\mathcal{X}, \mathcal{Y}) \) with the set of all compact operators \( \mathcal{K}(\mathcal{X}, \mathcal{Y}) \) equipped with the norm

\[
\|C\|_\infty := \sup_{n \in \mathbb{N}} s_n(C) = s_1(C).
\]

Note that \( \|C\|_\infty \) coincides with the ordinary operator norm \( \|C\| \). Hence \( B^\infty(\mathcal{X}, \mathcal{Y}) \) constitutes a closed subspace of \( B(\mathcal{X}, \mathcal{Y}) \) and thus a Banach space, too.

**Remark 2.2.** Evidently, if \( C \in B^p(\mathcal{X}, \mathcal{Y}) \) for some \( p \in [1, \infty] \) then the series (2.1) converges in the Schatten-\( p \)-norm.

The following results can be found in [5, Coro. XI.9.4 & Lemma XI.9.9].

**Lemma 2.3.** (a) Let \( p \in [1, \infty] \). Then for all \( S, T \in B(\mathcal{X}) \), \( C \in B^p(\mathcal{X}) \):

\[
\|SCT\|_p \leq \|S\| \|C\|_p \|T\|.
\]

(b) Let \( 1 \leq p \leq q \leq \infty \). Then \( B^p(\mathcal{X}, \mathcal{Y}) \subseteq B^q(\mathcal{X}, \mathcal{Y}) \) and \( \|C\|_p \geq \|C\|_q \) for all \( C \in B^p(\mathcal{X}, \mathcal{Y}) \).

Note that due to (a), all Schatten-classes \( B^p(\mathcal{X}) \) constitute—just like the compact operators—a two-sided ideal in the \( C^* \)-algebra of all bounded operators \( B(\mathcal{X}) \).

Now for any \( C \in B^1(\mathcal{X}) \), the trace of \( C \) is defined via

\[
\text{tr}(C) := \sum_{i \in I} \langle f_i, Cf_i \rangle,
\]

where \( (f_i)_{i \in I} \) can be any orthonormal basis of \( \mathcal{X} \). The trace is well-defined, as one can show that the right-hand side of (2.2) is finite and does not depend on the choice of \( (f_i)_{i \in I} \). Important properties are the following, cf. [5, Lemma XI.9.14].
LEMMA 2.4. Let \( C \in B^p(\mathcal{X}, \mathcal{Y}) \) and \( T \in B^q(\mathcal{Y}, \mathcal{X}) \) with \( p, q \in [1, \infty] \) conjugate. Then one has \( CT \in B^1(\mathcal{Y}) \) and \( TC \in B^1(\mathcal{X}) \) with
\[
\text{tr}(CT) = \text{tr}(TC) \quad \text{and} \quad |\text{tr}(CT)| \leq \|C\|_p \|T\|_q.
\]

In order to recap the well-known diagonalization result for compact normal operators, we first have to fix the term \textit{eigenvalue sequence} of a compact operator \( T \in \mathcal{K}(\mathcal{H}) \). In general, it is obtained by arranging the (necessarily countably many) non-zero eigenvalues in decreasing order with respect to their absolute value and each eigenvalue is repeated as many times as its algebraic multiplicity calls for. If only finitely many non-vanishing eigenvalues exist, then the sequence is filled up with zeros, see [11, Ch. 15]. For our purposes, we have to pass to a slightly \textit{modified eigenvalue sequence} as follows:

- If the range of \( T \) is infinite-dimensional and the kernel of \( T \) is finite-dimensional, then put \( \dim(\ker T) \) zeros at the beginning of the eigenvalue sequence of \( T \).
- If the range and the kernel of \( T \) are infinite-dimensional, mix infinitely many zeros into the eigenvalue sequence of \( T \).
- Because in Definition 2.12 arbitrary permutations will be applied to the modified eigenvalue sequence, we do not need to specify this mixing procedure further, cf. also [3, Lemma 3.6].
- If the range of \( T \) is finite-dimensional leave the eigenvalue sequence of \( T \) unchanged.

LEMMA 2.5 ([1], Thm. VIII.4.6). Let \( T \in \mathcal{K}(\mathcal{H}) \) be normal, i.e. \( T^\dagger T = TT^\dagger \). Then there exists an orthonormal basis \( (e_n)_{n \in \mathbb{N}} \) of \( \mathcal{H} \) such that
\[
T = \sum_{j=1}^{\infty} \lambda_j(T) \langle e_j, \cdot \rangle e_j
\]
where \( (\lambda_j(T))_{j \in \mathbb{N}} \) is the modified eigenvalue sequence of \( T \).

2.2. SET CONVERGENCE. In order to transfer results about convexity and star-shapedness of the \( C \)-numerical range from matrices to Schatten-class operators, we need a concept of set convergence. We will use the Hausdorff metric on compact subsets (of \( \mathbb{C} \)) and the associated notion of convergence, see, e.g., [13].

The distance between \( z \in \mathbb{C} \) and a non-empty compact subset \( A \subseteq \mathbb{C} \) is given by \( d(z, A) := \min_{w \in A} d(z, w) = \min_{w \in A} |z - w| \), based on which the Hausdorff metric \( \Delta \) on the set of all non-empty compact subsets of \( \mathbb{C} \) is defined via
\[
\Delta(A, B) := \max \left\{ \max_{z \in A} d(z, B), \max_{z \in B} d(z, A) \right\}.
\]

The following characterization of the Hausdorff metric is readily verified.

\[1\] By [11, Prop. 15.12], every non-zero element \( \lambda \in \sigma(T) \) of the spectrum of \( T \) is an eigenvalue of \( T \) and has a well-defined finite algebraic multiplicity \( \nu_0(\lambda) \), e.g., \( \nu_0(\lambda) := \dim(\ker(T - \lambda I)^{n_0}) \), where \( n_0 \in \mathbb{N} \) is the smallest natural number \( n \in \mathbb{N} \) such that \( \ker(T - \lambda I)^{n+1} = \ker(T - \lambda I)^n \).
Lemma 2.6. Let \( A, B \subset \mathbb{C} \) be two non-empty compact sets and let \( \varepsilon > 0 \). Then
\[
\Delta(A, B) \leq \varepsilon \text{ if and only if for all } z \in A, \text{ there exists } w \in B \text{ with } d(z, w) \leq \varepsilon \text{ and vice versa.}
\]

With this metric one can introduce the notion of convergence for sequences
\( (A_n)_{n \in \mathbb{N}} \) of non-empty compact subsets of \( \mathbb{C} \) such that the maximum- as well as the minimum-operator are continuous in the following sense.

Lemma 2.7. Let \( (A_n)_{n \in \mathbb{N}} \) be a bounded sequence of non-empty, compact subsets of \( \mathbb{R} \) which converges to \( A \subset \mathbb{R} \). Then the sequences of real numbers \( (\max A_n)_{n \in \mathbb{N}} \) and
\( (\min A_n)_{n \in \mathbb{N}} \) are convergent with
\[
\lim_{n \to \infty} (\max A_n) = \max A \quad \text{and} \quad \lim_{n \to \infty} (\min A_n) = \min A.
\]

Proof. Let \( \varepsilon > 0 \). By assumption, there exists \( N \in \mathbb{N} \) such that \( \Delta(A_n, A) < \varepsilon \) for all \( n \geq N \). Hence by Lemma 2.6 one finds \( a_n \in A_n \) with \( |\max A_n - a_n| < \varepsilon \) and thus \( \max A < a_n + \varepsilon < \max A_n + \varepsilon \). Similarly, there exists \( a \in A \) such that \( |\max A_n - a| < \varepsilon \), so \( \max A_n < a + \varepsilon < \max A + \varepsilon \). Combining both estimates, we get \( |\max A - \max A_n| < \varepsilon \). The case of the minimum is shown analogously.

2.3. The C-Numerical Range of Schatten-Class Operators.
In this subsection, we present a few approximation results and collect some material on the C-numerical range of Schatten-class operators which is of fundamental importance in Section 3. Because said results appeared only in an addendum to another publication on trace-class operators, we decided to sketch the proof in the appendix for the readers’ convenience.

Definition 2.8. Let \( p, q \in [1, \infty] \) be conjugate. Then for \( C \in \mathcal{B}^p(\mathcal{X}) \) and \( T \in \mathcal{B}^q(\mathcal{X}) \), the C-numerical range of \( T \) is defined to be
\[
\mathcal{W}_C(T) := \{ \text{tr}(CU^TU) \mid U \in \mathcal{U}(\mathcal{X}) \}.
\]

Following Lemma 1.2., for \( C \in \mathcal{B}^p(\mathcal{X}, \mathcal{Y}) \) and \( T \in \mathcal{B}^q(\mathcal{Y}, \mathcal{X}) \) with \( p, q \in [1, \infty) \) conjugate one may actually introduce the more general set (now invoking the unitary equivalence orbit \( UTU \) of \( T \) instead of the unitary similarity orbit \( U^TU \))
\[
\mathcal{S}_C(T) := \{ \text{tr}(CUV) \mid U \in \mathcal{U}(\mathcal{X}), V \in \mathcal{U}(\mathcal{Y}) \}.
\]

Note that all traces involved are well-defined due to Lemma 2.3 and 2.4.

Lemma 2.9. Let \( p \in [1, \infty] \), \( C \in \mathcal{B}^p(\mathcal{X}) \) and \( (S_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{B}(\mathcal{X}) \) which converges strongly to \( S \in \mathcal{B}(\mathcal{X}) \). Then one has \( S_nC \to SC \), \( CS_n^T \to CS^T \), and \( S_nCS_n^T \to SCS^T \) for \( n \to \infty \) with respect to the norm \( \| \cdot \|_p \).

Proof. The cases \( p = 1 \) and \( p = \infty \) are proven in [3, Lemma 3.2]. As the proof for \( p \in (1, \infty) \) is essentially the same, we sketch only the major differences. First, choose \( K \in \mathbb{N} \) such that
\[
\sum_{k=K+1}^{\infty} s_k(C) \leq \frac{\varepsilon^p}{(3K)^p},
\]
where \( \kappa > 0 \) satisfies \( \|S\| \leq \kappa \) and \( \|S_n\| \leq \kappa \) for all \( n \in \mathbb{N} \). The existence of the constant \( \kappa > 0 \) is guaranteed by the uniform boundedness principle. Then decompose \( C = \sum_{k=1}^{\infty} s_k(C) \langle \epsilon_k, \cdot \rangle f_k \) into \( C = C_1 + C_2 \) with \( C_1 := \sum_{k=1}^{K} s_k(C) \langle \epsilon_k, \cdot \rangle f_k \) finite-rank. By Lemma 2.3 one has

\[
\|SC - SnC\|_p \leq \|SC_1 - SnC_1\|_p + \|SC_2\|_p + \|Sn\||C_2\|_p
\]

\[
< \|SC_1 - SnC_1\|_p + \frac{2\varepsilon}{3}.
\]

Thus, what remains is to choose \( N \in \mathbb{N} \) such that \( \|SC_1 - SnC_1\|_p < \varepsilon/3 \) for all \( n \geq N \). To this end, consider the estimate

\[
\|SC_1 - SnC_1\|_p \leq \sum_{k=1}^{K} s_k(C) \|\langle \epsilon_k, \cdot \rangle (f_k - Snf_k)\|_p = \sum_{k=1}^{K} s_k(C) \|Sf_k - Snf_k\|.
\]

Then the strong convergence of \( (Sn)_{n \in \mathbb{N}} \) yields \( N \in \mathbb{N} \) such that

\[
\|Sf_k - Snf_k\| < \frac{\varepsilon}{3 \sum_{k=1}^{K} s_k(C)}
\]

for \( k = 1, \ldots, K \) and all \( n \geq N \). This shows \( \|SC - SnC\|_p \rightarrow 0 \) as \( n \rightarrow \infty \). All other assertions are an immediate consequence of \( \|A\|_p = \|A^\dagger\|_p \) for \( A \in B^p(\mathcal{X}) \) and

\[
\|SCS^\dagger - SnCS_n^\dagger\|_p \leq \|S\||C\|CS^\dagger - CS_n^\dagger\|_p + \|SC - SnC\|_p S_n\|
\]

\[
\leq \kappa \|CS^\dagger - CS_n^\dagger\|_p + \|SC - Sn\|_p .
\]

**Proposition 2.10.** Let \( C \in B^p(\mathcal{X}, \mathcal{Y}) \), \( T \in B^q(\mathcal{Y}, \mathcal{X}) \) with \( p, q \in [1, \infty] \) conjugate and let \( (C_n)_{n \in \mathbb{N}} \) and \( (T_n)_{n \in \mathbb{N}} \) be sequences in \( B^p(\mathcal{X}, \mathcal{Y}) \) and \( B^q(\mathcal{Y}, \mathcal{X}) \), respectively, such that \( \lim_{n \rightarrow \infty} \|C - C_n\|_p = \lim_{n \rightarrow \infty} \|T - T_n\|_q = 0 \). Then

\[
\lim_{n \rightarrow \infty} S_{C_n}(T_n) = S_C(T).
\]

If, additionally, \( \mathcal{X} = \mathcal{Y} \) then

\[
\lim_{n \rightarrow \infty} W_{C_n}(T_n) = W_C(T).
\]

**Proof.** W.l.o.g. let \( C_n, T_n \neq 0 \) for some \( n \in \mathbb{N} \)–else all the involved sets would be trivial–so we may introduce the positive but (as seen via the reverse triangle inequality) finite numbers

\[
\kappa := \sup \{ \|C\|_p, \|C_1\|_p, \|C_2\|_p, \ldots \} \quad \text{and} \quad \tau := \sup \{ \|T\|_q, \|T_1\|_q, \|T_2\|_q, \ldots \} .
\]

Let \( \varepsilon > 0 \). By assumption there exists \( N \in \mathbb{N} \) such that

\[
\|C - C_n\|_p < \frac{\varepsilon}{4\tau} \quad \text{as well as} \quad \|T - T_n\|_q < \frac{\varepsilon}{4\kappa}.
\]

for all \( n \geq N \). We shall first tackle (2.3), as (2.4) can be shown in complete analogy. The goal will be to satisfy the assumptions of Lemma 2.6 in order to show \( \Delta(S_C(T), S_{C_n}(T_n)) < \varepsilon \) for all \( n \geq N \).
Let \( w \in \overline{S_C(T)} \) so one finds \( U \in \mathcal{U}(\mathcal{X}), V \in \mathcal{U}(\mathcal{Y}) \) such that \( w' := \text{tr}(CU-TV) \) satisfies \( |w-w'| < \frac{\varepsilon}{2} \). Thus for \( w_n := \text{tr}(C_nUT_nV) \) by Lemma 2.3 and 2.4

\[
|w-w_n| \leq |w-w'| - |w'-w_n| < \frac{\varepsilon}{2} + |\text{tr}((C-C_n)UT_nV)| + |\text{tr}(V(C_nU(T-T_n)))|
\]

\[
\leq \frac{\varepsilon}{2} + \|C-C_n\|_p\|U\|_q\|V\| + \|V\|\|C_n\|_p\|U\|\|T-T_n\|_q
\]

\[
\leq \frac{\varepsilon}{2} + \|C-C_n\|_p\tau + \kappa\|T-T_n\|_q < \varepsilon
\]

for all \( n \geq N \).

Similarly, let \( n \geq N \). Then for \( v_n \in \overline{S_C(T_n)} \) one finds \( U_n \in \mathcal{U}(\mathcal{X}), V_n \in \mathcal{U}(\mathcal{Y}) \) such that \( v'_n := \text{tr}(C_nU(T_nV_n)) \) satisfies \( |v_n-v'_n| < \frac{\varepsilon}{2} \). Thus for \( \bar{v}_n := \text{tr}(CU_nTV_n) \) we obtain

\[
|v_n-\bar{v}_n| \leq |v_n-v'_n| - |v'_n-\bar{v}_n| < \frac{\varepsilon}{2} + |\text{tr}((C_n-C)U(T_nV_n))| + |\text{tr}(V(C_nU_n(T_n-T)))|
\]

\[
\leq \frac{\varepsilon}{2} + \|C-C_n\|_p\tau + \kappa\|T-T_n\|_q < \varepsilon.
\]

The preceding proposition together with Lemma 2.2 immediately entails the next result.

**Corollary 2.11.** Let \( C \in \mathcal{B}_p(\mathcal{H}), T \in \mathcal{B}_q(\mathcal{H}) \) with \( p, q \in [1,\infty] \) conjugate. Then \( \lim_{k \to \infty} W_C(\Pi_kTT_k) = W_C(T) \), where \( \Pi_k \) is the orthogonal projection onto the span of the first \( k \) elements of an arbitrarily chosen orthonormal basis \((\epsilon_n)_{n \in \mathbb{N}}\) of \( \mathcal{H} \).

Here we used the well-known fact that the orthogonal projections \( \Pi_k \) strongly converge to the identity \( \text{id}_\mathcal{H} \) for \( k \to \infty \), cf., e.g., [3] Lemma 3.2).

**Definition 2.12 (C-spectrum).** Let \( p, q \in [1,\infty] \) be conjugate. Then, for \( C \in \mathcal{B}_p(\mathcal{H}) \) with modified eigenvalue sequence \((\lambda_n(C))_{n \in \mathbb{N}}\) and \( T \in \mathcal{B}_q(\mathcal{H}) \) with modified eigenvalue sequence \((\lambda_n(T))_{n \in \mathbb{N}}\), the C-spectrum of \( T \) is defined via

\[
P_C(T) := \left\{ \sum_{n=1}^\infty \lambda_n(C)\lambda_{\sigma(n)}(T) \big| \sigma : \mathbb{N} \to \mathbb{N} \text{ is any permutation} \right\}.
\]

Hölder’s inequality and the standard estimate \( \sum_{n=1}^\infty |\lambda_n(C)|^p \leq \sum_{n=1}^\infty s_n(C)_p, \) cf. [11] Prop. 16.31, yield

\[
\sum_{n=1}^\infty |\lambda_n(C)\lambda_{\sigma(n)}(T)| \leq \left( \sum_{n=1}^\infty s_n(C)_p \right)^{1/p} \left( \sum_{n=1}^\infty s_n(T)_q \right)^{1/q} \leq \|C\|_p\|T\|_q,
\]

showing that the elements of \( P_C(T) \) are well-defined and bounded by \( \|C\|_p\|T\|_q \).

Now, if the operators \( C \) and \( T \) are particularly “nice”, one can connect the C-numerical range and the C-spectrum of \( T \) as follows:

**Theorem 2.13 ([11]).** Let \( C \in \mathcal{B}_p(\mathcal{H}) \) and \( T \in \mathcal{B}_q(\mathcal{H}) \) with \( p, q \in [1,\infty] \) conjugate. Then the following statements hold.
(a) If \( W_C(T) \) is star-shaped with respect to the origin.

(b) If either \( C \) or \( T \) is normal with collinear eigenvalues, then \( W_C(T) \) is convex.

(c) If \( C \) and \( T \) both are normal, then \( P_C(T) \subseteq W_C(T) \subseteq \text{conv}(P_C(T)) \). If, in addition, the eigenvalues of \( C \) or \( T \) are collinear then \( W_C(T) = \text{conv}(P_C(T)) \).

As stated in the beginning, a sketch of the proof can be found in Appendix A.

3. MAIN RESULTS

Considering the inequalities (1.1) and (1.3) from the introduction, it arguably is easier to generalize the former, i.e. to generalize von Neumann’s “original” trace inequality to Schatten-class operators. To start with we first investigate the finite-rank case.

**Lemma 3.1.** Let \( C \in \mathcal{F}(\mathcal{X}, \mathcal{Y}) \), \( T \in \mathcal{F}(\mathcal{Y}, \mathcal{X}) \) and \( k := \max\{\text{rk}(C), \text{rk}(T)\} \in \mathbb{N}_0 \). Then \( S_C(T) = K_r(0) \) where \( r := \sum_{i=1}^{k} s_i(C)s_i(T) \).

**Proof.** Defining \( k \) as above, Lemma 2.1 yields orthonormal systems \( (e_j)_{j=1}^{k}, (h_j)_{j=1}^{k} \) in \( \mathcal{X} \) and \( (f_j)_{j=1}^{k}, (g_j)_{j=1}^{k} \) in \( \mathcal{Y} \) such that

\[
C = \sum_{i=1}^{k} s_i(C) \langle e_j, \cdot \rangle f_j \quad \text{and} \quad T = \sum_{i=1}^{k} s_i(T) \langle g_j, \cdot \rangle h_j.
\]

Note that forcing both sums to have same summation range means that, potentially, some of the singular values have to be complemented by zeros, which is not of further importance.

“\( \subseteq \)” : Let any \( U \in \mathcal{U}(\mathcal{X}), V \in \mathcal{U}(\mathcal{Y}) \) be given. Then

\[
\text{tr}(CUTV) = \text{tr} \left( \sum_{i,j=1}^{k} s_i(T)s_j(C) \langle e_j, Uh_i \rangle \langle V^t g_i, \cdot \rangle f_j \right) = \sum_{i,j=1}^{k} s_i(T)s_j(C) \langle e_j, Uh_i \rangle \langle g_i, V f_j \rangle
\]

by direct computation. Now consider the subspaces

\[
Z_1 := \text{span}\{e_1, \ldots, e_k, Uh_1, \ldots, Uh_k\} \subset \mathcal{X}
\]
\[
Z_2 := \text{span}\{f_1, \ldots, f_k, V^t g_1, \ldots, V^t g_k\} \subset \mathcal{Y}
\]

so there exist orthonormal bases of the form

\[
e_1, \ldots, e_k, e_{k+1}, \ldots, e_N \quad \text{and} \quad f_1, \ldots, f_k, f_{k+1}, \ldots, f_N'
\]
of $Z_1$ and $Z_2$ for some $N, N' \geq k$, respectively. W.l.o.g. we can assume $N = N'$ and define
\[ a_j := ((e_i, U h_j))^N_{i=1} \in \mathbb{C}^N \quad \text{and} \quad b_j := ((f_i, V^* g_j))^N_{i=1} \in \mathbb{C}^N \]
for $j = 1, \ldots, k$. This yields $N \times N$ matrices
\[ C' = \text{diag}(s_1(C), \ldots, s_k(C), 0, \ldots, 0) \quad \text{and} \quad T' = \sum_{j=1}^k s_j(T) (b_j, \cdot) a_j \]
which satisfy $\text{tr}(C'T') = \sum_{i=1}^k s_i(T) s_j(C) \langle e_i, U h_i \rangle \langle g_i, V f_j \rangle$. By construction, one readily verifies that $(a_j)_{j=1}^k, (b_j)_{j=1}^k$ are orthonormal systems in $\mathbb{C}^N$ so $s_j(T') = s_j(T)$ for all $j = 1, \ldots, N$. Thus von Neumann’s original result \cite{11} yields
\[ |\text{tr}(CUV)| = |\text{tr}(C'T')| \leq \sum_{j=1}^N s_j(C) s_j(T') = \sum_{j=1}^k s_j(C) s_j(T). \]

“≥”: We first consider unitary operators $U_T \in B(\mathcal{X}), V_T \in B(\mathcal{Y})$ such that $U_T h_j = e_j$ and $V_T f_j = g_j$ for all $j = 1, \ldots, k$. This is always possible by completing the respective orthonormal systems $(e_j)_{j=1}^k, \ldots$ to orthonormal bases $(e_j)_{j=1}^k, \ldots$ which can then be transformed into each other via some unitary. This allows us to construct $\tilde{T} := U_T TV_T = \sum_{j=1}^N s_j(T) (f_j, \cdot) e_j$ such that
\[ \text{tr}(CU\tilde{T}V) = \sum_{j=1}^N s_j(C) s_j(T) \langle e_j, \tilde{U} e_i \rangle \langle f_i, \tilde{V} f_j \rangle \]
for any $\tilde{U} \in U(\mathcal{X}), \tilde{V} \in U(\mathcal{Y})$. Of course $S_C(T) = S_C(\tilde{T})$ and the latter satisfies
\begin{itemize}
  \item $r \in S_C(T)$: choose $\tilde{U} = \text{id}_\mathcal{X}, \tilde{V} = \text{id}_\mathcal{Y}$ and also
  \item $0 \in S_C(\tilde{T})$: choose $\tilde{U}, \tilde{V}$ as cyclic shift on the first $k$ basis elements, i.e.
\end{itemize}
\[ \tilde{U} : \mathcal{X} \to \mathcal{X}, \quad e_j \mapsto \begin{cases} e_{j+1} & j = 1, \ldots, k-1 \\ e_1 & j = k \\ e_j & j \in J \setminus \{1, \ldots, k\} \end{cases} \]
and similarly $\tilde{V}$ (on $\{f_1, \ldots, f_k\}$).

Now because the unitary group $U(\mathcal{Y})$ on any Hilbert space $\mathcal{Y}$ is path-connected \cite{11} and because the mapping $f : B(\mathcal{X}) \times B(\mathcal{Y}) \to \mathbb{C}, (U, V) \mapsto \text{tr}(CUV)$ is continuous, the image $f(U(\mathcal{X}) \times U(\mathcal{Y}))$ has to be path-connected as well. In particular, 0 and $r$ are path-connected within $S_C(T)$, i.e. for every $s \in [0, r]$ there exists $\phi(s) \in [0, 2\pi)$ such that $se^{i\phi(s)} \in S_C(\tilde{T}) = S_C(T)$.

\textbf{2}This can be done for example by sufficiently expanding the “smaller” orthonormal systems in $\mathcal{X}$ or $\mathcal{Y}$ and possibly passing to new subspaces $Z'_1 \supset Z_1$ or $Z'_2 \supset Z_2$ which is always doable because we are in infinite dimensions. The particular choice of $Z'_1$ and $Z'_2$ is irrelevant because we only need the orthonormal systems which represent $C$ and $T$ to be contained within these finite-dimensional subspaces.

\textbf{3}The standard argument for this goes as follows, cf. \cite{12} Proof of Thm. 12.37: For every $U \in U(\mathcal{Y})$ there exists self-adjoint $Q \in B(\mathcal{Y})$ such that $U = \exp(itQ)$. Then $t \mapsto T(t) := \exp(itQ)$ is a continuous mapping of $[0, 1]$ into $U(\mathcal{Y})$ with $T(0) = \text{id}_\mathcal{Y}$ and $T(1) = U$. Thus every unitary operator is path-connected to the identity which implies path-connectedness of $U(\mathcal{Y})$. 

Finally, we can use the fact that $S_C(T)$ is circular—which follows easily by replacing $U$ by $e^{i\varphi}U \in \mathcal{U(X)}$ with $\varphi \in [0, 2\pi]$—to conclude $S_C(T) \supseteq K_r(0)$ and thus $S_C(T) = K_r(0)$.

**Theorem 3.2.** Let $C \in \mathcal{B}^q(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{B}^p(\mathcal{Y}, \mathcal{X})$ with $p, q \in [1, \infty]$ conjugate. Then

\begin{equation}
\sup_{U \in \mathcal{U}(\mathcal{X}), V \in \mathcal{U}(\mathcal{Y})} |\text{tr}(CU'TV)| = \sum_{j=1}^{\infty} sj(C)s_j(T).
\end{equation}

In particular, one has $S_C(T) = K_r(0)$ with $r := \sum_{j=1}^{\infty} s_j(C)s_j(T)$.

**Proof.** By Lemma 2.1 $C = \sum_{j=1}^{\infty} s_j(C)\langle e_{j'}, \cdot \rangle f_j$, $T = \sum_{j=1}^{\infty} s_j(T)\langle g_{j'}, \cdot \rangle h_j$ for some orthonormal systems $(e_j)_{j \in \mathbb{N}}$, $(h_j)_{j \in \mathbb{N}}$ in $\mathcal{X}$ and $(f_j)_{j \in \mathbb{N}}$, $(g_j)_{j \in \mathbb{N}}$ in $\mathcal{Y}$. This allows us to define finite rank approximations $C_n := \sum_{j=1}^{n} s_j(C)\langle e_{j'}, \cdot \rangle f_j$ and $T_n := \sum_{j=1}^{n} s_j(T)\langle g_{j'}, \cdot \rangle h_j$. To pass to the original operators $C, T$, we use Remark 2.2 to see

\[ \lim_{n \to \infty} \|C_n - C\|_p = 0 \quad \text{and} \quad \lim_{n \to \infty} \|T_n - T\|_q = 0. \]

Because of this we may apply Proposition 2.10 and Lemma 3.1 to obtain

\[ \overline{S_C(T)} = \lim_{n \to \infty} \overline{S_{C_n}(T_n)} = \lim_{n \to \infty} K_r_n(0) \]

with $r_n := \sum_{j=1}^{n} s_j(C)s_j(T)$. Using the obvious fact $\Delta(K_r(0), K_r_n(0)) = |r - r_n|$ for all $n \in \mathbb{N}$ one readily verifies $\overline{S_C(T)} = \lim_{n \to \infty} K_r_n(0) = K_r(0)$ with $r = \sum_{j=1}^{\infty} s_j(C)s_j(T)$.

**Remark 3.3.** To see that the supremum in (3.1) is not necessarily a maximum, consider $\mathcal{H} = \ell_2(\mathbb{N})$ with standard basis $(e_j)_{j \in \mathbb{N}}$. Now the positive definite trace-class operator $C = \sum_{j=1}^{\infty} \frac{1}{2}\langle e_{j'}, \cdot \rangle e_j$ as well as the compact operator $T = \sum_{k=1}^{\infty} \frac{1}{2}\langle e_{k+1'}, \cdot \rangle e_{k+1}$ satisfy

\[ \text{tr}(CU'TV) = \sum_{j=1}^{\infty} \frac{1}{2}\langle e_{j'}, UTVe_j \rangle = \sum_{j,k=1}^{\infty} \frac{1}{2}\langle e_{j'}, UVe_{k+1} \rangle\langle e_{k+1}, Ve_j \rangle \]

for any $U, V \in \mathcal{U}(\mathcal{H})$. We know that $\sup_{U, V \in \mathcal{U}(\mathcal{H})} |\text{tr}(CU'TV)| = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^2$ but if this was a maximum, then by the above calculation $\langle e_{j'}, UVe_{k+1} \rangle = \langle e_{k+1}, Ve_j \rangle = \delta_{jk}$ for all $j, k \in \mathbb{N}$. The only operators which satisfy these conditions are the left- and the right-shift, respectively, both of which are not unitary—a contradiction.

Finally, we are prepared to extend inequality (1.3) to Schatten-class operators on separable Hilbert spaces.

**Theorem 3.4.** Let $C \in \mathcal{B}^p(\mathcal{H})$, $T \in \mathcal{B}^q(\mathcal{H})$ both be self-adjoint with $p, q$ conjugate and let the positive semi-definite operators $C^+, T^+$ and $C^-, T^-$ denote the positive and negative part of $C$, $T$, respectively (i.e. $C = C^+ - C^-$, $T = T^+ - T^-$). Then

\begin{equation}
\sup_{U \in \mathcal{U(\mathcal{H})}} \text{tr}(CU'TU) = \sum_{j=1}^{\infty} \left(\lambda_j^+(C^+)\lambda_j^+(T^+) + \lambda_j^-(C^-)\lambda_j^-(T^-)\right)
\end{equation}
as well as

\[(3.3) \quad \inf_{U \in \mathcal{U} \in \mathcal{H}} \text{tr}(CU^TU) = -\sum_{j=1}^{\infty} (\lambda_j^+(C^+)\lambda_j^-(T^-) + \lambda_j^+(C^-)\lambda_j^-(T^+) ) . \]

In particular, one has:

\[-\sum_{j=1}^{\infty} (\lambda_j^+(C^+)\lambda_j^-(T^-) + \lambda_j^+(C^-)\lambda_j^-(T^+) ) \leq \text{tr}(CT) \leq \sum_{j=1}^{\infty} (\lambda_j^+(C^+)\lambda_j^-(T^+) + \lambda_j^+(C^-)\lambda_j^-(T^-) ) \]

Proof. Let \( C \in B^p(\mathcal{H}) , \ T \in B^q(\mathcal{H}) \) both be self-adjoint with \( p,q \) conjugate and first assume that \( T \) has at most \( k \in \mathbb{N} \) non-zero eigenvalues. Then the following is straightforward to show:

\[
\begin{align*}
\max \text{conv} (\text{tr}(CT)) &= \sum_{j=1}^{k} \lambda_j^+(C^+)\lambda_j^-(T^+) + \sum_{j=1}^{k} \lambda_j^+(C^-)\lambda_j^-(T^-) \\
\min \text{conv} (\text{tr}(CT)) &= -\sum_{j=1}^{k} \lambda_j^+(C^+)\lambda_j^-(T^-) - \sum_{j=1}^{k} \lambda_j^+(C^-)\lambda_j^-(T^+) 
\end{align*}
\]

Note that in this case the (modified) eigenvalue sequences of \( T \) contains infinitely many zeros. Now let us address the general case. Choose any orthonormal eigenbasis \((e_n)_{n \in \mathbb{N}}\) of \( T \) with corresponding modified eigenvalue sequence (Lemma 2.5). Moreover, let \( \Pi_k \) be the projection onto the span of the first \( k \) eigenvectors of \( T \). Then \( \Pi_k T \Pi_k \) has at most \( k \) non-zero eigenvalues and our preliminary considerations combined with Corollary 2.11 and Theorem 2.13 (c) as well as Lemma 2.7 readily imply

\[
\begin{align*}
\sup_{U \in \mathcal{U} \in \mathcal{H}} \text{tr}(CU^TU) &= \max \text{tr}(\Pi_k T \Pi_k) = \max \text{tr}(\Pi_k T \Pi_k) \\
&= \text{tr}(\Pi_k T) = \text{tr}(\Pi_k T) \\
&= \lim_{k \to \infty} \text{tr}(\Pi_k T) = \lim_{k \to \infty} \text{tr}(\Pi_k T) \\
&= \lim_{k \to \infty} \left( \sum_{j=1}^{k} \lambda_j^+(C^+)\lambda_j^-(T^+) + \sum_{j=1}^{k} \lambda_j^+(C^-)\lambda_j^-(T^-) \right) \\
&= \lim_{k \to \infty} \left( \sum_{j=1}^{k} \lambda_j^+(C^+)\lambda_j^-(T^+) + \sum_{j=1}^{k} \lambda_j^+(C^-)\lambda_j^-(T^-) \right)
\end{align*}
\]

where we used the identity \( (\Pi_k T \Pi_k)^\pm = \Pi_k T \Pi_k \). Now, the last step is to show that \( \sum_{j=1}^{k} \lambda_j^+(C^+)\lambda_j^-(T^+) \) converges to \( \sum_{j=1}^{\infty} \lambda_j^+(C^+)\lambda_j^-(T^+) \). Let \( \epsilon > 0 \) and \( \text{w.l.o.g.} \ T \neq 0 \). As \( \lambda_j^+(C^+) \) is a sequence in \( \ell^1_{\infty}(\mathbb{N}) \) we find \( N \in \mathbb{N} \) with

\[
\left( \sum_{j=N+1}^{\infty} \lambda_j^+(C^+) \right)^{\frac{1}{p}} < \frac{\epsilon}{2\|T\|_q}
\]

where for \( p = \infty \), the left-hand side becomes \( \sup_{n \geq N} \lambda_n^+(C^+) = \lambda_{N+1}^+(C^+) \).

Either way, associated to this \( N \) one can choose \( K \geq N \) such that the first \( N \) largest eigenvalues of \( T^+ \) are listed in \( (\lambda_j^+(\Pi_k T^+ \Pi_k))_{j \in \mathbb{N}} \) and thus \( \lambda_j^+(T^+) = \lambda_j^+(\Pi_k T^+ \Pi_k) \) for all \( j = 1, \ldots , N \). Putting things together and using Hölder’s
inequality yields
\[
\left| \sum_{j=1}^{K} \lambda_j^+ (C^+) \lambda_j^+ (\Pi_K T^+ \Pi_K) - \sum_{j=1}^{\infty} \lambda_j^+ (C^+) \lambda_j^+ (T^+) \right|
\leq 2 \| T^+ \|_q \left( \sum_{j=N+1}^{\infty} \left( \lambda_j^+ (C^+) \right)^p \right)^{1/p} < 2 \| T \|_q \frac{\varepsilon}{2 \| T \|_q} = \varepsilon.
\]

The case of \( C^-, T^- \) as well as the infimum-estimate are shown analogously which concludes the proof.

Therefore if \( C, T \) are self-adjoint (i.e. \( W_C(T) \subseteq \mathbb{R} \)), a path-connectedness argument similar to the proof of Lemma 3.1 shows \( (a, b) \subseteq W_C(T) \subseteq [a, b] \) with \( a \) (\( \leq 0 \)) given by (3.3) and \( b \) (\( \geq 0 \)) given by (3.2). In particular, \( W_C(T) = [a, b] \).

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A. PROOF OF THEOREM 2.13. The overall idea is to transfer properties of $W_C(T)$ from finite to infinite dimensions via the set convergence introduced in Section 2.2. However, we first need two auxiliary results to characterize the star-center of $W_C(T)$ later on.

**Lemma 4.1.** Let $T \in \mathcal{K}(\mathcal{X})$ and $(e_k)_{k \in \mathbb{N}}$ be any orthonormal system in $\mathcal{X}$. Then

(a) $\sum_{k=1}^{n} |\langle e_k, Te_k \rangle| \leq \sum_{k=1}^{n} s_k(T)$ for all $n \in \mathbb{N}$ and

(b) $\lim_{k \to \infty} \langle e_k, Te_k \rangle = 0$.

**Proof.**

(a) Consider a Schmidt decomposition $\sum_{m=1}^{\infty} s_m(T) \langle f_m, \cdot \rangle g_m$ of $T$ so

$$\sum_{k=1}^{n} |\langle e_k, Te_k \rangle| \leq \sum_{m=1}^{\infty} s_m(T) \left( \sum_{k=1}^{n} \langle e_k, f_m \rangle \langle g_m, e_k \rangle \right).$$

Defining $\lambda_m := \sum_{k=1}^{n} |\langle e_k, f_m \rangle \langle g_m, e_k \rangle|$ for all $m \in \mathbb{N}$, using Cauchy-Schwarz and Bessel’s inequality one gets

$$\lambda_m \leq \left( \sum_{k=1}^{n} |\langle e_k, f_m \rangle|^2 \right)^{1/2} \left( \sum_{k=1}^{n} |\langle g_m, e_k \rangle|^2 \right)^{1/2} \leq 1.$$
for all $m \in \mathbb{N}$. On the other hand, said inequalities also imply

$$\sum_{m=1}^{\infty} \lambda_m \leq \sum_{k=1}^{n} \left( \sum_{m=1}^{\infty} |\langle e_k, f_m \rangle|^2 \right)^{1/2} \left( \sum_{m=1}^{\infty} |\langle g_m, e_k \rangle|^2 \right)^{1/2} \leq \sum_{k=1}^{n} \|e_k\|^2 = n.$$ 

Hence, because $(s_m(T))_{m \in \mathbb{N}}$ is decreasing by construction, an upper bound of $\sum_{m=1}^{\infty} s_m(T) \lambda_m$ is obtained by choosing $\lambda_1 = \ldots = \lambda_n = 1$ and $\lambda_j = 0$ whenever $j > n$. This shows the desired inequality. A proof of (b) can be found, e.g., in [11, Lemma 16.17].

**Lemma 4.2.** Let $C \in B^p(\mathcal{H})$ with $p \in (1, \infty]$ and let $q \in [1, \infty)$ such that $p, q$ are conjugate. Furthermore, let $(e_n)_{n \in \mathbb{N}}$ be any orthonormal system in $\mathcal{H}$. Then

$$\lim_{n \to \infty} \frac{1}{n^{1/p}} \sum_{k=1}^{n} \langle e_k, Ce_k \rangle = 0.$$ 

**Proof.** First, let $p = \infty$, so $q = 1$. As $C$ is compact, by Lemma 4.1(b) one has $\lim_{k \to \infty} \langle e_k, Ce_k \rangle = 0$, hence the sequence of arithmetic means converges to zero as well. Next, let $p \in (1, \infty)$ and $\epsilon > 0$. Moreover, we assume w.l.o.g. $C \neq 0$ so $s_1(C) = \|C\| \neq 0$. As $C \in B^p(\mathcal{H})$, one can choose $N_1 \in \mathbb{N}$ such that $\sum_{k=N_1+1}^{\infty} s_k(C)^p < \frac{\epsilon^p}{N}$ and moreover $N_2 \in \mathbb{N}$ such that $\frac{\epsilon}{2} \sum_{k=N_1+1}^{n} \frac{1}{n^{1/q}} < \frac{\epsilon}{2} \sum_{k=N_1+1}^{n} s_k(C)$ for all $n \geq N_2$. Then, for any $n \geq N := \max\{N_1 + 1, N_2\}$, Lemma 4.1 and Hölder’s inequality yield the estimate

$$\left| \frac{1}{n^{1/q}} \sum_{k=1}^{n} \langle e_k, Ce_k \rangle \right| \leq \frac{1}{n^{1/q}} \sum_{k=1}^{n} s_k(C) + \frac{1}{n^{1/q}} \sum_{k=N_1+1}^{n} s_k(C)
\leq \frac{1}{n^{1/q}} \sum_{k=1}^{n} s_k(C) + \left( \sum_{k=N_1+1}^{n} s_k(C)^p \right)^{1/p} \left( \sum_{k=1}^{n} \frac{1}{n} \right)^{1/q}
< \frac{\epsilon}{2} + \left( \sum_{k=N_1+1}^{\infty} s_k(C)^p \right)^{1/p} \left( \frac{n-N_1}{n} \right)^{1/q} \leq \epsilon.$$ 

What we also need is some mechanism to associate bounded operators on $\mathcal{H}$ with matrices. In doing so, let $(e_n)_{n \in \mathbb{N}}$ be some orthonormal basis of $\mathcal{H}$ and let $(\hat{e}_i)_{i=1}^{n}$ be the standard basis of $\mathbb{C}^n$. For any $n \in \mathbb{N}$ we define $\Gamma_n : \mathbb{C}^n \to \mathcal{H}$, $\hat{e}_i \mapsto \Gamma_n(\hat{e}_i) := e_i$ and its linear extension to all of $\mathbb{C}^n$. With this, let

$$(4.1) \quad [\cdot]_n : \mathcal{B}(\mathcal{H}) \to \mathbb{C}^{n \times n}, \quad A \mapsto [A]_n := \Gamma_n^* A \Gamma_n$$

be the operator which “cuts out” the upper $n \times n$ block of (the matrix representation of) $A$ with respect to $(e_n)_{n \in \mathbb{N}}$. The key result now is the following:

**Proposition 4.3.** Let $C \in B^p(\mathcal{H})$, $T \in B^q(\mathcal{H})$ with $p, q \in [1, \infty]$ conjugate be given. Furthermore, let $(e_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be arbitrary orthonormal bases of $\mathcal{H}$. Then

$$\lim_{n \to \infty} W_{[(\cdot)]_n} \left( [T]_{2n}^g \right) = W_C(T)$$
where \([ \cdot ]_k^C\) and \([ \cdot ]_k^N\) are the maps given by (4.1) with respect to \((e_n)_{n \in \mathbb{N}}\) and \((\varphi_n)_{n \in \mathbb{N}}\), respectively. Moreover, if \(C\) and \(T\) both are normal then

\[
\lim_{n \to \infty} P_{|C|_k^N}(|T|_n^2) = P_{C}(T).
\]

where \((e_n)_{n \in \mathbb{N}}\) and \((\varphi_n)_{n \in \mathbb{N}}\) are the orthonormal bases of \(\mathcal{H}\) which diagonalize \(C\) and \(T\), respectively.

Proof. For \(p = 1, q = \infty\) (or vice versa) proofs are given in [3] Thm. 3.1 & 3.6] which can be adjusted to \(p, q \in (1, \infty)\) by minimal modifications. ■

With these preparations we are ready for proving our main result about the C-numerical range of Schatten-class operators.

Proof of Theorem 2.7.3: (a): For arbitrary orthonormal bases \((e_n)_{n \in \mathbb{N}}\) and \((\varphi_n)_{n \in \mathbb{N}}\) of \(\mathcal{H}\) as well as any \(n \in \mathbb{N}\), it is readily verified that

\[
\frac{\text{tr}(|C|_n^{2n}) \text{tr}(|T|_n^{2n})}{2n} = \frac{\text{tr}(|C|_n^{2n}) \text{tr}(|T|_n^{2n})}{(2n)^{1/p}} = \left(\frac{1}{(2n)^{1/q}} \sum_{j=1}^{2n} (e_j, C e_j)\right) \cdot \left(\frac{1}{(2n)^{1/p}} \sum_{j=1}^{2n} (\varphi_j, T \varphi_j)\right).
\]

Both factors converge and, by Lemma 4.2, at least one of them goes to 0 as \(n \to \infty\). Moreover, \(W_{|C|_n^{2n}}\) is star-shaped with respect to \((\text{tr}(|C|_n^{2n}) \text{tr}(|T|_n^{2n})/(2n))\) for all \(n \in \mathbb{N}\), cf. [2] Thm. 4. Because Hausdorff convergence preserves star-shapedness [3] Lemma 2.5 (d), Proposition 4.3 implies that \(W_{C}(T)\) is star-shaped with respect to \(0 \in \mathbb{C}\).

For what follows let \((e_n)_{n \in \mathbb{N}}\) and \((\varphi_n)_{n \in \mathbb{N}}\) be the orthonormal bases of \(\mathcal{H}\) which diagonalize \(C\) and \(T\), respectively.

(b): W.l.o.g. let \(C\) be normal with collinear eigenvalues. Since \(C\) is compact (i.e. its eigenvalue sequence is a null sequence) there exists \(\varphi \in [0, 2\pi)\) such that \(e^{i\varphi} C\) is self-adjoint and by Proposition 4.3 we obtain

\[
W_{C}(T) = W_{e^{i\varphi} C}(e^{-i\varphi} T) = \lim_{n \to \infty} W_{|e^{i\varphi} C|_n^{2n}}(e^{-i\varphi} T)_n^{2n}.
\]

Moreover, as \([e^{i\varphi} C]_n^{2n} \in \mathbb{C}^{2n \times 2n}\) is hermitian for all \(n \in \mathbb{N}\) we conclude that \(W_{e^{i\varphi} C}(e^{-i\varphi} T)_n^{2n}\) is convex, cf. [15]. The fact that Hausdorff convergence preserves convexity [3] Lemma 2.5 (c)] then yields the desired result.

(c): The inclusion \(P_{C}(T) \subseteq W_{C}(T)\) is shown exactly like [3] Thm. 3.4–first inclusion. For the second inclusion, we note that by assumption \(|C|_n^{2n}\) and \(|T|_n^{2n}\) are diagonal and thus normal for all \(n \in \mathbb{N}\). Hence [14] Coro. 2.4] tells us

\[
W_{|C|_n^{2n}}(|T|_n^{2n}) \subseteq \text{conv}(P_{|C|_n^{2n}}(|T|_n^{2n}))
\]
for all $n \in \mathbb{N}$. Using that Hausdorff convergence preserves inclusions \cite{3} Lemma 2.5 (a), \cite{4.2} together with Proposition \cite{4.3} yields

$$W_C(T) \subseteq W_C(T) = \lim_{n \to \infty} W_{C|\mathbb{C}^2_n}([T]_{2n}^2) \subseteq \lim_{n \to \infty} \text{conv}(P_{C|\mathbb{C}^2_n}([T]_{2n}^2)) = \text{conv}(P_C(T)).$$

Finally, applying the closure and the convex hull to the inclusions $P_C(T) \subseteq W_C(T)$ yields $\text{conv}(P_C(T)) \subseteq \text{conv}(W_C(T)) = W_C(T)$, where the last equality is due to (b), and thus $W_C(T) = \text{conv}(P_C(T))$. \hfill \blacksquare

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