Multistability in dynamics of an encapsulated bubble contrast agent: coexistence of three attractors

I R Garashchuk¹, D I Sinelshchikov² and N A Kudryashov³
Department of Applied Maths, NRNU MEPhI, Kashirskoe shosse 31, Moscow, Russia
E-mail: ¹ ivan.mail4work@yandex.ru, ² disine@gmail.com, ³ nakudr@gmail.com

Abstract. In this work we discuss complex dynamics arising in a model describing behavior of an encapsulated bubble contrast agent oscillating close to an elastic wall. We demonstrate presence of three coexisting attractors in the system. We propose an efficient numerical procedure based on the continuation method that can be used to locate the area of coexistence of these attractors in the parameters space. We provide area of coexistence of three attractors obtained by means of the proposed procedure.

1. Introduction

Nowadays there is a rising interest towards physical models with multistability and hidden oscillations [1]. It is known that multistability is possible in models, describing microbubble contrast agent dynamics [2, 3], which are encapsulated micrometer size gas bubbles. Contrast agents are used in medical ultrasound imaging in order to improve echogenicity and more complicated applications related with targeted drug delivery and non-invasive therapy. Modern models of contrast agents are based on the Rayleigh–Plesset equation [4]. They are modified accordingly to take into account the influence of the shell, external ultrasound pressure field and some other effects like a blood vessel wall close to the oscillating bubble. Thus, the dynamical system being discussed has a form of a forced non–linear oscillator. Such system has no fixed points and according to classification in [1] all the attractors are hidden.

We believe that multistability is not beneficial for contrast agent applications, because it can lead to unpredictable behavior [2]. Locating the multistability areas can be useful, so they can be avoided in applications. Finding all the coexisting attractors is a hard problem. Direct computation of the attractors’ basins is a very CPU intense task. Locating a multistability area with the help of this approach would lead to such calculations for a lot of parameter values, which can be very time consuming. Thus, it would be interesting to have another method for localizing areas of multistability. A good candidate is the continuation method [1, 2, 5]. In earlier works on microbubble’s dynamics only two coexisting attractors were presented [2, 3]. Here we focus on an area of multistability with three coexisting attractors.

2. The method applied

Here we consider the following model of an encapsulated gas bubble oscillating in the vicinity of an elastic wall [4]:
\[
\left(1 - \frac{1}{2}W - \frac{\dot{R}}{c}\right)R\ddot{R} + \frac{3}{2} \left(1 - \frac{2}{3}W - \frac{\dot{R}}{3c}\right)\dot{R}^2 = \frac{1}{\rho_1} \left[1 + \frac{\dot{R}}{c} + \frac{R}{c} \frac{d}{dt}\right] (P_r - P_{ac} \sin(\omega t)),
\]

where

\[
W = \frac{(\rho_1 - \beta) R}{(\rho_1 + \beta) d} + \frac{(\beta - \rho_3) R}{(\beta + \rho_3) (d + h)} - \frac{(\rho_1 - \beta) (\beta - \rho_3) R}{(\rho_1 + \beta) (\beta + \rho_3) h},
\]

\[
P_r = \left(P_0 + \frac{2\sigma}{R_0}\right) \left(\frac{R_0}{R}\right)^{3\gamma} - \frac{4\eta\dot{R}}{R} - \frac{2\sigma}{R} - P_0,
\]

$R$ is the bubble radius, $P_{stat} = 100$ kPa is the static pressure, $P_v = 2.33$ kPa is the vapor pressure, $P_0 = P_{stat} - P_v$, $P_{ac}$ is the magnitude of the pressure of the external field, $\sigma = 0.0725$ N/m is the surface tension, $\rho_1 = 1000$ kg/m$^3$ is the density of the liquid inside the blood vessel, $\rho_2 = 1060$ kg/m$^3$ is the blood vessel wall density, $\rho_3 = 1000$ kg/m$^3$ is the density of the fluid surrounding the blood vessel, $\eta = 0.001$ Ns/m$^3$ is the viscosity of the liquid, $c = 1500$ m/s is the sound speed, $\gamma = 4/3$ is the polytropic exponent, $\nu = 0.5$ is the Poisson’s ratio for an elastic wall, $\beta = \rho_2 \nu / (1 - \nu)$ is a characteristic of the wall, $h = 1$ mm is the thickness of the wall, $d$ denotes the distance between the wall and the center of the bubble and is close to $R_0$, shell parameters: $\chi = 0.22$ N/m and $\kappa_S = 2.5 \cdot 10^{-9}$ kg/s that correspond to the SonoVue contrast agent with equilibrium radius $R_0 = 1.72 \mu$m [6]. For numerical calculations we use the following non-dimensional variables: $r = R/R_0$, $\tau = \omega_0 t$, where $\omega_0^2 = 3\kappa P_0 / (\rho R_0^2) + 2(3\kappa - 1)\sigma / R_0 + 4\chi / R_0$ is the natural frequency of bubble oscillations [7]. We denote non-dimensional radial speed by $u$.

**Figure 1.** Three coexisting attractors at $P = 2.35$ MPa, $\omega = 5.567$ s$^{-1}$: chaotic (Poincare section) and two 3-periodic attractors (phase portraits, bold dots represent points in the corresponding Poincare sections). We refer to them as follows: left – “first”, middling - “second”, right – “third”.

Phase portraits of three coexisting attractors are given in Fig. 1. For convenience below we will refer to these attractors as “first”, “second” and “third” in the ordering from left to right respectively as they are placed in Fig 1. Different attractors can be effectively distinguished by the value of the maximal Lyapunov exponent (below we will refer to it as $\lambda_{max}$), corresponding to each attractor. Basins of attraction corresponding to attractors in Fig. 1 are presented in Fig. 2. One can observe that these basins have quite complicated shapes. Consequently, random picking of several points for initial conditions could easily lead to finding not all of them.

Once we have found these attractors we can use the continuation tool to determine the initial conditions leading to each attractor for the other parameter values. To establish the borderline of coexistence of all three attractors, we can calculate the maximal of the Lyapunov exponents.
Figure 2. Bassins of attractors presented in Fig. 1: red – the first attractor (chaotic), black – the second attractor, gray - the third one.

of the attractors for consecutive values of \((P,\omega)\), while moving towards border in the parameter space. For the third attractor the bifurcation of disappearance is accompanied by the condition \(\lambda_{\text{max}} = 0\). For the chaotic attractor when passing the point of disappearance the condition \(\lambda_{\text{max},n} \cdot \lambda_{\text{max},n+1} < 0\) is met. Let us describe the numerical procedure of determining this borderline, which allows us to perform calculations only in a neighborhood of the borderline instead of scanning the entire parameters area and computing bassins of attractors.

Let us assume we have found a point \((P_0^0,\omega_0^0)\) which is close to the borderline of disappearance of one of the attractors. Such points can be obtained by using the very same procedure (which is described below) with larger step size by parameters and more rough restrictions on the precision of satisfying conditions of disappearance of the attractor. It allows us to get rough estimations of some points of the desired borderline; initially acquired attractors (see Fig. 1) can be the starting point to get these estimates:

(i) Start from the point \((P_0^0,\omega_0^0)\) taking initial conditions from the attractor, which is assumed to disappear when moving through the borderline (see illustration in Fig. 3)
(ii) Calculate \(\lambda_{\text{max}}\) on every step \((P_i^0,\omega_i^0)\) and move towards border with small steps while taking the initial conditions for each following step by the continuation method
(iii) Stop when the disappearance condition is met with precision high enough and save the final point \((P_f^0,\omega_f^0)\) as the point lying on the borderline
(iv) Choose \((P_1^0,\omega_1^0)\) close to \((P_0^0,\omega_0^0)\) along the desired borderline. Move from \((P_0^0,\omega_0^0)\) to \((P_1^0,\omega_1^0)\) using the continuation method in order to stay on the same attractor.
(v) Start the procedure from step one treating \((P_1^0,\omega_1^0)\) as the starting point

It allows us to move along the borderline to a desirable point \((P^K,\omega^K)\) and eventually outline the entire area. It is important that all the points \((P^K_0,\omega^K_0)\) are chosen inside the area of coexistence of all the attractors, and they do not fall out of this area on any step. This procedure allows us to avoid a lot of long computations of basins of attractors.

The area of coexistence of three attractors in parameters space \((P,\omega)\) obtained by this algorithm is shown in Fig. 3. Note that on the right borderline (blue) the first attractor
disappears, while on the others the disappearing attractor is the third one. Also, attractors presented in Fig. 1 do not stay the same in this entire area, but can deform quite significantly. For example, in the upper-left corner of this area the first attractor goes through reversed period-doubling cascade and becomes periodic (e.g. there exist values of \((P, \omega)\) for which this attractor is 4-periodic, while two others are still 3-periodic).

Although the region of parameters shown in Fig. 3 is small in comparison with the one investigated in [2], it shows that coexistence of more than two attractors is possible in this model. It is also possible that there are more small areas of coexistence of multiple attractors which have not been found yet. Note that the third attractor exists in a large region of parameters in higher pressures zone. In this work we are not interested in this area of parameters, because only two attractors coexist there (chaotic attractor disappears earlier) and we do not demonstrate it. The third attractor found here was missed in earlier work [2] due to its complicated area of existence (some areas of multistability with this attractor were mentioned as zones with only one attractor in [2]).

3. Conclusion
In this work we have found three coexisting hidden attractors in the model of a bubble contrast agent oscillating near an elastic wall. Phase portraits and basins of attraction are provided for certain value of parameters (see Fig.-s 1, 2). We suggest a numerical procedure that can help to establish the area of multistability with three coexisting attractors in \((P, \omega)\) parameters space. They key feature of the algorithm is that once we have localized all the coexisting attractors for one value of the parameters, we do not need to scan the initial conditions space for other parameters values anymore, but we can use the continuation approach instead. Thus we avoid very time consuming computations by means of the continuation method. The area of coexistence of the three attractors obtained by using this algorithm is shown in Fig. 3.
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