INTRODUCING CAFEIN, A NEW COMPUTATIONAL TOOL FOR STELLAR PULSATIONS AND DYNAMIC TIDES

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Draft version December 12, 2013

ABSTRACT

Here we present CAFEin, a new computational tool for investigating radiative dissipation of dynamic tides in close binaries and of non-adiabatic, non-radial stellar oscillations in isolated stars in the linear regime. For the latter, CAFEin computes the non-adiabatic eigenfrequencies and eigenfunctions of detailed stellar models. The code is based on the so-called Riccati method, a numerical algorithm that has been successfully applied to a variety of stellar pulsators, and which doesn’t suffer of the major drawbacks of commonly used shooting and relaxation schemes. Here we present an extension of the Riccati method to investigate dynamic tides in close binaries. We demonstrate CAFEin’s capabilities as a stellar pulsation code both in the adiabatic and non-adiabatic regime, by reproducing previously published eigenfrequencies of a polytrope, and by successfully identifying the unstable modes of a stellar model in the β Cephei/SPB region of the Hertzsprung-Russell diagram. Finally, we verify CAFEin’s behavior in the dynamic tides regime by investigating the effects of dynamic tides on the eigenfunctions and orbital and spin evolution of massive Main Sequence stars in eccentric binaries, and of hot Jupiter host stars. The plethora of asteroseismic data provided by the NASA’s Kepler satellite, some of which include the direct detection of tidally excited stellar oscillations, make CAFEin quite timely. Furthermore, the increasing number of observed short-period detached double white dwarfs (WD) and the observed orbital decay in the tightest of such binaries open up a new possibility of investigating WD interiors through the effects of tides on their orbital evolution.

Subject headings: (stars:) binaries: general – stars: interiors – stars: oscillations – dynamic tides

1. INTRODUCTION

The current state and evolution of binary systems is affected by a wide range of physical processes, the understanding of which is important in interpreting observations. In this work we focus on dynamic tides, the tidal regime in which the free oscillation modes (eigenmodes) of one of the binary components can be excited by the companion’s periodic tidal potential, with the driving frequency being comparable to the stellar eigenfrequencies. For this purpose, we have developed CAFEin (Code for non-Adiabatic, non-Radial Forced stEllar oscillatIoNs), a novel computational tool to investigate in detail the impact of dissipation of dynamic tides in close binaries. The efficiency of non-adiabatic, dynamic tides in exchanging angular momentum between the binary orbit and the component spins in a binary depends on the tidal energy dissipation mechanism and on its strength. In addition, this new dynamic-tides tool can be used for the study of non-forced stellar pulsations.

Thanks to NASA Kepler’s unprecedented photometric accuracy, the effects of dynamic tides have become readily visible in electromagnetic data ([Welsh et al. 2011]; [Thompson et al. 2012]). A phenomenal example among Kepler’s Objects of Interest (KOI) is KOI 54, a highly eccentric binary hosting two A stars; their light curves clearly reveal that the free oscillation modes of one or both stars are tidally excited ([Welsh et al. 2011]). The theoretical modeling of such features not only allows to further constrain the stellar and binary properties, but also potentially provides a probe to the stellar interiors, otherwise hidden to direct electromagnetic observations ([Burkart et al. 2012]; [Fuller & Lai 2012c]).

Pioneering investigations targeting the observed circularization periods of massive (O, B, F) binaries identified radiative damping as the main mechanism for the dissipation of dynamic tides. In particular, Zahn (1975) was the first to invoke radiative damping of tidally excited g-modes in these massive Main Sequence (MS) binaries. A systematic comparison between the circularization periods predicted by Zahn’s theory and the observations showed that many systems circularized well above the theoretically predicted period, showing evidence for a more efficient tidal dissipation mechanism ([Giuricin et al. 1984]; [North & Zahn 2003]; [Mazeh et al. 2006]). A highly promising explanation to this discrepancy was provided by Witte & Savonije (2001). The authors followed simultaneously the exchange of angular momentum between the stellar spins and the orbit due to tides and the evolution of the star’s eigenfrequencies due to natural stellar evolution. This detailed analysis demonstrated that a binary can be locked into a resonant state for a prolonged period of time (the so-called resonance locking). Such long-lasting resonances can dramatically speed up dissipation and hence tidal evolution of a binary’s orbit, yielding better agreement with the observed circularization periods. Here we note that such resonance locking is unlikely to be present in solar-type stars, as it requires that the modes form standing waves. In fact, it has been shown (e.g. [Goodman & Dickson 1998]; [Barker & Ogilvie 2010]) that the waves resonantly excited by the tide in these stars are highly nonlinear in the core and break. They are therefore traveling waves,
which suggests that the resonant locking mechanism of Witte & Savonije cannot operate in these systems.

Beyond the extensive work on tides in non-degenerate systems (see Zahn 2008 for a review), recent studies have focused on the effects of dynamic tides on the orbital evolution of detached binaries hosting white dwarfs (WD) and, in particular, double WDs (DWD). These binaries are widely recognized as important gravitational wave (GW) sources: they are the most numerous and guaranteed sources for the next-generation of space-based detectors sensitive to low-frequency GWs (e.g. LISA, Danzmann & the LISA study team, 1999, and references therein, and eLISA/NGO, Amaro-Seoane et al. 2012; see also Nelemans et al. 2001a, 2004; Liu et al. 2010; Rupetz et al. 2010 for theoretical predictions), and are currently observed electromagnetically. In the past few years, the Extremely Low Mass WD survey (ELM, Brown et al. 2010, 2012; Kilic et al. 2010, 2011) successfully quintupled the number of known detached DWDs expected to merge within a Hubble time, bringing the number of systems to 24 (see Table 4 by Kilic et al. 2012 for a summary of the currently known systems) and discovering DWDs with periods down to $\sim 12$ min (SDSS J065133.33$+$284243.3, hereafter J0651, Brown et al. 2011; Hermes et al. 2012). The exciting J0651 system harbors a tidally deformed He WD eclipsing a C/O WD in a detached binary. Since its orbital decay was recently measured (Hermes et al. 2012), J0651 is one of the cleanest astrophysical laboratories to test our understanding of tidal dissipation in these sources, and WD interiors.

Recent work has focused on the adiabatic tidal excitation of free oscillation modes in C/O WDs. Rathore et al. (2005) used polytropic models to represent C/O WDs in eccentric binaries and focused on the $f$-mode. They found that adiabatic tides can drive the modes to high amplitudes potentially becoming nonlinear. Similar conclusions were reached by Fuller & Lai (2011), who targeted adiabatic, dynamic tides in circular binaries, using detailed C/O WD models and focusing on the excitation of $g$-modes in the close passage through a resonance. These authors also found that mode excitations can cause significant deviations in the orbital evolution of DWDs from the pure point-mass assumption and are very important in the spin synchronization process. Their analysis, being limited to the adiabatic regime, does not include any dissipation, which can limit the non-linear growth of the resonant modes. As a follow-up on the violation of linearity found in the adiabatic treatment, Fuller & Lai (2012b) considered the tidal excitation of gravity waves in C/O DWDs treating dissipation via the so-called “outgoing wave boundary condition” (BC). Such a BC implicitly assumes the waves are damped at the WD’s surface via radiative damping or non-linear effects; as a result the formation of standing waves is prevented. They obtain results similar to Fuller & Lai (2011). Using a similar approach, Fuller & Lai (2012a) extended their investigation to He WDs, focusing mainly on the effect of tidal heating and its observational signatures. The authors found that tidal heat is likely deposited in the outer layers of the WD and that it can dominate the WD’s luminosity for the shortest orbital period binaries ($\lesssim 15$ min). Recent investigations by Valsecchi et al. (2012) and Burkart et al. (2012a) are more focused on the effect of dynamic tides on the orbital and spin evolution of DWDs. Valsecchi et al. (2012) applied CAFein to a He WD model representative of the He component in J0651 to investigate the effect of linear, dissipative (non-adiabatic) dynamic tides on its orbital evolution. Valsecchi et al. (2012) calculated the full tidal response of the WD as multiple modes are excited simultaneously for a wide range of driving frequencies. Burkart et al. (2012a) studied the effects of linear and non-linear dynamic tides in DWDs with circular orbits hosting both He and C/O WDs. In this study the dynamical tide is approximated as a superposition of standing waves and the WD response is treated as a simple harmonic oscillator with driving and dissipation. In the linear regime, the damping processes considered are thermal diffusion and turbulent convection, while in the non-linear regime the damping time for traveling waves is set by the $g$-mode group travel times. The radiative damping rate is also approximated in the quasi-adiabatic limit (from the star’s adiabatic eigenfunctions, instead of the full non-adiabatic eigenfunctions). We refer to Valsecchi et al. (2012) for a discussion and comparison with the results presented by Burkart et al. (2012a).

We have developed CAFein, a novel code to compute both non-adiabatic, non-radial stellar oscillations and isolated stars and forced stellar oscillations in close binaries. Following our understanding of tidal dissipation in non-degenerate stars, which is now able to explain the observed circularization periods in open cluster binaries (Zahn 1972; Witte & Savonije 2001), we consider radiative damping to be the main mechanism to dissipate dynamic tides in stars with radiative envelopes.

In this paper we describe in detail the mathematical and numerical implementation of CAFein and we present comparisons to past results found in the literature. In § 2 we introduce CAFein as a stellar oscillation code. In § 2.1 we briefly summarize the equations governing non-adiabatic and non-radial stellar oscillations and in § 2.2 we give a detailed description of the Riccati method implemented to solve such equations. In § 2.3 we test CAFein’s results and accuracy by calculating eigenfrequencies and eigenfunctions of different stellar models. In § 2.3.1 we calculate the adiabatic eigenfrequencies of a polytrope and check how the results change if the relevant parameters entering the Riccati method are varied. In § 2.3.2 we compare these eigenfrequencies with previously published results. In § 2.3.3 we verify the reliability of CAFein in the non-adiabatic regime by identifying the unstable modes of a stellar model in the $\beta$ Cephei/SPB region of the Hertzprung-Russell (HR) diagram. In § 2.4 we move onto dynamic tides and their implementation in CAFein. In § 3.1 we introduce the tide-generating potential. In § 3.2 we summarize the equations governing tidally excited stellar pulsations and the secular evolution of the orbital elements and stellar spin. In § 3.3 we describe the modifications applied to the Riccati method to solve the stellar pulsation equations when the tide-generating potential is included, and test such extension in § 3.4. In § 3.4.1 we reproduce the results presented by Pollfiet & Smeevers (1990) and show that a dynamical tide can be approximated as the sum of the

$\footnote{SPB = slowly pulsating B-type stars}$
equilibrium tide and another part reflecting the oscillatory properties of the star itself. In §3.4.1 we reproduce the results presented by Willems et al. (2003) on the orbital and spin evolution timescales due to dynamic tides for an eccentric binary hosting a 5 $M_\odot$ MS star and a neutron star. In §3.4.3 we compute the orbital and spin evolution timescales for a binary hosting a 1.5$M_\odot$ star and a hot Jupiter. We conclude in §1.

and a hot Jupiter. We conclude in §1 Cantiello et al. (2009). which could affect or excite non-radial stellar pulsations of outer convective regions (Stothers & Chin 1993), 

associated with Fe and He ionization lead to the formation of massive stars, which have convective cores and predominantly radiative envelopes. Furthermore, the comparisons we present with earlier published models are with studies where convection is neglected as well. However, here we note that peaks in the opacity associated with Fe and He ionization lead to the formation of outer convective regions (Stothers & Chin 1993), which could affect or excite non-radial stellar pulsations (Cantiello et al. 2009).

In the star’s frame we take the equilibrium state to be spherically symmetric and assume that the spatial and temporal part of the small perturbations can be written in Eulerian form as $f'(r, t) = f'(r)Y_l^m(\theta, \phi) e^{i \omega t}$ and similarly for the Lagrangian perturbations, denoted by $\delta f$. Here $\sigma$ is an eigenfrequency (below we will denote with $\omega = \sqrt{\sigma^2 R^3 (GM)^{-1}}$ its dimensionless counterpart), $Y_l^m(\theta, \phi)$ is a spherical harmonic, $l$ is the harmonic degree, and $m$ the azimuthal order.

Following the standard procedure, we then apply a small perturbation to the unperturbed star. Perturbing and linearizing the basic Eqs. (1)-(5) and introducing

$$
y_1 = \frac{\xi}{r}, \quad y_2 = \frac{1}{gr} \left( \frac{\rho'}{\rho} + \Phi' \right) = \frac{\sigma^2 r \xi_0}{g r} \quad (6)
$$

$$
y_3 = \frac{1}{gr} \Phi', \quad y_4 = \frac{1}{g} \frac{d \Phi'}{dr} \quad (7)
$$

$$
y_5 = \frac{\delta S_{\eta}}{c_p} \quad y_6 = \frac{\delta L_R}{L_R} \quad (8)
$$

yields

$$
r \frac{dy_1}{dr} = (v_g - 3) y_1 + \left[ \frac{l(l + 1)}{c_1 u^2} - V_R \right] y_2 + V_g y_3 + v_t y_5 \quad (9)
$$

$$
r \frac{dy_2}{dr} = (c_1 \omega^2 - A^2) y_1 + (A^2 - U + 1) y_2 - A^2 y_3 + v_t y_5 \quad (10)
$$

$$
r \frac{dy_3}{dr} = (1 - U) y_3 + y_4 \quad (11)
$$

$$
r \frac{dy_4}{dr} = U A^2 y_1 + U V_R y_3 + \left[ \frac{l(l + 1)}{c_1 u^2} - U \right] y_4 - U v_t y_5 \quad (12)
$$

$$
r \frac{dy_5}{dr} = V \left[ \nabla_{ad}(U - c_1 u^2) - 4 \nabla_{ad} - \nabla c_2 \right] y_1 + \left[ V \left( \nabla_{ad} - 4 \nabla_{ad} - c_2 \right) + 1 \right] y_2 + V c_2 y_3 + V V_{ad} y_4 + V \nabla (4 - k_3) y_5 - V \nabla y_6 \quad (13)
$$

$$
r \frac{dy_6}{dr} = \left[ l(l + 1) \frac{\nabla_{ad} - \nabla}{\nabla} - \epsilon_{ad} c_3 V \right] y_1 + \left[ \epsilon_{ad} c_3 V + l(l + 1) \left( \frac{\nabla_{ad} + c_3}{c_1 u^2} \right) \right] y_2 + \left[ \epsilon_{ad} c_3 V + \left( l(l + 1) - \epsilon_{ad} c_3 V \right) y_3 + \epsilon_{ad} c_3 - i w c_3 \right] y_5 - \frac{d \ln L_R}{d \ln \tau_{dyn}} y_6. \quad (14)
$$

Eqs. (9)-(14) represent the system of equations describing non-adiabatic and non-radial stellar pulsations. Here, $\xi$ is the displacement of a fluid element from the unperturbed position ($\xi_1$ and $\xi_0$ are its radial and orthogonal components, respectively), $g$ is the local gravity, $\omega^2 \equiv \sigma^2 R^3 (GM)^{-1}$ is the dimensionless squared eigenfrequency, $L_R$ is the radiative luminosity, and $c_4$ is the ratio of thermal to dynamical timescale ($\tau_{th}/\tau_{dyn}$). We will see below that the latter is used to determine the degree of adiabaticity. The remaining terms are summarized in Table I. Recall that the number of radial nodes in $\xi_1$ determines the radial order $n$ of each mode.

The homogeneous system of Eqs. (9)-(14) with the proper BCs constitute a well-posed eigenvalue problem with complex eigenvalue $\omega$. The real and imaginary part of the eigenvalue $\omega = (\omega_R + i \omega_I)$, respectively, represent the oscillation frequency and the linear growth ($\omega_I < 0$) or damping ($\omega_I > 0$) rate, respectively.

In the star’s interior, the thermal timescale is much longer than the oscillation period and the oscillation be-
Table 1

Terms entering the equations governing non-adiabatic and non-radial stellar pulsations: $M_r$ is the mass contained within a radius $r$, $c_p$ is the specific heat at constant pressure.

| Symbol | Expression |
|--------|------------|
| $\nabla$ | $\left( \frac{\partial n}{\partial r} \right)_S$ |
| $\nabla_{ad}$ | $\frac{\partial n}{\partial r} + \frac{\partial S}{\partial r} \frac{\partial n}{\partial S} \frac{\partial S}{\partial r}$ |
| $\Gamma_1$ | $\left( \frac{\partial n}{\partial r} \right)_S$ |
| $c_1$ | $\frac{\partial n}{\partial r} - \frac{\partial n}{\partial r}$ |
| $V$ | $\frac{\partial n}{\partial r} + \frac{\partial S}{\partial r} \frac{\partial n}{\partial S} \frac{\partial S}{\partial r}$ |
| $V_b$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $U$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $A^*$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $k_T$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $k_\rho$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $k_{ad}$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $c_p$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $\epsilon_T$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $\epsilon_\rho$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $\epsilon_n$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $\epsilon_{ad}$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $c_2$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $c_3$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $c_4$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $\frac{\partial n}{\partial r}$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $\frac{\partial n}{\partial r}$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |
| $\frac{\partial n}{\partial r}$ | $\frac{\partial n}{\partial r} = \frac{\partial n}{\partial r}$ |

The homogeneous system of Eqs. (9)-(14) greatly simplifies in the adiabatic case, since the terms involving the perturbation of the entropy ($y_s$) and radiative luminosity ($y_p$) are neglected. Furthermore, the remaining system of four Eqs. (9)-(12) forms an eigenvalue problem with real eigenvalue $\omega$. When computing adiabatic stellar oscillations with CAFein we use the zero-boundary limit in which the density and pressure vanish at the stellar surface, and we substitute Eq. (15) with $y_1 - y_2 + y_3 = 0$. Here we note that neither the equations nor the BCs involve the azimuthal order $m$, therefore the eigenvalue is $(2l+1)$-fold degenerate with respect to $m$.

The radial distribution of the modal families inside a star is determined by the run of the Brunt-Väisälä ($N$) and Lamb ($L_1$) frequencies, as they characterize the local vibrational properties of a star. The Lamb frequency is the inverse of the horizontal sound-crossing timescale

$$L_1^2 = \frac{l(l+1)c_s^2}{2}$$

where $c_s = \sqrt{\Gamma_1 p/\rho}$ is the isentropic sound speed. The Brunt-Väisälä frequency is the frequency of buoyancy oscillations

$$N^2 = g \left( \frac{1}{\Gamma_1} \frac{\partial n}{\partial r} - \frac{\partial n}{\partial r} \right).$$

In this work we follow the prescription from Brassard et al. [1991], which accounts for the buoyancy due to the gradient in composition. The high frequency oscillations ($\omega^2 > L_1^2, N^2$) have locally the characteristics of acoustic waves ($p$-modes). The low frequency oscillations ($\omega^2 < L_1^2, N^2$) have locally the characteristics of gravity waves ($g$-modes).

2.2. The Riccati Method

Here we describe the numerical method we have implemented in CAFein to solve the system of Eqs. (9)-(14). This so-called Riccati method, as introduced by Scotti [1973] and extended by Davis [1974], was applied for the first time to the stellar pulsation problem by Gautschy & Glatzel [1990]. Commonly, the system of differential Eqs. (9)-(14) is solved using relaxation and shooting schemes. The Riccati method differs from such techniques mainly in the type of equations that have to be solved; from a technical point of view it is really a shooting method. According to the Riccati method the linear first-order ordinary differential system describing a boundary eigenvalue problem is transformed into a numerically stable, non-linear initial value problem. This initial-value problem is then solved using a shooting method, where the eigenfrequency is the only shooting parameter to be iterated.

For a linear two-point boundary value problem in which the solutions change very rapidly, like the problem describing stellar oscillations, the advantages of the Riccati method become most clear. As pointed out by Takata & Löffler [2004], if commonly-used shooting methods are adopted, it is difficult to satisfy the matching condition in a numerically stable manner, as the eigenfunctions at the fitting point are strongly dependent on their values at the star’s boundaries. On the other hand, if Henyey-type relaxation methods are implemented in CAFein, the system of Eqs. (9)-(14) is solved using relaxation and shooting schemes.
From Eq. (25) we can derive two separate equations for \( u \) and \( v \). Substituting this into Eqs. (26) and (27) it is straightforward to show that \( R \) satisfies
\[
\frac{dR}{dr} = B + AR - RD - RCR.
\]
Equation (29) denotes the new system of non-homogeneous and non-linear differential equations that will be integrated instead of the original stellar pulsation problem. Equation (29) has to be solved together with \( J \) homogeneous BCs at both extrema of the integration interval. The most general form of the BCs can be written as
\[
Pu_{\text{Ric}} = Qv_{\text{Ric}}
\]
where \( P \) and \( Q \) are \( J \times J \) matrices. As either \( u_{\text{Ric}} \) or \( v_{\text{Ric}} \) can be considered as arbitrary, Eq. (30) uniquely determines \( R \). Given the form of the non-adiabatic BCs at the star’s center (15) and surface (18), we chose the vectors \( u_{\text{Ric}} \) and \( v_{\text{Ric}} \) as
\[
\begin{align*}
\begin{pmatrix}
 u_{\text{Ric}} \\
 v_{\text{Ric}}
\end{pmatrix} &= \begin{pmatrix}
 y_1 \\
 y_2 \\
 y_3 \\
 y_4 \\
 y_5 \\
 y_6
\end{pmatrix},
\end{align*}
\]
(31)
The non-adiabatic initial conditions on the Riccati matrix \( R \) at the star’s center and surface thus become
\[
R_c = \begin{pmatrix}
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
(32)
\[
R_s = \begin{pmatrix}
 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]
(33)

2.2.1. The Riccati Equation

We start by writing the original system of Eqs. (9)-(13) in the form
\[
\frac{dy}{dr} = My = \begin{pmatrix} A & B \\ C & D \end{pmatrix} y
\]
(23)
If the number of elements in the vector \( y \) is \( N \), then \( M \) is a square matrix of size \( N \times N \), while \( A, B, C, \) and \( D \) are square matrices of size \( J \), with \( N = 2J \). Next, we introduce two vectors \( u_{\text{Ric}} \) and \( v_{\text{Ric}} \) of size \( J \) which store the first and last \( J \) components of \( y \), respectively:
\[
y = \begin{pmatrix} u_{\text{Ric}} \\ v_{\text{Ric}} \end{pmatrix}
\]
(24)
Then, equation (23) can be rewritten as:
\[
\frac{d}{dr} \begin{pmatrix} u_{\text{Ric}} \\ v_{\text{Ric}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u_{\text{Ric}} \\ v_{\text{Ric}} \end{pmatrix}
\]
(25)
From Eq. (25) we can derive two separate equations for \( u_{\text{Ric}} \) and \( v_{\text{Ric}} \):
\[
\frac{du_{\text{Ric}}}{dr} = Au_{\text{Ric}} + Bv_{\text{Ric}}
\]
(26)
\[
\frac{dv_{\text{Ric}}}{dr} = Cu_{\text{Ric}} + Dv_{\text{Ric}}
\]
(27)
Defining the Riccati matrix \( R \) as
\[
u_{\text{Ric}} = Rv_{\text{Ric}}
\]
(28)
from Eqs. (26) and (27) it is straightforward to show that \( R \) satisfies
\[
\frac{dR}{dr} = B + AR - RD - RCR.
\]
(29)
det[\mathbf{R}^{in}(r_{fit}) - \mathbf{R}^{out}(r_{fit})] as initial guesses for the iteration of the exact eigenfrequencies. During this iteration we use a complex secant method to find the exact values of \( \omega_R \) and \( \omega_1 \) for which both the real and imaginary part of condition (39) are satisfied. Following Takata & Löffler (2004) we chose the fitting point \( r_{fit} \) based on the behavior of the Brunt-Väisälä and Lamb frequencies. Specifically, for a given frequency \( \omega_R \) we pick the innermost point where \( \bar{N}^2 \) or \( L^2 \approx \omega_R^2 \). For low frequency \( g \)-modes or high frequency \( p \)-modes, the fitting point where \( \bar{N}^2 \approx \omega_R^2 \) or \( L^2 \approx \omega_R^2 \), respectively, is very close to the star’s center. Scanning the parameter space in \( \omega \) following the procedure described above yields all eigenfrequencies.

2.2.3. Reimbedding

During the calculation of the eigenfrequencies (§2.2.2), \( \mathbf{v}_{Ric} \) might vanish and therefore \( \mathbf{R} \) becomes singular (recall Eq. (28)). In this case, we avoid the singularity by re-defining \( \mathbf{u}_{Ric} \) and \( \mathbf{v}_{Ric} \) according to the so-called “re-imbedding” procedure, as described by Takata & Löffler (2004). The new \( \mathbf{u}_{Ric}' \) and \( \mathbf{v}_{Ric}' \) are a transformation of the original \( \mathbf{u}_{Ric} \) and \( \mathbf{v}_{Ric} \) and are given by

\[
\begin{pmatrix}
\mathbf{u}_{Ric}'
\mathbf{v}_{Ric}'
\end{pmatrix}
= \mathbf{T}
\begin{pmatrix}
\mathbf{u}_{Ric}
\mathbf{v}_{Ric}
\end{pmatrix}
= \begin{pmatrix}
\mathbf{T}_{00} & \mathbf{T}_{01}
\mathbf{T}_{10} & \mathbf{T}_{11}
\end{pmatrix}
\begin{pmatrix}
\mathbf{u}_{Ric}
\mathbf{v}_{Ric}
\end{pmatrix}
\]  

(40)

where \( \mathbf{T} \) is a square \( N \times N \) matrix, while \( \mathbf{T}_{ij} \) are square \( J \times J \) submatrices (recall \( N = 2J \)). For the transformed Riccati matrix \( \mathbf{R}' \) we can write

\[
\mathbf{R}' = (\mathbf{T}_{00} \mathbf{R} + \mathbf{T}_{01})(\mathbf{T}_{10} \mathbf{R} + \mathbf{T}_{11})^{-1},
\]  

(41)

choosing matrix \( \mathbf{T} \) such that

\[
det(\mathbf{T}_{10} \mathbf{R} + \mathbf{T}_{11}) \neq 0.
\]  

(42)

The transformed Riccati matrix \( \mathbf{R}' \) is still a solution for Eq. (29), provided that \( \mathbf{A}, \mathbf{B}, \mathbf{C}, \) and \( \mathbf{D} \) are replaced by their prime (‘) counterparts.

\[
\begin{pmatrix}
\mathbf{A}' & \mathbf{B}'
\mathbf{C}' & \mathbf{D}'
\end{pmatrix}
= \mathbf{T}
\begin{pmatrix}
\mathbf{A} & \mathbf{B}
\mathbf{C} & \mathbf{D}
\end{pmatrix}
\]  

(43)

During the calculation of the eigenfrequencies, we apply this procedure whenever the Euclidean norm of \( \mathbf{R} \) (\( ||\mathbf{R}|| \)) = \( \sqrt{\sum_{i=0}^{J} \sum_{j=0}^{J} \mathbf{R}_{ij}^2} \) goes above a certain predefined value, switching to the permuted \( \mathbf{R}' \) with the lowest \( ||\mathbf{R}'|| \) (see §2.2.4 for the permutation criterion adopted during the calculation of the eigenfunctions). Here we note that for the fully non-adiabatic case, the integration variables \( y_i \) (\( i = 1 \to 6 \)) are complex. In this configuration, \( \mathbf{u}_{Ric} \) and \( \mathbf{v}_{Ric} \) have size 6, and we chose them so that

\[
\mathbf{u}_{Ric} = \begin{pmatrix}
y_1.R \\
y_4.R \\
y_5.R \\
y_1.1 \\
y_4.1 \\
y_5.1
\end{pmatrix}
= \begin{pmatrix}
\mathbf{u}_{Ric,R} \\
\mathbf{u}_{Ric,I}
\end{pmatrix}
\]  

and

\[
\mathbf{v}_{Ric} = \begin{pmatrix}
y_2.R \\
y_3.R \\
y_6.R \\
y_2.1 \\
y_3.1 \\
y_6.1
\end{pmatrix}
= \begin{pmatrix}
\mathbf{v}_{Ric,R} \\
\mathbf{v}_{Ric,I}
\end{pmatrix}.
\]  

(44)

The resulting \( \mathbf{R} \) has a size \( 6 \times 6 \) and the search for the permutation yielding \( \mathbf{R}' \) with the minimum norm would result in long computational times. However, given that

\[
\begin{pmatrix}
\mathbf{u}_{Ric,R} \\
\mathbf{u}_{Ric,I}
\end{pmatrix}
= \begin{pmatrix}
\mathbf{R}_1 & \mathbf{R}_2 \\
\mathbf{R}_3 & \mathbf{R}_4
\end{pmatrix}
\begin{pmatrix}
\mathbf{V}_{Ric,R} \\
\mathbf{V}_{Ric,I}
\end{pmatrix}
\]  

(45)

we minimize the numbers of trials finalized to find the permutation yielding \( \mathbf{R}' \) with the minimum norm by only permuting the (dominant) real part of \( \mathbf{R} \) (\( \mathbf{R}_1 \) and \( \mathbf{R}_4 \)) and applying that same permutation to the imaginary part (\( \mathbf{R}_2 \) and \( \mathbf{R}_3 \)). In the adiabatic case, instead, only Eqs. (40)-(42) are solved (the terms containing \( y_5 \) and \( y_6 \) are neglected) and the eigenfrequency is purely real. This reduces the size of the Riccati matrix to \( 2 \times 2 \) and we scan on all possible permutations to minimize \( ||\mathbf{R}'|| \).

2.2.4. The Calculation of the Eigenfunctions

Once the eigenfrequencies of interest have been determined, the calculation of the eigenfunctions for a particular eigenfrequency proceeds as follows. Eq. (29) is integrated as described in § 2.2.2 and the components of \( \mathbf{R} \) (\( \mathbf{R}_{ij} \)) are stored together with the permutations applied. If the adiabatic eigenfunctions are computed, we permute the Riccati matrix as described in § 2.2.3. In the non-adiabatic case, we track the behavior of the Euclidian norm of \( \mathbf{R} \) during the integration and apply a permutation every time \( ||\mathbf{R}|| \) has a maximum. Once \( \mathbf{R}_{ij} \) have been evaluated both for \( \mathbf{R}^{in}(r) \) and \( \mathbf{R}^{out}(r) \), the eigenfunctions are readily calculated by solving

\[
\frac{d\mathbf{v}_{Ric}}{dr} = (\mathbf{CR} + \mathbf{D})\mathbf{v}_{Ric}
\]  

(46)

together with Eq. (28). It is straightforward to derive Eq. (46) from Eqs. (27) and (28). Eq. (16) is integrated twice, from the fitting point \( r_{fit} \) to the star’s center and from \( r_{fit} \) to the surface. At this stage, we do not integrate \( \mathbf{R} \) again, but we interpolate the already calculated \( \mathbf{R}_{ij} \) using linear interpolation in the adiabatic regime and a third order polynomial (Steffen 1990) in the non-adiabatic regime, and apply the same permutations used during their calculation. The initial value of \( \mathbf{v}_{Ric} \) is given by a non-trivial solution of Eq. (38). Clearly, during the integration from \( r_{fit} \) to the center (\( r_{fit} \) to the surface) \( \mathbf{R}^{out} \) (\( \mathbf{R}^{in} \)) must be used. This numerical scheme where \( \mathbf{v}_{Ric} \) and \( \mathbf{R} \) are not integrated together and are integrated in the opposite directions is necessary for numerical stability (Sloan 1977).
2.3. Examples of Free Stellar Oscillations and Tests

In this section, we present a series of tests for CAFein both internal and against published results in the literature. For the latter, we compute the eigenfrequencies of a polytropic model and compare them with results in the literature which rely both on the Riccati method (Takata & Löffler 2004) and on standard relaxation and shooting techniques (Christensen-Dalsgaard & Mullan 1994, hereafter CDM94). In what follows, we refer to the eigenfrequency of the purely adiabatic problem with $\omega_\text{Ad}$, while we denote with $\omega$ the non-adiabatic eigenfrequency. The subscripts “R” and “I” have their usual meaning.

2.3.1. Testing CAFein’s numerical accuracy on a Polytrope: the Adiabatic Case

In this section, we apply CAFein to a polytropic model to compute its eigenfrequencies. We then test the numerical accuracy of our calculation by changing some of the relevant parameters entering the computation of the Riccati matrix and analyzing how the eigenfrequencies vary. In what follows, when we mention the integrator accuracy adopted in our calculation, we refer both to the absolute and relative accuracies.

We create a polytrope with index $n = 3$, assuming an ideal gas with $\Gamma_1 = 5/3$. Below, we indicate the polytropic index with $n_p$ to avoid confusion with the index denoting the radial order of a mode. We first solve the Lane-Emden equation using a variable step 4th order Runge-Kutta integrator with an accuracy requirement of $10^{-12}$. We then iterate the integration until we reach a resolution between two consecutive mesh points of $\Delta r \leq 5 \times 10^{-5}$ (here we are following TL04, but adopting a slightly higher resolution). We conveniently pass onto dimensionless quantities by expressing the physical quantities in the units listed in Table 2. This table also lists the units that will be used in the non-adiabatic regime. During the calculation of the adiabatic eigenfrequencies, we use again a variable step 4th order Runge-Kutta integrator and we interpolate the various polytropic parameters using a third order polynomial (Steffen 1990). As an example, we report some of the calculated eigenfrequencies $\omega_\text{Ad}$ in Table 3. Here we set the required integration accuracy to $10^{-12}$, the limit on the Euclidian norm of $R$ ($||R||_\text{limit}$) used for reembedding (§ 2.2.3) to 10, and performed 1000 integrations in $\omega$ on a randomly chosen interval $[\omega_\text{Ad}-0.1, \omega_\text{Ad}+0.1]$ across each eigenfrequency. In what follows, we will refer to the eigenfrequencies listed in Table 3 as our “fiducial” ones.

Next, we test the stability of CAFein with respect to the parameters mentioned above. We re-compute the same eigenfrequencies for different integrator’s accuracies ($10^{-9}$ and $10^{-12}$), $||R||_\text{limit}$ (10, 100, and 1000), and number of integrations across each eigenfrequency (50, 200, and 1000), fixing the width of each integration interval as described above, and compare them with our fiducial values. The outcome of this test is summarized in Fig. 1 where it is clear that our results are sensitively affected only by the resolution adopted during the scan of the parameter space in $\omega$ (i.e. the eigenfrequency scan resolution, see top plot). Varying $||R||_\text{limit}$ while keeping the number of iterations and integrator accuracy fixed, the relative difference with our fiducial eigenfrequencies is between $10^{-13}$ and $10^{-11}$ (middle plot in Fig. 1). Decreasing the integrator accuracy by 3 order of magnitudes, the relative difference with our fiducial eigenfrequencies is $\lesssim 10^{-8}$ (bottom plot in Fig. 1). The same test per-
formed on the $l = 3$ eigenfrequencies might yield similar results.

Once the eigenfrequencies are known, the eigenfunctions can be readily calculated as described in § 2.2.4. The eigenfunctions should be orthogonal. In fact, as Fuller & Lai (2011) point out, the numerical determination of an eigenfunction might be contaminated by other eigenfunctions

\[(\xi_{\alpha})_{\text{num}} = h_{\alpha} \xi_{\alpha} + h_{0} \xi_{0} + h_{1} \xi_{1} + \ldots \]  

(47)

where the subscript $\alpha$ denotes the order of the mode and the displacement from the equilibrium position $\xi_{\alpha}$ for the mode $\alpha$ can be written as

\[\xi_{\alpha} = [\xi_{r,\alpha}(r)e_{r} + \xi_{\delta,\alpha}(r)e_{\delta} \nabla]Y_{m}^{l}(\theta, \phi). \]  

(48)

The terms with subscript “0” in Eq. (47) refer to the $f$-mode and the various coefficients are given by

\[h_{l} = < \xi_{l} | \xi_{\alpha} > = \int_{0}^{1} r^{2} [\xi_{r,\alpha}(l+1) + l(1+l)\xi_{h,\alpha}]dr \]  

(49)

where the various terms in Eq. (49) are the ones calculated numerically. To derive Eq. (49), we used the normalization of the spherical harmonics as given by Unno et al. (1969):

\[\int_{0}^{2\pi} \int_{0}^{\pi} Y_{m}^{l}(\theta, \phi)Y_{m}^{l'}(\theta, \phi) \sin \theta d\theta d\phi = \delta_{ll'}\delta_{mm'} \]  

(50)

where $\delta_{ll'}$ and $\delta_{mm'}$ are the Kronecker deltas. Since the $f$-mode gives the dominant contribution in Eq. (17), we normalize the eigenfunctions so that $h_{l} = < \xi_{l} | \xi_{\alpha} > = 1$ and we take $(\xi_{\alpha})_{\text{num}}$ to accurately represent the actual $\xi_{\alpha}$ if $|h_{0}| \ll 1$. The values of the coefficient $h_{0}$ for the $f$-mode and the first five $p$- and $g$-modes of harmonic degree $l = 2$ for the polytropic model considered are reported in Table 3. The results show that orthogonality is satisfied to the expected accuracy of our eigenfrequencies, $\sim 10^{-12}$.

Recall that CAFeIn has been developed to investigate dynamic tides in close binaries. Our focus on the harmonic degree $l = 2$ will become clear in § 3.1 where we introduce the tide-generating potential.

2.3.2. Comparing Our Fiducial Polytrope Eigenfrequencies to Polytrope Results in the Literature

In this section, we compare our fiducial eigenfrequencies calculated in the previous section with results in the literature which rely both on the Riccati method (TL04) and on standard relaxation and shooting techniques (CDM94). Our fiducial eigenfrequencies are listed in Table 3 together with the eigenfrequencies calculated by TL04 and CDM94. In Fig. 2 we show the relative difference between our fiducial values and TL04 and CDM94 results (with filled circles), and the relative difference between TL04 and CDM94 (with “x”). The relative difference between our fiducial values and TL04 (CDM94) is between $10^{-11}$ and $10^{-8}$ ($10^{-8}$ and $10^{-7}$) both for $l = 2$ and $l = 3$. The upper end of this intervals agrees with the relative difference between TL04 and CDM94. In particular, the better agreement with TL04, results in a relative difference between our fiducial values and CDM94 that is nearly the same as the one between TL04 and CDM94 (orange filled circles overlap with “x”). This behavior occurs for both the $l = 2$ and $l = 3$ eigenfrequencies. Among the factors that might be contributing to the small differences between the various results presented here are: the different accuracies adopted during the calculation of the polytropic model among the different studies, the accuracy adopted during the calculation of the eigenfrequencies, and round-off errors, as suggested by TL04 (§ 5 of their paper).

Hence forward we focus on the harmonic degree $l = 2$.

2.3.3. Testing CAFeIn on a Zero Age Main Sequence Star: the Non-Adiabatic Case

In this section we test the behavior of CAFeIn in the non-adiabatic regime by applying it to a Zero Age Main Sequence (ZAMS) model in the $\beta$ Cephei/SPB region of the HR diagram. Our goal here is only to verify the reliability of CAFeIn in identifying unstable modes; a detailed study of $\beta$ Cepheis and SPBs is beyond the scope of this work. We refer to, e.g., Gautschy & Said (1993, 1996) for a review of pulsating stars and to Cox et al. (1992), Moskalik & Dziembowski (1992), Kiriakidis et al. (1992), and Dziembowski et al.
create a ZAMS model of 7 $M_\odot$ at metallicity Z = 0.03 and X = 0.7. We then increase the number of mesh points by interpolating the model with a third order polynomial (Steffen 1990), to reach a resolution between two adjacent mesh points of $\Delta r \lesssim 5 \times 10^{-5}$. Our stellar model has a luminosity, effective temperature, and radius of log($L/L_\odot$) = 3.249, log(T$_{\text{eff}}/K$) = 4.309, and log($R/R_\odot$) = 0.530, respectively. The H abundance at the center is 0.67. For comparison, the same properties for one of the models used by SC80 are log($L/L_\odot$) = 3.246, log(T$_{\text{eff}}/K$) = 4.311, and log($R/R_\odot$) = 0.523, while the H abundance at the center is 0.7. We conveniently pass onto dimensionless quantities by expressing the physical quantities in the units listed in Table 2.

To have a sense of where the modes can propagate inside the star and whether non-adiabatic effects are significant, we calculate a few adiabatic eigenfrequencies and place them on the propagation diagram (see Fig. 3). This diagram shows that $p$–modes can propagate all the way to the surface, while $g$–modes are more confined towards the star’s interior, especially the low-order modes. To investigate the importance of non-adiabatic effects for this stellar model and for the modes of interest, we consider the ratio of the star’s thermal to dynamical timescale ($\tau_{\text{th}}/\tau_{\text{dyn}}$). Since the fundamental oscillation timescale (defined as the travel time of a sound wave from the center to the surface) is of the same order of magnitude as $\tau_{\text{dyn}}$, non-adiabatic effects are relevant when $\tau_{\text{th}}/\tau_{\text{dyn}}$ is small (see e.g. Unno et al. 1989). The dashed-dot line in Fig. 3 shows that even though $\tau_{\text{th}} >> \tau_{\text{dyn}}$ through the bulk of the star, non-adiabaticity becomes relevant approaching the surface, where $p$-modes can propagate.

Next, we follow the procedure outlined in §2.2.2 and calculate the non-adiabatic eigenfrequencies. As the inclusion of non-adiabatic effects renders the stellar pulsation equations stiff, at this stage of the calculation we use a variable step implicit Bulirsch-Stoer integrator with an accuracy requirement of $10^{-13}$. The results are summarized in Table 5. From a test on CAFEin’s numerical sta-

![Figure 3. Propagation diagram and degree of non-adiabaticity for the 7 $M_\odot$ ZAMS star described in the text. Left y-axis: the solid lines denote the Brunt-Väisälä (N) and Lamb (L) frequencies squared, while the dots represent the zeros of the radial part of the eigenfunctions of the modes (from the top to the bottom) $p_2, p_1, g_1$, $g_0$. The harmonic degree was set to $l = 2$ and the units in Table 2 are used. Right y-axis: the dashed-dotted line shows ratio of the thermal timescale ($\tau_{\text{th}}$) to the dynamical timescale ($\tau_{\text{dyn}}$). Non-adiabatic effects become significant when the two timescales are comparable (see text).](image-url)
Table 5
Non-adiabatic l = 2 eigenfrequencies for the 7 M⊙ ZAMS described in the text.

| mode | \(\omega_R\) | \(\omega_I\) |
|------|-------------|-------------|
| p2   | 5.1635      | 1.02 \times 10^{-4} |
| p1   | 4.01794     | 5.86 \times 10^{-7} |
| f    | 3.18136     | 1.13 \times 10^{-3} |
| g1   | 1.86456     | 1.71 \times 10^{-7} |
| g2   | 1.27851     | 3.75 \times 10^{-7} |
| g3   | 9.66801 \times 10^{-1} | 5.85 \times 10^{-7} |
| g4   | 7.71481 \times 10^{-1} | 7.92 \times 10^{-7} |
| g5   | 6.39310 \times 10^{-1} | 9.66 \times 10^{-7} |
| g6   | 5.45175 \times 10^{-1} | 1.06 \times 10^{-5} |
| g7   | 4.76419 \times 10^{-1} | 9.79 \times 10^{-7} |
| g8   | 4.25210 \times 10^{-1} | 4.20 \times 10^{-7} |
| g9   | 3.84928 \times 10^{-1} | 1.11 \times 10^{-6} |

The purpose of this section was to prove that CAFEin can identify unstable modes in a β Cephei/SPB variable star, if the OPAL opacity tables (Rogers & Iglesias 1992; Iglesias et al. 1992; Seaton et al. 1994; Kiriakidou et al. 1992; Moskalik & Dziembowski 1992) are used. However, we neglected the effect of convection, which might affect or even excite non-adiabatic pulsations, as mentioned in §2.4.1. The thin convective regions at the star’s surface are visible in the propagation diagram shown in Fig. 4, where \(N^2 < 0\).

3. INVESTIGATING DYNAMIC TIDES WITH CAFEIN

Before describing the extension of the Riccati method developed to investigate tidally excited stellar oscillations, we briefly outline the basic assumptions adopted in this work and introduce the various parameters entering the dynamic tides theoretical framework.

3.1. The Tide-Generating Potential

We consider a close binary system of stars with masses \(M_1\) (primary) and \(M_2\) (secondary) orbiting around one another in a Keplerian orbit. We assume that the primary has a radius \(R_1\) and that it rotates uniformly around an axis orthogonal to the orbital plane with angular velocity \(\Omega_1\) in the sense of the orbital motion, while we treat the companion as a point mass. Furthermore, we assume \(\Omega_1\) to be small enough so that the Coriolis force and the centrifugal force can be neglected. Under these assumptions, the tides raised by the photon absorption by the \(L\)-shell of Fe and photoionization from the \(K\)-shell of C, O, and Ne (Rogers & Iglesias 1992). It is the \(k\)-mechanism associated with this second bump at a temperature of \(\simeq 2 \times 10^5\) K which drives the pulsations observed in \(\beta\) Cephei variables.
compartment can be treated as small forced perturbations applied on a spherically symmetric star in hydrostatic equilibrium. Following the general procedure, we can express the tide-generating potential in spherical coordinates \( r = (r, \theta, \phi) \) with respect to an orthogonal frame corotating with the star and expand it in Fourier series as (e.g. Polfliet & Smeyers 1990, hereafter PS90)

\[
\epsilon_T W(r, t) = -\epsilon_T \sum_{l=2}^{4} \sum_{m=-l}^{l} \sum_{k=-\infty}^{\infty} c_{l,m,k} \left( \frac{r}{R_1} \right)^l Y_l^m(\theta, \phi) \times \exp[i(\sigma_{l,m,k} t - k\Omega_{orb} t)]
\]

where the polar angle \( \theta \) is measured from the rotational angular velocity vector, while the azimuthal angle \( \phi \) is measured in the orbital plane and in the sense of the orbital motion. At time \( t = 0 \), the angle \( \phi = 0 \) marks the position of the periastron of the binary orbit.

The tide-generating potential is a solution to Laplace's equation. The main properties of the Fourier coefficients are described by Smeyers et al. (1998), PS90, and Willems et al. (2010). Briefly, \( c_{l,m,k} \) are symmetric with respect to \( m \) and \( k \) \((c_{l,m,k} = c_{l,-m,-k}) \) and are equal to zero for odd values of \( l + m \) since \( P_l^m(0) = 0 \) for odd values of \( l + m \). Furthermore, the binomial theorem implies that \( c_{l,m,0} \) is equal to \( R_1 \), \( l = 2 \), \( m \), and \( 1 \). For a given orbital eccentricity, the absolute value of \( c_{l,m,k} \) decreases with increasing \( k \), although the decrease is stronger for highly eccentric orbits (Willems 2003; Willems et al. 2008; Smeyers et al. 1998). This implies that the number of \( c_{l,m,k} \) terms with non-trivial contributions to the tide-generating potential is finite, though it increases with increasing eccentricity. Given the dependence of \( c_{l,m,k} \) on \( (R_1/a)^{l-2} \), investigations on dynamic tides are often restricted to the terms belonging to \( l = 2 \), as they are dominant.

It is clear from the expansion (51) of the tide-generating potential that the tidal action from the companion induces in the primary an infinite number of forcing angular frequencies \( \sigma_{m,k} \). The terms associated with \( \sigma_{m,k} = 0 \) (the time-independent terms in \( \epsilon_T W \)) give rise to static tides, while the terms associated with \( \sigma_{m,k} \neq 0 \) (the time-dependent terms in \( \epsilon_T W \)) give rise to dynamic tides.

In the limit of an infinite orbital period, tides are referred to as equilibrium tides (e.g. PS90, Willems et al. 2010).

3.2. The Equations For Tidally Excited Stellar Pulsations and The Secular Evolution of the Orbital Elements

If the tides raised by the companion are treated as a small perturbation applied on a spherically symmetric star in hydrostatic equilibrium, the equations describing forced stellar oscillations are still derived from Eqs. (1)-(5), provided that the \( \epsilon_T W \) is added to Eq. (2) of momentum conservation (e.g. PS90, Willems et al. 2010). Following the standard procedure, we take the unperturbed solution to be axisymmetric and assume that the spatial and temporal part of a small perturbation can be written in Eulerian form as \( f_T(r, \theta, \phi, t) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \sum_{k=-\infty}^{\infty} f_{l,m,k}(r) Y_l^m(\theta, \phi) e^{i\sigma_{l,m,k} t} \) or similarly for the Lagrangian form, denoted with \( \delta \). Since the tide-generating potential is a solution to Laplace's equations, perturbing and linearizing the basic Eqs. (1)-(5) with the new momentum for equation conservation yields, for each set of \( (l, m, k) \) in the expansion of the tide-generating potential, a system of equations which is formally identical to Eqs. (9)-(14), with the following modifications. The perturbation of the star's gravitational potential and the tide-generating potential are grouped into the total perturbation of the gravitational potential defined as \( \Psi = \Psi_p + \epsilon_T W \) (e.g. Zahn 1975). PS90, Willems et al. (2010), where \( \Psi_p \) (denoted in §2.1 as \( \Phi \)) is the perturbation of the star's potential of self-gravitation due to the tidal action of the companion. Furthermore, the new integration variables retain the same form, provided that \( \Phi \) is substituted with \( \Psi \). A final modification to Eqs. (9)-(14) concerns the BCs at the star's surface. As the gravitational potential and its first derivative must be continuous at \( r = R_1 \), BC (19) becomes (e.g. PS90)

\[
y_3 + (l+1)y_3 + \frac{4\pi\rho}{g}y_1 + \frac{\epsilon_T(2l+1)c_{l,m,k}}{g} = 0
\]

where \( \rho \) and \( g \) are made dimensionless via the units listed in Table 2. Therefore, the introduction of \( \Psi \) keeps the tidally excited stellar pulsation Eqs. (9)-(14) homogeneous, but it renders the BCs non-homogeneous. Because of the non-homogeneous term in Eq. (53), the solutions to Eqs. (9)-(14) are proportional to \( \epsilon_T c_{l,m,k} \). Furthermore, even though the system of equations is complex, the dimensionless tidal forcing frequency \( \omega_{m,k} \) is purely real (recall that \( \omega_{m,k}^2 = \sigma_{m,k}^2 R_1^3 (GM)^{-1} \)). In what follows, we refer to the solution of Eqs. (9)-(14) with BCs (15)-(17) at the star’s center, and BCs (18), (53), and (20) at the star’s surface with “tidal eigenfunctions”.

From the tidal eigenfunctions, the timescales for the secular evolution of the orbital elements and stellar spin due to dynamic tides can be readily calculated.

The evolution of the orbital separation and eccentricity is due to the primary’s tidal deformation, which in turn perturbs the external gravitational field and therefore the Keplerian motion of the binary components. Energy dissipation in the surface layers causes a phase shift between the perturbation of the gravitational potential and the companion’s position. This phase shift results in a torque exerted from the secondary on the tidally deformed primary, which affects the primary’s spin. The rates of secular evolution for \( a \), \( e \), and \( \Omega_1 \) are given by
(e.g. Willems et al. 2010)

\[
\begin{align*}
\left(\frac{da}{dt}\right)_\text{sec} &= \frac{8\pi}{F_{\text{orb}}M_1} \sum_{l=1}^{l+m} \sum_{m=1}^{\infty} \sum_{k=0}^{l} \frac{R_l}{a} \, \kappa_{l,m,k} \, |F_{l,m,k}| \sin \gamma_{l,m,k} \, G^{(2)}_{l,m,k}(e) \\
\left(\frac{de}{dt}\right)_\text{sec} &= \frac{8\pi}{F_{\text{orb}}M_1} \sum_{l=1}^{l+m} \sum_{m=1}^{l} \sum_{k=0}^{l} \frac{R_l}{a} \, \kappa_{l,m,k} \, |F_{l,m,k}| \sin \gamma_{l,m,k} \, G^{(3)}_{l,m,k}(e) \\
\left(\frac{d\Omega_l}{dt}\right)_\text{sec} &= \frac{8\pi}{F_{\text{orb}}M_1} \left( \frac{GM_1 M_2}{M_1 + M_2} \right)^{1/2} \frac{M_2 a^{1/2}}{M_1 I_1} \sum_{l=1}^{l+m} \sum_{m=1}^{l} \sum_{k=0}^{l} \frac{R_l}{a} \, \kappa_{l,m,k} \, |F_{l,m,k}| \sin \gamma_{l,m,k} \, G^{(4)}_{l,m,k}(e)
\end{align*}
\]

where \( I_1 \) is the star’s moment of inertia and Eq. (56) is derived assuming solid-body rotation and that the tidal deformation does not affect \( I_1 \). In the above equations, the dimensionless \( F_{l,m,k} \) measure the response of the star to the various tidal forcing frequencies and are given by

\[
F_{l,m,k} = \frac{1}{2} \left[ \frac{R_l}{a} \, \Psi_{l,m,k}(R_l) + 1 \right].
\]

These coefficients are independent of \( \epsilon_{T_{l,m,k}} \) because \( \Psi_{l,m,k}(R_l) \propto \epsilon_{T_{l,m,k}} \) (§3.2). In the units of Table 2 expression (57) reduces to

\[
F_{l,m,k} = -\frac{1}{2} \left[ \frac{(y_3)_l}{\epsilon_{T_{l,m,k}}} \frac{g(1)}{(y_3)_l} + 1 \right] \equiv |F_{l,m,k}| \epsilon_{\gamma_{l,m,k}}
\]

where the last equality comes from the complex nature of the tidal eigenfunctions. For the various properties of symmetry obeyed by \( F_{l,m,k} \) and for the definition of \( \kappa_{l,m,k} \) (not to be confused with the opacity), \( G^{(2)}_{l,m,k}(e), G^{(3)}_{l,m,k}(e), \) and \( G^{(4)}_{l,m,k}(e), \) and their properties we refer to Willems et al. (2003, 2010). Here we just note that \( G^{(3)}_{l,m,k}(e) \) are all zero for a binary with a circular orbit.

Eqs. (53) - (56) take the same form as the equations for the rate of secular change of orbital separation, eccentricity, and spin derived by Zahn (1977, 1978), Hut (1981), and Ruymaekers (1992), in the limiting case of weak damping and small forcing angular frequencies (see Appendix D of Willems et al. 2010 for a derivation).

In what follows, we omit the subscripts \( l, m, \) and \( k \) from the components of the tidal displacement field and the perturbed stellar structure quantities and we denote the tidal forcing frequency by \( \omega_T \).

### 3.3. Extending the Riccati Method to Forced Stellar Oscillations

As described in §2.2, the Riccati method relies in going from a homogeneous system of ordinary differential equations to a non-homogeneous one. However, as explained in §5.2 even though the equations describing the tidally excited stellar oscillations are homogeneous, the same is not true for the BCs at the star’s surface. We make the Riccati method viable for investigating dynamic tides by introducing two new variables \( y_7 \) and \( y_8 \), such that BC (53) at the star’s surface becomes homogeneous

\[
y_4 + (l + 1)y_3 + \frac{4\pi\rho y_1}{g} + \frac{\epsilon_T(2l + 1)c_{mk}y_8}{g} = 0
\]

The introduction of two variables instead of one is required to keep all the matrices entering the Riccati method square. We take \( y_7 \) and \( y_8 \) to be a solution of the following differential equation

\[
\frac{dy_7}{dr} = \frac{dy_8}{dr} = 0
\]

with BCs at the center and at the surface given by

\[
y_7 = y_8.
\]

Once the tidal eigenfunctions are determined, we normalize them so that \( y_8 = 1 \) at the star’s surface. This choice of normalization causes Eq. (59) to reduce to the original BC (53). Here we note that changing the form of Eq. (60) does not affect our results, but it can affect the running time. With the introduction of \( y_7 \) and \( y_8 \), the new definition of vectors \( v_{\text{Ric}} \) and \( v_{\text{Ric}} \) (see §2.2) at the star’s boundaries becomes

\[
u_{\text{Ric}} = \begin{pmatrix} y_1 \\ y_4 \\ y_5 \\ y_7 \end{pmatrix}, \quad v_{\text{Ric}} = \begin{pmatrix} y_2 \\ y_3 \\ y_6 \\ y_8 \end{pmatrix}
\]

and the new initial conditions on the Riccati matrices at the star’s center and surface are derived accordingly. Even though the introduction of the new variables increases the size of the various matrices, it does not affect the running time significantly. In particular, as far as reimbedding (§2.2.3) is concerned, during the search for the permutation yielding \( R' \) with the minimum Euclidean norm, \( y_7 \) and \( y_8 \) are kept fixed. This trick yields the same number of trials as in the non-adiabatic stellar pulsation problem.

The tidal eigenfunctions are calculated as described in §2.2.4.

### 3.4. Testing the Extension of the Riccati Method to Investigate Dynamic Tides

Here we test whether our extension of the Riccati method to treat tidally excited stellar pulsation works as expected. First, we compare the tidal eigenfunctions calculated with CAFein with the work presented by PS90 in the adiabatic regime. Next, we compare the orbital and spin evolution timescales due to dynamic tides (§5.2) computed with CAFein with the results presented by Willems et al. (2003) (WVHS03, hereafter). Finally, we compute these same timescales for a binary hosting a 1.5 \( M_\odot \) Main Sequence star and a hot Jupiter. The purpose of this last exercise is to demonstrate that CAFein can handle the high-order modes involved in these binaries.

#### 3.4.1. Tidal Eigenfunctions of a 5 \( M_\odot \) MS star

Here, we investigate the effect of non-adiabatic dynamic tides on a MS star of 5 \( M_\odot \) (primary) and metallicity \( Z = 0.018 \), by studying the variation of the radial component of the tidal displacement at the surface \( |\xi_r(1)| \) as a function of the tidal forcing frequency. Our
main goal here is to verify numerically that a dynamical tide can be approximated as the sum of the equilibrium tide and another part reflecting the oscillatory properties of the star itself, in agreement with what asymptotic theories have shown (e.g., Zahn 1975; Smeyers 1997; Smeyers & Willems 1998). The equilibrium tide associated with the limiting case of an infinite orbital period is given by \( \xi_{\text{eq}} = -\Psi/g \) (e.g., PS90).

As before, we create the stellar model with MESA and increase the number of mesh points as described in § 2.3.3. The radius of the model adopted here is \( R_1 \approx 2.62 \, R_\odot \), and the convective core extends out to \( r/R_1 \approx 0.185 \). For comparison, these same parameters for the model used by PS90 are \( R_1 \approx 2.52 \, R_\odot \) and \( r/R_1 \approx 0.18 \), respectively. We set \((l, m, k) = (2, 0, 1)\), we take the companion to be a point mass of \( 5.0 \, M_\odot \), and we set \( e = 0.4 \). We then vary the orbital period so that the tidal forcing frequency becomes comparable to the \( g_0 \) - and \( g_{10} \)-mode frequencies. Here we note that, since \( m = 0 \) and \( \sigma_{m, k} \approx k \Omega_{\text{orb}} + m \Omega_1 \), the star’s spin does not need to be specified.

The behavior of the radial component of the displacement from the equilibrium position at the surface \( |\xi_r(1)| \) is shown in Fig. 5 as a function of the tidal forcing frequency (and orbital frequency). This Figure can be compared with Fig. 1 in PS90 or Savonije & Papaloizou (1983). The difference between the Figures presented by these two studies is that Savonije & Papaloizou (1983) performed a non-adiabatic calculation, while in the adiabatic calculation of PS90 \( \xi_r(1) \) changes discontinuously between \( \pm \infty \). Fig. 5 shows that, close to a resonance, the amplitude of the displacement at the surface varies greatly for small changes in \( \omega_1 \), while away from a resonance changes in the amplitude occur at a much slower rate.

In Fig. 6 we show the radial and orthogonal components of the tidal displacement and the total perturbation of the gravitational potential as a function of the radial coordinate, for tidal forcing frequencies \( \omega \approx 0.42778 \rightarrow 0.42813 \). The nine zeros displayed by \( \xi_r(1)/r \) indicate that we are close to a resonance with the \( g_0 \)-mode. Our Fig. 6 can be compared with Figs. 2 and 3 in PS90, where the discontinuity in \( \xi_r(1) \) is again a result of the adiabatic treatment. As PS90 pointed out, the behavior of \( (\xi_r/r)_{9} \) at the surface is determined by BC (18) and, therefore, by the competition of the orthogonal component of the tidal displacement and the total perturbation of the potential. This can be seen by considering that, close to a resonance with the \( g_0 \)-mode, \( \omega_1 \approx 0.2 \) and \( \omega_T^2 \approx 5.5 \), while the other term \( V \) entering BC (18) is \( V \approx 1.5 \times 10^4 \), so \( V^{-1} \approx 6.7 \times 10^{-4} \). Therefore, for \( l = 2 \), it can be shown that BC (18) reduces to

\[
y_1 \approx y_2 - y_3
\]  

As shown in Fig. 6 close to resonance the behavior of \( \xi_r(1) \) is related to the trend of \( \xi_{9}(1) \), as the contribution from the total perturbation of the potential remains small. Moving along the steep slope in Fig. 6 towards smaller \( \omega_T \), the rapid decrease in \( \xi_r(1) \) is related to the rapid decrease of \( \xi_{9}(1) \) (as PS90 pointed out referring to Fig. 1 in their paper). This can be seen from the decrease in \( \xi_r(1) \) in going from \( \omega_T = 0.42778 \) to 0.4259 (from the dot-dashed line in Fig. 6 to the top plot in Fig. 7) to be compared with the behavior presented by PS90 in going from their Fig. 3 to 4). At \( \omega_T = 0.4259 \) the nine zeros in \( \xi_r(1)/r \) are still visible. Along the nearly horizontal line of Fig. 6 as \( \omega_T \) decreases further, the increasingly significant total perturbation of the potential affects the behavior of \( \xi_r(1) \). As shown in the middle-top panels in Fig. 7 (to compare with Fig. 5 in PS90), at \( \omega_T = 0.4057, \xi_r(r)/r \) close to the star’s surface shifts from the horizontal dotted line marking the position of \( \xi_{9}(r)/r = 0 \). Decreasing further the tidal forcing frequency to \( \omega_T = 0.3971 \) (middle-bottom panel in Fig. 7), the radial component of the displacement makes a turn upward at the surface (to compare with Fig. 6 in PS90). This behavior is related to the change in sign of \( \xi_{9}(1) \). Finally, at \( \omega_T = 0.3926 \) (bottom panel in Fig. 7), the contribution from the or-
of the model adopted here is consistent with the results of Smeyers & Willems (1998).

Oscillatory theories have shown (e.g. Zahn 1975; Smeyers 1997; Savonije & Papaloizou 1998) that the oscillatory properties of the star itself, as asymptotic component of the displacement and the total perturbation of the potential become comparable, and the resulting amplitude of $\xi_1(r)$ is zero (to compare with Fig. 7 in PS90). The radial component of the displacement from equilibrium is approaching the $g_{10}$-mode.

From the behavior of $\xi_1(r)/r$ displayed in Fig. 8, it is clear that away from resonances a dynamical tide can be approximated as the sum of the equilibrium tide $-(\Psi/g)$ divided by $r$ (e.g. PS90). In the top and bottom plots, $\omega_R$ is close to the $g_9$ and $g_{10}$ eigenfrequency, respectively (see text).

As before, we use MESA to create a stellar model of a $5M_\odot$ MS star at solar metallicity and increase the number of mesh points as described in § 2.3.3. The radius of the model adopted here is $R_1 \simeq 2.66R_\odot$, its dynamical ($\tau_{\text{dyn}}$), thermal ($\tau_{\text{th}}$) and nuclear ($\tau_{\text{nuc}}$) timescales are 323.9 min, $2.71 \times 10^5$ yr and $9.20 \times 10^7$ yr, respectively. For comparison, these same parameters for the model used by WVHS03 are $R_1 \simeq 2.8 R_\odot$, $\tau_{\text{dyn}} = 54.8$ min, $\tau_{\text{th}} = 4.88 \times 10^7$ yr, and $\tau_{\text{nuc}} = 8.67 \times 10^7$ yr, respectively. In both cases the H fraction at the center is $X_C = 0.7$. We take the companion to be a point mass of $1.4M_\odot$, the eccentricity $e = 0.5$, and we take the spin of the primary to be 50% of the companion’s orbital angular velocity at periastron.

Similarly to WVHS03, we consider the dominant terms in the expansion of the tide-generating potential and fix $l = 2$ and $m = -2$. We then calculate the orbital and spin evolution timescales for orbital periods ($P_{\text{orb}}$) ranging from 2 to 5 days, taking into account several terms in the expansion of the tide-generating potential and considering $k$ up to 20 in Eq. (3). The results are summarized in Fig. 9 which should be compared with Fig. 3 in WVHS03 (we used the same range for the $x$- and $y$-axis). It is clear that, even though the resolution used by WVHS03 during the scan of the parameter space in $P_{\text{orb}}$ is higher than the one adopted here, the magnitude and trend of the timescales due to dynamic tides both in and out of resonance are in good agreement (and, as pointed out by WVHS03, also in agreement with what previous investigations have found, e.g. Savonije & Papaloizou 1983). For the results presented by WVHS03, the resonantly excited eigenmodes in the range $P_{\text{orb}} = 2-5$ days are $g$-modes of radial order $n$ from $1$ to $12$. In Table 4 we list the real part of the non-adiabatic eigenfrequencies for the modes $g_1 - g_{12}$ of the stellar model adopted in this work. As our stellar model differs from the one used by WVHS03, a direct comparison between their eigenfrequencies (see Table 1 of WVHS03) and the one calculated here is not possible. However, making WVHS03’s eigenfrequencies dimensionless, $\omega_R$ for the $g_1$- and $g_{12}$-modes calculated by WVHS03 are $2.20672$ and $0.36723$, respectively. This suggests that the resonantly excited modes in our stellar model are in the range between $g_1$ to $g_{11}$, in agreement between the two studies.

### 3.4.3. Dynamic Tides Timescales in a Binary Hosting a 1.5 $M_\odot$ MS star and a Hot Jupiter

As of Spring 2013, $\approx 850$ planets have been confirmed and $\approx 2,700$ new candidates have been provided by NASA’s Kepler satellite (Batalha & Kepler Team 2012). Among the confirmed exoplanets, $\approx 80$ have a mass $\gtrsim M_J$ and an orbital period $P_{\text{orb}} \lesssim 5$ d. This number increases with a less stringent mass and period constraints and, potentially, many more hot Jupiters exist among...
generating potential and fix (consider the dominant term in the expansion of the tide-generating velocity, and we take the orbit to be circular. We take the companion to be a point mass of $1 M_{\odot}$, we set $a_0 = 0.5$, and its age is $\tau_0 = 3.3 \times 10^8$ yr. We take the companion to be a point mass of $1 M_{\odot}$, we set the star’s spin to be 50% of the companion’s orbital angular velocity, and we take the orbit to be circular. We consider the dominant term in the expansion of the tide-generating potential and fix $(l, m, k) = (2, -2, 2)$. We then calculate the orbital and spin evolution timescales for orbital periods ranging from 1 to 5 days. The results are summarized in Fig. 8. The orbital and spin periods correspond to a range in tidal forcing frequencies between $\omega_T \approx 0.04 - 0.18$, which for the stellar model adopted here spans modes between $\gtrsim g_{200} - g_{2000}$.

Even though our goal here was to demonstrate only that CAFein can handle the high-order modes involved when dynamic tides in a star hosting a hot Jupiter are considered, we note that we currently neglect the effect of rotation on the eigenfrequency spectrum. Depending on its magnitude, rotation is expected to affect the magnitude of the timescales computed, as it enriches the eigenfrequencies spectrum leading to more resonances (Witte & Savonije 1999a). The horizontal dotted line represents the logarithm of the star’s age. The gap in the timescales at $\approx 2.5$ days is due to the resolution adopted during the calculation.

The inclusion of the tide-generating potential in the stellar pulsation equations formally does not change the system of equations that have to be integrated, and thus the applicability of the Riccati method. However, it renders the BCs at the star’s surface non-homogeneous. We made the Riccati method viable for solving the tidally excited stellar pulsation problem by introducing two new variables and corresponding differential equations to make the BCs homogeneous.

We tested CAFein as a pure stellar pulsation code in hydrostatic equilibrium and the linear approximation is adopted. The horizontal dotted line represents the logarithm of the star’s age. The gap in the timescales at $\approx 2.5$ days is due to the resolution adopted during the calculation.

4. SUMMARY, DISCUSSION, AND CONCLUSIONS

Here we have presented CAFein, a new computational tool for calculating non-adiabatic stellar pulsations in isolated stars and tidally excited stellar oscillations in close binaries, particularly in the dynamic tides regime, where the driving frequencies are comparable to the stellar eigenfrequencies. Tides are considered as a small perturbation applied on a spherically symmetric star in hydrostatic equilibrium and the linear approximation is adopted. CAFein is based on the so-called Riccati method, a numerical algorithm that has been extensively and successfully applied to a variety of stellar pulsators and which does not suffer from the major drawbacks of commonly-used shooting and relaxation schemes. Even though the Riccati method is formally a shooting method, it relies on transforming the linear first-order boundary eigenvalue problem describing stellar oscillations into a numerically stable, non-linear initial value problem. This initial-value problem is then solved using a shooting method, where the eigen-frequency is the only shooting parameter to be iterated.

The inclusion of the tide-generating potential in the stellar pulsation equations formally does not change the system of equations that have to be integrated, and thus the applicability of the Riccati method. However, it renders the BCs at the star’s surface non-homogeneous. We made the Riccati method viable for solving the tidally excited stellar pulsation problem by introducing two new variables and corresponding differential equations to make the BCs homogeneous.
for two different applications. In the adiabatic regime, we first calculated the eigenfrequencies of a polytrope and verified that the results are not significantly affected if some of the relevant parameters entering the Riccati method are varied, thus demonstrating CAFein’s numerical stability. Next, we compared the computed eigenfrequencies with previously published results which relied both on the Riccati method (TL04) and on other commonly-used shooting and relaxation techniques (CDM94). The comparison yielded very good agreement and the orthogonality of low order eigenfunctions was also successfully verified. In the non-adiabatic regime, we considered a stellar model in the β Cephei/SPB instability strip of the HR diagram and successfully recovered the unstable modes in the part of the parameter space examined.

We showed that the extension of the Riccati method to treat tidally excited stellar pulsations works as expected by successfully reproducing the work presented by PS90 and showing that a dynamical tide can be approximated as the sum of the equilibrium tide and another part reflecting the oscillatory properties of the star itself. Furthermore, we successfully reproduced the magnitude and trend of the orbital and spin evolution timescales due to dynamic tides both in and out of resonance presented by WVHS03. Here we stress the fact that WVHS03 did not solve numerically the fully non-adiabatic problem but found the perturbed stellar quantities via semi-analytical solutions. Finally, we applied CAFein to a 1.5 \( M_\odot \) Main Sequence star hosting a hot Jupiter and showed that this code can handle the high-order modes involved in these binaries.

In this paper we have explored CAFein’s performance in the dynamic tides regime, where the tidal forcing frequencies are close to the star’s eigenfrequencies and the latter can be resonantly excited by tides. This code could be used to explore cases that are closer to the quasi-static tides limit, in which the tidal forcing frequencies are much smaller compared to the inverse of the WD’s dynamical time scale (almost synchronized components or long orbital and rotational periods). However, as noted by Savonije & Papaloizou (1983), it is numerically challenging to calculate low-frequency tides by integrating the full set of tidal oscillation equations because of the short-wavelength components entering the tidal response. For this reasons, the system of equations is usually reduced using perturbation theory (e.g. Smeyers 1997; Willems et al. 2003, 2010).

Even though the physics included in CAFein makes it suitable for investigations of stars with envelopes that are mostly radiative we intend to use this novel code to investigate a variety of binaries and stars. To this purpose, we are currently upgrading it to account for the effect of rotation on the eigenmode spectrum in the so-called traditional approximation (Unno et al. 1989) and the effect of turbulent friction acting on the equilibrium tide (Terquem et al. 1998; Savonije & Witte 2002; Willems et al. 2010).

We are grateful to A. Barker, Y. Lithwick, Lars Bildsten, Bill Paxton, and the MESA community for useful discussions during the development of CAFein. Simulations were performed on the computing cluster Fugu available to the Theoretical Astrophysics group at Northwestern and partially funded by NSF grant PHY-0619274 to VK. This work was supported by NASA Awards NNX09AJ56G to V.K. and NNX12AI86G to F.A.R.
REFERENCES

Amaro-Seoane, P., et al. 2012, Classical and Quantum Gravity, 29, 124016

Barker, A. J., & Ogilvie, G. I. 2010, MNRAS, 404, 1849

Batalha, N. M., & Kepler Team. 2012, in American Astronomical Society Meeting Abstracts, Vol. 220, American Astronomical Society Meeting Abstracts #220, 306.01

Brassard, P., Fontaine, G., Wesemael, F., Kawaler, S. D., & Tassoul, M. 1991, ApJ, 367, 601

Brown, W. R., Kilic, M., Allende Prieto, C., & Kenyon, S. J. 2010, ApJ, 723, 1072

—. 2012, ApJ, 744, 142

Brown, W. R., Kilic, M., Hermes, J. J., Allende Prieto, C., Kenyon, S. J., & Winget, D. E. 2011, ApJ, 737, L23+

Burkart, J., Quataert, E., Arras, P., & Weinberg, N. N. 2012a, ArXiv e-prints

—. 2012b, MNRAS, 421, 983

Cantiello, M., et al. 2009, A&A, 499, 279

Christensen-Dalsgaard, J., & Mullan, D. J. 1994, MNRAS, 270, 124016

Cox, A. N., Morgan, S. M., Rogers, F. J., & Iglesias, C. A. 1992, ApJ, 393, 272

Danzmann, K., & the LISA study team. 1996, Classical and Quantum Gravity, 13, 247

Davey, A. 1977, Journal of Computational Physics, 24, 331

Dziembowski, W. A., Moskalik, P., & Pamyatnykh, A. A. 1992, MNRAS, 253, 509

Fuller, J., & Lai, D. 2011, MNRAS, 412, 1331

—. 2012a, ArXiv e-prints

—. 2012b, MNRAS, 421, 426

Gautschy, A., & Loffler, W. 1996, Delta Scuti Star Newsletter, 10, 13

Gautschy, A., Ludwig, H.-G., & Freytag, B. 1996, A&A, 311, 493

Gautschy, A., & Saio, H. 1995, ARA&A, 33, 75

Glatzel, W., & Gautschy, A. 1992, MNRAS, 256, 805

Giuricin, G., Mardirossian, F., & Mezzetti, M. 1984, A&A, 134, 365

Glatzel, W., & Glatzel, W. 1999a, MNRAS, 254, 154

Goodman, J., & Dickson, E. S. 1998, ApJ, 502, 788

Hellier, C., et al. 2009, Nature, 460, 938

Henyey, L. G., Forbes, J. E., & Gould, N. L. 1964, ApJ, 139, 306

Hermes, J. J., et al. 2012, ArXiv e-prints

Hut, P. 1981, A&A, 99, 126

Iglesias, C. A., Rogers, F. J., & Wilson, B. G. 1992, ApJ, 397, 717

Kilic, M., Brown, W. R., Allende Prieto, C., Aguiros, M. A., Heinke, C., & Kenyon, S. J. 2011, ApJ, 727, 3

Kilic, M., Brown, W. R., Allende Prieto, C., Kenyon, S. J., Heinke, C. O., Aguiros, M. A., & Kleinman, S. J. 2012, ApJ, 751, 141

Kilic, M., Brown, W. R., Allende Prieto, C., Kenyon, S. J., & Panei, J. A. 2010, ApJ, 716, 122

Kiriakisidis, M., El Eid, M. F., & Glatzel, W. 1992, MNRAS, 255, 1P

Liu, J., Han, Z., Zhang, F., & Zhang, Y. 2010, ApJ, 719, 1546

löfler, W. 2000, in Astronomical Society of the Pacific Conference Series, Vol. 234, L. Szabados & D. Kurtz, 447–448

Mazeh, T., Tamuz, O., & North, P. 2006, MNRAS, 367, 1531

Moskalik, P., & Dziembowski, W. A. 1992, A&A, 256, L5

Nelemans, G., Portegies Zwart, S. F., Verbunt, F., & Yuan, L. R. 2001a, A&A, 368, 939

Nelemans, G., Yungelson, L. R., & Portegies Zwart, S. F. 2004, MNRAS, 349, 181

Nelemans, G., Yungelson, L. R., Portegies Zwart, S. F., & Verbunt, F. 2001b, A&A, 365, 491

Paxton, B., Bildsten, L., Dotter, A., Herwig, F., Lesaffre, P., & Timmes, F. 2011, ApJS, 192, 3

Pollfiet, R., & Smeers, P. 1990, A&A, 237, 110

Rathore, Y., Blandford, R. D., & Broderick, A. E. 2005, MNRAS, 357, 834

Rogers, F. J., & Iglesias, C. A. 1992, ApJS, 79, 507

Ruiter, A. J., Belczynski, K., Benacquista, M., Larson, S. L., & Williams, G. 2010, ApJ, 717, 1006

Ruymaekers, E. 1992, A&A, 259, 349

Sao, H., & Cox, J. P. 1980, ApJ, 236, 549

Savonije, G. J., & Papaloizou, J. C. B. 1983, MNRAS, 203, 581

Savonije, G. J., & Witte, M. G. 2002, A&A, 386, 211

Schon, K., & Gautschi, A. 1998, in Astronomical Society of the Pacific Conference Series, Vol. 135, A Half Century of Stellar Evolution Interpretation, ed. P. A. Bradley & T. A. Guzik, 116

Scott, M. R. 1973, Journal of Computational Physics, 12, 334

Seaton, M. J., Yan, Y., Mihalas, D., & Pradhan, A. K. 1994, MNRAS, 266, 805

Sloan, D. M. 1977, Journal of Computational Physics, 24, 320

Smeers, P., & Willems, B. 1998, A&A, 336, 539

Smeers, P., Willems, B., & Van Hooft, T. 1998, A&A, 335, 622

Steffen, M. 1990, A&A, 239, 443

Stothers, R. B., & Chin, C.-W. 1993, ApJ, 408, L85

Takata, M., & löfler, W. 2004, PASJ, 56, 645

Terquem, C., Papaloizou, J. C. B., Nelson, R. P., & Lin, D. N. C. 1998, ApJ, 502, 788

Thompson, S. E., et al. 2012, ApJ, 753, 86

Unno, W., Osaki, Y., Ando, H., Sao, H., & Shibahashi, H. 1989, Nonradial oscillations of stars, ed. Unno, W., Osaki, Y., Ando, H., Sao, H., & Shibahashi, H.

Valsecchi, F., Farr, W. M., Willems, B., & Kalogera, V. 2012, ArXiv e-prints

Welsh, W. F., et al. 2011, ApJS, 197, 4

Willems, B. 2003, MNRAS, 346, 968

Willems, B., Deloye, C. J., & Kalogera, V. 2010, ApJ, 713, 239

Willems, B., van Hooft, T., & Smeers, P. 2003, A&A, 397, 973

Witte, M. G., & Savonije, G. J. 1999a, A&A, 341, 842

—. 1999b, A&A, 350, 129

—. 2001a, A&A, 366, 840

Zahn, J.-P. 1975, A&A, 41, 329

Zahn, J.-P. 1977, A&A, 57, 383

Zahn, J.-P. 2008, in EAS Publications Series, Vol. 29, EAS Publications Series, ed. M.-J. Goupil & J.-P. Zahn, 67–90

Zahn, J.-R. 1978, A&A, 67, 162