Inhomogeneous quantum Lie algebras

P. P. Kulish

St.Petersburg Department of the Steklov Mathematical Institute,
Fontanka 27, St.Petersburg, 191011, Russia
(kulish@pdmi.ras.ru)

A. I. Mudrov

Department of Theoretical Physics, Institute of Physics, St.Petersburg State
University, Ulyanovskaya 1, St.Petergof, St.Petersburg, 198904, Russia
(aimudrov@dg2062.spb.edu)

Abstract

We study quantization of a class of inhomogeneous Lie bialgebras which are crossproducts in dual sectors with Abelian invariant parts. This class forms a category stable under dualization and the double operations. The quantization turns out to be a functor commuting with them. The Hopf operations and the universal R-matrices are given in terms of generators. The quantum algebras obtained appear to be isomorphic to the universal enveloping Poisson-Lie algebras on the dual groups.

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1 Introduction

Inhomogeneous Lie groups such as those belonging to the Cayley-Klein series, including Poincaré and Galilei, play an important role in classical physics and geometry. They realize the maximal sets of (continuous) symmetries of the simply connected (pseudo) Riemannian spaces of the zero curvature. The generalization of the semi-direct product of classical groups in the framework of non-commutative geometry is the bicrossproduct \([1]\) of two Hopf algebras \(A\) and \(B\) characterized by actions of \(A\) on \(B\) and \(B^*\) on \(A^*\). Nowadays, there are numerous examples of bicrossproducts known, including those among quantum deformations of the Cayley-Klein algebras \([2]\). Unfortunately, contractions of the quantum orthogonal algebras leading to those solutions result in poles in their classical r-matrices, however disappearing from the skew-symmetric part. Thus a Lie bialgebra survives, whereas the quasitriangular structure is broken. The canonical (and the simplest) example of the bicrossproduct construction is the second (non-standard) quantization of the Borel subalgebra \(b(2) \subset sl(2)\). At the same time, this algebra is the result \([3]\) of Drinfeld’s twist \([4, 5]\) of the universal enveloping algebra \(U(b(2))\). Another examples of twisted bicrossproduct Hopf algebra are the null-plane quantized Poincaré algebra \([6]\) and extended jordanian deformations of \(U(sl(N))\). These quantizations involve special non-degenerate 1-cocycles on Lie groups \([7, 8, 9]\). All those algebras are twist-equivalent to classical universal enveloping algebras, and that equivalence holds for their representation theories. Quasitriangular bicrossproduct Hopf algebras with non-unitary R-matrices were found in \([10]\) via the quantum double construction in the framework of the matched pairs of finite groups. The present work is devoted to the study of “continuous” bicrossproduct Hopf algebras with Abelian invariant subalgebras. In the classical differential geometry these correspond to inhomogeneous Lie groups, containing sets of commutative translations. Quantum version of the theory appears to possess a number of remarkable features, for example, invariance of the category of interest with respect to dualization and the double procedures. Explicitly built Hopf operations allows us to conduct the detailed study of quantum doubles, construct canonical elements and R-matrices for generic
quasitriangular Hopf algebras from the category under investigation. Our approach
relies on a kind of ”universality” of the double construction, proved by Radford [11]
and meaning the following. Every quasitriangular Hopf algebra contains a minimal
quasitriangular Hopf subalgebra which is a quotient of the quantum double of another
Hopf subalgebra.

The Hopf algebras studied in this paper are related to the matched pairs of con-
tinuous groups, that explains appearance of objects inherent to classical differential
geometry, such as Lie group 1-cocycles. In fact, quantum commutation relations turn
out to be just the Poisson brackets on the dual Lie group, and the quantum symme-
tries form the universal enveloping algebra of the corresponding Poisson-Lie algebra of
functions.

2 Quantum double and quasitriangularity

The purpose of this preliminary section is to present, for completeness, to prove that
every quasitriangular Hopf algebra contains a subalgebra which is a quotient of the
quantum double [11]. We start with the following elementary proposition from the
linear algebra.

Lemma 1 Let $L$ be a vector space and $r \in L \otimes L$. Consider the subspaces $L_+ = r(L^*)$, $r(x) = \langle x \otimes id, r \rangle$ and $L_- = r^*(L^*)$. Then $L_+^* \sim L_-^*$ and the element $r$ coincides with the image of the canonical element under the induced map $L_+^* \otimes L_+ \to L_- \otimes L_+ \subset L \otimes L$

identical on the second tensor factor.

The first part of the statement follows from the commutative diagrams

1. \[
\begin{array}{ccc}
L^* & r & L \\
\downarrow r & \r & \downarrow r \\
L_+ & \downarrow i & \downarrow i \\
0 & 0 & 0
\end{array}
\]

2. \[
\begin{array}{ccc}
L & r^* & L^* \\
\downarrow r^* & \r & \downarrow r^* \\
L_+^* & \downarrow i^* & \downarrow i^* \\
0 & 0 & 0
\end{array}
\]

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where the isomorphism $L_+^* \to L_-$ is given by the map $\tilde{r}^*$. Let us prove that the map $\tilde{r}^* \otimes id$ brings the canonical element $l \in L_+^* \otimes L_+$ right to $r \in L_- \otimes L_+ \subset L \otimes L$. Indeed, for every $x, y \in L^*$ we have $\langle (\tilde{r}^* \otimes id)(l), x \otimes y \rangle = \langle l, r(x) \otimes y \rangle = \langle r(x), y \rangle = \langle r, x \otimes y \rangle$. Here we used the characteristic of the canonical element, $\langle l, r(x) \otimes id \rangle = r(x)$.

Now consider the quasi-classical situation when $L$ is a Lie bialgebra. Then $L_\pm$ and their linear sum are themselves sub-bialgebras. Moreover, $L_+^* + L_-$ is the minimal quasitriangular Lie sub-bialgebra, where the classical r-matrix lives in fact. Since $r^*$ is a coalgebra homomorphism but an algebra anti-homomorphism, it also can be regarded as a morphism in the Lie bialgebra category, $L_+^*$ being endowed with the opposite bracket. Let us consider the double $D(L_+)$ built on the linear sum of $L_\pm$ and including them as Lie sub-bialgebras [12]. In an evident way the mapping $D(L_+) \to L_+^* + L_- \subset L$ is defined, which is just the identification embedding on each addends. Its restrictions on $L_\pm$ preserves the Lie structures separately. Let us prove the same assertion with respect to the commutator $[L_+, L_-]$. For arbitrary $x, y, z \in L^*$ consider the classical Yang-Baxter equation

$$\langle [r_{12}, r_{13}], x \otimes y \otimes z \rangle + \langle [r_{13}, r_{23}], x \otimes y \otimes z \rangle + \langle [r_{12}, r_{23}], x \otimes y \otimes z \rangle = 0.$$

Having introduced the notations $x_+ = r(x) \in L_+, x_- = r^*(x) \in L_-$, where $x \in L^*$, rewrite this equality as

$$\langle [y_-, z_-], x \rangle + \langle [x_+, y_+], z \rangle + \langle [x_+, z_-], y \rangle = 0.$$

This, in its turn, is equivalent to

$$[x_+, z_-] = (x_+ \triangleright z)_- - (z_- \triangleright x)_+,$$

where $\triangleright = -ad^*|_L$ - is the conjugate to the adjoint representation. As the mapping $i^*: L^* \to L_+^* \sim L_-$ is a homomorphism of $L_\pm$-modules (and the same is the case with replacement $\pm \to \mp$), the latter expression can be rewritten in the form

$$[x_+, z_-] = x_+ \triangleright z_- - z_- \triangleright x_+,$$

where $\triangleright$ is already considered as $-ad^*|_{L_\pm}$. But this is exactly the definition of the Lie bracket in $D(L_+)$.
Now the similar result is formulated for abstract quasitriangular Hopf algebras (strictly speaking, finite dimensional). Recall that a Hopf algebra $\mathcal{H}$ is quasitriangular \cite{12} if there exists an element $\mathcal{R} \in \mathcal{H}^{\otimes 2}$ (the universal R-matrix) such that

\[
(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},
\]

\[
\mathcal{R}\Delta(h) = \Delta'(h)\mathcal{R},
\]

where the prime denotes the opposite coproduct and the subscripts indicate the way of embedding into the tensor cube. It follows from here that $\mathcal{R}$ satisfies the Yang-Baxter equation

\[
\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.
\]

R-matrix defines two algebra and anti-coalgebra homomorphisms from $\mathcal{H}^*$ to $\mathcal{H}$, $\eta \rightarrow \langle \eta \otimes \text{id}, \mathcal{R} \rangle$ and $\eta \rightarrow \langle \text{id} \otimes \eta, \mathcal{R}^{-1} \rangle = \langle \text{id} \otimes \eta, (S \otimes \text{id})(\mathcal{R}) \rangle$, their images denoted $\mathcal{H}_+$ and $\mathcal{H}_-$. Hopf algebra $\mathcal{H}_-$ is isomorphic to $\mathcal{H}_+^*$ taken with the opposite multiplication. Let us consider the double $\mathcal{D}(\mathcal{H}_+)$, which is built on the tensor product of $\mathcal{H}_+$ and $\mathcal{H}_+^{*, \text{op}}$ embedded there as sub-bialgebras. The relations between these two factors are encoded in the Yang-Baxter equation on the canonical element $I = h^i \otimes h_i \in \mathcal{H}_+^{*, \text{op}} \otimes \mathcal{H}_+$. The map $\mathcal{D}(\mathcal{H}_+) \rightarrow \mathcal{H}$ defined as identical on $\mathcal{H}_+ \otimes 1$ and the isomorphism $1 \otimes \mathcal{H}_+^{*, \text{op}} \rightarrow \mathcal{H}_-$ respects the bialgebra structures when restricted to these sub-bialgebras. The image of the canonical element under this mapping is the R-matrix (cf. Lemma \cite{13}), and the cross-relations in the quantum double go over into the quantum Yang-Baxter equation fulfilled by the R-matrix. Hence the map of concern is a homomorphism. Its surjective image includes sub-Hopf algebras $\mathcal{H}_\pm$ and is exactly that subalgebra in $\mathcal{H}$ where the R-matrix actually lies. Thus we finish the proof.

The subspaces $L_+$ and $L_-$ glue over the Cartan subalgebra in a standard (Drinfeld-Jimbo) semisimple Lie bialgebra. Its quantization belongs to the class of the factorizable Hopf algebras introduced in \cite{13}. For that type of algebras, the “universality” property of the double was stated therein. Alternative examples are triangular bialgebras with skew-symmetric r-matrices, where $L_+$ coincides with $L_-$. The simplest case of the double of the triangular quantized $sl(2)$-Borel subalgebra was studied in \cite{14, 13}. 

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3 Inhomogeneous Lie bialgebras and their quantization

We introduce the bicrossproduct structure on a Lie bialgebra by means of an involution $\sigma$ which is assumed to be an anti-automorphism of the algebra and an automorphism of coalgebra, demanding that $\sigma$- and $(-\sigma^*)$- invariant subspaces should be Lie subalgebras in $L$ and $L^*$ and therefore commutative subalgebras. The exact description, in terms of generators $H_i \in H$ and $X^\mu \in V$, $L = H \triangleright V$, is as follows.

\begin{align}
[H_i, H_k] &= C^m_{ik} H_m, \\
[H_i, X^\mu] &= A^\mu_{\nu i} X^\nu, \\
[X^\mu, X^\nu] &= 0, \\
\delta(X^\mu) &= \gamma^{\mu}_{\rho \sigma} (X^\rho \otimes X^\sigma), \\
\delta(H_i) &= \alpha^k_{\rho i} (X^\rho \otimes H_k - H_k \otimes X^\rho).
\end{align}

(1)

The tensors $C^m_{ik}$ and $\gamma^{\mu}_{\rho \sigma}$ are skew-symmetric and satisfy the Jacobi identity. Matrices $A^\mu_{\nu i}$ and $\alpha^k_{\rho i}$ realize representations of $H$ on $V$ and $V^*$ on $H^*$, respectively.

To match Lie bialgebra cocycle condition, Lie structures on $L$ and $L^*$ should be consistent:

\begin{align}
A^\mu_{\nu i} \gamma^\nu_{\rho \sigma} - \gamma^\mu_{\nu \rho} A^\nu_{i \sigma} - \gamma^\mu_{\rho \nu} A^\nu_{i \sigma} &= A^\mu_{k \sigma} \alpha^k_{\rho i} - A^\mu_{k \rho} \alpha^k_{\sigma i}, \\
\alpha^k_{\mu m} C^m_{ij} - C^k_{im} \alpha^m_{\mu j} - C^k_{mj} \alpha^m_{\mu i} &= \alpha^k_{\nu j} A^\nu_{i \mu} - \alpha^k_{\nu i} A^\nu_{j \mu}.
\end{align}

(2) (3)

Bialgebras of such a type form a category which we denote $\mathcal{B}$. Its morphisms are those respecting Lie products on $L$ and $L^*$ and commuting with the involution $\sigma$.

Our quantization strategy relies on the quantum duality principle [13, 12] as applied to the problem of ”exponentiating” bialgebras of concern. Following this principle, we consider a quantum algebra as a variety of noncommutative functions on the group $\exp(L^*)$. In accordance with the dual group method of building quantum deformations
[17], we fix the coproduct

\[
\begin{align*}
\Delta(1) &= 1 \otimes 1, \\
\Delta(X^\mu) &= D^\mu(X \otimes 1, 1 \otimes X) \\
\Delta(H_i) &= (e^{\alpha \cdot X})_i^k \otimes H_k + H_i \otimes 1.
\end{align*}
\]

(4)

just exponentiating the Lie bracket on \( L^* \). We use notations \( D(.,.) \) for the Campbell-Hausdorff series corresponding to the Lie structure constants \( \gamma_{\rho\sigma}^{\mu} \) in the Lie algebra \( V^* \), and \( \alpha \cdot X \) for the matrix with entries \( \alpha^i_{\rho k} X^\rho \). The coproduct is evidently coassociative, as the elements \( X^\mu \) commute. The problem boils down to evaluating the full set of quantum commutation relations consistent with (4). We will search for them in the form

\[
\begin{align*}
[X^\mu, X^\nu] &= 0, \\
[H_i, H_k] &= C(X)^m_{ik} H_m, \\
[H_i, X^\mu] &= A(X)^{\mu}_{i}.
\end{align*}
\]

(5)

treating quantum structure constants as formal series in commutative generators \( X^\mu \).

**Theorem 1** There exists the unique quantization of the bialgebra \((L, L^*)\) with coproduct (4) and commutation relations (5), such that

\[
C(0)^m_{ik} = C^m_{ik}, \quad \frac{\partial A(0)^{\mu}_{i}}{\partial X^\nu} = A^{\mu}_{i\nu}.
\]

It is a functor from the category \( \mathcal{B} \) onto the sub-category \( \mathcal{H} \) of Hopf algebras.

**Proof.** Substituting (4) into \([\Delta(H), \Delta(X)] = \Delta([H, X])\) we come to the equation

\[
A(D(X', X''))^\mu_i = (e^{\alpha \cdot X'})^k_i \partial'_\nu D^\mu(X', X'')A(X'')^{\mu}_{k} + \partial'_\nu D^\mu(X', X'')A(X')^{\mu}_{i},
\]

(6)

where primes distinguish tensor factors. Regarding \( X^\mu \) as the coordinate functions on the Lie group \( \exp(V^*) \) we can consider \( A(X)^{\mu}_{i} \) as a set of vector fields in the normal neighborhood of the identity, labeled by index \( i \). Then equation (6) is nothing else than

\[
A(\xi \circ \zeta)^{\mu}_{i} = (e^{\alpha \cdot \xi})^k_i L_\zeta A(\zeta)^{\mu}_{k} + R_\zeta A(\xi)^{\mu}_{i}, \quad \xi, \zeta \in V^*,
\]

(7)
where \( L_\xi, R_\xi \) stand for the left and right actions of the group \( \exp(V^*) \) on the vector fields. Note that both these actions commute with the action specified by the matrices \( \alpha_\mu \). Transition to the functions \( \hat{A}(\xi) = R_\xi^{-1}A(\xi) \) leads to the group 1-cocycle equation

\[
\hat{A}(\xi \circ \zeta)_{\nu}^{\mu} = (e^{\alpha \cdot \xi})_{\mu}^{\kappa} Ad(\xi) \hat{A}(\zeta)_{\kappa}^{\mu} + \hat{A}(\xi)_{\nu}^{\mu},
\]

which has the unique solution, provided the differential \( d\hat{A}(0) \) is a corresponding 1-cocycle of the Lie algebra \( V^* \). That is a part of the Lie bialgebra consistency conditions \( (\mathfrak{g}) \) on the pair \( (\mathbf{L}, \mathbf{L}^*) \). The explicit formula for the functions \( A(X)_{\nu}^{\mu} \) is

\[
A(X)_{\nu}^{\mu} = \left( \frac{\gamma \cdot X}{\epsilon^{\gamma \cdot X} - 1} \right)^{\mu}_{iv} A_{\nu i}^{\kappa} X^{\kappa}. \tag{9}
\]

Here \( (\gamma \cdot X)^{\mu}_{\nu} = \gamma^{\mu}_{\nu} X^\sigma \) specifies the adjoint representation of the Lie algebra \( V^* \). We mark the matrices with primes to stress that they act on the different groups of indices.

Note, that formula (9) is simplified in the case of Abelian \( V^* \): then \( A(X)_{\nu}^{\mu} \) takes the form

\[
A(X)_{\nu}^{\mu} = \left( \frac{e^{\alpha \cdot X} - 1}{\alpha \cdot X} \right)^{\mu}_{iv} A_{\nu i}^{\kappa} X^{\kappa}.
\]

Requirement \( [\Delta(H), \Delta(H)] = \Delta([H, H]) \) leads to the following two equations:

\[
C(D(X', X''))_{ij}^{k} = C(X')_{ij}^{k}, \tag{10}
\]

meaning that \( C(X)_{jk} \) are actually constant, and

\[
(e^{\alpha \cdot X})_{m}^{n} C_{ij}^{n} = C_{mn}^{k} C_{ij}^{m} = [H_i, (e^{\alpha \cdot X})_{j}^{k}] = [H_j, (e^{\alpha \cdot X})_{i}^{k}]. \tag{11}
\]

Lie algebra representation by matrices \( \alpha_\mu \) induces an anti-homomorphism of \( \exp(V^*) \) into the linear group \( \text{Lin}(\mathbf{H}, \mathbf{H}) \). The expressions on the right-hand side of (11) are the vector fields \( A(X)_i \), transferred by that map to \( \text{Lin}(\mathbf{H}, \mathbf{H}) \). In terms of matrices \( a = e^{\alpha \cdot X} \), we rewrite (11) as

\[
a_{m}^{k} C_{ij}^{m} a_{i}^{n} a_{j}^{n} = A(a)_{ij}^{k} - A(a)_{ji}^{k} \tag{12}
\]

or

\[
aC(a^{-1} \otimes a^{-1}) - C = A(a) \pi(a^{-1} \otimes a^{-1}), \tag{13}
\]
where we introduced the anti-symmetrizer $\pi$, $\pi^{ij}_{kl} = \delta^i_k \delta^j_l - \delta^i_l \delta^j_k$. The left-hand side of this equation is a coboundary 1-cocycle on the linear group, so we must prove that for the right-hand side. Then, since group 1-cocycles are uniquely determined by their derivatives at the identity, (13) will follow from (3). By virtue of (7), we have

$$A(ba)\pi((ba)^{-1} \otimes (ba)^{-1}) = A(b)(a \otimes a) + b A(a)(a^{-1} \otimes a^{-1})\pi(b^{-1} \otimes b^{-1})$$

$$= A(b)\pi(b^{-1} \otimes b^{-1}) + b\{A(a)\pi(a^{-1} \otimes a^{-1})\}(b^{-1} \otimes b^{-1}),$$

as required.

The counit is evident: $\epsilon(H_i) = \epsilon(X^\mu) = 0$. The antipode is determined on the generators by the coproduct: $S(X^\mu) = -X^\mu, S(H_i) = -(e^{-\alpha \cdot X})^k_i H_k$. Let us prove that it is extended over the whole algebra anti-homomorphically. It is trivial in the commutative $X$-sector. Further,

$$[S(H_i), S(X^\mu)] = (e^{-\alpha \cdot X})^k_i[H_k, X^\mu] = (e^{-\alpha \cdot X})^k_i A(X)^\mu_k$$

$$= -A(-X)^\mu_i = S([X^\mu, H_i]),$$

as immediately follows from formula (11). Condition $[S(H_i), S(H_j)] = S([H_j, H_i])$ boils down to verification of

$$C^k_{mn} a^{-1^m}_i a^{-1^n}_j + a^{-1^m}_i [H_m, a^{-1^n}_j] + a^{-1^n}_j [a^{-1^m}_i, H_n] = a^{-1^k}_m C^m_{ij}, \quad a = e^{\alpha \cdot X}.$$

We represent it as

$$S\left(C^k_{mn} a^m_i a^n_j\right) - S\left(H_i a^n_j\right) - S\left(a^m_i H_j\right) = S\left(a^k_m C^m_{ij}\right),$$

which holds true in view of (12).

Thus we described the Hopf structure of quantum algebras from $\mathcal{H}$. We have yet to check that the quantization $\mathcal{B} \rightarrow \mathcal{H}$ is a natural map of categories. Let $\phi$ be a Lie bialgebra morphism $L \rightarrow L'$ such that $\phi \sigma = \sigma' \phi$. This implies that $\phi(H) \subset H'$ and $\phi(V) \subset V'$. We define the map $\Phi: U_q(L) \rightarrow U_q(L')$ by the same formulas on the generators as $\phi$. Note that $X^\mu$ and $H_i$ in the quantum algebra are, normally, not the same as the classical generators. We do not interested in relations between them,
although in some cases like twisted algebras \[7, 8\] it is possible to give the explicit formulas. Linear map \( \Phi \) on the generators can be extended over the whole quantum algebras as a Hopf homomorphism. It is evident for the coproduct because it is given by the composition in the dual Lie groups, and \( \phi \) is a Lie bialgebra homomorphism. That can be shown for the commutation relations as well. Indeed, value of the quantum commutator \((9)\) differs from the classical one by involvement of the matrices \((\alpha \cdot X)_i^k\) and \((\gamma \cdot X)_\mu^\nu\). They specify the adjoint representation of the subalgebra \( V^* \subset L^* \).

Because \( \phi^* \) is a homomorphism of the dual Lie algebras, and preserves \( \sigma \)-invariant subspaces, matrices \( \Phi_i^k \) and \( \Phi^\mu_\nu \) are pulled through \( \alpha \cdot X \) and \( \gamma \cdot X \) properly, e.g. \( (\alpha' \cdot X')\Phi = \Phi(\alpha \cdot \Phi(X)) \), so the proof becomes immediate.

We denote quantization of \( U(L) \) as \( U_q(L) \) although there is no deformation parameter involved so far. It can be introduced by substitution \( \alpha \rightarrow \ln(q)\alpha, \gamma \rightarrow \ln(q)\gamma \) for the structure constants of the dual Lie algebra but irrelevant for our study. Algebra \( U_q(L) \) contains two classical objects: universal enveloping algebra \( U(H) \) and the commutative algebra of functions on the Lie group \( \exp(V^*) \). In accordance with our convention, we may assume \( \text{Fun}(\exp(V^*)) \cong U_q(V) \). Actually \( U_q(L) \) is a bicrossproduct Hopf algebra \( U(H) \triangleright U_q(V) \), with the coaction on \( U(H) \) given by \( H_i \rightarrow (e^{\alpha \cdot X})_i^k \otimes H_k \).

4 \hspace{1em} \textbf{Duality- and double-invariance}

Lie bialgebra category \( \mathcal{B} \) is evidently self-dual, the involution \( \sigma \) going over into \(-\sigma^*\). Let us prove the analogous assertion for \( \mathcal{H} \) and deduce explicitly the canonical element.

As a linear space \( U_q(L) \) is the tensor product \( U_q(V) \otimes U(H) \). There are two natural algebra maps from \( U^*(H) \) and \( U(V^*) \) into \( U_q^*(L) \): we set \( \eta \rightarrow \varepsilon_V \otimes \eta \) and \( \zeta \rightarrow \zeta \otimes \varepsilon_H \), correspondingly. It is straightforward that

\[
\langle \eta \zeta, \varphi(X)\psi(H) \rangle = \langle \eta \otimes \zeta, \varphi(\Delta(X))\psi(\Delta(H)) \rangle = \langle \eta, \psi(H) \rangle \langle \zeta, \varphi(X) \rangle.
\]

Here we do not make difference between functionals \( \zeta, \eta \) and there images in \( U_q^*(L) \). Just proved, the factorization property justifies such an abuse of notations. It means that linear spaces \( U^*(H) \) and \( U(V^*) \) are isomorphically embedded into \( U_q^*(L) \) (in fact,
these are isomorphisms of associative algebras, see Appendix), and the induced map
\( U(\mathbf{V}^*) \otimes U^*(\mathbf{H}) \to U(\mathbf{V}^*)U^*(\mathbf{H}) \) is a linear bijection on \( U_q^*(\mathbf{L}) \).

Let us choose the bases \( \eta^i \in U^*(\mathbf{H}) \) and \( \zeta_\mu \in U(\mathbf{V}^*) \) dual to \( H_i \) and \( X^\mu \) as generators of \( U_q^*(\mathbf{L}) \). It can be shown that they have coproducts of the form (1), and commutation relations similar to (1), of course, after interchanging \( \mathbf{L} \) and \( \mathbf{L}^* \). That is done in Appendix. Then Theorem 1 states that \( \mathbf{L}^* \) admits the unique quantization \( U_q(\mathbf{L}^*) \sim U_q^*(\mathbf{L}) \) belonging to \( \mathcal{H} \).

Because of the factorization property and due to the fact that the pairings \( \langle \eta, \psi(H) \rangle \) and \( \langle \zeta, \varphi(X) \rangle \) are the same as if \( H_i \) and \( \zeta_\mu \) were primitive, we can easily write down the canonical element \( \mathcal{T} \in U_q(\mathbf{L}^*) \otimes U_q(\mathbf{L}) \). Nevertheless, it is convenient to deal with the opposite algebra \( U_q(\mathbf{L}^*)_{op} \); moreover, it is that algebra which takes part in construction of the double, the subject of our further interest. In Appendix we prove the following result:

\[
\mathcal{T} = \exp(\zeta_\mu \otimes X^\mu) \exp(\eta^i \otimes H_i),
\]

using expressions for the canonical elements of the classical universal enveloping Hopf algebras. Summation over repeating indices is assumed.

Now we proceed to the study of the double in the category \( \mathcal{H} \). First recall that the double of a Lie bialgebra \( \mathcal{D}(\mathbf{L}) \) is a unique Lie bialgebra such that \( \mathbf{L} \) and \( \mathbf{L}^* \), taken with the opposite Lie bracket, are embedded as sub-bialgebras, and the canonical pairing between them gives rise to a non-degenerate invariant symmetric bilinear form on \( \mathcal{D}(\mathbf{L}) \). The double procedure preserves category \( \mathcal{B} \). Indeed, the classical commutation
relations followed from the definition are

\[ [H_i, H_k] = C_{ik}^m H_m, \]

\[ [\zeta_{\mu}, \zeta_{\nu}] = \gamma_{\mu\nu}^{\sigma} \zeta_{\sigma}, \]

\[ [H_i, \zeta_{\mu}] = -\alpha_{\mu}^k H_k - A_{\mu}^\nu \zeta_{\nu} \]

\[ [X^\mu, X^\nu] = 0, \]

\[ [\eta^i, \eta^j] = 0, \]

\[ [X^\mu, \eta^i] = 0, \]

\[ [H_i, X^\mu] = A_{\mu i}^\nu X^\nu, \]

\[ [H_i, \eta^j] = C_{jki}^\nu \eta^k + \alpha_{\mu}^i X^\mu, \]

\[ [\zeta_{\mu}, \eta_i] = -\alpha_{i\mu}^k \eta_k, \]

\[ [\zeta_{\mu}, X^\nu] = -\gamma_{\sigma\mu}^\nu X^\sigma - A_{\mu}^\nu \eta^i, \]

that proves the assertion.

It is thus natural to expect the analogous statement in the quantum case.

**Theorem 2** Quantum double preserves category $\mathcal{H}$. Moreover, $\mathcal{D}(U_q(\mathbf{L})) = U_q(\mathcal{D}(\mathbf{L}))$.

As a coalgebra, the double coincides with the tensor product of $U_q(\mathbf{L})$ and $U_q(\mathbf{L}^*)_{\text{op}}$, which are at the same time subalgebras. Therefore, to prove the theorem, it suffices to show that the cross-relations have the appropriate form. Then we will satisfy conditions of Theorem [1] which states the uniqueness of the quantization and provides its explicit form. The cross-relations are deduced from the Yang-Baxter equation on the canonical element and can be written as

\[ e_\mu e^\nu = e_\alpha e_{\beta} m_{\gamma\alpha\beta} m_{\mu}^{\rho\beta\sigma} S^\gamma_{\rho}, \]

where $e_\mu \in U_q(\mathbf{L})$, $e^\nu \in U_q(\mathbf{L}^*)$, $m_{\gamma\alpha\beta}$ and $m_{\mu}^{\rho\beta\sigma}$ denote the iterated coproduct structure constants, and $S^\gamma_{\rho}$ is the matrix of the antipode. Using the explicit formulas for the
we get the required result. For example,

\[ H\zeta = \langle e^{-\alpha \cdot X}, \zeta \rangle H + \langle e^{-\alpha \cdot X}, e^{A \cdot \eta} \rangle \zeta H + \langle H, e^{-A \cdot \eta} \rangle \zeta \]

(only non-vanishing terms retained). Thus we obtain the expression for the commutator

\[ [H_i, \zeta_\mu] = -\alpha_{\mu i}^k H_k - A_{\mu \nu}^i \zeta_\nu. \]

Similarly we prove that \( X^\mu \) and \( \eta^i \) form a commutative algebra, invariant under the adjoint action of \( H_i \) and \( \zeta_\mu \).

Now consider a Lie bialgebra from \( \mathcal{B} \) which possesses a classical r-matrix. To use the advantages of functorial property of the quantization, let us demand that the r-matrix viewed as a Lie bialgebra morphism be that in \( \mathcal{B} \). It means that \( r \) must commute with the involution \( \sigma \) (we remind that for dual bialgebra \( \sigma \) goes over into \(-\sigma^*\)). The general form of \( r \) is then

\[ r = P^i_\mu H_i \otimes X^\mu + Q^i_\mu X^\mu \otimes H_i. \]

We may assume that summation is performed over elements of the bases of \( L_+ \) and \( L_- \). Then the universal R-matrix is found from (14):

\[ R = \exp(P^i_\mu H_i \otimes X^\mu) \exp(Q^i_\mu X^\mu \otimes H_i). \]

In this form this is a generalization of the result of [14] obtained for the simplest case of the double of the jordanian Borel quantum algebra.

5 Discussions

Drinfeld’s conjecture of the possibility to quantize an arbitrary Lie bialgebra was proved by Etingof and Kazhdan [18]. Although there are numerous examples of quantum algebras, the problem of exponentiating a Lie bialgebra in every particular case remains
highly nontrivial. In the present paper, we do it for a class of algebras playing significant role in the classical differential geometry and physics, inhomogeneous Lie algebras. This class forms a nice category invariant under dualization and quantum double operations. The quantization built is a functor from the Lie bialgebra category of concern into the category of Hopf algebras. This functor commutes with the functors of dualization and double. We showed that the quantization of inhomogeneous Lie algebras possessing classical r-matrix itself contains the solution to the quantum Yang-Baxter equation which is the universal R-matrix for its quasitriangular Hopf subalgebra. This statement is based on the fact that every minimal quasitriangular Hopf algebra is a quotient of the double of its subalgebra, and, also, on the functorial property of the quantization.

The class of Hopf algebras considered in the present article includes twisted universal enveloping Lie algebras taking part in the null-plane quantization of Poincaré algebra and extended jordanian deformations of $sl(N)$. ([19, 7, 8]). They are characterized by the identification $V \sim H^*$ and involve non-degenerate Lie algebra 1-cocycles in building crossproduct $H \rtimes H^*$. The resulting quantum algebras were triangular and twist-equivalent to the classical universal enveloping algebras. Here we have studied the general case.

The appearance of Lie group 1-cocycles in construction of quantization is quite understandable. According to Drinfeld [12], a group 1-cocycles with values in Lie algebra external square defines Poisson-Lie structures on the group. Generators $H_i \in H$ and $X^\mu \in V$ are the coordinate functions on the group $H^* \ltimes \exp(V^*)$, and the quantum commutation relations among them represent nothing else than the Poisson bracket. Along with the coproduct (4), this implies that the quantum algebra $U_q(L)$ is just universal enveloping algebra of the Lie-Poisson algebra on the group $\exp(L^*)$. Indeed, the classical commutation relations of the types $[H,H]$ and $[X,X]$ are given by the Poisson structure on the Abelian group $H^*$ and the trivial Poisson bracket on the group $\exp(V^*)$. Further, the Poisson bracket of the type $\{H,X\}$ must satisfy the
\[ \Delta(\{H,X\}) = \{ e^{\alpha \cdot X} \otimes H + H \otimes 1, D(X \otimes 1, 1 \otimes X) \} \]
\[ + \{ H \otimes 1, D(X \otimes 1, 1 \otimes X) \} \]
\[ \Delta(\{H,H\}) = \{ e^{\alpha \cdot X} \otimes H + H \otimes 1, e^{\alpha \cdot X} \otimes H + H \otimes 1 \} = \]
\[ e^{\alpha \cdot X} e^{\alpha \cdot X} \otimes \{ H, H \} + \{ H, H \} \otimes 1 + \]
\[ e^{\alpha \cdot X}, H \} \otimes H + \{ H, e^{\alpha \cdot X} \} \otimes H \] (17)

(we drop all the indices for the reason of transparency). These expressions involve the Poisson bracket and the multiplication in the function algebra on \( \exp(L^*) \). It is seen that only the product in \( \text{Fun}(\exp(V^*)) \) really matters, which is commutative and coincides with that on \( U_q(V) \subset U_q(L) \). So, one can consider equations (17) as in the quantum algebra.

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**Appendix**

The aim of this section is to exhibit the details of the proof of the formula (14). The canonical element for the universal enveloping algebra \( U(V^*) \) is (12)

\[ e^{\zeta_{\mu} \otimes X^\mu} = \sum_{n} \sum_{\vec{\mu}} \left( \zeta_{\mu_1}, \ldots, \zeta_{\mu_n} \right) \otimes X^{\mu_1} \ldots X^{\mu_n}, \]

where \( \vec{\mu} = (\mu_1, \ldots, \mu_n) \) stands for the ordered multiindex of length \( n \), and the parentheses denote symmetrized monomials \( \left( \zeta_{\mu_1}, \ldots, \zeta_{\mu_n} \right) = \frac{1}{n! s(\mu)} \sum_{\sigma} \zeta_{\sigma(\mu_1)} \ldots \zeta_{\sigma(\mu_n)} \) of degree \( n \). Here \( \sigma \) belongs to the symmetric group \( S_n \) and \( s(\mu) \) is equal to the order of the stability subgroup under permutations of \( \mu \). Similarly we have

\[ e^{\eta^i \otimes H_i} = \sum_{n} \sum_{i} \eta^{i_1} \ldots \eta^{i_n} \otimes (H_{i_1}, \ldots, H_{i_n}) \]
for the algebra $U(H)$. Hence, due to the factorization of the matrix elements of the canonical pairing (Section 4), the element $e^{\zeta_\mu \otimes X^\mu} e^{\eta^i \otimes H_i}$ is canonical for $U_q(L)^*_{op}$. What remains is to show that the coproduct and commutation relations for $\zeta_\mu$ and for $\eta^i$ have the proper form

\[
\langle \eta^i \eta^j, X^{\mu_1} \ldots X^{\mu_n}(H_{i_1}, \ldots, H_{i_m}) \rangle = \varepsilon(X^{\mu_1} \ldots X^{\mu_n}) \langle \eta^i \otimes \eta^j, \Delta((H_{i_1}, \ldots, H_{i_m})) \rangle = \varepsilon(X^{\mu_1} \ldots X^{\mu_n}) \langle \eta^i \otimes \eta^j, \Delta_0((H_{i_1}, \ldots, H_{i_m})) \rangle.
\]

The last equality, where $\Delta_0$ is the classical cocommutative comultiplication in $U(H)$, is due to that one can push all the factors $(e^{\alpha \cdot X})^k_i$ in $\Delta(H_{i_k})$ to the left, where they are reduced to 1, as though they commute with $H$’s. That is because $X$’s generate an ideal, and, once appeared, the terms containing them will be annihilated by $\eta^i$. So, the generators $\eta^i$ commute. Further,

\[
\langle \zeta_\mu \zeta_\nu, X^{\mu_1} \ldots X^{\mu_n}(H_{i_1}, \ldots, H_{i_m}) \rangle = \langle \zeta_\mu \otimes \zeta_\nu, \Delta(X^{\mu_1} \ldots X^{\mu_n}(H_{i_1}, \ldots, H_{i_m})) \rangle = \langle \zeta_\mu \otimes \zeta_\nu, \Delta(X^{\mu_1} \ldots X^{\mu_n})(H_{i_1}, \ldots, H_{i_m} \otimes 1) \rangle = \varepsilon(H_{i_1}, \ldots, H_{i_m}) \langle \zeta_\mu \otimes \zeta_\nu, \Delta(X^{\mu_1} \ldots X^{\mu_n}) \rangle,
\]

and therefore $\zeta$’s have the classical product of the universal enveloping algebra $U(V^*)$. Among the matrix elements

\[
\langle \zeta_\mu \eta^i, X^{\mu_1} \ldots X^{\mu_n}(H_{i_1}, \ldots, H_{i_m}) \rangle = \langle \zeta_\mu \otimes \eta^i, (X^{\mu_1} \ldots X^{\mu_n} \otimes 1)\Delta(H_{i_1}, \ldots, H_{i_m}) \rangle
\]

only those survive where $n \leq 1$. Developing products of $\Delta(H)$’s we see that monomials $1 \otimes H_{i_1} \ldots H_{i_k}$ turn out to be symmetrized automatically, hence we can retain terms of the first degree in $1 \otimes H_i$ only. And furthermore, if $n = 1$ then with necessity $m = 1$. The non-vanishing pairings are

\[
\langle \zeta_\mu \eta^i, X^{\mu_1}(H_{i_1}, \ldots, H_{i_m}) \rangle = \langle \zeta_\mu \otimes \eta^i, (X^{\mu_1} \ldots X^{\mu_n} \otimes 1)\Delta(H_{i_1}, \ldots, H_{i_m}) \rangle = \langle \zeta_\mu \otimes \eta^i, (e^{\alpha \cdot X})^k_{i_m} \otimes H_k \rangle = \langle \zeta_\mu \otimes \eta^i, \varphi(X) \otimes H_k \rangle
\]

where $\varphi(X)$ is a result of pulling the exponential to the left. Thus we state that the difference $\zeta_\mu \eta^i - \eta^i \zeta_\mu$ does not vanish only on the elements $(H_{i_1}, \ldots, H_{i_m})$, therefore it depends solely on $\eta^i$. 

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We have yet to find the coproduct. It is straightforward that $\Delta(\eta)$ survives on the elements with no $X$’s and therefore is expressed by the Campbell-Hausdorff series corresponding to $U(H)$. For $\Delta(\zeta)$ the nontrivial pairing is with elements $X^\mu \otimes 1$ and $(H_{i_1}, \ldots, H_{i_m}) \otimes X^\mu$. While pulling $H$’s to the right we can assume that they commute with $X$’s with the classical relations, that is the commutator is linear in $X$ because the higher degrees will annihilate. Thus we come to the desired formula

$$\Delta(\zeta_\mu) = (e^{A_\eta})_\mu^\nu \otimes \zeta_\nu + \zeta_\mu \otimes 1.$$ 

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