Generating nonequilibrium stationary state from ground state condensate through an almost-adiabatic cycle

Atushi Tanaka,1,∗ Takaaki Nakamura,2 and Taksu Cheon2

1Department of Physics, Tokyo Metropolitan University, Hachioji, Tokyo 192-0397, Japan
2Laboratory of Physics, Kochi University of Technology, Tosa Yamada, Kochi 782-8502, Japan

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It is shown that the ground state of weakly interacting Bose particles in a quasi one-dimensional box trap can be converted into an excited stationary state by an adiabatic cyclic operation that involves a quench of interaction strength: A sharp impurity potential is applied and its strength is varied during the cycle, which induces a nonequilibrium stationary state exhibiting the inversion of population. This process is robust in the sense that the resultant stationary state is almost independent of the details of the cycle, such as the position of the impurity as long as the cycle is far enough from critical regions. The case of the failure of the population inversion due to the strong interparticle interactions is also examined.

I. INTRODUCTION

According to the principle of the equilibrium thermodynamics, a quasi-static adiabatic cycle is trivial, in the sense that the initial and final states are identical [1]. Once we slightly relieve the quasi-static adiabatic condition, or the thermodynamic condition, however, a cyclic operation may transform a stationary state into another one. A promising system to realize such stationary state transformations in a many-body, nearly thermodynamic setting is cold atoms. This is because it is possible to manipulate the system with its quantum coherence intact, as it can be well-isolated from environmental degrees of freedom.

An example [2] can be found in the Lieb-Liniger model [3, 4], which describes Bose particles confined in a one-dimensional ring. A stationary state of the Lieb-Liniger model is delivered to another stationary state through a cyclic operation where the strength of interparticle interaction is adiabatically increased except at a point: Once the interparticle interaction becomes infinitely repulsive to make the Tonks-Girardeau regime [5], the interparticle interaction strength is flipped to infinitely attractive to form the super-Tonks-Girardeau regime [6, 7]. During the cyclic operation, the stationary state is smoothly deformed even at the strength flipping point, where normalizable stationary states are ensured to be kept unchanged.

The state transformations in the Lieb-Liniger model is an example of exotic quantum holonomy in adiabatic cycles of microscopic, non-thermal systems: Although one may expect that an adiabatic cycle brings no change in stationary states up to a phase factor, it may transform a stationary state into another, for example, in Floquet systems through the winding of quasienergy [8–10] and Hamiltonian systems with level crossings [11, 12]. A similar, but distinct concept to the exotic quantum holonomy, Wilczek-Zee’s holonomy, where an adiabatic cycle offers a transformation of degenerated stationary states [13–15], is also utilized to control quantum states [16–18].

We note that the adiabatic state transformation in the Lieb-Liniger model [2] heavily depends on the particularity of the model. The number of particles is required to be specified precisely. Also, the theoretical argument in Ref. [2] depends essentially on the solvability of the system. Hence it seems difficult to extend this result to other quantum many-body systems.

In this paper, we examine an adiabatic state transformation in a simple quantum many-body system, which might be extensible to various ways. We here examine cyclic operations on Bose particles confined in a quasi one-dimensional box trap [19–22] to generate a nonequilibrium stationary state from the ground state. Applying an almost-adiabatic cycle that involves a quench of the strength of a sharp impurity potential, we obtain a population-inverted state, in which the Bosons occupy a single-particle excited state. The population inversion has been utilized to achieve negative temperature [23], for example, and is related to dark solitons in the studies of Bose-Einstein condensates [24, 25].

Our starting point is the analysis of cyclic operations for noninteracting Bose particles [26, 27]. We define a cycle using a sharp impurity potential, which is often utilized to manipulate condensates both experimentally and theoretically [28–33]. In order to incorporate the interparticle interaction, we suppose that the system is described by the time-dependent Gross-Pitaevskii equation. We show that the interparticle interaction can significantly modify the almost-adiabatic processes due to the appearance of bifurcations in the solution of time-independent Gross-Pitaevskii equation such as the swallowtail structure [25, 34–38].

The plan of this paper is the following. In Sec. 2, we introduce a cyclic operation that involves a flip of the potential strength which is analogous to the cyclic operation introduced in Ref. [26] for non-interacting systems. In Sec. 3, we numerically examine the cyclic operation in the repulsively interacting Bose particles using the Gross-Pitaevskii equation, where it is shown that the

∗ http://researchmap.jp/tanaka-atushi/
strong nonlinearity invalidates the population inversion. In Sec. 4, a theoretical interpretation for the numerical result is shown. Sec. 5 concludes this paper with summary and outlook. Appendix A offers details of the linear stability analysis (the Bogoliubov analysis) at the quench point.

II. CYCLE FOR NON-INTERACTING PARTICLES

We look at a cyclic operation for particles confined in a quasi one-dimensional boxed trap with an impurity potential. We illustrate how this cycle works for non-interacting particles [26], which offers a basis for examining the case of interacting Bosons.

We assume that a particle is described by the one-dimensional time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(x,t) = -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x^2} \Psi(x,t) \quad (1)$$

under the boundary condition $\Psi(0,t) = \Psi(L,t) = 0$, where $M$ is the mass of the particle, and $L$ is the size of the box trap. In the following, we assume $\hbar = 1$, $M = 1/2$ and $L = 2\pi$.

After the system is prepared to be in a stationary state, a sharp impurity potential is placed at $x = X$ to realize cyclic operations, where the strength $v$ is varied. We assume that the impurity potential $V_i(x;v)$ is described by the Dirac’s delta function:

$$V_i(x;v) = v\delta(x-X). \quad (2)$$

There are two building blocks of the cyclic operation: One is the smooth and monotonic variations of parameter: $C_s((v',v''),(v''',v'''))$, which denotes the variation of $v$ from $v'$ to $v'''$, while keeping the value of $X$. The other is the discontinuous operation $C_d((v',v''),(v''',v'''))$ in which the value of $v$ is changed from $v'$ to $v'''$, which resembles the process that is utilized to create the super-Tonks-Girardeau gas from the Tonks-Girardeau gas [7].

We define the almost-adiabatic cyclic operation $C(X)$ (see, Fig. 1), which involves a quench of the impurity potential placed at $X$. This cycle is a succession of three operations $C_d(0,\infty)$, $C_d(\infty,-\infty)$ and $C_d(-\infty,0)$.

We show that an initial stationary state $\Psi_n(x)$ can be transformed to another stationary state after the completion of a cycle. Here we assume that the parameters are varied adiabatically during the smooth operations. Hence the system is governed by the adiabatic theorem [39]. Since the relevant eigenenergy and eigenfunction are continuous during the quench, the parametric dependence of eigenenergies tells us the final stationary state for the almost-adiabatic cycle $C(X)$. We depict an example of the parametric evolution of eigenenergies in Fig. 2.

After the completion of the almost-adiabatic cycle $C(X)$, the final state is $\Psi_{n+1}(x)$, as long as $X$ does not coincide with the node of the initial wavefunction $\Psi_n(x)$. This is because the eigenenergies are increased monotonically during $C_d(0,\infty)$ and $C_d(-\infty,0)$ as $v$ is increased monotonically [26, 40], and are continuous at the quench process $C_d(\infty,-\infty)$ [26].

III. CYCLE FOR AN INTERACTING BOSE SYSTEM

We here examine the almost-adiabatic cycle $C(X)$ in a many-body setting. We assume that Bose particles are
confined in the quasi one-dimensional box and the interparticle interaction is repulsive. First, we examine the parametric evolution of stationary states along the cycle, which suggests that the population inversion is indeed possible if the interparticle interaction is weak. Also, it is shown that the population inversion breaks down when the interparticle interaction is strong enough. Second, we numerically integrate the time evolution equation to confirm the picture obtained through the parametric evolution of the stationary states. We provide a theoretical explanation based on a perturbation theory to these observations in the next section.

We assume that the Bose particles are described by the time-dependent one-dimensional Gross-Pitaevskii equation

$$i \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\partial^2}{\partial x^2} \Psi(x, t) + g|\Psi(x, t)|^2 \Psi(x, t), \quad (3)$$

where \( \hbar = 1 \), \( M = 1/2 \) and \( L = 2\pi \) are assumed, and \( g \geq 0 \) represents the strength of the effective interparticle interaction. We impose the boundary condition \( \Psi(0, t) = \Psi(L, t) = 0 \) and the normalization condition \( \int_0^L |\Psi(x, t)|^2 dx = 1 \). Let \( E_n(g) \) and \( \Psi_n(x; g) \) \((n = 1, 2, \ldots)\) denote the \( n \)-th eigenenergy (chemical potential) and the corresponding stationary state for Eq. (3). We suppose that the system is initially in \( n \)-th stationary state \( \Psi_n(x; g) \). During the cycle, we impose the sharp impurity potential (Eq. (2)) to Eq. (3) and vary its strength \( v \) slowly except at the quench point.

To find the stationary states of the system where the position \( X \) and strength \( v \) of the impurity potential (Eq. (2)) are “frozen”, we examine the time-independent Gross-Pitaevskii equation with the impurity potential

$$\left(-\frac{\partial^2}{\partial x^2} + g|\Psi(x)|^2 + V_i(x; v)\right) \Psi(x) = E \Psi(x). \quad (4)$$

Examples of parametric evolution of \( n \)-th eigenenergy along the cycle \( C(X) \) are shown in Fig. 3. Corresponding parametric evolution of stationary states are shown in Figs. 4 and 5.

When the interparticle interaction \( g \) is small enough, the quasi-adiabatic cyclic operation induces the population inversion. The parametric evolution of eigenenergy along the cycle connects the initial eigenenergy \( E_1(g) \) with \( E_2(g) \) (see, Fig. 3 (a)). The connection is equivalent to the case of noninteracting particles (see, Fig. 2). This allows us to infer the parametric evolution of eigenfunction whose initial condition is the ground state \( \Psi_1(x; g) \) of Eq. (3), i.e., at \( g = 0 \) (see, Fig. 4 (A)). While the stationary state is nodeless during the strength of the sharp impurity potential \( v \) is positive and finite, the state becomes localized at a side of the impurity as \( v \) become larger (Fig. 4 (B)). Immediately before the quench, i.e., \( v = \infty \), the localization completes [41]. The state is unchanged during the flip of the potential strength from \( v = \infty \) to \( v = -\infty \) [26]. As \( v \) is slightly increased from \(-\infty\), the stationary state extends to the other side of the impurity to produce a node (Fig. 4 (C)). The resultant stationary state has a single node while \( v \) is finite (Fig. 4 (D)). This is the reason why the destination of the stationary state at the end of the cycle is the first excited state \( \Psi_2(x; g) \).

On the other hand, when the interparticle interaction is large, discrepancies from the linear case become significant. In particular, as is seen in Figs.3 (c) and (d), the parametric evolution of a stationary energy involves a loop structure that emanates from the quench point \( |v| = \infty \) in \( C(X) \). We note that loop structures are often observed in the studies of time-independent Gross-Pitaevskii equation [25, 34–37].

The corresponding parametric evolution of the stationary state along the loop is explained in Fig. 5. In the vicinity of the quench, the wavefunction extends to both sides of the sharp impurity potential, which is a distinctive feature of the case with stronger interparticle interaction (B±). Across the quench point the wavefunction smoothly connect to the lower branch of the loop to acquire two nodes (B±). As \( v \) increased, the lower branch arrives the extremum point (C) to connect the uppermost branch, where the wavefunction localizes at a side of the
impurity potential, where the stationary state mimics the one in the linear system. As \( v \) decreased to follow the uppermost branch, the stationary state become localized at a side of the impurity (\( D_\pm \)). At the quench point in the uppermost branch, the localization completes. Across the quench, the number of the nodes of the stationary decreases from 2 (\( D_- \)) to 1 (\( D_+ \)). Then the uppermost branch arrives another extremum point (\( E \)), where the state delocalize again to connect the final branch at (\( F \)), which smoothly connects the first excited state \( \Psi_2(x; g) \), see (\( G \)).

We expect that the loop structure disturbs the adiabatic evolution, as reported in Refs. [34, 35], and thus hinders the population inversion. This is because the stationary state is transformed into a non-stationary state when the adiabatic time evolution has to departs from the parametric evolution by having a loop.

To clarify whether the adiabatic time evolution along \( C(X) \) really occurs, we numerically examine the linear stability of the stationary states in \( C(X) \) by diagonalizing the Bogoliubov equation [42, 43] corresponding to the stationary solutions. Also, an analytical study on the linear stability for the quench point is shown in Appendix A.

When the interparticle interaction is small (see, Figs. 3 (a) and 4), we find that the stationary states are linearly stable along \( C(X) \). Hence we may expect that the adiabatic time evolution remains intact for the weakly interacting case.

Meanwhile, when the interparticle interaction is larger to form the loop structure as shown in Figs. 3 (d) and 5, we find that the stationary state is linearly stable within the intervals from \( v = 0 \) through \( v = \pm \infty \), i.e., from \( A \) to \( B \) and from \( G \) to \( F \) in Fig. 3 (d), and the uppermost branch of the loop (from \( C \) to \( E \)). On the other hand, the stationary state is linearly unstable at the lower part of the loop (i.e., from \( B \) to \( C \) and \( E \) to \( F \)).

The result of the linear stability analysis for stronger interparticle interaction suggests that the adiabatic time evolution whose initial condition is the ground state \( \Psi_1 \) remains intact until the quench point, i.e., within the interval \( A \) to \( B \). After the system passes the quench point, the adiabatic time evolution breaks down during the interval \( B \) to \( C \) due to the linear instability. We note that the emergence of the unstable stationary state at the lower branch of the loop structure in the Brillouin zone is reported in Refs. [44, 45]. We also note that this is a distinctive point from the instabilities in the conventional studies [34, 35] of adiabatic time evolution along the loop structure, where the linearly unstable region appears only at the uppermost of the loop structure. After the point \( C \), the adiabatic time evolution is impossible since the stationary solution cannot be adiabatically extended anymore [34, 35].

![Fig. 4](image1.png)

**FIG. 4.** (Color online) Parametric evolution of stationary state \( \Psi(x) \) of weakly interacting Bose particles along the cycle \( C(X = 0.42L) \) and \( g = 1 \). The corresponding points in \((g, E)\)-plane is shown in Fig. 3 (a). (A) \( \Psi(x) \) is the ground state initially (i.e., \( v = 0 \)); (B) and (C) correspond to the case immediately before and after the quench. (D) the final state is the first excited state. The phase of \( \Psi(x) \) is chosen so that \( \Psi(x) \) is positive at right hand side of the sharp impurity.

![Fig. 5](image2.png)

**FIG. 5.** (Color online) Parametric evolution of stationary state \( \Psi(x) \) along an eigenenergy loop associated with the cycle \( C(X = 0.42L) \) at \( g = 4 \), which correspond to the case shown in Fig. 3 (d). The loop connects the initial point \( A \) and the final point \( G \) of the cycle \( C(X) \). Around the quench points \( B \) and \( D \), the sign of \( v \) is indicated by the suffix \( \pm \).

We test the scenario above through numerical integration of the time-dependent Gross-Pitaevskii equation along the almost-adiabatic cycle \( C(X) \). We show our numerical result for various values of \( X \) in Fig 6, where the initial condition is prepared to be in the ground state \( \Psi_1(x; g) \) of Eq. (3). We numerically evaluate the fidelity for the population inversion \(|\langle \Psi_2(g)|\Psi\rangle|^2\), where \( \Psi \) is a state after the completion of the cycle. Since the final states may not be stationary, we depict the time-average of the fidelity probability after the completion of the cycles.

From Fig 6, we conclude that the population inversion fails if the value of \( g \) exceeds a critical value \( g_c \) which depends on the position of the impurity potential.
Moreover, when we restrict the case \(0 < X < L/2\), \(g_c\) becomes larger as \(X\) become smaller.

We make a remark on the integration of time-dependent Gross-Pitaevskii equation along \(C(X)\), where we introduce an approximation for the quench of impurity potential. We keep \(v\), the strength of the impurity potential. finite. Namely, \(v\) is increased from 0 to \(v_{\text{max}}\) with a finite velocity \(dv/dt\) during the first process \(C_s(0, v_{\text{max}})\). At the quench, \(v\) is suddenly changed from \(v_{\text{max}}\) to \(-v_{\text{max}}\). Then, during \(C_s(-v_{\text{max}}, 0)\), the value of \(v\) is increased from \(-v_{\text{max}}\) to 0 with non-zero velocity \(dv/dt\). Although this induces a tiny nonadiabatic error during the quench, as is seen from Fig. 6, the error is far less important than the nonlinear effect.

![FIG. 6. (Color online) Population inversion probability from the ground state through \(C(X)\). Vertical dashed lines indicate the critical point predicted by the two-mode approximation (see, Sec. 4).](image)

**IV. TWO-MODE APPROXIMATION AT THE QUENCH POINT**

We discuss our numerical results in the previous section with an approximate theory. In particular, we would like to clarify the reason why the population inversion breaks down as the strength of the interparticle interaction becomes stronger (Fig. 6). A key ingredient must be the emergence of the loop structure in \(g(E)\)-plane (Fig. 3). To identify the loop structure, we examine the quench point \(|\psi| = \infty\), because a loop emanates from a point at \(|\psi| = \infty\) in \((g(E))\)-plane. This analysis allows us to infer the loop structure, as long as the loop is small enough.

In the following, we utilize a two-mode approximation. Namely, we assume that the stationary wavefunction \(\psi(x)\) is a superposition of two eigenfunctions \(\psi_j(x)\) \((j = 0, 1)\) of the noninteracting system. Since the infinitely strong impurity divide the box completely [41], as suggested in Fig. 3, we utilize the unperturbed eigenfunction \(\psi_j(x)\) that is localized at the left or right side of the impurity.

For example, to examine the stationary states that are associated with the ground state at the initial point of the cycle, we assume that \(\psi_0(x)\) and \(\psi_1(x)\) describe the ground state of a single particle confined within the right and left boxes, respectively, i.e.,

\[
\psi_0(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq X \\ \sqrt{\frac{2}{X}} \sin \frac{x}{X} & \text{for } X \leq x \leq L' \\ 0 & \text{for } X \leq x \leq L, \end{cases}
\]

whose eigenenergies are \(E_0 = E_g/r^2\) and \(E_1 = E_g/r^2\), respectively, where \(r = X/L\), \(r' = (L - X)/L\), and \(E_g = \hbar^2\pi^2/2ML^2\).

In the following, we assume \(0 < X < L/2\), which implies \(E_0 < E_1\), i.e., the ground state \(\psi_0(x)\) under the presence of the infinitely strong impurity localizes at the right side of the impurity.

From the two-mode assumption that \(\Psi(x,t) = \Psi_0(t)\psi_0(x) + \Psi_1(t)\psi_1(x)\) satisfies the time-dependent Gross-Pitaevskii equation, we obtain the time-evolution equation for the amplitudes \(\Psi_j(t) \ (j = 0, 1)\):

\[
i \frac{d}{dt} \Psi_j(t) = E_j \Psi_j(t) + g \int_0^L \psi_j^*(x) |\Psi(x,t)|^2 \Psi(x,t) dx.
\]

Because \(\psi_0(x)\) and \(\psi_1(x)\) have no overlap in the position space, i.e., \(\psi_0(x)\psi_1(x) = 0\) holds, and are real, we obtain

\[
i \frac{d}{dt} \Psi_j = E_j \Psi_j + g \int_0^L dx \{\psi_j(x)\}^4 |\Psi_j|^2 \Psi_j.
\]

Hence the nonlinear Schrödinger equation for \(\Psi_j\) is

\[
i \frac{d}{dt} \Psi_j = (E_j + gc_j|\Psi_j|^2) \Psi_j,
\]

where \(c_0 = 3g/(2(L - X))\) and \(c_1 = 3g/(2X)\). We also impose the normalization condition \(|\Psi_0|^2 + |\Psi_1|^2 = 1\).

The stationary solutions of Eq. (9) are classified into two groups. First, there are two localized solutions \((\Psi_0, \Psi_1) = (1, 0)\) and \((0, 1)\), whose eigenenergies are

\[
E_0(g) = E_0 + gc_0, \quad \text{and} \quad E_1(g) = E_1 + gc_1,
\]

respectively.

Second, the other two solutions \(\Psi_{\pm}\) are

\[
\left[\begin{array}{c} \Psi_{\pm,0} \\ \Psi_{\pm,1} \end{array} \right] = \begin{cases} \sqrt{r'}(E_0 - E_1)(2Lr')/(3g) \\ \pm \sqrt{r}(E_0 - E_1)(2Lr')/(3g) \end{cases},
\]

which are delocalized to both sides of the impurity. The corresponding eigenenergies are doubly degenerate

\[
E_{\pm}(g) = r'E_0 + r E_1 + \frac{3g}{2L}.
\]
We explain the condition that the stationary states Eq. (11) are physical, i.e., $0 \leq |\Psi_0|^2, |\Psi_1|^2 \leq 1$ holds. First, we introduce the critical points

$$ g_0 = \frac{2(L - X)}{3}(E_1 - E_0) \quad (13) $$

$$ g_1 = -\frac{2X}{3}(E_1 - E_0), \quad (14) $$

where $|\Psi_j|^2 = 1$ holds if $g = g_j$ ($j = 0, 1$). Since we assume $X < L/2$, the physical condition for the stationary solution Eq. (11) is summarized as $g \geq g_0$ or $g \leq g_1$. Also, as we restrict the case that the interparticle interaction is repulsive, i.e., $g \geq 0$, the delocalized solutions Eq. (11) exist only when $g \geq g_0$.

From the linear stability analysis (Bogoliubov analysis) of Eq. (9), whose details are shown in Section A, we find that the localized solutions $(\Psi_0, \Psi_1) = (1, 0)$ and $(0, 1)$ are stable. On the other hand, the delocalized solutions Eq. (11) are marginally stable.

With the two mode approximation at $|v| = \infty$, we recapturate the parametric evolution of stationary states and eigenenergies along $C(X)$. We assume that the system is initially in the ground state $\Psi_1(0; g)$.

First, we revisit the case that the interparticle interaction is weak enough, i.e., $g$ is smaller than the critical value $g_0$. The stationary state is localized at $|v| = \infty$, which is consistent with Fig. 4. The corresponding estimation of eigenenergy at $|v| = \infty$ is given by Eq. (10), which is indicated in Fig. 3. Also, the stationary state at $|v| = \infty$ is stable, according to the linear stability analysis. Hence the stationary state must be stable during $C(X)$, which is also consistent with the numerical result that $C(X)$ induces the population inversion for smaller $g$ (Fig. 6).

Second, when the strength of the interparticle interaction $g$ exceeds the critical value $g_0$, the localized and delocalized stationary solutions coexist at the quench point: the degenerated eigenenergies of the delocalized solutions $E_+ (g)$ (Eq. (12)) is lower than the one of the localized solution (Eq. (10)), as shown in Fig. 3. The delocalized nodeless solution $\Psi_+ (\text{Fig } 5 \text{ B}_\pm)$ is connected with the initial ground state $\Psi_1$ through $C_\mu (0, \infty)$, the former half of the whole cycle. After $\Psi_+$ is evolved along the loop, the stationary solution arrives at the localized solution $(\Psi_0, \Psi_1) = (1, 0)$ at $|v| = \infty$ (see, Fig 5 D$_\pm$). After the completion of the loop, the stationary solution becomes the other delocalized solution $\Psi_-$ with a node (Fig 5 F). The latter half of the cycle $C_\mu (-\infty, 0)$ smoothly connects $\Psi_-$ with the first excited state $\Psi_2$. In this sense, the parametric evolution of the stationary solution smoothly deforms the ground state into the first excited state. Meanwhile the adiabatic population inversion is hindered by the presence of the loop structure due to the instability of the stationary state at the lower branch of the loop, as explained in the previous section.

In Fig. 6, we indicate the critical interparticle interaction strength $g_0$, which is estimated by the two-mode approximation for each value of $X$ by vertical lines. Hence we conclude that the two-mode approximation qualitatively describes the breakdown of the population inversion.

We depict how the eigenenergies at the quench point depend on the position $X$ of the sharp impurity in Fig. 7, using the two-mode approximation. This helps us to understand the cycle $C(X)$ for a given value of $g$. For example, at $X = 0.42L$, the ground branch connected with a loop whose section at $v = |x|$ has two delocalized and a localized stationary states, which predicts the breakdown of the population inversion to the first excited states. Also, the first (the second) excited state at the initial point of the cycle is connected with a localized stationary state, which suggests that the population inversion to the second (the third) excited state may be possible.

![FIG. 7. (Color online) X-dependence of stationary energies at the quench $|v| = \infty$ and $g = 2$ under the two-mode approximation. The solid (dashed) lines correspond to the stable stationary states localized at the right (left) side of the infinitely strong impurity. The dotted lines correspond to the delocalized stationary states that are marginally stable.](image)

**V. SUMMARY AND OUTLOOK**

We have shown that the adiabatic cyclic operation with a quench $C(X)$ induces the population inversion of Bose particles described by the Gross-Pitaevskii equation confined in a quasi one-dimensional box, if the Bose particles are initially prepared to be in the ground state and the strength $g$ of the interparticle interaction is weak enough. An estimation of the critical value of $g$ where the population inversion is broken is also shown. We find that these results are consistent with our numerical investigation through the time-dependent Gross-Pitaevskii equation.

We note that the time evolution generated by the almost-adiabatic operation $C(X)$ can confine the system within a family of stationary states in the weak interaction regime. Namely, the adiabatic time evolution can be realized in spite of the presence of the flip of the potential strength at the quench point. This “adiabatic” cycle converts the ground state of Bose particles into a nonequilibrium stationary state. The present study offers
an example of the subtle difference between the adiabatic processes in thermodynamic systems and non-thermal, mechanical systems.

We believe that the present result offers an experimentally feasible method to produce a nonequilibrium stationary state. Application of the acceleration of the adiabatic scheme to the present procedure (e.g., Ref. [46, 47]) should be also interesting. We note that the preparation of condensates in a quasi one-dimensional box trap is experimentally achieved in Ref. [19], which motivates theoretical studies, e.g., on solitonic excitations [22]. We also remark that the box trap may be useful to investigate quantum information processing through, e.g. the Szilard engine in the quantum regime [48, 49]. To extend these works to many-body settings, our analysis on the infinitely strong impurity should be applicable.

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Appendix A: Linear stability analysis for the two-mode system (9) at the quench point

We explain the linear stability analysis (Bogoliubov analysis) of the stationary states at the quench point in $C(X)$. To carry this out, the nonlinear Schrödinger equation (9) under the two-mode approximation is cast into a nonlinear Bloch equation (Eq. (A3) below). The components of Bloch vector $S = (S_x, S_y, S_z)$ are the expectation values of Pauli matrices $\sigma_j$’s for a normalized state ($\Psi_0, \Psi_1$), e.g., $S_z = |\Psi_0|^2 - |\Psi_1|^2$.

To find the time evolution equation of $S$, we first obtain a matrix form of Eq. (9). The system is described by an effective nonlinear Hamiltonian $H_2 = \Delta_+ I + \Delta_- \sigma_z$, where $I$ is the identity matrix, and

$$\Delta_{\pm}(S) = \frac{1}{2} \left\{ (E_0 + g_0 \frac{1 + S_z}{2}) \pm (E_1 + g_1 \frac{1 - S_z}{2}) \right\}.
$$

(A1)

Hence $S$ experiences the effective magnetic field

$$B(S) = \Delta_-(S) e_z,
$$

(A2)

where $e_z = (0, 0, 1)$. Namely, $S$ obeys the nonlinear Bloch equation

$$\frac{d}{dt} S = S \times B(S).
$$

(A3)

A stationary state of Eq. (9) corresponds to a stationary solution $S_*$ of Eq. (A3), where $S_\times B(S_*) = 0$ holds.

We proceed to the linear stability analysis for a stationary solution $S_*$ to examine a slightly perturbed Bloch vector $S = S_* + \delta S$. We expand $\delta S$, using a orthogonal system $e_0 = S_*$, $e_1 = e_y (= (0, 1, 0))$, and $e_2 = e_0 \times e_1$, as

$$\delta S = \alpha_1 e_1 + \alpha_2 e_2,
$$

(A4)

where small coefficients $\alpha_j$’s are taken up to a first order ($j = 1, 2$). The absence of the $e_0$ component in Eq. (A4) is consistent with the normalization condition up to the first order. The linearized equation for $\alpha_j$’s are

$$\frac{d}{dt} \alpha_j = e_j \cdot \{g \delta B | e_z \times S_* + B_e e_z \times \delta S \},
$$

(A5)

where $B_e \equiv \Delta_s (S_*)$, $\delta B = g (c_0 + c_1) S_z e_2 / 4$, and $S_{j} = S_* \cdot e_j$ ($j = x, y, z$), from Eqs. (A2) and (A1). Hence we obtain

$$\frac{d}{dt} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = M \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}
$$

(A6)

where

$$M \equiv \begin{bmatrix} 0 & -B_e S_{zz} + \frac{g_{c_0 + c_1}}{4} S_{xz}^2 \\ B_e S_{zz} & 0 \end{bmatrix}.
$$

(A7)

We examine $M$ for each stationary solution $S_*$. First, we examine the localized solutions $S_* = \pm e_z$, where

$$M = \pm B_e \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
$$

(A8)

and $B_e$ is non-zero, except at the bifurcation point. Since the eigenvalues of $M$ are purely imaginary, the perturbation $\delta S$ evolves oscillatory, and doesn’t grows up exponentially in time. Hence we conclude that the stationary solutions $S_* = \pm e_z$ are linearly stable.

Second, we examine the delocalized solution Eq. (11), which implies $B_e = 0$. We find

$$M = \frac{g_{c_0 + c_1}}{4} S_{xz}^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
$$

(A9)

which is non-zero, except at the bifurcation point. Namely, $M$ has a non-trivial Jordan block and cannot be diagonalized. In terms of dynamical systems, the stability of the the delocalized solutions Eq. (11) are marginally stable. Although the perturbation $\delta S$ does not grows up exponentially fast, it grows up linearly in $t$.

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