THE SELF-SCREENING HAWKING ATMOSPHERE* †

A different approach to quantum black hole microstates

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Abstract

A model is proposed in which the Hawking particles emitted by a black hole are treated as an envelope of matter that obeys an equation of state, and acts as a source in Einstein’s equations. This is a crude but interesting way to accommodate for the back reaction. For large black holes, the solution can be given analytically, if the equation of state is $p = \kappa \rho$, with $0 < \kappa < 1$. The solution exhibits a singularity at the origin. If we assume $N$ free particle types, we can use a Hartree-Fock procedure to compute the contribution of one such field to the entropy, and the result scales as expected as $1/N$. A slight mismatch is found that could be attributed to quantum corrections to Einstein’s equations, but can also be made to disappear when $\kappa$ is set equal to one. The case $\kappa = 1$ is further analysed.

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† This paper was modified in order to attribute credit to Ref 6, where similar calculations were reported.
1. Introduction.

Hawking radiation is usually treated by means of a general coordinate transformation between the frame of an infalling observer and that of a stationary observer outside. What is obtained is the so-called Hartle-Hawking state, a state in which the local energy-momentum tensor does not differ substantially from that of the vacuum. This Hartle-Hawking state, however, does not yield any clues concerning the nature of the quantum mechanical microstates of a black hole, even if arguments from standard thermodynamics and statistical physics tell us what the density of these microstates should be expected to be.

Recent results in string theory, complemented with the higher-dimensional structures called D-branes, do show the appearance of quantum microstates, but a detailed understanding of these in terms of general properties of a horizon seems to be missing. What we intend to do is to establish to what extent one can stretch more conventional versions of quantum gravity to yield these quantum states.

The result of Ref, concerning the energy-momentum tensor generated by Hawking radiation, will be interpreted in an unusual fashion. According to these authors, an anomaly in this tensor exactly cancels out the contribution of Hawking radiation near the horizon. In this paper, we will assume that this anomaly is not present in the energy-momentum tensor observed by the outside observer, but only when this tensor is observed by observers in an infalling (inertial) frame. Further discussion of this issue is postponed to Section 6.

For studying the complete set of quantum states, one must consider other modes than just the Hartle-Hawking state. The modes to be studied in this paper are stationary in time. In some sense, they are unphysical. This is because a black hole in equilibrium with an external heat bath cannot be stable; it has a negative heat capacity, and it is not difficult to deduce that thermal oscillations therefore diverge. Thus, the states that will be discussed in this paper, being stationary in time, are good candidates for the quantum microstates, but do not represent the Hartle-Hawking state. This way we claim to be able to justify the calculations carried out in this paper: we simply take the energy-momentum tensor generated by Hawking radiation as if the Hawking particles represented an 'atmosphere'. This atmosphere is taken as the source for the gravitational field equations. We ignore its instability, which is relevant only for much larger time scales. Thermal, non-interacting, massless particles obey the equation of state,

\[ p = \frac{1}{3} \varrho, \]  

where \( p \) is the pressure and \( \varrho \) the energy density. If we have \( N \) massless particle types,
at temperature $T = 1/\beta$, we have

$$\rho = 3p = \frac{\pi^2 N}{30 \beta^4}. \quad (1.2)$$

In particular, when $N$ is large, the Hawking radiation is intense, and, at least far away from the black hole, its effect on the space-time curvature should be non-negligible.

In this paper, we regard Eq. (1.1) or (1.2) as describing matter near the black hole, and at a later stage we will substitute the true Hawking temperature for $\beta^{-1}$. Even if the reader finds there are reasons to criticise these starting points, one may still be interested in knowing the results of this model exercise. Taking Hawking radiation as a description of the boundary condition at some distance from the horizon, we continue the solution of Einstein’s equations combined with the equation of state as far inwards as we can. What is found is that Hawking radiation produces radical departures from the pure Schwarzschild metric near the horizon. In fact, the horizon will be removed entirely, but eventually a singularity is reached at the origin ($r = 0$). This should have been expected; our system is unphysical in the sense that, when $M$ is sufficiently large, the Chandrasekhar limit is violated, so that a solution that is regular everywhere, including the origin, cannot exist.

In physical terms, what happens is the following (see Fig. 1). A blanket of Hawking particles near the (would-be) horizon is the source of a gravitational field that is stronger than that of the entire black hole, and therefore there will be gravitational field lines beyond the blanket, pointing outwards, from the origin, rather than inwards. On the one hand, these field lines cause the gravitational potential to rise when we follow it further inwards, so that the matter particles are confined to stay close to the blanket, and on the other hand, in the immediate vicinity of the origin, the field lines accumulate to form a singularity there. The singularity will be recognised as a negative-mass singularity, and we will be able to compute precisely the value of this negative mass (Eqs. (3.13) and (3.14)).

At this point it should be mentioned that after completion of an earlier draft of this paper, the author learned that the differential equations for this system, the so-called Tolman-Oppenheimer-Volkoff equations $^5$, have been studied before by Zurek and Page $^6$, and although they use slightly different variables, their conclusions were essentially identical to the ones presented here. In view of this, we replaced the original Appendix (the contents of which can also be found in Ref $^6$) by a more detailed elaboration of a special case.

The negative-mass singularity is the only way in which this approach departs (radically) from earlier Einstein-matter calculations $^7$. But for the remainder of our considerations this singularity is harmless. It being repulsive, all particles will keep a safe distance from this singularity. This then, enables us to compute quantum effects in the metric
obtained. Thus, in the second part of our paper, we compute the entropy due to one of the scalar fields, using a WKB approximation. We find that this entropy is finite, so that the Hawking ‘blanket’ itself apparently acts as a soft alternative to the ‘brick wall’ introduced in Ref. 8. Furthermore, we find that the entropy per particle type decreases as $N$ increases, simply because $N$ occurs in the source described by Eq. (1.2). The total entropy of all particles becomes independent of $N$, but is found to slightly overshoot Hawking’s value. We take this as a sign that the equation of state (1.1) was too much of a simplification.

It so happens that the metric can also be calculated if Eq. (1.1) is replaced by

$$p = \kappa \rho,$$

where $\kappa$ is a coefficient ranging anywhere between 0 and 1. Moreover, the total entropy can also be calculated in this case. One may decide to adjust the value of $\kappa$ such that the entropy calculation matches precisely, but this could be premature, because here one cannot ignore the quantum corrections to Einstein’s equations, since we are operating in the Plankian regime. It may nevertheless be of interest to note that a complete match is achieved if $\kappa$ is set equal to 1, which is the case that will be elaborated further in the Appendix. It is a rather singular and unphysical case. We do conclude that, with interactions taken into account, it may well be possible to obtain a self-consistent approach towards microcanonical quantization of black holes, using ordinary Hartree-Fock methods in standard gravity theories. We do stress that the price paid was a (mild) singularity at the origin. A more thorough analysis of the exact role played by this singularity in a more comprehensive theory of quantum gravity is still to be performed.
2. The equations for $\kappa = \frac{1}{3}$

We only consider spherically symmetric, stationary metrics in $3 + 1$ dimensions, of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2d\Omega^2,$$

having as a material source a perfect fluid with pressure $p(r)$ and energy density $\rho(r)$. The Einstein equations read\(^\dagger\)

$$1 - \partial_r \left( \frac{r}{B} \right) = 8\pi r^2 \rho,$$  \hspace{1cm} (2.2)

$$\frac{\partial_r (AB)}{AB^2 r} = 8\pi (\rho + p),$$  \hspace{1cm} (2.3)

where $\partial_r$ stands for $\partial/\partial r$. The relativistic Euler equations for a viscosity-free fluid can be seen here to amount to\(^\circ\)

$$\partial_r p = -(\rho + p)\partial_r \log \sqrt{A(r)},$$  \hspace{1cm} (2.4)

and the relation between $p$ and $\rho$ is governed by an equation of state. We will concentrate on the case

$$p = \kappa \rho,$$  \hspace{1cm} (2.5)

where $\kappa$ is a fixed coefficient. Massless, non-interacting particles in thermal equilibrium have $\kappa = \frac{1}{3}$. Although the calculations described here can be performed for any choice of $\kappa$ between 0 and 1, the special choice $\kappa = \frac{1}{3}$, appearing to be the most important case, will be taken here first, because it shortens the expressions considerably. For the general case we refer to Ref.\(^6\). If $\kappa = 1$, however, complications arise, so this case is given special attention in the Appendix.

Our liquid will be viscosity-free and free of vortices, so that Eq. (2.4) can be integrated to yield

$$8\pi \rho A^2 = C,$$  \hspace{1cm} (2.6)

where $C$ is a constant, later to be called $3\lambda^2/(2M)^2$. After inserting this equation into Eqs (2.2)–(2.5), the latter can be cast in a Lagrange form. We will not discuss this Lagrangian however, but focus on integrating the remainder of the equations. What is needed there is to observe the scaling behavior as a function of $r$. Scale-independent variables are $X$ and $Y$, defined by

$$X = A/r,$$  \hspace{1cm} (2.7)

$$\text{and} \hspace{1cm} Y = B.$$  \hspace{1cm} (2.8)

\(^\dagger\) Here, units are chosen such that $G = 1$.  

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This turns the equations into:

\[
\begin{align*}
\frac{r \partial_r X}{X} &= \frac{CY}{3X^2} + Y - 2, \\
\frac{r \partial_r Y}{Y} &= 1 - Y + \frac{CY}{X^2}.
\end{align*}
\] (2.9) (2.10)

and this allows us to eliminate \( r \), obtaining a first order, non-linear differential equation relating \( X \) and \( Y \).

The result of a numerical analysis of this equation is presented in Fig. 2. Here, the constant \( C \) has been scaled to one, by rescaling \( X \). Each curve now represents a solution. For large \( r \), all solutions spiral towards the point \( \Omega \). For small \( r \), only one curve allows \( B \) to approach the value one, so that, according to Eq. (2.2), the density \( \varrho \) stays finite, and the metric (2.1) stays locally flat. This is the only regular solution. Further away from the origin, this solution requires \( \varrho \) always to be large.

![Fig. 2. Solutions to Eqs. (2.9) – (2.10). Solutions entering the shaded region must have a singularity at \( r = 0 \).](image-url)
We are interested in a very different class of solutions, the ones which, far from the origin, approach a black hole surrounded by a very tenuous cloud of matter, Hawking radiation. This is the case where \( Y \approx 1 \) and \( C \ll 1 \), or, after rescaling, \( X \gg 1 \). We now observe that for very large \( r \), all solutions will eventually spiral into the point \( \Omega \):

\[
\Omega = \left( X = \sqrt{\frac{7C}{3}}, \ Y = \frac{7}{4} \right).
\]  

(2.11)

Physically, this means that, since the universe is actually filled with radiation, the curvature at large distances becomes substantial. For the study of Hawking radiation, this large distance effect is immaterial and will henceforth be ignored.

Thus, our boundary condition far from the origin will be chosen to be

\[
A(r) = 1 - \frac{2M}{r},
\]

(2.12)

\[
B(r) = \left( 1 - \frac{2M}{r} \right)^{-1},
\]

(2.13)

\[
8\pi \varrho(r) = \frac{3\lambda^2}{(2MA)^2},
\]

(2.14)

in the region

\[
\lambda \ll \frac{r}{2M} - 1 \ll \frac{1}{\lambda}.
\]

(2.15)

Here, in Eq. (2.14), the constant \( C \) was replaced by \( 3\lambda^2/(2M^2) \), since \( \varrho \) has dimension \((\text{mass}) \times (\text{length})^{-3} = (\text{mass})^{-2}\), and the factor 3 is for later convenience. In the mathematical construction of the equations, \( \lambda \) will be dimensionless. If there are \( N \) species of radiating fields, we have

\[
\lambda^2 = \frac{4\pi^3}{45} \frac{N}{\beta^4} (2M)^2,
\]

(2.16)

where \( \beta \) is the inverse temperature. In Planck units, Hawking radiation has

\[
\beta = 8\pi GM = 8\pi M.
\]

(2.17)

So,

\[
2M\lambda = \frac{1}{24} \sqrt{\frac{N}{5\pi}},
\]

(2.18)

and hence for large black holes, \( \lambda \) is very small. The approximation (2.15) holds over a huge domain.

When \( r \) approaches the horizon, the effects of \( \varrho \) are nevertheless felt, and the solution becomes more complicated. Actally, \( A \) never goes to zero, so there is no horizon at all.
Eventually, at small $r$, $A$ diverges as a constant$/r$, which means that there is a negative-mass singularity, as sketched in Figure 1.

Following the line of the solution of interest in Figure 2 (see also Fig. 4), one gets the impression that, for small enough $\lambda$, the solution may be found analytically. This is indeed the case, as will be shown in the next section.

3. The solution for small $\lambda$

Eqs. (2.9) and (2.10) become slightly easier if we substitute $X$ and $Y$ by $P$ and $Q$:

$$P = (3/C)X^2/Y;$$
$$Q = 1/Y;$$

hence $$X = (C/3)^{1/2}P^{1/2}Q^{-1/2}. \quad (3.3)$$

Defining $L = \log r$, the equations become

$$\frac{\partial P}{\partial L} = \frac{3P}{Q} - 5P - 1; \quad (3.4)$$
$$\frac{\partial Q}{\partial L} = -\frac{3Q}{P} - Q + 1. \quad (3.5)$$

In all limiting cases of interest to us, the r.h.s. of these equations will simplify sufficiently to make them exactly soluble.

Taking Eqs (2.12) – (2.14) to be valid in the region (2.15), for sufficiently small $\lambda$, we can now integrate the equations down towards $r \to 0$. To do this, we have to glue together different regions, where different approximations are used. All in all, we cover the line $0 < r \ll \frac{2M}{\lambda}$ with six overlapping regions, as depicted in Figure 3.
The region of Eq. (2.15) is region (1). The effects of $\varrho$ on the metric are negligible, and so $A$ and $B$ still obey Eqs. (2.12) and (2.13). These expressions agree with Eqs. (3.4) and (3.5) if we take $P \gg Q$, so that (3.5) is readily integrated.

In region (2) we have:

\[(2) : \quad \lambda \ll \frac{r}{2M} - 1 \ll \lambda^{2/3}. \quad (3.6)\]

It lies entirely inside region (1), so here we have also:

\[r = 2M + \sigma; \quad A = \frac{\sigma}{2M}; \quad B = \frac{2M}{\sigma}; \quad 8\pi \varrho = \frac{3\lambda^2}{\sigma^2}. \quad (3.7)\]

Region 2 is defined by the requirement $Q \ll P \ll 1$.

Region 3 has $P \ll 1$ and $Q \ll 1$. Here we can use $\frac{P}{Q} \equiv \omega^2$ as a new variable. The differential relation between $L$ and $\omega$ yields

\[(3) : \quad r = 2M e^{\lambda(\omega - 1/\omega)}; \quad A = \lambda\omega; \quad 8\pi \varrho = \frac{3\lambda^2}{\omega^2(2M)^2}; \quad B = \frac{\omega^3}{\lambda(1 + \omega^2)^2}. \quad (3.8)\]

The approximation corresponds to

\[\left| \frac{r}{2M} - 1 \right| \ll \lambda^{2/3}. \quad (3.9)\]

It is important to note that there are two integration constants in Eqs. (3.8). These have been adjusted such that agreement with region (1) is obtained where the two regions overlap, which is region (2).

Region (4) is defined by $P \ll Q \ll 1$, and here the expressions (3.8) simplify to

\[(4) : \quad r = 2M(1 - P); \quad A = \frac{\lambda^2}{P}; \quad B = \frac{\lambda^2}{P^3}; \quad 8\pi \varrho = \frac{3P^2}{\lambda^2(2M)^2}. \quad (3.10)\]

This overlaps with region (5), defined by $P \ll Q$ and/or $Q \gg 1$). In (5), integrating the equations yields

\[(5) : \quad \frac{r}{2M} = (1 + 5P)^{-1/5}; \quad A = \frac{\lambda^2}{P} \left( \frac{2M}{r} \right)^6; \quad B = \frac{\lambda^2}{P^3} \left( \frac{2M}{r} \right)^{14}; \quad 8\pi \varrho = \frac{3P^2}{\lambda^2(2M)^2} \left( \frac{r}{2M} \right)^{12}. \quad (3.11)\]
Again the constants of integration have been carefully adjusted so as to obtain agreement in region (4), where (5) overlaps with (3).

Finally, in region (6) we have $P \gg 1$ and $Q \gg 1$, and our solution simplifies into

\begin{align*}
P &= \frac{1}{5} \left( \frac{2M}{r} \right)^5 ; \quad A = 5\lambda^2 \frac{2M}{r} ; \\
B &= 125\lambda^2 \frac{r}{2M} ; \quad 8\pi\rho = \frac{3}{25\lambda^2(2M)^2} \left( \frac{r}{2M} \right)^2 .
\end{align*}

(3.12)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{Sketch (not to scale) of the path followed in the $X Y$ plane.}
\end{figure}

It is important that, throughout this calculation, the integration constants in the different regions were carefully matched. So, now we can read off the strength of the singularity at the origin. The space-like curvature (determined by the $B$ coefficient) has the singularity of a “black hole” with mass

$$2M' = -\frac{2M}{125\lambda^2} .$$

(3.13)

The gravitational potential, determined by $A$, is so much stretched in the time direction that it corresponds to an object whose mass is only

$$2M'' = -5\lambda^2 \cdot 2M .$$

(3.14)

Both masses are negative.
4. The Rindler space limit

Let us now consider the limit where the black hole mass $M$ becomes large. Assuming the surrounding matter to correspond to Hawking radiation, we take $\lambda$ to obey Eq. (2.18). From the inequalities in Figure 3 it may seem that, for $M$ large, $\lambda$ small, region (5) is a much larger stretch than region (3), but, if the geodesic distances are considered, the converse turns out to be true. The distance between the point $r = 2M$ (more or less in the middle of region (3)) and the origin at $r = 0$ follows from Eqs. (3.8):

$$\int_{0}^{2M} \sqrt{B(r)}dr = \int_{0}^{1} 2M \sqrt{\lambda} \frac{d\omega}{\omega} = 4M \sqrt{\lambda} = \sqrt{\frac{M}{3}} \left( \frac{N}{5\pi} \right)^{1/4}, \quad (4.1)$$

where, in the end, we inserted Eq. (2.18). Notice, that the numbers are in Plank units, so, if the black hole is large, the geodesic distance between the apparent horizon ($r = 2M$) and our singularity at the origin is small in comparison to the length scale of the black hole, but large in Planck units.

In contrast, region (5) is small, in spite of the fact that $r$ runs from 0 to nearly $2M$. Let $\alpha$ be a number slightly bigger than 1. Then the geodesic length of the region between $r = 0$ and $r = 2M/\alpha$ is

$$\int_{0}^{2M/\alpha} dr \sqrt{B(r)} = 2M \lambda \sqrt{5} \int_{\alpha^{5}}^{\infty} \frac{y^{1/5}}{(y-1)^{3/2}} dy, \quad (4.2)$$

and this distance is of order of the Planck length, independently of the mass $M$ of the black hole.

We now wish to consider the Rindler limit of our space-time, with which we mean the limit $2M \to \infty$, $2M\lambda$ fixed. In this limit, in region (3), space-time becomes flat (indeed, the matter density $\rho$ in Eq. (3.8) tends to zero). Only in region (5) we get a deviation from the usual flat Rindler metric. Let us replace the coordinate $r$ by a more convenient coordinate $y$,

$$y = \left( \frac{2M}{r} \right)^{5}. \quad (1 \leq y \leq \infty) \quad (4.3)$$

Furthermore, we replace the angular coordinate $\Omega$ by a transverse coordinate $2M\tilde{x}$. The metric, described by (3.11), then becomes

$$ds^2 = -\frac{5\lambda^2 y^{6/5}}{y-1} dt^2 + 5(2M\lambda)^2 \frac{y^{2/5}}{(y-1)^3} dy^2 + y^{-2/5} d\tilde{x}^2. \quad (4.4)$$

Substituting (2.18), and rescaling the time $t$, shows that this metric will be universal for all black holes.
The gravitational potential $A(r)$ takes an absolute minimum inside this region (5):
\[
\frac{\partial A}{\partial r} = 0 \quad \rightarrow \quad y = 6 \quad , \quad \frac{r}{2M} = \frac{1}{\sqrt{6}},
\] (4.5)
and here, the matter density takes the extreme value
\[
\varrho_{\text{extr}} = \frac{3(y - 1)^2}{200\pi(2M\lambda)^2} y^{-12/5} = \frac{5 \cdot 6^{3/5}}{N}.
\] (4.6)
Observe, that this is inversely proportional to the number of fields $N$ contributing to the Hawking radiation.

5. Free energy and entropy of a spectator field in this metric

In a metric of the form
\[
ds^2 = -A(y)dt^2 + G(y)dy^2 + H(y)d\tilde{x}^2,
\] (5.1)
a real scalar field with mass $m$ has a Lagrangian
\[
\mathcal{L} = \frac{1}{2} H \sqrt{AG} \left( A^{-1}\varphi^2 - G^{-1}(\partial_y \varphi)^2 - H^{-1}(\partial_x \varphi)^2 - m^2 \varphi^2 \right).
\] (5.2)
Our calculation of the entropy goes as in Ref.\(^8\). In the WKB approximation, at energy $E$ and transverse momentum $\tilde{k}$, the oscillating part of the field $\varphi(y)$ behaves as
\[
\varphi(y) \approx e^{i \int k_y(y) dy},
\] (5.3)
with
\[
\frac{k_y^2(y)}{G} = \frac{E^2}{A} - \frac{\tilde{k}^2}{H} - m^2.
\] (5.4)
This expression transforms covariantly with general coordinate transformations in the radial variable $y$, so the details of the radial coordinate chosen are irrelevant. We count the number $\nu$ of solutions with energy below $E$, in a region with transverse surface area $\Sigma$:
\[
\pi \nu = \sum_{\tilde{k}} \int_{Y} k_y dy,
\] (5.5)
where $Y$ is the domain of $y$ values in which the r.h.s. of Eq. (5.4) is positive. Writing $x = \tilde{k}^2$, we find
\[
\sum_{\tilde{k}} = \Sigma \cdot \int \frac{d^2 \tilde{k}}{(2\pi)^2} = \Sigma \int_0^\infty \frac{\tilde{k} |d\tilde{k}|}{2\pi} = \frac{\Sigma}{4\pi} \int_0^\infty dx,
\] (5.6)
and
\[ \nu(E) = \frac{\Sigma}{(2\pi)^2} \int dx \sqrt{G \left( \frac{E^2}{A} - \frac{x}{H} - m^2 \right)} . \] (5.7)

The integral over \( x \) is over those values for which the argument of the root is positive,
\[ \int_0^{P/Q} dx \sqrt{P - Qx} = \frac{2}{3} \frac{P^{3/2}}{Q}, \] (5.8)

so that
\[ \nu(E) = \frac{\Sigma}{6\pi^2} \frac{H \sqrt{G}}{A^{3/2}} (E^2 - m^2 A)^{3/2} . \] (5.9)

The free energy \( F \) at inverse temperature \( \beta \) is given by
\[ \beta F = \sum \log(1 - e^{-\beta E}) = \int_0^\infty d\nu(E) \log(1 - e^{-\beta E}) = -\int_0^\infty dE \frac{\beta \nu(E)}{e^{\beta E} - 1} \\
= -\frac{\Sigma \beta}{6\pi^2} \int dy \int_{m\sqrt{A}}^\infty dE \frac{H \sqrt{G} (E^2 - m^2 A)^{3/2}}{A^{3/2} (e^{\beta E} - 1)} . \] (5.10)

The entropy \( S \) will be given by
\[ S = \beta^2 \frac{\partial F}{\partial \beta} = \frac{\beta \Sigma}{6\pi^2} \int dy \int_{m\sqrt{A}}^\infty dE \frac{H \sqrt{G} (4E^2 - m^2 A) \sqrt{E^2 - m^2 A}}{e^{\beta E} - 1} . \] (5.11)

So-far, we left the mass of this field to be a free parameter, but from here on, a considerable simplification is achieved by putting this mass equal to zero (which is the case we are most interested in anyway). We can then proceed analytically. The integral over \( E \) gives
\[ m = 0 \quad \rightarrow \quad S = \frac{2\pi^2 \Sigma}{45\beta^3} \int dy \frac{H \sqrt{G}}{A^{3/2}} . \] (5.12)

In the Rindler limit, this integral will only receive a sizable contribution from region (5). Substituting the values for \( A, G \) and \( H \) that correspond to the metric (4.4), we find that
\[ S = \frac{2\pi^2 \Sigma}{45\beta^3} \frac{2M}{5\lambda^2} \int_1^\infty \frac{dy}{y^2} , \] (5.13)

and with the Hawking temperature \( \beta = 8\pi M \), this becomes
\[ S = \frac{2}{5N} \Sigma . \] (5.14)

If we add the contribution of all \( N \) field types, the entropy per unite of area becomes
\[ \frac{S}{\Sigma} = \frac{2}{5} . \] (5.15)
This number should now be compared with what should have been expected, the entropy of a black hole is

\[
\frac{S}{\Sigma} = \frac{1}{4},
\]

so we have a mismatch factor of

\[
\frac{8}{5}.
\]

This number (which was also mentioned in Ref \textsuperscript{6}) is quite robust. It does not change if we change the number \(N\) of the fields, or replace them by vector fields. It does change if either we add quantum corrections to Einstein’s equations, or if we modify the equation of state. The latter is an interesting exercise. It turns out that the entire calculation given above can easily be generalised into the case for general \(\kappa\), as long as \(0 < \kappa < 1\). Thus we reobtained the results of Ref \textsuperscript{6}. The quantum modes of the individual fields cannot be counted as easily as in the non-interacting case, but it is not difficult to convince oneself that all we really did was to integrate the entropy density of the material over the curved space of our metric. Now, the entropy density \(s\) of matter at any \(\kappa\) value, is according to local observers,

\[
s = \beta(r)(1 + \kappa)\varrho.
\]

Here, however, one has to substitute the locally observed temperature, which is, due to redshift, given by

\[
\beta(r) = \sqrt{A(r)} \beta,
\]

where \(\beta\) is the inverse temperature as experienced by the distant observer.

The entropy for general \(\kappa\) was also calculated by the author, but it also can be deduced from Ref \textsuperscript{6}, who use the coefficient \(\gamma = \kappa + 1\), or \(n = 1 + \frac{1}{\kappa}\). One finds for the total entropy \(S\):

\[
\frac{S}{\Sigma} = \frac{\kappa + 1}{7\kappa + 1},
\]

and this equals the desired value \(1/4\) if \(\kappa = 1\). It is difficult to imagine ordinary matter with such a high \(\kappa\) value. Free massless fields generate the entropy density

\[
s = C N T^3,
\]

where \(N\) is the number of non-interacting field species. \(C\) is a universal constant. It appears that, at very high temperature, this should be replaced by

\[
s = C T^{1/\kappa}.
\]

A striking feature of the result of Eq. (5.20) is the independence on \(N\). So, if \(N\) is made to depend strongly on temperature, this of course affects the equation of state. If a strong
kind of unification takes place at Planckian temperatures, such that, at those temperatures $N$ suddenly decreases strongly, one could imagine an effective increase of $\kappa$ beyond the canonical value $1/3$. It is more likely, however, that this argument is still far too naive, and that our approach must merely be seen as a rough approximation. An error of 60% is perhaps not so bad. On the other hand, it is tempting to speculate that the case $\kappa \to 1$ has physical significance. This is a highly peculiar case, as is shown by the calculation in the (new) Appendix. In this case, the total entropy receives its main contribution from a very tiny region in space where the matter density reaches values diverging exponentially with the small parameter $\lambda$.

6. Discussion and Conclusions

Usually, it is argued \cite{6,10} that Hawking radiation produces an energy momentum tensor that does not diverge at the horizon. The state of affairs beyond the horizon, however, can be described in many ways, and depends delicately on the nature of the quantum state considered \cite{11}. What we would like to see, is an unambiguous prescription of the black hole quantum states in terms of a general coordinate transformation of the observations made by an inertial observer there.

The aim of this paper was to illustrate a point. Suppose one assumed that the majority of quantum states does show a strong back reaction of Hawking particles upon the surrounding matter, just as if these particles form a kind of “atmosphere” \cite{12}. The price paid is a singularity, which in the spherically symmetric case resides at the origin. But this singularity is repulsive, and therefore its ‘quantum’ effects are completely harmless. One can calculate the entropy of this self-screening envelope of matter, and the result is finite. Apparently, Hawking radiation can provide for its own ‘brick wall’ \cite{8}, in this case rather a ‘soft wall’. The situation is nearly stable; the entropy is larger than what it is supposed to be, but only by a relatively insignificant numerical coefficient, $8/5$. It is suggestive to speculate that this coefficient can be made to match, just by adding non-trivial interactions. This would then provide us with a highly non-trivial consistency check for the interactions up to the Planck region, albeit that the check consists of just one numerical coefficient.

If the number of fields, $N$, is large, one may note that the distance scale of the effects that lead to our entropy is large compared to the Planck length (the length scale grows as $\sqrt{N}$), so that a fairly consistent theory of black holes may result in the $N \to \infty$ limit.

The role of the singularity should be further investigated. The picture we get seems to tell us that particles with a large, negative masses may have to be introduced in quantum gravity. A confinement mechanism must assure that this negative mass is always surrounded by positive mass material, so that we end up with a stable configuration.
The rules for these particles under general coordinate transformations should be further investigated.

The question, ‘what does an ingoing observer see?’, will be posed again and again. Our (preliminary) answer is the following. Let us assume that, as seen in the frame where the black hole was formed, and also in the frame of the observer, indeed this observer survives his journey through the horizon, until he or she hits the usual Schwarzschild singularity. But this is the black hole in the Hartle-Hawking state, and, as we argued in the Introduction, this state fluctuates beyond control. Now, since we put the black hole in a stationary, non-fluctuating state, we are dealing with a superposition of many different Hartle-Hawking states, and this superposition is determined by measurements made in the observer’s distant future. Particles created by these measurements are likely to kill the observer at the very moment he/she crosses the horizon. The complete set of quantum states that serves as a basis in our configuration has a vast majority of modes with energetic particles at the horizon that obstruct a safe journey beyond. Ordinary measurements are therefore not possible there, and so it can happen that an entirely different description of this region is called for. This description is the one presented in this paper.

An elegant way to formulate the situation may also be the following. Since Hilbert space, as it is experienced by the infalling observer, is constructed using states that are entirely different from the ones in Hilbert space of the observer outside, it may be best to view the time parameter $t$ in the Schrödinger equation as a parameter with multiple “branches”, see Figure 5. The Hamiltonian operator allows one to move from one point to a neighboring point, but if one wishes to relate what is seen by the outside observer, $A$, to the observations of the infalling observer, $B$, one has to follow the Hamiltonian all around the forking curve connecting the two observers. Now, the energy-momentum operator is directly related to energy, and the energy of a state is obtained by Fourier transforming with respect to time. Since the Fourier transform of a time-dependent configuration along the curve connecting $S$ to $A$ may be different from the Fourier transform along the curve to $B$, the expressions for the energy-momentum tensor for the two observers may be different. This holds for energy localized in some region (expressed by the energy momentum tensor.
in that region); of course total energy is the same everywhere since it is conserved. This way one may justify that the anomaly has to be taken into account for observer $B$ but not for observer $A$.

The case $\kappa = 1$ is treated in detail in the Appendix. It is important to observe what happens in this case. The integral that yields the total entropy, Eq. (A.19), receives contributions that are fairly evenly distributed over the range $0 < r < 2M$. But notice that the integrand is the product of an exponentially large factor, the entropy density (behaving as $\sqrt{Q}$), and an exponentially small factor, the space volume element, $\sqrt{B}$. These functions both contain the factor $1/\lambda^2$ in the exponent (see Eqs. (A.16)), and they both peak near the region $r \approx 0$. Thus, we have ventured deeply into the trans-Planckian region, where the physical basis for these equations is extremely weak.

**APPENDIX. The equations for $0 < \kappa < 1$, and the calculation for the case $\kappa = 1$**

For general $\kappa$, Eqs. (2.6) and (2.7) are replaced by

\[
8\pi \varrho = \frac{\lambda^2}{\kappa(2M)^2} A^{\frac{1+\kappa}{2\kappa}}, \quad X = A r^{\frac{4\kappa}{1+\kappa}}, \quad (A.1)
\]

and (2.9), (2.10) become

\[
\frac{r \partial_r X}{X} = \frac{\lambda^2 Y}{(2M)^2} X^{\frac{1+\kappa}{2\kappa}} + Y - \frac{1 + 5\kappa}{1 + \kappa}, \quad (A.2)
\]

\[
\frac{r \partial_r Y}{Y} = 1 - Y + \frac{\lambda^2 Y}{\kappa(2M)^2} X^{-\frac{1+\kappa}{2\kappa}}. \quad (A.3)
\]

It is convenient to define

\[
P = \frac{(2M)^2}{\lambda^2} X^{\frac{1+\kappa}{2\kappa}}; \quad Q = 1/Y. \quad (A.4)
\]

The field equations (3.4) and (3.5) become

\[
\frac{dP}{dL} = \frac{3\kappa + 1}{2\kappa} P - \frac{7\kappa + 1}{2\kappa} P - \frac{1 - \kappa}{2\kappa}, \quad (A.6)
\]

\[
\frac{dQ}{dL} = 1 - Q - \frac{Q}{\kappa P}, \quad (A.7)
\]

and  \[8\pi \varrho = \frac{Q}{\kappa P r^2}. \quad (A.8)\]
The solution of these equations can be obtained along the lines of Sects. 3-5, but we now refer to Ref. \textsuperscript{6}. The identification relating the variables $u$ and $v$ of Ref \textsuperscript{6} with the $P$ and $Q$ used here is:

$$u = \frac{1 - Q}{2}, \quad v = \frac{Q}{2P}, \tag{A.9}$$

and we find complete agreement with their Eq. 7, which corresponds to our Eq. (5.20). The black hole entropy is matched if one puts $\kappa = 1$. Let us now concentrate on this case.

When $\kappa = 1$, Eqs. (A.6) and (A.7) simplify into

$$\frac{dP}{dL} = \frac{2P}{Q} - 4P; \quad \frac{dQ}{dL} = 1 - Q - \frac{Q}{P}. \tag{A.10}$$

We have

$$A = \lambda^2 \left( \frac{r}{2M} \right)^2 \frac{P}{Q} \quad \text{and} \quad B = \frac{1}{Q}. \tag{A.11}$$

At sufficient distance from the black hole (but not too far away from it) we have region 1, where one may assume that $P \gg Q$. There, we have Schwarzschild’s solution:

$$P = \left( \frac{2M}{\lambda} \right)^2 \frac{1}{r^2} \left( 1 - \frac{2M}{r} \right)^2, \quad Q = A = 1 - \frac{2M}{r}. \tag{A.12}$$

Near to the would-be horizon, region 1 overlaps with region 2, where $Q \ll P \ll 1$, which in turn overlaps with region 3: $P \ll 1$, $Q \ll 1$, and there, one obtains

$$Q = \frac{\lambda P}{\sqrt{P} - \lambda}, \quad L = \lambda \sqrt{P} + \lambda^2 \log \left( \frac{\sqrt{P} - \lambda}{\lambda} \right) \approx 0; \quad A = \lambda (\sqrt{P} - \lambda). \tag{A.13}$$

Since in region 2 we have $P \gg \lambda^2$, we find that (A.12) and (A.13) agree in that region. Region 3 has overlap with region 4, defined by $P \ll Q \ll 1$, which in turn overlaps with region 5, defined by $P \ll 1$ and $P \ll Q$. Here, it is convenient to use $Q$ as independent variable, while solving for $P$ and $L$:

$$P = \frac{\lambda^2 Q}{Q - \lambda^2 (4Q \log Q + 2)}, \quad L = -\lambda^2 \int \frac{dQ}{Q - \lambda^2 (4Q \log Q + 2)} \approx 0. \tag{A.14}$$

It overlaps with region 6 ($P \ll 1$, $Q \gg 1$), which in turn overlaps with region 7, defined by $Q \gg 1$ and $P \ll Q$. Taking $P$ as the independent variable, one finds in region 7:

$$Q^4 = Pe^{\frac{1}{\lambda^2}} - \frac{1}{\lambda^2} \quad \text{and} \quad L = -\frac{1}{4} \log \left( \frac{P}{\lambda^2} \right). \tag{A.15}$$
from which

\[ P = \lambda^2 \left( \frac{2M}{r} \right)^4, \quad Q = \sqrt{\lambda} \left( \frac{2M}{r} \right) e^{\frac{1}{2\lambda}} \left[ 1 - \left( \frac{r}{2M} \right)^4 \right], \]

\[ A = \lambda^{\frac{7}{2}} \left( \frac{2M}{r} \right) e^{\frac{1}{2\lambda}} \left[ \left( \frac{r}{2M} \right)^4 - 1 \right], \quad B = \lambda^{-\frac{1}{2}} \left( \frac{r}{2M} \right) e^{\frac{1}{2\lambda}} \left[ \left( \frac{r}{2M} \right)^4 - 1 \right]. \]  

(A.16)

Region 7 includes the singularity at the origin.

The entropy now cannot be computed directly by counting states, but we can integrate the entropy density. From (5.18) and (5.19) we find that the local entropy density is

\[ s = \frac{\lambda^2 \beta}{8\pi \kappa (2M)^2} (1 + \kappa) A^{-\frac{1}{2\lambda}}, \]

(A.17)

so that in our case,

\[ \frac{S}{\Sigma} = \lambda \int \left( \frac{r}{2M} \right)^2 \frac{dL}{\sqrt{P}}, \]

(A.18)

and this yields

\[ \frac{S}{\Sigma} = \int_0^{2M} \left( \frac{r}{2M} \right)^3 \frac{dr}{2M} = \frac{1}{4}. \]

(A.19)

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