Treatment of the background error in the statistical analysis of Poisson processes

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Abstract

The formalism that allows to take into account the error \( \sigma_b \) of the expected mean background \( \bar{b} \) in the statistical analysis of a Poisson process with the frequentistic method is presented. It is shown that the error \( \sigma_b \) cannot be neglected if it is not much smaller than \( \sqrt{\bar{b}} \). The resulting confidence belt is larger that the one for \( \sigma_b = 0 \), leading to larger confidence intervals for the mean \( \mu \) of signal events.

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I. INTRODUCTION

The statistical treatment of experimental results obtained in a Poisson process with background and a small signal is difficult and controversial. Two methods are accepted by the Particle Data Group [1]: the Bayesian Method and the Unified Approach, which is a frequentist method proposed recently by Feldman and Cousins [2] that allows the unified calculation of confidence intervals and upper limits with the correct coverage (see [3]).

The Unified Approach represents a major breakthrough for a satisfactory statistical treatment of processes with small signals with the frequentist method. However, as already noted by Feldman and Cousins [2], when the number of observed events in a Poisson process with mean \( \mu \) is smaller than the expected background, the upper limit for \( \mu \) obtained with the Unified Approach decreases rapidly when the background increases. Hence, by observing less events than the expected background an experiment can establish a very stringent upper bound on \( \mu \) even if it is not sensitive to such small values of \( \mu \). This problem has been further discussed in Ref. [4], where an alternative frequentist method has been proposed. This method yields confidence intervals and upper limits with all the desirable properties of those calculated with the Unified Approach and in addition minimizes the effect of the observation of less background events than expected. In the following this method will be called “Alternative Unified Approach”. The basic features of the Unified Approach and the Alternative Unified Approach, which are necessary for the understanding of the present paper, are reviewed in Section II.

The original formulation of the Unified Approach [2] and of the Alternative Unified Approach [4] for a Poisson process with background assumed a precise knowledge of the expected mean background. The aim of this paper is the presentation of the extension of these approaches to the case in which the background is known with a non-negligible error. This is done in Sections III and IV, where the probability to observe a number \( n \) of events in a Poisson process consisting in signal events with mean \( \mu \) and background events with known mean \( b \) is derived (in Section III we consider the simpler case \( \sigma_b \ll b/3 \) and in Section IV this constraint is removed), and in Section V, where the method for deriving the corresponding confidence intervals in the Unified Approach and in the Alternative Unified Approach is presented. Conclusions are drawn in Section VI.

II. POISSON PROCESSES WITH BACKGROUND

The probability to observe a number \( n \) of events in a Poisson process consisting in signal events with mean \( \mu \) and background events with known mean \( b \) is

\[
P(n|\mu; b) = \frac{1}{n!} (\mu + b)^n e^{-(\mu+b)}.
\]

(2.1)

The classical frequentist method for obtaining the confidence interval for the unknown parameter \( \mu \) is based on Neyman’s method to construct a confidence belt [3]. This confidence belt is the region in the \( n-\mu \) plane lying between the two curves \( n_1(\mu; b, \alpha) \) and \( n_2(\mu; b, \alpha) \) such that for each value of \( \mu \)
where $\alpha$ is the desired confidence level. The two curves $n_1(\mu; b, \alpha)$ and $n_2(\mu; b, \alpha)$ are required to be monotonic functions of $\mu$ and can be inverted to yield the corresponding curves $\mu_1(n; b, \alpha)$ and $\mu_2(n; b, \alpha)$. Then, if a number $n_{\text{obs}}$ of events is measured, the confidence interval for $\mu$ is $[\mu_2(n_{\text{obs}}; b, \alpha), \mu_1(n_{\text{obs}}; b, \alpha)]$. This method guarantees by construction the correct coverage, i.e. the fact that the resulting confidence interval $[\mu_2(n_{\text{obs}}; b, \alpha), \mu_1(n_{\text{obs}}; b, \alpha)]$ is a member of a set of confidence intervals obtained with an ensemble of experiments that contain the true value of $\mu$ with a probability $\alpha$ (in other words, $100\alpha\%$ of the confidence intervals in the set contain the true value of $\mu$).

As noted by Cousins in Ref. [3], Neyman himself pointed out [5] that the usefulness of classical confidence intervals lies in the fact that the experiments in the ensemble do not need to be identical, but can be real, different experiments. One can see this fact in a simple way by considering, for example, two different experiments that measure the same quantity $\mu$. The $100\alpha\%$ classical confidence interval obtained from the results of each experiment belongs to a set of confidence intervals which can be obtained with an ensemble of identical experiments and contain the true value of $\mu$ with probability $\alpha$. It is clear that the sum of these two sets of confidence intervals is still a set of confidence intervals that contain the true value of $\mu$ with probability $\alpha$.

In the case of a Poisson process, since $n$ is an integer, the relation (2.2) can only be approximately satisfied and in practice the chosen acceptance intervals $[n_1(\mu; b, \alpha), n_2(\mu; b, \alpha)]$ are the smallest intervals such that

$$P(n \in [n_1(\mu; b, \alpha), n_2(\mu; b, \alpha)]|\mu; b) \geq \alpha.$$  

This choice introduces an overcoverage for some values of $\mu$ and the resulting confidence intervals are conservative. As emphasized in Ref. [2], conservativeness is an undesirable but unavoidable property of the confidence intervals in the case of a Poisson process (it is undesirable because it implies a loss of power in restricting the allowed range for the parameter $\mu$).

The construction of Neyman’s confidence belt is not unique, because in general there are many different couples of curves $n_1(\mu; b, \alpha)$ and $n_2(\mu; b, \alpha)$ that satisfy the relation (2.2). Hence, an additional criterion is needed in order to define uniquely the acceptance intervals $[n_1(\mu; b, \alpha), n_2(\mu; b, \alpha)]$. The two common choices are

$$P(n < n_1(\mu; b, \alpha)|\mu; b) = P(n > n_2(\mu; b, \alpha)|\mu; b) = \frac{1-\alpha}{2},$$  

which leads to central confidence intervals and

$$P(n < n_1(\mu; b, \alpha)|\mu; b) = 1-\alpha,$$  

which leads to upper confidence limits. Central confidence intervals are appropriate for the statistical description of the results of experiments reporting a positive result, i.e. the measurement of a number of events significantly larger than the expected background. On
the other hand, upper confidence limits are appropriate for the statistical description of the results of experiments reporting a negative result, i.e. the measurement of a number of events compatible with the expected background. However, Feldman and Cousins [2] noticed that switching from central confidence level to upper confidence limits or vice-versa on the basis of the experimental data (“flip-flopping”) leads to undercoverage for some values of $\mu$, which is a serious flaw for a frequentist method.

Feldman and Cousins [2] proposed an ordering principle for the construction of the acceptance intervals that is based on likelihood ratios and produces an automatic transition from central confidence intervals to upper limits when the number of observed events in a Poisson process with background is of the same order or less than the expected background, guaranteeing the correct frequentist coverage for all values of $\mu$. The acceptance interval for each value of $\mu$ is calculated assigning at each value of $n$ a rank obtained from the relative size of the likelihood ratio

$$R_{UA}(n|\mu; b) = \frac{P(n|\mu; b)}{P(n|\mu_{\text{best}}; b)},$$

(2.6)

where $\mu_{\text{best}} = \mu_{\text{best}}(n; b)$ (for a fixed $b$) is the non-negative value of $\mu$ that maximizes the probability $P(n|\mu; b)$:

$$\mu_{\text{best}}(n; b) = \max[0, n - b].$$

(2.7)

For each fixed value of $\mu$, the rank of each value of $n$ is assigned in order of decreasing value of the ratio $R_{UA}(n|\mu; b)$: the value of $n$ which has bigger $R_{UA}(n|\mu; b)$ has rank one, the value of $n$ among the remaining ones which has bigger $R_{UA}(n|\mu; b)$ has rank two and so on. The acceptance interval for each value of $\mu$ is calculated by adding the values of $n$ in increasing order of rank until the condition (2.3) is satisfied.

The automatic transition from two-sided confidence intervals to upper confidence limits for $n \lesssim b$ is guaranteed in the Unified Approach by the fact that $\mu_{\text{best}}$ is always non-negative. Indeed, since $\mu_{\text{best}}(n \leq b; b) = 0$, the rank of $n \leq b$ for $\mu = 0$ is one, implying that the interval $0 \leq n \leq b$ for $\mu = 0$ is guaranteed to lie in the confidence belt.

As already noticed by Feldman and Cousins [3], when $n \lesssim b$ the upper bound $\mu_1(n; b, \alpha)$ decreases rather rapidly when $b$ increases and stabilizes around a value close to 0.8 for large values of $b$. Hence, a stringent upper bound for $\mu$ obtained with the Unified Approach by an experiment that has observed a number of events significantly smaller than the expected background is not due to the fact that the experiment is very sensitive to small values of $\mu$, but to the fact that less background events than expected have been observed.

The Alternative Unified Approach proposed in Ref. [4] allows the construction of a classical confidence belt which has all the desirable features of the one in the Unified Approach (i.e. an automatic transition with the correct coverage from two-sided confidence intervals to upper confidence limits when the observed number of events is of the order or less than the expected background) and in addition minimizes the decrease of the upper confidence limit $\mu_1(n; b, \alpha)$ for a given $n$ as the mean expected background $b$ increases. The Alternative Unified Approach is based on an ordering principle for the construction of a classical confidence belt that is implemented as the Feldman and Cousins ordering principle in the Unified Approach, but for each value of $\mu$ the rank of each value of $n$ is calculated from the relative size of the likelihood ratio
where the reference value $\mu_{\text{ref}} = \mu_{\text{ref}}(n; b)$ is taken to be the bayesian expected value for $\mu$:

$$\mu_{\text{ref}}(n; b) = \int_0^\infty \mu P(\mu|n; b) \, d\mu = n + 1 - \left( \sum_{k=0}^{n} \frac{k^k}{k!} \left( \sum_{k=0}^{n} \frac{k^k}{k!} \right)^{-1} \right. \ .$$  \hspace{1cm} (2.9)

Here $P(\mu|n; b)$ is the bayesian probability distribution for $\mu$ calculated assuming a constant prior for $\mu \geq 0$ (see, for example, [6]):

$$P(\mu|n; b) = \frac{1}{n!} (\mu + b)^n e^{-\mu} \left( \sum_{k=0}^{n} \frac{b^k}{k!} \right)^{-1} .$$  \hspace{1cm} (2.10)

The assumption of a constant prior is arbitrary, but it seems to be the most natural choice if $\mu$ is the parameter under investigation and there is no prior knowledge on its value. Notice also that the arbitrariness induced by the choice of the prior is of “second order” with respect to the dominant arbitrariness induced by the choice of the method for constructing the confidence belt.

The obvious inequality $\sum_{k=0}^{n} k^k / k! \leq n \sum_{k=0}^{n} b^k / k!$ implies that $\mu_{\text{ref}}(n; b) \geq 1$. Therefore, $\mu_{\text{ref}}(n; b)$ represents a reference value for $\mu$ that not only is non-negative, as desired in order to have an automatic transition from two-sided intervals to upper limits, but is even bigger or equal than one. This is a desirable characteristic in order to obtain a weak decrease of the upper confidence limit for a given $n$ when the expected background $b$ increases. Indeed, it has been shown in Ref. [4] that for $n \lesssim b$ the upper bound $\mu_1(n; b, \alpha)$ decreases rather weakly when $b$ increases and stabilizes around a value close to 1.7 for large values of $b$. This behaviour of $\mu_1(n; b, \alpha)$ is more suitable for the physical interpretation of experimental results than the behaviour of $\mu_1(n; b, \alpha)$ in the Unified Approach. Furthermore, as shown by the example in Ref. [4], the upper limits $\mu_1(n; b, \alpha)$ obtained with the Alternative Unified Approach for $n \lesssim b$ are are in reasonable agreement with those obtained with the Bayesian Approach. Hence, the Alternative Unified Approach extends the approximate agreement between the Bayesian and frequentist methods from $n \gg b$ to $n \lesssim b$ (although the statistical interpretations of the confidence intervals is different in the two methods).

III. BACKGROUND WITH SMALL ERROR

Let us consider an experiment that measures a Poisson process with an expected background $b = \bar{b} \pm \sigma_b$ and a normal probability distribution function for the mean expected background $b$:

$$f(b; \bar{b}, \sigma_b) = \frac{1}{\sqrt{2\pi} \sigma_b} \exp \left[ -\frac{(b - \bar{b})^2}{2\sigma_b^2} \right] .$$  \hspace{1cm} (3.1)

The importance of $\sigma_b$ can be estimated by comparing it with $\sqrt{b}$, which represents the rms fluctuation of the number of background events if $b = \bar{b}$. If $\sigma_b \ll \sqrt{b}$ the uncertainty of
the value of the background is much smaller than the typical fluctuation of the number of observed events induced by the background and can be safely neglected. Here we consider the possibility that \( \sigma_b \) is not much smaller than \( \sqrt{b} \) and its contribution cannot be neglected.

For simplicity, in this section we assume that \( \sigma_b \lesssim b/3 \) and we consider \( b \) varying from \(-\infty\) to \(+\infty\), neglecting the small error introduced by considering negative values of \( b \). This approximation allows a simple analytic solution of all the integrals involved in the calculation. The general case with arbitrarily large \( \sigma_b \) and \( b \) restricted in the interval \([0, +\infty)\) is treated in Section IV.

If \( \mu \) is the mean of true signal events, the probability \( P(n|\mu; \bar{b}, \sigma_b) \) to observe \( n \) events is given by

\[
P(n|\mu; \bar{b}, \sigma_b) = \int P(n|\mu; b) f(b; \bar{b}, \sigma_b) \, db ,
\]

with the Poisson probability \( P(n|\mu; b) \) given in Eq.(2.1). With the change of variable \( x = (b - \bar{b} + \sigma_b^2)/\sigma_b \), the probability \( P(n|\mu; \bar{b}, \sigma_b) \) can be written as

\[
P(n|\mu; \bar{b}, \sigma_b) = \frac{1}{n!} \left( \mu + \bar{b} - \sigma_b^2 \right)^n \exp \left[ -\left( \mu + \bar{b} \right) + \frac{\sigma_b^2}{2} \right] I_n(\mu, \bar{b}, \sigma_b) ,
\]

where

\[
I_n(\mu, \bar{b}, \sigma_b) = \sum_{k=0}^{n} \frac{n!}{k!} \left( \frac{\sigma_b}{\mu + \bar{b} - \sigma_b^2} \right)^k m_k.
\]

Here \( m_k \) is the \( k^{th} \) central moment of the normal distribution with unit variance,

\[
m_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^k e^{-x^2/2} \, dx .
\]

Taking into account that \( \int x e^{-x^2/2} \, dx = -e^{-x^2/2} \), the integral in Eq.(3.3) can be calculated by parts, yielding

\[
m_k = \frac{k!}{(k/2)! \, 2^{k/2}}
\]

for \( k \) even and \( m_k = 0 \) for \( k \) odd. From Eqs.(3.4) and (3.6), we obtain

\[
I_n(\mu, \bar{b}, \sigma_b) = \frac{n/2}{\sum_{k=0}^{n/2} \left( \frac{n!}{(n-2k)! \, k! \, 2^k \left( \frac{\sigma_b}{\mu + \bar{b} - \sigma_b^2} \right)^{2k} \right)} .
\]

Equation (3.3) gives the formula for the probability \( P(n|\mu; \bar{b}, \sigma_b) \) to observe a number \( n \) of events in a Poisson process consisting in signal events with mean \( \mu \) and background events with known mean \( b = \bar{b} \pm \sigma_b \), i.e. it replaces Eq.(2.1) if the error \( \sigma_b \) of the calculated mean background is not negligible. The expression (3.7) for \( I_n(\mu, \bar{b}, \sigma_b) \) is valid only if \( \sigma_b \lesssim \bar{b}/3 \), but, as we will see in the next section, with an appropriate redefinition of \( I_n(\mu, \bar{b}, \sigma_b) \) the formula (3.3) for \( P(n|\mu; \bar{b}, \sigma_b) \) is valid for any value of \( \sigma_b \).
IV. BACKGROUND WITH LARGE ERROR

In this section we present the formalism that allows treatment of cases in which $\sigma_b$ is arbitrarily large and $b$ is restricted to the interval $[0, +\infty)$. The gaussian probability distribution function of the mean expected background $b$ normalized in the interval $[0, +\infty)$ is

$$f(b; \bar{b}, \sigma_b) = \frac{N}{\sqrt{2\pi} \sigma_b} \exp\left[-\frac{(b - \bar{b})^2}{2 \sigma_b^2}\right] \quad (b \geq 0), \quad (4.1)$$

with the normalization factor $N$ given by

$$N^{-1} = \frac{1}{2} \left[ 1 + \text{erf}\left(\frac{\bar{b}}{\sqrt{2} \sigma_b}\right) \right]. \quad (4.2)$$

Apart from the error function that must be evaluated numerically, the integral over $db$ in Eq.(3.2) can still be solved analytically. Indeed, Eqs.(3.3) and (3.4) are still valid, with

$$m_k = \frac{N}{\sqrt{2\pi}} \int_{x_{\min}}^{+\infty} x^k e^{-x^2/2} \, dx, \quad (4.3)$$

where

$$x_{\min} = -\frac{\bar{b} - \sigma_b^2}{\sigma_b}. \quad (4.4)$$

The moments (4.3) can be calculated by parts, yielding

$$m_k = \frac{N}{2} \left[ 1 + \text{erf}\left(\frac{x_{\min}}{\sqrt{2}}\right) \right] \frac{k!}{(k/2)! 2^{k/2}} \frac{N}{\sqrt{2\pi}} e^{-x_{\min}^2/2} \frac{k!}{(k/2)!} \left[ \sum_{\ell=0}^{(k/2)-1} \frac{\left(\frac{k}{2} - \ell\right)!}{(k - 2\ell)!} \frac{x_{\min}^{k-2\ell-1}}{2^\ell} \right]$$

for $k$ even and

$$m_k = \frac{N}{\sqrt{2\pi}} e^{-x_{\min}^2/2} \left(\frac{k - 1}{2}\right)! \left[ \sum_{\ell=0}^{(k-1)/2} x_{\min}^{-\frac{k-2\ell}{2}} \left(\frac{k-1}{2} - \ell\right)! \frac{2^\ell}{\ell!} \right], \quad (4.5)$$

for $k$ odd. Therefore, the probability $P(n|\mu; \bar{b}, \sigma_b)$ to observe a number $n$ of events is given by the formula in Eq.(3.3) with

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1The error function is defined by $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$. 

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\[
I_n(\mu, \overline{b}, \sigma_b) = \frac{N}{2} \left[ 1 + \text{erf} \left( \frac{\overline{b} - \sigma_b^2}{\sqrt{2} \sigma_b} \right) \right] \left[ \sum_{k=0}^{n/2} \frac{n!}{(n-2k)!} \frac{1}{k!} \left( \frac{\sigma_b}{\mu + \overline{b} - \sigma_b^2} \right)^{2k} \right] + \frac{N}{\sqrt{2\pi}} \exp \left[ -\frac{(\overline{b} - \sigma_b^2)^2}{2 \sigma_b^2} \right] \times \left\{ \sum_{k=0}^{(n-1)/2} \frac{n!}{(n-2k-1)!} \frac{k!}{(2k+1)!} \left( \frac{\overline{b} - \sigma_b^2}{\mu + \overline{b} - \sigma_b^2} \right)^{2k+1} \sum_{\ell=0}^{k} \frac{2\ell}{(k-\ell)!} \left( \frac{\sigma_b}{\overline{b} - \sigma_b^2} \right)^{2\ell+1} \right. \\
- \sum_{k=0}^{n/2} \frac{n!}{(n-2k)!} \frac{k!}{k!} \left( \frac{\overline{b} - \sigma_b^2}{\mu + \overline{b} - \sigma_b^2} \right)^{2k} \sum_{\ell=0}^{k-1} \frac{(k-\ell)!}{2(2k-\ell)!} \frac{2\ell}{(k-\ell)!} \left( \frac{\sigma_b}{\overline{b} - \sigma_b^2} \right)^{2\ell+1} \left\} \right. .
\] (4.7)

These quantities have a cumbersome expression, but their numerical evaluation with a computer is not much more difficult than that of the corresponding quantities in Eq. (3.1) (however, the calculation of \(I_n(\mu, \overline{b}, \sigma_b)\) is rather difficult if \(\sigma_b^2 > \overline{b}\) because the addenda in Eq. (4.7) have alternating signs and the roundoff errors introduced by subtracting large numbers become crucial).

V. CONFIDENCE INTERVALS

The construction of the confidence belt for the probability (4.3) follows the same procedure described in Section II but now the confidence interval for \(\mu\) corresponding to a number \(n_{\text{obs}}\) of observed events is \([\mu_2(n_{\text{obs}}; \overline{b}, \sigma_b, \alpha), \mu_1(n_{\text{obs}}; \overline{b}, \sigma_b, \alpha)]\), i.e. it depends on \(\overline{b}\) and \(\sigma_b\). The acceptance intervals can be constructed following the same principles discussed in Section II and one can construct the confidence belt for central confidence intervals or upper confidence limits, or the confidence belt in the Unified Approach or in the Alternative Unified Approach. This section is devoted to the presentation of the formalism for the implementation of the Unified Approach and of the Alternative Unified Approach. As an example, we will consider \(\overline{b} = 3\) and \(\sigma_b = 0, 1, 1.8\).

The quantity \(\mu_{\text{best}}(n; \overline{b}, \sigma_b)\) in the Unified Approach is the value of \(\mu\) that maximizes \(P(n|\mu; \overline{b}, \sigma_b)\) and the acceptance interval for each value of \(\mu\) is calculated assigning at each value of \(n\) a rank obtained from the relative size of the ratio

\[
R_{UA}(n|\mu; \overline{b}, \sigma_b) = \frac{P(n|\mu; \overline{b}, \sigma_b)}{P(n|\mu_{\text{best}}; \overline{b}, \sigma_b)}. \tag{5.1}
\]

The value of \(\mu_{\text{best}}(n; \overline{b}, \sigma_b)\) can be easily calculated by hand for \(n = 0, 1, 2\), whereas for higher values of \(n\) it can be calculated numerically. The resulting 90% CL confidence belts for \(\overline{b} = 3\) and \(\sigma_b = 0, 1, 1.8\) are plotted in Fig. 1. We have checked that the confidence belt for \(\sigma_b \lesssim 0.2\) practically coincides with the one for \(\sigma_b = 0\), confirming the prediction that the contribution of \(\sigma_b\) is negligible if \(\sigma_b \ll \sqrt{\overline{b}}\).

In Fig. 1, the confidence belt for \(\sigma_b = 1\) has been obtained with the formulas presented in Section III that are valid for \(\sigma_b \lesssim \overline{b}/3\), whereas the confidence belt for \(\sigma_b = 1.8\) has been obtained with the formulas presented in Section IV which are valid for any value of \(\sigma_b\). We
have checked that the confidence belt for $\sigma_b = 1$ calculated with the formulas presented in Section IV practically coincides with the one shown in Fig. 1.

From Fig. 1 one can see that the broadness of the confidence belt increases with $\sigma_b$. This is due to the fact that the integral in Eq. (3.2) has the effect of flattening the probability $P(n|\mu, \bar{b}, \sigma_b)$ as a function of $n$ for fixed $\mu$ with respect to $P(n|\mu, 0, \sigma_b = 0)$ and this flattening effect increases with the size of $\sigma_b$. The shift of the borders of the confidence belt as $\sigma_b$ increases is not always monotonic because of the unavoidable overcoverage caused by the fact that $n$ is an integer (see Section II).

The lower value of $\mu$ for which $n = 0$ is out of the confidence belt in Fig. 1 is lower for $\sigma_b = 1.8$ than for $\sigma_b = 0$ and $\sigma_b = 1$. This is caused by the fact that the ratio (5.1) for $n = 0$ does not depend on $\sigma_b$. Indeed, from Eqs. (3.3) and (4.7) we have

$$P(n = 0|\mu, \bar{b}, \sigma_b) = \frac{N}{2} \left[ 1 + \text{erf}\left( \frac{\bar{b} - \sigma_b^2}{\sqrt{2} \sigma_b} \right) \right] \exp \left[ -(\mu + \bar{b}) + \frac{\sigma_b^2}{2} \right]. \quad (5.2)$$

Therefore, $\mu_{\text{best}}(n = 0; \bar{b}, \sigma_b) = 0$ and

$$R_{UA}(n = 0|\mu, \bar{b}, \sigma_b) = e^{-\mu}. \quad (5.3)$$

On the other hand, the ratio $R_{UA}(n|\mu, \bar{b}, \sigma_b)$ for $n > 0$ increases with $\sigma_b$ because of the flattening of $P(n|\mu, \bar{b}, \sigma_b)$ as a function of $n$. Hence, the rank of $n = 0$ for each value of $\mu$ decreases with the increasing of $\sigma_b$, causing the peculiar behaviour of the upper bound $\mu_1(n = 0; \bar{b}, \sigma_b, \alpha)$ as a function of $\sigma_b$ exemplified in Fig. 1. Since the possibility to set a smaller upper bound on $\mu$ for larger $\sigma_b$ as a consequence of the observation of $n = 0$ events is undesirable from the physical point of view, we think that in this case the physical interpretation of the experimental result should be very cautious, waiting for a better understanding of the background.

In the Alternative Unified Approach the acceptance interval for each value of $\mu$ is calculated assigning at each value of $n$ a rank obtained from the relative size of the ratio

$$R_{AUA}(n|\mu, \bar{b}, \sigma_b) = \frac{P(n|\mu, \bar{b}, \sigma_b)}{P(n|\mu_{\text{ref}}; \bar{b}, \sigma_b)}, \quad (5.4)$$

where the reference value $\mu_{\text{ref}} = \mu_{\text{ref}}(n; \bar{b}, \sigma_b)$ is the bayesian expected value for $\mu$. In order to calculate analytically the value of $\mu_{\text{ref}}(n; \bar{b}, \sigma_b)$, it is convenient to write the probability (3.3) as

$$P(n|\mu, \bar{b}, \sigma_b) = \exp \left[ -(\mu + \bar{b}) + \frac{\sigma_b^2}{2} \right] \sum_{k=0}^{n} \frac{\mu^{n-k}}{(n-k)!} J_k(\bar{b}, \sigma_b), \quad (5.5)$$

with

$$J_k(\bar{b}, \sigma_b) \simeq \sum_{j=0}^{k/2} \frac{(\bar{b} - \sigma_b^2)^{k-2j} \sigma_b^{2j}}{(k-2j)! \cdot 2^j} \quad (5.6)$$

for $\sigma_b \lesssim \bar{b}/3$ and
\[ J_k(\bar{b}, \sigma_b) = \frac{N}{2} \left[ 1 + \text{erf} \left( \frac{\bar{b} - \sigma_b^2}{\sqrt{2} \sigma_b} \right) \right] \left( \sum_{j=0}^{k/2} \frac{(\bar{b} - \sigma_b^2)^{k-2j}}{(k-2j)! \, j! \, 2^j} \sigma_b^{2j} \right) \]

\[ + \frac{N}{\sqrt{2\pi}} \exp \left[ -\frac{(\bar{b} - \sigma_b^2)^2}{2 \sigma_b^2} \right] \times \left\{ \sum_{j=0}^{(k-1)/2} \frac{j!}{(2j+1)! \, (k-2j-1)!} \sum_{\ell=0}^{j} \frac{2\ell \, (\bar{b} - \sigma_b^2)^{k-2\ell-1}}{\sigma_b^{2\ell+1} \, (j-\ell)!} \right. \]

\[ - \sum_{j=0}^{k/2} \frac{1}{(k-2j)! \, j!} \sum_{\ell=0}^{j-1} \frac{(j-\ell)! \, (\bar{b} - \sigma_b^2)^{k-2\ell-1}}{(2(j-\ell))! \, 2^\ell} \sigma_b^{2\ell+1} \right\} \]  

(5.7)

for arbitrarily large \( \sigma_b \). For the bayesian probability distribution function for \( \mu \) with a constant prior,

\[ P(\mu|n; \bar{b}, \sigma_b) = \frac{P(n|\mu; \bar{b}, \sigma_b)}{\int_0^\infty P(n|\mu; \bar{b}, \sigma_b) \, d\mu} , \]  

(5.8)

one obtains

\[ P(\mu|n; \bar{b}, \sigma_b) = e^{-\mu} \left( \sum_{k=0}^{n} \frac{\mu^{n-k}}{(n-k)!} J_k(\bar{b}, \sigma_b) \right) \left( \sum_{k=0}^{n} J_k(\bar{b}, \sigma_b) \right)^{-1} . \]  

(5.9)

Hence, the reference value \( \mu_{\text{ref}}(n; \bar{b}, \sigma_b) \), which is the bayesian expected value for \( \mu \), is given by

\[ \mu_{\text{ref}}(n; \bar{b}, \sigma_b) = n + 1 - \left( \sum_{k=0}^{n} k \, J_k(\bar{b}, \sigma_b) \right) \left( \sum_{k=0}^{n} J_k(\bar{b}, \sigma_b) \right)^{-1} . \]  

(5.10)

If \( \sigma_b \lesssim \bar{b}/3 \) the quantities \( J_k(\bar{b}, \sigma_b) \) are given by Eq. (5.6) and one can see that they are all positive. Hence, the inequality \( \sum_{k=0}^{n} k \, J_k(\bar{b}, \sigma_b) \leq n \sum_{k=0}^{n} J_k(\bar{b}, \sigma_b) \) implies that \( \mu_{\text{ref}}(n; \bar{b}, \sigma_b) \geq 1 \) as in the case \( \sigma_b = 0 \) (see Eq. (2.9) and the following discussion). On the other hand, the general formula (5.7) allows \( J_k(\bar{b}, \sigma_b) \) to be negative and \( \mu_{\text{ref}}(n; \bar{b}, \sigma_b) \) is not guaranteed to be larger than one if \( \sigma_b \gtrsim \bar{b}/3 \).

The 90\% CL confidence belts in the Alternative Unified Approach for \( \bar{b} = 3 \) and \( \sigma_b = 0, 1, 1.8 \) are plotted in Fig. 4. One can see again that the broadness of the confidence belt increases with \( \sigma_b \). The behaviour of the upper bound \( \mu_1(n = 0; \bar{b}, \sigma_b, \alpha) \) as a function of \( \sigma_b \) is similar to the one obtained in the Unified Approach and the same caveats apply to the physical interpretation of the observation of \( n = 0 \) events.

VI. CONCLUSIONS

We have presented the formalism that allows the error \( \sigma_b \) of the calculated mean background \( \bar{b} \) in the statistical analysis of a Poisson process with the frequentistic method to be
taken into account. This error must be taken into account if it is not much smaller than $\sqrt{b}$, which represents the rms fluctuation of the number of background events.

We have considered in particular the Unified Approach \cite{2} and the Alternative Unified Approach \cite{4}, that guarantee by construction a correct frequentist coverage. We have shown that the broadness of the classical confidence belt increases with $\sigma_b$, leading to an increase of the confidence intervals for the mean $\mu$ of signal events. The only exception to this behaviour is represented by the upper bound $\mu_1(n = 0; \bar{b}, \sigma_b, \alpha)$, which decreases with the increasing of $\sigma_b$ for large values of $\sigma_b$ in both approaches. Hence, the physical interpretation of the observation of $n = 0$ events when $\sigma_b$ is large should be very cautious and the effort towards a better understanding of the background should receive high priority.

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FIGURES

Fig. 1. 90% CL confidence belts in the Unified Approach for $\bar{b} = 3$ and $\sigma_b = 0, 1, 1.8$.

Fig. 2. 90% CL confidence belts in the Alternative Unified Approach for $\bar{b} = 3$ and $\sigma_b = 0, 1, 1.8$. 
Figure 1

Unified Approach

σ_{b} = 0
σ_{b} = 1.0
σ_{b} = 1.8

n

Figure 1
Figure 2

Alternative Unified Approach

\( \sigma_b = 0 \)
\( \sigma_b = 1.0 \)
\( \sigma_b = 1.8 \)