The Mittag-Leffler Phillips Curve

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Abstract

In this paper, a mathematical model containing two-parameter Mittag-Leffler function in its definition is proposed to be used for the first time to describe the relation between unemployment rate and inflation rate, also known as the Phillips curve. The Phillips curve is in the literature often represented by an exponential-like shape. On the other hand, Phillips in his fundamental paper used a power function in the model definition. Considering that the ordinary as well as generalized Mittag-Leffler function behaves between a purely exponential function and a power function it is natural to implement it in the definition of the model used to describe the relation between the data representing the Phillips curve. For the modeling purposes the data of two European economies, France and Germany, were used.

Keywords: Econometric modeling; Identification; Phillips curve; Mittag-Leffler function

1 Introduction

It is “because of” or “thanks to” Paul Anthony Samuelson and Robert Merton Solow (Samuelson and Solow 1960), that the economists all around the world call the negative correlation between the rate of wage change (or the price inflation rate) and unemployment rate the Phillips curve (PC). It is lesser-known, that the idea occurred more than 30 years before publishing the famous paper of Alban William Housego Phillips (Phillips
in the work by Irving Fisher (Fisher 1926), and Fisher was not the only one who would deserve such an important “discovery” be named after him. Arthur Joseph Brown in his book (Brown 1955), published 3 years before Phillips paper, precisely described the inverse relation between the wage and price inflation and the rate of unemployment. Also Richard George Lipsey (Lipsey 1960) played an important role by the birth, creation of the theoretical foundations and popularisation of the PC. Samuelson and Solow in their paper (Samuelson and Solow 1960) for the first time mention policy implications. In the empirical studies (Brown 1955; Phillips 1958; Lipsey 1960) for the United Kingdom, and (Bowen 1960; Samuelson and Solow 1960; Bodkin 1966) for the United States, the inverse relationship between the rate of wage change and the unemployment rate was proven. The PC was in its beginnings widely used by the policy-makers to exploit the trade-off to reduce unemployment at a small cost of additional inflation - “sacrifice ratio”.

Since then the PC has been studied, extended and re-formulated by many authors. One of the “modern” forms of the PC is represented by models in which expectations are not anchored in backward-looking behavior but can jump in response to current and anticipated changes in policy - the New Keynesian theory of the output-inflation tradeoff. The model called the New Keynesian Phillips Curve (NKPC) builds, among others, on the works of Taylor (Taylor 1979, 1980) and Calvo (Calvo 1983), where the models of staggered contracts were developed, and on the quadratic price adjustment cost model of Rotemberg and Woodford (Rotemberg 1982; Rotemberg and Woodford 1997), all of which have a similar formulation as the expectations-augmented PC of Friedman and Phelps (Friedman 1968; Phelps 1968). The work of Clarida et al. (Clarida et al. 1999) illustrates the widely usage of this model in theoretical analysis of monetary policy. With the focus shifted from the unemployment rate to the output gap, Phillips’ relationship has become an aggregate supply curve, but the idea remains the same. McCallum (McCallum 1997) has called it “the closest thing there is to a standard specification”. The NKPC stayed popular also in the late 90’s and at the beginning of the 21st century as a theory for understanding inflation dynamics. In the works (Galí and Gertler 1999; Galí et al. 2001, 2005) the NKPC was transformed into a hybrid version, that relates inflation to expected future inflation, lagged inflation and real marginal costs.

When Magnus Gustaf Mittag-Leffler in his works (Mittag-Leffler 1903a,b) proposed a new function $E_\alpha(x)$, he surely did not expect how important generalization of the exponential function $e^x$ he developed. The ML function and its generalizations interpolate between a purely exponential law and a power-law-like behavior, and they arise naturally in the solution of fractional-order integro-differential equations, random walks, Lévy flights, the study of complex systems, and in other fields. In numerous works the properties, generalizations and applications of the ML type functions
were studied e.g. (Hille and Tamarkin 1930; Dzhrbashyan 1966; Caputo and Mainardi 1971; Blair 1974; Torvik and Bagley 1984; Samko et al. 1993; Gorenflo and Vessella 1991; Gorenflo and Mainardi 1996, 1997; Gorenflo et al. 2002, 2014; Kilbas and Saigo 1995, 1996; Kilbas et al. 2004; Mainardi and Gorenflo 1996, 2000; Srivastava and Saxena 2001; Srivastava and Tomovski 2009; West et al. 2003; Haubold et al. 2011; Garrappa 2015b), and computation procedures for evaluating the ML function were developed e.g. in (Gorenflo et al. 2002; Chen 2008b,a; Podlubny 2005, 2011; Podlubny et al. 2012; Garrappa 2015a; Matychyn 2017).

The Mittag-Leffler (ML) function become of great use and importance not only for mathematicians, but because of its special properties and huge potential by solving applied problems it found its applicability also in the fields such as psychorheology (Blair 1974), electrotechnics (Capelas de Oliveira et al. 2011; Petras et al. 2012; Sierociuk et al. 2013), modeling of processes such as diffusion (Mainardi 2018), combustion (Samuel et al. 2016), universe expansion (Zeng et al. 2015), etc. The idea to use the fractional-order calculus and the ML function for modelling phenomenons from the fields of economics and econophysics was elaborated by several authors (Scalas et al. 2000; Gorenflo et al. 2001; Mainardi et al. 2000, 2002; Cartea and del-Castillo-Negrete 2007; Vilela Mendes 2008; Garibaldi and Scalas 2010; Tarasov 2011; Skovranek et al. 2012; Tarasov and Tarasova 2017; Tarasova and Tarasov 2018).

In this paper the two-parameter ML function is for the first time used to model the relation between the unemployment rate and the inflation rate - the Phillips curve. French and German economic data are taken for the period 1980 - 2011 from the portal EconStats™ (Econstats.com 2015) to identify the PC of these economies. The data are first pre-processed following the algorithm used in the Phillips’ paper (Phillips 1958), where the average values are calculated for different levels of unemployment rate and corresponding inflation rates. These average values are used for the fitting purposes. To evaluate the efficiency of the proposed model, two other models are considered, that in their definition use the exponential function, power function, respectively. The performance of all three models is compared based on the fitting-criterion, the least squares error between the fitting curve and the “average” data-set. To have a better insight into the fitting performance, the least squares error between the fitting curve and the “original” data-set (all data) is also calculated for all three concurrent models.
2 Preliminaries: Mittag-Leffler function and its generalizations

In 1903 M. G. Mittag-Leffler (Mittag-Leffler 1903a,b) introduced a new function \( E_\alpha(x) \), a generalization of the classical exponential function \( e^x \), which is till today known as the one-parameter ML function. Using Erdélyi’s notation (Erdélyi et al. 1955), where \( z \) is used instead of \( x \), the function can be written as:

\[
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha \in \mathbb{C}, \text{ Re}(\alpha) > 0, \ z \in \mathbb{C},
\]

where \( \Gamma \) denotes the (complete) Gamma function, having the property \( \Gamma(n) = (n - 1)! \). The one-parameter ML function and its properties were further investigated in (Mittag-Leffler 1904, 1905; Wiman 1905a,b; Pollard 1948; Agarwal 1953; Humbert 1953; Humbert and Agarwal 1953) followed by the generalization to a two-parameter function of the ML type, by some authors called the Wiman’s function (some give the credit to Agarwal). Following the Erdélyi’s handbook the formula has the form:

\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \text{ Re}(\alpha) > 0, \text{ Re}(\beta) > 0, \ z \in \mathbb{C}.
\]

The main properties of the above mentioned functions, and other ML type functions, can be found in the book by Erdélyi et al. (Erdélyi et al. 1955), and a detailed overview in the book by Dzhrbashyan (Dzhrbashyan 1966). To demonstrate the concept of generality of the ML type functions let us point out, that the ML function for one parameter (1), i.e. if we substitute \( \beta = 1 \) in (2).

Accordingly, the classical exponential function is a special case of the one-parameter ML function, where \( \alpha = 1 \):

\[
E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \equiv E_{\alpha}(z),
\]

\[
E_{1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k + 1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.
\]

In 1971 the generalization of the two-parameter function of the ML type (2) was introduced by T.R. Prabhakar (Prabhakar 1971) in terms of the series representation:

\[
E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta) k!}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \text{ Re}(\alpha) > 0, \text{ Re}(\beta) > 0, \ z \in \mathbb{C},
\]

where \((\gamma)_k\) is Pochhammer’s symbol (Rainville 1960), defined by:

\[
(\gamma)_k = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)} = \begin{cases} 
1, & (k = 0, \ \gamma \neq 0), \\
\gamma(\gamma + 1)\ldots(\gamma + k - 1), & (k \in \mathbb{N}, \ \gamma \in \mathbb{C}).
\end{cases}
\]
The function defined in (3) is a natural generalization of the exponential function $e^z$, the one-parameter ML function $E_\alpha(z)$ and the Wiman’s function $E_{\alpha,\beta}(z)$. In 2007 Shukla and Prajapati (Shukla and Prajapati 2007) proposed and investigated the function $E_{\alpha,\beta}^{\gamma,q}(z)$, defined as:

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)^q_k z^k}{\Gamma(\alpha k + \beta) k!}, \quad \alpha, \beta, \gamma \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0, z \in \mathbb{C},$$

where $q \in (0, 1) \cup \mathbb{N}$, and $(\gamma)^q_k$ denotes the generalized Pochhammer’s symbol:

$$(\gamma)^q_k = \frac{\Gamma(\gamma + qk)}{\Gamma(\gamma)},$$

which in particular reduces to:

$$q_k = \prod_{r=1}^{q} \left(1 + \frac{\gamma - 1}{q} - \frac{r}{q}\right)^k, \quad \text{if } q \in \mathbb{N}.$$ 

Also other authors introduced and investigated further generalizations of the ML function, but for the purpose to demonstrate the potential of the ML function these four generalizations of the exponential function are sufficient.

3 Modeling the Phillips curve

As in many fields of science and applications so in economics, to describe a relation between two variables the regression analysis is often used. One can use different regression models from simple linear type, throughout exponential and power type models, to polynomial ones, and many other more complex and sophisticated. The discussion on the linearity or nonlinearity, and on the convex/concave shape of the PC, if it is supposed to be nonlinear, is still ongoing. Some authors are in favour of convex shape (Cover 1992; Karras 1996; Nobay and Peel 2000; Schaling 2004), some of concave (Stiglitz 1997), and some of their combination (Filardo 1998). The application of the ML type function to describe the PC perfectly fits into this discussion.

3.1 The Original Phillips Curve

Phillips in (Phillips 1958) used British economic data - the percentage unemployment data, provided by the Board of Trade and the Ministry of Labour (calculated by Phelps Brown and Sheila Hopkins (Brown and Hopkins 1950)), and corresponding percentage employment data, quoted in Beveridge, Full Employment in a Free Society, Table 22. But the economic data were first preprocessed, i.e. the average values of the rate of change of money wage rates and of the percentage unemployment for 6 different levels
of the unemployment (0-2, 2-3, 3-4, 4-5, 5-7, 7-11) were calculated. The crosses in the Fig. [1] refer to these average values. Each cross gives an approximation to the rate of change of wages which would be associated with the indicated level of unemployment if unemployment were held constant at that level. Finally, Phillips fitted a curve to the crosses using a model in the form:

\[
y + a = bx^c \Rightarrow \log (y + a) = \log b + c \log x,
\]

where \( y \) being the rate of change of wage rates and \( x \) the percentage unemployment. The constants \( b \) and \( c \) were estimated using the least squares to fit four crosses laying between 0-5 % of unemployment, and constant \( a \) was chosen to fit the remaining two crosses laying in the interval 5-11 % of unemployment. Based on this “fitting criterion” Phillips identified the parameters of the model (4) as follows:

\[
y + 0.900 = 9.638 x^{-1.394} \Rightarrow \log (y + 0.900) = 0.984 - 1.394 \log x.
\]

Figure 1: Original Phillips curve (Phillips 1958)

### 3.2 The Mittag-Leffler Phillips Curve

The idea to use a ML type function to describe the economic data (representing the Phillips curve) leads naturally from observation of two facts:
• the simplicity of the model used by Phillips in his paper (Phillips 1958) given in (4),
where a power type regression is used to fit the economic data, where the model can be defined in the form:

\[ y(x) = b x^c - a, \quad a, b, c \in \mathbb{R}, \]

(5)

• the usual shape of the PC, used in the literature, which reminds on the exponential type function:

\[ y(x) = b e^{c x} - a, \quad a, b, c \in \mathbb{R}, \]

(6)

where for both cases, (5) and (6), \( x \) stands for the unemployment rate and \( y \) for the inflation rate.

Based on these facts, the two-parameter ML function appears to be a general model to fit the PC relation, as it behaves between a purely exponential function and a power function. Some of the possible manifestations of the ML function are shown in Fig. 2 (figures generated using the Matlab demo published by Igor Podlubny (Podlubny 2011)).

The two-parameter ML function defined in (2), which includes the special cases when \( \beta = 1 \) called the one-parameter ML function (1), and the classical exponential function when \( \beta = \alpha = 1 \), is used to model the economic data under study. Generally, the proposed fitting model can be written as follows:

\[ y(x) = C x^{\beta-1} E_{\alpha,\beta} (a x^\alpha), \quad \alpha, \beta \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, a, C \in \mathbb{R}, \]

(7)

where the parameters \( \alpha, \beta, a, C \) are subject to optimization procedure minimizing the squared sum of the vertical offsets between the data points and the fitting curve.

For the evaluation of the two-parameter ML function, and for performing the “Mittag-Leffler fitting”, the Matlab functions created by Podlubny (Podlubny 2005, 2011) are implemented. To identify the parameters of the ML model \( (\alpha, \beta, a, C) \), and of the power-function model, and the exponential-function model \( (a, b, c) \), the MATLAB optimization function \texttt{FMINSEARCH} was used, where the squared sum of the vertical distances of the points lying on the fitting-curve to the “average” data-set was minimized. It is worth remembering that the Matlab function \texttt{FMINSEARCH} is based on simplex search method, which may sometimes lead to finding a local minimum instead of the global one; changing the initial guess is a standard approach in such situations.

Here it should be also noted, that in this paper a similar procedure as the one used by Phillips is applied, \textit{i.e.} first the average values for different levels of the unemployment rate are calculated (with the step 1 for the levels), and then the average values of the inflation rate corresponding to these levels of unemployment rate are evaluated. These points to be fitted will further be called the “average” data-set. The results are given
Figure 2: Mittag-Leffler fitting using different functions $y(x)$ for generating data [Podlubny 2011].

In the form of tables, where the least squares error (the squared sum of the vertical distances) between the fitting curve and the “average” data-set and also the least squares error between the fitting curve and all the economic data on hand (further called the “original” data-set), are listed. This-way it is be possible to compare the goodness-of-fit of the three used models:

- the power-function model (5),
- the exponential-function model (6),
- the Mittag-Leffler model (7),

not only to the average values but also to the original data-points.
4 Numerical results and discussion

The procedure detailed in the previous section was used to describe the relation between the unemployment rate and the inflation rate of selected countries. The economic data were obtained from the *EconStats*™ portal [Econstats.com](http://Econstats.com) [2015], and the whole list of the used data can be found in the Tab. A.1. As representatives, two European countries (France and Germany) were chosen to demonstrate the fitting procedure. The period under focus was selected based on the availability of the data.

Unemployment rate and inflation rate of the French economy were taken for the period 1980-2011. Following the procedure of pre-processing the data used by Phillips, the average values for different levels of unemployment rate, in this case 6 levels (6-7, 7-8, 8-9, 9-10, 10-11, 11-12, with lower bound included), and corresponding inflation rate averages, were calculated, to obtain the “average” data-set to be fitted (Tab. A.2). Afterwards the data were fitted using three different models: the power-function model (5), the exponential-function model (6), and the ML model (7), respectively, based on minimizing the fitting criterion, the least squares error between the “average” data-set and the fitting curve (obtained as the solution of the used model).

The result of the data fitting is shown in Fig. 3 and the evaluated errors, the least squares error for the fitting curve to the “average” data-set and the least squares error for the fitting curve to all given data (“original” data-set), as well as the identified parameters of the above mentioned models, are listed in the Tab. 1.

![Figure 3: Fitting the French economic data](image)

Identically as in the French case, the German economy data (unemployment rate and
Table 1: Fitting the French economic data

|                          | Mittag-Leffler model | Exponential model | Power model |
|--------------------------|----------------------|-------------------|-------------|
| LS error to “average” data-set | 1.1843               | 1.5780            | 1.7704      |
| LS error to “original” data-set | 189.1845             | 195.9090          | 198.3482    |
| Model definition         | \( y(x) = C x^{\beta-1} E_{\alpha,\beta}(a x^{\alpha}) \) | \( y(x) = b e^{c x} - a \) | \( y(x) = b x^{c} - a \) |
| Identified parameters    | \( \alpha = 1.5317 \) | \( a = -0.1507 \) | \( a = 1.7578 \) |
|                          | \( \beta = 1.9470 \) | \( b = 220.3057 \) | \( b = 1933.2 \) |
|                          | \( a = -0.3209 \)   | \( c = -0.4449 \) | \( c = -2.6297 \) |
|                          | \( C = 13.9419 \)   |                   |             |

inflation rate) were taken for the period 1980-2011, and the average values for 9 different levels of the unemployment rate (3-4, 4-5, 5-6, 6-7, 7-8, 8-9, 9-10, 10-11, 11-12, always including the lower bound) were evaluated with corresponding inflation rate averages. This-way again an “average” data-set to be fitted was obtained (Tab. A.2), and these data were fitted using the power-function model, the exponential-function model, and the ML model, respectively. The fitting criterion used here was the same as in the French case, i.e. the least squares error between the fitting curve and the “average” data-set. The results of the data fitting displayed in Fig. 4 are also given in the form of the evaluated errors, the least squares error of the fitting curve to the average values and the least squares error of the fitting curve to the “original” data-set. The errors as well as the identified parameters of the three models used to fit the German economic data, are listed in the Tab. 2.

In both fitting cases, using the French economic data as well as the German economic data, the model using the ML function in its definition shows better performance from the viewpoint of the least squares errors in comparison to the models using exponential function, power function, respectively. It can be explained as a consequence of the possibility of the two-parameter ML function to adapt between exponential and power-like law functions.

In the French case (shown in Fig. 3), the ML model has similar shape as the power-function model and the exponential-function model, but with the rising unemployment rate it has a growing tendency, while the other two models are still decaying on the given
The behavior of the model using ML function in the case of fitting the German economic data (shown in Fig. 4) reminds on oscillations (a combination of convex and concave shape), while the models, that use exponential function, power function, respectively, are on the observed interval strictly decaying, what does not correspond to the last samples from neither “average” nor “original” data-set. The numerical results show, that the

|                      | Mittag-Leffler model | Exponential model | Power model |
|----------------------|----------------------|-------------------|-------------|
| LS error to “average” data-set | 6.0313               | 7.0219            | 7.3879      |
| LS error to “original” data-set | 45.6200              | 48.1316           | 49.7545     |
| Model definition     | $y(x) = C x^{\beta-1} E_{\alpha,\beta}(a x^\alpha)$ | $y(x) = b e^{c x} - a$ | $y(x) = b x^{c} - a$ |
| Identified parameters| $\alpha = 1.382$     | $a = 0.0381$      | $a = 32.1245$ |
|                      | $\beta = 1.7055$     | $b = 12.1022$     | $b = 43.8469$ |
|                      | $a = -0.3167$        | $c = -0.2007$     | $c = -0.1141$ |
|                      | $C = 4.6929$         |                   |             |
usage of the ML function in the model offers higher precision and more possibilities by
describing the economic data on hand, in comparison to the cases, where exponential
function or power function is used in the model definition. Especially in the German case
it is possible to observe that the fitting curve representing the solution of the ML model
can combine the convex and concave shape. This could be a useful property not only
by modeling the Phillips curve, but also by fitting the linkage between other economic
variables, e.g. the fiscal multiplier effect, the link between inflation and money growth,
etc.

5 Conclusion

In the present paper for modeling the relation between the unemployment rate and the
inflation rate, in the literature known as the Phillips curve, a mathematical model was
proposed using the two-parameter ML function in its definition. To determine the perform-
ance of the proposed model, two other models were introduced to fit the given data, one
using the exponential function, the other using the power function. These two concur-
rent models were chosen from simple reasons, exponential-like shape can be observed in
most of the works when modeling PC, and the power-law model was used in the original
Phillips paper.

All three models, the ML model, exponential, and the power-function model, were
applied to fit the French and German economic data. In both cases the ML model has
proved better performance from the viewpoint of the fitting criterion - the least squares
error of the fitting curve to the “average” data-set. In the Tab. 1 and Tab. 2 the values
of the least squares error of the fitting curve to the “average” data-set and the least
squares error of the fitting curve to the “original” data-set are listed, together with the
identified parameters for all three compared models. The property of the two-parameter
ML function to describe different data relations (see Fig. 2) was exploited by modeling the
French and German Phillips curve, Fig. 3 and Fig. 4 respectively, where it can be seen,
that the model using the ML function is not restricted to an exponential-like or power-like
shape, but can also manifest itself in the form of oscillations, or damped oscillations, etc.

In the further work the attention will be focused on better understanding and ex-
planation of the parameters used in the ML model, in the context of other economic
indicators, and on application of such model for modelling other economic phenomenons.
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### Appendix

Table A.1: The “original” data-set: Economic data for years 1980 – 2011 (Econstats.com 2015)

| Year | Unemployment rate [%] | Inflation rate [%] | Unemployment rate [%] | Inflation rate [%] |
|------|-----------------------|--------------------|-----------------------|--------------------|
| 1980 | 6.349                 | 13.060             | 3.359                 | 5.447              |
| 1981 | 7.438                 | 13.330             | 4.831                 | 6.324              |
| 1982 | 8.069                 | 11.980             | 6.734                 | 5.256              |
| 1983 | 8.421                 | 9.460              | 8.099                 | 3.284              |
| 1984 | 9.771                 | 7.674              | 8.058                 | 2.396              |
| 1985 | 10.230                | 5.831              | 8.124                 | 2.084              |
| 1986 | 10.360                | 2.539              | 7.834                 | -0.125             |
| 1987 | 10.500                | 3.289              | 7.843                 | 0.242              |
| 1988 | 10.010                | 2.701              | 7.735                 | 1.274              |
| 1989 | 9.396                 | 3.498              | 6.790                 | 2.778              |
| 1990 | 8.975                 | 3.380              | 6.155                 | 2.687              |
| 1991 | 9.467                 | 3.217              | 5.470                 | 3.474              |
| 1992 | 9.850                 | 2.366              | 6.575                 | 5.046              |
| 1993 | 11.120                | 2.106              | 7.833                 | 4.476              |
| 1994 | 11.680                | 1.661              | 8.433                 | 2.717              |
| 1995 | 11.150                | 1.778              | 8.275                 | 1.729              |
| 1996 | 11.580                | 2.084              | 8.950                 | 1.193              |
| 1997 | 11.540                | 1.283              | 9.692                 | 1.533              |
| 1998 | 11.070                | 0.667              | 9.433                 | 0.602              |
| 1999 | 10.460                | 0.562              | 8.625                 | 0.635              |
| 2000 | 9.083                 | 1.827              | 8.000                 | 1.400              |
| 2001 | 8.392                 | 1.781              | 7.883                 | 1.904              |
| 2002 | 8.908                 | 1.938              | 8.700                 | 1.355              |
| 2003 | 8.900                 | 2.169              | 9.783                 | 1.031              |
| 2004 | 9.233                 | 2.342              | 10.520                | 1.790              |
| 2005 | 9.292                 | 1.900              | 11.210                | 1.920              |
| 2006 | 9.242                 | 1.912              | 10.190                | 1.784              |
| 2007 | 8.367                 | 1.607              | 8.783                 | 2.276              |
| 2008 | 7.808                 | 3.159              | 7.600                 | 2.754              |
| 2009 | 9.500                 | 0.103              | 7.742                 | 0.234              |
| 2010 | 9.802                 | 1.736              | 7.058                 | 1.150              |
| 2011 | 9.675                 | 2.293              | 5.983                 | 2.482              |
Table A.2: The “average” data-set: Economic data for different unemployment levels

| Unemployment level [%] | Unemployment rate [%] | Inflation rate [%] | Unemployment rate [%] | Inflation rate [%] |
|------------------------|-----------------------|--------------------|-----------------------|--------------------|
| 3 – 4                  | 3.359                 | 5.447              |                       |                    |
| 4 – 5                  | 4.831                 | 6.324              |                       |                    |
| 5 – 6                  |                       |                    | 5.727                 | 2.978              |
| 6 – 7                  | 6.349                 | 13.060             | 6.564                 | 3.942              |
| 7 – 8                  | 7.623                 | 8.245              | 7.691                 | 1.489              |
| 8 – 9                  | 8.576                 | 4.616              | 8.405                 | 1.907              |
| 9 – 10                 | 9.483                 | 2.624              | 9.636                 | 1.055              |
| 10 – 11                | 10.312                | 2.984              | 10.355                | 1.787              |
| 11 – 12                | 11.357                | 1.597              | 11.210                | 1.920              |

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