Abstract: In a previous study [Inaba et al., Physica D (2020), web on-line], we discovered nested mixed-mode oscillations (MMOs) generated by a Bonhoeffer–van der Pol (BVP) oscillator under weak periodic perturbations near a subcritical Hopf bifurcation point. The dynamics of BVP oscillators are equivalent to FitzHugh–Nagumo dynamics and have been studied extensively for more than five decades. In this study, we focus on the singly nested MMOs that occur between the $1^4$- and $1^5$-generating regions in a piecewise-smooth BVP oscillator with an idealized diode where $1^s$ indicates alternating time-series waveforms that consist of a large amplitude oscillation followed by $s$ small peaks, and we confirm 400 nested mixed-mode oscillation-incrementing bifurcations (MMOIBs). Our numerical results suggest that the universal constant converges to one, which was predicted because MMOIBs increment and terminate toward an MMO increment-terminating tangent bifurcation point and the gradient of the tangent points is one.

Key Words: nested mixed-mode oscillations, mixed-mode oscillation-incrementing bifurcations, forced Bonhoeffer–van der Pol oscillator, grazing–sliding dynamics

1. Introduction
In a previous study [1], we investigated nested mixed-mode oscillations (MMOs) generated by a piecewise-smooth forced constrained Bonhoeffer–van der Pol (BVP) oscillator that includes an idealized diode with grazing–sliding dynamics [2, 3]. BVP oscillators are known to be equivalent to FitzHugh–Nagumo models [4, 5] and have been extensively studied for many years [6]. In this study, we analyze nested MMOs generated by the dynamics and confirm that they appear and disappear alternately at least 400 times. These simple and nested MMOs are sandwiched by chaos in one-parameter bifurcation diagrams. We assert that the universal constant could converge to one, because nested MMOs are terminated by a tangent bifurcation. In this paper, we numerically confirm that the universal constant for successive nested MMOs could converge to one.

MMOs are a phenomenon that was first found in chemical experiments [7–11]. They are known to occur in extended canard-generating ordinary differential equations (ODEs). They consist of
$L$-large amplitude oscillations and $s$-small peaks and are abbreviated by the symbol “$L_s$,” which indicates alternating time-series waveforms which consist of $L$ large amplitude oscillations followed by $s$ small peaks. At first glance, the definition of MMOs are ambiguous; however, MMOs are ones of universal phenomena observed in extended canard-generating dynamics [18–21]. Canards were found and investigated using nonstandard analysis from 1980s [12–17]. Canards occur in a simple two-variable slow-fast dynamics such as a van der Pol and BVP electric oscillators. However, historically, MMOs were observed over 100 years ago [11] although MMOs are a topic next to canards. Recent studies have clarified that they occur in forced or extended three-dimensional (3D) ODEs that can exhibit canards [11, 18–20, 32], and MMOs have been the subject of intense research in recent years [18–41].

The mechanisms behind the generation for simple MMOs were clarified by several researchers [20, 22, 23, 36, 37, 42, 43] recently. However, MMO bifurcations are known to generate much more complex phenomena with evidently strong order, such as MMO period-adding phenomena, that were observed in several chemical [8–10] and electronic-circuit [6, 33, 35] experiments. Such phenomena have become intense research interest, and have been investigated numerically in autonomous [18, 21, 26–28, 35] and nonautonomous [1, 6, 24, 25, 32, 33, 44] ODEs. The MMO period-adding phenomena were reported by Kawczyński et al. [26–28], where they showed that the corresponding period-adding phenomena occur in a similar way to the period-adding bifurcations generated by the circle map. Shimizu et al. [34, 35] discovered the simplest MMO period-adding bifurcations denoted by $[1^2, 1^3 \times n]_{n+1}$ in the forced BVP oscillator where the subscript indicates the forcing term periods per MMO sequence, and called the resulting bifurcations mixed-mode oscillation-incrementing bifurcations (MMOIBs) [32]. These MMOIB-generated MMOs consist of a sequence $1^2$ followed by $n$ times $1^3$ sequences in $(n+1)$-number of periods of the forcing term. Since MMOIBs increment and terminate by saddle-node bifurcations, which we call an MMO increment-terminating tangent bifurcation, their universal constants could converge to one, as has been confirmed by numerical experiments [32, 35]. Kousaka et al. [6] proposed a constrained BVP oscillator under weak periodic perturbation that includes an idealized diode with grazing–sliding characteristics [2, 3], and rigorously derived one-dimensional (1D) Poincaré return maps. The return maps resemble the circle maps, explaining their successive occurrences of MMOIBs precisely and confirming more than 50 MMOIBs.

Furthermore, we considered the same dynamics as Kousaka’s [6], focusing on a constrained BVP oscillator with an infinitely large slope in the diode’s ON region, and discovered nested MMOIB-generated MMOs [1]. The dynamical equations that include an infinitely large slope are called grazing–sliding dynamics in the field of mathematics [2, 3]. Using the grazing–sliding dynamics, Poincaré return maps can be derived as 1D, and rigorous analysis becomes possible. Using these return maps, simple (un-nested) MMOIBs generate $[A_0, B_0 \times n]_{n+1}$ where $A_0 = [1^s]_1$ and $B_0 = [1^{s+1}]_1$ [6, 32], i.e., these MMOIB-generated MMOs consist of a sequence $1^s$ followed by $n$ times of $1^{s+1}$ sequences in $(n+1)$-number of periods of the forcing term. Nested MMOIBs generate MMO sequences denoted by $[A_1, B_1 \times n]$ that consist of $A_1 = [A_0, B_0 \times m]_{m+1}$ sequence followed by $n$ times $B_1 = [A_0, B_0 \times (m + 1)]_{m+2}$ sequences where $m$ is particular integers. We believe that nested MMOs could be a universal phenomenon, because they are fascinating and evidently have strong order. However, in Ref. [1], we only confirmed six nested MMOIBs because the forcing term periods are already $(m+2)n+(m+1) = 20$ for $m = 1$ and $n = 6$, which is very large to distinguish the sequences by observing the solution on the Poincaré return maps. Thus, it is difficult to track further.

In this study, we consider the same constrained dynamics that include an idealized diode with grazing–sliding characteristics [2, 3] as Kousaka proposed [6], and track successive nested MMOIBs. We confirm 400 nested MMOIBs and verify numerically that the universal constant could converge to one, because nested MMOIBs also increment and terminate toward an MMO increment-terminating tangent bifurcation point [1].
2. BVP oscillator under weak periodic perturbations

Figure 1 shows the circuit diagram for the driven BVP oscillator proposed in [6], which generates simple and nested MMOIB-generated MMOs [1,6]. The circuit consists of an inductor $L$, capacitor $C$, linear resistor $k_1$, DC voltage source $B_0$, sinusoidal voltage source $B_1 \sin \omega \tau$, and only one nonlinear negative conductor $g$.

Note that the BVP oscillator consists of only two-terminal elements and voltage sources. Such circuits belong to the category of “natural circuits” because the voltages generated across and the currents flowing through the two-terminal elements play a crucial role in the dynamics and they never act as analog computers that exhibit input- and output-voltage terminals [45].

Via rescaling, $L$ can be normalized to one. In addition, the capacitor $C$ is assumed to be small and is represented by a small parameter $\varepsilon$. The voltage generated across $C$ and the current flowing through $L$ are denoted by $x$ and $y$, respectively. If $g$ is a function of $x$ in the entire domain, the governing equation is mathematically written by a system of two nonautonomous ODEs as follows:

\[
\begin{align*}
\frac{dx}{d\tau} &= y - g(x), \\
\frac{dy}{d\tau} &= -x - k_1 y + B_0 + B_1 \sin \omega \tau.
\end{align*}
\] (1)

In the following discussions, we set the constant parameters as $\varepsilon = 0.1$, $k_1 = 0.9$, $B_0 = 0.207$, $B_1 = 0.01$ and select $\omega$ as the bifurcation parameter.

However, analyzing MMO bifurcations is difficult because of the following reasons. Usually, chaos and MMOs can be generated in two- or higher-dimensional nonautonomous and three- or higher-
dimensional autonomous ODEs. Although a mapping method is conventionally used in the analysis, the resulting Poincaré return maps are still two- or higher dimensional, and the analysis for two- or higher dimensional discrete dynamical systems are usually very difficult [47–50]. Therefore, in Refs. [1, 6], we assume that \( g \) includes an idealized diode for which the characteristics is expressed by a nonlinear conductance with a complete saturation as shown in Fig. 2. In this case, the governing equation is represented by a constrained equation including a diode with grazing–sliding dynamics [1–3, 6] as follows:

1. \[
\begin{align*}
\varepsilon \dot{x} &= y - g(x) \\
\dot{y} &= -x - k_1 y + B_0 + B_1 \sin \omega \tau , \quad \text{for diode OFF}
\end{align*}
\]  
2. \[
\begin{align*}
x &= \alpha \\
y &= -\alpha - k_1 y + B_0 + B_1 \sin \omega \tau , \quad \text{for diode ON},
\end{align*}
\]

where we assume \( g(x) = -x + x^3 \), and these two equations are connected (forward in time only) by the following transition conditions:

1. diode OFF (2) \( \rightarrow \) 2. diode ON (3) : \( x = \alpha \),
2. diode ON (3) \( \rightarrow \) 1. diode OFF (2) : \( y = -\alpha + \alpha^3 \).

Since the dynamical Eq. (3) is represented by a piecewise one-variable nonautonomous equation, the Poincaré return maps can be constructed as an exactly 1D map [1, 6]. It is known that fruitful discussions have been obtained for 1D maps [51–56]. Such grazing–sliding dynamics [2, 3, 46] including an idealized diode in \( RLC \) circuits were proposed by Inaba et al. [57–59] to rigorously analyze chaos and torus breakdown in a nonautonomous and extended autonomous van der Pol oscillators. Since \( RLC \) circuits with a diode are natural, the governing equation is similar to simple mechanical oscillators, e.g., stick–slip dynamics with dry friction [2, 3, 46].

Here, we derived Eq. (5) as follows. When the solution is in the diode’s OFF region and the voltage \( x \) increases to \( \alpha \), the diode turns ON. By contrast, when the solution is constrained to \( x = \alpha \), the diode turns off when the current flowing through \( g(x) \) decreases to equal \( -\alpha + \alpha^3 \), i.e., \( y = g(\alpha) \), because the current through the capacitor is zero (recall that \( C \frac{dv}{dt} = C \frac{d\alpha}{dt} = 0 \) because \( \alpha \) is constant.). Throughout this study, we set \( \alpha = 0.8 \).

Before defining the 1D Poincaré return maps, we explain the geometric structure of the vector fields [1]. Figure 3 shows the structure that consists of \( x \)-nullcline (purple), \( y \)-nullcline (black), a stable focus, a stable relaxation oscillation (blue), and an unstable periodic solution (green). The intersection point between \( x \)-nullcline and \( y \)-nullcline is a stable focus for the constant parameter values. The magnified view is shown in Fig. 3(b). The stable relaxation oscillation coexists with
the stable focus and an unstable periodic solution because of a subcritical Hopf bifurcation. The subcritical Hopf bifurcation in BVP oscillators are discussed in detail in Ref. [31].

Now, we explain the construction of the 1D Poincaré return maps [1, 6]. First, define half plane Π and line segment Σ₁ as

\[
\Pi = \{ (\tau, x, y) | x - \alpha = 0 \},
\]

\[
\Sigma_1 = \{ (\tau, x, y) | x - \alpha = 0, y = g(x) \}.
\]

Consider a flow that is initially located on Σ₁ at point marked τ₀ in Fig. 4. The solution leaving an initial point τ₀ enters the diode OFF region, strikes Π at a point marked P, and returns to Σ₁ at a point marked τ₁. Therefore, 1D Poincaré return map T that transforms point at τ = τ₀ to a point at which the solution leaving Σ₁ returns at τ = τ₁ can be defined as follows:

\[
T : \Sigma_1 \rightarrow \Sigma_1, \theta_0 \mapsto \theta_1 = T(\theta_0),
\]

where \( \theta_0 = \frac{\tau_0}{2\pi} \) and \( \theta_1 = \frac{\tau_1}{2\pi} \mod 1 \).

The following results have been obtained by employing the fourth-order Runge-Kutta method with the step size \( 2\pi/(2048\omega) \) for Eq. (2) and the analytical solution for Eq. (3). The floating-point variables in our code are declared as 80-bit long double type variables.

Figure 5(a) shows T and the corresponding trajectory \([1^4, 1^5 \times 3]_4\), which is obtained after three MMOIBs. As shown in the figure, MMOIB-generated MMOs emerge in the bottom left of the return map T. Figure 5(b) shows a global view of the one-parameter bifurcation diagram between the
regions generating \([1^4]_1\) and \([1^5]_1\). As seen in the figure, MMOIB-generated MMOs occur successively. MMOIB-generated MMO trajectory after 30 MMOIBs in the \(\theta_k-\theta_{k+1}\) plane is shown in Fig. 6(a), and the corresponding time-series waveform is shown in Fig. 6(c). The successive MMOIB-generated MMOs \([1^4, 1^5 \times n]_{n+1}\) for successive \(n\) increment and accumulate toward MNO increment-terminating tangent bifurcation point, as shown in Fig. 6(b). The MMOIBs occur in a similar way to the period-adding bifurcations generated by the circle map, and it suggests that MMOIBs could occur as many times as desired.

![Fig. 6.](image)

**Fig. 6.** (a) \(T\) and the corresponding MMO trajectory observed after 30 MMOIBs for \(\omega = 0.409\). (b) \(T\) at the MNO increment-terminating tangent bifurcation point for \(\omega = 0.4085085\). (c) Time-series waveform corresponding to (a).

### 3. Successive nested MMOs

In our previous study [1], we showed that MMOIB-generated MMOs can be nested. Namely, between the regions generating adjacent \([1^4, 1^5 \times n]_{m+1}\) and \([1^4, 1^5 \times (m + 1)]_{m+2}\) MMO sequences, nested MMOIB-generated \([1^4, 1^5 \times n]_{m+1}, [1^4, 1^5 \times (m + 1)]_{m+2} \times n]_{(m+2)n+(m+1)}\) MMOs occur for successive \(n\) and integers \(m\). They also occur in a classical BVP oscillator with a third-order smooth nonlinear term [44]. In this section, we explain the nested MMOs occurring for \(m = 1\) in more detail than those in Ref. [1].

Figure 7(a) shows a magnified view of the one-parameter bifurcation diagram shown in Fig. 5(b) between the regions generating \([1^4, 1^5 \times 1]_2\) and \([1^4, 1^5 \times 2]_3\), where successive nested MMOIB-generated \([1^4, 1^5 \times 1]_2, [1^4, 1^5 \times 2]_3 \times n]_{3n+2}\) MMOs can occur for successive \(n\). A highly magnified view of the one-parameter bifurcation diagram in Fig. 7(a) is shown in Fig. 7(b). Similar to the successive simple MMOIB-generated \([1^4, 1^5 \times n]_{n+1}\) MMOs, nested MMOs occur in succession. Time-series waveforms of the MMOIB-generated MMOs for \(n = 1, 2, 3, 4, 5,\) and 6 are shown in Figs. 8(a)–(f), respectively.

The nested MMOIB-generated MMOs evidently have strong order. However, it may be difficult to identify them from the time-series waveforms because they are very complex. Possibly, some re-
searchers may have observed nested MMO time-series waveforms in numerical experiments. However, they may have failed to identify the strong order of the sequences because it could be very difficult to understand the fascinating order behind the very complex time-series waveforms.

To identify nested MMOIB-generated MMOs, we observe the behaviors for the trajectories of $T$. It can become easier to find the evidently strong order if we observe the trajectories in the 1D Poincaré return maps [1]. Figures 9(a)–(f) ($n = 1, 2, 3, 4, 5,$ and 6) show 1D Poincaré return maps $T$ corresponding to the time-series waveforms shown in Figs. 8(a)–(f).

The cases $n = 7, 8, 9, 19,$ and 20 are shown in Figs. 10(g)–(k), respectively. Furthermore, $T$ applied 62 ($= 3 \times 20 + 2$) times for the 20th saddle-node bifurcation point at $\omega = 0.4270584587012863$ is shown in Fig. 10(l) at which $[[1^4, 1^5 \times 1]_2, [1^4, 1^5 \times 2]_3 \times 20]_{62}$ is born. The exact tangency of $T^{62}(\theta)$ to the diagonal line at the saddle-node bifurcation point in Fig. 10(l) shows the justifiability of our numerical computations. However, it is a limit of our effort to distinguish nested MMOIB-generated MMOs through the observation of the trajectories on the Poincaré return map as seen in
Fig. 8. Time-series waveforms of nested MMOB-generated \([1^4, 1^5 \times 1]^2, [1^4, 1^5 \times 2]^3 \times n \) MMOs, showing (a) \(n = 1\) for \(\omega = 0.4305\), (b) \(n = 2\) for \(\omega = 0.4288\), (c) \(n = 3\) for \(\omega = 0.42815\), (d) \(n = 4\) for \(\omega = 0.4278\), (e) \(n = 5\) for \(\omega = 0.4276\), and (f) \(n = 6\) for \(\omega = 0.42747\).
Fig. 9. 1D Poincaré return maps $T$ and the corresponding trajectories of nested MMOIB-generated $[[1^4, 1^5 \times 1]_2, [1^4, 1^5 \times 2]_3 \times n]_{3n+2}$ MMOs, showing (a) $n = 1$ for $\omega = 0.4305$, (b) $n = 2$ for $\omega = 0.4288$, (c) $n = 3$ for $\omega = 0.42815$, (d) $n = 4$ for $\omega = 0.4278$, (e) $n = 5$ for $\omega = 0.4276$, and (f) $n = 6$ for $\omega = 0.42747$.

Fig. 10. 1D Poincaré return maps $T$ and the corresponding trajectories of nested MMOIB-generated $[[1^4, 1^5 \times 1]_2, [1^4, 1^5 \times 2]_3 \times n]_{3n+2}$ MMOs, showing (g) $n = 7$ for $\omega = 0.42738$, (h) $n = 8$ for $\omega = 0.427315$, (i) $n = 9$ for $\omega = 0.42726$, (j) $n = 19$ for $\omega = 0.4270655$, (k) $n = 20$ for $\omega = 0.427058$. (l) $T$ applied 62 times for the saddle-node bifurcation point with $\omega = 0.4270584587012863$.

the trajectories shown in Fig. 10. In the next section, we track the successive saddle-node bifurcation points at which nested MMOIB-generated $[[1^4, 1^5 \times 1]_2, [1^4, 1^5 \times 2]_3 \times n]_{3n+2}$ MMOs are generated.
4. Saddle-node bifurcation points for successive nested MMOs and the universal constant

In this section, we calculate saddle-node bifurcation points at which successive nested MMOIB-generated MMOs appear and show that the constants defined by

\[
\delta_n = \frac{\omega_{SNn+2} - \omega_{SNn+1}}{\omega_{SNn+1} - \omega_{SNn}}
\]  

(8)

approach to one where \( \omega_{SNn} \) is an \( n \)th saddle-node bifurcation point at which \([1^4, 1^5 \times 1]_2, [1^4, 1^5 \times 2]_3 \times n\) MMO sequence emerges. The nested MMOIB-generated \([1^4, 1^5 \times 1]_2, [1^4, 1^5 \times 2]_3 \times n\) MMOs can be found as many times as we attempt to track according to our numerical results. It appears that there could be no limitation for the increment of \( n \). Similar to the 1D Poincaré return map \( T \) shown in Fig. 6(b) at the MMO increment-terminating tangent bifurcation point, \( T \) applied thrice is tangent to the diagonal line at three points \( Q_{3,1}, Q_{3,2}, \) and \( Q_{3,3} \) at the MMO increment-terminating tangent bifurcation point at which nested MMOIB-generated \([1^4, 1^5 \times 1]_2, [1^4, 1^5 \times 2]_3 \times n\) MMO sequence increment and accumulate when \( n \) tends to \( \infty \) (see Fig. 7(b)) as shown in Fig. 11 at \( \omega = 0.4269778 \). Hence, it is expected that \( \delta_n \) converges to one when \( n \) tends to \( \infty \).

![Fig. 11. \( T \) applied thrice at the MMO increment-terminating tangent bifurcation point (\( \omega = 0.4269778 \)).](image)

The determination of the \( n \)th saddle-node bifurcation reduces to the simultaneous determination of \( \omega \) and \( \theta \) that fulfill the equations

\[
T^{3n+2}(\theta) - \theta = 0
\]  

and

\[
\frac{dT^{3n+2}(\theta)}{d\theta} - 1 = 0,
\]  

(10)

where the map \( T^{3n+2}(\theta) \) depends on \( \omega \). The derivative \( dT^{3n+2}(\theta)/d\theta \) can be evaluated using the linearized differential equation (variational equation) of Eqs. (2) and (3). To this end, let \( \delta x(\tau) \) and \( \delta y(\tau) \) be the state variables of the variational equation. The integration of the set of original (Eqs. (2) and (3)) and variational equations starts from the initial condition

\[
(x(\tau_0), y(\tau_0), \delta x(\tau_0), \delta y(\tau_0)) = (\alpha, -\alpha + \alpha^3, 0, 1),
\]  

(11)

where \( \tau_0 = 2\pi \theta_0/\omega \), and ends when the trajectory of the original equation returns to \( \Sigma_1 \) after \( 3n + 2 \) cycles, i.e., \( \tau = 3n + 2 \). Note that, during this integration, the value of \( \delta x \) should be reset to zero every time the diode turns on because \( x \) is constant in Eq. (3). The final value \( \delta y(\tau_{3n+2}) \) is closely related to the derivative \( dT^{3n+2}(\theta)/d\theta \) (evaluated at \( \theta = \theta_0 \)), and they exactly coincide at period \( 3n + 2 \) fixed points. It thus suffices to use the value of \( \delta y(\tau_{3n+2}) - 1 \) to evaluate the fulfillment of Eq. (10).
With the above setting, we employ a double-loop bisection method for the simultaneous determination of \( \omega \) and \( \theta \). In the conventional method \([47]\), Eqs. (9) and (10) can be simultaneously solved using Newton’s method, which would nevertheless require considerably fine initial estimates. On the other hand, the exactly 1D Poincaré return map (for a fixed \( \omega \) value) in the present study allows us to determine the value of \( \theta \) satisfying Eq. (10) independently via bisection search (the inner loop of the double-loop bisection method), which in turn allows us to evaluate the sign of the left-hand side of Eq. (9) for that \( \theta \) value. Then, in the outer loop of the double-loop bisection method, the value of \( \omega \) satisfying Eq. (9) is determined via another bisection search.

The primary difficulty here, especially for the case of large \( n \), lies in the fine estimation of the values of \( \omega \) and \( \theta \) that can ensure a high success rate of convergence of the bisection method. While such fine estimation can be achieved by a significantly small step scanning of \( \omega \) with long-duration observation for time-series periodicity, it drastically degrades the efficiency of the procedure. Thus, in our implementation, we exploit the existence of a period 3\( n + 2 \) superstable fixed point for an \( \omega \) value near the saddle-node bifurcation point. Estimation of such a superstable fixed point is relatively easy due to the short transient before settling into the fixed point. If we discard short transients after each stepwise change of \( \omega \) and observe the values of \( T^{3n+2}(\theta) - \theta \), we can easily capture a regular pattern for the number of zeros after the decimal point that reflects the distance from the superstable point. Once we have found an estimation of the superstable fixed point, we perform a double-loop pre-search for the values of \( \theta \) and \( \omega \) that are likely to ensure the convergence of the subsequent bisection method described above. It is worth noting that this double-loop pre-search has been successfully conducted by a combination of one-way (increasing only) search for \( \theta \) and \( \omega \). By carefully specifying several tolerances, the above procedure has allowed for a precision of 14 decimal digits for \( \omega_{SNn} \).

Figure 12 shows \( T \) applied 1202 (= 3 \times 400 + 2) times at the 400th saddle-node bifurcation point (\( \omega = 0.4269779897695446 \)). The exact tangency of \( T^{1202}(\theta) \) to the diagonal line at the saddle-node

![Fig. 12. \( T \) applied 1202 times at the 400th saddle-node bifurcation point (\( \omega = 0.4269779897695446 \)).](image)

Fig. 13. Convergence of the universal constant.

![Fig. 13. Convergence of the universal constant.](image)
bifurcation point shows the justifiability of our numerical computations also for the case of large $n$. Then finally we obtain Fig. 13, supporting our hypothesis that $\delta_n$ converges to one because $T$ applied thrice at the MMO increment-terminating tangent bifurcation point is tangent to the diagonal line [1] as shown in Fig. 11. However, the convergence speed to one could be slow because the termination process of nested MMOIBs is clarified to be a tangent bifurcation. The precision for $\omega_{SNn}$ attained in the present numerical procedure has allowed us to observe the smooth convergence curve in Fig. 13 even for $n$ as large as 400.

5. Conclusion
We have investigated fine structures of nested MMOIB-generated MMOs that occur between the $1^4$- and $1^5$-generating regions in a piecewise-smooth BVP oscillator with an idealized diode. Carefully conducted numerical analysis, based on the construction of the exactly 1D Poincaré map, has revealed that the nested MMOs appear and disappear alternately at least 400 times. Further, we have calculated saddle-node bifurcation points at which nested MMOs appear. High precision for the bifurcation points attained in our numerical analysis has allowed us to observe the smooth convergence curve of the universal constant, which strongly suggests that the universal constant converges to one. Future extensions of the present work would also uncover fine structures of other nested MMOs such as those in a classical BVP oscillator with a third-order smooth nonlinear term [44].

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