RETRACEMENTS OF H-SPACES

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Abstract. Stasheff showed that if a map between H-spaces is an H-map, then the suspension of the map is extendable to a map between projective planes of the H-spaces. Stasheff also proved the converse under the assumption that the multiplication of the target space of the map is homotopy associative. We show by giving an example that the assumption of homotopy associativity of the multiplication of the target space is necessary to show the converse. We also show an analogous fact for maps between A_n-spaces.

1. Introduction

Let X and Y be H-spaces, and f: X → Y a map. Stasheff [4] showed that if f is an H-map, then its suspension Σf: ΣX → ΣY is extendable to a map P_2f: P_2X → P_2Y between projective planes P_2X and P_2Y of X and Y, respectively. He also showed the converse under the assumption that the multiplication µ_Y of Y is homotopy associative. It has not been known if the converse holds without the assumption of the homotopy associativity of µ_Y. In this paper we show by giving an example that the assumption of homotopy associativity of µ_Y is necessary to show the converse.

Our example is the retraction r: J(X) → X for an H-space X. Here, J(X) is the reduced power space of X, which is defined as an identification space of \( \bigcup_{i \geq 1} X^i \). Then the map r is defined by

\[
r([x_1, \ldots, x_i]) = (\cdots ((x_1 \cdot x_2) \cdot x_3) \cdots) \cdot x_i,
\]

where [x_1, \ldots, x_i] is the class of \((x_1, \ldots, x_i) \in X^i\) and \(x \cdot y\) denotes the multiplication of \(x\) and \(y\). Our result is stated as follows.

Theorem 1.1. For any H-space X, there is an extension \(P_2r: P_2J(X) \to P_2X\) of \(\Sigma r: \Sigma J(X) \to \Sigma X\).

Stasheff showed the following

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Theorem 1.2 (Ⅲ). The retraction \( r \) is an \( H \)-map if and only if the multiplication of \( X \) is homotopy associative.

Thus in particular, if the multiplication of \( X \) is not homotopy associative, then \( r \) is not an \( H \)-map even though there exists a map between projective planes extending the suspension of \( r \).

Now, the above fact is a special case of the main result of this paper, which deals with the case that the \( H \)-space \( X \) is an \( A_n \)-space. An \( A_n \)-space is an \( H \)-space such that the multiplication satisfies higher homotopy associativity of order \( n \). For example, an \( A_2 \)-space is just an \( H \)-space, an \( A_3 \)-space is a homotopy associative \( H \)-space, and an \( A_\infty \)-space is a space with the homotopy type of a loop space.

Any \( A_n \)-space \( X \) has an associated space \( P_i X \) for each \( 1 \leq i \leq n \) which is called the projective \( i \)-space of \( X \). By definition, \( P_1 X \) is the suspension \( \Sigma X \), \( P_2 X \) is the projective plane, and \( P_\infty X \) is the classifying space of \( X \).

Maps preserving \( A_n \)-space structures are called \( A_n \)-maps. An \( A_2 \)-map is an \( H \)-map, and an \( A_\infty \)-map is a map homotopic to a loop map. See [1] for the definition. By definition, if \( f \): \( X \to Y \) is an \( A_n \)-map, then there are maps \( P_i f \): \( P_i X \to P_i Y \) (\( 1 \leq i \leq n \)) such that

\[
P_1 f = \Sigma f, \quad P_{i+1} f \big|_{P_i X} \simeq P_i f \quad (1 \leq i \leq n - 1).
\]

Then the problem is if the converse of the above fact holds. To state our result we call a map \( f \): \( X \to Y \) between \( A_n \)-spaces a quasi \( A_n \)-map if there are maps \( P_i f \): \( P_i X \to P_i Y \) (\( 1 \leq i \leq n \)) with (1.1). Then we prove the following

Theorem 1.3. Let \( X \) be an \( A_n \)-space for some \( n \geq 2 \). Then the retraction \( r \): \( J(X) \to X \) is a quasi \( A_n \)-map.

We notice that the above theorem for \( n = 2 \) is just Theorem 1.1.

We can show a fact analogous to Theorem 1.2 for \( A_n \)-spaces. Thus the existence of an \( A_{n+1} \)-space structure for \( X \) is essential for the quasi \( A_n \)-map \( r \): \( J(X) \to X \) to be an \( A_n \)-map. We discuss it in section 3.

2. Proof of the main theorem

First we recall some facts on the reduced product space given by James [2]. Let \( f \): \( Z \times J(X) \to Y \) be a map. Put \( f_n = f \circ (id_Z \times \nu_n) \): \( Z \times X^n \to Y \) for \( n \geq 1 \), where \( \nu_n \): \( X^n \to J(X) \) (\( n \geq 1 \)) is the canonical map. Then we have

\[
f_n \big|_{Z \times X^{i-1} \times * \times X^{n-i}} = f_{n-1} \quad \text{for } 1 \leq i \leq n,
\]

where \( X^{i-1} \times * \times X^{n-i} \) is identified with \( X^{n-1} \) by the obvious way.
On the other hand, if we have a sequence of maps \((f_n: Z \times X^n \to Y)_{n=1,2,...}\) with the above property, then there is a map \(f: Z \times J(X) \to Y\) such that \(f \circ (id_Z \times \nu_n) = f_n\). Such a sequence \((f_n)_{n=1,2,...}\) is called a compatible sequence of invariant maps.

The space \(J(X)\) has the homotopy type of \(\Omega \Sigma X\). A homotopy equivalence \(s: J(X) \to \Omega \Sigma X\) is defined by means of a compatible sequence of invariant maps \((s_n: X^n \to \Omega \Sigma X)_{n=1,2,...}\), where \(s_1: X \to \Omega \Sigma X\) is the adjoint of \(id_{\Sigma X}: \Sigma X \to \Sigma X\), and \(s_n (n \geq 2)\) is defined by using the loop multiplication of \(\Omega \Sigma X\) as

\[
s_n(x_1, \ldots, x_n) = (\cdots (s_1(x_1) \cdot s_1(x_2)) \cdots) \cdot s_1(x_n).
\]

Note that to make \((s_n)\) a compatible sequence of invariant maps we need to modify the loop multiplication so that the constant loop is the strict unit of the loop multiplication.

Let \(e: \Sigma \Omega \Sigma X \to \Sigma X\) be the evaluation map, that is, the adjoint of the \(id_{\Omega \Sigma X}: \Omega \Sigma X \to \Omega \Sigma X\). Then we prove the following

**Lemma 2.1.** Let \(X\) be an \(H\)-space and \(\varepsilon: \Sigma X \to P_2X\) the inclusion. Then \(\varepsilon \circ \Sigma r \simeq \varepsilon \circ e \circ \Sigma s\).

**Proof.** The projective plane \(P_2X\) is the mapping cone of the Dold-Lashoff construction \(q: X \cup \mu X \times CX \to \Sigma X\), where \(\mu: X \times X \to X\) is the multiplication of \(X\). Morisugi \([3, (1.3)]\) showed that there exists a homotopy equivalence \(X \cup \mu X \times CX \to \Sigma(X \land X)\) such that if we identify \(X \cup \mu X \times CX\) with \(\Sigma(X \land X)\) by this homotopy equivalence, then \(q\) is identified with a map \(q': \Sigma(X \land X) \to \Sigma X\) with

\[
q' \circ \Sigma \pi \simeq \Sigma p_1 + \Sigma p_2 - \Sigma \mu: \Sigma(X \times X) \to \Sigma X,
\]

where \(\pi: X \times X \to X \land X\) is the quotient map and \(p_i\) is the projection to the \(i\)-th factor. Thus,

\[
\varepsilon \circ \Sigma \mu \simeq \varepsilon \circ (\Sigma p_1 + \Sigma p_2).
\]

Put \(\mu_n = r \circ \nu_n: X^n \to X\). Then \(\mu_2 = \mu\) and \(\mu_n = \mu \circ (\mu_{n-1} \times id)\). We show that there are homotopies \(H_n: I \times \Sigma X^n \to P_2X\) \((n \geq 1)\) between \(\varepsilon \circ \Sigma \mu_n\) and \(\varepsilon \circ e \circ \Sigma s_n\) such that \(H_1 = \varepsilon \circ p_2\) and

\[
H_n|I \times \Sigma(X^{j-1} \times \ast \times X^{n-j}) = H_{n-1} \quad \text{for any } 1 \leq j \leq n.
\]

Then \((H_n)_{n=1,2,...}\) defines a homotopy between \(\varepsilon \circ \Sigma r\) and \(\varepsilon \circ e \circ \Sigma s\).

Now \(e \circ \Sigma s_2 = \Sigma p_1 + \Sigma p_2\) since the adjoint of the both maps are the same \(s_2\). Thus,

\[
\varepsilon \circ \Sigma \mu_2 \simeq \varepsilon \circ (\Sigma p_1 + \Sigma p_2) = \varepsilon \circ e \circ \Sigma s_2.
\]

We notice that the above homotopy \(H_2: I \times \Sigma X^2 \to P_2X\) can be chosen to be constant of \(I \times \Sigma(X \lor X)\).
Let \( n > 2 \). Suppose inductively that we have \( H_i \) for \( i < n \) with the desired properties. Then \( H_n \) is defined as the composition of homotopies as follows.

\[
\varepsilon \circ \Sigma \mu_n = \varepsilon \circ \Sigma \mu \circ \Sigma (\mu_{n-1} \times 1) \\
\simeq \varepsilon \circ (\Sigma p_1 + \Sigma p_2) \circ \Sigma (\mu_{n-1} \times 1) \\
= \varepsilon \circ \Sigma \mu_{n-1} \circ \Sigma p' + \varepsilon \circ e \circ \Sigma s_1 \circ \Sigma p_n \\
\simeq \varepsilon \circ e \circ \Sigma s_{n-1} \circ \Sigma p' + \varepsilon \circ e \circ \Sigma s_1 \circ \Sigma p_n \\
= \varepsilon \circ e \circ \Sigma s_n,
\]

where \( p' : X^n \to X^{n-1} \) is the projection to the first \( n-1 \)-factors, and the second homotopy is given by using \( H_{n-1} \). It is clear that we can modify \( H_n \) to satisfy (2.1). Thus we have \( H_n \) for all \( n \) by induction. \( \square \)

Now we prove Theorem 1.3. Theorem 1.1 is a special case of Theorem 1.3.

**Proof of Theorem 1.3.** Since \( J(X) \) is a topological monoid, we have the projective \( \infty \)-space \( P_\infty J(X) \). It is known that \( P_\infty J(X) \) has the homotopy type of \( \Sigma X \) such that the inclusion \( \Sigma J(X) \to P_\infty J(X) \) followed by the homotopy equivalence \( P_\infty J(X) \simeq \Sigma X \) is homotopic to \( e \circ \Sigma s \) (cf. [5, Proof of Theorem 4.8]).

Define \( P_i : P_i J(X) \to P_i X \) for \( 2 \leq i \leq n \) by the following composition

\[
P_i J(X) \subset P_\infty J(X) \simeq \Sigma X \xrightarrow{\varepsilon} P_2 X \subset P_i X.
\]

Then by Lemma 2.1 we have the result. \( \square \)

## 3. \( A_n \)-form of the retraction

In this section we show the following theorem which is analogous to Theorem 1.2.

**Theorem 3.1.** Let \( X \) be an \( A_n \)-space for some \( n \geq 2 \). Then the retraction \( r : J(X) \to X \) is an \( A_{n-1} \)-map. Moreover, if \( r \) is an \( A_n \)-map then the \( A_n \)-space structure of \( X \) is extendable to an \( A_{n+1} \)-space structure.

**Proof.** The idea of the proof is not so hard to understand. But, writing down the explicit proof is very complicated.

Let \( \{ \mu_i : K_i \times X^i \to X \}_{1 \leq i \leq n} \) be the \( A_n \)-form on \( X \). The second part of the theorem is a corollary to Iwase-Mimura [11, P10]. They claim that if \( f : X \to Y \) and \( g : Y \to X \) are maps between \( A_n \)-spaces such that \( g \circ f \simeq id_X \), and if one of \( f \) and \( g \) is an \( A_n \)-map, then the \( A_n \)-space
structure of $X$ is extendable to an $A_{n+1}$-space structure. In fact, in our case the extended $A_{n+1}$-form on $X$ is given as follows.

If $\{R_i: J_i \times J(X)^i \to X\}_{i \leq n}$ is the $A_n$-form on $r$, then we get $n - 1$ higher homotopies

$$R_n \circ (1 \times \nu_1^s \times \nu_2 \times \nu_1^{n-s-1}): J_n \times X^{n+1} \to X \quad (1 \leq s \leq n - 1).$$

Then by combining these higher homotopies, we can construct a map $\mu_{n+1}: K_{n+1} \times X^{n+1} \to X$ which extend $\{\mu_i\}_{i \leq n}$ to an $A_{n+1}$-form on $X$.

Next we consider the first part of Theorem 3.1. An $A_{n-1}$-form $\{R_i: J_i \times J(X)^i \to X\}_{2 \leq i \leq n-1}$ is defined by means of compatible sequences of invariant maps $(R_{i,j}: J_i \times J(X)^{i-1} \times X^j \to X)_{j=1,2,...}$.

First we define $R_{2,1}$ as the constant homotopy. For $j \geq 2$, $R_{2,j}$ is given as the composition of $\mu_2 \circ (R_{2,j-1} \times id_X)$ and $\mu_3 \circ (1 \times r \times r \circ \nu_{j-1} \times id_X)$.

We don’t give the explicit definition for $R_{i,j}$ with $i \geq 3$ since it is almost the same as the case of $R_{2,j}$. But we just give the following remark. The homotopy $R_{i,1}$ for $i \geq 3$ can not be a constant homotopy. For example, $R_{3,1}: J_3 \times J(X)^2 \times X \to X$ should be a map illustrated in Figure 1 where the double lines mean constant homotopies. By definition, the homotopy $R_{2}(t, x, y \cdot \nu_1(z))$ is given as the composition of two homotopies $R_{2}(t, x, y) \cdot z$ and $\mu_3(t, r(x), r(y), z)$. Thus $R_{3,1}$ can be defined by using a suitable degeneracy map $\delta: J_3 \to J_2$ as $R_{3,1}(\tau, x, y, z) = R_{2}(\delta(\tau), x, y \cdot \nu_1(z)).$
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