CURVES HAVING ONE PLACE AT INFINITY AND LINEAR SYSTEMS ON RATIONAL SURFACES

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Abstract. Denoting by $L_d(m_0, m_1, \ldots, m_r)$ the linear system of plane curves passing through $r + 1$ generic points $p_0, p_1, \ldots, p_r$ of the projective plane with multiplicity $m_i$ (or larger) at each $p_i$, we prove the Harbourne-Hirschowitz Conjecture for linear systems $L_d(m_0, m_1, \ldots, m_r)$ determined by a wide family of systems of multiplicities $m = (m_i)_{i=0}^r$ and arbitrary degree $d$. Moreover, we provide an algorithm for computing a bound of the regularity of an arbitrary system $m$ and we give its exact value when $m$ is in the above family. To do that, we prove an $H^1$-vanishing theorem for line bundles on surfaces associated with some pencils "at infinity".

1. Introduction

This paper deals with the problem of computing the dimension of linear systems on smooth projective surfaces. The main result provides, for any arbitrary number of generic points in the projective plane over the field of complex numbers, $\mathbb{P}^2$, a wide family of systems of multiplicities for which the Harbourne-Hirschowitz Conjecture holds. Moreover, we show an algorithm for computing upper bounds of the regularity of a system of multiplicities $m = (m_0, m_1, \ldots, m_r)$.

Our proofs of these results are based on Section 3 where we give an $H^1$-vanishing theorem for line bundles on those surfaces $X$ obtained from $\mathbb{P}^2$ eliminating (by means of successive blowing-ups) the indeterminacies of the rational map $f : \mathbb{P}^2 \cdots \to \mathbb{P}^1$ given by certain pencils of plane curves. These are the pencils "at infinity" associated with rational projective curves of $\mathbb{P}^2$ that have one place at infinity and are smooth in their affine parts. The set formed by the centers of the blowing-ups used to obtain such a surface $X$ turns out to be a $P$-sufficient configuration (this type of configurations has been introduced and studied in [26], [27] and [28]). This fact, together with the simplicity and good properties of the effective semigroup of $X$, leads up to the above mentioned vanishing theorem. Then, semicontinuity arguments will allow to deduce our main result.

Fixing $r + 1$ points $p_0, p_1, \ldots, p_r$ of $\mathbb{P}^2$ in generic position and given $r + 1$ non-negative integers $m_0, m_1, \ldots, m_r$, the linear system $L_d(m_0, m_1, \ldots, m_r)$ of plane projective curves of fixed degree $d$ having multiplicity $m_i$ (or larger) at $p_i$ for each $i$, has an expected dimension (attained when all the conditions being imposed are independent). Those systems whose dimension is larger than the expected one are called special. The Harbourne-Hirschowitz Conjecture intends to give a description of all special linear systems. Basically, it asserts that a linear system is special if and only if it has a multiple fixed component such that its strict transform on the surface obtained by blowing-up the points $p_0, p_1, \ldots, p_r$ is a $(-1)$-curve (that is, an integral curve with self-intersection equal to $-1$ and genus zero). This conjecture goes back to B. Segre [55], being reformulated by several authors (see [32], [29], [41], [33], [16], [17], and [14] for a survey).

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Different approaches have been applied to obtain partial results on the Harbourne-Hirschowitz Conjecture. It has been proved for \( r + 1 \leq 9 \) points (Castelnuovo was the first to deal with these cases \[12\], although modern proofs are due to Nagata \[38\], Gimigliano \[24\] and Harbourne \[33\]). Arbarello and Cornalba \[5\] treated the homogeneous case with multiplicity 2 (that is, \( m_0 = m_1 = \ldots = m_r = 2 \)) using infinitesimal deformation theory, and Hirschowitz \[40\] proved the conjecture for the homogeneous case with multiplicity not greater than 3, by using a specialization technique (the so-called Horace method). This result has been generalized by Ciliberto and Miranda (\[16\] and \[17\]) applying a different degeneration technique, showing that the Harbourne-Hirschowitz Conjecture is true for the quasihomogeneous case \( m_1 = m_2 = \ldots = m_r \leq 3 \) and \( m_0 \) arbitrary, and for the homogeneous case with multiplicity \( m \) up to 12 (the cases \( 13 \leq m \leq 20 \) are treated in \[15\] with the same technique and the help of a computer program). Using a similar approach, Seibert \[50\] proved the conjecture for the quasihomogeneous case with \( m_1 = m_2 = \ldots = m_r = 4 \) and, recently, Laface \[44\] has done it for \( m_1 = m_2 = \ldots = m_r = 5 \). Other advances have been done by Mignon \[45\] after proving the conjecture when \( m_i \leq 4 \) for all \( i \), and Évain \[21\], who proves it for the homogeneous case when the number of points \( r + 1 \) is a power of 4. Also, using a refinement of the Horace method (the so-called differential Horace method), Alexander and Hirschowitz \[4\] obtained a bound \( d_0 = d_0(m) \) (only depending on \( m \)) such that, for any \( d \geq d_0 \) and any system of multiplicities \((m_0, m_1, \ldots, m_r)\) with \( m_i \leq m \) for all \( i \), the linear system \( L_d(m_0, m_1, \ldots, m_r) \) is non-special. A result which shows that the conjecture holds whenever there exist sufficiently many small multiplicities \( m_i \leq 4 \), at least one of them being 1, is the one recently proved by Bunke and Lossen in \[8\] by applying the differential Horace method. More recent advances on the subject are the papers of S. Yang \[58\], who proves the conjecture when \( m_i \leq 7 \) for all \( i \), and M. Dumnicki and W. Jarnicki \[18\], who do so when \( m_i \leq 11 \) for all \( i \); also, in \[19\] the conjecture is proved for the homogeneous cases with multiplicity bounded by 42.

Our contribution to the study of linear systems \( L_d(m_0, m_1, \ldots, m_r) \) is made in Section 4. Using the iterated blowing-ups (introduced by Kleiman in \[12\] and \[13\], and also studied in \[35\], \[31\] and \[24\]) and results developed in Section 3, we deduce a sufficient condition for the non-speciality of a linear system of that type (Theorem 2). As a consequence, we determine, for any arbitrary number of points \( r + 1 \geq 2 \), a wide family of systems of multiplicities \((m_0, m_1, \ldots, m_r)\) for which the special linear systems of the form \( L_d(m_0, m_1, \ldots, m_r) \) are completely characterized, proving that the Harbourne-Hirschowitz Conjecture is true for them. This result has the particularity of providing, for each arbitrary integer \( r \geq 1 \), a large set of systems of multiplicities \( \mathbf{m} = (m_0, m_1, \ldots, m_r) \) satisfying the Harbourne-Hirschowitz Conjecture for which the possible \( m_i \) are unbounded. Moreover, we also determine, for whichever \( \mathbf{m} \) in the above set, the least degree \( d \) such that these multiplicities impose independent conditions to curves of degree \( d \) (that is, the regularity of \( \mathbf{m} \)).

There are many results giving upper bounds of the regularity of a system of multiplicities (see \[30\], \[13\], \[11\], \[10\], \[7\], \[59\], \[31\], \[37\], \[38\], \[18\] or \[36\] for a survey). In Section 4.3 we introduce a generalization of the algorithm given in \[52\], based on our results in Section 3 providing bounds of the regularity which, in many cases, are better than the existing ones (as far as the author knows).

Every variety \( X \) in this article will be considered over the field of complex numbers \( \mathbb{C} \). Moreover, \( K_X \) will denote a canonical divisor on \( X \).

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2. Preliminaries

2.1. Configurations. In this section we summarize some concepts and notations that will be used throughout the paper. We start with the definition of configuration.

An ordered configuration over \( \mathbb{P}^2 \) (a configuration in the sequel) will be a finite sequence \( \mathcal{K} = (p_0, p_1, \ldots, p_n) \) of closed points such that \( p_0 \) belongs to \( X_0 := \mathbb{P}^2 \) and, inductively, if \( i \geq 1 \) then \( p_i \) belongs to the blowing-up \( X_i \) of \( X_{i-1} \) at \( p_{i-1} \). Among the points of \( \mathcal{K} \) there is a natural partial ordering: \( p_i \leq p_j \) whenever \( p_i = p_j \) or the composition of blowing-ups \( X_j \rightarrow X_i \) maps \( p_j \) to \( p_i \). We will say that \( \mathcal{K} \) is a chain configuration when \( \leq \) be a total ordering.

Denote by \( \pi_K : Z_K \rightarrow \mathbb{P}^2 \) the morphism given by the composition of all the successive blowing-ups centered at the points of \( \mathcal{K} \). Each blowing-up at \( p_i \) gives rise to an exceptional divisor \( E_i \) whose total (resp., strict) transform on \( Z_K \) will be denoted by \( E_i^K \) (resp., \( \tilde{E}_i^K \)). In the same way, for each effective divisor \( C \) on \( X \), \( C^K \) (resp., \( \tilde{C}^K \)) will be the total (resp., strict) transform of \( C \) on \( Z_K \). Also, for each divisor \( D \) on \( Z_K \), \( [D] \) will denote its class in \( \text{Pic}(Z_K) \). The system \( \{[L^K], [E_0^K], [E_1^K], \ldots, [E_n^K]\} \) is a \( \mathbb{Z} \)-basis of \( \text{Pic}(Z_K) \), \( L \) denoting a general line on \( \mathbb{P}^2 \).

A point \( p_i \in \mathcal{K} \) is said to be proximate to another point \( p_j \in \mathcal{K} \) (in short, \( i \rightarrow j \) or \( p_i \rightarrow p_j \)) if either \( i = j + 1 \) and \( p_i \) belongs to the exceptional divisor \( E_j \), or \( i > j + 1 \) and \( p_i \) belongs to the strict transform on \( X_i \) of the exceptional divisor \( E_j \). The point \( p_i \) is said to be a free point if it is proximate to, at most, one point of \( \mathcal{K} \); otherwise, \( p_i \) is said to be a satellite point. The proximity relation among the points of \( \mathcal{K} \) is an equivalent datum to a matrix \( P_K = (q_{ij})_{0 \leq i, j \leq n} \), called proximity matrix of \( \mathcal{K} \), and defined as follows: \( q_{ij} = 1 \) if \( i = j \), \( q_{ij} = -1 \) if \( p_i \) is proximate to \( p_j \), and \( q_{ij} = 0 \) otherwise. For each \( j = 0, 1, \ldots, n \), the entries of its \( j \)th column are the coefficients of the expression of the divisor \( \tilde{E}_j^K \) as linear combination of the divisors \( E_0^K, E_1^K, \ldots, E_n^K \). The proximity relations can also be represented by means of a combinatorial object, the proximity graph. It will be denoted by \( G(\mathcal{K}) \) and it is a labelled graph whose vertices represent the points of \( \mathcal{K} \) and whose edges join vertices associated with proximate points. Each vertex is labelled with the subindex \( i \) of its associated point \( p_i \). An edge joining \( p_j \) and \( p_i \) \((i > j)\) is a continuous straight line whenever \( p_i \) is a minimal point of \( \mathcal{K} \)(with respect to the ordering \( \leq \)) which is proximate to \( p_j \), and it is a dotted curved line otherwise (the label of an edge is determined by its property of being continuous-straight or curved-dotted). For the sake of simplicity, when we will depict a proximity graph, we will not draw those edges which can be deduced from others. Notice that the subgraph consisting of the vertices and the continuous edges has a forest structure whose trees are rooted on the vertices corresponding to those points in the configuration which lie in \( \mathbb{P}^2 \). A proximity graph will be called unibranch if it is associated to a chain configuration. The proximity graph is an equivalent datum either to the Enriques diagram or the dual graph of \( \mathcal{K} \).

By a system of multiplicities we mean a finite sequence \( (m_0, m_1, \ldots, m_n) \) of non-negative integers. A weighted configuration (resp., weighted proximity graph) will be a pair \( (\mathcal{K}, \mathbf{m}) \) (resp., \( (G(\mathcal{K}), \mathbf{m}) \)), where \( \mathcal{K} = (p_i)_{i=0}^n \) is a configuration and \( \mathbf{m} = (m_0, m_1, \ldots, m_n) \) is a system of multiplicities. It can be seen as a map that assigns, to each point \( p_j \) of \( \mathcal{K} \)(resp., to the corresponding vertex of \( G(\mathcal{K}) \)), the non-negative integer (multiplicity) \( m_j \). The excesses of the weighted configuration \( (\mathcal{K}, \mathbf{m}) \) are defined to be the integers \( \rho_j(\mathcal{K}, \mathbf{m}) := m_j - \sum_{k=j}^n m_k, \ 0 \leq j \leq n \). Since the excesses only depend on the proximity relations among the points of the configuration, we can define the excesses of a given weighted proximity graph \( (G, \mathbf{m}) \) as those associated with every weighted configuration.
(\mathcal{K}, m) such that \mathcal{G}(\mathcal{K}) = \mathcal{G}; they will be denoted by \rho_j(\mathcal{G}, m). Similarly, the *proximity matrix associated with a proximity graph* \mathcal{G}, which will be denoted \mathbf{P}_\mathcal{G}, can be defined in an obvious way. If \mathcal{K} is a configuration of \(n + 1\) points and \(v = (v_0, v_1, \ldots, v_t)\) is a system of multiplicity such that \(t < n\), the pair \((\mathcal{K}, v)\) (resp., \((\mathcal{G}(\mathcal{K}), v)\)) will also be considered a weighted configuration (resp., weighted proximity graph), identifying \(v\) with the sequence of multiplicities of length \(n + 1\) obtained adding \(n - t\) zero components to \(v\), that is, \((v_0, v_1, \ldots, v_t, 0, 0, \ldots, 0)\).

2.2. **P-sufficient configurations.** Consider a configuration \(\mathcal{K} = (p_0, p_1, \ldots, p_n)\) and take the notations of Section 2.1. Denote by \((b_{ij})_{0 \leq i, j \leq n}\) the entries of the matrix \(\mathbf{P}^{-1}_\mathcal{K}\), whose columns contain the coefficients of the expressions of each divisor \(E_j^\mathcal{K}\) as linear combinations of the divisors \(\tilde{E}_0^\mathcal{K}, \tilde{E}_1^\mathcal{K}, \ldots, \tilde{E}_n^\mathcal{K}\). For each integer \(i\) such that \(0 \leq i \leq n\) we consider the divisor on \(Z_\mathcal{K}\) defined by \(D_i := \sum_{j=0}^i b_{ij} E_j^\mathcal{K}\), which has the following property: \(D_i \cdot E_j^\mathcal{K}\) equals \(-1\) if \(i = j\) and 0 otherwise. We define the square symmetric matrix \(G_\mathcal{K} = (g_{ij})_{0 \leq i, j \leq n}\) by

\[
g_{ij} = -9D_i \cdot D_j - (K_{Z_\mathcal{K}} \cdot D_i)(K_{Z_\mathcal{K}} \cdot D_j).
\]

Given an element \(x \in \mathbb{R}^{n+1}\), we set \(x > 0\) when all the coordinates of \(x\) are non-negative and at least one of them is positive. Recall \cite{25} that an \((n+1)\)-dimensional square symmetric matrix \(A\) is called to be *conditionally positive definite* if \(x A x^T > 0\) for all vector \(x \in \mathbb{R}^{n+1}\) such that \(x > 0\).

**Definition 1.** A configuration \(\mathcal{K}\) is called to be *P-sufficient* if the matrix \(G_\mathcal{K}\) is conditionally positive definite.

This type of configurations has been recently introduced in \cite{26} and \cite{27} and, in them, it is proved that the cone of curves of \(Z_\mathcal{K}\) is (finite) polyhedral whenever \(\mathcal{K}\) is a P-sufficient configuration. Recall that the *cone of curves* of a projective surface \(X\), which we will denote by \(NE(X)_\mathbb{R}\), is the convex cone of the real vector space \(\text{Pic}(X) \otimes_\mathbb{Z} \mathbb{R}\) spanned by the classes of the effective divisors on \(X\).

When \(\mathcal{K}\) is a chain configuration, checking whether it is P-sufficient or not is equivalent to checking a single condition \cite{27} Cor. 2: \(\mathcal{K}\) is P-sufficient if and only if \(-9D_n^2 - (K_{Z_\mathcal{K}} \cdot D_n)^2 > 0\).

The following result, whose proof can be found in \cite{28}, provides a property of the surfaces obtained from P-sufficient configurations which will be useful in Section 3.

**Proposition 1.** Let \(\mathcal{K} = (p_0, p_1, \ldots, p_n)\) be a P-sufficient configuration and \(D\) an effective divisor on \(Z_\mathcal{K}\) such that \(D_i^2 \geq 0\) and \(D \cdot \tilde{E}_i^\mathcal{K} \geq 0\) for all \(i = 0, 1, \ldots, n\). Then, \(K_{Z_\mathcal{K}} \cdot D < 0\).

2.3. **Plane curves having one place at infinity.** With the exception of Proposition 2 and Corollary 1 this section is expository and its aim is to summarize some facts related to plane curves having one place at infinity. This type of curves has been extensively studied by several authors (see, for instance, \cite{2, 17, 11, 54, 19, 27} or \cite{24}).

**Definition 2.** A projective curve \(C \hookrightarrow \mathbb{P}^2\) (which we will assume that is not a line) is said to have one place along a line \(H \hookrightarrow \mathbb{P}^2\) if the intersection \(C \cap H\) is a single point \(p\) and \(C\) is reduced and has only one analytic branch at \(p\). If \(H\) is viewed as the line of infinity in the compactification of the affine plane to \(\mathbb{P}^2\), we say that \(C\) has one place at infinity.

Throughout this section, we fix a projective curve \(C\) having one place at infinity, \(p\) being the intersection point of \(C\) with the line of infinity \(H\). Consider the infinite sequence of
morphism
\[ \cdots \to X_{i+1} \to X_i \to \cdots \to X_1 \to X_0 := \mathbb{P}^2, \]
where \( X_1 \to X_0 = \mathbb{P}^2 \) is the blowing-up of \( \mathbb{P}^2 \) at \( p_0 := p \) and, for each \( i \geq 1 \), \( X_{i+1} \to X_i \)
denotes the blowing-up of \( X_i \) at the unique point \( p_i \) which lies on the strict transform of \( C \) and on the exceptional divisor \( E_{i-1} \) created by the preceding blowing-up.

2.3.1. \( \delta \)-sequences. The unique branch at infinity of \( C \) corresponds to a normalized discrete valuation \( v \) of the field of rational functions of \( C \) over \( \mathbb{C} \). We define the semigroup at infinity (resp., Weierstrass semigroup) associated with \( C \), and we denote it by \( \Gamma_C \) (resp., \( S_C \)), as the subsemigroup of \( \mathbb{N} \) generated by all the integers of the form \(-v(g)\), where \( g \) belongs to the affine \( \mathbb{C} \)-algebra \( \mathcal{O}_C(C \setminus \{p\}) \) (resp., to the normalization of \( \mathcal{O}_C(C \setminus \{p\}) \)). Obviously, \( \Gamma_C \) is contained in \( S_C \) and they are equal if and only if \( C \setminus \{p\} \) is a smooth affine curve. Abhyankar and Moh proved in [2] the existence of a positive integer \( s \) and a sequence of positive generators \( \delta_0, \delta_1, \ldots, \delta_s \) of \( \Gamma_C \) such that:

I. If \( d_i = \gcd(\delta_0, \delta_1, \ldots, \delta_{i-1}) \), for \( 1 \leq i \leq s + 1 \) and \( n_i = d_i/d_{i+1}, 1 \leq i \leq s \), then \( d_{i+1} = 1 \) and \( n_i > 1 \) for \( 1 \leq i \leq s \).

II. For \( 1 \leq i \leq s \), \( n_i \delta_i \) belongs to the semigroup generated by \( \delta_0, \delta_1, \ldots, \delta_{i-1} \).

III. \( \delta_0 > \delta_1 \) and \( \delta_i < \delta_{i-1} n_{i-1} \) for \( i = 2, 3, \ldots, s \).

The sequence \( \{\delta_i\}_{i=0}^s \) can be obtained from an equation of the curve \( C \) using approximate roots [2 Chapter II, Sections 6,7]. We will refer to it as a \( \delta \)-sequence associated with \( C \).

Moreover, it turns out that any sequence \( (\delta_0, \delta_1, \ldots, \delta_s) \) satisfying the above conditions (I), (II) and (III) is a \( \delta \)-sequence associated with some curve having one place at infinity, which can be chosen of degree \( \delta_0 \) (see, for instance, [54] or [49]).

Associated with the branch at infinity of a curve having one place at infinity, there is a sequence of Newton polygons \( P_0, P_1, \ldots, P_{g-1} \) which determines the equisingularity class of that branch [9 3.4]. Assume that each \( P_i \) is the segment which joins the points \((0, e_i)\) and \((m_i, 0), e_i, m_i \in \mathbb{N} \). These Newton polygons can be explicitly recovered from a \( \delta \)-sequence \( (\delta_0, \delta_1, \ldots, \delta_s) \) associated with the curve:

If \( \delta_0 - \delta_1 \) does not divide \( \delta_0 \), then \( s = g \) and

\[
\begin{align*}
e_0 &= \delta_0 - \delta_1, & e_i &= d_{i+1} \\
m_0 &= \delta_0, & m_i &= n_i \delta_i - \delta_{i+1}
\end{align*}
\]

for \( 1 \leq i \leq s - 1 \). Otherwise, \( s = g + 1 \) and

\[
\begin{align*}
e_0 &= d_2 - \delta_0 - \delta_1, & e_i &= d_{i+2} \\
m_0 &= \delta_0 + m_1 \delta_1 - \delta_2, & m_i &= n_{i+1} \delta_{i+1} - \delta_{i+2}
\end{align*}
\]

for \( 1 \leq i \leq s - 2 \).

The above equalities are considered and used in [50] and the proximity relations among the infinitely near points \( p_0, p_1, \ldots \) can be easily deduced from the \( \delta \)-sequence, as we will describe next (see [9] for complete details):

Define \( s_0 = k_0 = 0 \) and let \( h_i, k_t \) and \( s_t \) (with \( 0 \leq i \leq s_g - 1 \) and \( 1 \leq t \leq g \)) be the positive integers obtained from the following continued fractions:

\[
m_j-1 \quad e_j-1 + k_{j-1} = h_{s_{j-1}} + \frac{1}{h_{s_{j-1}+1} + \cdots + \frac{1}{n_j}},
\]

for \( j = 1, 2, \ldots, g \). Also, for each \( n \in \{1, 2, \ldots, s_g\} \), define \( f(n) := k_t - 1 \) whenever \( n = s_t \) for some \( t \in \{1, 2, \ldots, g\} \), and \( f(n) := h_n \) otherwise. Then, the proximity relations are
the following: \( l \to l - 1 \) for each positive integer \( l \), and \( l \to \sum_{i=0}^{n-1} h_i - 1 \) for each pair \((n, l)\) such that \( 1 \leq n \leq s_g \) and \( \sum_{i=0}^{n-1} h_i < l \leq \sum_{i=0}^{n-1} h_i + f(n) \).

Thus, a \( \delta \)-sequence associated with a plane curve \( C \) having one place at infinity determines the equisingularity class of the branch of \( C \) at \( p \) and, therefore, the proximity graph of whichever configuration of the form \((p_0, p_1, \ldots, p_l)\) with \( l \in \mathbb{N} \) (in particular, that of the minimal embedded resolution of the branch).

2.3.2. Curves of Abhyankar-Moh-Suzuki type. In this paper we are mainly interested in a certain class of curves having one place at infinity: the so-called curves of Abhyankar-Moh-Suzuki type, which we define next.

**Definition 3.** A plane curve \( C \) having one place at infinity is said to be of Abhyankar-Moh-Suzuki type (AMS type for short) if it is rational and smooth in its affine part, that is, \( C \setminus H \) is isomorphic to \( \mathbb{C} \), \( H \) being the line of infinity.

Let \( H \) be the line of infinity in \( \mathbb{P}^2 \) and identify \( \mathbb{C}^2 \) with \( \mathbb{P}^2 \setminus H \). Recall that, by [3], a curve \( C \) is of AMS type if and only if it is the compactification in \( \mathbb{P}^2 \) of the zero locus of a component of a certain automorphism \( \phi : \mathbb{C}^2 \to \mathbb{C}^2 \). The embedding of \( \mathbb{C}^2 \) in \( \mathbb{P}^2 \) allows to extend \( \phi \) to a birational transformation \( \tilde{\phi} : \mathbb{P}^2 \to \mathbb{P}^2 \). The minimal embedded resolution of the singularity of \( C \) is closely related to the minimal resolution of the indeterminacy of \( \tilde{\phi} \), and the combinatorics of the last one can be described precisely, as we will show next. For details see [22] or [23].

First, we will define an associative operation \( \uparrow \) in the set of unibranched proximity graphs with two or more vertices (see Figure 1 for an example).

Let \( F_1 \) and \( F_2 \) be two proximity graphs of this type and assume that \( V_1 = \{v_0, v_1, \ldots, v_n\} \) (resp., \( V_2 = \{w_0, w_1, \ldots, w_m\} \)) is the set of vertices of \( F_1 \) (resp., \( F_2 \)) where, if \( \leq \) denotes the ordering induced in \( V_1 \) (resp., \( V_2 \)) by the natural ordering among the points of a configuration whose proximity graph is \( F_1 \) (resp., \( F_2 \)), it holds that \( v_0 < v_1 \ldots < v_n \) (resp., \( w_0 < w_1 \ldots < w_m \)). The graph \( F_1 \uparrow F_2 \) is the unibranched proximity graph such that:

- its set of vertices is \( V_{F_1} \cup V_{F_2} \);
- its set of edges is \( A \cup \{e_1, e_2\} \), where \( A \) is the union of the sets of edges of \( F_1 \) and \( F_2 \) and \( e_1, e_2 \) are two new edges such that \( e_1 \) is a continuous straight line joining \( v_n \) and \( w_0 \), and \( e_2 \) is a curved dotted line joining \( v_n \) and \( w_1 \);
- the vertex \( v_i \) (resp., \( w_i \)) is labelled with \( i \) (resp., \( n + i + 1 \)) for each \( i \) such that \( 0 \leq i \leq n \) (resp., \( 0 \leq i \leq m \)).

![Figure 1. Operation \( \uparrow \)](image_url)
For each integer $n \geq 2$ consider a chain configuration $(p_0, p_1, \ldots, p_{2n-2})$ such that $p_i$ is proximate to $p_0$ for all $i$ such that $1 \leq i \leq n-1$ and the remaining points $p_n, p_{n+1} \ldots p_{2n-2}$ are free. Define $G(n)$ to be the proximity graph of this configuration (see Figure 2).

![Figure 2. Proximity graph $G(n)$](image)

For each finite ordered sequence $(n_1, n_2, \ldots, n_r)$ of integers such that $r \geq 1$ and $n_i \geq 2$ for all $i$, we define a proximity graph, depending only on that sequence, by using the above considered associative operation:

$$G(n_1, n_2, \ldots, n_r) := G(n_1) \uparrow G(n_2) \uparrow \cdots \uparrow G(n_r).$$

Also, we denote by $G(n_1, n_2, \ldots, n_r)^-$ (resp., $G(n_1, n_2, \ldots, n_r)^+$) the proximity graph obtained from $G(n_1, n_2, \ldots, n_r)$ by deleting (resp., adding) the last $n_r-1$ vertices and the edges which are adjacent to them (resp., a new vertex with label $2\sum_{i=1}^{r} n_i - r$ and a new edge joining it with the vertex with label $2\sum_{i=1}^{r} n_i - r - 1$).

Now consider, as above, a curve $C$ of AMS type and an affine automorphism $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $C$ is the zero locus of a component of it. Let $\pi : X \to \mathbb{P}^2$ be the minimal resolution of the indeterminacy of $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ and let $\mathcal{K}$ be the configuration of centers of the blowing-ups involved in $\pi$. Then, there exists a sequence of integers $(n_1, n_2, \ldots, n_r)$ (with $n_i \geq 2$ for all $i$) such that $G(\mathcal{K}) = G(n_1, n_2, \ldots, n_r)$. Moreover, the strict transform on $X$ of the line of infinity, $\tilde{H}^\mathcal{K}$, is a $(-1)$-curve, that is, a smooth rational curve with self-intersection $-1$.

If $\mathcal{C}$ is the configuration such that $\pi_\mathcal{C} : X_\mathcal{C} \to \mathbb{P}^2$ induces the minimal embedded resolution of the singularity of $C$ at infinity, there are two possibilities: either $\pi_\mathcal{C}$ is the composition of all the blowing-ups of $\pi$ except the last $n_r-1$ of them (in this case, $G(\mathcal{C}) = G(n_1, n_2, \ldots, n_r)^-$), or it is the composition of the first $2\sum_{i=1}^{r-2} n_i + n_{r-1} - r + 2$ blowing-ups of $\pi$ (in this case, $G(\mathcal{C}) = G(n_1, n_2, \ldots, n_{r-1})^-$). The following proposition shows that, for each proximity graph of the form $G(n_1, n_2, \ldots, n_r)^-$, there exists a curve of AMS type such that the proximity graph associated to its minimal embedded resolution is this one. Then, the proximity graphs associated to minimal resolutions of curves of AMS type are exactly those of the form $G(n_1, n_2, \ldots, n_r)^-$.

**Proposition 2.** Let $(n_1, n_2, \ldots, n_r)$ be an ordered sequence of integers such that $n_i \geq 2$ for all $i = 1, 2, \ldots, r$. Then, there exists a curve $C$ of AMS type such that its degree is $n_1 n_2 \cdots n_r$ and the proximity graph associated with its minimal embedded resolution is $G(n_1, n_2, \ldots, n_r)^-$. 


Proof. Define the integers \( \delta_k = n_{k+1}n_{k+2} \cdots n_r \) for \( k = 0, 1, \ldots, r-1 \) and \( \delta_r = 1 \). It is obvious that the sequence \( (\delta_0, \delta_1, \ldots, \delta_r) \) satisfies the conditions (I), (II) and (III) which characterize the \( \delta \)-sequences and, therefore, there exists a curve \( C \) of degree \( \delta_0 = n_1n_2 \cdots n_r \) having one place at infinity with associated \( \delta \)-sequence \( (\delta_0, \delta_1, \ldots, \delta_r) \). From this sequence one can compute, using the formulae given in Section 2.3.1, the proximity relations among the points of the configuration which provides the minimal embedded resolution of the singularity of \( C \) and check that the proximity graph associated with this configuration is \( \mathcal{G}(n_1, n_2, \ldots, n_r)^- \). Finally, since \( \delta_r = 1 \), the Weierstrass semigroup of \( C \) and its semigroup at infinity are both equal to \( \mathbb{N} \) and, therefore, \( C \) is rational and smooth in its affine part. \( \square \)

A direct consequence of the above proposition and the genus formula is the following

**Corollary 1.** Let \( C \) be a curve of AMS type and \( n_1, n_2, \ldots, n_r \geq 2 \) integers such that the proximity graph associated with its minimal embedded resolution is \( \mathcal{G}(n_1, n_2, \ldots, n_r)^- \). Then, the degree of \( C \) is \( n_1n_2 \cdots n_r \).

### 3. Surfaces associated with pencils “at infinity”

Let \( C \) be a projective curve of \( \mathbb{P}^2 \) having one place at infinity and consider the notations of Section 2. Take projective coordinates \((X : Y : Z)\) on \( \mathbb{P}^2 \) such that \( Z = 0 \) be the equation of the line of infinity \( \mathcal{H} \) and let \( F(X, Y, Z) \) be an homogeneous polynomial in \( k[X, Y, Z] \) such that \( F(X, Y, Z) = 0 \) is an equation of \( C \). The pencil “at infinity” associated with \( C \), which we will denote by \( \mathcal{P}(C) \), will be the linear subspace of \( H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \) spanned by \( F \) and \( Z^d, d \) being the degree of \( F \). Let \( n \) be the smallest integer such that the composition of morphisms \( X_{n+1} \to X_n \to \cdots \to X_0 = \mathbb{P}^2 \) eliminates the indeterminacies of the rational map \( \mathbb{P}^2 \to \mathbb{P}^1 \) defined by \( \mathcal{P}(C) \). We will denote by \( \mathcal{K}_C \) the chain configuration \((p_0, p_1, \ldots, p_n)\) and by \( X_C \) the surface \( Z_{\mathcal{K}_C} = X_{n+1} \). It turns out that all the curves of \( \mathcal{P}(C) \), except the non-reduced one with equation \( Z^d = 0 \), are integral curves having one place at infinity, and the above morphism \( X_C \to \mathbb{P}^2 \) induces a simultaneous embedded resolution of all of them (see \( \ref{37} \)).

The objective of this section is to give a vanishing theorem for line bundles on the surface \( X_C \), when \( C \) is a curve of AMS type. This result will allow to determine the dimension of whichever complete linear system on \( X_C \).

The following proposition provides two characterizations of the curves of AMS type depending on the associated configuration \( \mathcal{K}_C \).

**Proposition 3.** Let \( C \) be a curve having one place at infinity. Then, the following conditions are equivalent:

(a) The configuration \( \mathcal{K}_C \) is \( P \)-sufficient.

(b) \( K_{X_C} \cdot \mathcal{C}^{\mathcal{K}_C} < 0 \).

(c) \( C \) is a curve of AMS type.

**Proof.** First, observe that the class of \( \mathcal{C}^{\mathcal{K}_C} \) in \( \text{Pic}(X_C) \) coincides with \([dI_K^C - D_n], d \) being the degree of \( C \) and \( D_n \) the divisor defined in Section 2.22 (associated with the configuration \( \mathcal{K}_C \)). The reason is that \( D_n = \sum_{i=0}^n u_i E_i^K \), \( u_i \) being the multiplicity of the strict transform of \( C \) at the point \( p_i, 0 \leq i \leq n \), because \( D_n \cdot E_i^K \) equals \(-1\) if \( i = n \) and \( 0 \) otherwise.

The equivalence between (a) and (b) is consequence of the equalities

\[
-9D_n^2 - (K_{X_C} \cdot D_n)^2 = 9d^2 - \left( \sum_{i=0}^n u_i \right)^2 = \left( 3d + \sum_{i=0}^n u_i \right) (-K_{X_C} \cdot \mathcal{C}^{\mathcal{K}_C}),
\]
Definition 4. For each curve $C$ having one place at infinity, the effective (resp., nef) semigroup of $X_C$, denoted by $NE(X_C)$ (resp., $P(X_C)$), is defined as the subsemigroup of $\text{Pic}(X_C)$ generated by the classes of all effective (resp., numerically effective) divisors on $X_C$.

Campillo, Piltant and Reguera described, in [10], the effective semigroup of $X_C$. They proved the following equality:

$$NE(X_C) = \mathbb{N}[\tilde{H}^K_C] \oplus \bigoplus_{i=0}^{n} \mathbb{N}[	ilde{E}^{KC}_i],$$

where $H$ denotes, as above, the line of infinity. Moreover, as a consequence of [10] Cor. 7 and Prop. 6 and Corollary 2 we have the following result:
Proposition 5. If $C$ is a curve of AMS type, then $P(X_C)$ coincides with the semigroup of classes in $\text{Pic}(X_C)$ of the form $[\tilde{D}^{K_C}]$, where $D \rightarrow \mathbb{P}^2$ is a projective curve whose support does not contain the line of infinity.

Now, we will state and prove the announced $H^1$-vanishing result for line bundles on $X_C$.

Theorem 1. Let $C$ be a curve of AMS type.

(a) If $D$ is a numerically effective divisor on $X_C$, then $h^1(X_C, \mathcal{O}_{X_C}(D)) = 0$.

(b) Let $D$ be an effective divisor on $X_C$ such that $D \cdot \tilde{E}_i^{K_C} \geq 0$ for all $i = 0, 1, \ldots, n$ and $D \cdot \tilde{E}_1^{K_C} \geq 1$. Then, $h^1(X_C, \mathcal{O}_{X_C}(D)) = 0$ if and only if $D \cdot \tilde{H}^{K_C} \geq -1$.

Proof. Firstly, notice that the configuration $K_C$ is $P$-sufficient, by Proposition 8. In order to prove (a), we will reason by contradiction. So, we assume the existence of a numerically effective divisor $D$ such that $h^1(X_C, \mathcal{O}_{X_C}(D)) > 0$. Since $[D]$ is an effective class (by Proposition 5), we can apply Proposition 11 and Lem. II.7, deducing that the complete linear system $|D|$ has fixed part. Moreover, again by Proposition 5, this fixed part has not exceptional components (that is, it has no divisor $\tilde{E}_i^{K_C}$ as a component).

Now, the integral fixed components of $|D|$ have negative self-intersection. Indeed, if we assume the existence of an integral fixed component $R$ such that $R^2 \geq 0$, then $K_{X_C} \cdot R < 0$ (by Proposition 11) and, applying the Riemann-Roch Formula, we get

$$h^0(X_C, \mathcal{O}_{X_C}(R)) \geq 1 + (R^2 - K_{X_C} \cdot R)/2 \geq 2,$$

which is false, since $h^0(X_C, \mathcal{O}_{X_C}(R)) = 1$.

Finally, Proposition 8 provides a contradiction, because the unique non-exceptional integral curve on $X_C$ with negative self-intersection is $\tilde{H}^{K_C}$.

To prove (b), we consider a divisor $D$ satisfying the hypotheses. First, we will assume the inequality $h^1(X_C, \mathcal{O}_{X_C}(D)) > 0$ and we will show that this implies $D \cdot \tilde{H}^{K_C} \leq -2$.

By applying (a) we have $D \cdot \tilde{H}^{K_C} \leq -1$ and, therefore, it only remains to prove that the inequality $D \cdot \tilde{H}^{K_C} = -1$ leads to a contradiction. Using the hypotheses and Remark 11, it can be deduced that $D - \tilde{H}^{K_C}$ is a numerically effective divisor. A similar reasoning to that given in the proof of (a) shows that $\tilde{H}^{K_C}$ is the unique possible integral fixed component of the linear system $|D - \tilde{H}^{K_C}|$ and, then, it must be fixed part free by Proposition 5. So, we have a decomposition $|D| = [\tilde{H}^{K_C}] + [T]$, where $T$ is an effective divisor such that $|T|$ is fixed part free. Applying Part (a) to the divisor $T$, we deduce that $h^1(X_C, \mathcal{O}_{X_C}(T)) = 0$. Taking into account this fact, Riemann-Roch Theorem and the equality $h^0(X_C, \mathcal{O}_{X_C}(D)) = h^0(X_C, \mathcal{O}_{X_C}(T))$, the following chain of equalities and inequalities holds:

$$0 < h^1(X_C, \mathcal{O}_{X_C}(D)) = h^0(X_C, \mathcal{O}_{X_C}(D)) - 1 - \frac{1}{2}(D^2 - K_{X_C} \cdot D) =$$

$$= h^0(X_C, \mathcal{O}_{X_C}(T)) - 1 - \frac{1}{2}(D^2 - K_{X_C} \cdot D) =$$

$$= 1 + \frac{1}{2}(T^2 - K_{X_C} \cdot T) - 1 - \frac{1}{2}(D^2 - K_{X_C} \cdot D) =$$

$$= \frac{1}{2}(-\tilde{H}^{K_C})^2 - 2\tilde{H}^{K_C} \cdot T + K_{X_C} \cdot \tilde{H}^{K_C}).$$

Hence, $(\tilde{H}^{K_C})^2 - K_{X_C} \cdot \tilde{H}^{K_C} < -2\tilde{H}^{K_C} \cdot T \leq 0$. But this is a contradiction, since $\tilde{H}^{K_C}$ is a $(-1)$-curve.
It only remains to prove that, if \( D \cdot H^{K_C} \leq -2 \), then \( h^1(X_C, \mathcal{O}_{X_C}(D)) > 0 \). But the inequality \( D \cdot H^{K_C} \leq -2 \) implies that the \((-1\)-curve \( H^{K_C} \) is a multiple fixed component of the linear system \([D]\), and it is easy to see that this fact implies that \( h^1(X_C, \mathcal{O}_{X_C}(D)) \) is positive (by a similar reasoning to that given in [46 pag. 197]). □

**Remark 2.** With the hypotheses of Theorem 4.1 Part (a) allows us to determine the dimension of all complete linear systems on \( X_C \). Indeed, let \( D \) be a divisor on \( X_C \) and consider the set \( S = \{[H^{K_C}], [E_0^{K_C}], [E_1^{K_C}], \ldots, [E_n^{K_C}]\} \subseteq \text{Pic}(X_C) \). If for some \( F \in S \) we have \( D \cdot F < 0 \), then it is obvious that \( h^0(X_C, \mathcal{O}_{X_C}(D)) = h^0(X_C, \mathcal{O}_{X_C}(D - F)) \). Therefore, we can perform the following process: check \( D \cdot F \) for each \( F \in S \), replace \( D \) by \( D - F \) whenever \( D \cdot F < 0 \) and continue with the new \( D \). The process ends when it gives rise to a divisor \( D' \) such that either it is obviously not effective (because either \( D' \cdot L^{K_C} < 0 \) or \( D' \cdot (L^{K_C} - E_1^{K_C}) < 0 \)) or \( D' \) is numerically effective. Since \( h^0(X_C, \mathcal{O}_{X_C}(D)) = h^0(X_C, \mathcal{O}_{X_C}(D')) \), in the first case the linear system \([D]\) is empty and in the second case its dimension is \((D'^2 - K_{X_C} \cdot D')/2\), by Part (a) of Theorem 4.1.

4. LINEAR SYSTEMS OF CURVES THROUGH GENERIC POINTS ON \( \mathbb{P}^2 \)

In this section we will use Theorem 4.1 and a specialization process to deduce some results about the dimension of linear systems of curves passing through a finite set of points of the plane in generic position.

**4.1. Special linear systems and the Harbourne-Hirschowitz Conjecture.** Given a projective curve \( C \) of \( \mathbb{P}^2 \), we will say that \( C \) goes through a weighted configuration \((\mathcal{K} = (p_i)_{i=0}^n, m = (m_i)_{i=0}^n)\) if and only if the divisor \( \mathcal{C}^{\mathcal{K}} - \sum_{i=0}^n m_i E_i^{\mathcal{K}} \) is effective. For any degree \( d \), denote by \( \mathcal{L}_d(\mathcal{K}, m) \) the set of projective curves on \( \mathbb{P}^2 \) of degree \( d \) going through \((\mathcal{K}, m)\). This is a linear system of \( \mathbb{P}^2 \) that is projectively isomorphic to the complete linear system \([D_{d,\mathcal{K}, m}]\) of \( Z_{\mathcal{K}}, D_{d,\mathcal{K}, m} \) being the divisor \( dL^{\mathcal{K}} - \sum_{i=0}^n m_i E_i^{\mathcal{K}} \). From Riemann-Roch Theorem one gets

\[
\dim \mathcal{L}_d(\mathcal{K}, m) = h^1(\mathcal{L}_d(\mathcal{K}, m)) = \frac{d(d + 3)}{2} - \sum_{i=0}^n \frac{m_i(m_i + 1)}{2},
\]

where \( \dim \mathcal{L}_d(\mathcal{K}, m) \) is the dimension of \( \mathcal{L}_d(\mathcal{K}, m) \) as projective space and \( h^1(\mathcal{L}_d(\mathcal{K}, m)) := h^1(Z_{\mathcal{K}}, \mathcal{O}_{Z_{\mathcal{K}}}(D_{d,\mathcal{K}, m})) \) will be called the superabundance of \( \mathcal{L}_d(\mathcal{K}, m) \). The independence of the linear conditions imposed by the weighted configuration \((\mathcal{K}, m)\) is equivalent to the vanishing of this superabundance.

If \( \mathcal{K} \) is a configuration whose points lie all in \( \mathbb{P}^2 \), the dimension and the superabundance of a linear system \( \mathcal{L}_d(\mathcal{K}, m) \) depend on the position of the points of \( \mathcal{K} \), and they reach their minimal values for a generic set of points. We will denote by \( \mathcal{K}_0(n) \) a configuration consisting of \( n + 1 \) closed points of \( \mathbb{P}^2 \) in generic position. For each integer \( d \geq 1 \) and for each system of multiplicities \( m = (m_0, m_1, \ldots, m_n) \) we will denote by \( \mathcal{L}_d(m) \) the linear system \( \mathcal{L}_d(\mathcal{K}_0(n), m) \). Also, we define the expected dimension of \( \mathcal{L}_d(m) \) to be the following number:

\[
\text{edim} \mathcal{L}_d(m) := \max \left\{ \frac{d(d + 3)}{2} - \sum_{i=0}^n \frac{m_i(m_i + 1)}{2}, -1 \right\}.
\]

**Definition 5.** We will say that a linear system \( \mathcal{L}_d(m) \) is special if and only if \( \text{edim} \mathcal{L}_d(m) > \text{edim} \mathcal{L}_d(m) \), that is, \( \mathcal{L}_d(m) \) is non-empty and the superabundance \( h^1(\mathcal{L}_d(m)) \) is positive.
Given a positive integer \(d\) and a system of multiplicities \(m = (m_i)_{i=0}^{n}\), it is easy to prove that, if there exists a curve \(C\) on \(\mathbb{P}^2\) such that its strict transform on \(Z_{K_0(n)}\) is a \((-1)\)-curve and \(D_{d,K_0(n),m} \cdot C_{K_0(n)} \leq -2\), then the linear system \(L_d(m)\) is special (see, for instance, [46, pag. 197]). One of the equivalent statements of the Harbourne-Hirschowitz Conjecture is just the converse assertion:

**Conjecture.** (Harbourne-Hirschowitz) If a linear system \(L_d(m)\) is special, then there exists a curve \(C\) on \(\mathbb{P}^2\) such that its strict transform on \(Z_{K_0(n)}\) is a \((-1)\)-curve and \(D_{d,K_0(n),m} \cdot C_{K_0(n)} \leq -2\).

### 4.2. A non-speciality result and some consequences.

For each positive integer \(n\), there exists a variety \(Y_n\) whose points are naturally identified with the configurations over \(\mathbb{P}^2\) with \(n+1\) points. These varieties, known as *iterated blowing-ups*, were introduced by Kleiman in [42] and [43] and they have also been treated in [35], [51] and [28] (see also [52]). There is a family of projective morphisms \(Y_{n+1} \to Y_n\) and relative divisors \(F_1, F_0, F_1, \ldots, F_n\) on \(Y_{n+1}\) such that the fiber over a given configuration \(K = (p_0, p_1, \ldots, p_n)\) (viewed as a point of \(Y_n\)) is isomorphic to the surface \(Z_K\) obtained by blowing-up the points in \(K\) and, if \(i \geq 0\) (resp., \(i = -1\)), the restriction of \(F_i\) to this fiber corresponds to the total transform \(E^0_i\) of the exceptional divisor appearing in the blowing-up centered at \(p_i\) (resp., the total transform of a general line of \(\mathbb{P}^2\)).

For each positive integer \(d\) and for each sequence of multiplicities \(m = (m_0, m_1, \ldots, m_n)\) we apply the Semicontinuity Theorem [39, III, 12.8] to the invertible sheaf \(O_{Y_{n+1}}(dF_1 - m_0 F_0 - m_1 F_1 - \ldots - m_n F_n)\), obtaining that the functions \(Y_n \to \mathbb{Z}\) given by

\[
K \mapsto h^i(Z_K, O_{Z_K}(D_{d,K,m})),
\]

for \(i \in \{0, 1\}\), are upper-semicontinuous.

For each proximity graph \(G\) with \(n+1\) vertices, we define \(U(G)\) as the subset of \(Y_n\) containing exactly the configurations \(K\) whose proximity graph is \(G\). This is an irreducible smooth locally closed subvariety ([51] and [28]). As a consequence of the upper-semicontinuity of the functions given in (2), for any positive integer \(d\) and any system of multiplicities \(m = (m_0, m_1, \ldots, m_n)\), the dimension and the superabundance of the linear systems \(L_d(K, m)\), for \(K\) varying in \(U(G)\), achieve the minimum value in a dense open subset of \(U(G)\).

**Definition 6.** We will say that a weighted configuration \((K, m) = (m_i)_{i=0}^{n}\) (resp., a weighted proximity graph \((G, m)\)) is *consistent* if all the excesses \(\rho_j(K, m)\) (resp., \(\rho_j(G, m)\)) are non-negative (see Section 2.1 for the definition of excesses). In this case, and provided that \(n \geq 1\), we associate with \((K, m)\) an integer, denoted by \(\epsilon(K, m)\) (or \(\epsilon(G, m)\), since it depends only on the weighted proximity graph) and defined to be either 1, if \(\rho_1(K, m) \geq 1\), or 0, if \(\rho_1(K, m) = 0\).

Given a weighted proximity graph \((G = G(K), m = (m_i)_{i=0}^{n})\), it is possible to obtain a unique system of multiplicities, which will be denoted by \(m^G = (m_0^G, m_1^G, \ldots, m_n^G)\), such that \((G, m^G)\) is consistent and the ideal sheaves given by \(\pi_K^* O_{Z_K}(-\sum_{i=0}^{n} m_i E^G_i)\) and \(\pi_K^* O_{Z_K}(-\sum_{i=0}^{n} m_i^G E^G_i)\) coincide. So, there exists a canonical bijection between the linear systems \([D_{d,G,m}^G]\) and \([D_{d,K,m}]\) for all integers \(d \geq 1\). The procedure used to obtain \(m^G\) is called *unloading* [11, 4.6] and it depends only on the proximity graph \(G\), and not on a special election of the configuration \(K\) associated with \(G\). In each step of the unloading procedure (unloading step) one must detect a point \(p_i\) of \(K\) such that its associated excess \(\rho_i(K, m)\) is negative; then, one replaces the system of multiplicities \(m\) by the system
\( \mathbf{m}' = (m'_0, m'_1, \ldots, m'_n) \) where \( m'_j = m_j + 1 \) for those indexes \( j \) such that \( p_j \) is proximate to \( p_i \), and \( m'_j = m_j \) otherwise (if some multiplicity in \( \mathbf{m}' \) is negative, it must be replaced by 0). Now, we must perform another unloading step from the new system \( \mathbf{m}' \), and so on. A finite number of unloading steps lead to the desired system of multiplicities \( \mathbf{m}^G \). An unloading step applied to a point \( p_i \) whose associated excess equals \(-1\) is called tame. Tame unloadings will be very useful for us, since they preserve independence of conditions, that is, if \( \mathbf{m}' \) is obtained from \( \mathbf{m} \) performing a tame unloading step, then \( h^1(Z_K, \mathcal{O}_{Z_K}(D_{d,K,m}')) = h^1(Z_K, \mathcal{O}_{Z_K}(D_{d,K,m})) \) for all positive integer \( d \) (this fact can be easily deduced from [11 4.7.1] and [11 4.7.3]).

**Definition 7.** We will say that a weighted configuration \((K, \mathbf{m})\) (resp., a weighted proximity graph \((G, \mathbf{m})\)) is almost consistent if either it is consistent or there exists a sequence of tame unloading steps leading from \( \mathbf{m} \) to \( \mathbf{m}^G(K) \) (resp., \( \mathbf{m}^G \)).

The following theorem gives a sufficient condition for the non-speciality of a linear system \( L_d(\mathbf{m}) \), when \( \mathbf{m} \) is a system of multiplicities such that \((K_C, \mathbf{m})\) is almost consistent, \( C \) being a curve of AMS type. Moreover, when this weighted configuration is consistent and the excess \( \rho_1(K_C, \mathbf{m}) \) is positive, it provides a characterization of such non-special linear systems which are not empty.

**Theorem 2.** Let \( d \) be a positive integer and \( \mathbf{m} = (m_0, m_1, \ldots, m_n) \) a system of multiplicities, with \( n \geq 1 \). Assume the existence of a curve \( C \) of AMS type such that \((K_C, \mathbf{m})\) is almost consistent. Then, the linear system \( L_d(\mathbf{m}) \) is non-special whenever \( d \geq m_0^G + m_1^G - \epsilon(G, \mathbf{m}^G) \), where \( G := G(K_C) \). Moreover, if \( L_d(\mathbf{m}) \) is not empty, \((K_C, \mathbf{m})\) is consistent and \( \rho_1(G, \mathbf{m}) \geq 1 \), then the following equivalence holds: \( L_d(\mathbf{m}) \) is non-special if and only if \( d \geq m_0 + m_1 - 1 \).

**Proof.** Set \( K_C = (p_0, p_1, \ldots, p_n) \) (adding null multiplicities to \( \mathbf{m} \), if it is necessary, we can assume that the cardinality of \( K_C \) is \( n + 1 \)). In order to prove the first assertion of the statement, we will reason by contradiction. So, assume that \( d \geq m_0^G + m_1^G - \epsilon(G, \mathbf{m}^G) \) and \( L_d(\mathbf{m}) \) is special. The subset \( U(G(K_0(n))) \) is dense in \( Y_n \) (see [12]) and then, as a consequence of the upper-semicontinuity of the functions given in [2], the following inequalities hold: \( \dim L_d(\mathbf{m}) \leq h^1(X_C, \mathcal{O}_{X_C}(D_{d,K_C,m})) - 1 \) and \( h^1(L_d(\mathbf{m})) \leq h^1(X_C, \mathcal{O}_{X_C}(D_{d,K_C,m})) \). Thus, the complete linear system on \( X_C \) given by \( |D_{d,K_C,m}| \) is not empty and \( h^1(X_C, \mathcal{O}_{X_C}(D_{d,K_C,m})) \) is positive. But, since \((K_C, \mathbf{m})\) is almost consistent, we have

\[
h^1(X_C, \mathcal{O}_{X_C}(D_{d,K_C,m})) = h^1(X_C, \mathcal{O}_{X_C}(D_{d,K_C,m})).
\]

The consistency of \((K_C, \mathbf{m}^G)\) implies that \( D_{d,K_C,m_0} \cdot \mathcal{E}_{i,K}^C \geq 0 \) for all \( i = 0, 1, \ldots, n \) and, therefore, we can apply Theorem [11] to deduce the inequality \( D_{d,K_C,m_0} \cdot \mathcal{H}^{K_C} \leq -1 - \epsilon(G, \mathbf{m}^G) \). But, taking into account that \( \mathcal{H}^{K_C} \) is a \((-1)\)-curve, this is equivalent to the condition \( d \leq m_0^G + m_1^G - 1 - \epsilon(G, \mathbf{m}^G) \), a contradiction.

For the last assertion, it only remains to prove that, assuming the consistency of \((G, \mathbf{m})\) and the inequality \( \rho_1(G, \mathbf{m}) \geq 1 \), the non-speciality of the linear system \( L_d(\mathbf{m}) \) implies the inequality \( d \geq m_1 + m_2 - 1 \). We will reason by contradiction. So, assume that \( L_d(\mathbf{m}) \) is non-special and \( d \leq m_1 + m_2 - 2 \). Again taking into account that \( \mathcal{H}^{K_C} \) is a \((-1)\)-curve, we have \( D_{d,K_C,m} \cdot \mathcal{H}^{K_C} \leq -2 \). If \( N \) denotes the line of \( \mathbb{P}^2 \) joining the two first points of the configuration \( K_0(n) \), then

\[
D_{d,K_0(n),m} \cdot \mathcal{N}_{K_0(n)} = d - m_0 - m_1 = D_{d,K_C,m} \cdot \mathcal{H}^{K_C} \leq -2,
\]
which is a contradiction with the non-speciality of $\mathcal{L}_d(\mathbf{m})$. □

As a consequence of Proposition 4 there is a bijection between the set of ordered sequences $(n_1, n_2, \ldots, n_r) \in (\mathbb{N} \setminus \{0\})^r$ (with $r \in \mathbb{N} \setminus \{0\}$) and the set of proximity graphs of the form $\mathbb{G}(K_C)$, $C$ being a curve of AMS type. Taking this fact into account, we obtain the following reformulation of Theorem 2 expressed in purely arithmetical terms:

**Corollary 3.** Let $d$ be a positive integer and $\mathbf{m} = (m_0, m_1, \ldots, m_n)$ a system of multiplicities with $n \geq 1$. Assume that there exist integers $n_1, n_2, \ldots, n_r \geq 2$ such that the weighted proximity graph $(\mathbb{G} = \mathbb{G}(n_1, n_2, \ldots, n_r)^+, \mathbf{m})$ is almost consistent. Then, the linear system $\mathcal{L}_d(\mathbf{m})$ is non-special whenever $d \geq m_0^2 + m_1^2 - \epsilon(\mathbb{G}, \mathbf{m})$. Moreover, if $\mathcal{L}_d(\mathbf{m})$ is not empty, $(\mathbb{G}, \mathbf{m})$ is consistent and $p_1(\mathbb{G}, \mathbf{m}) \geq 1$, then the following equivalence holds: $\mathcal{L}_d(\mathbf{m})$ is non-special if and only if $d \geq m_0 + m_1 - 1$.

Next, we will give two examples by applying Corollary 3 to two specific sequences of integers.

**Example 1.** Let $n \geq 2$ be an integer and consider the proximity graph $\mathbb{G} = \mathbb{G}(t + 1)^+$, where $t := \lfloor n/2 \rfloor$. The number of points $s + 1$ of whichever configuration whose proximity graph be $\mathbb{G}$ is $n + 1$ (resp., $n + 2$) if $n$ is odd (resp., if $n$ is even) and, moreover, the complete list of proximity relations among the points of such a configuration $(p_0, p_1, \ldots, p_s)$ are the following: $p_i \rightarrow p_{i-1}$ for all $i = 1, 2, \ldots, s$ and $p_j \rightarrow p_0$ for all $j = 2, 3, \ldots, t$. Then, applying Corollary 3 to the graph $\mathbb{G}$, we get the following result:

Let $\mathbf{m} = (m_0, m_1, \ldots, m_n)$ be a system of multiplicities such that $m_1 \geq m_2 \geq \ldots \geq m_n$ and $m_0 \geq \sum_{i=1}^{n} m_i$. A linear system $\mathcal{L}_d(\mathbf{m})$ is non-special whenever $d \geq m_0 + m_1 - \epsilon$, where $\epsilon = \min\{1, m_1 - m_2\}$. Moreover, if $\mathcal{L}_d(\mathbf{m})$ is not empty and $m_1 > m_2$, then $\mathcal{L}_d(\mathbf{m})$ is non-special if and only if $d \geq m_0 + m_1 - 1$.

**Example 2.** Let $k \geq 2$ be an integer. If $(p_0, p_1, \ldots, p_s)$ is whichever configuration whose proximity graph is $\mathbb{G}(2, 2, \ldots, 2)^+$ (where the number 2 appears $k$ times), one gets that $s = 3k$ and the proximity relations among the points are the following: $p_i \rightarrow p_{i-1}$ for all $i = 1, 2, \ldots, s$ and $p_{3j+1} \rightarrow p_{3j-1}$ for all $j = 1, 2, \ldots, k - 1$. Applying again Corollary 3 to this graph one gets the following result:

Let $\mathbf{m} = (m_0, m_1, \ldots, m_{3k})$ be a system of multiplicities such that $m_0 \geq m_1 \geq \ldots \geq m_{3k}$ and $m_{3i-1} \geq m_{3i} + m_{3i+1}$ for all $i = 1, 2, \ldots, k - 1$. Then, a linear system $\mathcal{L}_d(\mathbf{m})$ is non-special if $d \geq m_0 + m_1$. If, in addition, $\mathcal{L}_d(\mathbf{m})$ is not empty and $m_1 > m_2$, then $\mathcal{L}_d(\mathbf{m})$ is non-special if and only if $d \geq m_0 + m_1 - 1$.

The following direct consequence of Theorem 2 exhibits a wide range of cases in which the Harbourne-Hirschowitz Conjecture is satisfied.

**Corollary 4.** Let $\mathbf{m} = (m_0, m_1, \ldots, m_n)$ be a system of multiplicities such that $(K_C, \mathbf{m})$ is consistent and $p_1(K_C, \mathbf{m}) \geq 1$, $C$ being a curve of AMS type. Denote $K_0(n) = (p_0, p_1, \ldots, p_n)$ and set $N$ the line joining $p_0$ and $p_1$. If a linear system of the form $\mathcal{L}_d(\mathbf{m})$ is special, then $D_{d, K_0(n), \mathbf{m}} \cdot N^{K_0(n)} \leq -2$.

**Remark 3.** For a fixed positive integer $n$, let us denote by $\mathcal{S}_n$ the set of proximity graphs of the form $\mathbb{G}(K_C)$, where $C$ is a curve of AMS type, whose number of vertices is greater than or equal to $n + 1$. Each proximity graph $\mathbb{G} = \mathbb{G}(K_C) \in \mathcal{S}_n$ provides, by Corollary 4 an infinite family of systems of multiplicities $\mathbf{m} = (m_i)_{i=0}^n$ for which the Harbourne-Hirschowitz Conjecture is true. This family is given by the non-negative integer solutions
of the following system of linear inequalities in \( m_0, m_1, \ldots, m_n \):

\[
\begin{align*}
m_i - \sum_{j \leq n: p_j \rightarrow p_i} m_j & \geq 0, \quad i = 0, 2, 3, 4, \ldots, n \\
m_1 - \sum_{j \leq n: p_j \rightarrow p_i} m_j & \geq 1
\end{align*}
\]

where \( K_C = (p_0, p_1, \ldots, p_n) \). Two graphs \( G \) and \( G' \) in \( S_n \) give rise to the same system of inequalities if the proximity relations involving the first \( n + 1 \) vertices are the same for both graphs; in this case, we will say that \( G \) and \( G' \) are \( n \)-equivalent. Taking into account Proposition 4, a complete system of representants of the quotient set of \( S_n \) by this equivalence relation is given by the proximity graphs \( G(n_1, n_2, \ldots, n_r)^+ \) (with \( n_i \geq 2 \) for all \( i \)) such that either \( r = 1 \) and \( (n + 1)/2 \leq n_1 \leq n + 1 \), or \( r > 1 \), \( t := n - 2 \sum_{i=1}^{r-1} n_i + r > 0 \) and \( t/2 \leq n_r \leq t \). Thus, for a fixed positive integer \( n \), the set of distinct systems of linear inequalities in \( n + 1 \) variables provided by Corollary 4 (each of them satisfying that the Harbourne-Hirschowitz Conjecture is true for the solutions) is finite. In fact, they are in one-to-one correspondence with the \( n \)-equivalence classes.

**Definition 8.** We define the regularity of a system of multiplicities \( m = (m_0, m_1, \ldots, m_n) \) as the minimum integer \( d \) such that \( (K_0(n), m) \) imposes independent conditions to the curves of degree \( d \), and we will denote it by \( \tau(m) \).

The following result is another consequence of Theorem 2 and it allows to compute the exact value of the regularity of a wide range of systems of multiplicities:

**Corollary 5.** Let \( m = (m_0, m_1, \ldots, m_n) \) be a system of multiplicities, with \( n \geq 1 \). Assume the existence of a curve \( C \) of AMS type such that \( (K_C, m) \) is consistent and \( \rho_1(K_C, m) \geq 1 \). Then, \( \tau(m) = m_0 + m_1 - 1 \) if \( (m_0 + m_1 - 1)(m_0 + m_1 + 2) - \sum_{i=0}^{n} m_i(m_i + 1) \geq -2 \), and \( \tau(m) = m_0 + m_1 \) otherwise.

**Proof.** First, we will show that \( [D_{d,K_0(n)}, m] \) is an effective class, where \( d = m_0 + m_1 \). Indeed, the class \( [D_{d,K_C}, m] \) is numerically effective, due to the hypotheses and the fact that \( K_C \) is a \((-1)\)-curve. So, it is an effective class of Pic\((X_C)\) by Proposition 5 and \( h^1(L_d(K_C, m)) = h^1(X_C, O_C(D_{d,K_C}, m)) = 0 \), by Theorem 1. Therefore, we get

\[
\text{edim } L_d(m) = \text{dim } L_d(K_C, m) \geq 0.
\]

Hence, \( L_d(m) \) is non-empty and then, by Theorem 2, \( \tau(m) \leq d \).

If \( (d - 1)(d + 2) - \sum_{i=0}^{n} m_i(m_i + 1) \geq -2 \), then the inequality \( \text{edim } L_{d-1}(m) \geq -1 \) holds. So, the superabundance \( h^1(L_{d-1}(m)) \) must be zero, in virtue of Theorem 2, and then \( \tau(m) = d - 1 \) by (112).

Finally, if \( (d - 1)(d + 2) - \sum_{i=0}^{n} m_i(m_i + 1) < -2 \), then

\[
\text{dim } L_{d-1}(m) - h^1(L_{d-1}(m)) < -1
\]

and this implies that the superabundance \( h^1(L_{d-1}(m)) \) is positive. Therefore, in this case, \( \tau(m) = d \). \( \square \)

### 4.3. Bounding the regularity

In this section, it is described an algorithm, based on the unloading method, which provides an upper bound of the regularity of whichever system of multiplicities \( m = (m_0, m_1, \ldots, m_n) \). Although only the case of homogeneous multiplicities is explicitly treated (i.e., \( m_0 = m_1 = \cdots = m_n \)) this algorithm can be adapted without difficulty to the case of arbitrary multiplicities. In this section we introduce a generalization of this algorithm, based on our results in Section 3.
Let $m = (m_0, m_1, \ldots, m_n)$ be a sequence of multiplicities (with $n \geq 1$) such that $m_0 \geq m_1 \geq \cdots \geq m_n$. Take a sequence of integers $(n_1, n_2, \ldots, n_r)$ such that $n_i \geq 2$ for all $i = 1, 2, \ldots, r$ and $n + 1$ is not greater than the number of vertices of the graph $G := G(n_1, n_2, \ldots, n_r)^+$. By completing with zero multiplicities, if it is necessary, we will assume that this number of vertices coincides with $n + 1$. Denote by $G_i$ the proximity graph obtained from $G$ by deleting all curved-dotted edges involving some vertex with label greater than $i$. Let $(i_1, i_2, \ldots, i_w)$ be an increasing sequence of integers such that $G_{i_1}, G_{i_2}, \ldots, G_{i_w}$ are the distinct elements of the set $\{G_i \mid 1 \leq i < n\}$ and let $a$ be the maximum integer such that $0 \leq a < n$ and, if $(p_0, p_1, \ldots, p_a)$ denotes a configuration with proximity graph $G$, the cardinality of the set $\{j \mid a \leq j \leq n, \ p_j \rightarrow p_a \text{ and } m_j > 0\}$ is greater than 1 (if that integer does not exist, we will take $a = 0$).

**Example 3.** Let $G$ be the proximity graph $G(s)^+$, where $s > 1$ is an integer. The above described sequence of graphs $G_{i_1}, G_{i_2}, \ldots, G_{i_w}$ is, in this case, the sequence $G^1, G^2, \ldots, G^{s-1}$ where $G^1$ denotes the proximity graph of a chain of 2s free points and, for each $k = 2, 3, \ldots, s - 1$, $G^k$ stands for the graph depicted in Figure 3 of page 15\textsuperscript{a} taking $n = 2s - 1$; the integer $a$ is 0.

Set $m_1 := m$, which is consistent for $G_{i_1}$, and define recursively the systems of multiplicities $m_2, m_3, \ldots, m_w$ as follows. Suppose we have defined $m_k$ and perform the following two-steps algorithm applied to $v := m_k$, which will give rise to $m_{k+1}$:

**Step 1.** If $(G_{i_{k+1}}, v)$ is consistent, then define $m_{k+1} := v$. Otherwise, there exists a unique $j$ such that the excess $\rho_j(G_{i_{k+1}}, v)$ is negative. In this case, if $j = a$ and this excess equals $-1$, define also $m_{k+1} := v$; else, perform an unloading step at the point $(G_{i_k}, v)$ at the vertex which corresponds to that excess, replace $v$ by the obtained new system of multiplicities and go to Step 2.

**Step 2.** Replace $v$ by $v^G_{i_k}$ and return to Step 1.

Once we have computed $m_w$, we must consider the system of multiplicities $m' := m_w^G = (m'_0, m'_1, \ldots, m'_n)$. Let $C$ be a curve having one place at infinity whose associated proximity graph is $G_{i_w} = G$. Notice that $h^1(X_C, \mathcal{O}_{X_C}(D_{m'_0+1,kC,m'}) = 0$ (by Theorem 1), since $D_{m'_0+1,kC,m'}$ is a numerically effective divisor of $X_C$. We will compute the successive dimensions $h^0(X_C, \mathcal{O}_{X_C}(D_{m'_0+1,j,kC,m'}))$ for $j = 1, 2, \ldots$ (using the process described in Remark 2) until finding the minimum $j$ such that the mentioned dimension is positive. Finally, we will define $\beta(m) := m'_0 + m'_1 - j + 1$. Note that this process is independent of the chosen curve $C$; in fact, it only depends on the proximity graph $G$.

Now, we will justify that the obtained value $\beta(m)$ is an upper bound of the regularity $\tau(m)$. We start with a lemma whose proof is an adaptation of that of [53, Lem. 2.1] and we will omit it.

**Lemma 1.** Let $d$ be a positive integer, $K = \{p_0, p_1, \ldots, p_n\}$ a configuration and $m = (m_i)_{i=0}^n$ a system of multiplicities. Let $i \in \{0, 1, \ldots, n\}$ be such that $\rho_i(K, m) \geq -1$ and let $m' = (m'_0, m'_1, \ldots, m'_n)$ be the sequence of multiplicities obtained from $m$ by performing an unloading step at the point $p_i$. Then, $h^1(L_d(K, m)) = 0$ whenever $h^1(L_d(K, m')) = 0$.

**Theorem 3.** Let $m = (m_0, m_1, \ldots, m_n)$ be a sequence of multiplicities, let $(n_1, n_2, \ldots, n_r)$ be a sequence of integers such that $n_i \geq 2$ for all $i = 1, 2, \ldots, r$ and set $\beta(m)$ defined as above. Then, $\beta(m)$ is an upper bound of $\tau(m)$. 
Proof. Recall the notations of Section 11.2. Taking into account that the proximity graphs \( G_{i_1}, G_{i_2}, \ldots, G_{i_w} \) are associated with chain configurations and the matrix \( P^{-1}_{G_{i_k}} \cdot P_{G_{i_k}} \) has no negative entries for each \( k = 2, 3, \ldots, w \), it can be deduced, from (51), the existence of a chain of inclusions

\[
U(G) = U(G_{i_w}) \subseteq U(G_{i_{w-1}}) \subseteq \ldots \subseteq U(G_{i_1}).
\]

Let \( d = \beta(m) \) and, for each system of multiplicities \( v \), set \( h^1(d, G_{i_k}, v) \) the minimum of the superabundances \( h^1(L_d(K, v)) \) when \( K \) varies in \( U(G_{i_k}) \).

Take a plane curve \( C \) of AMS type such that \( G(K_C) = G(n_1, n_2, \ldots, n_r)^+ \). From the above description of the algorithm, it follows that

\[
h^1(L_d(K_C, m^l)) = h^1(X_C, O_X(D_{d,K_C,m^l})) = 0
\]

and, since the weighted proximity graph \( (G, m^l) \) is obtained from \( (G, m^w) \) by tame unloading steps, we get that the integer \( h^1(L_d(K_C, m^l)) \) vanishes and, hence, \( h^1(d, G_{i_w}, m^l) = 0 \).

Finally, for \( 2 \leq k \leq w \), we will show that the vanishing of \( h^1(d, G_{i_k}, m^l) \) implies that of \( h^1(d, G_{i_{k-1}}, m^l) \). In order to prove this assertion observe firstly that, if we assume that \( h^1(d, G_{i_k}, m^l) = 0 \), then \( h^1(d, G_{i_{k-1}}, m^l) = 0 \) by (3) and the upper-semicontinuity of the functions given in (2). Choose a configuration \( K \in U(G_{i_{k-1}}) \) such that \( h^1(L_d(K, m^l)) = 0 \). It is not hard to see that the unloading procedure of the Step 2 of the algorithm to obtain the sequence \( m_1, m_2, \ldots, m_w \) can be performed by means of tame unloading steps. From this fact and Lemma 11 the equality \( h^1(L_d(K, m^l)) = 0 \) is obtained and, therefore, \( h^1(d, G_{i_{k-1}}, m^l) = 0 \).

Now, it follows, by induction, that \( h^1(d, G_{i_1}, m^l) = 0 \). Finally, using again semicontinuity and taking into account the density of \( U(G(K_0(n))) \) in \( Y_n \), we get \( h^1(L_d(m^l)) = 0 \). Hence, \( d \) is an upper bound of \( \tau(m) \). \( \square \)

We conclude the paper with some remarks on the above described algorithmic bound.

First observe that, given a system of multiplicities \( m \), there is a bound \( \beta(m) \) for each one of a proximity graph \( G(n_1, n_2, \ldots, n_r)^+ \) with, at least, \( n + 1 \) vertices (that is, \( 2 \sum_{i=1}^{n} n_i - r \geq n \)). It is clear that \( n \)-equivalent proximity graphs give rise to the same bound (see Remark 5). Thus, one can apply the algorithm to all the graphs \( G(n_1, n_2, \ldots, n_r)^+ \) such that either \( r = 1 \) and \( (n+1)/2 \leq n_1 \leq n + 1 \), or \( r > 1 \), \( t := n - 2 \sum_{i=1}^{r-1} n_i + r > 0 \) and \( t/2 \leq n_r \leq t \), and then pick the best bound.

We will show that the algorithm given by Roé in (53) can be obtained as a particular case of the one we have described (essentially, it corresponds to a specific type of proximity graph \( G(n_1, n_2, \ldots, n_r)^+ \)). To apply his algorithm to a system of multiplicities \( m = (m_0, m_1, \ldots, m_n) \), he uses successive specializations, starting from a configuration of \( n + 1 \) general points of the plane and following with configurations corresponding to the sequence of proximity graphs \( G^1, \ldots, G^n \) where, for each \( k = 1, 2, \ldots, n \), \( G^k \) is the one shown in Figure 3 (assuming that \( G^1 \) has no curved-dotted edge). The knowledge of the dimensions of all complete linear systems on the surfaces obtained by blowing-up at the points of whichever configuration whose associated proximity graph is \( G^n \) allows him to deduce, using a similar reasoning to the one explained in the algorithm we present here (but adapted to the above mentioned specific sequence of specializations), an upper bound of the regularity of \( m \). From this explanation, it is easy to deduce that applying Roé’s algorithm to \( m \) is equivalent to computing our bound \( \beta(m) \) taking the graph \( G(n+1)^+ \), adding previously to \( m \) the suitable number of zeros. This proximity graph corresponds,
for instance, to the curve whose equation in projective coordinates \((X : Y : Z)\) is \(XZ^n + Y^{n+1} = 0\), \(Z = 0\) being the line of infinity.

Note that, for each integer \(n \geq 2\), the proximity graph \(G([n/2] + 1)\) (considered in Example 1) is the graph \(G^k\) of Figure 3 for \(k = [n/2]\), if \(n\) is odd, and the one obtained from the same graph adding a new vertex corresponding to a free point at the top, if \(n\) is even. The fact that this is one of the intermediate proximity graphs which appear in the sequence of specializations used in [53] and easy reasonings concerning semicontinuity imply that our bounds \(\beta(m)\), taking the above proximity graph, are either equal or lower than those obtained from [53]. For homogeneous systems of multiplicities \(m = (m, m, \ldots, m)\) and for a fixed value of \(n\), examples show that the difference between both bounds increases when the multiplicity does so. For instance, this is the behavior for \(n + 1 = 1000\) and \(m\) taking values between 1 and 100. In fact, when \(m \leq 38\) the two bounds coincide, \(\beta(m)\) is sometimes better when \(39 \leq m \leq 68\) (in which case, the difference is 1) and it is always better when \(69 \leq m \leq 100\) (the difference is 1 in all cases except for \(m = 98\), where it equals 2). Also, for \(m = 500\) (resp., \(m = 800\)) (resp., \(m = 1200\)), \(\beta(m) = 16014\) (resp., \(\beta(m) = 25617\)) (resp., \(\beta(m) = 38417\)) and Roé’s bound is 16021 (resp., 25629) (resp., 38436). However, [38] gives better values in all the checked cases where our bound is less than Roé’s one.

For quasihomogeneous systems of multiplicities, there are cases in which our bound seems to improve the existing ones (as far as the author knows). As an example, consider the system of multiplicities \(m = (4000, 1000, \ldots, m)\) (where the subindex is the number of occurrences). Taking the graph \(G(10)\), it is obtained the bound \(\beta(m) = 6009\). The Harbourne-Hirschowitz Conjecture predicts that the regularity of \(m\) is 5917 and the bounds provided in [11], [30], [13], [53], [6] and [59] are 8367, 8000, 8000, 6183, 11140 and 6238, respectively. Also, the bound 6015 is obtained by an algorithm based on the reduction method described in [18] and Cremona transformations (it has been computed by using the computer program provided in [20]), and the bound 7667 is obtained by using the algorithm given in [31] for computing the dimension of line bundles on an smooth rational surface \(X\) with anticanonical bundle having an irreducible and reduced global section \(D\), with the further assumption that the morphism \(\text{Pic}(X) \to \text{Pic}(D)\) induced by the inclusion \(D \subseteq X\) has trivial kernel. The bound provided by the algorithm of [38],
using the parameters \( r = 9 \) and \( d = 2 \), is 7667 (this algorithmic bound depends on the choice of two parameters, but it is not clear how to obtain the optimal values).

Now, consider the family of systems of multiplicities \( m(m) := (m, 1000) \) for \( m \geq 1000 \). By applying Corollary 3 to the graph \( G(10) \) (see also Example 11) it can be deduced that \( \tau(m(m)) = m + 1000 \) when \( m \geq 9000 \). Computing the above mentioned bounds of the regularity for the remaining values of \( m \) it holds that, when either \( m \in \{1619, 1622, 1623\} \) or \( 1625 \leq m \leq 7765 \), the value \( \beta(m(m)) \) (taking the graph \( G(10) \)) is less than all the non-parametric bounds given in \( [11], [30], [14], [53], [6] \) and \( [59] \). When either \( 3935 \leq m \leq 3939 \), \( 3944 \leq m \leq 4081 \), \( 4083 \leq m \leq 4085 \) or \( m \in \{3941, 3942, 4087, 4089, 4090, 4092\} \) it holds that the bound \( \beta(m(m)) \) is also better than the one provided in \( [18] \) and \( [20] \); moreover, in these cases, we have not found any pair of parameters \((r, d)\) for which the bound given in \( [35] \) improves \( \beta(m(m)) \). It is worth adding that, by looking at systems of multiplicities of the type \((m, h)\) with \( h \in \{1100, 1200, 1300, 1400, 1500\} \), we have observed an increasing tendency (when \( h \) grows) on the number of values of \( m \) for which \( \beta(m, h) \) seems to be the best bound.

Although, in order to establish comparisons, it is natural to look at homogeneous and quasihomogeneous cases, our algorithm can be applied to arbitrary systems of multiplicities. Finally we notice that, when the system of multiplicities \( m \) is either homogeneous or quasihomogeneous, examples suggest that the bound \( \beta(m) \) is better when the graph \( G([n/2] + 1)^+ \) is taken (where \( n + 1 \) is the length of \( m \)).

Remark 4. The results proved in Section 3 and the explanations given in the current section suggest that the algorithm provided in \( [57] \) for giving a lower bound of the least degree \( d \) such that a linear system \( L_d(m) \) is not empty can also be generalized. However, we have not found evidences of any significant improvement of this generalization with respect to the existing bounds.

References

[1] S. S. Abhyankar, Lectures on expansion techniques in Algebraic Geometry, Tata Institute of Fundamental Research Lectures on Mathematics and Physics 57, Tata Institute of Fundamental Research, Bombay (1977).

[2] S. S. Abhyankar, T. T. Moh, Newton-Puiseux expansion and generalized Tschirnhausen transformation, J. Reine Angew. Math. 260 (1973), 47—83 and 261 (1973), 29—54.

[3] S. S. Abhyankar, T. T. Moh, Embeddings of the line in the plane, J. Reine Angew. Math. 276 (1975), 148—166.

[4] J. Alexander, A. Hirschowitz, An asymptotic vanishing theorem for generic unions of multiple points. Invent. Math. 140, no. 2 (2000), 303—325.

[5] E. Arbarello, M. Cornalba, Footnotes to a paper of B. Segre, Math. Ann., 256 (1981), 341—362.

[6] E. Ballico, Curves of minimal degree with prescribed singularities, Illinois J. Math. 45 (1999), 672—676.

[7] E. Ballico, L. Chiantini, Nodal curves and postulation of generic fat points on \( P^2 \), Arch. Math. 71 (1998) no. 6, 501—504.

[8] F. Bunke, C. Lossen, An \( H^1 \)-vanishing theorem for generic fat points in \( P^2 \), Preprint Univ. Kaiserslautern (2002).

[9] A. Campillo, Algebroid curves in positive characteristic, Lecture Notes in Math. 813, Springer (1980).

[10] A. Campillo, O. Piltant, A. J. Reguera, Cones of curves and of line bundles on surfaces associated with curves having one place at infinity Proc. London Math. Soc. 84 (2002), 559—580.

[11] E. Casas-Alvero, Singularities of plane curves, London Math. Soc. Lecture Note Ser. 276, Cambridge University Press (2000).

[12] G. Castelnuovo, Ricerche generali sopra i sistemi lineari di curve piane, Mem. Accad. Sci. Torino, II 42 (1891).

[13] M. V. Catalisano, Linear systems of plane curves through fixed fat points of \( P^2 \), J. Algebra 142 (1991), no. 1, 81—100.

[14] C. Ciliberto, Geometric aspects of polynomial interpolation in more variables and of Waring’s Problem, European Congress of Mathematicians, Vol. I (Barcelona, 2000), 289—316, Progr. Math. 201, Birkhäuser, Basel (2001).
[50] A. J. Reguera, Semigroups and clusters at infinity, Algebraic geometry and singularities (La Rábida, 1991), 339–374, Progr. Math., 134, Birkhäuser, Basel (1996).
[51] J. Roé, Varieties of clusters and Enriques diagrams, Math. Proc. Cambridge Philos. Soc., 137 (2004), no. 1, 69—94.
[52] J. Roé, On the existence of plane curves with imposed multiple points, J. Pure App. Alg. 156 (2001), 115—126.
[53] J. Roé, Linear systems of plane curves with imposed multiple points, Illinois J. Math. 45 (2001), no. 3, 895—906.
[54] A. Sathaye, On planar curves, Amer. J. Math. 99 (1977), no. 5, 1105—1135.
[55] B. Segre, Alcune questioni su insiemi finiti di punti in geometria algebrica, Atti Convegno Intern. di Geom. Alg. di Torino (1961), 15—33.
[56] J. Seibert, The dimension of quasihomogeneous planar linear systems with multiplicity four, Comm. Algebra 29 (2001), no. 3, 1111—1130.
[57] M. Suzuki, Affine plane curves with one place at infinity, Ann. Inst. Fourier 49 (1999), no. 2, 375—404.
[58] S. Yang, Linear systems of plane curves with base points of bounded multiplicity, math.AG/0406591
[59] G. Xu, Ample line bundles on smooth surfaces, J. Reine Angew. Math. 469 (1995), 199—209.

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