Hawking-Like Radiation as Tunneling of the Cosmological Black Hole Solution in Modified Gravity: Semiclassical Approximation and Beyond

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Abstract

Hawking radiation is a controversial quantum phenomenon of a black hole that is specially attributed to the existence of an event horizon of the black hole. In this paper, using Hamilton-Jacobi and Parikh-Wilczek methods based on the semiclassical and beyond semiclassical approximation, we proved that there is indeed thermal Hawking-like radiation for the apparent horizons of the cosmological black hole in the FLRW background in the framework of STVG. The explicit forms of the three apparent horizons of the cosmological black hole were obtained and plotted for more details. As de Sitter and FLRW cases, the corresponding Hawking-like temperature in our setup as a function of inverse powers of apparent horizon radius was found. Also, we compare the outcomes with Hawking-like radiation of the apparent horizon of the McVittie spacetime, which can be obtained by eliminating the STVG parameter of the enhanced gravitational constant in the setup. Also, beyond semiclassical approximation, extra terms are added to the Hawking-like temperature of the apparent horizon of the cosmological black hole in the STVG theory because of considering higher-order quantum effects. We found all the quantum corrections to the semiclassical Hawking-like temperature for the cosmological black hole in the STVG framework.

PACS Numbers: 04.50.Kd, 04.70.-s, 04.70.Dy, 04.20.Jb
Key Words: Cosmological Black Hole, Modified Gravity, Hawking Radiation, Apparent Horizon, Semiclassical Approximation.

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1 Introduction

The most revolutionary theory to describe gravitational interaction and the concept of spacetime is General Relativity (GR), proposed by Albert Einstein in the early twentieth century. This theory had lots of successes in predicting and explaining astrophysical phenomena. Besides all the achievements of GR, it is not the ultimate gravitational theory. The problem of spacetime singularities, such as Big Bang and black hole ones, and also, lack of the prediction of the late-time accelerated expansion of the Universe [1–3] are two well-known failures of GR. Additionally, GR theory can not reproduce rotation curves of nearby galaxies [4, 5], mass profiles of galaxy clusters [6, 7], etc. There are two ways to modify GR: (1) modifying the mass-energy distribution of the theory on the right-hand side of Einstein equations; (2) reconstructing the geometry of the theory on the left-hand side of the Einstein equations. The first way is to consider two unknown mass-energy terms called dark matter and dark energy by postulating the cosmological constant term [8, 9], or an extra stress-energy tensor associated with a scalar field [10–12] in the background of spacetime. In a second way, one can follow several methods to change the geometric structure of the theory, one of which is the theory of Scalar-Tensor-Vector Gravity (STVG), also known as MOdified Gravity (MOG) [13], which modifies the right-hand side of Einstein equations, automatically.

In the framework of STVG theory, gravitational effects on the fabric of spacetime are expressed by three scalar fields and a massive Proca-type vector field in addition to a metric tensor field. In this sense, the scalar fields in STVG theory are the mass $\tilde{\mu}$ of the vector field in the setup, $G$ as the enhanced (effective) gravitational constant, and $\xi$. In general, all of these scalar fields can vary in space and time. In weak gravity regime, the field equations of STVG theory decrease in a modified acceleration law, which has two features: (1) an enhanced Newtonian parameter $G = G_N (1 + \alpha)$ where $G_N$ is the Newtonian gravitational constant, and $\alpha$ is a free dimensionless parameter quantified by the central mass-energy distribution; (2) a gravitationally repulsive force term with gravitational charge $Q = M \sqrt{\alpha G_N}$ that neutralizes the Newtonian acceleration law in which $M$ is the source mass. The first one permits the reproduction of the rotation curves of many galaxies [14–16] and the dynamics of galactic clusters [17–19] without dark matter.

There are three main classes of solutions of STVG field equations including vacuum and non-vacuum solutions for a matter distribution in an asymptotically flat spacetime [20–23], cosmological solutions [24, 25] and the solution representing an inhomogeneity in a dynamical universe, which is indeed a cosmological black hole solution [26]. In this paper, we want to focus on the latter solution. Generally, McVittie spacetime [27] was the first solution of Einstein field equations in GR expressing a central inhomogeneity embedded in a Friedmann-Lemaître-Robertson-Walker (FLRW) background, which is extensively investigated in Refs. [28–35]
and references therein. Studying the inhomogeneous spacetimes in various theories of gravity shows us that it is required to consider the cosmological expansion of the Universe in modeling the evolution of the structures in it. On the other hand, both McVittie spacetime [27] and the cosmological black hole solution in STVG [26] are the candidates for describing the gravitational fields of spherically symmetric mass distributions in expanding FLRW universe in theories of GR and STVG, respectively [29].

In 1973, Jacob Bekenstein [36] proposed that a black hole has entropy proportional to the area of its event horizon. In the next year, Stephen Hawking showed that black holes are indeed “radiating holes”, causing them not black [37]. Then, in 1975, in the seminal work [38] proved that black holes are some black body objects radiating a thermal emission, known as Hawking radiation from their event horizon with a temperature, known as Hawking temperature proportional to their event horizon surface gravity. Thus, it was found that the concept of event horizon plays a crucial role in black hole physics. Also, considering the back-reaction effects naturally leads to deviating Hawking radiation from the thermal spectrum [39]. On the other hand, in 1977, Gibbons and Hawking [40] discovered a Hawking temperature associated with the cosmological event horizon radius $l$ in the de Sitter universe to the form of $T_{dS} = \frac{\hbar}{2\pi l}$, similar to such a temperature corresponding to the event horizon of a black hole. In 2000, Parikh and Wilczek [41] represent Hawking radiation as a tunneling process of particles through the event horizon of a stationary black hole based on semiclassical approximation. Until now, lots of works focused on applying the Parikh-Wilczek approach (also known as the null geodesic method) to various black hole solutions [42–47] and studying quantum gravity effects on it [48]. Recently, the thermodynamics and Hawking radiation of non-commutative MOG black holes is studied in Ref. [49] following the Parikh-Wilczek method. On the other hand, Srinivasan et al. [50–52] proposed another method to derive Hawking radiation based on the tunneling process with semiclassical approximation in which the classical action of tunneling particles is calculated by the Hamilton-Jacobi equation. In both Parikh-Wilczek and Hamilton-Jacobi methods due to the semiclassical approximation, quantum corrections are generally not taken into account, because only the first semiclassical term of the tunneling particle action is considered. In 2008, Banerjee and Majhi presented a generalization to the semiclassical tunneling process [53]. By expanding the action of the tunneling particle in powers of $\hbar$ and taking into account all the higher order of quantum corrections to the semiclassical outcomes, they formulated the Hamilton-Jacobi method of tunneling beyond the semiclassical approximation. The Hawking and Hawking-like radiation as tunneling beyond the semiclassical approximation, respectively for BTZ black hole and FLRW universe are derived in Refs. [54–56].

The global concept of an event horizon in a spacetime, however, does not locally provide the possibility of locating an event horizon associated with the dynamical spacetime at a moment. This fact makes it difficult to investigate Hawking radiation in a non-stationary black hole. Due to the quasi-locally definition of apparent horizons, however, they do not refer to the global causal structure of a spacetime [33]. Accordingly, in pioneer works [57, 58] following the Hamilton-Jacobi approach, the authors studied the Hawking-like radiation of the apparent horizon of some non-stationary black holes. In the novel work [59] following the Hamilton-Jacobi method and Parikh-Wilczek approach [41], the authors for the first time showed that Hawking-like temperature corresponding with an inward thermal emission (Hawking-like radiation) radiated from the apparent horizon of FLRW universe is $T_{FLRW} = \frac{\hbar}{2\pi \tilde{r}_A}$, where $\tilde{r}_A$ is the apparent horizon radius of FLRW spacetime, which is measured by a Kodama observer inside the apparent horizon. As this interior Kodama observer is fixed inside the apparent horizon and in a time-dependent frame, the radiation cannot be pure Hawking radiation. So, one can call the radiated spectrum Hawking-like radiation [55, 60]. The Kodama vector [61] of the time-dependent black holes corresponding with the Kodama observer plays the same role as a Killing vector of stationary black holes [33].

In this paper, we aim to drive the Hawking-like temperature of the apparent horizons of the cosmological
black hole solution in STVG theory living in FLRW spacetime background [26] with and beyond semiclassical approximation. We use Hamilton-Jacobi and Parikh-Wilczek approaches for tunneling massive and massless particles, respectively based on semiclassical approximation. Similar to the FLRW case [59], the results in both approaches are the same and satisfy the correspondence principle. Then, we use the Hamilton-Jacobi method of tunneling beyond the semiclassical approximation for a massless scalar field as a tunneling particle to apply all higher-order quantum corrections to the previous semiclassical results. All of the quantities deduced in the paper tend to the corresponding ones of the McVittie universe in the limit $\alpha \to 0$. In the rest of the paper, we set $c = 1$, where $c$ is the speed of light. Also, all the figures in the paper are plotted using the scale factor of $\Lambda$ Cold Dark Matter ($\Lambda$CDM) model, which is $a(t) = \left(\frac{(1-\Omega_{\Lambda,0})}{\Omega_{\Lambda,0}} \sinh \left(\frac{t}{H_0 \sqrt{\Omega_{\Lambda,0}}}\right)\right)^{\frac{1}{3}}$, where $H_0 = 2.27 \times 10^{-18}$ $(s^{-1}) \approx 70$ $(km \ s^{-1} \ Mpc^{-1})$ and $\Omega_{\Lambda,0} = 0.7$ are the late-time Hubble factor and the cosmological constant density parameter, respectively. The paper is organized as follows. In Section 2 we briefly review the metric and properties of the cosmological black hole solution in STVG. Next, in Section 3 the Hamilton-Jacobi and Parikh-Wilczek methods based on the semiclassical approximation for massive and massless particles are studied, respectively. Then, section 4 includes discussing the Hamilton-Jacobi method beyond semiclassical approximation. Finally, in Section 5 we end with some conclusions.

\section{Review on the Cosmological Black Hole Solution in the STVG Theory}

The total action of STVG theory has four terms [13]. The first term is the well-known Einstein-Hilbert action as follows

$$S_{GR} = \frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} \frac{1}{G} R,$$

in which $\tilde{g}$ is the determinant of the metric tensor $\tilde{g}_{\mu\nu}$ of the background spacetime, $G(\tilde{x})$ is the enhanced Newtonian parameter as a scalar field, and $R$ is the scalar curvature. Next comes the matter action $S_M$ for possible matter fields. The third term is the vector field action for a massive Proca-type vector field $\phi^\mu$ which has the mass $\tilde{\mu}$ as follows

$$S_\phi = -\int d^4x \sqrt{-\tilde{g}} \left(\frac{1}{4} B^{\mu\nu} B_{\mu\nu} + V(\phi)\right) \xi,$$

in which $B_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$ and $V(\phi)$ denotes the potential of the vector field. Finally, the last term is the action of scalar fields as follows

$$S_S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{G^3} \left(\frac{1}{2} \tilde{g}^{\mu\nu} \nabla_\mu G \nabla_\nu G - V(G)\right) + \frac{1}{\tilde{\mu}^2 G} \left(\frac{1}{2} \tilde{g}^{\mu\nu} \nabla_\mu \tilde{\mu} \nabla_\nu \tilde{\mu} - V(\tilde{\mu})\right) + \frac{1}{G} \left(\frac{1}{2} \tilde{g}^{\mu\nu} \nabla_\mu \xi \nabla_\nu \xi - V(\xi)\right)\right],$$

where $\nabla_\mu$ shows the covariant derivative, $G(\tilde{x})$, $\xi(\tilde{x})$, and $\tilde{\mu}(\tilde{x})$ are three scalar fields in the setup, and also $V(G)$, $V(\xi)$ and $V(\tilde{\mu})$ are their corresponding potentials, respectively. Therefore, the total action of STVG theory is written in the form of

$$S_{tot} = S_{GR} + S_M + S_\phi + S_S.$$

The total stress-energy tensor in the setup is

$$T^{(tot)}_{\mu\nu} = T^{(M)}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + T^{(S)}_{\mu\nu}$$

in which $T^{(M)}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta S_M}{\delta g^{\mu\nu}}$ is the stress-energy tensor of ordinary matter distribution, and

$$T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta S_\phi}{\delta \tilde{g}^{\mu\nu}} = -\frac{1}{4} \left( B_{\sigma}^\mu B^{\mu\sigma} - \frac{1}{4} \tilde{g}_{\mu\nu} B^{\rho\lambda} B_{\rho\lambda}\right),$$

(2.4)
shows the stress-energy tensor corresponding with the vector field when \( V(\phi) = 0 \) \cite{21, 26}, and finally \( T^{(S)}_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{k_S}{\sqrt{2}} \) denotes the stress-energy tensor of the scalar fields contribution. Varying the total action \( S_{\text{tot}} \) with respect to \( \tilde{g}^{\mu\nu} \) results in the Einstein field equations \cite{13} to the form of

\[
G_{\mu\nu} + G \left( \nabla^\gamma \nabla_\gamma \frac{1}{\sqrt{-g}} \tilde{g}_{\mu\nu} - \nabla_\mu \nabla_\nu \frac{1}{\sqrt{-g}} \right) = 8\pi G T^{(\text{tot})}_{\mu\nu},
\]

\( (2.5) \) where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor. The extra term \( G \left( \nabla^\gamma \nabla_\gamma \frac{1}{\sqrt{-g}} \tilde{g}_{\mu\nu} - \nabla_\mu \nabla_\nu \frac{1}{\sqrt{-g}} \right) \) in the Einstein field equations \( (2.5) \) arises from boundary contributions \cite{13}.

To derive the cosmological black hole solution in STVG setup living in FLRW background, however, the authors in Ref. \cite{26} considered that \( \xi(\tilde{x}) = 1 \) and \( \tilde{\mu}(\tilde{x}) = 0 \) and also, they set \( Q = M \sqrt{\alpha G_N} \) as the gravitational source charge of the vector field, and \( G = G_N (1 + \alpha) \) in which \( \alpha \) is the renormalization constant, which modifies the nature of the gravitational field \cite{21, 62}. Hence, the total stress-energy tensor becomes to the form \( T^{(\text{tot})}_{\mu\nu} = T^{(M)}_{\mu\nu} + T^{(\phi)}_{\mu\nu} \) where

\[
T^{(M)}_{\mu\nu} = (\rho + p) u_\mu u_\nu + p \tilde{g}_{\mu\nu},
\]

\( (2.6) \) is considered as the stress-energy tensor of the cosmological perfect fluid in which \( \rho, p \), and \( u^\mu \) are the proper energy density, the proper pressure, and the 4-velocity of the fluid, respectively. Therefore, the Einstein field equations \( (2.5) \) take the simple form \( G_{\mu\nu} = 8\pi G \left( T^{(M)}_{\mu\nu} + T^{(\phi)}_{\mu\nu} \right) \). Finally, the authors in Ref. \cite{26} found the line element of the cosmological black hole solution in the STVG framework located in the FLRW spacetime expressed in isotropic coordinates \( \tilde{x}^\mu = (t, x, \theta, \phi) \) by setting \( G_N = 1 \) (for more details see Ref. \cite{26}) as follows

\[
ds^2 = -\frac{f(t, x)^2}{g(t, x)^2} dt^2 + a(t)^2 g(t, x)^2 \left( dx^2 + x^2 d\Omega^2 \right),
\]

\( (2.7) \) where \( t \) is cosmic time, \( a(t) \) is the scale factor,

\[
f(t, x) = 1 - \frac{M^2 (1 + \alpha)}{4a(t)^2 x^2},
\]

\( (2.8) \)

\[
g(t, x) = 1 + \frac{M(1 + \alpha)}{a(t)x} + \frac{M^2 (1 + \alpha)}{4a(t)^2 x^2},
\]

\( (2.9) \)

and the line element on the unit 2-sphere is \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \) and also, \( M \) is the gravitational mass of the central source. In the limit, \( a(t) \equiv 1 \), the line element \( (2.7) \) tends to be the line element of a Schwarzschild-MOG black hole, which is written in isotropic coordinates \cite{13, 21}, whereas in the limit \( M \to 0 \), Eq. \( (2.7) \) reduces to the line element of spatially flat FLRW model. As mentioned in the previous section, for \( \alpha \to 0 \), the McVittie spacetime in GR is recovered. Additionally, it is worth noting that by equating the gravitational charge \( Q \) in STVG setup with an electrical charge \( q \) in the charged McVittie solution as \( q = Q = M \sqrt{\alpha G_N} \), the line element of the charged McVittie spacetime becomes completely the same as the line element \( (2.7) \) of the cosmological black hole solution in the setup of STVG living in the FLRW background \cite{33}. The line element \( (2.7) \) has a scalar curvature singularity at those \( x \) values that satisfy the condition

\[
a(t)x = \frac{1}{2} M \sqrt{1 + \alpha}.
\]

\( (2.10) \)

This singularity can exist from the early cosmic time values. On the other hand, we focus on the spacetime events that are in the casual future of the singularity. The surface \( U(t, x) = a(t)x - \frac{1}{2} M \sqrt{1 + \alpha} \) at \( t = 0 \) is in the causal past of all these events. Hence, one can interpret it as a cosmological “Big-Bang” singularity. Kaloper et al. provided the same explanation for the curvature singularity of McVittie spacetime in GR \cite{29}.
Stationary black holes, which have metric coefficients independent of time can be characterized by the existence of event horizons. In non-stationary spacetimes, however, it is impossible to determine the location of an event horizon for a black hole since the entire spacetime manifold tends to future infinity. Instead, we can make use of the concept of the apparent horizon. Such a horizon is defined as the boundary between those light rays that are directed outwards and moving outwards, and those directed outward but moving inward. In other words, the apparent horizon is the boundary surface (usually, 3-surface) on which the null geodesic congruences change in their convergence properties. By definition, the following two conditions \( \theta_n = 0 \) and \( \theta_\ell > 0 \) determine the location of apparent horizons, where \( \theta_n \) and \( \theta_\ell \) are the expansions of the future-directed ingoing and outgoing null geodesics congruences, respectively \[33\]. The areal radius of the line element (2.7) is

\[
R(t, x) \equiv R = a(t)xg(t, x) = a(t)x \left( 1 + \frac{M(1 + \alpha)}{a(t)x} + \frac{M^2(1 + \alpha)}{4a(t)^2x^2} \right).
\]

Due to the spherical symmetry, one can rewrite the line element of the cosmological black hole solution (2.7) in the STVG theory in terms of areal radius as follows

\[
ds^2 = h_{jk}d\tilde{x}^j d\tilde{x}^k + R^2d\Omega^2,
\]

where \( \tilde{x}^j = (t, x) \), and

\[
h_{jk} = \text{diag} \left( -\frac{f(t, x)^2}{g(t, x)^2}, a(t)^2g(t, x)^2 \right).
\]

Consequently, by making use of the equation \( h^{jk}\partial_j \partial_k R = 0 \), which gives the location of apparent horizons in terms of areal radius, one can attain the apparent horizons of the line element (2.7) as the roots of the following quadratic equation

\[
H^2R^4 - R^2 + 4r_0R - 4r_1^2 = 0,
\]

where

\[
r_0 = \frac{M(1 + \alpha)}{2}, \quad r_1 = \frac{M\sqrt{\alpha(1 + \alpha)}}{2}.
\]

and \( H = \frac{a(t)}{a(t)} \) is the Hubble parameter in which ‘dot’ stands for time derivative. Increasing the values of the areal radius will lead to \( R \to \frac{1}{H} \) which is the value of the cosmological apparent horizon in the FLRW model. On the other hand, for \( H \to 0 \), Eq. (2.14) reduces to a quadratic equation whose two roots are the outer and the inner event horizons in the Schwarzschild-MOG black hole [21]. Again, as we pointed out in the previous section, for \( \alpha \to 0 \), Eq. (2.14) reduces to a cubic equation that gives the apparent horizons in McVittie spacetime in GR [29]. The fact that in the appropriate limits, the roots of Eq. (2.14) as the apparent horizons of line element (2.7) become a cosmological or a black hole event horizon is a vivid sign that the line element (2.7) is a cosmological black hole in the STVG framework. To be precise, at late cosmic time values, the positive Hubble factor shows that the line element (2.7) is a cosmological black hole in the theory of STVG [26]. Also, from Eq. (2.10) the location of the cosmological singularity in terms of areal radius (2.11) is in the form of

\[
R_{cs} = M \left( 1 + \alpha + \frac{\sqrt{1 + \alpha}}{M} \right).
\]

Eq. (2.16) demonstrates that growing \( M \) which leads to increasing \( \alpha \) results in appearing the cosmological singularity at some larger values of the areal radius.

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By solving the roots of Eq. (2.14) one can obtain the apparent horizons of the cosmological black hole solution in the STVG setup. Eq. (2.14) has three physical roots: \( R_+, R_-, \) and \( R_\ast \), so that \( R_\ast < R_- < R_+ \).

The explicit forms of these apparent horizons are

\[
R_\pm = \frac{\sqrt{A_1}}{2} \pm \frac{1}{2} \sqrt{\frac{2}{H^2} - A_1 - \frac{8r_0}{H^2\sqrt{A_1}}},
\]

(2.17)

and

\[
R_\ast = -\frac{\sqrt{A_1}}{2} + \frac{1}{2} \sqrt{\frac{2}{H^2} - A_1 + \frac{8r_0}{H^2\sqrt{A_1}}},
\]

(2.18)

in which we have defined

\[
A_1 \equiv \frac{\sqrt{2}}{3} \left( 1 - 48H^2 r_0^2 \right) + \frac{A_2}{3\sqrt{2}H^2} + \frac{2}{3H^2},
\]

(2.19)

and

\[
A_2 \equiv \left( \sqrt{(432H^2 r_0^2 - 288H^2 r_1^2 - 2)^2 - 4 (1 - 48H^2 r_1^2)^3 + 432H^2 r_0^2 - 288H^2 r_1^2 - 2} \right)^2.
\]

(2.20)

From early values of the cosmic time till a specific moment of it, there exist the cosmological singularity (2.16) and \( R_\ast \). Thereafter, the apparent horizons \( R_- \) and \( R_+ \) appear together in that specific value of cosmic time. Growing cosmic time \( t \) results in increasing \( R_+ \), so that it reaches the value of the cosmological apparent horizon in the FLRW model. Conversely, \( R_- \) becomes smaller by growing cosmic time, and for infinite values of cosmic time, it tends to the singularity. The apparent horizon \( R_\ast \) is always inside the singularity and separated from the exterior geometry (see Ref. [26] for more details). The situation is completely shown in Fig. 1. Accordingly, one can denote \( R_+ \) as the cosmological apparent horizon radius and \( R_- \) as the cosmological event horizon radius of the cosmological black hole solution (2.7) in STVG theory.

Fig. 2 is a three-dimensional illustration of three apparent horizons in addition to the singularity location of the cosmological black hole in the theory of STVG in terms of \((t,M)\) and \((t,\alpha)\). As mentioned above, Fig. 2 shows that the cosmological singularity will appear at some larger values of the areal radius by growing \( M \) which leads to an increase in \( \alpha \). From Fig. 2a we see that decreasing the central mass \( M \) of the black hole results in appearing \( R_- \) and \( R_+ \) in some earlier cosmic time together, so that for \( M = 0 \), they appear with \( R_\ast \) and the cosmological singularity at \( t = 0 \) simultaneously. Also, Fig. 2b illustrates the same behavior for \( R_- \) and \( R_+ \) in such a way that deceasing the STVG parameter \( \alpha \) lead to appear \( R_- \) and \( R_+ \) in some earlier cosmic time together, except that for \( \alpha = 0 \) (it is associated with McVittie solution) they appear at a specific value of cosmic time, which is not zero.

Due to the complexity of Eqs. (2.17)–(2.20) we cannot use the explicit form of these roots in the subsequent calculations to derive Hawking-like radiation as tunneling with and beyond semiclassical approximation for apparent horizons of the cosmological black hole in the setup of STVG. So, we rewrite Eq. (2.14) in the following form

\[
R^2 \equiv R^2_{AH} = \frac{1}{H^2} \left( 1 - \frac{4r_0}{R} + \frac{4r_1^2}{R^2} \right),
\]

(2.21)

in which we have made use of the following relation [26]

\[
\frac{f(t,x)^2}{g(t,x)} = 1 - \frac{4r_0}{R} + \frac{4r_1^2}{R^2}.
\]

(2.22)
Figure 1: The illustration of three physical apparent horizons of the cosmological black hole in the STVG theory versus cosmic time $t$ in which we have set $\alpha = 2.45$ for a supermassive black hole [17]. The blue, red, and orange thick lines are associated with the apparent horizons $R_-, R_+, R_*$ of the cosmological black hole in the STVG, respectively. Also, The black dashed line shows the location of the cosmological singularity $R_{cs}$.

Figure 2: The three-dimensional illustration of three physical apparent horizons of the cosmological black hole in the STVG theory versus; (a): $(t, M)$ and (b): $(t, \alpha)$. The blue, red, and orange surfaces are associated with the apparent horizons $R_-, R_+, R_*$ of the cosmological black hole in the STVG theory, respectively. Also, The gray transparent surface shows the location of cosmological singularity $R_{cs}$. 
Consequently, Eqs. (2.14) and (2.21) are equivalent, and Eq. (2.21) which contains all three physical roots $R_+, R_-$ and $R_\#$ can be applied to the rest of the calculations in this paper. Since we are interested in the spacetime events located in the casual future of the singularity, we focus only on $R_-$ and $R_+$ to express and plot the subsequent statements and figures.

3 Hawking-Like Radiation as Tunneling with Semiclassical Approximation

In this section, we confine all the calculations to the semiclassical approximation to eliminate higher-order quantum effects. In this sense, we follow both Hamilton-Jacobi [50–52] and Parikh-Wilczek [41] methods in the pseudo-Painlevé-Gullstrand coordinates system to study the Hawking-like radiation as tunneling for the apparent horizons of the cosmological black hole in STVG theory.

3.1 Hamilton-Jacobi Method for a Massive Particle

To consider the tunneling of a massive particle, we should use the coordinates system $(t, R, \theta, \varphi)$ to avoid the coordinate singularities of the spacetime metric (2.7). Therefore, line element (2.7) via the coordinates transformation $dx = \frac{dR}{a(t) f(t, x)} - H x dt$ can be rewritten in the following form [26]

$$ds^2 = - \left(1 - \frac{4r_0}{R} + \frac{4r_0^2}{R^2} - H^2 R^2\right) dt^2 - \frac{2RH}{\sqrt{1 - \frac{4r_0}{R} + \frac{4r_0^2}{R^2}}} dR dt + \frac{dR^2}{1 - \frac{4r_0}{R} + \frac{4r_0^2}{R^2}} + R^2 d\Omega^2,$$

(3.1)

which is the pseudo-Painlevé-Gullstrand form [33] of it. The corresponding Kodama vector [61] for the line element (3.1) is as follows

$$K^j \equiv -\epsilon_{jk} \nabla_k R = \left(\frac{\partial \chi}{\partial t}\right)^j,$$

(3.2)

where $\epsilon_{jk} = \sqrt{|\tilde{h}|} (dt)_j \wedge (dR)_k$ and $\tilde{h}$ are the volume form and the determinant of the 2-metric $\tilde{h}_{jk}$ (corresponding with $(t, R)$ sector of the line element (3.1)), respectively [33, 59, 61]. Using Eqs. (2.21) and (2.22), one can see $K^j K^j = -H^2 R_{AH}^2 \left(1 - \frac{4r_0^2}{R_{AH}^2}\right)$. The same result can be seen in McVittie spacetime and de Sitter solution in GR [33]. Consequently, this Kodama vector is null at $R = R_{AH}$ and is time-like and space-like at $R < R_{AH}$ and $R > R_{AH}$, respectively. So, the deduced Kodama vector in Eq. (3.2) is time-like inside the apparent horizons in the setup, where we focus on it. Note that the existence of the Kodama vector will play a key role in this study. To see the crucial role of the Kodama vector in this setup, we discuss some differences between a stationary black hole and a time-dependent spacetime. In static and stationary situations, a time-like Killing vector field exists outside the horizon and becomes null on it. By the time-like killing vector, one can define the conserved energy of a particle moving in the stationary black hole spacetime. In dynamical situations, however, there is no time-like Killing vector, but in spherically symmetric spacetimes, the Kodama vector mimics the features of a Killing vector and gives rise to a conserved current of a particle moving in the dynamical spacetime.

To study Hawking-like radiation as tunneling via the Hamilton-Jacobi method, one can suppose a radially moving particle with mass $m$ in the spacetime background (3.1). So, the Hamilton-Jacobi equation for the particle is as follows

$$g^{\mu\nu} \partial_\mu S \partial_\nu S + m^2 = 0,$$

(3.3)
in which $g_{\mu\nu}$ is the metric tensor corresponding with the line element (3.1) and $S$ is the particle action. One can use the Kodama vector (3.2) to define the energy $\omega$ and radial momentum $k_R$ associated with this radially tunneling particle, which are measured by an observer inside the apparent horizon, called Kodama observer

$$\omega = -K^i \partial_i S = -\partial_t S, \quad k_R = \left(\frac{\partial}{\partial R}\right)^i \partial_j S = \partial_n S.$$  (3.4)

Therefore, the action, $S$ can be written in the form of

$$S = -\int \omega dt + \int k_R dR.$$  (3.5)

Using the action (3.5), one can rewrite the $(t, R)$ sector of the Hamilton-Jacobi equation (3.3) to the following form

$$\left(1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2} - H^2 R^2\right) k_R + \frac{2HR\omega}{\sqrt{1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2}}} k_R + \left(m^2 - \frac{\omega^2}{1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2}}\right) = 0.$$  (3.6)

Eq. (3.6) has two roots for $k_R$ as follow

$$k_R = \frac{-\frac{HR}{R_{AH}} \pm \sqrt{1 - \frac{m^2}{\omega^2} \left(1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2} - H^2 R^2\right)}}{\left(1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2}\right) \left(1 - \frac{R^2}{R_{AH}^2}\right)} \omega,$$  (3.7)

in which the plus (minus) sign corresponds to the outgoing (incoming) motion. Since the energy and the radial momentum of the tunneling particle are measured by the fixed Kodama observer inside the apparent horizon, the Hawking-like radiation is, therefore, seen by the same observer, as we previously noted. The presence of the Kodama observer inside the apparent horizon necessitates us to consider the incoming motion, which is the same as the cases of the tunneling process in FLRW [59] and de Sitter spacetimes [63, 64]. This means that the particle tunnels from outside to inside the apparent horizon. From Eqs. (2.21) and (2.22), we can rewrite the radial momentum (3.7) of the incoming motion as

$$k_R = -\frac{R}{R_{AH}} + \frac{1 - \frac{m^2}{\omega^2} \left(1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2} \left(1 - \frac{R^2}{R_{AH}^2}\right)\right)}{\left(1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2}\right) \left(1 - \frac{R^2}{R_{AH}^2}\right)} \omega.$$  (3.8)

Since particle tunneling (through a classically forbidden region as a barrier) is responsible to produce the imaginary part of the action, $\text{Im} S = \text{Im} \int k_R dR$ we require only to calculate this part, while the left part is always real. Obviously, inserting Eq. (3.8) in the mentioned integral results in one pole in the action at the location of the apparent horizon $R_{AH}$. So, we have to deform the contour to gain the imaginary part of the incoming action, $S_{in}$ as follows

$$\text{Im} S_{in} = -\text{Im} \int \frac{R}{R_{AH}} + \frac{1 - \frac{m^2}{\omega^2} \left(1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2} \left(1 - \frac{R^2}{R_{AH}^2}\right)\right)}{\left(1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2}\right) \left(1 - \frac{R^2}{R_{AH}^2}\right)} \omega dR
$$

$$= \frac{\pi \omega R_{AH}}{1 - \frac{4\nu R}{R_{AH}} + \frac{4\nu^2 R^2}{R_{AH}^2}}.$$  (3.9)

To prove that the tunneling process in this dynamical setup is from outside to inside the apparent horizon, we can calculate the outgoing action, $S_{out}$ to find

$$\text{Im} S_{out} = \text{Im} \int -\frac{R}{R_{AH}} + \frac{1 - \frac{m^2}{\omega^2} \left(1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2} \left(1 - \frac{R^2}{R_{AH}^2}\right)\right)}{\left(1 - \frac{4\nu R}{R^2} + \frac{4\nu^2 R^2}{R^2}\right) \left(1 - \frac{R^2}{R_{AH}^2}\right)} \omega dR = 0.$$  (3.10)
Therefore, the action of outgoing motion has no imaginary part in the dynamical setup, while in the stationary black hole spacetimes, like Schwarzschild’s case in GR, the action of ingoing motion has no imaginary contribution \([41, 50–52, 59]\). Based on the WKB approximation, one can find the emission rate (transmission coefficient) of the tunneling particles as

\[
\Gamma \propto \exp \left( -2 \operatorname{Im} S_{in} \right). \quad (3.11)
\]

Again, by setting \(\alpha = 0\) in \(S_{in}\), one can find the corresponding emission rate for McVittie solution in GR.

Comparing Eq. (3.11) with the Boltzmann factor \(\Gamma \propto \exp \left( -\frac{\omega}{T} \right)\), we see that the emission rate has the temperature \(T\) in the form of

\[
T = \frac{h}{2\pi R_{AH}} \left( 1 - \frac{4r_0}{R_{AH}} + \frac{4r_1^2}{R_{AH}^2} \right) = \frac{1}{2\pi R_{AH}} \left( 1 - \frac{2M(1+\alpha)}{R_{AH}} + \frac{M^2\alpha(1+\alpha)}{R_{AH}^2} \right). \quad (3.12)
\]

This result is independent of the mass, \(m\) of the particle. Through tunneling of massive particles from outside to inside the apparent horizon, the interior Kodama observer will measure thermal radiation with temperature (3.12). Such a procedure expressed by the Hamilton-Jacobi method can be described as thermal Hawking-like radiation of the apparent horizons of the cosmological black hole in STVG in the same meaning of the particle tunneling procedure firstly suggested by Parikh and Wilczek \([41]\), which provides the Hawking radiation of the black hole as a tunneling process. Again, one can see that temperature (3.12) satisfies the correspondence principle, so that by setting \(\alpha = 0\), we can reach the corresponding Hawking-like temperature of the apparent horizons of McVittie spacetime, which is not in the literature, and also setting \(\alpha = M = 0\) results in Hawking-like temperature of the apparent horizon of the spatially flat case of FLRW universe \([59, 60]\).

### 3.2 Parikh-Wilczek Method for a Massless Particle

Now, we prove that the Hawking-like temperature (3.12) can be gained by making use of the Parikh-Wilczek method \([41, 59, 63, 64]\). The s-wave radiation (across the apparent horizon) of a tunneling massless particle moving along a radial null geodesic is considered to derive the Hawking-like radiation. Again, considering the semiclassical approximation, the transmission coefficient can be found as an exponential function of the imaginary part of the massless particle action without considering higher-order quantum effects.

In the cosmological black hole spacetime described by the pseudo-Painlevé-Gullstrand coordinates system introduced in the line element (3.1), the radial null geodesic can be found as follows \([26]\)

\[
\frac{dR}{dt} = \dot{R} = \pm \sqrt{1 - \frac{4r_0}{R} + \frac{4r_1^2}{R^2} \left( \sqrt{1 - \frac{4r_0}{R} + \frac{4r_1^2}{R^2} \pm HR} \right)} \quad (3.13)
\]

Again, one can rewrite Eq. (3.13) by using Eqs. (2.21) and (2.22) as

\[
\dot{R} = \pm \left( 1 - \frac{4r_0}{R} + \frac{4r_1^2}{R^2} \right) \left( 1 \pm \frac{R}{R_{AH}} \right), \quad (3.14)
\]

in which the plus (minus) sign is associated with an outgoing (ingoing) radial null geodesic. As previously explained, we choose the incoming mode since the particle tunneling is from outside to inside the apparent horizon. Also, as mentioned heretofore, we only need to calculate the imaginary part of the incoming action, which now can be written in the form

\[
\operatorname{Im} S_{in} = \operatorname{Im} \int_{R_i}^{R_f} p_R dR = \operatorname{Im} \int_{R_i}^{R_f} \int_0^{p_R} dp_R' dR', \quad (3.15)
\]
where $p_R$ is the canonical momentum of the tunneling particle, which is initially at $R_i$, somewhere slightly outside the apparent horizon, and then crosses it to $R_f$, somewhere slightly inside the apparent horizon. According to the Hamiltonian equation

$$\dot{R} = \frac{\partial \tilde{H}}{\partial p_R} = \left. \frac{d\tilde{H}}{dp_R} \right|_R,$$

(3.16)

where $\tilde{H}$ is the Hamiltonian of the tunneling particle, one can calculate the imaginary part of the incoming action (3.15) as follows

$$\text{Im} S_{in} = \text{Im} \int_{R_i}^{R_f} dR \int \frac{1}{R} d\tilde{H}$$

$$= \text{Im} \int_{R_i}^{R_f} \frac{\omega}{R} dR$$

$$= -\text{Im} \int_{R_i}^{R_f} \left( 1 - \frac{4\omega}{R} + \frac{4\omega^2}{R^2} \right) \left( 1 - \frac{R}{R_{AH}} \right) dR$$

$$= \frac{\pi \omega R_{AH}}{1 - \frac{4\omega}{R_{AH}} + \frac{4\omega^2}{R_{AH}^2}}.$$

(3.17)

It is worth noting that in our setup, like de Sitter spacetime $[63,64]$, the integration over Hamiltonian results in the energy, $\omega$ of the tunneling particle, which is measured by the interior Kodama observer associated with the Kodama vector (3.2). Again, by making use of the emission rate (3.11) and also comparing it with the Boltzmann factor, one can obtain the Hawking-like temperature

$$T = \frac{\omega \hbar}{2 \text{Im} S_{in}} = \frac{\hbar}{2\pi R_{AH}} \left( 1 - \frac{4\alpha}{R_{AH}} + \frac{4\alpha^2}{R_{AH}^2} \right) = \frac{\hbar}{2\pi R_{AH}} \left( 1 - \frac{2M(1+\alpha)}{R_{AH}} + \frac{M^2\alpha(1+\alpha)}{R_{AH}^2} \right).$$

(3.18)

Therefore, we proved that for massive or massless particle tunneling via Hamilton-Jacobi or Parikh-Wilczek methods with semiclassical approximation, an interior Kodama observer could measure the Hawking-like temperature (3.12) or (3.18) which are the same for the thermal Hawking-like radiation. The temperature is associated with the apparent horizons of the cosmological black hole solution in the STVG theory, which again satisfies the correspondence principle.

So far, we found out the emission rate of the incoming action and also, the Hawking-like temperature for the ingoing Hawking-like radiation of the apparent horizons $R_-$ and $R_+$ of the cosmological black hole in the STVG theory in both Hamilton-Jacobi or Parikh-Wilczek methods based on semiclassical approximation. We showed that in both methods, the emission rate and the Hawking-like temperature are the same, respectively. Also, we noted that for case $\alpha = 0$ the corresponding quantities of McVittie solution in GR can be achieved. Now, we can put Eq. (2.17) of the apparent horizons $R_-$ and $R_+$ of the cosmological black hole in the STVG theory in Eq. (3.11) and Eq. (3.12) or Eq. (3.17) and Eq. (3.18) to investigate the emission rate and Hawking-like temperature of these apparent horizons, respectively.

Fig. 3 illustrates the graph of the function $\ln(\Gamma)$ in terms of $t$ for both apparent horizons $R_-$ and $R_+$ of the cosmological black hole in the STVG theory and McVittie solution in GR corresponding with $\alpha = 0$. From Fig. 3 we see that since early cosmic time values till a specific moment of it because the apparent horizons do not exist, the emission rate $\Gamma$ of the radiated spectra is zero. For McVittie solution in GR, the emission rate appears sooner than the case of the cosmological black hole in STVG. Also, the function $\ln(\Gamma)$ for the case of the cosmological black hole in STVG tends to the case of McVittie solution in GR by increasing cosmic time. This is because the cosmological apparent horizon $R_+$ in both McVittie solution in GR and the cosmological black hole in STVG tend to the cosmological apparent horizon in FLRW universe by increasing cosmic time.
Figure 3: The illustration of the function \( \ln(\Gamma) \) versus cosmic time \( t \) for the apparent horizons \( R_- \) and \( R_+ \) of the cosmological black hole in the STVG theory and McVittie solution in GR in which we have set \( \omega = 1 \) and \( \alpha = 2.45 \) for a supermassive black hole [17]. The blue and red thick lines are respectively associated with \( R_- \) and \( R_+ \) of the cosmological black hole. The dashed brown and dot-dashed purple lines are respectively associated with \( R_- \) and \( R_+ \) of the McVittie solution.

Figure 4: The illustration of the temperature \( T \) versus cosmic time \( t \) for the apparent horizons \( R_- \) and \( R_+ \) of the cosmological black hole in the STVG theory and McVittie solution in GR in which we have set \( h = 1 \) and \( \alpha = 2.45 \) for a supermassive black hole [17]. The blue and red thick lines are respectively associated with \( R_- \) and \( R_+ \) of the cosmological black hole. The dashed brown and dot-dashed purple lines are respectively associated with \( R_- \) and \( R_+ \) of the McVittie solution.
Fig. 4 shows the graph of the temperature $T$ versus cosmic time $t$ for the apparent horizons $R_-$ and $R_+$ of the cosmological black hole in the STVG theory and the McVittie solution in GR corresponding with $\alpha = 0$. In Fig. 4, similar to Fig. 3, the temperature of the cosmological black hole in STVG theory becomes manifest later than in the McVittie case. The Hawking-like temperature associated with the cosmological apparent horizon $R_+$ in both McVittie spacetime in GR and the cosmological black hole in STVG has some bigger values, and again, the temperature of the case of the cosmological black hole in STVG approaches the corresponding case of the McVittie solution in GR by increasing cosmic time. Also, the temperature associated with $R_-$ of the cosmological black hole in STVG approaches the corresponding case in the McVittie solution.

### 4 Hawking-Like Radiation as Tunneling Beyond Semiclassical Approximation

In this section, we take into account all the higher-order quantum effects to study the Hawking-like radiation as tunneling beyond semiclassical approximation \cite{53} for the apparent horizons of the cosmological black hole in STVG theory. To do this, we just follow the Hamilton-Jacobi method in the pseudo-Painlevé-Gullstrand coordinates system and expand the action of the tunneling particle in the powers of $\hbar$ to apply all the quantum corrections to the semiclassical approximation.

The Klein-Gordon equation for a massless scalar field $\psi$ in the background spacetime (3.1) is as follows
\begin{equation}
-\frac{h^2}{\sqrt{-g}} \partial_\mu (g^{\nu\mu} \sqrt{-g} \partial_\nu) \psi = 0. \tag{4.1}
\end{equation}

Due to the spherical symmetry of the cosmological black hole in STVG theory, the $(t-R)$ sector of the spacetime (3.1) is considered to solve the Klein-Gordon equation (4.1). Thus, inserting the line element (3.1) into the Klein-Gordon equation (4.1) results in the following equation
\begin{equation}
-\frac{(\partial_t)^2 \psi}{1 - \frac{4 \alpha}{R^2} + \frac{4 \alpha^2}{R^4}} - \frac{2HR \partial_t \partial_R \psi}{1 - \frac{4 \alpha}{R^2} + \frac{4 \alpha^2}{R^4}}^{1/2} + \left(-2H^2R + \frac{4\alpha}{R^2} - \frac{8\alpha^2}{R^4}\right) \partial_R \psi + \left(1 - \frac{4\alpha}{R^2} + \frac{4\alpha^2}{R^4}\right) (\partial_t)^2 \psi
+ H \left(\frac{\left(\frac{4\alpha}{R^2} - \frac{8\alpha^2}{R^4}\right)}{2 \left(1 - \frac{4 \alpha}{R^2} + \frac{4 \alpha^2}{R^4}\right)^{3/2}} - \left(1 - \frac{4\alpha}{R^2} + \frac{4\alpha^2}{R^4}\right)^{-1/2}\right) \partial_t \psi = 0. \tag{4.2}
\end{equation}

From quantum mechanics, the wave function of the scalar field $\psi$ is given by the following standard ansatz
\begin{equation}
\psi (t, R) = \exp \left[\frac{i}{\hbar} S (t, R)\right], \tag{4.3}
\end{equation}

where again, $S (t, R)$ is the action of the tunneling particle. Hence adopting the ansatz (4.3) in Klein-Gordon equation (4.2) results in
\begin{equation}
\frac{\partial^2 S}{\partial t^2} + H \left(1 - \frac{4\alpha}{R^2} + \frac{4\alpha^2}{R^4}\right)^{3/2} \frac{\partial S}{\partial t} - \frac{1}{R} \left(1 - \frac{4\alpha}{R^2} + \frac{4\alpha^2}{R^4}\right) \left(H^2 R_{AH}^2 \left[1 - \frac{R^2}{R_{AH}^2}\right] + 1 - H^2 R^2 - \frac{4\alpha^2}{R^4}\right) \frac{\partial S}{\partial R}
+ 2HR \left(1 - \frac{4\alpha}{R^2} + \frac{4\alpha^2}{R^4}\right)^{1/2} \frac{\partial^2 S}{\partial t \partial R} - H^2 R_{AH}^2 \left(1 - \frac{R^2}{R_{AH}^2}\right) \left(1 - \frac{4\alpha}{R^2} + \frac{4\alpha^2}{R^4}\right) \frac{\partial^2 S}{\partial R^2}
- \left(\frac{i}{\hbar}\right) \left[\left(\frac{\partial S}{\partial t}\right)^2 - H^2 R_{AH}^2 \left(1 - \frac{R^2}{R_{AH}^2}\right) \left(1 - \frac{4\alpha}{R^2} + \frac{4\alpha^2}{R^4}\right) \left(\frac{\partial S}{\partial R}\right)^2 + 2HR \left(1 - \frac{4\alpha}{R^2} + \frac{4\alpha^2}{R^4}\right)^{1/2} \frac{\partial S}{\partial R} \frac{\partial S}{\partial t}\right] = 0. \tag{4.4}
\end{equation}
One can expand the action $S$ of the tunneling particle in the powers of $\hbar$ as follows

$$S(t, R) = S_0(t, R) + \hbar S_1(t, R) + \hbar^2 S_2(t, R) + \ldots = S_0(t, R) + \sum_{n} \hbar^n S_n(t, R) . \tag{4.5}$$

where $n = 1, 2, \ldots$ is the counter. In Eq. (4.5) the semiclassical term is $S_0$ while the other terms with the powers of $\hbar$ are the quantum corrections to this semiclassical value. Now, we can substitute Eq. (4.5) in Eq. (4.4) and then equate $\hbar$ powers on both sides to achieve an interesting result as follows

$$\begin{align*}
\hbar^0 : & \quad \frac{\partial S_0}{\partial t} = -H R \sqrt{1 - \frac{4r_0}{R} + 4r_0^2 R^2} \pm \left(1 - \frac{4r_0}{R} + 4r_0^2 R^2\right) \frac{\partial S_0}{\partial R} , \\
\hbar^1 : & \quad \frac{\partial S_1}{\partial t} = -H R \sqrt{1 - \frac{4r_0}{R} + 4r_0^2 R^2} \pm \left(1 - \frac{4r_0}{R} + 4r_0^2 R^2\right) \frac{\partial S_1}{\partial R} , \\
\hbar^2 : & \quad \frac{\partial S_2}{\partial t} = -H R \sqrt{1 - \frac{4r_0}{R} + 4r_0^2 R^2} \pm \left(1 - \frac{4r_0}{R} + 4r_0^2 R^2\right) \frac{\partial S_2}{\partial R} , \\
\vdots 
\end{align*} \tag{4.6}$$

Because all of the relations in the linear differential equations (4.6) are the same in their functional form, their solutions are dependent on each other. Hence, all $S_n$ terms can be written as functions of $S_0$. Therefore, one can rewrite Eq. (4.5) to the following form

$$S(t, R) = S_0(t, R) \left(1 + \sum_{n} \lambda_n \hbar^n\right) . \tag{4.7}$$

As we mentioned above, $S_0$ is the semiclassical term and the other terms $S_0(t, R) \left(\sum_{n} \lambda_n \hbar^n\right)$ are due to the higher-order quantum effects in which $\lambda_n$ are some proportionality constants. Based on the ansatz (4.3) and the expansion (4.5), one can see that $S_0$ has the dimension of $\hbar$. Consequently, $\lambda_n$ has the dimension of $(1/\hbar^n)$. Since we have set $G_N = c = 1$, the reduced Planck constant $\hbar$ has the order of the square of the Planck mass $M_{pl}$. Thus, the constants $\lambda_n$ have the dimension of $(1/E_{tot}^2)^n$ in which $E_{tot}$ is the total, physical mass-energy inside the apparent horizons of the cosmological black hole in STVG theory. This is because the cosmological black hole lives in the dynamic background. This mass-energy is determined with some quasi-local energy structure related to the apparent horizon. Due to the spherical symmetry of the line element (2.12), we can use the Misner-Sharp-Hernandez (MSH) mass, which is defined as

$$E_{tot} = M_{MSH} = \frac{R}{2G} \left(1 - \hbar^2 \delta j \partial_k R\right) \bigg|_{R_{AH}} . \tag{4.8}$$

Also, the generalized MSH mass is the Hawking-Hayward (HH) quasi-local energy in the absence of spherical symmetry [65, 66]. One can calculate MSH mass by inserting Eqs. (2.12) and (2.21) into Eq. (4.8) to find the following result

$$M_{MSH} = \frac{4}{3} \pi (1 + \alpha) \rho R_{AH}^3 + M - \frac{\alpha M^2}{2R_{AH}} = \frac{R_{AH}}{2(1 + \alpha)} , \tag{4.9}$$

where in the first equality, we use the relation $H^2 = 8\pi (1 + \alpha) \rho/3$ (see the proof in Ref. [26]). So, we can rewrite the action (4.7) to the form

$$S(t, R) = S_0(t, R) \left(1 + \sum_{n} \frac{4(1 + \alpha)^2}{R_{AH}^3} \eta_n \hbar^n\right) , \tag{4.10}$$

where $\eta_n$ are some dimensionless constants.
As in the previous section, again the Kodama vector (3.2) is used to define the energy \( \omega \) and the radial momentum \( k_R \) of the tunneling particle in Eq. (3.4). Therefore, one can read \( S_0 \) as

\[
S_0 = -\int \omega dt + \int k_R dR. \tag{4.11}
\]

Combining Eq. (4.11) and the first relation in Eq. (4.6) results in Eq. (3.7) for the radial momentum \( k_R \). Here we proceed with both outgoing and ingoing motion. Using Eqs. (3.7), (4.10), and (4.11), therefore, we can read the action of the tunneling particle for both outgoing and ingoing motion to the following forms, respectively

\[
S_{\text{out}}(t, R) = \left[ 1 + \sum_n \frac{4(1 + \alpha)^2}{R^2_{\text{AH}}} \eta_n h^n \right] \left[ -\int \omega dt + \int \omega \frac{-HR + \sqrt{1 - \frac{4\alpha^2}{R^2} + \frac{4r^2}{R^2}}}{\left( 1 - \frac{4\alpha^2}{R^2} + \frac{4r^2}{R^2} \right)^{1/2}} dR \right], \tag{4.12}
\]

and

\[
S_{\text{in}}(t, R) = \left[ 1 + \sum_n \frac{4(1 + \alpha)^2}{R^2_{\text{AH}}} \eta_n h^n \right] \left[ -\int \omega dt + \int \omega \frac{-HR - \sqrt{1 - \frac{4\alpha^2}{R^2} + \frac{4r^2}{R^2}}}{\left( 1 - \frac{4\alpha^2}{R^2} + \frac{4r^2}{R^2} \right)^{1/2}} dR \right]. \tag{4.13}
\]

In the dynamical spacetime of the cosmological black hole solution in the STVG theory, like the FLRW universe in GR, the Hawking-like radiation is detected by the interior Kodama observer using the Kodama vector (3.2). As previously mentioned, this is because the energy of the particle is defined by the Kodama vector, which is time-like, null, and space-like in outside, on, and inside the apparent horizon, respectively. So, there is a discrepancy between the Kodama vector of the interior and exterior regions of the apparent horizon. This discrepancy makes affects the temporal part of the action. Consequently, in Eqs. (4.12) and (4.13) the integral part has also an imaginary part in the action of the tunneling particle. Such a situation can be seen in the Schwarzschild black hole [67] in GR. Thus, using the well-known relation \( P = |\psi|^2 = |\exp \left[ \frac{i}{\hbar} \mathbf{S}(t, R) \right]|^2 \) in quantum mechanics, one can find the outgoing and ingoing probabilities as follows

\[
P_{\text{out}} = \exp \left[ -\frac{2}{\hbar} \left( 1 + \sum_n \frac{4(1 + \alpha)^2}{R^2_{\text{AH}}} \eta_n h^n \right) \left( -\text{Im} \int \omega dt + \text{Im} \int \omega \frac{-HR + \sqrt{1 - \frac{4\alpha^2}{R^2} + \frac{4r^2}{R^2}}}{\left( 1 - \frac{4\alpha^2}{R^2} + \frac{4r^2}{R^2} \right)^{1/2}} dR \right) \right]. \tag{4.14}
\]

and

\[
P_{\text{in}} = \exp \left[ \frac{2}{\hbar} \left( 1 + \sum_n \frac{4(1 + \alpha)^2}{R^2_{\text{AH}}} \eta_n h^n \right) \left( -\text{Im} \int \omega dt + \text{Im} \int \omega \frac{-HR - \sqrt{1 - \frac{4\alpha^2}{R^2} + \frac{4r^2}{R^2}}}{\left( 1 - \frac{4\alpha^2}{R^2} + \frac{4r^2}{R^2} \right)^{1/2}} dR \right) \right]. \tag{4.15}
\]

Dividing \( P_{\text{in}} \) by \( P_{\text{out}} \) results in the tunneling rate, \( \Gamma \) as follows

\[
\Gamma = \frac{P_{\text{in}}}{P_{\text{out}}} = \exp \left[ \frac{4}{\hbar} \omega \left( 1 + \sum_n \frac{4(1 + \alpha)^2}{R^2_{\text{AH}}} \eta_n h^n \right) \text{Im} \int \left( \frac{1}{1 - \frac{4\alpha^2}{R^2} + \frac{4r^2}{R^2}} \right) dR \right]. \tag{4.16}
\]

The integral in Eq. (4.16) has a pole at \( R = R_{\text{AH}} \). So, by deforming the contour, one can find the final form for the tunneling rate as follows

\[
\Gamma = \frac{P_{\text{in}}}{P_{\text{out}}} = \exp \left[ -\frac{2}{\hbar} \omega \left( 1 + \sum_n \frac{4(1 + \alpha)^2}{R^2_{\text{AH}}} \eta_n h^n \right) \frac{\pi R_{\text{AH}}}{1 - \frac{4\alpha^2}{R^2_{\text{AH}}} + \frac{4r^2}{R^2_{\text{AH}}}} \right]. \tag{4.17}
\]

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Then, the principle of “detailed balance” [50,51] gives us the following relation
\[ \Gamma = \frac{P_{in}}{P_{out}} = \exp \left( -\frac{\omega}{T_{BSA}} \right). \] (4.18)

Finally, we can find the Hawking-like temperature corresponding with the apparent horizons in the cosmological black hole solution within the STVG framework beyond the semiclassical approximation as follows
\[ T_{BSA} = T \left( 1 + \sum_n \frac{4(1 + \alpha)^2 \eta_n \hbar^n}{R_{AH}^2} \right)^{-1}, \quad T = \frac{\hbar}{2\pi R_{AH}} \left( 1 - \frac{2M(1 + \alpha)}{R_{AH}} + \frac{M^2\alpha(1 + \alpha)}{R_{AH}^2} \right), \] (4.19)
in which \( T \) is the semiclassical Hawking-like temperature, which we found in the previous section in Eqs. (3.12) and (3.18). Also, the other terms are the corrections due to consider the higher-order quantum effects. Accordingly, we have found all the quantum corrections to the semiclassical Hawking-like temperature derived in the previous section associated with the thermal Hawking-like radiation of apparent horizons of the cosmological black hole in the STVG theory. Again, all deduced results in the limit \( \alpha \to 0 \) tend to the corresponding ones in the McVittie solution in GR.

5 Summary and Conclusions

We used Hamilton-Jacobi and Parikh-Wilczek methods based on the semiclassical and beyond semiclassical approximation to derive the Hawking-like radiation as tunneling of the apparent horizons in the cosmological black hole solution in the framework of STVG theory and McVittie spacetime in GR. In semiclassical approximation, the deduced Hawking-like temperature of the apparent horizons of the cosmological black hole in the STVG theory completely satisfies the correspondence principle so that for \( \alpha = 0 \) it becomes the Hawking-like temperature of the apparent horizons of McVittie spacetime, which is not in the literature and also, by setting \( \alpha = 0, M = 0 \) it becomes the Hawking-like temperature of the apparent horizon of the spatially flat FLRW universe [59]. On the other hand, we took into account all the higher-order quantum effects to study the Hawking-like radiation as tunneling beyond semiclassical approximation for the apparent horizons of the cosmological black hole in the STVG theory, which again completely satisfies the correspondence principle. This resulted in some extra terms based on all the quantum corrections added to the semiclassical Hawking-like temperature. In addition, some remarks are in order. First, since in non-stationary spacetimes there is no possibility to determine Killing vectors, we computed the Kodama vector to define the energy of the tunneling particle. Therefore, the Hawking-like radiation is detected by a Kodama observer inside the apparent horizons of the cosmological black hole in the STVG theory and the McVittie spacetime in GR, and also, the Hawking-like temperature is attributed to the apparent horizons by this interior Kodama observer. Second, we can say that Hawking radiation is not always associated with an event horizon in spacetime. That is to conclude that the existence of event horizon is not a key cause of Hawking radiation, which was widely accepted before in the community of black hole physics. Third, the explicit forms of the three apparent horizons (\( R_{-}, R_{+}, \) and \( R_{*} \)) of the cosmological black hole in the STVG theory were obtained and plotted versus cosmic time to better understand their behavior. We came to the conclusion that \( R_{*} \) is always inside the singularity and separated from the exterior geometry but \( R_{-} \) and \( R_{+} \) appear together in a specific value of cosmic time and are denoted as the cosmological event horizon radius and the cosmological apparent horizon radius of the cosmological black hole solution in the STVG theory, respectively. Forth, we found out the emission rate of the incoming action and also, the Hawking-like temperature for the ingoing Hawking-like radiation for the apparent horizons \( R_{-} \) and \( R_{+} \) of the cosmological black hole in the STVG theory in both Hamilton-Jacobi or Parikh-Wilczek methods based on semiclassical approximation. We concluded that since early cosmic time values till a specific
moment of it because the apparent horizons do not exist, the emission rate $\Gamma$ of the radiated spectra is zero. For the McVittie solution in GR, the emission rate appears sooner than the case of the cosmological black hole in the framework of the STVG theory. Also, the function $\ln(\Gamma)$ for the case of the cosmological black hole in the STVG theory tends to the case of the McVittie solution in GR by increasing cosmic time. Furthermore, the temperature of the cosmological black hole in the STVG setup becomes manifest later than in the McVittie case. The Hawking-like temperature associated with the cosmological apparent horizon $R_+$ in both the McVittie spacetime in GR and the cosmological black hole in the framework of STVG has some bigger values and again, the temperature of the case of the cosmological black hole in the STVG theory approaches the corresponding case of the McVittie solution in GR by increasing cosmic time. This work can be a step towards a better understanding of cosmological black holes in the STVG setup and also, the nature of Hawking-like radiation of apparent horizons. In future work, we aim to write the thermodynamics of the cosmological black hole in the setup of the STVG framework based on the procedure performed in Refs. [35, 68–73].

References

[1] Perlmutter, S. et al. Measurements of $\Omega$ and $\Lambda$ from 42 high-redshift supernovae. *The Astrophysical Journal*, 517(2), 565 (1999).

[2] Riess, A. G. et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *The Astrophysical Journal*, 116(3), 1009 (1998).

[3] Garnavich, P. M. et al. Supernova limits on the cosmic equation of state. *The Astrophysical Journal*, 509(1), 74 (1998).

[4] Sofue, Y., and Rubin, V. Rotation curves of spiral galaxies. *Annual Review of Astronomy and Astrophysics*, 39(1), 137-174 (2001).

[5] Sofue, Y. Rotation and mass in the Milky Way and spiral galaxies. *Publications of the Astronomical Society of Japan*, 69(1) (2017).

[6] Ettori, S. et al. Mass profiles of Galaxy Clusters from X-ray analysis. *Space Science Reviews*, 177(1-4), 119-154 (2013).

[7] Voigt, L. M., and Fabian, A. C. Galaxy cluster mass profiles. *Monthly Notices of the Royal Astronomical Society*, 368(2), 518-533 (2006).

[8] Olive, K. A., and Pospelov, M. Evolution of the fine structure constant driven by dark matter and the cosmological constant. *Physical Review D*, 65(8), 085044 (2002).

[9] Alcaniz, J. S. Testing dark energy beyond the cosmological constant barrier. *Physical Review D*, 69(8), 083521 (2004).

[10] Copeland, E. J., Sami, M., and Tsujikawa, S. Dynamics of dark energy. *International Journal of Modern Physics D*, 15(11), 1753-1935 (2006).

[11] Magana, J., and Matos, T. A brief review of the scalar field dark matter model. In *Journal of Physics: Conference Series*, 378(1), 012012, IOP Publishing (2012, August).

[12] Arun, K., Gudennavar, S. B., and Sivaram, C. Dark matter, dark energy, and alternate models: A review. *Advances in Space Research*, 60(1), 166-186 (2017).

[13] Moffat, J. W. Scalar–tensor–vector gravity theory. *Journal of Cosmology and Astroparticle Physics*, 2006(03), 004 (2006).
[14] Brownstein, J. R., and Moffat, J. W. Galaxy rotation curves without nonbaryonic dark matter. *The Astrophysical Journal, 636*(2), 721 (2006).

[15] Moffat, J. W., and Rahvar, S. The MOG weak field approximation and observational test of galaxy rotation curves. *Monthly Notices of the Royal Astronomical Society, 436*(2), 1439-1451 (2013).

[16] Moffat, J. W., and Toth, V. T. Rotational velocity curves in the Milky Way as a test of modified gravity. *Physical Review D, 91*(4), 043004 (2015).

[17] Brownstein, J. R., and Moffat, J. W. Galaxy cluster masses without non-baryonic dark matter. *Monthly Notices of the Royal Astronomical Society, 367*(2), 527-540 (2006).

[18] Brownstein, J. R., and Moffat, J. W. The Bullet Cluster 1E0657-558 evidence shows modified gravity in the absence of dark matter. *Monthly Notices of the Royal Astronomical Society, 382*(1), 29-47 (2007).

[19] Moffat, J. W., and Rahvar, S. The MOG weak field approximation–II. Observational test of Chandra X-ray clusters. *Monthly Notices of the Royal Astronomical Society, 441*(4), 3724-3732 (2014).

[20] Cai, X. C., and Miao, Y. G. High-dimensional Schwarzschild black holes in scalar–tensor–vector gravity theory. *The European Physical Journal C, 81*(6), 1-12 (2021).

[21] Moffat, J. W. Black holes in modified gravity (MOG). *The European Physical Journal C, 75*(4), 175 (2015).

[22] Armengol, F. G. L., and Romero, G. E. Neutron stars in Scalar-tensor-vector gravity. *General Relativity and Gravitation, 49*(2), 27 (2017).

[23] Banerjee, S., Shankar, S., and Singh, T. P. Constraints on modified gravity models from white dwarfs. *Journal of Cosmology and Astroparticle Physics, 2017*(10), 004 (2017).

[24] Roshan, M. Exact cosmological solutions for MOG. *The European Physical Journal C, 75*(9), 1-8 (2015).

[25] Jamali, S., Roshan, M., and Amendola, L. On the cosmology of scalar-tensor-vector gravity theory. *Journal of Cosmology and Astroparticle Physics, 2018*(01), 048 (2018).

[26] Pérez, D., and Romero, G. E. Exact cosmological black hole solutions in Scalar Tensor Vector Gravity. *Classical and Quantum Gravity, 36*(24), 245022 (2019).

[27] McVittie, G. C. The mass-particle in an expanding universe. *Monthly Notices of the Royal Astronomical Society, 93*, 325-339 (1933).

[28] Faraoni, V., and Jacques, A. Cosmological expansion and local physics. *Physical Review D, 76*(6), 063510 (2007).

[29] Kaloper, N., Kleban, M., and Martin, D. McVittie’s legacy: black holes in an expanding universe. *Physical Review D, 81*(10), 104044 (2010).

[30] Carrera, M., and Giulini, D. Generalization of McVittie’s model for an inhomogeneity in a cosmological spacetime. *Physical Review D, 81*(4), 043521 (2010).

[31] Faraoni, V., Moreno, A. F. Z., and Prain, A. Charged McVittie spacetime. *Physical Review D, 89*(10), 103514 (2014).

[32] Nolan, B. C. Particle and photon orbits in McVittie spacetimes. *Classical and Quantum Gravity, 31*(23), 235008 (2014).

[33] Faraoni, V. Cosmological and black hole apparent horizons. (Vol. 907). *Cham: Springer International Publishing,* (2015).
[34] Antoniou, I., and Perivolaropoulos, L. Geodesics of McVittie spacetime with a phantom cosmological background. *Physical Review D*, 93(12), 123520 (2016).

[35] Akbar, M., Brahimi, T., and Qaisar, S. M. Thermodynamic analysis of cosmological black hole. *Communications in Theoretical Physics*, 67(1), 47 (2017).

[36] Bekenstein, J. D. Black holes and entropy. *Physical Review D*, 7, 2333 (1973).

[37] Hawking, S. W. Black hole explosions? *Nature*, 248(5443), 30–31 (1974).

[38] Hawking, S. W. Particle creation by black holes. *Communications in mathematical physics*, 43(3), 199–220 (1975).

[39] Medved, A. J. M., and Vagenas, E. C. On Hawking radiation as tunneling with back-reaction. *Modern Physics Letters A*, 20(32), 2449-2453 (2005).

[40] Gibbons, G. W., and Hawking, S. W. Cosmological event horizons, thermodynamics and particle creation. *Physical Review D*, 15(10), 2738 (1977).

[41] Parikh, M. K., and Wilczek, F. Hawking radiation as tunneling. *Physical Review Letters*, 85(24), 5042 (2000).

[42] Arzano, M., Medved, A. J. M., and Vagenas, E. C. Hawking radiation as tunneling through the quantum horizon. *Journal of High Energy Physics*, 2005(09), 037 (2005).

[43] Zhang, J., and Zhao, Z. Hawking radiation via tunneling from Kerr black holes. *Modern Physics Letters A*, 20(22), 1673-1681 (2005).

[44] Zhang, J., and Zhao, Z. Hawking radiation of charged particles via tunneling from the Reissner-Nordström black hole. *Journal of High Energy Physics*, 2005(10), 055 (2005).

[45] Jiang, Q. Q., Wu, S. Q., and Cai, X. Hawking radiation as tunneling from the Kerr and Kerr-Newman black holes. *Physical Review D*, 73(6), 064003 (2006).

[46] Wu, S. Q., and Jiang, Q. Q. Remarks on Hawking radiation as tunneling from the BTZ black holes. *Journal of High Energy Physics*, 2006(03), 079 (2006).

[47] Miao, Y. G., Xue, Z., and Zhang, S. J. Massive charged particles’ tunneling from spherical charged black hole *EPL (Europhysics Letters)*, 96(1), 10008 (2011).

[48] Nozari, K., and Saghafi, S. Natural cutoffs and quantum tunneling from black hole horizon. *Journal of High Energy Physics*, 2012(11), 5 (2012).

[49] Saghafi, S., Nozari, K., and Hajebrahimi, M. Thermodynamics of non-commutative scalar-tensor-vector gravity black holes. *International Journal of Geometric Methods in Modern Physics*, 18(2), 2150024 (2021).

[50] Srinivasan, K., and Padmanabhan, T. Particle production and complex path analysis. *Physical Review D*, 60(2), 024007 (1999).

[51] Shankaranarayanan, S., Padmanabhan, T., and Srinivasan, K. Hawking radiation in different coordinate settings: complex paths approach. *Classical and Quantum Gravity*, 19(10), 2671 (2002).

[52] Angheben, M. et al. Hawking radiation as tunneling for extremal and rotating black holes. *Journal of High Energy Physics*, 2005(05), 014 (2005).

[53] Banerjee, R., and Majhi, B. R. Quantum tunneling beyond semiclassical approximation. *Journal of High Energy Physics*, 2008(06), 095 (2008).
[54] Modak, S. K. Corrected entropy of BTZ black hole in tunneling approach. *Physics Letters B*, **671**(1), 167-173 (2009).

[55] Zhu, T., and Ren, J. R. Corrections to Hawking-like radiation for a Friedmann–Robertson–Walker universe. *The European Physical Journal C*, **62**(2), 413-418 (2009).

[56] Jiang, K. X., Feng, T., and Peng, D. T. Hawking radiation of apparent horizon in a FRW universe as tunneling beyond semiclassical approximation. *International Journal of Theoretical Physics*, **48**(7), 2112-2121 (2009).

[57] Hayward, S. A. et al. Local Hawking temperature for dynamical black holes. *Classical and Quantum Gravity*, **26**(6), 062001 (2009).

[58] Di Criscienzo, R. et al. On the Hawking radiation as tunneling for a class of dynamical black holes. *Physics Letters B*, **657**(1-3), 107-111 (2007).

[59] Cai, R. G., Cao, L. M., and Hu, Y. P. Hawking radiation of an apparent horizon in a FRW universe. *Classical and Quantum Gravity*, **26**(15), 155018 (2009).

[60] Zhu, T., Ren, J. R., and Singleton, D. Hawking-like radiation as tunneling from the apparent horizon in an FRW universe. *International Journal of Modern Physics D*, **19**(02), 159-169 (2010).

[61] Kodama, H. Conserved energy flux for the spherically symmetric system and the backreaction problem in the black hole evaporation. *Progress of Theoretical Physics*, **63**(4), 1217-1228 (1980).

[62] Moffat, J. W., and Toth, V. T. Fundamental parameter-free solutions in modified gravity. *Classical and Quantum Gravity*, **26**(8), 085002 (2009).

[63] Parikh, M. K. New coordinates for de Sitter space and de Sitter radiation. *Physics Letters B*, **546**(3-4), 189-195 (2002).

[64] Medved, A. J. M. Radiation via tunneling from a de Sitter cosmological horizon. *Physical Review D*, **66**(12), 124009 (2002).

[65] Hawking, S. W. Gravitational radiation in an expanding universe. *Journal of Mathematical Physics*, **9**(4), 598-604 (1968).

[66] Hayward, S. A. Quasilocal gravitational energy. *Physical Review D*, **49**(2), 831 (1994).

[67] Akhmedov, E. T., Pilling, T., and Singleton, D. Subtleties in the quasi-classical calculation of Hawking radiation. *International Journal of Modern Physics D*, **17**(13n14), 2453-2458 (2008).

[68] Sekiwa, Y. Thermodynamics of de Sitter black holes: thermal cosmological constant. *Physical Review D*, **73**(8), 084009 (2006).

[69] Cai, R. G., and Cao, L. M. Unified first law and the thermodynamics of the apparent horizon in the FRW universe. *Physical Review D*, **75**(6), 064008 (2007).

[70] Tian, D. W., and Booth, I. Apparent horizon and gravitational thermodynamics of the universe: solutions to the temperature and entropy confusions and extensions to modified gravity. *Physical Review D*, **92**(2), 024001 (2015).

[71] Bhattacharya, K., and Majhi, B. R. Temperature and thermodynamic structure of Einstein’s equations for a cosmological black hole. *Physical Review D*, **94**(2), 024033 (2016).

[72] Haldar, S., Bhattacharjee, S., and Chakraborty, S. Unified first law and some general prescription: a redefinition of surface gravity. *The European Physical Journal C*, **77**(9), 1-6 (2017).

[73] Ding, J. C. et al. Apparent horizon and gravitational thermodynamics of Universe in the Eddington-Born-Infeld theory. *Advances in High Energy Physics*, **2018**, 7801854 (2018).