Exact Solution of a Linear Wave Equation in Cosmological General Relativity

Firmin J. Oliveira †

January 1, 2022

† Joint Astronomy Centre, Hilo, Hawai‘i, U.S.A.
Email: firmin@jach.hawaii.edu
Address: 660 N. A‘ohoku Place, Hilo, Hawai‘i, U.S.A. 96720,
Telephone: 808-969-6539,
Fax: 808-961-6516.

Abstract

A linear second order wave equation is presented based on cosmological general relativity, which is a space-velocity theory of the expanding Universe. The wave equation is shown to be exactly solvable, based on the Gaussian hypergeometric function.

1 Introduction

In this paper a linear second-order wave equation is presented based on the space-velocity theory of cosmological general relativity (CGR) of Carmeli [1]. A spherically symmetric metric is used with comoving coordinates. The equation is separated by variables and exact solutions are found for all ensuing equations. The interesting radial equation is transformed into a Gaussian hypergeometric equation for which solutions are well known.

Note that this paper is more rigorous than [2], although the section on the derivation of the wave equation is essentially the same.

2 The Wave Equation

The CGR spherically symmetric, comoving space-velocity metric ([3], Eq. (A.5)) is defined by the line element

\[ ds^2 = \tau^2 dv^2 - e^\mu dr^2 - Q^2 \left( d\theta^2 + \sin^2(\theta)d\phi^2 \right) , \]

where coordinate \((v)\) is the radial velocity of expansion of the universe, and \(\tau\) is the Hubble-Carmeli time constant, its value is \(\tau = 12.486\) Gyr ([3], Eq. (A.66), p. 138). The functions \(\mu\) and \(Q\) are dependent only on the velocity \((v)\) and the
radial coordinate \((r)\). \((\theta, \phi)\) are the usual spherical coordinates. From Eq. 1 the non-zero elements of the metric \(g_{\mu\nu}\) are

\[
g_{00} = 1, \\
g_{11} = -e^\mu, \\
g_{22} = -Q^2, \\
g_{33} = -Q^2 \sin^2(\theta).
\]

A postulate of this cosmological theory is that the metric \(g_{\mu\nu}\) satisfies the Einstein field equations

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu},
\]

where \(R_{\mu\nu}\) is the Ricci tensor, \(R = g^{\alpha\beta}R_{\alpha\beta}\) is the scalar curvature, \(T_{\mu\nu}\) is the momentum-energy tensor, \(\kappa = \frac{8\pi G}{c^2}\), \(G\) is Newton’s constant and \(c\) is the speed of light in vacuo. (Compare this to the Einstein equation in General Relativity theory where \(\kappa = \frac{8\pi G}{c^4}\).) The momentum-energy tensor of a perfect fluid \([\ref{3}], \text{Eq. (A.10)}\) is

\[
T_{\mu\nu} = \rho_{\text{eff}} u_\mu u_\nu + p \left( u_\mu u_\nu - g_{\mu\nu} \right),
\]

where the effective mass density \(\rho_{\text{eff}} = \rho - \rho_c\), where \(\rho\) is the average mass density of the Universe and \(\rho_c\) is the critical mass density, a constant in CGR given by \(\rho_c = \frac{3}{8\pi G T^2}\). Also, \(p\) is the pressure, and \(u^\mu\) is the four-velocity

\[
u^\alpha = u_\alpha = (1, 0, 0, 0).
\]

The solution for Eq. \(\ref{3}\) was derived \([\ref{3}], \text{Appendix A.4}\) with the results,

\[
Q = r, \\
e^\mu = e^{\mu(r)} = \frac{1}{1 + f(r)}, \\
f(r) = \frac{1 - \Omega}{c^2 r^2}, \\
p = \frac{c(1 - \Omega)}{8\pi G T},
\]

where the density parameter \(\Omega = \rho/\rho_c\).

A linear second order wave equation is obtained from the dual \([\ref{4}]\) of Eq. \(\ref{4}\) in the form of the d’Alembertian for space-velocity

\[
\frac{\partial^2 \Psi}{\partial s^2} = \left\{ \frac{\partial^2}{\partial (\tau v)^2} - e^{-\mu(r)} \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{r^2}{r^2} \frac{\partial}{\partial r} \right) \\
- \frac{1}{r^2} \left[ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \phi^2} \right] \right\} \Psi.
\]

2
In CGR the condition for the expansion of the universe is defined by setting \( ds = 0 \). For the wave equation, it is assumed that the expansion of the universe corresponds to setting \( \frac{\partial^2 \Psi}{\partial s^2} = 0 \). With this condition, Eq. 13 becomes

\[
\frac{1}{\tau^2} \frac{\partial^2 \Psi}{\partial v^2} = e^{-\mu(r)} \frac{1}{\tau^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{\tau^2} \left[ \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 \Psi}{\partial \phi^2} \right].
\]

(14)

3 Solution of the Wave Equation

Solve Eq. 14 by a separation of variables. Assume

\[
\Psi(v, r, \theta, \phi) = \Psi_0(v) \Psi_1(r) \Psi_2(\theta) \Psi_3(\phi).
\]

(15)

By the well known process, the solutions of the equations for the \((v)\) and \((\theta, \phi)\) components are readily obtained with the result

\[
\Psi_0(v) = e^{\pm iD\tau v},
\]

(16)

\[
\Psi_2(\theta)\Psi_3(\phi) = P_l^m(\cos(\theta)) e^{+im\phi},
\]

(17)

where \( D \) is a constant of integration called the intrinsic curvature, and where \( P_l^m(\cos(\theta)) \) are the associated Legendre functions with

\[
l = 0, 1, 2, 3, \ldots ,
\]

(18)

\[
m = -l, -(l - 1), \ldots, 0, \ldots, l - 1, l.
\]

(19)

This leaves the radial equation. Substituting for \( e^{\mu(r)} \) from Eqs. 10 and 11, the equation for \( \Psi_1(r) \) is,

\[
\tau^2 (1 + Ar^2) \frac{d^2 \Psi_1(r)}{dr^2} + 2r (1 + Ar^2) \frac{d\Psi_1(r)}{dr} + \left[D^2 r^2 - l(l + 1)\right] \Psi_1(r) = 0,
\]

(20)

where

\[
A = \frac{1 - \Omega}{c^2 \tau^2},
\]

(21)

\[
0 \leq \Omega < \infty.
\]

(22)

This equation is transformed ([5], Sect. 2.1.2-6, Eq. 194) in terms of \( U(x) \) by defining

\[
x = -Ar^2,
\]

(23)
and the ansatz
\[ \Psi_1(x) = x^q U(x), \] (24)
where \( q \) is a solution to the quadratic equation
\[ q^2 - \frac{5}{2} q - \frac{1}{4} l (l + 1) = 0. \] (25)

Apply this transformation to Eq. 20 to obtain the Gaussian hypergeometric equation ([5], Sect. 2.1.2-5, Eq. 171)
\[ x (x - 1) \frac{d^2 U(x)}{dx^2} + [(\alpha + \beta + 1) x - \gamma] \frac{dU(x)}{dx} + \alpha \beta U(x) = 0, \] (26)
where
\[ q = \frac{5}{4} \pm \frac{1}{2} \sqrt{\frac{25}{4} + l (l + 1)}, \] (27)
\[ \alpha = q - \frac{5}{4} + \frac{1}{2} \sqrt{\frac{25}{4} - D^2 A}, \] (28)
\[ \beta = 2q - \frac{5}{2} - \alpha, \] (29)
\[ \gamma = 2q - \frac{3}{2}. \] (30)

To prove that \( \gamma \) is not an integer, substitute for \( q \) from Eq. 27 into Eq. 30 and obtain
\[ 25 = 4 (\gamma - 1)^2 - 4 l (l + 1), \] (31)
which is false for integers, since the l.h.s. is odd while the r.h.s. is even.

For \( \gamma \neq 0, -1, -2, -3, \ldots \), a solution to Eq. 26 is the hypergeometric series
\[ U(x) = F(\alpha, \beta, \gamma; x) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k (\beta)_k x^k}{(\gamma)_k k!}, \] (32)
\[ (\alpha)_k = \alpha (\alpha + 1) (\alpha + 2) \cdots (\alpha + k - 1), \] (33)
which, \( a \text{ fortiori} \), is convergent for
\[ |x| < 1. \] (34)

For \( \gamma \) not an integer, and for \( C_1 \) and \( C_2 \) arbitrary constants, the general solution of Eq. 26 is
\[ U(x) = C_1 F(\alpha, \beta, \gamma; x) + C_2 x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x). \] (35)
Consider the radial wave function \( \Psi_0(x) = x^q U(x) \), for \(|x| < 1\). As \(x \to 0\), only the first term in each series is relevant in Eq. 35,

\[
\Psi_0(x \to 0) \approx x^q \left( C_1 + C_2 x^{1-q} \right),
\]

(36)

\[
\approx C_1 x^q + C_2 x^{5/2-q}.
\]

(37)

where substitution for \(\gamma\) was made from Eq. 30. If \(q\) is positive then the second term of Eq. 37 will diverge as \(x \to 0\) for \(5/2 < q\), which requires that \(C_2 = 0\) for \(5/2 < q\). If \(q\) is negative then the first term of Eq. 37 will diverge as \(x \to 0\) unless \(C_1 = 0\). If \(q = 0\) there is no divergence.

4 Some Physical Aspects

From Eqs. 21, 23 and 34,

\[
|x| = -(1-\Omega) \frac{r^2}{c^2 \tau^2} < 1,
\]

(38)

which implies

\[
0 \leq r < R_\Omega,
\]

(39)

\[
R_\Omega = \sqrt{|R_\Omega^2|},
\]

(40)

\[
R_\Omega^2 = -\frac{\tau^2}{1-\Omega}.
\]

(41)

The radius of convergence \(R_\Omega\) is identical to the radius of curvature of the cosmological models for the expanding universe described in (6, Eqs. 2.4, 2.5, 5.8a and 5.8b). The conclusion is that the wave function solution \(\Psi(v, r, \theta, \phi)\) found here is valid for the entire expanding universe of radius \(R_\Omega\).

An expression for the intrinsic curvature is obtained by eliminating \(q\) between Eqs. 27 and 28,

\[
D = \pm \frac{\sqrt{\Omega - 1}}{c \tau} \left[ \left( 2 \alpha \pm \sqrt{\frac{25}{4} + l(l+1)} \right)^2 - \frac{25}{4} \right]^{1/2}
\]

(42)

Notice that \(D\) can be real or imaginary, depending on \(\Omega, \alpha\) and \(l\), which means \(\Psi_0(v)\) can be a sinusoidal wave or a decaying exponential.

The author is grateful to the Joint Astronomy Centre, Hilo, Hawai‘i for many years of employment and support, and to Professor Moshe Carmeli for his theories and communications.
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