The quantum black hole

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Abstract
The quantum nature of a black hole is revealed using the simplest terms that one learns in undergraduate and beginning graduate courses. The exposition demonstrates – vividly – the importance and power of the quantum oscillator in contemporary research in theoretical physics.

I. INTRODUCTION

“Finally, it is necessary to emphasize one major result of the whole investigation, namely that it must be taken as well established that the life-history of a star of small mass must be essentially different from the life-history of a star of large mass. For a star of small mass the natural white-dwarf stage is an initial step towards complete extinction. A star of large mass (> m) cannot pass into the white-dwarf stage, and one is left speculating on other possibilities.”, wrote Chandrasekhar in the observatory [1,2,3] in 1934. Thus originated the concept of a ‘black hole’, the collapsed state of stellar matter [4,5].

What does Einstein’s theory of general relativity (GR) say about this state of matter?
The answer was provided in a classic paper [6] by Oppenheimer and Snyder in 1939. They studied the collapse of a sufficiently massive star under the idealized conditions of spherical symmetry and pressureless stellar matter, and concluded that the star contracts under its own gravitational attraction, its boundary \( r_b \) necessarily approaching its gravitational radius \( r = \frac{2MG}{c^2} \) [now called the Schwarzschild radius, \( M \) being the mass of the star, \( G \) the Newton’s]
constant, and \( c \) the speed of light. All energy emitted outward from the surface of the star is reduced in escaping, by the Doppler effect from the receding source, by the large gravitational red-shift, \( (1 - \frac{r}{r_b})^{1/2} \), and by the gravitational deflection of light which prevents the escape of radiation except through a cone about the outward normal of progressively shrinking aperture as the star contracts. The star thus tends to close itself off from any communication with a distant observer; only its gravitation persists. From the point of view of a distant observer, it takes an infinite time for this asymptotic isolation to be established; for an observer comoving with the stellar matter this time \( \tau_0 \) [the proper time] is finite. After this time \( \tau_0 \) an observer comoving with the matter is not able to send a light signal from the star, as the cone within which a signal can escape closes entirely [7]. And one says that a black hole is formed with its surface at \( r = \frac{2MG}{c^2} \), a distant observer being unable to see inside this surface.

So what does the classical theory of general relativity tell an observer stationed at a distance greater than \( \frac{2MG}{c^2} \) about the nature of the black hole – the state of stellar matter inside the sphere of radius \( r = \frac{2MG}{c^2} \)? The answer is simple and emphatic, namely, absolutely nothing. In other words, the classical general relativity does not describe this collapsed state of stellar matter – the black hole. Is this state describable quantum mechanically? The answer is in the affirmative and the purpose of the present exposition is to give this description in the simplest terms that one learns in undergraduate and beginning graduate physics courses. No new concepts are needed. We briefly review, in the next few sections, those concepts that are necessary and sufficient for the tale to be told effectively. It is guaranteed that well before the story ends the readers would begin to see the “silver train” [8].

In the next section we go over (anew) the recipe that one learns to make quantum mechanics from Newtonian mechanics. The recipe is then used on Eq. (34), which ensues directly from classical general relativity, to obtain the quantum equation, Eq. (36), which quantizes mass as in Eq. (50), the fundamental quantum of mass being equal to twice the Planck mass. The use of Bose statistics (in section IX) reveals the quantum nature of the black hole.

II. MAKING QUANTUM MECHANICS FROM NEWTON’S SECOND LAW

Example (i)

According to Newton’s second law, the equation that describes a linear harmonic oscillator is

\[
m \frac{d^2x}{dt^2} + kx = 0, \tag{1}
\]

where \( m \) is the inertial mass, \( k \) the force constant, \( x \) one of the three Cartesian coordinates in Euclidean space, and \( t \) the Newtonian time which is the same in all coordinate frames. With \( \omega^2 = k/m \), Eq. (1) is rewritten as

\[
\frac{d^2x}{dt^2} + \omega^2 x = 0. \tag{2}
\]

It has sine and cosine as solutions, which are not straight lines. One gets straight line as a solution only when \( k = 0 \), that is when there is no force.

Now, to obtain a quantum mechanical equation from (1), one integrates it to obtain

\[
\frac{1}{2}m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2}kx^2 = E, \tag{3}
\]

where \( E \) is the constant of integration and is called the total energy. One then writes (3) as

\[
\frac{p_x^2}{2m} + \frac{1}{2}kx^2 = E \tag{4}
\]
with \( p_x \equiv m \frac{dx}{dt} \), and replaces \([9]\) \( p_x \) by \( -i \hbar \frac{\partial}{\partial x} \) and \( E \) by \( i \hbar \frac{\partial}{\partial t} \) to obtain the quantum Schrödinger equation

\[
-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + \frac{1}{2} k x^2 \psi = i \hbar \frac{\partial \psi}{\partial t}.
\]  

(5)

Since the replacement \( p_x \rightarrow -i \hbar \frac{\partial}{\partial x} \) is independent of \( m \), one might as well put \( m = 1 \) to obtain the quantum equation

\[
\left( -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} \omega^2 x^2 \right) \psi = i \hbar \frac{\partial \psi}{\partial t},
\]  

(6)

which corresponds to the Newtonian Eq. (2). With the time-dependence \( \psi(x,t) = \phi(x)e^{-i(E/\hbar)t} \), Eq. (6) becomes

\[
-\frac{\hbar^2}{2} \frac{d^2}{dx^2} \phi(x) + \frac{1}{2} \omega^2 x^2 \phi(x) = E \phi
\]  

(7)

which, when solved, gives for the energy eigenvalues \([10]\)

\[
E_n = (n + \frac{1}{2}) \hbar \omega, \quad n = 0, 1, 2, \ldots
\]  

(8)

Equation (8) says, ignoring the irrelevant constant \( \frac{1}{2} \hbar \omega \), that there is a fundamental quantum of energy equal to \( \hbar \omega \), and that the state \( \psi_1 \) contains just one such quantum, state \( \psi_2 \) two quanta, state \( \psi_3 \) three quanta, and so on. In other words, the quantum state of a harmonic oscillator is characterised by the number of energy quanta it contains. The importance of Eqs. (2) and (8) will manifest itself in section VI.

**Example (ii)**

Newton’s second law gives the equation \((\kappa = \text{constant})\)

\[
m \frac{d^2 \vec{r}}{dt^2} + \kappa \frac{\vec{r}}{r} = 0
\]  

(9)

for the so called Kepler-Coulomb problem. In Eq. (9) the Euclidean vector \( \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \), and \( t \) is the Newtonian time. To obtain the quantum equation that corresponds to (9), one integrates (9) to get

\[
\frac{1}{2} m \left( \frac{d\vec{r}}{dt} \right)^2 - \frac{\kappa}{r} = E
\]  

(10)

(with \( E \) the constant of integration) which is rewritten as

\[
\frac{\vec{p}^2}{2m} - \frac{\kappa}{r} = E,
\]  

(11)

which then, using the Schrödinger recipe \([9]\): \( \vec{p} \rightarrow -i \hbar \vec{\nabla} \), \( E \rightarrow i \hbar \frac{\partial}{\partial t} \), transforms into the quantum equation

\[
\left( -\frac{\hbar^2}{2m} \vec{\nabla}^2 - \frac{\kappa}{r} \right) \psi(\vec{r},t) = i \hbar \frac{\partial \psi}{\partial t} \psi(\vec{r},t).
\]  

(12)

With the time dependence \( \psi(\vec{r},t) = \phi(\vec{r})e^{-i(E/\hbar)t} \), Eq. (12) can be solved to give the well-known energy eigenvalues \([11]\)

\[
E_n = -\frac{me^4}{2(4\pi \varepsilon_0)^2 \hbar^2 n^2}, \quad n = 1, 2, \ldots
\]
for the hydrogen atom \( (\kappa = e^2/4\pi\varepsilon_0) \), but for our purpose (as will become evident later) we need the classical kinetic energy \( \vec{p}^2/2m \) in polar coordinates:

\[
\frac{\vec{p}^2}{2m} = \frac{1}{2}mr^2 + \frac{L^2}{2mr^2}.
\]

(13)

In Eq. (13) \( \dot{r} = \frac{dr}{dt} \) is the radial velocity and \( \vec{L} \) the angular momentum. For the special case when \( \vec{L} = 0 \), Eq. (11) becomes

\[
\frac{p_r^2}{2m} - \frac{\kappa}{r} = E
\]

(14)

by formally defining \( p_r \equiv m\dot{r} \). Proper calculation of \( p_r^2 \) using the Schrödinger prescription [12] gives for the differential operator for \( p_r^2 \) the expression \( -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \), and the eigenvalue equation that results from (14) is given by

\[
\left[ -\frac{1}{2mr^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{\kappa}{r} \right] \psi(r) = E\psi(r).
\]

(15)

Since the replacement \( p_r^2 \to -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) \) is independent of \( m \) in \( p_r = m\dot{r} \), we again put \( m = 1 \) in (15) to obtain the quantum equation

\[
\left[ -\frac{1}{2r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{\kappa}{r} \right] \psi(r) = E\psi(r)
\]

(16)

which corresponds to the Newton equation (9) with \( m = 1 \) in it. Eq. (16) will show its importance in section V.

III. STRAIGHT LINES IN MINKOWSKI SPACE

With the proclamation: “Henceforth space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.” by Minkowski [13], originated his 4-dimensional spacetime, now simply called Minkowski space in which the distance \( ds \) between two adjacent world points (events) is given by [14]

\[
ds^2 = dt^2 - dx^2 - dy^2 - dz^2.
\]

(17)

The proper time interval \( d\tau \) between two neighboring points for which \( dx = dy = dz = 0 \) is then given by

\[
d\tau^2 = ds^2 = dt^2 - dx^2 - dy^2 - dz^2,
\]

(18)

or

\[
d\tau = (1 - u^2)^{1/2} dt,
\]

(19)

and is an invariant quantity. The relation (17) is essentially non-Euclidean [15]. Nevertheless four-vectors like \( x^\mu (\mu = 0, 1, 2, 3) \), \( \frac{dx^\mu}{d\tau} \) (4-velocity), \( \frac{dx^\mu}{d\tau^2} \) (4-acceleration), \( p^\mu = m\frac{dx^\mu}{d\tau} \) (4-momentum), \( \frac{dp^\mu}{d\tau} \), \( \kappa^\mu \) (Minkowski 4-force) are constructed [16]. And special relativistic extension of Newton’s second law takes the form

\[
\frac{dp^\mu}{d\tau} = \kappa^\mu,
\]

(20)

or

\[
m\frac{d^2x^\mu}{d\tau^2} = \kappa^\mu.
\]

(21)
When $\kappa^\mu = 0$, only then
\[
\frac{d^2 x^\mu}{d\tau^2} = 0 \quad (22)
\]
irrespective of what the value of $m$ is.

The solutions of Eq. (22) are straight lines. Equation (22) then becomes the statement of Newton’s first law. It is to be noted that Eq. (21) can never have straight lines as their solutions, even in Minkowski space, as long as $\kappa^\mu \neq 0$.

IV. CASTING NEWTON’S SECOND LAW IN THE ‘FORM’ OF HIS FIRST LAW

But in the case when the force is Newton’s gravitational force, something very remarkable happens. In this case Newton’s second law says:
\[
m \frac{d^2 \vec{r}}{dt^2} + \frac{GM\vec{r}}{r^3} = 0. \quad (23)
\]
The remarkable thing is that the $m$ in the first term, usually called the inertial mass, is the same as the $m$ in the second term, usually called the gravitational mass. And Eq. (23) reduces to
\[
\frac{d^2 \vec{r}}{dt^2} + \frac{GM\vec{r}}{r^3} = 0, \quad (24)
\]
which in Minkowski space takes the form
\[
\frac{d^2 x^i}{d\tau^2} + \frac{GMx^i}{r^3} = 0 \quad (i = 1, 2, 3). \quad (25)
\]
The remarkable feature of Eq. (25) is that it is independent of $m$ just as Eq. (22). Note that Eq. (25) does not have straight lines as its solutions, whereas Eq. (22) does. Now the question arises whether one can cast Eq. (25) in a new form that has as its solutions, or curves which a world point traces, straight lines. The answer is a definite yes. All one has to do is go to what is called 4-dimensional Riemannian space [17] in which the distance $ds$, or the line element (as it is sometimes called), is given by
\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (\mu, \nu = 0, 1, 2, 3) \quad (26)
\]
with $g_{\mu\nu}(x^\mu)$ functions of $x^\mu$, and the equations of geodesics, the straightest lines [18], are
\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0. \quad (27)
\]
In Eqs. (27), $\tau$ is the proper time, and the affine connections $\Gamma^\mu_{\nu\lambda}$ are given by
\[
\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\sigma\mu} \left\{ \frac{\partial g_{\lambda\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\lambda} - \frac{\partial g_{\lambda\nu}}{\partial x^\sigma} \right\}. \quad (28)
\]
Thus once the $g_{\mu\nu}$ are determined from Einstein equation [17]
\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi T^{\mu\nu} \quad (29)
\]
for a given distribution of matter, the $\Gamma^\mu_{\nu\lambda}$ are obtained using (28), and the geodesic equations (27) are solved to obtain the geodesics or the “straightest world lines”.

5
The importance of Eqs. (27) lies in the fact that it was these equations which Einstein solved, though using approximation methods, to obtain the correct value for the advance of perihelion of mercury [19]. And it is precisely the integral of these equations which we are going to use in the next section for the special spherically symmetric case. The first integral of Eqs. (27) for time-like geodesics is

\[ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 1. \]  

(30)

V. MAKING QUANTUM MECHANICS FROM TIME-LIKE GEODESICS

The exact, static, spherically symmetric solution of Einstein equations for vacuum,

\[ R_{\mu\nu} = 0, \]  

(31)

was obtained by Karl Schwarzschild [20] in 1916 and is given by

\[ ds^2 = (1 - (2M/r))dt^2 - \left( \frac{dr^2}{1 - (2M/r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right). \]  

(32)

Note that Eq. (32) ensures that for \( r \to \infty \), it goes to the Minkowski line element

\[ ds^2 = dt^2 - (dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)). \]  

(33)

In (32) \( M \) is the Newtonian point mass located at the origin of coordinates \( r, \theta, \phi \). We want to emphasize that in solving Eqs. (31) \( M \) arises as a constant of integration and is identified with the Newtonian mass only when one falls back to the Newton’s gravitational law according to which the gravitational potential at a distance \( r \) from a point mass \( M \) is \(-GM/r\). In other words, mass or gravitation arises or is created out of the vacuum [21]. Put succinctly, Einstein equations (31) manifest only gravitation (mass) and nothing else.

However, with the expressions for \( f_{tt}, f_{rr}, f_{\theta\theta} \) and \( f_{\phi\phi} \) from (32), when the time-like geodesic Eqs. (27) are integrated there arise two more constants of integration [22] designated by \( E \) and \( L \), which, with the postulate [23] that test particles follow the time-like geodesic, are respectively identified with the energy per unit mass and angular momentum per unit mass of the test particle. But since we are interested only in pure gravitation (\( M \)) and nothing else, the values of these constants in our case of interest are necessarily zero. As a result only one equation remains [24], namely

\[ \frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 - \frac{M}{r} = -\frac{1}{2} \]  

(34)

which is characterized by only one parameter, the Newtonian mass \( M \) located at \( r = 0 \), and describes the state of pure gravitation in the region [25] \( r \leq 2M \), the first term being zero at \( r = 2M \), the Schwarzschild radius. Bigger the value of \( M \), larger is the Schwarzschild radius.

It should be apparent now to the reader, in view of section II and the fact that the proper time \( \tau \) has the same character as that of Newton’s time \( t \), how to make the quantum equation from (34): simply by replacing \( p_r^2 = \dot{r}^2 = (dr/d\tau)^2 \) by the differential operator \(-\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right)\). And one gets [26]

\[ -\frac{1}{2r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) - \frac{M}{r} \psi = -\frac{1}{2} \psi \]  

(35)
as the quantum equation which corresponds to the classical equation (34). It is obvious that Eq. (35) describes a bound state of binding energy $1/2$. With $U = r\psi$ and $M = \mu/4$, Eq. (35) reduces to the simple form

$$\left( \frac{1}{2} \frac{d^2}{dr^2} - \frac{\mu/4}{r} \right) U = -\frac{1}{2} U.$$  

(36)

We shall return to Eq. (36) in section VII.

VI. EMERGENCE OF PHOTONS FROM MAXWELL’S EQUATIONS

Maxwell’s equations for pure radiation (light waves) can be expressed in terms of only the spatial part $\vec{A}$ of the Minkowskian 4-potential $A_\mu$ ($A_0 = \phi = 0$), and takes the form

$$\nabla^2 \vec{A} - \frac{d^2 \vec{A}}{dt^2} = 0,$$  

(37)

$$\vec{\nabla} \cdot \vec{A} = 0,$$  

(38)

$\vec{A}$ being defined at all world points [27]. With periodic boundary conditions on $\vec{A}$, the general solution of (37) can be represented as a series of orthogonal ‘eigenvalues’:

$$\vec{A} = \sum_\lambda q_\lambda(t) \vec{A}_\lambda(\vec{r})$$  

(39)

where $\vec{A}_\lambda$ depends only upon the space coordinate and $q_\lambda$ only on the time coordinate. $\vec{A}_\lambda$ of course obeys the periodic boundary conditions. The $\vec{A}_\lambda$ then satisfy the wave equation

$$\nabla^2 \vec{A}_\lambda + K_\lambda^2 \vec{A}_\lambda = 0,$$  

(40)

with

$$\vec{\nabla} \cdot \vec{A}_\lambda = 0,$$  

(41)

and $q_\lambda$ satisfy the Newton’s equation for a harmonic oscillator

$$\frac{d^2 q_\lambda}{dt^2} + \omega_\lambda^2 q_\lambda = 0,$$  

for each wavelength $\lambda$. (42)

Thus arises the harmonic oscillator naturally from Maxwell’s equations; and one says that pure radiation is composed of them [harmonic oscillators]. Eq. (42), when quantized in the manner of section II, gives for energy eigenvalues

$$E_\lambda = (n_\lambda + \frac{1}{2})\hbar\omega_\lambda,$$  

(43)

and one says that a state $| n_{\lambda_1}, n_{\lambda_2}, \ldots \rangle$ of pure radiation is characterized by the numbers $n_{\lambda_1}, n_{\lambda_2}, \ldots$ of light quanta (photons) of respective energies $\hbar\omega_{\lambda_1}, \hbar\omega_{\lambda_2}, \ldots$.

VII. EMERGENCE OF PAIRED QUANTA OF PURE GRAVITATION

The radial Schrödinger equation for the $N'$-dimensional Coulomb or Kepler problem can be written [28,29] as

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} - \frac{(\ell' + N'/2 - 3/2)(\ell' + N'/2 - 1/2)}{r^2} - \frac{\alpha}{r} \right] U(r) = -BU(r),$$  

(44)
where the orbital quantum number $\ell'$ is a positive integer or zero and $B$ the Coulomb or Kepler binding energy. In $N$ dimensions the oscillator counterpart of Eq. (44) is

$$\left[ -\frac{1}{2} \left( \frac{d^2}{ds^2} - \frac{(\ell + N/2 - 3/2)(\ell + N/2 - 1/2)}{s^2} \right) + \frac{1}{2} \omega^2 s^2 \right] \phi_{n\ell}(s) = (2n + \ell + N/2) \omega \phi_{n\ell}(s), \quad (45)$$

where $n = 0, 1, \ldots$, and $\ell$ the corresponding angular momentum quantum number. Under the transformation \[29,30,31\]

$$r = \rho^2, \quad \text{and} \quad U = \rho^{1/2} \phi,$$

Eq. (44) becomes

$$\left[ -\frac{1}{2} \left( \frac{d^2}{d\rho^2} - \frac{(2\ell' + (2N' - 2)/2 - 3/2)(2\ell' + (2N' - 2)/2 - 1/2)}{\rho^2} \right) + 4B\rho^2 \right] \phi(\rho) = 4\alpha \phi(\rho). \quad (47)$$

Comparison of Eqs. (45) and (47) shows that there is a satisfactory mapping when \[31\]

$$\begin{cases} \ell = 2\ell', & N = 2N' - 2, \\ \frac{1}{2} \omega^2 = 4B, & (2n + \ell + N/2) \omega = 4\alpha, \end{cases} \quad (48)$$

Now we go back to Eq. (36) and note that it [Eq. (36)] is Eq. (44) with

$$\ell' = 0, \quad N' = 3, \quad B = 1/2, \quad \alpha = \mu/4; \quad (49)$$

and hence it represents a 4-dimensional harmonic oscillator with angular momentum $\ell = 0$, angular frequency $\omega = 2$, and $\mu$ given by

$$\mu_n = 2(n + 1)\omega, \quad n = 0, 1, 2, \ldots. \quad (50)$$

Thus the oscillator shows itself again. Though this time it is a four-dimensional one, as opposed to the one dimensional one in the case of pure radiation (see section VI) \[32\].

Equation (50) says that the fundamental quantum of pure gravitation is of frequency 2 or of mass (energy) twice the Planck mass (energy) \[33\]. But the quanta always come in pairs. And one says that the state $|n\rangle$ of pure gravitation is characterized by $n$ pairs of mass quanta.

**VIII. BLACK HOLE ENTROPY**

After the classic paper of Oppenheimer and Snyder \[6\] in 1939, the field of general relativity and black holes was dormant for about twenty years. Then there was a burst of activity. In 1963 Roy Kerr \[34\] found another exact, stationary, axisymmetric, solution of Einstein equations, characterised by two parameters that are identified with the mass and angular momentum of the so-called Kerr inner region \[25\]. During the 1970s, the mathematics of black holes was cast in a form that closely resembled that of thermodynamics \[35\]. In other words four laws of black hole mechanics were formulated which were analogous to the four laws of thermodynamics, the quantities $M$ (mass), $\kappa$ (surface gravity), $A$ (area) of a black hole being analogous, respectively, to the thermodynamic quantities $E$ (energy), $T$ (temperature), $S$ (entropy). However, Bekenstein \[36\], based on his calculations and gedanken experiments, went beyond a mere analogy. He proposed that the area $A$ of a black hole represents in actuality the physical entropy $S$ of a black hole, it being proportional to $A$ ($S = \alpha A$). But nobody believed him \[37\]. Then came Hawking \[38\] with his semi-classical calculation of scattering of a quantized scalar field of the boundary of a classical Schwarzschild black hole, and his interpretation of the results: that a
black hole emits particles, the emission spectrum being exactly the same as that of a black body of temperature \( T = \kappa/2\pi \). \( [\kappa = 1/4M \text{ for a Schwarzschild black hole giving its temperature } T = 1/(8\pi M).] \) This fixed the proportionality constant \( \alpha \) to be 1/4 giving the physical entropy \( S = A/4 \) for a black hole.

To obtain \( S = A/4 \) (the so-called Bekenstein-Hawking relation) theoretically has been a challenge for theoretical and mathematical physicists ever since 1975. Many a calculation have been done using a variety of mathematical techniques [39]. But none of these calculations has the simplicity and cleanliness possessed by the one given in the next section, a calculation that can be used in the classroom.

**IX. CALCULATION OF BLACK HOLE ENTROPY A LA BOSE**

Historically it was Planck who, in view of Kirchhoff’s theorem, used the one-dimensional oscillators (as the simplest mathematical convenience) with the discrete energy element \( \varepsilon = h\nu \), \( \nu \) being the frequency of the oscillators, to derive his black body radiation law [40], Einstein later interpreting \( \varepsilon = h\nu \) as the energy quantum of light itself [40,41]. However, the connection of a light quantum with the energy element of an oscillator was not formally established till after the advent of quantum mechanics in 1925 and its application to pure radiation, as described briefly in section VI. Despite this, Bose [42] in 1924 used the light quanta, coupled with the fact that they are indistinguishable from one another, to derive the entropy [along with the Planck spectral law] of pure radiation within a black body. Now that the connection of the fundamental energy element of a four-dimensional oscillator with the mass quantum of pure gravitation has been rigorously shown quantum mechanically, as described in section VII, it only behooves that we use Bose’s method to calculate the entropy \( S \) of pure gravitation within a black hole. We could simply use the formula for the entropy given in Bose’s paper, but we rederive it below so that it is expressed in terms of the area \( A \) of the black hole.

We saw in section VII that the quantum of pure mass is of \( \omega = 2 \) and that these mass (energy) quanta always come in pairs. So let the mass of a Schwarzschild black hole as in the paper of Oppenheimer and Snyder [6] be \( M_s \) and let it contain \( 2N \) quanta. Then \( M_s = 2N\omega \). Let us rewrite Eq. (50) as

\[
\epsilon_n = \frac{\mu_n}{2} = (n + 1)\omega, \quad n = 0, 1, 2, \ldots ,
\]

so that the \( n \)th \( \epsilon \)-state contains \( n \) quanta. The thermodynamic quantity \( E \) in Bose’s paper [42] is given by

\[
E = \frac{M_s}{2} = N\omega. \quad (52)
\]

This then allows us to literally apply Bose’s method to calculate the thermodynamic probability of the macroscopically defined state \((N, E)\).

Let \( p_0 \) be the number of vacant \( \epsilon \)-cells, \( p_1 \) the number of those cells which contain one quantum, \( p_2 \) the number of \( \epsilon \)-cells containing two quanta, and so on. Then the probability of the state defined by the \( p_r \) is

\[
W = \frac{P!}{p_0!p_1!\cdots} , \quad (53)
\]

where

\[
P = \sum_r p_r. \quad (54)
\]
is the total number of $\epsilon$-cells over which

$$N = \sum_r r p_r$$

(55) quanta are distributed. Since $p_r$ are large numbers we have, using Stirling’s formula,

$$\ln W = P \ln P - \sum_r p_r \ln p_r.$$  

(56) Then it is straightforward [42], by using the method of Lagrange multipliers, to maximize (56) satisfying the auxiliary conditions (52) and (55), and obtain

$$p_r = P (1 - e^{-\omega/\beta}) e^{-r \omega/\beta},$$

(57)

$$N = P (e^{\omega/\beta} - 1)^{-1},$$

(58)

and

$$S = \frac{E}{\beta} - P \ln(1 - e^{-\omega/\beta}).$$

(59)

From the condition $\frac{\partial S}{\partial E} = \frac{1}{T}$, one obtains $\beta = T$; and (59) becomes

$$S = \frac{E}{T} - P \ln(1 - e^{-\omega/T}).$$

(60)

As mentioned in section VIII, for a Schwarzschild black hole $T = \frac{1}{8 \pi M_s}$ and $A = 16 \pi M_s^2$, and (60) becomes [43]

$$S = A - M_s \left( e^{A/M_s} - 1 \right) \ln(1 - e^{-A/M_s}),$$

(61)

which, for large $M_s$, physically tends to

$$S = \frac{A}{4}$$

(62)

same as the Bekenstein-Hawking relation.

**X. THE REVELATION**

So what *is* the quantum nature of a black hole?

A black hole is a Bose-Einstein ensemble of quanta of mass equal to twice the Planck mass, confined in a sphere of radius twice the black hole mass.

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REFERENCES AND FOOTNOTES

[1] S. Chandrasekhar, “Stellar Configurations with degenerate cores”, The Observatory 57 373–377 (1934).

[2] For an historical recollection of the steps leading to the above statements, see S. Chandrasekhar, “The Richtmyer Memorial Lecture – Some Historical Notes”, Am. J. Phys. 37 577–584 (1969).

[3] The limiting mass $m$ is the so called Chandrasekhar limit.

[4] It was L. Landau, [ “On the theory of stars”, Phys. Z. Soviet 1, 285–287 (1932); reprinted in Neutron Stars, Black Holes and Binary X-ray sources, eds. H. Gursey and R. Ruffini, D. Reidel Publishing Co., Boston, 1975, pp. 271–273.] who stated that a star with mass $> m$ would collapse but did not think that such a state could exist in reality.

[5] The word black hole to describe such a collapsed state was coined in 1968 by J.A. Wheeler [“Our Universe: the known and the unknown”, American Scientist 56, 1–20, 1968].

[6] J.R. Oppenheimer and H. Snyder, “On continued gravitational contraction”, Phys. Rev. 56, 455–459 (1939).

[7] These conclusions are speculation free as they are founded on solid mathematical formulas obtained using Einstein’s equations of GR.

[8] J.S. Rigden, “Editorial: The fallacy of immediacy”, Am. J. Phys. 55, 395 (1987); B. Ram, “The silver train”, Am. J. Phys. 55, 968 (1987).

[9] J.J. Prentis and W.A. Fedak, “Energy Conservation in Quantum Mechanics”, Am. J. Phys. 72, 580–590 (2004).

[10] See, for example, L.I. Schiff, “Quantum mechanics”, 3rd edition (McGraw-Hill, New York, 1968), Chap. 4.

[11] See, for example, R. Eisberg and R. Resnick, Quantum Physics of Atoms, Molecules, Solids, Nucler, and Particles (Wiley, New York, 1985), chap. 7.

[12] In this connection we draw the reader’s attention to page 125 of W. Yourgrau and S. Mandelstam, Variational Principles in Dynamics and Quantum Theory, 3rd ed. (W.A. Saunders Co., Philadelphia, 1968).

[13] H. Minkowksi, “Space and time”, in The Principle of Relativity (Dover Publications, New York, 1923) p. 75.

[14] From now on we use units in which $\hbar$ (Planck’s constant) = $c$ (velocity of light) = $G$ (Newton’s gravitational constant) = $k$ (Boltzmann constant) = 1.

[15] See, for example, D.F. Lawden, An Introduction to Tensor calculus, Relativity and Cosmology, 3rd ed. (Dover Publications, New York, 2002), p. 15.

[16] Ref. 15, chap. 3.

[17] Ref. 15, chaps. 5 and 6.
A geodesic is a curve which has the property that the tangents at all its points are parallel. In Euclidean space such a curve is, of course, called a straight line.

In this connection, see L.I. Schiff, “On experimental tests of the general theory of relativity”, Am. J. Phys. 28, 340–343 (1960).

K. Schwarzschild, “Über das Gravitationsfeld eines Massenpunktes nach der Eisenstein-schen Theorie”, Sitzber. Deut. Akad. Wiss. Berlin, KI. Math.–Phys. Tech., 189–196 (1916); for an English translation see arXiv:physics/9905030.

For steps leading to this, see Ref. 15, pages 142-146.

See Ref. 15, pp. 147–148.

A. Einstein, The meaning of relativity, 5th ed. (Princeton University Press, 1974), p. 79.

With the Schwarzschild solution (32), Eq. (30), the first integral of geodesic equations (27), takes the form \( \frac{1}{2} \dot{r}^2 + \frac{L^2}{2r^2} - \frac{M}{r} - \frac{ML^2}{r^3} = \frac{1}{2}(E^2 - 1) \), which, with \( E = L = 0 \), reduces to Eq. (34).

Following S. Chandrasekhar, Truth and Beauty (University of Chicago Press, 1987), p. 71, we shall call this the inner region.

Eq. (35) was first obtained in B. Ram, “The mass quantum and black hole entropy”, Phys. Lett. A 265, 1–4 (2000).

For details see, W. Heitler, The Quantum Theory of Radiation, 3rd ed. (Dover, 1984), §§6 and 9.

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D.S. Bateman, C. Boyd and B. Dutta-Roy, “The mapping of the Coulomb problem into the oscillator,” Am. J. Phys. 60, 833–836 (1992).

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For the Kerr (Ref. 34) solution of Einstein equations, the first integral, Eq. (30), of time-like geodesics is given by

\[
\frac{1}{2} \dot{r}^2 - \frac{M}{r} + \frac{1}{2}(1 - E^2)(1 + a^2/r^2) + \frac{L^2}{2r^2} - \frac{M}{r^2}(L - aE)^2 = 0, \quad (K1)
\]

which, with \( E = L = 0 \), reduces to

\[
\frac{1}{2} \dot{r}^2 - \frac{M}{r} + \frac{a^2}{2r^2} = - \frac{1}{2}, \quad (K2)
\]

In the above two equations \( a \) is angular momentum of the Kerr inner region. It is left as an exercise for the reader to obtain the quantum equation that corresponds to (K2) and
then obtain the corresponding expression (analogous to Eq. (50)) for the eigenvalues of the mass $\mu$. (For hint, see B. Ram, “The mass quantum and black hole entropy II”, arXiv: gr-qc/0101056).

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