Disjointly homogeneous Orlicz spaces revisited

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Abstract
Let $1 \leq p \leq \infty$. A Banach lattice $X$ is said to be $p$-disjointly homogeneous or $(p-DH)$ (resp. restricted $(p-DH)$) if every normalized disjoint sequence in $X$ (resp. every normalized sequence of characteristic functions of disjoint subsets) contains a subsequence equivalent in $X$ to the unit vector basis of $\ell_p$. We revisit $DH$-properties of Orlicz spaces and refine some previous results of this topic, showing that the $(p-DH)$-property is not stable under duality in the class of Orlicz spaces and the classes of restricted $(p-DH)$ and $(p-DH)$ Orlicz spaces are different. Moreover, we give a characterization of uniform $(p-DH)$ Orlicz spaces and establish also closed connections between this property and the duality of the $DH$-property.

Keywords
Banach lattice · Symmetric space · Orlicz function · Orlicz space · $\ell_p$-spaces · Disjoint functions · Disjoint homogeneous symmetric space

Mathematics Subject Classification 46B70 · 46B42

1 Introduction

A Banach lattice $X$ is called disjointly homogeneous (briefly, $DH$) if two any normalized disjoint sequences in $X$ contain equivalent subsequences. In particular, given a $p \in [1, \infty]$, a Banach lattice $X$ is $p$-disjointly homogeneous (briefly, $(p-DH)$) if each normalized disjoint sequence in $X$ has a subsequence equivalent to the unit vector basis of $\ell_p$ ($c_0$ when $p = \infty$). These notions were introduced explicitly first in the paper [14] and proved to be very useful in studying the general problem of identifying Banach lattices $X$ such that the ideals of strictly singular and compact operators bounded in $X$ coincide (see also [11, 7, 16], the survey [13] and references therein). Results obtained in these papers can be treated as a continuation and substantial development of a classical theorem of V. D. Milman [26]
which states that every strictly singular operator in $L_p(\mu)$ has compact square. This was a motivation to find out how large is the class of $DH$ Banach lattices. As is known, it contains $L_p(\mu)$-spaces, $1 \leq p \leq \infty$, Lorentz function spaces $L_{q,p}$ and $\Lambda(W, p)$, a certain class of Orlicz function spaces, Tsirelson space and some others.

Another direction of research was connected with studying the $DH$-property itself and some its versions. In particular, a special attention has been paid to investigation of the duality problem in the class of $DH$ Banach lattices. (A systematic study of this subject was undertaken in the work [12].) First, if $X$ is an $(\infty - DH)$ Banach lattice, then $X^*$ is a $(1 - DH)$ Banach lattice as well [14, Theorem 2.2]. In contrast to that, in the same paper, it was showed that the Lorentz space $L_{p,1}[0,1], 1 < p < \infty$, is $(1 - DH)$ but the dual $L_{q,\infty}[0,1], 1/p + 1/q = 1$, fails to be an $(\infty - DH)$ space. What concerns with Orlicz spaces, in [12, p. 5863] and [13, p. 6], basing on [11, Theorem 4.1], the authors have asserted that if the Orlicz space $L_F$ is $DH$, then $L_F^*$ is $DH$ as well. Moreover, in [12, p. 5877] (see also Question 3 in the survey [13]), it was asked if the dual to a reflexive $p$-DH symmetric space on $[0,1]$ is $DH$. This question was answered in negative recently in [5].

One more issue is a weaker property of restricted disjoint homogeneity that was introduced (for $p = 2$) in the paper [16]. A symmetric space $X$ on $[0,1]$ is said to be restricted $p$-disjointly homogeneous (in brief, restricted $(p - DH)$) if every sequence of normalized characteristic functions of disjoint subsets contains a subsequence equivalent to the unit vector basis of $\ell_p$. Clearly, each $(p - DH)$ space is restricted $(p - DH)$. In [16], the authors have proved the converse for Orlicz spaces [16, Theorem 5.1] and also asked whether a symmetric space $X$ on $[0,1]$, which is restricted $(p - DH)$, must be $(p - DH)$. This question (repeated also in [13, p. 19]) was motivated by the fact that restricted $(p - DH)$ spaces have rather “good” properties (see [16] and [6]). Answering it, in [6], for every $1 \leq p < \infty$, various examples of restricted $(p - DH)$ symmetric spaces that are not $(p - DH)$ were given.

The first purpose of this paper is to revisit $DH$-properties of Orlicz spaces and to refine some previous results of this topic proved in the papers [11–13, 16]. Unfortunately, the proof of Theorem 4.1 from [11] contains a gap, and moreover, as we will see, this result and some others, which used it, turned to be not true in full generality. In particular, if $1 < p < \infty$, the classes of restricted $(p - DH)$ and $(p - DH)$ Orlicz spaces are in fact different (see Proposition 2, Theorem 1 and Corollary 5). Furthermore, $(p - DH)$-property is not stable under duality in the class of Orlicz spaces both in the reflexive ($1 < p < \infty$) and non-reflexive ($p = 1$) cases (see Theorems 4 and 6). Roughly speaking, the results obtained here indicate that the $DH$-properties of Orlicz spaces are much richer and more non-trivial than one can see in the above papers. A crucial role in recovering these issues will be played by an example of the Orlicz function given in Proposition 1.

Our another (“positive”) aim is to study the uniform $DH$-property for Orlicz spaces, which means that the constant of equivalence of subsequences in the definition of $DH$-property can be chosen uniformly for all normalized disjoint sequences. We give a characterization of uniform $(p - DH)$ Orlicz spaces in Theorem 2 when $1 \leq p < \infty$, and then in Corollary 3 for $p = \infty$. From these results, it follows that for every $1 \leq p < \infty$, there is a $(p - DH)$ Orlicz space, which is not uniformly $(p - DH)$ (see also Corollary 4). In a sharp contrast to that, each $(\infty - DH)$ Orlicz space is uniformly $(\infty - DH)$ as well (Corollary 3). We establish also closed connections between this property and the duality of $DH$-property in Theorem 3 (in the non-reflexive case) and in Theorem 5 (in the reflexive case). Observe that analogous results in a special case of Banach lattices ordered by basis have been obtained earlier in the paper [12].

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2 Preliminaries

In this section, we will briefly list the definitions and notions used throughout this paper. For more detailed information, we refer to the monographs [20, 21, 23–25].

2.1 Symmetric spaces, Orlicz functions and spaces

A Banach function lattice $X$ on the interval $[0, 1]$ is a Banach space of real–valued Lebesgue measurable functions (of equivalence classes) defined on $[0, 1]$, which satisfies the ideal property: if $x$ is a measurable function, $|x| \leq |y|$ almost everywhere (a.e.) with respect to the Lebesgue measure $m$ on $[0, 1]$ and $y \in X$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$.

A Banach function lattice $X$ on $[0, 1]$ is said to be a symmetric (or rearrangement invariant) space if from the conditions: $y \in X$, $x$ is a measurable function on $[0, 1]$, with $x^*(t) = y^*(t)$, $0 < t \leq 1$, it follows that $x \in X$ and $\|x\|_X = \|y\|_X$. Here, $x^*$ denotes the non-increasing, right-continuous rearrangement of a measurable function $x$ on $[0, 1]$ given by

$$x^*(t) := \inf\{ s \geq 0 : m\{u \in [0, 1] : |x(u)| > s\} \leq t \}, \quad t > 0.$$ 

For any symmetric space $X$ on $[0, 1]$ we have $L_\infty[0,1] \subseteq X \subseteq L_1[0,1]$. The fundamental function $\phi_X$ of a symmetric space $X$ is defined by $\phi_X(t) := \|\chi_{[0,t]}\|_X$, $0 \leq t \leq 1$. In what follows, $\chi_A$ is the characteristic function of a set $A$. If $X$ is a symmetric space, then $X^0$ is the closure of $L_\infty$ in $X$. Then, $X^0$ is a symmetric space, which is separable provided that $X \neq L_\infty$.

The Köthe dual (or the associated space) $X'$ of a symmetric space $X$ consists of all measurable functions $y$ such that

$$\|y\|_{X'} := \sup \left\{ \int_0^1 |x(t)y(t)| dt : x \in X, \|x\|_X \leq 1 \right\} < \infty.$$ 

If $X$ is separable, then $X'$ coincides with the (Banach) dual space $X^*$.

The most known and important symmetric spaces are the $L_p$-spaces, $1 \leq p \leq \infty$. Their natural generalization is the Orlicz spaces. Let $F$ be an Orlicz function, that is, an increasing convex continuous function on $[0,\infty)$ such that $F(0) = 0$ and $\lim_{t \to \infty} F(t) = \infty$. Denote by $L_F := L_F[0,1]$ the Orlicz space on $[0, 1]$ (see e.g. [20]) endowed with the Luxemburg–Nakano norm

$$\|f\|_{L_F} = \inf \left\{ \lambda > 0 : \int_0^1 F(|f(t)|/\lambda) dt \leq 1 \right\}.$$ 

In particular, if $F(u) = u^p$, $1 \leq p < \infty$, we obtain $L_p$. The fundamental function $\phi_{L_F}(u) = 1/F^{-1}(1/u)$, $0 < u \leq 1$, where $F^{-1}$ is the inverse function.

If $F$ is an Orlicz function, then the Young conjugate function $G$ is defined by

$$G(u) := \sup_{t>0} (ut - F(t)), \quad u > 0.$$ 

Note that $G$ is also an Orlicz function and the Young conjugate for $G$ is $F$.

Similarly, we can define an Orlicz sequence space. Specifically, the space $\ell_\varphi$, where $\varphi$ is an Orlicz function, consists of all sequences $(a_k)_{k=1}^\infty$ such that
\[(a_k)_{k=1}^{\infty} \in l_\varphi \iff \inf \left\{ u > 0 : \sum_{k=1}^{\infty} \varphi \left( \left| \frac{a_k}{u} \right| \right) \leq 1 \right\} < \infty.\]

An Orlicz function \( H \) satisfies the \( \Delta_2^\infty \)-condition \( (H \in \Delta_2^\infty) \) (resp. the \( \Delta_2^0 \)-condition \( (H \in \Delta_2^0) \)) if
\[
\limsup_{t \to \infty} \frac{H(2t)}{H(t)} < \infty \quad \text{(resp. } \limsup_{t \to 0} \frac{H(2t)}{H(t)} < \infty).\]

It is well-known that an Orlicz function space \( L_F \) on \([0, 1]\) (resp. an Orlicz sequence space \( \ell_\varphi \)) is separable if and only if \( F \in \Delta_2^\infty \) (resp. \( \varphi \in \Delta_2^0 \)). In this case, we have \( L_F^* = L_F^* = L_G \), where \( G \) is the Young conjugate function for \( F \) (resp. \( \varphi^* = \varphi^* = \varphi^* \), with the Young conjugate function \( \psi \) for \( \varphi \)).

One can easily see (cf. [23, Proposition 4.a.2]) that the canonical unit vectors \( e_n = (e_n^i) \), \( e_n^i = \delta_{n,i}, n, i = 1, 2, \ldots \), form a symmetric basis of an Orlicz sequence space \( \ell_\varphi \) provided if \( \varphi \in \Delta_2^0 \). Recall that a basis \( \{x_n\}_{n=1}^{\infty} \) of a Banach space \( X \) is called symmetric if there exists \( C > 0 \) such that for arbitrary permutation \( \pi : \mathbb{N} \to \mathbb{N} \) and any \( a_n \in \mathbb{R} \), we have
\[
C^{-1} \left\| \sum_{n=1}^{\infty} a_n x_{\pi(n)} \right\|_X \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\|_X \leq C \left\| \sum_{n=1}^{\infty} a_n x_n \right\|_X.
\]

Observe that the definition of an Orlicz sequence space \( \ell_\varphi \) depends (up to equivalence of norms) only on the behaviour of the function \( \varphi \) near zero. More precisely, if \( \varphi, \psi \in \Delta_2^0 \), the following conditions are equivalent: (1) \( \ell_\varphi = \ell_\psi \) (with equivalence of norms); 2) the unit vector bases of the spaces \( \ell_\varphi \) and \( \ell_\psi \) are equivalent; 3) there are \( C > 0 \) and \( t_0 > 0 \) such that for all \( 0 \leq t \leq t_0 \) it holds
\[
C^{-1} \varphi(t) \leq \psi(t) \leq C \varphi(t)
\]
(cf. [23, Proposition 4.a.5] or [25, Theorem 3.4]). In the case when \( \varphi \) is a degenerate Orlicz function, i.e., there is \( t > 0 \) such that \( \varphi(t) = 0 \), we have \( \ell_\varphi = \ell_\infty \) (with equivalence of norms).

Quite similarly, the definition of an Orlicz function space \( L_F \) on \([0, 1]\) depends only on the behavior of the function \( F \) for large values of the argument.

Let \( F \) be an Orlicz function, \( F \in \Delta_2^\infty \). Define the following subsets of the space \( C[0, 1] \):
\[
E^\infty_{F,A} := \left\{ G(x) = \frac{F(xy)}{F(y)} : y > A \right\} \quad (A > 0), \quad E^\infty_F := \bigcap_{A>0} E^\infty_{F,A}, \quad C^\infty_F := \text{conv} E^\infty_F,
\]
where \( \text{conv} U \) is the convex hull of a set \( U \) and the closure is taken in the space \( C[0, 1] \). All these sets are non-void, compact in \( C[0, 1] \) and consist of Orlicz functions [23, Lemma 4.a.6]. It is well-known that the sets \( E^\infty_F \) and \( C^\infty_F \) rather fully determine the structure of disjoint sequences of \( L_F \) (see [23, § 4.a], [22]).

Let \( F \) be an Orlicz function, \( \varphi \) be a function defined on \([0, 1]\). We will write \( E^\infty_F \cong \{ \varphi \} \) if for every \( H \in E^\infty_F \) there is a constant \( C = C(H) \) such that
\[
C^{-1} \varphi(t) \leq H(t) \leq C \varphi(t), \quad 0 < t \leq 1.
\]
In the case when this condition is fulfilled uniformly for all \( H \in E_F^\infty \), i.e., there is a constant \( C > 0 \) such that (1) holds for each \( H \in E_F^\infty \), we will write \( E_F^\infty \equiv \{ \varphi \} \). In a similar way, we understand the expressions \( C_F^\infty \equiv \{ \varphi \} \) and \( C_F^\infty \equiv \{ \varphi \} \).

Let \( E_F^\infty \equiv \{ \varphi \} \). Suppose that \( \varphi \) is a non-degenerate function, i.e., \( \varphi(t) > 0 \) for \( t > 0 \). Then \( F \) is quasi-multiplicative, that is, \( C^{-1} \leq F(uv)/F(u)F(v) \leq C \) for some \( C > 0 \) and all \( 0 \leq u, v \leq 1 \) (see the proof of Theorem 4.1 in [11]). Therefore, by a classical result of Polya, there is \( 1 \leq p < \infty \) such that \( \varphi(u) \) is equivalent on \([0, 1]\) to the function \( u^p \).

It is obvious that from \( E_F^\infty \equiv \{ \varphi \} \) it follows that \( C_F^\infty \equiv \{ \varphi \} \). What is about a weaker condition \( E_F^\infty \equiv \{ \varphi \} \)? We will see further that in this case may be both situations: \( C_F^\infty \equiv \{ \varphi \} \) and \( C_F^\infty \not\equiv \{ \varphi \} \).

An Orlicz function \( F \) such that \( F \in \Delta_2^\infty \) is said to be regularly varying at \( \infty \) (in sense of Karamata) if the limit \( \lim_{t \to \infty} F(tu)/F(t) \) exists for all \( u > 0 \) (in fact, it suffices that the limit exists when \( 0 < u \leq 1 \)). Then, there is a \( p, 1 \leq p < \infty \), such that \( \lim_{t \to \infty} F(tu)/F(t) = u^p \); in this case \( F \) is called regularly varying of order \( p \).

### 2.2 Properties of disjoint sequences in Banach lattices.

Following to [14], a Banach lattice \( X \) is said to be disjointly homogeneous (in brief, DH) if two arbitrary normalized disjoint sequences \( \{ x_n \} \) and \( \{ y_n \} \) from \( X \) contain subsequences \( \{ x_{n_k} \} \) and \( \{ y_{n_k} \} \) that are \( C \)-equivalent in \( X \) for some \( C > 0 \). As usual, this means that for all \( c_k \in \mathbb{R} \) we have

\[
C^{-1} \left\| \sum_{k=1}^{\infty} c_k x_{n_k} \right\|_X \leq \left\| \sum_{k=1}^{\infty} c_k y_{n_k} \right\|_X \leq C \left\| \sum_{k=1}^{\infty} c_k x_{n_k} \right\|_X.
\]

If there exists a constant \( C \), which is suitable for each pair of normalized disjoint sequences \( \{ x_n \} \) and \( \{ y_n \} \), then we will say that \( X \) is uniformly DH [12].

Clearly, every uniformly DH Banach lattice is DH. In the paper [12], it is shown that there are DH Banach lattices ordered by basis that are not uniformly DH (a Banach lattice ordered by basis is a Banach lattice \( X \) with a basis \( \{ x_n \} \) such that \( \sum_{n=1}^{\infty} a_n x_n \geq 0 \) if and only if \( a_n \geq 0 \) for all \( n \); see [12]).

In particular, given a \( p \in [1, \infty) \), a Banach lattice \( X \) is called \( p \)-disjointly homogeneous (briefly, \((p-DH))\) if each normalized disjoint sequence in \( X \) has a subsequence equivalent to the unit vector basis of \( \ell_p \) (\( c_0 \) when \( p = \infty \)). In the case when the constant \( C \) of this equivalence can be chosen uniformly for all normalidz disjoint sequences, \( X \) is called uniformly \((p-DH))\).

As given in Section 1, the DH property is enjoyed, in particular, by \( L_p(\mu) \)-spaces, \( 1 \leq p \leq \infty \), Lorentz spaces \( L_{q,p} \) and \( \Lambda(W,p) \), certain Orlicz spaces, Tsirelson space and some others. Moreover, by using the complex method of interpolation, it was proved, in [3], that for every \( 1 \leq p \leq \infty \) and any increasing concave function \( \varphi(u) \) on \([0, 1]\), which is not equivalent to neither \( 1 \) nor \( u \), there exists a \((p-DH)\) symmetric space on \([0, 1]\) with the fundamental function \( \varphi \).

In [16], for \( p = 2 \), the following weaker notion has been introduced. Let \( 1 \leq p \leq \infty \). A symmetric space \( X \) is said to be restricted \((p-DH)\) if for every sequence of pairwise disjoint subsets \( \{ A_n \}_{n=1}^{\infty} \) of \([0, 1]\) there is a subsequence \( \{ A_{n_k} \} \) such that \( \frac{1}{\| x_{n_k} \|_X} x_{n_k} \) is equivalent to the unit vector basis of \( \ell_p \) (\( c_0 \) when \( p = \infty \)). As above, we can define also uniformly restricted \((p-DH)\) symmetric spaces.
Clearly, every $(p - DH)$ symmetric space is restricted $(p - DH)$. On the other hand, for each $1 \leq p < \infty$, there exist restricted $(p - DH)$ symmetric spaces on $[0, 1]$, which fail to be $(p - DH)$ (see [6]).

A symmetric space $X$ is called *disjointly complemented* (in brief, DC) if every disjoint sequence in $X$ has a subsequence whose span is complemented in $X$ (see [12]).

For more detailed information related to the $DH$- and $(p - DH)$-properties, see [3, 5, 6, 11–14, 16].

The notation $A \asymp B$ means that there exists a constant $C > 0$ that does not depend on the arguments of $A$ and $B$ such that $C^{-1}A \leq B \leq CA$.

Finally, throughout the paper, we set $\log a := \log_2 a$, where $a > 0$, and $\bar{x}_A := x_A/\|x_A\|$.

### 3 A special Orlicz function.

The following result will play a key role in our further considerations.

**Proposition 1** There exists an Orlicz function $\Phi$ on $[0, \infty)$ such that $C^\infty_\Phi \cong \{t\}$ and $E^\infty_\Phi \not\cong \{t\}$.

**Proof** Let $\{r_i\}_{i=0}^\infty$ be an increasing sequence of positive integers such that $r_0 = 1$ and $\lim_{i \to \infty} r_{i-1}/r_i = 0$. Furthermore, for each $i = 1, 2, \ldots$ let $\{k_i^j\}_{j=1}^\infty$ be an increasing sequence of positive integers such that the intervals $A_i^j := [2^{k_i^j} - r_i, 2^{k_i^j}]$, $i, j = 1, 2, \ldots$, are pairwise disjoint and are separated in the sense that size of the gaps between them tends to $\infty$. More precisely, if $\{A^*_i\}_{i=0}^\infty$ is the sequence of these sets enumerated so that $A^*_1 < A^*_2 < \ldots$, then

$$\min A^*_{k+1} - \max A^*_k \to \infty \text{ as } k \to \infty. \tag{2}$$

We define also the functions

$$\varphi(s) := \sum_{i=1}^\infty (1 + r_{i-1}/r_i) \left( \sum_{j=1}^\infty x_{A_i^j}(s) \right) + x_{[0,\infty) \cup \bigcup_{i} A_i^j}(s),$$

$$f(t) := \int_0^t \varphi(s) ds,$$

and

$$F(x) := 2^{f(\log x)} x_{(1,\infty)}(x) + x_{[0,1]}(x).$$

The function $F$ need not to be convex. However, we will show that $F$ is equivalent on $[0, \infty)$ to the convex function $\Phi$ defined by

$$\Phi(t) := \int_0^t F(s)/s \, ds.$$

To prove the convexity of $\Phi$ it suffices to check that $F(s)/s$ increases, which is equivalent to the inequality

$$\frac{F(st)}{F(s)} \geq t \text{ for all } t \geq 1, s > 0. \tag{3}$$
If \( s \geq 1 \), then
\[
    f(\log t + \log s) - f(\log s) = \int_{\log s}^{\log t + \log s} \varphi(u) \, du \geq \log t,
\]
whence
\[
    \frac{F(st)}{F(s)} \geq 2^{\log t} = t.
\]
Suppose that \( 0 < s < 1 \). Then, if \( st < 1 \), by definition, we have \( F(st) = st = tF(s) \). Otherwise, if \( st \geq 1 \), then
\[
    f(\log(st)) = \int_{0}^{\log(st)} \varphi(u) \, du \leq \log(st),
\]
and therefore
\[
    F(st) = 2^{f(\log(st))} \geq st = tF(s).
\]
Thus, (3) is proved, which implies the convexity of \( \Phi \). Moreover, from (3), it follows that \( \Phi(t) \leq F(t) \) for all \( t > 0 \).

Show now that
\[
    F(t) \leq 4F(t/2), \quad t > 0. \tag{4}
\]
Indeed, if \( 0 < t \leq 1 \), we have \( F(t) = t = 2F(t/2) \). Since \( \varphi(u) \leq 2 \) for \( 1 < u \leq 2 \), it follows that, for \( 1 < t \leq 2 \),
\[
    f(\log t) = \int_{0}^{\log t} \varphi(u) \, du \leq 2 \log t,
\]
whence
\[
    F(t) = 2^{f(\log t)} \leq t^2 \leq 2t = 4F(t/2).
\]
Finally, if \( t > 2 \), then
\[
    f(\log t) - f(\log(t/2)) = \int_{\log(t/2)}^{\log t} \varphi(u) \, du \leq 2.
\]
Consequently,
\[
    \frac{F(t)}{F(t/2)} = 2^{f(\log t) - f(\log(t/2))} \leq 4.
\]
Summarizing these estimates, we obtain (4).

In turn, from inequalities (3) and (4), it follows that
\[
    \Phi(t) \geq \int_{t/2}^{t} \frac{F(u)}{u} \, du \geq F(t/2) \geq \frac{1}{4} F(t).
\]
Thus,
Let us prove next that \( E^\infty_\Phi \not\equiv \{t\} \). To this end, it suffices clearly to find functions \( H_i \in E^\infty_\Phi \), \( i = 1, 2, \ldots \), such that
\[
2^i H_i(2^{-r_i}) \to 0 \quad \text{as} \quad i \to \infty. \tag{6}
\]

By the definition of \( F \), for all \( i, j = 1, 2, \ldots \)
\[
\frac{F(2^{2^j-r_i})}{F(2^{2^j})} = 2^{f(2^{2^j}-r_i)-f(2^{2^j})}. \tag{5}
\]

Therefore, since
\[
2^i H_i(2^{-r_i}) = \frac{\int_{2^j}^{2^{2^j-r_i}} \Phi(s) \, ds - \int_{2^j}^{2^{2^j}} (1 + r_{i-1}/r_i) \, ds}{r_i - r_{i-1}},
\]
we have
\[
\frac{F(2^{2^j-r_i})}{F(2^{2^j})} = 2^{-n_i-n_{i-1}}, \quad i, j = 1, 2, \ldots
\]

Then, by (5), passing to an appropriate subsequence \( \{2^{2^j}\}_{j=1}^\infty \), for a function \( H_i \in E^\infty_\Phi \) we get
\[
H_i(2^{-r_i}) = \lim_{s \to \infty} \frac{\Phi(2^{2^j-r_i})}{\Phi(2^{2^j})} = 2^{-n_i-n_{i-1}}, \quad i = 1, 2, \ldots
\]
with some constant independent of \( i \). Thus, (6) is proved and hence \( E^\infty_\Phi \not\equiv \{t\} \).

It is left to show that \( C^\infty_\Phi \cong \{t\} \). Since \( H(t) \leq t \) for every \( H \in C^\infty_\Phi \), we need to prove only that \( H(t) \geq c t, 0 < t \leq 1 \), for some constant \( c > 0 \) depending on \( H \). Observe that it suffices to check that there exists \( c > 0 \) such that
\[
H(2^{-r}) \geq c 2^{-r}, \quad r = 0, 1, \ldots. \tag{7}
\]

Indeed, assuming that (7) holds, for each \( 0 < t \leq 1 \) we can find \( r = 0, 1, \ldots \) satisfying \( 2^{-r-1} < t \leq 2^{-r} \). Then, since \( H \) increases, it follows
\[
H(t) \geq H(2^{-r-1}) \geq c 2^{-r-1} \geq c_1 t,
\]
with \( c_1 := c/2 \). Moreover, one can readily see that it is sufficient to prove (7) only for \( r \geq m \), with some \( m \in \mathbb{N} \).

By definition, every function \( H \in C^\infty_\Phi \) can be represented as follows:
\[
H(u) = \lim_{l \to \infty} y_l(u), \quad \text{where} \quad y_l(u) := \sum_{s=1}^{n_l} \lambda^*_s \Phi(t^*_s u)/\Phi(t^*_s), \quad 0 < u \leq 1, \tag{8}
\]
where \( \lambda^*_s > 0, \sum_{s=1}^{n_l} \lambda^*_s = 1 \), and \( \beta_l := \min_{s=1,\ldots,n_l} t^*_s \to \infty \) as \( l \to \infty \).
Observe that if $y_l(2^{-r})2^r > \frac{1}{16}$ for every $r = 1, 2, \ldots$ and all sufficiently large $l$, then we have inequality (7) for $c = \frac{1}{16}$, and so everything is done. Otherwise, passing to a subsequence if it is necessary, we can assume that there is $m \in \mathbb{N}$ such that for all sufficiently large $l$\

$$y_l(2^{-m})2^m < \frac{1}{16}. \quad (9)$$

For a fixed $m$ satisfying (9), choose $i_0$ so that\

$$\frac{r_{i-1}}{r_i} \leq \frac{1}{m} \quad \text{for all } i \geq i_0. \quad (10)$$

Also, let\

$$\Delta(l, s, r) := [\log t_s^l - r, \log t_s^l], \quad s = 1, \ldots, n_l, \ l, r = 1, 2, \ldots$$

and\

$$I(l, r) := \{s = 1, \ldots, n_l : \Delta(l, s, r) \cap A_i^j \neq \emptyset \text{ for some } 1 \leq i \leq i_0 \text{ and } j = 1, 2, \ldots \},$$

$$I'(l, r) := \{1, 2, \ldots, n_l\} \setminus I(l, r), \ l, r = 1, 2, \ldots$$

Let us estimate from above the sum $\sum_{s \in I'(l, m)} \lambda_s^l$ for sufficiently large $l = 1, 2, \ldots$\

Suppose $s \in I'(l, m)$. In the case when $\Delta(l, s, m) \cap A_i^j = \emptyset$ for all $i, j = 1, 2, \ldots$, we have\

$$f(\log t_s^l - m) - f(\log t_s^l) = -\int_{\Delta(l,s,m)} du = -m,$$

whence\

$$\frac{F(t_s^l2^{-m})}{F(t_s^l)} = 2^{f(\log t_s^l-m)-f(\log t_s^l)} = 2^{-m}.$$\

Let now $s \in I'(l, m)$ and $\Delta(l, s, m) \cap A_i^j \neq \emptyset$ for some positive integers $i$ and $j$. Then, by definition of the set $I'(l, m)$, we deduce that $i > i_0$. Therefore, $\varphi(u) \leq 1 + \max_{i \geq i_0} r_{i-1}/r_i$, $u \in \Delta(l, s, m)$, and from (10), it follows\

$$f(\log t_s^l - m) - f(\log t_s^l) = -\int_{\Delta(l,s,m)} \varphi(u) du \geq -\int_{\Delta(l,s,m)} (1 + \max_{i \geq i_0} r_{i-1}/r_i) du$$

$$= -m(1 + \max_{i \geq i_0} r_{i-1}/r_i) \geq -m - 1.$$\

Thus, in this case,\

$$\frac{F(t_s^l2^{-m})}{F(t_s^l)} \geq 2^{-m-1}.$$\

Summarizing the above estimates and appealing to inequalities (5) and (9), we get\

$$\sum_{s \in I'(l, m)} \lambda_s^l \leq 2^{m+1} \sum_{s \in I'(l, m)} \lambda_s^l \frac{F(t_s^l2^{-m})}{F(t_s^l)} \leq 2^{m+3} \sum_{s \in I'(l, m)} \lambda_s^l \frac{\Phi(t_s^l2^{-m})}{\Phi(t_s^l)} \leq 2^{m+3} y_l(2^{-m}) \leq \frac{1}{2}.$$
Therefore, by definition of the set \( I(l, m) \), for all sufficiently large \( l \)

\[
\sum_{s \in I(l, m)} \lambda_s^l \geq \frac{1}{2}.
\]

Assuming that \( r \geq m \), we clearly have \( \Delta(l, s, m) \subset \Delta(l, s, r) \) for all \( s = 1, 2, \ldots, n_l \) and \( l = 1, 2, \ldots \). This implies, in turn, that \( I(l, m) \subset I(l, r) \), \( l = 1, 2, \ldots \). Thus, from the last inequality it follows that for every \( r > m \) and for all sufficiently large \( l \)

\[
\sum_{s \in I(l, r)} \lambda_s^l \geq \frac{1}{2}.
\] (11)

Moreover, choosing \( l \) sufficiently large, thanks to (2), we can assume that each interval \( \Delta(l, s, r) \) intersects, at most, one of the intervals \( A^l_i, i, j = 1, 2, \ldots \). In particular, if \( s \in I(l, r) \) the interval \( \Delta(l, s, r) \) intersects only a unique interval \( A^l_i \) for some \( 1 \leq i \leq i_0 \) and \( j = 1, 2, \ldots \). Then, denoting \( \alpha := m(\Delta(l, s, r) \cap A^l_i) \), we have \( 0 < \alpha \leq r_i \) and

\[
f(\log t_s^l - r) - f(\log t_s^l) = -\int_{\Delta(l, s, r)} \varphi(u) \, du = -r + \alpha - \int_{\Delta(l, s, r) \cap A^l_i} (1 + r_{i-1}/r_i) \, du
\]

\[
= -r + \alpha(1 + r_{i-1}/r_i) \geq -r - r_{i-1} \geq -r - r_{i_0-1},
\]

whence

\[
\frac{F(t^l_s - r)}{F(t^l_s)} \geq 2^{-r_{i_0-1}} \cdot 2^{-r}, \quad s \in I(l, r).
\]

Combining this together with inequalities (5) and (11), we deduce that for all sufficiently large \( l \)

\[
y(2^{-r}) \geq \sum_{s \in I(l, r)} \lambda_s^l \frac{\Phi(t^l_s - r)}{\Phi(t^l_s)} \geq \frac{1}{4} \sum_{s \in I(l, r)} \lambda_s^l \frac{F(t^l_s - r)}{F(t^l_s)} \geq 2^{-r_{i_0-2}} \cdot 2^{-r} \sum_{s \in I(l, r)} \lambda_s^l \geq 2^{-r_{i_0-3}} \cdot 2^{-r}.
\]

Observe that the index \( i_0 \) in (10) does not depend on \( r \) (it depends only on the given function \( H \)). Therefore, the last inequality implies estimate (7) for \( c = 2^{-r_{i_0-1}} \cdot 3 \). This completes the proof of the proposition.

\[\square\]

**Corollary 1** For every \( 1 < p < \infty \) there exists an Orlicz function \( \Phi_p \) on \([0, \infty)\) such that \( C^\infty_{\Phi_p} \cong \{t^p\} \) and \( E^\infty_{\Phi_p} \not\cong \{t^p\} \).

**Proof** Let \( \Phi \) be the Orlicz function from Proposition 1. We set \( \Phi_p(u) := \Phi(u^p) \), \( 0 \leq u \leq 1 \). Clearly, \( \Phi_p \) is an Orlicz function. Moreover, one can readily check that

\[H \in E^\infty_{\Phi_p} \text{ if and only if } H(u^{1/p}) \in E^\infty_{\Phi}, \]

and

\[H \in C^\infty_{\Phi_p} \text{ if and only if } H(u^{1/p}) \in C^\infty_{\Phi}. \]

Indeed, assuming that \( H \in E^\infty_{\Phi_p} \), we can find a sequence \( \{t_n\} \) such that \( t_n \uparrow \infty \) as \( n \to \infty \) and
Since $t_n^p \uparrow \infty$, the function $H(u^{1/p})$ belongs to the set $E_{\Phi}^\infty$. The converse can be shown in the same way. Since the result for the set $C_{\Phi}^\infty$ can be obtained similarly, the desired result follows now easily from Proposition 1. □

**4 (p – DH) Orlicz spaces, $1 \leq p < \infty$.**

We start this section with proving necessary and sufficient conditions, under which an Orlicz space is restricted $(p – DH)$. We provide the proof of this simple result for the reader’s convenience and as well as to track the constants in the inequalities for future purposes.

Recall that $\bar{\chi}_A := \chi_A/\|\chi_A\|_{L_p}$.

**Proposition 2** Let $1 \leq p < \infty$, and let $F$ be an Orlicz function. The following conditions are equivalent:

(a) $L_F$ is restricted $(p – DH)$;

(b) for every sequence of disjoint sets $E_n \subset [0, 1]$, $n = 1, 2, \ldots$, there exists a subsequence $E_{n_k}$, $k = 1, 2, \ldots$, such that

$$C^{-1} m^{1/p} \leq \left\| \sum_{k=1}^{m} \bar{\chi}_{E_{n_k}} \right\|_{L_p} \leq C m^{1/p}, \ m = 1, 2, \ldots \tag{12}$$

with some constant $C > 0$;

(c) $E_{\Phi}^\infty \cong \{t^p\}$.

**Proof** Let us show first that for every $H \in L_{\Phi}^\infty$ and any $\varepsilon > 0$ there is a sequence of disjoint sets $E_n \subset [0, 1]$, $n = 1, 2, \ldots$, such that for all $(c_n) \in \ell_H^\infty$

$$(1 – \varepsilon)\|c_n\|_{\ell_H^\infty} \leq \left\| \sum_{n=1}^{\infty} c_n \bar{\chi}_{E_n} \right\|_{L_p} \leq (1 + \varepsilon)\|c_n\|_{\ell_H^\infty}. \tag{13}$$

Since $H \in L_{\Phi}^\infty$, then there is a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \to \infty$ and

$$\left| \frac{F(t_n u)}{F(t_n)} – H(u) \right| < \frac{\varepsilon}{2^n}$$

for all $0 \leq u \leq 1$ and $n = 1, 2, \ldots$ This implies that for all $c_n \in \mathbb{R}$

$$\sum_{n=1}^{\infty} H(c_n) – \varepsilon \leq \sum_{n=1}^{\infty} \frac{F(t_n c_n)}{F(t_n)} \leq \sum_{n=1}^{\infty} H(c_n) + \varepsilon. \tag{14}$$

Without loss of generality, we may assume that $F(t_n) \geq 2^n$, $n = 1, 2, \ldots$. Thanks to that, there are disjoint sets $E_n \subset [0, 1]$ such that $m(E_n) = 1/F(t_n)$, $n = 1, 2, \ldots$. Denoting
\( f_n := \bar{\chi}_{E_n} \), suppose that \( \| \sum_{n=1}^{\infty} c_nf_n \|_{L_f} \leq 1 \). Since \( \| \chi_{E_n} \|_{L_f} = 1/F^{-1}(1/m(E_n)) = 1/t_n \) (see Section 2.1), it follows

\[
\sum_{n=1}^{\infty} \frac{F(t_n|c_n|)}{F(t_n)} = \int_0^1 F\left( \sum_{n=1}^{\infty} c_nf_n(u) \right) du \leq 1.
\]

By (14), this yields

\[
\sum_{n=1}^{\infty} H(|c_n|) \leq 1 + \varepsilon,
\]

and, as \( H \) is a convex function, \( \| (c_n) \|_{\ell_H} \leq 1 + \varepsilon \).

Conversely, if \( \| (c_n) \|_{\ell_H} \leq 1 + \varepsilon \), we have \( \sum_{n=1}^{\infty} H(|c_n|) \leq 1 \). Therefore, according to (14),

\[
\int_0^1 F\left( \sum_{n=1}^{\infty} c_nf_n(u) \right) du \leq 1 + \varepsilon.
\]

Hence, \( \| \sum_{n=1}^{\infty} c_nf_n \|_{L_f} \leq 1 + \varepsilon \). In view of the obtained estimates, applying the simple homogeneity argument, we come to inequality (13).

Proceeding with the proof of the proposition, note that the implication \((a) \Rightarrow (b)\) is obvious. Assuming now that \((b)\) holds, we prove \((c)\).

Let \( H \in E_f^{\infty} \) be arbitrary. Then, setting \( \varepsilon = \frac{1}{2} \), we can find disjoint sets \( E_n \subset [0, 1], n = 1, 2, \ldots \), such that for all \( (c_n) \in \ell_H \) we have \((13)\). In particular, for all \( m = 1, 2, \ldots \) it follows that

\[
\frac{1}{2} \left\| \sum_{n=1}^{m} e_n \right\|_{\ell_H} \leq \left\| \sum_{n=1}^{m} \bar{\chi}_{E_n} \right\|_{L_f} \leq 2 \left\| \sum_{n=1}^{m} e_n \right\|_{\ell_H},
\]

where \( e_n \) are the vectors of the unit basis of \( \ell_H \). On the other hand, according the hypothesis, passing to a subsequence (and preserving the notation), we get

\[
C^{-1}m^{1/p} \leq \left\| \sum_{n=1}^{m} \bar{\chi}_{E_n} \right\|_{L_f} \leq Cm^{1/p}, \quad m = 1, 2, \ldots
\]

with some constant \( C > 0 \). In consequence,

\[
(2C)^{-1}m^{-1/p} \leq H^{-1}(1/m) \leq (2C)^{-1}m^{-1/p}, \quad m = 1, 2, \ldots,
\]

and so \( H(u) \asymp u^{1/p}, 0 < u \leq 1 \), with a constant that depends only on \( C \). Thus, \( E_f^{\infty} \cong \{ v^p \} \), which completes the proof of \((c)\).

Finally, we will prove the implication \((c) \Rightarrow (a)\). Let \( E_n \subset [0, 1], n = 1, 2, \ldots \), be disjoint, \( f_n := \bar{\chi}_{E_n} \). We introduce the functions

\[
H_n(u) := \int_0^1 F(uf_n(v)) dv = F(uF^{-1}(1/m(E_n)))m(E_n).
\]

Clearly,

\[
H_n(u) = \frac{F(ut_n)}{F(t_n)}, \quad \text{where } t_n := F^{-1}(1/m(E_n)), \quad n = 1, 2, \ldots
\]
Since $m(E_n) \to 0$, then $t_n \to \infty$ as $n \to \infty$. Therefore, since $\{H_n\}$ is a relatively compact set in $C[0, 1]$ [23, Lemma 4.a.6 and subsequent Remark], there is a subsequence $\{H_{n_j}\}$, uniformly converging on $[0, 1]$ to some $H \in E_F^\infty$. We may assume that $|H_{n_j}(u) - H(u)| \leq 2^{-j}$ for all $0 \leq u \leq 1$ and $j = 1, 2, \ldots$, whence

$$\sum_{j=1}^{\infty} H(|c_j|) - 1 \leq \sum_{j=1}^{\infty} H_{n_j}(|c_j|) \leq \sum_{j=1}^{\infty} H(|c_j|) + 1$$

for all $c_j \in \mathbb{K}$. Since $E_F^\infty \cong \{\ell^p\}$, then $C^{-1} u^p \leq H(u) \leq C u^p$ for some $C \geq 1$ and all $0 \leq u \leq 1$. Consequently,

$$C^{-1} \sum_{j=1}^{\infty} |c_j|^p - 1 \leq \sum_{j=1}^{\infty} H_{n_j}(|c_j|) \leq C \sum_{j=1}^{\infty} |c_j|^p + 1,$$

or equivalently

$$C^{-1} \sum_{j=1}^{\infty} |c_j|^p - 1 \leq \int_0^1 F\left(\sum_{j=1}^{\infty} c_{f_{n_j}}(v)\right) dv \leq C \sum_{j=1}^{\infty} |c_j|^p + 1.$$

Hence, if $\|c_j\|_{\ell^p} \leq 1$, then $\|\sum_{j=1}^{\infty} c_{f_{n_j}}\|_{L^F} \leq C + 1$. Conversely, if $\|\sum_{j=1}^{\infty} c_{f_{n_j}}\|_{L^F} \leq 1$, then $\|c_j\|_{\ell^p} \leq (2C)^{1/p}$. Thus, the subsequence $\{f_{n_j}\}$ is $2C$-equivalent to the unit vector basis of $\ell^p$, and the proof is completed.

To prove a similar criterion for $DH$-property, we will need the next useful result, which is well-known in the separable case (see [22, Proposition 3]).

**Lemma 1** Let $F$ be an Orlicz function. Then, every normalized disjoint sequence from the Orlicz space $L_F$ contains a subsequence that is 6-equivalent to the unit vector basis of an Orlicz sequence space $\ell^n_H$ with some $H \in C^n_F$.

**Proof** Let $\{f_{n_j}\}_{j=1}^{\infty} \subset L_F$ be a disjoint normalized sequence. We claim that the lemma will be proved once we select a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that $f_{n_k} = u_k + v_k$, where the functions $u_k$ and $v_k$, $k = 1, 2, \ldots$, satisfy the conditions:

(a) $u_k \cdot v_i = 0$ for all $k, i = 1, 2, \ldots$;

(b) the sequence $\{u_k\}_{k=1}^{\infty}$ 4-equivalent to the unit vector basis of an Orlicz sequence space $l^\infty_H$ with some $H \in C^\infty_F$, i.e., for all $(a_k)_{k=1}^{\infty} \in l^\infty_H$

$$\frac{1}{4} \|\sum_{k=1}^{\infty} a_k u_k\|_{L^F} \leq \|\sum_{k=1}^{\infty} a_k u_k\|_{l_H} \leq 4 \|\sum_{k=1}^{\infty} a_k u_k\|_{L^F}.$$

(c) for each sequence $(a_k)_{k=1}^{\infty} \in c_0$, we have

$$\|\sum_{k=1}^{\infty} a_k v_k\|_{L^F} \leq 2 \|\sum_{k=1}^{\infty} a_k v_k\|_{c_0}.$$

Indeed, on the one hand, from (15) and (16) it follows that
\[ \left\| \sum_{k=1}^{\infty} a_k f_k \right\|_{L_F} \leq \left\| \sum_{k=1}^{\infty} a_k u_k \right\|_{L_F} + \left\| \sum_{k=1}^{\infty} a_k v_k \right\|_{L_F} \]
\[ \leq 4 \| (a_k) \|_{l_F} + 2 \| (a_k) \|_{l_0} \]
\[ \leq 6 \| (a_k) \|_{l_F}. \]

On the other hand, applying \((a)\) and once more \((15)\), we obtain the opposite inequality:
\[ \left\| \sum_{k=1}^{\infty} a_k f_k \right\|_{L_F} \geq \left\| \sum_{k=1}^{\infty} a_k u_k \right\|_{L_F} \geq \frac{1}{4} \| (a_k) \|_{l_F}. \]

Thus, our claim is proved. Consequently, it suffices to show that such a subsequence \([f_{n_k}]_{k=1}^{\infty}\) exists.

Since \(\|f_n\|_{L_F} = 1\), by definition of the Luxemburg-Nakano norm, for all \(n = 1, 2, \ldots\) we have
\[ \int_{0}^{1} F\left( \frac{|f_n(t)|}{2} \right) dt \leq 1 \quad \text{and} \quad \int_{0}^{1} F(2|f_n(t)|) dt > 1. \]

Hence, from the absolute continuity of Lebesgue integral, it follows that for each \(n = 1, 2, \ldots\) there is a constant \(M_n > 0\) such that for the functions \(x_n := f_n \chi_{\{f_n > M_n\}}\) and \(y_n := f_n \chi_{\{f_n \leq M_n\}}\) we have
\[ \int_{0}^{1} F\left( \frac{|x_n(t)|}{2} \right) dt \leq 2^{-n} \quad \text{and} \quad \int_{0}^{1} F(2|y_n(t)|) dt > 1. \]  
(17)

Observe that the functions \(y_n, n = 1, 2, \ldots\), are disjoint and \(y_n \in L_{\infty} \subset L_F^0\), where \(L_F^0\) is the separable part of the Orlicz space \(L_F\) (see Section 2.1). Moreover, \(\|y_n\|_{L_F} \leq \|f_n\|_{L_F} \leq 1\) and in view of the second inequality in \((17)\) we have \(\|y_n\|_{L_F} > 1/2\) for all \(n = 1, 2, \ldots\).

Thanks to these properties, reasoning precisely as in the proof of [22, Proposition 3], we can find a subsequence \([y_{n_k}] \subset [y_n]\), which is 4-equivalent to the unit vector basis of an Orlicz sequence space \(l_H\), where \(H \in C_F^\infty\). Therefore, the functions \(u_k := y_{n_k}, v_k := x_{n_k}, k = 1, 2, \ldots\), satisfy the conditions \((a), (b)\) and \(f_{n_k} = u_k + v_k, k = 1, 2, \ldots\)

Moreover, let \(\| (a_k) \|_{l_0} \leq 1\). Since \(x_{n_k}, k = 1, 2, \ldots\), are disjoint, from the first inequality in \((17)\), it follows
\[ \int_{0}^{1} F\left( \frac{\left| \sum_{k=1}^{\infty} a_k v_k(t) \right|}{2} \right) dt \leq \int_{0}^{1} F\left( \frac{\left| \sum_{k=1}^{\infty} v_k(t) \right|}{2} \right) dt = \sum_{k=1}^{\infty} \int_{0}^{1} F\left( \frac{|x_{n_k}(t)|}{2} \right) dt \leq \sum_{k=1}^{\infty} 2^{-n_k} \leq 1. \]

Applying now the homogeneity argument precisely in the same way as above, we get inequality \((16)\). Thus, the proof is completed. \(\square\)

**Remark 1** One can readily see that in the case when \([f_n]\) is a normalized sequence of characteristic functions of disjoint subsets of \([0, 1]\) the function \(H\) from the proof of Lemma 1 belongs to the smaller set \(E_F^\infty\).
Theorem 1 Let $1 \leq p < \infty$, and let $F$ be an Orlicz function. Then, the Orlicz space $L_F$ is $(p - DH)$ if and only if $C_F^\infty \cong \{ t^p \}$.

Proof Assume first that $L_F$ is a $(p - DH)$ space. Let $H \in C_F^\infty$. Then, arguing similarly as in the beginning of the proof of Proposition 2, we can find a sequence of normalized disjoint functions $\{ f_n \}$ equivalent to the unit vector basis of the sequence Orlicz space $\ell_H$. On the other hand, by the hypothesis, some subsequence $\{ f_{n_k} \} \subset \{ f_n \}$ is equivalent in $L_F$ to the unit vector basis of $\ell_p$. Since the unit vector basis of an arbitrary sequence Orlicz space is a symmetric basic sequence, we conclude that $H(u) \cong u^p$, $0 \leq u \leq 1$. Therefore, $C_F^\infty \cong \{ t^p \}$.

Conversely, let $C_F^\infty \cong \{ t^p \}$. By Lemma 1, every normalized disjoint sequence from $L_F$ contains a subsequence that is $6$-equivalent to the unit vector basis of an Orlicz sequence space $\ell_H$ for some $H \in C_F^\infty$. Since $H(u) \cong u^p$, $0 \leq u \leq 1$, then obviously this subsequence is equivalent in $L_F$ also to the unit vector basis of $\ell_p$. As a result, we conclude that $L_F$ is $(p - DH)$, and the proof is completed.

Theorem 2 Let $1 \leq p < \infty$, and let $F$ be an Orlicz function. The following conditions are equivalent:

(a) $L_F$ is uniformly $(p - DH)$;

(b) $L_F$ is uniformly restricted $(p - DH)$;

(c) there exists a constant $C > 0$ such that for every sequence of disjoint sets $E_n \subset [0, 1]$, $n = 1, 2, \ldots$, we have

$$C^{-1} m^{1/p} \leq \left\| \sum_{n=1}^m \tilde{\chi}_{E_n} \right\|_{L_F} \leq C m^{1/p}, \ m = 1, 2, \ldots;$$

(18)

(d) $E_F^\infty \equiv \{ t^p \}$;

(e) $C_F^\infty \equiv \{ t^p \}$.

Proof The implication (a) $\Rightarrow$ (b) is obvious. Next, an inspection of constants appearing in the proof of Proposition 2 yields immediately the equivalence (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). The implication (d) $\Rightarrow$ (e) follows easily from definition of the sets $E_F^\infty$ and $C_F^\infty$. Finally, appealing to the proof of Theorem 1 and taking into account that the constant of equivalence of a given sequence of normalized disjoint functions $\{ f_n \} \subset L_F$ to the unit vector basis of $\ell_p$ can be chosen uniformly for all such sequences, we see that (e) implies (a).

In the next section, we prove similar results for the (uniform) $(\infty - DH)$-property.

5 $(\infty - DH)$ Orlicz spaces.

Here, we will make use of results by Knaust and Odell about normalized weakly null sequences dominated by the unit vector basis of $c_0$ or $\ell_p$ from the papers [18] and [19] respectively (more general theorems of such a sort see in [15]). So, it will be convenient to adopt the terminology similar to that used in these papers.

\[ \text{Springer} \]
Let $1 < p \leq \infty$ and let $X$ be a Banach lattice. A disjoint sequence $\{x_n\}$ in $X$ is called a $C$-$u\ell_p$-sequence, where $C > 0$, if $\|x_n\|_X \leq 1$ for all $n = 1, 2, \ldots$ and $\{x_n\}$ satisfies the $C$-upper $\ell_p$-estimate, i.e., for all $(c_n) \in \ell_p$ we have

$$\left\| \sum_{n=1}^{\infty} c_n x_n \right\|_X \leq C \| (c_n) \|_{\ell_p}.$$ 

Moreover, $\{x_n\}$ is an $u\ell_p$-sequence if it is a $C$-$u\ell_p$-sequence for some $C > 0$. We say that a Banach lattice $X$ has property $(D_p)$ if every disjoint sequence $\{x_n\}$, $\|x_n\|_X \leq 1$, $n = 1, 2, \ldots$, in $X$ admits an $u\ell_p$-subsequence $\{x_{n_k}\}$. Also, $X$ has property $(UD_p)$, if there is a constant $C$ such that every disjoint sequence $\{x_n\}$, $\|x_n\|_X \leq 1$, $n = 1, 2, \ldots$, in $X$ contains a $C$-$u\ell_p$-subsequence.

We say that a sequence $\{x_n\}$ in $X$ is an $M$-bad $u\ell_p$-sequence for a constant $M > 0$ if $\{x_n\}$ is an $u\ell_p$-sequence, and no subsequence of $\{x_n\}$ is an $M$-$u\ell_p$-sequence. An array $\{x_i^n\}_{n,i=1}^{\infty}$ of elements in $X$ is called a bad $u\ell_p$-array, if each column $\{x_i^n\}$ is an $M$-$u\ell_p$-sequence for all $n \in \mathbb{N}$, where $M_n \to \infty$ as $n \to \infty$. An array $\{y_i^n\}_{n,i=1}^{\infty}$ is called a subarray of an array $\{x_i^n\}_{n,i=1}^{\infty}$ whenever each column of $\{y_i^n\}_{n,i=1}^{\infty}$ is a subsequence of $\{x_i^n\}_{n,i=1}^{\infty}$, for some sequence $k_1 < k_2 < \ldots$. Finally, let us say that a bad $u\ell_p$-array $\{x_i^n\}_{n,i=1}^{\infty}$ satisfies the $\ell_p$-array procedure if there exists a subarray $\{y_i^n\}_{n,i=1}^{\infty}$ of $\{x_i^n\}_{n,i=1}^{\infty}$ and there exist $a_n > 0$ with $\sum_{i=1}^{\infty} a_n \leq 1$ so that the elements $y_i = \sum_{n=1}^{\infty} a_n y_i^n$, $i = 1, 2, \ldots$ are disjoint and form a sequence having no $u\ell_p$-subsequence.

**Proposition 3** Let $1 < p \leq \infty$. A symmetric space $X$ on $[0, 1]$ with property $(D_p)$ has property $(UD_p)$.

**Proof** On the contrary, suppose that $X$ fails to have property $(UD_p)$. Then, it is plain that in $X$, there is a bad $u\ell_p$-array $\{x_i^n\}_{n,i=1}^{\infty}$. Since the space $X$ is symmetric, without loss of generality, we may assume that all the functions $x_i^n$, $n = 1, 2, \ldots$, are pairwise disjoint. Indeed, for each $n = 1, 2, \ldots$ the sequence $\{x_i^n\}_{i=1}^{\infty}$ consists of pairwise disjoint functions, and hence there exists $i_n \in \mathbb{N}$ such that $\sum_{i=i_n}^{\infty} m(\text{supp } x_i^n) < 2^{-n}$, $n = 1, 2, \ldots$. Now, one can easily see that in $[0, 1]$ it is enough of “room” to find pairwise disjoint functions $y_i^n$, $n = 1, 2, \ldots$, $i \geq i_n$, which are equimeasurable with $x_i^n$, $n = 1, 2, \ldots$, $i \geq i_n$ (that is, $(y_i^n) = (x_i^n)^{(t)}$ for all $n$, $i$ and $t \in [0, 1]$; see Section 2.1). Therefore, since $X$ is a symmetric space, for every $n = 1, 2, \ldots$ and all $c_i \in \mathbb{R}$ we have

$$\left\| \sum_{i=i_n}^{\infty} c_i y_i^n \right\|_X = \left\| \sum_{i=i_n}^{\infty} c_i x_i^n \right\|_{\ell_p},$$

and so the pairwise disjoint functions $y_i^n$, $n = 1, 2, \ldots$, $i \geq i_n$, form a bad $u\ell_p$-array as well.

Moreover, observe that each column $\{x_i^n\}_{i=1}^{\infty}$ is a disjoint sequence containing a subsequence admitting an upper $\ell_p$-estimate with $p > 1$. Hence, by Rosenthal’s $\ell_p$-theorem (see e.g. [1, Theorem 10.2.1]), it may be assumed also that the sequence $\{x_i^n\}_i$ is weakly null for each $n$. Then, by [19, Theorem 2], if $1 < p < \infty$ and, by [18, Theorem 3.3], if $p = \infty$ we conclude that this array satisfies the $\ell_p$-array procedure. Thus, there are $a_n > 0$, $\sum_{i=1}^{\infty} a_n \leq 1$, an increasing sequence of positive integers $\{k_n\}_{n=1}^{\infty}$, and subsequences $\{z_i^n\}_{i=1}^{\infty}$ of the sequences $\{x_i^n\}_{i=1}^{\infty}$, $n = 1, 2, \ldots$, such that the sequence $z_i := \sum_{n=1}^{\infty} a_n x_i^n$, $i = 1, 2, \ldots$, has no subsequence admitting an upper $\ell_p$-estimate. Observe that the elements $z_i$, $i = 1, 2, \ldots$, are pairwise disjoint and $\|z_i\|_X \leq 1$. Since this contradicts the hypothesis that $X$ possesses property $(D_p)$, the proof is completed. \qed
Corollary 2 Every \((\infty - DH)\) symmetric space is uniformly \((\infty - DH)\).

Proof Since for any disjoint normalized sequence \(\{x_n\}_{n=1}^\infty\) from a Banach lattice \(X\) and all \(c_n \in \mathbb{R}\), we have
\[
\left\| \sum_{n=1}^\infty c_n x_n \right\|_X \geq \| (c_n) \|_{c_0},
\]
then it suffices to apply Proposition 3.

Corollary 3 For every Orlicz function \(F\), the following conditions are equivalent:

(i) \(L_F\) is uniformly \((\infty - DH)\);

(ii) \(L_F\) is \((\infty - DH)\);

(iii) each function from the set \(C_F^\infty\) is degenerate.

Proof Note that a function \(H \in C_F^\infty\) is degenerate if and only if the unit vector basis in the Orlicz space \(l_H\) spans \(c_0\). Therefore, the equivalence \((ii) \Leftrightarrow (iii)\) can be obtained, by using Proposition 2 and Lemma 1, in the same way as a similar result for finite \(p\) in Theorem 1. Thus, the result follows from Corollary 2.

6 Uniform DH-property and duality of DH-property in the class of Orlicz spaces.

We start with duality results related to the non-reflexive case. The following lemma establishes a useful link between the uniform restricted \((1 - DH)\)-property of Orlicz spaces and the classical notion of a regularly varying function of order 1 at \(\infty\).

Lemma 2 Let \(F\) be an Orlicz function. Then, \(E_F^\infty \equiv \{t\}\) if and only if \(F\) is equivalent to an Orlicz function that is regularly varying of order 1 at \(\infty\).

Proof Assume first that \(F\) is equivalent to a regularly varying Orlicz function of order 1 at \(\infty\). Then, by [17, Lemma 1.6], there exists a constant \(c > 0\) such that for any \(0 < u_0 \leq 1\) there is \(t_0 > 0\) so for all \(t \geq t_0\) and \(u \in [u_0, 1]\), we have
\[
F(ut) \geq cu F(t).
\]
On the other hand, by definition, for an arbitrary \(H \in E_F^\infty\), there is a sequence \(t_n \uparrow \infty\) such that
\[
H(u) := \lim_{n \to \infty} \frac{F(ut_n)}{F(t_n)}.
\]
Combining this with the preceding inequality, we conclude that \(H(u) \geq cu\) for all \(u \in [u_0, 1]\). Since the constant \(c\) does not depend on \(u_0\), the last estimate may be extended to the whole interval \([0, 1]\). Taking into account that \(H(u) \leq u, 0 \leq u \leq 1\), we get \(E_F^\infty \equiv \{t\}\).
To prove the converse, we assume, on the contrary, that \( F \) is not equivalent to any regularly varying Orlicz function of order 1 at \( \infty \). Applying once more [17, Lemma 1.6], we get then that for arbitrary \( C > 0 \) there exists \( 0 < u_0 \leq 1 \) such that for every \( t_0 > 0 \) we can find \( t \geq t_0 \) and \( u \in [u_0, 1] \) satisfying the inequality:

\[
F(ut) \leq \frac{1}{C} uF(t).
\]

This assumption implies that for each \( n \in \mathbb{N} \), there are \( 0 < u_0^n \leq 1 \), a sequence \( \{ t_k^n \} \), \( \lim_{k \to \infty} t_k^n = \infty \), and \( u_k^n \in [u_0^n, 1] \) such that

\[
F(u_k^n t_k^n) \leq \frac{1}{n} u_k^n F(t_k^n), \quad n, k = 1, 2, \ldots
\]  

(19)

Passing to subsequences for every \( n \in \mathbb{N} \) (without changing the notation), we can assume that \( u_k^n \to u_n \in [u_0^n, 1] \) as \( k \to \infty \) and

\[
H_n(u) := \lim_{k \to \infty} \frac{F(u_k^n t_k^n)}{F(t_k^n)} \in E_F^\infty
\]

(with uniform convergence on \([0, 1]\)).

Fix \( n \in \mathbb{N} \). Then, since \( u_n > 0 \), there is a positive integer \( k_0 \) (depending on \( n \)) such that for all \( k \geq k_0 \) and \( 0 \leq u \leq 1 \) we have

\[
H_n(u) \leq \frac{F(u_k^n t_k^n)}{F(t_k^n)} + \frac{1}{n} u_n.
\]

Inserting in this inequality \( u = u_k^n, k \geq k_0 \), and using estimate (19), we obtain

\[
H_n(u_k^n) \leq \frac{F(u_k^n t_k^n)}{F(t_k^n)} + \frac{1}{n} u_n \leq \frac{1}{n} u_k^n + \frac{1}{n} u_n, \quad k \geq k_0.
\]

Observe that by continuity, \( H_n(u_k^n) \to H(u_n) \) as \( k \to \infty \) for each \( n \). Hence, taking in the last inequality the limit as \( k \to \infty \) yields

\[
H_n(u_n) \leq \frac{2}{n} u_n, \quad n = 1, 2, \ldots
\]

Since \( H_n \in E_F^\infty \), clearly, the last inequality implies that \( E_F^\infty \neq \{ t \} \). This contradiction completes the proof of the lemma.

It is known that if \( X \) is an \((\infty - \text{DH})\) Banach lattice, then the dual \( X^* \) is \((1 - \text{DH})\) [14, Theorem 2.2]. In the same paper, it was showed that the converse result, in general, is not true. In particular, the Lorentz space \( L_{p,1}[0, 1], 1 < p < \infty \), is \((1 - \text{DH})\) but its dual \( L_{q,\infty}[0, 1], 1/p + 1/q = 1 \), fails to be \((\text{DH})\), because it contains a disjoint sequence equivalent to the unit vector basis of \( \ell_p \). Moreover, we establish here lack of duality for the \((1 - \text{DH})\)-property even inside the class of Orlicz spaces. However, we begin with a positive result for the uniform \((1 - \text{DH})\) and \((\infty - \text{DH})\)-properties.

**Theorem 3** Let \( F \) and \( G \) be mutually Young conjugate Orlicz functions. Then, the following conditions are equivalent:
(i) there exists a constant $C_0 > 0$ such that
\[
\lim_{t \to \infty} \frac{G(C_0 t)}{G(t)} = \infty. \tag{20}
\]

(ii) $L_G$ is uniformly $(\infty - \text{DH})$;

(iii) $L_F$ is uniformly $(1 - \text{DH})$;

(iv) $L_F$ is uniformly restricted $(1 - \text{DH})$;

(v) $F$ is equivalent to a regularly varying Orlicz function of order 1 at $\infty$;

(vi) $E^\infty_F \equiv \{t\}$.

(vii) $L_G$ is uniformly restricted $(\infty - \text{DH})$;

Proof (i) $\Rightarrow$ (ii). Let $\{g_n\}_{n=1}^\infty$ be a normalized disjoint sequence in $L_G$. Reasoning similarly as in [2, Theorem 2.8], we show that
\[
\lim_{n \to \infty} \int_0^1 G(|g_n(t)|/(2C_0)) \, dt = 0. \tag{21}
\]
Indeed, by (20), $\lim_{t \to \infty} G(t/C_0)/G(t) = 0$. Therefore, given an $\varepsilon > 0$ there is $t_0 > 0$ such that for all $t \geq t_0$
\[
\frac{G(t/(2C_0))}{G(t/2)} < \frac{\varepsilon}{2}.
\]
Furthermore, denoting $E_n := \text{supp} \, g_n$, $n = 1, 2, \ldots$, we have $m(E_n) \to 0$ as $n \to \infty$, and hence there exists $N \in \mathbb{N}$ such that for $n \geq N$
\[
m(E_n) \leq \frac{\varepsilon}{2G(t_0/C_0)}.
\]
Since $\|g_n\|_{L_G} = 1$, then $\int_0^1 G(|g_n(t)|/2) \, dt \leq 1$, $n = 1, 2, \ldots$. Therefore, combining the above inequalities, we get for all $n \geq N$
\[
\int_0^1 G(|g_n(t)|/(2C_0)) \, dt \leq \int_{|g_n|>t_0} G(|g_n(t)|/(2C_0)) \, dt + \int_{|g_n|\leq t_0} G(|g_n(t)|/(2C_0)) \, dt \\
\leq G(t_0/C_0)m(E_n) + \frac{\varepsilon}{2} \int_0^1 G(|g_n(t)|/2) \, dt \leq \varepsilon,
\]
and (21) is established.

From (21), it follows the existence of a subsequence $\{g_{n_k}\} \subset \{g_n\}$ satisfying the condition:
\[
\int_0^1 G\left(\frac{\sum_{k=1}^{\infty} g_{n_k}(t)}{2C_0}\right) \, dt = \sum_{k=1}^{\infty} \int_0^1 G\left(\frac{|g_{n_k}(t)|}{2C_0}\right) \, dt \leq 1.
\]
Hence, $\sum_{k=1}^{\infty} g_{n_k} \in L_G \leq 2C_0$. Summarizing, we see that for every sequence $(b_k) \in c_0$
\[ \| (b_k) \|_{c_0} \leq \left\| \sum_{k=1}^{\infty} b_k g_{n_k} \right\|_{L_G} \leq 2C_0\| (b_k) \|_{c_0}. \]

Thus, \( \{g_n\} \) contains a subsequence \( \{g_{n_k}\} \) equivalent to the unit vector basis of \( c_0 \) with a constant independent of \( \{g_n\} \). This means that (ii) is proved.

(ii) \( \Rightarrow \) (iii). Let \( \{f_n\}_{n=1}^{\infty} \) be a normalized disjoint sequence from \( L_F \). Note that for any Orlicz function \( H \) and all non-negative \( h_n, h \in L_H, n = 1, 2, \ldots \), from the well-known properties of integral, the condition \( h_n \uparrow h \) a.e. implies that \( \|h_n\|_{L_H} \to \|h\|_{L_H} \) as \( n \to \infty \). Therefore, by [24, Proposition 1.b.18], the Köthe dual \( L_F^* = L_G \) is a norming subspace of \( L_F^* \). Consequently, we can find a disjoint sequence \( \{g_n\}_{n=1}^{\infty} \) from \( L_G \), \( \|g_n\|_{L_G} = 1 \), such that \( \int_0^1 f_n(t)g_n(t)\,dt = \delta_{n,m}, n, m = 1, 2, \ldots \). Then, by condition, there is a subsequence \( \{g_{n_k}\} \subset \{g_n\} \) such that

\[ \left\| \sum_{k=1}^{m} b_k g_{n_k} \right\|_{L_G} \leq C \| (b_k) \|_{c_0} \]

for a uniform constant \( C > 0 \) and all \( (b_k) \in c_0 \). Hence, for any \( m \in \mathbb{N} \) and all \( a_k \in \mathbb{R} \)
\[
\left\| \sum_{k=1}^{m} a_k f_{n_k} \right\|_{L_F} = \sup \left\{ \int_0^1 \left( \sum_{k=1}^{m} |a_k| f_{n_k}(t) \right) g(t)\,dt : g \in L_G, \|g\|_{L_G} \leq 1 \right\}
\]
\[
\geq C^{-1} \int_0^1 \left( \sum_{k=1}^{m} |a_k| f_{n_k}(t) \right) \left( \sum_{k=1}^{m} g_{n_k}(t) \right) \,dt
\]
\[
= C^{-1} \sum_{k=1}^{m} |a_k| \int_0^1 f_{n_k}(t)g_{n_k}(t)\,dt \geq C^{-1} \sum_{k=1}^{m} |a_k|.
\]

As a result, for every \( m \in \mathbb{N} \) and all \( a_k \in \mathbb{R}, k = 1, 2, \ldots, m \) it follows that
\[
C^{-1} \sum_{k=1}^{m} |a_k| \leq \left\| \sum_{k=1}^{m} a_k f_{n_k} \right\|_{L_F} \leq \sum_{k=1}^{m} |a_k|.
\]

Since the constant \( C \) does not depend on a normalized disjoint sequence \( \{f_n\}_{n=1}^{\infty} \subset L_F \) and \( m \in \mathbb{N} \) is arbitrary, we get the desired result.

The equivalence of conditions (iii), (iv), (v), and (vi) is obtained in Theorem 2 and Lemma 2.

(iv) \( \Rightarrow \) (vii). Let us show first that
\[
\frac{1}{2} \leq sF^{-1}(1/s)G^{-1}(1/s) \leq 1, \quad 0 < s \leq 1. \tag{22}
\]

Indeed, on the one hand, since \( \|X_{(0,s)}\|_{L_F} = 1/F^{-1}(1/s) \) and \( \|X_{(0,s)}\|_{L_G} = 1/G^{-1}(1/s) \), then for \( f = F^{-1}(1/s)X_{(0,s)}\), \( g = G^{-1}(1/s)X_{(0,s)} \), we have \( \|f\|_{L_F} = \|g\|_{L_G} = 1 \) and hence
\[
sF^{-1}(1/s)G^{-1}(1/s) = \int_0^1 f(t)g(t)\,dt \leq \|f\|_{L_F} \|g\|_{L_G} = 1,
\]
which implies the right-hand side inequality in (22). On the other hand, the facts that \( \|X_{(0,s)}\|_{L_F} \|X_{(0,s)}\|_{L_G^*} = s \) [21, Formula (4.39)] and \( \|g\|_{L_G} \leq \|g\|_{L_G^*} \leq 2\|g\|_{L_G}, \, g \in L_G \) [20, Formula (9.24)] imply the left-hand side inequality.
Let now $\{g_n\}_{n=1}^\infty$ be a normalized (in $L_G$) sequence of characteristic functions of disjoint sets $A_n \subset [0, 1]$, $n = 1, 2, \ldots$, that is, $g_n = G^{-1}(1/s_n)\chi_{A_n}$, where $s_n = m(A_n)$. If $f_n := F^{-1}(1/s_n)\chi_{A_n}$, $n = 1, 2, \ldots$, then by the hypothesis, there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that

$$\|(a_k)\|_{\ell^1} \leq C\left\| \sum_{k=1}^\infty a_k f_{n_k} \right\|_{L_F}$$

with some constant $C$ independent of given sets $A_n$, $n = 1, 2, \ldots$.

It is well-known (see e.g., [21, II.3.2]) that the projection

$$Pf := \sum_{k=1}^\infty \frac{1}{s_{n_k}} \int_{A_{n_k}} f(t) \, dt \cdot \chi_{A_{n_k}}$$

is bounded in $L_F$ and $\|P\|_{L_F} = 1$. Hence, from (23) and (22), it follows

$$\left\| \sum_{k=1}^\infty b_k g_{n_k} \right\|_{L_G} = \sup \left\{ \int_0^1 \left( \sum_{k=1}^\infty b_k g_{n_k}(t) \right) f(t) \, dt : \|f\|_{L_F} \leq 1 \right\}
\leq \sup \left\{ \sum_{k=1}^\infty b_k G^{-1}(1/s_{n_k}) \int_{A_{n_k}} f(t) \, dt : \|f\|_{L_F} \leq 1 \right\}
\leq \sup \left\{ \sum_{k=1}^\infty b_k g_{n_k}(t) F^{-1}(1/s_{n_k}) G^{-1}(1/s_{n_k}) : \|(a_k)\|_{\ell^1} \leq C \right\}
\leq C\|(b_k)\|_{c_0}.$$  

Thus, each normalized sequence $\{g_n\}$ of characteristic functions of disjoint sets contains a subsequence $\{g_{n_k}\}$ $C$-equivalent in $L_G$ to the unit vector basis of $c_0$ with a uniform constant $C$ for all $\{g_n\}$. As a result, the implication (iv) $\Rightarrow$ (vii) is proved.

(vii) $\Rightarrow$ (i). On the contrary, suppose that (i) does not hold. We need to prove that for each constant $C > 0$ there are disjoint subsets $A_n$, $n = 1, 2, \ldots$, of $[0, 1]$ such that the sequence $\{g_n\}$, where $g_n := G^{-1}(1/m(A_n))\chi_{A_n}$, does not contain any subsequence $C$-equivalent to the unit vector basis of $c_0$.

Let $C' > C$ be fixed. Since (20) fails for $C'$ we can find $M = M(C')$ such that for some $t_k \to \infty$, we have

$$G(C't_k) \leq MG(t_k), \quad k = 1, 2, \ldots$$

Then, for the inverse function $G^{-1}$, we obtain

$$C'G^{-1}(t_k/M) \leq G^{-1}(t_k), \quad k = 1, 2, \ldots, \tag{24}$$

where $t_k := MG(t_k) \to \infty$ as $k \to \infty$. Denoting $s_k := 1/t_k$ and passing to a subsequence, we can assume that $\sum_{k=1}^\infty s_k < 1$. Let $A_k \subset [0, 1]$, $k = 1, 2, \ldots$, be disjoint sets, $m(A_k) = s_k$ and $g_k := G^{-1}(1/s_k)\chi_{A_k}$. Assume that there is a subsequence $\{g_{n_k}\} \subset \{g_k\}$ such that
Then, in particular, for each \( m \in \mathbb{N} \), we have
\[
\left\| \sum_{i=1}^{\infty} b_i g_i \right\|_{L_G} \leq C \| (b_i) \|_{c_0}.
\] (25)

Combining this estimate with inequality (24) for \( \tau_k = 1/s_k, i = 1, 2, \ldots, m \) and taking into account that \( G \) increases, we get
\[
\sum_{i=1}^{m} G(G^{-1}(1/(Ms_k)))s_k = \frac{m}{M} \leq 1 \text{ for each } m = 1, 2, \ldots,
\]
which is a contradiction. Thus, (25) does not hold, and this completes the proof of the implication and the theorem.

The next result shows that the \((1 - DH)\)-property is not preserved under duality in the class of Orlicz spaces.

**Theorem 4** There exists a \((1 - DH)\) Orlicz space \( L_\Psi \) such that the dual space \( L_\Phi \) (\( \Psi \) is the Young conjugate Orlicz function for \( \Phi \)) is not DH.

**Proof** Let \( \Phi \) be the Orlicz function from Proposition 1. Then, in view of Lemma 1, every disjoint normalized sequence \( \{f_n\}_{n=1}^{\infty} \) from \( L_\Phi \) contains a subsequence \( \{f_{n_k}\}_{k=1}^{\infty} \) that is 6-equivalent to the unit vector basis of an Orlicz space \( \ell_H \) for some \( H \in C_\Phi^\infty \). Since \( C_\Phi^\infty \cong \{t\} \) by Proposition 1, we have that \( H(u) \asymp u, 0 \leq u \leq 1 \). Therefore, \( \{f_{n_k}\} \) is equivalent to the unit vector basis of \( \ell_1 \). This implies that \( L_\Phi \) is \((1 - DH)\).

On the other hand, \( E_\Phi^\infty \not\cong \{t\} \) and hence, by Theorem 3, \( L_\Psi \) fails to be uniformly \((\infty - DH)\). Thus, according to Corollary 3, \( L_\Psi \) is not \((\infty - DH)\). Moreover, arguing in the same way as in the proof of implication (iv) \( \Rightarrow \) (vii) of Theorem 3, one can readily check that \( L_\Psi \) is restrictive \((\infty - DH)\). Combining these facts, we conclude that \( L_\Psi \) fails to be \( DH \).

The next result shows that the \((1 - DH)\)-property is not preserved under duality in the class of Orlicz spaces.

**Corollary 4** There exists a \((1 - DH)\) Orlicz space \( L_\Phi \), which fails to be uniformly \((1 - DH)\).

Moreover, there is no constant \( c \) such that for every sequence of disjoint sets \( E_n \subset [0, 1], n = 1, 2, \ldots \), there is a subsequence \( E_{n_k}, k = 1, 2, \ldots \), satisfying the inequality
\[
\left\| \sum_{k=1}^{m} \bar{x}_{E_{n_k}} \right\|_{L_\Phi} \geq cm, \ m = 1, 2, \ldots
\] (26)
Theorem 5  Let $1 < p < \infty$, $1/p + 1/q = 1$, and let $F$ and $G$ be mutually Young conjugate Orlicz functions. Suppose that $L_F$ is a $(p - DH)$ space. Then, the following conditions are equivalent:

(i) $L_G$ is $(q - DH)$;

(ii) $L_G$ is a DC space;

(iii) $L_F$ is uniformly $(p - DH)$;

(iv) $E_f^\infty \equiv \{ t^\rho \}$.

Proof  First, from [13, Theorem 4.5] it follows that conditions (i) and (ii) are equivalent.

(i) $\Rightarrow$ (iii). By the hypothesis and Proposition 3, there are constants $C_1$ and $C_2$ such that every disjoint normalized sequence in $L_F$ (resp. $L_G$) contains a subsequence admitting the $C_1$-upper $\ell^p_q$-estimate (resp. $C_2$-upper $\ell^p_q$-estimate).

Let $\{ f_n \}_{n=1}^\infty$ be any disjoint normalized sequence in $L_F$. As above, we find a disjoint sequence $\{ g_n \}_{n=1}^\infty$ from $L_G$, $\| g_n \|_{L_G} = 1$, such that $\int_0^1 f_n(t) g_m(t) \, dt = \delta_{nm}$, $n, m = 1, 2, \ldots$. If $\{ f_n \}_{n=1}^\infty$ is a subsequence that admits the $C_2$-upper $\ell^p_q$-estimate, then we have

$$
\left\| \sum_{k=1}^\infty a_k f_n \right\|_{L_F} \geq \sup \left\{ \int_0^1 \left( \sum_{k=1}^\infty a_k f_n(t) \right) \left( \sum_{k=1}^\infty b_k g_n(t) \right) \, dt : \left\| \sum_{k=1}^\infty b_k g_n \right\|_{L_G} \leq 1 \right\}
\geq \sup \left\{ \sum_{k=1}^\infty a_k b_k : \| b_k \|_{\ell^p_q} \leq C_2^{-1} \right\} = C_2^{-1} \| (a_k) \|_{\ell^p_q}.
$$

Thus, every disjoint normalized sequence in $L_F$ contains a subsequence satisfying the $C_2^{-1}$-lower $\ell^p_q$-estimate. This completes the proof of the implication.

Since the implication (iii) $\Rightarrow$ (iv) is an immediate consequence of Theorem 2, we need only to prove that (iv) implies (i).

Let us assume that $E_f^\infty \equiv \{ t^\rho \}$. Then, arguing in the same way as in the proof of the implication (iv) $\Rightarrow$ (vii) of Theorem 3, we get that $E_f^\infty \equiv \{ t^\rho \}$. Combining this fact with Theorem 2, we obtain that $L_G$ is uniformly $(q - DH)$.  

$\square$
Remark 3 Following to [12], we say that a Banach lattice $X$ with a basis $\{x_n\}$ such that $\sum_{n=1}^{\infty} a_n x_n \geq 0$ if and only if $a_n \geq 0$ for all $n$ is a Banach lattice ordered by basis. Clearly, if $X$ is ordered by basis, then the vectors $x_n$, $n = 1, 2, \ldots$, are disjoint. In [12, Proposition 6], it is proved that if such a Banach lattice $X$ is $(p – DH)$ and $X^*$ is $(q – DH)$ $(1 < p < \infty$, $1/p + 1/q = 1)$, then $X$ is uniformly $(p – DH)$. Theorem 5 indicates that in the case of Orlicz spaces analogous result holds together with its converse for “usual” a.e. order of functions.

Now, we are able to show that the $(p – DH)$-property fails to be preserved under duality in the class of Orlicz spaces also in the case when $1 < p < \infty$ (cf. [12, p. 5863] and [13, p. 6], where the opposite is asserted; see Section 1).

Theorem 6 For every $1 < p < \infty$ there exists a $(p – DH)$ Orlicz space $L_{\Phi_p}$ such that the dual space $L_{\Psi_q}$, where $\Psi_q$ is the Young conjugate Orlicz function for $\Phi_p$, is restricted $(q – DH)$ but is not DH and not DC.

Proof Let $\Phi_p$ be the Orlicz function from Corollary 1. First, by Lemma 1, every disjoint normalized sequence $\{f_n\}_{n=1}^{\infty}$ from $L_{\Phi_p}$ contains a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which is 6-equivalent to the unit vector basis of an Orlicz space $\ell_H$ for some $H \in C_{\Phi_p}^{\infty}$. Since $C_{\Phi_p}^{\infty} \cong \{t^p\}$ by Corollary 1, then $H(u) \approx u^p$, $0 \leq u \leq 1$. Therefore, $\{f_{n_k}\}$ is equivalent to the unit vector basis. This implies that $L_{\Phi_p}$ is $(p – DH)$. In particular, $L_{\Phi_p}$ is restricted $(p – DH)$, and in the same way as in the proof of the implication (iv) $\Rightarrow$ (vii) of Theorem 3, it can be shown that the dual space $L_{\Psi_q}$ is restricted $(q – DH)$ as well (see also [16, Proposition 3.7]).

On the other hand, since $E_{\Phi_p}^{\infty} \neq \{t^p\}$, from Theorem 5 it follows that $L_{\Psi_q}$ is not $(q – DH)$. Since $L_{\Psi_q}$ is restricted $(q – DH)$, this yields that $L_{\Psi_q}$ fails to be DH. \hfill $\square$

Theorem 4.1 in the paper [11] reads that an Orlicz space $L_{\Phi}$ is DH if and only if $E_{\Phi}^{\infty} \cong \{\varphi\}$ for a certain function $\varphi$. Theorem 5.1 in [16], the proof of which is based on the above result, claims that every restricted $(2 – DH)$ Orlicz space is $(2 – DH)$. However, combining Theorem 6 with Proposition 2, we conclude that this assertion does not hold.

Corollary 5 Let $1 < q < \infty$. There exists an Orlicz function $\Psi_q$ such that $E_{\Psi_q}^{\infty} \cong \{t^q\}$ and the Orlicz space $L_{\Psi_q}$ is restricted $(q – DH)$, but $C_{\Psi_q}^{\infty} \neq \{t^q\}$ and $L_{\Psi_q}$ fails to be $(DH)$.

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