An alternative to Slepian functions on the unit sphere - A space-frequency analysis based on localized spherical polynomials

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15.05.2013

In this article, we present a space-frequency theory for spherical harmonics based on the spectral decomposition of a particular space-frequency operator. The presented theory is closely linked to the theory of ultraspherical polynomials on the one hand, and to the theory of Slepian functions on the 2-sphere on the other. Results from both theories are used to prove localization and approximation properties of the new band-limited yet space-localized basis. Moreover, particular weak limits related to the structure of the spherical harmonics provide information on the proportion of basis functions needed to approximate localized functions. Finally, a scheme for the fast computation of the coefficients in the new localized basis is provided.

AMS Subject Classification (2010): 42B05, 42C10, 45C05, 47B36

Keywords: Space-frequency analysis on the unit sphere, Slepian functions, spherical harmonics, ultraspherical polynomials, Jacobi matrices

1 Introduction

The roots of the space-frequency analysis studied in this article trace back to the works of Landau, Pollak and Slepian in the early 1960s ([22], [23], [24], [40], [41], [42]). They considered band-limited functions on $\mathbb{R}$ that have maximal $L^2$-energy inside a given interval. This optimization problem led them...
to the investigation of a particular integral equation with the so-called prolate spheroidal wave functions as optimally space-concentrated eigenfunctions. These satisfy a series of remarkable analytic properties. Among others, they form an orthogonal basis for the space of band-limited functions and emerge as solutions of the Helmholtz equation in prolate spheroidal coordinates.

If the underlying domain is the unit sphere $S_2$, an analogous solution of the spatio-spectral concentration problem was realized in [16]. The generalizations of the prolate spheroidal wave functions on $S_2$ were later on called Slepian functions and investigated thoroughly in several articles (see [16], [38], [39] and the references therein). The applications of the Slepian functions vary between spatio-spectral problems in geophysics ([11], [39]), planetary sciences ([45]) and medical imaging ([31]).

This article will consider an alternative approach to obtain a space-localized basis for spherical polynomials based on the idea of a preceding paper [8]. In [8], a time-frequency analysis for orthogonal polynomials on the interval $[-1,1]$ based on the spectral decomposition of a particular time-frequency operator was studied. In the weighted Hilbert space $L^2([-1,1], w)$, this operator was defined as the composition $P_n^m M_x P_n^m$ of a projection operator $P_n^m$ and a multiplication operator $M_x$. Using the orthogonal polynomials with respect to the weight function $w$, it was possible to show the unitary equivalence of $P_n^m M_x P_n^m$ with the Jacobi matrix $J_n^m$ of the orthonormal polynomial sequence. This connection made it possible to get explicit expressions for the eigenfunctions as well as to study their localization and approximation properties.

The idea of constructing a space-localized orthogonal basis as the eigenfunctions of a space-frequency operator linked to particular orthogonal polynomials is now transferred to the setting of the unit sphere $S_2$. As a spherical analogue of the time-frequency operator in the one-dimensional polynomial setting we consider now the space-frequency operator $P_n^m M_{\cos \theta} P_n^m$, where $P_n^m$ denotes the projection onto a space $\Pi_n^m$ of spherical harmonics and $M_{\cos \theta}$ the multiplication with $\cos \theta$. Here, $P_n^m$ plays the role of a band-limiting operator, whereas the aim of $M_{\cos \theta}$ is to measure the space-localization of a function $f$ with respect to the geodetic distance $\theta$ from a particular point on the 2-sphere (without loss of generality we will assume that this point is the north pole). In Section 3 we will further motivate the choice of the multiplication operator $M_{\cos \theta}$ and show that it is linked to a well-known uncertainty principle on the unit sphere ([5, 14, 29, 34]).

Using the multiplication operator $M_{\cos \theta}$ as a space-localization operator instead of a projection operator as in the Landau-Pollak-Slepian theory leads to some differences in the resulting space-frequency analysis. In order to compute the Slepian functions on the unit sphere efficiently, a second order differential operator is needed that commutes with the respective space-frequency operator (see [16], [39]). In the space-frequency theory given in this paper such a differential operator is no longer needed. In Section 3 it will turn out that the space-frequency operator $P_n^m M_{\cos \theta} P_n^m$ is unitarily equivalent to a tridiagonal
block diagonal matrix consisting of Jacobi matrices related to associated ultraspherical polynomials. This particular simple structure makes it possible to compute the eigenfunctions of the operator $P_m^m M_{\cos \theta} P_m^m$ very efficiently. Moreover, due to the connection of the operator to the ultraspherical polynomials, it is possible to derive a series of analytic properties for the spectrum as well as for the eigenfunctions of $P_m^m M_{\cos \theta} P_m^m$.

In the Landau-Pollak-Slepian theory, the eigenvalues of the space-frequency operator indicate whether the corresponding eigenfunction is concentrated in the examined sub-domain of $S_2$ (in this case the eigenvalue is close to one) or not (the eigenvalue is close to zero). The eigenvalues of the space-frequency operator $P_m^m M_{\cos \theta} P_m^m$ examined in this article provide a different information on the space-localization of the eigenfunctions. They give a measure on the mean geodetic distance from the north pole at which the corresponding eigenfunction is localized on $S_2$. In Section 4, it will further turn out that the eigenvalues are asymptotically uniformly distributed on $[-1, 1]$.

The article is structured as follows: In the next section, the necessary preliminaries concerning orthogonal expansions on the unit sphere and ultraspherical polynomials are given. In Section 3 the space-frequency operator $P_m^m M_{\cos \theta} P_m^m$ is introduced. Its spectral decomposition forms the mathematical groundwork for the new space-frequency analysis on $S_2$. The main result here is the spectral Theorem 3.4 in which the eigenvalues and eigenfunctions of the space-frequency operator $P_m^m M_{\cos \theta} P_m^m$ are given explicitly. The localization and approximation properties of the eigenfunctions are studied in Section 4. Here, error bounds for the approximation of space-localized polynomials in $\Pi_m^m$ are given and the distribution of the eigenvalues is studied. In the last section, we will give some considerations regarding the computation of the coefficients in the new space-localized basis. Due to the particular structure of the eigenfunctions and their relation to the ultraspherical polynomials, this can be done efficiently using algorithms based on the fast Fourier transform.

2 Preliminaries

In this preliminary section, we summarize all necessary notation on spherical harmonics and orthogonal polynomials. A general overview on spherical harmonics and approximation theory on the unit sphere can be found in the monographs [4, 11, 25, 26] and in [3, Section 2.1].

On the unit sphere $S_2 := \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$, every point $x \in S_2$ can be written in spherical coordinates as

$$x = (x_1, x_2, x_3) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$$ \hspace{1cm} (1)

where $\theta \in [0, \pi]$ denotes the polar angle and $\varphi \in [0, 2\pi)$ the azimuth angle. The
space of square-integrable functions on $S_2$ is defined as
\[
L^2(S_2) := \left\{ f : S_2 \rightarrow \mathbb{C} \mid \left( \int_{S_2} |f(x)|^2 \, d\omega(x) \right)^{\frac{1}{2}} < \infty \right\},
\] (2)
where $d\omega(x)$ denotes the scalar surface element on $S_2$. In spherical coordinates, it can be written as $d\omega(x) = \sin \theta \, d\theta \, d\varphi$. The inner product
\[
(f, g) := \frac{1}{4\pi} \int_{S_2} f(x) \overline{g(x)} \, d\omega(x) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi f(\theta, \varphi) \overline{g(\theta, \varphi)} \sin \theta \, d\theta \, d\varphi
\] (3)
turns $L^2(S_2)$ into a Hilbert space. This space can be decomposed as $L^2(S_2) = \bigoplus_{l=0}^\infty \text{Harm}_l$, where $\text{Harm}_l$ denotes the $2l+1$ dimensional space spanned by the spherical harmonics $Y^k_l$, $-l \leq k \leq l$, of order $l \in \mathbb{N}_0$. They can be written explicitly in spherical coordinates as
\[
Y^k_l(\theta, \varphi) := \sin|k| \theta p^{(|k|)}_l(\cos \theta) \, e^{ik\varphi}.
\] (4)
Here, the polynomials $p^{(\alpha)}_l(x)$ denote the ultraspherical polynomials of degree $l$ with positive leading coefficient and orthonormal on $[-1, 1]$ with respect to the inner product
\[
\langle f, g \rangle^{(\alpha)} := \frac{1}{2} \int_{-1}^1 f(x) \overline{g(x)} (1-x^2)^\alpha \, dx.
\] (5)
For a detailed treatise on ultraspherical polynomials in the context of general orthogonal polynomials, we refer to [2, Chapter 5], [13, Section 1.3.2], [17, Chapter 4] and [43, Chapter 4.7].

The orthonormal polynomials $p^{(\alpha)}_l(x)$ satisfy the three-term recurrence relation
\[
b^{(\alpha)}_{l+1} p^{(\alpha)}_{l+1}(x) = x p^{(\alpha)}_l(x) - b^{(\alpha)}_l p^{(\alpha)}_{l-1}(x), \quad l = 0, 1, 2, \ldots
\] (6)
\[
p^{(\alpha)}_{-1}(x) = 0, \quad p^{(\alpha)}_0(x) = \frac{1}{b^{(\alpha)}_0},
\] with the coefficients $b^{(\alpha)}_l = \left( \frac{l(l+2\alpha)}{(2l+2\alpha+1)(2l+2\alpha-1)} \right)^{\frac{1}{2}}, l > 0$, and $b^{(\alpha)}_0 = \left( \frac{\sqrt{\pi} \Gamma(\alpha+1)}{2 \Gamma(\alpha+\frac{3}{2})} \right)^{\frac{1}{2}}$.

Based on the three-term recurrence relation (6), one observes that
\[
J^{(\alpha)}_l \mathbf{v}_\alpha(x) = x \mathbf{v}_\alpha(x) - \begin{pmatrix}
0 \\
0 \\
\vdots \\
b^{(\alpha)}_{l+1} p^{(\alpha)}_{l+1}(x)
\end{pmatrix}
\] (7)
holds with $\mathbf{v}_\alpha(x) = (p^{(\alpha)}_0(x), p^{(\alpha)}_1(x), \ldots, p^{(\alpha)}_l(x))^T$ and the Jacobi matrix $J^{(\alpha)}_l$.
defined as

\[
J(\alpha)_{l} := \begin{pmatrix}
0 & b_{1}^{(\alpha)} & 0 & 0 & \cdots & 0 \\
b_{1}^{(\alpha)} & 0 & b_{2}^{(\alpha)} & 0 & \cdots & 0 \\
0 & b_{2}^{(\alpha)} & 0 & b_{3}^{(\alpha)} & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & b_{l-1}^{(\alpha)} & 0 & b_{l}^{(\alpha)} \\
0 & \cdots & \cdots & 0 & b_{l}^{(\alpha)} & 0 \\
\end{pmatrix}.
\]

The associated ultraspherical polynomials \(p_{l}^{(\alpha)}(x, m)\) are defined by the shifted recurrence relation (cf. [13, Section 1.3.4], [17, Section 5.7])

\[
b_{m+l+1}^{(\alpha)}p_{l+1}^{(\alpha)}(x, m) = x p_{l}^{(\alpha)}(x, m) - b_{m+1}^{(\alpha)}p_{l-1}^{(\alpha)}(x, m), \quad l = 0, 1, \ldots,
\]

\[
p_{-1}^{(\alpha)}(x, m) = 0, \quad p_{0}^{(\alpha)}(x, m) = \frac{1}{b_{0}^{(\alpha)}}.
\]

For \(m = 0\), the identity \(p_{l}^{(\alpha)}(x, 0) = p_{l}^{(\alpha)}(x)\) holds. In [2, III, Section 4] and [13, Section 1.3.4], the associated polynomials corresponding to an orthogonal polynomial sequence are called numerator polynomials.

For general \(m \in \mathbb{N}_{0}\), equation (7) can be written as

\[
J(\alpha)_{l_{i}+m}^{m}v_{\alpha}(x, m) = x v_{\alpha}(x, m) + \begin{pmatrix}
0 \\
0 \\
\vdots \\
b_{m+l+1}^{(\alpha)}p_{l+1}^{(\alpha)}(x, m)
\end{pmatrix}
\]

with \(v_{\alpha}(x, m) = (p_{0}^{(\alpha)}(x, m), p_{1}^{(\alpha)}(x, m), p_{2}^{(\alpha)}(x, m), \ldots, p_{l}^{(\alpha)}(x, m))^{T}\) and the truncated Jacobi matrix \(J(\alpha)_{l_{i}+m}^{m} = (J(\alpha))_{l_{i}+m}^{m+1}, m \in \mathbb{N}_{0}\)

**Remark 2.1.** It follows from (10) that the eigenvalues of \(J(\alpha)_{l_{i}+m}^{m}\) are exactly the \(l + 1\) roots \(x_{\alpha,i}, 1 \leq i \leq l + 1\) of the associated ultraspherical polynomial \(p_{l}^{(\alpha)}(x, m)\) with the eigenvectors

\[
v_{\alpha,i} := v_{\alpha}(x_{\alpha,i}, m) = (p_{0}^{(\alpha)}(x_{\alpha,i}, m), p_{1}^{(\alpha)}(x_{\alpha,i}, m), p_{2}^{(\alpha)}(x_{\alpha,i}, m), \ldots, p_{l}^{(\alpha)}(x_{\alpha,i}, m))^{T}.
\]

Furthermore, the polynomials \(p_{l}^{(\alpha)}(x, m)\) can be written as

\[
p_{l}^{(\alpha)}(x, m) = \frac{1}{b_{m}^{(\alpha)}b_{m+1}^{(\alpha)} \cdots b_{m+l}^{(\alpha)}} \det (x_{l} - J(\alpha)_{m+l-1}^{m}),
\]

where \(1_{l}\) denotes the \(l\)-dimensional identity matrix. This is easily seen by verifying that the right hand side of equation (11) satisfies the recurrence relation (9).
Finally, for $\alpha \in \mathbb{N}_0$, $\alpha \leq m$ and $x \neq y$ the following Christoffel-Darboux type identity holds ([7, Lemma 3.1]):

$$
\sum_{l=m}^{n} p_{l}^{(\alpha)}(x) p_{l-m}^{(\alpha)}(y, m - \alpha) = \frac{p_{m-\alpha-1}^{(\alpha)}(x)}{x - y} + b_{n-\alpha+1}^{(\alpha)} \frac{p_{n-\alpha}^{(\alpha)}(y, m - \alpha) - p_{n-m}^{(\alpha)}(y, m - \alpha)}{x - y}.
$$

(12)

For $\alpha = m$ the above formula reduces to the original Christoffel-Darboux formula, see [2, Theorem 4.5].

3 Spectral analysis of the space-frequency operator

The spherical harmonics $Y^k_l$, $m \leq l \leq n$, $-l \leq k \leq l$, form an orthonormal basis for the polynomial space

$$
\Pi^m_n := \bigoplus_{l=m}^{n} \text{Harm}_l.
$$

Due to their harmonic nature, the $L^2$-mass of the spherical harmonics $Y^k_l$ is distributed over the whole 2-sphere $S_2$. Spherical harmonics are therefore not well suited to decompose functions with mass concentrated in a specific sub-domain of $S_2$. The aim of this section is to obtain a set of space-localized basis functions in $\Pi^m_n$. To this end, we introduce and examine a particular space-frequency operator for $L^2$-functions on $S_2$ and derive its spectral decomposition. This spectral decomposition is the mathematical framework for a new space-localized basis in the space $\Pi^m_n$.

In comparison to other works (cf. [16, 27, 39]) dealing with spatio-spectral concentration on the unit sphere, we use the multiplication operator $M_{\cos \theta} : L^2(S_2) \to L^2(S_2)$, defined by

$$
(M_{\cos \theta} f)(\theta, \varphi) := \cos \theta f(\theta, \varphi),
$$

to measure the space-localization of a function $f \in L^2(S_2)$. Introducing the mean value

$$
\epsilon(f) := \langle M_{\cos \theta} f, f \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \cos \theta |f(\theta, \varphi)|^2 \sin \theta \, d\theta \, d\varphi,
$$

(13)

we can visualize the role of $M_{\cos \theta}$ as a descriptor of the localization of a function $f$ at the north pole of $S_2$. For a normalized function $f$ with $\|f\| = 1$, the mean value satisfies $-1 < \epsilon(f) < 1$. If $\epsilon(f)$ is close to 1, the mass of the $L^2(S_2)$-density $f$ has to be situated in the region in which $\theta$ is close to zero, i.e., at the north pole of $S_2$, in such a way that the influence of $\cos \theta$ in the integral [13] is compensated. On the other hand, a mean value $\epsilon(f)$ close to $-1$ is an indicator for a mass concentration of $f$ at the south pole of $S_2$. From now on, a normalized function $f$ is called space-localized at the north or the south pole if its mean value $\epsilon(f)$ is close to 1 or $-1$, respectively.
Remark 3.1. The particular choice of $\cos \theta$ in the multiplication operator is motivated by the particular structure of the spherical harmonics on $S_2$. Since $\cos \theta = Y_{1,0}(\theta, \varphi)$, the value $\varepsilon(f)$ can be considered as a first spherical moment of the $L^2$ density $f$. Further, $\cos \theta$ is used in the Fisher-von Mises distribution on $S_2$ to measure the distance between a point on $S_2$ and the north pole (corresponding to the mean point of the distribution), see [14, Chapter 7].

The mean value $\varepsilon(f)$ is also in further ways connected to space localization on $S_2$. In [5, 29, 34], the variance functional $\text{var}_{S_2}(f) = 1 - \varepsilon(f)^2 \varepsilon(f)^2$ is used to measure the space localization of a function $f$ and is an essential part of an uncertainty principle on $S_2$.

To measure the frequency localization of a function $f \in L^2(S_2)$, we consider its projection onto the finite dimensional space $\Pi_m^n$. The corresponding projection operator is defined as

$$P_n^m f(\theta, \varphi) := \sum_{l=m}^{n} \sum_{k=-l}^{l} \langle f, Y_{k}^{l} \rangle Y_{k}^{l}(\theta, \varphi).$$

The operator $P_n^m$ is bounded, self-adjoint and, since its range is a finite dimensional space, also compact. In the context of the space-frequency analysis discussed in this work, a polynomial $Q \in \Pi_m^n$ is called bandlimited.

As the main mathematical object for a space-frequency analysis on $S_2$, we consider the composite operator

$$P_n^m M \cos \theta P_n^m.$$

Due to the properties of $M \cos \theta$ and $P_n^m$, also $P_n^m M \cos \theta P_n^m$ is a compact and self-adjoint operator on $L^2(S_2)$. By the Hilbert-Schmidt theorem for compact and self-adjoint operators [33, Theorem VI.16], [44, Theorem VI.3.2], we have the general spectral decomposition

$$P_n^m M \cos \theta P_n^m f = \sum_{j=1}^{\infty} \lambda_j \langle f, e_j \rangle e_j, \quad f \in L^2(S_2),$$

with $\lambda_j \in \mathbb{R}$, $j \in \mathbb{N}$, denoting the eigenvalues of $P_n^m M \cos \theta P_n^m$ and $e_j$ the corresponding eigenfunctions.

The Hilbert-Schmidt theorem ensures that the eigenfunctions $\{e_j\}_{j \in \mathbb{N}}$ form a complete orthonormal system of $L^2(S_2)$. Since every function in $L^2(S_2) \setminus \Pi_m^n$ is in the kernel of $P_n^m M \cos \theta P_n^m$, it suffices to consider the operator $P_n^m M \cos \theta P_n^m$ restricted to the polynomial space $\Pi_m^n$. The problem at hand now consists in calculating the eigenvalues and the corresponding eigenfunctions of $P_n^m M \cos \theta P_n^m$ in the space $\Pi_m^n$. This is done by analyzing the behaviour of the operator $P_n^m M \cos \theta P_n^m$ in the frequency domain, i.e., the space spanned by the expansion coefficients corresponding to the spherical harmonics.

\[\text{We are very grateful to E. Grafarend for this hint.}\]
To this end, we need some further notation. First of all, we consider the expansion of $Q \in \Pi^m_n$ in the basis of the spherical harmonics, i.e.,

$$Q(\theta, \varphi) = \sum_{l=m}^{n} \sum_{k=-l}^{l} c_{l,k} Y^k_l(\theta, \varphi).$$  \hspace{1cm} (14)

The dimension of $\Pi^m_n$ is given by

$$N^m_n = \dim \Pi^m_n = (n + 1)^2 - m^2 = (n + m + 1)(n - m + 1).$$

Now, we sort the $N^m_n$ coefficients $c_{l,k}$ of the expansion \(14\) according to the row-index $k$, as illustrated in Figure 1. Therefore, we introduce the coefficient vectors

$$c_k := (c_{m,k}, c_{m+1,k}, \ldots, c_{n,k})^T, \quad -m \leq k \leq m,$$

$$c_k := (c_{|k|,k}, c_{|k|+1,k}, \ldots, c_{n,k})^T, \quad m + 1 \leq |k| \leq n,$$

and the subspaces

$$\Pi^m_{n,k} = \text{span}\{Y^k_l : l = \max(|k|, m), \ldots, n\}$$

with the dimension

$$N_k = \dim \Pi^m_{n,k} = n - \max(|k|, m) + 1.$$  

Further, we introduce the transition operators

$$T_k : \mathbb{C}^{N_k} \rightarrow \Pi^m_{n,k} : \quad T_k c_k = \sum_{l=m}^{n} c_{l,k} Y^k_l, \quad -m \leq k \leq m,$$

$$T_k : \mathbb{C}^{N_k} \rightarrow \Pi^m_{n,k} : \quad T_k c_k = \sum_{l=|k|}^{n} c_{l,k} Y^k_l, \quad m + 1 \leq |k| \leq n.$$  

Finally, we denote the complete vector of coefficients by

$$c := (c_n, c_{n-1}, \ldots, c_{-n})^T \in \mathbb{C}^{N^m_n}$$

and introduce the overall transition operator

$$T : \mathbb{C}^{N^m_n} \rightarrow \Pi^m_n, \quad Tc := \sum_{k=-n}^{n} T_k c_k = \sum_{k=-n}^{n} \sum_{l=m}^{n} c_{l,k} Y^k_l.$$  

Clearly, the linear operator $T$ maps the canonical basis of $\mathbb{C}^{N^m_n}$ onto the orthonormal basis of $\Pi^m_n$ and is therefore an unitary operator.

Now, we are able to represent the space-frequency operator $P^m_n M_{\cos \theta} P^m_n$ in the space of coefficient vectors.
Lemma 3.2.

The operator $P_n^m M \cos \theta P_n^m$ restricted to the polynomial space $\Pi_n^m$ is unitarily
equivalent to the block diagonal matrix $J_n^m \in \mathbb{C}^{N_n^m \times N_n^m}$ given by

$$J_n^m = \begin{pmatrix}
J(n)_0 \\
J(n-1)_1 \\
J(m)_{|k|}^{0} \\
J(0)_{|k|}^{m} \\
J(m)_{|k|}^{n-m} \\
J(n-1)_1 \\
J(n)_0
\end{pmatrix}$$

where $J(|k|)_n^m$ denote the Jacobi matrices of to the associated ultraspherical polynomials $p_{i}^{|k|}(x, m)$ defined in [8]. More precisely, $T^* (P_n^m M_{\cos \theta} P_n^m) T = J_n^m$ and

$$T_k^* (P_n^m M_{\cos \theta} P_n^m) T_k = J(|k|)_{n-|k|}^{|m-|k||}, \quad -m \leq k \leq m,$$  \hspace{1cm} (15)

$$T_k^* (P_n^m M_{\cos \theta} P_n^m) T_k = J(|k|)_{n-|k|}^{|m+1-|k||}, \quad m+1 \leq |k| \leq n.$$ \hspace{1cm} (16)

Further, if $Q = \sum_{l=m}^{n} \sum_{k=-l}^{l} c_{l,k} Y_l^k$, then

$$\varepsilon(Q) = \langle P_n^m M_{\cos \theta} P_n^m Q, Q \rangle$$

$$= \sum_{k=-m}^{m} c_k^H J(|k|)_{n-|k|}^{m-|k|} c_k + \sum_{|k|=m+1}^{n} c_k^H J(|k|)_{n-|k|} c_k = c^H J_n^m c.$$ \hspace{1cm} (17)

**Proof.** We consider an arbitrary polynomial $Q \in \Pi_n^m$. Since the projection operator $P_n^m$ is self-adjoint, we get for the mean value

$$\varepsilon(Q) = \langle M_{\cos \theta} Q, Q \rangle = \langle M_{\cos \theta} P_n^m Q, P_n^m Q \rangle = \langle P_n^m M_{\cos \theta} P_n^m Q, Q \rangle.$$ \hspace{1cm} (18)

Now using the expansion [14] of $Q$ in spherical harmonics and the transition
operators $T_k$, we obtain

$$
\langle P_m^{\cos} P_m^{\cos} Q, Q \rangle = \left\langle P_m^{\cos} P_m^{\cos} \left( \sum_{j=-n}^{n} T_j c_j \right), \sum_{k=-n}^{n} T_k c_k \right\rangle
$$

$$
= \sum_{k=-n}^{n} \sum_{j=-n}^{n} \langle P_m^{\cos} P_m^{\cos} T_j c_j, T_k c_k \rangle
$$

$$
= \sum_{k=-n}^{n} \sum_{j=-n}^{n} \langle \cos \theta T_j c_j, T_k c_k \rangle = \sum_{k=-n}^{n} \varepsilon(T_k c_k)
$$

$$
= \sum_{k=-n}^{n} \langle P_m^{\cos} P_m^{\cos} T_k c_k, T_k c_k \rangle,
$$

where the reduction in the third line is due to the orthogonality of the spherical harmonics $Y_{lm}^j$ and $Y_{lm}^k$ for different indices $j \neq k$. Thus, the operator $P_m^{\cos} P_m^{\cos}$ on $\Pi_n^m$ has a reducible block structure and the length of the $2n+1$ blocks is given by the number $N_k$ of coefficients $c_{l,k}$ in the row $k$ (see also Figure 1). To determine the behaviour of $P_m^{\cos} P_m^{\cos}$ on the subspaces $\Pi_n^m$, we consider the mean values $\varepsilon(T_k c_k)$ in more detail. Since the lengths $N_k$ of the subblocks differ, we have to consider two different cases and start with the case $-m \leq k \leq m$. By Definition 4 of the spherical harmonics $Y_{lm}^k$, the mean value $\varepsilon(T_k c_k)$ can be expressed as

$$
\varepsilon(T_k c_k) = \frac{1}{4\pi} \int_{0}^{2\pi} \int_{0}^{\pi} \cos \theta |T_k c_k(\theta, \varphi)|^2 \sin \theta \, d\theta \, d\varphi
$$

$$
= \frac{1}{4\pi} \int_{0}^{2\pi} \int_{0}^{\pi} \left( \sum_{l=-m}^{m} c_{l,k} \cos \theta \sin \left| k \right| \theta p_{l-|k|}^{(|k|)}(\cos \theta) e^{i\varphi} \right)
$$

$$
\cdot \left( \sum_{l=-m}^{m} \overline{c}_{l,k} \sin \left| k \right| \theta p_{l-|k|}^{(|k|)}(\cos \theta) e^{-i\varphi} \right) \sin \theta \, d\theta \, d\varphi.
$$

$$
= \frac{1}{2} \int_{0}^{\pi} \left( \sum_{l=-m}^{m} c_{l,k} \cos \theta p_{l-|k|}^{(|k|)}(\cos \theta) \right) \left( \sum_{l=-m}^{m} \overline{c}_{l,k} p_{l-|k|}^{(|k|)}(\cos \theta) \right) \sin 2\theta d\theta.
$$

Now, using the three-term recurrence relation and the orthonormality of the polynomials $p_l^{(|k|)}$, we can conclude

$$
\varepsilon(T_k c_k) = \frac{1}{2} \int_{0}^{\pi} \left( \sum_{l=-m}^{m} c_{l,k} \left( b_{l-|k|}^{(|k|)} + b_{l-|k|+1}^{(|k|)}(\cos \theta) \right) \right)
$$

$$
\cdot \left( \sum_{l=-m}^{m} \overline{c}_{l,k} p_{l-|k|}^{(|k|)}(\cos \theta) \right) \sin 2\theta d\theta
$$

$$
= \sum_{l=-m+1}^{m-1} c_{l,k} \overline{c}_{l-1,k} b_{l-|k|}^{(|k|)} + \sum_{l=m}^{n-1} c_{l,k} \overline{c}_{l+1,k} b_{l-|k|+1}^{(|k|)} = c_k^H J(|k|)^{m-|k|} c_k.
$$
In the case \( m + 1 \leq |k| \leq n \), we get by an analogous argumentation
\[
\varepsilon(T_k c_k) = \frac{1}{2} \int_0^\pi \left( \sum_{l = |k|}^n c_{l,k} \cos \theta p_{l-|k|}^{(k)}(\cos \theta) \right) \left( \sum_{l = |k|}^n c_{l,k} \cos \theta \right) \sin^{2|k|+1} \theta d\theta.
\]
\[
= \frac{1}{2} \int_0^\pi \left( \sum_{l = |k|}^n c_{l,k} \left( b_{l-|k|-1}^{(k)} + b_{l-|k|+1}^{(k)} \right) \cos \theta \right) \left( \sum_{l = |k|}^n c_{l,k} \cos \theta \right) \sin^{2|k|+1} \theta d\theta
\]
\[
= \sum_{l = |k|+1}^n c_{l,k} \bar{c}_{l-1,k} b_{l+1}^{(k)} + \sum_{l = |k|}^{n-1} c_{l,k} \bar{c}_{l+1,k} b_{l-1}^{(k)} + c_{l,k} \bar{c}_{l,k} \cdot \sum_{l = |k|}^n \tilde{c}_{l,k} \tilde{c}_{l,k}^{(k)} \cos \theta \sin^{2|k|+1} \theta d\theta
\]
\[
= \sum_{l = |k|+1}^n c_{l,k} \bar{c}_{l-1,k} b_{l+1}^{(k)} + \sum_{l = |k|}^{n-1} c_{l,k} \bar{c}_{l+1,k} b_{l-1}^{(k)} + c_{l,k} \bar{c}_{l,k} \cdot \sum_{l = |k|}^n \tilde{c}_{l,k} \tilde{c}_{l,k}^{(k)} \cos \theta \sin^{2|k|+1} \theta d\theta.
\]

In total, we can conclude
\[
\langle P_{\Pi_n,\cos}^m P_n^m Q, Q \rangle = \sum_{k = -m}^m c_{k}^{(k)} \mathcal{J}_{\Pi_n}^m \langle (|k|)^{m-k} c_k \rangle + \sum_{|k| = m+1}^n c_{k}^{(k)} \mathcal{J}_{\Pi_n}^m \langle (|k|)^{m-k} c_k \rangle,
\]
and, thus, that equation \([17]\) holds for any polynomial \( Q \in \Pi_n^m \). Therefore, by the uniqueness theorem for self-adjoint operators (see \([35, \text{Theorem 12.7}]\)), the two operators \( T^* (P_{\Pi_n,\cos}^m P_n^m) T \) and \( \mathcal{J}_{\Pi_n}^m \) coincide and for the sub-operators \( T_k (P_{\Pi_n,\cos}^m P_n^m) T_k \) the relations \([15]\) and \([16]\) hold.

**Remark 3.3.** In Lemma \([3.2]\) only the statement about the unitary equivalence of the two operators \( P_{\Pi_n,\cos}^m P_n^m \) and \( \mathcal{J}_{\Pi_n}^m \) is new. The characterization \([17]\) of \( \varepsilon(f) \) with help of the operator \( \mathcal{J}_{\Pi_n}^m \) is not new and already stated and proven in a generalized form in \([6, \text{Lemma 3.26}]\) and \([21]\). For the sake of completeness, we decided to formulate the proof here in a simplified form.

Now, we are able to state the spectral decomposition of the space-frequency operator \( P_{\Pi_n,\cos}^m P_n^m \) explicitly.

**Theorem 3.4.**
The space-frequency operator \( P_{\Pi_n,\cos}^m P_n^m \) on \( L^2(S_2) \) has the spectral decomposition
\[
P_{\Pi_n,\cos}^m P_n^m f = \sum_{k = -m}^m \sum_{i = 1}^{N_k} x_{|k|,i}^m f, \psi_{k,i}^m \psi_{k,i}.
\]
For \(-m \leq k \leq m\), the eigenvalues \( x_{|k|,i}^m, 1 \leq i \leq n - m + 1 \), denote the \( n - m + 1 \) roots of the associated polynomial \( p_{n-m+1}^{|k|}(x, m - |k|) \) and the eigenfunctions \( \psi_{k,i}^m \in \Pi_n^m \) have the explicit form
\[
\psi_{k,i}(\theta, \varphi) = \kappa_{k,i} b_{n-|k|+1}^{(|k|)}(\cos \theta) \frac{\sin |k| \cos \theta}{|k|} e^{ik\varphi}
\]
\[
+ \kappa_{k,i} \frac{\sin |k| \cos \theta}{|k|} e^{ik\varphi},
\]
(19)
Consequently, the normalized eigenfunctions of

\[ \kappa_{k,i} := \left( \sum_{l=m}^{n} |p_{l-m}^{(k)}(x_{|k|,i}, m - |k|)|^2 \right)^{- \frac{1}{2}}. \]  

(20)

For \( m + 1 \leq |k| \leq n \), the eigenvalues \( x_{|k|,i} \), \( 1 \leq i \leq n - |k| + 1 \), correspond to the \( n - |k| + 1 \) roots of the polynomials \( p_{n-|k|+1}^{(k)}(x) \) and the eigenfunctions \( \psi_{k,i} \in \Pi_{n}^{m} \) can be written as

\[ \psi_{k,i}(\theta, \varphi) = \kappa_{k,i} b_{n-|k|+1}^{(k)} P_{n-|k|}^{(k)}(x_{|k|,i}) \frac{\sin|k| \theta p_{n-|k|+1}(\cos \theta)}{\cos \theta - x_{|k|,i}} e^{ik\varphi}, \]  

(21)

with the normalizing constant

\[ \kappa_{k,i} := \left( \sum_{l=m}^{n} |p_{l-m}^{(k)}(x_{|k|,i}, m - |k|)|^2 \right)^{- \frac{1}{2}}. \]  

(22)

Proof. By Lemma [3.2], the operators \( P_{n}^{m} M_{cos} P_{n}^{m} \) and \( J_{n}^{m} \) are unitarily equivalent and, thus, exhibit the same spectrum. In particular, the spectrum of \( P_{n}^{m} M_{cos} P_{n}^{m} \) is composed of the eigenvalues of the Jacobi matrices \( J(|k|)_{m-|k|}^{n-|k|} \), \( 0 \leq |k| \leq m \), and \( J(|k|)_{n-|k|, m+1}^{n-|k|} \leq |k| \leq n \).

To determine the single eigenfunctions, we consider first the case \( -m \leq k \leq m \).

If \( v_{|k|,i} \) is an eigenvector of \( J(|k|)_{m-|k|}^{n-|k|} \) corresponding to the eigenvalue \( x_{|k|,i} \), then \( T_{k} v_{|k|,i} \) is an eigenfunction of \( P_{n}^{m} M_{cos} P_{n}^{m} \), since

\[ P_{n}^{m} M_{cos} P_{n}^{m} T_{k} v_{|k|,i} = T_{k} T_{k}^{*} P_{n}^{m} M_{cos} P_{n}^{m} T_{k} v_{|k|,i} = T_{k} J(|k|)_{m-|k|}^{n-|k|} v_{|k|,i} = x_{|k|,i} T_{k} v_{|k|,i}. \]

By Remark [2.1], the eigenvalues of \( J(|k|)_{m-|k|}^{n-|k|} \) are exactly the \( n + 1 \) roots \( x_{|k|,i} \), \( i = 1, \ldots, n - |k| + 1 \), of the associated polynomial \( p_{n-|k|+1}^{(k)}(x, m - |k|) \) with the corresponding eigenvectors

\[ v_{|k|,i} = (p_{0}^{(k)}(x_{|k|,i}, m - |k|), p_{1}^{(k)}(x_{|k|,i}, m - |k|), \ldots, p_{n-|k|+1}^{(k)}(x_{|k|,i}, m - |k|))^{T}. \]

Consequently, the normalized eigenfunctions of \( P_{n}^{m} M_{cos} P_{n}^{m} \) can be written as

\[ \psi_{k,i}(\theta, \varphi) = T_{k} \frac{v_{|k|,i}}{||v_{|k|,i}||_{2}} = \kappa_{k,i} \sum_{l=m}^{n} p_{l-m}^{(k)}(x_{|k|,i}, m - |k|) Y_{l}^{k}(\theta, \varphi) \]

\[ = \kappa_{k,i} \sin|k| \theta e^{ik\varphi} \sum_{l=m}^{n} p_{l-m}^{(k)}(x_{|k|,i}, m - |k|) p_{l-m}^{(k)}(\cos \theta) \]

\[ = \kappa_{k,i} \sin|k| \theta \frac{b_{n-|k|+1}^{(k)} P_{n-|k|}^{(k)}(x_{|k|,i}, m - |k|) p_{n-|k|+1}(\cos \theta)}{\cos \theta - x_{|k|,i}} e^{ik\varphi} \]

\[ + \kappa_{k,i} \sin|k| \theta \frac{p_{n-|k|+1}(\cos \theta)}{\cos \theta - x_{|k|,i}} e^{ik\varphi}, \]
by using the Christoffel-Darboux type formula \([12]\) (bearing in mind that 
\(p_m^{(k)}(x_{|k|,i}, m - |k|) = 0\) and defining the normalizing constant \(\kappa_{k,i}\) as given in \([20]\).

An analogous argumentation can be conducted for \(m + 1 \leq |k| \leq n\). By Remark \([2.1]\) the eigenvalues of \(J(|k|)_{n-|k|}\) are now the \(n - |k| + 1\) roots \(x_{|k|,i}\), 
\(i = 1, \ldots, n - |k| + 1\), of the ultrasperical polynomial \(p_m^{(|k|)}\) on the real line or on the unit circle the spectral analysis of the respective

Applying the transition operator \(T\) corresponding eigenvectors are given by

\[ p_m^{(|k|)}(x_{|k|,i}) \]

polynomials on \([37]\). The space-frequency analysis of such an operator for orthogonal

Remark \(3.5\). The spectral Theorem \([3.4]\) for the operator \(P_m^M\cos\theta P_m^n\) on the unit

sphere is completely novel in this work. However, there exist related versions of 

\(P_m^M\cos\theta P_m^m\) and Theorem \([3.4]\) in simpler settings. For orthogonal polynomials

on the real line or on the unit circle the spectral analysis of the respective operator leads to deep results for the theory of orthogonal polynomials itself

(see \([37]\)). The space-frequency analysis of such an operator for orthogonal polynomials on \([-1, 1]\) is studied in \([8]\).

4 Localization and approximation properties of the eigenfunctions

The Hilbert-Schmidt theorem ensures that the eigenfunctions \(\psi_{k,i}\) derived in Theorem \([3.4]\) form an orthonormal basis in the space \(\Pi_m^n\). In this section, we will investigate some more properties of the eigenfunctions \(\psi_{k,i}\) related to their space localization on the unit sphere \(S_2\).

First of all, it follows from the definition of the \(\psi_{k,i}\) as normalized eigenfunctions of the operator \(P_m^M\cos\theta P_m^m\) that the mean value \(\varepsilon(\psi_{k,i})\) coincides with the eigenvalue \(x_{|k|,i}\), i.e.,

\[ \varepsilon(\psi_{k,i}) = \langle P_m^M\cos\theta P_m^m \psi_{k,i}, \psi_{k,i} \rangle = x_{|k|,i} \langle \psi_{k,i}, \psi_{k,i} \rangle = x_{|k|,i}. \]

Using the correspondence of the eigenvalues \(x_{|k|,i}\) with the zeros of the associated ultrasperical polynomial \(p_m^{(|k|)}(x, m - |k|)\) or \(p_m^{(|k|)}(x)\), respectively, we can describe the localization regions of the single eigenfunctions \(\psi_{k,i}\).
Sorting all eigenvalues $x_{|k|,i}$, $0 \leq |k| \leq n$, $1 \leq i \leq N_k$, from a maximal eigenvalue $x_{\text{max}}$ to a minimal eigenvalue $x_{\text{min}}$, results in a hierarchy in the sequence of the eigenfunctions concerning the space localization measured by $\varepsilon(f)$. In this sense, the eigenfunction corresponding to $x_{\text{max}}$ is optimally localized at the north pole among all eigenfunctions. Then, examining the orthogonal complement $\Pi_m^0 \ominus \text{span}\{\psi_{\text{max}}\}$, the optimally localized eigenfunction is $\psi_{\text{max} - 1}$. Accordingly, one obtains a chain of eigenfunctions in which the $j$-th element is better localized with respect to the the north pole than the subsequent $(j - 1)$-th element.

There exists a series of relations between the different eigenvalues. In the following we list a few important ones.

a) For a fixed row-number $k$, the eigenvalues $x_{|k|,i}$, $1 \leq i \leq N_k$, are all in the interior of $[-1, 1]$ and pairwise distinct. This is a standard result from the theory of orthogonal polynomials (see [2 I. Theorem 5.2]). In the
following, we order the $N_k$ zeros in decreasing size such that

$$x_{k,1} > x_{k,2} > \cdots > x_{k,N_k-1} > x_{k,N_k}.$$ 

b) For $m \leq |k| < n$, the zeros $x_{|k|,i}$ and $x_{|k|+1,i}$ are interlacing (cf. [2] I. Theorem 5.3), i.e.

$$x_{k,i} > x_{k+1,i} > x_{k,i+1}, \quad 1 \leq i \leq N_k - 1.$$

c) If $|k| < m$, then $x_{k,1} > x_{k+1,1}$ and $x_{k,n-m+1} < x_{k+1,n-m+1}$. In particular, this statement implies that $x_{\max} = x_{0,1}$ and $x_{\min} = x_{0,n-m+1}$. The polynomial $Q \in \Pi_n^m$, $\|Q\| = 1$, that maximizes the functional $\varepsilon(Q)$ and that is optimally localized at the north pole is therefore given by the eigenfunction $\psi_{0,1}$. By the same argumentation, the eigenfunction that is best localized at the south pole is given by $\psi_{0,n-m+1}$. These results were deduced in [6, 9, 21]. Polynomials that are optimally localized with respect to more general localization functionals, were studied in [25].

The ordering of the different eigenfunctions for the polynomial space $\Pi_n^m$ is illustrated in Figure 2. Figure 3 shows some examples of the real part of the eigenfunctions.

![Figure 3](image)

Figure 3: Real part of the eigenfunctions $\psi_{k,i}$ for $n = 32$, $m = 0$ and their localization with respect to the north pole.

Now, given a space-localized basis for the polynomial spaces $\Pi_n^m$, we want to analyse the decomposition of a polynomials $Q$ in the new basis. The next theorems illustrate that for a good approximation of a space-localized polynomial
only the eigenfunctions are needed that are located in the region where the mass of $Q$ is concentrated.

**Theorem 4.1.**

Let $a > 0$, $I_a^− := [−1, −1 + a)$ and $I_a^+ := (1 − a, 1]$. If $Q ∈ Π^m_n$ with $∥Q∥ = 1$, the following error bounds hold:

\[
\left\|Q - \sum_{k=-n}^{n} \sum_{i:x|k,i| ∈ I_a^−} \langle Q, ψ_{k,i} \rangle ψ_{k,i}\right\|^2 ≤ \frac{1 + ε(Q)}{a},
\]

\[
\left\|Q - \sum_{k=-n}^{n} \sum_{i:x|k,i| ∈ I_a^+} \langle Q, ψ_{k,i} \rangle ψ_{k,i}\right\|^2 ≤ \frac{1 - ε(Q)}{a}.
\]

**Proof.** Consider a polynomial $Q ∈ Π^m_n$ with $∥Q∥ = 1$. Since the eigenfunctions $ψ_{k,i}$ form an orthonormal basis of $Π^m_n$, the polynomial $Q$ can be written as

\[
Q(θ, ϕ) = \sum_{k=-n}^{n} \sum_{i=1}^{N_k} (Q, ψ_{k,i})ψ_{k,i}(θ, ϕ).
\]

Therefore,

\[
\left\|Q - \sum_{k=-n}^{n} \sum_{i:x|k,i| ∈ I_a^−} \langle Q, ψ_{k,i} \rangle ψ_{k,i}\right\|^2 = \left\| \sum_{k=-n}^{n} \sum_{i:x|k,i| ∈ [-1+a,1]} \langle Q, ψ_{k,i} \rangle ψ_{k,i}\right\|^2
\]

holds by the Pythagorean theorem. For the eigenvalues $x|k,i|$, in the interval $[-1+a, 1]$ we have $(1+x|k,i|) ≥ a$. Consequently, we can derive the estimates

\[
\sum_{k=-n}^{n} \sum_{i:x|k,i| ∈ [-1+a,1]} |\langle Q, ψ_{k,i} \rangle|^2 \leq \frac{1}{a} \sum_{k=-n}^{n} \sum_{i:x|k,i| ∈ [-1+a,1]} |\langle Q, ψ_{k,i} \rangle|^2 (1+x|k,i|)
\]

\[
\leq \frac{1}{a} \left( \sum_{k=-n}^{n} \sum_{i=1}^{N_k} (1+x|k,i|)|\langle Q, ψ_{k,i} \rangle|^2 \right).
\]

By the Pythagorean theorem, we have

\[
\sum_{k=-n}^{n} \sum_{i=1}^{N_k} |\langle Q, ψ_{k,i} \rangle|^2 = \|Q\| = 1.
\]

On the other hand, the spectral Theorem 3.4 yields

\[
ε(Q) = \langle P^m_n M_\cos θ P^m_n Q, Q \rangle = \sum_{k=-n}^{n} \sum_{i=1}^{N_k} |\langle Q, ψ_{k,i} \rangle|^2 x|k,i|.
\]

Thus, we obtain the error bound (23). The second error bound is derived from the fact that $(1-x|k,i|) ≥ a$ holds for eigenvalues $x|k,i| ∈ [-1, 1] \setminus I_a^+$. Analogous steps as for the first bound then lead to equation (24). □
Remark 4.2. For a normalized polynomial \( Q \in \Pi^m_n \), we can define the discrete density function \( \rho \) on \( \mathbb{R} \) by

\[
\rho(x) = \sum_{i,k: x(\psi_k,i) = x} |\langle Q, \psi_{k,i} \rangle|^2
\]

supported on the set of eigenvalues \( x_{|x|,i} \in (-1,1) \). Then, the expectation value of a \( \rho \)-distributed random variable \( X \) is given by \( \varepsilon(Q) \) and the statement of Theorem 4.1 corresponds to Markov’s inequality (cf. [30, p. 114]) for the random variables \( 1 + X \) and \( 1 - X \).

Now, if we define the discrete variance

\[
\var_{\rho}(Q) := \sum_{k,i} |\langle Q, \psi_{k,i} \rangle|^2(x_{|x|,i} - \varepsilon(Q))^2 = \sum_{k,i} |\langle Q, \psi_{k,i} \rangle|^2(x_{|x|,i}^2 - \varepsilon(Q)^2)
\]

and use Chebyshev’s inequality ([30, p. 114]) for a \( \rho \)-distributed random variable, we immediately get the following result.

**Corollary 4.3.**

Let \( a > 0, Q \in \Pi^m_n, \|Q\| = 1 \) and \( I_a = (\varepsilon(Q) - a, \varepsilon(Q) + a) \). Then, the following error bound holds

\[
\|Q - \sum_{k=-n}^{n} \sum_{i:x_{|x|,i} \in I_a} \langle Q, \psi_{k,i} \rangle \psi_{k,i} \|^2 \leq \frac{\var_{\rho}(Q)}{a^2}.
\]

In Theorem 4.1 and Corollary 4.3, it is a priori not clear how many zeros \( x_{|x|,i} \) are contained in the intervals \( I_a, I^+_a \), or \( I^-_a \). Therefore, we want to analyze now the distribution of the zeros \( x_{|x|,i} \) on the interval \([-1,1]\). We give first an auxiliary result about particular operators related to the space-frequency operator \( P^m_n M_{\cos \theta} P^0_n \).

**Lemma 4.4.**

For \( j = 1, 2, 3, \ldots \), the operators

\[
A_j = (P^0_n M_{\cos \theta} P^0_n)^j - P^0_n M^j P^0_n,
\]

\[
B_j = P^0_n M_{\cos \theta}^j P^0_n - P^0_n M^j P^0_n,
\]

\[
C_j = (P^m_n M_{\cos \theta} P^m_n)^j - (P^0_n M_{\cos \theta} P^0_n)^j,
\]

have norm at most 2. The operator \( A_j \) is of rank at most \((2n + 1 - j)(j - 1)\), while the operators \( B_j \) and \( C_j \) are of rank at most \((2m - 1)(n + 1) - m(m - 1)\).

**Proof.** All the involved operators \( P^m_n M_{\cos \theta} P^m_n \) and \( (P^m_n M_{\cos \theta} P^m_n)^j, j = 1, 2, 3, \ldots \), have norm smaller than 1. Thus, the operators \( A_j, B_j \) and \( C_j \) have operator norm at most 2. The operators \( A_j, B_j \) and \( C_j \) map the orthogonal complement \( (\Pi^0_n)^\perp \) to zero. For the polynomial space \( \Pi^0_n \), we use the spherical harmonics \( Y^l_k \), \( 0 \leq l \leq n, -l \leq k \leq l \), as a basis. We have \( B_j Y^l_k = C_j Y^l_k = 0 \) if \( |k| \geq m \). Summing up all spherical harmonics \( Y^l_k \) with \( |k| < m \), we get the
upper bound \((2m-1)(n+1) - m^2 + m\) for the rank of the operators \(B_j\) and \(C_j\) (for \(m = 0\), we get of course 0). For the operators \(A_j\), we have \(A_j Y_l^k = 0\) for all \(Y_l^k\) with \(l \leq n - j + 1\). Summing up all possible spherical harmonics with index \(n \geq l \geq n - j\), we get \((2n + 1 - j)(j - 1)\) as an upper bound for the rank of the operator \(A_j\).

Now, we can state the following weak limit for the distribution of the zeros \(x_{k,l,i}\) as \(n\) tends to infinity.

**Theorem 4.5.**

Let \(f\) be a bounded, Riemann integrable function on \([-1, 1]\), then

\[
\lim_{n \to \infty} \frac{1}{N^m_n} \sum_{k=-n}^{n} \sum_{i=1}^{N_k} f(x_{k,l,i}) = \frac{1}{2} \int_{-1}^{1} f(x) dx. \tag{29}
\]

**Proof.** According to Theorem 3.4, the functions \(\psi_{k,i}, -n \leq k \leq n, 1 \leq i \leq N_k\), form a complete set of eigenfunctions for the operator \(P^m_n M_{\cos \theta} P^m_n\) on the space \(\Pi^m_n\). Thus, the functions \(\psi_{k,i}\) are also eigenfunctions of the operators \((P^m_n M_{\cos \theta} P^m_n)^j, j = 1, 2, 3, \cdots, \) and we get:

\[
\text{tr} \left( (P^m_n M_{\cos \theta} P^m_n)^j \right) = \sum_{k=-n}^{n} \sum_{i=1}^{N_k} \left( (P^m_n M_{\cos \theta} P^m_n)^j \psi_{k,i}, \psi_{k,i} \right) = \sum_{k=-n}^{n} \sum_{i=1}^{N_k} x^j_{k,i}.
\]

On the other hand, also the spherical harmonics \(Y_l^k, m \leq l \leq n, -l \leq k \leq l\), form an orthonormal basis of \(\Pi^m_n\). Using the addition theorem of the spherical harmonics (see [28, Theorem 2]), we obtain the identity:

\[
\text{tr} \left( P^m_n M_{\cos \theta} P^m_n \right) = \sum_{l=m}^{n} \sum_{k=-l}^{l} \left( (P^m_n M_{\cos \theta} P^m_n) Y_l^k, Y_l^k \right)
\]

\[
= \frac{1}{4\pi} \sum_{l=m}^{n} \sum_{k=-l}^{l} \int_{0}^{2\pi} \int_{0}^{\pi} (\cos \theta)^j |Y_l^k(\theta, \varphi)|^2 \sin \theta d\theta d\varphi
\]

\[
= \frac{1}{4\pi} \sum_{l=m}^{n} \int_{0}^{2\pi} \int_{0}^{\pi} (\cos \theta)^j \left( \sum_{k=-l}^{l} |Y_l^k(\theta, \varphi)|^2 \right) \sin \theta d\theta d\varphi
\]

\[
= \sum_{l=m}^{n} \frac{2l + 1}{2} \int_{0}^{\pi} (\cos \theta)^j \sin \theta d\theta
\]

\[
= \sum_{l=m}^{n} \frac{2l + 1}{2} \int_{-1}^{1} x^j dx = \frac{N^m_n}{2} \int_{-1}^{1} x^j dx.
\]

Now, by Lemma 4.4, we get the estimate

\[
\left| \frac{1}{N^m_n} \sum_{k=-n}^{n} \sum_{i=1}^{N_k} x^j_{k,i} - \frac{1}{2} \int_{-1}^{1} x^j dx \right| \leq \left| \text{tr} A_j + \text{tr} B_j + \text{tr} C_j \right| \left( n + 1 \right)^2 - m^2
\]

\[
\leq \left( n + 1 \right)^2 (j + 2m - 2) - \frac{j^2 + 1}{2} - m^2 + m.
\]
For every fixed \( j \), the term on the right hand side tends to zero as \( n \to \infty \). In this way, we have shown equation (29) for every polynomial on \([-1, 1]\). The statement for arbitrary bounded Riemann integrable functions \( f \) on \([-1, 1]\) follows from the theory of one-sided polynomial approximation developed by Freud (see [12]). □

**Corollary 4.6.**
For \( n \to \infty \), the distribution of the eigenvalues \( x_{|k|,i} \) converges weakly to the uniform distribution on \([-1, 1]\), i.e., for \(-1 \leq a < b \leq 1\) we have

\[
\lim_{n \to \infty} \frac{\sharp \{(k, i) : -n \leq k \leq n, \ 1 \leq i \leq N_k, \ x_{|k|,i} \in [a, b]\}}{N_n} = \frac{b - a}{2}.
\]

**Remark 4.7.** The idea for the proofs of Lemma 4.4 and Theorem 4.5 is taken from the works of Simon (see [37, Section 2.15], [36]). In these works, the corresponding weak limits are proven for orthogonal polynomials on the real line and on the unit circle.

## 5 Computational considerations

When applying the new localized basis for the analysis of functions on \( S_2 \), an important aspect is the numerical effort to compute the expansion coefficients as well as to reconstruct the function from the coefficients. In this section, we will show that due to the particular structure of the basis functions both can be done fast and efficiently from the expansion of the function in spherical harmonics.

In a first step, we investigate the relation

\[
Q = \sum_{l=m}^{n} \sum_{k=-l}^{l} c_{l,k} Y_l^k = \sum_{k=-n}^{n} \sum_{i=1}^{N_k} d_{k,i} \psi_{k,i}
\]

between an expansion in spherical harmonics and an expansion in the new localized basis. We use the notation

\[
d_k := (d_{k,1}, d_{k,2}, \ldots, d_{k,N_k})^T, \quad -n \leq k \leq n,
\]

and consider the relation between the spherical harmonics \( Y_l^k \) and the eigenfunctions \( \psi_{k,i} \). By the spectral Theorem 3.4, we have for \(-n \leq k \leq n\):

\[
\psi_{k,i} = T_k \frac{v_{|k|,i}}{\|v_{|k|,i}\|_2} = \left( Y_{n-N_k+1}^k, \ldots, Y_n^k \right) \cdot \frac{v_{|k|,i}}{\|v_{|k|,i}\|_2}, \quad 1 \leq i \leq N_k.
\]

Now, comparison of the two different expansions gives

\[
c_k = \left( \frac{v_{|k|,1}}{\|v_{|k|,1}\|_2}, \ldots, \frac{v_{|k|,N_k}}{\|v_{|k|,N_k}\|_2} \right) d_k, \quad -n \leq k \leq n.
\]
Since for fixed $k$ the eigenvectors $\mathbf{v}_{|k|,i}$, $1 \leq i \leq N_k$, of the symmetric matrices $\mathbf{J}(|k|)^{m-|k|}$ and $\mathbf{J}(|k|)^{n-|k|}$, are pairwise orthogonal, the matrices $\mathbf{V}_k$ are orthogonal and we get

$$c_k = \mathbf{V}_k \mathbf{d}_k, \quad d_k = \mathbf{V}_k^T c_k, \quad -n \leq k \leq n. \quad (30)$$

Once all the eigenvectors of the matrices $\mathbf{J}(|k|)^{m-|k|}$ and $\mathbf{J}(|k|)^{n-|k|}$ are computed, the coefficients $\mathbf{d}_k$ of the localized basis can be computed from the expansion coefficients $\mathbf{c}_k$ by a matrix-vector product in $(2N_k - 1)N_k$ arithmetic operations. For all $2n + 1$ involved blocks we then get a total amount of $\frac{1}{8}(n + m)(4n^2 + n(4m + 5) + 3 + m - 8m^2)$ arithmetic operations to conduct the change of basis.

Due to the particular structure of the transition matrices $\mathbf{V}_k$ and $\mathbf{V}_k^T$, the calculation of the coefficients $\mathbf{d}_k$ can be accelerated considerably by using algorithms based on the fast Fourier transform. To this end, we consider the entries of $\mathbf{V}_k^T$ in more detail. To simplify the representations, we will only consider the case $m \leq |k| \leq n$. In this case, we have

$$\begin{pmatrix} d_{k,1} \\ \vdots \\ d_{k,N_k} \end{pmatrix} = \begin{pmatrix} \kappa_{k,1} \\ \vdots \\ \kappa_{k,N_k} \end{pmatrix} \begin{pmatrix} p_0(|k|)(x_{|k|,1}) & \cdots & p_n(|k|)(x_{|k|,1}) \\ \vdots & \ddots & \vdots \\ p_0(|k|)(x_{|k|,N_k}) & \cdots & p_n(|k|)(x_{|k|,N_k}) \end{pmatrix} \begin{pmatrix} c_{1,k} \\ \vdots \\ c_{N_k,k} \end{pmatrix}.$$ 

Now, we consider transition matrices $\mathbf{B}_k$ (see [18, Section 1.4], [32]) that describe the change of basis from ultraspherical polynomials $p_l(|k|)(x)$ to Chebyshev polynomials $T_l(x) = \cos(l \arccos x)$ of the first kind, i.e.

$$\begin{pmatrix} T_0(x_{|k|,1}) & \cdots & T_{n-|k|}(x_{|k|,1}) \\ \vdots & \ddots & \vdots \\ T_0(x_{|k|,N_k}) & \cdots & T_{n-|k|}(x_{|k|,N_k}) \end{pmatrix} \mathbf{F}_k = \begin{pmatrix} p_0(|k|)(x_{|k|,1}) & \cdots & p_{n-|k|}(x_{|k|,1}) \\ \vdots & \ddots & \vdots \\ p_0(|k|)(x_{|k|,N_k}) & \cdots & p_{n-|k|}(x_{|k|,N_k}) \end{pmatrix} \mathbf{B}_k.$$ 

In this way, we can write the transforms in $(30)$ as

$$\mathbf{d}_k = \mathbf{K}_k \mathbf{F}_k \mathbf{B}_k \mathbf{c}_k, \quad \mathbf{c}_k = \mathbf{B}_k^T \mathbf{F}_k^T \mathbf{K}_k \mathbf{d}_k, \quad -n \leq k \leq n.$$ 

Using this representation, we can now use efficient algorithms to compute the single steps in a fast way. The matrix-vector product $\mathbf{B}_k \mathbf{c}_k$ can be computed efficiently using the fast polynomial transform described in [32] in $\mathcal{O}(N_k \log^2 N_k)$ arithmetic operations. Next, the multiplication with $\mathbf{F}_k$ can be conducted using a non-equispaced fast cosine transform in $\mathcal{O}(N_k \log N_k)$ arithmetic operations (cf. [10]). Finally, the application of $\mathbf{K}_k$ can be implemented cheaply (with $N_k$ arithmetic operations) by a point-wise vector-vector multiplication. In this way, the coefficients $\mathbf{d}_k$ can be computed from the coefficients $\mathbf{c}_k$ with a complexity
of $O(N_k \log^2 N_k)$. With $N_k \leq n - m + 1$, we get for all $2n + 1$ involved blocks a total complexity of

$$O(n(n - m) \log^2 (n - m))$$

arithmetic operations for the change in the new localized basis. From the order of complexity, this corresponds to the complexity of fast algorithms computing the Fourier transform on $S_2$ (cf. [20]). Implementations of the fast polynomial transform, the fast non-equispaced cosine transform and the fast spherical Fourier transform can be found in the software package NFFT3 documented in [19] and the references therein. An example plot of a computed decomposition can be found in Figure 4.

![Figure 4: Orthogonal decomposition of a function $f \in \Pi_{512}$ in two parts $f_1, f_2 \in \Pi_{512}$. The set $I \subset [-1, 1]$ is given by $[-1, -0.6] \cup [-0.2, 0.2] \cup [0.6, 1]$. The functions $f_1$ and $f_2$ are localized in the regions $\{p(\theta, \varphi) \in S_2 : \theta \in \arccos I\}$ and $\{p(\theta, \varphi) \in S_2 : \theta \notin \arccos I\}$ of $S_2$, respectively.](image)

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