Stochastic Gradient Descent on Nonconvex Functions with General Noise Models

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Abstract

Stochastic Gradient Descent (SGD) is a widely deployed optimization procedure throughout data-driven and simulation-driven disciplines, which has drawn a substantial interest in understanding its global behavior across a broad class of nonconvex problems and noise models. Recent analyses of SGD have made noteworthy progress in this direction, and these analyses have innovated important and insightful new strategies for understanding SGD. However, these analyses often have imposed certain restrictions (e.g., convexity, global Lipschitz continuity, uniform Hölder continuity, expected smoothness, etc.) that leave room for innovation. In this work, we address this gap by proving that, for a rather general class of nonconvex functions and noise models, SGD’s iterates either diverge to infinity or converge to a stationary point with probability one. By further restricting to globally Hölder continuous functions and the expected smoothness noise model, we prove that—regardless of whether the iterates diverge or remain finite—the norm of the gradient function evaluated at SGD’s iterates converges to zero with probability one and in expectation. As a result of our work, we broaden the scope of nonconvex problems and noise models to which SGD can be applied with rigorous guarantees of its global behavior.

1 Introduction

Stochastic gradient descent (SGD) is widely deployed throughout data science and adjacent fields to solve

$$\min_\theta F(\theta)$$  \hspace{1cm} (1)

where $F: \mathbb{R}^p \to \mathbb{R}$ and is defined to be the expectation of a function $f: \mathbb{R}^p \times X \to \mathbb{R}$, where $X$ is the range of a well-defined random variable $X$.\footnote{Hence, we can assume that there is a $\sigma$-Algebra defined on sets of $X$ that ensure it is a measure space that can support $X$. We can also define an appropriate push forward measure to specify a probability space. Consequently, we can define an expectation with respect to this probability space.} Owing to its broad usage, SGD’s global behavior on different classes of functions $f$ (and, hence, $F$) has been of substantial interest. While there are many works that have provided insight, understanding SGD’s global behavior has been notably advanced by several recent works [Asi and Duchi, 2019, Lei et al., 2019, Patel, 2020, Khaled and Richtárik, 2020] that we overview presently.

To explain the insights of these works, we will need some notation. We define $\tilde{F}(\theta)$ as the gradient of $F$ evaluated at the point $\theta \in \mathbb{R}^p$, and we define $\tilde{f}(\theta, X)$ as the gradient of $f$ with respect to its first argument evaluated at $(\theta, X)$, which follows the notation of Patel [2020]. We now describe these essential works.
1. Asi and Duchi [2019] show that when \( f(\cdot, x) \) is a closed, convex, subdifferentiable function for all \( x \in \mathcal{X} \), then SGD’s iterates are stable with probability one and will converge to a solution under some additional assumptions. Distinguishingly, Asi and Duchi [2019] allow \( E[\| f(\theta, X) \|_2^2] \) (i.e., the noise model) to grow arbitrarily with the distance between the current iterate and the solution set. To our knowledge, this is the most general assumption for the noise under which convergence has been demonstrated, and the proof relies intimately on convexity [see Asi and Duchi, 2019, Lemma 3.7].

2. Lei et al. [2019] prove that for uniformly lower bounded, nonconvex functions, \( f \), for which

\[
(\exists L > 0)(\forall \theta_1, \theta_2 \in \mathbb{R}^p)(\forall x \in \mathcal{X}) : \| \dot{f}(\theta_1, x) - \dot{f}(\theta_2, x) \|_2 \leq L \| \theta_1 - \theta_2 \|^2, \tag{2}
\]

with \( \alpha \in (0, 1) \), the objective function, \( F \), evaluated at the SGD iterates converges almost surely to a bounded random variable. Moreover, Lei et al. [2019] show that, when \( \alpha = 1 \), the expected value of the norm of the gradient function, \( \dot{F} \), evaluated at the SGD iterates converges to zero.

3. Patel [2020] shows that for a lower bounded, nonconvex objective function, \( F \), for which \( \dot{F} \) is globally Lipschitz continuous and for which

\[
(\exists C_1, C_2 \geq 0)(\forall \theta \in \mathbb{R}^p) : E \left[ \| \dot{f}(\theta, X) \|_2^2 \right] \leq C_1 + C_2 E \left[ \| \dot{F}(\theta) \|_2 \right]^2, \tag{3}
\]

the norm of the gradient function, \( \dot{F} \), evaluated at the SGD iterates converges to zero with probability one. Moreover, Patel [2020] allows for matrix-valued learning rates. The later global convergence work of Merikopoulou et al. [2020] offers similar conclusions under more stringent conditions, but also explores local properties such as local rates of convergence.

4. Khaled and Richtárik [2020] show that for a lower bounded, nonconvex objective function, \( F \), for which \( \dot{F} \) is globally Lipschitz continuous and for which

\[
(\exists C_1, C_2, C_3 \geq 0)(\forall \theta \in \mathbb{R}^p) : E \left[ \| \dot{f}(\theta, X) \|_2^2 \right] \leq C_1 + C_2 E \left[ \| \dot{F}(\theta) \|_2 \right]^2 + C_3 F(\theta), \tag{4}
\]

the smallest of all expected norms of the gradient evaluated at the SGD iterates converges to zero. Similar results are explored by Gower et al. [2020].

**Contributions**

Our goal here is to move towards a more general theory of convergence that combines all of these threads under a single analysis framework. Specifically, by innovating on the strategies of Lei et al. [2019] and Patel [2020], we will prove the following results for SGD with matrix-valued learning rates, which we state informally now and formalize later.

1. We prove that for a lower bounded, nonconvex objective function, \( F \), for which \( \dot{F} \) is locally \( \alpha \)-Hölder continuous and for which \( E[\| f(\theta, X) \|_2^2] \) is controlled by an arbitrary, non-negative upper semi-continuous function, either the iterates of SGD diverge to infinity, or they remain finite. See Theorems 1 and 2 in Section 3.

2. When the iterates remain finite, the objective function, \( F \), evaluated at the iterates converges to a finite random variable, and the gradient norm evaluated at the iterates converges to zero with probability one. With this result, we are able to relax the noise models of Bottou et al. [2018], Asi and Duchi [2019], Lei et al. [2019], Patel [2020], Khaled and Richtárik [2020]; relax the global, uniform Hölder continuity assumption of [Lei et al., 2019]; and relax the global Lipschitz continuity assumption of Bottou et al. [2018], Patel [2020], Khaled and Richtárik [2020]. See Theorem 2 in Section 3.

3. When the iterates diverge, we can also say something interesting under slightly stronger conditions. Specifically, by strengthening the local Hölder assumption to a global Hölder assumption of \( \dot{F} \) and restricting the noise model on \( \dot{f} \) to (4), we are able to show that, regardless of the iterate behavior, the objective function evaluated at the iterates converges with probability one to an integrable random variable, and the norm of the gradient function evaluated at the iterates converges to zero with probability one and in \( L^1 \). This result directly generalize the results of Bottou et al. [2018], Lei et al. [2019], Patel [2020], Khaled and Richtárik [2020], and a host of other more specialized results that are covered by these works. See Theorem 3 in Section 4.
To our knowledge, our results are the most general for the global analysis of SGD as they allow for rather general nonconvex functions (e.g., locally Hölder gradient function) and general noise models (e.g., arbitrary, upper semi-continuous bound on the second moment). As a result, our results broaden the scope of problems to which SGD can be used with rigorous guarantees of its asymptotic behavior.

Organization

In Section 2, Stochastic Gradient Descent (SGD) with matrix-valued learning rates are precisely specified. In Section 3, SGD’s iterates are shown to either diverge or remain finite over a general class of nonconvex functions and noise models, which are precisely specified in this section; moreover, when SGD’s iterates remain finite, then they are shown to converge to a stationary point with probability one. In Section 4, the gradient function evaluated at SGD’s iterates is shown to converge to zero with probability one and in $L^1$, under the stronger assumptions of global Hölder continuity and under a more restricted noise model. In Section 5, we conclude this work with a discussion of limitations and future work.

2 Stochastic Gradient Descent

We define Stochastic Gradient Descent to be the procedure that beings with an arbitrary $\theta_0 \in \mathbb{R}^p$ and generates $\{\theta_k : k \in \mathbb{N}\}$ according to the recursion

$$\theta_{k+1} = \theta_k - M_k \hat{f}(\theta_k, X_{k+1})$$

where $\{X_k : k \in \mathbb{N}\}$ are independent and are identically distributed to $X$; and $\{M_k : k+1 \in \mathbb{N}\} \subset \mathbb{R}^{p \times p}$ are matrices whose properties we specify momentarily. Let $F_0 = \sigma(\theta_0)$ and $F_k = \sigma(\theta_0, X_1, \ldots, X_k)$ for all $k \in \mathbb{N}$.

Remark. We note that if $\theta_0$ is random, then we will condition the results below on $F_0 = \sigma(\theta_0)$.

However, to avoid this additional notation, we will not state this explicitly.

First, we will require that

Property 1. $\{M_k : k + 1 \in \mathbb{N}\}$ are symmetric, positive definite matrices.

Property 1 is a natural extension to the scalar learning rate case in which the learning rate is required to be positive valued at each iterate.

Second, we consider the natural extension of the Robbins-Monro condition for $\alpha$-Hölder continuous functions with matrix-valued learning rates. Specifically, we require that $\{M_k\}$ satisfy

Property 2. $\sum_{k=1}^{\infty} \lambda_{\text{max}}(M_k)^{1+\alpha} =: S < \infty$ with $\alpha \in (0, 1]$.

and

Property 3. $\sum_{k=1}^{\infty} \lambda_{\text{min}}(M_k) = \infty$,

where $\lambda_{\text{max}}(\cdot)$ and $\lambda_{\text{min}}(\cdot)$ denote the largest and smallest eigenvalues of the given symmetric matrix.

For Section 4, we will require the following condition, which controls the relationship between $\lambda_{\text{max}}(M_k)$ and $\lambda_{\text{min}}(M_k)$. Note, such a condition is readily satisfied for scalar learning rates satisfying Property 2.

Property 4. $\lim_{m \to \infty} \lambda_{\text{max}}(M_k)^{\alpha} \kappa(M_k) = 0$ where $\kappa(M_k) = \|M_k\|_2 \|M_k^{-1}\|_2^{-1}$.

3 A Capture Theorem and Its Consequences

We will begin by defining a set of assumptions under which we will analyze SGD and we will discuss how it relates to the assumptions in the aforementioned works. We refer the reader to §2 of Patel [2020] for a review of common assumptions in the nonconvex landscape and their relationships.

We begin with a common assumption that ensure that minimizing the objective function is a reasonable effort.
**Assumption 1.** There exists \( F_{i,b} \in \mathbb{R} \) such that \( F_{i,b} \leq F(\theta) \) for all \( \theta \in \mathbb{R}^p \).

Indeed, this is the assumption made by Khaled and Richtárik [2020] and Patel [2020]. This assumption is implied by Lei et al. [2019]'s more stringent assumption that \( f(\theta, x) \geq 0 \) for all \( \theta \in \mathbb{R}^p \) and for all \( x \in \mathcal{X} \). Finally, this assumption is implied by Asi and Duchi [2019]'s assumptions that \( F \) is convex and that the optimization problem has a nonempty solution set.

We also require the common assumption that the stochastic gradient \( \dot{f}(\theta, X) \) are unbiased. We note that this assumption is common in most works, and can be relaxed as shown in §4 of Bottou et al. [2018]. Fortunately, this relaxation is rather easy to account for within our analysis.

**Assumption 2.** For all \( \theta \in \mathbb{R}^p \), \( \mathbb{E}[\dot{f}(\theta, X)] = \dot{F}(\theta) \).

We now come to our two less common, yet more general assumptions in comparison to what can be found in the literature. The first assumption is inspired by the noise model assumption of Asi and Duchi [2019], which allows the trace of the variance of \( \dot{f}(\theta, X) \) to grow with \( \theta \)'s distance from the assumed solution set. Here, we have no such luxury of having a guaranteed solution set, and so we require a more general assumption.

**Assumption 3.** Let \( G : \mathbb{R}^p \to \mathbb{R}_{\geq 0} \) be an upper semi-continuous function. For all \( \theta \in \mathbb{R}^p \), \( \mathbb{E}[\|\dot{f}(\theta, X)\|_2^2] \leq G(\theta) \).

We see that **Assumption 3** readily generalized the noise modeling assumptions of Khaled and Richtárik [2020] and Patel [2020]. Moreover, **Assumption 3** is implied under the more stringent conditions in Lei et al. [2019], specifically by using (2) and **Assumption 1** with an application of Young’s inequality.

For the last assumption, we recall that a function is locally \( \alpha \)-Hölder continuous if for any compact set \( \mathcal{K} \subset \mathbb{R}^p \), \( \exists L > 0 \) such that \( \forall \varphi_1, \varphi_2 \in \mathcal{K} \), \( \|\dot{F}(\varphi_1) - \dot{F}(\varphi_2)\|_2 \leq L\|\varphi_1 - \varphi_2\|_2 \).

**Assumption 4.** \( \dot{F} \) is locally \( \alpha \)-Hölder continuous for some \( \alpha \in (0, 1] \).

Again, **Assumption 4** is weaker than the global Lipschitz assumptions of Khaled and Richtárik [2020] and Patel [2020], and is implied by the global, uniform \( \alpha \)-Hölder continuity assumed in Lei et al. [2019]. Interestingly, **Assumption 4** is cleverly circumvented in Asi and Duchi [2019, Lemma 3.6] using the monotonicity of the gradient function and Young’s inequality, and one could argue that it would generalize **Assumption 4**. However, this argument would fall apart for matrix-valued learning rates, as the monotonicity of the gradient operator is no longer guaranteed even in the convex case.

With these assumption, we begin by defining a central property of SGD that is often overlooked or implicitly required, and has several immediate consequences and applications. The proof is a direct application of the Borel-Cantelli lemma and can be found in Appendix B.

**Theorem 1** (Capture Theorem). Let \( \widehat{\theta} \in \mathbb{R}^p \) be arbitrary. Let \( \{\theta_k\} \) be defined as in (5) and satisfy Properties 1 and 2. If **Assumption 3** holds, then for any \( R \geq 0 \),

\[
\mathbb{P} \left[ \left\| \theta_{k+1} - \widehat{\theta} \right\|_2 > R, \left\| \theta_k - \widehat{\theta} \right\|_2 \leq R \text{ i.o.} \right] = 0.
\]

(6)

**Theorem 1** has several immediate consequences. For example, **Theorem 1** is central to proving local rates of convergence as it ensures that the iterates are eventually captured within some basin of attraction, in which some sort of local analysis can be done; however, such a local analysis is not the focus of this work. For a global perspective, **Theorem 1** implies the following result.

**Theorem 2.** Let \( \{\theta_k\} \) be defined as in (5) and satisfy Properties 1 to 3. Suppose **Assumptions 1 to 4** hold. Let \( A_1 = \{\liminf_{k \to \infty} \|\theta_k\|_2 = \infty\} \) and \( A_2 = \{\lim_{k \to \infty} \|\theta_k\|_2 < \infty\} \). Then, the following statements hold

1. \( \mathbb{P}[A_1] + \mathbb{P}[A_2] = 1. \)

2. There exists a finite random variable, \( F_{\lim} \), such that, on \( A_2 \), \( \lim_{k \to \infty} F(\theta_k) = F_{\lim} \) and \( \lim_{k \to \infty} \|\dot{F}(\theta_k)\|_2 = 0 \) with probability one.

**Proof of Theorem 2.** Note, the referenced results can be found in Appendix B. Informally, **Theorem 1** implies that the limit supremum and limit infimum of \( \|\theta_k\|_2 \) cannot be distinct. As a
result, the limit of $\|\theta_k\|$ is either infinite or is finite with probability one. This is formalized in Corollary 1.

To show the remaining statement, we begin by constraining to the event $\{\sup_k \|\theta_k\|_2 \leq R\}$ for arbitrary $R \geq 0$. Then, on this event we can prove that $F(\theta_k)$ converges to a finite random variable. Similarly, we can prove that on this event $\|\hat{F}(\theta_k)\|_2$ must converge to zero. By taking the union over all $R \in \mathbb{N}$, we can conclude that these two statements hold on the event $\{\sup_k \|\theta_k\|_2 < \infty\}$ which is implied by $A_2$. These arguments are formalized in Corollary 2 for the objective function, and Corollary 3 for the gradient function statement. For the gradient function statement, the proof strategy is adapted from Patel [2020].

\textbf{Remark.} By Theorem 1, we have that $\{\lim_{k \to \infty} \|\theta_k\|_2 < \infty\}$ is equal to $\{\lim_{k \to \infty} \|\theta_k - \bar{\theta}\|_2 < \infty\}$ for an arbitrary choice of $\bar{\theta}$. Thus, by choosing $\bar{\theta}_1$ and $\bar{\theta}_2$ such that $\{0, \bar{\theta}_1, \bar{\theta}_2\}$ are not colinear, then we conclude by Theorem 1 and triangulation that $\theta_k$ converges to a finite random variable on $A_2$. Hence, by Theorem 2, $\{\theta_k\}$ converges to a finite random variable that takes value over the stationary points of the objective function on $A_2$. Or, to be more succinct, we will say that $\{\theta_k\}$ converges to a stationary point on $A_2$.

We note that the statement of $P[A_1] + P[A_2] = 1$ is not trivial. There is no \textit{apriori} guarantee that the limit supremum and limit infimum of $\|\theta_k\|_2$ must coincide (cf., a simple random walk with a positive one bias at each step, which will have its limit supremum as infinity and limit infimum as zero with probability one). Moreover, in the case that the limit exists, it is also nontrivial that the procedure converges to a stationary point.

One case that is not explored by Theorem 2 is the case of $A_1$. There are two possibilities here: either we want to allow $\|\theta_k\|_2$ to diverge as the gradient function can be zero in the limit, or we want to disallow the possibility of $A_2$ entirely (i.e., $P[A_2] = 0$). This first possibility is the focus of Section 4, while the second possibility will not be explored in this work.

\section{Global Hölder Continuity and Expected Smoothness}

Here, we consider the situation in which the iterates diverging (i.e., event $A_1$ in Theorem 2) might be meaningful to the underlying optimization problem. As a simple example of a situation where this may occur, consider optimizing the smooth rectifier function as described in Fig. 1. In this example, the optimizer would find $\{\theta_k\}$ with $\theta \to -\infty$.

![Figure 1: A plot of the smooth rectifier function $F(\theta) = \log(1 + \exp(\theta))$, where $\theta \in \mathbb{R}$.](image)

To address this case, we will need to strengthen Assumptions 3 and 4. We will begin by strengthening Assumption 3 with the following assumption, termed expected smoothness [Khaled and Richtárik, 2020].

\textbf{Assumption 5.} There exists $C_1, C_2 \geq 0$ and $C_3 \geq 1$ such that, $\forall \theta \in \mathbb{R}^p$,

$$
\mathbb{E} \left[ \left\| \hat{f}(\theta, X) \right\|_2^2 \right] \leq C_1 + C_2(\mathbb{E} \left[ F(\theta) \right] - F_{\theta_k}) + C_3 \left\| \hat{F}(\theta) \right\|_2^2. \tag{7}
$$
We note that Assumption 5 is implicitly making use of Assumption 1 owing to the term $F_{l,b}$, but this can be easily addressed by removing $F_{l,b}$ and ensuring that the upper bound is non-negative (e.g., by using $\max\{F(\theta),0\}$ in place of $F(\theta) - F_{l,b}$). We also note that the requirement $C_3 \geq 1$, in conjunction with Assumption 2, implies
\[
\mathbb{E}\left[\left\|\hat{f}(\theta,X) - \hat{F}(\theta)\right\|_2^2\right] \leq C_1 + C_2(F(\theta) - F_{l,b}) + (C_3 - 1) \left\|\hat{F}(\theta)\right\|_2^2, \tag{8}
\]
which allows for the variance to be well-specified (i.e., the upper bound is non-negative).

We now turn our attention to strengthening Assumption 4 as follows.

**Assumption 6.** $\hat{F}$ is globally $\alpha$-Hölder continuous for some $\alpha \in (0,1]$.

While Assumption 6 is more restrictive than Assumption 4 and precludes certain objective functions, it is rather natural for the case in which $\|\theta_k\| \to \infty$ may be meaningful for the optimization problem. Note, by Lemma 11, Assumptions 1, 5 and 6 together imply that there exists an integrable random variable, and the norm of the gradient function evaluated at the iterates converges to zero with probability one and in $L^1$. As a result, even if the iterates diverge, we see that they are still tending to regions where the gradient is zero. This result generalizes the global analysis results of Bottou et al. [2018], Lei et al. [2019], Patel [2020] and Khaled and Richtárik [2020].

**Theorem 3.** Let $\{\theta_k\}$ be defined as in (5) and satisfy Properties 1 to 4. Suppose Assumptions 1, 2, 5 and 6 hold. Then,

1. there exists an integrable random variable, $F_{\lim}$, (i.e., $\mathbb{E}[F_{\lim}] < \infty$) such that $\lim_{k \to \infty} F(\theta_k) = F_{\lim}$ with probability one.
2. Moreover, $\sup_{k+1 \in \mathbb{N}} \mathbb{E}[F(\theta_k)] < \infty$. Therefore, $\forall \gamma \in (0,1)$,
   \[
   \lim_{k \to \infty} \mathbb{E}\left[\left\|\left(F(\theta_k) - F_{l,b}\right)^\gamma - (F_{\lim} - F_{l,b})^\gamma\right\|\right] = 0.
   \]
3. Finally, $\lim_{k \to \infty} \left\|\hat{F}(\theta_k)\right\|_2 = 0$ with probability one and $\lim_{k \to \infty} \mathbb{E}\left[\left\|\hat{F}(\theta_k)\right\|_2\right] = 0$.

**Proof.** Note, the reference results can be found in Appendix C. The proofs follow a rather similar strategy to that of Theorem 2 for demonstrating convergence of $\{F(\theta_k)\}$ and $\{\|\hat{F}(\theta_k)\|_2\}$ with probability one. The details are supplied in Corollaries 4 and 6, respectively.

The proof for bounding $\sup_{k+1 \in \mathbb{N}} \mathbb{E}[F(\theta_k)]$ follows by setting up a recursive relationship between sequential objective function values and making use of Property 2 to show that they are uniformly bounded. The strategy is adapted from Lei et al. [2019]. Once we can bound the supremum, we have that $\{(F(\theta_k) - F_{l,b})^\gamma\}$ are uniformly integrable for $\gamma \in [0,1)$. By combining this observation with strong convergence, we have that $\{(F(\theta_k) - F_{l,b})^\gamma\}$ converges to $(F_{\lim} - F_{l,b})^\gamma$ in $L^1$. The details are supplied in Corollary 5.

Once the objective function is controlled, the supremum over the squared norm of the gradient function at the iterates can be controlled using well-known inequalities that make use of Assumption 1, Assumption 6 and the fundamental theorem of calculus. As a result, we have that the squared norm of the gradient function at the iterates is uniformly integrable. By combining this with strong convergence, we have that the norm of the gradient function evaluated at the iterates converges to zero in $L^1$. The details are supplied in Corollary 7.

**Remark.** If it is of interest, under the setting of Theorem 3, we can preclude divergence by requiring a radially coercive objective function (i.e., that the objective function tends to infinity as the norm of
its argument goes to infinity). To be specific, under this additional requirement, the event $A_1$ from Theorem 2 coincides with the event on which $(F(\theta_k) - F_{l.b.})^{1/2}$ diverges to infinity; however, by Markov’s inequality, the probability that $(F(\theta_k) - F_{l.b.})^{1/2}$ exceeds some value $\ell$ is bounded by the product of $\ell^{-1}$ and the supremum $E[(F(\theta_j) - F_{l.b.})^{1/2}]$, where this final quantity is finite by Theorem 3. Hence, $P[A_1]$ can be made arbitrarily small, which implies that it is zero.

5 Conclusion

In this work, we have, to our knowledge, provided results about the global behavior of SGD on the most general nonconvex functions and noise models currently in the literature. In particular, we prove that SGD’s iterates either diverge or remain finite, and, in the latter case, we prove that SGD’s iterates converge to a stationary point with probability one. Moreover, if we restrict the class of nonconvex functions to those that are globally Hölder continuous and the noise model to expected smoothness, then we prove that $\dot{F}$ evaluated at SGD’s iterates converges to zero with probability one and in $L^1$. With these results, we broaden the scope of problems to which SGD can be applied with rigorous guarantees of its asymptotic behavior.

Limitations

A key limitation of this work is that, under the general setting of local Hölder continuity and generic noise model (i.e., Theorem 2), we have not provided insight into what happens on the event that the iterates diverge. That is, in Theorem 2, we do not say anything about $A_1$ (the event on which the iterates diverge), except that it exists. Ideally, we would like to show that such an even has probability zero, but this is not necessarily true even for objective functions that are radially coercive, so long as the noise model grows sufficiently rapidly.

Future Work

Our primary goal for future work is to address the above limitations. Specifically, our goal is to determine general, reasonable conditions on $F$ that will imply that $P[\liminf_{k \to \infty} \|\theta_k\|_2 = \infty] = 0$. Once this limitation is address, we will look to integrate current local analyses (i.e., local rates of convergence) with the general class of nonconvex functions and noise models established in this work.
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A Technical Lemmas

The first lemma is a straightforward conclusion of Properties 1 and 4.

**Lemma 1.** Suppose \( \{M_k : k + 1 \in \mathbb{N}\} \) satisfy Properties 1 and 4, then \( \forall C > 0, \exists K \in \mathbb{N} \) such that \( \forall k \geq K \),

\[
\lambda_{\min}(M_k) - \frac{C}{2}\lambda_{\max}(M_k)^{1+\alpha} \geq \frac{1}{2}\lambda_{\min}(M_k).
\]

(10)

**Proof of Lemma 1.** Fix \( C > 0 \). By rearranging (10), it is equivalent to prove that \( \exists K \in \mathbb{N} \) such that for all \( k \geq K, 1/C \geq \lambda_{\max}(M_k)^{\alpha} \). This follows directly by Property 4.

Let \( B(R) \subset \mathbb{R}^p \) denote the open ball around zero with radius \( R \geq 0 \). Let \( \overline{B(R)} \) denote the closure of said ball. The following lemma is a standard consequence of the fundamental theorem of calculus and the continuity assumptions on \( F \).

**Lemma 2.** Suppose Assumption 4 holds. Then, for any \( R > 0 \), \( \exists L_R > 0 \) such that \( \forall \theta, \varphi \in \overline{B(R)} \) and \( \forall \tilde{L} \geq L_R \),

\[
F(\theta) \leq F(\varphi) + \tilde{F}(\varphi)'(\theta - \varphi) + \frac{\tilde{L}}{1+\alpha} \|\theta - \varphi\|_2^{1+\alpha}
\]

(11)

Suppose Assumption 6 holds. Then, \( \forall \theta, \varphi \in \mathbb{R}^p, \exists L > 0, \forall \tilde{L} \geq L, \)

\[
F(\theta) \leq F(\varphi) + \tilde{F}(\varphi)'(\theta - \varphi) + \frac{\tilde{L}}{1+\alpha} \|\theta - \varphi\|_2^{1+\alpha}.
\]

(12)

**Proof.** By fundamental theorem of calculus and Assumption 4, \( \exists L_R > 0 \) such that \( \forall \theta, \varphi \in \overline{B(R)} \),

\[
F(\theta) = F(\varphi) + \tilde{F}(\varphi)'(\theta - \varphi) + \int_0^1 \tilde{F}(\varphi + t(\theta - \varphi)) - \tilde{F}(\varphi) \|\theta - \varphi\|_2 dt
\]

(13)

\[
\leq F(\varphi) + \tilde{F}(\varphi)'(\theta - \varphi) + \int_0^1 \|\tilde{F}(\varphi + t(\theta - \varphi)) - \tilde{F}(\varphi)\|_2 \|\theta - \varphi\|_2 dt
\]

(14)

\[
\leq F(\varphi) + \tilde{F}(\varphi)'(\theta - \varphi) + L_R \|\theta - \varphi\|_2^{1+\alpha} \int_0^1 t^\alpha dt.
\]

(15)

Computing the integral gives the first result. The case for Assumption 6 is proved nearly identically.

The following lemma allows us to relate smaller moments of the norm of the stochastic gradients, \( \|f(\theta, X)\|_2 \), to the second moment.

**Lemma 3.** Let \( \alpha \in (0, 1] \). Let \( \mathcal{F} \) be a \( \sigma \)-algebra. Then, for all \( \theta \in \mathbb{R}^p \),

\[
\mathbb{E} \left[ \left\| f(\theta, X) \right\|_2^{1+\alpha} \middle| \mathcal{F} \right] \leq \mathbb{E} \left[ \left\| f(\theta, X) \right\|_2^2 \middle| \mathcal{F} \right] \left( \frac{1+\alpha}{2} \right) \mathbb{E} \left[ \left\| f(\theta, X) \right\|_2^2 \middle| \mathcal{F} \right] + \frac{1-\alpha}{2}
\]

(16)

**Proof.** If \( \alpha = 1 \), the result holds. Suppose \( \alpha \in (0, 1) \). Then \( 1 + \alpha < 2 \) and Hölder’s inequality implies

\[
\mathbb{E} \left[ \left\| f(\theta, X) \right\|_2^{1+\alpha} \middle| \mathcal{F} \right] \leq \mathbb{E} \left[ \left\| f(\theta, X) \right\|_2^{1+\alpha} \middle| \mathcal{F} \right] \left( \frac{1+\alpha}{2} \right) \mathbb{E} \left[ \left\| f(\theta, X) \right\|_2^2 \middle| \mathcal{F} \right] + \frac{1-\alpha}{2}
\]

(17)

Now, we recall Young’s inequality: \( uv = \frac{p^2}{q} + \frac{q}{p} \) where \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( u = \mathbb{E}[\|f(\theta, X)\|_2^2|\mathcal{F}]^{\frac{1}{1+\alpha}}, v = 1, p = \frac{2}{1+\alpha} \) and \( q = \frac{2}{1-\alpha} \). Then, Young’s inequality completes the result.
B Analysis of the Local Hölder Continuity and General Noise Model Case

We will begin with a proof of Theorem 1. Then, we split the proof of Theorem 2 into three pieces. The first piece deals with specifying $A_1$ and $A_2$, and showing the sum of their probabilities is one. The second piece analyzes the objective function behavior of the iterates on $A_2$. The third piece analyzes the gradient function behavior of the iterates on $A_2$.

B.1 The Capture Theorem

Recall that Theorem 1 states the following. Let $\bar{\theta} \in \mathbb{R}^p$ be arbitrary. Let $\{\theta_k\}$ be defined as in (5) and satisfy Properties 1 and 2. If Assumption 3 holds, then for any $\bar{R} \geq 0$,

$$\mathbb{P} \left[ \left\| \theta_{k+1} - \bar{\theta} \right\|_2 > R, \left\| \theta_k - \bar{\theta} \right\|_2 \leq R \text{ i.o.} \right] = 0. \quad (18)$$

Proof of Theorem 1. Fix $R \geq 0$ and let $\epsilon > 0$. Then,

$$\mathbb{P} \left[ \left\| \theta_{k+1} - \bar{\theta} \right\|_2 \geq R + \epsilon, \left\| \theta_k - \bar{\theta} \right\|_2 \leq R \right] \quad (19)$$

$$= \mathbb{P} \left[ \left\{ \left\| \theta_{k+1} - \bar{\theta} \right\|_2 - \left\| \theta_k - \bar{\theta} \right\|_2 \right\} + \left\{ \left\| \theta_k - \bar{\theta} \right\|_2 \leq R \right\} \geq R + \epsilon \right] \quad (20)$$

$$\leq \mathbb{P} \left[ \left\{ \left\| \theta_{k+1} - \bar{\theta} \right\|_2 - \left\| \theta_k - \bar{\theta} \right\|_2 \right\} \left\{ \left\| \theta_k - \bar{\theta} \right\|_2 \leq R \right\} \geq R + \epsilon \right] \quad (21)$$

$$\leq \mathbb{P} \left[ \left\{ \left\| \theta_{k+1} - \theta_k \right\|_2 \leq \bar{R} \right\} \left\{ \left\| \theta_k - \bar{\theta} \right\|_2 \leq \bar{R} \right\} \geq \epsilon \right] \quad (22)$$

$$\leq \mathbb{P} \left[ \left\{ M_k \hat{f}(\theta_k, X_{k+1}) \right\} \left\{ \left\| \theta_k - \bar{\theta} \right\|_2 \leq \bar{R} \right\} \geq \epsilon \right] \quad (23)$$

$$\leq \frac{1}{\epsilon^2} \| M_k \|_2^2 \mathbb{E} \left[ \left\| \hat{f}(\theta_k, X_{k+1}) \right\|_2^2 \mathcal{F}_k \right] \left\{ \left\| \theta_k - \bar{\theta} \right\|_2 \leq \bar{R} \right\} \quad (24)$$

$$\leq \frac{1}{\epsilon^2} \| M_k \|_2^2 \mathbb{E} \left[ G(\theta_k) \left\{ \left\| \theta_k - \bar{\theta} \right\|_2 \leq \bar{R} \right\} \right] \quad (25)$$

$$\leq \frac{1}{\epsilon^2} \| M_k \|_2^2 G_R, \quad (27)$$

where $G_R = \sup_{\theta : \| \theta - \bar{\theta} \|_2 \leq R} G(\theta) < \infty$ since $G$ is upper semi-continuous. By Property 2, we see that the sum of the probabilities is finite. Together with the Borel-Cantelli lemma, $\mathbb{P} \left[ \left\{ \left\| \theta_{k+1} - \bar{\theta} \right\|_2 \geq R + \epsilon \right\} \left\{ \left\| \theta_k - \bar{\theta} \right\|_2 \leq R \text{ i.o.} \right\} \right] = 0$. Since $\epsilon > 0$ is arbitrary, we can show that this statement holds for a countable sequence of $\epsilon_n \downarrow 0$. As the union of countably many measure zero sets has measure zero, the conclusion of the result holds. \hfill \square

B.2 Global Consequences of the Capture Theorem

We begin with a direct consequence of Theorem 1, which addresses the first component of Theorem 2.

Corollary 1. Suppose the setting of Theorem 1 holds. Let $A_1 = \{ \liminf_{k \to \infty} \| \theta_k \|_2 = \infty \}$ and $A_2 = \{ \lim_{k \to \infty} \| \theta_k \|_2 < \infty \}$. Then, $\mathbb{P} [A_1] + \mathbb{P} [A_2] = 1$.

Proof of Corollary 1. For any $R \geq 0$, define the event

$$A(R) = \left\{ \liminf_{k \to \infty} \| \theta_k \|_2 \leq R \right\} \cap \left\{ \limsup_{k \to \infty} \| \theta_k \|_2 > R \right\}. \quad (28)$$

Then $\{ \liminf_{k \to \infty} \| \theta_k \|_2 < \limsup_{k \to \infty} \| \theta_k \|_2 \} \subset \cup_{R \geq 0} A(R)$, where $Q_{\geq 0}$ is the set of non-negative rational numbers. By Theorem 1, $\mathbb{P} [A(R)] = 0$ for all $R \geq 0$. Since the countable union of measure zero sets has measure zero, $\mathbb{P} [\liminf_{k \to \infty} \| \theta_k \|_2 < \limsup_{k \to \infty} \| \theta_k \|_2] = 0$. Hence, we conclude that, with probability one, $\liminf_{k \to \infty} \| \theta_k \|_2$ is either finite and equal to $\limsup_{k \to \infty} \| \theta_k \|_2$ (i.e., $A_2$) or is infinite (i.e., $A_1$). Since $A_1$ and $A_2$ are mutually exclusive, the result follows. \hfill \square
Importantly, Corollary 1 says that the possibility that the limit supremum and limit infimum of \( \{\|\theta_k\|_2\} \) being distinct occurs with probability zero (cf., a simple random walk with a positive one bias at each step, which will have its limit supremum as infinity and limit infimum as zero with probability one). Thus, Corollary 1 provides us with two cases that we can study: (with probability one) \( \{\|\theta_k\|_2\} \) diverges or converges to a finite value. The remaining result explore what happens in the finite case.

### B.3 Asymptotic Behavior of the Objective Function

The following result applies Lemmas 2 and 3 under Assumptions 1 to 4 to produce a recursive relationship between the objective function evaluated at two sequential iterates.

**Lemma 4.** Let \( \{M_k\} \) be defined as in (5) satisfying Property 1. For all \( k + 1 \in \mathbb{N} \) and \( R \geq 0 \), let \( B_k(R) = \bigcap_{i=0}^{k} \{\|\theta_i\|_2 \leq R\} \). Suppose Assumptions 1 to 4 hold. Then, \( \forall R \geq 0 \), \( \exists L_{R+1} > 0 \), such that

\[
\mathbb{E} \left[ \left[ F(\theta_{k+1}) - F_{l,b.} \right] 1 \left[ B_{k+1}(R) \right] \right] \leq \left[ F(\theta_k) - F_{l,b.} \right] 1 \left[ B_k(R) \right] - \lambda_{\min}(M_k) \left\| F(\theta_k) \right\|_2^2 1 \left[ B_k(R) \right] + \frac{L_{R+1}}{1 + \alpha} \lambda_{\max}(M_k)^{1+\alpha} G_R^{\frac{1}{1+\alpha}},
\]

where \( G_R = \sup_{\theta \in \overline{B(R)}} \theta \) occurs, then \( G(\theta) < \infty \) with \( G(\theta) \) defined in Assumption 3; and \( \partial F_R = \sup_{\theta \in \overline{B(R)}} \left| \dot{F}(\theta) \right|_2 (1 + \alpha) < \infty \).

**Proof.** Fix \( R \geq 0 \). For any \( k + 1 \in \mathbb{N} \), Lemma 2 implies that \( \exists L_{R+1} > 0 \) such that

\[
\mathbb{E} \left[ \left[ F(\theta_{k+1}) - F_{l,b.} \right] 1 \left[ B_{k+1}(R + 1) \right] \right] \leq \left[ F(\theta_k) - F_{l,b.} \right] 1 \left[ B_k\left(R + 1\right) \right] + \left( \left[ F(\theta_k) - F_{l,b.} \right] 1 \left[ B_k\left(R \right) \right] + \frac{L_{R+1}}{1 + \alpha} \left\| \theta_{k+1} - \theta_k \right\|_2^{1+\alpha} \right) 1 \left[ B_{k+1}(R + 1) \right].
\]

Now, since \( \overline{B_k(R)} \subseteq \overline{B(R+1)} \), it also holds true that

\[
\left[ F(\theta_{k+1}) - F_{l,b.} \right] 1 \left[ B_{k+1}(R) \right] \leq \left[ F(\theta_k) - F_{l,b.} \right] 1 \left[ B_k\left(R \right) \right] + \left( \left[ F(\theta_k) - F_{l,b.} \right] 1 \left[ B_k\left(R \right) \right] + \frac{L_{R+1}}{1 + \alpha} \left\| \theta_{k+1} - \theta_k \right\|_2^{1+\alpha} \right) 1 \left[ B_{k+1}(R) \right].
\]

Our goal now is to replace \( B_{k+1}(R) \) on the right hand side by \( B_k(R) \). However, there is a technical difficulty which we must address. First, it follows from the preceding inequality that

\[
\left[ F(\theta_{k+1}) - F_{l,b.} \right] 1 \left[ B_{k+1}(R) \right] \leq \left( \left[ F(\theta_k) - F_{l,b.} \right] 1 \left[ B_k\left(R \right) \right] + \frac{L_{R+1}}{1 + \alpha} \left\| \theta_{k+1} - \theta_k \right\|_2^{1+\alpha} \right) \left( 1 \left[ B_{k+1}(R) \right] - 1 \left[ B_k(R) \right] \right) + \left[ F(\theta_k) - F_{l,b.} \right] 1 \left[ B_k\left(R \right) \right] + \frac{L_{R+1}}{1 + \alpha} \left\| \theta_{k+1} - \theta_k \right\|_2^{1+\alpha} 1 \left[ B_k(R) \right].
\]

The first term on the right hand side of the inequality only contributes meaningfully if it is positive. Since \( 1 \left[ B_k(R) \right] \geq 1 \left[ B_{k+1}(R) \right] \), then two statements hold: (i) \( 1 \left[ B_k(R) \right] 1 \left[ B_{k+1}(R) \right] = 1 \left[ B_{k+1}(R) \right]; \) and (ii) the first term of the right hand side of (32) is positive if and only if

\[
\left( \left[ F(\theta_k) - F_{l,b.} \right] 1 \left[ B_k\left(R \right) \right] + \frac{L_{R+1}}{1 + \alpha} \left\| \theta_{k+1} - \theta_k \right\|_2^{1+\alpha} \right) 1 \left[ B_k(R) \right] < 0.
\]

By the choice of \( L_{R+1} \), Assumption 1 and Lemma 2 imply that if (33) occurs, then \( \left\| \theta_{k+1} \right\|_2 > R + 1 \geq \left\| \theta_k \right\|_2 + 1 \). By the reverse triangle inequality and (5), if (33) occurs, then \( \left\| M_k \dot{F}(\theta_k, X_{k+1}) \right\|_2 \geq \frac{L_{R+1}}{1 + \alpha} \left\| \theta_{k+1} - \theta_k \right\|_2^{1+\alpha} \).
1. Hence,
\[
\left( [F(\theta_k) - F_{l,b.}] + \hat{F}(\theta_k)'(\theta_{k+1} - \theta_k) + \frac{L_{R+1}}{1 + \alpha} \|\theta_{k+1} - \theta_k\|_2^{1+\alpha} \right) \\
\times \left( 1 [B_{k+1}(R)] - 1 [B_k(R)] \right)
\leq \left( -[F(\theta_k) - F_{l,b.}] - \hat{F}(\theta_k)'(\theta_{k+1} - \theta_k) - \frac{L_{R+1}}{1 + \alpha} \|\theta_{k+1} - \theta_k\|_2^{1+\alpha} \right) \\
\times \left( 1 [B_k(R)] - 1 [B_{k+1}(R)] \right) 1 [B_k(R)] 1 \left( \|M_k \hat{f}(\theta_k, X_{k+1})\|_2 \geq 1 \right) \tag{34}
\]

We now compute another coarse upper bound for this inequality. Note, by Assumption 1 and Cauchy-Schwarz,
\[
\left( -[F(\theta_k) - F_{l,b.}] - \hat{F}(\theta_k)'(\theta_{k+1} - \theta_k) - \frac{L_{R+1}}{1 + \alpha} \|\theta_{k+1} - \theta_k\|_2^{1+\alpha} \right) \\
\times \left( 1 [B_k(R)] - 1 [B_{k+1}(R)] \right) 1 [B_k(R)] 1 \left( \|M_k \hat{f}(\theta_k, X_{k+1})\|_2 \geq 1 \right) \tag{35}
\]
\[
\leq \left\| \hat{F}(\theta_k) \right\|_2 \left\| M_k \hat{f}(\theta_k, X_{k+1}) \right\|_2^{1+\alpha} 1 [B_k(R)] \tag{36}
\]
\[
\leq \left\| \hat{F}(\theta_k) \right\|_2 \left\| M_k \hat{f}(\theta_k, X_{k+1}) \right\|_2^{1+\alpha} 1 [B_k(R)] \tag{37}
\]
\[
\leq \frac{\partial F_R}{1 + \alpha} \left\| M_k \hat{f}(\theta_k, X_{k+1}) \right\|_2^{1+\alpha} 1 [B_k(R)] \tag{38}
\]
where \( \partial F_R = \sup_{\theta \in \overline{B(R)}} \| \hat{F}(\theta) \|_2 (1 + \alpha) < \infty \) given that \( \| \hat{F}(\theta) \|_2 \) is a continuous function of \( \theta \).

Applying this inequality to (32), we conclude
\[
\left( [F(\theta_{k+1}) - F_{l,b.}] 1 [B_{k+1}(R)] \right) \\
\leq \left( [F(\theta_k) - F_{l,b.}] - \hat{F}(\theta_k)' M_k \hat{f}(\theta_k, X_{k+1}) + \frac{L_{R+1}}{1 + \alpha} \| M_k \hat{f}(\theta_k, X_{k+1}) \|_2^{1+\alpha} \right) \\
\times 1 [B_k(R)] \cdot \tag{39}
\]

By Assumption 2,
\[
E \left[ [F(\theta_{k+1}) - F_{l,b.}] 1 [B_{k+1}(R)] \left| F_k \right| \right] \\
\leq \left( \left( [F(\theta_k) - F_{l,b.}] - \hat{F}(\theta_k)' M_k \hat{f}(\theta_k, X_{k+1}) + \frac{L_{R+1}}{1 + \alpha} \| M_k \hat{f}(\theta_k, X_{k+1}) \|_2^{1+\alpha} \right) \right) \right) \tag{40}
\]

Using Property 1, Assumption 3 and Lemma 3,
\[
E \left( \left[ [F(\theta_{k+1}) - F_{l,b.}] - \hat{F}(\theta_k)' M_k \hat{f}(\theta_k) \right]_2^2 + \frac{L_{R+1}}{1 + \alpha} \left\| M_k \hat{f}(\theta_k, X_{k+1}) \right\|_2^{1+\alpha} \left| F_k \right| \right) \tag{41}
\]

By Assumption 3, \( G \) is upper semicontinuous and \( \overline{B(R)} \) is compact, which implies that \( G_R \) is well defined and finite. The result follows.

The following corollary to Lemma 4 proves the convergence of the objective function component for Theorem 2.

**Corollary 2.** Let \( \{\theta_k\} \) be defined as in (5) satisfying Properties 1 and 2. Suppose Assumptions 1 to 4 hold. Then, there exists a finite random variable \( F_{\lim} \) such that on the event \( \{ \sup_k \| \theta_k \|_2 < \infty \} \),
\[
\lim_{k \to \infty} F(\theta_k) = F_{\lim} \text{ with probability one.}
\]
Proof. By Lemma 4, for every $R \geq 0$,
\[
E \left[ |F(\theta_{k+1}) - F_{t,b,}] | B_{k+1}(R) | F_k \right]
\leq |F(\theta_k) - F_{t,b,}] | B_k(R) | + \frac{(L_{R+1} + \partial F_R)G_{\Delta}^R}{1 + \alpha} \lambda_{\max}(M_k)^{1+\alpha}.
\] (42)

By Neveu and Speed [1975, Exercise II.4] (cf. Robbins and Siegmund [1971]) and Property 2, \( \lim_{k \to \infty} |F(\theta_k) - F_{t,b,}] | B_k(R) | \) converges to a finite random variable with probability one. Since \( F_{t,b,] \) is a constant and \( R \geq 0 \) is arbitrary, we conclude that there exists a finite random variable \( F_{\lim} \) such that \( \{\sup_k \|\theta_k\|_2 \leq R\} \subset \{\lim_k F(\theta_k) = F_{\lim}\} \) up to a measure zero. Since the countable union of measure zero sets has measure zero,
\[
\{\sup_k \|\theta_k\|_2 < \infty\} = \bigcup_{R \in \mathbb{N}} \left\{\sup_k \|\theta_k\|_2 \leq R\right\} \subset \{\lim_{k \to \infty} F(\theta_k) = F_{\lim}\},
\] (43)
up to a measure zero set. The result follows. \( \square \)

B.4 Asymptotic Behavior of the Gradient

We now prove that the gradient norm evaluated at SGD’s iterates must, repeatedly, get arbitrarily close to zero. We adapt the strategy of Patel [2020].

Lemma 5. Let \( \{\theta_k\} \) be defined as in (5) satisfying Properties 1 to 3. For all \( k + 1 \in \mathbb{N} \) and \( R \geq 0 \), let \( B_k(R) = \bigcap_{j=0}^{\infty} \{\|\theta_k\|_2 \leq R\} \). Suppose Assumptions 1 to 4 hold. Then, \( \forall R \geq 0 \) and for all \( \delta > 0 \),
\[
\mathbb{P} \left[ \left\| \hat{F}(\theta_k) \right\|_2^2 \mathbb{1} [B_k(R)] \leq \delta, \ i.o. \right| F_0 \right] = 1, \ w.p.1.
\] (44)

Proof. By Lemma 4,
\[
\lambda_{\min}(M_k) E \left[ \left\| \hat{F}(\theta_k) \right\|_2^2 \mathbb{1} [B_k(R)] \right] \leq E \left[ |F(\theta_k) - F_{t,b,}] | B_k(R) | \right]
\] (45)
\[
- E \left[ |F(\theta_{k+1}) - F_{t,b,}] | B_{k+1}(R) | \right] + \frac{(L_{R+1} + \partial F_R)G_{\Delta}^R}{1 + \alpha} \lambda_{\max}(M_k)^{1+\alpha}.
\]

Taking the sum of this equation for all \( k \) from 0 to \( j \in \mathbb{N} \), we have
\[
\sum_{k=0}^{j} \lambda_{\min}(M_k) E \left[ \left\| \hat{F}(\theta_k) \right\|_2^2 \mathbb{1} [B_k(R)] \right| F_0 \right] \leq |F(\theta_0) - F_{t,b,}] | B_0(R) | \] (46)
\[
- E \left[ |F(\theta_{j+1}) - F_{t,b,}] | B_{j+1}(R) | \right| F_0 \right] + \frac{(L_{R+1} + \partial F_R)G_{\Delta}^R}{1 + \alpha} \sum_{k=0}^{j} \lambda_{\max}(M_k)^{1+\alpha}.
\]

By Assumption 1 and Property 2, the right hand side is bounded by
\[
|F(\theta_0) - F_{t,b,}] | B_0(R) | + \frac{(L_{R+1} + \partial F_R)G_{\Delta}^R}{1 + \alpha} S,
\] (47)
which is finite with probability one. Therefore, \( \sum_{k=0}^{\infty} \lambda_{\min}(M_k) E[\|\hat{F}(\theta_k)\|_2^2 \mathbb{1} [B_k(R)] | F_0] \) is finite almost surely. Furthermore, by Property 3, \( \lim \inf_k E[\|\hat{F}(\theta_k)\|_2^2 \mathbb{1} [B_k(R)] | F_0] = 0 \) with probability one.

Now, for any \( \delta > 0 \), Markov’s inequality implies that for all \( j + 1 \in \mathbb{N} \),
\[
\mathbb{P} \left[ \bigcap_{k=j}^{\infty} \left\{ \left\| \hat{F}(\theta_k) \right\|_2^2 \mathbb{1} [B_k(R)] > \delta \right\} \left| F_0 \right] \right] \leq \frac{1}{\delta} \min_{j \leq k} \mathbb{E} \left[ \left\| \hat{F}(\theta_k) \right\|_2^2 \mathbb{1} [B_k(R)] \right| F_0 \right],
\] (48)
where the right hand side is zero with probability one because \( \lim \inf_k E[\|\hat{F}(\theta_k)\|_2^2 \mathbb{1} [B_k(R)] | F_0] = 0 \) with probability one.
As the countable union of measure zero sets has measure zero, we conclude that for all \( \delta > 0 \),
\[
P \left( \left\| \hat{F}(\theta_k) \right\|_2^2 \mathbf{1} [B_k(R)] \leq \delta, \ i.o. \biggm| F_0 \right) = 1,
\]
with probability one.

Unfortunately, Lemma 5 does not guarantee that the gradient norm will be captured within a region of zero. In order to prove this, we first show that it is not possible (i.e., a zero probability event) for the limit supremum and limit infimum of the gradients to be distinct (cf., Theorem 1 for iterate distances).

**Lemma 6.** Let \( \{\theta_k\} \) be defined as in (5) satisfying Properties 1 and 2. For all \( k+1 \in \mathbb{N} \) and \( R \geq 0 \), let \( B_k(R) = \bigcap_{j=0}^{\infty} \{ \|\theta_k\|_2 \leq R \} \). Suppose Assumptions 1 to 4 hold. Then, \( \forall R \geq 0 \) and for all \( \delta > 0 \),
\[
P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 \mathbf{1} [B_{k+1}(R)] > \delta, \left\| \hat{F}(\theta_k) \right\|_2 \mathbf{1} [B_k(R)] \leq \delta, \ i.o. \biggm| F_0 \right) = 0,
\]
with probability one.

**Proof.** Let \( \epsilon > 0, L_R > 0 \), and \( G_R \) be defined as in Lemma 4. Then, for \( \delta > 0 \),
\[
P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 \mathbf{1} [B_{k+1}(R)] \mathbf{1} [B_k(R)] \leq \delta \biggm| F_0 \right) = 0,
\]
with probability one. Since this holds for any \( \epsilon > 0 \), it will hold for every value in a sequence \( \epsilon_n \downarrow 0 \).

By Property 2, the sum of the last expression over all \( k+1 \in \mathbb{N} \) is finite. By the Borel-Cantelli lemma, for all \( R \geq 0, \delta > 0 \) and \( \epsilon > 0 \),
\[
P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 \mathbf{1} [B_{k+1}(R)] > \delta + L_R \epsilon^a, \left\| \hat{F}(\theta_k) \right\|_2 \mathbf{1} [B_k(R)] \leq \delta, \ i.o. \biggm| F_0 \right) = 0,
\]
with probability one. Since the sum of this holds for any \( \epsilon > 0 \), it will hold for every value in a sequence \( \epsilon_n \downarrow 0 \).

We now put together Lemmas 5 and 6 to show that, on the event \( \{\sup_k \|\theta_k\|_2 < \infty\} \), \( \|\hat{F}(\theta_k)\|_2 \to 0 \) with probability one.

**Corollary 3.** Let \( \{\theta_k\} \) be defined as in (5) satisfying Properties 1 to 3. Suppose Assumptions 1 to 4 hold. Then, on the event \( \{\sup_k \|\theta_k\|_2 < \infty\} \), \( \lim_{k \to \infty} \|\hat{F}(\theta_k)\|_2 = 0 \) with probability one.

**Proof.** For any \( R \geq 0 \) and \( \delta > 0 \), Lemma 5 implies
\[
P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 \mathbf{1} [B_{k+1}(R)] > \delta, \ i.o. \biggm| F_0 \right) = 0,
\]
with probability one.
with probability one. We see that this latter event is exactly,

\[ \mathbb{P} \left[ \left\| \hat{F}(\theta_{k+1}) \right\|_2 1_{[B_{k+1}(R)]} > \delta, \left\| \hat{F}(\theta_k) \right\|_2 1_{[B_k(R)]} \leq \delta, \ i.o. \mid \mathcal{F}_0 \right] , \]  

which, by Lemma 6, is zero with probability one. Therefore,

\[ \mathbb{P} \left[ \left\| \hat{F}(\theta_{k+1}) \right\|_2 1_{[B_{k+1}(R)]} > \delta, \ i.o. \mid \mathcal{F}_0 \right] \]  
is zero with probability one. Letting \( \delta_n \downarrow 0 \) and noting that the countable union of measure zero sets has measure zero, we conclude

\[ \mathbb{P} \left[ \left\| \hat{F}(\theta_{k+1}) \right\|_2 1_{[B_{k+1}(R)]} > 0, \ i.o. \mid \mathcal{F}_0 \right] = 0 \]  
with probability one.

Therefore, for all \( R \geq 0 \), \( \{ \sup_k \| \theta_k \|_2 \leq R \} \subset \{ \lim_{k \to \infty} \| \hat{F}(\theta_k) \|_2 = 0 \} \) up to a measure zero set. Since \( \{ \sup_k \| \theta_k \|_2 < \infty \} = \cup_{R \in \mathbb{N}} \{ \sup_k \| \theta_k \|_2 \leq R \} \), the result follows.

\[ \Box \]

C Analysis of the Global Hölder Continuity and Expected Smoothness Case

We will divide the proof into four pieces. In Appendix C.1, we will begin by proving that \( \{ F(\theta_k) \} \) converges to an integrable random variable with probability one, which follows the same strategy used for Theorem 2. In Appendix C.2, we will then prove that \( \{ \mathbb{E}[F(\theta_k)] \} \) are bounded, which is an alternative way to imply that \( F_{\text{lim}} \) is integrable via Fatou’s lemma and which implies the \( L^1 \) convergence of \( F(\theta_k)^\gamma \) to \( F_{\text{lim}}^\gamma \) for \( \gamma \in [0, 1) \) by Hölder’s inequality and uniform integrability. In Appendix C.3, we will prove that \( \{ \| \hat{F}(\theta_k) \|_2 \} \) converges to zero with probability one. Finally, in Appendix C.4, we will prove that \( \sup_k \mathbb{E}[\| \hat{F}(\theta_k) \|_2^2] < \infty \), from which we can conclude that \( \mathbb{E}[\| \hat{F}(\theta_k) \|_2] \to 0 \) as \( k \to \infty \).

C.1 Asymptotic Behavior of the Objective Function

We begin with an analogue of Lemma 4 that allows us to use the global Hölder assumption to remove the indicator function that burdened Lemma 4

**Lemma 7.** Let \( \{ \theta_k \} \) be defined as in (5) satisfying Property 1. Suppose Assumptions 1, 2, 5 and 6 hold. Then,

\[ \mathbb{E} \left[ F(\theta_{k+1}) - F_{l.b.} \mid \mathcal{F}_k \right] \leq \left[ F(\theta_k) - F_{l.b.} \right] \left( 1 + \frac{LC_2}{2} \lambda_{\text{max}}(M_k)^{1+\alpha} \right) \]

- \( \left\| \hat{F}(\theta_k) \right\|_2 \left( \lambda_{\text{min}}(M_k) - \frac{LC_2}{2} \lambda_{\text{max}}(M_k)^{1+\alpha} \right) \]

+ \( \frac{L}{1+\alpha} \lambda_{\text{max}}(M_k)^{1+\alpha} \left( \frac{1+\alpha}{2} C_1 + \frac{1-\alpha}{2} \right) \)

If, in addition, Property 4 holds then \( \exists K \in \mathbb{N} \) such that for all \( k \geq K \),

\[ \mathbb{E} \left[ F(\theta_{k+1}) - F_{l.b.} \mid \mathcal{F}_k \right] \leq \left[ F(\theta_k) - F_{l.b.} \right] \left( 1 + \frac{LC_2}{2} \lambda_{\text{max}}(M_k)^{1+\alpha} \right) \]

- \( \frac{1}{2} \lambda_{\text{min}}(M_k) \left\| \hat{F}(\theta_k) \right\|_2^2 + \frac{L}{1+\alpha} \lambda_{\text{max}}(M_k)^{1+\alpha} \left( \frac{1+\alpha}{2} C_1 + \frac{1-\alpha}{2} \right) \)

**Proof.** By Lemma 2 and (5),

\[ F(\theta_{k+1}) - F_{l.b.} \leq F(\theta_k) - F_{l.b.} - \hat{F}(\theta_k)' M_k \hat{f}(\theta_k, X_{k+1}) + \frac{L}{1+\alpha} \left\| M_k \hat{f}(\theta_k, X_{k+1}) \right\|_2^{1+\alpha} \]

Now, taking conditional expectations, applying Assumptions 2 and 5 and Lemma 3,

\[ \mathbb{E} \left[ F(\theta_{k+1}) - F_{l.b.} \mid \mathcal{F}_k \right] \]

\[ \leq F(\theta_k) - F_{l.b.} - \hat{F}(\theta_k)' M_k \hat{f}(\theta_k) \]

+ \( \frac{L}{1+\alpha} \lambda_{\text{max}}(M_k)^{1+\alpha} \left[ \frac{1+\alpha}{2} \right] \left( C_1 + C_2 (F(\theta_k) - F_{l.b.}) + C_3 \left\| \hat{F}(\theta_k) \right\|_2^{1+\alpha} \right] \)
Finally, using Property 1 to show \( -\dot{F}(\theta_k) M_k \dot{F}(\theta_k) \leq -\lambda_{\min}(M_k) \|\dot{F}(\theta_k)\|_2^2 \) and rearranging the terms, the first part of the result follows.

By Lemma 1, there exists \( K \in \mathbb{N} \) such that for all \( k \geq K \),

\[
\lambda_{\min}(M_k) - \frac{LC_3}{2} \lambda_{\max}(M_k)^{1+\alpha} \geq \frac{1}{2} \lambda_{\min}(M_k). \tag{67}
\]

The result follows.

Corollary 4. Let \( \{\theta_k\} \) be defined as in (5) satisfying Properties 1, 2 and 4. Suppose Assumptions 1, 2, 5 and 6 hold. Then, there exists an integrable random variable \( F_{\lim} \) such that \( \lim_{k \to \infty} F(\theta_k) = F_{\lim} \) with probability one.

Proof. Lemma 7 implies \( \exists K \in \mathbb{N} \) such that \( k \geq K \),

\[
\mathbb{E}[F(\theta_{k+1}) - F_{l.b.}|F_k] \leq [F(\theta_k) - F_{l.b.}] \left( 1 + \frac{LC_2}{2} \lambda_{\max}(M_k)^{1+\alpha} \right) + \frac{L}{1+\alpha} \lambda_{\max}(M_k)^{1+\alpha} \left( \frac{1+\alpha}{2} C_1 + \frac{1-\alpha}{2} \right). \tag{68}
\]

By Neveu and Speed [1975, Exercise II.4] (cf. Robbins and Siegmund [1971]) and Property 2, \( \lim_{k \to \infty} [F(\theta_k) - F_{l.b.}] \) converges to an integrable random variable with probability one. The result follows.

C.2 Asymptotic Behavior of the Expected Objective Function

We now follow Lei et al. [2019] to prove that expected value of the objective function evaluated at the iterates remains bounded. This relies on the following recursive relationship.

Lemma 8. Let \( \{\theta_k\} \) be defined as in (5) satisfying Properties 1, 2 and 4. Suppose Assumptions 1, 2, 5 and 6 hold. There exists a \( K \in \mathbb{N} \) such that for all \( k \geq K \),

\[
\mathbb{E}[F(\theta_{k+1}) - F_{l.b.}|F_k] \leq [F(\theta_k) - F_{l.b.}] \left( 1 + \frac{LC_2}{2} \lambda_{\max}(M_k)^{1+\alpha} \right) + \frac{L}{1+\alpha} \lambda_{\max}(M_k)^{1+\alpha} \left( \frac{1+\alpha}{2} C_1 + \frac{1-\alpha}{2} \right) \sum_{j=k+1}^{\infty} \lambda_{\max}(M_j)^{1+\alpha}. \tag{69}
\]

Proof. Lemma 7 implies \( \exists K \in \mathbb{N} \) such that \( k \geq K \),

\[
\mathbb{E}[F(\theta_{k+1}) - F_{l.b.}|F_k] \leq [F(\theta_k) - F_{l.b.}] \left( 1 + \frac{LC_2}{2} \lambda_{\max}(M_k)^{1+\alpha} \right) + \frac{L}{1+\alpha} \lambda_{\max}(M_k)^{1+\alpha} \left( \frac{1+\alpha}{2} C_1 + \frac{1-\alpha}{2} \right) \left( 1 + \frac{LC_2}{2} \lambda_{\max}(M_k)^{1+\alpha} \right) \sum_{j=k}^{\infty} \lambda_{\max}(M_j)^{1+\alpha}. \tag{70}
\]

Since \( 1 + x \leq \exp(x) \) for \( x \geq 0 \),

\[
\mathbb{E}[F(\theta_{k+1}) - F_{l.b.}|F_k] \leq \exp \left( \frac{LC_2}{2} \lambda_{\max}(M_k)^{1+\alpha} \right) \times [F(\theta_k) - F_{l.b.}] \left( 1 + \frac{LC_2}{2} \lambda_{\max}(M_k)^{1+\alpha} \right). \tag{71}
\]

The result follows by Property 2, adding

\[
\left[ \frac{LC_1}{2} + \frac{L}{2} \left( \frac{1-\alpha}{1+\alpha} \right) \right] \sum_{j=k+1}^{\infty} \lambda_{\max}(M_j)^{1+\alpha} \tag{72}
\]

to both sides, and noting that \( \exp(x) \geq 1 \) for \( x \geq 0 \).
Corollary 5. Let \( \{ \theta_k \} \) be defined as in (5) satisfying Properties 1, 2 and 4. Suppose Assumptions 1, 2, 5 and 6 hold. Then, \( \sup_k \mathbb{E}[F(\theta_k) | \mathcal{F}_0] < \infty \) with probability one. Finally, for any \( \gamma \in [0, 1) \), \( \lim_{k \to \infty} \mathbb{E}[|F(\theta_k) - F_{l,b}|^\gamma - (F_{l_{\lim}} - F_{l,b})^\gamma| |\mathcal{F}_0| = 0 \) with probability one.

Proof. Applying Lemma 8 recursively,

\[
\mathbb{E}[F(\theta_{k+1}) - F_{l,b} | \mathcal{F}_k] + \left[ \frac{L C_1}{2} + \frac{L}{2} \left( \frac{1 - \alpha}{1 + \alpha} \right) \right] \sum_{j=k+1}^{\infty} \lambda_{\max}(M_j)^{1+\alpha} \\
\leq \exp \left( \frac{L C_2}{2} \sum_{j=K}^{k} \lambda_{\max}(M_j)^{1+\alpha} \right) \times \left[ F(\theta_K) - F_{l,b} + \left[ \frac{L C_1}{2} + \frac{L}{2} \left( \frac{1 - \alpha}{1 + \alpha} \right) \right] \sum_{j=K}^{\infty} \lambda_{\max}(M_j)^{1+\alpha} \right].
\]

(73)

By Property 2 and given that \( K \in \mathbb{N} \) is a constant,

\[
\mathbb{E}[F(\theta_{k+1}) - F_{l,b} | \mathcal{F}_0] + \left[ \frac{L C_1}{2} + \frac{L}{2} \left( \frac{1 - \alpha}{1 + \alpha} \right) \right] \sum_{j=k+1}^{\infty} \lambda_{\max}(M_j)^{1+\alpha} \\
\leq \exp \left( \frac{L C_2}{2} S \right) \mathbb{E}[F(\theta_K) - F_{l,b} | \mathcal{F}_0] + \left[ \frac{L C_1}{2} + \frac{L}{2} \left( \frac{1 - \alpha}{1 + \alpha} \right) \right] S,
\]

(74)

for which the right hand side is finite with probability one. Hence, \( \sup_k \mathbb{E}[F(\theta_k) | \mathcal{F}_0] < \infty \) with probability one. (Note, we can now apply Fatou’s lemma to prove \( \mathbb{E}[\bar{F}_{l\lim}] < \infty \), if it were not already provided for in Neveu and Speed [1975].)

For the final part of the proof, we note that \( \gamma = 0 \) is trivial. So, take \( \gamma \in (0, 1) \). Then, \( \{(F(\theta_k) - F_{l,b})^\gamma\} \) are bounded in \( L^{1/\gamma} \) (condition on \( \mathcal{F}_0 \)), as we have just shown. Thus, \( \{(F(\theta_k) - F_{l,b})^\gamma\} \) are uniformly integrable and, by Corollary 4, \( \{(F(\theta_k) - F_{l,b})^\gamma\} \) converges to \( (F_{l_{\lim}} - F_{l,b})^\gamma \) in \( L^1 \).

C.3 Asymptotic Behavior of the Gradient Function

Just as we did before, we now prove that the gradient norm evaluated at SGD’s iterates must, repeatedly, get arbitrarily close to zero. We use the strategy of Patel [2020].

Lemma 9. Let \( \{ \theta_k \} \) be defined as in (5) satisfying Properties 1 to 4. Suppose Assumptions 1, 2, 5 and 6 hold. Then, for all \( \delta > 0 \),

\[
\mathbb{P} \left[ \left\| \hat{F}(\theta_k) \right\|_2^2 \leq \delta, \ i.o., \mathcal{F}_0 \right] = 1, \ w.p.1.
\]

(75)

Proof. By Lemma 7, there exists \( K \in \mathbb{N} \) such that for all \( k \geq K \),

\[
\frac{1}{2} \lambda_{\min}(M_k) \left\| \hat{F}(\theta_k) \right\|_2^2 \leq |F(\theta_k) - F_{l,b}| - \mathbb{E}[F(\theta_{k+1}) - F_{l,b} | \mathcal{F}_k] \\
+ |F(\theta_k) - F_{l,b}| \frac{L C_2}{2} \lambda_{\max}(M_k)^{1+\alpha} + \frac{L}{1 + \alpha} \lambda_{\max}(M_k)^{1+\alpha} \left( \frac{1 + \alpha}{2} C_1 + \frac{1 - \alpha}{2} \right).
\]

(76)

Now, taking expectation with respect to \( \mathcal{F}_0 \) and applying Corollary 5 with \( M_0 := \sup_k \mathbb{E}[F(\theta_k) - F_{l,b} | \mathcal{F}_0] \),

\[
\frac{1}{2} \lambda_{\min}(M_k) \mathbb{E} \left[ \left\| \hat{F}(\theta_k) \right\|_2^2 \right] \leq \mathbb{E}[F(\theta_k) - F_{l,b} | \mathcal{F}_0] - \mathbb{E}[F(\theta_{k+1}) - F_{l,b} | \mathcal{F}_0] \\
+ M_0 \frac{L C_2}{2} \lambda_{\max}(M_k)^{1+\alpha} + \frac{L}{1 + \alpha} \lambda_{\max}(M_k)^{1+\alpha} \left( \frac{1 + \alpha}{2} C_1 + \frac{1 - \alpha}{2} \right).
\]

(77)
Summing from over all $k \geq K$, using Property 2, and Assumption 1,
\[ \frac{1}{2} \sum_{k=K}^{\infty} \lambda_{\min}(M_k) E \left[ \left\| \hat{F}(\theta_k) \right\|_2^2 \right] \leq M_0 + \frac{SL}{2} \left( M_0 C_2 + C_1 + \frac{1 - \alpha}{1 + \alpha} \right). \tag{78} \]
Therefore, by Property 3, we conclude that $\lim \inf_k E[\|\hat{F}(\theta_k)\|_2^2 | F_0] = 0$ with probability one. Now, for any $\delta > 0$, Markov’s inequality implies that for all $j \geq K$,
\[ P \left( \bigcap_{k=j}^{\infty} \left\{ \left\| \hat{F}(\theta_k) \right\|_2 > \delta \right\} \mid F_0 \right) \leq \frac{1}{\delta} \min_{j \leq k} E \left[ \left\| \hat{F}(\theta_k) \right\|_2^2 \mid F_0 \right] = 0, \tag{79} \]
with probability one. The countable union of measure zero sets has measure zero. Therefore, the conclusion follows. \[ \square \]

We now prove that the limit infimum and limit supremum of $\{\|\hat{F}(\theta_k)\|_2\}$ cannot be distinct.

**Lemma 10.** Let $\{\theta_k\}$ be defined as in (5) satisfying Properties 1, 2 and 4. Suppose Assumptions 1, 2, 5 and 6 hold. Then, for all $\delta > 0$,
\[ P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 > \delta, \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta, \text{i.o.} \mid F_0 \right) = 0, \tag{80} \]
with probability one.

**Proof.** Let $\epsilon > 0$. For $\delta > 0$,
\[ P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 > \delta \mid \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta \right) > \delta + L \epsilon, \tag{81} \]
\[ = P \left( \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 - \left\| \hat{F}(\theta_k) \right\|_2 \right) + \left\| \hat{F}(\theta_k) \right\|_2 \right) \leq \delta \mid \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta \right) > \delta + L \epsilon, \tag{82} \]
\[ \leq P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 - \left\| \hat{F}(\theta_k) \right\|_2 \right) \leq \delta \mid \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta \right) > \delta + L \epsilon, \tag{83} \]
\[ = P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 - \left\| \hat{F}(\theta_k) \right\|_2 \right) \leq \delta \mid \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta \right) > \epsilon, \tag{84} \]
\[ \leq \frac{1}{\epsilon^2} \left\| \hat{F}(\theta_{k+1}) \right\|_2^2 = \left\{ \left\| \hat{F}(\theta_k, X_{k+1}) \right\|_2 \right\} \left\{ \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta \right\} \mid F_0 \right) \tag{85} \]
\[ \leq \frac{1}{\epsilon^2} \left\| \hat{F}(\theta_{k+1}) \right\|_2^2 = \left\{ \left\| \hat{F}(\theta_k, X_{k+1}) \right\|_2 \right\} \left\{ \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta \right\} \mid F_0 \right), \tag{86} \]
where we make use of Assumption 5 in the last line. Moreover, by Corollary 5, we conclude
\[ P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 > \delta \mid \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta \right) > \delta + L \epsilon, \tag{87} \]
where $M_0 = \sup_k E[\|F(\theta_k) - F_{i,b}\|_2 | F_0]$ is finite. By Property 2, the sum of the last expression over all $k + 1 \in \mathbb{N}$ is finite. By the Borel-Cantelli lemma, for all $\delta > 0$ and $\epsilon > 0$,
\[ P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 > \delta + L \epsilon, \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta, \text{i.o.} \mid F_0 \right) = 0, \tag{88} \]
with probability one. Since this holds for any $\epsilon > 0$, it will hold for every value in a sequence $\epsilon_n \downarrow 0$. Since the countable union of measure zero events has measure zero,
\[ P \left( \left\| \hat{F}(\theta_{k+1}) \right\|_2 > \delta, \left\| \hat{F}(\theta_k) \right\|_2 \leq \delta, \text{i.o.} \mid F_0 \right) = 0, \tag{89} \]
with probability one. \[ \square \]

We now put the two preceding lemmas together to prove the result.
Corollary 6. Let $\{\theta_k\}$ be defined as in (5) satisfying Properties 1 to 4. Suppose Assumptions 1, 2, 5 and 6 hold. Then, $\lim_{k \to \infty} \| F(\theta_k) \|_2 = 0$ with probability one.

**Proof.** For any $\delta > 0$, Lemma 9 implies

$$
P \left[ \| F(\theta_{k+1}) \|_2 > \delta, \text{ i.o.} \mid \mathcal{F}_0 \right] = \mathbb{P} \left\{ \| F(\theta_{k+1}) \|_2 > \delta \right\} \cap \left\{ \| F(\theta_k) \|_2 \leq \delta, \text{ i.o.} \right\} \mathcal{F}_0],
$$

(90)

with probability one. We see that this latter event is exactly,

$$
P \left[ \| F(\theta_{k+1}) \|_2 > \delta, \| F(\theta_k) \|_2 \leq \delta, \text{ i.o.} \mid \mathcal{F}_0 \right],
$$

(91)

which, by Lemma 10, is zero with probability one. Therefore, $P \left[ \| F(\theta_{k+1}) \|_2 > \delta, \text{ i.o.} \mid \mathcal{F}_0 \right]$ is zero with probability one. Letting $\delta_n \downarrow 0$ and noting that the countable union of measure zero sets has measure zero, we conclude $P \left[ \| F(\theta_k) \|_2 > 0, \text{ i.o.} \mid \mathcal{F}_0 \right] = 0$ with probability one. In other words, $P \left[ \lim_{k \to \infty} \| F(\theta_k) \|_2 = 0 \mid \mathcal{F}_0 \right] = 1$ with probability one. \qed

**C.4 Asymptotic Behavior of the Expected Gradient Function**

We begin by proving that $\sup_k \mathbb{E}[\| F(\theta_k) \|_2^2 \mid \mathcal{F}_0]$ is finite with probability one. As a result, we will have that $\{ F(\theta_k) \}$ are uniformly integrable, which, with Corollary 6, implies $L^1$ convergence.

**Lemma 11.** Suppose Assumptions 1 and 6 hold. Then, for all $\phi \in \mathbb{R}^p$,

$$
\| F(\phi) \|_2^2 \leq \left( \frac{L \lambda (1 + \alpha)}{\alpha} | F(\phi) - F_{l.b.} | \right)^{\frac{2}{1 + \alpha}},
$$

(92)

where $2\alpha/(1 + \alpha) \leq 1$ for all $\alpha \in (0, 1]$.

Moreover, let $\{ \theta_k \}$ be defined as in (5) satisfying Properties 1, 2 and 4. Suppose Assumptions 1, 2, 5 and 6 hold. Then, $\sup_k \mathbb{E}[\| F(\theta_k) \|_2^2 \mid \mathcal{F}_0] < \infty$ with probability one.

**Proof.** By Lemma 2 and Assumption 1, for any $\phi, \theta \in \mathbb{R}^p$

$$
0 \leq F(\phi) - F_{l.b.} + F(\phi)'(\theta - \phi) + \frac{L}{1 + \alpha} \| \theta - \phi \|_2^{1 + \alpha}.
$$

(93)

We now find the $\theta$ that minimizes this upper bound, and plug it back into the upper bound. By rearranging, we conclude the result.

For the second part, by Corollary 5, $M_0 := \sup_k \mathbb{E}[F(\theta_k) - F_{l.b.} \mid \mathcal{F}_0] < \infty$ with probability one. By plugging $\theta_k$ into the first part of the result, taking expectations and applying Hölder’s inequality,

$$
\mathbb{E} \left[ \| F(\theta_k) \|_2^2 \mid \mathcal{F}_0 \right] \leq \left( \frac{L \lambda (1 + \alpha)}{\alpha} M_0 \right)^{\frac{2}{1 + \alpha}},
$$

(94)

with probability one. The result follows. \qed

**Corollary 7.** Let $\{ \theta_k \}$ be defined as in (5) satisfying Properties 1 to 4. Suppose Assumptions 1, 2, 5 and 6 hold. Then, $\lim_{k \to \infty} \mathbb{E}[\| F(\theta_k) \|_2 \mid \mathcal{F}_0] = 0$ with probability one.

**Proof.** By Lemma 11, $\{ \| F(\theta_k) \|_2 \}$ are bounded in $L^2$. Therefore, the sequence is uniformly integrable. In light of the uniform integrability of the sequence and Corollary 6, we can conclude the result. \qed