OPPOSITE ALGEBRAS OF GROUPOID $C^*$-ALGEBRAS

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ABSTRACT

We show that every groupoid $C^*$-algebra is isomorphic to its opposite, and deduce that there exist $C^*$-algebras that are not stably isomorphic to groupoid $C^*$-algebras, though many of them are stably isomorphic to twisted groupoid $C^*$-algebras. We also prove that the opposite algebra of a section algebra of a Fell bundle over a groupoid is isomorphic to the section algebra of a natural opposite bundle.

1. Introduction

Groupoids are among the most widely used models for operator algebras. It is therefore a basic question whether a given $C^*$-algebra $A$ can be realised as $C^*(G)$ or $C_r^*(G)$ for some locally compact topological groupoid $G$. Many classes of $C^*$-algebras have groupoid models: for example, graph $C^*$-algebras

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759
and higher-rank graph $C^*$-algebras, $C^*$-algebras of actions of inverse semi-groups, and $C^*$-algebras associated to foliations. Moreover, it follows from the main results in [5] that every UCT Kirchberg $C^*$-algebra (that is, every separable, simple, nuclear, purely infinite, UCT $C^*$-algebra) has an étale groupoid model.

We show in this paper that not every $C^*$-algebra has a groupoid model. We achieve this by showing that all groupoid $C^*$-algebras are self-opposite in the sense that they are isomorphic to their opposite $C^*$-algebras. Similar results for $L^p$-algebras of ample étale groupoids appear in [7, Corollary 6.10 and Remarks 6.8 and 6.13].

Several examples of non-self-opposite $C^*$-algebras are already known. The first, produced by Connes [2], is a non-self-opposite von Neumann factor. Later, examples of non-self-opposite separable $C^*$-algebras were found by Phillips in [11]. All of Phillips’ examples are continuous-trace $C^*$-algebras, hence nuclear. Simple and separable non-self-opposite $C^*$-algebras are constructed in [13, 12]; these examples are non-nuclear, though the one in [13] is exact. It remains open whether there exists a simple, separable and nuclear non-self-opposite $C^*$-algebra [3]. This is related to Elliott’s conjecture (see [17]) because the Elliott invariant (essentially $K$-groups) used in the conjecture cannot distinguish a $C^*$-algebra $A$ from its opposite $A^{\text{op}}$.

Although our result implies the existence of $C^*$-algebras with no groupoid model, it is still possible that such $C^*$-algebras can be realised as twisted groupoid $C^*$-algebras. That is, they could be isomorphic to $C^*(G, \Sigma)$ or $C^*_r(G, \Sigma)$, for some twist $\Sigma$ over a groupoid $G$. A twist over $G$ is essentially the same thing as a Fell line bundle $L$ over $G$, and $C^*(G, \Sigma)$ and $C^*_r(G, \Sigma)$ are then the corresponding full and reduced cross-sectional $C^*$-algebras $C^*(G, L)$ and $C^*_r(G, L)$. Renault proves in [16] that every $C^*$-algebra $A$ admitting a Cartan subalgebra $C_0(X) \subseteq A$ is isomorphic to $C^*_r(G, \Sigma)$ for some (second countable, locally compact Hausdorff) étale essentially principal groupoid $G$ with $G^0 = X$ and some twist $\Sigma$ on $G$; furthermore, the pair $(G, \Sigma)$ is uniquely determined by the Cartan pair $(A, C_0(X))$.

Kumjian, an Huef and Sims proved in [8] that every Fell $C^*$-algebra (in particular, every continuous-trace $C^*$-algebra) is Morita equivalent to one with a diagonal subalgebra in the sense of Kumjian [9]. These diagonal subalgebras are exactly the Cartan subalgebras (in the sense of Renault) with the unique extension property: every pure state of the Cartan subalgebra $C_0(X)$ extends
uniquely to $A$. The corresponding twist $(G, \Sigma)$ that describes $(A, C_0(X))$ is over a principal, not just essentially principal, groupoid $G$. After stabilisation, these results imply that all continuous-trace $C^*$-algebras have a twisted groupoid model—including the examples of Phillips in [11] that do not admit untwisted groupoid models. The point is that the opposite algebra of $C^*(G, \Sigma)$ arises as the $C^*$-algebra $C^*(G, \Sigma)$ of the conjugate twist, and this corresponds to taking the negative of the associated Dixmier–Douady invariant.

We elucidate the above phenomenon by describing the opposite $C^*$-algebras $C^*(G, A)^\text{op}$ and $C^*_r(G, A)^\text{op}$ of the cross-sectional algebras of arbitrary Fell bundles $A$ over locally compact groupoids. Specifically, given a Fell bundle $A$ over $G$, we construct an appropriate opposite bundle $A^\circ$ over $G$, and prove that $C^*(G, A)^\text{op} \cong C^*(G, A^\circ)$. This can also be described in terms of the conjugate Fell bundle $\bar{A}$. In the special case of a Fell line bundle $L$ (that is, a twist over $G$), this corresponds to the conjugate line bundle. When $L$ is the trivial line bundle, $L = L$, and $C^*_r(G; L)$ and $C^*(G; L)$ coincide with $C^*_r(G)$ and $C^*(G)$, so we recover our earlier result as a special case.

For a Fell bundle associated to an action $\alpha$ of a locally compact group $G$ on a $C^*$-algebra $A$, our result is equivalent to the statement that the opposite $C^*$-algebras of the full and reduced crossed products $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,r} G$ are isomorphic to $A^\text{op} \rtimes_{\alpha^\text{op}} G$ and $A^\text{op} \rtimes_{\alpha^\text{op},r} G$ (where $\alpha^\text{op}$ is the action of $G$ on $A^\text{op}$ determined by $\alpha$ upon identifying $A$ and $A^\text{op}$ as linear spaces); this was proved for full crossed products in [3].

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2. Groupoid $C^*$-algebras and their opposites

For background on groupoids and their $C^*$-algebras, we refer the reader to [15].

In this section we show that the full and reduced $C^*$-algebras of a locally compact, locally Hausdorff groupoid with Haar system are self-opposite. We first briefly recall how these $C^*$-algebras are defined.

Let $G$ be a locally compact and locally Hausdorff groupoid with Hausdorff unit space $G^0$ and a (continuous) left invariant Haar system $\lambda = \{\lambda^x\}_{x \in G^0}$. Let $\mathcal{E}_c(G, \lambda)$ be the $^*$-algebra of compactly supported, quasi-continuous sections,
that is, the linear span of continuous functions with compact support $f : U \to \mathbb{C}$ on open Hausdorff subsets $U \subseteq G$. These functions are extended by zero off $U$ and hence viewed as functions $G \to \mathbb{C}$. The continuity of $\lambda$ means that every such function is mapped to a continuous function $\lambda(f) : G^0 \to \mathbb{C}$ via

$$\lambda(f)(x) := \int_G f(g) d\lambda^x(g).$$

By definition, $\lambda^x$ is a Radon measure on $G$ with support $G^x := r^{-1}(x)$ for all $x \in G^0$.

Throughout the paper we follow the convention established by Renault that for $g \in G$ and $f_1, f_2 \in C_c(G)$, we abuse notation a little and write

$$\int_G f_1(h) f_2(h^{-1}g) d\lambda^{r(g)}(h)$$

rather than $\int_{G^{r(g)}} f_1(h) f_2(h^{-1}g) d\lambda^{r(g)}(h)$—strictly speaking, the integrand makes no sense for $h \notin G^{r(g)}$, but since the set of such $h$ has measure zero under $\lambda^{r(g)}$ it is clear what the integral means. Recall that the product and involution on $C_c(G, \lambda)$ are defined by

$$(f_1 \ast f_2)(g) := \int_G f_1(h) f_2(h^{-1}g) d\lambda^{r(g)}(h) \quad \text{and} \quad f^*(g) := \overline{f(g^{-1})}.$$  

Under these operations and the inductive-limit topology, $C_c(G, \lambda)$ is a topological $^*$-algebra. The $I$-norm on $C_c(G, \lambda)$ is defined by

$$\|f\|_I := \max\{\|\lambda(|f|)\|_\infty, \|\lambda(|f^*|)\|_\infty\}.$$  

The $L^1$-Banach $^*$-algebra of $G$ is the completion of $C_c(G, \lambda)$ with respect to $\|\cdot\|_I$; we denote it by $L^1_I(G, \lambda)$. The full $C^*$-algebra of $G$ is the universal enveloping $C^*$-algebra of $L^1_I(G, \lambda)$; in other words, it is the $C^*$-completion of $C_c(G, \lambda)$ with respect to the maximum $\|\cdot\|_I$-bounded $C^*$-norm:

$$\|f\|_u := \sup\{\|\pi(f)\| : \pi \text{ is an } I \text{-norm decreasing } ^*-\text{representation of } C_c(G, \lambda)\}.$$  

The regular representations of $(G, \lambda)$ are the representations

$$\pi_x : C_c(G, \lambda) \to B(L^2(G_x, \lambda_x)), \quad x \in G^0$$

given by $\pi_x(f)\xi(g) := (f \ast \xi)(g) = \int_G f(gh)\xi(h^{-1}) d\lambda^x(h)$. Here $\lambda_x$ is the image of $\lambda^x$ under the inversion map $G \to G$, $g \mapsto g^{-1}$; so it is a measure with support $G_x = s^{-1}(x)$. The system of measures $(\lambda_x)_{x \in G^0}$ is a right invariant Haar system on $G$. 

The regular representations of $G$ give rise to a $\| \cdot \|_I$-bounded $C^*$-norm called the \textbf{reduced $C^*$-norm}:
\[
\|f\|_r := \sup_{x \in G^0} \|\pi_x(f)\|.
\]
The \textbf{reduced $C^*$-algebra} of $G$ is the completion of $\mathcal{C}_c(G, \lambda)$ with respect to $\| \cdot \|_r$. It is denoted by $C^*_r(G, \lambda)$.

Given a groupoid $G$, we write $G^{\text{op}}$ for the opposite groupoid, equal to $G$ as a topological space, but with $G^{\text{op}}(2) = \{(h, g) : (g, h) \in G^{(2)}\}$ and composition given by $h^{\text{op}} \cdot g = gh$. We write $\lambda^{\text{op}}$ for the Haar system on $G^{\text{op}}$ defined as the image of $\lambda$ under the inversion map regarded as a homeomorphism of $G$ onto $G^{\text{op}}$.

**Theorem 2.1:** Let $(G, \lambda)$ be a locally compact, locally Hausdorff groupoid with Haar system. The inversion map $g \mapsto g^{-1}$ defines an isomorphism
\[
(G, \lambda) \cong (G^{\text{op}}, \lambda^{\text{op}})
\]
of topological groupoids with Haar systems. Given $f \in \mathcal{C}_c(G, \lambda)$, define $f^{\text{op}} : G \to \mathbb{C}$ by
\[
f^{\text{op}}(g) := f(g^{-1}).\]
Then $f \mapsto f^{\text{op}}$ is an isomorphism $\mathcal{C}_c(G, \lambda) \xrightarrow{\sim} \mathcal{C}_c(G, \lambda^{\text{op}})$ of topological $^*$-algebras. This isomorphism extends to a Banach $^*$-algebra isomorphism
\[
L^1_1(G, \lambda) \xrightarrow{\sim} L^1_1(G, \lambda^{\text{op}})
\]
and to $C^*$-algebra isomorphisms
\[
C^*(G, \lambda) \xrightarrow{\sim} C^*(G, \lambda^{\text{op}}) \quad \text{and} \quad C^*_r(G, \lambda) \xrightarrow{\sim} C^*_r(G, \lambda^{\text{op}}).
\]

**Proof.** The map $g \mapsto g^{-1}$ is a homeomorphism $G \to G^{\text{op}}$ and satisfies
\[
g^{-1} \cdot^{\text{op}} h^{-1} = h^{-1} g^{-1} = (gh)^{-1},
\]
so it is an isomorphism $G \xrightarrow{\sim} G^{\text{op}}$ of topological groupoids. The range (resp. source) map of $G^{\text{op}}$ is the source (resp. range) map of $G$, and the inversion map sends the left invariant Haar system $\lambda = (\lambda^x)_{x \in G^0}$ on $G$ to the right invariant Haar system $(\lambda^x)_{x \in G^0}$, which is precisely $\lambda^{\text{op}}$. This yields the isomorphism $(G, \lambda) \cong (G^{\text{op}}, \lambda^{\text{op}})$. The map
\[
f \mapsto f^{\text{op}}
\]
is a linear involution (in particular, a bijection) which is clearly a homeomorphism with respect to the inductive-limit topology. It is also clearly isometric for the \( \| \cdot \|_I \) -norms on \( \mathcal{C}_c(G, \lambda) \) and \( \mathcal{C}_c(G^{\text{op}}, \lambda^{\text{op}}) \). So to prove that it is topological *-algebra isomorphism \( \mathcal{C}_c(G, \lambda) \cong \mathcal{C}_c(G, \lambda)^{\text{op}} \) and extends to isomorphisms \( L_1^1(G, \lambda) \cong L_1^1(G, \lambda)^{\text{op}} \) and \( C^*(G, \lambda) \cong C^*(G, \lambda)^{\text{op}} \), it suffices to show that \( f \mapsto f^{\text{op}} \) is a *-homomorphism. For \( f \in \mathcal{C}_c(G, \lambda) \),

\[
(f^{\text{op}})^*(g) = \overline{f^{\text{op}}(g^{-1})} = \overline{f(g)} = f^*(g^{-1}) = (f^*)^{\text{op}}(g).
\]

So \( f \mapsto f^{\text{op}} \) preserves involution. If \( f_1, f_2 \in \mathcal{C}_c(G, \lambda) \), then

\[
(2.2) \quad (f_1 * f_2)^{\text{op}}(g) = (f_1 * f_2)(g^{-1}) = \int_G f_1(h)f_2(h^{-1}g^{-1}) \, d\lambda^{s(g)}(h),
\]

while

\[
(2.3) \quad (f_2^{\text{op}} * f_1^{\text{op}})(g) = \int_G f_2^{\text{op}}(h)f_1^{\text{op}}(h^{-1}g) \, d\lambda^{r(g)}(h)
\]

Making the change of variables \( h \mapsto gh \) and applying left invariance of \( \lambda \) shows that (2.2) and (2.3) are equal.

To prove that \( C^*_r(G, \lambda) \cong C^*_r(G, \lambda)^{\text{op}} \), observe that the map \( f \mapsto f^{\text{op}} \) gives an isomorphism \( L^2(G_x, \lambda_x) \cong L^2(G^x, \lambda^x) = L^2(G^x_\text{op}, \lambda^x_\text{op}) \) which induces a unitary equivalence between the regular representations

\[
\pi_x: \mathcal{C}_c(G, \lambda) \to \mathcal{B}(L^2(G_x, \lambda_x)) \quad \text{and} \quad \pi_x^{\text{op}}: \mathcal{C}_c(G^{\text{op}}, \lambda^{\text{op}}) \to \mathcal{B}(L^2(G^x_\text{op}, \lambda^x_\text{op}).
\]

This yields the equality

\[
\|f\|_r = \|f^{\text{op}}\|_r
\]

which shows that \( f \mapsto f^{\text{op}} \) extends to an isomorphism \( C^*_r(G, \lambda) \cong C^*_r(G^{\text{op}}, \lambda^{\text{op}}) \).

**Remark 2.4:** Similar arguments to those above show that the identity map on \( G \), regarded as an anti-multiplicative homeomorphism from \( G \) to \( G^{\text{op}} \), induces (by composition) an anti-multiplicative linear isomorphism \( \mathcal{C}_c(G, \lambda) \cong \mathcal{C}_c(G^{\text{op}}, \lambda^{\text{op}}) \), and therefore a topological *-algebra isomorphism \( \mathcal{C}_c(G, \lambda)^{\text{op}} \cong \mathcal{C}_c(G^{\text{op}}, \lambda^{\text{op}}) \). This latter extends to isomorphisms

\[
L_1^1(G, \lambda)^{\text{op}} \cong L_1^1(G^{\text{op}}, \lambda^{\text{op}}), \quad C^*(G, \lambda)^{\text{op}} \cong C^*(G^{\text{op}}, \lambda^{\text{op}})
\]

and \( C^*_r(G, \lambda)^{\text{op}} \cong C^*_r(G^{\text{op}}, \lambda^{\text{op}}) \).

Another way to prove Theorem 2.1 is to work with conjugate algebras. If \( A \) is a *-algebra, its **conjugate *-algebra** \( \bar{A} \) is the conjugate vector space of \( A \).
endowed with the same algebraic operations as $A$. Involution, $a \mapsto a^*$ is then a linear anti-multiplicative isomorphism $A \to \bar{A}$ and therefore an isomorphism $A^{\text{op}} \cong \bar{A}$. We have $\mathcal{C}_c(G, \lambda) \cong \overline{\mathcal{C}_c(G, \lambda)}$ via $\xi \mapsto \bar{\xi}$ and this extends to isomorphisms $L^1_1(G, \lambda) \cong \overline{L^1_1(G, \lambda)}$, $C^*(G, \lambda) \cong \overline{C^*(G, \lambda)}$ and $C^*_r(G, \lambda) \cong \overline{C^*_r(G, \lambda)}$.

**Corollary 2.5:** There are (nuclear, separable) $C^*$-algebras that are not isomorphic to either $C^*(G, \lambda)$ or $C^*_r(G, \lambda)$ for any locally compact, locally Hausdorff groupoid with Haar system.

**Proof.** It is known that there are examples of nuclear and separable $C^*$-algebras that are not self-opposite [11, 3].

Let us say that a $^*$-algebra $A$ is **self-opposite** if $A \cong A^{\text{op}}$. Our main result says that given a topological groupoid with Haar system $(G, \lambda)$, the $^*$-algebras $\mathcal{C}_c(G, \lambda)$, $L^1_1(G, \lambda)$, $C^*(G, \lambda)$ and $C^*_r(G, \lambda)$ are all self-opposite. Both the minimal and the maximal tensor product of self-opposite $C^*$-algebras are again self-opposite because $(A \otimes B)^{\text{op}} \cong A^{\text{op}} \otimes B^{\text{op}}$.

Let $\mathbb{K}$ denote the $C^*$-algebra of compact operators on a separable, infinite dimensional Hilbert space; writing $\mathcal{R}$ for the equivalence relation $\mathbb{N} \times \mathbb{N}$ regarded as a discrete principal groupoid, we have $\mathbb{K} \cong C^*(\mathcal{R}) = C^*_r(\mathcal{R})$. Hence the preceding paragraph shows that every self-opposite $C^*$-algebra is also stably self-opposite. The converse fails in general: Phillips constructs in [11] examples of (separable, continuous-trace) non-self-opposite $C^*$-algebras which are stably self-opposite. But Phillips also constructs examples of (separable, continuous-trace) $C^*$-algebras that are not stably self-opposite. This yields the following:

**Corollary 2.6:** There are separable continuous-trace $C^*$-algebras that are not stably isomorphic to any groupoid $C^*$-algebra.

**Remark 2.7:** By the Brown–Green–Rieffel theorem [1], Corollary 2.6 implies that there exist separable $C^*$-algebras that are not Morita equivalent to a separable (or even $\sigma$-unital) groupoid $C^*$-algebra. However, it is unclear whether these examples could be Morita equivalent to a non-$\sigma$-unital groupoid $C^*$-algebra.

In [6], in the framework of ZFC enriched with Jensen’s diamond principle (a strengthening of the continuum hypothesis), Farah and Hirshberg construct examples of non-separable approximately matricial algebras (uncountable direct limits of the CAR algebra) that are non-self-opposite, so we can also state:
Corollary 2.8: It is consistent with ZFC that there are non-separable approximately matricial (so simple, nuclear) $C^*$-algebras that are not isomorphic to a groupoid $C^*$-algebra.

Recall that the ordinary separable AF-algebras admit groupoid models: it is even known that they are always crossed products for a partial action of the integers, see [4].

By [8, Theorem 6.6(1)], every separable continuous-trace $C^*$-algebra (indeed, every Fell algebra) is Morita equivalent to a separable $C^*$-algebra with a diagonal subalgebra in the sense of Kumjian [9]. Kumjian shows in [9] that $C^*$-algebras containing diagonals are, up to isomorphism, the $C^*$-algebras obtained from twists on étale principal groupoids. More precisely, writing $T$ for the circle group, this means a locally compact Hausdorff central groupoid extension

$$T \times G^0 \hookrightarrow \Sigma \twoheadrightarrow G,$$

of a (second countable) locally compact Hausdorff groupoid $G$ by the (trivial) group bundle $T \times G^0$. To a twisted groupoid $(G, \Sigma)$ one can assign a full $C^*$-algebra $C^*(G, \Sigma)$ and a reduced $C^*$-algebra $C^*_r(G, \Sigma)$, and then every separable $C^*$-algebra containing a diagonal subalgebra has the form $C^*_r(G, \Sigma)$ for some twist $\Sigma$ over a principal groupoid $G$. Moreover, the pair $(G, \Sigma)$ is unique, up to isomorphism of twisted groupoids. This follows from the more general result, proved by Renault in [16], that isomorphism classes of Cartan subalgebras correspond bijectively to isomorphism classes of twisted essentially principal étale groupoids (meaning twisted groupoids where $G$ is not necessarily principal, but only essentially principal; see [16] for details). Using these results, we arrive at the following consequence:

Corollary 2.9: There are separable stable continuous-trace $C^*$-algebras that are not isomorphic to any groupoid $C^*$-algebra but which are isomorphic to the reduced $C^*$-algebra of a twisted principal étale groupoid.

Proof. Let $A$ be a separable continuous-trace $C^*$-algebra which is not stably isomorphic to any groupoid $C^*$-algebra as in Corollary 2.6. Let

$$B := A \otimes K$$

be the stabilisation of $A$. Then $B$ is a separable stable continuous-trace $C^*$-algebra which is not isomorphic to any groupoid $C^*$-algebra. By [8, Theorem 6.(1)]
A is Morita equivalent to $C^*_r(G, \Sigma)$, for some twisted principal étale groupoid $(G, \Sigma)$. It follows from the Brown–Green–Rieffel theorem that

$$B \cong C^*_r(G, \Sigma) \otimes \mathbb{K}.$$ 

To finish the proof we observe that, again writing $\mathcal{R}$ for the discrete equivalence relation $\mathbb{N} \times \mathbb{N}$, we have $C^*_r(G, \Sigma) \otimes \mathbb{K} \cong C^*_r(G \times \mathcal{R}, \Sigma \times \mathcal{R})$. \hfill \qed

3. Section $C^*$-algebras of Fell bundles and their opposites

Let $G$ be a locally compact and locally Hausdorff groupoid endowed with a continuous Haar system $\lambda$, which we fix throughout the rest of the section. In this section we generalise our previous result and describe the opposite $C^*$-algebras of the section $C^*$-algebras of Fell bundles over $G$. Our result generalises the observation in [3] that $(A \rtimes_\alpha G)^{\text{op}} \cong A^\text{op} \rtimes_{\alpha^\text{op}} G$ for any action $\alpha$ of a locally compact group $G$ on a $C^*$-algebra $A$.

Fell bundles over topological groupoids are defined in [10]. Only Hausdorff groupoids are considered there, but the same definition makes sense for locally Hausdorff groupoids. A Fell bundle over $G$ consists of an upper semicontinuous Banach bundle $\mathcal{A}$ over $G$ endowed with multiplications $A_g \times A_h \to A_{gh}$, $(a, b) \mapsto a \cdot b$, for every composable pair $(g, h) \in G^2$ and involutions $A_g \to A_g^{-1}$, $a \mapsto a^*$, for every $g \in G$. These operations are required to be continuous (with respect to the given topology on $\mathcal{A}$) and satisfy algebraic conditions similar to those in the definition of a $C^*$-algebra.

We next recall, briefly, how to define the full and reduced $C^*$-algebras of a Fell bundle. Consider the space $\mathcal{C}_c(G, \mathcal{A})$ of compactly supported continuous sections $\xi: U \to \mathcal{A}$ defined on open Hausdorff subspaces $U \subseteq G$ and extended by zero outside $U$ and hence viewed as sections $\xi: G \to \mathcal{A}$. The continuity of the algebraic operations on $\mathcal{A}$ implies that for $\xi, \eta \in \mathcal{C}_c(G, \mathcal{A})$, the formulas

$$(\xi \ast \eta)(g) := \int_G \xi(h) \cdot \eta(h^{-1}g) \, d\lambda^r(g)(h) \quad \text{and} \quad \xi^*(g) := \xi(g^{-1})^*$$

define elements $\xi \ast \eta, \xi^* \in \mathcal{C}_c(G, \mathcal{A})$ and so determine a convolution product $\ast$ and an involution $^*$ on $\mathcal{C}_c(G, \mathcal{A})$. Under these operations, $\mathcal{C}_c(G, \mathcal{A})$ is a $^*$-algebra; and indeed, a topological $^*$-algebra in the inductive-limit topology.
Since the norm function on $\mathcal{A}$ is upper semicontinuous, the function $g \mapsto \|\xi(g)\|$ from $G$ to $[0, \infty)$ is upper semicontinuous and hence measurable. So we can define the $I$-norm on $\mathcal{C}_c(G, \mathcal{A})$ by

$$\|\xi\|_I := \sup_{x \in G^{(0)}} \max \left\{ \int_{G^x} |\xi(g)| \, d\lambda^x(g), \int_{G^x} |\xi^*(g)| \, d\lambda^x(g) \right\}.$$ 

The $L^1$-Banach algebra of $\mathcal{A}$, denoted $L^1_f(G, \mathcal{A})$, is defined as the completion of $\mathcal{C}_c(G, \mathcal{A})$ with respect to $\| \cdot \|_I$. The full $C^*$-algebra $C^*(G, \mathcal{A})$ of $\mathcal{A}$ is defined as the universal enveloping $C^*$-algebra of $L^1_f(G, \mathcal{A})$: the completion of $\mathcal{C}_c(G, \mathcal{A})$ with respect to the $C^*$-norm

$$\|\xi\|_u := \sup\{\|\pi(\xi)\| : \pi \text{ is an } I\text{-norm decreasing } \ast\text{-representation of } \mathcal{C}_c(G, \mathcal{A})\}.$$ 

That this is indeed a norm on $\mathcal{C}_c(G, \mathcal{A})$, and not just a seminorm, follows from the existence of the following regular representations.

For each $x \in G^{(0)}$, let $L^2(G_x, \mathcal{A})$ be the right Hilbert $\mathcal{A}_x$-module completion of the space $\mathcal{C}_c(G_x, \mathcal{A})$ of quasi-continuous sections $G_x \to \mathcal{A}$ with respect to the norm induced by the $\mathcal{A}_x$-valued inner product

$$\langle \xi | \eta \rangle_{\mathcal{A}_x} := \int_G \xi(h) \ast \eta(h) \, d\lambda_x(h) = \int_G \xi(h^{-1}) \ast \eta(h^{-1}) \, d\lambda_x(h).$$ 

Then for each $x \in G^{(0)}$, the regular representation $\pi_x : \mathcal{C}_c(G, \mathcal{A}) \to \mathbb{B}(L^2(G_x, \mathcal{A}))$ is defined by

$$(\pi_x(\xi) \eta)(g) := \int_G \xi(g h) \eta(h^{-1}) \, d\lambda^x(h) = \int_G \xi(g h^{-1}) \eta(h) \, d\lambda_x(h),$$

for all $\xi \in \mathcal{C}_c(G, \mathcal{A})$, $\eta \in \mathcal{C}_c(G_x, \mathcal{A})$ and $g \in G_x$. The reduced $C^*$-norm on $\mathcal{C}_c(G, \mathcal{A})$ is defined by

$$\|\xi\| := \sup_{x \in G^0} \|\pi_x(\xi)\|.$$ 

This is, indeed, a norm: if $\pi_x(\xi) = 0$ then $(\xi \ast \eta)(g) = 0$ for all $\eta \in \mathcal{C}_c(G_x, \mathcal{A})$ and $g \in G_x$, so

$$\xi \ast \xi^*(x) = \int_G \xi(h) \xi(h)^* \, d\lambda^x(h) = 0 \quad \text{for all } x \in G^{(0)},$$

forcing $\xi|_{G^x} = 0$ for all $x$. A standard computation shows that $\|\xi\|_r \leq \|\xi\|_I$. Therefore $\| \cdot \|_u$ is also a $C^*$-norm and $\| \cdot \|_r \leq \| \cdot \|_u$. The completion of $\mathcal{C}_c(G, \mathcal{A})$ with respect to $\| \cdot \|_r$ is the reduced section $C^*$-algebra of $\mathcal{A}$, and is denoted by $C^*_r(G, \mathcal{A})$. 
Our goal is to describe the opposite $C^*$-algebras $C^*(G, \mathcal{A})^{\text{op}}$ and $C^*_r(G, \mathcal{A})^{\text{op}}$. We show that $C^*(G, \mathcal{A})^{\text{op}} \cong C^*(G, \mathcal{A}^\circ)$, for an appropriate opposite Fell bundle $\mathcal{A}^\circ$ over $G$ associated to $\mathcal{A}$. It is more natural to first define an opposite Fell bundle $\mathcal{A}^{\text{op}}$ over the opposite groupoid $G^{\text{op}}$ and then later use the canonical anti-isomorphism $G \cong G^{\text{op}}$ induced by the inversion map to obtain the desired Fell bundle $\mathcal{A}^\circ$ over $G$.

The opposite Fell bundle $\mathcal{A}^{\text{op}}$ over $G^{\text{op}}$ is defined as follows. As a Banach bundle, $\mathcal{A}^{\text{op}}$ does not differ from $\mathcal{A}$. In particular, the fibres are equal, $\mathcal{A}^{\text{op}}_g = \mathcal{A}_g$ for all $g \in G$, and also the topology on $\mathcal{A}^{\text{op}}$ is equal to that on $\mathcal{A}$. Moreover, $\mathcal{A}^{\text{op}}$ is also endowed with the same involution as $\mathcal{A}$, which makes sense because $G$ and $G^{\text{op}}$ carry the same inversion map. The only thing that changes in $\mathcal{A}^{\text{op}}$ is the multiplication: given $g, h \in G^{\text{op}}$ the condition $s^{\text{op}}(g) = r^{\text{op}}(h)$ means $r(g) = s(h)$, so we can use the multiplication map $\mu_{h,g} : \mathcal{A}_h \times \mathcal{A}_g \to \mathcal{A}_{hg}$ and define the multiplication maps

$$\mu^{\text{op}} : \mathcal{A}^{\text{op}}_g \times \mathcal{A}^{\text{op}}_h = \mathcal{A}_g \times \mathcal{A}_h \to \mathcal{A}^{\text{op}}_{g \cdot h} = \mathcal{A}_{hg} \quad \text{by} \quad \mu^{\text{op}}(a, b) := \mu(b, a).$$

In other words, $a \cdot^{\text{op}} b := b \cdot a$ if we use $\cdot$ and $\cdot^{\text{op}}$ to denote the multiplications on $\mathcal{A}$ and $\mathcal{A}^{\text{op}}$, respectively. It is straightforward to see that $\mathcal{A}^{\text{op}}$ is indeed a Fell bundle over $G^{\text{op}}$. Now we use the anti-isomorphism $G^{\text{op}} \cong G$ induced by the inversion map $g \mapsto g^{-1}$ to form the pullback Fell bundle of $\mathcal{A}^{\text{op}}$. In other words, $\mathcal{A}^\circ$ is a Fell bundle over $G$ with fibres $\mathcal{A}^\circ_g = \mathcal{A}_g^{-1}$ and the topology induced by the sections $\xi^\circ(g) := \xi(g^{-1})$ for $\xi : U \to \mathcal{A}$ a continuous section defined on a Hausdorff open subset $U \subseteq G$. The involution map $\mathcal{A}^\circ_g \to \mathcal{A}^\circ_{g^{-1}}$ is the involution map $\mathcal{A}_g^{-1} \to \mathcal{A}_g$ from $\mathcal{A}$ and the multiplication map $\mathcal{A}^\circ_g \times \mathcal{A}^\circ_h \to \mathcal{A}^\circ_{gh}$ is given by $(a, b) \mapsto b \cdot a$ for all $b \in \mathcal{A}^\circ_g = \mathcal{A}_g^{-1}$, $b \in \mathcal{A}^\circ_h = \mathcal{A}_h^{-1}$ and $g, h \in G$ with $s(g) = r(h)$.

**THEOREM 3.1:** Let $\mathcal{A}$ be a Fell bundle over a locally compact, locally Hausdorff groupoid with Haar system $(G, \lambda)$, and consider the Fell bundle $\mathcal{A}^\circ$ over $(G, \lambda)$ described above. The map $\xi \mapsto \xi^\circ$ defined by

$$\xi^\circ(g) := \xi(g^{-1})$$

gives an isomorphism of topological $*$-algebras $\mathcal{C}_c(G, \mathcal{A})^{\text{op}} \xrightarrow{\sim} \mathcal{C}_c(G, \mathcal{A}^\circ)$. Moreover, this isomorphism extends to an isomorphism of Banach $*$-algebras $L^1(G, \mathcal{A})^{\text{op}} \xrightarrow{\sim} L^1(G, \mathcal{A}^\circ)$ and $C^*$-algebras $C^*(G, \mathcal{A})^{\text{op}} \xrightarrow{\sim} C^*(G, \mathcal{A}^\circ)$ and $C^*_r(G, \mathcal{A})^{\text{op}} \xrightarrow{\sim} C^*_r(G, \mathcal{A}^\circ)$. 
Proof. We prove the equivalent assertion that \( \mathcal{C}_c(G, A)^{\text{op}} \cong \mathcal{C}_c(G^{\text{op}}, A^{\text{op}}) \) via the canonical linear isomorphism \( \mathcal{C}_c(G, A) \ni \xi \mapsto \xi^{\text{op}} := \xi \in \mathcal{C}_c(G^{\text{op}}, A^{\text{op}}) \), and that this isomorphism extends to isomorphisms

\[
L^1_I(G, A)^{\text{op}} \xrightarrow{\sim} L^1_I(G^{\text{op}}, A^{\text{op}}), \quad C^*(G, A)^{\text{op}} \xrightarrow{\sim} C^*(G^{\text{op}}, A^{\text{op}}), \quad \text{and}
C^*_r(G, A)^{\text{op}} \xrightarrow{\sim} C^*_r(G^{\text{op}}, A^{\text{op}}).
\]

Since the topologies on \( A \) and \( A^{\text{op}} \) are the same, the map \( \xi \mapsto \xi^{\text{op}} \) is clearly a linear bijection \( \mathcal{C}_c(G, A) \to \mathcal{C}_c(G^{\text{op}}, A^{\text{op}}) \) which is a homeomorphism with respect to the inductive-limit topologies. Also, this map preserves the involution; that is, \((\xi^{\text{op}})^* = \xi^* \) on \( \mathcal{C}_c(G, A) \) (which is the same as the involution on \( \mathcal{C}_c(G, A)^{\text{op}} \)), and on \( \mathcal{C}_c(G^{\text{op}}, A^{\text{op}}) \) because the involutions on \( A \) and on \( A^{\text{op}} \) are the same. It remains to check that the map is a homomorphism \( \mathcal{C}_c(G, A)^{\text{op}} \to \mathcal{C}_c(G^{\text{op}}, A^{\text{op}}) \). But, remembering that the left Haar system \( \lambda^{\text{op}} \) on \( G^{\text{op}} \) is the right Haar system \( (\lambda_x)_{x \in G^0} \) on \( G \), we get

\[
\xi^{\text{op}} \star \eta^{\text{op}}(g) = \int_{G^{\text{op}}} \xi(h)^{\text{op}} \eta(h^{-1}g) \, d(\lambda^{\text{op}})(\lambda^{\text{op}}(g))(h) = \int_G \eta(gh^{-1})\xi(h) \, d\lambda_s(h)(h) = \int_G \eta(gh)\xi(h^{-1}) \, d\lambda_s(g)(h) = (\eta \star \xi)(g)
\]

for all \( \xi, \eta \in \mathcal{C}_c(G, A) \) and \( g \in G \). This shows that the identity map is an anti-homomorphism \( \mathcal{C}_c(G, A) \to \mathcal{C}_c(G^{\text{op}}, A^{\text{op}}) \), that is, a homomorphism \( \mathcal{C}_c(G, A)^{\text{op}} \to \mathcal{C}_c(G^{\text{op}}, A^{\text{op}}) \), as desired. A similar computation shows that \( \|\xi^{\text{op}}\|_I = \|\xi\|_I \) and that therefore the identity map extends to an isomorphism \( L^1_I(G, A)^{\text{op}} \xrightarrow{\sim} L^1_I(G^{\text{op}}, A^{\text{op}}) \) and hence also to the corresponding universal enveloping \( C^* \)-algebras \( C^*(G, A)^{\text{op}} \xrightarrow{\sim} C^*(G^{\text{op}}, A^{\text{op}}) \).

Finally, to check that \( C^*_r(G, A)^{\text{op}} \cong C^*_r(G^{\text{op}}, A^{\text{op}}) \), fix \( x \in G^0 \). The regular representation \( \pi^{\text{op}}_x \) defines a representation of \( C^*(G^{\text{op}}, A^{\text{op}}) \) by adjointable operators on the right-Hilbert \( A^{\text{op}} \)-module \( L^2(G_x^{\text{op}}, A^{\text{op}}) \). Recall that a left-Hilbert module over a \( C^* \)-algebra \( C \) is a left \( C \)-module \( \mathcal{E} \) with an inner product \( C(\cdot, \cdot) \) in which \( C(c \cdot | \xi) = cC(|\xi|) \) for \( c \in C \) and \( \xi, \eta \in \mathcal{E} \). Any right Hilbert module \( \mathcal{E} \) over the opposite \( B^{\text{op}} \) of a \( C^* \)-algebra \( B \) determines a left Hilbert \( B \)-module \( \mathcal{E} \) with left \( B \)-action \( b \cdot \xi := \xi \cdot b \) and left \( B \)-valued inner product \( B(\xi | \eta) := \langle \eta | \xi \rangle_{B^{\text{op}}} \). This process preserves the \( C^* \)-algebras of adjointable operators, meaning that the identity map on \( \mathcal{E} \) yields an isomorphism \( \mathbb{B}(\mathcal{E}_{B^{\text{op}}}) \cong \mathbb{B}(B\mathcal{E}) \). Applying
this to the right Hilbert $\mathcal{A}_x^\text{op}$-module $L^2(G_{x}^\text{op}, \mathcal{A}^\text{op})$ we get a left Hilbert $\mathcal{A}_x$-module with left $\mathcal{A}_x$-action given by $a \cdot \xi = \xi \cdot \text{op} a$ for all $\xi \in L^2(G_{x}^\text{op}, \mathcal{A}^\text{op})$; the right hand side denotes the right $\mathcal{A}_x^\text{op}$-action on $L^2(G_{x}^\text{op}, \mathcal{A}^\text{op})$, so it is given by $(a \cdot \text{op} \xi)(g) = \xi(g) \cdot \text{op} a(s(g))(g) = a(r(g)) \cdot \xi(g)$. The left $\mathcal{A}_x$-valued inner product on $L^2(G_{x}^\text{op}, \mathcal{A}^\text{op})$ is given by

$$\mathcal{A}_x \langle \xi | \eta \rangle_{A_x^\text{op}} = \int_{G} \eta(h)^* \cdot \text{op} \xi(h) \, d\lambda_x^\text{op}(h) = \int_{G} \xi(h) \eta(h)^* \, d\lambda^x(h)$$

for all $\xi, \eta \in \mathcal{C}_c(G_{x}^\text{op}, \mathcal{A}^\text{op}) = \mathcal{C}_c(G^x, \mathcal{A})$. Therefore the left Hilbert $\mathcal{A}_x$-module obtained from the right Hilbert $\mathcal{A}_x^\text{op}$-module $L^2(G_{x}^\text{op}, \mathcal{A}^\text{op})$ in this way equals the left Hilbert $\mathcal{A}_x$-module $L^2(G_x, \mathcal{A})$ defined as the completion of $\mathcal{C}(G^x, \mathcal{A})$ with respect to the norm associated to the left $\mathcal{A}_x$-valued inner product given by the above formula and the left $\mathcal{A}_x$-action also defined above. Therefore we may view $\pi_{x}^\text{op}$ as a representation of $C^*(G_{x}^\text{op}, \mathcal{A}^\text{op})$ on

$$\mathcal{B}(\mathcal{A}_x L^2(G_{x}^\text{op}, \mathcal{A})) \cong \mathcal{B}(L^2(G_{x}^\text{op}, \mathcal{A}^\text{op}))_{\mathcal{A}_x^\text{op}}.$$

Under the isomorphism $C^*(G_{x}^\text{op}, \mathcal{A}^\text{op}) \cong C^*(G_x, \mathcal{A})^\text{op}$, this corresponds to the canonical representation $\tilde{\pi}_{x}$ of $C^*(G_x, \mathcal{A})^\text{op}$ on $\mathcal{A}_x L^2(G_{x}^\text{op}, \mathcal{A})$ via the formula

$$\tilde{\pi}_{x}(\xi)\eta(g) := (\eta \ast \xi)(g) = \int_{G} \eta(h) \xi(h^{-1}g) \, d\lambda^x(h)$$

for $\xi \in \mathcal{C}_c(G_x, \mathcal{A})$, $\eta \in \mathcal{C}_c(G^x, \mathcal{A})$ and $g \in G_x$. Straightforward computations show that the above formula defines a representation

$$\tilde{\pi}_{x} : C^*(G_x, \mathcal{A})^\text{op} \to \mathcal{B}(\mathcal{A}_x L^2(G_{x}^\text{op}, \mathcal{A}))$$

of the opposite $C^*$-algebra $C^*(G_x, \mathcal{A})^\text{op}$.

Given a left Hilbert $B$-module $\mathcal{E}$, let $\tilde{\mathcal{E}}$ denote the dual right Hilbert $B$-module of $\mathcal{E}$: as a vector space $\tilde{\mathcal{E}} = \{ \tilde{\xi} : \xi \in \mathcal{E} \}$ is the conjugate of $\mathcal{E}$ and the right $B$-action and right $B$-valued inner product are defined by

$$\tilde{\xi} \cdot b := (b^* \cdot \xi)^{\sim} \quad \text{and} \quad \langle \xi | \eta \rangle_B :=_{B} \langle \xi | \eta \rangle.$$

Then each representation $\pi : A^\text{op} \to \mathcal{B}(B \mathcal{E})$ of an opposite $C^*$-algebra $A^\text{op}$ on the $C^*$-algebra of adjointable operators $\mathcal{B}(B \mathcal{E})$ of a left Hilbert $B$-module $\mathcal{E}$ induces a representation

$$\pi^\text{op} : A \to \mathcal{B}(B \mathcal{E})^\text{op} \cong \mathcal{B}(\tilde{\mathcal{E}}_B).$$

The isomorphism $\mathcal{B}(B \mathcal{E})^\text{op} \cong \mathcal{B}(\tilde{\mathcal{E}}_B)$ we used above is induced by the involution; that is, it sends an operator $T \in \mathcal{B}(B \mathcal{E})$ to $\tilde{T} \in \mathcal{B}(\tilde{\mathcal{E}}_B)$ defined by

$$\tilde{T}(\tilde{\xi}) := (T^*(\xi))^{\sim}.$$
For \( \xi \in \mathcal{C}_c(G^x, A) \), the formula
\[
\xi^*(g) := \xi(g^{-1})^*
\]
determines an element \( \xi^* \in \mathcal{C}_c(G_x, A) \). The map \( \xi \mapsto \xi^* \) induces an isomorphism \( (L^2(G^x, A))_{\mathcal{A}_x} \cong L^2(G_x, A)_{\mathcal{A}_x} \) from the dual Hilbert \( \mathcal{A}_x \)-module of \( \mathcal{A}_x \) to the right Hilbert \( \mathcal{A}_x \)-module \( L^2(G_x, A) \) that carries the regular representation \( \pi_x : C^*(G, A) \to \mathbb{B}(L^2(G_x, A)_{\mathcal{A}_x}) \). This isomorphism intertwines the representations
\[
\pi_x : C^*(G, A) \to \mathbb{B}(L^2(G_x, A)_{\mathcal{A}_x})
\]
and
\[
\tilde{\pi}^\text{op} : C^*(G, A) \to \mathbb{B}(\mathcal{A}_x L^2(G^x, A))^{\text{op}} \cong \mathbb{B}((L^2(G^x, A))_{\mathcal{A}_x}).
\]
We conclude that
\[
\|\tilde{\pi}^\text{op}(\xi)\| = \|\tilde{\pi}_x(\xi^0)\| = \|\pi_x^\text{op}(\xi^0)\| = \|\pi_x(\xi^0)\|.
\]
Since \( x \in G^0 \) was arbitrary, we get the equality \( \|\xi^0\| = \|\xi\| \) and therefore the desired isomorphism
\[
C^*_{r}(G^\text{op}, \mathcal{A}^\text{op}) \cong C^*_{r}(G, A)^{\text{op}}.
\]

**Remark 3.2:** As in the case of groupoid \( C^* \)-algebras, we can rephrase the preceding result in terms of conjugate bundles as well. Let \( \mathcal{A} \) be a Fell bundle over a groupoid \( G \). For \( g \in G \), let \( \overline{\mathcal{A}}_g \) be the conjugate vector space of \( \mathcal{A}_g \); that is, \( \overline{\mathcal{A}}_g \) is a copy \{\( \overline{a} : a \in \mathcal{A}_g \)\} of \( \mathcal{A}_g \) as an abelian group under addition, but with scalar multiplication given by \( \lambda \overline{a} = \overline{\lambda a} \). Via the map \( a \mapsto \overline{a}, \) the operations on the Fell bundle \( \mathcal{A} \) induce operations on \( \overline{\mathcal{A}} := \bigsqcup_{g \in G} \overline{\mathcal{A}}_g : \overline{ab} = \overline{ab}, \) and \( \overline{a}^* = \overline{\overline{a}}. \) Under these operations, \( \overline{\mathcal{A}} \) is a Fell bundle over \( G \), called the **conjugate bundle** of \( \mathcal{A} \).

Let \( \mathcal{A}^\text{op} \) be the opposite bundle of \( \mathcal{A} \) defined above; so \( \mathcal{A}^\text{op}_g \cong A_g^{-1} \), and write \( \cdot_o \) for the multiplication in this bundle. Then the maps \( \mathcal{A}^\text{op}_g \ni a \mapsto \overline{a}^* \in \overline{\mathcal{A}}_g \) are linear isometries because the maps \( a \mapsto a^* \) and \( a \mapsto \overline{a} \) are both conjugate linear.

We have \( (a \cdot_o b)^* = (ba)^* = a^*b^* = a^*b^* \) and \( (\overline{a})^* = \overline{a} = (a^*)^* \), so \( a \mapsto \overline{a}^* \) determines an isomorphism \( \mathcal{A}^\text{op} \cong \overline{\mathcal{A}} \) of Fell bundles over \( G \). Thus Theorem 3.1 shows that there is a topological-*-algebra isomorphism \( \xi \mapsto \overline{\xi} \) from \( \mathcal{C}_c(G, A) \) to \( \mathcal{C}_c(G, \overline{\mathcal{A}}) \) given by \( \overline{\xi}(g) := \overline{\xi(g^{-1})}^* \) that extends to isomorphisms
\[
L^1_1(G, \mathcal{A}^\text{op}) \cong L^1_1(G, \overline{\mathcal{A}}), \quad C^*(G, \mathcal{A}^\text{op}) \cong C^*(G, \overline{\mathcal{A}}), \quad \text{and} \quad C^*_{r}(G, \mathcal{A}^\text{op}) \cong C^*_{r}(G, \overline{\mathcal{A}}).
\]

**Remark 3.3:** Remark 3.2 is closely related to the idea behind Phillips’ construction in [11] of non-self-opposite continuous-trace \( C^* \)-algebras \( A \); the observation underlying his construction is that the Dixmier–Douady class of the opposite
algebra $A^{\text{op}}$ is the inverse of the Dixmier–Douady class of $A$. To see how this relates to our results, fix a compact Hausdorff space $X$, and let $\mathcal{S}$ denote the sheaf of germs of continuous $\mathbb{T}$-valued functions on $X$. The Raeburn–Taylor construction [14] shows that (after identifying $\hat{H}^3(X, \mathbb{Z})$ with $H^2(X, \mathcal{S})$) any class $\delta \in H^2(X, \mathcal{S})$ is the Dixmier–Douady invariant of a twisted groupoid $C^*$-algebra $C^*(G, \sigma)$ associated to a continuous 2-cocycle $\sigma$ on a principal étale groupoid $G$ with unit space $G^{(0)} = \bigsqcup_{i, j} U_{ij}$ for some precompact open cover $\{U_i\}$ of $X$. The cocycle $\sigma$ determines, and is determined up to cohomology by, the Fell line-bundle $L_\sigma$ over $G$ given by $L_\sigma = G \times \mathbb{T}$ with twisted multiplication $(g, w)(h, z) = (gh, c(g, h)wz)$ and the obvious involution. Remark 3.2 shows that $C^*(G, \sigma)^{\text{op}}$ is given by the conjugate bundle $L_\sigma^\star$, so the corresponding class in $H^2(X, \mathcal{S})$ is determined by the pointwise conjugate of the class $\delta$; that is, $\delta(C^*(G, \sigma)) = \delta(C^*(G, \sigma)^{\text{op}})^{-1}$.

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