A counterexample of the birational Torelli problem via Fourier–Mukai transforms

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Abstract

We study the Fourier–Mukai numbers of rational elliptic surfaces. As its application, we give an example of a pair of minimal 3-folds with Kodaira dimensions 1, $h^1(O) = h^2(O) = 0$ such that they are mutually derived equivalent, deformation equivalent, but not birationally equivalent. It also supplies a counterexample of the birational Torelli problem.

1 Introduction

Let $X$ be a smooth projective variety over $\mathbb{C}$. The derived category $D^b(X)$ of $X$ is a triangulated category whose objects are bounded complexes of coherent sheaves on $X$. A Fourier–Mukai transform relating smooth projective varieties $X$ and $Y$ is a $\mathbb{C}$-linear equivalence of triangulated categories $\Phi : D^b(X) \to D^b(Y)$. If there exists a Fourier–Mukai transform relating $X$ and $Y$, we call $X$ a Fourier–Mukai partner of $Y$, or simply say that $X$ and $Y$ are derived equivalent.

It is an interesting problem to find a good characterization of Fourier-Mukai partners of given smooth projective varieties. For instance, it is known that two K3 surfaces are derived equivalent if and only if their Mukai lattices are Hodge isometric to each other (Or96). We also have a moduli-theoretic characterization of Fourier-Mukai partners of certain minimal elliptic surfaces due to Bridgeland and Maciocia (see Theorem 2.1).

Furthermore it is also interesting to study the cardinality of the set of isomorphism classes of Fourier–Mukai partners of $X$ (this set is often denoted by $\text{FM}(X)$ and its cardinality is called Fourier–Mukai number of $X$). Although it is predicted that the Fourier–Mukai numbers of any smooth projective varieties are finite (BM01, Ka02, To06), it is known that there are no universal bounds of Fourier–Mukai numbers for the families of K3 surfaces, abelian varieties, and rational elliptic surfaces respectively (Og02, Or02, HLOY03, Ue04).

Fourier–Mukai numbers of rational elliptic surfaces. In this article, we study the Fourier–Mukai numbers of rational elliptic surfaces over $\mathbb{C}$. 

Henceforth we consider only relatively minimal elliptic surfaces as elliptic surfaces.

Fix a rational elliptic surface $\pi_0 : B \to \mathbb{P}^1$ with a section and a point $s \in \mathbb{P}^1$, where the fiber of $\pi_0$ over $s$ is of type $I_n$ ($n \geq 0$). Choose an integer $m > 1$ and apply a logarithmic transformation to $B$ along $s$, and then we obtain a rational elliptic surface $\pi_1 : S \to \mathbb{P}^1$ with a multiple fiber of type $mI_n$ over $s$ (cf. Remark 2.2). Furthermore it is known that every rational elliptic surface $S$ has at most one multiple fiber, and it is obtained by applying a logarithmic transformation to its Jacobian $J(S)$, which is again rational.

We have a bound of the Fourier–Mukai numbers of $S$ from below as follows. In the statement, we denote by $\varphi(m)$ the Euler function.

**Theorem 1.1.** Fix $B$ and $s \in \mathbb{P}^1$ as above. Take an integer $m > 1$. Then we have a positive integer $n_0$, depending on $B$ and $s$, but not depending on $m$, satisfying

$$\frac{\varphi(m)}{n_0} \leq |FM(S)|$$

for any rational elliptic surfaces $S$ obtained from $B$ and $m$ by a logarithmic transformation along the point $s$. Consequently the Fourier–Mukai number of $S$ becomes larger as we take a larger $m$.

We remark that if $S$ has no multiple fiber or $S$ has a multiple fiber with multiplicity 2, then we readily know that $|FM(S)| = 1$ (see Remark 2.4).

We can apply our method to compute the Fourier–Mukai numbers of certain rational elliptic surfaces (see §2.5). Theorem 1.1 also produces counterexamples to Kawamata’s D-K conjecture ([Ka02]) as in [Ue04].

**Minimal 3-folds and the birational Torelli problem.** The second aim of this article is to study certain minimal 3-folds in the contexts of derived categories and the Torelli problem. Let us consider the fiber products $X$ of two rational elliptic surfaces over $\mathbb{P}^1$ with some properties. Using Theorem 1.1, we study Fourier–Mukai numbers of such 3-folds $X$. The precise statement is the following.

**Theorem 1.2.** Let $N$ be a given positive integer. Then there are smooth minimal 3-folds $X_i$ ($i = 1, \ldots, N$) satisfying the following properties:

(i) For all $i$, $\kappa(X_i) = 1$ and $X_i$’s have the following Hodge diamond:

```
1
0 0
0 19 0
1 19 19 1
0 19 0
0 0
1
```
(ii) \( X_i \) and \( X_j \) are not birationally equivalent for \( i \neq j \).

(iii) All \( X_i \)'s are mutually deformation equivalent and derived equivalent.

(iv) For all \( i, j \), we have Hodge isometries

\[
(H^3(X_i, \mathbb{Z})_{\text{free}}, Q_{X_i}) \cong (H^3(X_j, \mathbb{Z})_{\text{free}}, Q_{X_j}),
\]

where the polarizations are given by the intersection forms.

In particular, they supply a counterexample to the birational Torelli problem.

Bridgeland [Br02] shows that two smooth projective 3-folds connected by a sequence of flops are derived equivalent. Consequently birationally equivalent smooth minimal 3-folds are derived equivalent. Motivated by his result, Borisov and Căldărușu in [BC09] show that there is a pair of Calabi–Yau 3-folds such that they are derived equivalent but not birationally equivalent. Our theorem assures that a similar phenomenon happens for the case of Kodaira dimension 1. Furthermore in [Ca07] (see also [Sz04, Conjecture 0.2]), Căldărușu attempts to construct counterexamples to the birational Torelli problem for Calabi–Yau 3-folds. Another counterexample is discovered by Namikawa in [Na02] for irreducible symplectic manifolds. Our result says that the birational Torelli problem fails for the above minimal 3-folds.

The Iitaka fibrations of the above 3-folds \( X_i \)'s have multiple fibers. So the failure of the birational Torelli problem may not be very surprising because a similar phenomenon occurs for the 2-dimensional case ([Ch80], see also Remark 3.8).

Construction of this article. In §2 first we recall some general facts of Fourier–Mukai partners of elliptic surfaces and the Ogg–Shafarevich theory. After that we give the proof of Theorem 1.1. In §3 we first show some easy lemmas on Fourier–Mukai transforms between varieties of fiber products. Secondly by taking the fiber products of rational elliptic surfaces with certain properties, we construct minimal 3-folds in Theorem 1.2.

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Notations and conventions. All varieties are defined over \( \mathbb{C} \), and elliptic surface always means relatively minimal elliptic surface.

A point on a variety means a closed point unless specified otherwise.
Let $\pi : X \to Y$ be a surjective projective morphism between smooth projective varieties $X$ and $Y$. For a point $t \in Y$, we denote the scheme-theoretic fiber of $t$ by $X_t$ and the discriminant locus of $\pi$ by $\Delta(\pi)$.

We denote the diagonal with reduced structure in $X \times X$ by $\Delta_X$.

Let $\pi : B \to C$ be an elliptic surface with the 0-section and $s$ a point on $C$. We denote by $\text{Aut}_0B$ the group consisting of the automorphisms $\gamma$ of $B$ which fix the 0-section as a curve, and make the following diagram commutative for some automorphisms $\delta$ of $C$:

$$
\begin{array}{ccc}
B & \xrightarrow{\pi} & B \\
\downarrow{\pi} & & \downarrow{\pi} \\
C & \xrightarrow{\delta} & C
\end{array}
$$

Furthermore $\text{Aut}_0(B, s)$ (resp. $\text{Aut}_0(B/C)$) is the group consisting of $\gamma \in \text{Aut}_0B$ which induces $\delta \in \text{Aut} C$ fixing the point $s \in C$ (resp. all points in $C$).

For a set $I$, we denote by $|I|$ the cardinality of $I$.

2 Rational elliptic surfaces

2.1 Fourier–Mukai partners of elliptic surfaces

We need some standard notation and results before going further. Let $\pi : S \to C$ be an elliptic surface. For an object $E$ of $D^b(S)$, we define the fiber degree of $E$ as $d(E) = c_1(E) \cdot f$, where $f$ is a general fiber of $\pi$. Let us denote by $\lambda_{S/C}$ the highest common factor of the fiber degrees of objects of $D^b(S)$. Equivalently, $\lambda_{S/C}$ is the smallest number $d$ such that there is a holomorphic $d$-section of $\pi$. For integers $a > 0$ and $i$ with $i$ coprime to $a \lambda_{S/C}$, by [Br98] there exists a smooth, 2-dimensional component $J_S(a, i)$ of the moduli space of pure dimension one stable sheaves on $S$, the general point of which represents a rank $a$, degree $i$ stable vector bundle supported on a smooth fiber of $\pi$. There is a natural morphism $J_S(a, i) \to C$, taking a point representing a sheaf supported on the fiber $\pi^{-1}(x)$ of $S$ to the point $x$. This morphism is a minimal elliptic fibration (Br98). Put $J^i(S) := J_S(1, i)$. Obviously, $J^0(S) \cong J(S)$, the Jacobian surface associated to $S$, and $J^1(S) \cong S$. Moreover there is a natural isomorphism $J^i(S) \cong J^{i+\lambda_{S/C}}(S)$. Hence we may regard $i$ as an element of $\mathbb{Z}/\lambda_{S/C}\mathbb{Z}$, instead of $\mathbb{Z}$, when we consider the isomorphism classes of $J^i(S)$. We have a nice characterization of Fourier–Mukai partners of elliptic surfaces with non-zero Kodaira dimensions:
Theorem 2.1 (Proposition 4.4 in [BM01]). Let $\pi : S \to C$ be an elliptic surface and $T$ a smooth projective variety. Assume that the Kodaira dimension $\kappa(S)$ is non-zero. Then the following are equivalent.

(i) $T$ is a Fourier–Mukai partner of $S$.

(ii) $T$ is isomorphic to $J^b(S)$ for some integer $b$ with $(b, \lambda_{S/C}) = 1$.

2.2 Weil–Châtelet group

Fix an elliptic surface with a section $\pi_0 : B \to C$. Let $\eta = \text{Spec } k$ be the generic point of $C$, where $k = k(C)$ is the function field of $C$, and let $\overline{k}$ be an algebraic closure of $k$. Put $\eta = \text{Spec } k$. We define the Weil–Châtelet group $\text{WC}(B)$ by the Galois cohomology $H^1(G, B_\eta(\overline{k}))$. Here $G = \text{Gal}(\overline{k}/k)$ and $B_\eta(\overline{k})$ is the group of points of the elliptic curve $B_\eta$ defined over $k$.

Suppose that we are given a pair $(S, \varphi)$, where $S$ is an elliptic surface $S \to C$ and $\varphi$ is an isomorphism $J(S) \to B$ over $C$, fixing their 0-sections. Then we have a morphism $B_\eta \times_k S_\eta \to J(S_\eta) \times_k S_\eta \to S_\eta$. Here the first morphism is induced by $\varphi^{-1} \times id_S$ and the second is given by translation. We obtain a principal homogeneous space $S_\eta$ over $B_\eta$. Fix a point $p \in S_\eta(\overline{k})$ and take an element $g \in G$. Then the element $g(p) - p \in J(S_\eta)(\overline{k})$ can be regarded as an element of $B_\eta(\overline{k})$ via $\varphi$. The map $G \to B_\eta(\overline{k}) \quad g \mapsto g(p) - p$

is a 1-cocycle and changing a point $p$ replaces it by a 1-coboundary. Therefore this map defines a class in $\text{WC}(B)$. Since this correspondence is invertible (cf. [Se02]), we know that $\text{WC}(B)$ consists of all isomorphism classes of pairs $(S, \varphi)$. Here two pairs $(S, \varphi)$ and $(S', \varphi')$ are isomorphic if there is an isomorphism $\alpha : S \to S'$ over $C$, such that $\varphi' \circ \alpha_* = \varphi$, where $\alpha_* : J(S) \to J(S')$ is the isomorphism induced by $\alpha$ (fixing 0-sections).

$$
\begin{array}{ccc}
J(S) & \xrightarrow{\alpha_*} & J(S') \\
\varphi \downarrow & & \varphi' \downarrow \\
B & \overset{\varphi}{\longrightarrow} & B \\
\end{array}
$$

For any $\xi := (S, \varphi_1) \in \text{WC}(B), g \in G$ and $i \in \mathbb{Z}$, we obtain an element $i(g(p) - p) \in B_\eta(\overline{k})$, which can be regarded as an element of $J(J^i(S_\eta))(\overline{k})$. Therefore there is an isomorphism $\varphi_i : J(J^i(S)) \to B$ such that $(J^i(S), \varphi_i)$ defines the class $i\xi \in \text{WC}(B)$.

The group $\text{Aut}_0(B)$ acts on $\text{WC}(B)$ as follows: Let $\delta \in \text{Aut } C$ be the automorphism on $C$ induced by $\gamma \in \text{Aut}_0(B)$. Then for $\xi = (\pi_1 : S \to C, \varphi_1) \in \text{WC}(B)$,
we define
\[ \gamma \xi := (\delta \circ \pi_1 : S \to C, \gamma \circ \varphi_1). \quad (2.1) \]

Suppose that \( \kappa(S) \neq 0 \). Then Theorem 2.1 implies that the map
\[ \Phi : \{ i \xi \in WC(B) \mid i \in (\mathbb{Z}/\lambda_{S/C}\mathbb{Z})^* \} \to FM(S) \quad i \xi = (J^i(S), \varphi_i) \mapsto J^i(S) \quad (2.2) \]
is surjective. For the involution \( \gamma \in \text{Aut}_0(B/C) \), we shall see in §2.5 that
\[ \gamma \xi = -\xi. \]

In particular, the map \( \Phi \) is not injective whenever \( \lambda_{S/C} > 2 \). Therefore it is important to investigate the preimages of \( \Phi \) when we study the Fourier–Mukai number of \( S \).

### 2.3 Local invariant

Consider the completion \( \widehat{O}_{C,t} \) of the local ring \( O_{C,t} \) for \( t \in C \). We denote by \( K_t \) its field of fraction and put \( \widetilde{B}_t := B \times_C \text{Spec} K_t \) and \( \widetilde{S}_t := S \times_C \text{Spec} K_t \) for \( \xi = (S, \varphi) \in WC(B) \). If \( S_t \) is not a multiple fiber, \( S_t \) has a reduced irreducible component. Thus Hensel’s lemma implies that there is a section of \( \widetilde{S}_t \to \text{Spec} K_t \) and then \( \widetilde{S}_t \) is a principal homogeneous space over \( \widetilde{B}_t \). Put \( WC(B_t) := H^1(G_t, \mathcal{O}_{\widetilde{B}_t} ) \), where \( G_t \) is the Galois group of the local field \( K_t \) and \( \overline{K_t} \) is an algebraic closure of \( K_t \). Denote by \( \xi_t \) the class in \( WC(B_t) \) induced by \( \xi \). Then there is a group homomorphism
\[ WC(B) \to \bigoplus_{t \in C} WC(B_t) \quad \xi \mapsto (\xi_t)_{t \in C}, \quad (2.3) \]
which is compatible with the natural group action of \( \text{Aut}_0 B \) (cf. (2.1)). The element \( \xi_t \) is called local invariant at \( t \) and the kernel of the map (2.3) is called Tate–Shafarevich group and denote it by \( \text{III}(B) \). Namely, \( \text{III}(B) \) is the subgroup of \( WC(B) \) which consists of all isomorphism classes of pairs \((\pi_1 : S \to C, \varphi_1)\) such that \( \pi_1 \) has no multiple fibers.

It is known that for \( \xi = (\pi_1 : S \to C, \varphi_1) \in WC(B) \), \( \pi_1 \) has a multiple fiber of multiplicity \( m \) over \( s \in C \) if and only if \( \xi_s \) in (2.3) has order \( m \). Moreover the map in (2.3) is surjective if \( B \) is not the product \( C \times E \), where \( E \) is an elliptic curve (cf. [CD89, Theorem 5.4.1]).

We have the following isomorphisms ([Do81, page 124]);
\[ WC(B_s) \cong H^1(B_s, \mathbb{Q}/\mathbb{Z}) \quad (2.4) \]
\[ \cong \begin{cases} \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z} & \text{if } B_s \text{ is a smooth elliptic curve,} \\ \mathbb{Q}/\mathbb{Z} & \text{if } B_s \text{ is a singular fiber of type } I_n \ (n > 0), \\ 0 & \text{otherwise.} \end{cases} \quad (2.5) \]

The group \( \text{Aut}_0(B, s) \) naturally acts on both sides of (2.4) and the isomorphism in (2.4) is equivariant under these actions.
2.4 Proof of Theorem 1.1

Throughout in this subsection, we denote by \( \pi_0: S \to \mathbb{P}^1 \) a rational elliptic surface with a fiber of type \( m \mathbb{I}_n \) over a point \( s \in \mathbb{P}^1 \) for \( m > 1 \), and denote its Jacobian by \( \pi_0: B \to \mathbb{P}^1 \).

Remark 2.2. For a rational elliptic surface \( S, B = J(S) \) is again rational. Conversely, starting from a rational elliptic surface \( B \), we obtain a rational surface \( S \) by a logarithmic transformation along a single point \( s \). (To show these statements, see, for instance, [FM94, Proposition 1.3.23, Theorem 1.6.7].) In particular, \( J^i(S) \) is also rational, since \( J^i(S) \) is obtained from \( B \) by a logarithmic transformation.

Any automorphisms of \( B \) induce automorphisms of \( \mathbb{P}^1 \), since the rational surface \( B \) has a unique elliptic fibration. Hence we have natural homomorphisms

\[
\text{Aut } S \to \text{Aut}_0(B, s) \to \text{Aut } \mathbb{P}^1.
\]

Lemma 2.3. The group

\[
\text{Im}(\text{Aut}_0(B, s) \to \text{Aut } \mathbb{P}^1) \cong \frac{\text{Aut}_0(B, s)}{\text{Aut}_0(B/\mathbb{P}^1)}
\]

is finite.

Proof. Recall that the fiber of \( \pi_0 \) over the point \( s \) is of type \( I_n \) for \( n \geq 0 \). By a quick view of Persson’s list [Pe90], we know that \( |\Delta(\pi_0)\setminus\{s\}| \geq 2 \). Since the group \( \text{Im}(\text{Aut}_0(B, s) \to \text{Aut } \mathbb{P}^1) \) preserves the point \( s \) and the set \( \Delta(\pi_0)\setminus\{s\} \), it is finite. \( \square \)

We put

\[
N_1 := \text{Im}(\text{Aut}_0(B, s) \to \text{Aut } \mathbb{P}^1), \quad N_2 := \text{Coker}(\text{Aut } S \to N_1)
\]

and call their cardinalities \( n_1, n_2 \) respectively. We define \( \xi := (S, \text{id}_B) \in WC(B) \).

Now we are in position to show Theorem 1.1.

Proof of Theorem 1.1. First of all, we note that \( \lambda_{S/\mathbb{P}^1} = m \), since every \((-1)\)-curve on \( S \) is a \( m \)-section of \( \pi_1 \). By the definition of the map \( \Phi \) in (2.2), we have the natural one to one correspondence between the set \( \Phi^{-1}(J^i(S)) \) and the set

\[
I(i) := \{k \in (\mathbb{Z}/m\mathbb{Z})^* \mid J^i(S) \cong J^k(S)\}
\]

for any \( i \in (\mathbb{Z}/m\mathbb{Z})^* \).

Because there is an isomorphism \( J^i(J^i(S)) \cong J^{ij}(S) \) for \( i, j \in \mathbb{Z}/m\mathbb{Z}, \) \( S \) is isomorphic to a surface \( T \) if and only if \( J^i(S) \cong J^i(T) \) for some \( i \in \mathbb{Z}/m\mathbb{Z} \).
Then we have the equality $|\Phi^{-1}(J^1(S))| = |\Phi^{-1}(J^i(S))|$ for any $i \in (\mathbb{Z}/m\mathbb{Z})^*$. Therefore we know that

$$|FM(S)| = \frac{|(\mathbb{Z}/m\mathbb{Z})^*|}{|\Phi^{-1}(J^1(S))|} = \frac{\varphi(m)}{|I(1)|}.$$ 

Henceforth we identify $S$ and $J^1(S)$ by the natural isomorphism between them. For each $k \in I(1)$, we fix an isomorphism $\alpha_k : S \rightarrow J^k(S)$ because both of $S$ and $J^k(S)$ are rational, $\alpha_k$ induces an automorphism $\delta_k$ of $\mathbb{P}^1$ such that the following diagram is commutative:

Consider the following automorphism $\gamma_k \in \text{Aut}_0(B,s)$:

$$\gamma_k := \varphi_k \circ \alpha_{ks} : B = J(S) \rightarrow J(J^k(S)) \rightarrow B,$$

where we define $\varphi_k$ as $k\xi = (J^k(S), \varphi_k)$ holds (see (2.2)). Hence for any $k \in I(1)$, we can find an element $\gamma_k \in \text{Aut}_0(B,s)$ such that $\gamma_k\xi = k\xi$ (see (2.1)). In particular we have

$$I(1) = \{ k \in (\mathbb{Z}/m\mathbb{Z})^* \mid k\xi \in \text{Aut}_0(B,s)\xi \}$$

and hence

$$I(1) \leq |\text{Aut}_0(B,s)\xi|.$$ 

Next we show the following:

$$|\text{Aut}_0(B,s)\xi| \leq n_2 |\text{Aut}_0(B/\mathbb{P}^1)|.$$ (2.7)

Let us define two equivalent conditions on the group $\text{Aut}_0(B,s)$: For $\gamma_1, \gamma_2 \in \text{Aut}_0(B,s)$, define

$$\gamma_1 \sim_1 \gamma_2 \iff \gamma_1\xi = \gamma_2\xi,$$

$$\gamma_1 \sim_2 \gamma_2 \iff \gamma\gamma_1\xi = \gamma_2\xi \text{ for some } \gamma \in \text{Aut}_0(B/\mathbb{P}^1).$$

By the definition, we have

$$\gamma_1 \sim_2 \gamma_2 \iff \gamma\gamma_2^{-1}\gamma_1 = \alpha_s \text{ for } \alpha \in \text{Aut} S \text{ and } \gamma \in \text{Aut}_0(B/\mathbb{P}^1)$$

But these conditions are also equivalent to the fact that the automorphism of $\mathbb{P}^1$ induced by $\gamma_2^{-1}\gamma_1$ belongs to the group

$$\text{Im(\text{Aut} S \rightarrow \text{Aut} \mathbb{P}^1)}.$$
Therefore there is a one-to-one correspondence between the set \( \text{Aut}_0(B, s)/\sim_2 \) and the set \( N_2 \). By the definitions of \( \sim_1 \) and \( \sim_2 \), at most \( |\text{Aut}_0(B/\mathbb{P}^1)| \) elements in \( \text{Aut}_0(B, s)/\sim_1 \) correspond to a single element in \( \text{Aut}_0(B, s)/\sim_2 \). Hence (2.7) follows.

Put \( n_0 := n_1|\text{Aut}_0(B/\mathbb{P}^1)| \), then we obtain

\[
\frac{\varphi(m)}{n_0} \leq \frac{\varphi(m)}{n_2|\text{Aut}_0(B/\mathbb{P}^1)|} \leq \frac{\varphi(m)}{|I(1)|} = |\text{FM}(S)|.
\]

Obviously the integer \( n_0 \) is independent on the choice of \( m \).

**Remark 2.4.** For a minimal rational elliptic surface \( S \) without multiple fibers or with a multiple fiber of multiplicity 2, we can readily see from Theorem 2.1 that the Fourier–Mukai partner of \( S \) is only itself, i.e.

\[ |\text{FM}(S)| = 1. \]

**2.5 Examples**

Our method in the proof of Theorem 1.1 sometimes gives the Fourier–Mukai numbers of some rational elliptic surfaces. Here we use the same notation as in the previous subsection.

Since \( B \) is also rational, \( \text{III}(B) \) is trivial (cf. [FM94, Example 1.5.12], [CD89, Corollary 5.4.9]). In particular, the homomorphism (2.3) becomes an isomorphism

\[
WC(B) \cong \bigoplus_{t \in \mathbb{P}^1} WC(B_t) \quad \xi \mapsto (\xi_t)_{t \in \mathbb{P}^1} \quad (2.8)
\]

Because \( \pi_1 \) has a multiple fiber of type \( mI_n \) over the point \( s \in \mathbb{P}^1 \), the element \( \xi = (S, \text{id}_B) \in WC(B) \) is completely determined by an element \( \xi_s \in WC(B_s) \) of the form

\[
\xi_s = \begin{cases} 
(p/m, q/m) \in \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z} & \text{in the case } n = 0, \\
p/m & \text{in the case } n > 0.
\end{cases}
\]

Here we identify \( WC(B_s) \) with \( \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z} \) or \( \mathbb{Q}/\mathbb{Z} \) by the isomorphisms (2.4) and (2.5), and recall that the order of \( \xi_s \) is \( m \).

We put

\[
I' := \{ k \in (\mathbb{Z}/m\mathbb{Z})^* \mid k\xi_s \in \text{Aut}_0(B/\mathbb{P}^1)\xi_s \} \\
= \{ k \in (\mathbb{Z}/m\mathbb{Z})^* \mid k\xi \in \text{Aut}_0(B/\mathbb{P}^1)\xi \}.
\]
Case: \( B_s \) is a smooth elliptic curve. Observing the isomorphisms (2.4) and (2.5) carefully, we know that a generator \( \gamma \) of \( \text{Aut}_0(B/C) \) acts on \( WC(B_s) \cong \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z} \) as

\[
\gamma(a, b) = \begin{cases} 
(a, -b) & \text{if } \text{ord } \gamma = 2 \\
(-b, a) & \text{if } \text{ord } \gamma = 4 \\
(-b, a+b) & \text{if } \text{ord } \gamma = 6,
\end{cases}
\] (2.9)

where \( a, b \in \mathbb{Q}/\mathbb{Z} \).

Example 2.5. (i) Consider the case \( |\text{Aut}_0(B/P^1)| = 4 \) and take \( \xi_s = (1/5, 3/5) \). Then \( |I'| = 4 \).

(ii) Consider the case \( |\text{Aut}_0(B/P^1)| = 6 \) and take \( \xi_s = (1/7, 4/7) \). Then \( |I'| = 6 \).

Case: \( B_s \) is a singular fiber of type \( I_n \) \((n > 0)\). A generator \( \gamma \) of \( \text{Aut}_0(B/C) \cong \mathbb{Z}/2\mathbb{Z} \) acts on \( WC(B_s) \cong \mathbb{Q}/\mathbb{Z} \) as

\[
\gamma a = -a
\] (2.10)

for \( a \in \mathbb{Q}/\mathbb{Z} \). Therefore we obtain \( I' = \{1, -1\}(\subset (\mathbb{Z}/m\mathbb{Z})^*) \).

Assumption \( n_1 = 1 \). Assume that \( n_1 = 1 \), which also implies \( n_2 = 1 \). Then by the proof of Theorem 1.1, we know that the set \( \text{Aut}_0(B, s)/\sim_2 \) has a single element. Hence for any \( \gamma_1 \in \text{Aut}_0(B, s) \), there is an automorphism \( \gamma \in \text{Aut}_0(B/P^1) \) such that \( \gamma \xi = \gamma_1 \xi \). This implies

\[
I' = \{ k \in (\mathbb{Z}/m\mathbb{Z})^* \mid k \xi \in \text{Aut}_0(B, s)\xi \},
\]

and hence \( I' = I(1) \) by (2.8). Therefore we obtain the formula:

\[
|\text{FM}(S)| = \frac{\varphi(m)}{|I'|}.
\]

As mentioned above (cf. (2.8)), every rational elliptic surface \( \pi_1 : S \to \mathbb{P}^1 \) is determined by its local invariant \( \xi_s \) in \( WC(B_s) \). On the other hand, \(|I'|\) can be easily computed if we are given the local invariant \( \xi_s \). In particular, if \( n_1 = 1 \) for some fixed \( B \) and \( s \in \mathbb{P}^1 \), the Fourier-Mukai number \(|\text{FM}(S)|\) with \( B = J(S) \) is computable.

Example 2.6. We can actually find the following rational elliptic surfaces \( \pi_0 : B \to \mathbb{P}^1 \) in the Persson’s list [Pe90]. As we see below, they satisfy \( n_1 = 1 \) for some \( s \in \mathbb{P}^1 \). We have a lot of choices of \( B \) and \( s \in \mathbb{P}^1 \) satisfying \( n_1 = 1 \) by [Pe90].
(i) Suppose that $\pi_0$ has two singular fibers of types $\text{II}^*$ and $\text{II}$. Choose a point $s$ such that $B_s$ is smooth. Then any elements in the group $N_1$ should fix all three points in the set $\Delta(\pi_0) \cup \{s\}$. Therefore it must be the identity, i.e. $n_1 = 1$. In this case, the $J$-map has the constant value 0 and $|\text{Aut}_0(B/P^1)| = 6$. Define $\xi_s := (1/7, 4/7) \in WC(B_s)$ and denote by $S$ the rational elliptic surface corresponding to $\xi_s$. Then Example 2.5(ii) implies

$$|\text{FM}(S)| = \frac{\varphi(7)}{6} = 1.$$ 

(ii) Suppose that $\pi_0$ has a singular fiber of type $\text{II}^*$ and two singular fibers of type $\text{I}_1$. Take a point $s$ such that $B_s$ is of type $\text{I}_1$. Then by the same reason as above, we conclude $n_1 = 1$. In this case, $|\text{Aut}_0(B/P^1)| = 2$. Apply a logarithmic transformation along the point $s$ and then we obtain a rational elliptic surface $S_m$ whose Jacobian surface is $B$, and $S_m$ has a multiple fiber of type $m\text{I}_1$ over the point $s$ for some $m > 0$. Assume $m > 2$ to assure $1 \neq -1$ in $(\mathbb{Z}/m\mathbb{Z})^*$. Then we know

$$|\text{FM}(S_m)| = \frac{\varphi(m)}{2}.$$ 

### 3 Smooth minimal 3-folds with $\kappa(X) = 1$

#### 3.1 Fourier–Mukai transforms between varieties of fiber products

The results in this subsection must be well-known to specialists. Let $X$ and $Y$ be smooth projective varieties and $\mathcal{U}$ an object in $D^b(X \times Y)$. Then the object $\mathcal{U}$ determines the integral functor

$$\Phi^\mathcal{U} := \mathbb{R}\pi_*(\pi^\text{\text{\text{-}}} L) \otimes \mathcal{U} : D^b(X) \to D^b(Y),$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are projections. We call $\mathcal{U}$ the kernel of the functor. If $\Phi^\mathcal{U}$ gives an equivalence, we call it a Fourier–Mukai transform.

Next suppose that we are given a closed integral subvariety $\iota : Z \hookrightarrow X \times Y$ and a perfect object $\mathcal{U}_Z \in D^\text{perf}(Z)$. Moreover assume that the restrictions $\pi_X|_Z$ and $\pi_Y|_Z$ are flat. Then the projection formula yields that the functor

$$\mathbb{R}(\pi_Y|_Z)_*((\pi_X|_Z)^\text{\text{-}} L) \otimes \mathcal{U}_Z : D^b(X) \to D^b(Y)$$

is isomorphic to the functor $\Phi^{\iota_*\mathcal{U}_Z}$. By the abuse of notation, we also denote it by $\Phi^{\iota_*\mathcal{U}_Z}$. 

11
Let $X_0$ and $Y_0$ be smooth closed subvarieties of $X$ and $Y$ respectively and subvariety $Z_0(\subset Z)$ is a scheme-theoretic pull-back of $X_0$ and $Y_0$ by $\pi_X|_Z$ and $\pi_Y|_Z$. We consider the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_X|_Z} & Z \\
\downarrow{\iota_X} & & \downarrow{\iota_Z} \\
X_0 & \xleftarrow{\pi_X|_{Z_0}} & Z_0 \\
\end{array}
\quad
\begin{array}{ccc}
Z & \xrightarrow{\pi_Y|_{Z_0}} & Y \\
\downarrow{\iota_Z} & & \downarrow{\iota_Y} \\
Y_0 & \xleftarrow{\pi_Y|_{Z_0}} & Y_0 \\
\end{array}
\]

**Lemma 3.1.** The integral functor $\Phi^{U_0} : D^b(X_0) \to D^b(Y_0)$ is a Fourier–Mukai transform, where $U_0 := L\iota_0^*\pi_{Z_0}$. 

**Proof.** By the flat base change theorem and the projection formula, we obtain

\[
\Phi^{U_0}(\iota_X^*\alpha) \cong \iota_Y^*\Phi^{U_0}(\alpha)
\]

(3.1) for all $\alpha \in D^b(X_0)$. For a quasi-inverse functor $\Phi^{U_0'}$ of $\Phi^{U_0}$, a similar statement is true. Hence we know that

$$\Phi^{U_0'} \circ \Phi^{U_0}(\mathcal{O}_s) \cong \mathcal{O}_s,
\Phi^{U_0} \circ \Phi^{U_0'}(\mathcal{O}_t) \cong \mathcal{O}_t,$$

for all closed points $s \in X_0$ and $t \in Y_0$. By [BM98, 3.3] these conditions imply that

$$\Phi^{U_0} \circ \Phi^{U_0} \in \mathrm{Pic}
X_0,
\Phi^{U_0} \circ \Phi^{U_0} \in \mathrm{Pic}
Y_0$$

as autoequivalences of $D^b(X_0)$ and $D^b(Y_0)$. Combining these with (3.1), we conclude that

$$\Phi^{U_0'} \circ \Phi^{U_0} \cong \mathrm{id},
\Phi^{U_0} \circ \Phi^{U_0'} \cong \mathrm{id}.$$ 

\[\Box\]

Let $S_i \to C$ $(i = 1, \ldots, 4)$ be flat projective morphisms between smooth projective varieties. We denote various closed embeddings by

- $\iota_{ij} : S_i \times_C S_j \hookrightarrow S_j \times_S S_i$ $(i, j \in \{1, \ldots, 4\}),$
- $\iota : S_1 \times_C S_2 \times_C S_3 \to S_1 \times_S S_2 \times_S S_3 \times_S S_4,$
- $\iota_0 : S_1 \times_C S_2 \times_C S_3 \times_C S_4 \hookrightarrow S_1 \times_S S_2 \times_S S_3 \times_S S_4.$

**Lemma 3.2.** Suppose that perfect objects

- $\mathcal{P} \in D_{\mathrm{perf}}(S_1 \times_C S_3)$ and $\mathcal{Q} \in D_{\mathrm{perf}}(S_2 \times_C S_4)$

give Fourier–Mukai transforms

$$\Phi^\mathcal{P} : D^b(S_1) \to D^b(S_3) \quad \text{and} \quad \Phi^\mathcal{Q} : D^b(S_2) \to D^b(S_4)$$

12
respectively. Assume that both of $S_1 \times_C S_2$ and $S_3 \times_C S_4$ are smooth. Then the object
\[ P \boxtimes Q \in D_{\text{perf}}(S_1 \times_C S_2 \times S_3 \times_C S_4) \]
gives a Fourier–Mukai transform
\[ \Phi^{P \boxtimes Q}: D^b(S_1 \times_C S_2) \to D^b(S_3 \times_C S_4). \]

**Proof.** Under the assumptions in Lemma 3.2, $\Phi^{\iota_{13} \star P \boxtimes \iota_{24} \star Q}$ gives a Fourier–Mukai transform from $D^b(S_1 \times S_2)$ to $D^b(S_3 \times S_4)$ (see [Hu06, Exercise 5.20]). By the projection formula and the flat base change theorem, we have
\[ \iota_{13} \star P \boxtimes \iota_{24} \star Q = \iota_L \star \pi_{13}^* P \otimes \pi_{24}^* \iota_{24} \star Q \]
\[ = \iota_L \star (\pi_{13}^* P \otimes \iota_{R \star} \pi_{24}^* Q) \]
\[ = \iota_L \star (\pi_{13}^* P \otimes \iota_{R \star} \pi_{24}^* Q) \]
\[ = \iota_L \star (\pi_{13}^* P \otimes \iota_{R \star} \pi_{24}^* Q) \]
\[ = \iota_L \star (\Phi^{P \boxtimes Q}(P \boxtimes Q)), \]
where we consider the following diagram:

Apply Lemma 3.1 for
\[ X = S_1 \times S_2, \ Y = S_3 \times S_4, \ Z = S_1 \times C S_3 \times S_2 \times C S_4, \ U_Z = P \boxtimes Q \]
and
\[ X_0 = S_1 \times C S_2, \ Y_0 = S_3 \times C S_4, \ Z_0 = S_1 \times C S_2 \times C S_3 \times C S_4 \]
to get the conclusion. \[\square\]

### 3.2 Schoen’s construction

Fix a positive integer $N$ and choose an even integer $m \gg N$. Take a rational elliptic surface $\pi_0: B \to \mathbb{P}^1$ with a section. For a point $s \in \mathbb{P}^1$, let $\xi_s$ be an element of the order $m$ in $WC(B_s)$. We obtain from (2.8) a rational elliptic surface $\pi_1: S_1 \to \mathbb{P}^1$ admitting a unique multiple fiber over $s$ such that $J(S_1) \cong B$. Theorem 1.1 implies that the set $FM(S_1)$ contains at least $N$ elements $S_1, \ldots, S_N$. 

13
Lemma 3.3. For each $i, j$, there is a vector bundle $P_0$ on $S_i \times_{\mathbb{P}^1} S_j$ with $\text{rk} P_0 = 2$ such that the integral functor $\Phi^{P_0} : D^b(S_i) \to D^b(S_j)$ becomes a Fourier–Mukai transform.

Proof. By the choice of $S_i$ and $S_j$, we have $S_i \cong J^b(S_j)$ for some $b \in \mathbb{Z}$ such that $(b, m) = 1$ by Theorem 2.1. In particular, [BM01, Lemma 4.2] implies that $J^b_S(2, b) \cong J^b(S_j)$, where $J^b_S(2, b)$ is defined in §2.1. Take the universal sheaf $P_0$ on $J^b_S(2, b) \times_{\mathbb{P}^1} S_j(\cong S_i \times_{\mathbb{P}^1} S_j)$. Then $P_0$ satisfies the desired properties. See [BM01, §4] and [Br98] for the details.

Take another rational elliptic surface $\pi : S \to \mathbb{P}^1$. Let us denote the elliptic fibrations by $\pi_i : S_i \to \mathbb{P}^1$ and define $X_i$ to be the fiber product of $S$ and $S_i$ over $\mathbb{P}^1$ (cf. [Sc88]). We define $p_i, p, f_i$ as follows.

We also assume that

- $\Delta(\pi_1) \cap \Delta(\pi)$ is empty (equivalently, $\Delta(\pi_1) \cap \Delta(\pi)$ is empty for all $i$, since $\Delta(\pi_1) = \Delta(\pi_i)$), and
- the generic fibers of $\pi_i$ and $\pi$ are not isogenous each other.

The first condition implies that $X_i$’s are smooth and the second condition will be used when we apply the argument in [Na91]. The first condition is fulfilled by replacing $\pi$ with the composition of $\pi$ and some automorphism on $\mathbb{P}^1$. The second condition is satisfied by choosing general $S$ in the family of elliptic surfaces.

3.3 Proof of Theorem 1.2

In this subsection we inherit all notations in §3.2. Smooth projective varieties $X$ and $Y$ are said to be deformation equivalent if there is a smooth proper holomorphic map between connected complex analytic spaces $h : X \to T$ such that each irreducible component of $T$ is smooth and $X_s \cong X$ and $X_t \cong Y$ for points $s, t \in T$. 

14
Lemma 3.4. All $X_i$’s are minimal with $\kappa(X_i) = 1$ and they have the following Hodge diamond:

\[
\begin{array}{cccc}
  & 1 & & \\
 0 & 0 & & \\
 0 & 19 & 0 & \\
 1 & 19 & 19 & 1 \\
 0 & 19 & 0 & \\
 0 & 0 & & \\
 1 & & & \\
\end{array}
\]

Furthermore they are deformation equivalent and derived equivalent to each other.

Proof. Consider the morphism

$$g_i(:= \pi_i \times \pi) : S_i \times S \to \mathbb{P}^1 \times \mathbb{P}^1$$

and then we have $g_i^* \Delta_{\mathbb{P}^1} = X_i$. Therefore the adjunction formula says that $K_{X_i} = K_{S_i \times S} + X_i|_{X_i} = rF_i$ for a general fiber $F_i$ of $f_i$ and some $r \in \mathbb{Q}_{>0}$. This implies that $\kappa(X_i) = 1$. By the use of the flat base change theorem, we can show

$$\mathbb{R}f_{i*} \mathcal{O}_{X_i} \cong \mathbb{R}\pi_{i*} \mathcal{O}_{S_i} \otimes \mathbb{R}\pi_* \mathcal{O}_S$$

$$\cong (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)[-1]) \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)[-1])$$

and hence obtain

$$\mathbb{R}^1f_{i*} \mathcal{O}_{X_i} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

and $\mathbb{R}^2f_{i*} \mathcal{O}_{X_i} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$.

In particular, we have $h^1(X_i, \mathcal{O}_{X_i}) = h^2(X_i, \mathcal{O}_{X_i}) = 0$ and $h^3(X_i, \mathcal{O}_{X_i}) = 1$ by the Leray spectral sequence. The Euler number $e(X_i)$ should be 0, since the Euler number of every fiber of $f_i$ is 0.

The Picard number $\rho(X_i)(= h^{1,1} = h^{2,2})$ is 19; more precisely there is a short exact sequence

$$0 \longrightarrow \text{Pic} \mathbb{P}^1 \xrightarrow{\Pi} \text{Pic} S_i \times \text{Pic} S \xrightarrow{\pi_i^* \otimes p^*} \text{Pic} X_i \longrightarrow 0,$$

where $\Pi(M) = (\pi_i^* M, \pi^* M^{-1})$ for $M \in \text{Pic} \mathbb{P}^1$. The surjectivity of $\pi_i^* \otimes p^*$ is proved in the proof of [Na91 Proposition 1.1]. The inclusion

$$\ker(\pi_i^* \otimes p^*) \subset \text{im} \Pi$$

is proved as follows: First let us state the following claim due to Namikawa.
 Claim 3.5 (Proof of Proposition 1.1 in [Na91]). Take $L_i \in \text{Pic} S_i$ and $L \in \text{Pic} S$ and suppose that the line bundle $p^*_i L_i \otimes p^* L$ is an effective divisor on $X_i$ (henceforth we identify the isomorphism classes of line bundles with the linear equivalence classes of Cartier divisors). Then there is an integer $a \in \mathbb{Z}$ such that both of $L_i \otimes O_{\mathbb{P}^1}(-a)$ and $L \otimes O_{\mathbb{P}^1}(a)$ are effective divisors.

Take $(L_i, L) \in \text{ker}(p^*_i \otimes p^*)$. We conclude from Claim 3.5 that the pair $(L_i, L)$ is equal to a pair of some effective divisors in the group $(\text{Pic} S_i \times \text{Pic} S)/\text{Pic} \mathbb{P}^1$. In particular, to get (3.2), we may assume that $L_i$ and $L$ are effective divisors. Then we can easily deduce $(L_i, L) \in \text{im} \Pi$. The other inclusion of (3.2) is obvious.

We also know $h^1,2(X_i) = h^2,1(X_i) = 19$ by the equality $e(X_i) = 2(h^1,1 - h^{1,2})$.

Elliptic surfaces $S_i \to \mathbb{P}^1 (i = 1, \ldots, N)$ are deformation equivalent through elliptic surfaces ([FM94, Theorem 1.7.6]). In particular, all $X_i$’s are also deformation equivalent to each other.

The fact that $X_i$’s are mutually derived equivalent follows from Lemma 3.2. Hence the last assertion follows.

Before going further, we give a remark which is rather obvious from the above proof: In [Sc88], Schoen takes the fiber products of two rational elliptic surfaces without multiple fibers and then he obtains Calabi–Yau 3-folds as the result. In our construction, at least one of two elliptic surfaces has a multiple fiber. Consequently, our 3-folds have the Kodaira dimensions 1 as above.

Lemma 3.6. $X_i$ and $X_j$ are not birationally equivalent for $i \neq j$.

Proof. First we show that $X_i$ and $X_j$ are not isomorphic as follows. Suppose that there is an isomorphism $\varphi: X_i \to X_j$. Note that $f_i$ and $f_j$ are the Iitaka fibrations, that is, they are defined by the complete linear system of some multiple of canonical divisors $K_{X_i}$ and $K_{X_j}$. In particular, there is an automorphism $\delta$ on $\mathbb{P}^1$ such that $\delta \circ f_i = f_j \circ \varphi$. Moreover we can see that the relative Picard numbers $\rho(X_i/\mathbb{P}^1)$ and $\rho(X_j/\mathbb{P}^1)$ are 2. Thus $f_i$ (resp. $f_j$) factors through only in two ways; $f_i$ (resp. $f_j$) factors through $S_i$ (resp. $S_j$) or $S$. This is absurd by $S_i \not\cong S_j$.

Next we show that $X_i$ has no small contractions for all $i$. Suppose that $X_i$ has a small contraction contracting a curve $C$. Since $K_{X_i} \cdot C = 0$, $C$ is also contracted by the Iitaka fibration $f_i$. But this contradicts $\rho(X_i/\mathbb{P}^1) = 2$.

If minimal 3-folds $X_i$ and $X_j$ are birational, they are connected by a sequence of flops. But it is impossible by the facts proved above.

Lemma 3.7. There are Hodge isometries

\[ (H^3(X_i, \mathbb{Z})_{\text{free}}, Q_{X_i}) \cong (H^3(X_j, \mathbb{Z})_{\text{free}}, Q_{X_j}) \]
for all $i, j$, where the polarizations are given by the intersection forms.

Proof. Let $X$ and $Y$ be derived equivalent Calabi–Yau 3-folds. Căldăraru shows the existence of Hodge isometries

- between the free parts of $H^3(X, \mathbb{Z} \langle \frac{1}{2} \rangle)$ and $H^3(Y, \mathbb{Z} \langle \frac{1}{2} \rangle)$ ([Ca07, Proposition 3.1]), and
- between the free parts of $H^3(X, \mathbb{Z})$ and $H^3(Y, \mathbb{Z})$ under some additional assumptions ([Ca07, Proposition 3.4.]).

To get the conclusion as desired, we use and modify his argument in [Ca07]. Unlike 3-folds $X, Y$ there, our 3-folds $X_i$’s are not Calabi-Yau’s, and so $c_1(X_i)$ survives. We have to take care of it.

First we put $X = X_i$ and $Y = X_j$ to adapt our notations with the one in [Ca07].

**Step 1.** As is well-known, a Fourier–Mukai transform $\Phi^U: D^b(X) \to D^b(Y)$ with the kernel $U \in D^b(X \times Y)$ induces an isometry $\varphi := \pi_Y^*(\pi_X^*(-) \text{ch}(U) \sqrt{\text{td}(X \times Y)}): H^*(X, \mathbb{C}) \to H^*(Y, \mathbb{C})$.

More precisely, $\varphi$ preserves the odd cohomologies and the Hochshild graded pieces $\bigoplus_{q-p=k} H^{p,q}(X)$ and $\bigoplus_{q-p=k} H^{p,q}(Y)$ for all $k \in \mathbb{Z}$, which yields, as in the proof of [Ca07, ibid.], that $\varphi$ restricts to a Hodge isometry $\varphi|_{H^3(X, \mathbb{C})}: H^3(X, \mathbb{C}) \to H^3(Y, \mathbb{C})$.

So we want to get more information of $(\text{ch}(U) \sqrt{\text{td}(X \times Y)})^{3,3}$ for our purpose.

We have

$$\sqrt{\text{td}(X \times Y)} = 1 + \frac{1}{4} c_1(X \times Y) + \left( \frac{1}{96} c_1(X \times Y)^2 + \frac{1}{24} c_2(X \times Y) \right) + \frac{1}{96} c_1(X \times Y) c_2(X \times Y) + \text{higher order terms}.$$ 

Hence there are no $(3,3), (3,0), (0,3)$-components in $\sqrt{\text{td}(X \times Y)}$. In addition, because $H^1(X, \mathbb{Q}) = H^1(Y, \mathbb{Q}) = 0$ we have

$$(\text{ch}(U) \sqrt{\text{td}(X \times Y)})^{3,3} = (\text{ch}(U))^{3,3}(\sqrt{\text{td}(X \times Y)})^{0,0}.$$
In particular, we obtain
\[ \varphi|_{H^3(X,\mathbb{C})}(-) = \frac{1}{2} \pi_3^*(\pi^*_X(-)c_3(U))^{3,3}. \] (3.4)

**Step 3.** In this step, we show that \( \frac{1}{2} c_3(U)^{3,3} \in H^6(X \times Y, \mathbb{Z}) \) if we choose an object \( U \) appropriately. With \([118, \text{pp. 283}]\), this yields the conclusion. This time, we use ideas in the proof of \([118, \text{Proposition 3.4}]\).

Put \( Z = X \times_{\mathbb{P}^1} S \) and denote the closed embedding by \( \iota : Z \hookrightarrow X \times Y \). We choose a vector bundle \( P_0 \) on \( S_i \times_{\mathbb{P}^1} S_j \) with \( \text{rk} P_0 = 2 \) as in Lemma 3.3.

Define a sheaf \( \mathcal{U}_0 \) on \( Z \) as \( \mathcal{U}_0 := P_0 \boxtimes \mathcal{O}_{\Delta_S} \). Hence
\[ c_1(\mathcal{U}_0) = \text{rk}(P_0) \tilde{\pi}^* c_1(\mathcal{O}_{\Delta_S}) = 2\tilde{\pi}^* c_1(\mathcal{O}_{\Delta_S}), \]
where \( \tilde{\pi} : Z \to S \times_{\mathbb{P}^1} S \) is the projection. Then, as in the proof of Lemma 3.2, the object \( U := \iota_! \mathcal{U}_0 \) gives the kernel of the Fourier–Mukai transform \( \Phi^U \).

By the Grothendieck–Riemann–Roch theorem, we have
\[ \chi(U) = \iota_!(\chi(\mathcal{U}_0)) \text{td}(N_{Z/X \times Y}^{-1}) \] (3.5)
(cf. \([118, \text{pp. 283}]\)).

Define \( g := f_i \times f_j : X \times Y \to \mathbb{P}^1 \times \mathbb{P}^1 \). Then we have \( g^* \Delta_{\mathbb{P}^1} = Z \) and hence
\[ N_{Z/X \times Y} = g^* N_{\Delta_{\mathbb{P}^1} \times \mathbb{P}^1} = g^* \mathcal{O}_{\mathbb{P}^1}(2). \]
In particular, \( \text{td}(N_{Z/X \times Y})^{-1} = 1 - g^* c_1(\mathcal{O}_{\mathbb{P}^1}(1)). \)

Taking the (3,3)-components of both sides of (3.5), we obtain from (3.3)
\[
\frac{1}{2} c_3(U)^{3,3} = (\iota_!(\chi(\mathcal{U}_0)) \text{td}(N_{Z/X \times Y}^{-1}))^{3,3} \\
= (\iota_!(r(\mathcal{U}_0) + c_1(\mathcal{U}_0) + \frac{1}{2} c_2(\mathcal{U}_0)^2 - c_2(\mathcal{U}_0) + \text{h.o.t.})(1 - g^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))))^{3,3} \\
= (\iota_!(\frac{1}{2} c_1(\mathcal{U}_0)^2 - c_2(\mathcal{U}_0) - c_1(\mathcal{U}_0)g^* c_1(\mathcal{O}_{\mathbb{P}^1}(1))))^{3,3}.
\]

Therefore we obtain \( \frac{1}{2} c_3(U)^{3,3} \in H^6(X \times Y, \mathbb{Z}). \) \( \Box \)

**Proof of Theorem 1.2.** Combining Lemmas 3.4, 3.6 and 3.7, we obtain Theorem 1.2. \( \Box \)

**Remark 3.8.** In \([118, \text{Proposition 3.4}]\), Chakris gives a sketchy proof of the following result: Let \( S \) be a simply connected, minimal elliptic surface with \( p_g(S) \neq 0 \), having one or at most two multiple fibers. Then the period map has a positive dimensional fiber at the point corresponding to the surface \( S \).
Because it is known that the Fourier–Mukai number \(|\text{FM}(S)|\) is finite \([\text{BM01}]\), general elements in the fiber do not have equivalent derived categories. I am not sure whether we can show Lemma 3.7 without using derived equivalence.

**Remark 3.9.** Let \(X\) and \(Y\) be birationally equivalent smooth minimal 3-folds. Bridgeland theorem \([\text{Br02}]\) says that \(X\) and \(Y\) are derived equivalent. Assume furthermore that \(h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y) = 0\). Then Kollár \([\text{Ko89}]\) proves that there is a rational polarized Hodge isometry

\[
(H^3(X, \mathbb{Q}), Q_X) \cong (H^3(Y, \mathbb{Q}), Q_Y).
\]

Hence the non-birationality (Lemma 3.6) makes Theorem 1.2 novel.

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