G\textsubscript{2} Hitchin functionals at one loop

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Abstract
We consider the quantization of the effective target space description of topological M-theory in terms of the Hitchin functional whose critical points describe seven manifolds with a G\textsubscript{2} structure. The one-loop partition function for this theory is calculated and an extended version of it, that is related to the generalized G\textsubscript{2} geometry, is compared with the topological G\textsubscript{2} string. We relate the reduction of the effective action for the extended G\textsubscript{2} theory to the Hitchin functional description of the topological string in six dimensions. The dependence of the partition functions on the choice of background G\textsubscript{2} metric is also determined.

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1. Introduction

Topological string theory on Calabi–Yau manifolds has been the source of many recent insights in the structure of gauge theories and black holes. The traditional construction for topological strings is in terms of topologically twisted worldsheet A- and B-models, computing Kähler and complex structure deformations. The topological information these theories compute is encoded in Gromov–Witten invariants.

More recently a target space quantum foam reformulation of the A-model in terms of the Kähler structure has emerged \cite{1, 2}. The topological information computed are the Donaldson–Thomas invariants, providing a powerful reformulation of Gromov–Witten invariants. For topological string theories on Calabi–Yau manifolds there are additional well-developed computational tools using open–closed duality such as the topological vertex or matrix models.
In comparison, topological theories on $G_2$ manifold target spaces are much less explored. One motivation for considering such theories is since the $G_2$ structure couples Kähler and complex structure naturally, such a theory would couple topological A- and B-models non-perturbatively, a coupling which we expected to exist following recent work on topological string theory. A recent proposal for topological theories on $G_2$ manifolds that goes under the name of topological M-theory was given in [3].

The classical effective description of topological M-theory is in terms of a Hitchin functional [4]. Alternative topological theories on $G_2$ manifolds employing quantum worldsheet/worldvolume formulations have been proposed in terms of topological strings [5] and topological membranes [6–9]. The topological $G_2$ string and topological membrane theories [8] have the same structure of local observables associated with the de Rham cohomology of $G_2$ manifolds. The full quantum worldvolume formulation of these theories, especially the computation of the complete path integral is much more difficult though than for the usual topological theories on Calabi–Yau target spaces.

In this paper we attempt to understand the moduli space of topological M-theory in terms of a $G_2$ target space description. Our strategy is similar to the A-model quantum foam, where one considers fluctuations around a fixed background Kähler form. Here the quantum path integral is computed in terms of a topologically twisted six-dimensional Abelian gauge theory.

Analogously, the stable closed 3-form encoding the $G_2$ structure in seven dimensions can be understood as a perturbation around a fixed background associative 3-form. Locally the fluctuation can be regarded as the field strength of an Abelian 2-form gauge field. Unlike the A-model quantum foam, however, expanding the Hitchin functional to quadratic order around this fixed background gives a seven-dimensional gauge theory that is not quite topological but which is only invariant under diffeomorphisms of the $G_2$ manifold.

We will analyze the quantum structure of this theory by taking the 2-form gauge field to be topologically trivial. In practice this means we will neglect certain ‘total derivative’ terms in the expansion of the Hitchin functional involving components of the bare 2-form gauge field. This will allow us to generalize to seven dimensions the approach used by Pestun and Witten [11] to quantize the Hitchin functional for a stable 3-form in six dimensions to one-loop order. This approach is based on the powerful techniques developed by Schwarz [12] for evaluating the partition function of a degenerate quadratic action functional. The structure of the partition function here is most naturally understood by fixing the gauge symmetry of the action using the antifield-BRST method of Batalin and Vilkovisky [13]. See [3, 27] for possible alternatives to the perturbative quantization we consider here.

We will also investigate whether the one-loop agreement found by Pestun–Witten [11] between the partition functions of the extended Hitchin functional in six dimensions and the topological B-model has some analogy in seven dimensions. In particular we will repeat the one-loop partition function calculation for the extended Hitchin functional in seven dimensions to compare with the topological $G_2$ string. We find they are related only up to a multiplicative factor, corresponding to the Ray–Singer torsion invariant of the background $G_2$ manifold. It is not clear to us whether precise agreement could be obtained by a more careful analysis

\[4\] A topological version of F-theory on Spin(7) manifolds which are trivial torus fibrations over Calabi–Yau spaces was also considered in [10].

\[5\] The topological $G_2$ string partition function is only well understood below genus two. At genus zero it computes the Hitchin functional while its genus one contribution will be calculated in this paper. The topological membrane partition function is written only formally.

\[6\] For more conventional gauge theories such local ‘total derivative’ terms usually correspond to topological invariants computing certain characteristic classes for the gauge bundle from the patching conditions. Unlike in conventional Abelian gauge theory where the gauge field corresponds to a connection on a line bundle over the base space, the 2-form gauge field we have here corresponds to a connection on a gerbe.
incorporating the global topological structure of the local total derivative terms we have dropped. Nonetheless, it seems that the topological symmetry of such terms could potentially give rise to non-trivial one-loop determinants which we have ignored.

Our one-loop quantization of the generalized $G_2$ Hitchin functional is in terms of linear variations of a closed stable odd form in seven dimensions. However, the odd form can be parametrized nonlinearly in terms of other fields, that would be related to the dilaton, B-field, metric and RR flux moduli in compactifications of physical string theory on generalized $G_2$ manifolds. Hence an additional question is if we are using the appropriate degrees of freedom to describe the quantum theory. It would be interesting to see if our results could be checked by comparison with the couplings appearing in effective actions for generalized $G_2$ compactifications of physical string and M-theory.

A summary of the content of the paper is as follows. In section 2 we consider the expansion to cubic order of the Hitchin functional for a stable 3-form in seven dimensions around a fixed background $G_2$ manifold. It is only the quadratic term in the expansion that will contribute to the one-loop partition function. The local total derivative terms will be identified in this quadratic action. Section 3 begins with a brief summary of the Lagrangian antifield-BRST or BV formalism followed by a detailed analysis of the BV quantization of the quadratic Hitchin action in seven dimensions. Section 4 begins by summarizing the theory of elliptic resolvents used in [12]. We will then identify the resolvent that describes the BV quantized seven-dimensional quadratic Hitchin action. This will allow us to express the partition function in terms of determinants of elliptic operators in the resolvent. In section 5 we will repeat the aforementioned analysis for the generalized Hitchin functional for a stable odd form in seven dimensions. Section 6 contains a calculation of the one-loop partition function of the topological $G_2$ string, which will be compared with the Hitchin functional computations. In section 7 these results will be compared with the Pestun–Witten analysis in one dimension lower. Section 8 describes the dependence of these one-loop partition functions on the choice of background $G_2$ metric and proposes a seven-dimensional origin for the gravitational anomaly in the B-model. Section 9 contains our conclusions and a summary of interesting open questions related to this work.

2. Perturbation of the Hitchin functional

Consider the proposed classical effective action for topological M-theory [3] given by the Hitchin functional

$$\frac{1}{7} \int_M \Phi \wedge \ast_\Phi \Phi,$$

(2.1)

for a stable closed 3-form $\Phi$, whose extrema within a given cohomology class of $\Phi$ define metrics of $G_2$ holonomy. Recall that a Riemannian metric $g$ can be constructed from $\Phi$ using the formula

$$\sqrt{g}_{MN} = \frac{1}{144} \Phi_{MAB} \Phi_{NCD} \Phi_{EFG} \epsilon^{ABCDEF} = : \mathcal{G}_{MN},$$

(2.2)

and it is with respect to this metric that the Hodge-star $\ast_\Phi$ is defined. When both $\Phi$ and $\ast_\Phi \Phi$ are closed the Levi-Civita connection of the metric $g$ has holonomy in $G_2$ and (2.1) corresponds to the volume

$$\text{vol}_M = \int_M \sqrt{g} = \int_M (\det \mathcal{G})^{1/9},$$

(2.3)

of the $G_2$ manifold $M$. 
We will consider the expansion of the Hitchin functional \((2.1)\) around a fixed harmonic 3-form \(\phi\) that encodes the geometry of a background metric of \(G_2\) holonomy, that is
\[
d\phi = 0, \quad d^\ast \phi = 0.
\] (2.4)

We define \(M_0\) to be the seven manifold \(M\) equipped with this background metric. For simplicity we take \(\phi\) such that it reconstructs a flat metric although all subsequent formulae generalize in a straightforward way for curved \(G_2\) backgrounds.

Since \(\phi_1\) is closed then expanding around the fixed background \(\phi_1 = \phi + H\) \((2.5)\) implies that \(dH = 0\). The 3-form perturbation \(H\) is then understood as the field strength of an Abelian 2-form gauge field \(B\), which locally can be written as \(H = dB\).

One can expand
\[
G_{MN} = \delta_{MN} + A_{MN} + B_{MN} + C_{MN},
\] (2.6)
in powers of \(H\), such that
\[
A_{MN} = \frac{1}{2} H_{M AB} \phi_{N AB} + \frac{1}{2} H_{N AB} \phi_{M AB}
\]
\[
B_{MN} = \frac{1}{3} H_{MAB} H_{NCD} \phi^{ABC D}
\]
\[
C_{MN} = \frac{1}{144} H_{MAB} H_{NCD} H_{EFG} \epsilon^{ABCDEFG},
\] (2.7)

where indices are contracted using the flat background metric. To cubic order in \(H\), one finds
\[
(\det G)^{1/9} = 1 + \frac{1}{5} \tr A + \frac{1}{2} [\tr B - \frac{1}{2} \tr (A^2) + \frac{1}{15} (\tr A)^2]
\]
\[
+ \frac{1}{9} \left[ \tr C - \tr (AB) + \frac{1}{3} \tr A \tr B + \frac{1}{3} \tr (A^3) - \frac{1}{15} \tr A \tr (A^2) + \frac{1}{486} (\tr A)^3 \right].
\] (2.8)
The unit term corresponds to the volume measure for the flat background. The linear term
\[
\tr A = \frac{1}{2} \phi_{MNP} H_{MNP}
\]
is locally a total derivative which we will ignore in our analysis. The quadratic and higher order terms become more complicated and so we will discuss their structure separately.

2.1. Quadratic part

Let us first note some useful identities that will allow us to write the quadratic terms in the Hitchin functional in a convenient way. The projection operators in \((B.2)\) can be used to decompose the 3-form
\[
H_{MNP} = (P^I_{27} H)_{MNP} + (P^I_{2} H)_{MNP} + (P^I_{1} H)_{MNP}
\]
\[
= X_{MNP} + \phi_{MNP} Y^Q + \phi_{MNP} Z,
\] (2.9)

where \(Y_M = \frac{1}{32} \phi_{IJM} H^{IJK}\) and \(Z = \frac{1}{32} \phi_{IJK} H^{IJK}\). The first identity in \((A.2)\) then implies
\[
2X_{MNP} = -3X_{[MNP]} Y^Q + \phi_{MNP} Z.
\]
\[
(24)^2 Y_M Y^M + (42)^2 Z^2 = 6H_{MNP} H^{MNP} + 9\phi_{IJKL} H_{IJKM} H_{KL}.
\] (2.10)

Substituting the decomposition \((2.9)\) into the second term in the right-hand side of \((2.10)\) gives
\[
H_{MNP} H^{MNP} = X_{MNP} X^{MNP} + 24Y_M Y^M + 42Z^2.
\] (2.11)

This is useful because \(A_{MN} = \frac{1}{4} X_{M AB} \phi_{N AB} + 3\delta_{MN} Z\) (using the identity \(X_{M AB} \phi_{N AB} = X_{N AB} \phi_{M AB}\)) and explicit calculation gives
\[
\tr (A^2) = \frac{1}{4} X_{MNP} X^{MNP} + 63Z^2,
\] (2.12)
so that using the previous identity implies
\[
\text{tr } \mathcal{B} - \frac{1}{2} \text{tr}(\mathcal{A}^2) + \frac{1}{18} (\text{tr } \mathcal{A})^2 = -\frac{1}{4} |H_{MNP}|^2 + \frac{1}{18} |\phi_{MNPQ} H^{NPQ}|^2 + \frac{1}{72} |\phi_{MNP} H^{MPNQ}|^2.
\] (2.13)

One can check explicitly that the integral of this quadratic term is invariant under \( \delta \phi_{MNP} = 0, \delta B_{MN} = \phi_{MNP} v^P \) (i.e. under background-preserving diffeomorphisms \( \delta x^M = -v^M(\lambda) \)).

The quadratic term above is related to the metric on the moduli space of \( G_2 \) manifolds described by Hitchin in [20], this metric being just the second functional derivative of (2.1) with respect to \( \Phi \). In particular, if we define \( \delta \Phi = H \) then (2.13) can be used to write
\[
\frac{1}{9} \left[ \text{tr } \mathcal{B} - \frac{1}{2} \text{tr}(\mathcal{A}^2) + \frac{1}{18} (\text{tr } \mathcal{A})^2 \right] \delta \chi = \frac{1}{9} \delta(\Phi_\delta) \wedge \delta \Phi,
\] (2.14)

where
\[
\delta(\Phi_\delta) = \frac{1}{3} \ast P_1^2(\delta \Phi) + \ast P_1^2(\delta \Phi) - \ast P_{27}^2(\delta \Phi) =: \ast L(\delta \Phi), \tag{2.15}
\]
to linear order in \( \delta \Phi \), using the identities mentioned above and in appendix A (* being the Hodge dual with respect to flat background \( \phi \)). The linear combination \( L = \frac{1}{9} P_1^1 + P_1^1 - P_{27}^1 \) of 3-form projectors has been defined for notational convenience in forthcoming expressions. From this perspective it is clear that the \( G_2 \) moduli space metric has \((1 + b_3^1, b_{27}^1)\) signature. This corresponds to the Lorentzian signature \((1, b^3 - 1)\) for smooth compact seven manifolds with full \( G_2 \) holonomy.

Like the linear term, some terms in the quadratic part of the Hitchin functional are also local total derivatives. To see this let us locally decompose the 2-form gauge field into irreducible representations of \( G_2 \)
\[
B_{MN} = \tilde{B}_{MN} + \frac{1}{2} \phi_{MNP} A^P, \tag{2.16}
\]
using the 2-form projection operators defined in (B.2), where \( \tilde{B}_{MN} \in \Lambda_3^{14} \) and \( A^M = \phi_{MNP} B_{NP} \). Plugging this expression into (2.13) one finds
\[
\text{tr } \mathcal{B} - \frac{1}{2} \text{tr}(\mathcal{A}^2) + \frac{1}{18} (\text{tr } \mathcal{A})^2 = -\frac{1}{4} |\tilde{H}|^2 + \frac{1}{18} |\phi_{MNPQ} \tilde{H}^{NPQ}|^2
\]
\[
+ \partial M \left[ \frac{1}{2} A^M \partial N A^N \right] + \frac{1}{72} |\phi_{MNPQ} A_N \partial_P A_Q - \frac{1}{2} \phi_{MNP} \partial_P \tilde{H}^{NPQ} A^Q|,
\] (2.17)

where we define \( \tilde{H} = d\tilde{B} \). This is just for notational convenience, \( \tilde{H} \) is certainly not a gauge-invariant field strength. The integral of the second line in (2.17) is however gauge invariant under \( \delta \tilde{B} = P_{14}^1 d\lambda \). The third line in (2.17) involving the component \( A \) in the \( 7 \) irrep of \( G_2 \) is the aforementioned local total derivative. We will ignore these terms in the BV quantization in the next section. When \( B \) is topologically trivial we are justified in neglecting them and their omission can be understood in terms of fixing diffeomorphism symmetry (by relating \( A^M \) to the diffeomorphism generator \( u^M \)).

For the analysis in the next section it will be convenient to label the integral of the second line of (2.17) \( S_0 \) and rewrite it in form notation as
\[
S_0 = \frac{3}{2} \int \tilde{H} \wedge \ast L \tilde{H} = \frac{3}{2} \int \tilde{H} \wedge \ast (2P_1^1 \tilde{H} - \tilde{H}), \tag{2.18}
\]
which can be easily derived from (2.14) and (2.15) using the identity \( P_1^1 \tilde{H} = 0 \).

2.2. Cubic part

The terms of cubic order in the Hitchin functional are most conveniently expressed in terms of the projections \( X, Y \) and \( Z \) of \( \tilde{H} \) defined in the previous subsection. The new terms one must
calculate (that are not just polynomials of the quadratic and linear ones) are
\[
\text{tr}(A^3) = -\frac{1}{2} X_{MAB} X^{NAB} X^{MCD} \phi_{NCD} + X_{MNA} X_{NPB} X_{PMC} \phi^{ABC} \\
+ 3|X_{MNP}|^2 Z + 189 Z^3
\]
\[
\text{tr}(AB) = -\frac{1}{8} X_{MAB} X^{NAB} X^{MCD} \phi_{NCD} + \frac{1}{2} X_{MNA} X_{NPB} X_{PMC} \phi^{ABC} \\
+ \frac{3}{2}|X_{MNP}|^2 Z - \frac{1}{2} X_{MAB} \phi^{NAB} Y^M Y_N + \frac{1}{4} \text{tr}(A) \text{tr}(B)
\]
\[
\text{tr}C = -\frac{1}{12} X_{MAB} X^{NAB} X^{MCD} \phi_{NCD} + \frac{1}{4} X_{MNA} X_{NPB} X_{PMC} \phi^{ABC} \\
- \frac{1}{12} |X_{MNP}|^2 Z + X_{MAB} \phi^{NAB} Y^M Y_N + 6|Y_M|^2 Z + 7Z^3.
\]
(2.19)

Combining these expressions one finds
\[
\text{tr}C - \text{tr}(AB) + \frac{1}{2} \text{tr}A \text{tr}B + \frac{1}{4} \text{tr}(A^3) - \frac{1}{4} \text{tr}A \text{tr}(A^2) + \frac{1}{360} (\text{tr}A)^3 \\
= \frac{1}{8} X_{MAB} X^{NAB} X^{MCD} \phi_{NCD} + \frac{1}{2} X_{MNA} X_{NPB} X_{PMC} \phi^{ABC} \\
- \frac{1}{12} |X_{MNP}|^2 Z + 2|Y_M|^2 Z + \frac{7}{2} Z^3.
\]
(2.20)

This diffeomorphism invariant cubic term can be understood as a BRST invariant operator deforming the quadratic action calculated above. It will not effect the one-loop calculation of the partition function we are interested in here but is important when going beyond this order.

3. BV quantization

Before getting into the details of the BV quantization of the quadratic Hitchin action, it may be helpful to set up terminology by first giving a brief review of the Lagrangian antifield-BRST formalism, following the excellent lectures by Henneaux [14] where more details can be found.

3.1. Lagrangian antifield-BRST formalism

The basic idea is to implement the restriction of the configuration space spanned by functions of all fields \{φ\} ∈ I in a given field theory with classical action \(S_0[φ]\) to the physical subspace of functions of on-shell configurations \(\Sigma_1 \subset I\) modulo gauge equivalence, by means of constraints involving a nilpotent BRST operator \(Q\) on the former space giving the latter space as its cohomology. The construction of \(Q\) can be understood in terms of two preliminary nilpotent operators \(δ\) and \(d\) which are used to individually impose the on-shell and gauge-equivalence constraints, irrespective.

The zeroth homology group \(H_0(δ)\) is known as the \(Koszul–Tate\) resolution of \(C^∞(\Sigma)\). It turns out to be sufficient to consider \(H_0(δ)\) due to a consistency condition implying that all higher homology groups \(H_k>0(δ)\) must vanish. The grading of \(δ\) is called the antighost number (\{φ\} have the antighost number zero while \(δ\) itself has the antighost number −1 which is why the on-shell configuration space is a homology rather than the cohomology group).

In the absence of gauge symmetry, the structure of \(H_0(δ)\) is necessarily simple. The kernel (ker\(δ\)) \(= C^∞(I)\). The image (im\(δ\)) \(\subset H_0(δ)\) is the subset of functions whose vanishing defines all the equations of motion. Thus one has implicitly defined a set of so-called antifields \{φ∗\} such that \(δ\) acting on each antifield \(φ^∗\) is proportional to the field equation for \(φ\) (thus \{φ∗\} have antighost number 1).

In the presence of gauge symmetry one still has \(H_0(δ) = C^∞(\Sigma)\) but now one finds \(H_{k>0}(δ) \neq 0\) which contradicts the consistency condition. The reason for this is because gauge transformations of the antifields \{φ∗\} are closed under \(δ\) and thus lie in \(H_1(δ)\). The
trick is to introduce more so-called antighost fields \( \{C^*\} \) (with antighost number 2) such that each of the \( \delta \)-closed irreducible gauge variations above equals \( \delta C^* \) and is therefore trivial in \( H_1(\delta) \). This turns out to be sufficient if all gauge symmetries are irreducible (i.e. if no possible gauge transformations of fields vanish or are proportional to equations of motion). For each reducible gauge symmetry one can have a non-trivial element of \( H_2(\delta) \) which is again avoided by adding another antighost field \( \eta^* \) (with antighost number 3) to trivialize it. Obstructions to the triviality of \( H_{>2}(\delta) \) will not concern us here and we refer the interested reader to \([14]\) for a discussion of their resolution.

The zeroth cohomology group \( H^0(\delta) \) corresponds to the algebra of gauge-invariant functions on \( \Sigma \). Non-vanishing higher cohomology groups \( H^{k>0}(\delta) \) are not allowed. The grading of \( \delta \) is called pure ghost number \( (\Phi) \) have the pure ghost number zero while \( \delta \) itself has the pure ghost number 1). The action of the vertical exterior derivative along gauge orbits \( \delta \) is generated by the set of tangent vectors \( \{X\} \) at \( \phi \) on \( \Sigma \). The number of such tangent vectors equals the number of linearly independent gauge symmetries. It is convenient to define the set of one-forms or ghosts \( \{C\} \) as the dual of the tangent vectors \( \{X\} \) (thus \( \{C\} \) have pure ghost number 1). In the case where there exist reducible gauge symmetries, the set \( \{X\} \) form an overcomplete basis since its elements are subject to linear algebraic constraints on \( \Sigma \) (one constraint per reducible symmetry). It turns out one can enforce these constraints automatically by modifying the action of \( \delta \) on \( \{C\} \) in terms of additional ghost for ghost fields \( \eta \) (one per reducible symmetry with pure ghost number 2) to create a free differential algebra.

Collecting all these fields together we see that there is a perfect match between the number of (fields+ghosts) \( \Phi = \{\phi, C, \eta\} \) and anti(fields+ghosts) \( \Phi^* = \{\phi^*, C^*, \eta^*\} \). The physical BRST operator \( Q \) acts on both \( \Phi \) and \( \Phi^* \) and its cohomology can be understood as the cohomology of \( \delta \) on \( \Sigma \). Schematically one has \( Q = \delta + \delta + \) 'extra' and it turns out one can always choose 'extra' such that \( Q^2 = 0 \). For relatively simple Abelian gauge theories like the one we consider there are no 'extra' terms. The grading of \( Q \) is called ghost number which, from the formula above, is given by the pure ghost number minus the antighost number. Hence \( \{\eta^*, C^*, \phi^*, \phi, C, \eta\} \) have ghost numbers \(-3, -2, -1, 0, 1, 2\). Since \( Q \) is a fermionic nilpotent operator then \( \{\phi^*, C^*, \eta\} \) obey bosonic statistics while \( \{\phi^*, C, \eta^*\} \) obey fermionic ones (we have assumed the original fields \( \{\phi\} \) are all bosonic).

The pairing between \( \Phi \) and \( \Phi^* \) implies the existence of a graded symplectic structure on the space of fields given by the antibracket

\[
(A, B) = \frac{\delta^r A}{\delta \Phi} \frac{\delta^l B}{\delta \Phi^*} - \frac{\delta^r A}{\delta \Phi^*} \frac{\delta^l B}{\delta \Phi},
\]

(3.1)

where \( A \) and \( B \) are arbitrary functionals of both \( \Phi \) and \( \Phi^* \). The symbol \( \cdot \) denotes summation over all common indices of all fields in \( \Phi \) and \( \Phi^* \). For the theories we will consider, elements of \( \Phi \) will be in form representations and it will sometimes be more convenient henceforth to take elements in \( \Phi^* \) to be in the Hodge-dual representations to their partners in \( \Phi \). Thus we would understand the antibracket above as the coefficient of a top form in spacetime and replace the contraction of common indices \( \cdot \) with a wedge product. The superscripts on the functional derivatives denote a right (\( r \)) and left (\( l \)) action on \( A \) and \( B \), which is required by the grading.

The antibracket is useful because it allows the construction of the minimal BRST-invariant action \( S[\Phi, \Phi^*] \), involving ghosts and antifields, that includes \( S_0[\phi] \). This is achieved by solving the master equation

\[
Q F = (F, S),
\]

(3.2)

8 It must be stressed that the antibracket in the Lagrangian formalism is not induced from the Poisson bracket in the Hamiltonian formalism. The antibracket seems to be a purely auxiliary structure that is lost when one fixes a gauge.
for any functional $F$. Nilpotence of $Q$ ensures that $(S, S) = 0$. An immediate consequence of the master equation is that $Q\Phi = \frac{\delta S}{\delta \Phi}$ and $Q\Phi^* = -\frac{\delta S}{\delta \Phi^*}$. The proper solution $S$ is referred to as minimal because one can always introduce new variables $[\bar{C}, \pi]$ (and their respective antifields $\{\bar{C}^*, \pi^*\}$) that are cohomologically trivial in $H^0(Q)$ (i.e., $Q\bar{C} = \pi$, $Q\pi = 0$, $Q\pi^* = \bar{C}^*$, $Q\bar{C}^* = 0$) and which do not contribute to $H^{k=0}(Q)$. Thus one can add terms of the form $\int \bar{C}^* \cdot \pi$ to $S$ to obtain the most general solution of the master equation. Such non-minimal terms typically arise in the process of gauge fixing where the antifields $\{\bar{C}^*\}$ are related to the gauge-fixing functions and $\{\pi\}$ act as Lagrange multipliers imposing $\bar{C}^* = 0$.

The final step is to remove the degeneracy (i.e., gauge symmetry) in the action above in a way that preserves the BRST structure, which will allow a more straightforward evaluation of the path integral. If there are $2N$ fields in $\{\Phi, \Phi^*\}$ then this gauge fixing can be achieved by eliminating half of them via $N$ constraints $\{\Omega = 0\}$. Such constraints are guaranteed to preserve the BRST structure provided the antibracket of any two $\Omega$ in the set vanishes. A convenient way to satisfy the above constraint is to eliminate all the antifields by setting each $\Phi^* = \frac{\delta \Psi}{\delta \Phi}$ for some choice of gauge fermion functional $\Psi[\Phi]$ (this choice is by no means unique). It is evident from the definition that $\Psi$ must be fermionic and have ghost number $-1$. Note that this constraint has removed the antibracket structure for the gauge-fixed theory. The aforementioned gauge choice can be understood geometrically as restricting to a Lagrangian submanifold of the symplectic manifold parametrized by $\{\Phi, \Phi^*\}$ (these terms of course being used in the graded sense).

### 3.2. BV quantization of quadratic Hitchin action

We are now prepared to examine the quantum structure of (2.18) for $\bar{B} \in \Lambda^2_{14}$ following the logic of the previous subsection.

In addition to the 2-form gauge field $\bar{B}$ we need a one-form fermionic ghost $\psi$ and a 0-form bosonic ghost-for-ghost $\varphi$. The ghost comes from the one-form $\lambda$ which parametrizes the gauge symmetry $\delta \bar{B} = P_{14}^\lambda \, d\lambda$. This gauge symmetry is reducible when $\lambda = d\kappa$ for any 0-form $\kappa$, giving rise to ghost-for-ghost $\varphi$. The antifields for $\Phi = (\bar{B}, \psi, \varphi)$ are $\Phi^* = \{\bar{\chi}, \zeta, \xi\}$ which lie in the Hodge-dual irreps of $G_2$. That is $\bar{\chi}$ is a 5-form ferion whose Hodge dual is in $\Lambda^7_{14}$, $\zeta$ is a 6-form boson and $\xi$ is a 7-form fermion. The ghost numbers of $\{\bar{\xi}, \zeta, \bar{\chi}, \bar{B}, \psi, \varphi\}$ are $[-3, -2, -1, 0, 1, 2]$ respectively.

The global BRST transformations of the fields and ghosts $\Phi$ just follow from the residual local gauge transformations

$$Q \bar{B} = P_{14}^\varphi \, d\varphi, \quad Q\psi = d\varphi, \quad Q\varphi = 0$$

The master equation $Q\Phi = \frac{\delta S}{\delta \Phi}$ then fixes the terms one must add to the classical action $S_0$ (2.18) to be of the form $\int \Phi^* \wedge Q\Phi$. Thus the minimal solution to the master equation (3.2) is

$$S = \int \frac{3}{2} \bar{H} \wedge \ast(2P_{14}^2 - 1) \bar{H} + \bar{\chi} \wedge d\psi + \zeta \wedge d\varphi.$$  

The projector $P_{14}^\lambda$ in $Q \bar{B}$ has been absorbed by $\bar{\chi} \in \Lambda^7_{14}$ in the second term in (3.4). Using (3.4) in the other master equation $Q\Phi^* = -\frac{\delta S}{\delta \Phi^*}$ gives the antifield BRST transformations

$$Q\bar{\chi} = 3d \ast (2P_{14}^2 - 1) d\bar{B}, \quad Q\zeta = d\bar{\chi}.$$  

One can verify that the above BRST transformations indeed generate a symmetry of $S$ and obey $Q^2 = 0$. One can also check that the BRST transformation $Q \ast \bar{\chi}$ is in $\Lambda^7_{14}$ as required. Note that is their antibracket is invariant under canonical graded symplectic transformations which define the ambiguity in determining the minimal action $S$ from the master equation.
that any BRST transformation of $\xi$ will be a symmetry of $S$, and will be nilpotent provided it is BRST-trivial (i.e. $Q\xi = \mu, Q\mu = 0$).

To fix the gauge symmetry of the field $B$ and ghost $\psi$ via the constraints $d^1 \tilde{B} = 0$ and $d^1 \psi = 0$ it is appropriate to add to $S$ some non-minimal terms via the introduction of the pair of 6-forms $\{\gamma, u\}$ and 7-forms $\{\varepsilon, v\}$ (plus their antifield one-forms $\{\gamma^*, u^*\}$ and 0-forms $\{\varepsilon^*, v^*\}$) which are BRST-trivial (i.e. $Q\gamma = u, Qu = 0$ etc.). The appropriate gauge fermion in this case is given by

$$\Psi = \int \gamma \wedge d^1 \tilde{B} + \varepsilon \wedge d^1 \psi + \gamma \wedge d\theta. \quad (3.6)$$

The first two terms are as we would expect in order to gauge fix via coclosure of $\tilde{B}$ and $\psi$. The reason for the third term involving an additional BRST-trivial 0-form pair $[\theta, w]$ is because it fixes a residual gauge symmetry of the first term under $\delta \gamma = d^1 \rho$ for any fermionic 7-form $\rho$. There is no possible residual symmetry from the second term since $\delta \varepsilon$ cannot be coexact in seven dimensions. The corresponding BRST-invariant non-minimal addition to the action $S$ is

$$\int \gamma^* \wedge u + \varepsilon^* \wedge v + \theta^* \wedge w. \quad (3.7)$$

Thus $\{u, v, w\}$ are understood as auxiliary fields (with ghost numbers $\{0, -1, 1\}$) that will impose the gauge-fixing constraints after imposing $\Phi^* = \frac{d\gamma}{\gamma}$ on the antifields.

Including the non-minimal fields we have $\Phi = \{\tilde{B}, \psi, \varepsilon; \gamma, \varepsilon, \theta\}$ (with ghost numbers $\{0, 1, 2; -1, -2, 0\}$) and $\Phi^* = \frac{d\gamma}{\gamma}$ fixes the antifields to be

$$\tilde{\chi} = \ast P^2_{14} \ast d^1 \gamma, \quad \varepsilon = d^1 \varepsilon, \quad \xi = 0;$$

$$\gamma^* = d^1 \tilde{B} + d\theta, \quad \varepsilon^* = d^1 \psi, \quad \theta^* = d\gamma. \quad (3.8)$$

The antifields of the Lagrange multipliers $\{u, v, w\}$ all vanish. Integrating out these auxiliary fields in the non-minimal part of the action sets the three expressions in the second line of (3.8) equal to zero. This evidently gives the desired gauge fixing for $\psi$. Taking $d^1 \tilde{B} + d\theta = 0$ implies $\theta$ must be harmonic and thus equal to a constant (we assume $\theta$ is non-singular). Hence we also have $d^1 \tilde{B} = 0$.

In the topologically trivial case we are considering, these equations further imply the global constraints that $\gamma$ be exact and $\psi$ be coexact. This of course follows from the Poincaré lemma which for $B \in \Lambda^2_{14}$ is a bit more subtle. Indeed $d^1 \tilde{B} = 0$ still implies $\tilde{B} = d^1 \Xi$, for some 3-form $\Xi$, but now there is the additional constraint $P^2_{14} d^1 \Xi = 0$ so that the right-hand side is still in $\Lambda^2_{14}$. This is non-trivial because exterior derivatives do not commute with projection operators. As shown in appendix C, this leads to an expression for $\tilde{B}$ that is second order in derivatives. Let us then summarize the gauge-fixing constraints (just quoting the result shown in appendix C)

$$\tilde{B} = d^1 (2P^2_{14} - 1) \tilde{\omega}, \quad \psi = d^1 \beta,$$

$$\tilde{\chi} = \ast P^2_{14} \ast d^1 \omega, \quad \varepsilon = d^1 \varepsilon, \quad \xi = 0. \quad (3.9)$$

where $\tilde{\omega} \in \Lambda^2_{14}, \beta, \omega \in \Lambda^2$ and $\varepsilon \in \Lambda^7$. Plugging these expressions into the graded symplectic form

$$\int \Phi^* \wedge \Phi = \int \tilde{\chi} \wedge \tilde{B} + \xi \wedge \psi + \xi \wedge \varepsilon. \quad (3.10)$$

for the minimal fields implies it vanishes identically, thus defining a Lagrangian submanifold. From this we see how the strong constraint $\xi = 0$ is required by the fact that $\psi$ is completely unconstrained.

10 An analogous situation occurs in the Kähler geometry where the existence of a coclosed real $(1, 1)$-form $b_{11}$ (obeying $\partial b_{11} = 0$ and $\bar{\partial} b_{11} = 0$) implies $b_{11} = \bar{\partial}^\dagger \bar{\partial} b_{22}$ for some real $(2, 2)$-form $b_{22}$. This example is used by Pestun and Witten [11] in gauge fixing the quadratic Hitchin functional in six dimensions.
4. Resolvent and partition function

Having gauge fixed the quadratic part of the Hitchin action in a way that preserves the BRST structure, we are now almost prepared to evaluate its partition function. We will express the partition function in terms of determinants of elliptic operators using the theory of resolvents developed by Schwarz [12]. Again to introduce the necessary terminology it will be helpful to give a very brief account of the theory of resolvents as described in [12] wherein we defer for more a detailed exposition.

4.1. Resolvents

A resolvent is a generalization of a complex in algebraic geometry that is associated with an additional piece of data corresponding to a quadratic functional on one of the linear vector spaces in the complex. This functional will be understood as the classical action for a free field theory.

More precisely, given a quadratic functional \( S_0 \) on a linear space \( \Gamma_0 \), then a sequence of linear spaces \( \Gamma_i \) (\( i = 1, \ldots, n \)) and nilpotent linear operators \( T_i : \Gamma_i \to \Gamma_{i-1} \) obeying \( T_{i-1} T_i = 0 \) is defined to be the resolvent of \( S_0 \) if \( S_0[\phi + T_i C] = S_0[\phi] \) for all \( \phi \in \Gamma_0 \) and \( C \in \Gamma_1 \). This defines a complex when \( S_0 = 0 \). The notation reflects that used in section 3.1 to illustrate that \( \Gamma_i \) are to be understood as being spanned by all the descendent ghosts for classical bosonic fields \( \phi \) in the quantum theory. The resolvent property corresponds to BRST symmetry.

The existence of an inner product \( \langle \cdot, \cdot \rangle \) on the linear spaces \( \Gamma_0 \) and \( \Gamma_i \) will be assumed and adjoint linear operators \( T_i^\dagger : \Gamma_i \to \Gamma_{i+1} \) can be constructed from \( \langle x, T_i y \rangle = \langle T_i^\dagger x, y \rangle \) for any \( x \in \Gamma_i-1, y \in \Gamma_i \). It will also be assumed that the quadratic functional \( S_0 \) can be expressed schematically as

\[
S_0[\phi] = \langle \phi, K \phi \rangle = \langle K \phi, \phi \rangle,
\]

in terms of the self-adjoint ‘kinetic’ operator \( K : \Gamma_0 \to \Gamma_0 \).\(^{11}\) Note that the resolvent property implies \( K T_i = 0 \) and so \( K \) itself can be added to the complex associated with the resolvent as

\[
0 \to \Gamma_n \xrightarrow{T_n} \cdots \xrightarrow{T_1} \Gamma_0 \xrightarrow{K} \Gamma_0 \to 0.
\]

The resolvent of \( S_0 \) is said to be elliptic if the associated complex above is elliptic (i.e. the symbols of each \( T_i \) and \( K \) are invertible).

The partition function of \( S_0 \) with respect to the resolvent \( \{ \Gamma_i, T_i \} \) is defined as

\[
Z = (\det K)^{-1/2} \prod_{i=1}^n |(\det T_i)|^{-1/(i-1)}.
\]

The reason that this quantity corresponds to the physical partition function for theory with classical action \( S_0 \) explained in the appendix of the third reference in [12]. The \( (\det K)^{-1/2} \) factor of course just comes from the path integral of the free bosonic action \( S_0 \). In terms of the antifield-BRST formalism, the remaining ghost and antifield terms in the minimal action \( S \) solving the master equation take the form \( \langle \phi^*, T_i C \rangle + \langle C*, T_2 \eta \rangle \). That is schematically \( \sum_{i=1}^n (\Gamma_{i-1}^* T_i \Gamma_i) \), where elements of \( \Gamma_i \) have Grassmann parity \( (-1)^i \) and \( \Gamma_i^* \) is the same vector space as \( \Gamma_i \) but with elements of opposite Grassmann parity. This leads to a factor \( (\det T_i)^{(-1)^{i+1}/2} \) from the antifields in each \( \Gamma_i^* \) and a factor \( (\det T_i^\dagger)^{-(-1)^{i-1}/2} \) from the ghosts in each \( \Gamma_i \) which are combined to give (4.3).

\(^{11}\) A further technical requirement is that the operators \( K^2 \) and \( T_i^\dagger T_i \) be regular. We refer to [12] for the technical definition of regularity but the upshot is that this allows one to define the (regularized) determinant of such operators in a mathematically precise way.
4.2. $G_2$ resolvent for quadratic Hitchin action

Given the close relationship between complexes and resolvents one might expect the resolvent for the quadratic Hitchin action we are considering to be related to the two well-known Dolbeault complexes for $G_2$ manifolds

$$
\begin{align*}
\hat{D} : 0 &\longrightarrow \Lambda^0_1 \rightarrow \Lambda^1_7 \rightarrow \Lambda^2_7 \rightarrow \Lambda^3_1 \rightarrow 0 \\
\hat{D} : 0 &\longrightarrow \Lambda^2_{14} \rightarrow \Lambda^3_7 \oplus \Lambda^2_7 \rightarrow \Lambda^1_7 \oplus \Lambda^1_7 \rightarrow \Lambda^0_{14} \rightarrow 0,
\end{align*}
$$

(4.4)

As we will now see, the appropriate complex for the $G_2$ Hitchin action forms a subset of both these complexes.

Following the discussion in the previous subsection for the quadratic Hitchin action (2.18) we identify $\Gamma_0 = \Lambda^2_{14}$. For suitable normalization of $\hat{B}$, the kinetic operator in $S_0$ is

$$
K = -d^* Ld = \Delta^2_{14} - \frac{3}{2} P^2_{14} dd^1,
$$

(4.5)

which is self-adjoint and indeed maps $\Lambda^2_{14} \rightarrow \Lambda^2_{14}$. The latter fact follows from the right-hand side of the expression above or by noting the identity $P^2_{14} L d P^2_{14} = 0$. The structure of ghosts encountered in section 3.2 implies $n = 2$ with $T_1 = P^2_{14} d$ and $T_2 = d$ (their adjoints being just $T^*_1 = d^1$ and $T^*_2 = d^1$). Hence the appropriate complex is

$$
\hat{D} : 0 \longrightarrow \Lambda^0_1 \rightarrow \Lambda^1_7 \rightarrow \Lambda^2_7 \rightarrow \Lambda^3_1 \rightarrow 0,
$$

(4.6)

with the extension by $K$ included. The resolvent properties can be checked explicitly but of course just follow from the BRST structure. Note that the first two elements match those in the $\hat{D}$ complex in (4.4) while the third element corresponds to the first element in the $\hat{D}$ complex.

Using these identifications, the partition function (4.3) for (2.18) can be written as

$$
Z = (\det(\Delta^2_{14} - \frac{3}{2} P^2_{14} d_1 d_1^* P^2_{14}))^{-1/2} / |\det(P^2_{14} d_1)| |\det(d_0)|^{-1}
$$

$$
= (\det(\Delta^2_{14}))^{-1/2} / |\det(P^2_{14} d_1)|^2 (\det(\Delta^0_1))^{-1/2}
$$

$$
= (\det(\Delta^2_{14}))^{-1/2} (\det(\Delta^0_1)) (\det(\Delta^0_1))^{-3/2}.
$$

(4.7)

Superscripts (subscripts) denote the form degree ($G_2$ irrep) on which the Laplacian operator $\Delta^i = d^i_{+1} d_1 + d_{i-1} d^i_1$ acts. This action is invariant because $\Delta$ commutes with the projection operators on any $G_2$ manifold. The second equality has been obtained using the identity $\det(K + T_1 T^*_1) = (\det K) |T_1|^2$ which follows because $K T_1 T^*_1 = 0$ and $T_1 T^*_1 K = 0$. The final equality is obtained using a similar result $\det(T^*_1 T_1 + T_2 T^*_2) = |\det T_1|^2 |\det T_2|^2$ following from $T_1^* T_1 T^*_2 T_2 = 0$ and $T_2 T^*_2 T^*_1 T_1 = 0$.

We have omitted the infinite-volume normalization factors coming from the zero modes of the Laplacians above. Formally the multiplicative factor from these zero modes can be written as $\text{Vol}(H^2_{14}) \text{Vol}(H^1_1) / \text{Vol}(H^2_3)$ in terms of ‘volumes’ of the appropriate cohomology groups.

12 For any $\beta \in \Lambda^2_{14}$, one can use the identity $P^2_{14} d_1 = \frac{1}{2} (\phi \wedge d^1 \beta)$ to derive this result. It is obtained by first noting that $L d \beta = (2P^2_{14} - 1) d \beta$, then substituting $2d |P^2_{14} d \beta = -\frac{1}{2} (\phi \wedge dd^1 \beta) = -dd^1 \beta + \frac{1}{2} P^2_{14} dd^1 \beta$ that follows from taking $d_1^* \phi$ of the aforementioned identity. The second equality here follows from the 2-form projector identities $sP^2_{14} = \frac{1}{2} \phi \wedge P^2_{14} + \frac{1}{2} \phi \wedge P^2_{14}$, which can be derived from the expressions in appendix B.

13 This is not entirely obvious but can be deduced from the fact that $G_2$ manifolds are Ricci-flat $R_{MN} = 0$ and their Riemann tensor obeys $R_{MNFO} \phi^{FO} = 0$. The latter property can be deduced from the formula $[\nabla_M, \nabla_N] \xi = \frac{1}{4} R_{MNAB} \Gamma^{AB} \xi$ for covariant derivatives $\nabla_M$ acting on the spinor $\xi$. That is that the gamma matrices $\Gamma_{AB}$ on the right-hand side must generate the $G_2 \subset SO(7)$ holonomy group of the manifold and so only those $\{AB\}$ indices in the adjoint $14$ of $G_2$ should appear on the right-hand side of the commutator. Thus the 7 part of the $\{AB\}$ indices of $R_{MNAB}$ must vanish identically which gives the desired property.
5. Generalized Hitchin functional

In [11], one-loop computations in a theory based on the six-dimensional Hitchin functional were compared with one-loop computations in topological string theory, and the results were found to disagree. However, agreement was found once the Hitchin functional, which is a functional of a stable 3-form, was replaced by the generalized Hitchin functional [21], which is a functional of a generic stable form of odd degree. We will now repeat the analysis of the previous sections for an analogous generalization of the Hitchin functional in seven dimensions. The result of [11] suggests it is this theory that should be compared with the topological $G_2$ string.

5.1. The generalized Hitchin functional

The appropriate generalization of the Hitchin functional in seven dimensions was described and studied in [22–24]. Its critical points correspond to seven manifolds $M$ with the generalized $G_2$ structure. That is, the structure group $Spin(7, 7)$ of $TM \oplus T^*M$ is reduced to $G_2 \times G_2$. This $G_2 \times G_2$ is the stabilizer of a generic form of odd degree in seven dimensions under the action of the conformal structure group $Spin(7, 7) \times \mathbb{R}^*$. Each $G_2 \subset Spin(7)$ fixes a unit spinor on $M$. For a fixed embedding $Spin(7) \times Spin(7) \subset Spin(7, 7) \times \mathbb{R}^*$, one finds the generalized $G_2$ structure reduces to an ordinary one when these two spinors are parallel. We refer to [22–24] for a more detailed discussion.

The explicit construction of the generalized Hitchin functional proceeds as follows. One begins by writing a stable odd form $\varrho \in \Lambda^{odd} \cong \Lambda^1 \oplus \Lambda^3 \oplus \Lambda^5 \oplus \Lambda^{714}$ as\(^{14}\)

$$\varrho = e^{-\varphi} e^B \wedge (s\alpha - c\Phi - s*\Phi(\alpha \wedge \Phi) - s\sigma(\alpha \wedge *\Phi + c^2 \Phi \wedge *\Phi)),$$  \hspace{1cm} (5.1)

in terms of a scalar ‘dilaton’ $\varphi$, a 2-form $B$ (which is related to the $B$-field) and a 3-form $\Phi$. The Hodge star $*\Phi$ is defined in terms of the metric associated with $\Phi$, as in section 2. Furthermore, $s$ and $c$ are real numbers satisfying $s^2 + c^2 = 1$ and $\alpha$ is a unit one-form, i.e. $\int_M \alpha \wedge *\Phi \alpha = \int_M \frac{1}{2} \Phi \wedge *\Phi$. Despite the highly nonlinear structure in (5.1), a simple consistency check verifies that the number of independent components on the left- and right-hand side match. The odd form $\varrho$ has $\sum_{p=0}^{\infty} \binom{7}{2p+1} = 64$ components, while on the right-hand side there are two scalars, one unit one-form (with six independent degrees of freedom), plus a generic 2- and 3-form, which also adds up to a total of 64 independent components.

Geometrically, the metric constructed from $\Phi$ and the 2-form $B$ describes the embedding $Spin(7) \times Spin(7) \subset Spin(7, 7)$ while $\varrho$ parametrizes the conformal factor in $Spin(7, 7) \times \mathbb{R}^*$. The parameters $s$ and $c$ can be understood as the sine and cosine of the angle $\theta$ between the two unit spinors fixed under the action of each $G_2 \subset Spin(7)$ in the stabilizer.

The next step is to define an even form $\Box_{\varrho} \varrho = e^{-\varphi} e^B \wedge (c - c*\Phi + s*\Phi(\alpha \wedge *\Phi) - s\sigma(\alpha \wedge \Phi - s*\Phi \alpha)).$  \hspace{1cm} (5.2)

The generalized Hitchin functional is then defined as

$$\int_M \varrho \wedge \varrho,$$  \hspace{1cm} (5.3)

\(^{14}\) Stability here means that the orbit of $\varrho$ under $Spin(7, 7) \times \mathbb{R}^*$ forms an open subset of $\Lambda^{odd}$.

\(^{15}\) We define $e^B = 1 + B + \frac{1}{2} B \wedge B + \frac{1}{6} B \wedge B \wedge B$ in seven dimensions.
where the even form 
\[ \hat{\varrho} = \sigma(\square \varrho) = e^{-\varphi} e^{-\Phi} \wedge \ast \Phi \varrho_0, \] (5.4)
and we have defined \( \varrho_0 = e^{\varphi} e^{-\Phi} \wedge \varrho \) in the second equality.

It is easy to verify that
\[ \int_M \varrho \wedge \hat{\varrho} = \frac{8}{7} \int_M e^{-2\varphi} \Phi \wedge \ast \Phi. \] (5.5)
Thus the generalized Hitchin functional looks very similar to the ordinary one\(^{16}\). In particular, note that the generalized functional does not depend on either \( \alpha \) or \( B \) in the nonlinear parametrization (5.1). This invariance is similar to that found in the case of generalized Calabi–Yau manifolds \([29],[30]\), about which more will be discussed in the next section.

The first variation of (5.3) can be written as
\[ 2 \int_M \delta \varrho \wedge \hat{\varrho} = 2 \int_M \varrho \wedge \delta \hat{\varrho}. \] (5.6)
This can be obtained by direct calculation but also follows from \( \int_M \varrho \wedge \hat{\varrho} \) being ‘degree two’ in \( \varrho \). Recall that a similar property followed for the ordinary \( G_2 \) Hitchin functional from it being a homogeneous polynomial in \( \Phi \) of degree 7/3.\(^{17}\) Thus, for variations \( \delta \varrho = d\omega \) (for any \( \omega \in \Lambda^{even} \)) within a fixed cohomology class \([\varrho] \in H^{3,0}(M, \mathbb{R})\), the critical points of the generalized Hitchin functional correspond to generalized \( G_2 \) manifolds defined by \( d\varrho = 0 \), \( d\hat{\varrho} = 0 \).

As a quick consistency check of the variation above, note that the \( B \)-field variation in \( \delta \varrho \) appears in the form
\[ 2 \int_M \delta B \wedge \varrho \wedge \hat{\varrho}, \]
and one easily verifies that \( \varrho \wedge \hat{\varrho} \) has no 5-form component, so that the \( B \)-variation does not contribute. This agrees with the fact that the generalized Hitchin functional is independent of \( B \).

### 5.2. Perturbation

Having found the first variation of the generalized Hitchin functional (5.3), let us now proceed as in the previous sections and expand \( \int_M \varrho \wedge \hat{\varrho} \) to quadratic order in linear fluctuations of \( \varrho \) around a generalized \( G_2 \) manifold \( M_0 \), defined by a fixed background odd form \( \tilde{\varrho} \) that obeys \( d\tilde{\varrho} = 0 \) and \( d\hat{\tilde{\varrho}} = 0 \) (where \( \hat{\tilde{\varrho}} = \sigma(\square \tilde{\varrho}) \)). To simplify matters, we will choose this background to be an ordinary \( G_2 \) holonomy manifold, i.e. for which \( \varphi = B = s = 0 \) and \( \Phi = \phi \) is the associative 3-form.

We first expand \( \varrho = \tilde{\varrho} + \delta \varrho \), and likewise for \( \hat{\varrho} \), to linear order in variations of the parameters \( \varphi, B, \alpha, s \) and \( \Phi \). One finds the components
\[ \delta \varrho^1 = \delta (s \alpha) \]
\[ \delta \varrho^3 = \delta \varphi - \delta \Phi - \ast (\delta (s \alpha) \wedge \phi) \]
\[ \delta \varrho^5 = -\delta B \wedge \phi - \delta (s \alpha) \wedge \phi \]
\[ \delta \varrho^7 = -\frac{1}{7} \delta \varphi \wedge \phi + \frac{1}{7} \ast \phi \wedge \delta \Phi, \] (5.7)

\(^{16}\) Indeed it can be recast as the ordinary Hitchin functional \( \Phi \wedge \ast \Phi = e^{-2\varphi} \Phi \wedge \ast \phi \Phi \) in terms of the rescaled 3-form \( \tilde{\Phi} = e^{-\varphi}/7 \phi \).

\(^{17}\) Of course things are a bit more subtle in the generalized case with regard to homogeneity. One finds a well-defined notion of graded homogeneity exists for terms in \( \varrho \) and \( \hat{\varrho} \) if rescaling \( \Phi \) with weight 1 is accompanied by rescaling \( \alpha \) with weight 1/3 (all other fields have weight zero). This structure actually follows from scaling the constraint \( \int_M \alpha \wedge \ast \Phi = \int_M \frac{1}{7} \Phi \wedge \ast \Phi \).
for the first order variation of \( \varrho \) in (5.1), and

\[
\begin{align*}
\delta \varrho^0 &= -\delta \varphi \\
\delta \varrho^2 &= -\delta B - *(\delta(s\alpha) \wedge \varphi) \\
\delta \varrho^4 &= \delta \varphi \wedge \varphi - \delta(s\alpha) \wedge \varphi \\
\delta \varrho^6 &= \delta B \wedge \varphi + *(\delta(s\alpha)),
\end{align*}
\]

(5.8)

for \( \hat{\varrho} \).

In terms of these variations of \( \varrho \) and \( \hat{\varrho} \), the generalized Hitchin functional can be expanded as

\[
\int_M \varrho \wedge \hat{\varrho} = \int_M \varrho \wedge \hat{\varrho} + 2 \int_M \delta \varrho \wedge \delta \hat{\varrho}.
\]

(5.9)

Since we will be interested in only linear variations \( \delta \varrho = d\omega \) within a fixed cohomology class, we have not included the additional quadratic term \( \frac{1}{2} \int_M \delta^2 \varrho \wedge \hat{\varrho} = \frac{1}{2} \int_M \delta \varrho \wedge \delta^2 \hat{\varrho} \) which occurs when expanding \( \varrho \) and \( \hat{\varrho} \) as polynomials in linear variations of the parameters \( \varphi, B, \alpha, s \) and \( \Phi \). It must be stressed that we are assuming it as the linear variations \( \delta \varrho \) that describe the degrees of freedom of the quadratic Hitchin action here rather than those of the parameters \( \varphi, B, \alpha, s \) and \( \Phi \). It would be interesting to check this assumption by comparison with the degrees of freedom describing moduli in generalized \( G_2 \) compactifications of physical string and M-theory.

Plugging the linear variations of \( \varrho \) and \( \hat{\varrho} \) into the quadratic term \( S_0 = \int_M \delta \varrho \wedge \delta \hat{\varrho} \) gives

\[
S_0 = \int_M \delta \Phi \wedge \delta(\varphi B) - \frac{8}{3} \delta \varphi \delta \Phi \wedge \varphi + \frac{8}{7}(\delta \varphi)^2 \varphi \wedge \varphi + \delta \Phi \wedge \delta \varphi + \delta B \wedge \varphi + \delta \varphi \wedge \varphi.
\]

(5.10)

Note that demanding linear variations of \( \varrho \) has inevitably led to a dependence on the parameters \( \alpha \) and \( B \) in the quadratic part of the generalized Hitchin functional above. This dependence would only be removed by including the term \( \int_M \delta^2 \varrho \wedge \hat{\varrho} \) involving nonlinear variations of \( \varrho \).

To express \( S_0 \) in terms of \( \delta \varrho \) components, one must invert (5.7) to write variations of the parameters in terms of \( \delta \varrho \). This gives

\[
\begin{align*}
\delta \varphi &= \frac{2}{7} P_2 (\delta \varrho^3) + \frac{1}{7} *(\delta \varrho^7) \varphi \\
\delta \Phi &= \frac{2}{7} P_2 (\delta \varrho^3) - P_2 (\delta \varrho^4) - P_2 (\delta \varrho^5) + \frac{2}{7} *(\delta \varrho^7) \varphi - *(\delta \varrho^1 \wedge \varphi) \\
\delta B &= [1 - \frac{2}{7} P_2] * \delta \varrho^2 - \frac{1}{7} *(\delta \varrho^1 \wedge \varphi).
\end{align*}
\]

(5.11)

The identities \( ***P_2 = \frac{1}{7} \varphi \wedge P_2^2 \) and \( ***P_2^2 = -\varphi \wedge P_2^2 \) have been used in deriving the last expression above.

Substituting (5.11) into (5.10), and using (2.15), implies the quadratic part of the generalized Hitchin action can be written as

\[
S_0 = \int_M \delta \varrho^3 \wedge * \left[ \frac{3}{4} P_1^3 + P_1^3 - P_1^2 \right] \delta \varrho^3 - \frac{3}{4} \delta \varrho^7 \wedge \delta \varrho^3 - \frac{1}{2} *(\delta \varrho^7) \delta \varrho^3 \wedge \varphi
\]

\[
+ \delta \varrho^5 \wedge \left[ \frac{1}{2} P_2^3 - P_2^3 - P_1^2 \right] \delta \varrho^5 - \delta \varrho^1 \wedge \delta \varrho^5 \wedge \varphi - \frac{1}{2} \delta \varrho^1 \wedge \delta \varrho^1
\]

\[
= \int_M \delta \varrho^3 \wedge * \left[ \frac{3}{4} P_1^3 + P_1^3 - P_1^2 \right] \delta \varrho^3
\]

\[
- \frac{3}{4} \left( \delta \varrho^7 + \frac{1}{3} \delta \varrho^3 \wedge \varphi \right) \wedge * \left( \delta \varrho^7 + \frac{1}{3} \delta \varrho^3 \wedge \varphi \right)
\]

\[
+ \delta \varrho^5 \wedge * \left[ \frac{1}{2} P_2^3 - P_2^3 \right] \delta \varrho^5 - \delta \varrho^1 \wedge \delta \varrho^5 \wedge \varphi - \frac{1}{2} \delta \varrho^1 \wedge \delta \varrho^1.
\]

(5.12)
5.3. Quantization

Let us now take first order variations \( \delta \Omega = \delta \omega \) (for any \( \omega \in \Lambda^\text{even} \)) within a fixed cohomology class \([\tilde{\omega}] \in H^\text{odd}(M, \mathbb{R})\) of the background. The quadratic part \( S_0 \) of the generalized Hitchin action we have just obtained corresponds to the classical action to be quantized. If \( \omega \) is globally well defined then the linear term \( \int_M \delta \Omega \wedge \tilde{\omega} \) in the expansion vanishes since the integrand is a total derivative. The zeroth order term \( \int_M \delta \omega \wedge \tilde{\omega} = \frac{1}{2} \int_M \phi \wedge \ast \phi \) is just proportional to the volume of the background \( G_2 \) manifold.

Before embarking on this, we must take care that all the symmetries of the generalized Hitchin action are being accounted for. Recall that for the ordinary quadratic \( G_2 \) Hitchin action, background-preserving diffeomorphisms on \( \Phi \) gave rise to the symmetry under \( B \rightarrow B + \iota_v \phi \), for any vector field \( v \) on \( M \), in addition to the obvious gauge symmetry under \( B \rightarrow B + \ast d\lambda \).

However, it turned out that the part of \( B \) in \( \Lambda^2_1 \), only contributed a total derivative to the action and was ignored. Thus, since the diffeomorphism symmetry only acts on this component of \( B \), it was irrelevant in the quantization of the diffeomorphism-invariant \( \tilde{B} \in \Lambda^1_{14} \) part.

A similar story applies to the generalized \( G_2 \) Hitchin functional, but is somewhat more complicated. In this case one has background-preserving diffeomorphisms plus shifts by exact \( B \)-fields on \( TM \oplus T^*M \) for \( \varphi \) which give rise to the symmetry under \( \omega \rightarrow \omega + \iota_v \tilde{\varphi} + \xi \wedge \tilde{\varphi} \), for any vector field \( v \) and one-form \( \xi \) on \( M \), in addition to the gauge symmetry \( \omega \rightarrow \omega + \ast d\lambda \), for any \( \lambda \in \Lambda^\text{odd} \). The symmetry \( \omega \rightarrow \omega + \iota_v \tilde{\varphi} \) again corresponds to diffeomorphisms of \( M \) while \( \omega \rightarrow \omega + \xi \wedge \tilde{\varphi} \) can be understood as shifting \( B \rightarrow B + d\xi \) by an exact 2-form.

These two symmetries also correspond to the subset of \( \text{Spin}(7, 7) \) transformations that are automorphisms of the Courant bracket.

Let us first address the latter symmetry. Since \( \tilde{\varphi} = -\phi + \frac{1}{2} \phi \wedge \ast \phi \) for the background we have chosen, the only non-vanishing contribution to \( \xi \wedge \tilde{\varphi} \) is from the 4-form part \( -\xi \wedge \phi \). This corresponds to a transformation of the component \( \omega^4 \), where \( \delta \omega^4 = d\omega^4 \). In particular it acts only on the component of \( \omega^4 \) in the irreducible subspace \( \Lambda^4_2 \). However, one can check that the two terms involving \( \delta \omega^4 = d\omega^4 \) in \( S_0 \) only contain the component \( \omega^4 \in \Lambda^4_{27,0,1} \) with the 7 part dropping out as a total derivative, and so this symmetry is redundant.

The only contributions to \( \iota_v \tilde{\varphi} \) under diffeomorphisms come from its 2-form and 6-form parts. These correspond to the transformations \( \omega^2 \rightarrow \omega^2 + \iota_v \phi \) and \( \omega^6 \rightarrow \omega^6 + \ast v^6 \) of the components of \( \omega \), which certainly do appear in the action \( S_0 \) (\( v^6 \) denotes the one-form dual to vector \( v \) in the \( \omega^6 \) transformation). Note again that only the part of \( \omega^2 \) in \( \Lambda^2_1 \) transforms under diffeomorphisms. However, this part of \( \omega^2 \) does not just give a total derivative contribution to \( S_0 \). (Note the factor of \( 3/4 \) in the projector in square brackets in the first equality in (5.12), relative to the factor \( 4/3 \) in (2.15) that led to a total derivative contribution for the \( \Lambda^2_1 \) part.)

The trick here, highlighted by the second equality in (5.12), is to observe that although \( \omega^6 \) and the 7 part of \( \omega^2 \) individually transform under the diffeomorphism generated by \( v \), one can find a particular linear combination of them

\[
C := \omega^6 + \frac{1}{14} \omega^2 \wedge \ast \phi,
\]

that is diffeomorphism invariant. Therefore, up to total derivatives that we ignore, the action (5.12) can be written purely in terms of the diffeomorphism-invariant fields \( \tilde{B} := P_{14}^1 \omega^2, C := \omega^6 \) and \( \tilde{E} := P_{27,10,1}^1 \omega^4 \), and so we find this symmetry can also be ignored in our quantization.

Thus we are left with only the gauge symmetry under \( \delta \tilde{B} = P_{14}^1 \ast d\lambda, \delta C = d\mu \) and \( \delta \tilde{E} = P_{27,10,1}^1 dv \) in \( S_0 \), for any \( \lambda \in \Lambda^1, \mu \in \Lambda^5 \) and \( v \in \Lambda^3_{27,0,1} \). The required prefactor \( P_{27,10,1}^1 \) in the gauge transformation for \( \tilde{E} \) projects out any singlet component of \( v \) identically.
Before going on to consider the partition function for \( S_0 \), it will be convenient to illustrate how a field redefinition involving a shift by \( D \) of the singlet part of \( \tilde{E} \) can be used to remove the term \( \int_M \delta \phi^5 \wedge \ast \delta \phi^5 \wedge \ast \phi \) from the action. This works by first noting the identity
\[
\frac{1}{2} |P_7 \delta \phi^5 - \delta \phi^1 \wedge \ast \phi|^2 - 2 |\delta \phi^1|^2 = \frac{1}{2} |P_7 \delta \phi^5|^2 - \delta \phi^1 \wedge \ast \delta \phi^5 \wedge \ast \phi - \frac{1}{2} |\delta \phi^1|^2,
\]
which follows using \( |\xi \wedge \ast \phi|^2 = 3 |\xi|^2 \) for any one-form \( \xi \). In addition, one can check that \( P_{14} \delta \phi^5 = P_{14} \delta \phi \) projects out the singlet part of \( \tilde{E} \). Thus by redefining the singlet part \( P_1 \tilde{E} \to P_1 \tilde{E} - D + \phi \) one can rewrite the action (5.12) more conveniently as
\[
S_0 = \int_M \bar{d} \tilde{B} \wedge \ast \left( 2 P_7^2 - 1 \right) d \tilde{B} - \frac{3}{4} dC \wedge \ast dC - 2 dD \wedge \ast dD + d \tilde{E} \wedge \ast \left( \frac{3}{2} P_7^2 - 1 \right) d \tilde{E},
\]
where \( \tilde{E} \) is now the redefined field. This redefinition has therefore diagonalized the classical action. Note that the quadratic generalized Hitchin action (5.14) contains the ordinary quadratic Hitchin action we quantized in sections 3 and 4. In addition there are the decoupled actions for a free 6-form \( C \) and scalar \( D \), plus the somewhat more complicated action for the 4-form \( \tilde{E} \in \Lambda^4_{27 \oplus 1} \). The BV quantizations of \( C \) and \( \tilde{E} \) are detailed in appendices D and E respectively.

5.4. Partition function

To obtain the one-loop partition function for the generalized Hitchin action, we can just multiply the one-loop partition function found previously for the ordinary quadratic Hitchin action with those for the decoupled fields \( C \), \( D \) and \( \tilde{E} \).

The action for the scalar \( D \) is non-degenerate and so its partition function is simply \( Z_0 = (\det \Delta_0^0)^{1/2} \). The partition function for the 6-form \( C \) was calculated in appendix D and found to equal the reciprocal of the Ray–Singer torsion of the background \( G_2 \) manifold, \( Z_6 = I_{RS}^{-1} \). The calculation in appendix E also yielded \( Z_{27 \oplus 1}^{27 \oplus 1} = I_{RS}^{-1} \) for \( \tilde{E} \). Using the expression \( Z = Z_{14}^{14} \) in (4.7) for the partition function of the ordinary quadratic Hitchin action, the one-loop partition function \( Z_{gen} \) for the generalized Hitchin action can be written as
\[
Z_{gen} = Z_0 Z_6 Z_{27 \oplus 1}^{27 \oplus 1} Z_{14}^{14} = \left[ (\det \Delta_0^0)^{-1/2} \right] \times [ (\det \Delta_1^1)^{1/2} (\det \Delta_2^2)^{-3/2} (\det \Delta_3^3)^{1/2} (\det \Delta_4^4)^{-7/2} ] \\
\quad \times [ (\det \Delta_{27}^{27})^{-1/2} (\det \Delta_{27}^{14})^{1/2} (\det \Delta_{27}^{3})^{-3/2} (\det \Delta_{27}^{4})^{2} (\det \Delta_{27}^{0})^{3/2} ] \\
\quad \times [ (\det \Delta_{14}^{14})^{-1/2} (\det \Delta_{14}^{1})^{1/2} (\det \Delta_{14}^{0})^{3/2} ] \\
\quad = (\det \Delta_1)^{-8} (\det \Delta_7)^{4} (\det \Delta_{14})^{-17/2} (\det \Delta_{27}).
\]
The second equality follows using Hodge duality \( \det \Delta^p = \det \Delta^{7-p} \) to simplify \( Z_6 \). The final equality uses orthogonality of Laplacians acting on \( G_2 \) irreps to write \( \det \Delta_3 = (\det \Delta_1^1)(\det \Delta_2^2), \det \Delta_2 = (\det \Delta_1^2)(\det \Delta_2^1) \) and also the various \( G_2 \) isomorphisms to relate \( \det \Delta_1 = \det \Delta_1^1 = \det \Delta_3^1, \det \Delta_7 = \det \Delta_7^1 = \det \Delta_7^2 = \det \Delta_7^3 \) and also \( \det \Delta_{14} = \det \Delta_{14}^2 \).

6. Topological \( G_2 \) string at one loop

The genus-one free energy for a closed string theory is given by
\[
F_1 = \int \frac{d\tau d\bar{\tau}}{\tau_2} \text{Tr}((-1)^F F_L F_R e^{2\pi i H_L - 2\pi i H_R}).
\]
If we treat this as an integral over the upper half plane, rather than the fundamental domain of the torus complex structure \( \tau = \tau_1 + i \tau_2 \), then it simplifies to

\[
F_1 = \delta(H_L - H_R) \int \frac{d\tau_2}{\tau_2} \text{Tr}((-1)^F F_L F_R e^{2\pi i \tau_2 (H_L + H_R)})
= \delta(H_L - H_R) \log \left[ \prod_{F_L, F_R} \det(2\pi (H_L + H_R))^{-1/F_L F_R} \right].
\]  

Here \( F_1 \) and \( F_R \) are the right- and left-handed fermion number operators and \( F = F_L + F_R \). To evaluate this expression we need to know how the total Hamiltonian \( H_L + H_R \) acts on a general state \( A_{\alpha_1 \cdots \alpha_7}^{N_1 \cdots N_7}(X) \psi_{M_1}^{\alpha_1} \cdots \psi_{M_7}^{\alpha_7} \) in the Hilbert space of closed \( G_2 \) string states. Recall from [5] that the left- and right-moving sectors of the worldsheet each span a copy of the \( \tilde{D} \) complex in \( G_2 \) string Hilbert space, with the left- and right-moving BRST operators \( Q_L \) and \( Q_R \) identified with \( D \) acting on each of these two copies.

The Hamiltonians in the left- and right-moving sectors are \( H_L = \{ Q_L, Q_L \} \) and \( H_R = \{ Q_R, Q_R \} \). The \( G_2 \)-irreducible \( p \)-form spaces \( \Lambda^p_\alpha, \Lambda^p_\beta \) in the same \( G_2 \) irrep \( \alpha, \beta \) are isomorphic for different values of \( p \), and the action of the operators \( H_L \) and \( H_R \) on \( \Lambda^p_\alpha \) depends only on the dimension of the \( G_2 \) irrep. Thus we need only to determine how these operators act on the tensor products of the \( 1 \) and \( 7 \) (i.e. states with 0 or 1 fermion in the left and right sectors), since the action of \( H_L + H_R \) on the other states will follow from this. This is done in appendix F where the Hamiltonian is found to act as the Laplacian operator \( \Delta_{7,7} = \Delta_1^2 + \Delta_2^2 + \Delta_{27}^2 + \Delta_1^4 \) on states in \( 7 \otimes 7, \Delta_7 = \Delta_1^2 \) on states in \( 7 \otimes I \equiv I \otimes 7 \) and \( \Delta_1^2 = \Delta_3^2 \) on states in \( I \otimes I \).

Having obtained the action of \( H_L + H_R \) on the \( G_2 \) string spectrum, we are now prepared to evaluate the one-loop result

\[
\log \left[ \prod_{F_L, F_R} \det(2\pi (H_L + H_R))^{-1/F_L F_R} \right].
\]  

Recall that the fermion numbers \( F_L \) and \( F_R \) run from 0 to 3, each labeling elements of the respective left/right copy of the \( \tilde{D} \) complex \( \Lambda_1^\alpha \rightarrow \Lambda_2^\alpha \rightarrow \Lambda_{27}^\alpha \rightarrow \Lambda_1^3 \). The relevant \( G_2 \) irreps are thus \( 1, 7, 7 \) and \( 1 \) for 0, 1, 2 and 3, respectively. To compute all the contributions in the product above we determine \( (H_L + H_R)^{-1/F_L F_R} \) for all values of \( F_L \) and \( F_R \) in the following table:

| \( F_L/F_R \) | 0   | 1   | 2   | 3   |
|--------------|-----|-----|-----|-----|
| 0            | \( (\Delta_1)^0 \) | \( (\Delta_7)^0 \) | \( (\Delta_7)^0 \) | \( (\Delta_4)^0 \) |
| 1            | \( (\Delta_7)^0 \) | \( (\Delta_{7,7})^1 \) | \( (\Delta_{7,7})^{-2} \) | \( (\Delta_7)^3 \) |
| 2            | \( (\Delta_7)^0 \) | \( (\Delta_{7,7})^{-2} \) | \( (\Delta_{7,7})^{-4} \) | \( (\Delta_7)^{-6} \) |
| 3            | \( (\Delta_1)^0 \) | \( (\Delta_7)^0 \) | \( (\Delta_7)^{-6} \) | \( (\Delta_4)^0 \) |

Combining all these contributions gives \( (\det(\Delta_{7,7})) (\det(\Delta_7))^{-6} (\det(\Delta_4))^{-6} \) which can be further simplified by decomposing the first determinant in terms of Laplacians acting on \( G_2 \) irreps. That is \( \det(\Delta_{7,7}) = (\det(\Delta_{14})) (\det(\Delta_7)) (\det(\Delta_{27}) (\det(\Delta_1)) \) because composition of any two different irreducible Laplacians in \( \Delta_1^2 + \Delta_2^2 + \Delta_{27}^2 + \Delta_1^4 \) vanishes as a result of orthogonality of the \( G_2 \) projectors. Thus we have

\[
(\det(\Delta_{7,7})) (\det(\Delta_7))^{-6} (\det(\Delta_4))^{-6} = (\det(\Delta_1)) (\det(\Delta_7))^{-5} (\det(\Delta_{14}))(\det(\Delta_{27})).
\]  

It will be convenient to normalize such that the one-loop partition function for the \( G_2 \) string is \( Z_{\text{string}} = \exp(-\frac{1}{2} F_1) \), so that

\[
Z_{\text{string}} = (\det(\Delta_1))^{-5} (\det(\Delta_7))^{5/2} (\det(\Delta_{14}))^{-1/2} (\det(\Delta_{27}))^{-1/2}.
\]
Thus we conclude that $Z_{\text{string}} \neq Z_{\text{gen}}$ but their relation will be explained in more detail in section 8.

7. Dimensional reduction

Let us now consider a special $G_2$ background of the form $M_0 = CY_3 \times S^1$ and compactify the classical quadratic Hitchin functional (2.18) on the circle. The purpose of this reduction will be compared with previous work on quantizing Hitchin functionals in six dimensions.

The background and perturbation reduce to
\[ \phi = k \wedge dt + \rho \]
\[ \phi = \tilde{\rho} \wedge dt + \frac{1}{2} k \wedge k \]
\[ \tilde{B} = A \wedge dt + b, \] (7.1)
where $k$ is the Kähler form, $\rho$ and $\tilde{\rho}$ are the real and imaginary parts of the holomorphic $(3,0)$-form on $CY_3$ and $t$ is the $S^1$ coordinate. Recall that $\tilde{B} \in \Lambda^2_{14}$ and the constraint $\phi^{MN} B_{NP} = 0$ implies the one-form $A$ and 2-form $b$ in six dimensions are not linearly independent. In particular, the reduction of this constraint implies
\[ \rho_{mnp} b^{np} + 2k_{mn} A^m = 0, \quad k_{mn} b^{mn} = 0. \]

Since $\Lambda^2 = \Lambda^{20} \oplus \Lambda^{11} \oplus \Lambda^{02} = \Lambda^2 \oplus \Lambda^2 \oplus \Lambda^2 \oplus \Lambda^2$ in terms of $SU(3)$ representations, the equations above tell us that $b$ has no single singlet part and its $3 \oplus 3 \cong 6$ vector part is proportional to $A$. It will prove more convenient to remove this dependence by describing the reduction in terms of the redefined field
\[ \tilde{b}_{mn} = b_{mn} + \frac{1}{2} \tilde{\rho}_{mnp} A^p, \] (7.2)
which obeys $\rho_{mnp} \tilde{b}^{np} = 0, k_{mn} \tilde{b}^{mn} = 0$ and so $\tilde{b} \in \Lambda^2_6$.

The quadratic terms in the $G_2$ Hitchin action reduce to
\[ |\phi_{MN} \tilde{H}^{MNP}|^2 = 9 |k_{mn} F^{mn}|^2 + 6k_{ab} F_{ab} \rho_{mnp} h^{mnp} + |\rho_{mnp} F^{mnp}|^2 \]
\[ *|\phi_{MN} \tilde{H}^{NPO}|^2 = 18 |F_{mn1}|^2 - 18 F_{mn1} F_{pq1} k^{mnpq} - 18 \rho_{mab} F_{ab} k^{cd} h_{mcd} \]
\[ + |\rho_{mnp} F^{mnp}|^2 - 9 h_{abc} h_{mcd} k^{abcd}, \]
\[ |\tilde{H}_{MNP}|^2 = 3 |F_{mn1}|^2 + |\tilde{h}_{mnp}|^2, \] (7.3)
where $F = dA$ and $h = db$. Some algebraic identities for products of the background Calabi–Yau data $k$ and $\rho$ have been used, which follow by substituting (7.1) into the $G_2$ identities in appendix A. The first term vanishes due to $\phi^{MNP} \tilde{B}_{NP} = 0$. The second and third terms can be expressed in terms of $\tilde{h} = db$ and $F = dA$ and simplified.

The final result is that
\[ |\tilde{H}_{MNP}|^2 - \frac{1}{10} |\phi_{MNQ} \tilde{H}^{NPO}|^2 = |\tilde{h}_{mnp}|^2 - \frac{1}{3} |\tilde{h}_{mnp} F^{mnp}|^2 \]
\[ - 3 \left( \tilde{h}_{mnp} \rho_{mnp} F^p_q - \frac{3}{4} \tilde{h}_{mab} k^{abcd} F_{cd} \right) + \frac{2}{3} \left( |F_{mn1}|^2 - \frac{1}{3} |\rho_{mnp} F^{mnp}|^2 \right), \] (7.4)
up to total derivatives which we ignore. The integral of the left-hand side being proportional to the quadratic Hitchin action (2.18) in seven dimensions.

It is worth noting that this reduced action has quite a subtle gauge symmetry arising from reduction of the symmetry under $\delta \tilde{B} = P^2_{14} d\lambda$ in seven dimensions. It is invariant under the transformations
\[ \delta \tilde{b} = P^{2}_8 d\lambda, \quad \delta A_m = \frac{1}{2} \tilde{\rho}_{mnp} \rho^p \alpha + \frac{1}{2} \partial_m \alpha, \] (7.5)
where $\lambda$ and $\alpha$ are a one-form and a scalar in six dimensions. The $\alpha$ transformation leaving $F = dA$ invariant is the usual gauge symmetry but note that $A$ also transforms under the
where \( \hat{\text{resulting Lagrangian}} \) is proportional to making background-preserving diffeomorphisms a redundant symmetry of this action. The field \( b \) is the \( 1 \)-form field \( \Sigma \), which has the gauge transformation \( \delta b^{\mu} = \rho_m \delta^a \partial_\lambda b^{\mu} + 2 \rho_m \partial^a \alpha \). In terms of this dual field, the Lagrangian (7.4) becomes

\[
|\hat{h}_{\text{map}p}|^2 - \frac{1}{2} |\hat{h}_{\text{map}k^{np}}|^2 - \frac{1}{2} |\hat{h}_{\text{map}k^{np}}|^2 q_k \partial^q \partial^q + \frac{1}{2} |\partial^a \rho_{\text{map}p}|^2 + \frac{1}{16} |\rho_{\text{map}p} \beta^a|^2, \tag{7.6}
\]

up to total derivatives\(^{18}\). Note that \( \partial^a \beta_{\text{map}p} \) is invariant under the \( \alpha \) part of the gauge transformation while \( \rho_{\text{map}p} \beta^a \) is invariant under the full gauge transformation of \( \beta \). \( \partial^a \beta_{\text{map}p} \) transforms non-trivially under the \( \lambda \) part of the gauge transformation but the integral of the first line in (7.6) is fully gauge invariant.

We are now prepared to compare these results with the analysis of Pestun and Witten [11].

### 7.1. Comparison with Pestun–Witten

The field \( b \) we obtain from dimensional reduction of the \( G_2 \) Hitchin functional may look reminiscent of the 2-form appearing in the quantization of the quadratic Hitchin action for a stable 3-form in six dimensions [11]. Indeed one might expect the results of Pestun and Witten to follow as some kind of consistent truncation of the reduction of the \( G_2 \) theory. After all, the variations \( \rho \to \rho + \partial b' \) considered in [11] form a subset of the ones \( \phi \to \phi + \partial B \) we have used in seven dimensions (within which \( k \) is invariant). We will now show that this is indeed the case, though the relationship between the quadratic actions is not quite so straightforward.

Under variations \( \rho \to \rho + \partial b' \), the quadratic part of the Hitchin functional \( \int \rho \wedge \hat{\rho} \) for a stable 3-form in six dimensions is proportional to

\[
\int \Omega^3 \left[ |h_{\text{map}p}|^2 - \frac{3}{2} |h_{\text{map}k^{np}}|^2 \right] - \frac{1}{12} |h_{\text{map}p}|^2 \rho_{\text{map}p}^2 - \frac{1}{12} |\rho_{\text{map}p} \beta^a|^2, \tag{7.7}
\]

where \( h' = \partial b' \). This is just a rewriting of the classical action in equation (2.11) of [11] in real coordinates, and equals \( 6 \int h' \wedge J h' \), where

\[
J_{\text{map}p}^{qrs} = -\frac{1}{4} c_{\text{map}p}^{qrs} + \frac{1}{4} k_{\text{map}p}^q k_p^{rs} - \frac{1}{12} \partial_{\text{map}p} \rho_{\text{map}p}^{qrs} + \frac{1}{12} \rho_{\text{map}p} \beta_{\text{map}p}^{qrs}
\]

defines the action of the complex structure of the background on 3-forms (and indeed obeys \( J^2 = -1 \) and \( J \Omega = i \Omega, \Omega = \rho + i \hat{\rho} \)).

Let us now decompose \( b'_{\text{map}p} = b_{\text{map}p} + \frac{1}{2} \rho_{\text{map}p} a^p \), where \( a_{\text{map}p} = \rho_{\text{map}p} b^{np} \) and \( \hat{b} \in \Lambda_2^3 \otimes \Lambda_2^1 \) (\( a \) and \( \hat{b} \) correspond to \( b_{20} + b_{02} \) and \( b_{11} \) in [11] in complex coordinates). Just as in equation (2.11) of [11], one finds that all terms involving \( a \) drop out of (7.7) as total derivatives, making background-preserving diffeomorphisms a redundant symmetry of this action. The resulting Lagrangian is proportional to

\[
|\hat{h}_{\text{map}p}|^2 - \frac{3}{2} |\hat{h}_{\text{map}k^{np}}|^2, \tag{7.8}
\]

where \( \hat{b} = \partial b \), and its integral is invariant under the gauge transformation \( \delta \hat{b} = \partial b_{\text{map}p} + \frac{1}{2} k_{\text{map}p} \psi \) into irreducible representations of \( SU(3) \), where \( \psi = k^{\text{map}p} \) is its singlet part and \( b \in \Lambda_2^3 \otimes \Lambda_2^1 \) is its primitive component in the adjoint of \( SU(3) \), that we would like to relate to the 2-form gauge field appearing in (7.4).

\(^{18}\) The identities \( k_{\text{map}p} b_{\text{map}p} = k_{\text{map}p} b_{\text{map}p} \) and \( k_{\text{map}p} b_{\text{map}p} = -k_{\text{map}p} b_{\text{map}p} \), which hold for any \( b \in \Lambda_2^3 \) and \( \beta \in \Lambda_2^1 \), have been used. In addition, it is helpful in deriving (7.6) to use the identity \( 2 |F_{\text{map}p}|^2 = |\rho_{\text{map}p} \beta^a|^2 + |k^{\text{map}p} F_{\text{map}p}|^2 \) (up to total derivatives) and that \( \rho_{\text{map}p} F_{\text{map}p} = -\frac{1}{2} \hat{\rho}_{\text{map}p} \beta^a \) and \( k^{\text{map}p} F_{\text{map}p} = \frac{1}{2} \rho_{\text{map}p} \beta^a \).
Under this decomposition, \((7.8)\) reduces to
\[
|\hat{h}_{mnp}|^2 - \frac{1}{2} |\hat{h}_{mnp}k^{nmp}|^2 = |\hat{h}_{mnp}|^2 - \frac{1}{2} |\hat{h}_{mnp}k^{nmp}|^2 - \hat{h}_{mnp}k^{nmp} \hat{\varphi} - \frac{1}{4} |\partial_m \varphi|^2,
\]
and is invariant under the gauge transformations \(\delta \hat{b} = \hat{P}^a d \lambda, \delta \varphi = 2 k^{mn} \partial_m \lambda_a\).

Note that naïvely setting \(\varphi = 0\) would not identify this action with the first line of \((7.4)\). This could have been anticipated though since neither \(\varphi = 0\) or \(F_{mn} = 0\) are gauge-invariant equations. A better strategy is to integrate out \(\varphi\). The gauge-invariant equation of motion for \(\varphi\) is
\[
\frac{1}{2} \Box \varphi = - \hat{a}^{mn} (\hat{h}_{mnp} k^{nmp}).
\]
This implies the equation
\[
\frac{1}{2} \partial_m \varphi = - \hat{h}_{mnp} k^{nmp} + \hat{a}^m \delta_{mn},
\]
where the coexact term involves some locally defined 2-form \(\beta\). The equation above is only gauge invariant provided \(\partial \beta_{mn} = \rho_{mpq} \hat{P}^{npq} \partial_q \lambda_b + \rho_{mpq} \hat{\varphi} \gamma + \hat{\rho}_{mpq} \partial_q \epsilon\), for any scalars \(\gamma\) and \(\epsilon\). Thus we can identify \(\beta \in \Lambda^2 \) in the coexact term above with the dual variable to \(A\) introduced in the previous subsection (provided we set \(\gamma = -2 \alpha\) and \(\epsilon = 0\)).

Substituting the equation above into \((7.9)\), the Pestun–Witten Lagrangian becomes
\[
|\hat{h}_{mnp}|^2 - \frac{1}{2} |\hat{h}_{mnp}k^{nmp}|^2 = \frac{1}{2} |\hat{h}_{mnp}k^{nmp} \partial_q \beta_{mq} + \frac{1}{2} |\partial^p \delta_{mn}|^2,
\]
which agrees with the first line of \((7.6)\). The absence of the second line of \((7.6)\) in the Lagrangian above is due to the extra scalar gauge symmetry under \(\delta \beta_{mn} = \hat{\rho}_{mpq} \partial_q \varphi\) in the Pestun–Witten theory, which does not arise from reduction of the \(G_2\) theory in seven dimensions. However, the second line of \((7.6)\) has a nice interpretation from gauge fixing the extra \(\varepsilon\) symmetry in the Lagrangian above. That is, under \(\delta \beta_{mn} = \hat{\rho}_{mpq} \partial_q \varphi\), the dual one-form gauge field \(k_{mn}A^a\) has the canonical gauge transformation \(\partial_a \varepsilon\). Thus the Lorentz gauge-fixing term for this symmetry is proportional to \(|\partial^a (k_{mn}A^a)|^2\) which is exactly the square of \(k^{mn} F_{mn} = \frac{1}{2} \hat{\rho}_{mpq} \partial^m \beta^{np}\) appearing in the second line of \((7.6)\).

Thus we have found agreement between the local degrees of freedom arising from the reduction of the \(G_2\) theory and the Pestun–Witten theory describing variations of a stable 3-form in six dimensions. This may seem somewhat surprising since we were allowing variations of both \(k\) and \(\rho\) in the reduced theory. Indeed the premise of topological M-theory [3] is that classically the \(G_2\) Hitchin functional should encapsulate both Kähler and complex structure deformations of the A- and B-models in six dimensions. Thus, in addition to the Pestun–Witten theory, we might have expected the quadratic action for a stable 2-form, that is related to the quantum foam description of the A-model [1, 2], from the reduction. However, such a quadratic action would be proportional to \(\int k \wedge F \wedge F\) and so the Lagrangian corresponds to a locally defined total derivative. For general Calabi–Yau backgrounds this term corresponds to the non-trivial integral second Chern class of the \(U(1)\) gauge bundle with curvature \(F\). However, in the topologically trivial case we have considered, such terms have been dropped. It would be interesting to understand the global topological structure of the reduced theory in more detail, but this would require a more refined analysis than we are attempting here.

### 7.2. Dimensional reduction of generalized \(G_2\) theory

Having related the dimensional reduction of the quadratic \(G_2\) Hitchin functional to the corresponding quantity in six dimensions calculated in [11], we will now examine the reduction of the generalized \(G_2\) theory and its relation to the extended Hitchin functional used in [11]. To do this it will be helpful to begin with a brief review of generalized Calabi–Yau manifolds (see [29, 30] for more details).
7.2.1. Generalized Calabi–Yau manifolds. The critical points of the generalized Hitchin functional in six dimensions correspond to six-manifolds $N$ with generalized $SU(3)$ structure, so-called generalized Calabi–Yau manifolds [29]. The structure group $\text{Spin}(6, 6)$ of $T N \oplus T^* N$ here is reduced to an $SU(3) \times SU(3)$ subgroup in the following way. Under the action of the conformal structure group $\text{Spin}(6, 6) \times \mathbb{R}^+$ in six dimensions, the stabilizer of a generic form of either odd or even degree is $SU(3, 3)$. When acting on complex-valued odd/even forms, the conformal structure group is complexified to $\text{Spin}(12, \mathbb{C}) \times \mathbb{C}$ and its orbits correspond to the subspaces $\Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4 \subset \Lambda^{\text{odd}} \otimes \mathbb{C}$ and $\exp(\Lambda^0 \otimes \mathbb{C} \oplus \Lambda^2 \otimes \mathbb{C}) \subset \Lambda^{\text{even}} \otimes \mathbb{C}$ on $N$, each of which are fixed by an $SU(3, 3)$ subgroup. Both these $SU(3, 3)$-invariant orbits are 32-dimensional and, as vector spaces, are isomorphic to the real form subspaces $\Lambda^{\text{odd/even}} \subset \Lambda^{\text{odd/even}} \otimes \mathbb{C}$. Given two generic stable forms of odd and even degrees, a different $SU(3, 3)$ stabilizes each of them and it is only a common $SU(3) \times SU(3)$ subgroup that can fix them both simultaneously. An odd and an even form which are simultaneously stabilized by $SU(3) \times SU(3)$ in this way are said to be compatible. The existence of stable odd and even forms which are compatible defines a generalized Calabi–Yau structure. As noted in equation (2.102) of [30], any two stable forms $\chi_- \in \Lambda^{\text{odd}}$ and $\chi_+ \in \Lambda^{\text{even}}$ are guaranteed to be compatible provided they solve
\begin{equation}
(v + \xi) \cdot \chi_+ = \iota_v \chi_+ - \iota_v \chi_+ + \xi \wedge \chi_+ + \chi_+
+ \iota_v \chi_+ - \xi \wedge \chi_+ - \chi_+ = 0,
\end{equation}
for any vector $v$ and one-form $\xi$ on $N$. Since $v + \xi$ transforms as a vector under $\text{Spin}(6, 6)$, this condition is clearly necessary due to the absence of any singlets in the vector decomposition under $SU(3) \times SU(3) \subset \text{Spin}(6, 6)$. The operator $(v + \xi) = \iota_v + \xi \wedge$ gives the action of the Clifford algebra on odd/even forms (understood as Majorana–Weyl spinors of $\text{Spin}(6, 6)$). The bilinear map $\Lambda^{\text{odd/even}} \times \Lambda^{\text{odd/even}} \rightarrow \Lambda$, $\langle \omega^{\text{odd}}, \chi^{\text{odd}} \rangle = -\omega^0 \wedge \chi^5 + \omega^3 \wedge \chi^1 - \omega^5 \wedge \chi^4$
$\langle \omega^{\text{even}}, \chi^{\text{even}} \rangle = \omega^0 \wedge \chi^6 - \omega^2 \wedge \chi^4 + \omega^4 \wedge \chi^2 - \omega^6 \wedge \chi^0$,
called the Mukai pairing, represents the inner product of the isomorphic $\text{Spin}(6, 6)$ chiral spinors.

A special case where the generalized Calabi–Yau structure reduces to an ordinary one is when the stable odd form has no one-form and 5-form components but the even form is generic. The odd form is then a 3-form $\rho$, stabilized by $SL(3, \mathbb{C})$. If we call the complex 2-form $b + ik$ in the even-form orbit then the 12 generalized compatibility equations reduce to $\rho \wedge k = 0$ and solutions define an ordinary $SU(3)$ structure (corresponding to the common subgroup of odd/even-form stabilizers $SL(3, \mathbb{C})$ and $SU(3, 3)$).

7.2.2. Reduction of generalized $G_2$ theory. The parametrization given in [22–24] for the stable odd form $\rho$ (5.1) we used in seven dimensions is convenient for the reduction since, in an orthonormal frame, the one-form $\alpha$ defines the direction orthogonal to which the generalized Calabi–Yau structure is contained. Thus we will decompose the data with respect to the direction defined by $\alpha$ as
\begin{equation}
\Phi = \hat{\rho} + k \wedge \alpha, \quad \ast_9 \Phi = \hat{\rho} \wedge \alpha + \hat{k}, \quad \mathcal{B} = b + a \wedge \alpha.
\end{equation}

This is similar to the situation for ordinary Calabi–Yau structures in six dimensions, where the stable 2-form $k$ (fixed by $\text{Spin}(3, \mathbb{R}) \subset \text{GL}(6, \mathbb{R})$) and 3-form $\rho$ (fixed by $SL(3, \mathbb{C}) \subset \text{GL}(6, \mathbb{R})$) can only be simultaneously fixed by a common $SU(3)$ subgroup. These two forms are compatible if $k \wedge \rho = 0$. 

19
It will be convenient to write $e^{i\theta} = c + is$ and $\hat{\Omega} = \hat{\rho} + i\hat{\theta}$ to define the new 3-forms $\rho = \Re(e^{-i\theta}\hat{\Omega})$, $\hat{\rho} = \Im(e^{-i\theta}\hat{\Omega})$. It will also be convenient to take $\hat{k} = \frac{1}{2}k \wedge k$, anticipating the Calabi–Yau substructure that will occur in the reduction. In terms of the new data, (5.1) can be written as

$$\hat{\rho} = -e^{-i\nu}\rho \wedge e^{b\Lambda a/\alpha} + \Re(e^{-i\nu-i\theta} e^{b\Lambda i}) \wedge \alpha.$$  (7.15)

Note that the second term looks like it will give a stable even-form spanning $\exp(\Lambda^6 \otimes \mathbb{C} \oplus \Lambda^2 \otimes \mathbb{C}) \subset \Lambda^{even} \otimes \mathbb{C}$ in six dimensions. The non-$\alpha$ terms however seem to be missing a one-form component needed in order to reduce to a generic stable odd form.

If we now reduce by restricting attention to special generalized $G_2$ manifolds of the form $N \times S^1$, with the generalized Calabi–Yau structure on $N$ and coordinate $t \in S^1$, then one can identify $\alpha = dt + \zeta$, where $\zeta$ is an arbitrary harmonic one-form on $N$ (this preserves the constraints that $\alpha$ be closed and have unit norm with respect to the metric reconstructed from $\Phi$). It is $\zeta$ that will account for the missing one-form above. It is perhaps worth noting that if one dropped the harmonic constraint on $\zeta$, and only assumed it was closed, then one could always reobtain a harmonic representative in the cohomology class $[\zeta]$ via a suitable shift $\zeta \to \zeta + dy$ resulting from the seven-dimensional diffeomorphism generated by the 7-vector $X$ whose only non-vanishing component is $X' = y$ (i.e. a scalar on $N$). Similarly, one can use the freedom to shift $B \to B + d\alpha$ in seven dimensions to remove the term $\alpha \wedge d\alpha$ in the reduced 2-form $B$ by choosing $\varepsilon = ta$. We will not do this however since we do not yet want to fix any of the symmetries of the reduced theory.

The explicit expressions for the reduction of the stable forms in the generalized $G_2$ theory are

$$\hat{\rho} = e^{-i\nu}\left[ s\zeta + \left\{ -\rho + (sb - c k) \wedge \zeta \right\} + \left\{ -\rho \wedge b + \left( \frac{s}{2}(b^2 - k^2) - cb \wedge k - \rho \wedge a \right) \wedge \zeta \right\} \right]$$

$$+ e^{-i\nu} dt \wedge \left[ s + \left\{ sb - c k \right\} + \left\{ \frac{s}{2}(b^2 - k^2) - cb \wedge k - \rho \wedge a \right\} \right]$$

$$+ \left\{ \frac{s}{6}(b^3 - 3k^2 \wedge b) + \frac{c}{6}(k^3 - 3b^2 \wedge k) - \rho \wedge b \wedge a \right\}$$  (7.16)

$$\hat{\theta} = e^{-i\nu} dt \wedge \left[ ca + \left\{ \hat{\rho} - (sk + cb) \wedge a \right\} + \left\{ -\hat{\rho} \wedge b - \left( \frac{c}{2}(k^2 - b^2) - sb \wedge k \right) \wedge a \right\} \right]$$

$$+ e^{-i\nu} \left[ c + \left\{ -sk - cb - ca \wedge \zeta \right\} \right]$$

$$+ \left\{ \frac{c}{2}(b^2 - k^2) + sb \wedge k - \hat{\rho} \wedge \zeta + (sk + cb) \wedge a \wedge \zeta \right\}$$

$$+ \left\{ \frac{s}{6}(k^3 - 3b^2 \wedge k) - \frac{c}{6}(b^3 - 3k^2 \wedge b) \right\}$$

$$+ \left\{ \frac{c}{2}(k^2 - b^2) - sb \wedge k \right\} \wedge a \wedge \zeta + \hat{\rho} \wedge b \wedge \zeta \right\} \right].$$  (7.17)

This allows us to identify $\hat{\rho} = \chi_- + \chi_+ \wedge dt$ and $\hat{\theta} = \hat{\chi}_- \wedge dt + \hat{\chi}_+ \wedge dt$ in terms of odd forms $\chi_-, \chi_+$ and even forms $\hat{\chi}_-, \hat{\chi}_+$ on $N$. In fact, it will be more convenient to define $\hat{\rho} = e^{a \Lambda c} \wedge \chi_- + \chi_+ \wedge dt$ and $\hat{\theta} = \hat{\chi}_- \wedge dt + e^{a \Lambda c} \wedge \hat{\chi}_+$. The identities $\int_N \chi_- \wedge a \wedge \zeta \wedge \hat{\chi}_- = 0$ and $\int_N \chi_+ \wedge a \wedge \zeta \wedge \hat{\chi}_+ = 0$ ensure that either choice will give rise to the same reduced Hitchin
functional $\int_M \mathcal{Q} \wedge \hat{\mathcal{Q}} = \int_{N \times S^1} (\mathcal{Q}_- \wedge \hat{\mathcal{Q}}_- + \mathcal{Q}_+ \wedge \hat{\mathcal{Q}}_+) \wedge dt$. Thus we have

$$
\begin{align*}
\mathcal{Q}_- &= e^{-\varphi} e^b \wedge \left[ s \zeta - (\rho + c \zeta \wedge k) - \frac{1}{2} k^2 \wedge s \xi \right] \\
\mathcal{Q}_+ &= e^{-\varphi} e^b \wedge \left[ s - ck - \left( \rho \wedge a + \frac{s}{2} k^2 \right) + \frac{c}{6} k^3 \right] \\
\hat{\mathcal{Q}}_- &= e^{-\varphi} e^{-b} \wedge \left[ -ca - (\check{\rho} - sa \wedge k) + \frac{1}{2} k^2 \wedge ca \right] \\
\hat{\mathcal{Q}}_+ &= e^{-\varphi} e^{-b} \wedge \left[ c - sk - \left( \check{\rho} \wedge \zeta + \frac{c}{2} k^2 \right) + \frac{s}{6} k^3 \right].
\end{align*}
$$

Note that the odd forms are related by an anti-involution $\mathcal{Q}_- \to \hat{\mathcal{Q}}_-$ which is generated by the parameter transformations $(s, c, \rho, k, b, a, \zeta) \to (c, -s, \check{\rho}, -k, -b, -\zeta, -a)$. This is a symmetry of the odd-form functional $\int_{N \times S^1} \mathcal{Q}_- \hat{\mathcal{Q}}_- \wedge dt$ if $t$ is invariant. Similarly the even forms are related by an anti-involution $\mathcal{Q}_+ \to \hat{\mathcal{Q}}_+$, that is generated by $(s, c, \rho, k, b, a, \zeta) \to (c, -s, \check{\rho}, -k, -b, -\zeta, -a)$. This is a symmetry of the even-form functional $\int_{N \times S^1} \mathcal{Q}_+ \hat{\mathcal{Q}}_+ \wedge dt$ if $t \to -t$. In both cases the transformations of $(c, \rho)$ follow from a shift $\theta \to \theta + \pi/2$ of the angle between the two $G_2$-invariant unit spinors in seven dimensions (recalling that $\rho = c \check{\rho} + s \check{\rho}$ and $\check{\rho} = -s \check{\rho} + c \check{\rho}$ in terms of the original data).

These two anti-involutions just correspond to the action of the Hamiltonian vector field on the symplectic spaces spanned by $\chi_\pm + i \hat{\chi}_\pm$ that is defined respectively by the odd- and even-form functionals

$$
\begin{align*}
\chi_+ + i \hat{\chi}_+ &= \int_N e^{-2\varphi} \left[ \rho \wedge \check{\rho} + \frac{s}{6} k^3 \right] \\
\chi_- + i \hat{\chi}_- &= \int_N e^{-2\varphi} \left[ \frac{1}{2} k^2 \wedge \left( \frac{s}{2} k + a \right) \wedge \zeta \right],
\end{align*}
$$

using the constant symplectic form on the complexified stable odd- and even-form spaces, as described by Hitchin on p 16 in [29]. Thus the circle action $\chi_\pm + i \hat{\chi}_\pm \to e^{-i\theta} (\chi_\pm + i \hat{\chi}_\pm)$ generated by this Hamiltonian vector field has a nice interpretation via shifts $\theta \to \theta + \varphi$ in the angular separation of the generalized $G_2$ unit spinors.

As a quick consistency check, we see that the sum of these functionals

$$
\int_M \mathcal{Q} \wedge \hat{\mathcal{Q}} = \int_{N \times S^1} e^{-2\varphi} \left( \rho \wedge \check{\rho} + \frac{2}{3} k \wedge k \wedge k \right) \wedge dt
$$

(7.20)

corresponds to the generalized Calabi–Yau Hitchin functional [30], modulo the compatibility conditions that we will now discuss.

With the identification (7.18), the generalized Calabi–Yau compatibility conditions (7.13) for $\chi_\pm$ become

$$
\begin{align*}
& \frac{c}{s} \left( \rho + \frac{1}{c} k \wedge k \wedge \zeta \right) \wedge \left( k + \frac{s}{c} a \wedge \zeta \right) = 0 \\
& \left( \rho + c \zeta \wedge k \wedge a + \frac{s}{2} k^2 \right) \wedge \iota_v \rho = 0,
\end{align*}
$$

(7.21)

for any vector field $v$ (the arbitrary one-form $\xi$ has been factored out of the first equation). An additional term $[\epsilon \rho \wedge k + s \rho \wedge a \wedge \zeta + k^2 \wedge \xi \wedge \zeta] \wedge \epsilon \rho$ in the second equation vanishes as a result of the first equation, to completely remove the dependence on $b$ in (7.21). The second equation has also been simplified using $c \rho \wedge k \wedge \zeta = 0$ which follows from the first equation.
It will be useful to conclude this subsection by also noting the related compatibility conditions for $\hat{\chi}_{\pm}$

$$
\begin{align*}
& s \left( \hat{\rho} - \frac{1}{s} a \wedge k \right) \wedge \left( k + \frac{c}{s} a \wedge \xi \right) = 0 \\
& \left[ -\hat{\rho} + s a \wedge k \right] \wedge \left( \xi + \frac{c}{2} k^2 \right) \wedge \iota_\nu \hat{\rho} = 0.
\end{align*}
$$

(7.22)

7.2.3. Comparison with Pestun–Witten. Let us now consider how the quadratic part of the reduced functional above relates to the one considered in [11]. Recall that the extended functional used in [11] corresponds to the Hitchin functional for a stable odd form $\sigma$, fixed by $SU(3, 3)$. Since the reduction of the generalized $G_2$ Hitchin functional involves both a stable odd form $\chi_-$ and even form $\chi_+$, fixed by $SU(3) \times SU(3) \subset G_2 \times G_2$, we should not expect these theories to agree directly but will find that the theory of Pestun and Witten arises as a truncation of the generalized $G_2$ theory, with $\chi_-$ related to the odd form $\sigma$ in [11] after imposing the generalized Calabi–Yau compatibility equations.

When expanded around a Calabi–Yau background $N_0$, the quadratic part of the extended functional in [11] becomes

$$
\int_{N_0} \delta \sigma \wedge \delta \hat{\sigma} = \int_{N_0} \delta \sigma^3 \wedge J \delta \sigma^3 + 2 \delta \sigma^5 \wedge J \delta \sigma^1,
$$

(7.23)

where $\delta \hat{\sigma} = J \delta \sigma$ in terms of the background complex structure $J : \Lambda^\text{odd} \rightarrow \Lambda^\text{odd}$. The action of this background complex structure can be written as $J = J_1 + J_3 + J_5$, where $J_3$ acts on 3-forms via the map given below (7.7), $J_1$ acts on one-forms mapping $\xi_m \rightarrow k_m \equiv k_\nu$ and $J_5 = -J_1 \ast$ acting on 5-forms. Thus one has $J^2 = -1$. The identity $\sigma^1 \wedge J \delta \sigma^5 = \sigma^3 \wedge J \delta \sigma^1$ has been used above.

To relate this to the quadratic part of the odd-form functional for $\chi_-$ in (7.19) requires some more work. We begin by implementing the first compatibility equation in (7.21), (7.22) in (7.19) which gives

$$
\begin{align*}
& \int_N \chi_+ \wedge \hat{\chi}_- = \int_N e^{-2\psi} \left[ \rho \wedge \hat{\rho} - \frac{1}{sc} k^2 \wedge a \wedge \xi \right] \\
& \quad = \int_N e^{-2\psi} \left( \rho + \frac{1}{c} k \wedge k \right) \wedge \left( \hat{\rho} - \frac{1}{s} a \wedge k \right) \\
& \int_N \chi_+ \wedge \hat{\chi}_+ = \int_N e^{-2\psi} \left[ \frac{2}{3} k^3 + \frac{1}{sc} k^2 \wedge a \wedge \xi \right] \\
& \quad = \int_N e^{-2\psi} k \wedge \left( k + \frac{a}{c} \wedge \xi \right) \wedge \left( k + \frac{c}{s} a \wedge \xi \right),
\end{align*}
$$

(7.24)

for generic $s, c \neq 0$. Note, this has decoupled the terms involving $\rho$ and $\hat{\rho}$ from $a$ and $\xi$ in the odd-form functional.

Although we have not yet expanded around a fixed Calabi–Yau background it will be convenient to define the map $J_1 : \Lambda^1 \rightarrow \Lambda^1$ as $J_1(\xi) = -\ast \left( \frac{1}{2} k^2 \wedge \xi \right)$ and the map $J_5 = -\ast J_1 \ast : \Lambda^5 \rightarrow \Lambda^5$ which will reduce to their namesakes defined earlier in the quadratic expansion. The odd-form functional above can then be written more suggestively as

$$
\int_N \chi_+ \wedge \hat{\chi}_- = \int_N e^{-2\psi} \left[ \rho \wedge \hat{\rho} + \frac{2}{sc} a \wedge J_1 \xi \right].
$$

(7.25)

---

20 The sign here follows from the requirement that $J_5 \omega \wedge J_1 \xi = \omega \wedge \xi$ for any 5-form $\omega$ and one-form $\xi$. In particular, taking $\omega = \ast \xi$ then this follows from the fact that $\ast J_1 \xi \wedge J_1 \xi = \ast \xi \wedge \xi$ and that $\ast^2 = -1$ on odd forms in six dimensions.
From this one can read off more suitable expressions for the odd forms
\[ \chi_- = e^{-\varphi} e^b \wedge \left[ \frac{1}{\alpha} \varphi + \beta + \frac{1}{s} \right] \]
\[ \hat{\chi}_- = e^{-\varphi} e^{-b} \wedge \left[ \frac{1}{\alpha} \varphi + \beta + \frac{1}{s} \right], \]  \tag{7.26}
which are presumably related to those given in (7.18) by a suitable symplectic transformation on the space of stable odd forms (supplemented with the generalized Calabi–Yau compatibility constraints), since they both give rise to the same Hitchin functional\(^2\). One therefore has \( \delta \hat{\chi}_- = J \delta \chi_- \) for first order variations around a Calabi–Yau background with Kähler form \( k \) and complex structure \( \Omega = \rho + i \hat{\rho} \). One can recover the quadratic functional of Pestun and Witten by identifying the first order variations \( \delta \sigma^1 = \delta (e^{-\varphi} \rho) \) and \( \delta \sigma^5 = \# \delta (s^{-1} \alpha) \).

8. Background dependence

In this section we will investigate the dependence of the one-loop partition functions we have calculated on the choice of background metric. This can be deduced from theorems in [12] but we will derive it from first principles. This analysis will help us reconcile the results of sections 5 and 6, by showing how the one-loop partition functions \( Z_{\text{gen}} \) and \( Z_{\text{str}} \) are related. The pertinent quantity to calculate is the first order variation \( \delta g \) of the logarithm determinant of Laplacians acting on \( p \)-forms in \( D \) dimensions. For simplicity we will begin by assuming metric variations around a background with trivial cohomology so that there are no extra contributions from harmonic forms to concern us.

8.1. General formulae

Consider the variation of the canonical inner product
\[ \langle \omega, \xi \rangle_p = \int_M d^D x \sqrt{-g} \frac{1}{p!} g^{m_1 \ldots m_p} \omega_{m_1 \ldots m_p} \xi_{m_1 \ldots m_p}, \]  \tag{8.1}
between two \( p \)-forms on a \( D \)-dimensional Riemann manifold \( M \), with respect to the Riemannian metric \( g \). This can be written as
\[ \delta g(\omega, \xi)_p = \langle \omega, B_p \xi \rangle_p = \langle B_p \omega, \xi \rangle_p, \]  \tag{8.2}
in terms of the algebraic operator
\[ (B_p)_{m_1 \ldots m_p n_1 \ldots n_p} = p \delta g^{m_1 b} \delta g_{n_1 a} \delta g^{m_2 d} \delta g_{n_2 c} \ldots \delta g^{m_p f} \delta g_{n_p g} - \frac{1}{2} \delta g^{a b} \delta g_{m_1 \ldots m_p} e_{n_1 \ldots n_p}, \]  \tag{8.3}
which is a function of \( \delta g \) and \( g \), mapping \( \Lambda^p \to \Lambda^p \).

One can prove that \( \delta g(\ast \omega) = -B_{D-p} \ast \omega = \# B_{p} \omega \) for any \( \omega \in \Lambda^p \), from which one derives
\[ \delta g(d^1 \omega) = d^1 B_{p} \omega - B_{p-1} d^1 \omega. \]  \tag{8.4}

Using the formula \( \delta (\log \det X) = \delta (\det X)/|\det X| = \text{tr}(X^{-1} \delta X) \), for the variation of an elliptic operator \( X \), and \( \Delta^p = d_{p+1} ^1 d_p + d_{p-1} d^1_p \), one obtains the result
\[ \delta g(\log \det \Delta^p) = -2 \text{tr} \left( B_p + 2 \sum_{k=0}^{p-1} B_k \right) = - \left( \frac{D}{p} \right) \text{tr}(g^{-1} \delta g). \]  \tag{8.5}

\(^2\) For example, the expressions for \( \chi_- \) and \( \hat{\chi}_- \) inside the square brackets in (7.18) and (7.26) are related by the symplectic transformation \( \chi_- \to - \# \hat{\chi}_- \) and \( \hat{\chi}_- \to \# \chi_- \) that preserves \( \int_X \chi_- \wedge \hat{\chi}_- \) (followed by a multiplicative factor \( 1/2 \)) related to the generalized Calabi–Yau constraint.

25
Note that $\text{tr}(g^{-1} \delta g) = 2 \delta_g (\log \text{Vol}(M))$, where $\text{Vol}(M)$ is the volume of the Riemann manifold $M$. Some partition functions that are $\delta_g$-invariant around backgrounds with trivial cohomology develop a gravitational anomaly for variations around more general backgrounds. Topological symmetry can sometimes be restored in such cases by multiplying the original partition function by a compensating power of the volume of the background manifold. This has been shown for the B-model in [11] (and previously in unpublished work by Klemm and Vafa) and for certain Chern–Simons-type actions in [12].

A nice consistency check of this result is to verify the lemma of Schwarz [12] which states that the Ray–Singer torsion is a topological invariant in odd dimensions. Taking the log of this torsion, for $D = 2k + 1$, one finds that indeed

$$
\delta_g \left( -\frac{1}{2} \sum_{p=0}^{k} (-1)^p (2p + 1) \log \det \Delta^{k-p} \right) = \text{tr} \left( \sum_{p=0}^{k} (-1)^{k-p} B_p \right) = 0. \quad (8.6)
$$

The last equality simply follows from a combinatorial identity.

### 8.2. Metric dependence of one-loop Hitchin functionals

Using the expressions found in the last subsection, one can calculate the metric variation of the (log of the) one-loop $G_2$ partition functions $Z$, $Z_{\text{gen}}$, $Z_{\text{string}}$. None of them are $\delta_g$-invariant. This is to be expected though since the $G_2$ metric deformations contain both Kähler and complex structure deformations in the reduced theory. The B-model partition function is $\delta_g$-invariant since it does not depend on the Kähler moduli, but has the well known wavefunction behavior under variations of the complex structure of the Calabi–Yau.

To understand this in more detail, let us examine the structure of topological invariants built out of products of powers of $\det \Delta^p$ in $D = 7$. Since Hodge duality implies $\det \Delta^p = \det \Delta^{8-p}$, the only independent Laplacian determinants are for $p = 0, 1, 2, 3$. (Using the various $G_2$ isomorphisms already mentioned, all the one-loop partition functions we have calculated can be written as products of powers of these four Laplacian determinants.) Consider now the most general such product

$$(\det \Delta^a)^b (\det \Delta^b)^c (\det \Delta^c)^d,$$

specified by any four real numbers $a, b, c, d$. Demanding the log of this expression to be $\delta_g$-invariant implies $35a + 21b + 7c + d = 0$. The three independent numbers parametrizing the invariant can be recast as powers of 3 more basic topological invariants. A convenient choice for these three basis invariants is

$$
l_0 = (\det \Delta^1)^{-1/2} (\det \Delta^0)^{7/2} = (\det \Delta^0)^{-1/2} (\det \Delta^1)^{7/2},
$$

$$
l_1 = (\det \Delta^2)^{-1/2} (\det \Delta^1)^{3/2} = (\det \Delta^{14})^{-1/2} (\det \Delta^7),
$$

$$
l_2 = l_{\text{RS}} = (\det \Delta^3)^{-1/2} (\det \Delta^2)^{3/2} (\det \Delta^1)^{-5/2} (\det \Delta^0)^{7/2} = (\det \Delta^{27})^{-1/2} (\det \Delta^{14})^{3/2} (\det \Delta^7)^{-3/2} (\det \Delta^1)^{3}. \quad (8.7)
$$

Any $\delta_g$-invariant constructed from products of powers of Laplacian determinants in $D = 7$ can be written as $l_0^a l_1^b l_2^c$, for some choice of $a, b, c$. $Z$, $Z_{\text{gen}}$ and $Z_{\text{string}}$ cannot be factorized in this way. However, $Z$, $Z_{\text{gen}}$ and $Z_{\text{string}}$ can be written as

$$
Z = l_0^{-1} \times \text{Tor}(\tilde{D}) \times (\det \Delta^1)^{1/2},
$$

$$
Z_{\text{gen}} = l_0^{-2} l_0^{-1} \times \text{Tor}(\tilde{D})
$$

$$
Z_{\text{string}} = l_0^{-4} \times \text{Tor}(\tilde{D})^4, \quad (8.8)
$$

26
in terms of the invariants $b_0$, $h_1$, $h_2$, and a non-invariant object $\text{Tor}(\mathcal{D}) = (\det \Delta_T)^{-1/2} (\det \Delta_4)^{-3/2}$ corresponding to the analytic torsion of the $G_2$ Dolbeault $\mathcal{D}$ complex in (4.4). Recall that the $\mathcal{D}$ complex describes the spectrum of the topological $G_2$ string [5] in much the same way that the $\tilde{\partial}$ complex does for the B-model. Indeed $Z_{\text{string}} = \text{Tor}(\mathcal{D} \otimes \Lambda^1)/\text{Tor}(\mathcal{D})^2$, where $\text{Tor}(\mathcal{D} \otimes \Lambda^1) = (\det \Delta_7 \otimes \mathcal{D})^{-1/2} (\det \Delta_7)^{3/2} = (\det \Delta_7)^{-1/2} (\det \Delta_4)^{-3/2} (\det \Delta_7) (\det \Delta_4)^{-1/2}$ is the analytic torsion of the $\mathcal{D}$ complex for $\Lambda^1_2$-valued forms. $Z_{\text{string}}$ is therefore not $\delta_\epsilon$-invariant due to the identity $\text{Tor}(\mathcal{D} \otimes \Lambda^1) = \text{I}_3 \text{I}_0^6 \times \text{Tor}(\mathcal{D})^7$.

The expressions above show that $Z_{\text{gen}} = I_{\text{RS}}^{-9/4} Z_{\text{string}}^{-1/4}$. Therefore, although not identical, the one-loop partition functions for the generalized $G_2$ Hitchin functional and the topological $G_2$ string seem to be related up to a power of the Ray–Singer torsion invariant of the background manifold.

Let us now perform a similar analysis in $D = 6$ on a Calabi–Yau manifold, to reconcile the results above with those found in [11]. Hodge duality in $D = 6$ again implies that the only independent Laplacian determinants are for $p = 0, 1, 2, 3$. However, since we are considering a Calabi–Yau background, we can use the Hodge decomposition $\Lambda^3 = \Lambda^{30} \oplus \Lambda^{21} \oplus \Lambda^{12} \oplus \Lambda^{03}$ and vector space isomorphisms $\Lambda^{30} = \Lambda^{03} = \Lambda^{00}, \Lambda^{21} = \Lambda^{12} = \Lambda^{11}$ to relate 3-form Laplacian determinants to those for forms of lower degree. Thus the most general product is of the form

$$(\det \Delta_3)^{a} (\det \Delta_1)^{b} (\det \Delta_0)^{c}.$$ 

This is $\delta_\epsilon$-invariant if $15a + 6b + c = 0$. A convenient choice of two basis invariants, whose powers are parametrized by these two independent numbers, is

$$I_0 = (\det \Delta_1)^{-1/2} (\det \Delta_0)^{3/2} = (\det \Delta_1)^{-1/2} (\det \Delta_0)^{3/2}$$

$$I_1 = (\det \Delta_1)^{-1/2} (\det \Delta_1)^{5/4} = (\det \Delta_1)^{-1/2} (\det \Delta_0)^{3/2}.$$ 

These are precisely the holomorphic Ray–Singer torsions $I_0 = I_{\text{RS}}^{03}$ and $I_1 = I_{\text{RS}}^{12}$ of the Dolbeault complex (for $\Lambda^{30}$- and $\Lambda^{12}$-valued $(0, q)$-forms), used in [11]. The one-loop partition function for the stable 3-form Hitchin functional in $D = 6$ is $I_1/I_0$ while that for the extended Hitchin functional (and B-model) is $I_1/I_0^3$. Thus we see that both are topological invariants.

### 8.3. B-model gravitational anomaly from seven dimensions?

Note that the coefficient in (8.5) corresponds to the dimension of the vector space $\Lambda^p$. For backgrounds with non-trivial cohomology, $\det \Delta_p$ is defined by removing the zero modes of $\Delta_p$, corresponding to harmonic $p$-forms, from the determinant. Let us naively assume the results in the redefined coefficient $(\frac{p}{2}) \rightarrow (\frac{p}{2}) - b_p$ in (8.5), where $b_p = \dim(H_p(M, \mathbb{R}))$ are the Betti numbers. This will imply that partition functions that are invariant under deformations of a background metric with trivial cohomology can develop gravitational anomalies for variations around more general backgrounds. Furthermore, such anomalies will be proportional to some power of the volume of the background manifold. This power being some linear sum of Betti numbers.

For the B-model, the particular linear sum of Betti numbers appearing corresponds to the Euler number $\chi = 2(h_{11} - h_{12})$, in terms of the Hodge numbers of the Calabi–Yau background. Let us now consider a $G_2$ background of the form $CY_3 \times S^1$ and ask
if there is any combination of Laplacian determinants in seven dimensions, whose metric
variation correctly reduces to this B-model gravitational anomaly? Using the Kunneth formula
\( b_p(CY_3 \times S^1) = b_p(CY_3) + b_{p-1}(CY_3) \), we see that \( b_3 = 2(1 + h_{12}) + h_{11}, b_2 = h_{11} \) and
\( b_1 = b_0 = 1 \) on \( CY_3 \times S^1 \). Thus we can write
\[ \chi = 2(h_{11} - h_{12}) = -b_3 + 3b_2 + (2 - a)b_1 + ab_0, \]
for any real number \( a \). This coefficient appears from the metric variation of
\[ (\det \Delta^3)^{-1/2}(\det \Delta^2)^{3/2}(\det \Delta^1)^{1-a/2}(\det \Delta^0)^{a/2} = I_{RS} \times \left( \frac{\det \Delta^1}{\det \Delta^0} \right)^{(7-a)/2}, \]
(possibly to some overall power). Of course, we also want this to be \( \delta_\tau \)-invariant around
backgrounds with trivial cohomology and so must choose \( a = 7 \). Hence the gravitational
anomaly of the B-model on \( CY_3 \) has a simple interpretation from metric variations of the
Ray–Singer torsion on \( CY_3 \times S^1 \).

9. Conclusions and open questions

In this paper we have attempted to understand more about the quantum structure of topological
M-theory [3] from perturbative quantization of the (generalized) \( G_2 \) Hitchin functional.

We computed the one-loop partition function of the ordinary \( G_2 \) Hitchin functional and
agreement was found between the local degrees of freedom for the reduction of this theory on
a circle and the corresponding theory of Pestun and Witten [11], obtained from the Hitchin
functional for a stable 3-form in six dimensions.

The calculation was repeated for the generalized \( G_2 \) Hitchin functional and a certain
truncation of the circle reduction of this theory was related to the extended Hitchin functional
in six dimensions, whose one-loop partition function was equated with the topological B-
model in [11]. The one-loop partition function for the topological \( G_2 \) string [5] was also
computed here and found to agree with the generalized \( G_2 \) theory only up to a power of the
Ray–Singer torsion of the background \( G_2 \) manifold.

There are however a number of subtleties involved in this calculation. First it is not obvious
to us that the linear variations \( \delta \rho \) of the stable odd form in seven dimensions constitute the
appropriate degrees of freedom describing the quantum theory. That is \( \rho \) is a nonlinear function
of parameters which seem more naturally related to stringy moduli. To clarify this point as
well as for physical applications of our results it would be important to understand whether
our computation could be related to effective actions for generalized \( G_2 \) compactifications
of physical string and M-theory. This could also help determine the fundamental degrees of
freedom of topological M-theory.

Another important issue is whether the gauge field components we have ignored in the
quantization, because they do not appear in the quadratic action (i.e. they are projected out or
neglected as total derivatives), have a non-trivial contribution to the partition function. For
instance they could be important in defining an appropriate path integral measure and give
rise to non-trivial one-loop determinants that would modify our results. It is possible that
resolving these subtleties could lead to a more precise agreement between the generalized \( G_2 \)
Hitchin functional and topological \( G_2 \) string at one-loop.

Certainly it would be interesting to understand the global structure of the one-loop
\( G_2 \) Hitchin functionals for topologically non-trivial gauge fields and their reduction to six
dimensions. According to the philosophy of topological M-theory, this would provide a non-
trivial gauge-theoretic description of the coupling between the Pestun–Witten description of
the B-model [11] and the quantum foam description of the A-model [1, 2]. The observables
of this theory could compute interesting gerbe invariants.

28
Higher order diffeomorphism-invariant terms in the expansion of the $G_2$ Hitchin functional can be understood as BRST-invariant operators deforming the quadratic theory. For example, in the reduction $B = b + A \wedge dt$, the cubic term in the expansion that we calculated in section 2.2 contains the $\int F \wedge F \wedge F$ deformation of the quadratic term $\int k \wedge F \wedge F$ in the A-model quantum foam [1, 2]. It would be interesting to understand the effect of such higher order deformations in seven dimensions.

Finally, since general background $G_2$ metric variations contain complex structure variations in six dimensions, it is natural to ask whether the wavefunction behavior of B-model has a nice interpretation in seven dimensions? Indeed this was one of the original motivations for the proposal of topological M-theory in [3]. It is possible that this could be understood from the structure of partition functions we have calculated here although we have not investigated this idea.

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Appendix A. $G_2$ and CY identities

A seven-dimensional Riemann manifold $M$ is guaranteed to have holonomy in the subgroup of $G_2 \subset SO(7)$ by the existence of a harmonic 3-form $\Phi$. In our conventions $\Phi$ and its Hodge dual $\ast \Phi$ can be written as

$$\Phi = e^{123} - e^{147} - e^{156} - e^{246} + e^{257} + e^{345} + e^{367}$$

$$\ast \Phi = e^{1245} + e^{1267} + e^{1346} - e^{1357} - e^{2347} - e^{2356} + e^{4567},$$

with respect to an orthonormal basis $e^I$ (where $e^{I_1 \cdots I_r} = e^{I_1} \wedge \cdots \wedge e^{I_r}$).

Some useful identities for products of the components of $\Phi$ and $\ast \Phi$ are as follows:

$$\ast \Phi_{IJK} \Phi^{PQR} = 6\delta^P_{[I} \delta^Q_{J} \delta^R_{K]} + 9\delta^R_{[I} \ast \Phi^{PQR}] - \Phi_{IJK} \Phi^{PQR}$$

$$\ast \Phi_{IJK} \Phi^{PQA} = -6\delta^P_{[I} \Phi_{JK]} Q\} (A.2)$$

$$\Phi_{IJK} \Phi^{PQA} = 2\delta^P_{[I} \delta^Q_{J} \delta^A_{K]} + \ast \Phi_{IJK} PQ.$$ 

These identities can be proven in the orthonormal basis above but it is clear they are also valid in any coordinate basis by simply acting on the formulae with the appropriate combination of vielbeins. Other required identities follow by taking contractions or Hodge dualizations of the ones above. For example,

$$\ast \Phi_{IJK} \Phi^{PAB} = 8\delta^P_{[I} \delta^A_{J} \delta^B_{K]} + 2\ast \Phi_{IJK} PQ$$

$$\Phi_{IJK} \Phi^{PAB} = 4\Phi_{IJK} P.$$ $$(A.3)$$

One can deduce the corresponding Calabi–Yau identities from dimensional reduction of the $G_2$ ones above, with $\Phi = \rho + k \wedge dt$ and $\ast \Phi = \hat{\rho} \wedge dt + \frac{i}{2} k \wedge k$ ($\rho$ and $\hat{\rho}$ being the
real and imaginary parts of the holomorphic (3,0)-form $\Omega$ and $k$ being the Kähler form), as in (7.1). In an orthonormal basis $e^m$ for the Calabi–Yau we can write $\Omega = -i d z^1 \wedge d z^2 \wedge d z^3$ and $k = \frac{1}{2} d z^m \wedge d z^m$, where $d z^{2m} = e^m + i e^{m+3}$. Written out explicitly, these expressions give
\[ \rho = e^{126} - e^{135} + e^{234} - e^{456} \]
\[ \hat{\rho} = -e^{123} + e^{156} - e^{246} + e^{345} \] (A.4)
and follow from the aforementioned reduction of (A.1) after relabeling 1 ↔ 4.

Some useful identities for products of components of $\rho$, $\hat{\rho}$ and $k$ are
\[ k_{mn} k^{pq} + \rho_{mn} \rho^{pq} = 2 \delta^q_{[m} \delta^p_{n]} + \frac{1}{2} (k \wedge k)_{mn}^{pq} \]
\[ \rho_{mn} \rho^{pq} = \hat{\rho}_{mn} \hat{\rho}^{pq} = 2 \delta^q_{[m} \delta^p_{n]} - 2 k_{[m}^{\rho} k^{p]}_{n} \]
\[ \rho_{mn} \hat{\rho}^{pq} = 2 \delta^q_{[m} k_{n]}^{p} - 2 \delta^q_{[m} k^{p]}_{n} \]
\[ \rho_{mn} \rho^{qrs} + \hat{\rho}_{mn} \hat{\rho}^{qrs} = 6 \delta^q_{[m} \delta^r_{n]} \delta^s_{p]} - 18 \delta^q_{[m} k_{n]}^{r} k_{p]}^{s} \]
\[ k_{mn} k^{pq} = -\delta^p_m \]
\[ k_{mn} \rho^{pq} = -\hat{\rho}_{mn}^{pq} \]
\[ k_{mn} \hat{\rho}^{pq} = \rho_{mn}^{pq} \]
\[ \rho_{mn} \rho^{pq} = \hat{\rho}_{mn} \hat{\rho}^{pq} = 4 \delta_m^q \]
\[ \epsilon_{mnqrs} \rho_{qrs} = -6 \rho_{mnq} \]
\[ \epsilon_{mnqrs} \hat{\rho}_{qrs} = 6 \rho_{mnq} \]
\[ \epsilon_{mnqrs} k_{rs} = 6 k_{[mn}^{[p} k_{p]}^{q]} \]
\[ \frac{1}{3} \rho \wedge \hat{\rho} = \frac{1}{6} k \wedge k \wedge k = *1 \]
\[ 5 \rho_{[mn} \hat{\rho}_{qrs]} = 15 k_{[mn}^{[p} k_{p]}^{q} k_{rs]} = \epsilon_{mnqrs}. \] (A.5)

### Appendix B. $G_2$ cohomology

The de Rham cohomology groups on a seven manifold $M$ with holonomy in $G_2$ have the following decompositions
\[ H^0(M, \mathbb{R}) = \mathbb{R} \]
\[ H^1(M, \mathbb{R}) = H^1_1(M, \mathbb{R}) \]
\[ H^2(M, \mathbb{R}) = H^2_2(M, \mathbb{R}) \oplus H^2_4(M, \mathbb{R}) \]
\[ H^3(M, \mathbb{R}) = H^3_1(M, \mathbb{R}) \oplus H^3_2(M, \mathbb{R}) \oplus H^3_3(M, \mathbb{R}). \] (B.1)

Similar decompositions follow for the remaining cohomology groups by Hodge duality. The subscripts in $H^n_i$ denote the irreducible representations $n$ of $G_2$ that the I-form components occupy. The non-trivial projection operators $P^I_n$ onto these irreducible subspaces are given by
\[ (P^2_2)_{IJK}^{PQ} = \frac{1}{2} \Phi_{IJK} \Phi^{PQA} = \frac{1}{2} \delta^{[P \delta^{Q]}_{[I} + \frac{1}{2} \Phi_{IJK} \Phi^{PQA} = \frac{2}{3} \delta^{[P \delta^{Q]}_{[I} - \frac{1}{2} \Phi_{IJK} \Phi^{PQA} \]
\[ (P^2_4)_{IJK}^{PQR} = \frac{1}{2} \Phi_{IJK} \Phi^{PQR} \]
\[ (P^2_{14})_{IJK}^{PQR} = \frac{1}{2} \Phi_{IJK} \Phi^{PQA} \]
\[ (P^2_{27})_{IJK}^{PQR} = \frac{1}{2} \Phi_{IJK} \Phi^{PQA} \]
(B.2)
These can be checked using the $G_2$ identities in appendix A.
Smooth compact $G_2$ manifolds have a somewhat simpler cohomology due to the fact that all $H_2^j = 0$ (that is when the holonomy is the full $G_2$ and not a proper subgroup thereof). The only independent non-trivial cohomology groups in this case are $H_1^3, H_2^7$ and $H_1^4$. A useful way to analyze the first two is to observe the isomorphism
\[ \alpha_{IJK} = 3\Phi_{[I}^{\alpha} \xi_{K]} \alpha, \] (B.3)
between the components $\alpha_{IJK}$ in $\Lambda_{1}^{3} \oplus \Lambda_{27}^{3}$ and the symmetric tensor representation $\xi_{IJ} \equiv \xi_{IJ}$ of $G_2$. The traceless part $\xi_{IJ} - \frac{1}{3} g_{IJ} \xi^K$ of $\xi_{IJK}$ is isomorphic to $\Lambda_{1}^{3}$ while its trace part $\frac{1}{3} \xi_{IJK} \delta^K$ is isomorphic to the singlet representation $\Lambda_{1}^{3}$. Thus the only elements of $H_1^3$ are constant multiples of $\Phi$. Furthermore one can show that if the 3-form $\alpha$ defined above is closed and coclosed (i.e. harmonic) then it follows that $\xi$ obeys
\[ \xi_i^I = 0, \quad \nabla^I \xi_{IJ} = 0, \quad \Phi_{J}^{KL} \nabla_{K} \xi_{LJ} = 0. \] (B.4)
The equations above are precisely those satisfied by the (linearly independent) small variations $\xi_{IJ} = \delta g_{IJ}$ of a $G_2$ holonomy metric $g_{IJ}$ in order that the new metric $g_{IJ} + \delta g_{IJ}$ also has $G_2$ holonomy. Thus elements of $H_2^7$ correspond to such $G_2$ holonomy preserving deformations. Finally, any element of $\Lambda_{1}^{3}$ can be written as $\Phi_{14}^{2} \beta$ for some 2-form $\beta$ on $M$. Such elements have no other special properties, to the best of our knowledge, except that closure $d(\Phi_{14}^{2} \beta) = 0$ of $\Phi_{14}^{2} \beta$ implies coclosure $d^\dagger(\Phi_{14}^{2} \beta) = 0$ identically.

Appendix C. Poincaré lemma for $\Lambda_{14}^{3}$

Given a 2-form $\tilde{B}$ on $\mathbb{R}^7$ in the 14 irrep of $G_2$ that is coclosed $d^\dagger \tilde{B} = 0$ then the Poincaré lemma can be used to deduce
\[ \tilde{B} = d^\dagger \Xi, \quad P_3^2 d^\dagger \Xi = 0, \] (C.1)
for some 3-form $\Xi$. One can decompose $\Xi$ into irreps of $G_2$ as $\Lambda_{3} = \Lambda_{1}^{3} \oplus \Lambda_{27}^{3} \oplus \Lambda_{27}^{7}$ using the 3-form projection operators in appendix B, such that
\[ \Xi_{MNP} = \Phi_{MNP}^{a} a + \Phi_{MNPQ}^{b} b_{Q} + \delta_{MNP}. \] (C.2)
where $a = \frac{1}{3} \Phi_{MNP} \Xi_{MNP}, b_{Q} = \frac{1}{3} \Phi_{MNPQ} \Xi_{MNP}$ and $c = \Lambda_{1}^{3}$.

The identity $P_3 \dagger \Xi = 0$ together with (C.1) implies that we can neglect $a$ in $\Xi$ because it will drop out of $\tilde{B} = P_3 \tilde{B} = P_3 \dagger \Xi$.

The identities in appendix A can be used to rewrite the second equation in (C.1) in components as
\[ \Phi_{MNP} \partial_Q \Xi^{NP} = - \frac{2}{3} \Phi_{NPQ} \partial_N (\ast \Phi_{PQM} \partial_R b^R - c_{PQM}) = 0, \] (C.3)
where the identities $\Phi_{M1JC}^{IJ} = \Phi_{N1JC}^{IJ}$ and $\Phi_{MNP} \delta_{MN} = 0$ have also been used. Thus an equivalent form of this equation reads
\[ P_3^2 d (P_3^2 - P_{27}^3 \Xi = 0, \] (C.4)
a solution of which is
\[ (P_3^2 - P_{27}^3 \Xi = d \tilde{a}, \] (C.5)
where $\tilde{a} \in \Lambda_{14}^{3}$. The reason that $\tilde{a}$ is not a general 2-form is because of the identity $P_3^2 dP_{14}^2 = 0$ implying $d \tilde{a}$ is automatically in $\Lambda_{3}^{3} \oplus \Lambda_{27}^{7}$ as required by the equation.

Since we have assumed $P_3^2 \Xi = 0$ then acting on the equation above with $(P_3^2 - P_{27}^3)$ gives
\[ \Xi = (P_3^2 - P_{27}^3) d \tilde{a} = (2P_3^3 - 1) d \tilde{a}, \] (C.6)
and hence
\[ \tilde{B} = d^\dagger (2P_3^3 - 1) d \tilde{a}. \] (C.7)
Appendix D. BV quantization of a free 6-form in seven dimensions

The classical action for a free Abelian $p$-form $\omega_p$ in $n$ dimensions is $S_0 = \frac{1}{2} \int_n d\omega_p \wedge \ast d\omega_p$. We will now describe the BV quantization of this action for the special case of $p = 6$ and $n = 7$, where $\omega_0 = C$.

The classical action for $\omega_6$ is degenerate under the gauge transformations $\delta \omega_6 = d\lambda_5$. Thus we must introduce a fermionic ghost $\omega_5$ for this symmetry. The gauge symmetry is reducible for gauge parameters $\lambda_5 = d\lambda_4$. This necessitates the addition of a ghost-for-ghost bosonic field $\omega_4$. Continuing this line of reasoning leads to a tower of descendent $p$-form ghosts $\omega_p$, associated to $\omega_6$, with $0 \leq p < 6$ and Grassmann parity $(-1)^p$. Thus we have the collection $\Phi = \{\omega_p, p = 0, \ldots, 6\}$ of fields+ghosts with associated BRST transformations

$$Q \omega_p = d\omega_{p-1},$$

such that $Q\omega_0 = 0$. The corresponding set of anti(fields+ghosts) is $\Phi^* = \{\chi^*_{7-p}, p = 0, \ldots, 6\}$, where $\chi^*_{7-p}$ is a $(7-p)$-form with Grassmann parity $(-1)^{p+1}$. The master equation $Q \Phi = \delta S/\delta \Phi$ then fixes the form $\int \Phi^* \wedge Q\Phi$ of the minimal contribution to the classical action from these fields. The minimal solution of the master equation therefore corresponds to the action

$$S = S_0 + \sum_{p=0}^{5} \int \chi_{6-p} \wedge d\omega_p,$$

from which one derives the BRST transformations

$$Q\chi_1 = d \ast d\omega_0, \quad Q\chi_p = d\chi_{p-1} (p = 2, \ldots, 6),$$

for the antifields. The BRST transformation of $\chi_7$ can be an arbitrary BRST-invariant function. These transformations are indeed nilpotent and generate a global symmetry of $S$.

To fix all the residual gauge symmetries in a systematic way requires the introduction of quite an elaborate set of non-minimal fields. We will not need to get into the details of their BRST structure but let us just note that the appropriate gauge fermion here is

$$\Psi = \sum_{k=1}^{6} \int \gamma_{8-k} \wedge d^1 \omega_k + \sum_{k=2}^{6} \int \gamma_{8-k} \wedge d\theta_{k-2} + \sum_{k=3}^{6} \int \theta_{k-2} \wedge d^1 \alpha_{10-k}$$

$$+ \sum_{k=4}^{6} \int \alpha_{10-k} \wedge d\beta_{k-4} + \int \beta_1 \wedge d^1 \xi_7 + \beta_2 \wedge d^1 \xi_6 + \int \xi_6 \wedge d\phi_0.$$

The form degree and parity of all the non-minimal fields appearing here should be implicit. Imposing the gauge fermion constraint $\Phi^* = \delta \Psi/\delta \Phi$ and integrating out the Lagrange multiplier fields in the non-minimal terms in the action then leads to the following antifield constraints

$$\chi_k = d^1 \gamma_{k+1} (k = 1, \ldots, 6), \quad \chi_7 = 0$$

$$\gamma^*_{k} = d^1 \omega_{k+1} + d\theta_{k-1} = 0 \quad (k = 0, \ldots, 5)$$

$$\theta^*_{k} = d\gamma_{k-1} + d^1 \alpha_{k+1} = 0 \quad (k = 3, \ldots, 7)$$

$$\alpha^*_{k} = d^1 \theta_{k+1} + d^1 \beta_{k-1} = 0 \quad (k = 0, 1, 2, 3)$$

$$\beta^*_k = d\alpha_{k-1} + d^1 \xi_{k+1} = 0 \quad (k = 5, 6, 7)$$

$$\xi^*_k = d^1 \beta_{k+1} + d\phi_{k-1} = 0 \quad (k = 0, 1)$$

$$\phi^*_k = d\phi_6 = 0.$$
Solving these equations implies the non-minimal fields \((\phi_1, \epsilon_{6,7}, \beta_{0,1,2}, \alpha_{4,5,6,7}, \theta_{0,1,2,3,4})\) are harmonic, \(\gamma_{2,3,4,5,6,7}\) are closed and \(\omega_{1,2,3,4,5,6}\) are coclosed. The latter condition corresponds to the expected gauge-fixing constraint for \(p\)-forms.

Imposing these constraints in the non-minimal action solving the master equation leads us to the gauge-fixed action

\[
S = \frac{1}{2} \int \omega_6 \wedge * \Delta \omega_6 + \sum_{k=1}^{6} \int \gamma_{6+k} \wedge \Delta \omega_{6-k}.
\]

One can also verify that the aforementioned constraints solve the equation \(\int \Phi^* \wedge \Phi = \sum_{k=1}^{6} \chi_k \wedge \omega_{7-k} = 0\), defining the graded Lagrangian submanifold in configuration space.

Using the techniques of Schwarz that were reviewed in section 4, we are now ready to compute the partition function for the free 6-form. The resolvent for the classical action \(S_0\) here has the associated differential complex

\[
0 \to \Lambda^0 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^6 \xrightarrow{d} \Lambda^6 \to 0,
\]

where \(n = 6, \Gamma_i = \Lambda^0 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^6 \xrightarrow{d} \Lambda^6\). This allows us to identify \(Z_6\) as the reciprocal of the Ray–Singer torsion \(R_{RS}\) of the 7-manifold (see e.g. equation (2.21) in [11] for explicit identification). For our purposes it will be more convenient to write \(Z_6\) in terms of determinants of Laplacian operators \(\Delta = d^d + d^d\). This can be easily achieved using standard properties of determinants (see [11, 12]) to give

\[
Z_6 = (\det d_2^{1} \det d_6)^{-1/2} \left| \begin{array}{ccc} \det (d_3) \det (d_4) \det (d_1) \\ \det (d_4) \det (d_5) \det (d_0) \end{array} \right|.
\]

The leading term can be written in a similar form to the other terms using the identities \(\det d_6 = (\det d_1)^{1/2} = (\det d_2)^{1/2}\). This allows us to identify \(Z_6\) as the reciprocal of the analytic torsion of the complex (D.7) and not the de Rham complex (despite the fact they are identical up to the last term). Nonetheless, we may still have expected a topological invariant given that we are describing the special case of a free 6-form in seven dimensions—which has no local on-shell degrees of freedom. Indeed this result generalizes to any classical action \(S_0 = \frac{1}{2} \int_n d\omega_{n-1} \wedge * d\omega_{n-1}\) describing a free \((n-1)\)-form in \(n\) dimensions.

We end this appendix by noting a nice relation between the partition function for a free \(p\)-form gauge field \(\omega\) and a free \((n-p-2)\)-form gauge field \(\tilde{\omega}\), in odd dimensions \(n = 2k + 1\), involving the Ray–Singer torsion. Recall that such gauge fields describe equivalent local degrees of freedom in that the field equation \(d' G = 0\) and Bianchi identity \(dG = 0\) for \(\omega\) (where \(G = d\omega\)) can also be written as \(dG = 0\), \(dG = 0\), in terms of \(G = * d\tilde{\omega}\). Without loss of generality, we will now assume \(p = 2r + 1\) (the dual field will then always have even degree in odd dimensions). The partition functions for \(\omega\) and \(\tilde{\omega}\) are

\[
Z_{\omega} = (\det \Delta^{2r+1})^{-1/2} (\det \Delta^{2r}) (\det \Delta^{2r-1})^{-3/2} \cdots (\det \Delta^0)^{r+1},
\]

\[
Z_{\tilde{\omega}} = (\det \Delta^{2(k-r)-2})^{-1/2} (\det \Delta^{2(k-r)-3}) (\det \Delta^{2(k-r)-4})^{-3/2} \cdots (\det \Delta^0)^{(k-r-1)-1/2}.
\]
Some algebra and use of Hodge duality \( \det \Delta^\nu = \det \Delta^{\nu - \rho} \) then implies that the ratio
\[
Z_\omega / Z_\omega = \prod_{i=0}^{k} (\det \Delta^i)^{(-1)^i ((k-i)+1/2)} = (\mathcal{I}_\mathcal{R}_\mathcal{S})^{(-1)^i i!}.
\] (D.11)

### Appendix E. BV quantization of \( \vec{E} \)

Following the discussion of resolvents in section 4, we identify \( \Gamma_0 = \Lambda_{27\oplus 1}^4 \) in the action
\[
\int_{M_0} d\vec{E} \wedge * \left( \frac{3}{5} \mathcal{P}_7^3 - 1 \right) d\vec{E}
\]
for \( \vec{E} \) in (5.14). For suitable normalization of \( \vec{E} \), the kinetic operator in this action is
\[
K = -d! M d = \left[ \Lambda^4_{27} - \mathcal{P}^4_{27} d! \mathcal{P}^4_{27} + \frac{9}{2} \mathcal{P}^1_{27} d! \mathcal{P}^1_{27} \right] - \frac{1}{2} \left[ \Delta^1 - d! \mathcal{P}^1_1 \right],
\] (E.1)
where \( M = \frac{1}{2} \mathcal{P}^1_7 - \mathcal{P}^4_7 \). This kinetic operator is self-adjoint and indeed maps \( \Lambda_{27\oplus 1}^4 \rightarrow \Lambda_{27\oplus 1}^4 \), which follows from the identity \( \mathcal{P}^4_{27} d! \mathcal{P}^4_{27} = 0 \) using \( \mathcal{P}^2_{27} d! \mathcal{P}^2_{27} = 0 \).

The classical action for \( \vec{E} \) above is invariant under the gauge transformation \( \delta \vec{E} = \mathcal{P}^4_{27} d! \nu \), for any \( \nu \in \Lambda_{27\oplus 1}^4 \) (the singlet part of \( \nu \) is projected out of the gauge transformation). Furthermore this gauge symmetry is reducible for \( \nu = \mathcal{P}^3_{27\oplus 7} d! \epsilon \), for any 2-form \( \epsilon \). The projection operators do not commute with the exterior derivative so this statement is not obvious, but follows by noting \( \mathcal{P}^3_{27\oplus 7} d! \epsilon = d! \mathcal{P}^2_7 + \mathcal{P}^3_{27\oplus 7} d! \mathcal{P}^2_7 \) and using that \( d! \mathcal{P}^2_7 d! \mathcal{P}^2_7 \in \Lambda^4_7 \).

The remaining reducibilities are for \( \epsilon = d! \xi \), for any one-form \( \xi \), and \( \xi = d! \gamma \), for any scalar \( \gamma \).

In the notation of section 4, we therefore have a resolvent with \( n = 4 \) and \( T_1 = \mathcal{P}^4_{27\oplus 4} d_3 \), \( T_2 = \mathcal{P}^4_{27\oplus 7} d_2 \), \( T_3 = d_1 \), \( T_4 = d_0 \) (their adjoints being just \( T^1 = d^{123} \)). The appropriate complex is
\[
0 \longrightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{\mathcal{P}^4_{27\oplus 7} \mathcal{P}^4_{27\oplus 4} d} \Lambda^4_{27\oplus 7} \xrightarrow{\mathcal{P}^4_{27\oplus 4} d} \Lambda^4_{27\oplus 1} \longrightarrow 0,
\] (E.2)
with the extension by \( K \) included.

Using these identifications, the partition function (4.3) for \( \int_{M_0} d\vec{E} \wedge * \left( \frac{3}{5} \mathcal{P}_7^3 - 1 \right) d\vec{E} \) can be written as
\[
Z_{\mathcal{R}_\mathcal{S}}^{27\oplus 1} = (\det K)^{-1/2} |\det(\mathcal{P}^4_{27\oplus 1} d_3)||\det(\mathcal{P}^3_{27\oplus 7} d_2)|^{-1/2} |\det(\mathcal{P}^1_{27\oplus 7} d_1)||\det(\mathcal{P}^1_{27\oplus 4} d_0)|^{-1/2}
= (\det \Delta_{27\oplus 1}^4)^{-1/2} (\det \Delta_{27\oplus 7}^3)(\det \Delta_{27\oplus 1}^1)^{-3/2} (\det \Delta_{27\oplus 4}^0)^{-5/2}
= (\det \Delta_{27\oplus 1}^1)^{1/2} (\det \Delta_{27\oplus 7}^3)^{-3/2} (\det \Delta_{27\oplus 1}^4)^{3/2} (\det \Delta_{27\oplus 4}^{-3})^{-1} = I_{\mathcal{R}_\mathcal{S}}^{-1}.
\] (E.3)

The second equality has been obtained using \( \det(\mathcal{K} + T_1 T_1^\dagger) = (\det \mathcal{K})|\det T_1|^2 \) and similar descendent identities for the resolvent. The final equality uses the various \( G_2 \) isomorphisms described in section 5.4. Thus we conclude that the partition function for \( \vec{E} \) is also equal to the inverse Ray–Singer torsion.

### Appendix F. Hamiltonian action on \( G_2 \) string states

In order to determine the action of Hamiltonian operators \( H_L \) and \( H_R \) on the spectrum of the \( G_2 \) string, the two main examples to consider are states of the form \( \mathcal{A}_M(X) \psi^M_L \) in \( 7 \oplus 1 \cong 7 \) and \( B_{MN}(X) \psi^M_L \psi^N_R \) in \( 7 \oplus 7 \cong 14 \oplus 7 \oplus 27 \oplus 1 \). Note that the latter state need have no definite (anti)symmetry properties since \( \psi_L \) and \( \psi_R \) live in different sectors of the worldsheet

23 Actually \( T_1^\dagger = \mathcal{P}^4_{27\oplus 7} d_4 \) but this is identical to \( d_4 \) when acting on elements of \( \Lambda_{27\oplus 1}^4 \) since \( \mathcal{P}^4_{27\oplus 4} d_4 = 0 \).
theory. Following appendix B, the symmetric part $7 \otimes 7$ is isomorphic to $\Lambda^2_7 \otimes \Lambda^0_7$ while the antisymmetric part $7 \otimes 7$ corresponds to a general 2-form in $\Lambda^2_{14\otimes 7}$. All other cases can be mapped into these two examples by using the isomorphisms between the various $G_2$ irreps in the exterior algebra.

We start with the $7 \otimes 7$ case, since the other example follows easily from this one. We know from [5] that the action of $Q_L$ on this state is given by

$$Q_L : B_{M_1 N_1} \rightarrow (P^2_7)_{M_1 M_2}^{M_3 M_4} \nabla_{M_3} B_{M_4 N_1}, \quad (F.1)$$

i.e. $\tilde{D}$ acting only on the left indices. The state $Q_L B$ is an element of $\Lambda^2_7 \otimes \Lambda^0_7$. To define $H_L$, we also need to understand the action of the adjoint operator $Q^\dagger_L$. With respect to the standard inner product $\langle \omega, \xi \rangle = \int \sqrt{|g|} A^i B^i \omega^A R^B \xi^B \omega^A \xi^B$ of rank $n$ tensors, the adjoint of $Q_L$ acting on $B$ is defined as

$$\langle \Omega, Q_L B \rangle = \langle Q_L^\dagger \Omega, B \rangle, \quad (F.2)$$

for any $\Omega \in \Lambda^2_7 \otimes \Lambda^0_7$. The left-hand side of this equation is given by

$$\langle \Omega, Q_L B \rangle = 6 \int \sqrt{|g|} \nabla_{M_1} \Omega_{M_1 M_2} (P^2_7)_{M_1 M_2}^{M_3 M_4} \nabla_{M_3} B_{M_4 N_1}$$

$$\quad = -6 \int \sqrt{|g|} \left( \nabla_{M_1} \Omega_{M_1 M_2} (P^2_7)_{M_1 M_2}^{M_3 M_4} \right) B_{M_4 N_1}$$

$$\quad = \langle Q_L^\dagger \Omega, B \rangle,$$

from which one reads off

$$Q_L^\dagger : \Omega_{M_1 M_2} \rightarrow 6 \nabla_{M_1} \Omega_{M_1 M_2} (P^2_7)_{M_1 M_2}^{M_3 M_4}. \quad (F.3)$$

Thus we can now compute

$$Q_L^\dagger Q_L B_{M_1 N_1} = 6 \nabla_{M_1} \Omega_{M_1 M_2} (P^2_7)_{M_1 M_2}^{M_3 M_4} \nabla_{M_3} B_{M_4 N_1}$$

$$\quad = 6 \nabla_{M_1} \Omega_{M_1 M_2} (P^2_7)_{M_1 M_2}^{M_3 M_4} \nabla_{M_3} B_{M_4 N_1}$$

$$\quad = 6 \nabla_{M_1} \Omega_{M_1 M_2} (P^2_7)_{M_1 M_2}^{M_3 M_4} \nabla_{M_3} B_{M_4 N_1}$$

using $(P^2_7)^2 = P_7^2$. Now substituting the explicit form $(P^2_7)_{M_1 M_2}^{M_3 M_4} = \frac{1}{2} (\delta^{M_3}_{M_1} \delta^{M_4}_{M_2} + \frac{1}{2} \phi^{M_3 M_4}_{M_1 M_2})$ of the projector we find

$$Q_L^\dagger Q_L B_{M_1 N_1} = - \nabla^2 B_{M_1 N_1} + \nabla_{M_1} \nabla_{M_2} B_{M_2 N_1} + \nabla_{M_3} \nabla_{M_4} B_{M_4 N_1} \phi^{M_3 M_4}_{M_1 M_2}. \quad (F.4)$$

Note that the right-sector index of $B$ has just gone along for the ride in the calculation above.

To get $H_L$ we still need to compute $Q_L Q_L^\dagger B$. It is easy to show that

$$Q_L^\dagger : B_{M_1 N_1} \rightarrow - \nabla_{M_1} B_{M_1 N_1}, \quad (F.5)$$

and then

$$\langle Q_L Q_L^\dagger B \rangle_{M_1 N_1} = - \nabla_{M_1} \nabla_{M_2} B_{M_2 N_1}. \quad (F.6)$$

Putting these results together gives

$$H_L B_{M_1 N_1} = - \nabla^2 B_{M_1 N_1} - [\nabla_{M_1}, \nabla_{M_2}] B_{M_2 N_1} + [\nabla_{M_2}, \nabla_{M_3}] B_{M_3 N_1} \phi^{M_3 M_2}_{M_1 M_4}$$

$$\quad = - \nabla^2 B_{M_1 N_1} - R_{M_1 M_2 N_1 N_2} B_{M_2 N_1} + R_{M_2 M_3 N_1 N_2} B_{M_3 N_1} \phi^{M_3 M_2}_{M_1 M_4}$$

$$\quad = - \nabla^2 B_{M_1 N_1} - 3 R_{M_1 M_2 N_1 N_2} B_{M_2 N_1}, \quad (F.7)$$

where we have used $\phi^{A M N} R_{M N P Q} = 0$ on a $G_2$ manifold (i.e. the curvature 2-form must transform in the adjoint 14 of $G_2$). Consequently one finds that $\phi^{A M} R_{A B P Q} = - 2 R_{A M N P Q}$

24 The extra factor of 6 comes via the identity $\phi_{M A B} \phi^{N A B} = 6 \delta^N_M$, which leads to the different normalizations of the $\Lambda^2_7 \otimes \Lambda^0_7$ and $\Lambda^1_7 \otimes \Lambda^1_7$ inner products.
whose trace implies Ricci-flatness $R_{MN} = 0$ by virtue of the Bianchi identity $R_{M[NPQ]} = 0$. Both these results have also been used above.

Let us now decompose $B_{M,N}$ into symmetric and antisymmetric components and consider the action of $H_L$ on each component. If we take $B_{M,N}$ to be symmetric then (F.7) corresponds to the Lichnerowicz Laplacian acting on a metric deformation of the $G_2$ manifold [5, 28]. We can map the symmetric tensor $B$ to a 3-form $\omega$ in $\Lambda^3_{7\otimes 1}$ via the isomorphism
\[
\omega_{IJK} = 3\phi^A_{IJK}B_KA.
\]
described in appendix B. Thus, multiplying (F.7) with $\phi$ followed by appropriate contraction and antisymmetrization, one obtains
\[
H_L\omega_{IJK} = -\nabla^2\omega_{IJK} - \frac{3}{7}R_{[IJK]}^A\omega_KAB.
\]
The expression above follows using the $G_2$ curvature identity $\phi^A_{IJK}R_KAB = 0$ (this again follows from both pairs of indices of the Riemann tensor being in the $14$ irrep of $G_2$).

Recalling the Weitzenboch formula
\[
(\Delta^p\omega)_{I_1...I_p} = -\nabla^2\omega_{I_1...I_p} - \frac{p}{(p-1)!}R_{[I_1...I_p]}\omega^A_{I_1...I_p} - \frac{1}{4} \frac{p(p-1)}{(p-2)!} R_{AB[I_1...I_p]}\omega^A_{I_1...I_p},
\]
for $p$-forms we see that, under the map from symmetric tensors to 3-forms, $H_L$ maps to the ordinary 3-form Laplacian $\Delta^3 = d\delta + \delta d$ on a $G_2$ manifold.

On the other hand, if $B$ is antisymmetric then one can easily check that (F.7) just reduces to the Weitzenboch formula for 2-forms. Putting this together we have shown that $H_L = H_R = \Delta^2 + \Delta^3_{7\otimes 1}$ on states in the $7\otimes 7$ representation. It will be convenient to refer to this as $\Delta_{7\otimes 1} = \Delta^2 + \Delta^3_{7\otimes 1} + \Delta^3_7$.

We now wish to determine the action of $H_L + H_R$ on states of the form $A_M(X)\psi^M_L$ in $7\otimes 1$. Repeating the calculation above for this simpler case we find
\[
\begin{align*}
&\{Q_L, Q^L_R\}A_M = -\frac{1}{2} (\nabla^2 A_M - \frac{1}{2} \phi^{ABC} R_{ABC} A_M) = -\frac{1}{4} \nabla^2 A_M, \\
&\{Q_R, Q^L_R\}A_M = -\nabla^2 A_M,
\end{align*}
\]
using again Ricci-flatness $R_{MN} = 0$ and the Bianchi identity $R_{M[NPQ]} = 0$. Thus $H_L = H_R = \Delta^1 = \Delta^1$, up to normalization.

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