**Topological Micro-Electro-Mechanical Systems**

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We investigate the topological aspect of dynamics in a micro-electro-mechanical system (MEMS), which is a combination of an electric-circuit system and a mass-spring system. A simplest example is a sequential chain of capacitors and springs. It is shown that such a chain exhibits novel topological dynamics with respect to its oscillation modes. On one hand, when it undergoes free oscillation, the system is governed by the Su-Schrieffer-Heeger model, and the topological charge is given by a winding number. Topological and trivial phases are differentiated by measuring the dynamics of the outermost plate. On the other hand, when it undergoes periodical motion in time, the system is governed by an inversion-symmetric-trimer model, and the topological phases are characterized by an inversion-symmetry indicator. There emerge topological edge states, which are well signaled by measuring electromechanical-impedance resonance. Our results will open an attractive field of theoretical MEMS.

**Introduction:** Topological physics is one of the most important developments made over the past few decades. It has been explored mainly in the context of electron systems in solid. However, it is now expanded to artificial topological systems such as photonic[2,3] acoustic[4,5] mechanical[1,10] and electric-circuit[20,21] systems. For example, in electric circuits, the circuit Laplacian is identified with the Hamiltonian of the system, and the topological edge states are well signaled by impedance resonance[20,21].

Micro-electro-mechanical systems (MEMS) are combined systems of electric circuits and mechanical systems, which is a key technology of current industry. Actuators convert a mechanical motion to an electric signal. Among them, the parallel-plate electrostatic actuator is a simplest example, where the plates of a capacitor are connected with a spring. Since the energy stored in a capacitor depends on the separation between two plates constituting the capacitor, the mechanical motion of the parallel-plate is transformed into an electric signal. It is a basis of microphone.

In this work, we reveal the presence of a novel topological structure in the dynamics of MEMS. We propose a simplest model consisting of a sequential connection of capacitors and springs. It is intriguing that such a system exhibits entirely different dynamical behaviors as a function of a system parameter. We explain it in terms of topological phase transitions described by the Su-Schrieffer-Heeger model or the trimer model depending on free or forced oscillation.

First, we study a free oscillation, which occurs without applying external force and electric field. It is shown that the oscillation dynamics is described by the Su-Schrieffer-Heeger model (SSH) model, where the topological charge is the winding number. Topological and trivial phases are differentiated experimentally by measuring the dynamics of the outermost plate. Second, we study a forced oscillation, driving the system by external force or electric field. It is shown that the oscillation dynamics is described by a trimer model with inversion symmetry. The topological charge is given with the use of an inversion-symmetry indicator, which counts the emergent topological edge states. The topological edge states are well observed experimentally by measuring electromechanical-impedance resonance. It is interesting that two different topological systems emerge by changing the experimental setup in a single MEMS.

**Micro-electro-mechanical systems (MEMS):** We consider a system where capacitors and springs are connected sequentially as shown in Fig.1 where the spring has elastic constant $\kappa$. A capacitor is made of two plates, where each plate has mass $m$. Each plate is bound to its equilibrium position by a spring with elastic constant $u_0$. We focus on the $j$th capacitor. Let the displacement of the left (right) parallel plate measured from its equilibrium position be $u_{2j}$ and $u_{2j+1}$, where we use a convention such that $u_{2j} > 0$ ($u_{2j+1} > 0$) for the leftward (rightward) displacement. The actual distance between the two plates is $\ell_{\text{cap}} = X_{\text{cap}} + u_{2j+1} + u_{2j}$ with $X_{\text{cap}}$ being the equilibrium distance. On the other hand, the length of the spring is $\ell_{\text{spr}} = X_{\text{spr}} - u_{2j} - u_{2j+1}$ with $X_{\text{spr}}$ being the equilibrium length. The springs are insulating, where there is no charge transfer between adjacent capacitors. Finally, $q_j$ is the charge deviation from the equilibrium charge $Q_0$ in the $j$th capacitor. We attach a battery for each capacitor in order to charge up $Q_0$. The system shows a coupled oscillation. We analyze the dynamics of the system as a function of $Q_0$.  

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**FIG. 1:** (a) Illustration of MEMS. Parallel metallic plates of capacitors can move, where adjacent plates are connected by insulating springs. The displacement $u_j$ of the $j$th plate is measured from its equilibrium position, while the charge $q_j$ is measured from the equilibrium charge $Q_0$ of the capacitor. Each capacitor is connected by a battery in order to charge up. One-dimensional tight-binding model (b) for a free-end chain and (c) for a fix-end chain, where the unit cell contains three sites marked in magenta, orange and cyan.
FIG. 2: (a) Eigen spectra of the part \((K - \kappa_0 I)\) of the dynamical matrix for a free-end chain and (b) that for a fix-end chain in the case of free oscillation. The emergence of topological edge states marked in magenta differentiates the topological and the trivial phases. The horizontal axis is \(Q/\kappa\). Only the region \(Q/\kappa > 0\) is physical.

The Lagrangian of the system is given by

\[
L = \sum_j \left[ \frac{m}{2} (v_{j+1}^2 + v_j^2) - \frac{\kappa_0}{2} (u_{j+1}^2 + u_j^2) - \frac{\kappa}{2} (X_{\text{spr}} - u_{j+1} - u_j)^2 - \frac{(Q_0 + q_j)^2}{2C(u)} \right],
\]

(1)

where \(v_j = du_j/dt\) is the velocity of the plate, while \(C(u) = \varepsilon_0 S/(X_{\text{cap}} + u_{j+1} + u_j)\) is the capacitance with area \(S\) and permittivity \(\varepsilon_0\). The Euler-Lagrange equations are

\[
F_0 + f_j = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}_j} \right) - \frac{\partial L}{\partial v_j},
\]

(2)

\[
E_0 + e_j = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}_j} \right) - \frac{\partial L}{\partial u_j},
\]

(3)

where \(f_j\) is the force acting on the \(j\)th plate of the capacitor measured from the equilibrium force \(F_0\), and \(e_j\) is the electric field between the two plates of the \(j\)th capacitor measured from the equilibrium electric field \(E_0\). \(I_j = dq_j/dt\) is the current flowing the \(j\)th capacitor, which is absent in the Lagrangian and we have \(\partial L/\partial I_j = 0\). The equilibrium conditions reads

\[
\kappa X_{\text{spr}} = Q_0^2/(2\varepsilon_0 S) \quad \text{and} \quad E_0 = X_{\text{cap}} Q_0/(\varepsilon_0 S),
\]

where we have assumed \(F_0 = 0\).

We explicitly write down the Euler-Lagrange equations \(^2\) and \(^3\), and expand them to the linear order in \(u_{j+1}, u_j\) and \(q_j\). Making the Fourier transformation from time \(t\) to frequency \(\omega\), we obtain

\[
f_{2j} = -M_0 u_{2j} + \kappa u_{j-1} + Q q_j,
\]

(4)

\[
e_j = Q (u_{j+1} + u_j) + X q_j,
\]

(5)

\[
f_{2j+1} = -M_0 u_{2j+1} + \kappa u_{j+2} + Q q_j,
\]

(6)

where \(Q = Q_0/(\varepsilon_0 S), M_0 = m \omega^2 - \kappa - \kappa_0\) and \(X = X_{\text{cap}}/(\varepsilon_0 S)\).

For a spatially periodic system, by making the Fourier transformation from the site index \(j\) to momentum \(k\) \((-\pi < k \leq \pi)\), a set of equations \(^4\), \(^5\) and \(^6\) is transformed into

\[
\begin{pmatrix}
  f_L \\
  e \\
  f_R
\end{pmatrix} =
\begin{pmatrix}
  -M_0 & Q & \kappa e^{-ik} \\
  Q & X & Q \\
  \kappa e^{ik} & Q & -M_0
\end{pmatrix}
\begin{pmatrix}
  u_L \\
  q \\
  u_R
\end{pmatrix},
\]

(7)

where \(u_L(k)\) and \(u_R(k)\) are the Fourier components of \(u_{2j}\) and \(u_{2j+1}\), while \(f_L(k)\) and \(f_R(k)\) are the Fourier components of \(f_{2j}\) and \(f_{2j+1}\).

We represent Eq.\(^7\) in the form of

\[
\Phi_{fe} = J \Phi_{eq},
\]

(8)

where \(J\) is a \(3 \times 3\) matrix. We call \(J\) the MEMS Laplacian as in the case of the circuit Laplacian in electric circuits.

A finite length chain may have either free ends or fixed ends. In the present problem, the free-end (fixed-end) oscillation occurs when a chain is terminated at the positions of the capacitor planes \(u_L\) and \(u_R\) (charge \(q\) as in Fig.\(^1\b\) [c]). Note that the free-end chain respects the unit cell, but the fixed-end chain does not.

**Free oscillation:** First, we study the case of free oscillation in the absence of external force \(f\) and electric field \(e\), \(\Phi_{fe} = 0\), where Eq.\(^8\) becomes \(J \Phi_{eq} = 0\). The charge is solved as

\[
q = -Q (u_L + u_R)/X.
\]

We use it to eliminate \(q\) from Eq.\(^7\), and obtain the kinetic equation,

\[
m \frac{d^2}{dt^2} \begin{pmatrix}
  u_L \\
  u_R
\end{pmatrix} = -K \begin{pmatrix}
  u_L \\
  u_R
\end{pmatrix},
\]

(9)

with the dynamical matrix

\[
K = \begin{pmatrix}
  \kappa + \kappa_0 - Q & I_2 - H_{SSH} \\
  Q & X
\end{pmatrix},
\]

\[
H_{SSH} = \begin{pmatrix}
  0 & Q - \kappa e^{-ik} \\
  Q_\kappa e^{ik} & 0
\end{pmatrix}.
\]

(10)

The spring constant \(\kappa\) becomes \(\kappa + \kappa_0 - Q\) for \(H_{SSH} = 0\). This is known as the soft spring effect in the context of MEMS. We can tune \(\kappa_0\) to make all the eigenvalues of \(K\) positive.

We construct the phase diagram in term of a dimensionless parameter \(Q/\kappa\) instead of \(Q_0\). The SSH model is topological...
We have set $M = 1.5\kappa$. The horizontal axis is $Q/\kappa$, while the vertical axis is the admittance of the MEMS Laplacian. Topological edge states are marked in magenta. (c) Analytically obtained bulk admittance for $k = 0$ and $\pi$. The signs $\eta_j^s = +$ and $-\eta_j^s$ indicate the sign of the inversion-symmetry indicator. (d) Admittance spectrum for various $k$, where the color is ranging from $k = 0$ marked in magenta to $k = \pi$ marked in cyan. Only the region $Q/\kappa > 0$ is physical.

Forced oscillation: We next investigate a forced oscillation driven by external force or electric field periodic in time. We introduce a new quantity $H_{in}$ by $H_{in} = J + M_0I_3$,

$$H_{in} = \begin{pmatrix} Q & \kappa e^{-ik} & 0 \\ \kappa e^{ik} & M - \kappa & Q \\ 0 & Q & 0 \end{pmatrix},$$

with $M = ma^2 - \kappa_0 + X$. Here, we have made explicit how $H_{in}$ depends on $Q$ and $\kappa$. It is a key observation that $H_{in}$ is identical to the Hamiltonian of a trimer model on one-dimensional lattice, where the unit cell contains three sites, as illustrated in Fig. 4(b) and (c).

The trimer-model Hamiltonian has inversion symmetry $PH_{in}(k)P^{-1} = H_{in}(-k)$, where

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ (12)

It plays a key role to define inversion-symmetry protected topological phases.

There are two inversion-symmetric momenta $k = 0$, $\pi$. At these points ($s = 0$, $\pi$), the eigenenergies $E_j^s$ and eigenfunctions $\psi_j^s$ are analytically obtained,

$$E_0^0 = -\kappa, \quad E_0^\pi = \frac{M}{2} \pm \frac{\sqrt{(M - 2\kappa)^2 + 8Q^2}}{2},$$ (13)

$$E_0^\pi = \kappa, \quad E_0^s = \frac{M}{2} - \kappa \pm \frac{\sqrt{M^2 + 8Q^2}}{2},$$ (14)

$$\psi_0^s = (-1, 0, 1)/\sqrt{3}, \quad \psi_0^\pi = (1, F_\pi^s, 1)/\sqrt{2 + F_\pi^2}.$$ (15)

We show the spectrum $E_j^s(Q)$ in Fig. 4(c), where the horizontal axis is a dimensionless quantity $Q/\kappa$ for a fixed value of $Q$. By solving $E_1^s(Q) = E_0^s(Q)$, we obtain a gap closing point $\pm Q^0$ with $Q^0 = \sqrt{\kappa M}$. By solving $E_1^\pi(Q) = E_0^\pi(Q)$, we obtain two gap closing points $\pm Q^\pi$ with $Q^\pi = \sqrt{\kappa (2\kappa - M)}$. They are the endpoints of the magenta curves in Fig. 4(a) and (b).

Topological edge states: The admittance is the eigenvalue of the MEMS Laplacian $J$. We show the admittance spectrum of a finite length chain as a function of $Q/\kappa$ in Fig. 4(a) and (b). The magenta curves express edge states isolated from the bulk spectra at each $\kappa$. Indeed, the local density of states (LDOS) is localized at the edges as shown in Fig. 5. As suggested by the bulk-edge correspondence, they would be topological edge states at $|Q| < Q^0$ for the lower band and $|Q| < Q^\pi$ for the upper band in Fig. 4(a), and for $|Q| > Q^0$ for the lower band and $|Q| > Q^\pi$ for the upper band in

FIG. 4: (a) Admittance spectrum for a finite free-end chain and (b) that for a finite fixed-end chain in the case of forced oscillation. We have set $M = 1.5\kappa$. The horizontal axis is $Q/\kappa$, while the vertical axis is the admittance of the MEMS Laplacian. Topological edge states are marked in magenta. (c) Analytically obtained bulk admittance for $k = 0$ and $\pi$. The signs $\eta_j^s = +$ and $-\eta_j^s$ indicate the sign of the inversion-symmetry indicator. (d) Admittance spectrum for various $k$, where the color is ranging from $k = 0$ marked in magenta to $k = \pi$ marked in cyan. Only the region $Q/\kappa > 0$ is physical.

FIG. 5: (a) LDOS of all eigen functions for a free-end chain with $Q = 0.25\kappa$ and (b) that for a fixed-end chain with $Q = 2.5\kappa$ in the case of forced oscillation. We have set $M = 0.5\kappa$. There are two edge states signaled by two peaks at each edge in both cases. They are topological edge states.
for the highest spectrum, that is, \( \zeta \) the lowest spectrum. We obtain \( \zeta \) enough to define the topological number  

\[ \text{circuits} = \text{a good signal to detect topological edge states in electric} \]

impedance well signals the emergence of the topological edge states. In the similar way, the number of the lower edge states is counted by the number \( \zeta \) for \( \zeta \) we obtain \( \zeta \) fine the inversion-symmetry indicator for the occupied band  

\[ \text{Inversion-symmetry indicator:} \]

One dimensional system  

\[ \text{with inversion symmetry is classified} \]

of the upper edge states is counted by the number \( \zeta \) in the trivial phase as shown in Fig.6(b). This is because the topological phase as shown in Fig.6(a). It reflects the fact  

\[ \text{Oscillation of a string is subject to damping in an actual} \]

sample. This effect may be included by adding the term \( \partial R/\partial v \) to the right-hand side of the Euler-Lagrange equation (2), where \( R \) is the Rayleigh dissipation function  

\[ \Phi_{uv} = Z \Phi_{fe}. \]

For a semi-infinite free-end chain, the left-upper most component of the Green function is \( G_{00} \), which is expressed in terms of a continued fraction \( G_{00} = \overline{G}/\kappa^2 \) as  

\[ G = \frac{-M_0 - \frac{1}{\kappa^2}}{X - \frac{Q^2}{M_0 - \frac{1}{\kappa^2}}} = \frac{-M_0 - \frac{1}{\kappa^2}}{X - \frac{Q^2}{M_0 - \frac{1}{\kappa^2}}}. \]

It is solved in a closed form,  

\[ G = \left\{ (\kappa^2 + M_0^2) X + 2M_0 Q^2 \pm \sqrt{M^2}/(2(Q^2 + M_0 X)), \right\} \]

with \( M = (M_0^2 - \kappa^2) [2Q^2 + M_0 X]^2 - \kappa^2 X^2 \).  

The impedance diverges at \( \omega = \sqrt{(\kappa - Q^2/X)/m} \), which is the solution of \( Q^2 + M_0 X = 0 \). We show the impedance \( Z_{00} \) for a finite free-end chain as a function of \( \omega \) in Fig.6. It measures the response of the plate displacement \( u_0 \) when we apply a force \( f_0 \) with frequency \( \omega \).

There is a single strong impedance resonance peak in the topological phase as shown in Fig.6(a). It reflects the fact that the left most plate \( u_1 \) is almost isolated in the topological phase. On the other hand, there are many small peaks in the trivial phase as shown in Fig.6(b). This is because the oscillation propagates to other plates. The number of peaks increases as the length increases.

Oscillation of a string is subject to damping in an actual sample. This effect may be included by adding the term \( \partial R/\partial v \) to the right-hand side of the Euler-Lagrange equation (2), where \( R \) is the Rayleigh dissipation function  

\[ R = \sum_j \frac{1}{\gamma_j^2}, \]

with \( \gamma \) being the damping factor. All formulas stand as they are provided the definition \( M_0 = m \omega^2 - \kappa - \kappa_0 \) is replaced by \( M_0 = m \omega^2 - \kappa - \kappa_0 - i\gamma \). Hence, \( M = X + m \omega^2 - \kappa - \kappa_0 - i\gamma \) in (11).

In this work we have revealed a novel topological structure in dynamics of MEMS. Our results will open an attractive field of topological MEMS.

The author is very much grateful to Y. Mita and N. Nagaosa for helpful discussions on the subject. This work is supported by the Grants-in-Aid for Scientific Research from MEXT KAKENHI (Grants No. JP17K05490 and No. JP18H03676). This work is also supported by CREST, JST (JPMJCR16F1 and JPMJCR20T2).

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