Chronological null complete spacetimes admit a global time

E. Minguzzi

Abstract. The result “chronological spacetimes without lightlike lines are stably causal” is announced and motivated. It implies that chronological spacetimes which are null geodesically complete and satisfy the null genericity and the null (averaged) energy condition admit a time function.

Keywords: Time function, stable causality, null incompleteness

PACS: 4.20.Aq; 04.20.Cv; 04.20.Dw; 04.20.Gz

INTRODUCTION

A time function, \( t : M \to \mathbb{R} \), is a continuous function on spacetime \((M, g)\) (a time oriented Lorentzian manifold) which increases on every future directed causal curve. If it exists, it provides a total ordering of the spacetime events which respects the notion of causal precedence.

In the work “The existence of cosmic time functions” \([1]\), Hawking pointed out the equivalence between the existence of a time function and the property of stable causality. Recall that a spacetime is stably causal if the light cones can be strictly widened everywhere without introducing closed timelike curves. The notion of stable causality is often regarded as the minimal causality requirement which allows to remove altogether from the spacetime any form of causality violation – from closed causal curves, to almost closed chains of causal curves. The result by Hawking then proves the equivalence of two very desirable features for a spacetime. Subsequent work has shown that the time function, whenever it exists, can be chosen smooth with a timelike gradient \([2]\) (see also \([3]\)).

The problem of reducing the existence of a time function to more direct, physically reasonable conditions was not addressed by Hawking and in fact it has remained open so far. In the conclusion of their 1979 review “Global structure of spacetimes” \([4]\) Geroch and Horowitz identified this problem as one of the most important open problems concerning the global aspects of general relativity together with that of proving the cosmic censorship conjecture. In fact, the proof of the existence of a time function may also be regarded as a first step towards the goal of proving the global hyperbolicity of spacetime starting from physically well motivated assumptions.

This work announces a result which, as I shall argue, solves Geroch and Horowitz’s problem. In order to state this result let me recall that a lightlike line is a inextendible achronal causal curve, in particular a lightlike line is a lightlike geodesic without conjugate points \([5]\) Chap. 10, Prop. 48]. In the forthcoming work \([6]\) I shall prove the theorem.
(C): “chronological spacetimes without lightlike lines are stably causal”. Here I shall just sketch the basic ideas underlying the proof, and comment on the physical consequences of this result. A first important observation is that the mentioned theorem involves only conformal invariant properties and hence it is a theorem on the causal structure of the spacetime.

Now, recall that if a spacetime is null geodesically complete, satisfies the null genericity condition and the null convergence condition then any inextendible lightlike geodesic admits a pair of conjugate points [7, Prop. 4.4.5] [8, Prop. 12.17], and hence any such spacetime does not admit lightlike lines. Finally, thanks to (C), provided the spacetime is chronological, one gets that it is also stably causal and hence admits a time function. Physically this result is very satisfactory in that the null convergence condition can even be replaced by the weaker averaged null convergence condition, [9, 10, 11, 12] which is the weakest among the energy conditions which are usually imposed on the stress energy (Ricci) tensor.

Note that the real universe could indeed be null geodesically complete while being timelike geodesically incomplete. It is easy to check that this case is compatible with all the singularity theorems. Even if global hyperbolicity holds one cannot conclude using Penrose’s singularity theorem [7] (1965) that the presence of a trapped surface would lead to a null incomplete geodesic, in fact this result holds only if the Cauchy hypersurfaces are non-compact (and, by the way, if one assumes global hyperbolicity there is no need to argue for the validity of stable causality by assuming null geodesic completeness). Thus the assumption of null geodesic completeness is compatible with the constraints given by the theory and by the observation. The mentioned theorem can then be used to substantiate the existence of a time function on the physical ground of this mild non-singularity requirement.

However, the theorem can also be used in the “negative” way as an aid to singularity theorems because it proves that under the same assumption of, say, Hawking and Penrose (1970) singularity theorem [7], the spacetime admits a time function and hence a foliation of partial Cauchy hypersurfaces. Thus one of the boundary assumptions of Hawking and Penrose (1970) singularity theorem, namely the existence of a compact partial Cauchy surface, is truly only a compactness requirement.

Finally, (C) can be regarded as a singularity theorem in its own right, in fact in the form “non-stably causal spacetimes either are non-chronological or admit lightlike lines” receives the following physical interpretation “if there is a form of causality violation on spacetime then either it is the worst possible, namely violation of chronology, or the spacetime is singular” a result which clarifies the influence of causality violations on singularities.

**SKETCH OF THE PROOF**

In this section I motivate the claim (C): chronological spacetimes without lightlike lines are stably causal. Recall Hawking’s result that a chronological spacetime without lightlike lines is strongly causal (1966 Adams prize essay, see [13]). The proof of (C) is based on the preliminary result that the absence of lightlike lines implies the transitivity of the causal relation $J^+$. Take two pairs $(x,y) \in J^+$ and $(y,z) \in J^+$ and two sequences of
causal curves $\sigma_n$ of endpoints $(x_n, y_n) \to (x, y)$, and $\gamma_n$ of endpoints $(y'_n, z_n) \to (y, z)$. The limit curve theorem states that each sequence has a subsequence that either converges to a connecting causal curve or converges to a (past in the $\sigma_n$ case, future in the $\gamma_n$ case) inextendible causal curve passing through $y$.

![Diagram of causal curves](image)

**FIGURE 1.** The potentially dangerous case in the proof of the transitivity of $J^+$. In the proof of $(x, z) \in \bar{J}^+$, the only potentially dangerous case is that in which neither subsequence converges to a connecting curve (see figure 1). By joining at $y$ the two limit curves $\sigma$ and $\gamma$ one gets an inextendible causal curve which by assumption is not a lightlike line. As a result $\gamma \circ \sigma$ admits two chronologically related events, $(\bar{x}, \bar{z}) \in I^+$, which without loss of generality, can be found so that $\bar{x} \in \sigma$ and $\bar{z} \in \gamma$. The fact that $I^+$ is open implies that a further subsequence can be found such that $(x_n, z_n) \in I^+$, and hence $(x, z) \in \bar{J}^+$. With a similar argument it is possible to show that the further assumption of chronology implies not only that $(M, g)$ is strongly causal but also that the relation $\bar{J}^+$ is antisymmetric (a property known as $A$-causality [14, 15]).

Now, note that if $\bar{J}^+$ is transitive then it is also the smallest closed and transitive relation which contains $J^+$, that is $K^+ = \bar{J}^+$, where $K^+$ is the causal relation introduced by Sorkin and Woolgar [16]. A spacetime is by definition, $K$-causal if $K^+$ is antisymmetric. Thus we have shown that a chronological spacetime without lightlike lines is $K$-causal.

It has long been suspected that $K$-causality may be equivalent to stable causality. Indeed, R. Low [16] suggested that since stable causality is equivalent to the antisymmetry of the Seifert relation [17] $J_S^+ = \bigcap_{g' > g} J_{g'}^+$ (a fact rigorously proved in [18] and [19]), and this relation is closed and transitive, one has $K^+ \subset J_S^+$, thus stable causality implies $K$-causality, and maybe the equality $J_S^+ = K^+$ holds which would imply that $K$-causality coincides with stable causality. However, the situation proved more complex. Indeed, it was later shown [19] that examples exist of spacetimes such that $J_S^+ \neq K^+$, but nevertheless no example is known of a $K$-causal spacetime which is not stably causal.

The last step of the proof of (C) would be provided by the proof of the equivalence between $K$-causality and stable causality. Recently I gave a proof of this result [20], but for for the sake of proving (C) it is possible to follow another simplified route [6] which avoids the direct proof of the equivalence between stable and $K$-causality. In this strategy one first define a property weaker that stable causality, which I termed compact
stable casuality and then takes advantage of the fact that the property “the spacetime is compactly stably causal and does not have lightlike lines” is invariant under suitable enlargement of the light cones over compact sets \cite{6}.

**ACKNOWLEDGMENTS**

This work has been partially supported by GNFM of INDAM.

**REFERENCES**

1. S. W. Hawking, *Proc. Roy. Soc. London, series A* **308**, 433–435 (1968).
2. A. N. Bernal, and M. Sánchez, *Commun. Math. Phys.* **257**, 43–50 (2005).
3. H. J. Seifert, *Gen. Relativ. Gravit.* **8**, 815–831 (1977).
4. R. Geroch, and G. T. Horowitz, *Global structure of spacetimes*, Cambridge University Press, Cambridge, 1979, vol. General relativity: An Einstein centenary survey, pp. 212–292.
5. B. O’Neill, *Semi-Riemannian Geometry*, Academic Press, San Diego, 1983.
6. E. Minguzzi, Chronological spacetimes without lightlike lines are stably causal (2008), preprint: 0806.0153.
7. S. W. Hawking, and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge, 1973.
8. J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian Geometry*, Marcel Dekker Inc., New York, 1996.
9. F. J. Tipler, *J. Diff. Eq.* **30**, 165–174 (1978).
10. F. J. Tipler, *Phys. Rev. D* **15**, 942–945 (1978).
11. C. Chicone, and P. Ehrlich, *Manuscripta Math.* **31**, 297–316 (1980).
12. A. Borde, *Class. Quantum Grav.* **4**, 343–356 (1987).
13. S. W. Hawking, and R. Penrose, *Proc. Roy. Soc. Lond. A* **314**, 529–548 (1970).
14. N. M. J. Woodhouse, *J. Math. Phys.* **14**, 495–501 (1973).
15. E. Minguzzi, *Class. Quantum Grav.* **25**, 015009 (2008).
16. R. D. Sorkin, and E. Woolgar, *Class. Quantum Grav.* **13**, 1971–1993 (1996).
17. H. Seifert, *Gen. Relativ. Gravit.* **1**, 247–259 (1971).
18. S. W. Hawking, and R. K. Sachs, *Commun. Math. Phys.* **35**, 287–296 (1974).
19. E. Minguzzi, *Class. Quantum Grav.* **25**, 015010 (2008).
20. E. Minguzzi, K-causality coincides with stable causality (2008), preprint: 0809.1214.