Properties of minimal charts and their applications VI: the graph $\Gamma_{m+1}$ in a chart $\Gamma$ of type $(m;2,3,2)$

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Abstract

Let $\Gamma$ be a chart, and we denote by $\Gamma_m$ the union of all the edges of label $m$. A chart $\Gamma$ is of type $(m;2,3,2)$ if $w(\Gamma) = 7$, $w(\Gamma_m \cap \Gamma_{m+1}) = 2$, $w(\Gamma_m \cap \Gamma_{m+2}) = 3$, and $w(\Gamma_{m+2} \cap \Gamma_{m+3}) = 2$ where $w(G)$ is the number of white vertices in $G$. In this paper, we prove that if there is a minimal chart $\Gamma$ of type $(m;2,3,2)$, then each of $\Gamma_{m+1}$ and $\Gamma_{m+2}$ contains one of three kinds of graphs. In the next paper, we shall prove that there is no minimal chart of type $(m;2,3,2)$.

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1 Introduction

Charts are oriented labeled graphs in a disk (see [1], [5], and see Section 2 for the precise definition of charts). From a chart, we can construct an oriented closed surface embedded in 4-space $\mathbb{R}^4$ (see [5] Chapter 14, Chapter 18 and Chapter 23]). A C-move is a local modification between two charts in a disk (see Section 2 for C-moves). A C-move between two charts induces an ambient isotopy between oriented closed surfaces corresponding to the two charts.

We will work in the PL category or smooth category. All submanifolds are assumed to be locally flat. In [16], we showed that there is no minimal chart with exactly five vertices (see Section 2 for the precise definition of minimal charts). Hasegawa proved that there exists a minimal chart with exactly six white vertices [2]. This chart represents a 2-twist spun trefoil. In [3] and [15], we investigated minimal charts with exactly four white vertices. In this paper, we investigate properties of minimal charts and need to prove that there is no minimal chart with exactly seven white vertices (see [6], [7], [8], [9], [10], [11], [12]).

Let $\Gamma$ be a chart. For each label $m$, we denote by $\Gamma_m$ the union of all the edges of label $m$.

Now we define a type of a chart: Let $\Gamma$ be a chart with at least one white vertex, and $n_1, n_2, \ldots, n_k$ integers. The chart $\Gamma$ is of type $(n_1, n_2, \ldots, n_k)$ if there exists a label $m$ of $\Gamma$ satisfying the following three conditions:

(i) For each $i = 1, 2, \ldots, k$, the chart $\Gamma$ contains exactly $n_i$ white vertices in $\Gamma_{m+i-1} \cap \Gamma_{m+i}$. 

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(ii) If $i < 0$ or $i > k$, then $\Gamma_{m+i}$ does not contain any white vertices.

(iii) Both of the two subgraphs $\Gamma_m$ and $\Gamma_{m+k}$ contain at least one white vertex.

If we want to emphasize the label $m$, then we say that $\Gamma$ is of type $(m; n_1, n_2, \ldots, n_k)$. Note that $n_1 \geq 1$ and $n_k \geq 1$ by the condition (iii).

We proved in [7, Theorem 1.1] that if there exists a minimal $n$-chart $\Gamma$ with exactly seven white vertices, then $\Gamma$ is a chart of type $(7), (5, 2), (4, 3), (3, 2, 2)$ or $(2, 3, 2)$ (if necessary we change the label $i$ by $n-i$ for all label $i$). In [10], we showed that there is no minimal chart of type $(3, 2, 2)$. In this paper and [11], we shall show the following.

**Theorem 1.1** ([11, Theorem 1.1]) There is no minimal chart of type $(2, 3, 2)$.

In the future paper [12], we shall show there is no minimal chart of type $(7), (5, 2), (4, 3)$. Therefore we shall show that there is no minimal chart with exactly seven white vertices.

An edge in a chart is called a terminal edge if it has a white vertex and a black vertex.

In our argument we often construct a chart $\Gamma$. On the construction of a chart $\Gamma$, for a white vertex $w \in \Gamma_m$ for some label $m$, among the three edges of $\Gamma_m$ containing $w$, if one of the three edges is a terminal edge (see Fig. 1(a) and (b)), then we remove the terminal edge and put a black dot at the center of the white vertex as shown in Fig. 1(c). Namely Fig. 1(c) means Fig. 1(a) or Fig. 1(b). We call the vertex in Fig. 1(c) a BW-vertex with respect to $\Gamma_m$.

![Figure 1: (a),(b) White vertices in terminal edges. (c) A BW-vertex.](image)

For example, the graph as shown in Fig. 2(a) means one of the four graphs as shown in Fig. 2(b),(c),(d),(e).

![Figure 2: Graphs with two white vertices.](image)

The three graphs in Fig. 3 are examples of graphs in $\Gamma_m$ for a chart $\Gamma$ and a label $m$. We call a $\theta$-curve, an oval, a skew $\theta$-curve the three graphs as shown in Fig. 3(a),(b),(c) respectively.

Let $X$ be a set in a chart $\Gamma$. Let

$$w(X) = \text{the number of white vertices in } X.$$
Let $\Gamma$ be a chart of type $(m; 2, 3, 2)$. Then $w(\Gamma) = 7$, $w(\Gamma_m \cap \Gamma_{m+1}) = 2$, $w(\Gamma_{m+1} \cap \Gamma_{m+2}) = 3$, $w(\Gamma_{m+2} \cap \Gamma_{m+3}) = 2$. Thus $w(\Gamma_{m+1}) = 5$ and $w(\Gamma_{m+2}) = 5$. First we shall show the following lemma.

**Lemma 1.2** Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. Then each of $\Gamma_{m+1}$ and $\Gamma_{m+2}$ contains one of nine graphs as shown in Fig. 4, or the union of a $\theta$-curve and a skew $\theta$-curve, or the union of an oval and a skew $\theta$-curve.

In this paper, the following is the main result.

**Theorem 1.3** If there exists a minimal chart $\Gamma$ of type $(m; 2, 3, 2)$, then each of $\Gamma_{m+1}$ and $\Gamma_{m+2}$ contains either the union of an oval and a skew $\theta$-curve, or one of two graphs as shown in Fig. 4(g),(h).

The paper is organized as follows. In Section 2, we define charts and minimal charts. In Section 3, we investigate connected components of $\Gamma_m$. 

![Figure 3](image-url)  
Figure 3: (a) A $\theta$-curve. (b) An oval. (c) A skew $\theta$-curve.

![Figure 4](image-url)  
Figure 4: (a),(b),(c) Graphs with three black vertices. (d),(e),(f) Graphs with one black vertex. (g),(h) Graphs with three black vertices. (i) A graph with one black vertex.
with five white vertices for a minimal chart $\Gamma$. We shall show Lemma 1.2.

In Section 4, we shall show that neither $\Gamma_m$ nor $\Gamma_{m+3}$ contains a $\theta$-curve for any minimal chart $\Gamma$ of type $(m; 2, 3, 2)$ (i.e. both of $\Gamma_m$ and $\Gamma_{m+3}$ contain ovals) (see Corollary 1.3). In Section 5, we investigate an oval of label $m$ for a minimal chart $\Gamma$. In Section 6, we investigate white vertices in an oval of label $m$ for a minimal chart $\Gamma$ of type $(m; 2, 3, 2)$. In Section 7, we shall show that for any minimal chart $\Gamma$ of type $(m; 2, 3, 2)$, the graph $\Gamma_{m+1}$ contains none of the five graphs as shown in Fig. 4(a),(d),(e),(f),(i), and neither does $\Gamma_{m+2}$. Moreover we shall show that neither $\Gamma_{m+1}$ nor $\Gamma_{m+2}$ contains a $\theta$-curve. In Section 8, we consider a minimal chart $\Gamma$ of type $(m; 2, 3, 2)$ such that $\Gamma_{m+1}$ contains either an oval, or one of the four graphs as shown in Fig. 4(b),(c),(g),(h). We investigate that the chart $\Gamma$ contains what kind of pseudo charts. In Section 9, we shall show that neither $\Gamma_{m+1}$ nor $\Gamma_{m+2}$ contains the graph as shown in Fig. 4(c) for any minimal chart $\Gamma$ of type $(m; 2, 3, 2)$. We obtain the main theorem (Theorem 1.3).

2 Preliminaries

In this section, we introduce the definition of charts and its related words.

Let $n$ be a positive integer. An $n$-chart (a braid chart of degree $n$ or a surface braid chart of degree $n$) is an oriented labeled graph in the interior of a disk, which may be empty or have closed edges without vertices satisfying the following four conditions (see Fig. 5):

(i) Every vertex has degree 1, 4, or 6.

(ii) The labels of edges are in $\{1, 2, \ldots, n-1\}$.

(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled $i$ and $i + 1$ alternately for some $i$, where the orientation and label of each arc are inherited from the edge containing the arc.

(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels $i$ and $j$ of the diagonals satisfy $|i - j| > 1$.

We call a vertex of degree 1 a black vertex, a vertex of degree 4 a crossing, and a vertex of degree 6 a white vertex respectively.

Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward (resp. outward) is called a middle arc at the white vertex (see Fig. 5(c)). For each white vertex $v$, there
are two middle arcs at \( v \) in a small neighborhood of \( v \). An edge is said to be middle at a white vertex \( v \) if it contains a middle arc at \( v \).

Let \( e \) be an edge connecting \( v_1 \) and \( v_2 \). If \( e \) is oriented from \( v_1 \) to \( v_2 \), then we say that \( e \) is oriented outward at \( v_1 \) and inward at \( v_2 \).

![Diagram](image.png)

Figure 5: (a) A black vertex. (b) A crossing. (c) A white vertex. Each arc with three transversal short arcs is a middle arc at the white vertex.

Now \( C\)-moves are local modifications of charts as shown in Fig. [6] (cf. [1], [5] and [17]). Two charts are said to be \( C\)-move equivalent if there exists a finite sequence of \( C\)-moves which modifies one of the two charts to the other.

An edge in a chart is called a free edge if it has two black vertices.

For each chart \( \Gamma \), let \( w(\Gamma) \) and \( f(\Gamma) \) be the number of white vertices, and the number of free edges respectively. The pair \((w(\Gamma), -f(\Gamma))\) is called a complexity of the chart (see [4]). A chart \( \Gamma \) is called a minimal chart if its complexity is minimal among the charts \( C\)-move equivalent to the chart \( \Gamma \) with respect to the lexicographic order of pairs of integers.

We showed the difference of a chart in a disk and in a 2-sphere (see [6, Lemma 2.1]). This lemma follows from that there exists a natural one-to-one correspondence between \{charts in \( S^2 \}/C\)-moves and \{charts in \( D^2 \}/C\)-moves, conjugations ([5 Chapter 23 and Chapter 25]). To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk.

**Assumption 1** In this paper, all charts are contained in the 2-sphere \( S^2 \).

We have the special point in the 2-sphere \( S^2 \), called the point at infinity, denoted by \( \infty \). In this paper, all charts are contained in a disk such that the disk does not contain the point at infinity \( \infty \).

Let \( \Gamma \) be a chart, and \( m \) a label of \( \Gamma \). A hoop is a closed edge of \( \Gamma \) without vertices (hence without crossings, neither). A ring is a simple closed curve in \( \Gamma_m \) containing a crossing but not containing any white vertices. A hoop is said to be simple if one of the two complementary domains of the hoop does not contain any white vertices.

We can assume that all minimal charts \( \Gamma \) satisfy the following four conditions (see [6,7,8,14]):

**Assumption 2** If an edge of \( \Gamma \) contains a black vertex, then the edge is a free edge or a terminal edge. Moreover any terminal edge contains a middle arc.
Assumption 3 All free edges and simple hoops in $\Gamma$ are moved into a small neighborhood $U_{\infty}$ of the point at infinity $\infty$. Hence we assume that $\Gamma$ does not contain free edges nor simple hoops, otherwise mentioned.

Assumption 4 Each complementary domain of any ring and hoop must contain at least one white vertex.

Assumption 5 The point at infinity $\infty$ is moved in any complementary domain of $\Gamma$.

In this paper for a set $X$ in a space we denote the interior of $X$, the boundary of $X$ and the closure of $X$ by Int$X$, $\partial X$ and Cl$(X)$ respectively.

3 Connected components of $\Gamma_m$

In this section, we investigate connected components of $\Gamma_m$ with five white vertices for a minimal chart $\Gamma$. We shall show Lemma 1.2

Lemma 3.1 ([10] Lemma 3.1) In a minimal chart $\Gamma$, for each BW-vertex in $\Gamma_m$, the two edges of label $m$ containing the BW-vertex are oriented inward or outward at the BW-vertex simultaneously if each of the two edges is not a terminal edge (see Fig. 7).
Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. A loop is a simple closed curve in $\Gamma_m$ with exactly one white vertex (possibly with crossings).

**Lemma 3.2** ([10, Lemma 3.1]) Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $G$ be a connected component of $\Gamma_m$. Then we have the following.

(a) If $1 \leq w(G)$, then $2 \leq w(G)$.

(b) If $1 \leq w(G) \leq 3$ and $G$ does not contain any loop, then $G$ is one of three graphs as shown in Fig. 3.

The following lemma is easily shown. Thus we omit the proof.

**Lemma 3.3** Let $G$ be a 3-regular graph in $S^2$. Then we have the following.

(a) The graph $G$ contains exactly an even number of vertices.

(b) If $G$ has at most four vertices, then $G$ is one of seven graphs as shown in Fig. 3(a) and Fig. 8.

![Figure 7: BW-vertices.](image)

![Figure 8: (a),(b) Graphs without loops. (c),(d),(e),(f) Graphs with loops.](image)

**Lemma 3.4** Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $G$ be a connected component of $\Gamma_m$. If $w(G) = 5$ and $G$ has no loop, then $G$ is one of nine graphs as shown in Fig. 4.
Proof. By Assumption 2, each terminal edge is middle at a white vertex. Thus each white vertex in $\Gamma_m$ is contained in at most one terminal edge of label $m$. Hence

(1) the graph $G$ is obtained from a simple closed curve or a 3-regular graph (possibly with loops) by adding BW-vertices.

Now we shall show that $G$ contains at least one black vertex. If not, then the graph $G$ is a 3-regular graph on $S^2$. By Lemma 3.3(a), the graph $G$ contains exactly an even number of white vertices. This contradicts the fact $w(G) = 5$. Hence $G$ contains at least one black vertex. Thus

(2) $G$ contains at least one BW-vertex.

Claim. The graph $G$ is obtained from a 3-regular graph by adding BW-vertices.

Suppose that all white vertices in $G$ are BW-vertices. Then the graph $G$ is obtained from a simple closed curve by adding BW-vertices. By Lemma 3.1 in a minimal chart, for each BW-vertex in $\Gamma_m$, the two edges of label $m$ containing the BW-vertex are oriented inward or outward at the BW-vertex simultaneously if each of the two edges is not a terminal edge. Hence the orientation of edges must change at BW-vertices. Thus $G$ contains exactly an even number of BW-vertices. This contradicts the fact $w(G) = 5$. Hence $G$ contains a white vertex not a BW-vertex. Thus by (1), Claim holds. □

By $w(G) = 5$, (2) and Claim, the graph $G$ is obtained by adding BW-vertices from a 3-regular graph with at most four vertices. Hence by Lemma 3.3(b), the graph $G$ is obtained by adding BW-vertices from one of seven graphs as shown in Fig. 3(a) and Fig. 8. Since $G$ has no loop with $w(G) = 5$, the graph $G$ is not obtained from the graphs as shown in Fig. 8(e),(f).

Now the graph $G$ is on the 2-sphere $S^2$. Hence if $G$ is obtained from the graph as shown in Fig. 3(a), then the graph $G$ is one of three graphs as shown in Fig. 4(a),(b),(c). If $G$ is obtained from the graph as shown in Fig. 8(a), then the graph $G$ is one of two graphs as shown in Fig. 4(d),(e). If $G$ is obtained from the graph as shown in Fig. 8(b), then the graph $G$ is the graph as shown in Fig. 4(f). If $G$ is obtained from the graph as shown in Fig. 8(c), then the graph $G$ is one of two graphs as shown in Fig. 4(g),(h). If $G$ is obtained from the graph as shown in Fig. 8(d), then the graph $G$ is the graph as shown in Fig. 4(i). Therefore $G$ is one of nine graphs as shown in Fig. 4. We complete the proof of Lemma 3.4. □

Lemma 3.5 Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. If $w(\Gamma_m) = 5$ and $\Gamma_m$ has no loop, then $\Gamma_m$ contains one of the following graphs:

(a) one of nine graphs as shown in Fig. 4 or
(b) the union of a \( \theta \)-curve and a skew \( \theta \)-curve, or

c) the union of an oval and a skew \( \theta \)-curve.

**Proof.** First we shall show that there exist at most two connected components of \( \Gamma_m \) with white vertices. Suppose that there exist at least three connected components \( G_1, G_2, G_3 \) of \( \Gamma_m \) with \( w(G_i) \geq 1 \) for each \( i = 1, 2, 3 \). Then by Lemma 3.2(a), we have \( w(G_i) \geq 2 \) for each \( i = 1, 2, 3 \). Thus

\[
5 = w(\Gamma_m) \geq w(G_1) + w(G_2) + w(G_3) \geq 2 + 2 + 2 = 6.
\]

This is a contradiction. Hence there exist at most two connected components of \( \Gamma_m \) with white vertices.

Suppose that there exists a connected component \( G_1 \) of \( \Gamma_m \) with \( w(G_1) = 5 \). Since \( \Gamma_m \) has no loop, by Lemma 3.4 the graph \( G_1 \) is one of nine graphs as shown in Fig. 4.

Suppose that there exists two connected components \( G_1, G_2 \) of \( \Gamma_m \) with \( w(G_1) \geq 1, w(G_2) \geq 1 \) and \( w(G_1) + w(G_2) = 5 \). Then by Lemma 3.2(a), we have \( w(G_1) \geq 2, w(G_2) \geq 2 \).

Without loss of generality we can assume \( 2 \leq w(G_1) \leq w(G_2) \). Since \( w(G_1) + w(G_2) = 5 \), we have \( w(G_1) = 2 \) and \( w(G_2) = 3 \). Since \( \Gamma_m \) has no loop, by Lemma 3.2(b) the graph \( G_1 \) is a \( \theta \)-curve or an oval, and the graph \( G_2 \) is a skew \( \theta \)-curve. \( \square \)

**Lemma 3.6** ([9, Theorem 1.1]) There is no loop in any minimal chart with exactly seven white vertices.

By Lemma 3.5 and Lemma 3.6, we have Lemma 1.2.

### 4 \( \theta \)-curves

In this section we shall show that neither \( \Gamma_m \) nor \( \Gamma_{m+3} \) contains a \( \theta \)-curve for any minimal chart \( \Gamma \) of type \((m; 2, 3, 2)\) (i.e. both of \( \Gamma_m \) and \( \Gamma_{m+3} \) contain ovals) (see Corollary 4.3).

Let \( \Gamma \) be a chart, and \( m \) a label of \( \Gamma \). Let \( L \) be the closure of a connected component of the set obtained by taking out all the white vertices from \( \Gamma_m \). If \( L \) contains at least one white vertex but does not contain any black vertex, then \( L \) is called an *internal edge of label \( m \)*. Note that an internal edge may contain a crossing of \( \Gamma \).

Let \( \Gamma \) be a chart. Let \( D \) be a disk such that

1. the boundary \( \partial D \) consists of an internal edge \( e_1 \) of label \( m \) and an internal edge \( e_2 \) of label \( m + 1 \), and

2. any edge containing a white vertex in \( e_1 \) does not intersect the open disk \( \text{Int} D \).
Note that $\partial D$ may contain crossings. Let $w_1$ and $w_2$ be the white vertices in $e_1$. If the disk $D$ satisfies one of the following conditions, then $D$ is called a lens of type $(m, m + 1)$ (see Fig. 9):

(i) Neither $e_1$ nor $e_2$ contains a middle arc.

(ii) One of the two edges $e_1$ and $e_2$ contains middle arcs at both white vertices $w_1$ and $w_2$ simultaneously.

![Figure 9: Lenses.](image)

**Lemma 4.1** ([7, Corollary 1.1]) There is no lens in any minimal chart with at most seven white vertices.

**Lemma 4.2** Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Suppose that $\Gamma_m$ contains a $\theta$-curve $G$. If $\Gamma$ has no lens, and if the two white vertices in $G$ are contained in $\Gamma_{m+\varepsilon}$ for some $\varepsilon \in \{+1, -1\}$, then $w(\Gamma_{m+\varepsilon}) \geq 6$.

**Proof.** Let $w_1, w_2$ be the white vertices in $G$, and $e$ the internal edge of label $m$ in $G$ middle at $w_1$. Without loss of generality we can assume that

1. the edge $e$ is oriented from $w_1$ to $w_2$.

Then the other two internal edges in $G$ are oriented from $w_2$ to $w_1$ (see Fig. 10(a)). Thus

2. the edge $e$ is middle at $w_1, w_2$.

The $\theta$-curve $G$ divides $S^2$ into three disks. Let $D_1, D_2$ be two of the three disks with $D_1 \cap D_2 = \partial D_1 \cap \partial D_2 = e$. Let $e_1, e_2$ be internal edges (possibly terminal edges) of label $m + \varepsilon$ at $w_1$ in $D_1, D_2$ respectively. Since $e$ is middle at $w_1$ by (2), neither $e_1$ nor $e_2$ is middle at $w_1$. By Assumption 2, neither $e_1$ nor $e_2$ is a terminal edge. Hence $e_1$ and $e_2$ contain white vertices different from $w_1$, say $w_3, w_4$.

We shall show $w_3 \neq w_2$ and $w_4 \neq w_2$. If $w_3 = w_2$, then the edge $e_1$ separates the disk $D_1$ into two disks. One of the two disks contains the edge $e$. By (2), the disk is a lens. This contradicts the condition that $\Gamma$ has no lens. Hence $w_3 \neq w_2$. Similarly we can show $w_4 \neq w_2$.

Let $e_1', e_2'$ be internal edges (possibly terminal edges) of label $m + \varepsilon$ at $w_2$ in $D_1, D_2$ respectively. By using (2), we can show similarly that $e_1', e_2'$ contain white vertices different from $w_2$, say $w_3', w_4'$.
We shall show that \( w(\Gamma_{m+\varepsilon} \cap \text{Int} D_1) \geq 2 \). There are two cases: \( w_3 \neq w'_3 \) and \( w_3 = w'_3 \).

If \( w_3 \neq w'_3 \), then \( w(\Gamma_{m+\varepsilon} \cap \text{Int} D_1) \geq 2 \).

Suppose \( w_3 = w'_3 \) (see Fig. 10(b)). Let \( e''_1 \) be an internal edge (possibly a terminal edge) of label \( m+\varepsilon \) at \( w_3 \) different from \( e_1, e'_1 \). By (1) and (2), the edge \( e_1 \) is oriented from \( w_1 \) to \( w_3 \) and the edge \( e'_1 \) is oriented from \( w_3 \) to \( w_2 \).

Thus \( e''_1 \) is not middle at \( w_3 \). Hence by Assumption 2, the edge \( e''_1 \) is not a terminal edge. Thus the edge \( e''_1 \) contains a white vertex different from \( w_3 \).

Thus \( w(\Gamma_{m+\varepsilon} \cap \text{Int} D_1) \geq 2 \).

Similarly we can show \( w(\Gamma_{m+\varepsilon} \cap \text{Int} D_2) \geq 2 \). Finally we have

\[
w(\Gamma_{m+\varepsilon}) \geq w(\Gamma_{m+\varepsilon} \cap G) + w(\Gamma_{m+\varepsilon} \cap \text{Int} D_1) + w(\Gamma_{m+\varepsilon} \cap \text{Int} D_2) \geq 2 + 2 + 2 = 6
\]

\( \square \)

\[\text{Figure 10: } \text{(a) The dark gray region is the disk } D_1, \text{ the light gray region is the disk } D_2. \text{ (b) Both of } e_1 \text{ and } e'_1 \text{ contain the white vertex } w_3.\]

**Corollary 4.3** Let \( \Gamma \) be a minimal chart of type \((m; 2, 3, 2)\). Then both of \( \Gamma_m \) and \( \Gamma_{m+3} \) contain ovals.

**Proof.** Since \( \Gamma \) is of type \((m; 2, 3, 2)\), we have \( w(\Gamma_m) = 2 \). Since the graph \( \Gamma_m \) does not contain any loop by Lemma 3.6, Lemma 3.2(b) implies that the graph \( \Gamma_m \) contains one of the two graphs as shown in Fig. 3(a) and (b). Hence the graph \( \Gamma_m \) contains a \( \theta \)-curve or an oval. If the graph \( \Gamma_m \) contains a \( \theta \)-curve, then we have \( w(\Gamma_{m+1}) \geq 6 \) by Lemma 4.2.

On the other hand, since \( \Gamma \) is of type \((m; 2, 3, 2)\), we have \( w(\Gamma_{m+1}) = 5 \). This is a contradiction. Thus the graph \( \Gamma_m \) contains an oval.

Similarly we can show that the graph \( \Gamma_{m+3} \) contains an oval. \( \square \)

### 5 Ovals

In this section we investigate an oval of label \( m \) for a minimal chart \( \Gamma \).

Let \( \Gamma \) be a chart, \( m \) a label of \( \Gamma \), \( D \) a disk with \( \partial D \subset \Gamma_m \), and \( k \) a positive integer. If \( \partial D \) contains exactly \( k \) white vertices, then \( D \) is called a \( k \)-angled disk of \( \Gamma_m \). Note that the boundary \( \partial D \) may contain crossings.
Lemma 5.1 Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $G$ be an oval of label $m$, and $D$ a 2-angled disk of $\Gamma_m$ with $\partial D \subset G$. Let $E$ be a disk in $D$ whose boundary consists of an internal edge in $G$ and an internal edge of label $m + \varepsilon$ ($\varepsilon \in \{+1, -1\}$) connecting the two white vertices of $G$. If $E$ does not contain the terminal edges in $G$, then $E$ is a lens of $\Gamma$.

Proof. Let $e$ be the internal edge of label $m + \varepsilon$ in $\partial E$. Let $v_1, v_2$ be the white vertices in $G$, and $e_1, e_2$ the terminal edges at $v_1, v_2$ in $G$ respectively. By Assumption 2.

(1) both of the two edges $e_1$ and $e_2$ contain middle arcs.

There are three cases: (i) neither $e_1$ nor $e_1$ is contained in $D$ (see Fig. 11(a)), (ii) only one of $e_1$ and $e_1$ is contained in $D$ (see Fig. 11(b)), (iii) both of $e_1$ and $e_1$ are contained in $D$ (see Fig. 11(c)).

Case (i). By (1), the edge $e$ is middle at both white vertices $v_1$ and $v_2$ simultaneously. Thus the disk $E$ is a lens.

Case (ii). Without loss of generality we can assume that

(2) the edge $e_1$ is oriented inward at $v_1$.

Then by (1), the other two internal edges in $G$ are oriented from $v_1$ to $v_2$. Thus

(3) the edge $e_2$ is oriented outward at $v_2$.

If $e_1 \subset D$ (see Fig. 11(b)), then by (1) and (2), the edge $e$ is oriented from $v_2$ to $v_1$. Hence $e$ is oriented outward at $v_2$. On the other hand, by (1) and (3), the edge $e$ is oriented inward at $v_2$. This is a contradiction.

Similarly if $e_2 \subset D$, then we have the same contradiction. Thus Case (ii) does not occur.

Case (iii). By (1), none of $e$ and the two internal edges in $G$ contain middle arcs. Thus the disk $E$ is a lens. \[\Box\]

Figure 11: The gray regions are disks $E$. (a) $e_1 \not\subset D, e_2 \not\subset D$. (b) $e_1 \subset D, e_2 \not\subset D$. (c) $e_1 \subset D, e_2 \subset D$.

Let $\Gamma$ be a chart. Suppose that an object consists of some edges of $\Gamma$, arcs in edges of $\Gamma$ and arcs around white vertices. Then the object is called a pseudo chart.

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Lemma 5.2 Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $G$ be an oval of label $m$. If for some $\varepsilon \in \{+1, -1\}$ there exists a 2-angled disk $D$ of $\Gamma_{m+\varepsilon}$ with $G \cap \partial D$ two white vertices, then there exists a lens of $\Gamma$.

Proof. Let $C$ be the simple closed curve in $G$. Since $G \cap \partial D = C \cap \partial D$ consists of two white vertices, the union $C \cup \partial D$ separates $S^2$ into four disks. One of the four disks, say $E$, is bounded by an internal edge of label $m$ in $G$ and an internal edge of label $m + \varepsilon$ in $\partial D$. Let $e$ be the internal edge of label $m + \varepsilon$ in $\partial E$, and $E'$ the disk with $\partial E' \subset C \cup e$ and $E' \cap E = e$. Then $E \cup E'$ is a 2-angled disk of $\Gamma_m$ whose boundary is contained in $G$.

If $E$ or $E'$ is a lens, then there exists a lens of $\Gamma$.

Suppose that neither $E$ nor $E'$ is a lens. Then by Lemma 5.1,

(1) each of $E$ and $E'$ contains a terminal edge in $G$.

Hence $E \cup E'$ contains one of the two pseudo charts as shown in Fig. 12.

Let $e'$ be the internal edge of label $m + \varepsilon$ in $\partial D$ different from $e$. Then $e' \not\subset E \cup E'$. Thus $e' \subset Cl(S^2 - (E \cup E'))$. Hence the edge $e'$ separates the disk $Cl(S^2 - (E \cup E'))$ into two disks. Thus neither of the two disks contains a terminal edge in $G$, because $E \cup E'$ contains the two terminal edges of $G$ by (1). Hence by Lemma 5.1 both of the two disks are lenses. \qed

Figure 12: The light gray region and the dark gray region are $E$ and $E'$.

Lemma 5.3 Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $G$ be an oval of label $m$, and $v_1, v_2$ the white vertices in $G$. Let $D$ be a 2-angled disk of $\Gamma_m$ with $\partial D \subset G$. If $D$ satisfies one of the following two conditions, then $\Gamma$ is not minimal.

(a) The disk $D$ does not contain terminal edges of $G$, but contains two internal edges $e_1, e_2$ of label $m + \varepsilon$ at $v_1, v_2$ respectively ($\varepsilon \in \{+1, -1\}$) such that $e_1 \cap e_2$ is a BW-vertex with respect to $\Gamma_{m+\varepsilon}$ in $\text{Int} D$ (see Fig. 13(a)).

(b) The disk $D$ contains exactly one terminal edge of $G$, and contains three internal edges $e_1, e_2, e_3$ of label $m + \varepsilon$ at $v_1, v_1, v_2$ respectively ($\varepsilon \in \{+1, -1\}$) such that $e_1 \cap e_2 \cap e_3$ is a white vertex in $\text{Int} D$ (see Fig. 13(b)).
Figure 13:  (a),(c) The gray regions are 2-angled disks not containing terminal edges of $G$.  (b),(d) The gray regions are 2-angled disks containing one terminal edge of $G$.

**Proof.** Suppose that $\Gamma$ is minimal.  Let $e, e'$ be the terminal edges at $v_1, v_2$ in $G$ respectively.  By Assumption 2

(1) each of the two edges $e$ and $e'$ is middle at a white vertex.

Without loss of generality we can assume that

(2) the edge $e$ is oriented inward at $v_1$.

Then by (1), the other two internal edges in $G$ are oriented from $v_1$ to $v_2$.  Thus

(3) the edge $e'$ is oriented outward at $v_2$.

If the disk $D$ satisfies the condition (a), then $e_1 \cap e_2$ is a BW-vertex with respect to $\Gamma_{m+\varepsilon}$.  Let $v_3 = e_1 \cap e_2$.  By (2), the edge $e_1$ is oriented from $v_1$ to $v_3$.  Thus the edge $e_1$ is oriented inward at the BW-vertex $v_3$.

On the other hand, by (3), the edge $e_2$ is oriented from $v_3$ to $v_2$.  Thus the edge $e_2$ is oriented outward at the BW-vertex $v_3$.  Hence for the BW-vertex $v_3$, the edge $e_1$ of label $m+\varepsilon$ is oriented inward at $v_3$, and the edge $e_2$ of label $m+\varepsilon$ is oriented outward at $v_3$ (see Fig. 13(c)).  This contradicts Lemma 3.1.  Thus $\Gamma$ is not minimal.

If the disk $D$ satisfies the condition (b), then $e_1 \cap e_2 \cap e_3$ is a white vertex.  Let $v_3 = e_1 \cap e_2 \cap e_3$.  By (1) and (2), both of $e_1$ and $e_2$ are oriented from $v_3$ to $v_1$.  Thus

(4) both of $e_1$ and $e_2$ are oriented outward at $v_3$.

On the other hand, by (3), the edge $e_3$ is oriented from $v_3$ to $v_2$.  Thus the edge $e_3$ is oriented outward at $v_3$.  Hence by (4), the three edges $e_1, e_2, e_3$ of label $m+\varepsilon$ are oriented outward at $v_3$ (see Fig. 13(d)).  This contradicts the condition (iii) of the definition of charts.  Thus $\Gamma$ is not minimal.  □
6 White vertices in the graphs $\Gamma_{m+1}$

In this section, we investigate white vertices in an oval of label $m$ for a minimal chart $\Gamma$ of type \((m; 2, 3, 2)\).

**Lemma 6.1** Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $w$ be a white vertex in a terminal edge of label $m$. Let $e_1, e_2$ be the two edges of label $m$ at $w$ different from the terminal edge. If $w$ is contained in a terminal edge of label $m + \varepsilon$ for some $\varepsilon \in \{+1, -1\}$, then both edges $e_1, e_2$ are contained in the closure of the same connected component of $S^2 - \Gamma_{m+\varepsilon}$.

**Proof.** Since $w$ is contained in a terminal edge of label $m$ and since $w$ is contained in a terminal edge of label $m + \varepsilon$, by Assumption 2 we can show that in a neighborhood of the vertex $w$, the chart $\Gamma$ contains the pseudo chart as shown in Fig. 14. Hence the edges $e_1$ and $e_2$ of label $m$ are contained in the closure of the same connected component $F$ of $S^2 - \Gamma_{m+\varepsilon}$. Thus we complete the proof of Lemma 6.1. \qed

![Figure 14](image.png)

Figure 14: A white vertex $w$ is contained in two terminal edges. The gray region is $F$.

From the above lemma, we have the following lemma:

**Lemma 6.2** Let $\Gamma$ be a minimal chart, and $m$ a label of $\Gamma$. Let $G$ be an oval of label $m$. If one of the two white vertices in $G$ is a BW-vertex with respect to $\Gamma_{m+\varepsilon}$ for some $\varepsilon \in \{+1, -1\}$, then the two internal edges in $G$ are contained in the closure of the same connected component of $S^2 - \Gamma_{m+\varepsilon}$.

**Lemma 6.3** Let $\Gamma$ be a minimal chart of type \((m; 2, 3, 2)\). Suppose that $\Gamma_m$ contains an oval $G$. Then we have the following.

(a) If $\Gamma_{m+1}$ contains one of the three graphs as shown in Fig. 4(a),(b),(c), then either $G$ contains two BW-vertices with respect to $\Gamma_{m+1}$, or $G$ does not contain any BW-vertex with respect to $\Gamma_{m+1}$.

(b) If $\Gamma_{m+1}$ contains one of the three graphs as shown in Fig. 4(d),(e),(f), then $G$ does not contain any BW-vertex with respect to $\Gamma_{m+1}$.
(c) If $\Gamma_{m+1}$ contains the union of a $\theta$-curve and a skew $\theta$-curve, then $G$ does not contain any BW-vertex with respect to $\Gamma_{m+1}$. Moreover the two white vertices in $G$ are contained in the $\theta$-curve or the skew $\theta$-curve simultaneously.

(d) If $\Gamma_{m+1}$ contains the union of an oval and a skew $\theta$-curve, then either $G$ contains two BW-vertices with respect to $\Gamma_{m+1}$, or $G$ does not contain any BW-vertex with respect to $\Gamma_{m+1}$.

Proof. Let $e_1, e_2$ be the internal edges of label $m$ in $G$.

Statement (a). The graph $\Gamma_{m+1}$ contains exactly three BW-vertices with respect to $\Gamma_{m+1}$, say $w_1, w_2, w_3$. Let $w_4, w_5$ be the other white vertices in $\Gamma_{m+1}$. It suffices to prove that if $G$ contains one of BW-vertices $w_1, w_2, w_3$, then $G$ contains two of $w_1, w_2, w_3$.

If $G$ contains one of BW-vertices $w_1, w_2, w_3$, then by Lemma 6.2 there exists a connected component $F$ of $S^2 - \Gamma_{m+1}$ with $e_1 \cup e_2 \subset Cl(F)$ (see Fig. 15(a)). Thus

1. for each white vertex $w$ in $G$, there exist two edges of label $m$ at $w$ contained in $Cl(F)$.

On the other hand, by the condition of Lemma 6.3(a), for each white vertex $w_i$ ($i = 4, 5$) there exists at most one edge $m$ at $w_i$ in $Cl(F)$ (see Fig. 15(a)). Hence by (1), the oval $G$ does not contain $w_4$ nor $w_5$. Thus $G$ contains two of BW-vertices $w_1, w_2, w_3$. Hence Statement (a) holds.

Statement (b). The graph $\Gamma_{m+1}$ contains exactly one BW-vertex with respect to $\Gamma_{m+1}$, say $w_1$. Let $w_2, w_3, w_4, w_5$ be the other white vertices in $\Gamma_{m+1}$.

Suppose that $G$ contains the BW-vertex $w_1$. Then by Lemma 6.2 there exists a connected component $F$ of $S^2 - \Gamma_{m+1}$ with $e_1 \cup e_2 \subset Cl(F)$ (see Fig. 15(b)). Thus for each white vertex in the oval $G$, there exist two edges of label $m$ at the vertex in $Cl(F)$. However, by the condition of Lemma 6.3(b), for each white vertex $w_i$ ($i = 2, 3, 4, 5$) there exists at most one edge of label $m$ at $w_i$ in $Cl(F)$. This is a contradiction. Hence the oval $G$ does not contain the vertex $w_1$. Hence Statement (b) holds.

Statement (c). Let $w_1, w_2$ be the white vertices of the $\theta$-curve in $\Gamma_{m+1}$. Let $w_3$ be the BW-vertex of the skew $\theta$-curve with respect to $\Gamma_{m+1}$, and $w_4, w_5$ the other white vertices of the skew $\theta$-curve.

By a similar way of the proof of Statement (b), we can show that the oval $G$ does not contain the BW-vertex $w_3$. Thus $G$ contains two of $w_1, w_2, w_4, w_5$.

Suppose that the oval $G$ contains one of the white vertices $w_1, w_2$, and one of the white vertices $w_4, w_5$. Without loss of generality we can assume $w_1, w_4 \in G$. Then

2. the two edges $e_1$ and $e_2$ of label $m$ connect the vertices $w_1$ and $w_4$. 

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Now, the $\theta$-curve in $\Gamma_{m+1}$ separates $S^2$ into three disks. One contains the skew $\theta$-curve in $\Gamma_{m+1}$, say $F$. Hence the vertex $w_4$ in the skew $\theta$-curve is contained in $F$. Thus by (2), we have $e_1 \cup e_2 \subset F$. Hence in $F$ there exist two edges of label $m$ at $w_1$. However, since $w_1$ is a white vertex of the $\theta$-curve in $\Gamma_{m+1}$, there exists at most one edge of label $m$ at $w_1$ in $F$ (see Fig. [15](c)). This is a contradiction. Hence $w_1, w_2 \in G$ or $w_4, w_5 \in G$. Thus Statement (c) holds.

Similarly we can show Statement (d).

![Figure 15: The gray regions are $F$. (a) The graph as shown in Fig. 4(b) with $w_2 \in G$. (b) The graph as shown in Fig. 4(d) with $w_1 \in G$. (c) The skew $\theta$-curve of label $m+1$ is contained in $F$.](image)

7 The graphs $\Gamma_{m+1}$ and $\Gamma_{m+2}$

In this section, we shall show that for any minimal chart $\Gamma$ of type $(m; 2, 3, 2)$, the graph $\Gamma_{m+1}$ contains none of the five graphs as shown in Fig. 4(a),(d),(e),(f),(i), and neither does $\Gamma_{m+2}$. Moreover we shall show that neither $\Gamma_{m+1}$ nor $\Gamma_{m+2}$ contains a $\theta$-curve.

**Lemma 7.1** Let $G$ be one of 12 graphs as shown in Fig. 3 and Fig. 4. If for any minimal chart $\Gamma$ of type $(m; 2, 3, 2)$, the graph $\Gamma_{m+1}$ does not contain the graph $G$, then the graph $\Gamma_{m+2}$ does not contain the graph $G$.

**Proof.** Suppose that the graph $\Gamma_{m+1}$ does not contain the graph $G$ for any minimal chart $\Gamma$ of type $(m; 2, 3, 2)$.

If there exists a minimal chart $\Gamma'$ of type $(m; 2, 3, 2)$ with $\Gamma_{m+2}' \supset G$, then let $\Gamma''$ be the chart obtained from $\Gamma'$ by changing labels $\cdots, m, m + 1, m + 2, m + 3, \cdots$ into $\cdots, m + 3, m + 2, m + 1, m, \cdots$, respectively. Then $\Gamma''$ is a chart of type $(m; 2, 3, 2)$ with $\Gamma''_{m+1} \supset G$. Hence $\Gamma''$ is not minimal. Thus $\Gamma''$ is C-move equivalent to a chart whose complexity is less than the complexity of $\Gamma''$. Hence by using the above C-moves, the chart $\Gamma'$ is also C-move equivalent to a chart whose complexity is less than the complexity $\Gamma'$. Thus $\Gamma'$ is not minimal. This is a contradiction. Hence if $\Gamma'$ is a minimal chart of type $(m; 2, 3, 2)$, then $\Gamma_{m+2}' \not\supset G$. □
Lemma 7.2 Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. Then neither $\Gamma_{m+1}$ nor $\Gamma_{m+2}$ contains the graph as shown in Fig. 4(a).

Proof. Suppose that $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(a). We use the notations as shown in Fig. 16(a) where $w_3, w_4, w_5$ are BW-vertices. By Corollary 4.3, the graph $\Gamma_m$ contains an oval $G$. Thus by Lemma 6.3(a), there are three cases: (i) $w_1, w_2 \in G$, (ii) $w_3, w_4 \in G$ or $w_4, w_5 \in G$ (see Fig. 16(b)), (iii) $w_3, w_5 \in G$ (see Fig. 16(c)).

Case (i). Since there exist two internal edges of label $m + 1$ connecting $w_1$ and $w_2$, there exists a 2-angled disk $D$ of $\Gamma_{m+1}$ with $w_1, w_2 \in \partial D$. Thus $w_1, w_2 \in G \cap \partial D$. Hence by Lemma 5.2 there exists a lens of $\Gamma$. This contradicts Lemma 4.1. Hence Case (i) does not occur.

Case (ii). By Lemma 6.2, the two internal edges $e_1, e_2$ of label $m$ in $G$ are contained in the closure of the same connected component $F$ of $S^2 - \Gamma_{m+1}$. Thus the curve $e_1 \cup e_2$ bounds a 2-angled disk of $\Gamma_m$ in $Cl(F)$, say $D$. Hence $Cl(S^2 - D)$ is also a 2-angled disk of $\Gamma_m$, and by Lemma 5.1 the disk $Cl(S^2 - D)$ contains a lens (see Fig. 16(b)). This contradicts Lemma 4.1. Thus Case (ii) does not occur.

Case (iii). Let $e_1, e_2$ be the two internal edges of label $m$ in $G$. Let $e_3, e_5$ be the internal edges of label $m + 1$ at $w_3, w_5$ containing $w_4$ respectively (see Fig. 16(c)).

By Lemma 6.2 the two edges $e_1, e_2$ are contained in the closure of the same connected component $F$ of $S^2 - \Gamma_{m+1}$ (see Fig. 16(c)). Without loss of generality we can assume that the terminal edge of label $m$ at $w_3$ is oriented inward at $w_3$. By Assumption 2 the terminal edge is middle at $w_3$. Thus

(1) the edge $e_3$ is oriented inward at $w_3$,

and the two edges $e_1, e_2$ are oriented from $w_3$ to $w_5$. Hence the edge $e_5$ is oriented outward at $w_5$ (see Fig 16(d)). Thus $e_5$ is oriented inward at the BW-vertex $w_4$. However by (1) the edge $e_3$ is oriented outward at the BW-vertex $w_4$. This contradicts Lemma 3.1. Hence Case (iii) does not occur.

Therefore $\Gamma_{m+1}$ does not contain the graph as shown in Fig. 4(a). By Lemma 7.1 we can show that $\Gamma_{m+2}$ does not contain the graph as shown in Fig. 4(a). \[\boxtimes\]

Figure 16: (a) $w_1, w_2, \ldots, w_5$ are white vertices. (b) $w_3, w_4 \in G$. (c),(d) $w_3, w_5 \in G$. 18
Lemma 7.3 Let \( \Gamma \) be a minimal chart of type \((m; 2, 3, 2)\). Then neither \( \Gamma_{m+1} \) nor \( \Gamma_{m+2} \) contains the graph as shown in Fig. 4(d).

Proof. Suppose that \( \Gamma_{m+1} \) contains the graph as shown in Fig. 4(d). We use the notations as shown in Fig. 17(a) where \( w_1 \) is a BW-vertex. By Corollary 4.3, the graph \( \Gamma_m \) contains an oval \( G \). Thus by Lemma 6.3(b), there are four cases: (i) \( w_2, w_3 \in G \) (see Fig. 17(b)), (ii) \( w_2, w_4 \in G \) or \( w_3, w_5 \in G \) (see Fig. 17(c)), (iii) \( w_2, w_5 \in G \) or \( w_3, w_4 \in G \) (see Fig. 17(d)).

Case (i). By Lemma 5.3(a), the chart \( \Gamma \) is not minimal. This is a contradiction. Hence Case (i) does not occur.

Case (ii). By Lemma 5.2 there exists a lens of \( \Gamma \). This contradicts Lemma 4.1. Hence Case (ii) does not occur.

Case (iii). By Lemma 5.3(b), the chart \( \Gamma \) is not minimal. This is a contradiction. Hence Case (iii) does not occur.

Case (iv). By Lemma 5.1 there exists a lens of \( \Gamma \). This contradicts Lemma 4.1. Hence Case (iv) does not occur.

Therefore \( \Gamma_{m+1} \) does not contain the graph as shown in Fig. 4(d). By Lemma 7.1 we can show that \( \Gamma_{m+2} \) does not contain the graph as shown in Fig. 4(d).

\[
\begin{align*}
\text{(a) } w_2 & \quad w_3 \\
\text{ } w_1 & \quad \text{ } w_4 & \quad \text{ } w_5 \\
m+1 & \quad m+1
\end{align*}
\]

Figure 17: (a) \( w_1, w_2, \ldots, w_5 \) are white vertices. (b) \( w_2, w_3 \in G \). (c) \( w_2, w_5 \in G \). (d) \( w_4, w_5 \in G \).

Lemma 7.4 Let \( \Gamma \) be a minimal chart of type \((m; 2, 3, 2)\). Then neither \( \Gamma_{m+1} \) nor \( \Gamma_{m+2} \) contains the graph as shown in Fig. 4(e).

Proof. Suppose that \( \Gamma_{m+1} \) contains the graph as shown in Fig. 18(e). We use the notations as shown in Fig. 18(a) where \( w_1 \) is a BW-vertex. By Corollary 4.3, the graph \( \Gamma_m \) contains an oval \( G \). Thus by Lemma 6.3(b), there are four cases: (i) \( w_2, w_3 \in G \) (see Fig. 18(b), (c), (d)), (ii) \( w_2, w_4 \in G \) or \( w_3, w_5 \in G \) (see Fig. 18(c)), (iii) \( w_2, w_5 \in G \) or \( w_3, w_4 \in G \) (see Fig. 18(f)), (iv) \( w_4, w_5 \in G \).

Case (i). Around the white vertex \( w_2 \), there are three internal edges of label \( m+1 \). Let \( e_1, e_2, e_3 \) be the three internal edges of label \( m+1 \) at \( w_2 \) containing \( w_1, w_3, w_4 \), respectively (see Fig. 18(a)). Let \( D \) be the 2-angled
disk of $\Gamma_m$ not containing the terminal edge of label $m$ at $w_2$. Then there are three cases: $e_1 \subset D$ or $e_2 \subset D$ or $e_3 \subset D$.

If $e_1 \subset D$ (see Fig. 18(b)), then by Lemma 5.3(a) the chart $\Gamma$ is not minimal. This is a contradiction. If $e_2 \subset D$ (see Fig. 18(c)), then by Lemma 5.1 there exists a lens of $\Gamma$. This contradicts Lemma 4.1. If $e_3 \subset D$ (see Fig. 18(d)), then by Lemma 5.1 there exists a lens in $\text{Cl}(S^2 - D)$. This contradicts Lemma 4.1. Hence Case (i) does not occur.

**Case (ii).** By Lemma 5.1 there exists a lens of $\Gamma$. This contradicts Lemma 4.1. Hence Case (ii) does not occur.

**Case (iii).** By Lemma 5.3(b), the chart $\Gamma$ is not minimal. This is a contradiction. Hence Case (iii) does not occur.

**Case (iv).** By Lemma 5.2, there exists a lens of $\Gamma$. This contradicts Lemma 4.1. Hence Case (iv) does not occur.

Therefore $\Gamma_{m+1}$ does not contain the graph as shown in Fig. 4(e). By Lemma 7.1, we can show that $\Gamma_{m+2}$ does not contain the graph as shown in Fig. 4(e).

\[\square\]

**Figure 18:** (a) $w_1, w_2, \cdots, w_5$ are white vertices. (b), (c), (d) $w_2, w_3 \in G$, the gray regions are 2-angled disks of $\Gamma_m$ not containing the terminal edges of $G$. (e) $w_2, w_4 \in G$. (f) $w_2, w_5 \in G$.

**Lemma 7.5** Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. Then neither $\Gamma_{m+1}$ nor $\Gamma_{m+2}$ contains the graph as shown in Fig. 4(f).

**Proof.** Suppose that $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(f). We use the notations as shown in Fig. 19(a) where $w_1$ is a BW-vertex. By Corollary 4.3, the graph $\Gamma_m$ contains an oval $G$. There are two cases: (i) $w_4 \in G$ or $w_5 \in G$, (ii) $w_4 \notin G$ and $w_5 \notin G$.

**Case (i).** If $w_4 \in G$, then by Lemma 6.3(b) the oval $G$ contains one of $w_2, w_3, w_5$. Thus there exists an internal edge of label $m + 1$ connecting the
two white vertices of $G$ (see Fig. 19(b)). Hence by Lemma 5.1 there exists a lens of $\Gamma$. This contradicts Lemma 4.1.

If $w_5 \in G$, then we have the same contradiction. Hence Case (i) does not occur.

**Case (ii).** By Lemma 6.3(b), we have $w_2, w_3 \in G$ (see Fig. 19(c)). By Lemma 5.3(a), the chart $\Gamma$ is not minimal. This is a contradiction. Hence Case (ii) does not occur.

Therefore $\Gamma_{m+1}$ does not contain the graph as shown in Fig. 4(f). By Lemma 7.1 we can show that $\Gamma_{m+2}$ does not contain the graph as shown in Fig. 4(f).

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**Lemma 7.6** Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. Then neither $\Gamma_{m+1}$ nor $\Gamma_{m+2}$ contains the graph as shown in Fig. 4(i).

**Proof.** Suppose that $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(i). We use the notations as shown in Fig. 20(a) where $w_1$ is a BW-vertex. By Corollary 4.3 the graph $\Gamma_m$ contains an oval $G$. There are four cases: (i) $G$ contains the BW-vertex $w_1$ (see Fig. 20(b), (c)), (ii) $w_2, w_3 \in G$ (see Fig. 20(d)), (iii) $w_3, w_4 \in G$ or $w_3, w_5 \in G$ (see Fig. 20(e)), (iv) $w_4, w_5 \in G$.

**Case (i).** Let $e_1, e_2$ be the internal edges in $G$, and $F_1, F_2$ the connected components of $S^2 - \Gamma_{m+1}$ with $w_1 \in Cl(F_1) \cap Cl(F_2)$ and $w_3 \notin Cl(F_1)$ (see Fig. 20(a)).

We shall show that $e_1 \cup e_2 \subset Cl(F_2)$. By Lemma 6.2 we have $e_1 \cup e_2 \subset Cl(F_1)$ or $e_1 \cup e_2 \subset Cl(F_2)$. If $e_1 \cup e_2 \subset Cl(F_1)$, then in $Cl(F_1)$ there exist two edges of label $m$ at $w_2$. However, since $w_2$ is a white vertex as shown in Fig. 20(a), there exists at most one edge of label $m$ at $w_2$ in $Cl(F_1)$. This is a contradiction. Hence $e_1 \cup e_2 \subset Cl(F_2)$.

Thus there are two cases: $w_1, w_2 \in G$ or $w_1, w_3 \in G$. If $w_1, w_2 \in G$ (see Fig. 20(b)), then by Lemma 5.1 there exists a lens of $\Gamma$. This contradicts Lemma 4.1. If $w_1, w_3 \in G$ (see Fig. 20(c)), then by Lemma 5.3(b) the chart $\Gamma$ is not minimal. This is a contradiction. Hence Case (i) does not occur.

**Case (ii) and Case (iii).** By Lemma 5.1 there exists a lens of $\Gamma$. This contradicts Lemma 4.1. Hence Case (ii) and Case (iii) do not occur.
Case (iv). By Lemma 5.2 there exists a lens of $\Gamma$. This contradicts Lemma 4.1. Hence Case (iv) does not occur.

Therefore $\Gamma_{m+1}$ does not contain the graph as shown in Fig. 4(i). By Lemma 7.1 we can show that $\Gamma_{m+2}$ does not contain the graph as shown in Fig. 4(i).

By using Lemma 1.2 and lemmata in this section, we obtain the following corollary:

**Corollary 7.8** If there exists a minimal chart $\Gamma$ of type $(m; 2, 3, 2)$, then each of $\Gamma_{m+1}$ and $\Gamma_{m+2}$ contains either the union of an oval and a skew $\theta$-curve, or one of four graphs as shown in Fig. 4(b),(c),(g),(h).

**8 RO-families of pseudo charts**

In this section, we consider a minimal chart $\Gamma$ of type $(m; 2, 3, 2)$ such that $\Gamma_{m+1}$ contains either an oval, or one of the four graphs as shown in Fig. 4(b),(c),(g),(h). We investigate that the chart $\Gamma$ contains what kind of pseudo charts.
Let $\Gamma$ be a chart, $D$ a disk, and $G$ a pseudo chart with $G \subset D$. Let $r : D \to D$ be a reflection of $D$, and $G^*$ the pseudo chart obtained from $G$ by changing the orientations of all of the edges. Then the set $\{G, G^*, r(G), r(G^*)\}$ is called the **RO-family of the pseudo chart** $G$.

In our argument, we often need a name for an unnamed edge by using a given edge and a given white vertex. For the convenience, we use the following naming: Let $e', e_i, e''$ be three consecutive edges containing a white vertex $w_j$. Here, the two edges $e'$ and $e''$ are unnamed edges. There are six arcs in a neighborhood $U$ of the white vertex $w_j$. If the three arcs $e' \cap U, e_i \cap U, e'' \cap U$ lie anticlockwise around the white vertex $w_j$ in this order, then $e'$ and $e''$ are denoted by $a_{ij}$ and $b_{ij}$ respectively (see Fig. 21). There is a possibility $a_{ij} = b_{ij}$ if they are contained in a loop.

![Figure 21](image)

**Figure 21:** The three edges $a_{ij}, e_i, b_{ij}$ are consecutive edges around the white vertex $w_j$.

**Lemma 8.1** Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. If $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(c), then $\Gamma$ contains one of the RO-family of the pseudo chart as shown in Fig. 22(a).

![Figure 22](image)

**Figure 22:** (a) A pseudo chart containing the graph as shown in Fig. 4(c). The light gray region is $D_1$, and the dark gray region is $D_2$. (b) $w_1, w_2, \ldots, w_5$ are white vertices. (c) $w_4, w_5 \in G$.

**Proof.** Suppose that $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(c). We use the notations as shown in Fig. 22(b) where $w_1, w_2, w_3$ are BW-vertices.
By Corollary 4.3, the graph $\Gamma_m$ contains an oval $G$. Thus by Lemma 6.3(a), there are two cases: (i) $w_4, w_5 \in G$ (see Fig. 22(c)), (ii) the oval $G$ contains two of BW-vertices $w_1, w_2, w_3$.

**Case (i).** By Lemma 5.3(a), the chart $\Gamma$ is not minimal. This is a contradiction. Hence Case (i) does not occur.

**Case (ii).** Without loss of generality we can assume $w_1, w_2 \in G$. Let $e_3$ be the terminal edge of label $m + 1$ at $w_3$. Let $e'_1, e''_1$ be the internal edges of label $m + 1$ at $w_1$ containing $w_4, w_5$ respectively (see Fig. 22(b)).

Now, the graph in $\Gamma_{m+1}$ as shown in Fig. 4(c) separates $S^2$ into three disks. One of the three disks contains internal edges of label $m$ in $G$, say $D_1$. One of the three disks contains the terminal edge $e_3$, say $D_2$. The last one is denoted by $D_3$. By Assumption 5, we can assume that the disk $D_3$ contains the point at infinity $\infty$.

If necessary we change the orientation of all the edges of $\Gamma$, we can assume that the terminal edge of label $m$ at $w_1$ is oriented inward at $w_1$. Then by Assumption 2

(1) the two edges $e'_1, e''_1$ are oriented inward at $w_1$.

Since $w_1, w_2 \in \Gamma_m \cap \Gamma_{m+1}$, $w_3, w_4, w_5 \in \Gamma_{m+1}$ and since $\Gamma$ is of type $(m; 2, 3, 2)$, we have $w_3, w_4, w_5 \in \Gamma_{m+1} \cap \Gamma_{m+2}$. Let $e_4, e_5$ be the internal edges (possibly terminal edges) of label $m + 2$ at $w_4, w_5$ in $D_2$ respectively. Then by (1), the two edges $e_4, e_5$ are oriented inward at $w_4, w_5$ respectively. Thus $\Gamma$ contains the pseudo chart as shown in Fig. 22(a).

**Lemma 8.2** Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. If $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(b), then $\Gamma$ contains one of the RO-families of the two pseudo charts as shown in Fig. 22.

![Figure 23: Pseudo charts containing the graph as shown in Fig. 4(b). The gray regions are the disk $D'_1$.](image)

**Proof.** Suppose that $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(b). We use the notations as shown in Fig. 24(a) where $w_1, w_2, w_3$ are BW-vertices.
By Corollary 4.3, the graph $\Gamma_m$ contains an oval $G$. Thus by Lemma 6.3(a), there are two cases: (i) $w_4, w_5 \in G$ (see Fig. 24(b), (c), (d)), (ii) $w_1, w_2 \in G$ or $w_1, w_3 \in G$. Hence Case (ii) does not occur.

**Case (i).** Let $e', e'', e'''$ be the internal edges of label $m + 1$ at $w_4$ containing $w_1, w_2, w_3$, respectively (see Fig. 24(a)). Let $D$ be the 2-angled disk of $\Gamma_m$ not containing the terminal edge of label $m$ at $w_4$. There are three cases: $e' \subset D$ or $e'' \subset D$ or $e''' \subset D$.

If $e' \subset D$ (see Fig. 24(b)), then by Lemma 5.3(a) the chart $\Gamma$ is not minimal. This is a contradiction. If $e'' \subset D$ (see Fig. 24(c)), then by Lemma 5.1 there exists a lens in $\text{Cl}(S^2 - D)$. This contradicts Lemma 4.1. If $e''' \subset D$ (see Fig. 24(d)), then by Lemma 5.1 there exists a lens in $D$. This contradicts Lemma 4.1. Hence Case (i) does not occur.

**Case (ii).** Without loss of generality we can assume that $w_1, w_2 \in G$. Let $e_1, e_2$ be the internal edges of label $m$ in $G$.

Now, the graph in $\Gamma_{m+1}$ as shown in Fig. 4(b) separates $S^2$ into three disks. One of the three disks contains both of $e_1$ and $e_2$.

Moreover, since $w_1, w_2 \in \Gamma_m \cap \Gamma_{m+1}$, $w_3, w_4, w_5 \in \Gamma_{m+1}$ and since $\Gamma$ is of type $(m; 2, 3, 2)$, we have $w_3, w_4, w_5 \in \Gamma_{m+1} \cap \Gamma_{m+2}$. Hence the chart $\Gamma$ contains the pseudo chart as shown in Fig. 24(e). We use the notations as shown in Fig. 24(e), where $e'_1, e''_1$ are internal edges of label $m + 1$ at $w_1$, $e'_2, e''_2$ are internal edges of label $m + 1$ at $w_2$ containing $w_3, w_4$ respectively, $e'_3$ is an internal edge of label $m + 1$ connecting $w_3, w_5$.

Without loss of generality we can assume that the terminal edge of label $m$ at $w_1$ is oriented inward at $w_1$. Thus by Assumption 2 the two edges $e'_1, e''_1$ are oriented inward at $w_1$, and the two edges $e_1$ and $e_2$ are oriented from $w_1$ to $w_2$. Hence the two edges $e'_2, e''_2$ are oriented outward at $w_2$. Thus the edge $e'_2$ is oriented from $w_2$ to the BW-vertex $w_3$. Hence by Lemma 3.1 the edge $e'_3$ is oriented from $w_3$ to $w_3$. Moreover, we have the orientation of other edges. Thus $\Gamma$ contains one of the two pseudo charts as shown in Fig. 24.

**Case (iii).** By Lemma 6.2 the two internal edges $e_1, e_2$ of label $m$ in $G$ are contained in the closure of the same connected component $F$ of $S^2 - \Gamma_{m+1}$ (see Fig. 24(f)). Thus the curve $e_1 \cup e_2$ bounds a 2-angled disk of $\Gamma_m$ in $\text{Cl}(F)$, say $D$. Hence $\text{Cl}(S^2 - D)$ is also a 2-angled disk of $\Gamma_m$, and by Lemma 5.1 the disk $\text{Cl}(S^2 - D)$ contains a lens. This contradicts Lemma 4.1. Hence Case (iii) does not occur.

Therefore $\Gamma$ contains one of the RO-families of the two pseudo charts as shown in Fig. 24. □

**Lemma 8.3** Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. If $\Gamma_{m+1}$ contain an oval, then $\Gamma$ contains one of the RO-families of the two pseudo charts as shown in Fig. 25(a) and (b).

**Proof.** By Corollary 4.3, the graph $\Gamma_m$ contains an oval $G$. Since $\Gamma_{m+1}$ contain an oval, by Lemma 1.2 the graph $\Gamma_{m+1}$ contains a skew $\theta$-curve. Let
$w_1, w_2, w_3$ be the white vertices in the skew $\theta$-curve such that $w_1$ is a BW-vertex with respect to $\Gamma_{m+1}$. Let $w_4, w_5$ be the white vertices in the oval of label $m+1$. Then $w_4, w_5$ are BW-vertices with respect to $\Gamma_{m+1}$. Hence by Lemma 6.3(d), there are three cases: (i) $w_1, w_4 \in G$ or $w_1, w_5 \in G$, (ii) $w_2, w_3 \in G$, (iii) $w_4, w_5 \in G$.

**Case (ii) and Case (iii).** By Lemma 5.2 there exists a lens of $\Gamma$. This contradicts Lemma 4.1. Hence Case (ii) and Case (iii) do not occur.

**Case (i).** Without loss of generality we can assume $w_1, w_4 \in G$. Since $w_1, w_4 \in \Gamma_m \cap \Gamma_{m+1}$, $w_2, w_3 \in \Gamma_{m+1}$ and since $\Gamma$ is of type $(m; 2, 3, 2)$, we have $w_2, w_3 \in \Gamma_{m+1} \cap \Gamma_{m+2}$ (see Fig. 25(c)). We use the notations as shown in Fig. 25(c), where $e$ is the terminal edge of label $m+1$ at $w_1$, $e'_1, e''_1$ are internal edges of label $m+1$ at $w_1$, and $e'_2, e''_2$ are internal edges of label $m+1$ connecting $w_2$ and $w_3$.

Without loss of generality, we can assume that

1. the terminal edge $e$ is oriented outward at $w_1$.

Since the terminal edge $e$ is middle at $w_1$ by Assumption 2, the two edges $e'_1, e''_1$ are oriented inward at $w_1$. If necessary we reflect the chart $\Gamma$, we can assume that the edge $e'_2$ is oriented from $w_2$ to $w_3$. Looking at edges around $w_2$, the edge $e''_2$ is oriented from $w_3$ to $w_2$. Hence we have the orientation of the other edges of label $m+2$. Let $e_1, e_2$ be the internal edges of label $m$ in $G$. Then by (1) the edges $e_1, e_2$ is oriented from $w_1$ to $w_4$. Hence $\Gamma$ contains the pseudo chart as shown in Fig. 25(d).

Let $D$ be the 2-angled disk of $\Gamma_m$ with $\partial D \ni w_1, w_4$ and $D \not\ni w_2$ (see Fig. 25(d)). Let $e_4$ be the terminal edge of label $m$ at $w_4$. There are two cases: $e_4 \not\subset D$ or $e_4 \subset D$. If $e_4 \not\subset D$, then the chart $\Gamma$ contains the pseudo chart as shown in Fig. 25(a). If $e_4 \subset D$, then the chart $\Gamma$ contains the pseudo chart as shown in Fig. 25(b). Therefore $\Gamma$ contains one of the RO-families of the two pseudo charts as shown in Fig. 25(a),(b). □
Figure 25: (a),(b) Pseudo charts containing a skew $\theta$-curve and an oval of label $m + 1$. (c) The skew $\theta$-curve of label $m + 1$. (d) The gray region is the disk $D$.

**Lemma 8.4** Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. If $\Gamma_{m+1}$ contains the graph as shown in Fig. 26(g), then $\Gamma$ contains one of the RO-families of the two pseudo charts as shown in Fig. 26(a),(b).

**Proof.** Suppose that $\Gamma_{m+1}$ contains the graph as shown in Fig. 26(g). We use the notations as shown in Fig. 26(c) where $w_1, w_4, w_5$ are BW-vertices. By Corollary 4.3, the graph $\Gamma_m$ contains an oval $G$. There are seven cases: (i) $w_1, w_2 \in G$, (ii) $w_1, w_3 \in G$ (see Fig. 26(d)), (iii) $w_1, w_4 \in G$ or $w_1, w_5 \in G$ (see Fig. 26(a)), (iv) $w_2, w_3 \in G$ (see Fig. 26(e)), (v) $w_2, w_4 \in G$ or $w_2, w_5 \in G$ (see Fig. 26(b)), (vi) $w_3, w_4 \in G$ or $w_3, w_5 \in G$ (see Fig. 26(f)), (vii) $w_4, w_5 \in G$ (see Fig. 26(g)).

**Case (i).** By Lemma 5.2, there exists a lens of $\Gamma$. This contradicts Lemma 4.1. Hence Case (i) does not occur.

**Case (ii).** By Lemma 5.3(b), the chart $\Gamma$ is not minimal. This is a contradiction. Thus Case (ii) does not occur.

**Case (iii).** We use the notations as shown in Fig. 26(c) where $e_1', e_1''$ are two internal edges at $w_1$, $e_2$ is the internal edge connecting $w_2$ and $w_3$, $e_4', e_4''$ are two internal edges at $w_4$, and $e_5'$ is the internal edge connecting $w_3$ and $w_5$.

If necessary we reflect the chart, we can assume that $w_1, w_4 \in G$. If necessary we change the orientation of all the edges, we can assume that the two edges $e_1', e_1''$ are oriented from $w_1$ to $w_2$. Then the edge $e_2$ is oriented from $w_2$ to $w_3$, and the two internal edges of label $m$ in $G$ are oriented from $w_4$ to $w_1$. Thus the two edges $e_4', e_4''$ are oriented inward at $w_4$. Hence the edge $e_5'$ is oriented from $w_3$ to $w_4$. Since $w_5$ is a BW-vertex with respect to $\Gamma_{m+1}$, by Lemma 3.1 the edge $e_5'$ is oriented from $w_5$ to $w_3$. Moreover we
Figure 26: (a),(b) Pseudo charts containing the graph as shown in Fig. 4(g). (c) $w_1, w_2, \ldots, w_5$ are white vertices. (d) $w_1, w_3 \in G$. (e) $w_2, w_3 \in G$. (f) $w_3, w_4 \in G$. (g) $w_4, w_5 \in G$.

have the orientation of the other edges.

Since $w_1, w_4 \in \Gamma_m \cap \Gamma_{m+1}$, $w_2, w_3 \in \Gamma_{m+1}$ and since $\Gamma$ is of type $(m; 2, 3, 2)$, we have $w_2, w_3 \in \Gamma_{m+1} \cap \Gamma_{m+2}$. Therefore $\Gamma$ contains the pseudo chart as shown in Fig. 26(a).

Case (iv). By Lemma 5.1 there exists a lens. This contradicts Lemma 4.1. Thus Case (iv) does not occur.

Case (v). If necessary we reflect the chart, we can assume that $w_2, w_4 \in G$. If necessary we change the orientation of all the edges, we can assume that the two internal edges $e'_1, e''_1$ of label $m+1$ at $w_1$ are oriented from $w_1$ to $w_2$. By a similar way as Case (iii), we can show that $\Gamma$ contains the pseudo chart as shown in Fig. 26(b).

Case (vi). By Lemma 6.2 the two internal edges $e_1, e_2$ of label $m$ in $G$ are contained in the closure of the same connected component $F$ of $S^2 - \Gamma_{m+1}$. Thus the curve $e_1 \cup e_2$ bounds a 2-angled disk $D$ of $\Gamma_m$ with $w_5 \in D$ (see Fig. 26(f)), and the disk $D$ contains a lens by Lemma 5.1. This contradicts Lemma 4.1. Thus Case (vi) does not occur.

Case (vii). By the similar way of Case (vi), there exists a lens. This contradicts Lemma 4.1. Thus Case (vii) does not occur.

Therefore $\Gamma$ contains one of the RO-families of the two pseudo charts as shown in Fig. 26(a),(b).

\[\square\]

Lemma 8.5 Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. If $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(h), then $\Gamma$ contains one of the RO-families of
the two pseudo charts as shown in Fig. 27(a), (b).

Figure 27: (a),(b) Pseudo charts containing the graph as shown in Fig. 4(h). (c) \(w_1, w_2, \ldots, w_5\) are white vertices. (d) \(w_1, w_5 \in G\). (e) \(w_2, w_3 \in G\). (f) \(w_2, w_4 \in G\).

Proof. Suppose that \(\Gamma_{m+1}\) contains the graph as shown in Fig. 4(h). We use the notations as shown in Fig. 27(c) where \(w_1, w_3, w_5\) are BW-vertices, \(e'_1, e''_1\) are two internal edges at \(w_1\), \(e'_2\) is the internal edge connecting \(w_2\) and \(w_3\), \(e'_4\) is the internal edge connecting \(w_3\) and \(w_4\), \(e'_5, e''_5\) are two internal edges at \(w_5\).

By Corollary 4.3 the graph \(\Gamma_m\) contains an oval \(G\). There are six cases: (i) \(w_1, w_2 \in G\) or \(w_5, w_4 \in G\), (ii) \(w_1, w_3 \in G\) or \(w_5, w_3 \in G\) (see Fig. 27(a)), (iii) \(w_1, w_4 \in G\) or \(w_5, w_2 \in G\) (see Fig. 27(b)), (iv) \(w_1, w_5 \in G\) (see Fig. 27(d)), (v) \(w_2, w_3 \in G\) or \(w_4, w_3 \in G\) (see Fig. 27(e)), (vi) \(w_2, w_4 \in G\) (see Fig. 27(f)).

Case (i). By Lemma 5.2 there exists a lens of \(\Gamma\). This contradicts Lemma 4.1. Hence Case (i) does not occur.

Case (ii). If necessary we reflect the chart, we can assume that \(w_1, w_3 \in G\). By Assumption 2 a neighborhood of \(w_3\) contains the pseudo chart as shown in Fig. 14.

If necessary we change the orientation of all the edges, we can assume that the two edges \(e'_1, e''_1\) are oriented from \(w_1\) to \(w_2\). Then the two internal edges of label \(m\) in \(G\) are oriented from \(w_3\) to \(w_1\), and the two edges \(e'_2, e'_4\) are oriented inward at \(w_3\). Thus the edge \(e'_4\) is oriented outward at \(w_1\). Hence by Lemma 3.1 the two edges \(e'_5, e''_5\) are oriented from \(w_5\) to \(w_4\). Thus we have the orientation of the other edges.

Since \(w_1, w_3 \in \Gamma_m \cap \Gamma_{m+1}\), \(w_2, w_4 \in \Gamma_{m+1}\) and since \(\Gamma\) is of type \((m; 2, 3, 2)\), we have \(w_2, w_4 \in \Gamma_{m+1} \cap \Gamma_{m+2}\). Therefore \(\Gamma\) contains the pseudo chart as shown in Fig. 27(a).
Case (iii). If necessary we reflect the chart, we can assume that \( w_1, w_4 \in G \). If necessary we change the orientation of all the edges, we can assume that the two edges \( e'_1, e''_1 \) are oriented from \( w_1 \) to \( w_2 \). By a similar way as Case (ii), we can show that \( \Gamma \) contains the pseudo chart as shown in Fig. 27(b).

Case (iv). If necessary we change the orientation of all the edges, we can assume that the two edges \( e'_1, e''_1 \) are oriented from \( w_1 \) to \( w_2 \). Then

(1) the edge \( e'_2 \) is oriented from \( w_2 \) to \( w_3 \) (i.e. the edge \( e'_2 \) is oriented inward at \( w_3 \)), and

the two internal edges of label \( m \) in \( G \) are oriented from \( w_5 \) to \( w_1 \). Thus the two edges \( e'_5, e''_5 \) are oriented from \( w_4 \) to \( w_5 \). Hence the edge \( e'_4 \) is oriented from \( w_3 \) to \( w_4 \). Thus the edge \( e'_4 \) is oriented outward at \( w_3 \) (see Fig. 27(d)). However by (1) and Lemma 3.1 we have a contradiction, because \( w_3 \) is a BW-vertex with respect to \( \Gamma_{m+1} \). Thus Case (iv) does not occur.

Case (v). Without loss of generality we can assume \( w_2, w_3 \in G \). If necessary we reflect the chart, by Assumption 2 (see Fig. 14) the chart \( \Gamma \) contains the pseudo chart as shown in Fig. 27(e). Hence by Lemma 5.1 there exists a lens. This contradicts Lemma 4.1. Thus Case (v) does not occur.

Case (vi). By Lemma 5.3(a), the chart \( \Gamma \) is not minimal. This is a contradiction. Thus Case (vi) does not occur.

Therefore \( \Gamma \) contains one of the RO-families of the two pseudo charts as shown in Fig. 27(a),(b).

\[ \square \]

9 IO-Calculation

In this section, we shall show that neither \( \Gamma_{m+1} \) nor \( \Gamma_{m+2} \) contains the graph as shown in Fig. 4(e) for any minimal chart \( \Gamma \) of type \((m; 2, 3, 2)\).

Let \( \Gamma \) be a chart, and \( v \) a vertex. Let \( \alpha \) be a short arc of \( \Gamma \) in a small neighborhood of \( v \) such that \( v \) is an endpoint of \( \alpha \). If the arc \( \alpha \) is oriented to \( v \), then \( \alpha \) is called an inward arc, and otherwise \( \alpha \) is called an outward arc.

Let \( \Gamma \) be an \( n \)-chart. Let \( F \) be a closed domain with \( \partial F \subset \Gamma_{k-1} \cup \Gamma_k \cup \Gamma_{k+1} \) for some label \( k \) of \( \Gamma \), where \( \Gamma_0 = \emptyset \) and \( \Gamma_n = \emptyset \). By Condition (iii) for charts, in a small neighborhood of each white vertex, there are three inward arcs and three outward arcs. Also in a small neighborhood of each black vertex, there exists only one inward arc or one outward arc. We often use the following fact, when we fix (inward or outward) arcs near white vertices and black vertices:

(*) The number of inward arcs contained in \( F \cap \Gamma_k \) is equal to the number of outward arcs in \( F \cap \Gamma_k \).

When we use this fact, we say that we use IO-Calculation with respect to \( \Gamma_k \) in \( F \). For example, in a minimal chart \( \Gamma \), consider the pseudo chart as shown in Fig. 28 where

\[ \square \]
Figure 28: The gray region is the disk $F$.

(1) $F$ is a 4-angled disk of $\Gamma_{k-1}$,

(2) $v_1, v_2, v_3, v_4$ are white vertices in $\partial F$ with $v_1 \in \Gamma_{k-2} \cap \Gamma_{k-1}$ and $v_2, v_3, v_4 \in \Gamma_{k-1} \cap \Gamma_k$,

(3) $e_1$ is a terminal edge of label $k-2$ at $v_1$,

(4) $e_3$ is a terminal edge of label $k-1$ oriented inward at $v_3$,

(5) for $i = 2, 4$, the edge $e_i$ of label $k$ is oriented inward at $v_i$.

Then we can show that $w(\Gamma \cap \text{Int} F) \geq 1$. Suppose $w(\Gamma \cap \text{Int} F) = 0$. By Assumption 2 the terminal edge $e_3$ contains a middle arc. Thus

(6) neither of edges $a_{33}, b_{33}$ of label $k$ is middle at $v_3$ (by Assumption 2 neither of them is a terminal edge).

Hence by (4),

(7) both of edges $a_{33}, b_{33}$ of label $k$ are oriented inward at $v_3$.

If both of $e_2$ and $e_4$ are terminal edges of label $k$, then by (5), (6), (7) the number of inward arcs in $F \cap \Gamma_k$ is four, but the number of outward arcs in $F \cap \Gamma_k$ is two. This contradicts the fact (*). Similarly if one of $e_2$ and $e_4$ is not a terminal edge of label $k$, then we have the same contradiction. Thus $w(\Gamma \cap \text{Int} F) \geq 1$. Instead of the above argument, we just say that

$w(\Gamma \cap \text{Int} F) \geq 1$ by IO-Calculation with respect to $\Gamma_k$ in $F$.

Lemma 9.1 ([6, Lemma 5.4]) If a minimal chart $\Gamma$ contains the pseudo chart as shown in Fig. 29, then the interior of the disk $D^*$ contains at least one white vertex, where $D^*$ is the disk with the boundary $e_3^* \cup e_1^* \cup e^*$.

Lemma 9.2 Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. Then neither $\Gamma_{m+1}$ nor $\Gamma_{m+2}$ contains the graph as shown in Fig. 4(c).
Figure 29: The gray region is the disk $D^*$. The label of the edge $e^*$ is $m$, and $\varepsilon \in \{+1, -1\}$.

Proof. Suppose that $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(c). By Lemma 8.1 the chart $\Gamma$ contains one of the RO-family of the pseudo chart as shown in Fig. 22(a). We use the notations as shown in Fig. 22(a) where $e_1, e_2$ are the internal edges of label $m$ with $e_1 \cap e_2 \ni w_1, w_2$, $e_3$ is the terminal edge of label $m + 1$ at $w_3$, and

(1) the two internal edges $e_4, e_5$ (possibly terminal edges) of label $m + 2$ are oriented inward at $w_4, w_5$, respectively.

Now, the graph in $\Gamma_{m+1}$ as shown in Fig. 4(c) separates $S^2$ into three disks. One of them contains $e_1$, say $D_1$, and one of them contains the terminal edge $e_3$, say $D_2$.

Claim 1. $w(\Gamma \cap \text{Int}D_1) \geq 2$ and $w(\Gamma \cap \text{Int}D_2) = 0$.

Proof of Claim 1. The curve $e_1 \cup e_2$ separates the disk $D_1$ into three disks. One of them contains $w_4$, say $D'_1$, and one of them contains $w_5$, say $D''_1$. Apply Lemma 9.1 considering as $D^* = D'_1$ and $w^*_3 = w_4$, we have $w(\Gamma \cap \text{Int}D'_1) \geq 1$. Similarly we can show that $w(\Gamma \cap \text{Int}D''_1) \geq 1$. Since $D_1 \supset D'_1 \cup D''_1$ and $\text{Int}D'_1 \cap \text{Int}D''_1 = \emptyset$, we have $w(\Gamma \cap \text{Int}D_1) \geq 2$.

Since $\Gamma$ is of type $(m; 2, 3, 2)$, we have $w(\Gamma) = 7$ and $w(\Gamma_{m+1}) = 5$. Thus

$7 = w(\Gamma) \geq w(\Gamma_{m+1}) + w(\Gamma \cap \text{Int}D_1) + w(\Gamma \cap \text{Int}D_2) \geq 5 + 2 + w(\Gamma \cap \text{Int}D_2)$.

Hence $w(\Gamma \cap \text{Int}D_2) = 0$. Thus Claim 1 holds.

Claim 2. The terminal edge $e_3$ is oriented outward at $w_3$.

Proof of Claim 2. Suppose that $e_3$ is oriented inward at $w_3$. Considering as $F = D_2$ and $k = m + 2$ in the example of IO-Calculation in Section 9, the condition (1) implies that we have $w(\Gamma \cap \text{Int}D_2) \geq 1$. This contradicts the second equation of Claim 1. Hence the terminal edge $e_3$ is oriented outward at $w_3$. Thus Claim 2 holds.

Let $a_{33}, b_{33}$ be the internal edges of label $m + 2$ at $w_3$ in $D_2$ such that $a_{33}, e_3, b_{33}$ lie anticlockwise around $w_3$ in this order (see Fig. 22(a)). By Assumption 2

(2) the terminal edge $e_3$ is middle at $w_3$.

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Claim 3. $a_{33} = e_5$ and $b_{33} = e_4$.

Proof of Claim 3. By (2) and Assumption 2, neither $a_{33}$ nor $b_{33}$ is a terminal edge. Moreover, by Claim 2, both of $a_{33}$ and $b_{33}$ are oriented outward at $w_3$. Thus by the second equation of Claim 1, we have $a_{33} = e_5$ and $b_{33} = e_4$. Hence Claim 3 holds.

Finally we shall show that there exists a lens of $\Gamma$. Let $e'_3$ be the internal edge of label $m + 1$ with $e'_3 \ni w_3, w_4$ (see Fig. 22(a)). By (2), neither $e'_3$ nor $b_{33}$ is middle at $w_3$. By (2) and Claim 2, the terminal edge $e_3$ is oriented outward at $w_3$ and middle at $w_3$. Hence the curve $e'_3 \cup b_{33}$ bounds a lens in $D_2$. This contradicts Lemma 4.1. Therefore $\Gamma_{m+1}$ does not contain the graph as shown in Fig. 4(c).

By Lemma 7.1, we can show that $\Gamma_{m+2}$ does not contain the graph as shown in Fig. 4(c). We complete the proof of Lemma 9.2.

\[\square\]

10 Shifting Lemma

In this section we shall show that neither $\Gamma_{m+1}$ nor $\Gamma_{m+2}$ contains the graph as shown in Fig. 4(b) for any minimal chart $\Gamma$ of type $(m; 2, 3, 2)$. Thus by Corollary 7.8 and Lemma 9.2, we obtain the main theorem.

Lemma 10.1 Let $\Gamma$ be a chart of type $(m; 2, 3, 2)$. If $\Gamma$ contains the pseudo chart as shown in Fig. 23(a), then $\Gamma$ is not minimal.

Proof. Suppose that $\Gamma$ is minimal. We use the notations as shown in Fig. 23(a). Here $e_1, e_2$ are internal edges of label $m$, and $e''_4, e''_5, a_{33}, b_{33}$ are internal edges (possibly terminal edges) of label $m + 2$ such that

1. $e''_4, e''_5$ are oriented inward at $w_4, w_5$, respectively,
2. $a_{33}, b_{33}$ are oriented outward at $w_3$.

Moreover none of $e''_4, e''_5, a_{33}, b_{33}$ are middle at $w_3, w_4$ or $w_5$. Thus by Assumption 2

3. none of $e''_4, e''_5, a_{33}, b_{33}$ are terminal edges.

Now, the graph $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(b). The graph in $\Gamma_{m+1}$ separates $S^2$ into three disks. One of them contains the edges $e_1$ and $e_2$ of label $m$, say $D_1$, and one of them contains the edge $e''_4$, say $D_2$. Moreover, the curve $e_1 \cup e_2$ separates the disk $D_1$ into three disks. One of them contains $w_4$, say $D'_1$. Apply Lemma 9.1 considering as $D^* = D'_1$ and $w_3^* = w_4$, we have
(4) \( w(\Gamma \cap \text{Int}D'_1) \geq 1 \).

There are three cases: (i) \( w(\Gamma \cap \text{Int}D_2) = 0 \), (ii) \( w(\Gamma \cap \text{Int}D_2) = 1 \), (iii) \( w(\Gamma \cap \text{Int}D_2) \geq 2 \).

**Case (i).** By using (1), (2) and (3), we have \( a_{33} = e''_4 \) and \( b_{33} = e''_5 \). Thus the curve \( b_{33} \cup e'_3 \) bounds a lens in \( D_2 \). This contradicts Lemma 4.1. Hence Case (i) does not occur.

**Case (ii).** Let \( v \) be the white vertex in \( \text{Int}D_2 \). Since the five white vertices \( w_1, w_2, \ldots, w_5 \) are in \( \Gamma_{m+1} \) and \( \Gamma \) is of type \( (m; 2, 3, 2) \), we have \( v \in \Gamma_{m+2} \cap \Gamma_{m+3} \). Thus by using (1),(2) and (3), we have a contradiction by IO-Calculation with respect to \( \Gamma_{m+2} \) in \( D_2 \). Hence Case (ii) does not occur.

**Case (iii).** Since \( \Gamma \) is of type \( (m; 2, 3, 2) \), we have \( w(\Gamma) = 7 \) and \( w(\Gamma_{m+1}) = 5 \). Thus by (4) and the condition \( w(\Gamma \cap \text{Int}D_2) \geq 2 \) of this case,

\[
7 = w(\Gamma) \geq w(\Gamma_{m+1}) + w(\Gamma \cap \text{Int}D'_1) + w(\Gamma \cap \text{Int}D_2) \geq 5 + 1 + 2 = 8.
\]

This is a contradiction. Hence Case (iii) does not occur.

Therefore the three cases do not occur. Hence \( \Gamma \) is not minimal. □

Let \( \Gamma \) and \( \Gamma' \) be C-move equivalent charts. Suppose that a pseudo chart \( X \) of \( \Gamma \) is also a pseudo chart of \( \Gamma' \). Then we say that \( \Gamma \) is modified to \( \Gamma' \) by C-moves keeping \( X \) fixed. In Fig. 30, we give examples of C-moves keeping pseudo charts fixed.

![Figure 30: C-moves keeping thicken figures fixed.](image)

Let \( \Gamma \) be a chart. Let \( \alpha \) be an arc in an edge of \( \Gamma_m \), and \( w \) a white vertex with \( w \notin \alpha \). Suppose that there exists an arc \( \beta \) in \( \Gamma \) such that its end points are the white vertex \( w \) and an interior point \( p \) of the arc \( \alpha \). Then we say that the white vertex \( w \) connects with the point \( p \) of the arc \( \alpha \) by the arc \( \beta \).

Let \( \alpha \) be a simple arc, and \( p, q \) points in \( \alpha \). We denote by \( \alpha[p,q] \) the subarc of \( \alpha \) whose endpoints are \( p \) and \( q \).

**Lemma 10.2 ([6, Lemma 4.2]) (Shifting Lemma)** Let \( \Gamma \) be a chart and \( \alpha \) an arc in an edge of \( \Gamma_m \). Let \( w \) be a white vertex of \( \Gamma_k \cap \Gamma_h \) where \( h = k + \varepsilon, \varepsilon \in \{+1, -1\} \). Suppose that the white vertex \( w \) connects with a point \( r \) of the arc \( \alpha \) by an arc in an edge \( e \) of \( \Gamma_k \). Suppose that one of the following two conditions is satisfied:

1. \( h > k > m \).
2. \( h < k < m \).
Then for any neighborhood $V$ of the arc $e[w,r]$ we can shift the white vertex $w$ to $e - e[w,r]$ along the edge $e$ by C-I-R2 moves, C-I-R3 moves and C-I-R4 moves in $V$ keeping $\bigcup_{i<0} \Gamma_{k+i\varepsilon}$ fixed (see Fig. 31).

Figure 31: $k > m$ and $\varepsilon = +1.$

**Proposition 10.3** Let $\Gamma$ be a minimal chart of type $(m; 2, 3, 2)$. Then neither $\Gamma_{m+1}$ nor $\Gamma_{m+2}$ contains the graph as shown in Fig. 4(b).

**Proof.** Suppose that $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(b). By Lemma 8.2 and Lemma 10.1, the chart $\Gamma$ contains one of the RO-family of the pseudo chart as shown in Fig. 23(b). We use the notations as shown in Fig. 23(b). Here, $e_1, e_2$ are internal edges of label $m$, and $e'_4, e''_4, e'_5, a_{33}, b_{33}$ are internal edges (possibly terminal edges) of label $m + 2$ such that

1. $e'_4, e''_4$ are oriented inward at $w_4, w_5$ respectively,

2. $e'_5, a_{33}, b_{33}$ are oriented outward at $w_5, w_3, w_3$ respectively.

Moreover, none of $e'_4, e''_4, a_{33}, b_{33}$ are middle at $w_3, w_4$ or $w_5$. Thus by Assumption 2

3. none of $e'_4, e''_4, a_{33}, b_{33}$ are terminal edges.

Now, the graph $\Gamma_{m+1}$ contains the graph as shown in Fig. 4(b). The graph in $\Gamma_{m+1}$ separates $S^2$ into three disks. One of them contains the edges $e_1$ and $e_2$, say $D_1$, and one of them contains the edge $e'_4$, say $D_2$. Moreover, the curve $e_1 \cup e_2$ separates the disk $D_1$ into three disks. One of them contains $w_4$, say $D'_1$. Apply Lemma 9.1 considering as $D^* = D'_1$ and $w_4^* = w_4$, we have

4. $w(\Gamma \cap \text{Int} D'_1) \geq 1.$

**Claim 1.** $w(\Gamma \cap \text{Int} D_1) \geq 1$ and $w(\Gamma \cap \text{Int} D_2) \geq 1.$

**Proof of Claim 1.** By (4) and $D'_1 \subset D_1$, we have $w(\Gamma \cap \text{Int} D_1) \geq 1.$

By (3), neither $e''_4$ nor $e'_5$ is a terminal edge. Since $e'_4, e''_4$ are oriented inward at $w_4, w_5$ respectively by (1), we have $w(\Gamma \cap \text{Int} D_2) \geq 1$ by IO-Calculation with respect to $\Gamma_{m+2}$ in $D_2.$  

□

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Claim 2. \(w(\Gamma \cap \text{Int}D_1) = 1\) and \(w(\Gamma \cap \text{Int}D'_1) = 1\).

Proof of Claim 2. Suppose that \(w(\Gamma \cap \text{Int}D_1) \geq 2\). Since \(\Gamma\) is of type \((m; 2, 3, 2)\), we have \(w(\Gamma) = 7\) and \(w(\Gamma_{m+1}) = 5\). Thus by the second inequality of Claim 1

\[
7 = w(\Gamma) \geq w(\Gamma_{m+1}) + w(\Gamma \cap \text{Int}D_1) + w(\Gamma \cap \text{Int}D_2) \geq 5 + 2 + 1 = 8.
\]

This is a contradiction. Thus \(w(\Gamma \cap \text{Int}D_1) \leq 1\). Hence by the first inequality of Claim 1, we have \(w(\Gamma \cap \text{Int}D_1) = 1\).

Thus by (4) and \(D'_1 \subset D_1\), we have \(w(\Gamma \cap \text{Int}D'_1) = 1\). Hence Claim 2 holds. \(\square\)

Let \(w_6\) be the white vertex in \(\text{Int}D'_1\). Since the five white vertices \(w_1, w_2, \ldots, w_5\) are in \(\Gamma_{m+1}\) and \(\Gamma\) is of type \((m; 2, 3, 2)\), we have

\[
(5) \ w_6 \in \Gamma_{m+2} \cap \Gamma_{m+3}.
\]

Claim 3. \(a_{33} \ni w_6\) or \(b_{33} \ni w_6\).

Proof of Claim 3. First we shall show that \(a_{33} = e'_4\) or \(a_{33} \ni w_6\). By (3), the edge \(a_{33}\) is not a terminal edge. Moreover, by (2), we have \(a_{33} \neq e'_5\) and \(a_{33} \neq b_{33}\). Thus by Claim 2, we have \(a_{33} = e'_4\) or \(a_{33} \ni w_6\).

Similarly we can show that \(b_{33} = e'_4\) or \(b_{33} \ni w_6\).

If \(a_{33} \neq w_6\) and \(b_{33} \neq w_6\), then \(a_{33} = e'_4\) and \(b_{33} = e'_4\). This is a contradiction. Therefore \(a_{33} \ni w_6\) or \(b_{33} \ni w_6\). Thus Claim 3 holds. \(\square\)

If \(a_{33} \ni w_6\), let \(e = a_{33}\), otherwise let \(e = b_{33}\).

Claim 4. We can move the white vertex \(w_6\) from the disk \(D'_1\) to the outside of \(D'_1\).

Proof of Claim 4. Since the edge \(e\) connects the vertex \(w_6\) in \(\text{Int}D'_1\) and the vertex \(w_3\) in the outside of \(D'_1\), the edge \(e\) intersects the boundary \(\partial D'_1\). Let \(x\) be the point in the edge \(e\) with \(e[w_6, x] \cap \partial D'_1 = x\). Since the edge \(e\) is of label \(m+2\) and since \(\partial D'_1\) consists of the edge \(e_1\) of label \(m\) and two internal edges of label \(m+1\), we have

\[
e[w_6, x] \cap e_1 = e[w_6, x] \cap \partial D'_1 = x.
\]

Thus by (5), the white vertex \(w_6 \in \Gamma_{m+2} \cap \Gamma_{m+3}\) connects with the point \(x\) in the edge \(e_1\) of label \(m\) by the arc \(e[w_6, x]\) of label \(m+2\). Hence by Shifting Lemma (Lemma 10.2), we can shift the white vertex \(w_6\) by C-I-R2 moves, C-I-R3 moves and C-I-R4 moves in a neighborhood of the arc \(e[w_6, x]\) keeping \(\bigcup_{\epsilon < 0} \Gamma_{m+2+i}\) fixed. Thus we can shift the white vertex \(w_6\) to the outside of \(D'_1\) by C-moves keeping \(\partial D'_1\) fixed. Therefore Claim 4 holds. \(\square\)

By Claim 2 and Claim 4, we have \(w(\Gamma \cap \text{Int}D'_1) = 0\). However we have a contradiction by Lemma 9.1 considering as \(D^* = D'_1\) and \(w_3^* = w_4\). Therefore \(\Gamma_{m+1}\) does not contain the graph as shown in Fig. 4(b).
By Lemma 7.1, we can show that $\Gamma_{m+2}$ does not contain the graph as shown in Fig. 4(b). We complete the proof of Proposition 10.3.

By Corollary 7.8, Lemma 9.2 and Proposition 10.3 we have the main theorem (Theorem 1.3).

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**List of terminologies**

| **k**-angled disk | **p11** | middle at \( v \) | **p5** |
|-------------------|---------|-----------------|-------|
| BW-vertex         | **p2**  | minimal chart   | **p5** |
| C-move equivalent | **p5**  | outward         | **p5** |
| chart             | **p4**  | outward arc     | **p30** |
| complexity        | **p5**  | oval            | **p2** |
| free edge         | **p5**  | point at infinity \( \infty \) | **p5** |
| hoop              | **p5**  | pseudo chart    | **p12** |
| internal edge     | **p9**  | ring            | **p5** |
| inward            | **p5**  | RO-family       | **p23** |
| inward arc        | **p30** | simple hoop     | **p5** |
| IO-Calculation    | **p30** | skew \( \theta \)-curve | **p2** |
| keeping X fixed   | **p34** | terminal edge   | **p2** |
| lens              | **p10** | type \((m; n_1, n_2, \ldots, n_k)\) | **p1** |
| loop              | **p7**  | \( w \) connects with \( p \) by an arc \( \beta \) | **p34** |
| middle arc        | **p4**  | \( \theta \)-curve | **p2** |

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List of notations

\( \Gamma_m \) \hspace{1cm} \text{p1}
\( w(X) \) \hspace{1cm} \text{p2}
\( \text{Int}X \) \hspace{1cm} \text{p6}
\( \partial X \) \hspace{1cm} \text{p6}
\( \text{Cl}(X) \) \hspace{1cm} \text{p6}
\( a_{ij}, b_{ij} \) \hspace{1cm} \text{p23}
\( \alpha[p, q] \) \hspace{1cm} \text{p34}