The intersection numbers of the $p$-spin curves from random matrix theory

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Abstract

The intersection numbers of $p$-spin curves are computed through correlation functions of Gaussian ensembles of random matrices in an external matrix source. The $p$-dependence of intersection numbers is determined as polynomial in $p$; the large $p$ behavior is also considered. The analytic continuation of intersection numbers to negative values of $p$ is discussed in relation to SL(2,R)/U(1) black hole sigma model.

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1 Introduction

The intersection numbers for $p$-spin curves appear in the generalized Kontsevich matrix model \([1, 2, 3]\). The generating function for $p$-spin intersection number obeys the $p$-th KdV equation or Gelfand-Dikii equation. In a random matrix theory, the correlation functions at the edges of the spectrum, where one can tune a degeneracy of order $p$, are expressed through intersection numbers \([4, 5, 6, 7]\). In conformal field theory, the $p$-spin curve intersection theory is related to $\mathcal{N}=2$ superconformal minimal theory for Lie algebra $A_{p-1}$ type. It has been pointed out that it corresponds to a gauged Wess-Zumino-Witten (WZW) model of $SU(2)_k/U(1)$, where $k = p - 2$ is the level of the Kac-Moody algebra of Lie group $SU(2)$ \([8]\), and is related to $SL(2,R)/U(1)$ black hole sigma model when $k$ becomes negative \([9]\).

The free energy for the $p$-spin curve satisfies interesting universal equations, such as a string equation, dilaton equation, and WDVV equation, so called tautological equations or universal equations, and has been studied in the connection to Gromov-Witten theory \([10, 11, 12]\). Although the $p$-spin curve intersection numbers can be obtained through tautological equations in a recursive way, the actual computation for higher genuses is limited \([13, 14, 15]\).

In previous papers, we have derived explicit integral formula for the $p$-spin curve intersection numbers of the moduli space $\overline{M}_{g,n}$ valid for all order of genus $g$. We have shown that they are obtained analytically for a fixed number $n$ of marked points. Our formulation starts from simple Gaussian matrix models with an external matrix source and based upon a duality relation \([5, 6, 7]\), from which one recovers a generalized Kontsevich matrix model.

The intersection numbers for the spin moduli spaces with $n$-marked points are obtained from the $n$ $n$-point correlation functions $U(s_1, \ldots, s_n)$ of Gaussian random matrices in a scaling limit at critical edges \([16, 17]\). In a previous article \([18]\), we have computed explicitly the intersection numbers of moduli space of $p$-spin curves with one marked point, for arbitrary values of $p$, as polynomials in $p$. This allowed us to consider continuations in $p$; in particular the limit $p \to -1$ exhibits an interesting relation between the intersection numbers, and the orbifold Euler characteristics $\chi(\overline{M}_g, 1) = \zeta(1-2g)$, where $\zeta(x)$ is the Riemann zeta function) \([20, 21]\).

In this paper, we extend the evaluation of the intersection numbers beyond the one marked point ($n = 1$) for arbitrary $p$. The obtained intersection numbers are consistent with previously known results \([13, 14, 15]\) for small values of $p$. We pursue the large $p$ behavior, $p \to \infty$ limit. The $p$-spin curve intersection theory is equivalent to gauged WZW model. For this gauged WZW theory, in which $k = p - 2$ appears as overall factor, the large $k$ limit
may give a semi-classical solution [9, 23]. In the negative $k$, the gauged WZW model on $SU(2)/U(1)$ is changed to WZW model on non-compact $SL(2, R)/U(1)$, which is relevant to a black hole $\sigma$ model [9]. We will discuss the relation between the intersection numbers and the density of state of $SL(2, R)/U(1)$ black hole sigma model [24, 25, 27].

2 Generating function for $p$-spin Intersection numbers

The mathematical definition of the intersection numbers of the moduli space of $p$-spin curves with $s$-marked points is given by[3]

$$<\tau_{n_1}(U_{j_1})\cdots \tau_{n_s}(U_{j_s})> = \frac{1}{p^g} \int_{M_{g,s}} C_T(\nu) \prod_{i=1}^{s} (c_1(\mathcal{L}_i))^{n_i}$$  \quad (1)

where $U_j$ is an operator for the primary matter field (tachyon), related to top Chern class $C_T(\nu)$, and $\tau_n$ is a gravitational operator, related to the first Chern class $c_1$ of the line bundle $\mathcal{L}_i$ at the $i$th-marked point. We denote $\tau_n(U_j)$ by $\tau_{n,j}$, and $j$ represents the spin index ($j=0,\ldots,p-1$). The problem of definition (1) has been discussed extensively [11].

In a previous paper [18], we have shown that those intersection numbers (1) are expressed through the correlation functions $U(s_1, \ldots, s_n)$ as coefficients of powers of $s_j$,

$$U(s_1, s_2, \ldots, s_n) = <\text{tr}^{s_1M}\text{tr}^{s_2M} \cdots \text{tr}^{s_nM}>$$

$$= \int \prod_{l=1}^{m} d\lambda e^{\sum_{l=1}^{m} \delta(\lambda_l - M)} <\prod_{1}^{m} \text{tr}(\delta(\lambda_l - M))>$$ \quad (2)

where $s_l = it_l$; $M$ is an $N \times N$ Hermitian random matrix. The bracket stands for averages with the Gaussian probability measure

$$<X> = \frac{1}{Z} \int dM e^{-\frac{1}{2} trM^2 + M^TAX} X(M),$$ \quad (3)

in which $A$ is an $N \times N$ external Hermitian matrix source. By an appropriate tuning of the external source matrix $A$, we may obtain the desired singularity, which generates the $p$-spin curves. The relation to the generalized Kontsevich model is discussed in § 3.4 of [18].

An exact and useful integral representation for $U(s_1, \ldots, s_n)$ is known in presence of an arbitrary external matrix source $A$ with eigenvalues $a_{\alpha}$ [19]:

$$U(s_1, \ldots, s_n) = \frac{1}{N} <\text{tr}^{s_1M} \cdots \text{tr}^{s_nM}>$$

$$= e^{\sum_{i=1}^{n} s_i^2} \int \prod_{i=1}^{n} \frac{du_i}{2\pi i} e^{\sum_{i=1}^{n} u_i s_i} \prod_{\alpha=1}^{N} \prod_{i=1}^{n} \left(1 - \frac{s_i}{a_\alpha - u_i}\right) \text{det} \frac{1}{u_i - u_j + s_i}$$ \quad (4)
This representation involves contour integrals around \( u_i = a_\alpha \). In the large \( N \) limit, it is convenient to express the factors in the determinant as additional integrals. For instance, in the case of the two point correlation (\( n=2 \)), after the shift \( u_i \rightarrow u_i - \frac{s_i}{2} \), \( s_i \rightarrow \frac{s_i}{N} \), in the two point function, we have

\[
\frac{1}{u_1 - u_2 + \frac{1}{2N}(s_1 + s_2)} \frac{1}{u_1 - u_2 - \frac{1}{2N}(s_1 + s_2)} = \frac{N}{s_1 + s_2} \int_0^\infty dx e^{-x(u_1 - u_2)} \text{sh} \left( \frac{x}{2N}(s_1 + s_2) \right)
\]

(5)

Tuning now the \( a_\alpha \)'s, and taking the large \( N \) limit, we obtain

\[
U(s_1, s_2) = \frac{2N}{s_1 + s_2} \int_0^\infty dx \int du_1 du_2 \text{sh} \left( \frac{1}{2N}x(s_1 + s_2) \right)e^{-(u_1 - u_2)x} \times \exp \left[ -\frac{N}{p^2 - 1} \sum \frac{1}{a_\alpha + 1} \left( \sum (u_i + \frac{1}{2N}s_i)^{p+1} - \sum (u_i - \frac{1}{2N}s_i)^{p+1} \right) \right]
\]

(6)

For the three and four point correlations, similar useful formulae for the determinant part of (4) may be found in the appendices A and B of [18].

### 3 Intersection numbers for \( p = 3 \) with two marked points

The intersection numbers are obtained as coefficients of the power series in \( s_1, s_2 \) of \( U(s_1, s_2) \). In a previous paper [18], for \( p=3 \), we have computed the intersection numbers with two marked points or genus one case (\( g=1 \)) starting from (7). As an example, we compute the \( p = 3 \) case up to genus 3. The general expansion

\[
U(s_1, s_2) = \sum_{g,m,j} < \tau_{m_1,j_1, \tau_{m_2,j_2}} > g \Gamma(1 - \frac{1+j_1}{p})\Gamma(1 - \frac{1+j_2}{p}) s_1^{m_1} s_2^{m_2} \]  

(7)

with the condition,

\[
(p + 1)(2g - 2 + n) = \sum_{i=1}^{s}(pm_i + j_i + 1), \quad m'_k = m_k + \frac{1+j_k}{p}, \quad (k = 1, 2)
\]

(8)

is applied to the special case \( n = 2, p = 3 \) The gamma functions in (7) represent the spin factors.

After rescaling of the parameters,

\[
U(s_1, s_2) = \frac{2}{(s_1 + s_2)(3s_2)^{1/3}} \int_0^\infty dy \text{sh} \left( \frac{s_1 + s_2}{2}(3s_1)^{1/3}y \right) A_i(y - \frac{1}{4} \cdot 3^{1/3} s_1^{8/3}) \times A_i(-ay - \frac{1}{4} \cdot 3^{1/3} s_2^{8/3})
\]

(9)
in which \( a = (s_1/s_2)^{1/3} \), and the Airy function is

\[
A_i(y) = \int_{-\infty}^{+\infty} \frac{du}{2\pi} e^{\frac{3}{2}u^2 + iuy}
\] (10)

The Airy function satisfies the differential equation

\[
A''_i(y) = yA_i(y), \quad A''_i(-ay) = -a^3yA_i(-ay)
\] (11)

The genus one case \((g=1)\) has been studied in [7].

If one expands the hyperbolic sine function and the Airy functions in (9) up to relevant orders, we find a sum of six terms which, for \( g = 2 \), involve the following integrals:

\[
I_1 = \int_0^\infty dy \, y^5 A_i(y) A_i(-ay), \quad I_2 = \int_0^\infty dy \, y^6 A''_i(y) A_i(-ay),
\]

\[
I_3 = \int_0^\infty dy \, y A_i(y) A''_i(-ay), \quad I_4 = \int_0^\infty dy \, y A''_i(y) A'_i(-ay),
\]

\[
I_5 = \int_0^\infty dy \, y^3 A'_i(y) A_i(-ay), \quad I_6 = \int_0^\infty dy \, y^3 A_i(y) A'_i(-ay)
\] (12)

A repeated use of (11) plus integrations by parts allows us to write all these integrals in terms of

\[
A_i(0) = \frac{3^{-2/3}}{\Gamma(2/3)} = \frac{1}{2\pi^{1/3}} \Gamma(\frac{1}{3}), \quad A'_i(0) = -\frac{3^{-1/3}}{\Gamma(1/3)} = -\frac{1}{2\pi^{1/3}} \Gamma(\frac{2}{3})
\] (13)

plus the integral

\[
T = \int_0^\infty dy \, A_i(y) A'_i(-ay)
\] (14)

which cannot be reduced to \( A_i(0) \) or \( A'_i(0) \). For instance one finds

\[
(1 + a^3)I_2 = A_i(0)^2 - 2T
\] (15)

and so on. However, all the T-dependence cancels when we sum up all the terms relevant to \( g = 2 \) in \( U(s_1, s_2) \). For instance the sum of all terms of order \( s_2^{4/3} \) is given by

\[
\frac{1}{3!} \frac{1}{16} \left(1 + a^3\right)^{4/3} a^{5/3} s_2^{4/3} I_1 + \frac{1}{2} \left(1 + a^3\right)^{2} a^{17/3} s_2^{16/3} I_2
\]

\[
+ \frac{1}{2} \left(1 + a^3\right)^{2} a^{-1} s_2^{16/3} I_3 - \left(1 + a^3\right)^{2} a^{3/3} s_2^{16/3} I_4
\]

\[
- \frac{1}{3!} \frac{1}{16} \left(1 + a^3\right)^{3/3} a^{11/3} s_2^{16/3} I_5 + \frac{1}{3!} \frac{1}{16} \left(1 + a^3\right)^{2} a^{2} s_2^{16/3} I_6
\] (16)

and we add up the six terms and expand in powers of \( s_1 \) to the relevant orders we find
\[ U(s_1, s_2)|_{g=2} = \frac{(A_i(0))^2}{32 \cdot 3^{2/3}} \left( -s_1^{14/3} s_2^{2/3} - \frac{11}{5} s_1^{11/3} s_2^{5/3} - \frac{17}{5} s_1^{8/3} s_2^{3} - \frac{11}{5} s_1^{5/3} s_2^{11/3} - s_1^{2/3} s_2^{14/3} \right). \] (17)

From these results, we obtain the intersection numbers

\[ < \tau_{0,1} \tau_{4,1} >_{g=2} = \frac{1}{864} \]

\[ < \tau_{1,1} \tau_{3,1} >_{g=2} = \frac{11}{4320} \]

\[ < \tau_{2,1} \tau_{2,1} >_{g=2} = \frac{17}{4320} \] (18)

Rather than computing the exact dependence in \( a \) of the terms proportional to \( s_2^{16/3} \) and then re-expand in \( a \) to obtain the various terms of (17), we may proceed in a simpler way by expanding \( A_i(-ay), A_i'(ay), A_i''(-ay) \) for small \( a \):

\[ A_i(-ay) = A_i(0) - ayA_i'(0) + \frac{a^2}{2} y A_i''(0) + \cdots \] (19)

and we then recover (18).

In the genus- three case (g=3), we have again ten distinct integrals \( J_1 - J_{10} \) for the terms of order \( s_2^8 a^m \), in the small \( s_1, s_2 \) expansion of (9).

\[ J_1 = \int_0^\infty dyy^7 A_i(y) A_i(-ay), \quad J_2 = \int_0^\infty dyy^5 A_i'(y) A_i(-ay) \]

\[ J_3 = \int_0^\infty dyy^3 A_i(y) A_i'(y) A_i(-ay), \quad J_4 = \int_0^\infty dyy^3 A_i'(y) A_i'(y) A_i(-ay) \]

\[ J_5 = \int_0^\infty dyy^3 A_i''(y) A_i(-ay), \quad J_6 = \int_0^\infty dyy^3 A_i(y) A_i''(y) A_i(-ay) \]

\[ J_7 = \int_0^\infty dyy A_i''(y) A_i(-ay), \quad J_8 = \int_0^\infty dyy A_i'(y) A_i''(y) A_i(-ay) \]

\[ J_9 = \int_0^\infty dyy A_i''(y) A_i'(-ay), \quad J_{10} = \int_0^\infty dyy A_i'(y) A_i''(y) A_i(-ay) \] (20)

The genus 3 contribution for \( U(s_1, s_2) \) is then expressed as the sum of four terms, \( U^{(1)} - U^{(4)} \). The term \( U^{(1)} \), which is related to \( J_1 \), is

\[ U^{(1)} = \frac{9}{7! \cdot 64} s_1^{7/3} s_2^{17/3} (1 + a^3)^6 J_1 \]

\[ = \frac{3}{8960} s_1^8 (-15a^7 - 42a^8 + 90a^{10} + 63a^{11} + 63a^{13} + 90a^{14} - 42a^{16} - 15a^{17}) A_i(0) A_i'(0). \] (21)
The term $U^{(2)}$, which is related to $J_2$, is

$$U^{(2)} = -\frac{1}{2560} s^8 a^{13}(1 + a^3)^4 J_2$$

$$= -\frac{1}{2560} s^8 a^{13}(30 + 72a - 120a^3 - 90a^4 + 12a^6) A_i(0) A'_i(0). \quad (22)$$

The term $U^{(3)}$, which is related to $J_3$, is

$$U^{(3)} = \frac{1}{2560} s^8 a^4(1 + a^3)^4 J_3$$

$$= \frac{1}{2560} s^8 a^5(-12 + 90a^2 + 120a^3 - 72a^5 - 30a^6) A_i(0) A'_i(0). \quad (23)$$

The term $U^{(4)}$ from the sum of the contributions of $J_4$ to $J_{10}$. We have

$$J_{10} = \frac{1}{1 + a^3}(2a^3 K_1 - a^3 K_2)$$

$$J_9 = -K_2 - J_{10}$$

$$J_8 = \frac{a^3 + 2a^4 - 2a^6 - a^7}{(1 + a^3)^2} L$$

$$J_7 = \frac{1}{a^3} J_8$$

$$J_6 = \frac{6a^3}{1 + a^3} J_7$$

$$J_5 = \frac{1}{a^3} J_6$$

$$J_4 = a^3 J_5 - 3J_9 \quad (24)$$

with $L = A_i(0) A'_i(0)$, and $K_1, K_2$ are given below. We have also the following relations between $J_1, J_2$ and $J_3$,

$$J_1 = \frac{1}{1 + a^3}(30J_5 + 12J_3)$$

$$J_2 = \frac{1}{1 + a^3}(-5J_5 + 4J_4)$$

$$J_3 = \frac{1}{1 + a^3}(-5a^3 J_5 - 4J_4) \quad (25)$$

Thus $U^{(4)}$ becomes

$$U^{(4)} = \frac{1}{1152} s^8 (a + 5a^4 - 20a^7 + 23a^{10} + 16a^{13}$$

$$-19a^{16} + 13a^{19} + 2a^{22} + (2a^2 + 13a^5 - 19a^8$$

$$+ 16a^{11} + 23a^{14} - 20a^{17} + 5a^{20} + a^{23})) A_i(0) A'_i(0). \quad (26)$$
Since \( a = (s_1/s_2)^{1/3} \), the above expression for \( U(s_1, s_2) \) is a symmetric function in \( s_1 \) and \( s_2 \). Denoting
\[
s^{m+1/3} = t_{m,j}
\]
and dividing \( U(s_1, s_2) \) by \( 1/g^p \), i.e. \( 1/27 \) in this case, we obtain the intersection numbers \( <\tau_{m_1,j_1}\tau_{m_2,j_2}>_{g=3} \) as the coefficient of \( t_{m_1,j_1}t_{m_2,j_2} \) in \( U(s_1, s_2) \) taking into account the spin factors. The following spin factor appears as a
over all factor in \( U(s_1, s_2) \) at genus 3.

\[
A_i(0)A'_i(0) = -\frac{1}{(2\pi)^{23/3}}\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})
\]

where \( \Gamma(\frac{1}{3}), \Gamma(\frac{2}{3}) \) are spin 1 and spin 2 factors, respectively, as (7). All the integrals \( J_1, ..., J_{10} \) are expressed by (28), and there are no terms like (14), which appeared in the integrals for the \( g=1, g=2 \) cases. Finally we have to compute the following terms
\[
K_1 = \int dy A''_i(y)A_i(-ay) = -A_i(0)A'_i(0) - K_2
\]
\[
K_2 = \int dy A'_i(y)A'_i(-ay).
\]

For these integrals, we find
\[
K_1 = -\frac{1+a}{1+a^3}A_i(0)A'_i(0), \quad K_2 = \frac{a-a^3}{1+a^3}A_i(0)A'_i(0)
\]
and all the integrals reduces to the spin factor (28). Summing up the results
of \( U(1) \) to \( U(4) \), we obtain the intersection numbers for \( p = 3, g = 3 \),
\[
<\tau_{0,0}\tau_{7,1}>_{g=3} = \frac{1}{31104}, \quad <\tau_{0,1}\tau_{7,0}>_{g=3} = \frac{1}{15552}
\]
\[
<\tau_{1,0}\tau_{6,1}>_{g=3} = \frac{5}{31104}, \quad <\tau_{1,1}\tau_{6,0}>_{g=3} = \frac{19}{77760}
\]
\[
<\tau_{2,0}\tau_{5,1}>_{g=3} = \frac{103}{217728}, \quad <\tau_{2,1}\tau_{5,0}>_{g=3} = \frac{47}{77760}
\]
\[
<\tau_{3,0}\tau_{4,1}>_{g=3} = \frac{443}{544320}, \quad <\tau_{3,1}\tau_{4,0}>_{g=3} = \frac{67}{77760}
\]
\[
(31)
\]

The above results are in complete agreement with the previous results \([13, 15] \), which were obtained by recursion relations.

4 Intersection numbers for \( p > 3 \)

For higher multicritical points the algebra is similar, except that we have to deal with generalized Airy functions. For instance for \( p = 4 \) instead of \( A_i(x) \)
we have to work with \( \phi(x) \) defined as
\[
\phi(x) = \int_0^\infty dv e^{-\frac{1}{4}v^4+vx}
\]
which satifies
\[
\phi''(x) = x\phi.
\]
Then, similarly
\[
U(s_1, s_2) = \frac{2}{(s_1 + s_2)(4s_2)^{1/4}} \int_0^\infty dx \int_0^\infty dv_1dv_2sh(\frac{s_1 + s_2}{2}(4s_1)^{1/4}x)
\]
\[
e^{-\frac{(s_1^2 + s_2^2)}{4(s_1 s_2)^{1/4}}(\frac{s_1}{s_1^2})^{1/2}v_1^2 - \frac{s_1^2}{4(s_1 s_2)^{1/4}}v_2^2} e^{-\frac{v_1^2}{4}s_1 - \frac{v_2^2}{4}s_2 - axv_1v_2}
\]
where \( a = (s_1/s_2)^{1/4} \). In complete analogy with the \( p = 3 \) case, a repeated use of integration by parts and of (33) leads to the expansion of \( U(s_1, s_2) \). In the genus one case,
\[
U(s_1, s_2) = 1 = \frac{1}{4} (\phi''(0))^2 s_1^{1/4} s_2^{1/4} (s_1^2 + s_1 s_2 + s_2^2)
\]
\[
+ \frac{1}{12} (s_1 s_2)^{3/4} (s_1 + s_2) (\phi(0))^2
\]
with
\[
\phi''(0) = 2^{1/2}\Gamma(\frac{3}{4}), \quad \phi(0) = 2^{-1/2}\Gamma(\frac{1}{4})
\]
which provide the \( j = 0, j = 2 \) spin factors respectively. Replacing \( s_1, s_2 \) by \( t_m, j, (s^m+(1+j)/p = t_{m,j}) \),
\[
U(s_1, s_2) = \frac{1}{2} (t_2,0t_0,0 + t_1,0t_1,0 + t_0,0t_2,0) (\Gamma(\frac{3}{4}))^2
\]
\[
+ \frac{1}{24} (t_1,2t_{0,2} + t_0,2t_{1,2}) (\Gamma(\frac{1}{4}))^2
\]
Multiplying by a factor \( \frac{1}{p^2} \), we obtain the intersection numbers as coefficients of (37) for \( p = 4 \) in the genus one case,
\[
<\tau_0,0\tau_2,0>_{g=1} = \frac{1}{8}, \quad <\tau_1,0\tau_1,0>_{g=1} = \frac{1}{8}, \quad <\tau_0,2\tau_1,2>_{g=1} = \frac{1}{96}
\]
For \( g = 2, p = 4 \), from the term \( s_2^4 s_1^4 \), we have similarly
\[
<\tau_0,1\tau_4,1>_{g=2} = \frac{1}{320}
\]
For general \( p \) we have to deal with the generalized Airy functions \( \phi(x) \) for \( p > 2 \), which satisfy the differential equation,
\[
\phi^{(p)}(x) = x\phi(x)
\]
where $\phi^{(p)}(x)$ means the $p$-th derivative of $\phi(x)$. The generalized Airy function has an integral representation,

$$
\phi(y) = \int_0^\infty du \, e^{-\frac{u}{y} + yu}.
$$

As examples of what the method can provide we give a few results: for the case $p = 5$, we obtain

$$
<\tau_{1,3}\tau_{0,2} >_{g=1} = \frac{1}{60}, \quad <\tau_{1,2}\tau_{0,3} >_{g=1} = \frac{1}{6}, \quad <\tau_{0,1}\tau_{4,1} >_{g=2} = \frac{7}{1200}.
$$

For the case $p = 6$,

$$
<\tau_{0,3}\tau_{1,3} >_{g=1} = \frac{1}{36}, \quad <\tau_{0,2}\tau_{1,4} >_{g=1} = \frac{1}{48}, \quad <\tau_{0,4}\tau_{1,2} >_{g=1} = \frac{1}{48}.
$$

For the case $p = 7$,

$$
<\tau_{0,2}\tau_{1,5} >_{g=1} = \frac{1}{42}, \quad <\tau_{0,3}\tau_{1,4} >_{g=1} = \frac{1}{28}, \quad <\tau_{1,0}\tau_{3,1} >_{g=1} = \frac{1}{4}.
$$

### 5 The $p$ dependence of the intersection numbers

In a previous article [18], we have considered the intersection numbers with one marked point for arbitrary $p$, and found results such as

$$
<\tau_{1,0} >_{g=1} = \frac{p - 1}{24},
$$

$$
<\tau_{n,j} >_{g=2} = \frac{(p - 1)(p - 3)(1 + 2p)}{p \cdot 5! \cdot 4^2 \cdot 3} \frac{\Gamma(1 - \frac{3}{p})}{\Gamma(1 - \frac{1+j}{p})},
$$

$$
<\tau_{n,j} >_{g=3} = \frac{(p - 5)(p - 1)(1 + 2p)(8p^2 - 13p - 13)}{p^2 \cdot 7! \cdot 4^3 \cdot 3^2} \frac{\Gamma(1 - \frac{5}{p})}{\Gamma(1 - \frac{1+j}{p})},
$$

$$
<\tau_{n,j} >_{g=4} = \frac{(p - 7)(p - 1)(1 + 2p)(72p^4 - 298p^3 - 17p^2 + 562p + 281)}{p^3 \cdot 9! \cdot 4^4 \cdot 15} \frac{\Gamma(1 - \frac{7}{p})}{\Gamma(1 - \frac{1+j}{p})}.
$$
with \( n = 2g - 1 + \frac{2g-2-j}{p} \). In the large \( p \) limit, the intersection numbers \(<\tau_{n,j}>_g\) behave as

\[
<\tau_{n,j}>_g = \frac{B_g}{(2g)!(2g)}p^g + O(p^{g-1})
\]

(46)

with \( B_g \) is a Bernoulli number, \( B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, B_4 = \frac{1}{30} \). Note the well known relation to \( \zeta(2g) \) as

\[
\frac{B_g}{(2g)!2g} = \frac{1}{(2\pi)^{2g}}\zeta(2g)
\]

(47)

We have derived (46) from \( U(s) \) in the large \( p \) limit. The one point function \( U(s) \) has the following expression\[16\],

\[
U(s) = \frac{1}{Ns} \int \frac{du}{2i\pi} \exp\left(-\frac{c}{p+1}(u + \frac{1}{2}s)^{p+1} - (u - \frac{1}{2}s)^{p+1}\right)
\]

(48)

With \( s = \frac{\sigma}{p} \), and \( u^{p+1} = x^2 \), we have

\[
U(s) = \frac{2}{N\sigma} \int \frac{dx}{2i\pi} x^{-1+\frac{2}{p}} e^{-\frac{c}{p+1}x^2(e^{\sigma/2} - e^{-\sigma/2})}
\]

(49)

Thus we obtain

\[
U(s) = \frac{2}{N\sigma} \Gamma\left(\frac{2}{p}\right)\left(\frac{2c}{p+1}\text{sh}\frac{\sigma}{2}\right)^{-1/p}
\]

(50)

This may be written as

\[
U(s) = \frac{2}{N\sigma} \Gamma\left(\frac{2}{p}\right)\left(\frac{2c}{p+1}\right)^{-\frac{1}{p}}\left(\frac{\sigma}{2}\right)^{-\frac{1}{p}}\exp\left(-\frac{1}{p}\log\frac{\text{sh}\frac{\sigma}{2}}{2}\right)
\]

(51)

and expanding the exponent in \( \frac{1}{p} \), we find

\[
U(s) = \frac{2}{N\sigma} \Gamma\left(\frac{2}{p}\right)\left(\frac{2c}{p+1}\right)^{-\frac{1}{p}}\left(\frac{\sigma}{2}\right)^{-\frac{1}{p}}\left(1 - \frac{1}{p}\log\frac{\text{sh}\frac{\sigma}{2}}{2}\right)
\]

(52)

If we use the expansion

\[
\log\left(\frac{\text{sh}\frac{\sigma}{2}}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n\sigma^{2n}}{(2n)!2n}
\]

(53)

and drop the factors \( \left(\frac{2c}{p+1}\right)^{-\frac{1}{p}} \), \( (\sigma/2)^{-1/p} \) which are close to one in the large \( p \) limit, we obtain

\[
U(s) = \left(1 - \frac{1}{p}\sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_n\sigma^{2n}}{(2n)!2n}\right) \frac{p}{N\sigma} \Gamma\left(1 + \frac{2}{p}\right)
\]

(54)
Since the intersection numbers $<\tau_{n,j}>_g$ are related to $U(s)$ by [16]

$$U(s) = \sum_g <\tau_{n,j}>_g \frac{1}{N\pi} \Gamma(1 - \frac{1}{p}) s^{(2g-1)(1+p)/p} p^{g-1}$$  \hspace{1cm} (55)$$

with $(p+1)(2g-1) = pn + j + 1$, we have rederived the large $p$ behavior of (46).

From (53), taking a derivative with respect to $\sigma$, gives,

$$\frac{1}{e^\sigma - 1} + \frac{1}{2} - \frac{1}{\sigma} = \sum_{n=1}^\infty \frac{(-1)^{n-1} B_n}{(2n)!} \sigma^{2n-1},$$  \hspace{1cm} (56)$$

Using this relation one obtains

$$\frac{d}{d\sigma}(\sigma U(\sigma)) = \frac{1}{\sigma} - \frac{1}{2} - \frac{1}{e^\sigma - 1}$$  \hspace{1cm} (57)$$

The di-gamma function $\psi(z)$ has the following expression,

$$\psi(z) = \frac{d}{dz} \log\Gamma(z) = \log z - \frac{1}{2z} - \int_0^\infty d\sigma (\sigma U(\sigma)) e^{-\sigma z}.$$  \hspace{1cm} (58)$$

From (57) and (58) we find thus in the large $p$ limit,

$$\frac{d}{dz} \log\Gamma(z) = \log z - \frac{1}{2z} + \int_0^\infty d\sigma (\frac{d}{d\sigma}(\sigma U(\sigma))) e^{-\sigma z}$$  \hspace{1cm} (59)$$

$$= \log z - \frac{1}{2z} - \frac{z}{2z} \int d\sigma U(\sigma) e^{-\sigma z}$$

The last integral is related to the density of states. In (2), $s$ is replaced by $s = it$, and if we replace $z$ by $iE$, and take the imaginary part, we obtain the density of states $\rho(E)$. After integration by parts, we obtain

$$\rho(E) = \frac{d}{dE} \text{Im} \log\Gamma(iE) - \frac{\pi}{2} - \frac{1}{2E}$$  \hspace{1cm} (60)$$

We will discuss this expression in connection to the density of states of the $SL(2,R)/U(1)$ black hole sigma model in the next section.

Next we consider the two point correlation function $U(s_1, s_2)$. For general $p$, $U(s_1, s_2)$ is expressed as

$$U(s_1, s_2) = \frac{2}{(s_1 + s_2)(ps_1)^{1/p}} \int_0^\infty dx \int_0^\infty dv_1dv_2 \text{sh} \left( \frac{s_1 + s_2}{2} (ps_1)^{1/p} \right)$$

$$e^{-\frac{x^p}{p} + x_1 - \frac{(p-1)}{24}s_1^2(ps_1)^{2/p}} v_1^{2-p} \cdots e^{-\frac{v_1^p}{p} - \frac{(p-1)}{24}s_2^2(ps_2)^{2/p}} v_2^{2-p} \cdots$$  \hspace{1cm} (61)$$
The exponent of (61) follows from the binomial expansion,

\[(u + \frac{s}{2})^{p+1} = u^{p+1} + (p + 1)u^{p}\left(\frac{s}{2}\right) + (p + 1)p\frac{1}{2}u^{p-1}\left(\frac{s}{2}\right)^2 + \ldots\]  

(62)

and we use \(c(p + 1) = 1\), \(u^p s = t^p/p\). As in the case of \(p=3\), polynomials in \(a\) (21) give the intersection numbers. Therefore we expand (61) in power series of \(a\). At lowest order in \(a\), we obtain two terms from (61),

\[U_1 = \frac{1}{3!4}(s_1 + s_2)^2\frac{(ps_1)^2}{(ps_2)^2}\int dx x^3 \phi(x)\phi(-ax)\]

\[U_2 = -\frac{p(p-1)}{24}\frac{(s_1)^{1/2} s_2}{(s_2)^{1/2}}(ps_2)^{2-p}\int dx x \phi(x)\phi((p-2)(-ax)(a)^{2-p}\]  

(63)

From \(U_2\) we find a term proportional to \(as_2^{2+\frac{2}{p}}\), namely

\[\Delta U_2 = \frac{p - 1}{24}p^{\frac{2}{p}}a s_2^{2+\frac{2}{p}}(\phi^{(p-2)}(0))^2\]

(64)

with

\[\phi^{(p-2)}(0) = \int_0^\infty du u^{p-2}e^{-\frac{u^p}{p}}\]

\[= p^{-\frac{1}{p}} \Gamma(1 - \frac{1}{p}).\]

(65)

Since \(s_2^{2+\frac{1}{p}} s_1^{\frac{1}{p}} = t_2,0 t_{0,0}\), we obtain

\[< \tau_{0,0}^2 >_{g=1} = \frac{p - 1}{24}\]

(66)

From \(U_1\) and \(U_2\), we collect terms proportional to \(a^2 s_2^{2+\frac{2}{p}}\) and obtain

\[< \tau_{0,2}^2 >_{g=1} = \frac{p - 3}{24p}\]

(67)

This result agrees with those obtained previously for \(p = 4, 5, 6\) and 7 in (38), (42), (43) and (44). The intersection number (67) can be neglected in the large \(p\) limit in comparison with (66).

Similarly one obtains the \(g=2\) terms from the coefficients of \(a^m s_2^{4+\frac{m}{p}}\) (\(m=1,2,3\)),

\[< \tau_{0,0}^2 >_{g=2} = \frac{(p - 1)(p - 3)(2p + 1)}{5760p}\]

\[< \tau_{0,1}^2 >_{g=2} = \frac{(p - 1)(p - 2)(p + 2)}{2880p}\]

\[< \tau_{0,2}^2 >_{g=2} = \frac{(p - 1)(p - 3)(2p + 11)}{5760p}\]

(68)
For the particular values of $p = 3, 4, 5$, the above expressions agree with the previous results (18), (39) and (42) for the genus two case.

From the $a^5 s_2^{4+\frac{5}{p}}$ term, one finds

$$<\tau_{0,4}\tau_{3,p-2}>_{g=2}= \frac{2p^3 + 13p^2 - 158p + 215}{5760p^2}$$

(69)

which is valid for $p \geq 6$.

In the large $p$ limit, the three terms of (68) become equal, and coincide with the result for the one point intersection number (46).

$$<\tau_{0,m}\tau_{4,2-m}>_{g=2} = \frac{B_2p^2}{4! \cdot 4} \quad (p \to \infty)$$

(70)

Note that (69) is order $p$, and is negligible compared to (68).

From the terms $a^m s_4^{6+\frac{6}{p}}$ in the small $a$ expansion of $U(s_1, s_2)$, we obtain the $g=3$ (genus 3) terms. In the case $m=1$, we have

$$<\tau_{0,0}\tau_{6,4}>_{g=3} = \frac{(p - 1)(p - 5)(2p + 1)(8p^2 - 13p - 13)}{p^2 \cdot 7! \cdot 3^3} \quad (p > 5)$$

(71)

This is identical to $<\tau_{5,4}>_{g=3}$ in (45). The identity follows from the string equation, in which the insertion of $\tau_{0,0}$ reduces the intersection number from $s$ to $s - 1$ marked points:

$$<\tau_{0,0} \prod_{i=1}^{s} \tau_{n_i,j_i}>_{g} = \sum_{l=1}^{s} <\tau_{n_l-1,j_l} \prod_{i=1,i\neq l}^{s} \tau_{n_i,j_i}>_{g}$$

(72)

In our formulation, this string equation follows from the integral representation for the intersection numbers, when one collects the terms proportional to $a$. By explicit calculation of two marked points, we verified this string equation. It might be possible to verify this string equation for $n$-marked points by the taking account of the term of $a$.

From $a^2 s_2^{6+\frac{6}{p}}$, we have for $p > 5$,

$$<\tau_{0,1}\tau_{6,3}>_{g=3} = \frac{(p - 1)(p - 2)(p - 4)(p + 2)(2p + 1)}{p^2 \cdot 7! \cdot 8 \cdot 3^2}$$

(73)

From $a^3 s_2^{6+\frac{6}{p}}$,

$$<\tau_{0,2}\tau_{6,2}>_{g=3} = \frac{(p - 1)(p - 3)(16p^3 + 34p^2 - 155p - 129)}{p^2 \cdot 7! \cdot 64 \cdot 3^2}$$

(74)

In the large $p$ limit, these $g=3$ terms exhibit same behavior as in (46),

$$<\tau_{0,m}\tau_{6,4-m}>_{g=3} = \frac{B_3}{6! \cdot 6p^3} + O(p^2) \quad (p \to \infty)$$

(75)
6 Analytic continuation to negative $p$

One may analytically continue the integral representations of the correlation functions to negative values of $p$. This continuation was already examined in [18], and we recall some of the results here:

$$U(s) = \frac{1}{Ns} \int \frac{du}{2i\pi} e^{-c[(u+\frac{1}{2}s)p+1-(u-\frac{1}{2}s)p+1]}$$  \hspace{1cm} (76)$$

where $c = \frac{N}{p-1} \sum \frac{1}{a^+}$.

Expanding the exponent, we obtain

$$U(s) = \int \frac{du}{2i\pi} \text{exp}[-c(su^p + \frac{p(p-1)}{3!4} s^3 u^{p-2} + \frac{p(p-1)(p-2)(p-3)}{5!42} s^5 u^{p-4} + \cdots)].$$  \hspace{1cm} (77)$$

This integrals yield Gamma functions after the replacement $u = (\frac{t}{cs})^{1/p}$,

$$U(s) = \frac{1}{Ns^{1/p}} \left( \frac{1}{(cs)^{1/p}} \int_0^\infty dt \frac{1}{t} e^{-(t+\frac{p(p-1)}{3!4} s^2 + \frac{p(p-1)(p-2)(p-3)}{5!42} s^4 + \cdots) c^\frac{1}{p} t^{1-\frac{1}{p}}} \right)$$

$$= \frac{1}{Ns^{1/p}} \left( \frac{1}{(cs)^{1/p}} \left[ -\frac{p-1}{24} c^\frac{2}{p} y \Gamma(1 - \frac{3}{p}) + \frac{(p-1)(p-3)(1+2p)}{5! \cdot 4^2 \cdot 3} y^2 \Gamma(1 - \frac{3}{p}) \right] ight)$$

$$+ (p-5)(p-1)(1+2p)(8p^2 - 13p - 13) y^3 \Gamma(1 - \frac{5}{p})$$

$$+ (p-7)(p-1)(1+2p)(72p^4 - 298p^3 - 17p^2 + 562p + 281)$$

$$\times \frac{1}{9!4^315} y^4 \Gamma(1 - \frac{7}{p}) \cdots$$  \hspace{1cm} (78)$$

with $y = c^\frac{2}{p} s^{2+\frac{2}{p}}$.

From this expansion, we obtain the intersection numbers for one marked point as (45). The intersection number $<\tau_{n,j}>_g$ is obtained from the term $y^g \Gamma(1 - \frac{1}{p} - \frac{j}{p})$ in (78).

The continuation to $p < 0$ is straightforward. The $t$-integral in (78) can be changed to $v$ by $t = \frac{1}{v}, (0 < v < \infty)$, and one obtains the small $s$ expansion for negative $p$. Therefore the expression for the intersection numbers (45) can be analytically continued to negative $p$. This analytic continuation can also be done for two marked points, since we have computed them in the previous sections for general $p$. For instance, from (45), we have the intersection numbers for $p = -3$,

$$<\tau_{1,0}>_{g=1} = -\frac{1}{6}, \quad <\tau_{3,2}>_{g=2} = \frac{1}{144}$$

$$<\tau_{6,1}>_{g=3} = -\frac{35}{34992}$$  \hspace{1cm} (79)$$
In a previous article [18], we have computed the intersection numbers $<\tau_{1,0}>_{g}$ for the case of $p = -1$ from $U(s)$, which provides the orbifold Euler characteristics $\chi(M_{g,1})$ with one marked point,

$$<\tau_{1,0}>_{g} = \chi(M_{g,1}) = \zeta(1 - 2g) = -\frac{B_g}{2g}$$  \hspace{1cm} (80)

with the Bernoulli number $B_g$, ($B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, ...$). The $s$-point orbifold Euler characteristics $\chi(M_{g,s})$ may be obtained from the dilaton equation:

$$<\tau_{1,0}\tau_{n_1,j_1} \cdots \tau_{n_k,j_k}>_{g} = (2g - 2 + k) <\tau_{n_1,j_1} \cdots \tau_{n_k,j_k}>_{g}$$  \hspace{1cm} (81)

Since the Euler characteristics with $s$ marked points is $<\tau_{1,0} \cdots \tau_{1,0}>_{g}$, the dilaton equation yields from (80),

$$\chi(M_{g,s}) = <(\tau_{1,0})^s>_{g} = -\frac{2g - 1}{(2g)!} (2g + s - 3)!B_g$$  \hspace{1cm} (82)

This agrees with previous results obtained in [20, 21, 22].

For $p = -2$, we have considered previously the equivalence with the unitary matrix model in a matrix source [26].

The central charge of the gauged Wess-Zumino-Witten model with symmetry $SU(2)_k/U(1)$ is

$$C = 2 - \frac{6}{k + 2}$$  \hspace{1cm} (83)

Changing $p$ to $p' = k$ to $k' = -k' (p < 0, k < 0)$, we have $p' = k' - 2$, and the central charge $C'$ is given by

$$C = 2 + \frac{6}{k' - 2}$$  \hspace{1cm} (84)

The analytic continuation to negative $p$ yields a gauged WZW model for $SL(2,R)_{k'}/U(1)$. It is known that this model represents a black hole $\sigma$ model[9], in particular for the value $k' = 9/4 (p = -\frac{1}{4})$, for which the central charge $C$ becomes 26.

The density of states for the $SL(2,R)/U(1)$ black hole has been studied in [24, 25, 27],

$$\rho(E) = \frac{1}{\pi} \log \epsilon + \frac{1}{4\pi i} \frac{d}{dE} \log \frac{\Gamma(-iE + \frac{1}{2} - m)\Gamma(-iE + \frac{1}{2} + \tilde{m})}{\Gamma(+iE + \frac{1}{2} + \tilde{m})\Gamma(+iE + \frac{1}{2} - m)}$$  \hspace{1cm} (85)

in which $\epsilon$ is a regularization factor, and $m = \frac{1}{2}(n - kw)$, $\tilde{m} = -\frac{1}{2}(kw + n)$ are eigenvalues of $J^3_0$ and $\tilde{J}^3_0$ in CFT ($J^3_0 - \tilde{J}^3_0 = n, J^3_0 + \tilde{J}^3_0 = -kw$). If we neglect $m$, $\tilde{m}$, and the $\frac{1}{2}$ terms in the large $E$ limit, we obtain

$$\rho(E) = \frac{1}{\pi} \log \epsilon + \frac{1}{2\pi i} \frac{d}{dE} \log \frac{\Gamma(-iE)}{\Gamma(+iE)}$$  \hspace{1cm} (86)
or

\[
\rho(E) = \frac{2}{\pi} \frac{d}{dE} \text{Im} \log \Gamma(-iE)
\]  \hspace{1cm} (87)

This expression agrees with (60), obtained from the intersection numbers for large \( p \). We have scaled \( s = \sigma/p \), and the expression (60) is valid for small \( s \). Therefore, the Fourier transform of \( U(s) \) gives the large \( E \) behavior, in which the terms \( m, \bar{m} \) and \( 1/2 \) in (85) can be neglected.

7 Discussion

In this article, we have shown that the correlation functions \( U(s_1, s_2, \cdots, s_n) \) of a Gaussian matrix model in a tuned external source, provide the intersection numbers for \( p \)-spin curves. For instance, from the two point function \( U(s_1, s_2) \), in the case of \( p=3 \), the intersection numbers are computed up to genus 3.

We have also computed the intersection numbers for general \( p \). They are given by power series in \( a, a = \left( \frac{s_1}{s_2} \right)^{\frac{1}{p}} \). Then we have considered the large \( p \) behavior for the two point functions. The density of states \( \rho(E) \) becomes a di-gamma function in the large \( p \) limit, and this expression agrees with the density of states of a \( SL(2, R)_{k}/U(1) \) WZW model, which has been studied in the context of two dimensional black hole solutions. The n-point correlation functions \( U(s_1, \cdots, s_n) \) are known through the determinant of a kernel for the \( p \)-spin curve case. It will be interesting to investigate further the detailed comparison of those correlation functions, between \( SL(2, R)_{k}/U(1) \) WZW theory and the intersection numbers for negative \( p \)-spin curves.

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