Path Integrals, BRST Identities and Regularization Schemes in Nonstandard Gauges

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Abstract The path integral of a gauge theory is studied in Coulomb-like gauges. The Christ-Lee terms of operator ordering are reproduced within the path integration framework. In the presence of fermions, a new operator term, in addition to that of Christ-Lee, is discovered. Such kind of terms is found to be instrumental in restoring the invariance of the effective Lagrangian under a field dependent gauge transformation, which underlies the BRST symmetry. A unitary regularization scheme which maintains manifest BRST symmetry and is free from energy divergences is proposed for a nonabelian gauge field.

I. Introduction

The quantization of a gauge field in the continuum requires gauge fixing. The standard gauge for perturbative calculations is the covariant gauge, in which the theory can be regularized and renormalized systematically. On the other hand, noncovariant gauges, though computationally more involved, possess a number of advantages of their own. Take Coulomb gauge as an example, all degrees of freedom there are physical and unitarity is manifest. The Coulomb propagator alone gives rise to the renormalization of the coupling constant [1] [2]. In addition, the explicit solubility of the Gauss law constraint in Coulomb gauge might make it easier to construct variational
wave functional to explore the nonperturbative physics of a nonabelian gauge field.

The difficulties with noncovariant gauges of a nonabelian gauge theory include 1) the complication of the curvilinear coordinates; 2) the lack of manifest BRST and Lorentz invariances; 3) the additional divergence, the energy divergence, because of the instantaneous nature of the bare Coulomb propagator and the ghost propagator; 4) the lack of a unitary regularization scheme which can be applied to all orders and therefore the uncertainty of the renormalizability.

The operator ordering associated with the curvilinear coordinates has been investigated by Christ and Lee [3]. They derived the path integral in a noncovariant gauge from the Weyl ordering of the corresponding Hamiltonian and found that the effective Lagrangian contains two nonlocal terms, referred to as Christ-Lee terms, in addition to the classical Lagrangian and the ghost determinant. These terms start to show up at the two loop order and their diagrammatic implications have been discussed to that order [4][5]. Here, I approach the noncovariant gauge strictly within the path integral formulation, along the line of Refs. 6 and 7. Starting with the discrete time path integral in the time axial gauge, the change of the integration variables to other gauges and the continuous time limit are examined carefully and the Christ-Lee terms are reproduced. In the presence of quark fields, a new nonlocal term of the effective Lagrangian involving fermion bilinears, which starts to show up at the one loop order is discovered.

According to Feynman [8], a path integral of a quantum mechanical system is the $\epsilon \to 0$ limit of a multiple integral over canonical coordinates on a one dimensional lattice of time slices separated by $\epsilon$. The exponent of the weighting factor on each time slice is equal to $i\epsilon$ times the classical Lagrangian only if the canonical coordinates are cartesian [2]. This is the case of the time-axial gauge of a gauge theory. Even there, the velocities in the Lagrangian is the mean velocity between the neighboring time slices instead of instantaneous ones. When transforming the integration variables to curvilinear ones, or to other gauges, e.g. Coulomb gauge, one has to keep track of all the contributions to the limit $\epsilon \to 0$, which introduces the Christ-Lee operator ordering terms in addition to the classical Lagrangian in terms of the new coordinates and the corresponding Jacobian. For the same reason, the classical Lagrangian with the mean velocities is not exactly invariant under a gauge transformation which depends on canonical coordinates and the variation contributes to the limit $\epsilon \to 0$. The standard form of the BRST identity is only recovered after including the Christ-Lee terms. Alternatively, one may retain a nonvanishing $\epsilon$ and this lends us to a gauge theory with discrete time coordinates, which is manifest BRST invariant. For a field the-
ory, this discrete time formulation serves as a unitary regularization scheme, which regularizes the energy divergence and the ordinary ultraviolet divergence at one shot. It also possesses several technical advantages which may be helpful in higher orders.

This paper is organized as follows: I will illustrate the technique of the gauge fixing within the path integral formulation and the derivation of the BRST identity of the soluble model of Ref. [5] in the next two sections. The application to a nonabelian gauge field is discussed in the Sections IV and V. There I will also test the discrete time regularization scheme by evaluating the one loop correction to the Coulomb propagator. The comparison of my regularization scheme with others and some comments on the renormalizability will be discussed in the final section. Except for the fermionic operator ordering term, the interplay between the BRST identity and the operator ordering terms and the discrete time regularization scheme, all other results are not new. But it is instructive to see how the operator ordering terms come about without referring to the operator formulations. It is also amazing to see how similar in formulation the soluble model and the nonabelian gauge field are.

II. The Path Integral of a Soluble Model

The soluble model proposed by Friedberg, Lee, Pang and the author [5] provides a playground for the investigation of gauge fixing and the BRST invariance in some nonstandard gauges within the path integral formulation.

The Lagrangian of the soluble model is

\[ L = \frac{1}{2}[(\dot{x} + g\xi y)^2 + (\dot{y} - g\xi x)^2 + (\dot{z} - \xi)^2] - U(x^2 + y^2). \]  \hspace{1cm} (2.1)

It is invariant under the following gauge transformation

\[ x \rightarrow x' = x \cos \alpha - y \sin \alpha, \]  \hspace{1cm} (2.2)

\[ y \rightarrow y' = x \sin \alpha + y \cos \alpha, \]  \hspace{1cm} (2.3)

\[ z \rightarrow z' = z + \frac{1}{g} \dot{\alpha}, \]  \hspace{1cm} (2.4)

and

\[ \xi \rightarrow \xi' = \xi + \frac{1}{g} \dot{\xi}. \]  \hspace{1cm} (2.5)

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with $\alpha$ an arbitrary function of time. The Lagrangian \((2.1)\) does not contain
the time derivative of $\xi$ and the corresponding equation of motion reads
\[
\frac{\partial L}{\partial \xi} = g[y(\dot{x} + g\xi y) - x(\dot{y} - g\xi x)] - \dot{z} + \xi = 0, \tag{2.6}
\]
which is the analog of the Gauss law of a gauge field. In the following, we
shall review the canonical quantization in the time-axial g auge, i.e., $\xi = 0$,
convert it into a path integral and transform carefully the path integral into
the $\lambda$-gauge, i.e., $z = \lambda x$, an analog of the Coulomb gauge.

In the time-axial gauge, the Lagrangian \((2.1)\) becomes
\[
L = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - U(X^2 + Y^2). \tag{2.7}
\]
The canonical momenta corresponding to $X$, $Y$, and $Z$ are
\[
P_X = \dot{X} = -i \frac{\partial}{\partial X}, \tag{2.8}
\]
\[
P_Y = \dot{Y} = -i \frac{\partial}{\partial Y} \tag{2.9}
\]
and
\[
P_Z = \dot{Z} = -i \frac{\partial}{\partial Z}, \tag{2.10}
\]
and the Hamiltonian operator reads
\[
H = \frac{1}{2}(P_X^2 + P_Y^2 + P_Z^2) + U(X^2 + Y^2). \tag{2.11}
\]
The physical states in the Hilbert space are subject to the Gauss law con-
straint, i.e.
\[
[P_Z + g(XP_Y - YP_X)]\ = 0, \tag{2.12}
\]
as follows from \((2.6)\) and the operator $P_Z + g(XP_Y - YP_X)$ commutes with
$H$. In terms of polar coordin ates, $X = \rho \cos \Phi$ and $Y = \rho \sin \Phi$, the wave
function of a physical state takes the form
\[
<X, Y, Z| = \Psi(\rho, \Phi - gZ). \tag{2.13}
\]
For a harmonic oscillator potential, $U = \frac{1}{2}\omega^2(X^2 + Y^2)$, the energy spectrum
is given by
\[
E = \omega(n_+ + n_- + 1) + \frac{1}{2}g^2 (n_+ - n_-)^2 \tag{2.14}
\]
with $n_+, n_- $ non-negative integers, and the corresponding eigenfunction can
be expressed in terms of Laguerre polynomials.
Following Feynman, the transition matrix element \( < X, Y, Z | e^{-iHt} | > \) can be cast into a path integral
\[
< X, Y, Z | e^{-iHt} | > = \lim_{\epsilon \to 0} \left( \frac{1}{2i\pi \epsilon} \right)^{3N} \int \prod_{n=0}^{N-1} dX_n dY_n dZ_n
\times e^{i \epsilon \sum_{n=0}^{N-1} L(n)} < X_0, Y_0, Z_0 | > , \tag{2.15}
\]
where \( \epsilon = t/N \) and
\[
L(n) = \frac{1}{2} (\dot{X}_n^2 + \dot{Y}_n^2 + \dot{Z}_n^2) - U(X_n^2 + Y_n^2) \tag{2.16}
\]
with \( \dot{X}_n = (X_{n+1} - X_n)/\epsilon \) etc.. In the rest part of the paper, the limit sign and the normalization factors like \((2i\pi \epsilon)^{-\frac{3N}{2}}\) will not be displayed explicitly.

As was pointed out in [3] and [5], the path integral (2.15) picks up the velocity
\[
(\dot{X}_n, \dot{Y}_n, \dot{Z}_n) \text{ as large as } O\left(\epsilon^{-\frac{1}{2}}\right). \tag{2.17}
\]
In other words, the contribution to the path integral comes from paths which can be more zigzag than classical ones. This has to be taken into account in variable transformations. The magnitudes of \( X_n, Y_n \) and \( Z_n \), on the other hand, remains of the order one with a well defined initial wave function \( < X_0, Y_0, Z_0 | > \). To transform the path integral (2.15) to \( \lambda \)-gauge, i.e., \( z = \lambda x \), we insert the identity
\[
1 = \text{const.} \int \prod_{n=0}^{N-1} d\theta_n \mathcal{J}_n \delta(z_n - \lambda x_n), \tag{2.18}
\]
with
\[
x_n = X_n \cos \theta_n - Y_n \sin \theta_n, \tag{2.19}
y_n = X_n \sin \theta_n + Y_n \cos \theta_n, \tag{2.20}
z_n = Z_n + \frac{1}{g} \theta_n \tag{2.21}
\]
and
\[
\mathcal{J}_n = \frac{1}{g} + \lambda y_n, \tag{2.22}
\]
we have
\[
< X, Y, Z | e^{-iHt} | > = \text{const.} \int \prod_{n=0}^{N-1} dX_n dY_n dZ_n d\theta_n \mathcal{J}_n \delta(z_n - \lambda x_n) \times
\]
\[ \times e^{i \epsilon \sum_{n=0}^{N-1} L(n)} < X_0, Y_0, Z_0 | >, \]  
(2.23)

Introducing back \( \xi_n \) via
\[ \xi_n = \frac{1}{g} \dot{\theta}_n = \frac{\theta_{n+1} - \theta_n}{g \epsilon} \]  
(2.24)

and changing the integration variables from \( X_n, Y_n, Z_n \) and \( \theta_n \) to \( x_n, y_n, z_n \) and \( \xi_n \), we obtain that
\[ < X, Y, Z | e^{-i H t} > = \text{const.} \int \prod_{n=0}^{N-1} dx_n dy_n dz_n d\xi_n \delta(z_n - \lambda x_n) \times \]
\[ \times e^{i \epsilon \sum_{n=0}^{N-1} L'(n)} < x_0, y_0, z_0 | >, \]  
(2.25)

where
\[ L'(n) = L(n) - \frac{i}{\epsilon} \ln J_n \]  
(2.26)

with \( L(n) \) the same Lagrangian (2.16). Written in terms of the new variables, \( L(n) \) becomes
\[ L(n) = \frac{1}{2 \epsilon^2} (\bar{r}_{n+1} e^{-i g \xi_n \sigma_2} - \bar{r}_n) (e^{i g \xi_n \sigma_2} r_{n+1} - r_n) + \frac{1}{2} (\dot{z}_n - \xi_n)^2 - U(\bar{r}_n r_n), \]  
(2.27)

where we have grouped \( x_n \) and \( y_n \) into a 2 \times 1 matrix
\[ r_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \]  
(2.28)

and \( \sigma_2 \) is the second Pauli matrix.

For finite \( \epsilon \), (2.25) with (2.26)-(2.27) define a one-dimensional lattice gauge field and the limit \( \epsilon \to 0 \) corresponds to its continuum limit. As \( \epsilon \to 0 \), it follows from (2.17), (2.19)-(2.21) and (2.24) that
\[ (\dot{x}_n, \dot{y}_n, \dot{z}_n, \xi_n) = O(\epsilon^{-\frac{1}{2}}). \]  
(2.29)

Several terms beyond the naive continuum limit have to be kept when expanding the exponential \( e^{i g \xi_n \sigma_2} \) in (2.27) according to \( \xi_n \) [3]. In addition, the commutativity between time derivative and path integral contractions requires the Lagrangian to be written in terms of \( \dot{x}_n, \dot{y}_n, \dot{z}_n, \xi_n, \bar{x}_n, \bar{y}_n \) and \( \bar{z}_n \) with \( \bar{x}_n = (x_n + x_{n+1})/2 \) etc.. In another word, we write
\[ L'(n) \equiv L'(\dot{x}_n, \dot{y}_n, \dot{z}_n, \xi_n, x_n, y_n, z_n) = L'(\dot{x}_n, \dot{y}_n, \dot{z}_n, \xi_n, \bar{x}_n, \bar{y}_n, \bar{z}_n) + \delta L'(n) \]  
(2.30)
\[ \delta L'(n) = L'(\dot{x}_n, \dot{y}_n, \dot{z}_n, \xi_n, x_n, y_n, z_n) - L'(\dot{x}_n, \dot{y}_n, \dot{z}_n, \xi_n, \bar{x}_n, \bar{y}_n, \bar{z}_n). \] (2.31)

According to the estimate (2.29), the contribution from the potential energy \( U \) to the difference (2.31) vanishes in the limit \( \epsilon \to 0 \). But that from the kinetic energy and from the jacobian (2.22) do not. With this precaution, we rewrite \( L'(n) \) as

\[ L'(n) = L_{\text{eff}}(n) + \frac{i}{2\epsilon} \left( \ln \mathcal{J}_n + \ln \mathcal{J}_n \right), \] (2.32)

where

\[ L_{\text{eff}}(n) = L(n) - \frac{i}{\epsilon} \ln \bar{J}_n + \frac{i}{\epsilon} \left[ \ln \bar{J}_n - \frac{1}{2} \left( \ln \mathcal{J}_{n+1} + \ln \mathcal{J}_n \right) \right] \] (2.33)

with \( \bar{J}_n = \frac{1}{g} + \lambda \bar{y}_n \). (2.34)

The path integral (2.25) becomes then

\[ <X,Y,Z|e^{-iHt}|>= \text{const.} \mathcal{J}_N^{-\frac{1}{2}} \int \prod_{n=0}^{N-1} dx_n dy_n dz_n d\xi_n \delta(z_n - \lambda x_n) \times \]

\[ \times e^{i \epsilon \sum_{n=0}^{N-1} L_{\text{eff}}(n)} \mathcal{J}_0^{-\frac{1}{2}} <x_0, y_0, z_0|, \] (2.35)

Now it is the time to take the limit \( \epsilon \to 0 \). The small \( \epsilon \) expansion of \( L_{\text{eff}}(n) \) reads

\[ L_{\text{eff}}(n) = L_{\text{cl}}(n) - \frac{i}{\epsilon} \ln \bar{J}_n + \Delta L(n) + O(\epsilon^\frac{1}{2}), \] (2.36)

where

\[ L_{\text{cl}}(n) = \frac{1}{2} \left[ (\dot{x}_n + g\xi_n \bar{x}_n)^2 + (\dot{y}_n - g\xi_n \bar{y}_n)^2 + (\dot{z}_n - \xi_n)^2 \right] - U(x_n^2 + y_n^2) \] (2.37)

is the classical Lagrangian but with mean velocities and

\[ \Delta L(n) = -\frac{1}{8} \epsilon^2 g^2 \xi_n^2 (\dot{r}_n - ig\xi_n \sigma_2) \left( \dot{r}_n + \frac{i}{3} g\sigma_2 \xi_n \bar{r}_n \right) + \frac{i}{8} \epsilon \lambda^2 g^2 \bar{y}_n^2. \] (2.38)

The first term of \( \Delta L(n) \) comes from the kinetic energy, the second term from the jacobian. It follows from (2.29) that \( \Delta L(n) = O(1) \) and the terms not displayed all vanish as \( \epsilon \to 0 \).
It remains to convert $\Delta L(n)$ into an equivalent potential (to eliminate the explicit $\epsilon$ dependence). The recipe was given by Gervais and Jevicki [6], which we shall outline here. We assume that the integrations on the time slices $t = 0, \epsilon, 2\epsilon, \ldots, (n-1)\epsilon$ have been carried out and we are left with

$$<X, Y, Z|e^{-iHt}|> = \text{const.} \mathcal{J}_N^{-\frac{1}{2}} \int \prod_{m=n+1}^{N-1} dx_m dy_m dz_m d\xi_m$$

$$\times \delta(z_m - \lambda x_m) e^{i \epsilon \sum_{m=n+1}^{N-1} L_{\text{eff}}(m)}$$

$$\times \int dx_n dy_n dz_n d\xi_n \mathcal{J}_n^{\frac{1}{2}} \delta(z_n - \lambda x_n) e^{i e L_{\text{cl}}(n)} [1 + i \epsilon \Delta L(n)] <x_n, y_n, z_n|e^{-i\epsilon H}|>,$$

where the corresponding transition matrix element, $<x_n, y_n, z_n|e^{-i\epsilon H}|>$, is a smooth function of $x_n$, $y_n$ and $z_n$. The structure of $\Delta L(n)$ is

$$\Delta L(n) = \sum_l C_l(\bar{x}_n, \bar{y}_n) P_l(n) \epsilon^\frac{n_l}{2}$$

(2.40)

with $P_l(n)$ a product of $\dot{x}_n$, $\dot{y}_n$, and $\xi_n$, and $n_l$ the number of factors. Changing the integration variables from $x_n$, $y_n$ and $\xi_n$ to $\dot{x}_n$, $\dot{y}_n$, and $\xi_n$ while replacing $(x_n, y_n)$ by $(x_{n+1} - \epsilon \dot{x}_n, y_{n+1} - \epsilon \dot{y}_n)$ and $(\bar{x}_n, \bar{y}_n)$ by $(x_{n+1} - \epsilon \dot{x}_n / 2, y_{n+1} - \epsilon \dot{y}_n / 2)$, we have, upon a Taylor expansion in terms of $\epsilon \dot{x}_n$ and $\epsilon \dot{y}_n$, that

$$\int dx_n dy_n dz_n d\xi_n \mathcal{J}_n^{\frac{1}{2}} \delta(z_n - \lambda x_n) e^{i e L_{\text{cl}}(n)} [1 - i \epsilon \mathcal{V}(\bar{x}_n, \bar{y}_n) + O(\epsilon^{\frac{3}{2}})]$$

$$\times <x_n, y_n, z_n|e^{-i\epsilon H}|>,$$

(2.41)

where

$$\mathcal{V}(\bar{x}_n, \bar{y}_n) = - <\Delta L(n)>_{\text{Gauss}} = - \sum_l C_l(\bar{x}_n, \bar{y}_n) <P_l(n)>_{\text{Gauss}}$$

(2.42)

and the Gauss average $<...>_{\text{Gauss}}$ is defined to be

$$<F(n)>_{\text{Gauss}} = \frac{\int d\dot{x}_n d\dot{y}_n d\xi_n e^{i e L_{\text{cl}}(n)} F(n)}{\int d\dot{x}_n d\dot{y}_n d\xi_n e^{i e L_{\text{cl}}(n)}}$$

(2.43)

while regarding $\bar{x}_n$ and $\bar{y}_n$ constants. Such a procedure is valid even if a linear term of $\dot{x}_n$, $\dot{y}_n$ and $\xi_n$ with coefficients of the order one is added to
\( L_{\text{cl}}(n) \), as will be the case when external sources are introduced to generate various Green’s functions. Introduce a \( 3 \times 1 \) matrix

\[
\begin{pmatrix}
\zeta_{1n} \\
\zeta_{2n} \\
\zeta_{3n}
\end{pmatrix} = \begin{pmatrix}
\dot{x}_n + g\bar{y}_n\zeta_n \\
\bar{y}_n - g\bar{x}_n\zeta_n \\
\xi_n - \lambda\dot{x}_n
\end{pmatrix},
\]

(2.44)

we have

\[
\begin{pmatrix}
\dot{x}_n \\
\bar{y}_n \\
\xi_n
\end{pmatrix} = \frac{1}{1 + \lambda g y_n} \begin{pmatrix}
1 & 0 & -g\bar{y}_n \\
\lambda g\bar{x}_n & 1 + \lambda g\bar{y}_n & g\bar{x}_n \\
0 & 1 & -\lambda
\end{pmatrix} \begin{pmatrix}
\zeta_{1n} \\
\zeta_{2n} \\
\zeta_{3n}
\end{pmatrix}.
\]

(2.45)

It follows from (2.43) that

\[
<\zeta_{in}\zeta_{jm}>_{\text{Gauss}} = \frac{i}{\epsilon} \delta_{nm} \delta_{ij}.
\]

(2.46)

Working out the Wick contractions in (2.42) according to (2.45) and (2.46), we end up with

\[
\mathcal{V}(x, y) = -\frac{g^2(2 + 3\lambda^2) + \lambda^3 g^3 y}{8(1 + \lambda g y)^3} + \frac{\lambda^2 g^4 x^2(1 + \lambda^2)}{8(1 + \lambda g y)^4},
\]

(2.47)

which agrees with the result obtained via Weyl ordering. The effective Lagrangian \( L_{\text{eff}}(n) \) in (2.35) is then replaced by that of Christ-Lee type, i.e.

\[
\mathcal{L}(n) = L_{\text{cl}}(n) - \frac{i}{\epsilon} \ln \bar{J}_n - \mathcal{V}(n).
\]

(2.48)

Before closing this section, I would like to remark that the subtleties of the path integral depends strongly on the way in which the gauge condition is introduced. Consider a general linear gauge fixing with the insertion (2.18) replaced by

\[
1 = \text{const.} \int \prod_{n=0}^{N-1} d\theta_n J e^{-i\frac{\epsilon}{\lambda^2} \left( \zeta_{in} - \lambda x_n - \kappa \right)^2}
\]

(2.49)

with \( a \) and \( \kappa > 0 \) gauge parameters like \( \lambda \). This is the discrete version of the gauge fixing used in [9] and the \( \lambda \)-gauge, (2.18), corresponds to \( a = 0 \) and \( \kappa = 0 \). The analysis in Appendix A gives rise to the estimates in the table I for the typical magnitude of \( \xi \) in the path integral with different choices of the gauge parameters.

For \( \kappa \neq 0 \), the limit \( \epsilon \to 0 \) is trivial and the same estimates apply to the gauge fixing with \( \bar{x}_n \) and \( \bar{z}_n \) in (2.49) replaced by \( \bar{x}_n \) and \( \bar{z}_n \), an analog of the covariant gauge in a relativistic field theory. But the limit \( \kappa \to 0 \) with a continuous time will entail higher degrees of energy divergence for individual diagrams.
Table I. The typical magnitude of $\xi$

|       | $\kappa = 0$       | $\kappa \neq 0$ |
|-------|--------------------|------------------|
| $a = 0$ | $O(\epsilon^{-\frac{1}{2}})$ | $O(1)$           |
| $a \neq 0$ | $O(\epsilon^{-\frac{3}{2}})$ | $O(1)$           |

III. The BRST Identity of the Soluble Model

There are two approaches to the BRST identity of the soluble model (2.1). One can prove BRST invariance by introducing ghost variables and establish the identity with external sources. One may also start with the Slavnov-Taylor identity [10] and construct the BRST identity afterwards. It turns out the former is more straightforward for the path integral (2.25) with the lattice Lagrangian (2.26) and (2.27), while the latter is more convenient with the Christ-Lee type of path integral. We shall illustrate both approaches in the following.

III.1 Prior to the $\epsilon$-expansion

Introducing the ghost variables $c_n$ and $\bar{c}_n$ and an auxiliary field $b_n$, the path integral (2.25) can be cast into

$$<X,Y,Z|e^{-iHT}|=\text{const.}\int \prod_{n=0}^{N-1} dx_n dy_n dz_n d\xi_n db_n dc_n d\bar{c}_n \times$$

$$\times e^{i\epsilon \sum_{n=0}^{N-1} L_{\text{BRST}}(n)} <x_0,y_0,z_0|>$$  \hspace{1cm} (3.1)

where

$$L_{\text{BRST}}(n) = \frac{1}{2\epsilon^2}(\tilde{r}_{n+1} e^{-ig\xi_n \sigma_2} - \tilde{r}_n)(e^{ig\xi_n \sigma_2} r_{n+1} - r_n) + \frac{1}{2}(\dot{z}_n - \xi_n)^2 - U(\tilde{r}_n r_n)$$

$$+ b_n(z_n - \lambda x_n) - \bar{c}_n(1 + \lambda g y_n)c_n.$$  \hspace{1cm} (3.2)

The integration measure and the Lagrangian (3.2) is invariant under the following transformation

$$\delta r_n = -ig\theta_n \sigma_2 r_n, \hspace{1cm} (3.3)$$

$$\delta z_n = \theta_n, \hspace{1cm} (3.4)$$
\[ \delta \xi_n = \dot{\theta}_n, \quad (3.5) \]
\[ \delta b_n = 0, \quad (3.6) \]
\[ \delta c_n = 0 \quad (3.7) \]

and
\[ \delta \bar{c}_n = s_n b_n \quad (3.8) \]

with \( \theta_n = s_n c_n \) and \( s_n \) a Grassmann number. For \( n \)-independent \( s_n \equiv s \), an operator \( Q \) such that \( \delta = sQ \) can be extracted. It is straightforward to show that \( Q^2 = 0 \) and the transformation (3.3)-(3.8) is of BRST type.

To establish the BRST identity, we introduce the generating functional of connected Green’s functions,
\[
e^{iW(J,\zeta,u,\eta,\bar{\eta})} = \lim_{T \to \infty} \langle |e^{-iHT}| \rangle = \text{const.} \int \prod_{n=0}^{N} dx_n dy_n dz_n d\xi_n db_n dc_n d\bar{c}_n \times \]
\[
\times \langle |x_N, y_N, z_N| e^{i\epsilon \sum_n [L_{\text{BRST}}(n) + L_{\text{ext}}(n)]} < x_0, y_0, z_0 \rangle, \quad (3.9) \]

where \( L_{\text{ext}}(n) \) stands for the source term, i.e.
\[
L_{\text{ext}}(n) = \tilde{J}_n r_n + \zeta_n z_n + u_n \xi_n + \bar{\eta}_n c_n + \bar{c}_n \eta_n, \quad (3.10) \]

| \rangle denotes the ground state of the system, and the limits \( \epsilon \to 0 \) and \( N\epsilon = T \to \infty \) are understood for the right hand side. It follows from the transformations (3.3)-(3.8) that
\[
\langle \delta L_{\text{ext}}(n) \rangle_\eta = -ig \tilde{J}_n \sigma_2 < c_n r_n \rangle_\eta + (\zeta_n - \dot{u}_n) < c_n \rangle_\eta + < b_n \rangle_\eta \eta_n = 0. \quad (3.11) \]

This is the prototype of the BRST identity and can be converted into various useful forms.

**III.2 After the \( \epsilon \)-expansion**

After integrating out the ghost variables and carrying out the \( \epsilon \)-expansion, the path integral (3.9) becomes
\[
e^{iW(J,\zeta,u,\eta,\bar{\eta})} = \lim_{T \to \infty} \langle |e^{-iHT}| \rangle = \text{const.} \int \prod_{n=0}^{N} dx_n dy_n dz_n d\xi_n db_n \times \]
\[
\times \langle |x_N, y_N, z_N| e^{i\epsilon \sum_n [L(n) + L_{\text{ext}}(n)]} < x_0, y_0, z_0 \rangle, \quad (3.12) \]
where
\[ L(n) = L_{\text{cl}}(n) + b_n(z_n - \lambda x_n) - \frac{i}{\epsilon} \ln \tilde{J}_n - \mathcal{V}(n) \] (3.13)
is the Lagrangian of Christ-Lee type with \( L_{\text{cl}}(n) \) given by (2.37), \( \mathcal{V}(n) \) by (2.47) and \( L_{\text{ext}}(n) \) by (3.10) at \( \eta_n = \bar{\eta}_n = 0 \). One may introduce the ghost variables for the path integral (3.12), but they will be different from the ones in (3.9), since the argument of the jacobian \( J_n \) has been shifted from \( y_n \) to \( \bar{y}_n \). On the other hand, the BRST identity can be constructed from the Slavnov-Taylor identity and we shall adapt this strategy. We consider a field dependent gauge transformation
\[ \delta r_n = -i\chi_n \sigma_2 r_n, \] (3.14)
\[ \delta z_n = \frac{1}{g} \chi_n, \] (3.15)
\[ \delta \xi_n = \frac{1}{g} \dot{\chi}_n, \] (3.16)
where
\[ \chi_n = \frac{\varepsilon_n}{\frac{1}{g} + \lambda y_n} \] (3.17)
with \( \varepsilon_n \) an infinitesimal ordinary number. Keeping in mind that the velocities \( \dot{x}_n, \dot{y}_n \) and \( \dot{z}_n \), and the coordinates \( x_n, \bar{y}_n \) and \( \bar{z}_n \) follow strictly the discrete time definition and the variable transformation (3.14)-(3.16) is nonlinear, the variation of the Lagrangian \( L_{\text{cl}}(n) \) contributes to the path integral in the limit \( \varepsilon \to 0 \). We find that
\[ \delta L_{\text{cl}}(n) = \frac{1}{4} \varepsilon^2 \sigma_2 \varepsilon_n \chi_n (D\widetilde{r})_n \dot{r}_n = -\frac{1}{4} \varepsilon^2 \lambda y_n (D\widetilde{r})_n \dot{r}_n \varepsilon_n, \] (3.18)
where the terms containing \( \dot{\varepsilon}_n \) have been dropped since \( \varepsilon_n \) is assumed smooth with respect to \( n \) i.e. \( \dot{\varepsilon}_n = O(1) \). Furthermore, the combination \( dx_n dy_n dz_n d\xi_n d\bar{y}_n J_n \) ceases to be invariant like the combination \( dx_n dy_n dz_n d\xi_n d\bar{y}_n J_n \). We have, instead,
\[ \delta \left( dx_n dy_n dz_n d\xi_n \right) = dx_n dy_n dz_n d\xi_n \Delta_n \]
with
\[ \Delta_n = \left\{ \frac{1}{2} \left[ \frac{x_n}{(1 + \lambda y_n)^2} - \frac{x_{n+1}}{(1 + \lambda y_{n+1})^2} \right] + \frac{1}{4} \frac{\varepsilon^2 \lambda^2 y_n (D\widetilde{r})_n \dot{r}_n}{(1 + \lambda y_n)^3} - \frac{1}{2} \frac{\varepsilon \lambda y_n^2 (D\widetilde{r})_n \dot{r}_n}{(1 + \lambda y_n)^4} \right\} \varepsilon_n \] (3.19)
With these additional terms, the identity (3.11) is replaced by

\[
\sum_n \varepsilon_n \left[ -ig\tilde{J}_n\sigma_2 < \frac{1}{(1 + \lambda g\bar{y}_n)} r_n > + (\zeta_n - \dot{u}_n) < \frac{1}{1 + \lambda g\bar{y}_n} > + < b_n > \right]
\]

\[+ < \delta L_{\text{cl}}(n) - i\Delta_n - \delta \mathcal{V}(n) > \] = 0, \quad (3.20)

where the term \(< b_n >\) comes from the variation of the gauge fixing term of (3.13). Upon utilizing (2.45) and (2.46) for the Gauss average in the last section, we obtain that

\[
\sum_n \varepsilon_n < \delta L_{\text{cl}}(n) - i\Delta_n > = \sum_n \varepsilon_n < - \frac{1}{8} \lambda^3 g^4 \frac{\bar{x}_n}{(1 + \lambda g\bar{y}_n)^4} \]

\[+ \frac{3}{8} \lambda g^4 (2 + 3\lambda^2)\bar{x}_n - \lambda^2 g^5 (2 + \lambda^2)\bar{x}_n\bar{y}_n - \frac{1}{2} \lambda^3 g^6 (1 + \lambda^2) \frac{\bar{x}_n^3}{(1 + \lambda g\bar{y}_n)^6} + O(\epsilon^{\frac{1}{2}}) > \]

\[= \sum_n \varepsilon_n < \delta \mathcal{V}(n) > + O(\epsilon^{\frac{1}{2}}), \quad (3.21)\]

and the Slavnov-Taylor identity follows in the limit \(\epsilon \to 0\)

\[-ig\tilde{J}_n\sigma_2 < \frac{1}{(1 + \lambda g\bar{y}_n)} r_n > + (\zeta_n - \dot{u}_n) < \frac{1}{1 + \lambda g\bar{y}_n} > + < b_n > = 0. \quad (3.22)\]

This can also be obtained from (3.11) after integrating over \(c\) and \(\bar{c}\) at \(\eta = \bar{\eta} = 0\).

To construct the BRST identity, we introduce the ghost variables by rewriting (3.12) as

\[e^{iW(J,\zeta,u,\eta,\bar{\eta})} = \lim_{T \to \infty} |e^{-iHT}| = \text{const.} \int \prod_{n=0}^{N} dx_n dy_n dz_n d\xi_n db_n d\tilde{c}_n d\tilde{\bar{c}}_n \times \]

\[\times < x_N, y_N, z_N > e^{i\epsilon \sum_{n=0}^{N} [L'(n) + L'_{\text{ext.}}(n)]} < x_0, y_0, z_0 >, \quad (3.23)\]

where

\[L'(n) = L_{\text{cl}}(n) + b_n(z_n - \lambda x_n) - \tilde{c}_n'(1 + \lambda g\bar{y}_n)\xi_n - \mathcal{V}(n) \quad (3.24)\]

and

\[L_{\text{ext.}}(n) = \tilde{J}_n r_n + \zeta_n z_n - u_n\xi_n + \bar{\eta}_n\tilde{c}_n + \bar{\xi}_n\tilde{\bar{c}}_n. \quad (3.25)\]

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Note that the primed ghosts are different from the original ones. Denoting the average with respect to the path integral (3.23) by $< \ldots >_\eta$, we have, for a function of the integration variables, $F$,

$$< F >_\eta = \frac{\langle F e^{i\epsilon \sum_n (\eta_n c_n' + \bar{c}_n' \eta_n)} \rangle}{\langle e^{i\epsilon \sum_n (\eta_n c_n' + \bar{c}_n' \eta_n)} \rangle}. \quad (3.26)$$

Then it follows that

$$< c_n' r_n >_\eta = \frac{1}{1 + \lambda g \bar{y}_n} r_n > \eta, \quad (3.27)$$

$$< c_n' >_\eta = \frac{1}{1 + \lambda g \bar{y}_n} > \eta \quad (3.28)$$

and

$$< b_n >_\eta = < b_n >. \quad (3.29)$$

In the limit $\epsilon \to 0$, the difference of $\bar{y}_n$ in (3.27) from $y_n$ may be neglected. The Slavnov-Taylor identity (3.22) implies the following BRST identity

$$-ig \bar{J}_n \sigma^2 < c_n' r_n >_\eta + (\zeta_n - \dot{u}_n) < c_n' >_\eta + < b_n >_\eta \eta_n = 0, \quad (3.30)$$

which is equivalent to (3.11). As will be shown in Appendix B, the invariance of the Lagrangian under the field dependent transformation (3.14)-(3.17) is related to an symmetry of the corresponding Hamiltonian in the $\lambda$-gauge after factoring out the Gauss law constraint. There we shall present a derivation of the Slavnov-Taylor identity (3.22) from canonical formulations.

IV. The Path Integral of an Nonabelian Gauge Field in Coulomb Gauge

IV.1 Quantization in the time axial gauge

The Lagrangian density of a nonabelian gauge theory is

$$L = -\int d^3\vec{r} \left[ \frac{1}{4} V^l_{\mu\nu} V^l_{\mu\nu} + \psi^\dagger \gamma_4 (\gamma_\mu D_\mu + m) \psi \right]. \quad (4.1)$$

where

$$V^l_{\mu\nu} = \frac{\partial V^l_\mu}{\partial x_\mu} - \frac{\partial V^l_\mu}{\partial x_\nu} + gf^{lmn} V^m_\mu V^n_\nu \quad (4.2)$$
with $V_\mu^l$ the gauge potential and $f^{lmn}$ the structure constant of the Lie algebra of the gauge group. The fermion field $\psi$ carries both color and flavor indices and the mass matrix $m$ is diagonal with respect to the color indices. The gauge covariant derivative is $D_\mu = \partial_\mu - igT^lV_\mu^l$ with $T^l$ the generator of the gauge group in the representation to which $\psi$ belongs. The normalizations of $f^{lmn}$ and $T^l$ are given by $\text{tr} T^l T^l = \frac{1}{2} \delta_{ll'}$ and $f^{lmn} f^{l'mn} = C_2^2 \delta_{ll'}$ with $C_2$ the second Casimir of the gauge group. The Lagrangian (4.1) is invariant under the following gauge transformation

$$V_\mu \rightarrow V_\mu' = uV_\mu u^\dagger + \frac{i}{g} u \frac{\partial u^\dagger}{\partial x_\mu}$$

and

$$\psi \rightarrow \psi' = u\psi$$

with $V_\mu = V_\mu^l T^l$ and $u$ the transformation matrix in the representation of $\psi$.

The quantization of the gauge field is specified in the time axial gauge where $V_0 = 0$. The Lagrangian (4.1) becomes

$$L = \int d^3 \vec{r} \left[ \frac{1}{2} \dot{V}_j^l \dot{V}_j^l - \frac{1}{2} B_j^l B_j^l + i \Psi^\dagger \dot{\Psi} - \Psi^\dagger \gamma_4 (\gamma_j D_j + m) \Psi \right]$$

and the corresponding Hamiltonian reads

$$H = \int d^3 \vec{r} \left[ \frac{1}{2} \Pi_j^l \Pi_j^l + \frac{1}{2} B_j^l B_j^l + \Psi^\dagger \gamma_4 (\gamma_j D_j + m) \Psi \right]$$

where the canonical momentum

$$\Pi_j^l(\vec{r}) = \dot{V}_j^l(\vec{r}) = -i \frac{\delta}{\delta V_j^l(\vec{r})}$$

and $B_j^l(\vec{r}) = \frac{1}{2} \epsilon_{ijk} V_{jk}(\vec{r})$ is the color magnetic field. The Hamiltonian (4.6) commutes with the generator of time-independent gauge transformations, $G^l$ with

$$G^l = \frac{1}{g} (\delta^{lm} \nabla_j - g f^{lmn} V_j^m) \Pi_j^m + \Psi^\dagger T^l \Psi.$$  

A physical state in the Hilbert space is subject to the Gauss law constraint, i.e.

$$G^l | > = 0.$$  

The path integral in the time axial gauge can be readily written down

$$< V | e^{-iHt} | > = \text{const.} \int \prod_n [dV d\Psi d\bar{\Psi}]_n e^{it} \sum_n L(n) < V_j(0, \vec{r}) | >.$$  

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where
\[
[dVdΨd\bar{Ψ}]_n \equiv \prod_{\vec{r},j,l} dV_j^l(n, \vec{r}) dΨ(n, \vec{r}) d\bar{Ψ}(n, \vec{r}),
\] (4.11)

\[
L(n) = \int d^3\vec{r} \left[ \frac{1}{2} \dot{V}_j^l(n) \dot{V}_j^l(n) - \frac{1}{2} B_j^l(n) B_j^l(n)
+ i\bar{Ψ}(n)\gamma_4 Ψ(n) - \bar{Ψ}(n)(\gamma_j D_j(n) + m)Ψ(n) \right]
\] (4.12)

with
\[
V_j^l(n) = \frac{V_j^l(n+1) - V_j^l(n)}{\epsilon} = O(\epsilon^{-\frac{1}{2}}),
\] (4.13)

and the initial wave functional satisfies the Gauss law (4.9). The dependence
of the field amplitudes on \(\vec{r}\) has been suppressed in (4.12).

**IV.2 Transformation to Coulomb Gauge**

Inserting the following identity into the path integral (4.10),
\[
1 = \text{const.} \int \prod_{n,\vec{r}} du(n, \vec{r}) J(n) \delta(\nabla_j A_j^l(n, \vec{r})),
\] (4.14)

where
\[
A_j(n, \vec{r}) = u^\dagger(n, \vec{r}) V_j(n, \vec{r}) u(n, \vec{r}) + \frac{i}{g} u^\dagger(n, \vec{r}) \nabla_j u(n, \vec{r}),
\] (4.15)

\(u(n, \vec{r})\) is a representation matrix of the gauge group and
\[
J(n) = \det(-\nabla_j D_j(n))
\] (4.16)

with \(D_j^{lm} = \delta^{lm}\nabla_j - g f^{lnm} A_j^n\). Introducing
\[
u^\dagger(n, \vec{r}) u(n+1, \vec{r}) = e^{i g A_0(n, \vec{r})},
\] (4.17)
\[
\psi(n, \vec{r}) = u^\dagger(n, \vec{r}) Ψ(n, \vec{r})
\] (4.18)

and
\[
\bar{ψ}(n, \vec{r}) = \bar{Ψ}(n, \vec{r}) u(n+1, \vec{r}),
\] (4.19)

and transforming the integration variables from \(V_j^l(n, \vec{r})\) and \(u(n, \vec{r})\) into
\(A_j^l(n, \vec{r})\) and \(A_0^l(n, \vec{r})\), we obtain that
\[
<V | e^{-iHt} | > = \text{const.} \int \prod_{n}[dAdψd\bar{ψ}]_n \delta(\nabla_j A_j^l(n, \vec{r})) e^{iε \sum_{\epsilon} L'(n)} < A_j(0, \vec{r}) >,
\] (4.20)
where
\[ L'(n) = L(n) - \frac{i}{\epsilon} \ln \mathcal{J}(n) - \frac{i}{\epsilon} \delta^3(0) \int d^3 \vec{r} \ln h(n, \vec{r}) \] (4.21)
with \( L(n) \) given by (4.12) and \( h(n, \vec{r}) \) the Haar measure of the integration of the group element \( e^{iegA_0(n, \vec{r})} \) with respect to \( A_0(n, \vec{r}) \)
\[ h(n, \vec{r}) = 1 - \frac{e^2 g^2}{24} A'_0(n, \vec{r}) A'_0(n, \vec{r}) + O(e^3 g^3), \] (4.22)
which does not have an analog in the soluble model. In terms of the new variables, we have
\[ L(n) = \int d^3 \vec{r} \left[ \text{tr} [\mathcal{E}_j(n) \mathcal{E}_j(n) - \mathcal{B}_j(n) \mathcal{B}_j(n)] \right] \]
\[ \frac{i}{\epsilon} \bar{\psi}(n) [\psi(n + 1) - e^{-iegA_0(n)} \psi(n)] - \bar{\psi}(n) e^{-iegA_0(n)} [\gamma_j D_j(n) + m] \psi(n) \] (4.23)
with
\[ \mathcal{E}_j(n) = -\frac{1}{\epsilon} \left[ e^{iegA_0(n)} A_j(n + 1) e^{-iegA_0(n)} - A_j(n) + \frac{i}{g} e^{iegA_0(n)} \nabla_j e^{-iegA_0(n)} \right] \] (4.24)
and
\[ \mathcal{B}_j(n) = \frac{1}{2} \epsilon_{jki} \left[ \nabla_k A_i(n) - \nabla_i A_k(n) - ig[ A_k(n), A_i(n) ] \right]. \] (4.25)
The Lagrangian (4.23) with (4.24) and (4.25) defines a gauge theory in a spatial continuum and on a temporal lattice. The action \( \epsilon \sum_n L(n) \) coincides with the naive continuum limit of the spatial links of Wilson’s lattice action. But it comes naturally from the definition of a path integral and the procedure of gauge fixing. It follows from (4.13), (4.15) and (4.17) that [3]
\[ (\dot{A}_j'(n, \vec{r}), A'_0(n, \vec{r})) = O(\epsilon^{-\frac{1}{2}}) \] (4.26).
The path integral (4.20) can be rewritten as
\[ <V|e^{-iHt}|A(0)> = \text{const.} \mathcal{J}^{-\frac{1}{2}}(N) \int \prod_n [dAd\psi d\bar{\psi}]_n \]
\[ e^{i\epsilon \sum_n L_{\text{eff.}}(n)} \mathcal{J}^{\frac{1}{2}}(0) <A(0)> \] (4.27)
with
\[ L_{\text{eff.}}(n) = L(n) - \frac{i}{\epsilon} [\ln \mathcal{J}(n) + \ln h(n)] + \frac{i}{\epsilon} \left[ \ln \mathcal{J}(n) - \frac{1}{2} (\ln \mathcal{J}(n+1) + \ln \mathcal{J}(n)) \right]. \] (4.28)
where
\[ \mathcal{J}(n) = \det(-\nabla_j D_j(n)) \]  
(4.29)
with \( D_j^{ab} = \delta^{ab} \nabla_j - g f^{abc} A_c^j(n) \) and \( \tilde{A}_j^l(n) = \frac{1}{2} [A_j^l(n+1) + A_j^l(n)] \). The small \( \epsilon \) expansion of \( L_{\text{eff.}}(n) \) reads
\[
L_{\text{eff.}}(n) = L_{\text{cl.}}(n) - \frac{i}{\epsilon} \ln \mathcal{J}(n) + \Delta L(n),
\]
(4.30)
where
\[
L_{\text{cl.}}(n) = \int d^3 \tilde{r} \left[ \text{tr} [\tilde{\mathcal{E}}_j(n) \tilde{\mathcal{E}}_j(n) - B_j(n) B_j(n)] + i \tilde{\psi}(n) \left[ \gamma_4 (\tilde{\psi}(n) + ig A_0(n) \psi(n)) - (\gamma_j D_j(n) + m) \psi(n) \right] \right]
\]
(4.31)
with
\[
\tilde{\mathcal{E}}_j(n) = -\tilde{A}_j(n) - \nabla_j A_0(n) - ig [A_0(n), \tilde{A}_j(n)],
\]
(4.32)
and
\[
\Delta L(n) = \int d^3 \tilde{r} \left[ \frac{1}{8} g^2 \epsilon^2 f^{lm'l'} f^{akt} \hat{\mathcal{E}}_l^a(n) A_0^m(n) A_0^k(n) \left[ \tilde{A}_j^a(n) + \frac{1}{3} D_j^{ab}(n) A_0^b(n) \right] 
- \frac{i}{8} \epsilon g^2(\tilde{r}, l) \left[ \nabla_j D_j(n) \right]^{-1} t^m \tilde{A}_j^m(n) \nabla_j [\nabla_i D_i(n)]^{-1} t^{m'} \tilde{A}_j^{m'}(n) \nabla_i |\tilde{r}, l) 
+ \frac{i}{24} \delta^3(0) C_2 g^2 A_0^l(n) A_0^l(n) + \frac{i}{2} \epsilon g^2 \tilde{\psi}(n) \gamma_4 T^l T^m \psi(n) A_0^l(n) A_0^m(n) \right] + O(\epsilon^3)
\]
(4.33)
with \( t^l \) the generator in the adjoint representation, \((t^l)^{ab} = if^{alb} \). The first term of the integrand of \( \Delta L(n) \) comes from the \( \epsilon \) expansion of the color electric field (4.24), the second term from the shift of the Jacobian \( \mathcal{J}(n) \), the last term of (4.28), the third term comes from the Haar measure \( h \) and the last term from the \( \epsilon \) expansion of the fermionic part of (4.23). We may notice the close resemblance of the first two terms of (4.33) with (2.38).

**IV.3 Converting \( \Delta L \) into an equivalent potential**

Following the recipe of Section III, the potential energy which is equivalent to \( \Delta L(n) \) in the limit \( \epsilon \to 0 \) is

\[
\mathcal{V} = -< \Delta L(n) >_{\text{Gauss}}
\]
\[
\equiv -\frac{\int \prod_{\tilde{r}, j, l} d\tilde{A}_j^l(n,\tilde{r}) dA_0^l(n,\tilde{r}) \delta(\nabla_j A_j^l(n,\tilde{r})) e^{i\epsilon} \int d^3 \tilde{r} \tilde{\epsilon}_j(n,\tilde{r}) \tilde{\epsilon}_j(n,\tilde{r}) \Delta L(n)}{\int \prod_{\tilde{r}, j, l} d\tilde{A}_j^l(n,\tilde{r}) dA_0^l(n,\tilde{r}) \delta(\nabla_j A_j^l(n,\tilde{r})) e^{i\epsilon} \int d^3 \tilde{r} \tilde{\epsilon}_j(n,\tilde{r}) \tilde{\epsilon}_j(n,\tilde{r})}
\]
(4.34)
while regarding $\tilde{A}_j(n, \vec{r})$ constant. The Gauss average of a product of $\dot{A}_j$ and $A_0$ can be decomposed by Wick’s theorem. We have

$$< \mathcal{E}_i^a(\vec{r}) \mathcal{E}_j(\vec{r}') >_{\text{Gauss}} = \frac{i}{\epsilon} \delta_{ij}(\vec{r}, a|\vec{r}', b) = \frac{i}{\epsilon} \delta_{ij} \delta^{ab} \delta^3(\vec{r} - \vec{r}'), \quad \text{(4.35)}$$

$$< A_0^a(\vec{r}) A_0^b(\vec{r}') >_{\text{Gauss}} = -\frac{i}{\epsilon} (\vec{r}, a|G\nabla^2 G|\vec{r}', b), \quad \text{(4.36)}$$

$$< A_0^a(\vec{r}) \dot{A}_j^b(\vec{r}') >_{\text{Gauss}} = -\frac{i}{\epsilon} \left[ (\vec{r}, a|G\nabla_j G|\vec{r}', b) + (\vec{r}, a|G\nabla^2 G D_j|\vec{r}', b) \right], \quad \text{(4.37)}$$

$$< \dot{A}_i^a(\vec{r}) \dot{A}_j^b(\vec{r}') >_{\text{Gauss}} = \frac{i}{\epsilon} \left[ \delta_{ij} \delta^{ab} \delta^3(\vec{r} - \vec{r}') + (\vec{r}, a|\nabla_i G D_j|\vec{r}', b) \\
+ (\vec{r}, a|D_i G\nabla_j G|\vec{r}', b) + (\vec{r}, a|D_i G\nabla^2 G D_j|\vec{r}', b) \right], \quad \text{(4.38)}$$

where we have suppressed the $n$-dependence and $G = (-\nabla_j D_j)^{-1}$ with $D$ from here on to the end of the section defined at $\tilde{A}_j(n, \vec{r})$. Substituting (4.35)-(4.38) into (4.33), we obtain

$$V = -\frac{1}{24} C_2 g^2 \delta^3(0) \int d^3 \vec{r}(\vec{r}, l|G\nabla^2 G|\vec{r}, l)$$

$$+ \frac{1}{8} g^2 f^{kam} f^{nal} \int d^3 \vec{r}(\vec{r}, l|G\nabla_j G|\vec{r}, k)(\vec{r}, m|G\nabla_j G, n)$$

$$- \frac{1}{4} g^2 f^{km} f^{nbl} \int d^3 \vec{r} \int d^3 \vec{r}'(\vec{r}, l|G\nabla_i G|\vec{r}', k)(\vec{r}, n|G\nabla_j G|\vec{r}', m)(\vec{r}, b|D_j G\nabla_i G|\vec{r}', a)$$

$$+ \frac{1}{8} g^2 f^{kam} f^{nbl} \int d^3 \vec{r} \int d^3 \vec{r}'(\vec{r}, l|G\nabla_i G|\vec{r}', k)(\vec{r}', m|G\nabla_j G|\vec{r}, n)(\vec{r}', a|D_i G\nabla^2 G D_j|\vec{r}, b)$$

$$+ \frac{3}{8} C_2 g^2 \delta^3(0) \int d^3 \vec{r}(\vec{r}, m|G\nabla^2 G|\vec{r}, m)$$

$$+ \frac{1}{8} g^2 f^{kam} f^{nma} \int d^3 \vec{r}(\vec{r}, k|G\nabla_j G|\vec{r}, l)(\vec{r}, m|G\nabla_j G, n)$$

$$- \frac{1}{8} g^2 f^{kam} f^{nma} \int d^3 \vec{r}(\vec{r}, k|G\nabla_j G|\vec{r}, l)(\vec{r}, m|G\nabla_j G, n)$$

$$+ \frac{1}{12} g^2 f^{lmk} f^{ank} \int d^3 \vec{r}'[(\vec{r}, m|G\nabla_j G|\vec{r}', l)(\vec{r}, a|D_j G\nabla^2 G|\vec{r}', n)$$

$$(\vec{r}, a|D_j G\nabla_j G|\vec{r}', l)(\vec{r}, m|G\nabla^2 G|\vec{r}, n) + (\vec{r}, n|G\nabla_j G|\vec{r}, l)(\vec{r}, a|D_j G\nabla^2 G|\vec{r}, m)$$

$$- \frac{1}{2} g^2 \int d^3 \vec{r}(\vec{r}, l|G\nabla^2 G|\vec{r}, m) \bar{\psi}(\vec{r}) \gamma_4 T^l T^m \psi(\vec{r}). \quad \text{(4.39)}$$
The first term is the Wick contraction of the Haar measure term of (4.33), the second to the fourth terms are from the jacobian of the gauge fixing, i.e., the second term of (4.33), the fifth to the eighth terms are from the color electric field energy, i.e., the first term of (4.33) and the last term is from the fermion part. This lengthy expression can be simplified with the aid of the following two Jacobian identities, i.e.

\[ f_{abc} \int d^3 \mathbf{r} (\mathbf{r}, a|D_i|X)(\mathbf{r}, b|Y)(\mathbf{r}, c|Z) + (\mathbf{r}, a|X)(\mathbf{r}, b|D_j|Y)(\mathbf{r}, c|Z) \]

\[ + (\mathbf{r}, a|X)(\mathbf{r}, b|Y)(\mathbf{r}, c|D_j|Z) = 0, \]  
(4.40)

\[ f_{abl} f_{ckl} + f_{bcl} f_{akl} + f_{cal} f_{bkl} = 0. \]  
(4.41)

First of all, the seventh term of (4.39) is already of the form of Christ-Lee’s \( V_1 \). The covariant derivative \( D_j \) of the third term may be moved into the middle factor of the integrand according to (4.40), and the result will cancel with the second and the sixth terms through (4.41). Upon repeat applications of (4.40) and (4.41), the first, forth, fifth and eighth terms will combine into Christ-Lee’s \( V_2 \). We have finally

\[ V = V_1 + V_2 + V_3, \]  
(4.42)

where

\[ V_1 = \frac{1}{8} g^2 \int d^3 \mathbf{r} (\mathbf{r}, l'|G \nabla_j |\mathbf{r}, l)(\mathbf{r}, m|G \nabla_j t'|t'| |\mathbf{r}, m), \]  
(4.43)

\[ V_2 = -\frac{1}{8} g^2 \int d^3 \mathbf{r}' \int d^3 \mathbf{r}'' (\mathbf{r}'', l'|(\delta_{ii} + D_i G \nabla_i )|\mathbf{r}', n)(\mathbf{r}, l|(\delta_{ii} + D_i G \nabla_i )|\mathbf{r}'', n') \times (\mathbf{r}, n|G\nabla^2 G t''|t'', n') \]  
(4.44)

and

\[ V_3 = -\frac{1}{2} g^2 \int d^3 \mathbf{r} \bar{\psi}(\mathbf{r}) \gamma_4 T^l T^m \psi(\mathbf{r})(\mathbf{r}, l|G\nabla^2 G |\mathbf{r}, m). \]  
(4.45)

The terms \( V_1 \) and \( V_2 \) are the Christ-Lee operator ordering terms for a pure gauge theory. The term \( V_3 \) is new and its expansion in \( g \) reads

\[ V_3 = -\frac{1}{2} g^2 \int d^3 \mathbf{r} \left[ \delta_{lm} (\mathbf{r}|\nabla^{-2} |\mathbf{r}) + 3 g^2 \int_{l''k} f_{l''k} f_{klnm} (\mathbf{r}|\nabla^{-2} A_i l'' |\nabla^{-2} A_j |\nabla^{-2} A_j l'' |\nabla^{-2} |\mathbf{r}) \right. \]

\[ + O(g^3 A^3) \right] \bar{\psi}(\mathbf{r}) \gamma_4 T^l T^m \psi(\mathbf{r}), \]  
(4.46)

where the term linear in \( A_j^l \) vanishes because of the transversality. Operator-wise, this term stems from the normal ordering of the four fermion coupling
in the color Coulomb potential, which is necessary for the passage from the canonical formulation to the path integral. The details will be explained in Appendix C. The effective Lagrangian in the path integral (4.27) is replaced by the following Lagrangian of Christ-Lee type in the limit $\epsilon \to 0$

$$L(n) = L_{\text{cl}}(n) - \frac{i}{\epsilon} \ln \mathcal{J}(n) - V_1(n) - V_2(n) - V_3(n), \quad (4.47)$$

The formulation of this section for the Coulomb gauge can be easily generalized to an arbitrary noncovariant gauge introduced in [3]

$$\int d^3\!\mathbf{r}'(\mathbf{r}, l|\Gamma_j|\mathbf{r}', l') A^l_j(\mathbf{r}') = 0. \quad (4.48)$$

Since $\epsilon$ is the only dimensional parameter in the formal manipulation of this section, it would be expected that $A_0(n, \mathbf{r}) = O(\epsilon^{-1})$ on dimensional grounds, different from the estimate of $\xi$ for the soluble model and the estimate (4.26). On the other hand, the field theory in $D = 4$ suffers from the ultraviolet divergences which have to be regularized in order for the path integral to make sense. The validity of the estimate $A_0(n, \mathbf{r}) = O(\epsilon^{-\frac{1}{2}})$ as well as the Christ-Lee path integral depends on an implicit assumption that there is a fixed ultraviolet length, which makes the summation over all physical degrees of freedom finite, in the process $\epsilon \to 0$. If $\epsilon$ is identified with the ultraviolet length as in the discrete time regularization scheme of the next section, the $\epsilon$-expansion can no longer be truncated.

V. The BRST Identity and the Discrete Time Regularization

Neglecting fermion couplings, the Lagrangian of a nonabelian gauge field with discrete times reads

$$L(n) = \int d^3\!\mathbf{r} \text{tr}[\mathcal{E}_j(n)\mathcal{E}_j(n) - \mathcal{B}_j(n)\mathcal{B}_j(n)] \quad (5.1)$$

with $\mathcal{E}_j$ and $\mathcal{B}_j$ given by (4.24) and (4.25). The corresponding path integral is

$$< V|e^{-iHt}| > = \text{const.} \int \prod_n [dAdbdcd\mathcal{\bar{c}}]_n e^{i\epsilon \sum_n L_{\text{BRST}}(n)} < A_j(0)|, \quad (5.2)$$

where

$$[dAdbdcd\mathcal{\bar{c}}]_n = \prod_{\mathbf{r}, \mu, l} dA^l_\mu(n, \mathbf{r}) db^l(n, \mathbf{r}) dc^l(n, \mathbf{r}) d\mathcal{\bar{c}}^l(n, \mathbf{r}) h(n, \mathbf{r}) \quad (5.3)$$
\[ L_{\text{BRST}}(n) = L(n) + \int d^3\vec{r} \delta A_j^l(n, \vec{r}) \nabla_j A_j^l(n, \vec{r}) - \int d^3\vec{r} \delta \bar{c}^l(n, \vec{r}) [\nabla_j D_j(n, \vec{r})]^l r^l c^l(n, \vec{r}). \]  

(5.4)

The Lagrangian (5.4) and the integration measure of (5.3) are invariant under the following transformation:

\[ \delta A_j^l(n, \vec{r}) = s_n D_j^l(n, \vec{r}) c^l(n, \vec{r}), \]  

(5.5)

\[ \delta e^{ie^g A_0(n, \vec{r})} = ig s_n c^l(n, \vec{r}) T^l e^{ie^g A_0(n, \vec{r})} - ig s_{n+1} c^l(n + 1, \vec{r}) e^{ie^g A_0(n, \vec{r})} T^l, \]  

(5.6)

\[ \delta c^l(n, \vec{r}) = -\frac{1}{2} s_n g_{f^{lab} c^a(n, \vec{r}) c^b(n, \vec{r})}, \]  

(5.7)

\[ \delta \bar{c}^l(n, \vec{r}) = s_n b^l(n, \vec{r}), \]  

(5.8)

and

\[ \delta b^l(n, \vec{r}) = 0, \]  

(5.9)

where \( s_n \) is a Grassmann number. For a \( n \)-independent \( s_n \), a nilpotent charge operator can be extracted and the transformation (5.5)-(5.9) is therefore of BRST type. Introducing the generating functional of the connected Green’s functions via a source term, i.e.

\[ e^{iW(J, \eta, \bar{\eta})} = \lim_{T \to \infty} < | e^{-iHT} | > \]

\[ = \text{const.} \int \prod_n [dAdbdc\bar{c}]_n < | A(N) > e^{it \sum_n [L_{\text{BRST}}(n) + L_{\text{ext.}}(n)]} < | A(0) > \]  

(5.10)

with

\[ L_{\text{ext.}}(n) = 2 \int d^3\vec{r} \text{tr} [J_\mu(n, \vec{r}) A_\mu(n, \vec{r}) + \bar{\eta}(n, \vec{r}) c(n, \vec{r}) + \bar{c}(n, \vec{r}) \eta(n, \vec{r})]. \]  

(5.11)

The invariance of (5.3) and (5.4) under (5.5)-(5.9) implies the following BRST identity [11]

\[ \int d^3\vec{r} \text{tr} \bar{J}(n, \vec{r}) \cdot < \bar{D} c(n, \vec{r}) > - < D_0 J_0(n, \vec{r}) > + ig \bar{\eta}(n, \vec{r}) < c^2(n, \vec{r}) > 

+ < b(n, \vec{r}) > \eta(n, \vec{r})] = 0, \]  

(5.12)

where

\[ \bar{D} c(n, \vec{r}) = \bar{\nabla} c(n, \vec{r}) - ig [\bar{A}(n, \vec{r}), c(n, \vec{r})] \]  

(5.13)
\[ D_0 J_0(n, \vec{r}) = J_0(n, \vec{r}) + ig[A_0(n, \vec{r}), J_0(n, \vec{r})]. \] (5.14)

The transformation law of \( A_0(n, \vec{r}) \), deduced from (5.6),
\[ \delta A_0(n, \vec{r}) = -D_0 \theta(n, \vec{r}) + \frac{1}{12} g^2 \epsilon^2 [A_0(n, \vec{r}), [A_0(n, \vec{r}), \dot{\theta}(n, \vec{r})]] + \ldots \] (5.15)

has been utilized, only the first term of which contributes to the limit \( \epsilon \to 0 \).

The identity (5.12) can be cast into various useful forms [11].

Similar to the case of the soluble model, the BRST identity can also be constructed from the Slavnov-Taylor identity of the Christ-Lee type of path integral (4.27) with \( L_{\text{eff}}(n) \) replaced by \( L(n) \) of (4.47) in the limit \( \epsilon \to 0 \).

Unlike the soluble model, the field theory case suffers from an ultraviolet divergence which needs to be regularized and subtracted. Owing to its manifest BRST invariance, the discrete time Lagrangian (4.23) with (4.24) and (4.25) serves also as a gauge invariant regularization scheme with \( \epsilon \) an ultraviolet cutoff. There are several additional technical advantages with this regularization. 1) The energy integration with a continuum time is regularized by the summation over the Bloch momentum on the temporal lattice. This is particularly important for resolving the ambiguities associated with the energy divergence. 2) With fixed Bloch momenta on the temporal lattice, the integration over spatial momenta is less divergent. There is only a finite number of divergent skeletons and these can be handled by the dimensional regularization; 3) For fixed lattice momenta, the integrand of each Feynman diagram is a rational function of the spatial momenta and can be simplified with the aid of Feynman parametrization; 4) Manifest unitarity is maintained throughout the calculation. In what follows, we shall test this regularization by an evaluation of the one loop Coulomb propagator in the absence of the quark fields.

The expansion of the Lagrangian (5.3) according to the power of \( g^2 \) reads

\[ L_{\text{BRST}}(n) = L_{\text{cl}}(n) + \int d^3 \vec{r} \delta_{l}^j \nabla_j A_0^l(n, \vec{r}) - \int d^3 \vec{r} \bar{c}^l \nabla_j [D_j(n, \vec{r})] c^{l'}(n, \vec{r}) + R_n, \] (5.16)

where

\[ R_n = \int d^3 \vec{r} \left[ -\frac{1}{8} g^2 \epsilon^2 f_{lmn}^l f_{akl}^l \left( \dot{A}_j^l(n) A_0^m(n) A_0^k(n) \dot{A}_j^l(n) \right) + \frac{1}{3} \nabla_j A_0^l(n) A_0^m(n) A_0^k(n) \nabla_j A_0^l(n) \right] + \frac{i}{24} \epsilon \delta^3(0) C_2 g^2 A_0^l(n) A_0^l(n) \] (5.17)
where at the order $g^2$, only terms of an even number of $A_0$ factor are kept. The first term of (5.17) comes from the expansion of $e^{i\epsilon gA_0}$ in the color electric field and the second from the Haar measure. Both of them have been included in $\Delta L(n)$ of (4.33). For the reason we shall explain later, the perturbative expansion ought to be performed in Euclidean space, which amounts to the substitutions $\epsilon \rightarrow -i\epsilon$, $A_0 \rightarrow -iA_4$ and $\dot{A}_j \rightarrow i\frac{\partial A_j}{\partial x^4}$. The dressed Coulomb propagator reads

$$d_0'(k_0, \vec{k}) = \frac{1}{k^2 + \sigma(k_0, \vec{k})},$$

where the one loop contribution to $\sigma(k_0, \vec{k})$ is given by the amputated Feynman diagrams of Fig. 1. plus the contribution of (5.17), i.e.

$$\sigma(k_0, \vec{k}) = -\left( \text{Fig. 1a} + \text{Fig. 1b} + \text{Fig. 1c} \right) + \text{contribution from } R_n \quad (5.18)$$

with the relevant Feynman rules given in Fig. 2. A wavy line stands for a transverse gluon propagator and contributes a factor

$$\delta_{ll'} d_{ij}(\theta|\vec{k}) = \frac{\delta_{ll'} k^2}{k^2 + k^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \quad (5.19)$$

with $k_0 = \frac{2}{\epsilon} \sin \frac{\theta}{2}$ and $\theta \in (-\pi, \pi)$ a Bloch momentum; a dashed line stands for a bare Coulomb propagator and contributes a factor

$$\delta_{ll'} d_0(\vec{k}) = \frac{\delta_{ll'}}{k^2}. \quad (5.20)$$

A three point vertex of one Coulomb line and two transverse gluons with incoming momenta $(\theta_1, \vec{k}_1)$, $(\theta_2, \vec{k}_2)$ and $(\theta_3, \vec{k}_3)$ is associated with the factor

$$-i \frac{2}{\epsilon} g f^{lmn} \delta_{ij} \sin \frac{\theta_3 - \theta_2}{2}, \quad (5.21)$$
a three point vertex of two Coulomb lines and one transverse gluon with incoming momenta $(\theta_1, \vec{k}_1)$, $(\theta_2, \vec{k}_2)$ and $(\theta_3, \vec{k}_3)$ is associated with the factor

$$-igf^{lmn}(k_{2j} - k_{1j}) \cos \frac{\theta_3}{2}. \quad (5.22)$$

A four point vertex of two transverse gluons and two Coulomb lines is associated with the factor

$$-g^2(f^{la'a} f^{lb'b} + f^{la'b} f^{lb'a}) \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \delta_{ij}. \quad (5.23)$$

With these rules, we have

$$\text{Fig.1a} = \frac{1}{2} C_{2\ell''l'} g^2 \frac{4}{\epsilon^2} \int_{-\pi}^{\pi} \frac{d\theta'}{2\pi \epsilon} \sin^2 \theta + \theta' I, \quad (5.24)$$
\[ I = \int \frac{d^3 \vec{p}}{(2\pi)^3} d_{ij}(\theta | \vec{p}) d_{ij}(\theta' | \vec{p}') \]

\[ = 3! \int \frac{d^3 \vec{p}}{(2\pi)^3} \int_0^1 dx \int_0^1 dy \int_0^1 dz (1-z) \frac{\vec{p}^2 (\vec{p} + \vec{k})^2 + [(\vec{p} \cdot (\vec{p} + \vec{k})]^2}{[(\vec{p} + \vec{k}z)^2 + \vec{p}_0^2 x(1-z) + \vec{p}_0^2 yz + \vec{k}^2 z(1-z)]^4} \]  

(5.25)

with \( p_0 = \frac{2}{\epsilon} \sin \frac{\theta}{2} \), \( p'_0 = \frac{2}{\epsilon} \sin \frac{\theta'}{2} \) and \( \phi = \theta' - \theta \), \( \vec{k} = \vec{p}' - \vec{p} \) the external energy and momentum. Similarly,

\[ \text{Fig. 1b} = C_2 \delta^{ll'} g^2 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi \epsilon} \cos^2 \frac{\theta}{2} II, \]  

(5.26)

where

\[ II = \int \frac{d^3 \vec{p}}{(2\pi)^3} (p + 2k)_i (p + 2k)_j d_{ij}(\theta | \vec{p}) d_0(\vec{p}') \]

\[ = 8 \int \frac{d^3 \vec{p}}{(2\pi)^3} \int_0^1 dx \int_0^1 dz (1-z) \frac{\vec{p}^2 \vec{k}^2 - (\vec{p} \cdot \vec{k})^2}{[(\vec{p} + \vec{k}z)^2 + \vec{p}_0^2 x(1-z) + \vec{k}^2 z(1-z)]^3} \]  

(5.27)

and

\[ \text{Fig. 1c} = -C_2 \delta^{ll'} g^2 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi \epsilon} \cos^2 \frac{\theta}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} d_{jj}(\theta | \vec{p}) \]

\[ = -2C_2 \delta^{ll'} g^2 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi \epsilon} \cos^2 \frac{\theta}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{\vec{p}_0^2 + \vec{p}^2} \]  

(5.28)

and

\[ \text{Contribution of } R_n = -C_2 \delta^{ll'} g^2 \frac{26}{12} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi \epsilon} \int \frac{d^3 \vec{p}}{(2\pi)^3} \int_0^1 dx \frac{(24 \sin^2 \frac{\theta}{2} + \vec{k}^2 \epsilon^2) \vec{p}_0^2 + \vec{p}_0^2 \vec{k} \epsilon^2}{(\vec{p}^2 + \vec{p}_0^2 x)^2}. \]  

(5.29)

We shall not expose the details of the evaluation of (5.25)-(5.29), but only remark on a few key points which lead to the final answer. First of all, the \( \vec{p} \)-integrations in (5.27)-(5.29) are all linearly divergent, which upon the replacement

\[ \int \frac{d^3 \vec{p}}{(2\pi)^3} \to \int \frac{d^D \vec{p}}{(2\pi)^D}, \]  

(5.30)

give rise to Gamma functions with arguments of the form \( \frac{D}{2} + \text{integer} \), and therefore yield finite limits as \( D \to 3 \). After the \( \vec{p} \)-integration, the integrand for \( \theta \)-integration is of the dimension of a momentum. Because of the \( \epsilon \) of the denominator of the Bloch momentum \( p_0 \), the leading divergence as \( \epsilon \to 0 \) is
of the order of $\epsilon^{-2}$, which reflects the usual quadratic divergence. At $\vec{k} = 0$, we obtain that

\begin{align}
(5.24) &= \frac{2}{3\pi^2\epsilon^2} C g^2 \delta_{ll'}, \\
(5.26) &= 0, \\
(5.28) &= \frac{2}{3\pi^2\epsilon^2} C g^2 \delta_{ll'},
\end{align}

and

\begin{align}
(5.29) &= \frac{4}{3\pi^2\epsilon^2} C g^2 \delta_{ll'}. \quad (5.31)
\end{align}

If follows from (5.18) that

\begin{align}
\sigma(k_0, \vec{k}) &= 0,
\end{align}

which renders the net divergence logarithmic. After some manipulations, we obtain that

\begin{align}
\sigma(0, \vec{k}) &= -\frac{11}{24\pi^2} C g^2 \vec{k}^2 \left( \ln \frac{1}{k\epsilon} - \frac{74}{33} - \frac{91}{22} \ln 2 \right), \quad (5.36)
\end{align}

and the one loop renormalized Coulomb propagator reads

\begin{align}
d'_0(\vec{k}) &= \frac{Z}{k^2} \quad (5.37)
\end{align}

with

\begin{align}
Z = 1 + \frac{11}{24\pi^2} C g^2 \left( \ln \frac{1}{k\epsilon} - \frac{74}{33} - \frac{91}{22} \ln 2 \right), \quad (5.38)
\end{align}

the divergent part of which coincides with the charge renormalization [1] [2].

We end this section with two technical remarks:

1). Euclidean time is adapted for the above one loop calculation. This turns out to be necessary for the logarithmically divergent diagrams with the integration order we followed. Consider a simple integral with a Minkowski momentum $p = (p_0, \vec{p})$

\begin{align}
I = \int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^2 + m^2)^2}, \quad (5.39)
\end{align}

with $p^2 = \vec{p}^2 - p_0^2$. If the Wick rotation is performed before the spatial integration, the infinite arc, $p_0 = Re^{i\phi}$ with $R \to \infty$, $0 < \phi < \frac{\pi}{2}$ and $\pi < \phi < \frac{3\pi}{2}$ will not contribute. But if the spatial momentum is integrated first as we did, the Wick rotation then will pick up a term from the infinite arc. As a result, the renormalization constant will be complex unless we start with the Euclidean definition of the diagram.
2). It may looks puzzling that the very terms of \( R_n \) which help to cancel the quadratic divergence of the one loop diagrams of the Coulomb propagator are actually the same terms which contribute to the Christ-Lee anomalous vertices which are expected at two loop level. This paradox is tied to the identification \( \epsilon \) with the ultraviolet cutoff. If an independent ultraviolet cutoff is introduced for the integration over \( \vec{p} \) and the limit \( \epsilon \to 0 \) is taken before sending the cutoff to infinity, the contribution of \( R_n \) to the one loop Coulomb propagator, (5.28), will vanish as can be seen easily.

VI. Concluding Remarks

In this work, we have carefully traced all the subtleties of gauge fixing and variable transformation in a path integral of a gauge model, without resorting to the operator formalism. For a soluble quantum mechanical model in \( \lambda \)-gauge and for a nonabelian gauge field in Coulomb gauge, the well known operator ordering terms are reproduced exactly. In the presence of fermionic degrees of freedom, an additional operator ordering term is discovered. Because of the intrinsic nonlinearity of a BRST transformation, the operator ordering terms are found essential in restoring the simple form of the identity associated with this transformation. In the field theory case, a manifest BRST invariant and unitary regularization scheme is proposed and it does give rise to the correct \( \beta \)-function at one loop order.

Though this work does not attempt to prove the renormalizability of a nonabelian gauge theory in Coulomb gauge, I do not see any problems in applying the discrete time regularization scheme to higher orders. The only draw back is that the \( \epsilon \)-expansion of the temporal lattice Lagrangian can no longer be truncated since the ultraviolet cutoff is identified with \( \epsilon \).

Alternatively, one may try to renormalize the theory with a Christ-Lee type of path integral. Then one has to face the energy divergence and the ambiguities associated with it. The coupling with the ultraviolet divergence makes it difficult to organize the cancellation in higher orders. Several scenarios have been proposed but none of them \cite{12} goes smoothly beyond two loops. On the other hand, the energy divergence is an artifact of the path integral, since it is not there with canonical perturbation methods. In principle, one should be able to organize the energy integral before integrating spatial momenta and to reproduce the canonical perturbation series. But then no advantages of Feynman diagrams have been taken and the path integral seems unnecessary. What we need for the renormalization with a Christ-Lee type of path integral is an unambiguous scheme which regularize the spatial loop integral. The only feasible BRST invariant scheme is a
spatial lattice.

At this point, it is instructive to draw some connections of the path integral in the continuum with Wilson’s lattice formulation \[13\]. In the absence of quarks, the partition function of Wilson’s formulation on a four dimensional rectangular lattice reads

\[ Z = \int \prod_{<ij>} dU_{ij} e^{-\frac{1}{g^2} S_W[U]}, \quad (6.1) \]

where \( U_{ij} \) is a gauge group matrix on a nearest neighbor link. The simplest choice of the action is

\[ S_W[U] = \frac{1}{g^2} \left[ \frac{a_s}{a_t} \sum_{P_t} \text{tr} \left( 1 - \frac{1}{d} \text{Re} U_{P_t} \right) + \frac{a_t}{a_s} \sum_{P_s} \text{tr} \left( 1 - \frac{1}{d} \text{Re} U_{P_s} \right) \right], \quad (6.2) \]

where \( U_P = U_{ij} U_{jk} U_{kl} U_{li} \) for a plaquette \( Pijkl \) with the subscript \( s \) labeling the space-like one and \( t \) time-like one, \( a_s, a_t \) denote the spatial and temporal lattice spacings and \( d \) the dimension of \( U \)'s. The lattice Coulomb gauge condition can be imposed as in Ref. 14. The discrete time regularization scheme presented in Section 5 corresponds to the limit \( a_t \to 0 \) after \( a_s \to 0 \) and any regularization corresponding to Christ-Lee path integral follow from the limit \( a_s \to 0 \) after \( a_t \to 0 \).

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Appendix A

To estimate the typical contributions of \( \dot{x}, \dot{y}, \dot{z} \) and \( \xi \) to the path integral in the limit \( \epsilon \to 0 \), we may neglect the interaction term and consider the path integral with the free Lagrangian only,

\[ L_0(n) = \frac{1}{2} \left[ \dot{x}_n^2 + \dot{y}_n^2 + (\dot{z}_n - \xi_n)^2 - (\omega^2 - i0^+)(x_n^2 + y_n^2) - \frac{1}{a}(z_n - \lambda x_n - \kappa \dot{\xi}_n)^2 \right], \quad (A.1) \]
where the total time interval \( T = N\epsilon \to \infty \) with \( N \) the number of time slices between the time interval \( T \), and the infinitesimal imaginary part of \( \omega \) provides a converging factor of the integral. The last term of (A.1) is the gauge fixing term (2.49) with the gauge parameter \( a \). Defining the path integral average of an arbitrary function of \( x_n, y_n, z_n \) and \( \xi_n \) by

\[
\langle F \rangle = \frac{\prod_{n} dx_n dy_n dz_n d\xi_n e^{i\epsilon \sum_n L_0(n)} F}{\prod_{n} dx_n dy_n dz_n d\xi_n e^{i\epsilon \sum_n L_0(n)}}, \quad (A.2)
\]

we obtain the following expressions for various propagators:

\[
\langle x_n x_m \rangle = \langle y_n y_m \rangle = \frac{1}{\epsilon} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{i e^{-i(n-m)\theta}}{p^* p - \omega^2 + i0^+}, \quad (A.3)
\]

\[
\langle z_n z_m \rangle = \frac{1}{\epsilon} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} i \frac{\lambda^2 - (a - \kappa^2 p^* p)(p^* p - \omega^2)}{(1 + \kappa p^2)(1 + \kappa p^2)(p^* p - \omega^2 + i0^+)} e^{-i(n-m)\theta}, \quad (A.4)
\]

\[
\langle \xi_n \xi_m \rangle = \frac{1}{\epsilon} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} i \frac{\lambda^2 p^* p + (1 - a p^* p)(p^* p - \omega^2)}{(1 + \kappa p^2)(1 + \kappa p^2)(p^* p - \omega^2 + i0^+)} e^{-i(n-m)\theta}, \quad (A.5)
\]

\[
\langle x_n \xi_m \rangle = -\frac{1}{\epsilon} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{\lambda p^* e^{-i(n-m)\theta}}{(1 + \kappa p^2)(p^* p - \omega^2 + i0^+)}, \quad (A.6)
\]

\[
\langle x_n \xi_m \rangle = 1 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{\lambda e^{-i(n-m)\theta}}{(1 + \kappa p^2)(p^* p - \omega^2 + i0^+)} \quad (A.7)
\]

and

\[
\langle x_n \xi_m \rangle = -\frac{1}{\epsilon} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} i \frac{\lambda^2 p^* - (\kappa p + a p^* p)(p^* p - \omega^2)}{(1 + \kappa p^2)(1 + \kappa p^2)(p^* p - \omega^2 + i0^+)} e^{-i(n-m)\theta}, \quad (A.8)
\]

where \( p = \frac{i\epsilon^{-i\theta}}{\epsilon-1} \). According to the definition of \( \dot{x}_n \) and \( \dot{y}_n \), we have

\[
\langle \dot{x}_n \dot{x}_m \rangle = \langle \dot{y}_n \dot{y}_m \rangle = \frac{1}{\epsilon} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \frac{\lambda e^{-i(n-m)\theta}}{p^* p - \omega^2 + i0^+}. \quad (A.9)
\]

The squares of the typical magnitude of \( x_n, y_n, z_n, \dot{x}_n, \dot{y}_n, \dot{z}_n \) and \( \xi_n \) inside the path integral in the limit \( \epsilon \to 0 \) are of the same order as the expectation value of their squares, i.e. the propagators (A.3)-(A.5) at \( n = m \). It follows from (A.3) and (A.9) that

\[
\langle x_n^2 \rangle = \langle y_n^2 \rangle = \frac{1}{2\omega} + O(\epsilon^2), \quad (A.10)
\]
but
\[ <\dot{x}_n^2> = <\dot{y}_n^2> = \frac{i}{\epsilon} + \frac{\omega}{2} + O(\epsilon^2) \]  
(A.11)

for arbitrary \( \lambda, \kappa \) and \( a \). On the other hand, the \( \epsilon \to 0 \) limit of \( <z_n^2> \), \( <\dot{z}_n^2> \) and \( <\xi_n^2> \) are very delicate and we consider the following situations.

1) None of \( \kappa \) or \( a \) vanishes. It follows from (A.4) and (A.5) that

\[ <z_n^2> = \frac{\lambda^2}{2\omega(1 + \kappa \omega^2)^2} + \frac{i}{\kappa} \left[ \frac{\kappa - a}{1 + \kappa \omega^2} - \frac{2\lambda^2 \kappa}{(1 + \kappa \omega^2)^2} \right] + O(\epsilon^2), \]  
(A.12)

\[ <\xi_n^2> = \frac{\lambda^2 \omega}{2(1 + \kappa \omega^2)^2} + \frac{i}{4\kappa \sqrt{\kappa}} \left[ \frac{\kappa - a}{1 + \kappa \omega^2} + \frac{2\lambda^2 \kappa}{(1 + \kappa \omega^2)^2} \right] + O(\epsilon^2) \]  
(A.13)

and

\[ <\dot{z}_n^2> = \frac{i}{\epsilon} + \text{finite terms}. \]  
(A.14)

The small \( \epsilon \) expansion of the lattice Lagrangian (2.27) is trivial with such a gauge fixing. So is the case when \( x_n \) and \( z_n \) in the last term of (A.1) are replaced by \( \bar{x}_n \) and \( \bar{z}_n \).

2) \( \kappa = 0 \) and \( a \to 0 \) before \( \epsilon \to 0 \). This is the \( \lambda \)-gauge in the text. It is easy to show, using (A.3), (A.4) and (A.5) that

\[ <z_n^2> = \lambda^2 <x_n^2> = \frac{\lambda^2}{2\omega}, \]  
(A.15)

\[ <\dot{z}_n^2> = \lambda^2 <\dot{x}_n^2> = \frac{i\lambda^2}{\epsilon} + \text{finite terms} \]  
(A.16)

and

\[ <\xi_n^2> = \frac{i(1 + \lambda^2)}{\epsilon} + \text{finite terms}. \]  
(A.17)

3) \( \kappa = 0 \) but \( a \neq 0 \). This corresponds to the “smeared \( \lambda \)-gauge”. We obtain from (A.4) and (A.5) that

\[ <z_n^2> = -\frac{ia}{\epsilon} + \text{finite terms}, \]  
(A.18)

\[ <\dot{z}_n^2> = -2i \frac{a}{\epsilon^3} + \frac{i \lambda^2}{\epsilon} + \text{finite terms} \]  
(A.19)

and

\[ <\xi_n^2> = -2i \frac{a}{\epsilon^3} + i \frac{1 + \lambda^2}{\epsilon} + \text{finite terms}. \]  
(A.20)

The equations (A.12)-(A.20) give the announced estimates in Section II.
Appendix B

The Hamiltonian of the soluble model in the $\lambda$-gauge is given by [5]

$$H = \frac{1}{2J} \begin{pmatrix} x \quad y \end{pmatrix} \begin{pmatrix} \mathcal{M}^{-1}_{xx} \quad \mathcal{M}^{-1}_{xy} \\ \mathcal{M}^{-1}_{yx} \quad \mathcal{M}^{-1}_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + U(x^2 + y^2) \quad (B.1)$$

after solving the Gauss law constraint, where

$$\mathcal{M}^{-1}_{xx} = J^{-2}(y^2 + \frac{1}{g^2}), \quad (B.2)$$

$$\mathcal{M}^{-1}_{xy} = \mathcal{M}^{-1}_{yx} = J^{-2}x(\frac{\lambda}{g} - y), \quad (B.3)$$

$$\mathcal{M}^{-1}_{yy} = J^{-2}\left[(\frac{\lambda}{g} + \frac{1}{g})^2 + x^2(\lambda^2 + 1)\right] \quad (B.4)$$

and

$$J = \frac{1}{g} + \lambda y. \quad (B.5)$$

It was pointed out that the Hamiltonian (B.1) commutes with the operator

$$K = J^{-1}(xp_y - yp_x), \quad (B.6)$$

i.e. $[H, K] = 0$ [5]. With $U = e^{i\varepsilon K}$, we have

$$x_\varepsilon \equiv UXU^{-1} = x - \varepsilon J^{-1}y \quad (B.7)$$

and

$$y_\varepsilon \equiv UyU^{-1} = y + \varepsilon J^{-1}x \quad (B.8)$$

for infinitesimal $\varepsilon$. Adding the source term

$$h(t) = \kappa(t)K + J_x(t)x(t) + J_y(t)y(t) \quad (B.9)$$

to the Hamiltonian (B.1), the Schroedinger equation of the state is given by

$$i\frac{\partial}{\partial t} \ket{t} = h(t) \ket{t}, \quad (B.10)$$

where the operators follow the time development generated by $H$, e.g.

$$x(t) = e^{iHt}x(0)e^{-iHt} \quad (B.11)$$
and
\[ y(t) = e^{iHt}y(0)e^{-iHt}. \]  
(B.12)

The c-number sources \( \kappa(t), J_x(t) \) and \( J_y(t) \) are adiabatically switched on in the remote past and are switched off in the remote future. It can be shown that
\[ \left( \mathcal{J}^{\frac{1}{2}}K\mathcal{J}^{-\frac{1}{2}} \right)_W = \mathcal{J}^{\frac{1}{2}}K\mathcal{J}^{-\frac{1}{2}}. \]  
(B.13)

with the subscript \( W \) standing for the Weyl ordering. The general solution of (B.10) reads
\[ |t> = U(t,t_0)|t_0> \]  
(B.13)

with
\[ U(t,t_0) = T \exp \left( -i \int_{t_0}^{t} dt' h(t') \right). \]  
(B.14)

Define the generating functional of the connected Green’s functions, \( \mathcal{W}(\kappa, J) \) by
\[ e^{i\mathcal{W}(\kappa, J)} = <|U(\infty, -\infty)|> \]  
(B.15)

with \( |> \) the ground state of the Hamiltonian (B.1), the previously defined one, \( \mathcal{W}(J, \zeta, u, \eta, \bar{\eta}) \) in (3.12) at \( \zeta = u = \eta = \bar{\eta} = 0 \) corresponds to \( \mathcal{W}(0, J) \).

Effecting an infinitesimal transformation (B.7) and (B.8) with \( \varepsilon(t) \to 0 \) at \( t \to \pm \infty \), we have
\[ i\frac{\partial}{\partial t}|t>_\varepsilon = h_{\varepsilon}(t)|t>_\varepsilon, \]  
(B.16)

where
\[ |t>_\varepsilon = U(t)|t>, \]  
(B.17)

and
\[ h_{\varepsilon}(t) = (\kappa - \frac{\partial}{\partial t})K + J_x(t)x_{\varepsilon}(t) + J_y(t)y_{\varepsilon}(t). \]  
(B.18)

Consequently,
\[ |t>_{\varepsilon} = U_{\varepsilon}(t,t_0)|t_0>_{\varepsilon} \]  
(B.19)

with
\[ U_{\varepsilon}(t,t_0) = T \exp \left( -i \int_{t_0}^{t} dt' h_{\varepsilon}(t') \right). \]  
(B.20)

The invariance of the Hamiltonian \( H \) and its ground state under the transformation implies that
\[ <|U_{\varepsilon}(\infty, -\infty) - U(\infty, -\infty)|> = 0, \]  
(B.21)
which, to the linear power of $\varepsilon$ gives
\[
\int_{-\infty}^{\infty} dt \left\{ \frac{\partial \varepsilon}{\partial t} < K > t + \varepsilon(t) \left[ J_x(t) < \frac{gy}{1 + \lambda gy} >_t - J_y(t) < \frac{gx}{1 + \lambda gy} >_t \right] \right\} = 0, \tag{B.22}
\]
where the canonical average $< ... >_t$ is defined as
\[
< O>_t = \frac{< |T[U(\infty, -\infty)O(t)]| >}{< |U(\infty, -\infty)| >}. \tag{B.23}
\]
Converting $< |U(\infty, -\infty)| >$ into the path integral and denoting the path integral average by $< ... >$ without the subscript $t$, we have
\[
< \frac{gy}{1 + \lambda gy} >_t = < \frac{gy(t)}{1 + \lambda gy(t)} >, \tag{B.24}
\]
\[
< \frac{gx}{1 + \lambda gy} >_t = < \frac{gx(t)}{1 + \lambda gy(t)} > \tag{B.25}
\]
and
\[
< K >_t |_{\kappa=0} = -g < \frac{\dot{x}(t)y(t) - x(t)\dot{y}(t) + \lambda g[x^2(t) + y^2(t)]\dot{x}(t)}{1 + g^2[x^2(t) + y^2(t)]} > |_{\kappa=0}. \tag{B.26}
\]
The last equality requires some explanation. In the canonical form, we may write
\[
< K >_t |_{\kappa=0} = -\frac{\delta}{\delta \kappa(t)} \mathcal{W}(\kappa, J)|_{\kappa=0}. \tag{B.27}
\]
On the other hand, the operator $K$ contains the canonical momenta. Performing a Legendre transformation of the Hamiltonian $H + h$ [15], the term of the corresponding Lagrangian which is linear in $\kappa$ reads
\[
\kappa g \frac{\dot{y} - x \dot{y} + \lambda g(x^2 + y^2)\dot{x}}{1 + g^2(x^2 + y^2)} \tag{B.28}
\]
and the equality (B.26) follows from (B.27) with the path integral representation of $\mathcal{W}(\kappa, J)$. Putting back the $\xi$ and $z$, we find
\[
< K >_t |_{\kappa=0} = < \xi(t) - \dot{z}(t) > \tag{B.29}
\]
and the identity (B.22) becomes
\[
\int_{-\infty}^{\infty} dt \left\{ \frac{\partial \varepsilon}{\partial t} < \xi(t) - \dot{z}(t) > + \varepsilon(t) \left[ J_x(t) < \frac{gy(t)}{1 + \lambda gy(t)} > - J_y(t) < \frac{gx(t)}{1 + \lambda gy(t)} > \right] \right\} = 0, \tag{B.30}
\]
For an arbitrary function \( \varepsilon(t) \), the integration sign may be removed after a partial integral and the Slavnov-Taylor identity (3.22) at \( u = \zeta = 0 \) emerges. The relation
\[
\langle b(t) \rangle = \frac{d}{dt} \langle \xi(t) - \dot{z}(t) \rangle,
\]
which can be checked explicitly, is utilized in the final step.

Appendix C

To make the paper self-contained, we shall go through the path integral of fermionic degrees of freedom, following the coherent field treatment of the Ref. [7]. Consider a pair of fermion annihilation and creation operators, \( a \) and \( a^\dagger \), with anticommutator
\[
\{a, a^\dagger\} = 1,
\] (C.1)
The combination \( a^\dagger a a^\dagger a \) is not zero. But if we replace \( a \) and \( a^\dagger \) by a pair of Grassmann numbers \( z \) and \( \bar{z} \), the combination \( \bar{z}z\bar{z}z \) is always zero. Therefore there are ordering ambiguities when transforming the canonical formulation for fermionic operators to the path integral. The question is which order goes through to the path integral simply through the above replacements \( a \rightarrow z \) and \( a^\dagger \rightarrow \bar{z} \). We shall discuss the systematics in the following:

For a system of \( M \) fermionic degrees of freedom, represented by the annihilation and creation operators \( a_j \) and \( a_j^\dagger \) with
\[
\{a_i, a_j\} = 0
\] (C.2)
and
\[
\{a_i, a_j^\dagger\} = \delta_{ij},
\] (C.3)
we introduce two set of independent Grassmann numbers, \( z_1, z_2, \ldots, z_M \) and \( \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_M \). We also specify that they anticommute with the \( a \)'s, \( a^\dagger \)'s and commute with the ket or bra of the ground state in the Hilbert space. Furthermore, the following integration rule is imposed
\[
\int dz_j = \int d\bar{z}_j = 0
\] (C.4)
and
\[
\int z_i dz_j = \int d\bar{z}_i \bar{z}_j = \delta_{ij}.
\] (C.5)
Defining a coherent state by

\[ |z_1, z_2, \ldots, z_M \rangle \equiv e^{\sum_j a_j^\dagger z_j} |0 \rangle \]  \hspace{1cm} (C.6)

and its conjugate by

\[ < \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_M | \equiv 0 | e^{\sum_j \bar{z}_j a_j} \]  \hspace{1cm} (C.7)

with \(|0 \rangle\) the ground state. It follows that

\[ a_j |z_1, z_2, \ldots, z_M \rangle = z_j |z_1, z_2, \ldots, z_M \rangle, \]  \hspace{1cm} (C.8)

\[ < \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_M | a_j^\dagger = < \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_M | \bar{z}_j \]  \hspace{1cm} (C.9)

and

\[ < \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_M | z_1, z_2, \ldots, z_M \rangle = e^{\sum_j \bar{z}_j z_j}. \]  \hspace{1cm} (C.10)

Furthermore, we have the completeness relation

\[ \int |z_1, z_2, \ldots, z_M \rangle \prod_j dz_j d\bar{z}_j e^{-\sum_j \bar{z}_j z_j} < \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_M | = 1. \]  \hspace{1cm} (C.11)

Let the Hamiltonian of the system be

\[ H(a^\dagger, a) = \sum_{i,j} \omega_{ij} a_i^\dagger a_j + \frac{1}{2} \sum_{i'i',j'j} v_{ii',jj'} a_{i'}^\dagger a_{j'}^\dagger a_{j'} a_{i'} + \ldots \]

\[ + \frac{1}{M!} \sum_{i_1, \ldots, i_M, j_1, \ldots, j_1} v_{i_1 \ldots i_M, j_1 \ldots j_1} a_{i_1}^\dagger \ldots a_{i_M}^\dagger a_{j_1} \ldots a_{j_M}, \]  \hspace{1cm} (C.12)

where the normal ordering with respect to the state \(|0 \rangle\) is the crucial point. It follows from (C.8) and (C.9) that

\[ < \bar{z}_1, \ldots, \bar{z}_M | H | z_1, \ldots, z_M \rangle = H(\bar{z}, z) \]

\[ = \sum_{i,j} \omega_{ij} z_j z_j + \frac{1}{2} \sum_{i'i',j'j} v_{ii',jj'} \bar{z}_{i'} \bar{z}_{j'} z_j + \ldots \]

\[ + \frac{1}{M!} \sum_{i_1, \ldots, i_M, j_1, \ldots, j_1} \bar{z}_{i_1} \ldots \bar{z}_{i_M} z_{j_1} \ldots z_{j_M} \]  \hspace{1cm} (C.13)

and therefore

\[ < \bar{z}_1, \ldots, \bar{z}_M | e^{-i\varepsilon H} | z_1, \ldots, z_M \rangle = e^{\sum_j \bar{z}_j z_j} [1 - i\varepsilon H(\bar{z}, z) + O(\varepsilon^2)]. \]  \hspace{1cm} (C.14)
With the aid of the completeness relation (C.11), we end up with the following path integral representation of the fermionic system

\[
< z_1', ..., z_M' | e^{-itH} | z_1, ..., z_M > = \int [dz]_N \prod_{n=1}^{N-1} [d\bar{z}dz]_n [d\bar{z}]_0 e^{i \sum_n L(n)} \tag{C.15}
\]

where \( t = N\epsilon \) and \( \epsilon \to 0 \) at fixed \( t \), and we have made the abbreviation

\[
[d\bar{z}dz]_n = \prod_j d\bar{z}_j(n)dz_j(n), \tag{C.16}
\]

\[
[dz]_N = \prod_j dz_j(N) \tag{C.17}
\]

and

\[
[d\bar{z}]_0 = \prod_j d\bar{z}_j(0) \tag{C.18}
\]

The Lagrangian \( L(n) \) reads

\[
L(n) = i \sum_j \bar{z}_j(n)\dot{z}_j(n) - H(\bar{z}(n), z(n)) \tag{C.19}
\]

with \( \dot{z}_j(n) = \frac{1}{\epsilon}[z_j(n + 1) - z_j(n)] \).

Like bosonic operators, the ordering ambiguity here is also reflected in the difference between the Dyson-Wick contraction and the path integral contraction at equal time. Consider a free system whose Hamiltonian is given by (C.12) with \( \omega_{ij} = \omega \delta_{ij} \) and all \( v \)'s vanishing. The Dyson-Wick contraction gives

\[
\lim_{t \to 0^+} < 0 | T(a(t)a_0^\dagger(0)) | 0 >= 1 \tag{C.20}
\]

while

\[
\lim_{t \to 0^-} < 0 | T(a(t)a_0^\dagger(0)) | 0 >= 0. \tag{C.21}
\]

The path integral, on the other hand, gives rise to an unambiguous result at \( t = 0 \) since

\[
S_{ij} = \frac{\int [dz]_N \prod_{n=1}^{N-1} [d\bar{z}dz]_n [d\bar{z}]_0 z_i(m) \bar{z}_j(m) e^{i \epsilon \sum_n L(n)}}{\int [dz]_N \prod_{n=1}^{N-1} [d\bar{z}dz]_n [d\bar{z}]_0 e^{i \epsilon \sum_n L(n)}} = \delta_{ij} \frac{1}{\epsilon} \int_{-\pi}^{\pi} d\theta \frac{i}{2\pi \frac{e^{-i\theta}-1}{\epsilon} - \omega + i0^+} = 0. \tag{C.22}
\]
To illustrate the caution which is needed in transforming the canonical formulation to path integral, we consider a soluble gauge model whose Lagrangian is given by

$$L = \frac{1}{2}(\dot{z} - \xi)^2 + i\psi^\dagger(\dot{\psi} - ig\xi) - m\psi^\dagger\psi$$  \hspace{1cm} (C.23)

with $\psi, \psi^\dagger$ fermionic and $z, \xi$ bosonic. The gauge transformation reads

$$z \to z' = z + \frac{\alpha}{g},$$  \hspace{1cm} (C.24)

$$\xi \to \xi' = \xi + \frac{\dot{\alpha}}{g}$$  \hspace{1cm} (C.25)

and

$$\psi \to \psi' = e^{i\alpha}\psi$$  \hspace{1cm} (C.26)

with $\alpha$ an arbitrary function of time. In the time axial gauge where $\xi = 0$, the Hamiltonian corresponding to (C.23) is

$$H = -\frac{1}{2}\frac{\partial^2}{\partial Z^2} + m\Psi^\dagger\Psi$$ \hspace{1cm} (C.27)

and the Gauss law constraint is

$$\left( -i\frac{\partial}{\partial Z} + g\Psi^\dagger\Psi \right) | >= 0$$ \hspace{1cm} (C.28)

The constraint can be solved explicitly and the physical spectrum consists of two states with $\Psi^\dagger\Psi = 0, 1$ and the corresponding eigenvalue of $H = 0$ and $m + \frac{g^2}{2}$. Though trivial, we still follow the transformation of (C.27) and (C.28) to the gauge where $z = 0$, with the dynamical variables $\theta$ determined by $Z + \frac{2}{g} = 0$ and $\psi = e^{i\theta}\Psi$. The Hamiltonian (C.27) and the constraint (C.28) becomes

$$H = -\frac{g^2}{2}\frac{\partial^2}{\partial \theta^2} + m\psi^\dagger\psi$$  \hspace{1cm} (C.29)

and

$$\left( -i\frac{\partial}{\partial \theta} - \psi^\dagger\psi \right) | >= 0.$$ \hspace{1cm} (C.30)

Substituting the solution of (C.30) into (C.29), we obtain

$$H_{\text{eff.}} = m\psi^\dagger\psi + \frac{g^2}{2}(\psi^\dagger\psi)^2.$$ \hspace{1cm} (C.31)
Following the above recipe, we convert (C.28) into a path integral

\[ <\psi|e^{-iH_{\text{eff}}t}|\psi> = \int \prod_n dz_n d\xi_n d\psi_n d\bar{\psi}_n \delta(z_n)e^{i\epsilon \sum_n L_{\text{eff}}(n)} <\psi_0>, \quad (C.32) \]

where

\[ L_{\text{eff}}(n) = \frac{1}{2}(\dot{z}_n - \dot{\xi}_n)^2 + i\bar{\psi}_n(\psi_n - ig\xi_n \psi_n) - m\bar{\psi}_n \psi_n - \frac{g^2}{2}\bar{\psi}_n \psi_n \quad (C.33) \]

with the last term comes from the normal ordering of the four-fermion term of (C.31). The integration over \( \xi_n \) in (C.33) will not generate quartic terms since the combination \((\bar{\psi}_n \psi_n)^2\) vanishes. Applying the Feynman rules given by the path integral (C.32) at \( t \to \infty \), we have verified explicitly that the shift of the self-energy because of the interaction vanishes to one loop order, in agreement with the result of canonical quantization.

Finally, we come to the nonabelian gauge field. The four fermion Coulomb interaction term of Christ-Lee Hamiltonian in Coulomb gauge reads

\[ H_{\text{Coul}} = \frac{g^2}{2} \int d^3\vec{r}d^3\vec{r}' \psi_\dagger(\vec{r})T_l\psi(\vec{r})G(-\nabla^2)G|\vec{r}',l\rangle\psi_\dagger(\vec{r}')T_{l'}\psi(\vec{r}) \]

\[ = \frac{g^2}{2} \int d^3\vec{r}d^3\vec{r}' :\psi_\dagger(\vec{r})T_l\psi(\vec{r})G(-\nabla^2)G|\vec{r}',l\rangle\psi_\dagger(\vec{r}')T_{l'}\psi(\vec{r}) : \]

\[ + \frac{g^2}{2} \int d^3\vec{r}d^3\vec{r}' \psi_\dagger(\vec{r})T_lT_{l'}\psi(\vec{r}). \quad (C.34) \]

The last term becomes \( V_3 \) of (4.45).

The additional term stemming from the normal ordering of fermionic operators begins to show up at one loop level, unlike its bosonic counterpart. In the case of an abelian gauge theory, the term (4.45) corresponds to the self Coulomb energy of a fermion and is not observable, but here, for the nonabelian case, it carries the coupling to the gluon fields and may not be ignored.

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