Gauge Theory of Massive Tensor Field II
— Covariant Expressions —

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Abstract

Covariant forms are given to a gauge theory of massive tensor field. This is accomplished by introducing another auxiliary field of scalar type to the system composed of a symmetric tensor field and an auxiliary field of vector type. The situation is compared to the case of the theory in which a tensor field describes a scalar ghost as well as an ordinary massive tensor. In this case only an auxiliary vector field is needed to give covariant expressions for the gauge theory.

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§1. Introduction

In a previous paper\(^1\) (referred to as I) a massive tensor field theory with a smooth massless limit was constructed. We applied the Batalin-Fradkin (BF) algorithm\(^2\) to the pure-tensor (PT) model which describes a massive pure tensor of five degrees of freedom. By introducing an auxiliary vector field, we converted the original second-class constrained system into a first-class one. To the gauge-invariant system we obtained, massless-regular gauge-fixing was imposed. The resulting theory was found to have a smooth massless limit. Based on the Hamiltonian formalism, however, our formulation is non-covariant from the beginning. The final result has been left lacking in covariance.

The purpose of the present paper is to give covariant forms to the result. This is accomplished by introducing another auxiliary field of scalar type in addition to the auxiliary field of vector type. The situation is to be compared to the case of the additional-scalar-ghost (ASG) model where a symmetric tensor field describes an additional scalar ghost as well as the ordinary massive tensor. For the ASG model, an auxiliary vector field has also to be introduced to convert the original second-class constrained system into a first-class one. In this case, however, this is enough. It is seen that we can obtain covariant expressions without introducing any other auxiliary field.

In §2, canonical formalism of massive tensor field is presented. It is shown that the structures of constraints are different according to the value of a parameter \(a\) in mass terms. When \(a = 1\), which gives the PT model, we have five kinds of constraints, four second-class and one first-class. On the other hand, in the case of \(a \neq 1\), which corresponds to the ASG model, we have only four kinds of second-class constraints. In §3, the BF algorithm is applied to these systems. For both cases of \(a = 1\) and \(a \neq 1\), we can convert all the second-class constraints to first-class ones by introducing an auxiliary field of vector type. In §4, we investigate massless-regular gauge-fixings that allow to take smooth massless limits. Covariant expressions for the final results are given in §5. In the case of \(a \neq 1\), we can easily find covariant path integral expressions. When \(a = 1\), however, we have to introduce another auxiliary field of scalar type in order to write down the result in covariant forms. Section 6 gives summary.

§2. Canonical formalism
A massive tensor field is described by the Lagrangian

\[ L[h] = L[h, m = 0] - \frac{m^2}{2} \left( h_{\mu\nu} h^{\mu\nu} - ah^2 \right), \tag{2.1} \]

where \( L[h, m = 0] \) represents the Lagrangian for a massless tensor field

\[ L[h, m = 0] \equiv -\frac{1}{2} \left( \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \partial_\lambda h \partial^\lambda h \right) + \partial_\lambda h_{\mu\nu} \partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu} \partial_\nu h, \tag{2.2} \]

\( h \) stands for the trace of \( h_{\mu\nu} \left( h \equiv h^{\mu}_{\mu} \right) \), and \( a \) is a real parameter. Field equations are

\[ (\Box - m^2) h_{\mu\nu} - (2a - 1) \left( \partial_\mu \partial_\nu + \frac{m^2}{2} \eta_{\mu\nu} \right) h = 0, \tag{2.3} \]
\[ \partial^\nu h_{\mu\nu} - a \partial_\mu h = 0, \tag{2.4} \]
\[ 2(a - 1) \Box h + (4a - 1)m^2 h = 0. \tag{2.5} \]

For \( a = 1 \) (PT model), the field equations reduce to

\[ (\Box - m^2) h_{\mu\nu} = 0, \tag{2.6} \]
\[ \partial^\nu h_{\mu\nu} = 0, \tag{2.7} \]
\[ h = 0, \tag{2.8} \]

which show that this case purely describes a massive tensor field with five degrees of freedom.

For other arbitrary value of \( a \neq 1 \) (ASG model), the Lagrangian describes an additional scalar ghost as well as the ordinary tensor field. In particular \( a = \frac{1}{2} \) gives simple field equations

\[ (\Box - m^2) h_{\mu\nu} = 0, \tag{2.9} \]
\[ \partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h = 0. \tag{2.10} \]

To investigate the structure of constraints and Hamiltonian, we have to consider two cases of \( a \neq 1 \) and \( a = 1 \) separately.

2.1. The case of \( a \neq 1 \)

In this case we have two primary constraints

\[ \varphi^0 \equiv \pi^0 \approx 0, \tag{2.11} \]
\[ \varphi^m \equiv \pi^m + \partial_\mu h^{mn} - \partial^m h^\prime \approx 0, \tag{2.12} \]

\(^*\) In the present paper Greek indices run \( 0 - 3 \), while Latin indices \( 1 - 3 \). The metric is \( \eta^{\mu\nu} \equiv (-1, +1, +1, +1) \).
and two secondary constraints

\[ \varphi^0_1 \equiv \partial_m \partial^m h' - \partial_m \partial_n h^m \equiv -m^2[(1 - a)h_0 + ah'] \approx 0, \quad (2.13) \]
\[ \varphi^m_1 \equiv \partial_n \pi^{mn} + \frac{1}{2} \partial_n (\partial^m h^n - \partial^n h^m) + m^2 h^m \approx 0, \quad (2.14) \]

where \( \pi^0, \pi^m \) and \( \pi^{mn} \) are momenta conjugate to \( h^0 \equiv h_{00}, h^m \equiv h_{0m} \) and \( h^{mn} \) respectively, and \( h' \) denotes the three-dimensional trace \( h' \equiv h^m_m \). The Poisson brackets between these constraints are calculated as

\[ \{ \varphi^0_1(x), \varphi^0_1(x') \} = (1 - a)m^2 \delta^3(x - x'), \quad (2.15) \]
\[ \{ \varphi^m_1(x), \varphi^n_1(x') \} = -m^2 \eta^{mn} \delta^3(x - x'), \quad (2.16) \]
\[ \{ \varphi^0_1(x), \varphi^m_1(x') \} = a m^2 \delta^3(x - x'), \quad (2.17) \]

The Hamiltonian is

\[ H = H_0 + \left[ \frac{a}{2(1 - a)} \pi' + \partial^m h_m \right] \varphi^0 + \partial^m h^{mn} \partial_m (h_0 - h') \varphi^m, \quad (2.18) \]

where

\[ H_0 \equiv H_0(m = 0) + \frac{m^2}{2} \left[ (1 - a)h_0^2 - 2h_m h^m + h^{mn} h_m h^n + 2ah_0 h' - ah'^2 \right], \quad (2.19) \]

and

\[ H_0(m = 0) \equiv \frac{1}{2} \pi^{mn} \pi_{mn} - \frac{1}{4} \pi^2 + \frac{3}{4} \partial_m h_n \partial^m h^n - (\partial^m h_m)^2 + \partial_m h_0 \partial^n h^m - \partial_m h_0 \partial^n h^{mn} \]
\[ + \frac{1}{2} \left( \partial_t h_m \partial^m h^n - \partial_t h' \partial^m h^n \right) - \partial_t h_m \partial^n h^m + \partial_m h^{mn} \partial_t h'. \quad (2.20) \]

The Poisson brackets between the constraints and the Hamiltonian are

\[ \{ \varphi^0, H \} = \varphi^0_1 + a \partial_m \varphi^m, \quad (2.21) \]
\[ \{ \varphi^m, H \} = 2 \varphi^m_1 + \frac{1 - 2a}{1 - a} \partial^m \varphi^0, \quad (2.22) \]
\[ \{ \varphi^0_1, H \} = -\partial_m \varphi^m_1 + \frac{a}{2(1 - a)} (2 \partial_m \partial^m - 3am^2) \varphi^0, \quad (2.23) \]
\[ \{ \varphi^m_1, H \} = \frac{1}{2} \partial_n \partial^m \varphi^m + \frac{1}{2} (1 - 2a) \partial^m \partial_n \varphi^n. \quad (2.24) \]

Equations (2.15), (2.16) and (2.17) show that all the four constraints are of the second class.
2.2. The case of $a = 1$

This case was studied in I. The results are quoted here for the sake of comparison with the case of $a \neq 1$. ∗ Primary constraints are the same as (2.11) and (2.12). For secondary constraints, however, we have three in this case

\begin{align}
\varphi_1^0 & \equiv \partial_m \partial^m h' - \partial_m \partial^m h^{mn} - m^2 h' \approx 0, \quad (2.25) \\
\varphi_1^m & \equiv \partial_n \pi^{mn} + \frac{1}{2} \partial_n (\partial^m h^n - \partial^m h^m) + m^2 h^m \approx 0, \quad (2.26) \\
\varphi_2^0 & \equiv \pi' \approx 0. \quad (2.27)
\end{align}

The Poisson brackets are

\begin{align}
[\varphi^m(x), \varphi_1^0(x')] &= -m^2 \eta^{mn} \delta^3(x - x'), \\
[\varphi_1^0(x), \varphi_1^m(x')] &= m^2 \partial^m \delta^3(x - x'), \\
[\varphi^m(x), \varphi_2^0(x')] &= -2 \partial^m \delta^3(x - x'), \\
[\varphi_1^0(x), \varphi_2^0(x')] &= (2 \partial_m \partial^m - 3m^2) \delta^3(x - x'), \\
[\varphi_1^0(x), \varphi_1^0(x')] &= -\partial_m \varphi_1^m + \frac{m^2}{2} \varphi_2^0.
\end{align}

The others = 0.

The Hamiltonian is

\[ H = H_0 + \lambda_0 \varphi_2^0 + \partial^m h_{mn} \varphi^m + (h_0 - h') \varphi_1^0, \quad (2.32) \]

where

\[ H_0 \overset{d}{=} H_0(m = 0) + \frac{m^2}{2} \left(-2h_m h^m + h_{mn} h^{mn} + 2h_0 h' - h'^2\right), \quad (2.33) \]

with $H_0(m = 0)$ defined by (2.20), and $\lambda_0$ is an arbitrary coefficient. The Poisson brackets between the constraints and the Hamiltonian are

\begin{align}
[\varphi^0, H] &= 0, \quad (2.34) \\
[\varphi^m, H] &= 2 \varphi_1^m, \quad (2.35) \\
[\varphi_1^0, H] &= -\partial_m \varphi_1^m + \frac{m^2}{2} \varphi_2^0, \quad (2.36) \\
[\varphi_1^m, H] &= \frac{1}{2} \partial_n (\partial^m \varphi^n + \partial^n \varphi^m) + \partial^m \varphi_1^0, \quad (2.37) \\
[\varphi_2^0, H] &= \partial_m \varphi^m + 4 \varphi_1^0. \quad (2.38)
\end{align}

It is seen that $\varphi^0$ is a first-class constraint and the other four constraints are of the second class.

\* Some of the equations have minor differences from the corresponding ones in I. Since the differences come from total divergences in the Lagrangian, however, they have no essential effects.
§3. Batalin-Fradkin extension

In this section we convert the second-class constraints into first-class ones by applying the BF algorithm. It is seen that for both cases of $a \neq 1$ and $a = 1$, the introduction of an auxiliary vector field (BF field) $\theta_\mu$ and its conjugate momentum $\omega^\mu$ is sufficient to modify the constraints and the Hamiltonian.

3.1. The case of $a \neq 1$

By the use of $\theta_\mu$ and $\omega^\mu$, we define new field variables as follows,

$$\tilde{h}_0 \equiv h_0 + \theta_0,$$

$$\tilde{h}_m \equiv h_m + \theta_m - \frac{a}{1 - a m^2} \partial_m \omega^0,$$

$$\tilde{h}_{mn} \equiv h_{mn} - \frac{1}{2m^2} (\partial_m \omega_n + \partial_n \omega_m),$$

$$\tilde{\pi}^0 \equiv \pi^0 - \omega^0,$$

$$\tilde{\pi}^m \equiv \pi^m - \omega^m - \frac{1}{2m^2} \partial_n (\partial^m \omega^n - \partial^n \omega^m),$$

$$\tilde{\pi}^{mn} \equiv \pi^{mn} + \frac{1}{2} (\partial^m \theta^n + \partial^n \theta^m) - \eta^{mn} \partial^l \theta_l - \frac{a}{1 - a} \eta^{mn} \omega^0 - \frac{1}{m^2} (\partial^m \partial^n - \eta^{ln} \partial^l \partial^n) \omega^0.$$  

The constraints $\varphi^A \equiv (\varphi^0, \varphi^m, \varphi_1^0, \varphi_1^m)$ and the Hamiltonian are modified as

$$\tilde{\varphi}^A \equiv \varphi^A \left[ (h, \pi) \rightarrow (\tilde{h}, \tilde{\pi}) \right] \approx 0,$$

$$\tilde{H} \equiv H \left[ (h, \pi) \rightarrow (\tilde{h}, \tilde{\pi}) \right].$$

Their concrete forms are

$$\varphi^0 = \varphi^0 - \omega^0 \approx 0,$$

$$\varphi^m = \varphi^m - \omega^m \approx 0,$$

$$\varphi_1^0 = \varphi_1^0 - (1 - a)m^2 \theta_0 + a \partial_m \omega^m \approx 0,$$

$$\varphi_1^m = \varphi_1^m + m^2 \theta^m \approx 0,$$

and

$$\tilde{H} = H + \left[ \frac{1 - 2a}{1 - a} \partial^m \theta_m + \frac{2a}{1 - a} \left( \frac{1}{m^2} \partial_m \partial^m - \frac{3a}{4(1 - a)} \right) \omega^0 \right] \varphi^0$$

$$+ \left[ a \partial_m \theta_0 - \frac{1}{2m^2} \partial^m ((1 - 2a) \partial_m \omega_n + \partial_n \omega_m) \right] \varphi^m.$$
\[-\theta_0 \varphi^0 + \left( -2\theta_m + \frac{1}{1-a} \frac{1}{m^2} \partial_m \omega^0 \right) \tilde{\varphi}_1^m - \frac{1}{2} \frac{1}{m^2} \theta_0 \tilde{\varphi}_1^m + \frac{1}{2} \frac{1}{m^2} \omega^0 \partial^m \theta_m + m^2 \theta_m \theta^m + \frac{1}{4m^2} \left[ \partial_m \omega_n \partial^m \omega_n + (1 - 2a) (\partial_m \omega^m)^2 \right] \]

\[-\frac{a}{1-a} m^2 \partial_m \omega^0 \partial^m \omega^0 - \frac{3a^2}{4(1-a)^2} \omega^0. \quad (3.13)\]

These modified set of constraints and Hamiltonian gives indeed a first-class constrained system:

\[
\left[ \tilde{\varphi}^A(x), \tilde{\varphi}^B(x') \right] = 0, \quad (3.14)
\]

\[
\left[ \tilde{\varphi}^A, \tilde{H} \right] = 0. \quad (3.15)
\]

### 3.2. The case of \(a = 1\)

The results for this case have been given in I. We define the following new field variables:

\[
\tilde{h}_0 \overset{d}{=} h_0, \quad (3.16)
\]

\[
\tilde{h}_m \overset{d}{=} h_m + \theta_m - \frac{1}{m^2} \partial_m \omega^0, \quad (3.17)
\]

\[
\tilde{h}_{mn} \overset{d}{=} h_{mn} - \frac{1}{2m^2} (\partial_m \omega_n + \partial_n \omega_m) - \frac{1}{3} \left( \eta_{mn} + \frac{2}{m^2} \partial_m \partial_n \right) \theta_0, \quad (3.18)
\]

\[
\tilde{\pi}^0 \overset{d}{=} \pi^0, \quad (3.19)
\]

\[
\tilde{\pi}^m \overset{d}{=} \pi^m - \omega^m - \frac{1}{2m^2} \partial_n (\partial^m \omega^n - \partial^n \omega^m) - \frac{2}{3} \partial^m \theta_0, \quad (3.20)
\]

\[
\tilde{\pi}^{mn} \overset{d}{=} \pi^{mn} + \frac{1}{2} (\partial^m \theta^n + \partial^n \theta^m) - \eta^{mn} \partial^l \theta_l + \eta^{mn} \omega^0. \quad (3.21)
\]

The modification of the constraints and the Hamiltonian is carried out as (3.7) and (3.8), which gives

\[
\tilde{\varphi}^0 = \varphi^0 \approx 0, \quad (3.22)
\]

\[
\tilde{\varphi}^m = \varphi^m - \omega^m \approx 0, \quad (3.23)
\]

\[
\tilde{\varphi}_1^0 = \varphi_1^0 + \partial_m \omega^m + m^2 \theta_0 \approx 0, \quad (3.24)
\]

\[
\tilde{\varphi}_1^m = \varphi_1^m + m^2 \theta^m \approx 0, \quad (3.25)
\]

\[
\tilde{\varphi}_2^0 = \varphi_2^0 - 2 \theta^m \theta_m + 3 \omega^0 \approx 0, \quad (3.26)
\]

and

\[
\tilde{H} = H + \left[ -\frac{1}{3} \left( 1 + \frac{2}{m^2} \partial_n \partial^n \right) \partial_m \theta_0 - \frac{1}{2m^2} \partial^n (\partial_m \omega_n + \partial_n \omega_m) \right] \tilde{\varphi}^m
\]
\[ + \left[ \frac{1}{3} \left( 4 + \frac{2}{m^2} \partial_m \partial^m \right) \theta_0 + \frac{1}{m^2} \partial_m \omega^m \right] \tilde{\varphi}_1^0 \\
+ \left( -2 \theta_m + \frac{1}{m^2} \partial_m \omega^0 \right) \tilde{\varphi}_1^m - \frac{1}{2} \omega^0 \tilde{\varphi}_2^0 \\
+ \frac{1}{3} \partial_m \theta_0 \partial^m \theta_0 - \frac{2}{3} m^2 \theta_0^2 - \theta_0 \partial_m \omega^m + \frac{1}{8m^2} \left( \partial_m \omega_n - \partial_n \omega_m \right)^2 \\
+ m^2 \theta_m \theta^m + \frac{3}{4} \omega^0 \omega^2. \] (3.27)

§4. Gauge fixing

4.1. The case of \( a \neq 1 \)

In order to find a massless-regular theory, we impose the following gauge-fixing conditions \( \chi_A \equiv (\chi_0, \chi_m, \chi_{10}, \chi_{11}) \):

\[ \chi_0 \equiv h_0 \approx 0, \] (4.1)
\[ \chi_m \equiv h_m \approx 0, \] (4.2)
\[ \chi_{10} \equiv \pi' - \frac{3a}{1-a} \omega^0 \approx 0, \] (4.3)
\[ \chi_{11} \equiv \partial^m h_{mn} - \frac{1}{2} \partial_m h' \approx 0. \] (4.4)

The path integral is given by

\[ Z = \int \mathcal{D} \pi^0 \mathcal{D} \pi^m \mathcal{D} \pi^{mn} \mathcal{D} h_0 \mathcal{D} h_m \mathcal{D} h_{mn} \mathcal{D} \omega^0 \mathcal{D} \omega^m \mathcal{D} \theta_0 \mathcal{D} \theta_m \prod_A \delta(\tilde{\varphi}^A) \delta(\chi_A) \prod_i \text{Det} M \]
\[ \times \exp i \int d^4 x \left[ \pi^0 h_0 + \pi^m h_m + \pi^{mn} h_{mn} + \omega^0 \dot{\theta}_0 + \omega^m \dot{\theta}_m - \tilde{H} \right], \] (4.5)

where

\[ M \equiv \delta^\alpha_\beta \partial_m \partial^m \delta^3(x - x'). \quad (\alpha, \beta = 0 - 3) \] (4.6)

The integrations over \( \pi^0, \pi^m, h_0 \) and \( h_m \) are easily carried out. The \( \delta \)-functions \( \delta(\tilde{\varphi}_1^0), \delta(\tilde{\varphi}_1^m) \) and \( \delta(\chi_{10}) \) are exponentiated as

\[ \delta(\tilde{\varphi}_1^0) \delta(\tilde{\varphi}_1^m) \delta(\chi_{10}) = \int \mathcal{D} \lambda_0 \mathcal{D} \lambda_m \mathcal{D} \mu \exp i \int d^4 x \left[ \lambda_0 \tilde{\varphi}_1^0 + \lambda_m \tilde{\varphi}_1^m + \mu \chi_{10} \right]. \] (4.7)

We integrate over \( \pi^{mn} \), write \( 2h_m \) and \( h_0 \) over \( \lambda_m \) and \( \lambda_0 \) respectively, and further integrate with respect to \( \theta_0, \theta_m \) and \( \mu \). Replacing variables as \( \omega^m \rightarrow 2m^2 \theta_m \) and \( \omega^0 \rightarrow -2(1-a)m^2 \theta_0 \), we obtain

\[ Z = \int \mathcal{D} h_{\mu \nu} \mathcal{D} \theta_\mu \delta \left( \partial^m h_{mn} - \frac{1}{2} \partial_m h' \right) \prod_i \text{Det} M \]
\[ \times \exp i \int d^4 x \left[ L[h, \theta] + \frac{4}{3} \left( \partial^m h_m - \frac{1}{2} h' \right)^2 \right], \] (4.8)
where
\[ L[h, \theta] \equiv L[h, m = 0] - \frac{m^2}{2} \left( (h_{\mu\nu} - \partial_\mu \theta_\nu - \partial_\nu \theta_\mu)^2 - a (h - 2 \partial^\mu \theta_\mu)^2 \right). \] (4.9)

4.2. The case of \( a = 1 \)

For this case, we imposed the following gauge-fixing conditions in I.
\[
\begin{align*}
\chi_0 & \equiv h_0 \approx 0, \quad (4.10) \\
\chi_m & \equiv h_m \approx 0, \quad (4.11) \\
\chi_{10} & \equiv \pi' + 3 \omega^0 \approx 0, \quad (4.12) \\
\chi_{1m} & \equiv \partial^m h_{mn} - \frac{1}{2} \partial_m h' \approx 0, \quad (4.13) \\
\chi_{20} & \equiv \theta_0 \approx 0. \quad (4.14)
\end{align*}
\]

The final expression obtained in I was
\[
Z = \int \mathcal{D}h_{\mu\nu} \mathcal{D}\theta_\mu \delta \left( \partial^m h_m - \frac{1}{2} \pi' \right) \delta \left( \partial^m h_{mn} - \frac{1}{2} \partial_m h' \right) \prod_t \text{Det} M \\
\times \exp i \int d^4x \left[ L[h, \theta] + 4m^2 \dot{\theta}_0 \left( \partial^m \theta_m - \frac{1}{2} \pi' \right) \right],
\] (4.15)

where
\[ L[h, \theta] \equiv L[h, m = 0] - \frac{m^2}{2} \left( (h_{\mu\nu} - \partial_\mu \theta_\nu - \partial_\nu \theta_\mu)^2 - a (h - 2 \partial^\mu \theta_\mu)^2 \right). \] (4.16)

§5. Covariant expressions

5.1. The case of \( a \neq 1 \)

It is easy to obtain covariant expressions for the generating functional \( Z (4.8) \). In consideration of the fact that the Lagrangian \( L[h, \theta] \) is invariant under the gauge transformation with four arbitrary functions \( \varepsilon_\mu (x) \)
\[
\begin{align*}
\delta h_{\mu\nu} &= \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu, \\
\delta \theta_\mu &= \varepsilon_\mu,
\end{align*}
\] (5.1)

we can give various expressions for \( Z \). The situation is almost the same as in a massless tensor field. For example, for a ‘Coulomb-like gauge’, we have
\[
Z = \int \mathcal{D}h_{\mu\nu} \mathcal{D}\theta_\mu \delta \left( \partial^m h_m - \frac{1}{2} \pi' - f_0 \right) \delta \left( \partial^m h_{mn} - \frac{1}{2} \partial_m h' - f_m \right) \prod_t \text{Det} M
\]
\[ \times \exp i \int d^4xL[h, \theta] \]  
\[ = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\theta_\mu\delta \left( \partial^m h_{mn} - \frac{1}{2} \partial_m h' - f_\mu \right) \prod_t \text{Det}M \]  
\[ \times \exp i \int d^4x \left[ L[h, \theta] + \frac{1}{2\alpha} \left( \partial^m h_m - \frac{1}{2} h' \right)^2 \right], \]  
(5.2)

where \( f_\mu (\mu = 0 - 3) \) are arbitrary functions of \( x \), and \( \alpha \) is an arbitrary constant, gauge parameter. The expression (4.8) is a special case of (5.3). The covariant expressions are also obtained as follows:

\[ Z = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\theta_\mu\delta(\partial^\rho h_{\rho\mu\nu} - \frac{1}{2} \partial_\mu h - f_\mu) \text{Det}N \exp i \int d^4xL[h, \theta] \]  
(5.4)

\[ = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\theta_\mu\mathcal{D}B_\mu\mathcal{D}c_\mu\mathcal{D}\bar{c}_\mu \]  
\[ \times \exp i \int d^4x \left[ L[h, \theta] + B_\mu \left( \partial^\rho h_{\rho\mu\nu} - \frac{1}{2} \partial_\mu h + \frac{\alpha}{2} B_\mu \right) + i\bar{c}_\mu \Box c_\mu \right], \]  
(5.5)

where \( N \) is defined by

\[ N \equiv \delta_\beta^\alpha \partial^4(x - x'), \quad (\alpha, \beta = 0 - 3) \]  
(5.6)

and the Nakanishi-Lautrup (NL) field \( B_\mu \) and the Faddeev-Popov (FP) ghosts \( (c_\mu, \bar{c}_\mu) \) have been introduced.

5.2. The case of \( a = 1 \)

As the first step to obtain covariant expressions, we introduce another auxiliary field of scalar type \( \varphi(x) \) and define a gauge transformation with five arbitrary functions \( \varepsilon_\mu(x) \) and \( \varepsilon(x) \):

\[
\begin{align*}
\delta_\varepsilon h_{\mu\nu} &= \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu, \\
\delta_\varepsilon \theta_\mu &= \varepsilon_\mu + \partial_\mu \varepsilon, \\
\delta_\varepsilon \varphi &= \varepsilon.
\end{align*}
\]  
(5.7)

The expression (4.15) can be written as

\[ Z = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\theta_\mu\mathcal{D}\varphi \delta \left( \partial^m h_m - \frac{1}{2} h' \right) \delta \left( \partial^m h_{mn} - \frac{1}{2} \partial_m h' \right) \delta (\partial_m \partial^m \varphi) \prod_t \text{Det}M' \]  
\[ \times \exp i \int d^4x \left[ L[h, \theta] + 4m^2 \dot{\theta}_0 \left( \partial^m \theta_m - \frac{1}{2} h' \right) \right], \]  
(5.8)

where

\[ M' \equiv \delta_\beta^\alpha \partial_\alpha \partial^m \delta^3(x - x'). \quad (\alpha, \beta = 0 - 4) \]  
(5.9)
The Lagrangian \( L[h, \theta] \) can be decomposed into a manifestly gauge-invariant part \( L[h, \theta, \varphi] \) and the rest \( R[h, \theta, \varphi] \) as

\[
L[h, \theta] = L[h, \theta, \varphi] + R[h, \theta, \varphi],
\]

where

\[
L[h, \theta, \varphi] \overset{\text{def}}{=} L[h, m = 0] - \frac{m^2}{2} \left[ (h_{\mu\nu} - \partial_\mu \theta_\nu - \partial_\nu \theta_\mu + 2 \partial_\mu \varphi \partial_\nu \varphi)^2 - (h - 2 \partial^\mu \theta_\mu + 2 \Box \varphi)^2 \right],
\]

\[
R[h, \theta, \varphi] \overset{\text{def}}{=} 4m^2 \left[ (\partial^m h_m - \frac{1}{2} h') \varphi - \frac{1}{2} (\partial^m h_{mn} - \frac{1}{2} \partial_m h') \partial^m \varphi + \frac{1}{2} (h_0 - \frac{1}{2} h') \partial_m \partial^m \varphi \right].
\]

Because of the existence of three \( \delta \)-functions in (5.8), \( R \) has null effect. The path integral then reduces to

\[
Z = \int \mathcal{D}h_{\mu\nu} \mathcal{D}\theta_\mu \mathcal{D}\varphi \delta \left( \partial^m h_m - \frac{1}{2} h' \right) \delta \left( \partial^m h_{mn} - \frac{1}{2} \partial_m h' \right) \delta (\partial_m \partial^m \varphi) \prod_t \text{Det} M'
\times \exp i \int d^4x \left[ L[h, \theta, \varphi] + 4m^2 \theta_0 \left( \partial^m \theta_m - \frac{1}{2} h' \right) \right].
\]

As the next step, we define the following two gauge-invariant quantities:

\[
\Delta^{-1}[h, \theta, \varphi] \overset{\text{def}}{=} \int \mathcal{D}\varepsilon \delta \left( \partial^m h^\varepsilon_m - \frac{1}{2} h'^\varepsilon \right) \delta \left( \partial^m h^\varepsilon_{mn} - \frac{1}{2} \partial_m h'^\varepsilon \right) \delta (\partial_m \partial^m \varphi^\varepsilon),
\]

\[
\times \exp i \int d^4x \left[ 4m^2 \theta_0 \left( \partial^m \theta^\varepsilon_m - \frac{1}{2} h'^\varepsilon \right) \right],
\]

\[
\Delta'^{-1}[h, \theta, \varphi] \overset{\text{def}}{=} \int \mathcal{D}\varepsilon \delta \left( \partial^m h^\varepsilon_m - \frac{1}{2} h'^\varepsilon \right) \delta \left( \partial^m h^\varepsilon_{mn} - \frac{1}{2} \partial_m h'^\varepsilon \right) \delta \left( \partial^m \theta^\varepsilon_m - \frac{1}{2} h'^\varepsilon \right).
\]

In the above, \( \mathcal{D}\varepsilon \) stands for the invariant measure on the gauge group, having the property

\[
\mathcal{D}\varepsilon = \mathcal{D}(\varepsilon \varepsilon') = \mathcal{D}(\varepsilon' \varepsilon) = \mathcal{D}\varepsilon,
\]

where \( \varepsilon \) denotes the inverse of \( \varepsilon \). The quantities like \( h^\varepsilon \) indicate gauge-transformed ones of the respective fields. For \( \Delta'[h, \theta, \varphi] \) we simply have

\[
\Delta'[h, \theta, \varphi] = \prod_t \text{Det} M'.
\]

To evaluate \( \Delta[h, \theta, \varphi] \), we take a special gauge orbit \( \mathcal{O} \) that contains a configuration \( (h, \theta, \varphi) \) satisfying

\[
\begin{cases}
\partial^m h_m - \frac{1}{2} h' = 0, \\
\partial^m h_{mn} - \frac{1}{2} \partial_m h' = 0, \\
\partial^m \theta_m - \frac{1}{2} h' = 0.
\end{cases}
\]
For this configuration the quantity $\Delta[h, \theta, \varphi]$ is calculated as

$$\Delta^{-1}[h, \theta, \varphi] = \int \mathcal{D}\varepsilon \delta (\partial_m \partial^m \varepsilon_0) \delta (\partial_n \partial^n \varepsilon_m) \delta (\partial_m \partial^m \varphi + \partial_m \partial^m \varepsilon)$$

$$\times \exp i \int d^4x \left[ 4m^2 \left( \dot{\theta}_0 + \dot{\varepsilon}_0 + \ddot{\varepsilon}_0 \right) \partial_m \partial^m \varepsilon \right]$$

$$= \int \mathcal{D}\varepsilon \delta (\partial_m \partial^m \varepsilon_0) \delta (\partial_n \partial^n \varepsilon_m) \delta (\partial_m \partial^m \varphi + \partial_m \partial^m \varepsilon)$$

$$\times \exp i \int d^4x \left[ -4m^2 \left( \dot{\theta}_0 - \dot{\varphi}_0 \right) \partial_m \partial^m \varphi \right]$$

$$= (\prod_t \text{Det} M')^{-1} \exp i \int d^4x \left[ -4m^2 \left( \dot{\theta}_0 - \dot{\varphi}_0 \right) \partial_m \partial^m \varphi \right]. \quad (5.19)$$

The gauge invariance of this quantity tells that the expression (5.19) is valid for any configuration belonging to $O$. Since the configuration $(\bar{h}, \bar{\theta}, \bar{\varphi})$ such that

$$\left\{ \begin{array}{l}
\partial^m \bar{h}_m - \frac{1}{2} \dot{\bar{h}}' = 0, \\
\partial^m \bar{h}_{mn} - \frac{1}{2} \partial_m \bar{h}' = 0, \\
\partial_m \partial^m \bar{\varphi} = 0
\end{array} \right. \quad (5.20)$$

belongs to $O$, we can evaluate $\Delta[h, \theta, \varphi]$ for this configuration. We then have

$$\Delta[h, \theta, \varphi] = \Delta[\bar{h}, \bar{\theta}, \bar{\varphi}] = \prod_t \text{Det} M'. \quad (5.21)$$

Multiply Eq. (5.13) by $[\text{The right hand side of (5.15)}] \times \Delta'[h, \theta, \varphi] = 1$ to give

$$Z = \int \mathcal{D}h_{\mu\nu} \mathcal{D}\theta_\mu \mathcal{D}\varphi \delta \left( \partial^m h_m - \frac{1}{2} \dot{h}' \right) \delta \left( \partial^m h_{mn} - \frac{1}{2} \partial_m h' \right) \delta (\partial_m \partial^m \varphi) \prod_t \text{Det} M'$$

$$\times \exp i \int d^4x \left[ L[h, \theta, \varphi] + 4m^2 \hat{\theta}_0 \left( \partial^m \theta_m - \frac{1}{2} \partial_m h' \right) \right]$$

$$\times \int \mathcal{D}\varepsilon \delta \left( \partial^m \varepsilon_m - \frac{1}{2} \dot{\varepsilon}' \right) \delta \left( \partial^m \varepsilon_{mn} - \frac{1}{2} \partial_m \varepsilon' \right) \delta (\partial_m \partial^m \varepsilon) \Delta'[h, \theta, \varphi]. \quad (5.22)$$

Change variables from $(h, \theta, \varphi)$ to $(h', \theta', \varphi') \stackrel{d}{=} (h^\varepsilon, \theta^\varepsilon, \varphi^\varepsilon)$, take into account the gauge-invariance of $L[h, \theta, \varphi]$ and $\Delta'[h, \theta, \varphi]$ as well as that of the measure $\mathcal{D}h_{\mu\nu} \mathcal{D}\theta_\mu \mathcal{D}\varphi$, and remove prime signs. Then we have

$$Z = \int \mathcal{D}h_{\mu\nu} \mathcal{D}\theta_\mu \mathcal{D}\varphi$$
\[ \times \left\{ \left[ \int D\bar{\varepsilon} \delta \left( \partial^m h^\varepsilon_m - \frac{1}{2} \dot{h}^\varepsilon \right) \delta \left( \partial^m h^\varepsilon_{mn} - \frac{1}{2} \partial_m h^\varepsilon \right) \delta (\partial_m \partial^m \varphi^\varepsilon) \right. \right. \\
\times \exp i \int d^4x \left( 4m^2 \dot{\theta}^\varepsilon \left( \partial^m \theta^\varepsilon_m - \frac{1}{2} \dot{\theta}^\varepsilon \right) \right) \left. \right] \prod_i \text{Det} M' \right\} \\
\times \exp i \int d^4x L[h, \theta, \varphi] \\
\times \delta \left( \partial^m h_m - \frac{1}{2} \dot{h} \right) \delta \left( \partial^m h_{mn} - \frac{1}{2} \partial_m h' \right) \delta \left( \partial^m \theta_m - \frac{1}{2} h' \right) \Delta'[h, \theta, \varphi]. \tag{5.23} \]

In this expression, the factor enclosed in braces is equal to 1 as seen from (5.21). The path integral can be expressed as follows,

\[ Z = \int Dh_{\mu\nu} D\theta_\mu D\varphi \delta \left( \partial^\nu h^\mu - \frac{1}{2} \dot{h}_\mu \right) \delta \left( \partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h \right) \delta \left( \partial^\mu \theta - \frac{1}{2} h - f \right) \text{Det} N' \\
\times \exp i \int d^4x L[h, \theta, \varphi]. \tag{5.24} \]

Coming to this stage, it is an easy task to give covariant forms to \( Z \). That is

\[ Z = \int Dh_{\mu\nu} D\theta_\mu D\varphi \delta \left( \partial^\nu h^\mu - \frac{1}{2} \dot{h}_\mu \right) \delta \left( \partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h \right) \delta \left( \partial^\mu \theta - \frac{1}{2} h - f \right) \text{Det} N' \\
\times \exp i \int d^4x L[h, \theta, \varphi], \tag{5.25} \]

where \( N' \) is defined by

\[ N' \equiv \frac{d}{\alpha} \delta^4 (x - x'), \quad (\alpha, \beta = 0 - 4) \tag{5.26} \]

and \( f_\mu \) and \( f \) are arbitrary functions of \( x \). By introducing the Nakanishi-Lautrup fields \((B_\mu, B)\) and the Faddeev-Popov ghosts \((c_\mu, c)\) and \((\bar{c}^\mu, \bar{c})\), we arrive at the final form of the path integral

\[ Z = \int Dh_{\mu\nu} D\theta_\mu D\varphi DB_\mu DB_\nu Dc_\mu D\bar{c}^\mu Dc D\bar{c} \\
\times \exp i \int d^4x \left[ L[h, \theta, \varphi] + L_{\text{GF+FP}} \right], \tag{5.27} \]

\[ L_{\text{GF+FP}} = B^\mu \left( \partial^\nu h^\mu - \frac{1}{2} \partial_\mu h + \frac{\alpha}{2} B_\mu \right) + i\bar{c}^\mu \square c_\mu \\
+ B \left( \partial^\mu \theta - \frac{1}{2} h + \frac{\beta}{2} B \right) + i\bar{c} \square c, \tag{5.28} \]

where \( \alpha \) and \( \beta \) are arbitrary constants, gauge parameters.
§6. Summary

We have given the covariant path integral expressions to the gauge theories of massive tensor fields. It has turned out that in the case of the PT model a scalar field in addition to a vector field has to be introduced as auxiliary BF field, while only a vector field is necessary for the ASG model. The difference comes from that of the constraint structures in the two models.

To construct a complete nonlinear theory which smoothly reduces to general relativity in the massless limit is left for future study.

Acknowledgements

We would like to thank T. Kurimoto for discussion.

References

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