FINE STRUCTURE AND FRACTIONAL M/N AHARONOV-BOHM EFFECT

F.V. Kusmartsev\textsuperscript{1,2}

and

Minoru Takahashi \textsuperscript{2}

Department of Theoretical Physics
University of Oulu
SF-90570 Oulu, Finland\textsuperscript{1}

and

Institute for Solid State Physics, University of Tokyo
Roppongi, Minato-ku, Tokyo 106 \textsuperscript{2}

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We find a fine structure in the Aharonov-Bohm effect, which is characterized by the appearance of a new type of periodic oscillations having smaller fractional period and an amplitude, which may compare with the amplitude of the conventional Aharonov-Bohm effect. Specifically, at low density or strong coupling on a Hubbard ring can coexist along with the conventional Aaronov-Bohm oscillations with the period equal to an integer, measured in units of the elementary flux quantum, two additional oscillations with periods $1/N$ and $M/N$. The integers $N$ and $M$ are the particles number and the number of down-spin particles, respectively.

From a solution of the Bethe ansatz equations for $N$ electrons located on a ring in a magnetic field we show that the fine structure is due to electron-electron and Zeeman interactions. Our results are valid in the dilute density limit and for an arbitrary value of the Hubbard repulsion $U$.

For the appearance or the existence of integer-fractional $M/N$ flux quantum periodic oscillations the key role is played by the Zeeman energy. This is in agreement with an assumption by Choy about importance of Zeeman energy. Without Zeeman energy there is a coexistence of the fractional $1/N$ and half-flux quantum periodic oscillations, only. With increasing magnetic field, the Zeeman energy increases and the half-flux quantum period transforms to the integer one. This transformation is not continuous but via the appearance of fractional $M/N$ oscillations, i.e. by some singular way.

The fractional $1/N$ regime occurs when the value $N < (LU/t)^{1/4}$, where $L$ is the number of sites. The coexistence regime of integer-fractional $M/N$ and fractional $1/N$ AB oscillations occurs when $(LU/t)^{1/4} << N << (LU/t)$. The parity effect
disappears in the fractional regime, but returns in the integer-fractional regime. We discuss the relation of the fine structure or the coexistence regime to existing experiments on single rings and on array of rings, where integer and half flux quantum periods are, respectively, observed. The explanation for the fractional $1/4$ AB period, recently observed by Liu et al is proposed.
I. INTRODUCTION

In the limit of large repulsion, when $U/t \to \infty$ Kusmartsev [1] found that the Aharonov-Bohm(AB) effect is fractional. In units of elementary flux quantum its period is equal to $1/N$, where the $N$ is the number of electrons on the ring. This effect is drastically different from conventional integer or half-integer AB effects [2], [3], [4], [5], [6].

This result has since been confirmed in the investigation by Schofield, Wheatley and Xiang [7]. The first corrections were studied by Yu and Fowler [8], where they first recognized the importance of the magnon excitations for the Aharonov-Bohm effect. We extend their analysis and investigated the effects for the dilute system. We find a new type of oscillations, related to a magnon excitation spectrum. The new AB oscillations with the period $M/N$ arising with the first corrections to the strong coupling limit or to the dilute density limit appears when we take into account the polarized or partially polarized ring with $M$ down-spin electrons. That is for the effect it is very important a role of the Zeeman interaction causing the polatization effects. The importance of the Zeeman energy was noticed recently by Choy [9].

We show that for the very dilute system, when the filling factor tends to zero, as well as for strong-coupling limit, the fractional $1/N$ Aharonov-Bohm effect may coexist with the integer conventional, integer-fractional or half-flux quantum periodic AB effect. By the integer-fractional regime of AB effect we mean the appearance of a new type of oscillations on the envelope function of $1/N$ periodic oscillations, which display a new fractional period. The period of these oscillations is equal to $M/N$, where $M$ is the number of down-spin particles. The $M/N$ oscillations exist even when the $1/N$ oscillations are washed out. When the Zeeman interaction is neglected, i.e. the Zeeman energy is zero and $M = N/2$, there is
a coexistence of the fractional $1/N$ and half-flux quantum periodic oscillations. In contrast, when the system is almost polarized, i.e. $M = 1$, the fractional $1/N$ and integer periodic oscillations coexist.

In previous work [10] we found that finite scaling finite size effects were very important. There is a scaling behavior of the ground state energy which does not depend on the size $L$ or on $U$, but depends only on $N/UL = \text{const} = \alpha$, where $U$ is measured in units of $t$ and $L$ is measured in units of the lattice constant. Such scaling occurs only at small values of $\alpha$. Due to this scaling symmetry the fractional Aharonov-Bohm effect may arise even for small values of $U$, for the very dilute electron systems. In the present work, we shall show that, such a symmetry is related to the structure of the low energy excitations of the Hubbard Hamiltonian, namely, with the spectrum of the spinon or magnon excitations. In fact the parameter $\alpha$ defines the energy of the magnon excitations. Thus, the shape of the new AB oscillations completely depends upon the energy of the magnon excitations.

In fact, we get an amusing physical picture. First, let us discuss the case $M = N/2$. For an odd number of particles in the ground state of the Hubbard ring, there already exists a spinon excitation. The momentum of this excitation is associated with the top of the spin-wave spectrum. This excitation is needed because of the parity effect, discussed by Legget [11], Loss [12] and Kusmartsev [6], for the case of interacting spinless fermions.

Thus, again with the first correction to the strong coupling limit or to the dilute density limit we have shown that on the Hubbard ring there occurs the reconstruction of the parity effect, reflecting the structure of low-energy excitations of the Hubbard ring. However, the parity effect obtained is distinct from the parity effect for the mesoscopic ring with spinful electrons and disorder [13]. Inducing the flux the momentum of this spinon excitation, which exists in the ground state, changes. This gives rise to the correspondence between the spin
excitation spectrum and the Aharonov-Bohm fine structure. With the magnetic field the number of polarized electrons changes, i.e. the spin excitation spectrum changes, this gives rise to the change of the Aharonov-Bohm periodicity.

This occurs also for the general situation, with any number of particles on the ring and at different rational ratio $M/N$. In the latter case one may see an additional fractional $M/N$ AB oscillations, which are due to the structure of the spinon excitation spectrum; the picture is as follows. With the flux, the spinon changes its momentum, which is, in fact, proportional to the flux. The change in the momentum does not occur continuously, but via jumps related to the finite number $N$ of the particles on the ring. Therefore these jumps are associated with the $1/N$ fraction of elementary flux quantum. Thus, on the Hubbard ring with the flux the spinons behave similar to spinless fermions, where the momentum is also shifted with the flux, reconstructing the parity effects. The Fermi momentum of this fermions is related to the number of polarized electrons $M$. This gives the new scale and the new periodicity in Aharonov-Bohm effect, related to the ratio $M/N$. However, the fractional $M/N$ AB effects do not destroy the fractional $1/N$ effect associated with the bound state of $N$ particles. Both effects may coexist, reflecting the separation of spin and charge degrees of freedom.

II. MAIN EQUATIONS

To demonstrate this picture we have studied the Hubbard Hamiltonian

$$H = -t \sum_{<i,j>,\sigma} a_{i\sigma}^\dagger a_{j\sigma} + U \sum_{i=1}^{L} n_{i+} n_{i-}$$

which is parametrized by the electron hopping integral $t$, the on-site repulsive Coulomb potential $U$ and the number of sites $L$. The operator $a_{i\sigma}^\dagger (a_{i\sigma})$ creates (destroys) an electron.
with spin projection $\sigma (\sigma = + \text{ or } -)$, at the ring site $i$, and $n_{i\sigma}$ is the occupation number operator $a_{i\sigma}^\dagger a_{i\sigma}$. The summation in Eq. (1) extends over the ring sites $i$ or $-$ as indicated by $< i, j >, \sigma$ – over all distinct pairs of nearest-neighbor sites, around the ring with the spin projection $\sigma$. For the case of the magnetic field we use the same form of the wave function as in Refs [14], [16] and [18], i.e.

$$\psi(x_1, \ldots, x_N) = \sum_P [Q, P] \exp[i \sum_{j=1}^N k_{Pj}x_{Qj}], \quad (2.2)$$

where $P = (P_1, \ldots, P_N)$ and $Q = (Q_1, \ldots, Q_N)$ are two permutations of $(1, 2, \ldots, N)$, and $N$ is the number of electrons.

The coefficients $[Q, P]$, as well as $(k_1, \ldots, k_N)$, are determined from the Bethe equations, which in a magnetic field are changed by the addition of the flux phase $2\pi f$ [16], [18], [19], [1], [20], [21], and [22]

$$e^{i(k_jL-2\pi f)} = \prod_{\beta=1}^M \left( \frac{it \sin k_j - i\lambda_\beta - U/4}{it \sin k_j - i\lambda_\beta + U/4} \right) \quad (2.3)$$

and

$$-\prod_{j=1}^N \left( \frac{it \sin k_j - i\lambda_\alpha - U/4}{it \sin k_j - i\lambda_\alpha + U/4} \right) = \prod_{\beta=1}^M \left( \frac{i\lambda_\alpha - i\lambda_\beta + U/2}{i\lambda_\alpha - i\lambda_\beta - U/2} \right) \quad (2.4)$$

$f$ being the flux in units of the elementary quantum flux $\phi_0$. The explicit form of the Bethe equations in a magnetic field is [14], [17], [1], [20], [8]

$$Lk_j = 2\pi I_j + 2\pi f - \sum_{\beta=1}^M \theta(4(t \sin k_j - \lambda_\beta)/U) \quad (2.5)$$

$$-\sum_{j=1}^N \theta(4(t \sin k_j - \lambda_\beta)/U) = 2\pi J_\beta + \sum_{\lambda_\alpha=1}^M \theta(2(\lambda_\beta - \lambda_\alpha)/U) \quad (2.6)$$

where $\theta(x) = 2\arctan(x)$ and the quantum numbers $I_j$ and $J_\beta$, which are associated with the charge and spin degrees of freedom, respectively, are either integers or hal- odd integers, depending on the parities of the numbers of down and up-spin electrons, respectively, i.e.
\[ I_j = \frac{M}{2} \quad \text{(mod 1)} \quad \text{and} \quad J_\beta = \frac{N - M + 1}{2} \quad \text{(mod 1)}. \quad (2.7) \]

III. EXPANSION

In previous work [10] showned that in the dilute density limit there exists a small parameter \( \alpha \), which is

\[ \alpha = \frac{N t}{L U} = \rho t / U, \quad (3.8) \]

where \( \rho = N / L \) is the filling factor.

From an expansion in \( \alpha \) (\( \alpha \ll 1 \)) of the second Bethe equation (2.8), we get the equations

\[ N \theta(t_\beta) - \frac{8}{U} \frac{1}{(1 + t_\beta^2)} \sum_{j=1}^{N} \sin(k_j) = 2\pi J_\beta + \sum_{\lambda_\alpha=1}^{M} \theta((t_\beta - t_\alpha)/2), \quad (3.9) \]

where \( t_\beta = \frac{4\lambda_\beta}{U} \).

Next step, we notice that the second term on the left hand side the eq.(3.9) is small and therefore we may incorporate it into the argument \( x \) of \( \theta(x) \), giving

\[ N \theta\left[t_\beta - \frac{4}{NU} \sum_{j=1}^{N} \sin(k_j)\right] = 2\pi J_\beta + \sum_{\lambda_\alpha=1}^{M} \theta((t_\beta - t_\alpha)/2), \quad (3.10) \]

With the substitution \( x_\beta = t_\beta - \frac{4}{NU} \sum_{j=1}^{N} \sin(k_j) \), this equation is reduced to the Bethe equation for an isotropic Heisenberg antiferromagnet on a ring having \( N \) sites and \( M \) down spins:

\[ N \theta(x_\beta) = 2\pi J_\beta + \sum_{\alpha=1}^{M} \theta((x_\beta - x_\alpha)/2), \quad (3.11) \]

One sees that the solution for \( x_\alpha \) is independent of both the flux \( f \) and on its repulsion \( U \).
Let us substitute the variables \( \lambda_\beta = U(x_\beta + \frac{4}{NU} \sum_{j=1}^{N} \sin(k_j))/4 \) into equation (2.5) for the momenta \( k_j \) and study the resulting equation in the variables \( x_\beta \). Then, analogously to the expansion of eq.(2.6), expanding the eq.(2.5) in powers of \( \alpha \), we get a new form of the first Bethe equation, namely

\[
Lk_j = 2\pi I_j + 2\pi f + \sum_{\beta=1}^{M} \theta(x_\beta) + \frac{8 \sum_{l=1}^{N} (1 - N\delta_{lj}) \sin k_l}{NU} \sum_{\beta=1}^{M} \frac{1}{1 + x_\beta^2} \quad (3.12)
\]

For parameter \( \alpha \) small (dilute density), this system of linear equations may be solved, with the result for the momentum

\[
k_j = \frac{2\pi}{L}(I_j + f + \frac{1}{N} \sum_{\beta=1}^{M} J_\beta + \frac{8B}{UL}(\frac{1}{N} \sum_{l=1}^{N} I_l + f + \frac{1}{N} \sum_{\beta=1}^{M} J_\beta)) \quad (3.13)
\]

and for the ground state energy

\[
E_{\text{ground}} = -\tilde{D} \cos\left(\frac{2\pi}{L}\left(f - \frac{p}{N} + \frac{I_{\text{max}} + I_{\text{min}}}{2} + \frac{8B}{UL}(\frac{1}{N} \sum_{l=1}^{N} I_l + f - \frac{p}{N})\right)\right) \quad (3.14)
\]

where \( p = -\sum_{\beta=1}^{M} J_\beta \), \( I_{\text{max}} \) and \( I_{\text{min}} \) are the maximal and minimal charge quantum numbers,

\[
\tilde{D} = 2 \sin(\pi N/\tilde{L})/\sin(\pi/\tilde{L}),
\]

and

\[
\tilde{L} = L(1 + \frac{8B}{UL}).
\]

The magnitude

\[
-B = -\sum_{\beta=1}^{M} \frac{1}{1 + x_\beta^2},
\]

which is a real number, has the physical meaning of the total energy( ground state + excitations) of the antiferromagnetic Heisenberg Hamiltonian with the exchange constant equal to 1/4.
The variables $x_\alpha$ are satisfied by the Bethe ansatz equation for that Hamiltonian (3.11). Actually, the value $B(k)$ is a function associated with the excitations spectrum, where $k$ is the momentum of the spin-wave excitation of the Heisenberg Hamiltonian $k = 2\pi \sum_\alpha J_\alpha / N$. For the antiferromagnetic state of Heisenberg Hamiltonian the dependence $B(k)$ has been studied in detail in many works: first being estimated by Anderson [23], within a spin wave approximation and then by des Cloizeaux and Pearson [24] [25] in the framework of the Bethe ansatz obtaining the exact form of this spectrum.

At large number $N$ this dependence has the form:

$$B(k) = (N\log(2 - 1/4) - \frac{\pi}{2} |\sin k|)/4 \quad (3.15)$$

The ground state energy (first two terms) has been obtained by Hulthen [26] in the framework of the Bethe method [27]. It was seven years later after Bethe introduced his method (the Bethe ansatz), applying it to the ferromagnetic Heisenberg chain.

IV. SINGLE SPIN

Let us consider the case with $M = 1$, i.e. when the system is strongly polarized. Here, we find an analytical solution, permitting eq. (3.11) to be immediately solved yielding

$$x_\beta = \tan\left(\frac{\pi J_\beta}{N}\right) \quad (4.16)$$

The function $B(J_\beta)$ can then be calculated from this solution, giving

$$B = \cos^2\left(\frac{\pi J_\beta}{N}\right).$$

Now we must consider two independent cases, when $N$ is odd and when $N$ is even. Let $N$ be an even number, then $J_\beta$ is an integer, $p$ say. Then, for the ground state energy we get the explicit expression
\[ E_{\text{even}-N} = -2 \frac{\sin \left( \frac{\pi N}{L + \frac{p \pi}{N} \cos^2 (p \pi/N)} \right)}{\sin \left( \frac{\pi}{L + \frac{p \pi}{N} \cos^2 (p \pi/N)} \right)} \cos \left[ \frac{2\pi}{L} \left( f - \frac{p}{N} \right) \right]. \] (4.17)

The ground state energy-flux dependence is this a function with two types of oscillation. The period of the first of oscillation is equal to the single flux quantum, while the period of the second type of oscillations is equal to the fraction \(1/N\) of the elementary flux quantum. Let us describe this dependence. Each period of the single flux quantum then consists of \(N\)-parabolic like curves, where each of these curve is labelled by \(p\): \(0 \leq p \leq N\). In the limit of large \(N\) we may put as \(f = p/N\) to get an exact single flux periodic function, which has minima and maxima at integer flux values and at half-odd integer flux values respectively.

For an illustration on Fig.1 is presented the energy-flux dependence for the Hubbard ring having 1000 sites with 6 electrons and a single down spin at the coupling \(U/t = 10\). On this Figure one sees 7 parabolic curves having minima at \(p/6\) flux values, and 6 cusps associated with intersections between each two neighbour parabolas. The envelope of these parabolic curves is the single flux periodic function \(\cos^2 \pi f\).

Next we turn to the case of \(N\) odd, when the numbers \(J_\beta\) are half-odd integer. So, let us take \(J_\beta = p + N/2\), where \(p\) is an arbitrary integer. Given this, then the ground state energy may be expressed as

\[ E_{\text{odd}-N} = -2 \frac{\sin \left( \frac{\pi N}{L + \frac{p \pi}{N} \sin^2 (p \pi/N)} \right)}{\sin \left( \frac{\pi}{L + \frac{p \pi}{N} \sin^2 (p \pi/N)} \right)} \cos \left[ \frac{2\pi}{L} \left( f - \frac{p}{N} \right) \right], \] (4.18)

Again, we find that the ground state energy dependence has both the single flux quantum and the fractional flux quantum periodicity, described by the first and the second factors, respectively. In the region of the one period of the single flux quantum this dependence again consists of \(N\)-parabolic like curves, labelled by the integer \(p\): \(0 \leq p \leq N\). In the limit of large \(N\) we may put as \(f = p/N\) to get an exact single flux periodic function. However,
for an odd number of particles, this function now has maxima at integer flux values and minima at half-odd integer flux values.

For an illustration on Fig. 2 is presented the energy-flux dependence for the ring having 1000 sites with 11 electrons and a single down spin at the coupling $U/t = 10$. On this Figure one sees 12 parabolic curves having minima at $p/11$ flux values, and 11 cusps associated with intersections between each two neighbour parabolas. The envelope of these parabolic curves is the single flux periodic function $\sim - \sin^2 \pi f$.

Several conclusions can be drawn immediately. Firstly, the parity effect, which disappeared in the $\alpha$ small limit, is again recovered when the first correction is taken into account. In other words, the ground state energy-flux dependence is different for the even and odd numbers of fermions on the ring. To be more precise, the behavior of the flux-energy dependence for an even number of particles is shifted by a half flux quantum in comparison with the case of an odd number of particles. This is similar to the parity effect observed on a ring with interacting spinless fermions [11], [12]. However, in comparison with spinless fermions on the Hubbard ring the, cases of even and odd number of particles are reversed. The parity effect discussed might be related to the number of down-spin electrons, which is $N - 1$. In comparison with the case of spinless fermions, because of this shifting of the particle number $N$ by $N - 1$, the parity effect discussed might be reversed. In this case, again, one may connect the parity effect with the creation of the statistical flux $\pi$, due to Fermi statistics [3].
V. SPIN UP-DOWN EQUILIBRIUM

In this section we will follow the important suggestion by Yu and Fowler [8] about the possible role of spin excitations in the Aharonov-Bohm effect on the Hubbard ring in the limit of large $U/t$. Our results extend their analysis of the first corrections in the strong coupling limit as well as being applicable to systems with the low electron density at arbitrary coupling. Now, let us consider the general case, when the number of spin-up fermions is approximately equal to the number of spin-down fermions, i.e. it may differ by $\pm 1$.

To investigate the behavior of the flux-energy dependence for such a general situation, we must consider four independent cases, when the numbers $N$ and $M$ are even or odd. As we will see below (also, see, above the single spin case $M=1$) for the ground state energy flux dependence only the parity of the number $N$ will be important.

Let us consider the first of these cases, when $N$ and $M$ are even. The quantum numbers (which are half-odd integer) of the Heisenberg Hamiltonian, $J_\alpha$, for the ground state are distributed symmetrically with respect to origin, so that $\sum_\beta J_\beta = 0$. The quantum numbers of the charge degrees of freedom $I_\alpha$ for the Hubbard Hamiltonian are integer with $\sum I_j = -\frac{N}{2} = -M$.

However, at zero flux the minimum energy of the Hubbard Hamiltonian will correspond to the set of quantum number $\{J_\alpha\}$, but with nonvanishing sum $\sum J_\alpha = M$, which does not correspond to the ground state of the Heisenberg Hamiltonian. Such a state is associated with the single magnon excitation of the Heisenberg chain, having a momentum $k_0 = \frac{2\pi P_0}{N} = \pi$ and the excitation energy $B(k_0) = B_0$. The distribution of quantum numbers $\{J_\alpha\}$ associated with the ground state of the Hubbard ring at zero flux has a hole near the origin, i.e. is of the form:
\[ J_1, \ldots, J_M = -\frac{M-1}{2}, \ldots, -\frac{1}{2}, \text{hole}, \frac{3}{2}, \ldots, \frac{M-1}{2} \]  \tag{5.19}

With increasing flux this hole in the distribution is shifted to another place so that \( \sum \alpha J_\alpha = M - 1 \) and, therefore, the magnon momentum decreases. In other words with the flux the hole in the distribution moves to its left side. Therefore, with that motion of the hole, exactly speaking, with the new position of the hole the magnon excitation will have the new momentum \( k < k_0 \) and the energy \( B(k) < B_0 \). Exactly speaking, the obtained dependence \( B(k) \) describes the low-energy excitation of the discrete antiferromagnetic Heisenberg chain, as first discussed by des Cloiseaux and Pearson \[24, 25\].

With the dependence \( B(k) \) taken into account the appropriate ground state energy dependence of the Hubbard model on the flux may be written in the form:

\[ E_{\text{even}}^{M-N} = -\tilde{D} \cos \left[ \frac{2\pi}{L} (f - \frac{p}{N}) \right], \tag{5.20} \]

where the flux value \( f \) is changed in the region \( (2p - 1)/2N < f < (2p + 1)/N \) and \( \tilde{D}(k) = 2 \sin(\pi N/L(k))/\sin(\pi/L(k)) \), with \( k = 2\pi p/N \) being the magnon momentum of the finite Heisenberg chain with \( N \) sites, where \( p \) takes integer values.

The dependence consists of equidistant parabolic segments. The position of each parabolic segment along the vertical axis depends upon the value \( -\tilde{D}(k) \). As the integer \( p \) increases, the magnon momentum \( k \) and the energy of this magnon excitation \( B(k) \) also increases, so long as \( p \leq \frac{N}{2} \). The magnon energy then has a maximal value at \( p = \frac{N}{2} \).

This means that the value \( -\tilde{D}(k) \) associated with the position of bottom of the \( p \)-th parabolic segment along the vertical axis decreases monotonically. As a result, we see that the envelope curve which passes through the bottom of these parabolic segments is similar to the curve describing the spectrum of the spin-wave excitations of the antiferromagnetic Heisenberg chain with \( N \) sites. Note, that this curve is described by the equation \( E_{\text{gr}}(k) = \).
$-\tilde{D}(B(k))$. The flux dependence of this envelope curve can be obtained if we substitute $k = 2\pi f$. It is a half flux quantum periodic function.

Now, let us consider the second case when $N$ is even, but $M$ is an odd number. In this case, at zero flux, the ground state energy corresponds to half-integer numbers $I_j$ and integer numbers $J_\alpha$ distributed symmetrically around the origin (see, for example [6]). That is, here,

$$\sum_j I_j = \sum_\beta J_\beta = 0 \quad (5.21)$$

The analysis of this case is similar to that above. The given distribution of quantum numbers $J_\alpha$ corresponds to the ground state energy, both of the antiferromagnetic Heisenberg chain and the Hubbard ring. However, including the flux the ground state energy of the Hubbard ring will now correspond to the nonvanishing sum $\sum_\beta J_\beta$, which is negative. This means that there will be created a magnon excitation similar to that described above, associated with the appearance of a hole in the distribution of $J_\alpha$. With the increasing flux the hole will move from the left to the right of the distribution.

We, therefore, once again have the flux-energy dependence of the Hubbard ring determined via the low energy excitations of the Heisenberg chain. This dependence again consists of $N$ parabolic segments and is described by a similar formula to eq. (5.20). The envelope, which passes through the bottom of these parabolic segments is described with the aid of the same equations $-\tilde{D}(2\pi f)$, i.e. it has cusps at its minima, which are at integer and half-integer flux values and smooth maxima at the values of the flux exactly midway between the cusps.

Let us now consider the case when $N$ is odd and $M$ is even. The quantum numbers $I_j$ are integer and are symmetrically distributed around the origin, i.e. $\sum I_j = 0$. The quantum numbers $J_\alpha$ are also integers. The ground state energy of the Heisenberg Hamiltonian
corresponds to the dense distribution with the sum $\sum_\alpha J_\alpha = -\frac{M}{2}$. However, the ground state energy of the Hubbard ring with zero flux values must correspond to the state with $\sum_\alpha J_\alpha = 0$. This sum can equal zero when the associated state of the Heisenberg chain is not a ground state, at rather an excited state. This means that it is a state with the hole in the $\{J_\alpha\}$ distribution. One sees that in this case the hole will be created exactly in the origin:

$$J_1, \ldots, J_M := -\frac{M}{2}, \ldots, -1, \text{hole}, 1, \ldots, \frac{M}{2}$$ (5.22)

This state corresponds to the top in the spin-wave excitations of the Heisenberg chain, i.e. to the spin-wave excitation with momentum $k = \frac{\pi}{2}$. As the flux changes, the parabolic segment with its minimum at zero flux is exchanged with another the parabolic segment which has its minimum at the flux $f = \frac{1}{N}$.

This parabolic segment will correspond to the new set of $J_\alpha$, where the hole is in the position formerly occupied by $J_\alpha = 1$, so now $\sum_\alpha J_\alpha = -1$. The appropriate magnon excitation associated with this parabolic segment will have momentum $k = \frac{\pi(N-2)}{2N}$ and energy $B\left(\frac{\pi(N-2)}{2N}\right)$, which are lower than the momentum $k = \pi/2$ and the maximal energy $B(\pi/2)$, respectively.

With the next increasing the flux, there will occur a transition to a new parabolic segment, corresponding to a new set of $J_\alpha$ with the hole at the position $J_\alpha = 2$. This is the parabolic segment corresponding to the magnon excitations with momentum $k = \frac{\pi(N-4)}{2N}$ which has a lower energy than the magnon excitation in the previous case. Therefore, at this flux the minimum energy of the appropriate parabolic segment decreases. The absolute minimum is reached at the quart and $3/4$ of the flux quantum, if the flux is changed within a single flux quantum.
The envelope curve, which goes through the minima of these parabolic segments is a half flux quantum periodic function, which has smooth maxima at integer and half integer flux values. This curve may be described with the aid of the formula \( E_{\text{ground}} = -\tilde{D}(2\pi f) \).

Let us consider now the last case, which is \( N \) and \( M \) odd. Here, both the \( J_\alpha \) and \( I_j \) are half-odd integer. The energy of the Hubbard Hamiltonian, which corresponds to the dense distributions with \( \sum I_j = -\frac{N}{2} \) and with \( \sum J_\alpha = \frac{M}{2} \) and assumed to be a ground state is not a ground state of the Hubbard ring at zero flux. But this dense distribution of \( \{J_\alpha\} \) is the true ground state of the Heisenberg Hamiltonian.

The ground state of the Hubbard ring will correspond to the distribution of the quantum number \( \{J_\alpha\} \) with the hole at the position \( J_\alpha = -\frac{1}{2} \) and the sum \( \sum_{\alpha=1}^{M} J_\alpha = +\frac{2M+1}{2} = +\frac{N}{2} \). That is, to say, in the ground state of the Hubbard ring there already exists the magnon excitation with momentum \( k = \frac{2M+1}{N} \pi = \pi/2 \), which is, in fact, associated with the top in low energy spin wave excitation spectrum of the Heisenberg chain. The energy-flux dependence will again consist of parabolic segments as described above. The minima of these segments will be located at the flux \( f = \frac{f}{N} \). The energy behavior of the minima of these segments will be exactly the same as described in the previous case.

The picture discussed is illustrated in Figs 3 and 4. In Fig.3 one sees the coexistence of the half-integer and of the fractional \( 1/N \) Aharonov-Bohm effects. The ring has 7500 sites filled with 25 electrons. There are two types of oscillations present: half-flux quantum periodic, which has the largest amplitude and the other fractional - \( 1/25 \) - flux quantum periodic, the amplitude of which is much smaller. The envelope curve of the fractional oscillations is a half flux quantum periodic function, with smooth maxima at integer and half-integer flux quanta \( f = ..., -1/2, 0, 1/2, 1, ... \) and cusps, which are minima, in between. This is due to the parity effect, since the number of particles on the ring is odd.
For contrast, in Fig. 4 we present a case for an even number of electrons on the ring with $N = 10$, and $L = 1000$. Again, as in Fig. 3 the coexistence of the half-integer and fractional Aharonov-Bohm effects is observed and because there are 10 particles the fractional $1/10$ flux quantum oscillations arise. The envelope curve of these fractional oscillations is as in Fig. 3 a half flux quantum periodic function, which, however, has minima at integer and half-integer flux quanta.

In the comparing these two Figures, one notes that with the change of the number of particles from odd to even, the position of the maxima is shifted by a quarter-flux quantum. This is precisely due to the parity effect similar to the parity effect for spinless fermions [6], [12], [11]. That is on the Hubbard ring we have obtained the parity effect which is similar to one for spinless fermions. The effect is due to the contribution of the spinon excitation spectrum in the AB effect. Note, that in the second case (Fig. 4) the difference in amplitudes between the half-integer and fractional $1/N$ oscillations is smaller, indicating that the fractional effect is more important at small number of particles. With increasing $U$ or $L$, this difference may disappear.

**VI. FRACTIONAL $\frac{M}{N}$ AHARONOV-BOHM EFFECT.**

In previous sections we have shown that the AB periodicity for the cases of $M = 1$ and $M = N/2$ are different (single and half flux quantum, respectively). Therefore, there occurs the question: what kind of AB periodicity occurs at arbitrary $M$: $1 < M < N/2$? Let us now consider the case when $M$ is arbitrary, but $M << N$. For this general case, when the density of down-spin particles $M/N$ is much smaller, then the density of up-spin particles $1 - M/N$ can again be found an analytic solution. In this case we expand equation (3.11)
with in the parameter $M/N$. We can assume that the values of $x_\beta$ in eq. (3.11) are small.

Using these assumptions the eq. (3.11) may be written in the form

$$x_\beta + \sum_{\alpha=1}^{M} \frac{x_\alpha - x_\beta}{2N} = \frac{\pi J_\beta}{N} \quad (6.23)$$

i.e. we have obtained a system of linear equations, which can be immediately solved, giving

$$x_\beta = \frac{2\pi J_\beta}{N - 2M} - \sum_{\alpha=1}^{M} \frac{\pi J_\alpha}{(2N - M)N} \quad (6.24)$$

This solution looks like the momentum spectrum of a free particle on a chain of $2N - M$ sites (the first term of the right hand side of the equation), plus an additional fictitious flux associated with the second term of the right hand side of the equation. From the solution obtained one sees that the values $x_\beta$ are small if $\pi M/(2N - M) << 1$, which is equivalent to our small parameter $M/N << 1$, and therefore $x_\alpha << 1$ does, indeed, hold.

On the other hand this analysis has shown that the second term of the right hand side of eq. (3.11) is smaller that the first one in the parameter $M/N << 1$. Therefore, in that case the solution may be obtained in the form:

$$x_\alpha = \tan \frac{\pi J_\beta}{N} \quad (6.25)$$

The substitution of which into $B[x_\alpha]$ gives

$$B(k) = \frac{M}{2} + \frac{1}{2} \sum_{\beta=1}^{M} \cos \frac{2\pi J_\beta}{N} \quad (6.26)$$

The choice of the set $J_\alpha$ determines the dependence of the low lying excitations $B(k)$. Now we must consider four independent cases associated with the parity of numbers $N$ and $M$.

Let, first us consider the case, when $N$ is even and $M$ is odd. Then the ground state energy corresponds to the vanishing sums of the integers $J_\alpha$, $\sum_\alpha J_\alpha = 0$ and of the half-integers $I_j$, $\sum_j I_j = 0$. Both of these sets are densely distributed, symmetrically around the
origin. The excitation will correspond to the hole in the distribution of \( J_\alpha \). For example, when \( p = 1 = - \sum J_\alpha \), then the hole is created on the left hand side, from the left edge. For the value \( p = 2 \), the hole is moved on to the next position. In general, if the hole in the position \( J_\alpha \) then we can write

\[
p = - \sum_{\beta=1}^{M+1} J_\beta + J_\alpha
\]  

(6.27)

where the sum is taken over the dense distribution. Therefore, if \( p \leq M \), this sum equals \( \sum_{\beta=1}^{M+1} J_\beta = -(M+1)/2 \). In the similar way, by splitting into to terms, the sum over the dense distribution of \( M + 1 \) quantum numbers \( J_\beta \) and the term associated with the hole at the place of \( J_\alpha \) we find for the energy of such excitations:

\[
B(k) = \frac{1}{2} \left( \frac{M}{2} + \frac{\sin \frac{\pi(M+1)}{N}}{\sin \frac{\pi}{N}} \cos \frac{\pi}{N} - \cos \frac{2\pi J_\alpha}{N} \right)
\]  

(6.28)

Then, expressing \( J_\alpha \) from eq.(6.27) and substituting it into eq.(6.26) we get the dependence of \( B(2p/N) = B(k) \) where \( p \leq M \). Note, that the case of \( p = - \sum_{\beta=1}^{M} J_\beta = M \) corresponds to the ground state distribution of \( J_\alpha \) shifted by 1 to its left side. When \( p > M \), in this shifted distribution there will again appear the hole, initially, at the beginning on the left hand side, from the left edge.

With increasing \( p \) this hole will move from the left to the right side, once \( M < p \leq 2M \). At the value \( p = M \), the distribution of \( J_\alpha \) will be dense, but shifted by 2 to the left in comparison with the distribution associated with the value \( p = 0 \). Thus, for an arbitrary value of \( p \), we can write

\[
p = +l(M + 1) + \left( \frac{M + 1}{2} \right) - J_\alpha
\]  

(6.29)

The energy of the associated excitations is equal to:
\[ B(p) = \frac{1}{2} \left[ M + \frac{\sin \frac{\pi (M + 1)}{N}}{\sin \frac{2\pi l + \pi}{N}} \cos \left( \frac{2\pi l + \pi}{N} \right) - \cos \frac{2\pi}{N} (p - \frac{M + 1}{2} - l(M + 1)) \right] \]  

(6.30)

where \( l = 0 \), if \( 0 \leq p \leq M \), \( l = 1 \), if \( M \leq p \leq 2M \), \( l = 2 \), if \( 2M \leq p \leq 3M \) and so on.

Thus, the ground state energy has cuspidal minima (cusps) at \( p = nM \), where \( n \) is an integer. So, if we substitute \( p = lM \), we have for the envelope functions of these cusps:

\[ B(p) = \frac{1}{2} \left( M + \frac{\sin \left( \frac{\pi M}{N} \right)}{\sin \frac{2\pi l}{N}} \cos \frac{2\pi l}{N} \right) \]  

(6.31)

Since \( p \leq \frac{N}{2} \), the function \( B(l) \) is monotonically decreasing with \( l \), having a maximal value when \( l \sim \left[ \frac{N}{2M} \right] \), where the brackets indicate that we take the integer part of the value \( \frac{N}{2M} \). The spectrum of the excitations of the Heisenberg chain reflects the flux quantum periodicity of the Hubbard ring, as described above. Therefore, we come to the conclusion, that in addition to fractional \( 1/N \), and integer on the Hubbard ring there also occurs fractional \( M/N \) flux quantum periodicity.

For an illustration we give several examples for the ring with \( U/t = 10 \) and \( L = 10^4 \), presented in Fig.5. In the first example (see, Fig.5a) the energy-flux dependence is given for \( N = 20 \) and \( M = 3 \) in the region of the half flux quantum. Since the number of particles is not large one sees, the very pronounced \( 1/N \) oscillations associated with the single parabolic like segment. However, one sees already \( M/N \) oscillations with much smaller amplitude. So, the first, fourth and seventh parabolic like segments have minima lower than neighboiring parabolas. If the number of particles increases, for example, \( N = 30 \) (see, Fig.5b, where we taken \( M = 5 \)) the amplitude of \( M/N \) oscillations increases. In this case the amplitudes of \( 1/N \) and are \( M/N \) oscillations are, already, compared. With the next increasing number of particles \( N \), for example, when \( N = 40 \) and \( M = 7 \) presented in Fig.5c, the amplitude of \( M/N \) oscillation becomes larger than the amplitude of the \( 1/N \) oscillation.
Let us consider the second case, when $N$ is odd and $M$ is even, which corresponds to two types of integer quantum numbers $J_\alpha$ and $I_j$, with $\sum I_j = 0$ and $\sum_\alpha J_\alpha = -\frac{M}{2}$. However, the ground state of the Hubbard ring will correspond to an excited state of the Heisenberg chain with the hole in the origin, i.e. the ground state corresponds to the sum: $\sum_\alpha J_\alpha = 0$.

The excitation associated with the value $p = 1$ corresponds to the shift of this hole to the position $J_\alpha = 1$. So with the increasing $p$ the hole moves to right. Therefore, we can write $p = -\sum_{\alpha=1}^{M+1} J_\alpha + J_\beta$, where $p \leq M/2$ and the sum is taken over the dense distribution.

As $p$ is increased from $M/2$ the hole will be created on the left edge and moved to the right edge, where $p \leq (M + M/2)$. With the next increase of $p$ this process will be repeated and so on. Thus, we may write in general form that

$$p = +l(M + 1) + J_\beta,$$

where the first term related to the sum $\sum_{\alpha=1}^{M+1} J_\alpha$.

The envelope function of parabolic like curves associated with the ground state energy-flux dependence is calculated in a similar way and equals:

$$B(p) = \frac{1}{2}[M + \sin\frac{\pi(M+1)}{N}\cos\frac{2\pi l}{N} - \cos\frac{2\pi}{N}(p - l(M + 1))] .$$

(6.33)

where $l = 0$, if $-(M)/2 \leq p \leq (M)/2$; $l = 1$, if $(M)/2 < p \leq (M)/2 + M$, $l = 2$, if $(M)/2 + M \leq p \leq (M)/2 + 2M$ and so on.

This dependence is shifted by $\frac{M}{2N}$ is comparison with the previous case. It consists of cuspoidal minima and smooth maxima in midway between. In comparison with the previous case here the cuspoidal minima, one of which existed at the value $p = 0$, are shifted by the same value of the flux, too, and are exchanged in the positions with smooth maxima. Thus, the properties of the envelope function are the same as in the previous case. For an
illustration in Fig.6 we present the energy-flux dependence for the ring with $U/t = 10$. So, it is clearly coexist $6/19 \sim 1/3$ and $1/19$ flux quantum periodicities for the ring with 19 electrons having 6 down-spins for number of sites $L = 1000$ (see, Fig.6a). The parabolic like curves are associated with $1/N$ type oscillation. The envelope function of these curves is $6/19 \sim 1/3$ flux quantum periodic function. The cusps of this function are located at $3/19$ and $9/19$ flux quantum. With the increasing number of particles the amplitudes of both type of fractional oscillations decrease. This can be seen in Fig.6b, where the particles number was changed to 49 and the number of sites was taken as $10^4$. Although with the next increase of the number of particles at the fixed value of $M$ the amplitudes of both type of oscillation decrease, the fractional $1/N$ oscillations begin to wash out, however the fractional $M/N$ oscillations are still exist. This is shown in Fig.6c where the number of particles was taken as $N = 99$ on the ring with the same the number of sites, as in Fig.6b. On the other hand when both numbers, the number of particles $N$ and the number of down-spins $M$ increase, the amplitude of $M/N$ oscillation increases and it is compared with the amplitude of the main single flux quantum periodic AB oscillation. In the latter the fractional $1/N$ oscillation is completely washed out, although it is an origin for the fractional $M/N$ oscillation. The fractional $23/99$ AB oscillation is presented in Fig.6d, where we consider the ring with $M = 23$ and $N = 99$. One sees on this Figure that the fractional $1/99$ oscillation is practically washed out.

The other two cases only differ from these already considered ones only by an appropriate shift. In both of them the ground state of the Hubbard ring corresponds to the excited state of the Heisenberg chain with the value

$$p_0 = - \sum_{\alpha=1}^{M} J_\alpha = -\frac{N}{2}. $$

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Because of the $M$-periodicity, this state may be mapped into another one, where the sum $\sum_{\alpha=1}^{M} \tilde{J}_\alpha = 0$ vanishes, i.e. this can be obtained with the aid of the transformation:

$$\tilde{J}_\alpha = J_\alpha - l_1,$$

where $l_1$ is the integer part of the ratio $\frac{N}{2M}$. Therefore, we can write

$$\sum_\alpha \tilde{J}_\alpha = \sum_\alpha J_\alpha - l_1 M = q,$$

where $q < \left[\frac{M}{2}\right]$.

Thus, the problem has been reduced to the one, discussed above, but with the difference, that the ground state of the Hubbard ring will correspond to the excited state of the Heisenberg chain where the hole in the dense distribution of the set $J_\alpha$ is located at the $-q$-th position, which is determined by eq.(VI). The other excited state of the Heisenberg chain (or the ground state energy-flux dependence of the Hubbard ring) is associated with the shift of this hole to the right edge. Then all excitations may be described exactly by the same manner as above. That is we may write

$$\tilde{p} = -\sum_{\alpha=1,\text{dense}}^{M+1} \tilde{J}_\alpha + \tilde{J}_\beta$$

(6.34)

Now we must consider the two particular cases (1) $N$ and $M$ are even and 2) $N$ and $M$ are odd), independently.

In the first case, the values $J_\alpha$ or $\tilde{J}_\alpha$ are half-odd integer and from eq.(6.34) we get

$$\tilde{p} = \frac{M + 1}{2} + \tilde{J}_\beta.$$

(6.35)

The excitation energy is estimated as above yielding:

$$B(p) = \frac{1}{2} \left[M + \frac{\sin\frac{\pi(M+1)}{N}}{\sin\frac{\pi}{N}} \cos\left(\frac{2\pi l + \pi}{N}\right) - \cos\frac{2\pi \tilde{J}_\beta}{N}\right].$$

(6.36)

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From eq. (6.35) and making the inverse transformation from $\tilde{J}_\alpha$ to $J_\alpha$ we finally get

$$B(p) = \frac{1}{2} \left\{ M + \frac{\sin \frac{\pi(M+1)}{N}}{\sin \frac{\pi}{N}} \cos \left( \frac{2\pi l + \pi}{N} \right) - \cos \frac{2\pi}{N} \left[ p + \frac{N}{2} - \frac{M+1}{2} - l(M+1) \right] \right\} \quad (6.37)$$

where the value $l$ is chosen thus $l = 0$, if $-N/2 \leq p \leq -N/2 + M$; $l = 1$, if $-N/2 + M \leq p \leq -N/2 + 2M$, $l = 2$, if $-N/2 + 2M \leq p \leq -N/2 + 3M$ and so on.

For an illustration we present in Fig. 7 the dependence of the ground state energy for the ring with 1000 sites and $U/t = 10$. The number of electrons is taken as $N = 20$ amd $M = 6$.

One sees on this Figure that the dependence of the envelope function of the parabolic curves is monotonically decreased with the flux increases from zero. This dependence has cusps, which are repeated with the periodicity 6/20. So we have here a quasi $\sim 1/3$ flux quantum periodicity.

Similarly way we can obtain the excitation energy for the second case, when the parity of particles number is odd, i.e. $N$ and $M$ are odd:

$$B(p) = \frac{1}{2} \left\{ M + \frac{\sin \frac{\pi(M+1)}{N}}{\sin \frac{\pi}{N}} \cos \left( \frac{2\pi l + \pi}{N} \right) - \cos \frac{2\pi}{N} \left[ p + \frac{N}{2} - l(M+1) \right] \right\} \quad (6.38)$$

where the value $l$ is chosen thus $l = 0$, if $-N/2 - (M - 1)/2 \leq p \leq -N/2 + (M - 1)/2$; $l = 1$, if $-N/2 + (M - 1)/2 \leq p \leq -N/2 + (3M - 1)/2$, $l = 2$, if $-N/2 + (3M - 1)/2 \leq p \leq -N/2 + (5M - 1)/2$ and so on.

For an illustration we present in Fig. 8 the dependence of the ground state energy for the ring with 1000 sites and $U/t = 10$. The number of electrons is taken as $N = 19$ and $M = 5$.

One sees on this Figure that the dependence of the envelope function of the parabolic curves is monotonically decreased with the flux increases from zero. This dependence has cusps, which are repeated with the periodicity 5/19. So, here again as in the previous case we have a quasi-$\sim 1/3$- flux quantum periodicity.
Thus, for all four cases the excitation energy is as \( M/N \) periodic function, having cusps repeated with the same periodicity. To study the flux dependence we must substitute the flux value \( f \) instead of the value \( p/N \) of the excitations of Heisenberg ring. Generally speaking, in the last two cases the ground state energy-flux dependence of the Hubbard ring related to these excitations of the Heisenberg chain is shifted by half-flux quantum in comparison with two previous cases, respectively. But, the qualitative shape of these dependencies are not changed.

**VII. FINE STRUCTURE**

One general conclusion that can be drawn here that the complete picture of the Aharonov-Bohm effect must include a fine structure. The latter appears when we take into account the electron-electron interactions and is characterized by oscillations with an amplitude and a period smaller than the conventional one. In fact, there is a hierarchy of flux periodicities which depends implicitly on the Zeeman interaction.

The Fourier spectrum of the energy-flux dependence for the Hubbard ring mainly consists of two or three harmonics. The number and type of these harmonics depend on the contribution from the Zeeman interaction. One of these harmonics is conventional and has a single flux quantum period; the second one has the fractional \( 1/N \) flux quantum period; the third one has half-flux quantum period or fractional \( M/N \) period. The latter appears when the number of down spin particles \( M \) and the number of up-spin particles \( N - M \) are different.

Let us analyse the fine structure. The amplitude of the single-flux or the fractional \( M/N \) periodic oscillations is proportional to the parameter \( n^2 \alpha \), where \( n = \pi N/L \) and we
use dimensionless units. The amplitude of the fractional $M/N$ oscillation is smaller, by factor $M/N$, than the amplitude of oscillations with the integer period. Here, we assume that $M \ll N$. When $M \sim N/2$ the situation is reversed and the $M/N$ oscillation has a dominant amplitude. The amplitude of the fractional $1/N$ flux quantum periodic oscillations is proportional to $\frac{n^2}{N^3}$ and does not depend on the value $U$. One sees also that both types of oscillations only depend upon $U$ and $L$ through their product $UL$ or in other words they only depend upon $\alpha$. With increasing $N$ at constant $n$ and $\alpha$, the amplitude of the fractional $1/N$ oscillations vanishes.

The single flux quantum harmonic coexisted with fractional $M/N$ periodic harmonics show the parity dependence. That is there is a difference in the energy-flux dependencies for the cases of an even and an odd number of fermions. This is in contrast with the fractional $1/N$ periodicity, which has no the parity dependence. In the present case the parity effect for an even and an odd number of fermions is transposed with respect to the case of spinless fermions.

It is useful explicitly to write an envelope of the flux-dependence. For example in the case, where there is up spin- down spin equilibrium (i.e. $M=\frac{N}{2}$), for the two cases of an even or an odd number of particles on the ring the flux energy dependence has the form:

$$E_{erv} = \frac{n^3 t^2}{3 U} \left\{ \begin{array}{l} \sin 2\pi f \quad \text{if } N \text{ is even} \\ \cos 2\pi f \quad \text{if } N \text{ is odd} \end{array} \right\}. \quad (7.39)$$

For the other values of $M$ the prefactor will be the same, but the oscillation factor will be changed to reflect the fractional $M/N$ flux quantum periodicity.

On the other hand the amplitude of the fractional $1/N$ oscillations is estimated to be $\sim t n^2 / N^3$, which is rapidly decreases with increasing $N$. Comparing this amplitude with the amplitude of $E_{erv}$ we get the criterion for when the single flux quantum or $M/N$ flux
quantum periodicity becomes dominant over the fractional $1/N$ ones; this happens when

$$N \gg \sqrt[4]{UL}t$$  \hspace{1cm} (7.40)

However, to investigate this problem we have used the parameter $\alpha = \frac{Nt}{UL} \ll 1$. The region where the single or $M/N$ flux quantum periodicity is dominant is

$$\sqrt[4]{UL}t \ll N \ll \frac{UL}{t}$$  \hspace{1cm} (7.41)

One sees that at large $L$ the limits when such behavior exists may be easily recovered.

In conclusion, then in the dilute density limit or for strong coupling we have discovered two different regimes of the Aharonov-Bohm effect, namely, fractional $1/N$ and integer coexisting with fractional $M/N$ and $1/N$. Integer and fractional $M/N$ periodic oscillations appear simultaneously and may have comparable amplitudes, which are distinct by numerical factor. As a rule, in high magnetic fields the amplitude of $M/N$ oscillations is always smaller than the amplitude of integer ones.

Thus, a very surprising implicit role of the Zeeman interaction has been found, i.e. with increasing magnetic field, as the contribution of the Zeeman energy increases, in addition to the fractional $1/N$ flux quantum periodicity, there appears the integer-fractional $M/N$ flux quantum periodicity. With increasing the magnetic field, i.e. with decreasing $M$ the half flux quantum periodicity at $M = N/2$ transforms into fractional $M/N$ ones and integer one. Finally, in strong field, when the system is almost polarized, only the integer flux quantum periodicity remains.

This transformation is not continuous but singular. The period of this fine structure of the AB effect changes with each spin flip. For almost polarized system, we have the coexistence of integer and fractional $1/N$ flux quantum periodic oscillations, only. On the
ring with a small number of electrons $N \leq \sqrt{\frac{UL}{t}}$, there occurs the fractional $1/N$ Aharonov-Bohm effect. However with increasing $N$, when $N \gg \sqrt{\frac{UL}{t}}$, the single-fractional $M/N$ flux quantum periodicity becomes dominant over the fractional $1/N$ effect, although they may both coexist.

**VIII. CHANGE OF THE FINE STRUCTURE PERIOD WITH MAGNETIC FIELD**

The period of fractional oscillations $\frac{M}{N}$ related to the number of down-spin particles. However, with the increase of magnetic field on the ring there may occur spin-flip processes. As the result the number of the down-spin particles changes (decreases) and the magnetization of the ring related to the number $M$ decreases. This Zeeman magnetization may be described with the aid of the formula:

$$ (N - 2M) = \left\lfloor \frac{2\chi H}{\mu_B} \right\rfloor $$

where brackets means an integer part of the value. The value $\chi$ is a susceptibility of the Hubbard chain. First, in thermodynamic limit the value of $\chi$ has been calculated by Takahashi [28] for the case of the half-filling, then for an arbitrary filling the value of $\chi$ has been calculated by Shiba [29]. For the low density limit $\alpha \ll 1$ we discuss the value of $\chi$ may be estimated by the similar way. For the large values of $L$ the thermodynamic limit is acceptable. Therefore we use Shiba’s result

$$ \chi = \frac{3UL}{2\pi^4 t^2 n^2} $$

whence
\[ M = \frac{N}{2} - \left[ \frac{\chi H}{\mu_B} \right] \]  

(8.44)

One sees that with the field the value of \( M \), and, therefore, the period of the fine structure decreases

\[ f_T = \frac{1}{2} - \left[ \frac{\chi H}{\mu_B} \right] \frac{1}{N} \]  

(8.45)

Thus we came to the conclusion that in the Aharonov-Bohm effect of strongly correlated system such as the Hubbard ring may occur the oscillation of the smallest amplitude, which period decreases, when the field increases. In such an effect the spin orbit interaction might be important (see, for example, Ref. [30])

**IX. TOWARDS REALITY FROM EXACT SOLUTIONS**

In the present work we have presented the exact solution for the Hubbard ring, showing a fine structure in the Aharonov-Bohm effect. The mesoscopic rings in existing experiments have many channels, large sizes and many electrons (see, also, discussion in Ref. [31]). They also have disorder or impurities. The measurements are carried at finite temperatures. Of course, all these factors make a large difference between the presented model calculations and the realistic situations.

However on such rings the electron density is much smaller than in a 3D metal, where screening exists. Therefore the electron-electron interaction is very important. The effects of the strong electron-electron correlations, which give rise the finite structure of AB effect, of course, will appear there. But at present time there is no tool to take into account correctly the many-body effects related to this interaction. The Bethe ansatz method is only one single tool, where the effects of strong correlations may be studied for large number of electrons
and a system of a large size, but for some models only. So, using the Bethe ansatz we found, that the interaction between particles also may create the finite structure, which is characterized by several types of periodic oscillations with the amplitude and period smaller than one unit.

In the integer-fractional regime the effects of disorder and temperature will be to make the flux-energy dependence smoother leading to the disappearance of the fine structure. In experimental situations, where the disorder or the temperature always exist, but have a low level for the ring with large number of electrons one may see three types of oscillations in the flux energy dependence. With the next increase of the disorder or the temperature the oscillations having the smallest amplitude will disappear. So, for example, instead of integer-fractional $M/N$ periodic oscillations on the single ring, one may see only the integer flux quantum periodic oscillations.

The discovered single flux quantum periodicity, when the finite structure has disappeared, may be relevant to two recent experimental studies of the Aharonov-Bohm effect on single metallic and semiconductor rings [32], [33]. Recent experiments [32] have described only the full quantum period in the case of individual metallic rings. On a semiconductor single loop in the GaAs/GaAlAs system [34] such single flux periodicity has been detected. To explain this integer flux quantum periodicity we assume that the individual rings are sufficiently disordered. Recently, similar results [13] [35] were obtained that free electrons on the single ring with disorder or with the temperature may show only the full single flux quantum periodicity. Both these findings are consistent with existing experiments.

The interaction between particles may also support the full single quantum periodicity, and, therefore, it may also be responsible for the observed single flux quantum period. The prediction of fractional $1/N$ and fractional $M/N$ oscillations could be made for the ring
having lowest level of disorder. Once the resolution of the experimental studies has been improved such periodicities may be observed.

The other experiment [33], which is worth to mention is that the half-flux quantum periodicity simultaneously of $10^7$ rings has been measured. It is also consistent with our results. It is reasonable to assume that there is a disorder. So the single ring must show the single flux periodicity. For such large ensemble of the rings we must take also into account the parity effect. Then it is also reasonable that half of these rings has an even number of particles and the other half of these rings has an odd number of particles. Because of the discovered and described parity effect, averaging over the two cases of odd and even number of particles indicated above, immediately gives the $\Phi_0/2$ period. This interpretation is then natural for the experimental results concerning many rings [33]. Thus in the framework of our approach we have obtained the unified pictures why the $\Phi_0$ period is seen in individual rings and why $\Phi_0/2$ period is seen an ensemble of $10^7$ rings.

In addition to these explanations, we predict the existence of oscillations with periods smaller than $\Phi_0$ or $\Phi_0/2$. To check the predictions of our theory, it is important to perform the Fourier analysis of the persistent current oscillations in order to detect the fractional Aharonov-Bohm effect, which coexists with integer one, i.e. whether there are fractional oscillations on a background of integer ones.

Recently the fractional $1/4$–AB effect has been observed [36] in AuIn-rings prepared by $e$–beam lithography. Since this effect has been observed in the region of a superconducting phase transition, where the phase separation occurs. It seems that there the effective system reminding the quantum dot chain ring is created. Therefore, the discussed Hubbard model for the description of unpaired electrons in these droplets may be applicable.

In this case the fractional $1/4$– effect may be created as a result of averaging over rings
having different parity of electron’s number. This occurs only at a small magnetic field when Zeeman energy may be neglected. With the increasing magnetic field the electrons become partially polarized and such periodicity is destroyed. But here the fractional $M/N$ may appear instead.

Thus, our finding of a coexistence of the fractional $1/N$ effect with fractional $1/4$ or $1/2$ integer AB effects may not only to explain the observed $1/4$ flux quantum periodicity but also predicts smallest, namely $1/N$ and $M/N$ fractional AB effects, once the experimental resolution has been improved.

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permanent address:

*)L.D. Landau Institute for Theoretical Physics

Moscow, 117940, GSP-1, Kosygina 2, V-334, Russia

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Figure Captions

**Fig.1** The behavior of the ground state energy as a function of flux $f$ for 6 electrons with the single up spin at the values $L = 1000$ and $U = 10$ in the region of flux within the single fundamental flux quantum. The energy is expressed in the units $t10^5$. The zero energy corresponds to $-11.9993t$.

**Fig.2** The behavior of the ground state energy as a function of flux $f$ for 11 electrons at the values $L = 1000$ and $U = 10$ in the region of flux within the single fundamental flux quantum. One electron has down spin i.e. $M = 1$. The zero energy corresponds to $-21.995t$.

**Fig.3** The behavior of the ground state energy as a function of flux $f$ for 25 electrons at the values $L = 7500$ and $U = 10$ in the region of flux within the single fundamental flux quantum. 12 particles have up-spin and 13 particles have down-spin. The energy is expressed in the units $t10^6$. The zero energy corresponds to $-24.999544t$.

**Fig.4** The behavior of the ground state energy as a function of flux for 10 electrons at the values $L = 1000$ and $U = 10$ in the region of flux within the single fundamental flux quantum. Five particles have up-spin and five particles have down-spin. The energy is expressed in the units $t10^5$. The zero energy corresponds to $-9.9983t$.

**Fig.5** The behavior of the ground state energy as a function of flux for the ring with even number of electrons $N$ and with odd number for projection of angular momentum $M$, for the parameters $L = 10^4$ and $U = 10$. We present the cases when $N$ and $M$ are equal to: a) $N = 20$ and $M = 3$, energy is expressed in arbitrary units, the region of flux is the half
of fundamental flux quantum. b) \( N = 30 \) and \( M = 5 \), energy in units \( t10^6 \), the zero energy corresponds to \(-59.99911t\); the region of flux is the half of fundamental flux quantum. c) \( N = 40 \) and \( M = 7 \), energy is expressed in the units \( t10^7 \), the zero energy corresponds to \(-79.997896t\); the region of flux is the half of fundamental flux quantum.

**Fig.6** The behavior of the ground state energy as a function of flux for the ring with odd number of electrons \( N \) and with even number for projection of angular momentum \( M \), for the parameters \( L = 10^4 \) and \( U = 10 \). We present the cases when \( N \) and \( M \) are equal to: a) \( N = 19 \) and \( M = 6 \), energy is expressed in the units \( t10^6 \), the zero energy corresponds to \(-37.97752t\); the region of flux is the half of fundamental flux quantum. b) \( N = 49 \) and \( M = 6 \), energy is expressed in the units \( t10^7 \), the zero energy corresponds to \(-97.996131t\); the region of flux is the half of fundamental flux quantum. c) \( N = 99 \) and \( M = 6 \), energy is expressed in the units \( t10^6 \), the zero energy corresponds to \(-197.96809t\); the region of flux is the quarter of fundamental flux quantum. d) \( N = 99 \) and \( M = 23 \), energy is expressed in the units \( t10^6 \), the zero energy corresponds to \(-197.96809t\); the region of flux is the half of fundamental flux quantum.

**Fig.7** The behavior of the ground state energy as a function of flux for the ring with even number of electrons \( N \) and with even number of a projection of the angular momentum \( M \), for the parameters \( L = 10^3 \) and \( U = 10 \). We present the case when \( N = 20 \) and \( M = 6 \), energy is expressed in the units \( t10^6 \), the zero energy corresponds to \(-39.97377t\); the region of flux is the half of fundamental flux quantum.

**Fig.8** The behavior of the ground state energy as a function of flux for the ring with odd number of electrons \( N \) and projection of angular momentum \( M \), for the parameters \( L = 10^4 \) and \( U = 10 \). We present the case when \( N = 19 \) and \( M = 5 \), energy is expressed in the units \( t10^6 \), the zero energy corresponds to \(-39.97752t\); the region of flux is the half of fundamental flux quantum.
fundamental flux quantum.