1. Introduction

This paper is devoted to some fundamental calculation on 2-dimensional smoothable semi-log-terminal singularities. If we study minimal or canonical models of one parameter degeneration of algebraic surfaces, we must treat singularities that appear in the central fiber. Smoothable semi-log-terminal singularities are the singularities of the central fiber of the minimal model of degeneration, and the singularities of the central fiber of the canonical model of degeneration which may have large Gorenstein index. Kollár and Shepherd-Barron characterized these singularities in [K-SB], but for numerical theory of degeneration, we need more detailed information.

In this paper, we calculate general \( n \)-canonical divisors on these singularities, in other words, we calculate the full sheaves associated to the double dual of the \( n \)-th tensor power of the dualizing sheaves. And the application of this result, we bound the Gorenstein index by the local self-intersection number of the \( n \)-canonical divisor.

Notation: In this paper,

\[
[q_1, q_2, q_3, \ldots] = q_1 + \frac{1}{q_2} + \frac{1}{q_3} + \cdots
\]

\[
[[q_1, q_2, q_3, \ldots]] = q_1 - \frac{1}{q_2} - \frac{1}{q_3} - \cdots
\]

If \( p : \bar{X} \to X \) is a birational morphism and \( D \) is a divisor on \( X \), we denote by \( \bar{D} \) the proper transform of \( D \) on \( \bar{X} \).
2. Basic calculation

Let $Y$ be a cyclic quotient singularity of the form $\text{Spec}(\mathbb{C}[F_\infty,F_0])/(\alpha)$, where $\langle \alpha \rangle$ is a cyclic group of order $r$ and $\alpha$ acts on $\text{Spec}(\mathbb{C}[F_\infty,F_0])$ as $(\alpha^* z_1, \alpha^* z_2) = (\eta^* z_1, \eta z_2)$ in which $\eta$ is a primitive $r$-th root of unity, $(r,s) = 1$, and $0 < s < r$. Let $\frac{r}{s} = \lfloor [q_1,q_2,\ldots,q_k] \rfloor$ be an expansion into continued fraction, and $r_i$ be the $i$-th remainder of the Euclidean algorithm, i.e. $\{r_i\}_{i=0,1,\ldots,k+1}$ is a sequence determined by $r_0 = r, r_1 = s, r_{i-1} = q_i r_i - r_{i+1}$. Let $\frac{P_i}{Q_i}$ be the $i$-the convergent, i.e. $\{P_i\}_{i=-1,0,\ldots,k}$ is a sequence determined by $P_{-1} = 0, P_{0} = 1, P_{i} = q_{i} P_{i-1} - P_{i-2}$ and $\{Q_i\}_{i=-1,0,\ldots,k}$ is determined by $Q_{-1} = -1, Q_{0} = 0, Q_{i} = q_{i} Q_{i-1} - Q_{i-2}$. Let $f : \mathbb{C}[\infty] \rightarrow Y$ be a quotient map, and $p : \tilde{Y} \rightarrow Y$ the minimal desingularization. It is well known that the dual graph of the exceptional divisors of $p$ is a chain of rational curves $\cup_{1 \leq i \leq k} E_i$ such that $E_i^2 = -q_i$. We put

$$\lambda_i = P_{i-1}, \quad \mu_i = r_i, \quad \alpha_i^j = \begin{cases} \frac{1}{m} \mu_i \lambda_j & \text{for } j \leq i \\ \frac{1}{m} \lambda_i \mu_j & \text{for } i < j \end{cases}$$

Note that $c_1 z_1^{\lambda_1} + c_2 z_2^{\lambda_2}$ is a $\langle \alpha \rangle$-semi-invariant since $r_1 P_i - r_0 Q_i = r_{i+1}$.

**Lemma 2.1.** Let $C_i$ be a divisor on $Y$ such that $f^* C_i = (c_1 z_1^{\lambda_i} + c_2 z_2^{\mu_i} = 0)$ in which $c_1, c_2 \in \mathbb{C}^\times$. Then

(i) $\tilde{C}_i \cdot E_j = \delta_{i,j}$

(ii) $p^* C_i = \tilde{C}_i + \sum \alpha_i^j E_j$

**Proof.** If we write $Y = T_{\text{Nemb}}(\sigma)$, where $N = \mathbb{Z}\kappa_\infty + \mathbb{Z}\kappa_0$ and $\sigma = \mathbb{R}_{\geq 0} \kappa_\infty + \mathbb{R}_{\geq 0}[(\sim \sim) \kappa_\infty + \kappa_0]$, then $E_i$ corresponds to $(P_{i-1} - Q_{i-1})m_1 + P_{i-1}n_2$. The rest of proof is a direct calculation using the above description and the formula $P_i Q_{i-1} - Q_i P_{i-1} = -1$, and it can be easily done.

Next lemma is a easy fact on continued fraction. We denote by $(2,q)$ the sequence $(2,2,\ldots,2)$ of length $q$.

**Lemma 2.2.** Let $a, m$ be a natural number such that $(a,m) = 1$, $\frac{m}{2} < a < m$. Put $b = m - a$. Let $\frac{a}{b} = [q_1,q_2,\ldots,q_k]$. Then
Proof. Since this is elementary we left it for the reader.

In the rest of this section we shall index the exceptional divisors of the minimal (semi-) resolution smoothable of the semi-log-terimal singularity.

First we treat the normal case. Let $(a, d, m)$ be a triplet of positive integers such that $a < b$ and $a$ is prime to $m$. We denote by $X_{a,d,m}$ a 2-dimentional quotient singularity of the form $\text{Spec}(\mathbb{C}[F_\alpha, F_\beta])/\langle \alpha \rangle$, where $\langle \alpha \rangle$ is a cyclic group of order $dm^2$ and $\alpha$ acts on $\text{Spec}(\mathbb{C}[F_\alpha, F_\beta])$ as $\alpha^* (z_1, z_2) = (\varepsilon^{adm-1} z_1, \varepsilon z_2)$ in which $\varepsilon$ is a primitive $dm^2$-th root of unity. By [K-SB Proposition 3.10], a singularity of class T which is not RDP is analytically isomorphic to unity. By [K-SB Proposition 3.10], a singularity of class T which is not RDP is analytically isomorphic to $X_{a,d,m}$ for some $(a, d, m)$. Let $f : \mathbb{C}^\alpha \to X_{\alpha,a,b}$ be a quotient map and $p : \tilde{X}_{a,d,m} \to X_{a,d,m}$ the minimal desingularization. We assume $2a > m$ since $X_{a,d,m} \simeq X_{m-a,d,m}$. Put $b = m - a$, and let $\frac{a}{b} = [q_1, q_2, \ldots, q_k]$ be an expansion into continued fraction. Let $r_i$ be the $i$-th remainder of the Euclidean algorithm, i.e. $\{r_i\}_{i=0,1,\ldots,k+1}$ is a sequence determined by $r_0 = a, r_1 = m - a, r_{i-1} = q_i r_i + r_{i+1}$. Let $\frac{P_i}{Q_i}$ be an the $i$-th convergent, i.e. $\{P_i\}_{i=-1,0,\ldots,k}$ is a sequence determined by $P_{-1} = 0, P_0 = 1, P_i = q_i P_{i-1} + P_{i-2}$ and $\{Q_i\}_{i=-1,0,\ldots,k}$ is determined by $Q_{-1} = 1, Q_0 = 0, Q_i = q_i Q_{i-1} + Q_{i-2}$.

**Lemma 2.3.** Let $m, a, b, d$ be positive integers such that $m = a + b$, $a > b$, $(a, b) = 1$. Let $\frac{a}{b} = [q_1, q_2, \ldots, q_k]$ be an expansion into continued fraction. Then

$$\frac{dna - 1}{dmb + 1} = [q'_1, q'_2, \ldots, q'_k]$$
where \( q'_i \) is as follows.

(i) If \( d = 1 \) and \( k \) is even,

\[
q'_i = \begin{cases} 
q_1 & (i = 1) \\
q_i & (2 \leq i \leq k - 1) \\
q_k + 1 & (i = k) \\
q_k - 1 & (i = k + 1) \\
q_{2k-i+1} & (k + 2 \leq i \leq 2k - 1) \\
q_1 + 1 & (i = 2k = k')
\end{cases}
\]

(ii) If \( d = 1 \) and \( k \) is odd,

\[
q'_i = \begin{cases} 
q_1 & (i = 1) \\
q_i & (2 \leq i \leq k - 1) \\
q_k - 1 & (i = k) \\
q_k + 1 & (i = k + 1) \\
q_{2k-i+1} & (k + 2 \leq i \leq 2k - 1) \\
q_1 + 1 & (i = 2k = k')
\end{cases}
\]

(iii) If \( d \geq 2 \) and \( k \) is even,

\[
q'_i = \begin{cases} 
q_1 & (i = 1) \\
q_i & (2 \leq i \leq k) \\
d - 1 & (i = k + 1) \\
1 & (i = k + 2) \\
q_k - 1 & (i = k + 3) \\
q_{2k+3-i} & (k + 4 \leq i \leq 2k + 1) \\
q_1 + 1 & (i = 2k + 2 = k')
\end{cases}
\]

(iv) If \( d \geq 2 \) and \( k \) is odd,

\[
q'_i = \begin{cases} 
q_1 & (i = 1) \\
q_i & (2 \leq i \leq k - 1) \\
q_k - 1 & (i = k) \\
1 & (i = k + 1) \\
d - 1 & (i = k + 2) \\
q_{2k+3-i} & (k + 3 \leq i \leq 2k + 1) \\
q_1 + 1 & (i = 2k + 2 = k')
\end{cases}
\]

By Lemma 2.2 and 2.3, we can calculate the dual graph of the exceptional divisors of \( p \) in terms of the continued fraction expansion of
\[ \frac{a}{b}. \] (See [K-SB]) From now, we assume \( k \) is even since the calculation is the same for odd \( k \). We shall index the exceptional divisors in the following manner. Set the index set \( I_o, I_e, I \) as follows:

\[
I_o = \{(i,j)|1 \leq i \leq k + 1; i \text{ odd}; 1 \leq j \leq q_i (\text{for } i < k + 1), 1 \leq j \leq d (\text{for } i = k + 1)\}
\]

\[
I_e = \{(i,j)|1 \leq i \leq k + 1; i \text{ even}; 1 \leq j \leq q_i (\text{for } i < k), 1 \leq j \leq q_k - 1 (\text{for } i = k)\}
\]

\[ I = I_o \cup I_e \]

We define \( \rho^i_j, \bar{\lambda}^i_j, \lambda^i_j, \bar{\mu}^i_j, \mu^i_j \) for \((i,j) \in I\) as follows.

\[
\rho^i_j = r_{i-1} - (j - 1)r_i
\]

\[
\bar{\lambda}^i_j = \begin{cases} P_{i-2} + (j - 1)P_{i-1} & ((i,j) \in I_o) \\ -\{P_{i-2} + (j - 1)P_{i-1}\} + da\rho^i_j & ((i,j) \in I_e) \end{cases}
\]

\[
\hat{\lambda}^i_j = \begin{cases} Q_{i-2} + (j - 1)Q_{i-1} & ((i,j) \in I_o) \\ -\{Q_{i-2} + (j - 1)Q_{i-1}\} + db\rho^i_j & ((i,j) \in I_e) \end{cases}
\]

\[
\bar{\mu}^i_j = \begin{cases} -\{P_{i-2} + (j - 1)P_{i-1}\} + da\rho^i_j & ((i,j) \in I_o) \\ P_{i-2} + (j - 1)P_{i-1} & ((i,j) \in I_e) \end{cases}
\]

\[
\hat{\mu}^i_j = \begin{cases} -\{Q_{i-2} + (j - 1)Q_{i-1}\} + db\rho^i_j & ((i,j) \in I_o) \\ Q_{i-2} + (j - 1)Q_{i-1} & ((i,j) \in I_e) \end{cases}
\]

\[
\lambda^i_j = \bar{\lambda}^i_j + \hat{\lambda}^i_j, \quad \mu^i_j = \bar{\mu}^i_j + \hat{\mu}^i_j
\]

We can write \( Y = T_{\text{Nemb}}(\sigma) \), where

\[
N = \mathbb{Z}_{\times K_{\mathcal{E}}} + \mathbb{Z}_{\times K_{\mathcal{E}}}, \quad \sigma = \mathbb{R}_{>0} \times K_{\mathcal{E}} + \mathbb{R}_{>0}[(>1) \times K_{\mathcal{E}} + >\epsilon \times \mathcal{E}]
\]

We denote by \( E^i_j \) the exceptional divisor associated to \( \hat{\lambda}^i_j n_1 + \lambda^i_j n_2 \). Note that by Lemma 2.1, for \( \iota \in I \), the proper transform of \( \hat{C} = (c_1z_1^{\lambda_1} + c_2z_2^{\lambda_2})/\langle \alpha \rangle \in X_{a,d,m} \) intersects the exceptional locus transversely at \( E_\iota \). We define the order ‘\( \leq \)’ in the index set \( I \) by the lexicographic order.

Next we treat the non-normal case. We denote by \( NC^2 = \text{Spec} \mathbb{C}[F_{\mathcal{E}}, F_{\mathcal{E}}]/(F_{\mathcal{E}}F_{\mathcal{E}}) \) a 2-dimentional normal crossing point. Let \((a,m)\) be a pair of positive integers such that \( 0 < a < m \) and \( a \) is prime to \( m \). Put \( b = m - a \), and let \( a' \) and \( b' \) be integers such that \( aa' \equiv bb' \equiv 1 \pmod{m} \), \( 0 < a' < m \), and \( 0 < b' < m \). Let \( \langle \alpha \rangle \) be a cyclic group of order \( m \), and let \( \langle \alpha \rangle \) act on \( NC^2 \) as \( (\alpha^* z_1, \alpha^* z_2, \alpha^* z_3) = (\varepsilon^{a'} z_1, \varepsilon^{b'} z_2, \varepsilon z_3) \) where \( \varepsilon \) is a primitive \( m \)-th root of unity. We denote by \( X_{a,m} \) the quotient of \( NC^2 \)
by this \( \langle \alpha \rangle \)-action. By [K-SB], 2-dimentional smoothable semi-log-terminal singularity which is neither normal nor NC is analytically isomorphic to \( X_{a,m} \) for some \( (a, m) \). Put \( X = X_{a,m} \). Let \( f : NC^2 \to X \) be the quotient map and \( p : \bar{X} \to X \) the minimal semi-resolution. Let \( g : C^\infty \times C^\infty \to NC^\infty \), \( g_X : X_0 \times X_0 \to X \), and \( g_{\bar{X}} : \bar{X}_0 \times \bar{X}_0 \to \bar{X} \) be normalizations, where \( C^\infty = \text{Spec} \mathbb{C}[[F_{\mathfrak{m}}, F_{\mathfrak{n}}]] \), \( C^\infty = \text{Spec} \mathbb{C}[[F_{\mathfrak{n}}, F_{\mathfrak{m}}]] \), \( X_0 \) (resp. \( X_\sigma \)) is the quotient of \( C^\infty \) (resp. \( C^\infty \)), and \( X_0 \) (resp. \( \bar{X}_e \)) is the minimal resolution of \( X_0 \) (resp. \( X_e \)). We get the following diagram.

\[
\begin{array}{ccc}
C^\infty \times C^\infty & \xrightarrow{f_0 \times f_e} & X_0 \times X_e \\
g \downarrow & & \downarrow g_X \\
NC^2 & \xrightarrow{f} & X & \xleftarrow{p} & \bar{X}
\end{array}
\]

We denote by \( \Delta, \Delta' \), and \( \bar{\Delta} \) the double curve of \( X, NC^2 \), and \( \bar{X} \) respectively. Let \( \Delta_0 \) (resp. \( \Delta_\sigma \)) be the inverse image of \( \Delta \) in \( X_0 \) (resp. \( X_\sigma \)), and define \( \Delta_0', \Delta_\sigma', \bar{\Delta}_0 \), and \( \bar{\Delta}_\sigma \) similarly. We assume \( k \) is even. Set the index set \( I_0, I_e, I \) as follows:

\[
I_0 = \{(i,j)|1 \leq i \leq k+1; i \text{ odd}; 1 \leq j \leq q_i (\text{for } i < k), j = 1 (\text{for } i = k+1)\}
\]

\[
I_e = \{(i,j)|1 \leq i \leq k+1; i \text{ even}; 1 \leq j \leq q_i\}
\]

\[
I = I_0 \times I_e
\]

We define \( \lambda^i_j, \mu^i_j \) for \( (i,j) \in I \) as follows:

\[
\begin{align*}
\lambda^i_j &= \begin{cases}
P_{i-2} + Q_{i-2} + (j-1)(P_{i-1} + Q_{i-1}) & ((i,j) \in I_0) \\
q_{i-1} - (j-1)q_i & ((i,j) \in I_e)
\end{cases} \\
\mu^i_j &= \begin{cases}
q_{i-1} - (j-1)q_i & ((i,j) \in I_0) \\
P_{i-2} + Q_{i-2} + (j-1)(P_{i-1} + Q_{i-1}) & ((i,j) \in I_e)
\end{cases}
\end{align*}
\]

We define \( \tilde{\lambda}^i_j, \tilde{\mu}^i_j \) for \( (i,j) \in I_0 \) as follows:

\[
\tilde{\lambda}^i_j = P_{i-2} + (j-1)P_{i-1}, \quad \tilde{\mu}^i_j = Q_{i-2} + (j-1)Q_{i-1}
\]

We define \( \tilde{\mu}^i_j, \tilde{\mu}^i_j \) for \( (i,j) \in I_e \) as follows:

\[
\tilde{\mu}^i_j = P_{i-2} + (j-1)P_{i-1}, \quad \tilde{\mu}^i_j = Q_{i-2} + (j-1)Q_{i-1}
\]

We write \( X_0 = T_{N^0} \text{emb}(\sigma^0) \) and \( X_\sigma = T_{N^\infty} \text{emb}(\sigma^\infty) \) where

\[
N^0 = \mathbb{Z}\mathfrak{m} + \mathbb{Z}\mathfrak{m}, \quad \sigma^0 = \mathbb{R}_{\geq \mathfrak{m}} \mathfrak{m} + \mathbb{R}_{\geq \mathfrak{m}} (\mathfrak{m} \otimes \mathfrak{m})
\]

\[
N^\infty = \mathbb{Z}\mathfrak{m} + \mathbb{Z}\mathfrak{m}, \quad \sigma = \mathbb{R}_{\geq \mathfrak{m}} \mathfrak{m} + \mathbb{R}_{\geq \mathfrak{m}} (\mathfrak{m} \otimes \mathfrak{m})
\]
For $(i,j) \in I_o$ (resp. $I_e$), we denote by $E^i_j$ the exceptional divisor of $p_o$ (resp. $p_e$) which associated to $\lambda_j^1 n_1^i + \lambda_j^2 n_2^i \in N^o$ (resp. $\mu_j^1 n_1^i + \mu_j^2 n_2^i \in N^e$). Note that by Lemma 2.1, for $i \in I_o$ (resp.$I_e$), the proper transform of $C_i = (c_1 z_i^3 + c_2 z_i^3^\mu)/\langle \alpha \rangle \in X_o$ (resp. $C_i = (c_1 z_i^3 + c_2 z_i^3^\mu)/\langle \alpha \rangle \in X_e$) intersects the exceptional locus transversely at $E_i$. We define the order in $I$ as the same way as the normal case.

In the rest of this paper, we treat $X_{a,d,m}$ and $X_{a,m}$ simultaneously, otherwise we specifically state the normal or non-normal case.

3. $\lambda$-expansion and $\mu$-expansion

In this section we introduce the notion of $\lambda$-expansion and $\mu$-expansion, which is the key in this paper.

**Definition 3.1.** Let $L = (j_1, l_2, j_3, l_4, \ldots, j_{k-1}, l_k)$ be a sequence of non-negative integers which is not $(0,0,\ldots,0)$. We call $L$ a $\lambda$-sequence if it satisfies the following conditions.

(i) $l_i \leq q_i + 1$ if $i \neq k$, and $l_k \leq q_k$; $j_i \leq q_i$ for all odd $i$, and $j_i \neq 1$ if $i \neq 1$

(ii) If $l_{i_0} = q_{i_0} + 1$, then there exists odd $i_1$ and $i_2$ which satisfies the following conditions.

(a) $i_1 < i_0 < i_2 \leq k - 1$

(b) $\lambda_{i'} = q_{i'}$ for all even $i'$ such that $i_1 < i' < i_2$ and $i' \neq i_0$; $j_{i'} = 0$ for all odd $i'$ such that $i_1 \leq i' < i_2$

(c) $l_{i_1-1} < q_{i_1-1}$ if $i_1 \geq 3$ and $l_{i_2+1} < q_{i_2}$

(iii) If $l_{i_0} = q_{i_0}$ and $j_{i_0+1} \geq 2$, then there exists odd $i_3$ which satisfies the following conditions.

(a) $i_3 < i_0$

(b) $\lambda_{i'} = q_{i'}$ for all even $i'$ such that $i_3 < i' \leq i_0$; $j_{i'} = 0$ for all odd $i'$ such that $i_3 \leq i' < i_0$

(c) $l_{i_3-1} < q_{i_3-1}$ if $i_3 \geq 3$

**Definition 3.2.** Let $M = (l_1, j_2, l_3, j_4, \ldots, l_{k-1}, j_k)$ be a sequence of non-negative integers which is not $(0,0,\ldots,0)$. We call $M$ a $\mu$-sequence if it satisfies the following conditions.

(i) $l_i \leq q_i + 1$ for all odd $i$; $2 \leq j_i \leq q_i$ for all even $i$

(ii) If $l_{i_0} = q_{i_0} + 1$, then there exists even $i_1$ and $i_2$ which satisfies the following conditions.

(a) $0 \leq i_1 < i_0 < i_2 \leq k$

(b) $\lambda_{i'} = q_{i'}$ for all odd $i'$ such that $i_1 < i' < i_2$ and $i' \neq i_0$; $j_{i'} = 0$ for all even $i'$ such that $i_1 \leq i' \leq i_2$

(c) $l_{i_1-1} < q_{i_1}$ if $i_1 \geq 2$, $l_{i_2+1} < q_{i_2+1}$ if $i_2 \leq k - 2$
(iii) If \( l_{i_0} = q_{i_0} \) and \( j_{i_0+1} \geq 2 \), then there exists even \( i_3 \) which satisfies the following conditions.

(a) \( 0 \leq i_3 < i_0 \)

(b) \( l'_{i'} = q_{i'} \) for all odd \( i' \) such that \( i_3 < i' < i_0 \); \( j_{i'} = 0 \) for all even \( i' \) such that \( i_3 \leq i' < i_0 \)

(c) \( l_{i_3-1} < q_{i_{3-1}} \) if \( i_3 \geq 2 \)

We denote by \( S_\lambda \) (resp. \( S_\mu \)) the set of all \( \lambda \)-(resp. \( \mu \))-sequences.

Let \( L \) be a \( \lambda \)-sequence and \( h \) an integer such that \( 1 \leq h \leq k - 1 \). We say that the condition \( * (h) \) holds for \( L \) if the following conditions hold.

(i) If \( h \) is odd,

\[
\lambda_{j_1}^1 + \sum_{3 \leq i \leq h \atop i \text{ odd}} (l_{i-1} \lambda_1^i + \lambda_{j_i}^i) < P_h + Q_h
\]

(ii) If \( h \) is even,

\[
\lambda_{j_1}^1 + \sum_{3 \leq i \leq h-1 \atop i \text{ odd}} (l_{i-1} \lambda_1^i + \lambda_{j_i}^i) + l_h \lambda_1^{h+1} < P_{h-1} + Q_{h-1} + P_h + Q_h
\]

If \( l_h < q_h \) or \( j_{h+1} \geq 2 \) also hold,

\[
\lambda_{j_1}^1 + \sum_{3 \leq i \leq h-1 \atop i \text{ odd}} (l_{i-1} \lambda_1^i + \lambda_{j_i}^i) + l_h \lambda_1^{h+1} < P_h + Q_h
\]

**Lemma 3.1.** \(* (h) \) holds for all \( L \in S_\lambda \) and for all \( h = 1, 2, \ldots, k - 1 \).

**Proof.** We use the induction on \( h \). It is clear that \( *(1) \) holds. Assume that \( 2 \leq h \leq k - 1 \) and that \( * (\bar{h}) \) holds for all \( \bar{h} \) such that \( \bar{h} < h \).

First we treat that the case where \( h \) is odd.

If \( j_h \geq 2 \), then by \( * (h) \),

\[
\lambda_{j_1}^1 + \sum_{3 \leq i \leq h-2 \atop i \text{ odd}} (l_{i-1} \lambda_1^i + \lambda_{j_i}^i) l_{h-1} \lambda_1^h < P_{h-1} + Q_{h-1}
\]

Hence

\[
\lambda_{j_1}^1 + \sum_{3 \leq i \leq h \atop i \text{ odd}} (l_{i-1} \lambda_1^i + \lambda_{j_i}^i) < P_{h-1} + Q_{h-1} + \lambda_{j_h}^h
\]

\[
= P_{h-1} + Q_{h-1} + j_h (P_{h-1} + Q_{h-1})
\]

\[
\leq P_{h-2} + Q_{h-2} + q_h (P_{h-1} + Q_{h-1})
\]

\[
= P_h + Q_h
\]
If $j_h = 0$, by $*(h - 1)$

$$
\lambda^1_{j_1} + \sum_{3 \leq i \leq h \atop i \text{ odd}} (l_{i-1} \lambda^i_1 + \lambda^i_{j_i}) = \lambda^1_{j_1} + \sum_{3 \leq i \leq h - 2 \atop i \text{ odd}} (l_{i-1} \lambda^i_1 + \lambda^i_{j_i}) + l_{h-1} \lambda^h_1

< P_{h-2} + Q_{h-2} + P_{h-1} + Q_{h-1}

\leq P_h + Q_h
$$

Next we treat the case where $h$ is even. We divide the proof into four cases as follows

(1) $l_h < q_h$  (2) $l_h = q_h, j_h = 0$  (3) $l_h = q_h, j_h \geq 2$  (4) $l_h = q_h + 1$

(1) By $*(h - 1)$,

$$
\lambda^1_{j_1} + \sum_{3 \leq i \leq h-1 \atop i \text{ odd}} (l_{i-1} \lambda^i_1 + \lambda^i_{j_i}) < P_{h-1} + Q_{h-1}
$$

Hence

$$
\lambda^1_{j_1} + \sum_{3 \leq i \leq h-1 \atop i \text{ odd}} (l_{i-1} \lambda^i_1 + \lambda^i_{j_i}) + l_k \lambda^{h+1}_1 < P_{h-1} + Q_{h-1} + (q_h - 1)(P_{h-1} + Q_{h-1})

< P_h + Q_h
$$

(2) By $*(h - 1)$,

$$
\lambda^1_{j_1} + \sum_{3 \leq i \leq h-1 \atop i \text{ odd}} (l_{i-1} \lambda^i_1 + \lambda^i_{j_i}) + l_h \lambda^{h+1}_1 < P_{h-1} + Q_{h-1} + q_h(P_{h-1} + Q_{h-1})

< P_{h-1} + Q_{h-1} + P_h + Q_h
$$

(3) By (iii) in Definition 3.1, there exists odd $h'$ such that $l_{h'-1} < q_{h'-1}$ and

$$(j_{h'}, l_{h'+1}, j_{h'+2}, l_{h'+3}, \ldots, j_{h-1}, l_h) = (0, q_{h'+1}, 0, q_{h'+3}, \ldots, 0, q_h)$$
Hence by $\ast(h' - 1)$,
\[
\sum_{3 \leq i \leq h-1 \atop \text{i odd}}^\lambda l_{i-1} \lambda^i_j + \lambda^i_{j_i})l_h \lambda^{h+1}_1
\]
\[
-\lambda^l_j_1 + \sum_{3 \leq i \leq h' - 2 \atop \text{i odd}}^\lambda l_{i-1} \lambda^i_j + \lambda^i_{j_i}) + l_{h' - 1} \lambda^{h'}_1
\]
\[
+ \lambda^{h'}_{j_{h'}} + \sum_{h'+2 \leq i \leq h-1 \atop \text{i odd}}^\lambda l_{i-1} \lambda^i_j + \lambda^i_{j_i}) + l_h \lambda^{h+1}_1
\]
\[
< P_{h' - 1} + Q_{h' - 1} + \sum_{h'+2 \leq i \leq h-1 \atop \text{i odd}}^\lambda q_{i-2}(P_{i-2} + Q_{i-2})(q_{h} + 1)(P_{h-1} + Q_{h-1})
\]
\[
= P_{h-1} + Q_{h-1} + P_h + Q_h
\]

(4) By (ii) in Definition [3.1], there exists odd $h'$ such that $l_{h' - 1} < q_{h' - 1}$ and
\[
(j_{h'}, l_{h'+1}, j_{h'+2}, l_{h'+3}, j_{h'+4}, \ldots, l_{h-2}, j_{h-1}, l_h)
\]
\[
= (0, q_{h'+1}, 0, q_{h'+3}, 0, \ldots, q_{h-2}, 0, q_h)
\]

Hence by $\ast(h' - 1)$,
\[
\sum_{3 \leq i \leq h-1 \atop \text{i odd}}^\lambda l_{i-1} \lambda^i_j + \lambda^i_{j_i}) + l_h \lambda^{h+1}_1
\]
\[
-\lambda^l_j_1 + \sum_{3 \leq i \leq h' - 2 \atop \text{i odd}}^\lambda l_{i-1} \lambda^i_j + \lambda^i_{j_i}) + l_{h' - 1} \lambda^{h'}_1
\]
\[
+ \lambda^{h'}_{j_{h'}} + \sum_{h'+2 \leq i \leq h-1 \atop \text{i odd}}^\lambda l_{i-1} \lambda^i_j + \lambda^i_{j_i}) + l_h \lambda^{h+1}_1
\]
\[
< P_{h' - 1} + Q_{h' - 1} + \sum_{h'+2 \leq i \leq h-1 \atop \text{i odd}}^\lambda q_{i-2}(P_{i-2} + Q_{i-2})(q_{h} + 1)(P_{h-1} + Q_{h-1})
\]
\[
= P_{h-1} + Q_{h-1} + P_h + Q_h
\]

We define the order in $S_\lambda$ as $(j_1, l_2, \ldots, l_k) < (j'_1, l'_2, \ldots, l'_k)$ if and only if there exists $i$ such that $j_i < j'_i$ or $l_i < l'_i$ and that $j_h = j'_h, l_h = l'_h$ for all $h > i$.

Let $v$ be a map from $S_\lambda$ to $\mathbb{Z}$ defined by
\[
v(l_1, j_2, \ldots, l_{k-1}, j_k) = \lambda^l_j_1 + \sum_{3 \leq i \leq k-1 \atop \text{i odd}}^\lambda l_{i-1} \lambda^i_j + \lambda^i_{j_i}) + l_k \lambda^{k+1}_1
\]
Proposition 3.1. The map \( v \) is an order isomorphism from \( S_\lambda \) to \( \{ n \in \mathbb{Z} | k \leq \lambda \leq -k^2 \} \)

Proof. First we show that \( L < L' \) implies \( v(L) < v(L') \). Let \( L = (j_1, l_2, \ldots, j_{k-1}, l_k) \) and \( L' = (j'_1, l'_2, \ldots, j'_{k-1}, l'_k) \) be \( \lambda \)-sequences such that \( L < L' \). Put

\[
i_0 = \max \{ i | l'_i = l_i \text{ and } j'_i = j_i \text{ for all } i > j \}
\]

If \( i_0 \) is odd,

\[
v(L') - v(L) \geq \lambda^i_{j_0} - \lambda^i_{j_0} - \{ \lambda^1_{j_1} + \sum_{3 \leq i \leq i_0 - 2} (l_{i-1} \lambda^i_{j_1} + l_{i-1} \lambda^i_1) \}
\]

Note that

\[
\lambda^i_{j_0} - \lambda^i_{j_0} \geq \begin{cases} P_{i_0-1} + Q_{i_0-1} & \text{(if } j_0 \geq 2) \\ P_{i_0-2} + Q_{i_0-2} + P_{i_0-1} + Q_{i_0-1} & \text{(if } j_0 = 0) \end{cases}
\]

Hence by \(*i_0 - 1\), \( v(L') - v(L) > 0 \)

If \( i_0 \) is even,

\[
v(L') - v(L) \geq (l'_{i_0} - l_0)\lambda^1_{j_0} + \sum_{3 \leq i \leq i_0 - 1} (l_{i-1} \lambda^i_{j_1} + l_{i-1} \lambda^i_1) \}
\]

\[
\geq P_{i_0-1} + Q_{i_0-1} - \{ \lambda^1_{j_1} + \sum_{3 \leq i \leq i_0 - 1} (l_{i-1} \lambda^i_{j_1} + l_{i-1} \lambda^i_1) \}
\]

Hence by \(*i_0 - 1\), \( v(L') - v(L) > 0 \). Thus we have done.

Note that \( \max S_\lambda = (\pi, \Pi_\epsilon, \pi, \Pi_\Delta, \pi, \ldots, \Pi_\parallel - \epsilon, \pi, \Pi_\parallel) \). Thus \( v \) is an order-preserving injection into \( \{ n \in \mathbb{Z} | k \leq \lambda \leq -k^2 \} \). Hence the rest we must prove is that it is a injection into \( \{ n \in \mathbb{Z} | k \leq \lambda \leq -k^2 \} \). Note that \( 1 = \lambda^1_{i_1} < \lambda^1_{i_2} < \ldots < \lambda^1_{i_q} < \lambda^2_{i_1} < \lambda^2_{i_2} < \ldots \). Thus it is sufficient to show that \( \text{Im}(v) \supseteq \{ n | 1 \leq n < \lambda^1_{i_q} \} \) implies \( \text{Im}(v) \supseteq \{ n | 1 \leq n < \lambda^1_{i_q + 2} \} \). Let \( n \) be an integer such that \( \lambda^1_{i_q} \leq n \leq \lambda^1_{i_q + 2} \). We divide the proof into two cases.

1. \( \lambda^h_{l_1} < n < \lambda^h_{l_2} \) or \( q_h = 1 \) (II) \( \lambda^h_{l_2} < n < \lambda^h_{l_1} + 1 \)

1. Write \( n = l_{h-1} \lambda^h_{l_1} + n' \) such that \( 0 \leq n' < \lambda^h_{l_1} \). It is clear that \( l_{h-1} \leq q_h - 1 + 1 \). By the induction hypothesis, there exists \( L' = (j_1, l_2, \ldots, j_{h-3}, j_{h-2}, 0, \ldots, 0) \) such that \( v(L') = n' \). Put \( L = (j_1, l_2, \ldots, l_{h-3}, j_{h-2}, l_{h-1}, 0, \ldots, 0) \).

Assume that \( L \) is not a \( \lambda \)-sequence. Then by the definition of \( \lambda \)-sequence, we get \( l_{h-1} = q_{h-1} + 1 \) and there exists \( h' \) such that

\[
l_{h'} = q_{h'+1}, 0, q_{h'+2}, 0, q_{h'+4}, 0, \ldots, q_{h'-3}, 0, q_{h-1} + 1
\]
or
\[(jh', lh'+1, jh'+2, lh'+3, jh'+4, \ldots, lh'-3, jh'-2, lh-1)\]
\[= (jh', qh'+1, 0, qh'+3, 0, \ldots, qh-3, 0, qh-1 + 1)\]
and \(jh' > 0\). In the both cases it is easily checked
\[n \geq \begin{cases} 
\lambda_2^h & \text{(if } qh \geq 2) \\
\lambda_1^{h+2} & \text{(if } qh = 1) 
\end{cases}\]
and this is a contradiction. Thus \(L\) is a \(\lambda\)-sequence, so we have done.

(II) Put \(j_h = \max \{j \mid j \geq 2, \lambda_j^h \leq n\}\). Clearly \(j_h \leq q_h\). Since \(n - \lambda_j^h < q_h\), there exists \(L' = (j_1, l_2, \ldots, j_{h-2}, l_{h-1}, 0, \ldots, 0)\) such that \(v(L') = n - \lambda_j^h\) by the induction hypothesis and (I). Put \(L = (j_1, l_2, \ldots, j_{h-2}, l_{h-1}, j_h, 0, \ldots, 0)\)

We can check \(L \in \mathcal{S}_\lambda\) by the definition of \(\lambda\)-sequence.

When we write \(n = \lambda_j^1 + \sum_{2 \leq i \leq k} (l_{i-1} \lambda_i^1 + \lambda_j^i) + l_k \lambda_k^{k+1}\) where \((j_1, l_2, \ldots, l_k, j_k)\) is a \(\lambda\)-sequence, we call this expression a \(\lambda\)-expansion of \(n\). By the above proposition, \(n = 1, 2, \ldots, m - 1\) has unique \(\lambda\)-expansion. Note that the proof of Proposition 3.1 shows how to calculate \(\lambda\)-expansion of actual number.

When we write \(n = \sum_{2 \leq i \leq k} (l_{i-1} \mu_i^1 + \mu_j^i)\) where \((l_1, j_2, \ldots, l_{k-1}, j_k)\), we call this expression a \(\mu\)-expansion of \(n\). Similarly to \(\lambda\)-expansion, we can prove that \(n = 1, 2, \ldots, m - 1\) has unique \(\mu\)-expansion.

4. General \(n\)-canonical divisors

Let \((Y, y)\) be a 2-dimensional rational singularity, and \(p : \tilde{Y} \to Y\) be the minimal desingularization. Let \(M\) be a reflexive module of rank 1 on \(Y\), \(F(M)\) the full sheaf associated to \(M\). (For the definition of full sheaf, see [ESN].) In this situation,

**Definition 4.1.** Let \(D\) be a member of \(|M|\). We call \(D\) a general member of \(|M|\) if \(\tilde{D}\) is a member of \(|F(M)|\) and intersects the exceptional locus transversely.

Note that general members always exist since the full sheaf is generated by global sections. Let \(E = \bigcup_i E_i\) be the exceptional locus, and write \(p^* D = \tilde{D} + \sum_i \alpha(D)_i E_i\).

**Lemma 4.1.** Let \(D\) be a member of \(|M|\) such that \(\tilde{D}\) and \(E\) intersects transversely. Then \(D\) is a general member if and only if the inequality \(\alpha(D)_i \leq \alpha(D')_i\) holds for all \(D' \in |M|\) and all \(E_i\).
Proof. Suppose that $D$ is a general member of $|M|$ and $D'$ is a member of $|M'|$. The sequence
\[
H^0_E(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D})) \rightarrow \mathcal{H}'(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D})) \rightarrow \mathcal{H}'(\tilde{Y} \setminus \mathcal{E}, \mathcal{O}_{\tilde{Y}}(\tilde{D})) \rightarrow \mathcal{H}^\infty_E(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D}))
\]
is exact. Since $\mathcal{O}_{\tilde{Y}}(\tilde{D})$ is a full sheaf, $H^0_E(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D})) = \mathcal{H}^\infty_E(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{D}))$. Hence $D$ is linearly equivalent to $D'$ + (effective divisor). Hence the inequality holds since $\tilde{D}$ and $\tilde{D}' + \sum (\alpha(D')_i - \alpha(D)_i)E_i$ are linearly equivalent. Thus we have proved only if part. Next suppose that $D$ is a member of $|M|$ such that $\tilde{D}$ and $E$ intersect transversely and $\alpha(D)_i \leq \alpha(D')_i$ for all $D' \in |M|$. Let $D_0$ be a general member of $|M|$. By the assumption of $D$, $\alpha(D)_i \leq \alpha(D_0)_i$. By the fact that we have already showed, $\alpha(D)_i \geq \alpha(D_0)_i$. Hence $\alpha(D)_i = \alpha(D_0)_i$. Thus $D$ and $D_0$ are numerically equivalent, hence they are linearly equivalent since $Y$ is a rational singularity. Hence $\mathcal{O}_{\tilde{Y}}(\tilde{D})$ is a full sheaf. $\square$

Corollary 4.1. Let $M$ and $M'$ be reflexive modules of rank 1 on $Y$. Let $D_0 (\text{resp. } D'_0)$ be a general member of $|M|$ (resp. $|M'|$). Then
\[
D_0 \cdot D'_0 = \min\{D \cdot D'| D \in |M|, D' \in |M'|\}
\]
Proof. Let $D (\text{resp. } D')$ be a member of $|M|$ (resp. $|M'|$).
\[
D \cdot D' = (p^*D) \cdot \tilde{D}' \\
= (\tilde{D} + \sum \alpha(D)_i E_i) \cdot \tilde{D}' \\
\geq \sum \alpha(D)_i (E_i \cdot \tilde{D}') \\
\geq \sum \alpha(D_0)_i (E_i \cdot \tilde{D}') \\
= (\tilde{D}_0 + \sum \alpha(D_0)_i E_i) \cdot \tilde{D}' \\
= p^*D_0 \cdot \tilde{D}_0 = D_0 \cdot D'
\]
Similarly we can show $D_0 \cdot D'_0 \geq D_0 \cdot D'_0$. $\square$

In the rest of this section we shall calculate the general element of the $n$-canonical system of semi-log-terminal singularities. We denote by $\mathcal{L}$ (resp. $\mathcal{L}_t$ resp. $\mathcal{L}_l$) the set of all functions from $I$ (resp. $I_o$ resp. $I_e$) to $\mathbb{Z}$.

First we treat singularities of class T. We begin by purely arithmetical lemmas. Let $a, d, m$ and $n$ be positive integers such that $\frac{m}{2} < a < m$, $(a, m) = 1$ and $n < m$. Let $\frac{a}{m-a} = [q_1, q_2, \ldots, q_k]$ be
the expansion into continued fraction, \( r_i \) be the \( i \)-th remainder of the Euclidean algorithm, and \( \frac{P_i}{Q_i} \) be the \( i \)-th convergent.

**Lemma 4.2.** Let \( \{t_i\}_{i=1,2,...,k} \) be a sequence of non-negative integers such that \( t_i \leq q_i \) for \( i \leq k - 1 \) and \( t_k \leq q_k - 1 \). Assume that \( t_{i_0} \) be positive. Then

(i) If \( i_0 \) is even,

\[-r_{i_0-1} < \sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i < 0\]

(ii) If \( i_0 \) is odd,

\[0 < \sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i < r_{i_0-1}\]

**Proof.** We use the induction on \( k - i_0 \). If \( i_0 = k \), it is clear that the inequalities hold. Assume that \( i_0 < k \) and that the inequalities hold for all \( i_0' \) such that \( i_0 < i_0' \leq k \). Let \( i_0 \) be even. First we show

\[\sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i \leq -t_{i_0} r_{i_0} < 0\]

Thus we assume that there exists \( i' \) such that \( i' > i_0 \), \( i' \) is odd, and \( t_{i'} > 0 \). Let \( i_0' \) be the minimum of such \( i' \). By the induction hypothesis,

\[\sum_{i_0' \leq i \leq k} (-1)^{i-1} t_i r_i < r_{i_0'} - 1\]

Thus

\[\sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i \leq -t_{i_0} r_{i_0} + \sum_{i_0' \leq i \leq k} (-1)^{i-1} t_i r_i < -t_{i_0} r_{i_0} + r_{i_0' - 1} = 0\]

Next we show \( \sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i > -r_{i_0-1} \). If \( t_{i'} = 0 \) for all \( i' \) such that \( i' > i_0 \) and \( i' \) is even,

\[\sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i \geq -t_{i_0} r_{i_0} \geq -q_{i_0} r_{i_0} > -r_{i_0-1}\]

Thus we assume that there exists \( i' \) such that \( i' > i_0 \), \( i' \) is even, and \( t_{i'} > 0 \). By the induction hypothesis,

\[\sum_{i_0' \leq i \leq k} (-1)^{i-1} t_i r_i > -r_{i_0'} - 1\]
Thus
\[
\sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i \geq -t_{i_0} + \sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i
\]
\[
> -t_{i_0} - r_{i_0}'
\]
\[
\geq -q_{i_0} r_{i_0} - r_{i_0+1}
\]
\[
= -r_{i_0-1}
\]

The proof is similar for odd \(i_0\). \(\square\)

**Definition 4.2.** Let \((t_1, t_2, \ldots, t_k)\) be a sequence of non-negative integers which is not \((0, 0, \ldots, 0)\). We call this a \(\tau\)-sequence if it satisfies the following conditions.

(i) \(t_i \leq q_i\) if \(i \neq k\) and \(t_k < q_k\).

(ii) If \(t_{i_0-1} > 0\) and \(t_{i_0} = q_{i_0}\) for some \(i_0\) such that \(1 < i_0 < k\), then \(t_{i_0+1} = q_{i_0+1}\). If \(t_{k-2} > 0\) and \(t_{k-1} = q_{k-1}\), then \(t_k = q_k - 1\).

**Lemma 4.3.** Let \((0, 0, \ldots, 0, t_{i_0}, t_{i_0+1}, \ldots, t_k)\) be a \(\tau\)-sequence such that \(i_0\) is odd and \(t_{i_0} > 0\). Then

(i) If \(k\) is even,
\[
\sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i = 1 \Leftrightarrow i_0 = k-1, t_{k-1} = 1, t_k = q_k - 1
\]

(ii) If \(k\) is odd,
\[
\sum_{i_0 \leq i \leq k} (-1)^{i-1} t_i r_i = 1 \Leftrightarrow i_0 = k, t_k = 1
\]

**Proof.** It can be easily checked and we left it for the reader. \(\square\)

**Lemma 4.4.** For any integer \(t\) such that \(0 < t < m - 2\), there exists a \(\tau\)-sequence \((t_1, t_2, \ldots, t_k)\) such that \(t = \sum_{1 \leq i \leq k} t_i (P_{i-1} + Q_{i-1})\).

(We call this expression of \(t\) the \(\tau\)-expansion of \(t\).)

**Proof.** (Step 1) We can write \(t = \sum_{1 \leq i \leq k} t_i (P_{i-1} + Q_{i-1})\) where \(0 \leq t_i \leq q_i\) for \(i < k\) and \(0 \leq t_k < q_k\).

(proof) We use the induction on \(t\). By the induction hypothesis, we can write \(t - 1 = \sum_{1 \leq i \leq k} t_i (P_{i-1} + Q_{i-1})\), where \(0 \leq t_i \leq q_i\) for \(i < k\) and \(0 \leq t_k < q_k\). If \(t_1 < q_1\), we have done. Assume \(t_1 = q_1\). Note that \(\sum_{1 \leq i \leq k-1} q_i (P_{i-1} + Q_{i-1}) + (q_k - 1)(P_{k-1} + Q_{k-1}) = m - 2\). Hence there exists \(i_0\) such that \(i_0 < k\), \(t_i = q_i\) for all \(i \leq i_0\), and \(t_{i_0+1} < q_{i_0+1}\) (if
\( i_0 < k - 1 \); \( t_k \leq q_k - 2 \) (if \( i_0 = k - 1 \)). Thus

\[
\sum_{i \text{ even}}^{1 \leq i \leq k} q_i (P_{i-1} + Q_{i-1}) + (t_{i_0+1} + 1)(P_{i_0} + Q_{i_0}) + \sum_{i_0+2 \leq i \leq k} t_i (P_{i-1} + Q_{i-1})
\]

(i) \( i_0 \) is odd

\[
\sum_{i \text{ odd}}^{1 \leq i \leq k} q_i (P_{i-1} + Q_{i-1}) + (t_{i_0+1} + 1)(P_{i_0} + Q_{i_0}) + \sum_{i_0+2 \leq i \leq k} t_i (P_{i-1} + Q_{i-1})
\]

(ii) \( i_0 \) is even

(Step 2) We write \( t = \sum_{1 \leq i \leq k} t_i (P_{i-1} + Q_{i-1}) \) as in (Step 1). If there exists \( i_0 \) such that \( t_{i_0-1} > 0 \), \( t_{i_0} = q_{i_0} \), and \( t_{i_0+1} < q_{i_0+1} \) (if \( i_0 < k - 1 \)); \( t_{i_0} \leq q_{i_0+1} - 2 \) (if \( i_0 = k - 1 \), we transform \( (t_1^{(1)}, \ldots, t_k^{(1)}) \) to \( (t_1^{(2)}, \ldots, t_k^{(2)}) = (t_1, \ldots, t_{i_0-2}, t_{i_0-1} - 1, 0, t_{i_0+1} + 1, t_{i_0+2}, \ldots, t_k) \). We repeat this operation. Since \( \sum_{1 \leq i \leq k} t_i^{(j)} \) strictly decrease by this operation, we can get a \( \tau \)-sequence after finitely many operations. \( \square \)

Remark. We can prove that the \( \tau \)-expansion is unique.

Put \( T, v \) and \( T_{\min} \) as follows

\[
T = \{(s, t) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} | \sim + \{ \triangleright \triangleright - \triangleright \} \sim \triangleright \triangleleft \} \equiv \triangleright \triangleright - \triangleright \}
\]

\[
v = \min \{s + t| (s, t) \in T\}
\]

\[
T_{\min} = \{(s, t) \in T | f + \square = \square\}
\]

Proposition 4.1. \( i \) If \( d = 1 \) and \( k \) is even, then

\[
T_{\min} = \{(\nabla, \rho)\} \quad (n < m - (P_{k-1} + Q_{k-1}))
\]

\[
T_{\min} = \{(\nabla + P_{\parallel - \infty} + Q_{\parallel - \infty}, \nabla - \circ + P_{\parallel - \infty} + Q_{\parallel - \infty})\} \quad (n \geq m - (P_{k-1} + Q_{k-1}))
\]

(ii) If \( d = 1 \) and \( k \) is odd, then

\[
T_{\min} = \{(\nabla, \rho)\} \quad (n < m - (P_{k-1} + Q_{k-1}))
\]

\[
T_{\min} = \{(\nabla - \circ + P_{\parallel - \infty} + Q_{\parallel - \infty}, \nabla + P_{\parallel - \infty} + Q_{\parallel - \infty})\} \quad (n \geq m - (P_{k-1} + Q_{k-1}))
\]

(iii) If \( d \geq 2 \), then \( T_{\min} = \{(\nabla, \rho)\} \)

Proof. For a positive integer \( t \), put

\[
s_t = \min \{s| (s, t) \in T\}
\]

Since \( (n, n) \in T \), we get \( v \leq 2n \), thus

\[
v = \min \{s_t + t| 0 \leq t \leq 2n\}
\]

\[
T_{\min} = \{(\nabla, \parallel)| 0 \leq \parallel \leq \nabla, \nabla + \parallel = \square\}
\]

Hence we estimate \( s_t \) for \( t \) such that \( 0 \leq t \leq 2n \). It is clear that \( s_n = n \).

(Claim I) Assume \( t < n \). Then
(i) If $k$ is even, $d = 1$, $n \geq m - (P_{k-1} + Q_{k-1})$ and $t = n - m + (P_{k-1} + Q_{k-1})$, then $s_t = n + P_{k-1} + Q_{k-1}$

(ii) Otherwise $s_t + t > 2n$

(Proof of Claim I) It is clear that $s_0 = dma > 2n$, thus we assume $t > 0$. Put $t' = n - t$. Let $t' = \sum_{1 \leq i \leq k} t_i (P_{i-1} + Q_{i-1})$ be the $\tau$-expansion of $t'$. Put $i_0 = \min \{ i | t_i > 0 \}$

First let $i_0$ be even. We must prove $s + t > 0$ in this case since $i_0 = 1$ when $k$ is even and $t = n - m + P_{k-1} + Q_{k-1}$. Let $(s,t)$ be an element in $L$. We can write $s = n + \{ dm(m-a) - 1 \}t' - s'$ in which $s'$ is an integer. Note that $(m-a)t' - m \sum_{1 \leq i \leq k} t_i Q_{i-1} = \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i$. Assume $s' \geq \sum_{1 \leq i \leq k} t_i Q_i$. Then by Lemma 4.2.

$$s \leq n - t' + dm \{ (m-a)t' - m \sum_{1 \leq i \leq k} t_i Q_{i-1} \}$$

$$= n - t' + dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i$$

$$\leq n - t' - dm$$

$$< 0$$

This is a contradiction. Thus $s' \leq \sum_{1 \leq i \leq k} t_i Q_{i-1} - 1$.

Next put $s' = \sum_{1 \leq i \leq k} t_i Q_{i-1} - 1$. Then by Lemma 4.2.

$$s = n - t' + dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i + dm^2$$

$$> n - t' + dma$$

$$> 2n$$

Hence $s'_t = \sum_{1 \leq i \leq k} t_i Q_{i-1} + 1$ and $s_t > 2n$. Thus we have done for even $i_0$.

Secondly let $i_0$ be odd. Assume that $s' \geq \sum_{1 \leq i \leq k} t_i Q_{i-1} + 1$. Then by Lemma 4.2.

$$s \leq n - t' + dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i - dm^2$$

$$< n - t' + dma - dm^2$$

$$< 0$$
This is a contradiction, thus $s' \leq \sum_{1 \leq i \leq k} t_i Q_{i-1} + 1$. Next assume $s' = \sum_{1 \leq i \leq k} t_i Q_{i-1}$. Then by Lemma 4.2
\[ s = n - t' + dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i \geq n - t' + dm > 0 \]
Hence
\[ s_t = n - t' + dm \sum_{1 \leq i \leq k} t_i r_i \]
and
\[ s_t + t = 2t + dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i \]
If $d \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i \geq 2$, then $s_t + t = 2m > 2n$. If $d \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i = 1$, by Lemma 4.3,
(a) $d = 1$, $k$ is even, and $t' = m - (P_{k-1} + Q_{k-1})$.
or (b) $d = 1$, $k$ is odd, and $t' = P_{k-1} + Q_{k-1}$.
In the case (b),
\[ s_t + t = 2n - 2(P_{k-1} + Q_{k-1}) + m > 2n \]
In the case (a),
\[ t = n - m + P_{k-1} + Q_{k-1}, \quad s_t = n + P_{k-1} + Q_{k-1} \]
Thus we have done.

(Claim II) Assume $t > n$. Then
(i) If $k$ is odd, $d = 1$, $n \geq m - (P_{k-1} + Q_{k-1})$, and $t = n + P_{k-1} + Q_{k-1}$, then $s_t = n - m + P_{k-1} + Q_{k-1}$
(ii) Otherwise $s_t + t > 2n$
(Proof of Claim II) Put $t' = t - n$. If $t' = m - 1$, then $n = m - 1$ and $t = 2m - 2$. It is easy to see
\[ s_{2m-2} = dm(m - a) + m - 1 > 2n \]
Thus we assume $t' \leq m - 2$. Let $t' = \sum_{1 \leq i \leq k} t_i (P_{i-1} + Q_{i-1})$ be the $\tau$-expansion of $t'$, and put $i_0 = \min\{i | t_i > 0\}$. Write $s = n - \{dm(m - a) - 1\}t' + dm^2 s'$ for $(s, t) \in \mathcal{T}$.
For the case such that $i_0$ is even, we can get
\[ s_t = n + t' - dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i > 2n \]
by the similar way using Lemma 4.2. If $t = P_{k-1} + Q_{k-1}$, then $i_0 = k$, thus we have done for this case.
Assume that $i_0$ is odd, and let $(s, t) \in \mathcal{T}$. If $s' \leq \sum_{1 \leq i \leq k} t_i Q_{i-1} - 1$, we can get $s < 0$ by Lemma 4.2. Put $s' = \sum_{1 \leq i \leq k} t_i Q_{i-1}$. Then we get $s = n + t' - dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i$. If $d \sum_{1 \leq i \leq k} (-1)^{i-1} t_i r_i \geq 2$, then
\[ s_t = n + t' - dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_ir_i + dm^2, \text{ hence } s_t + t > 2n. \]

If \( d \sum_{1 \leq i \leq k} (-1)^{i-1} t_ir_i = 1 \), by Lemma 4.2

(a) \( d = 1 \), \( k \) is even, \( t' = m - (P_{k-1} + Q_{k-1}) \)

or (b) \( d = 1 \), \( k \) is odd, \( t' = P_{k-1} + Q_{k-1} \)

In the case (a), \( n + t' - dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_ir_i = n - (P_{k-1} + Q_{k-1}) \)

If \( n < P_{k-1} + Q_{k-1} \), then \( s'_t \geq \sum_{1 \leq i \leq k} t_iQ_i - 1 \), thus \( s_t + t > 2n \).

If \( n \geq P_{k-1} + Q_{k-1} \), then \( s_t = n - (P_{k-1} + Q_{k-1}) \),

thus \( s_t + t = 2n + m - 2(P_{k-1} + Q_{k-1}) > 2n \).

In the case (b),

\[ n + t' - dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_ir_i = n + P_{k-1} + Q_{k-1} - m \]

If \( n < m - (P_{k-1} + Q_{k-1}) \), then \( s_t = n + t' - dm \sum_{1 \leq i \leq k} (-1)^{i-1} t_ir_i + dm^2 \),

thus \( s_t + t > 2n \).

If \( n \geq m - (P_{k-1} + Q_{k-1}) \), then \( s_t = n - m + P_{k-1} + Q_{k-1} \).

Thus we have done.

Summarizing (I) and (II), we have proved the proposition. \( \square \)

We define \( \delta^i \in \mathcal{L} \) by \( \delta^i = 0 \) for \( \eta \neq i \) and \( \delta^i = 1 \). And for \( \nu \in \mathcal{L} \), we define \( \alpha(\nu) \in \mathcal{L} \otimes \mathbb{Q} \) by \( \alpha(\nu)_{\eta} = \sum_{i \in \nu} \alpha_i^i \nu_i \).

**Definition 4.3.** Let \( n \) be an integer such that \( 1 \leq n \leq m - 1 \). Let

\[
\begin{align*}
    n = \lambda_{j_i} + \sum_{3 \leq i \leq k-1 \atop i \text{ odd}} (l_{i-1}\lambda_1 + \lambda_{j_i}) + l_k\lambda_k^{k+1} = \sum_{2 \leq i \leq k \atop i \text{ even}} (l_{i-1}\mu_i + \mu_{j_i})
\end{align*}
\]

be the \( \lambda \)- and \( \mu \)- expansion of \( n \). We define \( \nu(n)^o \in I_o \), \( \nu(n)^e \in I_e \), \( \nu(n) \in I \) as follows:

\[
\begin{align*}
    \nu(n)^o = & \delta_{1,j_1} + \sum_{3 \leq i \leq k-1 \atop i \text{ odd}} (l_{i-1}\delta_{i,1} + \delta_{i,j_i}) + l_k\delta_{k+1,1} \\
    \nu(n)^e = & \sum_{2 \leq i \leq k \atop i \text{ even}} (l_{i-1}\delta_{i,1} + \delta_{i,j_i}) \\
    \nu(n) = & \nu(n)^o + \nu(n)^e
\end{align*}
\]

**Theorem 4.1.** Let \( n \) be an integer such that \( 1 \leq n \leq m - 1 \). Then

\[ \deg_{E_i} F(-nK_X) = \nu(n), \text{ for all } i \in I \]

**Proof.** Let \( \nu'(n) \) be an element of \( \mathcal{L} \) such that \( \nu'(n)_i = \deg_{E_i} F(-K_X) \).

Put \( d' = \lfloor \frac{d}{2} \rfloor \). Put \( I'_o \) and \( I'_e \) as follows

\[
\begin{align*}
    I'_o = & \{(i, j) \in I | i \text{ is odd and } j \leq d' \text{ if } i = k + 1 \} \\
    I'_e = & I \setminus I'_o
\end{align*}
\]

19
For $\nu \in \mathcal{L}$, define $s(\nu)$ and $t(\nu)$ as follows

$$s(\nu) = \sum_{i \in I_0^\prime} \nu_i \lambda_i, \quad t(\nu) = \sum_{i \in I_0^\prime} \nu_i \mu_i$$

Let $\mathcal{L}(\emptyset)$ be the set of elements of $\mathcal{L}$ satisfying the following conditions

(i) $\nu_i \geq 0$ for all $i \in I$

(ii) $s(\nu) + \{dm(m - a) - 1\}t(\nu) \equiv dm(m - a)n \pmod{dm^2}$

By Lemma [4.1], it is clear that $\nu'(n)$ is an element of $\mathcal{L}(\emptyset)$ which is characterized by the inequalities $\alpha(\nu'(n))_\eta = \alpha(\nu)_\eta$ for all $\nu \in \mathcal{L}$ and for all $\eta \in I$. First we show

$$s(\nu'(n)) = s(\nu(n)), \quad t(\nu'(n)) = t(\nu(n))$$

For $\nu \in \mathcal{L}$,

$$dm^2 \alpha(\nu)_{d+1} = \mu_{d+1} \sum_{i \in I_0^\prime} \nu_i \lambda_i + \lambda_{d+1} \sum_{i \in I_0^\prime} \nu_i \mu_i$$

$$= \{(d - d')m - (P_{k-1} + Q_{k-1})\}s(\nu) + t(\nu)$$

$$- \{(d - d')m - 2(P_{k-1} + Q_{k-1})\}t(\nu)$$

$$= : \alpha(s(\nu) + t(\nu)) - \beta t(\nu)$$

If $d \geq 2$,

$$dm^2 \alpha(\nu)_{d+1} = \{(d - d')m - (P_{k-1} + Q_{k-1})\}s(\nu) + t(\nu)$$

$$+ \{(2d' - d)m + 2(P_{k-1} + Q_{k-1})\}t(\nu)$$

$$= : \gamma(s(\nu) + t(\nu)) + \delta t(\nu)$$

Note that $\alpha$, $\beta$, $\gamma$, and $\delta$ are all positive. Thus by the Proposition [4.1] we have done for this case.

If $d = 1$,

$$dm^2 \alpha(\nu)_{1} = \{m - 2(P_{k-1} + Q_{k-1})\}(s(\nu) + t(\nu)) + 4(P_{k-1} + Q_{k-1})t(\nu)$$

Thus we can use the same argument as above.

Next we show $\nu(n)_\eta = \nu'(n)_\eta$ for $\eta \in I_o$ by induction. Let $\eta$ be an element of $I_o$ which is not $(1, 1)$. Assume $\nu(n)_\iota = \nu'(n)_\iota$, for all $\iota \in I_o$ such that $\iota > \eta$. For $\nu \in \mathcal{L}$,

$$dm^2 \alpha(\nu)_{\eta'} = -dm^2 \nu_{\eta} + (s(\nu) - \sum_{i > \eta} \nu_i \lambda_i)\mu_{\eta'}$$

$$+ (t(\nu) + \sum_{i > \eta} \nu_i \mu_i)\lambda_{\eta'}$$
Corollary 4.2. Let \( \sum \) be the \( \lambda \) be the \( \nu \) be an integer such that \( 1 \leq n \leq m - 1 \). Let
\[
n = \lambda_{j,1} + \sum_{3 \leq i \leq k-1 \atop i \text{ odd}} (l_{i-1} \lambda_1^i + \lambda_j^i) \quad \text{and put} \quad \nu = \sum_{2 \leq i \leq k \atop i \text{ even}} (l_{i-1} \mu_i^i + \mu_j^i)
\]
be the \( \lambda \)- and \( \mu \)- expansion.
Then
\[
C_{j,1}^1 + \sum_{3 \leq i \leq k-1 \atop i \text{ odd}} \left( \sum_{1 \leq h_{i-1} \leq l_{i-1}} C_{1,h_{i-1}}^i + C_{j,1}^i \right) + \sum_{1 \leq h_k \leq l_k} C_{1,h_k}^{k+1} \bigg|_{i \text{ even}}
\]
\[
+ \sum_{2 \leq i \leq k \atop i \text{ even}} \left( \sum_{1 \leq h_{i-1} \leq l_{i-1}} C_{1,h_{i-1}}^i + C_{j,1}^i \right)
\]
is a general member of \( | - nK_X | \).

Next we treat the non-normal case. It is easier than the normal case.

Theorem 4.2. Let \( n \) be an integer such that \( 1 \leq n \leq m - 1 \). Then
\[
\deg_{E_i} F(-n(K_{X_0} + \Delta_o)) = \nu_o(n)_i \quad \text{for all } i \in I_o
\]
\[
\deg_{E_i} F(-n(K_{X_e} + \Delta_e)) = \nu_e(n)_i \quad \text{for all } i \in I_e
\]
Proof. We only show the first equality since the proof is similar for the second one. Let \( \nu'_o(n)_i \) be an element of \( L_i \) such that \( \nu'_o(n)_i = \deg_{E_i} F(-n(K_{X_0} + \Delta_o)) \) for \( i \in I_o \). Put \( \sigma(\nu) = \sum_{i \in I_o} \nu_i \lambda_i \) for \( \nu \in L_i \), and put
\[
\mathcal{L}(\cap) = \{ \nu \in L_i | \nu \text{ is nef, } \sigma(\nu) \equiv \ (\mod \ \mathcal{F}) \}
\]
By Lemma 4[1], \( \nu'_o(n)_i \) is an element of \( \mathcal{L}(\cap) \) which is characterized by \( \alpha(\nu'_o(n))_{i} \leq \alpha(\nu)_{i} \) for all \( \nu \in \mathcal{L}_i \) and for all \( i \in I_o \). Since \( m \alpha(\nu)^{k+1} = \sigma(\nu) \) and \( \sigma(\nu_o(n)) = n \), we get \( \sigma(\nu'_o(n)) = \sigma(\nu_o(n)) \). Thus we can prove the theorem by the induction using the formula
\[
m \alpha(\nu)_q = -m \nu_q + \mu_q (\sigma(\nu) - \sum_{i \in I_o, q > i} \nu_i \lambda_i) + \lambda_q \sum_{i \in I_o, q > i} \nu_i \lambda_i
\]
similarly to the proof of Theorem 4[1]. \( \square \)
5. Local intersection number

As the application of the result of Section 4, we shall prove the following Theorem 5.1 in this section.

**Definition 5.1.** For a sequence of positive integers \((L_1, L_2, \ldots, L_J)\) and a positive integer \(N\), we define the sequence \((N_{-1}, N_0, N_1, \ldots, N_J)\) by

\[
N_{-1} = N_0 = N, \quad N_j = L_j N_{j-1} + N_{j-2} \quad (1 \leq j \leq J)
\]

We define \(B((L_1, L_2, \ldots, L_J), N)\) by

\[
B(((L_1, L_2, \ldots, L_J), N) = \sigma(N_J).
\]

**Theorem 5.1.** Let \((X, x)\) be a 2-dimensional smoothable semi-log-terminal singularity, and \(n\) a positive integer. Let \(D\) and \(D'\) be members in \(|nK_X|\) which do not have common components. Then

\[
\text{index}(X, x) \leq B(D \cdot D' + 1, n)
\]

We define \(Z\)-valued symmetric bilinear forms \(O\) and \(E\) on \(L\) by

\[
O(\delta^i, \delta^m) = \begin{cases} 
\lambda_i \lambda_\eta & (i, \eta \in I_o, i < \eta) \\
\lambda_i \lambda_\eta & (i, \eta \in I_o, i \geq \eta) \\
0 & (\text{otherwise})
\end{cases}
\]

\[
E(\delta^i, \delta^m) = \begin{cases} 
\mu_i \mu_\eta & (i, \eta \in \bar{I}_e, i < \eta) \\
\mu_i \mu_\eta & (i, \eta \in \bar{I}_e, i \geq \eta) \\
0 & (\text{otherwise})
\end{cases}
\]

where \(\bar{I}_e = I_e \cup \{(k + 1, d)\}\) in the normal case, and \(\bar{I}_e = I_e\) in the non-normal case.

**Lemma 5.1.** Let \(\nu = \nu^o + \nu^e\) and \(\tilde{\nu} = \tilde{\nu}^o + \tilde{\nu}^e\) be members in \(L\). Then in the normal case,

\[
dm^2(\nu \cdot \tilde{\nu}) = (dma - 1)\sigma(\nu^o)\sigma(\tilde{\nu}^o) + \sigma(\nu^o)\tau(\tilde{\nu}^e) + \tau(\nu^e)\sigma(\tilde{\nu}^o) - (dma + 1)\tau(\nu^e)\tau(\tilde{\nu}^e)
\]

\[
+ dm^2(E(\nu^i, \tilde{\nu}^i) - O(\nu^i, \tilde{\nu}^i))
\]

and in the non-normal case,

\[
m(\nu \cdot \tilde{\nu}) = a(\sigma(\nu^o)\sigma(\tilde{\nu}^o) - \tau(\nu^e)\tau(\tilde{\nu}^e)) + m(E(\nu^i, \tilde{\nu}^i) - O(\nu^i, \tilde{\nu}^i))
\]
**Proof.** First, we treat the normal case. We shall calculate the each term of the left hand side of
\[
\nu \cdot \nu = \nu^o \cdot \nu^o + \nu^o \cdot \nu^e + \nu^e \cdot \nu^o + \nu^e \cdot \nu^e
\]
We can easily get \(dm^2 \nu^o \cdot \nu^o = \sigma(\nu^o) \tau(\nu^e)\) and \(dm^2 \nu^e \cdot \nu^o = \tau(\nu^e) \sigma(\nu^o)\).

By the formula \(\mu_i = -\lambda_i + dm \rho_i\),
\[
dm^2 \nu^o \cdot \nu^e = \sum_{i \in I_o} \nu^o \iota \sum_{\eta \in I_o, \eta \geq i} \nu^o_{\eta} \mu_{\eta} + \mu_i \sum_{\eta \in I_o, \eta \leq i} \nu^0_{\eta} \lambda_{\eta}
\]
\[
= \sum_{i \in I_o} \nu^o \iota \left[-\lambda_i \sigma(\nu^o) + dm \rho_i \sum_{\eta \in I_o, \eta \leq i} \nu^0_{\eta} \lambda_{\eta} + \lambda_i \sum_{\eta \in I_o, \eta \geq i} \nu^0_{\eta} \rho_{\eta} \right]
\]
\[
= -\sigma(\nu^o) \sigma(\nu^e) + dm \sum_{i \in I_o} \nu^o \iota \rho_i \sum_{\eta \in I_o, \eta \leq i} \nu^0_{\eta} \lambda_{\eta} + \lambda_i \sum_{\eta \in I_o, \eta \geq i} \nu^0_{\eta} \rho_{\eta}
\]

By the formula \(\rho_i = -m \bar{\lambda}_i + a \lambda_i\), we can get
\[
\rho_i \sum_{\eta \in I_o, \eta < i} \nu^0_{\eta} \lambda_{\eta} + \lambda_i \sum_{\eta \in I_o, \eta \geq i} \nu^0_{\eta} \rho_{\eta}
\]
\[
= a \lambda_i \cdot \sigma(\nu^o) - m(\bar{\lambda}_i \sum_{\eta \in I_o, \eta < i} \nu^0_{\eta} \lambda_{\eta} + \lambda_i \sum_{\eta \in I_o, \eta \geq i} \nu^0_{\eta} \rho_{\eta}
\]

Hence we get
\[
dm^2 \nu^o \cdot \nu^e = (dma - 1) \sigma(\nu^o) \sigma(\nu^o) - dm^2 \mathcal{O}(\nu^o, \nu^e)
\]

Simiarly we get
\[
dm^2 \nu^e \cdot \nu^e = -(dma + 1) \tau(\nu^e) \tau(\nu^e) + dm^2 \mathcal{E}(\nu^e, \nu^e)
\]

Summarizing all the above formulas, we get the first equality.

Next for the non-normal case, we can get \(mv \cdot \nu = a \sigma(\nu) \nu(\nu) - m \mathcal{O}(\nu, \nu)\) for \(\nu, \eta \in L\), \(mv \cdot \nu = -a \tau(\nu) \nu(\nu) + m \mathcal{E}(\nu, \nu)\) for \(\nu, \overline{\nu} \in L\).

We leave the details for the reader. \(\square\)

Note that \(\nu \cdot \nu = \mathcal{E}(\nu^1, \nu^1) - \mathcal{O}(\nu^1, \nu^1)\) if \(\sigma(\nu^o) = \tau(\nu^e)\) and \(\sigma(\nu^o) = \tau(\nu^e)\) hold.

**Corollary 5.1.** If \(\nu^o \leq \nu^o, \nu^e \leq \nu^e, \sigma(\nu^o) = \tau(\nu^e), \sigma(\nu^o) = \tau(\nu^e)\) and \(\bar{\sigma}(\nu^o) = \bar{\tau}(\nu^e)\) hold, then \(\nu \cdot \nu = 0\).

**Proof.** This can be easily checked by the above lemma. \(\square\)

**Definition 5.2.** For \(i = (i, j) \in I\) such that \(i \neq k + 1\), put
\[
\varphi(i) = -\delta^{i,1} + \delta^{i,j} + (j - 1)\delta^{i+1,j}
\]
Lemma 5.2.

\[ \sigma(\varphi(i)^o) = \tau(\varphi(i)^e) = (j - 1)(P_{i-1} + Q_{i-1}) \]
\[ \bar{\sigma}(\varphi(i)^o) = \bar{\tau}(\varphi(i)^e) = (j - 1)P_{i-1} \]
\[ \varphi(i)^2 = j - 1 \]

**Proof.** We can get the first and the second formula by direct calculation. By the formula \( P_{i-2}Q_{i-1} - Q_{i-2}P_{i-1} = (-1)^i \), we get
\[ \mathcal{E}(\varphi(i)^1, \varphi(i)^1) = (| - \infty|^\infty \mathcal{P}_{j-\infty} + \mathcal{Q}_{j-\infty}) \]
and
\[ \mathcal{O}(\varphi(i)^1, \varphi(i)^1) = (| - \infty|^\infty \mathcal{P}_{j-\infty} + \mathcal{Q}_{j-\infty}) - | + \infty \]
Thus by the above corollary, we can get the third formula. \qed

**Definition 5.3.** Let \( i = (i_1, j_1) \), \( \eta = (i_2, j_2) \) be elements in \( I \) such that \( i_2 \neq k + 1 \), the parity of \( i_1 \) coincides the one of \( i_2 \) and \( i \leq (i_2, 1) \). For such pair \( (i, \eta) \), we define \( \psi(i, \eta) \) as follows
\[ \psi(i, \eta) = \begin{cases} -\delta^i + \delta_\ell + \sum_{i \text{ odd}, i \neq 1} q_{i-1} - \delta^1 - \delta^i, i \leq i \leq \eta & (i \in I_o) \\ -\delta^e + \delta_\ell + \sum_{i \text{ even}} q_{i-1} - \delta^1 - \delta^i, i \leq i \leq \eta & (i \in I_e) \end{cases} \]

**Lemma 5.3.** Let \( (i, \eta = (i'', 1)) \) be the pair for which \( \psi \) can be defined. Then
\[ \sigma(\psi(i, \eta)^o) = \tau(\psi(i, \eta)^e) = P_{i''+1} + Q_{i''+1} \]
\[ \bar{\sigma}(\psi(i, \eta)^o) = P_{i''+1}, \quad \bar{\tau}(\psi(i, \eta)^e) = \begin{cases} P_{i''-1} & (i \neq (2, 1)) \\ P_{i''-1} + 1 & (i = (2, 1)) \end{cases} \]
\[ \psi(i, \eta)^2 = \begin{cases} 1 + \sum_{i \text{ even}} q_{i-1} & (i \in I_o) \\ 1 + \sum_{i \text{ odd}} q_{i-1} & (i \in I_e) \end{cases} \]

**Proof.** Since the calculation is the same, we show the outline of it for the case \( i \in I_o \). We can easily calculate \( \sigma, \tau, \bar{\sigma}, \) and \( \bar{\tau} \). Hence
\[ \psi(i, \eta)^2 = \mathcal{E}(\psi(i, \eta)^1, \psi(i, \eta)^1) - \mathcal{O}(\psi(i, \eta)^1, (\psi(i, \eta)^1) \]
By the definition,
\[ \mathcal{E}(\psi(i, \eta)^1, \psi(i, \eta)^1) = \mathcal{P}_{i''-\infty}(\mathcal{P}_{j''-\infty} + \mathcal{Q}_{j''-\infty}) \]
To calculate \( \mathcal{O}(\psi(i, \eta)^1, (\psi(i, \eta)^1) \), note that
\[ -\lambda_i + \lambda_i + \sum_{i'' + 2 \leq i \leq h} q_{i-1} \lambda_i = P_{h-1} + Q_{h-1} \]
Corollary 5.2. Let $(i, \eta)$ be a pair for which $\psi$ can be defined. Then

$$\sigma(\psi(i, \eta)) = \tau(\psi(i, \eta)) = j''(P_{i''-1} + Q_i)$$
\[
\bar{\sigma}(\psi(\iota, \eta)\circ) = j''P_{i''-1}, \quad \bar{\tau}(\psi(\iota, \eta)\circ) = \begin{cases} j''P_{i''-1} & (i \neq (2, 1)) \\ j''P_{i''-1} + 1 & (i = (2, 1)) \end{cases}
\]

\[
\psi(\iota, \eta) = \begin{cases} j'' + \sum_{i \text{ even}}^{i''-1} q_i & (i \in I_o) \\ j'' + \sum_{i \text{ odd}}^{i''-1} q_i & (i \in I_e) \end{cases}
\]

**Proof.** Note that \( \psi(\iota, \eta) = \psi(\iota, (i'', 1)) + \varphi(\eta) \). Hence

\[
\sigma(\psi(\iota, \eta)\circ) = \sigma(\psi(\iota, \eta)\circ) + \sigma(\varphi(\eta)\circ) = j''(P_{i''-1} + Q_{i''-1})
\]

\( \tau, \bar{\sigma} \) and \( \bar{\tau} \) are similarly calculated. By Corollary 5.1, \( \psi(\iota, (i'', 1)) \cdot \varphi(\eta) = 0 \). Hence

\[
\psi(\iota, \eta)^2 = \psi(\iota, (i'', 1))^2 + \varphi(\eta)^2
\]

Thus the formula follows from Lemma 5.2 and 5.3. \( \square \)

**Definition 5.4.** For \((i, j) \in I\) such that \(1 \leq i \leq k - 1\) and \(1 \leq j \leq q_i\), we put

\[
\theta(\iota, \eta) = -\delta^{i, q_i-j+1} + \delta^{i+2,1} + j\delta^{i+1,1}
\]

**Lemma 5.4.**

\[
\sigma(\theta(\iota, j)\circ) = \tau(\theta(\iota, j)\circ) = j(P_{i-1} + Q_{i-1})
\]

\[
\bar{\sigma}(\theta(\iota, j)\circ) = \bar{\tau}(\theta(\iota, j)\circ) = jP_{i-1}
\]

\[
\theta(\iota, j) = j
\]

**Proof.** We will only show the outline of the calculation for odd \(i\) since it is similar for even \(i\). We can easily get the formulas for \(\sigma, \tau, \bar{\sigma}\) and \(\bar{\tau}\). Thus by Lemma 5.1,

\[
\theta(\iota, j)^2 = \mathcal{E}(\theta(\iota, |)^1, \theta(\iota, |)^1) - \mathcal{O}(\theta(\iota, |)^1, \theta(\iota, |)^1)
\]

By the definition,

\[
\mathcal{E}(\theta(\iota, |)^1, \theta(\iota, |)^1) = [\mathcal{E}_{\mathcal{P}_{-\infty}}(\mathcal{P}_{-\infty} + \mathcal{Q}_{-\infty})]
\]
and

\[
\mathcal{O}(\theta(), |\rangle, \theta(|\rangle)) = \lambda^i_{q_i-j+1}(\tilde{\lambda}^i_{q_i-j+1} - \tilde{\lambda}^i_{1}) + (-\lambda^i_{q_i-j+1} + \lambda^i_{1+2})\tilde{\lambda}^i_{1+2}
\]

\[= -jP_{i-1}\lambda^i_{q_i-j+1} + j(P_{i-1} + Q_{i-1})\tilde{\lambda}^i_{1+2}
\]

\[= j^2P_{i-1}(P_{i-1} + Q_{i-1}) + (P_{i-1} - Q_{i-1})P_{i-1}
\]

\[= j^2P_{i-1}(P_{i-1} + Q_{i-1}) - j
\]

Hence we have done. \(\Box\)

For a positive integer \(n\) which is smaller than \(m - (P_{k-1} + Q_{k-1})\), we put

\[i(n) = \max\{i | 0 \leq i \leq k - 1, P_i + Q_i \leq n\}\]

**Proposition 5.1.** If \(0 < n < m - (P_{k-1} + Q_{k-1})\),

\[\nu(n)^2 \leq \frac{n}{P_i(n) + Q_i(n)}\]

**Proof.** We use the induction on \(i(n)\). If \(i(n) = 0\), it can be easily checked that \(\nu(n)^2 = n\). Let \(i\) be an integer such that \(1 \leq i \leq k - 1\). Assume that the inequality holds for all \(n\) such that \(i(n) < i\). We will show that the inequality holds for \(n\) such that \(i(n) = i\) under this assumption. We also assume \(i\) is odd since the proof is similar for even \(i\).

Write \(n = j(P_i + Q_i) + n'\) such that \(0 \leq n' < P_i + Q_i\). If \(n' = 0\), we can check (by the definition of \(\lambda\) and \(\mu\)-expansion) that

\[\nu(n) = \psi((2, 1), (i + 1, j))\]

Thus by Corollary 5.2,

\[\nu(n)^2 \leq j\]

Hence we have done in this case. Thus we assume \(n' > 0\). We divide the proof into two cases as follows.

(i) Put

\[\eta = \min\{i \in \tilde{I}_e | i' < i \text{ for all } i' \in \tilde{I}_e \text{ such that } \nu(n')i' \neq 0\}\]

We can check (by the definition of \(\lambda\) and \(\mu\)-expansion)

\[\nu(n) = \nu(n') + \psi(\eta, (i + 1, j))\]

Since \(\nu(n')^e \leq \psi(\eta, (i + 1, j))^e\) and \(\nu(n')e \leq \psi(\eta, (i + 1, j))e\) hold, thus by Corollary 5.1,

\[\nu(n)^2 = \nu(n')^2 + \psi(\eta, (i + 1, j))^2\]
By Corollary 5.2 and the induction hypothesis,
\[
\nu(n)^2(P_i + Q_i) \geq \nu(n')^2(P_i + Q_i) + j(P_i + Q_i)
\geq \nu(n')^2(P_{i(n')} + Q_{i(n')}) + j(P_i + Q_i)
\geq n' + j(P_i + Q_i)
= n
\]
Thus we have done.
(ii) We can check
\[
\nu(n) = \nu(n') + \varphi(i + 1, j + 1)
\nu(n')^o \leq \varphi(i + 1, j + 1)^o, \quad \nu(n')^e \leq \varphi(i + 1, j + 1)^e
\]
Thus we can get the inequality by the similar way to (i) using Lemma 5.2 and the induction hypothesis, \(\square\)

For \(n\) such that \(m - (P_{k-1} + Q_{k-1}) \leq n \leq m - 1\), we define \(i(n)\) and \(j(n)\) as follows
\[
i(n) = \min\{i|0 \leq i \leq k - 1, m - n \leq P_i + Q_i\}
jk(n) = \left\lceil \frac{m - n}{P_{i(n)-1} + Q_{i(n)-1}} \right\rceil - 1
\]

**Lemma 5.5.** Let \(n, i, j\) be positive integers such that \(m - (P_{k-1} + Q_{k-1}) \leq n \leq m - 1\), \(i \leq k - 1\), \(j \leq q_i\) and \(i(n) \leq i - 1\). Then \(\nu(n) \cdot \theta(i, j) = j\).

**Proof.** We only show the proof for even \(i\). By Lemma 5.4,
\[
\nu(n) \cdot \theta(i, j) = \mathcal{E}(\nu(\{|\rangle, j\rangle\}) - \mathcal{O}(\nu(|\rangle, \theta(\langle)|\rangle))
\]
First we calculate \(\mathcal{E}(\nu(\{|\rangle, j\rangle\})\). Note that \(m - (P_{i-1} + Q_{i-1}) \leq n \leq m - 1\). Thus the \(\mu\)-expansion of \(n\) is as follows
\[
n = \sum_{\substack{2 \leq h \leq i-2 \atop h \text{ even}}} (l_{h-2}\mu_1^h + \mu_1^2) + l_{i-1}\mu_1^1 + \sum_{\substack{i+2 \leq h \leq k-2 \atop h \text{ even}}} q_{h-1}\mu_1^h + q_{k-1}\mu_1^k + \mu_{q_k}
\]
Since \(\sum_{\ell \geq (i+2, 1)} \nu(n)^e \mu_\ell = m - (P_{i-1} + Q_{i-1})\), we get
\[
\sum_{\substack{2 \leq h \leq i-2 \atop h \text{ even}}} \nu(n)^e \mu_\ell = n - m + P_{i-1} + Q_{i-1}
\]
Using this formula and the formula \(\sum_{\ell \geq (i+2, 1)} \nu(n)^e \bar{\mu}_\ell = a - P_{i-1}\), we get
Lemma 5.6.

\[ \nu(m - 1)^2 = \sum_{1 \leq h \leq k} q_h \]
Proof. Note that $\lambda$- and $\mu$- expansion of $m - 1$ is as follows

$$m - 1 = \lambda_0^1 + \sum_{3 \leq h \leq k-1, \ h \ odd} (q_{h-1} \lambda_1^h + \lambda_0^h) + q_k \lambda_1^{k+1}$$

$$= \sum_{2 \leq h \leq k-2, \ h \ even} (q_{h-1} \mu_1^h + \mu_0^h) + q_{k-1} \mu_1^k + \mu_1^k$$

We leave the rest of calculation for the reader’s exercise.

**Proposition 5.2.** Let $n$ be an integer such that $m - (P_{k-1} + Q_{k-1}) \leq n \leq m - 1$. Then

$$\nu(n)^2 \geq \sum_{i(n) \leq h \leq k} q_h - j(n)$$

Proof. We use the induction on $i(n)$. If $i(n) = 0$, then $n = m - 1$, Thus we have already done in the above lemma. Let $i$ be a positive integer and assume that the inequality holds for $n'$ such that $i(n') < i$. Let $n$ be an integer such that the inequality holds for this $n$. Put $n' = n + j(P_{i-1} + Q_{i-1})$. Then $i(n') < i$. We can check

$$\nu(n) = \nu(n') - \theta(i,j)$$

Thus by Lemma 5.4 and 5.5, we can get

$$\nu(n)^2 = \nu(n')^2 - 2\nu(n') \cdot \theta(i,j) + \theta(i,j)^2$$

$$= \nu(n')^2 - j$$

By the induction hypothesis,

$$\nu(n') \geq \sum_{i(n')+1 \leq h \leq k} q_h$$

Thus we have done.

(Proof of the Theorem 5.1)

From Proposition 5.1 and Proposition 5.2, we know the theorem holds if $D$ and $D'$ is general members in $|nK_X|$. Thus by Corollary 4.1, we have proved the theorem.

References

[ESN] H. Esnault, Reflexive modules on quotient surface singularities, J. Reine Angew. Math. 362 (1985), 63–71

[K-SB] J. Kollár and N. Shepherd-Barron, Threefolds and deformations of surface singularities, Invent. Math. 91 (1988), 299–338

30