Hyers-Ulam Stability of Quadratic Functional Equation Based on Fixed Point Technique in Banach Spaces and Non-Archimedean Banach Spaces

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Abstract: In this paper, the authors investigate the Hyers–Ulam stability results of the quadratic functional equation in Banach spaces and non-Archimedean Banach spaces by utilizing two different techniques in terms of direct and fixed point techniques.

Keywords: Hyers–Ulam stability; quadratic functional equation; fixed point

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1. Introduction and Preliminaries

The study of stability problems for functional equations is one of the essential research areas in mathematics, which originated in issues related to applied mathematics. The first question concerning the stability of homomorphisms was given by Ulam [1] as follows.

Given a group \((G, \ast)\), a metric group \((G', \cdot)\) with the metric \(d\), and a mapping \(f\) from \(G\) and \(G'\), does \(\delta > 0\) exist such that

\[d(f(x \ast y), f(x) \cdot f(y)) \leq \delta\]

for all \(x, y \in G\). If such a mapping exists, then does a homomorphism \(h : G \rightarrow G'\) exist such that

\[d(f(x), h(x)) \leq \epsilon\]

for all \(x \in G\)?

Hyers partially answered affirmatively with respect to the question of Ulam for Banach spaces [2]. By assuming an infinite Cauchy difference, Aoki [3] expanded Hyers’ Theorem for additive mappings and Rassias [4] for linear mappings. Gajda [5] discovered an affirmative answer to the issue \(p > 1\) by using the same approach as Rassias [4]. Gajda [5], as well as Rassias and Šemrl [6], showed that a Rassias’ type theorem cannot be established for \(p = 1\).

One of the most famous functional equations is the additive functional equation

\[f(x + y) = f(x) + f(y)\] (1)

In 1821, it was first solved by A.L. Cauchy in the class of continuous real-valued functions. It is often called the Cauchy additive functional equation in honor of A.L. Cauchy. The theory of additive functional equations is frequently applied to the development of theories of other functional equations. Moreover, the properties of additive functional
Theorem 1. Suppose that a complete generalized metric space \((V, d)\) and a mapping \(H : V \to V\) is strictly contractive with Lipschitz constant \(L < 1\). Then, for every \(v \in V\), either

\[
d(H^l v, H^{l+1}v) = \infty
\]

for all integers \(l > 0\) or there is an integer \(l_0 > 0\) satisfies the following:

1. \(d(H^l v, H^{l+1}v) < \infty\) for all \(l \geq l_0\);
2. The sequence \(\{H^l v\}\) converges to a fixed point \(u^*\) of \(H\);
3. \(u^*\) is the unique fixed point of \(H\) in \(W = \{u \in V \mid d(H^l v, u) < \infty\}\);
4. \(d(u, u^*) \leq \frac{1}{1-L}d(u, Hu)\) for all \(u \in W\).

In [13], Nazek Alessa et al. introduced a new type of generalized quadratic functional equation as the following:

\[
\sum_{1 \leq i < j \leq m} \phi(v_i + v_j) + \sum_{1 \leq i < j \leq m} \phi(v_i - v_j) = 2(m - 1) \sum_{1 \leq i \leq m} \phi(v_i)
\]
where \( m \geq 2 \), and derived its solution. A non-Archimedean \((n, \beta)\)-normed space was used to study the stability of the functional Equation (2) in terms of Hyers–Ulam.

In this paper, we study the Ulam-Hyers stability results of the generalized additive functional Equation (2) in Banach spaces and non-Archimedean Banach spaces by using different approaches of direct and fixed point techniques. This paper is structured as follows: In Sections 2 and 3, we investigate the Ulam–Hyers stability results in Banach spaces by using direct and fixed point techniques where we consider that \( V \) and \( W \) are normed spaces and Banach spaces, respectively. In Sections 4 and 5, we examined the Ulam–Hyers stability results in non-Archimedean Banach spaces by using direct and fixed point techniques where we consider that \( V \) is a non-Archimedean normed space, \( W \) is a non-Archimedean Banach space, and let \(|2| \neq 1\).

**Lemma 1** ([13]). If a mapping \( \phi : V \to W \) satisfies the functional Equation (2), then the mapping \( \phi : V \to W \) is quadratic.

For notational simplicity, we define \( \phi : V \to W \) by the following:

\[
\Lambda \phi(v_1, v_2, \ldots, v_m) = \sum_{1 \leq i < j \leq m} \phi(v_i + v_j) + \sum_{1 \leq i < j \leq m} \phi(v_i - v_j) - 2(m - 1) \sum_{1 \leq i \leq m} \phi(v_i).
\]

2. Stability Results in Banach Spaces: Direct Technique

**Theorem 2.** Let \( \zeta \in \{-1, 1\} \) and a mapping \( \chi : V^m \to [0, \infty) \) such that

\[
\lim_{l \to \infty} \frac{\chi(2^l v_1, 2^l v_2, \ldots, 2^l v_m)}{2^l} = 0
\]

(3)

for all \( v_1, v_2, \ldots, v_m \in V \). If a mapping \( \phi : V \to W \) with \( \phi(0) = 0 \), and it satisfies the below inequality:

\[
\|\Lambda \phi(v_1, v_2, \ldots, v_m)\| \leq \chi(v_1, v_2, \ldots, v_m)
\]

(4)

for all \( v_1, v_2, \ldots, v_m \in V \). Then, there exists a unique quadratic mapping \( Q_2 : V \to W \) such that

\[
\|\phi(v) - Q_2(v)\| \leq \frac{1}{2^l} \sum_{l=1}^{\infty} \frac{\chi(2^l v, 2^l v, 0, \ldots, 0)}{2^l}
\]

(5)

for all \( v \in V \). Then, the mapping \( Q_2(v) \) is defined by

\[
Q_2(v) := \lim_{l \to \infty} \frac{\phi(2^l v)}{2^l}
\]

for all \( v \in V \).

**Proof.** Assume that \( \zeta = 1 \). Replacing \( (v_1, v_2, \ldots, v_m) \) by \( (v, v, 0, \ldots, 0) \) in (4), we obtain

\[
\|2^l \phi(v) - \phi(2^l v)\| \leq \chi(v, v, 0, \ldots, 0)
\]

(6)

for all \( v \in V \). From inequality (6), we have

\[
\|\phi(2^l v) - 2^l \phi(v)\| \leq \frac{\chi(v, v, 0, \ldots, 0)}{2^l}
\]

(7)
for all \( v \in V \). By replacing \( v \) by \( 2v \) and dividing by \( 2^2 \) in (7) and then combining the resultant inequality with (7), we obtain

\[
\left\| \frac{\phi(2^2v)}{2^4} - \phi(v) \right\| \leq \frac{1}{2^2} \left\| \chi(v, v, 0, \ldots, 0) + \frac{\chi(2v, 2v, 0, \ldots, 0)}{2^2} \right\|
\]

for all \( v \in V \). We conclude for any non-negative integer \( p \) that one can easy to verify the following:

\[
\left\| \frac{\phi(2^pv)}{2^{2p}} - \phi(v) \right\| \leq \frac{1}{2^2} \sum_{l=0}^{p-1} \frac{\chi(2^lv, 2^lv, 0, \ldots, 0)}{2^{2l}}
\]

(8)

for all \( v \in V \). To show that the sequence \( \left\{ \frac{\phi(2^pv)}{2^{2p}} \right\} \) is converging, replacing \( v \) by \( 2^l v \) and dividing by \( 2^{2l} \) in (8) for \( p, l > 0 \), we obtain

\[
\left\| \frac{\phi(2^{p+l}v)}{2^{2(p+l)}} - \frac{\phi(2^lv)}{2^{2l}} \right\| = \frac{1}{2^2} \left\| \frac{\phi(2^{p+l}v)}{2^{2p}} - \frac{\phi(2^lv)}{2^{2l}} \right\|
\]

\[
\leq \frac{1}{2^2} \sum_{p=0}^{\infty} \frac{\chi(2^{p+l}v, 2^{p+l}v, 0, \ldots, 0)}{2^{2(p+l)}} \to 0 \text{ as } l \to \infty
\]

for all \( v \in V \). Hence, \( \left\{ \frac{\phi(2^pv)}{2^{2p}} \right\} \) is a Cauchy sequence. Since \( W \) is complete, there exists a mapping \( Q_2 : V \to W \) such that

\[
Q_2(v) = \lim_{l \to \infty} \frac{\phi(2^pv)}{2^{2l}}
\]

for all \( v \in V \). Taking limit \( l \) tending to \( \infty \) in (8), we can observe that (5) holds for all \( v \in V \). Next, we want to prove that the function \( Q_2 \) satisfies the functional Equation (2). By replacing \( (v_1, v_2, \ldots, v_m) \) by \( (2^l v_1, 2^l v_2, \ldots, 2^l v_m) \) and dividing by \( 2^{2l} \) in (4), we obtain

\[
\frac{1}{2^l} \left\| \Lambda \phi(2^l v_1, 2^l v_2, \ldots, 2^l v_m) \right\| \leq \frac{1}{2^l} \chi(2^l v_2, 2^l v_2, \ldots, 2^l v_m)
\]

for all \( v_1, v_2, \ldots, v_m \in V \). Allowing \( l \to \infty \) in the above inequality and using the definition of \( Q_2(v) \), we see that \( Q_2(v_1, v_2, \ldots, v_m) = 0 \). Hence, the function \( Q_2 \) satisfies the functional Equation (2) for all \( v_1, v_2, \ldots, v_m \in V \). Next, we want to show the uniqueness of \( Q_2 \). Consider another quadratic function \( R_2(v) \) which satisfies the functional Equation (2) and inequality (5), then

\[
\| Q_2(v) - R_2(v) \| \leq \frac{1}{2^l} \left\{ \| Q_2(2^l v) - \phi(2^l v) \| + \| \phi(2^l v) - R_2(2^l v) \| \right\}
\]

\[
\leq \frac{2}{2^l} \sum_{p=0}^{\infty} \frac{\chi(2^{p+l}v, 2^{p+l}v, 0, \ldots, 0)}{2^{2(p+l)}} \to 0 \text{ as } l \to \infty
\]

for all \( v \in V \). Hence, the function \( Q_2 \) is unique. On the other hand, for \( \zeta = -1 \), in the same manner, we can verify a similar sense of stability. The proof of the theorem is now complete.

\[ \square \]

**Corollary 1.** If a mapping \( \phi : V \to W \) with \( \phi(0) = 0 \) and it satisfies the following inequality:

\[
\left\| \Lambda \phi(v_1, v_2, \ldots, v_m) \right\| \leq A \left( \sum_{j=1}^{m} ||v_j||^2 \right)
\]
for all \(v_1, v_2, \ldots, v_m \in V\), where \(\lambda\) and \(\alpha\) are two non-negative real numbers with \(\alpha \neq 2\), then there exists a unique quadratic mapping \(Q_2 : V \rightarrow W\) such that

\[
\|\phi(v) - Q_2(v)\| \leq \frac{2\lambda \|v\|^\alpha}{2^2 - 2^\alpha}
\]

(9)

for all \(v \in V\).

**Proof.** If we replace \(\chi(v_1, v_2, \ldots, v_m) = \lambda \left(\sum_{j=1}^m \|v_j\|^\alpha + \prod_{j=1}^m \|v_j\|^\alpha\right)\) in Theorem 2, we obtain the result (9). \qed

**Corollary 2.** If a mapping \(\phi : V \rightarrow W\) with \(\phi(0) = 0\) satisfies the following inequality:

\[
\|\Lambda\phi(v_1, v_2, \ldots, v_m)\| \leq \lambda \left(\sum_{j=1}^m \|v_j\|^\alpha + \prod_{j=1}^m \|v_j\|^\alpha\right)
\]

for all \(v_1, v_2, \ldots, v_m \in V\), where \(\lambda\) and \(\alpha\) are two non-negative real numbers with \(\alpha \neq 2\), then there exists a unique quadratic mapping \(Q_2 : V \rightarrow W\) such that

\[
\|\phi(v) - Q_2(v)\| \leq \frac{2\lambda \|v\|^\alpha}{2^2 - 2^\alpha}
\]

(10)

for all \(v \in V\).

**Proof.** If we replace \(\chi(v_1, v_2, \ldots, v_m) = \lambda \left(\sum_{j=1}^m \|v_j\|^\alpha + \prod_{j=1}^m \|v_j\|^\alpha\right)\) in Theorem 2, we obtain the result (10). \qed

3. Stability Results in Banach Spaces: Fixed Point Technique

**Theorem 3.** Suppose a mapping \(\phi : V \rightarrow W\) with \(\phi(0) = 0\) for which there exists a mapping \(\chi : V^m \rightarrow [0, \infty)\) with the condition

\[
\lim_{l \to \infty} \frac{\chi(\sigma_1^j v_1, \sigma_1^j v_2, \ldots, \sigma_1^j v_m)}{\sigma_1^j} = 0
\]

(11)

where

\[
\sigma_1 = \begin{cases} 2, & \text{if } j = 0; \\ \frac{1}{2}, & \text{if } j = 1; \end{cases}
\]

satisfies the inequality (4). If there exists \(L = L(j)\) that satisfies the following:

\[
v \to \beta(v) = \chi \left(\frac{v}{2}, \frac{v}{2}, 0, \ldots, 0\right)
\]

and it has the following property:

\[
\frac{\beta(\sigma_1^j v)}{\sigma_1^j} = L\beta(v)
\]

for all \(v \in V\), then there exists a unique quadratic mapping \(Q_2 : V \rightarrow W\) satisfying the functional Equation (2) and such that

\[
\|\phi(v) - Q_2(v)\| \leq \frac{L^{1-j}}{1-L} \beta(v)
\]

for all \(v \in V\).
Proof. Consider the following set:

$$\Psi := \{ q : V \rightarrow W, \quad q(0) = 0 \}$$

and allow a general metric $d$ on $\Psi$ such that

$$d(p, q) = \inf \{ c \in (0, \infty) : \| p(v) - q(v) \| \leq c\beta(v), \quad \text{for all} \quad v \in V \}.$$ 

It is clear that $(\Psi, d)$ is complete. Define a mapping $F : \Psi \rightarrow \Psi$ by

$$Fp(v) = \frac{\phi(\sigma_j v)}{\sigma_j^2}, \quad v \in V.$$ 

For all $p, q \in \Psi$, we obtain

$$d(p, q) = c \Rightarrow \| p(v) - q(v) \| \leq c\beta(v)$$

$$\Rightarrow \| \frac{p(\sigma_j v)}{\sigma_j^2} - \frac{q(\sigma_j v)}{\sigma_j^2} \| \leq \frac{1}{\sigma_j^2} c\beta(\sigma_j v)$$

$$\Rightarrow \| Fp(v) - Fq(v) \| \leq \frac{1}{\sigma_j^2} c\beta(\sigma_j v)$$

$$\Rightarrow \| Fp(v) - Fq(v) \| \leq Lc\beta(v)$$

$$\Rightarrow d(Fp, Fq) \leq Ld(p, q).$$

As a result, a strictly contractive function $F$ on $\Psi$ with $L$ is obtained. It is clear from (6) that

$$\| 2^2 \phi(v) - \phi(2v) \| \leq \chi(v, v, 0, \cdots, 0)$$

for all $v \in V$. We have $j = 0$ by using the above inequality and definitions of $\beta(v)$.

$$\| \phi(v) - \frac{\phi(2v)}{2^2} \| \leq \frac{1}{2^2} \beta(v)$$

$$\Rightarrow \| \phi(v) - F\phi(v) \| \leq L\beta(v).$$

Hence, we obtain the following:

$$d(F\phi, \phi) \leq L = L^{1-j}$$

for all $v \in V$. Replacing $v$ by $\frac{v}{2}$ in (12), we obtain

$$\| 2^2 \phi\left(\frac{v}{2}\right) - \phi(v) \| \leq \phi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right)$$

for all $v \in V$. Using the definition of $\beta(v)$ in the above inequality (14) for $j = 0$, we have

$$\| 2^2 \phi\left(2^{-1}v\right) - \phi(v) \| \leq \beta(v)$$

$$\Rightarrow \| \phi(v) - F\phi(v) \| \leq \beta(v)$$

for all $v \in V$. Hence, we obtain

$$d(\phi, F\phi) \leq 1 = L^{1-j}$$
for all $v \in V$. Using (13) and (15), we can conclude that
\[
d(\phi, F\phi) \leq L^{1-j} < \infty
\]
for all $v \in V$. Now, in both cases, the fixed point alternative theorem suggests that exists a fixed point $Q_2$ of $F$ in $\mathcal{Y}$ such that
\[
Q_2(v) = \lim_{l \to \infty} \frac{\phi(\sigma^l_j v)}{d^l_j}
\]
for all $v \in V$. In order to prove that $Q_2 : V \to W$ satisfies (2), the proof follows a similar manner as Theorem 2. Since the function $Q_2$ is a unique fixed point of $F$ in the set $\Theta = \{\phi \in \mathcal{Y} / d(\phi, Q_2) < \infty\}$, thus, the function $Q_2$ is a unique function such that
\[
d(\phi, Q_2) \leq \frac{1}{1-L} d(\phi, F\phi)
\]
i.e., $\|\phi(v) - Q_2(v)\| \leq \frac{L^{1-j}}{1-L} \beta(v), \ v \in V.
\]
The proof of the Theorem is now complete. □

Corollary 3. If a mapping $\phi : V \to W$ with $\phi(0) = 0$ and such that
\[
\left\| \Lambda \phi(v_1, v_2, \cdots, v_m) \right\| \leq \begin{cases} 
\lambda; \\
\lambda \left( \sum_{j=1}^{m} v_j \right) \left( \sum_{j=1}^{m} v_j \right)^a; \\
\lambda \left( \sum_{j=1}^{m} v_j \right)^m + \Pi_{j=1}^{m} \left( v_j \right)^a,
\end{cases}
\]
for all $v_1, v_2, \cdots, v_m \in V$, where $\lambda$ and $a$ are two non-negative real numbers, then there exists a unique quadratic mapping $Q_2 : V \to W$ which satisfies the following:
\[
\left\| \phi(v) - Q_2(v) \right\| \leq \begin{cases} 
\frac{\lambda}{1 - \alpha^{1/a}}; \\
\frac{\lambda \left( \sum_{j=1}^{m} v_j \right)^a}{1 - \alpha^{1/a}}; \\
\frac{\lambda \left( \sum_{j=1}^{m} v_j \right)^m + \Pi_{j=1}^{m} \left( v_j \right)^a}{1 - \alpha^{1/a}},
\end{cases}
\]
for all $v \in V$.

Proof. We set
\[
\chi(v_1, v_2, \cdots, v_m) \leq \begin{cases} 
\lambda; \\
\lambda \left( \sum_{j=1}^{m} v_j \right)^a; \\
\lambda \left( \sum_{j=1}^{m} v_j \right)^m + \Pi_{j=1}^{m} \left( v_j \right)^a,
\end{cases}
\]
for all $v_1, v_2, \cdots, v_m \in V$. Now,
\[
\frac{\chi(\sigma^l_j v_1, \sigma^l_j v_2, \cdots, \sigma^l_j v_m)}{\sigma^l_j} = \begin{cases} 
\frac{\lambda}{\sigma^l_j}; \\
\frac{1}{\sigma^l_j} \left( \sum_{1 \leq i \leq m} \sigma^l_j v_i \right)^a; \\
\frac{1}{\sigma^l_j} \left( \sum_{1 \leq i \leq m} \sigma^l_j v_i \right)^m + \Pi_{1 \leq i \leq m} \left( \sigma^l_j v_i \right)^a;
\end{cases}
\]
\[
\to 0 \text{ as } l \to \infty;
\]
\[
\to 0 \text{ as } l \to \infty;
\]
\[
\to 0 \text{ as } l \to \infty,
\]
In other words, (11) holds. As such, we obtain the following.

$$\beta(v) = \chi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right) = \begin{cases} \frac{\lambda}{2} \|v\|^2; \\ \frac{2\lambda}{2m} \|v\|^m \end{cases}$$

Moreover,

$$\frac{1}{\sigma^j} \beta(\sigma^j v) = \begin{cases} \frac{\lambda}{\sigma^2}, \\ \frac{2\lambda \|v\|^2 \sigma^2}{2^{2m+2} \sigma^2}, \\ \frac{2\lambda \|v\|^{2m} \sigma^m}{2^{2m+2} \sigma^2}, \end{cases}$$

for all $v \in V$. Hence, Equation (2) holds for the following.

$$L = 2^{-2} \text{ if } j = 0 \text{ and } L = \frac{1}{2^{-2}} \text{ if } j = 1.$$ 

$$L = 2^{\alpha - 2} \text{ for } \alpha < 2 \text{ if } j = 0 \text{ and } L = \frac{1}{2^{\alpha - 2}} \text{ for } \alpha > 2 \text{ if } j = 1.$$ 

$$L = 2^{ma - 2} \text{ for } ma < 2 \text{ if } j = 0 \text{ and } L = \frac{1}{2^{ma - 2}} \text{ for } ma > 2 \text{ if } j = 1.$$ 

From the above conditions, we obtain our needed outcomes of (16). \(\square\)

4. Stability Results in Non-Archimedean Banach Spaces: Direct Technique

**Theorem 4.** Let a mapping $\chi : V^m \to [0, \infty)$ and $\phi : V \to W$ be a mapping that satisfies $\phi(0) = 0$ and (4) with

$$\lim_{n \to \infty} |2|^{2n} \chi\left(2^{-n} v_1, 2^{-n} v_2, \cdots, 2^{-n} v_m\right) = 0. \quad (17)$$

Then, there exists a unique quadratic mapping $Q_2 : V \to W$ that satisfies

$$\|\phi(v) - Q_2(v)\| \leq \sup_{n \in \mathbb{N}} \left\{ |2|^{2(n-1)} \chi\left(\frac{v}{2^n}, \frac{v}{2^n}, 0, \cdots, 0\right) \right\} \quad (18)$$

for all $v \in V$.

**Proof.** Switching $(v_1, v_2, \cdots, v_m)$ by $(v, v, 0, \cdots, 0)$ in (4), we obtain

$$\|\phi(2v) - 2^2 \phi(v)\| \leq \chi(v, v, 0, \cdots, 0) \quad (19)$$

for all $v \in V$. Thus,

$$\|\phi(v) - 2^2 \phi\left(\frac{v}{2}\right)\| \leq \chi\left(\frac{v}{2}, \frac{v}{2}, 0, \cdots, 0\right)$$
for all \( v \in V \). Hence, we have
\[
\left\| 2^j \phi \left( \frac{v}{2^j} \right) - 2^{2j} \phi \left( \frac{v}{2^j} \right) \right\| \\
\leq \max \left\{ \left\| 2^0 \phi \left( \frac{v}{2^0} \right) - 2^{2(0+1)} \phi \left( \frac{v}{2^{0+1}} \right) \right\| , \ldots , \left\| 2^{(p-1)} \phi \left( \frac{v}{2^{(p-1)}} \right) - 2^{2p} \phi \left( \frac{v}{2^p} \right) \right\| \right\} \\
\leq \left\{ 2^{|2^j|} \phi \left( \frac{v}{2^j} \right) - 2^{2j} \phi \left( \frac{v}{2^j} \right) \right\| , \ldots , \left\| 2^{|2^{p-1}|} \phi \left( \frac{v}{2^{p-1}} \right) - 2^{2p} \phi \left( \frac{v}{2^p} \right) \right\| \right\} \\
\leq \sup_{n \in \{ 1, 2, \ldots \}} \left\{ 2^{|2^n|} \chi \left( \frac{v}{2^n+1}, \frac{v}{2^n+2}, 0, \ldots , 0 \right) \right\} \\
\tag{20}
\]

for all \( p > l > 0 \) and for all \( v \in V \). As a result of (20), the sequence \( \left\{ 2^{2n} \phi \left( \frac{v}{2^n} \right) \right\} \) is a Cauchy sequence for every \( v \in V \). Since \( W \) is complete, the sequence \( \left\{ 2^{2n} \phi \left( \frac{v}{2^n} \right) \right\} \) converges. As a result, the mapping \( Q_2 : V \to W \) may be defined
\[
Q_2(v) := \lim_{l \to \infty} 2^{2l} \phi \left( \frac{1}{2^l} v \right)
\]
for all \( v \in V \). Taking \( l = 0 \) and the limit \( p \to \infty \) in (20), we obtain (18). As a result of (17) and (4), we have
\[
\left\| \Lambda Q_2(v_1, v_2, \ldots , v_m) \right\| = \lim_{n \to \infty} 2^{2n} \left\| \Lambda \phi \left( 2^{-n} v_1, 2^{-n} v_2, \ldots , 2^{-n} v_m \right) \right\| \\
\leq \lim_{n \to \infty} 2^{2n} \chi \left( 2^{-n} v_1, 2^{-n} v_2, \ldots , 2^{-n} v_m \right) = 0
\]
for all \( v_1, v_2, \ldots , v_m \in V \). Thus, we obtain
\[
\Lambda Q_2(v_1, v_2, \ldots , v_m) = 0.
\]

From Lemma 1, the mapping \( Q_2 : V \to W \) is quadratic. Now, consider another quadratic mapping \( R_2 : V \to W \) that satisfies inequality (18). Then, we obtain
\[
\left\| Q_2(v) - R_2(v) \right\| = \left\| 2^{2k} Q_2 \left( \frac{v}{2^k} \right) - 2^{2k} R_2 \left( \frac{v}{2^k} \right) \right\| \\
\leq \max \left\{ \left\| 2^{2k} Q_2 \left( \frac{v}{2^k} \right) - 2^{2k} \phi \left( \frac{v}{2^k} \right) \right\| , \left\| 2^{2k} R_2 \left( \frac{v}{2^k} \right) - 2^{2k} \phi \left( \frac{v}{2^k} \right) \right\| \right\} \\
\leq \sup_{n \in \mathbb{N}} \left\{ 2^{|2^{(k+n-1)}|} \chi \left( \frac{v}{2^{k+n}}, \frac{v}{2^{k+n+1}}, 0, \ldots , 0 \right) \right\} \\
\to 0 \text{ as } k \to \infty.
\]

Thus, we may infer that \( Q_2(v) = R_2(v) \) for all \( v \in V \). This proves the uniqueness of \( Q_2 \). As a result, the mapping \( Q_2 : V \to W \) is a unique quadratic mapping that satisfies (18). \( \square \)

**Corollary 4.** If a mapping \( \phi : V \to W \) with \( \phi(0) = 0 \) and such that
\[
\left\| \Lambda \phi(v_1, v_2, \ldots , v_m) \right\| \leq \alpha \left( \sum_{i=1}^{m} \left\| v_i \right\|^4 \right)
\tag{21}
\]
for all \( v_1, v_2, \ldots , v_m \in V \), then there exists a unique quadratic mapping \( Q_2 : V \to W \) that satisfies
\[
\left\| \phi(v) - Q_2(v) \right\| \leq \frac{2\alpha}{\left( 2^2 \right)^4} \left\| v \right\|^4
\]
for all \( v \in V \), where \( \lambda < 2 \) and \( \alpha \) are in \( \mathbb{R}^+ \).
Theorem 5. If a mapping $\phi : V \to W$ with $\phi(0) = 0$ and satisfies
\[
\left\| \Lambda \phi(v_1, v_2, \ldots, v_m) \right\| \leq \alpha \left( \sum_{j=1}^m \|v_j\|^{m\lambda} + \prod_{j=1}^m \|v_j\|^\lambda \right)
\] (22)
for all $v_1, v_2, \ldots, v_m \in V$, then there exists a unique quadratic mapping $Q_2 : V \to W$ such that
\[
\left\| \phi(v) - Q_2(v) \right\| \leq \frac{2\alpha}{2|m^\lambda} \|v\|^{m\lambda}
\]
for all $v \in V$, where $m\lambda < 2$ and $\alpha$ are in $\mathbb{R}^+$. 

Corollary 6. If there exists a mapping $\phi : V \to W$ with $\phi(0) = 0$ and satisfies the inequality (21), then there exists a unique quadratic mapping $Q_2 : V \to W$ such that
\[
\left\| \phi(v) - Q_2(v) \right\| \leq \frac{2\alpha}{|2|^\lambda} \|v\|^\lambda
\]
for all $v \in V$, where $\lambda > 2$ and $\alpha$ are in $\mathbb{R}^+$. 

Proof. It follows from (19) that
\[
\left\| \phi(v) - 2^{-2} \phi(2v) \right\| \leq \frac{1}{2^p} \chi(v, v, 0, \ldots, 0)
\]
for all $v \in V$. Hence,
\[
\begin{align*}
\left\| \frac{1}{2^p} \phi(2^p v) - \frac{1}{2^{p+1}} \phi(2^{p+1} v) \right\| &\leq \max \left\{ \left\| \frac{1}{2^p} \phi(2^p v) - \frac{1}{2^{p+1}} \phi(2^{p+1} v) \right\|, \ldots, \left\| \frac{1}{2^{p+l-1}} \phi(2^{p+l-1} v) - \frac{1}{2^p} \phi(2^p v) \right\| \right\} \\
&\leq \max \left\{ \left\| \frac{1}{2^p} \phi(2^p v) - \frac{1}{2^{p+1}} \phi(2^{p+1} v) \right\|, \ldots, \left\| \frac{1}{2^{p+l-1}} \phi(2^{p+l-1} v) - \frac{1}{2^p} \phi(2^p v) \right\| \right\} \\
&\leq \sup_{n \in \{p+l, \ldots\}} \left\{ \frac{1}{|2^{p+l+1}|} \chi(2^{p+l} v, 2^{p+l} v, 0, \ldots, 0) \right\}
\end{align*}
\] (24)
for all $p > l > 0$. As a result of (24), $\{\frac{1}{2^p} \phi(2^p v)\}$ is a Cauchy sequence.

Since $W$ is complete, $\{\frac{1}{2^p} \phi(2^p v)\}$ converges. Thus, we can define a mapping $Q_2 : V \to W$ by
\[
Q_2(v) := \lim_{n \to \infty} \frac{1}{2^p} \phi(2^p v)
\]
for all $v \in V$. Now, taking $l = 0$ and the limit $p \to \infty$ in (24), we obtain (23). The remaining part of the proof is similar to that of Theorem 4. 

Corollary 5. If there is a mapping $\phi : V \to W$ with $\phi(0) = 0$ and satisfies
\[
\left\| \Lambda \phi(v_1, v_2, \ldots, v_m) \right\| \leq \alpha \left( \sum_{j=1}^m \|v_j\|^{m\lambda} + \prod_{j=1}^m \|v_j\|^\lambda \right)
\] (22)
Corollary 7. If there exists a mapping \( \phi : V \to W \) with \( \phi(0) = 0 \) and it satisfies (22), then there exists a unique quadratic mapping \( Q_2 : V \to W \) such that
\[
\| \phi(v) - Q_2(v) \| \leq \frac{2\alpha}{2^2} \| v \|^{m\lambda}
\]
for all \( v \in V \), where \( m\lambda > 2 \) and \( \alpha \) are in \( \mathbb{R}^+ \).

5. Stability Results in Non-Archimedean Banach Spaces: Fixed Point Technique

Theorem 6. Let a mapping \( \chi : V^m \to [0, \infty) \) such that there is \( L < 1 \) with
\[
\chi \left( \frac{v_1}{2}, \frac{v_2}{2}, \cdots, \frac{v_m}{2} \right) \leq L \frac{1}{2^m} \chi(v_1, v_2, \cdots, v_m), \quad v_1, v_2, \cdots, v_m \in V. \tag{25}
\]
Let a mapping \( \phi : V \to W \) which satisfies \( \phi(0) = 0 \) and (4). Then, there exists a unique quadratic mapping \( Q_2 : V \to W \) such that
\[
\| \phi(v) - Q_2(v) \| \leq \frac{L}{2^m(1 - L)} \chi(v, v, 0, \cdots, 0)
\]
for all \( v \in V \).

Proof. Replacing \((v_1, v_2, \cdots, v_m)\) by \((v, v, 0, \cdots, 0)\) in (4), we obtain
\[
\| \phi(2v) - 2^2 \phi(v) \| \leq \chi(v, v, 0, \cdots, 0) \tag{26}
\]
for all \( v \in V \). Let us consider the set
\[
M := \{ q : V \to W, q(0) = 0 \}
\]
as well as the generalised metric \( d \) on \( M \):
\[
d(p, q) = \inf \left\{ \theta \in \mathbb{R}_+ : \| p(v) - q(v) \| \leq \theta \chi(v, v, 0, \cdots, 0), \text{ for all } v \in V \right\},
\]
where, as is typical, \( \inf \emptyset = +\infty \). It is simple to demonstrate that \( (M, d) \) is complete (see [20]). Now, we examine the linear mapping \( F : M \to M \), which has the following property:
\[
Fp(v) := 2^2 p \left( \frac{v}{2} \right)
\]
for all \( v \in V \). Let \( p, q \in M \) be given such that \( d(p, q) = \epsilon \). Then, we have
\[
\| p(v) - q(v) \| \leq \epsilon \chi(v, v, 0, \cdots, 0)
\]
for all \( v \in V \). Hence,
\[
\| Fp(v) - Fq(v) \| = \| 2^2 p \left( \frac{v}{2} \right) - 2^2 q \left( \frac{v}{2} \right) \| \\
\leq 2^2 \epsilon \chi \left( \frac{v}{2}, \frac{v}{2}, 0, \cdots, 0 \right) \\
\leq 2^2 \epsilon \frac{L}{2^m} \chi(v, v, 0, \cdots, 0) \\
\leq \epsilon L \chi(v, v, 0, \cdots, 0)
\]
for all \( v \in V \). Thus, \( d(p, q) = \epsilon \) implies that
\[
d(Fp, Fq) \leq \epsilon L.
\]
This means that
\[ d(Fp, Fq) \leq Ld(p, q) \]
for all \( p, q \in M \). It follows from (26) that
\[
\left\| \phi(v) - 2^2 \phi\left(\frac{v}{2}\right) \right\| \leq \Lambda\left(\frac{v}{2}, \frac{v}{2}, 0, \ldots, 0\right) \\
\leq \frac{L}{2^2} \chi(v, v, 0, \ldots, 0)
\]
for all \( v \in V \). Thus, \( d(\phi, F\phi) \leq \frac{L}{2^2} \). From Theorem 1, there exists a quadratic mapping \( Q_2 : V \rightarrow W \) satisfying the following:

1. \( Q_2 \) is a fixed point of \( F \)

\[ i.e., \quad Q_2(v) = 2^2 Q\left(\frac{v}{2}\right), \quad v \in V. \tag{27} \]

The function \( Q_2 \) is a unique fixed point of \( M \) in the set
\[ T = \{ p \in M : d(\phi, p) < \infty \}. \]

This yields that \( Q_2 \) is a unique function satisfying (27) such that there exists \( \theta \in (0, \infty) \) satisfying
\[
\left\| \phi(v) - Q_2(v) \right\| \leq \theta\Lambda\left(\frac{v}{2}, \frac{v}{2}, 0, \ldots, 0\right), \quad v \in V.
\]

2. \( d(F^n\phi, Q_2) \to 0 \) as \( n \to \infty \). This indicates the below equality
\[
\lim_{n \to \infty} 2^{2n}\phi(2^{-n}v) = Q_2(v), \quad v \in V.
\]

3. \( d(\phi, Q_2) \leq \frac{1}{1-\theta} d(\phi, F\phi) \), and it implies the following:
\[
\left\| \phi(v) - Q_2(v) \right\| \leq \frac{L}{2^2(1 - \theta)} \chi(v, v, 0, \ldots, 0)
\]
for all \( v \in V \). It follows from (25) and (4) that
\[
\left\| \Lambda Q_2(v_1, v_2, \ldots, v_m) \right\| = \lim_{n \to \infty} 2^{2n} \left\| \Lambda \phi(2^{-n}v_1, 2^{-n}v_2, \ldots, 2^{-n}v_m) \right\| \\
\leq \lim_{n \to \infty} 2^{2n} \chi(2^{-n}v_1, 2^{-n}v_2, \ldots, 2^{-n}v_m) = 0.
\]

Thus,
\[ \Lambda Q_2(v_1, v_2, \ldots, v_m) = 0, \quad v_1, v_2, \ldots, v_m \in V. \]

By Lemma 1, the mapping \( Q_2 : V \rightarrow W \) is quadratic. \( \square \)

**Corollary 8.** If a mapping \( \phi : V \rightarrow W \) satisfies \( \phi(0) = 0 \) and the following:
\[
\left\| \Lambda \phi(v_1, v_2, \ldots, v_m) \right\| \leq \alpha \left( \sum_{i=1}^{m} \| v_i \|^{\lambda} \right) \tag{28}
\]
for all \( v_1, v_2, \ldots, v_m \in V \), where \( \lambda < 2 \) and \( \alpha \) are two non-negative real numbers, then there exists a unique quadratic mapping \( Q_2 : V \rightarrow W \) such that
\[
\left\| \phi(v) - Q_2(v) \right\| \leq \frac{2\alpha \| v \|^{\lambda}}{2^{2\lambda} - 2^{2}}
\]
for all \( v \in V \).

**Proof.** The proof is based on Theorem 6 by allowing the following:

\[
\chi(v_1, v_2, \cdots, v_m) = a \left( \sum_{j=1}^{m} \|v_j\|^\lambda \right)
\]

for all \( v_1, v_2, \cdots, v_m \in V \). After that, we may use \( L = |2|^{2-\lambda} \) to obtain our desired result. \( \square \)

**Corollary 9.** If a mapping \( \phi : V \rightarrow W \) satisfies \( \phi(0) = 0 \) and the following:

\[
\left\| A\phi(v_1, v_2, \cdots, v_m) \right\| \leq 2\alpha \left( \sum_{j=1}^{m} \|v_j\|^{m\lambda} + \prod_{j=1}^{m} \|v_j\|^\lambda \right)
\]

for all \( v_1, v_2, \cdots, v_m \in V \), where \( m\lambda < 2 \) and \( \alpha \) are two non-negative real numbers, then there exists a unique quadratic mapping \( Q_2 : V \rightarrow W \) such that

\[
\left\| \phi(v) - Q_2(v) \right\| \leq \frac{2\alpha \|v\|^{m\lambda}}{2^{m\lambda} - |2|^2}
\]

for all \( v \in V \).

**Proof.** The proof is based on Theorem 6 by allowing the following:

\[
\chi(v_1, v_2, \cdots, v_m) = a \left( \sum_{j=1}^{m} \|v_j\|^\lambda \right)
\]

for all \( v_1, v_2, \cdots, v_m \in V \). After that, we may use \( L = |2|^{2-m\lambda} \) to obtain our desired result. \( \square \)

**Theorem 7.** Let a mapping \( \chi : V^m \rightarrow [0, \infty) \) such that there is \( L < 1 \) with the following:

\[
\chi(v_1, v_2, \cdots, v_m) \leq |2|^{2\lambda} L \chi(v_1, 2^{-1}v_2, \cdots, 2^{-1}v_m)
\]

for all \( v_1, v_2, \cdots, v_m \in V \). If a mapping \( \phi : V \rightarrow W \) satisfies \( \phi(0) = 0 \) and (4), then there exists a unique quadratic mapping \( Q_2 : V \rightarrow W \) such that

\[
\left\| \phi(v) - Q_2(v) \right\| \leq \frac{1}{|2|^{2\lambda}(1-L)} \chi(v, v, 0, \cdots, 0)
\]

for all \( v \in V \).

**Proof.** It follows from (26) that

\[
\left\| \phi(v) - \frac{1}{2^\lambda} \phi(2v) \right\| \leq \frac{1}{|2|^{2\lambda}} \chi(v, v, 0, \cdots, 0)
\]

for all \( v \in V \). Let \((M, d)\) denote the generalised metric space noted in Theorem 25. Now, let us consider the linear mapping \( F : M \rightarrow M \) that satisfies the following:

\[
Fp(v) := \frac{1}{2^\lambda} p(2v)
\]
for all \( v \in V \). This comes from (30) that
\[
d(\phi, F \phi) \leq \frac{1}{2|2|}.\]
Thus,
\[
\|\phi(v) - Q_2(v)\| \leq \frac{1}{|2|} \chi(v, v, 0, \ldots, 0)
\]
for all \( v \in V \). The remaining part of the proof is similar to that of Theorem 6. \( \square \)

**Corollary 10.** If a mapping \( \phi : V \to W \) satisfies \( \phi(0) = 0 \) and (28), then there exists a unique quadratic mapping \( Q_2 : V \to W \) such that
\[
\|\phi(v) - Q_2(v)\| \leq \frac{2\alpha \|v\|^\lambda}{|2|^2 - |2|^\lambda}
\]
for all \( v \in V \), where \( \lambda > 2 \) and \( \alpha \) are two positive real numbers.

**Proof.** The proof is based on Theorem 7 by allowing the following:
\[
\chi(v_1, v_2, \ldots, v_m) = \alpha \left( \sum_{j=1}^{m} \|v_j\|^\lambda \right).
\]
Then, we can take \( L = |2|^{\lambda-2} \), and we obtain our result. \( \square \)

**Corollary 11.** If a mapping \( \phi : V \to W \) satisfies \( \phi(0) = 0 \) and (29), then there exists a unique quadratic mapping \( Q_2 : V \to W \) such that
\[
\|\phi(v) - Q_2(v)\| \leq \frac{2\alpha \|v\|^{m\lambda}}{|2|^2 - |2|^{m\lambda}}
\]
for all \( v \in V \), where \( m\lambda > 2 \) and \( \alpha \) are two non-negative real numbers.

**Proof.** The proof is based on Theorem 7 by allowing the following:
\[
\chi(v_1, v_2, \ldots, v_m) = \alpha \left( \sum_{j=1}^{m} \|v_j\|^{m\lambda} + \prod_{j=1}^{m} \|v_j\|^{\lambda} \right).
\]
Then, we can take \( L = |2|^{m\lambda-2} \) and we obtain our result. \( \square \)

6. Conclusions

In this work, we studied the Ulam–Hyers stability results of the generalized additive functional Equation (2) in Banach spaces and non-Archimedean Banach spaces by using different approaches of direct and fixed point methods. In future works, the researcher can obtain the Ulam–Hyers stability results of this generalized additive functional equation in various normed spaces such as matrix paranormed spaces, quasi-\( \beta \)-normed spaces, fuzzy normed spaces, etc.

The results obtained and the methods adopted in this study would be useful for other researchers for carrying out further investigations. Since there are lot of applications of functions in various fields including physics, economics, business, medicine, digital image processing, chemistry, etc., the study of this type of equation has a lot of scope for other researchers.
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