GEOMETRIC K-HOMOLOGY OF FLAT D-BRANES

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Abstract. We use the Baum-Douglas construction of K-homology to explicitly describe various aspects of D-branes in Type II superstring theory in the absence of background supergravity form fields. We rigorously derive various stability criteria for states of D-branes and show how standard bound state constructions are naturally realized directly in terms of topological K-cycles. We formulate the mechanism of flux stabilization in terms of the K-homology of non-trivial fibre bundles. Along the way we derive a number of new mathematical results in topological K-homology of independent interest.

Introduction

One of the most exciting recent interactions between physics and mathematics has been through the realization that D-branes in string theory are classified by generalized cohomology theories such as K-theory. The charges of D-branes in Type II superstring theory are classified by the K-theory groups of the spacetime in which they live \[30, 49, 53, 54\]. In Type I superstring theory one uses instead KO-theory, while the charges of branes in orbifolds and orientifolds are classified by various equivariant K-theories, KR-theories, and extensions thereof \[9, 10, 31, 33, 53, 54\]. In curved backgrounds and in the presence of a non-trivial \(B\)-field, D-brane charge takes values in a twisted K-theory group \[14, 40, 64\]. In addition, the Ramond-Ramond fields which are typically supported on D-branes similarly take values in appropriate K-theory groups \[10, 50, 29\]. These realizations have prompted intensive investigations in both the mathematics and physics literature into the properties and definitions of various K-theory groups. At the heart of the excitement in these investigations is the fact that the correct physical picture of D-brane charges (and Ramond-Ramond fields) cannot be properly captured in general by ordinary cohomology but rather requires K-theory \[30, 24\], and conversely that the known physical properties of D-branes in string theory give insights into the rigorous characterizations of certain less widely explored generalized cohomology functors such as those of twisted K-theory \[46\].

In this paper we will elucidate in detail the observation that K-homology, the homological version of K-theory, is really the more appropriate arena in which to classify D-branes \[2, 34, 47, 54, 59, 60\]. We treat only the case of Type II D-branes in the absence of non-trivial \(B\)-fields. We will build on the classic Baum-Douglas construction of K-homology \[6, 7\] which is called topological K-homology in order to distinguish it from analytic K-homology, another homological realization of K-theory. In \[6, 7\] it is proven that this is indeed the homology theory dual to K-theory. For a unified treatment which works for a generic cohomology theory, see \[38\]. The main advantage of this geometric formulation is that the K-homology cycles encode the most primitive requisite objects that must be carried by any D-brane, such as a spin\(c\) structure and a complex vector bundle.

Generally, D-branes are much more complicated objects than just subspaces in an ambient spacetime and require a more abstract mathematical notion, such as that of a derived category \[26, 61\]. Nevertheless, the realization of D-branes in topological K-homology gives them a very natural robust definition in fairly general spacetime backgrounds and reveals various
important properties of their (quantum) dynamics that could not be otherwise detected if one classified the brane worldvolumes using only ordinary singular homology. As we will describe in detail in the following, such effects include important stability properties as well as the fact that D-branes do not always wrap submanifolds of the spacetime. Moreover, there is a natural relationship between the Baum-Douglas construction and the realization of certain D-branes as objects in a particular triangulated category.

The requisite mathematical material used for this investigation is surveyed in Section 1. We elaborate on various aspects of K-homology, giving all the relevant definitions (in terms of the Baum-Douglas approach) and describing some new results which to the best of our knowledge have not previously appeared in the literature. We also present different perspectives on K-homology: the spectrum based definition, which gives a more algebraic topological setting, allowing us to use general results in topology to investigate the structure of D-branes; and the analytic definition (based on the Baum-Douglas approach and the Brown-Douglas-Fillmore construction, although one can also use Kasparov’s approach), which relates the study of D-branes with the study of C*-algebras and their K-homology. This second approach makes way to the study of operator algebras, as well as giving a natural setting in which to express the famous index results. Of importance in this section is the set up of several methods to calculate the K-homology groups such as Poincaré Duality, the Universal Coefficient Theorem, and the Chern Character.

We then undertake the task of attempting to describe D-branes within this rigorous formalism in Section 2. Our goal throughout is two-fold. Firstly, we define and describe the physics of D-branes in a rigorously precise K-homological setting which we hope is accessible to mathematicians with little or no prior knowledge of string theory, while at the same time describing another mathematical framework in which to set and do string theory. Secondly, we emphasize the fact that the (sometimes surprising) physical properties of D-branes are completely transparent when the branes are defined and analysed within the mathematical framework of topological K-homology. Our basic aim will be to find generators for the pertinent K-homology groups which will in turn be identified geometrically with the D-branes of the spacetime. We give some general results on how to do this and some of our results on the description of the K-homology generators (Th. 2.1, 2.3) deal with the mathematical problem of representation of cycles in a homology theory. These results also relate K-homology with homology and spin^c bordism. Many non-trivial dynamical aspects of D-branes are then reformulated as the problem of finding appropriate changes of bases for the generators of the K-homology groups. Included in this list are the constructions of bound states of D-branes from both the “branes within branes” mechanism 29 and the dielectric effect 21, as well as the decay of unstable systems of D-branes into stable bound states via tachyon condensation on their worldvolumes 36, 53, 57, 64. While our constructions find their most natural interpretation in the physics of D-branes, the results may also be of independent mathematical interest. In this section we also relate our K-homological approach with the one using derived categories.

In Section 3 we start turning our attention to explicit examples, including a simple analysis of D-branes in spheres and projective spaces as well as a homological treatment of T-duality. We apply the structure results from Section 2 and introduce the Hurewicz homomorphism. This homomorphism is important in the problem of representation of cycles, as well as that of finding the K-homology generators.

In Section 4 we look at some more complicated examples of D-branes which carry torsion charges. In both the torsion and torsion-free cases, we study in detail the phenomenon of brane instability 21, 46, i.e. that some D-branes may be unstable even though they wrap non-trivial spin^c homology cycles of the spacetime. This problem becomes particularly transparent in topological K-homology. A number of explicit examples of torsion D-branes are presented, including those in Lens spaces, in all (even and odd dimensional) real projective spaces and
some products thereof, and in the basic Fermat quintic threefold and its mirror Calabi-Yau threelfold. We introduce another method of calculating the K-homology groups, namely the Atiyah-Hirzebruch-Whitehead (AHW) spectral sequence, and apply it to Lens spaces. We also deal with the problem of factorizing the Hurewicz homomorphism through K-homology. This last issue is non-trivial, since there is no map between the K-homology and the ordinary homology spectra. We give conditions and particular instances in which this occurs. We also calculate and give a geometric interpretation (to the best of our knowledge, for the first time) of the K-homology of the even real projective spaces, which is a good example of a non spin\(^c\) space, therefore of one that doesn’t allow for the application of the usual tools (i.e. Poincaré Duality).

Finally, in Section 5 we examine the problem of stabilizing certain D-branes even when they wrap homologically trivial worldvolumes. This is achieved by regarding the ambient spacetime as the total space of a non-trivial fibration. The characteristic class of the fibre bundle then acts as a source of stabilization and effectively renders the D-brane stable. We present a number of classes of examples of this type, and in each instance explicitly determine the topological K-homology groups. Our analysis includes as a special case the well-known example of spherical D-branes wrapping \(S^2 \subset S^3\) \cite{5}, which in our construction are rendered stable by virtue of the Hopf fibration over \(\mathbb{C}P^1\). The methods of this section are based in the Leray-Serre spectral sequence, of which the AHW spectral sequence is a special case. By the use of this spectral sequence we give general results in the structure of the K-homology groups of spaces which are total spaces of certain types of fibrations (coverings, spherical fibrations, and fibrations with fibre a sphere).

In summary, in this paper we present another way of viewing D-branes. In this setting a D-brane on a background spacetime \(X\) is a triple \((M, E, \phi)\), where \(M\) is a spin\(^c\) submanifold of \(X\) (usually interpreted as a D-brane in string theory), \(E\) is the Chan-Paton bundle on \(M\), and \(\phi : M \to X\) is the embedding. This contains more information than the usual picture of a D-brane as a submanifold of spacetime, since it also considers the specific embedding of \(M\) in \(X\) and its Chan-Paton bundle. D-brane charges are then classified by the K-homology group of \(X\), instead of its K-theory. The relations used to define the equivalence relation defining the group are physically meaningful and therefore natural. In general, the K-homology group may contain elements in which the generators aren’t as above (e.g., \(\phi\) may only be a continuous map instead of an embedding of manifolds), and the question of whether this is always the case relates to the mathematical problem of representation of cycles for which we have given some answers (for instance in Theorems 2.1, 2.3, the numerous examples given). In terms of string theory, one of the main advantages of this method is that it allows, through calculation, to obtain a complete description, up to equivalence, of all the D-branes that may appear in a specific spacetime. Several methods are given to calculate these, and abstract as well as specific examples are worked out in detail. Also, this paper recasts interpretations of D-brane phenomena done in the setting of K-homology of C\(^*\)-algebras into an algebraic topological framework, while supplying the necessary tools to work in this new topological framework. Although we lose the more rich structure of a cohomology theory, we think that the ability to obtain a full geometric characterization of D-branes more than compensates for that.

Another task we tried to fulfil with this paper was to create, as far as possible, a dictionary between string theory (and particularly D-brane theory) and mathematics, trying with this to make it easier for mathematicians not necessarily connected with mathematical physics to acquaint themselves with some of the mathematical problems that lye at the heart of string theory. Also, as an effort to put the physical notions in a mathematically rigorous setting as much as possible.

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1. K-Homology

In this section we shall develop the requisite mathematical material that will be required throughout this paper, postponing the start of our string theory considerations until the next section. We define geometric K-homology and describe some basic properties of the topological K-homology groups of a topological space. We also compare this homology theory with other formulations of K-homology as the dual theory to K-theory.

Unless otherwise stated, in this paper \( X \) will denote a finite CW-complex, i.e. a compact CW-complex (see [58] for a rigorous mathematical definition). For instance, compact manifolds are finite CW-complexes.

1.1. K-Cycles.

**Definition 1.1.** A K-cycle on \( X \) is a triple \((M,E,\phi)\) where

(i) \( M \) is a compact spin\(^c\) manifold without boundary;
(ii) \( E \) is a complex vector bundle over \( M \); and
(iii) \( \phi : M \to X \) is a continuous map.

There are no connectedness requirements made upon \( M \), and hence the bundle \( E \) can have different fibre dimensions on the different connected components of \( M \). It follows that disjoint union

\[
(M_1, E_1, \phi_1) \sqcup (M_2, E_2, \phi_2) := (M_1 \sqcup M_2, E_1 \sqcup E_2, \phi_1 \sqcup \phi_2)
\]

is a well-defined operation on the set of K-cycles on \( X \).

**Definition 1.2.** Two K-cycles \((M_1, E_1, \phi_1)\) and \((M_2, E_2, \phi_2)\) on \( X \) are isomorphic if there exists a diffeomorphism \( h : M_1 \to M_2 \) such that

(i) \( h \) preserves the spin\(^c\) structures;
(ii) \( h^*(E_2) \cong E_1 \); and
(iii) The diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{h} & M_2 \\
\phi_1 \downarrow & & \downarrow \phi_2 \\
& X &
\end{array}
\]

commutes.

The set of isomorphism classes of K-cycles on \( X \) is denoted \( \Gamma(X) \).

**Definition 1.3 (Bordism).** Two K-cycles \((M_1, E_1, \phi_1)\) and \((M_2, E_2, \phi_2)\) on \( X \) are bordant if there exist a compact spin\(^c\) manifold \( W \) with boundary, a complex vector bundle \( E \to W \), and a continuous map \( \phi : W \to X \) such that the two K-cycles \((\partial W, E|_{\partial W}, \phi|_{\partial W})\) and \((M_1 \sqcup (-M_2), E_1 \sqcup E_2, \phi_1 \sqcup \phi_2)\) are isomorphic. Here \(-M_2\) denotes \( M_2 \) with the spin\(^c\) structure on its tangent bundle \( TM_2 \) reversed.
1.2. Clutching Construction. Before proceeding with further definitions, we need a construction that will be instrumental in defining the topological K-homology groups. Let \( M \) be a compact spin\( ^c \) manifold and \( F \to M \) a \( C^\infty \) real spin\( ^c \) vector bundle with even-dimensional fibres and projection map \( \rho \). Let \( \mathbb{R} M := M \times \mathbb{R} \) denote the trivial real line bundle over \( M \). Then \( F \oplus \mathbb{R} M \) is a real vector bundle over \( M \) with odd-dimensional fibres. By choosing a \( C^\infty \) metric on it, we may define the unit sphere bundle
\[
(1.1) \quad \hat{M} = S (F \oplus \mathbb{R} M)
\]
by restricting the set of fibre vectors of \( F \oplus \mathbb{R} M \) to those which have unit norm. The spin\( ^c \) structures on \( TM \) and \( F \) induce a spin\( ^c \) structure on \( T\hat{M} \) by the exact sequence lemma \[6\], and hence \( \hat{M} \) is a spin\( ^c \) manifold. By construction, \( \hat{M} \) is a sphere bundle over \( M \) with even-dimensional spheres as fibres. We denote the bundle projection by
\[
(1.2) \quad \pi : \hat{M} \to M.
\]
Alternatively, we may regard the total space \( \hat{M} \) as consisting of two copies \( \mathbb{B}^\pm(F) \) of the unit ball bundle \( \mathbb{B}(F) \) of \( F \) (carrying opposite spin\( ^c \) structures) glued together by the identity map \( \text{id}_{\mathbb{S}(F)} \) on its boundary, so that
\[
(1.3) \quad \hat{M} = \mathbb{B}^+(F) \cup_{\mathbb{S}(F)} \mathbb{B}^-(F).
\]

For \( p \in M \), let \( 2n = \dim_F F_p \) with \( n \in \mathbb{N} \). The group \( \text{Spin}^c(2n) \) has two irreducible half-spin representations. The spin\( ^c \) structure on \( F \) associates to these representations complex vector bundles \( S_0(F) \) and \( S_1(F) \) of equal rank \( 2^{n-1} \) over \( M \). Their Whitney sum \( S(F) = S_0(F) \oplus S_1(F) \) is a bundle of Clifford modules over \( M \) such that \( \mathbb{C}^\ell(F) \otimes \mathbb{C} \cong \text{End} S(F) \), where \( \mathbb{C}^\ell(F) \) is the real Clifford algebra bundle of \( F \). Let \( S_+(F) \) and \( S_-(F) \) be the spinor bundles over \( F \) obtained from pullbacks to \( F \) by the bundle projection \( \rho : F \to M \) of \( S_0(F) \) and \( S_1(F) \), respectively. Clifford multiplication induces a bundle map \( F \otimes S_0(F) \to S_1(F) \) that defines a vector bundle map \( \sigma : S_+(F) \to S_-(F) \) covering \( \text{id}_F \) which is an isomorphism outside the zero section of \( F \). Since the ball bundle \( \mathbb{B}(F) \) is a sub-bundle of \( F \), we may form spinor bundles over \( \mathbb{B}^\pm(F) \) as the restriction bundles \( \Delta^\pm(F) = S_\pm(F)|_{\mathbb{B}^\pm(F)} \). We can then glue \( \Delta^+(F) \) and \( \Delta^-(F) \) along \( \mathbb{S}(F) = \partial \mathbb{B}(F) \) by the Clifford multiplication map \( \sigma \) giving a vector bundle over \( \hat{M} \) defined by
\[
(1.4) \quad H(F) = \Delta^+(F) \cup_\sigma \Delta^-(F).
\]
For each \( p \in M \), \( H(F)|_{\pi^{-1}(p)} \) is the Bott generator vector bundle over the even-dimensional sphere \( \pi^{-1}(p) \) \[6\]. Thus, starting from the triple \((M, F, \rho)\) we have constructed another triple \((\hat{M}, H(F), \pi)\).

**Definition 1.4.** If \((M, E, \phi)\) is a K-cycle on \( X \) and \( F \) is a \( C^\infty \) real spin\( ^c \) vector bundle over \( M \) with even-dimensional fibres, then the process of obtaining the K-cycle \((\hat{M}, H(F) \otimes \pi^*(E), \phi \circ \pi)\) from \((M, E, \phi)\) is called vector bundle modification.

1.3. **Topological K-Homology.** We are now ready to define the topological K-homology groups of the space \( X \).

**Definition 1.5.** The topological K-homology group of \( X \) is the group obtained from quotienting \( \Gamma(X) \) by the equivalence relation \( \sim \) generated by the relations of

(i) bordism;
(ii) direct sum: if \( E = E_1 \oplus E_2 \), then \((M, E, \phi) \sim (M, E_1, \phi) \amalg (M, E_2, \phi)\); and
(iii) vector bundle modification.

The group operation is induced by disjoint union of K-cycles. We denote this group by \( K_*^X(X) := \Gamma(X) / \sim \), and the homology class of the K-cycle \((M, E, \phi)\) by \([M, E, \phi] \in K_*^X(X)\).
The manifolds $MII N$ and $NII M$ are bordant through a bordism which clearly induces a bordism of the respective K-cycles. It follows that $[0,0,0]$ is the identity for the operation induced by disjoint union of K-cycles and $[-M,E,\phi]$ is the inverse of $[M,E,\phi]$, where $-M$ denotes $M$ with its spin$^c$ structure reversed. The operation is also clearly associative and commutative. Thus $K^j_1(X)$ is an abelian group. Since the equivalence relation on $\Gamma(X)$ preserves the parity of the dimension of $M$ in K-cycles $(M,E,\phi)$, one can define the subgroup $K^j_0(X)$ (resp. $K^j_1(X)$) consisting of classes of K-cycles $(M,E,\phi)$ for which all connected components $M_i$ of $M$ are of even (resp. odd) dimension. Then $K^j_0(X) = K^j_0(X) \oplus K^j_1(X)$ has a natural $\mathbb{Z}_2$-grading.

The geometric construction of K-homology is functorial. If $f : X \to Y$ is a continuous map, then the induced homomorphism

$$f_* : K^j_1(X) \to K^j_1(Y)$$

of $\mathbb{Z}_2$-graded abelian groups is given on classes of K-cycles $[M,E,\phi] \in K^j_1(X)$ by

$$f_*[M,E,\phi] := [M,E,\circ \phi].$$

One has $(id_X)_* = id_{K^j_1(X)}$ and $(f \circ g)_* = f_* \circ g_*$. Since vector bundles over $M$ extend to vector bundles over $M \times [0,1]$, it follows by bordism that induced homomorphisms depend only on their homotopy classes.

If $pt$ denotes a one-point topological space, then the only K-cycles on $pt$ are $(pt, pt \times \mathbb{C}^k, id_{pt})$ with $k \in \mathbb{N}$. Thus $K^0_0(pt) \cong \mathbb{Z}$ and $K^1_1(pt) \cong 0$. The collapsing map $\varepsilon : X \to pt$ then induces an epimorphism

$$(1.5) \quad \varepsilon_* : K^j_1(X) \twoheadrightarrow K^j_1(pt) \cong \mathbb{Z}.$$ 

The reduced topological K-homology group of $X$ is

$$(1.6) \quad \tilde{K}^j_1(X) := \ker \varepsilon_*.$$ 

Since the map (1.5) is an epimorphism with left inverse induced by the inclusion of a point $i : pt \hookrightarrow X$, one has $K^j_1(X) \cong \mathbb{Z} \oplus \tilde{K}^j_1(X)$ for any space $X$.

1.4. Computational Tools. Before adding further structure to this K-homology theory, we pause to describe some basic technical results which will aid in calculating the groups $K^j_1(X)$, particularly in the subsequent sections when we shall seek explicit K-cycle representatives for their generators. In what follows we shall use the notation $[n] := \{1, 2, \ldots, n\}$.

**Lemma 1.1.** $K^j_1(X)$ is generated by classes of K-cycles $[M,E,\phi]$ where $M$ is connected.

*Proof.* Let $\{M_i\}_{i \in I}$ be the set of connected components of $M$. Since $M$ is compact, $I$ is a finite set. Defining $E_i := E_{|M_i}$ and $\phi_i := \phi_{|M_i}$, we have $E = \coprod_{i \in I} E_i$ and $\phi = \coprod_{i \in I} \phi_i$ so that $[M,E,\phi] = \sum_{i \in I} [M_i,E_i,\phi_i]$. \qed

**Lemma 1.2.** If $\{X_j\}_{j \in J}$ is the set of connected components of $X$ then $K^j_1(X) = \bigoplus_{j \in J} K^j_1(X_j)$.

*Proof.* Let $[M,E,\phi] \in K^j_1(X)$ with $\{M_i\}_{i \in I}$ the set of connected components of $M$. As in the proof of Lemma 1.1 one has $[M,E,\phi] = \sum_{i \in I} [M_i,E_i,\phi_i]$. For any $i \in I$, $M_i$ is connected and $\phi_i$ is continuous, and so there exists $j_i \in J$ such that $\phi_i(M_i) \subset X_{j_i}$. Thus $[M_i,E_i,\phi_i] \in K^j_1(X_{j_i})$ and so $[M,E,\phi] \in \bigoplus_{j \in J} K^j_1(X_j)$. Conversely, let $[M_i,E_i,\phi_i] \in K^j_i(X_{j_i})$ for some $i \in [n]$ and $j_i \in J$. Defining $M := \coprod_{i \in [n]} M_i$, $E := \coprod_{i \in [n]} E_i$ and $\phi := \coprod_{i \in [n]} \phi_i$, one has $[M,E,\phi] \in K^j_1(X)$. The conclusion now follows by considering the image of the class $\sum_{i \in [n]} [M_i,E_i,\phi_i]$ in $K^j_1(X)$ under the homomorphism induced by the continuous map $\coprod_{i \in [n]} t_{j_i}$, where $t_{j_i} : X_{j_i} \hookrightarrow X$ are the canonical inclusions. \qed
Lemma 1.3. Let \((M, E, \phi)\) be a K-cycle on \(X\). Suppose that the degree 0 topological K-theory group \(K^0(M)\) of \(M\) is generated as a \(\mathbb{Z}\)-module by classes \([F_1], \ldots, [F_p]\) of complex vector bundles over \(M\). Then \([M, E, \phi]\) belongs to the \(\mathbb{Z}\)-submodule of \(K^1_2(X)\) generated by \([M, F_i, \phi]\) for \(i \in [p]\).

Proof. By hypothesis there exist integers \(n_1, \ldots, n_p\) such that \([E] = \sum_{i \in [p]} n_i [F_i]\). Without loss of generality we may suppose that \(n_j \geq 0\) for all \(1 \leq j \leq m\) while \(n_j < 0\) for all \(m + 1 \leq j \leq p\), for some \(m\) with \(1 \leq m \leq p\). Then

\[
[E] + \sum_{i=m+1}^{p} (-n_i) [F_i] = \sum_{i=1}^{m} n_i [F_i],
\]

which implies that there exists an integer \(k \geq 0\) such that

\[
E \oplus \left( \bigoplus_{i=m+1}^{p} (-n_i) F_i \right) \oplus \mathbb{Z}^k_M = \left( \bigoplus_{i=1}^{m} n_i F_i \right) \oplus \mathbb{Z}^k_M.
\]

Going down to classes in \(K^1_2(X)\) using the direct sum relation, we then have

\[
[M, E, \phi] + \sum_{i=m+1}^{p} (-n_i) [M, F_i, \phi] + [M, \mathbb{Z}^k_M, \phi] = \sum_{i=1}^{m} n_i [M, F_i, \phi] + [M, \mathbb{Z}^k_M, \phi]
\]

which implies that \([M, E, \phi] = \sum_{i \in [p]} n_i [M, F_i, \phi]\). \(\square\)

Corollary 1.1. The homology class of a K-cycle \((M, E, \phi)\) on \(X\) depends only on the K-theory class of \(E\) in \(K^0(M)\).

Lemma 1.4. The homology class of a K-cycle \((M, E, \phi)\) on \(X\) depends only on the homotopy class of \(\phi\) in \([M, X]\).

Proof. This follows from \([M, E, \phi] = [M, E, \phi \circ \text{id}_M]\) = \(\phi_*[M, E, \text{id}_M]\) and the fact that induced homomorphisms depend only on their homotopy classes. \(\square\)

Corollary 1.2. If \(X\) is a compact spin\(^c\) manifold without boundary, \(E \to X\) is a complex vector bundle and \(\phi : X \to X\) is a continuous map, then \([X, E, \phi]\) depends only on the homotopy class of \(\phi\) in \([X, X]\).

1.5. Cap Product. The \(\cap\) product is the \(\mathbb{Z}_2\)-degree preserving bilinear pairing

\[
\cap : K^0_2(X) \otimes K^1_2(X) \to K^1_2(X)
\]

given for any complex vector bundle \(F \to X\) and K-cycle class \([M, E, \phi] \in K^1_2(X)\) by

\[
[F] \cap [M, E, \phi] := [M, \phi^* F \otimes E, \phi]
\]

and extended linearly. It makes \(K^1_2(X)\) into a module over the ring \(K^0_2(X)\). Later on (see Sections 2.7 and 3.2) we will see that this product can be extended to a bilinear form

\[
\cap : K^1_2(X) \otimes K^1_2(X) \to K^1_2(\Sigma X),
\]

where we have denoted the mod 2 congruence class of an integer \(n \in \mathbb{Z}\) by

\[
e(n) := \begin{cases} 
0 & \text{if } n \text{ even} \\
1 & \text{if } n \text{ odd}
\end{cases}
\]

The construction utilizes Bott periodicity and the isomorphism \(K^1_2(X) \cong K^0_2(\Sigma X)\), where \(\Sigma X\) is the reduced suspension of the topological space \(X\). The product \(\cap : K^0_2(X) \otimes K^1_2(X) \to K^1_2(\Sigma X)\) is given by the pairing \(\cap : K^0_2(\Sigma X) \otimes K^1_2(\Sigma X) \to K^1_2(\Sigma X)\).
1.6. Exterior Product. If \( X \) and \( Y \) are spaces, then the exterior product

\[
\times : K^i(X) \otimes K^j(Y) \rightarrow K^{i+j}(X \times Y)
\]

is given for classes of K-cycles \([M, E, \phi] \in K^i(X)\) and \([N, F, \psi] \in K^j(Y)\) by

\[
[M, E, \phi] \times [N, F, \psi] := [M \times N, E \otimes F, (\phi, \psi)],
\]

where \( M \times N \) has the spin\(^c \) product structure uniquely induced by the spin\(^c \) structures on \( M \) and \( N \), and \( E \otimes F \) is the vector bundle over \( M \times N \) with fibres \((E \otimes F)_{(p,q)} = E_p \otimes F_q\) for \((p,q) \in M \times N\). This product is natural with respect to continuous maps and there is the following version (due to Atiyah) of the Künneth theorem in K-homology [48].

**Theorem 1.1.** If \( X \) is a CW-complex and \( Y \) is a compact topological space, then for each \( i = 0, 1 \) there is a natural short exact sequence

\[
0 \rightarrow \bigoplus_{e(j+l)=i} K^i(X) \otimes K^j(Y) \rightarrow K^i(X \times Y) \rightarrow \bigoplus_{e(j+l)=e(i+1)} \text{Tor}(K^i(X), K^j(Y)) \rightarrow 0.
\]

This sequence always splits, although unnaturally (see [12], [23] where a generalization to the case of C\(^*\)-algebras is given, and [50]).

1.7. Spectral K-Homology. Let \( X \) be a general (not necessarily compact) CW-complex. We then define

\[
K^i_t(X) := \lim_{\rightarrow} K^i(Y),
\]

for \( i = 0, 1 \), where the limit runs over the finite CW-subcomplexes \( Y \) of \( X \). By defining \( K^i_{2k}(X) := K^i_0(X) \) and \( K^i_{2k+1}(X) := K^i_1(X) \) for all \( k \in \mathbb{Z} \), one has that \( K^i_k(X) \) is a 2-periodic unreduced homology theory on the category of CW-complexes. On the other hand, K-theory is a 2-periodic cohomology theory which can be defined in terms of its spectrum \( K^U_\ast = (K^U_n)_{n \in \mathbb{Z}} \), where \( K^U_0 := Z \times BU(\infty) \) and \( K^U_{2k+1} := U(\infty) \) are the classifying spaces for \( K^0 \) and \( K^1 \), respectively. Thus we can define a homotopy theory related to \( K^t_\ast \) by the inductive limit

\[
K^t_i(X, Y) := \lim_{\rightarrow} \pi_{n+i}((X/Y) \wedge K^U_n)
\]

for all \( i \in \mathbb{Z} \), where \( Y \) is a closed subspace of the topological space \( X \) and \( \wedge \) denotes the smash product. Bott periodicity then implies that this is a 2-periodic homology theory.

For any finite CW-complex \( X \), we can construct a map

\[
\mu^s : K^i_t(X) \rightarrow K^s_t(X) := K^s_t(X, \emptyset)
\]

given by

\[
\mu^s([M, E, \phi]) := \phi_*([E] \cap [M]_s)
\]

on classes of K-cycles and extended by linearity. Here \( \cap : K^0_t(X) \otimes K^s_t(X) \rightarrow K^s_t(X) \) for \( i = 0, 1 \) is the spectrally defined cap product with \([M]_s\) the fundamental class of the manifold \( M \) in \( K^s_t(X) \) [58]. The transformation \( \mu^s \) is an isomorphism which is natural in \( X \), and so it defines a natural equivalence between the functors \( K^s_t \) and \( K^s_t \) [58]. It follows that \( K^s_t(X) \) is a realization of \( K^s_t(X) \). The map \( \mu^s \) is also compatible with cap products, i.e. \( \mu^s(\xi \cap \alpha) = \xi \cap \mu^s(\alpha) \) for all \( \xi \in K^s_t(X) \) and \( \alpha \in K^s_t(X) \), or equivalently there is a commutative diagram

\[
\begin{array}{c}
K^i_t(X) \otimes K^j_t(X) \xrightarrow{\cap} K^{i+j}_t(X) \\
\downarrow \text{id}_{K^i_t(X)} \otimes \mu^s \\
K^i_t(X) \otimes K^j_t(X) \xrightarrow{\mu^s} K^{i+j}_t(X)
\end{array}
\]
In particular, if $X$ is a compact connected spin$^c$ manifold without boundary, then
\[ \mu^s([X, \mathbb{I}_X^\mathbb{C}, \text{id}_X]) = (\text{id}_X)_*(\mathbb{I}_X^\mathbb{C} \cap [X]) = (\text{id}_X)_*([X]) = [X] \]
in $K_2^s(X)$, with $\mathbb{I}_X^\mathbb{C}$ the trivial complex line bundle over $X$. Since $\mu^s$ is a natural equivalence between $K_2^s$ and $K_2^s$ it follows that, within the framework of topological K-homology as the dual theory to K-theory, $[X, \mathbb{I}_X^\mathbb{C}, \text{id}_X]$ is the fundamental class of $X$ in $K_2^s(X)$.

One can give a definition of relative K-homology groups $K^i_2(X,Y)$ in such a way that there is also a map $\mu^s : K^i_2(X,Y) \to K^s_2(X,Y)$ which defines a natural equivalence between functors on the category of topological spaces having the homotopy type of finite CW-pairs $(X,Y)$ [33]. One can also give a bordism description of $K^i_2(X,Y)$ as follows. We consider the set of all triples $(M, E, \phi)$ where

(i) $M$ is a compact spin$^c$ manifold with boundary;
(ii) $E$ is a complex vector bundle over $M$; and
(iii) $\phi : M \to X$ is a continuous map with $\phi(\partial M) \subset Y$.

This set is quotiented by relations of bordism (modified from Definition 1.33 by the requirement that $M_1 \Pi(-M_2) \subset \partial W$ is a regularly embedded submanifold of codimension 0 with $\phi(\partial W \setminus M_1 \Pi (-M_2)) \subset Y$, direct sum and vector bundle modification. The collection of equivalence classes is a $\mathbb{Z}_2$-graded abelian group with operation induced by disjoint union of relative K-cycles [38].

Since K-homology is a generalized homology theory, there is a long exact homology sequence for any pair $(X,Y)$. Because $K^i_2$ is a 2-periodic theory, this sequence truncates to the six-term exact sequence

\[
\begin{array}{ccccccc}
K_0^i(Y) & \xrightarrow{i_*} & K_0^i(X) & \xrightarrow{c_*} & K_0^i(X,Y) \\
\partial & & \downarrow & & \downarrow & & \partial \\
K_1^i(X,Y) & \xleftarrow{c_*} & K_1^i(X) & \xleftarrow{c_*} & K_1^i(Y)
\end{array}
\]

where the horizontal arrows are induced by the canonical inclusion maps $i : Y \hookrightarrow X$ and $\varsigma : (X, \emptyset) \to (X,Y)$. In the bordism description, the connecting homomorphism is given by the boundary map

\[ \partial[M, E, \phi] := [\partial M, E|_{\partial M}, \phi|_{\partial M}] \]
on classes of K-cycles and extended by linearity. One also has the usual excision property. If $U \subset Y$ is a subspace whose closure lies in the interior of $Y$, then the inclusion $\varsigma^U : (X \setminus U, Y \setminus U) \hookrightarrow (X,Y)$ induces an isomorphism

\[ \varsigma^U_* : K^i_2(X \setminus U, Y \setminus U) \overset{\cong}{\xrightarrow{\sim}} K^i_2(X,Y) \]
of $\mathbb{Z}_2$-graded abelian groups.

1.8. Analytic K-Homology. We will now briefly describe the relationship between $K_2^s(X)$ and the analytic K-homology groups of a finite CW-complex $X$ (the construction of these groups follows verbatim to the case of a compact metrizable topological space – see [9]).

1.8.1. The Group $K_{0}^s(X)$. Let $\Omega_0(X)$ be the set of all quintuples $(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)$ where

(i) for each $i = 0, 1$, $\mathcal{H}_i$ is a separable Hilbert space;
(ii) for each $i = 0, 1$, $\psi_i : C(X) \to \mathcal{L}(\mathcal{H}_i)$ is a unital algebra $\ast$-homomorphism, where $C(X)$ is the $C^\ast$-algebra of continuous complex-valued functions on $X$ and $\mathcal{L}(\mathcal{H}_i)$ is the $C^\ast$-algebra of bounded linear operators on $\mathcal{H}_i$; and
(iii) $T : \mathcal{H}_0 \to \mathcal{H}_1$ is a bounded Fredholm operator such that the operator $T \circ \psi_0(f) - \psi_1(f) \circ T$ is compact for all $f \in C(X)$.
We can define on $\Omega_0(X)$ a direct sum operation and an equivalence relation generated by isomorphism, direct sum with a trivial object, and compact perturbation of Fredholm operators. The quotient set is, with direct sum, an abelian group $K_0^a(X)$ called the degree 0 analytic K-homology group of $X$. There is an epimorphism

$$\text{Index} : K_0^a(X) \rightarrow \mathbb{Z}$$

given by

$$\text{Index}[\mathcal{H}_0, \psi_0; \mathcal{H}_1, \psi_1, T] := \text{Index } T .$$

Suppose that $X$ is a closed $C^\infty$ manifold, $E_0$, $E_1$ are complex $C^\infty$ vector bundles over $X$ and $D : C^\infty(E_0) \rightarrow C^\infty(E_1)$ is an elliptic pseudo-differential operator on $X$. Then one can construct an element $[D] \in K_0^a(X)$ which depends only on $D$. All elements of $K_0^a(X)$ arise in this way, and in this case we have that

$$\text{Index}[D] = \text{Index } D$$

is the analytic index of $D$ regarded as a Fredholm operator \[6\].

1.8.2. The Group $K_1^a(X)$. Let $\Omega_1(X)$ be the set of all pairs $(\mathcal{H}, \tau)$ where

(i) $\mathcal{H}$ is a separable Hilbert space; and
(ii) $\tau : C(X) \rightarrow \mathcal{O}(\mathcal{H})$ is a unital algebra *-homomorphism, where $\mathcal{O}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra with $\mathcal{K}(\mathcal{H})$ the closed ideal in $\mathcal{L}(\mathcal{H})$ consisting of compact operators on $\mathcal{H}$. On $\Omega_1(X)$ we can define a direct sum operation and an equivalence relation using unitary equivalence and triviality. The quotient set is, with direct sum, an abelian group $K_1^a(X)$ called the degree 1 analytic K-homology group of $X$. It coincides with the Brown-Douglas-Fillmore group $\text{Ext}(X) := \text{Ext}(C(X), \mathcal{K})$ of equivalence classes of extensions of the $C^*$-algebra $C(X)$ by compact operators $\mathcal{K}[9]$, defined by $C^*$-algebras $A$ which fit into the short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow A \rightarrow C(X) \rightarrow 0 .$$

Suppose that $X$ is a closed $C^\infty$ manifold, $E$ is a complex $C^\infty$ vector bundle over $X$ and $A : C^\infty(E) \rightarrow C^\infty(E)$ is a self-adjoint elliptic pseudo-differential operator on $X$. Then one can construct an element $[A] \in K_1^a(X)$ which depends only on $A$. All elements of $K_1^a(X)$ arise in this way \[6\].

1.8.3. The Group $K_2^a(X)$. We define $K_2^a(X) := K_0^a(X) \oplus K_1^a(X)$ to be the analytic $K$-homology group of $X$. There is a natural notion of induced homomorphism $f_* : K_2^a(X) \rightarrow K_2^a(Y)$ for continuous maps $f : X \rightarrow Y$ such that $K_2^a$ is a 2-periodic homology theory. Let us now describe its explicit relation to the topological $K$-homology theory $K_2$. Let $(M, E, \phi)$ be a K-cycle on the finite CW-complex $X$ and $\mathcal{D}$ the Dirac operator on the spin$^c$ manifold $M$. Then the twisted Dirac operator $\mathcal{D}_E$ is an elliptic first order differential operator on $M$ (self-adjoint if $\dim \mathbb{R} M$ is odd; the "corner operator" that maps odd spinors to even spinors, and vice-versa, if $\dim \mathbb{R} M$ is even). Hence it determines an element $[\mathcal{D}_E] = [E] \cap [\mathcal{D}] \in K_2^a(M)$ with the degree preserved, and $\phi_*[\mathcal{D}_E] \in K_2^a(X)$. This element depends only on the K-homology class $[M, E, \phi] \in K_1^a(X)$, and so we get a well-defined map of $\mathbb{Z}_2$-graded abelian groups

$$\mu^a : K_1^a(X) \rightarrow K_2^a(X)$$

given by

$$\mu^a[M, E, \phi] := \phi_*[\mathcal{D}_E]$$
on classes of K-cycles. This map is an isomorphism which is natural. The index epimorphism and the epimorphism induced by the collapsing map together with the isomorphism \( \mu^a \) generate a commutative diagram

\[
\begin{array}{ccc}
K^i_a(X) & \xrightarrow{\mu^a} & K^i(X) \\
\downarrow & & \downarrow \\
K^i_b(X) & \xrightarrow{\varepsilon_x} & \mathbb{Z} .
\end{array}
\]

As a consequence of what was said in the previous and this subsections one has the following.

**Theorem 1.2.** Let \( X \) be a finite CW-complex. Then all three forms of K-homology are isomorphic, i.e.

\[
K^i_a(X) \cong K^i_b(X) \cong K^i_c(X)
\]

are isomorphic as graded groups.

1.9. **Poincaré Duality.** Let \( X \) be an \( n \)-dimensional compact manifold with (possibly empty) boundary, and \( \mathbb{B}(TX) \to X \) and \( S(TX) \to X \) the unit ball and sphere bundles of \( X \). An element \( \tau \in K^e(n)(\mathbb{B}(TX) \times S(TX)) \) is called a Thom class or an orientation for \( X \) if \( \tau|_{(\mathbb{B}(TX) \times S(TX))} \in K^e(n)(\mathbb{B}(TX) \times S(TX)) \cong K^i(\text{pt}) \) is a generator for all \( x \in X \). The manifold \( X \) is said to be \( K^e \)-orientable if it has a Thom class. In that case the usual cup product on the topological K-theory ring yields the Thom isomorphism

\[
\Theta_X : K^i_a(X) \xrightarrow{\sim} K^e(i+n)(\mathbb{B}(TX) \times S(TX))
\]

given for \( i = 0, 1 \) and \( \xi \in K^i_a(X) \) by

\[
\Theta_X(\xi) := \pi^*_\mathbb{B}(TX)(\xi) \cup \tau ,
\]

where \( \pi_\mathbb{B}(TX) : \mathbb{B}(TX) \to X \) is the bundle projection. This construction also works by replacing the tangent bundle of \( X \) with any \( O(r) \) vector bundle \( V \to X \), defining a Thom isomorphism

\[
\Theta_{X,V} : K^i_a(X) \xrightarrow{\sim} K^e(i+r)(\mathbb{B}(V) \times S(V))
\]

given by

\[
\Theta_{X,V}(\xi) := \pi^*_\mathbb{B}(V)(\xi) \cup \tau_V ,
\]

where the element \( \tau_V \in K^e(r)(\mathbb{B}(V) \times S(V)) \) is called the Thom class of \( V \).

Any \( K^e \)-oriented manifold \( X \) of dimension \( n \) has a uniquely determined fundamental class \( [X]_s \in K^e(n)(X, \partial X) \). One then has the Poincaré duality isomorphism

\[
\Phi_X : K^i_a(X) \xrightarrow{\sim} K^e(i+n)(X, \partial X)
\]

given for \( i = 0, 1 \) and \( \xi \in K^i_a(X) \) by taking the cap product

\[
\Phi_X(\xi) := \xi \cap [X]_s .
\]

In particular, if \( X \) is a compact spin \( c \) manifold of dimension \( n \) without boundary, then \( X \) is \( K^e \)-oriented and so in this case we also have a Poincaré isomorphism as above giving

\[
K^0_b(X) \cong K^e(n)(X) , \quad K^1_b(X) \cong K^e(n+1)(X) .
\]
1.10. **Universal Coefficient Theorem.** Let \( X \) be a compact \( n \)-dimensional spin\(^c\) manifold without boundary. In the framework of analytic K-homology, the six-term exact sequence in K-theory corresponding to an extension \( [\mathbb{J}] \) reduces to the short exact sequence

\[
0 \rightarrow K_0(X) \rightarrow K_0(A) \rightarrow K_1^0(X) \rightarrow 0
\]

and therefore defines an element of \( \text{Ext}(K_0^0(X), \mathbb{Z}) \) in homological algebra. Conversely, there is a universal coefficient theorem given by the short exact sequence \( [50,11,34] \) (here we’re assuming that this extension is in the kernel of \( \gamma_0 : \text{Ext}(X) \rightarrow \text{Hom}(K_1^1(X), \mathbb{Z}) \) – see \( [28] \) for a definition of this homomorphism)

\[
0 \rightarrow \text{Ext}(K_0^0(X), \mathbb{Z}) \rightarrow \text{Ext}(C(X), X) \rightarrow \text{Hom}(K_1^1(X), \mathbb{Z}) \rightarrow 0.
\]

This sequence splits, although not naturally. For definiteness, suppose that the degree 0 K-theory of \( X \) can be split as \( K_0^0(X) = \Lambda_{K_0^0(X)} \oplus \text{tor}_{K_0^0(X)} \), where the lattice \( \Lambda_{K_0^0(X)} = K_0^0(X) / \text{tor}_{K_0^0(X)} \) is the free part of the K-theory group and \( \text{tor}_{K_0^0(X)} = \bigoplus_{i=1}^{m} \mathbb{Z}_{n_i} \) is its torsion subgroup (such a split is neither unique nor natural). Since \( X \) is a finite CW-complex, the abelian group \( K_0^0(X) \) is finitely generated and we have

\[
\text{Ext}(K_0^0(X), \mathbb{Z}) = \text{Ext}(\text{tor}_{K_0^0(X)}, \mathbb{Z}) \cong \bigoplus_{i=1}^{m} \text{Ext}(\mathbb{Z}_{n_i}, \mathbb{Z}) \cong \bigoplus_{i=1}^{m} \mathbb{Z}_{n_i} = \text{tor}_{K_0^0(X)}
\]

from which it follows that

\[
K_1^1(X) \cong \text{Hom}(K_1^1(X), \mathbb{Z}) \oplus \text{tor}_{K_0^0(X)}.
\]

Although the torsion part of the dual homology group to the topological K-theory group \( K_1^1(X) \) can differ from that of the analytic K-homology \( K_1^1(X) \), Poincaré duality always asserts an isomorphism between the full groups \( K_2^0(X) \cong K_1^1(X) \) and \( K_2^0(X) \). Note that if \( \dim_{\mathbb{R}} X \) is even and \( K_0^0(X) \) is a free abelian group, then \( K_1^1(X) \cong K_1^1(X) \cong K_1^1(X) \cong \text{Hom}(K_1^1(X), \mathbb{Z}) \).

One can make a stronger statement (due to D. W. Anderson) which works for any CW-complex \( X \) with finitely generated K-theory. Since \( K^U \) is a CW-spectrum and a ring-spectrum, there is a universal coefficient theorem expressed by the (split) exact sequence \( [11,13,66] \)

\[
0 \rightarrow \text{Ext}(K_{e(i-1)}^1(X), \mathbb{Z}) \rightarrow K_1^1(X) \rightarrow \text{Hom}(K_1^1(X), \mathbb{Z}) \rightarrow 0
\]

for \( i = 0, 1 \). The epimorphism is given by the index map. As above, let us consider the splits \( K_1^1(X) = \Lambda_{K_1^1(X)} \oplus \text{tor}_{K_1^1(X)} \) and \( K_1^1(X) = \Lambda_{K_1^1(X)} \oplus \text{tor}_{K_1^1(X)} \). One then easily concludes that

\[
\text{Ext}(K_{e(i-1)}^1(X), \mathbb{Z}) \cong \text{Ext}(\text{tor}_{K_{e(i-1)}^1(X)}, \mathbb{Z}) \cong \text{tor}_{K_{e(i-1)}^1(X)} ,
\]

\[
\text{Hom}(K_1^1(X), \mathbb{Z}) \cong \text{Hom}(\Lambda_{K_1^1(X)}, \mathbb{Z}) \cong \Lambda_{K_1^1(X)}.
\]

By the universal coefficient theorem it follows that \( \text{tor}_{K_{e(i-1)}^1(X)} \cong \text{tor}_{K_1^1(X)} \) and \( \Lambda_{K_1^1(X)} \cong \Lambda_{K_1^1(X)} \).

We can also conclude the following from these general calculations.

**Proposition 1.1.** Let \( X \) be a finite CW-complex. Then its K-homology group is finitely generated.

**Proof.** Since \( X \) is a finite CW-complex, its K-theory groups are finitely generated. The isomorphisms above then imply the conclusion. \( \square \)
1.11. Chern Character. There is a natural transformation $\mathrm{ch}_\bullet : K^*_T(X) \to H_*(X; \mathbb{Q})$ of $\mathbb{Z}_2$-graded homology theories called the (homology) Chern character which is defined in the following way. Recall the $\mathbb{Z}_2$-grading on singular homology given by $H^*_T(X; \mathbb{Q}) = H^\text{even}(X; \mathbb{Q}) \oplus H^\text{odd}(X; \mathbb{Q})$ with $H^\text{even}(X; \mathbb{Q}) = \bigoplus_{\epsilon(k) = 0} H_k(X; \mathbb{Q})$ and $H^\text{odd}(X; \mathbb{Q}) = \bigoplus_{\epsilon(k) = 1} H_k(X; \mathbb{Q})$. Given a K-cycle $(M, E, \phi)$ on $X$, let $\phi_* : H^t(M; \mathbb{Q}) \to H^t(X; \mathbb{Q})$ be the homomorphism induced on rational homology by $\phi$. Then $\mathrm{ch}^*(E) \cup \text{td}(TM) \cap [M]$ is the Poincaré dual on $M$ of the even degree cohomology class $\mathrm{ch}^*(E) \cap \text{td}(TM)$, where $\mathrm{ch}^* : K^*_T(-) \to H^\text{even}(-; \mathbb{Q})$ is the (cohomology) Chern character in K-theory, $\text{td}$ denotes the Todd class of a spin$^c$ vector bundle and $[M]$ is the orientation cycle of $M$ in $H^*_T(M; \mathbb{Q})$ induced by the spin$^c$ structure on $TM$. Then
\begin{equation}
\mathrm{ch}_\bullet(M, E, \phi) := \phi_*(\mathrm{ch}^*(E) \cup \text{td}(TM) \cap [M])
\end{equation}
is an element of $H^*_T(X; \mathbb{Q})$ which depends only on the K-homology class $[M, E, \phi] \in K^*_T(X)$. This map preserves the $\mathbb{Z}_2$-grading. The Chern characters $\mathrm{ch}^*$ and $\mathrm{ch}_\bullet$ preserve the cap product, i.e. for every topological space $X$ there is a $\mathbb{Z}_2$-degree preserving commutative diagram
\[
\begin{array}{ccc}
K^*_T(X) \otimes K^*_T(X) & \xrightarrow{\cap} & K^*_T(X) \\
\mathrm{ch}^* \otimes \mathrm{ch}_\bullet & & \mathrm{ch}_\bullet \\
H^*(X; \mathbb{Q}) \otimes H^*_T(X; \mathbb{Q}) & \xrightarrow{\cap} & H^*_T(X; \mathbb{Q})
\end{array}
\]
Since $X$ is a finite CW-complex, $K^*_T(X)$ is a finitely generated abelian group and $\mathrm{ch}_\bullet$ induces an isomorphism $K^*_T(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^*_T(X; \mathbb{Q})$ of $\mathbb{Z}_2$-graded vector spaces over $\mathbb{Q}$.

The Chern character can be used to give an explicit formula for the epimorphism $\varepsilon_*$ in terms of characteristic classes as $\varepsilon_*[M, E, \phi] = \mathrm{ch}^*(E) \cup \text{td}(TM)[M]$. Then the commutative diagram can be recast as the equality
\begin{equation}
\text{Index } \phi_*[\mathcal{P}_E] = \text{Index } \mu^a[M, E, \phi] = \varepsilon_*[M, E, \phi] = \mathrm{ch}^*(E) \cup \text{td}(TM)[M] .
\end{equation}
In the special case where $X$ is a point this becomes $\text{Index } \mathcal{P}_E = \mathrm{ch}^*(E) \cup \text{td}(TM)[M]$, which is a particular instance of the Atiyah-Singer index theorem.

2. K-Cycles and D-Brane Constructions

We will now bring string theory into the story. Many of our subsequent results find their most natural interpretations in terms of D-branes, with the mathematical formalism of K-homology leading to new insights into the properties of D-branes wrapping cycles in non-trivial spacetimes. We begin with some heuristic physical discussion aimed at motivating the interpretation of D-branes as K-cycles in topological K-homology. Then we move on to more mathematical computations explaining the interplay between K-homology and the properties of D-branes. The analysis will center around finding explicit K-cycle representatives for the generators of the K-homology groups, which will be interpreted as D-branes in the pertinent spacetime. For physical definitions and descriptions of D-branes in string theory, see [80] [53].

2.1. D-Branes. Consider Type II superstring theory on a spacetime $X$ with all background supergravity form fields turned off. $X$ is an oriented ten-dimensional spin manifold. A D-brane in $X$ is an oriented spin$^c$ submanifold $M \subset X$ together with a complex vector bundle $E \to M$ called the Chan-Paton bundle. $M$ itself is referred to as the worldvolume of the D-brane and when $\dim_{\mathbb{R}} M = p + 1$ we will sometimes refer to the brane as a D$p$-brane to emphasize its dimensionality. The presence of non-vanishing background form fields would mean that the classification of D-branes requires algebraic and/or twisted (co)homology tools. Their absence means that we can resort to topological methods, which thereby classify flat D-branes. While this works whenever the tangent bundle $TX$ over spacetime is stably trivial, in more general
cases the set-up would not describe a true background of string theory since the terminology
‘flat’ used here is not meant to imply that we consider a flat spacetime geometry. Nonetheless,
the topological description will provide geometric insight into the nature of D-branes in curved
backgrounds, even in this simplified setting. Thus a very crude definition of a D-brane is as a K-
cycle \((M, E, \phi)\) on the spacetime \(X\), with \(\phi = \iota : M \hookrightarrow X\) the natural inclusion (The remaining
elements of \(\Gamma(X)\) then ensure that the quotient \(\Gamma(X) / \sim\) really is \(K^*_1(X)\)). Sometimes we regard
D-branes as sitting in an ambient space which is a proper subspace of spacetime, for instance
when \(X\) is a product \(X = Q \times Y\) we may be interested in worldvolumes \(M \subset Q\). When there is
no danger of confusion we will also use the symbol \(X\) for this ambient space. In either case, \(X\)
is also customarily called the target space.

D-branes are generally more complicated objects than just submanifolds carrying vector bundles,
because in string theory they are realized as (Dirichlet) boundary conditions for a two-
dimensional superconformal quantum field theory with target space \(X\). Any classification based
solely on K-theory is expected to capture only those properties that depend on D-brane charge.
Nevertheless, the primitive definition of a D-brane as a K-cycle in topological K-homology is very
natural and carries much more information than its realization in the dual K-theory framework.
We shall see that the geometric description of K-homology is surprisingly rich and provides a
simple context in which non-trivial D-brane effects are exhibited in a clear geometrical fashion.

**Example 2.1 (B-Branes).** Let \(X\) be a (possibly singular) complex \(n\)-dimensional projective algebraic variety, and let \(O_X\) be the structure sheaf of regular functions on \(X\). Recall that a coherent sheaf on \(X\) is a sheaf of \(O_X\)-modules which is locally the cokernel of a morphism of holomorphic vector bundles over \(X\). The coherent algebraic sheaves on \(X\) form an abelian category denoted \(\text{coh}(X)\). The bounded derived category of coherent sheaves on \(X\), denoted \(D(X) := D(\text{coh}(X))\), is the triangulated category of topological B-model D-branes (or B-branes for short) in \(X\) [26, 61]. An object of this category is a bounded differential complex of coherent sheaves. It contains \(\text{coh}(X)\) as a full subcategory by identifying a coherent sheaf \(\mathcal{F}\) with the trivial complex

\[
\mathcal{F}^\bullet = (0 \to 0 \to \cdots \to 0 \to 0 \to 0 \to 0 \to \cdots \to 0)
\]

having \(\mathcal{F}\) in degree 0. For details of these and other constructions, see [3].

The relevant K-homology group, denoted \(K^0_\omega(X)\), is the Grothendieck group of coherent
algebraic sheaves on \(X\) obtained by applying the usual Grothendieck completion functor to the
abelian category \(\text{coh}(X)\). There is a natural transformation \(\alpha^\omega : D(X) \to K^0_\omega(X)\) which may
be described as follows. Let \(\mathcal{F}^\bullet \in D(X)\) be a complex. Using a locally-free resolution we may
replace \(\mathcal{F}^\bullet\) by a quasi-isomorphic complex of locally-free sheaves, each of which has associated to
it a holomorphic vector bundle. The virtual Euler class of the complex obtained by replacing
locally-free sheaves with their corresponding vector bundles is then the K-homology class \(\alpha^\omega(\mathcal{F}^\bullet)\)
that we are looking for. On the other hand, the underlying topological space of \(X\) is a finite CW-complex and has topological K-homology group \(K^*_1(X)\). We will now construct a natural
map \(\mu^\omega : K^0_\omega(X) \to K^*_1(X)\), which by composition gives a natural map from the derived category
\(\mu^\omega \circ \alpha^\omega : D(X) \to K^*_1(X)\) and gives an intrinsic description of B-branes in terms of K-cycles.

Consider triples \((M, E, \phi)\) where

(i) \(M\) is a non-singular complex projective algebraic variety;
(ii) \(E\) is a complex algebraic vector bundle over \(M\); and
(iii) \(\phi : M \to X\) is a morphism of algebraic varieties.

Two triples \((M_1, E_1, \phi_1)\) and \((M_2, E_2, \phi_2)\) are said to be isomorphic if there exists an isomorphism
\(h : M_1 \to M_2\) of complex projective algebraic varieties such that \(h^*(E_2) \cong E_1\) as complex
algebraic vector bundles over \(M_1\) and \(\phi_1 = \phi_2 \circ h\). The set of isomorphism classes of triples is
denoted \(\Gamma^\omega(X)\). Given such a triple \((M, E, \phi)\), the morphism \(\phi : M \to X\) induces the direct
image functor $\text{coh}(\phi) : \text{coh}(M) \to \text{coh}(X)$, defined by $\text{coh}(\phi)(\mathcal{F}) = \phi_*(\mathcal{F})$ for $\mathcal{F} \in \text{coh}(M)$, which is left exact and induces the $i$-th right derived functor $R^i \text{coh}(\phi) : D(M) \to D(X)$ for $i = 0, 1, \ldots, n$ as follows. We include the category of coherent sheaves into the category of quasi-coherent sheaves, replace a complex by a quasi-isomorphic complex of injectives, and apply the functor $\text{coh}(\phi)$ componentwise to the complex of injectives (If $X$ is singular this requires a resolution of its singularities). Then the induced map

$$\phi_* : K_0^\omega(M) \to K_0^\omega(X)$$

is given for coherent algebraic sheaves $\mathcal{F}$ on $M$ by

$$(2.1) \quad \phi_*[\mathcal{F}] := \sum_{i=0}^{n} (-1)^i \alpha^i \circ R^i \text{coh}(\phi)[\mathcal{F}^\bullet].$$

In particular, if $\mathcal{E}$ denotes the sheaf of germs of algebraic sections of $E \to M$, then $\phi_*[\mathcal{E}] \in K_0^\omega(X)$. By using a resolution of the singularities of $X$ if necessary and the fact that any coherent sheaf on a non-singular variety admits a resolution by locally free sheaves, one can show that the $\phi_*[\mathcal{E}]$ obtained from triples in $\Gamma^\omega(X)$ generate the abelian group $K_0^\omega(X)$ [6]. By forgetting some structure a triple $(M, E, \phi)$ becomes a K-cycle on $X$ and hence determines an element $[M, E, \phi] \in K_0^\omega(X)$.

Thus we get a well-defined map

$$\mu^\omega : K_0^\omega(X) \to K_0^\omega(X)$$

given on generators by

$$\mu^\omega \phi_*[\mathcal{E}] := [M, E, \phi].$$

This map is a natural transformation of the covariant functors $K_0^\omega$ and $K_0^\omega$, thus providing an extension of the Grothendieck-Riemann-Roch theorem. However, in contrast to the transformations $\mu^\ast$ and $\mu^\omega$ of the previous section, $\mu^\omega$ is not an isomorphism [6]. This suggests that the topological K-homology group $K_0^\omega(X)$ carries more information about the category of B-branes than the Grothendieck group $K_0^\omega(X)$.

One of the virtues of this mapping of elements in the derived category $D(X)$ to classes of K-cycles in $K_0^\omega(X)$ is that it allows one to compute B-brane charges even when the variety $X$ is singular. The collapsing map $\varepsilon : X \to \text{pt}$ induces, as before, an epimorphism $\chi : K_0^\omega(X) \to K_0^\omega(\text{pt})$. Since a coherent algebraic sheaf over a point is just a finite dimensional complex vector space, which can be characterized by its dimension, one has $K_0^\omega(\text{pt}) \cong \mathbb{Z}$. The charge of a B-brane represented by a coherent algebraic sheaf $\mathcal{F}$ on $X$ is the image of $[\mathcal{F}] \in K_0^\omega(X)$ in $\mathbb{Z}$ under the epimorphism $\chi$, which using (2.1) is given explicitly by the Euler number

$$\chi(X; \mathcal{F}) := \chi[\mathcal{F}] = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(X; \mathcal{F}),$$

where $H^i(X; \mathcal{F})$ is the $i$-th cohomology group of $X$ with coefficients in $\mathcal{F}$. Together with the epimorphism [1.5] and the transformation $\mu^\omega$, there is a commutative diagram similar to (1.10) given by

$$\begin{array}{ccc}
K_0^\omega(X) & \xrightarrow{\mu^\omega} & K_0^\omega(X) \\
\varepsilon_* \downarrow & & \downarrow \chi \\
\mathbb{Z} & \xrightarrow{\chi} & \mathbb{Z}.
\end{array}$$

For a B-brane represented by a K-cycle $(M, E, \phi)$, the characteristic class formula for $\varepsilon_*$ in (1.14) then gives the charge

$$\chi(X; \phi_*[\mathcal{E}]) = \text{ch}^\bullet(E) \cup \text{td}(TM)[M].$$
If \( X \) is non-singular and \( E \) is a complex vector bundle over \( X \), then this charge formula applied to the K-cycle \((X, E, \mathrm{id}_X)\) is just the Hirzebruch-Riemann-Roch theorem which computes the analytic index of the twisted Dolbeault operator \( \overline{\partial}_E \) on \( X \). See [4] for more details.

\[ \phi \]

2.2. Qualitative Description of K-Cycle Classes. We will now explain how the equivalence relations of topological K-homology, as spelled out in Definition [1.5], translate into physical statements about D-branes. Let us begin with bordism. Suppose that \( X \) is a locally compact topological space and \( \mathbb{pt} \not\in X \) contains a distinguished point called “infinity”. Let \( X^\infty := X \amalg \mathbb{pt} \) be the one-point compactification of \( X \). We are interested in configurations of D-branes in \( X \) which have finite energy. This means that they should be regarded as equivalent to the closed string vacuum asymptotically in \( X^\infty \). The charge of a D-brane \((M, E, \phi)\) is given through the index formula [11.14]. (In this paper we do not deal with the square root of the \( \hat{A} \)-genus which usually defines D-brane charges [49].) The condition that the D-brane should have vanishing charge at infinity is tantamount to requiring that its K-homology class have a trivialization at infinity, i.e. that there is a compact subset \( U \) of \( X \) such that \( E|_{X-U} \) is a trivial bundle and \( \phi|_{X-U} \) is homotopically trivial, so that Index \( \phi|_{[\mathbb{D}_E]} \) = 0 over \( X - U \). This is the physical meaning of the definition of the reduced K-homology group [11.9], which measures charges of D-branes relative to that of the vacuum. The K-homology of \( X \) in this case should then be defined by the relative K-homology group [11.7] as \( \mathrm{K}^i_t(X^\infty, \mathbb{pt}) \cong \mathrm{K}^i_t(X) \), where we have used excision. Since the six-term exact sequence for this relative K-homology has connecting homomorphism \( \partial: \mathrm{K}^i_t(X^\infty, \mathbb{pt}) \to \mathrm{K}^i_{t+i}(\mathbb{pt}) \) acting by \( \partial[W,F,\psi] = [\partial W,F|_{\partial W},\psi|_{\partial W}] \), boundaries \( \partial W \) of spin\(^c\) manifolds \( W \) map into \( \mathbb{pt} \) and so any D-brane \((M, E, \phi)\) which is bordant to the trivial K-cycle \((\emptyset, \emptyset, \emptyset, \emptyset)\) on \( X \) should carry the same charge as the vacuum. This is guaranteed by the bordism relation. The argument extends to a “compactification” of spacetime in which \( X = Q \times Y \), with \( Y \) locally compact and \( Y \) a compact topological space without boundary. D-branes will now either be located at particular points in \( Y \) or will wrap submanifolds of \( Y \). Requiring that such configurations be equivalent to the vacuum asymptotically in \( Q^\infty \) thus requires adding a copy of \( Y \) at infinity in \( Q^\infty \), and so we look for K-homology classes with a trivialization on \( Y \) at infinity. Using relative bordism and excision, the classes of D-brane K-cycles are now seen to live in the group \( \mathrm{K}^i_t(X,Y) \cong \mathrm{K}^i_t(X/Y, \mathbb{pt}) \cong \mathrm{K}^i_t(Q^\infty, \mathbb{pt}) \).

Let us now turn to direct sum. It represents “gauge symmetry enhancement for coincident branes” which occurs when several D-branes wrap the same submanifold \( M \subset X \) [83]. In this case, the Chan-Paton bundles \( E_i \to M \) of the constituent D-branes are augmented to the higher-rank Chan-Paton bundle \( E = \bigoplus_i E_i \) of the combined brane configuration. Such a combination is called a bound state of D-branes. The branes are bound together by open string excitations corresponding to classes of bundle morphisms \([f_{ij}] \in \mathrm{Hom}(E_i,E_j)\). Other open string degrees of freedom are described by the higher cohomology groups \( \mathrm{Ext}^p(E_i,E_j), p \geq 1 \).

Finally, let us look at vector bundle modification. Consider the K-cycle \((\mathbb{pt}, \mathbb{1}^C_{\mathbb{pt}}, \iota)\) on \( X \) where \( \mathbb{1}^C_{\mathbb{pt}} = \mathbb{pt} \times \mathbb{C} \) and \( \iota: \mathbb{pt} \hookrightarrow X \) is the inclusion of a point. Let \( F = \mathbb{pt} \times \mathbb{R}^{2n} \) with \( n \geq 1 \). Then \( \mathbb{M} = \mathbb{pt} \times S^{2n} \cong S^{2n} \) is a 2\( n \)-dimensional sphere, and \( \pi = \varepsilon : S^{2n} \to \mathbb{pt} \) is the collapsing map. From the clutching construction of Section [12.2] \( H(F) = H(F)_{|S^{2n}} = H(F)|_{\pi^{-1}(\mathbb{pt})} \) is the Bott generator for the K-theory of \( S^{2n} \). Using vector bundle modification one then has \([\mathbb{pt}, \mathbb{1}^C_{\mathbb{pt}}, \iota] = [S^{2n},H(F) \otimes \mathbb{1}_{\mathbb{pt}}, \iota \circ \pi] = [S^{2n},H(F),\varepsilon] \), where \( \iota \circ \pi = \varepsilon \) is a collapsing map. This equality represents the “blowing up” of a D\((-1)\)-brane (also known as a D-instanton) into a collection of spherical D\((2n-1)\)-branes and is a simple example of what is known in string theory as the “dielectric effect” [51]. This is described in more generality in Section [2.5] below. The equality also illustrates the crucial point that topological K-homology naturally encodes the fact that D-branes are typically not static objects but will “decay” into stable configurations of branes. Suppose that the spacetime has a split \( X = \mathbb{R} \times X' \), where \( \mathbb{R} \) is thought of as parametrizing “time” and \( X' \) is “space”. Consider a D-brane in \( X \) whose worldvolume is initially
of the form $M = \mathbb{R} \times M'$ with $M' \subset X'$. In the distant future (at large time), this brane will "decay" into stable D-branes of lower dimension which wrap non-trivial homology cycles of $X'$. In the example above, if we regard the $2n$-sphere $S^{2n}$ as the one-point compactification of the locally compact space $\mathbb{R}^{2n}$, then the spherical $D(2n - 1)$-brane can be thought of as decaying into a bound state of $D(-1)$-branes. There are no conserved charges for higher branes because $\mathbb{R}^{2n}$ is contractible. The precise formulation of this notion of "stability" in K-homology is one of the goals of the present paper.

Type II superstring theory on the spacetime manifold $X$ itself splits into two string theories, called Type IIA and Type IIB. The former contains stable supersymmetric $Dp$-branes for all even $0 \leq p \leq 8$ while the latter contains stable supersymmetric $Dp$-branes for all odd $-1 \leq p \leq 9$. Thus Type IIA D-branes are naturally classified by the group $K_{p}^{1}(X)$ while $K_{0}^{1}(X)$ classifies Type IIB branes. Since $\dim_{\mathbb{R}} X$ is even in the present case, by Poincaré duality this is in complete agreement with the corresponding K-theory classification. We shall now proceed to examine properties of these D-branes by giving more rigorous mathematical calculations in K-homology, beginning with an accurate statement of what is meant by "stable" in the above discussion.

2.3. Stability. In the absence of background form fields, stable supersymmetric D-branes are known to wrap the non-trivial spin\(^{c}\) homology cycles of the spacetime manifold $X$ in which they live. In K-homology, this is asserted by the following fundamental result that will play an important role throughout the rest of this paper.

**Definition 2.1.** Let $X$ be a space and $Y \subset X$. We say that a D-brane $[M, E, \phi] \in K_{p}^{1}(X)$ wraps $Y$ if $\phi(M) \subset Y$.

The idea behind this definition is that the condition above determines a continuous map $\psi : M \to Y$ such that $i \circ \psi = \phi$, where $i : Y \to X$ is the canonical inclusion and $i_{*}([M, E, \psi]) = [M, E, \phi]$, and so we can interpret the initial D-brane as one over $Y$.

On the other hand, the notion of stability is, mathematically, a heuristic one, since we interpret that a D-brane $[M, E, \phi] \in K_{p}^{1}(X)$ is unstable if it is decomposable (non-trivially) in the form

$$[M, E, \phi] = \sum_{p=0}^{n} \sum_{i=1}^{m_{p}} d_{p}(M, E) [M_{i}^{p}, \mathbb{I}_{M_{i}^{p}}, \phi_{i}^{p}],$$

where the branes $[M_{i}^{p}, \mathbb{I}_{M_{i}^{p}}, \phi_{i}^{p}]$ are linearly independent generators of $K_{p}^{1}(X)$. Namely, the stable objects are the possible linearly independent generators.

**Theorem 2.1.** Let $X$ be a compact connected finite CW-complex of dimension $n$ whose rational homology can be presented as

$$H_{p}(X; \mathbb{Q}) = \bigoplus_{p=0}^{n} \bigoplus_{i=1}^{m_{p}} [M_{i}^{p}] \mathbb{Q},$$

where $M_{i}^{p}$ is a $p$-dimensional compact connected spin\(^{c}\) submanifold of $X$ without boundary and with orientation cycle $[M_{i}^{p}]$ given by the spin\(^{c}\) structure. Suppose that the canonical inclusion map $i_{*}^{p} : M_{i}^{p} \hookrightarrow X$ induces, for each $i, p$, a homomorphism $(i_{*}^{p})_{*} : H_{p}(M_{i}^{p}; \mathbb{Q}) \to H_{p}(X; \mathbb{Q}) \cong \mathbb{Q}^{m_{p}}$ with the property

$$\left((i_{*}^{p})_{*} [M_{i}^{p}] = \kappa_{ip} [M_{i}^{p}] \right)$$

for some $\kappa_{ip} \in \mathbb{Q} \setminus \{0\}$. Then the lattice $\Lambda_{K_{p}^{1}(X)} := K_{p}^{1}(X) / \text{tor}_{K_{p}^{1}(X)}$ is generated by the classes of $K$-cycles

$$[M_{i}^{p}, \mathbb{I}_{M_{i}^{p}}, \phi_{i}^{p}], \quad 0 \leq p \leq n, \quad 1 \leq i \leq m_{p}.$$
Proof. Fixing 0 \leq p \leq n, 1 \leq i \leq m_p, let \{x_{ab}^{ip}\}_{bc[a]} be cohomology classes in degree \(a\) generating the rational cohomology group \(H^a(M^p_1; Q) \cong Q^{n^{ip}_a}\) for each \(a = 0, 1, \ldots, p\). Since \(M^p_1\) is oriented and connected, one has \(H^p(M^p_1; Q) \cong Q \cong H^0(M^p_1; Q)\) and hence \(n^{ip}_p = 1 = n^{ip}_0\). Without loss of generality we may assume that \(x_{01}^{ip} = 1\), and we set \(x_{ip} := x_{ip}^{1p}\). The rational cohomology ring of the submanifold \(M^p_1 \subset X\) can thus be presented as

\[
H^p(M^p_1; Q) = 1 \oplus \left( \bigoplus_{a=0}^{p-1} \bigoplus_{b=1}^{n^{ip}_a} x_{ab}^{ip} \right) \oplus x_p^{ip} Q.
\]

In particular, the Todd class \(td(TM^p_1) \in H^{even}(M^p_1; Q)\) may be expressed in the form

\[
\begin{align*}
\text{td}(TM^p_1) &= 1 + \sum_{a=1}^{p-1} \sum_{b=1}^{n^{ip}_a} d^{ip}_{ab} x_{ab}^{ip} + \delta_{c(p),0} x_p^{ip}
\end{align*}
\]

for some \(d^{ip}_{ab} \in Q\) with \(d^{ip}_{ab} = 0\) whenever \(a\) is odd. Let us now use the Chern character \([1.13]\) to compute

\[
\text{ch} \cdot \left( M^p_1, \mathbb{C}^{\mathbb{C}}_M, \iota^{ip}_p \right) = (i^{ip}_p)_* \left( \text{td}(TM^p_1) \cap [M^p_1] \right)
\]

\[
= (i^{ip}_p)_* \left( [M^p_1] + \sum_{a=1}^{p-1} \sum_{b=1}^{n^{ip}_a} d^{ip}_{ab} x_{ab}^{ip} \cap [M^p_1] + r_{ip} [pt] \right)
\]

\[
= \kappa_{ip} [M^p_1] + \sum_{a=1}^{p-1} \sum_{b=1}^{n^{ip}_a} d^{ip}_{ab} \left( i^{ip}_p \right)_* (x_{ab}^{ip} \cap [M^p_1]) + r_{ip} \left( i^{ip}_p \right)_* [pt]
\]

(2.3)

for some \(r_{ip} \in Q\) with \(r_{ip} = 0\) for \(p\) odd. We have used \(\text{ch} \cdot (\mathbb{C}^{\mathbb{C}}_M) = 1\) and \([2.2]\). For each \(1 \leq a < p, 1 \leq b \leq n^{ip}_a\) one has \((i^{ip}_p)_* (x_{ab}^{ip} \cap [M^p_1]) \in H_{p-a}(X; Q)\).

The ordered collection

\[
\vec{c} = \left( [pt]\ [M^1_1]\ldots[M^1_{m_1}] \cdots [M^p_1]\ldots[M^p_{m_p}] \right)
\]

of homology cycles is a basis of \(H^\sharp_+(X; Q)\) as a rational vector space. On the other hand, if we set

\[
\vec{h} = \left( \text{ch} \cdot (\mathbb{C}^{\mathbb{C}}_M, i^{ip}_p) \ldots \text{ch} \cdot (\mathbb{C}^{\mathbb{C}}_{M^1_{m_1}}, i^{ip}_{11}) \ldots \text{ch} \cdot (\mathbb{C}^{\mathbb{C}}_{M^p_{m_p}}, i^{ip}_{p1}) \right) = \left( \text{ch} \cdot (\mathbb{C}^{\mathbb{C}}_M, i^{ip}_p) \ldots \text{ch} \cdot (\mathbb{C}^{\mathbb{C}}_{M^p_{m_p}}, i^{ip}_{p1}) \right)
\]

then from \([2.3]\) it follows that \(\vec{h} = \vec{c} \Omega\), where \(\Omega\) is an upper triangular matrix whose diagonal elements are the non-zero rational numbers \(\kappa_{ip}\). Thus \(\text{det} \Omega \neq 0\) and so the collection \(\vec{h}\) is also a basis of \(H^\sharp_+(X; Q)\) as a rational vector space. Since \(X\) is a finite CW-complex, \(\text{ch} \cdot \otimes \text{id}_Q : K^\sharp_+(X) \otimes Q \to H^\sharp_+(X; Q)\) is an isomorphism and hence \(\left( \text{ch} \cdot \otimes \text{id}_Q \right)^{-1} \vec{h}\) is a set of generators for \(K^\sharp_+(X) \otimes Q\). \(\square\)

Remark 2.1. Some comments are in order concerning the statement of the previous Theorem. The inclusions considered are topological embeddings (which, between manifolds, may turn out to be smooth maps), since \(X\) is not assumed to be a manifold. Since embedding manifolds into other manifolds can be very troublesome, our assumptions are of a purely combinatorial nature, stated by the shape of the rational homology of \(X\) and the behavior of the maps induced in homology by the inclusion maps. One instance in which this is a straightforward result is when \(X\) has a CW-structure given by manifolds satisfying the stated properties whose skeletons have big enough gaps, for instance if \(X^{2k} = X^{2k+1}\), for all \(k\), as is the case for projective spaces other than real projective spaces. \(\diamondsuit\)
Remark 2.2. We do not know if this theorem can be proven by replacing the assumption of a weaker condition. The crucial issue is whether or not there is a non-trivial linear relation $\sum_{p,i} n_{pi} [M^p_i, \Sigma^C_{M^p_i}, t^p_i] = 0$ over $\mathbb{Z}$ among the “lifts” $[M^p_i] \rightarrow [M^p_i, \Sigma^C_{M^p_i}, t^p_i]$ of the non-trivial homology cycles of $X$ to K-homology. If such a relation exists, then the lifts of some non-trivial singular homology classes in $K^i_1(X)$ are 0. This means that some D-brane state is unstable, even though it wraps a non-trivial spin$^c$ homology cycle. It either decays into the closed string vacuum state, or is not completely unstable but decays into other D-branes according to the solutions of the linear equation $\sum_{p,i} n_{pi} [M^p_i, \Sigma^C_{M^p_i}, t^p_i] = 0$ over $\mathbb{Z}$. The same argument as that used in Section 2.4 below shows that such a decay is always into branes wrapped on manifolds of lower dimension than that of the original D-brane. The condition guarantees that this does not occur. We shall analyse this feature from a different perspective in Section 4.1. This analysis illustrates the fact that D-branes need not generally simply correspond to subspaces of spacetime.

Another perspective on the problem of finding the K-homology generators is given by Spin$^c$ bordism. Let $\text{MSpin}^c_2(X)$ be the Spin$^c$ bordism group of a finite CW-complex $X$. This group is the set of equivalence classes, through bordism, of pairs $(M, f)$, where $M$ is a compact Spin$^c$ manifold without boundary and $f : M \to X$ is a continuous map. Disjoint union induces an abelian group structure on $\text{MSpin}^c_2(X)$. This group is $\mathbb{Z}$-graded by the dimension of $M$.

This construction gives a generalized homology theory $\text{MSpin}^c_2(\cdot)$ on the category of finite CW-complexes, which can also be defined in terms of a spectrum $\text{MSpin}^c$. For more details see.

Conner and Floyd (20) constructed an orientation map $\text{MU} \to K^U$ between the complex bordism and the K-theory spectra which was later generalized to a map $\text{MSpin}^c \to K^U$ by Atiyah-Bott-Shapiro (18). Hence, if $X$ is a finite CW-complex, this translates into a homomorphism $\text{MSpin}^c_2(X) \to K^U_1(X)$, given by $[M, f] \mapsto [M, \Sigma^C_M, f]$ which is a natural transformation of homology theories inducing an $\text{MSpin}^c_2(pt)$-module structure on $K^U_1(pt)$. The relevant result for our purposes is the following theorem by Hopkins and Hovey (34).

**Theorem 2.2.** The map $\text{MSpin}^c_2(X) \otimes_{\text{MSpin}^c_2(pt)} K^U_1(pt) \to K^U_1(X)$ induced by the Atiyah-Bott-Shapiro orientation is an isomorphism of $K^U_1(pt)$-modules.

This immediately implies the following result, reducing the problem of calculating the K-homology generators to the analogous problem in Spin$^c$ bordism.

**Theorem 2.3.** Let $X$ be a finite CW-complex. Suppose $[M_i, f_i]$, $1 \leq i \leq m$ are generators of $\text{MSpin}^c_2(X)$ as an $\text{MSpin}^c_2(pt)$-module. Then $[M_i, \Sigma^C_{M_i}, f_i]$, $1 \leq i \leq m$, generate $K^U_1(X)$ as a $K^U_1(pt)$-module.

Namely, for all $n = 0,1$, $K^U_1(X)$ is generated by the elements $[M_i, \Sigma^C_{M_i}, f_i]$, $1 \leq i \leq m$, such that $\dim M_i = n$. In the case where $X$ is also a manifold, we can consider differentiable bordism, namely we can consider bordism of pairs $(M, f)$, where $M$ is a compact Spin$^c$ manifold without boundary and $f : M \to X$ is a differentiable map. We then obtain another abelian group $\text{MSpin}^{c\text{Diff}}_2(X)$, which is canonically isomorphic to $\text{MSpin}^c_2(X)$ (21).

### 2.4. Branes within Branes.

A D$p$-brane also generally has, in addition to its $p$-brane charge, lower-dimensional $q$-brane charges with $q = p - 2, p - 4, \ldots$ which depend on the Chan-Paton bundle over its worldvolume (25). Let $M \subset X$ be a compact connected spin$^c$ manifold without boundary and let $E$ be a complex vector bundle over $M$. Then $[M, E, \iota] \in K^U_1(X)$ where $\iota : M \to X$ is the natural inclusion. Under the assumptions of Theorem 2.1 if the brane is torsion-free...
then it has an expansion in terms of the lifted homology basis for the lattice $\Lambda_{\kappa_f}^{\ast}(X)$ of the form

$$[M, E, \iota] = \sum_{p=0}^{n} \sum_{i=1}^{m_p} d_{pi}(M, E) \left[ M_p^R, \mathbb{I}_{M_p^R}, \iota_p^R \right]$$

with $d_{pi}(M, E) \in \mathbb{Z}$. The crucial point here is that the branes on the right-hand side of (2.4) are of lower dimension than the original brane on the left-hand side, and have even codimension with respect to the worldvolume $M$.

**Lemma 2.1.** If $[M_p^R, \mathbb{I}_{M_p^R}, \iota_p^R] \in \text{im}(\iota_{p}^R)$, for each $0 \leq p \leq n$, $1 \leq i \leq m_p$, then $d_{pi}(M, E) = 0$ for all $p \neq \dim_R(M) - 2j$ with $j = 0, 1, \ldots, \lceil \dim_R(M) \rceil$.

**Proof.** We apply the Chern character (1.13) to both sides of (2.4). Then $\text{ch}$ with respect to the worldvolume $M$ is a sum of homology cycles in $H_\ast(X; \mathbb{Q})$ for $p$ even (resp. odd) of degree at most $p$. Since $[M, E, \iota] = \iota_*[M, E, \text{id}_M]$, the conclusion then follows from the commutative diagram

$$
\begin{array}{ccc}
\text{K}_f(M) & \xrightarrow{\iota_*} & \text{K}_f(X) \\
\downarrow \text{ch}_* & & \downarrow \text{ch}_* \\
\text{H}_\ast(M; \mathbb{Q}) & \xrightarrow{\iota_*} & \text{H}_\ast(X; \mathbb{Q})
\end{array}
$$

Remark 2.3. This relationship represents the possibility of being able to construct stable states of D-branes as bound states in higher-dimensional branes by placing non-trivial Chan-Paton bundles on the higher-dimensional worldvolume. Note that this works even when $E$ is a non-trivial line bundle. In general, it is difficult to determine the lower $(p-1)$-brane charges $d_{pi}(M, E) \in \mathbb{Z}$ in (2.3) explicitly. We will return to this issue in Section 2.1.

2.5. **Polarization.** There is an “opposite” effect to the one just described wherein a Dp-brane can expand or “polarize” into a higher dimensional brane $[51]$. This is the dielectric effect which was mentioned in Section 2.2 and it is intrinsically due to the nonabelian structure groups that higher rank Chan-Paton bundles possess. We will now describe this process in more detail. Let $M \subset X$ be a compact spin$^c$ manifold without boundary. Then $M$ is $\text{K}_f^2$-oriented. Let $F$ be a real $C^\infty$ spin$^c$ vector bundle over $M$ of even rank $2r$ with structure group $O(2r)$ and Thom class $\tau_F$. Then $F \oplus \mathbb{R}_M^2$ is a smooth $O(2r + 1)$ vector bundle which admits a nowhere zero section $\Omega^F : M \rightarrow F \oplus \mathbb{R}_M^2$ given by $\Omega^F(x) := 0_x \oplus 1$ for $x \in M$. The map $\Omega^F$ may be regarded as a section of the sphere bundle defined by (1.3). Then the Poincaré duality isomorphism yields a functorial homomorphism

$$\Omega_{f}^F : K_i^f(M) \longrightarrow K_i^f(\hat{M})$$

given for $i = 0, 1$ and $\xi \in K_i^f(M)$ by

$$\Omega_{f}^F(\xi) := \Phi_{\hat{M}}^{-1} \circ \Omega^F \circ \Phi_M(\xi).$$

This is called the **Gysin homomorphism**, and by using (1.3) it may be represented as the composition $\Omega_{f}^F : K_i^f(M) \xrightarrow{\pi_{M,F}} K_i^f(\mathbb{B}^+(F), S(F)) \xrightarrow{\text{exp}} K_i^f(\hat{M}, \mathbb{B}^-(F)) \xrightarrow{\iota_*} K_i^f(\hat{M}).$

where the first map is the Thom isomorphism of $F$, the second map is excision and the third map is restriction induced by the inclusion $\varsigma : (\hat{M}, \emptyset) \hookrightarrow (\hat{M}, \mathbb{B}^-(F))$.

Consider now the complex vector bundle $H(F) \rightarrow \hat{M}$ defined in (1.4). With $\pi$ the bundle projection (1.2), $\left[H(F)\right]|_{\pi^{-1}(x)} = \left[H(F)\right]|_{\pi^{-1}(x)}$ is the Bott generator of $K_0^\text{B}(\mathbb{B}(F)_x / S(F)_x)$ for
all \( x \in M \). It follows that the K-theory class \([H(F)] \in K^0_t(M)\) is related to the Thom class of \( F \) by \( \xi^* \circ \text{exc}(\tau_F) = [H(F)] \). Since \( \xi^* \) and \( \text{exc} \) are ring isomorphisms, from the explicit expression for the Thom isomorphism \([1.11]\) one has \( \Omega^F[E] = \pi^*[E] \cup [H(F)] = [\pi^*(E) \otimes H(F)] \) for any complex vector bundle \( E \to M \). Thus given a K-cycle \((M, E, \phi)\) on \( X \), we can rewrite vector bundle modification as the equivalence relation

\[
(M, E, \phi) \sim (\hat{M}, \Omega^F[E], \phi \circ \pi).
\]

The charge of the D-brane \([M, E, \phi] \in K^*_X(X)\) is, by definition, the index \([1.11]\). From \(2.5\) we see that we can rewrite it using

\[
\mu^a[M, E, \phi] = \mu^a[\hat{M}, \Omega^F[E], \phi \circ \pi]
\]

to get

\[
\text{Index } \phi_*[\mathcal{D}_E] = \text{ch}^* \Omega^F i \cup \text{td}(T \hat{M})[\hat{M}].
\]

Thus the charge of a polarized \( D_p \)-brane can be expressed entirely in terms of characteristic classes associated with the higher-dimensional spherical \( D(p + 2r) \)-brane into which it has dissolved to form a bound state. A similar formula was noted in a specific context in \([23]\). This is a general feature of bound states of D-branes, and they can always be expressed entirely in terms of quantities intrinsic to the ambient space \( X \) \([19]\), as we will now proceed to show.

2.6. Tachyon Condensation. The constructions we have thus far presented involve stable states of D-branes. We can also consider configurations of branes which are a priori unstable and decay into stable D-branes. This requires us to start considering virtual elements in K-theory, with the stable states being associated to the positive cone of the K-theory group. Using these elements we can construct an explicit set of generators for \( K^*_X(X) \), including its torsion subgroup. The decay mechanism is then represented as a change of basis for the topological K-homology group after we compare these generators with those obtained in Theorem \([2.4]\).

**Proposition 2.1.** Let \( X \) be an \( n \)-dimensional compact connected spin\(^c\) manifold without boundary whose degree 0 reduced topological K-theory group can be presented as the split

\[
\tilde{K}^0_\varepsilon(X) = \left( \bigoplus_{j=1}^m \xi_j \mathbb{Z} \right) \oplus \left( \bigoplus_{i=1}^k \bigoplus_{l=1}^{p_i} \gamma^l \mathbb{Z}_{n_i} \right),
\]

where \( n_i \geq 2, p_i \geq 1 \) and \( \xi_j, \gamma^l \in \tilde{K}^0_\varepsilon(X) \) for each \( 1 \leq j \leq m, 1 \leq i \leq k, 1 \leq l \leq p_i \). Choose representatives for the generators \( \xi_j = [E_j] - [F_j] \) and \( \gamma^l = [G^l] - [H^l] \) in terms of complex vector bundles over \( X \). Then \( \tilde{K}^0_\varepsilon(X) \) is generated as an abelian group by the elements

\[
[X, E_j, \text{id}_X] - [X, F_j, \text{id}_X], \quad 1 \leq j \leq m,
\]

\[
[X, G^l, \text{id}_X] - [X, H^l, \text{id}_X], \quad 1 \leq i \leq k, \quad 1 \leq l \leq p_i.
\]

**Proof.** We explicitly construct the Poincaré duality isomorphism \( \Phi^\varepsilon_X : \tilde{K}^0_\varepsilon(X) \to \tilde{K}^0_\varepsilon(X) \) induced from \([1.12]\) in this case. Recall from Section \([1.7]\) that the K-cycle class \([X, 1_X, \text{id}_X] = \text{fundamental class of X in K}^0_\varepsilon(X) \) and that the isomorphism \( \mu^\varepsilon \) is compatible with cap products. It follows that the map \( \Phi^\varepsilon_X := (\mu^\varepsilon)^{-1} \circ \Phi_X \) is given explicitly by

\[
\Phi^\varepsilon_X(\xi_j) = ([E_j] - [F_j]) \cap [X, 1_X, \text{id}_X]
\]

\[
= [X, 1_X \otimes E_j, \text{id}_X] - [X, 1_X \otimes F_j, \text{id}_X] = [X, E_j, \text{id}_X] - [X, F_j, \text{id}_X].
\]

The conclusion now follows by Poincaré duality. \( \square \)
Remark 2.4. Suppose that $X$ satisfies the conditions of both Theorem 2.1 and Proposition 2.1. Let $\left[ M^p_i, \mathbb{C}_{M^p_i}^e, i^p_1 \right]$ be generators of the lattice $K^{e(n)}(X) = K^{e(n)}(X)/\text{tor}_{K^{e(n)}(X)}$ given by Theorem 2.1 and $\left[ X, E_j, \text{id}_X \right] - \left[ X, F_j, \text{id}_X \right]$ the generators given by Proposition 2.1. Since these are bases of the same free $\mathbb{Z}$-module $K^{e(n)}(X)$, there are uniquely defined integers $a^j_{pi}$ such that

\[(2.6) \quad \left[ X, E_j, \text{id}_X \right] - \left[ X, F_j, \text{id}_X \right] = \sum_{p=1}^{n} \sum_{e(p)=e(n)}^{m_p} a^j_{pi} \left[ M^p_i, \mathbb{C}_{M^p_i}^e, i^p_1 \right]\]

for $1 \leq j \leq m$. In the string theory setting, $X$ is a ten-dimensional spin manifold and represents a change of basis on $K^{e(n)}(X)$. The right-hand side is an expansion in terms of the stable torsion-free Type IIB D-branes which wrap the non-trivial spin homology cycles of $X$ in even degree. The left-hand side is the difference between a pair of spacetime-filling D9-branes wrapping the entire ambient space $X$. The relative sign difference indicates that one of these branes should be regarded as oppositely charged relative to the other, i.e. it is an antibrane. The left-hand side thus represents a brane-antibrane system. It is unstable and describes its decay into lower-dimensional stable D-branes in $X$. Note that any stable D-brane of even degree can be constructed from such 9-brane pairs. We have thereby reproduced the construction of Type IIB stable supersymmetric D-branes from spacetime filling brane-antibrane pairs [53, 57, 64]. As before, however, it is in general quite difficult to explicitly determine the $(p-1)$-brane charges $a^j_{pi} \in \mathbb{Z}$ in (2.6). We remark on the analogous construction in Type IIA string theory in Section 2.7.$\diamond$

Example 2.2 (ABS Construction). Let $X$ be a compact connected ten-dimensional $C^\infty$ spin manifold without boundary. Let $(M, E, \phi)$ be a K-cycle on $X$ such that $\phi : M \rightarrow X$ is a proper imbedding with $M$ connected and of even codimension $2k$ in $X$ (so that the corresponding K-homology class describes a D$p$-brane in $X$ with $p = 9 - 2k$). Choosing a Riemannian metric on $X$, the normal bundle $NM \rightarrow M$ of $M$ in $X$ fits into an exact sequence of real vector bundles as

\[0 \rightarrow TM \xrightarrow{\phi^*} TX \rightarrow NM \rightarrow 0 .\]

Let $w_i(F) \in H^i(M; \mathbb{Z}_2)$ denote the $i$-th Stiefel-Whitney class of a real vector bundle $F \rightarrow M$ with $w_0(F) = 1$. If the metric on $X$ restricts non-degenerately to the worldvolume $M$, then one has the Whitney sum formula

\[\phi^* w_i(TX) = \sum_{j=0}^{i} w_j(TM) \cup w_{i-j}(NM) .\]

Since $X$ and $M$ are orientable, we have $w_1(TM) = w_1(TX) = 0$ and hence $w_1(NM) = 0$. Since $X$ is spin, we also have $w_2(TX) = 0$ and hence $w_2(NM) = w_2(TM)$. Thus endowing $M$ with a spin$^c$ structure is equivalent to endowing its normal bundle with a spin$^c$ structure. It follows that $NM \rightarrow M$ is a real spin$^c$ $C^\infty$ vector bundle with structure group $SO(2k)$. Applying the clutching construction of Section 1.2 to $F = NM$, vector bundle modification then identifies the D-branes

\[(2.7) \quad \left[ \hat{M}, H(NM) \otimes \pi^*(E), \phi \circ \pi \right] = \left[ M, E, \phi \right] .\]

To proceed further we need the following elementary result [18].

Lemma 2.2. Let $W$ be a compact manifold and $Z$ a connected manifold which are both non-empty and have the same dimension. Then any embedding $f : W \rightarrow Z$ is a diffeomorphism.

Proof. Let $V = Z \setminus f(W)$. Since $W$ and $Z$ have the same dimension, $f(W)$ is open in $Z$. On the other hand, since $W$ is compact, $f(W)$ is compact in $Z$ and hence closed. Thus one concludes...
that $V$ is both open and closed in $Z$. Since $Z$ is connected, it follows that either $V = \emptyset$ or $V = Z$. Since $W$ and $Z$ are non-empty, one has $V = \emptyset$. 

Let us apply this lemma to the sphere bundle $\widehat{M}$, which in the present case is a compact ten-dimensional submanifold of $X$. It follows that $\widehat{M} \cong X$ and so the left-hand side of (2.7) represents a configuration of spacetime-filling D9-branes. To see that it is an element of the basis set provided by Proposition 2.4, we appeal to the Atiyah-Bott-Shapiro (ABS) construction in topological K-theory [4]. For this, we use the metric on $X$ to construct a tubular neighbourhood $M'$ of $M$ in $X$. Let $\overline{M}$ denote the closure of $M'$ in $X$ and $M'$ its boundary. The neighbourhood $M'$ may be identified with the total space of the normal bundle $NM \to M$. Without loss of generality, we can identify $M'$ with the interior of the unit ball bundle $B(NM) \setminus S(NM)$ (whose fibres consist of normal vectors with norm $< 1$). Then we have the identifications $\overline{M} = B(NM)$ and $M' = S(NM)$. Let $\rho : M' \to M$ be the retraction of the regular neighbourhood $M'$ onto $M$, and denote the twisted spinor bundles over $\overline{M}$ by $\Delta_E^\pm := \Delta^\pm(NM) \otimes \rho^*(E)$. Set $X' = X \setminus M'$.

Suppose first that the bundle $\Delta_E^+$ admits an extension over $X'$, also denoted $\Delta_E^-$. Via the (extended) Clifford multiplication map $\sigma_E := \sigma \otimes \text{id}_{\rho^*(E)}$, the bundle $\Delta_E^+$ is isomorphic to $\Delta_E^-$ on $M'$, and so it can also be extended over $X'$ by declaring that it be isomorphic to $\Delta_E^-$ over $X'$. This gives a pair of bundles $\Delta_E^+ \to X$ that determine an element $[\Delta_E^+ - \Delta_E^-]_E$ of the reduced K-theory group $K^0_t(X)$ which vanishes on $X'$. On the other hand, from (1.4) we see that this K-theory element is just the Gysin homomorphism

$$[\Delta_E^+] - [\Delta_E^-] = \Omega_t^{NM}[E] = \pi^*[E] \cup [H(NM)] .$$

Using this fact along with the diffeomorphism $\widehat{M} \cong X$, the identification (2.4) becomes

$$[X, \Delta_E^+, \text{id}_X] - [X, \Delta_E^-, \text{id}_X] = \pm [M, E, \phi]$$

(2.8)

where the sign depends on whether or not the spin$^c$ structures on $\widehat{M}$ and $X$ coincide. This is the standard construction of a Type IIB D-brane $[M, E, \phi]$ in terms of spacetime-filling brane-antibrane pairs [53, 64]. In this context the Clifford multiplication map $\sigma_E$ is called the tachyon field and the decay mechanism (2.8) is known as tachyon condensation. If the bundle $\Delta_E$ does not admit an extension over $X'$, we use Swan’s theorem to construct a complex vector bundle $G \to M$ such that $G \oplus (S_1(NM) \otimes E)$ is trivial over $M$, and hence whose pullback $\rho^*(G) \oplus \Delta_E^-$ is trivial over $\overline{M}$. Then $\rho^*(G) \oplus \Delta_E^-$ can be extended over the whole of $X$ as a trivial bundle. The bundle $\rho^*(G) \oplus \Delta_E^+$ is isomorphic to $\rho^*(G) \oplus \Delta_E^-$ on $M'$ under the vector bundle map $\text{id}_{\rho^*(G)} \oplus \rho_E$, and so it can also be extended over $X$ by setting it equal to $\rho^*(G) \oplus \Delta_E^-$ over $X'$. The resulting K-theory class is again trivial over $X'$ (but not over $X$), and by the direct sum relation the Poincaré dual K-homology class coincides with (2.8). 

2.7. Unstable 9-Branes. The crux of the constructions of Section 2.6 is that one can use virtual elements which signal instability of the given configurations of branes. Using Corollary 1.1 one can replace $\Gamma(X)$ with the collection of isomorphism classes of triples $(M, \xi, \phi)$, where $M$ and $\phi$ are as in Definition 1.1 and $\xi \in K^0_\xi(M)$ is a class in the degree 0 topological K-theory of $M$. Clearly both definitions lead to the same group $K^0_\xi(M)$. One can further extend the definition to triples $(M, \xi, \phi)$ with $\xi \in K^0_\xi(M) = K^0_\xi(M) \oplus K^1_\xi(M)$ [35]. The $\mathbb{Z}_2$-grading on $K^1_\xi(X)$ is then defined by taking $K^0_\xi(X)$ (resp. $K^1_\xi(X)$) to be the subgroup given by classes of K-cycles $(M, \xi, \phi)$ such that $\xi|M_i \in K^0_{\xi_i}(M)$ for some $i_0 = 0, 1$ and $\dim_{\mathbb{R}}(M_i) + i_0$ is an even (resp. odd) integer for all connected components $M_i$ of $M$. Vector bundle modification is now generically described as the equivalence relation (2.5) using the Gysin homomorphism associated to a real spin$^c$ vector bundle $F \to M$ whose rank has the same parity as that of the dimension of (the connected components of) $M$. 


To see that this definition is in fact equivalent to our previous one, let \([M, \xi, \phi] \in K^1_t(X)\) with \(i = 0, 1, M\) connected, and \(\xi\) a non-zero element of \(K^i_t(M)\) for some \(j = 0, 1\). If \(m := \dim_k(M)\), then one has \(m + j \equiv i \mod 2\) by definition. Consider the trivial spin\(^c\) bundle \(F = \mathbb{R}^{2i+m+1}_M\) and the associated Gysin map \(\Omega_t^F : K_t^i(M) \to K_t^{j+i+m}(\hat{M})\). Since \(j + i + m \equiv j + m + j + m \mod 2 \equiv 0 \mod 2\), one has \(K_{t}^{j+i+m}(\hat{M}) \cong K_t^0(\hat{M})\). It follows that there are complex vector bundles \(E, H \to M\) with \(\Omega_t^F(\xi) = [E] - [H]\), and by vector bundle modification one has \([M, \xi, \phi] = [\hat{M}, E, \phi \circ \pi] - [\hat{M}, H, \phi \circ \pi]\) in \(K_t^1(X)\). Notice that by using the usual cup product on the K-theory ring \(K_t^i(X)\), the cap product \((\ref{cap})\) may now be alternatively defined by \(\hat{\iota} \cap [M, \xi, \phi] = [M, \phi^*\hat{\iota} \cup \xi, \phi]\) for \(\hat{\iota} \in K_t^i(X)\) and K-cycle classes \([M, \xi, \phi] \in K_t^j(X)\).

Suppose that \(X\) is a compact spin\(^c\) manifold without boundary obeying the conditions of Theorem 2.1. Let \(E\) be a complex vector bundle over \(X\) and \(\alpha\) an automorphism of \(E\). This defines a degree 1 K-theory class \([E, \alpha] \in K_t^1(X)\) which we assume to be torsion-free. Applying the Poincaré duality isomorphism as before then gives the analog of the expansion (2.6) as

\[
[X, (E, \alpha), \mathrm{id}_X] = \sum_{\epsilon(p) = \epsilon(n+1)} \sum_{i=1}^{n} m_p b_{pi}(E, \alpha) \left[ M^p_i, \mathbb{I}^{\epsilon(t)}_{M^p_i}, \iota^p_{M^p_i} \right]
\]

in \(\Lambda_{K_t^1(X)}\). In the string theory setting, this is a relation in \(\Lambda_{K_t^1(X)}\) expressing the decay of an unstable D9-brane into stable Type IIA D-branes \([36, 53]\). \(E \to X\) is the Chan-Paton bundle on the 9-brane and now the automorphism \(\alpha : E \to E\) plays the role of the tachyon field. As an explicit example of the decay mechanism \((\ref{vb_def})\), we may construct the Type IIA version of the ABS construction of Example 2.2. Now we consider a K-cycle \((M, E, \phi)\) on \(X\) with \(M\) of odd codimension \(2k + 1\) in \(X\), so that the corresponding normal bundle \(NM \to M\) is an SO(2k + 1) vector bundle. By Lemma 2.2 one again has a diffeomorphism \(\hat{M} \cong X\). Define \(\Delta_E := \Delta(NM) \otimes \rho^*(E)\), where \(\Delta(NM) = S(NM)\) is the pull-back of the unique irreducible spinor bundle \(S(NM)\) over \(NM\), and assume that it admits an extension over \(X\). Then the Gysin homomorphism gives \(\Omega[E] = [\Delta_E, \exp \sigma_E]\) in \(K_t^1(X)\), and so by vector bundle modification one has the identification

\[
[X, (\Delta_E, \exp \sigma_E), \mathrm{id}_X] = \pm [M, E, \phi].
\]

This is the standard construction of a Type IIA D-brane \([M, E, \phi] \in K_t^1(X)\) in terms of unstable spacetime-filling 9-branes \([36, 53]\).

Remark 2.5. It is important to realize that one can stick to our original definition and thus avoid \(K_t^1\)-classes entirely. One of the great advantages of the geometric formulation of K-homology, in contrast to other homology theories, is that it is naturally defined in terms of stable objects and one need never consider virtual elements. While brane-antibrane systems are straightforward to construct, the unstable D-branes defined by virtual K-theory classes in degree 1 are not so natural in this framework. This reflects the difficulties encountered in the description of these D-brane states directly in string theory.

To illustrate this point further, let \(X\) be as in Proposition 2.1 and consider the Gysin homomorphism in K-theory \(\Omega_t^{F_n} : K_t^1(X) \to K_t^{e(i+n)}(\hat{X})\), where \(F_n = X \times \mathbb{R}^n\). Then \(\hat{X} = X \times S^n\) and the Gysin homomorphism becomes a map

\[
\Omega_t^{F_n} : K_t^1(X) \to K_t^{e(i+n)}(X \times S^n).
\]

The sphere bundle projection \(\pi : X \times S^n \to X\) in this case is the projection onto the first factor. Since \(\pi_t \circ \Omega_t^{F_n} = \mathrm{id}_{K_t^1(X)}\), it follows that \(\Omega_t^{F_n}\) is a monomorphism and \(\pi_t\) is an epimorphism. By definition one has \(\hat{\Phi}_X = (\Phi_{X_\times S^n})^{-1} \circ \hat{\Phi}_n \circ \Phi_X^t\) and \(\pi_t = (\Phi_X^t)^{-1} \circ \pi_X \circ \Phi_X^t\). Because the
Poincaré duality maps are isomorphisms, one concludes that the induced maps $\Omega^F_n$ and $\pi_*$ in K-homology are also a monomorphism and an epimorphism, respectively.

Assume that the degree 1 topological K-theory group of $X$ admits a split

$$K^1_t(X) = \left( \bigoplus_{j=1}^m \xi_j \mathbb{Z} \right) \oplus \left( \bigoplus_{i=1}^k \bigoplus_{l=1}^{p_i} \gamma^j_l \mathbb{Z}_{n_i} \right).$$

There is a commutative diagram

$$\begin{array}{ccc}
K^1_t(X) & \xrightarrow{\Omega^F_1} & K^0_t(X \times \mathbb{S}^1) \\
\Phi_X \downarrow & & \downarrow \Phi_{X \times \mathbb{S}^1} \\
K^1_t(e^{-1})(X) & \xrightarrow{\Omega^F_1} & K^1_t(e^{-1})(X \times \mathbb{S}^1).
\end{array}$$

Let $\Omega^F_1(\xi_j) = [E_j] - [F_j]$ and $\Omega^F_1(\gamma^j_l) = [G^j_l] - [H^j_l]$ in terms of complex vector bundles over $X \times \mathbb{S}^1$. Then for all $1 \leq j \leq m$ one has

$$[X \times \mathbb{S}^1, E_j, \pi] - [X \times \mathbb{S}^1, F_j, \pi] = \pi_* \left( [X \times \mathbb{S}^1, E_j, \text{id}_{X \times \mathbb{S}^1}] - [X \times \mathbb{S}^1, F_j, \text{id}_{X \times \mathbb{S}^1}] \right)$$

for some $\tau_j \in K^0_1(X)$. Since $\pi_* \circ \Omega^F_* = \text{id}_{K^1_1(X)}$, by Poincaré duality it follows that

$$[X \times \mathbb{S}^1, E_j, \pi] - [X \times \mathbb{S}^1, F_j, \pi], \quad 1 \leq j \leq m$$

is a set of generators for the torsion-free part of the K-homology group $K^1_t(e^{-1})(X)$. One can use the same procedure for the other set of generators of $K^1_1(X)$ to conclude that

$$[X \times \mathbb{S}^1, G^i_l, \pi] - [X \times \mathbb{S}^1, H^i_l, \pi], \quad 1 \leq i \leq k, \quad 1 \leq l \leq p_i$$

is a set of generators for the torsion subgroup of $K^1_t(e^{-1})(X)$. In the string theory setting, this gives an “M-theory” realization of unstable Type IIA 9-branes in terms of brane-antibrane systems on an 11-dimensional extension $X \times \mathbb{S}^1$ of the spacetime manifold $X$ \cite{33,34}. More generally, one can start from any real spin$^c$ line bundle $F \to X$ and describe these unstable D-brane states in terms of a spin$^c$ circle bundle over $X$.

3. TORSION-FREE D-BRANES

In this section we will describe some elementary applications of the formalism of the previous section. While for the most part we will arrive at the anticipated results, this simple analysis will illustrate how the known properties of D-branes arise within a mathematically precise formalism. We will only look at examples of torsion-free K-homology groups, deferring the analysis of torsion D-branes to the next section.

3.1. SPHERICAL D-BRANES. An important role will be played by the D-branes which wrap images of $n$-dimensional spheres $S^n$ in $X$. We will first consider the case where $X$ is an arbitrary finite CW-complex. Let $n \geq 0$ and let $E$ be a complex vector bundle over $S^n$. Using Lemma \ref{lemma1} we can construct a homomorphism

$$\gamma_{n,E} : [S^n, X] \to K^1_1(X)$$

given by

$$\gamma_{n,E}[\psi] := [S^n, E, \psi].$$

We can also construct a homomorphism

$$h_{n,E} : \pi_n(X) \to K^1_1(X)$$
given by
\[ h_{n,E}[f] := [S^n, E, f]. \]

The subgroup of \( K^n_1(X) \) generated by the K-cycle classes of the form \([S^n, E, \psi]\) for \( n \geq 1 \) and \([pt, F, \phi]\) is denoted \( S^n_1(X) \). It has a natural \( \mathbb{Z}_2 \)-grading \( S^n_2(X) = S^n_0(X) \oplus S^n_1(X) \) by the parity \( e(n) \) of the sphere dimensions \( n \).

**Proposition 3.1.** Let \( n \geq 0 \) and let \( E \) be a complex vector bundle over \( S^n \).

(a) If \( X \) is path connected and simply connected, then \( \text{im} \gamma_{n,E} = \text{im} h_{n,E} \) in \( K^n_1(X) \).

(b) If \( E = \mathbb{I}_{S^n} \), then \( h_n := h_{n,\mathbb{I}_{S^n}} \) is the Hurewicz homomorphism in K-homology.

**Proof.** (a) follows immediately from the fact that \([S^n, X] = \pi_n(X)\) in this case \[22\]. (b) follows from the fact that \([S^n, E, f] = f_*[S^n, \mathbb{I}_{S^n}, \text{id}_{S^n}]\) with \([S^n, \mathbb{I}_{S^n}, \text{id}_{S^n}]\) the fundamental class of \( S^n \) in \( K^n_1(S^n) \). \( \square \)

**Remark 3.1.** Let \( f : S^1 \to S^1 \) be a continuous map with \( f(s_0) = x_0 \neq s_0 \). Regarding \( S^1 \subset \mathbb{C} \), let \( \theta \in (0, 1) \) be defined by \( x_0 = e^{2\pi i \theta} s_0 \). Define \( f_0 : S^1 \to S^1 \) by \( f_0(z) := e^{-2\pi i \theta} f(z) \). Then \( f_0 \) is continuous with \( f_0(s_0) = s_0 \), and so it is a based map at \( s_0 \). The map \( H : [0, 1] \times S^1 \to S^1 \) defined by \( H(t, z) := e^{-2\pi i t \theta} f(z) \) is a homotopy between \( f \) and \( f_0 \). Since homotopy of maps is an equivalence relation, we conclude that the map \( G : [S^1, S^1] \to \pi_1(S^1) \) given by the assignment \([f] \mapsto [f_0]\) is well-defined and a bijection. Thus we may consider \( \gamma_{1,E} : [S^1, X] \to K^1_1(X) \) as a homomorphism \( \gamma_{1,E} : \pi_1(X) \to K^1_1(X) \) for any topological space \( X \).

We shall now specialize to the case \( X = S^n \), whose cellular structure consists of a single \( k \)-cell in dimensions \( k = 0, n \). We will find generators for the subgroup of \( K^1_1(S^n) \) generated by the lower dimensional spheres \( S^k \) with \( 1 \leq k \leq n \) and \( pt \).

**Proposition 3.2.** \( K^1_1(S^0) = S^1_1(S^0) \cong \mathbb{Z} \oplus \mathbb{Z} \) with \( S^0_0(S^0) \cong \mathbb{Z} \) and \( S^1_1(S^0) \cong 0 \).

**Proof.** By definition of the group \( S^1_1(pt) \) it follows immediately that \( S^1_1(pt) \) is the free abelian group generated by the trivial K-cycle class \([0, 0, 0]\), i.e. \( S^1_1(pt) = 0 \). A complex vector bundle \( E \) over \( pt \) is just a finite dimensional vector space, i.e. there exists an integer \( m > 0 \) such that \( E = pt \times \mathbb{C}^m = \bigoplus_{i=1}^m \mathbb{I}_{pt}^i \). Since the unique map \( \phi : pt \to pt \) is the identity, it follows that \([pt, E, \phi] = m [pt, \mathbb{I}_{pt}^0, \text{id}_{pt}] \). Thus \( S^0_0(pt) \cong [pt, \mathbb{I}_{pt}^0, \text{id}_{pt}] \mathbb{Z} \cong \mathbb{Z} \) and the conclusion now follows from Lemma 1.2. \( \square \)

As a consequence of Lemma 1.3, Theorem 2.1 and Proposition 3.2 we have the following result.

**Lemma 3.1.** If \( n \geq 1 \) then \( S^n_2(S^n) \) is generated by classes of the form \([S^k, E, \phi]\) and \([pt, \mathbb{I}_{pt}^0, i]\) where \( 1 \leq k \leq n \), \( E \) is a generating vector bundle for the K-theory of \( S^k \), and \( i : pt \to S^n \) is the inclusion of a point.

Recall that the (complex) K-theory of the spheres is given by
\[
K^0_1(S^n) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases}, \quad K^1_1(S^n) \cong \begin{cases} 0 & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases}.
\]

The trivial line bundle \( \mathbb{I}_{S^n} = S^n \times \mathbb{C} \) is always a degree 0 generator, given by the monomorphism \( K^0_1(pt) \to K^0_1(S^n) \) induced by the inclusion of a point. The non-trivial generator of \( K^0_1(S^2) \) is obtained from the homeomorphism \( \mathbb{C}P^1 \cong S^2 \) by taking the class of the canonical line bundle \( \mathcal{L}_1 \) over the complex projective line \( \mathbb{C}P^1 \). The non-trivial generator \([\mathcal{L}_1]^p\) of \( K^0_1(S^{2p}) \), \( p \in \mathbb{N} \) is
obtained from $[\mathcal{L}_1]$ by using the K-theory cup product. The $K^t_i$-groups are obtained through suspension $S^{n+1} \cong \Sigma S^n = S^n \wedge S^1$. By Poincaré duality one concludes that

$$
K_0(S^n) \cong K^e(n)(S^n) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z}, & n \text{ even} \\
\mathbb{Z}, & n \text{ odd}
\end{cases}
$$

$$
K_1(S^n) \cong K^e(n-1)(S^n) \cong \begin{cases} 
0, & n \text{ even} \\
\mathbb{Z}, & n \text{ odd}
\end{cases}
$$

for $n \geq 1$, and hence

$$K^t_i(S^n) \cong \mathbb{Z} \oplus \mathbb{Z}.$$ 

**Proposition 3.3.** Let $n \geq 1$.

(a) $K^t_i(S^n)$ is generated by the classes $[pt, \mathbb{C}^n_{pt}, i]$ and $[S^n, \mathbb{C}^n_{S^n}, id_{S^n}]$.

(b) $S^t_i(S^n) = K^t_i(S^n)$ as $\mathbb{Z}_2$-graded abelian groups.

**Proof.** (a) follows from calculating the Chern characters of the classes. Since $ch^*(\mathbb{C}^n_X) = 1$ for any space $X$ and $td(Tpt) = 1$, it follows that $ch^*(pt, \mathbb{C}^n_{pt}, i) = \iota_\ast (ch^*(\mathbb{C}^n_{pt}) \cup td(Tpt) \cap [pt]) = \iota_\ast [pt] = 1$. We also have $TS^n \oplus \mathbb{C}^n_{S^n} = \mathbb{C}^{n+1}_{S^n}$, so that $1 = td(\mathbb{C}^{n+1}_{S^n}) = td(TS^n) \cup td(\mathbb{C}^n_{S^n}) = td(\mathbb{C}^n_{S^n})$ and $ch^*(S^n, \mathbb{C}^n_{S^n}, id_{S^n}) = ch^*(\mathbb{C}^n_{S^n}) \cup td(TS^n) \cap [S^n = [S^n]$. Thus $ch^*$ maps the pertinent classes to distinct non-torsion elements of $H^2(S^n; \mathbb{Z})$, and the conclusion follows by Poincaré duality. (b) follows from the fact that the generators of $K^t_i(S^n)$ are in $S^t_i(S^n)$.

**Remark 3.2.** Proposition 3.3(a) is just a special case of Theorem 2.1 and it allows us to conclude, without the assumption of Poincaré duality, that the torsion-free part of $K^t_i(S^n)$ is generated by the said classes. This is also true of the equality $K^t_i(pt) = S^t_i(pt)$ considered in Proposition 3.2.

**Corollary 3.1.** Let $n \geq 0$.

(a) $S^t_0(S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by the classes $[pt, \mathbb{C}^n_{pt}, i]$ and $[S^n, \mathbb{C}^n_{S^n}, id_{S^n}]$, while $S^t_1(S^n) \cong 0$.

(b) The Hurewicz homomorphism in reduced K-homology $h_n : \pi_n(S^n) \to K^t_1(S^n)$ is a bijection.

**Proof.** (a) follows immediately along the lines of the proof of Proposition 3.3 since $ch^*$ is an isomorphism in this case. (b) follows from the fact that $h_n = ch^{-1}_\ast \circ \rho_n$, where $\rho_n : \pi_n(S^n) \xrightarrow{\approx} H_n(S^n; \mathbb{Z})$ is the Hurewicz isomorphism.

**Remark 3.3.** From the discussion of Section 2.2 we see that the classes $[S^{2k}, \mathbb{C}^n_{S^{2k}}, \iota_{S^{2k}}]$, with $2k < n$ and $\iota_{S^{2k}} : S^{2k} \hookrightarrow S^n$ the inclusion, are all identified in $K^t_i(S^n)$ through vector bundle modification. Looking at the proof of Proposition 3.3 one has $[S^n, \mathbb{C}^n_{S^n}, id_{S^n}] = [S^n, \mathbb{C}^n_{S^n}, \psi_n]$ where $\psi_n : S^n \to S^n$ is a map of winding number $n$. In particular, by Corollary 3.1(b) a K-cycle class $[S^n, \mathbb{C}^n_{S^n}, \phi] \in K^t_1(S^n)$ depends only on the degree of the map $\phi$. Finally, from (1.14) the charge of the D-brane $[S^n, \mathbb{C}^n_{S^n}, id_{S^n}]$ is $ch^*(\mathbb{C}^n_{S^n}) \cup td(TS^n)[S^n] = 1$, and similarly for the “vacuum” D-brane $[pt, \mathbb{C}^1_{pt}, i]$. Thus the mathematical analysis above reproduces the well-known physical property that D-branes in flat space carry no lower-dimensional D-brane charges and thus have a simple additive charge.

### 3.2. T-Duality

Using the reduced version of the exterior product of Section 1.6 and the consequent Künneth theorem, we can investigate the relationship between the groups $\tilde{K}^t_0(S^n X)$ and $K^t_{e(n)}(X)$, where $S^n X = S^n \wedge X$ is the $n$-th reduced suspension of the topological space $X$. By Bott periodicity and induction one immediately concludes that $\tilde{K}^t_0(S^{2n} X) \cong \tilde{K}^t_0(X) = \tilde{K}^t_{e(2n)}(X)$. On the other hand, a simple application of the Künneth theorem in its reduced
version yields \( \tilde{K}_0^1(\Sigma^1 X) \cong \tilde{K}_0^1(X) \otimes \tilde{K}_1^1(S^1) \cong \tilde{K}_1^1(X) \), and by induction we conclude that \( \tilde{K}_0^t(\Sigma^{2n+1} X) \cong \tilde{K}_1^t(X) = \tilde{K}_{e(2n+1)}^t(X) \).

Let us now consider the group \( K_0^t(X \times S^1) \). The Künneth theorem gives \( K_0^t(X \times S^1) \cong K_0^1(X) \oplus K_1^1(X) \), and therefore

\[
\tilde{K}_0^t(X \times S^1) \cong \tilde{K}_0^1(X) \oplus \tilde{K}_1^1(X).
\]

The inclusion \( \iota : X \cong X \times pt \hookrightarrow X \times S^1 \) induces a homomorphism \( \iota_* : \tilde{K}_0^1(X) \to \tilde{K}_0^1(X \times S^1) \). From the decomposition \( (3.1) \) it follows that \( \iota_*(\alpha) = \alpha \otimes 0 \) for all \( \alpha \in \tilde{K}_0^1(X) \), and hence \( \text{im} \ i_* = \tilde{K}_0^1(X) \) and coker \( \iota_* = \tilde{K}_0^1(X \times S^1)/\tilde{K}_0^1(X) \cong \tilde{K}_1^1(X) \) where we have identified \( \tilde{K}_1^1(X) \) (resp. \( \tilde{K}_0^1(X) \)) with the subgroup of \( \tilde{K}_0^1(X \times S^1) \) consisting of K-cycle classes \([M, E, \phi]\) such that up to homotopy \( \phi(M) \not\subseteq X \times pt \) (resp. \( \phi(M) \subseteq X \times pt \)). This construction can be used to provide an alternative “M-theory” definition of the unstable 9-branes in Type IIA superstring theory introduced in Section 2.7 which does not require virtual K-theory elements. For \( X \) a ten-dimensional compact spin manifold without boundary, they are identified with the classes \([X, E, \phi]\) on the 11-dimensional space \( X \times S^1 \) for which \( E \not\cong E_X^C \) and \( \phi(X) \not\subseteq X \times pt \). This is consistent with the construction presented in Remark 2.4.

Another application of these simple observations is to the description of T-duality in topological K-homology. Let \( K_0^t(\mathbb{T}^n) \cong (S^1)^n \) be an \( n \)-dimensional torus. By the Künneth theorem one has \( K_1^1(\mathbb{T}^n) \cong K_1^1(S^1) \otimes \mathbb{Z}^{2n-1} \cong \mathbb{Z}^{2n-1} \) for \( i = 0, 1 \). Generalizing the computation of \( (3.1) \) thus gives the isomorphisms

\[
K_0^t(Q \times \mathbb{T}^n) \cong (\tilde{K}_0^1(Q) \oplus K_1^1(Q) \oplus \mathbb{Z})^{2n-1},
\]

\[
K_1^1(Q \times \mathbb{T}^n) \cong (\tilde{K}_1^1(Q) \oplus \tilde{K}_0^1(Q) \oplus \mathbb{Z})^{2n-1},
\]

and therefore

\[
K_1^1(Q \times \mathbb{T}^n) \cong K_1^1(Q \times \mathbb{T}^n).
\]

This isomorphism describes a relationship between Type IIB and Type IIA D-branes on the spacetime \( X = Q \times \mathbb{T}^n \) called T-duality. From the identifications above we see that the isomorphism exchanges wrapped D-branes \([M, E, \phi]\) (having \( \phi(M) \not\subseteq Q \times pt \)) with unwrapped D-branes (having \( \phi(M) \subseteq Q \times pt \)). The powers of \( 2^{n-1} \) give the expected multiplicity of Dp-brane charges arising from wrapping all higher stable D-branes on various cycles of the torus \( \mathbb{T}^n \).

A more geometrical derivation of the T-duality isomorphism \( (3.2) \) may be given as follows. Let \( \Lambda \cong \mathbb{Z}^n \) be a lattice of rank \( n \) in a real vector space \( V \) of dimension \( n \), and let \( \Lambda^\perp \subseteq V^\perp \) be the dual lattice. Consider the real torus \( \mathbb{T}^n = V/\Lambda \) and the corresponding dual torus \( \mathbb{T}^n = V^\perp/\Lambda^\perp \). The lattices \( \Lambda \) and \( \Lambda^\perp \) may then be identified with the first homology lattices \( H_1(\mathbb{T}^n; \mathbb{Z}) \) and \( H_1(\mathbb{T}^n; \mathbb{Z}) \), while the first homology lattice of \( \mathbb{T}^n \times \mathbb{T}^n \) coincides with \( \Lambda \otimes \Lambda^\perp \). There is a unique line bundle \( \mathcal{P} \) over the product space \( \mathbb{T}^n \times \mathbb{T}^n \), called the Poincaré line bundle, such that for any point \( t \in \mathbb{T}^n \) the restriction \( \mathcal{P}_t = \mathcal{P}|_{\mathbb{T}^n \times \{t\}} \) represents an element of the Picard group of \( \mathbb{T}^n \) corresponding to \( t \), and such that the restriction \( \mathcal{P}|_{\{0\} \times \mathbb{T}^n} \) is the trivial complex line bundle over \( \mathbb{T}^n \). This bundle defines a class in \( K_0^0(\mathbb{T}^n \times \mathbb{T}^n) \) which is a K-theory cup product of odd degree generators for the K-theory of the tori \( \mathbb{T}^n \) and \( \mathbb{T}^n \).

Consider now the projections

\[
\begin{array}{ccc}
Q \times \mathbb{T}^n & \xrightarrow{p} & Q \times \mathbb{T}^n \\
\downarrow & & \downarrow \bar{p} \\
Q \times \mathbb{T}^n & \rightarrow & Q \times \mathbb{T}^n.
\end{array}
\]
The T-duality isomorphism in topological K-theory is given by

\[ T_{i} : K_{i}^{t}(Q \times \mathbb{T}^{n}) \cong K_{i}^{t(i+n)}(Q \times \bar{\mathbb{T}}^{n}) \]

is given for \( \xi \in K_{i}^{t}(Q \times \mathbb{T}^{n}) \) and \( i = 0, 1 \) by

\[ T_{i}(\xi) = \tilde{p}_{n}(p^{*}\xi \otimes \mathcal{P}) , \]

where \( \tilde{p}_{n} : K_{i}^{t}(Q \times \mathbb{T}^{n} \times \mathbb{T}^{n}) \rightarrow K_{i}^{t(i+n)}(Q \times \mathbb{T}^{n}) \) is the push-forward map in K-theory which is given by the topological index. Since we assume that the spacetimes \( X = Q \times \mathbb{T}^{n} \) and \( \bar{X} = Q \times \bar{\mathbb{T}}^{n} \) are spin (equivalently \( Q \) is spin), they are \( K_{t}^{2} \)-oriented and thus obey Poincaré duality. The K-homology of \( Q \times \mathbb{T}^{n} \) thereby has a set of generators given by \( [Q \times \mathbb{T}^{n}, \xi, \text{id}_{Q \times \mathbb{T}^{n}}] \) where \( \xi \in K_{2}^{t}(Q \times \mathbb{T}^{n}) \) is a generator, and similarly for \( Q \times \bar{\mathbb{T}}^{n} \). It follows that the map

\[ T_{i} : K_{i}^{t}(Q \times \mathbb{T}^{n}) \rightarrow K_{i}^{t(i+n)}(Q \times \mathbb{T}^{n}) \]

given by

\[ T_{i}[Q \times \mathbb{T}^{n}, \xi, \text{id}_{Q \times \mathbb{T}^{n}}] = [Q \times \mathbb{T}^{n}, \tilde{p}_{n}(p^{*}\xi \otimes \mathcal{P}), \text{id}_{Q \times \mathbb{T}^{n}}] \]

is a well-defined group homomorphism. Since \( T_{i} = \Phi^{t}_{Q \times \mathbb{T}^{n}} \circ T_{i}(\Phi^{t}_{Q \times \mathbb{T}^{n}})^{-1} \), it is an isomorphism. This isomorphism is the T-duality isomorphism in topological K-homology. While this map is defined in terms of virtual K-theory elements, one can straightforwardly obtain a picture with only stable isomorphism classes appearing by applying vector bundle modification along the lines explained in Section 2.7. If \( n \) is even, the T-duality isomorphism maps a spacetime-filling brane-antibrane pair on \( X \) to a spacetime-filling brane-antibrane pair on the 11-dimensional “M-theory” extension \( \bar{X} \times S^{1} \), as spelled out by the construction of Remark 2.5. In particular, this description can be used to provide a more general construction of T-duality in the case of a spin\(^c\) torus bundle over \( Q \). The construction also thereby provides a topological K-homology realization of brane descent relations among D-branes.

### 3.3. Projective D-Branes.

The simplest example of flat D-branes in a curved background is provided by the complex projective spaces \( \mathbb{C}P^{n} \) of real dimension \( 2n \). Being complex manifolds they are automatically spin\(^c\). The cellular structure in this case may be described by the stratification of \( \mathbb{C}P^{n} \) into linearly embedded subspaces as

\[ \mathbb{C}P^{0} \subset \mathbb{C}P^{1} \subset \cdots \subset \mathbb{C}P^{n-1} \subset \mathbb{C}P^{n} \]

where \( \mathbb{C}P^{0} = \text{pt} \). Since \( \mathbb{C}P^{n} \) satisfies the hypotheses of Theorem 2.1, a set of generators for its reduced topological K-homology group \( K_{2}(\mathbb{C}P^{n}) \) is given by

\[ [\mathbb{C}P^{k}, \mathbb{C}P^{k}, \iota_{k}] , \quad 1 \leq k \leq n \]

where \( \iota_{k} : \mathbb{C}P^{k} \hookrightarrow \mathbb{C}P^{n} \) is the canonical inclusion. On the other hand, let \( \mathcal{L}_{n} \) denote the canonical line bundle over \( \mathbb{C}P^{n} \) and \( \mathcal{L}_{n}^{\vee} \) its dual line bundle. The reduced K-theory of \( \mathbb{C}P^{n} \) is then given by \( \overline{K}_{2}^{t}(\mathbb{C}P^{n}) = \overline{K}_{0}^{t}(\mathbb{C}P^{n}) = \bigoplus_{i=1}^{n} (\mathbb{I}_{\mathbb{C}P^{n}} - [\mathcal{L}_{n}^{\vee}])^{i} \mathbb{Z} \). From Proposition 2.3 it follows that the K-cycle classes \( (\mathbb{I}_{\mathbb{C}P^{n}} - [\mathcal{L}_{n}^{\vee}])^{i} \cap [\mathbb{C}P^{n}, \mathbb{I}_{\mathbb{C}P^{n}}, \text{id}_{\mathbb{C}P^{n}}] \) describe spacetime-filling D-branes on complex projective space and we arrive at the following result.

**Proposition 3.4.** For \( n \geq 1 \), \( \overline{K}_{2}^{t}(\mathbb{C}P^{n}) = \overline{K}_{0}^{t}(\mathbb{C}P^{n}) \cong \mathbb{Z}^{\oplus n} \) is the free abelian group with generators

\[ \sum_{k=0}^{2} \left[ \mathbb{C}P^{n}, \bigoplus_{l=0}^{i} (\mathcal{L}_{n}^{\vee})^{\otimes (i-k)}, \text{id}_{\mathbb{C}P^{n}} \right] - \sum_{k=0}^{2} \left[ \mathbb{C}P^{n}, \bigoplus_{l=0}^{i} (\mathcal{L}_{n})^{\otimes (i-k)}, \text{id}_{\mathbb{C}P^{n}} \right] , \quad 1 \leq i \leq n \]
Remark 3.4. The decay of the brane-antibrane system provided by Proposition 3.4 into the stable D-branes described by the K-cycle classes is rather intricate to describe. Since $T\mathbb{C}P^n \oplus \mathbb{H}^C_{\mathbb{C}P^n} \cong (\mathcal{L}_n)^{(n+1)}$, one has $\text{td}(T\mathbb{C}P^n) = f(-c_1(\mathcal{L}_n))^{n+1}$ where $c_1(\mathcal{L}_n)$ is the first Chern class of the canonical line bundle $\mathcal{L}_n \to \mathbb{C}P^n$ and

$$f(x) = 1 + \frac{x}{2} + \sum_{k \geq 1} (-1)^{k-1} \frac{B_k}{2k!} x^{2k}$$

with $B_k \in \mathbb{Q}$ the $k$-th Bernoulli number. This fact may be used to attempt to find the change of basis map between these two sets of generators of $\tilde{K}^1_\sharp(\mathbb{C}P^n)$ via the Chern character and the homology of $\mathbb{C}P^n$. However, both the Chern character and the Todd class lead directly to a strictly rational-valued change of basis matrix. The obstruction to this explicit procedure is encoded in whether or not the Chern character admits an integral lift, i.e. an extension of the usual map into rational homology to a map into integer homology. With the exception of the projective line $\mathbb{C}P^1 \cong \mathbb{S}^2$, one easily checks that such a lift is not possible on complex projective spaces. The problem of integral lifts is discussed more thoroughly in the next Section.\[\Box\]

Remark 3.5. These results generalize straightforwardly to all three spin$^c$ projective spaces $\mathbb{K}P^n$, where $\mathbb{K}$ is one of the three division algebras generated by the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ or the Cayley numbers $\mathbb{O}$. Let $r := \dim_\mathbb{R}(\mathbb{K})$, so that $\mathbb{K}P^n \cong \mathbb{S}^r$. Then the only non-trivial reduced homology groups of $\mathbb{K}P^n$ are $\tilde{H}_{r+k}(\mathbb{K}P^n; \mathbb{Z}) \cong \mathbb{Z}$, $1 \leq k \leq n$. The cellular structure is determined by a stratification of $\mathbb{K}P^n$ into $(r+k)$-cells for $k = 0, 1, \ldots, n$ described by linearly embedded subspaces, analogously to (3.3). The K-theory ring is generated by $\mathbb{K}^0_{\mathbb{K}P^n} = [\mathbb{C}\mathbb{P}^n] - [\mathcal{L}_{\mathbb{K}P^n}]$, with $\mathcal{L}_{\mathbb{K}P^n} \to \mathbb{K}P^n$ the canonical complex line bundle. In all three instances one arrives at the isomorphism $\tilde{K}^1_\sharp(\mathbb{K}P^n) \cong \tilde{H}^r(\mathbb{K}P^n; \mathbb{Z})$, with generators constructed as above. The equivalence between K-homology and singular homology in these cases can be understood through the appropriate spectral sequence and the sparseness of the cellular structure of $\mathbb{K}P^n$. Spectral sequences will play an important role in the investigations of subsequent sections. The real projective spaces $\mathbb{R}P^n$, based on the algebraically open field $\mathbb{K} = \mathbb{R}$, have a more intricate K-homology and will be treated in the next section.\[\Box\]

4. Torsion D-Branes

Let us now turn to the somewhat more interesting situation in which D-branes are described by K-cycles which generally produce torsion elements in K-homology. In these cases one can encounter K-homology groups which do not coincide with the corresponding integral homology groups, and here the K-homology classification makes some genuine predictions that cannot be detected by ordinary homological methods. We will first consider the general problem of finding explicit homology cycles in the spacetime which are wrapped by D-branes. This analysis extends that of Section 2.3 to examine general circumstances under which a spin$^c$ homology cycle has a non-trivial lift to K-homology and hence is wrapped by a stable D-brane. Then we turn to a number of explicit examples illustrating how the K-cycle representatives of torsion charges are constructed in practice.

4.1. Stability. Since $\tilde{K}_\sharp^1$ is a homology theory defined by means of a ring spectrum, it satisfies the wedge axiom [58]. One consequence of this fact is that we can immediately obtain the groups $\tilde{K}_\sharp^1(\bigvee_\alpha \mathcal{S}_\alpha^n)$ using the K-homology of the spheres, since then $\tilde{K}_\sharp^1(\bigvee_\alpha \mathcal{S}_\alpha^n) = \bigoplus_\alpha \tilde{K}_\sharp^1(\mathcal{S}_\alpha^n)$. Another consequence is that we can use the Atiyah-Hirzebruch-Whitehead (AHW) spectral sequence [58] to compute the $\tilde{K}_\sharp^1$-groups of CW-complexes.

Let $X$ be a connected finite CW-complex, and let $X^{[n]}$ denote its $n$-skeleton with $X^{[0]} = \text{pt}$. By the Whitehead cellular approximation theorem, the inclusion $\iota_n : X^{[n]} \hookrightarrow X$ induces an
isomorphism in integral homology up to degree \( n - 1 \). Consider the AHW spectral sequence \( \{E^r_{p,q}, d^r\}_{r \in \mathbb{N}; p,q \in \mathbb{Z}} \) for reduced K-homology satisfying

\[
E^2_{p,q} = \tilde{H}_p(X; K_{\gamma(q)}(pt)) \Rightarrow \tilde{K}^t_{\gamma(p+q)}(X)
\]

(4.1)

with \( \tilde{H}_p(X; \mathbb{Z}) := H_p(X, pt; \mathbb{Z}) \). Convergence of the spectral sequence means that there is a filtration \( \{F_{p,n-p}\}_{p \in \mathbb{N}_0} \) of \( \tilde{K}^t_{\gamma(n)}(X) \) given by

\[
0 = F_{0,n} \subset F_{1,n-1} \subset \cdots \subset F_{p,n-p} \subset \cdots \subset F_{n,0} \subset \cdots \subset \bigcup_{p+q=n} F_{p,q} = \tilde{K}^t_{\gamma(n)}(X)
\]

where \( F_{p,q} = \text{im}(t_p) : \tilde{K}^t_{\gamma(p+q)}(X[\mathbb{Z}]) \to \tilde{K}^t_{\gamma(p+q)}(X) \) and \( F_{p+1,n-p-1}/F_{p,n-p} \cong E_\infty^p_{p+1,n-p-1} \).

For each \( j = 1, 2, 3 \) there is a natural epimorphism

\[
\beta_j^X : H_j(X; \mathbb{Z}) = E^2_{j,0} \twoheadrightarrow E^\infty_{j,0} = F_{j,0}/F_{j-1,1}.
\]

In particular, since \( F_{0,1} = 0 = F_{1,1} \) the cases \( j = 1, 2 \) yield an epimorphism

\[
\beta_j^X : H_j(X; \mathbb{Z}) = E^2_{j,0} \twoheadrightarrow E^\infty_{j,0} = F_{j,0} \twoheadrightarrow \tilde{K}^t_{\gamma(j)}(X).
\]

By analysing the spectral sequence, one concludes that \( \beta_j^X \) is injective if and only if no non-zero differential \( d^r : E^r_{p,q} \to E^r_{p-r,q+r-1} \) reaches \( E^r_{j,0} \) for \( k \geq 2 \). Thus if the reduced singular homology \( \tilde{H}_j(X; \mathbb{Z}) \) is concentrated in odd (resp. even) degree, except for possibly \( H_2(X; \mathbb{Z}) \) (resp. \( H_1(X; \mathbb{Z}) \) and \( H_3(X; \mathbb{Z}) \)), then \( \beta_1^X \) (resp. \( \beta_2^X \)) is injective.

From these considerations one concludes that if \( X \) is a connected CW-complex of dimension \( \leq 4 \), then there are natural short exact sequences

\[
0 \to H_1(X; \mathbb{Z}) \to \tilde{K}^t_1(X) \to H_3(X; \mathbb{Z}) \to 0,
\]

\[
0 \to H_2(X; \mathbb{Z}) \to \tilde{K}^t_2(X) \to H_4(X; \mathbb{Z}) \to 0.
\]

The latter sequence splits, yielding an isomorphism

\[
\text{ch}_{\gamma}^Z : \tilde{K}^t_0(X) \cong \tilde{H}_0(X; \mathbb{Z}).
\]

If \( X \) is simply connected, then the map \( \text{ch}_{\gamma}^Z : \tilde{K}^t_1(X) \to H_3(X; \mathbb{Z}) \) is a bijection and we thereby obtain an isomorphism

\[
\text{ch}_{\gamma}^Z : \tilde{K}^t_1(X) \cong \tilde{H}_4(X; \mathbb{Z})
\]

of \( \mathbb{Z}_2 \)-graded abelian groups such that the diagram

\[
\begin{array}{ccc}
\tilde{K}^t_1(X) & \xrightarrow{\text{ch}_{\gamma}^Z} & \tilde{H}_4(X; \mathbb{Z}) \\
\downarrow{\text{ch}_{\gamma}} & & \downarrow{\Lambda} \\
\tilde{H}_4(X; \mathbb{Q}) & & \tilde{H}_4(X; \mathbb{Q})
\end{array}
\]

commutes, where \( \Lambda : \tilde{H}_4(X; \mathbb{Z}) \to \tilde{H}_4(X; \mathbb{Q}) \) is the homomorphism induced by the inclusion of abelian groups \( \mathbb{Z} \hookrightarrow \mathbb{Q} \). (These calculations were first carried out in \([42]\).)

The isomorphism (4.2) is called an integral lift of the Chern character in K-homology. It extends the usual Chern character map between stable D-branes and non-trivial homology cycles in \( X \) to include torsion classes. For \( X \) of dimension \( \leq 3 \), this isomorphism even exists without the assumption of simple connectivity \([15]\). For CW-complexes \( X \) of higher dimension, the problem of determining which homology cycles lift to stable D-branes is much more difficult, because then the analysis of the spectral sequence is not so clear cut. For instance, in general \( F_{2,1} \cong H_1(X; \mathbb{Z}) \) is non-trivial, thereby making this kind of analysis generically impossible.
Remark 4.1. The filtration groups \( F_{p,q} \) approximating the full K-homology group consist of D-branes \([M, E, \phi]\) in \( X \) whose worldvolumes are supported in the \( p \)-skeleton, i.e. \( \phi(M) \subseteq X^{[p]} \). The extension groups \( E_{p,q}^\infty \) between successive approximants consist of those D-branes in the \( p \)-skeleton which are not supported on the \((p-1)\)-skeleton, i.e. \( \phi(M) \nsubseteq X^{[p-1]} \), or in other words \( E_{p,q}^\infty \) consists of \( D(p-1) \)-branes which carry no lower-dimensional brane charges. By definition, the approximations \( E_{p,q}^r \) for \( r \geq 2 \) compute the homology of the differential \( d^{r-1} \) \( (E_{p,q}^1 \) is the group of singular \( p \)-chains on \( X \) with values in \( K_{c(q)}(pt) \) with \( d^1 \) the usual simplicial boundary homomorphism).

Let \( M \subset X \) be a \( p \)-dimensional compact spin\(^c \) manifold without boundary which defines a non-trivial homology class \([M] \in E_{p,q}^2 \) in \([4,1]\). If \([M] \) extends through the spectral sequence as a non-trivial element of all homology groups \( E_{p,q}^r \), then it can represent a non-trivial element of \( E_{p,q}^\infty \) and hence have a non-trivial lift to K-homology. In this case there exists a D-brane \([M, E, \phi]\) wrapping \( M \) on the \( p \)-skeleton of \( X \) which is stable and carries no lower brane charges. Conversely, suppose that

\[
[M] = d^r \omega
\]

for some \( r \in \mathbb{N} \) and \( \omega \in \tilde{H}_q(X; \mathbb{Z}) \). Then the homology class \([M] \) can be lifted to K-homology, but the lift is trivial as it vanishes in \( E_{p,q}^\infty \). This means that there exists a D-brane wrapping \( M \) in \( X^{[p]} \) with no lower brane charges, but this D-brane is unstable. Thus the AHW spectral sequence in this context keeps track of the possible obstructions for a homology cycle of \( p,q \) required to get the filtration groups \( F_{p,q} \). The Lens space \( L \) is a compact connected \( C^\infty \) manifold without boundary which defines a \( 2n \)-dimensional compact spin\(^c \) manifold without boundary. The corresponding quotient space \( \mathbb{L}(p; q) \) is a compact connected \( C^\infty \) manifold of dimension \( 2n+1 \) called a Lens space. For definiteness we will consider only the case \( n = 1 \). The corresponding D-branes are then a particular instance of topological A-model D-branes (or A-branes for short) which are mirror duals to the B-branes described in Example [24] and belong to the derived Fukaya category of the spacetime \([3]\). The mirror manifold to the algebraic variety \( X \) is taken to be the non-compact Calabi-Yau threefold which is the total space of the rank 2 complex vector bundle \((\mathcal{L}_n)^{\otimes p} \oplus (\mathcal{L}_1)^{(p-2)} \to \mathbb{C}P^1 \). For \( q = 1 \) the Lens space \( \mathbb{L}(p; 1) \) may be identified with the boundary of \((\mathcal{L}_1)^{\otimes p} \). Higher-dimensional Lens spaces are similarly identified with the boundaries of the total spaces of the line bundles \((\mathcal{L}_n)^{\otimes p} \to \mathbb{C}P^n \).

The Lens space \( \mathbb{L}(p; q) \) is a compact connected spin three-manifold which admits a CW-complex structure with one \( n \)-cell for each dimension \( n = 0, 1, 2, 3 \). Its singular homology is given by

\[
\begin{align*}
\text{H}_0(\mathbb{L}(p; q); \mathbb{Z}) & \cong \mathbb{Z} \\
\text{H}_1(\mathbb{L}(p; q); \mathbb{Z}) & \cong \mathbb{Z}_p \\
\text{H}_2(\mathbb{L}(p; q); \mathbb{Z}) & \cong 0.
\end{align*}
\]
Since we know the singular homology of \( \mathbb{L}(p; q) \), we can work out the spectral sequence in this case and thus calculate the topological K-homology \( K^*_t(\mathbb{L}(p; q)) \).

**Proposition 4.1.** \( K_0^t(\mathbb{L}(p; q)) \cong \mathbb{Z} \), \( K_1^t(\mathbb{L}(p; q)) \cong \mathbb{Z} \oplus \mathbb{Z}_p \).

**Proof.** There exists an AHW spectral sequence \( \{ E^r_{n,m} \} \) converging to \( K^*_t(\mathbb{L}(p; q)) \) with

\[
E^r_{n,m} = H_n(\mathbb{L}(p; q); \mathbb{K}^t_{e(m)}(pt)) \cong \begin{cases} H_n(\mathbb{L}(p; q); \mathbb{Z}) & , \text{m even} \\ 0 & , \text{m odd} \end{cases}.
\]

From [4.4] it follows that the only non-zero groups are \( E^2_{0,2k} \cong \mathbb{Z} \cong E^3_{3,2k} \) and \( E^2_{1,2k} \cong \mathbb{Z}_p \) with \( k \in \mathbb{Z} \). The next sequence of homology groups of the differential module is defined by

\[
E^3_{n,m} := \ker d^2 : E^2_{n,m} \longrightarrow E^2_{n-2,m+1} / \text{im } d^2 : E^2_{n+2,m-1} \longrightarrow E^2_{n,m}.
\]

If \( m \) is odd, \( n \geq 4 \) or \( n < 0 \), then \( E^2_{n,m} = 0 \) so that \( \ker d^2 = 0 = \text{im } d^2 \) and hence \( E^3_{n,m} = 0 \). For the remaining cases with \( m = 2k \) and \( n = 0, 1, 2, 3 \), the pertinent part of the differential bicomplex is of the form

\[
\cdots \xrightarrow{d^2} E^2_{n+2,2k-1} \xrightarrow{d^2} E^2_{n,2k} \xrightarrow{d^2} E^2_{n-2,2k+1} \xrightarrow{d^2} \cdots,
\]

implying that \( \text{im } d^2 = 0 \) and hence \( E^3_{n,2k} = \ker d^2 \cong H_n(\mathbb{L}(p; q); \mathbb{Z}) \).

By induction we conclude from this data that \( E^r_{0,2k} \cong \mathbb{Z} \cong E^r_{3,2k} \) and \( E^r_{1,2k} \cong \mathbb{Z}_p \) for every \( r \geq 2 \), with all other homology groups vanishing. We therefore have

\[
E^\infty_{n,m} = \lim_{\rightarrow r} E^r_{n,m} \cong \begin{cases} H_n(\mathbb{L}(p; q); \mathbb{Z}) = \mathbb{Z} & , \text{m even , } n = 0, 3 \\ H_1(\mathbb{L}(p; q); \mathbb{Z}) = \mathbb{Z}_p & , \text{m even , } n = 1 \\ H_2(\mathbb{L}(p; q); \mathbb{Z}) = 0 & , \text{otherwise} \end{cases}.
\]

For each \( l \in \mathbb{Z} \) let \( F_{0,l} = E^\infty_{0,l} \). Then solving the extension problems

\[
0 \longrightarrow F_{n-1,1-n} \longrightarrow F_{n,-n} \longrightarrow E^\infty_{n,-n} \longrightarrow 0,
\]

\[
0 \longrightarrow F_{n-1,2-n} \longrightarrow F_{n,1-n} \longrightarrow E^\infty_{n,1-n} \longrightarrow 0
\]

for every \( n \in \mathbb{N} \) will produce groups \( F_{n,-n} \) and \( F_{n,1} \) such that \( \{ F_{n,-n} \}_{n \in \mathbb{N}_0} \) (resp. \( \{ F_{n,1} \}_{n \in \mathbb{N}_0} \)) is a filtration of \( K_0^t(\mathbb{L}(p; q)) \) (resp. \( K_1^t(\mathbb{L}(p; q)) \)). Starting from the data above, it is straightforward to compute \( F_{n,-n} = \mathbb{Z} \) for all \( n \in \mathbb{N}_0 \), and hence \( K_0^t(\mathbb{L}(p; q)) = \mathbb{Z} \). Furthermore, one finds \( F_{0,0} \cong 0 \), \( F_{1,0} \cong \mathbb{Z}_p \cong F_{2,-1} \) and \( F_{n,1} \cong \mathbb{Z} \oplus \mathbb{Z}_p \) for all \( n \geq 3 \), so that \( K_1^t(\mathbb{L}(p; q)) = \mathbb{Z} \oplus \mathbb{Z}_p \). \( \square \)

Let us now work out D-brane representatives for the K-homology groups of Proposition 4.1. Since \( \mathbb{K}_0^t(\mathbb{L}(p; q)) = 0 \), it is immediate that \( [pt, \mathbb{K}^t_{e(p,t)}] \) is the generator of \( K_0^t(\mathbb{L}(p; q)) \) = \( \mathbb{Z} \).

Furthermore, since \( \mathbb{L}(p; q) \) is an odd-dimensional spin manifold, it is \( K^2 \)-orientable and so it has a fundamental class \( [\mathbb{L}(p; q), \mathbb{K}^3_{\mathbb{L}(p; q)}, \text{id}_{\mathbb{L}(p; q)}] \) which is the free generator of \( K_1^t(\mathbb{L}(p; q)) = \mathbb{Z} \oplus \mathbb{Z}_p \).

If we take \( q = 1 \) and let \( \mathcal{L}_1 \) denote as before the canonical line bundle over \( \mathbb{C}P^1 \), then we can identify the sphere bundle of \( \mathcal{L}^\otimes_1 \) with the Lens space \( \mathbb{L}(p; 1) \). In this case, from the K-theory of \( \mathbb{L}(p; 1) \) [1] we can identify the torsion generator of \( K_1^t(\mathbb{L}(p; 1)) \) with the K-cycle class

\[
\mathbb{L}(p; 1), \mathbb{K}^3_{\mathbb{L}(p; 1)}, \text{id}_{\mathbb{L}(p; 1)}] - [\mathbb{L}(p; 1), \pi^*(\mathcal{L}_1^\gamma), \text{id}_{\mathbb{L}(p; 1)}]
\]

where \( \pi : \mathcal{S}(\mathcal{L}^\otimes_1) \rightarrow \mathbb{C}P^1 \) is the bundle projection.

To describe the decay of the spacetime-filling brane-antibrane pair [13] into stable D-branes, we note that \( H_1(\mathbb{L}(p; q); \mathbb{Z}) = \mathbb{Z}_p \cong \pi_1(\mathbb{L}(p; q)) \) and that the Hurewicz homomorphism \( \rho_1 : \)}
\[ \pi_1(\mathbb{L}(p; q)) \to H_1(\mathbb{L}(p; q); \mathbb{Z}) \text{ given by } \rho_1[f] = f_1[s^1] \text{ is a bijection.} \] In addition, the Hurewicz homomorphism in K-homology \( h_1 : \pi_1(\mathbb{L}(p; q)) \to K^*_1(\mathbb{L}(p; q)) \) is given by \( h_1[f] = f_1[s^1, \mathbb{C}^\infty_1, \text{id}_{\mathbb{C}^\infty_1}] = [S^1, \mathbb{C}^\infty_1, f] \). Since \( L(p; q) \) is a compact three-dimensional manifold, the homological Chern character admits an integral lift (4.3), fitting into the commutative diagram (4.3) for \( X = L(p; q) \). Furthermore, \( \text{ch} \mathbb{Z} \) character admits an integral lift (4.2) fitting into the commutative diagram (4.3) for \( \pi_1(\mathbb{L}(p; q); \mathbb{Z}) \to \text{Tor}_{K^*_1}(\mathbb{L}(p; q)) \) which fits into the commutative diagram (4.3).

\[ \begin{array}{ccc}
\pi_1(\mathbb{L}(p; q)) & \xrightarrow{h_1} & \text{Tor}_{K^*_1}(\mathbb{L}(p; q)) \\
\rho_1 \downarrow & & \beta_1 \\
H_1(\mathbb{L}(p; q); \mathbb{Z}) & & \end{array} \]

It follows that \( h_1 : \pi_1(\mathbb{L}(p; q)) \to \text{Tor}_{K^*_1}(\mathbb{L}(p; q)) \) is an isomorphism. The generator of the fundamental group \( [f] \in \pi_1(\mathbb{L}(p; q)) = \mathbb{Z}_p \) may be taken to be any loop obtained by projecting a path on the universal cover \( S^3 \to \mathbb{L}(p; q) \) connecting two points on \( S^3 \) that are related by the \( \mathbb{Z}_p \)-action defining the Lens space. Then \( [S^1, \mathbb{C}^\infty_1, f] \) is the torsion generator of \( K^*_1(\mathbb{L}(p; q)) \). For \( q = 1 \) it coincides with the generator (4.5).

**Remark 4.2.** Examining the proof of Proposition 4.1, we see that this construction of the stable D-brane states in \( \mathbb{L}(p; q) \) follows from the form of the homology groups \( E_{n,m}^\infty \) of the differential module in the AHW spectral sequence. In the present example, all homology cycles have nontrivial lifts to K-homology and are thus wrapped by stable states of D-branes.

### 4.3. Projective D-Branes

We will now complete the calculation initiated in Section 4.3 by exhibiting the D-branes in the fourth and final real projective spaces \( \mathbb{R}P^m \), which arise in certain orbifold spacetimes of string theory [23]. They can be realized as the quotient of the \( m \)-sphere \( S^m \) by the antipodal map. Let \( q_m : S^m \to \mathbb{R}P^m \) be the quotient map. With the exception of the projective line \( \mathbb{R}P^1 \cong S^1 \), the corresponding K-homology groups contain torsion subgroups. Analogously to (4.3), the CW-complex structure of \( \mathbb{R}P^m \) may be given by the stratification provided by linearly embedded subspaces, so that its set of \( k \)-cells consists of the single element \( \mathbb{R}P^k \) for \( k = 0, 1, \ldots, m \) with \( \mathbb{R}P^k/\mathbb{R}P^{k-1} \cong S^k \).

The singular homology of \( \mathbb{R}P^m \) is given by

\[ H_0(\mathbb{R}P^{2n+1}; \mathbb{Z}) \cong \mathbb{Z} \cong H_m(\mathbb{R}P^m; \mathbb{Z}) , \]

\[ H_{m-2i}(\mathbb{R}P^m; \mathbb{Z}) \cong \mathbb{Z}_4 , \quad i = 1, \ldots, \left[ \frac{m}{2} \right] . \]

Let \( L_{\mathbb{R}P^m} = S^m \times \mathbb{R}/\mathbb{Z}_4 \) be the canonical flat line bundle over \( \mathbb{R}P^m \), and \( \iota_k : \mathbb{R}P^k \to \mathbb{R}P^m \) the inclusion of the \( k \)-cell.

### 4.3.1. \( \mathbb{R}P^{2n+1} \)

We begin with the odd-dimensional cases \( m = 2n + 1 \). In this instance \( \mathbb{R}P^{2n+1} \) is a spin\(^c \) manifold. Thus it is \( K^*_1 \)-oriented and we can apply Poincaré duality to compute its topological K-homology \( K^*_1(\mathbb{R}P^{2n+1}) \) from its known K-theory \( K^*_1(\mathbb{R}P^{2n+1}) \) [31].

**Proposition 4.2.** \( K^*_0(\mathbb{R}P^{2n+1}) \cong \mathbb{Z} \), \( K^*_1(\mathbb{R}P^{2n+1}) \cong \mathbb{Z} \oplus \mathbb{Z}_{2^n} \).

Applying Proposition 2.1 to the example at hand, one finds that the generating D-brane of \( K^*_0(\mathbb{R}P^{2n+1}) \) is \([pt, \mathbb{C}^{\infty}_1, \iota] \), while the spacetime-filling D-brane \([\mathbb{R}P^{2n+1}, \mathbb{C}^{\infty}_{\mathbb{R}P^{2n+1}}, \text{id}_{\mathbb{R}P^{2n+1}}] \) is the free generator of \( K^*_1(\mathbb{R}P^{2n+1}) \). The torsion generator of \( K^*_1(\mathbb{R}P^{2n+1}) \) is the spacetime-filling brane-antibrane pair

\[ \begin{align*}
[\mathbb{R}P^{2n+1}, \mathbb{C}^{\infty}_{\mathbb{R}P^{2n+1}}, \iota] & \sim [\mathbb{R}P^{2n+1}, L_{\mathbb{R}P^{2n+1}} \otimes \mathbb{C}, \text{id}_{\mathbb{R}P^{2n+1}}] ,
\end{align*} \]
As in the examples of Lens spaces, the decay products of the brane-antibrane system \([4.6]\) cannot be determined through Theorem 2.1 due to the torsion. The difference between K-homology and singular homology here can be understood by appealing to the AHW spectral sequence. After some calculation one finds the filtration groups \(F_2n−3.4.2n \cong Z_{2n−1}\) and \(F_2n−1.2.2n \cong Z_{2n} [16, 55]\), which thereby alter the additive structure in \(H_{odd}(R^P2n+1; Z)\). The lift of the generator \([R^P2n−1] \in H_{2n−1}(R^P2n+1; Z) \cong Z_2\) to K-homology is the stable D-brane \(\omega_0 = [R^P2n−1, I_{R^P2n−1}, I_{2n−1}] \in K_1^1(R^P2n+1)\). While \([R^P2n−1]\) is of order 2 in \(H_{odd}(R^P2n+1; Z)\), \(\omega_0\) is of order \(2^n\) in \(K_1^1(R^P2n+1)\) and is thus equal to \([4.6]\). For every \(k = 0, 1, \ldots, n\), \(\omega_k := 2^k \omega_0 = [R^P2n−2k−1, I_{R^P2n−2k−1}, I_{2n−2k−1}]\) (with \(\omega_n := 0\)) corresponds to the order 2 generator \([R^P2n−2k−1] \in H_{2n−2k−1}(R^P2n+1; Z) \cong Z_2\). These associations illustrate that an integral lift \(ch_{odd}\) of the Chern character, along the lines described in Section 4.1, does not generally exist in the present class of examples.

**Remark 4.3.** This example furnishes a nice illustration of D-brane decay \([16\). Placing together \(2^k \, D(2n - 2)\)-branes wrapping \(R^P2n−1 \subset R^P2n+1\) creates an unstable state that decays into a \(D(2n - 2k)\)-brane wrapping \(R^P2n−2k-1 \subset R^P2n−1\), due to the triviality of the singular homology classes \(2^k \, [R^P2n−2k−1] \) in \(H_{odd}(R^P2n+1; Z)\). Similarly, stacks of \(2^j\) of these \(D(2n - 2k)\)-branes for \(1 \leq j < n − k\) decay into a \(D(2n - 2k - 2j)\)-brane, and so on.

**Remark 4.4.** For \(n = 1\) this construction of the torsion generator of \(K_1^1(R^P3)\) coincides with the construction which is completely analogous to that used in Section 4.2 to construct the torsion D0-brane wrapping \(S^1 \cong R^P1\) on the Lens spaces \(L(p; q)\).

### 4.3.2. \(R^P2n\).

The even-dimensional real projective spaces \(R^P2n\) are more difficult to deal with because they are not orientable. In particular, they are not spin\(^c\) and so most of the techniques used thus far cannot be applied to this case. In fact, this space provides an exotic example whereby not only does the K-homology differ from singular homology, but also where Poincaré duality breaks down and the K-homology differs from the dual K-theory which in the present case is given by \(K_0(R^P2n) \cong Z \oplus Z_{2n−1}\), \(K_1^1(R^P2n) \cong 0\).

**Proposition 4.3.** \(K_0^0(R^P2n) \cong Z\), \(K_1^1(R^P2n) \cong Z_{2n−1}\).

**Proof.** A simple application of the AHW spectral sequence shows that \(K_0^0(R^P2n) = 0\). Since \(H_{odd}(R^P2n; Z) = 0\), via the Chern character we conclude that \(K_1^1(R^P2n)\) has no free part. Finally, by applying the universal coefficient theorem of Section 1.10 to \(X = R^P2n\) one concludes that \(\text{tor}K_1^1(R^P2n) \cong \text{tor}K_0^0(R^P2n) \cong Z_{2n−1}\). \(\square\)

As always, the generator of \(K_0^0(R^P2n) \cong Z\) is the D-instanton \([pt., I_{pt}, t]\). The remaining torsion generators of \(K_1^1(R^P2n) \cong Z_{2n−1}\) are more difficult to find. They may be constructed as follows. By excision, the quotient map \(p_{2n} : (R^P2n, R^P2n−1) \to (R^P2n/R^P2n−1, pt)\) induces an isomorphism \((p_{2n})_* : K_1^1(R^P2n, R^P2n−1) \to K_1^1(R^P2n/R^P2n−1, pt)\) giving

\[
K_1^1(R^P2n, R^P2n−1) \cong K_1^1(R^P2n/R^P2n−1, pt) \cong K_1^1(S^2n)
\]

and one concludes that

\[
(4.7) \quad K_0^0(R^P2n, R^P2n−1) \cong Z, \quad K_1^1(R^P2n, R^P2n−1) \cong 0.
\]

The six-term exact sequence associated to the pair \((R^P2n, R^P2n−1)\) is given by

\[
\begin{array}{ccc}
K_0^0(R^P2n−1) & \xrightarrow{(i_{2n−1})_*} & K_0^0(R^P2n) \\
\downarrow & & \downarrow \\
K_1^1(R^P2n, R^P2n−1) & \xrightarrow{\sim} & K_1^1(R^P2n−1/R^P2n−1)
\end{array}
\]
The homomorphism \((\iota_{2n-1})_*: K^1_0(\mathbb{R}P^{2n-1}) \rightarrow K^1_0(\mathbb{R}P^{2n})\) is induced by the inclusion of the \((2n-1)\)-skeleton in \(\mathbb{R}P^{2n}\). Since both groups \(K^1_0(\mathbb{R}P^{2n-1}) = K^1_0(\mathbb{R}P^{2n}) = \mathbb{Z}\) are generated by \([pt, \mathbb{I}^C_{\text{pt}}, \iota]\), it follows that \((\iota_{2n-1})_*[pt, \mathbb{I}^C_{\text{pt}}, \iota] = [pt, \mathbb{I}^C_{\text{pt}}, \iota]\) and hence \((\iota_{2n-1})_*\) is an isomorphism. Combining this fact with (4.7), we conclude that the six-term exact sequence truncates to the short exact sequence given by

\[
0 \rightarrow K^1_0(\mathbb{R}P^{2n}, \mathbb{R}P^{2n-1}) \xrightarrow{\partial} K^1_0(\mathbb{R}P^{2n-1}) \xrightarrow{(\iota_{2n-1})_*} K^1_0(\mathbb{R}P^{2n}) \rightarrow 0.
\]

It finally follows that

\[
(4.8) \quad K^1_0(\mathbb{R}P^{2n}) \cong K^1_0(\mathbb{R}P^{2n-1}) / \text{im} \partial \cong \mathbb{Z} \oplus \mathbb{Z}_{2n-1} / \text{im} \partial.
\]

Comparing with Proposition 4.3, we conclude that the connecting homomorphism has range \(\text{im} \partial \cong \mathbb{Z}\). The torsion D-branes in this case are thus supported in the \((2n-1)\)-skeleton which is a spin$^c$ submanifold of \(\mathbb{R}P^{2n}\). Their explicit K-cycle representatives, along with the pertinent decay products, can be constructed exactly as in our previous example above. Note that there are no spacetime filling branes in \(\mathbb{R}P^{2n}\).

**Remark 4.5.** To understand the geometrical meaning of the quotient in (4.8), consider the commutative diagram

\[
\begin{array}{ccc}
\vdash & (S^{2n}, \text{pt}) & \vdash \\
\downarrow & & \downarrow \\
(p_{2n}) & \vdash (\mathbb{B}^{2n}, S^{2n-1}) \xrightarrow{f_{2n}} \vdash (\mathbb{R}P^{2n}, \mathbb{R}P^{2n-1})
\end{array}
\]

where \(f_{2n}\) is the characteristic map of the \(2n\)-cell. This induces the commutative diagram in K-homology given by

\[
\begin{array}{ccc}
\vdash & K^1_0(S^{2n}) & \vdash \\
\downarrow & & \downarrow \\
(p_{2n})_* \vdash K^1_0(\mathbb{B}^{2n}, S^{2n-1}) \xrightarrow{(f_{2n})_*} \vdash K^1_0(\mathbb{R}P^{2n}, \mathbb{R}P^{2n-1})
\end{array}
\]

where the induced maps \((f_{2n})_*\) and \((p_{2n})_*\) are isomorphisms. It follows that \([\mathbb{B}^{2n}, \mathbb{I}^C_{\mathbb{B}^{2n}}, f_{2n}]\) is the generator of \(K^1_0(\mathbb{R}P^{2n}, \mathbb{R}P^{2n-1}) \cong \mathbb{Z}\) with \(\partial[\mathbb{B}^{2n}, \mathbb{I}^C_{\mathbb{B}^{2n}}, f_{2n}] = [S^{2n-1}, \mathbb{I}^C_{\mathbb{B}^{2n-1}}, q_{2n-1}] = h_{2n-1}[q_{2n-1}]\), where \(h_{2n-1} : \pi_{2n-1}(\mathbb{R}P^{2n-1}) \rightarrow K^1_0(\mathbb{R}P^{2n-1})\) is the Hurewicz homomorphism in K-homology and the quotient map \(q_{2n-1} : S^{2n-1} \rightarrow \mathbb{R}P^{2n-1}\) is the generator \([q_{2n-1}] \in \pi_{2n-1}(\mathbb{R}P^{2n-1}) \cong \mathbb{Z}\). We thus conclude that \(\text{im} \partial = \text{im} h_{2n-1}\), and hence the quotient by the image of the boundary homomorphism in (4.8) projects out the integrally charged D-brane \([S^{2n-1}, \mathbb{I}^C_{\mathbb{B}^{2n-1}}, q_{2n-1}]\) which fills the entire \((2n-1)\)-cell of \(\mathbb{R}P^{2n}\).

4.3.3. \(\mathbb{R}P^{2n+1} \times \mathbb{R}P^{2k+1}\). When dealing with torsion K-homology groups, the structure of D-branes on product manifolds becomes an interesting problem. Let us first consider the representative example \(\mathbb{R}P^{2n+1} \times \mathbb{R}P^{2k+1}\) wherein the factors each support torsion D-branes.

**Proposition 4.4.** \(K^1_0(\mathbb{R}P^{2n+1} \times \mathbb{R}P^{2k+1}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_k \oplus \mathbb{Z}_{2n} \oplus \mathbb{Z}_{2p} \cong K^1_0(\mathbb{R}P^{2n+1} \times \mathbb{R}P^{2k+1})\) where \(p = \gcd(n, k)\).
Proof. We apply the Künneth theorem of Section 4.6 (Theorem 1.1). The torsion extension for the \( K_0 \)-group is given by

\[
\bigoplus_{i+j=1} \text{Tor}(K_i^1(\mathbb{RP}^{2n+1}), K_j^1(\mathbb{RP}^{2k+1})) = \text{Tor}(Z, Z \oplus Z_2) \oplus \text{Tor}(Z \oplus Z_2, Z) = 0,
\]

and so there is an isomorphism

\[
K_0(\mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1}) = (K_i^1(\mathbb{RP}^{2n+1}) \otimes K_0^1(\mathbb{RP}^{2k+1})) \oplus (K_1^1(\mathbb{RP}^{2n+1}) \otimes K_1^1(\mathbb{RP}^{2k+1}))
\]

(4.9)

On the other hand, the torsion extension for the \( K_1 \)-group is \( Z_2 \). Again the short exact sequence of Theorem 1.1 for the present space splits and we find the same isomorphism as in (4.9) for the \( K_1 \)-homology group \( K_1^1(\mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1}) \).

The generating D-branes are straightforward to work out as before. For the various subgroups of \( K_0(\mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1}) \) given by Proposition 4.4 one finds the generators

\[
\begin{align*}
Z \oplus Z &: \left[ \text{pt} \times \mathbb{C}^0 \right], \left[ \mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1} \times \mathbb{C}, \mathbb{C} \right], \\
Z_{2n} &: \left[ \mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1}, \mathbb{C} \right], \left[ \mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1} \times \mathbb{C}, \mathbb{C} \right], \\
Z_{2k} &: \left[ \mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1}, \mathbb{C} \right], \left[ \mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1} \times \mathbb{C}, \mathbb{C} \right], \\
Z_{2p} &: \left[ \mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1}, \mathbb{C} \right], \left[ \mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1} \times \mathbb{C}, \mathbb{C} \right],
\end{align*}
\]

The \( 2^n \)-torsion and \( 2^k \)-torsion charges come from stable spacetime-filling D-branes on \( \mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1} \). Since they carry non-trivial line bundles on their worldvolumes, they can be decomposed into lower-dimensional D-branes carrying trivial line bundles along the lines of Section 2.2. The precise nature of the constituent D-branes can again be deduced upon careful examination of the AHW spectral sequence. The decomposition of the \( 2^n \)-torsion spacetime-filling brane-antibrane pairs is analogous to that of the \( \mathbb{RP}^{2n+1} \) example studied earlier.

For the first four subgroups of \( K_1^1(\mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1}) \) one finds the generators

\[
\begin{align*}
Z \oplus Z &: \left[ \mathbb{RP}^{2n+1}, \mathbb{C} \right], \left[ \mathbb{RP}^{2k+1}, \mathbb{C} \right], \\
Z_{2n} &: \left[ \mathbb{RP}^{2n+1}, \mathbb{C} \right], \left[ \mathbb{RP}^{2n+1}, \mathbb{C} \right], \\
Z_{2k} &: \left[ \mathbb{RP}^{2k+1}, \mathbb{C} \right], \left[ \mathbb{RP}^{2k+1}, \mathbb{C} \right],
\end{align*}
\]

The \( 2^n \)-torsion class is more difficult to determine in this case because it arises from the torsion extension in the Künneth formula. It can be found by again comparing to singular homology and identifying it as the lift of the remaining cycles in \( H_{\text{odd}}(\mathbb{RP}^{2n+1} \times \mathbb{RP}^{2k+1}; Z) \) after the other decay products have been determined from the AHW spectral sequence along the lines of the \( \mathbb{RP}^{2n+1} \) example above.

4.3.4. \( \mathbb{RP}^{2n+1} \times S^k \). For our final example of projective D-branes we consider a product spacetime in which one factor carries only torsion-free D-branes. For the representative example \( \mathbb{RP}^{2n+1} \times S^k \) we proceed exactly as in the previous case.

Proposition 4.5. \( K_0^1(\mathbb{RP}^{2n+1} \times S^k) \cong \left\{ \begin{array}{ll} Z \oplus Z, & k \text{ even}, \\
Z \oplus Z \oplus Z_2, & k \text{ odd} \end{array} \right\} \quad K_1^1(\mathbb{RP}^{2n+1} \times S^k) \cong \left\{ \begin{array}{ll} Z \oplus Z \oplus Z_2 \oplus Z_2, & k \text{ even}, \\
Z \oplus Z \oplus Z_2, & k \text{ odd} \end{array} \right\} \).
The generators of the $K^t_0$-groups are given by
\[
\begin{cases}
\{ \mathrm{pt}, \mathbb{C}_{\mathrm{pt}}, \ell \}, & \{ \mathbb{R}^{2n+1} \times S^k, \mathbb{C}_{\mathbb{R}^{2n+1} \times S^k}, \mathrm{id}_{\mathbb{R}^{2n+1} \times S^k} \}; & k \text{ even} \\
\{ \mathbb{R}^{2n+1} \times S^k, \mathbb{C}_{\mathbb{R}^{2n+1} \times S^k}, \mathrm{id}_{\mathbb{R}^{2n+1} \times S^k} \}; & \{ \mathbb{R}^{2n+1} \times S^k, \mathbb{C}_{\mathbb{R}^{2n+1} \times S^k}, \mathrm{id}_{\mathbb{R}^{2n+1} \times S^k} \}; & k \text{ odd}
\end{cases}
\]
while the $K^t_1$-groups are generated by the D-branes
\[
\begin{cases}
\{ \mathbb{R}^{2n+1} \times S^k, \mathbb{C}_{\mathbb{R}^{2n+1} \times S^k}, \mathrm{id}_{\mathbb{R}^{2n+1} \times S^k} \}; & \{ \mathbb{R}^{2n+1} \times S^k, \mathbb{C}_{\mathbb{R}^{2n+1} \times S^k}, \mathrm{id}_{\mathbb{R}^{2n+1} \times S^k} \}; & k \text{ even} \\
\{ \mathbb{R}^{2n+1} \times S^k, \mathbb{C}_{\mathbb{R}^{2n+1} \times S^k}, \mathrm{id}_{\mathbb{R}^{2n+1} \times S^k} \}; & \{ \mathbb{R}^{2n+1} \times S^k, \mathbb{C}_{\mathbb{R}^{2n+1} \times S^k}, \mathrm{id}_{\mathbb{R}^{2n+1} \times S^k} \}; & k \text{ odd}
\end{cases}
\]

4.4. D-Branes on Calabi-Yau Spaces. We conclude this section by further indicating how our analysis of torsion D-branes can be applied to the topological A-model and B-model examples studied earlier, and ultimately to a K-cycle description of mirror symmetry \([\textit{[3]}\). For definiteness, let us work with the Fermat quintic threefold $Y$ defined by
\[
Y = \left\{ (z_1, \ldots, z_5) \in \mathbb{C}^4 \left| \sum_{i=1}^{5} z_i^5 = 0 \right. \right\}.
\]
This is a simply-connected complex projective algebraic variety of real dimension 6. It is therefore a spin$^c$ manifold and satisfies Poincaré duality. Let $\mathcal{H}$ be the hyperplane line bundle on $Y$, i.e. the restriction to $Y$ of the line bundle which is associated with any hyperplane $\mathcal{H}$ in $\mathbb{C}^4$. Let $D$ be the corresponding hyperplane divisor whose zero set in $\mathbb{C}^4$ is the original hyperplane $\mathcal{H}$. Let $C \cong \mathbb{C}^4$ be a degree 1 rational curve on $Y$. Then $K^t_0(Y) \cong \mathbb{Z}^{204}$ \([\textit{[17]}\textit{[19]}\) and by Poincaré duality we have $K^t_0(Y) \cong \mathbb{Z}^{204}$. From the known K-theory generators we can thus identify the generating A-branes
\[
[Y, \mathcal{H}_Y, \mathrm{id}_Y] = [Y, \mathbb{C}, \mathrm{id}_Y] = \left[ D, \mathbb{C}_{D}, \ell_D \right], \quad [Y, (\ell_C), (\mathbb{C}_{C}, \mathrm{id}_Y)] = [C, \mathbb{C}, \ell_C].
\]
In addition, one has $K^t_1(Y) \cong \mathbb{Z}^{204}$ so that $K^t_1(Y) \cong \mathbb{Z}^{204}$. The corresponding D-branes wrap the 204 independent three-cycles of $H_3(Y; \mathbb{Z})$ and are constructed using Theorem 2.1. As expected, the A-branes all wrap lagrangian submanifolds with flat line bundles.

The corresponding B-branes live in the multiply-connected non-singular Calabi-Yau threefold $X$ obtained by quotienting $Y$ by the $\mathbb{Z}_5$-action generated by $z_i \rightarrow \zeta^{-i} z_i$, $i = 1, \ldots, 5$ where $\zeta^5 = 1$ \([\textit{[3]}\textit{[32]}\textit{[62]}\). This is also a complex projective algebraic variety of real dimension 6, and hence a spin$^c$ manifold. Let $\mathcal{H}_X$ be the hyperplane line bundle restricted to $X$, and let $\mathcal{L}_X \rightarrow X$ be the flat line bundle $\mathcal{L}_X = Y \times \mathbb{C}/\mathbb{Z}_5$ with respect to the $\mathbb{Z}_5$-action above. Then by Poincaré duality one has $K^0_0(X) \cong K^0_0(X) \cong \mathbb{Z}^{204} \oplus \mathbb{Z}_5$. Using the known K-theory generators \([\textit{[19]}\) we may write down the D-branes corresponding to the free part as
\[
\begin{cases}
[X, \mathcal{H}_X, \mathrm{id}_X] - [X, \mathbb{C}, \mathrm{id}_X], & [X, \mathcal{H}_X \otimes \mathcal{H}_X \otimes \mathbb{C}_X, \mathrm{id}_X] - [X, \mathcal{H}_X \otimes \mathcal{H}_X, \mathrm{id}_X], \\
[X, (\mathcal{H}_X)^{\otimes 3} \oplus (\mathcal{H}_X)^{\otimes 3 \oplus 3}, \mathrm{id}_X] - [X, (\mathcal{H}_X \otimes \mathcal{H}_X)^{\otimes 3} \oplus \mathbb{C}_X, \mathrm{id}_X]
\end{cases}
\]
while the torsion generator is the spacetime-filling brane-antibrane pair
\[
[X, \mathcal{L}_X, \mathrm{id}_X] - [X, \mathbb{C}_{\mathbb{C}_X}, \mathrm{id}_X].
\]
This brane-antibrane system decays into a stable torsion D4-brane at the Gepner point of the given Calabi-Yau moduli space \([\textit{[1]}\). Furthermore, one has $K^t_1(X) \cong K^t_1(X) \cong \mathbb{Z}^{44} \oplus \mathbb{Z}_5$, with the free part generated by lifting the 44 three-cycles in $H_3(X; \mathbb{Z})$ and the torsion generated by applying the Hurewicz homomorphism to the generator of the fundamental group $\pi_1(X) \cong \mathbb{Z}_5$. 
5. Flux Stabilization of D-Branes

In this final section we shall consider D-branes which live on the total space $X$ of a fibration

$$
\begin{array}{c}
F \\
\downarrow \\
B
\end{array}
\xrightarrow{X}
$$

where we assume that the base space $B$ is a simply connected finite CW-complex (we can consider $B$ to be only path-connected, which then requires the use of local coefficients). There are many such situations in which one is interested in the classification of D-branes in $X$. In fact, many of the examples we have considered previously fall into this category. For instance, both the Lens spaces $L(p; q_1, \ldots, q_n)$ and the real projective spaces $\mathbb{R}P^{2n+1}$ are circle bundles over $\mathbb{C}P^n$. Furthermore, the very application of vector bundle modification identifies those D-branes such that one is a spherical fibration over the other. Both our descriptions of “M-theory” definitions of unstable 9-branes and T-duality also have natural extensions to torus bundles.

Since K-homology satisfies both the wedge and weak homotopy equivalence axioms for finite CW-complexes, we may apply the Leray-Serre spectral sequence to calculate the topological K-homology groups in these instances. The Leray-Serre theorem \[58\] states that there is a spectral sequence \(\{E^r_{p,q}, d^r\}_{r \in \mathbb{N}, p,q \in \mathbb{Z}}\) converging to \(K^*_t(X)\) and satisfying

\[E^2_{p,q} = H_p(B; K_{(q)}^t(F))\]

for all \(p,q \in \mathbb{Z}\). This spectral sequence relates D-branes on $X$ to homology cycles of the base and fibres. Stability criteria can be formulated along the lines of Section 4.1 to determine which homology classes in \(H^*_p(B; K_{(q)}^t(F))\) lift to non-trivial classes in \(K^*_t(X)\). Part of the homology basis of $X$ wrapped by stable D-branes may then contain cycles of the base $B$ embedded in $X$ as the zero section of the fibre bundle along with cycles from the inclusion of some of the fibres $F$.

The worldvolumes will typically be labelled by the characteristic class of the fibre bundle. Some of these singular homology cycles may be trivial in the homology of $X$ (but not in the homology of $B$ or $F$), and on its own a D-brane wrapping the given cycle in $X$ would be unstable. However, regarding $X$ as the total space of a non-trivial fibration can effectively render the D-brane state stable in a process somewhat reverse to the decay of branes wrapping non-trivial homology cycles that we studied earlier (Sections 2.3 and 4.1). We refer to this process as “flux stabilization” \[5\,13\,46\], with the characteristic class of the fibration playing the role of a “flux” on the D-brane worldvolume. These fluxes act as conserved topological charges on the D-branes which give an obstruction for them to decay to the vacuum state. For example, a circle bundle $X$ over $B$ is classified entirely by its first Chern class $c_1(X) \in H^2(B; \mathbb{Z})$.

5.1. Spherically-Fibred D-Branes. We begin with the class of fibrations wherein the fibre spaces $F = S^{2n}$ are even-dimensional spheres.

**Proposition 5.1.** Let $S^{2n} \hookrightarrow X \rightarrow B$ be a spherical fibration such that the base space $B$ is a simply connected finite CW-complex with freely generated singular homology concentrated in even degree. Then

\[K^*_0(X) \cong H^*_z(B; \mathbb{Z}) \oplus H^*_z(B; \mathbb{Z}) , \quad K^*_1(X) \cong 0 .\]

**Proof.** The second term of the Leray-Serre spectral sequence is given by

\[E^2_{p,q} \cong \begin{cases} H_p(B; \mathbb{Z} \oplus \mathbb{Z}) & \text{, } q \text{ even} \\
0 & \text{, } q \text{ odd} \end{cases} .\]
By the universal coefficient theorem for singular homology one has $H_p(B; \mathbb{Z}^{\oplus n}) \cong H_p(B; \mathbb{Z})^{\oplus n}$, and so

$$E^p_{2q} \cong \begin{cases} H_p(B; \mathbb{Z}) \oplus H_p(B; \mathbb{Z}) & , \ q \ \text{even} \\ 0 & , \ q \ \text{odd} \end{cases}.$$ 

Under the assumptions on the homology of $B$, one has $E^p_{2q} = 0$ if either $p$ or $q$ are odd and hence $E^p_{2q} \cong E^p_{2q}$ for all $r \geq 2$. It follows that $H^\times_{2q} \cong E^p_{2q}$. Then it is easy to see that $F_{p,1-p} = F_{0,1} = F_{0,1}^\times = 0$ for all $p \geq 0$, and so $K^*_1(X) = 0$. Furthermore, one has the filtration groups

$$F_{p,-p} \cong \begin{cases} H_q(B; \mathbb{Z}) \oplus H_q(B; \mathbb{Z}) & , \ p \ \text{even} \\ \bigoplus_{q=0}^{p-1} H_q(B; \mathbb{Z}) \oplus H_q(B; \mathbb{Z}) & , \ p \ \text{odd} \end{cases}$$

and hence $K^*_0(X) = H_2(B; \mathbb{Z}) \oplus H_2(B; \mathbb{Z})$.

5.2. **Fractional Branes.** Next we look at the cases where the fibre $F$ is a finite set of points. In this case, stable D-branes again wrap cycles in the base $B$. Any such D-brane is also accompanied by a set of $|F|$ mirror images called fractional branes [27] which live on the various leaves of the covering space $X$.

**Proposition 5.2.** Let $F \hookrightarrow X \to B$ be a covering such that the base space $B$ is a simply connected finite CW-complex with freely generated singular homology concentrated in even degree. Then

$$K^*_0(X) \cong H_2(B; \mathbb{Z})^{\oplus |F|}, \quad K^*_1(X) \cong 0.$$ 

**Proof.** Since the functor $K^*_2$ satisfies the infinite wedge axiom [38], the second term of the Leray-Serre spectral sequence is given by

$$E^2_{p,q} \cong \begin{cases} H_p(B; \bigoplus_{a \in F} \mathbb{Z}) & , \ q \ \text{even} \\ 0 & , \ q \ \text{odd} \end{cases}.$$ 

Applying the universal coefficient theorem allows us to rewrite this term as

$$E^2_{p,q} \cong \begin{cases} H_p(B; \mathbb{Z})^{\oplus |F|} & , \ q \ \text{even} \\ 0 & , \ q \ \text{odd} \end{cases}.$$ 

Under the stated assumptions on the homology of $B$ we thereby conclude that

$$E^\times_{p,q} \cong E^2_{p,q} \cong \begin{cases} H_p(B; \mathbb{Z})^{\oplus |F|} & , \ q \ \text{even}, \ p \geq 0 \ \text{even} \\ 0 & , \ q \ \text{odd} \end{cases}.$$ 

One easily shows that $F_{p,1-p} = F_{0,1} = E^\times_{p,0} = 0$ for all $p \geq 0$, and so $K^*_1(X) = 0$. One also concludes that

$$F_{p,-p} \cong \bigoplus_{a \in F} \bigoplus_{q=0}^{p-1} H_q(B; \mathbb{Z}) \oplus H_q(B; \mathbb{Z}) \oplus H_q(B; \mathbb{Z}) \oplus H_q(B; \mathbb{Z}) \oplus H_q(B; \mathbb{Z}), \quad p \ \text{even}$$

and hence $K^*_0(X) = \bigoplus_{a \in F} H_2(B; \mathbb{Z})$.

There is a relative version of the Leray-Serre spectral sequence of the form

$$E^2_{p,q} = \tilde{H}_p(B; K^t_q(F)) \implies K^t_{(p+q)}(X, F).$$

Using an analysis similar to the one just made allows one to determine D-brane states analogous to those of Section 5.1 in the case where the homology of $B$ is supported complimentarily to that above and the fractional branes are all identified with one another.
Proposition 5.3. Let $F \hookrightarrow X \to B$ be a covering such that the base space $B$ is a simply connected finite CW-complex with freely generated reduced singular homology concentrated in odd degree. Then
\[
K_0^i(X, F) \cong 0, \quad K_1^i(X, F) \cong H_2(B; \mathbb{Z}) \oplus H_2(B; \mathbb{Z}) .
\]

Corollary 5.1. Let $F \hookrightarrow X \to B$ be a covering such that the base space $B$ is a simply connected finite CW-complex with freely generated reduced singular homology concentrated in degree $n \mod 2$. Then
\[
\tilde{K}_{e(n)}^i(X) \cong \tilde{H}_2(B; \mathbb{Z}) , \quad \tilde{K}_{e(n+1)}^i(X) \cong 0 .
\]

Remark 5.1. Since $B$ is path-connected, an application of these results to the trivial fibration $id_B : B \to B$ gives as a corollary that $K_0^i(B)$ (resp. $K_1^i(B)$) is isomorphic to $H_2(B; \mathbb{Z})$ while $K_1^i(B)$ (resp. $K_0^i(B)$) is trivial when $B$ has non-trivial homology only in even (resp. odd) degree. \hfill \Box

5.3. Spherically-Based D-Branes. For our final general class of fibre bundles, we consider the case where the base space $B$ is a sphere. In such instances the stable D-branes on $X$ are determined by images of $K$-cycles on the fibre $F$. Our first result completely determines the case of coverings of even-dimensional spheres.

Proposition 5.4. Let $F \hookrightarrow X \to \mathbb{S}^{2n}$, $n \geq 1$ be a fibration over the $2n$-sphere such that the topological $K$-homology group of the fibre $K_2^i(F) = K_0^i(F)$ is freely generated. Then
\[
K_0^i(X) \cong K_0^i(F) \oplus K_0^i(F) , \quad K_1^i(X) \cong 0 .
\]

Proof. The second term of the Leray-Serre spectral sequence is given by
\[
E_{p,q}^2 \cong \left\{ \begin{array}{ll}
K_{e(q)}^i(F) & , \quad p = 0, 2n \\
0 & , \quad \text{otherwise }
\end{array} \right.
\]
Since $K_1^i(F) = 0$, this becomes
\[
E_{p,q}^2 \cong \left\{ \begin{array}{ll}
K_0^i(F) & , \quad q \text{ even}, p = 0, 2n \\
0 & , \quad \text{otherwise }
\end{array} \right.
\]
Since $E_{p,q}^2 = 0$ if either $p$ or $q$ is odd, it follows that $d' = 0$ for all $r, p, q$ and so $E_{p,q}^\infty = E_{p,q}^2$. One easily concludes that $F_{p,1-p} = F_{0,1} = E_{0,1}^\infty = K_1^1(F) = 0$ for all $p$, and so $K_1^1(X) = 0$. On the other hand, for $p \geq 2n-1$ one has $F_{p,p} = F_{0,0} = E_{0,0}^\infty = K_0^0(F)$ and the only extension problem arises in the exact sequence
\[
0 \to F_{2n-1,1-2n} \to F_{2n-2n} \to E_{2n-2n}^\infty \to 0 .
\]
Since $\text{Ext}(K_0^0(F), K_0^0(F)) = 0$ it follows that $F_{2n-2n} = K_0^0(F) \oplus K_1^1(F)$. For $p > 2n$ one has $F_{p,p} = F_{2n-2n}$, and so we conclude that $K_0^0(X) = K_0^0(F) \oplus K_1^1(F)$. \hfill \Box

Proposition 5.5. Let $F \hookrightarrow X \to \mathbb{S}^1$ be a fibration over the circle such that the topological $K$-homology group of the fibre obeys $\text{Ext}(K_1^i(F), K_0^e(F)) = 0$ for $i = 0, 1$. Then
\[
K_0^i(X) = K_1^i(X) = K_0^0(F) \oplus K_1^1(F) .
\]

Proof. The second term in the Leray-Serre spectral sequence is given by
\[
E_{p,q}^2 \cong \left\{ \begin{array}{ll}
K_{e(q)}^i(F) & , \quad p = 0, 1 \\
0 & , \quad \text{otherwise }
\end{array} \right.
\]
The spectral sequence collapses at the second level, and so
\[
E_{p,q}^\infty \cong E_{p,q}^2 \cong \left\{ \begin{array}{ll}
K_{e(q)}^i(F) & , \quad p = 0, 1 \\
0 & , \quad \text{otherwise }
\end{array} \right.
\]
We therefore have $F_{0,q} \cong E_{0,q}^2 \cong K_{e(q)}^t(F)$. Since $\text{Ext}(K_{e}^1(F), K_{e(i+1)}^t(F)) = 0$, $i = 0, 1$ one finds $F_{p,-p} = K_{0}^t(F) \oplus K_{1}^t(F) = F_{p,1-p}$ and the conclusion follows. \hfill $\Box$

**Remark 5.2.** For a fibration $F \hookrightarrow X \to \mathbb{S}^{2n+1}$ over a generic odd-dimensional sphere, with the additional assumption that $\text{im} : E_{2n+1,q-2n}^{2n+1} \to F_{2n+1}^{2n+1} = 0 = \ker d^{2n+1}$ one can derive an analogous result to the one of Proposition 5.5.

Using the relative version of the Leray-Serre spectral sequence and performing an analysis analogous to those just made allows one to conclude the following result.

**Proposition 5.6.** Let $F \hookrightarrow X \to \mathbb{S}^n$, $n \geq 1$ be a fibration over the n-sphere. Then for $i = 0, 1$ one has

$$K_i^t(X,F) \cong K_{e(i+n)}^t(F).$$

5.4. **Hopf Branes.** Intimately related to the four classes of projective D-branes studied earlier are the Hopf fibrations. Let $r := \dim_{\mathbb{K}}(\mathbb{K})$, where $\mathbb{K}$ is one of the four normed division algebras over the field of real numbers given by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. Then the projective plane $\mathbb{K}P^2$ is the mapping cone of the Hopf fibration

$$\mathbb{S}^{r-1} \longrightarrow \mathbb{S}^{2r-1} \longrightarrow \mathbb{S}^r.$$

The total space $X = \mathbb{S}^{2r-1}$ of this fibre bundle is a sphere in $\mathbb{K}^2$, while its base $B = \mathbb{K}P^1 \cong \mathbb{S}^r$ is the one-point compactification $\mathbb{K}^\infty$. The Hopf fibrations are the free generators of the fundamental groups $\pi_{2r-1}(\mathbb{S}^r)$. For the case $r = 1$ ($\mathbb{K} = \mathbb{R}$) the corresponding D-branes are represented as solitonic kinks. For $r = 2$ ($\mathbb{K} = \mathbb{C}$) the D-branes are Dirac monopoles corresponding to the magnetic monopole bundle over $\mathbb{C}P^1$. For $r = 4$ ($\mathbb{K} = \mathbb{H}$) the D-branes are SU(2) Yang-Mills instantons corresponding to the holomorphic vector bundle of rank 2 over $\mathbb{C}P^3$. Finally, the case $r = 8$ ($\mathbb{K} = \mathbb{O}$) realizes D-branes as Spin(8) instantons of the Hopf bundle over $\mathbb{R}P^8$. These characterizations [52, 53] are asserted by computing the topological K-homology groups using the relative version of the Leray-Serre spectral sequence. In fact, they are special cases of a more general result.

**Proposition 5.7.** For any spherical fibration of the form $\mathbb{S}^{2n+1} \hookrightarrow X \to \mathbb{S}^{2m}$ one has

$$(X,\mathbb{S}^{2n+1}) \cong \tilde{H}_{2m}(\mathbb{S}^{2m};\mathbb{Z}) = \mathbb{Z}$$

for $i = 0, 1$.

**Proof.** The second term in the relative Leray-Serre spectral sequence is given by

$$E_{2}^{p,q} \cong \begin{cases} \tilde{H}_{2m}(\mathbb{S}^{2m};\mathbb{Z}) & , \ p = 2m \\ 0 & , \ otherwise \end{cases}.$$ 

The spectral sequence collapses at the second level so that

$$E_{\infty}^{p,q} \cong E_{2}^{p,q} \cong \begin{cases} \tilde{H}_{2m}(\mathbb{S}^{2m};\mathbb{Z}) & , \ p = 2m \\ 0 & , \ otherwise \end{cases}.$$ 

Since $E_{\infty}^{0,-p} = 0$ unless $p = 2m$, one has $F_{p,-p} = F_{2m,-2m} = \tilde{H}_{2m}(\mathbb{S}^{2m};\mathbb{Z})$ and we conclude that $K_{0}^t(X,\mathbb{S}^{2n+1}) \cong \tilde{H}_{2m}(\mathbb{S}^{2m};\mathbb{Z})$. Furthermore, one has the filtration groups $F_{p,1-p} = F_{2m,1-2m} = \tilde{H}_{2m}(\mathbb{S}^{2m};\mathbb{Z})$ and it follows that also $K_{1}^t(X,\mathbb{S}^{2n+1}) = \tilde{H}_{2m}(\mathbb{S}^{2m};\mathbb{Z})$. \hfill $\Box$
Remark 5.3. Proposition 5.7 shows that the only stable D-branes (in addition to the usual point-like D-instantons) in any of the four non-trivial Hopf fibrations above wrap a spherical submanifold $S^r$ embedded into $S^{2r-1}$ as the zero section of the fibre bundle, even though these submanifolds are homologically trivial. The worldvolume spheres $S^r$ are labelled by the classifying map of the Hopf fibration, which is the generator of $\pi_{r-1}(\text{Spin}(r))/\pi_{r-1}(\text{Spin}(r-1)) \cong \pi_{r-1}(S^{r-1})$, and they are stabilized by the flux given by the $r$-th Chern class $c_r(S^{2r-1}) \in H^r(S^r; \mathbb{Z}) \cong \mathbb{Z}$. For example, for $r = 2$ this construction reproduces the well-known result that the stable branes in $S^3$ are spherical D2-branes wrapping $S^2 \subset S^3$ with integral charge labelled by $\pi_1(S^1) \cong \mathbb{Z}$ [5]. The stabilizing flux in this case is the magnetic charge $c_1(S^3) \in H^2(S^3; \mathbb{Z}) \cong \mathbb{Z}$ given by the first Chern class of the monopole bundle.

References

[1] J.F. Adams, Stable Homotopy and Generalised Homology (The University of Chicago Press, 1974).
[2] T. Asakawa, S. Sugimoto and S. Terashima, “D-Branes, Matrix Theory and K-Homology”, J. High Energy Phys. 0203 (2002) 034 [arXiv:hep-th/0108085].
[3] P.S. Aspinwall, “D-Branes on Calabi-Yau Manifolds”, arXiv:hep-th/0403166.
[4] M.F. Atiyah, R. Bott and A. Shapira, “Clifford Modules”, Topology (1964) 3–38.
[5] C. Bachas, M.R. Douglas and C. Schweigert, “Flux Stabilization of D-Branes”, J. High Energy Phys. 0005 (2000) 049 [arXiv:hep-th/0005037].
[6] P. Baum and R.G. Douglas, “K-Homology and Index Theory”, Proc. Symp. Pure Math.
[7] P. Bordalo, S. Ribault and C. Schweigert, “Flux Stabilization in Compact Groups”, J. High Energy Phys. 0110 (2001) 036 [arXiv:hep-th/0108201].
[8] P. Bouwknegt, J.R. Douglas and P.A. Fillmore, “Extensions of C*-Algebras and K-Homology”, Ann. Math. 105 (1977) 265–324.
[9] O. Bergman, E.G. Gimon and P. Horava, “Brane Transfer Operations and T-Duality of Non-BPS States”, J. High Energy Phys. 9904 (1999) 010 [arXiv:hep-th/9902160].
[10] O. Bergman, E.G. Gimon and S. Sugimoto, “ Orientifolds, RR Torsion and K-Theory”, J. High Energy Phys. 0105 (2001) 047 [arXiv:hep-th/0103183].
[11] B. Blackadar, K-Theory for Operator Algebras (Springer-Verlag, 1986).
[12] C.F. Bödigheimer, “Splitting the Künneth Sequence in K-theory”, Math. Ann. 242 (1979), no. 2, 159-171.
[13] P. Bordalo, S. Ribault and C. Schweigert, “Flux Stabilization in Compact Groups”, J. High Energy Phys. 0110 (2001) 036 [arXiv:hep-th/0108201].
[14] P. Bouwknegt and V. Mathai, “D-Branes, B-Fields and Twisted K-Theory”, J. High Energy Phys. 0003 (2000) 007 [arXiv:hep-th/0002023].
[15] P. Bouwknegt, J. Evslin and V. Mathai, “T-Duality: Topology Change from H-Flux”, Commun. Math. Phys. 249 (2004) 383–415 [arXiv:hep-th/0306062].
[16] P. Bouwknegt, J. Evslin, B. Jurco, V. Mathai and H. Sati, “Flux Compactifications of Projective Spaces and the S-Duality Puzzle”, arXiv:hep-th/0501110.
[17] V. Braun, “K-Theory Torsion”, arXiv:hep-th/0005103.
[18] T. Bröcker and K. Jänich, Introduction to Differential Topology (Cambridge University Press, 1982).
[19] I. Brunner and J. Distler, “Torsion D-branes in Nongeometrical Phases”, Adv. Theor. Math. Phys. 5 (2002) 265–309 [arXiv:hep-th/0102018].
[20] P.E. Conner, E.E. Floyd, “The Relation of Cobordism to K-theories, Lecture Notes in Mathematics 28 (1966), Springer.
[21] P.E. Conner, E.E. Floyd, “Differentiable Periodic Maps, 1964, Springer.
[22] J.F. Davis and P. Kirk, Lecture Notes in Algebraic Topology (American Mathematical Society, 2001).
[23] A. Deutz, “The Splitting of the Künneth sequence in K-theory for C*-algebras”, PhD Dissertation, Wayne State University, 1981.
[24] D.-E. Diaconescu, G.W. Moore and E. Witten, “E_8 Gauge Theory and a Derivation of K-Theory from M-Theory”, Adv. Theor. Math. Phys. 6 (2003) 1031–1134 [arXiv:hep-th/0005090].
[25] M.R. Douglas, “D-branes within Branes”, in: Strings, Branes and Dualities (Kluwer, 1999) 267–275 arXiv:hep-th/9512077.
[26] M.R. Douglas, “D-Branes, Categories and N = 1 Supersymmetry”, J. Math. Phys. 42 (2001) 2818–2843 [arXiv:hep-th/0011017].
[27] M.R. Douglas and G.W. Moore, “D-Branes, Quivers and ALE Instantons”, arXiv:hep-th/9603167.
[28] R.G. Douglas, “C*-algebra Extensions and K-homology”, Annals of Mathematics Studies, Princeton University Press, 1980.
[29] D.S. Freed and M.J. Hopkins, “On Ramond-Ramond Fields and K-Theory”, J. High Energy Phys. 0005 (2000) 044 [arXiv:hep-th/0002027].
[30] D.S. Freed and E. Witten, “Anomalies in String Theory with D-Branes”, Asian J. Math. 3 (1999) 819 [arXiv:hep-th/9907189].
[31] H. Garcia-Compean, “D-Branes in Orbifold Singularities and Equivariant K-Theory”, Nucl. Phys. B 557 (1999) 480–504 [arXiv:hep-th/9902226].
[32] B.R. Greene and M.R. Plesser, “Duality in Calabi-Yau Moduli Space”, Nucl. Phys. B 338 (1990) 15–37.
[33] S. Gukov, “K-Theory, Reality and Orientifolds”, Commun. Math. Phys. 210 (2000) 621–639 [arXiv:hep-th/9901042].
[34] J.A. Harvey and G.W. Moore, “Noncommutative Tachyons and K-Theory”, J. Math. Phys. 42 (2001) 2765–2780 [hep-th/0009030].
[35] M.I. Hopkins, M.A. Hovey, “Spin Cobordism Determines Real K-theory, Math. Z. 210 (1992), 181-196.
[36] P. Horava, “Type IIA D-Branes, K-Theory and Matrix Theory”, Adv. Theor. Math. Phys. 2 (1999) 1373–1404 [arXiv:hep-th/9812135].
[37] K. Hori, “D-Branes, T-Duality and Index Theory”, Adv. Theor. Math. Phys. 3 (1999) 281–342 [arXiv:hep-th/9902102].
[38] M. Jakob, “A Bordism-Type Description of Homology”, Manuscr. Math. 96 (1998) 67–80.
[39] C.V. Johnson, D-Branes (Cambridge University Press, 2003).
[40] A. Kapustin, “D-Branes in a Topologically Non-Trivial B-Field”, Adv. Theor. Math. Phys. 4 (2000) 127–154 [arXiv:hep-th/9909089].
[41] M. Karoubi, K-Theory: An Introduction (Springer-Verlag, 1978).
[42] J. Kaminker, C. Schochet, “Topological Obstructions to Perturbations of Pairs of Operators, K-Theory and Operator Algebras”, Lecture Notes in Mathematics 575, 1975, Springer.
[43] O. Lechtenfeld, A.D. Popov and R.J. Szabo, “Noncommutative Instantons in Higher Dimensions, Vortices and Topological K-Cycles”, J. High Energy Phys. 0312 (2003) 022 [arXiv:hep-th/0310267].
[44] I. Madsen and J. Rosenberg, “The Universal Coefficient Theorem for Equivariant K-Theory of Real and Complex C*-Algebras”, Contemp. Math. 70 (1988) 145–173.
[45] J.M. Maldacena, G.W. Moore and N. Seiberg, “Geometrical Interpretation of D-Branes in Gauged WZW Models”, J. High Energy Phys. 0107 (2001) 046 [arXiv:hep-th/0105038].
[46] J.M. Maldacena, G.W. Moore and N. Seiberg, “D-Branes Instantons and K-Theory Charges”, J. High Energy Phys. 0111 (2001) 062 [arXiv:hep-th/0108100].
[47] Y. Matsuo, “Topological Charges of Noncommutative Soliton”, Phys. Lett. B 499 (2001) 223–228 [arXiv:hep-th/0006002].
[48] M. Matthey, “Mapping the Homology of a Group to the K-Theory of its C*-Algebra”, Illinois Math. J. 46 (2002) 953–977.
[49] G.W. Moore and E. Witten, “Self-Duality, Ramond-Ramond Fields and K-Theory”, J. High Energy Phys. 9711 (1997) 002 [arXiv:hep-th/9710230].
[50] G.W. Moore and E. Witten, “Anomalies and Non-BPS Brane Charges”, J. High Energy Phys. 0005 (2000) 032 [arXiv:hep-th/9912279].
[51] R.C. Myers, “Dielectric-Branes”, J. High Energy Phys. 9912 (1999) 022 [arXiv:hep-th/9910053].
[52] K. Olsen and R.J. Szabo, “Brane Descent Relations in K-Theory”, Adv. Theor. Math. Phys. 4 (2000) 889–1025 [arXiv:hep-th/9904153].
[53] K. Olsen and R.J. Szabo, “Constructing D-Branes from K-Theory”, Adv. Theor. Math. Phys. 4 (2000) 889–1025 [arXiv:hep-th/9907140].
[54] V. Periwal, “D-Brane Charges and K-Homology”, J. High Energy Phys. 0007 (2000) 041 [arXiv:hep-th/9906223].
[55] J. Polchinski, String Theory, Vol. 2 (Cambridge University Press, 1998).
[56] J. Rosenberg, C. Schochet, “The K¨ unneth Theorem and the Universal Coefficient Theorem for Kasparov’s Generalized K-functor”, Duke Math. J. 55, no 2 (1987), 431-474.
[57] A. Sen, “Tachyon Condensation on the Brane-Antibrane System”, J. High Energy Phys. 9808 (1998) 012 [arXiv:hep-th/9805170].
[58] R.M. Switzer, Algebraic Topology: An Introduction (Springer-Verlag, 1978).
[59] R.J. Szabo, “Superconnections, Anomalies and Non-BPS Brane Charges”, J. Geom. Phys. 43 (2002) 241–292 [arXiv:hep-th/0108043].
[60] R.J. Szabo, “D-Branes, Tachyons and K-Homology”, Mod. Phys. Lett. A 17 (2002) 2297–2316 [arXiv:hep-th/0209210].
[61] E. Witten, “Chern-Simons Gauge Theory as a String Theory”, Prog. Math. 133 (1995) 637–678 [arXiv:hep-th/9207094].
[62] E. Witten, “Phases of N = 2 Theories in Two Dimensions”, Nucl. Phys. B 403 (1993) 159–222 [arXiv:hep-th/9301042].
[63] E. Witten, “Bound States of Strings and p-Branes”, Nucl. Phys. B 460 (1996) 335–350 [arXiv:hep-th/9510135].
[64] E. Witten, “D-Branes and K-Theory”, J. High Energy Phys. 9812 (1998) 019 [arXiv:hep-th/9810188].

[65] U. Würgler, “Riemann-Roch Transformationen und Kobordismen”, Comm. Math. Helv. 46 (1971) 414–424.

[66] Z. Yosimura, “Universal Coefficient Sequences for Cohomology Theories of CW-Spectra”, Osaka J. Math. 16 (1979) 201–217.

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