Abstract

Different types of domination on the Sierpiński graphs \( S(K_n, t) \) will be studied in this paper. More precisely, we propose a minimal dominating set for \( S(K_n, t) \) so that the exact values of their domination numbers, Roman domination numbers, and double Roman domination numbers are given. As applications, some previous bounds and results are confirmed to be tight and further generalized.

Keywords: Domination number, Roman domination number, double Roman domination number, Sierpiński graphs.

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1 Introduction and preliminary

Let \( [n] = \{1, 2, \ldots, n\} \) be the set of positive integers at most \( n \). For every pair of positive integers \( n \) and \( t \), the Sierpiński graph \( S(K_n, t) \) is defined as the simple graph with vertices set \([n]^t = \{v_1v_2\ldots v_t \mid v_i \in [n] \text{ for } 1 \leq i \leq t\} \), in which \( u_1u_2\ldots u_t \) and \( v_1v_2\ldots v_t \) are adjacent if and only if there exists \( s \in [t] \) satisfying

\[
\begin{cases}
  u_j = v_j & \text{if } j < s; \\
  u_s \neq v_s; \\
  u_j = v_s \text{ and } v_j = u_s & \text{if } j > s.
\end{cases}
\]

In short, the consecutively repeated entries in a vertex are often written together. For example, the vertex \( u_1u_2u_3\ldots u_3 \) can be denoted by \( u_1u_2u_3^{t-2} \). See Figure 1 as an example of \( S(K_n, t) \) when \( n = 4 \) and \( t = 3 \).
Figure 1: The Sierpiński graph $S(K_4, 3)$.

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. For each $v \in V$, $N_G(v)$ denotes the set of vertices adjacent to $v$ in $G$, and $N_G[v] = N_G(v) \cup \{v\}$. A set $D \subseteq V$ is said to be dominating in $G$ if $\bigcup_{v \in D} N_G[v] = V$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality among all dominating sets of $G$. It is well-known, for example, see [7], that testing whether $\gamma(G) \leq k$ or not for some input $k$ is an NP-complete problem. A Roman dominating function on $G$ is defined as a function $f : V \rightarrow \{0, 1, 2\}$ such that every vertex $u \in V$ with $f(u) = 0$ has at least a neighbor $v \in N_G(u)$ satisfying $f(v) = 2$. The weight of $f$ is realized as $f(V) = \sum_{v \in V} f(v)$, and the Roman domination number of $G$, denoted by $\gamma_R(G)$, is the minimum weight among all Roman dominating functions of $G$. A double Roman dominating function on $G$ is defined as a function $f : V \rightarrow \{0, 1, 2, 3\}$ such that

(i) every vertex $u \in V$ with $f(u) = 0$ has at least one neighbor $v \in N_G(u)$ satisfying $f(v) = 3$ or at least two distinct neighbors $w, x \in N_G(u)$ satisfying $f(w) = f(x) = 2$, and

(ii) every vertex $u \in V$ with $f(u) = 1$ has at least a neighbor $v \in N_G(u)$ satisfying $f(v) \geq 3$.

Similarly, the weight of $f$ is realized as $f(V) = \sum_{v \in V} f(v)$, and the double Roman domination number of $G$, denoted by $\gamma_{dR}(G)$, is the minimum weight among all double Roman dominating functions of $G$.

Klavžar and U. Milutinović introduced the graph $S(K_n, t)$ in [11], and noticed that as $n = 3$ those graphs are exactly the Tower of Hanoi graphs. Later
in [12], $S(K_n, t)$ have been called Sierpiński graphs and studied from many aspects. The concept of Sierpiński graphs was generalized so that $S(G, t)$ can be constructed for every simple graph $G$. One can refer to [8, 9, 10, 13, 16, 14] for more details. The Roman domination was defined in [13, 17] and has been studied by many authors. The authors in [5, 6] proposed inspiring properties and problems involved with Roman domination in graphs. In 2016, Beeler et al. pioneered the study of double Roman domination in [4]. The decision of double Roman domination numbers was verified to be NP-complete for some families of graphs in [1]. Some upper and lower bounds for $\gamma_{dR}(G)$ were given in [2, 3, 18] in terms of the number of vertices and various parameters in a graph.

The following result given in [4] will be useful in this paper.

**Proposition 1.1.** In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the values 1.

In [14], Ramezani et al. made a progress in Roman domination numbers of Sierpiński graphs.

**Theorem 1.2.** For any integers $n \geq 2$ and $t \geq 1$,

$$\gamma_R(S(K_n, t)) \leq \begin{cases} \frac{2n^t}{n+1} & \text{if } t \text{ is odd;} \\ \frac{2n^t-1}{n+1} & \text{if } t \text{ is even.} \end{cases}$$

When $t = 2$, Theorem 1.2 was verified to be tight in [3]. Moreover, the authors in [3] also gave exact values of $\gamma_{dR}(S(K_n, 2))$. In this paper, we will show that the bounds in Theorem 1.2 are tight and determine the precise values of $\gamma_{dR}(S(K_n, t))$ for each positive integer $t$.

This paper is organized as follows. Basic definitions and previous results are mentioned in Section 1. In Section 2 a dominating set $D_{n,t}$ of the Sierpiński graph $S(K_n, t)$ are constructed, whose cardinality is $|D_{n,t}| = \gamma(S(K_n, t))$. With the aid of $D_{n,t}$, in Section 3 the domination number $\gamma(S(K_n, t))$, Roman domination number $\gamma_R(S(K_n, t))$, and double Roman domination $\gamma_{dR}(S(K_n, t))$ of $S(K_n, t)$ will be attained. The results are reviewed as a concluding remark in Section 4.

## 2 Dominating sets $D_{n,t}$ of $S(K_n, t)$

We propose a subset $D_{n,t}$ of vertices in $S(K_n, t)$ for every pair of positive integers $n \geq 2$ and $t$ in this section. It will be shown that $D_{n,t}$ is a dominating set for $S(K_n, t)$.

**Definition 2.1.** For positive integers $n$ and $t$, let $D_{n,t}$ be a subset of the vertex set $V_{n,t}$ of $S(K_n, t)$ such that $D_{n,1} = \{1\}$ and $D_{n,2} = \{11, 21, \ldots, n1\}$. When $t \geq 3$, for each $v = v_1v_2\ldots v_{t-2} \in D_{n,t-2}$, let

$$E_1(v) = \{v_1v_2\ldots v_{t-3}v_{t-2}a | a \in [n]\},$$

$$E_2(v) = \{v_1v_2\ldots v_{t-3}a\alpha v_{t-2} | a, \alpha \in [n] \setminus \{v_{t-2}\}\}.$$
and, if the entries of \( v \) are not constant, let \( \ell = \ell(v) \) denote the largest number in \([t - 3]\) satisfying \( v_\ell \neq v_{\ell+1} \) and

\[
E_3(v) = \{ v_1 v_2 \ldots v_{\ell-1} v_\ell v_{\ell+1}^{t-\ell-2} \alpha v_\ell \mid \alpha \in [n] \setminus \{v_\ell\} \}.
\]

Then, we define \( D_{n,t} \) as follows.

(i) If \( t \geq 3 \) is odd, then

\[
D_{n,t} = E_1(1^{t-2}) \cup E_2(1^{t-2}) \cup \bigcup_{v \in D_{n,t-2} \setminus \{1^{t-2}\}} E_1(v) \cup E_2(v) \cup E_3(v).
\]

(ii) If \( t \geq 4 \) is even, then

\[
D_{n,t} = \{1^{t-2} \alpha 1 \mid \alpha \in [n]\} \cup \bigcup_{v \in D_{n,t-2} \setminus \{1^{t-2}\}} E_1(v) \cup E_2(v) \cup E_3(v).
\]

The sets \( D_{n,t} \) are constructed inductively by odd and even \( t \), respectively. For example, one can see the subset \( D_{3,4} \) of vertices in \( S(K_3,4) \) in Figure 2.
Figure 2: The set $D_{3,4}$ of filled vertices in $S(K_3, 4)$.

In the rest of this section, we aim to describe the properties of $D_{n,t}$.

**Remark 2.2.** Some quick observations involved with $D_{n,t}$ are given below.

(i) We verify that $1^t \in D_{n,t}$ while all of the vertices $\{\alpha^t \mid \alpha \in [n] \setminus \{1\}\}$ are not in $D_{n,t}$ so that Definition 2.1 is well-defined. Since the vertices in $E_2(v)$ and $E_3(v)$ are obviously with non-constant entries, we focus on $E_1(v)$. Clearly, $E_1(v)$ contains a vertex in $D_{n,t}$ of constant entry if and only if $v$ is a vertex in $D_{n,t-2}$ of constant entry. Moreover, 1 and 11 are the only vertices in $D_{n,1}$ and $D_{n,2}$ of constant entry, respectively. The claim immediately follows by induction.

(ii) Directly from Definition 2.1, $|E_1(v)| = n$ and $|E_2(v)| = (n - 1)^2$ for $v \in D_{n,t-2}$ (if applicable), and $|E_3(v)| = n - 1$ for $v \in D_{n,t-2} \setminus \{1^{t-2}\}$.

To simplify the notation, let $D_{n,t}^*$ denote the set $D_{n,t} \setminus \{1^t\}$ throughout this paper.

**Lemma 2.3.** Let $n, t$ be positive integers not less than 3. If $t$ is odd, then the sets $E_1(u), E_2(v)$, and $E_3(w)$ are pairwise disjoint for $u, v \in D_{n,t-2}$, and $w \in D_{n,t-2}^*$. If $t$ is even, then the sets $E_1(u), E_2(v), E_3(w)$, and $\{1^{t-2} \alpha \mid \alpha \in [n]\}$ are pairwise disjoint for $u, v, w \in D_{n,t-2}^*$. Additionally, for each $1 \leq i \leq 3$, $E_i(u)$ and $E_i(v)$ are disjoint if $u \neq v$ in their proper domain.

**Proof.** By Definition 2.1, every vertex in $E_1(u)$ has the same entries in the last two entries, while different for each vertex in $E_2(v) \cup E_3(w)$. Therefore, $E_1(u)$ and $E_2(v) \cup E_3(w)$ have no intersection. Also, the $(t - 2)$-th and last entries are identical for every vertex in $E_3(w)$, while distinct for any vertex in $E_2(v)$. Hence, $E_2(v)$ and $E_3(w)$ have no intersection. Then, we deal with the set $\{1^{t-2} \alpha \mid \alpha \in [n]\}$ as $n$ is even. For $1 \leq i \leq 3$ and $u \in D_{n,t-2}^*$, we notice that all vertices in $E_i(u)$ do not have ones on all entries other than the $(t - 1)$-th position, and thus $E_i(u) \cap \{1^{t-2} \alpha \mid \alpha \in [n]\}$ is empty. Lastly, for $1 \leq i \leq 3$ and $u \neq v$ in their proper domain, the fact $E_i(u) \cap E_i(v) = \emptyset$ can be attained directly from the definition of $E_i$. The result follows.

Immediately from Lemma 2.3 we count the number of vertices in $D_{n,t}$.

**Lemma 2.4.** For positive integers $n \geq 2$ and $t$, the cardinality of $D_{n,t}$ is

$$|D_{n,t}| = \left\lceil \frac{n^t}{n + 1} \right\rceil.$$

**Proof.** We fix $n$ and prove the result by induction on $t$ in 2 cases: $t$ is odd and $t$ is even.
When $t$ is odd, we have $|D_{n,1}| = |\{1\}| = 1 = \lceil n/(n+1) \rceil$. Suppose that $|D_{n,t-2}| = \lceil n^{t-2}/(n+1) \rceil$ for some odd $t \geq 3$. Then by Definition 2.1(i),

$$|D_{n,t}| = n + (n-1)^2 + n^2 \cdot (|D_{n,t-2}| - 1)$$

$$= n + (n-1)^2 + n^2 \cdot \left( \frac{n^{t-2} + 1}{n+1} - 1 \right)$$

$$= \frac{n^t + 1}{n+1} = \left\lceil \frac{n^t}{n+1} \right\rceil,$$

where the equality in (1) can be referred to Remark 2.2(ii).

Let $t$ be even. Then $|D_{n,2}| = |\{\alpha 1 \mid \alpha \in [n]\}| = n = \lceil n^2/(n+1) \rceil$. Assume that $|D_{n,t-2}| = \lceil n^{t-2}/(n+1) \rceil$ for some even $t \geq 4$. Then by Definition 2.1(ii),

$$|D_{n,t}| = n + n^2 \cdot (|D_{n,t-2}| - 1)$$

$$= n + n^2 \cdot \left( \frac{n^{t-2} + n}{n+1} - 1 \right)$$

$$= \frac{n^t + n}{n+1} = \left\lceil \frac{n^t}{n+1} \right\rceil,$$

where the equality in (2) is referred to Remark 2.2(ii). The result follows.

In Lemma 2.5, we show that all vertices are distinct in $E_1$, $E_2$, and $E_3$. Moreover, by observing the neighborhood, they are separated far away. Recall that the distance between two vertices in a simple graph is the number of edges in a shortest path connecting them.

**Lemma 2.5.** Let $n \geq 2$ and $t$ be positive integers.

(i) If $t$ is odd, then every pair of distinct vertices in $D_{n,t}$ have distance at least 3 in $S(K_n,t)$.

(ii) If $t$ is even, then every pair of distinct vertices in $D_{n,t}$ have distance at least 3 in $S(K_n,t)$.

**Proof.** For a vertex $v$ in $S(K_n,t)$, let $N_{n,t}[v]$ denote the set containing the vertex $v$ and its neighbors in $S(K_n,t)$. Equivalently, two vertices $u$ and $v$ have distance at least 3 if and only if $N_{n,t}[u]$ and $N_{n,t}[v]$ have no intersection. In the following, we prove the result by induction on $t$ as $t$ is odd and even, respectively.

For (i), when $t$ is odd, the result holds for $D_{n,1} = \{1\}$, and we assume that it holds for $D_{n,t-2}$ for some odd $t \geq 3$. By observing Definition 2.1 for each $1 \leq i \leq 3$ if $x \in E_i(u)$, then the first $t-2$ entries of a vertex in $N_{n,t}[x]$ must be a vertex in $N_{n,t-2}[u]$. Therefore, for two distinct vertices $x \in E_i(u)$ and $y \in E_j(v)$ in $D_{n,t}$, where $1 \leq i,j \leq 3$, if $u$ and $v$ are distinct vertices in $D_{n,t-2}$ then $N_{n,t}[x]$ and $N_{n,t}[y]$ must have no intersection, or otherwise the first $t-2$ entries of an element in $N_{n,t}[x] \cap N_{n,t}[y]$ will be a vertex in $N_{n,t-2}[x] \cap N_{n,t-2}[y]$ so that the distance of $u$ and $v$ is less than 3, which contradicts to the induction
hypothesis. Now, suppose that \( x \) and \( y \) are two distinct vertices in \( D_{n,t} \) such that \( x \in E_i(u) \) and \( y \in E_j(u) \), for some \( 1 \leq i, j \leq 3 \). Additionally, for a non-constant vertex \( v = v_1 \ldots v_t \) in \( S(K_n,t) \), let \( v^* \) denote the unique neighbor \( v_1 \ldots v_{t-1}v_{t+1}v_t \ldots v_t \) of \( v \) obtained by flipping the entries, where \( 1 \leq t < t \) is the largest integer such that \( v_t \neq v_{t+1} \). Note that \((v^*)^* = v \), and hence \( x = y \) if and only if \( x^* = y^* \). The discussion can be partitioned into the following cases.

**Case 1.** \( i = 1 \) and \( j = 1 \).

Assume that \( x = \{u_1 \ldots u_{t-2} \alpha \} \) and \( y = \{u_1 \ldots u_{t-2} \beta \} \), where \( \alpha \neq \beta \). Then \( n_{n,t}[x] = \{u_1 \ldots u_{t-2} \alpha' \mid \alpha' \in [n]\} \cup \{x^*\} \), where \( x^* \) exists if \( x \) is not the only vertex. Therefore, if \( n_{n,t}[x] \cap n_{n,t}[y] \) is nonempty, then we may assume

\[
x^* = u_1 \ldots u_{t-3} \alpha u_t \ldots u_t = u_1 \ldots u_{t-3} \beta \alpha \in n_{n,t}[y],
\]

where \( \ell \leq t-2 \) is the largest number satisfying \( u_\ell \neq \alpha \) and \( \lambda \in [n] \). However, the \( \ell \)-th entry in (3) implies \( u_\ell = \alpha \), which is a contradiction.

**Case 2.** \( i = 1 \) and \( j = 2 \).

Assume that \( x = \{u_1 \ldots u_{t-2} \alpha \} \) and \( y = \{u_1 \ldots u_{t-3} \beta \lambda u_{t-2} \} \), where \( \beta, \lambda \in [n] \setminus \{u_{t-2}\} \). Since every vertex in \( n_{n,t}[y] \) is of the \((t-2)\)-th entry \( \beta \), hence, if \( n_{n,t}[x] \) and \( n_{n,t}[y] \) have intersection then the only possibility is

\[
x^+ = u_1 \ldots u_{t-1} \alpha u_t \ldots u_t = u_1 \ldots u_{t-3} \beta \lambda \delta \in n_{n,t}[y],
\]

where \( \ell \leq t-2 \) is the largest number satisfying \( u_\ell \neq \alpha \) and \( \delta \in [n] \). However, if \( \ell = t-2 \) then a contradiction occurs because the \((t-1)\)-th entry in (3) tells that \( u_{t-2} = \lambda \); if \( \ell < t-2 \) then the \( \ell \)-th entry in (4) implies \( u_\ell = \alpha \), which attains a contradiction also.

**Case 3.** \( i = 2 \) and \( j = 2 \).

Assume that \( x = \{u_1 \ldots u_{t-3} \beta \alpha u_{t-2} \} \) and \( y = \{u_1 \ldots u_{t-3} \lambda u_{t-2} \} \), where \( \alpha, \beta, \lambda, \delta \in [n] \setminus \{u_{t-2}\} \) with \( (\alpha, \beta) \neq (\lambda, \delta) \). Notice that that the \((t-3)\)-th entries of every vertex in \( n_{n,t}[x] \) and \( n_{n,t}[y] \) are \( \alpha \) and \( \lambda \), respectively. Hence, if \( n_{n,t}[x] \cap n_{n,t}[y] \) is not empty then \( \alpha = \lambda \) so that \( \beta \neq \delta \), and we may assume

\[
x^* = u_1 \ldots u_{t-3} \alpha u_{t-2} \beta = u_1 \ldots u_{t-3} \lambda \delta \epsilon \in n_{n,t}[y],
\]

where \( \epsilon \in [n] \). However, a contradiction happens since the \((t-1)\)-th entry in (5) says that \( \delta = u_{t-2} \).

**Case 4.** \( i = 1 \) and \( j = 3 \).

For the following cases involved with \( j = 3 \), let \( u \in D_{n,t}^* \) and \( h < t-2 \) be the largest number satisfying \( u_h \neq u_{h+1} \). Assume that \( x = \{u_1 \ldots u_{t-2} \alpha \} \) and \( y = \{u_1 \ldots u_{h-1} u_{h+1} u_h \ldots u_{t-2} \alpha \} \), where \( \beta \in [n] \setminus \{u_h\} \). It is clear that the \( h \)-th entry of each vertex in \( n_{n,t}[y] \) is \( u_{h+1} \). Thus, if there exists an element in \( n_{n,t}[x] \cap n_{n,t}[y] \) then it is

\[
x^* = u_1 \ldots u_{t-1} \alpha u_t \ldots u_t = u_1 \ldots u_{h-1} u_{h+1} u_h \ldots u_{t-2} \beta \lambda \in n_{n,t}[y],
\]

where \( \ell \leq t-2 \) is the largest number satisfying \( u_\ell \neq \alpha \) and \( \lambda \in [n] \). Consequently, the only possibility is \( \ell = h \) and the last \( t-h \) entries are all \( u_h \), which contradicts to \( \beta \neq u_h \).
Case 5. \(i = 2\) and \(j = 3\).
Assume that \(x = \{u_1 \ldots u_{t-3}\alpha \beta u_{t-2}\}\) and \(y = \{u_1 \ldots u_{t-1}u_h+1u_h \ldots u_h\lambda u_h\}\), where \(\alpha, \beta \in [n] \setminus \{u_{t-2}\}\) and \(\lambda \in [n] \setminus \{u_h\}\). From the fact that any vertex in \(N_{n,t}[x]\) has \(h\)-th entry \(u_h\), while \(u_{h+1}\) for those in \(N_{n,t}[y]\), it is obvious that \(N_{n,t}[x] \cap N_{n,t}[y]\) is the empty set.

Case 6. \(i = 3\) and \(j = 3\).
Assume that \(x = \{u_1 \ldots u_{h-1}u_{h+1}u_h \ldots u_h\alpha u_h\}\) and \(y = \{u_1 \ldots u_{h-1}u_{h+1}u_h \ldots u_h\beta u_h\}\), where \(\alpha, \beta \in [n] \setminus \{u_h\}\) with \(\alpha \neq \beta\). One can see that both of \(x\) and \(y\) have distinct last 2 entries, and they are different in the \((t-1)\)-th entry only. Therefore, if \(N_{n,t}[x] \cap N_{n,t}[y]\) is nonempty, then we may assume
\[
x^* = u_1 \ldots u_{h-1}u_{h+1}u_h \ldots u_h\alpha = u_1 \ldots u_{h-1}u_{h+1}u_h \ldots u_h\beta \lambda \in N_{n,t}[y],
\]
where \(\lambda \in [n]\). However, the \((t-1)\)-th entry in \((7)\) indicates \(\beta = u_h\), which is a contradiction.

For (ii), as \(t\) is even, we can check that the result holds for \(D^*_{n,2} = \{\alpha 1 \mid \alpha \in [n] \setminus \{1\}\}\), and assume that it holds for \(D^*_{n,t-2}\) for some even \(t \geq 4\). Similar argument can be made between \(E_1, E_2,\) and \(E_3\) as what we did for odd \(t\), but there is another set of vertices \(F = \{1^{t-2}\alpha 1 \mid \alpha \in [n] \setminus \{1\}\}\) in \(D^*_{n,t}\). Nevertheless, for every element \(x\) in \(F\), the vertices in \(N_{n,t}[x]\) are of all ones in the first \(t-2\) entries, while for any \(y \in E_i(u)\) where \(u \in D^*_{n,t-2}\) and \(i \in \{1, 3\}\), the vertices in \(N_{n,t}[y]\) are not constant in the first \(t-2\) entries since \(u\) is not constant by Remark 2.2(i). Moreover, we can see that \(D^*_{n,t-2}\) and \(\{1^{t-3}\beta \mid \beta \in [n] \setminus \{1\}\}\) have no intersection, or otherwise \(1^{t-2}\) and \(1^{t-3}\beta\) are of distance 1 in \(S(K_n, t-2)\), which violates the induction hypothesis. Therefore, for \(y \in E_2(u)\) where \(u \in D^*_{n,t-2}\), the first \(t-2\) entries of any element from \(N_{n,t}[y]\) are not all ones so that \(N_{n,t}[x] \cap N_{n,t}[y] = \emptyset\) for all \(x \in F\). The proof is completed.

We bring out the main property of the vertices set \(D_{n,t}\).

**Theorem 2.6.** For positive integers \(n \geq 2\) and \(t\), \(D_{n,t}\) forms a dominating set of \(S(K_n, t)\).

**Proof.** By Remark 2.2(i), each vertex in \(D^*_{n,t}\) is of degree \(n\), while \(1^t\) is of degree \(n-1\). If \(t\) is odd, then from Lemma 2.3(i), \(N_{n,t}[v]\) are pairwise disjoint for all \(v \in D_{n,t}\). Therefore,
\[
\left| \bigcup_{v \in D_{n,t}} N_{n,t}[v] \right| = \sum_{v \in D_{n,t}} N_{n,t}[v] = (n+1) \cdot \left( \left\lceil \frac{n^t}{n+1} \right\rceil - 1 \right) + n \cdot 1 = (n+1) \cdot \left( \frac{n^t + 1}{n+1} - 1 \right) + n = n^t,
\]
where (8) is from Lemma 2.4.
Next, assume that $t$ is even. By Lemma 2.5(ii), $N_{n,t}[v]$ are pairwise disjoint for all $v \in D_{n,t}^*$. Furthermore, none of the vertices in $D_{n,t}^*$ is adjacent to $1^t$ in $S(K_n,t)$, since by Definition 2.1(ii) the first $t-1$ entries of each vertex in $D_{n,t}^*$ are not all ones. As a result,

$$\left| \bigcup_{v \in D_{n,t}} N_{n,t}[v] \right| = \left| \bigcup_{v \in D_{n,t}^*} N_{n,t}[v] \right| + \left| N_{n,t}[1^t] \setminus \bigcup_{v \in D_{n,t}^*} N_{n,t}[v] \right|$$

$$\geq \left( \sum_{v \in D_{n,t}} N_{n,t}[v] \right) + |\{1^t\}| \text{ nonumber} \quad (9)$$

$$= (n + 1) \cdot \left( \left\lceil \frac{n^t}{n + 1} \right\rceil - 1 \right) + 1 \quad (10)$$

$$= (n + 1) \cdot \left( \frac{n^t + n}{n + 1} - 1 \right) + n^t,$$

where (10) is from Lemma 2.5(i).

The above argument indicates that $D_{n,t}$ and their neighbors include all $n^t$ vertices in $S(K_n,t)$, since $\bigcup_{v \in D_{n,t}} N_{n,t}[v]$ is a subset of the vertex set of $S(K_n,t)$. The result follows. \hfill \Box

3 Domination in $S(K_n,t)$

In this section, the exact values of domination numbers $\gamma(S(K_n,t))$, Roman domination numbers $\gamma_R(S(K_n,t))$, and double Roman domination numbers $\gamma_{dR}(S(K_n,t))$ of the Sierpiński graphs $S(K_n,t)$ are given.

3.1 Domination numbers

The vertices set $D_{n,t}$ is verified to be a dominating set for $S(K_n,t)$ in Theorem 2.6. Its cardinality $|D_{n,t}|$ is also obtained in Lemma 2.4. Therefore, we may attain the domination number $\gamma(S(K_n,t))$ as follows.

**Theorem 3.1.** For every positive integers $n \geq 2$ and $t$, the domination number of the Sierpiński graph $S(K_n,t)$ is

$$\gamma(S(K_n,t)) = \left\lceil \frac{n^t}{n + 1} \right\rceil.$$ 

**Proof.** Firstly, since the maximum vertex degree in $S(K_n,t)$ is $n$, it is straightforward to see that

$$\gamma(S(K_n,t)) \geq \left\lceil \frac{n^t}{n + 1} \right\rceil,$$
since there are \( n^t \) vertices in \( S(K_n,t) \).

Next, since the set \( D_{n,t} \) given in Definition 2.1 is shown to be a dominating set for \( S(K_n,t) \) in Theorem 2.6, we have

\[
\gamma(S(K_n,t)) \leq |D_{n,t}| = \left\lceil \frac{n^t}{n+1} \right\rceil,
\]

in which the cardinality of \( D_{n,t} \) is counted in Lemma 2.4. The proof is completed. 

\[\square\]

### 3.2 Roman domination numbers

In this subsection, we extend the results and proof in Theorem 3.1 and obtain the Roman domination numbers of \( S(K_n,t) \).

**Theorem 3.2.** For every positive integers \( n \geq 2 \) and \( t \), the Roman domination number of the Sierpiński graph \( S(K_n,t) \) is

\[
\gamma_R(S(K_n,t)) = \begin{cases} 
2 \left\lceil \frac{n^t}{n+1} \right\rceil & \text{if } t \text{ is odd;} \\
2 \left\lceil \frac{n^t}{n+1} \right\rceil - 1 & \text{if } t \text{ is even.}
\end{cases}
\]

**Proof.** Let \( f : V(S(K_n,t)) \to \{0, 1, 2\} \) be a Roman dominating function on \( S(K_n,t) \), where \( V(S(K_n,t)) \) is the set of vertices in \( S(K_n,t) \). Suppose that \( V_1 \) and \( V_2 \) are the sets of vertices in \( S(K_n,t) \) that are valued with 1 and 2 in \( f \), respectively. We have

\[
n^t = |V(S(K_n,t))| = |V_1 \cup \bigcup_{v \in V_2} N_{n,t}(v)|
\]

\[
\leq |V_1| + \sum_{v \in V_2} |N_{n,t}(v)|
\]

\[
\leq |V_1| + (n+1)|V_2|,
\]

where (11) is from the fact that the maximal vertex degree in \( S(K_n,t) \) is \( n \).

Then, it comes to a linear program: finding \( \min\{|V_1| + 2|V_2|\} \) provided that the nonnegative integers \( |V_1| \) and \( |V_2| \) satisfying \( |V_1| + (n+1)|V_2| \geq n^t \). By comparing the slopes, \( \min\{|V_1| + 2|V_2|\} \) can be attained if we make \( |V_1| \) as small as possible. Therefore,

\[
\gamma_R(S(K_n,t)) \geq \min\{|V_1| + 2|V_2|\}
\]

\[
= \begin{cases} 
1 \cdot 0 + 2 \cdot \frac{n^t}{n+1} = 2 \left\lceil \frac{n^t}{n+1} \right\rceil & \text{if } t \text{ is odd;} \\
1 \cdot 1 + 2 \cdot \frac{n^t-1}{n+1} = 2 \left\lceil \frac{n^t}{n+1} \right\rceil - 1 & \text{if } t \text{ is even.}
\end{cases}
\]

(12)
where (12) is obtained by letting $(|V_1|,|V_2|) = (0, (n^t + 1)/(n + 1))$ if $t$ is odd, and $(|V_1|,|V_2|) = (1, (n^t - 1)/(n + 1))$ if $t$ is even.

On the other hand, although the upper bound has been shown in Theorem 1.2, we derive a Roman dominating function from the set $D_{n,t}$ and reprove it. If $t$ is odd, let
\[
f(v) = \begin{cases} 2 & \text{if } v \in D_{n,t}; \\
0 & \text{if } v \notin D_{n,t}
\end{cases}
\]
which achieves a Roman domination since $D_{n,t}$ is a dominating set of $S(K_n, t)$.

Also, $f$ sums to $\sum_v f(v) = 2|D_{n,t}| = 2[n^t/(n + 1)]$. If $t$ is even, let
\[
f(v) = \begin{cases} 2 & \text{if } v \in D^*_{n,t}; \\
1 & \text{if } v = 1^t; \\
0 & \text{if } v \notin D_{n,t}
\end{cases}
\]
which attains a Roman domination, since in the latter part of proof in Theorem 2.6 we mention that $1^t$ is not a neighbor of any vertex in $D_{n,t}$. In this case, we have $\sum_v f(v) = 2|D^*_{n,t}| + 1 = 2[n^t/(n + 1)] - 1$.

Since the lower bound and upper bound meet in the above argument, the result follows. \qed

### 3.3 Double Roman domination numbers

In this subsection, we give the exact values of the double Roman domination numbers $\gamma_{dR}(S(K_n, t))$ for arbitrary $n, t$, which generalize the result [3, Theorem 3.2] stating that $\gamma_{dR}(S(K_n, t)) = 3n - 1$. The methods of finding double Roman domination numbers of $S(K_n, t)$ will be similar to those for Roman domination numbers.

**Theorem 3.3.** For every positive integers $n \geq 2$ and $t$, the double Roman domination number of the Sierpiński graph $S(K_n, t)$ is

\[
\gamma_{dR}(S(K_n, t)) = \begin{cases} 3 \left\lceil \frac{n^t}{n+1} \right\rceil & \text{if } t \text{ is odd}; \\
3 \left\lceil \frac{n^t}{n+1} \right\rceil - 1 & \text{if } t \text{ is even}.
\end{cases}
\]

**Proof.** By Proposition 1.1 we narrow our discussion by letting $f : V(K_n, t) \to \{0, 2, 3\}$ be a double Roman dominating function, where $V(K_n, t)$ is the set of vertices in $S(K_n, t)$. Suppose that $V_2$ and $V_3$ are the set of vertices in $S(K_n, t)$ that are valued with 2 and 3, respectively. Since a vertex valued with 0 can be “guarded” by 2 neighbors valued with 2 in $f$, we may assume that a vertex valued with 2 in $f$ guards at most $1 + n/2$ vertices in a graph. Therefore, the equation (11) becomes

\[
n^t = |V(S(K_n, t))| \leq \left(1 + \frac{n}{2}\right)|V_2| + (n + 1)|V_3|.
\]
A new linear program appears as follows: finding min\{2|V_2| + 3|V_3|\} provided that nonnegative integers |V_2| and |V_3| satisfying (13). We can see that the absolute value of slope in (13) is \(2 - \frac{2}{n+2}\) ≥ \(\frac{3}{2}\). Therefore, min\{2|V_3|+3|V_3|\} can be attained if |V_2| is as small as possible. We have

\[
\gamma_{dR}(S(K_n, t)) \geq \min\{2|V_2| + 3|V_3|\}
\]

\[
= \begin{cases} 
2 \cdot 0 + 3 \cdot \frac{n^t+1}{n+1} = 3 \left\lceil \frac{n^t}{n+1} \right\rceil & \text{if } t \text{ is odd;} \\
2 \cdot 1 + 3 \cdot \frac{n^t}{n+1} = 3 \left\lceil \frac{n^t}{n+1} \right\rceil - 1 & \text{if } t \text{ is even}
\end{cases}
\]

(14)

where (14) is obtained by letting (|V_2|, |V_3|) = (0, (n^t + 1)/(n + 1)) if t is odd, and (|V_2|, |V_3|) = (1, (n^t - 1)/(n + 1)) if t is even.

On the other hand, we verify the upper bound by giving a double Roman dominating function. If t is odd let

\[
f(v) = \begin{cases} 
3 & \text{if } v \in D_{n,t}; \\
0 & \text{if } v \notin D_{n,t}
\end{cases}
\]

with \(\sum_v f(v) = 3|D_{n,t}| = 3\lceil n^t/(n + 1) \rceil\), and if t is even let

\[
f(v) = \begin{cases} 
3 & \text{if } v \in D_{n,t}^*; \\
2 & \text{if } v = 1; \\
0 & \text{if } v \notin D_{n,t}
\end{cases}
\]

with \(\sum_v f(v) = 3|D_{n,t}^*| + 2 = 3\lceil n^t/(n + 1) \rceil - 1\). Similar to the proof in Theorem 3.2 each of the above two cases reaches a double Roman domination in \(S(K_n, t)\).

Thus, the lower bound and upper bound meet. We have the proof.

\[\square\]

4 Concluding remark

In this study, based on the properties of \(D_{n,t}\) defined in Definition 2.1, we obtain the precise values of domination numbers \(\gamma(S(K_n, t))\), Roman domination numbers \(\gamma_R(S(K_n, t))\) and double Roman domination numbers \(\gamma_{dR}(S(K_n, t))\) of Sierpiński graphs \(S(K_n, t)\). As applications, we improve Theorem 1.2 given in [14] by showing that the equality hold for any pair of n and t. Moreover, since \(\gamma_R(S(K_n, 2))\) and \(\gamma_{dR}(S(K_n, 2))\) have been obtained in [5], our work also extend the their results to arbitrary t. To conclude this paper, one can refer to the following table.
Sierpiński graphs
$S(K_n, t)$

| Numbers | Dominating sets or functions |
|---------|-----------------------------|
| $\gamma(S(K_n, t)) = \frac{n^t}{n+1}$ | $D_{n,t}$ |

Roman domination
$\gamma_R(S(K_n, t))$

| When $t$ is odd, |
|-----------------|
| $f(v) = \begin{cases} 2 & \text{if } v \in D_{n,t} \\ 0 & \text{if } v \notin D_{n,t} \end{cases}$ |

| When $t$ is even, |
|------------------|
| $f(v) = \begin{cases} 2 & \text{if } v \in D_{n,t}^* \\ 1 & \text{if } v = 1^t \\ 0 & \text{if } v \notin D_{n,t} \end{cases}$ |

Double Roman domination
$\gamma_{dR}(S(K_n, t))$

| When $t$ is odd, |
|-----------------|
| $f(v) = \begin{cases} 3 & \text{if } v \in D_{n,t} \\ 0 & \text{if } v \notin D_{n,t} \end{cases}$ |

| When $t$ is even, |
|------------------|
| $f(v) = \begin{cases} 3 & \text{if } v \in D_{n,t}^* \\ 2 & \text{if } v = 1^t \\ 0 & \text{if } v \notin D_{n,t} \end{cases}$ |

Table 1: Domination in the Sierpiński graphs $S(K_n, t)$.

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