COUNTING MAXIMALLY BROKEN MORSE TRAJECTORIES ON ASPHERICAL MANIFOLDS

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ABSTRACT. We prove a lower bound on the number of maximally broken trajectories of the negative gradient flow of a Morse-Smale function on a closed aspherical manifold in terms of integral (torsion) homology.

1. INTRODUCTION

A fundamental relationship between a Morse function $f : M \to \mathbb{R}$ on a closed smooth manifold $M$ and its homology are the Morse inequalities. They relate the number $\nu_p(f)$ of critical points of index $p$ to the Betti numbers $b_p(M)$ of $M$ by the inequalities

$$\sum_{p=0}^{n} (-1)^{p+n} \nu_p(f) \geq \sum_{p=0}^{n} (-1)^{p+n} b_p(M)$$

for every $n \leq \text{dim}(M)$.

The Morse inequalities follow by easy homological algebra on the cellular chain complex of a CW-complex $X$ that is homotopy equivalent to $M$ and has $\nu_p(f)$ many $p$-cells. Such a CW-complex always exists. If $f$ is a Morse-Smale function, i.e. a Morse function satisfying the Morse-Smale condition with respect to a Riemannian metric on $M$, then one can directly associate a chain complex with $f$, the Morse-Smale-Witten complex which computes the homology of $M$.

The goal of this paper is to describe the following relation between a Morse function $f$ on a closed smooth aspherical manifold and its homology which does not seem obtainable from the homological algebra of the Morse-Smale-Witten complex alone (see Remark 2.4).

**Theorem 1.1.** Let $M$ be a $d$-dimensional closed orientable aspherical Riemannian manifold. Let $f : M \to \mathbb{R}$ be a Morse-Smale function. Then the number of maximally broken trajectories of $f$ is at least

$$\frac{1}{2^d 2(d+2)!} \cdot \sum_{p=0}^{d} \text{size}(H_p(M; \mathbb{Z})).$$

Let us recall and define the notions appearing in the statement. If $A$ is an abelian group we define its size $\text{size}(A) \in \mathbb{N} \cup \{\infty\}$ by

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size$(A) = \text{rank}(A) + \log |\text{tors } A|$, the sum of its rank and the logarithm of the cardinality of its torsion subgroup.

An $n$-part broken trajectory of $f$ is a sequence of critical points $x_n, \ldots, x_0$ where $i$ is the index of $x_i$, and a sequence of (unparametrized) flow lines $\gamma_n, \ldots, \gamma_1$ of $-\nabla f$ where each $\gamma_i$ runs from $x_i$ to $x_{i-1}$. The number of (unparametrized) flow lines between critical points of index $i$ and $i-1$ is always finite. See [4] as a background reference for the relevant Morse theory. If $M$ is $d$-dimensional, then a $d$-part broken trajectory is also called a \textit{maximally broken trajectory}.

The theorem above is inspired by the following result of H. Alpert:

\textbf{Theorem 1.2} (Alpert [1, Theorem 18]). \textit{Let $M$ be a closed oriented aspherical Riemannian manifold. Let $f: M \to \mathbb{R}$ be a Morse-Smale function. Then the number of maximally broken trajectories of $f$ is at least $\|M\|$, where $\|M\|$ denotes the simplicial volume of $M$.}

With Gromov’s and Thurston’s computation of simplicial volume of hyperbolic manifolds [6] Alpert concludes:

\textbf{Corollary 1.3} ([1, Theorem 1]). \textit{Let $M$ be a closed oriented hyperbolic $d$-manifold. Then the number of maximally broken trajectories of a Morse-Smale function on $M$ is at least $\text{vol}(M)/v_d$, where $v_d$ is the volume of a regular ideal $d$-simplex in hyperbolic $d$-space.}

We follow Alpert’s methods closely. In fact, this paper grew out of a seminar in which we read Alpert’s paper and realized that her proof works integrally and can be adjusted to show the lower (torsion) homology bound for the number of maximally broken trajectories.

In the next section we discuss some examples where Theorem 1.1 but not Theorem 1.2 can be applied. Section 4 is devoted to the comparison between the norms on the singular chain complex and the so-called oriented chain complex of a topological space. Section 4 introduces the key lemma of the method in the version adapted to our case, namely the integral reduction lemma (Lemma 4.4). Section 5 deals with properties of norms related to cellular decompositions of manifolds coming from Morse-Smale functions. The proof of Theorem 1.1 is concluded in Section 6.

We would like to thank our former colleague Federico Franceschini with whom we started working on Alpert’s methods.

2. Examples

We discuss three geometric examples and applications. Each example exhibits a sequence $(M_n)\_n$ of closed aspherical 3-manifolds such that for any choice of Morse-Smale function $f_n$ on $M_n$ the number of maximally broken trajectories of $f_n$ tends to infinity. The first example is taken from Alpert’s paper and can be concluded from Corollary 1.3. The other two examples follow from Theorem 1.1.

\textbf{Example 2.1} ([1, Proposition 2]). \textit{Let $\phi: \Sigma \to \Sigma$ be a pseudo-Anosov diffeomorphism of a closed surface $\Sigma$ of genus at least two. Then the mapping tori...}
$M_n = T(\phi^n)$ are closed hyperbolic 3-manifolds that admit a Morse-Smale function $f_n: T(\phi^n) \to \mathbb{R}$ such that the number of critical points of $f_n$ is uniformly bounded; but $\text{vol}(T(\phi^n)) \to \infty$ and hence the number of maximally broken trajectories of $f_n$ tends to $\infty$.

**Example 2.2.** Let $K$ be the figure eight knot in $S^3$. The knot complement $M = S^3 - K$ is a complete hyperbolic 3-manifold of finite volume. By various Dehn fillings on the torus boundary of $M$ one obtains a sequence of closed hyperbolic 3-manifolds $M_n$ with $|\text{tors}(H_1(M_n; \mathbb{Z}))| \geq n$ and uniformly bounded volume (see [3, Theorem 1.7]). In fact, by Thurston’s hyperbolic Dehn filling theorem the volume of $M_n$ is at most the volume of $M$. By Theorem 1.1 the number of maximally broken trajectories of $f_n$ for any sequence of Morse-Smale functions $f_n: M_n \to \mathbb{R}$ grows at least like $\log(n)$.

**Example 2.3.** For every $n \in \mathbb{N}$ consider the group $\Gamma_n = \langle x, y, z \mid [x, z] = [y, z] = 1, [x, y] = z^n \rangle$.

One verifies that $\Gamma_n$ is a torsion-free nilpotent group with Malcev completion $\mathbb{R}^3$: This can be seen from the fact that $\Gamma_n$ is just the variant of the integral 3-dimensional Heisenberg group where the $(1, 2)$-matrix entry is in $n\mathbb{Z}$. Let $M_n = \mathbb{R}^3/\Gamma_n$ be the associated nil-manifold. In particular, each $M_n$ is a 3-dimensional closed aspherical smooth 3-manifold. The first homology group of $M_n$ equals the abelianization

$$\Gamma_n/\Gamma_n \cong \mathbb{Z}^2 \oplus \mathbb{Z}/n.$$ 

In particular, the Betti numbers of each $M_n$ are $b_0 = 1, b_1 = 2, b_2 = 2, b_3 = 1$ by Poincaré duality and thus uniformly bounded. Fix a Riemannian metric on each $M_n$ and choose Morse-Smale functions $f_n: M_n \to \mathbb{R}$ with respect to these metrics. The number of maximally broken trajectories of $f_n$ grows at least like $\log(n)$ by Theorem 1.1. Note also that the simplicial volume of each $M_n$ vanishes since its fundamental group $\Gamma_n$ is amenable [6].

Finally, we would like to discuss why Theorem 1.1 is not an obvious consequence of the Morse-Smale-Witten complex alone, as was pointed out before the statement of Theorem 1.1.

**Remark 2.4.** Since the Morse-Smale-Witten complex $W_\ast(M)$ associated to a Morse-Smale function $f: M \to \mathbb{R}$ computes the singular homology of $M$ it is tempting to extract the bound in Theorem 1.1 from it by homological algebra. Each chain group $W_i(M)$ is a free abelian group whose rank is the number of critical points of index $i$. Let $p$ and $q$ be critical points of index $i$ resp. $i - 1$. The $(p, q)$-matrix entry of the differential $W_i(M) \to W_{i-1}(M)$ is the signed number of flow lines from $p$ to $q$. If we could bound all these numbers we would obtain bounds on the norm of the differentials and hence bounds on the (torsion) homology (cf. [8, Lemma 3.1]). But it could be a priori possible that the number of maximally broken trajectories is small but there are still two critical points $p, q$ as above with a large number of flow lines from $p$ to $q$. This prevents us from deducing Theorem 1.1 directly from the Morse-Smale-Witten complex.

Other consequences of Theorem 1.1 are more elementary and can be deduced more directly. It follows from Theorem 1.1 for example, that on a
closed aspherical smooth manifold any Morse-Smale function has a critical point of index $p$ for every $p \in \{0, \ldots, \dim(M)\}$. This, however, can be shown directly by basic homotopy theory:

It suffices to show that any CW-structure on a closed aspherical manifold $M$ has cells in every degree $\leq \dim M$. Let $X$ be a CW-structure on $M$. Suppose for $i < \dim X = \dim M$ we have $X^{(i)} = X^{(i+1)}$, i.e. there are no $(i+1)$-cells. We can suppose that $i > 0$; otherwise we had trivial $\pi_1(M)$ and $M$ were of dimension 0. We can also suppose that $i > 1$; if $i = 1$ then $\pi_1(M)$ would be free. If it is free abelian, $M = S^1$ and the claim is verified; if not we have a contradiction, as there is no closed aspherical manifold with free nonabelian fundamental group since the homology of such a group does not satisfy Poincaré duality. Hence $i \geq 2$ and the inclusion $X^{(i)} \to X$ is a $\pi_1$-isomorphism. Let $p : \tilde{X} \to X$ be the universal covering. Note that $\tilde{X}^{(i)} = p^{-1}(X^{(i)})$ is the universal covering of $X^{(i)}$. Since $\tilde{X}^{(i)} = \tilde{X}^{(i+1)} \to \tilde{X}$ is $(i+1)$-connected the homology satisfies

$$H_{\ell}(\tilde{X}^{(i)}) = 0 \text{ for all } 0 < \ell \leq i.$$ 

On the other hand, $H_{\ell}(\tilde{X}^{(i)}) = 0$ for $\ell > i$ since $\tilde{X}^{(i)}$ is $i$-dimensional. By Whitehead’s theorem $X^{(i)}$ is contractible and thus $X^{(i)}$ is aspherical with fundamental group $\pi_1(X) = \pi_1(M)$. Hence the inclusion $X^{(i)} \to X$ is a homotopy equivalence. Since the $\mathbb{Z}/2$-homology of $X$ does not vanish in degree $\dim M$ and $i \leq \dim M$, this implies that $i = \dim M$.

3. Norms on integral chains

The integral singular chain complex of a topological space $X$ is denoted by $C_*(X)$. The integral norm $\|c\|_Z$ of a chain $c = \sum_\sigma a_\sigma \sigma \in C_p(X)$, where $\sigma$ runs over the singular $p$-simplices of $X$, is defined as

$$\|c\|_Z = \sum_\sigma |a_\sigma|.$$ 

The symmetric group $S(p+1)$ in $p+1$ letters acts on the set $S_p(X)$ of singular $p$-simplices of $X$ through the affine-linear extension of the permutation action on the vertices. Let $O_p(X) \subset C_p(X)$ be the submodule generated by the union of 

$$\{ g\sigma - \text{sign}(g)\sigma \mid \sigma \in S_p(X), g \in S(p+1) \}$$

and 

$$\{ \sigma \in S_p(X) \mid \text{there is a transposition } t \text{ with } t\sigma = \sigma \}.$$ 

It is straightforward to verify that $O_*(X)$ is a subcomplex [Proposition 2.2]. We denote by $C^*_p(X)$ the quotient complex $C_*(X)/O_*(X)$. Obviously, $C^*_p$ provides a functor from topological spaces to chain complexes of abelian groups. The natural projection $\text{pr}_*: C_* \to C^*_p$ is a natural chain homomorphism. For an element $c \in C^*_p(X)$ we let the integral oriented norm $\|c\|_Z^{\text{or}}$ be defined as

$$\|c\|_Z^{\text{or}} = \min \{ \|c'\|_Z \mid \text{pr}_p(c') = c \}.$$ 

Further, both the integral and the integral oriented norm induce functions on $H_p(C_*(X)) = H_p(X)$ and $H_p(C^*_p(X))$, respectively, by taking the minimum among representing cycles; we also call these functions integral norm.
and integral oriented norm, respectively, and denote them by the same symbols as their analogues on the chain complexes.

**Definition 3.1.** The integral simplicial volume of a closed oriented manifold $M$ is the integral norm of its fundamental class. It is denoted by $\|M\|_\mathbb{Z}$.

The oriented singular chain complex $C_*^\sigma$ historically predates the singular chain complex $C_*$ and was introduced in 1944 by Eilenberg. A comparison between the two should have been discussed a long time ago but was only addressed in writing in a 1995 paper by Barr where the following result is proved \[\text{[5, Theorem 1.1]}\].

**Theorem 3.2.** The projection $\text{pr}_\sigma$ induces an isomorphism in homology.

To obtain the next theorem we basically have to reprove Theorem 3.2 with greater care.

**Theorem 3.3.** Let $c$ be a singular $p$-cycle and $[c]$ its homology class. Then

$$\|\langle \text{pr}_p(c) \rangle \|_\mathbb{Z}^\sigma \leq \|c\|_\mathbb{Z} \leq (p + 1)! \cdot \|\langle \text{pr}_p(c) \rangle \|_\mathbb{Z}^\sigma.$$ 

**Proofs of Theorems 3.2 and 3.3.** For a topological space $X$ and $x \in X$ let $P_\sigma(X)$ be the space of continuous paths in $X$ starting from $x$ endowed with the compact-open topology. Let $G(X)$ be the topological sum of $P_\sigma(X)$ running over all points $x$ in $X$. In a natural way, $X \to G(X)$ becomes an endofunctor of the category of topological spaces. Evaluation at 1 yields a natural transformation $\text{ev}$ from $G$ to the identity functor.

For any space $Y$ let $\epsilon: C_0(Y) \to \mathbb{Z}$ denote the augmentation map sending each singular 0-simplex to 1. Note that $C_0^\epsilon(Y) = C_0(Y)$. We set $C_{-1}(Y) = C_{-1}^\epsilon(Y) = \mathbb{Z}$.

To make sense of the next statement we will think of $S(p + 1)$ being embedded into $S(p + 2)$ and consisting of those permutations of $\{0, \ldots, p + 1\}$ that fix $p + 1$. According to \[\text{[5, Proposition 4.1]}\] there is a natural and $S(p+1)$-equivariant chain contraction

$$s_p: C_p(G(X)) \to C_{p+1}(G(X)), \ p \geq -1,$$

of the (augmented) chain complex

$$\cdots \to C_2(G(X)) \to C_1(G(X)) \to C_0(G(X)) \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$ 

By connectedness, a singular $p$-simplex $\sigma \in S(G(X))$ lies in exactly one component $P_\sigma(X)$ of $G(X)$. We say that $x$ is the base point of $\sigma$ and denote it by $b_\sigma = x$. We may view $\sigma$ as a continuous map $\Delta^p \times [0, 1] \to X$. We use barycentric coordinates for the points of a standard simplex. The explicit formula for $s$ applied to a singular $p$-simplex in \[\text{[5]}\] is

$$s_p(\sigma)(t_0, \ldots, t_{p+1}; u) = \begin{cases} \sigma\left(\frac{t_0}{t_{p+1}}, \ldots, \frac{t_p}{t_{p+1}}; (1 - t_{p+1})u\right) & \text{if } t_{p+1} \neq 1, \\ b_\sigma & \text{otherwise,} \end{cases}$$

provided $p > 0$. There is a similar expression for $p = 0$. Therefore we see that $s_p(\sigma)$ is a singular $(p + 1)$-simplex for every singular $p$-simplex $\sigma$. By naturality we obtain that

$$\|s_p(c)\|_\mathbb{Z} \leq \|c\|_\mathbb{Z}$$

(3.1)

---

\[1\] As the MathSciNet reviewer of \[\text{[5]}\] puts it: *This paper discusses a strange historical lacuna.*
for every chain \( c \in C_p(G(X)) \).

Further, for every \( p \geq 0 \) there is a natural and \( S(p+1) \)-equivariant homomorphism \( \theta_p : C_p(G(X)) \to C_p(G(X)) \) such that \( \theta_p \) followed by \( C_p(\text{ev}) \) is the identity [5, Proposition 4.2]. Note that we do not claim that \( \theta_p \) is a chain map. The explicit formula for \( \theta_p(\sigma) \) where \( \sigma \in \mathcal{S}_p(X) \) is

\[
\theta_p(\sigma)(t_0, \ldots, t_p; u) = \sigma \left( ut_0 + \frac{1-u}{p+1}, \ldots, ut_p + \frac{1-u}{p+1} \right).
\]

So the basepoint of \( \theta_p(\sigma) \) is the barycenter of \( \sigma \). We see that \( \theta_p \) maps every singular \( p \)-simplex of \( X \) to a singular \( p \)-simplex of \( G(X) \). As above, this implies an estimate

\[
\|\theta_p(c)\|_Z \leq \|c\|_Z \tag{3.2}
\]

for every chain \( c \in C_p(X) \). For every \( p \geq 0 \) we have

\[ s_p(O_p(G(X))) \subset O_{p+1}(G(X)) \]

as an immediate consequence of equivariance. Hence \( s_* \) descends to a natural chain contraction \( s_*^\circ \) of

\[
\cdots \to C^\circ_q(G(X)) \to C^\circ_q(G(X)) \to C^\circ_q(G(X)) \xrightarrow{\delta} Z \to 0.
\]

Moreover, each \( \theta_p \) descends to a natural homomorphism

\[
\theta_p^\circ : C^\circ_p(X) \to C^\circ_p(G(X))
\]

that is a left inverse of \( C_p(\text{ev}) \). The 1-Lipschitz property in (3.1) or (3.2) holds true for \( s_*^\circ \) or \( \theta_p^\circ \) as well.

Next we construct a natural chain homomorphism \( \phi : C_*(X)^\circ \to C_*(X) \) that is a chain homotopy inverse to the projection \( \text{pr}_* \) such that

\[
\|\phi_p(c)\|_Z \leq (p+1)! \cdot \|c\|_Z^\circ.
\]

This will finish the proof. Suppose that we have for any space \( X \) and every \( 0 \leq i \leq p \) natural chain homomorphisms \( \phi_i : C_i(X)^\circ \to C_i(X) \) such that

\[
\begin{align*}
C^\circ_p(X) & \xrightarrow{\text{pr}} C^\circ_{p-1}(X) \xrightarrow{\partial_p^\circ} \cdots \xrightarrow{\partial_p^\circ} C^\circ_i(X) \xrightarrow{\partial_p^\circ} C^\circ_0(X) \xrightarrow{\text{id}} Z \\
C_p(X) & \xrightarrow{\partial_p} C_{p-1}(X) \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_p} C_1(X) \xrightarrow{\partial_p} C_0(X) \xrightarrow{\text{id}} Z
\end{align*}
\]

commutes and such that \( \phi_i \) is \((i+1)!\)-Lipschitz for every \( i \in \{0, \ldots, p\} \). We define \( \phi_{p+1} \) as the composition

\[
\begin{align*}
C^\circ_{p+1}(X) & \xrightarrow{\phi_{p+1}} C^\circ_{p+1}(G(X)) \xrightarrow{\partial_{G(X)}^\circ} C^\circ_p(G(X)) \xrightarrow{\phi_p} C_p(G(X)) \\
& \xrightarrow{s_p} C_p(G(X)) \xrightarrow{C_p(\text{ev})} C_{p+1}(X).
\end{align*}
\]
One verifies that $\phi_{p+1}$ adds another commutative square to (3.3) by the following computation:

$$
\partial_X \circ \phi_{p+1} = \partial_X \circ C_{p+1}(ev) \circ s_p \circ \phi_p \circ \partial_{G(X)} \circ \theta_{p+1}^o \\
= C_p(ev) \circ \partial_{G(X)} \circ s_p \circ \phi_p \circ \partial_{G(X)} \circ \theta_{p+1}^o \\
= C_p(ev) \circ (\text{id} - s_{p-1} \circ \partial_{G(X)}) \circ \phi_p \circ \partial_{G(X)} \circ \theta_{p+1}^o \\
= C_p(ev) \circ \phi_p \circ \partial_{G(X)} \circ \theta_{p+1}^o - C_p(ev) \circ s_{p-1} \circ \partial_{G(X)} \circ \phi_p \circ \partial_{G(X)} \circ \theta_{p+1}^o \\
= 0 \text{ since } \partial_{G(X)} \circ \phi_p = \phi_{p-1} \circ \partial_{G(X)}.
$$

Further, since the boundary homomorphism $C_{p+1}(G(X)) \to C_p(G(X))$ is $(p + 2)$-Lipschitz and $\phi_p$ is $(p + 1)!$-Lipschitz and the other homomorphisms appearing in (3.4) are 1-Lipschitz the map $\phi_{p+1}$ is $(p + 2)!$-Lipschitz.

Finally, to show that there are natural chain homotopies for $\text{pr}_* \circ \phi_*$ and $\phi_* \circ \text{pr}_*$ is similar to constructing $\phi_*$. It is also a consequence of [5, Theorem 1.1].

4. INTEGRAL ORIENTED REDUCTION LEMMA

In this section we prove the integral oriented version of the amenable reduction lemma [1, Lemma 4, Corollary 5], [2, Lemma 4]. It will allow us to estimate the integral oriented norm of cycles in terms of some conditions on the shape of the simplices composing them.

By [2, Lemma 2] (which attributes it to [6, p. 48]) the singular chain complex of an aspherical space $Z$ admits a straightening operator

$$\text{str}_*: C_*(Z) \to C_*(Z),$$

i.e. a chain homomorphism $\text{str}_*$ such that

1. $\text{str}_*$ is chain homotopic to the identity,
2. $\text{str}_p$ is equivariant with regard to the action of the symmetric group $S(p + 1)$ on $C_p(Z)$, and
3. If the singular $p$-simplices $\sigma$ and $\sigma'$ have the same sequence of vertices and their lifts to the universal cover have the same sequence of vertices then $\text{str}_p(\sigma) = \text{str}_p(\sigma')$.

By the second property the straightening operator descends to a well-defined chain homomorphism

$$\text{str}_*: C_*(Z) \to C_*(Z).$$

Remark 4.1. For any cycle $c \in Z_p^o(Z)$, we have $[\text{str}_p^o(c)] = [c] \in H_p(Z)$.

Indeed, by Theorem 3.2 the map $\text{pr}_*$ induces an isomorphism on homology. Therefore there exists $\tilde{c} \in Z_p(Z)$ such that $\text{pr}_*([\tilde{c}]) = [c]$. Denote by $h_*: C_*(Z) \to C_{*+1}(Z)$ the homotopy between $\text{str}_*$ and the identity on
$C_*(Z)$. Then

$$[\text{str}_p^c(\ell)] - [c] = [\text{str}_p^c \text{pr}_p(\ell)] - [\text{pr}_p(\ell)] = [\text{pr}_p(\text{str}_p(\ell) - \ell)] = [\text{pr}_p(\partial h_p(\ell) - h_{p-1}(\ell))] = [\partial^p(\text{pr}_p(h_p(\ell)))) = 0.$$

**Definition 4.2.** Let $X$ be a topological space. We call a triple $(\Sigma, c_\Sigma, \rho)$ consisting of a $p$-dimensional $\Delta$-complex $\Sigma$ which is a pseudomanifold, a simplicial fundamental cycle $c_\Sigma$ on $\Sigma$ and a continuous map $\rho: \Sigma \rightarrow X$ a **simplicial triple** of $X$. For a singular $p$-chain $c$ on $X$ we say that a simplicial triple $(\Sigma, c_\Sigma, \rho)$ is a simplicial $c$-triple or a simplicial triple representing $c$ if $\rho_*(c_\Sigma) = c$.

It is well known that singular cycles can always be represented by simplicial triples [1] Section 2, [2] p. 108.

**Definition 4.3.** A **partial coloring** of a $\Delta$-complex $\Sigma$ is a coloring of a subset of its vertices. The induced partial coloring on edges of the $\Delta$-complex colors an edge with color $\ell$ if and only if both of its endpoints are colored with $\ell$. A simplicial triple is called partially colored if its underlying $\Delta$-complex is endowed with a partial coloring. Given a partial coloring of the simplicial triple $(\Sigma, c_\Sigma, \rho)$ of $X$, we say that a simplex $\Delta$ of $\Sigma$ is non-essential if either

1. $\Delta$ has two distinct vertices with the same color, or
2. $\Delta$ has two vertices with the same $\rho$-image and the $\rho$-image of their connecting edge is a homotopically trivial loop in $X$.

If neither of these hold, $\Delta$ is called essential. Finally, we call a singular simplex appearing in $\rho_*(c_\Sigma)$ essential if it is the $\rho$-image of an essential simplex.

The following statement is the integral analogue of [1] Lemma 4, Corollary 5 and of [2] Lemma 4.

**Lemma 4.4** (Integral reduction lemma). Let $\bar{c} = \sum_{\sigma} a_{\sigma} \sigma \in C_p(X)$ be a $p$-cycle represented by a partially colored simplicial triple $(\Sigma, c_\Sigma, \rho)$. Let $\alpha: X \rightarrow Z$ be a continuous map to an aspherical space $Z$ such that the restriction of $\alpha \circ \rho: \Sigma \rightarrow Z$ to every component of every 1-dimensional subcomplex given by edges of the same color induces the trivial map on fundamental groups. Let $c = \text{pr}_p(\bar{c}) \in C_p^\alpha(X)$. Then we have

$$\|\alpha_*[c]\|_Z^\alpha \leq \sum_{\sigma \text{ essential}} |a_{\sigma}|.$$  \hspace{1cm} (4.1)

**Proof.** In the first part of the proof we reduce the statement to the special case where $\rho$ maps all vertices of $\Sigma$ to the same point $x_0 \in X$.

By introducing new colors for the connected components we may assume that each 1-dimensional subcomplex of edges of the same color is connected. This does not change the subset of essential singular simplices.

We will homotope the map $\rho: \Sigma \rightarrow X$ to a map $\rho': \Sigma \rightarrow X$ such that $\rho'$ maps all vertices of $\Sigma$ to the same point of $X$ and $\alpha \circ \rho'$ maps every colored edge to a nullhomotopic loop in $Z$. 

To this end, we pick for every color \( \ell \) a spanning tree \( T(\ell) \) of the 1-dimensional subcomplex \( \Sigma(\ell) \subset \Sigma \) of \( \ell \)-colored edges in \( \Sigma \). Choose a vertex \( v(\ell) \) in \( T(\ell) \) for every color \( \ell \). We pick paths from every \( \rho(v(\ell)) \) and from the \( \rho \)-image of every uncolored vertex to \( x_0 \). These paths define a homotopy on the subset of these vertices to \( X \) which we extend to a homotopy \( G: \Sigma \times [0,1] \to X \) using the fact that a subcomplex in a simplicial complex is a cofibration.

Since \( T(\ell) \subset \Sigma(\ell) \) is a contractible subset and an inclusion which is a cofibration, the projection \( \text{pr}(\ell): \Sigma(\ell) \to \Sigma(\ell)/T(\ell) \) is a homotopy equivalence and thus has a homotopy inverse \( j(\ell) \) which maps the basepoint given by \( T(\ell) \) to \( v(\ell) \). Again by the cofibration property we can extend the homotopy

\[
\left( \prod_{\ell} \Sigma(\ell) \prod \{\text{uncolored vertices}\} \right) \times [0,1] \to \Sigma(\ell) \hookrightarrow \Sigma
\]

which is constant \( \rho \) on the uncolored vertices and is a homotopy between \( \rho \) and \( \prod \rho \circ j(\ell) \circ \text{pr}(\ell) \) on \( \Sigma(\ell) \) to a homotopy \( H: \Sigma \times [0,1] \to \Sigma \) with \( H_0 = \rho \). Then the concatenation of the homotopies \( H \) and \( G \) is a homotopy from \( \rho \) to a map \( \rho' \) that maps all vertices to \( x_0 \). It is immediate from the construction that the restriction of \( \rho' \) to \( \Sigma(\ell) \) is still trivial on fundamental groups. The simplicial triple \( (\Sigma, \Sigma, \rho') \) represents \( \epsilon' = \rho'_*((\Sigma)) \). The left and right hand sides of (4.1) are unchanged if we replace \( c \) by \( \epsilon' \). So we might as well assume the original map \( \rho \) maps all vertices of \( \Sigma \) to the same point \( x_0 \in X \).

Consider the straightened cycle \( \text{str}_p(\alpha_*(\tilde{c})) \in C_p(Z) \). It is homologous to \( \alpha_*\epsilon \) in \( C_*(Z) \); by Remark 4.1 \( \text{str}_p^o(\alpha_*c) \) is homologous to \( \alpha_*\epsilon \) in \( C^o_p(Z) \). So we will rather estimate \( \|\text{str}_p^o(\alpha_*(\tilde{c}))\|_Z^{\alpha} \).

The result follows once we prove that for every non-essential singular simplex \( \sigma \) in the linear combination \( \tilde{c} = \sum_\sigma a_\sigma \sigma \) we have

\[
\text{str}_p^o(\alpha_*(\text{pr}_p(\sigma))) = \text{pr}_p(\text{str}_p(\alpha_*(\sigma))) = 0. \tag{4.2}
\]

Let \( \sigma_0 \) be a non-essential simplex of \( \tilde{c} \). To show (4.2) we distinguish two cases:

1. \( \sigma_0 \) maps the distinct vertices \( m, n \in \{0, \ldots, p\} \) to the same point in \( X \) with the loop of the corresponding edge being null-homotopic in \( X \), thus in \( Z \):

   Let \( (n, m) \in S(p+1) \) be the transposition. Then

   \[
   \text{str}_p(\alpha_*(\sigma_0)) = \text{str}_p((n, m)\alpha_*(\sigma_0)) = (n, m)\text{str}_p(\alpha_*(\sigma_0))
   \]

   by the second and third property of the straightening operator. Hence \( \text{str}_p(\alpha_*(\sigma_0)) \in O_p(Z) \) and (4.2) is implied.

2. \( \sigma_0 \) has two distinct vertices of the same color:

   Then the loop in \( Z \) resulting from the edge connecting these vertices is null-homotopic by assumption, and we conclude \( \text{str}_p(\alpha_*(\sigma_0)) \in O_p(Z) \) as before.

\( \square \)

5. Norms from the Cellular Structure of a Morse-Smale Function

On a CW-complex we consider the smallest partial order \( \leq \) on the set of open cells such that \( e \cap e' \neq \emptyset \) for open cells \( e \) and \( e' \) implies \( e \leq e' \). Two open cells are incomparable if they are incomparable with respect to \( \leq \).
Definition 5.1. We define four conditions on a cycle $c \in C_\ast(X)$ in a CW-complex $X$. Let $(\Sigma, c_\Sigma, \rho)$ be a simplicial $c$-triple.

1. The **cellular** condition requires that for each simplex of $\Sigma$, the image of the interior of each face (of any dimension) must be contained in one cell of $X$.
2. The **order** condition requires that the image of each simplex of $\Sigma$ must be contained in a totally ordered chain of cells; that is, the simplex does not intersect any two incomparable cells.
3. The **internality** condition requires that for each simplex of $\Sigma$, if the boundary of a face (of any dimension) maps into a cell $e \subset X$, then the whole face maps into $e$.
4. The **loop** condition requires that if a simplex in $\Sigma$ maps two vertices to the same point in $X$ then it maps the corresponding edge to that point.

It is easy to see that these conditions do not depend on the choice of simplicial triple.

Definition 5.2. Let $X$ be a CW-complex. Let $(\Sigma, c_\Sigma, \rho)$ be a simplicial triple representing a cycle $c = \sum a_\sigma \sigma \in C_p(X)$. Each open cell in $X$ is assigned its own color, and we color a vertex $v$ of $\Sigma$ with the color of the open cell containing $\rho(v)$. We define the **essential cellular norm** of $c$ by

$$\|c\|_{Z, \text{ess}}^\text{CW} = \sum_{\sigma \text{ essential}} |a_\sigma|.$$ 

The **essential cellular norm** of a (relative) homology class $h$ of $X$, which we also denote by $\|h\|_{Z, \text{ess}}^\text{CW}$, is defined by taking the infimum of the essential cellular norms of chains that represent $h$ and satisfy the cellular, order, internality and loop conditions.

Lemma 5.3. Let $X$ be a finite CW-complex. Let $Z$ be any aspherical topological space and let $\alpha : X \to Z$ be a continuous map. For any homology class $h \in H_p(X)$, let $h_{\text{rel}}$ denote the corresponding relative homology class in $H_p(X, X^{p-1})$. Then

$$\|\alpha_* h\|_Z^{\text{or}} \leq \|h_{\text{rel}}\|_{Z, \text{ess}}^\text{CW}.$$ 

Proof. The statement is the analog of Lemma 3 in Alpert’s paper [1] when one replaces the simplicial norm of $\alpha_* h$ by its integral oriented norm and considers the (generalized) stratification of $X$ by open cells. We follow Alpert’s proof closely and just point out the necessary modifications. Let $A = X^{p-1}$. Let $c_{\text{rel}}$ be a relative cycle representing $h_{\text{rel}}$ that satisfies the cellular, order, internality and loop conditions. By adding an integral singular chain $c_A$ in $A$ we obtain a cycle $c_{\text{rel}} + c_A \in C_p(X)$. From the chains $c_{\text{rel}}$ and $c_A$ Alpert produces chains $c_\delta, c_1, c_2 \in C_p(X)$ with $c = c_{\text{rel}} + c_\delta + c_1 + c_2 \in C_p(X)$ being a cycle and a partial coloring of a simplicial $c$-triple $(\Sigma, c_\Sigma, \rho)$ such that

- no singular simplex appearing in $c_\delta, c_1$ or $c_2$ is essential, and
- every essential singular simplex in $c_{\text{rel}}$ is also essential with regard to the coloring in Definition 5.2.

Having thus produced a representing cycle for $h$ that satisfies the conditions required in Lemma 4.4, we apply this result. □
In the following let $M$ be a closed Riemannian $d$-manifold. Let $f : M \to \mathbb{R}$ be a Morse-Smale function that is, in addition, Euclidean. The latter is a technical assumption; see [9, Definition 2.16] and [1, Section 3] for more details on this notion. If $f : M \to \mathbb{R}$ is Morse-Smale and Euclidean, then the set of descending manifolds $\{D(p) \mid p \in M \text{ critical}\}$ is a CW-decomposition of $M$ \cite[Lemma 11; 9, Section 3.4]{1}.

**Definition 5.4.** Let $f : M \to \mathbb{R}$ be a Euclidean Morse-Smale function on a closed Riemannian $d$-manifold. The essential cellular norm of a homology class $h \in H_p(M)$ with regard to the CW-structure of descending manifolds is denoted by $\|h\|_f$.

The following lemma is the crucial point in which the number of maximally broken trajectories is related to the essential cellular norm of the descending manifolds $D(p)$ of the Morse function. It is proved in detail in \cite{1}.

**Lemma 5.5 (\cite[Lemma 11]{1}).** Let $M$ be a closed Riemannian $d$-manifold, and let $f : M \to \mathbb{R}$ be an Euclidean Morse-Smale function. For each descending manifold $D(p)$ of dimension $d$, let $[D(p)] \in H_d(M, M^{d-1})$ be the corresponding relative homology class. Then the number of maximally broken trajectories of $f$ starting at the critical point $p$ is at least $\|D(p)\|_f$.

### 6. Proof of the main theorem

**Theorem 6.1.** Let $M$ be a closed oriented Riemannian $d$-manifold. Let $f : M \to \mathbb{R}$ be a Morse-Smale function. Let $Z$ be any aspherical topological space, and let $\alpha : M \to Z$ be a continuous map. Then the number of maximally broken trajectories of $f$ is at least $\|\alpha_*[M]\|_{\mathbb{Z}}$.

**Proof.** By Theorem 10 and its subsequent remark in \cite{1} we may and will assume that $f$ is in addition Euclidean.

Consider the CW-structure on $M$ given by the descending manifolds of $f$. The relative class $[M]_{\text{rel}} \in H_d(M, M^{d-1})$ corresponding to $[M]$ is equal to the sum of $d$-cells

$$[M]_{\text{rel}} = \sum_{\text{ind}(p) = d} [D(p)].$$

So by triangle inequality and Lemma 5.5 we have

$$\|\alpha_*[M]\|_{\mathbb{Z}} \leq \sum_{\text{ind}(p) = d} \|D(p)\|_f \leq \|\alpha_*[M]\|_{\mathbb{Z}}.$$

Applying Lemma 5.3 we obtain

$$\|\alpha_*[M]\|_{\mathbb{Z}} \leq \|\text{rel}_f\|.$$

**Proof of Theorem 1.1.** Let $M$ be a closed orientable aspherical Riemannian $d$-manifold with a Morse-Smale function $f : M \to \mathbb{R}$. We apply Theorem 6.1 with $Z = M$ and $\alpha$ being the identity map. Hence the number of maximally broken trajectories is at least $\|\text{rel}_f\|$, where $[M]$ denotes the fundamental class. By Theorem 3.3 we have

$$\|\text{rel}_f\| \leq (d + 1)! \cdot \|\text{rel}_f\|_{\mathbb{Z}}.$$
By [8, Theorem 3.2] we have
\[ \log |\text{tors}(H_p(M;\mathbb{Z})| \leq \log(d+1) \binom{d+1}{p+1} \|M\|_Z \]
for every \( p \in \{0, \ldots, d\} \). Further, by [8, Theorem 3.5],
\[ \text{rank}(H_p(M;\mathbb{Z})) \leq \binom{d+1}{p} \|M\|_Z. \]
Now we can conclude the theorem from
\[ \sum_{p=0}^{d} \text{size}(H_p(M;\mathbb{Z})) \leq \sum_{p=0}^{d} \left( \log(d+1) \binom{d+1}{p+1} + \binom{d+1}{p} \right) \|M\|_Z \]
\[ \leq \log(d+1)2^{d+2} \|M\|_Z \]
\[ \leq \log(d+1)2^{d+2}(d+1)! \cdot \|M\|_{Z}^{\text{or}} \]
\[ \leq 2^{d+2}(d+2)! \cdot \|M\|_{Z}^{\text{or}}. \]

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