LIMIT THEOREMS FOR SINGULAR SKOROHOD INTEGRALS

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Abstract. In this paper we prove the convergence in distribution of sequences of Itô and Skorohod integrals with integrands having singular asymptotic behavior. These sequences include stochastic convolutions and generalize the example \( \sqrt{n} \int_0^1 t^n B_t dB_t \) first studied by Peccati and Yor in 2004.

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1. Introduction

The main objective of this paper is to study the limit in law of sequences of random variables defined by Skorohod integrals

\[
F_n = \int_0^1 \phi_n(t) u_t \delta B_t^H.
\]

Here \( u_t \) is a continuous process, \( B^H \) is fractional Brownian motion with Hurst parameter \( H \) lying in the range \((0,1)\), and \( \phi_n \) is a sequence of deterministic kernels converging (in some sense) to a delta function based at 1 (hence the “singular” in the title of the paper). We show, under suitable conditions on the \( \phi_n \) and \( u \), that the limit law of the couple \((B^H, F_n)\) has the form \((B^H, cu_1 Z)\), where \( Z \) is a \( N(0,1) \) random variable, independent of \( B^H \).

Our study of limit problems of this type was motivated by the special case \( H = \frac{1}{2} \), \( u_t = B^H_t \) and \( \phi_n(t) = \sqrt{n} t^n \), introduced in Proposition 2.1 of [8], and studied in Proposition 18 of [7], and Example 4.2 in [3]. Then, \( B^H_t \) is standard Brownian motion and the integrals are of classical Itô type. Quantitative bounds for such integrals in the case \( H > \frac{1}{2} \) have been established by Nourdin, Nualart & Peccati [5] using estimates derived from Malliavin calculus and, more recently, by Pratelli & Rigo in [6] for \( H \in (1/4,1) \), using more a elementary (but nonetheless intricate) argument.

In this article, we provide a new approach to this problem, valid for a more general class of integrands exhibiting singular asymptotic behavior at the right-hand endpoint. The approach is based on the following observation. The singular behavior of the kernels \( \phi_n \) in (1.1) as \( n \to \infty \) implies that the limit law of \( F_n \) is determined by the behavior of integrals over arbitrarily time intervals \([1-\delta,1]\). This makes it possible to study the limit law via the more tractable sequence of random variables

\[
G_n = u_{\alpha_n} \int_{\alpha_n}^1 \phi_n(t) \delta B_t^H,
\]

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where $\alpha_n$ is a sequence of times chosen such that $\alpha_n \uparrow 1$ at a carefully chosen rate. We show that $F_n$ and $G_n$ have that same limit in $L^2$, and hence in law. Furthermore, the Skorohod integrals $I_n$ in (1.2) are Gaussian, as is the process $B^H$. It thus suffices to show that the sequence $I_n$ has a convergent variance and is asymptotically uncorrelated with $B^H$.

In Section 3.1, we implement this argument in the case where $B^H$ is standard Brownian motion (denoted here by $B$) and the process $u_t$ is progressively measurable. In this case the integrals are Itô integrals and the argument is technically easier. The basic result in this section is Theorem 3.1. As a special case of this theorem, we obtain the limit in law of the aforementioned sequence

$$\sqrt{n} \int_0^1 t^n B_t dB_t.$$ Theorem 3.1 is extended in Theorems 3.3 and 3.4 to double, and multiple, integrals respectively. In Section 3.2 we discuss the problem for stochastic convolutions.

In Section 4, we study the case of fractional Brownian motion ($H \neq 1/2$). Here it turns out to be more convenient to work with the approximating sequence

$$G_n = \int_0^1 \phi_n(t) u_1 \delta B^H_t.$$ As is usual in this subject, the cases $H \in (1/2, 1)$ and $H \in (0, 1/2)$ seem to require slightly different hypotheses and analyses, with the latter proving more involved. Analogues of Theorem 3.1 are presented in Theorems 4.1 and 4.3 for these two cases. The proof involves the use the divergence operator on Wiener space and thus has the flavor of Malliavin calculus. As an example of these theorems, we obtain the main result of [6] concerning the limit law of the sequence $n^H \int_0^1 t^n B^H_t dB^H_t$, for $H$ in the range $(1/4, 1)$.

2. Preliminaries

Fractional Brownian motion with Hurst parameter $H \in (0, 1)$, $B^H = \{B^H_t, t \in [0, 1]\}$ is a zero mean Gaussian process with a covariance function given by

$$R_H(t, s) := E[B_t B_s] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right),$$ where $s, t \in [0, 1]$. The Hilbert space $\mathfrak{H}$ is defined as the closure of the space of step functions $\mathcal{E}$ on $[0, 1]$ endowed with the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathfrak{H}} = R_H(t, s).$$ Then the mapping $1_{[0,t]} \rightarrow B^H_t$ can be extended to a linear isometry between $\mathfrak{H}$ and the Gaussian space $\mathcal{H}_1$ spanned by $B^H$. When $H = \frac{1}{2}$, $B^H$ is just a standard Brownian motion and $\mathfrak{H} = L^2([0, 1]^2)$.

When $H \in (\frac{1}{2}, 1)$, the inner product of two step functions $\phi, \psi \in \mathcal{E}$ can be expressed as

$$\langle \phi, \psi \rangle_{\mathfrak{H}} = \alpha_H \int_0^1 \int_0^1 \phi(s) \psi(t) |t - s|^{2H-2} ds dt,$$ where $\alpha_H = H(2H - 1)$. The space of measurable functions $\phi$ on $[0, 1]$, such that

$$\|\phi\|_{\mathfrak{H}} := \alpha_H \int_0^1 \int_0^1 |\phi(s)||\phi(t)||t - s|^{2H-2} ds dt < \infty,$$
denoted by $|\mathcal{F}|$, is a Banach space and we have the continuous embeddings $L^\frac{1}{2}([0,1]) \subset |\mathcal{F}| \subset L^2$.

When $H \in (0, \frac{1}{2})$, the covariance of the fractional Brownian motion $B^H$ can be expressed as

$$R_H(t,s) = \int_0^{s \wedge t} K_H(s,u)K_H(t,u)du,$$

where $K_H(t,s)$ is a square integrable kernel defined as

$$K_H(t,s) = d_H\left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2})s^{\frac{3}{2} - H}\int_s^t u^{H-\frac{3}{2}}(v-s)^{H-\frac{1}{2}}dv\right),$$

for $0 < s < t$, with $d_H$ being a constant depending on $H$. The kernel $K_H$ satisfies the following estimates

$$(2.2) \quad |K_H(t,s)| \leq c_H \left( (t-s)^{H-\frac{1}{2}} + s^{H-\frac{1}{2}} \right),$$

and

$$(2.3) \quad \left| \frac{\partial K_H}{\partial t}(t,s) \right| \leq c_H'(t-s)^{H-\frac{3}{2}},$$

for all $s < t$ and for some constants $c_H, c_H'$. Define a linear operator $K^*_H$ from $\mathcal{E}$ to $L^2([0,1])$ as follows

$$(2.4) \quad (K^*_H\phi)(s) = \left( K_H(1,s)\phi(s) + \int_s^1 (\phi(t) - \phi(s))\frac{\partial K_H}{\partial t}(t,s)dt \right).$$

The operator $K^*_H$ can be extended to a linear isometry between the Hilbert space $\mathcal{F}$ and $L^2([0,1])$, that is, for any $\phi, \psi \in \mathcal{F}$, we have

$$(2.5) \quad \langle \phi, \psi \rangle_{\mathcal{F}} = \langle K^*_H\phi, K^*_H\psi \rangle_{L^2([0,1])}.$$ 

The space of Hölder continuous functions of order $\gamma > \frac{1}{2} - H$ is included in $\mathcal{F}$.

Next, we introduce the derivative operator and its adjoint, the divergence. Consider a smooth and cylindrical random variable of the form $F = f(B^H_{t_1}, \ldots, B^H_{t_d})$, where $f \in C^\infty_c(\mathbb{R}^d)$ ($f$ and its partial derivatives are all bounded). We define its Malliavin derivative as the $\mathcal{F}$-valued random variable $DF$ given by

$$D_sF = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(B^H_{t_1}, \ldots, B^H_{t_d})1_{[0,t_i]}(s).$$

For any real number $p \geq 1$, we define the Sobolev space $\mathbb{D}^{1,p}$ as the closure of the space of smooth and cylindrical random variables with respect to the norm $\| \cdot \|_{1,p}$ given by

$$\|F\|_{1,p}^p = \mathbb{E}(|F|^p) + \mathbb{E} \left( \|DF\|_{\mathcal{F}}^p \right).$$

Similarly, if $\mathbb{W}$ is a general Hilbert space, we can define the Sobolev space of $\mathbb{W}$-valued random variables $\mathbb{D}^{1,p}(\mathbb{W})$.

The adjoint of the Malliavin derivative operator $D$, denoted as $\delta$, is called the divergence operator. A random element $u$ belongs to the domain of $\delta$, denoted as Dom $\delta$, if there exists a positive constant $c_u$ depending only on $u$ such that

$$|E(\langle DF, u \rangle_{\mathcal{F}})| \leq c_u \|F\|_{L^2(\Omega)}.$$
for any $F \in \mathbb{D}^{1,2}$. If $u \in \text{Dom} \delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$E(F\delta(u)) = E(\langle DF, u \rangle_{\mathbb{S}}),$$

for any $F \in \mathbb{D}^{1,2}$. We make use of the notation $\delta(u) = \int_0^\infty u \delta B^H_t$ and call $\delta(u)$ the Skorohod integral of $u$ with respect to the fractional Brownian motion $B^H$. The Skorohod integral satisfies the following isometry property for any element $u \in \mathbb{D}^{1,2}$(H) $\subset \text{Dom} \delta$:

$$E(\delta(u)^2) = E(\|u\|_{\mathbb{S}}^2) + E(\langle Du, (Du)^* \rangle_{\mathbb{S} \otimes \mathbb{S}}),$$

where $(Du)^*$ is the adjoint of $Du$. As a consequence, we have

$$E(\delta(u)^2) \leq E(\|u\|_{\mathbb{S}}^2) + E(\|Du\|_{\mathbb{S} \otimes \mathbb{S}}^2).$$

We will make use of the following result.

**Lemma 2.1.** Let $F \in \mathbb{D}^{1,2}$ and let $g \in \mathfrak{H}$. Then the process $Fg$ belongs to the domain of $\delta$ and

$$\int_0^1 Fg_t \delta B^H_t = F \delta(g) + \langle DF, g \rangle_{\mathbb{S}}.$$

We refer to [5] and the references therein for a more detailed account of the properties of the fractional Brownian motion and its associated Malliavin calculus (and to [1] for an introduction to the latter subject).

We will make use of the following property of the Gamma function.

**Lemma 2.2.** For any $a, b$ positive

$$\lim_{n \to \infty} \frac{\Gamma(n + a)n^{b-a}}{\Gamma(n + b)} = 1.$$

**Proof.** This is a direct application of Stirling’s formula. \hfill \square

Throughout the paper we will make use of the notion of stable convergence provided in the next definition. Suppose that the fractional Brownian motion $B^H$ is defined in a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the $P$-completion of the $\sigma$-field generated by $B^H$.

**Definition 2.3.** Fix $d \geq 1$. Let $F_n$ be a sequence of random variables with values in $\mathbb{R}^d$, all defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $F$ be a $\mathbb{R}^d$-valued random variable defined on some extended probability space $(\Omega', \mathcal{F}', P')$. We say that $F_n$ converges stably to $F$, if

$$\lim_{n \to \infty} E[Z e^{i(\lambda, F_n)_{\mathbb{S}^d}}] = E'[Z e^{i(\lambda, F)_{\mathbb{S}^d}}]$$

for every $\lambda \in \mathbb{R}^d$ and every bounded $\mathcal{F}$-measurable random variable $Z$.

Condition (2.7) is equivalent to saying that the couple $(B^H, F_n)$ converges in law to $(B^H, F)$ in the space $C([0, \infty)) \times \mathbb{R}$ (see, for instance, [2, Chapter 4]).

3. **Singular limits of sequences of Itô integrals**

Let $B = \{B_t, t \geq 0\}$ be a standard Brownian motion. Denote by $\mathcal{F}_t$ the natural filtration generated by $B$. In this section we will study the asymptotic behavior of two types of sequences of Itô integrals. First, we discuss a class of integrals on $[0, 1]$ that include a sequence of deterministic kernels $\phi_n$ converging to a delta function based at 1. Secondly, we apply our argument to stochastic convolutions with this type of asymptotic behavior.
3.1. **Stochastic integrals concentrating at** $t = 1$. Consider a sequence of bounded non-negative functions $\phi_n(t)$ on $[0, 1]$, that satisfies the following conditions:

**(h1)**: There is a sequence $\alpha_n \uparrow 1$ such that

$$
\lim_{n \to \infty} \int_{\alpha_n}^1 \phi_n^2(t) \, dt = \lim_{n \to \infty} \int_0^1 \phi_n^2(t) \, dt = L > 0.
$$

**(h2)**: For any $\delta \in [0, 1)$, $\sup_{0 \leq t \leq \delta} \phi_n(t) \to 0$ as $n \to \infty$.

The aim of this section is to study the asymptotic behavior of the sequence of Itô integrals

$$
F_n := \int_0^1 \phi_n(t) u_t \, dB_t, \quad n \geq 1,
$$

where $u = \{u_t, t \in [0, 1]\}$ is a progressively measurable process such that $\int_0^1 E(u_t^2) \, dt < \infty$.

**Theorem 3.1.** Suppose that the process $u$ is continuous in $L^2(\Omega)$ at $t = 1$ and the sequence $\phi_n$ satisfies conditions (h1) and (h2). Then, the sequence $F_n$ introduced in (3.1) converges stably, as $n \to \infty$ to $\sqrt{Lu_1} Z$, where $Z$ is a $N(0,1)$ random variable independent of the process $B$.

**Proof.** Define

$$
G_n := u_{\alpha_n} \int_{\alpha_n}^1 \phi_n(t) dB_t,
$$

where $\alpha_n$ is the sequence appearing in condition (h1). Then, as $n \to \infty$,

$$
E[G_n^2] = E[u_{\alpha_n}^2] \int_{\alpha_n}^1 \phi_n^2(t) \, dt \to E(u_1^2)L.
$$

Moreover, as $n \to \infty$,

$$
E[F_n G_n] = E\left[ \int_{\alpha_n}^1 \phi_n(t) u_t \, dB_t \int_{\alpha_n}^1 \phi_n(t) u_{\alpha_n} \, dB_t \right] = \int_{\alpha_n}^1 \phi_n^2(t) E(u_t u_{\alpha_n}) \, dt
$$

$$
= \int_{\alpha_n}^1 \phi_n^2(t) E[u_{\alpha_n} (u_t - u_{\alpha_n})] \, dt + \int_{\alpha_n}^1 \phi_n^2(t) E(u_{\alpha_n}^2) \, dt \to E(u_1^2) L,
$$

because $\int_{\alpha_n}^1 \phi_n^2(t) E(u_{\alpha_n}^2) \, dt \to E(u_1^2) L$ and

$$
\left| \int_{\alpha_n}^1 \phi_n^2(t) E[u_{\alpha_n} (u_t - u_{\alpha_n})] \, dt \right| \leq \frac{E(u_{\alpha_n}^2)}{2} \sup_{\alpha_n \leq t \leq 1} E((u_t - u_{\alpha_n})^2) \frac{1}{2} \int_{\alpha_n}^1 \phi_n^2(t) \, dt \to 0
$$

by the $L^2$-continuity of $u$ at $t = 1$. On the other hand, for any fixed $\delta \in [0, 1)$,

$$
E[F_n^2] = \int_0^1 \phi_n^2(t) E[u_t^2] \, dt
$$

$$
= \int_0^\delta \phi_n^2(t) E(u_t^2) \, dt + \int_0^1 \phi_n^2(t) E(u_t^2 - u_1^2) \, dt + \int_0^1 \phi_n^2(t) E(u_1^2) \, dt.
$$

As $n \to \infty$, the third term in (3.4) has limit $E(u_1^2) L$ and the first term converges to zero in view of hypothesis (h2), because

$$
\left| \int_0^\delta \phi_n^2(t) E(u_t^2) \, dt \right| \leq \sup_{0 \leq t \leq \delta} \phi_n^2(t) \frac{1}{5} \int_0^1 E(u_t^2) \, dt.
$$
Moreover, the second converges to zero as $\delta \uparrow 1$, uniformly in $n$, because we can write
\[
\left| \int_{\delta}^{1} \phi_n^2(t) E(u_t^2 - u_{t1}^2) \, dt \right| \leq \sup_{\delta \leq t \leq 1} E(|u_t^2 - u_{t1}^2|) \int_{0}^{1} \phi_n^2(t) \, dt.
\]
Therefore,
\[
(3.5) \quad \lim_{n \to \infty} E[F_n^2] = E(u_1^2)L.
\]

Now, (3.2), (3.3) and (3.5) imply that $F_n - G_n \to 0$ in $L^2$, and hence also in law. Finally, notice that the sequence $(B, \int_{\alpha_n}^{1} \phi_n(t) \, dB_t)$ converges in law in the space $C([0,1]) \times \mathbb{R}$ to $(B, \sqrt{LZ})$, where $Z$ is a $N(0,1)$ random variable independent of $B$. This completes the proof. □

An example of a sequence of functions satisfying conditions (h1) and (h2) with $L = \frac{1}{2}$ is
\[
\phi_n(t) = \sqrt{nt^n}.
\]
Indeed condition (h2) holds trivially and condition (h1) holds taking, for instance, $\alpha_n = 1 - \frac{\log n}{n}$, because $\alpha_n \sim \frac{1}{n}$ and, therefore, as $n \to \infty$,
\[
n \int_{\alpha_n}^{1} t^{2n} \, dt = \frac{n(1 - \alpha_n^{2n})}{2n + 1} \to \frac{1}{2}.
\]
Thus we have proved the following.

**Proposition 3.2.** The sequence of Itô integrals
\[
\sqrt{n} \int_{0}^{1} t^n B_t dB_t
\]
converges stably, as $n \to \infty$, to $\frac{1}{\sqrt{2}} B_1 Z$, where $Z$ is a $N(0,1)$ random variable independent of the process $B$.

**Remarks:**

(i) We note that Proposition 3.2 was obtained by Nourdin, Nualart & Peccati in [2, Proposition 3.7] as a corollary of a theorem proved by integration by parts on Wiener space.

(ii) If we assume that $E(u_t^2)$ is bounded on $[0,1]$, then it is easy to show that we can remove condition (h2) in Theorem 3.1.

The next result is an extension of Theorem 3.1 to the case of double stochastic Itô integrals, which is proved by similar arguments. We need the following condition on the sequence $\phi_n$, which is stronger than (h2):

(h3): For any $\delta \in [0,1)$, $\left( \sup_{0 \leq t \leq \delta} \phi_n(t) \right)$ $\left( \sup_{0 \leq t \leq 1} \phi_n(t) \right) \to 0$ as $n \to \infty$.

**Theorem 3.3.** Let $u = \{u_{s,t}, 0 \leq s \leq t \leq 1\}$ be a two-parameter process satisfying the following conditions:

(i) $u_{s,t}$ is $\mathcal{F}_s$-measurable for $s \leq t$.

(ii) $\int_{0}^{1} \int_{s}^{t} E(u_{s,t}^2) \, ds \, dt < \infty$.

(iii) $u_{s,t}$ is continuous at $(1,1)$ in the $L^2(\Omega)$ sense.
Consider the sequence of iterated Itô integrals

\[ F_n := \int_0^1 \int_0^t \phi_n(s)\phi_n(t)u_{s,t}dB_s dB_t, \quad n \geq 1, \]

where the sequence \( \phi_n \) satisfies conditions (h1) and (h3). Then \( F_n \) converges stably, as \( n \to \infty \) to \( \frac{1}{2}Lu_{1,1}H_2(Z) \), where \( Z \sim N(0, 1) \) is independent of the process \( B \), and \( H_2 = x^2 - 1 \) is the second Hermite polynomial.

Proof. Define

\[ G_n := u_{\alpha_n, \alpha_n} \int_{\alpha_n}^1 \int_{\alpha_n}^t \phi_n(s)\phi_n(t)dB_s dB_t, \]

where \( \alpha_n \) is the sequence appearing in condition (h1). By (h1) we have, as \( n \to \infty \),

\[ E[G_n^2] = E[u_{\alpha_n, \alpha_n}^2] \int_{\alpha_n}^1 \int_{\alpha_n}^t \phi_n^2(s)\phi_n^2(t) ds \, dt \to \frac{L^2}{2}E(u_{1,1}^2). \]

Also

\[ E[F_n^2] = \int_0^1 \int_0^t \phi_n^2(s)\phi_n^2(t)E[u_{s,t}^2] ds \, dt. \]

As in the proof of Theorem 3.1, fix \( 0 < \delta < 1 \) and consider the decomposition

\[ E[F_n^2] = \int_0^1 \int_0^{t+\delta} \phi_n^2(s)\phi_n^2(t)E[u_{s,t}^2] ds \, dt \]

\[ + \int_0^1 \int_0^t \phi_n^2(s)\phi_n^2(t)E[u_{s,t}^2 - u_{1,1}^2] ds \, dt + E[u_{1,1}^2] \int_0^1 \int_0^t \phi_n^2(s)\phi_n^2(t) ds dt. \]

The first term of (3.7) converges to zero as \( n \to \infty \) by condition (h3) and the third term converges to \( \frac{L^2}{2}E(u_{1,1}^2) \). For the second term we have the estimate

\[ \left| \int_0^1 \int_0^t \phi_n^2(s)\phi_n^2(t)E[u_{s,t}^2 - u_{1,1}^2] ds \, dt \right| \leq \sup_{\delta \leq s \leq t \leq 1} |E[u_{s,t}^2 - u_{1,1}^2]| \int_0^1 \int_0^t \phi_n^2(s)\phi_n^2(t) ds dt, \]

which shows that this term converges to zero as \( \delta \uparrow 1 \), due to the continuity of \( u \) and \( (1, 1) \), uniformly in \( n \). In this way, we obtain

\[ \lim_{n \to \infty} E(F_n^2) = \frac{L^2}{2}E(u_{1,1}^2). \]

Also

\[ E[F_n G_n] = \int_0^1 \int_0^t \phi_n^2(s)\phi_n^2(t)E(u_{s,t}u_{\alpha_n, \alpha_n}) ds dt \]

\[ = \int_0^1 \int_0^t \phi_n^2(s)\phi_n^2(t)E \left[ (u_{s,t} - u_{\alpha_n, \alpha_n}) u_{\alpha_n, \alpha_n} \right] ds dt \]

\[ + \int_0^1 \int_0^t \phi_n^2(s)\phi_n^2(t)E \left[ u_{\alpha_n, \alpha_n}^2 \right] ds dt. \]
At this point similar calculations to (3.6) show that the second term in (3.9) has limit \( \frac{L^2}{2} E(u_{1,1}^2) \), while the first term of (3.9) converges to 0 due again to Cauchy-Schwartz inequality, condition (h1) and the continuity of \( u \) at (1,1) in \( L^2 \). Consequently, as \( n \to \infty \),

\[
E[F_n G_n] \to \frac{L^2}{2} E(u_{1,1}^2).
\]

Thus (3.6), (3.8) and (3.10) imply that \( F_n - G_n \to 0 \) in \( L^2 \), and hence also in law. Finally to see that the limit of \( G_n \) has the desired form note that

\[
\int_{\alpha_n}^1 \int_{\alpha_n}^t \phi_n(s)\phi_n(t) \, dB_s \, dB_t = \frac{1}{2} I_2(\phi_n^2 1_{[\alpha_n,1]^2}),
\]

where \( I_2 \) denotes the double Itô-Wiener integral. Then by using the fact that multiple stochastic integrals of this form can be written in terms of Hermite polynomials, we can write

\[
G_n = \frac{1}{2} u_{\alpha_n,\alpha_n} \|\phi_n 1_{[\alpha_n,1]}\|^2_{L^2} H_2 \left( \int_{\alpha_n}^1 \phi_n(t) \, dB_t \right) \left( \frac{\int_{\alpha_n}^1 \phi_n(t) \, dB_t}{\|\phi_n 1_{[\alpha_n,1]}\|_{L^2([0,1])}} \right).
\]

Then the conclusion follows because \( \|\phi_n 1_{[\alpha_n,1]}\|_{L^2([0,1])} \to \sqrt{L} \) as \( n \to \infty \), \( u_{\alpha_n,\alpha_n} \) converges to \( u_{1,1} \) in \( L^2(\Omega) \) as \( n \to \infty \), and the sequence \( (B, \int_{\alpha_n}^1 \phi_n(t) \, dB_t) \) converges in law in the space \( C([0,1]) \times \mathbb{R} \) to \( (B, \sqrt{L} Z) \), where \( Z \) is a \( N(0,1) \) random variable independent of \( B \). This implies that the limit law of \( G_n \) has the stated form and completes the proof. \( \square \)

Remarks:

(i) The previous theorem applies to the particular case \( \phi_n(t) = \sqrt{n} t^n \), as before.

(ii) One can consider the more general situation of a sequence of bounded symmetric functions \( \phi_n(s,t) \) on \([0,1]^2\), satisfying the following conditions:

(h12): There is a sequence \( \alpha_n \uparrow 1 \) such that

\[
\lim_{n \to \infty} \int_{\alpha_n}^1 \int_{\alpha_n}^1 \phi_n^2(s,t) \, ds \, dt = \lim_{n \to \infty} \int_0^1 \int_0^1 \phi_n^2(s,t) \, ds \, dt = L > 0.
\]

(h22): For any \( \delta \in [0,1) \), \( \sup_{0 \leq s \leq \delta, 0 \leq t \leq 1} |\phi_n(s,t)| \to 0 \) as \( n \to \infty \).

In this case we need to compute the limit in law of \( I_2(\phi_n 1_{[\alpha_n,1]^2}) \), which is a more complicated problem that requires additional conditions on the sequence \( \phi_n \). We will not treat this problem here.

Theorem 3.3 can be extended to higher dimensions. The proof is similar and omitted. We need the following condition on the sequence \( \phi_n \), which is stronger than (h2):

(h3m): For any \( \delta \in [0,1) \), \( \left( \sup_{0 \leq t \leq \delta} \phi_n(t) \right) \left( \sup_{0 \leq t \leq 1} \phi_n(t) \right)^{m-1} \to 0 \) as \( n \to \infty \), where \( m \) is the number of parameters.

**Theorem 3.4.** Let \( u = \{u_{t_1,\ldots,t_m}, 0 \leq t_1 \leq \cdots \leq t_m \leq 1\} \) be an \( m \)-parameter stochastic process satisfying the following properties.

i) \( u_{t_1,\ldots,t_m} \) is \( F_{t_1} \)-measurable.
3.2. Asymptotic behavior of stochastic convolutions.

Consider the sequence of iterated Itô integrals

\[ F_n := \int_0^1 \int_0^{t_1} \cdots \int_0^{t_2} \phi_n(t_1) \cdots \phi_n(t_m) u_{t_1} \cdots u_{t_m} dB_{t_1} \cdots dB_{t_m}, \quad n \geq 1, \]

where the sequence \( \phi_n \) satisfies conditions (h1) and (h3m). Then, \( F_n \) converges stably, as \( n \to \infty \), to \( (m!)^{-1} L_m^2 u_{t_1} \cdots H_m(Z) \), where \( Z \sim N(0, 1) \) is independent of the process \( B \) and \( H_m \) is the \( m \)th Hermite polynomial.

### Theorem 3.5.

Assume \( u = \{u_t, t \geq 0\} \) is an adapted, square integrable process, continuous at a fixed time \( t \geq 0 \) in the \( L^2(\Omega) \) sense. Consider a nonnegative continuous and bounded function \( \psi(x) \) on \( \mathbb{R} \), such that \( \int_{-\infty}^{\infty} \psi^2(x) dx = 1 \). Let \( \psi_n(x) = \sqrt{n} \psi(nx) \). Then \( \psi_n^2 \) is an approximation of the identity.

Let \( u = \{u_t, t \geq 0\} \) be an adapted and square integrable process. Define the stochastic convolution

\[ (u * B \psi_n) t = \int_0^\infty u_s \psi_n(t - s) dB_s, \quad t \geq 0. \]

In this subsection we are interested in the asymptotic behavior of \( (u * B \psi_n) \) as \( n \) tends to infinity. The limit in law will have the form \( u_t Z_t \), where \( Z \) is a Gaussian process independent of \( B \).

The following theorem is the main result of this subsection.

**Theorem 3.5.** Assume \( u = \{u_t, t \geq 0\} \) is an adapted, square integrable process, continuous at a fixed time \( t \geq 0 \) in the \( L^2(\Omega) \) sense. Consider a nonnegative continuous and bounded function \( \psi(x) \) on \( \mathbb{R} \), such that \( \int_{-\infty}^{\infty} \psi^2(x) dx = 1 \) and \( \psi^2(x) = o(|x|^{-1}) \) as \( x \to \infty \). Then, the stochastic convolution \( (u * B \psi_n)_t \) converges stably to \( u_t Z_t \) as \( n \to \infty \), where \( Z \) is a standard Gaussian random variable independent of \( B \).

**Proof.** Let \( \alpha_n \) be a sequence decreasing to 0 so that \( n \alpha_n \to \infty \). For \( t \geq 0 \), set

\[ G_n = u_{(t-\alpha_n)_+} S_n, \]

where \( S_n = \int_{R_n(t)}^{\infty} \psi_n(t - s) dB_s \) with \( R_n(t) = \{ s \geq 0 : |t - s| \leq \alpha_n \} \). Then we can write

\[ E(S_n^2) = \int_{R_n(t)}^{\infty} \psi_n^2(t - s) ds = \int_{|r| \leq \alpha_n, r \leq t} \psi_n^2(r) dr = \int_{|z| \leq n \alpha_n, z \leq nt} \psi^2(z) dz \to 1 \quad \text{as} \quad n \to \infty. \]

Moreover, since \( u_{(t-\alpha_n)_+} \) is \( F_{(t-\alpha_n)_+} \) measurable we can write

\[ G_n = \int_{R_n(t)}^{\infty} u_{(t-\alpha_n)_+} \psi_n(t - s) dB_s \]

and therefore

\[ E(G_n^2) = \int_{R_n(t)}^{\infty} E(u^2_{(t-\alpha_n)_+}) \psi_n^2(t - s) ds = E(u^2_{(t-\alpha_n)_+}) \int_{|s| \leq \alpha_n, s \leq t} \psi_n^2(s) ds \to E(u^2). \]
On the other hand, by Itô’s isometry property we can write

$$E \left( (u \ast_B \psi_n)^2 \right) = \int_0^\infty E(u_s^2)\psi_n^2(t-s)\,ds.$$ 

That means, $E((u \ast_B \psi_n)^2)$ is the convolution of $s \rightarrow E(u_s^2)$ with $\psi_n^2$, and by Theorem 9.9 in [9], we deduce

$$\lim_{n \to \infty} E \left( (u \ast_B \psi_n)^2 \right) = E(u_t^2).$$

Finally, by Itô’s isometry and the $L^2$-continuity of $u$ at $t$

$$E \left( (u \ast_B \psi_n) G_n \right) = \int_{R_n(t)} E(u_s u_{(t-\alpha_n)_+})\psi_n^2(t-s)\,ds$$

$$= \int_{R_n(t)} E(u_{(t-\alpha_n)_+}(u_s-u_{(t-\alpha_n)_+}))\psi_n^2(t-s)\,ds$$

$$+ E(u_{(t-\alpha_n)_+}) \int_{R_n(t)} \psi_n^2(t-s)\,ds \to E(u_t^2),$$

as $n \to \infty$. Thus $(u \ast_B \psi_n)t - G_n \overset{L^2(\Omega)}{\to} 0$ as $n \to \infty$ and hence in law. Finally, note that for each $n$, $u_{t-\alpha_n}$ and $S_n$ are independent random variables such that $u_{t-\alpha_n}$ converges to $u_t$ and $\text{Var}(S_n^2) \to 1$. This implies that the limit law of $G_n$ has the stated form and completes the proof. \hfill \Box

As in the proof of Theorem 3.5, if $\alpha_n$ is a sequence decreasing to 0 so that $n\alpha_n \to \infty$, we can consider for each $t \geq 0$ the sequence of random variables

$$S_n^t := \int_{[t-r] \leq \alpha_n} \psi_n(t-r)dB_r.$$ 

The next lemma establishes the asymptotic behavior of the sequence of processes $\{S_n^t, t \geq 0\}$.

**Lemma 3.6.** The finite-dimensional distributions of the process $\{S_n^t, t \geq 0\}$ introduced in (3.11) converge stably to those of a centered Gaussian process $\{Z_t, t \geq 0\}$ independent of $B$ and with covariance function given by

$$E(Z_tZ_s) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases}$$

**Proof.** Let $0 \leq t_1 < t_2 < \cdots < t_k$. We need to prove the convergence in law

$$(B, S_n^{t_1}, \ldots, S_n^{t_k}) \overset{\text{Law}}{\rightarrow} (B, Z_{t_1}, \ldots, Z_{t_k})$$

in the space $C(\mathbb{R}_+) \times \mathbb{R}^k$. We can choose $N$ large enough so that for $n \geq N$, the Gaussian random variable $S_n^{t_i}$ become uncorrelated and hence independent. Then as in the proof of Theorem 3.5, it holds that

$$(S_n^{t_1}, \ldots, S_n^{t_k}) \overset{\text{Law}}{\rightarrow} (Z_{t_1}, \ldots, Z_{t_k}),$$

where the random vector $(Z_{t_1}, \ldots, Z_{t_k})$ has a standard Gaussian distribution on $\mathbb{R}^k$ and is independent of $B$. This completes the proof. \hfill \Box
Notice that we cannot expect that the convergence in Proposition 3.7 holds in \( C(0, \infty) \). Indeed, although under some mild conditions the stochastic convolution has a continuous version, the process \( Z \) does not have a continuous version.

The following proposition establishes the convergence of the stochastic convolution as a process in the sense of the finite-dimensional distributions.

**Proposition 3.7.** Under the assumptions of Theorem 3.5, suppose that the process \( u \) is continuous in \([0, \infty)\) in the \( L^2 \) sense. Then the finite-dimensional distributions of the process \( \{(X * B \psi_n)_t, t \geq 0\} \) converges stably to those of \( \{u_t Z_t, t \geq 0\} \), where \( \{Z_t, t \geq 0\} \) is a Gaussian process independent of \( B \) with covariance function given by (3.12).

**Proof.** Let \( 0 < t_1 < t_2 < \cdots < t_k \). We want to show that

\[
(B, (u * B \psi_n)_{t_1}, (u * B \psi_n)_{t_2}, \ldots, (u * B \psi_n)_{t_k}) \xrightarrow{\text{Law}} (B, u_{t_1} Z_{t_1}, u_{t_2} Z_{t_2}, \ldots, u_{t_k} Z_{t_k}),
\]

where the random vector \((Z_{t_1}, \ldots, Z_{t_k})\) has a standard Gaussian distribution on \( \mathbb{R}^k \) and is independent of \( B \). As in the proof of Theorem 3.5, if \( \alpha_n \) is a sequence decreasing to 0 such that \( n\alpha_n \to \infty \), we can consider for each \( t \geq 0 \) the sequence of random variables \( S_n^t \) defined in (3.11). Then, we have that, by the proof of theorem 3.5, for each \( i = 1, \ldots, k \),

\[
(u * B \psi_n)_{t_i} - u_{(t_i - \alpha_n) + S_n^t_i} \xrightarrow{L^2} 0.
\]

Also, by the \( L^2 \)-continuity of \( u \) and the Cauchy-Schwartz inequality, we can write

\[
u_{(t_i - \alpha_n) + S_n^t_i} - u_{t_i} S_n^t_i \xrightarrow{L^1} 0.
\

In particular the above convergence holds also in probability, so that

\[
A_n^i := (u * B \psi_n)_{t_i} - u_{t_i} S_n^t_i \xrightarrow{P} 0
\]

for \( i = 1, \ldots, k \). As a consequence,

\[
(A_n^1, A_n^2, \ldots, A_n^k) \xrightarrow{P} (0, 0, \ldots, 0).
\]

Then by Slutsky’s theorem (3.13) follows from the convergence in law

\[
(B, u_{t_1} S_n^{t_1}, \ldots, u_{t_k} S_n^{t_k}) \xrightarrow{\text{Law}} (B, u_{t_1} Z_{t_1}, \ldots, u_{t_k} Z_{t_k}),
\]

which is a consequence of Lemma 3.6. This completes the proof. \( \square \)

4. **Skorohod integrals with respect to fractional Brownian Motion**

Consider a fractional Brownian motion \( B^H = \{B^H_t, t \in [0, 1]\} \) with Hurst parameter \( H \in (0, 1) \). That is, \( B^H \) is a zero mean Gaussian process with covariance function (3.12). In this section we will study the asymptotic behavior as \( n \to \infty \) of a sequence of Skorohod integrals of the form

\[
F_n = \int_0^1 \phi_n(t) u_t \, \delta B^H_t, \quad n \geq 1,
\]

where \( u \) is a stochastic process verifying some suitable conditions. We split our study in two cases according to whether \( H > 1/2 \) or \( H < 1/2 \).
Case $H > 1/2$. We will assume the following conditions on the sequence $\phi_n$ of nonnegative and bounded functions:

(h4): $\lim_{n \to \infty} \|\phi_n\|_0^2 = L > 0$.

(h5): $\lim_{n \to \infty} \|\phi_n\|_r = 0$ for some $r < \frac{1}{H}$ (where here, and in the sequel, $\| \cdot \|_r$ denotes the $L^r$-norm on $[0, 1]$).

We are now ready to state and prove the main results of this section.

**Theorem 4.1.** Assume $B^H$ is a fractional Brownian motion with Hurst parameter $H > 1/2$. Consider a sequence of nonnegative and bounded functions $\phi_n$ on $[0, 1]$ satisfying conditions (h3), (h4) and (h5). Let $u$ be a stochastic process satisfying the following conditions:

(i) For any $t \in [0, 1]$, $u_t \in \mathbb{D}^{1,2}$ and the mapping $t \to \|u_t\|_{1,2}$ belongs to $\mathcal{F}$.

(ii) $u_t$ is continuous in $\mathbb{D}^{1,2}$ at $t = 1$.

(iii) $\int_0^1 (E[|D_s u_1|])^p ds < \infty$ where $\frac{1}{p} + \frac{1}{r} = 2H$, and $r$ is the number appearing in condition (h5).

Consider the sequence of Skorohod integrals introduced in (4.1). Then $F_n$ converges stably as $n \to \infty$ to $u_1 \sqrt{L}Z$, where $Z$ is a $N(0, 1)$ random variable independent of $B^H$.

**Proof.** Notice first that conditions (i) and (ii) imply that $\phi_n(t)u_t$ belongs to $\mathbb{D}^{1,2}(\mathcal{F}) \subset \text{Dom} \delta$.

Set $G_n := \int_0^1 \phi_n(t)u_t \, dB^H_t$. Denoting $\alpha_H = H(2H - 1)$, in view of (2.6) we can write

$$E \left[ (F_n - G_n)^2 \right] \leq E(\|\phi_n(t)(u_t - u_1)\|_0^2) + E(\|\phi_n(t)D(u_t - u_1)\|_0^2)$$

$$= \alpha_H \int_0^1 \int_0^1 \phi_n(t)\phi_n(s)E\left[ (u_t - u_1)(u_s - u_1) \right] |t - s|^{2H-2} dsdt$$

$$+ \alpha_H \int_0^1 \int_0^1 \phi_n(t)\phi_n(s)E\left[ (D(u_t - u_1), D(u_s - u_1))_\delta \right] |t - s|^{2H-2} dsdt$$

$$= A_{1,n} + A_{2,n}. \quad (4.2)$$

Both terms in (4.2) are handled similarly and we will show the details only for the second one. Let $0 < \delta < 1$. Then, separating the second term in two integrals, yields

$$A_{2,n} = \alpha_H \int_0^1 \int_0^1 1_{\{s \land t \leq \delta\}} \phi_n(t)\phi_n(s)E\left[ (D(u_t - u_1), D(u_s - u_1))_\delta \right] |t - s|^{2H-2} dsdt$$

$$+ \alpha_H \int_0^1 \int_0^1 \phi_n(t)\phi_n(s)E\left[ (D(u_t - u_1), D(u_s - u_1))_\delta \right] |t - s|^{2H-2} dsdt. \quad (4.3)$$

At this step note that by condition (i)

$$\int_0^1 \int_0^1 E(\langle D(u_t - u_1), D(u_s - u_1) \rangle_\delta |t - s|^{2H-2} dsdt$$

$$\leq \int_0^1 \int_0^1 \|u_t - u_1\|_{1,2} \|u_s - u_1\|_{1,2} |t - s|^{2H-2} dsdt < \infty.$$ 

So there is a constant $C$ such that the first term in (4.3) is bounded by

$$C \sup_{s \land t \leq \delta} \phi_n(s)\phi_n(t),$$

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which converges to 0 as $n \to \infty$ by condition (h3).

On the other hand, for the second term in (4.3), it follows from Cauchy-Schwartz inequality that
\[
\int_0^1 \int_0^1 \phi_n(t)\phi_n(s)E\left[\langle D(u_t - u_1), D(u_s - u_1)\rangle_S\right]|t - s|^{2H-2}dsdt \\
\leq \sup_{t \in [0,1]} E\left[\|D(u_t - u_1)\|_{\mathcal{S}}^2\right] \left(\int_0^1 \int_0^1 \phi_n(s)\phi_n(t)|t - s|^{2H-2}dsdt\right).
\]

By condition (h4), the sequence $\int_0^1 \int_0^1 \phi_n(s)\phi_n(t)|t - s|^{2H-2}dsdt$ is bounded and by condition (ii) the first factor tends to zero as $\delta \to 1$. This shows that $A_{2,n}$ tends to zero as $n \to \infty$. Repeating the same argument, we obtain that $A_{1,n}$ tends to zero as $n \to \infty$.

We have shown that $F_n - G_n \to 0$ in $L^2$, and hence also in law. All is left is to show that the limit of $G_n$ has the desired form. To this end note that, applying Lemma 2.1, $G_n$ can also be written as
\[
G_n = u_1 \int_0^1 \phi_n(t)\delta B_t^H + \alpha_H \int_0^1 \int_0^1 |t - s|^{2H-2}\phi_n(t)D_su_1dsdt
\]
\[
=: u_1 B_{1,n} + \alpha_H B_{2,n}.
\]

Let $p$ be as in the statement of the theorem and note that $p > 1$. Applying Hölder’s inequality with $\frac{1}{p} + \frac{1}{q} = 1$, yields
\[
E[\|B_{2,n}\|] \leq \left(\int_0^1 (E[|D_su_1|]^p)_sds\right)^{\frac{1}{p}} \left(\int_0^1 \left(\int_0^1 |t - s|^{2H-2}\phi_n(t)dt\right)^q ds\right)^{\frac{1}{q}}.
\]

The second factor is the $L^q$-norm of the fractional integral of order $2H - 1$ of the function $\phi_n$ on $[0,1]$. By the Hardy-Littlewood inequality, this factor is bounded by a constant times $\|\phi_n\|_{L^r([0,1])}$, where $\frac{1}{r} = \frac{1}{q} + 2H - 1 = 2H - \frac{1}{p}$. Taking into account conditions (iii) and (h5), we deduce
\[
\lim_{n \to \infty} E[\|B_{2,n}\|] = 0.
\]

In order to complete the proof of the theorem it suffices to show that $(B^H, B_{1,n})$ converges in law in the space $C([0,1]) \times \mathbb{R}$ to $(B^H, \sqrt{L}Z)$, where $Z$ is a $\mathcal{N}(0,1)$ random variable independent of $B^H$. In view of the fact that $B^H$ is a Gaussian process, this will follow from the next two properties:

(a): $\lim_{n \to \infty} E[B^H_{1,n}] = L$, which follows from property (h4).

(b): For any $t_0 \in [0,1]$, $\lim_{n \to \infty} E[B_{1,n}B^H_{t_0}] = 0$. In fact, using property (h5), we obtain for $\frac{1}{r'} + \frac{1}{q'} = 1$,
\[
E[B_{1,n}B^H_{t_0}] = \alpha_H \int_0^1 \int_0^{t_0} \phi_n(t)|t - s|^{2H-2}dsdt \\
\leq \alpha_n \|\phi_n\|_r \left(\int_0^1 \left(\int_0^{t_0} |t - s|^{2H-2}ds\right)^{q'}/r'\right) \to 0,
\]
as $n \to \infty$. $\square$
Theorem 4.1 can be applied to the example \( \phi_n(t) = n^H t^n \), and in this case, \( L = H \Gamma(2H) \). Indeed, condition \( (h3) \) is obvious. Condition \( (h4) \) follows from Lemma 4.2 below. Condition \( (h5) \) holds for any \( r < \frac{1}{H} \). This means that in condition (iii) it suffices to show that the integral is bounded for some \( p > \frac{1}{H} \).

**Lemma 4.2.** For any \( n, m \in \mathbb{N} \) and \( r > -1 \)

\[
\int_0^1 \int_0^1 t^n s^m |t - s|^r \, ds \, dt = \frac{\Gamma(m + 1) \Gamma(r + 1)}{(n + m + r + 2) \Gamma(2 + m + r)} + \frac{\Gamma(n + 1) \Gamma(r + 1)}{(n + m + r + 2) \Gamma(2 + n + r)}.
\]

In particular for \( H > 1/2 \)

\[
\lim_{n \to \infty} n^{2H} \int_0^1 \int_0^1 x^n y^m |x - y|^{2H-2} \, dy \, dx = \Gamma(2H - 1).
\]

**Proof.** First of all, note that using \( y = zx \) yields

\[
\int_0^1 y^m(x-y)^r \, dy = x^{m+1+r} \int_0^1 z^m(1-z)^r \, dz = x^{m+1+r} B(m+1,r+1),
\]

where \( B \) denotes the Beta function. Then

\[
\int_0^1 \int_0^1 t^n s^m |t - s|^r \, ds \, dt
\]

\[
= \int_0^1 \int_0^t t^n s^m (t-s)^r \, ds \, dt + \int_0^1 \int_0^s t^n s^m (s-t)^r \, dt \, ds
\]

\[
= \int_0^1 t^{n+m+r+1} B(m+1,r+1) \, dt + \int_0^1 s^{n+m+r+1} B(n+1,r+1) \, dt
\]

\[
= \frac{B(m+1,r+1) + B(n+1,r+1)}{n + m + r + 2}.
\]

The first part of the lemma now follows from the well-known relationship between the Beta and Gamma functions. The second part follows by taking \( n = m \) and using Lemma 3.6. \( \Box \)

**Case \( H < 1/2 \).** We assume the following conditions on the sequence \( \phi_n \) of nonnegative and bounded functions:

\( (h6) \): \( \sup_n \int_0^1 (s^{2H-1} + (1-s)^{2H-1}) \phi_n^2(s) \, ds < \infty \).

\( (h7) \): For any \( \delta \in [0,1) \), we have

\[
\lim_{n \to \infty} \int_0^\delta \left( \int_s^\delta |\phi_n(t) - \phi_n(s)|(t-s)^{H-\frac{3}{2}} \, dt \right)^2 \, ds = 0.
\]

\( (h8) \): \( \lim_{n \to \infty} \int_0^1 |(K_H^* \phi_n)(s)|^p \, ds = 0 \) for some \( p > 1 \).

**Theorem 4.3.** Assume \( B^H \) is a fractional Brownian motion with Hurst parameter \( 0 < H < 1/2 \). Consider a sequence of nonnegative and bounded functions \( \phi_n \) on \([0,1]\) satisfying conditions \( (h3), (h4), (h6), (h7) \) and \( (h8) \). Let \( u \) be a stochastic process satisfying the following conditions:

(i) For all \( t \in [0,1] \), \( u_t \in \mathbb{D}^{1.2} \).
(ii) The mapping $t \rightarrow u_t$ is Hölder continuous of order $\gamma > 1/2 - H$ from $[0,1]$ into $\mathbb{D}^{1,2}$.

(iii) We have
\[ \int_0^1 E((K^*_H Du_1 (s))^q) ds < \infty, \]
where $\frac{1}{p} + \frac{1}{q} = 1$ and $p$ is the exponent appearing in condition (h8).

Consider the sequence of Skorohod integrals introduced in (4.1). Then $F_n$ converges stably as $n \to \infty$ to $u_1 \sqrt{LZ}$, where $Z$ is a $N(0,1)$ random variable independent of $B^H$.

Proof. We divide the proof into 3 steps.

Step 1: We need to compute the variance of the random variable $\int_0^1 \phi_n(t) \delta B^H_t$. Condition (h4) implies that
\[ \lim_{n \to \infty} E\left(\left(\int_0^1 \phi_n(t) \delta B^H_t\right)^2\right) = L. \]

Step 2: Showing $F_n - G_n \xrightarrow{L^2(\Omega)} 0$, where
\[ G_n := \int_0^1 \phi_n(t) u_1 \delta B^H_t. \]

As in the proof of theorem 4.1, we can write
\[ E\left[(F_n - G_n)^2\right] \leq E(\|\phi_n(t)(u_t - u_1)\|_{\mathcal{H}}^2) + E(\|\phi_n(t)D(u_t - u_1)\|_{\mathcal{H} \otimes \mathcal{H}}^2) =: C_{1,n} + C_{2,n}. \]

We only work with $C_{2,n}$, the analysis of $C_{1,n}$ being similar by changing $\phi_n(t)D(u_t - u_1)$ and $\mathcal{H}$ appropriately by $\phi_n(t)(u_t - u_1)$ and $\mathbb{R}$ in the argument below. We have, using (2.5) and (2.4),
\[ C_{2,n} = E\left(\| \int_0^1 K_H(1,s)\phi_n(s)D(u_s - u_1) \right) \]
\[ + \int_0^1 (\phi_n(t)D(u_t - u_1) - \phi_n(s)D(u_s - u_1)) \left( \frac{\partial K_H}{\partial t}(t,s) dt \right)_{L^2([0,1];\mathcal{H})}^2. \]

Since
\[ (\phi_n(t)D(u_t - u_1) - \phi_n(s)D(u_s - u_1)) = \phi_n(s)D(u_t - u_s) + (\phi_n(t) - \phi_n(s))D(u_t - u_1), \]
we obtain
\[ C_{2,n} \leq 9E \left( \| K_H(1,s)\phi_n(s)D(u_s - u_1) \|_{L^2([0,1];\delta)}^2 \right) \]
\[ + 9E \left( \left\| \int_s^1 \phi_n(s)D(u_t - u_s)\frac{\partial K_H}{\partial t}(t,s)dt \right\|_{L^2([0,1];\delta)}^2 \right) \]
\[ + 9E \left( \left\| \int_s^1 (\phi_n(t) - \phi_n(s))D(u_t - u_1)\frac{\partial K_H}{\partial t}(t,s)dt \right\|_{L^2([0,1];\delta)}^2 \right) \]
\[ =: R_{1,n} + R_{2,n} + R_{3,n}. \]

To handle the term \( R_{1,n} \) we note that, by (2.2), there is a constant \( d_H \) such that
\[ K(1,s)^2 \leq d_H((1-s)^{2H-1} + s^{2H-1}). \]
We will denote by \( C \) a generic constant that may vary from line to line. Then by Minkowski’s inequality and condition (ii) for any \( \delta \in [0,1) \) we obtain
\[ R_{1,n} \leq 9E \left( \int_0^1 K_H(1,s)^2\phi_n^2(s)\|D(u_s - u_1)\|_{L^2([0,1];\delta)}^2 ds \right) \]
\[ \leq C \int_0^1 K_H(1,s)^2\phi_n^2(s)\|D(u_s - u_1)\|_{L^2([0,1];\delta)}^2 ds \]
\[ \leq C \int_0^\delta K_H(1,s)^2\phi_n^2(s)(1-s)^{2\gamma} ds \]
\[ + C \int_\delta^1 K_H(1,s)^2\phi_n^2(s)(1-s)^{2\gamma} ds \]
\[ =: R_{12,n} + R_{22,n}. \]

The term \( R_{12,n} \) can be estimated as follows
\[ R_{12,n} \leq C \sup_{0 \leq s \leq \delta} \phi_n^2(s) \int_0^1 K_H(1,s)^2(1-s)^{2\gamma} ds, \]

Taking into account that \( \int_0^1 K_H(1,s)^2(1-s)^{2\gamma} ds < \infty \), we deduce from condition (h3) that \( R_{12,n} \) converges to zero as \( n \to \infty \). For \( R_{22,n} \) we can write
\[ R_{22,n} \leq C(1-\delta)^{2\gamma} \int_0^1 K_H(1,s)^2\phi_n^2(s) ds. \]

From (4.5) and condition (h6), we deduce that \( \sup_n R_{22,n} \to 0 \) as \( \delta \uparrow 1 \). Therefore, we have proved that
\[ \lim_{n \to \infty} R_{1,n} = 0. \]
Concerning the term $R_{2,n}$, using Minkowski’s inequality, the estimate (2.3) and condition (ii), we obtain

$$R_{2,n} = 9E \left( \left\| \int_0^1 \phi_n(s) D(u_t - u_s) \frac{\partial K_H}{\partial t} (t, s) dt \right\|_{L^2([0,1];\delta)}^2 \right) \leq C \int_0^1 \left( \int_0^1 \phi_n(s) \left\| D(u_t - u_s) \right\|_{L^2(\Omega;\delta)} \left| \frac{\partial K_H}{\partial t} (t, s) \right| dt \right)^2 ds \leq C \int_0^1 \phi_n^2(s) \left( \int_0^1 (t-s)^\gamma (t-s)^{-3/2} dt \right)^2 ds \leq C \int_0^1 \phi_n^2(s)(1-s)^{2\gamma+2H-1} ds.$$  

Then, for any $\delta \in [0,1)$, the integral $\int_0^\delta \phi_n^2(s)(1-s)^{2\gamma+2H-1} ds$ converges to zero as $n \to \infty$ due to condition (h3), whereas, by condition (h6),

$$\int_0^1 \phi_n^2(s)(1-s)^{2\gamma+2H-1} ds \leq (1-\delta)^{2\gamma} \int_0^1 \phi_n^2(s)(1-s)^{2H-1} ds \leq C(1-\delta)^{2\gamma} \to 0,$$  

as $\delta \uparrow 1$. Therefore, we have proved that

$$\lim_{n \to \infty} R_{2,n} = 0.$$  

Finally for $R_{3,n}$ taking $0 < \delta < 1$, it follows from Minkowski’s inequality that

$$R_{3,n} = 9E \left( \left\| \int_0^1 (\phi_n(t) - \phi_n(s)) D(u_t - u_1) \frac{\partial K_H}{\partial t} (t, s) dt \right\|_{L^2([0,1];\delta)}^2 \right) \leq C \int_0^1 \left( \int_0^1 (\phi_n(t) - \phi_n(s)) \left\| D(u_t - u_1) \right\|_{L^2(\Omega;\delta)} \left| \frac{\partial K_H}{\partial t} (t, s) \right| dt \right)^2 ds \leq C \int_0^\delta \left( \int_0^\delta (\phi_n(t) - \phi_n(s)) \left\| D(u_t - u_1) \right\|_{L^2(\Omega;\delta)} \left| \frac{\partial K_H}{\partial t} (t, s) \right| dt \right)^2 ds \hspace{1cm} + \hspace{1cm} C \int_0^\delta \left( \int_0^\delta (\phi_n(t) - \phi_n(s)) \left\| D(u_t - u_1) \right\|_{L^2(\Omega;\delta)} \left| \frac{\partial K_H}{\partial t} (t, s) \right| dt \right)^2 ds \hspace{1cm} + \hspace{1cm} C \int_0^\delta \left( \int_0^\delta (\phi_n(t) - \phi_n(s)) \left\| D(u_t - u_1) \right\|_{L^2(\Omega;\delta)} \left| \frac{\partial K_H}{\partial t} (t, s) \right| dt \right)^2 ds =: T_{1,n} + T_{2,n} + T_{3,n}.$$  

At this step we study each term separately. For both $T_{2,n}$ and $T_{3,n}$ note that $\delta \leq t$ so condition (ii) gives

$$\left\| D(u_t - u_1) \right\|_{L^2(\Omega;\delta)} \leq C(1-\delta)^\gamma.$$
Step 3: We show that the limit in law of $G_n$ has the desired form. To this end note that $G_n$ can also be written as

\[(4.9) \quad G_n = u_1 \int_0^1 \phi_n(t) \delta B^H_t + \langle \phi_n, Du_1 \rangle_{\mathcal{H}}.\]

First we will show using condition (iii) that

\[(4.10) \quad E(\langle \phi_n, Du_1 \rangle_{\mathcal{H}}) \to 0\]

as $n \to \infty$. Fix $\delta \in [0, 1)$. We can write, by Hölder’s inequality

\[
E(\langle \phi_n, Du_1 \rangle_{\mathcal{H}}) = E(\langle (K_H^* \phi_n), (K_H^* Du_1) \rangle_{L^2([0,1])}) \leq \left( \int_0^1 |(K_H^* \phi_n)(s)|^p ds \right)^{\frac{1}{p}} \left( \int_0^1 E(|(K_H^* Du_1)(s)|^q) ds \right)^{\frac{1}{q}}.
\]
The first factor converges to zero as $n \to \infty$ by property (h8) and the second one is bounded by condition (iii). Therefore, (4.10) holds.

It remains to show that $(B^H, \int_0^1 \phi_n(t) \delta B^H_t)$ converges in law in the space $C([0,1]) \times \mathbb{R})$ to $(B^H, \sqrt{L}Z)$, where $Z$ is a $N(0,1)$ random variable independent of $B^H$. This claim follows from Step 1 and the fact that for any $t_0 \in [0,1]$, $\lim_{n \to \infty} E \left[ B^H_{t_0} \int_0^1 \phi_n(t) \delta B^H_t \right] = 0$. Indeed, we can write

$$E \left[ B^H_{t_0} \int_0^1 \phi_n(t) \delta B^H_t \right] = \langle 1_{[0,t_0]}, \phi_n \rangle_{\delta}$$

and

$$\|\langle 1_{[0,t_0]}, \phi_n \rangle_{\delta} \| = \left| \int_0^1 (K^*_H \phi_n)(s)(K^*_H 1_{[0,t_0]})(s) ds \right| \leq \| K^*_H \phi_n \|_{L^p([0,1])} \| K^*_H 1_{[0,t_0]} \|_{L^q([0,1])},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then the result follows from property (h8) and the fact that

$$\| K^*_H 1_{[0,t_0]} \|_{L^q([0,1])} < \infty.$$  

\[\square\]

Theorem 4.3 can be applied to the example $\phi_n(t) = n^H t^n$, when $H > \frac{1}{4}$. Indeed, condition (h4), again with $L = H \Gamma(2H)$, holds by Lemma 4.4 below. Condition (h3) is obvious. Property (h6) follows from the following computations:

$$\int_0^1 ((1-s)^{2H-1} + s^{2H-1}) \phi_n^2(s) ds = n^{2H} \int_0^1 ((1-s)^{2H-1} s^{2n} + s^{2H-1+2n}) ds = \frac{n^{2H} \Gamma(2H) \Gamma(2n+1)}{\Gamma(2n+2H+1)} + \frac{n^{2H} \Gamma(2n+2H)}{\Gamma(2n+2H+1)},$$

which is uniformly bounded by Lemma 3.6. In order to show property (h7), we write, for any $\delta \in [0,1]$,

$$n^{2H} \int_0^\delta \left( \int_0^\delta (t^n - s^n)(t-s)^{H-\frac{3}{2}} dt \right)^2 ds = n^{2H} \int_0^\delta \left( \int_0^\delta \sum_{k=0}^{n-1} t^k s^{n-1-k}(t-s)^{H-\frac{3}{2}} dt \right)^2 ds \leq n^{2H+2} \delta^{2n-2} \int_0^\delta \left( \int_0^\delta (t-s)^{H-\frac{1}{2}} dt \right)^2 ds = C n^{2H+2} \delta^{2n-2},$$

which converges to zero as $n \to \infty$.  

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To show property (h8), we write

\[
\int_0^1 |K^*_H \phi_n|^p(s) ds \leq C_n^{pH} \int_0^1 |K(1,s)|^p s^p ds + C_n^{pH} \int_0^1 \left| \int_s^1 (t^n - s^n)(t - s)^{H - \frac{3}{2}} dt \right|^p ds \\
\leq C_n^{pH} \int_0^1 ((1 - s)^{p(\frac{H}{2} - \frac{1}{2})} + s^{p(\frac{H}{2} - \frac{1}{2})}) s^p ds \\
+ C_n^{pH+2} \int_0^1 \left| \int_s^1 t^{n-1}(t - s)^{H - \frac{1}{2}} dt \right|^p ds \\
=: B_{1,n} + B_{2,n}.
\]

For the term \( B_{1,n} \), we have

\[
B_{1,n} \leq C \left( \frac{n^{pH} \Gamma(p(H - \frac{1}{2}) + 1) \Gamma(np + 1)}{\Gamma(p(H + \frac{1}{2}) + 2 + np)} + \frac{n^{pH}}{p(H - \frac{1}{2} + n + 1)} \right)
\]

By Lemma 3.6, this term converges to zero as \( n \to \infty \), provided \( p < 2 \). The same conclusion can be deduced for the term \( B_{2,n} \) using Young’s inequality.

**Lemma 4.4.** For any \( H \in (0, 1/2) \), we have

\[
\lim_{n \to \infty} n^H \|t^n\|_S = \sqrt{H \Gamma(2H)}.
\]

**Proof.** Using the operator \( K^*_H \) and integrating by parts, we can write

\[
n^{2H} \|t^n\|_S^2 = n^{2H} \|K^*_H(t^n)\|_{L^2([0,1])}^2 \\
= n^{2H} \left( \int_0^1 (K_H(1,s) s^n + \int_s^1 (t^n - s^n) \frac{\partial K_H}{\partial t}(t,s) dt)^2 ds \\
= n^{2H} \left( \int_0^1 (K_H(1,s) - n \int_s^1 t^{n-1} K_H(t,s) dt)^2 ds \\
= n^{2H} \int_0^1 K_H(1,s)^2 ds - 2n^{2H+1} \int_0^1 \int_s^1 t^{n-1} K_H(t,s) K_H(1,s) dt ds dt \\
+ n^{2H+2} \int_0^1 \left( \int_s^1 t^{n-1} K_H(t,s) dt \right)^2 ds \\
= A_{1,n} + A_{2,n} + A_{3,n}.
\]

(4.11)

At this step we work each term in (4.11) separately. Since

\[
R_H(t, s) = \int_0^{t\wedge s} K_H(t,u) K_H(s,u) du,
\]
the first term is $A_{1,n} = n^{2H} R(1,1) = n^{2H}$. Changing the order of integration in the second term yields

$$A_{2,n} = 2n^{2H+1} \int_0^1 \int_0^t t^{n-1} K_H(t,s)K_H(1,s) \, ds \, dt$$

$$= 2n^{2H+1} \int_0^1 t^{n-1} R(1,t) \, dt$$

$$= n^{2H+1} \int_0^1 t^{n-1}(1 + t^{2H} - (1-t)^{2H}) \, dt$$

$$= n^{2H} + \frac{n^{2H+1} \Gamma(n) \Gamma(2H+1)}{n + 2H} - \frac{n^{2H+1} \Gamma(n) \Gamma(2H+1)}{\Gamma(n + 2H + 1)}.$$

Writing the third term as a triple integral, changing the order of integration and using Lemma 4.2 gives

$$A_{3,n} = n^{2H+2} \int_0^1 \int_0^t \int_0^u t^{n-1} K_H(t,s)u^{n-1}K_H(u,s) \, du \, ds \, dt$$

$$= n^{2H+2} \int_0^t \int_0^u t^{n-1} K_H(t,s)u^{n-1}K_H(u,s) \, du \, ds \, dt$$

$$+ n^{2H+2} \int_0^1 \int_0^t t^{n-1} K_H(t,s)u^{n-1}K_H(u,s) \, du \, ds \, dt$$

$$= n^{2H+2} \int_0^1 \int_0^t t^{n-1} u^{n-1} R_H(t,u) \, du \, dt + n^{2H+2} \int_0^1 \int_0^t t^{n-1} u^{n-1} R_H(t,u) \, du \, dt$$

$$= \frac{n^{2H+1}}{2(n + 2H)} + \frac{n^{2H+1}}{2(n + 2H)} - \frac{n^{2H+2}}{2} \int_0^1 \int_0^t t^{n-1} u^{n-1} |t-u|^{2H} \, du \, dt$$

$$= \frac{n^{2H+1}}{(n + 2H)} - \frac{n^{2H+2}}{2(n + 2H)} \frac{\Gamma(n) \Gamma(2H+1)}{\Gamma(n + 1 + 2H)}.$$

Thus (4.11) simplifies to

$$\frac{n^{2H+1} \Gamma(n) \Gamma(2H+1)}{\Gamma(n + 2H + 1)} - \frac{n^{2H+2} \Gamma(n) \Gamma(2H+1)}{2(n + H) \Gamma(n + 1 + 2H)},$$

which, due to Lemma 3.6, converges to

$$\Gamma(2H+1) - \frac{\Gamma(2H+1)}{2} = \frac{\Gamma(2H+1)}{2} = H \Gamma(2H).$$

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