ULRICH BUNDLES ON NON–SPECIAL SURFACES
WITH $p_g = 0$ AND $q = 1$

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Abstract. Let $S$ be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample line bundle $\mathcal{O}_S(h)$ such that $h^1(S, \mathcal{O}_S(h)) = 0$. We show that such an $S$ supports families of dimension $p$ of pairwise non–isomorphic, indecomposable, Ulrich bundles for arbitrary large $p$. Moreover, we show that $S$ supports stable Ulrich bundles of rank 2 if the genus of the general element in $|h|$ is at least 2.

1. Introduction and Notation

Throughout the whole paper we will work on an uncountable algebraically closed field $k$ of characteristic 0 and $\mathbb{P}^N$ will denote the projective space over $k$ of dimension $N$. The word surface will always denote a projective smooth connected surface.

If $X$ is a smooth variety, then the study of vector bundles supported on $X$ is an important tool for understanding its geometric properties. If $X \subseteq \mathbb{P}^N$, then $X$ is naturally polarised by the very ample line bundle $\mathcal{O}_X(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$: in this case, at least from a cohomological point of view, the simplest bundles $\mathcal{F}$ on $X$ are the ones which are Ulrich with respect to $\mathcal{O}_X(h)$, i.e. such that

$$h^i(X, \mathcal{F}(-ih)) = h^j(X, \mathcal{F}(-(j+1)h)) = 0$$

for each $i > 0$ and $j < \dim(X)$.

The existence of Ulrich bundles on each variety is a problem raised by D. Eisenbud and F.O. Schreyer in [19] (see [10] for a survey on Ulrich bundles). There are many partial results (e.g. see [2], [3], [7], [8], [9], [11], [12], [13], [15], [17], [18], [26], [27], [28], [31]). Nevertheless, all such results and those ones proved in [20] seem to suggest that Ulrich bundles exist at least when $X$ satisfies an extra technical condition, namely that $X$ is arithmetically Cohen–Macaulay, i.e. projectively normal and such that

$$h^i(X, \mathcal{O}_S(th)) = 0$$

for each $i = 1, \ldots, \dim(X) − 1$ and $t \in \mathbb{Z}$. When $X$ is not arithmetically Cohen–Macaulay, the literature is very limited (e.g. see [9] and [14]).

Now let $S \subseteq \mathbb{P}^N$ be a surface and set $p_g(S) := h^2(S, \mathcal{O}_S)$, $q(S) := h^1(S, \mathcal{O}_S)$, whence $\chi(\mathcal{O}_S) := 1 − q(S) + p_g(S) = 0$. Thanks to the Enriques–Kodaira classification of surfaces, we know that $\kappa(S) \leq 1$ and $K^2 \leq 0$ (see [3], Theorem X.4 and Lemma VI.1). In what follows we will denote by $\text{Pic}(S)$ the Picard group of $S$: it

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is a group scheme and the connected component \( \text{Pic}^0(S) \subseteq \text{Pic}(S) \) of the identity is an abelian variety of dimension \( q(S) \) parameterizing the line bundles algebraically equivalent to \( O_S \).

In this paper we first rewrite the proof of Proposition 6 of [10], in order to be able to extend its statement to a slightly wider class of surfaces.

Our modified statement (which holds also without the hypothesis that \( k \) is uncountable) is as follows: recall that \( O_S(h) \) is called special if \( h^1(S, O_S(h)) \neq 0 \), non–special otherwise.

**Theorem 1.1.** Let \( S \) be a surface with \( p_g(S) = 0 \), \( q(S) = 1 \) and endowed with a very ample non–special line bundle \( O_S(h) \).

If \( O_S(\eta) \in \text{Pic}^0(S) \backslash \{ O_S \} \) is such that \( h^0(S, O_S(K_S + \eta)) = h^1(S, O_S(h + \eta)) = 0 \), then for each general \( C \in |O_S(h)| \) and each general set \( Z \subseteq C \) of \( h^0(S, O_S(h)) \) points, there is a rank 2 Ulrich bundle \( E \) with respect to \( O_S(h) \) fitting into the exact sequence

\[
0 \longrightarrow O_S(h + K_S + \eta) \longrightarrow E \longrightarrow I_{Z|S}(2h + \eta) \longrightarrow 0.
\]

As pointed out in [10], Proposition 6, when \( S \) is a bielliptic surface then each very ample line bundle \( O_S(h) \) is automatically non–special and there always exists a non–trivial \( O_S(\eta) \in \text{Pic}^0(S) \) of order 2 satisfying the above vanishings: thus the bundle \( E \) defined in Theorem 1.1 is actually special, i.e. \( c_1(E) = 3h + K_S \). We can argue similarly if \( S \) is either anticanonical, i.e. \( |− K_S| \neq \emptyset \), or geometrically ruled.

A condition forcing the indecomposability of a coherent sheaf \( F \) on an \( n \)–dimensional variety \( X \) is its stability. Recall that the slope \( \mu(F) \) and the reduced Hilbert polynomial \( p_F(t) \) of \( F \) with respect to the very ample polarisation \( O_X(h) \) are

\[
\mu(F) = c_1(F) h^{n−1} / \text{rk}(F), \quad p_F(t) = \chi(F(th)) / \text{rk}(F).
\]

The coherent sheaf \( F \) is called \( \mu \)–semistable (resp. \( \mu \)–stable) if for all subsheaves \( G \) with \( 0 < \text{rk}(G) < \text{rk}(F) \) we have \( \mu(G) \leq \mu(F) \) (resp. \( \mu(G) < \mu(F) \)).

The coherent sheaf \( F \) is called semistable (resp. stable) if for all \( G \) as above \( p_G(t) \leq p_F(t) \) (resp. \( p_G(t) < p_F(t) \)) for \( t \gg 0 \).

On an arbitrary variety we have the following chain of implications

\( F \) is \( \mu \)–stable \( \Rightarrow \) \( F \) is stable \( \Rightarrow \) \( F \) is semistable \( \Rightarrow \) \( F \) is \( \mu \)–semistable.

Nevertheless, when we restrict our attention to Ulrich bundles, the two notions of (semi)stability and \( \mu \)–(semi)stability actually coincide.

A priori, it is not clear whether the bundles constructed in Theorem 1.1 are stable. In Section 4 we deal with their stability as follows. The sectional genus of \( S \) with respect to \( O_S(h) \) is defined as the genus of a general element of \( |h| \). By the adjunction formula

\[
\pi(O_S(h)) := \frac{h^2 + hK_S}{2} + 1.
\]

Notice that the equality \( \pi(O_S(h)) = 0 \) would imply the rationality of \( S \) (e.g. see [11] and the references therein), contradicting \( q(S) = 1 \). Thus \( \pi(O_S(h)) \geq 1 \) in our setup.
Theorem 1.2. Let $S$ be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non–special line bundle $O_S(h)$. If $\pi(O_S(h)) \geq 2$, then the bundle $\mathcal{E}$ constructed in Theorem 1.1 from a very general set $Z \subseteq C \subseteq S$ of $h^0(S, O_S(h))$ points is stable.

Once that the existence of Ulrich bundles of low rank is proved, one could be interested in understanding how large a family of Ulrich bundles supported on $S$ can actually be. In particular we say that a smooth variety $X \subseteq \mathbb{P}^N$ is Ulrich–wild if it supports families of dimension $p$ of pairwise non–isomorphic, indecomposable, Ulrich bundles for arbitrary large $p$.

The last result proved in this paper concerns the Ulrich–wildness of the surfaces we are dealing with.

Theorem 1.3. Let $S$ be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non–special line bundle $O_S(h)$. Then $S$ is Ulrich–wild.

In Section 2 we list some general results on Ulrich bundles on polarised surfaces. In Section 3 we prove Theorem 1.1. In Section 4 we first recall some easy facts about the stability of Ulrich bundles, giving finally the proof of Theorem 1.2. In Section 5 we prove Theorem 1.3.

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2. General results

In general, an Ulrich bundle $\mathcal{F}$ on $X \subseteq \mathbb{P}^N$ collects many interesting properties (see Section 2 of [19]). The following ones are particularly important.

- $\mathcal{F}$ is globally generated and its direct summands are Ulrich as well.
- $\mathcal{F}$ is initialized, i.e. $h^0(X, \mathcal{F}(-h)) = 0$ and $h^0(X, \mathcal{F}) \neq 0$.
- $\mathcal{F}$ is aCM, i.e. $h^i(X, \mathcal{F}(th)) = 0$ for each $i = 1, \ldots, \dim(X) - 1$ and $t \in \mathbb{Z}$.

Let $S$ be a surface. The Serre duality for $\mathcal{F}$ is

$$h^i(S, \mathcal{F}) = h^{2-i}(S, \mathcal{F}^\vee(K_S)), \quad i = 0, 1, 2,$$

and the Riemann–Roch theorem is

$$h^0(S, \mathcal{F}) + h^2(S, \mathcal{F}) = h^1(S, \mathcal{F}) + \text{rk}(\mathcal{F})\chi(O_S) + \frac{c_1(\mathcal{F})c_1(\mathcal{F}) - K_S}{2} - c_2(\mathcal{F}).$$

Proposition 2.1. Let $S$ be a surface endowed with a very ample line bundle $O_S(h)$. If $\mathcal{E}$ is a vector bundle on $S$, then the following assertions are equivalent:

1. $\mathcal{E}$ is an Ulrich bundle with respect to $O_S(h)$;
2. $\mathcal{E}^\vee(3h + K_S)$ is an Ulrich bundle with respect to $O_S(h)$;
3. $\mathcal{E}$ is an aCM bundle and
   $$c_1(\mathcal{E})h = \text{rk}(\mathcal{E}) \frac{3h^2 + hK_S}{2},$$
   $$c_2(\mathcal{E}) = \frac{c_1(\mathcal{E})^2 - c_1(\mathcal{E})K_S}{2} - \text{rk}(\mathcal{E})(h^2 - \chi(O_S));$$
(4) $h^0(S, \mathcal{E}(-h)) = h^0(S, \mathcal{E}'(2h + K_S)) = 0$ and Equalities (3) hold.

Proof. See [14], Proposition 2.1.

The following corollaries are immediate consequences of the above characterization.

**Corollary 2.2.** Let $S$ be a surface endowed with a very ample line bundle $\mathcal{O}_S(h)$. If $\mathcal{O}_S(D)$ is a line bundle on $S$, then the following assertions are equivalent:

1. $\mathcal{O}_S(D)$ is an Ulrich bundle with respect to $\mathcal{O}_S(h)$;
2. $\mathcal{O}_S(3h + K_S - D)$ is an Ulrich bundle with respect to $\mathcal{O}_S(h)$;
3. $\mathcal{O}_S(D)$ is an aCM bundle and

$$D^2 = 2(h^2 - \chi(\mathcal{O}_S)) + DK_S, \quad Dh = \frac{1}{2}(3h^2 + hK_S);$$

$$h^0(S, \mathcal{O}_S(D - h)) = h^0(S, \mathcal{O}_S(2h + K_S - D)) = 0$$ and Equalities (4) hold.

Proof. See [14], Corollary 2.2.

**3. Existence of rank 2 Ulrich bundles**

We start this section by recalling that if $S$ is any surface, then the connected component $\text{Pic}^0(S)$ of the identity inside $\text{Pic}(S)$ is an abelian variety of dimension $q(S)$ called Picard variety of $S$. The quotient is a finitely generated abelian group called Néron–Severi group of $S$.

Now, let $S$ be a surface with $p_g(S) = 0$ and $q(S) = 1$. Then $\text{Pic}^0(S)$ is an elliptic curve: in particular $\text{Pic}^0(S)$ contains three pairwise distinct non–trivial divisors of order 2.

In order to prove Theorem 1.1 we will make use of the Hartshorne–Serre correspondence on surfaces. We recall that a locally complete intersection subscheme $Z$ of dimension zero on a surface $S$ is Cayley–Bacharach (CB for short) with respect to a line bundle $\mathcal{O}_S(A)$ if, for each $Z' \subseteq Z$ of degree $\deg(Z) - 1$, the natural morphism $H^0(S, \mathcal{I}_{Z'|S}(A)) \rightarrow H^0(S, \mathcal{I}_{Z'|S}(A))$ is an isomorphism.

**Theorem 3.1.** Let $S$ be a surface and $Z \subseteq S$ a locally complete intersection subscheme of dimension 0.

Then there exists a vector bundle $\mathcal{F}$ of rank 2 on $S$ fitting into an exact sequence of the form

$$0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{F} \rightarrow \mathcal{I}_{Z'|S}(A) \rightarrow 0,$$

if and only if $Z$ is CB with respect to $\mathcal{O}_S(A + K_S)$.

Proof. See Theorem 5.1.1 in [23].

We now prove Theorem 1.1 stated in the introduction. As we already noticed therein, its proof for $hK_S = 0$ coincides with the one of Proposition 6 in [10] because in this case the vanishing $h^1(S, \mathcal{O}_S(h \pm \eta)) = 0$ follows immediately from the Kodaira vanishing theorem as we will show below in Corollary 3.3.
Proof of Theorem 3.1. Recall that by hypothesis \( p_g(S) = h^1(S, \mathcal{O}_S(h)) = 0 \) and \( q(S) = 1 \). It follows that \( \chi(\mathcal{O}_S) = 0 \) and
\[
h^2(S, \mathcal{O}_S(h)) = h^0(S, \mathcal{O}_S(K_S - h)) \leq h^0(S, \mathcal{O}_S(K_S)) = 0,
\]
thus \( S \subseteq \mathbb{P}^N \), where
\[
N := h^0(S, \mathcal{O}_S(h)) - 1 = \frac{h^2 - hK_S}{2} - 1 \geq 4,
\]
because \( q(S) = 0 \) for each surface \( S \subseteq \mathbb{P}^3 \).

Let \( C := S \cap H \mid h \) be a general hyperplane section and let \( i : C \to S \) be the inclusion morphism. The curve \( C \) is non-degenerate in \( \mathbb{P}^{N-1} \cong H \subseteq \mathbb{P}^N \). Indeed the exact sequence
\[
0 \to \mathcal{I}_S|_{\mathbb{P}^N}(1) \to \mathcal{O}_{\mathbb{P}^N}(1) \to \mathcal{O}_S(h) \to 0
\]
implies \( h^0(\mathbb{P}^N, \mathcal{I}_S|_{\mathbb{P}^N}(1)) = h^1(\mathbb{P}^N, \mathcal{I}_S|_{\mathbb{P}^N}(1)) = 0 \). Thus, the exact sequence
\[
0 \to \mathcal{I}_S|_{\mathbb{P}^N}(1) \to \mathcal{I}_C|_{\mathbb{P}^N}(1) \to \mathcal{I}_C(h) \to 0
\]
implies \( h^0(\mathbb{P}^N, \mathcal{I}_C|_{\mathbb{P}^N}(1)) = 1 \), because \( \mathcal{I}_C(h) \cong \mathcal{O}_S \). Finally the exact sequence
\[
0 \to \mathcal{I}_{H|\mathbb{P}^N}(1) \to \mathcal{I}_C|_{\mathbb{P}^N}(1) \to \mathcal{I}_{C|H}(1) \to 0
\]
and the isomorphism \( \mathcal{I}_{H|\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{P}^N} \) yields \( h^0(C, \mathcal{I}_{C|H}(1)) = 0 \).

It follows the existence of a reduced subscheme \( Z \subseteq C \subseteq S \) of degree \( N + 1 \) whose points are in general position inside \( H \cong \mathbb{P}^{N-1} \). Thus \( Z \) is CB with respect to \( \mathcal{O}_S(h) \), hence there exists Sequence (3) with \( \mathcal{O}_S(A) \cong \mathcal{O}_S(h - K_S) \), thanks to Theorem 3.1.

Let \( \mathcal{O}_S(\eta) \in \text{Pic}^0(S) \setminus \{ \mathcal{O}_S \} \) be such that \( h^0(S, \mathcal{O}_S(K_S \pm \eta)) = h^1(S, \mathcal{O}_S(h \pm \eta)) = 0 \) and set \( \mathcal{E} := \mathcal{F}(h + K_S + \eta) \). The bundle \( \mathcal{E} \) fits into Sequence (1) and satisfies Equalities (3). If we show that \( h^0(S, \mathcal{E}(-h)) = h^0(S, \mathcal{E}^\vee(2h + K_S)) = 0 \), then we conclude that \( \mathcal{E} \) is Ulrich thanks to Proposition 2.1 above. Notice that the second vanishing is equivalent to \( h^0(S, \mathcal{E}(-h - 2\eta)) = 0 \) because \( c_1(\mathcal{E}) = 3h + K_S + 2\eta \).

The vanishing \( h^0(S, \mathcal{O}_S(K_S \pm \eta)) = 0 \) implies
\[
h^0(S, \mathcal{E}(-h)) \leq h^0(S, \mathcal{I}_Z|\mathcal{S}(h + \eta)), \quad h^0(S, \mathcal{E}(-h - 2\eta)) \leq h^0(S, \mathcal{I}_Z|\mathcal{S}(h - \eta)).
\]
The exact sequence
\[
0 \to \mathcal{I}_C|\mathcal{S} \to \mathcal{I}_Z|\mathcal{S} \to \mathcal{I}_Z|\mathcal{S} \to 0
\]
and the isomorphisms \( \mathcal{I}_C|\mathcal{S} \cong \mathcal{O}_S(-h) \) and \( \mathcal{I}_Z|\mathcal{S} \cong \mathcal{O}_C(-Z) \) imply
\[
h^0(S, \mathcal{I}_Z|\mathcal{S}(h \pm \eta)) \leq h^0(C, \mathcal{O}_C(-Z) \otimes \mathcal{O}_S(h \pm \eta))
\]
because \( h^0(S, \mathcal{O}_S(\pm\eta)) = 0 \). Thanks to the general choice of the points in \( Z \), the Riemann–Roch theorem on \( C \) and the adjunction formula \( \mathcal{O}_C(K_C) \cong i^*\mathcal{O}_S(h + K_S) \) on \( S \) give
\[
h^0(C, \mathcal{O}_C(-Z) \otimes \mathcal{O}_S(h \pm \eta)) = h^0(C, i^*\mathcal{O}_S(h \pm \eta)) - \deg(Z) = h^2 + 1 - \pi(\mathcal{O}_S(h)) - \deg(Z) + h^1(C, i^*\mathcal{O}_S(h \pm \eta)) = h^0(C, i^*\mathcal{O}_S(K_S \mp \eta)) = 0.
\]
The exact sequence
\[ 0 \rightarrow \mathcal{O}_S(-h) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_C \rightarrow 0 \]
implies the existence of the exact sequence
\[ H^0(S, \mathcal{O}_S(K_S \mp \eta)) \rightarrow H^0(C, i^* \mathcal{O}_S(K_S \mp \eta)) \rightarrow H^1(S, \mathcal{O}_S(h \mp h \mp \eta)) \cong H^1(S, \mathcal{O}_S(h \pm \eta)). \]
Thus the hypothesis on $\mathcal{O}_S(K_S \pm \eta)$ and $\mathcal{O}_S(h \pm \eta)$ forces $h^0(S, i^* \mathcal{O}_S(K_S \mp \eta)) = 0$.

It is natural to ask when the vanishings $h^1(S, \mathcal{O}_S(K_S \pm \eta)) = h^1(S, \mathcal{O}_S(h \pm \eta)) = 0$ actually occur. We list below some related result.

**Corollary 3.2.** Let $S$ be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non–special line bundle $\mathcal{O}_S(h)$.

Then $S$ supports Ulrich bundles of rank $r \leq 2$.

**Proof.** Since each direct summand of an Ulrich bundle is Ulrich as well, it follows from Theorem [1.1] that it suffices to prove the existence of $\mathcal{O}_S(\eta) \in \text{Pic}^0(S) \setminus \{ \mathcal{O}_S \}$ such that $h^0(S, \mathcal{O}_S(K_S \pm \eta)) = h^1(S, \mathcal{O}_S(h \pm \eta)) = 0$.

Let $\mathcal{P}$ be the Poincaré line bundle on $S \times \text{Pic}(S)$. Recall that (e.g. see [29], Lecture 19), if $p: S \times \text{Pic}(S) \rightarrow \text{Pic}(S)$ is the projection on the second factor and $\mathcal{L} \in \text{Pic}(S)$, then the restriction of $\mathcal{P}$ to the fibre $p^{-1}(\mathcal{L}) \cong S$ is isomorphic to the line bundle $\mathcal{L}$. The line bundle $\mathcal{P}$ is thus flat on $\text{Pic}(S)$.

Let $\mathcal{P}_0$ be the restriction of $\mathcal{P}$ to $S \times \text{Pic}^0(S)$, $A \subseteq S$ a divisor, $s: S \times \text{Pic}(S) \rightarrow S$ the projection on the first factor. The line bundle $\mathcal{P}_0 \otimes s^* \mathcal{O}_S(A)$ is flat over $\text{Pic}^0(S)$ and parameterizes the line bundles on $S$ algebraically equivalent to $\mathcal{O}_S(A)$. Thus the semicontinuity theorem (e.g. see Theorem III.12.8 of [22]) applied to the sheaf $\mathcal{P}_0 \otimes s^* \mathcal{O}_S(A)$ and the map $p_0: S \times \text{Pic}^0(S) \rightarrow \text{Pic}^0(S)$ imply that for each $i = 0, 1, 2$ and $c \in \mathbb{Z}$ the sets
\[ V^i_A(c) := \{ \eta \in \text{Pic}^0(S) \mid h^i(S, \mathcal{O}_S(A \pm \eta)) > c \}, \]
are closed inside $\text{Pic}^0(S)$. In particular $V := V^0_\emptyset(0) \cup V^0_{K_S}(0)$ is closed.

By definition $\mathcal{O}_S \in \text{Pic}^0(S) \setminus V \neq \emptyset$. Thus for each general $\mathcal{O}_S(\eta) \in \text{Pic}^0(S)$, the hypothesis $h^0(S, \mathcal{O}_S(K_S \pm \eta)) = h^1(S, \mathcal{O}_S(h \pm \eta)) = 0$ is satisfied and the statement is then completely proved. \[ \square \]

Notice that the above result guarantees the existence of an Ulrich bundle $\mathcal{E}$ with $c_1(\mathcal{E}) = 3h + K_S + 2\eta$ fitting into Sequence [11]. Such bundle is special if and only if $\mathcal{O}_S(\eta)$ has order 2. It is not clear if such a choice can be done in general. Anyhow in some particular cases we can easily prove an existence result also for special Ulrich bundles: we start from Beauville’s result for bielliptic surfaces, i.e. minimal surfaces $S$ with $p_g(S) = 0$, $q(S) = 1$ and $\kappa(S) = 0$ (see Proposition 6 of [10]).

**Corollary 3.3.** Let $S$ be a bielliptic surface endowed with a very ample line bundle $\mathcal{O}_S(h)$.

Then $\mathcal{O}_S(h)$ is non–special and $S$ supports special Ulrich bundles of rank 2.
Proof. If \( \kappa(S) = 0 \), then \( K_S \) is numerically trivial, hence \( h - K_S \) is ample for each choice of \( \mathcal{O}_S(\eta) \in \text{Pic}^0(S) \), thanks to the Nakai criterion. Thus the vanishing \( h^1(S, \mathcal{O}_S(h \pm \eta)) = 0 \) follows from the Kodaira vanishing theorem: in particular \( \mathcal{O}_S(h) \) is non-special.

We can find \( \mathcal{O}_S(\eta) \in \text{Pic}^0(S) \setminus \{ \mathcal{O}_S, \mathcal{O}_S(\pm K_S) \} \) of order 2, because there are three non-trivial and pairwise non-isomorphic elements of order 2 in \( \text{Pic}^0(S) \). Thus \( h^0(S, \mathcal{O}_S(K_S \pm \eta)) = 0 \) because \( K_S \pm \eta \) is not trivial by construction, hence the statement follows from Theorem 1.1. \( \square \)

The surface \( S \) is anticanonical if \( |-K_S| \neq 0 \); in particular \( p_g(S) = 0 \). The ampleness of \( \mathcal{O}_S(h) \) implies \( hK_S < 0 \) in this case.

Corollary 3.4. Let \( S \) be an anticanonical surface with \( q(S) = 1 \) and endowed with a very ample line bundle \( \mathcal{O}_S(h) \).

Then \( \mathcal{O}_S(h) \) is non-special and \( S \) supports special Ulrich bundles of rank 2.

Proof. If \( A \in |-K_S| \), then \( \omega_A \cong \mathcal{O}_A \) by the adjunction formula. We have \( h^1(A, \mathcal{O}_S(h \pm \eta) \otimes \mathcal{O}_A) = h^0(A, \mathcal{O}_S(-h \mp \eta) \otimes \mathcal{O}_A) \), for each \( \mathcal{O}_S(\eta) \in \text{Pic}^0(S) \).

On the one hand, if \( h^1(A, \mathcal{O}_S(-h \mp \eta) \otimes \mathcal{O}_A) > 0 \), then \(-hC \geq 0 \) for some irreducible component \( C \subset A \). On the other hand \( \mathcal{O}_S(h) \) is ample, hence \( hC > 0 \).

The contradiction implies \( h^0(A, \mathcal{O}_S(-h \mp \eta) \otimes \mathcal{O}_A) = 0 \), hence the cohomology of the exact sequence

\[
0 \longrightarrow \mathcal{O}_S(h + K_S \mp \eta) \longrightarrow \mathcal{O}_S(h \mp \eta) \longrightarrow \mathcal{O}_S(h \mp \eta) \otimes \mathcal{O}_A \longrightarrow 0
\]

and the Kodaira vanishing theorem yield \( h^1(S, \mathcal{O}_S(h \mp \eta)) = 0 \). In particular \( \mathcal{O}_S(h) \) is non-special. Finally \( hK_S < 0 \), hence \( h^0(S, \mathcal{O}_S(K_S \pm \eta)) = 0 \).

The statement then follows from Theorem 1.1 by taking any non-trivial \( \mathcal{O}_S(\eta) \in \text{Pic}^0(S) \) of order 2. \( \square \)

Recall that a geometrically ruled surface is a surface \( S \) with a surjective morphism \( p: S \rightarrow E \) onto a smooth curve such that every fibre of \( p \) is isomorphic to \( \mathbb{P}^1 \). If \( S \) is geometrically ruled, then \( p_g(S) = 0 \) and \( q(S) \) is the genus of \( E \) (see [22], Chapter V.2 for further details).

Remark 3.5. Let \( S \) be a geometrically ruled surface on an elliptic curve \( E \) so that \( p_g(S) = 0 \) and \( q(S) = 1 \). Thanks to the results in [22], Chapter V.2, we know the existence of a vector bundle \( \mathcal{H} \) of rank 2 on \( E \) such that \( h^0(E, \mathcal{H}) \neq 0 \) and \( h^0(E, \mathcal{H}(-P)) = 0 \) for each \( P \in E \) and \( S \cong \mathbb{P}(\mathcal{H}) \). Then \( p \) can be identified with the natural projection map \( \mathbb{P}(\mathcal{H}) \rightarrow E \). The group \( \text{Pic}(S) \) is generated by the class \( \xi \) of \( \mathcal{O}_{\mathbb{P}(\mathcal{H})}(1) \) and by \( p^* \text{Pic}(E) \). If we set \( \mathcal{O}_E(h) := \det(\mathcal{H}) \) and \( e := -\deg(\mathcal{H}) \), then \( e \geq -1 \) (see [30]). Moreover, \( K_S = -2\xi + p^*h \).

There exists an exact sequence

\[
0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{H} \longrightarrow \mathcal{O}_E(h) \longrightarrow 0.
\]

The symmetric product of Sequence (9) yields

\[
0 \longrightarrow \mathcal{H}(-h) \longrightarrow S^2\mathcal{H}(-h) \longrightarrow \mathcal{O}_E(h) \longrightarrow 0.
\]
Sequence (9) splits if and only if $\mathcal{H}$ is decomposable. Thus, if this occurs, then $S^2\mathcal{H}(-h)$ contains $\mathcal{O}_E$ as direct summand, whence

\begin{equation}
 h^0(S, \mathcal{O}_S(-K_S)) \geq h^0(E, \mathcal{O}_E) = 1.
\end{equation}

because $h^0(S, \mathcal{O}_S(-K_S)) = h^0(E, S^2\mathcal{H}(-h))$, thanks to the projection formula.

Assume that $\mathcal{H}$ is indecomposable. Then either $\mathcal{O}_E(h) = \mathcal{O}_E$ or $\mathcal{O}_E(h) \neq \mathcal{O}_E$.

In the first case the cohomology of Sequences (9) and (10) again implies Inequality (11).

If $\mathcal{O}_E(h) \neq \mathcal{O}_E$, then Lemma 22 of [4] implies that $S^2\mathcal{H}(-h)$ is the direct sum of the three non–trivial elements of order 2 of $\text{Pic}(E)$, hence $h^0(S, \mathcal{O}_S(-K_S)) = 0$.

We conclude that a geometrically ruled surface on an elliptic curve is anticanonical if and only if $e \geq 0$.

Thanks to the above remark and Corollary 3.4 we know that each geometrically ruled surface $S$ with $q(S) = 1$ and $e \geq 0$ supports special Ulrich bundles of rank 2 with respect to each very ample line bundle $\mathcal{O}_S(h)$. We can extend the result also to the case $e = -1$.

**Corollary 3.6.** Let $S$ be a geometrically ruled surface with $q(S) = 1$ and endowed with a very ample line bundle $\mathcal{O}_S(h)$.

Then $\mathcal{O}_S(h)$ is non–special and $S$ supports special Ulrich bundles of rank 2.

**Proof.** We have to prove the statement only for $e = -1$. If $\mathcal{O}_S(h) = \mathcal{O}_{\mathcal{H}}(a\xi + p^*b)$, then $\deg(b) > -a/2$ (see [22], Proposition V.2.21). Thus the Table in Proposition 3.1 of [21] implies that $h^1(S, \mathcal{O}_S(h + \eta)) = 0$ for each $\eta \in \text{Pic}^0(S)$.

Again the statement follows from Theorem 1.1 by taking any non–trivial $\mathcal{O}_S(\eta)$ of order 2.

**Remark 3.7.** The corollary above extends Propositions 3.1, 3.3 and Theorem 3.4 of [2] to the range $e \leq 0$, when $g = 1$.

Recall that an embedded surface $S \subseteq \mathbb{P}^N$ is called non–degenerate if it is not contained in any hyperplane.

**Corollary 3.8.** Let $S \subseteq \mathbb{P}^4$ be a non–degenerate non–special surface with $p_g(S) = 0$. Then $S$ supports special Ulrich bundles of rank 2.

**Proof.** The cohomology of Sequence (8) tensored by $\mathcal{O}_S(h)$ implies $h^1(C, i^*\mathcal{O}_S(h)) = 0$. In particular such surfaces are sectionally non–special (see [24] for details). Non–special and sectionally non–special surfaces are completely classified in [24] and [25]. They satisfy $q(S) \leq 1$ and, if equality holds, then they are either quintic scrolls over elliptic curves, or the Serrano surfaces (these are very special bielliptic surfaces of degree 10: see [32]). The results above and Section 4 of [14] yields the statement.

**Remark 3.9.** Linearly normal non–special surface $S \subseteq \mathbb{P}^4$ with $p_g(S) = 0$ satisfy $3 \leq h^2 \leq 10$ (see [24] and [25]). If $h^2 \leq 6$, such surfaces are known to support Ulrich line bundles: see [27] for the case $q(S) = 0$ and [10], Assertion 2) of Proposition 5 for the case $q(S) = 1$. 
4. Stability of Ulrich bundles

We start this section by recalling the following result: see [13], Theorem 2.9 for its proof.

**Theorem 4.1.** Let $X$ be a smooth variety endowed with a very ample line bundle $\mathcal{O}_X(h)$.
If $\mathcal{E}$ is an Ulrich bundle on $X$ with respect to $\mathcal{O}_X(h)$, the following assertions hold:

1. $\mathcal{E}$ is semistable and $\mu$–semistable;
2. $\mathcal{E}$ is stable if and only if it is $\mu$–stable;
3. if $0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{M} \rightarrow 0$
is an exact sequence of coherent sheaves with $\mathcal{M}$ torsion free and $\mu(\mathcal{L}) = \mu(\mathcal{E})$, then both $\mathcal{L}$ and $\mathcal{M}$ are Ulrich bundles.

We now prove Theorem 1.2 stated in the introduction.

**Proof of Theorem 1.2.** Recall that $\mathcal{E}$ is constructed as follows. First we choose $C := S \cap H \in |h|$ where $H \cong \mathbb{P}^{N-1}$ is a general hyperplane: from now on we denote by $i: C \rightarrow S$ the inclusion morphism. The Hilbert scheme $\mathcal{H}_C$ of 0–dimensional subschemes of degree $N + 1$ on $C$ has dimension $N + 1$ and contains an open non–empty subset $\mathcal{R} \subseteq \mathcal{H}_C$ corresponding to reduced schemes of $N + 1$ points in general position in $H$. If we choose a general $Z \in \mathcal{R}$, then we finally construct $\mathcal{E}$ from $Z$ by means of Theorem 3.1.

We now show that if $Z$ is very general inside $\mathcal{R}$, i.e. it is in the complement of a countable union of suitable proper closed subsets, then $\mathcal{E}$ is stable.

To this purpose, let $\mathcal{O}_S(D)$ be an Ulrich line bundle on $S$ (if any). By hypothesis $\pi(\mathcal{O}_S(h)) \geq 2$, then

$$(h + \eta - D)h = -\frac{h^2 + hK_S}{2} = 1 - \pi(\mathcal{O}_S(h)) \leq -1,$$

hence

$$(12) \quad h^0(S, \mathcal{I}_C|S(2h + \eta - D)) = h^0(S, \mathcal{O}_S(h + \eta - D)) = 0,$$

i.e. there are no divisors in $|2h + \eta - D|$ containing $C$. Thus the cohomology of Sequence (8) tensored by $\mathcal{O}_S(2h + \eta - D)$ yields the injectivity of the restriction map

$h^0(S, \mathcal{O}_S(2h + \eta - D)) \rightarrow H^0(C, i^*\mathcal{O}_S(2h + \eta - D)).$

Since $(2h + \eta - D)h = N + 1$, it follows that each $Z \subseteq A \in |2h + \eta - D|$ containing a point the Hilbert scheme $\mathcal{H}_C$ of subschemes of degree $N + 1$ on $C$, is actually cut out on $C$ by $A$.

Thus, if $\mathcal{Z}_D$ denotes the closed subset of $\mathcal{H}_C$ of points $Z$ such that $h^0(C, \mathcal{I}_Z|C \otimes i^*\mathcal{O}_S(2h + \eta - D)) \geq 1$, then

$$\dim(\mathcal{Z}_D) = h^0(C, i^*\mathcal{O}_S(2h + \eta - D)) - 1.$$
On the one hand, if \( i^* \mathcal{O}_S(2h + \eta - D) \) is special, then the Clifford theorem and the second Equality (1) imply
\[
h^0(C, i^* \mathcal{O}_S(2h + \eta - D)) \leq \frac{(2h + \eta - D)h}{2} + 1 = \frac{N + 3}{2} \leq N,
\]
because \( N \geq 4 \) (see Inequality (3)). On the other hand, if \( i^* \mathcal{O}_S(2h + \eta - D) \) is non–special, the Riemann–Roch theorem on \( C \) and the second Equality (4) return
\[
h^0(C, i^* \mathcal{O}_S(2h + \eta - D)) = N + 2 - \pi(\mathcal{O}_S(h)) \leq N,
\]
because \( \pi(\mathcal{O}_S(h)) \geq 2 \). It follows from the above inequalities that \( \dim(Z_D) \leq N - 1 \).

Since \( q(S) = 1 \) and the Néron–Severi group of \( S \) is a finitely generated abelian group, it follows that the set \( \mathcal{D} \subseteq \text{Pic}(S) \) of Ulrich line bundles is contained in a countable disjoint union of a fixed elliptic curve. In particular there is
\[
Z \in \mathcal{R} \setminus \bigcup_{\mathcal{O}_S(D) \in \mathcal{D}} Z_D
\]
because \( \dim(\mathcal{R}) = N + 1 \). Let \( \mathcal{E} \) be the corresponding bundle.

Assume that \( \mathcal{E} \) is not stable: then it is not \( \mu \)–stable, thanks to Theorem 4.1. In particular there exists a line subbundle \( \mathcal{O}_S(D) \subseteq \mathcal{E} \) such that \( \mu(\mathcal{E}) = \mu(\mathcal{O}_S(D)) \).

Again Theorem 4.1 implies that \( \mathcal{O}_S(D) \) is Ulrich.

On the one hand, if \( \mathcal{O}_S(D) \) is contained in the kernel \( \mathcal{K} \cong \mathcal{O}_S(h + K_S + \eta) \) of the map \( \mathcal{E} \to \mathcal{I}_{Z|S}(2h + \eta) \) in Sequence (1), then \( h^0(S, \mathcal{O}_S(h + K_S + \eta - D)) \neq 0 \). On the other hand, Equality (4) and Inequality (6) imply
\[
(h + K_S + \eta - D)h = -\frac{h^2 - hK_S}{2} = 1 - N \leq -3,
\]
whence \( h^0(S, \mathcal{O}_S(h + K_S + \eta - D)) = 0 \).

We deduce that \( \mathcal{O}_S(D) \not\subseteq \mathcal{K} \), hence the composite map \( \mathcal{O}_S(D) \subseteq \mathcal{E} \to \mathcal{I}_{Z|S}(2h + \eta) \) should be non–zero, i.e. \( h^0(S, \mathcal{I}_{Z|S}(2h + \eta - D)) \geq 1 \). The cohomology of Sequence (7) tensored by \( \mathcal{O}_S(2h + \eta - D) \) and Equality (12) then would imply
\[
h^0(C, \mathcal{I}_{Z|C} \otimes \mathcal{O}_S(2h + \eta - D)) \geq h^0(S, \mathcal{I}_{Z|S}(2h + \eta - D)) \geq 1,
\]
contradicting our choice of \( Z \): thus the bundle \( \mathcal{E} \) is necessarily stable. \( \square \)

**Remark 4.2.** If \( \pi(\mathcal{O}_S(h)) = 1 \), then \( S \) is a geometrically ruled surface embedded as a scroll by \( \mathcal{O}_S(h) \cong \mathcal{O}_S(\xi + p^*b) \) over an elliptic curve, thanks to [11], Theorem A (here we are using the notation introduced in Remark 3.5).

Moreover \( (h + \eta - D)h = 0 \), hence the argument in the above proof does not lead to any contradiction when \( \mathcal{O}_S(D) \cong \mathcal{O}_S(h + \eta) \). Such a line bundle is actually Ulrich, because one easily checks that it satisfies all the conditions of Corollary 2.2.

In [16], via a slightly different but similar construction, we are able to show the existence of special Ulrich bundles of rank 2 on elliptic scrolls.

Let \( S \) be a surface with \( p_g(S) = 0 \), \( q(S) = 1 \) and endowed with a very ample non–special line bundle \( \mathcal{O}_S(h) \). Let
\[
c_1 := 3h + K_S + 2\eta, \quad c_2 := \frac{5h^2 + 3hK_S}{2},
\]
where $\mathcal{O}_S(\eta) \in \text{Pic}^0(S) \setminus \{0\}$ satisfies
\[
h^0(S, \mathcal{O}_S(K_S \pm \eta)) = h^1(S, \mathcal{O}_S(h \pm \eta)) = 0.
\]
If $\pi(\mathcal{O}_S(h)) \geq 2$, then the coarse moduli space $\mathcal{M}_S^U(2; c_1, c_2)$ parameterizing isomorphism classes of stable rank 2 bundles on $S$ with Chern classes $c_1$ and $c_2$ is non-empty (see Theorem [7.2]). The locus $\mathcal{M}_S^{s,U}(2; c_1, c_2) \subseteq \mathcal{M}_S^U(2; c_1, c_2)$ parameterizing stable Ulrich bundles is open as pointed out in [13].

**Proposition 4.3.** Let $S$ be a surface with $p_g(S) = 0$, $q(S) = 1$ and endowed with a very ample non-special line bundle $\mathcal{O}_S(h)$.

If $\pi(\mathcal{O}_S(h)) \geq 2$, then there is a component $\mathcal{C}_S(\eta)$ of dimension at least $h^2 - K_S^2$ in $\mathcal{M}_S^{s,U}(2; c_1, c_2)$ containing all the points representing the stable bundles $\mathcal{E}$ constructed in Theorem [11].

**Proof.** Let us denote by $\mathcal{H}_S$ the Hilbert flag scheme of pairs $(Z, C)$ where $C \in |\mathcal{O}_S(h)|$ and $Z \subseteq C$ is a 0-dimensional subscheme of degree $N + 1$. The general $C \in |\mathcal{O}_S(h)|$ is smooth and its image via the map induced by $\mathcal{O}_S(h)$ generate a hyperplane inside $\mathbb{P}^N$. Thus the set $\mathcal{H}_U^U \subseteq \mathcal{H}_S$ of pairs $(Z, C)$ corresponding to sets of points $Z$ in a smooth curve $C \subseteq \mathbb{P}^N$ which are in general position in the linear space generated by $C$ is open and non-empty.

We have a well-defined forgetful dominant morphism $\mathcal{H}_S \to |\mathcal{O}_S(h)|$ whose fibre over $C$ is an open subset of the $(N + 1)$–symmetric product of $C$. In particular $\mathcal{H}_S^U$ is irreducible of dimension $2N + 1$. Let $(Z, C)$ represent a point of $\mathcal{H}_S^U$: the Ulrich bundles associated to such a point via the construction described in Theorem [11] correspond to the sections of
\[
\text{Ext}^1_0(\mathcal{I}_{Z|S}(h - K_S), \mathcal{O}) \cong H^1(S, \mathcal{I}_{Z|S}(h))^\vee.
\]
By definition of $\mathcal{H}_S^U$, we have $h^0(C, \mathcal{I}_{Z|C}(h)) = 0$, hence the cohomology of the exact sequence
\[
0 \to \mathcal{I}_{Z|C}(h) \to \mathcal{O}_C(h) \to \mathcal{O}_Z(h) \to 0
\]
and the Riemann–Roch theorem for $\mathcal{O}_C(h)$ yield $h^1(C, \mathcal{I}_{Z|C}(h)) = \text{deg}(Z) - \chi(\mathcal{O}_C(h)) = 0$. Sequence [7], the isomorphism $\mathcal{I}_{C|S} \cong \mathcal{O}_S(-h)$ and the hypothesis $q(S) = p_g(S) = 0$ finally return $h^1(S, \mathcal{I}_{Z|S}(h)) = 1$. Thus we have a family $\mathcal{C}$ of Ulrich bundles of rank 2 with Chern classes $c_1$ and $c_2$ parameterized by $\mathcal{H}_S^U$.

If $\pi(\mathcal{O}_S(h)) \geq 2$, then the bundles in the family are also stable for a general choice of $Z$. Since stability is an open property in a flat family (see [23], Proposition 2.3.1 and Corollary 1.5.11), it follows the existence of an irreducible open subset $\mathcal{H}_S^{s,U} \subseteq \mathcal{H}_S^U \subseteq \mathcal{H}_S$ of points corresponding to stable bundles.

Thus, we have a morphism $\mathcal{H}_S^{s,U} \to \mathcal{M}_S^{s,U}(2; c_1, c_2)$ whose image parameterizes the isomorphism classes of stable bundles constructed in Theorem [11]. In particular such bundles, correspond to the points of a single irreducible component $\mathcal{U}_S(\eta) \subseteq \mathcal{M}_S^{s,U}(2; c_1, c_2)$.

Theorems 4.5.4 and 4.5.8 of [23] imply that $\dim(\mathcal{U}_S(\eta)) \geq 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$. Taking into account the definitions of $c_1$ and $c_2$, simple computations finally yield $\dim(\mathcal{U}_S(\eta)) \geq h^2 - K_S^2$. \qed
If we have some extra informations on the surface \( S \), then we can describe \( \mathcal{U}_S(\eta) \) as the following proposition shows.

**Proposition 4.4.** Let \( S \) be an anticanonical surface with \( p_g(S) = 0, q(S) = 1 \) and endowed with a very ample line bundle \( \mathcal{O}_S(h) \).

If \( \pi(\mathcal{O}_S(h)) \geq 2 \), then \( \mathcal{U}_S(\eta) \) is non–rational and generically smooth of dimension \( h^2 - K_S^2 \).

**Proof.** Thanks to Corollary 3.4 we know that \( \mathcal{O}_S(h) \) is non–special. Let \( A \in |-K_S| \):

the cohomology of

\[
0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S \to \mathcal{O}_A \to 0
\]

tensored with \( \mathcal{E} \otimes \mathcal{E}^\vee \) yields the exact sequence

\[
0 \to H^0(S, \mathcal{E} \otimes \mathcal{E}^\vee(K_S)) \to H^0(S, \mathcal{E} \otimes \mathcal{E}^\vee) \to H^0(A, \mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{O}_A).
\]

Since \( \mathcal{E} \) is stable (see Theorem 1.2), then it is simple, i.e. \( h^0(S, \mathcal{E} \otimes \mathcal{E}^\vee) = 1 \) (see [23], Corollary 1.2.8), hence the map

\[
H^0(S, \mathcal{E} \otimes \mathcal{E}^\vee) \to H^0(A, \mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{O}_A)
\]

is injective. We deduce that \( h^2(S, \mathcal{E} \otimes \mathcal{E}^\vee) = h^0(S, \mathcal{E} \otimes \mathcal{E}^\vee(K_S)) = 0 \).

Thus \( \mathcal{E} \) corresponds to a smooth point of \( \mathcal{U}_S(\eta) \) and \( \dim(\mathcal{U}_S(\eta)) = h^2 - K_S^2 \), thanks to Corollary 4.5.2 of [23]. Finally, being \( q(S) = 1 \), then \( \mathcal{U}_S(\eta) \) is irregular thanks to [5] as well.

Remark 3.5 and the above proposition yield the following corollary.

**Corollary 4.5.** Let \( S \) be a geometrically ruled surface with \( q(S) = 1, e \geq 0 \) and endowed with a very ample line bundle \( \mathcal{O}_S(h) \).

If \( \pi(\mathcal{O}_S(h)) \geq 2 \), then \( \mathcal{U}_S(\eta) \) is non–rational and generically smooth of dimension \( h^2 \).

5. Ulrich–wildness

Let \( S \) be a surface with \( p_g(S) = 0 \) and \( q(S) = 1 \). Moreover \( \pi(\mathcal{O}_S(h)) \geq 1 \) because \( S \) is not rational, as pointed out in the introduction.

We will make use of the following result.

**Theorem 5.1.** Let \( X \) be a smooth variety endowed with a very ample line bundle \( \mathcal{O}_X(h) \).

If \( \mathcal{A} \) and \( \mathcal{B} \) are simple Ulrich bundles on \( X \) such that \( h^1(X, \mathcal{A} \otimes \mathcal{B}^\vee) \geq 3 \) and \( h^0(X, \mathcal{A} \otimes \mathcal{B}^\vee) = h^0(X, \mathcal{B} \otimes \mathcal{A}^\vee) = 0 \), then \( X \) is Ulrich–wild.

**Proof.** See [20], Theorem 1 and Corollary 1.

An immediate consequence of the above Theorem is the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Recall that \( S \) is a surface with \( p_g(S) = 0, q(S) = 1 \) and endowed with a very ample non–special line bundle \( \mathcal{O}_S(h) \). We have \( \pi(\mathcal{O}_S(h)) \geq 1 \), \( \chi(\mathcal{O}_S) = 0 \) and \( K_S^2 \leq 0 \).

If \( \pi(\mathcal{O}_S(h)) \geq 2 \), then Theorems 1.1 and 1.2 yield the existence of a stable special Ulrich bundle \( \mathcal{E} \) of rank 2 on \( S \).
The local dimension of $\mathcal{M}_S^s(2; c_1, c_2)$ at the point corresponding to $E$ is at least $4c_2 - c_1^2 = h^2 - K_S^2 \geq 1$. Thus, there exists a second stable Ulrich bundle $G \not\cong E$ of rank 2 with $c_i(G) = c_i$, for $i = 1, 2$. Both $E$ and $G$, being stable, are simple (see [23, Corollary 1.2.8]).

Due to Proposition 1.2.7 of [23] we have $h^0(F, E \otimes G^\vee) = h^0(F, G \otimes E^\vee) = 0$, thus

$$h^1(F, E \otimes G^\vee) = h^2(F, E \otimes G^\vee) - \chi(E \otimes G^\vee) \geq -\chi(E \otimes G^\vee).$$

Equality [2] with $F := E \otimes G^\vee$ and the equalities $\text{rk}(E \otimes G^\vee) = 4$, $c_1(E \otimes G^\vee) = 0$ and $c_2(E \otimes G^\vee) = 4c_2 - c_1^2$ imply

$$h^1(F, E \otimes G^\vee) \geq 4c_2 - c_1^2 = h^2 - K_S^2 \geq 3.$$

because surfaces of degree up to 2 are rational. We conclude that $S$ is Ulrich–wild, by Theorem 5.1.

Finally let $\pi(O_S(h)) = 1$. In this case, $S$ is a geometrically ruled surface on an elliptic curve $E$ thanks to Theorem A of [11] embedded as a scroll by $O_S(h)$. Using the notations of Remark 3.3 we can thus assume that $O_S(h) = O_S(\xi + p^e b)$, where $\deg(b) \geq e + 3$.

Assertion 2) of Proposition 5 in [10] yields that for each $\vartheta \in \text{Pic}^0(E) \setminus \{ O_E \}$ the line bundle $L := O_S(h + p^e \vartheta) \cong O_S(\xi + p^e b + p^e \vartheta)$ is Ulrich. It follows from Corollary 2.2 that $M := O_S(2h + K_S - p^e \vartheta) \cong p^e O_E(2b + h - \vartheta)$ is Ulrich too.

Trivially, such bundles are simple and $h^0(S, L \otimes M^\vee) = h^0(S, M \otimes L^\vee) = 0$ because $L \not\cong M$. Since $L \otimes M^\vee \cong O_S(\xi - p^e b - p^e b + 2\vartheta)$ and $e = -\deg(h) \geq -1$, it follows from Equality [2] that

$$h^1(S, L \otimes M^\vee) \geq -\chi(L \otimes M^\vee) = 2\deg(b) - e \geq e + 6 \geq 5.$$

The statement thus again follows from Theorem 5.1. \]

The following consequence of the above theorem is immediate, thanks to Corollaries 3.3, 3.4, 3.6.

**Corollary 5.2.** Let $S$ be a surface endowed with a very ample line bundle $O_S(h)$. If $S$ is either bielliptic, or anticanonical with $q(S) = 1$, or geometrically ruled with $q(S) = 1$, then it is Ulrich–wild.

The following corollary strengthens the second part of the statements of Theorems 4.13 and 4.18 in [27].

**Corollary 5.3.** Let $S \subseteq \mathbb{P}^4$ be a non–degenerate linearly normal non–special surface of degree at least 4 with $p_g(S) = 0$. Then $S$ is Ulrich–wild.

**Proof.** As pointed out in the proof of Corollary 3.8 the surface $S$ satisfies $q(S) \leq 1$ and if equality holds it is either an elliptic scroll or a bielliptic surface. Theorem 13.3 above and Section 5 of [14] yields that $S$ is Ulrich–wild. \]

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