Quantization of affine bodies. Theory and applications in mechanics of structured media.

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Abstract

Discussed is kinematics and dynamics of bodies with affine degrees of freedom, i.e., homogeneously deformable ”gyroscopes”. The special stress is laid on the status and physical justification of affine dynamical invariance. On the basis of classical Hamiltonian formalism the Schrödinger quantization procedure is performed. Some methods of the partial separation of variables, analytical treatment and search of rigorous solutions are developed. The possibility of applications in theory of structured media, nanophysics, and molecular physics is discussed.

Keywords: affine degrees of freedom, molecular dynamics, nanophysics, quantized media, structured media.

Introduction

The mechanics of affine bodies was a subject of many papers [6, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 30, 37, 38, 39, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 80]. It has been a field of intensive studies in our group at the Institute of Fundamental Technological Research in Warsaw. Up to our knowledge, for the first time the idea of objects with affine degrees of freedom in mechanics appeared in papers of Eringen [26, 27, 28, 29] devoted to structured continua, to be more precise in his theory of micromorphic media. Micromorphic continuum is an affine extension of the micropolar Cosserat continuum. Roughly speaking, the Cosserat medium is a deformable continuum of infinitesimal gyroscopes. Similarly, the micromorphic body is a deformable continuum of infinitesimal homogeneously deformable gyroscopes. Affine model of collective degrees of freedom was also used in the theory of collective phenomena in atomic nuclei [17]. The idea of affine body is interesting in itself from the
point of view of analytical mechanics and theory of dynamical systems. It is an
instructive example of systems with degrees of freedom ruled by Lie groups. In
mechanics of non-constrained continua the configuration space may be identified
with the group of all diffeomorphisms of the physical space (volume-preserving
diffeomorphisms in mechanics of ideal incompressible fluids). It is rather diffi-
cult to be rigorous with such infinite-dimensional groups. Affine model is placed
between rigid-body mechanics and the general theory of deformable continua,
i.e., it involves deformations but at the same time one deals there with a finite
number of degrees of freedom. The Lie-group background of the geometry of the
configuration space offers the possibility of the effective use of powerful analytic
techniques. One can realize certain finite-dimensional generalizations when the
configuration space geometry is ruled, e.g., by the projective or conformal group.
Also other finite-dimensional discretized approaches are useful but of course the
models based on geometric transformation groups are particularly interesting
and efficient.

The range of applications of affine model of collective and internal degrees of
freedom is very wide and has to do with various scales of physical phenomena:

- macroscopic elastic problems when the length of excited waves is compar-
able with the linear size of the body.
- purely computational and engineering problems connected with the finite
elements methods. A mixture of analytic and numerical procedures.
- structured bodies, e.g., micromorphic continua and molecular crystals.
- vibrations of astrophysical objects (stars, concentrations of the cosmic
dust), theory of the shape of the Earth.
- molecular vibrations.
- nuclear dynamics.

Obviously, the last two subjects must be based on the quantized version of
the theory. Quantum description is also necessary in various problems concern-
ing the nanoscale phenomena, fullerens, etc. It is a new fascinating subject
where one deals with the very intriguing convolution of the classical and quan-
tum levels, perhaps also with some yet non-solved paradoxes from the realm of
quantum-mechanical foundations like decoherence, etc.

Quantization as a purely mathematical procedure is connected with certain
ambiguities which may be solved only a posteriori, on the bases of experimental
data. There are some well-known problems with the ordering of operators.
In models with a firm group-theoretic background there are some canonical
procedures, usually confirmed by experiments. Because of this, the extensive
geometric introduction presented below, almost a treatise as a matter of fact,
is a constitutive element of the theory, motivated by deeper reasons than the
purely mathematical curiosity or artificial sophistication.

Affine models of degrees of freedom of the structured elements is very natural.
When one deals with fullerens, macromolecules, microdefects, affine modes of
motion are certainly the most relevant ones. There are molecules, e.g., $P_4$, which have no other degrees of freedom; there are also such ones for which non-affine behaviour is a merely small correction. As mentioned, affine modes have also to do with finite elements, when the body is described as an aggregate of small affine objects.

Perhaps the quantization of such an approach might be a procedure alternative to the phonon description based on the quantized plane elastic waves.

And finally, one of the most important things. The group-theoretical description of internal and collective modes is really effective when the dynamics is invariant (or in some sense almost invariant) with respect to the group underlying kinematics of the problem. And this is not the case in all models of affine bodies met in literature. Kinematics is there affine but the group of dynamical symmetries is broken to the isometry group. Because of this, there is no full use and the full profit of Lie-group techniques. Unlike this, we formulate here affinely-invariant dynamics, where elastic interactions may be encoded in appropriate kinetic energy models without (or "almost" without) any use of potential energy terms. This procedure is similar to that following from the Maupertuis variational principle. There are indications that just such models may be useful in condensed matter theory, where the structural elements are more sensitive to the geometry of a surrounding piece of the body, e.g., to the Cauchy deformation tensor than to the "true" metric tensor of the physical space. This is something similar to the effective mass tensors of electrons in crystals.

There is an interesting link between our models and theories of integrable lattices like Calogero-Moser, Sutherland, and others \[8, 40, 41\]. On the quantum level the deformation invariants behave like indistinguishable, exotically parastatistical one-dimensional "particles".

Obviously, the real world, the arena of mechanical phenomena, is three-dimensional. However, certain important invariance and other problems are explained in a more lucid way when described with non-physical generality, i.e., in \(n\) dimensions. By the way, two-dimensional problems are also interesting not only in "Flatland" \[1\] but also in some realistic physical problems. At the same time, they are computationally simple due to some exceptional, so to speak pathological, feature of $GL(2, \mathbb{R})$ among all $GL(n, \mathbb{R})$ ($SO(2, \mathbb{R})$ is Abelian, whereas $SO(n, \mathbb{R})$ for $n > 2$ are semisimple).

1 Classical preliminaries

Let us briefly describe various models of the configuration space of affinely rigid body. It depends on the particular problem under consideration which of them is more convenient. The possibility and usefulness of many choices of geometric structures underlying physically the same degrees of freedom was pointed out by Capriz \[9, 10, 11, 12, 33, 34, 35\]. Various descriptions differ in assuming some auxiliary geometric objects.

We begin with some elementary concepts of affine geometry, just to fix the language and notation. Affine space is given by a triple \((X, E, \to)\), where \(X\) is
a point set, just the "space itself", $E$ is a linear space of translations in $X$, and the arrow $\rightarrow$ denotes a mapping from the Cartesian product $X \times X$ onto $E$; the vector assigned to $(p, q) \in X \times X$ is denoted by $pq$. The arrow operation satisfies some axioms, namely,

(i) $pq + qr + rp = 0$ for any $p, q, r \in X$,

(ii) for any $p \in X$ and $v \in E$ there exists exactly one $q \in X$ such that $pq = v$; we write $q = tv(p)$.

For any $v \in E$, $tv : X \to X$ is a one-to-one mapping of $X$ onto $X$, the translation by $v$. And obviously, $tvu = tuv = t_vt_u = id_X$, $t_u^{-1} = t^{-1}u$.

In this way, $E$ considered as an additive-rule Abelian group acts freely and transitively on $X$.

Any linear space $E$ may be considered as an affine space $(E, E, -)$, i.e., $a = v - u$.

The axiom (i) implies that $pp = 0$, $pq = -qp$ for any $p, q \in X$.

Let $\Omega$ be an arbitrary set, in general, structure-less one. The set of all mappings from $\Omega$ into $X$, denoted by $X^\Omega$, is simply the $\Omega$-indexed Cartesian product of $X$. For any mapping $f : \Omega \to X$, the image $f(\omega) \in X$ is interpreted as the $\omega$-th component of $f$. When $\Omega$ is a finite $N$-element set, e.g., $\Omega = \{1, 2, \ldots, N\}$, this is just the familiar finite Cartesian product $X^N$.

The set $X^\Omega$ is in a natural way an affine space. Its translation space is identical with $E^\Omega$, the set of all mappings from $\Omega$ into $X$. If $F, G$ are mappings from $\Omega$ into $X$, then the corresponding translation vector $FG \in E^\Omega$ is simply given by

$$\left( \begin{array}{c} \overline{F} \\ \overline{G} \end{array} \right)(\omega) := \overline{F}(\omega)\overline{G}(\omega),$$

for any $\omega \in \Omega$. One can easily show that all axioms of affine geometry are satisfied then.

Affine mappings, by definition, preserve all affine relationships between figures and points. So, if $(N, U, \rightarrow), (M, V, \rightarrow)$ are affine spaces, then we say that $\Phi : N \to M$ is affine if there exists such a linear mapping $L[\Phi] : U \to V$ denoted also by $D\varphi$ that for any $p, q \in N$ the following holds:

$$\Phi(p)[\Phi(q)] = L[\Phi]pq.$$

The mapping $L[\Phi] : U \to V$ is referred to as a linear part of $\Phi$. The set of all affine mappings from $N$ to $M$ will be denoted by $\text{Af}(N, M)$; similarly, $L(U, V)$ denotes the set of linear mappings. If $\Phi_1 \in \text{Af}(P, M)$ and $\Phi_2 \in \text{Af}(N, P)$, then $\Phi_1 \circ \Phi_2 \in \text{Af}(N, M)$ and $L[\Phi_1 \circ \Phi_2] = L[\Phi_1]L[\Phi_2]$.

Dimension of the translation space $E$ is referred to as the dimension of $X$ itself. Any fixed point $p \in M$ establishes the bijection of $M$ onto $V$ given by $M \ni q \mapsto pq \in V$. Such $V$-valued charts establish in $M$ the structure of analytical differential manifold just of dimension $\dim V$. The manifold of affine
injections from $N$ into $M$ will be denoted by $\text{AfI}(N, M)$, and the corresponding set of linear injections from $U$ into $V$ by $\text{LI}(U, V)$. They are open submanifolds of $\text{Af}(N, M)$, $L(U, V)$, respectively. Obviously, they are non-empty if and only if $\text{dim} \ M \geq \text{dim} \ N$. If $\text{dim} \ M = \text{dim} \ N$, they become respectively the manifolds of affine and linear isomorphisms.

If $N = M$ and $U = V$, i.e., when we work within some fixed affine space $(M, V, \rightarrow)$, then some simplified notation is used, namely,

$$\text{L}(V, V), \quad \text{Af}(M, M), \quad \text{LI}(V, V), \quad \text{AfI}(M, M)$$

are denoted respectively by

$$\text{L}(V), \quad \text{Af}(M), \quad \text{GL}(V), \quad \text{GAf}(M).$$

Obviously, the last two sets are groups, respectively, the general linear and affine groups in $V$. Translations are affine isomorphisms; their set $T[V] = \{t_v : v \in V\}$ is a normal subgroup of $\text{GAf}(M)$. This subgroup is the kernel of the group epimorphism:

$$\text{GAf}(M) \ni \phi \mapsto L[\phi] \in \text{GL}(V).$$

The quotient group $\text{GAf}(M)/T(V)$ is isomorphic with $\text{GL}(V)$ but in a non-canonical way; any choice of centre $o \in M$ gives rise to some isomorphism.

The set of affine mappings from $(N, U, \rightarrow)$ to $(M, V, \rightarrow)$, i.e., $\text{Af}(N, M)$, is an affine subspace of $M^N$ in the sense of (1); the translation space is identified with $V^N$.

If in the space $N$ some origin point $O \in N$ is chosen then the manifold $\text{Af}(N, M)$ may be simplified to the Cartesian product $M \times L(U, V)$. Namely, with any $\Phi \in \text{Af}(N, M)$ we associate a pair $(x, \varphi)$ in the $(M, V, \rightarrow)$ such that $x = \Phi(O)$ and $\Phi(O)\Phi(a) = \varphi \cdot Oa$. When we restrict ourselves to the open submanifold of affine isomorphisms $\text{AfI}(N, M) \subset \text{Af}(N, M)$, then $\varphi$ in the above expression runs over the open submanifold $\text{LI}(U, V) \subset L(U, V)$.

And finally, let us fix some linear frame, i.e., an ordered basis in $U$, $E = (E_1, \ldots, E_A, \ldots, E_n)$, $n = \text{dim} \ U = \text{dim} \ V$. When it is kept fixed, any linear mapping $\varphi \in L(U, V)$ may be identified with the system $e = (e_1, \ldots, e_A, \ldots, e_n)$, where $e_A = \varphi E_A$, $A = 1, \ldots, n$. When $\Phi \in \text{Af}(N, M)$, i.e., $\varphi \in L(U, V)$, then $e$ is a linear frame in $V$. In this way $\text{LI}(U, V)$ is identified with $F(V)$, the manifold of linear frames in $V$. And $\text{Af}(N, M)$ is identified with $M \times F(V)$, the manifold of affine frames in $M$ (the pairs consisting of points in $M$ and ordered bases in $V$).

Fixing an affine frame $(O, E)$ in $N$ we turn it into the arithmetic space $\mathbb{R}^n$. Linear isomorphisms of $U$ onto $V$ become then linear frames in $V$; their inverse isomorphisms are identified with the dual co-frames: $\overline{e} = (e^1, \ldots, e^A, \ldots, e^n)$, $\langle e^A, e_B \rangle = \delta_A^B$.

As frames and dual co-frames mutually determine each other, $\text{AfI}(N, M)$ may be as well identified with $M \times F(V^*) = M \times F(V)^*$; here $F(V^*)$ is the manifold of frames in the dual space $V^*$ denoted also as $F(V)^*$. 
If in addition some affine frame \((o, \mathcal{E}) = (o; \mathcal{E}_1, \ldots, \mathcal{E}_A, \ldots, \mathcal{E}_n)\) in \(M\) is fixed, then also \(V\) becomes identified with \(\mathbb{R}^n\). The manifold \(LI(U, V)\) is then analytically identified with the general linear group \(\text{GL}(n, \mathbb{R})\), and \(\text{Aff}(N, M)\) may be identified with the semi-direct product \(\text{GAf}(n, \mathbb{R}) \simeq \text{GL}(n, \mathbb{R}) \times_s \mathbb{R}^n\).

The manifold \(\text{Aff}(N, M)\) is a homogeneous space of affine groups \(\text{GAf}(M)\), \(\text{GAf}(N)\) acting respectively on the left and on the right:

\[
A \in \text{GAf}(M) : \text{Aff}(N, M) \ni \Phi \mapsto A \circ \Phi, \tag{2}
\]

\[
B \in \text{GAf}(N) : \text{Aff}(N, M) \ni \Phi \mapsto \Phi \circ B. \tag{3}
\]

Similarly, linear groups \(\text{GL}(V)\), \(\text{GL}(U)\) act transitively on \(LI(U, V)\):

\[
\alpha \in \text{GL}(V) : LI(U, V) \ni \varphi \mapsto \alpha \varphi, \tag{4}
\]

\[
\beta \in \text{GL}(U) : LI(U, V) \ni \varphi \mapsto \varphi \beta. \tag{5}
\]

Let us observe that although \(\text{GL}(V)\), \(\text{GL}(U)\) are logically distinct disjoint sets, the corresponding transformation groups intersect nontrivially. Namely, dilatations belong to both of them, the left and right actions of \(\alpha = \lambda \text{Id}_V\), \(\beta = \lambda \text{Id}_U\) result in multiplying \(\varphi\) by \(\lambda\), i.e., \(\varphi \mapsto \lambda \varphi\).

When some origin \(O \in N\) is fixed and \(\text{Aff}(N, M)\) is identified with \(M \times LI(U, V)\), the left-acting transformation groups \(\text{GAf}(M)\), \(\text{GL}(V)\) may be represented as follows:

\[
A \in \text{GAf}(M) : M \times LI(U, V) \ni (x, \varphi) \mapsto (A(x), L(A)\varphi), \tag{6}
\]

\[
\alpha \in \text{GL}(V) : M \times LI(U, V) \ni (x, \varphi) \mapsto (x, \alpha \varphi). \tag{7}
\]

The origin \(O\) enables one to identify \(\text{GAf}(N)\) with the semi-direct product \(\text{GL}(U) \times_s U\). Namely, \(B \in \text{GAf}(N)\) is represented by the pair \((L(B), \overrightarrow{OB}(O))\).

And conversely the pair \((\beta, u) \in \text{GL}(U) \times_s U\) gives rise to the mapping \(B \in \text{GAf}(N)\) such that for any \(a \in N\):

\[
\overrightarrow{OB}(a) = \beta \cdot \overrightarrow{Oa} + u.
\]

The right action \((3)\) of \(B\) on \(\text{Aff}(N, M)\) is represented in \(M \times LI(U, V)\) as follows:

\[
(\beta, u) \in \text{GL}(U) \times_s U : (x, \varphi) \mapsto (t_{\varphi u})(x), \varphi \beta). \tag{8}
\]

If we put \(u = 0\), then the group \(\text{GL}(U)\) itself acts only on the second component:

\[
\beta \in \text{GL}(U) : (x, \varphi) \mapsto (x, \varphi \beta). \tag{9}
\]

The standard language of continuum mechanics is based on the use of two affine spaces: the physical and material ones. We denote them respectively by \((M, V, \rightarrow)\) and \((N, U, \rightarrow)\). If we deal with the infinite continuum medium filling up the whole physical space, configurations are described by diffeomorphisms of \(N\) onto \(M\). The smoothness class of these diffeomorphisms depends on peculiarities of the considered problem. The manifold \(N\) is interpreted as the set
of material points. In configuration given by \( \Phi : N \to M \), the material point \( a \in N \) occupies the spatial position \( \Phi(a) \in M \). Diffeomorphism groups \( \text{Diff}(M) \) and \( \text{Diff}(N) \) give rise to transformation groups acting on the configuration space \( \text{Diff}(N, M) \), i.e.,

\[
A \in \text{Diff}(M) : \text{Diff}(N, M) \ni \Phi \mapsto A \circ \Phi, \quad (10)
\]

\[
B \in \text{Diff}(N) : \text{Diff}(N, M) \ni \Phi \mapsto \Phi \circ B. \quad (11)
\]

They are referred to respectively as spatial and material transformations. Obviously, spatial and material transformations mutually commute. In continuum mechanics they have to do with symmetries of space and material itself.

Obviously, when one deals with realistic bounded bodies, this description should be modified, e.g., manifolds with boundary become a better model of the material space. Another possibility is to use a smooth smeared-out model of the boundary, i.e., to describe the bounded body as a non-bounded, one however, with the mass density quickly vanishing outside the real object.

Deeper modifications are necessary when describing continua with degenerate dimension like membranes, strings, infinitesimally thin shells, rods, etc. And obviously, for discrete systems the description based on the affine space \( N \) as a material body is not applicable in the literal sense, unless some tricks like smeared out density functions and so on are used. It would be a good thing, especially when dealing with affine systems in microscopic applications (molecular, microstructural, etc.) to start from some general formulation applicable both to discrete and continuous systems of various kinds. There is also some more subtle point to that, namely, the material space is primarily the abstract set of material points or their labels, so-to-say "identification cards". A priori this set is structure-less; it is a kind of "powder" of material points. Let us denote it by \( \Omega \). Configurations are mappings from \( \Omega \) to \( M \), i.e., elements of \( M^\Omega \) (the usual finite Cartesian product \( M^N \) when one deals with an \( N \)-particle system). More precisely, \( M^\Omega \) is the set of singular configurations, i.e., ones admitting coincidences of different material points at the same spatial point. To avoid such a "catastrophe" one must decide that the "true" configuration space is the set of injections from \( \Omega \) into \( M \), i.e., \( \text{Inj}(\Omega, M) \). As far as \( \Omega \) is structure-less the only well-defined set of material transformations is \( \text{Bij}(\Omega) \), the set of bijections of \( \Omega \) onto \( \Omega \). They act on configurations according to the following rule:

\[
B \in \text{Bij}(\Omega) : \text{Inj}(\Omega, M) \ni \Phi \mapsto \Phi \circ B. \quad (12)
\]

These "permutations" of material points are just the only admissible material transformations on this yet amorphous stage. Transformations of \( M \) onto itself, in particular diffeomorphisms of \( M \) onto itself of an appropriate class, act on \( \text{Inj}(\Omega, M) \) according to the following rule:

\[
A \in \text{Bij}(M) : \text{Inj}(\Omega, M) \ni \Phi \mapsto A \circ \Phi. \quad (13)
\]

In general, this action is not transitive and splits into orbits, i.e., transitively classes. Any fixed class carries over geometric structures from \( M \) to \( \Omega \). For
example, if $\Omega$ has the continuum cardinal number and only the bijections of $\Omega$ onto $M$ are admitted as configurations, then any fixed orbit of the left-acting diffeomorphism group $\text{Diff}^r(M)$ induces in $\Omega$ some structure of $C^r$-class differentiable manifold. The powder of material points becomes the continuous body and its configuration space is identified with $\text{Diff}^r(\Omega, M)$, i.e., the set of $C^r$-class diffeomorphisms of $\Omega$ onto $M$. If some orbit of the left-acting affine group $\text{GAf}(M)$ is fixed, $\Omega$ becomes endowed with the induced structure of affine space. And the one can sensibly tell about affine mappings from $\Omega$ onto $M$ and about affine relationships between material points in $\Omega$.

Different orbits induce structures in $\Omega$, which are literally different, although usually isomorphic.

In general, when dealing with constrained systems of material points, it is not the total group $\text{Bij}(M)$ or the diffeomorphism group $\text{Diff}^r(M)$, but some rather peculiar proper subgroup $G \subset \text{Bij}(M)$ that rules geometry of degrees of freedom and perhaps also the dynamics. Configuration spaces are constructed as orbits of such groups. Let us assume that some orbit $Q$, i.e., some particular model of degrees of freedom is fixed. By the very definition, $G$ acts transitively on $Q$, i.e., $Q$ is a homogeneous space of the action (13). The point is how to define some right-hand-side action analogous to (11) or (3). In general, the transformation group $\text{Bij}(\Omega)$ acting through (12) is too poor. For example, when one deals with a finite system of material points, $\text{Bij}(\Omega)$ is the permutation group of $\Omega$ and nothing like continuous groups of material transformations, e.g., (11), (3) can be constructed on the basis of $\text{Bij}(\Omega)$. One feels intuitively that there is something non-satisfactory here. And indeed, it is possible to define some rich, in general continuous, group of material transformations acting on the right on $Q$.

The construction is more lucid when one forgets for a moment about details and considers an abstract homogeneous space with the underlying point set $Q$ and the group $G$ acting transitively on $Q$ on the left. For simplicity, we denote the action of $G$ by

$$g \in G : \quad Q \ni q \mapsto gq \in Q.$$  

This graphical convention is well-suited to the left-hand-side nature of this action:

$$(g_1 g_2)q = g_1(g_2 q).$$

The action is assumed to be effective, i.e., the group identity $e \in G$ (its neutral element) is the only element of $G$ satisfying the condition $eq = q$ for any $q \in Q$. Roughly speaking, $G$ is a "proper" transformation group, not something homomorphically (with a non-trivial kernel) mapped into the group $\text{Bij}(Q)$. This is exactly the case in problems described previously. Let $q_0 \in Q$ denote some arbitrarily fixed points, and $H(q_0) \subset G$ denote its isotropy group, i.e., the set of elements which do not move $q$:

$$H(q_0) := \{ g \in G : gg_0 = q_0 \}.$$
It is well-known that $Q$ may be identified in a one-to-one way with the set of left cosets, i.e., with the quotient space $G/H(q_0)$. The action of $G$ on $G/H(q_0)$ is represented by left translations, namely, the coset $xH(q_0) := \{xh : h \in H(q_0)\}$ is transformed by $q \in G$ into $ gxH(q_0) $: obviously, the result does not depend on the particular choice of $x$ within its coset, i.e., on the replacement $x \mapsto xh$, $h \in H$.

The question is now whether there exist some right translations of representa"nts, $x \mapsto xg$, admitting an interpretation in terms of transformations acting on $G/H(q_0)$. It is easy to see that the answer is affirmative.

Namely, let $N(H(q_0)) \subset G$ denote the maximal subgroup of $G$ for which $H(q_0) \subset N(H(q_0))$ is a normal subgroup. It is easy to see that for every $n \in N(H(q_0))$ the corresponding right regular translation $G \ni x \mapsto xn \in G$ is projectable to the manifold of left cosets $G/H(q_0)$. Indeed, cosets are transformed onto cosets:

$$(xH(q_0))n = xH(q_0)n = x(nH(q_0)n^{-1})n = xnH(q_0) = (xn)H(q_0).$$

In this way transformations may be performed on representa"nts, $G \ni x \mapsto xn \in G$. Obviously, the choice of representa"nts does not matter because

$$(xh)nH(q_0) = (xhn)H(q_0) = xnn^{-1}hnH(q_0),$$

and for any $h \in H(q_0)$, $n^{-1}hn \in H(q_0)$, thus $n^{-1}hnH(q_0) = H(q_0)$, and finally $((xh)n)H(q_0) = (xn)H(q_0)$. The non-effectiveness kernel of the right action of $N(H(q_0))$ on the coset manifold coincides with the group $H(q_0)$ itself, thus the true group of right-acting transformations is given by

$$N(H(q_0))/H(q_0) = H(q_0) \setminus N(H(q_0)).$$

The above construction of right-acting transformations pre-"sumes some choice of the reference point $q_0 \in Q$. The question arises as to what extent does the presented prescription depend on $q_0$. It turns out that the constructed transformation group itself is well-defined and the particular choice of $q_0$ influence only the "parametri"zation", so-to-say identification labels of the group elements. Let $q_1, q_2 \in Q$ be two arbitrarily chosen reference point. The subset of $G$ consisting of elements $k$ transforming $q_1$ into $q_2$, i.e., $kq_1 = q_2$, will be denoted by $H(q_1, q_2)$. Obviously, $H(q_1, q_2)$ is simultaneously the left and right coset of the subgroups $H(q_1)$, $H(q_2)$, respectively; if $k$ is an element of $H(q_1, q_2)$, then so is $h_2kh_1$ for any $h_1 \in H(q_1)$, $h_2 \in H(q_2)$. (Incidently, let us notice: perhaps it would be convenient to write $H(q_1, q_1)$, $H(q_2, q_2)$ instead of $H(q_1)$, $H(q_2)$.) Obviously, for any $k \in H(q_1, q_2)$ we have

$$H(q_2) = kH(q_2)k^{-1}, \quad N(q_2) = kN(q_1)k^{-1}.$$  

Any choice of $k \in H(q_1, q_2)$ fixes some isomorphisms of $H(q_1)$, $N(q_1)$ respectively onto $H(q_2)$, $N(q_2)$.

Let us take some $q_1 \in G$ and the point $q \in Q$ produced by it from $q_1 \in Q$, i.e.,

$$q = gq_1.$$  

9
When \( q \) is fixed, \( g_1 \) is defined up to the gauging \( g_1 \mapsto g_1 h, h \in H(q_1) \). And now we transform \( q \) by the right action of \( n_1 \in N(H(q_1)) \):

\[
q \mapsto q' = g_1 n_1 q_1.
\]

The result does not depend on the gaugings \( g_1 \mapsto g_1 h, n_1 \mapsto \chi_1 n_1 \chi_1 \), where \( h, \chi_1, \chi_2 \in H(q_1) \). Let us now express this action in terms of the reference point \( q_2 = k q_1 \)

\[
q' = g_1 n_1 q_1 = g_1 n_1 k^{-1} q_2 = (g_1 k^{-1}) (kn_1 k^{-1}) q_2.
\]

Now \( q \) is produced from the reference point \( q_2 \) by \( g_2 = g_1 k^{-1}; q = g_2 q_2 \). And its representing group elements \( g_2 \) is affected on the right by \( n_2 = k n_1 k^{-1} \in N(H(q_2)) \) the \( k \)-conjugation of \( n_1 \in N(H(q_1)) \). In this way different right-hand-side actions in \( G \), i.e.,

\[
G \ni x \mapsto x n_1, \quad G \ni x \mapsto x n_2,
\]

describe the same transformation in \( Q \). They are different "labels" of this transformation corresponding to various choices of reference points \( q_1, q_2 \in Q \) and various choices of \( k \in H(q_1, q_2) \).

We are particularly interested in situations when the action of \( G \) on \( Q \) is free, i.e., when the isotropy groups are trivial, \( H(q) = \{e\} \) for any \( q \in Q \). Then \( H(q_1, q_2) \) are one-element sets, \( H(q_1, q_2) = \{k\} \), i.e., the above \( k \)-element is unique. The "labelling" of the right-acting transformation group by elements of \( G \) depends only on the choice of the reference point. It is clear that for any \( q_0 \in Q, N(H(q_0)) = G \), i.e., the right-acting transformation group is isomorphic with \( G \) itself. Choosing some reference point \( q_0 \) we automatically fix one of these isomorphisms.

The extended affinely rigid body is defined as a system of material points constrained in such a way that all affine relationships between constituents remain frozen during any admissible motion. Summarizing the above remarks we can formulate a few geometric models of its configuration space.

1. If we use the standard terms of continuum mechanics based on the affine physical and material spaces \((M, V, \rightarrow), (N, U, \rightarrow)\), then the configuration space is given by \( \text{Aff}(N, M) \), i.e., the manifold of affine isomorphisms of \( N \) onto \( M \). Affine groups \( \text{GAf}(M), \text{GAf}(N) \) act on \( \text{Aff}(N, M) \) according to the rules \( (2), (3) \) and describe respectively spatial and material transformations (kinematical symmetries). If we formally admitted singular configurations with degenerate dimension, the configuration space would be given by \( \text{Aff}(N, M) \), i.e., the set of all affine mappings of \( N \) into \( M \), in general non-invertible ones. By the way, \( \text{Aff}(N, M) \) is also an affine space with \( \text{Aff}(N, V) \) as the translation space; this is just the special case of \( (1) \).

2. If some material origin \( O \in N \) is fixed, the configuration space may be identified with

\[
Q = Q_{\text{tr}} \times Q_{\text{int}} = M \times \text{LI}(U, V);
\]
the first and second factors refer respectively to translational and internal (relative) motion. And again, when singular configurations are admitted, \( L(U, V) \) is replaced by its linear shell \( L(U, V) \). The natural groups of affine symmetries act on \( Q \) according to (6), (7), (8), (9). When translational motion is neglected, the configuration space reduces to \( Q_{\text{int}} = L(U, V) \), or simply to \( L(U, V) \) when singular internal configurations are admitted.

3. When in addition to \( O \in \mathcal{N} \) some linear basis \( E = (E_1, \ldots, E_A, \ldots, E_n) \) is chosen, i.e., when an affine frames \((O, E)\) is fixed in \( \mathcal{N} \), the configuration space becomes identified with \( Q = Q_{\text{tr}} \times Q_{\text{int}} = M \times F(V) \); \( F(V) \) denotes as previously the manifold of linear frames in \( V (n = \dim V) \). When we are not interested in translational motion, simply \( Q_{\text{int}} = F(V) \) is used as the configuration space. If, for any reason, singular configurations are admitted, we extend \( F(V) \) to \( V^n = V \times \cdots \times V \) (\( n \) Cartesian factors).

Just as in the model \( Q = M \times L(U, V) \) transformation groups act essentially according to the rules (6), (7), (8), (9), the linear space \( U \) being replaced by \( \mathbb{R}^n \) (any choice of \( E \in F(U) \) identifies \( U \) with \( \mathbb{R}^n \)). More precisely, spatial transformations are given by

\[
A \in \text{GAf}(M) : M \times F(V) \ni (x; \ldots, e_K, \ldots) \mapsto (A(x); \ldots, L(A)e_K, \ldots),
\]

\[
\alpha \in \text{GL}(V) : M \times F(V) \ni (x; \ldots, e_K, \ldots) \mapsto (x; \ldots, \alpha e_K, \ldots).
\]

The frame \((O, E)\) identifies \( \mathcal{N} \) with \( \mathbb{R}^n \); namely, the point \( a \in \mathcal{N} \) is identified with its coordinates \( a^K(a) \) with respect to \((O, E)\):

\[
\overline{Oa} = a^K E_K.
\]

Therefore, \( \text{GAf}(\mathcal{N}), \text{GL}(U) \) are identified respectively with \( \text{GL}(n, \mathbb{R}) \times \mathbb{R}^n \), \( \text{GL}(n, \mathbb{R}) \). Their right-hand-side actions on \( Q = M \times F(V) \) are respectively described as follows:

\[
(\beta, u) \in \text{GL}(n, \mathbb{R}) \times \mathbb{R}^n : (x; \ldots, e_K, \ldots) \mapsto (t_{eu}(x); \ldots, e_L\beta^L_{-K}, \ldots),
\]

\[
\beta \in \text{GL}(n, \mathbb{R}) : (x; \ldots, e_K, \ldots) \mapsto (x; \ldots, e_L\beta^L_{-K}, \ldots),
\]

where \( eu \in V \) denotes the vector the coordinates of which with respect to the basis \( e \) coincide with \( u^K, \ k = \frac{1}{n}, \)

\[
eu = u^K e_K.
\]

4. Model with the structure-less material space. This is just the model based on orbits and homogeneous spaces, described in some details above. So, \( \Omega \) is the (structure-less) set of material points and \( \text{Inj}(\Omega, M) \) denotes the
set of injections of $\Omega$ into $M$. The spatial affine group $\text{GAf}(M)$ acts on $\text{Inj}(\Omega, M)$ through (13):

$$A \in \text{GAf}(M) : \text{Inj}(\Omega, M) \ni \Phi \mapsto A \circ \Phi.$$  

(14)

Any orbit of this action may be chosen as the configuration space of affinely rigid body. Different orbits are related to each other by non-affine transformations. More precisely, we usually concentrate on such orbits $Q$ that for any $\Phi \in Q$ $\Phi(\Omega) \subset M$ is not contained in any proper affine subspace of $M$. Therefore, the body is essentially $n$-dimensional ($n = \dim M$), although, obviously, it need not be so in the rigorous topological sense (e.g., when $\Omega$ is finite the body is topologically zero-dimensional). Let us mention, however that there are interesting applications of the model of singular affine body. The configuration space $Q$ is then such an orbit of (14) that for any $\Phi \in Q$ the subset $\Phi(\Omega) \subset M$ is contained in an affine subspace of $M$ of dimension $k < n$. The right-acting partner of (14) is then constructed as described above for the general homogeneous space $(Q, G)$. Now $G = \text{GAf}(M)$ and $Q$ is an orbit of (14) consisting of injections $\Phi$ with $n$-dimensional affine shells of $\Phi(\Omega)$ ($n = \dim M$).

There is another way of fixing the configurations of an extended affine body with the structure-less material space. Let us assume that the body is non-degenerate. There exists then an $(n + 1)$-element subset $B \subset \Omega$ of material points such that for any $\Phi \in Q$ $\Phi(B) \subset M$ is not contained in any proper affine subspace, i.e., its affine shell coincides with $M$. Let us take the elements of $B$ in some peculiar order ($\omega_1, \ldots, \omega_{n+1}$). Every configuration $\Phi \in Q$ is uniquely fixed by position ($\Phi(\omega_1), \ldots, \Phi(\omega_{n+1})$) of the ordered system ($\omega_1, \ldots, \omega_{n+1}$). The current positions $\Phi(\omega)$ of all other material points $\omega \in \Omega$ are uniquely determined by ($\Phi(\omega_1), \ldots, \Phi(\omega_{n+1})$). The reason is that all affine relationships between positions of material points, i.e., all linear equations satisfied by vectors $\vec{y}_A\vec{y}_B$ ($\omega, \omega'$ being arbitrary elements of $\Omega$) are invariant during any admissible (affinely constrained) motion. In other words, they depend only on $\omega, \omega'$ but are independent of $\Phi \in Q$.

In this way, configurations are identified with elements of the Cartesian product $M^{n+1} = M \times \cdots \times M$ ($(n + 1)$ copies of $M$). When the body is non-singular, it is not the total $M^{n+1}$ that is admitted but its open subset consisting of such $(n + 1)$-tuples $(y_1, \ldots, y_A, \ldots, y_{n+1})$, which are not contained in any proper affine subspace of $M$ (therefore, the affine shell of $y_A$, $A = 1, (n+1)$), coincides with $M$. This means that the vectors $\vec{y}_A\vec{y}_A$, $A = 2, (n+2)$, are linearly independent.

For $k$-dimensional degenerate body ($k \leq n = \dim M$) the configuration space $Q$ (an orbit of $\text{GAf}(M)$) may be identified with $M^{k+1} = M \times \cdots \times M$ ($(k + 1)$ Cartesian factors) or rather with its open subset consisting of $(k + 1)$-tuple $(y_1, \ldots, y_A, \ldots, y_{k+1})$ with $k$-dimensional affine shells.

Non-degenerate ordered $(n + 1)$-tuples $y \in M^{n+1}$ may be interpreted as affine bases (affine frames) in $M$. For any pair of such bases $y =$
There exists exactly one affine transformation \( \Phi \in \text{GAf}(M) \) such that \( y'_A = \Phi(y_A) \), \( A = 1, (n+1) \). Any such basis may be interpreted as a reference configuration.

5. Finally, if some affine frames \((O, E), (o, e)\) are fixed both in \(N\) and \(M\), these affine spaces become identified with \(\mathbb{R}\), and the numerical affine group \(\text{GAf}(n, \mathbb{R}) \simeq \text{GL}(n, \mathbb{R}) \times \mathbb{R}^n\) may be used as the configuration space. Spatial and material transformations become then respectively left and right regular translations.

Obviously, all the above models are mutually equivalent and their formal utility and practical usefulness depend on the kind of considered problems. For example, when describing discrete systems we shall use the model (4) with the structure-less material space. In certain microstructural applications and in fundamental physics one uses internal degrees of freedom which are not interpreted in terms of composed multi-particle systems and perhaps by some principal reasons do not admit such interpretation at all (cf. the concept of spin of elementary particles). Obviously, then the model (3) based on the manifold of linear frames \(FM\) is the most adequate one. When one deals with the motion of structured bodies in non-Euclidean spaces, this is practically the only adequate approach if one wishes to remain within the framework of finite-dimensional analytical mechanics [56, 57, 58, 59, 60, 61, 70, 71, 72].

There are some delicate points concerning the connectedness of the configuration space. Obviously, within the standard continuum treatment the singular situations are forbidden, \( \Phi \) and \( \varphi \) must be bijections, and \( e \) is a linear frame, not an arbitrary \( n \)-tuple of vectors. Therefore, the genuine configuration space is then one of the two connected components of \( \text{Aff}(U, V), \text{LI}(U, V), \text{F}(V) \). Otherwise one would have to pass through the forbidden "singular configurations". Only the connected components of group unity, i.e., orientation preserving subgroups \( \text{GAf}^+(M), \text{GL}^+(V), \text{GAf}^+(N), \text{GL}^+(U), \text{GL}^+(n, \mathbb{R}) \) are admitted as transformation groups. However, it is not the case when one deals with discrete affine bodies. Then there is nothing catastrophic in passing through "singular" situations when at some instant of time \( \Phi(\Omega) \) is contained in a proper affine subspace of \( M \). And there is nothing bad in mirror-reflected configurations forbidden in continuum mechanics. Obviously, the above description of configuration spaces must be modified then, e.g., \( \tilde{Q} \) is not any longer the homogeneous space of \( \text{GAf}(M) \); rather, it may be a union of various transitivity orbits corresponding to all possible dimensions \( k \leq n \) of affine shells of \( \Phi(\Omega) \).

Analytical formulas will be usually expressed in terms of rectilinear Cartesian coordinates \( x^i, a^K \) respectively in \( M \) and \( N \). They are fixed by affine frames \((o, e), (O, E)\)

\[
\begin{align*}
\vec{a} = x^i(x)e_i, & \quad \vec{O}a = a^K(a)E_K.
\end{align*}
\]

Coordinates \( x^i, a^K \) induce parameterization of configurations. Eulerian and Lagrangian coordinates (spatial and material variables) are related to each other by the formula:

\[
y^i = \Phi^i(a) = x^i + \varphi^i_K a^K,
\]
where \( y^i \) are coordinates of the spatial position of the \( a \)-th material point, and \( x^i \) are coordinates of the spatial position \( \Phi(\mathcal{O}) \) of the fixed reference point \( \mathcal{O} \in \mathbb{N} \). The quantities \((x^i, \varphi^i_K)\) are labels of \( \Phi \) and may be used as generalized coordinates \( q^a, \alpha = 1, n(n + 1) \) on the configuration space \( Q \).

The reference point \( \mathcal{O} \in \mathbb{N} \), i.e., origin of Lagrange coordinates \((\alpha^K(\mathcal{O}) = 0)\) was chosen here in a completely arbitrary way. In practical problems the choice of \( \mathcal{O} \) is, as a rule, physically motivated. If there exist additional constraints due to which some material point is immovable (i.e., the body is pinned at it), then the material origin \( \mathcal{O} \) usually is chosen just at this point. There is no translational motion then. If translations are non-constrained, the centre of mass is usually chosen as the material reference point. Let us remember that in situations other than continuous medium filling up the whole space, centre of mass may happen to be placed "in vacuum". And even if it is not the case, the centre of mass is something else than the material point coinciding with it.

Let the reference mass distribution be described by some positive regular measure \( \mu \) on \( \mathbb{N} \); this means that the mass of the sub-body \( B \subset \mathbb{N} \) is given by
\[
\int_B d\mu.
\]
Centre of mass \( C(\mu) \) in \( \mathbb{N} \) is the only point satisfying
\[
\int \overrightarrow{C(\mu)a}d\mu(a) = 0;
\]
the dipole moment of \( \mu \) with respect to \( C(\mu) \) vanishes. Any configuration \( \Phi \) gives rise to the \( \Phi \)-transported measure \( \mu_\Phi \) on \( M \),
\[
\mu_\Phi(\Phi(B)) = \mu(B), \quad \mu_\Phi(A) = \mu(\Phi^{-1}(A)).
\]
The measure \( \mu_\Phi \) describes the Eulerian mass distribution on \( M \) (current mass distribution). The current centre of the mass distribution \( C(\mu_\Phi) \in M \) is given by the formula:
\[
\int \overrightarrow{C(\mu_\Phi)y}d\mu_\Phi(y) = 0.
\]
It is well-known that the centre of mass is an invariant of affine transformations; by the way, affine transformations may be defined just as those preserving centres of mass. Therefore,
\[
\Phi(C(\mu)) = C(\mu_\Phi).
\]
And besides, for any affine transformation \( A \in \text{GAf}(M) \),
\[
A(C(\mu_\Phi)) = C(\mu_{A\Phi}).
\]
This is not true for non-affine configurations and transformations.

If the material reference point \( \mathcal{O} \) is chosen as \( C(\mu) \), i.e., Lagrangian centre of mass, then generalized coordinates \((x^i, \varphi^i_K)\) on \( Q \) are especially convenient because \( x^i \) are spatial (Eulerian) coordinates of the instantaneous position of
the centre of mass in $M$. The variables $\varphi^i_K$ refer to the purely relative (internal) motion.

The physical quantity $\mu$ (Lagrangian mass distribution) is at the same time an auxiliary geometric object underlying the convenient models $M \times LI(U, V)$, $M \times F(V)$ of the configuration space of the affine body. Practically all over this treatment we put the origin of Lagrangian coordinates at the Lagrangian centre of mass.

The centre of mass is defined as such a point with respect to which the dipole moment of the mass distribution vanishes. The monopole moment equals the total mass of the body,

$$m = \mu(N) = \int_N d\mu.$$

Higher order multipole moments given an account of inertial properties of extended bodies. Thus, the Lagrangian second-order moment $J \in U \otimes U$ is given by

$$J^{KL} := \int a^K a^L d\mu(a).$$

As mentioned, the origin of $a^K$-coordinates is placed at the Lagrangian centre of mass $C(\mu)$. The above object $J^{KL}$ is algebraically equivalent to the usual co-moving tensor of inertia known from the rigid body mechanics. One can $\Phi$-transport it to the physical space $M$. Obviously, the result $J(\varphi) \in V \otimes V$ is non-constant; it depends explicitly on the configuration $\Phi$ but only through its internal part $\varphi$,

$$J(\varphi)^{ij} = \varphi^i_J \varphi^j_L J^{KL}.$$  \hspace{1cm} (15)

It is clear that

$$J(\varphi)^{ij} = \int (y^i - x^i)(y^j - x^j) d\mu_\Phi(y).$$  \hspace{1cm} (16)

Obviously, the tensors $J$, $J(\varphi) = \Phi_* \cdot J$ are symmetric and positively definite. The most convenient choices of the material reference frames $E$ are those diagonalizing $J$.

One can as well define higher-order multipoles, i.e.,

$$J^{K_1 \cdots K_i} = \int a^{K_1} \cdots a^{K_i} d\mu(a),$$

$$J(\varphi)^{i_1 \cdots i_l} = \int (y^{i_1} - x^{i_1}) \cdots (y^{i_l} - x^{i_l}) d\mu_\Phi(y).$$

Obviously, in affine motion

$$J(\varphi)^{i_1 \cdots i_l} = \varphi^{i_1}_{K_1} \cdots \varphi^{i_l}_{K_l} J^{K_1 \cdots K_l}.$$  

Inertial multipoles of the order $l > 2$ do not occur in mechanics of affine bodies, nevertheless they are useful in other problems of continuum mechanics [13].

As yet we have used above only affine concepts, i.e., we remained on the ascetic level of Tales geometry. No metric concepts like distances and angular
were used. The admissible configurations $\Phi \in \text{Af}(N, M)$ are homogeneous in the sense that for any $a \in N$ the placement $D_a \Phi \in \text{LI}(U, V)$ takes on the same value $\varphi = L[\Phi]$. Nevertheless, it would be incorrect to say that deformations are homogeneous, because without metric (Euclidean) geometry there is no deformation concept at all. The only well-defined "deformation" is then violation of affine geometry, i.e., non-constancy of the mapping $a \mapsto D_a \Phi$.

Let us now introduce metrical concepts. The spatial and material metric tensors will be denoted respectively by $g \in V^* \otimes V^*$, $\eta \in U^* \otimes U^*$. By definition they are symmetric and positively definite. Their contravariant inverses are denoted by $\tilde{g} \in V \otimes V$, $\tilde{\eta} \in U \otimes U$, but in analytical expressions we use the same kernel symbols; the distinction is indicated by the use of lower- and upper-case indices, $g^{ij}$, $g_{ij}$, $\eta^{AB}$, $\eta_{AB}$, $g^{ik}g_{kj} = \delta^i_j$, $\eta^{AC}\eta_{CB} = \delta^A_B$.

For any configuration $\Phi \in \text{GAf}(M)$ we define Green and Cauchy tensors $G[\Phi] \in U^* \otimes U^*$, $C[\Phi] \in V^* \otimes V^*$,

$$G[\Phi] = \varphi^* g, \quad C[\Phi] = \varphi^{-1*} \eta,$$

i.e., analytically:

$$G[\Phi]_{AB} = g_{ij} \varphi^A_i \varphi^j_B, \quad C[\Phi] = \eta_{AB} \varphi^{-1A} \varphi^{-1B}.$$

In these formulas, as usual, $\varphi$ denotes the linear part of $\Phi$, $\varphi = L[\Phi] = D\Phi$. For general non-affine configurations these tensors become fields respectively on $N$, $M$, namely:

$$G[\Phi]_a = D_a \Phi^* \cdot g, \quad C[\Phi]_y = D_y \Phi^{-1*} \cdot \eta.$$

Analytically:

$$G_{AB} = g_{ij} \frac{\partial y^i}{\partial a^A} \frac{\partial y^j}{\partial a_B}, \quad C_{ij} = \eta_{AB} \frac{\partial a^A}{\partial y^i} \frac{\partial a^B}{\partial y^j}.$$

Obviously, Green and Cauchy tensors are symmetric and positively definite. $G$ is built of the spatial metric tensor $g$ and is independent of the material metric $\eta$. And conversely, $C$ is independent of $g$ and explicitly depends on $\eta$. Therefore, the traditional term "deformations tensors" is rather non-adequate here; the deformation concept presumes comparison of two metrics, whereas $G$ and $C$ are well-defined even if respectively $\eta$ and $g$ are not fixed at all. Let us assume they are both fixed and denote the corresponding Euclidean space structures by $(N, U, \rightarrow, \eta)$, $(M, V, \rightarrow, g)$. When some configuration $\Phi$ is fixed, any of these two spaces is endowed by two metric-like tensors, respectively, $G, \eta \in U^* \otimes U^*$, $C, g \in V^* \otimes V^*$. The Lagrange and Euler deformation tensors are respectively given by

$$E[\Phi] = \frac{1}{2} (G[\Phi] - \eta), \quad e[\Phi] = \frac{1}{2} (g - e[\Phi]).$$
They vanish in the non-deformed configurations, i.e., when $\Phi$ are isometries. Obviously, these are usual (metrically) rigid body configurations. Their manifold will be denoted by $\text{Is}(N, \eta; M, g) \subset \text{Aff}(N, M)$. The isometry groups $\text{Is}(N, \eta) \subset \text{GAff}(N)$, $\text{Is}(M, g) \subset \text{GAff}(M)$ act on $\text{Is}(N, \eta; M, g)$ respectively on the right and on the left in the sense of (2), (3). Obviously, in realistic classical mechanics of (metrically) rigid body mirror-reflected coordinates are excluded and the genuine configuration space is given by some connected component of $\text{Is}(N, \eta; M, g)$. When orientations are fixed in $N, M$, this will be the manifold $\text{Is}^+(N, \eta; M, g)$ of orientation-preserving isometries. Physically admissible symmetries are given by the connected subgroups $\text{Is}^+(N, \eta) \subset \text{GAff}(N)$, $\text{Is}^+(M, g) \subset \text{GAff}(M)$ of orientation-preserving transformations (obviously, these group themselves do not assume any fixed orientation; they preserve separately both of them).

When translational degrees of freedom are neglected, configurations of internal (relative) motion are elements of $\text{D}(U, \eta; V, g)$, i.e., the manifold of linear isometries of $(U, \eta)$ onto $(V, g)$. The corresponding spatial and material transformations are respectively elements of the subgroup $\text{O}(V, g) \subset \text{GL}(V)$, $\text{O}(U, \eta) \subset \text{GL}(U)$; $g$- and $\eta$-orthogonal transformation groups. And again in classical rigid body mechanics one should restrict ourselves to one of the two connected components of $\text{O}(U, \eta; V, g)$. When orientations in $U, V$ are fixed, this is $\text{SO}(U, \eta; V, g) \subset \text{L}^+(U, \eta; V, g)$, i.e., the manifold of orientation-preserving linear isometries. The connected components are ruled by the proper orthogonal groups $\text{SO}(V, g)$, $\text{SO}(U, \eta)$ consisting of isometries with positive, thus plus-one-determinants (no orientations in $V, U$ needed for fixing these subgroups). When using the matrix representations we describe configurations by elements of the orthogonal group $\text{O}(n, \mathbb{R})$, or in classical problems, by elements of the proper rotation group $\text{SO}(n, \mathbb{R})$.

Let us now fix some metric tensors $\eta$, $g$. The geometric structure becomes more rich and certain additional object may be defined. For example, the material reference frame may be made less arbitrary and more based on physical concepts. Let us define two tensor objects built of the inertial tensor $J \in U \otimes U$ namely $\tilde{J} \in U^* \otimes U^*$ and $\tilde{J} \in U \cap U^* \simeq L(U)$, analytically given by the formulas:

$$\tilde{J}_{AC} J^{CB} = \delta_A^B, \quad \tilde{J}^A_B = J^{AC} \eta_{CB}.$$ 

Obviously, the object $\tilde{J}$ is non-metrical, but $\tilde{J}$ depends explicitly on the material metric tensor $\eta$. Now we have a well-defined eigenproblem in $U$:

$$\tilde{J} E = l E.$$

In the generic non-degenerate case there are $n$ mutually distinct eigenvalues $l_A$, $A = 1, \ldots, n$, and $n$ mutually orthogonal eigendirections. The directions are determined by vectors $E_A$ which may be chosen $\eta$-normalized to unity and such that the orthonormal frame $E = (E_1, \ldots, E_A, \ldots, E_n)$ is oriented positively with respect to the fixed orientation in $U$. When the eigenvalues $l_A$ are ordered by convention in increasing order then $E_A$ are unique up to multiplying some
of them by minus-one-factors. And the inertial tensor $J$ is then represented as follows:

$$J = \sum_{A=1}^{n} J^A E_A \otimes E_A. \quad (17)$$

This is just the best choice of the material reference frame. Obviously, it is no longer unique when degeneracy occurs, but these are non-generic situations. The extreme degeneracy correspond to the $\eta$-spherical body, when $J$ is proportional to $\eta$

$$J^{AB} = \mu \eta^{AB}. \quad \text{(Remark)}$$

If $\varphi$ is not an isometry, then obviously the co-moving vectors $e_A = \varphi E_A$ are not orthonormal in the $g$-sense, however, they are orthonormal with respect to the Cauchy tensor $C$ used as a "metric" in $V$. And then:

$$J[\varphi] = \sum_{A=1}^{n} J^A e_A \otimes e_A$$

with the same values $J^A$ which occur in (17).

Remark: For the sake of economy of symbols we could as well denote $\tilde{J}^{AB}$ by $J^A$, following the convention used in Euclidean and Riemannian geometry. But it is not the case with $\tilde{J}_{AB}$. Writing it as $J^A$ would suggest $\eta^{AC} \eta^{BD} J_{CD}$, quite incorrectly, this is not the $\eta$-lowering of indices. Typical notational shorthands may be misleading. Only for the $\eta$-manipulation of indices and for the contravariant inverse of $\eta (\eta^{AC} \eta^{BD} = \delta_{CD})$ we can safely use the index manipulation with non-modified kernel symbols.

When metric tensors are fixed, we can discuss the problem of interaction between rotations and deformations. In any of linear spaces $U, V$ we are given two symmetric positively definite tensors: $\eta \in U^* \otimes U^*$, $G \in U^* \otimes U^*$; $g \in V^* \otimes V^*$, $C \in V^* \otimes V^*$. Obviously, $\eta, g$ are fixed whereas $G, C$ are configuration-dependent. Just as previously, we can define the byproduct-objects $\tilde{G} \in U \otimes U$, $\tilde{G} \in U \otimes U \simeq L(U)$, $C \in V \otimes V$, $\tilde{C} \in V \otimes V \simeq L(V)$. Analytically they are given by

$$\tilde{G}^{AC} G_{CB} = \delta^A_B, \quad \tilde{G}^A_B = \eta^{AC} G_{CB}, \quad \tilde{C}^{ik} C_{jk} = \delta^i_j, \quad \tilde{C}^i_j = g^{ik} C_{kj}.$$ 

Again the same care must be taken as to the upper- and lower-case indices.

And now $G[\varphi], C[\varphi]$ may be expressed in terms of their $\eta$- and $g$-orthonormal bases $(\ldots, F_a[\varphi], \ldots), (\ldots, f_a[\varphi], \ldots)$ in $U, V$

$$G[\varphi] = \sum_{a=1}^{n} \lambda_a[\varphi] F^a[\varphi] \otimes F^a[\varphi],$$

$$C[\varphi] = \sum_{a=1}^{n} \frac{1}{\lambda_a[\varphi]} f^a[\varphi] \otimes f^a[\varphi];$$

obviously, $(\ldots, F^a[\varphi], \ldots), (\ldots, f^a[\varphi], \ldots)$ are the dual orthonormal bases of $U^*, V^*$. When there is no danger of misunderstanding, the label $\varphi$ at $\lambda_a$, $F^a$, $f^a$, $F_a$, $f_a$ may be omitted.
The quantities \( \lambda_a[\varphi] \) are deformation invariants in the sense that they do not feel spatial and material linear isometries,

\[
\lambda_a[\alpha\varphi\beta] = \lambda_a[\varphi]
\]

for any \( \alpha \in \text{O}(V, g) \), \( \beta \in \text{O}(U, \eta) \). The more so they are non-sensitive with respect to the spatial and material affine isometries (because translations evidently do not affect them). Unlike this, the Green and Cauchy deformation tensors are non-sensitive only with respect to spatial and material isometries,

\[
G[\alpha\varphi] = G[\varphi], \quad C[\varphi\beta] = C[\varphi]
\]

for any \( \alpha \in \text{O}(V, g) \), \( \beta \in \text{O}(U, \eta) \), but not conversely. \( F^a[\varphi] \in U \), \( f^a[\varphi] \in V \) are normalized eigenvectors respectively for \( \hat{G}[\varphi] \), \( \hat{C}[\varphi] \):

\[
\hat{G}[\varphi]F_a[\varphi] = \lambda_a F_a[\varphi], \quad \hat{C}[\varphi]f_a[\varphi] = \frac{1}{\lambda_a} f_a[\varphi],
\]

and they are essentially unique for the non-degenerate spectra.

The fixed bases \( (\ldots, F_a[\varphi], \ldots), (\ldots, f_a[\varphi], \ldots) \), being orthonormal, give rise some isometry \( U[\varphi] \in \text{O}(U, \eta; V, g) \), namely such one that

\[
U[\varphi]F_a[\varphi] = f_a[\varphi], \quad a = 1, \ldots, n.
\]

The \( \varphi \) itself may be written in the form

\[
\varphi = U[\varphi]A[\varphi], \quad (18)
\]

where \( A[\varphi] : U \rightarrow U \) is \( \eta \)-symmetric and positively definite, i.e.,

\[
\eta(A[\varphi]u, v) = \eta(u, A[\varphi]v), \quad \eta(A[\varphi]u, u) > 0,
\]

for arbitrary \( u, v \in U \) and arbitrary \( u \neq 0 \) in the inequality. Analytically:

\[
\eta_{AC}A[\varphi]^C_B = \eta_{BC}A[\varphi]^C_A,
\]

and the matrix \( [A[\varphi]^A_B] \) has positive eigenvalues, coinciding, by the way, with square roots of deformation invariants \( \lambda_a \) (which, obviously, are also positive).

Obviously, (18) is a geometric interpretation of the polar decomposition, and analytically, when we put \( U = V = \mathbb{R}^n \) it exactly coincides with the usual polar decomposition known from the matrix theory. It may be also alternatively written in the form:

\[
\varphi = B[\varphi]U[\varphi], \quad (19)
\]

where \( B[\varphi] : V \rightarrow V \) is \( g \)-symmetric and positively definite, and obviously:

\[
B[\varphi] = U[\varphi]A[\varphi]U[\varphi]^{-1}.
\]
These are simply the left and right polar decomposition known from the matrix theory. It is clear that Green and Cauchy tensor satisfy respectively:

\[ \hat{G}(\varphi) = A[\varphi]^2, \quad \hat{C}(\varphi) = B[\varphi]^{-2}. \]

Let us mention, there are also other possible choices of deformation invariants; every system of \( n \) functionally independent functions of \( \lambda_a, a = 1, \ldots, n \) may be used as a basic system of invariants. Let us remind a few popularly used system e.g.,

\[ K_a(\varphi) = \text{Tr} \left( \hat{G}(\varphi)^a \right), \quad a = 1, \ldots, n. \]

Obviously,

\[ K_a = \sum_{i=1}^{n} (\lambda_i)^a. \]

Another possibility is the system of coefficients of the characteristic polynomial of \( \hat{G}(\varphi) \), \( I_p(\varphi) \):

\[ \det \left[ \hat{G}(\varphi)^A_B - \lambda \delta^A_B \right] = \sum_{k=0}^{n} (-1)^k I_{n-k}(\varphi) \lambda^k. \]

Obviously, \( I_0 = 1 \), and for \( p = 1, \ldots, n \), \( I_p \) is the sum of all possible products of \( p \) quantities \( \lambda_a \) with different (but not necessarily disjoint) sets of labels \( a = 1, \ldots, n \), e.g.,

\[ I_1 = \sum_{i=1}^{n} \lambda_i = \text{Tr} \left( \hat{G} \right), \quad I_n = \lambda_1 \cdots \lambda_n = \det \left( \hat{G} \right). \]

In the physical three-dimensional case

\[ I_2 = \lambda_2 \lambda_3 + \lambda_3 \lambda_1 + \lambda_1 \lambda_2. \]

Let us observe that orthonormal bases \( (\ldots, f_a[\varphi], \ldots), (\ldots, F_a[\varphi], \ldots) \) represent formally configurations of two fictitious rigid bodies, respectively in \( (V, g) \) and \( (U, \eta) \). They refer respectively to the eigenaxes of the Cauchy and Green deformations tensors, therefore, they tell us how the deformation state is oriented with respect to \( V, U \) (what are instantaneous positions of deformation ellipsoids). Unlike this, deformation invariants contain only the scalar deformation about the deformation state (how large are stretchings). The manifold of scalar deformation states (parametrized by deformation invariants) may be considered as double-coset-space of \( \text{LI}(U, V) \) with respect to the left and right actions of \( \text{O}(V, g), \text{O}(U, \eta) \),

\[ \text{Inv} = \text{O}(V, g) \backslash \text{LI}(U, V) / \text{O}(U, \eta), \]

or, when reflections are excluded,

\[ \text{Inv} = \text{SO}(V, g) \backslash \text{LI}^+(U, V) / \text{SO}(U, \eta). \]
Let us observe that, as usual linear frames $F[\varphi], f[\varphi]$ may be naturally identified with linear isomorphisms: $R[\varphi] : \mathbb{R}^n \rightarrow U, L[\varphi] : \mathbb{R}^n \rightarrow V$. As they are orthonormal they are linear isometries of $(U, \eta)$ onto $(\mathbb{R}^n, \delta)$ and of $(V, g)$ onto $(\mathbb{R}^n, \delta)$; $\delta$ denotes here the natural Descartes-Kronecker metric of $\mathbb{R}^n$. Similarly, the dual co-frames $\tilde{F}[\varphi], \tilde{f}[\varphi]$ may be identified with the inverse mappings $R[\varphi]^{-1} : U \rightarrow \mathbb{R}^n, L[\varphi]^{-1} : V \rightarrow \mathbb{R}^n$; obviously, they are also linear isometries. One can show that $\varphi$ may be represented as

$$\varphi = L[\varphi]D[\varphi]R[\varphi]^{-1},$$

where the linear mapping $D[\varphi] : \mathbb{R}^n \rightarrow \mathbb{R}^n$, i.e., simply a matrix, is diagonal,

$$D[\varphi] = \begin{bmatrix} D_1[\varphi] & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_n[\varphi] \end{bmatrix} = \text{Diag}(D_1[\varphi], \ldots, D_n[\varphi]).$$

If there is no danger of misunderstanding we shall omit the label $\varphi$ at the quantities $L, D, R$. So, finally, we write

$$\varphi = LDR^{-1}$$

and in this way any internal (relative) configuration $\varphi$ is formally identified with the configuration of two fictitious (metrically) rigid bodies and $n$ purely oscillatory degrees of freedom of the stretching state.

If we put $U = V = \mathbb{R}^n$, then the above decomposition (two-polar decomposition, triple decomposition) is formally obtained from the polar one. Namely, for any $\varphi = \text{GL}(n, \mathbb{R})$ one starts from the polar decomposition

$$\varphi = UA, \quad U \in \text{O}(n, \mathbb{R}), \quad A \in \text{Sym}^+(n, \mathbb{R})$$

and then $A$ is orthogonally diagonalized,

$$A = RDR^{-1}, \quad D \in \text{Diag}(n, \mathbb{R}), \quad R \in \text{O}(n, \mathbb{R}),$$

so, finally,

$$\varphi = LDR^{-1}, \quad L = UR \in \text{O}(n, \mathbb{R}).$$

Unlike the polar decomposition, the two-polar one is non-unique. In the non-degenerate case, when all diagonal elements of $D$ are pairwise distinct, this non-uniqueness is discrete and controlled by the permutation group $S(n)$ inter-changing deformation invariants. When degeneracy occurs the non-uniqueness is more catastrophic, in a sense continuous, and resembles the singularity of spherical coordinates at $r = 0$ (although, one must say, it is much more complicated).

Let us finish with some kinematical concepts. Generalized velocity of an affine body is given by the pair $(v, \xi) \in V \times \text{L}(U, V)$ consisting of the translational velocity $v$ and the internal one $\xi$. On a given classical motion $\mathbb{R} \ni t \mapsto (x(t), \varphi(t))$ it is analytically given by the system

$$\left( \frac{dx^i}{dt}, \ldots, \frac{d\varphi^i_A}{dt}, \ldots \right).$$
When \( U = \mathbb{R}^n \), i.e., \( Q = M \times F(V) \), then velocities are elements of \( V \times V^* = V^{n+1} \).

It is convenient to use affine velocities in the spatial and material (co-moving) representations, \( \Omega \in \text{L}(V), \widehat{\Omega} \in \text{L}(U) \), namely:

\[
\Omega = \xi \varphi^{-1} = \frac{d\varphi}{dt} \varphi^{-1}, \quad \widehat{\Omega} = \varphi^{-1} \xi = \varphi^{-1} \frac{d\varphi}{dt}.
\]

They are interrelated as follows:

\[
\Omega = \varphi \widehat{\Omega} \varphi^{-1}.
\]

Analytically:

\[
\Omega^i_j = \frac{d\varphi^i_A}{dt} \varphi^{-1} A_j, \quad \widehat{\Omega}^A_B = \varphi^{-1} A^i_j \frac{d\varphi^i_B}{dt}, \quad \Omega^i_j = \varphi^i_A \widehat{\Omega}^A_B \varphi^{-1} B_j.
\]

Eringen in his micromorphic theory \cite{27, 28, 29} uses them the term "gyration". They are Lie-algebraic objects related respectively to the right-invariant and left-invariant vector fields and differential forms,

\[
X[E]_\varphi = E \varphi, \quad X[\widehat{E}]_\varphi = \varphi \widehat{E},
\]

\[
\omega = d\varphi \varphi^{-1}, \quad \widehat{\omega} = \varphi^{-1} d\varphi.
\] (20)

In very rough, formal terms we would say that \( \omega, \widehat{\omega} \) are obtained from \( \Omega, \widehat{\Omega} \) via the \( dt \)-multiplying. More rigorously, they are respectively \( L(V) \)-valued and \( L(U) \)-valued differential one-forms on \( \text{LI}(U, V) \). Their evaluations on vectors tangent to trajectories just coincide with \( \Omega, \widehat{\Omega} \).

The right and left invariance is meant obviously in the sense of transformations (4), (5). They become right and left regular group translations when \( U = V = \mathbb{R}^n \) and \( \text{LI}(U, V) \) is identified with \( \text{GL}(n, \mathbb{R}) \). In the above formulas for vector fields \( E \) and \( \widehat{E} \) are respectively fixed elements of \( \text{L}(V), \text{L}(U) \).

Affine velocities are non-holonomic in the sense that there no generalized coordinates for which they would be time derivatives. This is due to the non-commutativity of the full linear group.

In continuum mechanics \( \Omega \) may be interpreted in terms of the Euler velocity field. Namely, the material point which at a given instant of time passes the spatial point \( y \in M \) has the velocity:

\[
v(y) = v + \Omega \vec{y};
\]

analytically,

\[
v^i(y) = v^i + \Omega^i_j \left( y^j - x^j \right).
\]

When the motion is metrically-rigid, i.e., gyroscopic, then affine velocity becomes skew-symmetric with respect to the appropriate metric tensor,

\[
\Omega^i_j = -g^{ia} g_{jb} \Omega^b_a = -\Omega^i_j, \quad \widehat{\Omega}^A_B = -\eta^{AK} \eta_{BL} \widehat{\Omega}^L_K = -\widehat{\Omega}^A_B.
\]
These objects are the usual angular velocities, respectively in the spatial and co-moving representation. In the physical three-dimensional case they are identified in a standard way the usual pseudo-vectors of angular velocity; namely, in orthonormal coordinates

\[
\Omega^i_{\ j} = -\varepsilon^i_{\ jk} \Omega^k, \quad \hat{\Omega}^A_{\ B} = -\varepsilon^A_{\ BCD} \hat{\Omega}^C, \\
\Omega^i = -\frac{1}{2} \varepsilon^i_{\ jk} \Omega^j_{\ k}, \quad \hat{\Omega}^A = -\frac{1}{2} \varepsilon^A_{\ BCD} \hat{\Omega}^B_{\ C}, \quad \hat{\Omega}^i = \varphi^i_{\ A} \hat{\Omega}^A.
\]

Obviously, \(\varepsilon\) denotes here the totally antisymmetric Ricci symbol and indices are trivially shifted with the use of Kronecker-delta (orthonormal coordinates).

Obviously, in the two-dimensional case (also physically interesting), angular velocities are one-dimensional objects, \(\Omega\) and \(\hat{\Omega}\) numerically coincide and in orthonormal coordinates:

\[
\begin{bmatrix}
\Omega^i_{\ j} \\
[\varphi^i_{\ A}]
\end{bmatrix} = \begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & -1
\end{bmatrix} = \frac{d\theta}{dt} \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix},
\]

For the fictitious rigid bodies corresponding to the polar and two-polar decompositions we also introduce the corresponding angular velocities:

\[
\begin{align*}
\omega &= \frac{dU}{dt} U^{-1} \in SO(V, g)' \subset L(V), \\
\hat{\omega} &= U^{-1} \frac{dU}{dt} \in SO(U, \eta)' \subset L(U), \\
\chi &= \frac{dL}{dt} L^{-1} \in SO(V, g)' \subset L(V), \\
\hat{\chi} &= L^{-1} \frac{dL}{dt} \in SO(n, \mathbb{R})' \subset L(n, \mathbb{R}), \\
\vartheta &= \frac{dR}{dt} R^{-1} \in SO(U, \eta)' \subset L(U), \\
\hat{\vartheta} &= R^{-1} \frac{dR}{dt} \in SO(n, \mathbb{R})' \subset L(n, \mathbb{R}).
\end{align*}
\]

These objects are elements of the of the indicated Lie algebras of orthogonal groups. So, \(\omega, \chi\) are \(g\)-skew-symmetric, \(\hat{\omega}, \vartheta\) are \(\eta\)-skew-symmetric, and \(\hat{\chi}, \hat{\vartheta}\) are skew-symmetric in the usual Kronecker sense.

In certain problems it is convenient to use co-moving representation of the translational velocity,

\[
\hat{v}^A := \varphi^{-1A} v^i = \varphi^{-1A} \frac{dx^i}{dt}.
\]

Canonical moments, i.e. linear functional on generalized velocities are pairs \((p, \pi) \in V^* \times L(V, U)\), i.e., analytically \((\ldots, p_i, \ldots; \ldots, p^A, \ldots)\). Their evaluations on virtual velocities are given by

\[
\langle (p, \pi), (v, \xi) \rangle = \langle p, v \rangle + \text{Tr} (\pi \xi) = p_i v^i + p^A \xi^i_A.
\]
When \( U = \mathbb{R}^n \) and \( Q = M \times F(V) \), then canonical momenta are elements of \( V^* \times V^{*n} = V^{*n+1} \).

The duality between \( \pi \) and \( \xi \) may be expressed in terms of the duality between affine spin and affine velocity. More rigorously, affine spin \( \Sigma \in L(V) \) and its co-moving representation \( \hat{\Sigma} \in L(U) \) are defined as:

\[
\Sigma = \varphi \pi, \quad \hat{\Sigma} = \pi \varphi.
\]

Analytically

\[
\Sigma^i_j = \varphi^i_A p^A_j, \quad \hat{\Sigma}^A_B = p^A_i \varphi^i_B.
\]

One uses also the term "hypermomentum". The quantities \( \Sigma, \hat{\Sigma} \) are respectively Hamiltonian generators of the transformation groups \( (4), (5) \).

Linear momentum \( p_i \) generates spatial translations. In many problems it is convenient to use its co-moving representation

\[
\hat{p}_A = p_i \varphi^i A
\]

which has to do with the material translations.

One uses also the orbital affine momentum and the total affine momentum \( \Lambda, J \), given respectively by

\[
\Lambda^i_j = x^k p_j, \quad J^i_j = \Lambda^i_j + \Sigma^i_j.
\]

Unlike \( \Sigma \) the quantities \( \Lambda, J \) depend on the choice of the origin \( o \in M \) of affine coordinates in \( M \). And \( J \), more precisely \( J(o) \) is a Hamiltonian generator (momentum mapping) of the centre-affine subgroup \( GAf(M, o) \subset GAf(M) \) (affine transformations preserving \( o \)) acting through \( (2) \).

The doubled skew-symmetric parts of hypermomenta,

\[
S^i_j = \Sigma^i_j - g^i_{kl} \Sigma^l k, \\
L^i_j = x^i p_j - g^i_{kl} x^l p_k, \\
\Omega^i_j = L^i_j + S^i_j,
\]

are the usual angular momenta: the internal (spin), the orbital, and the total ones. They are Hamiltonian generators (momentum mapping) of the corresponding isometry groups. The quantity

\[
V^A_B := \hat{\Sigma}^A_B - \eta^{AL} \eta_{BK} \hat{\Sigma}^K_L,
\]

called by Dyson "vorticity", is the Hamiltonian generator of the right-acting rotation group \( (9) \). When \( \varphi \) is not an isometry, then \( V \) is not the co-moving representation of \( S \),

\[
S^i_j \neq \varphi^i_A V^A_B \varphi^{-1} B.
\]

\( S \) and \( V \) generate respectively spatial and material rotations of internal degrees of freedom. \( \Omega \) and \( p \) together are Hamiltonian generators of spatial isometries.

In a sense, \( \Sigma \) and \( \hat{\Sigma} \) are non-holonomic canonical momenta; non-holonomic, because their Poisson brackets do not vanish.
The pairing between internal canonical momenta and velocities may be now expressed as follows:

\[ \langle \Sigma, \Omega \rangle = \langle \hat{\Sigma}, \hat{\Omega} \rangle = \text{Tr} (\Sigma \Omega) = \text{Tr} (\hat{\Sigma} \hat{\Omega}) = \text{Tr} (\pi \xi). \]

Analytically:

\[ \langle \Sigma, \Omega \rangle = \langle \hat{\Sigma}, \hat{\Omega} \rangle = \sum_{j} \Omega_{ij} = \hat{\Sigma}^{B} A \hat{\Omega}^{A}_{B}. \]

Just as \( \Omega, \hat{\Omega} \) themselves, \( \Sigma, \hat{\Sigma} \) may be interpreted in terms of right- and left-invariant vector fields or differential forms.

We use the standard conventions of differential geometry according to which vector fields with components \( Z^{i} \) related to local coordinates \( z^{i} \) are identified with first-order differential operators

\[ Z = Z^{i} \frac{\partial}{\partial z^{i}}. \]

Then \( \Sigma, \hat{\Sigma} \) as systems of vector fields dual to systems of Pfaff forms \[20\] (more precisely, to \( \text{L}(V) \)- and \( \text{L}(U) \)-valued differential one-forms) are given by

\[ E^{j}_{i} = \varphi_{j}^{A} \frac{\partial}{\partial \varphi_{i}^{A}}, \quad \hat{E}^{A}_{B} = \varphi^{i}_{B} \frac{\partial}{\partial \varphi^{i}_{A}}. \quad (23) \]

In other words, at the point \( \varphi \in \text{LI}(U, V) \), the \( \{k_{A}\} \)-th component of \( E^{i}_{j} \) equals \( \varphi^{i}_{A} \delta^{k}_{j} \), and the \( \{i_{C}\} \)-th component of \( \hat{E}^{A}_{B} \) equals \( \varphi_{B}^{i} \delta^{A}_{C} \).

Interpreted as invariant forms \( \Sigma \)- and \( \hat{\Sigma} \)-objects become respectively the following fields on \( \text{LI}(U, V) \):

\[ Y[F]_{\varphi} = \varphi^{-1} F, \quad Y[\hat{F}]_{\varphi} = \hat{F} \varphi^{-1}, \]

where \( F, \hat{F} \) are arbitrarily fixed elements of \( \text{L}(V), \text{L}(U) \), and we remember that the dual space \( \text{L}(U, V)^{\ast} \) is canonically isomorphic with \( \text{L}(U, V) \) through the formula \[22\].

Let us quote the obvious transformation rules of \( \Omega, \hat{\Omega}, \Sigma, \hat{\Sigma} \) under transformations \[4\], \[5\] of internal degrees of freedom:

\[
\begin{align*}
\alpha & \in \text{GL}(V) : & \quad \Omega & \mapsto \alpha \Omega \alpha^{-1}, \quad \hat{\Omega} & \mapsto \hat{\Omega} \\
\beta & \in \text{GL}(U) : & \quad \Omega & \mapsto \Omega, \quad \hat{\Omega} & \mapsto \beta^{-1} \hat{\Omega} \beta \\
\alpha & \in \text{GL}(V) : & \quad \Sigma & \mapsto \alpha \Sigma \alpha^{-1}, \quad \hat{\Sigma} & \mapsto \hat{\Sigma} \\
\beta & \in \text{GL}(U) : & \quad \Sigma & \mapsto \Sigma, \quad \hat{\Sigma} & \mapsto \beta^{-1} \hat{\Sigma} \beta
\end{align*}
\]

Now let us quote the basic Poisson brackets. The most important of them, namely those involving the above generators, are determined by the structure
constants of the linear and affine groups.

\begin{align*}
\{\Sigma^i_j, \Sigma^k_l\} &= \delta^i_l \Sigma^k_j - \delta^k_j \Sigma^i_l, & \left\{\tilde{\Sigma}^A_B, \tilde{\Sigma}^C_D\right\} &= \delta^C_B \tilde{\Sigma}^A_D - \delta^A_D \tilde{\Sigma}^C_B, \\
\{J^i_j, \tilde{J}^{k^i}_l\} &= \delta^i_l \tilde{J}^{k^i}_j - \delta^{k^i}_j \tilde{J}^i_l, & \left\{\tilde{\Sigma}^i_j, \tilde{\Sigma}^A_B\right\} &= 0, \\
\{A^i_j, \Lambda^k_l\} &= \delta^i_l A^k_j - \delta^k_j A^i_l, \quad & \left\{\tilde{\Sigma}^A_B, \tilde{p}_C\right\} &= \delta^A_C \tilde{p}_B, \\
\{J^i_j, \tilde{p}_k\} &= \{\Lambda^i_j, p_k\} &= \delta^i_k p_j.
\end{align*}

For any function \(F\) depending only on generalized coordinate \(x^i, \varphi^i_A\) we have:

\begin{align*}
\{\Sigma^i_j, F\} &= -E^i_j F = -\varphi^i_A \frac{\partial F}{\partial \varphi^j_A}, \\
\{\Lambda^i_j, F\} &= -x^i \frac{\partial F}{\partial \varphi^j_A}, \\
\{J^i_j, F\} &= -x^i \frac{\partial F}{\partial \varphi^j_A} - \varphi^A_i \frac{\partial F}{\partial \varphi^i_A}, \\
\{\tilde{\Sigma}^A_B, F\} &= -\tilde{E}^A_B F = -\varphi^B_A \frac{\partial F}{\partial \varphi^i_A}.
\end{align*}

These Poisson brackets are in principle sufficient for obtaining equations of motion in the form:

\[
\frac{dF}{dt} = \{F, H\},
\]

where \(H\) is the Hamilton function.

It is convenient to introduce in addition to vector fields \(E^i_j, \tilde{E}^A_B\) and differential one-forms \(\omega^i_j, \tilde{\omega}^A_B\) some others objects, namely, the vector fields \(H_a, \tilde{H}_A\) and differential one-forms \(\theta^a, \tilde{\theta}^A\), all defined on the configuration space \(Q = M \times \text{LI}(U, V)\). In terms of affine coordinates \(x^i, \varphi^i_A\) but result is coordinate-independent they are given by

\[
\theta^a = dx^a, \quad H_a = \frac{\partial}{\partial x^a}, \quad \tilde{\theta}^A = \varphi^{-1} A_i dx^i, \quad \tilde{H}_A = \varphi^i_A \frac{\partial}{\partial x^i}.
\]

The following duality relations hold among them:

\[
\langle \tilde{\omega}^A_B, E^C_D \rangle = \delta^A_D \delta^C_B, \quad \langle \tilde{\omega}^A_B, \tilde{H}_C \rangle = 0, \quad \langle \tilde{\omega}^A_B, \tilde{H}_C \rangle = \delta^A_B, \quad \langle \tilde{\omega}^A_B, \tilde{H}_C \rangle = 0,
\]

and similarly,

\[
\langle \omega^a_b, E^c_d \rangle = \delta^a_d \delta^c_b, \quad \langle \omega^a_b, H_c \rangle = 0, \quad \langle \theta^a, E^c_d \rangle = 0, \quad \langle \theta^a, H_b \rangle = \delta^a_b.
\]

Therefore, they are mutually dual fields of non-holonomic frames and co-frames on \(Q\). As seen from the structure of Poisson brackets, these fields are geometrically important. In the theory of principal fibre bundles of linear frames
or co-frames they are known as structural fields and standard horizontal fields [32] [33].

Additional important Poisson brackets:

\[
\{ p_a, F \} = -H_a F, \quad \{ \hat{p}_A, F \} = -\hat{H}_AF
\]

(26)

for any function \( F \) depending only on generalized coordinate.

Let us finish the above description of classical geometry of degrees of freedom with a brief review of symmetry problems underlying the polar and two-polar decomposition. They are very important for quantization problems. First of all we shall modify slightly our notation. We introduce new generalized coordinates parameterizing deformation invariants. In many problems it is convenient to denote the diagonal elements of \( D \) by \( Q_a, a = 1, \ldots, n \),

\[
Q_a := D_{aa}, \quad \lambda_a = (D_{aa})^2 = (Q_a)^2.
\]

And many formulas become remarkably simplified when the logarithmic scale is used for parametrizing deformation invariants, \( q^a = \log Q_a \), thus:

\[
Q^a = D_{aa} = \exp(q^a)\lambda_a = \exp(2q^a).
\]

Deformation parameters \( q^a \) run over the total real range \( \mathbb{R} \). They are fictitious ”material points” moving along the real axis. As such they are essentially identical and indistinguishable; this has to do with the mentioned non-uniqueness of the two-polar decomposition. This non-distinguishability is essentially striking and interesting in the quantized version of the theory. The volume extension ratio is given by \( \det D = \exp(q^1 + \cdots + q^n) \), thus it may be measured in a convenient way by the sum \( q^1 + \cdots + q^n \). It is often convenient to split \( D \) into the isochoric (incompressible) and the purely dilatational parts,

\[
D = l\Delta, \quad \det \Delta = 1, \quad l \in \mathbb{R}^+.
\]

The factor \( l \) is the linear size extension ratio; obviously,

\[
l = \sqrt[n]{\det D} = \exp\left(\frac{1}{n}(q^1 + \cdots + q^n)\right).
\]

The logarithmic measure of this ratio:

\[
q = \log l = \frac{1}{n}(q^1 + \cdots + q^n)
\]

is simply the ”centre of mass” of the mentioned ”material points”. The isochoric part \( \Delta \) depends only on the ratios \( Q^i/Q^j \), i.e., logarithmically, on the ”relative positions” \( q^i - q^j \).

The splitting of internal configurations \( \varphi \) into dilatational and isochoric parts may be written down as follows:

\[
\varphi = l\Psi = \exp(q)\Psi = \exp(q)L\Delta R^{-1}.
\]

27
where $\Delta$ as previously is diagonal and isochoric ($\det \Delta = 1$). The term $\Psi$ refers to the shear-rotational degrees of freedom. It is convenient to use the isochoric affine velocities

$$\nu = \frac{d\Psi}{dt} \Psi^{-1}, \quad \hat{\nu} = \Psi^{-1} \frac{d\Psi}{dt} = \Psi^{-1} \nu \Psi.$$  

They are trace-less, i.e. $\nu \in \text{SL}(V)'$, $\hat{\nu} \in \text{SL}(U)'$ (elements of the Lie algebras of $\text{SL}(V)$, $\text{SL}(U)$). The total affine velocities may be expressed as follows:

$$\Omega = \nu + \frac{dq}{dt} \text{Id}_V, \quad \hat{\Omega} = \hat{\nu} + \frac{dq}{dt} \text{Id}_U,$$

(27)

where $\text{Id}_V$, $\text{Id}_U$ are identity transformations in $V$, $U$.

Similarly, the affine spin may be decomposed as follows:

$$\Sigma = \sigma + \frac{p}{n} \text{Id}_V, \quad \hat{\Sigma} = \hat{\sigma} + \frac{p}{n} \text{Id}_V,$$

(28)

where $\sigma \in \text{SL}(V)'$, $\hat{\sigma} \in \text{SL}(U)'$ (traceless) and $p$ is the dilatational canonical momentum,

$$\text{Tr}(\Sigma) = \text{Tr}\left(\hat{\Sigma}\right) = p.$$  

(29)

This momentum is canonically conjugate to the above $q$-variable (logarithmic size variable). The pairing between velocities and momenta may be expressed as follows:

$$\text{Tr}(\Sigma \Omega) = \text{Tr}\left(\hat{\Sigma} \hat{\Omega}\right) = \text{Tr}(\sigma \omega) + \frac{p}{n} \dot{q} = \text{Tr}(\hat{\sigma} \hat{\omega}) + \frac{p}{n} \dot{q}.$$

Poisson brackets for the components of $\sigma$ are based on the structure constants of $\text{SL}(V)$. The same based for $\hat{\sigma}$ with the only precise that the signs are reverse. The mutual Poisson brackets $\{\sigma, \hat{\sigma}\}$ vanish. And obviously, $\{q, p\} = 1$, and the dilatational phase-space variables have vanishing Poisson brackets with the shear-rotational quantities $\Psi, \sigma, \hat{\sigma}$.

Dilatational canonical momentum $p$ may be interpreted as the total linear momentum of the one-dimensional $q^n$-particles.

The two-polar decomposition identifies (modulo some non-uniqueness) internal configuration with the triplets $(L; q^1, \ldots, q^n; R)$, where $(L, R)$ is the pair of rigid bodies (Cauchy and Green deformation tensors principal axes), and the fictitious one-dimensional material points $q^1, \ldots, q^n$ are deformation invariants (in logarithmic scale). The formulas (21) suggest us to make use of two possible systems of non-holonomic velocities:

$$\left(\hat{\chi}; \ldots, q^n; \ldots; \vartheta\right), \quad (\chi; \ldots, q^n; \ldots; \vartheta).$$

(30)

As expected from the (metrically) rigid body mechanics the first subsystem is more effective in analysis of dynamical models.

Similarly, when the polar decompositions are used, we have at disposal the following natural systems of non-holonomic velocities:

$$\left(\hat{\omega}, \hat{A}\right), \quad \left(\hat{\omega}, \hat{B}\right), \quad \left(\omega, \hat{A}\right), \quad \left(\omega, \hat{B}\right),$$

(31)

28
For the qualitative analysis of practically important dynamical models the \((\hat{\omega}, \hat{A})\)-system is most convenient.

Non-holonomic canonical momenta conjugate to (30) are respectively denoted by 
\[
(\hat{\rho}; \ldots, p_a; \ldots; \hat{\tau}), \quad (\rho; \ldots, p_a; \ldots; \tau),
\]
where 
\[
\hat{\rho} \in \text{SO}(n, \mathbb{R})', \quad \hat{\tau} \in \text{SO}(n, \mathbb{R})', \quad g \in \text{SO}(V, g)', \quad \tau \in \text{SO}(U, \eta)', \quad (32)
\]
and \(p_a\) are canonical momenta conjugate to \(q^a\).

Canonical spin variables of the Green and Cauchy gyroscope and their dual angular velocities are considered as elements of the same linear spaces; this is due to the natural isomorphisms between orthogonal Lie algebras and their duals. So, the corresponding pairings are given by
\[
\langle (\hat{\rho}, \hat{\sigma}), (\hat{\omega}, \dot{\hat{A}}) \rangle = p_a \dot{q}^a + \frac{1}{2} \text{Tr} (\hat{\rho} \hat{\sigma}) + \frac{1}{2} \text{Tr} (\hat{\tau} \hat{\omega}),
\]
\[
\langle (\rho, \hat{\mu}), (\chi, \dot{\tau}) \rangle = p_a \dot{q}^a + \frac{1}{2} \text{Tr} (\rho \chi) + \frac{1}{2} \text{Tr} (\mu \tau).
\]
Similarly, the dual objects of (31) will be denoted by 
\[
(\widehat{\sigma}, \alpha), \quad (\widehat{\mu}, \beta), \quad (\sigma, \alpha), \quad (\mu, \beta),
\]
where 
\[
\hat{\sigma}, \hat{\mu} \in \text{SO}(U, \eta)', \quad \sigma, \mu \in \text{SO}(V, g)',
\]
and \(\alpha \in \text{L}(U), \beta \in \text{L}(V)\) are respectively \(\eta\)- and \(g\)-symmetric. The corresponding pairings are given by 
\[
\langle (\hat{\sigma}, \alpha), (\hat{\omega}, \dot{A}) \rangle = \frac{1}{2} \text{Tr} (\hat{\sigma} \hat{\omega}) + \text{Tr} (\alpha \dot{A}),
\]
e tc., an analogous way for other combinations.

Let us remind that in the physical three-dimensional case the skew-symmetric tensors are identified with the axial vectors, e.g.,
\[
\chi^i_j = -\varepsilon^i_{jk} \chi^k, \quad \chi^i = -\frac{1}{2} \varepsilon^i_{jk} \chi^j \chi^k
\]
in orthonormal coordinates. For the dual angular momentum quantities we have the reversed-sign-convention, e.g.
\[
\rho^i_j = \varepsilon^i_{jk} \rho^k, \quad \rho^i = \frac{1}{2} \varepsilon^i_{jk} \rho^j \rho^k.
\]
The shift of indices here is meant in the cosmetic Kronecker-delta-sense. Then the former formulas are compatible with the standard \(\mathbb{R}^3\)-conventions, e.g., 
\[
\frac{1}{2} \text{Tr} (\rho \chi) = \rho_i \chi^i,
\]
and similarly for other angular velocity and angular momentum quantities.

The quantities $\rho$, $\tau$ coincide respectively with spin $S$ and negative vorticity $-V$. They are Hamiltonian generators of transformations:

$$\varphi \mapsto A\varphi, \quad \varphi \mapsto \varphi C^{-1}, \quad A \in \text{SO}(V, g), \quad C \in \text{SO}(U, \eta),$$

i.e., in terms of the two-polar decomposition:

$$L \mapsto AL, \quad R \mapsto CR.$$

The objects $\hat{\rho}$, $\hat{\tau}$ are Hamiltonian generators of transformations

$$L \mapsto LA, \quad R \mapsto RC, \quad A, C \in \text{SO}(n, \mathbb{R}).$$

Similarly, if we use the polar decomposition, $\sigma$ coincides with spin $S$, because it generates transformations $\varphi \mapsto W\varphi$, $W \in \text{SO}(V, g)$, i.e.,

$$U \mapsto WU, \quad \varphi = UA \mapsto WUA = W\varphi.$$

The quantity $\hat{\sigma}$ is the Hamiltonian generator of

$$\varphi = UA \mapsto UKA, \quad K \in \text{SO}(U, \eta).$$

Similarly, the objects $\hat{\mu}$ generates transformations:

$$C \in \text{SO}(U, \eta) : \quad \varphi = BU \mapsto BUC = \varphi C.$$

Therefore, it coincides with the canonical vorticity $V$.

And finally, $\mu$ generates the transformation group:

$$W \in \text{SO}(V, g) : \quad \varphi = BU \mapsto BWU.$$

Everything said above implies that

$$\chi^i_j = L^i_a \hat{\chi}^a_b L^{-1b}_j, \quad \psi^A_B = R^A_a \hat{\psi}^a_b R^{-1b}_B, \quad \omega^i_j = U^i_A \hat{\omega}^A_B U^{-1B}_j$$

and similarly

$$\rho^i_j = L^i_a \hat{\rho}^a_b L^{-1b}_j, \quad \tau^A_B = R^A_a \hat{\tau}^a_b R^{-1b}_B, \quad \sigma^i_j = U^i_A \hat{\sigma}^A_B U^{-1B}_j.$$

In situations where the variables $Q^i$ are more convenient than $q^i = \ln Q^i$, the canonical momenta $P_i$ conjugate to $Q^i$ will be used instead $p_i$ (conjugates of $q^i$). The relationship is as follows:

$$P_i = p_i \exp (-q^i) = \frac{p_i}{Q^i}.$$
Having in view applications on the fundamental level, including the atomic and molecular structure, we concentrate here on the quantization procedure. Because of this, on the classical level we are interested mainly in Hamiltonian models. Even if dissipative phenomena are taken into account, they are considered as a correction to the Hamiltonian background. In any case, the primary concept is that of the kinetic energy. If we assume that the mechanism of affine constraint is compatible with the d’Alembert principle, then the kinetic energy is obtained by restriction of the primary multiparticle kinetic energy to the tangent bundle of the constraints manifold. After easy calculations one obtains:

\[ T = T_{tr} + T_{int} = \frac{m}{2} \sum g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} \sum g_{ij} \frac{d\varphi^i_A}{dt} \frac{d\varphi^j}_B J_{AB}, \]

where \( m, J \) denote, as previously, the total mass and the second-order moment of the mass distribution. They characterize respectively the translational and internal inertia. It is instructive to quote two alternative formulas:

\[ T = \frac{m}{2} G_{AB} \hat{\varphi}^A \hat{\varphi}^B + \frac{1}{2} G_{AB} \hat{\Omega}^A_K \hat{\Omega}^B_L J^{KL}, \]

\[ T = \frac{m}{2} g_{ij} v^i v^j + \frac{1}{2} \sum g_{ij} \Omega^i \Omega^j |J|^{kl}. \]

Legendre transformation may be written in any of the following equivalent forms:

\[ p_i = \frac{\partial T}{\partial v^i} = m g_{ij} v^j, \quad p^A_i = \frac{\partial T}{\partial \xi^i_A} = g_{ij} \xi^j_B J^{BA}, \]

\[ \hat{p}_A = \frac{\partial T}{\partial \hat{\varphi}^A} = m G_{AB} \hat{\varphi}^B, \quad \hat{\Sigma}^A_B = \frac{\partial T}{\partial \hat{\Omega}^A_B} = G_{BC} \hat{\Omega}^C_D J^{DA}, \]

\[ \hat{\Sigma}^i_j = \frac{\partial T}{\partial \hat{\Omega}^i_j} = g_{jk} \Omega^i_k J|\varphi|^{li}. \]

Obviously, it is assumed here that there is no generalized potential depending on velocities (e.g., no magnetic forces). Otherwise we would have to replace the kinetic energy \( T \) by the total Lagrangian. Inverting the above formulas and substituting them to the kinetic energy expressions we obtain there formulas for the kinetic Hamiltonian:

\[ H = T = T_{tr} + T_{int} = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} J_{AB} \hat{p}^A \hat{p}^B g^{ij}, \]

\[ H = T = \frac{1}{2m} \sum G^{AB} \hat{\varphi}^A \hat{\varphi}^B + \frac{1}{2} J_{AB} \hat{\Sigma}^A_K \hat{\Sigma}^B_L \hat{G}^{KL}, \]

\[ H = T = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} \sum J|\varphi| \hat{\Omega}^i_k \hat{\Omega}^j_l g^{kl}, \]

where the "tilda" objects are reciprocal tensors.

For Lagrangians \( L = T - V(x, \varphi) \), the resulting Hamiltonians have the form \( H = T + V(x, \varphi) \).
The usual kinetic energy quadratic in velocities is geometrically equivalent to some Riemann structure on the configuration space,
\[ T = \frac{1}{2} \Gamma_{\mu\nu}(q) \left( \frac{dq^\mu}{dt} \frac{dq'^\nu}{dt} \right), \quad \Gamma = \Gamma_{\mu\nu}(q) dq^\mu \otimes dq'^\nu, \]
or, in traditional notation using the arc element:
\[ ds^2 = \Gamma_{\mu\nu}(q) dq^\mu dq^\nu. \]

For the kinetic Hamiltonian we have:
\[ T = \frac{1}{2} \tilde{\Gamma}_{\mu\nu}(q) p_\mu p_\nu, \quad \tilde{\Gamma} = \tilde{\Gamma}_{\mu\nu}(q) \frac{\partial}{\partial q^\mu} \otimes \frac{\partial}{\partial q'^\nu}. \]

Using the previously introduced symbols we can express the metric tensor underlying our kinetic energy in any of the following equivalent forms:
\[ \Gamma = mg_{ij} dx_i \otimes dx_j + g_{ij} J_{AB} d\varphi^i_A \otimes d\varphi^j_B, \] (36)
\[ \Gamma = mG_{AB} \tilde{\theta}^A \otimes \tilde{\theta}^B + G_{AB} J_{KL} \tilde{\omega}^A_K \otimes \tilde{\omega}^B_L, \]
\[ \Gamma = mg_{ij} \theta^i \otimes \theta^j + g_{ij} J[\varphi]^{kl} \omega^i_k \otimes \omega^j_l. \]

Similarly, for the inverse metric \( \tilde{\Gamma} \) we have the following equivalent expressions:
\[ \tilde{\Gamma} = \frac{1}{m} g^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} + g^{ij} \tilde{J}_{AB} \frac{\partial}{\partial \varphi^i_A} \otimes \frac{\partial}{\partial \varphi^j_B}, \]
\[ \tilde{\Gamma} = \frac{1}{m} \tilde{G}^{AB} \tilde{H}_A \otimes \tilde{H}_B + \tilde{G}^{AB} \tilde{J}_{KL} \tilde{E}^K_A \otimes \tilde{E}^L_B, \]
\[ \tilde{\Gamma} = \frac{1}{m} g^{ij} H_i \otimes H_j + g^{ij} \tilde{J}[\varphi]_{kl} E^i_k \otimes E^j_l. \]

It is clear that the above kinetic energies (metric tensors) are invariant under the group of spatial isometries \( Is(M, g) \) acting through \( \Box \). It is also invariant under \( O(U, J) \) acting through \( \mathcal{L} \), i.e., the subgroup of \( GL(U) \) preserving \( J \).

In particular, it is materially isotropic, i.e., invariant under \( O(U, \eta) \) when the inertial tensor is spherical, \( J = \mu \tilde{\eta} \). However, there is no total affine invariance either in the spatial or material sense. This kinematical symmetry is broken by the tensors \( g, J \). Therefore, the traditional d’Alembert model of the kinetic energy does not belong to the framework of left- or right- (or two-side-) invariant geodetic systems on Lie groups or their group spaces, i.e., to the theory developed by Hermann and Arnold on the basis of rigid body or incompressible ideal fluid dynamics. By the way, with the above metric tensors the geodetic systems are non-physical, because they predict the unlimited contraction and expansion of the body. And when some extra potential is introduced as a dynamical model of deformative vibrations, then, except some very special potential shape, none or rather small profit is gained from the group-theoretical model of degrees of freedom.

So, it is a tempting idea to formulate dynamical models unifying two things: geodetic description (no potential as far as possible) and affine invariance. It
turns out that to some extent this may be successfully done: not only the inertia but also interactions are encoded in some affinely-invariant kinetic energy forms (metrics on the configuration space).

The most general and reasonable class of dynamical geodetic models invariant under the spatial affine group \( \text{GAf}(M) \) acting through (2) is given by the metric tensor:

\[
\Gamma = m \eta_{AB} \hat{\theta}^A \otimes \hat{\theta}^B + \mathcal{L}^B_{\ A\ C} \hat{\omega}^A \otimes \hat{\omega}^C, \tag{37}
\]

where \( \mathcal{L}^B_{\ A\ C} \) are constant and symmetric in their bi-indices,

\[
\mathcal{L}^B_{\ A\ D\ C} = \mathcal{L}^D_{\ C\ A\ B},
\]

The inverse metric is given by

\[
\tilde{\Gamma} = \frac{1}{m} \eta_{AB} \hat{H}^A \otimes \hat{H}^B + \mathcal{L}^B_{\ D\ A\ C} \hat{E}^A \otimes \hat{E}^C, \tag{38}
\]

with similar properties of constants \( \mathcal{R} \):

\[
\mathcal{R}^j_{\ i\ k} = \mathcal{R}^k_{\ j\ i}.
\]

The inverse contravariant metric (underlying the kinetic Hamiltonian) is given by

\[
\tilde{\Gamma} = \frac{1}{m} g^{ij} \hat{v}^i \otimes \hat{v}^j + \tilde{\mathcal{R}}^j_{\ i\ k} \hat{\omega}^i \otimes \hat{\omega}^k, \quad \tilde{\mathcal{R}}^j_{\ i\ k} \mathcal{R}^{k\ a\ b} = \delta^j_{\ b} \delta^a_{\ i}. \tag{40}
\]

The corresponding explicit expressions for kinetic energies and kinetic Hamiltonians are:

\[
T = \frac{m}{2} \eta_{AB} \hat{\theta}^A \hat{\theta}^B + \frac{1}{2} \mathcal{L}^B_{\ A\ D\ C} \hat{\omega}^A \hat{\omega}^C, \tag{39}
\]

\[
T = \frac{1}{2m} \eta_{AB} p_A p_B + \frac{1}{2} \mathcal{L}^B_{\ A\ D\ C} \Sigma^A \otimes \Sigma^C, \tag{40}
\]

In certain problems it is convenient to use another equivalent expressions for the translational parts. They are respectively given by

\[
T_{tr} = \frac{m}{2} C_{ij} v^i v^j, \quad T_{tr} = \frac{1}{2m} \tilde{\mathcal{G}}_{ij} p_i p_j; \tag{41}
\]

\[
T_{tr} = \frac{m}{2} \tilde{G}^{AB} \hat{v}^A \hat{v}^B, \quad T_{tr} = \frac{1}{2m} \tilde{G}^{AB} \tilde{p}_A \tilde{p}_B. \tag{42}
\]

33
Let us observe that expressions for GAf($M$)-invariant may be interpreted in the following way. No metric tensor $g$ in the physical space is assumed and even if it exists (as it does in reality) it does not enter the kinetic energy expression (if it did, affine symmetry would be broken and restricted to isometric one). The role of the metric tensor in contraction of tensorial indices is played by the Cauchy tensor $C$.

There is no kinetic energy model invariant simultaneously under spatial and material affine transformations. More precisely, any symmetric twice covariant tensor field on $Q = M \times \operatorname{LI}(U, V)$ must be degenerate. This is due to the very malicious non-semisimplicity of the affine group. However, if we neglect the translational motion, then there exist internal metric on $\operatorname{LI}(U, V)$ invariant under both spatial and material (homogeneous) affine transformations (4), (5). They are given by

$$\Gamma^0_{\text{int}} = A\omega^K_L \otimes \omega^L_K + B\omega^K_K \otimes \omega^L_L$$

$$= A\omega^I_i \otimes \omega^j_I + B\omega^k_k \otimes \omega^l_l,$$

$A, B$ denoting constants. Their inverses have the form:

$$\tilde{\Gamma}^0_{\text{int}} = \frac{1}{A} \hat{E}^K_L \otimes \hat{E}^L_K - \frac{B}{A(A + nB)} \hat{E}^K_K \otimes \hat{E}^L_L$$

$$= \frac{1}{A} E^k_i \otimes E^l_k - \frac{B}{A(A + nB)} E^k_k \otimes E^l_l.$$

Such a metric is never positively-definite. The reason is that $\operatorname{SL}(n, \mathbb{R})$ is non-compact and semisimple. $\Gamma^0_{\text{int}}$ becomes the usual Killing metric when $A = 2n$, $B = -2$. But this is the pathological situation, because $\Gamma^0_{\text{int}}$ is degenerate for $A/B = -n$ (due to the dilatational centre in $\operatorname{SO}(n, \mathbb{R})$).

The corresponding kinetic energies are given by

$$T^0_{\text{int}} = \frac{A}{2} \operatorname{Tr} (\hat{\Omega}^2) + \frac{B}{2} \left( \frac{\operatorname{Tr} \hat{\Omega}}{A(A + nB)} \right)^2$$

$$= \frac{A}{2} \operatorname{Tr} (\Omega^2) + \frac{B}{2} (\operatorname{Tr} \Omega)^2,$$

$$T^0_{\text{int}} = \frac{1}{2A} \operatorname{Tr} (\hat{\Sigma}^2) - \frac{B}{2A(A + nB)} (\operatorname{Tr} \hat{\Sigma})^2$$

$$= \frac{1}{2A} \operatorname{Tr} (\Sigma^2) - \frac{B}{2A(A + nB)} (\operatorname{Tr} \Sigma)^2.$$

(43) (44) (45) (46)

The $B$-controlled term in $T_{\text{int}}$ above is a merely correction. The main term ($A$-controlled one) has the hyperbolic signature $(n(n + 1)/2 + n(n - 1)/2 -)$, where the "plus" contribution corresponds to the non-compact dimensions and the "minus" one to the compact dimensions in $\operatorname{GL}(V)$, $\operatorname{GL}(U)$. This is the highest possible symmetry of $T_{\text{int}}$, an affine counterpart of the spherical top. One is rather reluctant to non-positive "kinetic energies". However, one can
show that in the above model the lock of positive definiteness is not essentially embarrassing; on the contrary, the negative contributions may encode the attractive part of the deformation dynamics. By the way, the same effect may be obtained within the framework of positive Riemannian structures on $Q$, when we use a slightly modified version of (44).

Let us observe that translational kinetic energy (39), (41) is affinely-invariant in the physical space and isometry-invariant in the material space. And conversely, (40), (42) is isometry-invariant (homogeneous and isotropic) in the physical space and affinely invariant in the material space. This focuses our attention on Riemannian structures on $Q=M \times \text{L} \text{I}(U, V)$ invariant under the spatial affine group $\text{G} \text{A} \text{f}(M)$ and the group of material isometries $\text{I} \text{s}(N, \eta)$; the opposite models are those invariant under spatial isometries $\text{I} \text{s}(M, g)$ and material affine transformations $\text{G} \text{A} \text{f}(N)$. The corresponding metric tensors are respectively given by

$$\Gamma = m \eta_{KL} \hat{\theta}^K \otimes \hat{\theta}^L + I \eta_{KN} \hat{\eta}^M \hat{\omega}^K_N + \Gamma^0_{\text{int}},$$

$$\Gamma = m g_{ij} \theta^i \otimes \theta^j + I g_{kl} \omega^i_j \otimes \omega^k_l + \Gamma^0_{\text{int}},$$

where the constants $I, A, B$ are generalized moments of inertia.

The corresponding contravariant inverses have the form

$$\tilde{\Gamma} = \frac{1}{m} \eta^{KL} \tilde{H}_K \otimes \tilde{H}_L + \frac{1}{I} \eta^{KN} \hat{\eta}^M \hat{\omega}^K_N \hat{\omega}^L_N + \frac{1}{A} \hat{\omega}^K_K \otimes \hat{\omega}^L_L,$$

$$\tilde{\Gamma} = \frac{1}{m} g^{ij} H_i \otimes H_j + \frac{1}{I} g^{kl} E^i_j \otimes E^k_l + \frac{1}{A} E^j_j \otimes E^i_i,$$

where the inertial constants $\tilde{I}, \tilde{A}, \tilde{B}$ are given by

$$\tilde{I} = \frac{1}{I} (I^2 - A^2), \quad \tilde{A} = \frac{1}{A} (A^2 - I^2), \quad \tilde{B} = -\frac{1}{B} (I + A) (I + A + n B).$$

The corresponding kinetic energies are explicitly given by

$$T = T_{\text{tr}} + T_{\text{int}} = \frac{m}{2} \eta_{AB} \tilde{v}^A \tilde{v}^B + \frac{1}{2} \eta_{KN} \hat{\eta}^M \hat{\omega}^K_N \hat{\omega}^L_N$$

$$+ \frac{A}{2} \text{Tr} \left( \hat{\omega}^2 \right) + \frac{B}{2} \left( \text{Tr} \hat{\omega} \right)^2,$$

$$T = T_{\text{tr}} + T_{\text{int}} = \frac{m}{2} g_{ij} v^i v^j + \frac{1}{2} g_{kl} \Omega^{ij} \Omega^{kl}$$

$$+ \frac{A}{2} \text{Tr} \left( \Omega^2 \right) + \frac{B}{2} \left( \text{Tr} \Omega \right)^2.$$

Obviously, the last two terms in both expressions coincide because $\text{Tr} (\Omega^p) = \text{Tr} \left( \tilde{\Omega}^p \right)$ for any natural $p$. 35
The corresponding kinetic Hamiltonians have the following form:

\[ T = T_{\text{tr}} + T_{\text{int}} = \frac{1}{2m} \eta^{AB} \dot{p}_A \dot{p}_B + \frac{1}{2I} \eta_{KL} \tilde{\Sigma}^K_M \tilde{\Sigma}^L_N \eta^{MN} \]  

\[ + \frac{1}{2A} \text{Tr} \left( \tilde{\Sigma}^2 \right) + \frac{1}{2B} \left( \text{Tr} \tilde{\Sigma} \right)^2 , \]  

\[ T = T_{\text{tr}} + T_{\text{int}} = \frac{1}{2m} \dot{g}^{ij} p_i p_j + \frac{1}{2I} g_{ik} \Sigma^i_j \Sigma^k_l g^{jl} \]  

\[ + \frac{1}{2A} \text{Tr} \left( \Sigma^2 \right) + \frac{1}{2B} \left( \text{Tr} \Sigma \right)^2 . \]  

(49)  

Let us observe that the metrical (\( g \)- and \( \eta \)-dependent) parts of kinetic energies may be alternatively written down in terms of the Cauchy and Green tensors:

\[ \frac{m}{2} C_{ij} v^i v^j + \frac{1}{2m} G_{AB} \tilde{\Sigma}^A \tilde{\Sigma}^B, \]  

\[ \frac{1}{2m} G_{ij} p_i p_j + \frac{1}{2m} G_{AB} \tilde{\Sigma}^A \tilde{\Sigma}^B \]  

\[ \frac{1}{2m} \tilde{G}^{ij} \dot{p}_i \dot{p}_j + \frac{1}{2m} \tilde{G}^{AB} \dot{p}_A \dot{p}_B + \frac{1}{2I} \tilde{G}^{AB} \tilde{\Sigma}^A \tilde{\Sigma}^B \]  

\[ = \frac{m}{2} C_{ij} v^i v^j + \frac{1}{2m} G_{AB} \tilde{\Sigma}^A \tilde{\Sigma}^B, \]  

\[ \frac{1}{2m} \tilde{G}^{ij} \dot{p}_i \dot{p}_j + \frac{1}{2m} \tilde{G}^{AB} \dot{p}_A \dot{p}_B + \frac{1}{2I} \tilde{G}^{AB} \tilde{\Sigma}^A \tilde{\Sigma}^B \]  

\[ + \frac{1}{2A} \text{Tr} \left( \Sigma^2 \right) + \frac{1}{2B} \left( \text{Tr} \Sigma \right)^2 . \]  

(50)  

It is important that in a certain open range of triples \( (I, A, B) \in \mathbb{R}^3 \) the above kinetic energies are positively definite and at the same time they have all geometrical and analytical advantages of invariant geodetic system on the group manifolds.

One can show that the spatially affine and materially metrical model (49), or more precisely, its internal part, may be expressed as follows:

\[ T_{\text{int}} = \frac{1}{2\alpha} \text{Tr} \left( \tilde{\Sigma}^2 \right) + \frac{1}{2\beta} \left( \text{Tr} \tilde{\Sigma} \right)^2 + \frac{1}{2\mu} \| V \|^2 , \]  

(51)  

where

\[ \alpha = I + A, \quad \beta = -\frac{1}{B} (I + A) (I + A + nB), \quad \mu = \frac{1}{I} (I^2 - A^2) , \]  

(52)

and \( \| V \| \) denotes the magnitude of the vorticity,

\[ \| V \|^2 = -\frac{1}{2} \text{Tr} \left( V^2 \right) . \]

Denoting the \( k \)-th order Casimir invariant built of generators by \( C(k) \),

\[ C(k) = \text{Tr} \left( \Sigma^k \right) = \text{Tr} \left( \tilde{\Sigma}^k \right) , \]

we can write simply:

\[ T_{\text{int}} = \frac{1}{2\alpha} C(2) + \frac{1}{2\beta} C(1)^2 + \frac{1}{2\mu} \| V \|^2 . \]  

(53)

Similarly, for the spatially metrical and materially affine model (50) we have

\[ T_{\text{int}} = \frac{1}{2\alpha} C(2) + \frac{1}{2\beta} C(1)^2 + \frac{1}{2\mu} \| S \|^2 , \]  

(54)
with the same as previously convention concerning the magnitude of spin:
\[ \|S\|^2 = -\frac{1}{2} \text{Tr} \left( S^2 \right). \]

Let us note that \( \|V\|, \|S\| \) are simply second-order Casimir built of vorticity and spin.

For the model (43) affinely invariant both in the physical and material space we have:
\[ T_{\text{int}}^0 = \frac{1}{2A} C(2) + \frac{1}{2A(n + A/B)} C(1)^2. \]  

It is very convenient to separate dilatational and incompressible motions, especially when affinely invariant kinetic energies (metrics on \( Q \)) are used.

One can easily show that for the affine-affine model (43) we have
\[ T_{\text{int}}^0 = \frac{A}{2} \text{Tr} \left( \nu^2 \right) + \frac{n(A + nB)}{2} \dot{q}^2 = T_{\text{sh}}^0 + T_{\text{dil}}^0 \]  

cf. (27). Performing the Legendre transformation we obtain
\[ T_{\text{int}}^0 = \frac{1}{2A} \text{Tr} \left( \sigma^2 \right) + \frac{1}{2n(A + nB)} p^2 = T_{\text{sh}}^0 + T_{\text{dil}}^0 \]  

cf. (28), (29). Similarly, for the affine-metrical and metrical-affine models one obtains respectively the following expression:
\[ T_{\text{int}}^0 = \frac{1}{2(I + A)} C_{\text{SL}(n)}(2) + \frac{1}{2n(I + A + nB)} p^2 + \frac{I}{2(I^2 - A^2)} \|V\|^2, \]  
\[ T_{\text{int}}^0 = \frac{1}{2(I + A)} C_{\text{SL}(n)}(2) + \frac{1}{2n(I + A + nB)} p^2 + \frac{I}{2(I^2 - A^2)} \|S\|^2, \]

where \( C_{\text{SL}(n)}(k) \) are Casimir invariants built of \( \sigma \),
\[ C_{\text{SL}(n)}(k) = \text{Tr} \left( \sigma^k \right) = \text{Tr} \left( \hat{\sigma}^k \right). \]

Let us observe, the only difference is that concerning the last, third term. And the both expressions reduce to (57) when we substitute \( I = 0 \). And conversely, they may be obtained from (57) by replacing: \( A \mapsto (I + A) \) and introducing the mentioned terms,
\[ T_{\text{int}}^{\text{aff} \rightarrow \text{met}} = T_{\text{int}}^0[A \mapsto I + A] + \frac{I}{2(I^2 - A^2)} \|V\|^2, \]  
\[ T_{\text{int}}^{\text{met} \rightarrow \text{aff}} = T_{\text{int}}^0[A \mapsto I + A] + \frac{I}{2(I^2 - A^2)} \|S\|^2. \]

Our philosophy is to base the dynamics as for as possible on geodetic affinely-invariant models. In particular, geodetic affine-isometric and isometric-affine models are of special interest. They are "as affine as possible" and at the same time compatible with the positive definiteness demand. Nevertheless some
models with potentials are still of interest, and, for non-affine models they are just unavoidable. So, we shall consider also potential models

\[ H = T + V, \]

where \( V \) depends only on the configuration variables \((x, \varphi)\). What concerns inertial properties we concentrated on highly-symmetric models; in any case they are always spatially- and usually materially-isotropic (one can be general in formulation, but no so much in effective analysis). It is natural to assume that the potential energy \( V \) is compatible with these invariance properties of the kinetic term. So, \( V \) is invariant under internal spatial rotations if and only if it depends on \( \varphi \) through the Green tensor \( G \). It is invariant under material spatial rotations if and only if it depends on \( \varphi \) through the Cauchy tensor.

And finally, \( V \) is both spatially and materially isotropic in internal degrees of freedom if and only if it depends on \( \varphi \) through the determinant \( \det \varphi \). If we use the logarithmic scale of deformation invariants, this means that \( V \) is function of \( q = (q^1 + \cdots + q^n)/n \), the "centre of mass" of logarithmic deformation invariants \( q^i, i = 1, \ldots, n \). In kinetic energy models \((57), (58), (59)\) dilatational and shear-rotational degrees of freedom (incompressible motion) are mutually orthogonal; there is no interaction between them. This suggests us to concentrate also on adapted potentials where these degrees of freedom are explicitly separated,

\[ V (q^1, \ldots, q^n) = V_{\text{dil}}(q) + V_{\text{sh}} (\ldots, q^i - q^j, \ldots); \]

(60)

the labels "dil" and "sh" refer respectively to "dilatation" and "shear". The most natural scheme for \( V_{\text{sh}} \) is that of "binary interactions" between deformation invariants

\[ V_{\text{sh}} = \sum_{i \neq j} f_{ij} (q^i - q^j). \]

(61)

For isotropic models \( H_{\text{int}} = T_{\text{int}} + V (q^1, \ldots, q^n) \) with \( T_{\text{int}} \) given by \((49), (51), (58)\) the vorticity \( V \) is a constant of motion and the third term in \((58)\) has also the vanishing Poisson brackets with all terms of \((58)\). The structure of Poisson brackets and equations of motion \((24)\) implies that the evolution of variables \( \Sigma^j, q^a \), ruled by the above Hamiltonian \( H_{\text{int}} \), is the same as one ruled by

\[ H^0_{\text{int}} = T^0_{\text{int}} [A \mapsto I + A] + V (q^1, \ldots, q^n), \]

where \( T^0_{\text{int}} [A \mapsto I + A] \) is obtained from \((45)\) \( T^0_{\text{int}} \) by substituting \((I + A)\) instead \( A \). The difference occurs only in degrees of freedom ruled by \( \text{SO}(V, g), \text{SO}(U, \eta) \), i.e., in the time evolution of quantities \( L, R \) describing the orientation of principal axes of deformation tensors \( C, G \). If \( V \) depends only on the dilatational invariant \( q \), than the total motion in \( Q \) is a direct product of two independent things: the geodetic incompressible motion and the autonomous dynamics of
the $q$-variable. The deviator
\[
\sigma^i_j = \Sigma^i_j - \frac{1}{n} \Sigma_a^a \delta^i_j
\]
is then a constant of motion. The general solution for geodetic models based on $T^0_{\text{int}}$ (45) is explicitly given by exponential mapping. Roughly speaking, it is produced from initial conditions by one-parameter subgroups of $\text{GL}(V)$, $\text{GL}(U)$. And it may be shown on the basis of the properties of matrix exponents that for the incompressible geodetic affine-affine model
\[
T^0_{\text{int}} = \frac{A}{2} \text{Tr} (\nu^2), \quad T^0_{\text{int}} = \frac{A}{2} \text{Tr} (\sigma^2)
\]
(with constrains $q = 0$) the general solution contains an open-subset of bounded (oscillating) motions and an open subset of unbounded (escaping, dissociated) motions. When dilatations are allowed, then for any Hamiltonian
\[
H = T^0_{\text{int}} + V(q)
\]
with $T^0_{\text{int}}$ given by (57) and $V(q)$ stabilizing dilatations, there exists also an open subfamily of bounded motions (and an open subfamily of unbounded motions if $\sup V < \infty$). The same remains true for the general geodetic affine-metrical model (44), (49), (53) with incompressibility constraints $q = 0$, and similarly, without such constraints but with dilatations - stabilizing potential $V(q)$.

The same arguments may be applied to dilatationally stabilized geodetic models in (48), (50), (54) invariant under $O(V, g) \times \text{GL}(U)$ or purely geodetic isochoric models with the symmetry group $O(V, g) \times \text{SL}(U)$ (materially special-affine and spatially material models). On the level of state variables $\hat{S}^A B, q^a$ the time evolution is exactly identical with that based on the affine-affine model of $T_{\text{int}}$ again with $A$ in (53) replaced by $I + A$.

Let us summarize the main message. Incompressible affine-affine, affine-material and metrical-affine models may encode the dynamics of elastic vibrations without any extra potential used, because their general solutions contain open subset of bounded motions. When no incompressibility constraints are imposed, the same may be achieved by introducing some dilatations-stabilizing potential $V(q)$, e.g., some potential well oscillator $V(q) = (k/2)q^2$, etc.

The bounded or unbounded character of motion has to do only with the time evolution of $q^a$-variables, and from this point of view the mentioned three models are essentially identical. The difference appears only on to level of $L, R$ - degrees of freedom, but these gyroscopic variables with compact topology cannot influence the property of trajectories to be bounded or escaping.

To finish this classical description we describe everything in terms of the two-polar decomposition. It is convenient to combine the non-holonomic canonical momenta $\hat{\rho}, \hat{\tau}$ in the following way:
\[
M := -\hat{\rho} - \hat{\tau}, \quad N := \hat{\rho} - \hat{\tau}.
\]
This provides a partial diagonalization of the kinetic energy and elimination of certain interference terms. Namely, after some calculations one obtain the following expressions for Casimirs:

\[
C(2) = \sum_a p_a^2 + \frac{1}{16} \sum_{a,b} \frac{(M_{ab})^2}{\text{sh}^2 \frac{q_a - q_b}{2}} - \frac{1}{16} \sum_{a,b} \frac{(N_{ab})^2}{\text{ch}^2 \frac{q_a - q_b}{2}},
\]

and, obviously,

\[
C(1) = p = \sum_a p_a.
\]

The first term in \(C(2)\) may be suggestively decomposed into the "relative" and the over-all ("centre-of-mass") parts:

\[
\frac{1}{2n} \sum_{a,b} (p_a - p_b)^2 + \frac{p^2}{n}.
\]

This enables one to separate the incompressible and purely dilatational parts.

It is interesting that \(C(2)\) and therefore the kinetic energy itself has a characteristic lattice structure known from the theory of one-dimensional many-body system. Here the logarithmic deformation invariants \(q_a\) are positions of \(n\) indistinguishable fictitious "material points". Unlike in the usual Sutherland, hyperbolic-Sutherland and Calogero-Moser lattices where all binary coupling parameters were identical, now the quantities \(M^i j, N^i j\) are not only non-identical, but also non-constant. Moreover, they are state variables subject, together with other ones, to some closed system of evolution equations \([21]\), where \(F\) runs over the quantities \(q^i, p_i, L, R, M^i j, N^i j\).

Obviously, one should substitute to \([21]\) the following basic Poisson brackets

\[
\{q^a, p_b\} = \delta^a b, \\
\{q^a, M^c d\} = \{p_a, M^c d\} = \{q^a, N^c d\} = \{p_a, N^c d\} = 0, \\
\{M_{ab}, N_{cd}\} = \{N_{ab}, M_{cd}\} = M_{cb}g_{ad} - M_{ad}g_{cb} + M_{ac}g_{db} - M_{db}g_{ac}, \\
\{M_{ab}, N_{cd}\} = N_{cb}g_{ad} - N_{ad}g_{cb} + N_{ac}g_{db} - N_{db}g_{ac},
\]

where the shift of indices is meant in the \(q\)-sense. Obviously, usually Cartesian orthonormal coordinates are used and then simply \(g_{ab} = \delta_{ab}\). The Poisson brackets for \(M = -\hat{\rho} - \hat{\tau}, N = \hat{\rho} - \hat{\tau}\) follow from the following ones for \(\hat{\rho}, \hat{\tau}\):

\[
\{\hat{\rho}_{ab}, \hat{\rho}_{cd}\} = -\hat{\rho}_{cb}g_{ad} + \hat{\rho}_{ad}g_{cb} - \hat{\rho}_{ac}g_{db} + \hat{\rho}_{db}g_{ac}, \\
\{\hat{\tau}_{ab}, \hat{\tau}_{cd}\} = -\hat{\tau}_{cb}g_{ad} + \hat{\tau}_{ad}g_{cb} - \hat{\tau}_{ac}g_{db} + \hat{\tau}_{db}g_{ac}, \\
\{\hat{\rho}_{ab}, \hat{\tau}_{cd}\} = 0.
\]

And these brackets are based on the structure constants of \(\text{SO}(n, \mathbb{R})\), because \(\hat{\rho}, \hat{\tau}\) (similarly like \(S, -V\)) are corresponding Hamiltonian generators of \(\text{SO}(n, \mathbb{R})\).

In affine kinetic energies \([57, 58, 59]\) the second-order \(\text{SL}(n, \mathbb{R})\)-Casimir invariant has the form:

\[
C_{\text{SL}(n)} = \frac{1}{2n} \sum_{a,b} (p_a - p_b)^2 + \frac{1}{16} \sum_{a,b} \frac{(M_{ab})^2}{\text{sh}^2 \frac{q_a - q_b}{2}} - \frac{1}{16} \sum_{a,b} \frac{(N_{ab})^2}{\text{ch}^2 \frac{q_a - q_b}{2}}.
\]
It is seen from (53), (54), (55), (57), (58), (59), (62), (63), (64) that the $M$-term describes some effective centrifugal repulsion of deformation invariants, whereas the $N$-term is a model of "centrifugal attraction" between $q^a$-"particles". This has nothing to do with any potential $V(q^1,\ldots,q^n)$; this attraction is due only to the negative contribution to the affine-affine kinetic energy (57). In this way an apparently "embarrassing" turns out to be just desirable.

The affine-metrical and metrical-affine models (58), (59) may be respectively written as follows:

$$T^{\text{aff-met}}_{\text{int}} = T^{\text{aff-aff}}_{\text{int}} + \frac{1}{2} \frac{(I^2 - A^2)}{S} \|V\|^2, \quad (66)$$

$$T^{\text{met-aff}}_{\text{int}} = T^{\text{aff-aff}}_{\text{int}} + \frac{1}{2} \frac{(I^2 - A^2)}{\|S\|^2}, \quad (67)$$

where $T^{\text{aff-aff}}_{\text{int}}$ here is just (57) but with $A$ replaced by $(I + A)$. Let us repeat the explicit formula for $T^{\text{aff-aff}}_{\text{int}}$ in terms of the two-polar parametrization and the shear-dilatation splitting:

$$T^{\text{aff-aff}}_{\text{int}} = \frac{1}{4(I + A)n} \sum_{a,b} (p_a - p_b)^2 + \frac{1}{32(I + A)} \sum_{a,b} \frac{(M_{ab})^2}{\text{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{32(I + A)} \sum_{a,b} \frac{(N_{ab})^2}{\text{ch}^2 \frac{q^a - q^b}{2}} + \frac{p^2}{2n(I + A + nB)} \quad (68)$$

It is interesting that for Hamiltonians of the form

$$H = T + V(q^1,\ldots,q^n),$$

with potentials depending on deformation invariants only, the models (66), (67), (68) give exactly the same evolution equations for the system of state variables:

$$(\ldots,q^a,\ldots; p_a,\ldots; M_{ab},\ldots; N_{ab},\ldots).$$

This follows from the basic Poisson brackets quoted above. The only distinction between these three model appear on the level of variables $L, R$, i.e., the principal axes of the Cauchy and Green deformation tensors. These degrees of freedom have compact topology, thus they do not influence anything concerning the bounded or unbounded character of motion.

It is roughly seen from (66), (67), (68) and may be rigorously shown that: the incompressible sector of our state variables admits an open family of bounded motions and an open family of unbounded ones even in the purely geodetic models (without potential). And this is interesting because invariant geodetic systems on Lie groups $(SL(n,\mathbb{R})$ this time) may be successfully analyzed in terms of the exponential mapping and special functions on groups. Obviously, the dilatational sector violates these nice features. Without potential energy the dilatational parameter $q$ moves uniformly in time and the total motion is
unbounded. The only bounded solutions $q = const$ are exponentially unstable on the level of physical $\varphi$-variables. Therefore, the “maximally geodetic” affinely-invariant systems have the form:

$$H = T + V(q),$$

where $T$ stabilizes dilatations. There is no interaction between dilatational and shear-rotational degrees of freedom. Dilatational parameter $q$ is subject to the one-dimensional dynamics ruled by the Hamiltonian

$$H_{\text{dil}} = \frac{p^2}{2n(I + A + nB)} + V(q).$$

The same is true in a more general situation when the potential energy depends also on the shear variables (non-geodetic models) and has the explicitly separated form

$$V(q^1, \ldots, q^n) = V_{\text{dil}}(q) + V_{\text{sh}}(\ldots, q^i - q^j, \ldots)$$

cf (60), (61); usually the effective models of $V_{\text{sh}}$ will have the binary structure (60).

Finally, let us quote the two-polar representation of the doubly-isotropic d’Alembert model:

$$T_{\text{int}} = \frac{1}{2I} \sum_a P_a^2 + \frac{1}{8I} \sum_{a,b} \frac{(M_{ab})^2}{(Q^a - Q^b)^2} + \frac{1}{8I} \sum_{a,b} \frac{(N_{ab})^2}{(Q^a - Q^b)^2}. \quad (69)$$

As mentioned earlier, on the purely geodetic level it would be completely non-physical. Here it is seen explicitly that $T_{\text{int}}$ is purely repulsive on the level of $Q$-variables. All realistic models should be based on some potential term,

$$H = T_{\text{int}} + V(Q^1, \ldots, Q^n)$$

The binary structure of $T_{\text{int}}$ resembles the Calogero-Moser lattices. And in fact, the general scattering solution of the Calogero-Moser chain is a subfamily of the general solution of the geodetic model (69).

2 General ideas of quantization

After all above classical preliminaries we can formulate the general ideas of quantization. We practically restrict ourselves to models based on Riemannian structures in the configuration space and on kinetic energies quadratic in velocities. Let us mention in connection with this the idea of Capriz [9] about kinetic energies of more general type, i.e., non-quadratic ones. Such models in fact appear in relativistic problems and may be useful in complicated problems of condensed matter theory, defects dynamics, etc. However, it may be very difficult to use them in quantization problems, because they may need the use
of pseudo-differential operators; this may be hopelessly difficult in curved configuration spaces. So, we remain within the traditional Schrödinger framework.

Let us assume that the classical problem is based (as above) on the Riemann structure \( \Gamma \), i.e., on the kinetic energy form

\[
T = \frac{1}{2} \Gamma_{\mu\nu}(q) \frac{dq^\mu}{dt} \frac{dq^\nu}{dt},
\]

or, if canonical language is used, on the kinetic Hamiltonian

\[
\mathcal{T} = \frac{1}{2} \Gamma^{\mu\nu}(q) p_\mu p_\nu.
\]

The Riemannian volume element is given by

\[
d\mu_\Gamma(q) = \sqrt{|\det \Gamma_{\mu\nu}|} dq^1 \cdots dq^f.
\]

Quantum-mechanical formulation is based on the Hilbert space \( L^2(Q, \mu_\Gamma) \) of \( \mathbb{C} \)-valued functions with the scalar product

\[
\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi}_1(q) \Psi_2(q) d\mu_\Gamma(q).
\]

Quantum operator of the kinetic energy is given by

\[
\mathbf{T} = -\hbar^2 \frac{1}{2} \Delta (\Gamma),
\]

where \( \hbar \) is the "crossed" Planck constant and \( \Delta (\Gamma) \) denotes the Laplace-Beltrami operator of \( \Gamma \):

\[
\Delta (\Gamma) = \frac{1}{\sqrt{|\Gamma|}} \sum_{\mu,\nu} \partial_\mu \left( \sqrt{|\Gamma|} \Gamma^{\mu\nu} \partial_\nu \right) = \Gamma^{\mu\nu} \nabla_\mu \nabla_\nu.
\]

Obviously, \( \nabla \) denotes the Levi-Civita covariant derivative induced by the \( \Gamma \)-metric. This means that the quantum kinetic energy is obtained from the classical one by the formal replacing of \( p_\mu \) in \( \mathcal{T} \) by the operator \( p_\mu = (\hbar/\imath) \nabla_\mu \). Parallel transports preserve \( \Gamma \) and \( \sqrt{|\Gamma|} \), thus, \( p_\mu \) is formally self-adjoint in \( L^2(Q, \mu_\Gamma) \). When the classical problem is non-geodetic and based on some potential \( V(q^1, \ldots, q^f) \), i.e., on the Hamiltonian \( H = \mathcal{T} + V \), then the corresponding quantum Hamiltonian is given by

\[
H = \mathbf{T} + \mathbf{V},
\]

where \( \mathbf{V} \) denotes the operator multiplying wave functions by the potential \( V \), i.e., \( \mathbf{V} \Psi = V \Psi \); usually we do not distinguish them graphically. Velocity-dependent generalized (magnetic) potentials are not considered here.

Strictly speaking, from the very principal point of view wave functions are not scalars but scalar densities of weight 1/2, and the squared moduli \( \overline{\Psi} \Psi \) are
scalar densities of weight one. But in all realistic models, and we do not go outside this scope, some Riemann structure is used and all tensor densities are factorized into tensors and standard densities built of $\Gamma$. In particular, $1/2$-densities $\Psi$ describing pure quantum states are factorizing as $\Psi = \Psi \sqrt{|\Gamma|}$, where $\Psi$ are just the aforementioned scalar wave functions.

Let us also mention that despite some current views, the one-valuedness of wave functions is nota fundamental assumption of quantum mechanics. There are at least some situations where multivalued amplitudes seem to be acceptable. First of all, it is so when the configuration space $Q$ is multiply-connected and has a finite homotopy group. Then it is natural to define the wave functions on the covering manifold $\overline{Q}$. They need not project onto $Q$ as one-valued amplitudes but it seems natural to demand that, according to the statistical interpretation, the squared moduli $\Psi\Psi$ are uniquely projectable. This has to do with the projective representation. And just such situations are interesting in our model, where the configuration spaces of rigid and affinely-rigid bodies in dimensions $n \geq 3$ have two-element homotopy groups. This point was stressed, e.g., in [2, 3, 4, 5, 43, 46], where the possibility of doubly-valued wave functions for quantized rigid body was pointed out.

Hilbert spaces may be constructed without calculating the complicate coordinate expressions for the metric tensors $\Gamma_{\mu\nu}$ and their densities $\sqrt{|\Gamma|}$. Namely, our configuration spaces may be in a sense identified with Lie groups (more precisely their group spaces), therefore we can simply use Haar measures, which are explicitly known and given by simple expressions. We are usually dealing with left- and right-invariant metrics $\Gamma$, thus, the corresponding measures $\mu_{\Gamma}$ are also invariant, and just coincide with the invariant Haar measures, because the latter ones are unique (modulo normalization).

First of all, let us observe that our configuration space $Q = M \times \text{LI}(U, V)$ as an affine space (with the translation space $V \times \text{LI}(U, V)$) is endowed with the natural Lebesgue measure a unique up to normalization. Fixing metric tensors $g \in V^* \otimes V^*$, $\eta \in U^* \otimes U^*$ and some adapted Cartesian coordinates (orthonormal with respect to these tensors) $x^i$, $a^K$, $\varphi^1_K$, we can normalize $l$ as follows:

$$da(x, \varphi) = dx^1 \cdots dx^n d\varphi^{1}_1 \cdots d\varphi^{n}_n.$$  

When translational degrees of freedom are neglected, we use the usual Lebesgue measure $l$ on $\text{LI}(U, V)$ as an open subset of the linear space $L(U, V)$:

$$dl(\varphi) = d\varphi^{1}_1 \cdots d\varphi^{n}_n.$$  

These measures are invariant under translations in the affine space $M \times \text{LI}(U, V)$ and under spatial and material isometries. They are, however, non-invariant under spatial and material affine transformations. To achieve the affine invariance
we must use the following Haar measures $\alpha, \lambda$ on $Q = M \times LI(U, V)$ and $Q_{\text{int}} = LI(U, V)$ induced from the affine group $\text{GAf}(n, \mathbb{R}) \simeq \text{GL}(n, \mathbb{R}) \times \mathbb{R}^n$ and the linear group $\text{GL}(n, \mathbb{R})$: 

$$
\begin{align*}
\alpha(\varphi, x) &= (\det \varphi)^{-n-1} d\alpha(x, \varphi) = (\det \varphi)^{-n-1} dx_1 \cdots dx_n d\varphi_1 \cdots d\varphi_n, \\
\lambda(\varphi) &= (\det \varphi)^{-n} d\lambda(\varphi) = (\det \varphi)^{-n} d\varphi_1 \cdots d\varphi_n.
\end{align*}
$$

Expressing the measure $l$ in terms of the two-polar decomposition $\varphi = LDR^{-1}$ we obtain 

$$
\begin{align*}
\lambda(\varphi) &= \lambda(L; q^a; R) = \prod_{i \neq j} \left| \text{sh} \left( q^i - q^j \right) \right| dq_1 \cdots dq_n d\mu(L) d\mu(R),
\end{align*}
$$

where $\mu$ is the left- and right-invariant Haar measure on the manifolds of linear isometries $LI(\mathbb{R}^n, \delta; V, g)$, $LI(\mathbb{R}^n, \delta; U, \eta)$. Obviously, when $LI(U, V)$ is identified with $\text{GL}(n, \mathbb{R})$ and the mentioned manifolds of isometries are identified with the orthogonal group $\text{SO}(n, \mathbb{R})$, then $\mu$ becomes simply the literally understood Haar measure on $\text{SO}(n, \mathbb{R})$. As the manifolds $LI$s and the group $\text{SO}(n, \mathbb{R})$ are compact, the measure $\mu$ may, although need not, be normalized to unity (the manifold volume equals the unity). In certain formulas it is convenient to use the symbol 

$$
P_\lambda := \prod_{i \neq j} \left| \text{sh} \left( q^i - q^j \right) \right|, \quad (71)
$$

thus, 

$$
\lambda(\varphi) = P_\lambda dq_1 \cdots dq_n d\mu(L) d\mu(R).
$$

One can also obtain the following convenient expression for the Lebesgue measure $l$:

$$
dl = P_\ell dq^1 \cdots dq^n d\mu(L) d\mu(R),
$$

where 

$$
P_\ell = \prod_{i \neq j} (Q^{i2} - Q^{j2}) = \prod_{i \neq j} (Q^i + Q^j) (Q^i - Q^j), \quad (72)
$$

and, as we remember, 

$$
Q^a = \exp \left( q^a \right).
$$

The Haar measure on the internal configuration space of the isochoric (incompressible) affinely-rigid body may be expressed in terms of the Dirac distribution 

$$
\lambda_{\text{SL}}(\varphi) = P_\lambda \left( q_1^1, \ldots, q^n^a \right) \delta \left( q^1 + \cdots + q^n \right) dq_1 \cdots dq_n d\mu(L) d\mu(R).
$$

Our quantum-mechanical models will be based on Hilbert spaces $L^2(Q, a)$, $L^2(Q_{\text{int}}, l)$, $L^2(Q, \alpha)$, and $L^2(Q_{\text{int}}, \lambda)$. Obviously, for affinely-invariant models $L^2(Q, \alpha)$, $L^2(Q_{\text{int}}, \lambda)$ are more convenient. Similarly, for the usual d’Alembert models $L^2(Q, a)$, $L^2(Q_{\text{int}}, l)$ are more natural. Nevertheless, it is a matter of convenience; one should stress that both types of models may be formulated in terms of any of these Hilbert spaces.
The spatial and material actions of $\text{GAf}(M)$ and $\text{GAf}(N)$ \((2), (3)\) on the configuration space $Q$ preserve the Haar measure $\alpha$. Similarly, \((4), (5)\) preserve the Haar measure $\lambda$ on the internal configuration space $Q^\text{int}$. On the other hand, except isometries, they do not preserve the usual Lebesgue measures on affine spaces, i.e., $a, l$. The latter ones are invariant, however, under the usual affine translations given analytically by

\[
(\ldots, x^i, \ldots, \varphi^i_A, \ldots) \mapsto (\ldots, x^i + \xi^i, \ldots, \varphi^i_A + \xi^i_A, \ldots)
\]

just the usual additive translations in $M \times L(U, V)$ and $L(U, V)$ as affine spaces. \(\text{Remark: on } Q \text{ and } Q^\text{int}, \text{ when } L(U, V) \text{ is replaced by its open subset } L^I(U, V), \text{ then these translations act only locally.}\) On the other hand, these additive translations in general do not preserve the Haar measures $\alpha, \lambda$.

All the mentioned groups act argument-wise on wave functions. When they preserve the measure on $Q$ or $Q^\text{int}$, the resulting transformations of wave functions preserve the corresponding $L^2$-spaces and are unitary, i.e., they preserve the scalar products too.

Let us quote explicitly some expressions, at least to fix the notation used later on.

For any $A \in \text{GAf}(M)$ we define the operation $A$ which transforms the wave function $\Psi : \text{AfI}(N, M) \to \mathbb{C}$ into such one that

\[(A \Psi)(\Phi) = \Psi(A \circ \Phi).\]  \(\text{(73)}\)

Similarly, for any $A \in \text{GAf}(N)$ we define the operator $B$ such that

\[(B \Psi)(\Phi) = \Psi(\Phi \circ B).\] \(\text{(74)}\)

If translational degrees of freedom are neglected and we deal with wave functions $\Psi : L^I(U, V) \to \mathbb{C}$, then for any $A \in \text{GL}(V)$, $B \in \text{GL}(U)$ we define

\[(A \Psi)(\varphi) = \Psi(A \varphi),\] \(\text{(75)}\)

\[(B \Psi)(\varphi) = \Psi(\varphi B).\] \(\text{(76)}\)

Obviously, for any $A, B$ the operators $A, B$ are unitary in $L^2(Q, \alpha), L^2(Q^\text{int}, \lambda)$, because the measures $\alpha, \lambda$ are invariant under regular translations. Unlike this, they are not unitary in $L^2(Q, \alpha), L^2(Q^\text{int}, \lambda)$, unless $A, B$ are volume-preserving mappings, i.e., elements of $\text{SAf}(M), \text{SAf}(N), \text{SL}(V), \text{SL}(U)$ (more precisely, unimodularity is sufficient, i.e., $\det L(A) = \det L(B) = \pm 1$). Obviously, the differential operators (vector fields) $H_a, E^i_{jk}$ defined in \((25), (23)\) are generators of unitary groups defined in \((73), (75)\), therefore, they formally anti-self-adjoint in $L^2(Q, \alpha), L^2(Q^\text{int}, \lambda)$, or rather in the subspaces of smooth functions. Being non-bounded (non-continuous) they are not anti-Hermitian in the rigorous mathematical sense. However, they are so in rough terms used in physics. They possess anti-Hermitian extensions. The following differential operators:

\[
P_a = \frac{\hbar}{i} H_a = \frac{\hbar}{i} \frac{\partial}{\partial x^a}, \quad \Sigma^a_{\ b} = \frac{\hbar}{i} E^a_{\ b} = \frac{\hbar}{i} \varphi^a_{\ K} \frac{\partial}{\partial \varphi^b_{\ K}}\]
are formally Hermitian. They are respectively quantum linear momentum and hyperspin operators. One can also introduce the operator of the total affine momentum (hypermomentum)

\[ J^a_{\ b} = x^a P_a + \Sigma^a_{\ b} = \Lambda^a_{\ b} + \Sigma^a_{\ b}. \]

The ordering of non-commuting operators meant just as written above; it follows from their geometric nature as group generators. Obviously, the coordinate operators are defined in the usual way,

\[ (x^a \Psi)(x, \varphi) = x^a \Psi(x, \varphi), \quad (\varphi^a_K \Psi)(x, \varphi) = \varphi^a_K \Psi(x, \varphi). \]

If we define quantum Poisson bracket in the usual way,

\[ \{ A, B \} := \frac{1}{i \hbar} [A, B] = \frac{1}{i \hbar} (AB - BA), \]

then the above basic quantities satisfy the rules identical with the classical ones.

The same concerns the co-moving representants based on differential operators \( \hat{H}_A, \hat{E}^A_{\ B} \). The corresponding formally Hermitian operators

\[ \hat{P}_A = \frac{\hbar}{i} \hat{H}_A = \frac{\hbar}{i} \varphi^a_A \frac{\partial}{\partial x^a}, \quad \hat{\Sigma}^A_{\ B} = \frac{\hbar}{i} \hat{E}^A_{\ B} = \frac{\hbar}{i} \varphi^a_B \frac{\partial}{\partial \varphi^a_A}, \]

and

\[ \hat{J}^{K}_{\ L} = a^K \hat{P}_L + \hat{\Sigma}^{K}_{\ L} = \hat{\Lambda}^{K}_{\ L} + \hat{\Sigma}^{K}_{\ L} \]

are quantum generators of \( \text{GAf}(N), \text{GL}(U) \).

Operators of angular momenta are given by the doubled g-skew-symmetric parts of affine momenta:

\[ S^a_{\ b} = \Sigma^a_{\ b} - g^{ac} g_{bd} \Sigma^d_{\ c}, \quad L^a_{\ b} = \Lambda^a_{\ b} - g^{ac} g_{bd} \Lambda^d_{\ c}, \]

and

\[ J^a_{\ b} = J^a_{\ b} - g^{ac} g_{bd} J^d_{\ c} = L^a_{\ b} + S^a_{\ b}. \]

They are respectively spin, orbital, and the total angular momentum. Similarly, the quantum vorticity operator is given by

\[ V^A_{\ B} = \hat{\Sigma}^A_{\ B} - \eta^A_{\ CD} \eta_{BD} \hat{\Sigma}^D_{\ C}. \]

Canonical linear momentum conjugate to \( \varphi^i_A \) is on the quantum level represented by the operator

\[ P^A_{\ i} = \frac{\hbar}{i} \frac{\partial}{\partial \varphi^i_A}. \]

Important: it is not formally Hermitian in \( L^2(Q, \alpha), L^2(Q_{\text{int}}, \lambda) \). Indeed, classically it generates additive translations in \( L(U, V) \):

\[ \varphi^i_A \rightarrow \varphi^i_A + \xi^i_A. \]
And those do not preserve the Haar measures \( \alpha, \lambda \). But they preserve the Lebesgue measures \( a, l \), therefore, \( \mathbf{P}^A_{i} \) is formally Hermitian in \( L^2(Q, a) \) and \( L^2(Q_{\text{int}}, l) \). Because of this the Hilbert spaces are more convenient for describing quantization of models based on the usual d’Alembert principle.

However, in all kinds of models the ”non-usual” Hilbert spaces may be also applied, simply the definition of some operators must be modified. For example, affine models may be as well quantized in \( L^2(Q, a) \) and \( L^2(Q_{\text{int}}, l) \), but the operators \( \mathbf{A}, \mathbf{B} \) in (73), (75) are to be replaced by \( \mathbf{A}', \mathbf{A}' \) given by

\[
\mathbf{A}' := \det L[A]^{-(n+1)/2} \mathbf{A}, \quad \mathbf{B}' := \det B^{-n/2} \mathbf{B}.
\]

More explicitly,

\[
(A' \Psi)(x, \varphi) = \det L[A]^{-(n+1)/2} \Psi(A(x), L[A] \varphi),
\]

\[
(B' \Psi)(x, \varphi) = \det B^{-n/2} \Psi(x, \varphi B).
\]

Similarly, when considering only the action of \( A \in \text{GL}(V) \) on internal degrees of freedom, we define

\[
A' := \det A^{-n/2} \mathbf{A},
\]

i.e.,

\[
(A' \Psi)(x, \varphi) = \det A^{-n/2} \Psi(x, A \varphi).
\]

Due to the multiplicative terms, the above operators are unitary in Hilbert spaces based on the Lebesgue measures. Their infinitesimal generators are then modified by additive correction terms due to which they become formally Hermitian in \( L^2(Q, a) \) and \( L^2(Q_{\text{int}}, l) \). For example, \( \Sigma^a_{b}, \Lambda^a_{b}, \hat{\Sigma}^A_{B} \) are respectively replaced by

\[
\Sigma'^a_{b} = \Sigma^a_{b} + \frac{\hbar n}{2i} \delta^a_{b}, \quad \Lambda'^a_{b} = \Lambda^a_{b} + \frac{\hbar}{2i} \delta^a_{b}, \quad \hat{\Sigma}'^A_{B} = \hat{\Sigma}^A_{B} + \frac{n}{2i} \delta^A_{B}.
\]

It is easy to see that the linear momentum \( \mathbf{P}_{a} \), spin \( \mathbf{S}^a_{b} \), vorticity \( \mathbf{V}^A_{B} \) remain unchanged. The finite actions generated by them preserve the Lebesgue measures.

Similarly, when quantizing the d’Alembert models with the use of non-usual for them \( L^2(Q, \alpha) \) and \( L^2(Q_{\text{int}}, \lambda) \), we would have to modify \( \mathbf{P}^A_{i} = (\hbar/i) \partial/\partial \varphi^i_{A} \), but we shall not do this here. Nevertheless, it must be stressed that quantizations in terms of ”non-usual” Hilbert spaces may be convenient when one is interested in comparison between various models.

As mentioned, the best tool when quantizing the d’Alembert model is offered by the geometry of Hilbert spaces \( L^2(Q, a) \) and \( L^2(Q_{\text{int}}, l) \). Then the quantized version of thee kinetic energy (33) is given by the operator

\[
\mathbf{T} = \mathbf{T}_{\text{tr}} + \mathbf{T}_{\text{int}} = \frac{1}{2m} g^{ij} p_i p_j + \frac{1}{2} J_{AB} \mathbf{P}^A_{i} \mathbf{P}^B_{j} g^{ij}.
\]

Explicitly, this is a kind of ”Laplace operator” in the \( n(n+1) \)-dimensional Euclidean space:

\[
\mathbf{T} = \frac{\hbar^2}{2m} g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} - \frac{\hbar^2}{2} J_{AB} g^{ij} \frac{\partial^2}{\partial \varphi^i_{A} \partial \varphi^j_{B}}.
\]
Unfortunately, geodetic (potential free) models are non-physical because they predict only escaping, non-bounded classical motion and the purely continuous positive spectrum after quantization (no bounded states). And for realistic potentials the variables \((x^i, \varphi^i_A)\) are rather non-adequate. The classical expressions (34), (35) are non-convenient for quantization because they suffer from the embarrassing problem of the ordering of operators.

There are no such problems with models based on the affine invariance, i.e., (39), (40). Let us remind that the first of them is affinely invariant in the physical space and isometries-invariant in the material space. On the contrary, the other one is isometries-invariant in the physical space and affinely invariant in the body. There are no ordering problems and the quantum operators of the kinetic energy may be immediately obtained via the simple replacement of the classical linear momentum and affine spin by the operators just written down. Therefore, for the quantized versions of (39), (40) we respectively obtain

\[
\mathbf{T} = \frac{1}{2m} \eta^{AB} p_A \hat{p}_B + \frac{1}{2} \tilde{\mathbf{L}}^{B A C D} \dot{\Sigma}^A B \dot{\Sigma}^C D,
\]

\[
\mathbf{T} = \frac{1}{2m} g^{ij} p_i \hat{p}_j + \frac{1}{2} \tilde{R}^{ij}_{k l} \dot{\Sigma}^i_j \dot{\Sigma}^k_l,
\]

where, as previously, \(\tilde{C}\) denotes the inverse Cauchy deformation tensor. The ordering of operators \(\varphi, \partial/\partial \varphi\) is essential, therefore, there appear first-order differential operators, respectively,

\[
-\frac{\hbar^2}{2} \tilde{L}^{A C \varphi^k B} \frac{\partial}{\partial \varphi^k C} = -\frac{\hbar i}{2} \tilde{L}^{A C \varphi^k B} \tilde{\Sigma}^C B,
\]

\[
-\frac{\hbar^2}{2 \tilde{R}^{ij}_{k l} \varphi^i_A \varphi^j_A} \frac{\partial}{\partial \varphi^k B} = -\frac{\hbar i}{2} \tilde{R}^{ij}_{k l} \Sigma^i j.
\]

The second-order terms are obvious:

\[
-\frac{\hbar^2}{2} \tilde{L}^{A C \varphi^k B} \varphi^j_D \frac{\partial^2}{\partial \varphi^k A \partial \varphi^j C}, \quad -\frac{\hbar^2}{2} \tilde{R}^{ij}_{k l} \varphi^i_A \varphi^j_A \varphi^k B \varphi^l B \frac{\partial^2}{\partial \varphi^j A \partial \varphi^l B}.
\]

There "curved" structure is obvious. Geometrically this is due to the fact that the metric tensors on \(Q\) given by (37), (38) define there essentially Riemannian structures with non-vanishing curvature tensors. Unlike this, the d'Alembert model is based on the evidently flat, Euclidean geometry with the metric tensor (44). All this has to do with strong nonlinearity encoded in geodetic terms of classical affine Hamiltonians. And this strong nonlinearity follows from the "large" group of assumed symmetries.

As mentioned in the classical part, there are good reasons to concentrate the attention on those metric tensors on \(Q = M \times LI(U, V)\) (those models of kinetic energy) which are:
1. affinely-invariant in the physical space and simultaneously isometry-invariant (homogeneous and isotropic) in the material space,

2. and conversely, homogeneous and isotropic in the physical space (isometry-invariant) and simultaneously affinely-invariant in the material space.

It is impossible to satisfy simultaneously both demands \(1\) and \(2\). However, if translational degrees of freedom are neglected, there exist metrics on \(Q_{\text{int}} = LI(U, V)\) affinely (or rather centro-affinely) invariant both in the space and in the body. They are always pseudo-Riemannian, i.e., have the non-definite hyperbolic signature. Obviously, \((78), (79)\) are formally Hermitian in \(L^2(Q, \alpha)\). Just like \((77)\) is so in \(L^2(Q, a)\).

The operators of translational kinetic energy \(T_{tr}\) are exactly like in general models \((78), (79)\), so we concentrate on the internal parts \(T_{\text{int}}\). And they are just the very special cases of those in the general formulas \((78), (79)\). Due to their very peculiar features it is instructive to quote them explicitly.

So, for internal degrees of freedom the quantized versions of \((49), (50)\) are obtained by the literal substitution of \(\hat{\Sigma}, \Sigma\) instead of their classical counterparts, so, respectively,

\[
T_{\text{int}} = \frac{1}{2I} \eta_{KL} \eta^{MN} \hat{\Sigma}^K M \hat{\Sigma}^L N + \frac{1}{2A} \hat{\Sigma}^K L \hat{\Sigma}^L K + \frac{1}{2B} \hat{\Sigma}^K K \hat{\Sigma}^L L, \quad (80)
\]

\[
T_{\text{int}} = \frac{1}{2I} g_{ik} g_{jl} \Sigma^i j \Sigma^k l + \frac{1}{2A} \Sigma^i k \Sigma^k i + \frac{1}{2B} \Sigma^i k \Sigma^k i \quad (81)
\]

with the same meaning of inertial constants \(\tilde{I}, \tilde{A}, \tilde{B}\) like previously, \((1)\). And again the second terms of both expressions are identical; the same is true of the third ones. Let us write explicitly

\[
\hat{\Sigma}^A B \hat{\Sigma}^B A = \Sigma^k i \Sigma^i k = -\hbar^2 \varphi^k B \varphi^l A \frac{\partial^2}{\partial \varphi^k A \partial \varphi^l B} - i\hbar \varphi^k A \frac{\partial}{\partial \varphi^k A}.
\]

The geometric interpretation of the affine spin, usual spin, and vorticity as generators of transformation groups implies that many quantum expressions involving them may be, as it was just seen, directly obtained from classical formulas by simple substitution of appropriate operators instead of the corresponding classical phase-space quantities, so that the difficult ordering problems are avoided.

For example, the very convenient classical expressions \((53), (54)\) remain valid on the operator level as alternative expressions respectively for \((80), (81)\) free of the inconvenient transposition term (one with \(\tilde{I}\)):

\[
T_{\text{int}} = \frac{1}{2\alpha} C(2) + \frac{1}{2\beta} C(1)^2 + \frac{1}{2\mu} \|V\|^2 \quad (82)
\]

(affine-metrical model), and

\[
T_{\text{int}} = \frac{1}{2\alpha} C(2) + \frac{1}{2\beta} C(1)^2 + \frac{1}{2\mu} \|S\|^2 \quad (83)
\]
(metrical-affine model), where $\alpha$, $\beta$, $\mu$ are the same constants as from classical formulas, and the operator Casimirs $C(k)$, $\|V\|^2$, $\|S\|^2$ are given by

$$C(1) = \hat{\Sigma}^A_A = \Sigma^a_a, \quad C(2) = \hat{\Sigma}^A_B \hat{\Sigma}^B_A = \Sigma^a_b \Sigma^b_a,$$

and analogously for $k > 2$ (till $k = n$), and

$$\|V\|^2 = -\frac{1}{2} V^A_B V^B_A, \quad \|S\|^2 = -\frac{1}{2} S^a_b S^b_a.$$

The more so the affine-affine model retains its structure when quantized (transformed to the operator form):

$$T_{\text{int}} = \frac{1}{2A} C(2) + \frac{1}{2A(n + A/B)} C(1)^2.$$  \hspace{1cm} (84)

And similarly, the splitting of the kinetic energy into incompressible and dilatational parts survives smoothly the quantization procedure. Decomposition of the affine spin into the spin-shear and dilatation parts has the following form:

$$\Sigma^a_b = s^a_b + \frac{1}{n} \mathbf{p} \delta^a_b, \quad \hat{\Sigma}^A_B = \hat{s}^A_B + \frac{1}{n} \mathbf{p} \delta^A_B,$$

where the trace-less parts are given by

$$s^a_b = \Sigma^a_b - \frac{1}{n} \Sigma^d_d \delta^a_b, \quad \hat{s}^A_B = \hat{\Sigma}^A_B - \frac{1}{n} \hat{\Sigma}^D_D \delta^A_B.$$

They are formally Hermitian operators generating the unitary actions of $\text{SL}(V)$, $\text{SL}(U)$ on $L^2(Q_{\text{int}}, \lambda)$ in the sense of (73), $A \in \text{SL}(V) : (A \Psi)(\varphi) = \Psi(A \varphi)$,

$$B \in \text{SL}(U) : (B \Psi)(\varphi) = \Psi(\varphi B).$$

Incidentally, they are also formally Hermitian in $L^2(Q_{\text{int}}, l)$, because the above actions of $\text{SL}(V)$, $\text{SL}(U)$ are there unitary.

The operator $\mathbf{p}$ has the following form:

$$\mathbf{p} = \Sigma^a_a = \hat{\Sigma}^A_A = \frac{\hbar}{i} \frac{\partial}{\partial q}.$$

It is formally Hermitian in $L^2(Q_{\text{int}}, l)$ (but not in $L^2(Q_{\text{int}}, l)$) and generates the one-parameter unitary group of dilatations. This is the group induced by the additive translations of logarithmic deformation invariants $q^a$, in particular, by the additive translations of their "centre of mass" $q$. This is the group which acts as follows:

$$\Lambda = e^{\xi} \in \mathbb{R}^+ : (\Lambda \Psi)(\varphi) = \Psi(\lambda \varphi), \quad q \mapsto q + \xi.$$

Let us denote the second-order Casimir operator for $\text{SL}(n, \mathbb{R})$ by $C_{\text{SL}(n)}(2)$:

$$C_{\text{SL}(n)}(2) = s^a_b s^b_a = \hat{s}^A_B \hat{s}^B_A.$$
and similarly for higher-order ones $C_{\text{SL}(n)}(k)$ (but of course $C_{\text{SL}(n)}(1) = 0$; for orthogonal groups all of the odd orders vanish).

On the quantized level the structure of affine-affine, affine-metrical, metrical-affine models (respectively, (57), (58), (59)) beautifully survives in operator language. Namely, one obtains, respectively,

\[
T_{\text{int}}^0 = \frac{1}{2A} C_{\text{SL}(n)}(2) + \frac{1}{2n(A + nB)} p^2 = T_{\text{sh}}^0 + T_{\text{dil}}^0,
\]

\[
T_{\text{int}} = \frac{1}{2(I + A)} C_{\text{SL}(n)}(2) + \frac{1}{2n(I + A + nB)} p^2 + \frac{I}{2(I^2 - A^2)} ||V||^2;
\]

\[
T_{\text{int}} = \frac{1}{2(I + A)} C_{\text{SL}(n)}(2) + \frac{1}{2n(I + A + nB)} p^2 + \frac{I}{2(I^2 - A^2)} ||S||^2.
\]

Obviously,

\[
p^2 = -\hbar^2 \frac{\partial^2}{\partial q^2}.
\]

All these formulas are automatically obtained from the corresponding classical expressions (57), (58), (59) by the formal substitution of operators instead of phase-space quantities. It is so because one deals here with generators of the underlying transformation groups, quantities of profound geometric interpretation.

Just as in the classical case the quantum unbounded dilatational motion should be stabilized by some potential $V(q)$ if the model is to describe quantum elastic vibrations. The Hamilton operator splits then into two independent mutually commuting parts

\[
H = H_{\text{sh}} + H_{\text{dil}}.
\]

The same is true for more general doubly isotropic potentials separating explicitly the shape and dilatation dynamics:

\[
V = V_{\text{sh}}(\ldots, q^i - q^j, \ldots) + V_{\text{dil}}(q).
\]

We have then

\[
H_{\text{sh}} = \frac{1}{2A} C_{\text{SL}(n)}(2) + V_{\text{sh}},
\]

\[
H_{\text{sh}} = \frac{1}{2(I + A)} C_{\text{SL}(n)}(2) + \frac{I}{2(I^2 - A^2)} ||V||^2 + V_{\text{sh}},
\]

\[
H_{\text{sh}} = \frac{1}{2(I + A)} C_{\text{SL}(n)}(2) + \frac{I}{2(I^2 - A^2)} ||S||^2 + V_{\text{sh}}
\]

respectively for the affine-affine, affine-metrical, and metrical-affine models. The first two of them reduce to the third one when we put $I = 0$.

And obviously

\[
H_{\text{dil}} = \frac{1}{2n(I + A + nB)} p^2 + V_{\text{dil}}(q) = -\frac{\hbar^2}{2n(I + A + nB)} \frac{\partial^2}{\partial q^2} + V_{\text{dil}}(q).
\]
\[ V_{\text{dil}} \] may be chosen as some qualitatively satisfactory phenomenological model, e.g., \( V_{\text{dil}} = (\kappa/2)q^2 \), the finite or infinite potential well, etc.

When \( V_{\text{sh}} = 0 \), we are dealing with purely geodetic affinely-invariant Hamiltonians built entirely of the group generators. In such situations one can expect solutions of the eigenproblem based completely on some purely algebraic ladder procedure.

Finally, it is interesting to express everything in terms of the two-polar decomposition. And now an unpleasant surprise, namely, the automatic replacing of classic quantities by seemingly natural operators does not work any longer. Namely, although \( p \) may be automatically substituted by \( p = (\hbar/i)\partial/\partial q \), it is not the case with \( p_a \), they are not replace by \( (\hbar/i)\partial/\partial q^a \) and \( \sum_a p_a^2 \) in \( (63) \) is not "quantized" to

\[ -\hbar^2 \Delta [q^a] = -\hbar^2 \sum_a \frac{\partial^2}{\partial q^a}. \]

The point is that the additive translations of logarithmic deformation invariants are not geometrically fundamental operations. So, whereas \( (52), (53), (54) \) are automatically \((-\hbar^2/2)\)-multiples of the corresponding Laplace-Beltrami operators, there is no such automatism with the two-polar expression of these operators. Fortunately, there are no problems with the spin and vorticity operators \( S_{ij}, V^{A B} \) and with operators \( \hat{r}_{ab}, \hat{t}_{ab} \) corresponding to the classical quantities \( (32) \). The reason is again their group-theoretic interpretation: spin and vorticity generate respectively spatial and material rotations. And the operators \( \hat{r}_{ab}, \hat{t}_{ab} \) are their representations in terms of the principal axes of the Cauchy and Green deformation tensors,

\[ \hat{r}_{ab} = L^a_i L^b_j S_{ij}, \quad \hat{t}_{ab} = -R^a_AR^{B}B V^{A B}; \]

the ordering of operators just as written here. Just as in the classical theory, \( \hat{r}_{ab}, \hat{t}_{ab} \) are generators (in the quantum-Poisson-bracket sense) of the right action of \( \text{SO}(n, \mathbb{R}) \) on the quantities \( L : \mathbb{R}^n \to V, \quad R : \mathbb{R}^n \to U \)

\[ L \mapsto LU, \quad R \mapsto RU, \quad U \in \text{SO}(n, \mathbb{R}). \]

Just as \( \hat{r}_{ab}, \hat{t}_{ab} \), the operators \( S_{ij}, V^{A B} \) act only on generalized coordinates \( x^\mu, y^\mu \) parameterizing respectively \( L \) and \( R \) (some Euler angles, rotation vectors, first-kind canonical coordinates, and so on). Any of the mentioned classical quantities \( S_{ij}, V^{A B}, \hat{r}_{ab}, \hat{t}_{ab} \) has the following form:

\[ f^\mu(x)p(x)_\mu, \quad g^\mu(y)p(y)_\mu, \]

where \( p(x)_\mu, p(y)_\mu \) are canonical momenta conjugate respectively to \( x^\mu, y^\mu \). Due to the group-theoretical meaning of the mentioned quantities, the corresponding quantum operators are given respectively by the operators

\[ \frac{\hbar}{i} f^\mu(x) \frac{\partial}{\partial x^\mu}, \quad \frac{\hbar}{i} g^\mu(y) \frac{\partial}{\partial y^\mu}; \]

the ordering just as explicitly written.
In analogy to (62) we introduce the operators

\[ M = -\hat{r} - \hat{t}, \quad N = \hat{r} - \hat{t}. \]

The kinetic energy operator for affine-affine model is then given as

\[
T_{\text{int}}^{\text{aff-aff}} = -\frac{\hbar^2}{2A} D_\lambda + \frac{\hbar^2 B}{2A(A + nB)} \frac{\partial^2}{\partial q^2} + \frac{1}{32A} \sum_{a,b} \frac{M^2_{ab}}{\text{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{32A} \sum_{a,b} \frac{N^2_{ab}}{\text{ch}^2 \frac{q^a - q^b}{2}},
\]

where

\[ D_\lambda = \frac{1}{P_\lambda} \sum_a \frac{\partial}{\partial q^a} P_\lambda \frac{\partial}{\partial q^a} = \sum_a \frac{\partial^2}{\partial q^a} + \sum_a \frac{\partial \ln P_\lambda}{\partial q^a} \frac{\partial}{\partial q^a}. \]

and \( P_\lambda \) is given by (71). The expression (85) exactly equals \(-\frac{\hbar^2}{2} \Delta (\Gamma^0)\), where the Laplace-Beltrami operator \( \Delta (\Gamma^0) \) is built of the configuration metric \( \Gamma^0 \), i.e., corresponds to the classical expression (43), (45). It is seen that \( D_\lambda \) differs from the \( R^n \)-Laplace operator \( \sum_a \frac{\partial^2}{\partial q^a} \) by some first-order differential operator. This is just the mentioned breakdown of the naive classical analogy between \( p_a \) and \((\hbar/i) \frac{\partial}{\partial q^a}\). The reason of this breakdown is that the additive translations \( q^a \mapsto q^a + u^a \) do not preserve the measures \( \lambda, \alpha \). Because of this their argument-wise action on wave functions is not unitary in \( L^2(Q, \alpha), L^2(Q_{\text{int}}, \lambda) \). Incidentally, it is not unitary in \( L^2(Q, a), L^2(Q_{\text{int}}, l) \) either. And infinitesimal generators \( (\hbar/i) \frac{\partial}{\partial q^a}, (\hbar/i) \frac{\partial}{\partial Q^a} \) are not formally self-adjoint in these Hilbert spaces.

The affine-metric and metric-affine models are respectively given by

\[
T_{\text{int}}^{\text{aff-met}} = -\frac{\hbar^2}{2\alpha} D_\lambda - \frac{\hbar^2}{2\beta} \frac{\partial^2}{\partial \eta^2} + \frac{1}{32\alpha} \sum_{a,b} \frac{M^2_{ab}}{\text{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{32\alpha} \sum_{a,b} \frac{N^2_{ab}}{\text{ch}^2 \frac{q^a - q^b}{2}} + \frac{1}{2\mu} \| V \|^2,
\]

\[
T_{\text{int}}^{\text{met-aff}} = -\frac{\hbar^2}{2\alpha} D_\lambda - \frac{\hbar^2}{2\beta} \frac{\partial^2}{\partial \eta^2} + \frac{1}{32\alpha} \sum_{a,b} \frac{M^2_{ab}}{\text{sh}^2 \frac{q^a - q^b}{2}} - \frac{1}{32\alpha} \sum_{a,b} \frac{N^2_{ab}}{\text{ch}^2 \frac{q^a - q^b}{2}} + \frac{1}{2\mu} \| S \|^2.
\]

Quite similarly, quantizing the doubly-isotropic d’Alembert model we obtain

\[
T_{\text{int}}^{d'A} = -\frac{\hbar^2}{2f} D_l + \frac{1}{8f} \sum_{a,b} \frac{M^2_{ab}}{(Q^a - Q^b)^2} + \frac{1}{8f} \sum_{a,b} \frac{N^2_{ab}}{(Q^a + Q^b)^2},
\]

54
where
\[ D_t = \frac{1}{P_l} \sum_a \frac{\partial}{\partial Q^a} P_l \frac{\partial}{\partial Q^a} = \sum_a \frac{\partial^2}{\partial Q^a} + \sum_a \frac{\partial \ln P_l}{\partial Q^a} \frac{\partial}{\partial Q^a}, \] (87)
and the weight factor \( P_l \) is given by (72).

The ordering of non-commuting operators \( D_\lambda, D_l \) is just as explicitly written here. There are no other ordering problems because the operators \( M_{ab}, N_{ab} \) (equivalently \( \hat{r}_{ab}, \hat{t}_{ab} \)) do commute with deformation invariants \( q^a, Q^a \).

Let us observe that the first-order differential operators in (86), (87) may be eliminated by introducing modified amplitudes \( \varphi \) given respectively by
\[ \varphi = \sqrt{P_\lambda} \Psi, \quad \varphi = \sqrt{P_l} \Psi. \] (88)

Then the action of \( D \)-operators on \( \Psi \) is represented by the action of operators \( \tilde{D} \) given respectively by
\[ -\frac{\hbar^2}{2A} \tilde{D}_\lambda = -\frac{\hbar^2}{2A} \sum_a \frac{\partial^2}{\partial q^a} + \tilde{V}_\lambda, \quad -\frac{\hbar^2}{2I} \tilde{D}_l = -\frac{\hbar^2}{2I} \sum_a \frac{\partial^2}{\partial Q^a} + \tilde{V}_l, \]
where \( \tilde{V}_\lambda, \tilde{V}_l \) are auxiliary "artificial" potentials
\[ \tilde{V}_\lambda = -\frac{\hbar}{2A P_\lambda} + \frac{\hbar^2}{4A P_\lambda} \sum_a \left( \frac{\partial P_\lambda}{\partial q^a} \right)^2, \]
\[ \tilde{V}_l = -\frac{\hbar}{2I P_l^2} + \frac{\hbar^2}{4I P_l} \sum_a \left( \frac{\partial P_l}{\partial Q^a} \right)^2. \]

In other words
\[ \tilde{D} \varphi = \sqrt{P} D \Psi. \]

All other terms of the kinetic energy operators commute with \( \sqrt{P} \) interpreted as a position-type operator. Obviously, the same concerns usual potential terms \( V \).

Therefore, finally, the Hamilton operator \( H = T + V \) is represented in \( \varphi \)-terms by \( \tilde{H} \),
\[ \tilde{H} \varphi = \sqrt{P} H \Psi, \]
where analytically the action of \( \tilde{H} \) differs from the action of \( H \) in that the \( \tilde{D} \)-operators are replaced by the usual \( \mathbb{R}^n \)-Laplace operators and additional \( \tilde{V} \)-potentials appear and are combined with the "true" potentials \( V \). The stationary Schrödinger equation, i.e., eigenequation
\[ H \Psi = E \Psi \]
is equivalent to
\[ \tilde{H} \varphi = E \varphi. \]

This eigenproblem is meant in the Hilbert space based on the modified scalar product without the weight factor \( P \) in the integration element,
\[ (\varphi_1 | \varphi_2) = \int \overline{\varphi_1} \varphi_2 dq^1 \cdots dq^n d\mu(L)d\mu(R) \]
in affine models, and
\[(\varphi_1|\varphi_2) = \int D_1\varphi_2dQ^1 \cdots dQ^n d\mu(L)d\mu(R)\]
in d’Alembert models. Obviously,
\[(\varphi_1|\varphi_2) = (\Psi_1|\Psi_2).\]

There is no real simplification in replacing \(\Psi\) by \(\varphi\) because instead of the complicated first-order differential operator the equally so complicated potential \(\tilde{V}\) appears.

In geodetic problems and in problems with the doubly isotropic potentials \(V(q^1, \ldots, q^n)\), in particular, with the stabilizing dilatation potentials \(V_{\text{dil}}(q)\), the above Schrödinger equations may be reduced to ones involving only coordinates \(q^1, \ldots, q^n\), because the action of \(H\) on the \((L, R)\)-dependence of wave functions may be algebraized. This is based on the generalized Fourier analysis on the compact group \(\text{SO}(n, \mathbb{R})\).

To simplify the treatment we identify analytically \(Q_{\text{int}}\) with \(\text{GL}^+(n, \mathbb{R})\) and use the matrix form of the two-polar decomposition \(\varphi = LDR^{-1}\). According to the Peter-Weyl theorem, the wave functions may be expanded in \((L, R)\)-variables with respect to matrix elements of irreducible unitary representations of \(\text{SO}(n, \mathbb{R})\). Their expansion coefficients are functions of deformation invariants \(q^a\), or equivalently, \(Q^a\). Let \(\Omega\) denote the set of irreducible unitary representations of \(\text{SO}(n, \mathbb{R})\) (more precisely, the set of their equivalence classes). Obviously, due to the compactness of the group \(\text{SO}(n, \mathbb{R})\) these representations are finite-dimensional; their dimensions will be denoted by \(N(\alpha)\). In the physical three-dimensional case \(\Omega\) is the set of all non-negative integers \(s = 0, 1, 2, \ldots\) and \(N(s) = 2s + 1\). If for some reasons we replace the rotation groups by their universal coverings \(\text{SO}(n, \mathbb{R})\) and so admit half-integer angular momenta, then \(\Omega\) is the set of all non-negative half-integers and integers \(s = 0, 1/2, 1, 3/2, \ldots\) and again \(N(s) = 2s + 1\). Obviously, in the planar case \(\Omega = \mathbb{Z}\) is the set of all integers and \(N(m) = 1\) for any \(m \in \mathbb{Z}\) (Abelian group).

Let \(D^\alpha\) be \(N(\alpha) \times N(\alpha)\) matrices of irreducible representations. Then the mentioned expansion has the following form:
\[
\Psi(\varphi) = \Psi(L, D, R) = \sum_{\alpha, \beta \in \Omega} \sum_{m,n=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} D^\alpha_{mn}(L)f^\alpha_{\beta kl}(D)D^\beta_{kl}(R^{-1}). \quad (89)
\]

The non-uniqueness of the polar decomposition implies that the deformation invariants \(q^1, \ldots, q^n\) \((Q^1, \ldots, Q^n)\) are very complicated indistinguishable parastatistical ”particles” in \(\mathbb{R}\). There is no place here to get into more details. The point is that the reduced amplitudes \(f^\alpha_{\beta kl}\) as functions of \(q^1, \ldots, q^n\) must satisfy certain conditions due to which the resulting \(\Psi\) as a function of \(L, D, R\) does not distinguish triplets \((L, D, R)\) representing the same configuration \(\varphi\), i.e., \(\Psi(L_1, D_1, R_1) = \Psi(L_2, D_2, R_2)\) if \(L_1D_1R_1^{-1} = L_2D_2R_2^{-1}\). This is simply the condition for \(\Psi\) to be a one-valued function on the configuration space \(Q_{\text{int}}\).
One can consider the matrix elements $D_{mk}^a$ as explicitly known. And in fact, they are deeply investigated special functions on the orthogonal groups $\text{SO}(n, \mathbb{R})$. In the physical case $n = 3$ they are well-known functions $D_{mk}^j$ found by Wigner. Here $j = 0, 1, 2, \ldots$ or, if we replace $\text{SO}(3, \mathbb{R})$ by its universal covering $\text{SU}(2)$ (for the general $n$, $\text{SO}(n, \mathbb{R})$ is replaced by the group $\text{Spin}(n)$), $j = 0, 1/2, 1, 3/2, \ldots$. And according to the standard convention $m = -j, -j + 1, \ldots, j - 1, j, k = -j, -j + 1, \ldots, j - 1, j$; for the fixed $j$, $m$, and $k$ have $(2j+1)$-element integer range with jumps by one both for the integer and half-integer $j$.

The operators $S^i_j$, $V^A_B$, $\tilde{r}^a_b$, $\tilde{t}^a_b$ when acting on functions $D_{mk}^j$ may be replaced by some standard algebraic operations. This enables one to reduce the Schrödinger equation for the wave functions $\Psi$ depending on $n^2$ variables $\varphi_A$ to some eigenproblems for the multi-component amplitudes $f^{\alpha\beta}$ depending only on the $n$ deformation invariants $q^n$. Therefore, in a sense, the problem may be reduced to the Cartan subgroup of diagonal matrices $\varphi$ (the maximal Abelian subgroup in $\text{GL}(n, \mathbb{R})$).

In geodetic models and in models with doubly isotropic potentials (ones depending only on deformation invariants; dilatations-stabilizing potentials $V(q)$ provide the simplest example), the labels $m, n$ in $\text{SO}(n)$ are good quantum numbers. The Hamilton operator $H$ commutes with the operators of spin and vorticity, i.e., $S^i_j$, $V^A_B$. Also the representation labels $\alpha, \beta \in \Omega$ are good quantum numbers. They are equivalent to the systems of eigenvalues of the Casimir invariants built of $S, V$:

$$C(S, p) = S^k_m S^m_k \cdots S^r_z S^z_r, \quad C(V, p) = V^K_m V^K_m V^R_Z V^Z_A$$

($p$ factors). These eigenvalues will be denoted respectively by $C^n(p)$, $C^0(p)$. Obviously,

$$C(S, p) = C(\tilde{r}, p), \quad C(V, p) = C(\tilde{t}, p) .$$

The above Casimir invariants vanish trivially for the odd values of $p$; so in the physical case $n = 3$ there is only one possibility: $C(S, 2)$, $C(V, 2)$. Due to the peculiarity of dimension three, where skew-symmetric tensors may be identified with axial vectors, it is more convenient to use

$$\|S\|^2 = \frac{1}{2} S^a_b S^b_a, \quad \|V\|^2 = \frac{1}{2} V^A_B V_B^A,$$

i.e., $(-1/2)$-multiples of $C(S, 2)$, $C(V, 2)$. The point is that for $n = 3$

$$\|S\|^2 = S^1_1 + S^2_2 + S^3_3, \quad \|V\|^2 = V^1_1 + V^2_2 + V^3_3,$$

where

$$S_a = \frac{1}{2} \varepsilon_{abc} S^b_c, \quad V_A = \frac{1}{2} \varepsilon_{ABC} V^B_C.$$

The raising and lowering of indices is meant here in the sense of orthonormal coordinates (Kronecker-delta trivial operation). The same convention is used for $\tilde{r}^a_b$, $\tilde{t}^a_b$, i.e.,

$$\tilde{r}_a = \frac{1}{2} \varepsilon_{abc} \tilde{r}^b_c, \quad \tilde{t}_a = \frac{1}{2} \varepsilon_{abc} \tilde{t}^b_c.$$
Obviously,
\[ ||\mathbf{F}||^2 = ||\mathbf{S}||^2, \quad ||\mathbf{t}||^2 = ||\mathbf{V}||^2. \]

The corresponding eigenvalues are given by
\[ C(s, 2) = \hbar^2 s(s + 1), \quad C(j, 2) = \hbar^2 j(j + 1), \]
where \( s, j \) are non-negative integers or non-negative integers and positive half-integers when \( \text{GL}^+(3, \mathbb{R}), \text{SL}(3, \mathbb{R}) \) are replaced by their coverings \( \text{GL}^+(3, \mathbb{R}), \text{SU}(2) \).

It is convenient to use multi-component wave functions with values in the space of complex \( N(\alpha) \times N(\beta) \) matrices \((2s + 1) \times (2j + 1)\) in the physical case \( n = 3 \):
\[ \Psi(\varphi) = \Psi^{\alpha\beta}(L, D, R) = \mathcal{D}^\alpha(L) f^{\alpha\beta}(D) \mathcal{D}^\beta(R), \tag{91} \]
where \( f^{\alpha\beta} \) are complex \( N(\alpha) \times N(\beta) \) matrices-reduced wave amplitudes depending only on the deformation invariants. In the physical three-dimensional case, when \( \mathcal{D}^\alpha_{mn} \) are Wigner special functions \( \mathcal{D}^s_{mn} \), we, as usual, take \( m, n \) running from \(-s\) to \(s\) and jumping by one (also in the "spinorial" case when \( s \) is half-integer). And then
\[ ||\mathbf{S}||^2 \Psi^{sj} = \hbar^2 s(s + 1) \Psi^{sj}, \quad ||\mathbf{V}||^2 \Psi^{sj} = \hbar^2 j(j + 1) \Psi^{sj}, \]
\[ \mathbf{S}_3 \Psi^{sj}_{ml} = \hbar m \Psi^{sj}_{ml}, \quad \mathbf{V}_3 \Psi^{sj}_{ml} = \hbar l \Psi^{sj}_{ml}. \]

And similarly, when the values \( n, k \) in the superposition (89) are kept fixed and we retain only the corresponding single term, for the resulting \( \Psi \) we have
\[ \hat{\mathbf{r}}_3 \Psi^{sj}_{nk} = \hbar n \Psi^{sj}_{nk}, \quad \hat{\mathbf{t}}_3 \Psi^{sj}_{nk} = \hbar k \Psi^{sj}_{nk}. \]

Let us now describe in a few words the afore-mentioned algebraization procedure in the sector of \((L, R)\)-degrees of freedom. If a compact group counterpart of the usual Fourier-transform algebraization, where \( \partial/\partial x^a \) is represented by the point-wise multiplication of the Fourier transform of \( f(\bar{x}), \hat{f}(\bar{k}) \) by \( ik^a \).

Let us introduce some auxiliary symbols.

The group \( \text{SO}(n, \mathbb{R}) \) may be parameterized by the first-kind canonical coordinates \( \omega \), namely,
\[ W(\omega) = \exp \left( \frac{1}{2} \omega_a E^b_a \right), \]
where the basic matrices of the Lie algebra \( \text{SO}(n, \mathbb{R})' \) are given by
\[ (E^b_a)_c^d = \delta^b_d \delta^c_a - \delta^b_c \delta^c_d, \]
and the matrix \( \omega \) is skew-symmetric in the "cosmetic" Kronecker sense. Therefore, independent coordinates may be chosen as \( \omega^a b, a < b \), or conversely. However, for the symmetry reasons it is more convenient to use the representation with the summation extended over all possible \( \omega^a b \).
To be more "sophisticated", the groups \( SO(V, g) \), \( SO(U, \eta) \) are parameterized as follows:

\[
W(\omega) = \exp \left( \frac{1}{2} \omega^{ij} E_{ij} \right), \quad W(\omega) = \exp \left( \frac{1}{2} \tilde{E}_{AB} \tilde{E}^{AB} \right),
\]

where \( E_{ij} \), \( \tilde{E}_{AB} \) are basic matrices of Lie algebras \( SO(V, g) \), \( SO(U, \eta) \) given by

\[
(\hat{E}^i_j)_l = \delta^i_l \delta^k_j - g^{ik} g_{jl}, \quad \left( \tilde{E}_{AB} \right)^C_D = \delta^A_D \delta^C_B - \eta^{AC} \eta_{BD}.
\]

The skew-symmetry of \( \omega \) in the above exponential formulas is meant respectively as follows:

\[
\omega^a_b = -g^{ac} g_{bd} \omega^d_c, \quad \hat{\omega}^A_B = -\eta^{AC} \eta_{BD} \hat{\omega}_C.
\]

Now matrices of irreducible representations \( D^\alpha \) are given by

\[
D^\alpha (L(l)) = \exp \left( \frac{1}{2} l^{ab} M^\alpha_{ab} \right), \quad D^\alpha (R(r)) = \exp \left( \frac{1}{2} r^{ab} M^\alpha_{ab} \right),
\]

where \( l \) and \( r \) denote the \( \omega \)-parameters, respectively, for the \( L \)- and \( R \)-factors of the two-polar decomposition. The anti-Hermitian matrices \( M^\alpha \) will be expressed by the Hermitian ones \( S^\alpha \),

\[
S^{\alpha a b} = \frac{i}{\hbar} M^{\alpha a b}.
\]

The commutation rules for \( M^{\alpha a b} \) are expressed through the structure constants of \( SO(n, \mathbb{R}) \),

\[
[M^{ab}_{cd}, M^{cd}_{ef}] = -g_{cd} M^{ae}_{bc} + g_{bc} M^{af}_{cd} - g_{bd} M^{af}_{ac} + g_{ac} M^{ae}_{bd},
\]

and therefore

\[
\frac{1}{i\hbar} [S^i_{ab}, S^j_{cd}] = g_{ad} S^j_{cb} - g_{cb} S^j_{ad} + g_{bd} S^j_{ac} - g_{ac} S^j_{bd}.
\]

Indices here are shifted with the help of \( g_{ab} \); as a rule we use orthonormal coordinates when \( g_{ab} = \delta_{ab} \).

In the physical three-dimensional case when we put

\[
S^i_{a} := \frac{1}{2} \varepsilon^{-1}_{abc} S^i_{bc},
\]

we obviously have

\[
\frac{1}{i\hbar} [S^i_{a}, S^j_{b}] = \varepsilon^{abc} S^j_{bc}.
\]

From the fact that \( D^\alpha \) are representations and \( (i/\hbar) \mathbf{S}^k_i \), \( (i/\hbar) \mathbf{V}^A_B \), \( (i/\hbar) \tilde{\mathbf{V}}^a_b \), \( (i/\hbar) \tilde{\mathbf{S}}^{ab}_b \) are infinitesimal generators of left and right orthogonal actions on the \( (L, R) \)-variables it follows immediately that

\[
\mathbf{S}^i_j \Psi^{\alpha \beta} = \mathbf{S}^{\alpha i} \Psi^{\alpha \beta}, \quad \tilde{\mathbf{V}}^a_b \Psi^{\alpha \beta} = D^\alpha (L) \mathbf{S}^{\alpha a} f^{\alpha \beta} (D) \mathbf{D}^\beta (R^{-1}),
\]

\[
\mathbf{V}^A_B \Psi^{\alpha \beta} = \Psi^{\alpha \beta} \mathbf{S}^{AB}, \quad \tilde{\mathbf{S}}^{ab}_b \Psi^{\alpha \beta} = D^\alpha (L) f^{\alpha \beta} (D) \mathbf{S}^{ab} \mathbf{D}^\beta (R^{-1}).
\]
Therefore, spin and vorticity act on the wave amplitudes $\Psi^{\alpha\beta}$ as a whole, and in a purely algebraic way. On the other hand, to describe in an algebraic way the action of $\tilde{r}^a_b$, $\tilde{t}^a_b$, one must extract from $\Psi^{\alpha\beta}$ the reduced amplitudes $f^{\alpha\beta}(q^1, \ldots, q^n)$. And it is only this amplitude that is affected by the action of $\tilde{r}^a_b$, $\tilde{t}^a_b$ according to the following rules:

$$\tilde{r}^a_b : \quad f^{\alpha\beta} \mapsto S^{\alpha a}_b f^{\alpha\beta}, \quad \tilde{t}^a_b : \quad f^{\alpha\beta} \mapsto f^{\alpha\beta} S^{\beta a}_b.$$  

It is very convenient to use the following notation:

$$S^{\alpha a}_b f^{\alpha\beta} := S^{\alpha a}_b f^{\alpha\beta}, \quad S^{\beta a}_b f^{\alpha\beta} := f^{\alpha\beta} S^{\beta a}_b.$$

As $D^\alpha$ are irreducible, the matrices

$$C(S^\alpha, p) := S^{\alpha a}_b S^{\alpha b}_c \cdots S^{\alpha u}_w S^{\alpha w}_a$$  

($p$ factors) are proportional to the $N(\alpha) \times N(\alpha)$ identity matrix,

$$C(S^\alpha, p) = C^\alpha(p) \mathbb{I}_{N(\alpha)},$$

where $C^\alpha(p)$ are eigenvalues of (90).

In particular, in the physical case $n = 3$ we have

$$\|S\|^2 \Psi^{s j} = \|\tilde{r}\|^2 \Psi^{s j} = \hbar^2 s(s + 1) \Psi^{s j}, \quad S^a \Psi^{s j} = S^a \Psi^{s j},$$

$$\|V\|^2 \Psi^{s j} = \|\tilde{t}\|^2 \Psi^{s j} = \hbar^2 j(j + 1) \Psi^{s j}, \quad V^A \Psi^{s j} = \Psi^{s j} S^A,$$

where $S^a$ are standard Wigner matrices of the angular momentum with the squared magnitude $\hbar^2 s(s + 1)$. Multiplying them by $(i/\hbar)$ we obtain standard bases of irreducible representations of the Lie algebra $SO(3, \mathbb{R})'$. For the standard Wigner representation the following is also true:

$$S^a_3 \Psi^{s j}_{m l} = \hbar m \Psi^{s j}_{m l}, \quad V_A \Psi^{s j}_{m l} = \hbar l \Psi^{s j}_{m l}.$$  

Similarly, the action of $\tilde{r}, \tilde{t}$ operators is represented by the following operations on the reduced amplitudes:

$$\tilde{r}_a : \quad f^{s j} \mapsto S^a \Psi^{s j} = S^a f^{s j}, \quad \tilde{t}_a : \quad f^{s j} \mapsto f^{s j} S^j_a = S^j_a f^{s j}.$$  

In particular,

$$\tilde{r}_3 : \quad \left[ f^{s j}_{m l} \right] \mapsto \left[ \hbar m f^{s j}_{m l} \right], \quad \tilde{t}_3 : \quad \left[ f^{s j}_{m l} \right] \mapsto \left[ \hbar l f^{s j}_{m l} \right].$$

Using the well-known orthogonality relations for the matrix elements of irreducible unitary representations $D^{\alpha mn}$ [31, 79] we can rewrite the scalar product in the following form:

$$\langle \Psi_1 | \Psi_2 \rangle = \sum_{\alpha, \beta \in \Omega} \frac{1}{N(\alpha)N(\beta)} \int \text{Tr} \left( f^{\alpha\beta+}_1 (q^1) f^{\alpha\beta}_2 (q^n) \right) P_\lambda dq^1 \cdots dq^n, \quad (92)$$

60
where \( P_\lambda \) is the weight factor given by (71), and the argument symbols like \( q^a \) are abbreviations for the system \((q^1, \ldots, q^n)\). The trace operation is meant in the sense of matrix two-indices:

\[
\text{Tr} \left( f_1^{\alpha\beta+} f_2^{\alpha\beta} \right) = \sum_{n,m=1}^{N(\alpha)} \sum_{k,l=1}^{N(\beta)} f_{n_1 m_1}^{\alpha\beta} f_{n_2 m_2}^{\alpha\beta}.
\]

If no superposition over \( m, l \) in (89) is performed and we use the matrix-valued wave functions (91), the trace operation is meant in the usual sense. Obviously, if we use the modified wave functions \( \varphi \) (88), then the scalar product expression is free of the weight factor \( P_\lambda \),

\[
(\varphi_1 | \varphi_2) = \sum_{\alpha, \beta} \frac{1}{N(\alpha)N(\beta)} \int \text{Tr} \left( g_1^{\alpha\beta+} (q^a) g_2^{\alpha\beta} (q^b) \right) dq^1 \cdots dq^n,
\]

where, obviously, \( g = \sqrt{P_\lambda} f \). Quite analogous formulas are true for the d’Alembert models; simply \( P_\lambda \) is replaced then by \( P_l \) (72) and instead \( q^a \) we use their exponential functions \( Q^a \),

\[
\langle \Psi_1 | \Psi_2 \rangle = \sum_{\alpha, \beta} \frac{1}{N(\alpha)N(\beta)} \int \text{Tr} \left( f_1^{\alpha\beta+} (Q^a) f_2^{\alpha\beta} (Q^b) \right) P_l dQ^1 \cdots dQ^n,
\]

\[
(\varphi_1 | \varphi_2) = \sum_{\alpha, \beta} \frac{1}{N(\alpha)N(\beta)} \int \text{Tr} \left( g_1^{\alpha\beta+} (Q^a) g_2^{\alpha\beta} (Q^b) \right) dQ^1 \cdots dQ^n.
\]

Remark: it is implicit assumed in the above formulas that the Haar measure on the \((L, R)\)-manifolds is normalized to unity (the total "volume" of the corresponding manifolds equals one).

We restrict ourselves to Hamiltonians of the form \( H = T + V \) with some doubly-isotropic potentials \( V(q^1, \ldots, q^n) \), in particular, with some dilatation-stabilizing potentials \( V(q) \) (affinely-invariant geodetic incompressible models). The energy eigenproblem, i.e., stationary Schrödinger equation

\[
H \Psi = E \Psi
\]

is equivalent to the infinite sequence of eigenequations for the reduced multi-component amplitudes \( f^{\alpha\beta} \):

\[
H^{\alpha\beta} \Psi^{\alpha\beta} = E^{\alpha\beta} \Psi^{\alpha\beta}.
\]

The simultaneous spatial and material isotropy imply in the \( N(\alpha) \times N(\beta) \)-fold degeneracy, i.e., for every component of the \( N(\alpha) \times N(\beta) \)-matrix amplitude \( f^{\alpha\beta} \) there exists an \( N(\alpha) \times N(\beta) \)-dimensional subspace of solutions, just as seen from the symbol \( f_{\alpha\beta}^{\alpha\beta} \) used in (89).

The reduced Hamiltonians

\[
H^{\alpha\beta} = T^{\alpha\beta} + V
\]

61
are $N(\alpha) \times N(\beta)$ matrices built of second-order differential operators.

For the affine-affine model of the kinetic energy we have

$$T^{\alpha \beta} f_{\alpha \beta} = -\frac{\hbar^2}{2A} D f^{\alpha \beta} + \frac{1}{32A} \sum_{a,b} \frac{(S^{3a}_{ab} - S^{3a}_{ba})^2}{\text{sh}^2 \frac{q^a - q^b}{2}} f^{\alpha \beta}$$

$$- \frac{1}{32A} \sum_{a,b} \frac{(S^{3a}_{ab} + S^{3a}_{ba})^2}{\text{ch}^2 \frac{q^a - q^b}{2}} f^{\alpha \beta} + \frac{\hbar^2}{2A(A + nB)} \partial^2 f^{\alpha \beta}.$$ (93)

For the spatially metrical and materially affine model we obtain

$$T^{\alpha \beta} f_{\alpha \beta} = -\frac{\hbar^2}{2\mu} D f^{\alpha \beta} + \frac{1}{2\mu} C(\alpha, 2) f^{\alpha \beta} + \frac{1}{32\alpha} \sum_{a,b} \frac{(S^{3a}_{ab} - S^{3a}_{ba})^2}{\text{sh}^2 \frac{q^a - q^b}{2}} f^{\alpha \beta}$$

$$- \frac{1}{32\alpha} \sum_{a,b} \frac{(S^{3a}_{ab} + S^{3a}_{ba})^2}{\text{ch}^2 \frac{q^a - q^b}{2}} f^{\alpha \beta} - \frac{\hbar^2}{2\beta} \partial^2 f^{\alpha \beta},$$ (94)

where $C(\alpha, 2)$ is the $\alpha$-th eigenvalue of the rotational Casimir $\|S\|^2$, thus,

$$-\frac{1}{2} S_{\alpha i} S^{\alpha j} = C(\alpha, 2) \mathbb{I}_{N(\alpha)}.$$

Obviously, for the physical dimension $n = 3$, $f^{\alpha \beta} = f^{a^2}$, we have $C(s, 2) = \hbar^2 s (s + 1)$. And similarly for the spatially affine and materially metrical model we have

$$T^{\alpha \beta} f_{\alpha \beta} = -\frac{\hbar^2}{2\alpha} D f^{\alpha \beta} + \frac{1}{2\mu} C(\beta, 2) f^{\alpha \beta} + \frac{1}{32\alpha} \sum_{a,b} \frac{(S^{3a}_{ab} - S^{3a}_{ba})^2}{\text{sh}^2 \frac{q^a - q^b}{2}} f^{\alpha \beta}$$

$$- \frac{1}{32\alpha} \sum_{a,b} \frac{(S^{3a}_{ab} + S^{3a}_{ba})^2}{\text{ch}^2 \frac{q^a - q^b}{2}} f^{\alpha \beta} - \frac{\hbar^2}{2\beta} \partial^2 f^{\alpha \beta},$$ (95)

where $C(\beta, 2)$ appears as the $\beta$-th eigenvalue of the vorticity Casimir $\|V\|^2$, and just as previously for $n = 3$, $f^{\alpha \beta} = f^{a^2}$, we have $C(j, 2) = \hbar^2 j (j + 1)$. It is so as if the doubly affine background ($T$ affinely-invariant in the physical and material space) was responsible for some fundamental part of the spectra, perturbated by some internal rotations of the body itself or of the deformation axes. This perturbation and the resulting splitting of energy levels becomes remarkable when $\mu$ is small, i.e., when the inertial constants $I, A$ differ slightly.

The suggestive terms

$$\frac{\hbar^2}{2\mu} s(s + 1), \quad \frac{\hbar^2}{2\mu} j(j + 1)$$
as contributions to energy levels are very interesting and seem to be supported by experimental data in various ranges of physical phenomena.

Finally, let us quote the corresponding form of $T^{\alpha \beta}$ for the quantized d’Alembert model:

$$T^{\alpha \beta} f_{\alpha \beta} = -\frac{\hbar^2}{2I} D_{\alpha \beta \gamma} f_{\alpha \beta \gamma} + \frac{1}{8I} \sum_{a,b} \left( S^{\alpha \beta}_{a b} - S^{\alpha \beta}_{a b} \right)^2 f_{\alpha \beta}$$

$$+ \frac{1}{8I} \sum_{a,b} \left( S^{\alpha \beta}_{a b} + S^{\alpha \beta}_{a b} \right)^2 f_{\alpha \beta}.$$  

In this way the problem has been successfully reduced from $n^2$ internal degrees of freedom (physically 9, sometimes 4) to the $n$ purely deformative degrees of freedom (physically 3, sometimes 2). The price one pays for that is the use of multi-component wave functions subject to the strange parastatistical conditions in the reduced $q^a$-variables. The particular values of labels $\alpha, \beta$ and the corresponding matrices $S^{\alpha}_{a b}, S^{\beta}_{a b}$ describe the influence of quantized rotational degrees of freedom on the quantized dynamics of deformation invariants. It is interesting that on the classical level there is no simple way to perform such a dynamical reduction to deformation invariants.

For any reduced problem with $\alpha, \beta$ labels the quantity

$$\rho(q^i) := \text{Tr} \left( f_{1}^{\alpha \beta} f_{2}^{\alpha \beta} \right) P_{\lambda}(q^c)$$

is the probability density for finding the object in the state of deformation invariants $(q^1, \ldots, q^n)$. More precisely, $\rho(q^1, \ldots, q^n) dq^1 \cdots dq^n$ is the probability that the values of deformation invariants will be detected in the infinitesimal range $dq^1 \cdots dq^n$ about the values $(q^1, \ldots, q^n)$.

Similarly, performing the integration

$$\rho(L, R) = \int \Psi(L; q^a; R) \Psi(L; q^a; R) P_{\lambda}(q) dq^1 \cdots dq^n$$

one obtains the probability density for detecting the "gyroscopic" degrees of freedom $L, R$ (equivalently, the Cauchy and Green deformation tensors) in some range of the configuration space. Obviously, this distribution is meant in the sense of the Haar measure $\mu$. The integrals

$$p^{\alpha \beta}_{m l} = \int f_{\alpha \beta}^{n k} (q^c) f_{n k}^{\alpha \beta} (q^c) P_{\lambda}(q) dq^1 \cdots dq^n$$

are probabilities of detecting the particular indicated values of angular momenta and vorticities, $C(\alpha, 2), C(\beta, 2), h m, h l, h n, h k$. In the physical three-dimensional case they are respectively $\hbar^2 s(s + 1), \hbar^2 j(j + 1), h m, h l, h n, h k$, where $s, j$ are non-negative integers, $m, n = -s, \ldots, s, k, l = -j, \ldots, j$, jumping by one. Particularly interesting are

$$p^{\alpha \beta}_{m l} = \sum_{n, k} p^{\alpha \beta}_{n k}$$

63
because they refer to the constants of motion $S^j_a$, $V^A_B$ and to "good" quantum numbers $\alpha, \beta, m, l$. Except the special case $n = 2$, $\hat{r}^a_b$, $\hat{t}^a_b$ are not constants of motion, and $k, l$ are not "good" quantum numbers. Quite analogous statements are true for the quantized d’Alembert model; the only formal difference is that the integration element in the manifold of invariants is given by $P_l(Q^1, \ldots, Q^n)dQ^1 \cdots dQ^n$.

Let us now write down the explicit formulas for the physical three-dimensional case. For the affine-affine model (93) we have now

$$T_{\text{aff-aff}}^{ij} f^{sj} = - \frac{\hbar^2}{2A} D_{\lambda} f^{sj} + \frac{\hbar^2 B}{2A(A + 3B)} \frac{\partial^2}{\partial q^2} f^{sj} + \frac{1}{16A} \sum_{a=1}^{3} \frac{(S^a_s)^2 f^{sj} - 2S^a_s f^{sj} S^j_a + f^{sj}(S^j_a)^2}{\text{sh}^2 q^a - q^a} - \frac{1}{16A} \sum_{a=1}^{3} \frac{(S^a_s)^2 f^{sj} + 2S^a_s f^{sj} S^j_a + f^{sj}(S^j_a)^2}{\text{ch}^2 q^a - q^a}.$$  \quad (96)

where $S^j_a$ are the standard Wigner matrices for $j$-angular momentum, i.e., $\hbar^2 j(j + 1)$-magnitude, and for any $a$-th term of both summation we have obviously $b \neq a, c \neq a, b \neq c$. Obviously, it does not matter in what an ordering $q^b, q^c$ are written, because the denominators are sign-non-sensitive. $D_\lambda$ is given by (80), where explicitly

$$P_l = \left| \text{sh} (q^2 - q^3) \text{sh} (q^3 - q^1) \text{sh} (q^1 - q^2) \right|.$$  \quad (80)

And similarly, using the abbreviated form, we can write for the metrical-affine (94) and affine-metrical (95) models, respectively, as follows:

$$T_{\text{met-aff}}^{ij} = T_{\text{aff-aff}}^{ij} [A \mapsto I + A] + \frac{I}{2(I^2 - A^2)} \hbar^2 s(s + 1),$$

$$T_{\text{aff-met}}^{ij} = T_{\text{aff-aff}}^{ij} [A \mapsto I + A] + \frac{I}{2(I^2 - A^2)} \hbar^2 j(j + 1),$$

where, obviously, $T_{\text{aff-aff}}^{ij} [A \mapsto I + A]$ is obtained from $T_{\text{aff-aff}}^{ij}$ (96) simply by replacing $A$ with $\alpha = I + A$.

The doubly-isotropic d’Alembert model in three dimensions has the following form:

$$T_{\text{aff-aff}}^{ij} f^{sj} = - \frac{\hbar^2}{2I} D_I f^{sj} + \frac{1}{4I} \sum_{a=1}^{3} \frac{(S^a_s)^2 f^{sj} - 2S^a_s f^{sj} S^j_a + f^{sj}(S^j_a)^2}{(Q^b - Q^a)^2} + \frac{1}{4I} \sum_{a=1}^{3} \frac{(S^a_s)^2 f^{sj} + 2S^a_s f^{sj} S^j_a + f^{sj}(S^j_a)^2}{(Q^b + Q^a)^2}.$$  \quad (97)

with the same convention as previously, $D_I$ given by (87), and explicitly

$$P_l = \left| \left( (Q^2)^2 - (Q^3)^2 \right) \left( (Q^3)^2 - (Q^1)^2 \right) \left( (Q^1)^2 - (Q^2)^2 \right) \right|.$$  \quad (87)

64
Let us mention that in principle half-integer angular momentum of extended objects may be formally introduced by replacing the group $GL(3, \mathbb{R})$ by its universal covering $\tilde{GL}(3, \mathbb{R})$. There are some indications that the physical usefulness of such models is not excluded. Formally, the procedure is as follows. In (89) specialized to $n = 3$ we replace the group $SO(3, \mathbb{R})$ by its covering $SU(2)$ and write the following expression involving the known Wigner matrices $D_{smn}$:

$$\Psi(u, D, v) = \sum_{s,j} \sum_{m,n=-s} D_{smn}(u) f_{mn}^{s,j}(D) D^{j kl}(v^{-1}), \quad (98)$$

where $u, v \in SU(2)$, $D \in \text{Diag}(3, \mathbb{R}) \subset GL(3, \mathbb{R})$, and both the integer and half-integer values of $s, j$ are admissible. However, if the function of triples $(u, D, v)$ is to represent a function on $GL^+(3, \mathbb{R})$, then the values of $s, j$ in the above series must have the same "halfness", i.e., either both $s, j$ in (98) are integers or both are non-integers. And no superposition between elements of these two function spaces is admitted (a kind of superselection rule). The point is that for such "halfness-mixing" superpositions the squared modulus $\Psi^2$ would be two-valued from the point of view of $SO(3, \mathbb{R})$. This would be violation of the probabilistic interpretation of $\Psi$ in $GL^+(3, \mathbb{R})$. If there is no mixing, then in the case of superposing over half-integer $s, j$ in (98) the resulting $\Psi$ is two-valued on $GL^+(3, \mathbb{R})$, i.e., it does not project from $GL^+(3, \mathbb{R})$ to $GL^+(3, \mathbb{R})$ but $\Psi^2$ does project, i.e., it is single-valued in $GL^+(3, \mathbb{R})$.

The simplest possible situation in (96), (97) is $s = j = 0$, i.e., purely scalar amplitude $f_{00}$. Then $T_{00}$ reduces respectively to

$$T_{00} = -\frac{\hbar^2}{2 A} D_\lambda + \frac{\hbar^2 B}{2 A(A + 3 B)} \frac{\partial^2}{\partial q^2}, \quad T^2_{00} = -\frac{\hbar^2}{2 I} D_l,$$

i.e., there is no direct contribution from internal degrees of freedom.

If we admit half-integers, then the next simple situation is $s = j = 1/2$. Then $S_{1/2 a} = (\hbar/2) \sigma_a$, where $\sigma_a$ are Pauli matrices. Therefore, $(S_{1/2 a}^2) = (\hbar^2/4) I_2$.

Finally, let us briefly describe the two-dimensional situation, i.e., "Flatland" $\Pi$, $n = 2$. Obviously, it may have some direct physical applications when we deal with flat molecules or other structural elements. But besides, the two-dimensional models shed some light on the general situation and enable one to make it more comprehensible and lucid. Indeed, let us observe that the expressions (65) and (68) (without the last $p^2$-term) are superpositions of two-dimensional clusters corresponding to all possible $\mathbb{R}^2$-subspaces in $\mathbb{R}^n$. Obviously, these terms in general are non-disjoint and for $n = 3$ they simply cannot be disjoint (all two-dimensional linear subspaces in $\mathbb{R}^3$ have intersections of dimension higher than null; if different, they always intersect along one-dimensional linear subspaces).

There are some very exceptional features of the dimension $n = 2$. They are very peculiar, in a sense pathological. But nevertheless the resulting simplifications generate some ideas and hypotheses concerning the general dimension. Of
course, later on they must be verified on the independent basis. Let us begin with the classical description.

The one-dimensional group of planar rotations $\text{SO}(2, \mathbb{R})$ is Abelian, therefore, $\hat{\rho} = \rho = S, \hat{\tau} = \tau = -V$. In doubly-isotropic models $S$ and $V$ are constants of motion and so are $\hat{\rho}, \hat{\tau}, M, N$ if $n = 2$. It is not the case for $n > 2$, where, as always in isotropic models, $S, V$ are constants of motion but $\hat{\rho}, \hat{\tau}$ do not equal $S, -V$ and are non-constant. But it is exactly the use of $\hat{\rho}, \hat{\tau}$ and their combinations $M, N$ that simplifies the problem and leads to a partial separation of variables. In two-dimensional space these things coincide and the problem may be effectively reduced to the dynamics of two-deformation invariants both on the classical and quantum level. The two-polar decomposition $\varphi = LDR^{-1}$ will be parameterized in a standard way; using the matrix language we have

$$L = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}, \quad R = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix},$$

$$D = \begin{bmatrix} Q^1 & 0 \\ 0 & Q^2 \end{bmatrix} = \begin{bmatrix} \exp q^1 & 0 \\ 0 & \exp q^2 \end{bmatrix}.$$ 

To separate the dilatational and incompressible motion we introduce new variables:

$$q = \frac{1}{2} (q^1 + q^2), \quad x = q^2 - q^1.$$ 

Their conjugate momenta are given by

$$p = p_1 + p_2, \quad p_x = \frac{1}{2} (p_2 - p_1).$$

Angular velocities are given by the following matrices:

$$\chi = \frac{dL}{dt} L^{-1} = L^{-1} \frac{dL}{dt} = \hat{\chi} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\vartheta = \frac{dR}{dt} R^{-1} = R^{-1} \frac{dR}{dt} = \hat{\vartheta} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ 

Spin and vorticity essentially coincide with canonical conjugate momenta $p_\alpha, p_\beta$, i.e.,

$$S = \rho = \hat{\rho} = p_\alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad V = -\tau = -\hat{\tau} = p_\beta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$ 

With this convention the pairing between velocities and momenta has the form:

$$p_\alpha \frac{d \alpha}{dt} = \frac{1}{2} \text{Tr}(S \chi), \quad p_\beta \frac{d \beta}{dt} = \frac{1}{2} \text{Tr}(V \vartheta).$$

The diagonalizing quantities $M = -\hat{\rho} - \hat{\tau}, N = \hat{\rho} - \hat{\tau}$ are also expressed by matrices

$$M = m \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad N = n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

66
where

\[ m = p_\beta - p_\alpha, \quad n = p_\beta + p_\alpha. \]

In some formulas it is convenient to use modified variables

\[ \gamma = \frac{1}{2}(\beta - \alpha), \quad \delta = \frac{1}{2}(\beta + \alpha). \]

Their conjugate momenta just coincide with the above \( m, n \), i.e.,

\[ p_\gamma = m, \quad p_\delta = n. \]

The magnitudes of \( S, V \) have the form:

\[ \| S \| = |p_\alpha| = \frac{1}{2}|n - m|, \quad \| V \| = |p_\beta| = \frac{1}{2}|n + m|. \]

As mentioned, \( p_\alpha, p_\beta, m, n \) are constants of motion because in doubly isotropic models \( \alpha, \beta \) are cyclic variables. The corresponding affine-affine, metrical-affine, and affine-metrical kinetic energies of internal degrees of freedom are respectively given by

\[
T_{\text{aff-aff}} = \frac{1}{2A} \left( p_1^2 + p_2^2 \right) - \frac{B}{2A(A + 2B)} p^2 + \frac{1}{16A \text{sh}^2 \frac{x}{2}} m^2 - \frac{1}{16A \text{ch}^2 \frac{x}{2}} n^2,
\]

\[
T_{\text{met-aff}} = T_{\text{aff-aff}} [A \mapsto I + A] + \frac{I}{8(I^2 - A^2)} (n - m)^2,
\]

\[
T_{\text{aff-met}} = T_{\text{aff-aff}} [A \mapsto I + A] + \frac{I}{8(I^2 - A^2)} (n + m)^2,
\]

where, as usual, \( T_{\text{aff-aff}} [A \mapsto I + A] \) denotes \( T_{\text{aff-aff}} \) with \( A \) replaced by \( I + A \).

Separating dilatational and incompressible motion we obtain respectively the following expressions:

\[
T_{\text{int-aff}} = \frac{p^2}{4(A + 2B)} + \frac{p_x^2}{A} + \frac{(p_\alpha - p_\beta)^2}{16A \text{sh}^2 \frac{x}{2}} - \frac{(p_\alpha + p_\beta)^2}{16A \text{ch}^2 \frac{x}{2}},
\]

\[
T_{\text{int-met}} = T_{\text{int-aff}} [A \mapsto I + A] + \frac{I p_\alpha^2}{I^2 - A^2},
\]

\[
T_{\text{int-met}} = T_{\text{aff-aff}} [A \mapsto I + A] + \frac{I p_\beta^2}{I^2 - A^2}.
\]

Canonical momenta \( p_\alpha, p_\beta \), or equivalently, \( m, n \), are constants of motion and their Poisson brackets with the variables \( q, x, p, p_x \). Therefore, if we are interested only in the evolution of variables \( q, x \) but not in that of \( \alpha, \beta \), we can simply replace \( p_\alpha, p_\beta, m, n \) in the above expressions by constants characterizing a given family of solutions. The are effective coupling constants for the interaction between deformation invariants \( q^1, q^2 \). The sh\(^{-2}(x/2)\)-term
controlled by \( m \) is always repulsive and singular at the coincidence \( x = 0 \) (non-deformed shape but dilatation admitted). The \( \chi^{-2}(x/2) \)-term controlled by \( n \) is attractive and finite at \( x = 0 \). At large "distances" of deformation invariants, \( |x| \to \infty \), attraction prevails if and only if \( |m| > |n| \), i.e., if \( p_\alpha, p_\beta \) have the same signs, \( p_\alpha p_\beta > 0 \). If \( |m| < |n| \), i.e., \( p_\alpha p_\beta < 0 \), then the time evolution of \( x \) is unbounded. This is just the very special \( (n = 2) \) example of that was said formerly, namely that in the incompressible and affinely-invariant geodetic regime there exists an open family of bounded motions ("elastic vibrations") and an open family of unbounded motions ("dissociation", decay). If the total deformative motion is to be bounded, then some dilatations-stabilizing potential \( V(q) \) must be included into Hamiltonian. But even if there is no \( x \)-dependent potential, our affine geodetic model in the non-compact configuration space of incompressible motion may encode bounded elastic vibrations. The same is true for \( n > 2 \), however the situation is more complicated then because \( M_{ab}, N_{ab} \) are not constants of motion and also undergo some vibrations.

Analogous statements are true on the quantum level. The Haar measure in our coordinates is given by

\[
d\lambda (\alpha; q, x; \beta) = \|\text{sh}x\| dqdx d\alpha d\beta,
\]

its weight factor equals \( P_{\lambda} = \|\text{sh}x\| \). The wave functions \( \Psi \) are expanded in the double Fourier series:

\[
\Psi (\alpha; q, x; \beta) = \sum_{m, n \in \mathbb{Z}} f_{mn}(q, x) e^{im\alpha} e^{in\beta}.
\]

This is obviously the Peter-Weyl theorem specialized to the two-dimensional torus group \( T^2 \). Our integers \( m, n \in \mathbb{Z} \) are just the labels \( \alpha, \beta \) from the general theory.

For the affine-affine model the reduced operator of the kinetic energy is given by

\[
T^{mn}_{\text{aff-aff}} f^{mn} = -\frac{\hbar^2}{A} D_\lambda f^{mn} - \frac{\hbar^2}{4(A + 2B)} \frac{\partial^2 f^{mn}}{\partial q^2} + \frac{\hbar^2(n - m)^2}{16 A \text{sh}^2 \frac{x}{2}} f^{mn} - \frac{\hbar^2(n + m)^2}{16 A \text{ch}^2 \frac{x}{2}} f^{mn},
\]

where now \( D_\lambda \) is expressed as follows:

\[
D_\lambda = \frac{1}{\|\text{sh}x\|} \frac{\partial}{\partial x} \left( \|\text{sh}x\| \frac{\partial}{\partial x} \right).
\]

Similarly for the metric-affine and affine-metric models we obtain respectively

\[
T^{mn}_{\text{met-aff}} = T^{mn}_{\text{aff-aff}} [A \mapsto I + A] + \frac{I \hbar^2 m^2}{T^2 - A^2},
\]

\[
T^{mn}_{\text{aff-met}} = T^{mn}_{\text{aff-aff}} [A \mapsto I + A] + \frac{I \hbar^2 n^2}{T^2 - A^2},
\]

68
To avoid the purely continuous spectrum one must include into Hamiltonian at least some dilatations-stabilizing potential \( V(q) \). The problems is then (as usual) separable in \((x, q)\)-variables. And just as on the classical level, for affinely-invariant incompressible dynamics, i.e., for the \(x\)-sector of the above operators, there exists discrete spectrum if \(|n + m| > |n - m|\), i.e., if \(mn > 0\). This is the quantum bounded motion.

In three-dimensional problems the above condition will be replaced by some more complicated one between quantum numbers labelling the reduced amplitudes \( f \).

Let us observe that in more general, not necessarily geodetic, problems in two dimensions with explicitly separable potentials \( V(q, x) = V_{\text{dil}}(q) + V_{\text{sh}}(x) \), the Schrödinger equation

\[
H_{\text{mn}} f_{\text{mn}} = E f_{\text{mn}},
\]

where \( H_{\text{mn}} = T_{\text{mn}} + V_{\text{dil}}(q) + V_{\text{sh}}(x) \) with any of the above \( T_{\text{mn}} \), splits into two one-dimensional Schrödinger equations. The reduced wave function is sought in the form

\[
f_{\text{mn}}(q, x) = \varphi_{\text{mn}}(x) \chi(q),
\]

where \( \varphi_{\text{mn}}, \chi \) satisfy the following eigenequations:

\[
H_{\text{sh}} \varphi_{\text{mn}} = E_{\text{sh}} \varphi_{\text{mn}}, \quad H_{\text{dil}} \chi = -\frac{\hbar^2}{4(A + 2B)} \frac{d^2 \chi}{dq^2} + V_{\text{dil}} \chi = E_{\text{dil}} \chi.
\]

Here the shear-rotational Hamiltonian operator \( H_{\text{sh}}^{\text{mn}} \) is given by

\[
H_{\text{sh}}^{\text{mn}} = -\frac{\hbar^2}{A} D\lambda + \frac{\hbar^2(n-m)^2}{16A^2 \hbar^2 x^2} - \frac{\hbar^2(n+m)^2}{16A^2 \hbar^2 x^2} + V_{\text{sh}}(x)
\]

in the affine-affine model, and by

\[
H_{\text{sh}}^{\text{mn}} = H_{\text{sh}}^{\text{mn}} + \frac{\hbar^2 m^2}{L^2 - A^2} + \frac{\hbar^2 n^2}{L^2 - A^2} = H_{\text{sh}}^{\text{mn}} + \frac{\hbar^2 m^2}{L^2 - A^2} + \frac{\hbar^2 n^2}{L^2 - A^2}
\]

respectively in the metric-affine and affine-metric models. Obviously, the total energy equals \( E = E_{\text{sh}} + E_{\text{dil}} \). It is seen that the main point of the analysis is the affine-affine model because with fixed \( m, n \) the other ones differ from it by \((m, n\)-dependent\) \( c \)-numbers.

Just as in the classical model, for \(|n + m| > |n - m|\), i.e., \(nm > 0\), the "centrifugal" term

\[
V_{\text{cfg}} := \frac{\hbar^2(n-m)^2}{16A^2 \hbar^2 x^2} - \frac{\hbar^2(n+m)^2}{16A^2 \hbar^2 x^2}
\]

is singular repulsive at \( x = 0 \) and finite-attractive for \(|x| \to \infty \). And then even for the purely geodetic incompressible model \( V_{\text{sh}} = 0 \) there exist bounded states and discrete energy spectrum for \( E_{\text{sh}} \). For \( nm < 0 \) the energy spectrum is continuous (scattering states).
Finally, let us quote the corresponding formulas for the quantized d’Alembert model. Obviously, using the same notation as above we have the following expression for the classical kinetic Hamiltonian:

\[
T_{\text{int}} = \frac{1}{2I} (P_1^2 + P_2^2) + \frac{m^2}{4I (Q^1 - Q^2)^2} + \frac{n^2}{4I (Q^1 + Q^2)^2}.
\]

With fixed values of \(m, n\) the problem reduces again to the dynamics of deformation invariants \(Q^1, Q^2\). In the two-polar coordinates the Lebesgue measure element is given by

\[
dl(\alpha, Q^1, Q^2; \beta) = P_l(Q^1, Q^2) dQ^1 dQ^2 d\alpha d\beta,
\]

where

\[
P_l = \left| (Q^1)^2 - (Q^2)^2 \right| = \left| (Q^1 + Q^2) (Q^1 - Q^2) \right|.
\]

The reduced amplitudes \(f^{mn}\) satisfy the eigenequations

\[
H^{mn} f^{mn} = T^{mn} f^{mn} + V(Q^1, Q^2) f^{mn} = E^{mn} f^{mn}
\]

with

\[
T^{mn} f^{mn} = -\frac{\hbar^2}{2I} D_l f^{mn} + \frac{\hbar^2 m^2}{4I (Q^1 - Q^2)^2} f^{mn} + \frac{\hbar^2 n^2}{4I (Q^1 + Q^2)^2} f^{mn},
\]

where

\[
D_l = \frac{1}{P_l} \frac{\partial}{\partial Q^1} \left( P_l \frac{\partial}{\partial Q^1} \right) + \frac{1}{P_l} \frac{\partial}{\partial Q^2} \left( P_l \frac{\partial}{\partial Q^2} \right).
\]

The coordinates \(Q^1, Q^2\) are very badly non-separable even in the very kinetic energy expression. There are however other coordinates on the plane of deformation invariants, much better from this point of view. The simplest ones are coordinates \(Q^+, Q^-\) obtained from \(Q^1, Q^2\) by the rotation by the angle \(\pi/4\),

\[
Q^+ := \frac{1}{\sqrt{2}} (Q^1 + Q^2), \quad Q^- := \frac{1}{\sqrt{2}} (Q^1 - Q^2),
\]

where \(Q^+\) and \(Q^-\) may be expressed in terms of polar and elliptic coordinates respectively as

\[
Q^+ = r \cos \varphi, \quad Q^- = r \sin \varphi,
\]

and

\[
Q^+ = \text{ch} \rho \cos \lambda, \quad Q^- = \text{sh} \rho \sin \lambda.
\]

In all these variables the Hamilton-Jacobi and Schrödinger equations without potential are separable. Obviously, the geodetic d’Alembert model \(T = (I/2)\text{Tr}(\dot{\varphi}^T \dot{\varphi})\) is completely non-physical. However, the coordinate systems \((Q^+, Q^-), (r, \varphi), (\rho, \lambda)\) enable one to find a class of potentials which are physically realistic and at the same time both the Hamilton-Jacobi and Schrödinger equations are separable for the corresponding Hamiltonians.
The reduced Schrödinger eigenproblem (99) with doubly isotropic potentials is separable if

\[ V(Q^1, Q^2) = V_+(Q^+) + V_-(Q^-). \]

Namely, we have then \( H^{mn} = H^+_m + H^-_m \), where

\[
H^+_m = -\frac{\hbar^2}{2I} \left( \frac{\partial^2}{\partial Q^2} + \frac{1}{Q^+} \frac{\partial}{\partial Q^+} \right) f^+_m + \left( \frac{\hbar^2 (m-n)^2}{8IQ^+} + V_+ \right) f^+_m, \\
H^-_m = -\frac{\hbar^2}{2I} \left( \frac{\partial^2}{\partial Q^{-2}} + \frac{1}{Q^-} \frac{\partial}{\partial Q^-} \right) f^-_m + \left( \frac{\hbar^2 (m+n)^2}{8IQ^-} + V_- \right) f^-_m,
\]

where

\[ H^+_m f^+_m = E^+_m f^+_m, \quad H^-_m f^-_m = E^-_m f^-_m, \]

and

\[ f^{mn}(Q^1, Q^2) = f^+_m(Q^+) f^-_m(Q^-), \quad E^{mn} = E^+_m + E^-_m. \]

Obviously, the volume element is given by

\[
dl(\alpha; Q^+, Q^-; \beta) = 2 |Q^+| |Q^-| dQ^+ dQ^- d\alpha d\beta.
\]

The doubly isotropic models separable in coordinates \((r, \phi)\) are based on potentials of the form

\[ V(r, \phi) = V_r(r) + \frac{1}{r^2} V_\phi(\phi). \]

The wave functions are factorized as follows:

\[ \Psi = e^{im\alpha} e^{in\beta} f^{mn}(r, \phi) = e^{im\alpha} e^{in\beta} R^{mn}(r) \Phi^{mn}(\phi) \]

and then of course

\[ S\Psi = \hat{r}\Psi = \hbar m \Psi, \quad V\Psi = -\hbar\phi = \hbar n \Psi. \]

The reduced Hamiltonian has the form:

\[ H^{mn} = H^r_m + \frac{1}{r} H^\phi_m, \]

where

\[ H^r_m = -\frac{\hbar^2}{2I} \left( \frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} \right) + V_r \]

is as a matter of fact independent of \( m, n \); unlike this, \( H^\phi_m \) depends explicitly on \( m, n \):

\[ H^\phi_m = \frac{\hbar^2}{2I} \left( \frac{\partial^2}{\partial \phi^2} + 2\cot(2\phi) \frac{\partial}{\partial \phi} \right) + \frac{\hbar^2 m^2 + 2mn \cos(2\phi) + n^2}{4} \frac{\sin^2(2\phi)}{\sin^2(2\phi)} + V_\phi. \]

The functions \( \Phi^{mn}, R^{mn} \) satisfy eigenvalues

\[ H^\phi_m \Phi^{mn} = E^{mn}_\phi \Phi^{mn}, \quad \left( H^r_m + \frac{1}{r} E^\phi_m \right) R^{mn} = E R^{mn}. \]
The $\Phi$-equation (100) is to be solved as first. Then the resulting quantized values of $E_{mn}^\phi$, labelled by an additional quantum number $k$ are to be substituted to (101) and because of this the labels $m, n$ appear in $R$ although the radial operator $H_{mn}^r$, in spite of the used notation, is independent of $m, n$. There are also two additional quantum numbers in $R$, namely $k$ itself appearing through $E_{mnk}^\phi$ and the proper radial quantum number $\mu$, thus, $E$ obtained from (101) will be denoted by $E_{mnk\mu}$.

Let us quote some very interesting model qualitatively compatible with standard demands of the macroscopic nonlinear elasticity,

$$V = \frac{2\kappa}{r^2 \cos(2\phi)} + \frac{\kappa}{2} r^2 = \kappa \left( \frac{1}{D_1 D_2} + \frac{D_1^2 + D_2^2}{2} \right).$$

In the natural state of elastic equilibrium $r = 0, \phi = 0$. We do not quote more complicated and rather non-useful one-dimensional equations for the elliptic coordinates $\rho, \lambda$. Let us only mention the general shape of separable doubly-isotropic potentials

$$V(\rho, \lambda) = \frac{V_\rho(\rho)}{2 (\cosh^2 \rho - \cos^2 \lambda)} + \frac{V_\lambda(\lambda)}{2 (\cosh^2 \rho - \cos^2 \lambda)}.$$

Finally, we quote a three-parameter family of doubly-isotropic potentials for which both the classical and quantum problems are simultaneously separable in all the aforementioned coordinate systems:

$$V = \frac{A}{Q_1^2} + \frac{B}{Q_2^2} + C \left( Q_1^{-2} + Q_2^{-2} \right)$$

$$= \frac{2A}{(Q_1^2 + Q_2^2)^2} + \frac{2B}{(Q_1^2 - Q_2^2)^2} + C \left( (Q_1^2)^2 + (Q_2^2)^2 \right).$$

Here $A, B, C$ are arbitrary constants. It is well-known that the simultaneous separability of Hamilton-Jacobi and Schrödinger equations in a few coordinate systems has to do with degeneracy and hidden symmetries.

Two-dimensional models are interesting not only from the philosophical point of view of the "Flatland" geometry. They may be practically useful in the theory of surfaces of structured bodies and in the dynamics of elongated molecules or other structure elements.

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