Equality on all #CSP Instances Yields Constraint Function Isomorphism via Interpolation and Intertwiners

Ben Young*
benyoung@cs.wisc.edu

Abstract
A fundamental result in the study of graph homomorphisms is Lovász’s theorem [22] that two graphs are isomorphic if and only if they admit the same number of homomorphisms from every graph. A line of work extending Lovász’s result to more general types of graphs was recently capped by Cai and Govorov [6], who showed that it holds for graphs with vertex and edge weights from an arbitrary field of characteristic 0. In this work, we generalize from graph homomorphism – a special case of #CSP with a single binary function – to general #CSP by showing that two sets $F$ and $G$ of arbitrary constraint functions are isomorphic if and only if the partition function of any #CSP instance is unchanged when we replace the functions in $F$ with those in $G$. We give two very different proofs of this result. First, we demonstrate the power of the simple Vandermonde interpolation technique used in [6] by extending it to general #CSP. Second, we give a proof using the intertwiners of the automorphism group of a constraint function set, a concept from the representation theory of compact groups. This proof is a generalization of a classical version of the recent proof of the Lovász-type result in [25] relating quantum isomorphism and homomorphisms from planar graphs.

1 Introduction

Graph homomorphisms. A homomorphism from graph $K$ to graph $X$ is an adjacency-preserving map from $V(K)$ to $V(X)$. Since graph homomorphisms’ introduction in [22], counting the number of homomorphisms from $K$ to $X$ has emerged as a well-studied problem in theoretical computer science and combinatorics. The number of homomorphisms from graph $K$ to $X$, denoted $\text{hom}(K, X)$, can be computed as the evaluation of a partition function, parameterized by $X$, on $K$ – the sum over all maps $\phi : V(K) \to V(X)$ of the product $\prod_{(u,v) \in E(K)} (A_X)_{\phi(u),\phi(v)}$, where $A_X$ is the adjacency matrix of $X$. The partition function perspective leads to extensions of graph homomorphism to more general types of graphs, as well as a natural view of graph homomorphism as a special case of counting constraint satisfaction problems, or #CSP.

One such more general type of graph has a real weight assigned to each edge and a nonnegative real weight assigned to each vertex. In the partition function formulation of graph homomorphism to such a graph $X$, we simply use the weighted adjacency matrix $A_X$, and add factors for the vertex weights. The problem of counting homomorphisms to such weighted graphs was studied in [18, 23, 24]. These works prove their results by studying graph algebras of formal $C$-combinations of $k$-labeled graphs (called quantum graphs, which we generalize in discussion above Theorem 5), using the $k$-labeled graph product extended by our Definition 4 below. In [23], Lovász extended to these weighted graphs his result, proved forty years prior in [22], that two graphs are isomorphic if and only if they admit the same number of homomorphisms from every graph. Throughout this paper, we will refer to such generalizations of Lovász’s original isomorphism theorem as “Lovász-type results”.

Still making use of quantum and $k$-labeled graphs, but applying invariant theory and the Nullstellensatz from algebraic geometry, Schrijver [27] studies homomorphisms to graphs with complex edge weights but without vertex weights, and proves a Lovász-type result for such complex-edge-weighted graphs. Using similar proof methods, Regts [26] studies “vertex-coloring models” – homomorphisms to graphs with arbitrary vertex and edge weights, provided that no nonempty subset of vertex weights sums to zero.

*Department of Computer Sciences, University of Wisconsin-Madison
Finally, Cai and Govorov \cite{cai2022novel} prove a Lovász-type result for graphs with any vertex and edge weights from an arbitrary field \( \mathbb{F} \) of characteristic 0 (as discussed in \cite{cai2022novel}, a similar, slightly weaker result can also be obtained from the results of \cite{cai2022novel}, proved using \( k \)-labeled graph algebras and matroid invariants), and show this is the most general possible Lovász-type result for graph homomorphism. Cai and Govorov obtain this generality, overcoming the algebraic approaches' technical difficulties of vertex weights summing to 0, by applying a simple, direct \textit{Vandermonde interpolation} technique, dependent only on no algebraic results aside from the fact that a Vandermonde matrix with distinct roots is nonsingular. It is remarkable that such a simple tool unifies all previous Lovász-type results, and in \textsection 3, we give a further demonstration of its power by using it to extend Cai and Govorov’s results to \#CSP.

Another line of work studies homomorphisms \textit{from} restricted classes of graphs rather than homomorphisms \textit{to} expanded classes as above, and uses invariance of homomorphism counts from restricted classes of graphs to characterize relaxations of graph isomorphism. In \cite{bohler2017constraint}, using the techniques of \cite{dvorak2013constraint}, Dvořák showed that homomorphism count from 2-degenerate graphs suffices to determine a graph up to isomorphism, and that homomorphism count from graphs of treewidth at most \( k \) determines graphs up to their \( k \)-degree refinements, but not up to isomorphism. Then in \cite{bohler2017constraint} it was shown that two graphs admit the same number of homomorphisms from all graphs of treewidth at most \( k \) if and only if they are indistinguishable by the \( k \)-dimensional Weisfeiler-Leman algorithm.

Most notably, Mančinska and Roberson showed in \cite{mancinska2019quantum} that two graphs are \textit{quantum isomorphic} if and only if they admit the same number of homomorphisms from all planar graphs. Quantum isomorphism is defined using \textit{quantum permutation groups}, and its characterization by planar graph homomorphisms is achieved using the \textit{intertwiner space} of the quantum automorphism group of a graph, a quantum permutation group analogous to the graph’s classical automorphism group. A key component of the proof is a ‘quantum’ version of Woronowicz’s Tannaka-Krein duality \cite{woronowicz1987duality}, which implies that a quantum permutation group is uniquely determined by its intertwiner space. A ‘classical’ version of Tannaka-Krein duality (Theorem 4) similarly applies to the intertwiner space of the classical automorphism group of a graph, or, more generally, of a set of \#CSP constraint functions. Using this, in \textsection 3, we give a classical version of Mančinska and Roberson’s proof, generalized to sets of real-valued \#CSP functions – the same result as proved via Vandermonde interpolation in \textsection 3, but restricted to \( \mathbb{R} \) rather than general fields. In an upcoming work, we also generalize Mančinska and Roberson’s original ‘quantum’ result to \#CSP.

**Counting complexity and \#CSP.** A \#CSP(\(\mathcal{F}\)) problem is parameterized by a set \(\mathcal{F}\) of \(\mathbb{F}\)-valued constraint functions on one or more inputs from a finite domain \(V(\mathcal{F})\). The problem input is a \#CSP instance, consisting of a set of constraints, each applying a constraint function to a subset of variables. The output is the value of the partition function, the sum over all variable assignments of the product of the constraint evaluations. Letting \(\mathcal{F} = \{\lambda_X\}\), \(V(\mathcal{F}) = V(X)\), the variable set be \(V(K)\), and the constraint set be \(E(K)\) with each edge-constraint applying \(\lambda_X\) to its two endpoints, one can see from the partition function formulation of graph homomorphism above that counting homomorphisms from \(K\) to \(X\) is the special case of \#CSP(\(\mathcal{F}\)) on instance \(K\) where \(\mathcal{F}\) contains a single binary (arity-2) constraint function \(\lambda_X\).

Counting graph homomorphisms is a central problem in counting complexity, both in its own right and as a special case of \#CSP. Both settings have seen many significant dichotomy theorems classifying the partition function as either tractable of \#P-hard to compute, depending on \(X\) or \(\mathcal{F}\), respectively. Graph homomorphism dichotomies were established for unweighted graphs in \cite{cohen2009graph}, nonnegative-real-weighted graphs in \cite{cohen2009graph, cohen2012nonnegative}, real-weighted graphs in \cite{cohen2012nonnegative}, and finally complex-weighted graphs in \cite{cohen2012nonnegative}. For \#CSP, dichomies were established for sets of 0-1 valued constraint functions in \cite{cai1998complexity, cai1999complexity}, nonnegative-real-valued constraint functions in \cite{cai2001complexity}, and complex-valued constraint functions in \cite{cai2004complexity}.

Extending the notion of graph isomorphism, we say two constraint functions \(F_1\) and \(F_2\) of the same arity \(n\) on the same domain \(V(F)\) are isomorphic if there is a permutation \(\sigma\) of \(V(F)\) such that \(F_1(x_1, \ldots, x_n) = F_2(\sigma(x_1), \ldots, \sigma(x_n))\) for all \(x_1, \ldots, x_n \in V(F)\). Two sets of constraint functions \(\mathcal{F}\) and \(\mathcal{G}\) are isomorphic if there is a common isomorphism between each \(F \in \mathcal{F}\) and a corresponding \(G \in \mathcal{G}\). Some similar concepts exist: in \cite{bohler2017constraint, bohler2017constraint}, Böhler et al. study “constraint isomorphism” between Boolean \#CSP instances (rather than constraint functions) that involves permuting variables (rather than domain elements). One can also view an \(n\)-ary constraint function \(F\) as a tensor in \(\mathbb{F}^{V(F)^n}\); from this perspective the notion of tensor isomorphism in \cite{cai2022novel} is a relaxation of constraint function isomorphism from permutations to invertible linear transformations on each dimension.
Our results. Our main result is the following theorem, which to our knowledge is the first Lovász-type result of any kind for #CSP.

Theorem (Theorem 1, informal). For field $\mathbb{F}$ of characteristic $0$, sets $\mathcal{F}$ and $\mathcal{G}$ of $\mathbb{F}$-valued constraint functions are isomorphic if and only if the partition function of every #CSP($\mathcal{F}$) instance is preserved when we replace every constraint function in $\mathcal{F}$ with the corresponding function in $\mathcal{G}$.

We prove Theorem 1 in the style of [1] (Vandermonde interpolation) in Section 3 and in the style of [2] (intertwiner spaces) in Section 4. The former actually proves a more general result (Theorem 2) applying to $k$-labeled #CSP instances and constraint function sets with domain/vertex weights. By the above discussion, one can see that the Lovász-type result, proved in [1], that $\mathbb{F}$-weighted graphs $X$ and $Y$ are isomorphic if and only if $\text{hom}(K,X) = \text{hom}(K,Y)$ for all graphs $K$, is the special case of Theorem 1 where $\mathcal{F} = \{A_X\}$ and $\mathcal{G} = \{A_Y\}$. We carry out the latter proof in the Holant framework from counting complexity. While [26] does not explicitly make use of the Holant framework, we find it a very natural setting. Roughly, the main idea is to express the intertwiner space of Aut($\mathcal{F}$) as the span of the signature matrices of Holant gadgets, which one can view as #CSP instances with free/input variables, via a decomposition (Theorem 3) of any gadget into fundamental ‘building block’ gadgets and an analogous characterization (Lemma 4) of the intertwiner space of Aut($\mathcal{F}$). While the version of Theorem 1 in Section 4 is restricted to constraint functions over $\mathbb{R}$, rather than over general fields as in Section 3 we believe that the combinatorial reasoning in Section 3 is more intuitive than the interpolation technique of Section 3 and [1], as well as the algebraic proofs of [18, 22, 24, 27, 29] discussed above. The intertwiner proof also demonstrates the surprisingly natural application of the powerful representation theoretic tools of intertwiner spaces and Tannaka-Krein duality to #CSP and Holant theory. We hope it inspires further applications of representation theory to theoretical computer science.

2 Preliminaries

For notational brevity, following [14] and others working with $n$-ary structures, write $x_j^i$ to mean $(x_1, \ldots, x_j)$ if $j \geq i$, and the empty list if $i > j$. When the index range is clear, we simply write $x = (x_1, \ldots, x_r)$. For any $q \in \mathbb{N}$, write $[q] = \{1, 2, \ldots, q\}$ and $[0, q] = \{0, 1, \ldots, q - 1\}$. For sets $A$ and $B$, $A^B$ denotes the set of functions from $B$ to $A$. For a set $B$ with an implicit linear order and $a_b \in A$ for every $b \in B$, $(a_b)_{b \in B}$ denotes a tuple of elements of $A$ indexed and ordered by $B$. We will view $(a_b)_{b \in B}$ as an element of $A^B$, and will abbreviate it as simply $(a_b)$ if the index set $B$ is clear from context. Let $S_q$ be the symmetric group of permutations on $[q]$. Throughout, let $\mathbb{F}$ be a field of characteristic 0.

Counting Constraint Satisfaction Problems. Any function $F : [q]^{n_F} \rightarrow \mathbb{F}$ on $n_F \geq 1$ variables taking values in $[q]$ is a constraint function with domain $[q]$ and arity $n_F$. When $n_F = 2$, one can view $F$ as a $q \times q$ matrix with entries in $\mathbb{F}$, the adjacency matrix of an $\mathbb{F}$-weighted graph $\mathcal{G}$. Denote sets of constraint functions by calligraphic letters such as $\mathcal{F}$ and $\mathcal{G}$. It is assumed that all constraint functions in a set $\mathcal{F}$ have the same domain, denoted by $V(\mathcal{F})$ ($V$ stands for ‘vertices’, terminology inherited from the weighted graph special case) and that all constraint function sets are finite.

Definition 1 (#CSP, $Z_\mathcal{F}$). A #CSP problem #CSP($\mathcal{F}$) is parameterized by a set $\mathcal{F}$ of constraint functions. A #CSP($\mathcal{F}$) instance $K = (V,C)$ is defined by a set $V$ of variables and a multiset $C$ of constraints. Each constraint $(F, v_{i_1}, \ldots, v_{i_{n_F}})$ consists of a constraint function $F \in \mathcal{F}$ and an ordered tuple of variables to which $F$ is applied.

The partition function $Z_\mathcal{F}$, on input #CSP($\mathcal{F}$) instance $K = (V,C)$, outputs

$$Z_{\mathcal{F}}(K) = \sum_{\phi : V \rightarrow V(\mathcal{F})} \prod_{(F, v_{i_1}, \ldots, v_{i_{n_F}}) \in C} F(\phi(v_{i_1}), \ldots, \phi(v_{i_{n_F}})).$$

Definition 2 (Compatible constraint function sets, $K_\mathcal{F} \rightarrow \mathcal{G}$). Constraint function sets $\mathcal{F} = \{F_j\}_{j \in [t]}$, $\mathcal{G} = \{G_j\}_{j \in [t]}$ are compatible if $t_j = t_g = t$ and $F_j : [q_j]^{n_j} \rightarrow \mathbb{F}$ and $G_j : [q_j]^{n_j} \rightarrow \mathbb{F}$ for all $j \in [t]$ (in other words, equal-indexed constraint functions have the same arity). We call $F_j$ and $G_j$ corresponding constraint functions.
For compatible constraint function sets $\mathcal{F}$ and $\mathcal{G}$ and any $\#\text{CSP}(\mathcal{F})$ instance $K_{\mathcal{F}\rightarrow\mathcal{G}}$ by replacing every constraint in $K$ with the corresponding constraint function in $\mathcal{G}$ applied to the same variable tuple.

More generally, we may define a domain-weighted $\#\text{CSP}$ problem.

**Definition 3** ($\#\text{CSP}(\mathcal{F}, \alpha)$, $Z_{\mathcal{F}, \alpha}$). The problem $\#\text{CSP}(\mathcal{F}, \alpha)$ is parameterized by a set $\mathcal{F}$ of constraint functions with domain $V(\mathcal{F}) = [q]$, and a vector of domain weights $\alpha \in (\mathbb{F} \setminus \{0\})^q$. The partition function $Z_{\mathcal{F}, \alpha}$, defined on $\#\text{CSP}(\mathcal{F})$ instances $K = (V, C)$ as above, is

$$Z_{\mathcal{F}, \alpha}(K) = \sum_{\phi: V \rightarrow [q]} \prod_{v \in V} \alpha_{\phi(v)} \prod_{(F, v_1, \ldots, v_{n_F}) \in C} F(\phi(v_1), \ldots, \phi(v_{n_F})).$$

In particular, $Z_{\mathcal{F}, 1} = Z_{\mathcal{F}}$, where 1 is the all-ones vector. We use “$\#\text{CSP}(\mathcal{F})$ instance” and “$\#\text{CSP}(\mathcal{F}, \alpha)$ instance” interchangeably, since a $\#\text{CSP}(\mathcal{F}, \alpha)$ instance does not depend on the domain weights – it is identical to a $\#\text{CSP}(\mathcal{F})$ instance.

**Definition 4** ($k$-labeled $\#\text{CSP}$ instance (product), $\mathcal{PLL}[\mathcal{F}; k]$, $\mathcal{PLL}^{\text{imp}}[\mathcal{F}; k]$). A $\#\text{CSP}$ instance $K = (V, C)$ is $k$-labeled if $k$ variables are labeled by $1, 2, \ldots, k$. A single variable cannot be labeled more than once. Define the product $K_1 K_2$ of two $k$-labeled $\#\text{CSP}(\mathcal{F})$ instances $K_1 = (V_1, C_1)$, $K_2 = (V_2, C_2)$ as follows. For $i \in [k]$, let $u_i \in V_1, v_i \in V_2$ be the variables labeled $i$ in $V_1$ and $V_2$, respectively. Define a new variable set $V$ by starting with $V_1 \sqcup V_2$, then for each $i \in [k]$ merging $u_i$ and $v_i$ into a new variable $w_i$, and label $w_i$ by $i$. Then define a new constraint multiset $C$ by starting with $C_1 \cup C_2$ (multiset union), then for every $i \in [q]$ replacing every occurrence of $u_i$ or $v_i$ in each constraint with $w_i$. Then take $K_1 K_2 = (V, C)$.

Define $\mathcal{PLL}[\mathcal{F}; k]$ to be the set of $k$-labeled $\#\text{CSP}(\mathcal{F})$ instances. Let $U_k = (V, \emptyset) \in \mathcal{PLL}[\mathcal{F}; k]$, where $V$ contains exactly $k$ vertices labeled $1, \ldots, k$. The $k$-labeled instance product is commutative and associative and has identity $U_k$, so $\mathcal{PLL}[\mathcal{F}; k]$ forms a commutative monoid under this product. Let $\mathcal{PLL}^{\text{imp}}[\mathcal{F}; k]$ denote the submonoid of $\mathcal{PLL}[\mathcal{F}; k]$ consisting of simple instances – those where the variables in any constraint $c \in C$ are distinct, the multiplicity of every constraint in $C$ is 1 up to permutation of the order of its variables, and no constraint contains only labeled variables.

Observe that, for $K_1, K_2 \in \mathcal{PLL}[\mathcal{F}; k]$, $(K_1)_{\mathcal{F}\rightarrow\mathcal{G}}(K_2)_{\mathcal{F}\rightarrow\mathcal{G}} = (K_1 K_2)_{\mathcal{F}\rightarrow\mathcal{G}} \in \mathcal{PLL}[\mathcal{G}; k]$.

**Definition 5** ($Z_{\mathcal{F}, \alpha}^\psi$). For $K = (V, C) \in \mathcal{PLL}[\mathcal{F}; k]$ and a map $\psi : [k] \rightarrow [q]$ fixing, or pinning, the values of the labeled variables, define

$$Z_{\mathcal{F}, \alpha}^\psi(K) = \sum_{\phi: V \rightarrow [q] \text{ extends } \psi} \alpha_{\phi} \prod_{(F, v_1, \ldots, v_{n_F}) \in C} F(\phi(v_1), \ldots, \phi(v_{n_F})),$$

where

$$\alpha_{\phi} = \prod_{v \in V} \alpha_{\phi(v)} \text{ and } \alpha_{\psi} = \prod_{i \in [k]} \alpha_{\psi(i)},$$

and $\phi$ extends $\psi$ means $\phi$ assigns value $\psi(i)$ to the variable labeled $i$. Then we have

$$Z_{\mathcal{H}, \alpha}(K) = \sum_{\psi: [k] \rightarrow [q]} \alpha_{\psi} Z_{\mathcal{F}, \alpha}^\psi(K).$$

The $k$-labeled instance product $K_1 K_2$ merges the labeled variables, and the unlabeled variables of $K_1$ and $K_2$ both still appear in constraints from $K_1$ and $K_2$ with the combined labeled variables. The unlabeled variables of $K_1$ take values independently of the unlabeled variables of $K_2$ (i.e. they appear in no constraints with each other). Hence

$$Z_{\mathcal{F}, \alpha}^\psi(K_1 K_2) = Z_{\mathcal{F}, \alpha}^\psi(K_1) Z_{\mathcal{F}, \alpha}^\psi(K_2). \tag{1}$$

For fixed $\mathcal{F} = \{F_1, \ldots, F_i\}$ with common domain $[q]$, let

$$\mathcal{J}(\mathcal{F}) = \{(j, x, r) \mid j \in [t], x \in [q]^{n_j - 1}, r \in [n_j]\} \tag{2}$$
(recall that $n_j$ is the arity of $F_j \in \mathcal{F}$). If $n_j = 1$ ($F_j$ is unary), then say $[q]_{n_j}^{-1} = \{0\}$ (the set containing the empty tuple), so $x \in [q]_{n_j}^{-1}$ means $x = ()$. $\mathcal{J}(\mathcal{F})$ represents all ‘configurations’ in which we may fill in the remaining arguments of an application a function $f$ in $\mathcal{F}$ when given a single distinguished argument. Note that the length of $x$ and the domain of $r$ both depend on $j$ (the choice of $F_j \in \mathcal{F}$). Domain elements $i, i' \in [q]$ are twins if

$$F_j(x_1^{r-1}, i, x_r^{n_j-1}) = F_j(x_1^{r-1}, i', x_r^{n_j-1})$$

for every $(j, x, r) \in \mathcal{J}(\mathcal{F})$.

If $n_j = 1$ and $x = ()$, then $F_j(x_1^{r-1}, i, x_r^{n_j-1}) = F_j(i)$. If every $F \in \mathcal{F}$ is symmetric, meaning $F$ is invariant under permutations of the order of its inputs, then say $\mathcal{F}$ is symmetric, and $i, i' \in [q]$ are twins if $F_j(i, x) = F_j(i', x)$ for every $j \in [t]$ and $x \in [q]_{n_j}^{-1}$, where we abbreviate $F_j(i, x) = F_j(x_1^{n_j-1})$. If $n_j = 1$ and $x = ()$, then $F_j(i, x) = F_j(i)$. $\mathcal{F}$ is twin-free if no two domain elements are twins. Equivalently, $\mathcal{F}$ is twin-free iff the tuples

$$\left( F_j(x_1^{r-1}, i, x_r^{n_j-1}) \right)_{(j, x, r) \in \mathcal{J}(\mathcal{F})}$$

are pairwise distinct for $i \in [q]$. If $\mathcal{F}$ is symmetric, then $\mathcal{F}$ is twin free iff the tuples $\left( F_j(i, x) \right)_{j \in [t], x \in [q]_{n_j}^{-1}}$ are pairwise distinct for $i \in [q]$.

For any $\mathcal{F}$ and $\alpha$, let $I_1, \ldots, I_s$ be the partition of $[q]$ under the twin relation. Define the twin-contracted domain weight $Z_{\mathcal{F}, \alpha}(K) = 1$ for all $K \in \mathcal{P}LZ[\mathcal{F}; k]$. If every $F \in \mathcal{F}$ is symmetric, then $\mathcal{F}$ is twin free iff the tuples $\left( F_j(i, x) \right)_{j \in [t], x \in [q]_{n_j}^{-1}}$ are pairwise distinct for $i \in [q]$.

We now have the notation to state our main theorem.

**Theorem 1.** Let $\mathcal{F}$ be a field of characteristic 0, and let $\mathcal{F}$ and $\mathcal{G}$ be compatible $\mathbb{F}$-valued constraint function sets. Then $\mathcal{F} \cong \mathcal{G}$ if and only if $Z_{\mathcal{F}}(K) = Z_{\mathcal{G}}(K_{\mathcal{F} \to \mathcal{G}})$ for every $\#\text{CSP}(\mathcal{F})$ instance $K$.

**Vandermonde Interpolation.** Next, we introduce the useful Vandermonde interpolation technique from [6], which is essentially the only technique used to prove our main result. The basis for the technique is the following simple lemma.

**Lemma 1.** Let $n, m \geq 0$ and $a_i, x_i \in \mathbb{F}$ for $1 \leq i \leq n$, and suppose $\sum_{i=1}^{n} a_i x_i^j = 0$ for all $0 \leq j < n$. Then, for any function $f: \mathbb{F} \to \mathbb{F}$, we have $\sum_{i=1}^{n} a_i f(x_i) = 0$. 


Corollary 1 (Corollary 4.2). Let \( I \) and \( J \) be finite index sets, and \( a_i, b_{i,j} \in \mathbb{F} \) for all \( i \in I, j \in J \). Further, let \( I = \bigcup_{\ell \in [s]} I_\ell \) be the partition of \( I \) into equivalence classes defined by relation \( \sim \), where \( i \sim i' \) iff \( b_{i,j} = b_{i',j} \) for all \( j \in J \). If \( \sum_{i \in I} a_i \prod_{j \in J} b_{i,j}^{p_j} = 0 \), for all choices of \((p_j)_{j \in J}\) where each \( 0 \leq p_j < |I| \), then \( \sum_{i \in I_\ell} a_i = 0 \) for every \( \ell \in [s] \).

\( I \) (and \( J \)) will often be the set of all \( m \)-tuples whose entries range over \([q]\), for some \( m \) and \( q \), and the product of \( b_{i,j}^{p_j} \)'s for a fixed tuple will have \( i \) range over all the tuple's entries, rather than refer to the tuple itself. In this case, we have the following corollary, used implicitly in [6].

Corollary 2. Let \( J \) be a finite index set and \( q, m \geq 1 \). Let \( a_i \in \mathbb{F} \) for \( i \in [q]^m \) and \( b_{i,j} \in \mathbb{F} \) for \( i \in [q], j \in J \), and for \( i, i' \in [q] \), say \( i \sim i' \) iff \( b_{i,j} = b_{i',j} \) for all \( j \in J \). Let \( [q]^m = \bigcup_{\ell \in [s]} I_\ell \) be a partition of \([q]^m \) into equivalence classes defined by relation \( \approx \), where \( i \approx i' \) if \( i_h \approx i'_h \) for all \( h \in [m] \). If

\[
\sum_{i \in [q]^m} a_i \prod_{j \in J} b_{i,j}^{p_{h,j}} = 0
\]

for every choice of \((p_{h,j})_{h \in [m], j \in J}\) where each \( 0 \leq p_{h,j} < 1 \). Then \( \sum_{i \in I_\ell} a_i = 0 \) for every \( \ell \in [s] \).

Proof. Separating the sum over \( i_m \), which we rename to \( i \), we have

\[
\sum_{i \in [q]} \left( \sum_{i_{m-1} \in [q]} a_i \prod_{j \in J, h \in [m-1]} b_{i_{h,j}}^{p_{h,j}} \right) \left( \prod_{j \in J} b_{i,j}^{p_{m,j}} \right) = 0. \tag{3}
\]

Applying Corollary 1 with

\[
I := [q], \quad a_i := \sum_{i_{m-1} \in [q]} a_i \prod_{j \in J, h \in [m-1]} b_{i_{h,j}}^{p_{h,j}} \quad \text{for } i \in [q], \quad \text{and } p_j := p_{m,j},
\]

we obtain

\[
\sum_{i \in I_{i_1}} \left( \sum_{i_{m-1} \in [q]} a_i \prod_{j \in J, h \in [m-1]} b_{i_{h,j}}^{p_{h,j}} \right) = 0. \tag{4}
\]

for every \( \ell_1 \in [s'] \), where \([q] = \bigcup_{\ell \in [s']} I_\ell \) is a partition of \([q]\) into the equivalence classes of \( \approx \). Renaming \( i_{m-1} \) to \( i \), the LHS of (4) is equal to

\[
\sum_{i \in [q]} \left( \sum_{i_{m-2} \in [q]} \left( \sum_{i_{m-1} \in I_{i_1}} a_i \prod_{j \in J, h \in [m-2]} b_{i_{h,j}}^{p_{h,j}} \right) \prod_{j \in J} b_{i,j}^{p_{m-1,j}} \right),
\]

which has a similar form to (3), but with the \( m \)th index removed. After \( m \) repetitions, we eliminate the outer sum and both products and obtain the result. \( \square \)

3 The Interpolation Proof

3.1 The Symmetric Ternary Case

For clarity of exposition, we first prove the special case where all constraint functions are symmetric and ternary. The general proof requires more sophisticated indexing but is not fundamentally different from the following proof of this special case.

Proposition 1. Let \( \mathcal{F} = \{ F_j \mid j \in [\ell] \} \) and \( \mathcal{G} = \{ G_j \mid j \in [\ell] \} \) be compatible constraint function sets with domains \([q_f]\) and \([q_g]\), with \( q_f \geq q_g \), such that every \( F_j \in \mathcal{F} \) and \( G_j \in \mathcal{G} \) are symmetric and have arity 3, and assume \( \mathcal{F} \) is twin-free. Let \( \alpha, \beta \) be the domain weights associated with \( \mathcal{F} \) and \( \mathcal{G} \), respectively.
Let $\varphi : [2k] \to [q_f]$ and $\psi : [2k] \to [q_g]$ for $k \geq 0$, and for every $x, y \in [q_f]$, let

$$I_{xy} = \{ a \in [k] \mid \varphi(a) = x \land \varphi(a + k) = y \}. \quad (5)$$

Assume $\varphi$ is well-balanced—that is, for every $x, y \in [q_f]$, $|I_{xy}| \geq 2q_g$. If $Z^\psi_{F,\alpha}(K) = Z^\psi_{G,\beta}(K \rightarrow \varphi)$ for every $K \in \mathcal{PLI}$, then $q_f = q_g = q$ and there is a domain-weighted isomorphism $\sigma : [q] \to [q]$ from $(F, \alpha)$ to $(G, \beta)$ such that $\psi = \sigma \circ \varphi$.

Proof. Consider 2k-labeled variable set $V_1 = \{ v, u_1, \ldots, u_{2k} \}$, where each $u_\ell$ is labeled $\ell$. For a matrix $\chi \in \{0,1\}^{k \times t}$, define the following set of constraints on $V_1$:

$$C_\chi = \{ (F_j, v, u_a, u_{a+k}) \mid a \in [k], j \in [t], \chi(a, j) = 1 \},$$

define 2k-labeled #CSP($F$) instance $K_\chi = (V_1, C_\chi) \in \mathcal{PLI}_{\text{simp}}[F;2k]$.

Now construct a certain family of $\chi$. $|I_{xy}| \geq 2q_f^2 \geq 2q_f q_g^2$ for every $x, y \in [q_f]$, so, by the pigeonhole principle, there is a function $s : [q_f]^2 \to [q_g]^2$ such that for every $x, y \in [q_f]$, there exists a $J_{xy} \subseteq I_{xy}$ such that $|J_{xy}| \geq 2q_f$ and for every $a \in J_{xy}$, $(\psi(a), \psi(a + k)) = s(x, y)$. For a choice of $(p_{xyj})_{x,y \in [q_f], j \in [t]} \in [0,2q_f]^{|q_f|^2 \times [t]}$, for every $x, y \in [q_f]$ choose an arbitrary $P_{xyj} \subset J_{xy}$ of cardinality $p_{xyj}$ for every $j \in [t]$ and define $\chi = \chi((P_{xyj})_{x,y \in [q_f], j \in [t]})$ by $\chi(a,j) = 1$ for all $x,y \in [q_f], j \in [t]$ and $a \in P_{xyj}$. Set the remaining entries of $\chi$ to 0. Let $R$ be the set of all such matrices $\chi$ for all choices of $(p_{xyj}) \in [0,2q_f]^{[q_f]^2 \times [t]}$.

To recap, for $j \in [t]$, if $a \in P_{xyj} \subset J_{xy} \subset I_{xy} \subset [k]$ for $x, y \in [q_f]$, then $(\varphi(a), \varphi(a + k)) = (x, y)$ and $(\psi(a), \psi(a + k)) = s(x, y)$. Hence the variables $(u_a, u_{a+k})$ take values $(x,y)$ and $(s(x,y))$ under $\varphi$ and $\psi$, respectively. These values are independent of the choice of a within $P_{xyj}$. By construction, $K_\chi$ contains a constraint $(F_j, v, u_a, u_{a+k})$ for every $a \in P_{xyj}$. Therefore $Z_{F,\alpha}(K_\chi) = Z_{G,\beta}((K_\chi) \rightarrow \varphi)$ for every $\chi \in R$ is equivalent to: for all $(p_{xyj}) \in [0,2q_f]^{[q_f]^2 \times [t]}$,

$$\sum_{i=1}^{q_f} \alpha_i \prod_{x,y \in [q_f], j \in [t]} F_j(i,x,y)^{p_{xyj}} = \sum_{i=1}^{q_g} \beta_i \prod_{x,y \in [q_g], j \in [t]} G_j(i,s(x,y))^ {p_{xyj}},$$

where we write $G_j(i,s(x,y))$ to mean $G_j(i,s(x,y)_1, s(x,y)_2)$. The sum over $i$ corresponds to the choice of assignment for the only free variable $v$. Subtracting the RHS, we are left with a sum of $q_f + q_g \leq 2q_f$ terms on the LHS. Treating $[q_f]$ and $[q_g]$ as disjoint, apply Corollary 1 to this sum with

$$I := [q_f] \cup [q_g], \quad J := [q_f]^2 \times [t], \quad \alpha_i := \begin{cases} \alpha_i & i \in [q_f] \\ \beta_i & i \in [q_g] \end{cases}, \quad \beta_i := \begin{cases} F_j(i,x,y) & i \in [q_f] \\ G_j(i,s(x,y)) & i \in [q_g] \end{cases}. \quad (6)$$

$F$ is twin-free, so the tuples $(F_j(i,x,y))_{x,y \in [q_f], j \in [t]}$ are pairwise distinct for $i \in [q_f]$. Hence no equivalence class $I_\ell$ contains more than one element of $[q_f]$. However, every $\alpha_i \neq 0$ by definition, so no equivalence class contains only a single element of $[q_f]$. Thus there is a function $\sigma : [q_f] \to [q_g]$ such that $i \in [q_f]$ is in an equivalence class with $\sigma(i) \in [q_g]$—that is

$$(F_j(i,x,y))_{x,y \in [q_f], j \in [t]} = (G_j(\sigma(i), s(x,y)))_{x,y \in [q_g], j \in [t]} \quad \text{ for } i \in [q].$$

Since no two elements of $[q_f]$ are in the same equivalence class, $\sigma$ is injective, hence bijective, as $q_f \geq q_g$. Thus $q_f = q_g = q$, and we view $\sigma$ as a function $[q] \to [q]$.

Next, define another family of #CSP($F$) instances. Fix $F \in \mathcal{F}$ and corresponding $G \in \mathcal{G}$. Define a new 2k-labeled variable set

$$V_2 = \{ v, v', v'', u_1, \ldots, u_{2k} \},$$

where each $u_\ell$ is labeled $\ell$ (equivalent to the previous $V_1$, but with two new free variables $v'$ and $v''$). For $\chi, \chi', \chi'' \in R$, define the following set of constraints on $V_2$:

$$C_{\chi,\chi',\chi''} = \{ (F, v, v', v'') \} \cup \{(F_j, v, u_a, u_{a+k}) \mid \chi(a,j) = 1 \} \cup \{(F_j, v', u_a, u_{a+k}) \mid \chi'(a,j) = 1 \} \cup \{(F_j, v'', u_a, u_{a+k}) \mid \chi''(a,j) = 1 \},$$

$$7$$
and define \( K_{X,X',X''} = (V_2, C_{X,X',X''}) \in \mathcal{P} \mathcal{L} \mathcal{I}^{\text{imp}}[\mathcal{F}; 2k] \). Every \((\chi, \chi', \chi'') \in R^3\) corresponds to three sequences of subsets \((P_{xyj})\), \((P'_{xyj})\), \((P''_{xyj})\) and three sequences of integers \((p_{xyj})\), \((p'_{xyj})\), \((p''_{xyj})\) \in [0, 2q]^{[q] \times [t]}\), where \(p_{xyj}, p'_{xyj}, p''_{xyj}\) are the cardinalities of \(P_{xyj}, \ P'_{xyj}, \ P''_{xyj}\), respectively, \(P_{xyj}, P'_{xyj}, P''_{xyj} \subseteq I_{xy}\), and \(\chi, \chi', \chi''\) are 1 at entry \((a, j)\) for \(x, y \in [q], j \in [t]\), \(a \in P_{xyj}, P'_{xyj}, P''_{xyj}\), respectively, and are 0 elsewhere.

Now the assumption \( Z_{F,\alpha}^{\psi}((K_{X,X',X''})_{F \rightarrow G}) \) for every \((\chi, \chi', \chi'') \in R^3\) is equivalent to: for all \((p_{xyj}), (p'_{xyj}), (p''_{xyj}) \in [0, 2q]^{[q] \times [t]}\),

\[
\sum_{i,i',i''=1}^{q} \alpha_{i,i',i''} \chi_{i,i',i''} F(i,i',i'') \prod_{x,y \in [q], j \in [t]} F_j(i,x,y)^{p_{xyj}} F_j(i',x,y)^{p'_{xyj}} F_j(i'',x,y)^{p''_{xyj}} = 0.
\]

\((F_j(i,x,y))_{xyj}\) are pairwise distinct for \(i \in [q]\), so the tuples \((F_j(i,x,y), F_j(i',x,y), F_j(i'',x,y))_{xyj}\) are distinct for distinct \((i,i',i'')\). Applying Corollary 2 with

\[
m := 3, J = [q]^2 \times [t], a_{i,i',i''} := \alpha_{i,i',i''} F(i,i',i'') - G(\sigma(i), \sigma(i'), \sigma(i''))\]

\(b_{i,xyj} := F_j(i,x,y), p_{1,xyj} := p_{xyj}, p_{2,xyj} := p'_{xyj}, p_{3,xyj} := p''_{xyj}\)

we obtain \(\alpha_{i,i',i''} F(i,i',i'') - G(\sigma(i), \sigma(i'), \sigma(i'')) = 0\) for all \(i, i', i''\) (each \(p_{xyj}\) ranges over \([0, 2q] \cap [0, q]\). Since each \(\alpha_i \neq 0\) and our choice of \(F\) and \(G\) is arbitrary, this implies

\[
F(i,i',i'') = G(\sigma(i), \sigma(i'), \sigma(i'')) \quad \text{for every} \quad i,i',i'' \in [q] \quad \text{and every corresponding} \quad F \in \mathcal{F}, G \in \mathcal{G}. \quad (7)
\]

Combined with (6), (7) implies that \(\sigma\) is a domain-weighted isomorphism between \((\mathcal{F}, \alpha)\) and \((\mathcal{G}, \beta)\). Since \(\mathcal{F}\) is twin-free, (7) also implies \(\mathcal{G}\) is also twin-free.

It remains to show that \(\psi = \alpha \circ \varphi\). Again let \(F \in \mathcal{F}\) and \(G \in \mathcal{G}\) be corresponding constraint functions. Define a third family of \#CSP(\mathcal{F}) instances. Fix \(c \in [2k]\). Define a \(2k\)-labeled variable set

\[
V_3 = \{v, v', u_1, \ldots, u_{2k}\},
\]

where each \(u_\ell\) is labeled \(\ell\) (equivalent to the previous \(V_2\), but we have removed the free variable \(v''\)). For \((\chi, \chi') \in R^2\), define the following set of constraints on \(V_3:\)

\[
C_{X,X'} = \{(F, u_{c}, v, v') \cup \{F_j, v, u_a, u_{a+k}\} \mid \chi(a,j) = 1\} \cup \{(F_j, v', u_a, u_{a+k}) \mid \chi'(a,j) = 1\},
\]

and define \(K_{X,X'} = (V_3, C_{X,X'}) \in \mathcal{P} \mathcal{L} \mathcal{I}^{\text{imp}}[\mathcal{F}; 2k]\). Now \(Z_{F,\alpha}^{\psi}((K_{X,X'})_{F \rightarrow G})\) for every \((\chi, \chi') \in R^2\) is equivalent to: for all \((p_{xyj}), (p'_{xyj}) \in [0, 2q]^{[q] \times [t]}\),

\[
\sum_{i,i'=1}^{q} \alpha_{i,i'} F(\varphi(c), i,i') \prod_{x,y \in [q], j \in [t]} F_j(i,x,y)^{p_{xyj}} F_j(i',x,y)^{p'_{xyj}} = 0.
\]

Subtracting the RHS and applying (6) and (5) gives

\[
\sum_{i,i'=1}^{q} \alpha_{i,i'} F(\varphi(c), i,i') - G(\psi(c), \sigma(i), \sigma(i')) \prod_{x,y \in [q], j \in [t]} F_j(i,x,y)^{p_{xyj}} F_j(i',x,y)^{p'_{xyj}} = 0.
\]
As above, the tuples \((F_j(i, x, y), F_j(i', x, y))_{xyj}\) are distinct for distinct \((i, i')\), so by a similar application of Corollary 2 with \(m = 2\), we have \(F(\varphi(c), i, i') = G(\psi(c), \sigma(i), \sigma(i'))\) for all \(i, i' \in [q]\). This holds for any corresponding pair \(F \in \mathcal{F} \) and \(G \in \mathcal{G}\), so, by \(\square\),

\[
G_j(\sigma(\varphi(c)), \sigma(i), \sigma(i')) = F_j(\varphi(c), i, i') = G_j(\psi(c), \sigma(i), \sigma(i'))
\]

for all \(i, i' \in [q] \) and \(j \in [t]\). Since \(\mathcal{G}\) is twin-free and \(\sigma\) is a bijection, this gives \(\sigma(\varphi(c)) = \psi(c)\). We chose \(c \in [2k]\) arbitrarily, so \(\psi = \sigma \circ \varphi\).

### 3.2 The General Case

We now extend Proposition 1 to general sets of arbitrary arity, non-necessarily-symmetric constraint functions, containing at least one non-unary symmetric constraint function.

**Lemma 2.** Let \(\mathcal{F} = \{F_j \mid j \in [t]\}\) and \(\mathcal{G} = \{G_j \mid j \in [t]\}\) be compatible constraint function sets with domains \([q_j]\) and \([q_j]\), with \(q_j \geq q_g\) and assume \(\mathcal{F}\) is twin-free. Let \(\alpha, \beta\) be the domain weights associated with \(\mathcal{F}\) and \(\mathcal{G}\), respectively. Let \(n\) be the maximum arity among all functions in \(\mathcal{F}\), and assume \(n \geq 2\). Suppose \(\varphi : [(n-1)k] \rightarrow [q_j]\) and \(\psi : [(n-1)k] \rightarrow [q_g]\) for \(k \geq 0\), and for every \(x \in [q_j]^{n-1}\), let

\[
I_x = \{a \in [k] \mid \varphi(a) + (d-1)k = x_d \text{ for all } d \in [n-1]\}.
\]

Assume \(\varphi\) is well-balanced -- that is, for every \(x \in [q_j]^{n-1}\), \(|I_x| \geq 2nq_j^n\). If \(Z_{\mathcal{F}, \alpha}(K) = Z_{\mathcal{G}, \beta}(K_{\mathcal{F} \rightarrow \mathcal{G}})\) for every \(K \in \mathcal{P}\mathcal{L}\mathcal{T}_{\text{simp}}^{\text{imp}}[\mathcal{F}; (n-1)k]\), then \(q_j = q_g = q\) and there is a domain-weighted isomorphism \(\sigma : [q] \rightarrow [q]\) from \((\mathcal{F}, \alpha)\) to \((\mathcal{G}, \beta)\) such that \(\psi = \sigma \circ \varphi\).

**Proof.** Consider \((n-1)k\)-labeled variable set \(V_1 = \{v\} \cup \{u_a^{(d)}\}_{a \in [k], d \in [n-1]}\), where \(u_a^{(d)}\) is labeled \(a + (d-1)k\).

For a matrix \(\chi \in \{0, 1\}^{k \times n}\) satisfying \(\chi(\ast, j) \subseteq [n] \cup \{\bot\}\), define the following set of constraints on \(V_1\):

\[
C^{\chi}_V = \{(F_j, u_a^{(1)}, \ldots, u_a^{(r-1)}, v, u_a^{(r)}, \ldots, u_a^{(n)}) \mid a \in [k], j \in [t], r \in [n], \chi(a, j) = r\}
\]

Add no constraints for \(\chi(a, j) = \bot\). Define a \((n-1)k\)-labeled \#CSP(\(\mathcal{F}\)) instance \(K_\chi = (V_1, C^{\chi}_V) \in \mathcal{P}\mathcal{L}\mathcal{T}_{\text{simp}}^{\text{imp}}[\mathcal{F}; (n-1)k]\).

Now construct a certain family of \(\chi\). \(|I_x| \geq 2nq_j^n \geq 2nq_j q_g^{n-1}\) for every \(x \in [q_j]^{n-1}\), so, by the pigeonhole principle, there is a function \(s : [q_j]^{n-1} \rightarrow [q_g]^{n-1}\) such that for every \(x \in [q_j]^{n-1}\), there exists a \(J_x \subseteq I_x\) such that \(|J_x| \geq 2nq_j^n\) and, for every \(a \in J_x\), \(\varphi(a + (d-1)k))_{d \in [n-1]} = s(x)\). For \(n_1 \leq n\) and \(x \in [q_j]^{n_1-1}\), let \(\text{ext}(x) \in [q_j]^{n_1-1}\) denote the extension of \(x\), defined by

\[
\text{ext}(x)_i = \begin{cases} 
  x_i & i \leq n_1 - 1 \\
  1 & i > n_1 - 1
\end{cases}.
\]

The choice of \(1 \in [q_j]\) is arbitrary. If \(n_1 = 1\) and \(x = ()\), then \(\text{ext}(x)\) is the all-ones vector, though this is completely arbitrary, as in this case \((F_j, u_a^{(1)}, \ldots, u_a^{(r-1)}, v, u_a^{(r)}, \ldots, u_a^{(n)}) = (F_j, v)\) and \(F_j(x_r^{1-1}, i, x_r^{n_1-1}) = F_j(i)\), so one will see below that the entries of \(\text{ext}(x)\) are irrelevant.

Extend \(s\) to a function on \(\bigcup_{d=1}^{n-1} [q_j]^d\) by \(s(x) := s(\text{ext}(x))\). For every \((j, x, r) \in \mathcal{J}(\mathcal{F})\) \(\square\), fix an arbitrary subset \(P_{j,x,r} \subset J_{\text{ext}(x)}\) with cardinality \(p_{j,x,r}\), such that, for fixed \(x\) and \(j\), \(P_{j,x,r}\) are disjoint for distinct \(r \in [n]\). This is possible for values of \(p_{j,x,r}\) up to \(2q_f\) because \(|J_{\text{ext}(x)}| \geq 2nq_f\) and \(r\) can take at most \(n\) distinct values. For a fixed choice of \((p_{j,x,r}) \in [0, 2q_f]^{\mathcal{J}(\mathcal{F})}\), define \(\chi = \chi((P_{j,x,r})_{(j,x,r) \in \mathcal{J}(\mathcal{F})})\) by \(\chi(a, j) = r\), for all \((j, x, r) \in \mathcal{J}(\mathcal{F})\) and \(a \in P_{j,x,r}\). Set the remaining entries of \(\chi\) to \(\bot\). Let \(R\) be the set of all such matrices \(\chi\) for all choices of \((p_{j,x,r}) \in [0, 2q_f]^{\mathcal{J}(\mathcal{F})}\).

To recap, for \(j \in [t]\), if \(a \in P_{j,x,r} \subset J_{\text{ext}(x)}\) for \(x \in [q_j]^{n_1-1}\) and \(r \in [n]\), then \(\varphi(a + (d-1)k) = x_d\) and \(\psi(a + (d-1)k) = s(\text{ext}(x))_d = s(x)_d\) for \(d \in [n_1 - 1]\). Hence the variable \(u_a^{(d)}\) takes value \(x_d\) and \(s(x)_d\) under \(\varphi\) and \(\psi\), respectively, for \(d \in [n_1 - 1]\). These values are independent of the choice of \(a\) within \(P_{j,x,r}\). By construction, \(K_\chi\) contains a constraint \((F_j, u_a^{(1)}, \ldots, u_a^{(r-1)}, v, u_a^{(r)}, \ldots, u_a^{(n)})\) for every \(a \in P_{j,x,r}\). Therefore \(Z_{\mathcal{F}, \alpha}(K_\chi) = Z_{\mathcal{G}, \beta}(K_{\mathcal{F} \rightarrow \mathcal{G}})\) for every \(\chi \in R\) is equivalent to: for all \((p_{j,x,r}) \in [0, 2q_f]^{\mathcal{J}(\mathcal{F})}\),

\[
\sum_{i=1}^{q_j} \alpha_i \prod_{(j, x, r) \in \mathcal{J}(\mathcal{F})} F_j(x_r^{1-1}, i, x_r^{n_1-1})^{p_{j,x,r}} = \sum_{i=1}^{q_g} \beta_i \prod_{(j, x, r) \in \mathcal{J}(\mathcal{F})} G_j(s(x)_r^{1-1}, i, s(x)_r^{n_1-1})^{p_{j,x,r}}.
\]
where the sum over \( i \) corresponds to the choice of assignment for the only free variable \( v \). Subtracting the RHS, we are left with a sum of \( q_f + q_g \leq 2q_f \) terms on the LHS. Treating \([q_f]\) and \([q_g]\) as disjoint, apply \textbf{Corollary 1} to this sum with

\[
I := [q_f] \cup [q_g], \quad J := \mathcal{J}(\mathcal{F}), \quad a_i := \left\{ \begin{array}{ll}
\alpha_i & i \in [q_f] \\
\beta_i & i \in [q_g]
\end{array} \right., \quad b_{i,j,r} := \left\{ \begin{array}{ll}
F_j(x_1^{r-1}, i, x_r^{n_j-1}) & i \in [q_f] \\
G_j(s(x)^{r-1}, i, s(x)^{n_j-1}) & i \in [q_g]
\end{array} \right.
\]

\( \mathcal{F} \) is twin-free, so the tuples \((F_j(x_1^{r-1}, i, x_r^{n_j-1}))(j,x,r) \in \mathcal{J}(\mathcal{F})\) are pairwise distinct for \( i \in [q_f] \). Hence no equivalence class \( I_i \) contains more than one element of \([q_f]\). However, every \( \alpha_i \neq 0 \) by definition, so no equivalence class contains only a single element of \([q_f]\). Thus there is a function \( \sigma : [q_f] \to [q_g] \) such that every \( i \in [q_f] \) in an equivalence class with \( \sigma(i) \in [q_g] \) — that is

\[
(F_j(x_1^{r-1}, i, x_r^{n_j-1}))(j,x,r) \in \mathcal{J}(\mathcal{F}) = (G_j(s(x)^{r-1}, i, s(x)^{n_j-1}))(j,x,r) \in \mathcal{J}(\mathcal{F}).
\]

(8)

Since no two elements of \([q_f]\) are in the same equivalence class, \( \sigma \) is injective, hence bijective, as \( q_f \geq q_g \). Thus \( q_f = q_g = q \), and we view \( \sigma \) as a function \([q_f] \to [q_g]\). \textbf{Corollary 1} then gives

\[
\alpha_i = \beta_{\sigma(i)} \text{ for } i \in [q_f].
\]

(9)

Next, define another family of \#CSP(\( \mathcal{F} \)) instances. Fix \( F \in \mathcal{F} \) and corresponding \( G \in \mathcal{G} \), with common arity \( n_F \). Define a new \((n-1)k\)-labeled variable set

\[
V_2 = \{v_h \mid h \in [n_F]\} \cup \{u_a^{(d)}\}_{a \in [k], d \in [n-1]}
\]

where \( u_a^{(d)} \) is labeled \( a + (d-1)k \). For \( \chi_1, \ldots, \chi_{n_F} \in R \), define the following set of constraints on \( V_2 \):

\[
C_{\chi_1^{n_F}} = \{(F, v_1, \ldots, v_{n_F})\}
\]

\[
\cup \{(F, u_a^{(1)}, \ldots, u_a^{(r-1)}, v_h, u_a^{(r)}, \ldots, u_a^{(n_j)}) \mid a \in [k], j \in [t], r \in [n_j], h \in [n_F], \chi_h(a, j) = r\}.
\]

Let \( K_{\chi_1^{n_F}} = (V_2, C_{\chi_1^{n_F}}) \in \mathcal{PLT}^{\text{simp}}[\mathcal{F}; (n-1)k] \). Every \( \chi_h \in R \) corresponds to a sequence of subsets \((P_{h,j,x,r})_{(j,x,r) \in \mathcal{J}(\mathcal{F})}\) and sequence of integers \((p_{h,j,x,r})_{j,x,r} \in [0,2q)^{\mathcal{J}(\mathcal{F})}\) such that \( p_{h,j,x,r} \) is the cardinality of \( P_{h,j,x,r} \), \( P_{h,j,x,r} \subset J_{\text{ext}(x)} \subset I_{\text{ext}(x)} \), and \( \chi_h(a, j) = r \) for \( j \in [t], x \in [q]^{n_j-1} \), and \( a \in P_{h,j,x,r} \), and is \( \perp \) elsewhere.

Now the assumption \( Z_{\mathcal{F},\alpha}^{\mathcal{G}}(K_{\chi_1^{n_F}}) = Z_{\mathcal{G},\beta}^{\mathcal{G}}((K_{\chi_1^{n_F}})_{\mathcal{F},\mathcal{G}}) \) for every \((\chi_1^{n_F}) \in R^{n_F}\) is equivalent to: for all \((p_{h,j,x,r})_{j,x,r} \in [0,2q)^{\mathcal{J}(\mathcal{F})}\),

\[
\sum_{i \in [q]^{n_F}} \left( \prod_{h=1}^{n_F} \alpha_{ih} \right) F(i) \prod_{(j,x,r) \in \mathcal{J}(\mathcal{F}), h \in [n_F]} F_j(x_1^{r-1}, i, x_r^{n_j-1})^{p_{h,j,x,r}} = \sum_{i \in [q]^{n_F}} \left( \prod_{h=1}^{n_F} \beta_{ih} \right) G(i) \prod_{(j,x,r) \in \mathcal{J}(\mathcal{F}), h \in [n_F]} G_j(s(x)^{r-1}, i, s(x)^{n_j-1})^{p_{h,j,x,r}}
\]

where the sum over \( i \) corresponds to the choice of assignment for the free variables \( v_1^{n_F} \). Subtracting the RHS and applying (9) and (8) gives

\[
\sum_{i \in [q]^{n_F}} \left( \prod_{h=1}^{n_F} \alpha_{ih} \right) (F(i) - G(\sigma(i))) \prod_{(j,x,r) \in \mathcal{J}(\mathcal{F}), h \in [n_F]} F_j(x_1^{r-1}, i, x_r^{n_j-1})^{p_{h,j,x,r}} = 0.
\]

\((F_j(x_1^{r-1}, i, x_r^{n_j-1}))(j,x,r) \in \mathcal{J}(\mathcal{F})\) are distinct for \( i \in [q] \), so the tuples \((F_j(x_1^{r-1}, i, x_r^{n_j-1}))_{h \in [n_F], (j,x,r) \in \mathcal{J}(\mathcal{F})}\) are distinct for distinct \( i \in [q]^{n_F} \). Applying \textbf{Corollary 2} with

\[
m := n_F, J = \mathcal{J}(\mathcal{F}), a_i := \left( \prod_{h=1}^{n_F} \alpha_{ih} \right) (F(i) - G(\sigma(i))), \text{ and } b_{i,j,x,r} := F_j(x_1^{r-1}, i, x_r^{n_j-1}),
\]

10
we obtain \((\prod_{h=1}^{n_F} \alpha_{i_h})(F(i) - G(\sigma(i))) = 0\) for all \(i\) (each \(p_{h,j}x_j\) ranges over \([0,2q] \supset [0,q]\)). Since each \(\alpha_i \neq 0\) and our choice of \(F\) and \(G\) was arbitrary, this implies

\[
F(i) = G(\sigma(i)) \quad \text{for every } i \in [q]^{n_F} \text{ and every corresponding } F \in \mathcal{F}, G \in \mathcal{G}.
\]

(10)

Combined with (9), (10) implies that \(\sigma\) is a domain-weighted isomorphism between \((\mathcal{F}, \alpha)\) and \((\mathcal{G}, \beta)\). Since \(\mathcal{F}\) is twin-free, (10) also implies \(\mathcal{G}\) is also twin-free.

It remains to show that \(\psi = \sigma \circ \varphi\). Again let \(F \in \mathcal{F}\) and \(G \in \mathcal{G}\) be corresponding constraint functions with common arity \(n_F\). Fix \(c \in [(n-1)k]\). We aim to show that \(F(\varphi(c), i) = G(\psi(c), \sigma(i))\) for all \(i \in [q]^{n_F-1}\). If \(n_F = 1\), let \(K \in \mathcal{PLT}^{\text{simp}}[\mathcal{F}; (n-1)k]\) be an instance with no unlabeled/free variables and a single constraint \((F,v_c)\), where \(v_c\) is the variable labeled \(c\). Then by assumption we have

\[
F(\varphi(c)) = Z_{\mathcal{F}, \alpha}^p(K) = Z_{\mathcal{G}, \beta}^p(K_{\mathcal{F} \rightarrow \mathcal{G}}) = G(\psi(c)),
\]

(11)

as desired (recall \(i \in [q]^0 \implies i = (())\)).

Otherwise, if \(n_F \geq 2\), define a third family of \(#\text{CSP}(\mathcal{F})\) instances as follows. Define a \((n-1)k\)-labeled variable set

\[
V_3 = \{v_h \mid h \in [n_F - 1]\} \cup \{u_{a}^{(d)}\}_{a \in [k], d \in [n-1]}
\]

where \(u_{a}^{(d)}\) is labeled \(a + (d-1)k\) (equivalent to the previous \(V_2\), but we have removed the free variable \(v_{n_F}\)). Write \(c = ac + (d_c - 1)k\) (so that \(u_{a_c}^{(d_c)}\) is labeled \(c\)). For \(\chi_1, \ldots, \chi_{n_F-1} \in R\), define the following set of constraints on \(V_3\):

\[
C_{\chi_i}^{n_F-1} = \{(F, u_{a_c}^{(d)}), v_1, \ldots, v_{n_F-1}\} \cup \{(F_j, u_a^{(1)}, \ldots, u_a^{(r-1)}, v_h, u_a^{(r)}, \ldots, u_a^{(n_i)}) \mid a \in [k], j \in [t], r \in [n_j], h \in [n_F - 1], \chi_h(a, j) = r\}.
\]

Let \(K_{\chi_i}^{n_F-1} = (V_3, C_{\chi_i}^{n_F-1}) \in \mathcal{PLT}^{\text{simp}}[\mathcal{F}; (n-1)k]\).

Now the assumption \(Z_{\mathcal{F}, \alpha}^p(K_{\chi_i}^{n_F-1}) = Z_{\mathcal{G}, \beta}^p((K_{\chi_i}^{n_F-1})_{\mathcal{F} \rightarrow \mathcal{G}})\) for every \(\chi_i^{n_F-1} \in R^{n_F-1}\) is equivalent to:

for all \((p_{h,j}x_j)_{h,j \in \mathcal{J}(\mathcal{F})},[0,2q]^{n_F-1} \times \mathcal{J}(\mathcal{F})\),

\[
\sum_{i \in [q]^{n_F-1}} \left(\prod_{h=1}^{n_F-1} \alpha_{i_h}\right) F(\varphi(c), i) \prod_{(j, x, r) \in \mathcal{J}(\mathcal{F}), h \in [n_F-1]} F_j(x_1^{r-1}, i_h, x_r^{n_j-1})^{p_{h,j}x_j} = \sum_{i \in [q]^{n_F-1}} \left(\prod_{h=1}^{n_F-1} \beta_{i_h}\right) G(\psi(c), i) \prod_{(j, x, r) \in \mathcal{J}(\mathcal{F}), h \in [n_F-1]} G_j(s(x)_1^{r-1}, i_h, s(x)_r^{n_j-1})^{p_{h,j}x_j}.
\]

Subtracting the RHS and applying (9) and (10) gives

\[
\sum_{i \in [q]^{n_F-1}} \left(\prod_{h=1}^{n_F-1} \alpha_{i_h}\right) (F(\varphi(c), i) - G(\psi(c), \sigma(i))) \prod_{(j, x, r) \in \mathcal{J}(\mathcal{F}), h \in [n_F-1]} F_j(x_1^{r-1}, i_h, x_r^{n_j-1})^{p_{h,j}x_j} = 0.
\]

As above, the tuples \((F_j(x_1^{r-1}, i_h, x_r^{n_j-1}))_{h \in [n_F-1], (j, x, r) \in \mathcal{J}(\mathcal{F})}\) are distinct for distinct \(i \in [q]^{n_F-1}\). Hence by a similar application of Corollary 2 with \(m := n_F - 1\), we have \(F(\varphi(c), i) = G(\psi(c), \sigma(i))\) for all \(i \in [q]^{n_F-1}\).

This holds for any corresponding pair \(F \in \mathcal{F}\) and \(G \in \mathcal{G}\). Additionally, the reasoning is independent of the input order in \(F(\varphi(c), i)\) and \(G(\psi(c), \sigma(i))\). Hence, by (9),

\[
G_j(\sigma(i)_1^{r-1}, \varphi(c), \sigma(i)_r^{n_j-1}) = F_j(i_1^{r-1}, \varphi(c), i_r^{n_j-1}) = G_j(\sigma(i)_1^{r-1}, \psi(c), \sigma(i)_r^{n_j-1})
\]

for all \((j, i, r) \in \mathcal{J}(\mathcal{G})\). Since \(\mathcal{G}\) is twin-free and \(\sigma\) is a bijection, this gives \(\sigma(\varphi(c)) = \psi(c)\). We chose \(c \in [(n-1)k]\) arbitrarily, so \(\psi = \sigma \circ \varphi\).
Theorem 2. Let $F$ and $G$ be compatible constraint function sets with domains $[q_f]$ and $[q_g]$, with $q_f \geq q_g$, and $\alpha$ and $\beta$ are the domain weights associated with $F$ and $G$, respectively. Assume $F$ is twin-free. Let $k \geq 0$ and $\varphi : [k] \to [q_f]$ and $\psi : [k] \to [q_g]$. If $Z_{F,\alpha}^\psi(K) = Z_{G,\beta}^\varphi(K_{F \to G})$ for every $K \in \mathcal{PL}\mathcal{L}^{\text{simp}}[F;k]$, then $q_f = q_g = q$ and there is an domain-weighted isomorphism $\sigma : [q] \to [q]$ between $(F,\alpha)$ and $(G,\beta)$ such that $\psi = \sigma \circ \varphi$.

Proof. First handle the case where $F$ and $G$ contain only unary constraint functions, where Lemma 2 does not apply. Say $F \equiv G = 0$. For every $p \in [2q_f]^l$, let $K_p \in \mathcal{PL}\mathcal{L}^{\text{simp}}[F;k]$ be the instance defined by ignoring the labeled variables, and there is an isomorphism $\varphi$ and $\psi$, and there is a domain-weighted isomorphism $\sigma : [q] \to [q]$ between $(F,\alpha)$ and $(G,\beta)$. Now the unary function argument in the third step of the proof of Lemma 2 concluding with $\sigma$ gives $F \circ \varphi = G \circ \psi$ for every $j \in [l]$. Hence $G \circ \sigma \circ \varphi = F \circ \varphi = G \circ \psi$ for every $j \in [l]$, so since $G$, being isomorphic to $F$, is twin-free, $\sigma \circ \varphi = \psi$.

Otherwise, if $F$ and $G$ contain a function with arity $\geq 2$, the proof is a simple generalization of the proof of [6, Theorem 3.1]. We generalize $F$-weighted graphs $H$ and $H'$ to constraint function sets $F$ and $G$, $\ell$-labeled graphs $G \in \mathcal{PL}\mathcal{L}^{\text{simp}}[\ell]$ to $\ell$-labeled #CSP($\ell$) instances $K \in \mathcal{PL}\mathcal{L}^{\text{simp}}[\ell]$, and $\text{hom}_\mu(G,H)$ and $\text{hom}_\mu(G,H')$ to $Z_{F,\alpha}^\psi(K)$ and $Z_{G,\beta}^\varphi(K_{F \to G})$, respectively. The #CSP generalizations satisfy analogous properties to the special case of graph homomorphisms. In particular we use our Lemma 2 in place of [6, Lemma 6.1] (with “well-balanced” in place of “super-surjective”) and our $\text{hom}_\mu$ – the multiplicativity of $Z_{F,\alpha}^\psi$ in place of the multiplicativity of $\text{hom}_\mu(\cdot,H)$.

Next, we introduce domain weights to constraint function sets with unit domain weights (equivalently, no domain weights) to remove the twin-free requirement. We have the following generalization of [6, Corollary 6.2]

Corollary 3. Let $F$ and $G$ be compatible constraint function sets with domains $[q_f]$ and $[q_g]$. Let $k \geq 0$, $\varphi : [k] \to [q_f]$, and $\psi : [k] \to [q_g]$. If $Z_{F}^\varphi(K) = Z_{G}^\psi(K_{F \to G})$ for every $K \in \mathcal{PL}\mathcal{L}^{\text{simp}}[F;k]$, then $q_f = q_g = q$ and there is an isomorphism $\sigma : [q] \to [q]$ between $F$ and $G$ such that $\psi = \sigma \circ \varphi$, where $\psi(i)$ is a twin of $\varphi(i)$ for every $i \in [k]$.

Proof. The constraint function set twin-contraction procedure $F \mapsto \tilde{F}$ described in Section 2 is a generalization of and satisfies the same properties as the $F$-weighted graph contraction $H \mapsto \tilde{H}$ in [6]. Hence the proof is a simple generalization of the proof of [6, Corollary 6.2], where we use Theorem 2, $F$, $G$, $[q_f]$, $[q_g]$, and $\mathcal{PL}\mathcal{L}^{\text{simp}}[F;k]$ in place of [6, Theorem 3.1], $H$, $H'$, $V(H)$, $V(H')$, and $\mathcal{PL}\mathcal{L}^{\text{simp}}[k]$, respectively.

Finally, we have the following result for ordinary (unlabeled) #CSP instances, a slightly stronger version of Theorem 1. Say an unlabeled #CSP($\mathcal{F}$) instance $K$ is simple if $K \in \mathcal{PL}\mathcal{L}^{\text{simp}}[\mathcal{F};0]$ (equivalently, the corresponding bipartite variable-constraint incidence graph has no multiedges and the multiplicity of every constraint in $C$ is 1 up to permutation of its variable order).

Corollary 4. Let $F$ and $G$ be compatible constraint function sets. Then $F \equiv G$ if and only if $Z_{F}(K) = Z_{G}(K_{F \to G})$ for every simple #CSP($\mathcal{F}$) instance $K$.

Proof. We only need the backward direction, which is the $k = 0$ case of Corollary 3.

The next observation is a generalization of [6, Remark 2].

Remark 1. For compatible constraint function sets $F$ and $G$ with common domain $[q]$ and $\varphi, \psi : [k] \to [q]$, Theorem 2 asserts that if there is no isomorphism $\sigma$ between $F$ and $G$ satisfying $\psi = \sigma \circ \varphi$, then there is some witness instance $K \in \mathcal{PL}\mathcal{L}^{\text{simp}}[F;k]$ such that $Z_{F}^\varphi(K) \neq Z_{G}^\psi(K_{F \to G})$. The proofs of Lemma 2 and
where \( F \) is the finite list constructed as follows. In the proof of Theorem 2 (see [6]), we extend \( \varphi \) to a well-balanced map with domain \([\ell]\), where \( \ell \leq k + 2nq^{2n-1} \) (\( n \) is the maximum arity among functions in \( F \)). Let

\[
S = \{(V_1, C_\chi) \mid \chi \in R\} \cup \{(V_2, C_{\chi_{1,F}^p}) \mid F \in \mathcal{F}, \chi_{1,F}^p \in R\}
\]

\[
\cup \{(V_3, C_{\chi_{1,F}^{p-1}}) \mid F \in \mathcal{F}, \chi_{1,F}^{p-1} \in R\} \subset \mathcal{P}\mathcal{L}\mathcal{T}\text{imp}[\mathcal{F}; \ell]
\]

be the (finite) set of all \#CSP(\( \mathcal{F} \)) instances constructed in the three steps of the proof of Lemma 2. The proof of Theorem 2 constructs the finite set

\[
\mathcal{P} = \left\{ \prod_{k \in S} K^{h_k} \mid \text{each } 0 \leq h_k < 2q^\ell \right\} \subset \mathcal{P}\mathcal{L}\mathcal{T}\text{imp}[\mathcal{F}; \ell],
\]

where \( K^{h_k} \) is a product in \( \mathcal{P}\mathcal{L}\mathcal{T}\text{imp}[\mathcal{F}; \ell] \). The proof shows that there is an isomorphism \( \sigma \) between \( \mathcal{F} \) and \( \mathcal{G} \) satisfying \( \psi = \sigma \circ \varphi \) if and only if \( Z_\mathcal{F}^\psi(K) = Z_\mathcal{G}^\varphi(K_{\mathcal{F} \rightarrow \mathcal{G}}) \) for every \( K \in \pi_{[k]}(\mathcal{P}) \subset \mathcal{P}\mathcal{L}\mathcal{T}\text{imp}[\mathcal{F}; k] \), where \( \pi_{[k]} : \mathcal{P}\mathcal{L}\mathcal{T}\text{imp}[\mathcal{F}; \ell] \rightarrow \mathcal{P}\mathcal{L}\mathcal{T}\text{imp}[\mathcal{F}; k] \) erases the labels \( k + 1, \ldots, \ell \).

4 The Intertwiner Proof

In this section, we give an alternate proof of Theorem 1 for the case \( \mathcal{F} = \mathbb{C} \). Throughout this section, we also assume constraint function sets \( \mathcal{F} \) over \( \mathbb{C} \) are conjugate closed, meaning \( F \in \mathcal{F} \iff \overline{F} \in \mathcal{F} \), where \( \overline{F} \) is the entrywise conjugate of \( F \). In particular, any set of real-valued constraint functions is conjugate-closed.

The following construction ‘flattens’ a constraint function into a matrix.

**Definition 7** \((F^{m,d,f})\). For constraint function \( F \) of domain \([q]\) and arity \( n \) and any \( m, d \geq 0, m + d = n \), define \( F^{m,d} \in \mathbb{R}^{q^n \times q^d} \) by \( F^{m,d}_{x_1 \ldots x_m \ldots x_{m+1}} = F(x) \), where \( x_1 \ldots x_m \in \mathbb{N} \) is the base-\( q \) integer with most significant digit \( x_1 \), and similarly for \( x_n \ldots x_{m+1} \). Abbreviate \( f := F^{n,0} \in \mathbb{R}^n \), called the signature vector of \( F \).

Note that the bits of the column index of \( F^{m,d} \) are reversed. This is done so that the definition matches Definition 9 of a gadget signature matrix below.

4.1 Holant Problems and Gadgets

The proof is carried out in the Holant framework, a generalization of \#CSP. Like a \#CSP problem, a Holant problem Holant(\( \mathcal{F} \)) is is parameterized by a set \( \mathcal{F} \) of constraint functions, all on the same domain \( V(\mathcal{F}) \), called signature functions or signatures. The input to Holant(\( \mathcal{F} \)) is a signature grid \( \Omega \), which consists of an underlying multigraph with vertex set \( V \) and edge set \( E \), along with an assignment to each \( v \in V \) a signature \( F_v \in \mathcal{F} \) of arity \( \text{deg}(v) \). The incident edges \( E(v) \) to \( v \) are given an order and serve as the input variables to \( F_v \), taking values in \( V(\mathcal{F}) \). The output on input \( \Omega \) is

\[
\text{Holant}_\Omega(\mathcal{F}) = \sum_{\sigma: E \rightarrow V(\mathcal{F})} \prod_{v \in V} F_v(\sigma|_{E(v)}),
\]

where \( F_v(\sigma|_{E(v)}) \) is the evaluation of \( F_v \) on the ordered tuple \( \sigma|_{E(v)} \), the restriction of \( \sigma \) to \( E(v) \). For example, counting perfect matchings or proper edge colorings are expressed by assigning the \textsc{Exact-One} or \textsc{Disequality} function to each vertex, respectively. For sets \( \mathcal{F} \) and \( \mathcal{G} \) of signatures define the problem Holant(\( \mathcal{F} \mid \mathcal{G} \)), which takes as input a signature grid with a bipartite underlying multigraph with partition \( V = V_1 \cup V_2 \) such that the vertices in \( V_1 \) and \( V_2 \) are assigned signatures from \( \mathcal{F} \) and \( \mathcal{G} \), respectively.

We next define some particular constraint functions that we will use throughout this section.

**Definition 8** \((E_n, E^{m,d}, \mathcal{E}\mathcal{Q})\). Define the \( n \)-ary \textit{equality} constraint function \( E_n \) by \( E_n(x_1, \ldots, x_n) = 1 \) if \( x_1 = \ldots = x_n \), and 0 otherwise. Write \( E^{m,d} := (E_n)^{m,d} \), as we must have \( m + d = n \). Define \( \mathcal{E}\mathcal{Q} = \bigcup_n E_n \).
Let $\mathcal{F}$ be a set of constraint functions. To each $\#\text{CSP}(\mathcal{F})$ instance $K = (V, C)$ we associate a signature grid $\Omega_K$ in the context of Holant($\mathcal{F} | \mathcal{E} \mathcal{Q}$) defined as follows: For every constraint $c \in C$, if $c$ applies function $F$ of arity $n$, create a degree-$n$ vertex $u_c$, assigned $F$, called a constraint vertex. For each variable $v \in V$, if $v$ appears in constraints $C_v \subseteq C$, create a degree-$|C_v|$ vertex $u_v$, called an equality vertex, assigned $E_{C_v}$, and edges $(u_v, u_c)$ for every $c \in C_v$ (if $v$ appears in no constraints, the corresponding vertex is isolated). Assign the order of edges incident to $u_v$ to match the order of variables in $c$. Any edge assignment $\sigma$ must assign all edges incident to an equality vertex the same value (or else the term corresponding to $\sigma$ is 0), so we can view $\sigma$ as $\#\text{CSP}$ variable assign. Hence $Z_\mathcal{F}(K) = \text{Holant}_{\Omega_K}(\mathcal{F} | \mathcal{E} \mathcal{Q})$.

**Definition 9** (Gadget, $T(K)$, $\Theta(k, \ell)$, $\Theta_F(k, \ell)$, $\Theta_S(k, \ell)$). A gadget is a Holant signature grid equipped with an ordered set of dangling edges (edges with only one endpoint), defining external variables.

Let $K$ be a gadget with $n$ dangling edges and containing signatures of domain size $q$. For any $k, \ell \geq 0$, $\ell + k = n$, define $K$’s $(k, \ell)$-signature matrix $T(K) \in \mathbb{C}^{q^k \times q^\ell}$ by setting $T(K)_{x,y}$ to be the Holant value when the first $k$ dangling edges (called output dangling edges) are assigned $x_1, \ldots, x_k$ and the last $\ell$ dangling edges (called input dangling edges) are assigned $y_{\ell}, \ldots, y_n$. Draw the output/input dangling edges to the left/right of the gadget, respectively, in cyclic order (outputs from top to bottom and inputs from bottom to top).

Let $\Theta(k, \ell)$ be the collection of all gadgets with $k$ output and $\ell$ input dangling edges, and $\Theta_F(k, \ell) \subseteq \Theta(k, \ell)$ be the subcollection of gadgets in the context of Holant($\mathcal{F} | \mathcal{E} \mathcal{Q}$) and with all dangling edges incident to equality vertices. Let $\Theta_S = \bigcup_{k,\ell} \Theta_F(k,\ell)$.

$K$’s input dangling edges receive their inputs in reverse order in the definition of $T(K)$. This is done so that the output and input dangling edges both receive their inputs in order from top to bottom, so that the dangling edges merged in the composition operation below line up when we draw the gadgets being composed.

**Definition 10** (Gadget $\circ$, $\otimes$, $\ast$).

- Given $K_1 \in \Theta(j, k), K_2 \in \Theta(k, \ell)$, define the composition $K_1 \circ K_2 \in \Theta(j, \ell)$ by placing $K_2$ to the right of $K_1$, and merging the $i$th input dangling edge of $K_1$ with the $k - (i - 1)$st output dangling edge of $K_2$, for $i \in [k]$. If composition makes vertices assigned $E_a, E_b \in \mathcal{E} \mathcal{Q}$ adjacent, contract the edge between them and assign the resulting merged vertex $E_{a+b-2}$. This does not change the Holant value.

- For gadgets $K_1 \in \Theta(k_1, \ell_1), K_2 \in \Theta(k_2, \ell_2)$, define the tensor product $K_1 \otimes K_2 \in \Theta(k_1 + k_2, \ell_1 + \ell_2)$ by taking the disjoint union of the multigraphs underlying $K_1$ and $K_2$, placing $K_1$ above $K_2$.

- For $K \in \Theta(k, \ell)$, define the conjugate transpose $K^\ast \in \Theta(k, \ell)$ by reflecting $K$’s underlying multigraph horizontally, and replacing every signature $F$ with $\overline{F}$.

It is well known applying the $\circ, \otimes, \ast$ operations to gadgets corresponds to applying these operations to their signature matrices. See e.g. [8].

A $(k + \ell)$-labeled $\#\text{CSP}(\mathcal{F})$ instance $K \in P\mathcal{L}I[\mathcal{F}; k + \ell]$ corresponds to a gadget $K \in G_F(k, \ell)$ with dangling edges incident to the equality vertices constructed from the labeled variables. For a map $\psi : [k + \ell] \to V(\mathcal{F})$ assigning the labeled variables $x_1, \ldots, x_k, y_1, \ldots, y_{\ell}$ (i.e. $\psi([k + \ell]) = (\psi(1), \ldots, \psi(k + \ell)) = (x_1, \ldots, x_k, y_1, \ldots, y_{\ell}))$, we have $T_F(K)_{x,y} = Z^\psi(K)$, since giving an equality vertex an input $x$ along a dangling edge forces all of its adjacent edges to take value $x$, pinning the corresponding variable to $x$.

**Definition 11** ($\mathcal{E}^{m,d} \ast \mathcal{I} \ast \mathcal{F}$). For $m, d \geq 0$, let $\mathcal{E}^{m,d}$ be the gadget consisting of a single vertex, assigned $E_{m+d}$, with $m$ output and $d$ input dangling edges. Define $\mathcal{I} = \mathcal{E}^{1,1}$. For $n$-ary signature function $F$ let $\mathcal{F}$ be the gadget consisting of a degree-$n$ vertex assigned $F$ and $n$ output dangling edges, with the ith dangling edge serving as the ith input to $F$. See [Figure 2] for illustrations.

Since the signature $E_{m+d}$ is symmetric in the order of its inputs, we do not have to specify which input to $E_{m+d}$ each dangling edge corresponds to. Observe that $T(\mathcal{E}^{m,d}) = \mathcal{E}^{m,d}$ and $T(\mathcal{F}) = \mathcal{F}^{m,d,0} = f$.

**Definition 12** ($\mathcal{S}_\sigma$, $\mathcal{S}_\sigma$, $\mathcal{S}_\sigma$, $\mathcal{S}$). For permutation $\sigma \in S_k$, let $\mathcal{S}_\sigma \in \Theta(k, k)$ be the gadget formed from $\mathcal{I}^\otimes k$ by permuting the dangling ends of the input dangling edges according to $\sigma$—that is, the $i$th output dangling
edge is incident to the same $E_2$ vertex as the $\sigma(i)$th input dangling edge. Here we consider both input and output dangling edges in top-to-bottom order, as we do for signature matrices.

Define a $2k$-ary constraint function $S_\sigma$ by

$$S_\sigma(x_1, \ldots, x_k) = \begin{cases} 1 & x_i = y_{\sigma(i)} \text{ for all } i \in [k] \\ 0 & \text{otherwise} \end{cases}.$$ 

Then we have $T(S_\sigma) = S_\sigma^{k,k}$. We will make particular use of $S_{(1\ 2)} \in \mathcal{G}(2, 2)$, so we abbreviate $S := S_{(1\ 2)}$ and $S := S_{(1\ 2)}$ (so that $S^{2,2} = T(S)$).

We will compose $S_\sigma$ with other gadgets to permute their dangling edges. We generally treat a gadget containing $E_2$ vertices as equal to the gadget created by erasing these vertices from the edge they lie on, since this has no effect on the Holant value. Hence we can also view $S_\sigma$ as a gadget composed solely of two-sided dangling edges, with the $i$th output and $\sigma(i)$th input dangling edges being the same edge, or as a braid where we ignore the crossing order. Indeed, analogous to generating the braid group by crossing adjacent strands, we can construct any $S_\sigma$ using only $I$ and $S$:

**Lemma 3.** For any $k \in \mathbb{N}$ and $\sigma \in S_k$, we have $S_\sigma \in \langle I, S \rangle_{\circ, \otimes, *}$.

**Proof.** Decompose $\sigma$ into adjacent transpositions as $\sigma = (a_1\ a_1 + 1)(a_2\ a_2 + 1)\ldots(a_s\ a_s + 1)$. Then, since $S$ swaps the position of adjacent dangling edges, we have

$$S_\sigma = \bigcirc_{i=1}^s (I^{\otimes a_i - 1} \otimes S \otimes I^{\otimes k - a_i - 1}).$$

See Figure 1 for an illustration. \qed

![Figure 1: Illustrating the decomposition of $S_\sigma$ given by Lemma 3 with $\sigma = (1\ 3)(2\ 4) = (2\ 3)(1\ 2)(3\ 4)(2\ 3)$.](image)

**Theorem 3.** For any conjugate-closed constraint function set $\mathcal{F}$, $\mathcal{G}_\mathcal{F} = \langle \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, S, \{F \mid F \in \mathcal{F}\} \rangle_{\circ, \otimes, *}$.

**Proof.** Since $\mathcal{F}$ is conjugate closed and $\mathbf{E}^{1,0}, \mathbf{E}^{1,2}$, and $S$ are real-valued, the reverse inclusion $\subseteq$ is clear. To show the forward inclusion $\supseteq$, first observe that $I = \mathbf{E}^{1,2} \circ (\mathbf{E}^{1,2})^t = \mathbf{E}^{1,2} \circ \mathbf{E}^{2,1}$, $E^{m,d} = \bigcirc_{i=0}^{m-2} (\mathbf{E}^{2,1} \otimes I^{\otimes i}) \ldots \bigcirc_{i=0}^{d-2} (\mathbf{E}^{1,2} \otimes I^{\otimes i})$ for any $m, d \geq 2$, and $E^{m,d} = \mathbf{E}^{m,1} \circ \bigcirc_{i=0}^{d-2} (\mathbf{E}^{1,2} \otimes I^{\otimes i})$ for $m \in \{0, 1\}$, $d \geq 2$. Also $E^{0,0} = \mathbf{E}^{0,1} \circ \mathbf{E}^{1,0}$. Thus

$$E^{m,d} \in \langle \mathbf{E}^{1,2}, \mathbf{E}^{1,0} \rangle_{\circ, \otimes, *} \text{ for all } m, d \geq 0 \quad (13)$$

is also a recontextualization of $\langle \mathcal{G}_\mathcal{F}, \mathcal{G}_\mathcal{F} \rangle$. Consider a Holant($\mathcal{F} \mid \mathcal{Q}$) gadget $K \in \mathcal{G}_\mathcal{F}(M, D)$. We will construct $K$ from the fundamental gadgets. Suppose $K$ contains $r$ equality vertices, which we denote $e_1, \ldots, e_r$ in arbitrary order, and $s$ constraint vertices, denoted $c_1, \ldots, c_s$ in arbitrary order, with $c_j$ assigned signature $F_j \in \mathcal{F}$. For $i \in [r]$, suppose vertex $e_i$ is incident to $m_i$ output and $d_i$ input dangling edges in $K$, respectively, and has degree $m_i + d_i + t_i$. Let $T = \sum_{i=1}^r t_i = \sum_{j=1}^s \text{deg}(c_j)$, and we also have $M = \sum_{i=1}^r m_i$, $D = \sum_{i=1}^r d_i$, because by assumption all of
K’s dangling edges are incident to equality vertices. Let \( K_0 = \bigotimes_{i=1}^{r} \mathbf{E}_{m_i,d_i+t_i} \in \mathfrak{G}_F(M,D+T) \) and identify \( e_i \) with the vertex in \( \mathbf{E}_{m_i,d_i+t_i} \). By the bipartite structure of \( K \), for all \( k \in \{ \text{arity}(F_1) \} = [\text{deg}(c_1)] \), the \( k \)th input edge to \( c_1 \) is incident to some equality vertex \( e_{i_k} \), so \( t_{i_k} > 0 \). Thus, identifying \( c_1 \) with the vertex in \( F_1 \), there is a permutation \( \sigma_1 \in S_{D+T} \) such that, in the gadget

\[
K_1 = K_0 \circ S_{\sigma_1} \circ (F_1 \otimes I^{D+T-\text{arity}(F_1)}) \in \mathfrak{G}_F(M,D+T-\text{arity}(F_1)),
\]

c_1’s \( k \)th incident edge is merged with the proper edge incident to \( e_{i_k} \) for every \( k \in \{ \text{arity}(F_1) \} \). Similarly, for each \( j \in [s] \), let \( \sigma_j \in S_{D+T-\sum_{\ell=1}^{j-1} \text{arity}(F_\ell)} \) be the permutation that matches \( c_j \)’s incident edges with the proper equality vertices. Then

\[
K_s = K_0 \circ \bigodot_{j=1}^{s} \left( S_{\sigma_j} \circ (F_j \otimes I^{D+T-\sum_{\ell=1}^{j-1} \text{arity}(F_\ell)}) \right) \in \mathfrak{G}_F(M,D),
\]
is a gadget with the same internal structure as \( K \), but with its input and output dangling edges permuted by some \( \tau \in S_M \) and \( v \in S_D \), respectively, relative to \( K \). Then \( K = S_{\tau^{-1}} \circ K_s \circ S_v \). By Lemma 3 and (13), we have \( K \in \{ \mathbf{E}^{1,0}, \mathbf{E}^{1,2}, S, \{ F \mid F \in \mathcal{F} \} \}_{\otimes,k,s} \).

See Figure 2 for an illustration. The proof of Theorem 3 was inspired by the proof sketch of [25, Theorem 8.4], which is roughly Theorem 3 restricted to unweighted graph homomorphism (the case where \( \mathcal{F} \) contains a single binary symmetric 0-1 valued constraint function).

| \( K \) | \( S_{(1 \ 2 \ (3))} \) | \( S_{(1 \ 4 \ 5 \ 6 \ 3 \ (2 \ (7))} \) | \( S_{(1 \ 2 \ 3 \ 4)} \) |
|---|---|---|---|
| \( E_4 \) | \( F_1 \) | \( F_2 \) | \( E_1 \) |

Figure 2: Illustrating the Theorem 3 decomposition of a \( K \in \mathfrak{G}(3,2) \). We draw dangling edges thinner than internal edges, and use circles for equality vertices and squares for constraint vertices. A diamond on an edge marks this edge as the first input to the incident constraint vertex, and inputs proceed counterclockwise.

### 4.2 Intertwiner Spaces

Let \( G \) be a subgroup of the symmetric group \( S_q \). We identify elements \( \sigma \in G \) with the associated permutation matrix \( P_{\sigma} \in \{0,1\}^{q \times q} \). The \((k,\ell)\)-interwinder space of \( G \) is

\[
C_G(k,\ell) = \{ T \in \mathbb{C}^d^k \times \mathbb{C}^d^\ell \mid \forall \sigma \in G : P_{\sigma}^{\otimes k} T = T P_{\sigma}^{\otimes \ell} \}.
\]

Define \( C_G = \bigcup_{k,\ell} C_G(k,\ell) \) to be the space of all intertwiners of \( G \). If \( P_{\sigma} \) is a \( q \times q \) permutation matrix, then, for vector \( v \in \mathbb{C}^n \), \( P_{\sigma}^{\otimes n} v \) is the vector obtained by permuting \( v \)’s entries according to the natural action of \( \sigma \) on \( [q]^n \) (the action \( \sigma(x) = (\sigma(x_1), \ldots, \sigma(x_n)) \)). Hence two indices in \( [q]^n \) are in the same orbit of the action of \( G \) if and only if \((n,0)\) intertwiner takes equal values on the two indices — that is, for \( x, y \in [q]^n \),

There exists a \( \sigma \in G \) such that \( \sigma(x) = y \) if and only if \( v_x = v_y \) for every \( v \in C_G(n,0) \) (14)
(to see the reverse direction, suppose there is no such \( \sigma \) and consider the \( v \) which is 1 on the orbit containing \( x \) and 0 elsewhere).

It is well-known (see e.g. [1]) that for any \( G \subset S_2 \), \( C_G \) is a symmetric tensor category with duals, meaning each \( C_G(k, \ell) \) is a vector space over \( \mathbb{C} \) and \( C_G \) is closed under matrix multiplication, tensor product, and conjugate transpose, and satisfies \( I = E^{1,1} \in C_G(1,1), E^{2,0} \in C_G(2,0), \) and \( S^{2,2} \in C_G(2,2) \).

The next result is a version of classical Tannaka-Krein duality, proved by Woronowicz in [28], and expressed in this form in [12], [1], and elsewhere. It is the key result underlying our alternate proof of Theorem 1.

**Theorem 4.** The mapping \( G \mapsto C_G \) induces a bijection between subgroups \( G \subset S_2 \) and symmetric tensor categories with duals \( C \) satisfying \( D \subset C \).

The next lemma is an extension of [12, Proposition 3.5], which states that for graph \( X \) (equivalently, a symmetric binary 0-1 valued constraint function), \( C_{\text{Aut}(X)} = \langle E^{1,0}, E^{1,2}, S^{2,2}, A_X \rangle_{+1,0,0} \) (the assumption in [12] that \( X \) is vertex-transitive is not necessary – see also [23, Theorem 2.17]). The proof in [12], which makes implicit use of Theorem 4, easily extends to a proof of the next lemma by generalizing the statement \( \forall \sigma \in \text{Aut}(X) : P_{\sigma} A_X = A_X P_{\sigma} \) for every \( \sigma \in \text{Aut}(F) : P_{\sigma}^{|\text{arity}(F)} f = f \), which is equivalent to \( \forall F \in F : f \in C_{\text{Aut}(F)}(\text{arity}(F),0) \).

**Lemma 4.** \( C_{\text{Aut}(F)} = \langle E^{1,0}, E^{1,2}, S^{2,2}, \{ f \mid F \in F \} \rangle_{+1,0,0} \).

Define a \((k, \ell)\)-quantum \( F \)-gadget to be a formal \( C \)-linear combination of gadgets in \( \mathfrak{O}_F(k, \ell) \). In the context of graph homomorphism, where \( F = \{ F \} \), a binary constraint function, since a gadget in \( \mathcal{G}(F)(k, \ell) \) corresponds to a \((k+\ell)\)-labeled \#CSP\(\{F\}\) instance, a \((k, \ell)\)-quantum \( F \)-gadget is equivalent to a \((k+\ell)\)-labeled quantum graph [18, 23, 24]. Let \( \mathcal{Q}_F(k, \ell) \) be collection of all \((k, \ell)\)-quantum \( F \)-gadgets. We extend the signature matrix function \( T \) linearly to \( \mathcal{Q}_F(k, \ell) \). Observe that, for a fixed \((k, \ell)\), the set on the RHS of Lemma 4 is the span of the signature matrices of the gadgets in the set on the RHS of Theorem 3. Hence we have the following theorem.

**Theorem 5.** \( C_{\text{Aut}(F)}(k, \ell) = \{ T(Q) \mid Q \in \mathcal{Q}_F(k, \ell) \} \) for every \( k, \ell \in \mathbb{N} \).

By [14] and the equivalence between \((k, \ell)\)-quantum \( F \)-graphs and \((k+\ell)\)-labeled quantum graphs, the case \( \ell = 0 \) of Theorem 3 is a generalization (without domain weights) from graph homomorphisms to \#CSP of [23, Lemma 2.5] and [6, Theorem 9.3] (the latter restricted to \( \mathbb{R} \) rather than an arbitrary charactistic-0 field).

The next result is a similar generalization of [23, Lemma 2.4]. It is also a version of Theorem 2 without domain weights and restricted to \( F = G \) and \( k > 0 \), but without the twin-free assumption.

**Lemma 5.** Let \( k > 0 \) and \( \varphi, \psi : [k] \to V(F) \). If \( Z^\varphi_F(K) = Z^\psi_F(K) \) for every \( K \in \mathcal{P}[\mathcal{L}[F;k]] \), then there is a \( \sigma \in \text{Aut}(F) \) satisfying \( \psi = \sigma \circ \varphi \).

**Proof.** View \( K \) as a gadget \( K \in \mathfrak{O}_F(k,0) \), so by assumption \( (T(K))_{\varphi([k])} = Z^\varphi_F(K) = Z^\psi_F(K) = (T(K))_{\psi([k])} \) for every \( K \in \mathfrak{O}_F(k,0) \), hence \( (T(Q))_{\varphi([k])} = (T(Q))_{\psi([k])} \) for every \( Q \in \mathcal{Q}_F(k,0) \). Thus, by Theorem 5 \( v_{\varphi([k])} = v_{\psi([k])} \) for every \( v \in C_{\text{Aut}(F)}(k,0) \), so by [14], there is a \( \sigma \in \text{Aut}(F) \) satisfying \( \sigma(\varphi([k])) = \psi([k]) \). In other words, \( \sigma \circ \varphi = \psi \).

The final step is to use Lemma 5 to prove Theorem 1 for \( F = \mathbb{C} \) and \( \mathbb{C} F \) and \( G \). We begin with the following definition.

**Definition 13** (\( \oplus \)). Let \( F \in \mathbb{F}^n(F) \), \( G \in \mathbb{F}^n(G) \) be constraint functions of arity \( n > 1 \), and assume \( V(F) \cap V(G) = \emptyset \). For \( n > 1 \), the direct sum \( F \oplus G = \mathbb{F}^n(G) \cup \mathbb{F}^n(F) \) of \( F \) and \( G \) is defined by

\[
(F \oplus G)(x) = \begin{cases} 
F(x) & x \in V(F)^n \\
G(x) & x \in V(G)^n \\
0 & \text{otherwise}
\end{cases}
\]

for \( x \in (V(G) \cup V(F))^n \). For constraint function sets \( F \) and \( G \) of size \( t \), define \( \oplus \mathcal{G} = \{ F_i \oplus G_i \mid i \in [t] \} \).
For $n = 2$, $F \oplus G$ is the adjacency matrix of the disjoint union of the $F$-weighted graphs with adjacency matrices $F$ and $G$.

**Definition 14** ($\sim, \approx$, connected constraint function). For $n > 1$ and $F \in \mathbb{F}^{V(F)^n}$, define an equivalence relation $\approx$ on $V(F)$ as the transitive closure of the relation $\sim$, where $x \sim y$ if there is some tuple $x \in V(F)^n$ containing $x$ and $y$ such that $F(x) \neq 0$. Say $F$ is connected if $\approx$ has exactly one equivalence class, and is disconnected otherwise.

If $I, J \subseteq V(F)$ are distinct equivalence classes of $\approx$ (‘connected components’) and $\sigma \in \text{Aut}(F)$ satisfies $\sigma(i) = j$ for some $i \in I$ and $j \in J$, it follows that $\sigma$ is an isomorphism between the subtensor of $F$ induced by $I$ and the subtensor of $F$ induced by $J$. In particular, if $F$ and $G$ are connected, then $V(F)$ and $V(G)$ are the two equivalence classes of $V(F \oplus G)$, so if $a \in \text{Aut}(F \oplus G)$ maps some $x \in V(X)$ to some $y \in V(Y)$, then $F \approx G$. For $n = 2$ and symmetric $F$ and $G$, the above statements are all equivalent to the corresponding well-known facts about graphs.

Now Lemma 5 gives another proof of our main result.

**Proof of Theorem 1** for $\mathbb{F} = \mathbb{C}$ and $F, G$ conjugate-closed. Assume $Z_F(K) = Z_G(K_{F \to G})$ for every $\#\text{CSP}(F)$ instance $K$. WLOG, we may assume $V(F)$ and $V(G)$ are disjoint. Let $0_F$ and $0_G$ be new domain elements. For each $F \in \mathcal{F}$, $G \in \mathcal{G}$ of arity $n \geq 2$, define constraint functions $F'$ and $G'$ on $V(F') := V(F) \cup \{0_F\}$ and $V(G') := V(G) \cup \{0_G\}$, respectively, by

$$F'(x) = \begin{cases} F(x) & x \in V(F)^n \\ 1 & \text{otherwise} \end{cases}, \quad G'(x) = \begin{cases} G(x) & x \in V(G)^n \\ 1 & \text{otherwise} \end{cases}$$

for $x \in V(F)^n$ and $x \in V(G)^n$, respectively. In other words, if any entry of $x$ is $0_F$, then $F'(x) = 1$, and similarly for $G'$. For unary $F \in \mathcal{F}$, $G \in \mathcal{G}$, define binary $F'$ and $G'$ by

$$F'(x, y) = \begin{cases} F(x) & x = y \in V(F) \\ 1 & x = 0_F \text{ or } y = 0_F \\ 0 & \text{otherwise} \end{cases}, \quad G'(x, y) = \begin{cases} G(x) & x = y \in V(G) \\ 1 & x = 0_G \text{ or } y = 0_G \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in V(F)$ and $x, y \in V(G)$, respectively. The arity increase is necessary because the direct sum is only sensibly defined for constraint functions with arity $> 1$. Let $\mathcal{F}' = \{F' \mid F \in \mathcal{F}\}$ and $\mathcal{G}' = \{G' \mid G \in \mathcal{G}\}$.

Let $K = (V, C) \in \mathcal{PCIT}[F \oplus G'; 1]$ be a 1-labeled $\#\text{CSP}(F \oplus G')$ instance, with labeled variable $v_0 \in V$. We will show that

$$Z_{F' \oplus G'}^0(K) = Z_{F' \oplus G'}^0(K).$$

If $K$ is not connected (i.e. the underlying graph of the Holant$(F \mid \mathcal{E} \mathcal{Q})$ signature grid corresponding to $K$ is not connected), then the components of $K$ that do not contain $v_0$ contribute to the partition regardless of the assignment to $v_0$. Hence, to establish (15), we may assume $K$ is connected. Any variable assignment $\phi : V \to V(F' \oplus G') = V(F') \cup V(G')$ satisfying $\phi(v_0) = 0_F$ maps some $S \subseteq V$ to $0_F$, with $v_0 \in S$. Furthermore, since $K$ is connected, $0_F \in V(F')$, and each $F' \oplus G'$ evaluates to 0 unless all its inputs are in $V(F')$ or all its inputs are in $V(G')$, if such a $\phi$ makes a nonzero contribution to $Z_{F' \oplus G'}^0(K)$, we must have $\phi(V) \subseteq V(F')$. For a fixed $S \subseteq V$, the remaining variables $V \setminus S$ take all values in $V(F') \setminus \{0_F\} = V(F)$ as $\phi$ ranges over $\{\phi \mid \phi^{-1}(0_F) = S\}$. Additionally, any constraint containing a variable in $S$ always contributes 1, regardless of the assignments to the other variables.

Construct a $\#\text{CSP}(F)$ instance $K_{F \setminus \{S\}}$ from $K$ as follows. First eliminate all variables in $S$ and all constraints containing any variable in $S$. Then, for each constraint applying $F' \oplus G'$, if $F$ and $G$ have arity $> 1$, replace $F' \oplus G'$ with $F$, and if $F$ and $G$ are unary, then merge the two variables to which the binary $F' \oplus G'$ is applied and replace the constraint with a constraint applying $F$ to the merged variable. Assuming all inputs to $F' \oplus G'$ are in $V(F)$, this variable merging procedure does not change the value of the partition function, since by construction $F' \oplus G'$ acts as the function $(x, y) \mapsto \delta_{xy} F_x$. Now by the discussion in the previous paragraph, the contribution to $Z_{F' \oplus G'}^0(K)$ of the assignments $\phi$ satisfying $\phi^{-1}(0_F) = S$ is $Z_F(K_{F \setminus \{S\}})$. Thus

$$Z_{F' \oplus G'}^0(K) = \sum_{S \subseteq V, S \ni v_0} Z(K_{F \setminus \{S\}}).$$
A similar expression holds for $Z_{\mathcal{F} \oplus \mathcal{G}}^0(K)$, with $K^\mathcal{G}_{V \setminus S} = (K^\mathcal{F}_{V \setminus S})_{\mathcal{F} \rightarrow \mathcal{G}}$ in place of $K^\mathcal{F}_{V \setminus S}$. Thus by assumption we have
\[
Z_{\mathcal{F} \oplus \mathcal{G}}^0(K) = \sum_{S \subset V, S \not\ni 0} Z_\mathcal{F}(K^\mathcal{F}_{V \setminus S}) = \sum_{S \subset V, S \not\ni 0} Z_\mathcal{G}(K^\mathcal{G}_{V \setminus S}) = Z_{\mathcal{F} \oplus \mathcal{G}}^0(K),
\]
proving \eqref{eq:main}. Now by Lemma 5 with $k = 1$, there is a $\sigma \in \text{Aut}(\mathcal{F} \oplus \mathcal{G})$ satisfying $\sigma(0_F) = 0_G$. Since the domain elements $0_F$ and $0_G$ satisfy $0_F \sim x$ for every $x \in V(\mathcal{F})$ and $0_G \sim y$ for every $y \in V(\mathcal{G})$, $\mathcal{F}$ and $\mathcal{G}$ are connected. Hence by the discussion after Definition 14, $\sigma |_{V(\mathcal{F})}$ is an isomorphism between $\mathcal{F}'$ and $\mathcal{G}'$ for every corresponding $F \in \mathcal{F}$ and $G \in \mathcal{G}$, so by construction and the fact that $\sigma(0_F) = 0_G$, $\sigma |_{V(\mathcal{F})}$ is an isomorphism between $F$ and $G$ (if $F$ and $G$ are unary, $\sigma |_{V(\mathcal{F})}$ is really an isomorphism between the functions $(x, y) \mapsto \delta_{xy}F_y$ and $(x, y) \mapsto \delta_{xy}G_y$, but this implies an isomorphism between $F$ and $G$, since unary functions are isomorphic if and only if they have the same multiset of entries). Thus $\mathcal{F}' \cong \mathcal{G}'$.

The proof of Theorem 1 is a generalization of Lovász’s proof of \cite{Lovasz79}, which is essentially restricted to real-weighted graphs ($\mathcal{F}$ and $\mathcal{G}$ contain a single binary constraint function). Both proofs use the idea of adding a universal vertex to connect the graph/constraint function, since for weighted such objects we cannot take the complement to assume connectedness.

5 Discussion

The interpolation proof is constructive (in the sense of Remark 1), applies to any set of constraint functions over any characteristic-0 field, and relies only on the simple idea in Lemma 1, but requires a very detailed presentation. The intertwiner proof has a cleaner presentation and demonstrates interesting new connections between Holant and representation theory, but only applies to CC constraint function sets over $\mathbb{C}$ and is nonconstructive. In the proof of Lemma 5, Tannaka-Krein duality (via Theorem 5) guarantees the existence of a witness $K \in \mathcal{PCL}[\mathcal{F};k]$ such that $Z_\mathcal{F}^0(K) \neq Z_\mathcal{F}^0(K)$ if there is no $\sigma$ satisfying $\sigma \circ \varphi = \psi$, but, unlike the interpolation proof, does not provide an explicit finite list of instances that must contain $K$. One desirable feature of a constructive proof, as discussed in \cite[Section 7]{Boehler}, is to make certain dichotomy theorems (e.g. \cite{Bulatov13}) effective, meaning there is algorithm that decides whether the problem is #P-hard (the dichotomy is decidable) and, if so, constructs a reduction from a #P-hard problem (rather than simply asserting such a reduction exists). Notably, the current complex-weighted #CSP dichotomy \cite{Bulatov13} is not even known to be decidable; our results could someday be used in the proof of a decidable dichotomy.

Acknowledgements

The author thanks Jin-Yi Cai and Austen Fan for helpful discussions.

References

[1] Teodor Banica and Roland Speicher. “Liberation of orthogonal Lie groups”. In: Advances in Mathematics 222.4 (Nov. 2009), pp. 1461–1501. issn: 0001-8708. doi: 10.1016/j.aim.2009.06.009
[2] Elmar Böhler et al. “Equivalence and isomorphism for Boolean constraint satisfaction”. In: International Workshop on Computer Science Logic. Springer. 2002, pp. 412–426.
[3] Elmar Böhler et al. “The complexity of Boolean constraint isomorphism”. In: Annual Symposium on Theoretical Aspects of Computer Science. Springer. 2004, pp. 164–175.
[4] Andrei Bulatov and Martin Grohe. “The complexity of partition functions”. In: Theoretical Computer Science. Automata, Languages and Programming: Algorithms and Complexity (ICALP-A 2004) 348.2 (Dec. 2005), pp. 148–186. issn: 0304-3975. doi: 10.1016/j.tcs.2005.09.011
[5] Andrei A. Bulatov. “The Complexity of the Counting Constraint Satisfaction Problem”. In: J. ACM 60.5 (Oct. 2013). issn: 0004-5411. doi: 10.1145/2528400.
[6] J. Y. Cai and A. Govorov. “On a Theorem of Lovász that hom(\cdot, H) Determines the Isomorphism Type of H”. In: ACM Transactions on Computation Theory 13.11 (2 2008), pp. 1–25.
[7] Jin-Yi Cai and Xi Chen. “A decidable dichotomy theorem on directed graph homomorphisms with non-negative weights”. In: *computational complexity* (2010), pp. 1–64.

[8] Jin-Yi Cai and Xi Chen. *Complexity Dichotomies for Counting Problems*. Vol. 1. Cambridge University Press, 2017, pp. 1–34. DOI: 10.1017/9781107177063.002

[9] Jin-Yi Cai and Xi Chen. “Complexity of Counting CSP with Complex Weights”. In: *J. ACM* 64.3 (June 2017). ISSN: 0004-5411. DOI: 10.1145/2822891

[10] Jin-Yi Cai, Xi Chen, and Pinyan Lu. “Graph homomorphisms with complex values: A dichotomy theorem”. In: *SIAM Journal on Computing* 42.3 (2013), pp. 924–1029.

[11] Jin-Yi Cai, Xi Chen, and Pinyan Lu. “Nonnegative Weighted #CSP: An Effective Complexity Dichotomy”. In: *SIAM J. Comput.* 45.6 (2016), pp. 2177–2198. DOI: 10.1137/15M1032314

[12] Arthur Chassaniol. *Study of quantum symmetries for vertex-transitive graphs using intertwiner spaces*. 2019. DOI: 10.48550/ARXIV.1904.00455

[13] Holger Dell, Martin Grohe, and Gaurav Rattan. “Lovász Meets Weisfeiler and Leman”. In: *ICALP*. 2018.

[14] Wieslaw A. Dudek and Kazimierz Glazek. “Around the Hosszú–Gluskin theorem for n-ary groups”. In: *Discrete Mathematics* 308.21 (2008). Chongqing 2004, pp. 4861–4876. ISSN: 0012-365X.

[15] Zdeněk Dvořák. “On recognizing graphs by numbers of homomorphisms”. en. In: *Journal of Graph Theory* 64.4 (2010). _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/jgt.20461_, pp. 330–342. ISSN: 1097-0118. DOI: 10.1002/jgt.20461

[16] Martin Dyer and Catherine Greenhill. “The complexity of counting graph homomorphisms”. In: *Random Structures & Algorithms* 17.3-4 (2000), pp. 260–289.

[17] Martin Dyer and David Richerby. “An Effective Dichotomy for the Counting Constraint Satisfaction Problem”. In: *SIAM Journal on Computing* 42.3 (2013), pp. 1245–1274. DOI: 10.1137/100811258 URL: https://doi.org/10.1137/100811258

[18] Michael Freedman, László Lovász, and Alexander Schrijver. “Reflection positivity, rank connectivity, and homomorphism of graphs”. en. In: *Journal of the American Mathematical Society* 20.1 (Apr. 2006), pp. 37–51. ISSN: 0894-0347, 1088-6834. DOI: 10.1090/S0894-0347-06-S0529-87.

[19] Leslie Ann Goldberg et al. “A Complexity Dichotomy for Partition Functions with Mixed Signs”. en. In: *SIAM Journal on Computing* 39.7 (Jan. 2010), pp. 3336–3402. ISSN: 0097-5397, 1095-7111. DOI: 10.1137/090757496

[20] Andrew Goodall, Guus Regts, and Lluís Vena. “Matroid invariants and counting graph homomorphisms”. en. In: *Linear Algebra and its Applications* 494 (Apr. 2016), pp. 263–273. ISSN: 0024-3795. DOI: 10.1016/j.laa.2016.01.022

[21] Joshua A Grochow and Youming Qiao. “On the complexity of isomorphism problems for tensors, groups, and polynomials I: Tensor Isomorphism-completeness”. In: *12th Innovations in Theoretical Computer Science Conference (ITCS 2021)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik. 2021.

[22] László Lovász. “Operations with structures”. In: *Acta Mathematica Hungarica* 18.3-4 (1967), pp. 321–328.

[23] László Lovász. “The rank of connection matrices and the dimension of graph algebras”. In: *European Journal of Combinatorics* 27.6 (2006), pp. 962–970. ISSN: 0195-6698.

[24] László Lovász and Balázs Szegedy. “Contractors and connectors of graph algebras”. en. In: *Journal of Graph Theory* 60.1 (Jan. 2009), pp. 11–30. ISSN: 03649024, 10970118. DOI: 10.1002/jgt.20343

[25] Laura Mančinska and David E. Roberson. “Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs”. In: *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. 2020, pp. 661–672. DOI: 10.1109/FCS46700.2020.00067

[26] Guus Regts. “Graph parameters and invariants of the orthogonal group”. Universiteit van Amsterdam, Nov. 2013.
[27] Alexander Schrijver. “Graph invariants in the spin model”. en. In: *Journal of Combinatorial Theory, Series B* 99.2 (Mar. 2009), pp. 502–511. issn: 0095-8956. doi:10.1016/j.jctb.2008.10.003

[28] S. L. Woronowicz. “Tannaka-Krein duality for compact matrix pseudogroups. TwistedSU(N) groups”. en. In: *Inventiones mathematicae* 93.1 (Feb. 1988), pp. 35–76. issn: 1432-1297.