Topological transitions at $T > 0$ in the euclidean 2d U(1)-Higgs model

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The two-dimensional U(1)-gauged Higgs model is studied on an euclidean lattice of size $L_1 \times L_2$, where the temperature $T = L_2^{-1}$ is of the order of the sphaleron mass. The simulation parameters are taken from zero temperature results [3]. By comparison with classical and semiclassical results I discuss, whether the sphaleron transition rate can be extracted from the behavior of the Chern-Simons number and from the formation of vortices in an euclidean simulation at high temperatures.

1. Motivation

In the 2d U(1)-Higgs model, as well as in the Standard Model, the transition between gauge equivalent vacua with integer values of the Chern-Simons number $N_{CS}$ is related to the anomalous violation of the axial U(1)-symmetry. The transition rate $\Gamma$ is usually evaluated by semiclassical methods [4] and classical real-time simulations [5]. It is desirable that an evaluation of $\Gamma$ by euclidean simulations gives control over the full quantum corrections to these calculations.

At low temperatures (inverse temperature $\beta \to \infty$) $\Gamma$ is given by the topological susceptibility

$$\chi_{\text{top}} = \frac{1}{\beta V} \left< \left| N_{CS}(\beta) - N_{CS}(0) \right|^2 \right> \to \frac{\Gamma}{V} \, ,$$

(1)

$V$ is the spatial volume. This relation makes use of the random walk of $N_{CS}$ for large $t$.

At high temperatures $\beta$ is too small to see this long-time behavior. In fact, since the topological charge $N_{CS}(\beta) - N_{CS}(0)$ must be an integer, a configuration contributing to $\chi_{\text{top}}$ is forced to change $N_{CS}$ by at least 1 in the short time interval given by $\beta$. This leads to an exponential suppression $\chi_{\text{top}} \sim e^{-c/\beta}$, whereas $\Gamma$ should be enhanced by high temperatures [3]. Can other observables do better?

2. The high temperature transition rate of the quantum pendulum

I shall address this question at first for the quantum pendulum, a particle of unit mass in the potential $V(x) = [1 + \cos(x)]$. An observable asking for paths from vacuum to vacuum will be suppressed for $\beta \to 0$, since $\Delta x = 2\pi$ in a time $\Delta t = \beta$ is required. I thus consider paths, which only need to cross one of the barrier tops $V(x) = V_{\text{max}}$ at $x_m = (2n + 1)\pi$. I define

$$\dot{\rho} = \frac{1}{\beta Z} \int D[x(t)] \chi[x(t)] e^{-S[x(t)]} \, ,$$

(2)

$$\chi[x(t)] = \begin{cases} 1 & \text{if } t \in [0, \beta] \text{ with } x(t) = x_m \\ 0 & \text{else} \end{cases}$$

For any periodic potential I find in the limit $\beta \to 0$

$$\dot{\rho} = (\pi/2) \Gamma_{\text{cl}} \, ,$$

(3)

the classical transition rate $\Gamma_{\text{cl}}$ is [3]

$$\Gamma_{\text{cl}} = \left< |p| \delta(x - x_m) \right> = \sqrt{\frac{2}{\pi \beta}} \frac{e^{-\beta V_{\text{max}}}}{\int_0^{2\pi} dx e^{-\beta V(x)}} \, .$$

(4)

$\dot{\rho}$ counts repeated fluctuations over the barrier only once. Otherwise it would not be well-defined due to small time fluctuations of any quantum path. However, for small $\beta$ one expects at most one physical transition to occur, thus $\dot{\rho}$ has the meaning of a transition density per unit euclidean time. For large $\beta$ and periodic boundary conditions $x(0) = x(\beta) + 2\pi n$ the non-transition probability $p = 1 - \dot{\rho} \beta$ decays as $p = e^{-\beta \rho}$. So, for the interpretation as a transition density I consider

$$\rho \equiv -\ln(1 - \dot{\rho} \beta)/\beta \, .$$

(5)

For large $\beta$ the behavior of $\rho$ can be compared with the $T = 0$ transition rate given by the analogue of the topological susceptibility

$$\Gamma_\infty = \frac{1}{4\pi^2 \beta} \left< |x(\beta) - x(0)|^2 \right> \, .$$

(6)
In Figure 1 the diamonds give the value of $2\rho/\pi$, the squares give the values of $\Gamma_\infty$, the dashed line shows the classical rate $\Gamma_{cl}$, the full line shows the rate $\Gamma_{Sph}$ in sphaleron approximation $\square$.

Figure 1. $2\rho/\pi$ and $\Gamma$ for the quantum pendulum.

There is a qualitative agreement of $2\rho/\pi$ and $\Gamma_\infty$ at $\beta \to \infty$. For higher temperatures $\Gamma_\infty$ becomes suppressed as expected, whereas $2\rho/\pi$ approaches the classical behavior of $\Gamma$. However, I yet didn’t manage to improve $\rho$ such that there is an exact correspondence to $\Gamma$ on all temperature scales. This question shall be addressed in a forthcoming paper.

3. Topological transitions in the 2d U(1)-Higgs model

Can a similar observable be defined in the 2d U(1)-Higgs model? Lattice formulation and scaling behavior of this model at $T = 0$ are described in $\square$. Here the quantities of interest are the link fields

$$B_x^\mu = -\omega_x + A_x^\mu + \omega_{x+\hat{\mu}} \in [-\pi, \pi], \quad (7)$$

$A_x^\mu$ is the gauge field, $\omega_x$ is the phase of the scalar field. The difficulty is the analogue of the sector boundaries. A naive guess is the requirement $N_{CS} = n + 1/2$ with the lattice definition

$$N_{CS}(0) = \sum_{x_1} \frac{A_1^1}{2\pi}, \quad \partial_t^+ N_{CS}(t) = \sum_{x_1} \frac{F_x}{2\pi}, \quad (8)$$

$F_x/(2\pi)$ is the topological density, see $\square$. In addition a topological transition shows a vortex

$$B_x^1 + B_{x+e_1}^2 - B_x^2 - B_{x+e_2}^1 \notin [-\pi, \pi]. \quad (9)$$

However, the above condition $N_{CS} = n + 1/2$ is in general not related to the formation of a vortex. Consider the related quantity

$$N_B(t) \equiv \frac{1}{2\pi} \sum_{x_1} B^1_{(x_1,t)} = N_{CS}(t) \mod 1. \quad (10)$$

In the continuum model with fixed scalar field length $\rho(x) = v$, $N_B(t)$ decouples from all other degrees of freedom. Its effective action reads

$$S_{eff}[N_B] = \frac{2\pi^2}{e^2 L_1} \int dt \left( \tilde{N}_B^2 + v^2 e^2 N_B^2 \right). \quad (11)$$

This leads to a probability distribution of the constant mode $\tilde{N}_{CS} = \int_0^\beta dt N_{CS}(t)/\beta$

$$P(\tilde{N}_{CS}) \sim \sum_k e^{-\frac{d^2}{\pi^2} (\tilde{N}_{CS}-k)^2}, \quad d = 2\pi^2 v^2. \quad (12)$$

With growing $L_1$ it becomes constant. Thus, even with fixed $\rho(x)$, i.e. with parameters which do not allow for the standard instanton or sphaleron solutions, configurations with $N_{CS} \simeq n + 1/2$ are not suppressed. I found this $L_1$-dependence of $P(\tilde{N}_{CS})$ also in lattice simulations with variable scalar field length, completely dominating the effects induced by vortices. Only the parameter $d$ had to be matched. So the crossing of $N_{CS}(t)$ through the points $N_{CS} = n + 1/2$ is no good condition for topological transitions, see also $\square$.

Another possibility is to consider the density of vortices per unit time and spatial volume

$$\rho_v = < N_{vortices} > / (\beta L_1). \quad (13)$$

Again it is important not to count nearby vortices and antivortices separately, which tend to occur in small clusters. I evaluated $\rho_v(\beta)$ in the ‘Higgs region’ of parameter space with the MC algorithm described in $\square$. The $T = 0$ results $am_H = 0.442(19), am_v = 0.258(6), v = 1.849(1)$ lead to a sphaleron energy $aE_{Sph} = 1.01(4)$.

A comparison of $\rho_v(\beta)$ (diamonds) with the transition rate per volume $\Gamma_{Sph}/L_1$ in sphaleron approximation $\square$ (full line with dashed error)
Figure 2. $\rho V$, $\rho_{\text{red}}$, and sphaleron rate $\Gamma_{\text{Sph}}/L_1$.

shows a plateau of $\rho V(\beta)$ for large $\beta$, see Figure 2. It exceeds the topological susceptibility at large $\beta$ (horizontal line) by far. This plateau is related to dislocation-like vortex pairs (DVPs). These objects have a size of a few lattice spacings, but fixed $\delta S$ for $a \to 0$ compared to a vacuum configuration. A typical example is shown in Figure 3. For this example I find $\delta S = 4\kappa \bar{\rho}^2$, compared with a vacuum configuration scalar field length $\rho_x = \bar{\rho}$, $B^\mu_x = 0$. The DVPs contribute to the expectation value $\rho V$ and therefore destroy its scaling behavior in the continuum limit. Thus it is crucial to separate the true topological transitions from these effects on the cut-off scale.

Under cooling dislocation-like objects should loose their energy density faster than physical objects of the size of the correlation length. In fact, there is a correlation between vortices and lumps in the energy density $\varepsilon(x)$ after some cooling sweeps. Figure 4 shows the distribution $p(\varepsilon_{\text{max}})$ of the local maxima of the energy density near a vortex. I define the minimal required energy density for a true topological transition in the valley between the two peaks, thus throwing away a part of the vortices, identified as the dislocation-like ones. The cooling parameters are fixed for all $\beta$-values, a detailed study of the behavior under cooling shall be given in a future publication.

The such reduced vortex density $\rho_{\text{red}}$ (squares in Figure 3) fits better to the sphaleron rate at small $\beta$ and to the topological susceptibility at large $\beta$. However, this is a rough estimate, far from giving quantitative results. Rather it gives a hint which quantities should be better understood even in this simple toy model for the high temperature physics of the Standard Model.

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