GENERALIZED P(X)-ELLIPTIC SYSTEM WITH NONLINEAR PHYSICAL DATA

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Abstract This paper considers the following Dirichlet problem of the form
\[-\text{div}(\Phi(Du - \Theta(u))) = v(x) + f(x, u) + \text{div}(g(x, u)),\]
which corresponds to a diffusion problem with a source \(v\) in moving and dissolving substance, the motion is described by \(g\) and the dissolution by \(f\).
By the theory of Young measure we will prove the existence result in variable exponent Sobolev spaces \(W_{1,p}^{1,p}(\Omega; \mathbb{R}^m)\).

Keywords \(p(x)\)-Laplacian systems, variable exponents, weak solutions, young measures.

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1. Introduction and main result

Let \(\Omega\) be a bounded open domain in \(\mathbb{R}^n\), \(n \geq 2\). Throughout this paper, we denote by \(\mathbb{M}^{m \times n}\) the set of real \(m\) by \(n\) matrices equipped with the inner product 
\[\langle \xi, \eta \rangle = \sum_{i,j} \xi_{ij} \eta_{ij} \]
For \(\xi \in \mathbb{M}^{m \times n}\), \(\|\xi\|\) is the norm of \(\xi\) when regarded as a vector of \(\mathbb{R}^{mn}\). In [6], the following quasilinear elliptic system was considered

\[
\begin{aligned}
-\text{div}(\Phi(Du - \Theta(u))) &= v \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(Du\) is the symmetric part of the gradient of \(u : \Omega \rightarrow \mathbb{R}^m\), \(v\) belongs to \(W_{-1,p}^{1,p}(\Omega; \mathbb{R}^m)\), \(\frac{1}{p} + \frac{1}{p'} = 1\), \(\Phi(\xi) = \|\xi\|^{p-2}\xi\) for all \(\xi \in \mathbb{M}^{m \times n}\) and \(1 < p < \infty\) (a constant). And \(\Theta : \mathbb{R}^m \rightarrow \mathbb{M}^{m \times n}\) is a continuous function satisfying

\[
\Theta(0) = 0 \quad \text{and} \quad |\Theta(a) - \Theta(b)| \leq c|a - b|, \quad \forall a, b \in \mathbb{R}^m,
\]

where the positive constant \(c\) satisfies

\[
c < \frac{1}{\text{diam}(\Omega)} \left( \frac{1}{2} \right)^\frac{1}{p}.
\]

Here \(\text{diam}(\Omega)\) is the diameter of \(\Omega\). The authors used the theory of Young measure and Galerkin method to prove that (1.1) had a weak solution \(u \in W_{0}^{1,p}(\Omega; \mathbb{R}^m)\) under the conditions (1.2) and (1.3). See also [7, 9] for related topics.

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When the exponent \( p \) is not constant, but depends on \( x \), i.e. \( p \equiv p(x) \), Azroul and Balaadich [8] established the existence result for (1.1), where \( v \) belongs to \( W^{-1,p'(x)}(\Omega; \mathbb{R}^m) \) and the constant \( c \) in (1.2) is assumed to satisfy
\[
c < \frac{1}{\text{diam}(\Omega)} \left( \frac{1}{2} \right)^{\frac{1}{p^+}}.
\]

Here \( p^+ \) is the essential sup of the variable exponent \( p(x) \) for \( x \in \Omega \). They used also the tool of Young measure in establishing their result.

The operator \(-\text{div}(\left| Du \right|^{p(x)-2} Du)\) is said to be the \( p(x) \)-Laplacian and becomes \( p \)-Laplacian when \( p(x) \equiv p \). When \( p(x) > 2 \) (resp. \( 1 < p(x) < 2 \)), then the problem (1.1) is nonlinear degenerate (resp. singular) elliptic systems. Partial differential equations with nonlinearities involving nonconstant exponents have attracted an increasing amount of attention in recent years. The impulse for this may come from the sound physical applications in play, or perhaps it is just the thrill of developing a mathematical theory where PDEs again meet functional analysis in a truly two-way street.

As we know, the \( p(x) \)-Laplacian is inhomogeneous. This implies that it possesses more complicated nonlinearities than the case of \( p \) constant. Problems with variable exponent appear in several domains. For example, in the mathematical modeling of stationary thermorheological viscous of the process filtration of an ideal barotropic gas through a porous medium (cf. [2, 3]). In image processing [18], to outline the borders of a true image and to eliminate possible noise, the variable nonlinearity find its applications. For the case of calculus of variations, the reader can see [1, 14] and references therein.

The authors in [10] considered the following \( p(x) \)-curl systems
\[
\begin{align*}
&\nabla \times (|\nabla \times u|^{p(x)-2} \nabla \times u) = \lambda g(x, u) - \mu f(x, u), \quad \nabla \cdot u = 0 \quad \text{in } \Omega \\
&|\nabla \times u|^{p(x)-2} \nabla \times u \times n = 0, \quad u \cdot n = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Here \( \nabla \times u \) is the curl of \( u = (u_1, u_2, u_3) \). They studied the existence and nonexistence of solutions. Note that the above system is arising in electromagnetism.

E. Azroul et al. [4] investigated a class of nonlinear \( p(x) \)-Laplacian problems, in the scalar case, of the form
\[
\begin{align*}
&-\text{div} \Phi(\nabla u - \Theta(u)) + |u|^{p(x)-2}u + \alpha(u) = f \quad \text{in } \Omega, \\
&\Phi(\nabla u - \Theta(u)).n + \gamma(u) = g \quad \text{on } \partial \Omega,
\end{align*}
\]
where the source term \( f \) was assumed to belong to \( L^1(\Omega) \). They used the techniques of entropy solutions to prove the existence of a solution. See also [5, 13].

In this paper we shall prove the existence of weak solutions for the following problem which is motivated by physics or geometry:
\[
-\text{div} \Phi(Du - \Theta(u)) = v(x) + f(x, u) + \text{div}(g(x, u)) \quad \text{in } \Omega, \tag{1.4}
\]
supplemented with the Dirichlet boundary condition \( u = 0 \) on \( \partial \Omega \). Here \( v \) belongs to \( W^{-1,p'(x)}(\Omega; \mathbb{R}^m) \), \( \Phi : M^{m \times n} \rightarrow M^{m \times n} \) is given in a simple form \( \Phi(\xi) = |\xi|^{p(x)-2}\xi \) for all \( \xi \in M^{m \times n} \) and \( \Theta : \mathbb{R}^m \rightarrow M^{m \times n} \) is a continuous function such that
\[
\Theta(0) = 0 \quad \text{and} \quad |\Theta(a) - \Theta(b)| \leq c|a - b|, \quad \forall a, b \in \mathbb{R}^m, \tag{1.5}
\]
where $c$ is a positive constant that satisfies $c < \frac{1}{\text{diam}(\Omega)} \left( \frac{1}{2} \right)^{\frac{1}{\gamma(x)}}$. Moreover, $f$ and $g$ satisfy the following continuity and growth conditions:

(\text{F0})(\text{Continuity}) $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function, i.e. $x \mapsto f(x, u)$ is measurable for every $u \in \mathbb{R}^m$ and $u \mapsto f(x, u)$ is continuous for a.e. $x \in \Omega$.

(\text{F1})(\text{Growth}) There exist $b_1 \in L^p(x)(\Omega)$ and $0 < \gamma(x) < p(x) - 1$ such that
\[
|f(x, u)| \leq b_1(x) + |u|^\gamma(x).
\]

(\text{G0})(\text{Continuity}) $g : \Omega \times \mathbb{R}^m \to \mathbb{M}^{m \times n}$ is a Carathéodory function in the sense of (F0).

(\text{G1})(\text{Growth}) There exist $b_2 \in L^p(x)(\Omega)$ and $0 < q(x) < p(x) - 1$ such that
\[
|g(x, u)| \leq b_2(x) + |u|^q(x).
\]

Remark 1.1. 1) The strict bound $p(x) - 1$ for $\gamma(x)$ and $q(x)$ in the growth conditions (F1) and (G1) ensures the coercivity of the operator $T$ introduced in Section 3.

2) The function $f$ may depend even on the Jacobin matrix $Du := \frac{1}{2} (\nabla u + (\nabla u)^t)$ and linear with respect to its variable $\xi \in \mathbb{M}^{m \times n}$, see Appendix.

Let $u : \Omega \to \mathbb{R}^m$ be a vector-valued function.

Definition 1.1. A measurable function $u \in W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$ is called a weak solution to problem (1.4) if
\[
\int_{\Omega} \Phi(\nabla u - \Theta(u)) : D\varphi dx = \langle v, \varphi \rangle + \int_{\Omega} f(x, u) \cdot \varphi dx - \int_{\Omega} g(x, u) : D\varphi dx
\]
holds for all $\varphi \in W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$. Here $\langle ., . \rangle$ is the duality pairing of $W^{-1,p'(x)}(\Omega; \mathbb{R}^m)$ and $W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$.

We shall prove the following existence theorem:

Theorem 1.1. Assume that (1.5), (F0), (F1), (G0) and (G1) hold true. Then there exists at least one weak solution to (1.4) in the sense of Definition 1.1.

The outline of the present paper is as follows: In section 2, we introduce the functional space and its properties, and a brief review on the theory of Young measure. Section 3 is devoted to construct the approximating solutions by the Galerkin method and we give the proof of the main result. This paper ends with an appendix.

2. Preliminaries

2.1. Variable exponent Lebesgue and Sobolev spaces

In order to discuss the solutions to the Dirichlet problem (1.4), we need some theories and basic properties of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega; \mathbb{R}^m)$, $W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$ respectively (see [16,18,21] and other references therein).

Let $\Omega$ be a bounded open domain of $\mathbb{R}^n$. We denote $C_+(\overline{\Omega})$ the following set
\[
C_+(\overline{\Omega}) = \{ p \in C(\overline{\Omega}) : p(x) > 1 \text{ for all } x \in \overline{\Omega} \}.
\]
Throughout this paper, 

$$p^- := \text{ess inf}_{x \in \Omega} p(x) \quad \text{and} \quad p^+ := \text{ess sup}_{x \in \Omega} p(x)$$

for every $p \in C_+ (\Omega)$. We define the modular of a measurable function $u : \Omega \to \mathbb{R}^m$ by

$$\rho_{p(\cdot)} (u) := \int_{\Omega} |u(x)|^{p(x)} dx.$$ 

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega; \mathbb{R}^m)$ is a Banach space that is the set of all measurable functions $u : \Omega \to \mathbb{R}^m$ such that its modular

$$\rho_{p(\cdot)} (u) < +\infty$$

is finite, equipped with the Luxemburg norm

$$\| u \|_{L^{p(\cdot)}(\Omega; \mathbb{R}^m)} := \inf \{ \beta > 0 : \rho_{p(\cdot)} \left( \frac{u}{\beta} \right) \leq 1 \}.$$ 

Note that the generalized Lebesgue space $L^{p(x)}(\Omega; \mathbb{R}^m)$ is a kind of Musielak-Orlicz space. If

$$1 \leq p^- \leq p^+ < \infty,$$

$L^{p(x)}(\Omega; \mathbb{R}^m)$ is separable and, in particular, $C_c^{\infty}(\Omega; \mathbb{R}^m)$ is dense in $L^{p(x)}(\Omega; \mathbb{R}^m)$. If we restrict $p(\cdot)$ to satisfy

$$1 < p^- \leq p^+ < \infty,$$

then $L^{p(x)}(\Omega; \mathbb{R}^m)$ becomes reflexive, and its dual is given for $p'(x) = p(x)/(p(x) - 1)$
by $L^{p'(x)}(\Omega; \mathbb{R}^m)$, where $p'(x)$ is the conjugate of $p(x)$. In these spaces, a version of Hölder’s inequality

$$\int_{\Omega} uv dx \leq \left( \frac{1}{p^-} + \frac{1}{p^+} \right) \| u \|_{p(x)} \| v \|_{p'(x)} \leq 2 \| u \|_{p(x)} \| v \|_{p'(x)}$$

is valid for $u \in L^{p(x)}(\Omega; \mathbb{R}^m)$ and $v \in L^{p'(x)}(\Omega; \mathbb{R}^m)$. For the relation between the modular $\rho_{p(\cdot)} (\cdot)$ and the norm $\| \cdot \|_{p(x)}$, we recall the following properties: if $u_k, u \in L^{p(x)}(\Omega; \mathbb{R}^m)$ and $1 < p^- \leq p^+ < \infty$, then:

- if $\| u \|_{p(x)} > 1 \Rightarrow \| u \|_{p^-} \leq \rho_{p(\cdot)} (u) \leq \| u \|_{p^+}$;
- if $\| u \|_{p(x)} < 1 \Rightarrow \| u \|_{p^-} \leq \rho_{p(\cdot)} (u) \leq \| u \|_{p^+}$;
- $\| u_k \|_{p(x)} \to 0$ (resp. $+\infty$) $\Leftrightarrow \rho_{p(\cdot)} (u_k) \to 0$ (resp. $+\infty$).

We define the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega; \mathbb{R}^m)$ as the set of all $u \in L^{p(x)}(\Omega; \mathbb{R}^m)$ such that $Du \in L^{p(x)}(\Omega; \mathbb{M}^{m \times n})$, which is a Banach space for the norm

$$\| u \|_{1,p(x)} = \| u \|_{p(x)} + \| Du \|_{p(x)}.$$ 

This space is again a special case of Sobolev-Orlicz spaces and inherits many of the properties of $L^{p(x)}(\Omega; \mathbb{R}^m)$. In particular, $W^{1,p(x)}(\Omega; \mathbb{R}^m)$ is separable if $1 \leq p^- \leq p^+ < \infty$, and is reflexive if $1 < p^- \leq p^+ < \infty$. Further, $W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$ is the closure of $C_c^{\infty}(\Omega; \mathbb{R}^m)$ in the norm of $W^{1,p(x)}(\Omega; \mathbb{R}^m)$. If $p(\cdot) \in C_+ (\Omega)$, then an
equivalent norm in $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ is $\|Du\|_{p(x)}$. The dual space of $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ can be identified with $W^{-1,p'(x)}(\Omega; \mathbb{R}^m)$ for $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. As in [19], the following Poincaré inequality: there exists a positive constant $\alpha = \text{diam}(\Omega)$ such that
\[
\|u\|_{p(x)} \leq \frac{\alpha}{2} \|Du\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m),
\]
together with Hölder’s inequality, are central in this paper.

Let us summarize the above properties in the following proposition:

**Proposition 2.1** ([18]).
1) $W^{1,p(x)}(\Omega; \mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ are Banach spaces which are separable if $p(.) \in L^\infty(\Omega)$ and reflexive if $1 < p^- \leq p^+ < \infty$.
2) If $q \in C_+(\overline{\Omega})$ with $q(x) < p^*(x) := \frac{np(x)}{n-p(x)}$ for all $p(x) < n$, then the following compact embedding $W^{1,p(x)} \hookrightarrow L^{q(x)}(\Omega; \mathbb{R}^m)$ holds true. In particular, since $p(x) < p^*(x)$ for all $x \in \Omega$ then
\[
W^{1,p(x)} \hookrightarrow L^{p(x)}(\Omega; \mathbb{R}^m).
\]
3) There exists a constant $C > 0$ with $\|u\|_{p(x)} \leq C \|Du\|_{p(x)}$ for all $u \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$, hence $\|Du\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are two equivalent norms on $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$.

### 2.2. Young measures

The weak convergence is a basic tool of modern nonlinear analysis, because it has the same compactness properties as the convergence in finite dimensional spaces as discussed in the paper of Evans [15]. However this convergence, sometimes, does not behave as we desire with respect to nonlinear functionals and operators. To solve this difficulties, we can use the tool of Young measure which we will use in this paper to prove the needed result.

For convenience of the reader not familiar with this concept, we recall some basic notions and properties (see eg. [11, 15, 17, 20] and references therein).

By $C_0(\mathbb{R}^m)$ we denote the set of functions $g \in C(\mathbb{R}^m)$ satisfying $\lim_{|\lambda| \to \infty} g(\lambda) = 0$. Its dual is the well known space of signed Radon measures with finite mass and denoted by $\mathcal{M}(\mathbb{R}^m)$. The duality pairing of these spaces is defined for $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ as
\[
\langle \nu, g \rangle = \int_{\mathbb{R}^m} g(\lambda)d\nu(\lambda).
\]

**Lemma 2.1** ([15]). Assume that the sequence $\{w_k\}_{k \geq 1}$ is bounded in $L^\infty(\Omega; \mathbb{R}^m)$. Then there exist a subsequence (still denoted by $\{w_k\}$) and a Borel probability measure $\nu_x$ on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for each $g \in C(\mathbb{R}^m)$ we have $g(w_k) \rightharpoonup^* \bar{g}$ weakly in $L^\infty(\Omega; \mathbb{R}^m)$ where
\[
\bar{g}(x) = \langle \nu_x, g \rangle = \int_{\mathbb{R}^m} g(\lambda)d\nu_x(\lambda) \quad \text{for a.e. } x \in \Omega.
\]

We call $\nu = \{\nu_x\}_{x \in \Omega}$ the family of Young measure associated with the subsequence $\{w_k\}$. In [11] it is shown that if for all $R > 0$
\[
\lim_{L \to \infty} \sup_{k \in \mathbb{N}} \left\{ x \in \Omega \cap B_R(0) : |w_k(x)| \geq L \right\} = 0,
\]
then for any measurable \( \Omega' \subset \Omega \)
\[
 h(., w_k) \rightharpoonup \langle \nu_x, h(x, .) \rangle = \int_{\mathbb{R}^m} h(x, \lambda) d\nu_x(\lambda)
\]
weakly in \( L^1(\Omega') \), for any Carathéodory function \( h : \Omega' \times \mathbb{R}^m \to \mathbb{R} \) provided that the sequence \( \{h(., w_k)\} \) is weakly precompact in \( L^1(\Omega') \).

Moreover, if \( |\Omega| < \infty \),
\[
 w_k \to w \quad \text{in measure} \iff \nu_x = \delta_{w(x)},
\]
The Young measure associated to the sequence \((y_k, w_k)\) is given by
\[
 \delta_{y(x)} \otimes \nu_x
\]
if \( y_k \to y \) in measure and if \( \nu_x \) is the Young measure associated to \( w_k \).

**Lemma 2.2 (\cite{8}).** Let \((w_k)\) be the sequence defined as above. Then the Young measure \( \nu_x \) generated by \( Dw_k \) in \( L^{p(x)}(\Omega; \mathbb{M}^{m \times n}) \) has the following properties:

(i) \( \nu_x \) is a probability measure, i.e. \( \|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m \times n})} = 1 \) for almost every \( x \in \Omega \).

(ii) The weak \( L^1 \)-limit of \( Dw_k \) is given by \( \langle \nu_x, id \rangle = \int_{\mathbb{M}^{m \times n}} \lambda d\nu_x(\lambda) \).

(iii) \( \nu_x \) satisfies \( \langle \nu_x, id \rangle = Dw(x) \) for almost every \( x \in \Omega \).

## 3. Approximating solutions

To construct the approximating solutions, we will use the Galerkin method. To this purpose, we consider the following map \( T : W_0^{1, p(x)}(\Omega; \mathbb{R}^m) \to W^{-1, q'(x)}(\Omega; \mathbb{R}^m) \) defined for arbitrary \( u \in W_0^{1, p(x)}(\Omega; \mathbb{R}^m) \), by
\[
 \langle T(u), \varphi \rangle = \int_{\Omega} \Phi(Du - \Theta(u)) : D\varphi dx - \langle v, \varphi \rangle = \int_{\Omega} f(x, u) \varphi dx + \int_{\Omega} g(x, u) : D\varphi dx
\]
for all \( \varphi \in W_0^{1, p(x)}(\Omega; \mathbb{R}^m) \). As a consequence, our problem (1.4) is then equivalent to find \( u \in W_0^{1, p(x)}(\Omega; \mathbb{R}^m) \) such that
\[
 \langle T(u), \varphi \rangle = 0 \quad \text{for all} \quad \varphi \in W_0^{1, p(x)}(\Omega; \mathbb{R}^m).
\]

**Lemma 3.1.** The mapping \( T(u) \) is well defined, linear and bounded.

**Proof.** For arbitrary \( u \in W_0^{1, p(x)}(\Omega; \mathbb{R}^m) \), \( T(u) \) is linear. For all \( \varphi \in W_0^{1, p(x)}(\Omega; \mathbb{R}^m) \),
\[
 |\langle T(u), \varphi \rangle| \leq \int_{\Omega} \Phi(Du - \Theta(u)) : D\varphi dx - \langle v, \varphi \rangle - \int_{\Omega} f(x, u) \varphi dx + \int_{\Omega} g(x, u) : D\varphi dx
\]
\[
 \leq \int_{\Omega} |Du - \Theta(u)|^{p(x)-1} |D\varphi| dx + |\langle v, \varphi \rangle|
\]
\[
 + \int_{\Omega} |f(x, u)| |\varphi| dx + \int_{\Omega} |g(x, u)| |D\varphi| dx.
\]

Note that since
\[
 |a + b|^r \leq 2^{r-1} (|a|^r + |b|^r) \quad \text{for} \quad r > 1,
\]
(3.1)
it follows by Hölder's inequality that
\[
I_1 := \int_\Omega |Du - \Theta(u)|^{p(x)-1} |D\varphi| dx \\
\leq \left( \int_\Omega |Du - \Theta(u)|^{p(x)} dx \right)^{\frac{1}{p(x)}} \|D\varphi\|_{p(x)} \\
\leq \left( \int_\Omega 2^{p(x)-1} (|Du|^{p(x)} + |\Theta(u)|^{p(x)}) dx \right)^{\frac{p(x)-1}{p(x)}} \|D\varphi\|_{p(x)} \\
\leq 2^{\frac{p(x)-1}{p(x)}} \left( \|Du\|^{p(x)}_{p(x)} + \|\Theta(u)\|^{p(x)}_{p(x)} \right)^{\frac{p(x)-1}{p(x)}} \|D\varphi\|_{p(x)}.
\]
The generalized Hölder inequality implies that
\[
I_2 := \|\varphi\|_{L^1_{p(x)}(\Omega)} \leq \|\varphi\|_{L^{1,p(x)}(\Omega)}.
\]
On the other hand, it follows from the growth condition (F1) (without loss of generality, we can assume that \(\gamma(x) = p(x) - 1\), that
\[
I_3 := \int_\Omega |f(x,u)| |\varphi| dx \leq \|b_1\|_{L^{p'(x)}(\Omega)} \|\varphi\|_{L^{p(x)}(\Omega)} + \|u\|^{p(x)-1}_{L^{p(x)}(\Omega)} \|\varphi\|_{L^{p(x)}(\Omega)}.
\]
Finally, the growth condition (G1) (without loss of generality, we may assume that \(q(x) = p(x) - 1\) allows to estimate (by application of the Hölder inequality)
\[
I_4 := \int_\Omega |g(x,u)| |D\varphi| dx \leq \|b_2\|_{L^{p'(x)}(\Omega)} \|D\varphi\|_{L^{p(x)}(\Omega)} + \|u\|^{p(x)-1}_{L^{p(x)}(\Omega)} \|D\varphi\|_{L^{p(x)}(\Omega)}.
\]
By virtual of the Poincaré inequality, the \(I_i\) for \(i = 1, \ldots, 4\) are finite, then \(T(u)\) is well defined. Moreover, for all \(\varphi \in W^{1,p(x)}_0(\Omega; \mathbb{R}^m)\)
\[
\|T(u)\|_{L^{p(x)}(\Omega)} \leq \sum_{i=1}^4 I_i \leq C \|D\varphi\|_{L^{p(x)}(\Omega)},
\]
and this implies that \(T(u)\) is bounded.

\[\square\]

**Lemma 3.2.** The restriction of \(T\) to a finite dimensional linear subspace \(V\) of \(W^{1,p(x)}_0(\Omega; \mathbb{R}^m)\) is continuous.

**Proof.** Let \(r\) be the dimension of \(V\) and \((e_i)_{i=1}^r\) a basis of \(V\). Let \((u_j = a^i e_i)\) be a sequence in \(V\) which converges to \(u = a^i e_i\) in \(V\) (with conventional summation). Then \(u_j \to u\) and \(Du_j \to Du\) almost everywhere. The continuity of \(\Theta, f\) and \(g\) implies that
\[
\Phi(Du_j - \Theta(u_j)) : D\varphi \to \Phi(Du - \Theta(u)) : D\varphi,
\]
\[
f(x,u_j)\varphi \to f(x,u)\varphi \quad \text{and} \quad g(x,u_j) : D\varphi \to g(x,u) : D\varphi
\]
almost everywhere. Since \(u_j \to u\) strongly in \(V\),
\[
\int_\Omega |u_j - u|^{p(x)} dx \to 0 \quad \text{and} \quad \int_\Omega |Du_j - Du|^{p(x)} \to 0.
\]
According to [12] (Chapter IV, Section 3, Theorem 3) there exist a subsequence of \( \{u_j\} \) still denoted by \( \{u_j\} \) and \( h_1, h_2 \in L^1(\Omega) \) such that \( |u_j - u|^{p(x)} \leq h_1 \), \( |Du_j - Du|^{p(x)} \leq h_2 \). By virtue of the Eq (3.1), we can write

\[
|u_j|^{p(x)} = |u_j - u + u|^{p(x)} \leq 2^{p(x) - 1}(|u_j - u|^{p(x)} + |u|^{p(x)})
\]

\[
\leq 2^{p(x) - 1}(h_1 + |u|^{p(x)}),
\]

from which (similarly) we get \( |Du_j|^{p(x)} \leq 2 h_2 + |Du|^{p(x)} \). Consequently, the sequences \( ||u_j||_{p(x)} \) and \( ||Du_j||_{p(x)} \) are bounded by a constant denoted \( C \). Now, if \( \Omega' \subset \Omega \) is a measurable subset and \( \varphi \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m) \), then by Poincaré’s inequality

\[
\int_{\Omega'} \Phi(Du_j - \Theta(u_j)) : Du_j dx \leq 2 \frac{(p^+ - 1)^2}{p} \left( \left( \int_{\Omega'} |Du_j|^{p(x)} dx \right)^{\frac{p(x)}{p}} + \left( \int_{\Omega'} |\Theta(u_j)|^{p(x)} dx \right)^{\frac{p(x)}{p}} \right)
\]

\[
\leq 2 \frac{(p^+ - 1)^2}{p} \left( \left( \int_{\Omega'} \left( ||u_j||_{p(x)} + |\Theta(u_j)|^{p(x)} \right) dx \right)^{\frac{p(x)}{p}} \right)
\]

where the small \( c \) is the constant in (1.5), and (without loss of generality, we can assume that \( \gamma(x) = p(x) - 1 \) and \( q(x) = p(x) - 1 \))

\[
\int_{\Omega'} |f(x, u_j)| \varphi dx \leq C \left( \left( \int_{\Omega'} \left( ||u_j||_{p(x)} + |\Theta(u_j)|^{p(x) - 1} \right) dx \right)^{\frac{1}{p(x)}} \right)
\]

and

\[
\int_{\Omega'} |g(x, u_j) : D\varphi| dx \leq \left( \left( \int_{\Omega'} \left( ||u_j||_{p(x)} + |\Theta(u_j)|^{p(x) - 1} \right) dx \right)^{\frac{1}{p(x)}} \right)
\]

Therefore, the sequences \( \Phi(Du_j - \Theta(u_j)) : Du_j \), \( f(x, u_j) : D\varphi \) and \( g(x, u_j) : D\varphi \) are equiintegrable, since \( \int_{\Omega'} |D\varphi|^{p(x)} dx \) is arbitrary small if the measure of \( \Omega' \) is chosen small enough. Applying the Vitali Theorem, it follows for all \( \varphi \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m) \) that \( \lim_{j \to \infty} (T(u_j), \varphi) = (T(u), \varphi) \) as we desire.

\[\square\]

**Lemma 3.3.** The mapping \( T \) is coercive.

**Proof.** Taking \( \varphi = u \) in the definition of \( T \), then

\[
\langle T(u), u \rangle = \int_{\Omega} \Phi(Du - \Theta(u)) : Du dx - \langle v, u \rangle - \int_{\Omega} f(x, u) u dx + \int_{\Omega} g(x, u) : Du dx.
\]

We know that

\[
|\xi|^{r-2}\xi \cdot (\xi - \eta) \geq \frac{1}{r} |\xi|^{r} - \frac{1}{r} |\eta|^{r},
\]

then for \( \xi = Du - \Theta(u) \) and \( \eta = -\Theta(u) \), we get

\[
J_1 := \int_{\Omega} |Du - \Theta(u)|^{p(x)-2}(Du - \Theta(u)) : (Du - \Theta(u) + \Theta(u)) dx
\]
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$$\geq \int_{\Omega} \frac{1}{p(x)}|Du - \Theta(u)|^{p(x)}dx - \int_{\Omega} \frac{1}{p(x)}|\Theta(u)|^{p(x)}dx.$$ 

On the other hand, by (3.1)

$$|Du|^{p(x)} = |Du - \Theta(u) + \Theta(u)|^{p(x)} \leq 2^{p^+ - 1} \left( |Du - \Theta(u)|^{p(x)} + |\Theta(u)|^{p(x)} \right),$$

thus

$$J_1 \geq \frac{1}{p^+} \left( \frac{1}{2^{p^+ - 1}} - \frac{2}{p^+} \frac{1}{2\alpha^{q^+}} \left( \frac{\alpha}{2} \right)^{q^+} \right) \int_{\Omega} |Du|^{p(x)}dx$$

by definition of the function $\Theta$ and Poincaré’s inequality. Next the Hölder inequality implies that

$$|J_2| = |\langle v, u \rangle| \leq ||v||_{-1,p'(x)} ||u||_{1,p(x)}.$$ 

Finally, it follows from the growth conditions (F1) and (G1) that

$$J_3 := \int_{\Omega} f(x, u)udx \leq \|b_1\|_{p'(x)}\|u\|_{p(x)} + \|u\|_{p(x)}^{\gamma(x)+1} \leq C\|b_1\|_{p'(x)}\|Du\|_{p(x)} + C^{q^+ + 1}\|Du\|_{p(x)}^{q(x)+1}$$

and

$$J_4 := \left| \int_{\Omega} g(x, u) : Du dx \right| \leq \|b_2\|_{p'(x)}\|Du\|_{p(x)} + C^{q^+}\|Du\|_{p(x)}^{q(x)+1}.$$ 

From these estimations it follows that

$$\langle T(u), u \rangle = J_1 - J_2 - J_3 + J_4 \rightarrow +\infty \text{ as } \|u\|_{1,p(x)} \rightarrow +\infty$$

since $p^+ > \max \{1, \gamma^+ + 1, q^+ + 1\}$. Hence $T$ is coercive. \hfill\Box

Now, let $V_1 \subset V_2 \subset \ldots \subset W^{1,p(x)}_0(\Omega, \mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\bigcup_{k \in \mathbb{N}} V_k$ is dense in $W^{1,p(x)}_0(\Omega, \mathbb{R}^m)$. Notice that such a sequence $(V_k)$ exists since $W^{1,p(x)}_0(\Omega, \mathbb{R}^m)$ is separable. Let $dim V_k = r$ and $e_1, \ldots, e_r$ is a basis of $V_k$ for a fixed $k$. To construct the approximating solution, we define the map

$$S : \mathbb{R}^r \rightarrow \mathbb{R}^r,$$

$$\begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^r \end{pmatrix} \mapsto \begin{pmatrix} (T(a^1 e_i), e_1) \\ (T(a^2 e_i), e_2) \\ \vdots \\ (T(a^r e_i), e_r) \end{pmatrix}.$$
Lemma 3.4. 1) The map $S$ is continuous.
2) For all $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that
\[ \langle T(u_k), \varphi \rangle = 0 \quad \text{for all} \quad \varphi \in V_k. \quad (3.3) \]
3) The sequence constructed in 2) is uniformly bounded in $W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$, i.e. there exists a constant $R > 0$ such that
\[ \|u_k\|_{1,p(x)} \leq R \quad \text{for all} \quad k \in \mathbb{N}. \quad (3.4) \]

Proof. 1) Since $T$ restricted to $V_k$ is continuous (see Lemma 3.2), then $S$ is continuous.
2) Let $a \in \mathbb{R}^r$ and $u = a^i e_i \in V_k$. Then $S(a).a = \langle Tu, u \rangle$. Notice that $\|a\|_{\mathbb{R}^r} \to +\infty$ is equivalent to $\|u\|_{1,p(x)} \to +\infty$. It follows by Lemma 3.3 that
\[ S(a).a \to +\infty \quad \text{as} \quad \|a\|_{\mathbb{R}^r} \to +\infty. \]
Hence, there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $S(a).a > 0$. The usual topological arguments (see eg. [22, Proposition 2.8]) allows to deduce that $S(x) = 0$ has a solution $x \in B_R(0)$. Therefore, for all $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that
\[ \langle T(u_k), \varphi \rangle = 0 \quad \text{for all} \quad \varphi \in V_k. \]
3) We have by Lemma 3.3, that $\langle Tu, u \rangle \to +\infty$ as $\|u\|_{1,p(x)} \to +\infty$. It result then the existence of $R > 0$ with the property, that $\langle Tu, u \rangle > 1$ whenever $\|u\|_{1,p(x)} > R$. Taking this into consideration and the sequence of Galerkin approximations $u_k \in V_k$ which satisfy $\langle Tu_k, u_k \rangle = 0$ (by 2)), it follows that $(u_k)$ is uniformly bounded. $\square$

Before we pass to the limit in the approximating sequences and so to prove Theorem 1.1, notice that since $(u_k)$ is bounded in $W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$ (see Lemma 3.4), it follows by Lemma 2.1 the existence of a Young measure $\nu_x$ generated by $Du_k$ in $L^p(x)(\Omega; \mathbb{M}^{m \times n})$ which satisfies the properties of Lemma 2.2.

Proof of Theorem 1.1. To apply the convergence described in Lemma 2.2 to our approximating problem, we need the convergence in measure of $u_k$ to $u$. To this purpose, consider $E_{k,\epsilon} = \{ x \in \Omega; |u_k(x) - u(x)| \geq \epsilon \}$. Since $(u_k)$ is bounded in $W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$, then for a subsequence still denoted $u_k$, $u_k \to u$ in $L^p(x)(\Omega; \mathbb{R}^m)$. Therefore
\[ \int_{\Omega} |u_k(x) - u(x)|^{p(x)} \geq \int_{E_{k,\epsilon}} |u_k(x) - u(x)|^{p(x)} \geq \epsilon^p |E_{k,\epsilon}|, \]
which implies that
\[ |E_{k,\epsilon}| \leq \frac{1}{\epsilon^p} \int_{\Omega} |u_k(x) - u(x)|^{p(x)} \to 0 \quad \text{as} \quad k \to \infty. \]
Hence the sequence $u_k$ converges in measure to $u$ on $\Omega$. On the other hand, since $\{Du_k - \Theta(u_k)\}$ is equiintegrable by the condition (1.5) and the boundedness of $(u_k)$, it result that
\[ Du_k - \Theta(u_k) \to \int_{\mathbb{M}^{m \times n}} (\lambda - \Theta(u)) d\nu_x(\lambda). \]
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\[ = \int_{\mathbb{M}^{m \times n}} \lambda d\nu_x(\lambda) - \Theta(u) \int_{\mathbb{M}^{m \times n}} d\nu_x(\lambda) =: Du(x) \]
\[ = Du - \Theta(u) \]

weakly in $L^1(\Omega)$, where we have used Lemma 2.2. Further, from the reflexivity of $L^{p(x)}(\Omega)$ and the boundedness of $\{\Phi(Du_k - \Theta(u_k))\}$, we deduce that $\Phi(Du_k - \Theta(u_k))$ converges in $L^{p(x)}(\Omega)$ and its weak $L^{p(x)}$-limit is given by $\Phi(Du - \Theta(u))$. Consequently

\[ \lim_{k \to \infty} \int_{\Omega} \Phi(Du_k - \Theta(u_k)) : D\varphi dx = \int_{\Omega} \Phi(Du - \Theta(u)) : D\varphi dx \quad \forall \varphi \in \cup_{k \in \mathbb{N}} V_k. \]

Moreover, since $u_k \to u$ in measure for $k \to \infty$, we may infer that, after extraction of a suitable subsequence, if necessary,

\[ u_k \to u \quad \text{almost everywhere for} \quad k \to \infty. \]

Hence, for arbitrary $\varphi \in W^{1, p(x)}_0(\Omega; \mathbb{R}^m)$, it follows from the continuity conditions (F0) and (G0), that

\[ f(x, u_k)\varphi \to f(x, u)\varphi \quad \text{and} \quad g(x, u_k) : D\varphi \to g(x, u) : D\varphi \]

almost everywhere. As in the proof of Lemma 3.2, we have $f(x, u_k)\varphi$ and $g(x, u_k) : D\varphi$ are equiintegrable, thus

\[ f(x, u_k)\varphi \to f(x, u)\varphi \quad \text{and} \quad g(x, u_k) : D\varphi \to g(x, u) : D\varphi \]

in $L^1(\Omega)$ by the Vitali Convergence Theorem. Consequently

\[ \lim_{k \to \infty} \int_{\Omega} f(x, u_k)\varphi dx = \int_{\Omega} f(x, u)\varphi dx \quad \forall \varphi \in \cup_{k \geq 1} V_k \]

and

\[ \lim_{k \to \infty} \int_{\Omega} g(x, u_k) : D\varphi dx = \int_{\Omega} g(x, u) : D\varphi dx \quad \forall \varphi \in \cup_{k \geq 1} V_k. \]

Since $\cup_{k \geq 1} V_k$ is dense in $W^{1, p(x)}_0(\Omega; \mathbb{R}^m)$, $u$ is then a weak solution of (1.4). \qed

**Appendix**

Consider the function $f$ depends on $\xi \in \mathbb{M}^{m \times n}$, i.e. $f : \Omega \times \mathbb{R}^n \times \mathbb{M}^{m \times n} \to \mathbb{R}^m$ and satisfies

\[ |f(x, s, \xi)| \leq b_1(x) + |s|^{\gamma(x)} + |\xi|^{s(x)}, \quad (3.5) \]

where $b_1 \in L^{p'(x)}(\Omega)$, $0 < \gamma(x) < p(x) - 1$ and $0 < s(x) < p(x) - 1$. By similar arguments as above (since $p^+ > \max \{1, \gamma^+ + 1, q^+ + 1, s^+ + 1\}$), it follows that

\[ \lim_{k \to \infty} \int_{\Omega} f(x, u_k, Du_k)\varphi dx = \int_{\Omega} f(x, u, Du)\varphi dx \quad \forall \varphi \in \cup_{k \geq 1} V_k \]
for all $\varphi \in \bigcup_{k \geq 1} V_k$. Now, assume that $\frac{\xi}{\partial^2 f(x, u, \xi)}$ is linear. We have $f(x, u_k, Du_k)$ is equiintegrable (by the growth condition (3.5)), this implies

$$f(x, u_k, Du_k) \to \int_{\mathbb{M}^m \times \mathbb{R}^n} f(x, u, \lambda) d\nu_x(\lambda)$$

$$= f(x, u) \int_{\mathbb{M}^m \times \mathbb{R}^n} \lambda d\nu_x(\lambda) =: Du(x)$$

weakly in $L^1(\Omega)$, by linearity of $f$.

To conclude this paper, let $\varphi \in W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$, since $\bigcup_{k \geq 1} V_k$ is dense in $W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$, then there exists a sequence $(\varphi_k) \subset \bigcup_{k \geq 1} V_k$ such that $\varphi_k \to \varphi$ in $W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$. According to the previous results, we get

$$\lim_{k \to \infty} \langle T(u_k), \varphi_k \rangle = \langle T(u), \varphi \rangle.$$

The equation (3.3) implies that $\langle T(u), \varphi \rangle = 0$ as we desire for all $\varphi \in W^{1,p(x)}_0(\Omega; \mathbb{R}^m)$.

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