Quantum circuit for three-qubit random states

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We explicitly construct a quantum circuit which exactly generates random three-qubit states. The optimal circuit consists of three CNOT gates and fifteen single qubit elementary rotations, parametrized by fourteen independent angles. The explicit distribution of these angles is derived, showing that the joint distribution is a product of independent distributions of individual angles apart from four angles.

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I. INTRODUCTION

Quantum information science (see e.g. \textsuperscript{1} and references therein) has received an increased attention in recent years due to the understanding that it enables to perform procedures not possible by purely classical resources. Experimental techniques to manipulate increasingly complex quantum systems are also rapidly progressing. One of the central issues is on the one hand to control and manipulate delicate complex quantum states in an efficient manner, but on the other hand at the same time to prevent all uncontrollable influences from the environment. In order to tackle such problems, one has to understand the structure and properties of quantum states. This can be done either through studies of particular states in a particular setting, or through focusing on the properties of the most generic states.

Random quantum states, that is states distributed according to the unitarily invariant Haar measure, are good candidates for describing generic states. Indeed, they are typical in the sense that statistical properties of states from a given Hilbert space are well described by those of random quantum states. Also, they describe eigenstates of sufficiently complex quantum systems \textsuperscript{2} as well as time evolved states after sufficiently long evolution. Not least, because random quantum states possess a large amount of entanglement they are useful in certain quantum information processes like quantum density coding and remote state preparation \textsuperscript{3,4}. They can be used to produce random unitaries needed in noise estimation \textsuperscript{5} and twirling operations \textsuperscript{6}. In addition, as random states are closely connected to the unitarily invariant Haar measure of unitary matrices, the unitary invariance makes theoretical treatment of such states simpler.

Producing such states therefore enables to make available a useful quantum resource, and in addition to span the space of quantum states in a well-defined sense. Therefore several works have recently explored different procedures to achieve this goal. It is known that generating random states distributed according to the exact invariant measure requires a number of gates exponential in the number of qubits. A more efficient but approximate way to generate random states uses pseudo-random quantum circuits in which gates are randomly drawn from a universal set of gates. As the number of applied gates increases the resulting measure gets increasingly close to the asymptotic invariant measure \textsuperscript{7}. Some bipartite properties of random states can be reproduced in a number of steps that is smaller than exponential in the number of qubits. Polynomial convergence bounds have been derived analytically for bipartite entanglement \textsuperscript{8,9} for a number of pseudo-random protocols. On the numerical side, different properties of circuits generating random states have been studied \textsuperscript{10}. In order to quantify how well a given pseudo-random scheme reproduces the unitarily invariant distribution, one can study averages of low-order polynomials in matrix elements \textsuperscript{11}. In particular, one can define a state or a unitary $k$-design, for which moments up to order $k$ agree with the Haar distribution \textsuperscript{12,13}. Although exact state $k$-designs can be built for all $k$ (see references in \textsuperscript{14}) they are in general inefficient. In contrast, efficient approximate $k$-designs can be constructed for arbitrary $k$ (for the specific case of 2-design see \textsuperscript{3}).

The pseudo-random circuit approach can yield only pseudo-random states, which do not reproduce exactly the unitarily invariant distribution. The method has been shown to be useful for large number of qubits, where exact methods are clearly inefficient. However, for systems with few qubits, the question of asymptotic complexity is not relevant. It is thus of interest to study specifically these systems and to find the most efficient way – in terms of number of gates – to generate random states distributed according to the unitarily invariant measure. This question is not just of academic interest since, as mentioned, few-qubit random unitaries are needed for e.g. noise estimation or twirling operations. Optimal circuits for small number of qubits could also be used as a basic building block of pseudo-random circuits for larger number qubits, which might lead to faster convergence. In addition, systems of few qubits are becoming available experimentally, and it is important to propose algorithms that could be implemented on
such small quantum processors, and which use as little quantum gates as possible. Indeed, quantum gates, and especially two-qubit gates, are a scarce resource in real systems which should be carefully optimized.

In this paper we therefore follow a different strategy from the more generally adopted approach of using pseudo-random circuits to generate pseudo-random states, and try and construct exact algorithms generating random states for systems of three qubits. In the language of k-designs such algorithms are exact ∞-designs. We present a circuit composed of one-qubit and two-qubit gates which produces exact random states in an optimal way, in the sense of using the smallest possible number of CNOT gates. The circuit uses in total 3 CNOT gates and 15 one-qubit elementary rotations. Our circuit needs 14 random numbers which should be classically drawn and used as parameters for performing the one-qubit gates. The probability distribution of these parameters is derived, showing that it factorizes into a product of 10 independent distributions of one parameter and a joint distribution of the 4 remaining ones, each of these distributions being explicitly given. Since we had to devise specific methods to compute these distributions, we explain the derivation in some details, as these methods can be useful in other contexts.

After presenting the main idea of the calculation in Section II, we start by treating the simple case of two-qubit states in Section III. We then turn to the three-qubit case and first show factorization of the probability distribution for a certain subset of the parameters (Section IV), the remaining parameters being treated in Section V. The full probability distribution for three qubits is summarized in Section VI.

II. THE QUANTUM CIRCUIT

Formally, a quantum state |ψ⟩ can be considered as an element of the complex projective space CP^{N−1}, with N = 2^n the Hilbert space dimension for n qubits [10]. The natural Riemannian metric on CP^{N−1} is the Fubini-Study metric, induced by the unitarily invariant Haar measure on U(N). It is the only metric invariant under unitary transformations. To parametrize CP^{N−1} one needs 2N−1 independent real parameters. Such parametrizations are well-known, for instance using Hurwitz parametrization of U(N) [17]. However, they do not easily translate into one and two-qubit operations, as desired in quantum information. In Ref. [13], optimal quantum circuits transforming the three-qubit state |000⟩ into an arbitrary quantum state were discussed. In the case of three qubits, a generic state can be parametrized up to a global phase by 14 parameters. The quantum circuit requiring the smallest amount of CNOT gates has three CNOTs and 15 one-qubit gates depending on 14 independent rotation angles. From [13] it is possible (see Appendix) to extract the circuit depicted in Fig. I expressed as a series of CNOT gates and single qubit rotations, where Z-rotation is \( Z_\theta = \exp(-i\sigma_z \theta) \) and Y-rotation is \( Y_\theta = \exp(-i\sigma_y \theta) \) with \( \sigma_{x,y} \) the Pauli matrices. The circuit allows to go from |000⟩ to any quantum state (up to an irrelevant global phase). It therefore provides a parametrization of a quantum state |ψ⟩ by angles \( \theta_1, \ldots, \theta_{14} \).

In order to generate random vectors distributed according to the Fubini-Study measure, it would of course be possible to use e.g. Hurwitz parametrization to generate classically a random state, and then use the procedure described in [13] to find out the consequent steps that allow to construct this particular vector from |000⟩. However this procedure requires application of a specific algorithm for each realization of the random vector. Instead, our aim here is to directly find the distribution of the \( \theta_i \) such that the resulting |ψ⟩ is distributed according to the Fubini-Study measure. This is equivalent to calculating the invariant measure associated with the parametrization provided by Fig. I in terms of the angles \( \theta_1, \ldots, \theta_{14} \). Geometrically, the Fubini-Study distance \( D_{FS} \) is the angle between two normalized states, \( \cos(D_{FS}) = |\langle \psi | \phi \rangle| \). The metric induced by this distance is obtained by taking |\psi⟩ = |ψ⟩ + |dv⟩, getting

\[
\text{ds}^2 = \frac{\langle \psi, \psi \rangle \langle dv, dv \rangle - \langle \psi, dv \rangle \langle dv, \psi \rangle}{\langle \psi, \psi \rangle^2}
\]

where \( \langle , \rangle \) is the usual Hermitian scalar product on CP^{N−1}. If a state |ψ⟩ is parametrized by some parameters \( \theta_1, \theta_2, \ldots \) then the Riemannian metric tensor \( g_{ij} \) is such that \( \text{ds}^2 = \sum g_{ij} \text{d}\theta_i \text{d} \theta_j \) and the volume form at each point of the coordinate patch, directly giving the invariant measure, is then given by \( \text{d}v = \sqrt{\text{det}(g)} \prod \text{d} \theta_i \). Thus the joint distribution \( P(\theta) \) of the \( \theta_i \) is simply obtained by calculating the determinant of the metric tensor given by \( g_{ij} \) with the parametrization |ψ⟩ = |ψ(\theta)⟩, \, \theta = (\theta_1, \ldots, \theta_{14}) \). Unfortunately the calculation of such a \( 14 \times 14 \) determinant for \( n = 3 \) qubits is intractable and one has to resort to other means. Let us first consider the easier case of \( n = 2 \) qubits, where by contrast the calculation can be performed directly.

III. A SIMPLE EXAMPLE: THE TWO-QUBIT CASE

A normalized random 2-qubit state |ψ⟩ depends, up to a global phase, on 6 independent real parameters. A circuit producing |ψ⟩ from an initial state |00⟩ is depicted in Fig. 2. One can easily calculate the parametrization of the final state |ψ(\theta_1, \ldots, \theta_6)⟩ in terms of all six angles, thus directly obtaining the metric tensor \( g_{ij} \) from (1). Square root of the determinant of \( g \) then gives an unnormalized probability distribution of the angles as

\[
P(\theta) = |\cos^2 2\theta_1 \sin 2\theta_1 \sin 2\theta_3 \sin 2\theta_5|
\]
distribution of gles fixed. We also numerically computed the marginal uniformly distributed random vectors. If the distribution appendix to find the angles corresponding to a sample of the cases next section we will complete the proof by dealing with 

Let us denote by \( C \) the circuit of Fig. 1 and by \( C(\theta) \) the unitary operator corresponding to it, so that \( |\psi(\theta)\rangle = C(\theta) |000\rangle \). Because circuits \( C \) span the whole space of 3-qubit states, any unitary 3-qubit transformation \( V \) maps parameters \( \theta \) to new parameters \( \tilde{\theta} \) such that \( V|\psi(\theta)\rangle = |\psi(\tilde{\theta})\rangle \). We denote by \( \tilde{C} \) the circuit parametrized by angles \( \tilde{\theta} \) corresponding to performing \( C \) followed by \( V \). It is associated with the unitary operator \( C(\tilde{\theta}) \) such that \( C(\tilde{\theta})|000\rangle = V C(\theta)|000\rangle \). Unitary invariance of the measure implies for \( P(\theta) \) that 

\[
P(\theta) = P(\tilde{\theta}) |J|,
\]

with \( J \) the Jacobian of the transformation \( \theta \mapsto \tilde{\theta} \) and \( |J| \) denotes the determinant. Note that Eq. (3) is not a simple change of variables, as the same function \( P \) appears on both sides of the equation. The Jacobian matrix \( J \) for transformation \( V \) from angles \( \theta \) to \( \tilde{\theta} \), \( V|\psi(\theta)\rangle = |\psi(\tilde{\theta})\rangle \), tells how much do the angles \( \tilde{\theta} \) of \( |\psi(\tilde{\theta})\rangle \) change if we vary angles \( \theta \) in \( |\psi(\theta)\rangle \) keeping transformation matrix \( V \) fixed. Choosing \( V \) that sets some angles \( \theta_j \) in circuit \( \tilde{C} \) to a fixed value, say zero, and at the same time showing that \( |J| \) depends only on these angles \( \theta_j \), would prove factorization of \( P(\theta) \) with respect to angles \( \theta_j \) through Eq. (3).

**IV. FACTORIZATION OF THE THREE-QUBIT DISTRIBUTION FOR ANGLES \( \theta_7 \) TO \( \theta_{14} \)**

Let us now turn to our main issue, which is the distribution of angles in the three-qubit case. In order to have an indication whether the distribution of an angle \( \theta_i \) factorizes, we numerically computed the determinant \( \det(g) \) of the metric tensor as a function of \( \theta_i \) with the other angles fixed. We also numerically computed the marginal distribution of \( \theta_i \) by using the procedure given in the appendix to find the angles corresponding to a sample of uniformly distributed random vectors. If the distribution for a given angle \( \theta_i \) factorizes, these two numerically computed functions should match (up to a constant factor). This is what we observed for all angles but four of them (angles \( \theta_3 \) to \( \theta_6 \)).

In order to turn this numerical observation into a rigorous proof, we are going to show in this section that the distributions for angles \( \theta_7 \) to \( \theta_{14} \) indeed factorize. In the next section we will complete the proof by dealing with the cases \( \theta_1 \) to \( \theta_6 \). The explicit analytical expression of the probability distribution for individual angles will be given in Section VI.

**A. Gates 7-12 and 14**

The simplest case is that of gates at the end of the circuit \( C \) of Fig. 1 e.g., gate \( Y_{\theta_{14}} \). For \( V \) we take \( Y \)-rotation by angle \(-u\) on the third qubit, \( V = Y_{-u} \). It defines a mapping \( \theta \mapsto \tilde{\theta} \) such that \( \tilde{\theta}_i = \theta_i \) for \( i \leq 13 \) and \( \tilde{\theta}_{14} = \theta_{14} - u \). Matrix elements of the Jacobian, i.e. partial derivatives \( J_{jk} = \partial \tilde{\theta}_j / \partial \theta_k \), are equal to the Kronecker symbol \( \delta_{jk} \). The Jacobian is equal to an identity matrix and its determinant is one. Equation (3) taken at \( u = \theta_{14} \) then gives \( P(\theta_1, \ldots, \theta_{13}, 0) = P(\theta_1, \ldots, \theta_{13}, \theta_{14}) \), from which one concludes that the distribution for \( \theta_{14} \) factorizes and is in fact uniform (unless noted otherwise \( P \)’s are not normalized). The same argument holds for the two other rotations by angles \( \theta_{12} \) and \( \theta_9 \) applied at the end of each qubit wire.

Proceeding to angle \( \theta_8 \) one could use \( V = Y_{-u_8} Z_{-u_9} \) applied on the first qubit and show that the Jacobian depends only on \( \theta_8 \) and \( \theta_9 \), while at \( u_8 = \theta_8 \) and \( u_9 = \theta_9 \) one

**FIG. 1:** Circuit \( C \) for three-qubit random state generation.

**FIG. 2:** Circuit for two-qubit random state generation.
gets \( \theta_8 = \theta_9 = 0 \), from which factorization of \( \theta_8 \) would follow from Eq. 3. There is however a simpler way. Observe that the three single-qubit gates with angles \( \theta_7, \theta_8 \) and \( \theta_9 \) on the first qubit span the whole SU(2) group. Therefore, for any one-qubit unitary \( V \), gates \( VZ_{\theta_9}Y_{\theta_8}Z_{\theta_7} \) can be rewritten as \( Z_{\theta_9}Y_{\theta_8}Z_{\theta_7} \), without affecting other \( \theta_i \)s. The distribution of these three angles must therefore be the same as the distribution of corresponding SU(2) parameters. Note that the same argument can be applied in the 2-qubit case of Section III. As a consequence, the distribution of angles for gates \( Z - Y - Z \) at the end of the circuit should be the same in both cases, that is the distribution of \( \theta_7 \) is uniform while that of \( \theta_8 \) is proportional to \( |\sin 2\theta_8| \). Similarly, one can show that the distribution for the angles \( \theta_{10} \) to \( \theta_{12} \) is the same as for angles \( \theta_7 \) to \( \theta_9 \).

### B. Gate 13

As opposed to gates 10-12, for gate 13 we cannot use the analogy with the 2-qubit circuit (Fig.2) because the two gates 13 and 14 on the third qubit do not span the whole SU(2) group. Therefore a different argument should be used. In what follows we show that the joint distribution for angles \( \theta_{13} \) and \( \theta_{14} \) can be factorized out of the full distribution. Since it has been shown in Subsection IV.A that angle \( \theta_{14} \) factorizes, this will prove that the distribution for \( \theta_{13} \) also factorizes.

Using \( V = Z_{u_{13}}Y_{u_{14}} \) on the third qubit we can set \( \theta_{13} = 0 \) to zero with the choice \( u_{13} = \theta_{13} \) and \( u_{14} = \theta_{14} \). Our goal is to show that \( |J| \) depends only on \( \theta_{13} \) and \( \theta_{14} \). We can formally consider each angle \( \theta_i \) as being a function \( \hat{\theta}_i(\theta; u_{13}, u_{14}) \) of the initial \( \theta \) as well as of the parameters \( u_{13}, u_{14} \) through \( C(\theta)|000\rangle = VC(\theta)|000\rangle \).

To calculate matrix elements of \( J \) for our choice of \( V \) evaluated at \( u_{13} = \theta_{13} \) and \( u_{14} = \theta_{14} \), we must obtain the first-order expansion in \( \epsilon \) of the quantities

\[
\hat{\theta}_j(\theta_1, \ldots, \theta_{k-1}, \theta_k + \epsilon, \theta_{k+1}, \ldots, \theta_{14}; \theta_{13}, \theta_{14}).
\]

Some angles \( \hat{\theta}_j \) are very simple. We immediately see that when varying angles \( \theta_k \), that is taking \( k \leq 12 \) in Eq. 4, angles \( \theta_{j \leq 12} \) do not change (i.e \( \hat{\theta}_j = \theta_j \)). The corresponding 12-12-dimensional subblock in \( J \) is therefore equal to an identity matrix. Similarly, taking \( k = 13 \) in Eq. 4, we see that \( \theta_{j \leq 13} \) do not change. The corresponding column in \( J \) is therefore zero apart from 1 on the diagonal. The Jacobian thus has a block structure of the form

\[
|J| = \begin{pmatrix} 1 & B \\ 0 & A \end{pmatrix} = |A|,
\]

where \( I \) is a 13 \times 13-dimensional identity matrix and \( A \) is a 1 \times 1-dimensional block with partial derivative \( \partial \theta_{14} / \partial \theta_{14} \). The angle \( \theta_{14} \) given by Eq. 4 is obtained by varying angle \( \theta_{14} \) by \( \epsilon \). The condition that \( C(\theta)|000\rangle = VC(\theta)|000\rangle \) is \( \hat{\theta}_{14} Z_{\theta_{13}} |x\rangle = Z_{-\theta_{14}} Y_{\theta_{14}} Z_{\theta_{13}} |x\rangle \), where \( |x\rangle = \text{CNOT}_{13} |\phi\rangle \) is a state after the third CNOT acts on \( |\phi\rangle \) (counting from the left in Fig. 1).

V. JOINT THREE-QUBIT PROBABILITY DISTRIBUTION FOR ANGLES \( \theta_1 \) TO \( \theta_6 \)

In the preceding section, we have shown that the distribution for angles \( \theta_1 \) to \( \theta_{14} \) factorizes. As was mentioned, numerical observations indicated us that the distribution for angles \( \theta_1 \) and \( \theta_2 \) should also factorize, but that it is not the case for the joint distribution of \( \theta_3, \ldots, \theta_6 \).

As we were not able to directly prove by the same methods as above that the distributions for \( \theta_1 \) and \( \theta_2 \) factorize, we use a different strategy. Namely, we first assume that this factorization is true, then we compute the distributions under this assumption, and the knowledge of the answer allows us to prove a posteriori that it is indeed the correct probability distribution.

If the factorization holds, the distribution for \( \theta_1 \) and \( \theta_2 \) is easily calculated from the matrix \( g \) using symbolic manipulation software, by replacing angles \( \theta_1, \theta_2 \geq 3 \), in \( g \) by suitably chosen simple values, so that the 14 \times 14 determinant giving the volume form can now be handled. This yields, up to a normalization constant,

\[
P_1(\theta_1) = \cos^5 \theta_1 \sin^3 \theta_1
\]

\[
P_2(\theta_2) = \cos^5 2 \theta_2 \sin^3 2 \theta_2.
\]
determinant $\det(g)$ of the metric tensor given by \( I \) still depends on 4 variables, which is too much for it to be evaluated by standard software. We thus proceed as follows. First one can show that $\det(g)$ can be put under the form

$$
\det(g) = \sum_{p=-10}^{10} \sum_{q=-8}^{6} \sum_{r=-8}^{8} \sum_{s=-6}^{6} a_{pqrs} \cos(2p\theta_3 + 2q\theta_4 + 2r\theta_5 + 2s\theta_6),
$$

(12)

with the sums running over all $q, r$ but only even values of $p$ and $s$. Because of the parity of $\cos$, there are $M = 8509$ independent coefficients $a_{pqrs}$. Evaluating numerically the determinant at $M$ random values of the angles one gets an $M \times M$ linear system that can be solved numerically. If the values of the coefficients of the matrix $g_{ij}$ are multiplied by a factor 4, then one is ensured (from inspection of $\det(g)$) that the $a_{pqrs}$ are rationals of the form $k/2^9$, $k \in \mathbb{Z}$. This allows to deduce their exact value from the numerical result. We are left with 6998 nonzero terms in $\det(g)$, and terms with odd $q$ or $r$ do not exist. We then suppose that $\sqrt{\det(g)}$ can be expanded as

$$
\sqrt{\det(g)} = \sum_{p=-5}^{5} \sum_{q=-3}^{3} \sum_{r=-4}^{4} \sum_{s=-3}^{3} b_{pqrs} e^{i(2p\theta_3 + 2q\theta_4 + 2r\theta_5 + 2s\theta_6)}.
$$

(13)

This assumption is validated a posteriori, since a solution of the form \( I \) can indeed be found. There are 4851 coefficients $b_{pqrs}$, which can be obtained by identifying term by term coefficients in the expansion of $\sqrt{\det(g)}^2$ and $\det(g)$. We have to solve a system of quadratic equations

$$
a_{10,6,8,6} = b_{5343}^2
$$

$$
a_{10,6,8,5} = b_{5342}b_{5343} + b_{5343}b_{5342}
$$

$$
a_{10,6,8,4} = b_{5341}b_{5343} + b_{5342}b_{5342} + b_{5343}b_{5341}
$$

$$
\ldots = \ldots
$$

The first equation is quadratic and fixes an overall sign. Equation $k + 1$ is linear once the values obtained from equations 1 to $k$ are plugged into it. Starting with the highest-degree term $(p, q, r, s) = (5, 3, 4, 3)$ one can thus recursively solve all equations. There are only 1320 non-zero coefficients $b_{pqrs}$. Gathering together terms $(\pm p, \pm q, \pm r, \pm s)$ one can simplify the sum \( I \) to a sum of 96 terms of the form $c_{pqrs} \cos(p\theta_3) \cos(q\theta_4) \cos(r\theta_5) \sin(s\theta_6)$. Expanding this expression in powers of $\cos(2\theta_5)$ and $\sin(2\theta_6)$ and simplifying separately each coefficient we finally get

$$
P(\theta_3, \theta_4, \theta_5, \theta_6) = \sin 2\theta_5 \sin 4\theta_3 \sin^2 \varphi_1 \cos \varphi_2,
$$

(15)

where $\langle \alpha | \beta \rangle = \cos \varphi_1$ and $| \beta \rangle = \cos \varphi_2$ with $| \alpha \rangle$, $| \beta \rangle$ given by Eq. \( I \). Recall that $| \beta \rangle$ is the bit-flip transform of $| \beta \rangle$, $| \beta \rangle = \sigma_x | \beta \rangle$. Note that $\sin 2\theta_3 = \langle \alpha | \beta \rangle$. Angles $\varphi_1, \varphi_2$ can be obtained from

$$
\cos^2 \varphi_1 = (c_1c_3c_6 - s_3s_6)^2 + (c_2c_4c_6 + c_3s_4c_6)^2
$$

$$
\cos \varphi_2 = -s_3s_6 + c_6(s_3c_4c_5 - c_3s_5),
$$

(16)

where $c_i = \cos \theta_i$ and $s_i = \sin \theta_i$. We do not have a general argument to explain this remarkable expression of the distribution in terms of the scalar products of $|\alpha\rangle$ and $|\beta\rangle$.

To complete the proof for the joint distribution $P(\theta_1, \ldots, \theta_6)$ it remains to be checked that the determinant of the metric tensor $g$ with angles $\theta_7$ to $\theta_{14}$ replaced by constants is indeed proportional to $P_1(\theta_1)P_2(\theta_2)P(\theta_3, \theta_4, \theta_5, \theta_6)$. This a posteriori verification is easier to handle symbolically than the full a priori calculation of the $14 \times 14$ determinant. Indeed, the determinant can first be reduced to an $8 \times 8$ determinant by Gaussian elimination. The remaining determinant can be expanded as a trigonometric polynomial. Although symbolic manipulation software do not allow to simplify the coefficients of this polynomial, they are able to check that these coefficients match those of the expected distribution. We proved in that way that the difference between the determinant $\det(g)$ and our expression is identically zero. This gives a computer-assisted but rigorous proof for the distribution of angles $\theta_1$ to $\theta_6$.

VI. TOTAL THREE-QUBIT PROBABILITY DISTRIBUTION FUNCTION

Gathering together the results of the previous sections we obtain that the joint distribution $P(\theta)$ can be factorized as

$$
P(\theta) = |P_1(\theta_1)P_2(\theta_2)P(\theta_3, \theta_4, \theta_5, \theta_6)\prod_{i=7}^{14} P_i(\theta_i)|.
$$

(17)

The joint distribution $P(\theta_3, \theta_4, \theta_5, \theta_6)$ has been derived in the previous section and is given by Eq. \( I \). The distribution for $\theta_1$ and $\theta_2$ is given by Eqs. \( I \) and \( I \). Given the factorization \( I \), it is easy to calculate the remaining $P_i(\theta_i)$ for each $i = 7, \ldots, 14$ as was done for $\theta_1$ and $\theta_2$ in the previous section: replacing angles $\theta_j$, $j \neq i$, in $g$ by suitably chosen simple values, the $14 \times 14$ determinant giving the volume form can be easily evaluated by standard symbolic manipulation. This yields, up to a normalization constant,

$$
\prod_{i=7}^{14} P_i(\theta_i) = \sin 2\theta_8 \sin 2\theta_{11} \cos 2\theta_{13}.
$$

(18)

The knowledge of the angle distribution \( I \) allows to easily generate random three-qubit vectors using the circuit of Fig. \( I \). Angles $\theta_1, \theta_2$ and $\theta_7$ to $\theta_{14}$ can be drawn classically according to their individual probability distribution. Angles $\theta_3, \ldots, \theta_6$ can be obtained classically from the joint distribution \( I \) by, for instance, Monte-Carlo rejection method (that is, drawing angles $\theta_3$ to $\theta_6$ and a parameter $x \in [0, p]$ at random, and keeping them if $P(\theta_3, \theta_4, \theta_5, \theta_6) < x$). Bounding $P(\theta_3, \theta_4, \theta_5, \theta_6)$ from above by $p = 0.85$ yields a success rate of about 12%.
VII. CONCLUSION

In this work, we constructed a quantum circuit for generating three-qubit states distributed according to the unitarily invariant measure. The construction is exact and optimal in the sense of having the smallest possible number of CNOT gates. The procedure requires a set of 14 random numbers classically drawn, which will be the angles of the one-qubit rotations, and whose distribution has been explicitly given. Remarkably, we have shown that the distribution of angles factorizes, apart from that of four angles. The circuit can be used as a three-qubit random state generator, thus producing pseudo-random quantum states on an arbitrary number of qubits. At last, it gives an example of a quantum algorithm producing interesting results which could be implemented on a few-qubit platform, using only 18 quantum gates, of which 15 are one-qubit elementary rotations much less demanding experimentally.

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APPENDIX: THE PARAMETRIZATION CORRESPONDING TO THE THREE-QUBIT CIRCUIT

In this Appendix, we explain how to obtain the angles \( \theta_i \) of the circuit (Fig. 1) for a given \( |\psi\rangle \), based on the discussion in [15]. This justifies the use of these angles as a parametrisation of the quantum states. We start from a state \( |\psi\rangle \), and transform it by the inverse of the different gates of Fig. 1 to end up with \( |000\rangle \), specifying how the angles \( \theta_i \) are obtained in turn. More details can be found in [15]. A generic three-qubit state \( |\psi\rangle \) can be written in a canonical form as a sum of two (not normalized) product terms [21],

\[
|\psi\rangle = |\omega_1\omega_2\omega_3\rangle + |\omega_1^+\rangle |\xi\rangle_{23}, \tag{A.1}
\]

where \( |\omega_i\rangle \) are one-qubit states, \( |\omega_1^+\rangle \) is a one-qubit state orthogonal to \( |\omega_1\rangle \) and \( |\xi\rangle_{23} \) is a two-qubit state of the second and third qubits. The angle \( \theta_0 \) is chosen such that the Z-rotation of angle \(-\theta_0\) eliminates a relative phase between the coefficients of the expansion of \( |\omega_1\rangle \) into \( |0\rangle \) and \( |1\rangle \). (Note that because we are using the circuit in the reverse direction the angles of rotations have opposite signs). A subsequent Y-rotation with angle \(-\theta_8\) results in the transformation \( |\omega_1\rangle \rightarrow |0\rangle \) (up to a global phase). Similarly, rotations of angles \(-\theta_{12}\) and \(-\theta_{11}\) rotate \( |\omega_2\rangle \) into \( |0\rangle \). After applying rotations of angles \(-\theta_8\), \(-\theta_9\), \(-\theta_{11}\) and \(-\theta_{12}\) the state has become of the form \( |\psi'\rangle = \chi |00\rangle + |1\rangle |0\gamma_1\rangle + |1\gamma_2\rangle \) (up to normalization). Two rotations on the third qubit of angles \(-\theta_{13}\) and \(-\theta_{14}\) are now chosen so as to rotate \( |\gamma\rangle \) into some new state \( |\gamma'\rangle \) while \( |\gamma_2\rangle \) is rotated up to normalization, into \( \sigma_3 |\gamma'\rangle \). It was shown in [15] that this can always be done by writing the normalized \( |\gamma_2\rangle \) as \( |\gamma_2\rangle = \cos \phi_{1,2} |0\rangle + e^{i\phi_{1,2}} \sin \phi_{1,2} |1\rangle \), and then \( \theta_{14} \) is a solution of

\[
-\tan (2\theta_{14}) = \frac{\cos 2\phi_1 + \cos 2\phi_2}{\sin 2\phi_1 \cos \xi_1 + \sin 2\phi_2 \cos \xi_2}, \tag{A.2}
\]

while \( \theta_{13} = -(\delta_1 + \delta_2)/4 \), where \( \delta_i \)'s are relative phases in \( Y_{-\theta_{14}} |\gamma_{1,2}\rangle = e^{i\delta_{1,2}} \cos \kappa |0\rangle + \sin \kappa |1\rangle \). Acting with a CNOT gate on the resulting state one obtains a quantum state for the three qubits of the form \( |\psi''\rangle = \chi_1 |00\rangle + \chi_2 |1\rangle \), with \( \chi_1 = |\gamma'\rangle \). The Z-rotation angle \(-\theta_{10}\) on the second qubit is now determined so as to eliminate a relative phase between the expansion coefficients of \( |\omega_1\rangle \), making them real up to a global phase. On the third qubit we now apply three rotations of angles \(-\theta_4\), \(-\theta_5\), and \(-\theta_9\) to bring \( |\chi_1\rangle \) to \( |\chi'\rangle \) and \( |\chi_2\rangle \) into \( \sigma_3 |\chi'\rangle \), eliminating also a relative phase. Then a CNOT gate is applied. At this point (after the second CNOT in Fig. 1) counting from right, but without the \( \theta_7 \) rotation, the state has become of the form \( |\psi''\rangle = \cos \theta (|00\omega_0\rangle + \tau |1\omega_1 \omega_2\rangle) \), where the one-qubit states \( |\omega_0\rangle \) and \( |\omega_1\rangle \) are normalized and real. With \( \theta_7 \) we now eliminate the relative phase \( \tau \), and with an Y-rotation of angle \(-\theta_3\) the third qubit is brought to the state \( |0\rangle \). Then the combination of two Y-rotation of angles \( \theta_2 \) and \(-\theta_2\) with a CNOT gate brings the second qubit to \( |0\rangle \), and the last rotation of angle \(-\theta_1\) on the first qubit yields the final state \( |000\rangle \). Note that in the circuit of Fig. 1 the two Z-rotations of angles \( \theta_7 \) and \( \theta_{10} \) commute with CNOT gates if they act on the control qubit. This is the reason why the rotation of angle \( \theta_7 \) can be applied at any point between \( \theta_1 \) and \( \theta_8 \), and similarly, \( \theta_{10} \) can be applied at any point between \( \theta_2 \) and \( \theta_{11} \).

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