Abstract

The modeling and simulation of electromagnetic wave propagations is often accompanied by a restriction to bounded domains and the introduction of artificial boundary conditions which should be chosen in order to minimize parasitic reflections. In this paper, we investigate a new type of transparent boundary condition and its implementation in a Discontinuous Galerkin Trefftz Finite Element Method. The choice of a particular set of basis functions allows us to split the electromagnetic field into components with a specified direction of propagation. The reflections at the artificial boundaries are then reduced by penalizing components of the field incoming into the space-time domain of interest. We formally introduce this concept, discuss its realization within the discontinuous Galerkin framework, and demonstrate the performance of the resulting approximations in comparison with commonly used absorbing boundary conditions. In our numerical tests, we observe spectral convergence in the $L_2$ norm and a dissipative behaviour for which we provide a theoretical explanation.

Keywords: transparent boundary conditions, discontinuous Galerkin method, finite element method, Trefftz methods, electrodynamics, wave propagation

1. Introduction

We consider the propagation of electromagnetic waves in a domain $\Omega$ filled with non-conducting dielectric media. In the absence of charges and source currents, the evolution of the electromagnetic fields is governed by the time-dependent Maxwell equations

$$
\begin{align*}
\mu \partial_t \mathbf{H} + \nabla \times \mathbf{E} &= 0 \quad \text{and} \quad \epsilon \partial_t \mathbf{E} - \nabla \times \mathbf{H} &= 0 \quad &\text{in} \quad \Omega \times \mathbb{R}_+.
\end{align*}
$$

(1)

The electric permittivity $\epsilon$ and the magnetic permeability $\mu$ are assumed to be piecewise constant. At time $t = 0$ the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ are prescribed by the initial conditions

$$
\begin{align*}
\mathbf{E}(0) &= \mathbf{E}_0 \quad \text{and} \quad \mathbf{H}(0) &= \mathbf{H}_0 \quad &\text{in} \quad \Omega.
\end{align*}
$$

(2)

If the fields satisfy the gauge conditions $\nabla \cdot (\epsilon \mathbf{E}_0) = 0$ and $\nabla \cdot (\mu \mathbf{H}_0) = 0$ in the beginning, then

$$
\begin{align*}
\nabla \cdot (\epsilon \mathbf{E}) &= 0 \quad \text{and} \quad \nabla \cdot (\mu \mathbf{H}) &= 0 \quad &\text{on} \quad \Omega \times \mathbb{R}_+,
\end{align*}
$$

(3)

which follows by taking the divergence in (1). The two gauge conditions in (3) express the absence of electric charges and magnetic monopoles, respectively. If the computational domain $\Omega$ is bounded, the system has to be complemented by appropriate boundary conditions. We will consider different conditions that all can be cast in the abstract form

$$
\begin{align*}
b(\mathbf{E}, \mathbf{H}) &= \mathbf{g} \quad &\text{on} \quad \partial \Omega \times \mathbb{R}_+.
\end{align*}
$$

(4)
where $Z$ is the intrinsic impedance of the medium, is widely used in practice. This condition mimics the Silver-Müller radiation condition \[20, 21\], and it is satisfied exactly by plane waves propagating in the outward normal direction. A brief inspection of the Poynting vector

$$n \cdot P = n \cdot (E \times H) = H \cdot (n \times E) = Z^{-1} |n \times H|^2$$

reveals that energy is dissipated at the boundary. This boundary conditions is therefore called first-order absorbing or Silver-Müller condition. The simple choice \[6\] can be improved in several ways: In \[22\], a more accurate absorbing boundary condition is formulated that still involves only first order derivatives of the fields; for a stability analysis, see also \[23\]. Other possibilities include the classical Bayliss-Turkel and Enquist-Majda conditions \[8, 9\] and \[10, 11\], which allow to systematically construct conditions or arbitrary order. Due to lack of stability, these are however hardly used in practice. Let us also mention more recent methods due to Givoli, Netan Hagstrom, Warburton, Hariharan and others \[12, 13, 14, 15, 16\], the pole condition for the Dirichlet-to-Neuman operator or the use of infinite elements. Another strategy to minimize reflections from the artificial boundaries is to add an exterior absorbing layer, in which the fields decay very fast. This approach, known as perfectly matched layers, has been used very widely and successfully \[6, 7\]. The appropriate choice of geometric and physical parameters of the absorbing layer is however not completely clear in practice. In some cases it may be necessary to extend the computational domain substantially. In principle, it is possible to formulate exact boundary conditions, e.g., by the coupling to a boundary integral formulation for the outside domain \[3\]. This treatment leads to boundary conditions that are non-local in space and/or time \[4, 5\], which substantially complicates numerical computations. For a review and a comparison of various kinds of non-absorbing, transparent, or non-reflecting boundary conditions, let us refer to \[17, 18\].

In this paper, we follow a different strategy for devising local transparent boundary conditions: Motivated by some of the approaches mentioned previously, we expand the electromagnetic fields locally at any point of the boundary as a superposition $(E, H) = \sum_j c_j (e_j, h_j)$ of plane waves propagating in specific directions $d_j$. To minimize reflections at the artificial boundary, we then require that

$$c_j = 0 \quad \text{for any direction } d_j \text{ with } n \cdot d_j < 0.$$

(7)

The intuition behind this approach is to suppress the components of the electromagnetic field corresponding to incoming waves which stem from non-physical reflections. A similar idea has been used previously in the context of finite difference Trefftz schemes \[24, 25\]. In this paper we formulate a related condition that can be incorporated systematically in a discontinuous Galerkin framework. To evaluate stability of such a boundary condition, let us again consider the energy flux

$$n \cdot P = \sum_{r,f} c_r c_f n \cdot (e_r \times h_f) = \sum_j c_j^2 (n \cdot d_j) + \sum_{r \neq f} c_r c_f n \cdot (e_r \times h_f)$$

across the boundary. Due to \[7\], the summation runs only over indices corresponding outgoing directions, i.e., such that $n \cdot d_r > 0$ and $n \cdot d_f > 0$. If the field at the boundary is mainly propagating in one outgoing direction, one
can argue that the last term is dominated by the first term on the right hand side, and under such an assumption one

can therefore guarantee outflow of energy over the boundary. In order to incorporate a boundary condition of this

form in a numerical method, one has to be able to split the approximation of the electromagnetic field locally into

plane waves. This can be realized within the framework of generalized finite elements \[ \text{[26–27]} \] or Trefftz finite
difference approximations \[ \text{[28–29]} \]. Another possibility is provided by the discontinuous Galerkin framework
\[ \text{[34–35]} \], which allows one to systematically couple almost arbitrary local approximations for the simulation
on the global level.

In this paper, we consider a space-time discontinuous Galerkin framework for simulation of Maxwell’s equations
similar to that introduced in \[ \text{[3–29]} \]. In contrast to these papers, we utilize for the local approximation polynomial
Trefftz functions which satisfy \[ \text{[1–3]} \] exactly on every element. The use of Trefftz functions for the numerical
approximation of partial differential equations has been proposed in \[ \text{[30]} \] and since then been investigated intensively;
see e.g. \[ \text{[31–32]} \], \[ \text{[33]} \]. For the problem under investigation, the Trefftz functions depend on space and time, and we
therefore automatically arrive at a space-time method; see \[ \text{[31–33]} \] for a review on the topic. For wave propagation
problems in the frequency domain, let us also refer to \[ \text{[38–39–40]} \]. One of the basic building blocks of our method is
the explicit construction of a basis for the local Trefftz spaces consisting of polynomial plane waves. This allows us
to obtain the required splitting of the discretized electromagnetic field. The second step then consists in describing a
variational form of the boundary condition \[ \text{[7]} \] that can be implemented within the discontinuous Galerkin framework.
To illustrate the benefits of our approach, we present numerical tests including a comparison with first order absorbing
boundary conditions. In our computations, we observe spectral convergence for some model problems indicating that
the boundary condition is formally accurate of arbitrary order.

The outline of the paper is as follows: In Section 2, we introduce the space-time discontinuous Galerkin framework
which is the basis for our numerical method. In Section 3, we then construct the plane wave basis for the local Trefftz
approximation spaces. We also sketch the construction for two dimensional problems. The implementation of the new
boundary conditions is discussed in detail in Section 4, and the remaining two sections report on numerical results for
some simple test problems.

2. A space-time Discontinuous Galerkin formulation

For the numerical simulation of the initial boundary value problem \[ \text{[1–4]} \], we consider a space-time discontinuous
Galerkin method in the spirit of \[ \text{[3–29–41]} \]. For the local approximations we utilize Trefftz polynomials which, by
definition, satisfy Maxwell’s equations exactly. An appropriate choice for the basis will allow us to expand the
numerical solution locally into polynomial plane waves and to apply our new transparent boundary condition. In this
section, we introduce the general framework of the method. The construction of a plane wave basis for the polynomial
Trefftz space and incorporation of the boundary conditions will be addressed in the following two sections.

2.1. Notation

Let \( \Omega_h = \{K\} \) be a non-overlapping partition of the domain \( \Omega \) into regular elements \( K \), e.g., tetrahedral, parallel-
epipeds, prisms, etc. We denote by \( \mathcal{T}_h = \{f = \partial K \cap \partial K’, K \neq K’ \in \Omega_h\} \) the set of element interfaces and by
\( \mathcal{F}_{bdr} = \{f = \partial \Omega \cap \partial \Omega, K \in \Omega_h\} \) the set of faces on the boundary. On an element interface \( f = \partial K_1 \cap \partial K_2 \), any
piecewise smooth function \( v \in C^0(\Omega_h)^3 \) takes on two values \( v_1 = v|_{K_1} \) and \( v_2 = v|_{K_2} \). We then denote by

\[
[v] = \frac{1}{2} (v_1 + v_2), \quad [n \times v] = n_1 \times v_1 + n_2 \times v_2,
\]

the average and the jump of the tangential component of \( v \) on \( f = \partial K_1 \cap \partial K_2 \), respectively. \( n \) denotes the outward
normal vector on the element \( K_i \). Now let \( 0 = t_0 < t_1 < \ldots \) be a partition of the time axis into intervals \( \mathbb{R} = [t_{n-1}, t_n] \).
For every space-time element \( K \times \mathbb{I} \) with \( K \in \Omega_h \), we denote by \( \mathcal{P}_p(K \times \mathbb{I}) \) the space of polynomials in four variables
with order up to \( p \). We assume that \( \epsilon \) and \( \mu \) are constant on \( K \) and call

\[
T_p(K \times \mathbb{I}) = \{ (E, H) \in [\mathcal{P}_p(K \times \mathbb{I})] : \epsilon \partial_t E - \nabla \times H = 0, \mu \partial_t H + \nabla \times E = 0, \nabla \cdot E = 0, \nabla \cdot H = 0 \}
\]

the space of local Trefftz polynomials; this is the space of vector valued polynomials up to order \( p \) satisfying Maxwell’s
equations \[ \text{[1]} \] and the gauge conditions \[ \text{[5]} \] exactly on the corresponding space-time element.
2.2. The space-time DG framework

For the discretization of the wave propagation problem (1)–(4), we consider a space-time discontinuous Galerkin method that we explain in the following. On every time slab $\Omega \times I^m$, we approximate the electromagnetic field $(E, H)$ by piecewise polynomial functions of the space

$$T_p(\Omega_0 \times I^m) := \{(E, H) : \Omega \times I^m \to \mathbb{R}^6 : (E, H)|_{K \times I^m} \in T_p(K \times I^m) \quad \text{for all } K \in \Omega_0\}.$$  \hfill (9)

Functions in this space will be called piecewise Trefftz polynomials in the sequel. The method we have in mind results from using $T_p(\Omega_0 \times I^m)$ as approximation space in a space-time discontinuous Galerkin framework for Maxwell’s equations similar to [3, 29, 41]. The basic discrete variational method can be formulated as follows.

**Method 1** (Space-time discontinuous Galerkin Trefftz method).

Set $E^0_h = E_{ib}, H^0_h = H_{0b}$. For $n \geq 1$ find $(E^n_h, H^n_h) \in T_p(\Omega_0 \times I^m)$ such that for all $(v^E, v^H) \in T_p(\Omega_0 \times I^m)$

$$
\sum_{K \in \Omega_0} \int_K e E^n_h \left( e^{p-1} \right) \cdot v^E \left( e^{p-1} \right) + \mu H^n_h \left( e^{p-1} \right) \cdot v^H \left( e^{p-1} \right) + \sum_{f \in \mathcal{F}_{\text{ex}}} \int_f [n \times H^n_h] \cdot \{v^E\} - [n \times E^n_h] \cdot \{v^H\} \\
+ \sum_{f \in \mathcal{F}_{\text{in}}} \int_f b^E \left( n \times E^n_h, n \times H^n_h \right) \cdot v^E - b^H \left( n \times E^n_h, n \times H^n_h \right) \cdot v^H \\
= \sum_{K \in \Omega_0} \int_K e E^n_h \left( e^{p-1} \right) \cdot v^E \left( e^{p-1} \right) + \mu H^n_h \left( e^{p-1} \right) \cdot v^H \left( e^{p-1} \right) \\
+ \sum_{f \in \mathcal{F}_{\text{in}}} \int_f r^E \left( n \times g \right) \cdot v^E - r^H \left( n \times (g \times n) \right) \cdot v^H.
$$

Note that we still have to specify the boundary conditions to complete the definition of Method 1; this will be done in Section 4. Before we proceed, let us make some general remarks about this numerical scheme.

**Remark 2.** Since the approximating functions satisfy Maxwell’s equations exactly on every element, the formulation only contains spatial and temporal interface terms, which penalize discontinuity of the fields. Under mild conditions on the boundary terms, the discrete variational problem for one time slab can be shown to be well-posed, i.e., to have a unique solution. This allows to advance the solution in time. Now assume that the true solution $(E, H)$ of problem (1)–(4) is sufficiently smooth and that the boundary terms are consistently chosen, e.g., such that $b^E \left( n \times E, n \times H \right) = r^E \left( n \times g \right)$ and $b^H \left( n \times E, n \times H \right) = r^H \left( n \times (g \times n) \right)$ holds for every point on the boundary. Under this assumption, the whole method is consistent, i.e., any smooth solution of the problem (1)–(4) also satisfies the discrete variational principle. To see this, let us have a closer look onto the discrete variational problem: by tangential continuity of the fields, the last term in the first line drops out. Due to continuity in time, the first terms of the first and third line cancel, whereas the boundary terms cancel by assumption. Under some further general assumptions on the boundary terms, the overall method can also be shown to be dissipative. By this we mean that the discrete electromagnetic energy defined by $\mathcal{E}_n(t^n) = \int_{\Omega} e \left( E^n_h(t^n) \right)^2 + \mu \left( H^n_h(t^n) \right)^2$ is monotonically decreasing in time. Note that the same holds for the true solution of the problem (1)–(4), if appropriate boundary conditions are used. We will come back on this issue in Section 4. For details and further properties of the resulting scheme, let us refer to [41]; similar results for related discontinuous Galerkin methods can be found in [3, 29].

2.3. Implementation

Method 1 yields an implicit time stepping scheme. To evolve the discrete solution from time step $n - 1$ to time step $n$, one has to solve the linear system corresponding to the discrete variational problem. Let us sketch the basic structure of this system: After choosing a basis $\{(e^j, h^j) : j = 1, \ldots, J\}$ for the piecewise Trefftz space $T_p(\Omega_0 \times I^m)$, we expand the discrete solution into $(E^n_h, H^n_h) = \sum_j \alpha^j (e^j, h^j)$. The discrete variational problem of Method 1 is then equivalent to a linear system

$$A^n \alpha^n = B^n \alpha^{n-1} + G^n$$
Lemma 3. Assume that either \( c = \text{speed of light} \) (the gauge conditions (3))

\[ \partial_t u = \sum_{j \in \mathcal{V}_h} \int_{f_j} \mathbf{e}^j \cdot \mathbf{u}^{(p^{-1})} + \mu \mathbf{h}^j \cdot \mathbf{u}^{(p^{-1})} + \sum_{j \in \mathcal{F}_h} \int_{f \times t} \left[ \mathbf{n} \times \mathbf{h}^j \cdot \{ \mathbf{e}^j \} - \{ \mathbf{n} \times \mathbf{e}^j \} \cdot \{ \mathbf{h}^j \} \right] + \sum_{j \in \mathcal{F}_h} \int_{f \times t} \mathbf{b}^f \left( \mathbf{n} \times \mathbf{e}^j, \mathbf{n} \times \mathbf{h}^j \right) \cdot \mathbf{e}^j - \mathbf{h}^j \left( \mathbf{n} \times \mathbf{e}^j, \mathbf{n} \times \mathbf{h}^j \right) \cdot \mathbf{h}^j \]

\[ \mathbf{B}_i^j = \sum_{k \in \mathcal{V}_h} \int_{f_k} \mathbf{e}^j \cdot \mathbf{e}^{(p^{-1})} + \mu \mathbf{h}^j \cdot \mathbf{e}^{(p^{-1})} + \sum_{j \in \mathcal{F}_h} \int_{f \times t} \mathbf{K} \sum_{\Omega} \mathbf{J} \cdot \mathbf{e}^{(p^{-1})} - \mathbf{h}^{(p^{-1})} \cdot \mathbf{h}^j \left( \mathbf{n} \times \mathbf{e}^j, \mathbf{n} \times \mathbf{h}^j \right) \cdot \mathbf{h}^j \]

\[ \mathbf{G}_i^j = \sum_{j \in \mathcal{F}_h} \int_{f \times t} \mathbf{F} \left( \mathbf{n} \times \mathbf{g} \right) \cdot \mathbf{e}^j - \mathbf{R} \left( \mathbf{n} \times (\mathbf{g} \times \mathbf{n}) \right) \cdot \mathbf{h}^j. \]

If the size of the time-step is kept constant, i.e., \( t^n = t^{n-1} + \tau \) for all \( n \) with some \( \tau > 0 \), then up to translation of time, the same basis can be used on every time slab. In this case, the matrices \( \mathbf{A}^n \) and \( \mathbf{B}^n \) become independent of \( n \). This situation is particularly convenient from a computational point of view, since a factorization of the matrix \( \mathbf{A} = \mathbf{A}^n \) may then be computed a-priori, and the update of the coefficient vectors \( \mathbf{e}^j \) from step \( n-1 \) to \( n \) only requires one matrix-vector multiplication and a forward-backward substitution. Even if no factorization of \( \mathbf{A} \) is available, the linear system can be solved with acceptable computational effort by some iterative method, as the matrix \( \mathbf{A} \) stems from discretization of a linear hyperbolic problem and therefore will have a moderate condition number. The overall method will thus be very efficient if the time step is kept fixed.

3. A basis for the space of Trefftz polynomials

For the local approximation of the electromagnetic fields on every space-time element \( K \times I^n \), we use vector valued polynomials satisfying Maxwell’s equations (1) and the gauge conditions (2) exactly. In this section, we construct a particular basis for this space of Trefftz polynomials consisting of polynomial plane waves, and we discuss some basic properties of this construction. Since we only consider single elements \( K \times I^n \), we assume throughout this section that the material parameters \( \epsilon \) and \( \mu \) are positive constants.

3.1. Polynomial plane wave functions

As basis functions for the local Trefftz space, we consider polynomial plane waves of the form

\[ \mathcal{F}_{p,i} (\mathbf{r}, t) = \left( \frac{\mathbf{e}_{p,i}}{Z^{-1} \mathbf{h}_{p,i}} \right) \mathbf{e}_{p,i} (\mathbf{r}, t) \quad \text{with} \quad \mathbf{e}_{p,i} (\mathbf{r}, t) = \left( \hat{\mathbf{a}}_{p,i} \cdot \mathbf{r} - \hat{\mathbf{c}} \right)^p. \]

Note that in this construction, the material properties enter implicitly via the intrinsic impedance \( Z = \sqrt{\mu/\epsilon} \) and the speed of light \( c = 1/\sqrt{\epsilon \mu} \). The hat symbols are used to denote constant vectors of unit length.

**Lemma 3.** Assume that either \( p = 0 \) or that \( p \geq 1 \) and \( \hat{\mathbf{e}}_{p,i}, \hat{\mathbf{h}}_{p,i}, \hat{\mathbf{d}}_{p,i} \) are mutually orthogonal with \( \hat{\mathbf{d}}_{p,i} = \hat{\mathbf{e}}_{p,i} \times \hat{\mathbf{h}}_{p,i} \). Then the function \( (\mathbf{E}, \mathbf{H}) = \mathcal{F}_{p,i} \) is a vector valued polynomial of degree \( p \) which satisfies Maxwell’s equations (1) and the gauge conditions (3).

**Proof.** By definition, the electric field component has the form \( \mathbf{E} = \hat{\mathbf{e}}_{p,i} (\hat{\mathbf{d}}_{p,i} \cdot \mathbf{r} - ct)^p \). One then easily verifies by direct computation that \( \partial_t \mathbf{E} = -\hat{\mathbf{c}}_{p,i} \hat{\mathbf{e}}_{p,i} (\hat{\mathbf{d}}_{p,i} \cdot \mathbf{r} - ct)^p \) and \( \nabla \times \mathbf{E} = \hat{\mathbf{e}}_{p,i} \times \hat{\mathbf{d}}_{p,i} \hat{\mathbf{h}}_{p,i} (\hat{\mathbf{d}}_{p,i} \cdot \mathbf{r} - ct) \); similar expressions are obtained for the magnetic field component. Maxwell’s equations (1) then reduce to the algebraic conditions

\[ p \left( \hat{\mathbf{d}}_{p,i} \cdot \mathbf{r} - ct \right)^{p-1} (\hat{\mathbf{e}}_{p,i} + \hat{\mathbf{h}}_{p,i} \times \hat{\mathbf{d}}_{p,i}) = 0 \quad \text{and} \quad p \left( \hat{\mathbf{d}}_{p,i} \cdot \mathbf{r} - ct \right)^{p-1} (-\hat{\mathbf{h}}_{p,i} - \hat{\mathbf{e}}_{p,i} \times \hat{\mathbf{d}}_{p,i}) = 0. \]

The two equations are satisfied if \( p = 0 \); for \( p \geq 1 \), we have to require that \( \hat{\mathbf{e}}_{p,i} = \hat{\mathbf{h}}_{p,i} \hat{\mathbf{d}}_{p,i} \) and that \( \hat{\mathbf{h}}_{p,i} = -\hat{\mathbf{e}}_{p,i} \hat{\mathbf{d}}_{p,i} \) which are the assumptions of the Lemma. Additionally, we have \( \nabla \cdot \mathbf{E} = p \left( \hat{\mathbf{d}}_{p,i} \cdot \mathbf{r} - ct \right)^{p-1} \hat{\mathbf{e}}_{p,i} \hat{\mathbf{d}}_{p,i} \). Therefore, also the gauge conditions are satisfied if either \( p = 0 \) or if \( p \geq 1 \) and the directions \( \hat{\mathbf{e}}_{p,i}, \hat{\mathbf{h}}_{p,i}, \hat{\mathbf{d}}_{p,i} \) are orthogonal. \( \square \)
3.2. The polynomial Trefftz space

We will now utilize the polynomial plane wave functions introduced in the previous section to define a special basis for the space $T_p(K \times I^o)$ of local Trefftz polynomials.

**Lemma 4.** Let $\mu, \epsilon > 0$ be constant. Then

1. $\dim T_p(K \times I^o) = \frac{1}{2} (p + 1) (p + 2) (2p + 9)$.
2. For $p \geq 0$ there exist $i_p = 2(p + 1)(p + 3)$ independent orthogonal vector triples $(\bar{d}_{p,i}, \bar{\epsilon}_{p,i}, \bar{h}_{p,i})$, $i = 1, \ldots, i_p$ of unit length and with $\bar{h}_{p,i} = \bar{d}_{p,i} \times \bar{\epsilon}_{p,i}$, such that the functions $\mathcal{F}_{p,i}$, $i = 1, \ldots, i_p$ are linearly independent.
3. The functions $\mathcal{F}_{p,i}$, $1 \leq i \leq i_p$, $0 \leq k \leq p$, form a basis of $T_p(K \times I^o)$.

**Proof.** 1. For polynomials of four variables we have $[\dim \mathbb{T}_p]^6 = (p + 1)(p + 2)(p + 3)(p + 4)/4$. The two Maxwell equations give $p(p + 1)(p + 2)(p + 3)/4$ conditions. Applying the divergence operator to Maxwell’s equations yields $\frac{\varepsilon_d}{\varepsilon} \nabla \cdot \mathbf{E} = -\nabla \cdot \nabla \times \mathbf{H} = 0$ and $\mu \frac{\varepsilon_d}{\varepsilon} \nabla \cdot \mathbf{H} = \nabla \cdot \nabla \times \mathbf{E} = 0$. The two gauge conditions thus only have to be required at one point in time and therefore give additional $p(p + 1)(p + 2)/3$ independent conditions. The assertion now follows by counting the dimensions.

2. By construction and Lemma 3, we know that the polynomials $\mathcal{F}_{p,i}$ have order $p$ and are members of $\mathbb{T}_p$. The fact that there exist $i_p$ such linear independent functions can easily be verified in practice for any $p$ required; we present a construction below. Note that by 1. there cannot exist more than $i_p$ linear independent Trefftz polynomials of order $p$.

3. Trefftz polynomials $\mathcal{F}_{p,i}$ with different orders are linearly independent. The third assertion then follows from 1. and 2. by counting the dimensions. \hfill $\Box$

**Remark 5.** According to Lemma 4, every local Trefftz polynomial can be split into a superposition of polynomial plane wave functions $\mathcal{F}_{p,i}$. This is the basic requirement for the implementation of the boundary condition (7). If we use the polynomial plane wave basis in the implementation of the method, then the required decomposition is readily available. Let us emphasize that the Trefftz polynomials have coupled electric and magnetic field components, they are functions of space and time, but do not have a tensor-product structure.

3.3. The two-dimensional setting

In cases of translational invariance along one direction, Maxwell’s equations can be cast in a quasi two dimensional form. For illustration and later reference, let us consider one such case in more detail: We assume that the domain and all fields are homogeneous in the $z$-direction and that the electric field is polarized in this direction. The electromagnetic fields then have the form $\mathbf{H} = (H_1, H_2, 0)$ and $\mathbf{E} = (0, 0, E)$ with $H_1, H_2, E$ only depending on $x$ and $y$. This is known as the TM mode in electrical engineering and as the s-mode in optics. The computational domain is $\Omega = \Omega' \times R$ with $\Omega' \subset \mathbb{R}^2$ being the relevant slice of the three dimensional domain at any fixed $z$. We denote by $\Omega_0 = \{k'\}$ a mesh of the two dimensional domain $\Omega'$ and define $\Omega_0 = \{k = K = K' \times R\}$. Note that this setup still describes a truly three-dimensional problem with symmetry in the $z$ direction. We now define

$$ T_p(K \times I^o) = \{ (\mathbf{E}, \mathbf{H}) : \mathbf{E} = (0, 0, E), \mathbf{H} = (H_1, H_2, 0), \text{ with } H_1, H_2, E \text{ independent of } z \}. \quad (11) $$

The prime is used here to distinguish this formulation from a full three-dimensional problem. The construction of a basis for the polynomial Trefftz space is similar to the general case. Here we consider functions of the form

$$ \mathcal{F}_{p,i}^* (\mathbf{r}, t) = \left( \frac{\hat{e}_{p,i}^z}{Z^{-1} \hat{h}_{p,i}^x} \right)^p \varphi_{p,i} (\mathbf{r}, t) \quad \text{with} \quad \varphi_{p,i} (\mathbf{r}, t) = \left( \hat{d}_{p,i}^y \cdot \mathbf{r} - ct \right)^p \quad (12) $$

where $\hat{e}_{p,i}^z = (0, 0, 1)$ and $\hat{d}_{p,i}^y, \hat{h}_{p,i}^x$ are orthogonal unit vectors lying in the $x$-$y$ plane. Note that under these assumptions the vector $\hat{h}_{p,i}^x$ is already fixed by the choice of $\hat{d}_{p,i}^y$ and the condition $\hat{d}_{p,i}^y = \hat{e}_{p,i}^z \times \hat{h}_{p,i}^x$.

**Lemma 6.** Let $\mu, \epsilon > 0$ be constant. Then

1. $\dim T_p(K \times I^o) = (p + 1)(p + 3)$. 

3.4. Choice of directions

The symbol $\nabla \times$ that can be used to define the polynomial plane wave basis. We start with the three dimensional setting, and proceed also utilize the alternative representation

It suffices to consider the field components $F$ Linear independence of the corresponding plane wave functions been used in [38] to generate a basis for the time harmonic problem. The analysis of [38] shows that even (almost) independent of the corresponding functions $\mathcal{F}_{p,i}$ $i = 1, \ldots, i'$ are linearly independent.

3. The functions $\mathcal{F}_{i,k}$, $1 \leq i \leq i'$, $0 \leq k \leq p$ form a basis of $\mathbb{T}_p(K \times P)$.

The proof follows by counting arguments as in the three dimensional case. A particular choice of directions for the two-dimensional setting will again be given below.

Remark 7. It suffices to consider the field components $H_1$, $H_2$, and $E$ as functions of $x$, $y$, and $t$. One could therefore also utilize the alternative representation

$$\mathbb{T}_p'(K' \times P') = \{ (H', E) \in [\mathbb{P}_p(K' \times P')]^3 : \varepsilon \partial_t E - \nabla \times H' = 0, \mu \partial_t H' + \nabla \times E = 0, \nabla \cdot H' = 0 \}.$$ (13)

The symbol $\nabla \times$ here denotes the vector-to-scalar and scalar-to-vector curl, respectively. Note that the gauge condition for $E$ is satisfied automatically since $E$ only depends on $x$ and $y$ and the corresponding field $E = (0, 0, E)$ points into $z$-direction. The space $\mathbb{T}_p'$ is isomorphic with $\mathbb{T}_p^*$ and the results stated in the previous lemma carry over verbatim.

3.4. Choice of directions

To complete the description of the construction of our basis, let us present particular choices of directions $(\hat{d}_{p,i}, \hat{e}_{p,i}, \hat{h}_{p,i})$ that can be used to define the polynomial plane wave basis. We start with the three dimensional setting, and proceed as follows:

1. We choose $p + 1$ distinct numbers $z_m, m = 0, \ldots, p - 1$ in the interval $(-1, 1)$ such that

$$\ldots < z_{p-4} < z_{p-2} < z_p < z_{p-1} < z_{p-3} < \ldots$$

The ordering will become clear in the sequel.

2. For every $z_m$, we choose $2m + 3$ equidistantly spaced points on the circle $\{(x, y, z_m) : x^2 + y^2 + z_m^2 = 1\}$. This yields in total $\sum_{m=0}^{p} (2m + 3) = (p + 1)(p + 3)$ different directions $\hat{d}_{p,2j-1}$, $j = 1, \ldots, (p + 1)(p + 3)$.

3. For every direction $\hat{d}_{p,2j-1}$ we set $\hat{d}_{p,2j} = \hat{d}_{p,2j-1}$ and choose two independent, e.g., mutually orthogonal, polarizations $\hat{e}_{p,2j-1}, \hat{e}_{p,2j}$ orthogonal to $\hat{d}_{p,2j-1}$.

4. For any pair $(\hat{d}_{p,i}, \hat{e}_{p,i})$, $i = 1, \ldots, 2(p + 1)(p + 3)$ we finally define $\hat{h}_{p,i} = \hat{e}_{p,i} \times \hat{d}_{p,i}$.

A possible choice of the directions $\hat{d}_{p,2j-1}$, $j = 1, \ldots, (p + 1)(p + 3)$ is depicted in Figure [1]. Note that the linear independence of the corresponding functions $\mathcal{F}_{p,i}$ can be verified numerically. A similar construction of directions has been used in [38] to generate a basis for the time harmonic problem. The analysis of [38] shows that even (almost) any random choice of directions $\hat{d}_{p,2j-1}$ will yield a linearly independent system of Trefftz functions.

In two dimensions, the construction of a suitable set of directions is even simpler. Here we proceed as follows:

1. We choose equidistantly spaced directions $\hat{d}_{p,i}$, $i = 0, 2p + 3$ on the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$.

2. We define $\hat{e}_{p,i} = (0, 0, 1)$ and $\hat{h}_{p,i} = \hat{d}_{p,i} \times \hat{e}_{p,i}$.

Linear independence of the corresponding plane wave functions $\mathcal{F}_{p,i}$ can again easily be verified numerically.
4. Incorporation of the boundary conditions

To complete the definition of the discontinuous Galerkin method, we now demonstrate how to incorporate various boundary conditions. We start by discussing two different implementations for the impedance boundary condition (5), which allow us to treat perfect-electric-conducting and perfect-magnetic-conducting conditions, as well as the first order absorbing boundary conditions (6). Our implementation of the transparent boundary condition (7) will turn out to have a very similar structure. In addition to the formulation of these conditions, we also comment on their stability.

4.1. PEC-like boundary conditions

Let us first consider the impedance boundary condition of the form

\[ \mathbf{n} \times \mathbf{E} - \beta \mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = \mathbf{n} \times \mathbf{g}. \] (14)

In this case, we may choose

\[ \mathbf{b}^{E} (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) = 0 \quad \text{and} \quad \mathbf{b}^{H} (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) = \mathbf{n} \times \mathbf{E} - \beta \mathbf{n} \times (\mathbf{H} \times \mathbf{n}), \]
and the right hand sides are chosen consistently as \( \mathbf{r}^H (\mathbf{n} \times \mathbf{g}) = \mathbf{n} \times \mathbf{g} \) and \( \mathbf{r}^F (\mathbf{n} \times \mathbf{g}) = 0 \). Next let us consider the energy balance: Testing with \( (\mathbf{v}^F, \mathbf{v}^H) = (\mathbf{E}, \mathbf{H}) \), this choice yields

\[
\int f \mathbf{b}^F (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) \cdot \mathbf{v}^F - \mathbf{b}^H (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) \cdot \mathbf{v}^H = \int f \beta |\mathbf{n} \times \mathbf{H}|^2 + \mathbf{n} \times \mathbf{E} \cdot \mathbf{H}.
\]

For \( \beta \geq 0 \), the first boundary term on the right hand side yields a positive definite contribution to the bilinear form of the variational problem defining Method 1. The second term is absorbed by the remaining terms of the bilinear form. In summary, one therefore obtains a dissipative boundary condition provided that \( \beta \geq 0 \). For \( \beta = 0 \) and \( \mathbf{g} = 0 \), we arrive at the condition for a perfect electric conductor, and the choice \( \beta = Z^{-1} \) and \( \mathbf{g} = 0 \) corresponds to the first-order absorbing boundary condition.

### 4.2. PMC-like boundary conditions

Taking the cross product with \( \mathbf{n} \) from the right in equation (14), it is possible to obtain an alternative equivalent form of the impedance boundary condition, namely

\[
\mathbf{n} \times \mathbf{H} + \beta' \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) = \mathbf{n} \times \mathbf{g}'.
\]  

This condition can be incorporated in the discontinuous Galerkin method by choosing

\[
\mathbf{b}^F (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) = \mathbf{n} \times \mathbf{H} + \beta' \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) \quad \text{and} \quad \mathbf{b}^H (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) = 0,
\]

with \( \mathbf{r}^F (\mathbf{n} \times \mathbf{g}) = \mathbf{n} \times \mathbf{g}' \) and \( \mathbf{r}^H (\mathbf{n} \times \mathbf{g}) = 0 \) for the right hand side terms. This again yields a consistent representation of the boundary conditions. Testing with \( (\mathbf{v}^H, \mathbf{v}^H) = (\mathbf{E}, \mathbf{H}) \), this choice now yields

\[
\int f \mathbf{b}^F (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) \cdot \mathbf{v}^F - \mathbf{b}^H (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) \cdot \mathbf{v}^H = \int f \beta' |\mathbf{n} \times \mathbf{E}|^2 - \mathbf{n} \times \mathbf{H} \cdot \mathbf{E}.
\]

For \( \beta' \geq 0 \), one again obtains a positive contribution and a term that can be absorbed by the remaining parts of the bilinear form. We therefore again obtain a dissipative boundary condition, provided that \( \beta' \geq 0 \). For \( \beta' = 0 \) and \( \mathbf{g}' = 0 \), we arrive at the perfect-magnetic-conducting condition, and the choice \( \beta = Z \) and \( \mathbf{g} = 0 \) yields again an implementation of the first-order absorbing boundary condition (5).

### 4.3. New transparent boundary conditions

The basic idea of our approach to constructing transparent boundary conditions introduced in Section [7] is to locally expand of the electromagnetic field \((\mathbf{E}, \mathbf{H})\) into a superposition of plane waves

\[
(\mathbf{E}, \mathbf{H}) = \sum_j c_j (\mathbf{e}_j, \mathbf{h}_j).
\]  

Due to our choice of basis, such a decomposition is readily available, and \((\mathbf{e}_j, \mathbf{h}_j) := \mathcal{F}_j\) are just the polynomial plane wave basis function for the local Trefftz space introduced in the previous section. For the approximation of the transparent boundary condition (7) within the discontinuous Galerkin framework, we then consider the choice

\[
\int f \mathbf{b}^F (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) \cdot \mathbf{v}^F = \int f (\mathbf{n} \times (\mathbf{E}_m) \cdot (\mathbf{n} \times \mathbf{v}_m^E) + \frac{1}{2} \mathbf{n} \times \mathbf{H}_m \cdot \mathbf{v}^E),
\]

and

\[
- \int f \mathbf{b}^H (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) \cdot \mathbf{v}^H = \int f (\mathbf{n} \times \mathbf{H}_m) \cdot (\mathbf{n} \times \mathbf{v}_m^H) - \frac{1}{2} \mathbf{n} \times \mathbf{E}_m \cdot \mathbf{v}^H,
\]

and the right hand sides are chosen as \( \mathbf{r}^F (\mathbf{n} \times \mathbf{g}) = \mathbf{r}^H (\mathbf{n} \times \mathbf{g}) = 0 \). Here \((\mathbf{E}_m, \mathbf{H}_m) = \sum_j c_j (\mathbf{e}_j, \mathbf{h}_j)\) denotes the electromagnetic field consisting only of the incoming components; summation therefore has to be taken only over indices \( j \) with \( \mathbf{d}_j \cdot \mathbf{n} < 0 \). The condition is motivated by the fact that we assume that the true solution does not have any incoming components. In that case, all terms in this formulation of the boundary condition vanish. Also observe
that the structure is very similar to the previous conditions. As before, let us now examine the energy balance for the new boundary condition: Testing with \( (\mathbf{v}^E, \mathbf{v}^H) = (\mathbf{E}, \mathbf{H}) \), we obtain

\[
\int_f \mathbf{b}^E (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) \cdot \mathbf{v}^E - \mathbf{b}^H (\mathbf{n} \times \mathbf{E}, \mathbf{n} \times \mathbf{H}) \cdot \mathbf{v}^H = \frac{1}{2} \int_f \mathbf{n} \times \mathbf{H} \cdot \mathbf{E} - \mathbf{n} \times \mathbf{E} \cdot \mathbf{H} + \int_f \mathbf{n} \times \mathbf{H}_{\text{out}} \cdot \mathbf{E}_{\text{out}} - \mathbf{n} \times \mathbf{E}_{\text{out}} \cdot \mathbf{H}_{\text{out}} + \mathbf{n} \times \mathbf{E}_{\text{in}} \cdot \mathbf{H}_{\text{in}} + |\mathbf{n} \times \mathbf{E}_{\text{in}}|^2 + |\mathbf{n} \times \mathbf{H}_{\text{in}}|^2.
\]

We can now argue the same way as on the continuous level: If the numerical solution is mainly directed into one outward direction, which will be the case if the continuous solution is so, then the first two terms in the second line will give a positive contribution, and the third and fourth term can then be absorbed into the first two and the last term via a Young’s inequality. In summary, we can thus expect a decay of the discrete energy, which is what we also observe in our numerical tests.

5. Test Problems

In this section we introduce two test problems that we use for a numerical validation of the new transparent boundary condition. To facilitate the illustration of our results, we restrict ourselves here to the two dimensional setting. The first problem investigates the propagation of a plane wave and its reflection at the transparent boundary. The availability of an analytical solution for this problem allows to investigate the convergence behavior and we can report about the convergence rates observed in our tests. In the second problem, we consider the propagation of a cylindrical wave. Here we compare the numerical solution on an artificially bounded domain with that obtained by simulation on a larger domain. This allows us to precisely evaluate the effect of truncation of the domain, which is what one is typically interested in in practice. In both examples, the electromagnetic wave hits the artificial boundaries at different angles, which allows to assess the stability of our results with respect to the angle of impingement.

5.1. Propagation of a Plane Wave

We consider a plane wave propagating in a homogeneous domain with \( \epsilon = \mu = 1 \) in direction \( \mathbf{k} = 1/\sqrt{2} (-1, -1, 0) \). Since plane wave propagation has an analytic solution, we use this example to evaluate the convergence behaviour later in Section 6.1. For the true solution, we set \( \mathbf{E} = (0, 0, \mathbf{E}) \) and \( \mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, 0) \) with

\[
\mathbf{E} = \exp \left( - (k_1 x + k_2 y - t + \alpha)^2 / \delta \right), \quad \mathbf{H}_1 = k_2 \mathbf{E}, \quad \text{and} \quad \mathbf{H}_2 = -k_1 \mathbf{E}.
\]

The parameters \( \delta \) and \( \alpha \) allow to vary the width and the offset of the wave. For our numerical tests, we choose \( \delta = 4 \) and \( \alpha = 12 \). The propagation of the electric field component with these parameters depicted in Fig. 2. Since the solution is homogeneous in \( z \) direction, it suffices to consider the two-dimensional setting. As a computational domain, we choose \( \Omega' = (0, 10) \times (0, 10) \) and we utilize the transparent boundary conditions at \((x, 0)\) and \((0, y)\), where the wave leaves the domain. On the ingoing boundaries at \((x, 10)\) and \((10, y)\) we use the impedance boundary condition [5]. These values and the initial condition are chosen such that they are satisfied exactly by the true solution.

5.2. Propagation of Cylinder Wave

As a second scenario, we consider the propagation of a cylindrical wave of the form \( \mathbf{E} = (0, 0, \mathbf{E}), \mathbf{H} = (\mathbf{H}_1, \mathbf{H}_2, 0) \) originating from an initial field distribution

\[
\mathbf{E}_0(x, y) = \exp \left( - (x^2 + y^2) / \delta \right), \quad \mathbf{H}_1^0 = \mathbf{H}_2^0 = 0
\]

in a homogeneous domain with \( \epsilon = \mu = 1 \). For our tests we choose the width \( \delta = 18 \).

Again the fields are homogeneous in \( z \) direction, at it suffices to consider the quasi two-dimensional setting. Although a semi-analytical solution could be obtained via d’Alembert’s formula, a closed form solution is not available for this two-dimensional setting. Instead, we construct a reference solution by simulation on a larger domain \( \Omega' \). The evolution of the simulated electric field is shown in Fig. 3.

For our numerical tests, we consider \( \Omega' = (0, 10) \times (0, 10) \) as a restriction of the larger domain \( \Omega' \). Transparent boundary conditions on the artificial boundary \( \partial \Omega' \). The simulated fields are then compared to the once obtained by simulation on the larger domain \( \Omega' \), which allows us to assess the effect of the artificial boundaries.
Figure 2: Electric field of a plane wave propagating in the spatial domain \((0, 10) \times (0, 10)\) filled with vacuum. At \(x = 0\) and \(y = 0\) the new transparent boundary conditions are applied whereas at \(x = 10\) and \(y = 10\) exact boundary conditions are enforced. The solution is displayed at four relevant time steps. The first picture (upper left) depicts the start of the wave propagation. The second picture (upper right) shows the time when the wave is impinging at the transparent boundaries. In the third picture (lower left) most parts of the original wave have left the domain through the transparent boundaries. The last picture (lower right) depicts the remnants of the wave.

Figure 3: Electric field of a cylindrical wave in the spatial domain \((-10, 10) \times (-10, 10)\) filled with vacuum. The first picture (upper left) depicts the start of the cylindrical wave propagation. The second picture (upper right) shows the time when the cylindrical wave is impinging at the transparent boundaries. In the third picture (lower left) most parts of the original cylindrical wave have left the domain through the transparent boundaries. The last picture (lower right) depicts the remnants of the cylindrical wave.
6. Results

In this section we present numerical results obtained for the two test problems. We first investigate the convergence behaviour of our method with respect of the spatial and temporal meshsize as well as the polynomial degree for the first test case. In particular, we compare results obtained with the new transparent boundary condition to those obtained with the first-order absorbing conditions. In the second series of tests, we evaluate the dissipative effect of the artificial boundary conditions by tracking the energy evolution over time for the second test example. For all tests we start with a uniform quadrilateral mesh of $\Omega'$ consisting of 100 cells for the plane wave scenario and 400 cells for the cylindrical wave scenario, respectively.

6.1. Error Convergence

As a first validation we investigate the convergence of the method when increasing the polynomial degree of approximation. In Fig. 4 we display the relative error in the space-time $L_2$-norm in dependence of $p$. Note that according to our considerations in Section 3, the total number of degrees of freedom behaves like $p^3$ when increasing the polynomial degree. The numerical tests are carried out for the plane wave problem discussed in Section 5.1.

The fields are prescribed at the boundaries where the wave enters the domain via an impedance boundary condition which is satisfied exactly by the true solution. For the boundaries where the wave leaves the domain, we utilize our new first order absorbing boundary conditions and we compare to the results with those obtained with the first-order absorbing boundary conditions and with impedance boundary conditions that are satisfied exactly by the true solution.

As can be seen from the convergence plot, we obtain spectral convergence for the new transparent boundary conditions (green boxes) and also with the exact impedance boundary conditions (gray dashed line), which in practice are of course not available. In fact, the transparent boundary condition performs almost as good. The first-order absorbing condition (blue circles) yields a saturation due to a systematic consistency error arising from the fact that the wave does not impinge on the transparent boundary at normal angle.

We proceed by evaluating the convergence order for fixed polynomial degrees under spatial and temporal refinement. The error decay is visualized in Fig. 5 and Fig. 6. We display the relative error in the $L_2$ norm as functions of spatial and temporal meshsize $\Delta r$ and $\Delta t$ on a loglog scale. In order to isolate the convergence rate under spatial refinement we choose the temporal meshsize to be sufficiently small, meaning one refinement step smaller than the...
Figure 5: Relative error of a two dimensional plane wave propagation (see Fig. 2) calculated in the $L_2$ as a function of the spatial cell size $\Delta r$ for the first four orders $p = 0, 1, 2, 3$. On the abscissa the spatial cell size is shown which we consequently split by factors of two. Here the temporal meshsize $\Delta t$ is chosen sufficiently small (i.e. 1 refinement step smaller than the current $\Delta r$ size).

Figure 6: Relative error of a two dimensional plane wave propagation (see Fig. 2) calculated in the $L_2$ as a function of the time step size $\Delta t$ for the first four orders $p = 0, 1, 2, 3$. On the abscissa the time step size is shown which we consequently split by factors of two. Here the spatial meshsize $\Delta r$ is chosen sufficiently small (i.e. 1 refinement step smaller than the current $\Delta t$ size).
The same is done for the convergence rate under temporal refinement. However, in this case we chose the respective \( \Delta r \) one step smaller than the considered \( \Delta t \).

In all simulations, we observe optimal convergence rates of order \( p + 1 \) for a simulation employing polynomials of order \( p \). This especially means that we obtain an optimal convergence rate under temporal refinement. Together with the spectral convergence result in Fig. 4, we can conclude that high order time integration stays an inherent feature of the method, even with Trefftz transparent boundary conditions.

### 6.2. Dissipative behaviour

With numerical test for the second test problem discussed in Section 5.2, we want to evaluate the effect of the artificial boundaries on the simulation results. To do so, we investigate the evolution of the total electromagnetic energy \( \mathcal{E}(t) = \int_{\Omega'} \epsilon |E(t)|^2 + \mu |H|^2 \) contained in the computational domain \( \Omega' = (0, 10) \times (0, 10) \). We compare the energies obtained for the reference simulation on a larger domain \( \tilde{\Omega}' = (-10, 20) \times (-10, 20) \) with those obtained by simulation on the truncated domain \( \Omega' \). On the artificial boundary \( \partial \Omega' \), we utilize the new transparent boundary conditions, and we again compare with the results obtained with the first-order absorbing boundary conditions.

The evolution of the energy is displayed in Fig. 7. The time evolution can be roughly split into several phases. First the wave is propagating freely inside the computational domain \( \Omega' \) and the energy stays constant. In the second phase, the wave front hits the boundary energy flows out of the system energy. In the third phase, tail part of the wave front hits the boundary, and in the last phase the remaining energy slowly flows out of the domain. The overall algebraic decay in energy can be explained analytically by d’Alembert’s formula. Note that the three-dimensional cylindrical wave can be understood as a superposition of spherical waves. At the two-dimensional cross-section, information thus travels with any speed between 0 and \( c \), and therefore some part of the energy stays in the domain for a long time. Note that the energy is strictly decreasing in time, which illustrates the dissipative behaviour of the new transparent boundary condition. The error in the energy can be reduced substantially, if the same order of approximation is also utilized for simulation on the truncated domain.

![Figure 7: Evolution of the total energy of a cylindrical wave corresponding to the test problem of Section 5.2. The reference solution (gray dashed line) is computed on a larger domain \((-30, 30) \times (-30, 30)\) with approximation order \( p = 3 \). The second result (solid green line) is computed on the truncated domain \((-10, 10) \times (-10, 10)\) with the new transparent boundary conditions. The black dots correspond to the fields displayed in Fig. 3. Using the d’Alembert representation, the energy of true solution can be shown to be strictly decreasing in time. This behaviour is also observed for the simulation with the transparent boundary conditions.](image-url)
7. Summary

In this paper we proposed a new type of transparent boundary condition and discussed its implementation in the framework of a space-time discontinuous Galerkin method using polynomial Trefftz functions for the local approximations. We constructed a particular basis for the local Trefftz spaces consisting of polynomial plane wave functions, which allowed us to split the electromagnetic fields locally into plane waves propagating in specific directions. The transparent boundary conditions were then obtained by appropriately penalizing the ingoing waves stemming from unphysical reflections. The general procedure is applicable to approximations of arbitrary order and, in fact, we could observe spectral convergence of the error in our numerical tests. Also optimal orders of convergence with respect to the spatial and temporal meshsize were observed. While the implementation of the new boundary conditions is similar to that of more standard conditions, like the first order absorbing boundary condition, the new condition performs much better in numerical tests. As illustrated, the amount of unphysical reflections could be reduced dramatically. In all our numerical tests, the new boundary conditions show a dissipative behaviour. Although we could give some theoretical explanation for this, a rigorous theoretical justification is still missing and this issue has to be investigated further.

Acknowledgments

The work of the first, second, and fifth author was partially by the German Research Foundation (DFG) under grant GSC 233. Sascha M. Schnepf acknowledges the support of the Alexander von Humboldt-Foundation through a Feodor-Lynen research fellowship. The work of Igor Tsukerman was supported in part by the National Science Foundation (NSF) under Grant No. 1216927.

References

[1] R. Hadley, Transparent boundary condition for the beam propagation method, IEEE J. Sel. Top. Quant. Electron. 28 (1992) 363–370.
[2] T. Milligan, Modern Antenna Design, Wiley, New Jersey, 2005.
[3] T. Griesmair, P. Monk, Discretization of the wave equation using continuous elements in time and a hybridizable discontinuous Galerkin method in space, J Sci Comput 58 (2014) 472–498.
[4] R. Hiptmair, Coupling of Finite Elements and Boundary Elements in Electromagnetic Scattering, SIAM J Numer Anal 41 (2003) 919–944.
[5] S. Kurz, S. Russenschuck, The application of the BEM-FEM coupling method for the accurate calculation of fields in superconducting magnets, volume 82, Springer-Verlag, 1999.
[6] J. Berenger, A perfectly matched layer for the absorption of electromagnetic waves, J Comput Phys (1994) 185–200.
[7] J. Berenger, Three-Dimensional Perfectly Matched Layer for the Absorption of Electromagnetic Waves, J Comput Phys (1996) 363–379.
[8] S. Kurz, S. Russenschuck, The application of the bem-fem coupling method for the accurate calculation of fields in superconducting magnets, Electrical Engineering 82 (1980) 1–10.
[9] A. Bayliss, C. Goldstein, E. Turkel, On accuracy conditions for the numerical computation of waves, J Comp Phys 59 (1985) 396–404.
[10] B. Engquist, A. Majda, Absorbing boundary conditions for the numerical simulation of waves, Math. Comp. (1977) 629–651.
[11] B. Engquist, A. Majda, Radiation boundary conditions for acoustic and elastic wave calculations, Comm. Pure Appl. Math. (1979) 313–357.
[12] T. Hagstrom, T. Warburton, A new auxiliary variable formulation of high-order local radiation boundary conditions: corner compatibility conditions and extensions to first-order systems., Wave Motion 39 (2004) 327–338.
[13] T. Hagstrom, M. De Castro, D. Givoli, D. Tzemach, Local high-order absorbing boundary conditions for time-dependent waves in guides, J Comput Acoust 15 (2007) 1–22.
[14] T. Hagstrom, S. Hariharan, A formulation of asymptotic and exact boundary conditions using local operators, Appl Numer Math 27 (1998) 403–416.
[15] R. Higdon, Absorbing boundary conditions for difference approximations to the multidimensional wave equation., Math Comp. 47 (1986) 437–459.
[16] R. Higdon, Numerical absorbing boundary conditions for the wave equation., Math. Comp. 49 (1987) 65–90.
[17] L. Zhiedrich, Transparent Boundary Conditions for Maxwell’s Equations: Numerical Concepts beyond the PML, Method, Ph.D. thesis, FU Berlin, 2009.
[18] J. Dea, High-order non-reflecting boundary conditions for the linearized euler equations, Ph.D. thesis, Naval Postgraduate School, Monterey, California, 2008.
[19] C. Muller, Randwertprobleme der Theorie elektromagnetischer Schwingungen., Math Z 56 (1952) 261–270.
[20] H. Barucq, B. Hanouzet, Asymptotic behavior of solutions to Maxwell’s system with absorbing Silver-Müller condition on the exterior boundary, Asymptotic Anal. (1997) 25–40.
[21] L. Li, S. Lanteri, R. Perrussel, A hybridizable discontinuous Galerkin method combined to a Schwarz algorithm for the solution of 3d time-harmonic Maxwell’s equation, J Comput Phys 256 (2014) 563–581.
[22] P. Joly, B. Mercier, A new second order absorbing boundary condition for Maxwell’s equations in dimension 3, INRIA Res. Report 1047 (1989).
[23] F. Assous, E. Sonnendrücker, Joly-Mercier boundary condition for the finite element solution of 3D Maxwell equations, Math Comput Model 51 (2010) 915–943.
[24] I. Tsukerman, A class of difference schemes with flexible local approximation, J Comput Phys 211 (2006) 659–699.
[25] I. Tsukerman, Computational Methods for Nanoscale Applications: Particles, Plasmons and Waves, Springer, 2008.
[26] I. Babuska, B. Guo, The h, p and h-p version of the finite element method: basis theory and applications, Adv Eng Softw 15 (1992).
[27] J. Melenk, I. Babuska, The partition of unity finite element method: Basic theory and applications, Comput Method Appl M 139 (1996) 289 – 314.
[28] I. Tsukerman, Electromagnetic applications of a new finite-difference calculus, Magnetics, IEEE Transactions on 41 (2005) 2206 – 2225.
[29] M. Lilienthal, S. Schnepp, T. Weiland, Non-dissipative space-time hp-discontinuous galerkin method for the time-dependent maxwell equations, Journal of Computational Physics 275 (2014) 589–607.
[30] E. Trefftz, Ein Gegenstueck zum Ritzschen Verfahren, 2, Internationaler Kongress fuer Technische Mechanik, Zurich, 1926.
[31] P. Runge, The complete Trefftz method, Acta Mech 78 (1978) 235–242.
[32] J. Jirousek, A. Zielinski, Survey of Trefftz-type element formulations, Comput Struct 63 (1997) 225 – 242.
[33] I. Herrera, Trefftz Method: A General Theory, Numer Methods Partial Differ Equ 16 (2000) 561–580.
[34] W. Reed, T. Hill, Triangular mesh methods for the neutron transport equation, Technical Report, Los Alamos Scientific Laboratory Report, 1973.
[35] P. LeSaint, P.-A. Raviart, On a finite element method for solving the neutron transport equation, Academic Press, New York, 1974.
[36] B. Cockburn, C. Shu, Runge–Kutta discontinuous Galerkin methods for convection-dominated problems, J Sci Comput 16 (2001) 173–261.
[37] L. Fezoui, S. Lanteri, S. Lohrengel, S. Piperno, Convergence and stability of a discontinuous Galerkin time-domain method for the 3D heterogeneous Maxwell equations on unstructured meshes, ESAIM-Math Model Num 39 (2005) 1149–1176.
[38] A. Moiola, Trefftz-discontinuous Galerkin methods for time-harmonic wave problems, Ph.D. thesis, ETH Zuerich, 2011.
[39] A. Moiola, R. Hiptmair, I. Perugia, Plane wave approximation of homogeneous Helmholtz solutions, Z Angew Math Phys 62 (2011) 809–837.
[40] R. Hiptmair, A. Moiola, I. Perugia, Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the p-version, SIAM J Numer Anal 49 (2011) 264–284.
[41] F. Kretzschmar, S. Schnepp, I. Tsukerman, T. Weiland, Discontinuous Galerkin methods with Trefftz approximations, J Comput Appl Math 270 (2014) 211–222.