On endomorphism algebras of functors with non-compact domain

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Abstract

As a development of [2] and [3], we construct a “VN-core” in Vect$_k$ for each $k$-linear split-semigroupal functor from a suitable monoidal category $C$ to Vect$_k$. The main aim here is to avoid the customary compactness assumption on the set of generators of the domain category $C$ (cf. [3]).

1 Introduction

We propose the construction of a VN-core associated to each ($k$-linear) split semigroupal functor $U$ from a suitable monoidal category $C$ to Vect$_k$, where all our categories, functors, and natural transformations are assumed to be $k$-linear, for a fixed field $k$. Essentially, the category $C$ must be equipped with a small “$U$-generator” $A$ carrying some extra duality information and with $UA$ still being finite dimensional for all $A$ in $A$.

We shall use the term “VN-core” (in Vect$_k$) to mean a (usual) $k$-semibialgebra $E$ together with a $k$-linear endomorphism $S$ such that

$$\mu(\mu \otimes 1)(1 \otimes S \otimes 1)(1 \otimes \delta)\delta = 1 : E \to E.$$ 

The VN-core is called “antipodal” if $S(xy) = SxSy$ (and $S(1) = 1$) for all $x, y \in E$. This minimal type of structure is introduced here in order to avoid compactness assumptions on the generator $A \subset C$ and, at the same time, retain the “fusion” operator, namely

$$(\mu \otimes 1)(1 \otimes \delta) : E \otimes E \to E \otimes E,$$

satisfying the usual fusion equation [7]. Note that here the fusion operator always has a partial inverse (see [1]).

In §2 we establish sufficient conditions on a functor $U$ in order that

$$\text{End}^U = \int^A (UA)^* \otimes UA$$

be a VN-core in Vect$_k$ (following [2]). This core can be completed to a VN-core $\text{End}^U \oplus k$ with a unit element. In §3 we give several examples of suitable functors $U$ for the theory.
2 The construction of $\text{End}^U$

Let $\mathcal{C} = (\mathcal{C}, \otimes, I)$ be a monoidal category and let $U : \mathcal{C} \to \text{Vect}$ be a functor with both a semigroupal structure, denoted $r = r_{C,D} : UC \otimes UD \to U(C \otimes D)$, and a cosemigroupal structure, denoted $i = i_{C,D} : U(C \otimes D) \to UC \otimes UD$, such that $ri = 1$.

We shall suppose also that there exists a small full subcategory $\mathcal{A}$ of $\mathcal{C}$ with the properties:

1. $UA$ is finite dimensional for all $A \in \mathcal{A}$,
2. $U$-density; the canonical map $\alpha_C : \int^A \mathcal{C}(A, C) \otimes UA \to UC$ is an isomorphism for all $C \in \mathcal{C}$,
3. there is an “antipode” functor $(-)^* : \mathcal{A}^{\text{op}} \to \mathcal{A}$ with a ("canonical") map $e_A : A \otimes A^* \otimes A \to A$ in $\mathcal{C}$ for each $A \in \mathcal{A}$,
4. there is a natural isomorphism $u = u_A : U(A^*) \xrightarrow{\cong} U(A)^*$,
5. the following diagrams defining $\tilde{\tau}$, $\tilde{\rho}$ both commute

\[ \begin{array}{ccc}
UA \otimes U(A^*) \otimes UA & \xrightarrow{\tilde{\tau}} & U(A \otimes A^* \otimes A) \\
\downarrow 1 \otimes u^{-1} \otimes 1 & & \downarrow e_{UA} \\
UA \otimes U(A^*) \otimes UA & \xrightarrow{\tilde{\rho}} & U(A \otimes A^* \otimes A) \\
\end{array} \]

and

\[ \begin{array}{ccc}
UA \otimes U(A^*) \otimes UA & \xrightarrow{\tilde{\rho}} & U(A \otimes A^* \otimes A) \\
\downarrow 1 \otimes u \otimes 1 & & \downarrow e_{UA} \\
UA \otimes U(A^*) \otimes UA & \xrightarrow{\tilde{\tau}} & U(A \otimes A^* \otimes A) \\
\end{array} \]

where $e_{UA} = 1 \otimes \text{ev}$ in $\text{Vect}$, and $r_3i_3 = 1$. 
We now define the semibialgebra structure \((\text{End}^\vee U, \mu, \delta)\) on

\[
\text{End}^\vee U = \int^A U(A)^* \otimes U A
\]
as in \([2]\) §2, with the isomorphism of \(k\)-linear spaces

\[
S = \sigma : \text{End}^\vee U \to \text{End}^\vee U
\]
given (as in \([2]\) §3) by the usual components

\[
\begin{array}{c}
U(A)^* \otimes U A \\
\downarrow^1\otimes\sigma \\
U(A)^* \otimes U(A)^{**} \otimes U (A^*)^*
\end{array}
\begin{array}{c}
\rightarrow
\sigma_A \\
\downarrow \\
\rightarrow
\end{array}
\begin{array}{c}
U(A)^* \otimes U(A^*) \\
\downarrow \\
U(A^*) \otimes U(A^*)^*
\end{array}
\]

where \(d\) is the canonical map from a vector space to its double dual. Furthermore, each map

\[
e_{UA} = 1 \otimes \text{ev} : U A \otimes U A^* \otimes U A \to U A
\]
satisfies both the conditions

\[
\begin{array}{c}
U A \otimes U A^* \otimes U A
\end{array}
\begin{array}{c}
\uparrow^n \otimes 1 \\
\downarrow^e_{UA}
\end{array}
\begin{array}{c}
U A \\
\rightarrow 1
\end{array}
\begin{array}{c}
\rightarrow U A
\end{array}
\]

(E1)

commutes, and

\[
\begin{array}{c}
U A^* \otimes U A \otimes U A^*
\end{array}
\begin{array}{c}
\uparrow^1 \otimes n \\
\downarrow^1 \otimes d \otimes 1
\end{array}
\begin{array}{c}
U A^* \\
\rightarrow e_{UA}
\end{array}
\begin{array}{c}
\rightarrow U A^* \otimes U A^{**} \otimes U A^*
\end{array}
\]

(E2)

commutes, where \(n = \text{coev} : 1 \to U A \otimes U A^*\) in \(\text{Vect}\).

Then we obtain:

**Theorem 2.1.** The structure \((\text{End}^\vee U, \mu, \delta, S)\) is a VN-core in \(\text{Vect}_k\) which can be completed to the VN-core \((\text{End}^\vee U) \oplus k\).

**Proof.** The von Neumann axiom

\[
\mu_3(1 \otimes S \otimes 1) \delta_3 = 1
\]
becomes the diagram (in which we have omitted “⊗”):

$$
\begin{array}{c}
\begin{array}{c}
U(A)^* \otimes U(A)^* \otimes U(A) \otimes U(A)^* \otimes U(A)^* \\
\downarrow U(A)^* \otimes U(A)^* \otimes U(A) \otimes U(A)^* \otimes U(A)^* \\
U(A)^* \otimes U(A)^* \otimes U(A) \otimes U(A)^* \otimes U(A)^*
\end{array}
\end{array}
\end{array}
$$

where (*\) is the exterior of the diagram

$$
\begin{array}{c}
\begin{array}{c}
U(A)^* \otimes U(A)^* \otimes U(A) \otimes U(A)^* \otimes U(A)^* \\
\downarrow U(A)^* \otimes U(A)^* \otimes U(A) \otimes U(A)^* \otimes U(A)^* \\
U(A)^* \otimes U(A)^* \otimes U(A) \otimes U(A)^* \otimes U(A)^*
\end{array}
\end{array}
\end{array}
$$

which commutes using (E2) and commutativity of
3 Examples

3.1 Example

The first type of example is derived from the idea of a (contravariant) involution on a (small) comonoidal category \( \mathcal{D} \). This includes the doubles \( \mathcal{D} = B^{\text{op}} + B \) and \( \mathcal{D} = B^{\text{op}} \otimes B \) with their respective “switch” maps (where \( B \) is a given small comonoidal \( \text{Vect}_k \)-category), or any small comonoidal and compact-monoidal \( \text{Vect}_k \)-category \( \mathcal{D} \) (such as the category \( \text{Mat}_k \) of finite matrices over \( k \)) with the tensor duals of objects now providing an antipode on the comonoidal aspect of the structure rather than on the monoidal part, or any \(*\)-algebra structure on a given \( k \)-bialgebra (e.g., a \( C^* \)-bialgebra) with the \(*\)-operation providing the antipode.

In each case, an even functor from \( \mathcal{D} \) to \( \text{Vect} \) is defined to be a (\( k \)-linear) functor \( F \) equipped with a (chosen) dinatural isomorphism

\[
F(D^*) \cong F(D).
\]

If we take the morphisms of even functors to be all the natural transformations between them then we obtain a category

\[
\mathcal{E} = \mathcal{E}(\mathcal{D}, \text{Vect}).
\]

Let \( \mathcal{A} = \mathcal{E}(\mathcal{D}, \text{Vect}_{\text{fd}})_{\text{fs}} \) be the full subcategory of \( \mathcal{E} \) consisting of the finitely valued functors of finite support. While this category is generally not compact, it has on it a natural antipode derived from those on \( \mathcal{D} \) and \( \text{Vect}_{\text{fd}} \), namely

\[
A^*(D) := A(D^*)^*.
\]

Of course, there are also examples where \( \mathcal{A} \) is actually compact, such as those where \( \mathcal{D} \) is a Hopf algebroid, in the sense of [4], with antipode \((-)^* = S \), in which case each \( A \) from \( \mathcal{D} \) to \( \text{Vect} \) has a symmetry structure on it.

Now let \( \mathcal{C} \) be the full subcategory of \( \mathcal{E} \) consisting of the small coproducts in \( \mathcal{E} \) of objects from \( \mathcal{A} \). This category \( \mathcal{C} \) is easily seen to be monoidal under the pointwise convolution structure from \( \mathcal{D} \), and the inclusion \( \mathcal{A} \subset \mathcal{C} \) is \( U \)-dense for the functor

\[
U : \mathcal{C} \to \text{Vect}_k
\]

given by

\[
U(C) = \sum_{D} C(D)
\]

which is split semigrouplar with \( UA \) finite dimensional for all \( A \in \mathcal{A} \). Moreover,

\[
U(A^*) = \bigoplus_{D} A^*(D)
\]
\[= \bigoplus_{D} A(D)^*
\]
\[= U(A)^*,
\]

for all \( A \in \mathcal{A} \). The conditions of (5) are easily verified if we define maps

\[
e : A \otimes A^* \otimes A \to A
\]
by commutativity of the diagrams

\[
\begin{array}{ccc}
A(D) \otimes A^*(D) \otimes A(D) & \xrightarrow{\varepsilon_D} & A(D) \\
\downarrow \cong & & \downarrow 1 \otimes \text{ev} \\
A(D) \otimes A(D)^* \otimes A(D), & & \\
\end{array}
\]

where the exterior of

\[
\begin{array}{ccc}
A^*(D) \otimes A(D) & \xrightarrow{\cong} & A^*(D) \otimes A(D) \\
\downarrow A(f) \otimes 1 & & \downarrow \varepsilon \\
A^*(E) \otimes A(D) & \xrightarrow{\cong} & A^*(E) \otimes A(D) \\
\downarrow 1 \otimes A(f) & & \downarrow \varepsilon \\
A^*(E) \otimes A(E) & \xrightarrow{\cong} & A^*(E) \otimes A(E) \\
\end{array}
\]

commutes for all maps \( f : D \to E \) in \( \mathcal{D} \) so that

\[
e = 1 \otimes \hat{e} : A \otimes A^* \otimes A \to A \otimes k \cong A
\]

is a genuine map in \( \mathcal{C} \) when \( \mathcal{C} \) is given the pointwise monoidal structure from \( \mathcal{D} \). This completes the details of the general example.

### 3.2 Example

In the case where \( k = \mathbb{C} \) and \( \mathcal{D} \) has just one object \( D \) whose endomorphism algebra is a \( C^* \)-bialgebra, we have a one-object co-monoidal category \( \mathcal{D} \) with a \( \mathbb{C} \)-conjugate-linear antipode given by the \( * \)-operation. Then the convolution

\[
[D, \text{Hilb}_{\mathbb{C}}] \subset [D, \text{Vect}_{\mathbb{C}}],
\]

is a monoidal category, with a \( \mathbb{C} \)-linear antipode given by

\[
F^*(D) = F(D^*)^\circ
\]

where \( H^\circ \) denotes the conjugate-transpose of \( H \in \text{Hilb}_{\mathbb{C}} \). We now interpret an even functor \( F \) to be a functor equipped with a dinatural isomorphism \( F(D^*) \cong F(D) \) in \( D \in \mathcal{D} \) which is \( \mathbb{C} \)-linear, so that \( F^*(D) \cong F(D^*)^\circ \) for such a functor.

Take \( \mathcal{A} = \mathcal{E}(\mathcal{D}, \text{Hilb}_{\mathbb{C}}) \) and let \( \mathcal{C} \) be the class of small coproducts in \( [\mathcal{D}, \text{Vect}_{\mathbb{C}}] \) of the underlying \( [\mathcal{D}, \text{Vect}_{\mathbb{C}}] \)-representations of \( A \)'s in \( \mathcal{A} \) (with the appropriate maps). Each map

\[
e : A \otimes A^* \otimes A \to A
\]

in \( \mathcal{C} \) is defined by the \( \mathbb{C} \)-linear components

\[
e : A(D) \otimes A^*(D) \otimes A(D) \xrightarrow{1 \otimes \hat{e}} A(D),
\]
where
\[ \hat{c} : A^*(D) \otimes A(D) \to C \]
in \textbf{Vect}_C comes from the \( C \)-bilinear composite of two maps which are both \( C \)-linear in the first variable and \( C \)-linear in the second, namely
\[
\begin{array}{c}
A^*(D) \times A(D) \\
\overset{\cong}{\longrightarrow} \\
A(D)^\circ \times A(D).
\end{array}
\]

The remainder of this example is as seen before in Example 3.1.

### 3.3 Example

Let \( V = (\mathcal{V}, \otimes, I) \) be a (small) braided monoidal category and let \( B \) be the \( k \)-linearization of \textbf{Semicoalg}(\( V \)) with the monoidal structure induced from that on \( V \). By analogy with [5], let \( \mathcal{X} \subset B \) be a finite full subcategory of \( B \) with \( I \in \mathcal{X} \) and \( \mathcal{X}^{\text{op}} \) promonoidal when
\[
p(x, y, z) = B(z, x \otimes y) \\
j(z) = B(z, I)
\]
for \( x, y, z \in \mathcal{X} \).

For example (cf. [5]), one could take \( \mathcal{X} \) to be a (finite) set of non-isomorphic “basic” objects in some braided monoidal category \( V \), where each \( x \in \mathcal{X} \) has a coassociative diagonal map \( \delta : x \to x \otimes x \). However, we won’t need the category \( \mathcal{X} \) to be discrete or locally finite in the following.

Now let \( C \) be the convolution \( [\mathcal{X}^{\text{op}}, \textbf{Vect}] \) and let \( A = [\mathcal{X}^{\text{op}}, \textbf{Vect}_{fd}] \). The functor
\[
U : C \to \textbf{Vect}
\]
is defined by
\[
U(C) = \bigoplus_x C(x),
\]
and the obvious inclusion \( A \subset C \) is \( U \)-dense. If there is a canonical (natural) retraction
\[
p(x, y, z) = B(z, x \otimes y) \\
\overset{r_{x, y}}{\longrightarrow} B(z, x) \otimes B(z, y),
\]
derived from the semicoalgebra structures on \( x, y, z \), then \( U \) becomes a split semigroupal functor via the structure maps
\[
\begin{array}{c}
U(C) \otimes U(D) \\
\overset{r}{\longrightarrow} U(C \otimes D) \\
\bigoplus_x C(x) \otimes \bigoplus_y D(y) \\
\Delta \Delta^* \\
\bigoplus_z C(z) \otimes D(z) \\
\overset{\cong}{\longrightarrow} \\
\bigoplus_z \int^{x,y} p(x, y, z) \otimes C(x) \otimes D(y) \\
\Delta^* \Delta \\
\bigoplus_z \int^{x,y} B(z, x) \otimes B(z, y) \otimes C(x) \otimes D(y),
\end{array}
\]
where the isomorphism follows from the Yoneda lemma, and \( ri = 1 \).

If \( \mathcal{X} \) also has on it a duality
\[
(-)^* : \mathcal{X} \to \mathcal{X}^{\text{op}}
\]
such that \( x \cong x^{**} \), then, on defining
\[
A^*(x) = A(x^*)^*,
\]
we obtain
\[
U(A^*) = \bigoplus_x A^*(x) = \bigoplus_x A(x^*)^* \cong \bigoplus_x A(x)^* \quad \text{since } x \cong x^{**} \cong U(A)^*,
\]
for \( A \in \mathcal{A} \), in accordance with the fourth requirement on \( U \).

Finally, to obtain a suitable map
\[
e = 1 \otimes \hat{e} : A \otimes A^* \otimes A \to A \otimes I \cong A,
\]
where \( \hat{e} : A^* \otimes A \to I \), we suppose each \( A \) in \( \mathcal{A} \) has on it a “dual coupling”
\[
\chi = \chi_{xy} : A(x)^* \otimes A(y) \to B(x^* \otimes y, I).
\]
By considering the Yoneda expansion
\[
A(x) \cong \int^z A(z) \otimes \mathcal{X}(x, z)
\]
of the various functors \( A \) in \( \mathcal{A} = [\mathcal{X}^{\text{op}}, \text{Vect}_{fd}] \), such a coupling exists on each \( A \) if we suppose merely that \( \mathcal{X} \) itself is “coupled” by a natural transformation
\[
\chi : \mathcal{X}(y, z) \to \mathcal{X}(x, z) \otimes B(x^* \otimes y, I);
\]
or simply
\[
\chi : \mathcal{X}(x, z)^* \otimes \mathcal{X}(y, z) \to B(x^* \otimes y, I),
\]
if \( \mathcal{X} \) is locally finite. Then, the composite natural transformation
\[
\begin{array}{c}
A(x^*)^* \otimes A(y) \otimes B(z, x \otimes y) \\
\downarrow \chi \otimes 1
\end{array}
\]
\[
\begin{array}{c}
B(x^{**} \otimes y, I) \otimes B(z, x \otimes y) \\
\cong \\
B(x \otimes y, I) \otimes B(z, x \otimes y) \\
\downarrow \text{comp’n}
\end{array}
\]
\[
B(z, I)
\]
yields the map

\[ A^* \otimes A \xrightarrow{\delta} I \]

\[ \int^{xy} A^*(x) \otimes A(y) \otimes p(x, y, -) \longrightarrow B(-, I) \]

because \( p(x, y, -) = B(-, x \otimes y) \) (by definition). Thus suitable conditions on the coupling \( \chi \) give (5).

**Remark.** Actually, this last example in which the basic promonoidal structure occurs as a canonical retract of a comonoidal structure is typical of many other examples which can be treated along similar lines.

**References**

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