The Secrecy Capacity of Cost-Constrained Wiretap Channels

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Abstract

In many information-theoretic communication problems, adding an input cost constraint to the operational setup amounts to restricting the optimization domain in the capacity formula. This paper shows that, in contrast to common belief, such a simple modification does not hold for the cost-constrained (CC) wiretap channel (WTC). The secrecy-capacity of the discrete memoryless (DM) WTC without cost constraints is described by a single auxiliary random variable. For the CC DM-WTC, however, we show that two auxiliaries are necessary to achieve capacity. Specifically, we first derive the secrecy-capacity formula, proving the direct part via superposition coding. Then, we provide an example of a CC DM-WTC whose secrecy-capacity cannot be achieved using a single auxiliary. This establishes the fundamental role of superposition coding over CC WTCs.

Index Terms

Cost constraint, physical layer security, secrecy-capacity, superposition coding, wiretap channel.

I. INTRODUCTION

Physical-layer security (PLS), rooted in information-theoretic principles, dates back to Wyner’s landmark 1975 paper [1], where the wiretap channel (WTC) was introduced. This model formulates reliable and secure communication over noisy channels in the presence of an eavesdropper (see Fig. 1). By harnessing randomness from the noisy channel and combining it with proper physical layer coding, Wyner characterized the fundamental limit of reliable and secure communication, termed the secrecy-capacity. For a memoryless WTC $P_{Y,Z|X}$, the secrecy-capacity is [2]

$$C_{WTC}(P_{Y,Z|X}) = \max_{P_{V,X}} I(V;Y) - I(V;Z),$$

(1)

where the joint distribution is $P_{V,X}$ $P_{Y,Z|X}$ (i.e., $V - X - (Y, Z)$ forms a Markov chain) and $V$ is an auxiliary random variable. As PLS guarantees protection against computationally-unbounded adversaries without using shared
keys, Wyner’s formulation inspired a myriad of followup works. These included multiuser extensions [2]–[5], multi-
antenna settings [6]–[9], state-dependent WTCs [10]–[15], Gaussian WTCs [16]–[20], WTCs with feedback [21]–
[24], active adversaries [25]–[31], upgraded security metrics [32]–[39], practical wiretap codes [40]–[45], and more.

In this work, we revisit the classic WTC with an input cost constraint, and show that two-layered coding is necessary for achieving its secrecy-capacity. In many information-theoretic communication problems, adding an input cost constraint amounts to restricting the optimization domain in the capacity expression (e.g., the set of feasible $P_{V,X}$ in (1)). We show that this reasoning is not valid for cost-constrained (CC) WTCs. To do so, we characterize the CC secrecy-capacity using two auxiliary variables and prove that a single-auxiliary formula is strictly suboptimal. For the latter, an example of a CC WTC is provided for which a two-layered scheme strictly outperforms any single-layered code. This establishes the fundamental role of two-layered (superposition) coding for the CC WTC.

The necessity of two auxiliaries to achieve the secrecy-capacity of CC WTCs is surprising. This is evident from the fact that non-exact expressions for it have been used in recent works [36, Corollary 2], [46, Theorem 3.7]. The requirement of two auxiliaries is even more remarkable when one considers the recently established analogy between WTC and Gelfand-Pinsker (GP) channels [14] without cost constraints. Indeed, for the GP channel, the CC and the unconstrained capacities are given by the same expression up to adding the proper restriction to feasible input distributions [47] [48, Claim 1] [49, Fact2]. To the best of our knowledge, the WTC is the only point-to-point communication scenario for which the capacity formula itself changes due to the addition of an input cost constraint.

A. Background

The secrecy-capacity of a degraded WTC was established in [1], under the so-called weak-secrecy metric\textsuperscript{1}. The secrecy-capacity of the general WTC was proved in [2]. The main result of [2] in fact accounted for a generalization of the WTC to the broadcast channel (BC) with confidential messages. The secrecy-capacity of the latter is characterized using a pair of auxiliary random variables, and the direct proof uses superposition wiretap coding. It was also shown therein that reducing the BC result to the WTC (by nullifying the common message) and requiring perfect secrecy on the private message, one of the auxiliaries can be taken to be a constant\textsuperscript{2}. This gave rise to the secrecy-capacity formula given in (1).

Interestingly, this formula is quite robust to the security metric being used. In recent years, upgrading weak-secrecy to more stringent metrics gained much popularity. Strong-secrecy removes the normalization by the blocklength from the weak-secrecy metric, requiring that the information leakage itself vanishes [32], [33]. A further strengthening to the semantic-security metric was introduced in [38]. While weak- and strong-secrecy both assume a uniform distribution on the messages (i.e., security on average), semantic-security demands a vanishing information leakage for all probability distributions over the message set (i.e., worst-case). Despite these increasingly stringent security

\textsuperscript{1}Weak-secrecy refers a vanishing information leakage rate $\frac{1}{n} I(M; Z^n)$ as the blocklength $n \to \infty$, where $M$ is secret message and $Z^n$ is the eavesdropper’s observation.

\textsuperscript{2}Namely, $U$ in [2, Corollary 2] can be taken to be a constant $u^*$, where $u^* = \arg \max_{u \in U} I(V; Y | U = u) - I(V; Z | U = u)$. However, this argument cannot be applied in the CC WTC setting, as taking a constant $U$ may violate the cost constraint.
requirement, the secrecy-capacity with strong and semantic-security metrics remains unchanged \cite[Theorem 17.11]{38} compared to \cite{1}.

In practice, transmitted signals are often bound to cost (e.g., power) constraints. Therefore, various communication scenarios originally explored without such constraints were later adapted to the CC case. This includes point-to-point channels \cite[Chapter 7]{51}, the GP channel \cite{48} and the multiple-access channel \cite[Problem 4.8]{52}, to name a few. For all these aforementioned settings, the capacity under a CC is given by the same expression as in the unconstrained case, but with an added restriction on the optimization domain. As will be shown herein, such a simple adaptation of \cite{1} to the CC case is not valid for the WTC.

B. Contributions

We consider a discrete and memoryless (DM) WTC with an input cost constraint and establish a single-letter characterization of its secrecy-capacity. In contrast to \cite{1}, our characterization uses two auxiliary random variables. We consider all three aforementioned security metrics, that is, weak-secrecy, strong-secrecy and semantic-security, and show that the secrecy-capacity is the same for them all. This is done by proving achievability under semantic-security (strongest among the three), while establishing the converse with respect to weak-secrecy.

The achievability proof uses a superposition wiretap code that carries the entire confidential message in its outer layer. The inner layer encodes only random bits purposed to confuse the eavesdropper. The cost, reliability and security analyses rely on standard random coding arguments. However, due to the presence of a positive cost constraint, the expected value analysis (over the codebook ensemble) does not automatically imply the existence of a deterministic codes sequence with the desired performance. We resolve this issue via a novel two-step expurgation technique that first prunes ‘bad’ codebooks, and only then disposes of ‘bad’ messages. A careful analysis shows that the inflicted rate loss is negligible, giving rise to a deterministic codebook that satisfies the desired cost, reliability and security requirements.

We then turn to show that two-auxiliaries are necessary to achieve the CC WTC secrecy-capacity. This is done by constructing an example for which superposition coding attains a strictly higher secrecy rate than standard wiretap coding. The necessity of two auxiliaries can be understood by viewing the inner layer auxiliary as a “time-sharing” variable that leaks no information about the message to the eavesdropper. In a time-shared scheme, the cost constraint needs to be satisfied only on average (over the participating schemes). Thus, individual schemes could possibly violate the cost constraint, and indeed, it may be beneficial to consider such schemes for achieving higher secrecy rates. In particular, such a situation could occur if the mutual information term $I(V;Y) - I(V;Z)$ from the secrecy-capacity expression from \cite{1} is a convex function over the CC optimization domain.

C. Organization

The remainder of this paper is organized as follows. Section \ref{sec:prelim} provides preliminary definitions and sets up the operational problem. The main results are stated and discussed in Section \ref{sec:results} while their proofs are furnished in Section \ref{sec:proofs}. Finally, concluding remarks are given in Section \ref{sec:conclusions}.
II. Preliminaries and Problem Setup

A. Notation

We use the following notation. \( \mathbb{N}, \mathbb{R} \) and \( \mathbb{R}_{\geq 0} \) denote the set of natural numbers, real numbers and non-negative real numbers, respectively. For \( a, b \in \mathbb{R}_{\geq 0} \), \( [a : b] := \{ n \in \mathbb{N} : a \leq n \leq b \} \). Calligraphic letters, e.g., \( \mathcal{X} \), denote sets while \( |\mathcal{X}| \) stands for its cardinality. For \( n \in \mathbb{N} \), \( \mathcal{X}^n \) denotes the \( n \)-fold Cartesian product of \( \mathcal{X} \), and \( x^n = (x_1, \cdots, x_n) \) denotes an element of \( \mathcal{X}^n \). Whenever the dimension \( n \) is clear from the context, bold-face letters denote vectors or sequences, e.g., \( x \) for \( x^n \). For \( i, j \in \mathbb{N} \) such that \( i \leq j \), \( x_i^j := (x_i, x_{i+1}, \cdots, x_j) \); the subscript is omitted when \( i = 1 \).

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, where \( \Omega \), \( \mathcal{F} \) and \( \mathbb{P} \) are the sample space, \( \sigma \)-algebra and probability measure, respectively. Random variables over \( (\Omega, \mathcal{F}, \mathbb{P}) \) are denoted by uppercase letters, e.g., \( X \), with similar conventions as above for random vectors. We use \( \mathbb{1}_A \) for the indicator function of \( A \in \mathcal{F} \). The set of all probability mass functions (PMFs) on a finite set \( \mathcal{X} \) (always endowed with the power set \( \sigma \)-algebra) is denoted by \( \mathcal{P}(\mathcal{X}) \).

The joint PMF of two discrete random variables \( X \) and \( Y \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \) is denoted by \( P_{X,Y} \); the corresponding marginals are \( P_X \) and \( P_Y \). The conditional PMF of \( X \) given \( Y \) is represented by \( P_{X|Y} \). Expressions such as \( P_{X,Y} = P_X P_{Y|X} \) are to be understood as pointwise equality, i.e., \( P_{X,Y}(x,y) = P_X(x)P_{Y|X}(y|x) \), for all \( (x,y) \in \mathcal{X} \times \mathcal{Y} \). When the joint distribution of a triple \( (X,Y,Z) \) factors as \( P_{X,Y,Z} = P_{X,Y}P_{Z|Y} \), these variable form a Markov chain \( X \perp \perp Y \perp Z \). When \( X \) and \( Y \) are statistically independent, we write \( X \perp \perp Y \). If the entries of \( x^n \) are drawn in an independent and identically distributed (i.i.d.) manner, i.e., if \( P_{X^n}(x^n) = \prod_{i=1}^n P_X(x_i), \forall x^n \in \mathcal{X}^n \), then the PMF \( P_{X^n} \) is denoted by \( P^n_X \). Similarly, if \( P_{Y^n|X^n}(y^n|x^n) = \prod_{i=1}^n P_{Y|X}(y_i|x_i) \), then we write \( P^n_{Y|X} \) for \( P_{Y^n|X^n} \). The conditional product PMF given a fixed \( x^n \in \mathcal{X}^n \) is designated by \( P^n_{Y|X}(\cdot|x^n) \).

For a discrete measurable space \( (\mathcal{X}, \mathcal{F}) \), the probability measure induced by a PMF \( P \in \mathcal{P}(\mathcal{X}) \) is denoted by \( \mathbb{P}_P \); namely \( \mathbb{P}_P(A) = \sum_{x \in A} P(x) \), for all \( A \in \mathcal{F} \). The corresponding expectation is designated by \( \mathbb{E}_P \). Similarly, mutual information and entropy with an underlying PMF \( P \) are denoted as \( I_P \) and \( H_P \), respectively. When the PMF is clear from the context, the subscript is omitted. We use \( \mathcal{T}_\delta^{(n)}(P) \) to denote the set of letter-typical sequences of length \( n \) with respect to a PMF \( P \in \mathcal{P}(\mathcal{X}) \) and a non-negative \( \delta \):

\[
\mathcal{T}_\delta^{(n)}(P) := \{ x \in \mathcal{X}^n : |\nu_X(x) - P(x)| \leq \delta P(x), \forall x \in \mathcal{X} \},
\]  

where \( \nu_X(x) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{x_i = x\}} \) is the empirical PMF of sequence \( x \in \mathcal{X}^n \). Finally, for a countable sample space \( \mathcal{X} \) and PMFs \( P, Q \in \mathcal{P}(\mathcal{X}) \), the Kullback-Leibler (KL) divergence between \( P \) and \( Q \) is

\[
\text{D}_{\text{KL}}(P\|Q) := \sum_{x \in \mathcal{X}} P(x) \log \left( \frac{P(x)}{Q(x)} \right),
\]

and the total variation is

\[
\delta_{\text{TV}}(P, Q) := \frac{1}{2} \sum_{x \in \mathcal{X}} |P(x) - Q(x)|.
\]
B. Problem Setup

Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be finite sets, $b \geq 0$ and $n \in \mathbb{N}$. Let $C : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be a real-valued non-negative function. The $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, P_{Y,Z|X}, C, b)$ CC DM-WTC (henceforth referred to as CC WTC) is shown in Fig. 1, where $P_{Y,Z|X} : \mathcal{X} \to \mathcal{P}(\mathcal{Y} \times \mathcal{Z})$ is the channel transition kernel, $C$ is the cost function, and $b$ is the cost constraint. The encoder chooses a message $m \in \mathcal{M}_n := [1 : 2^n R ]$, $R \geq 0$, and maps it onto a channel input sequence $x \in \mathcal{X}^n$. The codeword $x$ is transmitted over the $n$-fold WTC $P_{Y,Z|X}^{\otimes n}$, which outputs sequences $y \in \mathcal{Y}^n$ and $z \in \mathcal{Z}^n$. The decoder observes $y$, based on which it produces an estimate $\hat{m} \in \mathcal{M}_n$ of $m$. The eavesdropper observes $z$, from which it tries to extract information about the transmitted message $m$.

Definition 1 (Code). An $(n, R)$-code $c_n$ for a CC WTC $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, P_{Y,Z|X}, C, b)$ with a message set $\mathcal{M}_n$ is a pair of functions $(f_n, g_n)$ such that

1) $f_n : \mathcal{M}_n \to \mathcal{P}(\mathcal{X}^n)$ is a stochastic encoder that satisfies the per-message cost constraint given by

$$\mathbb{E}[C_n(X(m))] := \sum_{x \in \mathcal{X}^n} f_n(x|m)C_n(x) \leq b, \quad \forall \ m \in \mathcal{M}_n,$$

where, $X(m) \sim f_n(\cdot|m)$, and $C_n(x) := \frac{1}{n} \sum_{i=1}^{n} C(x_i)$, $\forall \ x \in \mathcal{X}^n$, is the $n$-fold extension of $C$;

2) $g_n : \mathcal{Y}^n \to \mathcal{M}_n$ is the decoding function.

Remark 2 (Minimal cost). The per-message constraint in (5) can be satisfied only if $b \geq c_{\min} := \min \{ C(x) : x \in \mathcal{X} \}$. Henceforth, we will assume this condition holds without further mention.

A message PMF $P_M \in \mathcal{P}(\mathcal{M}_n)$ and a code $c_n = (f_n, g_n)$ induces a PMF on $\mathcal{M}_n \times \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n \times \mathcal{M}_n$ given by

$$P^{(c_n)}(m, x, y, z, \hat{m}) = P_M(m) f_n(x|m) P_{Y,Z|X}^{\otimes n}(y,z|x) \mathbb{I}_{\{\hat{m} = g_n(y)\}}$$

The performance of $c_n$ is evaluated in terms of the maximal decoding error probability and a chosen security metric.

Definition 3 (Error Probability). The maximal error probability of an $(n, R)$-code $c_n$ is

$$e(c_n) := \max_{m \in \mathcal{M}_n} e_m(c_n),$$

(7a)
where for any \( m \in \mathcal{M}_n \),
\[
e_m(c_n) := \mathbb{P}_{P(c_n)}(\hat{M} \neq m | M = m) = \sum_{x \in \mathcal{X}^n} f_n(x|m) \sum_{\substack{y \in \mathcal{Y}^n: \ g_n(y) \neq m}} P_{y|X}^{\otimes n}(y|x),
\]

(7b)

**Definition 4 (Security metrics).** For a given \((n, R)\)-code \( c_n \) and message PMF \( P_M \), the information leakage to the eavesdropper is \( \ell(P_M, c_n) := I_{P(c_n)}(M; Z) \). The weak-secrecy, strong-secrecy and semantic-security metrics with respect to (w.r.t.) the code \( c_n \) are, respectively, defined as
\[
\ell_{\text{weak}}(c_n) := \frac{1}{n} \ell(\tilde{P}_M, c_n) \tag{8}
\]
\[
\ell_{\text{str}}(c_n) := \ell(\tilde{P}_M, c_n) \tag{9}
\]
\[
\ell_{\text{sem}}(c_n) := \max_{P_M \in \mathcal{P}(\mathcal{M}_n)} \ell(P_M, c_n), \tag{10}
\]
where \( \tilde{P}_M \) denotes the uniform distribution on \( \mathcal{M}_n \).

**Definition 5 (Achievability).** A rate \( R \geq 0 \) is said to be achievable with semantic-security for a CC WTC \((\mathcal{X}, \mathcal{Y}, \mathcal{Z}, P_{Y,Z|X}, C, b)\) if for every \( \epsilon > 0 \) and sufficiently large \( n \), there exists an \((n, R)\)-code \( c_n \) such that (5) is satisfied, and
\[
\max \{ e(c_n), \ell_{\text{sem}}(c_n) \} \leq \epsilon. \tag{11}
\]
Achievability w.r.t. the weak- or strong-secrecy metrics is defined by replacing \( \ell_{\text{sem}} \) with \( \ell_{\text{weak}} \) or \( \ell_{\text{str}} \), respectively.

**Definition 6 (Secrecy Capacity).** The semantic-security capacity of a CC WTC \((\mathcal{X}, \mathcal{Y}, \mathcal{Z}, P_{Y,Z|X}, C, b)\), denoted by \( C_{\text{sem}}(P_{Y,Z|X}, C, b) \), is the supremum of the set of all rates achievable with semantic-security. The strong-secrecy capacity \( C_{\text{str}}(P_{Y,Z|X}, C, b) \) and weak-secrecy capacity \( C_{\text{weak}}(P_{Y,Z|X}, C, b) \) are defined similarly w.r.t. the corresponding notion of achievability.

### III. Main Results

We give a single-letter characterization of the weak-secrecy, strong-secrecy and semantic-security capacities of the CC WTC, all of which are shown to be equal. The characterization involves two auxiliary random variables. Both auxiliaries are necessary to achieve capacity in general. We first state the capacity result, and then provide an example for which any single-auxiliary scheme is suboptimal.

**A. Secrecy-capacity results**

Let \( \mathcal{U} \) and \( \mathcal{V} \) be finite sets, and for any \( P_{U,V,X} \in \mathcal{P}(\mathcal{U} \times \mathcal{V} \times \mathcal{X}) \), set
\[
\tilde{C}(P_{U,V,X}, P_{Y,Z|X}) := I_P(V; Y | U) - I_P(V; Z | U), \tag{12}
\]
where the mutual information terms are taken w.r.t. \( P = P_{U,V,X} P_{Y,Z|X} \). Also, let
\[
\mathcal{H}(C, b) := \{ P_{U,V,X} \in \mathcal{P}(\mathcal{U} \times \mathcal{V} \times \mathcal{X}) : P_{U,V,X} = P_{U,V} P_X | V, \ \mathbb{E}_P[C(X)] \leq b \}, \tag{13}
\]
and define

\[ \bar{C}(P_{Y,Z|X}, C, b) := \sup_{P_{U,V,X} \in \mathcal{H}(C,b)} \tilde{C}(P_{Y,Z|X}, P_{U,V,X}). \]  

Our main result is given next. For simplicity of presentation, we will suppress \( P_{Y,Z|X} \) and \( C \) from the notation \( C_{\text{sem}}(P_{Y,Z|X}, C, b), C_{\text{str}}(P_{Y,Z|X}, C, b), C_{\text{weak}}(P_{Y,Z|X}, C, b), \tilde{C}(P_{Y,Z|X}, P_{U,V,X}), \mathcal{H}(C,b) \) and \( \bar{C}(P_{Y,Z|X}, C, b) \), henceforth denoting them by \( C_{\text{sem}}(b), C_{\text{str}}(b), C_{\text{weak}}(b), \tilde{C}(P_{U,V,X}), \mathcal{H}(b) \) and \( \bar{C}(b) \), respectively.

**Theorem 7** (Secrecy-capacity). The secrecy-capacity of a CC WTC \((X, Y, Z, P_{Y,Z|X}, C, b)\) under weak-secrecy, strong-secrecy and semantic-security is the same, and is given by

\[ C_{\text{sem}}(b) = C_{\text{str}}(b) = C_{\text{weak}}(b) = \bar{C}(b). \]  

The proof of Theorem 7 is given in Section IV-A. The achievability of (15) relies on a superposition wiretap coding, while the converse adapts the classic WTC converse to accommodate the cost constraint. We note that the CC WTC’s secrecy-capacity expression involves two auxiliary random variables \( U \) and \( V \). In contrast, the secrecy-capacity formula of a WTC (without a cost constraint), given in (1), uses only a single auxiliary. In Section III-B we show that a reduction of \( \bar{C}(b) \) to a single auxiliary is impossible, in general.

The following lemma provides additional properties of \( \bar{C}(b) \). These properties are used in the proof of Theorem 7.

**Lemma 8** (Structural properties). In the definition of \( \bar{C}(b) \) in (14), it suffices to consider auxiliary alphabets \( U \) and \( V \) with \( |U| \leq |X| \) and \( |V| \leq |X|^2 \). Moreover, \( \bar{C}(b) \) is a non-decreasing and concave (for \( b \geq c_{\text{min}} \)) function of \( b \), and the supremum in (14) is achieved, i.e.,

\[ \bar{C}(b) = \max_{P_{U,V,X} \in \mathcal{H}(b)} \tilde{C}(P_{U,V,X}) := \max_{P_{U,V,X} \in \mathcal{H}(b)} I_P(V; Y|U) - I_P(V; Z|U). \]  

The proof of Lemma 8 is provided in Appendix A for completeness.

**B. Two Auxiliaries are Necessary**

Comparing (1) and (16), one might ask whether a reduction to a single auxiliary random variable in Theorem 7 is possible. We show that the answer is negative in general. To this end, we provide an example of a CC WTC \( P_{Y,Z|X} \), for which

\[ \max_{P_{U,V,X} \in \mathcal{H}(b)} I_P(V; Y|U) - I_P(V; Z|U) > \max_{P_{V,X}: \mathcal{E}(P(V|X)) \leq b} I_P(V; Y) - I_P(V; Z), \]  

where the mutual information terms on the right hand side (RHS) are w.r.t. \( P_{V,X} P_{Y,Z|X} \). To explain briefly, the example incorporates a WTC setup in which the transmitter (encoder) is connected to the receiver (decoder) via a noiseless private data link. The transmitter can choose the content as well as timing of the transmission, however, it is constrained to use the link at most half of the time. The receiver observes the data (error-free) when transmission occurs, and random noise otherwise. On the other hand, the eavesdropper has no access to the data link, but perfectly knows the timing of the transmission. We next describe the details of the WTC setup.

Consider the \((X, Y, Z, P_{Y,Z|X}, C, b)\) CC WTC shown in Fig. 2 that is defined as follows:
Let $\tilde{X} = \mathcal{Y} = \mathcal{Z} = \{0, 1\}$, and $\mathcal{X} = \tilde{X} \times \mathcal{Z}$.

- The channel input is $X = (\tilde{X}, \tilde{Z}) \sim P_{\tilde{X}, \tilde{Z}}$, where $\tilde{X}$ and $\tilde{Z}$ take values in $\tilde{X}$ and $\mathcal{Z}$, respectively, and both are controlled by the encoder.

- Consider the cost function $C(x) = C(\tilde{x}, \tilde{z}) = \tilde{z}$, for all $x = (\tilde{x}, \tilde{z}) \in \tilde{X} \times \mathcal{Z}$, and set the cost constraint to $b = 0.5$. Thus, the input must satisfy $\mathbb{E}[C(X)] = \mathbb{E}[\tilde{Z}] \leq \frac{1}{2}$.

- Let $N \sim \text{Ber}(0.5)$ be independent of $X = (\tilde{X}, \tilde{Z})$ and set $Y = \tilde{X} \tilde{Z} + N(1 - \tilde{Z})$.

Let $P_{Y|X} = P_{Y|\tilde{X}, \tilde{Z}}$ denote the transition kernel from $X$ to $Y$ induced by the above relation.

- The WTC $P_{Y,Z|X}$ is given by $P_{Y,Z|\tilde{X}, \tilde{Z}} = P_{Y|\tilde{X}, \tilde{Z}} \mathbb{1}(Z = \tilde{Z})$.

We have the following proposition whose proof is given in Section IV-B.

**Proposition 9.** For the $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, P_{Y,Z|X}, C, 0.5)$ CC WTC described above,

$$\bar{C}(0.5) := \max_{P_{U,V,X} \in \mathcal{H}(0.5)} I_P(V; Y|U) - I_P(V; Z|U) \geq 0.5 > \max_{P_{V,X} \in \mathcal{P}(0.5)} I_P(V; Y) - I_P(V; Z). \quad (18)$$

For proving Proposition 9, we choose a $P_{U,V,X} \in \mathcal{H}(0.5)$ such that $I_P(V; Y|U) - I_P(V; Z|U) = 0.5$, thus establishing $\bar{C}(0.5) \geq 0.5$. This is done by selecting $U = \tilde{Z} \sim \text{Ber}(0.5)$, $\tilde{X} \sim \text{Ber}(0.5) \perp \tilde{Z}$, and $V = X := (\tilde{X}, \tilde{Z})$. Intuitively, such a choice of auxiliaries correspond to a communication scheme of transmitting data half of the time (say, every alternate channel use). Subsequently, we show that the RHS of (18) is strictly below 0.5. This is shown by starting with the assumption that there exists a $P_{V,X}$ such that $\mathbb{E}_P[C(X)] \leq 0.5$ and $I_P(V; Y) - I_P(V; Z) \geq 0.5$, and then arguing that it leads to a contradiction.

Achieving the CC WTC secrecy-capacity in the above example requires two auxiliary random variables. While a reduction in the number of auxiliaries is impossible in general, we next show that no auxiliaries are needed when the WTC is less noisy. Namely, the condition is that $Y$ is less noisy than $Z$, i.e., $I_P(U; Y) \geq I_P(U; Z)$, for all
Corollary 10. If $Y$ is less noisy than $Z$, then

$$\bar{C}(b) = \max_{P_x: \mathbb{E}_P[C(X)] \leq b} I_P(X; Y) - I_P(X; Z).$$

(19)

If $Z$ is less noisy than $Y$, then $\bar{C}(b) = 0$.

The proof of Corollary 10 is provided in Section IV-C.

IV. PROOFS

A. Proof of Theorem 7

We will show that $C_{\text{weak}}(b) \leq \bar{C}(b)$ and $C_{\text{sem}}(b) \geq \bar{C}(b)$ for any $b \geq 0$. This combined with the fact that $C_{\text{sem}}(b) \leq C_{\text{str}}(b) \leq C_{\text{weak}}(b)$ will imply the desired result.

1) Converse: Recall that $\bar{P}_M$ denotes the uniform distribution on $\mathcal{M}_n$. It suffices to show that for any $\delta > 0$, a rate $R$ achievable under the weak-secrecy metric satisfies $R \leq \bar{C}(b) + \delta$, for large enough $n$. Fix $\epsilon > 0$ and let $c_n$ be an $(n, R)$-code with $\max \{e(c_n), \ell_{\text{weak}}(c_n)\} \leq \epsilon$. Any such code must also satisfy the weaker constraint $\max \{\mathbb{E}_{\bar{P}_M}[e_M(c_n)], \frac{1}{n}\ell(\bar{P}_M, c_n)\} \leq \epsilon$. Accordingly, we may assume without loss of generality that $P_M = \bar{P}_M$. From Fano’s inequality \cite{53}, it follows that

$$H(M|Y) \leq 1 + \epsilon n R.$$  

(20)

Now, we can write

$$nR$$

$\overset{a}{=} H(M)$

$\overset{b}{\leq} I(M; Y) + 1 + \epsilon n R$

$\overset{c}{\leq} I(M; Y) - I(M; Z) + 1 + \epsilon n R + n \epsilon$

$$= \sum_{i=1}^{n} I(M; Y_i|Y_i^{i-1}) - I(M; Z_i|Z_{i+1}^{n_i}) + 1 + \epsilon n(R+1)$$

$\overset{d}{=} \sum_{i=1}^{n} I(M; Y_i|Y_i^{i-1}) - I(M; Z_i|Z_{i+1}^{n_i}) + I(Z_{i+1}^{n_i}; Y_i|M, Y_i^{i-1}) - I(Y_i^{i-1}; Z_i|M, Z_{i+1}^{n_i}) + 1 + \epsilon n(R+1)$

$$= \sum_{i=1}^{n} I(M, Z_{i+1}^{n_i}; Y_i|Y_i^{i-1}) - I(M, Y_i^{i-1}; Z_i|Z_{i+1}^{n_i}) + 1 + \epsilon n(R+1)$$

$\overset{e}{=} \sum_{i=1}^{n} I(M, Y_i^{i-1}, Z_{i+1}^{n_i}; Y_i|Y_i^{i-1}, Z_{i+1}^{n_i}) - I(M, Y_i^{i-1}, Z_{i+1}^{n_i}; Z_i|Y_i^{i-1}, Z_{i+1}^{n_i}) + 1 + \epsilon n(R+1)$

$$= \sum_{i=1}^{n} I(M, Y_i^{i-1}, Z_{i+1}^{n_i}; Y_i|Y_i^{i-1}, Z_{i+1}^{n_i}) - I(M, Y_i^{i-1}, Z_{i+1}^{n_i}; Z_i|Y_i^{i-1}, Z_{i+1}^{n_i}) + 1 + \epsilon n(R+1)$$

$\overset{f}{=} \sum_{i=1}^{n} I(M, Y_i^{i-1}, Z_{i+1}^{n_i}; Y_i|Y_i^{i-1}, Z_{i+1}^{n_i}) - I(M, Y_i^{i-1}, Z_{i+1}^{n_i}; Z_i|Y_i^{i-1}, Z_{i+1}^{n_i}) + 1 + \epsilon n(R+1)$

$= \sum_{i=1}^{n} I(M, Y_i^{i-1}, Z_{i+1}^{n_i}; Y_i|Y_i^{i-1}, Z_{i+1}^{n_i}) - I(M, Y_i^{i-1}, Z_{i+1}^{n_i}; Z_i|Y_i^{i-1}, Z_{i+1}^{n_i}) + 1 + \epsilon n(R+1)$
\( (e) \sum_{i=1}^{n} I(V_i; Y_i | U_i) - I(V_i; Z_i | U_i) + 1 + \epsilon n(R + 1) \)

\( (f) \leq \sum_{i=1}^{n} \bar{C}\left(\mathbb{E}[C(X_i)]\right) + 1 + \epsilon n(R + 1) \)

\( (g) \leq nC \left( \sum_{i=1}^{n} \frac{1}{n} \mathbb{E}[C(X_i)] \right) + 1 + \epsilon n(R + 1) \)

\( (h) n \left( \bar{C}(b) + \epsilon (R + 1) + \frac{1}{n} \right) \)

where

(a) follows from (19); 
(b) is because \( l(P_n | c_n) \leq nc \); 
(c) and (d) use the Csiszár-sum identity [52]; 
(e) is due to the auxiliary random variable identification \( U_i = (Y^{i-1}, Z_{i+1}^{n}) \) and \( V_i = (M, Y^{i-1}, Z_{i+1}^{n}) \); 
(f) follows from the definition of \( \bar{C}(\cdot) \) in [14] since \( U_i - V_i - X_i - (Y_i, Z_i) \) form a Markov chain with the above auxiliary variable identification; 
(g) is due to the concavity of \( \bar{C}(\cdot) \) proved in Lemma 8; 
(h) is because \( \bar{C}(\cdot) \) is non-decreasing and \( X^n \) satisfies \( \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} C(X_i) \right] \leq b \) due to (5). 

Thus,

\[ R \leq \bar{C}(b) + \epsilon + \frac{1}{1 - \epsilon} . \]

The claim follows by taking \( n \) sufficiently large and \( \epsilon > 0 \) small enough.

2) Achievability: By the continuity of \( \bar{C}(b) \), it suffices to show that for any \( \epsilon > 0 \), there exists an \( (n, R) \) code \( c_n \) that satisfies (11), provided that \( R < \bar{C}(b) \) and \( n \) is sufficiently large. To this end, we construct an ensemble of superposition wiretap codes and show that the expected (over the ensemble) cost, error probability and semantic-security metric satisfy average versions of the constraints. Then, through a sequence of codebook and message expurgation steps, we show the existence of a code \( c_n \) that satisfies (5) and (11), as required.

Fix \( \epsilon > 0 \) and a joint PMF \( P_{U,V,X,Y,Z} := P_{U,V} P_{X|V} P_{Y,Z|X} \in \mathbb{P}(\mathcal{U} \times \mathcal{V} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}) \) such that \( \mathbb{E}_{P} [C(X)] < b \).

Codebook \( B_n \): We use a superposition codebook such that the inner layer adds redundancy to confuse the eavesdropper, while the outer layer carries the information about the message.

Define the index sets \( \mathcal{I}_n := [1:2^{nR_1}] \) and \( \mathcal{J}_n := [1:2^{nR_2}] \). Let \( \mathbb{B}_U^{(n)} := \{ U(i), i \in \mathcal{I}_n \} \) be a random inner layer codebook such that each codeword \( U(i), i \in \mathcal{I}_n \), is a sequence of length \( n \) generated independently according to \( P_U^{\otimes n} \). Denote a realization of \( \mathbb{B}_U^{(n)} \) by \( B_U^{(n)} := \{ u(i), i \in \mathcal{I}_n \} \).

For a fixed \( B_U^{(n)} \) and each \( i \in \mathcal{I}_n \), let \( \mathbb{B}_V^{(n)}(i) := \{ V(i, j, m), (j, m) \in \mathcal{J}_n \times \mathcal{M}_n \} \) denote a collection of \( n \)-length random sequences, each generated independently according to \( P_V^{\otimes n}(\cdot | u(i)) \). Denote a realization of \( \mathbb{B}_V^{(n)}(i) \) by \( B_V^{(n)}(i) := \{ v(i, j, m), (j, m) \in \mathcal{J}_n \times \mathcal{M}_n \} \). Also, set \( \mathbb{B}_V^{(n)} := \{ B_V^{(n)}(i), i \in \mathcal{I}_n \} \), and denote its possible outcome

\[ \text{This follows from the concavity of } C(b); \text{ see Lemma 8} ]
For any \( P \), we will also denote the probability measure induced by \( P \). Henceforth, we use \( B \) denoted by \( \mu \).

We random from the set \( X \in \mathcal{P} \) and \( \gamma \) by \( V \).

Cost Analysis: We analyze the expectation (w.r.t. the random codebook) of the cost averaged over messages. For any \( m \in \mathcal{M}_n \), note that \( \mathbb{E}_P \left[ P_{X|M}^{(B_n)}(x|m) \right] = P_{X|M}^{(B_n)}(x), \forall x \in \mathcal{X}_n \), which readily implies that

\[
\mathbb{E}_P \left[ \mathbb{E}_{P_{X|M}^{(B_n)}}(m) \right] = \mathbb{E}_{P^{(B_n)}}[C_n(X)] = \mathbb{E}_{P_X}[C(X)] < b.
\]

It follows that for some \( \gamma' > 0 \) and all \( n \in \mathbb{N} \),

\[
\mathbb{E}_P \left[ \mathbb{E}_{P_{X|M}^{(B_n)}}(m) \right] \leq b - \gamma'.
\]

Average error probability analysis: We analyze the expected error probability averaged over messages. For any \( B_n \in \mathcal{B}_n \) and \( i, j, m \in \mathcal{I}_n \times \mathcal{J}_n \times \mathcal{M}_n \), let \( Y \sim P_{Y|I,J,M}^{(B_n)}(i,j,m) \), and define the following error events:
Due to the symmetry of the random codebook $\mathbb{B}_n$, encoder $f_n$ and decoder $g_n$, the expected error probability over $\mathbb{B}_n$, i.e., $\mathbb{E}_\mu \left[ P_{\theta_n}(\hat{M} \neq M) \right]$, is the same for any realization of $(I, J, M)$. Thus, we may assume without loss of generality that $(I, J, M) = (1, 1, 1)$. We have

$$
\mathbb{E}_\mu \left[ P_{\theta_n}(\hat{M} \neq M) \right] = \mathbb{E}_\mu \left[ P_{\theta_n}(\hat{M} \neq M \mid (I, J, M) = (1, 1, 1)) \right] \\
\leq \mathbb{E}_\mu \left[ P_{\theta_n}(E_1(1, 1, 1) \cup E_2(1, 1, 1) \cup E_3(1, 1, 1) \mid (I, J, M) = (1, 1, 1)) \right]. 
$$

(24)

To upper bound the RHS of (24), we use the following lemma whose proof is given in Appendix B.

**Lemma 11.** If $(R, R_1, R_2) \in \mathbb{R}_{\geq 0}^3$ satisfy

$$
R_2 + R < I_P(V; Y | U) \\
R_1 + R_2 + R < I_P(U, V; Y),
$$

(25)

(26)

then there exists a $\zeta(\delta) > 0$ such that

$$
\mathbb{E}_\mu \left[ P_{\theta_n}(E_1(1, 1, 1) \cup E_2(1, 1, 1) \cup E_3(1, 1, 1) \mid (I, J, M) = (1, 1, 1)) \right] \leq e^{-n\zeta(\delta)}. 
$$

(27)

Thus, from (24) and (27), it follows that

$$
\mathbb{E}_\mu \left[ P_{\theta_n}(\hat{M} \neq M) \right] \leq e^{-n\zeta(\delta)} \longrightarrow 0,
$$

(28)

provided (25) and (26) holds.

**Security analysis:** First consider

$$
I_{\theta_n}(M; Z) \leq I_{\theta_n}(M; I, U, Z) \\
= D_{KL} \left( P_{I,M,U,Z}^{(B_n)} \| P_M^{(B_n)} P_{I,U,Z}^{(B_n)} \right) \\
= D_{KL} \left( P_{I,M,U,Z}^{(B_n)} \| P_M^{(B_n)} P_{I,U,Z}^{(B_n)} \right) \\
\overset{(a)}{=} D_{KL} \left( P_{I,Z}^{(B_n)} P_{I,Z}^{(B_n)} \| P_{I,U,Z}^{(B_n)} P_{I,U,Z}^{(B_n)} \right) - D_{KL} \left( P_{I,Z}^{(B_n)} P_{I,Z}^{(B_n)} \| P_{I,U,Z}^{(B_n)} P_{I,U,Z}^{(B_n)} \right) \\
\overset{(b)}{=} D_{KL} \left( P_{I,Z}^{(B_n)} P_{I,Z}^{(B_n)} \| P_{I,U,Z}^{(B_n)} P_{I,U,Z}^{(B_n)} \right) \\
\overset{(c)}{\leq} \max_{m \in M_n} D_{KL} \left( P_{I,U,Z}^{(B_n)} \| P_{I,U,Z}^{(B_n)} \right), 
$$

(29a)

(29b)

where,
(a) and (c) is since \( M \perp (I, U) \);

(b) uses the non-negativity of KL divergence.

Maximizing w.r.t. \( P_M \in \mathcal{P}(\mathcal{M}_n) \) on both sides of (29b), we obtain that

\[
\max_{P_M \in \mathcal{P}(\mathcal{M}_n)} I_{P(B_n)}(M; Z) \leq \max_{m \in \mathcal{M}_n} D_{\text{KL}} \left( P_{I, U}^{(B_n)} I_{Z|M, I, U}^{(m)} \parallel P_{I, U}^{(m)} \right),
\]

(30)

Note that \( P_{I, U}^{(B_n)} I_{Z|M, I, U}^{(m)} \ll P_{I, U}^{(m)} \), where \( \ll \) denotes absolute continuity of measures. For PMFs \( P, Q \) with a finite support such that \( P \ll Q \), \( D_{\text{KL}}(P \parallel Q) \) can be upper bounded by the total variation distance \( \delta_{TV}(P, Q) \) [55, Equation 30]. Defining

\[
\theta(m, B_n) := \delta_{TV} \left( P_{I, U}^{(B_n)} I_{Z|M, I, U}^{(m)} \parallel P_{I, U}^{(m)} \right),
\]

and applying [13, Lemma 9], we obtain

\[
\max_{m \in \mathcal{M}_n} D_{\text{KL}} \left( P_{I, U}^{(B_n)} I_{Z|M, I, U}^{(m)} \parallel P_{I, U}^{(m)} \right) \leq \max_{m \in \mathcal{M}_n} \theta(m, B_n) \left( n \log |Z| - \log \theta(m, B_n) + n \log P_{Z|U}^{(\text{min})} \right),
\]

(31)

where \( P_{Z|U}^{(\text{min})} := \min \{ P_{Z|U}(z|u), (z, u) \in Z \times U : P_{Z|U}(z|u) > 0 \} \).

Thus, showing that there exist \( B_n \in \mathcal{B}_n \) and \( \gamma_1 > 0 \) such that \( \max_{m \in \mathcal{M}_n} \theta(m, B_n) \leq e^{-n\gamma_1} \) for large enough \( n \), is sufficient (by (30) and (31)) to get

\[
\max_{P_M \in \mathcal{P}(\mathcal{M}_n)} I_{P(B_n)}(M; Z) \rightarrow_n 0.
\]

The existence of such a \( B_n \) is implied by the following lemma. The lemma restates the outcome of the secrecy analysis from [13], providing a double-exponential bound on the probability of an exponentially small deviation of \( \max_{m \in \mathcal{M}_n} \theta(m, B_n) \) from zero.

**Lemma 12** (Lemma 4 from [13]). If

\[
R_2 > I_P(V; Z|U),
\]

(32)

then there exists \( \gamma_1, \gamma_2 > 0 \) such that for all sufficiently large \( n \),

\[
\mathbb{P}_\mu \left( \max_{m \in \mathcal{M}_n} \theta(m, B_n) > e^{-n\gamma_1} \right) \leq e^{-e^{n\gamma_2}}.
\]

(33)

Lemma 12 follows from the proof of Lemma 4 in [13], which is a stronger version of the superposition soft-covering lemma [55]. The double-exponential bound in (33) is an implication of Chernoff bound applied to the collection of an exponential number of i.i.d. codewords in the random superposition codebook.
Summary of random coding argument: Combining (23), (25) and (33), we have shown that

\[
\mathbb{E}_\mu \left[ E_{P_{\mathcal{X}^{(\mathcal{B}_n)}}} \left[ C_n(X) \right] \right] \leq b' := b - \gamma', \tag{34a}
\]

\[
\mathbb{E}_\mu \left[ P_{\mathcal{B}_n}(M \neq M) \right] \leq e^{-n\epsilon(\delta)}, \tag{34b}
\]

\[
\mathbb{E}_\mu \left[ 1_{\{\max_{m \in \mathcal{M}_n} \theta(m, \mathcal{B}_n) > e^{-n\gamma_2}\}} \right] \leq e^{-e^{n\gamma_2}}, \tag{34c}
\]

provided (25), (26) and (32) holds.

We next perform a sequence of expurgation steps: first, w.r.t. codebooks and then w.r.t. messages. At the end of this process, we deduce the existence of a single codebook \( \mathcal{B}_n \) that satisfies (5) and (11). Note that the selection lemma of [56] or [13] is not applicable here as the RHS of (34a) is a constant which does not vanish to zero as required by the lemma.

**Expurgation:** For any \( \mathcal{B}'_n \subseteq \mathcal{B}_n \), let

\[
\bar{\mu}(\mathcal{B}'_n) := \sum_{B_n \in \mathcal{B}'_n} \mu(B_n) \tag{35}
\]

be the probability measure on \( \mathcal{B}_n \) induced by \( \mu \). Our expurgation technique on the codebooks \( B_n \in \mathcal{B}_n \) is performed for each fixed \( n \) (sufficiently large) as described in the following steps:

1) Codebook expurgation to satisfy average (over messages) cost:

Expurgate codebooks \( B_n \in \mathcal{B}_n \) with the highest cost \( E_{P_{\mathcal{X}^{(\mathcal{B}_n)}}} \left[ C_n(X) \right] \) to obtain a set \( \mathcal{B}'_n \subseteq \mathcal{B}_n \) such that

\[
\frac{1}{n+2} \leq \bar{\mu}(\mathcal{B}'_n) < \frac{1}{n+1}. \tag{36}
\]

This is possible for large \( n \) since each codebook \( B_n \) has exponentially small probability.

We now show that all the codebooks \( B_n \in \mathcal{B}'_n \) satisfy \( E_{P_{\mathcal{X}^{(\mathcal{B}_n)}}} \left[ C_n(X) \right] \leq \frac{n+1}{n} b' \). Assume otherwise that there exists \( B'_n \in \mathcal{B}'_n \) such that \( E_{P_{\mathcal{X}^{(\mathcal{B}'_n)}}} \left[ C_n(X) \right] > \frac{n+1}{n} b' \). Then, we can write

\[
\mathbb{E}_\mu \left[ E_{P_{\mathcal{X}^{(\mathcal{B}_n)}}} \left[ C_n(X) \right] \right] = \sum_{B_n \in \mathcal{B}_n} \mu(B_n) E_{P_{\mathcal{X}^{(\mathcal{B}_n)}}} \left[ C_n(X) \right]
\]

\[
\geq \sum_{B_n \in \mathcal{B}_n \setminus \mathcal{B}'_n} \mu(B_n) E_{P_{\mathcal{X}^{(\mathcal{B}_n)}}} \left[ C_n(X) \right]
\]

\[
\geq \sum_{B_n \in \mathcal{B}_n \setminus \mathcal{B}'_n} \mu(B_n) E_{P_{\mathcal{X}^{(\mathcal{B}'_n)}}} \left[ C_n(X) \right]
\]

\[
\geq \sum_{B_n \in \mathcal{B}_n \setminus \mathcal{B}'_n} \mu(B_n) \frac{n+1}{n} b'
\]

\[
\geq \frac{n}{n+1} \frac{n+1}{n} b' = b',
\]

where

(a) is because by the expurgation procedure, \( E_{P_{\mathcal{X}^{(\mathcal{B}_n)}}} \left[ C_n(X) \right] \geq E_{P_{\mathcal{X}^{(\mathcal{B}'_n)}}} \left[ C_n(X) \right] \) for every \( B_n \in \mathcal{B}_n \setminus \mathcal{B}'_n \);

(b) is since \( \bar{\mu}(\mathcal{B}_n \setminus \mathcal{B}'_n) \geq \frac{n}{n+1} \).

Eqn. (36) contradicts (34a), and hence \( E_{P_{\mathcal{X}^{(\mathcal{B}_n)}}} \left[ C_n(X) \right] \leq \frac{n+1}{n} b' \) should hold for all \( B_n \in \mathcal{B}'_n \).
Define a PMF \( \mu_1 \in \mathcal{P}(\mathfrak{B}_n) \) and its induced probability measure \( \bar{\mu}_1 \) on \( \mathfrak{B}_n \) by

\[
\mu_1(B_n) := \begin{cases} 
\frac{\mu(B_n)}{\mu(\mathfrak{B}_n)}, & \text{if } B_n \in \mathfrak{B}_n', \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\bar{\mu}_1(\bar{\mathfrak{B}}_n) := \sum_{B_n \in \bar{\mathfrak{B}}_n} \mu_1(B_n), \quad \bar{\mathfrak{B}}_n \subseteq \mathfrak{B}_n,
\]

respectively. Then, we have

\[
\mathbb{E}_{P_X(n)}[C_n(X)] \leq \frac{n + \frac{1}{n} b'}{n}, \quad \forall B_n \in \mathfrak{B}_n',
\]

\[
\mathbb{E}_{\mu_1}[\mathbb{P}_{P(n)}(\hat{M} \neq M)] \leq \frac{1}{\bar{\mu}(\mathfrak{B}_n')} \mathbb{E}_{\mu}[\mathbb{P}_{P(n)}(\hat{M} \neq M)] \leq \frac{e^{-n\zeta(\delta)}}{\bar{\mu}(\mathfrak{B}_n')} \leq (n + 2)e^{-n\zeta(\delta)},
\]

\[
\mathbb{E}_{\mu_1} \left[ \mathbb{I}_{\{\max_{m \in \mathcal{M}_n} \theta(m, B_n) > e^{-n\gamma_1}\}} \right] \leq \frac{1}{\bar{\mu}(\mathfrak{B}_n')} \mathbb{E}_{\mu} \left[ \mathbb{I}_{\{\max_{m \in \mathcal{M}_n} \theta(m, B_n) > e^{-n\gamma_1}\}} \right] \leq e^{-n\gamma_2},
\]

2) Codebook expurgation to satisfy average cost, average error probability and semantic-security:

Expurgate codebooks \( B_n \in \mathfrak{B}_n' \) with the highest average error probability to obtain a set \( \mathfrak{B}_n'' \subseteq \mathfrak{B}_n' \) such that \( \frac{1}{2} \leq \bar{\mu}_1(\mathfrak{B}_n'') < \frac{1}{2} \). Then, it follows similarly to step 1 that

\[
\mathbb{P}_{P(n)}(\hat{M} \neq M) \leq 2(n + 2)e^{-n\zeta(\delta)}, \quad \forall B_n \in \mathfrak{B}_n''.
\]

Define another PMF \( \mu_2 \in \mathcal{P}(\mathfrak{B}_n) \) by

\[
\mu_2(B_n) := \begin{cases} 
\frac{\mu_1(B_n)}{\bar{\mu}_1(\mathfrak{B}_n')}, & \text{if } B_n \in \mathfrak{B}_n'', \\
0, & \text{otherwise}.
\end{cases}
\]

Then, we have

\[
\mathbb{E}_{P_X(n)}[C_n(X)] \leq \frac{n + \frac{1}{n} b'}{n}, \quad \forall B_n \in \mathfrak{B}_n'',
\]

\[
\mathbb{P}_{\mu_2}(\hat{M} \neq M) \leq (n + 2)e^{-n\zeta(\delta)}, \quad \forall B_n \in \mathfrak{B}_n'',
\]

\[
\mathbb{E}_{\mu_2} \left[ \mathbb{I}_{\{\max_{m \in \mathcal{M}_n} \theta(m, B_n) > e^{-n\gamma_1}\}} \right] \leq 3(n + 2)e^{-n\gamma_2}.
\]

Perform one more expurgation step similar to the previous step to obtain a non-empty set of codebooks \( \mathfrak{B}_n''' \) such that for each codebook \( B_n \in \mathfrak{B}_n''' \) and sufficiently large \( n \),

\[
\mathbb{E}_{P_X(n)}[C_n(X)] \leq \frac{n + \frac{1}{n} b'}{n}, \quad \forall B_n \in \mathfrak{B}_n'''
\]

\[
\mathbb{P}_{P(n)}(\hat{M} \neq M) \leq 2(n + 2)e^{-n\zeta(\delta)}, \quad \forall B_n \in \mathfrak{B}_n'''
\]

\[
\max_{m \in \mathcal{M}_n} \theta(m, B_n) \leq e^{-n\gamma_1}, \quad \forall B_n \in \mathfrak{B}_n'''
\]

3) Message expurgation to satisfy per-message cost, maximal error probability and semantic-security:
Now, fixing a codebook $B_n \in \mathcal{B}_n''$, perform expurgation on the set of messages $\mathcal{M}_n$ to obtain upper bounds on the per-message cost and maximal error probability, in place of the average (over messages) cost and average (over messages) error probability given in (37) and (38), respectively. Let $\alpha \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right)$. Similar to step 1, by expurgating a $(1 - \alpha)$ fraction of the messages $m \in \mathcal{M}_n$ with the highest cost $E_{\hat{P}(B_n)}[C_n(X)]$ to obtain a set $\mathcal{M}_n' \subset \mathcal{M}_n$, and defining for all $(m, i, j, u, v, x, y, z, \hat{m}) \in \mathcal{M}_n \times \mathcal{I}_n \times \mathcal{J}_n \times \mathcal{U}_n \times \mathcal{V}_n \times \mathcal{X}_n \times \mathcal{Y}_n \times \mathcal{Z}_n \times \hat{M}_n$, a PMF $\tilde{P}(B_n)$ given by

$$
\tilde{P}(B_n)_{M, I, J, U, V, X, Y, Z, \hat{M}}(m, i, j, u, v, x, y, z, \hat{m}) = \begin{cases} 
\frac{|M_n|}{|M_n'|}P(B_n)_{M, I, J, U, V, X, Y, Z, \hat{M}}(m, i, j, u, v, x, y, z, \hat{m}), & \text{if } m \in \mathcal{M}_n', \\
0, & \text{otherwise},
\end{cases}
$$

it follows that

$$
E_{\tilde{P}(B_n)}[C_n(X)] \leq \frac{(n + 1)^2}{n^2} b', \quad \forall \ m \in \mathcal{M}_n',
$$

$$
P_{\tilde{P}(B_n)}(\hat{M} \neq M) \leq 2(n + 2)^2 e^{-n \zeta(\delta)},
$$

$$
\max_{m \in \mathcal{M}_n'} \theta(m, B_n) \leq e^{-n \zeta(\delta)}.
$$

Finally, for $\beta \in \left[\frac{1}{3}, \frac{1}{2}\right)$, expurgating a $1 - \beta$ fraction of the messages $m \in \mathcal{M}_n'$ with the highest error probability $P_{\tilde{P}(B_n)}(\hat{M} \neq M | M = m)$ and denoting the resulting set by $\mathcal{M}_n''$ ($\mathcal{M}_n'' \subset \mathcal{M}_n'$), it follows similar to step 3 that

$$
E_{\tilde{P}(B_n)}[C_n(X)] \leq \frac{(n + 1)^2}{n^2} b', \quad \forall \ m \in \mathcal{M}_n'',
$$

$$
P_{\tilde{P}(B_n)}(\hat{M} \neq M) \leq 4(n + 2)^2 e^{-n \zeta(\delta)}, \quad \forall \ m \in \mathcal{M}_n'',
$$

$$
\max_{m \in \mathcal{M}_n''} \theta(m, B_n) \leq e^{-n \zeta(\delta)}.
$$

**Summary of expurgation steps:**

Note that $|\mathcal{M}_n''| = \beta \alpha |\mathcal{M}_n| \geq e^{n R}/3(n+2)$. Thus, for sufficiently large $n$, we have shown the existence of a codebook $B_n$ and a $(n, R - \frac{1}{n} \log(3n + 6))$ code $c_n(B_n) = (f_n, g_n)$ with message set $\mathcal{M}_n''$, such that

$$
E[C_n(X(m))] \leq \frac{(n + 1)^2}{n^2} b' = b - \gamma' + \frac{2n + 1}{n^2} (b - \gamma') < b, \quad \forall \ m \in \mathcal{M}_n'',
$$

and

$$
\max \left\{ \max_{m \in \mathcal{M}_n''} e_m(c_n(B_n)), \max_{P_M \in \mathcal{P}(\mathcal{M}_n'')} \ell(P_M, c_n(B_n)) \right\} \leq \epsilon,
$$

provided (25), (26) and (32) holds.

Eliminating $R_1$ and $R_2$ from (25), (26) and (32) via the Fourier-Motzkin elimination yields $R < I_P(V; Y | Z) - I_P(V; Z | U)$. The proof is completed by noting that $C_{\text{sem}}(b)$ is a closed set by definition, and $I_P(V; Y | Z) - I_P(V; Z | U)$ is a continuous function of $P$. 

Remark 13. Theorem 7 states the invariance of CC WTC secrecy-capacity to the employed secrecy metric. The converse proof further shows that capacity remains unchanged if the maximal error probability and/or per message cost constraints are relaxed to average (over messages) constraints of the form \( \mathbb{E}_{\bar{P}_M}[e_M(c_n)] \leq \epsilon \) and \( \mathbb{E}_{\bar{P}_M}[C_n(X(M))] \leq b \), respectively.

B. Proof of Proposition 9

Since the WTC transition kernel is \( P_{Y,Z|\tilde{X},\tilde{Z}} = P_{Y|\tilde{X},\tilde{Z}} \mathbb{1}_{\{Z = \tilde{Z}\}} \), \( Z = \tilde{Z} \) with probability one. We henceforth identify \( Z \) and \( \tilde{Z} \). We start by showing that

\[
\max_{P_{U,V,X} \in \mathcal{H}(0.5)} I_P(V; Y|U) - I_P(V; Z|U) \geq 0.5. \tag{42}
\]

Set \( U = \tilde{Z} \sim \text{Ber}(0.5), \tilde{X} \sim \text{Ber}(0.5), \tilde{X} \perp \tilde{Z}, \) and \( V = X = (\tilde{X}, \tilde{Z}) \). This choice satisfies \( P_{U,V,X} \in \mathcal{H}(0.5) \), since \( U - V - X = (\tilde{X}, \tilde{Z}) - (Y, \tilde{Z}) \) and \( \mathbb{E}_P[C(X)] := \mathbb{E}_P[\tilde{Z}] = 0.5 \). Moreover,

\[
I_P(V; Y|U) - I_P(V; \tilde{Z}|U) = I_P(X; Y|\tilde{Z}) - I_P(X; \tilde{Z}|\tilde{Z})
= I_P(X; Y|\tilde{Z})
= 0.5,
\tag{43}
\]

where (43) is because \( I_P(X; Y|\tilde{Z}) = P_{\tilde{Z}}(1)H(\tilde{X}) = 0.5 \). Hence, (42) holds.

Next, we establish that

\[
\max_{P_{V,X}: \mathbb{E}_P[C(X)] \leq 0.5} I_P(V; Y) - I_P(V; \tilde{Z}) < 0.5. \tag{44}
\]

For any \( P_{V,X} = P_{V;\tilde{X},\tilde{Z}} \) such that \( \mathbb{E}_P[\tilde{Z}] \leq 0.5 \), we have the following chain of inequalities:

\[
I(V; Y) - I(V; \tilde{Z}) = I(\tilde{X}, \tilde{Z}, V; Y) - I(\tilde{X}, \tilde{Z}; Y|V) - I(V; \tilde{Z})
\leq (a) I(\tilde{X}, \tilde{Z}; Y) - I(\tilde{X}, \tilde{Z}; Y|V) - I(V; \tilde{Z})
= I(\tilde{X}; Y|\tilde{Z}) - I(\tilde{X}; Y|\tilde{Z}, V) - I(V; \tilde{Z}|Y)
\leq (b) P_{\tilde{Z}}(0)I(\tilde{X}; N|\tilde{Z} = 0) + P_{\tilde{Z}}(1)I(\tilde{Y}; Y|\tilde{Z} = 1) - I(\tilde{X}; Y|\tilde{Z}, V) - I(V; \tilde{Z}|Y)
\leq (c) P_{\tilde{Z}}(1)H(Y|\tilde{Z} = 1) - I(\tilde{X}; Y|\tilde{Z}, V) - I(V; \tilde{Z}|Y)
\leq 0.5 H(Y|\tilde{Z} = 1) - I(\tilde{X}; Y|\tilde{Z}, V) - I(V; \tilde{Z}|Y)
\leq 0.5, \tag{45}
\]

where,

(a) is due to the Markov chain \( V - (\tilde{X}, \tilde{Z}) - (Y, \tilde{Z}) \);
(b) follows from the definition of \( Y \);
(c) is implied by the definition of \( \mathbb{E}_P[C(X)] \).

\footnote{We omit the subscript \( P \) in the subsequent mutual information and entropy terms as the PMF is \( P = P_{V,X} P_{Y|\tilde{X},\tilde{Z}} \) throughout.}
(c) is because $N \perp (\tilde{X}, \tilde{Z})$;
(d) uses the cost constraint $E[\tilde{Z}] \leq 0.5$;
(e) is by the non-negativity of mutual information and since $Y$ is binary.

Thus,

$$\max_{P_{V,X}: E[\tilde{C}(X)] \leq 0.5} I(V; Y) - I(V; \tilde{Z}) \leq 0.5. \quad (47)$$

Consequently, (44) is violated only if there exists some $X = (\tilde{X}, \tilde{Z})$ and a joint PMF $P_{V,X,Y;\tilde{Z}} = P_{V,X}P_{Y;\tilde{Z}|X}$ such that $E[\tilde{Z}] \leq 0.5$, and the inequalities in (45) and (46) hold with equality, i.e.,

$$I(V; Y) - I(V; \tilde{Z}) = 0.5. \quad (48)$$

For this to be possible, the following conditions must hold:

1) $P_{\tilde{Z}}(1) = 0.5$;
2) $H(Y|\tilde{Z} = 1) = 1$ which means that given $\tilde{Z} = 1$, $Y \sim \text{Ber}(0.5)$;
3) $I(\tilde{X}; Y|\tilde{Z}, V) = 0$ which implies that $\tilde{X} - (\tilde{Z}, V) - Y$ forms a Markov chain;
4) $I(V; \tilde{Z}|Y) = 0$ which implies that $V - Y - \tilde{Z}$ forms a Markov chain.

Now, notice that $P_{Y|\tilde{Z}=0} = P_{Y|\tilde{Z}=1} = \text{Ber}(0.5)$. Hence, $Y \perp \tilde{Z}$, which further implies via Condition (4) that $(V, Y) \perp \tilde{Z}$. Finally,

$$I(V; Y|\tilde{Z} = 1) \geq I(\tilde{X}; Y|\tilde{Z} = 1) = H(Y|\tilde{Z} = 1) = 1. \quad (49)$$

The inequality in (49) is due to Condition (3) above, while the last equality is due to Condition (2). Since $Y$ is binary, $I(V; Y|\tilde{Z} = 1) \leq 1$, and therefore the inequality in (49) is an equality.

To conclude, observe that $(V, Y) \perp \tilde{Z}$ (shown above) implies that

$$I(V; Y) - I(V; \tilde{Z}) = I(V; Y|\tilde{Z} = 1) - 0 = 1. \quad (50)$$

However, $I(V; Y) - I(V; \tilde{Z}) \leq 0.5$ from (46). This leads to a contradiction, and so (48) is invalid. Via (47), this implies that (44) holds. Combining (44) with (42) proves Proposition 9.

C. Proof of Corollary 10

Fix $P_{U;V,X,Y,Z} = P_{U,V}P_{X|V}P_{Y,Z|X}$, where $Y$ is less noisy than $Z$. We have

$$I_P(V; Y|U) - I_P(V; Z|U)$$

$\overset{(a)}{=} I_P(V; Y) - I_P(V; Z) - (I_P(U; Y) - I_P(U; Z))$

$\overset{(b)}{=} I_P(X; Y) - I_P(X; Z) - (I_P(X; Y|V) - I_P(X; Z|V)) - (I_P(U; Y) - I_P(U; Z))$

$\overset{(c)}{\leq} I_P(X; Y) - I_P(X; Z),$
where, (a) and (b) are due to the Markov chain $U - V - X - (Y, Z)$ that holds under PMF $P_{U,V,X,Y,Z}$; and (c) is due to the less noisy assumption which implies that $I_p(U; Y) - I_p(U; Z) \geq 0$ and $I_p(X; Y|V) - I_p(X; Z|V) \geq 0$ (due to $V - X - (Y, Z)$ under $P_{U,V,X,Y,Z}$).

Thus, it follows that

$$
\bar{C}(b) \leq \max_{P_X \in \mathcal{P}(\mathcal{X}) | \mathcal{C}(X)| \leq b} I_p(X; Y) - I_p(X; Z). \quad (51)
$$

The reverse inequality follows trivially by selecting $V = X$ and $U = \emptyset$ in (16), thus proving (19).

If $Z$ is less noisy than $Y$, then $I_p(V; Y|U) - I_p(V; Z|U) \leq 0$ for any $P_{U,V,X,Y,Z} = P_{U,V}P_X|V,P_Y|Z|X$ which gives $\bar{C}(b) = 0$, due to its non-negativity.

V. CONCLUDING REMARKS

This paper revisited the classical WTC setting with a cost constraint, and showed that achieving its secrecy-capacity generally requires two-layer coding. To do so, we proved optimality of superposition wiretap coding under cost constraint, and provided a WTC example for which single-layer coding is strictly suboptimal. This stands in contrast to the classic WTC secrecy-capacity result without a cost constraint, that is characterized using a single auxiliary random variable. In many other communication scenarios, imposing a cost constraint on the input amounts to a simple adaptation of the unconstrained case capacity; namely, restricting the optimization to those input distributions that satisfy the constraint in expectation. Our result provides an example where this commonly observed behavior does not hold, and a second auxiliary must be introduced as a result of the added cost constraint. Our main goal was to highlight this important fact and put forth the correct CC WTC secrecy-capacity characterization, for which non-exact expressions exist in the literature.

APPENDIX A

PROOF OF LEMMA 8

The proof of the cardinality bounds $|\mathcal{U}| \leq |\mathcal{X}|$ and $|\mathcal{V}| \leq |\mathcal{X}|^2$ follows via a standard application of the Eggleston-Fenchel-Carathéodory Theorem [57] Theorem 18], and is omitted.

Given that $|\mathcal{U}|$ and $|\mathcal{V}|$ are finite, the set $\mathcal{H}(b)$ is non-empty (whenever $b \geq c_{\min}$) and compact. Since $\bar{C}(b)$ is the supremum of a continuous function $I(V; Y|U) - I(V; Z|U)$ of $P_{U,V,X}$ over $\mathcal{H}(b)$, the supremum is achieved and thus a maximum exists. The fact that $\bar{C}(b)$ is monotonic and non-decreasing in $b$ follows by its definition.

Finally, to show the concavity of $\bar{C}(b)$ for $b \geq c_{\min}$, consider the following. For $i = 0, 1$, let $b_i \geq c_{\min}$ and $P_{U_i,V_i,X_i,Y_i,Z_i} \in \mathcal{P}(\mathcal{U} \times \mathcal{V} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{Z})$ be a PMF for which

$$
P_{U_i,V_i,X_i} \in \mathcal{H}(b_i), \quad (52)
$$

$$
P_{U_i,V_i,X_i,Y_i,z} := P_{U_i,V_i,X_i}P_{Y_i,z|x_i} := P_{U_i,V_i,X_i}P_{Y_i,Z|X_i},
$$

$$
\bar{C}(b_i) := I_p(V_i; Y_i|U_i) - I_p(V_i; Z_i|U_i). \quad (53)
$$
Also, let $\tau \in [0,1]$, $Q \sim \text{Ber}(\tau)$ with $Q = \{0,1\}$, $U_\tau := (U_Q, Q)$, $V_\tau := (V_Q, Q)$ and $X_\tau := X_Q$, and $P_{U_\tau, V_\tau, X_\tau, Y_\tau, Z_\tau} \in \mathcal{P}(U_\tau \times V_\tau \times X \times Y \times Z)$ be a PMF defined by

$$P_{U_\tau, V_\tau, X_\tau, Y_\tau, Z_\tau} := P_{U_\tau, V_\tau} P_{X_\tau|V_\tau} P_{Y_\tau, Z_\tau|X_\tau} := P_{U_\tau, V_\tau} P_{X_\tau|V_\tau} P_{Y, Z|X}. \quad (54)$$

Note that $U_\tau - V_\tau - X_\tau - (Y_\tau, Z_\tau)$ holds under $P_{U_\tau, V_\tau, X_\tau, Y_\tau, Z_\tau}$, and $P_{U_\tau, V_\tau, X_\tau} \in \mathcal{H}(\tau b_0 + (1 - \tau) b_1)$ since

$$E_P(\mathcal{C}(X_\tau)) \leq \tau b_0 + (1 - \tau) b_1. \quad (55)$$

Then, we have

$$(1 - \tau) \bar{C}(b_0) + \tau \bar{C}(b_1) = (1 - \tau) (I_P(V_0; Y_0|U_0) - I_P(V_0; Z_0|U_0)) + \tau (I_P(V_1; Y_1|U_1) - I_P(V_1; Z_1|U_1))$$

$$= I_P(V_Q, Q; Y_Q|U_Q, Q) - I_P(V_Q, Q; Z_Q|U_Q, Q)$$

$$= I_P(V_\tau; Y_\tau|U_\tau) - I_P(V_\tau; Z_\tau|U_\tau)$$

$$\leq \bar{C}(\tau b_0 + (1 - \tau) b_1),$$

which establishes the concavity of $\bar{C}(\cdot)$.

**APPENDIX B**

**PROOF OF LEMMA 11**

The proof is standard, however, we provide it for completeness. Let $P^{(\mu)}(\cdot) := E_\mu \left[ P^{(\mathbb{B}_n)}(\cdot) \right]$ and $P_{P^{(\mu)}}$ denote the random coding PMF and its induced probability measure, respectively. Also, define

$$\zeta_1^{(n)}(\delta) := \inf_{\nu_{U,V,Y}: (U,V,Y) \notin T_{\delta}^{(n)}(P_{U,V,Y})} D_{\text{KL}}(\nu_{U,V,Y}||P_{U,V,Y}) - \frac{|U||V||Y| \log(n + 1)}{n},$$

where $\nu_{U,V,Y}$ denotes the empirical PMF of $(U,V,Y) \in U^n \times V^n \times Y^n$. Note that for $\delta > 0$ and $n$ sufficiently large, $\zeta_1^{(n)}(\delta) > 0$.

First, consider the probability of the error event $E_1(1,1,1)$ averaged over the random codebook $\mathbb{B}_n$. We have

$$E_\mu \left[ P_{P^{(\mu)}}(E_1(1,1,1)| (I,J,M) = (1,1,1)) \right] = E_{P^{(\mu)}} \left[ (U(1), V(1,1,1), Y) \in T_{\delta}^{(n)}(P_{U,V,Y}) \right]$$

$$\leq e^{-n \zeta_1^{(n)}(\delta)} \rightarrow 0, \quad (56)$$

where the inequality in (56) follows from Lemmas 2.2 and 2.6 in [50].

Next, we analyze the probability of the error event $E_2(1,1,1)$ averaged over $\mathbb{B}_n$. Note that for $(j,m) \neq (1,1)$ and sufficiently large $n$,

$$P_{P^{(\mu)}} \left( (U(1), V(1,j,m), Y) \in T_{\delta}^{(n)}(P_{U,V,Y}) | (I,J,M) = (1,1,1) \right)$$

$$= \sum_{(U,V,Y) \notin T_{\delta}^{(n)}(P_{U,V,Y})} P^{(\mu)}(\mathbb{B}_n)(u,v,y) P^{(\mu)}(y|u)$$
Hence,
\[
\mathbb{E}_\mu \left[ \mathbb{P}_{P^{(\delta_n)}}(\mathcal{E}_2(1, 1, 1) | (I, J, M) = (1, 1, 1)) \right] \\
\leq \sum_{(u,v,y) \in T^{(n)}_\delta(P_{U,V,Y})} \mathbb{P}_{P^{(\delta_n)}} \left( (U(1), V(1, j, m), Y) \in T^{(n)}_\delta(P_{U,V,Y}) | (I, J, M) = (1, 1, 1) \right) \\
\leq e^{-n(H_P(U,V)+H_P(Y|U)−O(\delta))} \\
= e^{-n(I_P(V,Y|U)−O(\delta))}.
\]

Thus, it follows that for \(\delta \) sufficiently small and \(n\) large enough, there exists \(\zeta_2(\delta) > 0\) such that
\[
\mathbb{E}_\mu \left[ \mathbb{P}_{P^{(\delta_n)}}(\mathcal{E}_2(1, 1, 1) | (I, J, M) = (1, 1, 1)) \right] \leq e^{-n\zeta_2(\delta)} \xrightarrow{n \to \infty} 0, \quad (57)
\]
provided (25) holds.

Finally, consider the third error event \(\mathcal{E}_3(1, 1, 1)\). We have for \(i \neq 1\) and sufficiently large \(n\) that
\[
\mathbb{P}_{P^{(\delta_n)}} \left( (U(i), V(i, j, m), Y) \in T^{(n)}_\delta(P_{U,V,Y}) | (I, J, M) = (1, 1, 1) \right) \\
= \sum_{(u,v,y) \in T^{(n)}_\delta(P_{U,V,Y})} P_{\delta_n^{(n)}}^{U,V}(u,v)P_{\delta_n^{(n)}}^{Y}(y) \\
\leq \sum_{(u,v,y) \in T^{(n)}_\delta(P_{U,V,Y})} e^{-n(H_P(U,V)+H_P(Y)-O(\delta))} \\
\leq e^{-n(H_P(U,V)+H_P(Y)-H_P(U,V,Y)−O(\delta))} \\
= e^{-n(I_P(U,V,Y)−O(\delta))}.
\]

Hence,
\[
\mathbb{E}_\mu \left[ \mathbb{P}_{P^{(\delta_n)}}(\mathcal{E}_3(1, 1, 1) | (I, J, M) = (1, 1, 1)) \right] \\
\leq \sum_{(i,j,m) \in \mathcal{I}_n \times \mathcal{J}_n \times \mathcal{M}_n: \ i \neq 1} \mathbb{P}_{P^{(\delta_n)}} \left( (U(i), V(i, j, m), Y) \in T^{(n)}_\delta(P_{U,V,Y}) | (I, J, M) = (1, 1, 1) \right) \\
\leq e^{n(R_1+R_2+R-I_P(U,V,Y)+O(\delta))}.
\]

Thus, it follows that for \(\delta \) sufficiently small and \(n\) large enough, there exists \(\zeta_3(\delta) > 0\) such that
\[
\mathbb{E}_\mu \left[ \mathbb{P}_{P^{(\delta_n)}}(\mathcal{E}_3(1, 1, 1) | (I, J, M) = (1, 1, 1)) \right] \leq e^{-n\zeta_3(\delta)} \xrightarrow{n \to \infty} 0, \quad (58)
\]
provided (26) holds. The claim in the lemma follows from (56), (57) and (58) via the union bound on probability applied to the left hand side of (27).
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