From Here to Criticality: Renormalization Group Flow Between Two Conformal Field Theories

W. A. Leaf-Herrmann†

Jefferson Physical Laboratory
Harvard University
Cambridge, MA 02138

Using nonperturbative techniques, we study the renormalization group trajectory between two conformal field theories. Specifically, we investigate a perturbation of the $A_3$ superconformal minimal model such that in the infrared limit the theory flows to the $A_2$ model. The correlation functions in the topological sector of the theory are computed numerically along the trajectory, and these results are compared to the expected asymptotic behavior. Excellent agreement is found, and the characteristic features of the infrared theory, including the central charge and the normalized operator product expansion coefficients are obtained. We also review and discuss some aspects of the geometrical description of $N = 2$ supersymmetric quantum field theories recently uncovered by S. Cecotti and C. Vafa.
1. Introduction

The set of conformal field theories are the infrared or ultraviolet fixed points, or critical points, of renormalization group flow trajectories in the space of two-dimensional quantum field theories. At these fixed points, conformal invariance provides a set of constraints, realized by the infinite-dimensional Virasoro algebra, which often allow for the exact solution of the theory, in principle via the BPZ bootstrap \[1\], or in practice via explicit representations of the Virasoro algebra \[2\]. However, these methods generically allow us to solve the theory only at the critical point.

To better understand the structure of the space of two-dimensional field theories, and the special role played by the conformal field theories, we would like to be able to compute the correlation functions both on and off of the critical point. Typically, the best that can be done is to use conformal perturbation theory in the neighborhood of a fixed point, as demonstrated by Zamolodchikov \[3\] and others \[4–8\]. For example, one would like to be able to calculate the scaling behavior of the quantum fields in a conformal theory perturbed by some relevant field, under the action of renormalization group flow. The infrared limit of this theory should correspond to some (possibly trivial) conformal field theory. While this question can be addressed perturbatively in some cases, for instance the minimal models near \(c = 1\) \[3\], we generally require some nonperturbative techniques to answer the above question.

In the case of two-dimensional field theories with \(N = 2\) supersymmetry some of the requisite techniques have recently been developed \[9–13\], which allow for the nonperturbative calculation of a class of correlation functions both on and off the critical point. This class of correlation functions is known as the topological sector of \(N = 2\) field theories, and is closely related to the correlation functions of physical observables in topological quantum field theories \[14\]. The topological sector is composed of the expectation values of chiral fields evaluated between the set of supersymmetric ground states.

The equivalence between two-dimensional \(\sigma\)-models on Calabi-Yau spaces and certain \(N = 2\) superconformal models, first observed by Gepner \[15\], is well-known, and \(N = 2\) Landau- Ginsburg effective Lagrangians provide explicit realizations of this correspondence \[16–19\]. This equivalence has led to the application of geometrical methods in the characterization of \(N = 2\) superconformal field theories \[10,20,21\]. Using the quasi-topological nature of \(N = 2\) supersymmetry, S. Cecotti and C. Vafa have recently uncovered...
the generalization of these geometrical aspects to arbitrary $N = 2$ quantum field theories \cite{12,13}. In this paper we shall apply their results to the nonperturbative calculation of the topological sector of a theory which interpolates between two conformal field theories along the renormalization group trajectory connecting them.

In Section 2 we define and discuss the properties of the topological sector of $N = 2$ supersymmetric quantum field theories, and the relation between this sector and topological quantum field theories based on twisted $N = 2$ models. The basic geometrical framework needed to solve for the topological sector correlation functions is reviewed in Section 3. Section 4 describes how, by using the $N = 2$ non-renormalization theorem, these correlation functions may be calculated nonperturbatively along a renormalization group trajectory. We also discuss two quantities introduced in \cite{13} which serve to characterize the theory both at the conformal point, and off of criticality, known as the Ramond charge matrix and the algebraic $Q$-matrix. Both are easily computed in the topological sector. As a concrete application of this framework, we analyze the renormalization group flow between two conformal field theories, the $A_3$ $N = 2$ minimal model perturbed in such a way that the theory flows to the $A_2$ model in the infrared limit. The computation is discussed in Section 5, and we compare the results of the nonperturbative numerical calculation of the correlation functions of the interpolating theory with the expected asymptotic behavior in Section 6. Our conclusions are presented in Section 7.

2. Topological Sector of $N = 2$ Quantum Field Theories

We consider two-dimensional quantum field theories with an $N = 2$ supersymmetry. We assume the topology of the two-dimensional space to be a cylinder, or equivalently, a sphere with two punctures. The supercharges $Q^+, Q^-$, and their Hermitian conjugates $\bar{Q}^-, \bar{Q}^+$, obey the algebra:

$$\{Q^+, Q^-\} = -\partial, \quad \{\bar{Q}^+, \bar{Q}^-\} = -\bar{\partial}, \quad (2.1)$$

with all other (anti-)commutation relations vanishing. We impose periodic boundary conditions on the fermionic operators, so that the Witten index, $Tr (-1)^F$, where $F$ is the operator which counts fermion number, is well-defined. Hence we restrict to the Ramond sector. We shall restrict our study to theories in which supersymmetry is not spontaneously broken, so we shall assume that there are $\Delta$ supersymmetric ground states $|i\rangle$ satisfying

$$Q^\pm |i\rangle = \bar{Q}^\pm |i\rangle = H |i\rangle = 0, \quad 0 \leq i \leq \Delta - 1, \quad (2.2)$$
where $H$ is the Hamiltonian of the theory. In the special case of $N = 2$ Landau-Ginsburg theory $Tr (-1)^F = \Delta$.

The set of chiral superfields are those superfields $X_i$ which satisfy

$$[Q^+, X_i] = [\bar{Q}^+, X_i] = 0. \quad (2.3)$$

It follows as an immediate consequence of (2.1) and (2.3) that

$$-\partial X_i = \{Q^+, [Q^-, X_i]\} \quad (2.4a)$$

$$-\bar{\partial} X_i = \{Q^+, [\bar{Q}^-, X_i]\}. \quad (2.4b)$$

Together these relations have an important consequence for the set of Green functions involving chiral fields and the supersymmetric ground states. They are independent of the positions of the chiral fields:

$$-\frac{\partial}{\partial z_a} \langle \bar{j} | \cdots X_k (z_a) \cdots | i \rangle = \langle \bar{j} | \cdots \{Q^+, [Q^-, X_k (z_a)]\} \cdots | i \rangle = 0, \quad (2.5)$$

since $Q^+$ commutes with all the chiral fields and annihilates the ground states. This set of Green functions, composed of matrix elements involving chiral fields evaluated between supersymmetric ground states is known as the topological sector of $N = 2$ supersymmetric theories.

Since the above Green functions are independent of the positions of the chiral fields, it is apparent that the product of two chiral fields contains no short distance singularities. By defining the point-wise product of chiral fields as

$$X_i X_j (z_a) = \lim_{z_b \to z_a} X_i (z_a) X_j (z_b), \quad (2.6)$$

we see that the set of chiral fields forms a commutative ring $\mathcal{R}$, since the product of any two chiral fields is also chiral. In the context of the critical theory, this chiral ring essentially characterizes the conformal field theory [22]. We can extend the notion of a product of chiral fields to chiral fields which are located at different points using an equivalence relation. If we choose a basis for the set of chiral fields, say $\{\varphi_i\}$, then we have

$$\varphi_i \varphi_j \sim C_{ij}^k \varphi_k, \quad (2.7)$$
where the equivalence is modulo $Q^+$ and $\bar{Q}^+$ commutator terms, since by (2.4) above the difference between a chiral field evaluated at two different points is terms of this form.

Another important fact about $N = 2$ quantum field theories is that the chiral ring $\mathcal{R}$ is isomorphic to the vector space of supersymmetric ground states as $\mathcal{R}$-modules. This isomorphism is the spectral flow \cite{23}. Hence we may identify the supersymmetric ground states by the operation of a chiral field on the unique ground state, $|0\rangle$, which is the image under spectral flow of the identity operator in the chiral ring:

$$|i\rangle \sim \varphi_i |0\rangle. \quad (2.8)$$

Using the relation (2.7) for the chiral fields, we then have the following equivalence relation for chiral fields acting on the ground states:

$$\varphi_i |j\rangle \sim C_{ij}^k |k\rangle, \quad (2.9)$$

where this equivalence is modulo $Q^+$ and $\bar{Q}^+$ acting on some state.

Using the above relations, we can now reduce the calculation of any Green function in the topological sector to essentially matrix multiplication. For example, if we wish to compute the correlation function

$$\langle \bar{j} | \varphi_m \varphi_n | i \rangle,$$

then using (2.9), and the fact that any states which are $Q^+$- (or $\bar{Q}^+$-)exact annihilate $\langle \bar{j} |$, we find that this computation reduces to the product

$$\langle \bar{j} | \varphi_m \varphi_n | i \rangle = g_{ij} C_{mk} C_{ni}^k, \quad (2.10)$$

where we define $g_{ij}$ to be the Hermitian inner product for the supersymmetric ground states:

$$g_{ij} = \langle \bar{j} | i \rangle. \quad (2.11)$$

Hence the calculation of the Green functions in the topological sector can be reduced to the problem of calculating the $C_{ij}^j$ and the inner product $g_{ij}$ of the $N = 2$ quantum field theory.

The above topological sector is closely related to the set of correlation functions of physical observables in topological quantum field theory. For every $N = 2$ quantum field
theory, we can define a topological quantum field theory by twisting the energy-momentum tensor using the conserved $R$-current $J_\mu$ corresponding to fermion number:

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + \frac{1}{4} (\partial_\mu J_\nu + \partial_\nu J_\mu).$$

This is equivalent to redefining the coupling of the theory to two-dimensional gravity using the fermion numbers of the fields. We then define a nilpotent BRST operator:

$$Q_{BRST} = Q^+ \bar{Q}^+. \quad (2.13)$$

The physical states of the topological theory are then defined to be the BRST cohomology classes of this operator. The fields which commute with the BRST operator are precisely the chiral fields discussed above.

However, there is an important difference between topological quantum field theories and the topological sector of an $N = 2$ quantum field theory. In topological theories, the natural inner product is defined by:

$$\eta_{ij} = \langle j| i \rangle = \langle 0 | \varphi_j \varphi_i | 0 \rangle. \quad (2.14)$$

This symmetric inner product is truly a topological object, in the sense that it is independent of the representative of the BRST cohomology class chosen to define a state. The Hermitian inner product $g_{ij}$ does not share this property. This is because in $N = 2$ quantum field theory we define the adjoint of a state by

$$\langle | i \rangle^\dagger = \langle i | \sim \langle 0 | \bar{\varphi}_i,$$

where now the equivalence relation is modulo states which are $Q^-$- and $\bar{Q}^-$-exact. Since states created by applying chiral fields, $\varphi_i$, to the state $| 0 \rangle$ are not annihilated by $Q^-$ or $\bar{Q}^-$, the inner product $g_{ij}$ is defined using precisely the supersymmetric ground states. We could have defined a topological theory using the adjoint of the above BRST operator (2.13), and this would have produced a topological theory with the set of physical observables being represented by the anti-chiral fields, those superfields annihilated by $Q^-$ and $\bar{Q}^-$, instead. S. Cecotti and C. Vafa have described this construction as an anti-topological theory, thus leading to the notion of the topological sector of $N = 2$ field theory as being the fusion of a topological theory and an anti-topological theory.
Since the bases $|i\rangle$ and $|\bar{j}\rangle$ correspond to two different set of labels for the set of supersymmetric ground states, we can derive a useful relation between $g_{ij}$ and $\eta_{ij}$. We define $M$ to be the complex matrix which relates these two bases:

$$\langle \bar{i} | = \langle j | M_{i}^{\bar{j}}. \quad (2.16)$$

$M$ is known as the real structure. Using this definition of $M$, we find the following relationship between $g_{ij}$ and $\eta_{ij}$:

$$g_{ij} = \eta_{ik} M_{j}^{\bar{k}}. \quad (2.17)$$

The CPT-invariance of the quantum field theory implies that $M$ satisfies

$$MM^* = 1, \quad (2.18)$$

and this leads immediately to the following condition:

$$\eta^{-1} g (\eta^{-1} g)^* = 1. \quad (2.19)$$

This relation shall be referred to as the reality constraint.

3. $N = 2$ Landau-Ginsburg Models

A particularly interesting class of $N = 2$ supersymmetric quantum field theories are those which can be represented by Landau-Ginsburg models. These models are described by a Lagrangian density defined in terms of the chiral (and anti-chiral) superfields of the theory by

$$L = \frac{1}{2} \int d^4 \theta \; K (X_i, \bar{X}_i) + \left( \int d^2 \theta \; W (X_i) + h.c. \right), \quad (3.1)$$

where the first term, which is integrated over all superspace, is known as a $D$-term and includes the kinetic terms of the action, and the second term, known as an $F$-term, is a holomorphic function of the chiral superfields. This function, $W (X_i)$, is the superpotential of the theory. In the case of Landau-Ginsburg models we shall find that the correlation functions in the topological sector can be computed nonperturbatively, and depend only on the superpotential. The computation of the operator product expansion coefficients $C_{ij}^{\bar{k}}$ and the topological inner product $\eta_{ij}$ is quite easy in these models, and only the calculation of $g_{ij}$ is nontrivial.
The Green functions in the topological sector are independent of the kinetic term, $K \left( X_i, \bar{X}_i \right)$, essentially because due to the integral over all superspace, this term is $Q^+$-exact. Hence the effect of any variation of the kinetic term on the correlation functions in the topological sector will vanish:

$$\delta \langle \tilde{j} | \phi_k | i \rangle = - \int d^2 z d^4 \theta \langle \tilde{j} | \phi_k \delta K \left( X_l, \bar{X}_l \right) (z, \bar{z}) | i \rangle$$

$$= - \int d^2 z \langle \tilde{j} | \phi_k \left\{ Q^+, \Psi \left( X_l, \bar{X}_l \right) \right\} (z, \bar{z}) | i \rangle$$

$$= 0,$$

(3.2)

where we have used the fact that $Q^+$ commutes with the chiral field $\phi_k$ and annihilates the supersymmetric ground states. This formal argument can be made rigorous [10], but it establishes the result that the topological sector is essentially determined by the superpotential $W \left( X_i \right)$ alone, and does not depend on the details of the kinetic term.

The superpotential also simply encodes the behavior of the ring of chiral fields. Using the Lagrangian (3.1), we derive the following equation of motion for the chiral fields:

$$\partial_i W \left( X_j \right) = - \left\{ Q^+, \left[ Q^+, \partial_i K \left( X_j, \bar{X}_j \right) \right] \right\},$$

(3.3)

and this defines which products of chiral fields are $Q^+$-exact in the theory. We may then define the chiral ring as

$$\mathcal{R} = \mathbb{C} \left[ X_i \right] / \partial_j W.$$  

(3.4)

The computation of the operator product coefficients $C_{ij}^k$, as defined in (2.7), is simply a matter of polynomial multiplication modulo the ideal generated by $\{ \partial_i W \}$, which set the products of chiral fields equal to zero if they are $Q^+$-exact,

$$\partial_i W \left( X_j \right) \sim 0.$$

(3.5)

In particular, if one chooses a holomorphic basis for the chiral ring, i.e. one that depends holomorphically on the complex parameters in the superpotential, then the operator product coefficients will also be holomorphic functions of these parameters.

The topological inner product $\eta_{ij}$ is also easily computed using a result from the study of topological Landau-Ginsburg models. It can be shown [24] that

$$\eta_{ij} = \langle 0 | \phi_j \phi_i | 0 \rangle$$

$$= \text{Res}_W \left[ \phi_j \phi_i \right],$$

(3.6)
where $\text{Res}_W [F(X_k)]$ is defined using the Grothendieck residue by

$$\text{Res}_W [F(X_k)] = \frac{1}{(2\pi i)^n} \int dX_1 \wedge \cdots \wedge dX_n \ F(X_k) \ (\partial_1 W \cdots \partial_n W)^{-1},$$

where $n$ is the number of chiral superfields in the theory. In particular, one can show that the fundamental nonvanishing topological correlation function is given by

$$\text{Res}_W [H] = \mu,$$

where $H$ is the Hessian of the superpotential $W$, defined by

$$H = \det (\partial_i \partial_j W),$$

and $\mu$ is the criticality index of $W$, which is the same as the number of supersymmetric ground states of the theory, or equivalently, the number of elements in the chiral ring $\mathcal{R}$. All other correlation functions $\langle 0 | F(X_k) | 0 \rangle$ will vanish unless $F(X_k)$ contains some scalar multiple of the Hessian $H$, modulo the ideal generated by $\{\partial_i W (X_j)\}$. Again, in a holomorphic basis of the chiral ring $\mathcal{R}$, the inner product $\eta_{ij}$ will only depend holomorphically on the parameters in the superpotential.

When the superpotential $W(X_i)$ is a quasi-homogeneous polynomial of the chiral superfields, then it characterizes a conformal field theory [18, 19, 22]. This statement relies on assumption that the usual non-renormalization theorems for $N = 2$ supersymmetry, for which perturbative proofs exist, hold nonperturbatively as well. In the case of such quasi-homogeneous superpotentials, it is known how to directly compute the Hermitian inner product $g_{\hat{i} \hat{j}}$ using the superpotential. This in itself is a very interesting result, since it allows one to compute those correlation functions in the topological sector of the conformal theory without an explicit representation of the Virasoro algebra. The result for the inner product of two relevant chiral fields is given by [12, 13]

$$g_{\hat{i} \hat{j}} = \int \prod_{l=0}^{n} dX_l d\bar{X}_l \ \varphi_i (X_k) \ \bar{\varphi}_j (\bar{X}_\bar{k}) \ \exp (W - \bar{W}),$$

where $\varphi_i (X_k)$ is the element of the chiral ring which corresponds to the spectral flow of the ground state $|i\rangle$ and $n$ is the number of chiral superfields in the theory.

To summarize the situation, for Landau-Ginsburg models we can easily compute the operator product coefficient $C_{ij}^k$ as well as the topological inner product $\eta_{ij}$, and in an appropriate basis they are holomorphic functions of the parameters in the
superpotential $W(X_i)$. Furthermore, at the critical point, where the superpotential is quasi-homogeneous, we can calculate the Hermitian inner product $g_{i\bar{j}}$. To address the question of calculating the Green functions of the topological sector we need to know how $g_{i\bar{j}}$ depends on the parameters in the superpotential. Thus we must determine how the supersymmetric ground states $|i\rangle$ depend on the parameters in the superpotential. This is the question to which we now turn.

Let us start with some superpotential, say one corresponding to a conformal theory, for which we know the supersymmetric ground states $|i\rangle$. We now consider adding perturbations to the superpotential using the elements of the chiral ring $\mathcal{R}$:

$$\Delta \mathcal{L} = \int d^2\theta \ t^i \varphi_i(X_j) + h.c.,$$  

(3.11)

where $t^i$ are the perturbing couplings and the $\varphi_i(X_j)$, defined by

$$\varphi_i(X_j) = -\frac{\partial}{\partial t^i} W(X_j),$$  

(3.12)

are a basis for the chiral ring $\mathcal{R}$. Geometrically, we may think of the parameters $t^i$ as complex coordinates on the space of supersymmetric deformations of the superpotential. At each point in this space, we have an associated vector space defined by the set of supersymmetric ground states. Locally, the vacuum bundle is the product of these two spaces. A natural connection $A_i$ may be defined on this bundle using the partial derivatives of the ground states with respect to the parameters $t^i$ by:

$$\partial_i|j\rangle = A_{ij}^k |k\rangle + |\Psi\rangle,$$  

(3.13)

where $|\Psi\rangle$ is orthogonal to the space of ground states. The components of the connection are defined by:

$$A_{ij\bar{k}} = \langle \bar{k}|\partial_i|j\rangle = A_{ij|^l} g_{l\bar{k}}.$$  

(3.14)

Hence, one may view the connection $A_i$ as the projection operator of the variation of the ground states onto the Hilbert space orthogonal to them. It follows from the above definition that $A_i$ is a metric connection with respect to the Hermitian inner product $g_{i\bar{j}}$:

$$D_i g_{j\bar{k}} = (\partial_i \langle \bar{k}|j\rangle) - A_{ij|^l} g_{l\bar{k}} - g_{j\bar{i}} A_{i|^\bar{k}} = 0,$$  

(3.15)
where $D_i$ denotes the covariant derivative satisfying

$$\langle \bar{k} | D_i | j \rangle = \langle \bar{k} | (\partial_i - A_i) | j \rangle = 0,$$

(3.16)

and

$$A_{ij}^k = A_{ijl} g^{lk}, \quad A_i^j k = g^{jl} A_{ikl}.$$  

(3.17)

Here $g^{ij}$ are the components of $g^{-1}$.

Just as we have defined a connection and covariant derivative in terms of the variation of the supersymmetric ground states with respect to the holomorphic parameters in the superpotential $W(X_j)$, we may likewise define an anti-holomorphic connection $A_{\bar{i}}$ and covariant derivative $\bar{D}_i$, under which the metric $g_{ij}$ is also covariantly constant.

Having defined these covariant derivatives, we can compute the curvature of this connection. The result of this computation is:

$$[D_i, D_j] = [D_i, \bar{D}_j] = 0, \quad [D_i, \bar{D}_j] = -[C_i, \bar{C}_j],$$

(3.18)

where we define

$$C_j = g (C_j)^\dagger g^{-1}. \quad (3.19)$$

One also finds that

$$D_i C_j = D_j C_i, \quad \bar{D}_i \bar{C}_j = \bar{D}_j \bar{C}_i, \quad D_i \bar{C}_j = \bar{D}_i C_j = 0.$$  

(3.20)

These relations have been derived both from a direct analysis of the supersymmetric ground states of Landau-Ginsburg models [10], and from straightforward path integral arguments which apply to any $N = 2$ supersymmetric model [13]. Hence we see that the curvature is determined by the operator product coefficients, $(C_i)_j^k$, of the chiral ring $\mathcal{R}$, as defined in (2.7). By choosing a holomorphic basis for the chiral ring we can set

$$A_{ijl} = 0,$$

(3.21)

and this allows us to easily solve for the connection in terms of the metric $g_{ij}$:

$$A_{ij}^k = -g_{ji} (\partial_i g^{-1})^l g^{lk}.$$  

(3.22)
We then substitute this expression for the connection into the equation for the curvature, (3.18), to arrive at a differential equation for the metric,

$$\bar{\partial}_i (g \partial_i g^{-1}) = \left[ C_j, g (C_i)^\dagger g^{-1} \right],$$

(3.23)

which is valid in any holomorphic basis. From (3.20) we find that the operator product expansion coefficients satisfy

$$\partial_i C_j - \partial_j C_i + [g (\partial_i g^{-1}), C_j] - [g (\partial_j g^{-1}), C_i] = 0,$$

(3.24)

with a similar relation for the derivatives of the $\bar{C}_i$ matrices. This equation allows us to nonperturbatively calculate the metric $g_{ij}$. If we know its initial value at some point in the space of superpotential parameters, as well as the first derivatives of the metric at that point, then in principle we may calculate the metric everywhere by solving the above second-order partial differential equations.

4. Renormalization Group Flow and the Ramond Charge Matrix

We now turn to the question of how the $N = 2$ theory behaves under the action of the renormalization group. The renormalization group flow is a set of trajectories in the space of quantum field theories relating different quantum field theories via scale transformations. The parameter of these curves can be thought of as the ultra-violet cut-off of the theory, or some other mass scale used to define the quantum field theory. The action of the renormalization group on the correlation functions of the theory may be described by saying that the effect of a scale transformation on the theory is equivalent to an appropriate redefinition of the fields and the couplings in the theory, determined by the anomalous dimensions and $\beta$-functions. An important tool for understanding renormalization group flow in $N = 2$ Landau-Ginsburg models is the non-renormalization theorem. The statement of the non-renormalization theorem is that the only kind of renormalization that occurs in the superpotential is wave function renormalization. While this theorem has only been proven to all orders in perturbation theory, we shall assume it holds even nonperturbatively.

As an example of how this non-renormalization theorem works, consider the superpotential with a single chiral superfield $Y$:

$$\int d^2 z \, d^2 \theta \, W(Y) = \int d^2 z \, d^2 \theta \left( \frac{1}{4} Y^4 - \kappa Y^3 \right).$$

(4.1)
First consider the case with $\kappa = 0$. Under a scale transformation we have $z \to \lambda z$, $\theta \to \lambda^{-1/2}\theta$, so that the term involving the superpotential scales as

$$\int d^2 z d^2 \theta \ W(Y) \to \lambda \int d^2 z d^2 \theta \ W(Y).$$

(4.2)

We can eliminate the effect of this scale transformation by making the field redefinition $Y \to \lambda^{-1/4}Y$. This redefinition will of course change the kinetic term, but we see that the superpotential is invariant under a combined scale transformation and field renormalization. From this scaling argument we see that the field $Y$ has an anomalous dimension of one-fourth, and at the critical point it has conformal weights $(h, \bar{h}) = (1/8, 1/8)$.

This invariance argument holds for any quasi-homogeneous superpotential.

Now consider turning on the coupling $\kappa$ and treating it as a perturbation of the above superpotential. Under the above scale transformation and field renormalization we find that

$$W(Y) \to \frac{1}{4}Y^4 - \tilde{\kappa}Y^3,$$

(4.3)

where $\tilde{\kappa}$ is the renormalized coupling,

$$\tilde{\kappa} = \lambda^{1/4} \kappa.$$

(4.4)

Hence we see that the $\beta$-function of $\kappa$ is determined by the renormalization of the field $Y^3$. Once $\kappa$ is nonzero, the scaling dimensions of the fields will depend upon $|\kappa|$, but it is intuitively clear that in the large $\lambda$ limit $\tilde{\kappa}$ will grow and the $Y^3$ term will dominate the superpotential.

We can generalize this picture and take as the action of the renormalization group the rescaling of the superpotential by a factor $\lambda$. The limit $\lambda \to 0$ is the ultraviolet limit, and the $\lambda \to \infty$ limit corresponds to the infrared regime. Since the correlation functions in the topological sector only depend on the superpotential, we may then use this scaling argument to calculate their dependence upon the renormalization group parameter. In the case of Landau-Ginsburg theories, the operator product coefficients $C_{ij}^{\cdot k}$ clearly do not depend on the scaling parameter $\lambda$, since the equations of motion (3.5) are independent of $\lambda$. Therefore all the dependence on the renormalization group parameter is encoded in the metric $g_{i\bar{j}}$.

In the above discussion of the scale dependence of the superpotential, the parameter $\lambda$ that appears in front of the superpotential can be taken to be complex. However, the metric $g_{i\bar{j}}$ is independent of the phase of $\lambda$ since we can absorb the phase of $\lambda$ by redefinition
of the $\theta$ superspace coordinates in the measure. This of course leads to a redefinition of the fermionic components of the chiral superfield $X$, but the inner product $g_{ij}$ depends only on the lowest bosonic component of the superfield, hence it remains invariant under such a redefinition.

Consideration of how $g_{ij}$ depends on $|\lambda|$ near the critical point leads to the definition of an interesting quantity, the Ramond charge matrix. From (3.10) we have

$$g_{ij} = \int \prod_{l=0}^{n} dX_l d\bar{X}_\bar{l} \varphi_i(X) \bar{\varphi}_\bar{k}(\bar{X}) \exp(\lambda W - \bar{\lambda}\bar{W}) .$$  \hspace{1cm} (4.5)

At a critical point, the superpotential is quasi-homogeneous, and we find the dependence of the metric on $|\lambda|$ by a change of variables $X_i \rightarrow \lambda^{q_i} X_i$, giving

$$g_{ii} \propto \left(\lambda \bar{\lambda}\right)^{-\left(q_i - \frac{\hat{c}}{2}\right) - \frac{n}{2}} ,$$  \hspace{1cm} (4.6)

where $q_i$ denotes the Neveu-Schwarz $U(1)$ charge of the superfield $\varphi_i(X)$, $n$ is the number of chiral superfields, and $\hat{c} = c/3$ is one-thirds the value of the conformal anomaly of the theory. In the above expression we have also used the relation:

$$\hat{c} = \sum_l \left(1 - 2q_l\right) .$$  \hspace{1cm} (4.7)

We note that the quantity $q_i - \frac{\hat{c}}{2}$ is the $U(1)$ charge of the Ramond ground state which is the image of the Neveu-Schwarz state created by $\varphi_i(X)$ under the action of spectral flow.

We are thus led to define the Ramond charge matrix as

$$q_{ij} = g_{ik}\partial_k g^kj - \frac{n}{2} .$$  \hspace{1cm} (4.8)

The Ramond charge matrix $q$ has a simple field-theoretical interpretation in terms of the expectation value of a partially conserved charge between Ramond ground states [10,13]. For simplicity consider a superpotential involving one chiral superfield:

$$W(X) = \sum_{i=0}^{m} t_i X_i ,$$  \hspace{1cm} (4.9)

and the $R$-symmetry $X(z, \theta) \rightarrow e^{i\phi/m} X\left(z, e^{-i\phi/2}\theta\right)$, where $m$ is the largest power appearing in the superpotential. When the superpotential is homogeneous, so that $t_i = 0$ for $i < m$, then the current corresponding to this symmetry, $R_\mu$, is conserved, otherwise
it is partially conserved. The Ramond charge matrix is then given by the matrix elements of this partially conserved charge:

$$q_{kj} = \langle j | \oint \frac{d\sigma}{2\pi i} R_0 (\sigma) | k \rangle. \quad (4.10)$$

It can be shown that these matrix elements do not depend on the explicit form of the superpotential \[10,13\], (the dependence on $m$ of the integrated current is in fact $Q^+$-exact), and at the ultraviolet critical point this matrix is diagonal, the eigenvalues being the Ramond charges of the supersymmetric ground states, ranging from $-\hat{c}/2$ to $\hat{c}/2$. Therefore at the critical points the maximum eigenvalue of this matrix is one-sixth the value of the conformal anomaly. It can also be shown that at the conformal points, where the superpotential is quasi-homogeneous, $q$ is critical as a function of the coupling constants, and conversely, the criticality of $q$ implies that the superpotential is quasi-homogeneous.

These properties of the Ramond charge matrix have led to the speculation that the maximum eigenvalue of this matrix may provide a “c-function” on the space of $N = 2$ supersymmetric theories, similar to that defined by Zamolodchikov on the space of two-dimensional quantum field theories using the correlation functions of the energy-momentum tensor \[25\]. Both agree at the critical point and behave similarly in a neighborhood of the critical point. However we do not have a general argument that the maximum eigenvalue of $q$ is a non-increasing function of the renormalization group flow parameter, although this has been the case in all models studied so far. Also the function defined by $q$ is independent of the kinetic $D$-terms in the Lagrangian, since it is computed entirely in the topological sector, while the c-function defined by Zamolodchikov does depend on the details of the $D$-terms. Hence one might expect that these two definitions might agree for some particular choice of a $D$-term, but the precise relationship between these two natural functions on the space of $N = 2$ quantum field theories is still unknown.

Another interesting question is to determine the precise relationship between the metric on the space of two-dimensional quantum field theories defined by Zamolodchikov in terms of the two-point functions of the perturbing fields \[3,25\], and the inner product of the supersymmetric ground states $g_{ij\bar{j}}$. At the critical point the relation between these two metrics is known \[13\]. The inner product discussed above is then related to the two-point function of the lowest components of the perturbing superfields evaluated on the sphere in the Neveu-Schwarz sector:

$$\frac{g_{ij\bar{j}}}{g_{00\bar{0}}} = G_{ij} = \langle \bar{\varphi}_j^\dagger (1) \varphi_i \rangle. \quad (4.11)$$
while the metric defined by Zamolodchikov is given by

\[ G^Z_{ij} = \left\langle \int d^2\theta \bar{\Phi}_j(1) \int d^2\theta \Phi_i(0) \right\rangle, \quad (4.12) \]

where \( \phi_i \) is the lowest component of the superfield \( \Phi_i \). The factor of \( g_{00}^{-1} \), where the index 0 labels the identity, occurs in (4.11) to provide the correct normalization by dividing out the vacuum amplitude. At the conformal point one may directly relate these two metrics, and using the superconformal Ward identities one finds

\[ G^Z_{ij} = q_{i,L}q_{i,R}G_{ij}, \quad (4.13) \]

where \( q_{i,L} \) and \( q_{i,R} \) are the left and right \( U(1) \) charges. In particular, we see that the components of the Zamolodchikov metric involving the identity vanish, since perturbations by multiples of the identity are annihilated by the integration over superspace, and for marginal perturbations, satisfying \( q_{i,L} = q_{i,R} = 1 \), the metrics are identical.

These considerations lead to the definition of the algebraic \( Q \)-matrix \[13\],

\[ Q^{ij} = G_{i\bar{k}} \partial_\tau (G^{-1})^{\bar{k}j}, \quad (4.14) \]

where \( G \) is the normalized matrix defined in (4.11). At the critical point, the eigenvalues of \( Q \) are the \( U(1) \) charges of the chiral primary fields, ranging from 0 to \( \hat{c} = c/3 \). It was suggested in \[13\] that the maximum eigenvalue of this matrix might also be a candidate \( c \)-function, but we shall provide an example below in which the infrared critical limit of the maximum eigenvalue of \( Q \) is approached from below, and hence is not a non-increasing function of the renormalization group flow parameter.

It is interesting to note that the definition of the \( Q \)-matrix (4.14) is quite similar to the renormalization group equation for the two-point function \( \langle \bar{\phi}_j(1) \phi_i(0) \rangle \). Under the scale transformation \( x \to e^{t|x|}x \), the two-point function satisfies the equation

\[ \left( \frac{1}{2} \frac{\partial}{\partial |\tau|} + \hat{\Gamma} - \beta^a \frac{\partial}{\partial \ell^a} \right) \langle \bar{\phi}_j(1) \phi_i(0) \rangle = 0, \quad (4.15) \]

where \( \hat{\Gamma} \) is the anomalous dimension operator

\[ \hat{\Gamma} \phi_i = \gamma_i^j \phi_j, \quad \hat{\Gamma} \bar{\varphi}_i = \bar{\varphi}_j \gamma^j_{\bar{i}}, \quad (4.16) \]
and the coefficients $\beta^a$ are the $\beta$-functions related to the scale-dependent coupling constants by

$$\beta^a = \frac{1}{2} \frac{dt^a}{d|\tau|}.$$  (4.17)

The anomalous dimension matrix has been normalized such that its eigenvalues are the conformal weights at the critical point, equal to one-half the scaling dimension for Landau-Ginsburg models, and the sum over $a$ runs over all the coupling constants in the theory.

If we now conjecture that the identification in (4.11) holds away from the critical point as well, (perhaps this identification again corresponds to a specific choice of a kinetic $D$-term), then since the left-hand side of (4.11) depends only on the superpotential, we may restrict the sum over coupling constants to those in the superpotential alone. The choice of a holomorphic basis allows us to write

$$\beta^a \partial_a G_{ij} = \beta^k \partial_k G_{ij} + \beta^\bar{k} \partial_{\bar{k}} G_{i\bar{j}} = \beta^k A_{ki}^\dagger G_{\bar{i}j} + G_{i\bar{k}} \beta^\bar{k} A_{\bar{k}j},$$  (4.18)

where $A_{ij}$ is the metric connection with respect to $G_{ij}$:

$$A_{ij}^k = -G_{j\bar{i}} \left( \partial_i G^{-1} \right)^{\bar{l}k}.$$  (4.19)

If we rewrite the definition of $Q$, (4.14), as follows,

$$\partial_\tau G_{ij} + \frac{1}{2} \left( Q_k^i G_{kj} + G_{ik} Q_k^j \right) = 0,$$  (4.20)

then since

$$\partial_\tau G_{ij} = \frac{1}{2} \frac{\partial}{\partial |\tau|} G_{ij},$$  (4.21)

we are led to the identification:

$$\frac{1}{2} Q_i^j = \gamma_i^j - \beta^k A_{k\bar{i}}^j.$$  (4.22)

Hence the $Q$-matrix seems to be directly related to the anomalous dimensions of the chiral fields, the $\beta$-functions of the theory, and the field-theoretical connection with respect to the normalized metric $G_{ij}$. In particular, at the critical points, where the $\beta$-functions vanish, we see that one-half the eigenvalues of $Q$ are the conformal weights of the chiral primary fields, as expected.
5. Renormalization Group Flow Along a Critical Line

We shall now apply the above formalism to a specific model, characterized by the superpotential involving one chiral superfield:

\[ W(X) = \frac{1}{4}X^4 - \alpha X - \beta X^2, \]  

(5.1)

where \( \alpha \) and \( \beta \) are complex parameters. The chiral fields \( X \) and \( X^2 \) are two of the three relevant superfields in the theory, the third being the identity, which does not deform the superpotential. The cases for \( \alpha = 0 \) or \( \beta = 0 \) have previously been studied by Cecotti and Vafa [13]. These correspond to massive deformations of the \( N = 2 \) \( A_3 \) minimal model with \( \hat{c} = \frac{1}{2} \). In both cases, the deformation gives an integrable model, and there exists a basis of the chiral ring for which the metric \( g_{ij} \) is diagonal and can be parameterized by a single function. The second order differential equation that results from (3.23) in these cases turns out to be special cases of the third Painlevé equation.

In this section we shall instead consider a perturbation which retains a massless field in the infrared limit. Recall that the bosonic potential \( V(X) \) is given in terms of the superpotential by:

\[ V(X) = \left| \frac{\partial}{\partial X} W(X) \right|^2. \]  

(5.2)

The classical ground states correspond to the critical points of \( W \), where the partial derivative vanishes. A requirement of the superpotential for a massless field to exist is that at one of the critical points the Hessian, defined by (3.9), also vanishes, so that at least two critical points will be degenerate. Imposing this constraint on the above superpotential (5.1), and solving the two equations \( \partial W = H = 0 \) gives the following relation between \( \alpha \) and \( \beta \):

\[ \left( \frac{1}{2} \alpha \right)^2 = \left( \frac{2}{3} \beta \right)^3 \equiv \kappa^6. \]  

(5.3)

If we make the field redefinition \( \tilde{Y} = X + \kappa \), then aside from an irrelevant constant the superpotential takes the form

\[ \tilde{W} \left( \tilde{Y} \right) = \frac{1}{4} \tilde{Y}^4 - \kappa \tilde{Y}^3. \]  

(5.4)

This superpotential has one critical point at \( \tilde{Y} = 3\kappa \) and two which lie at \( \tilde{Y} = 0 \). In accord with our previous discussion of the renormalization group flow of this model, we expect the renormalized coupling \( \kappa \) to scale as \( \lambda^{1/4} \), and therefore in the limit \( \lambda \to \infty \) the two sets
of ground states should decouple, with the double degeneracy at $\tilde{Y} = 0$ being described by the $N = 2 A_2$ minimal model, with exponentially suppressed overlap with the massive theory at $\tilde{Y} = 3\kappa$.

We wish to calculate the metric $g_{i\bar{j}}$ as a function of the renormalization group scale $\lambda$, so it is convenient to perform one more field redefinition so that the dependence on $\lambda$ is explicit. Let $\kappa = \lambda^{1/4}$ and $Y = \lambda^{-1/4}\tilde{Y}$ so that the superpotential now takes the form

$$W(Y) = \lambda \left( \frac{1}{4}Y^4 - Y^3 \right).$$

(5.5)

Our strategy for calculating the metric $g_{i\bar{j}}$ is as follows. First we shall solve the constraint (2.19) and require that the metric be Hermitian by finding a convenient parameterization of the metric. We shall then calculate the metric as a function of the renormalization group flow parameter $\lambda$ by solving the differential equation (3.23) using the appropriate initial conditions on the metric at $\lambda = 0$. We may then calculate the correlation functions in the topological sector as functions of the scale parameter $\lambda$, and verify our expectations based on the $N = 2$ non-renormalization theorem concerning the infrared behavior of this model.

For our purposes, a convenient basis is given by

$$\{\varphi_i\} = \{1, (Y - 1), (Y^2 - 2Y - \frac{1}{2})\}, \quad i = 0, 1, 2.$$  

(5.6)

We shall call this basis the flat basis. In this basis the topological inner product has a very simple dependence on the parameters in the superpotential, and is simply given by

$$\eta = \lambda^{-2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$  

(5.7)

Using (2.7) we may calculate the matrices $C_i$:

$$C_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 3/2 & 0 & 1 \\ 2 & 3/2 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 3/2 & 0 \\ 9/4 & 2 & 0 \end{pmatrix},$$  

(5.8)

which appear in the topological correlation functions as in (2.10).

The key to solving the reality constraint is to notice that in the flat basis the above topological inner product (5.7) implies that (2.19) takes the form

$$g\eta g^T = \eta.$$  

(5.9)
so the metric $g_{i\bar{j}}$ is essentially an element of complexified $SO(2,1)$, and therefore may be parameterized by three complex Euler angles. The parameterization used in the calculation of the metric is given by

$$g = ST\tilde{g}T^\dagger S^\dagger,$$

where

$$\tilde{g} = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} \cos \rho & \sin \rho & 0 \\ -\sin \rho & \cos \rho & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\psi, \theta,$ and $\rho$ are three complex parameters. The matrices $T$ and $S$ are given by

$$T = T^{-1} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix},$$

and

$$S = \begin{pmatrix} \lambda^{-1/4} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-3/4} \end{pmatrix}.$$ 

Now requiring that $g$ be a Hermitian matrix requires that $\psi = -\rho$, and that $\theta$ be either real or purely imaginary. We shall see below that the initial conditions on the metric require that $\theta$ be real. Hence we have a parameterization of the metric in terms of three real functions of $\lambda$.

For the superpotential we are considering, (5.3), there is actually one remaining symmetry of the metric $g_{i\bar{j}}$. It is invariant under the interchange of $\varphi_i$ with $\bar{\varphi}_i$, since the only complex parameter in the superpotential is $\lambda$, and as discussed in Section 4, the metric does not depend on the phase of $\lambda$. Hence, in the flat basis above, the metric is not only Hermitian, but also symmetric. This symmetry implies that $\psi$ must be real, thus in our calculation the metric may be parameterized by just two real functions of $\lambda$.

We now wish to solve the for the dependence of the metric on the renormalization group scale $\lambda$ by solving the differential equation

$$\frac{\partial}{\partial \lambda} \left( g \frac{\partial}{\partial \lambda} g^{-1} \right) = \left[ C_\lambda, g \right] \left( C_\lambda \right)^\dagger g^{-1},$$

where the matrix $C_\lambda$ is the matrix representation of the superpotential itself:

$$C_\lambda = \frac{3}{4} \left( C_2 + 2C_1 + \frac{5}{2}C_0 \right).$$
since
\[-\frac{\partial W}{\partial \lambda} = - \left( \frac{1}{4} Y^4 - Y^3 \right) \sim \frac{3}{4} Y^2 \]
\[\sim \frac{3}{4} \left[ (Y^2 - 2Y - \frac{1}{2}) + 2(Y - 1) + \frac{5}{2} \right]. \tag{5.16}\]

Since the component of $C_\lambda$ proportional to the identity matrix vanishes in the commutation relation in the above differential equation, we may neglect it and simply evaluate
\[C_\lambda = \frac{3}{4} \begin{pmatrix} 0 & 2 & 1 \\ 5 & 3/2 & 2 \\ 25/4 & 5 & 0 \end{pmatrix}. \tag{5.17}\]

Using the fact that in the flat basis above the metric is only a function of $x = |\lambda|$ we finally arrive at the following second order differential equation for the metric:
\[\frac{1}{4x} \frac{d}{dx} \left( xg \frac{d}{dx} g^{-1} \right) = [C_\lambda, gC_\lambda^T g^{-1}]. \tag{5.18}\]

To solve this differential equation, we first must specify the initial conditions on the metric. Before determining the boundary conditions, it is instructive to first recall some aspects of the analysis of the two cases where $\alpha = 0$ or $\beta = 0$ described in [13]. In both these cases the differential equation which describes the dependence of the metric on the parameters $\alpha$ or $\beta$ is the third Painlevé transcendent equation:
\[f'' = \left( \frac{f'}{f} \right)^2 - \frac{f'}{x} + \frac{1}{x} (p_1 f^2 + p_2) + p_3 f^3 + \frac{p_4}{f}, \tag{5.19}\]

where the function $f$ is essentially given by one of the (diagonal) elements of the metric and $x$ is related to the perturbing coupling $\alpha$ or $\beta$. In the case $\alpha = 0$ the parameters $p_i$ take the values $p_1 = -p_4 = 1$, $p_2 = p_3 = 0$, and in the case where $\beta = 0$ we have $p_1 = p_2 = 0$, $p_3 = -p_4 = 1$. This equation has been studied using the isomonodromic deformation method in [26,27]. Using the results of these studies, S. Cecotti and C. Vafa found that the requirement that $g_{ij}$ be a nonsingular, positive-definite metric will uniquely determine the required boundary conditions of the solution to the above differential equation. This remarkable situation was also found to be the case in many other models they studied, all of which correspond to integrable deformations of the $N = 2$ minimal models and are related to quantum and classical affine Toda theories. Hence the physical requirement of the regularity of the solution, combined with the differential equations (3.23), may completely determine the metric, including its values at the critical point, which gives the normalized
values of the operator product expansion coefficients of the corresponding conformal field theory.

Since in the flat basis above the metric $g_{ij}$ is singular as $\lambda \to 0$, we shall discuss the initial conditions in the basis $\{1, X, X^2\}$ for the superpotential in the form (5.1) where the parameters $\alpha$ and $\beta$ satisfy the relation (5.3). The differential equations for the functions $\psi$ and $\theta$ which result from (5.18) were determined using a mathematical manipulation language, Maple. The differential equations which result are rather lengthy, and do not appear to have previously been studied in the literature, so we have not yet determined whether the requirement of regularity alone uniquely determines the initial data for the metric solution. However, using the known value of the metric for the unperturbed conformal theory, we have been able to determine the initial values of the first derivatives of the parameters $\psi$ and $\theta$ by requiring that the differential equations for these parameters are nonsingular at the origin $x = |\lambda| = 0$.

Using some results derived in [11], the relevant ones having been collected in Appendix A below, we find

$$g_{ij}^X (x = 0) = \begin{pmatrix} \gamma/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2/\gamma \end{pmatrix},$$

(5.20)

where $x = |\lambda|$ and $\gamma$ is the ratio of the gamma functions:

$$\gamma = \frac{\Gamma (1/4)}{\Gamma (3/4)}.$$

(5.21)

The initial conditions on the derivatives of the metric elements at $x = |\lambda| = 0$ are found to be:

$$\frac{d}{dx} \langle \bar{0}|0 \rangle = -\frac{9}{4} \langle \bar{2}|2 \rangle,$$

(5.22a)

$$\frac{d}{dx} \langle \bar{2}|2 \rangle = \frac{9}{4} \left[(\langle \bar{2}|2 \rangle)^3 + \langle \bar{0}|0 \rangle \right],$$

(5.22b)

$$\frac{d}{dx} \langle \bar{0}|2 \rangle = \frac{3}{4} x^{-1/2} e^{i2\phi} \langle \bar{0}|0 \rangle,$$

(5.22c)

$$\frac{d}{dx} \langle \bar{2}|0 \rangle = \frac{3}{4} x^{-1/2} e^{-i2\phi} \langle \bar{0}|0 \rangle,$$

(5.22d)

where the phase $\phi$ arises due to the trivial dependence of the metric elements on the phase of $\kappa = |\kappa| e^{i\phi}$ in the above basis. The metric elements referred to on the right-hand side of (5.22) are those of the metric at $x = 0$, (5.20). The first derivatives of all other elements vanish at $x = 0$. These initial conditions on the first derivatives of the metric agree with those found by Cecotti and Vafa in [13].
6. Numerical Solution and Comparison with the Asymptotic Solution

Applying the initial data on the metric given above, we have solved the differential equation (5.18) numerically, using the fourth-fifth order Runge-Kutta method implemented by Maple.

In fig. 1 we have plotted the eigenvalues of the Ramond charge matrix, calculated in the flat basis (5.6), as a function of the perturbing coupling $\kappa$ in the superpotential. Recall that the running coupling $\kappa$ is related to the renormalization group scale parameter $\lambda$ by $\kappa = \lambda^{1/4}$. At $\kappa = 0$ the maximum eigenvalue is one-fourth, equal to the value of $\hat{c}/2$ of the unperturbed theory. In the infrared limit we see that the maximum eigenvalue monotonically decreases to the value one-sixth, the value of $\hat{c}/2$ for the theory described by the superpotential $W(Y) = Y^3$. In this limit, the two eigenvalues approaching $\pm \frac{1}{6}$ correspond to the Ramond charges of the two supersymmetric ground states of the conformal theory, and the zero eigenvalue corresponds to the trivial massive theory which is decoupled. We observe from this graph that we expect the infrared limit of the theory to behave critically near the value $|\kappa| = 0.6$.

In fig. 2 we have plotted the eigenvalues of the $Q$-matrix, again calculated in the flat basis, as a function of $\kappa$. These eigenvalues exhibit a different behavior than those of the Ramond charge matrix. Using the asymptotic solution for the metric, discussed in more detail below, we may compute the infrared limit behavior of the $Q$-matrix and its eigenvalues. In the flat basis we find that as $|\kappa| \to \infty$:

$$Q \approx \frac{1}{36} \begin{pmatrix} 5 & \frac{(12|\kappa|^{4/3}x-10)}{(6|\kappa|^{4/3}x+1)} & 2 & 0 \\ 0 & \frac{(36|\kappa|^{4/3}x-6)}{(6|\kappa|^{4/3}x+1)} & -2 & 0 \\ \frac{(60|\kappa|^{4/3}x-2)}{(6|\kappa|^{4/3}x+1)} & \frac{(60|\kappa|^{4/3}x-2)}{(6|\kappa|^{4/3}x+1)} & -2 & 0 \end{pmatrix},$$

(6.1)

where

$$\chi = \frac{\Gamma(1/3)}{\Gamma(2/3)}.$$

(6.2)

The eigenvalues of this matrix approach the asymptotic values $(0, 1/6, 1/3)$ from below, and for $|\kappa| > 0.74$ are in good agreement with the calculated values. The eigenvalues $(0, 1/3)$ correspond to the $U(1)$ charges of the chiral fields in the infrared conformal theory, while the eigenvalue of one-sixth corresponds to the massive theory, one-sixth being the shift due to spectral flow. It is interesting to note that the asymptotic form of the algebraic
Q-matrix depends on explicitly on \( \kappa \), while the similar asymptotic form of the Ramond charge matrix, given by

\[
q_R \approx \frac{1}{36} \begin{pmatrix} -4 & 2 & 0 \\ 5 & 0 & -2 \\ 0 & -5 & 4 \end{pmatrix},
\]  
(6.3)
in the flat basis, is independent of \( \kappa \).

In order to compare the results of the computation of the components of the metric with the asymptotic solution, we have chosen to display the results of the calculation of the metric elements in the “point” basis. Physically, this is the basis in which each element of the basis corresponds to a ground state wave function localized near a critical point of the superpotential. When the superpotential is of the form \( \tilde{W}(\tilde{Y}) \), \((5.4)\), the point basis is given by

\[
\{ \varphi^P_i \} = \left\{ \frac{1}{3\kappa} \left( 3\kappa - \tilde{Y} \right), \frac{1}{3\kappa} \tilde{Y} \left( 3\kappa - \tilde{Y} \right), \frac{1}{9\kappa^2} \tilde{Y}^2 \right\}, \quad i = 0, 1, 2.
\]  
(6.4)

In this basis the matrices \( C_i \) and the topological inner product \( \eta_{ij} \) are in block diagonal form, the first two elements of this basis representing the doubly degenerate critical point at \( \tilde{Y} = 0 \) and the third element representing the isolated critical point at \( \tilde{Y} = 3\kappa \). Asymptotically the metric \( g_{ij} \) will be in block diagonal form with exponentially suppressed off-diagonal metric elements.

In the limit \( \kappa \to \infty \) the superpotential behaves as \( \tilde{W} \to -\kappa \tilde{Y}^3 \), and the first two elements of the point basis give a basis for the chiral ring of the infrared conformal theory:

\[
\left\{ \frac{1}{3\kappa} \left( 3\kappa - \tilde{Y} \right), \frac{1}{3\kappa} \tilde{Y} \left( 3\kappa - \tilde{Y} \right) \right\} \to \left\{ 1, \tilde{Y} \right\}.
\]  
(6.5)

Again using the results of Appendix A, we may use the above behavior of the first two elements of the point basis to determine the leading dependence on \( \kappa \) of the metric components \( \langle \bar{0}|0 \rangle_P \) and \( \langle \bar{1}|1 \rangle_P \) in the point basis. Then using the reality constraint \((2.19)\), where

\[
\tilde{\eta}_P = -\frac{1}{3\kappa} \begin{pmatrix} -1/3\kappa & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1/3\kappa \end{pmatrix},
\]  
(6.6)
in the point basis, we find that the asymptotic limit of the metric in this basis is

\[
\tilde{g}_P = \begin{pmatrix} \left| \kappa \right|^{-2/3} \chi + \left| \kappa \right|^{-10/3} \frac{1}{3\chi} & \left| \kappa \right|^{-7/3} e^{-i\phi} \frac{1}{18\chi} & 0 \\ -\left| \kappa \right|^{-7/3} e^{i\phi} \frac{1}{18\chi} & \left| \kappa \right|^{-4/3} \frac{1}{3\chi} & 0 \\ 0 & 0 & \left| \kappa \right|^{-2/3} \end{pmatrix},
\]  
(6.7)
where $\chi$ was defined above in (6.2), and $\kappa = |\kappa| e^{i\phi}$ is the coupling in the superpotential, and we have neglected exponentially suppressed terms.

In figs. 3–6 we have plotted the calculated values of the metric components in the point basis as functions of the coupling $\kappa$, as shown by the solid curves. The dashed curves shown are the asymptotic values of the metric, given in (6.7) above, and in every case we see excellent agreement for $|\kappa| > 0.6$.

We can also compare the off-diagonal metric components, $\langle \bar{2}|0 \rangle$ and $\langle \bar{2}|1 \rangle$, to the leading order semi-classical corrections to (6.7). These components of the metric may be viewed as arising from the probability to tunnel between the classical ground states localized near distinct critical points. It has been argued in [13] that in a basis in which the metric is diagonal to leading order, the leading off-diagonal semi-classical correction to the metric is a universal function of the mass of the soliton connecting the two distinct critical points, if it exists. In such a basis, this function is of the form

$$
\frac{g_{ij}}{(g_{ii}g_{jj})^{1/2}} \approx C_{ij} m_{ij}^{-1/2} \exp(-m_{ij}),
$$

(6.8)

where $C_{ij}$ is some complex coefficient and $m_{ij}$ is the mass of the soliton interpolating between the two vacua:

$$
m_{ij} = 2|\lambda W(X_i) - \lambda W(X_j)|,
$$

(6.9)

$X_i$ and $X_j$ being the values of the distinct critical points. The method of calculating the above semi-classical correction is discussed in Appendix B. In the model we are analyzing the mass of the soliton connecting the vacua at $\tilde{Y} = 0$ with $\tilde{Y} = 3\kappa$ is $m = 27|\kappa|^4/2$, and it is of order one for $|\kappa| \approx 0.52$, a result in excellent accord with the behavior of the metric components we have displayed.

We have plotted the logarithm of the final two components of the metric in figs. 7, 8, and have displayed the leading semi-classical correction by a dashed line. These semi-classical corrections are given in the point basis by:

$$
|g_{02}| = \frac{(0.76)}{162(\pi \chi)^{1/2}}|\kappa|^{-14/3} \left(6\chi |\kappa|^{4/3} - 1\right) F(m) \exp(-m),
$$

(6.10)

$$
|g_{12}| = \frac{(0.76)}{27(\pi \chi)^{1/2}}|\kappa|^{-11/3} F(m) \exp(-m),
$$

(6.11)

where $m$ is the mass of the soliton defined above, and $F(m)$ is a function defined in Appendix B. Unfortunately the method of calculating the form of these corrections does

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not give us the value of the constant factor $C_{ij}$, so the number 0.76 in (6.10) and (6.11) was determined by a fit to the calculation. In principle these corrections might also be calculated by using the WKB approximation to compute the overlap of the wavefunctions localized near distinct critical points, but unfortunately this method also becomes unreliable in the neighborhood of the critical points of the superpotential and cannot be used to determine this factor. Nevertheless, after fitting this one parameter we again see good agreement between calculation and the predicted value of the soliton mass.

There are two primary sources of error in the numerical computations we have presented. One is associated with the finite step width involved in the numerical solution of the differential equations. The relative and absolute error tolerances in the computed quantities were chosen to be no greater than $10^{-11}$. However, by varying the allowed tolerances over several orders of magnitude, it was found that the numerical results were quite insensitive to the step width involved in the range over which calculations were performed.

The other source of error is associated with the initial conditions on the metric and its first derivatives. In the parameterization of the metric used in the computations, while the initial values and the first derivatives of the parameters used are finite at the point $|\lambda_0| = 0$, the second derivatives of the parameters are singular at this initial point. For this reason, we were required to specify the initial data at some point off the origin, and the initial point was taken to be $|\lambda_1| = 10^{-6}$. Unfortunately, the precise value of the metric and its derivatives is not known away from the origin, so the initial data was corrected to first order in $|\lambda_1 - \lambda_0| = |\lambda_1|$ for the first derivatives and to second order in $|\lambda_1|$ for the initial values by solving the differential equations in the small $|\lambda|$ limit.

By studying the effect of variations on the initial data it was found that both the values and the general behavior of the calculated quantities are very sensitive to the precise initial conditions imposed. In fact, it appears that the major source of error is attributable to the error in the initial conditions. This is in accord with our general expectations based on the previously studied cases discussed above. Unless the boundary conditions agree with the requirement of regularity, then we would expect to find some singularities in the solution of our differential equations, which would drastically alter the its nature. The departure of the calculated values of the off-diagonal metric elements from the asymptotic limit in figs. [7][8], for $|\kappa| > 0.8$ is the first indication of this error. By making minor adjustments in the initial data it is possible to correct this departure from the asymptotic limit, however we have chosen not to do this, and instead accept the value $|\kappa| \approx 0.8$ as the limit of the reliability of these calculations.
7. Conclusions

We have demonstrated by explicit computation using nonperturbative methods that the correlation functions in the topological sector of a theory with $N = 2$ supersymmetry may be computed both on and off of the critical point. We studied a particular perturbation of the $A_3$ superconformal minimal model which in the infrared limit sends the theory to the $A_2$ model along the renormalization group trajectory. Using standard numerical techniques we essentially obtained all the characteristic features of the infrared conformal field theory, such as the value of the conformal anomaly and the normalized operator product expansion coefficients of the chiral ring.

We also calculated the amplitude for quantum “tunnelling” between the conformal theory described by the $A_2$ model and the massive theory which decouples in the infrared limit. These results were compared to the leading order semi-classical instanton corrections, and we found good agreement with the predicted value of the mass of the soliton which connects the two critical points. Hence we have exhibited the complete behavior of the topological sector of the theory which interpolates between two nontrivial conformal field theories along the renormalization group flow trajectory. We are currently in the process of extending these results to the full space of perturbations of the $A_3$ model [28].

Several interesting questions remain. The physical requirement of the regularity of the Hermitian metric seems to essentially fix the initial data of the second order differential equation which describes its dependence on the parameters of the theory. This was found to be the case in all the models studied in [13], and although it was used to fix only the first derivatives in the problem studied above, it almost certainly must fix the initial values of the metric as well. Is this a general feature of all $N = 2$ supersymmetric models, and what is the precise connection with the isomonodromic deformation techniques used by mathematicians?

It would also be interesting to determine the precise relation between Zamolodchikov’s field theoretical metric and the $c$-function, and the Hermitian inner product and Ramond charge matrix of the topological sector. This would lead to a better understanding of the connection between the usual first order field-theoretical differential equations describing the renormalization group flow and the second order partial differential equations used above to describe the behavior of the Hermitian inner product of the Ramond ground states.
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Appendix A. Metric Components of the Conformal Theory

In this appendix we collect the results derived in [11] which were used to calculate the initial values of the metric components, as well as their asymptotic limits, in the case when the superpotential is homogeneous. Consider the case when the superpotential is of the form $W(X) = \lambda X^m$. Using the fact that field theoretical computations in the topological sector may be reduced by dimensional reduction to computations in supersymmetric quantum mechanics, we may express the computation of the metric $g_{i\bar{j}}$ as the inner product between vacuum waveforms:

$$g_{i\bar{j}} = \langle \bar{j} | i \rangle = \int ^\ast \bar{\omega}_j \wedge \omega_i,$$

(A.1)

where $\omega_i$ is the differential form associated with the supersymmetric ground state wave function $|i\rangle$ in the standard fashion [29]. This representation of $g_{i\bar{j}}$ can be shown to be equivalent to that given in (3.10) for a quasi-homogeneous superpotential [13]. Choose an orthonormal basis for the ground state waveforms:

$$\omega_i = \alpha_i \lambda^{(i+1)/m} X^i dX, \quad i = 0, \ldots, m-2,$$

(A.2)

where the normalization coefficients $\alpha_i$ are fixed up to a phase by the requirement of orthonormality. This waveform corresponds to the basis $\{\varphi_i\} = \{\alpha_i \lambda^{(i+1)/m} X^i\}$ of the chiral ring $\mathcal{R}$.

From the real structure matrix $M$, defined in (2.16), we have the relation between a vacuum waveform and its complex conjugate:

$$\bar{\omega}_j = M^i_j \omega_i.$$

(A.3)

Letting $\tilde{\omega}$ denote the vacuum waveform corresponding to $\bar{\omega}$ via the real structure, we may write

$$\tilde{\omega}_j = \beta_j \lambda^{(m-j-1)/m} X^{m-j-2} dX, \quad j = 0, \ldots, m-2,$$

(A.4)
where we have essentially used the action of spectral flow to determine the power of $X$ which appears, and $\beta_j$ is determined by the real structure.

Using the Bochner-Martinelli theorem it can be shown that

$$g_{k\bar{j}} = \int *\tilde{\omega}_j \wedge \omega_k$$

$$= Res_W[\tilde{\varphi}_j(X)\varphi_k(X)],$$

and therefore the orthonormality condition implies that

$$\alpha_k \beta_k = m. \quad (A.6)$$

To determine the normalizations $\alpha$ we need one more condition, the reality condition expressed by S. Cecotti. If we define $W_1$ to be the points defined by $X^m = 1$:

$$W_1 = \{ X_j = \exp(2\pi i j/m) \}, \quad j = 0, \ldots, m-1, \quad (A.7)$$

then a basis for the relative homology group $H_1(C, W_1; \mathbb{Z})$ is given by the segments $\gamma_j$ connecting the points $X_{j+1}$ and $X_j$. Using the reality condition of [11],

$$\int_{\gamma_j} e^{-W} \omega_k = \left( \int_{\gamma_j} e^{-W} \tilde{\omega}_k \right)^*,$$

and the fact that

$$\int_{\gamma_j} e^{-W} \omega_k = \frac{2i}{m} \alpha_k e^{2\pi i k/m} \sin \left( \frac{k\pi}{m} \right) \Gamma \left( \frac{k}{m} \right), \quad (A.9)$$

we arrive at the condition that

$$\frac{\alpha_k}{\beta_k} = -\frac{\Gamma(1-k/m)}{\Gamma(k/m)}. \quad (A.10)$$

Combining this condition with (A.6) above we find that

$$\alpha_k = \sqrt{n\pi} \left[ \sqrt{\sin(k\pi/m)} \Gamma(k/m) \right]^{-1}. \quad (A.11)$$

For the superpotential

$$W(X) = \frac{1}{4} X^4, \quad (A.12)$$
the relation between the above orthogonal basis, $|k\rangle_O$, and that associated with the polynomial basis $\{1, X, X^2\}$, say $|j\rangle_X$, is given by

$$|j\rangle_X = U_j^k |k\rangle_O,$$

(A.13)

where

$$U = \text{diag} \left( 4^{1/4} \alpha_0^{-1}, 4^{1/2} \alpha_1^{-1}, 4^{3/4} \alpha_2^{-1} \right).$$

(A.14)

Hence in the polynomial basis we have

$$g^X = U g^O U^\dagger,$$

(A.15)

where $g^O$ is simply the identity matrix, due to the orthonormality of the basis $|k\rangle_O$. This equation directly gives (5.20). Similarly, for the superpotential $W(\tilde{Y}) = \kappa \tilde{Y}^3$, the above relations combined with the reality constraint (2.19) lead to the result given in (6.7).

### Appendix B. Leading Semi-Classical Corrections to the Metric

In this appendix we outline the method used to calculate the semi-classical corrections given in (6.10) and (6.11). For a comprehensive discussion of semi-classical considerations, see Appendix B of [13]. Beginning with the metric in the point basis (6.7), one first expresses it in the form

$$\tilde{g}_P = V \exp(\zeta) V^\dagger,$$

(B.1)

where the matrix $V$ is the holomorphic transformation to an orthonormal basis, and $\zeta$ represents the exponentially suppressed off-diagonal terms, so that classically $\zeta = 0$. In the large $\lambda$ limit the calculation of $\zeta$ may be approximated by the one-instanton contribution, neglecting other contributions which are exponentially suppressed with respect to this leading order correction. Hence we may work to first order in $\zeta$.

In this approximation (B.18) is given by

$$\frac{1}{4x} \frac{d}{dx} \left( x \frac{d}{dx} \zeta \right) = \left[ C_\lambda, \left[ C_\lambda^\dagger, \zeta \right] \right].$$

(B.2)

We are interested in computing the elements of $\zeta$ which represent the tunnelling probability between the vacua at $\tilde{Y} = 0$ and $\tilde{Y} = 3\kappa$, so restricting our analysis to these off-diagonal elements, and using the fact that in this basis $C_\lambda$ is diagonal with its components given by
the values of the superpotential at the critical points, $(C_\lambda)^k_j = W(X_j)\delta_j^k$, we find (B.2) gives
\[
\frac{1}{4x} \frac{d}{dx} \left( x \frac{d}{dx} \zeta_{jk} \right) = |W(X_j) - W(X_k)|^2 \zeta_{jk}.
\] (B.3)

Defining
\[
m_{jk} = 2|\lambda W(X_j) - \lambda W(X_k)|,
\] (B.4)
and treating $\zeta$ as a function of this variable, gives the following differential equation
\[
\frac{d}{dm} \left( m \frac{d}{dm} \zeta(m) \right) = m \zeta(m).
\] (B.5)

The general solution, which vanishes in the large $\lambda$ limit, to this differential equation is given by
\[
\zeta_{jk} = C_{jk} K_0(m_{jk}),
\] (B.6)
where $C_{jk}$ is some complex coefficient and $K_0(m)$ is given by
\[
K_0(m) = \int_{-\infty}^{\infty} \frac{dp}{2\sqrt{p^2 + m^2}} \exp \left( -\sqrt{p^2 + m^2} \right).
\] (B.7)

In the limit $m \to \infty$, $K_0(m)$ has the asymptotic expansion
\[
K_0(m) \approx \sqrt{\frac{\pi}{2m}} e^{-m} F(m),
\] (B.8)
where
\[
F(m) = 1 + \sum_{l=1}^{\infty} \frac{1}{l!} [(2l - 1)!!]^2 (-8m)^{-l}.
\] (B.9)

Since $h \sim |\lambda|^{-1}$, the authors of [13] have suggested that this expansion may be interpreted as loop corrections to the one-instanton process. In figs. 7, 8 the first seven terms in the above expansion were used to calculate the asymptotic behavior displayed.

Transforming this result for $\zeta$ back into the original point basis leads directly to the results given in (6.10) and (6.11).
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Figure Captions

Fig. 1. Eigenvalues of the Ramond charge matrix as a function of the coupling $\kappa$

Fig. 2. Eigenvalues of the algebraic $Q$-matrix as a function of the coupling $\kappa$

Fig. 3. Plot of $g_{0\bar{0}}$ (in the point basis) as a function of the coupling $\kappa$

Fig. 4. Plot of $|g_{0\bar{1}}|$ (in the point basis) as a function of the coupling $\kappa$

Fig. 5. Plot of $g_{1\bar{1}}$ (in the point basis) as a function of the coupling $\kappa$

Fig. 6. Plot of $g_{2\bar{2}}$ (in the point basis) as a function of the coupling $\kappa$

Fig. 7. Plot of $-\ln(|g_{0\bar{2}}|)$ (in the point basis) as a function of the coupling $\kappa$

Fig. 8. Plot of $-\ln(|g_{1\bar{2}}|)$ (in the point basis) as a function of the coupling $\kappa$