STRONG ULTRA-REGULARITY PROPERTIES FOR
POSITIVE ELEMENTS IN THE TWISTED
CONVOLUTIONS

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Abstract. We show that positive elements with respect to the
twisted convolutions, belonging to some ultra-test function space
of certain order at origin, belong to the ultra-test function space
of the same order everywhere. We apply the result to positive
semi-definite Weyl operators.

0. Introduction

Several issues in operator theory can be studied by means of the
twisted convolution. For example, composition and positivity ques-
tions can be carried over to related questions for the twisted convolu-
tion product by simple manipulations. We notice the simple structure
of the twisted convolution, since it essentially consists of a convolution
product, disturbed by a (symplectic) Fourier kernel. It is also com-
mon that boundedness and regularity conditions on operator kernels
often correspond to convenient conditions on related elements in the
twisted convolution. For example, operator kernels which belong to the
Schwartz space $\mathcal{S}$, or the Gelfand-Shilov spaces $\mathcal{S}_s$ or $\Sigma_s$ of Roumieu
and Beurling types, respectively, carry over to elements in the same
class in the twisted convolution. (See Section 1 for notations.)

In [8] it is shown that various kinds of singularities for positive el-
ements with respect to the twisted convolution are attained at the
origin. Furthermore, it is proved that regularity at origin for such ele-
ments impose global regularity and boundedness for these elements and
their Fourier transforms.

More precisely, if $a \in \mathcal{D}'$ is positive semi-definite with respect to the
twisted convolution, then it is proved that the following is true:

(1) $a \in \mathcal{S}'$ (cf. [8, Theorem 2.6]);
(2) if $\text{WF}_s(a)$ is any wave-front set of $a$ and $(0,Y) \notin \text{WF}_s(a)$,
then $(X,Y) \notin \text{WF}_s(a)$ and $(X,Y) \notin \text{WF}_s(\mathcal{F}_\sigma a)$. Here $\mathcal{F}_\sigma$ is
the symplectic Fourier transform (cf. [8, Theorem 4.14] and [9,
Theorem 4.1]);

Key words and phrases. ultra-distributions, twisted convolution, Hermite series
expansions, Weyl quantization.
(3) if \( a \) is continuous at origin, then \( a \) and its Fourier transform \( \hat{a} \) are continuous everywhere and belong to \( L^2 \) (cf. [8, Theorem 3.13]);

(4) if \( a \in C^\infty \) near origin, then \( a \in \mathcal{S} \) (cf. [8, Theorem 3.13]);

(5) if \( s \geq 0, a \in C^\infty \) near origin and

\[
|\partial^\alpha a(0)| \lesssim h^{|\alpha|} \alpha!^s
\]

for some \( h > 0 \) (for every \( h > 0 \)), then \( a \in \mathcal{S}_s \) (\( a \in \Sigma_s \)) (cf. [11, Theorem 4.1]).

We note that if (0.1) holds true with \( s < 1/2 \) in (5), then \( a \) is trivially equal to 0, since the Gelfand-Shilov spaces \( \mathcal{S}_s \) and \( \Sigma_s \) are trivial for such choices of \( s \).

In this paper we investigate related questions in background of Pilipović spaces, \( \mathcal{S}_s \) and \( \Sigma_s \) of Roumieu and Beurling type respectively, a family of function spaces which agrees with corresponding Gelfand-Shilov spaces when these are non-trivial (cf. [6,7]). We introduce the so-called twisted Pilipović spaces \( \mathcal{S}_{\sigma,s} \) and \( \Sigma_{\sigma,s} \) which are symplectic analogies of Pilipović spaces, and show that they are homeomorphic to \( \mathcal{S}_s \) and \( \Sigma_s \), respectively. We also show that

\[
\mathcal{S}_{\sigma,s} = \mathcal{S}_s = \Sigma_s
\]

when the right-hand side is non-trivial, and similarly for corresponding spaces of Beurling types.

We consider norm conditions of powers of a second order partial differential operator \( H_\sigma \) and its conjugate. These operators are symplectic analogies to certain partial harmonic oscillators. We show that \( H_\sigma \) and \( \tilde{H}_\sigma \) commute and can be used to characterize \( \mathcal{S}_{\sigma,s} \) and \( \Sigma_{\sigma,s} \) as

\[
a \in \mathcal{S}_{\sigma,s} \ \text{(} a \in \Sigma_{\sigma,s} \text{)} \iff \| H^N_\sigma \tilde{H}^N_\sigma a \|_{L^\infty} \lesssim h^N(\alpha!)^4s
\]

(0.2)

for some \( h > 0 \) (for every \( h > 0 \)). In Section 3 we show that if \( a \) is positive semi-definite with respect to the twisted convolution, then the relaxed condition

\[
|H^N_\sigma \tilde{H}^N_\sigma a(0)| \lesssim h^N(\alpha!)^4s
\]

of the right-hand of (0.2) is enough to ensure that \( a \) should belong to \( \mathcal{S}_{\sigma,s} \) or \( \Sigma_{\sigma,s} \).

1. Preliminaries

In the first part we recall definitions of twisted convolution, the Weyl quantization and positivity in operator theory, and discuss basic properties. The verifications are in general omitted since they can be found in e.g. [8]. Thereafter we recall the definitions of Gelfand-Shilov and Pilipović spaces and discuss some properties. Here we also consider related symplectic analogies of such spaces, defined in terms of Wigner
distributions of Hermite functions, considered by Wong in [12][13]. Finally we recall some results in [1] on positivity with respect to the twisted convolution.

1.1. Operators and positivity. Let \( a \) and \( b \) belong to \( \mathcal{S}(\mathbb{R}^{2d}) \), the set of Schwartz functions on \( \mathbb{R}^{2d} \). Then the twisted convolution of \( a \) and \( b \) is given by

\[
(a \ast_\sigma b)(X) = (2/\pi)^{d/2} \int_{\mathbb{R}^{2d}} a(X-Y)b(Y)e^{2i\sigma(X,Y)} \, dY.
\]

Here \( \sigma \) is the symplectic form on \( \mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d} \), given by

\[
\sigma(X,Y) \equiv \langle y,\xi \rangle - \langle x,\eta \rangle, \quad X = (x,\xi) \in \mathbb{R}^{2d}, \ Y = (y,\eta) \in \mathbb{R}^{2d}.
\]

The definition of \( \ast_\sigma \) extends in different ways. For example, the map \((a,b) \mapsto a \ast_\sigma b\) from \( C_0^\infty(\mathbb{R}^{2d}) \times C_0^\infty(\mathbb{R}^{2d})\) to \( C_0^\infty(\mathbb{R}^{2d})\) is uniquely extendable to a continuous map from \( \mathcal{S}'(\mathbb{R}^{2d}) \times \mathcal{S}'(\mathbb{R}^{2d}) \) to \( \mathcal{S}'(\mathbb{R}^{2d}) \), and from \( \mathcal{S}'(\mathbb{R}^{2d}) \times C_0^\infty(\mathbb{R}^{2d}) \) to \( \mathcal{S}'(\mathbb{R}^{2d}) \).

There are strong links between the twisted convolution, and continuity and composition properties in operator theory. This also include analogous questions in the theory of pseudo-differential operators.

In fact, by straight-forward computations it follows that

\[
A(a \ast_\sigma b) = (Aa) \circ (Ab),
\]

(1.1)

where \( A \) is the operator defined by the formula

\[
(Aa)(x,y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} a((y-x)/2,\xi)e^{-i(x+y,\xi)} \, d\xi.
\]

(1.2)

(Here and in what follows we identify operators with their kernels.) We note that

\[
(Aa)(x,\xi) = (\mathcal{F}^{-1}a((y-x)/2,\cdot))(\xi + y),
\]

where \( \mathcal{F} \) is the Fourier transform on \( \mathcal{S}'(\mathbb{R}^d) \) which takes the form

\[
\mathcal{F} f(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi} \, dx
\]

when \( f \in \mathcal{S}(\mathbb{R}^d) \). Alternatively we may reformulate this identity as

\[
(Aa)(x,\xi) = (\mathcal{F}_2^{-1}a)((y-x)/2,-(x+y)),
\]

where \( \mathcal{F}_2 \) is the partial Fourier transform of \( \Phi(x,\eta) \) with respect to the \( y \)-variable. Evidently, the mappings \( \mathcal{F}_2 \) and the pullback which takes \( \Phi(x,\xi) \) into

\[
\Phi((y-x)/2,-(x+y))
\]

are homeomorphisms on \( \mathcal{S}(\mathbb{R}^{2d}) \) and on \( \mathcal{S}'(\mathbb{R}^{2d}) \), and unitary on \( L^2(\mathbb{R}^{2d}) \). Hence similar facts hold true for \( A \).

From these mapping properties it follows that if \( a \in \mathcal{S}'(\mathbb{R}^{2d}) \), then \( Aa \) is a linear and continuous operator from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \). Furthermore, by the kernel theorem of Schwartz it follows that any linear
and continuous operator from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \) is given by \( Aa \), for a uniquely determined \( a \in \mathcal{S}'(\mathbb{R}^{2d}) \).

At this stage we also note that (1.1) remains true, if more generally, \( a \in \mathcal{S}'(\mathbb{R}^{2d}) \) and \( b \in \mathcal{S}(\mathbb{R}^{2d}) \), which follows by straight-forward computations.

The operator \( A \) can also in convenient ways be formulated in the framework of the Weyl calculus of pseudo-differential operators. More precisely, the Weyl quantization \( \text{Op}^w(a) \) of \( a \in \mathcal{S}(\mathbb{R}^{2d}) \) (the symbol) is the operator from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}(\mathbb{R}^d) \) given by

\[
\text{Op}^w(a)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a((x+y)/2, \xi) f(y) e^{i(x-y, \xi)} \, dy \, d\xi.
\]

The definition of \( \text{Op}^w(a) \) extends in continuous and similar ways as for \( Aa \) to any \( \mathcal{S}'(\mathbb{R}^{2d}) \), and then \( \text{Op}^w(a) \) is continuous from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \). This extension can also be performed by the relation

\[
\text{Op}^w(a) = (2\pi)^{-d/2} A(\mathcal{F}_\sigma a)
\]

which follows by straight-forward computations. Here \( \mathcal{F}_\sigma \) is the symplectic Fourier transform on \( \mathcal{S}'(\mathbb{R}^{2d}) \), which takes the form

\[
(\mathcal{F}_\sigma a)(X) = \pi^{-d} \int_{\mathbb{R}^{2d}} a(Y) e^{2i\sigma(X,Y)} \, dY
\]

when \( a \in \mathcal{S}(\mathbb{R}^{2d}) \).

From these facts it follow that the Weyl product \( \# \), defined by

\[
\text{Op}^w(a \# b) = \text{Op}^w(a) \circ \text{Op}^w(b)
\]

is given by

\[
a \# b = (2\pi)^{d/2} a \ast_\sigma (\mathcal{F}_\sigma b)
\]

which again links the twisted convolution to compositions in operator theory.

There are also strong links between positivity for the twisted convolution and positivity in operator theory. We recall that a continuous and linear operator \( T \) from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \) (from \( C_0^\infty(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \)) is called positive semi-definite, whenever \( (Tf, f) \geq 0 \) for every \( f \in \mathcal{S}(\mathbb{R}^d) \) (\( f \in C_0^\infty(\mathbb{R}^d) \)), and then we write \( T \geq 0 \). Since \( C_0^\infty(\mathbb{R}^d) \) is dense in \( \mathcal{S}(\mathbb{R}^d) \), it follows that an operator from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \) is positive semi-definite, if it is positive semi-definite as an operator from \( C_0^\infty(\mathbb{R}^d) \) to \( \mathcal{S}'(\mathbb{R}^d) \).

Positivity for the twisted convolution is defined in an analogous way. That is, an element \( a \in \mathcal{S}'(\mathbb{R}^{2d}) \) (\( a \in \mathcal{S}'(\mathbb{R}^{2d}) \)) is positive semi-definite with respect to the twisted convolution, whenever \( (a \ast_\sigma \varphi, \varphi) \geq 0 \) for every \( \varphi \in \mathcal{S}(\mathbb{R}^{2d}) \) (\( \varphi \in C_0^\infty(\mathbb{R}^{2d}) \)). As above it follows that \( a \in \mathcal{S}'(\mathbb{R}^{2d}) \) is positive semi-definite with respect to \( \ast_\sigma \), if it is positive semi-definite as an element in \( \mathcal{S}'(\mathbb{R}^{2d}) \).

The following proposition explains the links between positivity in operator theory and positivity for the twisted convolution. Here \( W_{f,g} \)
is the Wigner distribution of $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, given by
\[ W_{f,g} \equiv A^{-1}(\hat{f} \otimes \hat{g}). \] If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $W_{f,g}$ takes the form
\[ W_{f,g}(x, \xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x - y/2)g(x + y/2)e^{iy\cdot\xi} \, dy. \]

**Proposition 1.1.** Let $a \in \mathcal{S}'(\mathbb{R}^d)$. Then the following conditions are equivalent:

1. $a$ is positive semi-definite with respect to the twisted convolution;
2. $\text{Op}^w(\mathbb{F}_a)$ is a positive semi-definite operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$;
3. $\text{Op}^w(\mathbb{F}_a)$ is a positive semi-definite operator from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$;
4. $(\mathbb{F}_a, W_{f,f}) \geq 0$ for every $f \in \mathcal{S}(\mathbb{R}^d)$.

**1.2. Gelfand-Shilov spaces.** Let $h, s \in \mathbb{R}_+$ be fixed. Then $\mathcal{S}_{s,h}(\mathbb{R}^d)$ is the set of all $f \in C^\infty(\mathbb{R}^d)$ such that
\[ \|f\|_{S_{s,h}} \equiv \sup \frac{|x^\alpha \partial^\beta f(x)|}{h^{\alpha + \beta} (\alpha! \beta!)^s} \]
is finite. Here the supremum is taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$.

The set $\mathcal{S}_{s,h}(\mathbb{R}^d)$ is a Banach space which increases with $h$ and $s$, and is contained in $\mathcal{S}(\mathbb{R}^d)$. If $s > 1/2$, then $\mathcal{S}_{s,h}$ and $\bigcup_{h>0} \mathcal{S}_{1/2,h}$ are dense in $\mathcal{S}$. Hence, the dual $(\mathcal{S}_{s,h})'(\mathbb{R}^d)$ of $\mathcal{S}_{s,h}(\mathbb{R}^d)$ is a Banach space which contains $\mathcal{S}'(\mathbb{R}^d)$.

The **Gelfand-Shilov spaces** $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ are the inductive and projective limits respectively of $\mathcal{S}_{s,h}(\mathbb{R}^d)$ with respect to $h > 0$. Consequently
\[ \mathcal{S}_s(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma_s(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}_{s,h}(\mathbb{R}^d), \]
The space $\Sigma_s(\mathbb{R}^d)$ is a Fréchet space with semi norms $\| \cdot \|_{S_{s,h}}, h > 0$. Moreover, $\mathcal{S}_s(\mathbb{R}^d) \neq \{0\}$, if and only if $s \geq 1/2$, and $\Sigma_s(\mathbb{R}^d) \neq \{0\}$, if and only if $s > 1/2$.

If $\varepsilon > 0$ and $s > 0$, then
\[ \Sigma_s(\mathbb{R}^d) \subseteq \Sigma_s(\mathbb{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbb{R}^d). \]

The **Gelfand-Shilov distribution spaces** $\mathcal{S}'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$ are the projective and inductive limits respectively of $\mathcal{S}'_{s,h}(\mathbb{R}^d)$. Hence
\[ \mathcal{S}'_s(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}'_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}'_{s,h}(\mathbb{R}^d). \]

By [3], $\mathcal{S}'_s$ and $\Sigma'_s$ are the duals of $\mathcal{S}_s$ and $\Sigma_s$, respectively.

The Gelfand-Shilov spaces and their duals are invariant under translations, dilations, (partial) Fourier transformations and under several other important transformations. In fact, by straight-forward computations it follows that the properties and results in Subsection 1.1 hold
true with $\mathcal{S}_s$ and $\mathcal{S}'_s$ in place of $\mathcal{S}$ and $\mathcal{S}'$, respectively, when $s \geq 1/2$, or with $\Sigma_s$ and $\Sigma'_s$ in place of $\mathcal{S}$ and $\mathcal{S}'$, respectively, when $s > 1/2$.

1.3. The Pilipović spaces. We start to consider spaces which are obtained by suitable estimates of Gelfand-Shilov or Gevrey type when using powers of the harmonic oscillator $H = |x|^2 - \Delta$, $x \in \mathbb{R}^d$. In general we omit the arguments, since more thorough exposition is available in e.g. [11].

Let $s \geq 0$ and $h > 0$. Then $\mathcal{S}_{h,s}(\mathbb{R}^d)$ is the Banach space which consists of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$
\|f\|_{\mathcal{S}_{h,s}} \equiv \sup_{N \geq 0} \frac{\|H^N f\|_{L^\infty}}{h^N (N!)^{2s}} < \infty. \quad (1.3)
$$

If $h_\alpha$ is the Hermite function

$$
h_\alpha(x) = \pi^{-\frac{d}{4}} (-1)^{\lfloor |\alpha| \rfloor} (2^{|\alpha|} \alpha!)^{-\frac{1}{2}} e^{\frac{|x|^2}{4}} (\partial e^{-|x|^2}) \quad (1.4)
$$
on $\mathbb{R}^d$ of order $\alpha$, then $H h_\alpha = (2|\alpha| + d) h_\alpha$. This implies that $\mathcal{S}_{h,s}(\mathbb{R}^d)$ contains all Hermite functions when $s > 0$, and if $s = 0$ and $\alpha \in \mathbb{N}^d$ satisfies $2|\alpha| + d \leq h$, then $h_\alpha \in \mathcal{S}_{h,s}(\mathbb{R}^d)$.

We let

$$
\Sigma_s(\mathbb{R}^d) \equiv \bigcap_{h > 0} \mathcal{S}_{h,s}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{S}_s(\mathbb{R}^d) \equiv \bigcup_{h > 0} \mathcal{S}_{h,s}(\mathbb{R}^d),
$$

and equip these spaces by projective and inductive limit topologies, respectively, of $\mathcal{S}_{h,s}(\mathbb{R}^d)$, $h > 0$. (Cf. [4, 6, 7, 11].)

The space $\Sigma_s(\mathbb{R}^d)$ is called the Pilipović space (of Beurling type) of order $s \geq 0$ on $\mathbb{R}^d$. Similarly, $\mathcal{S}_s(\mathbb{R}^d)$ is called the Pilipović space (of Roumieu type) of order $s \geq 0$ on $\mathbb{R}^d$. Evidently, $\Sigma_0(\mathbb{R}^d)$ is trivially equal to $\{0\}$, while

$h_\alpha \in \mathcal{S}_s(\mathbb{R}^d)$, when $s \geq 0$ and $h_\alpha \in \Sigma_s(\mathbb{R}^d)$, when $s > 0$.

The dual spaces of $\mathcal{S}_{h,s}(\mathbb{R}^d)$, $\Sigma_s(\mathbb{R}^d)$ and $\mathcal{S}_s(\mathbb{R}^d)$ are denoted by $\mathcal{S}'_{h,s}(\mathbb{R}^d)$, $\Sigma'_s(\mathbb{R}^d)$ and $\mathcal{S}'_s(\mathbb{R}^d)$, respectively. We have

$$
\Sigma'_s(\mathbb{R}^d) = \bigcup_{h > 0} \mathcal{S}'_{h,s}(\mathbb{R}^d)
$$
when $s > 0$ and

$$
\mathcal{S}'_s(\mathbb{R}^d) = \bigcap_{h > 0} \mathcal{S}'_{h,s}(\mathbb{R}^d)
$$
when $s \geq 0$, with inductive respective projective limit topologies of $\mathcal{S}'_{h,s}(\mathbb{R}^d)$, $h > 0$ (cf. [11]).

\footnote{The boldface characters $\Sigma_s$, $\mathcal{S}_s$, etc. denote Pilipović spaces, and non-boldface characters $\Sigma_s$, $\mathcal{S}_s$, etc. denote analogous Gelfand-Shilov spaces.}
Let $s > 0$ and $\varepsilon > 0$. Then

\[
\mathcal{S}_0(\mathbb{R}^d) \subseteq \Sigma_0(\mathbb{R}^d) \subseteq \mathcal{S}_s(\mathbb{R}^d) \subseteq \Sigma_{s+\varepsilon}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d)
\]

\[
\subseteq \mathcal{S}'(\mathbb{R}^d) \subseteq \Sigma'_{s+\varepsilon}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d) \subseteq \Sigma'_{s}(\mathbb{R}^d) \subseteq \mathcal{S}'_0(\mathbb{R}^d).
\]

(1.5)

Furthermore, in [11] it is proved that $\mathcal{S}_0(\mathbb{R}^d)$ consists of all finite linear combinations of Hermite functions, while $\mathcal{S}'_0(\mathbb{R}^d)$ consists of all formal series

\[
f = \sum_{\alpha \in \mathbb{N}^d} c_\alpha h_\alpha, \quad c_\alpha = c_\alpha(f) = (f, h_\alpha)_{L^2}.
\]

(1.6)

The next propositions show that Pilipović spaces can be characterized by Hermite coefficients $c_\alpha$ given by (1.6). The proofs can be found in [2][11]. Here $H_1 U$ and $H_2 U$ are the partial harmonic oscillators given by

\[
H_1 U(x, y) = (|x|^2 - \Delta_x)U(x, y), \quad H_2 U(x, y) = (|y|^2 - \Delta_y)U(x, y).
\]

(1.7)

**Proposition 1.2.** Let $s \geq 0$ ($s > 0$) and $f \in \mathcal{S}'_0(\mathbb{R}^d)$ be given by (1.6). Then the following conditions are equivalent:

1. $f \in \mathcal{S}_s(\mathbb{R}^d)$ ($f \in \Sigma_s(\mathbb{R}^d)$);
2. $|c_\alpha(f)| \lesssim e^{-r|\alpha|}h$ for some $r > 0$ (for every $r > 0$).

**Proposition 1.3.** Let $p, q \in (0, \infty]$, $p_0 \in [1, \infty]$, $s \geq 0$ ($s > 0$), $U \in \mathcal{S}'_0(\mathbb{R}^{2d})$, and $H_1$ and $H_2$ be given by (1.7). Then the following conditions are equivalent:

1. $U \in \mathcal{S}_s(\mathbb{R}^{2d})$ ($U \in \Sigma_s(\mathbb{R}^{2d})$);
2. $\|H_1^{N_1} H_2^{N_2} U\|_{L^{p_0}} \lesssim h^{N_1 + N_2 (N_1! N_2)!} 2s$ for some $h > 0$ (for every $h > 0$);
3. $\|H_1^{N_1} H_2^{N_2} U\|_{M^{p,q}} \lesssim h^{N_1 + N_2 (N_1! N_2)!} 2s$ for some $h > 0$ (for every $h > 0$).

**Remark 1.4.** Let $\mathcal{S}_s$ and $\Sigma_s$ be the Gelfand-Shilov spaces of order $s \geq 0$. Then it is proved in [3][7] that

\[
\mathcal{S}_{s_1} = \mathcal{S}_{s_1}, \quad \Sigma_{s_2} = \Sigma_{s_2}, \quad s_1 \geq \frac{1}{2}, \quad s_2 > \frac{1}{2}
\]

and

\[
\mathcal{S}_{s_1} \neq \mathcal{S}_{s_1} = \{0\}, \quad \Sigma_{s_2} \neq \Sigma_{s_2} = \{0\}, \quad s_1 < \frac{1}{2}, \quad 0 < s_2 \leq \frac{1}{2}.
\]

**Remark 1.5.** In [11] it is proved that $\mathcal{S}_{s_1}$ and $\Sigma_{s_2}$ are not invariant under dilations when $s_1 < 1/2$ and $s_2 \leq 1/2$.

**Remark 1.6.** Let the hypothesis in Proposition 1.3 be fulfilled. By letting $N_1 = N_2 = N$ we get

\[
(2)' \quad \|H_1^N H_2^N U\|_{L^{p_0}} \lesssim h^{N N! 4s} \text{ for some } h > 0 \text{ (for every } h > 0);\]

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(3') \( \|H_1^N H_2^N U\|_{M^p,q} \lesssim h^N N^{14s} \) for some \( h > 0 \) (for every \( h > 0 \)).

The same arguments as in \[2\] imply that these conditions are equivalent. Furthermore, let \( \tilde{S}_s(R^{2d}) = \Sigma_s(R^{2d}) \) be the set of all \( U \in S'_0(R^{2d}) \) such that
\[
|c_\alpha(U)| \lesssim e^{-r(\langle \alpha_1 \rangle \langle \alpha_2 \rangle)} h^N N^4 s,
\]
for some \( r > 0 \) (for every \( r > 0 \)). Then it follows by similar arguments as in \[2\] that
\[
(2) \iff (3) \iff U \in \tilde{S}_s(R^{2d}) (U \in \tilde{\Sigma}_s(R^{2d})).
\]

We note that \( S_s \subseteq \tilde{S}_s \subseteq S_{2s} \) with strict inclusions.

2. Twisted Pilipović spaces and their properties

In this section we introduce twisted Pilipović spaces as the counter images of the operator \( A \) on Pilipović spaces, and deduce some basic properties. We also consider their distribution spaces.

We begin with some definitions.

Definition 2.1. The Hermite-Wong function of order \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^d \times \mathbb{N}^d \simeq \mathbb{N}^{2d} \) on \( R^{2d} \) is given by
\[
\varrho_\alpha \equiv A^{-1}(h_{\alpha_1} \otimes h_{\alpha_2}) = A^{-1}(h_{\alpha_1} \otimes \overline{h_{\alpha_2}}) = (-1)^{|\alpha_1|} W_{h_{\alpha_1}, h_{\alpha_2}}.
\]

The Hermite-Wong functions were studied in different ways by M. W. Wong in \[12,13\]. By the definition it follows that
\[
\mathcal{F}_\sigma \varrho_{\alpha_1, \alpha_2} = (-1)^{|\alpha_1|} \varrho_{\alpha_1, \alpha_2},
\]
which follows from the fact that \( \mathcal{F}_\sigma(W_{f,g}) = W_{\tilde{f},\tilde{g}} \) (see e.g. \[?5\]). Here \( \tilde{f}(x) = f(-x) \).

Definition 2.2. Let \( s > 0 \).

(1) The set \( S'_{\sigma,0}(R^{2d}) \) consists of all formal expansions
\[
a = \sum_{\alpha} c_\alpha \varrho_\alpha,
\]
where \( \{c_\alpha\}_{\alpha \in \mathbb{N}^{2d}} \subseteq \mathbb{C} \).

(2) The set \( S_{\sigma,0}(R^{2d}) \) consists of all expansions in \[2.1\] such that \( c_\alpha \) are non-zero for at most finite numbers of \( \alpha \).
(3) The set $\mathcal{S}_{\sigma,s}(\mathbb{R}^{2d})$ consists of all expansions in (2.1) such that

$$|c_\alpha| \lesssim e^{-c|\alpha|^s}$$

for some $c > 0$ (for every $c > 0$).

(4) The set $\mathcal{S}'_{\sigma,s}(\mathbb{R}^{2d})$ consists of all expansions in (2.1) such that

$$|c_\alpha| \lesssim e^{c|\alpha|^s}$$

for every $c > 0$ (for some $c > 0$).

The spaces in Definition 2.2 are equipped by topologies in similar way as for the Pilipović spaces in [11].

The set $\mathcal{S}_{\sigma,s}(\mathbb{R}^{2d})$ is called the twisted Pilipović space of Roumieu type (Beurling type) of order $s$. It follows that the sets $\mathcal{S}_{\sigma,s}(\mathbb{R}^{2d})$ and $\mathcal{S}'_{\sigma,s}(\mathbb{R}^{2d})$ are corresponding distribution spaces, since similar facts hold true for Pilipović space [11].

We extend the definition of $A$ on $\mathcal{S}$ by letting

$$Aa = \sum_\alpha c_\alpha h_{\alpha}$$

when $a \in \mathcal{S}'_{\sigma,0}(\mathbb{R}^{2d})$ is giving by (2.1). It follows that $A$ is a homeomorphism from $\mathcal{S}_{\sigma,s}(\mathbb{R}^{2d})$ to $\mathcal{S}_{\sigma,s}(\mathbb{R}^{2d})$, from $\mathcal{S}_{\sigma,s}(\mathbb{R}^{2d})$ to $\mathcal{S}_{s}(\mathbb{R}^{2d})$, and similarly for their duals. Since it is clear that $A$ is homeomorphism on any Fourier invariant Gelfand-Shilov spaces, we get

$$\mathcal{S}_{\sigma,s}(\mathbb{R}^{2d}) = \mathcal{S}_{\sigma,s}(\mathbb{R}^{2d}) = \mathcal{S}_{s}(\mathbb{R}^{2d}), \text{ when } s \geq 1/2$$

and

$$\Sigma_{\sigma,s}(\mathbb{R}^{2d}) = \Sigma_{\sigma,s}(\mathbb{R}^{2d}) = \Sigma_{s}(\mathbb{R}^{2d}), \text{ when } s > 1/2,$$

and similarly for corresponding distribution spaces.

Remark 2.3. Let $a \in \mathcal{S}'_{\sigma,0}(\mathbb{R}^{2d})$ be as in (2.1). Since $A$ is a homeomorphism on $\mathcal{S}(\mathbb{R}^{2d})$ and on $\mathcal{S}''(\mathbb{R}^{2d})$, it follows from [10] that $a$ belongs to $\mathcal{S}(\mathbb{R}^{2d})$ if and only if $c_\alpha \lesssim (\langle x \rangle)^{-N}$ for every $N \geq 0$. In the same way, $a \in \mathcal{S}''(\mathbb{R}^{2d})$ if and only if $c_\alpha \lesssim (\langle x \rangle)^N$ for some $N \geq 0$.

Next we discuss the partial harmonic oscillators $H_1$ and $H_2$ in Proposition [1.3] and their counter images under the operator $A$. We let $H_\sigma$ be the operator on $\mathcal{H}(\mathbb{R}^{2d})$, given by

$$H_\sigma = (|X|^2 - \frac{1}{4}\Delta_X) + \langle \xi, D_x \rangle - \langle x, D_\xi \rangle, \quad X = (x, \xi) \in \mathbb{R}^{2d},$$

and we let $T_\sigma = H_\sigma \circ \bar{H}_\sigma$. Here we note that

$$\bar{H}_\sigma = (|X|^2 - \frac{1}{4}\Delta_X) - \langle \xi, D_x \rangle + \langle x, D_\xi \rangle.$$

The following lemma explains some spectral properties of the considered operators.
Lemma 2.4. Let \( s \geq 0 \). Then the following is true:

1. the Hermite-Wong functions \( \varrho_\alpha \) are eigenfunctions to \( H_\sigma, \bar{H}_\sigma \) and \( T_\sigma \), and

\[
H_\sigma \varrho_{\alpha_1,\alpha_2} = (2|\alpha_1| + d) \varrho_{\alpha_1,\alpha_2}, \quad \bar{H}_\sigma \varrho_{\alpha_1,\alpha_2} = (2|\alpha_2| + d) \varrho_{\alpha_1,\alpha_2},
\]

and

\[
T_\sigma \varrho_{\alpha_1,\alpha_2} = (2|\alpha_1| + d)(2|\alpha_2| + d) \varrho_{\alpha_1,\alpha_2};
\]

2. \( H_\sigma \) and \( \bar{H}_\sigma \) restrict to homeomorphisms on \( S_{\sigma,s}(\mathbb{R}^{2d}) \) and on \( \Sigma_{\sigma,s}(\mathbb{R}^{2d}) \);

3. the definitions of \( H_\sigma \) and \( \bar{H}_\sigma \) extend uniquely to homeomorphisms on \( \mathcal{S}^p(\mathbb{R}^{2d}) \), \( S^*_\sigma,s(\mathbb{R}^{2d}) \) and on \( \Sigma^*_\sigma,s(\mathbb{R}^{2d}) \).

For the proof, we shall make use of the operators

\[
Z_{1,j} = \frac{1}{2} \partial_{z_j} + \bar{z}_j, \quad \bar{Z}_{1,j} = \frac{1}{2} \partial_{\bar{z}_j} - z_j,
\]

\[
Z_{2,j} = \frac{1}{2} \partial_{\bar{z}_j} + z_j, \quad \bar{Z}_{2,j} = \frac{1}{2} \partial_{z_j} - \bar{z}_j,
\]

where

\[
z_j = x_j + i\xi_j, \quad \bar{z}_j = x_j - i\xi_j,
\]

\[
\partial_{z_j} = \partial_{x_j} - i\partial_{\xi_j}, \quad \partial_{\bar{z}_j} = \partial_{x_j} + i\partial_{\xi_j},
\]

(see [13, Section 22]). By similar arguments as in the proof of Theorem 22.1 in [13] we get

\[
Z_{1,j} \varrho_{\alpha_1,\alpha_2} = (2|\alpha_2,j|)^{1/2} \varrho_{\alpha_1,\alpha_2-e_j},
\]

\[
\bar{Z}_{1,j} \varrho_{\alpha_1,\alpha_2} = -(2|\alpha_2,j| + 2)^{1/2} \varrho_{\alpha_1,\alpha_2+e_j},
\]

\[
Z_{2,j} \varrho_{\alpha_1,\alpha_2} = -(2|\alpha_1,j|)^{1/2} \varrho_{\alpha_1-e_j,\alpha_2},
\]

\[
\bar{Z}_{2,j} \varrho_{\alpha_1,\alpha_2} = (2|\alpha_1,j| + 2)^{1/2} \varrho_{\alpha_1+e_j,\alpha_2},
\]

(2.3)

where \( e_1, \ldots, e_d \) is the standard basis in \( \mathbb{R}^d \), i.e., \( e_j = (\delta_{1,j}, \ldots, \delta_{d,j}) \), \( j = 1, \ldots, d \), and \( \delta_{i,j} \) is the Kroniker’s delta function.

In view of (2.3), the operators \( Z_{1,j} \) and \( Z_{2,j} \) can be considered as symplectic analogies of annihilation operators, \( \bar{Z}_{1,j} \) and \( \bar{Z}_{2,j} \) as symplectic analogies of creation operators.

Proof. First we prove (1). By straight-forward computations, we obtain

\[
H_\sigma = -\frac{1}{2} \left( \sum_j Z_{2,j} \bar{Z}_{2,j} + \bar{Z}_{2,j} Z_{2,j} \right)
\]

and

\[
\bar{H}_\sigma = -\frac{1}{2} \left( \sum_j Z_{1,j} \bar{Z}_{1,j} + \bar{Z}_{1,j} Z_{1,j} \right).
\]
Hence, by (2.3) we get

\[ H_\sigma g_{\alpha_1,\alpha_2} = (2|\alpha_1| + d)g_{\alpha_1,\alpha_2}, \]

and

\[ H_\sigma g_{\alpha_1,\alpha_2} = (2|\alpha_2| + d)g_{\alpha_1,\alpha_2}, \]

and (1) follows.

By (2.2), it follows that \( H_\sigma \) and \( \bar{H}_\sigma \) restrict to homeomorphisms on \( S_{\sigma,s}(\mathbb{R}^{2d}) \) and on \( \Sigma_{\sigma,s}(\mathbb{R}^{2d}) \), which gives (2).

If \( a \in S_{\sigma,s}'(\mathbb{R}^{2d}) \) and \( b \in S_{\sigma,s}(\mathbb{R}^{2d}) \). We now let \( H_\sigma \) be defined by

\[ (H_\sigma a, b)_{L^2} = (a, \bar{H}_\sigma b)_{L^2}, \]

as usual, which extends the definitions of \( H_\sigma \) and \( \bar{H}_\sigma \) to \( S_{\sigma,s}'(\mathbb{R}^{2d}) \). The extensions of these operators to \( \Sigma_{\sigma,s}'(\mathbb{R}^{2d}) \) and \( \mathcal{S}'(\mathbb{R}^{2d}) \) are performed in similar ways. By (2.2), it follows that these extensions are unique. \( \square \)

The next lemma shows important links between the latter operators and partial harmonic oscillators.

**Lemma 2.5.** Let \( H_1 \) and \( H_2 \) be as in Proposition 1.3 and let \( a \in S_{\sigma,s}(\mathbb{R}^{2d}) \). Then \( H_\sigma \) and \( \bar{H}_\sigma \) are commuting to each other, and

\[ A(H_\sigma^{N_1} \bar{H}_\sigma^{N_2} a) = H_1^{N_1} H_2^{N_2} (Aa), \]

for every integer \( N_1, N_2 \geq 0 \). In particular, if \( \{f_k\}_{k=1}^{\infty} \) and \( \{g_k\}_{k=1}^{\infty} \) are sequences in \( l^2(N; L^2(\mathbb{R}^d)) \), and \( a \) is given by

\[ a = \sum_{k=0}^{\infty} A^{-1}(f_k \otimes g_k), \]

then

\[ A(T_\sigma^N a) = \sum_{k=0}^{\infty} (H^N f_k) \otimes (\bar{H}^N g_k), \]

where the series convergences in \( \mathcal{S}'(\mathbb{R}^{2d}) \).

**Proof.** The commutation between \( H_\sigma \) and \( \bar{H}_\sigma \) follows if we prove (2.4). We recall the operators

\[ P_j = \frac{1}{2i} \partial \xi_j - x_j, \quad \Pi_j = \frac{1}{2i} \partial x_j + \xi_j, \]

\[ T_j = \frac{1}{2i} \partial \xi_j + x_j, \quad \Theta_j = \frac{1}{2i} \partial x_j - \xi_j, \]

and the relations

\[ A(P_j^2 a) = x_j^2 Aa, \quad A(\Pi_j^2 a) = -\partial_x^2 (Aa), \]

\[ A(T_j^2 a) = y_j^2 Aa, \quad A(\Theta_j^2 a) = -\partial_y^2 (Aa), \]

from [1, Theorem 4.1].
By straightforward computations we get
\[(x_j^2 - \partial^2_{x_j})(Aa) = A((P_j^2 + \Pi_j^2)a) = A(H_{\sigma,j}a),\]
where \(H_{\sigma,j} = (X_j^4 - \frac{1}{4} \Delta X_j) + \xi_j D_{x_j} - x_j D \xi_j.\)

Summing up over all \(j\) gives
\[H_1(Aa) = A(H_{\sigma}a).\]

In the same way we get
\[H_2(Aa) = A(\bar{H}_{\sigma}a),\]
and the result follows by induction. \(\square\)

From these mapping properties, Proposition 1.3 can now be carried over to the case of twisted Pilipović spaces as follows.

**Proposition 2.6.** Let \(p, q \in (0, \infty)\) and \(p_0 \in [1, \infty]\) and let \(s \geq 0\) \((s > 0)\). Then the following conditions are equivalent.

1. \(a \in S_{\sigma,s}^{(\mathbb{R}^d)} (a \in \Sigma_{\sigma,s}^{(\mathbb{R}^d)});\)
2. \(\|H_{\sigma}^N \bar{H}_{\sigma}^N a\|_{L^{p_0}} \lesssim h^{N_1 + N_2} (N_1! N_2!)^{2s} \) for some \(h > 0\) \((\text{for every } h > 0)\);
3. \(\|H_{\sigma}^N \bar{H}_{\sigma}^N a\|_{M^{p,q}} \lesssim h^{N_1 + N_2} (N_1! N_2!)^{2s} \) for some \(h > 0\) \((\text{for every } h > 0)\).

**Proof.** Let \(U = Aa\). Since \(M^{p_1}(\mathbb{R}^d) \subseteq M^{p_2}(\mathbb{R}^d) \subseteq M^{p_0}(\mathbb{R}^d)\), when \(p_1 = \min(p, q)\) and \(p_2 = \max(p, q)\), we may assume that \(p = q\).

Since \(A\) is a homeomorphism on \(M^p(\mathbb{R}^d)\), we get
\[\|H_{\sigma}^N \bar{H}_{\sigma}^N a\|_{M^p} = \|A(H_{\sigma}^N \bar{H}_{\sigma}^N a)\|_{M^p} = \|H_1^N \bar{H}_2^N U\|_{M^p},\]
and the equivalence between (3) and Proposition 1.3 (3) follows. The equivalence between (1) and (3) now follows from Proposition 1.3 and the fact that \(A\) is a homeomorphism from \(S_{\sigma,s}^{(\mathbb{R}^d)}\) to \(S_{(\mathbb{R}^d)}\).

Finally by the embeddings
\[M^1(\mathbb{R}^d) \subseteq L^{p_0}(\mathbb{R}^d) \subseteq M^{\infty}(\mathbb{R}^d),\]
the equivalence between (2) and (3) now follows. \(\square\)

**Corollary 2.7.** If \(s \geq 0\) and \(a \in S_{\sigma,s}^{(\mathbb{R}^d)} (a \in \Sigma_{\sigma,s}^{(\mathbb{R}^d)}),\) then
\[\|T_{\sigma}^N a\|_{L^\infty} \lesssim h^{2N} (N_1!)^{4s}, \quad (2.7)\]
holds for some \(h > 0\) \((\text{for every } h > 0)\).

Remark 1.6 and Lemma 2.5 show that (2.7) is necessary but not sufficient in order for \(a \in S_{\sigma,s}^{(\mathbb{R}^d)}\) or \(a \in \Sigma_{\sigma,s}^{(\mathbb{R}^d)}\).
3. Twisted Pilipović space property for positive elements with respect to the twisted convolution

We study positive elements with respect to twisted convolution in $\mathcal{S}'$, having the twisted Pilipović space regularities near the origin. We show that such elements are in $\mathcal{S}_{\sigma,s}$ or in $\Sigma_{\sigma,s}$.

The following theorem shows that the condition of the form (2.7) at origin is sufficient that the converse of Corollary 2.7 holds when dealing with positive semi-definite elements with respect to the twisted convolution.

**Theorem 3.1.** Let $s \geq 0$, $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $(a *_{\sigma} \psi, \psi) \geq 0$ for every $\psi \in \mathcal{S}(\mathbb{R}^{2d})$. If

$$(T_{\sigma}^N a)(0,0) \lesssim h^{2N}(N!)^{4s},$$

holds for some $h > 0$ (for every $h > 0$), then $a \in \mathcal{S}_{\sigma,s}(\mathbb{R}^{2d})$ ($a \in \Sigma_{\sigma,s}(\mathbb{R}^{2d})$).

**Proof.** By the assumption, we may write $a = \sum_k A^{-1}(f_k \otimes f_k)$. By Lemma 2.5, we obtain

$$A(T_{\sigma}^N a) = \sum_k (H^N f_k \otimes H^N f_k),$$

for some sequence $\{f_k\}_{k=0}^{\infty}$.

Let $K = \sum_k f_k \otimes f_k$ be the kernel of $Aa$. Then

$$\|H^N f_k \otimes H^N f_k\|_{L^2} \leq \|H^N f_k \otimes H^N f_k\|_{Tr} \leq \|A(T_{\sigma}^N a)\|_{Tr} = \sum_k \|H^N f_k\|_{L^2} = (\pi/2)^{d/2} (T_{\sigma}^N a)(0,0).$$

Thus by the assumption, we get

$$\|H^N f_k \otimes H^N f_k\|_{L^2} \lesssim h^{2N}(N!)^{4s},$$

for some $h > 0$ (for every $h > 0$), giving that $K \in \mathcal{S}_s(\mathbb{R}^{2d})$ ($K \in \Sigma_s(\mathbb{R}^{2d})$) in view of Proposition 1.3 and Remark 1.6. Hence $a \in \mathcal{S}_{\sigma,s}(\mathbb{R}^{2d})$ ($a \in \Sigma_{\sigma,s}(\mathbb{R}^{2d})$).

**Proposition 3.2.** Let $s \geq 0$ be real, and let $a \in \mathcal{S}'(\mathbb{R}^{2d})$ be such that $\text{Op}^s(a) \geq 0$. If

$$(T_{\sigma}^N \mathcal{F}a)(0) \lesssim h^{2N}(N!)^{4s},$$

(3.1)

holds for some $h > 0$ (for every $h > 0$), then $a \in \mathcal{S}_{\sigma,s}(\mathbb{R}^{2d})$ ($a \in \Sigma_{\sigma,s}(\mathbb{R}^{2d})$).

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