Asymptotic approach for the rigid condition of appearance of the oscillations in the solution of the Painleve-2 equation

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Abstract

The asymptotic solution for the Painleve-2 equation with small parameter is considered. The solution has algebraic behavior before point \( t^* \) and fast oscillating behavior after the point \( t^* \). In the transition layer the behavior of the asymptotic solution is more complicated. The leading term of the asymptotics satisfies the Painleve-1 equation and some elliptic equation with constant coefficients, where the solution of the Painleve-1 equation has poles. The uniform smooth asymptotics are constructed in the interval, containing the critical point \( t^* \).

In this work a special asymptotic solution on \( \epsilon \) at \( \epsilon \to 0 \) for the equation Painleve-2 is constructed:

\[
\epsilon^2 u'' + 2u^3 + tu = 1. \tag{1}
\]

The constructed solution describes a rigid condition of origin of fast oscillations in some moment \( t^* \). At the left, at \( t < t^* \), the asymptotic solution is algebraic, and on the right, at \( t > t^* \), this asymptotics is fast oscillating. An asymptotic solution in a transitional layer (small neighborhood of a critical point \( t^* \)) is investigated explicitly. The phase and phase shift of the oscillating asymptotics are calculated.

The qualitative behaviour of solutions of second-order ordinary differential equations according to an additional parameter is explained, for example, in the book [1]. In [1] the various types of bifurcations for equilibrium positions of conservative second-order ordinary differential equations are described also.

Consider the autonomous equation obtained from (1). If we "freeze" a value of variable coefficient \( t = T \) then we obtain the equation:

\[
\epsilon^2 V'' + 2V^3 + TV = 1. \tag{2}
\]
This equation has three solutions which are according to equilibrium positions of dynamical system for this equation, if the parameter $T$ is less than some bifurcation value $t_\ast$. Two of this equilibrium positions are stable and one position is unstable. If the parameter $T > t_\ast$ then the dynamical system has only one equilibrium position. This position is stable. The other solutions of the equation are described by some elliptic function. The construction of asymptotics of a solution for this differential equation near to a bifurcation point of a parameter $T$ is reduced to a research of the elliptic function asymptotics.

The equation (1) is nonautonomous and therefore the existence of equilibrium positions depends on the variable $t$. In a critical point $t = t_\ast$ the bifurcation of a "saddle-node" type happens and at $t > t_\ast$ one of stable "an equilibrium position" disappears. The bifurcation of such type leads to the rigid loss of a stability [2]. The "saddle - node" is the elementary bifurcation for the second-order ordinary differential equations. The solution for the equation (1) in the part of the neighborhood for the point of the "saddle-node" bifurcation is described by the equation Painleve-1. The solution of the Painleve-1 equation has poles. But the asymptotic solution of the equation (1) have not the poles. Therefore, near the poles of the solution for the Painleve-1 equation we can’t use the Painleve-1 equation. In such areas of the variable $t$ the asymptotic solution is described by some autonomous nonlinear equation.

The next in complexity bifurcation is the doubling originating in the equation Painleve-2. Asymptotics with respect to a small parameter of the solutions for the equation Painleve-2 with zero in the right hand side in the equation (1) were investigated in works [3], [4]. In this case solution in an interior layer near to a bifurcation value of the parameter $t_\ast$ is determined by the equation Painleve-2, but already without a small parameter; and the problem, generally speaking, does not become simpler.

The structure of the constructed asymptotic solution is various in various ranges of values of a parameter $t$. The asymptotic solutions, suitable at $t < t_\ast$ and in a small neighborhood $t_\ast$, are constructed using the direct method of the theory of perturbations [7]. The fast oscillating asymptotics at $t > t_\ast$ is constructed by the Krylov-Bogolyubov’s method [8], explained for nonlinear second-kind ordinary equations in the work by G.E. Kuzmak: [9] and justified by M.V. Fedoryuk in [10]. The various types of asymptotics are matched by the matching method for the asymptotic expansions [11].

The asymptotics of the solutions for the Painleve equations with a leading term as an elliptic function with the modulated parameters were studied, for example, in works [12] –[20]. The neighborhood of the degenerated point of the elliptic anatz in these works was not investigated. The scaling limit passages of the equation Painleve-2 to the equation Painleve-1 and to the sine-elliptic equation [12], [19] are known also. However, the asymptotic solutions constructed by this way, for example, in [14] are non-uniform on the variable $t$. The qualitative analysis for the relation between the algebraic and fast oscillating asymptotic solutions of the equation (1) was done in the work [21].

Here the equation Painleve-2 is considered as a model example for study of a connection of various types of movements circumscribed by one solution of
the nonlinear equation. When choosing the equation we take in account only its simply form and the statement about boundedness of a solutions for this equation \(5\). In this work the properties of an integrability of the equation \(1\) by the inverse scattering transform \(6\) are not used anywhere.

In this work we do not prove any theorems about the justification of the constructed asymptotics for the equation \(1\). However, in that part of researched area with respect to variable \(t\), where the formal asymptotic solution is oscillates rapidly, the justification is based on the results of the work \(10\).

1 **Naive statement of the problem**

Let’s consider a cubic equation obtained at a rejection of the term with the small parameter in the equation \(1\):

\[
2u^3 + tu = 1.
\]

This equation has three real roots \(u_1(t) < u_2(t) < u_3(t)\) at \(T < t_*\) and one real root at \(t > t_*\); if \(t = t_*\) then the roots \(u_1(t)\) and \(u_2(t)\) stick together \(u_1(t_*) = u_2(t_*) = u_*\). The values \(u_*\) and \(t_*\) are easy for obtaining, by solving of the equations:

\[
2u_*^3 + t_*u_* - 1 = 0, \quad 6u_*^2 + t_* = 0.
\]

At \(t < t_*\) it is possible to construct an asymptotic solution of the equation \(1\) by taking as a leading term of asymptotics any of the solutions for the equation \(3\). Thus the asymptotic solution with the leading term \(u_1(t)\) or \(u_3(t)\) is stable and if the leading term is \(u_2(t)\) then the asymptotics is unstable. It follows from the analysis of the linearized equation \(1\):

\[
\varepsilon^2v'' + (6u_*^2(t) + t)v = 0, \quad j = 1, 2, 3.
\]

The asymptotic solution with the leading term \(u_3(t)\) keeps its structure in the neighborhood of the bifurcation point \(t_*\) and after this point: at \(t > t_*\), unlike it, the structure of the asymptotic solution with the leading term \(u_1(t)\) varies sufficiently at passage through the bifurcation point \(t_*\).

Our problem is to construct a smooth formal asymptotic solution of the equation \(1\) on a segment \([t_* - a, t_* + a]\), \(a = \text{const} > 0\) with the leading term \(u_1(t)\) at \(t < t_*\).

2 **The main results**

In this work the smooth formal asymptotic solution of the equation \(1\) on a segment \(L = [t_* - a, t_* + a]\), \(a = \text{const} > 0\) is constructed. This solution has a various structure in the different areas of the segment \(L\).

At \((t_* - t) \gg \varepsilon^{4/5}\) the formal asymptotic solution has the form

\[
\begin{align*}
u(t, \varepsilon) &= u(t) + \varepsilon \frac{1}{2} u_1(t) + O(\varepsilon^4(t - t_*)^{-9/2}).
\end{align*}
\]
The leading term is least of the roots of the cubic equation \( \mathfrak{B} \). The gauge sequence in this case is following: \( \varepsilon^{2n}, \ n = 0, 1, 2, \ldots \). The coefficients of the asymptotic expansion are the algebraic functions with respect to the parameter \( t \).

At \( |t - t_*| \ll 1 \) the asymptotics is defined by two various types of the asymptotic expansions. One of them has the form

\[
\begin{align*}
  u(t, \varepsilon) &= u_* + \varepsilon^{2/5} \frac{\partial}{\partial \tau} v(\tau) + O(\varepsilon^{4/5} \tau) + O((\tau - \tau_k)^{-4} \varepsilon^{4/5}).
\end{align*}
\]  

The function \( v(\tau) \) is defined as the solution of the equation Painleve-1:

\[
\frac{d}{d\tau} v(\tau) + 6u_* v^2 + u_* \tau = 0,
\]

with the pure algebraic asymptotics at \( \tau \to -\infty \):

\[
0 v(t) = -\sqrt{-\tau/6} + O(\tau^{-2}).
\]

Here the variable \( \tau \) is define by the formula \( \tau = (t - t_*) \varepsilon^{-4/5} \).

The function \( v(\tau) \) has poles in some points \( \tau_k > 0 \). Outside of these poles the expansion \( \mathfrak{B} \) is suitable.

Near the poles of the function \( v(\tau) \) in the areas \( |\tau - \tau_k||\tau_k|^{1/5} \ll 1 \) the asymptotic solution of the equation \( \mathfrak{B} \) has an another structure:

\[
\begin{align*}
  u(t, \varepsilon) &= u_* + \frac{\partial}{\partial \theta} w(\theta) + O(\varepsilon^{4/5} \theta^2 |\tau_k|),
\end{align*}
\]  

where \( \theta = (\tau - \tau_k) \varepsilon^{-1/5} \). The function \( \frac{\partial}{\partial \theta} w(\theta) \) is defined by the formula:

\[
0 w(\theta) = \frac{-16u_*}{4 + 16u_*^2 \theta^2}.
\]

At \( (t - t_*) \varepsilon^{-4/5} \gg 1 \) the asymptotics have the fast oscillating behavior:

\[
\begin{align*}
  u(t, \varepsilon) &= U(t_1, t) + O(\varepsilon).
\end{align*}
\]  

The leading term of the asymptotic solution satisfies the equation

\[
(S')^2 \frac{\partial}{\partial t_1} U^2 = -U^4 - t U^2 + 2U + E(t),
\]

where \( t_1 = S(t)/\varepsilon \). The function \( E(t) \) is defined by the equation

\[
\int_{\alpha(t)}^{\beta(t)} \sqrt{-x^4 - tx^2 + 2x + E(t)} dx = \pi,
\]

where \( \alpha(t) \) and \( \beta(t) \) were solutions of the equation \(-x^4 - tx^2 + 2x + E(t) = 0\).
The phase function $S(t)$ is the solution of the Cauchy problem:

$$T = S' \sqrt{2} \int_{\alpha(t)}^{\beta(t)} \frac{dx}{\sqrt{-x^4 - tx^2 + 2x + E(t)}}, \quad S|_{t=t^*} = 0,$$

where $T$ is some constant defined in the formula (34).

**Note about the matching of the asymptotics.** The areas of usefulness for the asymptotic expansions (4) and (5), (5) and (6), and also (5), (6) and (7) are intersected. It allows to matching this asymptotics.

**Note about rigorously.** In areas, where the asymptotic expansions (4) and (6) are suitable, it is possible to construct the full asymptotic expansions with respect to parameter $\varepsilon$ and, probably, to justify them. The justification of the fast oscillating asymptotics (7) follows from the work [10]. A more complicated situation is with the expansion (5). The problem about the construction of the full asymptotic expansion of the type (5) is tightly associated with the problem about the perturbation of the equation the Painleve-1. As far as it is known, the problems about the perturbation of the Painleve equations are investigated insufficiently. The detailed research of the perturbed equation Painleve-1 is possible probably on the basis of the monodromy-preserving deformations method [6], but it goes out for the framework of the present work.

### 3 The qualitative analysis of the problem and of the results

We assume, that the constructed asymptotic solution of the equation (1) is an asymptotics of some true solution. The equation (1) is nonautonomous, for the qualitative analysis of its solutions at $t < t^*$, $t \sim t^*$, $t > T^*$ we shall take advantage the equation with the "frozen" value of the coefficient $t$ in the equation (1) in some point $t = T$. That is, we shall consider the equation (2). Integrate it once with respect to $T$, in the result we shall get:

$$\varepsilon^2 (V')^2 = -V^4 - TV + 2V + E, \quad E = \text{const}.$$

Depending on the value of the parameter $T$ the potential for the equation (2) has one of graphs:
At $T < t_*$ the points $V_1$ and $V_3$ are the point of a stable equilibrium, and $V_2$ is the unstable equilibrium. At $T = t_*$ the points $V_1$ and $V_2$ stick together and $u_*$ is the point of unstable equilibrium. At $T > t_*$ there is only one point of the equilibrium. Thus, at $T < t_*$ on the phase plane of the equation (2) there are two centers and one saddle; at $T = t_*$ one of centers sticks together to the saddle; at $T > t_*$ there is only one center. It is the bifurcation of the ”saddle - node” type.

Let’s explain the connection of figure 1 with the equation (1). It is easy to see, that at $t = T < t_*$ the value of the leading term of the algebraic asymptotic solution (the function $u_1(t)$) coincides with $V_1$. At $T = t_*$ the points $V_1$ and $V_2$ stick together and the solution $u(t, \varepsilon)$ begins to change with an energy which is close to $E_* = u_1^4 + t_*u_1^2 - 2u_*$. On the greater part of the trajectory the movement is fast and the characteristic variable is $\theta \sim \varepsilon^{-4}t$. On the slanting part (near to the point $u_*$) the characteristic variable is decelerated: $\tau = (t - t_*)\varepsilon^{-4/5}$. At $t > t_*$ the movement is close to periodic and characteristic fast variable is $t_1 = S(t)/\varepsilon$.

The results of the numerical resolving of the equation (1) by the Runge-Kutta method for the Cauchy problem with the entry conditions corresponding to the algebraic asymptotics at $t = -7$ and $\varepsilon = 0.2$ give the similar solution.
4 The exterior algebraic asymptotics

The algebraic asymptotic solution of the equation (1), which is suitable at \( t < t^*_0 \), is constructed here and its asymptotics is investigated at \( t \to t^*_0 - 0 \).

For the algebraic asymptotic solution of the equation (1) we search as:

\[
 u(t, \varepsilon) = u_0(t) + \varepsilon^2 \frac{1}{2} u_1(t) + \varepsilon^4 \frac{1}{2} u_2(t) + \ldots.
\]  

(8)

Let’s formulate the result of this section. The formal asymptotic solution (8), where \( u_0(t) \) is least of the solutions of the equation (3), is suitable at \( (t_0 - t) \varepsilon^{-4/5} \gg 1 \).

Begin to obtain the coefficients of the asymptotics (8). Substitute the anzatz (8) to the equation (1). Let’s equate the coefficients at identical powers of \( \varepsilon \). In the result we obtain the recurrence sequence of the formulas for the definition \( u_k(t) \), \( k = 0, 1, 2, \ldots \).

\[
 2 \frac{0}{u_3(t)} + t \frac{0}{u(t)} = 1, \quad (6 \frac{0}{u_2(t)} + t) \frac{1}{u_1(t)} = -\frac{0}{u''(T)},
\]

\[
 (6 \frac{0}{u_2(t)} + t) \frac{0}{u_2(t)} = -6 \frac{0}{u(t)} \frac{1}{u_1(t)} - \frac{1}{u(t)}.
\]

The cubic equation for \( \frac{0}{u(t)} \) at \( t < t^*_0 \) has three real roots \( u_1(t) < u_2(t) < u_3(t) \). As the leading term of asymptotic expansion (8) we choose \( u_1(t) \). Let’s calculate the second derivative of \( \frac{0}{u}(t) \) and express \( \frac{0}{u}(t) \) through \( \frac{1}{u}(t) \):
It is easy to get the expressions for the following terms of the asymptotics (8). In an explicit form they are not reduced here, however, it is important to notice, that the power of the denominator \((6^0 u^2(t) + t)\) in the coefficients of the asymptotics grows with each next step. At \((6^0 u^2(t) + t) \to 0\) n-th term of the asymptotic expansion has the form

\[
\begin{align*}
\frac{u}{n} \to O((6^0 u^2(t) + t)^{-5n+1}). \tag{10}
\end{align*}
\]

Let's write out the asymptotics of the asymptotic expansion (8) at \(t \to t_\star\).

For this purpose we shall calculate the asymptotics of the expression \((6^0 u^2(t) + t)\):

\[
(6^0 u^2(t) + t)|_{t \to t_\star} = -2u_\star \sqrt{6 \sqrt{t_\star - t} + \frac{2}{3}(t_\star - t) + \frac{-5}{9 \sqrt{6u_\star}}(t - t_\star)^{3/2}} + O((t_\star - t)^2).
\]

Using this formula and (8), (9), we get:

\[
\begin{align*}
u(t, \varepsilon) &= u_\star - \frac{1}{\sqrt{6 \sqrt{t_\star - t} + \frac{2}{3}(t_\star - t) + \frac{-5}{9 \sqrt{6u_\star}}(t - t_\star)^{3/2}}} + \epsilon^2 \left[-\frac{1}{32^{1/3}}(t_\star - t)^{-2} - O((t_\star - t)^{-3/2})\right] + O\left(\epsilon^4(t_\star - t)^{-9/2}\right) + O((t_\star - t)^{3/2}).
\end{align*}
\]

The area of usefulness for this expansion at \(t \to t_\star - 0\) is determined from the relation \(\epsilon^2 \frac{n}{u(t_\star)} \ll 1\). It follows from the formula (10), that the expansion (8) is suitable at \((t_\star - t)\varepsilon^{-4/5} \gg 1\).

5 The interior asymptotics

In this section the asymptotic expansions which are suitable in the small neighborhood of a point \(t_\star\) are constructed. Following the terminology of the matching method [11], they are called "the interior asymptotic expansions".

5.1 First interior expansion

In the neighborhood of the point \(t_\star\) we make the stretch, which are dictated by the asymptotics at \(t \to t_\star\) of the exterior expansion (8):

\[
(u - u_\star) = \varepsilon^{2/5} v, \quad (t - t_\star) = \varepsilon^{4/5} \tau.
\]

As result the equation (11) we write as

\[
\frac{d^2 v}{d\tau^2} + 6u_\star v^2 + u_\star \tau = -\varepsilon^{2/5}(\tau v + 2v^3). \tag{11}
\]

In the limit at \(\varepsilon \to 0\) we obtain the equation Painleve-1. This asymptotic reduction is known as one of the scaling limits for the Painleve-2 equation [18].
The asymptotic solution of the equation (11) we build as:

$$v(\tau, \varepsilon) = 0 v(\tau) + \varepsilon^{2/5} \frac{1}{5} v(\tau),$$  \hspace{1cm} (12)

where the function $0 v(\tau)$ is the solution of the Painleve-1 equation.

Here it is shown, that the asymptotic solution (12) is suitable in the neighborhood of infinity (at $\tau \ll \varepsilon^{-4/5}$) and in the neighborhood of the poles for the function $0 v(\tau)$: $(t - \tau_k)\varepsilon^{-1/5} \gg 1$.

The coefficients of the asymptotics are calculated from the matching condition for the asymptotic expansion (8) at $t \to t_*$ and the expansion (12) at $\tau \to -\infty$. In particular, $0 v(\tau)$ has the algebraic asymptotics:

$$0 v(\tau)|_{\tau \to -\infty} = -\frac{1}{\sqrt{6}} \sqrt{-\tau} + O(((-\tau)^{-2})).$$  \hspace{1cm} (13)

In the book [22] it is shown, that there is the solution of the Painleve-1 equation with the asymptotics (13). In the work [23] it is proved, that the solution of the Painleve-1 equation with the asymptotics (13) has not poles at $\tau \leq 0$. The data of a monodromy for the solution of the Painleve-1 equation with the asymptotics (13) are calculated in the work [24].

The first correction in the asymptotics (12) satisfies the equation

$$\frac{d^2}{d\tau^2} \frac{1}{5} v + 12u_* \frac{0 v}{v} = -\tau \frac{0 v}{v} - 2 \frac{0 v^3}{v^3}.$$  \hspace{1cm} (14)

The asymptotics of the solution for this equation at $\tau \to -\infty$ has the form

$$\frac{1}{5} v(\tau) = \frac{\tau}{18u_*} + O((-\tau)^{-3/2}).$$

The requirement of fitness for the asymptotics is $\varepsilon^{2/5} \frac{1}{5} v / 0 v \ll 1$. It reduces to the condition $(-\tau) \ll \varepsilon^{-4/5}$.

On the positive semiaxis the function $0 v(\tau)$ has the poles. Let’s denote this poles by $\tau_k$. In the neighborhood of the pole $\tau = \tau_k$ the function $0 v(\tau)$ is defined by the converging power series [22]

$$0 v(\tau) = \frac{-1}{u_*(\tau - \tau_k)^2} + \frac{\tau_k u_*}{10} (\tau - \tau_k)^2 + \frac{u_*}{6} (\tau - \tau_k)^3 + c_4 (\tau - \tau_k)^4 + O((\tau - \tau_k)^5).$$  \hspace{1cm} (15)

The constants $\tau_k$ and $c_4$ are the parameters of this solution. In the review [13] it is marked, that the problem on the connection between the asymptotics of this solution at infinity and the constants $\tau_k$ and $c_4$ is not investigated yet. The points of the poles $\tau_k$ and appropriate constants $c_4$ can be obtained with the help of the numerical calculation using the given asymptotics at infinity [13].

The asymptotics $\frac{1}{5} v$ at $\tau \to \tau_k$ has the form

$$\frac{1}{5} v = \frac{-1}{(\tau - \tau_k)^4} + \frac{19\tau_k}{10u_*} - \frac{5}{24u_*^2} (\tau - \tau_k) + O((\tau - \tau_k)^2).$$  \hspace{1cm} (16)
Using the asymptotics (15) and (16) it is easy to see, that the asymptotic expansion (12) is suitable at

\[ |\tau - \tau_k| \gg \varepsilon^{1/5}. \]

### 5.2 Second interior expansion

For the construction of the uniform asymptotics in the neighborhood of the pole of the function \( v^0 \) it is necessary to make one more stretching of the independent variable and the function:

\[ (\tau - \tau_k) = \varepsilon^{1/5} \theta, \quad \varepsilon^{-2/5} v = w. \]

For function \( w \) we obtain the equation:

\[
\frac{d^2 w}{d\theta^2} + 6u_\ast w^2 + 2w^3 = -\varepsilon^{4/5} \tau_k (u_\ast + w) - \varepsilon \theta (u_\ast + w). \tag{17}
\]

We search the asymptotic solution of this equation as

\[ w(\theta, \varepsilon) = w^0 + \varepsilon^{4/5} w^1 + \varepsilon^2 w^2 + \ldots. \tag{18} \]

It is shown here, that the asymptotic expansion (18) is the formal asymptotic solution of the equation (17) at \( \theta \tau_k^{1/5} | \ll \varepsilon^{1/5}. \)

The solution of the equation for the leading term of the asymptotics (18) is not uniquely determined from the asymptotics at \( \tau \rightarrow \tau_k \) of the asymptotic expansion (12), which is exterior in relation to (18). This solution has the form

\[ w^0 (\theta) = \frac{-16u_\ast}{4 + 16u_\ast^2 \theta^2}. \tag{19} \]

The corrections in the expansion (18) satisfy the linearized equations

\[
\frac{d^2 w^1}{d\theta^2} + (12u_\ast w^0 + 6 w^2) w^1 = -\tau_k (u_\ast + w^0); \\
\frac{d^2 w^2}{d\theta^2} + (12u_\ast w^0 + 6 w^2) w^2 = \theta (u_\ast + w^0).
\]

The expression for \( w^0 \) can be used to obtain two linearly independent solutions of the homogeneous equation for the corrections:

\[ w_1 = \frac{8\theta}{(1 + 4u_\ast^2 \theta^2)^2}; \]

\[ w_2 = \left[ -\frac{1}{8} + 2u_\ast^2 \theta^2 - u_\ast \theta^4 + \frac{2}{5} \theta^6 + \frac{2}{7} \theta^8 \right] \frac{1}{(1 + 4u_\ast^2 \theta^2)^2}. \]
Using these solutions of the homogeneous equation, it is easy to get the solutions of the nonhomogeneous equations for the corrections. The asymptotics of the corrections at $\theta \to \infty$ has the form:

$$
\frac{1}{w} = \frac{\tau_k u_*}{10} \theta^2 \left( \frac{\tau_k u_*^2}{10} + \frac{1}{12u_*} \right) + O(\theta^{-2});
$$

$$
\frac{2}{w} = \frac{u_*}{6} \theta^3 + \frac{1}{u_*} \theta + O(\theta^{-5}).
$$

Using the asymptotics for the corrections and the leading term, we obtain, that the expansion (18) is suitable at $|\theta \tau_1/5| \ll \varepsilon^{-1/5}$. On the other hand, the expansion (12) is suitable at $|\theta| \gg 1$. Hence, the areas of the usefulness for the expansions (12) and (18) are intersected at realization of the condition $|\tau_k| \ll \varepsilon^{-1}$. If we take into account also the requirement of fitness of the asymptotic expansion (18), then get the restriction $|\tau_k| \ll \varepsilon^{-4/5}$. From this restriction it follows, that in this section the formal asymptotic expansions suitable at $|t - t_*| \ll 1$ are constructed.

### 5.3 The asymptotics of the interior expansions at $\tau \to \infty$

The asymptotics of the solution of the equation Painleve-1 at $\tau \to \infty$ can be obtained with the help of the monodromy-preserving method [6]. If the monodromy data are known, then the defining solution is not uniquely defined. In the correspondence with [24], in our case the monodromy data are constants $s_2$ and $s_3$ and these constants are equal to zero. The asymptotics of the function $0v(\tau)$ outside of the poles has the form [18]

$$
0v = \sqrt{\tau} \rho(\phi(\tau)) + O(\tau^{-\gamma}), \quad (20)
$$

Where $\gamma > 0$ is some constant, the function $\rho(x)$ satisfies the equation

$$
\rho'' + 6u_* \rho^2 + u_* = 0,
$$

The phase function $\phi(\tau) = \frac{4}{5}(\tau)^{5/4}$.

It is important to note, that in the formula (20) the shift of the phase function $\phi(\tau)$ is equal to zero. The asymptotics of $0v(\tau)$ is determined by the Weierstrass elliptic function, through which the function $\rho$ is expressed in [18].

In the second interior expansion the asymptotics at $\tau \to \infty$ corresponds to the asymptotics at $\tau_k \to \infty$. It is easy to see, that in this case the first correction grows. This growth limits the value $\tau_k$, at which the asymptotics (18) is correct: $|\tau_k| \ll \varepsilon^{-4/5}$.

### 6 Fast oscillating asymptotics
The Kuzmak’s approximation

The formulas obtained in the work \[9\] for the construction of the fast oscillating asymptotic solution of the second-order ordinary differential equation by the Krylov-Bogolyubov’s method are obtained here and the parameters of this solution are specified, at which it is degenerated in the point \( t = t^* \).

Let’s pass to construction of the oscillating asymptotic solution. Following \[10\] we search for it as

\[
0^U(t_1, t) + \varepsilon \frac{1}{0^U(t_1, t)} + \ldots \tag{21}
\]

As the argument \( t_1 \) we use expression \( S(t)/\varepsilon \), where \( S(t) \) is known function. The equations for the leading term and the first correction of the asymptotics (21) look like:

\[
(S')^2 0^2_{t_1} U + 2 0^3_{t_1} U + t = 1, \tag{22}
\]

\[
(S')^2 0^2_{t_1} U + (6 0^2_{t_1} + t) 0^1_{t_1} U = -2S'0^2_{t_1} U - S''0_{t_1} U. \tag{23}
\]

Integrate once with respect to \( t_1 \) the equation for \( 0^0_{t_1} U \) and we obtain in the result:

\[
(S')^2(0_{t_1} 0^0_{t_1} U)^2 = - 0^4_{t_1} U 0^2_{t_1} + 2 0^1_{t_1} U + E(t), \tag{24}
\]

where \( E(t) \) is the "constant of integration".

In \[9\] it is shown, that the condition of periodicity with respect to the parameter \( t_1 \) for the function \( U \) reduces to the equation for the function \( S(t) \):

\[
S' \int_0^T \left[ \partial_{t_1} 0^0_{t_1} U (t_1, t) \right]^2 dt_1 = c_0.
\]

Here \( T \) is the period of the oscillations, \( c_0 \) is the constant. Using the explicit expression for the derivative with respect to \( t_1 \) we can present this formula in a little bit other form:

\[
2 \int_{\alpha(t)}^{\beta(t)} \sqrt{-x^4 - tx^2 + 2x + E(t)} dx = c_0, \tag{24}
\]

where \( \alpha(t) \) and \( \beta(t) \) are the solutions of the equation \(-x^4 - tx^2 + 2x + E(t) = 0\). The function \( E(t) \) is not defined here. Its connection with the phase of the fast oscillations \( S(t) \) is given by the formula \[10\]:

\[
T = \sqrt{2S'} \int_{\alpha(t)}^{\beta(t)} \frac{dx}{\sqrt{-x^4 - tx^2 + 2x + E(t)}}. \tag{25}
\]

Note. A.N. Belogrudov has mentioned, that the integral in the left part \[23\] is hypergeometric function satisfying a system of equations in partial derivatives with respect to parameters \( \alpha \) and \( \beta \).
The equations (23)-(25) define to an accuracy of some constant $c_0$ the leading term of the asymptotics (21). Let’s remark, that we build the asymptotic solution of the equation (1) at $t > t_*$. Thus the polynomial of the fourth power on $U$ in the right hand side of the equation (23) can have no more two various real roots $\alpha(t)$ and $\beta(t)$. Hence this polynomial can be submitted as:

$$F(x, t) = (\alpha(t) - x)(x - \beta(t)) \left( (x - m(t))^2 + n^2(t) \right).$$

In the point $t = t_*$ the curve on Figure 1 has the inflection point. The degeneration of the elliptic integral at $t = t_*$ corresponds to the case $m(t_*) = \beta(t_*) = u_*$ and $n(t_*) = 0$, when one of the roots of the polynomial corresponds to the value of the polynomial in the inflection point. For this case it is easy to calculate the constant in the right hand side of the equation (24) $c_0 = \pi$ and the value of the parameter $E(t_*) = E_*$.

6.2 Degeneration of the fast oscillating asymptotics

We shown here, that the formal asymptotic solution (24) obtained in the previous subsection is suitable at $(t - t_*)^{-4/5} > 1$. The matching by this asymptotics with the interior asymptotics (12) and (18) is carried out. From the matching condition for the phase function $S(t)|_{t = t_*} = 0$ is obtained.

The oscillating solution is degenerated at $t \rightarrow t_* + 0$. Let’s construct the asymptotics of this solution in the neighborhood of the degeneration point. For this purpose we calculate the asymptotics of the phase function $S(t)$ and the function $E(t)$. Let’s write the equation (24) as:

$$\int_{\alpha}^{\beta} \sqrt{(\alpha - x)(\beta - x)((x - m)^2 + n^2)} dx = \pi,$$

(26)

where $\alpha, \beta, m, n$ are real functions at $t \geq t_*$. These functions satisfy the Vieta equations:

$$\begin{align*}
\alpha + \beta + 2m &= 0, \\
m^2 + n^2 + \alpha \beta + 2m(\alpha + \beta) &= t, \\
(\alpha + \beta)(m^2 + n^2) + 2m\alpha\beta &= 2, \\
\alpha\beta(m^2 + n^2) &= -E.
\end{align*}$$

(27)

The equation (25) and three equation from (27) define the dependency $\alpha, \beta, m, n$ with respect to the parameter $t$. The last equation in (27) defines the function $E(t)$. Let’s make changes of variables: $E = E_* + g_1, \ t = t_* + \eta, \ m = m_* + m_1$. After the simple transformations of the equations (27) we get:

$$\begin{align*}
2m_*[6m^2_1 - 2n^2 + \eta] + [2m^2_1 - 2n^2 + \eta]2m_1 &= 0, \\
m^2_1(12m^2_1 - 4n^2 + \eta) + 2m_*m_1(6m^2_1 - 2n^2 + \eta) + \\
(3m^2_1 - n^2 + \eta)(m^2 + n^2) &= -g_1.
\end{align*}$$

(28)
Construct the solution of this system at $t \to t^* + 0$ as:

$$m_1 = \mu \sqrt{\eta} + O(\eta),$$
$$n = \nu_1 \sqrt{\eta} + O(\eta),$$
$$g_1 = \gamma_1 \eta + O(\eta^{3/2}).$$

Let’s substitute these expressions in (28), equate the coefficients at the identical powers of $\eta$. In the result we obtain:

$$6\mu_1^2 - 2\nu_1^2 = -1, \quad \gamma_1 = m_1^2.$$  

To define the constants $\mu_1$ and $\nu_1$ it is necessary to construct the asymptotics at $\eta \to +0$ of the left hand side of the equation (26). The asymptotics of the outside the integral coefficient in the equation (26) has the form

$$(\alpha - \beta)^3 = 64|m|^3\left[1 - \frac{3}{2} \frac{\mu \sqrt{\eta}}{m_*} + \frac{3}{2} \frac{\nu^2 - \mu_1^2 - 1}{4m_*^2} \eta + O(\eta^{3/2})\right]. \quad (29)$$

The integral in the equation (26) is presented as

$$I(k, \delta) = \int_0^1 \sqrt{(1 - z)z} \sqrt{(z - k\delta)^2 + \delta^2},$$

where

$$\frac{m - \beta}{\alpha - \beta} = k\delta, \quad \delta = \frac{n^2}{(\alpha - \beta)^2}. \quad (30)$$

The value of the constant $k$ will be defined from an asymptotics below.

The asymptotics of an integral $I(k, \delta)$ at $\delta \to 0$ has the form

$$I(k, \delta) = \frac{\pi}{16} - k\delta \frac{\pi}{8} + \delta^2 \frac{\pi}{4} + c(k)\delta^{5/2} + O(\delta^3), \quad (31)$$

where

$$c(k) = -\frac{8}{5} \int_0^\infty du \frac{-ky + k^2 + 1}{[(y - k)^2 + 1]^{5/2}} y^{5/2}.$$  

First three terms in this formula are calculated by standard way. Let’s show as we can obtain the function $c(k)$. For this purpose the following reception (24) is applicable. Let’s calculate third derivative with respect to $\delta$ of the function $I(k, \delta)$:

$$\frac{\partial^3 I}{\partial \delta^3} = -3 \int_0^1 dz \sqrt{(1 - z)z} \frac{-kz + k^2\delta + \delta}{[(z - k\delta)^2 + \delta^{5/2}].}$$

On the right hand side we replace $z$ by $\delta y$ and we present the integral as

$$\frac{\partial^3 I}{\partial \delta^3} = -3\delta^{-1/2} \int_0^\infty dy y^{5/2} \frac{-k y + k^2 + 1}{[(y - k)^2 + 1]^{5/2}} + O(1). \quad (32)$$

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Solving the ordinary differential equation (32) in the neighborhood of \( \delta = 0 \), we get:

\[ I(k, \delta) = c_0 + \delta c_1 + \delta^2 c_2 + \delta^{5/2} \frac{8}{15} c_3(k) + O(\delta^3), \]

where

\[ C_3(k) = -3 \int_0^\infty du \frac{-ky + k^2 + 1}{[y^2 + 1]^{5/2}}. \]

After that it is easy to obtain the asymptotics (31).

To define the value of the number \( k \) we substitute the asymptotics (29) and (31) in (26) and equate to zero the coefficients at identical powers of \( \eta \). In the result we get at \( \eta^{5/4} \) the equation

\[ c(k) = 0. \]

This is the transcendental equation for the definition of the parameter \( k \).

The numerical solution gives \( k \sim 0.463 \). Using the formula (30), we get:

\[ \mu_1 = \frac{k}{3} |\nu|, \quad \nu_1 = \sqrt{\frac{3}{2(3-k^2)}}. \]

To construct the asymptotics of \( S(t) \) at \( t \to t_* + 0 \) we use the equation connecting the period of fast oscillations with its phase (11):

\[ T = \sqrt{2S'} \int_\alpha^\beta dx \sqrt{\frac{\alpha - x}{\alpha - \beta}} \sqrt{(x - m)^2 + n^2}. \quad (33) \]

Present the integral in the right hand side as

\[ J = \frac{1}{\alpha - \beta} \int_0^1 \frac{dz}{\sqrt{(1 - z)z[(z-k^2)^2 + \delta^2]}}. \]

After the same replacements, as at the construction of the asymptotics \( \vec{\partial}^3 I / \vec{\partial \delta}^3 \), at \( \delta \to 0 \) we get:

\[ J = \frac{\delta^{-1/2}}{\alpha - \beta} \int_0^\infty \frac{dy}{\sqrt{y[(y+k)^2 + 1]}} + O(1). \]

We substitute this expression in the equation (33), use the asymptotics \( \delta \) and \( (\alpha - \beta) \) at \( \eta \to +0 \) and in the result we get:

\[ S' = (t - t_*)^{1/4} S_*(k) + O((t - t_*)^{1/2}), \]

where

\[ S_*(k) = \frac{T}{\sqrt{2}} \left[ \frac{2|m_*|^{1/2}}{C_*(k)} \left( \frac{3}{6 - 2k^2} \right)^{1/4} \right], \quad C_*(k) = \int_0^\infty \frac{dy}{\sqrt{y[(y+k)^2 + 1]}}. \]
The period of the oscillations for the function $^0U(t_1, t)$ with respect to the variable $t_1$ in the Krylov-Bogolubov’s method is an arbitrary constant. Let’s choose it such, that $S_*(k) = 1$:

$$T = \frac{S_*(k)\sqrt{2C_*(k)}}{2|u_*|^{1/2}} \left( \frac{3}{6 - 2k^2} \right)^{1/4}. \quad (34)$$

In the result the phase of the oscillations at $t \to t_*$ has form

$$S(t) = \frac{4}{5}(t - t_*)^{5/4} + O((t - t_*)^{3/2}) + S_0, \quad (35)$$

where $S_0$ is some constant. Its value will be defined below at the matching of the asymptotics (21) and interior asymptotics (12), (37) at $t \to t_* + 0$.

Now we turn to the evaluation of the asymptotics for the function $^0U$ at $t \to t_* + 0$. Let’s search for the asymptotics of the function $^0U$ as

$$^0U(t_1, t) = u_* + W(\theta) + O((t - t_*)), \quad (36)$$

where $\theta = S(t)/\varepsilon^{-1}$.

Substitute this asymptotics to the equation for $^0U$ (23), in the result we get:

$$(\partial_\theta W)^2 = -W^4 + 4u_*W^3 + O((t - t_*)).$$

It is easy to see, that in the leading order we have the function $^0w(\theta)$, defining the second interior asymptotic expansion.

The formula (36) is suitable at $|W(\theta)| \gg |t - t_*|$. When $W(\theta)$ is small, we consider other asymptotic formula for the function $^0U(t_1, t)$:

$$^0U(t_1, t) = u_* + \sqrt{t - t_*}p(S(t)) + O((t - t_*)^2). \quad (37)$$

Substitute this formula to the second-order equation for the function $^0U(t_1, t)$ (22), in the result we get:

$$p'' + 6u_*p^2 + u_* = 0.$$ 

This equation coincides with the equation for the asymptotics of the first correction of the first interior asymptotic expansion. The boundary conditions for the function $p(S(t))$ is obtained from the condition of the matching (37) with the asymptotics of the expansion (36) at $|\theta| \to \infty$. The phase shift $S_0$ in the formula (37) is finally defined at the matching of the asymptotic expansions (36), (37) with asymptotics of the interior asymptotic expansions. This get: $S_0 = 0$.

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