FARTHEST POINT PROBLEM AND PARTIAL STATISTICAL CONTINUITY IN NORMED LINEAR SPACES

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Abstract. In this paper, we prove that if $E$ is an uniquely remotal subset of a real normed linear space $X$ such that $E$ has a Chebyshev center $c \in X$ and the farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is partially statistically continuous at $c$, then $E$ is a singleton. We obtain a necessary and sufficient condition on uniquely remotal subsets of uniformly rotund Banach spaces to be a singleton. Moreover, we show that there exists a remotal set $M$ having a Chebyshev center $c$ such that the farthest point map $F : \mathbb{R} \to M$ is not continuous at $c$ but is partially statistically continuous there in the multivalued sense.

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1. Introduction. Let $X$ be a real normed linear space and $G$ be a nonempty, bounded subset of $X$. Throughout our discussion, we consider only bounded subsets of $X$. For any $x \in X$, the farthest distance from $x$ to a set $G$ is denoted by $\delta(x, G)$ and is defined by

$$\delta(x, G) = \sup \{ \|x - e\| : e \in G \}.$$ 

If the distance is attained, then the collection of all such points of $G$ corresponding to $x \in X$ is denoted by $F(x, G)$ and defined by

$$F(x, G) = \{ e \in G : \|x - e\| = \delta(x, G) \}.$$
For a non-empty and bounded subset $G$ of $X$, let us define

$$r(G) = \left\{ x \in X : F(x, G) \neq \emptyset \right\}.$$ 

$G$ is said to be remotal if $r(G) = X$ and uniquely remotal if $r(G) = X$ and $F(x, G)$ is a singleton for each $x \in X$. A nonempty, bounded subset $G$ of $X$ is said to be densely remotal if $r(G)$ is norm dense in $X$. The farthest point problem (FPP) states that “Must every uniquely remotal set in a Banach space be a singleton?” The FPP was proposed by Motzkin, Starus and Valentine [9] in the context of the Euclidean space $E^n$. The FPP for Banach spaces was introduced by Klee [7] and he proved that every compact uniquely remotal subset of a Banach space is a singleton. In [2], Asplund solved the FPP for any finite dimensional Banach space with respect to a norm which is not necessarily symmetric. Very recently Yosef, Khalil and Mutabagani [16] proved the FPP for $\ell_1$. In [12], Sababheh and Khalil proved that every infinite dimensional reflexive Banach space $X$ has a closed, bounded and convex nonremotal subset. They also proved that for a reflexive Banach space, every closed, bounded and convex subset of $X$ is remotal if and only if $X$ is finite dimensional. At the end of their paper [12], they asked a question that if every closed, bounded and convex subset of a Banach space $X$ (not necessary reflexive) is remotal, then does it follow that $X$ is finite dimensional. In [8], Martin and Rao showed the existence of a closed, bounded and convex nonremotal subset for every infinite dimensional Banach space $X$. Another concept, strongly remotal set, was introduced by Khalil and Matar in [6] and they showed that all such sets are singleton. Recall that, a Chebyshev center of a subset $E$ of a normed linear space $X$ is an element $c \in X$ such that $\delta(c, E) = \inf_{x \in X} \delta(x, E)$. Chebyshev centers of sets have played a major role in the study of uniquely remotal sets, see [2], [10] for more details. In [11], it was proved that if $E$ is a uniquely remotal subset of a normed space $X$, admitting a Chebyshev center at $c \in X$ and if the farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is continuous at $c$, then $E$ is a singleton. This result was further improved by Sababheh Yousef and Khalil in [13] by replacing continuity by partial continuity.

Now, we give a brief motivation. It is clear that, if $X$ is a normed linear space and $M$ is a non-empty, bounded subset of $X$, then the farthest point map $F : X \to M$ is not always a single valued map, even if $M$ is a remotal subset of $X$. The farthest point map $F : X \to M$ is single valued only when $M$ is uniquely remotal. We introduce the notion of partial statistical continuity of a single valued function and provide an example to show that the notion of partial statistical continuity is weaker than continuity as well as partial continuity introduced by Sababheh et al. in [13]. We also introduce the notion of partial statistical continuity for a multivariate mapping and prove that if $M$ is a uniquely remotal subset of a real normed linear space $X$ such that $M$ has a Chebyshev center $c$ and the farthest point map $F : X \to M$ restricted to $[c, F(c)]$ is partially statistically continuous at $c$ then $E$ is a singleton. This improves the result in [11].

2. Main results. We first recall some definitions and notations.
DEFINITION 2.1. ([3]) Let $Y, Z$ be topological spaces. A mapping $F : Y \to Z$ is said to be multivalued if $F(x)$ is a subset of $Z$ for each $x \in Y$. The mapping $F$ can be thought of as a single valued function from $Y$ into $2^Z$, the power set of $Z$.

DEFINITION 2.2. ([3]) Let $F : Y \to Z$ be a multivalued mapping and $A \subseteq Y$, $B \subseteq Z$. Then

(i) $F(A) = \bigcup \{F(x) : x \in A\}$;
(ii) $F^{-1}(B) = \{x \in Y : F(x) \cap B \neq \emptyset\}$.

DEFINITION 2.3. ([3]) Let $F : Y \to Z$ be a multivalued mapping. Then

(i) $F$ is a usc-function (i.e, upper semi-continuous function) provided $F^{-1}(B)$ is closed in $Y$ for each closed $B \subseteq Z$;
(ii) $F$ is a lsc-function (i.e, lower semi-continuous function) provided $F^{-1}(V)$ is open in $Y$ for each open $V \subseteq Z$;
(iii) $F : Y \to Z$ is a continuous function provided that $F$ is a usc-function and lsc-function.

DEFINITION 2.4. ([5]) Let $F : Y \to Z$ be a multivalued mapping. Then a function $f : Y \to Z$ is said to be a selection or selector of $F : Y \to Z$ if $f(x) \in F(x)$ for each $x \in Y$.

Throughout this paper we call a selection by the name extracted single valued function.

It is easy to see that

PROPOSITION 2.5. If each extracted single valued function $f : Y \to Z$ is continuous then the multivalued map $F : Y \to Z$ is continuous.

DEFINITION 2.6. ([13]) Let $X$ be a real normed linear space and $M \subseteq X$. A function $F : M \to X$ is said to be partially continuous at $a \in M$ if there exists a non-constant sequence $\{x_n\}_{n \in \mathbb{N}} \subset M$ such that $\{x_n\}_{n \in \mathbb{N}}$ is convergent to $a$ and $\{F(x_n)\}_{n \in \mathbb{N}}$ is convergent to $F(a)$.

DEFINITION 2.7. ([15]) A real sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be statistically convergent to $x \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \left\{ k \leq n : |x_k - x| \geq \varepsilon \right\} = 0.$$

DEFINITION 2.8. Let $X$ be a real normed linear space and $M \subseteq X$. A function $G : M \to X$ is said to be partially statistically continuous at $a \in M$ if there exists a non-constant sequence $\{x_n\}_{n \in \mathbb{N}} \subset M$ such that $\{x_n\}_{n \in \mathbb{N}}$ is statistically convergent to $a$ implies $\{G(x_n)\}_{n \in \mathbb{N}}$ is statistically convergent to $G(a)$ i.e. for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \left\{ k \leq n : \|x_k - a\| \geq \varepsilon \right\} = 0.$$
implies
\[
\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : \|G(x_k) - G(a)\| \geq \varepsilon \right\} \right| = 0.
\]
If \(G : M \to X\) is partially statistically continuous at each \(x \in M\) then \(G\) is said to be partially statistically continuous on \(M\).

The next example shows that, the notion of partial statistical continuity is much weaker than continuity as well as partial continuity.

**Example 2.9.** Let \(f : [-1, 0] \to \mathbb{R}\) be defined by \(f(x) = [x]\) for all \(x \in [-1, 0]\). Here \([x]\) denotes the greatest integer not exceeding \(x\). It follows easily that this function is not partially continuous (hence not continuous) at the point \(x = 0\).

Claim: \(f\) is partially statistically continuous at \(x = 0\).

Let \(x_n = \begin{cases} 0 & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N}, \\ -1 + \frac{1}{n} & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}. \end{cases}\)

Note that, the sequence \(\{x_n\}_{n \in \mathbb{N}}\) is not convergent to 0 in the usual sense. Let \(\varepsilon > 0\). Now \(\{ k \in \mathbb{N} : |x_k - 0| \geq \varepsilon \} \subseteq \{ k \in \mathbb{N} : k = m^2 \text{ for some } m \in \mathbb{N} \}\).

We have,
\[
\frac{1}{n} \left| \{ k \leq n : k \in A \} \right| = \frac{\sqrt{n}}{n} \leq \frac{\sqrt{n}}{n} \to 0 \text{ as } n \to \infty.
\]

Hence
\[
\lim_{n \to \infty} \frac{1}{n} \left| \{ k \leq n : |x_k - 0| \geq \varepsilon \} \right| = 0.
\]

So, the sequence \(\{x_n\}_{n \in \mathbb{N}}\) is statistically convergent to 0.

Now
\[
f(x_n) = \begin{cases} 0 & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N} \\ -1 & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}. \end{cases}
\]

Proceeding similarly as \(\{x_n\}\), it can be shown that the sequence \(\{f(x_n)\}_{n \in \mathbb{N}}\) is statistically convergent to \(f(0) = 0\). So, \(f\) is partially statistically continuous at the point \(x = 0\).

**Definition 2.10.** Let \(X\) and \(Y\) be real normed linear spaces and \(F : X \to Y\) be a multivalued mapping. The mapping \(F : X \to Y\) is said to be partially statistically continuous on \(X\) if each extracted single valued function \(f : X \to Y\) of \(F\) is partially statistically continuous on \(X\).

We now give an example to show that partial statistical continuity is weaker than continuity for multivalued mappings.

**Example 2.11.** Let \(\mathbb{R}\) be the set of all real numbers and \(M = [-1, 1]\). It was proved in [12], that a closed and bounded set in a finite dimensional space is remotal, so it follows that \(M\) is remotal in \(\mathbb{R}\). But \(M = [-1, 1]\) cannot be uniquely remotal. Indeed,
\[
\delta(0, M) = \sup\{|x| : x \in M\} = 1.
\]
Also, $F(0, M) = \{-1, 1\}$. So, in this case, the farthest point map $F : \mathbb{R} \to M$ is multivalued. We show that $c = 0$ is a Chebyshev center of $M$.

Suppose that $x > 1$. So $|x - 0| > 1$. Then, $\delta(x, M) \geq |x - 0| > 1$. Now let $x < -1$. So $x = -y$ for some $y > 1$. Then, $\delta(x, M) \geq |x - 0| = |y| > 1$. Now, suppose that $x \in M$ and $x > 0$. This implies $\delta(x, M) = |x - (-1)| = |x + 1| > 1$. Similarly $\delta(x, M) > 1$ for $x < 0$. But

$$\delta(0, M) = \inf_{x \in \mathbb{R}} \delta(x, M) = 1.$$ 

So $c = 0$ is the Chebyshev center of $M$.

Now, we show that the multivalued map $F : \mathbb{R} \to M$ is not continuous, in the sense of the Definition 2.3. Here, we consider that $M$ has the subspace topology as a subset of $\mathbb{R}$. So the set, $V = (-1, 1]$ is open in $M$. But $F^{-1}(V) = (-\infty, 0]$, which is not open in $\mathbb{R}$. This shows that $F$ is not lower-semi continuous. Hence $F$ is not continuous. Now we show that the farthest point map $F : \mathbb{R} \to M$ is partially statistically continuous at $c = 0$. The map $F$ has two extracted single valued functions. Let $F^* : \mathbb{R} \to M$ be the extracted single valued function such that $F^*(0) = 1$ and $F^{**} : \mathbb{R} \to M$ be the extracted single valued function such that $F^{**}(0) = -1$. We show that both $F^*$ and $F^{**}$ are partially statistically continuous at $c = 0$.

Let us define a sequence \( \{x_n\}_{n \in \mathbb{N}} \) in $[-1, 1]$ by

$$x_n = \begin{cases} 
0 & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N}, \\
-1 + \frac{1}{n} & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}.
\end{cases}$$

The sequence \( \{x_n\}_{n \in \mathbb{N}} \) is statistically convergent to 0 since for each $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - 0| \geq \varepsilon \right\} \right| = 0.$$ 

Now $F^*(x_n) = 1$ for all $n \in \mathbb{N}$. So $\{F^*(x_n)\}_{n \in \mathbb{N}}$ is statistically convergent to $F^*(0) = 1$. This shows that the extracted single valued function $F^* : \mathbb{R} \to M$ is partially statistically continuous at $c = 0$. Similarly, we can show that the extracted single valued function $F^{**} : \mathbb{R} \to M$ is partially statistically continuous at $c = 0$. Hence the farthest point map $F : \mathbb{R} \to M$ is partially statistically continuous at $c = 0$.

**Theorem 2.12.** Let $E \subset X$. Suppose $E$ is uniquely remotal and $E$ has a Chebyshev center $c \in X$. If the farthest point map $F : X \to E$ restricted to $[c, F(c)]$ is partially statistically continuous at $c$, then $E$ is singleton.

**Proof.** Since $E$ is uniquely remotal, so for each $x \in X$ there exists unique $e \in E$ such that $||x - e|| = \delta(x, E)$ and the farthest point map $F : X \to E$ is well defined. If $E$ has a Chebyshev center at $c = 0$ then the set $E - \{c\} = \{e - c : e \in E\}$ has Chebyshev center at $c = 0$. So, without loss of generality, we assume that $E$ has Chebyshev center at $c = 0$.

Suppose $E$ is not singleton. So we have $F(0) \neq 0$. Since the farthest point map $F : X \to E$ restricted to $[0, F(0)]$ is partially statistically continuous at 0,
there exists a non constant sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \([0, F(0)]\) such that \( \{x_n\}_{n \in \mathbb{N}} \) is statistically convergent to 0 and \( \{F(x_n)\}_{n \in \mathbb{N}} \) is statistically convergent to \( F(0) \).

Since \( x_n \in [0, F(0)] \) so we have \( x_n = \mu_n F(0) \) with \( \mu_n > 0 \) \( \forall n \in \mathbb{N} \) and \( \{\mu_n\}_{n \in \mathbb{N}} \) is statistically convergent to 0 as \( n \to \infty \). By the Hahn-Banach theorem, for each \( n \in \mathbb{N} \) there exists \( \psi_n \in X^* \) such that \( \psi_n(F(x_n) - x_n) = ||F(x_n) - x_n|| \) and \( ||\psi_n|| = 1 \). Now,

\[
\psi_n(x_n) = \psi_n(F(x_n)) - \psi_n(F(x_n) - x_n) \\
\leq ||\psi_n|| ||F(x_n)|| - ||F(x_n) - x_n|| \\
= ||F(x_n)|| - ||F(x_n) - x_n|| \\
= ||F(x_n) - 0|| - ||F(x_n) - x_n|| \\
\leq \delta(0, E) - \delta(x_n, E) \\
\leq 0 \text{ (as 0 is the Chebyshev center).}
\]

So,

\[
\psi_n(x_n) \leq 0 \text{ for all } n \in \mathbb{N} \\
\implies \psi_n(\mu_n F(0)) \leq 0 \text{ for all } n \in \mathbb{N} \\
\implies \mu_n \psi_n(F(0)) \leq 0 \text{ for all } n \in \mathbb{N} \\
\implies \psi_n(F(0)) \leq 0 \text{ for all } n \in \mathbb{N} \text{ as } \mu_n > 0.
\]

Since the sequence \( \{F(x_n) - x_n\}_{n \in \mathbb{N}} \) is statistically convergent to \( F(0) \) so for each \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : ||F(x_k) - x_k - F(0)|| \geq \varepsilon \right\} \right| = 0.
\]

Now

\[
\left| ||F(x_k) - x_k|| - ||F(0)|| \right| \leq ||F(x_k) - x_k - F(0)||.
\]

Let \( \varepsilon > 0 \). So we have

\[
\left\{ k \in \mathbb{N} : ||F(x_k) - x_k|| - ||F(0)|| \geq \varepsilon \right\} \subset \left\{ k \in \mathbb{N} : ||F(x_k) - x_k - F(0)|| \geq \varepsilon \right\}.
\]

This implies that

\[
\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : ||F(x_k) - x_k|| - ||F(0)|| \geq \varepsilon \right\} \right| = 0.
\]

So the sequence \( \{||F(x_k) - x_k||\}_{n \in \mathbb{N}} \) is statistically convergent to \( ||F(0)|| \).

Also

\[
\psi_n(F(x_n) - x_n) - \psi_n(F(0)) = \psi_n(F(x_n) - x_n - F(0)) \\
\leq ||\psi_n|| ||F(x_n) - x_n - F(0)|| \\
= ||F(x_n) - x_n - F(0)||.
\]

Since the sequence in the right is statistically convergent to 0, so the sequence \( \{\psi_n(F(0))\}_{n \in \mathbb{N}} \) is statistically convergent to \( ||F(0)|| \). But this is possible only when \( F(0) = 0 \). This is a contradiction. This proves that the uniquely remotal set \( E \) is singleton. \( \square \)
Remark 2.13. Theorem 2.12 is a generalization and improvement of Proposition 5 in [11].

Definition 2.14. Recall, $\mathbb{X}$ is a uniformly rotund space if for any $0 < \epsilon \leq 2$, there exists some $\delta > 0$ such that for any two vectors $x, y \in \mathbb{X}$, such that $\|x\| = \|y\| = 1$, the condition $\|x - y\| \geq \epsilon$ implies $\|\frac{x + y}{2}\| \leq 1 - \delta$. The most natural examples of such spaces are Hilbert spaces and real $L^p$ spaces for $1 < p < \infty$.

We need the following result.

Theorem 2.15. ([4]) Let $\mathbb{X}$ be an uniformly rotund Banach space and $G \subseteq \mathbb{X}$ be bounded then $G$ has a unique Chebyshev center.

Corollary 2.16. Let $\mathbb{X}$ be an uniformly rotund Banach space and $G \subseteq \mathbb{X}$. If $G$ is uniquely remotal and the farthest point map $F : \mathbb{X} \to G$ is partially statistically continuous, then $G$ is a singleton.

Proof. The proof follows immediately from Theorem 2.15 and Theorem 2.12. \qed

We recall the following well known definition.

Definition 2.17. ([14]) Let $\mathbb{X}$ be a real normed linear space and $M$ be a nonempty bounded subset of $\mathbb{X}$. A sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq M$ is said to be maximizing if there exists $x \in \mathbb{X}$ such that $\|x_n - x\| \to \delta(x, M)$ as $n \to \infty$.

Definition 2.18. Let $\mathbb{X}$ be a real normed linear space and $M$ be a nonempty bounded subset of $\mathbb{X}$. A sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq M$ is said to be statistically maximizing if there exists $x \in \mathbb{X}$ such that $\{\|x_n - x\|\}_{n \in \mathbb{N}}$ is statistically convergent to $\delta(x, M) = \sup \left\{ \|x - y\| : y \in M \right\}$ as $n \to \infty$.

Proposition 2.19. Let $\mathbb{X}$ be a real normed linear space and $M$ be a nonempty bounded subset of $\mathbb{X}$. If $\{x_n\}_{n \in \mathbb{N}}$ is maximizing in $M$ then $\{x_n\}_{n \in \mathbb{N}}$ is statistically maximizing in $M$.

Proof. Let the sequence $\{x_n\}_{n \in \mathbb{N}}$ be maximizing. So there exists $x \in \mathbb{X}$ such that $\|x_n - x\| \to \delta(x, M)$ as $n \to \infty$. Let $\epsilon > 0$. So the set, $A = \left\{ k \in \mathbb{N} : \left| \|x_k - x\| - \delta(x, M) \right| \geq \epsilon \right\}$ is finite. Thus

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \|x_k - x\| - \delta(x, M) \right| \geq \epsilon \right\} \right| = 0.$$ 

This implies that $\{x_n\}_{n \in \mathbb{N}}$ is statistically maximizing. \qed

But the converse of Proposition 2.19 is not true. Here is an example.
Example 2.20. Let $\mathbb{R}$ be the set of all real numbers and $M = [-1, 1]$. Now

$$\delta(0, M) = \sup \left\{ |x| : x \in M \right\} = 1.$$ 

Let $\{x_n\}_{n \in \mathbb{N}}$ in $[-1, 1]$ be defined as

$$x_n = \begin{cases} 
0 & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}, \\
1 - \frac{1}{n} & \text{if } n \neq m^2 \text{ for all } m \in \mathbb{N}.
\end{cases}$$

$\{x_n\}_{n \in \mathbb{N}}$ is not maximizing. Indeed, for any $x \in \mathbb{R}$, the real sequence $\{|x_n - x|\}_{n \in \mathbb{N}}$ is not convergent in $\mathbb{R}$. $\{x_n\}_{n \in \mathbb{N}} = \{|x_n - 0|\}_{n \in \mathbb{N}}$ is statistically convergent to $\delta(0, M) = 1$. Indeed, let $0 < \varepsilon \leq 1$. Now

$$\frac{1}{n} \left| \left\{ k \leq n : |x_k - 1| \geq \varepsilon \right\} \right| \leq \frac{1}{n} \left( \sqrt{n} + d \right) \leq \frac{\sqrt{n}}{n} + \frac{d}{n},$$

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - 1| \geq \varepsilon \right\} \right| = 0.$$ 

Here $d$ is a finite positive integer. Hence the sequence $\{x_n\}_{n \in \mathbb{N}}$ is statistically maximizing in $M$.

Theorem 2.21. Let $M \subseteq X$. If $\{x_n\}_{n \in \mathbb{N}}$ is a statistically maximizing sequence in $M$ then $\{x_n\}_{n \in \mathbb{N}}$ is a statistically maximizing sequence in $\overline{M}$.

Proof. Let $X$ be a real normed linear space and $M$ be a nonempty bounded subset of $X$. Let $x \in X$. It can be easily seen that $\delta(x, M) = \delta(x, \overline{M})$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a statistically maximizing sequence in $M$. So $\{x_n\}_{n \in \mathbb{N}}$ is also a sequence in $\overline{M}$. Since $\{x_n\}_{n \in \mathbb{N}}$ is a statistically maximizing sequence so there exists $x \in X$ such that $\|x_n - x\|$ is statistically convergent to $\delta(x, M)$. Since $\delta(x, M) = \delta(x, \overline{M})$ so, we have $\|x_n - x\|$ is statistically convergent to $\delta(x, \overline{M})$. This implies $\{x_n\}_{n \in \mathbb{N}}$ is also a statistically maximizing sequence in $\overline{M}$. \qed

Definition 2.22. [1] Let $C \subseteq X$ and $T : C \to X$ be a mapping. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in $C$ is said to be an approximate fixed point sequence (a.f.p.s in short) for $T$ if $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$.

Theorem 2.23. Let $X$ be a finite dimensional real normed linear space and $C$ be a non-empty closed bounded subset of $X$. Let $T : C \to C$ be a mapping and $\{x_n\}_{n \in \mathbb{N}}$ be an a.f.p.s for $T$. Then $\{x_n\}_{n \in \mathbb{N}}$ is statistically maximizing if and only if $\{T(x_n)\}_{n \in \mathbb{N}}$ is statistically maximizing.
Proof. Let $X$ be a finite dimensional real normed linear space and $C$ be a non-empty closed bounded subset of $X$. Let $T : C \to C$ be a mapping and $\{x_n\}_{n \in \mathbb{N}}$ be an a.f.p.s for $T$. So $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$. Since every closed bounded subsets in a finite dimensional space $X$ is remotal [12, Theorem B.], we conclude that $C$ is remotal.

First, suppose that $\{x_n\}_{n \in \mathbb{N}}$ is statistically maximizing. So there exists $x \in X$ such that $\{\|x_n - x\|\}_{n \in \mathbb{N}}$ is statistically convergent to $\delta(x, C)$. Since $C$ is remotal, there exists $p \in C$ such that $\delta(x, C) = \|x - p\|$. Since $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$, we have, $\{\|x_n - T(x_n)\|\}_{n \in \mathbb{N}}$ is statistically convergent to 0. We show that $x \in X$ serves our purpose. Now,

$$\left| \|T(x_n) - x\| - \|x - p\| \right| \leq \|T(x_n) - x\| - \|x_n - x\| + \|x_n - x\| - \|x - p\| \right| \leq \|x_n - T(x_n)\| + \|x_n - x\| - \|x - p\|.$$  

Now let $\varepsilon > 0$. Since $\{\|x_n - T(x_n)\|\}_{n \in \mathbb{N}}$ is statistically convergent to 0 so for $\frac{\varepsilon}{2} > 0$, we have

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : \|x_k - T(x_k)\| \geq \frac{\varepsilon}{2} \right\} \right| = 0.$$  

Also since $\{\|x_n - x\|\}_{n \in \mathbb{N}}$ is statistically convergent to $\delta(x, C) = \|x - p\|$ so

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : \|x_k - x\| - \|x - p\| \geq \frac{\varepsilon}{2} \right\} \right| = 0.$$  

Now

$$\{ k \leq n : \|T(x_k) - x\| - \|x - p\| \geq \varepsilon \}$$  

$$\subseteq \left\{ k \leq n : \|x_k - T(x_k)\| \geq \frac{\varepsilon}{2} \right\} \cup \left\{ k \leq n : \|x_k - x\| - \|x - p\| \geq \frac{\varepsilon}{2} \right\}$$  

$$\implies \frac{1}{n} \left| \left\{ k \leq n : \|T(x_k) - x\| - \|x - p\| \geq \varepsilon \right\} \right| \leq \frac{1}{n} \left| \left\{ k \leq n : \|x_k - T(x_k)\| \geq \frac{\varepsilon}{2} \right\} \right|$$  

$$+ \frac{1}{n} \left| \left\{ k \leq n : \|x_k - x\| - \|x - p\| \geq \frac{\varepsilon}{2} \right\} \right|$$  

$$\implies \lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : \|T(x_k) - x\| - \|x - p\| \geq \varepsilon \right\} \right| = 0,$$

which implies that the sequence $\{\|T(x_n) - x\|\}_{n \in \mathbb{N}}$ is statistically convergent to $\|x - p\| = \delta(x, C)$. So $\{T(x_n)\}_{n \in \mathbb{N}}$ is statistically maximizing.

Similarly, one can show that if $\{T(x_n)\}_{n \in \mathbb{N}}$ is statistically maximizing, then the sequence $\{x_n\}_{n \in \mathbb{N}}$ is statistically maximizing.  \qed
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