PROJECTIONS AND THE KADISON-SINGER PROBLEM

PETE CASAZZA, DAN EDIDIN, DEEPTI KALRA, AND VERN I. PAULSEN

Abstract. We prove some new equivalences of the paving conjecture and obtain some estimates on the paving constants. In addition, we give a new family of counterexamples to one of the Akemann-Anderson conjectures.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and let $B(\mathcal{H})$ denote the bounded, linear operators on $\mathcal{H}$. By a MASA we mean a maximal, abelian subalgebra of $B(\mathcal{H})$. R. Kadison and I. Singer studied [18] whether or not pure states on a MASA extend uniquely to states on $B(\mathcal{H})$. In their original work on this subject [18], it was shown that this question has a negative answer if the MASA had any continuous part. The remaining case, whether or not pure states on discrete MASAs have unique extensions to states on $B(\mathcal{H})$ has come to be known as the Kadison-Singer problem. Their work showed that this problem was equivalent to certain questions about "paving" operators by projections. J. Anderson [2] developed this idea significantly into a series of so-called "paving" conjectures. Since that time there has been a great deal of research on these paving conjectures [1], [5], [6], [8], [10], [12], [14], [15] and [16].

In this paper, we begin by restating some of these paving conjectures and add a few new equivalent paving conjectures.

2. Some New Equivalences of the Paving Conjecture

Let us begin with the familiar.

Given $A \subseteq I$, where $I$ is some index set, we let $Q_A \in B(\ell^2(I))$ denote the diagonal projection defined by $Q_A = (q_{i,j}), q_{i,i} = 1, i \in A, q_{i,i} = 0, i \notin A$ and $q_{i,j} = 0, i \neq j$.

Definition 1. An operator $T \in B(\ell^2(I))$ is said to have an $(r, \epsilon)$-paving if there is a partition of $I$ into $r$ subsets $\{A_j\}_{j=1}^r$ such that $\|Q_{A_j} T Q_{A_j}\| \leq \epsilon$. A collection of operators $C$ is said to be $(r, \epsilon)$-pavable if each element of $C$ has an $(r, \epsilon)$-paving.
Note that in this definition, we do not require that the diagonal entries of the operator be 0.

Some classes that will play a role are:

- $C_\infty = \{ T = (t_{i,j}) \in B(\ell^2(\mathbb{N})) : \|T\| \leq 1, t_{i,i} = 0 \forall i \in \mathbb{N} \}$,
- $C = \bigcup_{n=2}^{\infty} \{ T = (t_{i,j}) \in M_n : \|T\| \leq 1, t_{i,i} = 0, i = 1, \ldots, n \}$,
- $S_\infty = \{ T \in C_\infty : T = T^* \}$,
- $S = \{ T \in C : T = T^* \}$,
- $R_\infty = \{ T \in S_\infty : T^2 = I \}$,
- $R = \{ T \in S : T^2 = I \}$,
- $P_{1/2}^\infty = \{ T = (t_{i,j}) \in B(\ell^2(\mathbb{N})) : T = T^* = T^2, t_{i,i} = 1/2, \forall i \in \mathbb{N} \}$,
- $P_{1/2} = \bigcup_{n=2}^{\infty} \{ T = (t_{i,j}) \in M_n : T = T^* = T^2, t_{i,i} = 1/2, i = 1, \ldots, n \}$.

Note that the operators satisfying, $R = R^*, R^2 = I$ are reflections and that for such an operator, $\sigma(R) = \{-1, +1\}$. Since, the traces of our matrices are 0, in the finite dimensional case, these types of reflections can only exist in even dimensions. If the space is $2n$-dimensional, then there exists an $n$-dimensional subspace that is fixed by $R$ and such that for any vector $x$ orthogonal to the subspace. $Rx = -x$.

J. Anderson’s [2] remarkable contribution follows.

**Theorem 2** (Anderson). The following are equivalent:

1. the Kadison-Singer conjecture is true,
2. for each $T \in C_\infty$, there exists $(r, \epsilon)$ (depending on $T$) $\epsilon < 1$, such that $T$ is $(r, \epsilon)$-pavable,
3. there exists $(r, \epsilon), \epsilon < 1$, such that $C_\infty$ is $(r, \epsilon)$-pavable,
4. there exists $(r, \epsilon), \epsilon < 1$, such that $C$ is $(r, \epsilon)$-pavable,
5. for each $T \in S_\infty$, there exists $(r, \epsilon), \epsilon < 1$ (depending on $T$), such that $T$ is $(r, \epsilon)$-pavable,
6. there exists $(r, \epsilon), \epsilon < 1$, such that $S_\infty$ is $(r, \epsilon)$-pavable,
7. there exists $(r, \epsilon), \epsilon < 1$, such that $S$ is $(r, \epsilon)$-pavable.

Generally, when people talk about the paving conjecture they mean one of the above equivalences of the Kadison-Singer problem. Also, generally, when one looks at operators on an infinite dimensional space, it is enough to find $(r, \epsilon)$ depending on the operator, but for operators on finite dimensional spaces it is essential to have a uniform $(r, \epsilon)$, for all operators of norm one. Finally, since $S_\infty \subset C_\infty$, people looking for counterexamples tend to study $C_\infty$, while people trying to prove the theorem is true, study $S_\infty$ or $S$. However, by the above equivalences, if a counterexample exists in one set then it must exist in the other as well.

In this spirit, we prove that the following smaller sets with “more structure” are sufficient for paving.

**Theorem 3.** Let $\epsilon < 1$, then the following are equivalent:

1. the set $S_\infty$ can be $(r_1, \epsilon)$-pavable,
2. the set $R_\infty$ can be $(r_1, \epsilon)$-pavable,
(3) the set $P_{1/2}$ can be $(r_2, \frac{1+\epsilon}{2})$-paved,

(4) the set $S$ can be $(r_1, \epsilon)$-paved,

(5) the set $R$ can be $(r_1, \epsilon)$-paved,

(6) the set $P_{1/2}$ can be $(r_2, \frac{1+\epsilon}{2})$-paved.

Proof. Since the reflections are a subset of the self-adjoint matrices, it is clear that (1) implies (2) and that (4) implies (5).

To see that (2) implies (1), let $A \in S$, and set

$$R = \begin{pmatrix} \frac{A}{\sqrt{I-A^2}} & \sqrt{I-A^2} \\ -A & -I \end{pmatrix},$$

then $R \in R_\infty$ and clearly any $(r, \epsilon)$-paving of $R$ yields an $(r, \epsilon)$-paving of $A$.

Thus, (1) and (2) are equivalent and similarly, (4) and (5) are equivalent.

To see the equivalence of (2) and (3), note that $R \in R_\infty$ (respectively, $R$) if and only if $P = (I+R)/2 \in P_{1/2}$ (respectively, $P_{1/2}$). Also, if $\|QAQR_A\| \leq \epsilon$, then $\|QA_PQA\| \leq (1+\epsilon)/2$. Thus, if $R_\infty$ can be $(r_1, \epsilon)$-paved, then $P_{1/2}$ can be $(r_1, \frac{1+\epsilon}{2})$-paved.

Conversely, given $R \in R_\infty$, let $P = (I + R)/2$. If $\|QA_PQA\| \leq (1+\epsilon)/2 = \beta$, then,

$$0 \leq Q_A PQA \leq \beta Q_A,$$

and since $R = 2P - I$, we have that

$$-Q_A \leq Q_A RQA \leq (2\beta - 1)Q_A = \epsilon Q_A.$$

Applying the same reasoning to the reflection $-R$, we get a new projection, $P_1 = (I - R)/2$, with a possibly different paving of $P_1$, such that $-Q_B \leq Q_B(-R)Q_B \leq \epsilon Q_B$. Thus, $-\epsilon Q_B \leq Q_BRQ_B$ and if $Q_C = QAQ_B$, we have that $-\epsilon Q_C \leq Q_C RQ_C \leq +\epsilon Q_C$. Therefore, we have that the set of all products of the $Q_A$’s and $Q_B$’s pave $R$. Thus, if $P_{1/2}$ can be $(r_2, \frac{1+\epsilon}{2})$-paved, then $R_\infty$ can be $(r_2, \epsilon)$-paved.

The proof of the equivalence of (5) and (6) is identical.

Finally, (1) and (4) are equivalent by the standard limiting argument. In particular, see [10, Proposition 2.2] and the proof of [10, Theorem 2.3].

\[\square\]

Corollary 4. The following are equivalent:

1. the Kadison-Singer conjecture is true,

2. for each $R \in R_\infty$ there is a $(r, \epsilon), \epsilon < 1$ (depending on $R$) such that $R$ can be $(r, \epsilon)$-paved,

3. there exists $(r, \epsilon), \epsilon < 1$, such that every $R \in R$ can be $(r, \epsilon)$-paved,

4. for each $P \in P_{1/2}$ there is a $(r, \epsilon), \epsilon < 1$ (depending on $P$) such that $P$ can be $(r, \epsilon)$-paved,

5. there exists $(r, \epsilon), \epsilon < 1$, such that every $P \in P_{1/2}$ can be $(r, \epsilon)$-paved.

We will need some results from frame theory in this paper. We refer the reader to [13] for these. We will briefly give the definitions we will be using.
If \( \{f_i\}_{i \in I} \) is a family of vectors in a Hilbert space \( \mathbb{H} \), the analysis operator of this family is \( T: \mathbb{H} \to \ell_2(I) \) given by \( T(f) = \{(f, f_i)\}_{i \in I} \), and the synthesis operator is \( T^*(\{a_i\}_{i \in I}) = \sum_{i \in I} a_i f_i \). If \( T \) is bounded, we call \( \{f_i\}_{i \in I} \) a Bessel sequence, if it is also onto we call this a frame, and if \( T \) is invertible it is a Riesz basis. The frame is equal-norm or uniform if the \( f_i \) all have the same norm and it is equianular if there is a constant \( c \) so that \( |\langle f_i, f_j \rangle| = c \) for all \( i \neq j \in I \). This is a Parseval frame if \( T \) is a partial isometry. In this case, the Gram matrix \( \langle f_i, f_j \rangle \) is an orthogonal projection of \( \ell_2(I) \) onto the range of the analysis operator (and this takes \( e_i \) to \( T(f_i) \) where \( \{e_i\}_{i \in I} \) is the unit vector basis of \( \ell_2(I) \)). If \( \dim \mathbb{H} = k \) and \( |I| = n \) we call this an \( (n,k) \)-frame.

A sort of meta-corollary or Theorem 3 means that the frame based conjectures that are known to be equivalent to the Kadison-Singer result can be reduced to the case of uniform Parseval frames of redundancy 2. Similarly, for most harmonic analysis analogues of paving, it is enough to consider say subsets \( E \subset [0,1] \) of Lebesgue measure 1/2. We state one such equivalence. The Feichtinger Conjecture in frame theory asserts that every unit norm Bessel sequence is a finite union of Riesz basic sequences. Casazza and Tremain [12] have shown that the Feichtinger conjecture is equivalent to Kadison-Singer.

**Theorem 5.** The Feichtinger conjecture is true if and only if for each Parseval frame \( \{f_n\}_{n \in \mathbb{N}} \) for a Hilbert space with \( \|f_n\|^2 = 1/2 \) \( \forall n \) there is a partition \( \{A_j\}_{j=1}^r \) of \( \mathbb{N} \) into \( r \) disjoint subsets (with \( r \) depending on the frame) such that for each \( j \), \( \{f_n\}_{n \in A_j} \) is a Riesz basis for the space that it spans.

**Proof.** Clearly, if the Feichtinger is true, then it is true for this special class of frames.

Conversely, assume that the above holds and let \( P \in \mathcal{P}_{1/2}^\infty \). Then there exists a Parseval frame \( \{f_n\}_{n \in \mathbb{N}} \) for some Hilbert space \( \mathcal{H} \), such that \( I - P = \langle f_j, f_i \rangle \) is their Grammian. Now let \( \{A_k\}_{k=1}^r \) be the partition of \( \mathbb{N} \) into \( r \) disjoint subsets as above and let \( \mathcal{H}_k = \overline{\text{span}} \{f_n : n \in A_k\} \) denote the closed linear span.

Since \( \{f_n : n \in A_k\} \) is a Riesz basis for \( \mathcal{H}_k \), there exists an orthonormal basis, \( \{e_n : n \in A_k\} \) for \( \mathcal{H}_k \) and a bounded invertible operator, \( S_k : \mathcal{H}_k \to \mathcal{H}_k \), with \( S_k(e_n) = f_n \).

We have that \( Q_{A_k}(I - P)Q_{A_k} = \langle f_j, f_i \rangle_{i,j \in A_k} = \langle S_k^*S_k e_j, e_i \rangle \geq c_k Q_{A_k} \) where \( S_k^*S_k \geq c_k Q_{A_k} \) for some constant \( 0 < c_k \leq 1 \) since \( S_k \) is invertible. Hence, \( Q_{A_k}PQ_{A_k} \leq (1 - c_k) Q_{A_k} \) and we have that, \( \max \{\|Q_{A_k}PQ_{A_k}\| : 1 \leq k \leq r\} < 1 \).

Hence, condition (5) of Corollary 4 is met and so Kadison-Singer is true and thus, by [12] Theorem 5.3, the Feichtinger conjecture is true. \( \square \)

### 3. Some Paving Estimates

In this section we derive some estimates on paving constants that give some basic relationships between \( r \) and \( \epsilon \). In particular, we will prove that \( \mathcal{P}_{1/2} \) cannot be \( (2, \epsilon) \)-paved for any \( \epsilon < 1 \).
We begin with a result on paving $\mathcal{R}$.

**Theorem 6.** Assume that $\mathcal{R}$ is $(r, \epsilon)$-pavable. Then $1 \leq re^2$.

**Proof.** Recall that an $n \times n$ matrix $C$ is a conference matrix if $C = C^*, c_{i,i} = 0, c_{i,j} = \pm 1, i \neq j$ and $C^2 = (n-1)I$. Such matrices exist for infinitely many $n$.

Set $A = \frac{1}{\sqrt{n-1}} C$, then $A$ is a unitary matrix with zero diagonal.

Assume that $\{1,\ldots,n\} = B_1 \cup \ldots \cup B_r$ is a partition such that $\|Q_{B_i} A Q_{B_i}\| \leq \epsilon$. Let $d = \max\{\text{card}(B_i)\}$ and let $B_j$ attain this max. Note that $d \geq \frac{n}{r}$. Set $A_j = Q_{B_j} A Q_{B_j}$, then the Schur product $A_j * A_j = \frac{1}{n-1}[J_d - I_d]$ where $J_d$ denotes the matrix of all 1's. Hence, $\frac{d-1}{n-1} = \|A_j * A_j\| \leq \|A_j\|^2 \leq \epsilon^2$. Thus, $\frac{n/r-1}{n-1} \leq \epsilon^2$, and the result follows by letting $n \to +\infty$.

**Proposition 7.** If every projection $P \in \mathcal{P}_{1/2}$ can be $(r, \epsilon)$-paved then every projection $Q$ with

$$\frac{1}{2} - \delta \leq \langle Q e_i, e_i \rangle \leq \frac{1}{2} + \delta,$$

can be $(r, \beta)$-paved, where

$$\beta = (1 + 2\delta)\epsilon,$$

and so $\beta < 1$ when $\delta$ is small enough.

**Proof.** Let $D$ be the diagonal of $Q$ and let $B = Q - D$. Then

$$\|B\| \leq \frac{1 + 2\delta}{2}.$$

To see this note that for any vector $x$

$$0 \leq \langle Bx, x \rangle + \langle Dx, x \rangle \leq 1,$$

since $Q$ is a projection. Hence,

$$-\langle Dx, x \rangle \leq \langle Bx, x \rangle \leq 1 - \langle Dx, x \rangle.$$

Hence,

$$\|B\| = \sup_{\|x\|=1} |\langle Bx, x \rangle| \leq \max\{|\langle Dx, x \rangle|, |1 - \langle Dx, x \rangle|\} \leq \frac{1}{2} + \delta = \frac{1 + 2\delta}{2}.$$

Let $R = R^*$ be the symmetry we get by dilating

$$\frac{2}{1 + 2\delta} B,$$

as in the proof of Theorem 3. Let $P = \frac{1}{2}(I + R)$ be the projection with $1/2'$s on the diagonal. If we can $(r, \epsilon)$-pave $P$ with $\{A_j\}_{j=1}^r$ then we have $(r, \epsilon)$-paved

$$\frac{1}{2} I + \frac{1}{1 + 2\delta} B.$$

Substituting $B = Q - D$ we have an $(r, \epsilon)$-paving of

$$\frac{1}{1 + 2\delta} Q + \frac{1}{2} I - \frac{1}{1 + 2\delta} D = \frac{1}{1 + 2\delta} \left( Q + \frac{1 + 2\delta}{2} I - D \right).$$
Now, for any $j = 1, 2, \ldots, r$ since
\[
\frac{1 + 2\delta}{2} I - D
\]
is a positive operator,
\[
\|Q_{A_j} Q_{A_j}\| \leq \|Q_{A_j}(Q + \frac{1 + 2\delta}{2} I - D)Q_{A_j}\| \leq (1 + 2\delta)\epsilon < 1.
\]
□

**Theorem 8.** Assume that $P_{1/2}$ can be $(r, \epsilon)$-paved. Then
\[
\frac{r}{2} (r - 1) \leq \epsilon.
\]

**Note:** When $r=2$ this implies that $\epsilon = 1$, and hence 2-paving is impossible.

**Proof.** Let $m > 2$ be an integer and consider a uniform, Parseval $(n,k)$-frame with $n = mr, k = m(r - 1) + 1$. This will give rise to a projection $Q$ with diagonal entries, $\frac{m(r-1)+1}{mr} = \frac{1}{2} + \delta$, where $\delta = \frac{m(r-2)+3}{2mr}$. To see this, let

\[
\delta = \frac{m(r-1)+1}{mr} - \frac{1}{2} = \frac{2[m(r-1)+1] - mr}{2mr} = \frac{m(r-1)+2}{2mr}.
\]

By the above result, $Q$ can be $(r, \beta)$-paved, where $\beta = (1 + 2\delta)\epsilon$.

However, for any $r$ paving of $Q$, one of the blocks must be of size at least

\[
n/r = m = n - k + 1,
\]

by the choice of $n$ and $k$. Since $Q$ is a rank $k$ projection, this block will have norm 1 by the eigenvalue inclusion principle or by the eigenvalue interlacing results. Hence $\beta \geq 1$. We solve for $\epsilon$:

\[
(1 + 2\delta)\epsilon \geq 1.
\]

So

\[
\epsilon \geq \frac{1}{1 + 2\delta} = \frac{mr}{m(2r - 2) + 2}.
\]

Letting $m \to +\infty$ yields

\[
\epsilon \geq \frac{r}{2(r-1)}.
\]

□

**Corollary 9.** The set $P_{1/2}$ is not 2-pavable.

**Corollary 10.** The set $R$ is not 2-pavable.

We now generalize the results of the last theorem.
Theorem 11. For each \( r, n \in \mathbb{N} \) with \( r > 1 \) there is an \( \epsilon_n > 0 \) so that whenever \( P \) is a projection on \( \ell^n_2 \) with \( \frac{1}{r} \leq \langle Pe_i, e_i \rangle \leq 1 - \frac{1}{r} \) for all \( i = 1, 2, \ldots, n \) then \( P \) is \( (r, 1 - \epsilon_n) \)-pavable.

Moreover, for any \( \delta > 0 \) there is an \( n \in \mathbb{N} \) and a projection \( P \) on \( \ell^{2n}_2 \) of rank \( n \) so that \( \frac{1}{r} - \delta \leq \langle Pe_i, e_i \rangle \leq 1 - \frac{1}{r} + \delta \) for all \( i = 1, 2, \ldots, 2n \) while \( P \) is not \( (r, \epsilon) \)-pavable for any \( \epsilon < 1 \).

Proof. Given our assumptions, we will check the Rado-Horn Theorem (see [1] and its references) to see that the row vectors of our projection can be divided into \( r \) linearly independent sets. Then the rest of the first part of the theorem follows by the same argument (adjusted for \( r \)) as in the last theorem. For any \( J \subset \{1, 2, \ldots, 2n\} \) let \( P_J \) be the orthogonal projection of \( \ell^{2n}_2 \) onto the span \( \{Pe_i\}_{i \in J} \). Now,

\[
\dim \text{span} \{Pe_i\}_{i \in J} = \sum_{i=1}^{2n} ||P_J Pe_i||^2 \geq \sum_{i \in J} ||Pe_i||^2 \geq |J| \frac{1}{r}.
\]

By the Rado-Horn Theorem we can now write \( \{Pe_i\}_{i=1}^{2n} \) as a union of \( r \)-linearly independent sets.

For the moreover part, choose a \( k \in \mathbb{N} \) so that

\[
\frac{1}{r} - \delta < \frac{k}{rk+1} \leq \frac{1}{r} \leq 1 - \frac{1}{r}.
\]

Now, choose an \( n \) so that

\[
\frac{1}{r} - \delta \leq \frac{n-rk}{2n-(rk+1)} \leq 1 - \frac{1}{r} + \delta.
\]

With \( \{e_i\}_{i=1}^{n} \) the unit vectors in \( \ell^n_2 \) we can choose an equal norm Parseval frame \( \{f_i\}_{i=1}^{rk+1} \) for \( \{e_i\}_{i=1}^{k} \). Next, choose an equal norm Parseval frame \( \{f_i\}_{i= rk+2n}^{n} \) for \( \{e_i\}_{i=k+1}^{n} \). Now,

\[
\frac{1}{r} - \delta \leq ||f_i||^2 = \frac{k}{rk+1} \leq \frac{1}{r} \leq 1 - \frac{1}{r},
\]

and

\[
\frac{1}{r} - \delta \leq \frac{n-rk}{2n-(rk+1)} \leq 1 - \frac{1}{r} + \delta.
\]

Taking the embedding of this Parseval frame with \( 2n \)-elements for \( \ell^n_2 \) into \( \ell^{2n}_2 \) we get a projection \( P \) on \( \ell^{2n}_2 \) which has rank \( n \) and looks like

\[
\begin{bmatrix}
||f_1||^2 & b_{1,2} & \cdots & b_{1,(rk+1)} & 0 & 0 & \cdots & 0 \\
b_{21} & ||f_1||^2 & \cdots & b_{2,(rk+1)} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
b_{(rk+1),1} & b_{(rk+1),2} & \cdots & ||f_1||^2 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & ||f_{rk+2}||^2 & a_{(rk+2),(rk+3)} & \cdots & a_{(rk+2),2n} \\
0 & 0 & \cdots & 0 & a_{(rk+3),(rk+2)} & ||f_{rk+2}||^2 & \cdots & a_{(rk+3),2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & a_{(2n),(rk+2)} & a_{(2n),(rk+3)} & \cdots & ||f_{rk+2}||^2 \\
\end{bmatrix}
\]
For this projection, for any $J \subset \{1, 2, \ldots, n\}$ with $|J| > r$ the family $\{e_i\}_{i \in J}$ is linealy dependent and so $PQ_A$ has a zero eigenvalue. Hence, $(I - P)Q_A$ has one as an eigenvalue and hence is not $\epsilon$-pavable for any $\epsilon > 0$. □

4. COUNTEREXAMPLES TO THE AKEMANN-ANDERSON CONJECTURE

In [1] Akemann and Anderson introduce two paving conjectures, denoted Conjecture A and Conjecture B. They prove that Conjecture A implies Conjecture B and that Conjecture B implies Kadison-Singer, but it is not known if either of these implications can be reversed. Weaver [22] provides a set of counterexamples to Conjecture A. Thus, if these three statements were all equivalent then Weaver’s counterexample would be the end of the story. However, it is generally believed that Conjecture A is strictly stronger than the Kadison-Singer conjecture.

In this section, we show that the Grammian projection matrices of any uniform, equiangular $(n,k)$-frame, with $n > 5k$ yield counterexamples to Conjecture A. It is known that infinitely many such frames exist for arbitrarily large $n$ and $k$. The significance of our new set of counterexamples is that by the results of J. Bourgain and L. Tzafriri [8], there exists $\epsilon < 1$, such that the family of self-adjoint, norm one, 0 diagonal matrices obtained from these frames is $(2, \epsilon)$-pavable.

Thus, these new examples drive an additional wedge between Conjecture A and Kadison-Singer.

We then turn our techniques to Conjecture B and derive some results that could lead to a counterexample to Conjecture B.

We now describe the Akemann-Anderson conjectures. Let $P = (p_{i,j}) \in M_n$ be the matrix of a projection and set $\delta_P = \max\{p_{i,i} : 1 \leq i \leq n\}$. By a diagonal symmetry we mean a diagonal matrix whose diagonal entries are $\pm 1$, that is, $S$ is a diagonal self-adjoint unitary.

**Conjecture A** [1] 7.1.1. For any projection, $P$ there exists a diagonal symmetry, $S$, such that $\|PSP\| \leq 2\delta_P$.

**Conjecture B** [1] 7.1.3. There exists $\gamma, \epsilon > 0$ (and independent of $n$) such that for any $P$ with $\delta_P < \gamma$ there exists a diagonal symmetry, $S$, such that $\|PSP\| < 1 - \epsilon$.

Weaver [22] states that a counterexample to Conjecture B would probably lead to a negative solution to Kadison-Singer. We believe that these two conjectures are really more closely related to 2-pavings and this is why we believe that counterexamples to Conjecture B should be close at hand.

Finally, note that Conjecture B is about paving projections with small diagonal. But our results show that Kadison-Singer is equivalent to paving projections with diagonal 1/2. This would also seem to put further distance between these Akemann-Anderson conjectures and the Kadison-Singer conjecture.
Proposition 12. Let $P = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a projection, written in block-form with $A$ an $m \times m$, $B$ an $m \times (m + l)$, $C$ a $(m + l) \times (m + l)$, where $l \geq 0$. Then there exists a $m \times m$ unitary $U_1$ and an $(m + l) \times (m + l)$ unitary $U_2$ such that, $U_1^*AU_1 = D_1, U_1^*BU_2 = (D_2, 0), U_2^*CU_2 = \begin{pmatrix} D_3 & 0 \\ 0 & D_4 \end{pmatrix}$ where each of the $D_i$'s is a diagonal matrix with non-negative entries, $D_1, D_2, D_3$ are all $m \times m$, $D_4$ is $l \times l$ with 1's and 0's for its diagonal entries and the 0's represent matrices of all zeroes that are either $m \times l$ or $l \times m$.

Proof. First note that since $P$ is a projection we have that $A^2 + BB^* = A, B^*B + C^2 = C$ and $AB + BC = B$. Also, since the rank of $B$ is at most $m$, the matrix $B^*B$ must have a kernel of dimension at least $l$.

Conjugating $P$ by a unitary of the form $U = \begin{pmatrix} I_m & 0 \\ 0 & U_2 \end{pmatrix}$, we may diagonalize $C$ and the new matrix, $P_1$, will still be a projection. Since $U_2^*B^*BU_2 = U_2^*(C - C^2)U_2$, we see that both sides of this equation are in diagonal form. Since at least $l$ of the diagonal entries of $U_2^*B^*BU_2$ are zeroes, after applying a permutation if necessary, we may assume that,

$U_2^*B^*BU_2 = \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}, U_2^*CU_2 = \begin{pmatrix} D_3 & 0 \\ 0 & D_4 \end{pmatrix},$

where $D_2, D_3, D_4$ are as claimed.

Now we may polar decompose the $m \times (m + l)$ matrix $BU_2 = W|BU_2| = W \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}$, where $W$ is a $m \times (m + l)$ partial isometry whose initial space is the range of $|BU_2|$. Thus, $W = (U_1, 0)$ where $U_1$ is an $m \times m$ partial isometry. Hence, we may extend $W_1$ to an $m \times m$ unitary $U_1$ with $W_1D_2 = U_1D_2$ and $BU_2 = (U_1, 0) \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix} = (U_1D_2, 0)$.

Conjugating $P_1$ by the unitary $\begin{pmatrix} U_1 & 0 \\ 0 & I_{m+l} \end{pmatrix}$ we arrive at a new projection of the form,

$\begin{pmatrix} U_1^*AU_1 & D_2 & 0 \\ D_2 & D_3 & 0 \\ 0 & 0 & D_4 \end{pmatrix}$.

Note that since this last matrix is a projection, $U_1^*AU_1D_2 + D_2D_3 = D_2$ and so, $U_1^*AU_1D_2$ is diagonal. If all of the entries of $D_2$ were non-zero, then this would imply that $U_2^*AU_2$ is diagonal. In general, this implies that $U_1^*AU_1$ (which is self-adjoint) is of the form a diagonal matrix direct sum with another matrix corresponding to the block where $D_2$ is 0. Conjugating $U_1^*AU_1$ by another unitary to diagonalize this lower block, yields the desired form.

Finally, note that since $D_4$ is a diagonal projection, all of its entries must be 0’s or 1’s. \qed
Lemma 13. Let \( P = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) be a non-zero projection with real entries and let \( S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Then \( \|PSP\| = |1 - 2c| \).

Proof. If \( P \) is rank 2 then \( P = I \) and the result is trivial. So assume that \( P \) is rank one. We have that \( PSP = \begin{pmatrix} a^2 - b^2 & ab - bc \\ ab - bc & b^2 - c^2 \end{pmatrix} \) and since \( P \) is a rank one projection, \( a + c = 1, b^2 + c^2 = c \). A little calculation shows that the characteristic polynomial of \( PSP \) is \( x^2 - Tr(PSP)x + Det(PSP) = x^2 - (1-2c)x \), and hence the eigenvalues are 0 and 1-2c, from which the result follows. \( \square \)

Note that when \( S \) is a diagonal symmetry, then \( \|PSP\| = \|P(-S)P\| \) and so we may and do assume in what follows that the number of \(-1\)'s is greater than or equal to the number of \(+1\)'s. Also, given a matrix \( A \), we let \( \sigma(A) \) denote the spectrum of \( A \) and set \( \sigma'(A) \equiv \sigma(A) \setminus \{0\} \).

Theorem 14. Let \( P = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \) be an \( n \times n \) projection and let \( S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \) be a diagonal symmetry. Then \( \|PSP\| \geq \max\{1 - 2\lambda : \lambda \in \sigma'(C) \cup \sigma'(A) \} \).

Proof. Given any unitary of the type in the above Proposition, we have that \( \|PSP\| = \|U^*PSPU\| = \|(U^*PU)(U^*SU)(U^*PU)\| = \|(U^*PU)S(U^*PU)\| \). Thus, we may and do assume that \( P \) has been replaced by \( U^*PU \). But this reduces the norm calculation to the direct sum of a set of \( 2 \times 2 \) matrices of the form of the lemma together with the diagonal projection \( D_4 \). Now if \( \lambda \in \sigma'(C) \), then this \( 2 \times 2 \) matrix is necessarily rank one and so the lemma applies. Note also that in this case the corresponding eigenvalue of \( D_1 \) is \( 1 - \lambda \) and that \( |1 - 2(1 - \lambda)| = |1 - 2\lambda| = |1 - 2\lambda| \) so the values of this function agree. When \( \lambda = 0 \), then this \( 2 \times 2 \) matrix is either the 0 matrix or it is rank 1 and the corresponding eigenvalue of \( D_1 \) is 1. \( \square \)

We now provide a counterexample to Conjecture A.

Theorem 15. Let \( \{f_1, \ldots, f_n\} \) be a uniform equiangular Parseval frame for \( \mathbb{C}^k \) with \( n > 2k \) and let \( P = (\langle f_i, f_j \rangle) \) be the correlation matrix. If there exists a diagonal symmetry, \( S \), such that, \( \|PSP\| \leq 2\delta_P = \frac{2k}{n} \), then \( (k - 1)n^2 \leq 4k^2(n - 1) \).

Proof. Without loss of generality we may assume that \( S \) is a diagonal symmetry with \( m \) diagonal entries that are \(+1\) and \( n - m \) diagonal entries that are \(-1\) and, \( m \leq n - m \). Putting \( P \) into the form of the Proposition, we see that since \( D_4 \) is a projection, if it is non-zero, then \( \|PSP\| = 1 \). So we may assume that \( D_4 = 0 \).

Similarly, if any of the diagonal entries of \( D_1 \) or \( D_3 \) are 1, then \( \|PSP\| = 1 \). Thus, when we put \( P \) into the form of the above Proposition, we obtain a
direct sum of $2 \times 2$ rank 1 projections, together with some matrices of all 0’s.

Let $0 < \lambda_1 \leq ... \leq \lambda_t < 1$, denote the non-zero diagonal entries of $D_1$, so that the corresponding diagonal entries of $D_3$ are $1 - \lambda_1, ..., 1 - \lambda_t$, and the remaining entries of $D_3$ are 0’s. By the above Theorem, we have that $\|PSP\| = \max\{|1 - 2\lambda_1|, |1 - 2\lambda_t|\} = \max\{1 - 2\lambda_1, 2\lambda_t - 1\} \leq \frac{2k}{n}$. Hence, $\frac{n-2k}{2n} \leq \lambda_1$ and $\lambda_t \leq \frac{n+2k}{2n}$.

Since $P$ is a rank $k$ projection, we have that $k = \text{Tr}(P) = \text{Tr}(D_1) + \text{Tr}(D_3) = t$. Since $\text{Tr}(D_1) = \text{Tr}(A) = mk/n$, we have that $0 < \lambda_1 \leq m/n \leq \lambda_t$. Hence, $\frac{n-2k}{2n} \leq m/n \leq \frac{n+2k}{2n}$ yielding $n \leq 2k + 2m$. Note also, that by the choice of $m$ we have that $2m \leq n$, so that the other inequality is automatically satisfied.

If we let, $\mu_1,...,\mu_k$ be the corresponding entries of $D_2$, then since each matrix, $\begin{pmatrix} \lambda_i & \mu_i \\ \mu_i & 1 - \lambda_i \end{pmatrix}$ is a rank one projection and since $\mu_i \geq 0$, we have that $\mu_i^2 = \lambda_i(1 - \lambda_i)$.

Since $P$ is the correlation matrix of a uniform equiangular $(n,k)$-frame, by [17], we have that every off-diagonal entry of $P$ is of constant modulus, $c = \sqrt{\frac{k(n-k)}{n^2(n-1)}}$. This yields,

$$\sum_{i=1}^{k} \mu_i^2 = \text{Tr}(B^*B) = m(n-m)c^2 \leq \frac{n^2c^2}{4} = \frac{k(n-k)}{4(n-1)}.$$

Now observe that the function $t(1 - t)$ is increasing on $[0,1/2]$ and decreasing on $[1/2,1]$. Thus, we have that $\min\{\lambda_1(1 - \lambda_1), \lambda_t(1 - \lambda_t)\} = \min\{\mu_1^2, ..., \mu_k^2\} \leq \text{Tr}(B^*B)/k \leq \frac{n-k}{4(n-1)}.$

However, since $\frac{n-2k}{2n} \leq \lambda_1$, we have $\frac{n-2k}{2n}(1 - \frac{n-2k}{2n}) = \frac{n^2-4k^2}{4n^2} \leq \lambda_1(1 - \lambda_1)$. Similarly, using the fact that $1/2 < \frac{n+2k}{2n}$, one sees that $\frac{n+2k}{2n}(1 - \frac{n+2k}{2n}) = \frac{n^2-4k^2}{4n^2} \leq \lambda_t(1 - \lambda_t)$.

Combining these inequalities, yields $\frac{n^2-4k^2}{4n^2} \leq \frac{n-k}{4(n-1)}$. Cross-multiplying and canceling like terms yields the result. \hfill $\square$

Note that the above inequality, for $n$ and $k$ large becomes asymptotically, $n \leq 4k$. Thus, any uniform, equiangular $(n,k)$-frame with $n/k >> 4$, and $n$ sufficiently large will yield a counterexample.

**Corollary 16.** There exist uniform, equiangular Parseval frames whose projection matrices are counterexamples to Conjecture A.

**Proof.** In [4, Example 6.4] a real uniform, equiangular $(276, 23)$-frame is exhibited and these values satisfy $(k-1)n^2 > 4k^2(n-1)$. In [19], uniform, equiangular $(n,k)$-frames are constructed using Singer difference sets of size,

$$n = \frac{q^{m+1} - 1}{q - 1}, \quad k = \frac{q^m - 1}{q - 1},$$
where \( q = p^r \) with \( p \) a prime. Note that \( n/k > q - 1 \). Since Singer difference sets are known to exist for infinitely large \( q \), these frames give a whole family of counterexamples.

We now turn our attention to Conjecture B. We let \( \gamma, \epsilon > 0 \) be as in the statement of the conjecture. For each partition of \( \{1, \ldots, n\} = R \cup T \) into two disjoint sets, \( R, T \), we let \( Q_R, Q_T \) denote the corresponding diagonal projections.

**Theorem 17.** Let \( \gamma, \epsilon > 0 \) be fixed, let \( \{f_1, \ldots, f_n\} \) be a uniform Parseval frame for \( \mathbb{R}^k \) with \( k/n < \min\{\gamma, \epsilon/2, 1/2\} \) and let \( P = ((f_i, f_j)) \) be the correlation matrix. If Conjecture B is true for the pair \( \gamma, \epsilon \), then there exists a partition \( \{1, \ldots, n\} = R \cup T \) such that

\[
\text{Tr}(Q_R P Q_T P Q_R) \geq \frac{k\epsilon(2-\epsilon)}{4}.
\]

**Proof.** Each such partition defines a diagonal symmetry as before and corresponding to such a partition we write

\[
\begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix}
\]

so that \( \text{Tr}(Q_R P Q_T P Q_R) = \text{Tr}(BB^*) \).

We have that \( \delta_P = k/n < \gamma \). We repeat the proof above, with \( m = \min\{|R|, |T|\} \).

Letting \( \lambda_1 \) be the minimum non-zero eigenvalue and \( \lambda_k \) the largest eigenvalue of \( A \) as before, we have \( 1 - \epsilon \geq \|PSP\| \geq \max\{|1 - 2\lambda_1|, |1 - 2\lambda_k|\} \)

and, hence, \( \lambda_1 \geq \epsilon/2 \) and \( 1 - \lambda_k \geq \epsilon/2 \).

Using the properties of the function \( t \to t(1-t) \) and the fact that \( \sum_{i=1}^{k} \lambda_i(1 - \lambda_i) = \text{Tr}(B^*B) \), we have that \( \epsilon/2(1 - \epsilon/2) \leq \min\{\lambda_1(1 - \lambda_1), \lambda_k(1 - \lambda_k)\} \leq 1/k\text{Tr}(B^*B) \), which yields the result.

Using equiangular frames we can obtain a relation between \( \gamma \) and \( \epsilon \) in Conjecture B.

**Theorem 18.** Assume that Conjecture B is true for a pair \( (\gamma, \epsilon) \) and let \( \{f_1, \ldots, f_n\} \) be a uniform, equiangular \((n,k)\)-frame with \( k/n \leq \gamma \). Then

\[
\epsilon(2 - \epsilon) \leq \frac{q-1}{q}.
\]

**Proof.** By the above theorem, we have that there exists a partition with \( |R| = m \), such that

\[
\frac{k\epsilon(2-\epsilon)}{4} \leq \text{Tr}(Q_R P Q_T P Q_R) = m(n - m)\epsilon^2 \leq \frac{n^2}{4} \epsilon^2 = \frac{k(n-k)}{4(n-1)}.\]

If we have that infinitely many uniform, equiangular \((n,k)\)-frames exist for which \( n \to +\infty \) and \( k/n \to \gamma \), then

\[
\frac{n-k}{n-1} = \frac{1 - k/n}{1 - 1/n} \to 1 - \gamma,
\]

and hence, \( \epsilon(2 - \epsilon) \leq 1 - \gamma \). If for a given prime \( p \), there are infinitely many Singer difference sets, with \( q = p^r \), and we choose \( 1/q \leq \gamma \) then we get that \( \epsilon(2 - \epsilon) < \frac{q-1}{q} \).
Unfortunately, there are no uniform Parseval \((n,k)\) frames which violate the trace inequality in Theorem 17, so that finding a counter-example to Conjecture B is more subtle. We will show this below.

First, let us change the notation. If \(\{f_i\}_{i=1}^n\) is a Parseval frame for \(l_k^2\) with analysis operator \(V\) then the frame operator is \(S = V^*V = I\) and \(P = VV^*\) is a projection on \(l_k^2\) onto the image of the analysis operator (which is now an isometry). Let \(\{R,T\}\) be a partition of \(\{1,2,\ldots,n\}\). If \(x = \sum_{i=1}^n a_ie_i\) then
\[
Q_Rx = \sum_{i \in R} a_ie_i.
\]
Next,
\[
PQ_Rx = \sum_{j=1}^n \left(\sum_{i \in R} a_if_i, f_j\right)e_j.
\]
Finally,
\[
Q_TPQ_Rx = \sum_{j \in T} \left(\sum_{i \in R} a_if_i, f_j\right)e_j.
\]
It follows that
\[
Q_TPQ_Re_i = \sum_{i \in R} \sum_{j \in T} \langle f_i, f_j \rangle e_j.
\]
Now we have:

**Lemma 19.** Given the conditions above we have
\[
\text{Trace}(Q_RPQT PQ_R) = \sum_{i \in R} \sum_{j \in T} |\langle f_i, f_j \rangle|^2.
\]

**Proof.** We compute:
\[
\sum_{i=1}^n \langle Q_RPQT PQ_Re_i, e_i \rangle = \sum_{i=1}^n \langle Q_TPQ_Re_i, PQ_Re_i \rangle
\]
\[
= \sum_{i=1}^n \langle Q_TPQ_Re_i, Q_TPQ_Re_i \rangle
\]
\[
= \sum_{i=1}^n \|Q_TPQ_Re_i\|^2
\]
\[
= \sum_{i \in R} \sum_{j \in T} |\langle f_i, f_j \rangle|^2.
\]

Now we need to recall a result of Berman, Halpern, Kaftal and Weiss [5].

**Theorem 20.** Let \((a_{ij})_{i,j=1}^n\) be a self-adjoint matrix with non-negative entries and with zero diagonal so that
\[
\sum_{m=1}^n a_{im} \leq B, \text{ for all } i = 1, 2, \ldots, n.
\]
Then for every \( r \in \mathbb{N} \) there is a partition \( \{ A_j \}_{j=1}^r \) of \( \{1, 2, \ldots, n\} \) so that for every \( j = 1, 2, \ldots, r \),

\[
\sum_{m \in A_j} a_{im} \leq \sum_{m \in A_\ell} a_{im}, \quad \text{for every } i \in A_j \text{ and } \ell \neq j.
\]

Now we are ready for our result.

**Proposition 21.** If \( \{ f_i \}_{i=1}^n \) is a uniform \((n,k)\)-Parseval frame, then there is a partition \( \{ R, T \} \) of \( \{1, 2, \ldots, n\} \) so that

\[
\text{Trace}(Q_R P Q_T P Q_R) \geq \frac{k}{4} (1 - \frac{k}{n}).
\]

In particular, if \( \frac{k}{n} \) is small then the trace inequality of Theorem 17 holds.

**Proof.** Applying 20 to the matrix of values \( (a_{ij})_{i,j=1}^n \) where \( a_{ii} = 0 \) and \( a_{ij} = |\langle f_i, f_j \rangle|^2 \) for \( i \neq j \) we can find a partition \( \{ R, T \} \) of \( \{1, 2, \ldots, n\} \) (and without loss of generality we may assume that \( |R| \geq \frac{n}{2} \)) satisfying for all \( i \in R \):

\[
\sum_{j \in T} |\langle f_i, f_j \rangle|^2 \leq \sum_{j \in T} |\langle f_i, f_j \rangle|^2.
\]

It follows that for all \( i \in R \):

\[
\frac{k}{n} = \sum_{j=1}^n |\langle f_i, f_j \rangle|^2 \leq \frac{k^2}{n^2} + \sum_{i \neq j \in R} |\langle f_i, f_j \rangle|^2 + \sum_{j \in T} |\langle f_i, f_j \rangle|^2 \leq \frac{k^2}{n^2} + 2 \sum_{j \in T} |\langle f_i, f_j \rangle|^2.
\]

It follows that for all \( i \in R \)

\[
\sum_{j \in T} |\langle f_i, f_j \rangle|^2 \geq \frac{1}{2} \left( \frac{k}{n} - \frac{k^2}{n^2} \right).
\]

Now,

\[
\sum_{i \in R} \sum_{j \in T} |\langle f_i, f_j \rangle|^2 \geq |R| \frac{1}{2} \left( \frac{k}{n} - \frac{k^2}{n^2} \right) \geq \frac{n}{2} \frac{1}{2} \left( \frac{k}{n} - \frac{k^2}{n^2} \right) = \frac{k}{4} \left( 1 - \frac{k}{n} \right).
\]

Now, given \( 0 < \epsilon < 1 \),

\[
\frac{\epsilon}{2} \left( 1 - \frac{\epsilon}{2} \right) < \frac{1}{4}
\]
So the trace inequality of Theorem 17 will hold provided
\[ \frac{k}{4} (1 - \frac{k}{n}) \geq \frac{k\epsilon(2 - \epsilon)}{4}, \]
which is true for \( k/n \) small enough. \( \square \)

In fact, as with the case of equiangular frames, we see that if \( 1 - \gamma \geq \epsilon(2 - \epsilon) \), then whenever, \( k/n \leq \gamma \), we have that
\[ \text{Trace}(QRPPQTQPQ) \geq \frac{k}{4} (1 - \frac{k}{n}) \geq \frac{k}{4} (1 - \gamma) \geq \frac{k}{4} \epsilon(2 - \epsilon). \]

5. A Family of Potential Counterexamples

It is still unknown if the paving conjectures are true even for a smaller family of operators known as the Laurent operators. In this section we introduce a family of Laurent operators that we believe are potential counterexamples to the paving conjecture. We also prove some results about these operators that lends credence to the belief that they might yield counterexamples. For the purposes of this section, it will be convenient to replace the countable index set \( \mathbb{N} \) by \( \mathbb{Z} \).

Recall that a matrix, \( A = (a_{i,j})_{i,j \in \mathbb{Z}} \) is called a Laurent matrix if it is constant on diagonals, i.e., \( a_{i,j} = a(i-j) \) and that in this case \( A \) determines a bounded operator on \( \ell^2(\mathbb{Z}) \) if and only if there exists \( f \in L^\infty[0,1] \) such that \( a(n) = \hat{f}(n) = \int_0^1 f(t)e^{-2\pi i nt} dt \) and in this case we set \( A = L_f \) and call it the Laurent operator with symbol \( f \). Indeed, the Laurent operator \( L_f \) is just the matrix representation of the operator of multiplication by \( f, M_f \) on the space \( L^2[0,1] \) with respect to the orthonormal basis, \( \{e^{2\pi int}\}_{n \in \mathbb{Z}} \). So, in particular, \( L_f \) is self-adjoint with diagonal 0 if and only if \( f \) is real-valued a.e. and \( \int_0^1 f(t) dt = 0 \).

The problem of paving Laurent operators was first studied in \( \cite{16} \) where it was shown that Laurent operators with Riemann integrable symbols can be paved. Further work on the relation between Laurent operators and the Feichtinger conjecture can be found in Bownik and Speegle \( \cite{9} \).

Note that \( L_f \) is a projection if and only if \( f = \chi_E \) for some measurable set \( E \) and \( L_f \) is a reflection if and only if \( f = 2\chi_E - 1 \), for some measurable set \( E \). This reflection will have 0 diagonal when \( m(E) = 1/2 \), where \( m \) denotes Lebesgue measure. Thus, modulo the change from \( \mathbb{N} \) to \( \mathbb{Z} \), the family of Laurent operators corresponding to our set \( \mathcal{R} \) is exactly the set of operators of the form \( L_f, f = 2\chi_E - 1, m(E) = 1/2 \) and to \( \mathcal{P}_{1/2} \) is the set of operators of the form \( L_f, f = \chi_E, m(E) = 1/2 \).

Hence, we are interested in the Laurent operators that arise from certain subsets \( E \) with \( m(E) = 1/2 \). It is known that for every \( t, 0 < t < 1 \), there exists a measurable set \( E = E_t \) with \( m(E) = t \), and such that for every \( 0 < a < b < 1, m(E \cap (a,b)) > 0 \) and \( m(E^c \cap (a,b)) > 0 \), where \( E^c = [0,1] \setminus E \). One way to construct such a set is as a countable union of fat Cantor sets.
We believe that the projections and reflections coming from such sets for $t = 1/2$, are good candidates for counterexamples to the paving conjectures and we outline our reasons below.

**Proposition 22.** Let $E$ be a set as above for any, $0 < t < 1$. If $f_1, f_2$ are continuous functions such that $f_1 \leq \chi_E \leq f_2$, a.e., then $f_1 \leq 0$ and $1 \leq f_2$.

**Proof.** Since $\chi_E$ is zero on a set of positive measure in every interval, $f_1 \leq 0$. Similarly, $\chi_E$ is one on a set of positive measure in every interval and hence, $1 \leq f_2$.

The above inequalities show that $\chi_E$ is far from Riemann integrable.

**Proposition 23.** Let $g, h \in L^\infty[0, 1]$, with $0 \leq h \leq 1$. If for every $f_1, f_2 \in C[0, 1]$, we have that $f_1 \leq g \leq f_2$, a.e., implies that $f_1 \leq 0, 1 \leq f_2$, then there exists a positive linear map, $\phi : L^\infty[0, 1] \to L^\infty[0, 1]$ such that $\phi(f) = f$ for every $f \in C[0, 1]$ and $\phi(g) = h$.

**Proof.** First define $\phi$ on the linear span of $C[0, 1]$ and $g$ by $\phi(f + ag) = f + ah$, and note that the inequalities imply that if $f + ag \geq 0$, then $f + ah \geq 0$. Hence, $\phi$ is a positive map. Now using the fact that $L^\infty[0, 1]$ is an abelian, injective operator system, this map has a (completely) positive extension to all of $L^\infty[0, 1]$.

**Proposition 24.** Let $g, h \in L^\infty[0, 1]$, with $0 \leq h \leq 1$. If for every $f_1, f_2 \in C[0, 1]$, we have that $f_1 \leq g \leq f_2$, a.e., implies that $f_1 \leq 0, 1 \leq f_2$, then there exists a completely positive linear map, $\phi : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}))$ such that $\phi(L_f) = L_f$ for every Laurent operator with continuous symbol, $f \in C[0, 1]$ and $\phi(L_g) = L_h$.

**Proof.** The identification of $L^\infty[0, 1]$ with the space of Laurent operators is a complete order isomorphism. Hence, there exists a completely positive projection of $B(\ell^2(\mathbb{Z}))$ onto the space of Laurent operators. The remainder of the proof now follows from the last Proposition.

**Theorem 25.** Let $E \subset [0, 1]$ be a measurable set with $m(E) = 1/2$ such that for every $0 < a < b < 1, m(E \cap (a, b)) > 0$ and $m(E^c \cap (a, b)) > 0$ and let $P$ denote the projection that is the Laurent operator with symbol $\chi_E$. Then there exist completely positive maps, $\phi, \psi : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}))$ such that $\phi(L_f) = \psi(L_f) = L_f$ for every Laurent operator with continuous symbol $f$, but $\phi(P) = 0, \psi(P) = I$.

**Proof.** Apply the above Proposition with $h = 0$ and $h = 1$, respectively.

Thus, for the Laurent reflection with 0 diagonal, $R = 2P - I$, we have that $\phi(R) = -I, \psi(R) = +I$ even though these maps fix all Laurent operators with continuous symbols. In this sense, the ”value” of the diagonal of $R$ is not very stationary under completely positive maps which fix all Laurent operators with continuous symbol. In fact, it follows from the theory of completely positive maps, that the maps $\phi$ and $\psi$ constructed
above are actually bimodule maps over the C*-algebra of Laurent operators with continuous symbol. That is, \( \phi(L_{f_1}X L_{f_2}) = L_{f_1} \phi(X) L_{f_2} \), and \( \psi(L_{f_1}X L_{f_2}) = L_{f_1} \psi(X) L_{f_2} \) for any continuous functions, \( f_1, f_2 \) and any \( X \in B(\ell^2(\mathbb{Z})) \).

One suspects that the fact that the diagonal of \( R \) can be altered so dramatically, while fixing so many other operators, might be an obstruction to \( R \) being paved. It is also intriguing that for a suitable choice of the set \( E \), one can actually compute the coefficients of the Laurent matrix for \( R \), albeit as power series.

**Acknowledgments**

The authors would like to thank Don Hadwin for several valuable conservations.

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Department of Mathematics, University of Missouri
E-mail address: pete@math.missouri.edu

Department of Mathematics, University of Missouri
E-mail address: edidin@math.missouri.edu

Department of Mathematics, University of Houston, Houston, Texas 77204-3476, U.S.A.
E-mail address: deepti@math.uh.edu

Department of Mathematics, University of Houston, Houston, Texas 77204-3476, U.S.A.
E-mail address: vern@math.uh.edu