UNIQUENESS OF RANKIN-SELBERG PERIODS

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ABSTRACT. Let \( k \) be a local field of characteristic zero. Rankin-Selberg’s local zeta integrals produce linear functionals on generic irreducible admissible smooth representations of \( \text{GL}_n(k) \times \text{GL}_r(k) \), with certain invariance properties. We show that up to scalar multiplication, these linear functionals are determined by the invariance properties.

1. INTRODUCTION

The goal of this paper is to prove uniqueness of certain local linear functionals that naturally occur in the study of Rankin-Selberg convolutions. We first treat the archimedean case, and leave the non-archimedean case to Section 6. So assume that \( k \) is an archimedean local field throughout this paper except for Section 6.

The representations that we will consider are so called Casselman-Wallach representations. They appear as archimedean components of automorphic representations. Recall that a representation of a real reductive group is called a Casselman-Wallach representation if it is smooth, Fréchet, of moderate growth, and its Harish-Chandra module has finite length. The reader may consult [Ca], [Wa, Chapter 11] or [BK] for details about Casselman-Wallach representations.

To explain the motivation of our paper, we first give a brief introduction to the local Rankin-Selberg integrals. The reader is referred to [CPS, Ja, JPS, JS2] for more details. Fix two integers \( n > r \geq 0 \). Let \( \pi \) and \( \sigma \) be generic irreducible Casselman-Wallach representations of \( \text{GL}_n(k) \) and \( \text{GL}_r(k) \), respectively. For abbreviation and as usual, we do not distinguish a representation with its underlying vector space. For every integer \( m \geq 0 \), write \( N_m(k) \) for the subgroup of \( \text{GL}_m(k) \) consisting of upper-triangular unipotent matrices.

Fix generators

\[
\lambda_n \in \text{Hom}_{N_n(k)}(\pi, \psi_n) \quad \text{and} \quad \lambda_r \in \text{Hom}_{N_r(k)}(\sigma, \psi_r^{-1})
\]

of one dimensional spaces, where \( \psi_n \) is a generic unitary character of \( N_n(k) \), and \( \psi_r \) is its restriction to \( N_r(k) \) through the embedding

\[
g \mapsto \begin{bmatrix} g & 0 \\ 0 & 1_{n-r} \end{bmatrix}
\]

(for every \( m \geq 1 \), \( 1_m \) denotes the identity matrix of size \( m \)).

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For every $u \in \pi$ and $v \in \sigma$, define the Whittaker functions $W_u : \text{GL}_n(k) \to \mathbb{C}$ and $W'_v : \text{GL}_r(k) \to \mathbb{C}$ by

$$W_u(g) := \lambda_n(g.u) \quad \text{and} \quad W'_v(g) := \lambda_r(g.v).$$

The Rankin-Selberg integral associated to the pair $(W_u, W'_v)$ is defined to be

$$\Psi(s, W_u, W'_v) := \int_{N_r(k) \setminus \text{GL}_r(k)} W_u \left( \begin{bmatrix} g & 0 \\ 0 & 1_{n-r} \end{bmatrix} \right) W'_v(g) |\det g|^{s-n-r/2} \, dg,$$

where $s \in \mathbb{C}$ and $dg$ denotes a right $\text{GL}_r(k)$-invariant positive Borel measure on $N_r(k) \setminus \text{GL}_r(k)$. The integral is absolutely convergent when the real part of $s$ is sufficiently large, and has a meromorphic continuation to the whole complex plane $\mathbb{C}$. Moreover, for every $s_0 \in \mathbb{C}$, we have a well-defined non-zero continuous linear functional

$$\Psi_{s_0} : \hat{\pi} \otimes \sigma \to \mathbb{C}, \quad u \otimes v \mapsto \Psi(s, W_u, W'_v) \big|_{s=s_0},$$

where $L(s, \pi \times \sigma)$ denotes the local L-function, and the symbol “$\hat{\otimes}$” stands for the completed projective tensor product.

Define the $r$-th Rankin-Selberg subgroup $R_r$ of $\text{GL}_n(k)$ to be

$$R_r := \left\{ \begin{bmatrix} g & u \\ 0 & h \end{bmatrix} \mid g \in \text{GL}_r(k), u \in M^0_{r,n-r}, h \in N_{n-r}(k) \right\},$$

where

$$M^0_{r,n-r} := \{ x \in M_{r,n-r}(k) \mid \text{the first column of } x \text{ is zero} \}.$$ 

Here and henceforth, for every pair $i, j$ of non-negative integers, $M_{i,j}(k)$ denotes the space of $i \times j$-matrices with coefficients in $k$.

View $\sigma$ as a representation of $R_r$ by the inflation through the obvious homomorphism $R_r \to \text{GL}_r(k)$. Then the tensor product $\pi \hat{\otimes} \sigma$ is also a representation of $R_r$. It is easily checked that the functional $\Psi_{s_0}$ of (2) has the following invariance property:

$$\Psi_{s_0}(g.w) = \chi_{s_0}(g)\Psi_{s_0}(w), \quad g \in R_r, \ w \in \pi \hat{\otimes} \sigma,$$

where $\chi_{s_0}$ denotes the following character on $R_r$:

$$\left[ \begin{array}{cc} g & u \\ 0 & h \end{array} \right] \mapsto |\det g|^{s_0-n-r/2} \psi_n \left( \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix} \right).$$

Note that every character on $R_r$ factors through the quotient map $R_r \to \text{GL}_r(k) \times N_{n-r}(k)$. We say that a character on $R_r$ is generic if its descent to $\text{GL}_r(k) \times N_{n-r}(k)$ has the form $\chi_r \otimes \psi$, where $\chi_r$ is a character on $\text{GL}_r(k)$, and $\psi$ is a generic unitary character on $N_{n-r}(k)$. It is clear that $\chi_{s_0}$ is a generic character on $R_r$.

The following uniqueness theorem implies that the Rankin-Selberg integral is characterized by the invariance property (4).
Theorem A. For all irreducible Casselman-Wallach representations \( \pi \) of \( GL_n(k) \), and \( \sigma \) of \( GL_r(k) \), and for all generic characters \( \chi \) of \( R_r \), one has that
\[
\dim \text{Hom}_{R_r}(\pi \widehat{\otimes} \sigma, \chi) \leq 1.
\]

Here \( \pi \widehat{\otimes} \sigma \) is viewed as a representation of \( R_r \) as before. Theorem A is well known in two extremal cases: if \( r = 0 \), then \( R_r = N_n(k) \), and Theorem A asserts the uniqueness of Whittaker models for \( GL_n(k) \) (see [Sha, CHM]); if \( r = n - 1 \), then \( R_r = GL_{n-1}(k) \), and Theorem A asserts the multiplicity one theorem for \( GL_n(k) \) (see [AG, AGRS, SZ]).

We remark that Rankin-Selberg method implies that the hom spaces of (5) and (7) are at least one dimensional when the representations involved are generic.

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σ⊗τ_s is viewed as a representation of P_{r,n-r} by the inflation through the obvious homomorphism P_{r,n-r} \to G_r \times G_{n-r}.

Combining Langlands classification with the result of Speh-Vogan ([SV]), we have

**Proposition 2.1.** The representation ρ_s of G_n is irreducible except for a measure zero set of s ∈ C.

As in the Theorem A let χ be a generic character of R_r. In order to prove Theorem A, replacing σ by its tensor product with a suitable character, we may (and do) assume that

(9) the decent of χ to G_r × N_{n-r} has the form 1 ⊗ ψ,

where 1 stands for the trivial character on G_r, and ψ is a generic unitary character on N_{n-r}. Fix a non-zero element

\[ \lambda \in \text{Hom}_{N_{n-r}}(\tau, \psi^{-1}). \]

Define a continuous linear map

\[ \Lambda : \sigma \otimes \tau_s = \sigma \otimes \tau \to \sigma, \quad u \otimes v \mapsto \lambda(v)u \quad (s \in C). \]

Let π be an irreducible Casselman-Wallach representation of G_n as in Theorem A and let

\[ \langle , \rangle_{\mu_\pi} : \pi \times \sigma \to \mathbb{C} \]

be a continuous bilinear map which represents an element \( \mu \in \text{Hom}_{R_r}(\pi \otimes \sigma, \chi) \).

Write e_1, e_2, \ldots, e_n for the standard basis of k^n. Note that the vector \( e_{r+1} \) is fixed by the group R_r. Recall the representation \( \omega \) of G_n from the Introduction. Using the discussion of [Tr, page 533], we identify \( \rho_s \otimes \omega \) with the space of \( \rho_s \)-valued Schwartz functions on k^n. Then (9) obviously implies the following

**Lemma 2.2.** For every \( s \in C, v \in \pi \) and \( \phi \in \rho_s \otimes \omega \), the function

\[ g \mapsto \langle g.v, \Lambda(\phi(g^{-1}e_{r+1})(g)) \rangle_{\mu_\pi} \]

on G_n is left R_r-invariant.

The group R_r is not unimodular in general. Nevertheless, we still have the following

**Lemma 2.3.** (cf. [Lo Theorem 33D]) Up to scalar multiplication, there exists a unique positive Borel measure \( dg \) on \( R_r \backslash G_n \) such that

\[ \int_{R_r \backslash G_n} \varphi(gg_0) \, dg = |\det(g_0)|_k^{r+1-n} \int_{R_r \backslash G_n} \varphi(g) \, dg, \]

for all \( g_0 \in G_n \) and all non-negative measurable function \( \varphi \) on \( R_r \backslash G_n \).
Lemma 2.2 allows us to define the following zeta integrals which play a key role in the proof of Theorem A:

\[ Z_\mu(v, \phi) := \int_{R \setminus G_n} \langle g,v \rangle_N \Lambda \left( \phi(g^{-1}v_{r+1}(g)) \right) d g, \]

where \( v \in \pi, \phi \in \rho \hat{\otimes} \omega, \) and \( d g \) is as in Lemma 2.3. It follows from Lemma 2.3 that this integral has the following invariance property:

\[ Z_\mu(g Set) = |det(g)|_{k}^{r+1-n} Z_\mu(v, \phi), \quad g \in G_n. \]

The remainder of this section is devoted to the proof of Theorem A. To this end, let us first state two results about the integral \( Z_\mu(v, \phi) \), which will be proved in Sections 4 and 5 respectively.

**Proposition 2.4.** If \( \mu \neq 0 \), then for every \( s \in \mathbb{C} \), there exist \( v \in \pi \) and \( \phi \in \rho \hat{\otimes} \sigma \) such that the integral \( Z_\mu(v, \phi) \) is absolutely convergent and non-zero.

**Proposition 2.5.** There is a real constant \( c_\mu \), depending on \( \pi, \sigma, \chi, \tau \) and \( \mu \), such that for every \( \mu \in \mathbb{R} \) whose real part \( > c_\mu \), the integrals in (10) are absolutely convergent, and produce a continuous linear functional in \( \text{Hom}_{G_n}(\pi \hat{\otimes} \rho \hat{\otimes} \sigma, |det|_{k}^{r+1-n}) \).

We are now in a position to complete the proof of Theorem A. Let \( F \) be a finite dimensional subspace of \( \text{Hom}_{R_r}(\pi \hat{\otimes} \sigma, \chi) \). By Proposition 2.4 and Proposition 2.5, we have an injective linear map

\[ F \rightarrow \text{Hom}_{G_n}(\pi \hat{\otimes} \rho \hat{\otimes} \omega, |det|_{k}^{r+1-n}), \quad \mu \mapsto Z_\mu, \]

for all \( s \in \mathbb{C} \) with sufficiently large real part. By Proposition 2.3 and Theorem B, the hom space in (11) is at most one dimensional except for a measure zero set of \( s \in \mathbb{C} \). Therefore \( F \) is at most one dimensional. This finishes the proof of Theorem A.

3. Preliminaries on Schwartz inductions

For the proof of Proposition 2.4, we recall in this section some properties of Schwartz inductions ([dC, Section 2]). We work in the setting of Nash manifolds and Nash groups. The reader is referred to [Shi] or [Su2] for the notions of Nash manifolds, Nash maps, Nash submanifolds, and the related notion of semialgebraic sets.

By a Nash group, we mean a group which is simultaneously a Nash manifold so that all group operations (the multiplication and the inversion) are Nash maps. Every semialgebraic subgroup of a Nash group is automatically closed and is called a Nash subgroup. Every Nash subgroup is canonically a Nash group.

An action of a Nash group \( G \) on a Nash manifold \( M \) is called a Nash action if the action map \( G \times M \rightarrow M \) is a Nash map. Likewise, a finite dimensional real representation \( V_K \) of a Nash group \( G \) is called a Nash representation if the action
map $G \times V_\mathbb{R} \to V_\mathbb{R}$ is a Nash map. A Nash group is said to be almost linear if it admits a Nash representation with finite kernel. Structures and basic properties of almost linear Nash groups are studied in detail in [Su2]. Recall that a Nash manifold is said to be affine if it is Nash diffeomorphic to a Nash submanifold of some $\mathbb{R}^n$. All almost linear Nash groups are affine as Nash manifolds.

Let $M$ be an affine Nash manifold. For each complex Fréchet space $V_0$, a $V_0$-valued smooth function $f \in C^\infty(M; V_0)$ is said to be Schwartz if

$$|f|_{D, \nu} := \sup_{x \in M} |(Df)(x)|_\nu < \infty$$

for all Nash differential operators $D$ on $M$, and all continuous seminorms $|\cdot|_\nu$ on $V_0$. Recall that a differential operator $D$ on $M$ is said to be Nash if $D\varphi$ is a Nash function whenever $\varphi$ is a ($\mathbb{C}$-valued) Nash function on $M$. Denote by $C^\infty(M; V_0) \subset C^\infty(M; \mathbb{C})$ the subspace of all Schwartz functions. Then both $C^\infty(M; V_0)$ and $C^\infty(M; \mathbb{C})$ are naturally Fréchet spaces, and the inclusion map $C^\infty(M; V_0) \hookrightarrow C^\infty(M; \mathbb{C})$ is continuous. Furthermore, we have that (cf. [Tr, page 533])

$$C^\infty(M; V_0) = C^\infty(M) \hat{\otimes} V_0 \quad \text{and} \quad C^\infty(M; \mathbb{C}) = C^\infty(M) \hat{\otimes} \mathbb{C},$$

where $C^\infty(M) := C^\infty(M; \mathbb{C})$ and $C^\infty(M)$ := $C^\infty(M; \mathbb{C})$.

Let $G$ be an almost linear Nash group, $S$ a Nash subgroup of $G$, and let $V_0$ be a smooth Fréchet representation of $S$ of moderate growth (for the usual notion of smooth Fréchet representation of moderate growth, see [IC, Definition 1.4.1] or [Su1, Section 2], for example). We define the un-normalized Schwartz induction

$$\text{ind}^G_S V_0$$

to be the image of the following continuous linear map:

$$i_{S, V_0} : C^\infty(G; V_0) \to C^\infty(G; V_0), \quad f \mapsto \left( g \mapsto \int_S s \cdot f(s^{-1}g) \, ds \right),$$

where $ds$ is a left invariant Haar measure on $S$. Under the quotient topology of $C^\infty(G; V_0)$ and under right translations, this is a smooth Fréchet representation of $G$ of moderate growth.

We will use the following basic properties of Schwartz inductions for almost linear Nash groups.

**Lemma 3.1.** Let $G$ be an almost linear Nash group, $S$ a Nash subgroup of $G$, and let $V_0$ be a smooth Fréchet representation of $S$ of moderate growth.

(a) For each Nash subgroup $S'$ of $G$ containing $S$, the map

$$\text{ind}^G_S V_0 \to \text{ind}^{G'}_S \text{ind}^G_S V_0,$$

$$f \mapsto (g \mapsto (s' \mapsto f(s'g)))$$

is well-defined and is an isomorphism of representations of $G$. 

(b). For each Nash subgroup $T$ of $G$ such that $ST$ is open in $G$, the map
\[ \text{ind}^T_{S \cap T}(V_0|_{S \cap T}) \to (\text{ind}^G_S V_0)|_T, \]
\[ \phi \mapsto \begin{cases} s.\phi(t), & \text{if } g = st \in ST; \\ 0, & \text{otherwise} \end{cases} \]
is well-defined and is an injective homomorphism of representations of $T$.

(c). Let $V$ be a smooth Fréchet representation of $G$ of moderate growth, and let $G \times M \to M$ be a transitive Nash action of $G$ on an affine Nash manifold $M$. Assume that $V$ is nuclear as a Fréchet space, and $S$ equals the stabilizer in $G$ of a point $x_0 \in M$. Then the map
\[ \text{ind}^G_S (V|_S) \to C^\infty(M; V), \]
\[ \phi \mapsto (g.x_0 \mapsto g.\phi(g^{-1})) \]
is well-defined and is an isomorphism of representations of $G$. Here the action of $G$ on $C^\infty(M; V)$ is given by
\[ (g.\psi)(x) := g.(\psi(g^{-1}.x)), \quad \psi \in C^\infty(M; V), \quad g \in G, \quad x \in M. \]

(d). Let $f \in C^\infty(G; V_0)$. If
\[ f(sg) = s.f(g), \quad s \in S, \quad g \in G, \]
and $f$ is compactly supported modulo $S$, then $f \in \text{ind}^G_S V_0$.

Proof. Part (a) of the lemma is [dC, Lemma 2.1.6], and part (b) follows by the arguments of [dC, Section 2.2]. Part (c) is a special case of [LS, Lemma 3.2], and part (d) is [LS, Lemma 3.1].

4. PROOF OF PROPOSITION 2.4

We continue with the notation of Section 2. We first present a measure decomposition for the measure $dg$ given in Lemma 2.3. For every $m \geq 1$, write
\[ P_m := \left\{ \begin{bmatrix} 1_r & * \\ 0 & * \end{bmatrix} \in G_m \right\} \quad \text{and} \quad Q_m := \left\{ \begin{bmatrix} * & 0 \\ * & 1_{m-1} \end{bmatrix} \in G_m \right\}. \]
Respectively write $G_{n-r}$, $P'_{n-r}$, $Q'_{n-r}$ and $N'_{n-r}$ for the image of $G_{n-r}$, $P_{n-r}$, $Q_{n-r}$ and $N_{n-r}$, under the embedding
\[ G_{n-r} \to G_n, \quad g \mapsto \begin{bmatrix} 1_r & 0 \\ 0 & g \end{bmatrix}. \]
Put
\[ U_r := \left\{ \begin{bmatrix} 1_r & u \\ 0 & 1_{n-r} \end{bmatrix} \mid u = \begin{bmatrix} * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{bmatrix} \right\}. \]
and
\[ \tilde{U}_{r,n-r} := \left\{ \begin{bmatrix} 1_r & 0 \\ * & 1_{n-r} \end{bmatrix} \right\}. \]

**Lemma 4.1.** The multiplication map
\[ (14) \quad U_r \times (N'_{n-r} \setminus P'_{n-r}) \times Q'_{n-r} \times \tilde{U}_{r,n-r} \to R_r \setminus G_n \]
is a well-defined open embedding, and its image has full measure in \( R_r \setminus G_n \). Moreover, the restriction of the measure \( dg \) (see Lemma 2.3) through the embedding (14) has the form \( du \otimes dp' \otimes dq' \otimes \tilde{du} \), where \( du, dp', dq' \) and \( \tilde{du} \) are positive Borel measures on \( U_r, N'_{n-r} \setminus P'_{n-r}, Q'_{n-r} \) and \( \tilde{U}_{r,n-r} \), respectively.

**Proof.** Note that the multiplication map
\[ (15) \quad (R_r \setminus P_{r,n-r}) \times \tilde{U}_{r,n-r} \to R_r \setminus G_n \]
is an open embedding whose image has full measure in \( R_r \setminus G_n \). The relative-invariance of \( dg \) implies that its restriction through (15) has the form \( dp \otimes \tilde{du} \), where \( \tilde{du} \) is a Haar measure on \( \tilde{U}_{r,n-r} \), and \( dp \) is a positive Borel measure on \( R_r \setminus P_{r,n-r} \) which is relative-invariant under right translations of \( G'_{n-r} \).

Note that
\[ u N'_{n-r} u^{-1} \subset R_r \quad \text{for all} \quad u \in U_r. \]
Therefore the multiplication map
\[ (16) \quad U_r \times (N'_{n-r} \setminus G'_{n-r}) \to R_r \setminus P_{r,n-r} \]
is well-defined. Easy calculation shows that it is in fact a diffeomorphism. By \( G'_{n-r} \)-relative-invariance, the pull back of \( dp \) through (16) has the form \( du \otimes dq' \), where \( du \) is a positive Borel measure on \( U_r \), and \( dq' \) is a relative-invariant positive Borel measure on \( N'_{n-r} \setminus G'_{n-r} \).

Now, the multiplication map
\[ (17) \quad (N'_{n-r} \setminus P'_{n-r}) \times Q'_{n-r} \to N'_{n-r} \setminus G'_{n-r} \]
is an open embedding whose image has full measure in \( N'_n \setminus G'_n \). The relative-invariance of \( dq' \) implies that its restriction through (17) has the form \( dp' \otimes dq' \), where \( dp' \) and \( dq' \) are as in the Lemma. This finishes the proof. \( \square \)

Denote by \( P^o_{r+1} \) the stabilizer of \( e_{r+1} \in k^n \) in \( G_n \). It consists all matrices in \( G_n \) whose \((r+1)\)-th column equals \( e_{r+1} \).

**Lemma 4.2.** There exists an injective homomorphism
\[ \eta : \text{ind}_{P_{r,n-r} \cap P^o_{r+1}}^{G_n} \sigma \otimes \tau_s \to C^\infty(k^n; \rho_s) \]
of representations of \( G_n \) such that
\[ (18) \quad (\eta(\phi)(g^{-1}e_{r+1}))(g) = \phi(g) \]
for every \( \phi \in \text{ind}_{P_{r,n-r} \cap P^o_{r+1}}^{G_n} \sigma \otimes \tau_s \) and \( g \in G_n \).
Proof. Applying part (a) of Lemma 3.1 we get the following isomorphism of representations of $G_n$: 
\[ \eta_1 : \text{ind}_{P_{r,n-r} \cap P_{r+1}}^{G_n} \sigma \hat{\otimes} \tau_s \rightarrow \text{ind}_{P_{r+1}^o \cap P_{r+1}}^{P_{r+1}} \sigma \hat{\otimes} \tau_s, \]
\[ \phi \mapsto (g \mapsto (p^o \mapsto \phi(p^o g))). \]
Since $P_{r,n-r} \cap P_{r+1}^o$ is open in $G_n$, it follows from part (b) of Lemma 3.1 that there is an injective homomorphism
\[ \eta_2 : \text{ind}_{P_{r+1}^o \cap P_{r+1}^o}^{G_n} \sigma \hat{\otimes} \tau_s \rightarrow \text{ind}_{P_{r+1}^o \cap P_{r+1}^o}^{P_{r+1}^o} (\text{ind}_{P_{r,n-r} \cap P_{r+1}^o}^{G_n} \sigma \hat{\otimes} \tau_s)|_{P_{r+1}^o}, \]
such that
\[ (((\eta_2(\phi))(g))(g_0) = \begin{cases} p.(\phi(g)(p^o)), & \text{if } g_0 = p^o \in P_{r,n-r} \cap P_{r+1}^o; \\ 0, & \text{otherwise}. \end{cases} \]
Notice that $G_n$ acts transitively on $k^n \setminus \{0\}$. Using part (c) of Lemma 3.1 we obtain the following isomorphism:
\[ \eta_3 : \text{ind}_{P_{r+1}^o}^{G_n} \rho_s|_{P_{r+1}^o} \rightarrow C^c(k^n \setminus \{0\}; \rho_s), \]
\[ \psi \mapsto (ge \mapsto g.\psi(g^{-1})). \]
Denote by $\eta$ the injective homomorphism
\[ \eta_4 \circ \eta_3 \circ \eta_2 \circ \eta_1 : \text{ind}_{P_{r,n-r} \cap P_{r+1}^o}^{G_n} \sigma \hat{\otimes} \tau_s \rightarrow C^c(k^n; \rho_s), \]
where $\eta_1 : C^c(k^n \setminus \{0\}; \rho_s) \rightarrow C^c(k^n; \rho_s)$ is the map obtained by extension by zero. Then, for every $\phi \in \text{ind}_{P_{r,n-r} \cap P_{r+1}^o}^{G_n} \sigma \hat{\otimes} \tau_s$ and $g \in G_n$, we have that
\[ (\eta(\phi)(g^{-1}e_{r+1}))(g) \rightarrow (g^{-1}.(((\eta_2 \circ \eta_1)(\phi))(g))(g) = (((\eta_2 \circ \eta_1)(\phi))(g))(1) = ((\eta_1(\phi))(g))(1) = \phi(g), \]
as desired. \qed

As in Proposition 2.4 let $\mu$ be a non-zero element in $\text{Hom}_{\mathcal{R}}(\pi \hat{\otimes} \sigma, \chi)$. Fix $v_\pi \in \pi$ and $v_\sigma \in \sigma$ such that
\[ \langle v_\pi, v_\sigma \rangle_{\mu} \neq 0. \]
Note that the multiplication map
\[ (P_{r,n-r} \cap P_{r+1}^o) \times U_r \times Q_{n-r}' \times \bar{U}_{r,n-r} \rightarrow G_n \]
is an open embedding of smooth manifolds. Let $\phi_0$ be a compactly supported smooth function on $U_r \times Q_{n-r}' \times \bar{U}_{r,n-r}$, and let $v_\tau$ be a vector in $\tau$. Define a $\sigma \hat{\otimes} \tau_s$-valued smooth function $\phi$ on $G_n$ by
\[ \phi(g) = \begin{cases} \phi_0(u, q', u') \cdot (p.(v_\sigma \otimes v_\tau)), & \text{if } g = puq'u' \\ 0, & \text{in } (P_{r,n-r} \cap P_{r+1}^o)U_rQ_{n-r}', \bar{U}_{r,n-r}; \end{cases} \]
otherwise.
Part (d) of Lemma 3.1 implies that $\phi \in \text{ind}_{P_{r,n-r} \cap P_{r+1}^o}^{G_n} \sigma \hat{\otimes} \tau_s$. 

It is easy to check that
\[ p'_{r-1}u'p'u^{-1} \in U_{r,n-r} \cap P_{r+1} \]
for all \( p' \in P'_{n-r}, u \in U_r \),
where \( U_{r,n-r} \) denotes the unipotent radical of \( P_{r,n-r} \). Therefore,
\[
\phi(u'p'^{-1}q'u') = \phi(p'_{r-1}u'p'u^{-1}) = \phi_0(u, q', \bar{u}) \cdot (p'.(v_\sigma \otimes v_r)),
\]
for all \( u \in U_r, p' \in P'_{n-r}, q' \in Q'_{n-r} \) and \( \bar{u} \in \breve{U}_{r,n-r} \).

For simplicity in notation, put
\[
\Omega := U_r \times (N'_{n-r} \backslash P'_{n-r}) \times Q'_{n-r} \times \breve{U}_{r,n-r}.
\]
Recall the homomorphism \( \eta \) of Lemma 4.2. Then we have that
\[
Z_\mu(v_\sigma, \eta(\phi))
= \int_{\Omega} \langle u'p'^{-1}q'u.v_\sigma, \lambda(\eta(\phi)(u'p'^{-1}q'u)) \rangle \mu \, du \, dp' \, dq' \, d\bar{u}
\]
(by Lemma 4.1)
\[
= \int_{\Omega} \langle u'p'^{-1}q'u.v_\sigma, \lambda \circ \phi(u'p'^{-1}q'u) \rangle \mu \, du \, dp' \, dq' \, d\bar{u}
\]
(by Lemma 4.2)
\[
= \int_{\Omega} \langle u'p'^{-1}q'u.v_\sigma, \lambda(p'.v_\sigma) \phi_0(u, q', \bar{u}) \rangle \mu \, du \, dp' \, dq' \, d\bar{u}.
\]
(by (19))

Here in the last term of (20), the representation \( \tau \) of \( G_{n-r} \) is viewed as a representation of \( G'_{n-r} \) via the obvious isomorphism \( G_{n-r} \cong G'_{n-r} \). Likewise, we view the character \( \psi \) of \( N_{n-r} \) as a character of \( N'_{n-r} \). Recall from [JS1, Section 3] that for each smooth function \( W \) on \( P'_{n-r} \), if
\[
W(np') = \psi(n)^{-1}W(p'), \quad n \in N'_{n-r}, p' \in P'_{n-r},
\]
and \( W \) has compact support modulo \( N'_{n-r} \), then there is a vector \( v \in \tau \) such that
\[
W(p') = \lambda(p'.v), \quad p' \in P'_{n-r}.
\]
Thus, by (20), the integral \( Z_\mu(v_\sigma, \eta(\phi)) \) is absolutely convergent and non-zero for appropriate \( v_\tau \) and \( \phi_0 \). This completes the proof of Proposition 2.4.

5. PROOF OF PROPOSITION 2.5

For every \( t = (t_1, \cdots, t_{n-r}) \in (\mathbb{R}^+)^{n-r} \) and \( s = (s_1, \cdots, s_r) \in \mathbb{k}^r \), put
\[
a_t := \text{diag}\{1, \cdots, 1, t_1, \cdots, t_{n-r}\} \in G'_{n-r}
\]
and
\[
b_s := \begin{bmatrix} 1_r & u_s \\ 0 & 1_{n-r} \end{bmatrix} \in U_r, \quad \text{where} \quad u_s = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_r & 0 & \cdots & 0 \end{bmatrix}.\]
Fix an arbitrary maximal compact subgroup $K_n$ of $G_n$. It is easy to check that there is a positive character $\gamma$ on $(\mathbb{R}_+^\times)^{n-r}$ such that

$$
(21) \quad \int_{R_n \backslash G_n} \varphi(g) \, dg = \int_{k^n \times (\mathbb{R}_+^\times)^{n-r} \times K_n} \gamma(t) \varphi(a_tb_s k) \, ds \, d^x t \, dk,
$$

for all non-negative continuous functions $\varphi$ on $R_n \backslash G_n$. Here $dk$ is the normalized Haar measure on $K_n$; and $ds$ and $d^x t$ are appropriate Haar measures on $k^r$ and $(\mathbb{R}_+^\times)^{n-r}$, respectively.

For each $g \in G_n$, write

$$
||g|| := \text{Tr}(t^* \bar{g} g) + \text{Tr}((t^* \bar{g} g)^{-1}),
$$

where $t^* \bar{g}$ denotes the conjugate transpose of $g$. For each $t = (t_1, \cdots, t_{n-r}) \in (\mathbb{R}_+^\times)^{n-r}$, put

$$
||t|| := \prod_{i=1}^{n-r} (t_i + t_i^{-1}).
$$

Note that there exists a constant $c$, which depends on $n$ only, such that

$$
||a_t|| \leq c ||t||^2, \quad t \in (\mathbb{R}_+^\times)^{n-r}.
$$

Take a positive integer $c_0$ such that

$$
(22) \quad \gamma(t) \leq ||t||^{c_0}, \quad t \in (\mathbb{R}_+^\times)^{n-r}.
$$

Since the bilinear form $\langle \cdot, \cdot \rangle_\mu$ is continuous, there exist continuous seminorms $| \cdot |_{\pi,1}$ and $| \cdot |_{\sigma}$ on $\pi$ and $\sigma$ respectively such that

$$
(23) \quad |\langle u, v \rangle_\mu| \leq |u|_{\pi,1} \cdot |v|_{\sigma}, \quad u \in \pi, \ v \in \sigma.
$$

Using the moderate growth condition on $\pi$, we get two positive integers $d$ and $c_1$, and a continuous seminorm $| \cdot |_{\pi,2}$ on $\pi$ so that

$$
(24) \quad |\langle a_tb_s k, u \rangle_{\pi,1}| \leq ||b_s||^d \cdot ||t||^{c_1} \cdot |u|_{\pi,2},
$$

for all $s \in k^r$, $t \in (\mathbb{R}_+^\times)^{n-r}$, $k \in K_n$ and $u \in \pi$. Since $U_r$ acts trivially on $\sigma \hat{\otimes} \tau_s$, one has that

$$
(25) \quad f(a_tb_s k) = a_tb_s f(k) = f(a_t k),
$$

for all $s \in k^r$, $t \in (\mathbb{R}_+^\times)^{n-r}$, $k \in K_n$ and $f \in \rho_s$. 
Now, for every \( u \in \pi, f \in \rho_s \) and \( \phi \in \omega \), we have that
\[ |Z_\mu(u, f \otimes \phi)| \]
\[ \leq \int_{R^r \setminus G_n} |\langle g, u \rangle (\Lambda \circ f)(g)\rangle_\mu \cdot |\phi(g^{-1}e_{r+1})| \, dg \]
\[ = \int_{k^r \times (R_+^\times)^{n-r} \times K_n} \gamma(t) \cdot |\langle (a_t b_k) u, (\Lambda \circ f)(a_t b_k)\rangle_\mu| \cdot |\phi((a_t b_k)^{-1}e_{r+1})| \, ds \, d^rk \text{ d} t k \hspace{1cm} (by \ (21)) \]
\[ \leq \int_{k^r \times (R_+^\times)^{n-r} \times K_n} |t|^{c_0} \cdot |(a_t b_k) u|_{\pi, 1} \cdot |(\Lambda \circ f)(a_t b_k)|_\sigma \cdot |\phi((a_t b_k)^{-1}e_{r+1})| \, ds \, d^rk \text{ d} t k \hspace{1cm} (by \ (22) \text{ and } (23)) \]
\[ \leq |u|_{\pi, 2} \cdot \int_{k^r \times (R_+^\times)^{n-r} \times K_n} |t|^{c_0 + c_1} \cdot |b_k|^{d} \cdot |(\Lambda \circ f)(a_t k)|_\sigma \cdot |\phi((a_t b_k)^{-1}e_{r+1})| \, ds \, d^rk \text{ d} t k. \hspace{1cm} (by \ (24) \text{ and } (25)) \]

Recall that the linear map \( \Lambda : \sigma \otimes \tau \rightarrow \sigma \) is continuous. Then it follows from the moderate growth condition on \( \sigma \otimes \tau \) that there exist a positive integer \( c_2 \) and a continuous seminorm \( |\cdot|_{\sigma \otimes \tau} \) on \( \sigma \otimes \tau \) such that
\[ |\Lambda(p, u)|_\sigma \leq |p|^{c_2} \cdot |u|_{\sigma \otimes \tau}, \quad p \in P_{r, n-r}, \ u \in \sigma \otimes \tau. \]

**Lemma 5.1.** For every positive integer \( N \), there is a continuous seminorm \( |\cdot|_{\sigma \otimes \tau, N} \) on \( \sigma \otimes \tau \) (which depends on \( |\cdot|_{\sigma \otimes \tau} \)) such that
\[ |\Lambda(a_t, u)|_\sigma \leq \xi(t)^{-N} \cdot |t|^{c_2} \cdot |u|_{\sigma \otimes \tau, N}, \quad u \in \sigma \otimes \tau, \quad t \in (R_+^\times)^{n-r} \]

where
\[ \xi(t) := \prod_{i=1}^{n-r-1} \left(1 + \frac{t_i}{t_i+1}\right) \text{ for } t = (t_1, \cdots, t_{n-r}). \]

**Proof.** This is proved in [JSZ, Lemma 6.2]. \( \square \)

Lemma 5.1 easily implies that
\[ |(\Lambda \circ f)(a_t k)|_\sigma \leq \Pi(t)^{-Res} \cdot \xi(t)^{-N} \cdot |t|^{c_2} \cdot |f(k)|_{\sigma \otimes \tau, N}, \]
for all \( f \in \rho_s \), \( t \in (R_+^\times)^{n-r} \) and \( k \in K_n \). Here
\[ \Pi(t) := \prod_{i=1}^{n-r} t_i \text{ for } t = (t_1, \cdots, t_{n-r}). \]

Let \( |\cdot|_{\sigma \otimes \tau, N} \) be as in Lemma 5.1. Define a continuous seminorm \( |\cdot|_{\rho_s, N} \) on \( \rho_s \) by
\[ |f|_{\rho_s, N} := \sup \{ |f(k)|_{\sigma \otimes \tau, N} \mid k \in K_n \}, \quad f \in \rho_s. \]

Set \( c_\mu := c_0 + c_1 + c_2. \)
Lemma 5.2. If \( \text{Re}\, s > c_\mu \), then for all large enough integer \( N \), one has that

\[
c_{s,N} := \int_{(\mathbb{R}_+^\times)^{n-r}} \| t \|^{c_\mu} \cdot \Pi(t)^{-\text{Res}} \cdot \xi(t)^{-N} \cdot (1 + t_1^{-1})^{-N} \, dt < \infty.
\]

Proof. The proof is similar to that of [LS, Lemma 6.2]. We omit the details. \( \square \)

For each positive integer \( N \), put

\[
| \phi |_{\omega,N} = \sup \{ (1 + |a_r+1|_k)^N \cdot (|a_1^2| + \cdots + |a_r^2|_k)^N \cdot | \phi (ka) | \mid a = \sum_{i=1}^n a_i e_i \in k^n, k \in K_n \}, \quad \phi \in \omega.
\]

This defines a continuous seminorm on \( \omega \).

For each \( s = (s_1, \cdots, s_r) \in k^r \), \( t = (t_1, \cdots, t_{n-r}) \in (\mathbb{R}_+^\times)^{n-r} \) and \( k \in K_n \), it is routine to check that

\[
(a_t b_s k)^{-1} e_{r+1} = k^{-1} (t_1^{-1} e_{r+1} - \sum_{i=1}^r s_i e_i).
\]

Together with (30), this implies that

\[
|s_1^2 + \cdots + |s_r^2|_k^N \cdot (1 + t_1^{-1})^{-N} \cdot | \phi ((a_t b_s k)^{-1} e_{r+1}) | \leq | \phi |_{\omega,N}
\]

for all \( \phi \in \omega \).

In what follows we assume that \( \text{Re}\, s > c_\mu \). Fix a positive integer \( N \) which is large enough so that (29) holds true, and that

\[
c_N' := \int_{k^r} \| b_s \|^d \cdot (|s_1^2 + \cdots + |s_r^2|_k^N \cdot \mathbf{1} \cdot | \phi (a_t b_s k)^{-1} e_{r+1}) | \cdot \| b_s \|^d \cdot (|s_1^2 + \cdots + |s_r^2|_k^N \cdot \mathbf{1}) \, ds < \infty. \quad (s = (s_1, \cdots, s_r))
\]

Recall the continuous seminorms \( \cdot \mid_{\sigma \otimes_{\tau} N} \) on \( \sigma \otimes_{\tau} \) (see Lemma 5.1), \( \cdot \mid_{\rho_s N} \) on \( \rho_s \) (see (28)) and \( \cdot \mid_{\omega,N} \) on \( \omega \) (see (30)). Then one has that

\[
|t|^{c_\mu} \cdot |b_s|^d \cdot (\Lambda \circ f) (a_t k) \|_{\sigma} \cdot | \phi ((a_t b_s k)^{-1} e_{r+1}) | \leq |f|_{\rho_s N} \cdot |t|^{c_\mu} \cdot \Pi(t)^{-\text{Res}} \cdot \xi(t)^{-N} \cdot |b_s|^d \cdot | \phi ((a_t b_s k)^{-1} e_{r+1}) | \quad \text{(by (27), (28))}
\]

\[
\leq | \phi |_{\omega,N} \cdot |f|_{\rho_s N} \cdot |t|^{c_\mu} \cdot \Pi(t)^{-\text{Res}} \cdot \xi(t)^{-N} \cdot (1 + t_1^{-1})^{-N} \cdot \| b_s \|^d \cdot (|s_1^2 + \cdots + |s_r^2|_k^N) \quad \text{(by (31))}
\]

for all \( s = (s_1, \cdots, s_r) \in k^r \), \( t = (t_1, \cdots, t_{n-r}) \in (\mathbb{R}_+^\times)^{n-r} \), \( k \in K_n \), \( f \in \rho_s \) and \( \phi \in \omega \).

Finally, one concludes from (26), (33), (29) and (32) that

\[
| Z_\mu (u, f \otimes \phi) | \leq c_N' \cdot s_{\omega,N} \cdot |u|_{\pi,2} \cdot |f|_{\rho_s N} \cdot | \phi |_{\omega,N}
\]

for all \( u \in \pi, f \in \rho_s \) and \( \phi \in \omega \). This proves Proposition 2.5.
6. THE NON-ARCHIMEDEAN CASE

Let \( k \) be a non-archimedean local field of characteristic zero throughout this section. Fix \( n > r \geq 0 \) as in the Introduction. We shall prove an analog of Theorem A in the non-archimedean case:

**Theorem C.** For all irreducible admissible smooth representations \( \pi \) of \( \text{GL}_n(k) \), and \( \sigma \) of \( \text{GL}_r(k) \), and for all generic characters \( \chi \) of \( \text{R}_r \), one has that
\[
\dim \text{Hom}_{\text{R}_r}(\pi \otimes \sigma, \chi) \leq 1.
\]

Here the Rankin-Selberg subgroup \( \text{R}_r \) of \( \text{GL}_n(k) \) is defined as in (3) and the generic characters of \( \text{R}_r \) are defined in the same way as in the archimedean case.

As in Introduction, let \( \omega \) denote the representation of \( \text{GL}_n(k) \) on the space of \( \mathbb{C} \)-valued Schwartz-Bruhat functions on \( k \). Similar to what we have done in the archimedean case, we shall prove Theorem C by reducing it to the following multiplicity one theorem:

**Theorem D.** (cf. [Su1, Theorem B]) For all irreducible admissible smooth representations \( \pi \) and \( \pi' \) of \( \text{GL}_n(k) \), and all characters \( \chi' \) on \( \text{GL}_n(k) \), one has that
\[
\dim \text{Hom}_{\text{GL}_n(k)}(\pi \otimes \pi' \otimes \omega, \chi') \leq 1.
\]

Now we turn to the proof of Theorem C. The method of our proof for Theorem A remains valid. However, we will exploit a different (and simpler) approach, which is developed in [GGP]. As in Section 2, we write \( G_m := \text{GL}_m(k) \) and \( N_m := \text{N}_m(k) \) for every non-negative integer \( m \). For each smooth representation \( \varrho \) of a closed subgroup of \( G_n \), write \( \varrho^\vee \) for its contragradient representation. In this section, we use “ind” to indicate the un-normalized smooth induction with compact support.

Let \( \pi, \sigma \) and \( \chi \) be as in Theorem C. Following [GGP, Section 15], we fix a supercuspidal representation \( \tau \) of \( G_{n-r} \) such that for all Levi subgroups \( M \) of \( G_r \) and all supercuspidal representations \( \mu \) of \( M \), \( \pi^\vee \) does not belong to the Bernstein component (see [Be, Section 2.2], for example) associated to \((M \times G_{n-r}, \mu \otimes \tau)\). Without loss of generality, we assume that \( \rho := \text{ind}_{P_{r,n-r}}^G \sigma \otimes \tau \) is irreducible (otherwise, using [Sa, Theorem 3.2], replace \( \tau \) by its twist by a suitable unramified character). Here \( P_{r,n-r} \) is the parabolic subgroup of \( G_n \) as in (8), and \( \sigma \otimes \tau \) is viewed as a representation of \( P_{r,n-r} \) by inflation through the homomorphism \( P_{r,n-r} \to G_r \times G_{n-r} \).

As in (9), we assume that the decent of \( \chi \) on \( G_r \times N_{n-r} \) has the form \( 1 \otimes \psi \). We claim that
\[
\text{Hom}_{R_r}(\pi \otimes \sigma, \chi) \cong \text{Hom}_{G_n}(\pi \otimes \rho \otimes \omega, |\det|_k^{r+1-n}).
\]
Using Theorem D this clearly implies Theorem C.

**Proof of the claim:** Write \( \omega^\circ \) for the space of \( \mathbb{C} \)-valued Schwartz-Bruhat functions on \( k \setminus \{0\} \), which is a smooth representation of \( G_n \) as in (6). Then we have an
The stabilizer of the representation of $H^n$ is obvious exact sequence
\[ 0 \to \omega^0 \otimes \rho \to \omega \otimes \rho \to \rho \to 0 \]
of smooth representations of $G_n$. Applying the functor
\[ \text{Hom}_{G_n}(\bullet, \mid \det |k|^{r+1-n} \otimes \pi^\vee) \]
to the above short exact sequence, we get an exact sequence
\[ 0 \to \text{Hom}_{G_n}(\rho, \mid \det |k|^{r+1-n} \otimes \pi^\vee) \to \text{Hom}_{G_n}(\omega \otimes \rho, \mid \det |k|^{r+1-n} \otimes \pi^\vee) \]
\[ \to \text{Hom}_{G_n}(\omega^0 \otimes \rho, \mid \det |k|^{r+1-n} \otimes \pi^\vee) \to \text{Ext}^1_{G_n}(\rho, \mid \det |k|^{r+1-n} \otimes \pi^\vee). \]
It follows from the assumption on $\tau$ that ([Be, Section 2.2])
\[ \text{Hom}_{G_n}(\rho, \mid \det |k|^{r+1-n} \otimes \pi^\vee) = \text{Hom}_{G_n}(\text{ind}^G_{P_{r,n-r}} \sigma \otimes \tau, \mid \det |k|^{r+1-n} \otimes \pi^\vee) = 0, \]
and that
\[ \text{Ext}^1_{G_n}(\rho, \mid \det |k|^{r+1-n} \otimes \pi^\vee) = \text{Ext}^1_{G_n}(\text{ind}^G_{P_{r,n-r}} \sigma \otimes \tau, \mid \det |k|^{r+1-n} \otimes \pi^\vee) = 0. \]
Therefore, one has that
\[ \text{Hom}_{G_n}(\omega \otimes \rho, \mid \det |k|^{r+1-n} \otimes \pi^\vee) = \text{Hom}_{G_n}(\omega^0 \otimes \rho, \mid \det |k|^{r+1-n} \otimes \pi^\vee). \]
Recall that the space $k^n$ of column vectors carries the standard action of $G_n$, and has the standard basis $e_1, e_2, \cdots, e_n$. Denote by $P_i^o$ the subgroup of $G_n$ fixing $e_i$ ($i = 1, 2, \cdots, n$). Then we have that (cf. [H, Exercise 6 (ii)])
\[ \omega^0 \otimes \rho \cong (\text{ind}^G_{P_{r+1}} C) \otimes \rho \cong \text{ind}^G_{P_{r+1}} (\rho|_{P_{r+1}}). \]
Now we analyze the representation $\rho|_{P_{r+1}}$. Denote by $w \in G_n$ the permutation matrix which exchanges $e_1$ and $e_{r+1}$, and fixes $e_i$ whenever $i \neq 1, r + 1$. Write $[1_n], [w] \in P_{r,n-r} \setminus G_n$ for the cosets which are respectively represented by $1_n$ and $w$. Note that under right translation, $P_{r+1}$ has two obits in $P_{r,n-r} \setminus G_n$: one is open and contains $[1_n]$; the other is closed and contains $[w]$. Set
\[ T_i := P_{r,n-r} \cap P_i^o, \quad i = 1, 2, \cdots, n. \]
Then the stabilizer of $[1_n] \in P_{r,n-r} \setminus G_n$ in $P_i^o$ is $T_{r+1}$; and the stabilizer of $[w] \in P_{r,n-r} \setminus G_n$ in $P_i^o$ is $T_i^w$. Here and henceforth, for every subgroup $H$ of $G_n$, we set $H^w := wHw$. For every smooth representation $\varrho$ of $H$, we denote by $\varrho^w$ the representation of $H^w$ which has the same underlying space as that of $\varrho$, and has the action
\[ \varrho^w(h) = \varrho(whw), \quad h \in H^w. \]
Using Mackey theory, we get a short exact sequence
\[ 0 \to \text{ind}^P_{P_{r+1}}(\sigma \otimes \tau)|_{T_{r+1}} \to \rho|_{P_{r+1}} \to \text{ind}^P_{T_i^w}(\sigma \otimes \tau)|_{T_i^w} \to 0 \]
of smooth representations of $P_{r+1}^O$. Inducing this short exact sequence to $G_n$, and using induction in stages, we get a short exact sequence

$$0 \to \text{ind}_{T_{r+1}}^{G_n} (\sigma \otimes \tau)|_{T_{r+1}} \to \text{ind}_{T_{r+1}}^{G_n} \rho|_{P_{r+1}} \to \text{ind}_{T_{r+1}}^{G_n} (\sigma \otimes \tau)^w|_{T_{r+1}} \to 0$$

of smooth representations of $G_n$.

Recall the mirabolic subgroup $P_m$ of $G_m$ as in (12). We view the representation $(\sigma \otimes \text{ind}_{P_1}^{G_r} \otimes \tau)$ of $G_r \times G_{n-r}$ as a representation of $P_{r,n-r}$ by inflation. Then one has the following isomorphisms of representations of $P_{r,n-r}$:

$$\text{ind}_{T_{r+1}}^{P_{r,n-r}} (\sigma \otimes \tau)^w|_{T_{r+1}} w$$

$$\cong (\sigma \otimes \tau)^w \otimes \text{ind}_{T_{r+1}}^{P_{r,n-r}} \otimes \text{C} \quad ([H, \text{Exercise 6 (ii)})]

\cong (\sigma \otimes \tau)^w \otimes (\text{ind}_{T_{r+1}}^{P_{r,n-r}} \otimes \text{C})^w

\cong ((\sigma \otimes \text{ind}_{P_1}^{G_r} \otimes \tau)^w

(P_r \setminus G_r \cong T_1 \setminus P_{r,n-r})

This implies

$$\text{ind}_{T_{r+1}}^{G_n} (\sigma \otimes \tau)^w|_{T_{r+1}} w$$

$$\cong \text{ind}_{P_{r,n-r}}^{G_n} \text{ind}_{T_{r+1}}^{P_{r,n-r}} (\sigma \otimes \tau)^w|_{T_{r+1}} w

\cong \text{ind}_{P_{r,n-r}}^{G_n} ((\sigma \otimes \text{ind}_{P_1}^{G_r} \otimes \tau)^w

\cong \text{ind}_{P_{r,n-r}}^{G_n} ((\sigma \otimes \text{ind}_{P_1}^{G_r} \otimes \tau).$$

Then our assumption on $\tau$ implies that ([Be, Section 2.2])

$$\text{Hom}_{G_n} (\text{ind}_{T_{r+1}}^{G_n} (\sigma \otimes \tau)^w|_{T_{r+1}}, | \det |^{r+1-n} \circ \pi') = 0$$

and

$$\text{Ext}_{G_r}^1 (\text{ind}_{T_{r+1}}^{G_n} (\sigma \otimes \tau)^w|_{T_{r+1}}, | \det |^{r+1-n} \circ \pi') = 0.$$

Therefore, applying the functor $\text{Hom}_{G_n} (\bullet, | \det |^{r+1-n} \circ \pi')$ to the short exact sequence (36), we get

$$\text{Hom}_{G_n} (\text{ind}_{T_{r+1}}^{G_n} \rho|_{P_{r+1}}, | \det |^{r+1-n} \circ \pi')$$

$$= \text{Hom}_{G_n} (\text{ind}_{T_{r+1}}^{G_n} (\sigma \otimes \tau)|_{T_{r+1}}, | \det |^{r+1-n} \circ \pi').$$

Combine this with (34) and (35), we have that

$$\text{Hom}_{G_n} (\omega \otimes \rho, | \det |^{r+1-n} \circ \pi')$$

$$= \text{Hom}_{G_n} (\text{ind}_{T_{r+1}}^{G_n} (\sigma \otimes \tau)|_{T_{r+1}}, | \det |^{r+1-n} \circ \pi').$$

Recall that $\tau$ is a supercuspidal representation of $G_{n-r}$. By the well-known result of Gelfand-Kazhdan ([BZ, Section 5.18]), one has that

$$\tau|_{P_{n-r}} \cong \text{ind}_{N_{n-r}}^{P_{n-r}} \psi^{-1}.$$
Note that $T_{r+1} = (G_r \times P_{n-r}) \ltimes U_r$ ($U_r$ is defined as in (13)). Then we have
\[
\text{ind}^{G_n}_{T_{r+1}} (\sigma \otimes \tau)|_{T_{r+1}} = \text{ind}^{G_n}_{T_{r+1}} (\sigma \otimes \tau)|_{P_{n-r}} \leq \text{ind}^{G_n}_{T_{r+1}} (\sigma \otimes \text{ind}^{P_{n-r}}_{P_{n-r}} \psi^{-1}) \quad \text{(by (38))}
\]
\[
\cong \text{ind}^{G_n}_{T_{r+1}} \text{ind}^{T_{r+1}}_{R_r} (\sigma \otimes \psi^{-1}) \quad (R_r \backslash T_{r+1} \cong N_{n-r} \backslash P_{n-r})
\]
\[
\cong \text{ind}^{G_n}_{R_r} (\sigma \otimes \chi^{-1}) \quad (\chi = 1 \otimes \psi).
\]
Together with (37), this implies
\[
\text{Hom}_{G_n} (\omega \otimes \rho, | \det|_{k}^{r+1-n} \otimes \pi^\vee) = \text{Hom}_{G_n} (\text{ind}^{G_n}_{R_r} (\sigma \otimes \chi^{-1}), | \det|_{k}^{r+1-n} \otimes \pi^\vee).
\]
Finally, by Frobenius reciprocity ([BZ, Proposition 2.29]), one concludes from the above equality that
\[
\text{Hom}_{G_n} (\omega \otimes \rho, | \det|_{k}^{r+1-n} \otimes \pi^\vee) = \text{Hom}_{R_r} (\pi, \sigma^\vee \otimes \chi).
\]
This finishes the proof of the claim.

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