Differentially private multivariate medians

Kelly Ramsay, Aukosh Jagannath, Shoja’eddin Chenouri

Abstract

Statistical tools which satisfy rigorous privacy guarantees are necessary for modern data analysis. It is well-known that robustness against contamination is linked to differential privacy. Despite this fact, using multivariate medians for differentially private and robust multivariate location estimation has not been systematically studied. We develop novel finite-sample performance guarantees for differentially private multivariate depth-based medians, which are essentially sharp. Our results cover commonly used depth functions, such as the halfspace (or Tukey) depth, spatial depth, and the integrated dual depth. We show that under Cauchy marginals, the cost of heavy-tailed location estimation outweighs the cost of privacy. We demonstrate our results numerically using a Gaussian contamination model in dimensions up to $d = 100$, and compare them to a state-of-the-art private mean estimation algorithm. As a by-product of our investigation, we prove concentration inequalities for the output of the exponential mechanism about the maximizer of the population objective function. This bound applies to objective functions that satisfy a mild regularity condition.

1 Introduction

Protecting user privacy is necessary for a safe and fair society. In order to protect user privacy, many large institutions, including the United States Census Bureau (Abowd et al., 2022), Apple (Differential Privacy Team, 2017), and Google (Guevara, 2021), utilize a strong notion of privacy known as differential privacy. Differential privacy is favored because it protects against many adversarial attacks while requiring minimal assumptions on both the data and the adversary (Dwork et al., 2017). It is now well-known that differential privacy is linked to robustness against contamination, see (Dwork and Lei, 2009; Avella-Medina, 2020; Liu et al., 2021a) and the references therein. Surprisingly, however, multivariate medians have not been systematically studied in the differential privacy literature. We study this problem here. Note that there are several variants of differential privacy. Here, we focus on the setting of pure differential privacy, which is the strictest variant of differential privacy.

To date, the literature on differentially private location estimation has largely focused on means. Foygel Barber and Duchi (2014) proved a lower bound on the minimax risk of differentially private mean estimation in terms of the number of moments possessed by the population measure. Other early works focused on the univariate setting (Karwa and Vadhan, 2017; Bun and Steinke, 2019). Kamath et al. (2018) and Bun et al. (2019) studied differentially private mean estimation for several parametric models, including the multivariate Gaussian model. Later, several authors studied differentially private mean estimation for sub-Gaussian measures (Cai et al., 2021; Biswas et al., 2020; Brown et al., 2021; Liu et al., 2021b). After which, Kamath et al. (2020); Liu et al. (2021a) and Hopkins et al. (2022) studied differentially private mean estimation for measures with a finite second moment. Within this framework, Kamath et al. (2020) considered heavy tails and Liu et al. (2021a)
and Hopkins et al. (2022) considered robustness in terms of contamination models. Kamath et al. (2020) quantified the sample complexity of their (purely) differentially private mean estimator in terms of the number of moments possessed by the population measure. Liu et al. (2021a) quantified the relationship between the accuracy of an (approximately) differentially private mean estimator and the level of contamination in the observed sample. On a similar note, Hopkins et al. (2022) showed that there exists a level of contamination under which the sample complexity of their (purely) differentially private mean estimator remains unchanged.

On the other hand, if one wishes to estimate location robustly, the canonical estimators are medians. The differential privacy literature on medians has focused on the univariate setting: Dwork and Lei (2009) introduced an approximately differentially private median estimator with asymptotic consistency guarantees, Avella-Medina and Brunel (2019a); Brunel and Avella-Medina (2020) then introduced several median estimators which achieve sub-Gaussian error rates, and Tzamos et al. (2020) introduced median estimators with minimax optimal sample complexity. Private estimation of multivariate medians, however, has not been carefully studied.

In this paper, we study differentially private, multivariate median estimation through the framework of depth functions. Depth-based median estimation is the standard approach to multivariate median estimation. Here, one has a function, called the depth function, which provides a measure of centrality. Its maximizer is then the median, which is generally robust. Popular depth-based medians include the halfspace (or Tukey) median (Tukey, 1974), the spatial median (Vardi and Zhang, 2000), and the simplicial median (Liu, 1990). For more on the motivation of the depth-based medians approach, see Section 2 below and Small (1990); Vardi and Zhang (2000); Serfling (2006). Our results apply to a broad class of depth functions, including those listed above, and we make minimal distributional assumptions. In particular, we do not require concavity of the depth function or that the population measure has moments of any order.

Our contributions are as follows: Our main result is a general finite-sample deviations bound for private multivariate medians based on the exponential mechanism (Theorem 1). We use this result to give upper bounds on the deviations (and sample complexities) of several, purely differentially private multivariate medians arising from depth functions. These bounds are essentially sharp given recent results of Kamath et al. (2018); Cai et al. (2021) and do not require any moment assumptions on the population measure. In addition, we provide a fast implementation of a smoothed version of the integrated dual depth-based median (Cuevas and Fraiman, 2009): we can compute the (non-private) median of ten thousand 100-dimensional samples in less than one second on a personal computer. We show that this smoothed version of the integrated dual depth satisfies desirable properties for a depth function and can be approximated in polynomial time with finite-sample performance guarantees. As a by-product of our analysis, we uncover a general but elementary concentration bound (Theorem 11) for the exponential mechanism. We give a novel regularity condition “(о,о)-regularity” (Condition 3) on the objective function. This regularity condition implies both an upper bound on the sample complexity of a draw from the exponential mechanism and an upper bound on the global sensitivity of the objective function. As a second by-product of our analysis, we uncover uniform non-asymptotic error bounds for several depth functions and a differentially private estimator of data depth values.

Before turning to our main results, it is important to note that several authors have recently used depth functions in the context of differential privacy, likely due to their robustness properties, with a heavy focus on the halfspace depth. Depth functions have been used in settings such as finding a point in a convex hull (Beimel et al., 2019; Gao and Sheffet, 2020), comparing k-norm mechanisms (Awan and Slavkovicic, 2021), and improving robustness for mean estimation for Gaussian models.
Another related work is that of [Ben-Eliezer et al. (2022)], who focus on various problems concerning high-dimensional quantile estimation. In particular, one problem Ben-Eliezer et al. (2022) consider is that of privately obtaining one or more points which have halfspace depth above a given threshold. Though this problem is closely related, it is distinct. For instance, their algorithm can be used to estimate a point with high halfspace depth. However, it could not be used to directly estimate the halfspace median, since that would require setting the threshold to be close to the depth of the empirical halfspace median, which depends on the sample. To our knowledge, none of the aforementioned works have directly addressed depth-based median estimation.

2 Robust private multivariate median estimation

In this section, we provide finite-sample deviation bounds for differentially private robust multivariate location estimation. Let us begin by briefly recalling the following essential notions and definitions from differential privacy. For a textbook introduction to the concept of differential privacy see [Dwork and Roth (2014)].

A dataset of size $n \times d$ is a collection of $n$ points in $\mathbb{R}^d$, $X_n = (X_{\ell})_{\ell=1}^n$, with repetitions allowed. Let $D_{n \times d}$ be the set of all datasets of size $n \times d$. For a dataset $Y_n$, let $D(Y_n, m) = \{Z_n \in D_{n \times d}: |Z_n \triangle Y_n| = m\}$, that is, $D(Y_n, m)$ is the collection of datasets of size $n \times d$ which differ from $Y_n$ in exactly $m$ points. Two datasets $Y_n$ and $Z_n$ are said to be adjacent if $Z_n \in D(Y_n, 1)$. Finally, for a dataset $Y_n$ with empirical measure $\hat{\mu}_Y$, define

$$\tilde{M}(Y_n) = \{\tilde{\mu} \in M_1(\mathbb{R}^d) : \tilde{\mu} = \mu_{Z_n} \text{ for some } Z_n \in D(Y_n, 1)\},$$

where, for a space $S$, $M_1(S)$ denotes the space of probability measures on $S$.

Let us now recall the notion of differential privacy. Suppose that we observe a dataset $X_n$ comprised of $n$ i.i.d. samples from some unknown measure $\mu \in M_1(\mathbb{R}^d)$ and produce a statistic, $\tilde{\theta}_n$, whose law conditionally on the samples depends on their empirical measure, $\hat{\mu}_n$. Let us denote this law by $P_{\tilde{\theta}_n}$. The goal of differential privacy is to produce a statistic such that $P_{\tilde{\theta}_n}$ is non-degenerate and, more precisely, obeys a certain “privacy guarantee” which is defined as follows. Following the privacy literature, we call the algorithm that takes $\hat{\mu}_n$ and returns the (random) statistic, $\tilde{\theta}_n$, a mechanism. We also follow the standard abuse of terminology and call $\tilde{\theta}_n$ the mechanism when it is clear from context.

Definition 1. A mechanism $\tilde{\theta}_n$ is $\epsilon$-differentially private if for any dataset $Y_n$, any $\tilde{\mu} \in \tilde{M}(Y_n)$, and any measurable set $B$, it holds that

$$P_{\tilde{\theta}_n}(B) \leq e^\epsilon P_{\tilde{\mu}}(B). \quad (2.1)$$

Here, $\epsilon > 0$ is the privacy parameter, for which smaller values enforce stricter levels of privacy. Many general purpose differentially private mechanisms rely on the concept of global sensitivity. The global sensitivity $GS_n$ of a function $\phi : \mathbb{R}^d \times M_1(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$GS_n (\phi) = \sup_{X_n \in D_{n \times d}} \sup_{x \in \mathbb{R}^d} |\phi(x, \hat{\mu}_n) - \phi(x, \tilde{\mu}_n)|.$$

The simplest differentially private mechanisms are the additive noise mechanisms, e.g., the Laplace and Gaussian mechanisms. These mechanisms require the non-private estimator to have
finite global sensitivity. Unfortunately, virtually all standard location estimators, including both univariate and multivariate medians, have infinite global sensitivity, see e.g., (Avella-Medina and Brunel, 2019b).

However, **depth functions**, the objective functions for multivariate medians, have finite global sensitivity. This fact makes them a good candidate for use with the **exponential mechanism**, which is a general mechanism used to produce differentially private estimators from non-private estimators that are maximizers of a given objective function. For a given \( \beta > 0 \), base measure (i.e., prior) \( \pi \in \mathcal{M}_1(\mathbb{R}^d) \) and objective function \( \phi : \mathbb{R}^d \times \mathcal{M}_1(\mathbb{R}^d) \to \mathbb{R} \), suppose that \( \int \exp(\beta \phi(\theta, \nu))d\pi(\theta) < \infty \) for all \( \nu \in \mathcal{M}_1(\mathbb{R}^d) \), then the exponential mechanism with base measure \( \pi \) is given by

\[
Q_{\hat{\mu}, \beta} \propto \exp(\beta \phi(\theta, \hat{\mu}_n))d\pi.
\] (2.2)

A sample from the exponential mechanism \( \tilde{\theta}_n \sim Q_{\hat{\mu}, \beta} \) then provides an estimate of the population value \( \theta_0 = \arg\max_{\theta \in \mathbb{R}^d} \phi(\theta, \mu) \). In regard to the choice of \( \beta \), it was shown by McSherry and Talwar (2007) that if \( \tilde{\theta}_n \) is produced by the exponential mechanism, with \( \beta \leq \epsilon/2\text{GS}_n(\phi) \) then \( \tilde{\theta}_n \) is \( \epsilon \)-differentially private.

When studying multivariate median estimation, one might try to naively extend prior work on univariate medians to the multivariate setting, i.e., the coordinate-wise median. This, however, is well-known to be an unsatisfactory as a measure of center (Small, 1990). Indeed, the coordinate-wise median can reside outside the convex hull of the data (Serfling, 2006). Instead, the most popular framework for multivariate median estimation is through maximizing a depth function.

Depth functions are functions of the form \( D : \mathbb{R}^d \times \mathcal{M}_1(\mathbb{R}^d) \to \mathbb{R}^+ \) which, given a measure and a point in the domain, assign a number. This number represents how central that point is with respect to the measure, it is called the depth of that point. Given a depth function, the corresponding median is then defined to be a maximizer of this depth, or simply the deepest point:

\[
\text{MED}(\mu; D) = \arg\max_{x \in \mathbb{R}^d} D(x, \mu).
\]

Depth-based medians are generally robust, in the sense that they are not affected by outliers. For instance, depth-based medians have a high breakdown point and favorable properties related to the influence function (Chen and Tyler, 2002; Zuo, 2004). This makes them an interesting direction of study in a differentially private setting. Examples of commonly used depth functions include the halfspace depth (or Tukey depth) (Tukey, 1974), the simplicial depth (Liu, 1988), the spatial depth (Vardi and Zhang, 2000; Serfling, 2002), the integrated dual depth (IDD) (Cuevas and Fraiman, 2009), and the integrated-rank-weighted (IRW) depth (Ramsay et al., 2019). For a precise definition of the depth functions we discuss here, as well as a discussion of their basic properties, see Section 5.

If, instead of the median, one is interested in computing depth values themselves, this can be done privately with an additive noise mechanism, see Section 7.

Our main result relies on a regularity condition, for which we need to introduce the following notation. Let \( \mathscr{B} \) denote the space of Borel functions from \( \mathbb{R}^d \) to \( [0, 1] \). For a family of functions, \( \mathscr{F} \subseteq \mathscr{B} \), define a pseudometric on \( \mathcal{M}_1(\mathbb{R}^d) \),

\[
d_{\mathscr{F}}(\mu, \nu) = \sup_{g \in \mathscr{F}} |\int gd(\mu - \nu)|, \]

where \( \mu, \nu \in \mathcal{M}_1(\mathbb{R}^d) \). Recall that many standard metrics on probability measures can be written in this fashion. For example, taking \( \mathscr{F} \) to be the set of indicator functions of Borel sets yields the Total Variation distance and taking \( \mathscr{F} \) to be the set of indicator functions of semi-infinite rectangles yields the Kolmogorov–Smirnov distance. We then define the following important regularity condition.
Definition 2. We say that D is \((K, \mathcal{F})\)-regular if there exists a class of functions \( \mathcal{F} \subset \mathcal{B} \) such that \( D(x, \cdot) \) is \( K \)-Lipschitz with respect to the \( \mathcal{F} \)-pseudometric uniformly in \( x \), i.e., for all \( \mu, \nu \in \mathcal{M}_1(\mathbb{R}^d) \)

\[
\sup_x |D(x, \mu) - D(x, \nu)| \leq K d_{\mathcal{F}}(\mu, \nu).
\]

The property \((K, \mathcal{F})\)-regularity can be thought of as a robustness condition in the sense that \((K, \mathcal{F})\)-regular functions are affected by extreme observations in a very limited manner, through the boundedness of \( g \in \mathcal{F} \). Specifically, one can show that \((K, \mathcal{F})\)-regularity implies the bound \( GS_n(D) \leq K/n \) on the global sensitivity, see Lemma 28. Thus, for the proposed private median estimator, we will assume that \( D \) is \((K, \mathcal{F})\)-regular (Condition 3) and take \( \beta = n\epsilon/2K \), which ensures \( \hat{\theta}_n \) is \( \epsilon \)-differentially private, see Lemma 27. In our applications, we will typically take \( \mathcal{F} \) to be a class of functions with low complexity (in the sense of Vapnik–Chervonenkis dimension).

We now state the necessary regularity conditions on the pair \((\mu, D)\) that are required for our main result. In the following, let \( \text{VC}(\mathcal{F}) \) denote the Vapnik–Chervonenkis dimension of \( \mathcal{F} \).

Condition 1. The function \( D(\cdot, \mu) \) has a maximizer.

Condition 2. The map \( x \mapsto D(x, \mu) \) is \( L \)-Lipschitz a.e. for some \( L > 0 \).

Condition 3. There is some \( K > 0 \) and some family of functions \( \mathcal{F} \subseteq \mathcal{B} \) with \( \text{VC}(\mathcal{F}) < \infty \) such that \( D \) is \((K, \mathcal{F})\)-regular.

To state our main result, we must also introduce the discrepancy function. Let \( E_{D, \mu} = \{ \theta : D(\theta, \mu) = \max_{x \in \mathbb{R}^d} D(x, \mu) \} \) denote the set of maximizers of \( D(\cdot, \mu) \) for fixed \( D \) and \( \mu \). For a set \( A \) and \( r > 0 \), let \( B_r(A) = \{ x : \min_{y \in A} \| x - y \| \leq r \} \). The discrepancy function of the pair \( (D, \mu) \) is

\[
\alpha(t) = D(\theta_0, \mu) - \sup_{x \in B_r(E_{D, \mu})} D(x, \mu),
\]

where \( \theta_0 \in E_{D, \mu} \). In Table 1 below, we explicitly calculate \( \alpha \) for several depth functions. Note that \( \alpha \) is a non-decreasing function because the optimization problem involved is over decreasing sets.

We are now ready to state our main result, which provides bounds on the finite-sample deviations of the proposed differentially private multivariate medians via depths. For concreteness, we present our results for two of the most popular choices of priors, namely Gaussian measures and the uniform measure on \( A_R(y) \), the \( d \)-dimensional cube of side-length \( R \) centered at \( y \). In the following, let \( d_{R, y}(x) \) denote the minimum distance from a point \( x = (x_1, \ldots, x_d) \) to a face of the cube \( A_R(y) \): \( d_{R, y}(x) = \min_{1 \leq i \leq d} |x_i - y_i \pm R/2| \). Similarly, denote the minimum distance from a set \( B \subset \mathbb{R}^d \) to a face of the cube \( A_R(y) \) by \( d_{R, y}(B) = \inf_{x \in B} d_{R, y}(x) \) and let \( d(x, B) = \inf_{y \in B} \| x - y \| \) denote the usual point-to-set distance. Define \( \alpha^{-1}(t) = \sup\{ r \geq 0 : \alpha(r) = t \} \), where one notes that \( \alpha^{-1} \) exists because \( \alpha \) is a monotone function. Lastly, we write \( a \lesssim b \) if \( a \leq C b \) for some universal constant \( C > 0 \).

Theorem 1. If Conditions 1, 2, 3 hold, then the following holds:

(i) Suppose that \( L \geq 1 \). If \( \pi = N(\theta_\pi, \sigma_\pi^2 I) \), then there exists a universal constant \( c > 0 \) such that
for all $n \geq 8K/\epsilon$, all $d > 2$ and all $0 < \gamma < 1$, with probability at least $1 - \gamma$, we have that

$$d(\mathbb{E}_{D,\mu}, \hat{\theta}_n) \lesssim \alpha^{-1} \left( cK \left[ \sqrt{\frac{\log(1/\gamma) \vee \text{VC}(\mathcal{F}) \log n}{n}} \right. \right.$$

$$\left. + \frac{\log(1/\gamma) \vee (\frac{d(\mathbb{E}_{D,\mu}, \hat{\theta}_n)^2}{\sigma^2} + d \log \left( \frac{2\pi L\nu}{K} \vee d \right))}{n\epsilon} \right].$$

(ii) If instead $\pi \propto 1 \{ x \in A_R(\theta_\pi) \}$ where $E_{D,\mu} \subset A_R(\theta_\pi)$, then there exists a universal constant $c > 0$ such that for all $n, d \geq 1$ and all $0 < \gamma < 1$, with probability at least $1 - \gamma$, we have that

$$d(\mathbb{E}_{D,\mu}, \hat{\theta}_n) \lesssim \alpha^{-1} \left( cK \left[ \sqrt{\frac{\log(1/\gamma) \vee \text{VC}(\mathcal{F}) \log n}{n}} \right. \right.$$

$$\left. + \frac{\log(1/\gamma) \vee d \log \left( \frac{R_n}{n\eta \mathbb{E}_{D,\mu}(\frac{R_n}{\eta})} \right)}{n\epsilon} \right].$$

The proof of Theorem 1 can be found in Section 4, and relies on an elementary concentration bound (Theorem 11), which we introduce in Section 3. We also provide a version of Theorem 1 in terms of sample complexity, see Corollary 7.

Remark 2 (Small $\alpha$ and the cost of privacy). We are mainly concerned about small deviations. For small $\alpha$, for many choices of $\mu$ and $D$, $\alpha^{-1}$ is often at most linear, see Section 2.3. From a technical perspective, this is convenient, since then we need not be concerned about inverting $\alpha$ directly. In addition, this fact coupled with the fact that many of the depth functions satisfy Condition 3 with $K = O(1)$ and $\text{VC}(\mathcal{F}) = O(1)$ recovers a bound similar to those obtained in the case of private mean estimation (Kamath et al., 2020; Cai et al., 2021), viz.

$$d(\mathbb{E}_{D,\mu}, \hat{\theta}_n) \lesssim \sqrt{\frac{\log(1/\gamma) \vee d \log n}{n}} \sqrt{\frac{\log(1/\gamma) \vee d \log \left( \frac{R_n}{n\eta \mathbb{E}_{D,\mu}(\frac{R_n}{\eta})} \right)}{n\epsilon}}.$$  

Under such $\mu$ and $D$, assuming a reasonable prior and ignoring logarithmic terms, the cost of privacy is proportional to $\sqrt{d/n\epsilon}$. For concrete examples, see Examples 1 and 2.

While we state our result for general depth functions satisfying the above assumptions, our results apply to many of the most popular depths, such as halfspace depth and, more broadly, Type D depths in the sense of Zuo and Serfling (2000), spatial depth, simplicial depth and both the integrated rank-weighted and integrated dual depths. Depth functions to which these results apply and the conditions under which they apply are discussed at length in Section 5. The conditions for specific popular depths are summarized in Table 2.

We note that our bound is essentially sharp, see Lemma 17. Given that if $\mu$ is Gaussian, then the population mean equals the population median, the bound on the sample complexity implied by Theorem 1 (see Corollary 7) matches the lower bound on the sample complexity for private Gaussian mean estimation given by Kamath et al. (2018), Theorem 6.5 and Hopkins et al. (2022), Theorem 7.2 up to logarithmic factors.

Alternatively, Cai et al. (2021) give a minimax lower bound for approximately differentially private sub-Gaussian mean estimation. Unsurprisingly, given that it is designed for the approximately
differentially private setting, this lower bound becomes \(-\infty\) if we apply it to the setting of pure differential privacy. However, it is still interesting to compare Theorem 1 to the bound of Cai et al. (2021) in some way. A trivial manipulation of the conclusion of Lemma 17 shows that the bound given in Theorem 1 matches the lower bound of Cai et al. (2021), up to logarithmic terms, when the “approximate differential privacy parameter”, i.e., \(\delta\) is taken to be at least \(n^{-k}\) for some \(k > 0\). Therefore, as stated above, the bound in Theorem 1 is essentially sharp. Furthermore, at least for the setting of Gaussian mean estimation, relaxing to the setting of approximate differential privacy will not yield faster rates than those given by Theorem 1.

Recall that estimating a parameter from an unbounded parameter space, i.e., unbounded estimation, is non-trivial in the setting of differential privacy, e.g., see Avella-Medina and Brunel (2019a). Theorem 1 shows that choosing \(\sigma_\pi\) relatively large allows one to perform unbounded estimation at the cost of an extra \(\log d\) in the second term of (2.4), or, equivalently, an extra \(\log d\) to the sample complexity. To elaborate, using a depth function in the exponential mechanism with a Gaussian prior constitutes unbounded estimation of the population median. Theorem 1 then shows that \(d(E_{D,\mu}, \theta_\pi)/\sigma_\pi = O(\sqrt{\log d})\) is sufficient for the deviations under a misspecified Gaussian prior to match the deviations under a correctly specified uniform prior, up to a logarithmic factor in \(d\). Thus, we only require \(d(E_{D,\mu}, \theta_\pi)\) to be polynomial in \(d\), i.e., choosing \(\sigma_\pi\) to be some large polynomial in \(d\) will ensure that \(\sigma_\pi \geq d(E_{D,\mu}, \theta_\pi)/\sqrt{\log d}\).

Remark 3 (Projection depth). One popular notion of depth not covered by our analysis is the projection depth (Zuo, 2003). This is because a private median drawn from the exponential mechanism based on the empirical versions of this depth function is inconsistent, see Lemma 29.

Remark 4 (Comparison to non-private bounds). The first term in the maximum in Theorem 1 is a deviations bound for non-private depth-based medians. If we assume that \(\theta_0 \in A_R(0)\) and choose the depth function to be halfspace depth, then this upper bound recovers the upper bound on the non-private estimation error of the halfspace median for the Gaussian measure, given by Theorem 3 of Zhu et al. (2020).

Remark 5 (Relaxing the Lipschitz requirement). The Lipschitz requirement for \(D\) can be relaxed to obtain more general deviation bounds, at the cost of complicating the presentation. Furthermore, all the depth functions we consider are Lipschitz functions a.e. under mild assumptions, see Theorem 22. It is also not necessary to require that \(E_{D,\mu} \subset A_R(\theta_\pi)\). It is sufficient to assume that \(E_{D,\mu} \cap A_R(\theta_\pi) \neq \emptyset\). In this case, one can replace \(d_{R,\theta_\pi}(E_{D,\mu})\) with \(d_{R,\theta_\pi}(E_{D,\mu} \cap A_R(\theta_\pi))\).

Remark 6 (The univariate setting). Tzamos et al. (2020) gave a lower bound on the minimax sample complexity for estimating the one-dimensional median of a population measure whose density is bounded below by a positive constant. Applying Theorem 1 when \(d = 1\) to such measures yields an upper bound with matches that lower bound \(2\). Therefore, in one-dimension, the proposed depth-based median is optimal.

Note that Theorem 1 has a straightforward restatement in terms of sample complexity.

Corollary 7. Suppose that Conditions 1, 2, 3 hold. Then the following holds

(i) Suppose that \(D(x, \mu) \leq 1\) and \(L \geq 1\). If \(\pi = N(\theta_\pi, \sigma_\pi^2 I)\), then there exists universal constants \(C, c > 0\) such that for all \(t \geq 0\), all \(d > 2\) and all \(0 < \gamma < 1\), we have that \(d(E_{D,\mu}, \hat{\theta}_n) < t\) with

\(\footnote{1}{\text{It should not be difficult to extend this conclusion to the more general case of symmetric sub-Gaussian median estimation.}}\)

\(\footnote{2}{\text{This upper bound recovers the one given by Karwa and Vadhan (2017) for one-dimensional Gaussian mean estimation.}}\)

7
cannot be applied in this setting. We demonstrate that, omitting logarithmic terms, the number of
measure Cauchy marginals. This setup is more difficult in the sense that the mean of the probability

\[ \| \theta_0 - \theta_n \| / \sigma_\pi \leq \sqrt{d}, \] 

the finite-sample deviations of the private median reduce to \( O(\sqrt{\lambda_1 d/n} \lor \sqrt{\lambda_1 \gamma} \lor \sigma_\pi) \). Here, \( \lambda_1 \) is the largest eigenvalue of the covariance matrix of \( \mu \).

Next, we consider estimating the location parameter of a population measure \( \mu \) made up of Cauchy marginals. This setup is more difficult in the sense that the mean of the probability measure \( \mu \) does not exist. For instance, theory concerning differentially private mean estimators cannot be applied in this setting. We demonstrate that, omitting logarithmic terms, the number of

\[ n > C K^2 \left( \frac{\log(1/\gamma) \lor [\text{VC}(\mathcal{F}) \log(R / \sigma_\pi / \alpha(t)) \lor 1]}{\alpha(t)^2} \lor \frac{\log(1/\gamma) \lor d \log \left( \frac{d_E(u, \theta)}{\sigma_\pi} \lor d \right)}{\alpha(t)} \right). \] 

(ii) If instead \( \pi \propto 1 \{ x \in A_R(\theta_\pi) \} \) where \( E_{D, \mu} \subset A_R(\theta_\pi) \), then there exists universal constants \( C, c > 0 \) such that for all \( t \geq 0 \), all \( d \geq 1 \) and all \( 0 < \gamma < 1 \), we have that \( d(E_{D, \pi}, \theta_\pi) \leq t \) with probability at least \( 1 - \gamma \) provided

\[ n > C K^2 \left( \frac{\log(1/\gamma) \lor [\text{VC}(\mathcal{F}) \log(R / \sigma_\pi / \alpha(t)) \lor 1]}{\alpha(t)^2} \lor \frac{\log(1/\gamma) \lor d \log \left( \frac{d_E(u, \theta)}{\sigma_\pi} \lor d \right)}{\alpha(t)c} \right). \] 

The proof is deferred to Appendix A.

2.1 Examples

Let us now turn to some concrete examples that illustrate our result. We first consider the canonical

Table 1: Table of the discrepancy function \( \alpha(t) \) for different depth functions and underlying
population measures. Recall that \( \mu \) is \( d \)-version symmetric about zero if for any unit vector \( u \),
\( X^\top u = a(u)Z \) where \( X \sim \mu \) and \( Z \) is a random variable such that \( Z \overset{d}{=} -Z \) and \( a(u) = a(-u) \). Here, \( v_1 \) is the eigenvector associated with the largest eigenvalue \( \lambda_1 \) of \( \Sigma \) and \( F_0(x) = F(x, u, \mu) \), see \ref{6.2}, and \( \nu \) is the uniform measure on the \((d - 1)\)-dimensional unit sphere.

| D/\mu: | Nd(\theta_0, \Sigma) | d-version symmetric |
|-------|-------------------|--------------------|
| HD    | \Phi \left( \frac{1}{\sqrt{\lambda_1}} \right) - \frac{1}{2} | \frac{1}{2} - \inf_{\|u\|=1} \sup_{\|u\|=1} F_0 \left( \frac{-u^\top u}{a(u)} \right) |
| IRW   | \int \left| \frac{1}{2} - \Phi \left( \frac{t v^\top u}{\sqrt{\sigma_\pi^2} \Sigma_u} \right) \right| d\nu | \inf_{\|u\|=1} \int \left| \frac{1}{2} - F_0 \left( \frac{t v^\top u}{a(u)} \right) \right| d\nu |
| IDD   | \frac{1}{4} - \int \Phi \left( \frac{-u^\top u}{\sqrt{\sigma_\pi^2} \Sigma_u} \right) \Phi \left( \frac{t v^\top u}{\sqrt{\sigma_\pi^2} \Sigma_u} \right) d\nu | \frac{1}{4} - \sup_{\|u\|=1} \int F_0 \left( \frac{t v^\top u}{a(u)} \right) F_0 \left( \frac{-t v^\top u}{a(u)} \right) d\nu |

\[ n > C K^2 \left( \frac{\log(1/\gamma) \lor [\text{VC}(\mathcal{F}) \log(R / \sigma_\pi / \alpha(t)) \lor 1]}{\alpha(t)^2} \lor \frac{\log(1/\gamma) \lor d \log \left( \frac{d_E(u, \theta)}{\sigma_\pi} \lor d \right)}{\alpha(t)c} \right). \] 

\[ (2.7) \]
samples required for non-private estimation $O(d^2)$ eclipses the cost of privacy, which is $O(d^{3/2}/\epsilon)$; if we have enough samples to perform “heavy tailed” estimation, then we have enough samples to ensure privacy.

Example 1 (Multivariate Gaussians with misspecified prior). Consider the problem of estimating the location parameter $\theta_0$ of a multivariate Gaussian measure $\mu = \mathcal{N}(\theta_0, \Sigma)$ and we wish to estimate $\theta_0$. For this example, let $\lambda_1 > \ldots > \lambda_d > 0$ be the ordered eigenvalues of the covariance matrix $\Sigma$. Suppose that in order to estimate $\theta_0$ we use the exponential mechanism paired with the halfspace depth $D = \text{HD}$ in conjunction with a misspecified Gaussian prior: $\pi = \mathcal{N}(\theta_\pi, \sigma_d^2 I)$.

In order to apply Theorem 1, we need to check Condition 1 and Lipschitz continuity of HD. First, note that Condition 1 holds for any bounded depth function as defined in Definition 1. Next, one can show that (see Theorem 22) the halfspace depth is $(1, \mathcal{F})$-regular where $\mathcal{F}$ is the set of closed halfspaces in $\mathbb{R}^d$, thus, $\text{VC}(\mathcal{F}) = d + 2$. Lastly, the map $x \mapsto \text{HD}(x, \mu)$ is $L$-Lipschitz with $L = (2\pi\lambda_d)^{-1/2}$, implying Condition 2 is satisfied. We may now apply Theorem 4 for which we must compute $\alpha(t)$. To this end, it holds that

$$||\theta_0 - \hat{\theta}_n|| \lesssim \frac{\sqrt{\lambda_1}}{CR} \left[ \frac{\log(1/\gamma) \lor d \log n}{n} \sqrt{\frac{\log (1/\gamma) \lor \sigma_d^2}{\|	heta_0 - \theta_\pi\|^2 / \sigma_d^2}} \lor d \log (n\epsilon^2 / \lambda_d \lor d) \right].$$

Suppose we would like to achieve a success rate of $1 - e^{-d}$, i.e., $\gamma = e^{-d}$. This reduces the deviations bound to

$$||\theta_0 - \hat{\theta}_n|| \lesssim \frac{\sqrt{\lambda_1}}{CR} \left( \frac{d \log n}{n} \lor \frac{\|	heta_0 - \theta_\pi\|^2 / \sigma_d^2}{\epsilon^2} \lor d \log (n\epsilon^2 / \lambda_d \lor d) \right).$$

Observe now that in order for the prior effect to be negligible, we require $||\theta_0 - \theta_\pi|| / \sigma_d \leq \sqrt{d \log d}$ and that $\sigma_d$ is at most polynomial in $d$. This suggests that, up to a point, a larger prior variance $\sigma_\pi^2$ produces lower deviations. In this case, when the error level, $\epsilon$ and $\Sigma$ are fixed, we only need $n$ to grow slightly faster than the dimension $n \geq C d \log d$ in order to maintain a consistent level of error. If the prior effect is negligible in the sense just described, then, omitting logarithmic terms for brevity, the deviations bound reduces to

$$||\theta_0 - \hat{\theta}_n|| \lesssim \frac{\sqrt{\lambda_1}}{CR} \cdot \left( \frac{\sqrt{d}}{n} \lor \frac{d}{n} \right), \quad (2.8)$$

We then have that the cost of privacy is eclipsed by the sampling error whenever $n \gtrsim d / \epsilon^2$. Therefore, differential privacy is free, provided that $d$ is not too large. Furthermore, (2.8), makes it is easy to see that in this case, the private medians achieve the minimax lower bound for approximately differentially private sub-Gaussian mean estimation \cite{Cai et al. 2021}, if $\delta \propto n^{-k}$.

Example 2 (Cauchy marginals). Suppose that $\mu$ is the product probability measure constructed from independent Cauchy marginals with location parameter 0 and scale parameters $\sigma_1, \ldots, \sigma_d$ and we would like to use the exponential mechanism to privately estimate the halfspace median of $\mu$. In this case, $\mu$ is $d$-version symmetric with $a(u) = \sum_{j=1}^d |\sigma_j| |u_j|$ where $u_j$ is the $j^{th}$ coordinate of $u$.

\footnote{Recall that $\mu$ is $d$-version symmetric about zero if for any unit vector $u$, $X^T u \overset{d}{=} a(u)Z$ where $X \sim \mu$ and $Z \sim \nu$ such that $Z \overset{d}{=} -Z$ and $a(u) = a(-u)$.}
Table 2: Table of values for which different depth functions satisfy Theorem 1. Letting $\mu_u$ be the law of $X^\top u$ if $X \sim \mu$, $f_u$ is the density of $\mu_u$ with respect to the Lebesgue measure and $L'$ denotes $\sup_y E \|y - X\|^{-1}$. Note $\mathcal{A}$ is the set of admissible measures for which $D$ is a depth function, in the sense of Definition 3. Here, $\mathcal{C} \subset \mathcal{M}_1(\mathbb{R}^d)$ denotes the set of centrally symmetric measures over $\mathbb{R}^d$ and $\mathcal{N} \subset \mathcal{M}_1(\mathbb{R}^d)$ denotes the set of angularly symmetric measures over $\mathbb{R}^d$ that are absolutely continuous with respect to the Lebesgue measure.

| $\mathcal{A}$ | HD | IDD | IRW | SMD | SD | MSD |
|---------------|----|-----|-----|-----|----|-----|
| $\text{VC}(\mathcal{F})$ | $O(d)$ | $O(d)$ | $O(d)$ | $O(d \log d)$ | $O(d)$ | $O(d)$ |
| $K$ | 1 | 3 | 4 | $d + 1$ | $d$ | 1 |
| $L$ | $\sup_u \|f_u\|$ | $3 \sup_u \|f_u\|$ | $2 \sup_u \|f_u\|$ | $\sup_u \|f_u\|$ | $2L'$ | $2\sqrt{d}L'$ |
| $\mathcal{A}$ | $\mathcal{M}_1(\mathbb{R}^d)$ | $\mathcal{C}$ | $\mathcal{C}$ | $\mathcal{N}$ | $\emptyset$ (see Remark 20) | $\emptyset$ |

A similar analysis as to that of Example 1 gives that halfspace depth (under this $\mu$) satisfies the conditions of Theorem 1. Note that in this setting, the Lipschitz constant scales with the dimension: $L = O(\sqrt{d})$.

The next step is to compute $\alpha(t)$. Let $\bar{\sigma} = d^{-1} \sum_{j=1}^{d} \sigma_j$. Observe that when the scale parameters are constant in $d$, $\sup_u a(u) = \sqrt{d/\bar{\sigma}}$. It follows that under the halfspace depth, $\alpha(t) = \frac{1}{\pi} \arctan \left( \frac{t}{\sqrt{d/\bar{\sigma}}} \right)$. If $t < R$ for some large $R$, then we can write $\alpha(t) \gtrsim t(\bar{\sigma} \sqrt{1 + R^2/d\bar{\sigma}^2})^{-1}$.

If we use a Gaussian prior with unit scale, then, omitting logarithmic terms, the deviations bound becomes

$$\|\hat{\theta}_n\| \lesssim \frac{d\bar{\sigma}}{\sqrt{n}} \vee \frac{d^{1/2} \|\theta_\pi\|^2 \vee d^{3/2}}{en}.$$ 

One can immediately see the difficulty of estimating location when $\mu$ is high-dimensional and heavy-tailed: both the non-private and private terms are now a factor of $\sqrt{d}$ larger. In this case, the number of samples required for non-private estimation $O(d^2)$ eclipses the cost of privacy, which is $O(d^{3/2}/\epsilon)$. Thus, if we have enough samples to perform heavy tailed estimation, then we have enough samples to ensure privacy.

In the Gaussian setting, the discrepancy function is relatively similar between the halfspace depth to the integrated depths. However, in the Cauchy setting, the discrepancy functions differ between the halfspace depth, the integrated-rank weighted depth and the integrated dual depth. To see this, we must evaluate the integrals in Table 1. For the IRW this integral is given by

$$\alpha(t) = \frac{1}{\pi} \inf_{\|v\|=1} \int \left| \arctan \left( \frac{tv^\top u}{\sum_{j=1}^{d} \sigma_j |u_j|} \right) \right| dv(u).$$

We evaluate these integrals via Monte Carlo simulation with 10,000 unit vectors. Figure 1 plots the logarithm of the inverted discrepancy function computed at $t = 1$, i.e., $\log(1/\alpha(1))$, for the three depth functions as the dimension increases. Observe that the halfspace depth’s discrepancy function decreases the slowest. Thus, the halfspace depth gives the smallest sample complexity for heavy tailed estimation when the dimension is large. The difference in accuracy between the halfspace depth and the integrated depths demonstrates the trade-off between computability and statistical accuracy.

---

4 We don’t lose generality here; we can arbitrarily shrink the scale of the $\mu$. 

10
Figure 1: Logarithm of the inverse of the value of the discrepancy function at \( t = 1, \log(1/\alpha(1)) \), as the dimension increases. Here, \( \mu \) is made up of \( d \) independent standard Cauchy marginals. Notice that halfspace depth has the smallest values of \( 1/\alpha(1) \) which implies it has a smaller bound on the sample complexity. This demonstrates the trade-off between accuracy and computability when choosing a depth function. For the integrated depths, the value of the integrand is approximated via Monte Carlo simulation with 10,000 unit vectors.

2.2 Simulation

To complement our theoretical results, we demonstrate the cost of privacy through simulation. In order to simulate a private median, we use discretized Langevin dynamics. We recognize that privacy guarantees are affected by the use of Markov chain Monte Carlo (MCMC) methods; the purpose of the current implementation is to investigate the accuracy of the methods. For a discussion on the level of privacy guaranteed by MCMC, see Seeman et al. [2021]. As a by-product of our work, we also provide a fast implementation of a modified version of the non-private integrated dual depth-based median. This implementation shows that this multivariate median can be computed quickly, even when the dimension is large. For example, we can compute the median of ten thousand 100-dimensional samples in less than one second on a personal computer. We call this modified version of the integrated dual depth the smoothed integrated dual depth, see Section 6.1 for more details. Our code is available on GitHub.

Aside from the private smoothed integrated dual depth median, we also computed the non-private smoothed integrated dual depth median and the private coin-press mean of Biswas et al. [2020]. When computing the coin-press mean, we used the existing GitHub implementation provided by the authors. The bounding ball for the coin-press algorithm had radius \( 10\sqrt{d} \) and was run for four iterations.

We simulated fifty instances with \( n = 10,000 \) over a range of dimensions from two to one hundred. The data were either generated from a standard Gaussian measure or contaminated standard Gaussian measure. In the contaminated model, 25\% of the observations had mean \((5, \ldots, 5)\). The privacy parameter \( \epsilon \) and the smoothing parameter \( s \) were both fixed at 10 and \( \pi = N(0, 25dI) \).

Figure 2 shows the empirical root mean squared error (ERMSE) for the different location estimators as the dimension increases. Notice that the private median ERMSE grows at a similar rate.

---

5The code was run on a desktop computer with an Intel i7-8700K 3.70GHz chipset and an Nvidia GTX 1660 super graphics card.
as that of the non-private ERMSE. The mean estimators perform very well in the uncontaminated setting, but poorly in the contaminated setting. The median estimators do not perform as well as the mean estimators in the standard Gaussian setting, but do not become corrupted in the contaminated setting. This demonstrates the usual trade-off between accuracy and robustness between the mean and the median.

2.3 Inverse discrepancy function for small $t$

Below, we give conditions for there to exist $C_\mu > 0$ such that $\alpha(t) \geq C_\mu t$ for small $t$. This implies that the $\alpha^{-1}(t)$ in Theorem 1 can be replaced by $t/C_\mu$, when $n$ is large enough, thus, simplifying the bound greatly. We first give general conditions under which $\alpha(t) \geq C_\mu t$ for an arbitrary depth function.

**Lemma 8.** Suppose that $\sup_{x \in \mathbb{R}^d} D(x, \mu) = D(\theta_0, \mu)$, $D(\cdot, \mu)$ is translation invariant, decreasing along rays, continuous and continuously differentiable on $B_r(\theta_0)$ for some $r > 0$. In addition, suppose that there exists $0 < a < R < r$ such that $u_* = \arg\max_{u \in S^{d-1}} D(u \cdot t, \mu)$ is constant in $t$ for $a \leq t \leq R$. Then, if for $a \leq t \leq R$, it holds that

$$g^*(t) = u_*^\top \left[ \frac{d}{dx} D(x, \mu) \big|_{x=t,u_*} \right]$$

is non-decreasing on $[a, R]$, we have that $\alpha(t) \geq t\alpha(R)/R$.

Note that the proofs of results in this section are deferred to Appendix A. We next use Lemma 8 to show a general bound for integrated depth functions.

**Lemma 9.** Suppose that $\mu$ is centrally symmetric about $\theta_0$ and $D(x, \mu) = \int_{S^{d-1}} g(x^\top u, \mu_u) d\nu$ where for all $u \in S^{d-1}$, we have that
1. $g(ay + b, a\mu_u + b) = g(y, \mu_u)$,
2. $g(\cdot, \mu_u)$ are bounded, continuous and continuously differentiable,
3. $g(\theta_0^T u, \mu_u) = 0$ and $g(\theta_0^T u - y, \mu_u) = g(\theta_0^T u + y, \mu_u)$,
4. there exists $R > 0$ such that $g(\theta_0^T u + y, \mu_u)$ is non-increasing for $0 \leq y \leq R$ and
5. there exists $v \in \mathbb{S}^{d-1}$ such that $g(c(\theta_0 + v)^T u, \mu_u) \geq g(c(\theta_0 + w)^T u, \mu_u)$ for all $c > 0$ and $w \in \mathbb{S}^{d-1}$,

then for $t \in [0, R]$ it holds that \[\alpha(t) \geq t\alpha(R)/R.\]

Observe that Items 1-4 are satisfied for the smoothed integrated dual depth and the integrated-rank-weighted depth provided the projected cumulative distribution functions $h(y, u) = \mu(\{X^T u \leq y\})$ are continuous and continuously differentiable. Item 5 depends on the distribution, but is fairly general. For instance, it is easy to see that item 5 is satisfied by the class of elliptical distributions, or anytime $\mu$ is such that the depth function has nested, convex contours.

In the univariate setting, deviations bounds for the univariate median often only require a lower bound on the density of the population measure in a neighborhood of the median. We next show a result of this flavor, for the halfspace depth. Recall that a measure $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is halfspace symmetric about $\theta_0 \in \mathbb{R}^d$ if for every closed halfspace $H$ which contains $\theta_0$, it holds that $\mu(X \in H) \geq 1/2$ (Zuo and Serfling, 2000).

**Lemma 10.** Suppose that $D = HD$, $\argmax_{x \in \mathbb{R}^d} HD(x, \mu) = \theta_0$ and suppose that there is some $R > 0$ such that for all $u \in \mathbb{S}^{d-1}$, $\mu_u$ are halfspace symmetric and absolutely continuous. If there exists $C_\mu > 0$ such that for all $u \in \mathbb{S}^{d-1}$, $\inf_{\theta_0^T u - R} f_{\mu_u}(x) \geq C_\mu$, then for all $\theta_0^T u - R \leq t \leq \theta_0^T u + R$, we have that $\alpha(t) \geq tC_\mu$.

### 3 An elementary concentration bound

In this section, we introduce the underlying concentration bound used to produce Theorem 4 which we imagine will be useful more broadly. Our result shows that for a general objective function $\phi$, $\theta_\alpha$ concentrates around $\theta_0 = \argmax_{\theta \in \mathbb{R}^d} \phi(\theta, \mu)$ with high probability. First, we must introduce a relaxed version of Condition 2 and some new important functions.

**Condition 4.** The map $x \mapsto \phi(x, \mu)$ is $\pi$-a.s. uniformly continuous with modulus of continuity $\omega$.

Replacing the depth function $D$ with $\phi$, we can define the generalized discrepancy function of the pair $(\phi, \mu)$ as $\alpha(t) = \phi(\theta_0, \mu) - \sup_{x \in B_t^r(E_{\phi, \mu})} \phi(x, \mu)$, where $\theta_0 \in E_{\phi, \mu}$. The calibration function of the triple $(\phi, \mu, \pi)$ is the function

$$\psi(\lambda) = \min_{t > 0} \left[ \lambda \cdot \omega(r) - \log \pi(B_t^r(E_{\phi, \mu})) \right]$$

Note that $\psi(\lambda)$ is increasing and tends to $\infty$ as $\lambda \to \infty$. The rate function of the prior is given by

$$I(t) = -\log \pi(B_t^r(E_{\phi, \mu})).$$

The concentration/entropy function of the prior is the rate at which the prior concentrates around the maximizers of the population risk. For large $t$, it is the rate of decay of the tails of $\pi$.
The concentration bound is given in the following theorem, which quantifies the accuracy of $\hat{\theta}_n$ drawn from $(2.2)$ as an estimate of $\theta_0$.

**Theorem 11.** Suppose that for the pair $(\phi, \mu)$, Conditions 1, 3 and 4 hold for some $K > 0$. Then, there are universal constants $c_1, c_2 > 0$ such that the following holds. For any $t, \beta > 0$, we have that

$$\Pr \left( \max_{\theta_0 \in E_{\phi, \mu}} \left\| \hat{\theta}_n - \theta_0 \right\| \geq t \right) \leq \left( \frac{c_1 n}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-c_2 n \left[ t(t) \vee \alpha(t) \right]^2 / K^2} + e^{-\beta \alpha(t)/2 - t^2/2 + \psi(\beta)}.$$

**Remark 12** (Comparison to McSherry and Talwar [2007]). It is helpful to compare our result to the widely used accuracy bound of McSherry and Talwar [2007]. The latter controls the accuracy of the estimator $\hat{\theta}_n$ as measured by the empirical objective value: it bounds the difference between the empirical objective value achieved by the exponential mechanism and the maximal empirical objective value. More precisely, if we let $S_1 = \{ x \in \mathbb{R}^d : \phi(x, \hat{\mu}_n) - \sup_{\theta} \phi(\theta, \hat{\mu}_n) \geq 2t \}$, the result of McSherry and Talwar [2007] says that

$$\Pr \left( \sup_{\theta} \phi(\theta, \hat{\mu}_n) - \phi(\theta, \hat{\mu}_n) > 2t \right) \leq e^{-t^2} / \pi(S_1).$$

As well, note that there is no analogous sample complexity bound because this approximation is not improving with the sample size. By contrast, $(3.1)$ concerns the consistency of $\hat{\theta}_n$ as an estimator, namely the distance between the estimator and any minimizer of the population objective, that is $\| \hat{\theta}_n - \theta_0 \|$. Furthermore, the quality of this concentration bound depends on $n$ and thus a sample complexity bound can be read off readily.

**Proof.** We first prove that the regularity conditions imply that the empirical objective function concentrates around the population objective function. By Condition 3, there are some $K > 0$ and $\mathcal{F}$ with $\text{VC}(\mathcal{F}) < \infty$ such that $\phi$ is $(K, \mathcal{F})$-regular, i.e.,

$$\sup_x |\phi(x, \mu) - \phi(x, \hat{\mu}_n)| \leq K \sup_{g \in \mathcal{F}} |\mathbb{E}_{\hat{\mu}_n} g(X) - \mathbb{E}_{\mu} g(X)|. \tag{3.3}$$

Now, for a probability measure $Q$ over $\mathcal{F}$, let $N(\tau, \mathcal{F}, L_2(Q))$ be the $\tau$-covering number of $\mathcal{F}$ with respect to the $L^2(Q)$-norm. By assumption, for any $g \in \mathcal{F}$, $\|g\|_\infty \leq 1$ and $\text{VC}(\mathcal{F}) < \infty$. This fact implies that there is some universal $c > 0$ such that for any $0 < \tau < 1$, it holds that,

$$\sup_Q N(\tau, \mathcal{F}, L^2(Q)) \lesssim \text{VC}(\mathcal{F}) \left( \frac{\tau}{c} \right)^{\text{VC}(\mathcal{F})}.$$

See Sen [2018, Theorem 7.11], or Kosorok [2008, Theorem 9.3]. A straightforward manipulation of Talagrand’s inequality (Talagrand, 1994, Theorem 1.1) implies that there is some universal $c > 0$ such that for all $n \geq 1$ and for all $t > 0$, it holds that

$$\Pr \left( \sqrt{n} \sup_{g \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n g(X_i) - \mathbb{E} g(X) \right| \geq t \right) \leq \left( \frac{cn}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-2t^2}.$$

Now, let $c_1^* = 2 / K^2$. For all $t > 0$, it holds that

$$\Pr \left( \sup_{x \in \mathbb{R}^d} \left| \phi(x, \hat{\mu}_n) - \phi(x, \mu) \right| > t \right) \leq \left( \frac{cn}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-2nt^2 / K^2}. \tag{3.4}$$
where to go from the second inequality to the third we multiplied both sides of the inequality by $\sqrt{n}$.

We can now use the concentration result (3.4) to prove the rest of the theorem. For brevity, let $D_{n,t} = \{ \max_{\theta_0 \in F_{\phi, \mu}} \| \hat{\theta}_n - \theta_0 \| > t \}$, and for any $y > 0$, let $A_{n,y} = \{ \| \phi(\cdot, \hat{\mu}_n) - \phi(\cdot, \mu) \|_\infty < y \}$. It follows from the law of total probability and (3.4) that for any $y > 0$ it holds that

$$
\Pr( D_{n,t} ) = \Pr( D_{n,t} \cap A_{n,y} ) + \Pr( D_{n,t} \cap A_{n,y}^c ) \leq \Pr( A_{n,y}^c ) + \Pr( D_{n,t} \cap A_{n,y} )
$$

$$
\leq \left( \frac{cn}{K \VC(F)} \right)^{\sqrt{c}} e^{-2ny^2/K^2} + \Pr( D_{n,t} \cap A_{n,y} ) .
$$

(3.5)

We now focus on the right-hand term above. By definition, we have that

$$
\frac{1}{\beta} \log \Pr( D_{n,t} \cap A_{n,y} ) = \frac{1}{\beta} \log \int_{A_{n,y}} \frac{\int_{B_t(\phi, \mu)} \exp(\beta \phi(x, \mu)) d\pi}{\int_{\mathbb{R}^{d+\ell}} \exp(\beta \phi(x, \mu)) d\pi} d\mu.
$$

(3.6)

On $A_{n,y}$ it holds that $\exp(\beta \phi(x, \mu)) \exp(-\beta y) \leq \exp(\beta \phi(x, \hat{\mu}_n)) \leq \exp(\beta \phi(x, \mu)) \exp(\beta y)$. Applying this to the right-hand side of (3.6) results in

$$
\frac{1}{\beta} \log \Pr( D_{n,t} \cap A_{n,y} ) \leq \frac{1}{\beta} \log \int_{B_t(\phi, \mu)} \exp(\beta \phi(x, \mu)) d\pi + y .
$$

(3.7)

We now bound $E_{\pi} \exp(\beta \phi(x, \mu))$ below. For any $r > 0$, Condition 4 implies that

$$
\frac{1}{\beta} \log \int_{\mathbb{R}^{d+\ell}} \exp(\beta \phi(x, \mu)) d\pi \geq \frac{1}{\beta} \log \int_{B_r(\phi, \mu)} \exp(\beta \phi(x, \mu)) d\pi
$$

$$
= \phi(\theta_0, \mu) + \frac{1}{\beta} \log \int_{B_r(\phi, \mu)} \exp(\beta(\phi(x, \mu) - \phi(\theta_0, \mu))) d\pi
$$

$$
\geq \phi(\theta_0, \mu) - \omega(r) + \beta^{-1} \log \pi(B_r(\phi, \mu)) .
$$

Maximizing the right-hand side of the above, and recalling the definition of the calibration function, $\psi$, from (3.1), yields

$$
\frac{1}{\beta} \log \int_{\mathbb{R}^{d+\ell}} \exp(\beta \phi(x, \mu)) d\pi \geq \phi(\theta_0, \mu) - \psi(\beta)/\beta .
$$

Plugging this lower bound into (3.7) yields

$$
\frac{1}{\beta} \log \Pr( D_{n,t} \cap A_{n,y} ) \leq \frac{1}{\beta} \log \int_{B_t(\phi, \mu)} \exp(\beta \phi(x, \mu)) d\pi - \phi(\theta_0, \mu) + \psi(\beta)/\beta + 2y
$$

$$
\leq \sup_{x \in B_t(\phi, \mu)} \phi(x) - \phi(\theta_0) + \frac{1}{\beta} \log \pi(B_t(\phi, \mu)) + \psi(\beta)/\beta + 2y
$$

$$
= -\alpha(t) - I(t)/\beta + \psi(\beta)/\beta + 2y ,
$$

where the last line follows from the definitions of $\alpha(t)$ and $I(t)$. Rewriting the last inequality gives that

$$
\Pr( D_{n,t} \cap A_{n,y} ) \leq \exp(-\beta \alpha(t) - I(t) + \psi(\beta) + 2\beta y) .
$$

(3.8)
Let \( c_4 = c_1^4 / 16 \). Now, setting \( y = f(t, \beta) = \alpha(t) / 2 \lor I(t) / 2 \beta \) and plugging (3.8) into (3.5) results in

\[
\Pr \left( \max_{\theta_0 \in E_{\phi, \mu}} \left\| \hat{\theta}_n - \theta_0 \right\| > t \right) \leq \left( \frac{cn}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-nf(t, \beta)^2 / (8K^2)} + \Pr \left( D_{n,t} \cap A_{n, f(t, \beta)} \right) \\
\leq \left( \frac{cn}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-nf(t, \beta)^2 / 8K^2} + e^{-n\alpha(t) - I(t) + \psi(\beta) + 2f(t, \beta)}. \tag{3.9}
\]

Now, note that \( \alpha(t) + I(t) / \beta - f(t, \beta) \geq \alpha(t) / 2 + I(t) / 2 \beta \). Plugging this inequality into (3.9) yields the desired result.

It is helpful to view the two terms in the above bounds separately. The first term measures the sample complexity of estimation and, in the absence of sampling considerations, recovers the concentration properties of classical (non-private) estimators. (In particular, the non-private bound can be obtained from the above by formally setting \( \beta = \infty \).) The second term is novel and is a measure of the cost of sampling and encodes the trade-off between the prior, \( \pi \), the parameter \( \beta \), and the objective function \( \alpha \).

Through taking \( \beta = n\epsilon / 2K \), Theorem 11 immediately provides a consistency bound (and thereby a sample complexity bound) for the exponential mechanism:

\[
\Pr \left( \max_{\theta_0 \in E_{\phi, \mu}} \left\| \hat{\theta}_n - \theta_0 \right\| \geq t \right) \leq \left( \frac{c_n}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-c_n [I(t) / (2\alpha(t) / 2K)]^2 + e^{-n\alpha(t) / 4K - I(t) / 2 + \psi(n\epsilon / 2K)}}. \tag{3.10}
\]

Let us pause here to interpret this result. The cost of consistent estimation is encoded in the first terms of (3.10) and, in particular, can be related to the non-private setting as well. The cost of privacy is encoded in the second term in (3.10). Finally, we note that the second term in (3.10) encodes a trade-off between an effect of the prior and the privacy parameter.

Equation (3.10) readily yields the sample complexity of an estimator drawn from the exponential mechanism. This yields Theorem 7, and, for instance, Lemma 14 below. We end this section with the following important remark.

## 4 Proof of Theorem 1

We now prove Theorem 1 with a series of lemmas. Let us first note the following fact, whose proof is deferred to Appendix A.

**Lemma 13.** Let \( a, b, c \in \mathbb{R}^+ \). The function \( h(x) = ax - b \log (cx / b) \) is increasing and positive for

\[
x \geq \frac{2b}{a} \left[ \log (c/a) \lor 1 \right].
\]

To see this briefly, assume for simplicity that \( \pi \) is the uniform measure on a hypercube whose side lengths are at most polynomial in \( d \). In addition, assume that \( \theta_0 \) is well captured by this hypercube, i.e., \( \theta_0 \) is at least \( e^{-d} \) far from any boundary of the cube. Suppose we want to estimate a maximizer of \( \phi(\cdot, \mu) \) within some error level \( t \) with probability at least \( 1 - e^{-d} \). Lemma 26 and Corollary 7 give that it is sufficient for \( \text{VC}(\mathcal{F}) \) to be polynomial in \( d \) to ensure that the sample complexity is polynomial in \( d \).
Lemma 14. Suppose that Conditions 1 and 3 hold with $\phi = D$ for some $K > 0$. In addition, suppose that the map $x \mapsto D(x, \mu)$ is $L$-Lipschitz for some $L > 0$. Then, there exists a universal constant $c > 0$, such that for all $n, d \geq 1$ and all $0 < \gamma < 1$, with probability at least $1 - \gamma$, it holds that

$$d(E_{D, \mu}, \tilde{\theta}_n) \lesssim \alpha^{-1} \left( cK \sqrt{\frac{\log(1/\gamma) \vee VC(\mathcal{F}) \log n}{n}} \right).$$

Proof. Note that all of the conditions of Theorem 11 hold, and so we can apply Theorem 11 with $\beta = n \epsilon / 2K$. Proving the deviation bound then amounts to finding a lower bound on $t$ which ensures that:

$$c_1 n \frac{VC(\mathcal{F})}{c_2 n^2 [2KI(t)/n + \alpha(t)]/K^2} e^{-\alpha^2(t)(K^2) / n} \leq e^{-c_2 n \alpha(t)^2 / K^2}.$$

If we can find a lower bound on $t$ which implies that

$$\left( \frac{c_1 n}{VC(\mathcal{F})} \right)^{VC(\mathcal{F})} e^{-\alpha^2(t)(K^2) / n} \leq e^{-c_2 n \alpha(t)^2 / K^2},$$

then such a lower bound, together with the bound $t \geq \alpha^{-1} \left( \sqrt{K^2 / \log(2/\gamma)} / n \right)$ yields that $I \leq \gamma / 2$. It remains to find $t$ such that (4.3) holds.

To this end, define the function $h(n) = c_2 n \alpha(t)^2 / 2K^2 - VC(\mathcal{F}) \log (c_1 n / VC(\mathcal{F})).$ The condition (4.3) is then equivalent to the condition $h(n) \geq 0$. Note that $h(n)$ is in the form of Lemma 13 with $a = c_2 n \alpha(t)^2 / 2K^2$, $b = VC(\mathcal{F})$ and $c = c_1$. Applying Lemma 13 yields that $h(n) \geq 0$ provided that

$$n \geq K^2 \frac{VC(\mathcal{F}) \log \left( \frac{K^2}{\alpha(t)^2} \right) \vee 1}{\alpha(t)^2},$$

for some universal $c > 0$. Now, provided that $a^2(t) \geq K^2 / n$, (4.4) is satisfied when

$$\alpha(t)^2 \geq K^2 VC(\mathcal{F}) \log n / n.$$
We now turn to finding $t$ such that $\mathcal{II} \leq \gamma / 2$. The $\mathcal{II} \leq \gamma / 2$ is satisfied when
\[
\alpha(t) > 2K\frac{\log (2/\gamma) - I(t)/2 + \psi(n\epsilon/2K)}{n\epsilon} > 2K\frac{\log (2/\gamma) \lor \psi(n\epsilon/2K)}{n\epsilon}.
\] (4.6)

It remains to simplify $\psi(n\epsilon/2K)$. To this end, given that by assumption D is $L$-Lipschitz, we have that $\omega(r) \leq Lr$. Plugging this in to the definition of $\psi$ yields:
\[
\psi(n\epsilon/2K) \leq \min_{r > 0} \left[ \frac{n\epsilon}{2K} \cdot L \cdot r - \log \pi(B_r(E_{\phi,\mu})) \right].
\] (4.7)

Next, taking the specific choice $r = 2K/Ln\epsilon$ yields $\psi(n\epsilon/2K) \leq 1 - \log \pi(B_{2K/Ln\epsilon}(E_{\phi,\mu}))$. Plugging this inequality into (4.6), and combining the result with (4.5) yields the desired result.

We now note the following lemmas whose proofs are deferred to Appendix A.

**Lemma 15.** If $\pi = N(\theta_\pi, \sigma^2_\pi I)$ with $\sigma^2_\pi \geq 1/4$, then for all $E \subset \mathbb{R}^d$, all $d > 2$ and all $R \leq \sigma_\pi$ it holds that
\[
-\log \pi(B_R(E)) \lesssim d(E, \theta_\pi)^2 + d \log \left( \frac{\sigma_\pi}{R} \lor d \right).
\] (4.8)

**Lemma 16.** Suppose that $\pi \propto 1\{x \in A_R(\theta_\pi)\}$ where $E \subset A_R(\theta_\pi)$, then for all $E \subset \mathbb{R}^d$, $d > 0$ and all $r > 0$ it holds that
\[
-\log \pi(B_r(E)) \lesssim d \log \left( \frac{R}{d_{R,\theta_\pi}(E) \lor r} \right).
\] (4.9)

We can now prove Theorem 1.

**Proof of Theorem 1.** First note that by assumption, all the conditions of Lemma 14 are satisfied. To prove (2.4) and (2.5), we apply Lemma 15 and Lemma 16, respectively, in conjunction with Lemma 14. To apply Lemma 15, first, observe that together, the assumptions that $L > 1$, $\sigma^2_\pi \geq 1/16$ and $n \geq 8K/\epsilon$ imply that $2K/Ln\epsilon \leq \sigma_\pi$. Applying Lemma 15 with $R = 2K/Ln\epsilon$ and $E = E_{D,\mu}$ in conjunction with (4.1), yields (2.4). Applying Lemma 16 with $r = 2K/Ln\epsilon$ and $E = E_{D,\mu}$ in conjunction with (4.1), yields (2.5) as desired.

4.1 Optimality of Theorem 1

**Lemma 17.** The upper bound on the deviation of $\hat{\theta}_n$ about $E_{D,\mu}$ given in Theorem 1 is sharp, up to logarithmic factors. The upper bound on the sample complexity of $\hat{\theta}_n$ given in Corollary 7 is also sharp, up to logarithmic factors.

**Proof.** We show that for a specific set of parameters, the upper bound given in Corollary 7 matches the lower bound of Theorem 6.5 of [Kamath et al., 2018] up to logarithmic factors in $d$ and $t^{-1}$. Specifically, set $D = HD$, $\theta_0 \in A_R(0)$, $\pi = A_R(0)$, that $\mu = N(\theta_0, \sigma^2 I)$. For $t \geq R$, the sample complexity is 1. We then focus on the case where $t \leq R$.

We now check the conditions of Corollary 7 under these parameters. First, sup, $D(x, \mu) = D(\theta_0, \mu)$ and so Condition 1 is satisfied. Next, HD satisfies Condition 3 with $K = 1$ and $\text{VC}(\mathcal{F}) = d + 2$, as shown in Lemma 35. Furthermore, we have that HD($\cdot, \mu$) is $\sigma^{-1}$-Lipschitz. Therefore, the
conditions of Corollary 7 are satisfied. Plugging these values into Corollary 7 and setting \( \gamma = 1/3 \), as in Theorem 6.5 of (Kamath et al., 2018), yields

\[
n \gtrsim \frac{d \log \left( \frac{K}{\alpha(t)} \vee e \right)}{\alpha(t)^2} + \frac{d \log \left( \frac{R/\sigma}{a_{R,\sigma}(\theta_0) \vee a(t)} \vee d \right)}{\alpha(t) \epsilon}.
\]

(4.8)

Note that in this case, we have that \( \alpha(t) = \Phi(t/\sigma) - 1/2 \). Let \( a(R) = (\Phi(R/\sigma) - 1/2)/R \). Now, since \( t \leq R \), concavity of \( \Phi \) on \( \mathbb{R}^+ \) implies that \( \alpha(t) \geq a(R)t \). Applying this fact in conjunction with (4.8) results in

\[
n \gtrsim \frac{d \log \left( \frac{1}{a(R)^2} \vee e \right)}{a(R)^2 \cdot t^2} + \frac{d \log \left( \frac{R/\sigma}{a_{R,\sigma}(\theta_0) \vee a(R)} \vee d \right)}{a(R) \cdot t \cdot \epsilon},
\]

(4.9)

which matches the bound given in Theorem 6.5 of (Kamath et al., 2018) up to logarithmic factors in \( d \) and \( t^{-1} \). A simple manipulation of the terms in (4.9) gives an equivalent deviations bound, which matches the minimax lower bounds given by Cai et al. (2021), up to logarithmic factors, if \( \delta \) is taken to be \( n^{-k} \) for some \( k > 0 \).

5 Depth functions and their properties

For the convenience of the reader, we collect here some basic facts about depth functions. We begin by recalling the definition of depth functions. (We work with a slightly weaker notion of depth than is common in the literature to unify notation. See Remark 18) We then review several common depth functions in the literature and their properties. We then introduce a new depth function, the smoothed integrated dual depth, which has desirable properties from a computational perspective and is used in our simulations. For proofs of the results from this section, see Section D.

5.1 Definition of a depth function

Recall that a measure \( \mu \in M_1(\mathbb{R}^d) \) is centrally symmetric about a point \( x \in \mathbb{R}^d \) if \( X - x \sim \mu \) for \( X \sim \mu \). In the following, for \( A \in \mathbb{R}^{d \times d} \) and \( b \in \mathbb{R}^d \), let \( A\mu + b \) be the law of \( AX + b \) if \( X \sim \mu \) and let \( \mathcal{C}_x \subset M_1(\mathbb{R}^d) \) denote the set of centrally symmetric measures about \( x \).

**Definition 3.** A function \( D: \mathbb{R}^d \times M_1(\mathbb{R}^d) \to \mathbb{R}^+ \) is a depth function with admissible set \( \mathcal{A} \subset M_1(\mathbb{R}^d) \) if, for all \( \mu \in \mathcal{A} \), the following four properties hold:

1. Similarity invariance: For all orthogonal matrices \( A \in \mathbb{R}^{d \times d} \) and \( b \in \mathbb{R}^d \) it holds that \( D(x, \mu) = D(Ax + b, A\mu + b) \).
2. Maximality at center: If \( \mu \in \mathcal{C}_{\theta_0} \) then \( D(\theta_0, \mu) = \sup_x D(x, \mu) \).
3. Decreasing along rays: Suppose \( D \) is maximized at \( \theta_0 \). For all \( p \in (0, 1) \), it holds that \( D(x, \mu) \leq D(p\theta_0 + (1 - p)x, \mu) \leq D(\theta_0, \mu) \).
4. Vanishing at infinity: \( \lim_{c \to \infty} D(cu, \mu) = 0 \) for any unit vector \( u \).

**Remark 18.** We use here a weaker definition of a depth function than the influential work of Zuo and Serfling (2000), which replaces similarity invariance with affine invariance. For a discussion of desirable properties for a depth function to possess, see Zuo and Serfling (2000), Liu et al. (2006), and Serfling (2019).
We now define several depth functions discussed in this paper, after which we give a short summary of their properties.

For $X \sim \mu$, define

$$F(x, u, \mu) = \mu \left( \mathbb{I} \left\{ X^\top u \leq x^\top u \right\} \right)$$

and

$$F(x-, u, \mu) = \mu \left( \mathbb{I} \left\{ X^\top u < x^\top u \right\} \right).$$

Halfspace depth, otherwise known as Tukey depth (Tukey, 1974), is defined as follows:

**Definition 4 (Halfspace depth).** The halfspace depth HD of a point $x \in \mathbb{R}^d$ with respect to $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is

$$\text{HD}(x, \mu) = \inf_{\|u\|=1} F(x, u, \mu).$$

The halfspace depth of a point $x \in \mathbb{R}^d$ is the minimum probability mass contained in a closed halfspace containing $x$. Another classical depth function is the simplicial depth (Liu, 1988, 1990).

**Definition 5 (Simplicial Depth).** Suppose that $Y_1, \ldots, Y_{d+1}$ are i.i.d. from $\mu \in \mathcal{M}_1(\mathbb{R}^d)$. The simplicial depth of a point $x \in \mathbb{R}^d$ with respect to $\mu$ is

$$\text{SMD}(x, \mu) = \Pr(\Delta(Y_1, \ldots, Y_{d+1}) \ni x).$$

The simplicial depth of a point $x \in \mathbb{R}^d$ is the probability that a random simplex defined by $d+1$ draws from $\mu$ contains $x$. The next depth function we introduce is spatial depth (Vardi and Zhang, 2000; Serfling, 2002), which is maximized at the spatial median (Vardi and Zhang, 2000). Define the spatial sign function some as $\text{SR}(y) = y/\|y\|$ (with $\text{SR}(0) = 0$). We can then define the spatial rank function as $\mathbb{E}_\mu \text{SR}(x-X)$. The norm of the spatial rank of $x \in \mathbb{R}^d$ is a measure of the outlyingness of $x$ with respect to $\mu$. With this in mind, the spatial depth is defined as follows:

**Definition 6.** The spatial depth SD of a point $x \in \mathbb{R}^d$ with respect to $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is

$$\text{SD}(x, \mu) = 1 - \|\mathbb{E}_\mu \text{SR}(x-X)\|^2.$$ 

We will see below that the spatial depth is $(K, \mathcal{F})$-regular, however, $K = O(d)$. Replacing the norm with its square eliminates this inconvenience, and still yields a depth function which is maximized at the spatial median. Therefore, we define the modified spatial depth as follows:

**Definition 7.** The modified spatial depth MSD of a point $x \in \mathbb{R}^d$ with respect to $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is

$$\text{MSD}(x, \mu) = 1 - \|\mathbb{E}_\mu \text{SR}(x-X)\|^2.$$ 

Lastly, we define the integrated depth functions. Here, the idea is to define the depth of a point $x \in \mathbb{R}^d$ as the average, univariate depth of $x^\top u$ over all directions $u$. The first integrated depth function is the integrated dual depth of Cuevas and Fraiman (2009), which, letting $\nu$ be the uniform measure on $S^{d-1}$, is defined as follows:

**Definition 8 (Integrated Dual Depth).** The integrated dual depth of a point $x \in \mathbb{R}^d$ with respect to $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is

$$\text{IDD}(x, \mu) = \int_{S^{d-1}} F(x, u, \mu) (1 - F(x, u, \mu)) \, dv(u).$$

The first integrated multivariate depth function, to be precise. The first integrated depth was developed for functional data by Fraiman and Muniz (2001).
The integrated dual depth of \( x \in \mathbb{R}^d \) is the average univariate simplicial depth of \( x^\top u \) over all directions \( u \in S^{d-1} \). The integrated rank-weighted depth (Ramsay et al., 2019) replaces the simplicial depth in (6.3) with a slightly modified version of halfspace depth:

**Definition 9 (Integrated Rank-Weighted Depth).** The integrated rank-weighted depth of a point \( x \in \mathbb{R}^d \) with respect to \( \mu \in \mathcal{M}_1(\mathbb{R}^d) \) is \( \text{IRW}(x, \mu) = 2E_{\nu} F(x, U, \mu) \wedge (1 - F(x, -U, \mu)) \).

We now turn to a brief discussion of these functions satisfy Definition 3. To this end, recall the following classes of measures. Let \( C \subset \mathcal{M}_1(\mathbb{R}^d) \) denote the set of centrally symmetric measures over \( \mathbb{R}^d \). A measure \( \mu \) is angularly symmetric about a point \( \theta_0 \in \mathbb{R}^d \) if \( \frac{X - \theta_0}{\|X - \theta_0\|} = \frac{\theta_0 - X}{\|\theta_0 - X\|} \) for \( X \sim \mu \) (Liu, 1990). Let \( \mathcal{N} \subset \mathcal{M}_1(\mathbb{R}^d) \) denote the set of angularly symmetric measures over \( \mathbb{R}^d \) that are absolutely continuous with respect to the Lebesgue measure. The following proposition collects existing results in the literature:

**Proposition 19.** The aforementioned depth functions satisfy the following:

- Halfspace depth is a depth function with admissible set \( \mathcal{M}_1(\mathbb{R}^d) \).
- Simplicial depth is a depth function with admissible set \( \mathcal{N} \).
- Spatial depth and the modified spatial depth satisfy properties (1), (2) and (4) for all \( \mu \in \mathcal{M}_1(\mathbb{R}^d) \).
- Integrated dual depth and integrated rank-weighted depth are depth functions with admissible set \( C \). Furthermore, they satisfy properties (1), (2) and (4) for any \( \mu \in \mathcal{M}_1(\mathbb{R}^d) \).

**Remark 20.** Spatial depth does not satisfy Definition 3 as property (3) fails. However, spatial depth satisfies spatial angle monotonicity and spatial rank monotonicity (Serfling, 2019, see Section 4.3), which together provide a weaker alternative to property (3). Simplicial depth and halfspace depth are affine invariant, which is a stronger version of property (1).

**Remark 21.** Both the halfspace depth and the simplicial depth function have an exact computational running time of \( O(n^{d-1}) \), see, e.g., (Dyckerhoff and Mozharovskyi, 2016; Afshani et al., 2015) and the references therein. By contrast, the spatial depth and the integrated depth functions are computable in high dimensions (Chaudhuri, 1996; Ramsay et al., 2019). For instance, the integrated depth functions can be approximated easily via sampling, say, \( m \) points from \( \nu \). One can then approximate the sample depth value of a given point in \( O(mn d) \) time (Ramsay et al., 2019).

The following theorem demonstrates that the depth functions defined in this section satisfy the conditions of Theorem 11. Let \( \mu_u \) be the law of \( X^\top u \) if \( X \sim \mu \).

**Theorem 22.** The aforementioned depth functions satisfy the following:

- Conditions 4 and 3 hold for each of the depth functions defined in Section 5.
- Suppose that \( \pi \) is absolutely continuous and that for all \( u \in S^{d-1} \), the measure \( \mu_u \) is absolutely continuous with density \( f_u \) and sup \( \|u\|_1 \|f_u\|_{\infty} = L < \infty \). Then there exists a universal constant \( C > 0 \) such that for all \( s \in (0, \infty] \), halfspace, simplicial, integrated rank-weighted and the smoothed integrated dual depth\(^8\) satisfy Condition 4 with \( \omega(r) = C \cdot L \cdot r \).

\(^8\)The smoothed integrated dual depth is introduced in Section 6.1 below and equals the integrated dual depth when \( s = \infty \).
Suppose that \( \mu \) has a bounded density and that \( \sup_y \mathbb{E} \| y - X \|^{-1} = L' \leq \infty \). Then modified spatial depth satisfies Condition 4 with \( \omega(r) = 2\sqrt{d} \cdot L' \cdot r \) and spatial depth satisfies Condition 4 with \( \omega(r) = 2 \cdot L' \cdot r \).

The above results are also summarized in Table 2.

6.1 The smoothed integrated dual depth

One issue with many of the aforementioned depth functions is that their empirical versions contain indicator functions, which are non-smooth. This fact is inconvenient from an optimization perspective, e.g., we cannot apply gradient descent to estimate the median. In addition, this non-smoothness implies that the empirical depth functions will possess flat contours. This can cause the estimated medians to perform poorly when the population measure has atoms near its center (Lalanne et al., 2023). Therefore, to resolve this issue, recall that

\[
\mathbf{1} \{ X^\top u \leq x^\top u \} \approx \sigma(s(x - X)^\top u),
\]

for large \( s \), where \( \sigma(x) = (1 + e^{-x})^{-1} \). This motivates the following definition of depth, which we use in our simulation study from Section 2.2.

**Definition 10 (Smoothed Integrated Dual Depth).** The smoothed integrated dual depth with smoothing parameter \( s \) of a point \( x \in \mathbb{R}^d \) with respect to \( \mu \in \mathcal{M}_1(\mathbb{R}^d) \) is

\[
\text{IDD}(x, \mu, s) = \int_{S^{d-1}} \mathbb{E} \sigma(s(x - X)^\top u) (1 - \mathbb{E} \sigma(s(x - X)^\top u)) \, d\nu(u). \tag{6.4}
\]

As \( s \to \infty \), the smoothed integrated dual depth converges to the integrated dual depth. However, the smoothed integrated dual depth is convenient in that its empirical version is differentiable. Furthermore, the smoothed integrated dual depth is a depth with admissible set \( \mathcal{C} \).

**Proposition 23.** For any \( s > 0 \), the smoothed integrated dual depth is a depth function with admissible set \( \mathcal{C} \). In addition, properties (1) and (4) hold for any \( \mu \in \mathcal{M}_1(\mathbb{R}^d) \).

In the simulations presented in Section 2.2, we approximate (6.4) with

\[
\widehat{\text{IDD}}(x, \mu, s) = \frac{1}{M} \sum_{m=1}^{M} \mathbb{E} \sigma(s(x - X)^\top u_m) (1 - \mathbb{E} \sigma(s(x - X)^\top u_m)) ,
\]

where \( u_1, \ldots, u_M \) are drawn i.i.d. from \( \nu \). One can show that the number of unit vectors needed to maintain a fixed error level with high probability is polynomial in \( n \) and \( d \).

**Proposition 24.** Let \( u_1, \ldots, u_M \) be i.i.d. from \( \nu \). Then for all \( d, s, t > 0 \) and all \( 0 < \gamma < 1 \), there exists a universal constant \( c_1 > 0 \) such that

\[
\sup_x \left| \text{IDD}(x, \hat{\mu}_n, s) - \text{IDD}(x, \mu, s) \right| \leq t,
\]

with probability \( 1 - \gamma \), provided that \( M \geq c_1 \log(1/\gamma) \lor dn \log(1/t \lor e) \cdot t^{-2} \).

Lastly, note that the median estimators are relatively insensitive to the choice of \( s \). For \( s \geq 10 \), the specific value of \( s \) appears to have minimal effect on the estimation error and convergence of the gradient descent algorithm, see Appendix B for more details. We recommend choosing \( s = 100 \).
7 Computing private depth values

The depth values themselves can be of interest. For example, they are used in visualization and a host of inference procedures (Li and Liu 2004; Chenouri et al. 2011). The theoretical analysis from the previous sections yields simple differentially private methods for estimating depth values. If we want to compute the depth of a known point, then we can use an additive noise mechanism, such as the Laplace mechanism or the Gaussian mechanism (Dwork et al., 2006), i.e., we can just add noise that is calibrated to ensure differential privacy. For example, for a non-private statistic $T(\hat{\mu}_n)$, the Laplace mechanism is defined as $e^{T(\hat{\mu}_n)} = T(\hat{\mu}_n) + W_1 \text{GS}_n(T)/\epsilon$, where $W_1 \sim \text{Laplace}(1)$. It is well known that $e^{T(\hat{\mu}_n)}$ is $\epsilon$-differentially private (Dwork et al., 2006). Theorem 11 implies the concentration of private depth values generated from the Laplace mechanism.

**Corollary 25.** Suppose that $D$ satisfies Condition 3. For $x \in \mathbb{R}^d$ chosen independently of the data, $\tilde{D}(x, \hat{\mu}_n) = D(x, \hat{\mu}_n) + W_1 K_n^{1/2}$, is $\epsilon$-differentially private. In addition, there exists universal constants $c_1, c_2 > 0$ such that for all $t \geq 0$ it holds that

$$\Pr(|\tilde{D}(x, \hat{\mu}_n) - D(x, \mu)| > t) \leq (c_1 n/\text{VC}(\mathcal{F}))^{\text{VC}(\mathcal{F})} e^{-c_2 nt^2/K^2} + e^{-\epsilon n t^2/2K}.$$  

Corollary 25 suggests taking $\epsilon \propto t/K$ when simulating private depth values. An obvious problem of interest is estimating the vector of depth values at the sample points, i.e., can we estimate $\hat{D}(\hat{\mu}_n) = (D(X_1, \hat{\mu}_n), \ldots, D(X_n, \hat{\mu}_n))$ privately? In general, $\text{GS}_n(\hat{D}(\hat{\mu}_n)) = O(1)$, see Lemma 30. As a result, if $\hat{D}$ is the vector of depth values generated from the Laplace mechanism then $||\hat{D}(\hat{\mu}_n) - D(\mu)|| = O_p(n^{1/2})$; the level of noise is greater than that of the sampling error. This result is intuitive; these vectors reveal more information about the population as $n$ grows, which differs markedly from the single depth value case, where the amount of information received is fixed in $n$. In fact, for large $n$ the vector of depth values at the sample points contains a significant amount of information about $\mu$; the depth function can, under certain conditions, characterize the distribution of the input measure $\mu$ (see Struyf and Rousseeuw 1999; Nagy 2021, and the references therein). In order to release so much information about the population privately, we must inject non-negligible noise.

We cannot, then, simply plug in the $n$ private sample depth values into an inference procedure and proceed. We can instead sample $N$ points from the exponential mechanism for the median given in Section 2 using a very small $\epsilon$. This will give points similar in nature to the original sample. We can then either apply Corollary 25 to these points or compute the sample depth function from those private points. The performance of these procedures in specific inference settings is an interesting topic of future research.

References

Abowd, J., Ashmead, R., Cumings-Menon, R., Garfinkel, S., Heineck, M., Heiss, C., Johns, R., Kifer, D., Leclerc, P., Machanavajjhala, A., Moran, B., Sexton, W., Spence, M., and Zhuravlev, P. (2022). The 2020 Census Disclosure Avoidance System TopDown Algorithm. Harvard Data Science Review, (Special Issue 2). https://hdsr.mitpress.mit.edu/pub/7e4v361i.

Afshani, P., Sheehy, D. R., and Stein, Y. (2015). Approximating the simplicial depth. arXiv e-prints. arXiv:1512.04856.

Avella-Medina, M. (2020). The role of robust statistics in private data analysis. CHANCE, 33(4):37–42.
Differentially private sub-Gaussian location estimators. arXiv e-prints. arXiv:1906.11923.

Differentially private sub-gaussian location estimators.

Structure and sensitivity in differential privacy: Comparing $k$-norm mechanisms. Journal of the American Statistical Association, 116(534):935–954.

Private center points and learning of halfspaces. In Proceedings of the Thirty-Second Conference on Learning Theory, volume 99, pages 269–282. PMLR.

Archimedes meets privacy: On privately estimating quantiles in high dimensions under minimal assumptions.

CoinPress: Practical private mean and covariance estimation. Advances in Neural Information Processing Systems, 33:14475–14485.

Covariance-aware private mean estimation without private covariance estimation. Advances in Neural Information Processing Systems, 34:7950–7964.

Propose, test, release: Differentially private estimation with high probability. arXiv e-prints. arXiv:2002.08774.

Private hypothesis selection. Advances in Neural Information Processing Systems, 32.

Average-case averages: Private algorithms for smooth sensitivity and mean estimation. Advances in Neural Information Processing Systems, 32.

The cost of privacy: Optimal rates of convergence for parameter estimation with differential privacy. The Annals of Statistics, 49(5):2825 – 2850.

On a geometric notion of quantiles for multivariate data. Journal of the American Statistical Association, 91:862–872.

The influence function and maximum bias of Tukey’s median. Annals of Statistics, 30(6):1737–1759.

Data depth-based nonparametric scale tests. Canadian Journal of Statistics, 39(2):356–369.

On depth measures and dual statistics. a methodology for dealing with general data. Journal of Multivariate Analysis, 100(4):753–766.

Learning with privacy at scale. Apple.

Differential privacy and robust statistics. Proceedings of the 41st annual ACM symposium on theory of computing - STOC ’09, page 371.
Dwork, C., McSherry, F., Nissim, K., and Smith, A. (2006). Calibrating noise to sensitivity in private data analysis. In *Theory of Cryptography Conference*, pages 265–284.

Dwork, C. and Roth, A. (2014). The algorithmic foundations of differential privacy. *Foundations and Trends in Theoretical Computer Science*, 9(3–4):211–407.

Dwork, C., Smith, A., Steinke, T., and Ullman, J. (2017). Exposed! A survey of attacks on private data. *Annual Review of Statistics and Its Application*, 4(1):61–84.

Dyckhoff, R. and Mozharovskyi, P. (2016). Exact computation of the halfspace depth. *Computational Statistics & Data Analysis*, 98:19–30.

Dümbgen, L. (1992). Limit theorems for the simplicial depth. *Statistics & Probability Letters*, 14(2):119–128.

Foygel Barber, R. and Duchi, J. C. (2014). Privacy and statistical risk: Formalisms and minimax bounds. *arXiv e-prints*. arXiv:1412.4451.

Fraiman, R. and Muniz, G. (2001). Trimmed means for functional data. *Test*, 10(2):419–440.

Gao, Y. and Sheffet, O. (2020). Private approximations of a convex hull in low dimensions. *arXiv e-prints*. arXiv:2007.08110.

Guevara, M. (2021). How we’re helping developers with differential privacy. Technical report, Google.

Hopkins, S. B., Kamath, G., and Majid, M. (2022). Efficient mean estimation with pure differential privacy via a sum-of-squares exponential mechanism. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1406–1417.

Kamath, G., Li, J., Singhal, V., and Ullman, J. (2018). Privately learning high-dimensional distributions. *arXiv e-prints*. arXiv:1805.00216.

Kamath, G., Singhal, V., and Ullman, J. (2020). Private mean estimation of heavy-tailed distributions. *arXiv e-prints*. arXiv:2002.09464.

Karwa, V. and Vadhan, S. (2017). Finite sample differentially private confidence intervals. *arXiv e-prints*. arXiv:1711.03908.

Koltchinskii, V. I. (1997). M-estimation, convexity and quantiles. *The Annals of Statistics*, 25:435–477.

Kosorok, M. R. (2008). *Introduction to empirical processes and semiparametric inference*. Springer.

Lalanne, C. S., Gastaud, C., Grislain, N., Garivier, A., and Gribonval, R. (2023). Private quantiles estimation in the presence of atoms.

Li, J. and Liu, R. Y. (2004). New nonparametric tests of multivariate locations and scales using data depth. *Statistical Science*, 19(4):686–696.

Liu, R. Y. (1988). On a notion of simplicial depth. *Proceedings of the National Academy of Sciences*, 85(6):1732–1734.
Liu, R. Y. (1990). On a notion of data depth based on random simplices. *Annals of Statistics*, 18(1):405–414.

Liu, R. Y., Serfling, R. J., and Souvaine, D. L. (2006). *Data depth: Robust multivariate analysis, computational geometry, and applications*, volume 72. American Mathematical Soc.

Liu, X., Kong, W., Kakade, S., and Oh, S. (2021a). Robust and differentially private mean estimation. *arXiv e-prints*. arXiv:2102.09159.

Liu, X., Kong, W., and Oh, S. (2021b). Differential privacy and robust statistics in high dimensions. *arXiv e-prints*. arXiv:2111.06578.

McSherry, F. and Talwar, K. (2007). Mechanism design via differential privacy. In *48th Annual IEEE Symposium on Foundations of Computer Science*, pages 94–103. IEEE.

Nagy, S. (2021). Halfspace depth does not characterize probability distributions. *Statistical Papers*, 62(3):1135–1139.

Ramsay, K., Durocher, S., and Leblanc, A. (2019). Integrated rank-weighted depth. *Journal of Multivariate Analysis*, 173:51 – 69.

Seeman, J., Reimherr, M., and Slavković, A. (2021). Exact privacy guarantees for markov chain implementations of the exponential mechanism with artificial atoms. *Advances in Neural Information Processing Systems*, 34:13125–13136.

Sen, B. (2018). A gentle introduction to empirical process theory and applications. *Lecture Notes, Columbia University*.

Serfling, R. (2002). A depth function and a scale curve based on spatial quantiles. In *Statistical Data Analysis Based on the L_1-Norm and Related Methods*, pages 25–38. Birkhäuser Basel.

Serfling, R. (2019). Depth functions on general data spaces, II. Formulation and maximality, with consideration of the Tukey, projection, spatial, and “contour” depths. Technical report, Serfling & Thompson Statistical Consulting.

Serfling, R. J. (2006). Depth functions in nonparametric multivariate inference. *Data Depth: Robust Multivariate Analysis, Computational Geometry, and Applications*, pages 1–16.

Small, C. G. (1990). A survey of multidimensional medians. *International Statistical Review / Revue Internationale de Statistique*, 58(3):263.

Struyf, A. and Rousseeuw, P. J. (1999). Halfspace depth and regression depth characterize the empirical distribution. *Journal of Multivariate Analysis*, 69:1355153.

Talagrand, M. (1994). Sharper bounds for gaussian and empirical processes. *The Annals of Probability*, 22(1):28–76.

Tukey, J. W. (1974). Mathematics and the picturing of data. In *Proceedings of the International Congress of Mathematicians*.

Tzamos, C., Vlatakis-Gkaragkounis, E.-V., and Zadik, I. (2020). Optimal private median estimation under minimal distributional assumptions. *Advances in Neural Information Processing Systems*, 33:3301–3311.
Acknowledgments

The authors acknowledge Gautam Kamath for his helpful comments and discussion, which improved the paper.

Funding

The authors acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC). Cette recherche a été financée par le Conseil de recherches en sciences naturelles et en génie du Canada (CRSNG), [DGECR-2023-00311, DGECR-2020-00199].

A Technical proofs

Proof of Corollary 7. First note that by assumption, all of the conditions of Lemma 26 are satisfied. To prove (2.6) and (2.7) we apply Lemma 15 and Lemma 16 respectively, in conjunction with Lemma 26. To apply Lemma 15 first, observe that together, the assumptions that $L > 1$, $\sigma_x^2 \geq 1/16$ and $D(x; \mu) \leq 1$ imply that $\alpha(t)/4L\sigma_\pi \leq 1$. Applying Lemma 15 with $R = \alpha(t)/4L$ and $E = E_{D,\mu}$ in conjunction with (A.11) yields that there exists a universal constant $c > 0$ such that for all $t \geq 0$, all $d \geq 2$ and all $0 < \gamma < 1$, we have that $d(E_{D,\mu}, \hat{\theta}_n) < t$ with probability at least $1 - \gamma$ provided that

$$n \gtrsim K^2 \left[ \frac{\log(1/\gamma) \lor (\text{VC}(\mathcal{F}) \log(K \alpha(t)) \lor 1)}{\alpha(t)^2} \lor 1 \right] \lor \left[ \frac{\log(1/\gamma) \lor \frac{d(E_{D,\mu}, \hat{\theta}_n)^2}{\sigma_x^2} \lor d \log \left( \frac{L \sigma_\pi \lor d}{\alpha(t)} \lor d \right)}{\epsilon \alpha(t)} \lor 1 \right].$$

To show (2.7), we apply Lemma 16 with $r = \alpha(t)/4L$ and $E = E_{D,\mu}$ in conjunction with (A.11), which yields that there exists a universal constant $c_1 > 0$ such that for all $t \geq 0$, all $d \geq 1$ and all $0 < \gamma < 1$, we have that $d(E_{D,\mu}, \hat{\theta}_n) < t$ with probability at least $1 - \gamma$ provided that

$$n \gtrsim K^2 \left[ \frac{\log(1/\gamma) \lor (\text{VC}(\mathcal{F}) \log(K \alpha(t)) \lor 1)}{\alpha(t)^2} \lor 1 \right] \lor \left[ \frac{\log(1/\gamma) \lor d \log \left( \frac{R}{\alpha(t) \epsilon} \lor \alpha(t)/L \right)}{\alpha(t) \epsilon} \lor 1 \right].$$

\[ \square \]
Proof of Lemma 3. Without loss of generality, assume that $\theta_0 = 0$. It suffices to show that $\alpha(t)$ is concave on $[a, R]$. This holds if and only if $g(t) = \sup_{\|x\| \geq t} D(x, \mu)$ is convex on $[0, R]$. By the decreasing along rays property and the fact that $D$ is maximized at 0, it holds that $g(t) = \sup_{\|u\| = 1} D(t \cdot u, \mu)$. Next, let $u_{*,t} = \arg\max_{\|u\| = 1} D(t \cdot u, \mu)$. Observe that by assumption $u_{*,t} = u_*$, i.e., $u_{*,t}$ is constant in $t$. Therefore, $\sup_{\|u\| = 1} D(t \cdot u, \mu) = D(t \cdot u_*, \mu)$. Next, it holds that

$$
\frac{d}{dt} g(t) = \frac{d}{dt} D(t \cdot u_*, \mu) = u_*^\top \left[ \frac{d}{dx} D(x, \mu) \right]_{x=t \cdot u_*},
$$

which is non-decreasing by assumption. This implies that $g(t)$ is convex on $[a, R]$. It follows that $\alpha(t) \geq \alpha(R)/R$. \hfill $\square$

Proof of Lemma 4. Again, without loss of generality, take $\theta_0 = 0$. We have that

$$
\frac{d}{dx} \int_{S_{d-1}} g(x^\top u, \mu) du = \int_{S_{d-1}} \frac{d}{dx} g(x^\top u, \mu) du = \int_{S_{d-1}} g^{(1)}(x^\top u, \mu) ud\nu.
$$

Now,

$$
u_*^\top \left[ \frac{d}{dx} D(x, \mu) \right]_{x=t \cdot u_*} = \int_{S_{d-1}} u_*^\top u g^{(1)}(t \cdot u_*^\top u, \mu) d\nu = 2 \int_{S_{d-1}} |u_*^\top u| g^{(1)}(t \cdot |u_*^\top u|, \mu) d\nu,
$$

which is non-decreasing by the assumptions on $g$. Now, assumptions 1–5 and the preceding observation imply the conditions of Lemma 3 are satisfied, and the result follows. \hfill $\square$

Proof of Lemma 5. Given that HD is translation invariant, there is no loss of generality in assuming that $\theta_0 = 0$. First, by definition and the decreasing along rays property of HD, we have that

$$
\alpha(t) = HD(0, \mu) - \sup_{\|x\| \geq t} HD(x, \mu) = 1/2 - \sup_{\|x\| = t} \inf_{\|u\| = 1} F(x, u, \mu). \quad (A.1)
$$

Next, we bound the term $\sup_{\|x\| = t} \inf_{\|u\| = 1} F(x, u, \mu)$ above. To this end, we have that

$$
\sup_{\|x\| = t} \inf_{\|u\| = 1} F(x, u, \mu) \leq \sup_{\|u\| = 1} \left( \frac{1}{2} - \frac{1}{2} - h(t, u) \right).
$$

This inequality, in combination with (A.1) and the mean value theorem yields that for all $0 \leq t \leq R$, it holds

$$
\alpha(t) \geq \frac{1}{2} - h(t, u) \geq tC_\mu. \hfill \square
$$

Proof of Lemma 6. First, we use the derivative test to determine when the function $h(x)$ is increasing. Doing so yields that $h(x)$ is increasing for $x \geq b/a$. Now, it remains to find $x \geq b/a$ such that $h(x) \geq 0$. In other words, for $y_r = rb/a$, we want to find $r \geq 1$ such that $h(y_r) \geq 0$. By definition,

$$
h(y_r) = b \left( r - \log \left( cr/a \right) \right).
$$

It follows that $h(y_r) \geq 0$ is satisfied for $r$ such that $r - \log r \geq \log(c/a)$. Applying the inequality $x - \log x \geq x/2$ for $x \geq 1$ yields that $r - \log r \geq \log(c/a)$ is satisfied when

$$
r \geq 2 \log (c/a).
$$
Therefore, $h(x)$ is positive and increasing for

$$x \geq 2 \frac{b}{a} \log \left( \frac{c}{a} \sqrt{e} \right).$$

\textit{Proof of Lemma 15.} Let $R_* = \frac{R^2}{\sigma^2}$, $\theta_0 \in E$ and let $W \sim \chi^2_\eta(\|\theta_0 - \theta_*\|^2 / \sigma^2)$. We can lower bound $\pi(\mathcal{B}_R(E))$ as follows:

$$\pi(\mathcal{B}_R(E)) \geq \pi(\|X - \theta_0\| \leq R) = \Pr(W \leq R^\ast).$$ (A.2)

Let $G(x, k, \lambda)$ be the cumulative distribution function for the non-central chi-squared measure with $k$ degrees of freedom and non-centrality parameter $\lambda$. Recall that for all $x, k, \lambda > 0$, it holds that

$$G(x, k, \lambda) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} G(x, k + 2j) \geq e^{-\lambda^2} G(x, k, 0).$$

Therefore,

$$\Pr(W \leq R^\ast) \geq e^{-\|\theta_0 - \theta_*\|^2/2 \sigma^2} G(R^\ast, d, 0).$$ (A.3)

We now lower bound $G(x, d, 0)$ for $x \leq 1$. Let $\gamma$ be the lower incomplete gamma function. It follows from the properties of the gamma function that

$$-\log G(x, d, 0) = -\log \gamma(x/2, d/2) + \log \Gamma(d/2) \lesssim -\log \gamma(x/2, d/2) + d \log d.$$

Note that the assumption $d > 2$ implies that $\sup_y \gamma(y, d/2) \geq 1$. This implies that $y^{d/2-1} e^{-y/2}$ is increasing on $(0, x)$. Thus, for any $0 < r < 1$ it holds that

$$-\log G(x, d, 0) \lesssim -\log \int_0^x (y/2)^{d/2-1} e^{-y/2} dy + d \log d$$

$$\lesssim -\log(x - r)(r/2)^{d/2-1}/2 + r/2 + d \log d.$$

Setting $r = x/2$ and using the fact that $x \leq 1$ results in

$$-\log G(x, d, 0) \lesssim \frac{x}{4} + \frac{d}{2} \log \left( \frac{4}{x} \right) + d \log d \lesssim d \log \left( \frac{\sigma^2}{R} \lor d \right).$$ (A.4)

Note that $R^2 = R^2/\sigma^2 \leq 1$ by assumption. Thus, applying (A.4) results in

$$-\log G(R, d, 0) \lesssim d \log \left( \frac{\sigma^2}{R} \lor d \right).$$ (A.5)

Combining (A.2), (A.3) and (A.5) implies that

$$-\log \pi(\mathcal{B}_R(E)) \lesssim \frac{\|\theta_0 - \theta_*\|^2}{\sigma^2} + d \log \left( \frac{\sigma^2}{R} \lor d \right).$$

Now, note that this bound holds for all $\theta_0 \in E$, therefore, it holds that

$$-\log \pi(\mathcal{B}_R(E)) \lesssim \frac{d(E, \theta_*)^2}{\sigma^2} + d \log \left( \frac{\sigma^2}{R} \lor d \right).$$
Proof of Lemma 16. First, note that for any ball \( B_r(x) \) such that \( B_r(x) \subset A_R(\theta_\pi) \), we have that

\[
\pi(B_r(x)) = \frac{r^d \pi^{d/2}}{R^d \Gamma(d/2 + 1)} \geq \left( \frac{r}{R} \right)^d (d/2 + 1)^{d/2 + 1} \pi^{d/2} e^{-d/2} \geq \left( \frac{r}{R} \right)^d \left( \frac{\pi d}{2e} \right)^{d/2}.
\]

Taking the negated logarithm of both sides implies that

\[
-\log \pi(B_r(x)) \lesssim d \log \left( \frac{R}{r} \right) - d \log \pi d/2 + d \lesssim d \log \left( \frac{R}{r} \right).
\]

(A.6)

If there exists \( \theta_0 \in E \) such that \( B_r(\theta_0) \subset A_R(\theta_\pi) \), then (A.6) implies that

\[
-\log \pi(B_r(\theta_0)) \lesssim d \log \left( \frac{R}{r} \right). \quad (A.7)
\]

If instead there exists \( \theta_0 \in E \) such that \( A_R(\theta_\pi) \subset B_r(\theta_0) \) then

\[
-\log \pi(B_r(\theta_0)) = 0. \quad (A.8)
\]

Suppose finally that for all \( \theta_0 \in E \) it holds that both \( B_r(\theta_0) \not\subset A_R(\theta_\pi) \) and \( A_R(\theta_\pi) \not\subset B_r(\theta_0) \). Let \( \theta_{0,j} \) and \( \theta_{p,j} \) be the \( j^{th} \) coordinate of \( \theta_0 \) and \( \theta_\pi \) respectively, where

\[
r = \min_{1 \leq j \leq d} (|\theta_{0,j} - \theta_{p,j} - R/2| \wedge |\theta_{0,j} - \theta_{p,j} + R/2|) = d_{R,\theta_\pi}(\theta_0).
\]

Now, by assumption, there exists \( \theta_0 \in E \) such that \( \theta_0 \in A_R \). For such \( \theta_0 \in A_R \), the definition of \( d_{R,\theta_\pi} \) immediately implies that \( B_{d_{R,\theta_\pi}(\theta_0)}(\theta_0) \subset A_R(\theta_\pi) \). Therefore, (A.6) implies that

\[
-\log \pi(B_r(E)) \leq -\log \pi(B_r(\theta_0)) \lesssim d \log \left( \frac{R}{d_{R,\theta_\pi}(\theta_0)} \right). \quad (A.9)
\]

It follows from (A.7), (A.8) and (A.9) that for all \( d > 0 \) and all \( 0 < r \) it holds that

\[
-\log \pi(B_r(E)) \lesssim d \log \left( \frac{R}{d_{R,\theta_\pi}(\theta_0) \wedge r} \right). \quad (A.10)
\]

Note that (A.10) holds for all \( \theta_0 \in E \), thus, for all \( d > 0 \) and all \( 0 < r > 0 \) it holds that

\[
-\log \pi(B_r(E)) \lesssim d \log \left( \frac{R}{d_{R,\theta_\pi}(E) \wedge r} \right). \quad \square
\]

Lemma 26. Suppose that Conditions 1 and 2 hold with \( \phi = D \) for some \( K > 0 \). In addition, suppose that the map \( x \mapsto D(x, \mu) \) is \( L \)-Lipschitz for some \( L > 0 \). There is a universal \( c > 0 \) such that for all \( t \geq 0 \), all \( d \geq 1 \) and all \( 0 < \gamma < 1 \), we have that \( \| \hat{\theta}_n - E_{D,\mu} \| < t \) with probability at least \( 1 - \gamma \) provided that

\[
n \gtrsim \left[ K^2 \log(1/\gamma) \vee (\text{VC}(\mathcal{F}) \log(\frac{K}{\alpha(t)}) \vee 1) \right] \sqrt{\frac{K \log(1/\gamma) \vee -\log \pi(B_{\alpha(t) / 4L}(E_{D,\mu}))}{\epsilon \alpha(t)}}. \quad (A.11)
\]
Proof. Note that all of the conditions of Theorem 11 hold, and so we can apply Theorem 11 with \( \beta = ne/2K \). Proving the sample complexity bound then amounts to finding \( n \) which ensures that

\[
\left( \frac{c_1 n}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-c_2 n [2KI(t)/ne \vee \alpha(t)]^2 / K^2} + e^{-ne \alpha(t)/2} \leq \gamma. \tag{A.12}
\]

We will focus on the two terms individually; we will find \( n \) such that both

\[
\left( \frac{c_1 n}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-c_2 n [2KI(t)/ne \vee \alpha(t)]^2 / K^2} \leq \gamma/2, \tag{A.13}
\]

and

\[
e^{-ne \alpha(t)/2} \leq \gamma/2, \tag{A.14}
\]

hold. Together, (A.13) and (A.14) imply (A.12).

We start with \( n \) such that (A.13) holds. First, note that

\[
\left( \frac{c_1 n}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-c_2 n [2KI(t)/ne \vee \alpha(t)]^2 / K^2} \leq \left( \frac{c_1 n}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-c_2 n \alpha(t)^2 / K^2}. \tag{A.15}
\]

The preceding inequality then implies that

\[
\left( \frac{c_1 n}{\text{VC}(\mathcal{F})} \right)^{\text{VC}(\mathcal{F})} e^{-c_2 n \alpha(t)^2 / K^2} \leq e^{-c_2 n \alpha(t)^2 / 2K^2}.
\]

from which (A.13) holds provided that

\[
n \geq \frac{K^2 \log(2/\gamma)}{\alpha(t)^2}. \tag{A.16}
\]

It remains to find \( n \) such that (A.15) holds.

To this end, define the function

\[
h(n) = c_2 n \alpha(t)^2 / 2K^2 - \text{VC}(\mathcal{F}) \log \left( \frac{c_1 n}{\text{VC}(\mathcal{F})} \right).
\]

The condition (A.15) is then equivalent to the condition \( h(n) \geq 0 \). Note that \( h(n) \) is in the form of Lemma 13 with \( a = c_2 \alpha(t)^2 / 2K^2 \), \( b = \text{VC}(\mathcal{F}) \) and \( c = c_1 \). Applying Lemma 13 yields that \( h(n) \geq 0 \) provided that

\[
n \geq \frac{\text{VC}(\mathcal{F}) \log \left( \frac{K^2}{c_2 \alpha(t)^2} \right) \lor 1}{\alpha(t)^2}, \tag{A.17}
\]

for some universal \( c > 0 \).
In light of (A.16), it follows that there exists a universal constant \( c > 0 \) such that (A.13) holds provided that
\[
n \gtrsim K^2 \log(1/\gamma) \vee \text{VC}(\mathcal{F}) \log \left( \frac{K}{\alpha(t)} \right) \vee 1 / \alpha(t)^2 .
\] (A.18)

We now turn to finding \( n \) such that (A.14) holds. The bound (A.14) is satisfied when
\[
n > 2K \frac{\log(2/\gamma) - I(t)/2 + \psi(ne/2K)}{\epsilon \alpha(t)}.
\]

In order to simplify this bound, we require that
\[
\psi(ne/2K) \leq ne\alpha(t)/4K.
\] (A.19)

Given that by assumption \( D \) is \( L \)-Lipschitz, we have that \( \omega(r) \leq Lr \). Plugging this in to the definition of \( \psi \) from (3.1), we have that
\[
\psi(ne/2K) \leq \min_{r>0} \left\{ \frac{ne}{2K} \cdot L \cdot r - \log \pi(B_r(E_{\phi,\mu})) \right\}.
\]

By taking the specific choice \( r = \alpha/4L \), we see that a sufficient condition for (A.19) to hold is
\[
\frac{ne\alpha(t)}{8K} - \log \pi(B_{\alpha(t)/4L}(E_{D,\mu})) \leq \frac{ne\alpha(t)}{4K},
\] (A.20)

which is equivalent to
\[
n > -4K \log \pi(B_{\alpha(t)/4L}(E_{D,\mu})) / \epsilon \alpha(t).\] (A.21)

Combining (A.18) and (A.21) yields that (A.12) is satisfied when
\[
n \gtrsim \left[ K^2 \log(1/\gamma) \vee \text{VC}(\mathcal{F}) \log \left( \frac{K}{\alpha(t)} \right) \vee 1 / \alpha(t)^2 \right] \vee \left[ K \frac{\log(1/\gamma) \vee (-\log \pi(B_{\alpha(t)/4L}(E_{D,\mu})))}{\epsilon \alpha(t)} \right].
\]

\[\Box\]

**B  On the smoothing parameter in the smoothed integrated dual depth**

We executed a small simulation study to assess the effect of the smoothing parameter \( s \) on the non-private estimation error of the smoothed integrated dual depth median, as well as the effect of \( s \) on the convergence of the gradient descent algorithm. The simulation set up was the same as described in Section 2.2. Figure 3 shows the empirical root-mean-square error for different values of \( s \) and the dimension \( d \). We see that the choice of \( s \) does not particularly depend on the dimension of the data. Furthermore, we see that choosing \( s \geq 10 \) produces the best results in terms balancing convergence and estimation error. Given that larger \( s \) theoretically gives a closer approximation to the integrated dual depth, we recommend choosing \( s = 100 \) in practice.
Figure 3: ERMSE of the non-private smoothed integrated dual depth median for different values of $s$ under Gaussian data (left) and contaminated Gaussian data (center) data. Convergence of one simulation run of the gradient descent algorithm in dimension 2 is also presented (right). It can be seen that choosing $s$ large gives the lowest error and best convergence, though any value $s \geq 10$ produces similar results.

C Auxiliary lemmas

Lemma 27. Suppose $U_n$ is an upper bound on $GS_n(\phi)$. Setting $\beta = \epsilon/2U_n$ implies $\tilde{\theta}_n$ is $\epsilon$-differentially private.

Proof. From the definition of differential privacy, it is immediate that for any $0 < \delta_1 < \delta_2$, we have that $\delta_1$-differential privacy implies $\delta_2$-differential privacy. Now, it remains to show that $\tilde{\theta}_n$ is $\delta$-differentially private for some $\delta < \epsilon$. To see this, note that $\epsilon/2U_n \leq \epsilon/2GS_n(\phi)$, which implies that there exists $\delta < \epsilon$ such that $\epsilon/2U_n = \delta/2GS_n(\phi)$. This implies that $\tilde{\theta}_n$ is $\delta$-differentially private and the result follows.

Lemma 28. If $\phi$ satisfies Condition 3 then $GS_n(\phi) \leq K/n$.

Proof. Definition 2 gives that

$$\sup_{X_n \in D_n \times d, \mu_n \in M(X_n)} \sup_{x} |\phi(x, \hat{\mu}_n) - \phi(x, \tilde{\mu}_n)| \leq K \sup_{X_n \in D_n \times d, g \in F \bar{\mu}_n \in M(X_n)} \sup_{g \in F} |gd(\hat{\mu}_n - \tilde{\mu}_n)| \leq K \sup_{g \in F} ||g|| / n = K/n.$$ \hfill \Box

Lemma 29. Let $D$ be the projection depth with location measure MED and scale measure MAD. If $\epsilon = O(1)$ and $\pi$ is not proportional to $1 \{x \in E_\mu\mu\}$, then $GS_n(D) = 1$ and a median drawn from the exponential mechanism based on $D$ with $\beta = \epsilon/2GS_n(D)$ is inconsistent $\mu$-a.s. .

Proof. The univariate sample median has infinite global sensitivity (Brunel and Avella-Medina, 2020), which implies that the projection depth with location measure MED and scale measure MAD has sensitivity 1. Zuo (2003) gives that $D(x, \hat{\mu}_n) \to D(x, \mu) \mu - a.s.$ as $n \to \infty$. If $\pi$ is not proportional to $1 \{x \in E_\mu\mu\}$, then there exists $t > 0$ such that $Q_{\mu,\beta}(d(E_\mu, X) \geq t) \neq 0$. Thus, there exists $t > 0$ such that

$$\Pr \left( d(E_\mu, \tilde{\theta}_n) \geq t \right) \to Q_{\mu,\beta}(d(E_\mu, X) \geq t) \neq 0 \mu - a.s. .$$ \hfill \Box

Lemma 30. Suppose that $D$ is non-negative and satisfies properties (2) and (4) in Definition 3. Then there exists a universal constant $C > 0$ such that $GS_n(D(\mu_n)) \geq C$. Furthermore, if $D$ satisfies Condition 3 then $GS_n(D(\mu_n)) = O(1)$.

33
Thus, \( \Phi(X_1, \hat{\mu}_n) = C > 0 \). Let \( \hat{\mu}_n(Y_1) \) be the empirical measure corresponding to \( Y_1, \ldots, X_n \) with \( Y_1 \neq X_1 \). Given that \( \hat{\mu}_n(Y_1) \in \mathcal{M}(X_n) \), property (4) implies that
\[
\text{GS}_n(\hat{D}(\hat{\mu}_n)) \geq \sup_{Y_1 \in \mathbb{R}^d} |\text{D}(Y_1, \hat{\mu}_n(Y_1)) - \text{D}(X_1, \hat{\mu}_n)| \geq \text{D}(X_1, \hat{\mu}_n) - \inf_{Y_1 \in \mathbb{R}^d} \text{D}(Y_1, \hat{\mu}_n(Y_1)) \geq C.
\]

Now, if \( \text{D} \) satisfies Condition 3, then Lemma 28 implies that
\[
\text{GS}_n(\hat{D}(\hat{\mu}_n)) \leq \frac{K^2}{n} + \sup_{\mu, \nu} |\text{D}(\mu, \hat{\mu}_n) - \text{D}(X_1, \hat{\mu}_n)| \leq \frac{K^2}{n} + \|\text{D}\| = O(1).
\]

**Lemma 31.** If \( \mu = N(\theta_0, \Sigma) \) then for all \( t > 0 \) and positive definite \( \Sigma \), \( \sup_{\|x\|=t} \text{HD}(x, \mu) = \Phi(-t/\sqrt{\lambda_1}) \).

**Proof.** From the fact that \( \text{HD} \) is affine invariant, without loss of generality we can set \( \theta_0 = 0 \). From the fact that \( \Phi \) is increasing, it holds that
\[
\sup_{\|x\|=t} \text{HD}(x, \mu) = \sup_{\|x\|=t} \inf_{\|u\|=1} \Phi(X) = 1 - \inf_{\|v\|=1, \|u\|=1} \Phi(t) = 1 - \Phi(t) = \Phi(-t/\sqrt{\lambda_1}).
\]

It suffices to compute
\[
\inf_{\|v\|=1, \|u\|=1} \frac{(v^T u)^2}{u^T \Sigma u}.
\]

First, note that
\[
\inf_{\|v\|=1, \|u\|=1} \frac{(v^T u)^2}{u^T \Sigma u} \leq \frac{(e_1^T v)^2}{e_1^T \Sigma e_1} \leq \frac{(e_1^T \Sigma^{-1/2} z)^2}{z^T \Sigma^{-1/2} z} \leq \frac{1}{\lambda_1}.
\]

Thus,
\[
\sup_{\|x\|=t} \text{HD}(x, \mu) = 1 - \Phi(t/\sqrt{\lambda_1}) \geq \Phi(-t/\sqrt{\lambda_1}).
\]

**D  Proofs of the results from Section 5**

We now prove the various results presented in Section 5.

**Proof of Proposition 19.** With the exception of the modified spatial depth, these results were shown by Zuo and Serfling (2000), Liu (1988), Serfling (2002), Cuevas and Fraiman (2009), Ramasay et al. (2019), respectively. For the modified spatial depth, properties (1), (2) and (4) follow from the fact that properties (1), (2) and (4) for spatial depth.
Proof of Proposition 23. We prove properties (1)-(4) in order. Let $A \in \mathbb{R}^{d \times d}$ be any orthogonal matrix and $b \in \mathbb{R}^d$. For any integrable function $h : \mathbb{S}^{d-1} \to \mathbb{R}^+$, it holds that
\[
\int_{\mathbb{S}^{d-1}} h(u)d\nu = \int_{\mathbb{S}^{d-1}} h(Au)d\nu. \tag{D.1}
\]
Given that $A$ is an orthogonal matrix, it also holds that
\[
(Ax + b - AX - b)^	op u = A^	op (x - X)^	op u = A^	op u (x - X).
\]
Let $Y \sim A\mu + b$. Applying the above identity implies that
\[
\mathbb{E}\sigma \left( s(Ax + b - Y)^	op u \right) = \mathbb{E}\sigma \left( sA^	op u (x - X) \right).
\]
The result follows from the definition of the smoothed integrated dual depth and (D.1).

For property (2), by definition $\operatorname{sup}_x \operatorname{IDD}(x, \mu, s) \leq 1/4$. It remains to show $\operatorname{IDD}(\theta_0, \mu, s) = 1/4$. We will use the fact that
\[
(1 - \sigma(x - x)) = \sigma(X - x). \tag{D.2}
\]
Now, consider the integrand in the definition of smoothed integrated dual depth at any fixed $u$. Central symmetry of $\mu$ and (D.2) yield that
\[
\mathbb{E}\sigma \left( s(\theta_0 - X)^	op u \right) (1 - \mathbb{E}\sigma \left( s(\theta_0 - X)^	op u \right)) = \left( \mathbb{E}\sigma \left( s(X - \theta_0)^	op u \right) \right)^2.
\]
Central symmetry of $\mu$ and (D.2) also yield that
\[
\mathbb{E}\sigma \left( s(\theta_0 - X)^	op u \right) (1 - \mathbb{E}\sigma \left( s(\theta_0 - X)^	op u \right)) = \left( \mathbb{E}\sigma \left( s(X - \theta_0)^	op u \right) \right)^2.
\]
These two equalities yield that
\[
\mathbb{E}\sigma \left( s(X - \theta_0)^	op u \right) = 1/2.
\]
Given that this holds for any $u \in \mathbb{S}^{d-1}$, it follows that $\operatorname{IDD}(\theta_0; \mu, s) = 1/4 = \operatorname{sup}_x \operatorname{IDD}(x; \mu, s)$.

For property (3), at any fixed $u$, central symmetry of $\mu$ and the fact that $\sigma$ is increasing yields that either
\[
\mathbb{E}\sigma \left( s(x - X)^	op u \right) \leq \mathbb{E}\sigma \left( s(p\theta_0 + (1 - p)x - X)^	op u \right) \leq 1/2,
\]
holds or
\[
1/2 \leq \mathbb{E}\sigma \left( s(p\theta_0 + (1 - p)x - X)^	op u \right) \leq \mathbb{E}\sigma \left( s(x - X)^	op u \right),
\]
holds. The result follows immediately from the fact that $f(x) = x(1 - x)$ is monotone on both $[0, 1/2]$ and $[1/2, 1]$.

Lastly, we show property (4). For any fixed $u$, the fact that $\lim_{x \to \infty} \sigma(x)\sigma(-x) = 0$ and the bounded convergence theorem imply that
\[
\lim_{c \to \infty} \operatorname{IDD}(cu; \mu, s) = \lim_{c \to \infty} \int_{\mathbb{S}^{d-1}} \left( \mathbb{E}\sigma \left( s(cu - X)^	op u \right) \right) (1 - \mathbb{E}\sigma \left( s(cu - X)^	op u \right)) d\nu(u)
= \int_{\mathbb{S}^{d-1}} \lim_{c \to \infty} \left( \mathbb{E}\sigma \left( s(cu - X)^	op u \right) \right) (1 - \mathbb{E}\sigma \left( s(cu - X)^	op u \right)) d\nu(u)
= 0.
\]
Proof of Proposition 24. Let
\[ Z = \sup_x \left| \hat{D}(x, \mu_n, s) - \hat{D}(x, \hat{\mu}_n, s) \right|. \]

We first prove a concentration result for \( Z \). To this end, let
\[ G_n(x, u, s) = \frac{1}{n} \sum_{i=1}^n \sigma \left( s(x - X_i) \top u \right) \]
\[ W_1 = \sup_x \left| \frac{1}{M} \sum_{m=1}^M G_n(x, u_m, s) - \int_{\mathbb{R}^d} G_n(x, u, s) dv(u) \right| \]
\[ W_2 = \sup_x \left| \frac{1}{M} \sum_{m=1}^M G_n(x, u_m, s)^2 - \int_{\mathbb{R}^d} G_n(x, u, s)^2 dv(u) \right|. \]

From the fact that \( Z \leq W_1 + W_2 \), we need only prove concentration results for \( W_1 \) and \( W_2 \). Let \( \mathcal{G} = \{ \sigma(y \top u + b) / n : y \in \mathbb{R}^d, b \in \mathbb{R} \} \).

Note that \( \text{VC}(\mathcal{G}) \leq d + 2 \). Therefore, it follows from Lemma 7.11 in (Sen, 2018) that there exists a universal constant \( K > 0 \) such that for all \( \epsilon \in (0, 1) \), it holds that
\[ \sup_Q N(\epsilon/n, \mathcal{G}, L_2(Q)) \leq \left( \frac{K}{\epsilon} \right)^{2d+4}. \]

Now, define
\[ \sum_{i=1}^n \mathcal{G}_i = \left\{ \frac{1}{n} \sum_{i=1}^n \sigma(y_i \top u + b_i) : y_1, \ldots, y_n \in \mathbb{R}^d, b_1, \ldots, b_n \in \mathbb{R} \right\}. \]

It follows from Lemma 7.19 in (Sen, 2018) that for all \( \epsilon \in (0, 1) \), it holds that
\[ \sup_Q N \left( 2\epsilon, \sum_{i=1}^n \mathcal{G}_i, L_2(Q) \right) \leq \left( \frac{K}{\epsilon} \right)^{2dn+4n}. \]

It follows from (Talagrand, 1994; Sen, 2018), that there exists a universal constant \( K' > 0 \) such that for all \( t > 0 \), it holds that
\[ \Pr \left( \sqrt{M} W_1 > t \right) \leq \left( \frac{K't}{\sqrt{2dn + 4n}} \right)^{2dn+4n} e^{-2t^2}. \]

However, note that \( |W_1| \leq 1/4 \), therefore, for \( t > \sqrt{M}/4 \), it holds that \( \Pr \left( \sqrt{M} W_1 > t \right) = 0. \) Therefore, it holds that
\[ \Pr \left( W_1 > t \right) \leq \left( \frac{K'M}{4dn} \right)^{dn} e^{-2Mt^2}. \]
The same logic applied to \( W_2 \) gives that there exists a universal constant \( K'' > 0 \) such that for all \( t > 0 \), it holds that

\[
\Pr(W_2 > t) \leq \left( \frac{K''M}{d_n} \right)^{d_n} e^{-2Mt^2}.
\]

Therefore, there exists universal constants \( K, K' > 0 \) such that for all \( t > 0 \), it holds that

\[
\Pr(Z > t) \leq \left( \frac{KM}{d_n} \right)^{d_n} e^{-K'Mt^2}.
\]

Now, to prove the sample complexity bound, we want to find \( M \) such that

\[
\left( \frac{KM}{d_n} \right)^{d_n} e^{-K'Mt^2} \leq \gamma,
\]

which is equivalent to the inequality

\[
-dn \log M + K'Mt^2 \geq \log(1/\gamma) + dn \log \left( \frac{K}{d_n} \right). \tag{D.3}
\]

First, we show that \( dn \log M \leq K'Mt^2/2 \). Note that the function \( h(M) = K'Mt^2/2 - dn \log M \) satisfies the conditions of Lemma 13. Applying Lemma 13 yields that \( h(M) \geq 0 \) for

\[
M \geq 2 \frac{2dn K'}{K't^2} \log \left( \frac{2}{K't^2} \vee e \right) \geq \frac{dn}{t^2} \log \left( \frac{1}{t} \vee e \right). \tag{D.4}
\]

When (D.4) is satisfied, we have that (D.3) is satisfied for

\[
M \geq 2 \frac{\log(1/\gamma) + dn \log \left( \frac{K}{d_n} \right)}{K't^2}. \tag{D.5}
\]

Thus, for all \( t > 0 \) and all \( 0 < \gamma < 1 \), there exists a universal constant \( c_1 > 0 \) such that

\[
\sup_x \left| \text{iDD}(x, \hat{\mu}_n, s) - \text{iDD}(x, \hat{\mu}_n, s) \right| \leq t,
\]

with probability \( 1 - \gamma \), provided that

\[
M \geq c_1 \frac{\log(1/\gamma) \vee dn \log \left( \frac{1}{t} \vee e \right)}{t^2}.
\]

We prove Theorem 22 with a series of Lemmas.

**Lemma 32.** Condition 4 holds for each of the depth functions defined in Section 6.

**Proof.** Recall that all of the depth functions satisfy property (4) given in Definition 3. Property (4) implies that there exists a compact set \( E \) such that \( \sup_{x \in \mathbb{R}^d} D(x, \mu) = \sup_{x \in E} D(x, \mu) \). It follows from boundedness of \( D(x, \mu) \) and compactness of \( E \) that there exists a point in \( y \in E \) such that \( D(y, \mu) = \sup_{x \in E} D(x, \mu) = \sup_{x \in \mathbb{R}^d} D(x, \mu) \). \( \square \)
Lemma 33. Suppose that \( \pi \) is absolutely continuous and that for all \( u \in \mathbb{S}^{d-1} \), the measure \( \mu_u \) is absolutely continuous with density \( f_u \) and \( \sup_{\|u\|=1} \|f_u\|_\infty = L < \infty \). Then there exists a universal constant \( C > 0 \) such that for all \( s \in (0, \infty] \), halfspace, simplicial, integrated rank-weighted and smoothed integrated dual depth satisfy Condition [\ref{eq:Con4}] with \( \omega(r) = C \cdot L \cdot r \).

Proof. We prove the result for each depth in turn.

**Halfspace depth:** Consider two points \( x, y \in \mathbb{R}^d \). By assumption, \( F(\cdot, u, \mu) \) is \( L \)-Lipschitz function a.e. . It follows that

\[
|\text{HD}(x, \mu) - \text{HD}(y, \mu)| = |\inf_u F(x, u, \mu) - \inf_u F(y, u, \mu)|
\leq \sup_u |F(x, u, \mu) - F(y, u, \mu)|
\leq L \cdot \|x - y\| \text{ a.e.}.
\]

**Integrated rank-weighted depth:** For integrated rank-weighted depth, note that absolute continuity of \( \mu_u \) implies that \( F(x, u, \mu) = F(x-, u, \mu) \). With this in mind, for any two points \( x, y \in \mathbb{R}^d \) the reverse triangle inequality yields that

\[
|\text{IRW}(x, \mu) - \text{IRW}(y, \mu)| \leq \int_{\mathbb{S}^{d-1}} 2|F(x, u, \mu) - F(y, u, \mu)|d\nu(u) \leq 2L \cdot \|x - y\| \text{ a.e.}.
\]

**Integrated dual depth:** For any \( h : [0, 1], h' : S \to [0, 1], \) for any \( x \in S \) it holds that

\[
|h(x)(1 - h(x)) - h'(x)(1 - h'(x))| \leq 3|h(x) - h'(x)|.
\]

(D.6)

Using this inequality,

\[
|\text{IDD}(x, \mu, s) - \text{IDD}(y, \mu, s)| \leq 3 \sup_u |E_{\mu} \sigma(s(x - X)^\top u) - E_{\mu} \sigma(s(y - X)^\top u)|.
\]

(D.7)

It remains to bound \( |E_{\mu} \sigma(s(x - X)^\top u) - E_{\mu} \sigma(s(y - X)^\top u)| \). To this end, first assume that \( s \leq \infty \) and let \( w, z \in \mathbb{R} \). Using the fact that \( \sigma \) is a \( 1/4 \)-Lipschitz function, it holds that

\[
|E_{\mu, u} \sigma(s(w - X)) - E_{\mu, u} \sigma(s(z - X))| = \left| \int \sigma(s(w - t)) - \sigma(s(z - t)) f_u(t) dt \right|
\leq \frac{1}{4} \int \sigma(v - s(w - t)) f_u(-v/s - w) dt
\leq |z - w|/4 \int |f_u(-v/s - w)| dt
\leq |z - w| \|f_u\|_1 / 4,
\]

where in the second line we use the substitution \( v = s(w - t) \). The preceding inequality and (D.7) imply that

\[
|\text{IDD}(x, \mu, s) - \text{IDD}(y, \mu, s)| \leq 3 \sup_u |E_{\mu} \sigma(s(x - X)^\top u) - E_{\mu} \sigma(s(y - X)^\top u)| \leq 3L \|x - y\|.
\]

38
Now assume that \( s = \infty \). It follows from absolute continuity of \( \mu_u \) and \([D.7]\) that
\[
|\text{IDD}(x, \mu) - \text{IDD}(y, \mu)| \leq 3 \sup_u |F(x, u, \mu) - F(y, u, \mu)| \leq 3L \|x - y\| \text{ a.e.}
\]

**Simplicial depth:** For simplicial depth, we must show that \( \Pr(x \in \Delta(X_1, \ldots, X_{d+1}) \) is Lipschitz continuous. It is easy to begin with two dimensions. Consider \( \Pr(x \in \Delta(X_1, X_2, X_3)) - \Pr(y \in \Delta(X_1, X_2, X_3)) \), as per \([\text{Liu} 1990]\), we need to show that \( \Pr(\overline{X_1X_2} \text{ intersects } \overline{xy}) \leq L \cdot \|x - y\| \). In order for this event to occur, we must have that \( X_1 \) is above \( xy \) and \( X_2 \) is below \( xy \), but both are projected onto the line segment \( \overline{xy} \) when projected onto the line running through \( \overline{xy} \). Affine invariance of simplicial depth implies we can assume, without loss of generality, that \( x \) and \( y \) lie on the axis of the first coordinate. Let \( x_1 \) and \( y_1 \) be the first coordinates of \( x \) and \( y \). Suppose that \( X_{11} \) is the first coordinate of \( X_1 \). It then follows from \( \text{a.e.} \) Lipschitz continuity of \( F(x, u, \mu) \) that
\[
\Pr(\overline{X_1X_2} \text{ intersects } \overline{xy}) \leq \Pr(x_1 < X_{11} < y_1) \leq L \cdot |x_1 - y_1| \leq L \|x - y\| \text{ a.e. }.
\]
In dimensions greater than two, a similar line of reasoning can be used. We can again assume, without loss of generality, that \( x \) and \( y \) lie on the axis of the first coordinate. It holds that
\[
\Pr(x \in \Delta(X_1, X_2, X_3)) - \Pr(y \in \Delta(X_1, X_2, X_3)) \leq \binom{d+1}{d} \Pr(A_d),
\]
where \( A_d \) is the event that the \( d - 1 \)-dimensional face of the random simplex, formed by \( d \) points randomly drawn from \( F \), intersects the line segment \( \overline{xy} \). It is easy to see that
\[
\Pr(A_d) \leq \Pr(x_1 < X_{11} < y_1) \leq \|f_u\|_{\infty} |x_1 - y_1| \leq L \|x - y\| \text{ a.e. }.
\]

**Lemma 34.** Suppose that \( \mu \) has a bounded density and that \( \sup_x \mathbb{E} \|y - X\|^{-1} = L' \leq \infty \). Then the modified spatial depth satisfies Condition \([\text{4}]\) with \( \omega(r) = 2 \sqrt{d} L' \cdot r \) and spatial depth satisfies Condition \([\text{4}]\) with \( \omega(r) = 2L' \cdot r \).

**Proof.** Taking the derivative of MSD and using the assumption of a bounded density results in
\[
\frac{d}{dy} \text{MSD}(y, \mu) = -2 \mathbb{E}_\mu (\text{SR}(y - X)) \mathbb{E}_\mu \left[ \frac{1_d}{\|y - X\|} - \frac{(y - X) \ast (y - X)^\top 1_d}{\|y - X\|^3} \right].
\]
Now,
\[
\|\frac{d}{dy} \text{MSD}(y, \mu)\| \leq 2 \sqrt{d} \sup_y \mathbb{E} \|y - X\|^{-1} = 2 \sqrt{d} L'.
\]
Thus, the map \( y \mapsto \text{MSD}(y, \mu) \) is \( 2 \sqrt{d} L' \)-Lipschitz.

Define the spatial rank function \( h(y, \mu) = \mathbb{E}_\mu (\text{SR}(y - X)) \). Proposition 4.1 of \([\text{Koltchinskii} 1997]\) says that if \( \mu \) has a bounded density, then the map \( y \mapsto h(y, \mu) \) is continuously differentiable in \( \mathbb{R}^d \) with derivative
\[
\frac{dh}{dy} = \int_{(x \neq y)} \frac{1}{\|y - x\|} \left[ I_d - \frac{(y - x)(y - x)^\top}{\|y - x\|^2} \right] d\mu(x).
\]
Now, 
\[ \|\frac{dh}{dy}\| \leq 2 \sup_{y} \|y - X\|^{-1} = 2L', \]
where \(\|\cdot\|\) denotes the operator norm. Thus, the map \(y \mapsto h(y, \mu)\) is \(2L'\)-Lipschitz, which implies that the map \(y \mapsto SD(y, \mu)\) is also \(2L'\)-Lipschitz.

**Lemma 35.** Condition 3 holds for each of the depth functions defined in Section 5.

**Proof.** We prove the result for each depth in turn.

**Halfspace depth:** Let 
\( \mathcal{F} = \{ \mathbf{1} \{ X^\top u \leq y \} : u \in S^{d-1}, y \in \mathbb{R} \} \)
and recall that the set of subgraphs of \( \mathcal{F} \) is the set of closed halfspaces in \( \mathbb{R}^d \). Furthermore, it holds that
\[ \sup_{x} |HD(x, \mu_1) - HD(x, \mu_2)| \leq \sup_{g \in \mathcal{F}} |E_{\mu_1} g(X) - E_{\mu_2} g(X)|. \]
Thus, it follows HD is \((1, \mathcal{F})\)-regular with \( VC(\mathcal{F}) = d + 2 \).

**Integrated rank-weighted-depth:** First observe that
\[ \sup_{x} |IRW(x, \mu_1) - IRW(x, \mu_2)| \leq 4 \sup_{u,x} |F(x, u, \mu_1) - F(x, u, \mu_2)| \vee |F(x-, u, \mu_1) - F(x-, u, \mu_2)|. \]
Define
\[ \mathcal{G} = \{ g : g(X) = \mathbf{1} \{ X^\top u < y \}, u \in S^{d-1}, y \in \mathbb{R} \}. \]
We see that the integrated rank-weighted depth function is \((4, \mathcal{G} \lor \mathcal{F})\)-regular. Furthermore, since the subgraphs of \( \mathcal{G} \) are contained in those of \( \mathcal{F} \), we have that \( VC(\mathcal{G} \lor \mathcal{F}) = d + 2 \).

**Integrated dual depth:** If \( s = \infty \) then it follows immediately from (D.6) and the analysis of halfspace depth that the integrated dual depth is \((3, \mathcal{F})\)-regular with \( VC(\mathcal{F}) = d + 2 \). Suppose that \( s < \infty \). It follows from (D.6) that
\[ \sup_{x} |IDD(x, \mu_2) - IDD(x, \mu_1)| \leq 3 \sup_{y \in \mathbb{R}, u \in S^{d-1}} |E_{\mu_1} \sigma(s(x - X)^\top u) - E_{\mu_2} \sigma(s(x - X)^\top u)|. \]
We have that
\[ \sup_{y \in \mathbb{R}, u \in S^{d-1}} |E_{\mu_1} \sigma(s(x - X)^\top u) - E_{\mu_2} \sigma(s(x - X)^\top u)| \leq \sup_{g \in \mathcal{F}', |E_{\mu_1} g(X) - E_{\mu_2} g(X)|}, \]
where
\[ \mathcal{F}' = \{ \sigma(AX_i + b) : A \in \mathbb{R}^d, b \in \mathbb{R} \}. \]
The class of functions \( \mathcal{F}' \) is a constructed from a monotone function applied to a finite-dimensional vector space of measurable functions. It follows from Lemma 7.14 and Lemma 7.17 of Sen (2018)
that \( \text{VC}(\mathcal{F}') = d + 2 \). Therefore, the smoothed integrated dual depth is \((3, \mathcal{F}')\)-regular with \( \text{VC}(\mathcal{F}') = d + 2 \).

**Simplicial Depth:** [Dümbgen (1992)] gives that
\[
\sup_x |\text{SMD}(x, \mu_1) - \text{SMD}(x, \mu_2)| \leq (d + 1) \sup_{A \in \mathcal{A}} |\mu_1(A) - \mu_2(A)|,
\]
where \( \mathcal{A} \) is the set of intersections of \( d \) open halfspaces in \( \mathbb{R}^d \). Define the set of Boolean functions \( \mathcal{A}' = \{ g(x) = 1 \{ x \in A \} : A \in \mathcal{A} \} \). Thus, simplicial depth is \((d + 1, \mathcal{A}')\)-regular with \( \text{VC}(\mathcal{A}') = O(d^2 \log d) \).

**Spatial Depth:** Applying the reverse triangle inequality yields
\[
\sup_x \left| \mathbb{E}_{\mu_1} \frac{x - X}{\| x - X \|} - \mathbb{E}_{\mu_2} \frac{x - X}{\| x - X \|} \right| \leq \sup_x \left| \mathbb{E}_{\mu_1} \frac{x - X}{\| x - X \|} - \mathbb{E}_{\mu_2} \frac{x - X}{\| x - X \|} \right| \leq \sum_{j=1}^d \left| \mathbb{E}_{\mu_1} \frac{x_j - X_j}{\| x - X \|} - \mathbb{E}_{\mu_2} \frac{x_j - X_j}{\| x - X \|} \right| \leq d \sup_{1 \leq j \leq d} \left| \mathbb{E}_{\mu_1} \frac{x_j - X_j}{\| x - X \|} - \mathbb{E}_{\mu_2} \frac{x_j - X_j}{\| x - X \|} \right|.
\]
It remains to compute the VC-dimension of
\[
\mathcal{G} = \left\{ g(X) = \frac{x_j - X_j}{\| x - X \|} : x \in \mathbb{R}^d, j \in [d] \right\}.
\]
Now, let \( E \) be the set of standard basis vectors for \( \mathbb{R}^d \). We can then write
\[
\mathcal{G} = \left\{ g(X) = \frac{e^\top (x - X)}{\| x - X \|} : x \in \mathbb{R}^d, e \in E \right\}.
\]
It is now easy to see that
\[
\mathcal{G} \subset \left\{ g(X) = y^\top (x - X) : x \in \mathbb{R}^d, y \in \mathbb{R}^d, j \in [d] \right\}.
\]
It follows that \( \text{VC}(\mathcal{G}) \leq O(d) \) and that spatial depth is \((d, \mathcal{G})\) regular.

**Modified Spatial Depth:** We may write
\[
\sup_x |\text{SD}(X, \mu_1) - \text{SD}(X, \mu_2)| = \sup_x \left| \mathbb{E}_{\mu_1} \left( \frac{x - X}{\| x - X \|} \right)^\top u - \mathbb{E}_{\mu_2} \left( \frac{x - X}{\| x - X \|} \right)^\top u \right| \leq \sup_u \sup_x \left| \mathbb{E}_{\mu_1} \left( \frac{x - X}{\| x - X \|} \right)^\top u - \mathbb{E}_{\mu_2} \left( \frac{x - X}{\| x - X \|} \right)^\top u \right| \leq \sup_x \sup_u \left| \mathbb{E}_{\mu_1} \left( \frac{x - X}{\| x - X \|} \right)^\top u - \mathbb{E}_{\mu_2} \left( \frac{x - X}{\| x - X \|} \right)^\top u \right|.
\]

41
It remains to compute the VC-dimension of
\[ \mathcal{G} = \left\{ g(X) = \frac{u^\top (x - X)}{\|x - X\|} : x \in \mathbb{R}^d, \ u \in \mathbb{S}^{d-1} \right\}. \]

It is now easy to see that
\[ \mathcal{G} \subset \{ g(X) = y^\top X + b : b \in \mathbb{R}, \ y \in \mathbb{R}^d \ j \in [d] \}. \]

It follows that \( \text{VC}(\mathcal{G}) \leq d + 2 \) and that the modified spatial depth is \((1, \mathcal{G})\) regular.

**Proof of Theorem 22.** The result follows from Lemmas 32-35.

**Proof of Corollary 25.** Together, Lemma 28 and the definition of the Laplace mechanism imply that \( \hat{D}(x, \hat{\mu}_n) \) is \( \epsilon \)-differentially private. Furthermore, the properties of the Laplace measure give that
\[
\Pr \left( \left| \hat{D}(x, \hat{\mu}_n) - D(x, \mu) \right| > t \right) \leq \Pr \left( \left| D(x, \hat{\mu}_n) - D(x, \mu) \right| > t/2 \right) + \Pr \left( \left| W_1 \frac{K}{m \epsilon} \right| > t/2 \right)
\leq \Pr \left( \left| D(x, \hat{\mu}_n) - D(x, \mu) \right| > t/2 \right) + e^{-n \epsilon t/2K}.
\]

Applying (3.4) yields
\[
\Pr \left( \left| \hat{D}(x, \hat{\mu}_n) - D(x, \mu) \right| > t \right) \leq e^{\text{VC}(\mathcal{F}) \log \left( c_1 \frac{\text{VC}(\mathcal{F})}{\epsilon} \right) - c_2 nt^2} + e^{-n \epsilon t/2K}.
\]

\[ \Box \]