Comment on “Torsion Cosmology and the Accelerating Universe”

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Cosmological solutions for homogeneous isotropic models in the framework of the Poincaré gauge theory of gravity based on gravitational Lagrangian adopted in the paper by Kun-Feng Shie, James M. Nester and Hwei-Jang Yo (Phys. Rev. D 78, 023522 (2008)) are discussed. Cosmological solutions for accelerating Universe obtained in referred paper are compared with corresponding solutions of standard \( \Lambda CDM \)-model and also with cosmological solution obtained by authors of this Comment.

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The referred paper [1] is devoted to investigation of isotropic cosmology in the frame of the Poincaré gauge theory of gravity (PGTG) based on gravitational Lagrangian, which contains besides scalar curvature also the term quadratic in the scalar curvature and three terms quadratic in the torsion tensor with some coefficients. The PGTG with such gravitational Lagrangian was adopted in previous works of authors as describing dynamical torsion field. According to [1] corresponding theory can give the solution of the problem of the acceleration of cosmological expansion at present epoch (dark energy problem of general relativity theory (GR)). Note that this problem was investigated in [2] in the frame of PGTG by using general expression for gravitational Lagrangian including both a scalar curvature and various terms quadratic in the curvature and torsion tensors with indefinite coefficients, where regular inflationary Big Bang scenario with accelerating stage of cosmological expansion at asymptotics was proposed. Because the behaviour of cosmological solutions for accelerating Universe obtained in [1] and [2] are essentially different, we will analyze below in what relation to cosmological solution obtained in [1] is the cosmological solution of [2]. With this purpose we will consider the application of gravitational equations of PGTG for homogeneous isotropic models (HIM) deduced in [2] in the case of gravitational Lagrangian used in [1].

The following expression for gravitational Lagrangian of PGTG was used in [2] (by applying notations and definitions of [2]) [11]:

\[
\mathcal{L}_g = f_0 F + F^{\alpha \beta \mu \nu} (f_1 F_{\alpha \beta \mu \nu} + f_2 F_{\alpha \mu \beta \nu} + f_3 F_{\mu \alpha \beta \nu} + f_4 F_{\alpha \nu \beta \mu}) + f_5 F^{\mu \nu} (f_1 F_{\mu \nu} + f_5 F_{\nu \mu}) + f_6 F^2 + S^{\alpha \mu \nu} (a_1 S_{\alpha \mu \nu} + a_2 S_{\mu \nu \alpha}) + a_3 S^{\alpha \mu} S_{\beta \mu \beta}.
\]

In general case any HIM in PGTG is characterized by three functions of time: the scale factor of Robertson-Walker metrics \( R \) and two torsion functions \( S_1 \) and \( S_2 \). Then not vanishing components of the curvature tensor can be expressed by the following 4 functions:

\[
\begin{align*}
A_1 &= \dot{H} + H^2 - 2HS_1 - 2\dot{S}_1, \\
A_2 &= \frac{k}{R^2} + (H - 2S_1)^2 - S_2, \\
A_3 &= 2(H - 2S_1)S_2, \\
A_4 &= \dot{S}_2 + HS_2,
\end{align*}
\]

(2)

where \( H = \dot{R}/R \) is the Hubble parameter and a dot denotes the differentiation with respect to time. The total system of gravitational equations of PGTG for HIM filled with spinless matter with energy density \( \rho \) and pressure \( p \) takes the following form [2]:

\[
\begin{align*}
a(H - S_1)S_1 - 2bS_2^2 - 2f_0A_2 + 4f(A_1^2 - A_2^2) + 2q_2 (A_3^2 - A_4^2) &= -\frac{\dot{\rho}}{\dot{\rho}}. \tag{3}
\end{align*}
\]

(3)

\[
\begin{align*}
a \left( \dot{S}_1 + 2HS_1 - S_1^2 \right) - 2bS_2^2 - 2f_0(2A_1 + A_2) - 4f(A_1^2 - A_2^2) - 2q_2(A_3^2 - A_4^2) &= p. \tag{4}
\end{align*}
\]

(4)

\[
\begin{align*}
f \left[ \dot{A}_1 + 2H(A_1 - A_2) + 4S_1A_2 \right] + q_2S_2A_3 - q_1S_2A_4 + \left( f_0 + \frac{a_2}{S} \right) S_1 &= 0, \tag{5}
\end{align*}
\]

(5)

\[
\begin{align*}
q_2 \left[ \dot{A}_4 + 2H(A_4 - A_3) + 4S_1A_3 \right] - 4fS_2A_2 - 2q_1S_2A_1 - (f_0 - b)S_2 &= 0, \tag{6}
\end{align*}
\]

(6)

where

\[
\begin{align*}
a &= 2a_1 + a_2 + 3a_3, \\
b &= a_2 - a_1, \\
f &= f_1 + \frac{f_2}{2} + f_3 + f_4 + f_5 + 3f_6, \\
q_1 &= f_2 - 2f_3 + f_4 + f_5 + 6f_6, \\
q_2 &= 2f_1 - f_2.
\end{align*}
\]


The system of gravitational equations (3)-(6) allows to obtain equations for the torsion functions $S_1$ and $S_2$, and also the generalization of Friedmann cosmological equations for HIM in the frame of PGTG in the case of different particular gravitational Lagrangians. By analyzing the system of equations (3)-(6) we will use the Bianchi identities in the Riemann-Cartan continuum, which are reduced in the case of HIM to two following relations [2]:

$$A_2 + 2H(A_2 - A_1) + 4S_1A_1 + 2S_2A_4 = 0,$$

(7)

$$A_3 + 2H(A_3 - A_4) + 4S_1A_4 - 2S_2A_1 = 0.$$  

(8)

In the case of gravitational Lagrangian used in [3] the following restrictions on indefinite parameters in equations (3)-(6) are valid: $q_2 = 0$ and $2f = q_1$. As result by using (8) we obtain from (6):

$$[2fF + 3(f_0 - b)]S_2 = 0,$$

(9)

where $F = 6(A_1 + A_2)$ is the scalar curvature. According to (9) there are two possibilities corresponding to two different cosmological solutions:

$$S_2 \neq 0 \quad \text{and} \quad F = -\frac{3(f_0 - b)}{2f},$$

(10)

$$S_2 = 0 \quad \text{and} \quad F \neq -\frac{3(f_0 - b)}{2f}.$$  

(11)

In the first case, when conditions (10) are valid, the system of equations (3)-(5) by using (7) and the definitions (2) for the curvature functions $A_2$ and $A_1$ leads to the following solution: $S_1 = 0$ and

$$S_2^2 = \frac{f_0(f_0 - b)}{4fb} + \frac{\rho - 3p}{12b},$$

(12)

$$\frac{k}{R^2} + H^2 = \frac{1}{6b} \left[ \rho + \frac{3(f_0 - b)^2}{4f} \right],$$

(13)

$$\dot{H} + H^2 = -\frac{1}{12b} \left[ \rho + 3p - \frac{3(f_0 - b)^2}{2f} \right].$$  

(14)

The expression (12) for the torsion function $S_2$ and cosmological equations (13)-(14) are identical to that at asymptotics obtained in [2]. These equations allow to explain observable acceleration of cosmological expansion by certain restrictions on indefinite parameters [2] (see also [3]); however, because the equations (13)-(14) have the form of Friedmann cosmological equations with cosmological constant, the problem of cosmological singularity remains in such theory.

The second possibility, when conditions (11) are valid, is considered in [4]. In this case gravitational equation (6) vanishes, and the system of gravitational equations is reduced to three equations (3)-(5) including two indefinite parameters: $a$ and $f$. Note that these equations take place always independently on restrictions for indefinite parameters of gravitational Lagrangian (1), if $S_2 = 0$. At first time these gravitational equations for HIM were deduced and investigated in [3]. The analysis of gravitational equations (3)-(5) by using the Bianchi identity (7) (with $S_2 = 0$) gives the following expressions for the curvature functions $A_2$, $A_1$ and the torsion function $S_1$ [3]:

$$F = \frac{1}{2} \left( f_0 + \frac{1}{8}a \right) + \frac{3}{2} \left( \frac{k}{R^2} + \dot{H} + 2H^2 \right),$$

$$A_2 = \frac{1}{6} \left( f_0 + \frac{1}{8}a \right)^{-1} \left[ \rho + \frac{\rho + \frac{3}{2}f^2 + \frac{1}{2}(k + R^2)R^{-2}}{1 + \frac{4}{3}fF(f_0 + \frac{1}{8}a)^{-1}}, \right],$$

$$S_1 = -\frac{1}{6} \left( f_0 + \frac{1}{8}a \right)^{-1} \left[ \frac{F}{1 + \frac{4}{3}fF(f_0 + \frac{1}{8}a)^{-1}}, \right],$$

$$A_1 = \frac{2}{4}F - A_2.$$  

(15)

Unlike GR, where gravitational equations for HIM are identical to Friedmann cosmological equations, discussed gravitational equations of PGTG for HIM include besides the scale factor $R$ also the torsion function $S_1$. However, gravitational equations allow to deduce cosmological equations for the scale factor $R$ without torsion function generalizing Friedmann cosmological equations. We obtain these equations by substituting the solution (15) into the definitions (2) of the curvature functions $A_2$ and $A_1$. The explicit form of the first cosmological equation is the following:

$$\frac{k}{R^2} + H^2 = \frac{1}{6f_0} \left( 1 + \frac{2f}{3f_0} \right)^{-1} \left[ \rho + \frac{1}{3}f^2 - 4fH\dot{F} + \frac{2f^2}{3(f_0 + a/8)} \left( 1 + \frac{2f}{3(f_0 + a/8)} \right)^{-1} F^2 \right],$$

(16)

where the scalar curvature $F$ is defined by (15). We do not write the second cosmological equation following from (16) by using the conservation law, which in the case of spinless matter minimally coupled with gravitation takes the same form as in GR:

$$\dot{\rho} + 3H(\rho + p) = 0.$$  

(17)

Cosmological solutions for metric quantities can be found by solving the system of equations (16)-(17) by given equation of state for gravitating matter and initial conditions for $(R, H, \dot{H}, \rho)$. At first note that if the parameter $a$ is not equal to zero (in [4] $a = 4d_0 > 0$ [11]), cosmological equations contain higher derivatives of the scale factor $R$; in particular, the cosmological equation (16) contains the third derivative of $R$ (the second cosmological equation is differential equation of the fourth order with respect to $R$) [12]. By given initial conditions we obtain two different solutions corresponding to two different initial values of $\dot{H}$ (or $S_1$). It is because the equation (16) contains the squared derivative $\dot{H}$ and
hence by given initial values of \((R, H, \dot{H}, \rho)\) there is two different values of \(\dot{H}\). Cosmological equations contain the parameter \(\frac{1}{3} \left( \frac{f}{f_{0} + \alpha} \right)^{2} \sim \alpha = \frac{f_{0}}{3 f_{0}} > 0\) with inverse dimension of energy density, and they transform into Friedmann cosmological equations of GR if \(f \to 0\) \((\alpha \to 0)\) \([4]\). The solutions asymptotics of cosmological equations, where energy density is small \(\alpha \rho \ll 1\), is close to the Friedmannian asymptotics. The behaviour of cosmological solutions at present epoch depends essentially on the value of parameter \(\alpha\). If the value of \(\alpha^{-1}\) corresponds to the scale of extremely high energy densities and hence at present epoch we have \(\alpha \rho \ll 1\), the behaviour of cosmological solutions is quasi-Friedmannian and such solutions can not describe cosmological acceleration.

If the value of \(\alpha^{-1}\) is comparable with the average energy density at present epoch \((\alpha \rho \sim 1)\), the solutions behaviour differs from that of GR and depends essentially on values of indefinite parameters \((f, a)\) and using initial conditions. With the purpose to compare cosmological solutions for flat model \((k = 0)\) with that of standard \(\Lambda CDM\)-model of GR, we will use the initial values for \((H, \dot{H}, \rho)\) obtained in the frame of GR and corresponding to observational data. Note that as result by given parameters \(f\) and \(a\) the initial values for \(F\) and \(S_{1}\) are determined and they can not be introduced independently from \(f\) and \(a\) (compare with \([4]\)).

Now let us to demonstrate these statements by numerical solution of Eqs. (16)-(17) in the case of flat model filled with the dust matter \((p = 0)\), for which \(\rho R^{3} = \text{const.}\). With this purpose dimensionless units for variables and parameters will be introduced by the following way: \(t \to \tilde{t} = t \sqrt{\frac{c_{s}^{3}}{3 f_{0}^{2}}}, \ H \to \tilde{H} = H \sqrt{\frac{b_{0}}{\rho_{c}}}\), \(\dot{H} \to \dot{\tilde{H}} = \frac{\dot{H} \rho_{c}}{3 f_{0}}\), where the value of \(\rho_{c} = 6 f_{0} H_{0}^{2}\) corresponds to average energy density in the Universe at present epoch in the frame of GR (the index "0" at physical quantities denotes its values at present epoch). In the frame of standard \(\Lambda CDM\)-model one supposes that the total value of energy density includes the contributions of three components: baryonic matter, dark matter and dark energy. By taking into account that one uses at present epoch for baryonic and dark matter the equation of state of dust \((\rho_{BM} = \rho_{DM} = 0)\), and for dark energy \(\rho_{DE} = -\rho_{DE}\), we write the Friedmann cosmological equations of GR in dimensionless form for considering case:

\[
\dot{\tilde{H}}^{2} = \tilde{\dot{H}} BM + \tilde{\dot{H}} DM + \tilde{H} DE,
\]

\[
\ddot{\tilde{H}}' + \tilde{H}^{2} = -2(\tilde{\dot{H}} BM + \tilde{\dot{H}} DM - 2\tilde{\rho} DE). \quad (18)
\]

In accordance with Eqs. (18) and observational data we have the following initial values for physical parameters: \(\tilde{H}_{0} = 1, \ \rho_{BM0} + \rho_{DM0} = 0.3, \ H_{0} = -0.45\). If one supposes that there is in the Universe only baryonic and dark matter \((\rho = \rho_{BM} + \rho_{DM})\), then we obtain the following initial condition for energy density \(\rho_{0} = 0.3\).

By using obtained initial conditions numerical solutions for dimensionless Hubble parameter and acceleration \(w = \dot{H}' + \dot{H}^{2}\) are presented in Fig. 1 - 2 for the following choice of parameters: \(\tilde{f} = 0.35\) and \(\tilde{a} = 2\).

\[\text{FIG. 1: Solution for the Hubble parameter by initial conditions: } \tilde{\rho}_{0} = 0.3, \ H_{0} = 1, \ H_{0}' = -0.45, \ H_{0}'' = -4.60 \text{ and } \tilde{a} = 2, \ \tilde{f} = 0.35.\]

\[\text{FIG. 2: Solution for acceleration } w \text{ by initial conditions: } \tilde{\rho}_{0} = 0.3, \ H_{0} = 1, \ H_{0}' = -0.45, \ H_{0}'' = -4.60 \text{ and } \tilde{a} = 2, \ \tilde{f} = 0.35.\]
erations function $w$ oscillates near $w = 0$. The character of solutions of eq. (16) with quasi-Friedmannien asymptotics in the future depends essentially on initial values for $\dot{H}$, $H'$ and $\dot{\rho}$. As illustration of this statement the solution of (16)-(17) corresponding to the following initial conditions: $\dot{\rho}_0 = 0.05$, $H_0 = 1$, $H'_0 = -0.45$, $H''_0 = -7.97$ and $\ddot{a} = 2$, $\dot{f} = 0.35$.

![FIG. 3: Solution for the Hubble parameter by initial conditions: $\dot{\rho}_0 = 0.05$, $H_0 = 1$, $H'_0 = -0.45$, $H''_0 = -7.97$ and $\ddot{a} = 2$, $\dot{f} = 0.35$.](image1)

![FIG. 4: Solution for acceleration $w$ by initial conditions: $\dot{\rho}_0 = 0.05$, $H_0 = 1$, $H'_0 = -0.45$, $H''_0 = -7.97$ and $\ddot{a} = 2$, $\dot{f} = 0.35$.](image2)

ized cosmological Friedmann equations obtained in [4] are physical solutions with usual Friedmannien asymptotics. These equations allow to build regular isotropic cosmology including inflationary cosmology [5, 6, 7]. Regular character of all cosmological solutions with respect to metrics. Hubble parameter, its time derivative and energy density is connected with gravitational repulsion effect provoked by the torsion function $S_1$ at extreme conditions in the beginning of cosmological expansion. However, the torsion function $S_1$ tends to zero at asymptotics, and such theory can not describe the accelerating Universe. As was shown in [8], regular HIM with quasi-Friedmannien accelerating cosmological expansion at present epoch contain two torsion functions $S_1$ and $S_2$, and the effect of gravitational repulsion at present epoch is connected with effective cosmological constant induced by the torsion function $S_2$. As it follows from our consideration given above, the gravitational Lagrangian adopted in [1] allows to build HIM only with one torsion function. In the case of HIM with $S_2 = 0$ the scalar curvature $F$ is not constant for physical solutions (see (15)). However, if the dominator in (15) is equal to zero (that corresponds to constant scalar curvature), gravitational equations do not allow to determine the torsion function $S_1$ and the curvature functions (compare with [1]). Gravitational equations have in this case the following specific solution: cosmological equations take the form of Friedmann cosmological equations with cosmological constant, in which instead of Newton’s gravitational constant figures the parameter inversely proportional to $(-a)$. If $a > 0$, this specific solution is not physical. In connection with this the sign of $a$ for this case was changed that leads to ”degenerate” case discussed in [1]. Note that specific not physical solution with constant scalar curvature does not appear if $a = 0$.

The investigation of isotropic cosmology in the frame of PGTG based on gravitational Lagrangian (1) shows that the PGTG can have principal meaning for theory of gravitational interaction [8, 9]. Physical consequences depend essentially on restrictions on indefinite parameters of $\mathcal{L}_g$. Because at present the quadratic part of $\mathcal{L}_g$ is unknown, we have to investigate the PGTG by using general expression (1) of gravitational Lagrangian and to obtain restrictions on indefinite parameters of $\mathcal{L}_g$ in order that physical consequences and mathematical structure of gravitational equations should be satisfactory.

As it follows from our consideration given above the PGTG based on gravitational Lagrangian used in [1] (and also in [10]) lead to isotropic cosmology, in the frame of which the evolution of the Universe differs essentially from that of standard $\Lambda$CDM-model. Unlike [1] cosmological equations deduced in [2] take at asymptotics the quasi-Friedmannien form, and terms related to dark energy (and possibly to dark matter [3]) in cosmological equations of general relativity theory for $\Lambda$CDM-model are connected in considered theory with the change of gravitational interaction provoked by spacetime torsion.

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[11] In order to compare equations in [1, 2] relations for indefinite parameters of gravitational Lagrangians used in [1, 2] are given below (parameters of [1] are denoted by means of the prime): $f_0 = -a'_0$, $f_6 = b'_2$, $a_1 = -a'_1$, $a_2 = -a'_1$, $a_3 = 2a'_1$, coefficients $f_i$ ($i = 1, 2, ..., 5$) are equal to zero in [1].

[12] Note that gravitational equations of PGTG are differential equations of the second order with respect to gravitational gauge field variables – the tetrad and the Lorentz connection. By using these variables gravitational equations (3)-(6) for HIM were obtained. Cosmological solutions can be found by direct integration of gravitational equations without deduction and consideration of cosmological equations for the scale factor $R$ generalizing Friedmann cosmological equations of GR. Cosmological equations are important for comparison of cosmology built in the frame of PGTG with Friedmann cosmology of GR. As our analysis shows, the character of cosmological solutions depends essentially on the order of cosmological equations as differential equations for the scale factor (see below).