Some aspects of harmonic analysis related to Gegenbauer expansions on the half-line

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Abstract In this paper we consider the generalized shift operator, generated by the Gegenbauer differential operator

\[ G = (x^2 - 1)^{\frac{1}{2} - \lambda} \frac{d}{dx} (x^2 - 1)^{\lambda + \frac{1}{2}} \frac{d}{dx}. \]

Maximal function (G−maximal function), generated by the Gegenbauer differential operator \( G \) is investigated. The \( L_{p,\lambda} \)-boundedness for the \( G \)-maximal function is obtained. The concept of potential of Riesz-Gegenbauer is introduced and for it the theorem of Sobolev type is proved.

Keywords: Generalized shift operator. Riesz-Gegenbauer potential. Maximal function. Morrey spaces. BMO spaces.

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Introduction The Hardy-Littlewood maximal function is an important tool of harmonic analysis. It was first introduced by Hardy and Littlewood in 1930 (see [21]) for 2\( \pi \)-periodical functions, and later it was extendend to the Euclidean spaces, some weighted measure spaces (see [4, 28, 29]), symmetric spaces (see [5, 26]), various Lie groups [9], for the Jacobitype hypergroups [6, 7], for Chebli-Trimeche hypergroups [1], for the one-dimensional Bessel-Kingman hypergroups [27], for the \( n \)-dimensional Bessel-Kingman hypergroups (\( n \geq 1 \)) [10, 11, 13], for Morrey-Bessel spaces [2, 3, 12, 14], for Laguerre hypergroup [15, 16, 22, 25]. The structure of the paper is as follows. In Section 1 we present some definitions, notation and auxiliary results. In Section 2 the \( L_{p,\lambda} \) boundedness of the \( G \)-maximal function is proved. In Section 3 we introduce and study some embeddings into the function spaces, associated with the Gegenbauer differential operator. In
Section 4 we introduce definition Riesz-Gegenbaur potential and is for to Sobolev type theorem is proved.

1 DEFINITIONS, NOTATION AND AUXILIARY RESULTS

Let \( H(x, r) = (x - r, x + r) \cap [0, \infty), r \in (0, \infty), x \in [0, \infty) \). For all measurable set \( E \subset [0, \infty) \) \( \mu E \equiv |E|_\lambda = \int_E sh^{2\lambda} t dt \). For \( 1 \leq p \leq \infty \) let \( L_p([0, \infty), G) \equiv L_{p,\lambda}[0, \infty) \) be the space of functions measurable on \([0, \infty)\) with the finite norm

\[
\|f\|_{L_{p,\lambda}} = \left( \int_0^\infty |f(ch t)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}}, 1 \leq p < \infty,
\]

\[
\|f\|_{\infty,\lambda} = \operatorname{ess sup}_{t \in [0,\infty)} |f(ch t)|, \quad p = \infty
\]

Analogy by [7] we define Gegenbauer maximal functions is as follows:

\[
M_G f(ch x) = \sup_{r>0} \frac{1}{|H(0, r)|_\lambda} \int_0^r A^\lambda_{ch t} |f(ch x)| \mu(t),
\]

\[
M_\mu f(ch x) = \sup_{r>0} \frac{1}{|H(x, r)|_\lambda} \int_{H(x,r)} |f(ch t)| \mu(t), \quad \mu(t) = sh^{2\lambda} t dt,
\]

\[
|H(0, r)|_\lambda = \int_0^r sh^{2\lambda} t dt, \quad |H(x, r)|_\lambda = \int_{H(x,r)} sh^{2\lambda} t dt.
\]

Here

\[
A^\lambda_{ch t} f(ch x) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^\pi f(ch xch t - sh xsh t cos \varphi)(\sin \varphi)^{2\lambda-1} d\varphi
\]

denote the generalized shift operator, associated with the Gegenbauer differential operator

\[
G = (x^2 - 1)^{1/2-\lambda} \frac{d}{dx} (x^2 - 1)^{\lambda+1/2} \frac{d}{dx}.
\]
By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

Further we’ll need some auxiliary assertions.

**Lemma 1** For $0 < \lambda < 1/2$ the following correlations are true:

$$|H(0, r)|_\lambda \sim \begin{cases} (sh \frac{r}{2})^{2\lambda+1}, & 0 < r \leq c; \\ (ch \frac{r}{2})^{4\lambda}, & c < r < \infty, \end{cases}$$

where $c$ denotes positive constant.

**Proof.** Let first $0 < r \leq c$, then

$$|H(0, r)|_\lambda = \int_0^r sh^{2\lambda} t dt = \int_0^r (sh t)^{2\lambda-1} d(ch t) = \int_0^r (ch^2 t - 1)\lambda^{-\frac{1}{2}} d(ch t)$$

$$= \int_1^{ch r} (t - 1)^{\lambda^{-\frac{1}{2}}}(t + 1)^{\lambda^{-\frac{1}{2}}} dt \geq (ch r + 1)^{\lambda^{-\frac{1}{2}}} \int_1^{ch r} (t - 1)^{\lambda^{-\frac{1}{2}}} dt$$

$$\geq (ch1 + 1)^{\lambda^{-\frac{1}{2}}}(t - 1)^{\lambda^{-\frac{1}{2}}} dt \geq \frac{2(ch r - 1)^{\lambda^{-\frac{1}{2}}}}{(2\lambda + 1)(1 + ch1)^{\frac{1}{2}-\lambda}}$$

$$= \frac{2^{2\lambda+2}}{(2\lambda + 1)(1 + ch1)^{\frac{1}{2}-\lambda}}(sh \frac{r}{2})^{2\lambda+1}. \quad (1)$$

On the other hand,

$$|H(0, r)|_\lambda = \int_0^r sh^{2\lambda} t dt = \int_1^{ch r} (t - 1)^{\lambda^{-\frac{1}{2}}}(t + 1)^{\lambda^{-\frac{1}{2}}} dt \leq 2^{\lambda^{-\frac{1}{2}}} \int_1^{ch r} (t - 1)^{\lambda^{-\frac{1}{2}}} dt$$

$$= \frac{2^{\lambda + \frac{1}{2}}}{2\lambda + 1}(t - 1)^{\lambda^{\frac{1}{2}}} \bigg|_1^{ch r} = \frac{2^{\lambda + \frac{1}{2}}}{2\lambda + 1}(ch r - 1)^{\lambda^{\frac{1}{2}}} = \frac{2^{2\lambda+1}}{2\lambda + 1} \left( sh \frac{r}{2} \right)^{2\lambda+1}. \quad (2)$$

Let, now $c < r < \infty$. Then

$$|H(0, r)|_\lambda = \int_0^r sh^{2\lambda} t dt = \int_0^r (sh t)^{2\lambda-1} d(ch t) = \int_0^r (ch^2 t - 1)\lambda^{-\frac{1}{2}} d(ch t)$$
\[
\int_1^r \frac{(t-1)^{\lambda-\frac{1}{2}}}{(t+1)^{\lambda+\frac{1}{2}-\lambda}} dt \geq \int_1^r \frac{(ch \ r + 1)^{\lambda-\frac{1}{2}}}{(ch \ r + 1)^{\lambda+\frac{1}{2}-\lambda}} dt
\]

\[
= (ch \ r + 1)^{\lambda-\frac{1}{2}} \frac{(t-1)^{\lambda+\frac{1}{2}}}{\lambda+\frac{1}{2}} \bigg|_1^r = \frac{2}{2\lambda + 1} \frac{(ch \ r - 1)^{\lambda+\frac{1}{2}}}{(ch \ r + 1)^{\lambda-\frac{1}{2}}}
\]

\[
\geq \frac{2^{2\lambda+1}}{(2\lambda + 1)3^{2\lambda+1}} (ch \ r/2)^{4\lambda} \iff \frac{2^{2\lambda+1}}{2\lambda + 1} (3sh \ r/2)^{2\lambda+1}
\]

\[
\geq \frac{2^{2\lambda+1}}{2\lambda + 1} \left( \frac{ch \ r}{2} \right)^{2\lambda+1} \iff 3sh \ r/2 \geq ch \ r/2 \iff 3(e^{r/2} - e^{-r/2})
\]

\[
\geq \frac{e^{r/2} + e^{-r/2}}{2} \iff 3(e^{r} - 1) \geq e^{r} + 1 \iff 2e^{r} \geq 4,
\]

which takes place for \( r \geq c \geq 1 \).

So,

\[
|H(0, r)|_{\lambda} \geq \frac{2^{4\lambda+1}}{(2\lambda + 1)3^{2\lambda+1}} \left( \frac{ch \ r}{2} \right)^{4\lambda}. \quad (3)
\]

Estimate above \( |H(0, r)|_{\lambda} \):

\[
|H(0, r)|_{\lambda} = \int_0^r sh^{2\lambda} \ t dt = \int_0^r \left( 2sh \ \frac{t}{2} \ ch \ \frac{t}{2} \right)^{2\lambda} \ dt
\]

\[
= 2^{2\lambda+1} \int_0^r \left( 2sh \ \frac{t}{2} \ ch \ \frac{t}{2} \right)^{2\lambda-1} \ d \left( sh \ \frac{t}{2} \ ch \ \frac{t}{2} \right) \leq 2^{2\lambda+1} \int_0^r \left( 2sh \ \frac{t}{2} \ ch \ \frac{t}{2} \right)^{4\lambda-1} \ d \left( sh \ \frac{t}{2} \ ch \ \frac{t}{2} \right)
\]

\[
= \frac{2^{2\lambda+1}}{4\lambda} \left( sh \ \frac{t}{2} \ ch \ \frac{t}{2} \right)^{4\lambda} \bigg|_0^r = \frac{4\lambda}{2\lambda} \left( sh \ \frac{r}{2} \ ch \ \frac{r}{2} \right)^{4\lambda} \leq \frac{4\lambda}{2\lambda} \left( \frac{2\lambda + 1}{3^{2\lambda+1}} \right)^{4\lambda}. \quad (4)
\]

Combine (1)-(4), we obtain assertion of lemma 1.

**Lemma 2** Let \( 0 < \lambda < 1/2 \) and \( x \in [0, \infty) \), \( r \in (0, \infty) \). Then the following estimates are true: for \( 0 < r \leq c \)

\[
|H(x, r)|_{\lambda} \leq c_{\lambda} \left\{ \begin{array}{ll}
r^{2\lambda+1}, & 0 \leq x \leq r \leq c; \\
ch^{2\lambda} x, & r < x < \infty \ (r \leq c < x < \infty).
\end{array} \right. \quad (a)
\]
And for $c < r < \infty$

$$|H(x, r)|_\lambda \leq c_\lambda \begin{cases} 
  c h^{2\lambda} r, & 0 < x \leq 2r \ (0 < x \leq 2c < 2r); \\
  c h^{2\lambda} x h^{2\lambda} r, & 2r < x < \infty \ (2c < 2r < x < \infty).
\end{cases}$$

(b)

Here and further $c_\lambda, c_{\alpha, \lambda}, c_{\alpha, \lambda, p}$ will denote some constants, depending only on subscribed indexes and generally speaking different in different formulas.

**Proof.** First we consider case when $0 < r < c$ and $x \in [0, \infty)$.

Let $0 \leq t \leq 2c$, Then we have

$$t \leq sh t \leq e^{2ct}. \quad (5)$$

We prove left-hand part of this estimate. We consider the function $f(t) = sh t - t$. As, $f'(t) = ch t - t \geq 0$, then $f(t)$ increases in $[0, \infty)$, and that takes the smallest valuer for $t = 0$, $f(0) = 0$, consequently $f(t) \geq 0$ equivalent to $sh t \geq t$.

We prove right-hand part of estimate (5).

$$\frac{e^t - e^{-t}}{2} \leq e^{2ct} \cdot t \Leftrightarrow e^{2t} - 1 \leq 2 \cdot e^{2ct} \cdot t \Leftrightarrow e^{2t} \leq 2 \cdot e^{2ct} \cdot t + 1.$$

We consider the function $f(t) = 2 \cdot e^{2ct} \cdot t + 1 - e^{2t}$.

$$f'(t) = 2 \cdot e^{2ct} + 2 \cdot e^{2ct} \cdot t - 2e^{2t} = 2e^t \left( e^{2c} + t \cdot e^{2c} - e^t \right) \geq e^{2c} (t + 1) - e^t \geq e^{2c} - e^t \geq 0, \text{ as, } t \leq 2c.$$

Thus, the estimate (5) is proved.

Hence follows that for $0 \leq x \leq r < c$

$$|H(x, r)|_\lambda = \int_0^{x+r} sh^{2\lambda} t dt \leq e^{2c} \int_0^{2r} t^{2\lambda} dt = \frac{e^{2c} \cdot 2^{2\lambda}}{2\lambda + 1} \cdot r^{2\lambda+1}. \quad (6)$$

$r < x \leq c$

$$|H(x, r)|_\lambda = \int_{x-r}^{x+r} sh^{2\lambda} t dt \leq e^{2c} \int_{x-r}^{x+r} t^{2\lambda} dt \leq e^{2c} \cdot r \cdot (x + r)^{2\lambda}.$$
\[ e^{2c \cdot r} \cdot (2x)^{2\lambda} \leq c_x h^{2\lambda} x. \quad (7) \]

Let, now \(0 < r \leq c \leq x < \infty\), then we have

\[ |H(x, r)| = \int_{x-r}^{x+r} sh^{2\lambda} t dt \leq 2r \cdot sh^{2\lambda}(x + r) = 2r(sh \cdot xch \cdot r + ch \cdot xsh \cdot r)^{2\lambda} \]

\[ \leq 2r(sh \cdot xch \cdot c + ch \cdot xsh \cdot c)^{2\lambda} \leq 2r(2ch \cdot xch \cdot c)^{2\lambda} \leq c_x h^{2\lambda} x. \quad (8) \]

Now, we consider case when \(c < r < \infty\), \(x \in [0, \infty)\).

Let, \(0 < x \leq 2c < 2r\). Entering as with proof of the estimate (4), we obtain

\[ |H(x, r)| = \int_{0}^{x+r} sh^{2\lambda} t dt = \frac{4^\frac{\lambda}{2}}{2\lambda} sh^{4\lambda} \cdot \frac{t}{2} \bigg|_{0}^{x+r} = \frac{4^\frac{\lambda}{2}}{2\lambda} sh^{4\lambda} \cdot \frac{x + r}{2} \]

\[ = \frac{4^\frac{\lambda}{2}}{2\lambda} \left( sh \cdot \frac{x}{2} ch \cdot \frac{r}{2} + ch \cdot \frac{x}{2} sh \cdot \frac{r}{2} \right)^{\frac{4\lambda}{2}} \leq c_x (sh \cdot cch \cdot \frac{r}{2} + ch \cdot csh \cdot \frac{r}{2})^{4\lambda} \leq c_x h^{4\lambda} \cdot \frac{r}{2}. \quad (9) \]

Let, now \(2c < 2r < x < \infty\), then

\[ |H(x, r)| = \int_{x-r}^{x+r} sh^{2\lambda} t dt \leq \frac{4^\frac{\lambda}{2}}{2\lambda} \left( sh^{4\lambda} \cdot \frac{x + r}{2} - sh^{4\lambda} \cdot \frac{x - r}{2} \right) \]

\[ \leq \frac{4^\frac{\lambda}{2}}{2\lambda} sh^{4\lambda} \cdot \frac{x + r}{2} \leq c_x ch^{4\lambda} \cdot \frac{x}{2} ch^{4\lambda} \cdot \frac{r}{2} \leq c_x h^{2\lambda} \cdot xch^{2\lambda} \cdot r. \quad (10) \]

From (9) and (10) follows that at \(c < r < \infty\) and \(0 < x < \infty\)

\[ |H(x, r)| \leq c_x \begin{cases} \begin{array}{ll} ch^{2\lambda} \cdot r, & 0 < x \leq 2c < 2r; \\ ch^{2\lambda} \cdot xch^{2\lambda} \cdot r, & 2c < 2r < x < \infty. \end{array} \end{cases} \quad (11) \]

Assertion of lemma 2 follows from (6)-(8), (11) and (12).
2 \( L_{p,\lambda} \)-boundedness of \( G \)-maximal operator

**Theorem 1** For \( 0 \leq x < \infty \) and \( 0 < r < \infty \) the inequality is valid

\[
M_G f(ch x) \leq c_\lambda M_p f(ch x),
\]

where \( c_\lambda \) is some positive constant.

**Proof.** Consider the integral

\[
I(x, r) = \int_0^r A_{ch t}^\lambda |f(ch x)| sh^{2\lambda} t dt
\]

\[
= \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \left\{ \int_0^\pi |f(ch x \cdot ch t - sh x \cdot sh t \cos \varphi| (\sin \varphi)^{2\lambda - 1} d\varphi \right\} sh^{2\lambda} t dt.
\]

Making in internal integral replacing

\( z = ch x \cdot ch t - sh x \cdot sh t \cos \varphi \), we get that

\[
\cos \varphi = \frac{ch x \cdot ch t - z}{sh x \cdot sh t}, \varphi = \arccos \frac{ch x \cdot ch t - z}{sh x \cdot sh t},
\]

\[
d\varphi = \frac{dz}{\sqrt{1 - (\frac{ch x \cdot ch t - z}{sh x \cdot sh t})^2 sh x \cdot sh t}}.
\]

\[
= (sh^2 x \cdot sh^2 t - ch^2 x \cdot ch^2 t + 2 \cdot z \cdot ch x \cdot ch t - z^2)^{-\frac{1}{2}} dz.
\]

As,

\[
sh^2 x \cdot sh^2 t - ch^2 x \cdot ch^2 t
\]

\[
= (ch^2 x - 1)sh^2 t - ch^2 x \cdot ch^2 t = ch^2 x \cdot sh^2 t - sh^2 t - ch^2 x \cdot ch^2 t
\]

\[
= -sh^2 t + ch^2 x (sh^2 t - ch^2 t) = -sh^2 t + ch^2 x,
\]

that

\[
d\varphi = (2z \cdot ch x \cdot ch t - sh^2 t - ch^2 x - z^2)^{-\frac{1}{2}} dz
\]

and

\[
(sin \varphi)^{2\lambda - 1} = (2z \cdot ch x \cdot ch t - sh^2 t - ch^2 x - z^2)^{\lambda - \frac{1}{2}} (sh x \cdot sh t)^{1 - 2\lambda}.
\]
Then \( I(x, r) \) makes a list of form

\[
I(x, r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} |f(z)| (2z \cdot ch \cdot t - sh^2 \cdot t - ch^2 \cdot x - z^2)^{\lambda - 1} \frac{1}{sh \cdot x} \int_0^r \int_{ch(x-t)}^{ch(x+t)} df(z) \left( z^2 - 1 \right)^{\lambda - 1} \frac{sh \cdot t}{sh \cdot x} dt.
\]

(13)

Transform expansion

\[
2z \cdot ch \cdot t - sh^2 \cdot t - ch^2 \cdot x - z^2
= 2z \cdot ch \cdot t - sh^2 \cdot t(ch^2 \cdot x - sh^2 \cdot x) - ch^2 \cdot x - z^2
= 2z \cdot ch \cdot t - sh^2 \cdot t \cdot ch^2 \cdot t + sh^2 \cdot t \cdot sh^2 \cdot x - ch^2 \cdot x - z^2
= 2z \cdot ch \cdot t + sh^2 \cdot t \cdot sh^2 \cdot x - (ch^2 \cdot t - 1)ch^2 \cdot x - ch^2 \cdot x - z^2
= 2z \cdot ch \cdot t + sh^2 \cdot x \cdot ch^2 \cdot t - sh^2 \cdot x - ch^2 \cdot t \cdot ch^2 \cdot x - z^2 \cdot ch^2 \cdot x - z^2 \cdot sh^2 \cdot x
= 2z \cdot ch \cdot t - sh^2 \cdot x - ch^2 \cdot t - z^2 \cdot ch^2 \cdot x - z^2 \cdot sh^2 \cdot x = (z^2 - 1)sh^2 \cdot x - (ch \cdot t - z \cdot ch \cdot x)^2
= (z^2 - 1) \cdot sh^2 \cdot x \left[ 1 - \left( \frac{ch \cdot t - z \cdot ch \cdot x}{\sqrt{z^2 - 1} \cdot sh \cdot x} \right)^2 \right].
\]

(14)

Taking into account (13) and (14) we get

\[
I(x, r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \int_{ch(x-t)}^{ch(x+t)} df(z) \left( z^2 - 1 \right)^{\lambda - 1} \frac{sh \cdot t}{sh \cdot x} dt.
\]

(15)

Note that

\[
\frac{sh \cdot t}{sh \cdot x} = (z^2 - 1)^{\frac{1}{2}} \frac{\partial}{\partial t} \left( \frac{ch \cdot t - z \cdot ch \cdot x}{\sqrt{z^2 - 1} \cdot sh \cdot x} \right).
\]
rewrite (15) of form

$$I(x, r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^r \int_{\text{ch}(x-t)}^{\text{ch}(x+t)} \frac{|f(z)|(z^2 - 1)^{\lambda - \frac{1}{2}}}{z} dz dt. \quad (16)$$

As, $\text{ch}(x-t) \leq z \leq \text{ch}(x+t)$, then

$$\begin{cases}
\text{ch}(x-r) \leq z \leq \text{ch} x \\
x - \text{arcch} z \leq t \leq r
\end{cases} \quad \text{and} \quad \begin{cases}
\text{ch} x \leq z \leq \text{ch}(x+r) \\
\text{arcch} z - x \leq t \leq r
\end{cases}$$

And that’s why changing the order of integration in (16), we get

$$I(x, r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \left( \int_{\text{ch}(x-r)}^{\text{ch}(x)} dz \int_{x-\text{arcch} z}^{r} dt + \int_{\text{ch} x}^{\text{ch}(x+r)} dz \int_{\text{arcch} z - x}^{r} dt \right). \quad (17)$$

Consider the integral

$$A(x, z, r) \equiv A(x, r)$$

$$= \int_{x-\text{arcch} z}^{r} \left[ 1 - \left( \frac{\text{ch} t - z \cdot \text{ch} x}{\sqrt{z^2 - 1} \cdot \text{sh} x} \right)^2 \right]^{\lambda - 1} \frac{\partial}{\partial t} \left( \frac{\text{ch} t - z \cdot \text{ch} x}{\sqrt{z^2 - 1} \cdot \text{sh} x} \right) dt.$$

Putting here $u = \frac{\text{ch} t - z \cdot \text{ch} x}{\sqrt{z^2 - 1} \cdot \text{sh} x}$, we get

$$A(x, z, r) \equiv A(x, r) = \int_{-1}^{\frac{\text{ch} r - z \cdot \text{ch} x}{\sqrt{z^2 - 1} \cdot \text{sh} x}} (1 - u^2)^{\lambda - 1} du. \quad (18)$$

On the power of even ch t

$$B(x, r) = \int_{\text{arcch} z - x}^{r} \left[ 1 - \left( \frac{\text{ch} t - z \cdot \text{ch} x}{\sqrt{z^2 - 1} \cdot \text{sh} x} \right)^2 \right]^{\lambda - 1} \frac{\partial}{\partial t} \left( \frac{\text{ch} t - z \cdot \text{ch} x}{\sqrt{z^2 - 1} \cdot \text{sh} x} \right) dt.$$
\[
\int_{-1}^{1} (1 - u^2)^{\lambda-1} du.
\]  

Taking into account (18) and (19) in (17), we have

\[
I(x, r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_{ch(x-r)}^{ch(x+r)} \frac{ch(z) - ch x}{\sqrt{z^2 - 1} \cdot sh z} \int_{-1}^{1} (1 - u^2)^{\lambda-1} du dz.
\]  

As, \(ch(x - r) \leq z \leq ch(x + r)\), then

\[
\int_{ch(x-r)}^{ch(x+r)} \frac{ch(r - z \cdot ch x)}{\sqrt{z^2 - 1} \cdot sh z} \geq \frac{ch(r - ch x \cdot ch(x + r))}{sh x \cdot sh(x + r)} = \frac{2ch(r - 2ch x \cdot ch(x + r))}{2sh x \cdot sh(x + r)}
\]

\[
= \frac{2ch r - ch(2x + r) - ch r}{ch(2x + r) - ch r} = \frac{ch r - ch(2x + r)}{ch(2x + r) - ch r} = -1
\]  

(21)

On the other hand for \(ch(x - r) \leq z \leq ch(x + r)\),

\[
\int_{ch(x-r)}^{ch(x+r)} \frac{ch(r - z \cdot ch x)}{\sqrt{z^2 - 1} \cdot sh z} \leq \frac{ch r - ch x \cdot ch(x - r)}{sh x |sh(x - r)|} = \frac{2ch r - 2ch x \cdot ch(x - r)}{2sh x \cdot sh(r - x)}
\]

\[
= \frac{2ch r - ch(2x - r) - ch r}{ch r - ch(2x - r)} = \frac{ch r - ch(2x - r)}{ch r - ch(2x - r)} = 1.
\]  

(22)

From (21) and (22) follows that for \(ch(x - r) \leq z \leq ch(x + r)\), and

\[
0 < x \leq r
\]

\[
-1 \leq \frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x} \leq 1.
\]  

(23)

From (23) follows that for \(0 < x \leq r \leq c\)

\[
A(x, r) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_{-1}^{1} (1 - u^2)^{\lambda-1} du \leq \int_{-1}^{1} (1 - u^2)^{\lambda-1} du = \frac{\Gamma(\frac{1}{2})\Gamma(\lambda)}{\Gamma(\lambda + \frac{1}{2})},
\]  

(24)

But then taking into account (24) and (20), we obtain that for \(0 < x \leq r \leq c\)

\[
I(x, r) \leq \int_{ch(x-r)}^{ch(x+r)} |f(z)| (z^2 - 1)^{\lambda-\frac{1}{2}} dz = \int_{x-r}^{x+r} |f(ch t)| sh^2 \lambda t dt.
\]  

(25)
Now, let $c < r < x < \infty$ and $ch (x - r) \leq z \leq ch (x + r)$ ($c < r < x < \infty$). Then we have
\[
\frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x} \leq \frac{ch x - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x} = \frac{(1 - z)ch x}{\sqrt{z^2 - 1} \cdot sh x} = -\frac{\sqrt{z} - 1ch x}{\sqrt{z} + 1sh x} \leq 0. \tag{26}
\]

From (21) follows, that
\[
\max(1 - u) \leq \max_{-1 \leq u \leq 0} (1 - u)^\lambda = \max(2^{\lambda - 1}, 1) = 1,
\]
\[-1 \leq u \leq \frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}.
\]

Taking into account this circumstance, for the integral $A(x, r)$ we obtain of (18)
\[
A(x, r) = \int_{-1}^{\frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}} (1 - u^2)^{\lambda - 1} du
\]
\[
\leq \int_{-1}^{\frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}} (1 + u)^{\lambda - 1} du = \frac{1}{\lambda} \left(1 + u\right)^\lambda \bigg|_{-1}^{\frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}} = \frac{1}{\lambda} \left(1 + \frac{ch r - z \cdot ch x}{\sqrt{z^2 - 1} \cdot sh x}\right)^\lambda
\]
\[
= \frac{1}{\lambda} \left(1 - \frac{z \cdot ch x - ch r}{\sqrt{z^2 - 1} \cdot sh x}\right)^\lambda \leq \frac{1}{\lambda} \left[1 - \left(\frac{z \cdot ch x - ch r}{\sqrt{z^2 - 1} \cdot sh x}\right)^2\right]^\lambda. \tag{27}
\]

We find extremum of the function
\[
f(z) = 1 - \left(\frac{z \cdot ch x - ch r}{\sqrt{z^2 - 1} \cdot sh x}\right)^2.
\]
\[
f'(z) = -2 \left(\frac{z \cdot ch x - ch r}{\sqrt{z^2 - 1} \cdot sh x}\right) \times \frac{(z^2 - 1)sh x \cdot ch x - z^2 sh x \cdot ch x + z \cdot ch r \cdot sh x}{(z^2 - 1)^{\frac{3}{2}}sh^2 x}
\]
\[
= -2 \left(\frac{z \cdot ch x - ch r}{\sqrt{z^2 - 1} \cdot sh x}\right) \frac{z \cdot ch r \cdot sh x - ch x \cdot sh x}{(z^2 - 1)^{\frac{3}{2}}sh^2 x}.
\]
\[
\begin{align*}
&= \frac{2(z \cdot ch x - ch r)(ch x - z \cdot ch r)}{(z^2 - 1)^2 sh^2 x}.
\end{align*}
\]

As, \(ch (x - r) \leq z \leq ch (x + r)\), then the function \(f(z)\) in point \(z = ch x/ch r\) has maximum equal

\[
\begin{align*}
&\quad \quad f_{\text{max}} \left( \frac{ch x}{ch r} \right) = 1 - \left( \frac{ch^2 x - ch^2 r}{\sqrt{ch^2 x - ch^2 r \cdot sh x}} \right)^2 \\
&= 1 - \frac{ch^2 x - ch^2 r}{sh^2 x} = \frac{ch^2 r - 1}{sh^2 x} = \left( \frac{sh r}{sh x} \right)^2.
\end{align*}
\]

Here from (27) we have

\[
A(x, r) \leq \frac{1}{\lambda} \left( \frac{sh r}{sh x} \right)^{2\lambda}.
\]

Let \(0 < r \leq c\), then, making into account lemmas 1(a) and 2(a), for (25) with \(0 \leq x \leq r \leq c\)

\[
M_{\text{GF}} f(ch x) = \sup_{0 < r \leq 1} \frac{1}{\mu H(0, r)} \int_0^r A_{ch t}^\lambda |f(ch x)| d\mu(t)
\]

\[
= \sup_{0 < r \leq 1} \frac{\mu H(x, r)}{\mu H(0, r)} \cdot \frac{1}{\mu H(x, r)} \int_{x-r}^{x+r} |f(ch t)| sh^{2\lambda} t dt
\]

\[
\leq c_\lambda \sup_{0 < r \leq 1} \frac{1}{\mu H(x, r)} \int_{H(x,r)} |f(ch t)| d\mu(t) = c_\lambda M_{\mu} f(ch x).
\]

And for \(c < r < x < \infty\) from (28) and (20) we obtain

\[
M_{\text{GF}} f(ch x) \leq \sup_{0 < r \leq 1} \frac{A(x, r)\mu H(x, r)}{\mu H(0, r)\mu H(x, r)} \int_{x-r}^{x+r} |f(ch t)| sh^{2\lambda} t dt
\]

\[
\leq c_\lambda \sup_{0 < r \leq 1} \frac{r \cdot ch^{2\lambda} x \cdot sh^{2\lambda} r}{\mu H(x, r) (sh \frac{r}{x})^{2\lambda+1} sh^{2\lambda} x} \int_{x-r}^{x+r} |f(ch t)| d\mu(t)
\]
\[
\leq c_\lambda \left( \frac{\text{ch} \, x}{\text{sh} \, x} \right)^{2 \lambda} \sup_{0 < r \leq 1} \text{ch}^{2 \lambda} \frac{r}{2} \frac{1}{\mu H(x, r)} \int_{x-r}^{x+r} \left| f(\text{ch} \, t) \right| d\mu(t) \\
\leq c_\lambda \left( \frac{e^x + e^{-x}}{e^x - e^{-x}} \right)^{2 \lambda} \text{ch}^{2 \lambda} \frac{1}{2} \sup_{0 < r \leq 1} \frac{1}{\mu H(x, r)} \int_{H(x, r)} \left| f(\text{ch} \, t) \right| d\mu(t) \\
\leq c_\lambda \cdot 4^\lambda \cdot e \cdot M_\mu f(\text{ch} \, x),
\]

as \(\frac{e^{2x} + 1}{e^{2x} - 1} \leq 2 \iff e^{2x} + 1 \leq 2e^{2x} - 2 \iff e^{2x} \geq 3\) at \(x \geq 1\).

From (29) and (30) follows, that

\[
M_G f(\text{ch} \, x) \leq c_\lambda M_\mu f(\text{ch} \, x), 0 < r \leq c, \quad 0 \leq x < \infty.
\]  

(31)

Now, we consider case, when \(c < r < \infty\).

Point, that for \(\text{ch} \, (x - r) \leq z \leq \text{ch} \, (x + r)\) and \(x \geq 2r\) the function \(f(z) = \frac{\text{ch} \, r - z\text{ch} \, x}{\sqrt{z^2 - 1} \text{sh} \, x}\) has maximum equal \(\frac{\sqrt{\text{ch}^2 \, x - \text{ch}^2 \, r}}{\text{sh} \, x}\).

Really,

\[
f'(z) = -\frac{\sqrt{z^2 - 1} \text{sh} \, x \text{ch} \, x + \frac{z}{\sqrt{z^2 - 1}} \text{sh} \, x (\text{ch} \, r - z \text{ch} \, x)}{(z^2 - 1) \text{sh}^2 \, x} = \frac{\text{ch} \, x - z \text{ch} \, r}{(z^2 - 1) \frac{z}{\text{sh} \, x}} = 0 \iff z = \frac{\text{ch} \, x}{\text{ch} \, r}.
\]

In this point of the function \(f(z)\) has maximum equal

\[
f_{\max}(z) = f\left( \frac{\text{ch} \, x}{\text{ch} \, r} \right) = \frac{\text{ch}^2 \, r - \text{ch}^2 \, x}{\sqrt{\text{ch}^2 \, x - \text{ch}^2 \, r \cdot \text{sh} \, x}} = -\frac{\sqrt{\text{ch}^2 \, x - \text{ch}^2 \, r}}{\text{sh} \, x} \\
= -\frac{\text{ch} \, x}{\text{sh} \, x} \sqrt{1 - \left( \frac{\text{ch} \, r}{\text{ch} \, x} \right)^2} \sim -\frac{\text{sh} \, x}{\text{ch} \, x},
\]

as

\[
\lim_{x \to \infty} \frac{\text{sh} \, x}{\text{ch} \, x} = \lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1.
\]
From (27) and (32) we obtain
\[
A(x, r) \leq \int_{-1}^{\frac{1}{\sqrt{\lambda^2 - 1} \ sh x}} (1 + u)^{\lambda - 1} du \leq \int_{-1}^{\frac{-\sqrt{\lambda^2 - 1} \ ch^2 x}{sh x}} (1 + u)^{\lambda - 1} du
\]
\[
\sim \int_{-1}^{\frac{sh x}{ch x}} (1 + u)^{\lambda - 1} du = \frac{1}{\lambda} \left(1 - \frac{sh x}{ch x}\right)^\lambda \leq \frac{1}{\lambda} \left(1 - \frac{sh^2 x}{ch^2 x}\right)^\lambda
\]
\[
= \frac{1}{\lambda} (ch x)^{-2\lambda}, \ x \to \infty. \tag{33}
\]

Now, taking into account lemmas 1(b) and 2(b), also inequalities (24) and (33), we get

\[
\frac{|H(x, r)|}{|H(0, r)|} \leq c_\lambda \begin{cases} \frac{ch^{2\lambda} r}{c^{4\lambda} \ x ch^{2\lambda} r} & \leq c_\lambda, \ c < r < \infty. \tag{34} \\
\frac{ch^{2\lambda} \ x ch^{4\lambda} r}{2} & \end{cases}
\]

Applying (34) we easy obtain
\[
M_G f(ch x) = \sup_{r > c} \frac{1}{|H(0, r)|} \int_0^r A_{ch t}^\lambda |f(ch x)| d\mu(t)
\]
\[
= \sup_{r > c} \frac{|H(x, r)|}{|H(0, r)|} \cdot \frac{1}{|H(x, r)|} \int_0^x |f(ch t)| sh^{2\lambda} t dt
\]
\[
\leq c_\lambda \frac{1}{|H(x, r)|} \int_{H(x,r)} |f(ch t)| d\mu(t) = c_\lambda M_\mu f(ch x). \tag{35}
\]

Combine (31) and (35), we get
\[
M_G f(ch x) = \sup_{0 < r < \infty} \frac{1}{|H(0, r)|} \int_0^r A_{ch t}^\lambda |f(ch x)| d\mu(t)
\]
\[ \leq \sup_{0 < r \leq c} \frac{1}{|H(0, r)|} \int_0^r A^\lambda_{ch} t |f(ch x)| d\mu(t) \]

\[ + \sup_{r > c} \frac{1}{|H(0, r)|} \int_0^r A^\lambda_{ch} t |f(ch x)| d\mu(t) \leq c_\lambda M \mu f(ch x). \]

Theorem 1 is proved.

**Theorem 2** a) If \( f \in L_{1,\lambda}[0, \infty) \), then for all \( \alpha > 0 \)

\[ |\{ x : M_G f(ch x) > \alpha \}| \leq \frac{c_\lambda}{\alpha} \int \int |f(ch t)| \, sh^{2\lambda} \, t dt = \frac{c_\lambda}{\alpha} \| f \|_{L_{1,\lambda}[0, \infty)}, \]

where \( c_\lambda > 0 \) and depends only on \( \lambda \).

b) If \( f \in L_{p,\lambda}[0, \infty) \), \( 1 < p < \infty \), then \( M_G f(ch x) \in L_{p,\lambda}[0, \infty) \) and

\[ \| M_G f \|_{L_{p,\lambda}[0, \infty)} \leq c_\lambda \| f \|_{L_{p,\lambda}[0, \infty)}. \]

**Corollary 1** If \( f \in L_{p,\lambda}[0, \infty) \), \( 1 \leq p \leq \infty \), then

\[ \lim_{r \to 0} \frac{1}{|H(0, r)|} \int_{H(0, r)} A^\lambda_{ch} t f(ch x) sh^{2\lambda} \, t dt = f(ch x), \]

for a.e. \( x \in [0, \infty) \).

**Proof.** We need to introduce one maximal function defined on a space of homogeneous type. We mean a topological space \( X \) equipped with a continuous preudometric \( \rho \) and a positive measure \( \mu \), satisfying the doubling condition

\[ \mu(E(x, 2r)) \leq C \mu(E(x, r)), \quad (36) \]

with a constant \( C \) independent of \( x \) and \( r > 0 \).

Here \( E(x, r) = \{ y \in X : \rho(x, r) = |x - y| < r \} \).

Let \( (X, \rho, \mu) \) is a space of homogeneous type. Let us define

\[ M_\mu f(x) = \sup_{r > 0} \frac{1}{\mu E(x, r)} \int_{E(x, r)} |f(t)| d\mu(t). \]
It is well known that the maximal function $M_\mu$ is weak $(1, 1)$ and is bounded on $L_p(X, d\mu)$ for $1 < p < \infty$ (see [19]). The measure of maximal function $M_\mu f(ch x)$ introduced at the beginning of Section 1

$$\mu H(x, r) = |H(x, r)|_\lambda = \int_{H(x, r)} \text{sh}^{2\lambda} t dt,$$

where

$$H(x, r) = \begin{cases} (x - r, x + r), & x - r > 0; \\ (0, x + r), & x - r < 0, \end{cases}$$

it is clear that this measure satisfies the condition (36), but then the confirmation of Theorem 2 follows from Theorem 1.

**The proof of the Corollary 1.** First show that for any function $f \in L_{p,\lambda}[0, \infty), 1 \leq p \leq \infty$, representation $ch t \mapsto A_{ch, t}^\lambda f$ from $\mathbb{R}$ into $L_{p,\lambda}$ continuous, that is

$$\|A_{ch, t}^\lambda f - f\|_{L_{p,\lambda}} \to 0 \text{ при } t \to 0. \quad (37)$$

Let $f(x)$ is a continuous function defined for $[a, b] \subset [0, \infty)$. Consider the function

$$y(t, x, \varphi) = ch t ch x - sh t sh x cos \varphi$$

Hence we have

$$|y(t, x, \varphi) - y(0, x, \varphi)| = |ch t ch x - sh t sh x cos \varphi - ch x|$$

$$= |(ch t - 1)ch x - sh t sh x cos \varphi - ch x| \leq 2sh^2 \frac{t}{2} ch x + 2sh \frac{t}{2} sh x$$

$$\leq 2sh \frac{t}{2} \left( sh \frac{t}{2} ch x + ch \frac{t}{2} sh x \right) = 2sh \frac{t}{2} sh \left( \frac{t}{2} + x \right)$$

$$\leq 2sh \frac{t}{2} sh \left( \frac{t}{2} + b \right) \to 0 \text{ при } t \to 0. \quad (38)$$
On the strength of uniformly continuous of function $f(x)$ on segment $[a, b]$ for any $\varepsilon > 0$ one may choose the number $\delta > 0$, such that

$$|f[y(t, x, \varphi)] - f[y(0, x, \varphi)]| < \varepsilon,$$

have only

$$|y(t, x, \varphi) - y(0, x, \varphi)| < \delta,$$ (that follows from (38)).

Then we have

$$|A_{ch \ t}^\lambda f(ch \ x) - f(ch \ x)|$$

$$\leq \frac{\Gamma \left(\lambda + \frac{1}{2}\right)}{\Gamma \left(\frac{1}{2}\right) \Gamma (\lambda)} \int_0^\pi |f[y(t, x, \varphi)] - f[y(0, x, \varphi)]|(\sin \varphi)^{2\lambda-1} d\varphi < \varepsilon.$$

It follows, that

$$\|A_{ch \ t}^\lambda f - f\|_{\infty, \lambda} = \sup_{x \in [a, b]} |A_{ch \ t}^\lambda f(ch \ x) - f(ch \ x)| < \varepsilon.$$

And for $1 \leq p < \infty$

$$\|A_{ch \ t}^\lambda f - f\|_{L_p, \lambda[a, b]} = \left(\int_a^b |A_{ch \ t}^\lambda f(ch \ x) - f(ch \ x)|^p \sinh 2\lambda x dx\right)^{\frac{1}{p}}$$

$$< \varepsilon \left(\int_a^b \sinh 2\lambda x dx\right)^{\frac{1}{p}} < c_{p, \lambda} \varepsilon.$$ 

Thus for any continuous function definite by segment $[a, b] \subset [0, \infty)$ and for any number $\varepsilon > 0$ the inequality is valid:

$$\|A_{ch \ t}^\lambda f - f\|_{L_p, \lambda[a, b]} < \varepsilon \text{ при } 1 \leq p \leq \infty.$$ (39)

It is known the set of all continuous functions with compact support in $[0, \infty)$ is dense in $L_{p, \lambda}[0, \infty)$. Therefore for any number $\varepsilon > 0$ there exists a continuous function with compact support in $[0, \infty)$, such that

$$\|f - f_{\varepsilon}\|_{L_p, \lambda[0, \infty)} < \varepsilon.$$ (40)
We denote $g_\varepsilon = f - f_\varepsilon$. Then $g_\varepsilon \in L_{p,\lambda}(0, \infty)$ and

$$
\|g_\varepsilon\|_{L_{p,\lambda}(0,\infty)} < \varepsilon.
$$

(41)

Thus, if $f \in L_{p,\lambda}[0, \infty)$, then for any number $\varepsilon > 0$ there exists a continuous function $f_\varepsilon$ with the compact support and function $g_\varepsilon \in L_{p,\lambda}[0, \infty)$ with condition $\|g_\varepsilon\|_{L_{p,\lambda}(0,\infty)} < \varepsilon$, such that $f = f_\varepsilon + g_\varepsilon$.

Hence we have $A_{ch}^\lambda t f(chx) = A_{ch}^\lambda t f_\varepsilon(chx) + A_{ch}^\lambda t g_\varepsilon(chx) - f(chx) + f_\varepsilon(chx) - f_\varepsilon(chx)$, from which follows, that

$$
\|A_{ch}^\lambda t f - f\|_{L_{p,\lambda}(0,\infty)} \leq \|A_{ch}^\lambda t f_\varepsilon - f_\varepsilon\|_{L_{p,\lambda}(0,\infty)}
$$

$$
+ \|f - f_\varepsilon\|_{L_{p,\lambda}(0,\infty)} + \|A_{ch}^\lambda t g_\varepsilon\|_{L_{p,\lambda}(0,\infty)}.
$$

Today, taking into account, that (see [17], lemma 2)

$$
\|A_{ch}^\lambda t f_\varepsilon\|_{L_{p,\lambda}(0,\infty)} \leq \|f\|_{L_{p,\lambda}(0,\infty)}, t \in [0, \infty), 1 \leq p \leq \infty
$$

and also the inequality (39), (40) и (41), we get

$$
\|A_{ch}^\lambda t f_\varepsilon - f\|_{L_{p,\lambda}(0,\infty)} \leq 3\varepsilon,
$$

from which follows (37).

By virtue of the locality of the problem, one can account that $f \in L_{1,\lambda}(0, \infty)$. In general case one can multiply $f$ by characteristic function of ball $B(0, r)$ and obtain required convergence almost everywhere interior to this ball and the tending $r$ to infinity could be got on ball interval $[0, \infty)$.

Suppose for any $r > 0$ and for any $x \in [0, \infty)$.

$$
f_r(chx) = \frac{1}{|B(0, r)|_\lambda} \int_{B(0, r)} A_{ch}^\lambda t f(chx) sh^{2\lambda} x dx.
$$

Let $r_0 > 0, B = B(0, r_0)$. According to Minkowski generalized inequality and discount (37), we obtain

$$
\|f_r - f\|_{L_{1,\lambda}(B)} = \frac{1}{|B(0, r)|_\lambda} \int_{B(0, r)} \left(A_{ch}^\lambda t f(chx) - f(chx)\right) sh^{2\lambda} t dt\|_{L_{1,\lambda}(B)}
$$
\[
\leq \frac{1}{|B(0,r)|_\lambda} \int_{B(0,r)} \|A_{\lambda, \varepsilon}^\lambda f - f\|_{L_1(B)}^2 h^{2\lambda} \, dt
\]

\[
\leq \sup_{|t| \leq r_0} \|A_{\lambda, \varepsilon}^\lambda f - f\|_{L_1(B)} \to 0, \text{ при } r_0 \to +0.
\]

It means that there exists such sequence \( r_k \), that \( r_k \to +0, (k \to \infty) \) and \( \lim_{k \to \infty} f_{r_k}(ch x) = f(ch x) \) almost everywhere in \( x \in [0, \infty) \).

Now, let’s prove that \( \lim_{r \to +0} f_{r}(ch x) \) exists almost everywhere. For this purpose for any \( x \in [0, \infty) \).

\[
\Omega f(ch x) = \left| \lim_{r \to +0} f_{r}(ch x) - \lim_{r \to +0} f_{r}(ch x) \right|
\]

the oscillation of \( f_r \) at the point \( x \) as \( r \to +0 \).

If \( g \) is a continuous function with compact support on \( [0, \infty) \), then \( g_r \) is convergent to \( g \) and consequently \( \Omega g \equiv 0 \) is identically equal to zero in this case.

Further, if \( g \in L_{1,\lambda}[0, \infty) \), then, according to the statement of Theorem 2

\[
| \left\{ x \in [0, \infty) : M_{G} g(ch x) > \varepsilon \right\} |_\lambda \leq \frac{c}{\varepsilon} \| g \|_{L_1,\lambda}[0, \infty), \ g \in L_{1, \lambda}[0, \infty).
\]

On the other hand it is obvious that \( \Omega g(ch x) \leq 2M_{G} g(ch x) \). Thus

\[
| \left\{ x \in [0, \infty) : \Omega g(ch x) > \varepsilon \right\} |_\lambda \leq \frac{2c}{\varepsilon} \| g \|_{L_1,\lambda}[0, \infty), \ g \in L_{1, \lambda}[0, \infty).
\]

How it was evidence above, any function \( f \in L_{p,\lambda}[0, \infty) \) can be written in form \( f = h + g \), where \( h \) is continuous function and has compact support on \([0, \infty)\), and \( g \in L_{p,\lambda}[0, \infty) \), moreover \( \| g \|_{L_{p,\lambda}[0, \infty)} < \varepsilon \), for any \( \varepsilon > 0 \). But \( \Omega \leq \Omega_h + \Omega_g \) и \( \Omega_h \equiv 0 \), however is continuous \( h \). Therefore it follows that

\[
| \left\{ x \in [0, \infty) : \Omega g(ch x) > \varepsilon \right\} |_\lambda \leq \frac{c}{\varepsilon} \| g \|_{L_1,\lambda}[0, \infty).
\]

Taking in inequality \( \| g \|_{L_1,\lambda}[0, \infty) < \varepsilon \) number \( \varepsilon \) chosen arbitrary small, we get \( \Omega f = 0 \) almost everywhere on \([0, \infty)\). Consequently, \( \lim_{r \to 0} f_r(ch x) \) exists almost everywhere on \([0, \infty)\), what was confirmed.
Remark 1 Theorem 2 was proved earlier by W. C. Connett and A. L. Schwartz [7] for the Jacobi-type hypergroups.

Corollary 2 If \( f \in L_{1,\lambda}([0, \infty)) \), then (see [18], Theorem 1)

\[
\lim_{r \to 0} \frac{1}{(sh \frac{r}{2})^{2\lambda+1}} \int_0^r |A_{ch} \lambda t f(ch \ x) - f(ch \ x)| sh^{2\lambda} t \ dt = 0,
\]

almost everywhere on \( x \in [0, \infty) \).

From here it follows that for any \( \varepsilon > 0 \) find such \( \delta > 0 \), that for all \( r < \delta \) the inequality is just:

\[
\frac{1}{(sh \frac{r}{2})^{2\lambda+1}} \int_0^r |A_{ch} \lambda t f(ch \ x) - f(ch \ x)| sh^{2\lambda} t \ dt < \varepsilon.
\]

But then discount of lemma 1 (a), we obtain

\[
\left| \frac{1}{|H(0, r)|^{\lambda}} \int_{H(0, r)} \left[ A_{ch} \lambda t f(ch \ x) - f(ch \ x) \right] sh^{2\lambda} t \ dt \right| \leq \frac{1}{(sh \frac{r}{2})^{2\lambda+1}} \int_0^r |A_{ch} \lambda t f(ch \ x) - f(ch \ x)| sh^{2\lambda} t \ dt < \varepsilon,
\]

for all \( r < \delta \), that means approval of Corollary 1.

3 Some Morrey embeddings, associated with the Gegenbauer expansion

We shall define function spaces, generated by the Gegenbauer expansion \( G \).

Definition 1. [12] Let \( 1 \leq p < \infty, \ 0 \leq \gamma \leq 2\lambda+1, \ [r]_1 = \min \{1, r\} \). We denote by \( L_{p,\lambda,\gamma}([0, \infty)) \) Morrey-Gegenbauer spaces (G-Morrey spaces) and by \( \tilde{L}_{p,\lambda,\gamma}([0, \infty)) \) modified G-Morrey spaces which are the sets of
functions $f$ locally integrable on $[0, \infty)$ with finite norms

$$
\|f\|_{L^p, \lambda, \gamma([0, \infty), G)} = \sup_{x, r \in (0, \infty)} \left( \left( \text{sh} \frac{r}{2} \right)^{-\gamma} \int_{H(0,r)} (A^{\lambda}_{ch} t |f(ch x)|)^p \text{sh}^{2\lambda} \, dt \right)^{\frac{1}{p}}
$$

$$
\|f\|_{\tilde{L}^p, \lambda, \gamma([0, \infty), G)} = \sup_{x, r \in (0, \infty)} \left( \left( \text{sh} \frac{r}{2} \right)^{-\gamma} \int_{H(0,r)} (A^{\lambda}_{ch} t |f(ch x)|)^p \text{sh}^{2\lambda} \, dt \right)^{\frac{1}{p}}.
$$

**Definition 2** [10] We denote by $\text{BMO} ([0, \infty), G)$ the $\text{BMO}$-Gegenbauer spaces ($G$-$\text{BMO}$ space) as the set of functions locally integrable on $[0, \infty)$, with finite norm

$$
\|f\|_{*,G} = \sup_{x, r \in (0, \infty)} \frac{1}{|H(0,r)|_{\lambda}} \int_{H(0,r)} \left| A^{\lambda}_{ch} t f(ch x) - f(ch x) \right| \text{sh}^{2\lambda} \, dt.
$$

where

$$
f_{H(0,r)}(ch x) = \frac{1}{|H(0,r)|_{\lambda}} \int_{H(0,r)} A^{\lambda}_{ch} t |f(ch x)| \text{sh}^{2\lambda} \, dt.
$$

Note that

$$
\tilde{L}^p_{\lambda, \gamma}([0, \infty), G) \subseteq L^p_{\lambda, \gamma}([0, \infty), G) \subseteq L^p_{\lambda, \gamma}([0, \infty), G) = L^{\infty, \lambda}.
$$

**Lemma 3** Let $1 \leq p < \infty$, $0 \leq \gamma \leq 2\lambda + 1$ and $\alpha p = 2\lambda + 1 - \gamma$. Then

$$
L^p_{\lambda, \gamma}([0, \infty), G) \subseteq L^1_{1, \lambda, 2\lambda + 1 - \alpha}([0, \infty), G) \text{ and } \|f\|_{L^p_{\lambda, \gamma}([0, \infty), G)} \leq \|f\|_{L^p_{1, \lambda, 2\lambda + 1 - \alpha}([0, \infty), G)}.
$$

**Proof.** Let $f \in L_{\lambda, \gamma}([0, \infty), G)$, $1 \leq p < \infty$, $0 \leq \gamma \leq 2\lambda + 1$, $1/p + 1/q = 1$ and $\alpha p = 2\lambda + 1 - \gamma$.

Applying Holder’s inequality, we have

$$
\int_{H(0,r)} A^{\lambda}_{ch} t |f(ch x)| \text{sh}^{2\lambda} \, dt
$$

$$
\leq \left( \int_{H(0,r)} (A^{\lambda}_{ch} t |f(ch x)|)^p \text{sh}^{2\lambda} \, dt \right)^{\frac{1}{p}} \left( \int_{H(0,r)} \text{sh}^{2\lambda} \, dt \right)^{\frac{1}{q}}. \quad (42)
$$
From Lemma 1 (b) it follows that for $r > c$

$$|H(0, r)|_\lambda \leq c_\lambda ch^{4\lambda} \frac{r}{2} < c_\lambda \left( ch \frac{r}{2} \right)^{2\lambda+1} \leq c_\lambda \left( 3sh \frac{r}{2} \right)^{2\lambda+1} = c_\lambda \left( sh \frac{r}{2} \right)^{2\lambda+1}. \quad (43)$$

From Lemma 1 (a) and (43) it follows that for any $0 < r < \infty$

$$|H(0, r)|_\lambda \leq c_\lambda \left( sh \frac{r}{2} \right)^{2\lambda+1}. \quad (44)$$

Taking into account (44) and (42), we obtain

$$\int_{H(0, r)} A^{\lambda}_{ch, t} |f(ch x)| sh^{2\lambda} t dt$$

$$\leq c_{\lambda,p} \left( sh \frac{r}{2} \right)^{\frac{2\lambda+1}{q}} \left( \int_{H(0, r)} (A^{\lambda}_{ch, t} |f(ch x)|)^p sh^{2\lambda} t dt \right)^\frac{1}{p}.$$ 

Further,

$$\left( sh \frac{r}{2} \right)^{\alpha-2\lambda-1} \int_{H(0, r)} A^{\lambda}_{ch, t} |f(ch x)| sh^{2\lambda} t dt$$

$$\leq c_{\lambda,p} \left( sh \frac{r}{2} \right)^{\alpha-2\lambda-1+\frac{2\lambda+1}{q}} \left( \int_{H(0, r)} (A^{\lambda}_{ch, t} |f(ch x)|)^p sh^{2\lambda} t dt \right)^\frac{1}{p}$$

$$= c_{\lambda,p} \left( sh \frac{r}{2} \right)^{\alpha-2\lambda-1+(2\lambda+1)(1-\frac{1}{p})} \left( \int_{H(0, r)} (A^{\lambda}_{ch, t} |f(ch x)|)^p sh^{2\lambda} t dt \right)^\frac{1}{p}$$

$$= c_{\lambda,p} \left( sh \frac{r}{2} \right)^{\alpha-2\lambda-1+\frac{2\lambda+1}{p}} \left( \int_{H(0, r)} (A^{\lambda}_{ch, t} |f(ch x)|)^p sh^{2\lambda} t dt \right)^\frac{1}{p}$$

$$= c_{\lambda,p} \left( sh \frac{r}{2} \right)^{-\gamma} \int_{H(0, r)} (A^{\lambda}_{ch, t} |f(ch x)|)^p sh^{2\lambda} t dt \right)^p = c_{\lambda,p} \|f\|_{L_{p,\lambda,\gamma}}.$$ 

Thus

$$f \in L_{1,\lambda,2\lambda+1-\alpha}[0, \infty) \quad \text{and} \quad \|f\|_{L_{1,\lambda,2\lambda+1-\alpha}} \leq c_{\lambda,p} \|f\|_{L_{p,\lambda,\gamma}}.$$

Lemma 3 is proved.
4 RIESZ-GEGENBAUER POTENTIAL (\((R-G)\)-POTENTIAL)

In this Section the concept of potential of Riesz-Gegenbauer associated with
the Gegenbauer differential operator \(G\) is introduced and its presentation of
integrals is found. Moreover, for it the theorem of Sobolev type is proved. For
the function \(f, g \in L_{1,\lambda}[1,\infty)\) in [22] of Gegenbauer transformation is defined
for function \(P^\lambda(\gamma)\) and \(Q^\lambda(\gamma)\) which are eigenfunctions of this operator \(G\).

\[
F_P[f(t)] \mapsto \hat{f}_P(\gamma) = \int_1^\infty f(t) P^\lambda(\gamma) (t^2 - 1)^{\lambda - \frac{1}{2}} dt, \quad (45)
\]

\[
F_Q[f(t)] \mapsto \hat{f}_Q(\gamma) = \int_1^\infty f(t) Q^\lambda(\gamma) (t^2 - 1)^{\lambda - \frac{1}{2}} dt. \quad (46)
\]

The inverses Gegenbauer transformations are defined by the formulas

\[
F^{-1}_P[\hat{f}_P(\alpha)] \mapsto f(x) = c^*_\lambda \int_1^\infty \hat{f}_P(\gamma) Q^\lambda(\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma, \quad (47)
\]

\[
F^{-1}_Q[\hat{f}_Q(\alpha)] \mapsto f(x) = c_\lambda \int_1^\infty \hat{f}_Q(\gamma) P^\lambda(\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma, \quad (48)
\]

where

\[
c^*_\lambda = \frac{2^{\frac{4}{2} - \lambda} \sqrt{\pi} \Gamma(\lambda + 1) \Gamma \left(\frac{1}{2} - \gamma\right) \Gamma \left(\frac{3+2\lambda}{4}\right) \Gamma \left(\lambda + \frac{1}{2}\right) \Gamma \left(\frac{5-2\lambda}{4}\right) \cos \pi \lambda^{-1}}{F \left(1, \frac{1}{2} - \lambda; \frac{5-2\lambda}{4} ; \frac{1}{2}\right) - F \left(1, \frac{1}{2} - \lambda; \frac{5-2\lambda}{4} ; -\frac{1}{2}\right)}.
\]

Preliminary we prove one lemma

**Lemma 4.** Let \(f, g \in L_{1,\lambda}[1,\infty) \cap L_{2,\lambda}[1,\infty)\). Then the equality is just

\[
\int_1^\infty f(x) A^\lambda_t g(x) (x^2 - 1)^{\lambda - \frac{1}{2}} dx = c^*_\lambda \int_1^\infty \hat{f}_P(\gamma) \left(A^\lambda_t g\right)_P(\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma. \quad (49)
\]

**Proof.** From (49) we have

\[
\int_1^\infty f(x) A^\lambda_t g(x) (x^2 - 1)^{\lambda - \frac{1}{2}} dx = c^*_\lambda \int_1^\infty A^\lambda_t g(x) (x^2 - 1)^{\lambda - \frac{1}{2}} dx
\]
\[ \times \int_{1}^{\infty} \hat{f}_P(\gamma) Q^\lambda_\gamma(x) \left( \gamma^2 - 1 \right)^{\lambda - \frac{1}{2}} d\gamma. \]  

(56)

Such (see the proof of Lemma 8 in [22])

\[ \int_{1}^{\infty} \hat{f}_P(\gamma) Q^\lambda_\gamma(x) \left( \gamma^2 - 1 \right)^{\lambda - \frac{1}{2}} d\gamma \lesssim \| f \|_{L_{2,\lambda}}, \]

then making the inequality (see [22], Lemma 2)

\[ \| A^\lambda_t g \|_{L_{1,\lambda}} \leq \| g \|_{L_{1,\lambda}}, \]

we obtain

\[ \left| \int_{1}^{\infty} A^\lambda_t g(x) \left( x^2 - 1 \right)^{\lambda - \frac{1}{2}} dx \int_{1}^{\infty} \hat{f}_P(\gamma) Q^\lambda_\gamma(x) \left( \gamma^2 - 1 \right)^{\lambda - \frac{1}{2}} d\gamma \right| \]

\[ \leq \| f \|_{L_{2,\lambda}} \int_{1}^{\infty} \left| A^\lambda_t g(x) \right| \left( x^2 - 1 \right)^{\lambda - \frac{1}{2}} dx = \| f \|_{L_{2,\lambda}} \| A^\lambda_t g \|_{L_{2,\lambda}} \leq \| f \|_{L_{2,\lambda}} \| g \|_{L_{2,\lambda}}. \]

In accord of theorem of Fubini

\[ c^* \int_{1}^{\infty} A^\lambda_t g(x) \left( x^2 - 1 \right)^{\lambda - \frac{1}{2}} dx \int_{1}^{\infty} \hat{f}_P(\gamma) Q^\lambda_\gamma(x) \left( \gamma^2 - 1 \right)^{\lambda - \frac{1}{2}} d\gamma \]

\[ = c^* \int_{1}^{\infty} A^\lambda_t g(x) \left( x^2 - 1 \right)^{\lambda - \frac{1}{2}} dx \int_{1}^{\infty} \hat{f}_P(\gamma) \left( \gamma^2 - 1 \right)^{\lambda - \frac{1}{2}} d\gamma \]

\[ = c^* \int_{1}^{\infty} \left( \hat{A}^\lambda_t g \right)_Q(\gamma) \hat{f}_P(\gamma) \left( \gamma^2 - 1 \right)^{\lambda - \frac{1}{2}} d\gamma. \]  

(51)

Taking into account (51) in (50), we obtain (49).

Lemma 4 is proved.

**Definition 3** For \( 0 < \alpha < 2\lambda + 1 \) Riesz-Gegenbauer potential \( (R - G) \)-potential \( I^\alpha_G f(ch\, x) \) defined by the equality

\[ I^\alpha_G f(ch\, x) = G^{-\frac{\alpha}{2}} f(ch\, x). \]  

(52)
Such (see [8], p. 1933)
\[ GP_\gamma^\lambda (ch \, x) = \gamma (\gamma + 2\lambda) P_\gamma^\lambda (ch \, x), \]
then making selfadjoint of operator \( G \) (see [17], Lemma 4), we obtain for (45)
\[ \left( \widehat{G_\lambda f} \right)_P (\gamma) = \int_1^\infty P_\gamma^\lambda (ch \, x) Gf (ch \, x) \, sh^{2\lambda} \, x \, dx \]
\[ = \int_0^\infty f (ch \, x) (G_\lambda P_\gamma^\lambda (ch \, x)) \, sh^{2\lambda} \, x \, dx \]
\[ = \gamma (\gamma + 2\lambda) \int_\lambda^\infty f (ch \, x) P_\gamma^\lambda (ch \, x) \, sh^{2\lambda} \, x \, dx = \gamma (\gamma + 2\lambda) \hat{f}_P (\gamma). \]

Obviously, that by induction
\[ \left( \widehat{G^k_\lambda f} \right)_P (\gamma) = (\gamma (\gamma + 2\lambda))^k \hat{f}_P (\lambda), \quad k = 1, 2, \ldots. \]

This formula is naturally spread for the fractional indexes in the following form:
\[ \left( \widehat{G_{\lambda}^{-\alpha} f} \right)_P (\gamma) = (\gamma (\gamma + 2\lambda))^{-\alpha} \hat{f}_P (\lambda). \quad (53) \]

But then for (52) and (53) we have
\[ \left( \widehat{I_{\alpha}^G f} \right)_P (\gamma) = (\gamma (\gamma + 2\lambda))^{-\alpha} \hat{f}_P (\lambda). \quad (54) \]

\textbf{Lemma 5.} Let \( h_r (ch \, x) \) is the kernel associated with \( G \) and \( 0 < \alpha < 2\lambda + 1. \) Then
\[ I_{\alpha}^G f (ch \, t) = \frac{1}{\Gamma (\frac{\alpha}{2})} \int_0^\infty \left( \int_0^{\frac{2}{\lambda} - 1} h_r (ch \, x) \, dr \right) A_{ch \, t}^\lambda f (ch \, x) \, sh^{2\lambda} \, x \, dx. \quad (55) \]

\textbf{Proof.} Let
\[ \left( \hat{h}_r \right)_Q (\gamma) = e^{-\gamma (\gamma + 2\lambda) r}, \]
then from (48) it follows, that

\[ h_r (ch \ x) = \int_1^\infty e^{-\gamma (\gamma + 2\lambda)^r} P_\gamma^\lambda (ch \ x) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma. \]

At by Lemma 4

\[ \int_0^\infty h_r (ch \ x) A_{ch \ x}^\lambda f (ch \ x) sh^{2\lambda} x dx = c_\lambda^* \int_1^\infty e^{-\gamma (\gamma + 2\lambda)^r} \left( \frac{\hat{A}_{ch \ t} f}{P_\gamma} \right)_P (\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma. \]

Thus we have

\[ \int_0^\infty \int_0^\infty r^{\frac{\alpha}{2} - 1} h_r (ch \ x) A_{ch \ x}^\lambda f (ch \ x) sh^{2\lambda} x dx dr \]

\[ = c_\lambda^* \int_1^\infty \left( \int_0^\infty r^{\frac{\alpha}{2} - 1} e^{-\gamma (\gamma + 2\lambda)^r} dr \right) \left( \frac{\hat{A}_{ch \ t} f}{P_\gamma} \right)_P (\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma \]

\[ = \left| \gamma (\gamma + 2\lambda) r = t, dr = \frac{dt}{\gamma (\gamma + 2\lambda)} \right| \]

\[ = c_\lambda^* \int_1^\infty \left( \int_0^\infty e^{-t^{\frac{\alpha}{2} - 1}} dt \right) (\gamma (\gamma + 2\lambda))^{\frac{1}{2}} \left( \frac{\hat{A}_{ch \ t} f}{P_\gamma} \right)_P (\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma \]

\[ = c_\lambda^* \Gamma \left( \frac{\alpha}{2} \right) \int_1^\infty (\gamma (\gamma + 2\lambda))^{\frac{1}{2}} \left( \frac{\hat{A}_{ch \ t} f}{P_\gamma} \right)_P (\gamma) (\gamma^2 - 1)^{\lambda - \frac{1}{2}} d\gamma. \]

Taking into account that (see [23], lemma 2)

\[ \left( \frac{\hat{A}_{ch \ t} f}{P_\gamma} \right)_P (\gamma) = \hat{f}_P (\gamma) Q_\gamma^\lambda (ch \ t) \]
for (54) and (47) we obtain
\[
\int_0^\infty \int_0^\infty r^{\frac{\alpha}{2} - 1} h_r (ch x) A_{ch_t f}^\lambda (ch x) sh^{2\lambda} x dx dr
\]
\[
= c_\lambda \Gamma \left( \frac{\alpha}{2} \right) \int_1^\infty (\gamma (\gamma + 2\lambda))^{-\frac{\alpha}{2}} \hat{f}_P (\gamma) Q^\lambda_\gamma (ch t) (\gamma^2 - 1)^{\lambda - \frac{\alpha}{2}} d\gamma
\]
\[
= \Gamma \left( \frac{\alpha}{2} \right) \int_0^\infty \left( \int_1^\infty \left( \int_0^\infty \int_0^\infty \right) \right) \left( \int_0^\infty r^{\frac{\alpha}{2} - 1} h_r (ch x) dr \right) A_{ch_t f}^\lambda (ch x) sh^{2\lambda} x dx.
\]
from it and for (47) it follows, that
\[
I_{G}^\alpha f (ch t) = \frac{1}{\Gamma (\frac{\alpha}{2})} \int_0^\infty \left( \int_0^\infty r^{\frac{\alpha}{2} - 1} h_r (ch x) dr \right) A_{ch_t f}^\lambda (ch x) sh^{2\lambda} x dx.
\]
Lemma 5 is proved.

**Corollary 2.** The following inequality is valid
\[
|I_{G}^\alpha f (ch t)| \lesssim \int_0^\infty \left| A_{ch_t f}^\lambda (ch x) \right| (sh x)^{\alpha - 2\lambda - 1} sh^{2\lambda} x dx. \quad (56)
\]
Really, from formula (see [8], p. 1933)
\[
P^\lambda_\gamma (ch x) = \frac{\Gamma (\gamma + 2\lambda) \cos \pi \lambda}{\Gamma (\gamma) \Gamma (\gamma + \lambda + 1)} (2ch x)^{-\gamma - 2\lambda}
\]
\[
\times F \left( \frac{\gamma}{2} + \lambda, \frac{\gamma}{2} + \lambda + \frac{1}{2}; \gamma + \lambda + 1; \frac{1}{ch^2 x} \right)
\]
we have
\[
|P^\alpha_\gamma (ch x)| \lesssim (ch x)^{-\gamma - 2\lambda},
\]
Since the function of Gauss \( F (\alpha, \beta; \gamma; x) \) is convergence by appointed importance of parameters on the interval \([0, \infty)\), (see [20], p. 1054).

Taking into account the last inequality, we estimate from above \( h_r (ch x) \)
\[
|h_r (ch x)| \lesssim \int_1^\infty e^{-\gamma (\gamma + 2\lambda) r (ch x)^{-\gamma - 2\lambda}} (\gamma^2 - 1)^{\lambda - \frac{\alpha}{2}} d\gamma
\]
\begin{align*}
\int_{0}^{\infty} e^{-r (ch x)^{-2\lambda-1}} e^{-\frac{1}{2} \gamma} (ch x)^{-\gamma-\frac{1}{2}} d\gamma \\
\lesssim e^{-r (ch x)^{-2\lambda-1}} \int_{0}^{\infty} (ch x)^{-\gamma} d\gamma = \left| \frac{1}{ch x} \right| \lesssim \frac{e}{e^{x+1}} \\
\int_{0}^{\infty} e^{-r (ch x)^{-2\lambda-1}} e^{-\frac{1}{2} \gamma} (ch x)^{-\gamma-\frac{1}{2}} d\gamma = \left| (x+1) \gamma = u \right| \\
\int_{0}^{\infty} e^{-\frac{1}{2} \gamma} (ch x)^{-\gamma} d\gamma = \frac{1}{\Gamma \left( \frac{\lambda+1}{2} \right)} \frac{1}{\left( \frac{1}{2} \right)^{\lambda+1}}.
\end{align*}

Hence we have

\begin{align*}
\int_{0}^{\infty} r^\frac{\alpha}{2} h_r (ch x) dr \lesssim (ch x)^{-2\lambda-1} \int_{0}^{\infty} r^\frac{\alpha}{2} e^{-r} dr \\
= \Gamma \left( \frac{\alpha}{2} \right) (ch x)^{-2\lambda-1} \leq \Gamma \left( \frac{\alpha}{2} \right) (sh x)^{\alpha-2\lambda-1} \leq \Gamma \left( \frac{\alpha}{2} \right) (sh x)^{\alpha-2\lambda-1}.
\end{align*}

Taking into account this inequality on (55), we obtain our approval.

**Sobolev type theorem for Riesz-Gegenbauer potential.**

We consider the Riesz-Gegenbauer fractional integral

\[ \mathcal{I}_G^\alpha f(ch x) = \int_{0}^{\infty} A_{ch t}(sh x)^{\alpha-2\lambda-1} f(ch t)sh^{2\lambda} tdt, \quad 0 < \alpha < 2\lambda + 1. \]

For \((R-G)\)-potential the following analogue of Hardy-Littlewood-Sobolev theorem is valid.

**Theorem 3** Let \(0 < \alpha < 2\lambda + 1, 1 \leq p < \frac{2\lambda + 1}{\alpha} \) and \(1 - \frac{1}{q} = \frac{\alpha}{2\lambda + 1} \).

a) If \(f \in L_{p,\lambda}[0,\infty)\), then the integral \(\mathcal{I}_G^\alpha f\) is convergence absolutely for any \(x \in [0, \infty)\).

b) If \(1 < p < \frac{2\lambda + 1}{\alpha}\), \(f \in L_{p,\lambda}[0,\infty)\), then \(\mathcal{I}_G^\alpha \in L_{q,\lambda}[0,\infty)\) and

\[ \|I_G^\alpha f\|_{L_{q,\lambda}[0,\infty)} \leq \|\mathcal{I}_G^\alpha f\|_{L_{q,\lambda}[0,\infty)} \leq c_{\alpha,\lambda,p} \|f\|_{L_{p,\lambda}[0,\infty)}, \quad (57) \]
where $c_{\alpha,\lambda,p}$—positive constant, depending only on subscribed indexes.

c) If $f \in L_{1,\lambda}[0, \infty), \frac{1}{q} = 1 - \frac{\alpha}{2\lambda + 1}$, then

$$|\{t \in [0, \infty) : \Im_{\alpha}Gf(ch t) > \beta\}|_{\lambda} \leq \left(\frac{c_{\alpha,\lambda}}{\beta} \|f\|_{L_{1,\lambda}[0, \infty)}\right)^{\frac{1}{q}}, \beta > 0. \quad (58)$$

**Proof.** Let $f \in L_{p,\lambda}[0, \infty), 1 \leq p < \frac{2\lambda+1}{\alpha}$, $f_1(ch x) = f(ch x)\chi_{(0,1)}(x), f_2(ch x) = f(ch x) - f_1(ch x)$, where $\chi_{(0,1)}(x) = \begin{cases} 1, & x \in (0,1), \\ 0, & x \in [1, \infty). \end{cases}$

Then

$$\Im_{\alpha}Gf(ch x) = \Im_{\alpha}Gf_1(ch x) + \Im_{\alpha}Gf_2(ch x) = \Im_{1}(ch x) + \Im_{2}(ch x).$$

We estimate above $\Im_{1}(ch x)$.

$$|\Im_{1}(ch x)| \leq \int_{0}^{1} (sh x)^{\alpha-2\lambda-1} A_{ch t}^{\lambda} |f(ch x)| sh^{2\lambda} t dt$$

$$= \int_{0}^{\infty} (sh x)^{\alpha-2\lambda-1} \chi_{(0,1)}(t)A_{ch t}^{\lambda} |f(ch x)| sh^{2\lambda} t dt.$$

By Young inequality [23]

$$\|\Im_{1}(ch)(\cdot)\|_{L_{p,\lambda}[0, \infty)} \leq \|f(ch)(\cdot)\|_{L_{p,\lambda}[0, \infty)} \cdot \left\|\cdot|^{\alpha-2\lambda-1} \chi_{(0,1)}\right\|_{L_{1,\lambda}[0, \infty)}. \quad (59)$$

Here

$$\left\|\cdot|^{\alpha-2\lambda-1} \chi_{(0,1)}\right\|_{L_{1,\lambda}} = \int_{0}^{1} (sh t)^{\alpha-2\lambda-1} sh^{2\lambda} t dt$$

$$\leq \int_{0}^{1} (sh t)^{\alpha-1} ch t dt = \int_{0}^{1} (sh t)^{\alpha-1} d(sh t) = \frac{1}{\alpha} sh^{\alpha}1. \quad (60)$$

From (58) and (59) it follows, that $\Im_{1}(ch x)$ for any $x \in [0, \infty)$ is convergence absolutely.
By using the Holder inequality

\[ |\Im_2(ch \ x)| \leq \int_1^\infty (sh \ t)^{\alpha-2\lambda-1} A_{ch \ t}^\lambda |f(ch \ x)| sh^{2\lambda} t \ dt \]

\[ \leq \|A_{ch \ t}^\lambda f\|_{L_{p,\lambda}} \cdot \left( \int_1^\infty (sh \ t)^{(\alpha-2\lambda-1)q} sh^{2\lambda} t \ dt \right)^{\frac{1}{q}} \]

\[ \leq \|f\|_{L_{p,\lambda}} \left( \int_1^\infty (sh \ t)^{(\alpha-2\lambda-1)q+2\lambda} \ ch \ t \ dt \right)^{\frac{1}{q}} \]

\[ = \|f\|_{L_{p,\lambda}} \left( \int_1^\infty (sh \ t)^{(\alpha-2\lambda-1)q+2\lambda} d(sh \ t) \right)^{\frac{1}{q}} \]

\[ = \left( \frac{(sh1)^{(\alpha-2\lambda-1)q+2\lambda+1}}{(2\lambda+1-\alpha)q-2\lambda-1} \right)^{\frac{1}{q}} \cdot \|f\|_{L_{p,\lambda}} = c_{\alpha,\lambda,p} \|f\|_{L_{p,\lambda}}, \]

from here follows the absolutely convergence \( \Im_2(ch \ x) \) for all \( x \in [0, \infty) \).

Thus, for all \( f \in L_{p,\lambda}[0, \infty), \ 1 \leq p < \frac{2\lambda+1}{\alpha} \) (\( R - G \))- potential \( \Im_{G}^\alpha f(ch \ x) \) is convergence absolutely for all \( x \in [0, \infty) \).

b) We have

\[ \Im_{G}^\alpha f(ch \ x) = \left( \int_0^r + \int_r^\infty \right) A_{ch \ t}^\lambda f(ch \ x)(sh \ t)^{\alpha-2\lambda-1} sh^{2\lambda} t \ dt \]

\[ = A_1(x,r) + A_2(x,r). \quad (61) \]

We consider \( A_1(x,r) \).

\[ |A_1(x,r)| \leq \int_0^r |A_{ch \ t}^\lambda f(ch \ x)| (sh \ t)^{2\lambda}(sh \ t)^{\alpha-2\lambda-1} dt \]

\[ \leq \sum_{k=0}^\infty \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} A_{ch \ t}^\lambda |f(ch \ x)| sh^{2\lambda} t \ dt \| \left( sh \ \frac{r}{2^{k+1}} \right)^{\alpha} \left( sh \ \frac{r}{2^{k+1}} \right)^{-2\lambda-1} \]

30
× \int_0^{\frac{r}{2}} A^\lambda_{ch t}|f(ch x)| \, sh^{2\lambda} \, t \, dt \leq \left( sh \, \frac{r}{2} \right)^{\alpha} M_{G f(ch x)}.

(62)

We consider $A_2(x, r)$. By H"older inequality

$$ |A_2(x, r)| \leq \left( \int_r^\infty |A^\lambda_{ch t} f(ch x)|^p \, sh^{2\lambda} \, t \, dt \right)^{\frac{1}{p}} \left( \int_r^\infty (sh t)^{(\alpha-2\lambda-1)q} \, sh^{2\lambda} \, t \, dt \right)^{\frac{1}{q}}$$

$$ \leq \|A^\lambda_{ch t} f\|_{L_{p,\lambda}} \left( \int_{r/2}^\infty (sh t)^{(\alpha-2\lambda-1)q+2\lambda} \, ch \, t \, dt \right)^{\frac{1}{q}}
$$

$$ \leq \|f\|_{L_{p,\lambda}} \left( \frac{sh \, \frac{r}{2}}{2\lambda+1-\alpha} \right)^{\alpha-2\lambda-1+(2\lambda+1)(\frac{1}{q} - \frac{\alpha}{2\lambda+1})}
$$

$$ \leq \|f\|_{L_{p,\lambda}} \left( sh \, \frac{r}{2} \right)^{(2\lambda+1)(\frac{1}{q} - 1) - \frac{2\lambda+1}{q}} = c_{\alpha,\lambda,p} \|f\|_{L_{p,\lambda}} \left( sh \, \frac{r}{2} \right)^{-\frac{2\lambda+1}{q}}.$$

(63)

Taking into account (62) and (63) in (61), we obtain

$$ |\Im^\alpha_{G f(ch x)}| \leq c_{\alpha,\lambda,p} \left( \left( sh \, \frac{r}{2} \right)^{\alpha} M_{G f(ch x)} + \left( sh \, \frac{r}{2} \right)^{-\frac{2\lambda+1}{q}} \|f\|_{L_{p,\lambda}} \right).$$

(64)

Minimum of right-hand of the inequality (64) is reach for

$$ sh \, \frac{r}{2} = \left( \frac{\alpha q}{2\lambda+1} \cdot \frac{\|f\|_{L_{p,\lambda}}}{M_{G f(ch x)}} \right)^{2\lambda+1}.$$

Then for (64) we have

$$ |\Im^\alpha_{G f(ch x)}| \leq c_{\alpha,\lambda,p} \left\{ \left( \frac{\|f\|_{L_{p,\lambda}}}{M_{G f(ch x)}} \right)^{2\lambda+1} M_{G f(ch x)} + \left( \frac{\|f\|_{L_{p,\lambda}}}{M_{G f(ch x)}} \right)^{-\frac{2\lambda+1}{q}} \right\}.$$
(as for the condition \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda + 1} \) \( \Rightarrow \quad 1 - \frac{p}{q} = \frac{\alpha p}{2\lambda + 1} \)) = 
\[ c_{\alpha,\lambda,p}(M_G f(ch x))^q \| f \|_{L_{p,\lambda}}^{1 - \frac{p}{q}}. \]

From here we have

\[
\int_0^\infty |\Im^\alpha_G f(ch t)|^q \sinh^{2\lambda} t dt \leq c_{\alpha,\lambda,p} \| M_G f(ch(\cdot)) \|_{L_{p,\lambda}}^p \cdot \| f \|_{L_{p,\lambda}}^{q-p},
\]

from here it follows

\[ \| \Im^\alpha_G f \|_{L_{p,\lambda}} \leq c_{\alpha,\lambda,p} \| f \|_{L_{p,\lambda}}. \]

c) Let \( f \in L_{1,\lambda}[0, \infty) \). Denote

\[ |\{ x : |\Im^\alpha_G f(ch x)| > 2 \beta \}|_\lambda \leq |\{ x : |A_1(x, r)| > \beta \}|_\lambda + |\{ x : |A_2(x, r)| > \beta \}|_\lambda. \]

From inequality (63) and Theorem 2 we have

\[ \beta |\{ x \in [0, \infty) : |A_1(x, r)| > \beta \}|_\lambda = \beta \int_{\{x \in [0, \infty) : |A_1(x, r)| > \beta \}} \sinh^{2\lambda} x dx \]

\[ \leq \beta \int_{\{x \in [0, \infty) : c_{\alpha,\lambda}(\sinh^{\alpha} \frac{r}{2}) M_G f(ch x) > \beta \}} \sinh^{2\lambda} x dx \]

\[ = \beta \left| \left\{ x \in [0, \infty) : M_G f(ch x) > \frac{\beta}{c_{\alpha,\lambda}\sinh^{\alpha} \frac{r}{2}} \right\} \right|_\lambda \]

\[ \leq \beta \cdot \frac{c_{\alpha,\lambda}}{\beta} \sinh^{\alpha} \frac{r}{2} \int_0^\infty |f(ch x)| \sinh^{2\lambda} x dx = c_{\alpha,\lambda} \sinh^{\alpha} \frac{r}{2} \| f \|_{L_{1,\lambda}}, \]

and also

\[ |A_2(x, r)| \leq \int_r^\infty |A_{ch t}^\alpha f(ch x)| (sh t)^{2\lambda - 1} \sinh^{2\lambda} t dt \]

\[ \leq \int_r^\infty \frac{|A_{ch t}^\alpha f(ch x)| \sinh^{2\lambda} t dt}{(sh t)^{2\lambda + 1 - \alpha}} \leq \int_r^\infty \frac{|A_{ch t}^\alpha f(ch x)| \sinh^{2\lambda} t dt}{(sh \frac{t}{2})^{2\lambda + 1 - \alpha}}. \]
\[
\leq \left( \text{sh} \frac{r}{2} \right)^{\alpha - 2\lambda - 1} \int_r^\infty |A_{ch t} f(ch x)| \text{sh}^{2\lambda} t dt \leq \left( \text{sh} \frac{r}{2} \right)^{\frac{2\lambda + 1}{q}} \|f\|_{L_{1,\lambda}}.
\]

Suppose \( \left( \text{sh} \frac{r}{2} \right)^{-\frac{2\lambda + 1}{q}} \|f\|_{L_{1,\lambda}} = \beta \), we obtain \( |A_2(x, r)| \leq \beta \) and consequently \( \{x \in [0, \infty) : |A_2(x, r)| > \beta \}_\lambda = 0 \).

As last,
\[
\left\{ x \in [0, \infty) : |\mathcal{A}_{2}(x, r)| > \beta \right\}_\lambda \leq \beta \text{ and consequently } \left\{ x \in [0, \infty) : \mathcal{G}_{\lambda} \mathcal{A}_{2}(x, r) > \beta \right\}_\lambda = 0.
\]

Thus, \( f \mapsto \mathcal{G}_{\lambda} f \) is weak type \((1, q)\).

Theorem is proved,

**Theorem 4** Let \( 0 < \alpha < 2\lambda + 1, p\alpha = 2\lambda + 1, f \in L_{p,\lambda}[0, \infty), \frac{1}{p} + \frac{1}{q} = 1 \).

Then \( \mathcal{G}_{\lambda} f \in BMO[0, \infty) \) and the inequality
\[
\left\| \mathcal{G}_{\lambda} f \right\|_{BMO} \leq c_{\alpha,\lambda,p} \|f\|_{L_{p,\lambda}},
\]

is fair, where \( c_{\alpha,\lambda,p} > 0 \) — constant, depending only on written out indexes.

**Proof.** We suppose
\[
f_{1}(ch x) = f(ch x) \chi_{(0,r/4)}(ch x), f_{2}(ch x) = f(ch x) - f_{1}(ch x),
\]
where \( \chi_{(0,r/4)}(ch x) \) — is the characteristic function of the interval \([0, \infty)\),

that is,
\[
\chi_{(0,r/4)}(ch x) = \begin{cases} 
1, & 0 \leq x \leq \frac{r}{4}; \\
0, & x > \frac{r}{4}.
\end{cases}
\]

Then
\[
\mathcal{G}_{\lambda} f (ch x) = \mathcal{G}_{\lambda} f_{1} (ch x) + \mathcal{G}_{\lambda} f_{2} (ch x) = F_{1} (ch x) + F_{2} (ch x),
\]

where
\[
F_{1} (ch x) = \int_{0}^{r/4} \left( A_{ch t}(ch x)^{\alpha-2\lambda-1} - (sh t)^{\alpha-2\lambda-1} \chi_{(\frac{1}{4}, \infty)}(ch t) \right) f(ch t) \text{sh}^{2\lambda} t dt,
\]

\[
F_{2} (ch x) = \int_{r/4}^{\infty} A_{ch t}(ch x)^{\alpha-2\lambda-1} f(ch t) \text{sh}^{2\lambda} t dt.
\]
$$F_2(ch\ x) = \int_0^{r/4} A_{ch\ t}^\lambda (sh\ x)^{\alpha-2\lambda-1} - (sh\ t)^{\alpha-2\lambda-1} \chi_{(\frac{1}{4}, \infty)}(ch\ t) f(ch\ t)\ sh^{2\lambda} t dt.$$ 

As, function $f_1(ch\ x)$ has compact support, then the number

$$a_1 = - \int_{(0,r/4)/(0,\min\{\frac{1}{4}, \frac{r}{4}\})} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\ sh^{2\lambda} t dt$$

is finite. And that is why we can write

$$F_1(ch\ x) - a_1 = \int_0^{r/4} A_{ch\ t}^\lambda (sh\ x)^{\alpha-2\lambda-1} f(ch\ x)\ sh^{2\lambda} t dt -$$

$$- \int_{(0,r/4)/(0,\min\{\frac{1}{4}, \frac{r}{4}\})} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\ sh^{2\lambda} t dt +$$

$$+ \int_{(0,r/4)/(0,\min\{\frac{1}{4}, \frac{r}{4}\})} (sh\ t)^{\alpha-2\lambda-1} f(ch\ t)\ sh^{2\lambda} t dt =$$

$$= \int_0^{r/4} A_{ch\ t}^\lambda (sh\ x)^{\alpha-2\lambda-1} f(ch\ t)\ sh^{2\lambda} t dt =$$

$$= \int_0^\infty A_{ch\ t}^\lambda (sh\ x)^{\alpha-2\lambda-1} f_1(ch\ t)\ sh^{2\lambda} t dt.$$ (65)

Consider the integral

$$A_{ch\ t}^\lambda f_1(ch\ x) = c_\lambda \int_0^\pi |f(ch\ xch\ t - sh\ xsh\ t\ cos\ \varphi)|$$

$$\times \chi_{(0,r/4)}(ch\ xch\ t - sh\ xsh\ t\ cos\ \varphi)(sin\ \varphi)^{2\lambda-1} \ d\varphi.$$ 

So far as,

$$ch\ (x - t) \leq ch\ xch\ t - sh\ xsh\ t\ cos\ \varphi \leq ch\ (x + t),$$

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then for $|x - t| > \frac{r}{4}$

$$
\chi_{(0,r/4)}(ch \ xch \ t - sh \ xsh \ t \cos \ \varphi) = 0,
$$

and that is why

$$
A_{\alpha \lambda} f_1 (ch \ x) = c_{\lambda} \int_{\{\varphi \in [0,\pi],|x-t|\leq r/4\}} f(ch \ xch \ t - sh \ xsh \ t \cos \ \varphi) (\sin \ \varphi)^{2\lambda - 1} \ d\varphi
$$

$$
= A_{\alpha \lambda} f (ch \ x).
$$

But then for (65) we have

$$
|F_1 (ch \ x) - a_1| \leq \int_{\{t \in [0,\infty):|x-t|\leq r/4\}} (sh \ t)^{\alpha - 2\lambda - 1} A_{\alpha \lambda} |f (ch \ x)| \ sh^{2\lambda} \ t dt.
$$

Study the estimation (66).

Let $(x - \frac{r}{4}, x + \frac{r}{4}) \cap [0, \infty) = (0, x + \frac{r}{4})$, then $0 \leq x \leq r/4$ and we have for (66)

$$
|F_1 (ch \ x) - a_1| \leq \int_{0}^{x+r/4} (sh \ t)^{\alpha - 2\lambda - 1} A_{\alpha \lambda} |f (ch \ x)| \ sh^{2\lambda} \ t dt
$$

$$
\leq \int_{0}^{r} (sh \ t)^{\alpha - 2\lambda - 1} A_{\alpha \lambda} |f (ch \ x)| \ sh^{2\lambda} \ t dt \leq c_{\alpha,\lambda} \left( sh \ \frac{r}{2} \right)^{\alpha} M_G f (ch \ x). \quad (67)
$$

Let now $(x - \frac{r}{4}, x + \frac{r}{4}) \cap [0, \infty) = (x - \frac{r}{4}, x + \frac{r}{4})$, then $x > \frac{r}{4}$. Consider case, when $\frac{r}{4} \leq x \leq \frac{3r}{4}$. Then

$$
|F_1 (ch \ x) - a_1| \leq \int_{x-r/4}^{x+r/4} (sh \ t)^{\alpha - 2\lambda - 1} A_{\alpha \lambda} |f (ch \ x)| \ sh^{2\lambda} \ t dt
$$

$$
\leq \int_{0}^{r} (sh \ t)^{\alpha - 2\lambda - 1} A_{\alpha \lambda} |f (ch \ x)| \ sh^{2\lambda} \ t dt \leq c_{\alpha,\lambda} \left( sh \ \frac{r}{2} \right)^{\alpha} M_G f (ch \ x). \quad (68)
$$
Finally, let, \( \frac{3}{4} \leq x < \infty \), then by Holder inequality, we have

\[
|F_1(ch\ x) - a_1| = \int_{x - \frac{r}{4}}^{x + \frac{r}{4}} (sh\ t)^{\alpha - \lambda - 1} A_{ch\ t}^\lambda |f(ch\ x)| sh^{2\lambda} tdt
\]

\[
\leq \| A_{ch\ t}^\lambda f \|_{L_p,\lambda} \left( \int_{x - \frac{r}{4}}^{x + \frac{r}{4}} (sh\ t)^{(\alpha - 2\lambda - 1)q} sh^{2\lambda} tdt \right)^{\frac{1}{q}}
\]

\[
\leq \| f \|_{L_p,\lambda} \left( sh(x - \frac{r}{4}) \right)^{\alpha - \lambda - 1} \left( \int_{x - \frac{r}{4}}^{x + \frac{r}{4}} (sh\ \frac{t}{2})^{2\lambda} \left( ch\ \frac{t}{2} \right)^{2\lambda - 1} d\left( sh\ \frac{t}{2} \right) \right)^{\frac{1}{q}}
\]

\[
\leq c_{\lambda, p} \| f \|_{L_p,\lambda} \left( sh\ \left( x - \frac{r}{4} \right) \right)^{\alpha - \lambda - 1} \left( \int_{x - \frac{r}{4}}^{x + \frac{r}{4}} (sh\ \frac{t}{2})^{2\lambda} \left( ch\ \frac{t}{2} \right)^{2\lambda - 1} d\left( sh\ \frac{t}{2} \right) \right)^{\frac{1}{q}}
\]

\[
\leq c_{\lambda, p} \| f \|_{L_p,\lambda} \left( sh\ \left( x - \frac{r}{4} \right) \right)^{\alpha - \lambda - 1} \left( sh\ \left( \frac{x}{2} + \frac{r}{8} \right) \right)^{2\lambda + 1 q}
\]

\[
\leq c_{\lambda, p} \| f \|_{L_p,\lambda} \left( sh\ \left( \frac{x}{2} + \frac{r}{8} \right) \right)^{\alpha - \lambda - 1 + (2\lambda + 1) \left( 1 - \frac{\alpha}{2\lambda + 1} \right)} = c_{\lambda, p} \| f \|_{L_p,\lambda}.
\]

Combine (67), (68) and (69), we obtain that

\[
|F_1(ch\ x) - a_1| \leq c_{\alpha, \lambda} \left( sh\ \frac{r}{2} \right)^\alpha M_G f(ch\ x) + c_{\lambda, p} \| f \|_{L_p,\lambda}.
\]

From here it follows, that

\[
\sup_{r > 0} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r |A_{ch\ t}^\lambda (F_1(ch\ x) - a_1)| sh^{2\lambda} tdt
\]

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We suppose

\[
a_2 = \int_{(0, \max\{\frac{\alpha}{1}, \frac{t}{2}\})} (sh \ t)^{\alpha-2\lambda-1} f (ch \ t) \ sh^{2\lambda} \ t \ dt.
\]

We estimate above the difference

\[
|F_2 (ch \ x) - a_2|
\]

\[
= \left| \int_{\frac{\alpha}{1}}^{\infty} \left( A_{\chi t}^\lambda (sh \ x)^{\alpha-2\lambda-1} - (sh \ t)^{\alpha-2\lambda-1} \ N(\frac{\alpha}{1}, (ch \ t)) \right) f (ch \ t) \ sh^{2\lambda} \ t \ dt \right|
\]

\[
- \int_{(0, \max\{\frac{\alpha}{1}, \frac{t}{2}\})} (sh \ t)^{\alpha-2\lambda-1} f (ch \ t) \ sh^{2\lambda} \ t \ dt \bigg|_{(0, \max\{\frac{\alpha}{1}, \frac{t}{2}\})}^{\infty}
\]

\[
= \left| \int_{\frac{\alpha}{1}}^{\infty} \left( A_{\chi t}^\lambda (sh \ x)^{\alpha-2\lambda-1} - (sh \ t)^{\alpha-2\lambda-1} \right) f (ch \ t) \ sh^{2\lambda} \ t \ dt \right|
\]

\[
\leq \int_{\frac{\alpha}{1}}^{\infty} |f (ch \ t)| B (x, t) \ sh^{2\lambda} \ t \ dt = \mathfrak{F} (x, r).
\]

(71)
We consider expansion

\[ B(x, t) = \left| A_{ch, t}^\lambda (sh x)^{\alpha - 2\lambda - 1} - (sh t)^{\alpha - 2\lambda - 1} \right| \]

\[ = c_\lambda \left| \int_0^\pi \left( (ch xch t - sh xsh t \cos \varphi)^2 - 1 \right)^{\frac{\alpha - 2\lambda - 1}{2}} - (sh t)^{\alpha - 2\lambda - 1} \right| (\sin \varphi)^{2\lambda - 1} d\varphi \]

\[ \leq c_\lambda \int_0^\pi \left( \max \left( \{|sh (x + t), |sh (x - t)|\} \right)^{\alpha - 2\lambda - 1} - (sh t)^{\alpha - 2\lambda - 1} \right| (\sin \varphi)^{2\lambda - 1} d\varphi. \]

We estimate above the value \( B(x, t) \). Easy to notice, that

\[ B(x, t) \lesssim \max \left( \left\{ |sh (x + t), |sh (x - t)| \right\} \right)^{\alpha - 2\lambda - 1} - (sh t)^{\alpha - 2\lambda - 1} \equiv V(x, t). \]  

(72)

I Let \( 0 < t < x - t < \infty \), then \( 0 < t < \frac{x}{2} < x + t \).

From here it follows, that

\[ (sh t)^{\alpha - 2\lambda - 1} > (sh (x + t))^{\alpha - 2\lambda - 1}. \]  

(73)

II Let \( 0 < x - t < t < \infty \), then \( \frac{x}{2} < t < x < x + t \), and in this case the inequality (73) is just.

III Let \( 0 < t - x < \infty \), then \( x < t < x + t < \infty \).

Again the inequality (73) takes place.

IV Let \( 0 < x + t < \infty \), as \( t < x + t \), then and here (73) is fair.

Combine all these cases, we obtain, that

\[ V(x, t) = (sh t)^{\alpha - 2\lambda - 1} - (sh (x + t))^{\alpha - 2\lambda - 1}. \]

Applying the Lagrange formula to segment \([t, x + t]\), we obtain

\[ V(x, t) \equiv V_\xi(x, t) = \frac{(2\lambda + 1 - \alpha) xch \xi}{(sh \xi)^{2\lambda + 2 - \alpha}}, \quad t < \xi < t + x. \]

From here we have

\[ \nu_\xi(x, t) \lesssim \begin{cases} x (sh t)^{\alpha - 2\lambda - 2}, & \text{if } \xi < 1, \\ x (sh t)^{\alpha - 2\lambda - 1}, & \text{if } \xi \geq 1 \end{cases} \]  

(74)

(75)
At first we consider the case, when $\xi < 1$.

Applying the Holder inequality and also (74) and (72), from (71) for $x \leq r$ we obtain.

$$
\tau(x, r) = \int_{r/4}^{\infty} \left| f(\text{ch } t) \right| B(x, t) \text{sh}^{2\lambda} t dt \lesssim \|f\|_{L_{p,\lambda}} \frac{\int_{r/4}^{\infty} \text{sh}^{2\lambda} t dt}{(\text{sh } t)^{(2\lambda+2-\alpha)q}} \left( \frac{1}{q} \right)
$$

$$
\lesssim \|f\|_{L_{p,\lambda}} \left( \int_{r/4}^{\infty} (\text{sh } t)^{(2\lambda+1)q+2\lambda} d\text{sh } t \right)^{\frac{1}{q}} = \|f\|_{L_{p,\lambda}} \frac{r}{\text{sh } r} \lesssim \|f\|_{L_{p,\lambda}}.
$$

As according to condition of theorem $\alpha - 2\lambda - 2 + (2\lambda + 1) / q = \alpha - 2\lambda - 2 + (2\lambda + 1 - \alpha) = -1$.

Now we consider the case, when $\xi \geq 1$.

Let at first $0 < x \leq 8$. Taking into account (75) and acting how above for $x \leq r$ we obtain

$$
\tau(x, r) = \int_{r/4}^{\infty} \left| f(\text{ch } t) \right| B(x, t) \text{sh}^{2\lambda} t dt \lesssim \|f\|_{L_{p,\lambda}} \frac{\int_{r/4}^{\infty} \text{sh}^{2\lambda} t dt}{(\text{sh } t)^{(2\lambda+1-\alpha)q}} \left( \frac{1}{q} \right)
$$

$$
\lesssim \|f\|_{L_{p,\lambda}} \left( \int_{r/4}^{\infty} \frac{(\alpha-2\lambda-1)q+4\lambda-1}{(2\text{sh } t\text{ch } t)^{2(\lambda+1-\alpha)q}} \right)^{\frac{1}{q}} \lesssim \|f\|_{L_{p,\lambda}} \left( \int_{r/4}^{\infty} \frac{\text{sh } t\text{ch } t}{(\text{sh } t)^{(2\lambda+1-\alpha)q}} \right)^{\frac{1}{q}}
$$

$$
= \|f\| \left( \int_{r/4}^{\infty} \frac{\alpha-2\lambda-1+4\lambda}{(\text{sh } t)^{2\lambda+1-\alpha}q} \right)^{\frac{1}{q}} \lesssim \|f\|_{L_{p,\lambda}} \left( \frac{x}{8} \right)^{\alpha-2\lambda-1+4\lambda/(2\lambda+1)} = \|f\|_{L_{p,\lambda}} \left( \frac{x}{8} \right)^{\frac{\alpha-2\lambda-1+4\lambda}{2\lambda+1}}
$$

$$
\lesssim \|f\|_{L_{p,\lambda}} \left( \frac{x}{8} \right)^{\frac{\alpha-2\lambda-1+2\lambda}{2\lambda+1-\lambda}} \lesssim \|f\|_{L_{p,\lambda}}.
$$

(77)
Let now $8 < x < \infty$. Then, how above, we obtain

$$
\tau (x, r) \lesssim \| f \|_{L^p, \lambda} x \left( \int_{x/4}^{\infty} \frac{\text{sh}^{2\lambda} t dt}{(\text{sh} t)^{(2\lambda + 1 - \alpha)q}} \right)^{\frac{1}{q}} \lesssim \| f \|_{L^p, \lambda} x \left( \text{sh} \left( \frac{x}{8} \right) \right)^{\frac{1 - 2\alpha}{1 + 2\lambda} + 2\lambda - 1}
$$

$$
= \| f \|_{L^p, \lambda} \frac{x}{(\text{sh} \left( \frac{x}{8} \right))^{1 - 2\lambda - \frac{2\alpha}{1 + 2\lambda}}} = \| f \|_{L^p, \lambda} \frac{x}{(2\text{sh} \left( \frac{x}{16} \right) \text{ch} \left( \frac{x}{16} \right))^{1 - 2\lambda - \frac{2\alpha}{1 + 2\lambda}}} \lesssim \| f \|_{L^p, \lambda} \frac{x}{(2^n \text{sh} \left( \frac{x}{2^n} \right))^{2^n \left(1 - 2\lambda - \frac{2\alpha}{1 + 2\lambda} \right)}} \leq \| f \|_{L^p, \lambda} \frac{8}{\left( \frac{x}{8} \right)^{2n \left(1 - 2\lambda - \frac{2\alpha}{1 + 2\lambda} \right)}} \lesssim \| f \|_{L^p, \lambda},
$$

so far as for anough greater $n = n_0$

$$
2^{n_0} \left(1 - 2\lambda - \frac{2\alpha}{1 + 2\lambda} \right) - 1 \geq 0 \iff \frac{1 - 2\alpha}{1 + 2\lambda} \leq 1 - 2\lambda - \frac{2n_0}{1 + 2\lambda}
$$

$$
\frac{1 - 2\lambda}{1 + 2\lambda} < 1 - 2\lambda \iff \alpha < 2\lambda + 1.
$$

Combine the estimates (76), (77) and (78), for $0 < x \leq r$ on (71) we obtain

$$
| F_2 (\text{ch} x) - a_2 | \lesssim \| f \|_{L^p, \lambda}.
$$

Hence we have

$$
| A_{\text{ch} t}^\lambda F_2 (\text{ch} x) - a_2 | \leq A_{\text{ch} t}^\lambda | F_2 (\text{ch} x) - a_2 | \lesssim \| f \|_{L^p, \lambda}.
$$

(79)

From (79) it follows, that

$$
\sup_{r > 0} \frac{1}{| H (0, r) |_\lambda} \int_0^r \left| A_{\text{ch} t}^\lambda F_2 (\text{ch} x - a_2) \right| \text{sh}^{2\lambda} t dt
$$

$$
\leq \sup_{r > 0} \frac{1}{| H (0, r) |_\lambda} \int_0^r A_{\text{ch} t}^\lambda | F_2 (\text{ch} x) - a_2 | \text{sh}^{2\lambda} t dt
$$

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Denote by

\[ a_f = a_1 + a_2 = \int_{(0, \max\{\frac{1}{2}, \frac{r}{4}\})} (sh \, t)^{\alpha - 2\lambda - 1} f (ch \, t) \, sh^{2\lambda} \, t \, dt. \]

At last, from (70) and (80) finally we obtain

\[
\begin{align*}
\sup_{r > 0} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r |A_{ch \, t}^\lambda \tilde{\mathcal{I}}\alpha \mathcal{G} f (ch \, x) - a_f| \, sh^{2\lambda} \, t \, dt \\
\leq \sup_{r > 0} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r |A_{ch \, t}^\lambda F_1 (ch \, x) - a_1| \, sh^{2\lambda} \, t \, dt \\
+ \sup_{r > 0} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r |A_{ch \, t}^\lambda F_2 (ch \, x) - a_2| \, sh^{2\lambda} \, t \, dt \lesssim \|f\|_{L_p, \lambda},
\end{align*}
\]

from here it follows, that

\[
\|\tilde{\mathcal{I}}\alpha \mathcal{G} f\|_{BMO} \leq 2 \sup_{x, r} \frac{1}{|H(0, r)|_{\lambda}} \int_0^r |A_{ch \, t}^\lambda \tilde{\mathcal{I}}\alpha \mathcal{G} f (ch \, x) - a_f| \, sh^{2\lambda} \, t \, dt \lesssim \|f\|_{L_p, \lambda}.
\]

Theorem 4 is proved.

**Corollary 3** Let \( \alpha p = 2\lambda + 1, \ 0 < \alpha < 2\lambda + 1, \ f \in L_{p, \lambda} [0, \infty) \). If integral \( \tilde{\mathcal{I}}\alpha \mathcal{G} f \) is convergence absolutely, then \( \tilde{\mathcal{I}}\alpha \mathcal{G} f \in BMO [0, \infty) \) and the inequality

\[
\|\tilde{\mathcal{I}}\alpha \mathcal{G} f\|_{BMO} \lesssim \|f\|_{L_p, \lambda}
\]

is valid.
Список литературы

[1] Miloud Assal, Hacen Ben Abdallah, Generalized Besov type spaces on the Laguerre hypergroups, Ann. Math. Blaise Pascal. 12 (1) (2005), 117-145.

[2] V.R. Bloom, Z. Xu, The Hardy-Littlewood maximal function for Chebli-Trimeche hypergroups, Contemp. Math. 183 (1995), 45-75.

[3] V.I. Burenkov, H.V. Guliyev, Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces, Doklady Ross. Akad. Nauk 391 (2003), 591-594.

[4] V.I. Burenkov, V.S. Guliyev, Necessary and sufficient conditions for boundedness of the Riesz potential in the local Morrey-type spaces. Potential Anal 30, 211-249 (2009).

[5] F. Charenza, M. Frasca, Morrey spaces and Hardy-Littlewood maximal functions, Roud. Mat. E. Appl. 7 (3, 4) (1987), 237-279.

[6] I.I. Clerc and E.M. Stein, $L^p$—multipliers for noncompact symmetric spaces, Proc. Nat. Akad. Sci. USA 71 (1974), 3911-3912.

[7] W.C. Connett and A.L. Schwartz, The Littlewood-Paley theory for Jacobi expansions, Trans. Amer. Soc. 251 (1979), 219-234.

[8] W.C. Connett and A.L. Schwartz, A Hardy-Littlewood maximal inequality for Jacobi-type hypergroups, Proc. Amer. Soc. 107 (1989), 137-143.

[9] L. Durand, P.M. Fishbane, L.M. Simmons, Expansion formulas and addition theorems for Gegenbauer functions, J. Math. Phys. 17 (11) (1976), 1993-1948.

[10] G. Gaurdy, Guilini, A. Hulaniski and A.M. Mantero, Hardy-Littlewood maximal function on some solvable Lie groups, J. Austral. Math. Soc A. 45 (1988), 78-82.

[11] V.S. Guliyev, Sobolev theorems for B—Riesz potentials, Dokl. RAN 358(4) (1998), 450-451.
[12] V.S. Guliyev, *On maximal function and fractional integral, associated with the Bessel differential operator*, Math. Inequal. Appl. **6**(2) (2003), 317-330.

[13] V.S. Guliyev, *Sobolev theorems for anisotropic Riesz-Bessel potentials on Morrey-Bessel spaces*, Dokl. RAN **367** (1999), 155-156.

[14] V.S. Guliyev, *Some properties of the anisotropic Riesz-Bessel potential*, Anal. Math. **26** (2000), 99-118.

[15] V.S. Guliyev, J.J. Hasanov and Y. Zeren, *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal. **5**(4) (2011), 491-506.

[16] V.S. Guliyev, M. Assal, *On maximal function on the Laguerre hypergroup*, Fract. Calc. Appl. Anal. **9**(3) (2006), 1-12.

[17] V.S. Guliyev, M.N. Omarova, *On fractional maximal function and fractional integral on the Laguerre hypergroup*, J. Math. Anal. Appl. **340** (2008), 1058-1068.

[18] V.S. Guliyev, E.J. Ibrahimov, *On equivalent normalizations of functional spaces associated with the generalized Gegenbauer shift*, Anal. Math. **34** (2008), 83-103.

[19] V.S. Guliyev, E.J. Ibrahimov, *On estimating the approximation of locally summable functions by Gegenbauer singular integrals*, Georgian Math. J. **15** (2) (2008), 251-262.

[20] R.R. Goifman, G. Weiss, *Analyse harmonique non commutative sur certains espaces homogenes*, Lecture Notes in Math, 242, Springer-Verlag, Berlin, 1971.

[21] I.S. Grad and I.M. Ryzhick, *The tables of integrals, sums, series and derivatives*, M. 1971 (Russian).

[22] G.H. Hardy and J.E. Littlewood, *A maximal theorem with function-theoretic application*, Acta Math. **54** (1930), 81-116.
[23] Jizheng Huang and Heping Liu, *The weak Type (1,1) estimates of maximal functions on the Laguerre hypergroup*, Canad. Math. Bull. **53** (1) (2010), 491-502.

[24] V.M. Kokilashvili, *On Hardy’s inequalities in weighted spaces*, (Russian) Soobshch. Akad. Nauk Gruzin. SSR. **96** (2) (1979), 37-40.

[25] V.M. Kokilashvili, M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces*, World Scientific, Singapore (1991).

[26] E.J. Ibrahimov, *On gegenbauer transformation on the half-line*, Georgian Math. J. **18** (2011), 497-515.

[27] Jizheng Huang, Sobolev space, *Besov space and Triebel-Lizorkin space on the Laguerre hypergroup*, Appl. **190** (2012), 1-19.

[28] J-O. Stomberg, *Week type $L^1$-estimates for maximal functions on noncompact symmetric spaces*, Ann. of Math. **114** (1985), 115-126.

[29] K. Stempak, *La theorie de Littlewoo-Paley pour la transformation de Fourier-Bessel*, C. R. Acad. Sci. Paris Ser. I Math. **303** (1986), 15-18.

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