Diffusion equations from master equations  
— A discrete geometric approach —

Shin-itiro GOTO and Hideitsu HINO  
The Institute of Statistical Mathematics, 
10-3 Midori-cho, Tachikawa, Tokyo 190-8562, Japan  
February 4, 2020

Abstract

In this paper, continuous-time master equations with finite states employed in nonequilibrium statistical mechanics are formulated in the language of discrete geometry. In this formulation, chains in algebraic topology play roles, and master equations are described on graphs that consist of vertexes representing states and of directed edges representing transition matrices. It is then shown that master equations under the detailed balance conditions are equivalent to discrete diffusion equations, where the Laplacians are defined as self-adjoint operators with respect to introduced inner products. An isospectral property of these Laplacians is shown for non-zero eigenvalues, and its applications are given. The convergence to the equilibrium state is shown by analyzing this class of diffusion equations. In addition, a systematic way to derive closed dynamical systems for expectation values is given. For the case that the detailed balance conditions are not imposed, master equations are expressed as a form of a continuity equation.

1 Introduction

Master equations are vital in the study of nonequilibrium statistical mechanics[1, 2], since they are mathematically simple and show relaxation processes towards to equilibrium states under some conditions[3]. These equations describe the time evolution of probabilities, and they are first order differential or difference equations. In addition, these equations are used in Monte Carlo simulations[4]. Quantum mechanical case can be considered by extending classical systems[5]. Thus there have been a variety of applications in mathematical sciences, and its progress continues to attract attention in the literature[6, 7, 8].

Algebraic topology and graph theory have been applied to a variety of sciences and mathematical engineering. Several topological approaches to master equations exist in the literature[9, 10, 11]. Not only master equations, but also random walks on lattices[12], electric circuits[13, 14, 15], and so on, have been studied from the viewpoint of algebraic topology. By introducing inner products for functions on graphs together with coboundary operator, one can define the adjoint of the coboundary operator, and Laplacians as self-adjoint operators[12]. Corresponding operators in the continuous case are useful as proven in the literature of functional analysis[16]. An amalgamation of these mathematical disciplines may be called discrete geometry. It is then of interest to explore how above mentioned operators in discrete geometry can be used for master equations. Also, although discrete diffusion equations have been derived from master equations in some cases, the condition when such diffusion equations can be derived is not known. Since the knowledge of discrete diffusion equations has been accumulated, clarifying such a condition is expected to be fruitful for the study of master equations.

In this paper continuous-time master equations are formulated in terms of functions of chains, where states and transition matrices for master equations are described by chains used in algebraic topology. In particular, probability distribution function is regarded as a function of 0-chain, and transition matrix a function of 1-chain, where a discrete state is expressed as a vertex or a 0-chain. After introducing some
inner products for functions on chains and current as a function on 1-chain, master equations are shown to be written in terms of the adjoint of the coboundary operator acting on the current:

**Claim.** Master equations can be written as a form of a continuity equation (See Theorem 3.1 for details). In addition, under the assumption that the detailed balance conditions hold, an equivalence between master equations and discrete diffusion equations is shown, where the Laplacians are constructed by choosing appropriate measures for inner products:

**Claim.** Master equations under the detailed balance conditions are equivalent to discrete diffusion equations (See Theorem 3.2 for details).

By applying this statement, it is shown that probability distribution functions relax to the equilibrium state (See Corollary 3.1 and Proposition 3.2). By contrast, it is shown that discrete diffusion equations yield master equations (See Proposition 3.1). Also, an isospectral property for non-zero eigenvalues of these Laplacians is given (See Theorem 3.3), and this can be referred to as a supersymmetry [17]. With this supersymmetry dynamical systems for expectation variables are derived without any approximation (See Propositions 3.5 and 3.6).

These theorems, corollary, and so on should be compared with those in the existing literature. In the study of random walks on lattices, chains, functions and their Laplacians are also used [12]. In the literature the probability distribution functions are identified with functions on 1-chains, which is different to the present formalism. The differences appear since transition matrices are introduced for describing state transitions for the present study only. On the other hand, similarities between the present study and existing studies appear due to the use of Laplacians. Laplacian is a self-adjoint operator with respect to an inner product, and brings several properties as well as the case of the standard Riemannian geometry.

In Section 2 some preliminaries are provided in order to keep this paper self-contained. In Section 3 master equations are formulated on graph, and the main claims of this paper and their consequences are provided. Section 4 summarizes this paper and discusses some future studies.

## 2 Preliminaries

Let $G = (V, E)$ be a directed graph with $V$ a vertex set and $E$ an edge set. Throughout this paper, every graph is finite ($\# E < \infty$), connected, and allowed to have loop edges. However parallel edges that will be defined later are excluded from this contribution.

### 2.1 Standard operators

In this subsection, most of notions and notations follow Ref. [12]. However, some variants are also introduced.

For a given edge $e \in E$, the inverse of $e$, the terminus of $e$, and the origin of $e$ are denoted by $\overline{e}$, $t(e) \in V$, and $o(e) \in V$, respectively:

\[
\begin{align*}
\bullet & \xrightarrow{e} \bullet, \quad \text{and} \quad \bullet & \xleftarrow{\overline{e}} \bullet.
\end{align*}
\]

Then, it follows that

\[
t(\overline{e}) = o(e), \quad \text{and} \quad o(\overline{e}) = t(e).
\]

A **loop edge** $e \in E$ is such that $o(e) = t(e)$, and **parallel edges** $e_1, e_2 \in E$ are such that $e_1 \neq e_2$ with $o(e_1) = o(e_2)$ and $t(e_1) = t(e_2)$. Parallel edges are not assumed to exist in any graph in this paper.

Then one defines the groups of 0-**chains** and 1-**chains** on a graph $G$ with coefficients $\mathbb{R}$

\[
\begin{align*}
C_0(G, \mathbb{R}) & := \left\{ \sum_{x \in V} a_x x \biggm| a_x \in \mathbb{R} \right\}, \\
C_1(G, \mathbb{R}) & := \left\{ \sum_{e \in E} a_e e \biggm| a_e \in \mathbb{R} \right\},
\end{align*}
\]
respectively. The spaces of functions on \(C^0(G, \mathbb{R})\) and \(C^1(G, \mathbb{R})\) are denoted by
\[
C^0(G, \mathbb{R}) := \{ f : C_0(G, \mathbb{R}) \to \mathbb{R} \},
\]
\[
C^1(G, \mathbb{R}) := \{ \omega : C_1(G, \mathbb{R}) \to \mathbb{R} \}.
\]
Elements of the subset of \(C^0(G, \mathbb{R})\) being dual to \(C_0(G, \mathbb{R})\) are referred to as 0-cochains. Similarly, elements of the subset of \(C^1(G, \mathbb{R})\) being dual to \(C_1(G, \mathbb{R})\) are referred to as 1-cochains. The set \(C^0(G, \mathbb{R})\) is also denoted by \(\Lambda^0(G, \mathbb{R})\) in this paper. The subsets \(\Lambda^1(G, \mathbb{R}) \subset C^1(G, \mathbb{R})\) and \(S^1(G, \mathbb{R}) \subset C^1(G, \mathbb{R})\) are defined by
\[
\Lambda^1(G, \mathbb{R}) := \{ \omega \in C^1(G, \mathbb{R}) \mid \omega(\partial) = -\omega(\epsilon) \},
\]
\[
S^1(G, \mathbb{R}) := \{ \mu \in C^1(G, \mathbb{R}) \mid \mu(\partial) = \mu(\epsilon) \},
\]
where \(\omega(\epsilon)\) is understood as \(\omega(\epsilon) \in \mathbb{R}\). In what follows, \(C_0(G, \mathbb{R})\) and \(C_1(G, \mathbb{R})\) are often abbreviated as \(C_0(G)\) and \(C_1(G)\), respectively. Similar abbreviations will be adopted.

The boundary operator is defined as
\[
\partial : C_1(G) \to C_0(G), \quad \text{so that} \quad \partial(\epsilon) := t(\epsilon) - o(\epsilon),
\]
and the linearity holds:
\[
\partial(\epsilon + \zeta) = \partial \epsilon + \partial \zeta, \quad \text{and} \quad \partial(\alpha \epsilon) = \alpha \partial \epsilon, \quad \epsilon, \zeta, \alpha \in C_1(G).
\]
The dual of the boundary operator, the coboundary operator, is defined by
\[
d : \Lambda^0(G) \to \Lambda^1(G), \quad \text{so that} \quad (df)(\epsilon) := f(t(\epsilon)) - f(o(\epsilon)),
\]
and the linearity holds:
\[
d(f_1 + f_2) = df_1 + df_2, \quad \text{and} \quad d(af) = adf, \quad a, f, f_1, f_2 \in \Lambda^0(G).
\]
If \(f\) is a 0-cochain, then it follows that \((df)(\epsilon) = f(\partial \epsilon)\). For any loop edge \(\epsilon \in E\), it follows from \(f(t(\epsilon)) = f(o(\epsilon))\) that \((df)(\epsilon) = 0\). This \(df\) is indeed an element belonging to \(\Lambda^1(G)\), as shown below. For any \(f \in \Lambda^0(G)\) and \(\epsilon \in E\), one has that
\[
(df)(\partial \epsilon) = f(t(\partial \epsilon)) - f(o(\partial \epsilon)) = f(o(\epsilon)) - f(t(\epsilon)) = -(df)(\epsilon).
\]
To define inner products,
\[
\langle \cdot, \cdot \rangle_V : \Lambda^0(G) \times \Lambda^0(G) \to \mathbb{R} \quad \text{and} \quad \langle \cdot, \cdot \rangle_E : \Lambda^1(G) \times \Lambda^1(G) \to \mathbb{R},
\]
one introduces some measures. Let \(m_V\) and \(m_E\) be elements of \(C^0(G)\) and \(S^1(G)\), such that
\[
m_V(x) > 0, \quad \forall \ x \in V,
\]
and
\[
m_E(\epsilon) = m_E(\partial \epsilon) > 0, \quad \forall \ \epsilon \in E.
\]
This \(m_E \in S^1(G)\) is referred to as a reversible measure.

The following are inner products
\[
\langle f_1, f_2 \rangle_V := \sum_{x \in V} f_1(x) f_2(x) m_V(x), \quad \forall \ f_1, f_2 \in \Lambda^0(G), \quad (1)
\]
\[
\langle \omega_1, \omega_2 \rangle_E := \frac{1}{2} \sum_{\epsilon \in E} \omega_1(\epsilon) \omega_2(\epsilon) m_E(\epsilon), \quad \forall \ \omega_1, \omega_2 \in \Lambda^1(G). \quad (2)
\]
Associated with this set of inner products, one defines the adjoint of the coboundary operator $d^\dagger : \Lambda^1(G) \to \Lambda^0(G)$. This $d^\dagger$ is also referred to as co-derivative in this paper, which is explicitly written as

$$ (d^\dagger \omega)(x) = \frac{-1}{m_V(x)} \sum_{e \in E_x} \omega(e) m_E(e), \quad (3) $$

where

$$ E_x := \{ e \in E \mid o(e) = x \}. $$

It follows from (3) that the linearity holds:

$$ d^\dagger (\omega_1 + \omega_2) = d^\dagger \omega_1 + d^\dagger \omega_2, \quad \text{and} \quad d^\dagger (a \omega) = a d^\dagger \omega, \quad a \in \mathbb{R}, \quad \omega, \omega_1, \omega_2 \in \Lambda^1(G). $$

This operator $d^\dagger$ is indeed the adjoint one of $d$ as shown below.

**Lemma 2.1.**

$$ \langle d f, \omega \rangle_E = \langle f, d^\dagger \omega \rangle_V. $$

**Proof.** Straightforward calculations yield

$$ (d f, \omega)_E = \frac{1}{2} \sum_{e \in E} (d f)(e) \omega(e) m_E(e) = \frac{1}{2} \sum_{e \in E} [f(t(e)) - f(o(e))] \omega(e) m_E(e) $$

$$ = \frac{-1}{2} \sum_{e \in E} [f(o(e)) \omega(e) - f(t(e)) \omega(e)] m_E(e) $$

$$ = \frac{-1}{2} \sum_{e \in E} \{f(o(e)) \omega(e) m_E(e) + f(o(\overline{\tau})) \omega(\overline{\tau}) m_E(\overline{\tau})\} $$

$$ = \sum_{x \in V} f(x) (d^\dagger \omega)(x) m_V(x) = \langle f, d^\dagger \omega \rangle_V. $$

$\square$

**Remark 2.1.** A operator analogous to $-d^\dagger$ in the continuous standard Riemannian geometry is referred to as divergence. Thus, $-d^\dagger$ can be referred to as divergence on graph[18]. This operator $\Lambda^1(G) \to \Lambda^0(G)$ is denoted by

$$ \text{div} := -d^\dagger. \quad (4) $$

The Laplacian acting on $\Lambda^0(G)$, $\Delta_V : \Lambda^0(G) \to \Lambda^0(G)$, is defined as

$$ \Delta_V := -d^\dagger d. \quad (5) $$

The explicit form of its action is obtained as follows. By putting $\omega = df$ for $f \in \Lambda^0(G)$, one has

$$ (\Delta_V f)(x) = -(d^\dagger df)(x) = -(d^\dagger \omega)(x) $$

$$ = \frac{1}{m_V(x)} \sum_{e \in E_x} \omega(e) m_E(e) = \frac{1}{m_V(x)} \sum_{e \in E_x} (df)(e) m_E(e). $$

Since $d$ and $d^\dagger$ are linear operators, $\Delta_V$ is also a linear operator:

$$ \Delta_V (f_1 + f_2) = \Delta_V f_1 + \Delta_V f_2, \quad \text{and} \quad \Delta_V (a f) = a \Delta_V f, \quad a \in \mathbb{R}, \ f, f_1, f_2 \in \Lambda^0(G). $$

Moreover, it follows that

$$ (\Delta_V f)(x) = \frac{1}{m_V(x)} \sum_{e \in E_x} [f(t(e)) - f(o(e))] m_E(e). \quad (6) $$
To see a link between (6) and a well-known form of discrete Laplacian, choose \( m_V(x) = \delta \) and \( m_E(e) = 1 \) for all \( x \in V \) and \( e \in E \), where \( \delta > 0 \) is a constant. Then consider the graph

\[
\begin{array}{ccc}
  & \bullet & \\
  \bullet & \leftrightarrow & \bullet \\
  x' & \leftrightarrow & x \\
  & \leftrightarrow & \\
  e' & \leftrightarrow & e'' \\
  x & \leftrightarrow & x'' \\
\end{array}
\]

For this graph, one has the well-known form:

\[
(\Delta_V f)(x) = \frac{1}{\delta} \left[ f(x') - 2f(x) + f(x'') \right].
\]

By taking the limit \( \delta \to 0 \) appropriately, one has the second derivative of \( f \). See Ref. [20] for another link between this Laplacian \( \Delta_V \) and the so-called adjacency operator.

The operator \( \Delta_V \) is self-adjoint as shown below.

**Lemma 2.2.**

\[
\langle \Delta_V f_1, f_2 \rangle_V = \langle f_1, \Delta_V f_2 \rangle_V.
\]

**Proof.** It follows that

\[
\langle \Delta_V f_1, f_2 \rangle_V = \langle -d^\dagger d f_1, f_2 \rangle_V = \langle df_1, -d f_2 \rangle_E = \langle f_1, -d^\dagger d f_2 \rangle_V = \langle f_1, \Delta_V f_2 \rangle_V.
\]

The Laplacian acting on \( \Lambda^1(G) \), \( \Delta_E : \Lambda^1(G) \to \Lambda^1(G) \), is defined as

\[
\Delta_E := -d d^\dagger.
\]  

This operator is self-adjoint as shown below.

**Lemma 2.3.**

\[
\langle \Delta_E \omega_1, \omega_2 \rangle_E = \langle \omega_1, \Delta_E \omega_2 \rangle_E.
\]

**Proof.** It can be proven by straightforward calculations. To this end, we put \( f_1 = d^\dagger \omega_1 \in \Lambda^0(G) \) and then it follows that

\[
\langle \Delta_E \omega_1, \omega_2 \rangle_E = \langle -d d^\dagger \omega_1, \omega_2 \rangle_E = \langle -d f_1, \omega_2 \rangle_E = \langle f_1, -d^\dagger \omega_2 \rangle_V = \langle d^\dagger \omega_1, -d^\dagger \omega_2 \rangle_V = \langle \omega_1, -d d^\dagger \omega_2 \rangle_E = \langle \omega_1, \Delta_E \omega_2 \rangle_E.
\]

Most of the operators and their properties discussed so far are well-known[12]. On the other hand, those discussed in the next subsection are not standard ones.

### 2.2 Operators for master equations

To discuss master equations, one introduces

\[
\begin{align*}
C^1_A(G, \mathbb{R}) & := \left\{ \omega \in C^1(G, \mathbb{R}) \mid \omega(\overline{e}) = -(\varphi(e))^{-1} \omega(e), \quad \varphi \in C^1_R(G, \mathbb{R}) \right\}, \\
C^1_R(G, \mathbb{R}) & := \left\{ \varphi \in C^1(G, \mathbb{R}) \middle| \varphi(\overline{e}) = \frac{1}{\varphi(e)} \right\}.
\end{align*}
\]  

(8)

The operator associated with \( \varphi \in C^1_R(G) \) is defined as

\[
d_\varphi : C^0(G, \mathbb{R}) \to C^1_A(G, \mathbb{R}), \quad \text{such that} \quad (d_\varphi f)(e) := \varphi(e) f(t(e)) - f(o(e)),
\]

(9)
and the linearity in \( f \) holds:

\[
\mathsf{d}_\varphi (f_1 + f_2) = \mathsf{d}_\varphi f_1 + \mathsf{d}_\varphi f_2, \quad \text{and} \quad \mathsf{d}_\varphi (af) = a \mathsf{d}_\varphi f, \quad a \in \mathbb{R}, \quad f, f_1, f_2 \in \mathcal{C}_0^0(G).
\]

Condition in \( \text{Remark 2.2} \) is to guarantee the property \( \mathsf{d}_\varphi f \in \mathcal{C}_A^1(G) \) for \( f \in \mathcal{C}_0^0(G) \),

\[
(\mathsf{d}_\varphi f)(\mathcal{T}) = \frac{-1}{\varphi(e)} (\mathsf{d}_\varphi f)(e).
\]

The above equality is verified as

\[
(\mathsf{d}_\varphi f)(\mathcal{T}) = \varphi(\mathcal{T}) f(t(\mathcal{T})) - f(o(\mathcal{T})) = \varphi(\mathcal{T}) f(o(e)) - f(t(e)) = \frac{1}{\varphi(e)} f(o(e)) - f(t(e)) = \frac{-1}{\varphi(e)} (\mathsf{d}_\varphi f)(e).
\]

**Remark 2.2.** In the case that \( \varphi \) is such that \( \varphi(e) = 1 \) for any \( e \in E \), one has that \( \mathsf{d}_\varphi f = df \) for any \( f \in \mathcal{C}_0^0(G) \). The operator \( \mathsf{d}_\varphi \) is an analogue of the one introduced in Ref. [19].

As well as the case for \( \langle \ , \ \rangle_E \), one introduces an *inner product* on \( \mathcal{C}_A^1(G) \)

\[
\langle \ , \ \rangle^\varphi_E : \mathcal{C}_A^1(G) \times \mathcal{C}_A^1(G) \to \mathbb{R},
\]

associated with a reversible measure \( m_E \in \mathcal{S}_1(G) \). The value of the inner product is the same as \( [2] \):

\[
(\omega_1, \omega_2)^\varphi_E := \frac{1}{2} \sum_{e \in E} \omega_1(e) \omega_2(e) m_E(e) = (\omega_1, \omega_2)_E.
\]

With this inner product, one introduces the *co-derivative* on \( \mathcal{C}_R^1(G) \), \( \mathsf{d}_\varphi^\dagger : \mathcal{C}_A^1(G) \to \mathcal{C}_0^0(G) \) with \( \varphi \in \mathcal{C}_R^1(G) \) such that

\[
(\mathsf{d}_\varphi^\dagger \omega)(x) := \frac{-1}{m_V(x)} \sum_{e \in E_x} \omega(e) m_E(e).
\]

**Lemma 2.4.** For \( \varphi \in \mathcal{C}_R^1(G) \), \( \omega \in \mathcal{C}_A^1(G) \), and \( f \in \mathcal{C}_0^0(G) \), one has

\[
(\mathsf{d}_\varphi f, \omega)^\varphi_E = \langle f, \mathsf{d}_\varphi^\dagger \omega \rangle_V,
\]

**Proof.** Substituting

\[
\varphi(e) = \frac{1}{\varphi(e)}, \quad f(t(e)) = f(o(e)), \quad \omega(e) = -\varphi(\mathcal{T}) \omega(\mathcal{T}), \quad \text{and} \quad m_E(e) = m_E(\mathcal{T}),
\]

into

\[
(\mathsf{d}_\varphi f, \omega)^\varphi_E = \frac{1}{2} \sum_{e \in E} [\varphi(e) f(t(e)) - f(o(e))] \omega(e) m_E(e)
\]

\[
= \frac{-1}{2} \sum_{e \in E} [f(o(e)) \omega(e) - \varphi(e) f(t(e)) \omega(e)] m_E(e),
\]

one has

\[
(\mathsf{d}_\varphi f, \omega)^\varphi_E = \frac{-1}{2} \sum_{e \in E} \left[ f(o(e)) \omega(e) - \frac{1}{\varphi(e)} f(o(e)) (-\varphi(\mathcal{T}) \omega(\mathcal{T})) \right] m_E(e)
\]

\[
= \frac{-1}{2} \sum_{e \in E} [f(o(e)) \omega(e) m_E(e) + f(o(\mathcal{T})) \omega(\mathcal{T}) m_E(\mathcal{T})] = \langle f, \mathsf{d}_\varphi^\dagger \omega \rangle_V.
\]

\[ \square \]
Master equations are written as

\[ 3 \] Master equations

Introduce a graph

\[ 3.1 \] Graph formulation

Main claims of this paper and their consequences are then provided.

Although the following operators will not be used in Section 3 the Laplacian is discussed for the sake of completeness. The Laplacian \( \Delta^\varphi_V : C^0(G) \to C^0(G) \) is defined as

\[ \Delta^\varphi_V := -d^\varphi_v^dd^\varphi_v. \]

The explicit form of its action is obtained as follows. By putting \( \omega = d^\varphi f \) for \( f \in C^0(G) \), one has

\[
(\Delta^\varphi_V f)(x) = - (d^\varphi_v^dd^\varphi_v f)(x) = - (d^\varphi_v^d \omega)(x)
= \frac{1}{m_V(x)} \sum_{e \in E_x} \omega(e) m_E(e)
= \frac{1}{m_V(x)} \sum_{e \in E_x} (d^\varphi_v f)(e) m_E(e)
= \frac{1}{m_V(x)} \sum_{e \in E_x} [\varphi(e) f(t(e)) - f(o(e))] m_E(e).
\]

This operator is self-adjoint:

\[ \langle \Delta^\varphi_V f_1, f_2 \rangle_V = \langle f_1, \Delta^\varphi_V f_2 \rangle_V, \]

since

\[ \langle \Delta^\varphi_V f_1, f_2 \rangle_V = \langle -d^\varphi_v^d \varphi_v f_1, f_2 \rangle_V = \langle d^\varphi_v f_1, -d^\varphi_v f_2 \rangle_V = \langle f_1, -d^\varphi_v f_2 \rangle_V = \langle f_1, \Delta^\varphi_V f_2 \rangle_V. \]

3 Master equations

Master equations are written as

\[
\frac{d}{dt} p_t(x) = - \sum_{x' \neq x} w_{x \to x'} p_t(x) + \sum_{x' \neq x} w_{x' \to x} p_t(x'),
\]

(10)

where \( \{x\} \) are discrete states, \( \{w_{x \to y}\} \) a transition matrix that describes the transition rate from a state \( x \) to another \( y \). The equilibrium state at \( x \) is denoted by \( p^\text{eq}(x) \). In what follows, these objects, \( \{x\}, \{w_{x \to y}\}, \) and \( \{p^\text{eq}(x)\} \), are treated as given data.

In this section, a graph formulation of master equations is shown in terms of objects developed in Section 2.

Main claims of this paper and their consequences are then provided.

3.1 Graph formulation

Introduce a graph \( G = (V, E) \) associated with the given data \( \{x\} \) and \( \{w_{x \to y}\} \) so that

- \( x \in V \)
- \( w(e) = w_{x \to y} \) for \( e \in E \) such that \( t(e) = y \) and \( o(e) = x \)
- \( \forall e \in E \). Also, if \( w(\varpi) = w_{y \to x} \) does not exist, then let \( w \) be such that \( w(\varpi) = 0 \).

Then, regard \( p_t \) and \( w \) as follows.

- \( p_t \in A^0(G) \) so that \( p_t(x) \in \mathbb{R} \) for any \( x \in V \).
- \( w \in C^1(G) \) so that \( w(e) \in \mathbb{R} \) for any \( e \in E \).

The master equations (10) can be written in terms of these graph terminologies as

\[
\frac{d}{dt} p_t(x) = \sum_{e \in E_x} \left[-p_t(o(e)) w(e) + p_t(t(e)) w(\varpi)\right].
\]

An example of this set of the procedures is given below.
Example 3.1. Consider the set of master equations whose total number of states is 2,
\[ \frac{d}{dt} p_t(x_1) = -w_{1\to 2} p_t(x_1), \quad \frac{d}{dt} p_t(x_2) = w_{1\to 2} p_t(x_1), \]
where \( w_{1\to 2} \) is a positive constant. This set of equations can be written in terms of the graph
\[
\bullet \leftrightarrow e' \leftrightarrow \bullet \quad , \quad \text{where} \quad w(e') = w_{1\to 2}, \quad \text{and} \quad w(e'') = 0.
\]
A consistency between this graph expression and the given set of master equations is verified as follows. Since \( \{e'\} = E_1, \{e''\} = E_2, x_1 = o(e') = t(e''), x_2 = t(e') = o(e''), \) and \( e'' = e' \), (11) yields
\[
\frac{d}{dt} p_t(x_1) = -p_t(o(e')) w(e') + p_t(t(e')) w(e),
\]
and
\[
\frac{d}{dt} p_t(x_2) = -p_t(o(e'')) w(e'') + p_t(t(e'')) w(e) = p_t(x_1) w_{1\to 2}.
\]

In what follows, master equations are rewritten. To this end, one defines \( J_t \in C^1(G) \) and \( I_t \in \Lambda^1(G) \) so that
\[
J_t(e) := -p_t(o(e)) w(e), \quad I_t(e) := -J_t(e) + J_t(\bar{e}).
\]
One verifies that \( J_t(\bar{e}) = -J_t(e) \), and that \( -I_t(e) \) is the summand in the right hand side of (11),
\[
-I_t(e) = J_t(e) - J_t(\bar{e}) = -p_t(o(e)) w(e) + p_t(t(e)) w(\bar{e}) = -p_t(o(e)) w(e) + p_t(t(e)) w(\bar{e}),
\]
and then,
\[
\frac{d}{dt} p_t(x) = -\sum_{e \in E_x} I_t(e).
\]
The physical meaning of the case of \( I_t(e) = 0 \) for a given \( e \in E \) is that the probability flow or current is locally balanced. If the condition \( w(\bar{e}) = w(e) \) is satisfied for all \( e \in E \), then one has
\[
I_t(e) = -w(e) (dp_t)(e).
\]

To describe some measures, introduce
\[
1_V \in \Lambda^0(G), \quad \text{and} \quad 1_E \in S^1(G) \quad \text{such that} \quad 1_V(x) = 1 \quad \text{and} \quad 1_E(e) = 1, \quad \forall x \in V, \forall e \in E.
\]
In terms of these objects, one has the following.

Theorem 3.1. (Continuity equation for master equations). Choose \( m_V \in \Lambda^0(G) \) and \( m_E \in S^1(G) \) to be
\[
m_V = 1_V, \quad \text{and} \quad m_E = 1_E.
\]
Then (11) is identical to
\[
\frac{d}{dt} p_t = d^\dagger I_t.
\]

Proof. With the choices \( m_V \) and \( m_E \), the co-derivative \( d^\dagger \) defined in (3) reduces to the one such that
\[
(d^\dagger \omega)(x) = -\sum_{e \in E_x} \omega(e), \quad \omega \in \Lambda^1(G).
\]
From this and (11), one has
\[
\frac{d}{dt} p_t(x) = -\sum_{e \in E_x} I_t(e) = (d^\dagger I_t)(x),
\]
for any \( x \in V \). Thus, the desired expression is obtained. □
It is worth mentioning that (13) is written as a form of a continuity equation in continuum mechanics

$$\frac{d}{dt} p_t + \text{div} \mathcal{I}_t = 0,$$

where $\text{div} : \Lambda^1(G) \to \Lambda^0(G)$ has been defined in [4].

Theorem 3.1 deals with $p_t \in \Lambda^0(G)$. On the other hand, an equality on $\Lambda^1(G)$ associated with the master equations is obtained from (13) as

$$\frac{d}{dt} d^\dagger I_t = - \Delta_E I_t,$$  \hspace{1cm} (14)

where $\Delta_E$ has been defined in [7].

### 3.1.1 Expectation values

Much attention is devoted to expectation values with respect to $p_t$ in applications of master equations. To discuss such expectation values, one introduces $O_0 \in \Lambda^0(G)$ that is to be summed over vertexes at fixed time. In this subsection, $m_V \in \Lambda^0(G)$ and $m_E \in S^1(G)$ are chosen as [12].

The expectation value of $O_0$ with respect to $p_t$ is denoted by

$$E_p[O_0] := \sum_{x \in V} p_t(x) O_0(x),$$

which is written in terms of the inner product with (12) as

$$E_p[O_0] = \langle p_t, O_0 \rangle_V.$$

From the master equations [13] and Lemma 2.1, the time-development of

$$\langle O_0 \rangle_V := E_p[O_0],$$

is described by

$$\frac{d}{dt} \langle O_0 \rangle_V = \langle d^\dagger I_t, O_0 \rangle_V = \langle I_t, dO_0 \rangle_E.$$  \hspace{1cm} (16)

**Remark 3.1.** Combining the sum over vertexes at fixed time $t$,

$$E_p[1_V] = \langle p_t, 1_V \rangle_V = \sum_{x \in V} p_t(x),$$

the identity

$$(d1_V)(e) = 1_V(t(e)) - 1_V(o(e)) = 1 - 1 = 0,$$

and [10], one verifies that the derivative of $\langle 1_V \rangle_V = E_p[1_V]$ with respect to time vanishes:

$$\frac{d}{dt} E_p[1_V] = \frac{d}{dt} \langle 1_V \rangle_V = \langle \dot{p}_t, 1_V \rangle_V = \langle I_t, d1_V \rangle_V = 0,$$

where $\dot{p}_t$ denotes derivative of $p_t$ with respect to time $t$, $\dot{p}_t = dp_t/dt$.

**Remark 3.2.** For $O' = O + c 1_V \in \Lambda^0(G)$ with $c \in \mathbb{R}$ being constant, one has

$$\frac{d}{dt} \langle O' \rangle_V = \langle \dot{I}_t, dO' \rangle_E = \frac{d}{dt} \langle O \rangle_V.$$  

The inner product for $O_1^A \in \Lambda^1(G)$ and $O_1^B \in \Lambda^1(G)$ is $\langle O_1^A, O_1^B \rangle_E$. In the case where $O_1^B = d\dot{p}_t$ and $O_1^A$ that does not depend on time $t$, one uses (14) to obtain

$$\frac{d}{dt} \langle O_1^A, d\dot{p}_t \rangle_E = \langle O_1^A, d\dot{p}_t \rangle_E = - \langle O_1^A, \Delta_E I_t \rangle_E.$$
3.2 Detailed balance conditions

Let $p^{eq}$ be a probability distribution function so that

$$\sum_{x} p^{eq}(x) = 1.$$  

Impose for any connected states $x$ and $y$

$$p^{eq}(x) w_{x \to y} = p^{eq}(y) w_{y \to x}, \quad (17)$$

which are known as the detailed balance conditions. For the stationary solution $p^{eq}(x)$ of the master equations, $dp^{eq}(x)/dt = 0$, one has from (10) that

$$\sum_{x'} \left[ w_{x \to x'} p^{eq}(x) - w_{x' \to x} p^{eq}(x') \right] = 0. \quad (18)$$

If (17) holds, then (18) is satisfied. In this subsection (17) is assumed to hold.

These conditions are written in the graph theoretic language as

$$\sum_{x \in V} p^{eq}(x) = 1,$$

and

$$p^{eq}(o(e)) w(e) = p^{eq}(t(e)) w(\overline{e}). \quad (19)$$

The later leads to

$$w(\overline{e}) = \pi(e) w(e), \quad \text{where} \quad \pi(e) := \frac{p^{eq}(o(e))}{p^{eq}(t(e))} = \frac{1}{\pi(\overline{e})}.$$  

Thus, $\pi(e) = 1$ for any loop edge $e \in E$, and

$$\pi \in C^{1}_{R}(G).$$

Also, the $m_{E} \in S^{1}(G)$ specified such that

$$m_{E}(e) = p^{eq}(o(e)) w(e),$$

has the property that $m_{E}(e) = m_{E}(\overline{e}) > 0$ due to (19). Thus, this $m_{E} \in S^{1}(G)$ can be used for a reversible measure defining (2).

The master equations (11) under the detailed balance conditions are written as

$$\frac{d}{dt} p_{t}(x) = \sum_{e \in E_{x}} w(e) \left[ \pi(e) p_{t}(t(e)) - p_{t}(o(e)) \right]. \quad (20)$$

This can be written in terms of $d_{\pi}$, that is (9) with $\varphi = \pi$, as

$$\frac{d}{dt} p_{t}(x) = \sum_{e \in E_{x}} w(e) \left[ (d_{\pi} p_{t})(e) \right] = - \sum_{e \in E_{x}} w(\overline{e}) \left[ (d_{\pi} p_{t})(\overline{e}) \right]. \quad (21)$$

The above equations involve $d_{\pi}$. In (21), there is no loop edge contribution since $\pi(e) = 1$ and $(d_{\pi} p_{t})(e) = 0$ for any loop edge $e \in E$, (See Remark 2.2). Although there is no self-adjoint operator in (21), there exists a way to express master equations in terms of a Laplacian, which is accomplished by a change of variables.

The following is the main theorem in this paper.

**Theorem 3.2. (Diffusion equations from master equations).** Consider the case that
1. the stationary solution does not vanish, that is, \( p^{\text{eq}}(x) \neq 0 \), for all \( x \in V \), and

2. the detailed balance conditions are satisfied, that is, \( w(e) \) satisfies (19).

Choose the measures \( m_V \in \Lambda^0(G) \) and \( m_E \in S^1(E) \) to be

\[
m_V(x) = p^{\text{eq}}(x), \quad \text{and} \quad m_E(e) = w(e) p^{\text{eq}}(o(e)),
\]

respectively. Then introduce \( \psi_t \in \Lambda^0(G) \) such that

\[
p_t(x) = p^{\text{eq}}(x) \psi_t(x), \quad \forall x \in V.
\]

This function \( \psi_t \) satisfies the diffusion equations

\[
\frac{d}{dt} \psi_t = \Delta_V \psi_t,
\]

where \( \Delta_V \) has been defined in (6).

**Proof.** Substituting (23) into (20), and using the definition of the Laplacian (5), one can complete the proof. The details are as follows.

From (23) and (20), and \( dp^{\text{eq}}(x)/dt = 0 \), one has

\[
p^{\text{eq}}(x) \frac{d}{dt} \psi_t(x) = \sum_{e \in E_x} w(e) \left[ \frac{p^{\text{eq}}(t(e))}{p^{\text{eq}}(t(e))} p^{\text{eq}}(t(e)) \psi_t(t(e)) - p^{\text{eq}}(o(e)) \psi_t(o(e)) \right] = \sum_{e \in E_x} w(e) \left[ p^{\text{eq}}(o(e)) \psi_t(t(e)) - p^{\text{eq}}(o(e)) \psi_t(o(e)) \right],
\]

from which

\[
\frac{d}{dt} \psi_t(x) = \frac{1}{p^{\text{eq}}(x)} \sum_{e \in E_x} w(e) p^{\text{eq}}(o(e)) \left[ \psi_t(t(e)) - \psi_t(o(e)) \right].
\]

Since the action of the Laplacian (6) with (22) on \( \psi_t \) is

\[
(\Delta_V \psi_t)(x) = \sum_{e \in E_x} w(e) \left[ p^{\text{eq}}(o(e)) \psi_t(t(e)) - \psi_t(o(e)) \right],
\]

one obtains (24).

**Remark 3.3.** The explicit expression (25) reduces further. It follows from \( o(e) = x \) for \( e \in E_x \) that

\[
(\Delta_V \psi_t)(x) = \sum_{e \in E_x} w(e) \left[ \psi_t(t(e)) - \psi_t(o(e)) \right] = \sum_{e \in E_x} w(e) (d\psi_t)(e).
\]

**Remark 3.4.** The conservation law of the sum

\[
\sum_{x \in V} p_t(x) = 1, \quad \forall t \in \mathbb{R}
\]

can be shown below. With (24), one can show

\[
\frac{d}{dt} \sum_{x \in V} p_t(x) = \sum_{x \in V} p^{\text{eq}}(x) \left( \frac{d}{dt} \psi_t \right)(x) = \sum_{x \in V} p^{\text{eq}}(x) (\Delta_V \psi_t)(x)
\]

\[
= \langle \Delta_V \psi_t, 1_V \rangle_V = \langle \psi_t, \Delta_V 1_V \rangle_V = 0.
\]

**Remark 3.5.** Theorem (3.2) and (6) indicate that (24) does not involve any loop edge. This property also holds for the equations written in terms of \( p_t(x) \).
The following is an example of how this formulation can be applied to a model studied in nonequilibrium statistical mechanics.

**Example 3.2.** (Kinetic Ising model without spin-coupling, [21]). Let \( \sigma = \pm 1 \) be a spin variable, and \( p_{\text{Ising}}(\sigma) \) an equilibrium distribution of \( \sigma \). Consider the master equations

\[
\frac{d}{dt} p_{t, \text{Ising}}(\sigma) = -w_{\sigma \rightarrow -\sigma} p_{t, \text{Ising}}(\sigma) + w_{-\sigma \rightarrow \sigma} p_{t, \text{Ising}}(-\sigma), \quad \sigma = \pm 1,
\]

where the detailed balance condition is satisfied:

\[
w_{\sigma \rightarrow -\sigma} p_{\text{Ising}}^{\text{eq}}(\sigma) = w_{-\sigma \rightarrow \sigma} p_{\text{Ising}}^{\text{eq}}(-\sigma).
\]

This set of master equations can be written in terms of the graph

\[
\text{\includegraphics[width=\textwidth]{graph}}
\]

where the detailed balance condition is satisfied:

\[
w_{\sigma \rightarrow -\sigma} p_{\text{Ising}}^{\text{eq}}(\sigma) = w_{-\sigma \rightarrow \sigma} p_{\text{Ising}}^{\text{eq}}(-\sigma).
\]

Introduce \( \psi_{t, \text{Ising}} \) such that

\[
p_{t, \text{Ising}}(\sigma) = p_{\text{Ising}}^{\text{eq}}(\sigma) \psi_{t, \text{Ising}}(\sigma), \quad \sigma = \pm 1.
\]

With these variables and the detailed balance condition

\[
w_{\sigma \rightarrow -\sigma} = w_{-\sigma \rightarrow \sigma} \frac{p_{\text{Ising}}^{\text{eq}}(\sigma)}{p_{\text{Ising}}^{\text{eq}}(-\sigma)},
\]

one has that

\[
p_{\text{Ising}}^{\text{eq}}(\sigma) \frac{d}{dt} \psi_{t, \text{Ising}}(\sigma) = w_{\sigma \rightarrow -\sigma} \left[ -p_{\text{Ising}}^{\text{eq}}(\sigma) \psi_{t, \text{Ising}}(\sigma) + p_{\text{Ising}}^{\text{eq}}(\sigma) p_{\text{Ising}}^{\text{eq}}(-\sigma) \psi_{t, \text{Ising}}(-\sigma) \right],
\]

from which

\[
\frac{d}{dt} \psi_{t, \text{Ising}}(\sigma) = w_{\sigma \rightarrow -\sigma} \left[ \psi_{t, \text{Ising}}(-\sigma) - \psi_{t, \text{Ising}}(\sigma) \right], \quad \sigma = \pm 1.
\]

This derived set of equations is consistent with (26).

To investigate the long-time behavior of the system

\[
\frac{d}{dt} \psi_t = F(\psi_t), \quad \text{where} \quad F(\psi) := \Delta V \psi, \quad \in \Lambda^0(G)
\]

that is (24), the following Lemmas will be used.

**Lemma 3.1.** The non-trivial solutions, \( \psi(0)(x) \neq 0 \), to the equations \( \Delta V \psi(0)(x) = 0 \) for any \( x \in V \) are \( \psi(0)(x) = \psi_0(0) \), where \( \psi_0(0) \) is constant.

**Proof.** For general \( \psi \), it follows that

\[
\langle \Delta V \psi, \psi \rangle_V = -\langle d\psi, d\psi \rangle_E \leq 0.
\]

The equality holds only when \( d\psi = 0 \). It implies that

\[
\Delta V \psi(0) = 0 \quad \iff \quad d\psi(0) = 0.
\]

Since

\[
(\psi(0))(e) = \psi(0)(t(e)) - \psi(0)(o(e)) = 0, \quad \forall e \in E,
\]

and the assumption that the graph is connected, the solution is \( \psi(0)(x) = \psi_0(0) \) with \( \psi_0(0) \) being constant. \( \square \)
Lemma 3.1 states that the steady state \( \psi(0) \) for (24), \( d\psi(0)/dt = 0 \), is \( \psi(0) = \psi_{0}^{(0)}1_{V} \). In other words, this \( \psi_{0}^{(0)}1_{V} \) forms a fixed point set for this system. The following describes the stability for \( \psi = \psi_{0}^{(0)}1_{V} \).

**Lemma 3.2.** The state \( \psi = \psi_{0}^{(0)}1_{V} \) is asymptotically stable for (29).

**Proof.** Introduce the dynamical system for \( \tilde{\psi} \in \Lambda_{0}(G) \),

\[
\frac{d}{dt} \tilde{\psi}_{t} = \tilde{F} \left( \tilde{\psi}_{t} \right), \quad \tilde{F} \left( \tilde{\psi} \right) := \Delta_{V} \left( \tilde{\psi} + \psi_{0}^{(0)}1_{V} \right) \in \Lambda_{0}(G) \tag{30}
\]

where \( \tilde{\psi} = \psi_{0}^{(0)}1_{V} \) forms a fixed point set for (29). It follows from Lemma 3.1 that this \( \tilde{F}(\tilde{\psi}) \) has the properties that

\[
\tilde{F} \left( \tilde{\psi} \right) = F \left( \tilde{\psi} + \psi_{0}^{(0)}1_{V} \right), \quad \text{and} \quad \tilde{F}(0) = F(\phi_{0}^{(0)}1_{V}) = \Delta_{V}(\psi_{0}^{(0)}1_{V}) = 0_{V},
\]

where \( 0_{V} \) is the element of \( \Lambda_{0}(G) \) such that \( 0_{V}(x) = 0 \) for any \( x \in V \). The later states that the fixed point set for (30) is expressed by \( \tilde{\psi} = 0_{V} \).

Then define \( \mathcal{L} \) acting on \( \Lambda_{0}(G) \) such that

\[
\mathcal{L} \left( \tilde{\psi}_{t} \right) = \frac{1}{2} \left\langle \tilde{\psi}_{t}, \tilde{\psi}_{t} \right\rangle_{V}.
\]

It immediately follows that \( \mathcal{L}(\tilde{\psi}_{t}) \geq 0 \) for any \( t \in \mathbb{R} \). In addition, one has that \( d\mathcal{L}(\tilde{\psi}_{t})/dt \leq 0 \) for any \( t \in \mathbb{R} \), since

\[
\frac{d}{dt} \mathcal{L} \left( \tilde{\psi}_{t} \right) = \left\langle \frac{d}{dt} \tilde{\psi}_{t}, \tilde{\psi}_{t} \right\rangle_{V} = \left\langle \Delta_{V} \left( \tilde{\psi}_{t} + \psi_{0}^{(0)}1_{V} \right), \tilde{\psi}_{t} \right\rangle_{V} = -\left\langle d^{\dagger}d \tilde{\psi}_{t}, \tilde{\psi}_{t} \right\rangle_{V} = -\left\langle d\tilde{\psi}_{t}, d\tilde{\psi}_{t} \right\rangle_{E} \leq 0.
\]

From these properties, \( \mathcal{L}(\tilde{\psi}_{t}) \) is a Lyapunov function. Applying these statements to the Lyapunov stability theorem, one has that \( \tilde{\psi}_{t} = 0_{V} \) in the dynamical system (30) is asymptotically stable. This yields that \( \psi_{0}^{(0)}1_{V} \) is asymptotically stable for (29).

Then one has the following.

**Corollary 3.1.** (Convergence of master equations).

\[
\psi_{0}^{(0)} = 1, \quad \text{and} \quad \lim_{t \to \infty} p_{t}(x) = p_{eq}(x), \quad \forall x \in V.
\]

**Proof.** It follows from Lemmas 3.1 and 3.2 that

\[
\lim_{t \to \infty} p_{t}(x) = p_{eq}(x), \quad \lim_{t \to \infty} \psi_{t}(x) = p_{eq}(x) \psi_{0}^{(0)}.
\]

This and the normalization condition \( \sum_{x \in V} p_{eq}(x) = 1 \) lead to

\[
\sum_{x \in V} \lim_{t \to \infty} p_{t}(x) = \psi_{0}^{(0)} \sum_{x \in V} p_{eq}(x) = \psi_{0}^{(0)}.
\]

On the other hand, the conservation law (27) holds. Thus, the left hand side of the equation above is unity:

\[
\psi_{0}^{(0)} = 1.
\]

Therefore,

\[
\lim_{t \to \infty} p_{t}(x) = p_{eq}(x).
\]
It has been shown that discrete diffusion equations are derived from master equations under the conditions that the detailed balance conditions are satisfied. Its converse statement under these conditions also holds.

**Proposition 3.1.** (From diffusion equations to master equations). Let \( w \in C^1(G) \) be a transition matrix, and \( p^\text{eq} \in \Lambda^0(G) \) an equilibrium distribution function. Assume that the detailed balance conditions [14] hold. Choose \( m_V \in \Lambda^0(G) \) and \( m_E \in S^1(G) \) so that \( m_V(x) = p^\text{eq}(x) \) and \( m_E(e) = w(e)p^\text{eq}(o(e)) \) as in [22]. Then, the diffusion equations [22] yield the master equations.

**Proof.** Introduce the inner products (1) and (2). Also, introduce (22). Then, the diffusion equations (24) yield the master equations.

Proof. Introduce the inner products (11) and (2). Also, introduce \( p_t \in \Lambda^0(G) \) that depends on \( t \in \mathbb{R} \) such that \( p_t(x) = p^\text{eq}(x)\psi_t(x) \). Then multiplying both sides of the diffusion equations by \( p^\text{eq}(x) \), one has

\[
p^\text{eq}(x) \frac{d}{dt} \psi_t(x) = \sum_{e \in E_x} p^\text{eq}(o(e)) w(e) \left[ \psi_t(t(e)) - \psi_t(o(e)) \right], \quad \forall \ x \in V.
\]

With (19), the equations above can be written as

\[
p^\text{eq}(x) \frac{d}{dt} \psi_t(x) = \sum_{e \in E_x} p^\text{eq}(t(e)) w(\tau) \psi_t(t(e)) - p^\text{eq}(o(e)) w(e) \psi_t(o(e)).
\]

This can also be written in terms of \( p_t(x) = p^\text{eq}(x)\psi_t(x) \) as

\[
\frac{d}{dt} p_t(x) = \sum_{e \in E_x} \left[ -p_t(o(e)) w(e) + p_t(t(e)) w(\tau) \right].
\]

\( \square \)

So far, basic statements of the system have been given. These include asymptotic behaviour of the diffusion equations, Lemma[5,2]. Furthermore, if the eigenvalues and eigenfunctions of the Laplacian \( \Delta_V \) are known, then an explicit solution for the initial value problem of the diffusion equations [24] can be obtained. Since the diffusion equations are linear, it is enough to consider the case where the solution space is a linear space of \( \Lambda^0(G) \). Also, from Theorem[6,2] one has the equation on \( \Lambda^1(G) \) as

\[
\frac{d}{dt} \psi_t = \Delta_E \psi_t.
\]

Since (31) is linear, eigenvalues and eigenfunctions of \( \Delta_E \) play roles. Thus in what follows \( \Lambda^0(G) \) and \( \Lambda^1(G) \) denote linear spaces. Then how to construct an explicit solution to the diffusion equations is argued below.

Let \( A \) be a linear operator acting on a finite-dimensional vector space \( \mathcal{U} \), and \( \lambda_A \) a non-zero number satisfying

\[
A \phi = \lambda_A \phi,
\]

where \( \phi \in \mathcal{U} \) is not 0. Then \( \lambda_A \) and \( \phi \) are referred to as a non-zero eigenvalue and its associated or corresponding eigenfunction, respectively. For a self-adjoint operator \( A \), it is known that all the eigenvalues are real, \( \lambda_A \in \mathbb{R} \). In addition, if all the non-zero eigenvalues are positive, then \( A \) is referred to as a positive operator. If \( A \phi = 0 \) with non-zero \( \phi \), then \( \phi \) is referred to as the eigenfunction associated with the zero-eigenvalue.

Since the Laplacian \( \Delta_V \) is a linear operator acting on \( \Lambda^0(G) \) and is self-adjoint as shown in Lemma[24], one can discuss if \( -\Delta_V \) is a positive operator or not. Then one has the following.

**Lemma 3.3.** The negative of the Laplacian acting on \( \Lambda^0(G) \), \( -\Delta_V \), is a positive operator.

**Proof.** Let \( \tilde{\lambda}_V^{(s)} \) be a non-zero eigenvalue labeled by \( s \), and \( \tilde{\phi}_V^{(s)} \) an associated eigenfunction of \( -\Delta_V \) so that \( -\Delta_V \tilde{\phi}_V^{(s)} = \tilde{\lambda}_V^{(s)} \tilde{\phi}_V^{(s)} \). Then it follows that

\[
\tilde{\lambda}_V^{(s)} \langle \tilde{\phi}_V^{(s)}, \tilde{\phi}_V^{(s)} \rangle_V = \langle \tilde{\lambda}_V^{(s)} \tilde{\phi}_V^{(s)}, \tilde{\phi}_V^{(s)} \rangle_V = \langle -\Delta_V \tilde{\phi}_V^{(s)}, \tilde{\phi}_V^{(s)} \rangle_V = \langle d\tilde{\phi}_V^{(s)}, d\tilde{\phi}_V^{(s)} \rangle_E \geq 0,
\]

14
Lemma 3.4. The normalized zero-eigenfunction is $\phi^{(0)}_V = 1_V$:

$$\Delta_V \phi^{(0)}_V = 0, \quad \text{and} \quad \left\langle \phi^{(0)}_V, \phi^{(0)}_V \right\rangle_V = 1.$$ 

Proof. It follows that $\Delta_V 1_V = -d^1d1_V = 0$. Then, the normalization condition is verified as

$$\left\langle \phi^{(0)}_V, \phi^{(0)}_V \right\rangle_V = (1_V, 1_V)_V = \sum_{x \in V} p^{eq}(x) = 1.$$

Remark 3.6. Repeating similar arguments in the proof of Lemma 3.3 one has that all the eigenvalues of $\Delta_V$ are negative or zero,

$$\Delta_V \phi^{(s)}_V = \lambda^{(s)}_V \phi^{(s)}_V, \quad \lambda^{(s)}_V \leq 0,$$

where $\phi^{(s)}_V \in \Lambda^0(G)$ is the eigenfunction associated with $\lambda^{(s)}_V$ for $s \neq 0$. For $s = 0$, $\phi^{(0)}_V$ denotes the eigenfunction associated with the zero-eigenvalue $\lambda^{(0)}_V = 0$.

Moreover, given a self-adjoint operator $A$ acting on a linear space $U$, it is known that an element of $U$ has an orthonormal decomposition. By applying this, one has the decomposition

$$\psi_t = \sum_{s} a^{(s)}(t) \phi^{(s)}_V = \sum_{s \in \mathcal{N}_V} a^{(s)}(t) \phi^{(s)}_V + a^{(0)}(t) \phi^{(0)}_V, \quad \text{with} \quad \left\langle \phi^{(s)}_V, \phi^{(s')}_V \right\rangle_V = \delta_{ss'}, \quad (32)$$

where $s$ denotes a label for eigenfunction, $\mathcal{N}_V$ the totality of labels for non-zero eigenfunctions, $a^{(s)}$ the real-valued function labeled by $s$ of $t$, and $\delta_{ss'}$ the Kronecker delta, giving unity if $s = s'$ and zero otherwise.

The normalized eigenfunction associated with the zero-eigenvalue $\phi^{(0)}_V$ is obtained as follows.

Lemma 3.4. The normalized zero-eigenfunction is $\phi^{(0)}_V = 1_V$:

$$\Delta_V \phi^{(0)}_V = 0, \quad \text{and} \quad \left\langle \phi^{(0)}_V, \phi^{(0)}_V \right\rangle_V = 1.$$ 

Proof. It follows that $\Delta_V 1_V = -d^1d1_V = 0$. Then, the normalization condition is verified as

$$\left\langle \phi^{(0)}_V, \phi^{(0)}_V \right\rangle_V = (1_V, 1_V)_V = \sum_{x \in V} p^{eq}(x) = 1.$$

With this Lemma, the solution to (24), $\psi_t$, is obtained as follows.

Proposition 3.2. (Spectrum decomposition). Let $\{\lambda^{(s)}_V\}$ be the totality of non-zero eigenvalues of $\Delta_V$, and $\{\phi^{(s)}_V\}$ that of the corresponding eigenfunctions. Then

$$\psi_t = \sum_{s \in \mathcal{N}_V} a^{(s)}(0) e^{-|\lambda^{(s)}_V| t} \phi^{(s)}_V + 1_V, \quad (33)$$

is a solution to the diffusion equations (24) derived from the master equations.

Proof. From (32), it follows that

$$\left\langle \frac{d}{dt} \psi_t, \phi^{(s)}_V \right\rangle_V = \frac{d a^{(s)}(t)}{dt}, \quad \text{and} \quad \left\langle \Delta_V \psi_t, \phi^{(s)}_V \right\rangle_V = \lambda^{(s)}_V a^{(s)}(t).$$

With these equations and (24), one has

$$\left\langle \frac{d}{dt} \psi_t - \Delta_V \psi_t, \phi^{(s)}_V \right\rangle_V = \frac{d a^{(s)}(t)}{dt} - \lambda^{(s)}_V a^{(s)}(t) = 0,$$

from which $a^{(s)}(t) = a^{(s)}(0) \exp(\lambda^{(s)}_V t)$. Taking into account Remark 3.6 one can write $\lambda^{(s)}_V = -|\lambda^{(s)}_V| \leq 0$. Combining these arguments, one has

$$\psi_t = \sum_{s \in \mathcal{N}_V} a^{(s)}(0) e^{-|\lambda^{(s)}_V| t} \phi^{(s)}_V + a^{(0)}(0) 1_V.$$ 

Finally it follows from Corollary 3.1 that $a^{(0)}(0)$ above is unity. This completes the proof. □
Notice that
\[
\frac{da^{(s)}}{dt} = \lambda^{(s)} a^{(s)}, \quad \text{and} \quad \frac{d}{dt} \psi_t = \sum_s \frac{d a^{(s)}(t)}{dt} \phi^{(s)} = \sum_s \lambda^{(s)} a^{(s)}(t) \phi^{(s)} = \sum_{s \in \mathcal{N}} \lambda^{(s)} a^{(s)}(t) \phi^{(s)}. \quad (34)
\]

An example of the spectrum decomposition of \( \psi_t \) is given below.

**Example 3.3.** (Eigen system for kinetic Ising model without spin-coupling). Consider Example 3.2. The matrix form of (28) is immediately obtained as
\[
\frac{d}{dt} \begin{pmatrix} \psi_{t, \text{Ising}}(-1) \\ \psi_{t, \text{Ising}}(+1) \end{pmatrix} = M_{\text{Ising}} \begin{pmatrix} \psi_{t, \text{Ising}}(-1) \\ \psi_{t, \text{Ising}}(+1) \end{pmatrix}, \quad \text{where} \quad M_{\text{Ising}} := \begin{pmatrix} -w_{-1\to+1} & w_{-1\to+1} \\ w_{+1\to-1} & -w_{+1\to-1} \end{pmatrix}.
\]

Two eigenvalues of \( M_{\text{Ising}} \) are obtained as
\[
\lambda^{(0)}_{\text{Ising}} = 0, \quad \text{and} \quad \lambda^{(1)}_{\text{Ising}} = -(w_{+1\to-1} + w_{-1\to+1}).
\]

Their eigenfunctions are
\[
\phi^{(0)}_{\text{Ising}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \phi^{(1)}_{\text{Ising}} = c^{(1)}_{\text{Ising}} \begin{pmatrix} w_{-1\to+1} \\ -w_{+1\to-1} \end{pmatrix},
\]

where
\[
c^{(1)}_{\text{Ising}} := \left[ (w_{-1\to+1})^2 p^{eq}_{\text{Ising}}(-1) + (w_{+1\to-1})^2 p^{eq}_{\text{Ising}}(1) \right]^{-1/2}.
\]

The orthonormality is verified as
\[
\begin{align*}
\left< \phi^{(0)}_{\text{Ising}} , \phi^{(0)}_{\text{Ising}} \right>_V &= \sum_{\sigma = \pm 1} p^{eq}_{\text{Ising}}(\sigma) = 1, \\
\left< \phi^{(0)}_{\text{Ising}} , \phi^{(1)}_{\text{Ising}} \right>_V &= \sum_{\sigma = \pm 1} \phi^{(0)}_{\text{Ising}}(\sigma) \phi^{(1)}_{\text{Ising}}(\sigma) p^{eq}_{\text{Ising}}(\sigma) \\
&= c^{(1)}_{\text{Ising}} \left[ w_{-1\to+1} p^{eq}_{\text{Ising}}(-1) - w_{+1\to-1} p^{eq}_{\text{Ising}}(1) \right] = 0, \\
\left< \phi^{(1)}_{\text{Ising}} , \phi^{(1)}_{\text{Ising}} \right>_V &= \sum_{\sigma = \pm 1} \phi^{(1)}_{\text{Ising}}(\sigma) \phi^{(1)}_{\text{Ising}}(\sigma) p^{eq}_{\text{Ising}}(\sigma) \\
&= c^{(1)}_{\text{Ising}}^2 \left[ (w_{-1\to+1})^2 p^{eq}_{\text{Ising}}(-1) + (w_{+1\to-1})^2 p^{eq}_{\text{Ising}}(1) \right] = 1,
\end{align*}
\]

where the normalization of \( p^{eq}_{\text{Ising}} \) and the detailed balance condition have been used. In Ref. 21, the equilibrium distribution \( p^{eq}_{\text{Ising}} \) and \( w_{\sigma\to-\sigma} \) were chosen so that they depend on parameters \( \theta \in \mathbb{R} \) and \( \gamma \in \mathbb{R}_+ \) as
\[
p^{eq}_{\text{Ising}}(\sigma; \theta) = \frac{\exp(\theta \sigma)}{2 \cosh \theta}, \quad \text{and} \quad w_{\sigma\to-\sigma}(\theta, \gamma) = \frac{\gamma}{2} (1 - \sigma \tanh \theta).
\]

The physical meaning of \( \theta \) is a quantity that is proportional to the inverse temperature, and that of \( \gamma \) is a characteristic relaxation time. The non-zero eigenvalue and its corresponding eigenfunction for this case are
\[
\lambda^{(1)}_{\text{Ising}} = -\gamma, \quad \text{and} \quad \phi^{(1)}_{\text{Ising}} = c^{(1)}_{\text{Ising}}(\theta, \gamma) \frac{\gamma}{2} \left( 1 + \tanh \theta \right).
\]

So far eigenvalues and eigenfunctions of \( \Delta_V \) have been investigated. There exists a link between eigenvalues of \( \Delta_V \) and those of \( \Delta_E \). To state this, notations are introduced as follows. Let \( \mathcal{N}_V \) be the totality of labels for non-zero eigenfunctions of \( \Delta_E \), \( \text{Spec} \mathcal{D}_V \) and \( \text{Spec} \mathcal{D}_E \) the totalities of eigenvalues in \( \mathcal{N}_V \) and \( \mathcal{N}_E \), respectively. Also, let
\[
\begin{align*}
\ker \Delta_V := \{ \phi_V \in \Lambda^0(G) \mid \Delta_V \phi_V = 0 \}, & \quad \ker \Delta_E := \{ \phi_E \in \Lambda^1(G) \mid \Delta_E \phi_E = 0 \}, \\
\ker d := \{ \phi_V \in \Lambda^0(G) \mid d\phi_V = 0 \}, & \quad \ker d^\dagger := \{ \phi_E \in \Lambda^1(G) \mid d^\dagger \phi_E = 0 \}.
\end{align*}
\]

First, one has the following.
Lemma 3.5.
\[ \ker d = \ker \Delta_V. \]

**Proof.** This proof can be split into two steps. First, \( \ker d \subseteq \ker \Delta_V \) is shown. Then, \( \ker \Delta_V \subseteq \ker d \) is shown. From these, one completes the proof.

(Proof of \( \ker d \subseteq \ker \Delta_V \)): Take \( \phi_V^{(0)} \in \ker d \), i.e., \( d\phi_V^{(0)} = 0 \). Then it follows that
\[
\Delta_V \phi_V^{(0)} = -d\dagger d\phi_V^{(0)} = 0,
\]
from which \( \ker d \subseteq \ker \Delta_V \).

(Proof of \( \ker \Delta_V \subseteq \ker d \)): Take \( \phi_V^{(0)} \in \ker \Delta_V \), i.e., \( \Delta_V \phi_V^{(0)} = 0 \). Then it follows from
\[
0 = \left\langle \Delta_V \phi_V^{(0)}, \phi_V^{(0)} \right\rangle_V = -\left\langle d\dagger d\phi_V^{(0)}, \phi_V^{(0)} \right\rangle_V = -\left\langle d\phi_V^{(0)}, d\phi_V^{(0)} \right\rangle_E \leq 0
\]
that \( d\phi_V^{(0)} = 0 \). Thus, \( \ker \Delta_V \subseteq \ker d \).

These two steps yield \( \ker d = \ker \Delta_V \).

Similar to Lemma 3.5, one has the following.

Lemma 3.6.
\[ \ker d\dagger = \ker \Delta_E. \]

**Proof.** A way to prove this is analogous to that of Lemma 3.5. □

Then one has the following property, referred to as **supersymmetry** [17].

**Theorem 3.3.** (Supersymmetry of Laplacians).
\[ \text{Spec}'\Delta_V = \text{Spec}'\Delta_E. \]

**Proof.** This proof can be split into two steps. First, \( \text{Spec}'\Delta_V \subseteq \text{Spec}'\Delta_E \) is shown. Then, \( \text{Spec}'\Delta_E \subseteq \text{Spec}'\Delta_V \) is shown. From these, one completes the proof.

( Proof of \( \text{Spec}'\Delta_V \subseteq \text{Spec}'\Delta_E \)): Assume that \( \Delta_V \phi_V^{(s)} = \lambda_V^{(s)} \phi_V^{(s)} \), \( (s \in N_V) \). Then, \( \{\lambda_V^{(s)}\} = \text{Spec}'\Delta_V \). From this assumption and Lemma 3.5, it follows that \( d\phi_V^{(s)} \neq 0 \). Since
\[
\Delta_E(d\phi_V^{(s)}) = d\Delta_V\phi_V^{(s)} = \lambda_V^{(s)}(d\phi_V^{(s)}) \quad \text{with} \quad d\phi_V^{(s)} \neq 0,
\]
one has that \( \lambda_V^{(s)} \in \text{Spec}'\Delta_E \) for each \( s \in N_V \). This yields that \( \text{Spec}'\Delta_V \subseteq \text{Spec}'\Delta_E \).

( Proof of \( \text{Spec}'\Delta_E \subseteq \text{Spec}'\Delta_V \)): Assume that \( \Delta_E \phi_E^{(s)} = \lambda_E^{(s)} \phi_E^{(s)} \), \( (s \in N_E) \). Then, \( \{\lambda_E^{(s)}\} = \text{Spec}'\Delta_E \). From this assumption and Lemma 3.6, it follows that \( d\dagger \phi_E^{(s)} \neq 0 \). Since
\[
\Delta_V(d\dagger \phi_E^{(s)}) = d\dagger \Delta_E \phi_E^{(s)} = \lambda_E^{(s)}(d\dagger \phi_E^{(s)}) \quad \text{with} \quad d\dagger \phi_E^{(s)} \neq 0,
\]
one has that \( \lambda_E^{(s)} \in \text{Spec}'\Delta_V \) for each \( s \in N_E \). This yields that \( \text{Spec}'\Delta_E \subseteq \text{Spec}'\Delta_V \).

These two steps yield the desired equality. □

Due to Theorem 3.3, non-zero eigenvalues \( \lambda_V^{(s)} \), \( (s \in N_V) \) and \( \lambda_E^{(s)} \), \( (s \in N_E) \) are not distinguished. In what follows they are denoted \( \lambda^{(s)} \).
3.2.1 Expectation values

For the case where the detailed balance conditions are not imposed, how expectation values are described has been argued in Section 3.1.1. It is also of interest to formulate how expectation values are described for the case that the detailed balance conditions are satisfied. To establish such a formulation, a relation between an expectation value and the inner product on \( \Lambda^0(G) \) is shown first. Second, inequalities for sums involving \( \psi_t \) over vertexes are shown. Finally some identities for such sums are shown.

The expectation value of \( O^0 \in \Lambda^0(G) \) is denoted by \( E_p[O^0] \) as in (15). This expectation value is written in terms of the inner product with (22) as

\[
E_p[O^0] = \sum_{x \in V} O^0(x) p_t(x) = \langle O^0, \psi_t \rangle_V,
\]

where \( p_t(x) = p^\text{eq}(x) \psi_t(x) \) in (23) has been used. The time-development of

\[
\langle O^0 \rangle_V := E_p[O^0],
\]

for \( O^0 \in \Lambda^0(G) \) is then described by

\[
\frac{d}{dt} \langle O^0 \rangle_V = \langle O^0, \dot{\psi}_t \rangle_V = \langle O^0, \Delta_V \psi_t \rangle_V = \langle \Delta_V O^0, \psi_t \rangle_V.
\]

Entropy defined by

\[
H[p_t] := E_p[- \ln p_t] = \langle - \ln p_t, \psi_t \rangle_V,
\]

plays a central role in information theory. It is well-known that

\[
\frac{d}{dt} H[p_t] > 0.
\]

It is then of interest to explore similar inequalities for sums involving \( \psi_t \) over vertexes. Define \( S_t \in \Lambda^0(G) \) to be

\[
S_t := - \ln \psi_t.
\]

Then one has the following.

**Proposition 3.3. (Inequality for \( S_t \)).**

\[
\langle S_t, 1_V \rangle_V \geq 0, \quad \forall t \in \mathbb{R}.
\]

**Proof.** By applying the inequality

\[
\exp(-c) \geq 1 - c, \quad c \in \mathbb{R},
\]

to the equality

\[
1 = \sum_{x \in V} p_t(x) = \sum_{x \in V} p^\text{eq}(x) \exp(-S_t(x)) = \langle \exp(-S_t), 1_V \rangle_V,
\]

one has

\[
\langle S_t, 1_V \rangle_V \geq 0.
\]

\( \square \)

Also one defines

\[
H[\psi_t] := E_p[S_t] = E_p[- \ln \psi_t] = \langle S_t, \psi_t \rangle_V.
\]

Then one has the following.

**Proposition 3.4. (Inequality for time-derivative of \( H[\psi_t] \)).**

\[
\frac{d}{dt} H[\psi_t] \geq \sum_s |\lambda^{(s)}| (\alpha^{(s)}(t))^2 > 0.
\]

18
Proof. Substituting the inequality
\[ 1 + \ln \psi_t(x) \leq \psi_t(x), \quad \forall x \in V \]
and (24) into (35), one has
\[ -\frac{d}{dt} H[\psi_t] = \sum_{x \in V} p^{eq}(x) \frac{d}{dt} \left[ \psi_t(x) \ln \psi_t(x) \right] = \sum_{x \in V} p^{eq}(x) \left( 1 + \ln \psi_t(x) \right) \dot{\psi}_t(x) \]
\[ \leq \sum_{x \in V} p^{eq}(x) \psi_t(x) \dot{\psi}_t(x) = \langle \psi_t, \Delta V \psi_t \rangle_V. \]
From this and (32), it follows that
\[ -\frac{d}{dt} H[\psi_t] \leq \sum_{s} \sum_{s'} \lambda(s) a(s)(t) a(s')(t) \left\langle \phi_V^{(s)}, \phi_V^{(s')} \right\rangle_V = \sum_{s} \lambda(s) \left( a(s)(t) \right)^2, \]
from which
\[ \frac{d}{dt} H[\psi_t] \geq - \sum_{s} \lambda(s) \left( a(s)(t) \right)^2 = - \sum_{s \in \mathcal{N}_V} \lambda(s) \left( a(s)(t) \right)^2. \]
Since \( \lambda(s) = -|\lambda(s)| < 0 \) for \( s \in \mathcal{N}_V \) (See Remark 3.6), one has that
\[ \frac{d}{dt} H[\psi_t] \geq \sum_{s \in \mathcal{N}_V} |\lambda(s)| \left( a(s)(t) \right)^2 > 0. \]

There exist closed dynamical systems for expectation variables \( E_p[\mathcal{O}^0] \) with some appropriately chosen \( \mathcal{O}^0 \). Such systems are given as follows.

**Proposition 3.5.** (Dynamical systems for expectation variables 1). Consider the case where \( \mathcal{O}^0 = \phi_V^{(s)}, \quad s \in \mathcal{N}_V \).

Then,
\[ \left\langle \phi_V^{(s)} \right\rangle_V(t) := E_p[\phi_V^{(s)}], \]
follows the dynamical system
\[ \frac{d}{dt} \left\langle \phi_V^{(s)} \right\rangle_V = \lambda(s) \left\langle \phi_V^{(s)} \right\rangle_V. \]

Proof. It follows from (34) and (32) that
\[ \left\langle \phi_V^{(s)} \right\rangle_V = \left\langle \phi_V^{(s)}, \psi_t \right\rangle_V = a(s)(t). \]
From this and (34), one has the closed form
\[ \frac{d}{dt} \left\langle \phi_V^{(s)} \right\rangle_V = \dot{a}(s)(t) = \lambda(s) \left\langle \phi_V^{(s)} \right\rangle_V. \]

In the context of dynamical systems theory in information geometry, the dynamical system
\[ \frac{d}{dt} \eta_j = - \{ \eta_j - \eta_j' \}, \]
has been studied in \[22, 23, 24, 25\]. The variables \(\{\eta_j\}\) in \[37\] denote the expectation values for an exponential family. Notice that the system \(37\) with \(\{\eta_j'\}\) = 0 is obtained by replacing \(\lambda(s)\) with \(-1\) in \[50\].

Similar to this proposition, closed equations on \(\Lambda^1(G)\) are obtained as follows. Define for \(\mathcal{O}^1 \in \Lambda^1(G)\),

\[
\langle \mathcal{O}^1 \rangle_E := \langle \mathcal{O}^1, d\psi_t \rangle_E.
\]

Differentiating this equation with respect to \(t\) and using \(31\), one has

\[
\frac{d}{dt} \langle \mathcal{O}^1 \rangle_E = \langle \mathcal{O}^1, \Delta_E d\psi_t \rangle_E = \langle \Delta_E \mathcal{O}^1, d\psi_t \rangle_E.
\]

Then one has the following.

**Proposition 3.6.** *(Dynamical systems for expectation variables 2).* Consider the case where

\[
\mathcal{O}^1 = d\phi_V^{(s)}, \quad s \in N_E.
\]

Then, one has the closed dynamical system

\[
\frac{d}{dt} \langle d\phi_V^{(s)} \rangle_E = \lambda^{(s)} \langle d\phi_V^{(s)} \rangle_E.
\]

**Proof.** It follows from that

\[
\frac{d}{dt} \langle d\phi_V^{(s)} \rangle_E = \langle \Delta_E d\phi_V^{(s)}, d\psi_t \rangle_E = \langle \lambda^{(s)} d\phi_V^{(s)}, d\psi_t \rangle_E = \lambda^{(s)} \langle d\phi_V^{(s)} \rangle_E.
\]

\[
\square
\]

\section{4 Conclusions}

This paper offers a viewpoint that master equations employed in nonequilibrium statistical mechanics can be analyzed in discrete geometry. As the main theorem in this paper, when the detailed balance conditions are satisfied, master equations induce diffusion equations, where Laplacians are associated with chosen inner products. These diffusion equations can analytically be solved and then explicit solutions have been obtained. With the self-adjoint property of the Laplacians, the orthonormal decomposition of the solution and closed equations for expectation variables have been obtained. Furthermore, the coboundary operator and its adjoint enable us to simplify various calculations including expectation values, and to show the so-called supersymmetry for two classes of Laplacians. Also some examples have been shown.

The significance of this discrete geometric description of master equations is stated here. Although various graph theoretic approaches to master equations exist in the literature, none of them have yielded reduced equations such as diffusion equations without any approximation, which should be compared to the present study. A crucial point of the derivation of the diffusion equations is to introduce freedom to choose weights defining inner products for functions on graphs. Here the introduction of weights, or equivalently the additional freedom, has been incorporated in discrete geometry\[12\], and has then induced wider classes of Laplacians on graphs whose eigenvalues are real due to the self-adjoint property. The usefulness of this self-adjoint property of Laplacians has been employed throughout this paper, which should be compared with the existing studies.

There are numbers of extensions that follow from this study. One of them is to apply the present approach to quantum systems, chemical reactions, and economical systems. Also, some infinite graph case and discrete-time case should be addressed, and their applications to mathematical engineering, including Monte-Carlo simulations, should be considered as future works.

**Acknowledgments**

The author S.G. is partially supported by JSPS (KAKENHI) grant number JP19K03635, also grateful to Shuhei MANO for fruitful discussions. The other author H.H. is partially supported by JSPS (KAKENHI) grant number JP17H01793.
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