The role of electron-phonon interaction in a magnetically driven mechanism for superconductivity

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We use the renormalization group method to examine the effect of phonon mediated interaction on d-wave superconductivity, as driven by spin fluctuations in a quasi-one-dimensional electron system. The influence of a tight-binding electron-phonon interaction on the spin-density-wave and d-wave superconducting instability lines is calculated for arbitrary temperature, phonon frequency and antinesting of the Fermi surface. The domain of electron-phonon coupling strength where spin-density-wave order becomes unstable against the formation of a bond-order-wave or Peierls state is determined at weak antinesting. We show the existence of a positive isotope effect for spin-density-wave and d-wave superconducting critical temperatures which scales with the antinesting distance from quantum critical point where the two instabilities merge. We single out a low phonon frequency zone where the bond-oder-wave ordering gives rise to triplet f-wave superconductivity under nesting alteration, with both orderings displaying a negative isotope effect. We also study the electron-phonon strengthening of spin fluctuations at the origin of extended quantum criticality in the metallic phase above superconductivity. The impact of our results on quasi-one-dimensional organic conductors like the Bechgaard salts where a Peierls distortion is absent and superconductivity emerges near a spin-density-wave state under pressure is emphasized.

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I. INTRODUCTION

Since the discovery of superconductivity (SC) in the Bechgaard salts [(TMTSF)_{2}X] series\textsuperscript{31}, much of the attention paid to the mechanism of Cooper pairing has mostly focused on models of electrons with purely repulsive interactions\textsuperscript{2–13}. On empirical grounds, this has been amply supported by the ubiquity of spin-density-wave (SDW) correlations near the superconducting state when either pressure\textsuperscript{14–17}, temperature\textsuperscript{18,19}, or even magnetic field is varied\textsuperscript{20,21}. As one moves along the temperature axis for example and enters the metallic state, important SDW fluctuations are found to govern properties of the normal phase, giving rise for instance to a huge enhancement of the NMR relaxation rate and the linear-T resistivity term over a wide temperature interval above the critical temperature $T_c$ for superconductivity\textsuperscript{22–24}.

Besides the nesting of the Fermi surface, repulsive interactions are an essential component of SDW correlation\textsuperscript{25–28}. They have become inescapable ingredients of the model description of superconductivity in these materials. In this regard, the quasi-one dimensional electron gas model with the aid of the renormalization group (RG) method have played an important part in the description of these low dimensional electron systems. In the repulsive sector, it proved particularly generic of the SDW to d-wave SC (SCd) sequence of instabilities when the amplitude of the next-to-nearest neighbour interchain hopping, $t'_1$, called the anti nesting parameter, is tuned to simulate pressure effects on spin fluctuations responsible for superconducting pairing interaction\textsuperscript{29–32}. The approach has also shown how the constructive interference between spin fluctuations and Cooper pairing can explain the existence of a Curie-Weiss temperature dependence of the SDW correlation length, which is a key factor in the enhancement of the NMR relaxation rate and the linear-T component in resistivity over the whole pressure interval where superconductivity is present\textsuperscript{33,34}.

However, in view of the complex molecular structure of systems like the Bechgaard salts, the repulsive electron gas model must be regarded as an idealization. It ignores primarily the interaction of electrons with low energy phonon modes of the lattice. Early X-ray diffuse scattering experiments in (TMTSF)$_2$PF$_6$ and (TMTSF)$_2$ClO$_4$ compounds did reveal the existence of such a coupling, under the guise of lattice fluctuations at the 1D wave vector $2k_F$ of the electron gas ($k_F$ being the longitudinal Fermi wave vector)\textsuperscript{35–38}. The lattice fluctuations remain regular in temperature for the Bechgaard salts, in contrast to so many molecular chain systems where it terminates in a Peierls – bond-order-wave (BOW) – distorted state. Although the reason for this remains for a large part unexplained\textsuperscript{39–41}, the presence of $2k_F$ lattice fluctuations is a direct evidence of a finite coupling between electrons and phonons, a consequence of the modulation of tight-binding electron band parameters by lattice vibrations.

This points at the impact a retarded – phonon mediated (Ph-M) – interaction can have on the properties of the electron gas when the mechanism for Cooper pairing is magnetically driven. Whether it is detrimental to SDW and SCd correlations, as one would naturally expect if the electron-phonon interaction was taken in isolation\textsuperscript{41}, or on the contrary, if it becomes a factor of reinforcement when it is subordinate to repulsive interactions. The latter possibility can provide new insight as to the conditions prevailing in weakly dimerized systems.
like the Bechgaard salts that make SDW coming out on top of the Peierls phenomena. It can further clarify as to how electron-phonon interaction can be actively involved in the occurrence of superconductivity near magnetism. It can also shed light on the possibility of a positive isotope effect for the temperature scale of instabilities against SDW/SCd orderings as a function of phonon frequency. Reinforcement could also extend relatively far in the metallic phase by enhancing spin fluctuations as quantum critical effects due to interfering SCd and SDW instabilities.

These possibilities found a rather large echo in the context of other unconventional superconductors, in particular high-$T_c$ cuprates, where they framed a significant part of the debate surrounding the relative importance of Coulomb and electron-phonon interactions when superconductivity takes place in the proximity of antiferromagnetism and charge-density-wave ordering. Its transposition in quasi-one-dimensional superconductors like the Bechgaard salts close to a SDW instability has remained essentially unexplored since the very first attempts to reconcile electron-electron and electron-phonon interactions in the framework of mean-field theory of competing magnetism and superconductivity.

In this work we shall address this problem in the weak coupling framework of the RG approach to quasi-1D electron gas model. The model is extended to include both direct and Ph-M electron-electron interactions in the study of interfering (electron-electron) Cooper and (electron-hole) density-wave pairings at arbitrary phonon frequency $\omega_D$. The RG calculations will be carried out at finite temperature $T$, which bring additional difficulties in the presence of retarded interactions. This turns out to be required when antinesting is present. Actually, a finite $t'_\perp$ breaks the usual correspondence between $T$ and the scaled cut-off energy $\Lambda(\ell)$ from the Fermi surface that generates the RG flow. The flow will be then conducted at arbitrary temperature for interactions with momentum dependence along the Fermi surface and a finite set of Matsubara frequencies. This finite-$T$ RG procedure with momentum and frequency variables has been worked out recently for systems where Ph-M interactions are predominant, a situation relevant to competing charge-density-wave and s-wave SC instabilities away from half-filling. It is extended here to weakly dimerized chains systems like the Bechgaard salts where repulsive interactions are dominant and half-filling Umklapp scattering is finite.

The results put forward below show that the modulation of tight-binding electron band by acoustic lattice vibrations introduces effective Ph-M interactions with a very characteristic dependence on longitudinal momentum transfer of scattered electrons. The dependence affects the RG flow and produces a low-energy downward screening of the repulsive backward scattering and an enhancement of the repulsive Umklapp term. Both effects are $\omega_D$ dependent, concurring to boost antiferromagnetic exchange between itinerant electrons and reinforce both the SDW and magnetically driven SCd instability lines of the phase diagram. The impact of retardation generates a positive isotope effect whose amplitude peaks at the critical strength of antinesting where SDW and SCd instabilities lines meet and their constructive interference is the strongest. Above a definite strength of electron-phonon interaction, the SDW becomes unstable against the formation of a BOW distorted state and triplet $f$-wave superconductivity if antinesting and retardation are sufficiently high. The latter states are both characterized by a negative isotope effect, as a result of antiadiabaticity.

The boost of interference by electron-phonon interaction is not limited to the transition lines, but is also manifest in the metallic phase where it feeds deviations to Fermi liquid behaviour at the origin of extended quantum criticality in the normal phase. The latter can be followed through the reinforcement of the Curie-Weiss behaviour of the SDW susceptibility which is correlated to $\omega_D$ and antinesting $t'_\perp$ in the whole range where superconductivity is present.

In section II we introduce the quasi-1D electron gas model which is extended to include tight-binding electron-phonon interaction term. In Sec. III the one-loop RG flow equations for the different electron-electron vertices and relevant response functions are given and integrated in the determination of the phase diagram at arbitrary anti nesting and phonon frequency. Their integration is carried out in Sec. IV leads to the determination of the phase diagrams, isotope effects and spin fluctuations in the normal state. In Sec. V, we discuss the implications of our results in the description of unconventional superconductors like the Bechgaard salts and conclude this work.

II. THE MODEL

For a linear array of $N_L$ chains of length $L$, the Hamiltonian of the quasi-1D electron gas with electron-phonon coupling is given by

$$H = H^0_p + H_{ep} + \sum_{p,k,\sigma} E_p(k) c^\dagger_{p,k,\sigma} c_{p,k,\sigma} + \frac{\pi\nu_F}{L N_1} \sum_{\{k,\sigma\}} \left[ g_1 c^\dagger_{\pm k_4,\sigma_1} c^\dagger_{k_3,\sigma_2} c_{k_2,\sigma_2} c_{-k_1,\sigma_1} + g_2 c^\dagger_{-k_4,\sigma_1} c_{-k_3,\sigma_2} c_{-k_2,\sigma_2} c_{k_1,\sigma_1} + \frac{1}{2} g_3 \left( c^\dagger_{k_4,\sigma_1} c_{-k_3,\sigma_1} c_{k_2,\sigma_2} c_{-k_1,\sigma_1} + c^\dagger_{-k_4,\sigma_1} c_{k_3,\sigma_1} c_{-k_2,\sigma_2} c_{k_1,\sigma_1} \right) + H.c. \right] \delta_{k_1+k_2=\tilde{k}_3+k_4}(\pm G).$$

In the purely electronic part that has been made explicit, the operator $c^\dagger_{p,k,\sigma}$ ($c_{p,k,\sigma}$) creates (destroys) a right ($p = +$) and left ($p = -$) moving electron of wave vector $k = (k, k_\perp)$ and spin $\sigma$. The free part is modeled by the anisotropic one-electron energy spectrum in two...
dimensions,
\[ E_p(k) = v_F(pk - k_F) + \epsilon(k_\perp), \]
where
\[ \epsilon(k_\perp) = -2t_\perp \cos k_\perp - 2t'_\perp \cos 2k_\perp. \]

The longitudinal part has been linearized around the longitudinal Fermi wave vector \( pk_F; v_F \) is the longitudinal Fermi velocity, \( t_\perp \) is the nearest-neighbor hopping integral in the perpendicular direction while \( t'_\perp \) is a second nearest-neighbor hopping paramaterizing deviations to perfect nesting at \( q_0 = (2k_F, \pi) \), which simulates the most important effect of pressure in our model. The quasi-1D anisotropy of the spectrum is \( E_F \simeq 15t_\perp \), where \( E_F = v_Fk_F \simeq 3000K \) is the longitudinal Fermi energy congruent with the range found in the Bechgaard salts.\(^{[19,22]} \) \( E_F \) is taken as half the bandwidth cutoff \( E_0 = 2E_F \) in the model. The interacting part of the Hamiltonian is described by the bare backward, \( g_1 \equiv g_1(\pm k_F, -k_F); -k_F, -k_F) \), and forward, \( g_2 \equiv g_2(\pm k_F, -k_F); -k_F, +k_F) \), scattering amplitudes between right and left moving electrons defined on the 1D Fermi surface. The half-filling character of the band - a consequence of a small dimerization of the chains - gives rise to Umklapp scattering of bare amplitudes \( g_3 \equiv g_3(\pm k_F, \mp k_F; \mp k_F, \pm k_F) \), and for which momentum conservation involves the longitudinal reciprocal lattice vector \( G = (4k_F, 0) \). All couplings are normalized by \( \pi v_F \) and are initially independent of transverse momenta \( k_\perp \), but acquire such an dependence along the RG flow.

Regarding the values taken by the interaction parameters throughout the present calculations, we shall take \( g_1 = g_2/2 \simeq 0.32 \) and \( g_3 \simeq 0.025 \), which follows from the phenomenological analysis of previous works that fixes their amplitude from different experiments in the weakly dimerized systems like the Bechgaard salts.\(^{[28,29]} \)

The electron-phonon part of the hamiltonian\(^1\) follows from the modulation of the longitudinal hopping integral by acoustic phonons in the tight-binding approximation.\(^{[22]} \) It reads

\[
H_{op} + H_{ep} = \sum_{q,\nu} \omega_{q,\nu} \left( b_{q,\nu}^\dagger b_{q,\nu} + \frac{1}{2} \right) \\
+ \left( LN_\perp \right)^{-1} \sum_{p,\sigma,\nu} \sum_{k,q} g_\nu(k,q)c_{p,k+q,p,\sigma}^\dagger c_{p,k,\sigma} (b_{q,\nu} + b_{-q,\nu})
\]

where \( \nu \) is related to the different polarization of acoustic phonons. For phonons of interest propagating parallel to the chains \( a \)-axis, we have

\[
\omega_{q,\nu} = \omega_\nu |\sin \frac{q}{2}|, \quad g_\nu(k,q) = i 4 \frac{\lambda_\nu}{\sqrt{2M \omega_\nu}} \sin \frac{q}{2} \cos \left( k + \frac{q}{2} \right)
\]

where the coupling amplitude \( \lambda_\nu = \nabla t \cdot e_\nu \) is expressed in terms of the longitudinal hooping integral, \( t \), and the unit vector \( e_\nu \) of the lattice displacement; \( \omega_\nu = 2\sqrt{k_\nu/M} \) is the Deybe frequency for the acoustic branch \( \nu \), and \( M \) is the mass of molecular unit. The bandwidth of acoustic branches in the molecular systems like the Bechgaard salts does not exceed \( \omega_\nu \sim 100K^{[53,55]} \). We will consider in the following the interval normalized phonon frequency \( 0 < \omega_D/t_\perp \leq 0.5 \).

For the partition function \( Z \), it is straightforward to proceed to the partial trace of harmonic phonon degrees of freedom and express the partition function,

\[
Z = \int \mathcal{D}\psi \mathcal{D}\psi^* e^{S_0 + S_I},
\]

as a functional integral over the fermion anti commuting fields \( \psi^{(s)} \). The bare action in the Matsubara-Fourier space is given by

\[
S_0[\psi^*, \psi] = \sum_{k,p,\sigma} \left[ G_{p}^0(k) \right]^{-1} \psi_{p,\sigma}^*(k) \psi_{p,\sigma}(k)
\]

where \( \bar{k} = (k, \omega_n = \pm \pi T, \pm 3\pi T, \ldots) \) and

\[
G_{p}^0(k) = [i\omega_n - E_p(k)]^{-1}
\]

is the bare fermion propagator. The interacting part of the action is of the form
\[ S_t[\psi^*, \psi] = -\frac{T}{LN_\perp} \pi v_F \sum_{(k, \sigma)} \left\{ g_1(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) \psi^*_{+, \sigma_1}(\bar{k}_4) \psi_{+, \sigma_2}(\bar{k}_3) \psi_{-, \sigma_2}(\bar{k}_2) \psi_{-, \sigma_1}(\bar{k}_1) \\
+ g_2(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) \psi^*_{+, \sigma_1}(\bar{k}_4) \psi_{-, \sigma_2}(\bar{k}_3) \psi_{+, \sigma_2}(\bar{k}_2) \psi_{-, \sigma_1}(\bar{k}_1) \\
+ \frac{1}{2} [g_3(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) \psi^*_{+, \sigma_1}(\bar{k}_4) \psi_{-, \sigma_2}(\bar{k}_3) \psi_{+, \sigma_2}(\bar{k}_2) \psi_{-, \sigma_1}(\bar{k}_1) + \text{c.c.}] \delta_{k_1+k_2+k_3+k_4(\neq 0)} \right\} \]

where \( \bar{k}_i \equiv (k_{1,i}, \omega_{n_i}) \) and \( \bar{G} = (4k_F, 0, 0) \) for Umklapp scattering. The amplitude of the bare effective backscattering is given by

\[ g_1(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) = g_1 - \sum_\nu \frac{2}{\pi v_F \omega_\nu} g_{\nu}(k_F, -2k_F) g_{\nu}(-k_F, 2k_F) \]

\[ \equiv g_1 + \frac{g_{\text{ph}}}{1 + (\omega_{n_3} - \omega_{n_1})^2 / \omega_D^2} \tag{10} \]

where the electron-phonon matrix element has been evaluated on the 1D Fermi points \( \pm k_F \). This leads to an attractive contribution from all acoustic branches of normalized amplitude

\[ g_{\text{ph}} = -4 \sum_\nu \lambda_\nu^2 / (\pi v_F \kappa_\nu). \tag{11} \]

Here we have defined the Debye frequency \( \omega_D = (\omega_{2k_F}) \), as the average phonon frequency over the different branches at the zone edge. As for the amplitude of the effective forward scattering, we have

\[ g_2(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) = g_2 - \frac{2}{\pi v_F} \sum_\nu \omega_{\nu} g_{\nu}(k_F, 0) g_{\nu}(-k_F, 0) \]

\[ \equiv g_2 \tag{12} \]

which remains unaffected by phonons at vanishing momentum transfer. Finally, for the bare Umklapp term in the presence of phonons, we have

\[ g_3(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) = g_3 - \sum_\nu \frac{2}{\pi v_F \omega_\nu} g_{\nu}(k_F, 2k_F) g_{\nu}(k_F, -2k_F) \]

\[ \equiv g_3 + \frac{\eta g_{\text{ph}}}{1 + (\omega_{n_3} - \omega_{n_1})^2 / \omega_D^2} \tag{13} \]

which, in contrast to normal backscattering, gives rise to a retarded repulsive contribution. Here \( \eta \) is a reduction factor that takes into account the weak dimerization of the chains. For simplicity we shall take \( \eta = g_3/g_1 (= \Delta_D / E_F \ll 1) \) (see also Ref.1).

The dependence of the above bare retarded couplings on the longitudinal momentum transfer will play an important role in their RG flow at low energy.

\[ \partial_t g_1(\bar{k}_1, \bar{k}_2, \bar{k}_3, \bar{k}_4) = \frac{1}{2\pi} \int dk_\perp I_P(k_\perp, q_P) \]

\[ \times \left[ \epsilon_P(g_1(\bar{k}_1, \bar{k}, \bar{k}_P, \bar{k}_4)g_1(\bar{k}_P, \bar{k}_2, \bar{k}_3, \bar{k})) \\
+ \epsilon_{P, \nu}(g_2(\bar{k}_1, \bar{k}, \bar{k}_4, \bar{k}_P)g_1(\bar{k}_P, \bar{k}_2, \bar{k}_3, \bar{k})) \\
+ \epsilon_{P, \nu}(g_1(\bar{k}_1, \bar{k}, \bar{k}_P, \bar{k}_4)g_2(\bar{k}_P, \bar{k}_2, \bar{k}_3, \bar{k})) \right] \\
+ \epsilon_P(g_3(\bar{k}_1, \bar{k}, \bar{k}_3, \bar{k}_P)g_3(\bar{k}_P, \bar{k}_2, \bar{k}_4, \bar{k})) \\
+ \epsilon_{P, \nu}(g_3(\bar{k}_1, \bar{k}, \bar{k}_P, \bar{k}_3)g_3(\bar{k}_P, \bar{k}_2, \bar{k}_4, \bar{k})) \\
+ \frac{1}{2\pi} \int dk_\perp I_C(k_\perp, q_C) \]

\[ \times \left[ \epsilon_C(g_1(\bar{k}_1, \bar{k}_2, \bar{k}_C, \bar{k}_4)g_2(\bar{k}, \bar{k}_C, \bar{k}_3, \bar{k})) \\
+ \epsilon_C(g_2(\bar{k}_1, \bar{k}_2, \bar{k}_C, \bar{k})g_1(\bar{k}, \bar{k}_C, \bar{k}_3, \bar{k})) \right], \tag{14} \]
\[
\partial_t g_2(k_1, k_2, k_3, k_4) = \frac{1}{2\pi} \int dk I_P(k_{\perp}, q_P') \times \left[ \epsilon_{P,1}(g_2(k_1, k_2, k_3, k_4')g_2(k_5', k_2, k_4)) + \epsilon_{P,1}(g_3(k_1, k_2, k, k_4)) g_3(k_5', k_2, k_3, k) \right] + \frac{1}{2\pi} \int dk I_C(k_{\perp}, q_C) \times \left[ \epsilon_C(g_1(k_1, k_2, k_3, k_C)g_1(k_5, k_4, k)) + \epsilon_C(g_2(k_1, k_2, k_3, k_C)g_2(k_5, k_3, k_4)) \right],
\]
(15)

and

\[
\partial_t g_3(k_1, k_2, k_3, k_4) = \frac{1}{2\pi} \int dk I_P(k_{\perp}, q_P) \times 2 \left[ \epsilon_P(g_1(k_1, k_2, k_3, k_4')g_3(k_5', k_2, k_4)) + \epsilon_{P,v}(g_1(k_1, k_2, k_3, k_4')g_3(k_5', k_2, k_4)) + \epsilon_{P,v}(g_3(k_1, k_2, k_3, k_4')g_3(k_5, k_2, k_4)) \right] + \frac{1}{2\pi} \int dk I_P(k_{\perp}, q_P') \times 2 \epsilon_{P,1}(g_2(k_1, k_2, k_3, k_4')g_3(k_5', k_2, k_4)) \]
(16)

These consist of closed loop (\(\epsilon_P = -2\)), vertex corrections (\(\epsilon_{P,v} = 1\)) and ladder (\(\epsilon_{P,1} = 1\)) diagrams of the \(Q_p\) electron-hole (Peierls) pairing, which combine with the ladder diagrams (\(\epsilon_C = -1\)) of the electron-electron (Cooper) pairing. Here \(k_P = k + q_P\), \(k_P' = k + q_P'\) and \(k_C = -k + q_C\), where \(q_P, C = (g_{\perp, P,C}, \omega_{P,C})\) corresponds to the Peierls \(q_P = k_1 - k_4, q_P' = k_1 - k_3\) and Cooper \(q_C = k_2 + k_1\) variables. In the above equations, each diagram singles out a discrete frequency convolution of the form \(D_{P,C} = \sum_{\omega} g_1 \circ g_2 \circ L_{P,C}\), between the coupling products and the Peierls (Cooper) loop derivative \(L_{P,C} = T_{0C}(G_0^{\perp}(k + q_P,C)G_0^{\perp}(\pm k))\). The exact frequency summation at arbitrary \(T\) is computationally out of reach. It can be approximated, however, using a mean field scheme in which \(D_{P,C} \rightarrow (g_1 \circ g_2) \sum_{\omega} L_{P,C}\), where \(\langle \cdots \rangle = \delta^{-1} \sum_{\cdots} \cdots \) stands as an average of the couplings over \(\omega\), the internal loop frequency variable. The product of couplings averaged over \(\omega\) is thus considered constant in the exact evaluation of the \(\lambda\) derivative of the Cooper and Peierls loops \(I_{P,C} = \sum_{\nu = -\infty}^{\infty} L_{P,C} \) at temperature \(T\), which read

\[
I_{P,C}(k_{\perp}, q_{P,C}) = \sum_{\nu = \pm 1} \theta[|E_0(\ell)/2 + \nu A_{P,C}| - E_0(\ell)/2] \times \frac{1}{4} \left[ \tan \frac{E_0(\ell) + 2\nu A_{P,C}}{4T} + \tan \frac{E_0(\ell)}{4T} \right] \times \frac{(E_0(\ell) + \nu A_{P,C})E_0(\ell)}{(E_0(\ell) + \nu A_{P,C})^2 + \omega_{P,C}^2},
\]
(17)

where \(\omega_P = \omega_{n3} - \omega_{n1}\), and

\[
A_P = -\varepsilon(k_{\perp}) - \varepsilon(k_{\perp} + q_{\perp} C),
\]
(18)

for the Peierls channel; \(\omega_C = \omega_{n1} + \omega_{n2}\), and

\[
A_C = -\varepsilon(k_{\perp}) + \varepsilon(k_{\perp} + q_{\perp} C),
\]
(19)

for the Cooper channel. Here \(\theta[x]\) is the step function \((\theta[0] = 1/2)\). At finite temperature the above decoupling scheme with the number of frequencies retained represent a good compromise between exacting computing time and reproducing the results known for either the non-retarded case in quasi-1 \(\text{[27, 29]}\) or the quantum corrections to the BOW ordering in pure electron-phonon problem in one dimension \(\text{[31]}\).

The nature of instabilities of the electron gas and their critical temperatures, \(T_{\mu}\), are best studied from the susceptibilities \(\chi_{\mu}\). For the coupled electron-phonon model under consideration, only superconducting and staggered density-wave susceptibilities present a singularity as a function of antinesting and electron-phonon interaction strength. In the static limit, these are defined by

\[
\pi v_{P,C} \chi_{\mu}(q_0) = \frac{1}{2\pi} \int dk \int \langle \epsilon_{P,C} g_3(k, k', k'' \pm q_0) \rangle I_{P,C}(k_{\perp} + q_0) dl,
\]
(20)

where the vertex parts \(z_\mu\) are governed by one-loop flow equations. In the density-wave channel, we shall consider

\[
\partial_t z_{SDW}(k + q_0) = \frac{1}{2\pi} \int dk' I_{P,C}(k_{\perp}, q_P') z_{SDW}(k' + q_0') \times \langle [\epsilon_{P,C} g_3(k, k', k'' \pm q_0') + \epsilon_{P,C} g_2(k' + q_0', k, k', k, k' + q_0')] \rangle,
\]
(21)

and

\[
\partial_t z_{BOW}(k + q_0) = \frac{1}{2\pi} \int dk' I_{P,C}(k_{\perp}, q_P') z_{BOW}(k' + q_0') \times \langle [\epsilon_{P,C} g_3(k, k', k'' \pm q_0') + \epsilon_{P,C} g_3(k', k'' \pm q_0', k, k, k, k'' \pm q_0')] \rangle - \epsilon_{P,C} g_3(k', k'' \pm q_0', k, k, k, k' + q_0') \rangle \]
(22)

for the static \(\mu = SDW\) and BOW susceptibilities, respectively at \(q_0 = (\pi, 0)\). In the superconducting channel, we shall examine

\[
\partial_t \xi_{\mu}(-k + q_0) = \frac{1}{2\pi} \int dk' I_{C}(k_{\perp}, q_C) z_{SC}(-k' + q_0) \times \Delta_{\mu}(k_{\perp}) \langle \epsilon_{C} g_1(-k + q_0, k, -k + q_0, k) \rangle + g_2(-k + q_0, k, -k + q_0, k) \rangle \]
(23)

for the static SC susceptibility at \(q_0 = 0\), which \(\Delta_{\mu}(k_{\perp})\) is the form factor for the SC order parameter. For SCd and triplet-f wave (SCf) correlations, we have \(\Delta_{SCd}(k_{\perp}) = \sqrt{2} \cos k_{\perp}\) and \(\Delta_{SCf}(k_{\perp}) = (\text{sgn} k) \sqrt{2} \cos k_{\perp}\), whereas for conventional singlet pairing (SS), we have \(\Delta_{SS}(k_{\perp}) = 1\).

Before embarking on the solution of the above equations, it is instructive at this stage to examine their basic
features as a function of the different energy scales of the model. At high temperature where $T \gg \omega_D$ and the phonons are classical, the contribution of Ph-M interaction to all open diagrams – ladder and vertex corrections – becomes strongly dampened for all $\Lambda(\ell)$, as a result of retardation that reduces the summations over intermediate frequency transfer in such diagrams. In this temperature range, the Ph-M part contribute more appreciably to the closed loop diagram of the Peierls channel which does not have an intermediate sum over transfer frequency, and this on equal footing with the direct Coulomb part in Eqs. (14-16). On the other hand, when entering in the low temperature domain at $T < \omega_D$, retardation effects are reduced which progressively strengthen the contribution of electron-phonon interaction to open diagrams. This increases mixing or interference between all diagrams of the Peierls and Cooper scattering channels.

For the range of parameters considered in the model, the temperature scale $T_\mu$ of instabilities of the electron gas that are considered below all fall in the temperature range $T_\mu \ll t'_\perp$. This is where the transverse electron motion and the warping of the Fermi surface are coherent, making the electron gas effectively two-dimensional, albeit strongly anisotropic in this temperature domain. This is known to affect the interference in a particular way depending on the energy distance $\Lambda(\ell)$ from the Fermi surface in the RG flow. At high energy, when $\Lambda(\ell) \gg t'_\perp$, the flow essentially coincides with the 1D limit where the interference is maximum, although subjected to the above conditions between $T$ and $\omega_D$. When $\Lambda(\ell) \ll t'_\perp$, the interference between the Peierls and Cooper channels is affected by the coherent warping of the Fermi surface and ultimately nesting alterations at $\Lambda(\ell) < t'_\perp$. Both generate a momentum dependence of the coupling constants $\{14,16\}$ which reflects in the end the nature of the electron gas instability at $T_\mu$.

IV. RESULTS

A. Instabilities for weak phonon-mediated interaction

The integration of the RG equations up to $\ell \to \infty$ for the couplings $\{14,16\}$ and pair vertices $\{21,23\}$ leads to the temperature dependence of the selected susceptibilities as a function of antinesting, $t'_\perp/t'_\perp$, phonon frequency, $\omega_D/t'_\perp$, normalized by $t$ interchain hopping; and weak Ph-M interaction parameterized by the ratio

$$\tilde{g}_{\text{ph}} \equiv |g_{\text{ph}}|/g_1,$$  \hspace{1cm} (24)

here normalized by the strength of non retardated repulsive interaction $g_1$. The main features the influence of weak Ph-M coupling has on the temperature dependence of relevant susceptibilities are summarized in Fig.1 at small and intermediate antinesting parameter $t'_\perp$, and different $\omega_D$. In Fig.1a, $t'_\perp$ is taken sufficiently small so that

nesting promotes a singularity in $\chi_{\text{SDW}}$, indicating an instability against the onset of SDW order at $T_{\text{SDW}}$. As for the correlations in the BOW and SCd channels, the related susceptibilities are non singular and remain small. According to Fig.1a, the presence of an even small $|\tilde{g}_{\text{ph}}|$ at sizeable $\omega_D$ is sufficient to cause a noticeable increase of $T_{\text{SDW}}$ compared to the purely electronic limit.

At the outset, the strengthening of SDW instability takes its origin in the momentum transfer dependence of Ph-M interaction in Eqs. (10-13) at $\ell = 0$, resulting in a reduction of the backscattering and an increase of Umklapp term. As discussed in more detailed in Sec. IV C 1 both concur to an increase of antiferromagnetic spin exchange between itinerant spins. The above effects on scattering amplitudes are magnified by the RG flow. Moreover, the reinforcement of SDW becomes the most efficient in the temperature range $T < \omega_D$ owing to the reduction of retardation. It is where the Ph-M part acts progressively as non retarded contributions in all open diagrams such as the ladder and vertex corrections of $\{14,16\}$ and $\{21\}$. The influence of Ph-M coupling on SDW correlations will then naturally depend on the value of phonon frequency $\omega_D$. Fig.1c shows indeed that lowering $\omega_D$ reduces the enhancement of $T_{\text{SDW}}$ at low antinesting, an indication of a positive isotope effect on SDW (see Sec. IV C 1).

![Figure 1](image-url)
At large enough $t'_\perp$, nesting turns to be sufficiently poor to prevent the occurrence of SDW. The instability of the metallic state no longer takes place in the density-wave channel, but rather shows up by interference in the metallic state no longer takes place in the density-channel. The instability of the spin fluctuations is singled out. The growth of pronounced, though non singular, enhancement of $\chi_{SDW}$ is shown in Fig. 1-b where at non zero $|g_{ph}|$ a more pronounced, though non singular, enhancement of $\chi_{SDW}$ is found above $T_c$. This feature signals that the reinforcement of spin fluctuations persists relatively deep in the normal state. In Fig. 1-d the effect of $\omega_D$ on both $T_c$ and normal state spin fluctuations is singled out. The growth of $T_c$ with $\omega_D$ is correlated with the increase of spin correlations above $T_c$. In this part of the Figure, we note that the onset spin fluctuations reinforcement takes place at $T < \omega_D$ where $\chi_{SDW}$ clearly separates from the static $\omega_D \to 0$ limit; it signals the growth of ladder and vertex corrections following a reduction of retardation. The enhancement of spin fluctuations in the normal phase will be analyzed in Sec. IV D where it is found to follow a Curie-Weiss temperature dependence, which is comparatively more pronounced than the one occurring in the purely electronic limit.

Concerning BOW correlations, Fig. 1 shows that for weak $|g_{ph}|$, these remain weakly enhanced. However, as it will be shown next, the situation qualitatively changes when $|g_{ph}|$, though still small, reaches some critical value.

### B. Phase diagrams

#### 1. Spin-density-wave versus d-wave superconductivity

We now consider the sequence of instabilities of the metallic state as a function of $t'_\perp$ in order to construct the phase diagrams at weak Ph-M couplings. This is shown in Fig. 2a. At small $|g_{ph}|$ and for a sizeable $\omega_D$, the system remains unstable to the formation of a SDW state with a $T_{SDW}$ that displays the characteristic monotonic decrease with increasing $t'_\perp$ [37,47,52,56].

At the approach of a well defined antinesting threshold $t'_\perp^*$, however, $T_{SDW}$ undergoes a critical drop that terminates at $t'_\perp^*$ where SCd begins at its peak value denoted by $T'^*_{c}$. Above, $T_c$ shows a continuous decrease with $t'_\perp$ that correlates with the reduction of SDW fluctuations as the source of Cooper pairing.

As stressed above, Fig. 2a confirms that the Ph-M coupling, albeit small, reinforces both $T_{SDW}$ and $T_c$ for all $t'_\perp$, including the critical value $t'_\perp^*$ at which superconductivity emerges. We also note from Fig. 2a that this reinforcement reduces the sharpness of its critical drop at the approach of $t'_\perp^*$, an effect that carries over in the superconducting sector where the reduction of $T_c$ with $t'_\perp$ turns to be less rapid.

Also shown in the Figure are the instability lines in the static, $\omega_D \to 0$ limit (continuous lines of Fig. 2a). Retardation effects are found to be very important at the approach of the critical value $t'_\perp^*$ for (a), the SDW/SCd and (b) BOW/SCd sequences of instabilities at $\omega_D/t'_\perp = 0.4$. In (a), the continuous lines correspond to the instabilities lines in the adiabatic $\omega_D \to 0$ limit and the dashed lines show the variation of the Curie-Weiss scale $\Theta$ of $\chi_{SDW}$ [Eq. (26)] as a function of $t'_\perp$ in the superconducting region.

#### 2. Bond-order-wave versus superconductivity

By increasing further the strength of Ph-M coupling for the same $\omega_D$, the Fig. 2b shows that the SDW-SCd sequence of instabilities as a function of $t'_\perp$ is only maintained up to a critical, $|g^c_{ph}|$ ($\approx 0.52$ for the parameters used), above which SDW turns out to be no longer stable and replaced by the onset of a non magnetic BOW state.
at $T_{\text{BOW}}$. The typical variation of relevant susceptibilities in the BOW sector of the phase diagram are given in Fig. 3a. The BOW instability that takes place from the metallic state corresponds to the onset of a Peierls, though correlated, lattice distorted state. A remarkable feature of the phase diagrams of Fig. 2b is that above $|\tilde{g}_{\text{ph}}|$ and at not too small $\omega_D$, the BOW instability continues to be followed by SCd superconductivity at $t'_{\perp} > t_{\ast}^*_{\perp}$. In these conditions, however, $T_c$ becomes a decreasing function of $|\tilde{g}_{\text{ph}}|$. This is depicted in Fig. 4 where it behaves so after having reached its maximum at the boundary $|\tilde{g}_{\text{ph}}|$ where SDW, BOW are found to be essentially degenerate and at their maximum strength. It is worth noticing that at the boundary $T_c$, has increased by a factor of four or so compared to the purely electronic case. Despite the presence of a Peiersl lattice distorted state, the essential role played by spin fluctuations in the emergence of SCd at $t'_{\perp} \geq t_{\ast}^*_{\perp}$ remains. This is confirmed in Fig. 3b where $\chi_{\text{SDW}} > \chi_{\text{BOW}}$ over a large temperature interval at the approach of $T_c$ in the normal state.

Another surprising feature of the phase diagram in the $|\tilde{g}_{\text{ph}}| > |\tilde{g}_{\text{ph}}^c|$ is found at low phonon frequency. Fig. 5 shows that in the small $\omega_D$ range, the BOW ordering at $t'_{\perp} \geq t_{\ast}^*_{\perp}$ is followed by a triplet SCf instability instead of a SCd one. Since small phonon frequency increases retardation, it reinforces closed loop diagrams in the RG flow, related to density or charge fluctuations. Bond charge correlations are then increased with respect to their spin counterpart and for dominant repulsive interactions, this leads to SCf type of superconductivity. The triplet-singlet competition is in a way reminiscent of the one found when a weak repulsive (non retarded) interchain interaction is added to the quasi-1D electron gas model. The latter coupling is also known to boost exclusively charge fluctuations, in a way similar the electron-phonon interaction does in the present case when strong retardation is present; the same interchain coupling is also known to promote a SDW to BOW crossover in the density-wave instabilities at low antinesting. Cranking up $\omega_D$ results in the progressive enhancement of open diagrams which are responsible for spin fluctuations and d-wave superconductivity. Although from Fig. 5 the BOW ordering is weakly affected, whereas a SCf $\rightarrow$ SCd crossover is indeed found to occur at small $\omega_D/t_{\perp}$ ($\sim 0.1$ for the parameters used).

### C. Isotope effects

#### 1. Spin-density-wave and d-wave superconductivity

In the preceding paragraphs we mentioned on several occasions the positive influence of raising $\omega_D$ on the strength of SDW and SCd instabilities. This result ob-
tained by varying the molecular mass \( M \) at fixed elastic constant \( \kappa \) [\( g_{\text{ph}} \) kept constant according to Eq. (11)], corresponds to a positive isotopic effect. The reinforcement can be understood as a modification of the effective antiferromagnetic exchange by retardation. For itinerant electrons, the total scattering amplitudes \( g_2 \) and \( g_3 \) in the action \( S_I \) contribute an exchange term of the form

\[
S_I^\text{ex} = \pi v_F \frac{T}{\mathcal{L} N_\perp} \sum_{(k),q_P} \frac{1}{2} (g_2 + g_3) \overline{\bar{S}}_{k, q_P} \bar{S}_{k, -q_P}, \tag{25}
\]

where \( \bar{S}_{k, q_P} = \frac{1}{2} \bar{\psi}_{+a,b}^\dagger (k + q_P) \sigma^{a\beta} \bar{\psi}_{-,b}(k) + \text{c.c.} \) is the Fourier-Matsubara component of the SDW spin density. Thus in weak coupling, the combination \( \frac{1}{2} (g_2 + g_3) \) corresponds to a momentum and frequency dependent antiferromagnetic exchange interaction. It is the same exchange term that governs the enhancement of the vertex part \( g_{\text{SDW}} \) for the SDW susceptibility [See Eq. (21)]. Its growth with decreasing \( \Lambda(\ell) \) results from the multiple exchange scattering of virtual \( q_P \) electron-hole pairs carried by ladder and vertex corrections in the flow equations [15,16]. As to the backscattering term, \( g_1 \), its role is indirect. This coupling carries a large longitudinal momentum transfer corresponding to a repulsive short-range contribution along the chains, which tends to damp the amplitude of both \( g_2 \) and \( g_3 \), reducing the exchange scattering and then SDW correlations.

Therefore the combined influence of a \( g_1 \) reduction and a \( g_3 \) increase by Ph-M interactions in [10] and [13] will boost \( g_2 \) and in turn \( g_3 \) and antiferromagnetic exchange. As mentioned earlier, however, this additional and positive input of Ph-M interaction reaches its maximum impact in the temperature domain \( T < \omega_D \), namely where retardation effects on virtual electron-hole pair scattering processes become small, hence the isotopic effect on SDW.

The increase of \( T_{\text{SDW}} \) with \( \omega_D \) is illustrated in Fig. 6 for \( |\tilde{g}_{\text{ph}}| = 0.1 \) and different values of \( t'_\perp \) in the SDW part of the phase diagram. At relatively small \( t'_\perp \), that is well into the SDW sector, the \( T_{\text{SDW}} \) undergoes a monotonic but weak increase over all the frequency range of phonons, a consequence of ladder and vertex corrections to the antiferromagnetic exchange that grow in importance by increasing \( \omega_D \). It is worth noticing that in the adiabatic limit, \( T_{\text{SDW}} \mid_{\omega_D \to 0} \) is found to be slightly larger than the \( T_{\text{SDW}} \mid_{\omega_D = \infty} \) obtained in the absence of Ph-M interaction [see Fig. 2a]. This indicates that static phonons still have a positive influence on the exchange interaction (25) and the strength of SDW correlations. This adiabatic effect finds a certain echo in the strong coupling – Hubbard interaction – case where dynamical mean field theory calculations do predict an enhancement of antiferromagnetic exchange between localized spins by zero frequency phonons.13 Here the static enhancement essentially results from the mixing of Ph-M interaction to the non retarded Coulomb terms in the RG flow; it vanishes by taking \( q_1 \to 0 \) in Eqs (10), (12) and (13), a result found in the limit of pure the electron-phonon coupling.

When \( t'_\perp \) increases and approaches the critical domain where, the drop in \( T_{\text{SDW}} \) becomes according to Fig. 2a essentially vertical, and the isotope effect becomes huge as traced in Fig. 6a. Close to \( t'_\perp \), the reinforcement of SDW correlations by an even small increase in \( \omega_D \) gives rise to a large increase of \( T_{\text{SDW}} \). This is not the consequence of nesting improvement, but rather the result of stronger nesting deviations needed to counteract the reinforcement of SDW instability by Ph-M interactions. For \( t'_\perp \) slightly above \( t'_\perp^* \), Fig. 6a features the interesting possibility of a Scd to SDW transition as a function of \( \omega_D \).

The positive isotopic effect carries over into the Scd side of the phase diagram where \( T_c \) is found to increase with \( \omega_D \) at different \( t'_\perp \), as shown in Fig. 6b. This is directly associated with the \( \omega_D \) dependent reinforcement of spin correlations in the normal state as already pointed out in Fig. 4d, which strengthens the pairing interaction in the Scd channel. Although the isotopic effect is slightly larger in amplitude near the critical \( t'_\perp^* \), it remains of comparable size at arbitrary value of anisotuning with a power law \( T_c \sim \omega_D^\alpha \) that takes place at intermediate frequency with an exponent \( \alpha \approx 0.24(\equiv d \ln T_c/d \ln \omega_D) \), a value virtually independent of \( t'_\perp \) [see Fig. 6b] and \( |\tilde{g}_{\text{ph}}| \). At high phonon frequency where the ratio \( \omega_D/T_c \) becomes very large, retardation effects become negligible and \( T_c \) tends to level off with frequency. This saturation probably reflects the limitation of using a finite number of Matsubara frequencies in the mean-field approximation of the loop convolution over frequency.

2. Bond order wave versus superconductivity

In the BOW regime above \( |\tilde{g}_{\text{ph}}| \), the isotope effect on \( T_{\text{BOW}} \) has the opposite sign. At low \( t'_\perp \) for instance, Fig. 7a shows that \( T_{\text{BOW}} \) decreases monotonically with
the reduction becomes increasingly large with $t'_\perp$ which also softens the lattice distortion through nesting alteration. A reduction of $T_{\text{bow}}$ with $\omega_D$ is a consequence of the growth of non adiabaticity of the phonon field, a well known factor to be at play in the reduction of the Peierls distortion gap in purely electron-phonon models in one dimension. From a diagrammatic point of view, non adiabaticity is a quantum effect again tied to the unlocking of Ph-M interaction to open diagrams and thus to quantum interference between electron-hole and Cooper pairing at the one-loop level. In contrast to the SCd-SDW mixing, the interference is in the present case destructive: Cooper pairing contributions have opposite sign and this reduces the temperature scale of BOW ordering. The onset of a quantum to classical crossover for the BOW state is perceptible at $\omega_D/2T_{\text{bow}}|_{\omega_D\to0} \sim 1$, as it is found to occur in the pure electron-phonon limit.

Above $t'_\perp$, but for small $\omega_D$, we still observe an inverse isotope effect for the $T_c$ of triplet, SCf superconductivity, as shown in Fig. 7b. This confirms the role of BOW fluctuations in the existence of SCc ordering at repulsive coupling. This is further supported when $\omega_D$ increases and crosses the critical value at which SCd reappears in Fig. 5. Then the isotope effect becomes once again positive as a consequence of the growth of antiferromagnetic exchange and spin fluctuations that govern the $d$-wave Cooper pairing.

**D. Normal state**

Now that the increase of the influence of electron-phonon interactions on the temperature scales for ordering has been examined, one can turn our attention on the influence of a weak Ph-M interaction on spin correlations of the normal phase above $T_c$. This is done for the SDW-SCd sequence of instabilities. In Fig. 8a, we show the temperature dependence of the inverse SDW susceptibility at small $|\tilde{g}_{ph}|$ and various strengths of antinesting. At sufficiently high $t'_\perp > t^*_\perp$, $\chi_{\text{SDW}}$ decays essentially linearly from the high temperature region and extrapolates towards a critical point at a finite $T_{SDW}$. However, as the temperature is lowered at $T < t'_\perp$, nesting deviations becomes coherent and the susceptibility undergoes a change of regime and ceases to be critical. Nevertheless, according to Fig. 5, $\chi_{\text{SDW}}$ keeps decreasing and extrapolates to a non zero intercept at $T = 0$ and a finite slope at the end point $T_c$.

This non singular growth of spin correlations in the metallic state, which persist down to $T_c$, can be well described by a Curie-Weiss form (continuous lines in Fig. 8):

$$\chi_{\text{SDW}} = \frac{C}{T + \Theta},$$

extending up to the temperature $T_{CW}$ for the onset of the Curie-Weiss regime, which is about ten times $T_c$ in temperature at the frequency used in the Figure ($T_{CW}$ decreases when $\omega_D$ is lowered [see Fig. 4d]). Here the Curie-Weiss scale $\Theta$ stands as a characteristic energy for SDW fluctuations, which is defined positive when $t'_\perp > t^*_\perp$. The Curie-Weiss behaviour has been already found in the purely electronic case. It results from the positive feedback of SCd pairing on SDW correlations, a consequence of constructive interference between these channels of correlations. The presence of Ph-M interactions clearly reinforces this behavior. As shown in Fig. 8b, cranking up $|\tilde{g}_{ph}|$ leads to the decrease of the Curie-Weiss scale $\Theta$, and an increase of the constant $C$. This is consistent with an increase of the SDW correlation length $\xi \sim (T + \Theta)^{-1/2}$, in tune with the increase of $T_c$ discussed above. The softening of $\Theta$ in Fig. 8 carries on until $t'_\perp$ reaches $t^*_\perp$ where $\Theta \to 0$. There the system would then become quantum critical with $\chi_{SDW} \sim 1/T$ and $T_{SDW} \to 0$, had it not been the presence of superconductivity at a finite $T_c$ that prevents to reach the SDW quantum critical point. Below $t^*_\perp$, $\Theta < 0$ and the system enters in the SDW sector with a finite $T_{SDW}(\equiv -\Theta) > T_c$.

At the approach of $t^*_\perp$, $\Theta$ is well fitted by the quantum scaling form

$$\Theta \approx A(t'_\perp - t^*_\perp)^{\eta},$$

with an exponent $\eta \simeq 1$, consistently with product $\eta = \nu z$ of the correlation length ($\nu = 1/2$) and the dynamical ($z = 2$) exponents for SDW at the one-loop level. The linear profile of $\Theta$ near $t^*_\perp$ is illustrated in Fig. 2a. From the latter Figure and Fig. 8b, the coefficient $A$ decreases relatively quickly with $|\tilde{g}_{ph}|$.

**V. DISCUSSION AND CONCLUSION**

In this work we used a weak coupling RG approach to examine the influence of the tight-binding electron-
phonon interaction on the interplay between magnetism and superconductivity in quasi-one-dimensional conductors. When the phonon-mediated interaction remains weak and subordinate to the direct Coulomb terms of the electron gas, the RG flow of scattering amplitudes is found to be distorted for particular momentum transfers. This reinforces the antiferromagnetic exchange between itinerant spins and yields an increase of the temperature scale of SDW ordering. By introducing enough nesting deviations in the electron kinetics, SDW ordering is inhibited, but magnetic reinforcement by the electron-phonon interaction persists and shifts by interference in the superconducting channel. D-wave Cooper pairing and $T_c$ then becomes enhanced compared to the purely electronic situation. These properties were found to be affected by retardation effects linked to the exchange of low energy acoustic phonons that modulate the strength of virtual electron-hole scattering entering in the antiferromagnetic exchange term of the electron gas. This gives rise to a positive isotope effect on SDW ordering temperature, which carries over beyond critical antinesting $t_1^*$ where d-wave superconductivity is found.

Our results also revealed that such an increase for $T_c$ is preceded by the strengthening of spin fluctuations in the normal phase. This is manifest in a more pronounced Curie-Weiss SDW susceptibility compared to the purely electronic situation, a consequence of self-consistency between d-wave Cooper pairing and spin fluctuations, a positive interference effect whose amplitude scales with $T_c$.

We have also established the range of electron-phonon interaction beyond which SDW ordering is no longer stable against the BOW or Peierls distorted state. In these conditions, the Peierls ordering was found to be followed above critical antinesting by either d-wave or amazingly triplet f-wave superconductivity depending if retardation effects are weak or strong, respectively. Isotope effect which is found to be negative in the triplet SCf sector and positive in SCd reflects the origin of pairing interaction in both situations, namely BOW fluctuations in the former case and SDW ones in the latter.

The relevance of the above results for concrete materials like the Bechgaard salts is of interest. Superconductivity emerges in these systems where SDW state ends under pressure. Their normal state is characterized by important spin fluctuations over a large temperature interval above $T_c$ whose amplitude scales with the one of spin correlations under pressure, as made abundantly clear by NMR experiments.

Our findings show that intrachain repulsive interactions are dominant in these materials. While repulsive interactions are known to be able to generate on their own the sequence of SDW-SCd instabilities as a function $t_1^*$ in the quasi-1D electron gas model, the present results show, however, that the addition of relatively small tight-binding electron-phonon interaction, which would be compatible with diffuse X-ray scattering experiments, are far from being an obstacle for superconductivity. When subordinate to the purely electronic repulsion, the phonon-mediated interaction can indeed play a very active part in assisting anti ferromagnetism in the emergence of d-wave superconductivity with a stronger $T_c$.

Although the typical range of values taken by the electron-phonon matrix element has not been determined with great accuracy in materials like the Bechgaard salts [see for example Ref.], the results of the present paper suggest that it should be small in amplitude compared to direct interactions. This is supported by the stability of the SDW state against the Peierls distortion, which from the above results is found to be assured only within a finite interval of weak phonon-mediated interaction at essentially arbitrary retardation. Therefore the absence of the Peierls phenomena in the Bechgaard salts may be viewed as a mere consequence of the weakness of the electron-phonon coupling constant in these materials. This view would be consistent with previous estimations made from optics, and also from the fact that the only few materials showing a lattice distorted phase belong to the more correlated isostructural compounds of the (TMTTF)$_2$X series, the so-called Fabre salts. A compound like (TMTTF)$_2$PF$_6$, for instance, is well known to undergo a spin-Peierls transition within a strongly correlated Mott state. Less than 10 kbars of pressure is sufficient to weaken the coupling of phonons to electrons and transform this state into one with antiferromagnetic Néel order. 30 kbars separate the latter from the sequence of SDW-SC instabilities found in the prototype compound (TMTSF)$_2$PF$_6$ of the Bechgaard salts, in line with a coupling to phonons that remains in the background of direct Coulomb terms.

As to the possible experiments able to disentangle the part played by phonon-mediated interaction on the SDW-SC sequence of instabilities seen in molecular ma-

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**FIG. 8:** The temperature dependence of the normal phase inverse SDW susceptibility at different antinesting (a) and electron-phonon interaction strength (b). The straight lines correspond to the Curie-Weiss fit [Eq. 26].
terials like the Bechgaard salts, isotope effect measurements would be certainly of interest, especially near the quantum critical point where the present results show that it becomes huge at the approach of $t'^*_1$ on the SDW side of the phase diagram. While isotope effect in molecular materials proves to be difficult to realize in practice due to the complications of controlling all other parameters following a change in the mass $M$ of molecular units (volume of the unit cell, disorder, etc.), the $^{13}$C enrichment of the TMTSF molecular units stands probably as the best way to limit these side effects and to test some of the results obtained here. According to Fig. 2a, for instance, a finite reduction in $\omega_D$ would induce a decrease in the critical $t'^*_1$ at which superconductivity occurs. Practically, one should therefore expect a downward shift of the critical pressure for the emergence of superconductivity and a decrease in the maximum $T'_c$ at that point and beyond on the pressure axis.

Another possible signature of the reinforcement of anti ferromagnetism by electron-phonon interaction in the Bechgaard salts may be found in its influence on the Curie-Weiss behaviour of SDW susceptibility which governs the enhancement of the NMR spin-lattice relaxation rate observed down to $T[100K][57,58]$ . While the quasi-1D electron gas model with purely electronic interactions does predict a critical linear suppression of the Curie-Weiss scale $\Theta$ for spin fluctuations as $t'_1 \rightarrow t'^*_1$, its slope [coefficient $A$ of Eq. (27)] proves to be significantly larger than the one seen in experiments. In this regard, we have found that adding a small $|g_{ph}|$ is sufficient to reduce the downslope of $\Theta$ to values congruent with experiment, and this over a large range of retardation. This supports the view of an active role played by the electron-phonon interaction in the properties of the metallic state, especially those associated to quantum criticality at $t'^*_1$.

In this paper, we have dealt exclusively with the coupling of correlated electrons to low energy acoustic phonons within the tight-binding scheme for the electronic structure, a coupling well known to be responsible of electronically driven structural instabilities in low dimensional molecular materials. We did not consider intramolecular – Holstein – phonon modes, yet also well known to be present. Their classification alongside their small - coupling to electrons in (TMTSF)$_2$X have been obtained from infrared optical studies. These molecular phonons are characterized by relatively large energies and weak retardation effects compared to acoustic branches considered above. Their influence can then in first approximation be incorporated through a redefinition of the non retarded terms, amounting to a small and same down shift of the couplings $g_i$ of the electron gas model. Since the latter couplings were taken as phenomenological constants whose range were fixed by experiments, the values taken in the present work should embody to some extent the influence of intramolecular phonons.

In conclusion, we have performed a finite temperature renormalization group analysis of the quasi-1D electron gas model with non retarded electron-electron couplings and phonon-mediated interactions of the tight-binding electronic structure. For a phonon-mediated interaction that is weak compared to non retarded terms, we found a reinforcement of anti ferromagnetism and its transition toward superconductivity under bad nesting conditions of the electron spectrum. The weakness of phonon-mediated interactions acts as a decisive factor for the stability of anti ferromagnetism against the Peierls phenomena in low dimensional conductors. It is likely that these retarded interactions have also a built-in positive impact in the observation of organic superconductivity on the verge of anti ferromagnetism in the Bechgaard salts.

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