STATIONARITY OF ENTRANCE MARKOV CHAINS AND
OVERSHOOTS OF RANDOM WALKS

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Abstract. Consider the Markov chain of overshoots over the zero level of a one-dimensional random walk \( S \) that oscillates between \(-\infty\) and \(+\infty\). We establish an explicit formula for an invariant measure of this chain and prove its uniqueness and ergodicity. The main corollary is a limit theorem for the number of level crossings on a large time interval for walks with zero mean and finite variance of increments.

In the context of these results about the invariant measure, it is natural to consider the random walk \( S \) as a “stationary” Markov chain starting under the Lebesgue measure on \( \mathbb{R} \) and use the technique of inducing from infinite ergodic theory. We develop this approach in a general setting and apply it to the chain of overshoots, which is a particular case of the entrance Markov chain obtained by sampling an arbitrary Markov chain \( Y \) (in a Polish space, and possibly transient) at the moments when \( Y \) enters a fixed set \( A \) from its complement \( A^c \). In addition, we consider the exit and induced Markov chains, obtained by sampling \( Y \) at the exit times from \( A^c \) into \( A \) and restricting \( Y \) to \( A \), respectively. This paper provides a framework for analysing invariant measures of such entrance, exit, and induced chains in the case when the initial chain \( Y \) has a known invariant measure. We find invariant measures explicitly, and then study their uniqueness (and ergodicity) assuming that the chain \( Y \) is topologically recurrent and weak Feller.

1. Introduction

1.1. Introduction and description of the main results. Let \( S = (S_n)_{n \geq 0} \) with \( S_n = S_0 + X_1 + \ldots + X_n \) be a one-dimensional random walk with independent identically distributed (i.i.d.) increments \( X_1, X_2, \ldots \) and the starting point \( S_0 \) that is a random variable independent of the increments. Assume that the walk oscillates, that is \( \lim \sup S_n = -\lim \inf S_n = \infty \) a.s. as \( n \to \infty \). In particular, \( S \) can be transient. Define crossing times of the zero level as

\[ T_0 := 0, \quad T_n := \inf \{ k > T_{n-1} : S_{k-1} < 0, S_k \geq 0 \text{ or } S_{k-1} \geq 0, S_k < 0 \}, \quad n \in \mathbb{N}, \]

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and let
\[ O_n := S_{T_n}, \quad U_n := S_{T_n-1}, \quad n \in \mathbb{N} \] (1)
be the corresponding overshoots and undershoots; put \( O_0 = U_0 := S_0 \). The choice of zero is arbitrary and can be replaced by any fixed level.

The sequence of overshoots \( O = (O_n)_{n \geq 0} \) is a Markov chain. The sequence of undershoots \( U = (U_n)_{n \geq 0} \) also forms a Markov chain. The latter statement is less intuitive than the former one but it is still easy to check. We are mainly interested in overshoots but our methods will also yield results for undershoots. The chain \( O \) has periodic structure since its values at consecutive steps must have different signs. Essentially, it suffices to study the non-negative Markov chain \( O = (O_n)_{n \geq 0} \) of overshoots at up-crossings, defined by \( O_n := O_{2n-3(S_0<0)} \) for \( n \geq 1 \) and starting at \( O_0 := S_0 \).

This paper was motivated by our interest in stationarity and stability properties of the Markov chain of overshoots \( O \). First, we will show (Theorem 1) that this chain has an invariant measure \( \pi \) of the following explicit form. Let the state space of the walk \( S \), denoted by \( \mathcal{Z} \), be the minimal closed (in the topological sense) subgroup of \((\mathbb{R}, +)\) containing the topological support of the distribution of \( X_1 \), and let \( \lambda \) be the Haar measure on \((\mathcal{Z}, +)\). Put
\[ \pi(dx) := \frac{c_1}{2} [\mathbf{1}_{[0, \infty)}(x)\mathbb{P}(X_1 > x) + \mathbf{1}_{(-\infty, 0)}(x)\mathbb{P}(X_1 \leq x)] \lambda(dx), \quad x \in \mathcal{Z}, \]
where either \( c_1 := 1 \) if \( \mathbb{E}[X_1] = \infty \) or \( c_1 := 2/\mathbb{E}[X_1] \) otherwise, and \( \lambda \) is normalized such that \( \lambda(\mathcal{Z} \cap [0, x)) = x \) for positive \( x \in \mathcal{Z} \). Note that \( \pi \) is a finite measure (and a probability) if and only if \( \mathbb{E}[X_1] < \infty \), in which case by the assumption of oscillation we must have \( \mathbb{E}[X_1] = 0 \). Second, we will show (cf. Theorem 4) that under the additional assumption of topological recurrence of the walk \( S \), \( \pi \) is the unique up to a multiplicative factor locally finite invariant measure of the chain \( O \), and this chain is ergodic when started under \( \pi \).

We set aside the delicate question of convergence of \( O_n \) towards \( \pi \) as \( n \to \infty \) (when \( \mathbb{E}[X_1] = 0 \)). This problem will be considered, under additional smoothness assumptions on the distribution of increments of \( S \), in the companion paper [23] using entirely different methods. The main difficulty is that no standard criteria of convergence apply to \( O \); in particular, this chain in general is neither weak Feller (Remark 3) nor \( \psi \)-irreducible. By the same reasoning, a priori it is unclear if the chain \( O \) has a stationary distribution regardless of moment assumptions on \( S \).

As for the proofs, we first derived the shape of the stationary distribution \( \pi \) of the chain of overshoots \( O \) in a special case and then established the invariance of \( \pi \) in general via a purely probabilistic argument based on reversibility. We will present this proof here since it reveals a notable distributional symmetry hidden in the trajectory of arbitrary non-symmetrically distributed random walk, and also illustrates the difficulty of checking the invariance directly. On the other hand, although the reversibility argument appears to allow generalizations, it does not explain how to find the invariant measure.

Then we found an entirely different approach which allowed us to construct invariant measures and prove their uniqueness and ergodicity in a much more general context than level-crossings of one-dimensional random walks. The Markov chain of overshoots \( O \) at up-crossings of the zero level is obtained by sampling the random walk \( S \) at the moments of entrance to the set \([0, \infty)\) from \((-\infty, 0)\). Similarly, we can consider the entrance Markov
chain, denoted by $Y \rightarrow A$, constructed from an arbitrary Markov chain $Y$ sampled at the moments of entry into an arbitrary fixed set $A$ from its complement $A^c$. We also define the exit Markov chain, denoted by $Y^A \rightarrow$, obtained by sampling $Y$ at exit times from $A^c$ to $A$. In this notation, we have $S \rightarrow [0, \infty) = O$. The exit chain $S(-\infty, 0) \rightarrow$ corresponds to the chain of undershoots at up-crossings.

We will show (Theorem 2) that for any Borel set $A$ and any Markov chain $Y$ taking values in a Polish space with a known $\sigma$-finite invariant measure $\mu$ and satisfying certain mild assumptions, the entrance chain $Y \rightarrow A$ and the exit chain $Y^A \rightarrow$ have respective Borel invariant measures

$$
\mu_{\text{entr}}^A(dx) = P_x(\hat{Y}_1 \in A^c)\mu(dx) \text{ on } A, \quad \mu_{\text{exit}}^A(dx) = P_x(Y_1 \in A)\mu(dx) \text{ on } A^c,
$$

where $\hat{Y}$ is a dual Markov chain for $Y$ with respect to $\mu$ (cf. Section 5.1). The symmetry of these formulas is due to the fact the exit chain $Y^A \rightarrow$ of $Y$ from $A^c$ into $A$ can be regarded (equals in distribution when started under $\mu$) as the entrance chain $\hat{Y} \rightarrow A^c$ of the dual chain $\hat{Y}$ from $A$ into $A^c$. As for the assumptions on the chain $Y$, we either assume recurrence or, essentially, that the chains $Y$ and $\hat{Y}$ visit both sets $A$ and $A^c$ infinitely often $P_x$-a.s. for $\mu$-a.e. $x \in A$ and $x \in A^c$, respectively. Let us stress that under the latter assumptions $Y$ can be transient: examples include random walks that oscillates with $A = [0, \infty)$.

The main approach of this paper is built on inducing, a basic tool of ergodic theory, introduced by S. Kakutani in 1943. We will need to use infinite ergodic theory since in many cases of interest the invariant measure $\mu$ of $Y$ is infinite, e.g. the Haar measure $\lambda$ on $\mathcal{Z}$ for the random walk $S$. In this case, the measure $\lambda_{\text{entr}}^{[0, \infty)} + \lambda_{\text{exit}}^{(-\infty, 0)}$, which equals $2c_1^{-1}\pi$ by (2), is invariant for the chain of overshoots $\mathcal{O}$. The same approach works for random walks in any dimension (Section 6). As for ergodicity, the method of inducing easily implies ergodicity of the entrance and the exit chains from that of the chain $Y$.

Our further result (Theorem 3) implies that under the topological assumptions of recurrence, irreducibility, and weak Feller property of the chain $Y$, and non-emptiness of the interiors of $A$ and $A^c$, the invariant measure exists and is unique in the class of locally finite Borel measures simultaneously for the chains $Y, Y^A \rightarrow, Y^A \rightarrow$. In particular, since the Haar measure is known to be a unique locally finite invariant measure of the topologically recurrent random walk $S$ on $\mathcal{Z}$, this yields uniqueness of the invariant measure $\pi$ for the chain of overshoots $\mathcal{O}$. On the other hand, the result can be useful (Proposition 5) for proving existence of a locally finite invariant measure of the weak Feller chain $Y$ when its path empirical distributions are not tight and so the classical Bogolubov–Krylov theorem does not apply.

It is worth noting that some of our results on the entrance and exit chains can be also obtained by considering the induced chain on the product space formed by the pair of two consecutive values of the chain $Y$ sampled only when it belongs to the set $A^c \times A$. For completeness of exposition, we also give versions of the above statements on existence and uniqueness for general induced chains (obtained by sampling $Y$ when it is in an arbitrary set $A$), as this takes a little effort using the technique developed.

We are not aware of any applications of inducing in problems related to random walks. In the context of level-crossings by one-dimensional random walks, the classical and universal probabilistic tool is the Wiener–Hopf factorization. The literature is vast; let us mention
Baxter [3], Borovkov [8], and Kemperman [21], which are the most relevant works to the questions considered here. These papers rely fully on the Wiener–Hopf factorization (which does not yield much for our problem), in contrast to our entirely different approach. Moreover, to the best of our knowledge, there are no works dedicated specifically to random walks based on the idea of considering them as “stationary” processes starting from the Haar measure. Finally, we do not know of any works on entrance and exit Markov chains.

We conclude the paper with a number of results on level-crossings of random walks presented in Section 7. The main result there is the limit theorem for the number of level-crossings motivated below in Section 1.2. We also present several formulas for expected occupation times between level-crossings.

1.2. Motivation and related questions. There are several good reasons to study overshoots of random walks besides purely theoretical interest. First, there is connection to local time. Namely, the Markov chain $O$ features in Perkins’s [27] definition of the local time of a random walk. Note that there is no conventional definition of this notion, see Csörgő and Révész [11] for other versions. Denote by

$$L_n := \max\{k \geq 0 : T_k \leq n\}$$

the number of zero-level crossings of the walk by time $n$, and let $\ell_0$ be the local time at level 0 at time $1$ of a standard Brownian motion. Perkins [27, Theorem 1.3] proved that for a zero mean random walk $S$ with finite variance $\sigma^2 := \mathbb{E}X_1^2$ and starting at $S_0 = 0$, one has

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{L_n} |O_k| \xrightarrow{d} \ell_0.$$  \hspace{1cm} (3)

To the best of our knowledge, all other limit theorems for local time (under either definition) of a random walk with finite variance require additional smoothness assumptions on the distribution of increments.

Under the above assumptions on the random walk, by ergodicity of the Markov chain $O$, we have $\frac{1}{n} \sum_{k=1}^{n} |O_k| \rightarrow \int_{\mathbb{R}} |x| \pi(dx)$ a.s. for $\pi$-a.e. starting point $S_0 = x \in \mathbb{Z}$. We will show that this convergence actually holds for every $x$, hence [3] immediately gives a limit theorem for the number of level crossings $L_n$ divided by $\sqrt{n}$ (Theorem 6). Under the optimal moment assumption $\mathbb{E}X_1^2 < \infty$, limit theorems of such type were first obtained by Borodin in the early 1980s, who studied more general questions of convergence of additive functionals of consecutive steps of random walks; see Borodin and Ibragimov [7, Chapter V] and references therein. However, Borodin’s method assumes that the distribution of increments of the walk is either aperiodic integer-valued or has a square-integrable characteristic function, and hence it is (Kawata [20, Theorem 11.6.1]) absolutely continuous. We stress that our result, Theorem 6 does not require any smoothness assumptions.

Second, the Markov chain $O$ appeared in studying the probabilities that integrated random walks stay positive, see Vysotsky [34, 35]. The main idea of the approach of [34, 35] is in a) splitting the trajectory of the walk into consecutive “cycles” between the up-crossing times; and b) using that for certain particular distributions of increments, e.g. in the case when the distribution $\mathbb{P}(X_1 \in \cdot | X_1 > 0)$ is exponential, the overshoots $(O_n)_{n \geq 1}$ are stationary (actually, i.i.d.) regardless of the starting point $S_0$. The current paper was
originally motivated by the question whether this approach can be extended to the general case for arbitrary random walks.

Third, there is a close connection to the so-called oscillating random walks (which should not be confused with random walks that oscillate in the sense of Section 1.1). Define the oscillating ladder times

\[ T'_0 := 0, \quad T'_n := \begin{cases} \inf\{k > T'_{n-1} : S_k < S_{T'_n}\}, & \text{if } S_{T'_n} \geq 0, \\ \inf\{k > T'_{n-1} : S_k > S_{T'_n}\}, & \text{if } S_{T'_n} < 0, \end{cases} \quad n \in \mathbb{N}. \]

and the oscillating ladder heights \( Z_n := T'_{T'_n}, n \geq 0 \). The random sequence \( Z = (Z_n)_{n \geq 0} \) belongs to a special type of Markov chains called oscillating random walks, whose distributions of increments depend only on the sign of current position. More precisely, the transition probabilities \( P(x, dy) \) of such chain \( Y \) are of the form \( P(x, dy) = P_{\text{sign} x}(dy - x) \) for \( x \neq 0 \) and \( P(0, dy) = \alpha P_+(dy) + (1 - \alpha)P_-(dy) \), where \( P_+ \) and \( P_- \) are two probability distributions and \( \alpha \in [0, 1] \). In the symmetric case when \( P_+(dy) = P_-(dy) \) and \( \alpha = \frac{1}{2} \), the sequence \( (|Y_n|)_{n \geq 0} \) is called a reflected random walk. It can be written in the explicit simple form \( |Y_{n+1}| = ||Y_n| - B_{n+1}| \), where the random variables \( (B_n)_{n \geq 1} \) are i.i.d. distributed as \( P_- \).

Oscillating random walks were introduced by Kemperman [21] and further considered in a few works including Borovkov [8]. Reflected random walks received much more attention, see Peigné and Woess [26] for generalizations to processes of iterated i.i.d. random continuous mappings and the most recent and comprehensive list of references. The main relevance to the present paper is that the overshoots over the zero level of the oscillating ladder heights chain \( Z \) coincide with those of the random walk \( S \). We will explore this connection and use our inducing approach to study general oscillating random walks in the separate paper [36]. Recently we learned that the invariant measure for reflected random walks sampled at the moments of reflections at zero was found in the unpublished work Peigné and Woess [25]. In the above terminology, these random sequences are absolute values of overshoots over the zero level of oscillating random walks with \( P_+(dy) = P_-(dy) \) and \( \alpha = \frac{1}{2} \). One can check using the Wiener–Hopf factorization [10] that the invariant measure of [25] coincides with \( \pi(| \cdot |) \) in the special case of the oscillating ladder heights chain \( Z \) generated by a random walk \( S \) with symmetric non-arithmetic distribution of increments.

### 1.3. Structure of the paper.

In Section 2 we introduce the formal notation for random walks, present a heuristic derivation of the invariant measure \( \pi \) of the chain of overshoots, and also give a nice probabilistic interpretation of \( \pi \) (Proposition 1). In Section 3 we prove invariance of \( \pi \) using the method based on reversibility. In Section 4 we give basics for the use of inducing in infinite ergodic theory. These are applied in Section 5 to obtain general results on stationarity of induced, entrance, and exit Markov chains. The applications of these results to random walks in arbitrary dimension, including the answers to our initial questions on overshoots of one-dimensional random walks, are in Section 6. We conclude the paper with several results on level-crossings of random walks presented in Section 7.
2.1. Notation. Consider the random walk $S = (S_n)_{n \geq 0}$ from Section 1.1 and define its version $S' = (S'_n)_{n \geq 0}$ with $S'_n := S_n - S_0$ which always starts at zero. We assume that $S$, as well as all other random elements considered in this paper, are defined on a generic measurable space with a “basic” probability $\mathbb{P}$. We also equip it with probability measures $\{\mathbb{P}_x\}_{x \in \mathbb{R}}$ such that $\mathbb{P}_x(S \in \cdot) = \mathbb{P}(x + S' \in \cdot)$. For any Borel measure $\mu$ on $\mathbb{R}$, define the measure $\mathbb{P}_\mu(\cdot) := \int_\mathbb{R} \mathbb{P}_x(\cdot) \mu(dx)$. We do not necessarily assume that $\mu$ is a probability but we prefer to (ab)use probabilistic notation $\mathbb{P}_\mu$ and terms “law”, “expectation”, “random variable”, etc., by which we actually mean the corresponding notions of general measure theory. Under the measure $\mathbb{P}_\mu$, the starting point $S_0$ of the random walk $S$ follows the “law” $\mu$. Denote by $\mathbb{E}$, $\mathbb{E}_x$, $\mathbb{E}_\mu$ the respective expectations under $\mathbb{P}$, $\mathbb{P}_x$, $\mathbb{P}_\mu$. All the measures considered in the paper are Borel, that is defined on Borel $\sigma$-algebras.

Recall that the state space $\mathcal{Z}$ of the random walk $S$ is defined as the minimal closed subgroup of $(\mathbb{R}, +)$ containing the support of the distribution of $X_1$. Let us give a different representation for $\mathcal{Z}$. For any $h \in [0, \infty)$, let $\mathcal{Z}_h$ be the real line $\mathbb{R}$ if $h = 0$ and the integer lattice $\mathbb{Z}$ multiplied by $h$ if $h > 0$:

$$
\mathcal{Z}_h := \begin{cases} 
\mathbb{R}, & \text{if } h = 0, \\
h\mathbb{Z}, & \text{if } h > 0.
\end{cases}
$$

These are the only closed subgroups of $(\mathbb{R}, +)$. We equip $\mathcal{Z}_h$ with the discrete topology if $h > 0$ and the Euclidean topology if $h = 0$. Define the span of the distribution of increments of $S$ by

$$
d := \sup \{ h \in [0, \infty) : \mathbb{P}(X_1 \in \mathcal{Z}_h) = 1 \}, \tag{4}
$$

and note that $d \in [0, \infty)$. Then $\mathcal{Z} = \mathcal{Z}_d$, and in what follows we will use the latter notation for the state space. The distribution of increments of $S$ is called arithmetic (with span $d$) if $d > 0$ and is called non-arithmetic if $d = 0$. We always assume that the random walk starts in $\mathcal{Z}_d$, hence

$$
\mathbb{P}(S_0, S_1, \ldots \in \mathcal{Z}_d) = 1.
$$

Finally, denote $\mathcal{Z}_d^+ := \mathcal{Z}_h \cap [0, \infty)$ and $\mathcal{Z}_d^- := \mathcal{Z}_h \cap (-\infty, 0)$.

We will assume throughout that the random walk $S$ oscillates, that is $\limsup_{n \to \infty} S_n = \infty$ a.s. and $\liminf_{n \to \infty} S_n = -\infty$ a.s. The classical trichotomy states that every non-degenerate random walk either drifts to $\infty$, drifts to $-\infty$, or oscillates; see Feller [14, Section XII.2]. Hence oscillation of $S$ is necessary and sufficient for $S$ to cross the zero level infinitely often a.s. The trichotomy also implies that any symmetrically distributed non-degenerate random walk oscillates. Oscillation of $S$ is known to be equivalent to divergence of both series $\sum_{n=1}^{\infty} n^{-1} \mathbb{P}_0(S_n > 0)$ and $\sum_{n=1}^{\infty} n^{-1} \mathbb{P}_0(S_n < 0)$.

In particular, oscillation holds when the random walk $S$ is topologically recurrent on $\mathcal{Z}_d$, which means that $\mathbb{P}_0(S_n \in G \text{ i.o.}) = 1$ for every open neighbourhood $G \subset \mathcal{Z}_d$ of 0. Note that for such random walks, this equality is in fact true for every non-empty open set $G \subset \mathcal{Z}_d$; see Guivarc’h et al. [15, Theorem 24]. The converse does not hold: there are symmetric random walks (which always oscillate if non-degenerate) that are not topologically recurrent. By Spitzer [31, Section 8] and Ornstein [24], topological recurrence of $S$ is equivalent to divergence of the integral $\int_{-a}^{a} \text{Re}((1 - \mathbb{E}e^{iX_1})^{-1}) dt$ for all $a > 0$. In particular, this integral diverges when the random walk $S$ has zero-mean non-degenerate increments. On the other
where \( c \) and arbitrarily heavy tails; see Shepp [29].

The up-crossings times of the zero level are given by

\[
T_0 := 0, \quad T_n := \inf\{k > T_{n-1} : S_{k-1} < 0, S_k \geq 0\}, \quad n \in \mathbb{N}, \tag{5}
\]

while the overshoots and the undershoots at up-crossings are

\[
O_n := S_{T_n}, \quad U_n := S_{T_{n-1}}, \quad n \in \mathbb{N}; \tag{6}
\]

put \( O_0 = U_0 := S_0 \). The random sequences in (5) are defined on the event that all crossing times \( T_n \) are finite. This event occurs almost surely under \( \mathbb{P} \) and under \( \mathbb{P}_\mu \) with an arbitrary non-zero measure \( \mu \) on \( \mathbb{Z}_d \) since \( S \) oscillates. We will also need the down-crossings times of the zero level,

\[
T_0^\downarrow := 0, \quad T_n^\downarrow := \inf\{k > T_{n-1}^\downarrow : S_{k-1} \geq 0, S_k < 0\}, \quad n \in \mathbb{N},
\]

and the corresponding overshoots and undershoots at down-crossings:

\[
O_n^\downarrow := S_{T_n^\downarrow}, \quad U_n^\downarrow := S_{T_{n-1}^\downarrow}, \quad n \in \mathbb{N}, \tag{7}
\]

and \( O_0^\downarrow = U_0^\downarrow := S_0 \).

The Markov chains of overshoots at up-crossings \( O = (O_n)_{n \geq 0} \) and at down-crossings \( O^\downarrow = (O_n^\downarrow)_{n \geq 0} \) take values in \( \mathbb{Z}_d^+ \) and \( \mathbb{Z}_d^- \), respectively, with probability 1. Both chains are started at \( O_0 = O_0^\downarrow = S_0 \in \mathbb{Z}_d \). Our consideration mostly concerns \( O \), which for shortness will be called the chain of overshoots if there is no risk of confusion with the chain \( O \) considered in the Introduction.

Note that there is asymmetry at zero in the above definitions. Namely, since \( -\mathbb{Z}_d^+ \neq \mathbb{Z}_d^- \), the down-crossing times \( T_n^\downarrow \) (and also \( O_n^\downarrow \) and \( U_n^\downarrow \)) may not be equal to the up-crossing times \( T_n \) (and, respectively, \( -O_n \) and \( -U_n \)) for the dual random walk \((-S_n)_{n \geq 0}\). Therefore, in general, results on up-down-crossings do not follow directly from the ones on up-crossings.

Define the measure \( \lambda_d \) on \( \mathbb{Z}_d \) as follows: for any \( B \in \mathcal{B}(\mathbb{Z}_d) \), put

\[
\lambda_d(B) := \begin{cases} 
\lambda_0(B), & \text{if } d = 0, \\
\#B & \text{if } d > 0,
\end{cases}
\]

where \( \lambda_0 \) denotes the Lebesgue measure on \( \mathbb{R} \) and \# denotes the number of elements in a set. Hence, for \( x \in \mathbb{Z}_d^+ \) positive we have \( \lambda_d([0, x) \cap \mathbb{Z}_d) = x \), and for negative \( x \) we have \( \lambda_d([x, 0) \cap \mathbb{Z}_d) = -x \). Note that \( \lambda_d \) is the normalized Haar measure on the additive group \( \mathbb{Z}_d = \mathbb{Z} \), as it was defined in the Introduction.

Define the measures \( \lambda_d^+(dy) := \mathbf{1}_{\mathbb{Z}_d^+}(y) \lambda_d(dy) \) and \( \lambda_d^-(dy) := \mathbf{1}_{\mathbb{Z}_d^-}(y) \lambda_d(dy) \) on \( \mathbb{Z}_d \). Put

\[
\pi_+(dx) := c_1 \mathbb{P}(X_1 > x) \lambda_d^+(dx), \quad x \in \mathbb{Z}_d,
\]

and

\[
\pi_-(dx) := c_1 \mathbb{P}(X_1 \leq x) \lambda_d^-(dx), \quad x \in \mathbb{Z}_d,
\]

where \( c_1 = 1 \) if \( \mathbb{E}|X_1| = \infty \) and \( c_1 = 2/\mathbb{E}|X_1| \) if \( \mathbb{E}|X_1| < \infty \), and recall that

\[
\pi = \frac{1}{2} \pi_+ + \frac{1}{2} \pi_-.
\]
We are ready to state our first main result.

**Theorem 1.** Let $S$ be any random walk that oscillates. Then the measure $\pi_+$ is invariant for the Markov chains of overshoots $O$ and sign-changed shifted undershoots $(-U_n - d)_{n \geq 0}$ at up-crossings of the zero level. Similarly, $\pi_-$ is invariant for the chains $O^\uparrow$ and $(-U_n^\uparrow - d)_{n \geq 0}$.

**Corollary.** For any random walk $S$ that oscillates, the measure $\pi$ is invariant for the Markov chain of overshoots $O$ in either direction. Similarly, $\pi(-\cdot + d)$ is invariant for the chain of undershoots $U$.

Note that since oscillation is necessary and sufficient for $S$ to cross the zero level infinitely often a.s., the assumption of the theorem is absolutely minimal and cannot be relaxed.

**Remark 1.** Moreover, we will show that at stationarity, the laws of overshoots and undershoots of the zero level at consecutive down- and up-crossings are related as follows:

\[
\mathbb{P}_{\pi_+}(O_1 \in \cdot) = \pi_-, \quad \mathbb{P}_{\pi_-}(O_1 \in \cdot) = \pi_+, \quad \mathbb{P}_{\pi_+}(-U_1^\uparrow - d \in \cdot) = \pi_-, \quad \mathbb{P}_{\pi_-}(-U_1 - d \in \cdot) = \pi_+.
\]

Probability measures of the same form as $\pi_+$ appear as limit distributions for the following stochastic processes closely related to random walks. First, $\pi_+$ is the unique stationary distribution of the reflected random walk driven by an i.i.d. sequence $(B_n)_{n \geq 1}$ with the common non-arithmetic distribution $\mathbb{P}(X_1 \in \cdot | X_1 > 0)$; see Feller [14], Section VI.11 and Knight [22]. Second, $\pi_+$ is known as the limit distribution, as well as the stationary distribution, for the residual lifetime (decreased by $d$) in a renewal process with inter-arrival times distributed according to $\mathbb{P}(X_1 \in \cdot | X_1 > 0)$; see Asmussen [2, Section V.3.3] or Gut [16, Theorem 2.6.2]. For random walks, this can be interpreted in terms of the distribution of the overshoot over "infinitely remote" level as follows.

Denote by $H_+$ and $H_-$, respectively, the first strict ascending and the first strict descending ladder heights of the random walk $S'$, i.e. its positions at the first entrance to $(0, \infty)$ and $(-\infty, 0)$. Denote by $\hat{H}_-$ the first weak (non-strict) descending ladder height of $S'$, i.e. the position at the first return to $(-\infty, 0]$. These three random variables are integrable if and only if $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 < \infty$; see Feller [14, Sections XVIII.4 and 5]. In such case, by the results of renewal theory (say, [16, Theorem 2.6.2]), we have

\[
\mathbb{P}_x(O_1 \in dy) \xrightarrow{d} \frac{1}{\mathbb{E}H_+} \mathbb{P}(H_+ > y)\lambda_d(dy), \quad y \in \mathbb{Z}_d^+, \quad (8)
\]

and

\[
\mathbb{P}_x(O_1^- \in dy) \xrightarrow{d} \frac{1}{-\mathbb{E}H_-} \mathbb{P}(H_- \leq y)\lambda_d(dy), \quad y \in \mathbb{Z}_d^-.
\]

The r.h.s.’s of (8) and (9) are referred to as the distributions of overshoots of the walk $S$ over “infinitely remote” levels at $+\infty$ and $-\infty$, respectively. The overshoot over the level $+\infty$ in (8) is non-strict, and it is distributed as the corresponding strict one decreased by $d$. Notice that the invariant measure $\pi_+$, which is a probability if $\mathbb{E}|X_1| \in (0, \infty)$, is defined only in terms of the right tail of the distribution of the walk’s increments. One would expect that $\pi_+$ and $\pi$ shall be related to the limit distributions in (8) and (9), and in fact, we obtain such a representation below. Both limit distributions exist only for zero-mean random walks with finite variance, therefore a priori is not clear why the chain of overshoots should have a stationary distribution if the walk has infinite variance.
Lemma 1. For any random walk \( S \) that oscillates, we have
\[
\pi_+ = c_1 \mathbb{P}(\tilde{H}_- \neq 0) \left[ \mathbb{P}(H_- \leq x) \lambda_\mu(d x) \right] \ast \mathbb{P}(H_+ \in \cdot) \quad \text{on } \mathbb{Z}_d^+.
\]
Similarly,
\[
\pi_- = c_1 \mathbb{P}(\tilde{H}_- \neq 0) \left[ \mathbb{P}(H_+ > x) \lambda_\mu(d x) \right] \ast \mathbb{P}(H_- \in \cdot) \quad \text{on } \mathbb{Z}_d^-.
\]
Remark 2. If \( \mathbb{E}X_1^2 < \infty \) (which implies that \( \mathbb{E}X_1 = 0 \)), this can be interpreted as
\[
\pi_+(dy) = \mathbb{P}(R_- + H_+ \in dy | R_- + H_+ \geq 0),
\]
where \( R_- \) is a random variable having the distribution of the overshoot of \( S \) over “infinitely remote” level at \(-\infty\) (given by the r.h.s. of \([9]\)) and independent with \( H_+ \). Moreover,
\[
\mathbb{P}(R^- + H_+ \geq 0) = -\frac{1}{c_1 \mathbb{P}(H_- \neq 0) \mathbb{E}H_-} = -\frac{1}{c_1 \mathbb{E}H_-}.
\]
Combining this with the analogous probabilistic interpretation of \( \pi_- \) (and in the case \( d = 0 \) using that distribution functions are continuous a.e. with respect to the Lebesgue measure \( \lambda_0 \)) gives the following unexpected representation of \( \pi \).

Proposition 1. For any random walk \( S \) satisfying \( \mathbb{E}X_1 = 0 \) and \( 0 < \mathbb{E}X_1^2 < \infty \), we have
\[
\mathbb{E} \mathbb{P}_{\rho x}\left( S_{\tau_2} \in \cdot \big| |S_{\tau_1}| < |S_{\tau_1+1}|, \ldots, |S_{\tau_1}| < |S_{\tau_2-1}| \right) \xrightarrow{d} \pi,
\]
where \( \rho \) is a random variable independent with \( S \) and such that \( \mathbb{P}(\rho = 1) = \mathbb{P}(\rho = -1) = \frac{1}{2} \).

Proof of Lemma 1. From the Wiener–Hopf factorization
\[
\mathbb{P}(X_1 \in \cdot) = \mathbb{P}(H_+ \in \cdot) + \mathbb{P}(\tilde{H}_- \in \cdot) - \mathbb{P}(H_+ \in \cdot) \ast \mathbb{P}(\tilde{H}_- \in \cdot)
\]
(Feller [14, Chapter XII.3]) it follows that for any \( y \in \mathbb{Z}_d^+ \),
\[
\mathbb{P}(X_1 > y) = \mathbb{P}(H_+ > y) - \int_{(y, \infty)} \mathbb{P}(\tilde{H}_- > y-z) \mathbb{P}(H_+ \in dz) = \int_{(y, \infty)} \mathbb{P}(\tilde{H}_- \leq y-z) \mathbb{P}(H_+ \in dz).
\]
Then from the identity \( \mathbb{P}(\tilde{H}_- \leq u) = \mathbb{P}(\tilde{H}_- \neq 0) \mathbb{P}(H_- \leq u) \) for \( u \in \mathbb{Z}_d^- \), we get
\[
c_1 \mathbb{P}(X_1 > y) = c_1 \mathbb{P}(\tilde{H}_- \neq 0) \int_{\mathbb{Z}_d^-} \mathbb{P}(H_- \leq y-z) 1_{\mathbb{Z}_d^-}(y-z) \mathbb{P}(H_+ \in dz).
\]
The l.h.s. is the density of \( \pi_+ \) with respect to \( \lambda_\mu \), and for the r.h.s. it remains to use the following formula \([11]\) for the density of convolutions. For any random variable \( H \) supported on \( \mathbb{Z}_d \) and any measure \( \mu \) on \( \mathbb{Z}_d \) with a bounded density \( g \) with respect to \( \lambda_\mu \), we have
\[
(\mu \ast \mathbb{P}(H \in \cdot))(dx) = [\mathbb{E}g(x - H)] \lambda_\mu(dx), \quad x \in \mathbb{Z}_d.
\]
This is evident for \( d > 0 \). For the absolutely continuous case \( d = 0 \), see e.g. Cohn [10, Proposition 10.1.12].
2.2. Derivation of $\pi_+$. Let us present a simple probabilistic argument that we used to guess the shape of $\pi_+$. Assume that $E X_1 = 0$, the variance of increments $\sigma^2 = E X_1^2$ is finite and positive, and the random walk $S$ is integer-valued and aperiodic, i.e. the distribution of $X_1 - a$ is arithmetic with span 1 for every $a \in \mathbb{Z}$. In this case $\mathbb{Z}_d^+ = \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N}_0 \cup \{0\}$.

Consider the number of up-crossings of the zero level by time $n$:

$$L_n^+ := \sum_{i=0}^{n-1} 1(S_i < 0, S_{i+1} \geq 0) = \max\{k \geq 0 : T_k \leq n\}.$$ 

Assume that the chain $O$ has an ergodic stationary distribution $\mu$. Then by the ergodic theorem, for any $x, y \in \{z \in \mathbb{N}_0 : \mathbb{P}(X_1 > z) > 0\}$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1(O_i = y) = \lim_{n \to \infty} \frac{1}{L_n^+} \sum_{i=1}^{L_n^+} 1(O_i = y) = \mu(y), \quad \mathbb{P}_x\text{-a.s.} \quad (12)$$

On the other hand,

$$\mathbb{E}_x \left[ \frac{L_n^+}{\sqrt{n}} \cdot \frac{1}{L_n^+} \sum_{i=1}^{L_n^+} 1(O_i = y) \right] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \mathbb{P}_x(S_i < 0, S_{i+1} = y)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \sum_{k=1}^{\infty} \mathbb{P}_x(S_i = -k) \mathbb{P}(X_1 = y + k)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(X_1 = y + k) \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \mathbb{P}_x(S_i = -k).$$

By the local central limit theorem, there exists a constant $c > 0$ such that for every integer $i$ and $k \geq 1$ we have $\mathbb{P}_x(S_i = -k) \leq c/\sqrt{n}$, and also $\mathbb{P}_x(S_i = -k) \sim \frac{1}{\sqrt{2\pi i \sigma}}$ as $i \to \infty$. Hence from (12) and the dominated convergence theorem, we obtain

$$\mu(y) \lim_{n \to \infty} \mathbb{E}_x \left[ \frac{L_n^+}{\sqrt{n}} \right] = \sum_{k=1}^{\infty} \mathbb{P}(X_1 = y + k) \left( \lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \frac{1}{\sqrt{2\pi i \sigma}} \right) = \sqrt{\frac{2}{\pi \sigma^2}} \mathbb{P}(X_1 > y).$$

Thus, $\mu = \pi_+$ in the special case considered above.

Therefore it is feasible that the distribution $\pi_+$ is stationary for the chain of overshoots $O$ for general random walks but of course we need to prove this directly.

3. Invariant measure of overshoots via reversibility

In this section we present a purely probabilistic method of proving the invariance of $\pi_+$ asserted in Theorem [1]. This method is entirely different from the derivation of Section 2.2. It provides information on the whole trajectory of the random walk between up-crossings. The main result of the section, Proposition[3], reveals the distributional symmetry hidden in the trajectory of arbitrary non-symmetrically distributed random walk.
3.1. The setup and results. Define new Markov transition kernels $P$ and $Q$ on $\mathbb{Z}_d$:

$$P(x, dy) := \mathbb{P}_x(-U_1 - d \in dy), \quad Q(x, dy) := \mathbb{P}(X_1 - d \in dy + x | X_1 - d \geq x), \quad x, y \in \mathbb{Z}_d,$$

with the convention that $Q(x, dy) := \delta_0(dy)$ in the case when $\mathbb{P}(X_1 - d \geq x) = 0$; the choice of the delta measure is arbitrary and will not be relevant for what follows. The kernel $P$ is defined in terms of the sign-changed first undershoot $U_1$, given in (10) above, which is shifted by $d$ to ensure that $-U_1 - d$ can take value zero in the arithmetic case. The kernel $Q$ corresponds to up-crossings of the zero level by the walk $S$. Clearly, for every $x \in \mathbb{Z}_d$, the transition probabilities $P(x, dy)$ and $Q(x, dy)$ are supported on $\mathbb{Z}_d^+$.

The transition kernel of the Markov chains of overshoots $(O_n)_{n \geq 0}$ and shifted undershoots $(U_n - d)_{n \geq 0}$ equal $PQ$ and $QP$, respectively. More precisely, for any probability measure $\mu$ on $\mathbb{Z}_d$ and any $n \in \mathbb{N}$,

$$\mathbb{P}_\mu(O_n \in dy) = [\mu(PQ)^n](dy), \quad \mathbb{P}_\mu(-U_n - d \in dy) = [\mu(P(QP)^{n-1})](dy), \quad y \in \mathbb{Z}_d. \quad (14)$$

Here for any transition kernel $T$ on $\mathbb{Z}_d$, by $\mu T$ we denoted the measure on $\mathbb{Z}_d$ given by $\mu T(dy) := \int_{\mathbb{Z}_d} T(z, dy) \mu(dz)$, and put $T^0(x, dy) = \delta_x(dy)$.

We will show that $\lambda_d(dx)\mathbb{P}(X_1 - d \in dy + x)$ and $\lambda_d(dy)\mathbb{P}(X_1 - d \in dx + y)$ are equal as measures on $\mathbb{Z}_d \times \mathbb{Z}_d$. Combined with the equality of measures $\mathbb{P}(X_1 - d \geq z)\lambda_d(dz) = \pi_{+}(dz)$ on $\mathbb{Z}_d$, this implies that the transition kernel $Q$ is reversible with respect to $\pi_{+}$. Put differently, the detailed balance condition

$$\pi_{+}(dx)Q(x, dy) = \pi_{+}(dy)Q(y, dx), \quad x, y \in \mathbb{Z}_d$$

holds true for the measures on $\mathbb{Z}_d \times \mathbb{Z}_d$ (which are supported on $\mathbb{Z}_d^+ \times \mathbb{Z}_d^+$). Surprisingly, the kernel $P$ shares the same property. Put together, we have the following statement.

**Proposition 2.** For any random walk $S$ that oscillates, the kernels $P$ and $Q$ are reversible with respect to $\pi_{+}$.

A direct corollary of this proposition is invariance of the measure $\pi_{+}$ for the Markov chains $(O_n)_{n \geq 0}$ and $(-U_n - d)_{n \geq 0}$ asserted by Theorem 1. A similar argument yields invariance of $\pi_{-}$ for the chains $(O_n^+)_{n \geq 0}$ and $(-U_n^+)_{n \geq 0}$ (use (17) from Section 3.2 below and a kernel decomposition for theses chains analogous to (14)). This proves Theorem 1 given Proposition 2.

We will obtain Proposition 2 as a direct corollary Proposition 3 in the next section.

3.2. The time reversal argument. We now present a result concerning the full trajectory of the random walk between up-crossings of the level zero. Our proof is based on generalisation of the time-reversal argument from Vysotsky [35, Lemma 1]. It can be regarded as an illustration of the conclusion of Remark 5 in Section 5.2 on general Markov chains.

**Proposition 3.** For any random walk $S$ that oscillates, for any $m \in \mathbb{N}$ we have

$$\mathbb{P}_{\pi_{+}}\left( (S_0, S_1, \ldots, S_{T_m-1}, 0, \ldots) \in \cdot \right) = \mathbb{P}_{\pi_{+}}\left( (-S_{T_m-1} - d, -S_{T_m-2} - d, \ldots, -S_0 - d, 0, \ldots) \in \cdot \right) \quad (15)$$
and
\[
\mathbb{P}_{\pi_+}((S_0, S_1, \ldots, S_{T_m-1}, 0, \ldots) \in \cdot) = \mathbb{P}_{\pi_-}((-S_{T_m-1} - d, -S_{T_m-2} - d, \ldots, -S_0 - d, 0, \ldots) \in \cdot). \tag{16}
\]

The choice of the value 0 in the random sequences in (15) and (16) is arbitrary and could be substituted by any constant. However, we stress that the equalities in (15) and (16) cease to hold if the constant value is substituted by the remaining part of the path of \(S\). Note that (15) can be stated more elegantly as
\[
(S_0, S_1, \ldots, S_{T_m-1}) \overset{d}{=} (-S_{T_m-1} - d, -S_{T_m-2} - d, \ldots, -S_0 - d) \text{ under } \mathbb{P}_{\pi_+}.
\]

**Remark 3.** Similarly, we have
\[
\mathbb{P}_{\pi_-}((S_0, S_1, \ldots, S_{T_m-1}, 0, \ldots) \in \cdot) = \mathbb{P}_{\pi_-}((-S_{T_m-1} - d, -S_{T_m-2} - d, \ldots, -S_0 - d, 0, \ldots) \in \cdot) \tag{17}
\]
and
\[
\mathbb{P}_{\pi_-}((S_0, S_1, \ldots, S_{T_m-1}, 0, \ldots) \in \cdot) = \mathbb{P}_{\pi_+}((-S_{T_m-1} - d, -S_{T_m-2} - d, \ldots, -S_0 - d, 0, \ldots) \in \cdot). \tag{18}
\]

We first prove two simple corollaries of Proposition 3.

**Proof of Propositions 2.** Reversibility of the \(P\)-kernel follows immediately by (15) with \(m = 1\) since \(U_1 = S_{T_1-1}\).

As explained above, reversibility of the \(Q\)-kernel follows from the equalities of measures
\[
\lambda_d(dx)\mathbb{P}(X_1 - d \in dy + x) = \lambda_d(dy)\mathbb{P}(X_1 - d \in dx + y) \tag{19}
\]
on \(Z_d \times Z_d\) and \(\mathbb{P}(X_1 - d \geq z)\lambda_+^\pi(dx) = \pi_+(dz)\) on \(Z_d\). The latter equality is trivial so we need to prove the former. According to (27) below, for any random variable \(Z\) supported on \(Z_d\) we have \(\lambda_d(dx)\mathbb{P}(x + Z \in -dy) = \lambda_d(dy)\mathbb{P}(y + Z \in -dx)\) on \(Z_d \times Z_d\). Equivalently, \(\lambda_d(dx)\mathbb{P}(-Z \in dy + x) = \lambda_d(dy)\mathbb{P}(-Z \in dx + y)\), and (19) follows if we take \(-Z = X_1 - d\). \(\square\)

Recall that Remark 1 asserts that
\[
\mathbb{P}_{\pi_+}(O_1^+ \in \cdot) = \pi_-, \quad \mathbb{P}_{\pi_-}(O_1 \in \cdot) = \pi_+, \quad \mathbb{P}_{\pi_+}(-U_1^+ - d \in \cdot) = \pi_-, \quad \mathbb{P}_{\pi_-}(-U_1 - d \in \cdot) = \pi_+.
\]

**Proof of Remark 1.** Fix \(m = 1\). By (13), the random variables \(-O_1^+ - d = S_{T_1^+} - d\) and \(U_1^+ = S_{T_1^+ - 1}\) have the same law under \(\mathbb{P}_{\pi_+}\), hence \(\mathbb{P}_{\pi_+}(O_1^+ \in \cdot) = \mathbb{P}_{\pi_+}(-U_1^+ - d \in \cdot)\).

By (16), the law of \(-U_1^+ - d = S_{T_1^+ - 1} - d\) under \(\mathbb{P}_{\pi_+}\) is the same as the law of \(S_0\) under \(\mathbb{P}_{\pi_-}\), i.e. \(\mathbb{P}_{\pi_+}(-U_1^+ - d \in \cdot) = \pi_+\), and hence \(\mathbb{P}_{\pi_+}(O_1^+ \in \cdot) = \pi_+\). Similarly, by (17), we find \(\mathbb{P}_{\pi_-}(O_1 \in \cdot) = \mathbb{P}_{\pi_-}(-U_1 - d \in \cdot)\). Finally, by (18), we have \(\mathbb{P}_{\pi_-}(-U_1 - d \in \cdot) = \pi_+\). \(\square\)

We now prove the main statement of the section.
Proof of Proposition 3. Consider equality (15) in the case $m = 1$. Pick an arbitrary $k \in \mathbb{N}$ and define the time-reversal mapping $R_k : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ by

$$R_k(x_0, \ldots, x_k) := (-x_k - d, \ldots, -x_0 - d).$$

Introduce the random vector $K := (S_0, \ldots, S_k)$ and note that (15) follows if we establish the equality of measures on $(Z_d)^{k+1}$:

$$\mathbb{P}_{\pi_+}(K \in \cdot, T_1 = k + 1) = \mathbb{P}_{\pi_+}(R_k(K) \in \cdot, T_1 = k + 1). \tag{20}$$

Put

$$\tilde{Z}_d^+ := \begin{cases} Z_d^+ \setminus \{0\}, & \text{if } d = 0, \\ Z_d^+, & \text{if } d > 0, \end{cases}$$

and denote $C_k := \bigcup_{i=0}^{k-1} (\tilde{Z}_d^+)i \times (Z_d^-)^{k-1-i}$. Then $C_k' := \tilde{Z}_d^+ \times C_k \times Z_d^-$ is the set of sequences of length $k + 1$ that start from $\tilde{Z}_d^+$, down-cross the level zero exactly once, and in the non-arithmetic case have no zeroes.

Note that $R_k$ is an invertible mapping on $\mathbb{R}^{k+1}$, and it is an involution. Further, $R_k(C_k) = C_k$ since $\tilde{Z}_d^+ - d = Z_d^-$ in both cases $d = 0$ and $d > 0$. Similarly, $R_k(C_k') = C_k'$, implying that $R_k(\mathbb{R}^{k+1}\setminus C_k') = \mathbb{R}^{k+1}\setminus C_k'$. This gives

$$\mathbb{P}_{\pi_+}(R_k(K) \in \mathbb{R}^{k+1}\setminus C_k', T_1 = k + 1) = \mathbb{P}_{\pi_+}(K \in \mathbb{R}^{k+1}\setminus C_k', T_1 = k + 1) = 0. \tag{21}$$

The second equality is trivial in the arithmetic case. In the non-arithmetic case, it is due to the fact that $K$ has density with respect to the Lebesgue measure on $\mathbb{R}^{k+1}$, which in turn holds true since in this case the measure $\pi_+$ has density with respect to the Lebesgue measure on $\mathbb{R}$.

By (21), if suffices to check equality (20) on rectangles of the form $B_0 \times B \times B_k$ with Borel sides $B_0 \subset \tilde{Z}_d^+, B_k \subset Z_d^-$ and $B \subset C_k$. Using the definition of $\pi_+$ and the fact that $X_{k+1}$ is independent with $K$ under $\mathbb{P}_{x_0}$ for every $x_0 \in Z_d$, we obtain

$$\mathbb{P}_{\pi_+}(K \in B_0 \times B \times B_k, T_1 = k + 1)$$

$$= \int_{B_0} \mathbb{P}_{x_0}((S_1, \ldots, S_k) \in B \times B_k, T_1 = k + 1) \pi_+(dx_0)$$

$$= \int_{B_0} \int_{B_k} \left[ \mathbb{P}_{x_0}((S_1, \ldots, S_{k-1}) \in B, T_1 = k + 1|S_k = x_k)\mathbb{P}(X_1 > x_0) \right] \mathbb{P}_{x_0}(S_k \in dx_k) \lambda_d(dx_0)$$

$$= \int_{B_0 \times B_k} f_B(x_0, x_k) \mathbb{P}_{\lambda_d}((S_0, S_k) \in dx_0 \otimes dx_k), \tag{22}$$

where

$$f_B(x_0, x_k) := \mathbb{P}_{x_0}((S_1, \ldots, S_{k-1}) \in B|S_k = x_k)\mathbb{P}(X_1 > x_0)\mathbb{P}(X_1 \geq -x_k), \quad (x_0, x_k) \in \tilde{Z}_d^+ \times Z_d^-.$$
Then we use equality (22) to get
\[ \mathbb{P}_{s_+}(R_k(K) \in B_0 \times B \times B_k, T_1 = k + 1) \]
\[ = \mathbb{P}_{s_+}(K \in (-B_k - d) \times R_{k-2}(B) \times (-B_0 - d), T_1 = k + 1) \]
\[ = \int_{(-B_k - d) \times (-B_0 - d)} f_{R_{k-2}(B)}(x_0, x_k) \mathbb{P}_{\lambda_d}(\{S_0, S_k\} \in dx_0 \otimes dx_k) \]
\[ = \int_{B_0 \times B_k} f_{R_{k-2}(B)}(R_1(x_0, x_k)) \mathbb{P}_{\lambda_d}(R_1(S_0, S_k) \in dx_0 \otimes dx_k), \tag{23} \]
where in the last equality we used the change of variables formula and the facts that \( R_k \) is an involution and \((-B_k - d) \times (-B_0 - d) = R_1(B_0, B_k)\).

Let us simplify the integrand under the last integral in (23). We have
\[ \mathbb{P}_{-x_{k-d}}((S_1, \ldots, S_{k-1}) \in R_{k-2}(B)|S_k = -x_0 - d) \]
\[ = \mathbb{P}_0((S_1 - x_k - d, \ldots, S_{k-1} - x_k - d) \in R_{k-2}(B)|S_k = x_k - x_0) \]
\[ = \mathbb{P}_0((R_{k-2}(S_1 - S_k - x_0 - d, \ldots, S_{k-1} - S_k - x_0 - d) \in B|S_k = x_k - x_0) \]
\[ = \mathbb{P}_0((S_k - S_1 + x_0, \ldots, S_k - S_{k-1} + x_0) \in B|S_k + x_0 = x_k). \tag{24} \]
By the definition of \( f_B \), this gives
\[ f_{R_{k-2}(B)}(R_1(x_0, x_k)) = f_{R_{k-2}(B)}(-x_k - d, -x_0 - d) \]
\[ = \mathbb{P}_{x_0}((S_1, \ldots, S_{k-1}) \in B|S_k = x_k) \mathbb{P}(X_1 > -x_k - d) \mathbb{P}(X_1 \geq -x_0 - d). \]
Thus, since a distribution function can have at most countably many jumps, we get
\[ f_B(x_0, x_k) = f_{R_{k-2}(B)}(R_1(x_0, x_k)), \quad \mathbb{P}_{\lambda_d}((S_0, S_k) \in \cdot) \text{-a.e.} (x_0, x_k). \tag{25} \]
Hence by (22), (23), and (25) combined with (21), equality (20) follows in we show the equality of measures on \( \mathbb{Z}_d^+ \times \mathbb{Z}_d^+ \):
\[ \mathbb{P}_{\lambda_d}((S_0, S_k) \in \cdot) = \mathbb{P}_{\lambda_d}(R_1(S_0, S_k) \in \cdot). \tag{26} \]
Translation invariance of \( \lambda_d \) under shifts of \( \mathbb{Z}_d \) implies that
\[ \mathbb{P}_{\lambda_d}(R_1(S_0, S_k) \in \cdot) = \mathbb{P}_{\lambda_d}((-S_k - d, -S_0 - d) \in \cdot) = \mathbb{P}_{\lambda_d}((-S_k, -S_0) \in \cdot), \]
and thus the claim (26) reduces to
\[ \mathbb{P}_{\lambda_d}((S_0, S_k) \in \cdot) = \mathbb{P}_{\lambda_d}((-S_k, -S_0) \in \cdot), \tag{27} \]
which will be shown to hold for measures on the larger set \( \mathbb{Z}_d^2 = \mathbb{Z}_d \times \mathbb{Z}_d \).
It suffices to check (27) only on rectangular sets with Borel sides \( A_0, A_k \subset \mathbb{Z}_d \). By Fubini’s theorem and the properties of central symmetry and shift invariance of \( \lambda_d \),

\[
\mathbb{P}_{\lambda_d}((S_0, S_k) \in A_0 \times A_k) = \int_{\mathbb{Z}_d^2} 1(x_0 \in A_0, x_0 + x'_k \in A_k) \lambda_d(dx_0) \otimes \mathbb{P}(S'_k \in dx'_k)
\]

\[
= \int_{\mathbb{Z}_d} \lambda_d(A_0 \cap (A_k - x'_k)) \mathbb{P}(S'_k \in dx'_k)
\]

\[
= \int_{\mathbb{Z}_d} \lambda_d(-A_0 - x'_k \cap (-A_k)) \mathbb{P}(S'_k \in dx'_k)
\]

\[
= \mathbb{P}_{\lambda_d}((S_0, S_k) \in -A_k \times (-A_0)),
\]

where the last equality follows from the first two. This implies (27).

Thus, (15) is proved for \( m = 1 \). The general case \( m \in \mathbb{N} \) follows analogously, with the only difference that the set \( C'_k \) shall account for \( 2m - 1 \) crossings of the level zero.

Consider now (16). We need to prove that the law of \((S_0, S_1, \ldots, S_{T_m}^+)\) under \( \mathbb{P}_{\pi_+}\) equals the law of \((-S_{T_{m-1}} - d, -S_{T_{m-2}} - d, \ldots, -S_0 - d)\) under \( \mathbb{P}_{\pi_-}\). Similarly to the proof of (15), by the duality principle for random walks this reduces to the equality

\[
\mathbb{P}_{\pi_+}((S_0, S_k) \in \cdot, S_{k+1} < 0) = \mathbb{P}_{\pi_-}(R_1(S_0, S_k) \in \cdot, S_{k+1} \geq 0)
\]

of measures on \( \mathbb{Z}_d^+ \times \mathbb{Z}_d^+ \) for \( k \in \mathbb{N}_0 \). Use the definitions of \( \pi_+ \), \( \pi_- \), and \( R_1 \) to write this as

\[
\mathbb{P}_{\lambda_d}((S_0, S_k) \in dx_0 \otimes dx_k) \mathbb{P}(X_1 > x_0) \mathbb{P}(X_1 < -x_k)
\]

\[
= \mathbb{P}_{\lambda_d}(R_1(S_0, S_k) \in dx_0 \otimes dx_k) \mathbb{P}(X_1 \geq x_0 + d) \mathbb{P}(X_1 \leq -x_k - d).
\]

This equality holds true by (26) and the fact that \( \mathbb{P}(X_1 > x) = \mathbb{P}(X_1 \geq x + d) \) for \( \lambda_d \)-a.e. \( x \).

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4. Induced transformations in infinite ergodic theory

In this section we present basic results on inducing of measure preserving transformations of infinite measure spaces; see Aaronson [11, Chapter 1] for the introduction to infinite ergodic theory and Kaimanovich [18, Section 1] for an account of related probabilistic results on invariant Markov shifts. We present few variations of Kakutani’s classical results of 1943, mainly to cover inducing on sets of infinite measure. The statements we need are not directly available in [11] and surprisingly, we did not find any references.

Let \( T \) be a transformation of some measurable space \((X, \mathcal{F})\). For any set \( A \in \mathcal{F} \), consider the first hitting time \( \tau_A \) of \( A \) and the first hitting mapping \( \varphi_A \) defined by

\[
\tau_A(x) := \inf\{n \geq 1 : T^n x \in A\}, x \in X, \quad \text{and} \quad \varphi_A(x) := T^{\tau_A(x)} x, x \in \{\tau_A < \infty\}, \quad (28)
\]

and the first return or induced mapping \( T_A := (\varphi_A)|_A \) defined on \( A \cap \{\tau_A < \infty\} \). All these functions are measurable.

From now on and till the end of Section 4 we assume that the transformation \( T \) is measure preserving on the \( \sigma \)-finite measure space \((X, \mathcal{F}, m)\). We say that a set \( A \in \mathcal{F} \) is recurrent for \( T \) if \( \tau_A \) is finite \( m \)-a.e. on \( A \), which is the case when \( A \subset \cup_{k \geq 1} T^{-k} A \mod m \). If
A is recurrent for $T$, from invariance of $m$ it follows by simple induction that all iterations of the mapping $\varphi_A$ are defined $m$-a.e. on $A$ (see [11 Section 1.5]), that is
\[ m|_A(\tau_A = \infty) = 0 \implies m|_A(\{T^n \in A\text{ i.o.}\}^c) = 0. \] (29)

The following result on induced transformations essentially is in [11 Proposition 1.5.3].

**Lemma 2’.** Let $T$ be a measure preserving transformation of a measure space $(X, F, m)$, and $A \in F$ be any set recurrent for $T$ such that $0 < m(A) < \infty$. Then the induced mapping $T_A$ is a measure preserving transformation of the induced space $(A, F \cap A, m|_A)$.

Relaxing the condition $m(A) < \infty$ results in additional assumptions described in the next two statements.

**Lemma 2.** Let $T$ be an invertible measure preserving transformation of a $\sigma$-finite measure space $(X, F, m)$. Let $A \in F$ be any set such that $m(A) > 0$, $\{T^n A\}_{n \geq 1} \subset F$, $A \subset \cup_{k \geq 1} T^k A \mod m$, and $A$ is recurrent for $T$. Then the induced mapping $T_A$ is a measure preserving transformation of the induced space $(A, F \cap A, m|_A)$.

**Remark 4.** Since the mapping $T$ is invertible, the mapping $T^{-1}$ is always measurable (and hence $\{T^n A\}_{n \geq 1} \subset F$) if $(X, F)$ is a standard measurable space, that is $X$ is a Polish space and $F = B(X)$ is its Borel $\sigma$-algebra; see [11 Theorem 1.0.3]. In this case, the condition $A \subset \cup_{k \geq 1} T^k A \mod m$ simply means recurrence of $A$ for the mapping $T^{-1}$, which preserves $m$.

The $\sigma$-finiteness of $m$ can be relaxed to $\sigma$-finiteness of $m|_A$, and the same is valid for Lemma 3 below.

Let us comment on the assumptions of Lemma 2. If $T^{-1}$ is measurable (and hence measure preserving) and $m(A) < \infty$, then $A$ is recurrent for $T$ if and only if $A$ is recurrent for $T^{-1}$, that is the assumptions $A \subset \cup_{k \geq 1} T^{-k} A \mod m$ and $A \subset \cup_{k \geq 1} T^k A \mod m$ are equivalent; see Kaimanovich [12 Proposition 1.3]. However, the following example shows what can go wrong if we require only the former one for a set $A$ of infinite measure: if $T$ is the shift on $\mathbb{Z}$ equipped with the counting measure and $A = \mathbb{N}$, then $T_A$ is not measure preserving on $A$ since $T_A^{-1}(\{1\}) = \emptyset$. In this example $T_A$ is not surjective mod $m$ but it must be so if $A \subset \cup_{k \geq 1} T^k A \mod m$.

**Proofs of Lemmas 2 and 2’.** We need to show that $m(T_A^{-1} B) = m(B)$ for any measurable set $B \subset A$. By monotonicity, it suffices to prove this only for $B$ of finite measure since $m|_A$ is $\sigma$-finite. The rest is a standard argument (see the proof of [11 Proposition 1.5.3]), which we present here for convenience of the reader. Since the set $A$ is recurrent for $T$, we have
\[ m(T_A^{-1} B) = \sum_{n=1}^\infty m(A \cap \{T^{-n} B = \emptyset\}) = \sum_{n=1}^\infty m(A \cap T^{-n} B) \setminus \bigcup_{k=1}^{n-1} T^{-k} A) = \sum_{n=1}^\infty m(A \cap T^{-1} B - n), \]
where $B_0 := B$ and $B_n := T^{-n} B \setminus \bigcup_{k=1}^{n-1} T^{-k} A$ for $n \geq 1$. The set $T^{-1} B_n$ of finite measure is a disjoint union of $A \cap T^{-1} B_n$ and $B_{n+1}$, hence $m(A \cap T^{-1} B_n) = m(B_n) - m(B_{n+1})$. Then
\[ m(T_A^{-1} B) = \sum_{n=1}^\infty m(A \cap T^{-1} B_{n-1}) = \sum_{n=1}^\infty (m(B_{n-1}) - m(B_n)) = m(B) - \lim_{n \to \infty} m(B_n), \]
and this gives (only under recurrence of the set $A$ for $T$ !) $m(T_A^{-1}B) \leq m(B)$ and also

$$m(T_A^{-1}B) = m(B) \iff \lim_{n \to \infty} m(B_n) = 0.$$  \hfill (30)

Under the assumptions of Lemma 2, that is $m(A) < \infty$, we also have $m(T_A^{-1}(A \setminus B)) \leq m(A \setminus B)$ since in the inequality $m(T_A^{-1}B) \leq m(B)$ the set $B$ can be an arbitrary subset of $A$ of finite measure. Then

$$m(T_A^{-1}A) - m(T_A^{-1}B) = m(A) - m(T_A^{-1}B) \leq m(A) - m(B)$$

since $T_A^{-1}A = A \mod m$. Thus $m(T_A^{-1}B) \geq m(B)$, and so $m(B) = m(T_A^{-1}B)$.

Under the assumptions of Lemma 2 by invertibility of $T$ we have

$$B_n = T^{-n}B \setminus \bigcup_{k=0}^{n-1} T^{-k}A = T^{-n}B \setminus [T^{-n}(\bigcup_{k=1}^{n} T^kA)] = T^{-n}(B \setminus (\bigcup_{k=1}^{n} T^kA)).$$

Hence

$$m(T^{-n}B \setminus \bigcup_{k=0}^{n-1} T^{-k}A) = m(B \setminus (\bigcup_{k=1}^{n} T^kA)) \to m(B \setminus (\bigcup_{k=1}^{\infty} T^kA)) = 0,$$ \hfill (31)

and so $m(B) = m(T_A^{-1}B)$ follows from (30).

We say that $T$ is conservative if every measurable set is recurrent for $T$. Equivalently, every wandering set, that is a $W \in \mathcal{F}$ such that the sets $\{\theta^kW\}_{k \geq 0}$ are disjoint, is $m$-null. This gives, by Lemma 2 and Poincaré’s recurrence theorem that every measure preserving transformation of a finite measure space is conservative, the following criterion ([18 Proposition 1.2]).

**Conditions for conservativity.** A measure preserving transformation $T$ of a $\sigma$-finite measure space $(X, \mathcal{F}, m)$ is conservative if there exist a sequence of sets $\{A_n\}_{n \geq 1} \subset \mathcal{F}$, all of finite measure and recurrent for $T$, such that $X = \bigcup_{n \geq 1} A_n \mod m$. In particular, this holds if $X = \bigcup_{k \geq 1} T^{-k}A \mod m$, i.e. $\tau_A < \infty$ a.e., for some measurable set $A$ of finite measure.

The latter statement is known as Maharam’s recurrence theorem.

Next we give a version of Lemma 2 for conservative transformations. The additional statement on ergodicity is in [11 Propositions 1.2.2 and 1.5.2].

**Lemma 3.** Let $T$ be a measure preserving conservative transformation of a $\sigma$-finite measure space $(X, \mathcal{F}, m)$, and $A \in \mathcal{F}$ be any set with $m(A) > 0$. Then $T_A$ is a measure preserving conservative transformation of the induced space $(A, \mathcal{F} \cap A, m|_A)$. Moreover, if $T$ is ergodic, then $T_A$ is ergodic and $X = \bigcup_{k \geq 1} T^{-k}A \mod m$.

**Proof.** First of all, $T_A$ is well defined since $A$ is recurrent for $T$ by the conservativity. The latter property of $T$ trivially implies conservativity of $T_A$; see [11 Proposition 1.5.1]. Since $m$ is $\sigma$-finite, by monotonicity it suffices to check that $m(T_A^{-1}B) = m(B)$ for any measurable set $B \subset A$ of finite positive measure. But $T_B^{-1}B = B \mod m$ by conservativity of $T$, so $m(T_B^{-1}B) = m(B)$ and hence $\lim_{n \to \infty} m(T^{-n}B \setminus \bigcup_{k=0}^{n-1} T^{-k}B) = 0$ by (30). Since $B \subset A$, this gives $\lim_{n \to \infty} m(T^{-n}B \setminus \bigcup_{k=0}^{n-1} T^{-k}A) = 0$, which by (30) implies $m(T_A^{-1}B) = m(B)$.

For invertible $T$, inducing can be reversed under additional assumption $X = \bigcup_{k \geq 1} T^{-k}A \mod m$ using so-called suspensions (Kakutani towers). More generally, certain invariant measures of the induced transformation can be lifted to invariant measures of the original transformation as follows. Denote $m_A := m|_A$. 


Lemma 4. Let $T$ be a measure preserving transformation of a $\sigma$-finite measure space $(X, \mathcal{F}, m)$, and let $A \in \mathcal{F}$ be any set recurrent for $T$ such that $m(A) > 0$. Then for any $\sigma$-finite $T_A$-invariant measure $\nu$ on $(A, \mathcal{F} \cap A)$ such that $\nu \ll m_A$, the measure

$$\bar{\nu}(B) := \int_A \left[ \sum_{k=0}^{\tau_A(x)-1} 1(T^k x \in B) \right] \nu(dx), \quad B \in \mathcal{F},$$

is invariant for $T$ and satisfies $\bar{\nu}|_A = \nu$. Moreover, if $X = \bigcup_{k \geq 1} T^{-k}A \mod m$ and the assumptions of either Lemma 2 or 3 are satisfied, then $\bar{m}_A = m$.

The assumption $\nu \ll m_A$ is to ensure that $\tau_A$ is finite $\nu$-a.e. on $A$. In the conservative case, equation (32) with $B = X$ is known as Kac’s formula.

Proof. The equality $\bar{\nu}|_A = \nu$ is trivial. Invariance of $\bar{\nu}$ is standard; see the proof of [1, Proposition 1.5.7] or a similar argument in [2] below.

It remains to prove the statement on $\bar{m}_A = m$. By monotonicity and $\sigma$-finiteness of $m$, this can be checked only on sets of finite measure $m$. For any measurable set $B \subset X$,

$$\bar{m}_A(B) = \int_A \left[ \sum_{n=1}^{\infty} 1(\tau_A(x) = n) \times \sum_{k=0}^{\tau_A(x)-1} 1(T^k x \in B) \right] m(dx)$$

$$= \int_A \left[ \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} 1(T^k x \in B, \tau_A(x) = n) \right] m(dx)$$

$$= \sum_{k=0}^{\infty} m(A \cap T^{-k}B \cap \{\tau_A > k\}),$$

and therefore, assuming that $m(B) < \infty$, we get

$$\bar{m}_A(B) = \sum_{k=0}^{\infty} m(A \cap T^{-k}B \setminus \bigcup_{n=1}^{k} T^{-n}A) = m(A \cap B) + \sum_{k=1}^{\infty} m(A \cap T^{-1}B'_k - B'_k),$$

where $B'_k := T^{-k}B \setminus \bigcup_{n=0}^{k} T^{-n}A$ for $k \geq 0$. The set $T^{-1}B'_k$ has finite measure and it is a disjoint union of $A \cap T^{-1}B'_k$ and $B'_{k+1}$, hence $m(A \cap T^{-1}B'_k) = m(B'_k) - m(B'_{k+1})$. Hence the sequence $m(B'_k)$ is decreasing, and

$$\bar{m}_A(B) = m(A \cap B) + m(B'_0) - \lim_{k \to \infty} m(B'_k) = m(B) - \lim_{k \to \infty} m(B'_k).$$

(34)

It remains to show that the limit in the above formula is zero.

Under the assumptions of Lemma 2, $T_A$ is surjective as follows from $A \subset \bigcup_{k \geq 1} T^kA \mod m$. Hence $a := T^{-1}(\varphi_A(x)) = T_A^{-1}(T^{\tau_A(x)} x)$ satisfies $T^k(a) = x$ for some integer $k \geq 1$ for m-a.e. $x \in X \setminus A$. For $x \in A$, simply take $a := T^{-1}(x)$. All together, we get $X = \bigcup_{k \geq 1} T^kA \mod m$, and the equality $\lim_{k \to \infty} m(B'_k) = 0$ follows easily as in (31). Then $\bar{m}_A(B) = m(B)$ by (34).

Under the assumptions of Lemma 2, we have $m(A) < \infty$, and since $X = \bigcup_{k \geq 1} T^{-k}A \mod m$ by assumption, the transformation $T$ is conservative by Maharam’s recurrence theorem. Then the equality $\bar{m}_A = m$ is by [1, Lemma 1.5.4]. Since the argument there is not easy
to follow and it does not directly apply in the case \( m(A) = \infty \), which is possible under the assumptions of Lemma \( \text{3} \) and therefore needs to be covered, we give a complete proof here, based on a similar idea.

For any integer \( N \geq 1 \), denote \( B^{(N)} := B \cap (\cup_{n=1}^{N} T^{-n} A) \) and notice that
\[
T^{-k}(B^{(N)}) \setminus \cup_{n=0}^{k} T^{-n} A \subset \{ k - N < \tau_{B^{(N)}} \leq k \}, \quad k \geq N.
\]
Hence for \( k \geq N \),
\[
m(B'_{k}) = m(T^{-k}(B \setminus \cup_{n=1}^{N} T^{-n} A) \setminus \cup_{n=1}^{k} T^{-n} A) + m(T^{-k}(B^{(N)}) \setminus \cup_{n=1}^{k} T^{-n} A)
\leq m(T^{-k}(B \setminus \cup_{n=1}^{N} T^{-n} A)) + N \sup_{n>k-N} m(\tau_{B^{(N)}} = n)
= m(B \setminus \cup_{n=1}^{N} T^{-n} A) + N \sup_{n>k-N} m(T^{-n}(B^{(N)}) \setminus \cup_{i=1}^{n-1} T^{-i} B^{(N)}).
\]
The first term in the last line can be made as small as necessary by choosing \( N \) to be large enough, and the second term vanishes as \( k \to \infty \) for any fixed \( N \) by \( \text{(30)} \) applied with \( B^{(N)} \) substituted for \( A \) and \( B \). Hence \( \lim_{k \to \infty} m(B'_{k}) = 0 \), and by \( \text{(34)} \), this yields the required equality \( m_{A}(B) = m(B) \).

We say that the transformation \( T \) is \text{ergodic} if its invariant \( \sigma \)-algebra \( \mathcal{I}_T := \{ A \in \mathcal{F} : T^{-1} A = A \mod m \} \) is \( m \)-trivial, i.e. for every \( A \in \mathcal{I}_T \) either \( m(A) = 0 \) or \( m(A^{c}) = 0 \). Recall the following classical result; see Zweimüller \[37\].

**Hopf’s ratio ergodic theorem.** Let \( T \) be a conservative ergodic measure preserving transformation of a \( \sigma \)-finite measure space \((X, \mathcal{F}, m)\). Then for any functions \( f, g \in L^{1}(X, \mathcal{F}, m) \) with non-zero \( g \geq 0 \),
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} f \circ T^{k}}{\sum_{k=1}^{n} g \circ T^{k}} = \frac{\int_{X} f \, dm}{\int_{X} g \, dm}, \quad m\text{-a.e.}
\]

5. **Stationarity of induced and entrance Markov chains**

In this section we present general results on stationarity of entrance and exit Markov chains using the result of general ergodic theory developed above in Section \[4\]. For completeness of exposition, we also give analogous statements for closely related induced chains. It takes a little effort to make such addition, and it is worth to say that some of our assertions on entrance and exit chains can be also proved with inducing the chain formed by the pair of two consecutive values of the chain observed; see \[16\] below.

5.1. **The setup and notation.** Let \( \mathcal{X} \) be a topological space equipped with the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathcal{X}) \). All the measures on \( \mathcal{X} \) considered here will be Borel. Let \( Y = (Y_{n})_{n \geq 0} \) be a time-homogeneous Markov chain taking values in \( \mathcal{X} \), with a transition kernel \( P \) on \( \mathcal{X} \). Formally, we assume that \( Y \) is defined on some generic measurable space equipped with a family of probability measures \( \{ P_{x} \}_{x \in \mathcal{X}} \) such that: \( Y \) is a Markov chain with the transition kernel \( P \) under every \( P_{x} ; \ P_{x}(Y_{0} = x) = 1 \) for every \( x \in \mathcal{X} \); and the function \( x \mapsto P_{x}(Y \in B) \) is measurable for any set \( B \subset \mathcal{B}(\mathcal{X})^{\otimes \mathbb{N}_{0}} \). Recall that \( \mathbb{N}_{0} = \mathbb{N} \cup \{ 0 \} \). By the Ionescu Tulcea theorem, such family of measures always exists for any transition kernel. For any Borel \( \sigma \)-finite measure \( \mu \) on \( \mathcal{X} \), denote \( P_{\mu} := \int_{\mathcal{X}} P_{x}(\cdot) \mu(dx) \), so \( Y_{0} \) has the “distribution” \( \mu \) under \( P_{\mu} \),
and put \( P_Y^\mu := \mathbb{P}_\mu(Y \in \cdot) \) for the distribution of \( Y \) on \( \mathcal{B}(\mathcal{X})^\otimes N_0 \) under \( \mu \). We will further admit this abuse of probabilistic terminology, and denote by \( \mathbb{E}_x \) and \( \mathbb{E}_\mu \) the respective expectations (integrals) over \( \mathbb{P}_x \) and \( \mathbb{P}_\mu \).

For the rest of Section 5.1 we assume that \( \mu \) a \( \sigma \)-finite invariant measure of \( Y \), that is \( \mu = \mathbb{P}_\mu(Y_1 \in \cdot) \). For any \( x \in \mathcal{X}^N_0 \), denote by \( x_i \) the \( i \)-th coordinate of \( x \). Let \( \theta \) be the one-sided shift operator on \( \mathcal{X}^N_0 \) defined by \( (\theta x)_i := x_{i+1} \) for \( i \geq 0 \). Then \( \theta \) is a measure preserving transformation of the \( \sigma \)-finite measure space \( (\mathcal{X}^N_0, \mathcal{B}(\mathcal{X})^\otimes N_0, \mathbb{P}_\mu^Y) \).

For any \( k \geq 1 \) and \( B \in \mathcal{B}(\mathcal{X})^\otimes k \), define the cylindrical set
\[
C_B := \{ x \in \mathcal{X}^N_0 : (x_0, \ldots, x_{k-1}) \in B \}.
\]
Use the short notation \( \tau_B^\prime := \tau_{C_B} \), which matches in the case \( k = 1 \) the traditional probabilistic notation for the hitting time of \( B \). For arbitrary \( k \), we can think of \( \tau_B \) as of the hitting time of \( B \) by the coordinate Markov chain \( (Y_n, \ldots, Y_{n+k-1})_{n \geq 0} \).

The Markov chain \( Y \) is called recurrent starting under (its \( \sigma \)-finite invariant measure) \( \mu \) if for every Borel set \( B \subset \mathcal{X} \) such that \( \mu(B) > 0 \) we have \( \mathbb{P}_\mu(B)(\tau_B^\prime(Y) = \infty) = 0 \). It is easy to see from invariance of \( \mu \) that this equality is equivalent to \( \mathbb{P}_\mu^Y(\{Y_n \in B \text{ i.o.}\}) = 0 \). We say that \( Y \) is topologically recurrent if \( \mathbb{P}_x(\tau_B^\prime(G) < \infty) = 1 \) for every open set \( G \subset \mathcal{X} \) and every \( x \in G \). Warning: in Markov chains literature they usually take \( \tau_{C_B} \) instead of \( \tau_{C_B} \).

Further, if the shift \( \theta \) is ergodic, then we say that \( Y \) is ergodic starting under \( \mu \) or, synonymously, that \( \mu \) is an ergodic invariant measure of \( Y \). The chain \( Y \) is called irreducible starting under \( \mu \) if every invariant set of \( Y \) is \( \mu \)-trivial, that is for any \( A \subset \mathcal{B}(\mathcal{X}) \), the equality \( \mathbb{P}_x(Y_1 \in A) = \mathbb{1}_A(x) \mod \mu \) implies that either \( \mu(A) = 0 \) or \( \mu(A^c) = 0 \). Equivalently, every \( \theta \)-invariant cylindrical set \( C_A \) with the one-dimensional base \( A \) is \( \mathbb{P}_\mu^Y \)-trivial. We say that \( Y \) is topologically irreducible if \( \mathbb{P}_x(\tau_{C_B}^\prime(G) < \infty) > 0 \) for every \( x \in \mathcal{X} \) and every non-empty open set \( G \subset \mathcal{X} \). For example, so is any recurrent one-dimensional random walk \( S \) on its state space \( \mathbb{Z}_d \); see Section 2.1. We have the following criteria for recurrence and ergodicity.

**Conditions for recurrence and ergodicity.** Let \( Y \) be a Markov chain that takes values in a topological space \( \mathcal{X} \) and has a \( \sigma \)-finite invariant Borel measure \( \mu \).

1) \( Y \) is recurrent starting under \( \mu \) if there exists a sequence of sets \( \{B_n\}_{n \geq 1} \subset \mathcal{B}(\mathcal{X}) \) such that \( \mathbb{P}_\mu^Y(\tau_{B_n}^\prime(Y) = \infty) = 0 \) and \( \mu(B_n) < \infty \) for every \( n \geq 1 \) and \( \mathcal{X} = \bigcup_{n \geq 1} B_n \mod \mu \).

2) \( Y \) is recurrent starting under \( \mu \) if for some \( k \geq 1 \) there exists a set \( B \in \mathcal{B}(\mathcal{X})^\otimes k \) such that \( \mathbb{P}_\mu(\tau_B^\prime(Y) = \infty) = 0 \) and \( \mathbb{P}_\mu((Y_1, \ldots, Y_k) \in B) < \infty \). In particular, this holds if \( \mu \) is finite or if \( \mathbb{P}_x(\tau_{C_B}^\prime(Y) < \infty) = 1 \) for some non-empty open set \( G \subset \mathcal{X} \) of finite measure \( \mu \) and every \( x \in \mathcal{X} \).

3) \( Y \) is ergodic if it is irreducible and recurrent, all the three properties starting under \( \mu \).

The third assertion is by Kaimanovich [18, Proposition 1.7]. As for the first two assertions, the one-sided shift \( \theta \) on \( (\mathcal{X}^N_0, \mathcal{B}(\mathcal{X})^\otimes N_0, \mathbb{P}_\mu^Y) \) is conservative by the conditions of Section 4. In fact, in the first part, \( \mathcal{X}^N_0 \) is exhausted by the cylindrical sets \( C_B \), which all have finite measure \( \mathbb{P}_\mu^Y \) and are recurrent for \( \theta \), and in the second part, \( \tau_{C_B}^\prime < \infty \mathbb{P}_\mu^Y \)-a.e. and \( \mathbb{P}_\mu^Y(C_B) = \mathbb{P}_\mu((Y_1, \ldots, Y_k) \in B) < \infty \). By the same argument of exhausting \( \mathcal{X}^N_0 \) based on
\(\sigma\)-finiteness of \(\mu\), conservativity of the shift is equivalent to recurrence of \(Y\) starting under \(\mu\). Note that in general, a Borel measure on \(\mathcal{X}\) may be infinite on every non-empty open set: for example, take a sum of \(\delta\)-measures at points of a dense countable subset of \(\mathcal{X}\). See Lemma \[6\] below for conditions to exclude such pathological examples.

We will also study non-recurrent Markov chains, in which case we need to work with invertible measure preserving transformations. The shift \(\theta\) on one-sided sequences in \(\mathcal{X}^{\mathbb{N}_0}\) is not invertible, therefore we shall extend time to negative integers. This can be done carefully as follows; apparently, in ergodic theory this is called natural extension.

Now assume that \(\mathcal{X}\) is Polish space. Recall that in this case \(\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{X} \times \mathcal{X})\), \(\mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{X}^{\mathbb{N}_0}) = \mathcal{B}(\mathcal{X}^{\mathbb{N}_0})\), etc.; we always endow direct products of topological spaces with the product topology. For any \(x \in \mathcal{X}^\mathbb{Z}\), denote by \(x_i\) the \(i\)-th coordinate of \(x\), and put \(x^+ := (x_0, x_1, \ldots)\). For any non-empty set \(I \subset \mathbb{Z}\), denote by \(x_I\) the projection of \(x\) on \(\mathcal{X}^I\). For sets, write \(B^+ := \{x^+ : x \in B\}\) for \(B \subset \mathcal{X}^\mathbb{Z}\); and denote \(\bar{B} := \mathcal{X}^{-\mathbb{N}} \times B\) for \(B \subset \mathcal{Z}^{\mathbb{N}_0}\). The two-sided shift operator \(\theta\) on \(\mathcal{X}^\mathbb{Z}\) is defined by \((\theta x)_i := x_{i+1}\).

There exists a unique \(\sigma\)-finite measure on \(\mathcal{B}(\mathcal{X}^\mathbb{Z}) = \mathcal{B}(\mathcal{X}^{-\mathbb{N}}) \otimes \mathcal{B}(\mathcal{X}^{\mathbb{N}_0})\), which we denote by \(\bar{P}_\mu^Y\) and call the extended law of \(Y\), such that \(\bar{P}_\mu^Y(\bar{B}) = P_\mu^Y(B)\) for \(B \in \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{X}^{\mathbb{N}_0})\). This standard extension (Doob [13, Chapter X.1]) is possible by Kolmogorov’s existence theorem in Polish spaces (extended to \(\sigma\)-finite measures by additivity): for cylindrical sets \(B \in \mathcal{B}(\mathcal{X}^\mathbb{Z})\) with a base with finite number of negative coordinates, we put \(\bar{P}_\mu^Y(B) := P_\mu^Y((\hat{\theta}^k B)^+) = \mathbb{P}_\mu(Y \in (\hat{\theta}^k B)^+)\) for any \(k\) large enough such that all coordinates of the base of \(\hat{\theta}^k B\) are non-negative. This definition is consistent by “stationarity” of \(Y\) under \(\mu\). Now the two-sided shift \(\hat{\theta}\) is an invertible measure preserving transformation of the \(\sigma\)-finite measure space \((\mathcal{X}^\mathbb{Z}, \mathcal{B}(\mathcal{X}^\mathbb{Z}), \bar{P}_\mu^Y)\).

Let us describe the extended law \(\bar{P}_\mu^Y\). Let \(\hat{Y} = (\hat{Y}_n)_{n \geq 0}\) be a Markov chain on \(\mathcal{X}\) that is dual to \(Y\) with respect to \(\mu\), i.e. satisfying \(\mathbb{P}_\mu(\hat{Y}_0 \in \cdot) = \mu\) and the time-reversal relation \(\mathbb{P}_\mu((Y_0, Y_1) \in \cdot) = \mathbb{P}_\mu((\hat{Y}_1, \hat{Y}_0) \in \cdot)\) for Borel measures on \(\mathcal{X} \times \mathcal{X}\). For convenience, we will simply assume that \(\hat{Y}_0 = Y_0\). Under our assumptions, such a dual chain always exists and its transition kernel is unique mod \(m\). Indeed, if \(\mu\) is a probability measure, then this statement is nothing but the disintegration theorem combined with existence of regular conditional distributions for probability measures on Polish spaces; see Kallenberg [19] Theorems 6.3 and 6.4] or Aaronson [11, Theorem 1.0.8]. By additivity, this extends to \(\sigma\)-finite measures.

For any \(x \in \mathcal{X}^\mathbb{Z}\), define the time-reversal operator \(R(x) := (\ldots, x_1, x_0, x_{-1}, \ldots)\). We have the following representation for the extended law \(\bar{P}_\mu^Y\) on \(\mathcal{B}(\mathcal{X}^\mathbb{Z})\): for any rectangle \(B = B_1 \times B_0 \times B_2\) with Borel sides \(B_1 \subset \mathcal{X}^{-\mathbb{N}}, B_0 \subset \mathcal{X}^{(0)}, B_2 \subset \mathcal{X}^{\mathbb{N}}\),

\[
\bar{P}_\mu^Y(B) = \int_{B_0} P_\mu^Y(R(B)^+)P_\mu^Y(B^+)\mu(dx_0).
\]

(35)

Here the set \(B^+ = B_0 \times B_2\) is measurable, and by the measurable image theorem [11, Theorem 1.0.3] so is the set \(R(B)^+\). In fact, it satisfies \(R(B)^+ = B_0 \times V(B_1)\), where \(V : (\ldots, x_2, x_{-1}) \mapsto (x_1, x_2, \ldots)\) is a measurable one-to-one mapping between standard measurable spaces \((\mathcal{X}^{-\mathbb{N}}, \mathcal{B}(\mathcal{X}^{-\mathbb{N}}))\) and \((\mathcal{X}^{\mathbb{N}}, \mathcal{B}(\mathcal{X}^{\mathbb{N}}))\).
To prove (35), use the Markov property of $Y$ combined with the fact that $\mathbb{P}_\mu((Y_0, \ldots, Y_k) \in \cdot) = \mathbb{P}_\mu((\hat{Y}_k, \ldots, \hat{Y}_0) \in \cdot)$ for every integer $k \geq 1$. For any cylindrical set $B_1$, for a large enough $k$ this gives

$$
\tilde{P}^Y_\mu(B) = \mathbb{P}_\mu(Y \in (\hat{\theta}^k B)^+) = \mathbb{P}_\mu((Y_0, \ldots, Y_{k-1}) \in B_1, Y_k \in B_k, (Y_{k+1}, Y_{k+2}, \ldots) \in B_2) = \int_{B_0} \mathbb{P}_x((\hat{Y}_1, \hat{Y}_2, \ldots) \in V(B_1))\mathbb{P}_x((Y_1, Y_2, \ldots) \in B_2)\mu(dx_k).
$$

Thus, (35) is valid for cylindrical sets $B = B_1 \times B_0 \times B_2$ with finite number of non-trivial negative coordinates. By an approximation argument, this extends to arbitrary Borel sets $B_1$ if we consider both sides of (35) as measures of $\cdot \times B_0 \times B_2$ on $\mathcal{X}^{-\mathbb{N}}$.

We finish this section introducing some notation. Consider a non-empty Borel set $A \subset \mathcal{X}$. The induced sequence $Y^A$ is obtained by restricting the chain $Y$ to $A$. Formally, we define it using the one-sided shift $\theta$ acting on sequences in $\mathcal{X}^\mathbb{N}_0$:

$$
Y_n^A := (\varphi^n_{C_A}(Y))_0 \quad \text{on} \quad \{Y_k \in A \text{ i.o.}\}, \quad n \geq 0;
$$

in particular, $Y_1^A = Y_{i.o}(Y)$ on the above event. Define the entrance times to $A$ from $A^c$ of the chain $Y$ as

$$
T_0^{\rightarrow A} := 0, \quad T_n^{\rightarrow A} := \inf\{k > T_{n-1}^{\rightarrow A} : Y_{k-1} \not\in A, Y_k \in A\}, \quad n \in \mathbb{N},
$$

and the positions of entrances to $A$ from $A^c$ and exits from $A^c$ to $A$ (cf. (5) and (6)):

$$
Y_n^{\rightarrow A} := Y_{T_n^{\rightarrow A}}, \quad Y_n^{A^c \rightarrow} := Y_{T_n^{A^c \rightarrow} - 1} \quad \text{on} \quad \{Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}\}, \quad n \in \mathbb{N}.
$$

In other terms, for $n \geq 1$ we have

$$
(Y_n^{A^c \rightarrow}, Y_n^{\rightarrow A}) = (\varphi^{n-1}_{C_{A^c} \times A}(Y))_{(0,1)} \quad \text{on} \quad \{Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}\}, \quad (37)
$$

and also

$$
T_1^{\rightarrow A} = \tau_{C_{A^c} \times A}(Y) \cdot \mathbb{1}((Y_0, Y_1) \not\in A^c \times A) + 1 \quad \text{on} \quad \{Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}\}. \quad (38)
$$

For any Borel set $B \subset \mathcal{X}$, denote

$$
N_B := \{x \in \mathcal{X} : \mathbb{P}_x(Y_k \in B \text{ i.o.}) = 1\}.
$$

We will be interested in the cases when the sets $N_A$ and $N_A \cap N_{A^c}$ are non-empty. By the strong Markov property, these are invariant sets for the chain $Y$, in the sense that $\mathbb{P}_x(Y_1 \in N_A) = \mathbb{1}_{N_A}(x)$ or, equivalently, the cylindrical set $C_{N_A}$ is $\theta$-invariant mod $\mathbb{P}_x$ for every $x \in \mathcal{X}$. Hence the induced sequence $Y^A$ belongs to $A \cap N_A$ (and thus has infinitely many terms) $\mathbb{P}_x$-a.s. for every $x \in A \cap N_A$. Similarly, the sequences $Y^{A^c \rightarrow}$ and $Y^{\rightarrow A}$ belong respectively to $A^c \cap N_A \cap N_{A^c}$ and $A \cap N_A \cap N_{A^c}$ and thus have infinitely many terms $\mathbb{P}_x$-a.s. for every $x \in N_A \cap N_{A^c}$. By the strong Markov property, these three random sequences are infinite time-homogeneous Markov chains on $A \cap N_A$, $A^c \cap N_A \cap N_{A^c}$, and $A \cap N_A \cap N_{A^c}$, respectively. We refer to them as to the induced chain, the entrance chain, and the exit chain. For convenience, we define the induced chain $Y^A$ on the whole set $A$ by putting $Y_k^A := Y_k^A$ for $k \geq 1$ if $Y_0^A \in A \setminus N_A$, and extend the entrance chain to $A$ and the exit chain from $A^c$ analogously.
We say that a Borel measure \( \nu \) on \( A \) (equipped with the subspace topology induced from \( \mathcal{X} \)) is **proper** for the induced chain \( Y^A \) if \( \nu(A \setminus N_A) = 0 \). We will use this term analogously in relation to initial distributions of the entrance and the exit chains. For example, if \( Y \) is topologically recurrent and \( A \) is open, then every measure on \( A \) is proper since in this case \( A \subset N_A \). Properness of the measure \( \nu \) for the induced chain means that \( Y^A \) starting under \( \nu \) cannot behave trivially by staying fixed.

The exit chain \( Y^{A^c} \rightarrow \) takes values in the measurable set

\[
(A^c)_\text{exit} = A^c_{\text{ex}} := \{ x \in A^c : \mathbb{P}_x(Y_1 \in A) > 0 \}.
\]

We specify that the transition kernel of this chain is given by

\[
(x, \cdot) \mapsto \int_A \mathbb{P}_y(Y_1^{A^c} \in \cdot)\mathbb{P}_x(Y_1 \in dy | Y_1 \in A), \quad x \in A^c_{\text{ex}} \cap N_A \cap N_{A^c},
\]

and \((x, \cdot) \mapsto \delta_x(\cdot)\) for \( x \in A \setminus (A^c_{\text{ex}} \cap N_A \cap N_{A^c}) \); cf. [14]. We will need the set of possible values of the entrance chain \( Y^A \rightarrow A \) only when \( \mathcal{X} \) is a Polish space and there is a \( \sigma \)-finite invariant Borel measure \( \mu \) of the chain \( Y \) such that \( \mathbb{P}_\mu(Y_0 \in A^c, Y_1 \in A) > 0 \): put

\[
A_{\text{entr}}(\mu) = A_{\text{en}} := \{ x \in A : \mathbb{P}_x(\hat{Y}_1 \in A^c) > 0 \}.
\]

Notice that this set is defined only mod \( \mu \) since the dual transition kernel is unique mod \( \mu \). However, the measure \( \mathbb{P}_{\mu|\Lambda_{A^c}} \) is the same regardless of which modification of \( A_{\text{en}} \) one takes. The measure \( \mathbb{P}_{\mu|\Lambda^c}(Y_1 \in A \setminus \cdot) \) is supported on \( A_{\text{en}} \) in the usual sense \( \mathbb{P}_{\mu|\Lambda^c}(Y_1 \in A \setminus A_{\text{en}}) = 0 \).

### 5.2. Invariant measures of induced and entrance Markov chains.

**Theorem 2.** Let \( Y \) be a Markov chain that takes values in a Polish space \( \mathcal{X} \) and has a \( \sigma \)-finite invariant Borel measure \( \mu \). Let \( A \subset \mathcal{X} \) be a Borel set such that \( \mu(A) > 0 \).

1) The induced chain \( Y^A \) on \( A \) has a proper invariant measure \( \mu_A = \mu|_A \) if either of the following conditions is true:
   a) \( Y \) is recurrent starting under \( \mu \);
   b) \( \mathbb{P}_{\mu_A}(\tau_A(Y) = \infty) = 0 \) and \( \mathbb{P}_{\mu_A}(\tau_A(\hat{Y}) = \infty) = 0 \);
   c) \( \mathbb{P}_{\mu_A}(\tau_A(Y) = \infty) = 0 \) and \( \mu(A) < \infty \);

2) The entrance chain \( Y^A \rightarrow A \) and the exit chain \( Y^{A^c} \rightarrow \) have respective proper Borel invariant measures \( \mu_A^{\text{entr}}(dx) = \mathbb{P}_x(\hat{Y}_1 \in A^c)\mu(dx) \) on \( A \) and \( \mu^{\text{exit}}(dx) = \mathbb{P}_x(Y_1 \in A)\mu(dx) \) on \( A^c \) if \( \mathbb{P}_{\mu|\Lambda^c}(Y_1 \in A) > 0 \) and either of the following conditions is true:
   a) \( Y \) is recurrent starting under \( \mu \);
   b) \( \mathbb{P}_{\mu|\Lambda_{ex}^c}(\tau_{A_{\text{en}}}^c(Y) = \infty) = \mathbb{P}_{\mu|\Lambda_{en}}(\tau_{A_{\text{en}}}^c(Y) = \infty) = 0 \) and the same for \( Y \) replaced by \( \hat{Y} \);
   c) \( \mathbb{P}_{\mu|\Lambda_{ex}^c}(\tau_{A_{\text{en}}}^c(Y) = \infty) = \mathbb{P}_{\mu|\Lambda_{en}}(\tau_{A_{\text{en}}}^c(Y) = \infty) = 0 \) and \( \mathbb{P}_{\mu|\Lambda^c}(Y_1 \in A) < \infty \).

Moreover, if \( Y \) is recurrent (resp. recurrent and ergodic) starting under \( \mu \), then \( Y^A \) is recurrent (resp. recurrent and ergodic) starting under \( \mu_A \), and the same is true for \( Y^A \rightarrow A \) and \( Y^{A^c} \rightarrow \) starting respectively under \( \mu_A^{\text{entr}} \) and \( \mu^{\text{exit}} \) if additionally \( \mathbb{P}_{\mu|\Lambda^c}(Y_1 \in A) > 0 \).

The result of Part 1a with finite \( \mu \) is well-known in probabilistic community.

The assumption of Part 2 ensures that \( Y \) can get from \( A^c \) to \( A \) (starting under \( \mu \)). The measures \( \mu_A^{\text{entr}} \) and \( \mu^{\text{exit}} \) are supported on the respective sets \( A_{\text{en}} = A_{\text{en}}(\mu) \) and \( A_{\text{ex}}^c \). Hence

\[
\mu_A^{\text{entr}}(A_{\text{en}}) = \mu^{\text{exit}}(A_{\text{ex}}^c) = \mathbb{P}_{\mu|\Lambda^c}(Y_1 \in A).
\]
It is easy to express $\mu_A^{\text{entr}}$ directly in terms of the chain $Y$:

$$\mu_A^{\text{entr}}(B) = \int_{A^\infty} \mathbb{P}_x(Y_1 \in B) \mu(dx), \quad B \in \mathcal{B}(A). \quad (40)$$

Recall that sufficient conditions for recurrence and ergodicity of $Y$ were given in Section 5.1 above. We stress the chain $Y$ does not need to be recurrent starting under $\mu$ (i.e. can be \textit{transient}) under the assumptions of Parts 1b and 2b. The cost is checking the corresponding conditions for the dual chain $\hat{Y}$.

\textit{Remark 5.} Each of the assumptions of Part 2 on the chain $Y$ and the set $A$ is true if and only if it is true for the dual chain $\hat{Y}$ substituted for $Y$ and the set $A^c$ substituted for $A$. For Part 2a this follows from the conditions for recurrence in Section 5.1 and \cite[Proposition 1.3]{[18]}. Since the above substitutions interchange the sets $A^c_{\text{ex}}$ and $A_{\text{en}}$, we arrive at the conclusion:

\textit{The exit chain $Y^{A^c} \to$ and the entrance chain $\hat{Y}^{\to A^c}$ have the same laws under $\mu$.}

We already implicitly used it in the time-reversal argument of Section 3.2 for random walks.

\textit{Remark 6.} All the assertions of Theorem 2 except Parts 1b and 2b are valid if $\mathcal{X}$ is merely a topological space, in which case the formula for the measure $\mu_A^{\text{entr}}$ shall be replaced by (10) and the equalities assumed in Part 2c shall be replaced by

$$\mathbb{P}_{\mu|A^c_{\text{ex}}} (\tau_A'(Y) = \infty) = \mathbb{P}_{\mu|A^c_{\text{ex}}} (Y_1 \in A, \tau_{A^c}(Y) = \infty) = 0.$$ 

This is because the dual chain $\hat{Y}$ and the set $A_{\text{en}}$ are defined when the space $\mathcal{X}$ is Polish.

\textbf{Proof.} 1) a) As we explained in Section 5.1 recurrence of $Y$ is equivalent to conservativity of the measure preserving shift $\theta$ on $(\mathcal{X}^{N_0}, \mathcal{B}(\mathcal{X}^{N_0}), \mu_Y)$. We have $\mu_Y(C_A) = \mu(A) > 0$, and from Lemma 3, the induced transformation $\theta_{C_A}$ of the induced space $(C_A, \mathcal{B}(\mathcal{X}^{N_0}) \cap C_A, (\mu_Y)|_{C_A})$ is measure preserving and conservative. Hence for any Borel set $B \subset A$,

$$\mu(A)(B) = \mathbb{P}_\mu(Y_1 \in B) = (\mu_Y)|_{C_A}(C_B) = (\mu_Y)|_{C_A}(x \in \mathcal{X}^{N_0} : \theta_{C_A}(x) \in C_B) = \mathbb{P}_{\mu_A} ((\theta_{C_A}(Y))_0 \in B) = \mathbb{P}_{\mu_A} ((\psi_{C_A}(Y))_0 \in B) = \mathbb{P}_{\mu_A}(Y_1^A \in B),$$

where in the second line we used the definitions (36) and $\theta_{C_A} = (\psi_{C_A})|_{C_A}$. Thus, the measure $\mu_A$ is invariant for the induced chain $Y^A$. This measure is proper for $Y^A$ by implication (20) and the fact that $(\mu_Y)|_{C_A}(\tau_{C_A} = \infty) = 0$ (which holds by conservativity of $\theta$). Recurrence of $Y^A$ starting under $\mu_A$ follows trivially from recurrence of $Y$ under $\mu$.

It remains to infer ergodicity of the induced chain from ergodicity and recurrence of $Y$. Use representation (36) to write the law of the induced chain $Y^A$ starting under $\mu_A$ as $P_{\mu_A} = (\mu_Y)|_{C_A} \circ \psi^{-1}$, where $\psi : C_A \to \mathcal{X}^{N_0}$ is defined by $\psi(x) := (x_0, (\theta_{C_A}(x))_0, (\theta_{C_A}(x))_1, \ldots)$. Note that $\psi(\theta_{C_A}(x)) = \theta(\psi(x))$ for every $x \in C_A$, implying

$$\theta_{C_A}^{-1}(\psi^{-1}B) = \psi^{-1}(\theta^{-1}B), \quad B \in \mathcal{B}(A)^{\otimes N_0}. \quad (41)$$

In particular, this yields that $\theta$ is a measure preserving transformation of $(A^{N_0}, \mathcal{B}(A)^{\otimes N_0}, P_{\mu_A}^{Y_A})$, the fact we already know since $\mu_A$ is an invariant measure for the chain $Y^A$. To show ergodicity of $\theta$ on $A^{N_0}$, consider an invariant set $B \in \mathcal{B}(A)^{\otimes N_0}$, that is $\theta^{-1}B = B \mod P_{\mu_A}^{Y_A}$.
or, equivalently, \( \psi^{-1}(\theta^{-1}B) = \psi^{-1}B \mod (P^Y_\mu)_{\mid C_A} \). By (29), this gives \( \theta^A \mid C_A \left( \psi^{-1}B \right) = \psi^{-1}B \mod (P^Y_\mu)_{\mid C_A} \), meaning that \( \psi^{-1}B \) is an invariant set for \( \theta \mid C_A \) on \( C_A \). Since \( \theta \mid C_A \) is ergodic, the set \( \psi^{-1}B \) is \( (P^Y_\mu)_{\mid C_A} \)-trivial, implying that \( B \) is \( P^A_\mu \)-trivial. This proves ergodicity of the induced Markov chain \( Y^A \) starting under \( \mu_A \).

b) The two-sided shift \( \bar{\theta} \) is a measure preserving transformation of the standard measure space \( (\mathcal{X}^Z, B(\mathcal{X}^Z), \bar{P}^Y_\mu) \). Denote \( \bar{C}_A := \bar{C}_A \). This is a cylindrical set in \( \mathcal{X}^Z \) with no constraints on negative coordinates. Hence \( \bar{P}^Y_\mu(\bar{C}_A) = P^Y_\mu(C_A) = \mu(A) > 0 \) and

\[
\bar{P}^Y_\mu(\bar{C}_A \setminus \cup_{k \geq 1} \bar{\theta}^{-k}(\bar{C}_A)) = \bar{P}^Y_\mu(x \in \mathcal{X}^Z : x_0 \in A, \tau_{C_A} = \infty) = \mathbb{P}_{\mu_A}(\tau^A_{\bar{\theta}}(Y) = \infty) = 0.
\]

In particular, by (29) this implies that the measure \( \mu_A \) is proper for \( Y^A \). Similarly, use representation (35) to get

\[
\mu_A(B) = (P^Y_\mu)_{\mid \bar{C}_A}(\bar{C}_B) = (\bar{P}^Y_\mu)_{\mid \bar{C}_A}(x : \bar{\theta}_{\bar{C}_A}(x) \in \bar{C}_B) = (P^Y_\mu)_{\mid \bar{C}_A}(x : \theta_{\bar{C}_A}(x) \in C_B).
\]

The last probability was already shown in the proof of Part 1a to be \( \mathbb{P}_{\mu_A}(Y_1^A \in B) \).

c) We have \( P^Y_\mu(C_A) > 0 \) and \( P^Y_\mu(x \in C_A, \tau_{C_A} = \infty) = 0 \) arguing as in Part 1b, and the claim follows as in Part 1a if we apply Lemma 2 instead of Lemma 3.

2) a) We argue as above in Part 1a. The cylindrical set \( C_{A^c \times A} \) with two-dimensional base \( A^c \times A \) satisfies \( P^Y_\mu(C_{A^c \times A}) = \mathbb{P}_{\mu_{A^c}}(Y_1 \in A) > 0 \). The induced transformation \( \theta_{C_{A^c \times A}} \) on \( (C_{A^c \times A}, B(\mathcal{X}^{Y_1})) \cap C_{A^c \times A}, (P^Y_\mu)_{\mid C_{A^c \times A}} \) is measure preserving and conservative. In particular, implication (29) and the equality \( (P^Y_\mu)_{\mid C_{A^c \times A}}(\theta_{C_{A^c \times A}} = \infty) = 0 \) (which holds by conservativity of \( \theta \)) give us

\[
0 = (P^Y_\mu)_{\mid C_{A^c \times A}}(\{\theta^k \in C_{A^c \times A} \text{ i.o.}\}) = \mathbb{P}_\mu(\{Y_0 \in A^c, Y_1 \in A\} \cap \{Y_k \in A \text{ i.o.}, Y_k \in A^c \text{ i.o.}\}).
\]

Hence, by the definitions of the sets \( N_A \) and \( N_{A^c} \) and the measure \( \mu^{exit}_{A^c} \), and formula (40) for \( \mu^{entr}_{A^c} \),

\[
0 = \mathbb{P}_\mu(Y_0 \in A^c \setminus (N_A \cap N_{A^c}), Y_1 \in A) + \mathbb{P}_\mu(\{Y_0 \in A^c, Y_1 \in A \setminus (N_A \cap N_{A^c}))
\]

\[
= \mu^{exit}_{A^c}(A^c \setminus (N_A \cap N_{A^c})) + \mu^{entr}_{A^c}(A \setminus (N_A \cap N_{A^c})).
\]

Thus, the measures \( \mu^{exit}_{A^c} \) and \( \mu^{entr}_{A^c} \) are proper for the respective chains \( Y^{A^c} \) and \( Y^{A^c} \).

Now compute, by Lemma 3 for any Borel set \( B \subset C \times A \),

\[
\mathbb{P}_\mu((Y_0, Y_1) \in B) = (P^Y_\mu)_{\mid C_{A^c \times A}}(C_B) = (P^Y_\mu)_{\mid C_{A^c \times A}}(x \in \mathcal{X}^{Y_1} \setminus \theta_{C_{A^c \times A}}(x) \in C_B)
\]

\[
= \mathbb{P}_\mu((Y_0, Y_1) \in A^c \times A, (\theta_{C_{A^c \times A}}(Y))_{\{0, 1\}} \in B).
\]

Combining this with (37) and the identity \( \theta_{C_{A^c \times A}} = (\varphi_{C_{A^c \times A}})_{\mid C_{A^c \times A}} \), we obtain

\[
\mathbb{P}_\mu((Y_0, Y_1) \in B) = \mathbb{P}_\mu((Y_0, Y_1) \in A^c \times A, (Y_{2^{A^c \rightarrow A}}, Y_{2^{A^c \rightarrow A}}) \in B).
\]
By the Markov property of $Y$, for any Borel set $B_1 \subset A$,
\[
\mu_{A}^{\text{entr}}(B_1) = \mathbb{P}_\mu((Y_0, Y_1) \in A^c \times A, Y_2^{\to A} \in B_1) \\
= \int_{A^c} \mu(dx_0) \int_A \mathbb{P}_{x_0}(Y_2^{\to A} \in B_1 | Y_1 \in dx_1) \mathbb{P}_{x_0}(Y_1 \in dx_1) \\
= \int_{A^c} \mu(dx_0) \int_A \mathbb{P}_{x_1}(Y_1^{\to A} \in B_1) \mathbb{P}_{x_0}(Y_1 \in dx_1) \\
= \mathbb{P}_{\mu_{A}^{\text{entr}}}(Y_1^{\to A} \in B_1).
\]

(43)

Thus, the measure $\mu_{A}^{\text{entr}}$ is invariant for the entrance chain $Y_1^{\to A}$. Similarly, since the measure $\mu_{A^c}^{\text{exit}}$ is supported on $A^c_{ex}$, for any Borel set $B_2 \subset A^c$,
\[
\mu_{A^c}^{\text{exit}}(B_2) = \int_{A^c_{ex}} \mu(dx_0) \int_A \mathbb{P}_{x_1}(Y_1^{A^c_{ex} \to} \in B_2) \mathbb{P}_{x_0}(Y_1 \in dx_1) \\
= \int_{A^c_{ex}} \mathbb{P}_{x_0}(Y_1 \in A) \mu(dx_0) \int_A \mathbb{P}_{x_1}(Y_1^{A^c_{ex} \to} \in B_2) \mathbb{P}_{x_0}(Y_1 \in dx_1 | Y_1 \in A) \\
= \mathbb{P}_{\mu_{A^c}^{\text{exit}}}(Y_1^{A^c_{ex} \to} \in B_2),
\]

(44)

where in the last equality we used (39) and the fact that the measure $\mu_{A^c}^{\text{exit}}$ is proper for the exit chain $Y_1^{A^c \to}$. Thus, $\mu_{A^c}^{\text{exit}}$ is invariant for $Y_1^{A^c \to}$.

Recurrence of the entrance chain trivially follows from conservativity of the induced transformation $\theta_{C_{A^c \times A}}$ if we argue as above in (43) to start $Y^{\to A}$ under $\mu_{A}^{\text{entr}}$. Similarly, the exit chain is recurrent starting under $\mu_{A^c}^{\text{exit}}$, cf. (44).

As for ergodicity, define the functions $\psi_0 : C_{A^c \times A} \to (A^c)^N$ and $\psi_1 : C_{A^c \times A} \to A^N$ by
\[
\psi_i(x) := (x_i, (\theta_{C_{A^c \times A}}(x))_i, (\theta_{C_{A^c \times A}}^2(x))_i, \ldots), \quad x \in C_{A^c \times A}, i \in \{0, 1\}.
\]

The entrance chain $Y^{\to A} = (Y_n^{\to A})_{n \geq 1}$ starts from $Y_1^{\to A}$, which equals $Y_1$ on the "event" $\{Y_0 \in A^c, Y_1 \in A\}$. Since for any Borel set $B \subset A$,
\[
\mu_{A}^{\text{entr}}(B) = \mathbb{P}_{\mu}(Y_0 \in A^c, Y_1 \in B) = \mathbb{P}_{\mu}^{Y}(x \in C_{A^c \times A} : x_1 \in B),
\]

we can write the law $Y^{\to A}$ with $Y_1^{\to A}$ distributed according to $\mu_{A}^{\text{entr}}$ as $(\mathbb{P}_{\mu}^{Y})|_{C_{A^c \times A}} \circ \psi_1^{-1}$. Similarly, the law of the exit chain $Y^{A^c \to}$ with $Y_1^{A^c \to}$ following $\mu_{A^c}^{\text{exit}}$ is $(\mathbb{P}_{\mu}^{Y})|_{C_{A^c \times A}} \circ \psi_0^{-1}$. Then ergodicity of the chains $Y^{\to A}$ and $Y^{A^c \to}$ follows from ergodicity of $Y$ exactly as in Part 1.

b) We prove as in Part 1b. The cylindrical set $\bar{C}_{A^c \times A}$ satisfies $\bar{\mathbb{P}}_{\mu}^{Y}(\bar{C}_{A^c \times A}) = \mathbb{P}_{\mu|A^c}(Y_1 \in A) > 0$. Arguing similarly to (42) and using the equality $\tau_{A^c \times A}^t(Y) = \tau_{A^c_{ex}}^t(Y) \mod \mathbb{P}_{\mu}$,
we get
\[
\bar{P}_\mu (\bar{C}_{A^c \times A} \setminus \cup_{k \geq 1} \bar{\theta}^{-k}(\bar{C}_{A^c \times A})) \\
= P_\mu (Y_0 \in A^c, Y_1 \in A, \tau_{A^c \times A} (Y) = \infty) \\
= P_\mu (Y_0 \in A_{ex}^c, Y_1 \in A_{en}, Y_2, Y_3, \ldots \not\in A_{ex}^c) \\
+ \sum_{k=1}^{\infty} P_\mu (Y_0 \in A_{ex}^c, Y_1 \in A_{en}, Y_2, \ldots, Y_k \not\in A_{ex}^c, Y_{k+1} \in A_{ex}^c, Y_{k+2}, Y_{k+3}, \ldots \not\in A_{en}) \\
\leq P_{\mu | A_{en}} (\tau'_{A_{ex}^c} (Y) = \infty) + \infty \cdot P_{\mu | A_{ex}^c} (\tau'_{A_{en}} (Y) = \infty) = 0. \tag{45}
\]

In particular, this implies (as in Part 1b) that the measures \(\mu_{A^c}^{exc}\) and \(\mu_{A^c}^{entr}\) are proper for the respective chains \(Y \rightarrow A^c\) and \(Y \rightarrow A^c\).

Further, using representation (35) and invariance of the two-sided shift \(\bar{\theta}\),
\[
\bar{P}_\mu (\bar{C}_{A^c \times A} \setminus \cup_{k \geq 1} \bar{\theta}^{-k}(\bar{C}_{A^c \times A})) \\
= \bar{P}_\mu (x \in \mathcal{X}^Z : (x_0, x_1) \in A^c \times A, (x_{-1}, x_0) \not\in A^c \times A, (x_{-2}, x_{-1}) \not\in A^c \times A, \ldots) \\
= \bar{P}_\mu (x \in \mathcal{X}^Z : (x_0, x_{-1}) \in A \times A^c, (x_{-1}, x_{-2}) \not\in A \times A^c, (x_{-2}, x_{-3}) \not\in A \times A^c, \ldots) \\
= P_\mu (Y_0 \in A, \bar{Y}_1 \in A^c, \tau'_{A \times A^c} (\bar{Y}) = \infty).
\]
The last expression is zero. This follows from the equality \(\tau'_{A \times A^c} (\bar{Y}) = \tau'_{A_{en} \times A_{ex}^c} (\bar{Y}) \mod P_\mu \) exactly as in (45).

From Lemma 2 and Remark 4, the induced transformation \(\bar{\theta}_{\bar{C}_{A^c \times A}}\) of the standard measure space \((\bar{C}_{A^c \times A}, \mathcal{B}(\mathcal{X}^Z) \cap \bar{C}_{A^c \times A}, \bar{P}_\mu \mid_{\bar{C}_{A^c \times A}})\) is measure preserving. Then \((P_\mu \mid_{\bar{C}_{A^c \times A}}) (\bar{C}_{B}) = (\bar{P}_\mu \mid_{\bar{C}_{A^c \times A}}) (\bar{C}_{B})\) holds for any Borel set \(B \subset A^c \times A\), and thus
\[
P_\mu (Y_0, Y_1 \in B) = (P_\mu \mid_{\bar{C}_{A^c \times A}}) (\bar{\theta}_{\bar{C}_{A^c \times A}} \in C_B).
\]
This equality implies the required invariance, as already shown in Part 2a.

c) We have \(\bar{P}_\mu (C_{A^c \times A}) > 0\) and \(P_\mu (x \in C_{A^c \times A}, \tau_{C_{A^c \times A}} = \infty) = 0\) arguing as in Part 2b, and the claim follows as in Part 2a if we apply Lemma 2 instead of Lemma 3. \(\square\)

Proof of Remark 6. The one-sided shift is measure-preserving on \((\mathcal{X}^{N_0}, \mathcal{B}(\mathcal{X}) \otimes \mathcal{N}_0, P_\mu)\) if \(\mathcal{X}\) is a topological space, so we simply replace the measure space in the proofs of Parts 1a, 1c, 2a, 2c of Theorem 2 accordingly. \(\square\)

Note that we could have proved Part 2 applying directly the result of Part 1 to the coordinate chain \(Z = ((Y_n, Y_{n+1}))_{n \geq 0}\), which takes values in the Polish space \(\mathcal{X} \times \mathcal{X}\), has an invariant measure \(P_\mu ((Y_0, Y_1) \in \cdot)\), and satisfies the relation
\[
Z_{n-1}^{A^c \times A} = (Y_n^{A^c \rightarrow}, Y_n^{A^c \rightarrow}) \mod \{Y_0 \in A^c, Y_1 \in A\}, \quad n \geq 1 \tag{46}
\]
following from (37).

Next we present occupation time formulas for lifting invariant measures of the induced and entrance chains to recover the invariant measure of the underlying Markov chain. The assumptions of the following result are stronger than those of respective parts of Theorem 2.
Proposition 4. Let $Y$ be a Markov chain that takes values in a Polish space $\mathcal{X}$ and has a $\sigma$-finite invariant Borel measure $\mu$. Let $A \subset \mathcal{X}$ be a Borel set such that $P_{\mu}(\tau'_A(Y) = \infty) = 0$. 

1) 
\[
P_{\mu}(Y \in E) = E_{\mu_A}\left[\sum_{k=0}^{\tau'_A(Y)-1} 1((Y_k, Y_{k+1}, \ldots) \in E)\right], \quad E \in \mathcal{B}(\mathcal{X}^{\mathbb{N}_0}) \tag{47}
\]
if either of the following conditions is true:

a) $Y$ is recurrent starting under $\mu$;

b) $P_{\mu_A}(\tau'_A(Y) = \infty) = 0$;

2) 
\[
P_{\mu}(Y \in E) = E_{\mu_{\text{entr}}}\left[\sum_{k=0}^{T_{\text{entr}}-1} 1((Y_k, Y_{k+1}, \ldots) \in E)\right], \quad E \in \mathcal{B}(\mathcal{X}^{\mathbb{N}_0}) \tag{48}
\]
if $P_{\mu}(\tau'_A(Y) = \infty) = 0$ and either of the following conditions is true:

a) $Y$ is recurrent starting under $\mu$;

b) $P_{\mu_{\Lambda_{\text{entr}}}}(\tau'_{\Lambda_{\text{entr}}}(Y) = \infty) = P_{\mu_{\Lambda_{\text{entr}}}}(\tau'_{\Lambda_{\text{entr}}}(Y) = \infty) = 0$.

Thus, we can recover $\mu$ from either $\mu_A$ or $\mu_{\text{entr}}$ using (47) and (48) applied for $E = C_B$ with $B \in \mathcal{B}(\mathcal{X})$.

Remark 7. Similarly to Remark 6, the results of Parts 1a and 2a with $\mathcal{B}(\mathcal{X}^k)$ and $\mathcal{B}(\mathcal{X}^{\mathbb{N}_0})$ replaced respectively by $\mathcal{B}(\mathcal{X})^\otimes k$ and $\mathcal{B}(\mathcal{X})^\otimes \mathbb{N}_0$ are still valid if $\mathcal{X}$ is a topological space.

Proof. Assume that $\mu$ is non-zero, otherwise the statements are trivial. Then the equality $P_{\mu}(\tau'_A(Y) = \infty) = 0$ ensures that $\mu(A) > 0$ by

\[
0 < \mu(\mathcal{X}) = \sum_{k=1}^{\infty} P_{\mu}(\tau'_A(Y) = k) \leq \sum_{k=1}^{\infty} P_{\mu}(Y_k \in A) = \infty \cdot \mu(A). \tag{49}
\]

Thus, the assumptions of Parts 1a and 1b of the proposition are stronger than those of Parts 1a and 2b of Theorem 2.

1) b) As we seen in the proof of Part 1b of Theorem 2, the two-sided shift $\tilde{\theta}$ is an invertible measure preserving transformation of $(\mathcal{X}^{\mathbb{Z}}, \mathcal{B}(\mathcal{X}^{\mathbb{Z}}), P_{\mu}^Y)$ satisfying the assumptions of Lemma 2. For the cylindrical set $C_A$, we have $P_{\mu}^Y(C_A) = \mu(A) > 0$ and $P_{\mu}^Y(\tau_{C_A} = \infty) = P_{\mu}(\tau'_A(Y) = \infty) = 0$. Thus, all the assumptions of Lemma 4 are satisfied. From (32) and (35) it follows that for any $E \subset \mathcal{B}(\mathcal{X}^{\mathbb{N}_0})$,

\[
P_{\mu}^Y(E) = P_{\mu}^Y(\tilde{E}) = \int_{C_A} \left[\sum_{k=0}^{\tau_{C_A}(x)-1} 1(\theta^k x \in \tilde{E})\right] P_{\mu}^Y(dx) = \int_{C_A} \left[\sum_{k=0}^{\tau_{C_A}(x)-1} 1(\theta^k x \in E)\right] P_{\mu}^Y(dx).
\]

This implies (47) by the equality $\tau_{C_A} = \tau'_A$.

a) As we seen in the proof of Part 1a of Theorem 2, the one-sided shift $\theta$ is a measure preserving conservative transformation of $(\mathcal{X}^{\mathbb{N}_0}, \mathcal{B}(\mathcal{X}^{\mathbb{N}_0}), P_{\mu}^Y)$. For the cylindrical set $C_A$, we have $P_{\mu}^Y(C_A) = \mu(A) > 0$ and $P_{\mu}^Y(\tau_{C_A} = \infty) = P_{\mu}(\tau'_A(Y) = \infty) = 0$. Thus, all the
assumptions of Lemmas 3 and 4 are satisfied for the one-sided shift $\theta$ and the set $C_A$, and (17) follows directly from formula (32) as shown in the proof of Part 1a above.

2) Note that $\mathbb{P}_\mu(Y_0 \in A^c, Y_1 \in A) > 0$. In fact, relation (19) applied for $A^c$ instead of $A$ and equality $\mathbb{P}_\mu(\tau_{A^c}(Y) = \infty) = 0$ imply that $\mu(A^c) > 0$. Then the assertion follows from the equality $\mathbb{P}_\mu(\tau_{A^c}(Y) = \infty) = 0$ by an argument analogous to (19). Thus, the assumptions of Parts 2a and 2b of the proposition are stronger than those of Parts 2a and 2b of Theorem 2.

Then for the cylindrical set $C_{A^c \times A}$, we have $\mathbb{P}_\mu(Y)(C_{A^c \times A}) = \mathbb{P}_\mu(Y_0 \in A^c, Y_1 \in A) > 0$ and $\mathbb{P}_\mu(\tau_{C_{A^c \times A}} = \infty) = 0$. The latter equality follows from the assumptions $\mathbb{P}_\mu(\tau_{A^c}(Y) = \infty) = 0$ and $\mathbb{P}_\mu(\tau_{A^c}(Y) = \infty) = 0$ by the same argument as in (19).

a) All the assumptions of Lemmas 3 and 4 are satisfied for the one-sided shift $\theta$ on $(X^{\mathbb{N}_0}, \mathcal{B}(X^{\mathbb{N}_0}), P^\theta_Y)$ and the set $C_{A^c \times A}$. Using (32) and invariance of $(P^\theta_Y)_{C_{A^c \times A}}$ under $\theta_{C_{A^c \times A}}$, for any $E \in \mathcal{B}(X^{\mathbb{N}_0})$ we obtain

$$
P^\theta_Y(E) = \int_{C_{A^c \times A}} \sum_{k=1}^{\tau_{C_{A^c \times A}}(x)} 1(\theta^k x \in E) \mu(dx_0) = \int_{A^c} \mathbb{E}_{x_0} \left[ \sum_{k=1}^{\tau_{C_{A^c \times A}}(Y)} 1(\theta^k Y \in E, Y_1 \in A) \right] \mu(dx_0)
$$

(see (32) below for more details on the first equality). Then use the Markov property to get

$$
P^\theta_Y(E) = \int_{A^c} \mu(dx_0) \int_A \mathbb{E}_{x_1} \left[ \sum_{k=0}^{\tau_{C_{A^c \times A}}(Y)} 1(\theta^k Y \in E) \right] \mathbb{P}_{x_0}(Y_1 \in dx_1),
$$

which implies (18) by (32) and the definition of the measure $\mu^\text{entr}_A$.

b) The two-sided shift $\bar{\theta}$ on $(X^\mathbb{Z}, \mathcal{B}(X^\mathbb{Z}), P^\theta_Y)$ and the set $C_{A^c \times A}$ satisfy the assumptions of Lemmas 2 and 4. Then (48) follows by a computation similar to those in the proofs of Parts 1b and 2a above.

5.3. Existence and uniqueness of invariant measures for induced and entrance chains. Recall some definitions. A Borel measure on a topological space $\mathcal{X}$ is called locally finite if every point of $\mathcal{X}$ admits an open neighbourhood of finite measure. Such measures are finite on compact sets. Locally finite measures on Polish spaces are often called Radon. It is not hard to show that every locally finite measure on a separable metric space is $\sigma$-finite: the space can be represented as countable union of open balls of finite measure. A Markov chain taking values in a metric space is weak Feller if its transition kernel is weakly continuous in the starting point.

In the statements of this Section 5.3 we always assume that the subsets $A$ and $A^c$ of $\mathcal{X}$ are equipped with the subspace (induced) topology.

**Theorem 3.** Let $Y$ be a topologically irreducible topologically recurrent weak Feller Markov chain that takes values in a Polish space $\mathcal{X}$. Let $A \subset \mathcal{X}$ be a Borel set with $\text{Int}(A) \neq \emptyset$.

1) The mapping $\mu \mapsto \mu_A$ is a bijection between the sets of locally finite Borel invariant measures of the chain $Y$ on $\mathcal{X}$ and of the induced chain $Y^A$ on $A$.

2) Assume that there exists an $x \in \text{Int}(A^c)$ such that $\mathbb{P}_x(Y_1 \in \text{Int}(A)) > 0$. Then the mappings $\mu \mapsto \mu_A^\text{entr}$ and $\mu \mapsto \mu_A^\text{exit}$ (defined in (21)) are bijections between the sets of
locally finite Borel invariant measures of the chain $Y$ on $X$ and, respectively, of the entrance chain $Y \rightarrow A$ on $A$ and the exit chain $Y^{A^c} \rightarrow$ on $A^c$.

Note that under the assumptions of the theorem, the chain $Y$ may have two non-proportional invariant measures even if the space $X$ is compact, as in the nice example of Carlsson [9, Theorem 1] whose conditions are satisfied by (51) below.

The condition $P_x(Y_1 \in \text{Int}(A)) > 0$ is to exclude the case when the chain enters $A$ from $A^c$ only through $\partial A$.

The theorem implies that the questions of existence and uniqueness (up to multiplication by constant) of non-zero locally finite invariant measure are equivalent for the chains $Y$, $Y^A$, $Y^{A^c}$. The main use of this result is when the initial chain $Y$ has a known unique invariant measure. Our main application, given below in Section 6, is for random walks, whose invariant measure is the Lebesgue measure. On the other hand, in some cases Theorem 3 can be used to prove existence of invariant measure for the chain $Y$. We will give a result in this direction (Proposition 5), which easily follows from the next statement.

As for references, there are many existence and uniqueness results under much stronger assumptions than weak Fellerness, including strong Feller or Harris properties or $\psi$-irreducibility. These are not directly related to our paper and therefore are not listed here. After this paper was finished, we found the work of Skorokhod [30] on weak Feller chains. His uniqueness result [30, Theorem 3] is essentially equivalent to Part 1 of our Theorem 3, and his existence result is stronger than our Proposition 5. The approach of [30] is similar to inducing used in this paper.

Lemma 5. Under the respective assumptions of Theorem 3, the induced chain $Y^A$ and the entrance chain $Y \rightarrow A$ on $A$ are weak Feller if $P_x(Y_1 \in \partial A) = 0$ for every $x \in X$.

Corollary. For any topologically recurrent random walk $S$ on $\mathbb{R}$ with continuous distribution of increments, the chain $O$ of overshoots at up-crossings of the zero level is weak Feller.

We postpone the proof of the lemma for a moment. The Corollary follows immediately.

Remark 8. In general, the chains $Y^A$ and $Y^{A^c}$ may not be weak Feller. For example, consider a recurrent random walk $S$ on $\mathbb{R}$ whose distribution of increments is continuous except of an atom at $-1$. Take $A = [0, \infty)$. Then $P_1(S_1^A = 0) = P_1(S_1 = 0) = 1$ but for any $0 < x < 1$, the distribution of $S_1^A$ is continuous and thus $P_x(S_1^A = 0) = 0$. Similar examples can be constructed for the chain $O = S^{\rightarrow A}$.

Remark 9. The assertion of Part 1 of Theorem 3 remains valid without topological irreducibility of $Y$ if instead we assume that the chain $Y$ has a unique (non-zero) locally finite Borel invariant measure and either the set $A$ is open or $P_x(Y_1 \in \partial A) = 0$ for every $x \in X$. Similarly, the assertion of Lemma 5 on $Y^A$ remains valid without topological irreducibility of $Y$ if $A$ is open. In both statements, completeness of $X$ is not needed.

Combining Theorem 3 with Lemma 5 and the Bogolubov–Krylov theorem applied to the chain induced to an appropriately relatively compact open set, we immediately get an existence result. We give a slightly better version without topologically irreducibility; see Remark 6 and its proof below.
Proposition 5. Let $Y$ be a topologically recurrent weak Feller Markov chain that takes values in a locally compact Polish space $\mathcal{X}$. Assume that there exists a relatively compact non-empty open set $A \subset \mathcal{X}$ such that $\mathbb{P}_x(Y_1 \in \partial A) = 0$ for every $x \in \mathcal{X}$. Then the chain $Y$ has a locally finite non-trivial invariant measure.

When this paper was finished, we found a stronger result by Skorokhod [30], which is not stated explicitly but follows directly from [30, Lemmata 4 and 6] and Cohn [10, Proposition 7.1.9]: the chain $Y$ always has a non-zero locally finite invariant measure if it is weak Feller, satisfies (51), and the space $\mathcal{X}$ is locally compact separable Hausdorff.

Before proceeding to the proof of Theorem 3 we give a simple auxiliary result.

Lemma 6. Let $Y$ be a topologically irreducible weak Feller Markov chain taking values in a metric space $\mathcal{X}$. An invariant Borel measure $\mu$ of $Y$ is locally finite if and only if it is finite on some non-empty open set.

Proof. The necessary condition is trivial. To prove the sufficient one, assume that $G$ is a non-empty open subset of $\mathcal{X}$ satisfying $\mu(G) < \infty$. By topological irreducibility of $Y$, for any $x \in \mathcal{X}$ there exists an $n = n(x) \geq 1$ such that $\mathbb{P}_x(Y_n \in G) > 0$. It is easy to show, using the Chapman–Kolmogorov equation, that weak Fellerness of $Y$ implies that the $n$-step transition kernel $\mathbb{P}_x(Y_n \in \cdot)$ is weakly continuous in $x$. Hence there is an open neighbourhood $U_x$ of $x$ such that $\mathbb{P}_y(Y_n \in G) \geq \frac{1}{2} \mathbb{P}_x(Y_n \in G)$ for every $y \in U_x$. By invariance of $\mu$, this gives

$$\infty > \mu(G) = \int_{\mathcal{X}} \mathbb{P}_y(Y_n \in G) \mu(dy) \geq \int_{U_x} \mathbb{P}_y(Y_n \in G) \mu(dy) \geq \frac{1}{2} \mathbb{P}_x(Y_n \in G) \mu(U_x),$$

(50)

implying finiteness of $\mu(U_x)$. \hfill \Box

Proof of Theorem 3. First we claim that

$$\mathbb{P}_x(\tau_G^\mathcal{X}(Y) < \infty) = 1 \text{ for every } x \in \mathcal{X} \text{ and non-empty open } G \subset \mathcal{X}. \quad (51)$$

In fact, as in the proof of Lemma 6 by topological irreducibility and weak Fellerness of $Y$ we can find an open neighbourhood $U$ of $x$ such that $\inf_{y \in U} \mathbb{P}_y(\tau_G^\mathcal{X}(Y) < \infty) > 0$. The claim now follows by topological recurrence and the strong Markov property of the chain $Y$, which returns to $U$ $\mathbb{P}_x$-a.s.

Let $\mu$ be a non-zero locally finite Borel invariant measure of the Markov chain $Y$. Then the measure $\mu_A$ is Borel and locally finite on $A$, and so are the measures $\mu_{A^{\text{entr}}}^\mathcal{X}$ on $A$ and $\mu_{A^{\text{exit}}}^\mathcal{X}$ on $A^c$ as follows from the inequalities $\mu_{A^{\text{entr}}}^\mathcal{X} \leq \mu_A$ and $\mu_{A^{\text{exit}}}^\mathcal{X} \leq \mu_A^\mathcal{X}$. Further, $\mu$ is $\sigma$-finite as a locally finite measure on a Polish space. By choosing an open set $G$ in (51) of finite measure, we conclude that $Y$ is recurrent starting under $\mu$; see the conditions for recurrence in Section 5.1. Then Theorem 2 applies, and the measure $\mu_A$, $\mu_{A^{\text{entr}}}^\mathcal{X}$, and $\mu_{A^{\text{exit}}}^\mathcal{X}$ are invariant for the respective chains $Y^A$, $Y \to_A$, and $Y^{A^{\to}}$.

1) Let us prove surjectivity of the mapping $\mu \mapsto \mu_A$. Let $\nu$ be a locally finite non-zero Borel invariant measure of the induced chain $Y^A$ on $A$. As in (47) for one-dimensional cylindrical sets $E = C_B$, we can “lift” $\nu$ from $A$ to a measure on $\mathcal{X}$:

$$\tilde{\nu}(B) := \mathbb{E}_\nu\left[\sum_{k=0}^{\tau_A(Y)\mathcal{X}^{-1}} 1(Y_k \in B)\right] = \int_A \mathbb{E}_y\left[\sum_{k=0}^{\tau_A(Y)\mathcal{X}^{-1}} 1(Y_k \in B)\right] \nu(dy), \quad B \in \mathcal{B}(\mathcal{X}).$$
Note the difference with equality \( (32) \) in Lemma 4 where we considered measures on the trajectory space \( \mathcal{X}^\mathbb{N}_0 \) instead of measures on \( \mathcal{X} \) as here. The crucial observation is that \( \tau'_A(Y) \) is finite \( \mathbb{P}_\nu \)-a.e., although we do not require that \( \nu \ll \mu_A \) as in Lemma 4. In fact, we have \( \mathbb{P}_\nu(\tau'_A(Y) \leq \tau'_{\text{Int}(A)}(Y) < \infty) = 1 \) for every \( y \in \mathcal{X} \).

Then by the same argument as in \( (33) \), from the equality \( \mathbb{P}_\nu(\tau'_A(Y) = \infty) = 0 \) we obtain

\[
\bar{\nu}(B) = \sum_{k=0}^{\infty} \mathbb{P}_\nu(Y_k \in B, \tau'_A(Y) > k), \quad B \in \mathcal{B}(\mathcal{X}),
\]

and then

\[
\mathbb{P}_\nu(Y_1 \in B) = \int_\mathcal{X} \mathbb{P}_y(Y_1 \in B) \bar{\nu}(dy) = \sum_{k=0}^{\infty} \int_\mathcal{X} \mathbb{P}_y(Y_1 \in B) \mathbb{P}_\nu(Y_k \in dy, \tau'_A(Y) > k)
\]

\[
= \sum_{k=0}^{\infty} \mathbb{P}_\nu(Y_{k+1} \in B, \tau'_A(Y) \geq k + 1) = \mathbb{E}_\nu \left[ \sum_{k=1}^{\tau'_A(Y)} 1(Y_k \in B) \right] = \bar{\nu}(B), \quad k \geq 1
\]

where the second to the last inequality is again analogous to \( (33) \) and in the last equality we used the relation \( Y_{\tau'_A(Y)} = Y^A_1 \) and the assumed invariance of \( \nu \) for the induced chain \( Y^A \).

Thus, \( \bar{\nu} \) is an invariant measure for the chain \( Y \) and it clearly satisfies \( \bar{\nu}|_A = \nu \). Since \( \nu \) is locally finite in the subspace topology, we can find a set \( G \subset \text{Int}(A) \) open in this topology such that \( 0 < \nu(G) < \infty \). By Lemma 6 this implies that \( \bar{\nu} \) is locally finite since \( G \) is also open in the topology of \( \mathcal{X} \). Thus, the mapping \( \mu \mapsto \mu^\text{entr} \) is surjective.

2) We first consider the mapping \( \mu \mapsto \mu^\text{entr} \). To prove its surjectivity, let \( \nu \) be a locally finite non-zero Borel invariant measure of the entrance chain \( Y^A \) on \( A \). The Borel measure

\[
\mu_1(B) := \mathbb{E}_\nu \left[ \sum_{k=0}^{T^A_{\tau'_A(Y)}} 1(Y_k \in B) \right], \quad B \in \mathcal{B}(\mathcal{X}),
\]

is invariant for \( Y \), which can be checked exactly as in the proof of Part 1 above. In fact, we have \( Y^A = Y_{T^A_{\tau'_A(Y)}} \) by definition of the entrance chain, where \( \mathbb{P}_\nu \)-a.s. finiteness of \( T^A_{\tau'_A(Y)} \) follows from the strong Markov property of \( Y \) combined with the equalities

\[
\mathbb{P}_y(\tau'_A(Y) \leq \tau'_{\text{Int}(A)}(Y) < \infty) = \mathbb{P}_y(\tau'_{\text{entr}}(Y) \leq \tau'_{\text{entr} \text{Int}(A)}(Y) < \infty) = 1, \quad y \in \mathcal{X}.
\]

Further, for any Borel set \( B \subset A \), we have

\[
\int_\mathcal{X} \mathbb{P}_y(Y_1 \in B) \mu_1(dy) = \sum_{k=0}^{\infty} \int_\mathcal{X} \mathbb{P}_y(Y_1 \in B) \mathbb{P}_\nu(Y_k \in dy, T^A_{\tau'_A(Y)} > k)
\]

\[
= \sum_{k=0}^{\infty} \mathbb{P}_\nu(Y_{k+1} \in B, T^A_{\tau'_A(Y)} = k + 1)
\]

\[
= \mathbb{P}_\nu(Y^A_1 \in B) = \nu(B).
\]

By the assumption we have \( \mathbb{P}_x(Y_1 \in \text{Int}(A)) > 0 \) for some \( x \in \text{Int}(A^c) \), and it follows that there exists a \( z \in \text{Int}(A) \) such that for any open set \( U \) satisfying \( z \in U \subset \text{Int}(A) \) we have \( \mathbb{P}_x(Y_1 \in U) > 0 \). In fact, if this was not true, then every \( z \in \text{Int}(A) \) would admit
an open neighbourhood \( U_z \subset \text{Int}(A) \) such that \( \mathbb{P}_x(Y_1 \in U_z) = 0 \). Hence the Borel measure \( \mathbb{P}_x(Y_1 \in \cdot \cap \text{Int}(A)) \) is zero on compact sets, and by inner regularity of finite Borel measures on Polish spaces (Bogachev [6, Theorem 7.1.7]), we arrive at \( \mathbb{P}_x(Y_1 \in \text{Int}(A)) = 0 \), which is a contradiction.

By local finiteness of \( \nu \), choose an open set \( U \) satisfying \( z \in U \subset \text{Int}(A) \) such that \( \nu(U) < \infty \). It holds \( \mathbb{P}_x(Y_1 \in U) > 0 \) and by the weak Feller property of \( Y \), we can find an open set \( U_x \) such that \( x \in U_x \subset \text{Int}(A^c) \) and \( \mathbb{P}_y(Y_1 \in U) \geq \frac{1}{2}\mathbb{P}_x(Y_1 \in U) \) for every \( y \in U_x \). By (54) and exactly the same argument as in (50), this gives \( \mu_1(U_x) < \infty \). Hence the measure \( \mu_1 \) on \( \mathcal{X} \) is locally finite by Lemma [4] and so the mapping \( \mu \mapsto \mu_A^{\text{entr}} \) is surjective. Its injectivity follows immediately from Part 2a of Proposition [4].

Now consider the mapping \( \mu \mapsto \mu_A^{\text{exit}} \). To prove its surjectivity, let \( \nu^{\text{exit}} \) be a locally finite non-zero Borel invariant measure of the exit chain \( Y^{A^c} \) on \( A^c \). Then the Borel measure \( \nu := \int_{A^c} \mathbb{P}_y(Y_1 \in \cdot | Y_1 \in A) \nu^{\text{exit}}(dy) \) on \( A \) is invariant for the entrance chain \( Y^A \) from \( A^c \) to \( A \), and the measure \( \mu_1 \) introduced in (53) is invariant for the chain \( Y \). Moreover, we have the following equality of Borel measures on \( A^c \):

\[
\mathbb{P}_y(Y_1 \in A)\mu_1(dy) = \sum_{k=0}^{\infty} \mathbb{P}_y(Y_1 \in A)\mathbb{P}_\nu(Y_k \in dy, T_1^{A^c} > k) = \mathbb{P}_\nu(Y_{T_1^{A^c} - 1} \in dy) = \mathbb{P}_\nu(Y_1^{A^c} \rightarrow) = \nu^{\text{exit}}(dy), \quad y \in A^c. \tag{55}
\]

Then, if \( x \in A^c \) is such that \( \mathbb{P}_x(Y_1 \in \text{Int}(A)) > 0 \), by weak Fellerness of \( Y \) and local finiteness of \( \nu^{\text{exit}} \) we can choose an open set \( U \) such that \( x \in U \subset \text{Int}(A^c) \), \( \nu^{\text{exit}}(U) \) is finite, and \( \mathbb{P}_y(Y_1 \in \text{Int}(A)) \geq \frac{1}{2}\mathbb{P}_x(Y_1 \in \text{Int}(A)) \) for every \( y \in U \). By (55), this gives

\[
\mu_1(U) = \int_U \frac{\nu^{\text{exit}}(dy)}{\mathbb{P}_y(Y_1 \in A)} \leq \int_U \frac{\nu^{\text{exit}}(dy)}{\mathbb{P}_y(Y_1 \in \text{Int}(A))} \leq \frac{2\nu^{\text{exit}}(U)}{\mathbb{P}_x(Y_1 \in \text{Int}(A))} < \infty,
\]

hence the measure \( \mu_1 \) on \( \mathcal{X} \) is locally finite by Lemma [4]. So the mapping \( \mu \mapsto \mu_A^{\text{exit}} \) is surjective. Also, by the equality \( \nu = \int_{A^c} \mathbb{P}_y(Y_1 \in \cdot | Y_1 \in A) \mu_1(dy) \) of measures on \( A \), \( \nu \) is locally finite since \( \mu_1 \) is so, as we proved earlier. By the established injectivity of the mapping \( \mu \mapsto \mu_A^{\text{entr}} \), this implies injectivity of the mapping \( \mu \mapsto \mu_A^{\text{exit}} \).

**Proof of Remark 9.** Let \( \mu \) be the unique non-zero locally finite Borel invariant measure of \( Y \). We have \( \mathcal{X} = \cup_{n \geq 1} G_n \) for some sequence of open balls \( G_n \) of finite measure \( \mu \). Then \( Y \) is recurrent starting under \( \mu \) by topological recurrence and the conditions for recurrence of Section 7.1.

For the other direction, we need to show that if \( \nu \) is a locally finite non-zero Borel invariant measure of \( Y^A \) on \( A \), then the invariant measure \( \bar{\nu} \) of \( Y \) is locally finite on \( \mathcal{X} \). Under either of the assumptions, \( \nu \) must be supported on \( \text{Int}(A) \). From the definition of the measure \( \bar{\nu} \) and recurrence of the open set \( \text{Int}(A) \), we can see (cf. (53)) that \( \bar{\nu} \) is supported on the set \( N := \{ x \in \mathcal{X} : \sum_{n=0}^{\infty} \mathbb{P}_x(Y_n \in \text{Int}(A)) > 0 \} \). Then for every \( x \in N \) we have \( \mathbb{P}_x(Y_n \in \text{Int}(A) \cap G_k) > 0 \) for some integer \( k, n \geq 1 \), and by the same argument as in the proof of Lemma [6], \( x \) has an open neighbourhood of finite measure \( \bar{\nu} \). Finally, by uniqueness of locally finite invariant measure of \( Y \), the measures \( \bar{\nu} \) and \( \mu \) are proportional, and so are \( \nu \) and \( \mu_A \) as required.
Proof of Lemma 5.} Consider the induced chain $Y^A$. Note that if the set $A$ is open, then this chain is well defined even without topological irreducibility of $Y$ since $Y$ returns to $A$ infinitely often by topological recurrence. We need to show that the mapping $x \mapsto \mathbb{E}_x f(Y^A_1)$ is continuous on $A$ for every continuous bounded function $f$ on $A$. Define the extension $\bar{f}$ of $f$ on $X$ by putting $\bar{f} := 0$ on $A^c$. Then for every $x \in A$,

$$\mathbb{E}_x f(Y^A_1) = \sum_{k=1}^{\infty} \mathbb{E}_x \left[ \bar{f}(Y_k) 1(Y_1, \ldots, Y_{k-1} \in A^c) \right],$$

and by the dominated convergence theorem and the fact that $\mathbb{P}_y(\tau_A'(Y) < \infty) = 1$ for $y \in A$, it suffices to prove continuity in $x$ for every term. We use a simple inductive argument. It follows from the weak Feller property of $Y$ that the first term $\mathbb{E}_x \bar{f}(Y_1)$ is continuous at every $x \in \mathcal{X}$ since the bounded function $\bar{f}$ is continuous on $(\partial A)^c$ and therefore continuous $\mathbb{P}_x(Y_1 \in \cdot)$-a.s. For every $k \geq 1$, by the Chapman–Kolmogorov equation,

$$\mathbb{E}_x \left[ \bar{f}(Y_{k+1}) 1(Y_1, \ldots, Y_k \in A^c) \right] = \int_{\mathcal{X}} 1_{A^c}(y) \cdot \mathbb{E}_{y} \left[ \bar{f}(Y_k) 1(Y_1, \ldots, Y_{k-1} \in A^c) \right] \mathbb{P}_x(Y_1 \in dy).$$

Now we see that the above is a continuous function of $x$ by the weak Feller property of $Y$ and $\mathbb{P}_x(Y_1 \in \cdot)$-a.s. continuity (in $y$) of the integrand, whose second factor is continuous by assumption of induction.

Similarly, the weak Feller property of the entrance chain $Y \to A$ follows from the identity

$$\mathbb{E}_x f(Y^A_1) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mathbb{E}_x \left[ \bar{f}(Y_{n+k}) 1(Y_1, \ldots, Y_{n-1} \in A, Y_n, \ldots, Y_{n+k-1} \in A^c) \right]$$

by induction on $n$ using the continuity $\mathbb{E}_x f(Y^A_1)$ obtained above for the basis step $n = 1$. \(\Box\)

6. Applications to random walks

In this section we apply the ideas developed in Section 5 to random walks in arbitrary dimension. In particular, we answer our initial questions on stationarity properties of the chain of overshoots of a one-dimensional random walk over the zero level. Although the results of this section follow easily from those of Section 3 we state them as separate theorems.

Assume now that the random walk $S$ takes values in $\mathbb{R}^k$ with arbitrary $k \geq 1$. As in the Introduction, let $\lambda$ be the Haar measure on $\mathcal{Z}$, the minimal closed subgroup of $(\mathbb{R}^k, +)$ containing the topological support of the distribution of $X_1$. Then $\lambda$ is invariant for the walk $S$ on $\mathcal{X} = \mathcal{Z}$. We can say more.

**Lemma 7.** Any topologically recurrent random walk $S$ on its state space $\mathcal{Z}$ is recurrent and ergodic starting under $\lambda$, which is a unique (up to multiplication by constant) locally finite Borel invariant measure of $S$ on $\mathcal{Z}$.

**Proof.** The uniqueness is by Proposition I.45 in Guivarc’h et al. [15], which states that the right Haar measure on a locally compact Hausdorff topological group $G$ with countable base is a unique invariant Radon Borel measure for any topologically recurrent right random walk on $G$ such that no proper closed subgroup of $G$ contains the support of the distribution of increments of the walk.
To infer the ergodicity, note that uniqueness of invariant measure implies irreducibility of \( S \) starting under \( \lambda \). In fact, if there is a \( \lambda \)-non-trivial invariant set \( A \in \mathcal{B}(\mathbb{Z}) \) of \( S \), then the locally finite measure \( 1_A \lambda \) is invariant for \( S \), which contradicts the uniqueness. Further, by [15, Theorem 24], topological recurrence of \( S \) implies that \( \mathbb{P}_x(\tau_A^c(S) < \infty) = 1 \) for every \( x \in \mathbb{Z}_d \) and every non-empty open set \( G \subset \mathbb{Z} \). Hence \( S \) is recurrent starting under \( \lambda \) by the conditions for recurrence of Section 5.1. Therefore, \( S \) is ergodic by irreducibility and recurrence, all the three properties starting under \( \lambda \) (Kaimanovich [18, Proposition 1.7]). \( \square \)

We say that a Borel set \( A \subset \mathbb{Z} \) is massive for the random walk \( S \) if \( \mathbb{P}_x(\tau_A^c(S) < \infty) = 1 \) for \( \lambda \)-a.e. \( x \in \mathbb{Z} \). In particular, if \( S \) is topologically recurrent, then any Borel set of positive measure \( \lambda \) is massive, as follows (Aaronson [1, Proposition 1.2.2]) from ergodicity and recurrence of \( S \) starting under \( \lambda \) (Lemma [7]). By the Chung–Fuchs theorem, \( S \) is topologically recurrent in dimension \( k = 2 \) if \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}\|X_1\|^2 < \infty \). If \( S \) is transient (i.e. not topologically recurrent), no set of finite measure can be massive. For walks on \( \mathbb{Z} = \mathbb{Z}^k \) with \( k \geq 3 \) and satisfying \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}\|X_1\|^2 < \infty \), there is a necessary and sufficient condition for massiveness of a set (Wiener’s test), stated in terms of capacity, by Itô and McKean [17] and Uchiyama [33]. Easily verifiable sufficient conditions for massiveness in \( k = 3 \) are due to Doney [12]. For example, any “line” in \( \mathbb{Z}^3 \) is massive. Under the above assumptions, a set is massive for every such a walk if it is massive for a simple random walk, and so this is a property of a set rather than of a walk. We are not aware of any explicit results for non-lattice random walks. It appears (based on the estimates of Green’s function in Uchiyama [32, Section 8]) that these should be fully analogous to the lattice ones for walks with \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}\|X_1\|^2 < \infty \) if the distribution of \( X_1 \) has density with respect to the Lebesgue measure. The case of heavy-tailed random walks on \( \mathbb{Z}^k \), including transient walks in dimensions \( k = 1, 2 \), is considered by Bendikov and Cygan [4, 5].

Since \( -A \) is massive for \( S \) if and only if \( A \) is massive for \( -S \), and the random walk \( -S \) is dual to \( S \) with respect to the measure \( \lambda \) (see (27)), Theorem 2 immediately implies the following result.

**Theorem 4.** Assume that the sets \( A, -A, A^c, -A^c \) are massive for a random walk \( S \) on its state space \( \mathbb{Z} \), where \( \mathbb{Z} \subset \mathbb{R}^k, k \geq 1 \). Then the measures \( \lambda^\text{entr}_A(dx) = \mathbb{P}(X_1 \in x - A^c)\lambda(dx) \) on \( A \) and \( \lambda^\text{exit}_{A^c}(dx) = \mathbb{P}(X_1 \in A - x)\lambda(dx) \) on \( A^c \) are invariant for the respective entrance chain \( S \rightarrow A \) and exit chain \( S^A \rightarrow \).

If \( k = 1 \) and the walk \( S \) oscillates, this yields Theorem 1. In fact, the chain of overshoots \( O \) at up-crossings of the zero level is the entrance chain \( S \rightarrow \mathbb{Z}_d^+ \) into \( \mathbb{Z}_d^+ = \mathbb{Z} \cap [0, \infty) \) from \( \mathbb{Z}_d^- = \mathbb{Z} \cap (-\infty, 0) \). Then for \( \lambda = \lambda_d \), the measure

\[
\lambda^\text{entr}_{\mathbb{Z}_d^+}(dx) = \mathbb{P}_x(-S_1 \in \mathbb{Z}_d^-)\lambda_d(dx) = \mathbb{P}(X_1 > x)\lambda_d(dx) = c_1^{-1}\pi_+(dx), \quad x \in \mathbb{Z}_d^+
\]

is invariant for \( O \). Similarly, the chain of undershoots at up-crossings is \( U = S \mathbb{Z}_d^- \rightarrow \), hence

\[
\lambda^\text{exit}_{\mathbb{Z}_d^-}(dx) = \mathbb{P}_x(S_1 \in \mathbb{Z}_d^+)\lambda_d(dx) = \mathbb{P}(X_1 \geq -x)\lambda_d(dx) = c_1^{-1}\pi_+(-dx - d), \quad x \in \mathbb{Z}_d^-
\]

is invariant for \( U \). Therefore the measure \( \pi_+ \) is invariant for the chain \((-U_n - d)_{n \geq 0}\).
Remark 10. If \( Z = \mathbb{Z}^k \) with \( k \geq 3 \), \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}\|X_1\|^2 < \infty \), then the assumptions on \(-A\) and \(-A^c\) in Theorem 4 are not required since a set is massive for \( S \) whenever it is massive for a simple random walk, which is self-dual. It is not clear if such reduction is possible for arbitrary \( S \).

We now state a uniqueness result.

**Theorem 5.** Let \( S \) be any topologically recurrent random walk on its state space \( Z \), where \( Z \subset \mathbb{R}^k \) with \( k \in \{1, 2\} \), and let \( A \subset Z \) be any \( \lambda \)-non-trivial Borel set with \( \lambda(\partial A) = 0 \). Then the entrance chain \( S^{\to A} \) is ergodic and recurrent starting under \( \lambda^{\text{entr}}_A \), which is the unique (up to multiplication by constant) locally finite Borel invariant measure of \( S^{\to A} \). The same is true for \( S^{A^c} \to \) starting under \( \lambda^{\text{exit}}_A \).

This follows by combining Theorems 2, 3 and Lemma 7 with the fact that the transition kernel of any random walk is weak Feller and the inequality
\[
\mathbb{P}_{\lambda^{\text{entr}}_A} (S_1 \in \text{Int}(A)) > 0,
\]
which is true since otherwise the \( \lambda \)-non-trivial set \( \text{Cl}(A) \) is invariant for \( S \).

**Corollary.** If a one-dimensional random walk \( S \) is topologically recurrent on \( \mathbb{Z}_d \), then the chains of overshoots \( O, O^\downarrow, O^\uparrow \) are ergodic and recurrent starting, respectively, under their unique invariant measures \( \pi_+, \pi_- \), and \( \pi = \frac{1}{2} \pi_+ + \frac{1}{2} \pi_- \).

7. Further results on level-crossings by random walks

7.1. The limit theorem for the number of level-crossings. Recall that \( L_n \) denotes the number of zero-level crossings of the one-dimensional random walk \( S \) by time \( n \). Combining Theorem 5 on ergodicity of the chain of overshoots with a result by Perkins [27] on convergence of local times of random walks, we obtain the following central limit theorem for \( L_n \). To the best of our knowledge, all other results of this type require some smoothness assumptions for the distribution of increments of the walk.

**Theorem 6.** For any random walk \( S \) such that \( \mathbb{E}X_1 = 0 \) and \( \sigma^2 := \mathbb{E}X_1^2 \in (0, \infty) \), we have
\[
\lim_{n \to \infty} \mathbb{P}_x \left( \frac{L_n}{\sqrt{n}} \leq y \right) = 2\Phi \left( \frac{\sigma y}{2\mathbb{E}|X_1|} \right) - 1, \quad y \geq 0, \quad x \in \mathbb{Z}_d,
\]
where \( \Phi \) denotes the distribution function of a standard normal random variable.

We will need the next auxiliary result, the law of large numbers for the chain \( O \). It does not follow directly from ergodicity of \( O \) (given by the Corollary to Theorem 5) since Birkhoff’s ergodic theorem implies convergence of the time averages only for \( \pi \)-a.e. \( x \).

**Proposition 6.** Let \( S \) be any random walk such that \( \mathbb{E}X_1 = 0 \) and \( \sigma^2 := \mathbb{E}X_1^2 \in (0, \infty) \). Then for any \( x \in \mathbb{Z}_d \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |O_k| = \int_{\mathbb{Z}_d} |y| \pi(dy) = \frac{\sigma^2}{2\mathbb{E}|X_1|}, \quad \mathbb{P}_x \text{-a.s.} \tag{56}
\]
Proof of Theorem 6. According to Perkins [27, Theorem 1.3], if the random walk $S$ starts at $x = 0$, then

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{L_n} |O_k| \xrightarrow[n \to \infty]{d} \sigma \ell_0,$$  

(57)

where $\ell_0$ is local time at 0 at time 1 of a standard Brownian motion. Since Perkins’s definition of crossing times is slightly different from the one of ours, his result shall be applied to the random walk $-S$ in order to get (57). The proof of [27], which uses a (highly non-trivial and unusual) blend of probability and non-standard analysis, transfers without any changes to the case of an arbitrary starting point $x$. This follows (28) from [27, Lemma 3.2], which is the key ingredient in the proof of [27, Theorem 1.3].

By Lévy’s theorem, $\ell_0$ has the same distribution as the absolute valued of a standard normal random variable. Then the theorem follows by (57) and Proposition 6. □

Proof of Proposition 6. One can easily check that

$$\int_{Z_d} y \pi_+(dy) = \frac{2}{E|X_1|} \int_{d/2}^{\infty} (y - d/2) P(X_1 > y) dy = \frac{2}{E|X_1|} \int_{0}^{\infty} (y - d/2) P(X_1 > y) dy$$

(58)

and, similarly,

$$-\int_{Z_d} y \pi_-(dy) = \frac{2}{E|X_1|} \int_{0}^{\infty} (y + d/2) P(X_1 > y) dy.$$  

(59)

Using that $E|X_1| = 0$ and integrating the above equality by parts, we find that the first absolute moment of $\pi = \frac{1}{2} \pi_+ + \frac{1}{2} \pi_-$ is $\sigma^2/(2E|X_1|)$. Therefore, since $\pi_+$ is a probability measure, by Birkhoff’s ergodic theorem and ergodicity of the chain of overshoots $O$ asserted in the Corollary to Theorem 5, the convergence in (56) takes place for $\pi$-a.e. $x \in Z_d$. We need to prove this for every $x \in Z_d$.

Denote by $N$ the set of points $x \in \text{supp} \pi$ that satisfy (56). It suffices to prove that $N = \text{supp} \pi$. In fact, regardless of the starting point, the chain $O$ hits the support of $\pi$ (which is an interval, possibly infinite) at the first step. The equality needed is trivial in the arithmetic case $d = 0$, where $Z_d$ is discrete. In the non-arithmetic case, we will use that $N$ is dense in $\text{supp} \pi$. Our argument goes as follows.

For real $y_1, y_2$, define the functions

$$g(y_1, y_2) := 1(y_1 < 0, y_2 \geq 0 \text{ or } y_1 \geq 0, y_2 < 0), \quad f(y_1, y_2) := |y_2| g(y_1, y_2).$$

We claim that for any $x \in \text{supp} \pi$ and any $\varepsilon \in (0, 1)$, there exists a $y \in N$ such that

$$\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{k=1}^{n} f(y + S'_{k-1}, y + S'_k) - \frac{1}{n} \sum_{k=1}^{n} f(x + S'_{k-1}, x + S'_k) \right| \leq \varepsilon, \quad \mathbb{P}\text{-a.s.}$$

(60)

This will imply that $x \in N$ and hence prove Proposition 6, since

$$\mathbb{P} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} g(y + S'_{k-1}, y + S'_k) = \frac{\sigma^2}{2E|X_1|} \right) = \mathbb{P}_y \left( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |O_k| = \frac{\sigma^2}{2E|X_1|} \right) = 1.$$  

(61)
Here the second equality holds by the definition of the set $N$ and fact that $\lim_{n \to \infty} L_n = \infty$ $\mathbb{P}$-a.s., where, recall, $L_n$ denotes the number of zero level crossings of the random walk $S$ by time $n$.

From the identity $\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_1}{b_1} (1 - \frac{b_1}{b_2} \cdot \frac{a_2}{a_1})$ for $a_1, a_2, b_1, b_2 > 0$, we see that (61) follows if we show that for any $x \in \text{supp} \pi$ and any $\varepsilon \in (0, 1)$, there exists a $y \in N$ such that $\mathbb{P}$-a.s.,

$$
\limsup_{n \to \infty} \left[ \left| \sum_{k=1}^{n} f(x + S_{k-1}, x + S_k') \right| - 1 \right] \leq \frac{\varepsilon \mathbb{E}|X_1|}{\sigma^2}.
$$

(62)

For any $\delta > 0$, $k \geq 1$, and any $y \in N$ such that $|x - y| \leq \delta$, we have

$$
|f(x + S_{k-1}', x + S_k') - f(y + S_{k-1}', y + S_k')| 
\leq \delta g(y + S_{k-1}', y + S_k') + (|y + S_k'| + \delta) \mathbb{1}(|y + S_{k-1}'| \leq \delta \text{ or } |y + S_k'| \leq \delta)
$$

and

$$
g(y + S_{k-1}', x + S_k') - g(y + S_{k-1}', y + S_k') \leq \mathbb{1}(|y + S_{k-1}'| \leq \delta \text{ or } |y + S_k'| \leq \delta).
$$

Note that the r.h.s.'s of these inequalities (say, for $k = 0$) are integrable with respect to $\mathbb{P}_{\lambda_0}$; recall that $\int_{\mathbb{R}} \mathbb{P}_y(\cdot) \lambda_0(dy) = \mathbb{P}_{\lambda_0}$. Moreover, the functions $f(y + S_{k-1}', y + S_k')$ are also integrable with respect to $\mathbb{P}_{\lambda_0}$, cf. (58) and (59). We obtain

$$
\left| \sum_{k=1}^{n} f(x + S_{k-1}', x + S_k') \right| - 1 
\leq \frac{\sum_{k=1}^{n} \delta g(y + S_{k-1}', y + S_k') + (|X_k| + 2\delta) \mathbb{1}(|y + S_{k-1}'| \leq \delta) + 2\delta \mathbb{1}(|y + S_k'| \leq \delta)}{\sum_{k=1}^{n} f(y + S_{k-1}', y + S_k')}
$$

(63)

and

$$
\left| \sum_{k=1}^{n} g(x + S_{k-1}', x + S_k') \right| - 1 
\leq \frac{\sum_{k=1}^{n} \mathbb{1}(|y + S_{k-1}'| \leq \delta) + \mathbb{1}(|y + S_k'| \leq \delta)}{\sum_{k=1}^{n} g(y + S_{k-1}', y + S_k')}
$$

(64)

As we saw above in the proof of Theorem 5, the topologically recurrent random walk $S$ on $Z_0 = \mathbb{R}$ is recurrent and ergodic starting under the Lebesgue measure $\lambda_0$. As we explained in Section 5.1, recurrence of $S$ starting under $\lambda_0$ implies conservativity of the measure preserving one-sided shift $\theta$ on $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \mathbb{P}^S_{\lambda_0})$. Then by Hopf’s ratio ergodic theorem (see Section 4), for every fixed $\delta > 0$, the ratios in the r.h.s.’s of (63) and (64) converge $\mathbb{P}$-a.s. as $n \to \infty$ to the respective constants

$$
\delta \mathbb{E}|X_1| + 2\delta (\mathbb{E}|X_1| + 2\delta) + 4\delta^2 \quad \text{and} \quad \frac{4\delta}{\mathbb{E}|X_1|}\n$$

(65)

for $\lambda_0$-a.e. $y$. Denote by $N_\delta$ the set of such $y$’s.

Choose a $\delta > 0$ such that the sum of the constants in (65) smaller than $\varepsilon \mathbb{E}|X_1|/(2\sigma^2)$. The Borel set $N \cap N_\delta$ has full measure $\lambda_0 |_{\text{supp} \pi}$ and hence is dense in $\text{supp} \pi$. Therefore we can pick a $y \in N \cap N_\delta$ that satisfies $|x - y| \leq \delta$. Then (62) follows from (63) and (64). Recall that (62) implies (60), which proves the proposition by (61).
7.2. **Expected occupation times between level-crossings.** In the rest of the section we present several identities for occupation times, which are direct corollaries of Proposition 4 on general Markov chains.

**Proposition 7.** For any random walk \( S \) that oscillates, for any Borel set \( B \subset \mathbb{Z}_d \) we have

\[
c_1 \lambda(B) = \mathbb{E}_{\pi_+} \left[ \sum_{k=0}^{T_1-1} 1(S_k \in B) \right] = \mathbb{E}_{\pi_-} \left[ \sum_{k=0}^{T_1-1} 1(S_k \in B) \right] = 2 \mathbb{E}_\pi \left[ \sum_{k=0}^{T_1-1} 1(S_k \in B) \right].
\]

We have not seen these formulas in the random walks literature with exception of one particular case when \( S_1 \) is a symmetric simple random walk. Here \( \pi_+ = \delta_0 \) and the first formula above is described by Feller [14, Section XII.2, Example (b)], who praised “the fantastic nature of this result”. In this case, the above formulas are very similar to the formula above is described by Feller [14, Section XII.2, Example (b)], who praised “the fantastic nature of this result”. In this case, the above formulas are very similar to the identity

\[
1 = \mathbb{E}_0 \left[ \sum_{k=0}^{\tau(0)_{\pi} - 1} 1(S_k \in B) \right], \quad B \in \mathcal{B}(\mathbb{Z}_d),
\]

which holds for any arithmetic recurrent random walk \( S \) with span \( d \) and corresponds to \( A = \{0\} \) and \( E = C_B \) in [47]. The analogue of this identity for non-arithmetic recurrent walks with a general Borel set \( A \) of positive Lebesgue measure instead of \( \{0\} \) is in Ornstein [24, Theorem 5].

**Proof.** The Markov chain \( S \) on \( \mathbb{Z}_d \), its invariant measure \( c_1 \lambda \), and the set \( \mathbb{Z}_d^+ \) satisfy the assumptions of Part 2b of Proposition 4; see Section 6. Then the first equality follows from the fact that \( \pi_+ = c_1 \mu^{\text{entr}}_{\mathbb{Z}_d^+} \) and formula (48) applied to \( E = C_B \). The second equality is analogous. The third one follows by applying the first two to the sets \( B \cap \mathbb{Z}_d^+ \) and \( B \cap \mathbb{Z}_d^- \). \( \square \)

Further, for the **number of up-crossings** of arbitrary level \( a \) by time \( n \geq 1 \), defined as

\[
L_n^+(a) := \sum_{i=0}^{n-1} 1(S_i < a, S_{i+1} \geq a), \quad a \in \mathbb{Z}_d,
\]

we have the following surprising result.

By taking \( E = \{ x \in \mathbb{Z}_{d}^{\text{entr}} : x_0 < a, x_1 \geq a \} \) and \( A = \mathbb{Z}_d^+ \) in (48), we obtain a surprising result that the expected number of up-crossings by time \( T_1 \) does not depend on the level if \( S \) is started under 1.

**Proposition 8.** For any non-degenerate random walk \( S \) satisfying \( \mathbb{E}X_1 = 0 \), we have

\[
\mathbb{E}_{\pi_+} L_{T_1}^+(a) = 1 \quad \text{and} \quad \mathbb{E}_{\pi_-} L_{T_1}^+(a) = 1 \quad \text{for any } a \in \mathbb{Z}_d.
\]

Thus, the expected number of up-crossings by time \( T_1 \) does not depend on the level (if \( S \) is started under \( \pi_+ \) or \( \pi_- \), i.e. at stationarity of either chain \( O \) or \( O^\dagger \)), and therefore equals 1 since \( L_{T_1}^+(0) = 1 \) by the definition of \( T_1 \).

**Proof.** For the first equality, take \( E_a = \{ x \in \mathbb{Z}_d^{\text{entr}} : x_0 < a, x_1 \geq a \} \) and \( A = \mathbb{Z}_d^+ \) in (48), and use the facts that \( \mathbb{P}_{\lambda_0}(S \in E_a) = \mathbb{P}_{\lambda_0}(S \in E_0) = c_1^{-1} \) (where the first equality follows since
both $\lambda_d$ and the transition kernel of $S$ are shift-invariant) and $\pi_+ = c_1 \mu_{Z_d}^{entr}$. The second equality is analogous.

From the idea that it is more natural to start the random walk from 0 rather than under $\pi_+$, we can use Proposition 8 to find $\mathbb{E}_0 L_{T_1}^\uparrow (a)$ for two specific types of distributions of increments. We say that $X_1$ has upward exponential distribution if the conditional distribution $\mathbb{P}(X_1 > \cdot | X_1 > 0)$ is exponential. For every distribution of this type we have $\pi_+(\cdot) = \mathbb{P}(X_1 < \cdot | X_1 > 0)$, which by Proposition 8 and the memoryless property of exponential distributions easily implies (we omit the computations) that

$$\mathbb{E}_0 L_{T_1}^\uparrow (a) = \frac{\mathbb{P}(X_1 > 0)}{\mathbb{P}(X_1 \neq 0)} + \mathbb{P}(X_1 \geq a | X_1 > 0), \quad a > 0.$$  

We say that $X_1$ has upward skip-free distribution if $\mathbb{P}(X_1 \in \{1, 0, -1, \ldots\}) = 1$. If the random walk $S$ has such increments, then $\pi_+ = \delta_0$ and thus $\mathbb{E}_0 L_{T_1}^\uparrow (a) = 1$ for every real $a$.

The main application of random walks with upward exponential distributions is in queuing theory, where they feature in the Lindley formula for the waiting times in GI/M/1 queues with exponential service times; see Asmussen [2, Section III.6]. The main application of random walks with skip-free distributions is in theory of branching processes; they also appear in queuing theory [2, Section III.6].

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