Symplectic resolutions: deformations and birational maps

D. Kaledin

Abstract

We study projective birational maps of the form $\pi : X \to Y$, where $Y$ is a normal irreducible affine algebraic variety over $\mathbb{C}$, and $X$ is a smooth holomorphically symplectic resolution of the variety $Y$. Under these assumptions, we prove several facts on the geometry of the resolution $X$. In particular, we prove that the map $\pi$ must be semismall. We also prove that any two such resolutions $X_1 \to Y$, $X_2 \to Y$ are deformationally equivalent. Finally, we prove the existence part of the log-flip conjecture for an extremal contraction $\pi : X \to Y$ of a holomorphically symplectic projective algebraic manifold $X$, and prove that the associated log-flip $X'$ is also symplectic and smooth.

Contents

1 Statements of the results 4
2 Geometry 8
  2.1 Semismallness 8
  2.2 Birational maps 11
3 Deformation theory 13
4 Twistor deformations 17
  4.1 The construction 17
  4.2 Algebraization 19
  4.3 The generic fiber 21

*Partially supported by CRDF grant RM1-2087.
Introduction

In the recent papers [H1], [H2], [H4] D. Huybrechts has established some new and important facts about the geometry of compact hyperkähler manifolds. In particular, he proved the following result:

- If two simple projective holomorphically symplectic manifolds $X_1$, $X_2$ are birationally equivalent, then they are diffeomorphic and correspond to non-separated points in the moduli space. In other words, for such a pair $X_1$, $X_2$ there exist one-parameter deformations $X_1$, $X_2$ over the base $S = \text{Spec } \mathbb{C}[[t]]$ whose generic fibers are isomorphic as complex manifolds.

Here simple means that $H^{2,0}(X_1) = \mathbb{C}$ is generated by the holomorphic symplectic form $\Omega \in \Omega^2(X_1)$, and the same is true for $X_2$.

In this paper we try to establish a local counterpart of Huybrechts’ results. Namely, we consider an affine complex variety $X$ which admits two smooth projective resolutions $\pi_1 : X_1 \to Y$, $\pi_2 : X_2 \to Y$, and we assume that the manifolds $X_1$, $X_2$ are holomorphically symplectic. In this situation, we would like to prove that there exist one-parameter deformations $X_1$, $X_2$, $Y$ over the base $S = \text{Spec } \mathbb{C}[[t]]$ and projective maps $\pi_1 : X_1 \to Y$, $\pi_2 : X_2 \to Y$, such that

- The fibers over the special point $o \in S$ of the varieties $X_1/Y$, $X_2/Y$ are isomorphic to the given resolutions $X_1/Y$, $X_2/Y$.

- The generic fibers of the varieties $X_1$, $X_2$ are isomorphic.

Unfortunately (and contrary to what has been claimed in the first version of this paper) we are only able to prove this result under a very restrictive technical assumption (Condition 5.1). Roughly speaking, we need certain graded algebra to be finitely generated. We do believe that this condition is always satisfied, so that our result holds in full generality. However, at
present we do not have a complete proof of this fact. We plan to return to
this matter in a subsequent paper.

Assuming Condition 5.1, we will prove slightly more. Firstly, it turns out
that the generic fibers of the deformations \( X_1, X_2 \) are not only isomorphic
to each other, but they are both isomorphic to the generic fiber of the
deformation \( Y \) (in particular, all these generic fibers are affine). Secondly, we
can drop the smoothness assumption on the variety \( X_2 \): it suffices to require
that the resolution \( \pi_2 : X_2 \to Y \) is crepant (see Section 1, in particular
Theorem 1.4 for precise statements). Moreover, in some situations one can
derive the smoothness \textit{a posteriori} (see Theorem 1.4, proved in Section 1).

In the course of proving our main result, we establish some general facts
on the geometry of holomorphically symplectic manifolds \( X \) equipped with a
proper map \( \pi : X \to Y \). We believe that these results may be of independent
interest. In particular, we prove that such a map \( \pi \) must necessarily be
semismall (see Proposition 1.2).

We believe that our main Theorem 1.4 is of interest for two reasons.
In the first place, it seems that smooth crepant resolutions of singulari-
ties which occur in complex symplectic geometry, while non-compact, share
many properties with compact holomorphically symplectic manifolds and
deserve detailed study in their own right. This is, for example, the point of
view adopted by A. Beauville in [B]. Also, several important purely local
results have been recently proved by Y. Namikawa [N1], [N2]. The local ver-
sion of Huybrechts’ results that we prove here seems to further substantiate
this point of view.

Secondly, while the geometric picture established in [H1], [H2] never
caused any doubts, there is a serious technical mistake in [H2] which invalid-
dates much of the proof (see the erratum in [H3]). Recently D. Huybrechts
was able to fix his proof (see [H4]). To do this, he used a strong new theorem
of J.-P. Demailly and M. Paun [DP] which requires hard analysis. We feel
that the unexpected complexity of the result makes it worthwhile to consider
alternative approaches, especially those that are more algebraic. Moreover,
the combination of our methods with those of D. Huybrechts’ might yield
new information.

We would like to note that the first version of this paper did not contain
Condition 5.1 and claimed the main result in full generality. After it was
completed and posted to the e-prints server \texttt{arXiv.org}, the author had
an opportunity to visit the University of Cologne, at a kind invitation of
Prof. D. Huybrechts, and to give a long talk on his results. In the course
of subsequent discussions, several gaps in the proof were indicated to the
author by D. Huybrechts, M. Lehn and J. Wierzba. All of them but one were minor and are fixed here; however, the remaining gap is serious and forces us to introduce Condition 5.1. This invalidates an application of our result to a conjecture of V. Ginzburg’s which comprised Section 6 in the first version of the paper. In this revised version, everything about the Ginzburg’s conjecture has been removed. Everything else claimed in the first version, in particular Proposition 1.2, still stands. The last Section 6 is now taken up with a new smoothness result found by the author in the course of fixing the proofs.

The reader will find a detailed outline of the contents of the paper and precise formulations of all the results in Section 1.

Acknowledgments. This paper grew out of joint work with M. Verbitsky – his ideas and suggestions were invaluable, and he deserves equal credit for all the results (while all the mistakes are mine alone). I would also like to thank E. Amerik, F. Bogomolov, O. Biquard, P. Gauduchon, V. Ginzburg, D. Huybrechts, M. Lehn, T. Pantev, V. Shokurov and J. Wierzba for interesting and useful discussions, both in person and by e-mail. I am particularly grateful to J. Wierzba, D. Huybrechts and M. Lehn, who have suggested several improvements in the proofs and helped me to notice some gaps. Parts of the paper were prepared during the author’s stays at the Ecole Polytechnique in Paris and at the University of Cologne. The hospitality and the stimulating atmosphere of these institutions are gratefully acknowledged.

1 Statements of the results.

Notations. Throughout the paper, we work over the field \( \mathbb{C} \) of complex numbers. However, apart from an occasional reference to Hodge theory, all the proofs are algebraic, and everything works over an arbitrary algebraically closed field \( k \) of characteristic \( \text{char} k = 0 \). Birational will always mean dominant and generically one-to-one. Holomorphically symplectic will in fact mean algebraically symplectic. – in other words, equipped with a non-degenerate closed algebraic 2-form. We use the term “holomorphically symplectic” since it is more traditional.

The main subject of this paper will be what we will call symplectic resolutions.
Definition 1.1. A symplectic resolution is a pair $\langle X, Y \rangle$ of a normal irreducible affine complex algebraic variety $Y$ and a smooth irreducible holomorphically symplectic variety $X$ equipped with a projective birational map $\pi : X \to Y$.

Note that by the Zariski’s connectedness Theorem, these assumptions imply that

$$ Y = \text{Spec} \, H^0(X, \mathcal{O}_X), $$

so that both $Y$ and the map $\pi : X \to Y$ can be recovered from the manifold $X$. We will sometimes drop the explicit reference to the variety $Y$ and simply call the manifold $X$ itself a symplectic resolution. From this viewpoint, the projectivity of the map $\pi : X \to Y$ is a condition on the manifold $X$.

Symplectic resolutions occur naturally in many contexts – both in the local study of birational maps $X \to Y$ between compact algebraic varieties and independently, as smooth resolution of various affine singular spaces $Y$. In this regard, we note that our notion of symplectic resolution is stronger than the notion of a symplectic singularity recently introduced by A. Beauville. The difference is the following: we require the symplectic form $\Omega$ on $X$ to be non-degenerate everywhere, while in Beauville’s case it suffices that $\Omega$ is non-degenerate outside of the exceptional locus of the map $\pi : X \to Y$. On the other hand, our notion is very close to the notion of a symplectic contraction introduced by J. Wierzba.

The condition of projectivity of the map $\pi : X \to Y$ is probably too strong. We feel that most of our statements should hold in the case when $\pi$ is proper. However, we have not been able to prove anything in this more general case.

We can now give an overview of the paper and formulate the main results.

In Section 2, we study geometry of a general symplectic resolution $\pi : X \to Y$ by elementary methods. Apart from several technical results which we need later on, we also prove here the following statement, which might be of independent interest. The proof is in Subsection 2.1.

**Proposition 1.2.** Every symplectic resolution $\pi : X \to Y$ is semismall. In other words, for every closed irreducible subvariety $Z \subset Y$ we have

$$ 2 \text{codim } Z \geq \text{codim } \pi(Z). $$

Section 3 presents an overview of the deformation theory for symplectic manifolds that has been developed recently by the author jointly with M. Verbitsky (see [KV]).
In Section 4, we use the general deformation machinery of Section 3 to construct one particular deformation of a symplectic resolution \( X \). This deformation \( X/S \) is over the one-dimensional formal disc \( S = \text{Spec} \mathbb{C}[[t]] \), with the closed point \( o \in S \) and the generic point \( \eta \in S \) (note that \( \eta \) is geometrically not a point, but a punctured disc. We call the deformation \( X/S \) the twistor deformation. Analysing its properties, we prove the first of the main results of this paper (for the proof see Proposition 4.6).

**Theorem 1.3.** Let \( X \) be a symplectic resolution. Then there exists flat deformations \( X/S, \mathcal{Y}/S \) and an \( S \)-map \( \pi : X \rightarrow \mathcal{Y} \) such that

(i) the fibers \( X_o, \mathcal{Y}_o \) over the closed point \( o \subset S \) are isomorphic, respectively, to \( X \) and \( \mathcal{Y} \),

(ii) the isomorphisms \( X_o \cong X, \mathcal{Y}_o \cong \mathcal{Y} \) are compatible with the maps \( \pi : X \rightarrow \mathcal{Y}, X \rightarrow \mathcal{Y}, \) and

(iii) the map \( \pi : X \rightarrow \mathcal{Y} \) induces an isomorphism \( \pi : X_\eta \cong \mathcal{Y}_\eta \) between fibers \( X_\eta, \mathcal{Y}_\eta \) over the generic point \( \eta \in S \).

Section 5 is the main section of the paper. Here we partially prove a generalization of Theorem 1.3, which may be thought of as a local version of D. Huybrechts’ results [H2], [H1]. To formulate this statement, recall that a birational map \( \pi : X \rightarrow Y \) between normal irreducible algebraic varieties \( X, Y \) which admit canonical bundles \( K_X, K_Y \) is called crepant if the canonical map

\[ \pi^*K_Y \rightarrow K_X \]

defined over the non-singular locus extends to an isomorphism over the whole manifold \( X \). Every symplectic resolution \( \pi : X \rightarrow Y \) is automatically crepant. Indeed, the canonical bundle \( K_X \) is trivial – a trivialization is given by the top power of the symplectic form \( \Omega \). Since \( Y \) is normal, the map \( \pi : X \rightarrow Y \) is an isomorphism over the complement \( U_Y \subset Y \) to a subvariety of \( \text{codim} \geq 2 \). This implies that \( K_Y \) is also trivial (in other words, \( Y \) is Gorenstein). Therefore we have \( K_X \cong \pi^*K_Y \).

The main result of the paper is the following.

**Theorem 1.4.** Let \( \pi : X \rightarrow Y \) be a symplectic resolution. Assume given another normal variety \( X' \) and a crepant projective map \( \pi' : X' \rightarrow Y \).

Assume also that the pair \( X, X'/Y \) satisfies Condition 5.1.

Then there exist flat families \( X/S, \mathcal{Y}/S, X'/S \) over \( S = \text{Spec} \mathbb{C}[[t]] \) and projective family maps \( \pi : X \rightarrow \mathcal{Y}, \pi' : X' \rightarrow Y \) such that
(i) The special fibers $X_o, X_o', Y_o$ are isomorphic, respectively, to $X, X', Y$, and these isomorphisms are compatible with the maps $\pi, \pi'$.

(ii) The rational map

$$f = \pi^{-1} \circ \pi' : X' \rightarrow X$$

is an isomorphism between the generic fibers $X_\eta, X'_\eta$.

This includes Theorem 1.3 if we set $X' = Y$ and $\pi' = \text{id}$ (the identity map).

Finally, in the last Section 3 we give an application of our results to a problem in the Minimal Model Program for holomorphically symplectic manifolds. To formulate it, we need to recall some definitions. We will only give simple versions sufficient for our purposes; the reader is advised to consult a standard reference on the Minimal Model Program such as the paper [KMM] (in particular [KMM, §3-2]).

**Definition 1.5.** Let $X$ be a normal irreducible algebraic variety with trivial canonical divisor, and let $\Delta$ be an effective $\mathbb{Q}$-Cartier divisor on $X$. A birational projective map $\pi : X \rightarrow Y$ from $X$ to a normal irreducible algebraic variety $Y$ is called an extremal contraction if

(i) The $\mathbb{Q}$-Picard group $\text{Pic}(Y) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-vector subspace of codimension 1 in the $\mathbb{Q}$-Picard group $\text{Pic}(X) \otimes \mathbb{Q}$.

(ii) The $\mathbb{Q}$-Cartier divisor $\Delta$ is anti-ample with respect to the map $\pi : X \rightarrow Y$.

An extremal contraction $\pi : X \rightarrow Y$ is said to be of flipping type if it is an isomorphism outside of a closed subset $Z \subset Y$ of codimension $\text{codim} Z \geq 2$.

A normal irreducible algebraic variety $X'$ with trivial canonical divisor equipped with a projective birational map $\pi' : X' \rightarrow Y$ is called the flop of the flipping-type extremal contraction $\pi : X \rightarrow Y$ if the strict transform $\Delta' \subset X'$ of the divisor $\Delta \subset X$ is $\mathbb{Q}$-Cartier and ample with respect to the map $\pi' : X' \rightarrow Y$.

It is known ([KMM, Proposition 5-1-11(2)]) that a flop is unique, provided it exists. In fact, we must have

$$X' = \text{Proj}_Y \bigoplus_k \pi_* \mathcal{O}(mk[\Delta])$$
for some integer $m \geq 1$, and a flop exists if and only if the sheaf of graded $O(Y)$-algebras on the right-hand side is finitely generated. It is conjectured ([KMM, Conjecture 5-1-10]) that a flop always exists.

We can now state our last result, proved in Section 6.

**Theorem 1.6.**

(i) Let $X$ be a smooth quasiprojective holomorphically symplectic manifold, let $\pi : X \to Y$ be an extremal contraction of flipping type, and assume that the algebraic variety $Y$ is affine. Let $\Delta$ be an arbitrary divisor on $X$ anti-ample with respect to the map $\pi : X \to Y$.

Then the one-parameter deformation $X \to Y$ of the contraction $X \to Y$ provided by Theorem 1.3 is also an extremal contraction of flipping type, and the divisor $\Delta$ extends canonically to a divisor on $X$.

(ii) Assume in addition that both contractions $X \to Y$, $X' \to Y$ admit flops $\pi' : X' \to Y$, $\pi' : X' \to Y$ with respect to the divisor $\Delta$.

Then the contraction $\pi' : X' \to Y$ is flat over the base of the deformation $X \to Y$, and its special fiber coincides with the contraction $X' \to Y$. Moreover, both algebraic varieties $X'$ and $X'$ are smooth, and the variety $X'$ is symplectic.

2 Geometry.

2.1 Semismallness. Let $\pi : X \to Y$ be a symplectic resolution in the sense of Definition 1.1. As we have already noted, the canonical bundles $K_X, K_Y$ of the varieties $X, Y$ are trivial. Since $X$ is smooth, this implies that $Y$ has canonical singularities. By [E], this in turn implies that the singularities of the variety $Y$ are rational. However, the latter can be proved directly, without using the general theorem of [E].

**Lemma 2.1.** The higher direct images $R^i \pi_* O_X$, $i > 0$ are trivial.

**Proof.** Since $K_X \cong O_X$, we have

(2.1) \[ R^i \pi_* O_X = R^i \pi_* K_X = 0 \]

for $i \geq 1$ by the vanishing theorem of Grauert and Riemenschneider [GR]. \[ \square \]

We see that the singularities of the variety $Y$ are indeed rational. It is well-known that this implies that the variety $Y$ is Cohen-Macaulay, and
that the first relative cohomology sheaf $R^1\pi_*\mathbb{C}_X$ with coefficients in the constant sheaf $\mathbb{C}_X$ vanishes. Another corollary is that $H^k(X,\mathcal{O}_X) = 0$ for $k \geq 1$. Since $Y$ is affine, this follows directly from (2.1) by the Leray spectral sequence.

We will now turn to the non-trivial geometric property of $X/Y$, namely, Proposition 1.2. The proof proceeds along the standard lines used, for example, in the paper [V1]. Proposition will be derived from the following fact.

**Lemma 2.2.** Let $\pi : X \to Y$ be symplectic resolution. Denote by $\Omega \in \Omega^2(X)$ the symplectic form on the manifold $X$. Let $\sigma : Z \to U$ be a smooth map of smooth algebraic manifolds, and assume given a commutative square

$$
\begin{array}{ccc}
Z & \xrightarrow{\eta} & X \\
\sigma \downarrow & & \downarrow \pi \\
U & \longrightarrow & Y.
\end{array}
$$

Then there exists a 2-form $\Omega_U \in \Omega^2(U)$ on $U$ such that

$$
\sigma^*\Omega_U = \eta^*\Omega.
$$

Before we prove this Lemma, we would like to make the following remark. The conclusion of the Lemma is essentially a condition on the restriction $\eta^*\Omega$ – namely, the claim of the Lemma holds if and only if $\eta^*\Omega(\xi_1, \xi_2) = 0$ for every two tangent vectors $\xi_1, \xi_2$ to $Z$ at least one of which is vertical with respect to the map $\sigma : Z \to U$. Since the map $\sigma : Z \to U$ is smooth, this is an open condition. Therefore it suffices to prove the claim generically on $U$. Shrinking $U$ if necessary, we may assume that the fibered product $X \times_Y U$ admits a projective simultaneous resolution $\hat{Z}$, so that we have a commutative diagram

$$
\begin{array}{ccc}
\hat{Z} & \longrightarrow & X \times_Y U \\
& & \downarrow \pi \\
& & X \\
& & \downarrow \\
U & \longrightarrow & Y.
\end{array}
$$

Then the pair $\hat{Z} \to U$ also satisfies the assumptions of the Lemma. Moreover, to prove the claim for an arbitrary $Z$, it obviously suffices to consider the case $\hat{Z} = Z$. Therefore without any loss of generality we may impose an additional assumption on the map $\sigma : Z \to U$: 9
• The induced map \( \eta : Z \to X \times_Y U \) is generically one-to-one.

**Proof of Lemma 2.2.** The short exact sequence

\[
0 \longrightarrow \sigma^*\Omega^1(U) \longrightarrow \Omega^1(Z) \longrightarrow \Omega^1(Z/U) \longrightarrow 0
\]

of relative differentials for the map \( \sigma : Z \to U \) induces a three-step filtration

\[
\sigma^*\Omega^2(U) \subset \mathcal{F} \subset \Omega^2(Z), \quad \text{with} \quad \Omega^2(Z)/\mathcal{F} \cong \Omega^2(Z/U) \quad \text{and} \quad \mathcal{F}/\sigma^*\Omega^2(U) \cong \sigma^*\Omega^1(U) \otimes \Omega^1(Z/U).
\]

We have to prove that the 2-form \( \eta^*\Omega \) in fact is a section of the subsheaf \( \sigma^*\Omega^2(U) \subset \Omega^2(Z) \). We will do it in two steps.

**Step 1:** \( \eta^*\Omega^2(U) \in \mathcal{F} \). The proof of this step is an application of an idea of J. Wierzba [W]. It suffices to prove that for every point \( u \in U \), the restriction of the form \( \eta^*\Omega \) to the fiber

\[
Z_u = Z \times_U u \subset Z
\]

vanishes. To see this, consider the complex-conjugate \((0, 2)\)-form \( \overline{\Omega} \). Since \( H^2(X, \mathcal{O}_X) = 0 \), the Dolbeault cohomology class \([\overline{\Omega}]\) vanishes. Therefore the Dolbeault cohomology class

\[
\eta^* \left[ \overline{\Omega} \right]_{|Z_u}
\]

also vanishes. But the manifold \( Z_u \) is compact, smooth and projective. By Hodge theory, we obtain \( \eta^*\Omega = 0 \) on \( Z_u \).

**Step 2:** \( \eta^*\Omega \in \sigma^*\Omega^2(U) \). We have proved that \( \eta^*\Omega \) is a section of the sheaf \( \sigma^*\Omega^1(U) \otimes \Omega^1(Z/U) \). Therefore for every point \( u \in U \) and for every Zariski tangent vector \( \xi \in T_u U \) at the point \( u \) we have a well-defined 1-form

\[
\alpha = \eta^*\Omega \wedge \xi
\]

on the fiber \( Z_u \). We have to prove that \( \alpha = 0 \) for every \( u \in U \), \( \xi \in T_u U \).

Since \( Z_u \) is a smooth projective manifold, the 1-form \( \alpha \) is closed, and it suffices to prove that the cohomology class \( [\alpha] \in H^1(Z_u, \mathbb{C}) \) vanishes.

Let \( X_u = X \times_Y u \) be the (possibly singular) fibered product. By our additional assumption, the map \( \eta : Z_u \to X_u \subset X \) is generically one-to-one.

By construction, the form \( \alpha \) vanishes on every non-trivial fiber \( F \subset Z_u \) of the map \( \eta : Z_u \to X_u \). Indeed, for every tangent vector \( \xi' \) to the manifold \( F \) we have

\[
\alpha(\xi') = \Omega(\xi, d\eta(\xi')) = 0
\]
since \( dH(\xi') = 0 \). Therefore the cohomology class \([\alpha]\) also vanishes on every fiber \( F \in Z_u \). This implies that \([\alpha] = \eta_u^*[\alpha]'\) for some class \([\alpha]' \in H^1(X_u, \mathbb{C})\). But since \( Y \) has rational singularities, \( R^1\pi_*(\mathbb{C}) = 0 \) and \( H^1(X_u, \mathbb{C}) = 0 \). \( \square \)

Proposition 1.2 is deduced from Lemma 2.2 by a standard argument which we omit to save space (see [V1, Proposition 4.16] or [W, Section 3]).

2.2 Birational maps. Let now \( \pi : X \to Y \) be a symplectic resolution, and let \( \pi' : X' \to Y \) be a different, possibly singular crepant partial resolution of the same variety \( Y \). Proposition 1.2 imposes a strong constraint on the geometry of \( X \), but it does not say anything about the geometry of \( X' \). Our approach is to use the geometry of \( X \) to obtain results on the geometry of \( X' \). This becomes possible because of the following lemma. It is probably a standard fact in birational geometry; we have included a proof for the convenience of the reader. The argument is borrowed from [H1, 2.2]. The statement is in fact more general, since it does not require \( X \) to be symplectic – we only need it to be smooth and crepant over \( Y \).

Lemma 2.3. Let \( X, X' \) be two normal algebraic varieties equipped with crepant proper birational maps \( \pi : X \to Y \), \( \pi' : X' \to Y \) into the same normal irreducible Gorenstein algebraic variety \( Y \). Let

\[
f = (\pi')^{-1} \circ \pi : X \dashrightarrow X' \quad f^{-1} = \pi^{-1} \circ \pi' : X' \dashrightarrow X.
\]

be the natural rational maps.

(i) The rational map \( f^{-1} : X' \dashrightarrow X \) is defined and injective on a complement \( U \subset X' \) to a closed subset \( Z \subset X' \) of codimension \( \text{codim } Z \geq 2 \).

(ii) The pullback \( f^* : \text{Pic}(X') \to \text{Pic}(X) \) is an embedding from the Picard group of \( X' \) to the Picard group of \( X \).

(iii) For every line bundle \( L \) on \( X' \) the canonical map

\[
f^* : H^0(X', L) \to H^0(X, f^*L)
\]

is an isomorphism.

Proof. To prove (i), note that every birational map is \textit{defined} on the complement \( U \subset X' \) to a closed subvariety of codimension \( \geq 2 \); the problem is to prove that \( f^{-1} : U \to X \) is an embedding.

By assumption, the variety \( X' \) is normal. Therefore, shrinking \( U \subset X' \) if necessary, we can further assume that the definition subset \( U \subset X' \) is
smooth. Since the maps \( X \to Y, X' \to Y \) are crepant, the canonical bundles \( K_X, K_{X'} \) are both isomorphic to the pull-back of the canonical bundle \( K_Y \). This means that the Jacobian of the map \( f^{-1} : U \to X \) is a section of the trivial line bundle – in other words, a function. Denote this function by \( \Delta \).

Being a function, \( \Delta \) must extend to the whole normal variety \( X' \). By the Zariski Connectedness Theorem this implies that \( \Delta \) is lifted from a function \( \Delta \) on the variety \( Y \).

But both maps \( \pi' \) and \( \pi \) are invertible over the complement \( U_Y \subset Y \) to some closed subset \( X \subset Y \) of codimension \( \text{codim } Z \geq 2 \). This means that \( \Delta \) is also invertible on \( U_Y \), which implies that it is invertible everywhere.

This shows that the map \( f^{-1} : U \to X \) is \'{e}tale, hence an embedding.

To prove (ii), it suffices to notice that \( f \circ f^{-1} \) is defined and identical on an open subset \( U_0 \subset U \) whose complement is of codimension \( \geq 2 \). Therefore \( f^* \) induces an embedding from \( \text{Pic}(U_0) \) to \( \text{Pic}(X) \). Since \( X' \) is normal, \( \text{Pic}(X') \) is a subgroup in \( \text{Pic}(U_0) \). This proves (ii). Moreover, we have injective maps \( (f^{-1})^* : H^0(X, f^*L) \to H^0(X', L) \), \( f^* : H^0(X', L) \to H^0(X, f^*L) \), and the composition

\[
(f^{-1})^* \circ f^* = (f^{-1} \circ f)^* = \text{id}^* : H^0(X', L) \to H^0(X, f^*L) \to H^0(X', L)
\]

is the identity map. This proves (iii). \( \square \)

Lemma 2.3 (i) applies to the particular situation of a symplectic resolution \( \pi : X \to Y \) and gives a constraint on the map \( f^{-1} : X' \to X \). We can use it to get a constraint which goes in the other direction.

Lemma 2.4. Let \( \pi : X \to Y \) be a symplectic resolution, and let \( \pi' : X' \to Y \) be a projective crepant generically one-to-one map. The birational map

\[ f = (\pi')^{-1} \circ \pi : X \dashrightarrow X' \]

is defined on the preimage \( \pi^{-1}(U) \subset X \) of the complement \( U = Y \setminus Z \subset Y \) to a closed subset \( Z \subset Y \) of codimension \( \text{codim } Z \geq 3 \).

Proof. Let \( W \subset X \) be the indeterminacy locus of the map \( f : X \dashrightarrow X' \), and let \( Z = \pi(W) \subset Y \). To prove the Lemma, it suffices to prove that \( \text{codim } Z \geq 3 \). Since \( \text{codim } W \geq 2 \), the only possibility that we have to exclude is the following:

- There exists an irreducible component \( W_0 \subset W \subset X \) of codimension \( \text{codim } W_0 = 2 \) which is generically finite over its image \( \pi(W_0) \subset Y \).
Let $W_0 \subset W \subset X$ be such a component. Consider the graph $\Gamma \subset X \times Y$ of the birational map $f : X \dashrightarrow X'$ and let $\pi_1 : \Gamma \to X$, $\pi_2 : \Gamma \to X_2$ be the canonical projections. Since $W_0$ lies in the indeterminacy locus $W \subset X$, its preimage $\pi_1^{-1}(W) \subset \Gamma$ is of dimension strictly greater than $\dim W_0$. Then $\text{codim } W_0 = 2$ implies that $\pi_1^{-1}(W) \subset \Gamma$ is a divisor.

Denote by $E = \pi_2(\pi_1^{-1}(W_0)) \subset X'$ the image of the divisor $\pi_1^{-1}(W_0) \subset \Gamma$ under the second projection $\pi_2 : \Gamma \to X'$. By definition $\pi_1^{-1}(W)$ lies in the fibered product $W_0 \times_Y X'$. Therefore the fibers of the map $\pi_2 : \pi_1^{-1}(W_0) \to E \subset X'$ are closed subschemes of the fibers of the map $\pi : W_0 \to \pi(W_0) \subset Y$. Since $W_0$ is generically finite over $\pi(W_0)$, this implies that $\pi_1^{-1}(W_0)$ is generically finite over $E$. Thus $E \subset X'$ is also a divisor. This implies that the rational map $X' \dashrightarrow X$ is defined in the generic point of $E$, in other words, that the projection $\pi_2 : \pi_1^{-1}(W_0) \to E$ is generically one-to-one.

But by Lemma 2.3 (i) the rational map $X' \to X$ is not only defined in the generic point of the divisor $E \subset X'$, but also injective in this generic point. Therefore the projection $\pi_1 : \pi_1^{-1}(W_0) \to W_0$ is generically injective, which is a contradiction. □

3 Deformation theory.

In order to study symplectic resolutions, we need an appropriate version of Kodaira-Spencer deformation theory. We will use the theory proposed recently by the author jointly with M. Verbitsky [KV]. We will not use the results of [KV] in full generality, but only in a very particular case. To make the present paper self-contained, we have decided to state here all the necessary facts, while omitting proofs and unnecessary details. In order to make the material more accessible by placing it in a familiar context, we have also included a reminder on some known facts from deformation theory, in particular on some results of Z. Ran. The reader will find all the omitted details and much more in the paper [KV].

**Notation** For every integer $n \geq 0$, denote by $S_n = \text{Spec } \mathbb{C}[t]/(t^{n+1})$ the $n$-th order infinitesimal neighborhood of the closed point $o \in S$ of the formal disk $S = \text{Spec } \mathbb{C}[[t]]$. A deformation of order $n$ of a complex manifold $X$ is by definition a flat variety $X_n/S_n$ whose fiber over $o \in S_n$ is identified with $X$. Since $S_k \subset S_n$ for $k \leq n$, every deformation $X_n/S_n$ of order $n$ defines by restriction a deformation $X_k/S_k$ of order $k$ for every $k \leq n$. By a compatible system of deformations we will understand a collection of
deformations \(X_n/S_n\), one for each \(n \geq 1\), such that \(X_n/S_n\) restricted to order \(k \leq n\) coincides with the given \(X_k/S_k\).

In [Bo], F. Bogomolov has proved the following remarkable fact:

• Let \(X\) be a smooth projective holomorphically symplectic complex manifold. Then every first-order deformation \(X_1/S_1\) of \(X\) extends to a compatible system of deformations \(X_n\) of all orders \(n \geq 1\).

This fact has been generalized to Calabi-Yau manifolds by G. Tian [T] and A. Todorov [To]. Later, a purely algebraic proof has been given by Z. Ran.

We will now describe this proof. Recall that the general deformation theory associates to an \(n\)-th order deformation \(X_n/S_n\) a certain class \(\theta_n \in H^1(X_n, T(X_n/S_n) \otimes \Omega^1(S_n))\), called the Kodaira-Spencer class (here \(T(X_n/S_n)\) is the relative tangent bundle, and \(\Omega^1(S_n)\) is the \(\mathcal{O}(S_n)\)-module of Kähler differentials over \(\mathbb{C}\)).

The class \(\theta_n\) measures the non-triviality of the deformation \(X_n/S_n\). It would be natural to try describe all the deformations \(X_n/S_n\) in terms of the associated Kodaira-Spencer classes \(\theta_n\). However, this is not possible, since the cohomology group that contains \(\theta_n\) itself depends on the deformation \(X_n/S_n\).

The key observation in Ran’s approach is that this difficulty can be circumvented if one studies deformations step-by-step, by going through the embeddings \(i_n : S_n \hookrightarrow S_{n+1}\). Indeed, assume given a deformation \(X_n/S_n\) of order \(n\), and assume that the deformation \(X_n\) is further extended to a deformation \(X_{n+1}/S_{n+1}\). Then we can restrict the Kodaira-Spencer class \(\theta_{n+1}\) to \(S_n\) and obtain a cohomology class

\[
\hat{\theta}_n = i_n^* \theta_{n+1} \in H^1(X_n, T(X_n/S_n) \otimes i_n^* \Omega^1(S_{n+1})).
\]

Applying the canonical projection \(i_n^* \Omega^1(S_{n+1}) \to \Omega^1(S_n)\) to this class \(\hat{\theta}_n\) gives the order-\(n\) Kodaira-Spencer class \(\theta_n\). However, the \(\mathcal{O}(S_n)\)-module \(i_n^* \Omega^1(S_{n+1})\) is strictly bigger than the module \(\Omega^1(S_n)\) – we have a short exact sequence

\[
0 \longrightarrow \mathbb{C} \cdot d(t^{n+1}) \longrightarrow i_n^* \Omega^1(S_{n+1}) \longrightarrow \Omega^1(S_n) \longrightarrow 0.
\]

Thus on one hand, the class \(\hat{\theta}_n\) lies in a cohomology group defined entirely in terms of \(X_n\), while on the other hand, this class carries information about the next order deformation \(X_{n+1}\).

Formalizing this, Ran proves the following (where “lifting” is taken with respect to the canonical surjection in (3.1)).
Lemma 3.1 (Ran’s criterion). Assume given an \( n \)-th order deformation \( X_n/S_n \) of the complex manifold \( X \). Denote by 
\[
\theta_n \in H^1(X_n, \mathcal{T}(X_n/S_n) \otimes \Omega^1(S_n))
\]
the associated Kodaira-Spencer class.

Then the deformation \( X_n/S_n \) extends to a deformation \( X_{n+1}/S_{n+1} \) of order \( n+1 \) if and only if the class \( \theta_n \) lifts to a class 
\[
\tilde{\theta}_n \in H^1(X_n, \mathcal{T}(X_n/S_n) \otimes i^*_n \Omega^1(S_{n+1})).
\]
Moreover, such liftings are in one-to-one correspondence with the isomorphism classes of the extended deformations \( X_{n+1}/S_{n+1} \). \( \square \)

Remark 3.2. Note that for \( n = 1 \) this Lemma reduces to the standard fact: deformations of order 1 are classified, up to an isomorphism, by cohomology classes in the group
\[
H^1(X_0, \mathcal{T}(X_0/S_0) \otimes i^*_0 \Omega^1(S_1)) = H^1(X, \mathcal{T}(X)).
\]

Lemma 3.1 corresponds to what Ran called \( T_1 \)-lifting property. Having established this property, Ran then uses Hodge theory to construct all the necessary liftings. This proves the theorem.

In the paper [KV], this proof of the Bogolomov Theorem has been generalized to a certain class of non-compact holomorphically symplectic manifolds. We will not need the full generality of [KV]. It suffices to know that the results apply to holomorphically symplectic manifolds \( X \) with \( H^k(X, \mathcal{O}(X)) = 0 \) for \( k \geq 1 \). By Lemma 2.1, this holds for all symplectic resolutions \( \pi : X \to Y \).

To pay for such a generalization, one has to change the notion of a deformation. Namely, introduce the following.

Definition 3.3. A symplectic deformation \( \mathcal{X}/S \) of a holomorphically symplectic complex algebraic manifold \( \langle X, \Omega \rangle \) over a local Artin base \( S \) is a smooth family \( \mathcal{X}/S \) equipped with a closed non-degenerate relative 2-form
\[
\Omega \in \Omega^2(\mathcal{X}/S),
\]
such that the fiber over the closed point \( o \in S \) is identified with \( \langle X, \Omega \rangle \).
In the theory of symplectic deformations, the tangent sheaf $\mathcal{T}(\mathcal{X}/S)$ is replaced by a certain complex $\mathcal{K}^\ast(\mathcal{X}/S)$ of sheaves on $\mathcal{X}$. This complex

$$\mathcal{K}^\ast(\mathcal{X}/S) = F^1(\Omega^\ast(\mathcal{X}/S))[-1]$$

is the first term of the Hodge (=stupid) filtration of the relative de Rham complex $\Omega^\ast(\mathcal{X}/S)$, shifted by $-1$, – in other words, we set

$$K^i(\mathcal{X}/S) = \begin{cases} \Omega^{i+1}(\mathcal{X}/S), & i \geq 0, \\ 0, & i < 0, \end{cases}$$

with the de Rham differential. The cohomology of the complex $\mathcal{K}^\ast(\mathcal{X}/S)$ can be included into a long exact sequence

$$H^k_{DR}(\mathcal{X}/S) \longrightarrow H^k(\mathcal{X}, \mathcal{O}(\mathcal{X})) \longrightarrow H^k(\mathcal{X}, \mathcal{K}^\ast(\mathcal{X}/S)) \longrightarrow \ldots$$

where $H^k_{DR}(\mathcal{X}/S)$ is the relative de Rham cohomology of $\mathcal{X}$ over $S$. By assumption the coherent cohomology $H^k(\mathcal{X}, \mathcal{O}(\mathcal{X}))$ vanish for $k \geq 1$. Therefore for $k \geq 1$ we have isomorphisms

$$H^k(\mathcal{X}, \mathcal{K}^\ast(\mathcal{X}/S)) \cong H^k_{DR}(\mathcal{X}/S).$$

The Gauss-Manin connection $\nabla$ thus induces a connection on the higher cohomology groups of the complex $\mathcal{K}^\ast(\mathcal{X}/S)$.

The role of the Kodaira-Spencer class in the symplectic theory is played by the class

$$\theta = \nabla(\Omega) \in H^1(\mathcal{X}, \mathcal{K}^\ast(\mathcal{X}/S)) \otimes \Omega(S)$$

obtained by applying the Gauss-Manin connection to the class of the relative symplectic form $\Omega \in \Omega^2(\mathcal{X}/S)$. In particular, for an order-$n$ symplectic deformation $X_n/S_n$ we obtain a canonical class

$$\theta_n \in H^1(X_n, \mathcal{K}^\ast(X_n/S_n) \otimes \Omega^1(S_n)).$$

In the sequel, the only results of [KV] that we will need are the following two properties of the class $\theta_n$.

**Lemma 3.4.**

(i) The image $\tau(\theta_n) \in H^1(X_n, \mathcal{T}(X_n/S_n) \otimes \Omega^1(S_n))$ of the class $\theta_n$ under the canonical map

$$\tau : \mathcal{K}^\ast(X_n/S_n) \to \mathcal{K}^0(X_n/S_n) \cong \Omega^1(X_n/S_n) \cong \mathcal{T}(X_n/S_n))$$

is the usual Kodaira-Spencer class for the deformation $X_n/S_n$.  

16
The Ran’s Criterion (Lemma 3.1) holds literally for symplectic deformations, with the relative tangent bundle $T(X_k/S_k)$ replaced with the complex $\mathcal{K}^*(X_k/S_k)$. □

4 Twistor deformations.

4.1 The construction. Let $X$ be a smooth projective symplectic resolution. To construct symplectic deformations of the manifold $X$, we need a supply of cohomology classes in the group

$$H^1(X, \mathcal{K}^*(X)).$$

Such a supply is provided by Chern classes of line bundles on $X$. Indeed, recall that the Chern class $c_1(L) \in H^{1,1}(X)$ of a line bundle $L$ on $X$ equals

$$c_1(L) = d \log([L]),$$

where $[L] \in H^1(X, \mathcal{O}_X^*) = \text{Pic}(X)$ is the class of $L$ in the Picard group. But the $d \log$ map obviously factors through a map

$$d \log : \mathcal{O}_X^* \to \mathcal{K}^*(X) \to \mathcal{K}^0(X) = \Omega^1(X).$$

Therefore for every line bundle $L$ we have a canonical Chern class

$$c_1(L) \in H^1(X, \mathcal{K}^*(X))$$

which projects to the usual Chern class under the map $\tau : \mathcal{K}^*(X) \to \Omega^1(X)$. We will call it the symplectic Chern class $c_1(L)$ of the line bundle $L$. More generally, if we are given a deformation $X_n/S_n$ of order $n$ and a line bundle $L$ on $X_n$, then this construction gives a canonical class

$$c_1(L) \in H^1(X_n, \mathcal{K}^*(X_n/S_n))$$

in relative cohomology.

Definition 4.1. Let $\pi : X \to Y$ be a symplectic resolution, and let $L$ be an ample line bundles on $X$. A twistor deformation associated to the line bundle $L$ is a compatible system of symplectic deformations $X_n/S_n$ equipped with line bundles $L$ such that

(i) The fiber over $o \in S_n$ of the system $(X_n, L)$ is identified with the pair $(X, L)$.
(ii) The Kodaira-Spencer class

$$\theta_n \in H^1(X_n, \mathcal{K}_{\ast}(X_n/S_n) \otimes \Omega^1(S_n))$$

of the symplectic deformation $X_n/S_n$ is equal to $c_1(L) \otimes dt$.

This notion is useful because of the following.

**Lemma 4.2.** Let $X$ be a symplectic resolution. For every deformation $X_n/S_n$ of an arbitrary order $n$, every line bundle $L$ on $X$ extends to a line bundle $L$ on $X_n$, and this extension is unique up to an isomorphism.

**Sketch of the proof** (compare [H1, 2.3] and [H1, Corollary 4.3].) The deformation theory of the pair $\langle X, L \rangle$ is controlled by the (first and second) cohomology of the Atiyah algebra $A(L)$ of the line bundle $L$, while the deformation theory of the manifold $X$ itself is controlled by the (first and second) cohomology of the tangent bundle $T(X)$. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow A(L) \longrightarrow T(X) \longrightarrow 0,$$

which induces a long exact sequence of the cohomology groups. But since $X$ is a symplectic resolution, we have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Therefore cohomology groups $H^i(X, A(L))$ and $H^i(X, T(X))$ are isomorphic

$$H^i(X, A(L)) \cong H^i(X, T(X))$$

in every degree $i \geq 1$. □

This lemma shows that the (isomorphism classes of the) line bundles $L$ on $X_n$ are completely defined by their restrictions to the special fiber $X \subset X_n$. Combining it with Lemma 3.4 (ii), we immediately get the following.

**Lemma 4.3.** For every symplectic resolution $X$ and an ample line bundle $L$ on $X$, there exists a twistor deformation $X/S$ associated to the pair $(X, L)$. Moreover, such a twistor deformation is unique up to an isomorphism.

**Proof.** Lemma 3.4 (ii) and induction on $n$. □
4.2 Algebraization. Taking the direct limit

\[ X_\infty = \lim_{\to} X_n \]

of the order-\( n \) twistor deformations \( X_n/S_n \), we obtain a formal scheme \( X_\infty \) over the power series algebra \( \mathbb{C}[[t]] \). This formal scheme is smooth over \( \mathbb{C}[[t]] \). It carries a line bundle \( L \) and a relative symplectic form \( \Omega \in \Omega^2(X_\infty/S) \).

We can now use the ampleness condition on \( L \) to obtain not just the formal scheme, but an actual deformation over the formal disc \( S = \text{Spec} \mathbb{C}[[t]] \).

**Proposition 4.4.** There exists a Noetherian affine scheme \( Y/S \) which is normal, irreducible and flat over the base \( S \), a smooth scheme \( X/S \) also flat over \( S \), a projective birational \( S \)-scheme map \( \pi : X \to Y \) of finite type, a line bundle \( L \) on \( X \) very ample with respect to the map \( \pi \) and a relative symplectic form \( \Omega \in \Omega^2(X/S) \) such that the formal scheme \( \langle X_\infty, L \rangle \) with the bundle \( L \) is the completion of the pair \( \langle X, L \rangle \) near the fiber over the special point \( o \in S \), and the form \( \Omega \) on \( X_\infty \) is the completion of the form \( \Omega \in \Omega^2(X/S) \).

**Proof.** For every \( n \geq 0 \), the embedding \( X_n \subset X_{n+1} \) induces a short exact sequence

\[ 0 \to t^{n+1}O(X) \to O(X_{n+1}) \to O(X_n) \to 0 \]

of sheaves on \( X \), which in turn induces a cohomology long exact sequence

\[ H^0(X, O(X_{n+1})) \to H^0(X, O(X_n)) \to H^1(X, O(X)) \to \]

Since \( H^1(X, O(X)) = 0 \) by Lemma 2.1, the canonical map

\[ H^0(X, O(X_{n+1})) \to H^0(X, O(X_n)) \]

is surjective for every \( n \geq 0 \). Consider the \( \mathbb{C}[[t]] \)-algebra

\[ A = \lim_{\leftarrow} H^0(X_n, O(X_n)) \]

and let \( Y = \text{Spec} A \). The algebra \( A \) is complete with respect to the \((t)\)-adic topology. The associated graded quotient \( \text{gr} A \) with respect to the filtration by powers of the ideal \( t \cdot A \subset A \) is isomorphic

\[ \text{gr} A \cong O(Y)[[t]] \]
to the algebra of polynomials over the algebra $\mathcal{O}(Y)$ of functions on the affine variety $Y$. Therefore the algebra $A$ is Noetherian ([EGA, 0, Corollaire 7.2.6]). Moreover, $A$ obviously has no $t$-torsion, so that $Y$ is flat over $S = \text{Spec } \mathbb{C}[[t]]$.

Let $Y_\infty$ be the completion of $Y$ near the special fiber $Y \subset Y$. Then the formal scheme $X_\infty$ is proper over the formal scheme $Y_\infty$, and the line bundle $L$ extends to a bundle on the formal scheme $\mathcal{X}$. Therefore we can apply the Grothendieck Algebraization Theorem ([EGA, III, Théorème 5.4.5]) which claims the existence of the scheme $\mathcal{X}/Y$ and the bundle $L$ satisfying the conditions of the Proposition, except possibly for the smoothness of $\mathcal{X}$ and the normality of $Y$. Moreover, we have

$$Y = \text{Spec } H^0(\mathcal{X}, \mathcal{O}(X)),$$

which implies that $\pi : \mathcal{X} \rightarrow Y$ is birational and that $\pi_* \mathcal{O}(\mathcal{X}) \cong \mathcal{O}(Y)$.

It remains to prove that $\mathcal{X}$ is smooth and that $Y$ is normal. Since $\pi_* \mathcal{O}(\mathcal{X}) \cong \mathcal{O}(Y)$, the second follows from the first. Thus we have to prove that the scheme $\mathcal{X}$ is regular at every point $x \in \mathcal{X}$. It is well known that regularity is stable under generization. Therefore it suffices to consider only closed points $x \in \mathcal{X}$.

Fix such a point $x \in \mathcal{X}$. If the point $x$ lies in the special fiber $X \subset \mathcal{X}$, then the regularity on $\mathcal{X}$ at $x$ is equivalent to the regularity of the formal scheme $X_\infty$ at $x$. This immediately follows from the construction of $X_\infty$. If the point $x$ lies in the generic fiber $X_\eta \subset \mathcal{X}$, then it has no direct relation to the formal scheme, and a priori there is no reason for $\mathcal{X}$ to be regular at $x \in \mathcal{X}$.

However, the latter situation cannot occur. Indeed, since the map $\pi : \mathcal{X} \rightarrow Y$ is proper, the point $\pi(x) \subset Y$ is also closed. Therefore the function $t$ either vanishes at the point $x$, or it maps to an invertible element in the residue field $k(\pi(x))$. In the latter case, the ideal $t\mathcal{O}_Y \subset Y$ maps to the whole $k(\pi(x))$ under the projection $\mathcal{O}_Y \rightarrow k(\pi(x))$, so that we have $1 = at \in k(\pi(x))$ for some function $a \subset \mathcal{O}_Y$. Then $1 - at \in \mathcal{O}_Y$ vanishes at $x$. But since the ring $\mathcal{O}_Y$ is complete with respect to the $(t)$-adic topology, the function $1 - at$ is invertible in $\mathcal{O}_Y$ for every $a \in \mathcal{O}_Y$. This is a contradiction. We conclude that the function $t$ vanishes at every closed point $x \in \mathcal{X}$, which means that the point $x \in \mathcal{X}$ lies in the special fiber $X \subset \mathcal{X}$.

Finally, the existence of the form $\Omega = \Omega^2(\mathcal{X}/S)$ satisfying the conditions of the Proposition is immediate from [EGA, III, Proposition 5.1.2]. □

Note that this statement is not a mere formality – on the contrary, it has direct geometric meaning. Indeed, while the formal scheme $X_\infty$ is
concentrated set-theoretically on the special fiber \( X \subset X_\infty \), the scheme \( \mathcal{X}/S \) has a perfectly well-defined generic fiber over the generic point \( \eta \in S \). To pay for this, we have to introduce the scheme \( \mathcal{Y} \) which is *not of finite type over* \( S \). This has certain counterintuitive corollaries. For instance, the construction of the twistor deformation is essentially local on \( Y \) – if we have an open affine subset \( Y_0 \subset Y \), then the pullback \( X_0 = X \times_Y Y_0 \) admits a twistor deformation \( \mathcal{X}_0/\mathcal{Y}_0 \), and we have canonical comparison maps
\[
\mathcal{X}_0 \to \mathcal{X}, \quad \mathcal{Y}_0 \to \mathcal{Y}.
\]
However, it is unlikely that these maps are open embeddings. Thus one cannot hope to obtain twistor deformations over non-affine varieties \( Y \) by working locally on \( Y \) and then gluing the pieces together.

**Remark 4.5.** The name *twistor deformation* comes from hyperkähler geometry: if the holomorphically symplectic manifold \( X \) admits a compatible hyperkähler metric, then the twistor deformation \( X_\infty \) is simply the completion of the twistor space \( \mathcal{Tw}(X) \) along the fiber over the point \( 0 \in \mathbb{CP}^1 \). It might be possible to reverse the construction and obtain the whole twistor space \( \mathcal{Tw}(X) \) from the twistor deformation \( X_\infty \), thus giving a purely algebraic construction of a hyperkähler metric on the non-compact manifold \( X \). Proposition 4.4 can be considered as the first step in this direction. However, at present it is unclear how to proceed any further.

### 4.3 The generic fiber

We can now prove the following Proposition, which immediately implies Theorem 1.3.

**Proposition 4.6.** Let \( \pi : X \to Y \) be a symplectic resolution equipped with an ample line bundle \( L \). Consider the twistor deformation \( X_\infty \) associated to the line bundle \( L \), and let \( \pi : \mathcal{X} \to \mathcal{Y} \) be the family of schemes over \( S = \text{Spec} \mathbb{C}[[t]] \) constructed from \( X_\infty \) in Proposition 4.4.

Then the map \( \pi \) induces an isomorphism
\[
\pi : \mathcal{X}_\eta \to \mathcal{Y}_\eta
\]
between the fibers over the generic point \( \eta \in S \).

**Proof.** The generic fiber \( \mathcal{Y}_\eta = \mathcal{Y} \times_S \eta \) is open in the normal scheme \( \mathcal{Y} \) (indeed, it is the complement to the closed special fiber \( Y \subset \mathcal{Y} \)). Since the

---

1. The argument uses a standard trick which probably goes back to [F]. Compare [H1, Proposition 4.1] and [N1, Section 2, Claim 3 on p. 18].
scheme \( Y \) is normal, the open subscheme \( Y_\eta \subset Y \) is also normal. Since the map \( \pi : X \to Y \) is birational, it suffices to prove that it is finite over the generic point \( \eta \in S \). Moreover, by construction the map \( \pi : X_\eta \to Y_\eta \) is projective. Therefore by [EGA, IV, Théorème 8.11.1] it suffices to prove that it is quasifinite – in other words, that its fibers do not contain any compact curves.

Let \( \iota : C \to X_\eta \) be an arbitrary map from a compact curve \( C/\eta \) to \( X_\eta \). Replacing \( C \) with its normalization, we can assume that the curve \( C/\eta \) is connected and smooth. Since the point \( \eta = \text{Spec} \mathbb{C}((t)) \) is in fact the punctured disk, the curve (or rather, the family of curves) \( C/\eta \) has a well-defined Kodaira-Spencer class

\[
\theta_C \in H^1(C, \mathcal{T}(C/\eta) \otimes \Omega^1(\eta/\mathbb{C})) \cong H^1(C, \mathcal{T}(C/\eta)).
\]

Moreover, by functoriality of the Kodaira-Spencer classes, for every \( k \)-form \( \alpha \in \Omega^k(X/S) \) we have

\[
\iota^* \alpha \cup \theta_C = \iota^* (\alpha \cup \theta_\infty) \in H^1(C, \Omega^{k-1}(C/\eta)),
\]

where \( \theta_\infty \in H^1(X, \mathcal{T}(X_\infty/S)) \) is the Kodaira-Spencer class of the family \( X/S \). In particular, we have

\[
\iota^* (\Omega \cup \theta_\infty) = \iota^* \Omega \cup \theta_C,
\]

where \( \Omega \) is the symplectic form. Since \( C \) is a curve, we have \( \iota^* \Omega = 0 \). We deduce that \( \iota^* (\Omega \cup \theta_\infty) = 0 \). But by definition of the twistor deformation we have

\[
\Omega \cup \theta_\infty = \tau(\theta_\infty) = c_1(L).
\]

Therefore \( \iota^* c_1(L) = 0 \). Since the line bundle \( L \) is ample, this is possible only if \( \iota : C \to X_\eta \) is a projection onto a point in the generic fiber \( X_\eta \).

\[\square\]

**5 The main theorem.**

We now turn to the proof of our main Theorem 1.4.

**5.1 The setup and the construction.** The setup for proving Theorem 1.4 is as follows. Assume given a projective symplectic resolution \( \pi : X \to Y \) and a different irreducible normal variety \( X' \) equipped with a projective birational crepant map \( \pi' : X' \to Y \). Fix a relatively very
ample line bundle $L$ on $X/Y$ and a relatively very ample line bundle $L'$ on $X'/Y$, so that we have

$$X = \mathcal{P}roj_Y \bigoplus_{k \geq 0} \pi_* L^{\otimes k}, \quad X' = \mathcal{P}roj_Y \bigoplus_{k \geq 0} \pi'_* (L')^{\otimes k},$$

where $\mathcal{P}roj$ is taken on the variety $Y$.

By virtue of Lemma 2.3 (ii), the Picard group $\text{Pic}(X')$ is canonically embedded into the Picard group $\text{Pic}(X)$. Therefore every line bundle on $X'$ defines a line bundle on $X$. By abuse of notation, we will denote both by the same letter. Then by Lemma 2.3 (iii) we have

$$(5.1) \quad X' = \mathcal{P}roj_Y \bigoplus_{k \geq 0} \pi'_* (L')^{\otimes k} = \mathcal{P}roj_Y \bigoplus_{k \geq 0} \pi_* (L')^{\otimes k}.$$ 

In other words, global sections of the bundles $(L')^{\otimes k}$ are the same, whether computed on $X$ or on $X'$.

We denote by $\pi : \mathcal{X} \to \mathcal{Y}$ the twistor deformation of the pair $(X, L)$, whose existence is guaranteed by Proposition 4.4. We know that $\mathcal{Y}$ is a Noetherian affine scheme, flat over $S = \text{Spec } \mathbb{C}[[t]]$, $\mathcal{X}$ is smooth over $S$ and projective over $\mathcal{Y}$, and the fiber of the map $\pi : \mathcal{X} \to \mathcal{Y}$ over the special point $o \in S$ is identified with the given symplectic resolution $\pi : X \to Y$. Moreover, the line bundle $L$ extends to a relative very ample line bundle $\mathcal{L}$ on $\mathcal{X}$, so that we have

$$\mathcal{X} = \mathcal{P}roj_Y \bigoplus_{k \geq 0} \pi_* \mathcal{L}^{\otimes k}.$$ 

The fiber $\pi : \mathcal{X}_\eta \to \mathcal{Y}_\eta$ of the map $\pi : \mathcal{X} \to \mathcal{Y}$ over the generic point $\eta \in S$ is an isomorphism.

We can now introduce the technical condition that we need in order to prove Theorem 1.4. By Lemma 4.2 the line bundle $L'$ on $X$ canonically extends to a line bundle $\mathcal{L}'$ on the twistor deformation $\mathcal{X}$.

**Condition 5.1.** Consider the line bundle $\mathcal{L}'$ on the twistor deformation $\mathcal{X}$. The graded algebra

$$\bigoplus_{k \geq 0} H^0(\mathcal{X}, \mathcal{L}'^{\otimes k})$$

is finitely generated over the algebra $H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}) = \mathcal{O}(\mathcal{Y})$. 

23
This will be our standing assumption from now on and till the end of this Section.

To prove Theorem 1.4 under Condition 5.1, we first construct some partial resolution $X' \to Y$, then prove that it is indeed the resolution that we need. The construction is as follows. Consider the scheme

$$X'_0 = \operatorname{Proj}_Y \bigoplus_{k \geq 0} H^0(\mathcal{X}', \mathcal{L}' \otimes k)$$

In other words, let $X'_0$ be the image of the (rational) projective embedding of the scheme $X'$ defined by the line bundle $\mathcal{L}'$. Notice that since $\pi : \mathcal{X} \to \mathcal{Y}$ is an isomorphism over the generic point $\eta \in S$, the scheme $X'_0/Y$ is isomorphic to $Y$ over the generic point $\eta \in S$.

Lemma 5.2. The scheme $X'_0$ is reduced and flat over $S$.

Proof. Since the scheme $X'$ is irreducible, the $\mathcal{O}(Y)$-algebra

$$\bigoplus_{k \geq 0} \pi_* \mathcal{L}' \otimes k$$

has no zero divisors. This proves that $X'_0$ is reduced. Moreover, this proves that the structure sheaf $\mathcal{O}(X'_0)$ has no $t$-torsion, which implies that $X'_0$ is flat over $S = \mathbb{C}[|t|]$.

Lemma 5.3. The fiber $X'_{0, o} \subset X'_0$ of the scheme $X'_0/S$ over the special point $o \in S$ is reduced and irreducible.

Proof. Since $Y$ is affine, it suffices to prove that the graded algebra

$$\bigoplus_{k \geq 0} \pi_*(\mathcal{L}') \otimes k / t$$

has no zero divisors. By the base change theorem for the flat map $\mathcal{X} \to S$, the canonical map

$$\bigoplus_{k \geq 0} H^0(\mathcal{X}, (\mathcal{L}') \otimes k) / t \to \bigoplus_{k \geq 0} H^0(X, (L') \otimes k)$$

is injective. Therefore it suffices to prove that the algebra on the right-hand side has no zero divisors. But this is immediate, since the scheme $X$ is reduced and irreducible.
We now let $\mathcal{X}'$ be the normalization of the scheme $\mathcal{X}_0'$, and denote by $X'_0 \subset \mathcal{X}'$ the special fiber of the scheme $\mathcal{X}'/S$.

By construction, $\mathcal{X}'$ is a normal reduced irreducible scheme equipped with a canonical projective birational map $\pi' : \mathcal{X}' \to \mathcal{Y}$. This projection factors through a finite generically one-to-one map $\nu : \mathcal{X}' \to \mathcal{X}_0'$. By abuse of notation, we will denote by $L' = \nu_* O(1)$ the preimage of the relative $O(1)$-sheaf on $\mathcal{X}_0'/\mathcal{Y}$ (this is the same letter $L'$ as used for the pullback of the bundle $O(1)$ to the scheme $\mathcal{X}/\mathcal{Y}$). Then $L'$ is a relatively ample line bundle on the scheme $\mathcal{X}'/\mathcal{Y}$.

**Lemma 5.4.** The scheme $\mathcal{X}'$ is Cohen-Macaulay and has trivial canonical bundle $K_{\mathcal{X}'} \cong O_{\mathcal{X}'}$. In particular, the map $\pi' : \mathcal{X}' \to \mathcal{Y}$ is crepant.

**Proof.** We obviously have $K_{\mathcal{X}} \cong O_{\mathcal{X}}$, which yields $K_{\mathcal{Y}} \cong O_{\mathcal{Y}}$. To prove that $K_{\mathcal{X}'} \cong O_{\mathcal{X}'}$, it suffices to prove that the canonical rational map

$$f = \pi^{-1} \circ \pi' : \mathcal{X}' \to \mathcal{X}$$

is defined and injective outside of a subvariety $Z \subset \mathcal{X}'$ of codimension $\geq 2$. In other words, we have to show that the map $f : \mathcal{X}' \to \mathcal{X}$ is injective in the generic point of an arbitrary Weil divisor $E \subset \mathcal{X}'$.

Moreover, since both $\pi$ and $\pi'$ are isomorphisms over the generic point $\eta \in S$, it suffices to consider divisors $E \subset \mathcal{X}'$ which lie in the special fiber $X'_0 \subset \mathcal{X}'$.

Let $E$ be such a divisor. Consider the normalization map $\nu : \mathcal{X}' \to \mathcal{X}_0'$. Since the map $\nu$ is finite, the image $\nu(E) \subset X'_{0,\nu} \subset \mathcal{X}_0'$ is also a divisor. But the special fiber $X'_{0,\nu} \subset \mathcal{X}'$ is irreducible by Lemma 5.3, therefore such a divisor $\nu(E)$ must coincide with the whole special fiber $X'_{0,\nu} \subset \mathcal{X}'$. Thus the generic point of the divisor $E \subset \mathcal{X}'$ lies over the generic point of the variety $Y$. We are done, since the maps $\pi : X \to Y$, $\pi' : X'_0 \to Y$ are by construction both generically one-to-one.

Thus $K_{\mathcal{X}'}$ is trivial. This means that the map $\pi' : \mathcal{X}' \to \mathcal{Y}$ is crepant. Since $\mathcal{Y}$ has a smooth crepant resolution $\pi : \mathcal{X} \to \mathcal{Y}$, the singularities of $\mathcal{Y}$, hence of $\mathcal{X}'$, are canonical. This implies that these singularities are rational by [E, II, Théorème 1]. Hence $\mathcal{X}'$ (and $\mathcal{Y}$) are Cohen-Macaulay. □

We can now prove that in fact $\mathcal{X}' \cong \mathcal{X}_0'$.  

25
Lemma 5.5. We have

\[ X' \cong X'_0 = \text{Proj}_Y \bigoplus_k \pi'_*(L')^\otimes k. \]

Proof. Since the line bundle \( L' \) is relatively ample on \( X'/Y \), we have

\[ X' = \text{Proj}_Y \bigoplus_k \pi'_*(L')^\otimes k. \]

We now apply Lemma 2.3 to the pair \( \pi : X \to Y, \pi' : X' \to Y \) and conclude that

\[ \bigoplus_k \pi'_*(L')^\otimes k \cong \bigoplus_k \pi'_*(L')^\otimes k. \]

Therefore Lemma 5.2 and Lemma 5.3 apply to \( X' \). Combining them with Lemma 5.4, we conclude that the normal irreducible Cohen-Macaulay scheme \( X' \) is flat over \( S \) and crepant over \( Y \), and that the special fiber \( X'_o \subset X' \) is irreducible and reduced.

5.2 The special fiber. We have proved all the statements of Theorem 1.4 that deal with the scheme \( X'/Y \). To finish the proof, we now have to consider the special fiber \( X'_o \subset X' \). We have to prove that \( X'_o \) is a normal variety crepant over \( Y \), and that

\[ X'_o \cong X' = \text{Proj}_Y \bigoplus_k \pi'_*(L')^\otimes k. \]

Lemma 5.6. The special fiber \( X'_o \subset X' \) is normal.

Proof. Since \( X' \) is Cohen-Macaulay, the hypersurface \( X'_o \subset X' \) is also Cohen-Macaulay. Thus by Serre’s criterion, it suffices to prove that \( X'_o \) is non-singular in codimension 1. This splits into two parts.

(i) The scheme \( X' \) is non-singular in codimension \( \leq 2 \).

(ii) The function \( t \) on \( X' \) which cuts the hypersurface \( X'_o \subset X' \) has no critical points in codimension \( \leq 2 \) (in other words, the differential \( dt \) is a non-vanishing 1-form on a smooth open subset \( U \subset X' \) whose complement is of codimension \( \text{codim} X' \setminus U \geq 3 \)).
To prove these claims, we will use the technique of valuations (see, e.g., [K2, Section 1] for an overview). Let $Z' \subset X'_o \subset X'$ be an irreducible subvariety of codimension $\text{codim} \ Z' = 2$. Consider the valuation $v$ of the fraction field $\mathbb{C}(X') = \mathbb{C}(X) = \mathbb{C}(Y)$ associated to the exceptional divisor $E$ of the blow-up of $X'$ in $Z'$.

It is well-known that the discrepancy of the divisor $E$ in fact depends only on the variety $X'$ and on the valuation $v$, not on the particular geometric realization of this valuation. We will denote it by $\text{disc}(v, X')$. Moreover, since both $X$ and $X'$ are crepant over $Y$, the discrepancy $\text{disc}(v, X') = \text{disc}(v, X') = \text{disc}(v, Y) = \text{disc}(v)$ is the same when computed with respect to $X$, $X'$ or $Y$.

Since $K_{X'}$ is trivial, the number $\text{disc}(v) = \text{disc}(v, X') = \text{disc}(v, X')$ is either 1 or 0. Since the variety $X$ is smooth, the center of the valuation $v$ in the variety $X$ is a closed subvariety $Z \subset X$ of codimension $\text{codim} \ Z \leq \text{disc}(v) + 1 \leq 2$. Moreover, since $\pi(Z) = \pi'(Z') \subset Y$, the subvariety $Z \subset X$ must lie entirely in the special fiber $X \subset X$. This excludes the possibility $\text{codim}(Z) = 1$ - it would mean $Z = X \subset X$, hence $\pi'(Z') = \pi(Z) = Y \subset Y$ and $Z' = X'_o$, which contradicts $\text{codim} \ Z' = 2$.

Therefore we must have $\text{codim} \ Z = 2$ and $\text{disc}(v) = 1$. Since $X$ is smooth, this in turn implies that $v$ coincides with the valuation associated to the exceptional divisor of the blow-up of $X$ in $Z$. In particular, since we know that $X$ is smooth over $S$, this implies that $v(t) = 1$ for the parameter function $t \in \mathcal{O}(S)$.

Localizing at the generic point $z' \in Z' \subset X'$ and using the theory of algebraic surfaces, we see that $\text{disc}(v) = 1$ implies that $X'$ is non-singular at $z'$. This proves (i). Then (ii) immediately follows from $v(t) = 1$. \hfill \Box

**Remark 5.7.** Since the scheme $X'$ is not of finite type over $S$ (nor over $\mathbb{C}$), the notion of the canonical bundle causes some problems. Indeed, the sheaf $\Omega^1(X'/S)$ of relative Kähler differentials is not of finite rank. To make sense of its “top exterior power” $K_{X'}$, one has to replace $\Omega^1(X'/S)$ with the quotient sheaf that classifies all derivations compatible with the $(t)$-adic topology on the local rings of the scheme $X'$. This brings no difficulties with the notion of the discrepancy of a valuation. Indeed, all the valuations that we consider are centered at the special fiber $X'_o \subset X'$, hence compatible with the topology. The same applies to the schemes $X$, $Y$.

---

\[2\text{This is probably a standard fact, but an interested reader can find a proof in [K2, Corollary 2.3].}\]
Lemma 5.8. The canonical bundle $K_{X'_o}$ of the normal irreducible variety $X'_o$ is trivial, $K_{X'_o} \cong \mathcal{O}(X')$. In particular, the variety $X'_o$ is crepant over $Y$.

Proof. The canonical bundle $K_{X'}$ is trivial by Lemma 5.4. The adjunction formula gives an isomorphism

$$K_{X'_o} \cong \mathcal{N}(X'_o),$$

where $\mathcal{N}(X'_o)$ is the normal bundle to the hypersurface $X'_o \subset X'$. The differential $dt$ of the parameter $t \subset \mathcal{O}(S)$ defines a non-trivial global section of the conormal bundle to the hypersurface $X'_o \subset X'$. Moreover, by the statement (ii) in the proof of Lemma 5.6, this section is non-vanishing outside of a subset $Z \subset X'_o$ of codimension $\geq 2$. We conclude that the conormal bundle to $X'_o \subset X'$ is trivial. Therefore $K_{X'_o} \cong \mathcal{N}(X')$ is also trivial. □

5.3 Identification of the line bundle. We see that the variety $X'_o$ is normal and irreducible, and the projective map $\pi' : X'_o \rightarrow Y$ is crepant and generically one-to-one. To prove Theorem 1.4, it remains to establish the identification (5.2).

Denote by $L'_o$ the restriction of the ample line bundle $L'$ on $X'$ to the special fiber $X'_o \subset X'$. We have

$$X'_o \cong \text{Proj}_Y \bigoplus_{k \geq 0} \pi'_* L'_o \otimes^k.$$

Moreover, we know that $X'_o/Y$ is crepant. Therefore, if we consider the birational map

$$f_o = (\pi'_o)^{-1} \circ \pi : X \dashrightarrow X'_o,$$

then Lemma 2.3 shows that

$$\bigoplus_{k \geq 0} \pi'_* L'_o \otimes^k \cong \bigoplus_{k \geq 0} \pi_* L_o \otimes^k,$$

where

$$L_o = f_o^* L'_o$$

is the line bundle on $X$ corresponding to the line bundle $L'_o$ on $X'_o$. Thus to finish the proof of Theorem 1.4 it suffices to identify the line bundles $L_o$ and $L'$ on the variety $X$.

To make the statement we have to prove more clear, denote by $i : X \hookrightarrow \mathcal{X}$, $i' : X'_o \hookrightarrow \mathcal{X}'$ the embeddings of the special fibers. We have a rational
map \( f : X \to X' \) and a rational map of the special fibers \( f_o : X \to X_o' \). These maps are compatible with the embeddings: \( i' \circ f_o = f \circ i : X \to X' \). We have an ample line bundle \( L' \) on \( X' \). By definition of the projective scheme \( X' \), the line bundle \( f^*L' \) on \( X' \) is the extension to the scheme \( X' \) of the given line bundle \( L' \) on the special fiber \( X \). In other words, we have \( L' \cong i'f^*L' \). On the other hand, by construction we have \( L_o = f_o^*L'_o = f_o^*(i')^*L' \). The statement that we have to prove is
\[
f_o^*(i')^*L' \cong i^*f^*L'.
\]
This would have been trivial if rational maps \( f_o \) and \( f \) were defined everywhere. As things stand, we only have an isomorphism on the open subset in \( X \) where both \( f_o \) and \( f \) are defined.

Let \( Z \subset Y \) be the closed subset of codimension \( \geq 3 \) that is provided by Lemma 2.4 – namely, the subset such that the line bundle \( L' \) is relatively generated with respect to the map \( \pi : X \to Y \) over the complement \( Y \setminus Z \subset Y \). By Proposition 1.2 the preimage \( \pi^{-1}(Z) \subset X \) is of codimension \( \geq 2 \). Therefore to show that \( L_o \cong L' \) on \( X \), it suffices to show that \( L_o \cong L' \) outside of the subset \( \pi^{-1}(Z) \). Thus it suffices to show that both rational maps \( f, f_o \) are defined outside of \( \pi^{-1}(Z) \subset X \subset X' \).

The special fiber \( \Gamma_o \subset \Gamma \) of the graph \( \Gamma \subset X \times Y \) of the rational map \( f : X \to X' \) can have several irreducible components. One of these components is the graph of the rational map \( f_o : X \to X_o' \). Therefore the indeterminacy locus of the map \( f_o \) is a subset of the indeterminacy locus of the map \( f \).

We conclude that it suffices to consider the map \( f \). Thus to finish the proof of Theorem 1.4, it suffices to prove that the rational map \( f : X \to X' \) is defined on the complement to the closed subset \( \pi^{-1}(Z) \subset X \subset X' \). By construction of the scheme \( X' \), this is equivalent to the following lemma.

**Lemma 5.9.** The base locus \( B(L') = B(X, L') \) of the line bundle \( L' \) on \( X \) lies in the closed subset \( \pi^{-1}(Z) \subset X \subset X' \).

**Proof.** We want to show that the image \( \pi(B(L')) \subset \mathcal{Y} \) of the base locus \( B(L') \) under the projection \( \pi : X \to \mathcal{Y} \) lies in the closed subset \( Z \subset Y \subset \mathcal{Y} \). Since \( \mathcal{Y} \) is affine, the claim is local on \( \mathcal{Y} \) – in other words, the base locus \( B(L') \subset X \) coincides with the support of the cokernel of the canonical sheaf map
\[
\pi^*\pi_*L' \to L'.
\]
Denote by \( \mathcal{U} = \mathcal{Y} \setminus Z \) the open complement to the closed subset \( Z \subset \mathcal{Y} \), and let \( U = Y \setminus Z = \mathcal{U} \cap Y = Y \setminus Z \) be the complement to \( Z \) in the special fiber.
$Y \subset \mathcal{Y}$. We have to show that the canonical map $\pi^*\pi_*\mathcal{L}' \to \mathcal{L}'$ is surjective on the preimage $\pi^{-1}(U) \subset \mathcal{X}$. Since the projection $\pi : \mathcal{X} \to \mathcal{Y}$ is one-to-one outside of the special fiber $X \subset \mathcal{X}$, the map $\pi^*\pi_*\mathcal{L}' \to \mathcal{L}'$ is tautologically surjective outside of the special fiber. Therefore it suffices to prove that this map is surjective on the special fiber $\pi^{-1}(U) = \pi^{-1}(U) \cap X \subset \pi^{-1}(U)$. More precisely, we have to prove that the map

$$i^*\pi^*\pi_*\mathcal{L}' \to i^*\mathcal{L}'$$

is surjective on the subset $\pi^{-1}(U) \subset Y$, where $i : X \to \mathcal{X}$, $i : Y \to \mathcal{Y}$ denotes the embedding of the special fiber.

By definition we have $i^*\mathcal{L}' = \mathcal{L}'$, and we tautologically have $i^*\pi^*\pi_*\mathcal{L}' \cong \pi^*i^*\pi_*\mathcal{L}'$. But the canonical bundle $K_X$ is trivial, and the line bundle $\mathcal{L}'$ is by assumption relatively generated over $U \subset Y$. By Grauert-Riemenschneider vanishing this implies that

$$R^k\pi_*\mathcal{L}' = R^k\pi_*K_X \otimes \mathcal{L}' = 0$$

on $U \subset Y$ for all $k \geq 1$. Then the base change gives a line bundle isomorphism

$$\pi^*i^*\pi_*\mathcal{L}' \cong \pi^*\pi_*i^*\mathcal{L}' \cong \pi^*\pi_*\mathcal{L}',$$

and the map $\pi^*\pi_*\mathcal{L}' \to \mathcal{L}'$ is by assumption surjective on the open subset $\pi^{-1}(U) \subset X$. □

**Remark 5.10.** Lemma 5.9 is the first place in our construction where we have used the assumptions on the line bundles $\mathcal{L}'$ other than Condition 5.1. In fact, we can start with an arbitrary line bundle $\mathcal{L}'$. Then everything up to Lemma 5.9 works just as well. However, Lemma 5.9 does break down. If the original line bundle does not satisfy some form of Grauert-Riemenschneider vanishing, then the indeterminacy locus of the rational map $f : \mathcal{X} \dashrightarrow \mathcal{X}'$ may contain a divisor $E \subset X$ in the special fiber $X \subset \mathcal{X}$. In this case the graph $\Gamma \subset \mathcal{X} \times \mathcal{Y} \mathcal{X}'$ of the map $f$ has at least two irreducible components of dimension $\dim X$ in the special fiber $\Gamma_o \subset \Gamma$: one is the graph of the rational map $f_o : X \dashrightarrow \mathcal{X}_o'$, and the other lies entirely over the divisor $E \subset X$.

### 6 An application – smoothness of flops

We finish the paper with one concrete geometric corollary of Theorem 1.4 – both because it is interesting in its own right, and in order to convince the reader that even in spite of the very restrictive Condition 5.1, our result
does have immediate applications. This application is Proposition 1.6—the smoothness of flops. To save space, we only give a sketch of the proof (omitting exact references for deformation theory statements and the like).

### 6.1 Smoothness of flops.

We note that Proposition 1.6 (i) is an immediate corollary of Lemma 4.2; the real problem is to prove (ii). We will actually prove a slightly more general statement. For this we return to the setting of Theorem 1.4.

**Proposition 6.1.** In the setting of Theorem 1.4, assume that the scheme $X'$ is $\mathbb{Q}$-factorial. Then the scheme $X'$ and the special fiber $X' \subset X'$ are smooth, and the special fiber $X' \subset X'$ is symplectic.

**Proof.** By definition of the twistor deformation (Definition 4.1), the scheme $X$ carries an ample line bundle $L$ such that $c_1(L) = \Omega \cup \theta_X$, where $\Omega \in \Omega^2(X/S)$ is the relative symplectic form on the twistor deformation $X/S$, $\theta_X \in H^1(X, T(X/S))$ is the Kodaira-Spencer class of the deformation $X/S$, and $c_1(L) \in H^1(X, \Omega^1(X/S))$ is the relative first Chern class of the line bundle $L$.

The rational map $f : X' \to X$ is defined on an open subset $U \subset X'$ whose closed complement $X' \setminus U \subset X'$ is of codimension $\geq 2$. Moreover, as the proof of Lemma 5.3 shows, we can actually assume that $f$ is injective on $U \subset X'$. In particular, $U$ is smooth over $S$, and we have

$$c_1(f^*L) = f^*\Omega \cup \theta_U,$$

where $\theta_U \in H^1(U, T(U/S))$ is the Kodaira-Spencer class of the deformation $U/S$. Since $X'$ is by assumption $\mathbb{Q}$-factorial, some multiple of the line bundle $f^*L$ extends to the whole $X'$. Therefore the Chern class $c_1(f^*L) \in H^1(U, \Omega^1(U/S))$ extends to a class in $H^1(X', \Omega^1(X'/S))$.

The maps $\pi : X \to Y$, $\pi' : X' \to Y$ are one-to-one over the generic point $\eta \in S$. Therefore it suffices to prove that the schemes $X'$ and $X'$ are regular in every closed point $x \in X'$.

Let $x \in X'$ be such a point, and let $V \subset X$ be an open affine neighborhood of the point $x \in X' \subset X'$. Since $V$ is affine, every class in the cohomology group $H^1(X', \Omega^1(X'/S))$ vanishes after restriction to $V \subset X'$. This
applies in particular to the class $c_1(f^*L)$. We conclude that $c_1(f^*L) = 0$ on the intersection $U \cap V$. Since the relative symplectic form $f^*\Omega \in \Omega^2(U/S)$ is non-degenerate, this implies that the Kodaira-Spencer class

$$\theta_{U\cap V} \in H^1(U \cap V, \mathcal{T}(U/S)) = \text{Ext}^1_{U\cap V}(\Omega^1(U/S), \mathcal{O}(U \cap V))$$

also vanishes.

But the scheme $\mathcal{X}'$ is normal, and the complement to the subset $U \cap V \subset V$ is of codimension $\geq 2$. This means that if we denote by $j : U \cap V \hookrightarrow V$ the embedding, then the direct image $j_*\mathcal{O}(U \cap V)$ coincides with the structure sheaf,

$$j_*\mathcal{O}(U \cap V) \cong \mathcal{O}(V).$$

Therefore for every complex $\mathcal{F}$ of coherent sheaves on $V$ which is concentrated in non-positive degrees we have

$$\text{Ext}^1_V(\mathcal{F}, \mathcal{O}(V)) = \text{Ext}^1_V(\mathcal{F}, j_*\mathcal{O}(U \cap V)) \subset \text{Ext}^1_V(\mathcal{F}, R^j j_*\mathcal{O}(U \cap V)) = \text{Ext}^1_{U\cap V}(j^*\mathcal{F}, \mathcal{O}(U \cap V)).$$

Applying this to the relative cotangent complex $\Omega^\ast(V/S)$ of the flat deformation $V/S$, we conclude that the Kodaira-Spencer class

$$\theta_V = \theta_{U\cap V} \in \text{Ext}^1(\Omega^\ast(V/S), \mathcal{O}(V)) \subset \text{Ext}^1_{U\cap V}(\Omega^\ast(V/S), \mathcal{O}(V \cap U))$$

of this deformation is trivial. Therefore for every integer $n \geq 0$, the associated order-$n$ infinitesimal deformation $V_n$ of the special fiber $V \cap X'$ is trivial. Moreover, we can choose a compatible system of projections $V_n \to V_0$ onto the special fiber $V_0 = V \cap X' \subset \mathcal{X}'$.

Passing to the limit, we obtain a projection $\tau : V \to V_0$, hence a product decomposition $V = V_0 \times S$. Let $\bar{x} : S \to x \times S \subset V_0 \times S$ be the horizontal section of the map $\mathcal{X}' \to S$ which passes through the point $x \subset X'$. Then the conormal sheaf $\mathcal{N}(\bar{x})$ to this section is a constant vector bundle on $S$. Its rank is equal to $\dim T_x X'$, the dimension of the Zariski tangent space $T_x X'$ to $X'$ at $x$.

But since $V$ is by assumption smooth over the generic point $\eta \in S$, the rank of the normal bundle $\mathcal{N}(\bar{x})$ must coincide with $\text{codim}(\bar{x}, \mathcal{X}') = \dim(X')$, and we have $\dim T_x X' = \dim(X')$. This means that the scheme $X'$ is regular at the point $x \subset X'$. Since the point $x \subset X'$ was arbitrary, we conclude that both $\mathcal{X}'$ and $X'$ are smooth algebraic varieties.

---

3The scheme $V$ is not a priori smooth over $S$. However, one can still define the Chern class $c_1(f^*L) \in \Omega^1(\mathcal{X}'/S)$ — for instance, by applying $d\log$ to the corresponding class in $\text{Pic}(V) = H^1(V, \mathcal{O}_V^\ast)$. 32
Finally, we note that for every \( k \geq 0 \) the spaces of \( k \)-forms are the same for all smooth birational models of the same algebraic variety \( X \) (it suffices to prove this for blow-ups with smooth centers, where it immediately follows from a direct computation). Therefore the symplectic form \( f^*\Omega \) extends from the open subset \( U \cap X' \subset X' \) to the whole \( X' \).

\[ \square \]

Proof of Proposition 1.6. By the uniqueness of flops ([KMM, Proposition 5-1-11(2)]), we can assume that the flop \( \mathcal{X}' \) of the twistor deformation \( \mathcal{X}/\mathcal{Y} \) is the same scheme as the one provided by Theorem 1.4 (Condition 5.1 is satisfied precisely because the flop \( \mathcal{X}' \) is assumed to exist). Then the scheme \( \mathcal{X}' \) is \( \mathbb{Q} \)-factorial by [KMM, Proposition 5-1-11(1)], so that we can apply Proposition 6.1.

\[ \square \]

References

[B] A. Beauville, *Symplectic singularities*, math.AG/9903070, 1999.

[Bo] F. Bogomolov, *Hamiltonian Kähler manifolds*, Sov. Math. Dokl. 19 (1978), 1462–1465.

[DP] J.-P. Demailly and M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, math.AG/0105176, 2001.

[E] R. Elkik, *Rationalité des singularités canoniques*, Invent. Math. 64 (1981), 1–6.

[F] A. Fujiki, *On primitively symplectic compact Kähler V-manifolds*, in *Classification theory of algebraic and analytic manifolds*, Progress in Math. 39, Birkhäuser, 1983, 71–250.

[G] V. Ginzburg, a talk at the ICM Satellite Conference, Essen, 1998.

[GR] H. Grauert and O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. 11 (1970), 263–292.

[EGA] A. Grothendieck, *Éléments de Géométrie Algébrique, I, III, IV*, Publ. Math. IHES 4, 11, 24, 28.

[H1] D. Huybrechts, *Birational symplectic manifolds and their deformations*, J. Differential Geom. 45 (1997), no. 3, 488–513.
D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. **135** (1999), no. 1, 63–113.

D. Huybrechts, *The Kähler cone of a compact hyperkähler manifold*, math.AG/9909109, 1999.

D. Huybrechts, *Erratum to the paper: Compact hyperkaehler manifolds: basic results*, math.AG/0106014, 2001.

D. Kaledin, *Dynkin diagrams and crepant resolutions of quotient singularities*, math.AG/9903157, 1999.

D. Kaledin, *McKay correspondence for symplectic quotient singularities*, math.AG/9907087, 1999.

D. Kaledin, M. Verbitsky, *Period map for non-compact holomorphically symplectic manifolds*, math.AG/0005007, 2000.

Y. Kawamata, *Unobstructed deformations – A remark on a paper of Z. Ran*, J. Alg. Geom. **1** (1992), 183–190.

Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the Minimal Model Program*, Adv. Studies in Pure Math. **10** (1987), 283–360.

Y. Namikawa, *Deformation theory of singular symplectic n-folds*, preprint.

Namikawa, *Extension of 2-forms and symplectic varieties*, preprint.

Z. Ran, *Deformations of manifolds with torsion or negative canonical bundles*, J. Alg. Geom. **1** (1992), 279–291.

M. Schlessinger, *Rigidity of quotient singularities*, Inv. Math. **14** (1971), 17–26.

G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, in *Math. Aspects of String Theory*, S.-T. Yau, ed., Worlds Scientific, 1987, 629–646.

A. Todorov, *The Weil-Petersson geometry of the moduli space of SU(n ≥ 3) (Calabi-Yau) manifolds*, Comm., Math. Phys. **126** (1989), 325–346.
[V1] M. Verbitsky, Trianalytic subvarieties of the Hilbert scheme of points on a K3 surface, GAFA, 8 (1998), 732–782.

[V2] M. Verbitsky, Holomorphic symplectic geometry and orbifold singularities, math.AG/9903175, 1999.

[W] J. Wierzba, Contractions of Symplectic Varieties, math.AG/9910130, 1999.

E-mail address: kaledin@mccme.ru