A NOTE ON THE MATLIS DUAL OF A CERTAIN INJECTIVE HULL

PETER SCHENZEL

ABSTRACT. Let \((R, m)\) denote a local ring with \(E = E_R(R/m)\) the injective hull of the residue field. Let \(p \in \text{Spec } R\) denote a prime ideal with \(\text{dim } R/p = 1\), and let \(E_R(R/p)\) be the injective hull of \(R/p\). As the main result we prove that the Matlis dual \(\text{Hom}_R(E_R(R/p), E)\) is isomorphic to \(\hat{R}_p\), the completion of \(R_p\), if and only if \(R/p\) is complete. In the case of \(R\) a one dimensional domain there is a complete description of \(Q \otimes_R \hat{R}\) in terms of the completion \(\hat{R}\).

Dedicated to Hans-Bjoern Foxby

1. INTRODUCTION

Let \(R\) denote a commutative Noetherian ring. For injective \(R\)-modules \(I, J\) it is well-known that \(\text{Hom}_R(I, J)\) is a flat \(R\)-module. In order to understand them the first case of interest is when \(I, J\) are indecomposable (as follows by Matlis’ Structure Theory (see e.g. [7] or [3])).

Let \((R, m)\) denote a local ring with the injective hull \(E = E_R(R/m)\) of the residue field \(k = R/m\). In this situation it comes down to understand the Matlis dual \(\text{Hom}_R(I, E)\) of an injective \(R\)-module, in particular for \(I = E_R(R/p)\), the injective hull of \(R/p\) for \(p \in \text{Spec } R\). It was shown (see [3, 3.3.14] and [3, 3.4.1 (7)]) that

\[
\text{Hom}_R(E_R(R/p), E) \simeq E_R(R/p, E) \simeq E_R(R/p)_{\mu_p} \simeq \hat{R}_p^{\mu_p}.
\]

Moreover it follows (see [3, 3.10]) that

\[
\mu_p = \text{dim}_{k(p)} \text{Hom}_R(k(p), E).
\]

Therefore \(\text{Hom}_R(E_R(R/p), E)\) is the completion of a free \(R_p\)-module of rank \(\mu_p\).

Here we shall prove – as the main result of the paper – the following result on the Matlis dual of a certain \(E_R(R/p)\).

**Theorem 1.1.** Let \((R, m)\) denote a local ring. Let \(p\) denote a one dimensional prime ideal. Then \(\text{Hom}_R(E_R(R/p), E) \simeq \hat{R}_p\) (i.e. it is the completion of a free \(R_p\)-module of rank one) if and only if \(R/p\) is complete.

Let \(p\) denote a one dimensional prime ideal in a local ring \((R, m)\). The equality \(\mu_p = 1\) was proved in [4] resp. in [5] in the case of \(R\) a complete Gorenstein domain resp. in the case of \(R\) a complete Cohen-Macaulay domain. The proofs are based on the use of the dualizing module of a complete Cohen-Macaulay domain. Note that the dualizing module is isomorphic to \(\hat{R}\) in the case of a complete Gorenstein domain.

Here we use as a basic ingredient Matlis Duality and - as a main step - the reduction to the case of \(\text{dim } R = 1\) suggested by one of the reviewer’s. In the case of a one dimensional domain there is a complete description of \(\text{Hom}_R(E_R(R), E)\) and \(Q \otimes_R \hat{R}\) in terms of the completion \(\hat{R}\) (see Theorem 2.5 for the precise formulation).

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2. Proofs

In the following \((R, \mathfrak{m})\) denotes always a local ring with \(E = E_R(R/\mathfrak{m})\) the injective hull of the residue field \(R/\mathfrak{m}\). Then \(D_R(\cdot) = \text{Hom}_R(\cdot, E)\) denotes the Matlis duality functor.

**Remark 2.1.** (A) Let \(X\) be an arbitrary \(R\)-module. There is a natural homomorphism

\[ X \rightarrow D(D(X)) \]

that is always injective. If \((R, \mathfrak{m})\) is complete it is an isomorphism whenever \(X\) is an Artinian \(R\)-module resp. a finitely generated \(R\)-module (see [7, p. 528] and [7, Corollary 4.3]). Moreover it follows that the map is an isomorphism if and only if there is a finitely generated \(R\)-submodule \(Y \subset X\) such that \(X/Y\) is an Artinian \(R\)-module. For the proof we refer to [8] and also to [1] for a generalization.

(B) Let \(M\) denote a finitely generated \(R\)-module. Then there is a natural isomorphism \(M \otimes_R \hat{R} \simeq D(D(M))\). That is, \(M\) is Matlis reflexive if and only if it is complete.

(C) Let \(X\) denote an \(R\)-module with \(\text{Supp}_R X \subset \{\mathfrak{m}\}\). Then \(X\) admits the structure of an \(\hat{R}\)-module compatible with its \(R\)-module structure such that \(X \otimes_R \hat{R} \rightarrow X\) is an isomorphism (see e.g. [8, (2.1)]). Let \(M\) denote an \(R\)-module and \(N\) an \(\hat{R}\) module. Then \(\text{Ext}_R^i(M, N), i \in \mathbb{Z}\), has the structure of an \(\hat{R}\)-module. Moreover, here are natural isomorphisms

\[ \text{Ext}_R^i(M, N) \simeq \text{Ext}_{\hat{R}}^i(M \otimes_R \hat{R}, N) \]

for all \(i \in \mathbb{Z}\) since \(\hat{R}\) is a flat \(R\)-module.

As a technical tool we shall need the short exact sequence of the following trivial Lemma.

**Lemma 2.2.** Let \((R, \mathfrak{m})\) denote a one dimensional domain. Then there is a short exact sequence

\[ 0 \rightarrow R \rightarrow Q \rightarrow H^0_{\mathfrak{m}}(R) \rightarrow 0 \]

where \(Q = \mathbb{Q}(R)\) denotes the quotient field of \(R\).

**Proof.** We start with the following short exact sequence \(0 \rightarrow R \rightarrow Q \rightarrow Q/R \rightarrow 0\). The long exact cohomology sequence provides an isomorphism \(Q/R \simeq H^0_{\mathfrak{m}}(R)\). To this end recall that \(H^0_{\mathfrak{m}}(Q) = 0\) for all \(i \in \mathbb{Z}\). This proves the statement. \(\Box\)

As one of the main ingredients of the proof we start with a reduction to the one dimensional case suggested by one of the reviewers.

**Lemma 2.3.** Let \(p\) be a prime ideal in a local ring \((R, \mathfrak{m})\). Let

\[ \text{Hom}_R(E_R(R/p), E) \simeq \hat{R}^\mu_p \]

with \(\mu_p = \text{dim}_k(p)\text{Hom}_R(k(p), E) = \text{dim}_k(p)\text{Hom}_R(R/p)(k)\).

**Proof.** Since \(k(p)\) is an \(R/p\)-module the adjunction formula gives the following isomorphisms

\[ \text{Hom}_R(k(p), E) \simeq \text{Hom}_{R/p}(k(p), \text{Hom}_R(R/p, E)) \simeq \text{Hom}_{R/p}(k(p), E_{R/p}(k)). \]

For the last isomorphism note that \(\text{Hom}_{R}(R/p, E) \simeq E_{R/p}(k)\). \(\Box\)

Now we are prepared for the main result in the one dimensional case.

**Theorem 2.4.** Let \(R\) denote a one dimensional local domain and \(Q = \mathbb{Q}(R)\) its quotient field. There are isomorphisms

\[ D_{\hat{R}}(D_R(Q)) \simeq Q \otimes_R \hat{R} \simeq Q \oplus \hat{R}/R \text{ and } \hat{R}/R \simeq \text{Ext}_R^1(Q, R). \]

Thus \(\hat{R}/R\) has a natural structure as a \(Q\)-vector space, and so it is injective as an \(R\)-module. Moreover, \(R\) is complete if and only if \(\text{Ext}_R^1(Q, R) = 0\), if and only if \(D_R(Q) \simeq Q\).
Proof. Consider the short exact sequence of Lemma 2.1 and apply \( \cdot \otimes_R \hat{R} \). It induces a commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & R & \longrightarrow & Q & \longrightarrow & H^1_m(R) & \to & 0 \\
& \downarrow & & \downarrow & & \| & & \\
0 & \to & \hat{R} & \to & Q \otimes \hat{R} & \to & H^1_m(\hat{R}) & \to & 0.
\end{array}
\]

The vertical homomorphism at the right is an isomorphism since \( H^1_m(R) \) is an Artinian \( R \)-module (see Remark 2.1). Because the vertical homomorphisms are injective the snake lemma implies an isomorphism \( \hat{R}/R \simeq (Q \otimes R \hat{R})/Q \). Whence there is the short exact sequence

\[
0 \to Q \to Q \otimes_R \hat{R} \to \hat{R}/R \to 0.
\]

By virtue of Remark 2.1 there is an isomorphism \( D_R(Q) \simeq D_R(Q \otimes_R \hat{R}) \). So Matlis Duality implies that \( D_R(D_R(Q)) \simeq Q \otimes_R \hat{R} \). Since \( Q = E_R(R) \) is an injective \( R \)-module it follows that

\[
D_{\hat{R}}(D_R(Q)) \simeq Q \otimes_R \hat{R} \simeq Q \oplus \hat{R}/R.
\]

This proves the first isomorphisms. Moreover \( \hat{R}/R \simeq \text{Hom}_R(Q, \hat{R}/R) \) since \( \hat{R}/R \) admits the structure of a \( Q \)-vector space.

Next we claim that \( \text{Ext}^i_{\hat{R}}(Q, \hat{R}) = 0 \) for all \( i \in \mathbb{Z} \). By Matlis Duality and adjointness there are the following isomorphisms

\[
\text{Ext}^i_{\hat{R}}(Q, \hat{R}) \simeq \text{Ext}^i_R(Q, \text{Hom}_R(E, E)) \simeq \text{Hom}_R(\text{Tor}^R_i(Q, E), E).
\]

So it will be enough to show that \( \text{Tor}^R_i(Q, E) = 0 \) for all \( i \in \mathbb{Z} \). This follows since \( Q \) is a flat \( R \)-module and \( Q \otimes_R E = 0 \).

With this in mind the long exact cohomology sequence of \( \text{Ext}^i_{\hat{R}}(Q, \cdot) \) applied to the short exact sequence \( 0 \to R \to \hat{R} \to \hat{R}/R \to 0 \) induces the isomorphism

\[
\text{Hom}_R(Q, \hat{R}/R) \simeq \text{Ext}^1_R(Q, R).
\]

This provides the second isomorphism of the statement and finishes the proof of the first equivalence. For the second equivalence note that \( \dim_Q \text{Hom}_R(Q, E) = 1 \) implies \( D_R(D_R(Q)) \simeq Q \) and therefore \( \hat{R} = R \) by view of the short exact sequence of Lemma 2.1 and \( D_R(D_R(R)) \simeq \hat{R} \). \( \square \)

In the following we consider the general case of a one dimensional domain. To this end let \( \text{Ass} \hat{R} = \{ q_1, \ldots, q_r \} \) denote the set of associated prime ideals of the completion \( \hat{R} \) of the domain \( R \). Then \( q_i \cap R = (0) \) and \( \dim \hat{R}/q_i = 1 \) for \( i = 1, \ldots, r \).

**Theorem 2.5.** Let \( (R, m) \) denote a one dimensional domain. Then there is are isomorphisms

\[
\text{Hom}_R(Q, E) \simeq \bigoplus_{i=1}^r E_{\hat{R}}(\hat{R}/q_i) \quad \text{and} \quad Q \otimes_R \hat{R} \simeq \bigoplus_{i=1}^r \hat{R}_{q_i},
\]

where \( \hat{R}_{q_i} \) denotes the completion of \( \hat{R}_{q_i}, i = 1, \ldots, r \).

**Proof.** It is known that \( \text{Hom}_R(H^1_m(R), E) \simeq \text{Hom}_{\hat{R}}(H^1_{m\hat{R}}(\hat{R}), E) \) is the dualizing module \( \omega_{\hat{R}} \) of \( \hat{R} \). Its minimal injective resolution as \( \hat{R} \)-module has the following form

\[
0 \to \omega_{\hat{R}} \to \bigoplus_{i=1}^r E_{\hat{R}}(\hat{R}/q_i) \to E \to 0
\]

(for these results on the dualizing module see e.g. [2, Section 3.3]). By applying the Matlis dual functor \( D_R(\cdot) \) to the short exact sequence of Lemma 2.2 it provides a short exact sequence of \( \hat{R} \)-modules

\[
0 \to \omega_{\hat{R}} \to \text{Hom}_R(Q \otimes_R \hat{R}, E) \to E \to 0
\]
of \( \hat{R} \)-modules. Whence there is a commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \to & \omega_{\hat{R}} \\
\| & & \| \\
0 & \to & \omega_{\hat{R}} \\
\| & & \| \\
\end{array}
\]

\[
\begin{array}{ccc}
& \to & \operatorname{Hom}_{\hat{R}}(Q \otimes_{\hat{R}} \hat{R}, E) \to E \to 0 \\
\| & & \| \\
& \to & \oplus_{i=1}^{r} E_{\hat{R}}(\hat{R}/q_i) \to E \to 0.
\end{array}
\]

By the snake lemma it yields an isomorphism \( \oplus_{i=1}^{r} E_{\hat{R}}(\hat{R}/q_i) \simeq \operatorname{Hom}_{\hat{R}}(Q \otimes_{\hat{R}} \hat{R}, E) \) and therefore \( \operatorname{Hom}_{\hat{R}}(Q, E) \simeq \oplus_{i=1}^{r} E_{\hat{R}}(\hat{R}/q_i) \). By Matlis Duality (see Remark 2.1) there are the following isomorphisms

\[
Q \otimes_{\hat{R}} \hat{R} \simeq D(\hat{R})(Q \otimes_{\hat{R}} \hat{R}) \simeq \oplus_{i=1}^{r} \operatorname{Hom}_{\hat{R}}(E_{\hat{R}}(\hat{R}/q_i), E).
\]

By Lemma 2.3 and Theorem 2.4 it provides that \( \operatorname{Hom}_{\hat{R}}(E_{\hat{R}}(\hat{R}/q_i), E) \simeq \hat{R}_{q_i}, i = 1, \ldots, r \). This proves the second statement.

\[\Box\]

**Proof of Theorem 1.1.** By view of Lemma 2.3 we may reduce the computation of \( \mu_q \) to the case of the one dimensional domain \( R/p \). Then the statements are a consequence of Theorem 2.4.

\[\Box\]

### 3. Remarks

We conclude with a few discussions on the previous results.

**Remark 3.1.** Theorem 1.1 does not hold for a prime ideal \( p \subset R \) in a complete local ring \( R \) with \( \dim R/p > 1 \). To this end let \((R, m)\) a complete local two dimensional domain. Then \( E_{\hat{R}}(R) = Q \), the quotient field of \( R \). But now \( \operatorname{Hom}_{\hat{R}}(Q, E) \simeq Q \) can not be true. Assume that it holds. Then the natural map \( Q \to D(D(Q)) \) is an isomorphism too. This can not be the case as follows by view of Remark 2.1.

Let \( k \) denote a field and \( x \) an indeterminate over \( k \). Consider the situation of \( k[x]_{(x)} \) and its completion \( k[[x]] \). Then their quotient fields are \( k(x) \) and \( k((x)) \) resp. Then

\[
k(x) \otimes_{k[x]_{(x)}} k[[x]] \simeq k(x) \oplus k[[x]]/k[x]_{(x)}, \quad \text{and} \quad \operatorname{Hom}_{k[x]_{(x)}}(k(x), k(x)/k[x]_{(x)}) \simeq k((x))
\]

as follows by Theorems 2.4 and 2.5.

**Problem 3.2.** Let \( p \) denote a one dimensional prime ideal in a local ring \((R, m)\). Suppose that \( R/p \) is complete. We know that \( \operatorname{Hom}_{\hat{R}}(R/p, E) \) is again an injective \( R \)-module. Since the natural homomorphism \( E_{\hat{R}}(R/p) \to I = D(D(E_{\hat{R}}(R/p))) \) is injective it turns out that \( E_{\hat{R}}(R/p) \) is a direct summand of \( I \). By view of Remark 2.1 it can not be an isomorphism.

By the Matlis Structure Theorem it follows that \( I \simeq \oplus_{q \in \operatorname{Spec} R} E_{\hat{R}}(R/q)\mu(q, I) \) where

\[
\mu(q, I) = \dim_{k(q)} \operatorname{Hom}_{R_{k(q)}}(k(q), I_q)
\]

denotes the multiplicities of the occurrence of \( E_{\hat{R}}(R/q) \) in \( I \) (see e.g. [3, Section 3.3]). We know that \( \mu(q, I) = 0 \) for all \( q \not\subset p \) and \( \mu(p, I) \geq 1 \). It is not clear to us whether \( \mu(p, I) \) is finite or even 1? Which of the \( \mu(q, I) \) are not zero?

**Remark 3.3.** Let \((R, m)\) denote a one dimensional domain. One might ask whether the \( Q \)-rank of \( Q \otimes_{\hat{R}} \hat{R} \) is finite only if \( \hat{R} = R \). This is not true as it follows by Nagata’s Example (E3.3) (see [6, page 207]).

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Martin-Luther-Universität Halle-Wittenberg, Institut für Informatik, D — 06 099 Halle (Saale), Germany
E-mail address: peter.schenzel@informatik.uni-halle.de