COMPLEXITY ONE HAMILTONIAN SU(2) AND SO(3) ACTIONS

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ABSTRACT. We consider compact connected six dimensional symplectic manifolds with Hamiltonian SU(2) or SO(3) actions with cyclic principal stabilizers. We classify such manifolds up to equivariant symplectomorphisms.

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1. Introduction

Let $(M, \omega)$ be a symplectic manifold and $G$ be a compact connected Lie group that acts effectively on $M$ by symplectic transformations. A moment map $\Phi : M \to g^*$ is a $G$-equivariant map such that for every $\xi$ in the Lie algebra $g$ of $G$,

$$\iota(\xi_M)\omega = -d\langle \Phi, \xi \rangle$$

where $\xi_M : M \to TM$ denotes the induced vector field of $\xi$ on $M$. If there is a moment map, we say that the action is Hamiltonian. The triple $(M, \omega, \Phi)$ is called a Hamiltonian $G$-manifold or Hamiltonian $G$-action. An isomorphism between two such manifolds is an equivariant symplectomorphism that respects the moment maps. We usually assume that $M$ is connected, that $G$ acts effectively on $M$, and that the moment map $\Phi$ is proper.

For a point $\alpha$ in the dual of the Lie algebra of $G$, the symplectic quotient or reduced space at $\alpha$ is the topological space $M_{\alpha} = \Phi^{-1}(G \cdot \alpha)/G = \Phi^{-1}(\alpha)/G_{\alpha}$,
where $G \cdot \alpha$ is the coadjoint orbit through $\alpha$ and $G_\alpha$ denotes the stabilizer of $\alpha$. If $\alpha$ is a regular value of the moment map $\Phi$, the reduced space $M_\alpha$ is a symplectic orbifold. In general, the reduced space is a symplectic stratified space [LS, BL]. The complexity of $(M, \omega, \Phi)$ is half the dimension of the reduced space $M_\alpha$ at a generic value $\alpha$ in the moment image $\Phi(M)$.

Suppose a torus $T$ acts on a symplectic manifold $M$ in a Hamiltonian fashion. Its complexity is $\frac{1}{2} \dim M - \dim T$. In particular, the complexity is zero exactly if the dimension of the torus is half the dimension of the manifold. The space is then called a toric manifold. These spaces are classified by their moment images [D1]. Karshon and Tolman have studied complexity one torus actions in arbitrary dimensions [KT1, KT2]. Special cases are compact symplectic four-manifolds with circle actions which are classified by Karshon [K1], also see Audin, and Ahara-Hattori [Au1, Au2, AH].

As for SO(3) actions, there are two distinct cases: the principal stabilizer is $S^1$ or the principal stabilizer is $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. The former case is characterized by Iglesias [Ig]. In any dimension, the manifold $M$ is isomorphic to the product of $S^2$ by the symplectic orbit manifold $M/\text{SO}(3)$. The latter case in dimension four has complexity zero and is classified by Iglesias [Ig]. The only compact Hamiltonian SO(3)-manifolds are $\mathbb{C}P^2$ and $S^2 \times S^2$. The first is equipped with the natural action induced by SU(3) and the second can be equipped with different SO(3) actions indexed by $\mathbb{N}$. Complexity zero Hamiltonian actions of more general nonabelian groups have been studied by Delzant, Woodward, and Knop [D2, W, Kn].

In the algebraic and smooth categories, Lie group actions of complexity zero or one have been studied in [T1, T2, F, OW].

In this paper we study complexity one SU(2) and SO(3) actions. After Iglesias’s work, it remains to classify compact connected six dimensional symplectic manifolds equipped with Hamiltonian SU(2) or SO(3) actions with cyclic principal stabilizers $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. When the moment image does not contain zero, the manifold is of the form $G \times S^1 X$ where $G$ is SU(2) or SO(3) and $X$ is a symplectic four-manifold with a (possibly noneffective) Hamiltonian circle action. This can be viewed as an immediate corollary from the classification of circle actions on four-manifolds. Therefore, we emphasize the case when zero is in the moment image.

Generalizing techniques established by Karshon and Tolman [KT1, KT2], we proceed by first studying the basic building blocks: the preimages under the moment map of sufficiently small open subsets in $g^*$. In this context, the classification applies not only to a compact manifold but also to a noncompact manifold with a proper moment map. In Sections 2–12, we provide a complete set of invariants for the preimage of a neighborhood of $0 \in g^*$. In Sections 13–16, we discuss local invariants for the preimage of any neighborhood in $g^*$ away from zero. We then show that if two spaces are locally isomorphic, they are globally isomorphic.

We now describe the invariants.

The Duistermaat-Heckman function is a real function defined on the dual of the Lie algebra that takes the value of the symplectic volume of the reduced space. That is, $f: g^* \to \mathbb{R}$ such that

$$f(\alpha) = \text{Vol}(M_\alpha).$$

For $x \in M$, the set $G_x = \{ g \in G \mid gx = x \}$ of elements of $G$ leaving $x$ fixed is called the stabilizer of $x$. $G_x$ is a closed subgroup of $G$. It acts linearly on the tangent space $T_x M$, and thus $T_x M$ is a representation space of $G_x$. This
representation is called the **isotropy representation** at $x$. The stabilizers of points in the same orbit are conjugate, and their isotropy representations are linearly symplectically isomorphic. We call this conjugacy class the **stabilizer**, and the isomorphism class the **isotropy representation** of the orbit.

For convenience, we call the level set of the moment map $\Phi^{-1}(\alpha)$ the **moment fiber** at $\alpha$ or simply the $\alpha$ fiber. An orbit is **exceptional** if it has a strictly larger stabilizer than any nearby orbit in the same moment fiber. In particular, if $\Phi^{-1}(\alpha)$ contains only one $G_\alpha$ orbit, that orbit is exceptional. Since the moment map is proper, each moment fiber is compact, and it has finitely many exceptional orbits. The **isotropy data** at $\alpha$ consist of the unordered list of isotropy representations of the exceptional orbits in $\Phi^{-1}(\alpha)$.

If $M$ is a $G$-manifold with connected orbit space $M/G$, there exists an open dense subset of $M$ in which all stabilizers are conjugate. This conjugacy class is called the **principal stabilizer** of $M$. A similar notion exists for the zero fiber of the moment map.

If $\Phi^{-1}(\alpha)$ consists of a single $G_\alpha$-orbit, it is called a **short fiber**; otherwise, it is **tall**. In the complexity one case, if $\Phi^{-1}(\alpha)$ is tall, we show that its reduced space $M_\alpha = \Phi^{-1}(\alpha)/G_\alpha$ is topologically a closed connected oriented surface. We call its genus the **genus at $\alpha$**. Defining the genus of a point to be zero, we show that the genus is independent of $\alpha$ for any $\alpha \in \Phi(M)$ and is called the **genus** of the Hamiltonian $G$-manifold $(M,\omega,\Phi)$.

For any real $n$ dimensional vector bundle $\pi: W \to X$, there exists an associated orientation bundle $p: \tilde{X} \to X$, whose fiber over a point $x$ is the two ways to orient $\pi^{-1}(x)$. This is a two-sheeted covering, and Čech cocycles provide a convenient way to construct it. Choose an open cover $U = \{U_a\}$ of $X$ with trivialization maps $\varphi_a: U_a \times \mathbb{R}^n \to \pi^{-1}(U_a)$. The Jacobian determinants of the change of fiber coordinates from $\mathbb{R}^n$ to $\mathbb{R}^n$ have a locally constant sign, which gives a locally constant function from $U_a \cap U_b$ to $\mathbb{Z}_2$. The chain rule for Jacobians implies that this is a cocycle. It determines an element $w_1(W) \in H^1(X;\mathbb{Z}_2)$, called the **first Stiefel-Whitney class** of $W$. In a similar fashion, we can construct the associated orientation bundle and define the first Stiefel-Whitney class of any fiber bundle $W$ over a manifold $X$ provided that the fiber of $W$ is connected and orientable.

In particular, when the zero fiber of a Hamiltonian $SO(3)$-manifold is tall, and when the principal stabilizer of the zero fiber is $S^1$, the zero fiber $\Phi^{-1}(0)$ is a sphere bundle over the reduced space off the exceptional orbits. Let $E = \{E_j\}$ denote the set of exceptional orbits in the zero fiber. Let $M_0^{reg}$ denote the smooth part of the symplectic quotient at 0, i.e., $M_0^{reg} = (\Phi^{-1}(0) \setminus E)/G$. Then $M_0^{reg}$ is diffeomorphic to $\Sigma \setminus \{\text{finitely many points}\}$, where $\Sigma = \Phi^{-1}(0)/G$ is a closed connected oriented surface. Through the orientations on the fiber spheres, $\Phi^{-1}(0)$ induces an associated orientation bundle on $M_0^{reg}$, and the first Stiefel-Whitney class in $H^1(M_0^{reg};\mathbb{Z}_2)$.

Sections 2–12 are devoted to prove the local uniqueness over 0:

**Theorem A** (Local Uniqueness over 0). Let $G$ be $SU(2)$ or $SO(3)$. Let $(M,\omega,\Phi)$ and $(M',\omega',\Phi')$ be compact connected six dimensional Hamiltonian $G$-manifolds such that $0 \in \Phi(M) = \Phi'(M')$. There exists an invariant neighborhood $V$ of 0 in $\mathfrak{g}^*$ over which the Hamiltonian $G$-manifolds are isomorphic if and only if

- their Duistermaat-Heckman functions coincide on $V$,
- their isotropy data and genus at 0 are the same,
• their principal stabilizers of the zero fibers are the same,
• if the zero fibers are tall with principal stabilizer $S^1$, the first Stiefel-Whitney classes of $\Phi^{-1}(0)$ and $\Phi'^{-1}(0)$ in $H^1(M_0^{reg}, \mathbb{Z}_2)$ and $H^1(M'_0^{reg}, \mathbb{Z}_2)$ are equal (under an identification of $M_0$ and $M'_0$ that respects the isotropy data).

**Remark 1.1.** For $G = SU(2)$, a tall zero fiber has no exceptional orbits. In this case, Theorem A follows from the equivariant symplectic embedding theorem of [W1].

In Sections 13-16, we adapt the idea of symplectic cross-sections introduced by Guillemin and Sternberg [GS2]. It allows us to apply our techniques and determine when two Hamiltonian $G$-manifolds are locally isomorphic. We then define compatible invariants to construct a global isomorphism from the local isomorphisms.

Let $E$ denote the set of exceptional orbits in $M$. We consider the projections $M \to M/G$ and $g^* \to g^*/G$, and the map $\overline{\Phi}$ induced by the moment map $\Phi$. The **isotropy skeleton** is the space $E/G$ where each point is labeled by its isotropy representation, together with the map $\Phi: E/G \to g^*/G$. Two isotropy skeletons are considered the same if there exists a homeomorphism $f: E/G \to E'/G$ that sends each point to a point with the same isotropy representation and such that $\Phi = \Phi' \circ f$.

We have the following global uniqueness theorem:

**Theorem B.** Let $G$ be $SU(2)$ or $SO(3)$. Let $(M, \omega, \Phi)$ and $(M', \omega', \Phi')$ be compact connected six dimensional Hamiltonian $G$-manifolds such that $\Phi(M) = \Phi'(M)$. Then $M, M'$ are isomorphic if and only if they have the same Duistermaat-Heckman function, the same genus, the same isotropy skeleton, the same principal stabilizers of the manifolds and of the zero fibers, and the same first Stiefel-Whitney class of the zero fibers when applicable.

## 2. The zero fiber

We begin by stating some properties of the zero fiber of the moment map. If a compact Lie group acts on a symplectic manifold and if the action is Hamiltonian, the Local Normal Form Theorem of Marle, Guillemin and Sternberg [M, GS1] provides a nice description of the neighborhood of any orbit in the zero fiber.

**Theorem 2.1 (Local Normal Form).** Let a compact Lie group $G$ act on a symplectic manifold $(M, \omega)$ with a moment map $\Phi: M \to g^*$. Let $x$ be a point in the zero fiber of the moment map, $G \cdot x$ be its orbit in $M$, $H$ be the stabilizer of $x$, and $V$ be the symplectic slice $(T_x(G \cdot x))^\omega / T_x(G \cdot x)$ at $x \in M$. Given a choice of an $H$-equivariant splitting $g = h \oplus m$, there exists a $G$-invariant symplectic form on the **local model** $Y = G \times_H (h^0 \times V)$ such that

1. a neighborhood of $G \cdot x$ is equivariantly symplectomorphic to a neighborhood of the zero section in $Y$, and
2. the action of $G$ on $Y$ is Hamiltonian and the moment map is given by

$$\Phi_Y([g, \mu, v]) = Ad^!(g)(\mu + \pi^*\Phi_V(v)),$$

where $h^0$ is the annihilator of $h$, $Ad^!$ is the coadjoint action, $\pi^*: h^* \to g^*$ is induced by the projection $\pi: g \to h$, and $\Phi_V: V \to h^*$ is the moment map for the slice representation.

A special case of Theorem A follows immediately from the Local Normal Form Theorem:
Proposition 2.2. Let $G = \text{SU}(2)$ or $\text{SO}(3)$. Let $(M, \omega, \Phi)$ and $(M', \omega', \Phi')$ be Hamiltonian $G$-manifolds such that $0 \in \Phi(M) = \Phi'(M')$. Assume $\Phi^{-1}(0)$ and $\Phi'^{-1}(0)$ consist of one single orbit each and that these orbits have the same isotropy representation. Then there exists a neighborhood $V$ of 0 in $\mathfrak{g}^*$ over which $M$ and $M'$ are isomorphic.

Proof. Since $\Phi^{-1}(0)$ and $\Phi'^{-1}(0)$ consist of one single orbit each with the same stabilizer representation, we can find $x \in M$ and $x' \in M'$ with the same stabilizer such that $G \cdot x = \Phi^{-1}(0)$ and $G \cdot x' = \Phi'^{-1}(0)$. It follows from the Local Normal Form Theorem that there exist neighborhoods $U$ of $G \cdot x$ and $U'$ of $G \cdot x'$ and an equivariant symplectomorphism $\varphi: U \to U'$ such that $\varphi(x) = x'$ and $\Phi' \circ \varphi = \Phi$.

Since the moment maps $\Phi$ and $\Phi'$ are proper, there exist neighborhoods $W$ and $W'$ of 0 in $\mathfrak{g}^*$ such that $\Phi^{-1}(W) \subset U$ and $\Phi'^{-1}(W') \subset U'$. We can then take $V = W \cap W'$. □

If we ought to understand all the possible local models $G \times_H (\mathfrak{g}^0 \times V)$, we first have to understand all the isotropy representations. The isotropy representation is a direct sum of the coadjoint action of $H \subset G$ on $\mathfrak{g}^0 \subset \mathfrak{g}^*$ and the slice representation of $H$ on $V$. Therefore, we need to know all the possible stabilizers, slice representations, and coadjoint actions for $\text{SU}(2)$ and $\text{SO}(3)$.

Up to conjugacy, the finite subgroups of $\text{SO}(3)$ include the trivial group $\{1\}$, the cyclic groups $Z_k$, $k = 2, 3, \ldots$, the dihedral groups $D_{2k}$, $k = 2, 3, \ldots$, the tetrahedral, octahedral and icosahedral groups. Any of these finite subgroups will be denoted by $\Gamma$ if it doesn’t need to be specified.

Up to conjugacy, $\text{SO}(3)$ has two infinite one dimensional closed subgroups: the maximal abelian subgroup $S^1$, and its normalizer, which is isomorphic to $O(2)$ and will be denoted as $N_{\text{SO}(3)}(S^1)$, or simply $N(S^1)$ if there is no possible confusion.

Stabilizers are closed subgroups. Therefore, the possible stabilizers of an $\text{SO}(3)$ action, up to conjugacy, are the subgroups listed above and $\text{SO}(3)$ itself.

Similarly, up to conjugacy, the closed subgroups of $\text{SU}(2)$ are: a collection of finite subgroups, again denoted by $\Gamma$, the maximal torus $S^1$, the normalizer of the maximal torus $N_{\text{SU}(2)}(S^1)$, and $\text{SU}(2)$ itself. Note that $N_{\text{SU}(2)}(S^1)$ is no longer isomorphic to $O(2)$.

The slice representations we need to consider are linear symplectic representations of $H$ on $\mathbb{C}^n$ and those of $G$ on $\mathbb{C}^3$, where $G$ denotes either $\text{SU}(2)$ or $\text{SO}(3)$ and $H$ denotes $S^1$ or $N_G(S^1)$. The linear symplectic representations on a complex vector space $\mathbb{C}^n$ are equivalent to the unitary representations. Therefore, we know that the representations of $S^1$ on $\mathbb{C}$ are characterized by the weights $n \in \mathbb{Z}$, and that there is only one effective representation each for $\text{SU}(2)$ and $\text{SO}(3)$ on $\mathbb{C}^3$.

A slice representation $\rho: N_G(S^1) \to U(\mathbb{C}) = S^1$ is an analytic homomorphism. Because $S^1$ is abelian, the commutator group of $N_G(S^1)$ is in the kernel of $\rho$. So the kernel of $\rho$ is either $N_G(S^1)$, or $S^1$. The former implies that the slice representation is trivial; the latter implies that the slice representation reduces to a $\mathbb{Z}_2$ action such that $h \cdot z = z$ for $h \in S^1$ and $h \cdot z = -z$ otherwise.

We fix an inner product on the Lie algebra $\mathfrak{g}$ of $G = \text{SU}(2)$ or $\text{SO}(3)$. This determines a projection $\mathfrak{g} \to \mathfrak{h}$ and the induced inclusion $\mathfrak{h}^* \to \mathfrak{g}^*$ for any $\mathfrak{h} \subseteq \mathfrak{g}$. We also identify $\mathfrak{h}^*$ with its image in $\mathfrak{g}^*$. Since maximal tori in the same group are conjugate to each other, we use \[
\begin{pmatrix}
  e^{i\theta} & 0 \\
  0 & e^{-i\theta}
\end{pmatrix}
\] to represent $S^1$ in $\text{SU}(2)$ and
Therefore \( \Phi \) be a six dimensional Hamiltonian SU(2)-manifold with a moment map \( \Phi : M \rightarrow \mathfrak{su}(2)^* \simeq \mathbb{R}^3 \). Assume the action is effective. Then for any \( x \in \Phi^{-1}(0) \), the local model for the orbit \( G \cdot x \) is one of the following:

1. \( Y = \text{SU}(2) \times_{\Gamma} \mathbb{R}^3 = \{ [g, \mu] | g \in \text{SU}(2), \mu \in \mathbb{R}^3, \{ [g^{-1}, \text{Ad}^l(\mu)] \}_{\forall \mu} \} \) with \( \Phi_Y([g, \mu]) = \text{Ad}^l(g)\mu \), where \( \Gamma \) does not contain \( \mathbb{Z}_2 = \pm 1 \).
2. \( Y = \text{SU}(2) \times_{S^1} (\mathbb{R}^2 \times \mathbb{C}) \), where \( S^1 \) acts on \( \mathbb{C} \) with an odd numbered weight \( n \), and \( \Phi_Y([g, \mu, z]) = \text{Ad}^l(g)(\mu + \frac{n}{2}|z|^2) \).
3. \( Y = \mathbb{C}^2 \times \mathbb{C} = \{ (u, v, w) | u, v, w \in \mathbb{C} \} \), where \( \text{SU}(2) \) acts on \( \mathbb{C}^2 \) as the standard unitary transformations of \( \mathbb{C}^2 \) to itself. The moment map on \( Y \) is \( \Phi(u, v, w) = \left( \frac{|u|^2-|v|^2}{2}, \text{Re}(u \bar{v}), \text{Im}(u \bar{v}) \right) \).

**Corollary 2.4.** Let \( (M, \omega, \Phi) \) be a six dimensional Hamiltonian SO(3)-manifold with a moment map \( \Phi : M \rightarrow \mathfrak{so}(3)^* \simeq \mathbb{R}^3 \). Assume the action is effective. Then for any \( x \in \Phi^{-1}(0) \), the local model for the orbit \( G \cdot x \) is one of the following:

1. \( Y = \text{SO}(3) \times_{\Gamma} \mathbb{R}^3 \), and \( \Phi_Y([g, \mu]) = \text{Ad}^l(g)\mu \).
2. \( Y = \text{SO}(3) \times_{S^1} (\mathbb{R}^2 \times \mathbb{C}) \), where \( S^1 \) acts on \( \mathbb{C} \) with weight \( n \), and \( \Phi_Y([g, \mu, z]) = \text{Ad}^l(g)(\mu + \frac{n}{2}|z|^2) \).
3. \( Y = \text{SO}(3) \times_{S^1} \mathbb{R}^2 \times \mathbb{C} \), where \( S^1 \) acts trivially on \( \mathbb{C} \), and \( \Phi_Y([g, \mu, z]) = \text{Ad}^l(g)\mu \).
4. \( Y = \text{SO}(3) \times_{N(S^1)} (\mathbb{R}^2 \times \mathbb{C}) \), where \( N(S^1) \) acts on \( \mathbb{C} \) as \( N(S^1)/S^1 \simeq \mathbb{Z}_2 \), and \( \Phi_Y([g, \mu, z]) = \text{Ad}^l(g)\mu \).
5. \( Y = \text{SO}(3) \times_{N(S^1)} \mathbb{R}^2 \times \mathbb{C} \), where \( N(S^1) \) acts trivially on \( \mathbb{C} \), and \( \Phi_Y([g, \mu, z]) = \text{Ad}^l(g)\mu \).
6. \( Y = \mathbb{C}^3 = \{ q + \sqrt{-1}p | q, p \in \mathbb{R}^3 \} \simeq T^*\mathbb{R}^3 \), where \( \text{SO}(3) \) acts on \( \mathbb{C}^3 = T^*\mathbb{R}^3 \) through its standard action on \( \mathbb{R}^3 \), and the moment map is the vector cross product, i.e. \( \Phi_Y(q, p) = q \times p \).

**Remark 2.5.** For convenience, from now on, we will refer to the local models using the expressions appeared in the above corollaries. In particular, the isotropy representations are implied when we use different expressions. For instance, the local model \( \mathbb{C}^2 \times \mathbb{C} \) has an \( \text{SU}(2) \) action on the first \( \mathbb{C}^2 \) while the local model \( \mathbb{C}^3 \) has a canonical \( \text{SO}(3) \) action. In addition, \( G \times H (\mathbb{R}^2 \times \mathbb{C}) \) always has a nontrivial action of \( H \) on \( \mathbb{C} \) for \( G = \text{SU}(2) \) or \( \text{SO}(3) \) and \( H = S^1 \) or \( \mathbb{R} \).

We can read out information from the local models. For example, since any moment fiber is connected, and since any orbit is closed, if there is only one orbit \( O \in \Phi^{-1}(0) \) sitting inside a local model, there is only one orbit in \( \Phi^{-1}(0) \) and therefore \( \Phi^{-1}(0) \) is a short fiber.

**Corollary 2.6.** Let \( G \) be \( \text{SU}(2) \) or \( \text{SO}(3) \). Let \( (M, \omega, \Phi) \) be a six dimensional Hamiltonian \( G \)-manifold. Assume the zero fiber is short. The local model for the single orbit in the zero fiber is either \( G \times \mathbb{R}^3 \), or \( G \times S^1 (\mathbb{R}^2 \times \mathbb{C}) \).
Corollary 2.7. Let $G$ be SU(2) or SO(3). Let $(M,\omega,\Phi)$ be a six dimensional Hamiltonian $G$-manifold. Assume an orbit $O$ in $\Phi^{-1}(0)$ is nonexceptional. Then the local model for $O$ is either $\text{SO}(3) \times_H \mathbb{R}^2 \times \mathbb{C}$ with $H = S^1$ or $\tilde{N}_{\text{SO}(3)}(S^1)$, or $\mathbb{C}^2 \times \mathbb{C}$.

The following theorem from Lerman and Sjamaar \cite{LS} states that the reduced space at 0 is stratified:

Theorem 2.8. Let $G$ act on a symplectic manifold $M$ with a proper moment map. The reduced space $M_0 = \Phi^{-1}(0)/G$ can be decomposed into a disjoint union of symplectic manifolds with respect to the stabilizers, i.e.

$$M_0 = \bigsqcup_{H < G} (M_0)_H$$

where $(M_0)_H$ denotes all the orbits in the zero fiber whose stabilizer is conjugate to $H$.

Moreover, there exists a unique piece $(M_0)_H$ which is open, connected, and dense in the reduced space.

Definition 2.9. The stabilizer $H$ of the unique piece $(M_0)_H$ stated in Theorem 2.8 is called the principal stabilizer of the zero fiber.

Corollary 2.10. Let $G$ be SU(2) or SO(3). Let $(M,\omega,\Phi)$ be a six dimensional Hamiltonian $G$-manifold. Assume that the zero fiber $\Phi^{-1}(0)$ is tall. Then the principal stabilizer of the zero fiber is either

- $\text{SU}(2)$, when $G = \text{SU}(2)$,
- $S^1$, or $\text{SO}(3)$ when $G = \text{SO}(3)$.

Corollary 2.11. Let $G$ be SU(2) or SO(3). Let $(M,\omega,\Phi)$ be a six dimensional Hamiltonian $G$-manifold. Assume the zero fiber $\Phi^{-1}(0)$ is tall. Then

- there is no exceptional orbit in the zero fiber when $G = \text{SU}(2)$;
- when $G = \text{SO}(3)$, if there exists an exceptional orbit in the zero fiber, the local model for the exceptional orbit is either $\text{SO}(3) \times N_G(S^1) \mathbb{R}^2 \times \mathbb{C}$ or $\mathbb{C}^3$.

In this case, the principal stabilizer of the zero fiber is $S^1$.

The list of local models and their properties is important and useful later on. We use Table 1 for an easy reference.

3. Eliminating the symplectic form

An equivariant symplectomorphism between two Hamiltonian $G$-manifolds preserves the orientations. Using Moser's method \cite{W1}, we show that under appropriate circumstances, we can recover the symplectic form from an orientation preserving equivariant diffeomorphism that respects the moment maps. This enables us to use the techniques in differential topology in later sections.

Definition 3.1. Let a compact Lie group $G$ act on oriented manifolds $M$ and $M'$ with $G$-equivariant maps $\Phi: M \to g^*$ and $\Phi': M' \to g^*$. A $\Phi$-$G$-diffeomorphism $\Psi: M \to M'$ is an orientation preserving equivariant diffeomorphism such that $\Psi^*(\Phi') = \Phi$. 
Here we assume a technical condition:

\[\text{The restriction map } H^i(N/G) \to H^i((\Phi^{-1}(G \cdot \alpha))/G), \quad i = 1, 2\]

\[\text{(3.1)}\]

is one-to-one with \(\mathbb{Z}\) or \(\mathbb{Z}_2\) coefficient

for some neighborhood \(N\) of the zero fiber and \(\forall \alpha \in \Phi(N) \subset g^*\).

We will prove later in this paper that such a neighborhood \(N\) always exists and this condition is satisfied.

Remark 3.2. Throughout this paper, when we mention a neighborhood \(N\) of the zero fiber, we always assume that \(N\) is small enough so that it can be covered by invariant open subsets of the local models in Table 1.

**Proposition 3.3.** Let \(G\) be SU(2) or SO(3). Let \((M, \omega, \Phi)\) and \((M', \omega', \Phi')\) be neighborhoods of the zero fibers in six dimensional Hamiltonian \(G\)-manifolds such that \(0 \in \Phi(M) = \Phi(M')\). Assume that \(M\) and \(M'\) satisfy Condition 3.1 and that their Duistermaat-Heckman functions coincide. Then there exist neighborhoods \(N \subset M\) and \(N' \subset M'\) of the zero fibers such that there exists an equivariant symplectomorphism from \(N\) to \(N'\) if and only if there exists a \(\Phi\)-G-diffeomorphism from \(N\) to \(N'\).

We prove this with two lemmas. First we recall that a differential form \(\beta\) on \(M\) is basic if it is \(G\)-invariant and \(i(\xi_M)\beta = 0\) for every \(\xi \in g\) and \(\xi_M\) its induced vector field on \(M\). The basic forms on \(M\) give rise to a differential complex whose cohomology coincides with the Čech cohomology of the topological quotient \(M/G\) (see [K2]).

**Lemma 3.4.** Let \(G\) be a compact Lie group. Let \((N, \omega, \Phi)\) and \((N', \omega', \Phi')\) be neighborhoods of the zero fibers in six dimensional Hamiltonian \(G\)-manifolds such that \(0 \in \Phi(N) = \Phi'(N')\). Assume that \(N\) and \(N'\) satisfy Condition 3.1 and that
their Duistermaat-Heckman functions coincide. Then for any $\Phi$-$G$-diffeomorphism $f: N \to N'$, there exists a basic one-form $\beta$ such that $d\beta = f^*\omega' - \omega$.

Proof. Consider the closed two-form $\Omega = f^*\omega' - \omega$. Since $f$ commutes with the group action and both $\omega$ and $\omega'$ are $G$-invariant, $\Omega$ is also invariant. Using the fact that $f^*\omega'$ and $\omega$ have the same moment map $\Phi$ and $\iota(\xi_N)\omega = -d(\Phi, \xi)$, we have $\iota(\xi_N)\Omega = 0$ for all $\xi \in \mathfrak{g}$. So $\Omega$ is basic and is a pull-back of a two-form $\tilde{\omega}$ on $N/G$.

By Condition (3.1), it suffices to show that the restriction of $\tilde{\Omega}$ to the reduced space $\Phi^{-1}(G \cdot \alpha)/G$ is exact at some regular value $\alpha \in U$. Since the reduced space is two dimensional, it is enough to show that the integral of $\tilde{\Omega}$ over the reduced space is zero. That is, the symplectic volumes of the reduced spaces are the same. This follows from the fact that the Duistermaat-Heckman functions coincide at $\alpha$.

So $\tilde{\Omega} = d\tilde{\beta}$ for some $\tilde{\beta}$, and we pull back to obtain $\Omega = \pi^*d\tilde{\beta} = d(\pi^*\tilde{\beta}) = d\beta$, where $\beta = \pi^*\tilde{\beta}$ is a basic one-form. \[\square\]

Lemma 3.5. Let $G = SU(2)$ or $SO(3)$ act effectively on a six dimensional manifold $M$. Let $\omega_0$ and $\omega_1$ be two $G$-invariant symplectic forms on $M$ with the same moment map $\Phi$ and assume that $\omega_0$ and $\omega_1$ induce the same orientation. Then the $G$-invariant two-form $\omega_t = (1 - t)\omega_0 + t\omega_1$ is nondegenerate for all $0 \leq t \leq 1$ on a neighborhood of the zero fiber $\Phi^{-1}(0)$.

Proof. Nondegeneracy is a local condition. To show that $\omega_t$ is nondegenerate on a neighborhood of the zero fiber, it is enough to show that it is nondegenerate on the zero fiber. Consider a point $x \in \Phi^{-1}(0)$ such that the local model for the orbit $G \cdot x$ is $Y = G \times_H (\mathfrak{h}^0 \times \mathbb{C}^n)$. The tangent space at $x$ splits as $T_xY = \mathfrak{g}/\mathfrak{h} \times \mathfrak{h}^0 \times \mathbb{C}^n$. The two-form $\omega_t$ at $x$ is then of the form

$$
\begin{pmatrix}
0 & I & 0 \\
-1 & * & * \\
0 & * & \tilde{\omega}_t
\end{pmatrix}
$$

where $\tilde{\omega}_t$ is an $H$-invariant linear symplectic form on $\mathbb{C}^n$ and $I$ is the natural pairing between the tangent space $\mathfrak{g}/\mathfrak{h}$ and its dual $\mathfrak{h}^0$. So the two-form $\omega_t$ is nondegenerate if and only if the corresponding $\tilde{\omega}_t$ is nondegenerate.

Case 1: When the local model is $G \times_{A^t} \mathbb{R}^3$, the two-form $\omega_t = \begin{pmatrix} 0 & I \\ -I & * \end{pmatrix}$ is nondegenerate for all $t$.

Case 2: When the local model is $G \times_H (\mathbb{R}^2 \times \mathbb{C})$ or $G \times_H \mathbb{R}^2 \times \mathbb{C}$ with $H = S^1$ or $N(S^1)$, the two-form $\tilde{\omega}_t$ is a linear symplectic form defined on $\mathbb{C}$ and therefore is $A^tdz \wedge d\bar{z}$ for some constant $A^t$. By definition, $A^t = (1 - t)A_0^t + tA_1^t$. Since $\omega_0$ and $\omega_1$ induce the same orientation, $A_0^tA_1^t > 0$. So $A^t$ never vanishes for $0 \leq t \leq 1$, and $\tilde{\omega}_t$ is nondegenerate for $0 \leq t \leq 1$.

Case 3: When the local model is $\mathbb{C}^3$, we can translate $\tilde{\omega}_t$ to any other point in $\mathbb{C}^3$. In particular we can translate it to an $x$ in the zero fiber with $S^1$ stabilizer; see Table 1. Case 2 then applies.

Case 4: The local model $Y$ is $\mathbb{C}^2 \times \mathbb{C}$. Let $\omega_0$ and $\omega_1$ be $SU(2)$-invariant symplectic forms on $\mathbb{C}^2 \times \mathbb{C}$ that have the same moment map and induce
Consider \( \omega_t = (1 - t)\omega_0 + t\omega_1 \). Then \( \omega_t \) can be written as

\[
\omega_t = \sqrt{-1} \sum_{i=1}^{3} A_i^t dz_i \wedge d\bar{z}_i \\
+ \sqrt{-1} \sum_{i<j} \left( B_{ij}^t dz_i \wedge dz_j + B_{ij}^t d\bar{z}_i \wedge d\bar{z}_j \right) \\
+ \sqrt{-1} \sum_{i<j} \left( C_{ij}^t dz_i \wedge d\bar{z}_j + C_{ij}^t d\bar{z}_i \wedge dz_j \right)
\]

Because \( \omega_t \) is invariant under SU(2) action, it is invariant under the transformations \((0 \ 0), (\cos \theta \ \sin \theta), \text{ and } (\cos \theta \ -i \sin \theta)\) for any \( \theta \). It follows that \( A_1^t = A_2^t \), and \( B_{ij}^t = C_{ij}^t = 0 \) for \( i < j = 3 \). The induced vector fields for the above circle actions are

\[
X_{\xi_1} = \sqrt{-1} \left( z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial \bar{z}_2} + \bar{z}_3 \frac{\partial}{\partial \bar{z}_2} \right) \\
X_{\xi_2} = \left( z_2 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_2} - z_3 \frac{\partial}{\partial \bar{z}_1} - \bar{z}_3 \frac{\partial}{\partial \bar{z}_1} \right) \\
X_{\xi_3} = \sqrt{-1} \left( z_3 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial \bar{z}_1} + \bar{z}_2 \frac{\partial}{\partial \bar{z}_1} \right)
\]

respectively. Because \( \iota(X_{\xi_i})\omega_t = -d\Phi_t^{\xi_i} \), direct computation shows that the coefficients \( A_1^t, A_2^t, B_{12}^t, \text{ and } C_{12}^t \) are determined by the moment map and thus independent of \( t \).

Note \( A_3^t = (1 - t)A_3^0 + tA_3^1 \). Since \( \omega_0 \) and \( \omega_1 \) induce the same orientation, \( A_3^0 A_3^1 > 0 \). So \( A_3^t \) never vanishes and \( \omega_t \) is nondegenerate.

**Proof of Proposition 3.3.** Suppose \( f \) is a \( \Phi \)-G-diffeomorphism from \( (M, \omega, \Phi) \) to \( (M', \omega', \Phi') \). By Lemma 3.4, there exists a basic one-form \( \beta \) such that \( d\beta = f^* \omega' - \omega \). Consider \( \omega_t = (1 - t)\omega + t f^* \omega' \) for \( 0 \leq t \leq 1 \). Then \( \omega_0 = \omega \) and \( \omega_1 = f^* \omega' \) have the same moment map \( \Phi \) and induce the same orientation. By Lemma 3.5, there exist a neighborhood of the zero fiber \( N \subset M \) on which \( \omega_t \) is nondegenerate. We can solve \( \iota(X_t)\omega_t = -\beta \) for the time dependent vector field \( X_t \). Denote \( \varphi^t \) the flow of this vector field \( X_t \) satisfying \( \varphi^0 = \text{id} \). Because \( \langle d\Phi(X_t), \xi \rangle = -\omega_t(\xi_N, X_t) = (\iota(X_t)\omega_t)(\xi_M) = -\beta(\xi_M) = 0 \), the flow \( \varphi^t \) preserves the fibers of the moment map \( \Phi \). Since \( \Phi \) is proper, \( \varphi^t \) exists for all \( 0 \leq t \leq 1 \).

Define \( F_t = f \circ \varphi^t \). Since \( F_t^* \Phi' = \Phi' \circ F_t = \Phi' \circ f \circ \varphi^t = \Phi \circ \varphi^t = \Phi \), we know that \( F_t: N \to N' \) respects the moment maps for \( N' = f(N) \). Since \( \omega_t \) and \( \beta \) are invariant, \( X_t \) is invariant. So \( \varphi^t \) and hence \( F_t \) is \( G \)-equivariant. Finally

\[
\frac{d}{dt} (f^* \omega_t) = \varphi^*_t L_{X_t} \omega_t + \varphi^*_t \left( \frac{d}{dt} \omega_t \right) \\
= \varphi^*_t \left( L_{X_t} \omega_t + \frac{d}{dt} \omega_t \right) \\
= \varphi^*_t (dL(X_t)\omega_t - \omega + f^* \omega') \\
= \varphi^*_t (d(-\beta) + d\beta) \\
= 0
\]
So $\varphi^* \omega_1 = \varphi^* \omega_0 = \varphi^* (f \circ \varphi_1)^* \omega' = F_1^* \omega' = \omega$ and $F_1 : N \to N'$ is an equivariant symplectomorphism that respects the moment maps. $\square$

4. Passing to the Quotient

In this section, we show that it is enough to work with a specific class of diffeomorphisms of the quotients rather than $\Phi$-$G$-diffeomorphisms.

Let $G = SU(2)$ or $SO(3)$ act on a manifold $M$. With quotient topology, the quotient $M/G$ has a natural smooth structure; a function is smooth if and only if its pull-back to $M$ is smooth. A smooth function between quotients is a map $f : M/G \to M'/G$ that pulls back smooth functions to smooth functions. A smooth function $f$ between quotients is a diffeomorphism if it is smooth and has a smooth inverse. If $M$ and $M'$ are oriented, the smooth parts of the quotients $M/G$ and $M'/G$ can be oriented with a choice of the orientation on $G$. Whether or not a diffeomorphism $f : M/G \to M'/G$ preserves the orientation is independent of that choice.

While this notion of diffeomorphism is natural, we will use a stronger notion of $\Phi$-diffeomorphisms which allows us to have a better control over neighborhoods of the exceptional orbits.

First we observe that when $G = SU(2)$ or $SO(3)$, we can identify the dual of its Lie algebra $g^*$ with $\mathbb{R}^3$ and its orbit space $g^*/G$ under the coadjoint action with $\mathbb{R}_+ \simeq [0, \infty)$. Since the moment map $\Phi$ is equivariant, it induces a map $\Phi : M/G \to \mathbb{R}_+$ such that the following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\Phi} & g^* = \mathbb{R}^3 \\
\downarrow{\pi} & & \downarrow{p} \\
M/G & \xrightarrow{\Phi} & g^*/G = \mathbb{R}_+
\end{array}
$$

where $p$ is the norm square $|\xi|^2$ and $\Phi([m]) = |\Phi(m)|^2$.

**Definition 4.1.** Let $G$ be $SU(2)$ or $SO(3)$. Let $M$ and $M'$ be oriented manifolds with $G$ actions and $G$-equivariant maps $\Phi : M \to g^*$ and $\Phi' : M' \to g^*$. A $\Phi$-diffeomorphism from $M/G$ to $M'/G$ is an orientation preserving diffeomorphism $\psi : M/G \to M'/G$ such that

1. $\psi^* \Phi' = \Phi$.
2. $\psi$ and $\psi^{-1}$ lift to a $\Phi$-$G$-diffeomorphism in a neighborhood of each exceptional orbit.

We start with a series of lemmas.

**Lemma 4.2.** Let $G$ be $SU(2)$ or $SO(3)$ and $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold. Let $Y$ be a local model for a nonexceptional orbit in the zero fiber of the moment map. Let $W$ and $W'$ be invariant open subsets of $Y$. Let $f : W/G \to W'/G$ be a $\Phi$-diffeomorphism. Then $f$ lifts to a $\Phi$-$G$-diffeomorphism $F : W \to W'$.

**Proof.** When $G = SU(2)$, the only possible local model for a nonexceptional orbit in the zero fiber is $Y = \mathbb{C}^2 \times \mathbb{C}$. The quotient $Y/G$ can be identified as $\mathbb{C}^2 / G \times \mathbb{C}$. The $\Phi$-diffeomorphism $f : Y/G \to Y'/G$ necessarily has the form $f([w], z) =$
([w, \varphi([w], z)) for some diffeomorphism \varphi: \mathbb{C}^2/G \times \mathbb{C} \to \mathbb{C}. It locally lifts to a \Phi-G-diffeomorphism \text{F}(w, z) = (w, \varphi([w], z)) for w \in \mathbb{C}^2 and z \in \mathbb{C}.

When G = \text{SO}(3), the local model for a nonexceptional orbit is G \times \mathbb{R}^2 \times \mathbb{C} with H = S^1 or N_G(S^1). The quotient Y/G can be identified as \mathbb{R}^2/H \times \mathbb{C}. Then the \Phi-diffeomorphism \text{f}: Y/G \to Y'/G can be written as \text{f}([\mu], z) = ([\mu], \varphi([\mu], z)) for some \varphi: \mathbb{R}^2/H \times \mathbb{C} \to \mathbb{C}. It locally lifts to a \Phi-G-diffeomorphism \text{F}(g, \mu, z) = ([g, \mu], \varphi([\mu], z)).

\textbf{Lemma 4.3.} Let G = \text{SU}(2) act effectively on a six dimensional symplectic manifold (M, \omega). Assume the action is Hamiltonian and the moment map is \Phi. Consider the local model Y = \mathbb{C}^2 \times \mathbb{C} for a nonexceptional orbit in the zero fiber. Let \text{F}: Y \to Y be an equivariant diffeomorphism that preserves the orbits and respects the moment maps. Extend the \text{SU}(2) action to U(2). Then there exists a smooth invariant function h: Y \to S^1 such that \text{F}(y) = h(y) \cdot y where the S^1 action on Y is induced by U(2) on \mathbb{C}^2.

\textbf{Proof.} Let \text{F}: Y \to Y be an equivariant diffeomorphism that preserves the orbits and respects the moment maps. Identify the local model for a nonexceptional orbit as Y = \mathbb{C}^2 \times \mathbb{C}. Then \text{F} takes (w, z) to some (w', z') and the following diagram commutes,

\begin{align*}
\begin{array}{ccc}
\mathbb{C}^2 \times \mathbb{C} & \xrightarrow{\text{F}} & \mathbb{C}^2 \times \mathbb{C} \\
\downarrow & & \downarrow \\
\mathbb{C}^2/G \times \mathbb{C} & \xrightarrow{\text{id}} & \mathbb{C}^2/G \times \mathbb{C}
\end{array}
\end{align*}

where \pi(w, z) = ([w], z). Therefore z' = z, and w' = hw for some h \in U(2). Since \text{F} is equivariant, hw = ghw for all g in \text{SU}(2). And hence h belongs to the center of U(2), i.e., h \in S^1.

\textbf{Lemma 4.4.} Let G denote \text{SU}(2). Let (N, \omega, \Phi) and (N', \omega', \Phi') be neighborhoods of all zero fibers in six dimensional Hamiltonian G-manifolds. Assume that N and N' satisfy Condition 3.1 and their Duistermaat-Heckman functions are the same. Then every \Phi-diffeomorphism from N/G \to N'/G that locally lifts to a \Phi-G-diffeomorphism globally lifts to a \Phi-G-diffeomorphism.

\textbf{Proof.} Assume \psi: N/G \to N'/G is a \Phi-diffeomorphism. Choose an open invariant cover \mathcal{U} of N such that \text{U}_i \cap \Phi^{-1}(0) \neq \emptyset for each \text{U}_i \in \mathcal{U}. Take a refinement if necessary, we can assume that each \text{U}_i is an invariant open subset of the local model \mathbb{C}^2 \times \mathbb{C}; see Table 1. By Lemma 4.2, \psi locally lifts to a \Phi-G-diffeomorphism \Psi_i: \text{U}_i \to N' for each \text{U}_i. By Lemma 4.3, there exists smooth invariant functions \text{f}_{ij}: \text{U}_i \cap \text{U}_j \to S^1 such that \Psi_i = \text{f}_{ij} \cdot \Psi_j. The collection of \{\text{f}_{ij}\} form a \check{C}ech cocycle on \mathcal{U} with coefficient in S^1. If the corresponding class \{g\} \in \check{H}^1(N/G, S^1) is trivial, \psi lifts to a global \Phi-G-diffeomorphism.

The short exact sequence 0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0 determines a long exact sequence in cohomology. Since there exists a smooth partition of unity on N/G, \check{H}^i(N/G; \mathbb{R}) = 0 for all i > 0. In other words, \check{H}^1(N/G; S^1) = \check{H}^2(N/G; \mathbb{Z}). According to Condition 3.1, the restriction \check{H}^2(N/G; \mathbb{Z}) \to \check{H}^2(S; \mathbb{Z}) is one-to-one, where S = \Phi^{-1}(G \cdot \alpha)/G = \Phi^{-1}(\alpha)/S^1 is a regular reduced space; we only need to show that the image of \{g\} in \check{H}^2(S; \mathbb{Z}) vanishes. Let \{\lambda_j\} be a partition of unity.
subordinate to the cover \( U \cap \Phi^{-1}(\alpha) \). The Čech-de Rham isomorphism takes the image of \( [g] \) to the cohomology class of the basic differential two-form

\[
\Sigma_j d\lambda_j g^{-1}_{ij} dg_{ij}
\]

when restricted to the open set \( U_j \cap \Phi^{-1}(\alpha) \). It is equal to the difference between the curvature form \( d\Theta \) and the pullback \( \Psi^*d\Theta' \), where \( \Theta' \) is a connection one-form on \( \Phi^{-1}(\alpha) \) and \( \Theta = \sum \lambda_i \Psi_i \Theta' \) is a connection one-form on \( \Phi^{-1}(\alpha) \). The integrals of the curvature forms over the reduced spaces are equal to the slopes of the Duistermaat-Heckman functions at \( \alpha \). Since the Duistermaat-Heckman functions are the same, (4.2) is exact as a basic form and \( [g] = 0 \).

**Lemma 4.5.** Let \( G = SO(3) \) act effectively on a six dimensional symplectic manifold \((M, \omega)\). Assume the action is Hamiltonian and the moment map is \( \Phi \). Consider the local models for nonexceptional orbits of the zero fiber \( Y = SO(3) \times_H \mathbb{R}^2 \times \mathbb{C} \) with \( H = S^1 \) or \( N(S^1) \). Let \( F: Y \to Y \) be an equivariant diffeomorphism that preserves the orbits and respects the moment maps. Then there exists a smooth invariant function \( h: Y \to N(S^1)/H \) such that \( F(y) = h(y) \cdot y \) where the action is induced by the extension of the \( H \) action on \( SO(3) \times \mathbb{R}^2 \times \mathbb{C} \) to an action of \( N(S^1) \).

**Proof.** Let \( F: Y \to Y \) be an equivariant diffeomorphism that preserves the orbits and respects the moment maps. Identify the local model for a nonexceptional orbit as \( Y = (SO(3) \times_H \mathbb{R}^2) \times \mathbb{C} \) where \( H = S^1 \) or \( N(S^1) \). Then \( F \) takes \( ([g, \mu], z) \) to some \( ([g', \mu'], z') \) and the following diagram commutes,

\[
\begin{array}{ccc}
(SO(3) \times_H \mathbb{R}^2) \times \mathbb{C} & \xrightarrow{F} & (SO(3) \times_H \mathbb{R}^2) \times \mathbb{C} \\
\pi & & \pi \\
\mathbb{R}^2/H \times \mathbb{C} & \xrightarrow{id} & \mathbb{R}^2/H \times \mathbb{C}
\end{array}
\]

where \( \pi([g, \mu], z) = ([\mu], z) \). Since \( F \) is equivariant, \( g' = gh \) for some \( h \) in \( SO(3) \). \( F \) respects the moment maps, so \( Ad^\mu h \mu = Ad^{g'}(g'h') \) and \( \mu' = g^{-1} g \mu = h^{-1} \mu g \mu = h^{-1} \mu. \) Since \( [\mu'] = [\mu] = [h^{-1} \mu], h \in N(S^1) \). Now \( [gh, h^{-1} \mu] = [g, \mu] \) if \( h \in H \). Hence \( F([g, \mu], z) = ([gh, h^{-1} \mu], z) \) where \( h: Y \to N(S^1)/H \).

**Definition 4.6.** For any real \( n \) dimensional vector bundle \( \pi: W \to X \), there exists an associated orientation bundle \( p: \tilde{X} \to X \), whose fiber over a point \( x \) is the two ways to orient \( \pi^{-1}(x) \). This is a two-sheeted covering, and Čech cocycles provide a convenient way to construct it. Choose an open cover \( \mathcal{U} = \{U_a\} \) of \( X \) with trivialization maps \( \varphi_a: U_a \times \mathbb{R}^n \to \pi^{-1}(U_a) \). The Jacobian determinants of the change of fiber coordinates from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) have a locally constant sign, which gives a locally constant function from \( U_a \cap U_b \to \mathbb{Z}_2 \). The chain rule for Jacobians implies that this is a cocycle. It determines an element \( w_1(W) \in H^1(X; \mathbb{Z}_2) \), called the **first Stiefel-Whitney class** of \( W \). In a similar fashion, we can construct the associated orientation bundle and define the first Stiefel-Whitney class of any fiber bundle \( W \) over a manifold \( X \) provided that the fiber of \( W \) is connected and orientable.

In particular, when the zero fiber of a Hamiltonian \( SO(3) \)-manifold is tall, and when the principal stabilizer of the zero fiber is \( S^1 \), the zero fiber \( \Phi^{-1}(0) \) is a sphere bundle over the reduced space off the exceptional orbits. Let \( E = \{E_j\} \) denote the set of exceptional orbits in the zero fiber. Let \( M^\text{reg}_0 \) denote the smooth
Lemma 4.7. Let $G$ denote $\text{SO}(3)$. Let $(N, \omega, \Phi)$ and $(N', \omega', \Phi')$ be neighborhoods of zero fibers which satisfy Condition 3.1 in six dimensional Hamiltonian $G$-manifolds. Assume that the zero fibers are tall with the same principal stabilizer, and that they have no exceptional orbits. Then every $\Phi$-diffeomorphism from $N/G \to N'/G$ that locally lifts to a $\Phi$-$G$-diffeomorphism globally lifts to a $\Phi$-$G$-diffeomorphism if one of the following conditions holds:

1. the principal stabilizer of the zero fibers is $N(S^1)$; or
2. the principal stabilizer of the zero fibers is $S^1$ and the first Stiefel-Whitney class on $\Phi^{-1}(0) \to \Phi^{-1}(0)/G$ equals the pull-back of the first Stiefel-Whitney class on $\Phi'^{-1}(0) \to \Phi'^{-1}(0)/G$.

Proof. Assume $\psi: N/G \to N'/G$ is a $\Phi$-diffeomorphism. Choose an open invariant cover $\mathcal{U}$ of $N$ such that $U_i \cap \Phi^{-1}(0) \neq \emptyset$ for each $U_i \in \mathcal{U}$. Take a refinement if necessary, we can assume that each $U_i$ is an invariant open subset of the local model $G \times_H \mathbb{R}^2 \times \mathbb{C}$ where $H$ is $S^1$ or $N(S^1)$; see Table 1. By Lemma 4.2, $\psi$ locally lifts to a $\Phi$-$G$-diffeomorphism $\Psi_i: U_i \to N'$. By Lemma 4.5, there exists smooth invariant functions $f_{ij}: U_i \cap U_j \to N(S^1)/H$ such that $\Psi_i = f_{ij} \cdot \Psi_j$ where $H = S^1$ or $N(S^1)$.

Case 1: $H = N(S^1)$. Then $f_{ij}$ is the identity for all $i, j$ and $\Psi_i = \Psi_j$ on $U_i \cap U_j$. Therefore $\psi$ lifts globally.

Case 2: $H = S^1$. The set $\{f_{ij}\}$ defines a Čech cocycle $f \in \check{C}^1(\mathcal{U}; \mathbb{Z}_2)$. Then there exists a global lift if the class $[f] \in \check{H}^1(N/G; \mathbb{Z}_2)$ is trivial. By Condition 3.1, the restriction $H^1(N/G; \mathbb{Z}_2) \to H^1(\Phi^{-1}(0)/G; \mathbb{Z}_2)$ is one-to-one. It suffices to show that the image of $[f]$ in $H^1(\Phi^{-1}(0)/G; \mathbb{Z}_2)$ is trivial.

Let $X = \Phi^{-1}(0)/G$ and $X' = \Phi'^{-1}(0)/G$. Let $\pi: \tilde{X} \to X$ and $\pi': \tilde{X}' \to X'$ be the associated orientation bundles described in Definition 4.6. Refine the cover if necessary so that we have a good cover (simply connected, locally path connected, and each intersection is connected), again denoted by $\mathcal{U}$, on $\Phi^{-1}(0)/G$ and $\mathcal{U}' = \psi(\mathcal{U})$ also a good cover on $\Phi'^{-1}(0)/G$. Let $\varphi_i: \pi^{-1}(U_i) \to U_i \times \mathbb{Z}_2$ and $\varphi'_i: \pi'^{-1}(U'_i) \to U'_i \times \mathbb{Z}_2$ be the trivialization maps for these two bundles subject to the good covers. Let $\{g_{ij}\}$ and $\{g'_{ij}\}$ be the transition functions for $\{\varphi_i\}$ and $\{\varphi'_i\}$. Then $\{g_{ij}\}$ is a Čech cocycle for the first Stiefel-Whitney class $w_1(\Phi^{-1}(0))$ and $\{g'_{ij}\}$ for $w_1(\Phi'^{-1}(0))$.

The proof of Lemma 4.5 implies that there exists a constant function $h_i: U_i \to \mathbb{Z}_2$ induced by the restriction of a local $\Phi$-$G$-diffeomorphism $\Psi_i$ to the zero fiber. $h_i$ maps $(x, t) \in U_i \times \mathbb{Z}_2$, a local chart of the associated orientation bundle, to $(\psi(x), h_i + t) \in U'_i \times \mathbb{Z}_2$. Then $h_i + g'_{ij} \psi - h_j - g_{ij} = f_{ij}$. The class $[f]$ vanishes exactly when the first Stiefel-Whitney classes are the same.

□
Lemma 4.9. Let \( U = M \times \mathbb{Z}_2 \) be determined by \( M \). The zero fiber is determined by \( w \) and \( k \). The zero fiber is \( P \times \mathbb{Z}_2 \) where \( w \) is determined by \( M \). The zero fiber is determined by \( w_1 \) and \( k \). The zero fiber is \( P \times \mathbb{Z}_2 \), a sphere bundle that determines \( w_1 \), and the quotient \( M/\text{SO}(3) = \Sigma \times \mathbb{R}_+ \) determines the genus.

Lemma 4.9. Let \( G \) denote \( \text{SO}(3) \). Let \((N, \omega, \Phi)\) and \((N', \omega', \Phi')\) be neighborhoods of zero fibers which satisfy Condition 3.1 in six dimensional Hamiltonian \( G \)-manifolds. Assume that the zero fibers are tall and that there exist exceptional orbits \( E = \{E_j\} \) on the zero fiber of \( \Phi \) and \( E' = \{E'_j\} \) of \( \Phi' \). Every \( \Phi \)-diffeomorphism from \( N/G \to N'/G \) that locally lifts to a \( \Phi \)-\( G \)-diffeomorphism globally lifts to a \( \Phi \)-\( G \)-diffeomorphism if the first Stiefel-Whitney class for \( \Phi^{-1}(0) \setminus E \) equals the pull-back of the first Stiefel-Whitney class for \( \Phi'^{-1}(0) \setminus E' \).

Proof. The assumption that there exist exceptional orbits implies that the principal stabilizers of the zero fibers are \( S^1 \). After passing to a suitable finer covering \( U = \{U_i\} \) if necessary, we can assume that on \( \Phi^{-1}(0)/G \), each \( U_i \) is simply connected and locally path connected, and that \( U_i \cap U_j \) contains no exceptional orbits. We then apply a similar argument as before on \( \Phi^{-1}(0) \setminus E \).

Remark 4.10. There are two types of exceptional orbits that can occur in a tall zero fiber with \( S^1 \) stabilizer.

The first type of the exceptional orbits are the isolated fixed points in \( \Phi^{-1}(0) \). It corresponds to \( \{0\} \) in the local model \( Y = \mathbb{C}^3 \simeq T^* \mathbb{R}^3 = \{(q,p)\} \). Define the map

\[
\eta: \Phi_Y^{-1}(0) \setminus \{0\} \to \frac{\Phi^{-1}(0) \setminus \{0\}}{\text{SO}(3)} \to \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}
\]

by \((q,p) \mapsto \text{orbit}(q,p) = \text{orbit}(u,0,0,v,0,0) \mapsto [u,v] = [r,\theta] \mapsto (r,2\theta)\). Let \( I \) denote the interval \([0,\pi]\). Then there exists a commutative diagram

\[
\begin{array}{ccc}
I \times S^2 & \xrightarrow{f} & \eta^{-1}(S^1) \\
\downarrow g & & \downarrow \eta^{-1}(S^1) \\
(0,x) \sim (\pi,-x) & \xrightarrow{\eta^{-1}(S^1)} &
\end{array}
\]

where \( f(t,x) = ((\cos t)x,(\sin t)x) \) induces a homeomorphism \( g \). In other words, the product of the values of the \( \{f_{ij}\} \) defined in Lemma 4.5 as one moves along a loop around \( E/G \) in \( \Phi^{-1}(0)/G \) is the nontrivial element in \( \mathbb{Z}_2 \).

The second type of the exceptional orbits are those whose stabilizer is \( N(S^1) \) and whose local model is \( \text{SO}(3) \times_{\text{SO}(S^1)} (\mathbb{R}^2 \times \mathbb{C}) \). The local model immediately implies that the product of the values of the \( \{f_{ij}\} \) defined in Lemma 4.5 as one moves along a loop around \( E/G \) in \( \Phi^{-1}(0)/G \) is the nontrivial element in \( \mathbb{Z}_2 \).

In fact, blowing up an isolated fixed point gives us an exceptional orbit of the second type.
Proposition 4.11. Let $G$ denote SU(2) or SO(3). Let $(N, \omega, \Phi)$ and $(N', \omega', \Phi')$ be neighborhoods of tall zero fibers in six dimensional Hamiltonian $G$-manifolds. Assume that $N$ and $N'$ satisfy Condition 3.1 and that their Duistermaat-Heckman functions are the same. Assume that the zero fibers $\Phi^{-1}(0)$ and $\Phi'^{-1}(0)$ have the same principal stabilizer and the same first Stiefel-Whitney class when applicable. Then every $\Phi$-diffeomorphism from $N/G$ to $N'/G$ globally lifts to a $\Phi$-$G$-diffeomorphism.

Proof. This is a direct result from Lemmas 4.2, 4.4, 4.7, and 4.9. □

5. The topology of the quotient

Let $G$ be SU(2) or SO(3). Let $(N, \omega, \Phi)$ and $(N', \omega', \Phi')$ be neighborhoods of tall zero fibers in six dimensional Hamiltonian $G$-manifolds. By Propositions 3.3 and 4.11, we have shown that as long as $N$ and $N'$ have the same Duistermaat-Heckman function, and their zero fibers have the same principal stabilizer as well as the same first Stiefel-Whitney class, $N$ and $N'$ are isomorphic if their quotients $N/G$ and $N'/G$ are $\Phi$-diffeomorphic. In the following sections, we will show that the isotropy data and the genus determine the quotients up to $\Phi$-diffeomorphisms. First, we describe the topology of the quotients.

Proposition 5.1. Let $G$ be SU(2) or SO(3) and let $(N, \omega, \Phi)$ be a $G$-invariant neighborhood of the zero fiber in a six dimensional Hamiltonian $G$-manifold. Assume that the zero fiber is tall. The quotient $N/G$ is topologically a manifold with boundary. In particular, each reduced space $\Phi^{-1}(G \cdot \alpha)/G$ is topologically a closed connected oriented surfaces for $\alpha \in \Phi(N)$.

Proof. By the Local Normal Form Theorem 2.1 and Lemma 5.2 below, the quotient $N/G$ is topologically a manifold with boundary.

By the Local Normal Form Theorem 2.1 and Corollary 5.3, the reduced space is a topological surface. Since the moment map is proper and since every moment fiber is connected, $\Phi^{-1}(G \cdot \alpha)/G$ is closed and connected. It is oriented since the symplectic structure induces an orientation. □

Lemma 5.2. Let $G$ be SU(2) or SO(3). Identify $g^*/G$ with $\mathbb{R}^+$. Let $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold with a tall zero fiber. For any local model $Y$ of an orbit in $\Phi^{-1}(0)$, there exists a map $F_Y = (\overline{\Phi}_Y, \overline{P}_Y): Y/G \to \mathbb{R}^+ \times \mathbb{C}$ homeomorphic into its image, where $\overline{\Phi}_Y: Y/G \to \mathbb{R}^+$ is induced by the moment map, and $\overline{P}_Y$ is induced from a $G$-invariant map $P_Y: Y \to \mathbb{C}$.

We call such an $F_Y$ a trivializing homeomorphism of the local model $Y$.

Corollary 5.3. The restriction of $F_Y$ to the reduced space is a homeomorphism for all $\alpha$ such that $|\alpha|^2 \in \text{image } \overline{\Phi}_Y = \text{image } |\Phi_Y|^2$.

Proof of Lemma 5.2. Table 2 gives one collection of trivializing homeomorphisms for all possible local models when the zero fiber is tall (cf. Table 1). It is easy to verify that each $F_Y$ consists of the norm square of the moment map and a map $\overline{P}_Y: Y/G \to \mathbb{C}$ induced from a $G$-invariant map $P_Y: Y \to \mathbb{C}$.

For the case $Y = \mathbb{C}^3 = \{ (x, y, z) = q + p\sqrt{-1} | x, y, z \in \mathbb{C}, q = \text{Re}(x, y, z), p = \text{Im}(x, y, z) \}$, it is easier to consider $F_Y$ as a composition of two functions. Consider
the map $F_1 : \mathbb{C}^3 / \text{SO}(3) \to \mathbb{R}^3$ given by $F_1((q,p)) = (||q||^2, ||p||^2, \langle q,p \rangle)$ and the map $F_2 : \mathbb{R}^3 \to \mathbb{R}_+ \times \mathbb{C}$ given by $F_2(\alpha,\beta,\gamma) = (\alpha \beta - \gamma^2, \alpha - \beta, 2\gamma)$. We define $F_Y = F_2 \circ F_1$ so that $F_Y((q,p)) = (||q||^2, \langle q,p \rangle)$, or equivalently, $F_Y([x,y,z]) = (||x\bar{z} - \bar{y}z, z\bar{x} - \bar{z}x, x\bar{y} - \bar{x}y||^2, x^2 + y^2 + z^2)$.

The component functions of $F_1$ generate the $\text{SO}(3)$-invariant functions on $\mathbb{C}^3$. It is one-to-one, continuous and proper. Hence it is a homeomorphism from $\mathbb{C}^3 / \text{SO}(3)$ into its image in $\mathbb{R}^3$, which is a solid half cone $\alpha \geq 0, \beta \geq 0, \alpha \beta \geq \gamma^2$. Since $F_2^{-1}(a,b,c) = \left(\frac{1}{2}(b + \sqrt{a^2 + b^2 + c^2}), \frac{1}{2}(-b + \sqrt{4a + b^2 + c^2}), \frac{1}{2}\right)$ is continuous, $F_2$ is a homeomorphism from the solid cone to its image in $\mathbb{R}_+ \times \mathbb{C}$. Therefore $F = F_2 \circ F_1$ is also a homeomorphism.

In each of the other cases, routine checks show that $F_Y$ is well-defined, bijective, continuous and proper. It follows that $F_Y$ is a homeomorphism. \qed

6. The smooth structure on the quotient

In this section we study the smooth structure on the quotient. By a theorem of Schwarz [Sch1], any invariant smooth function can be expressed as a smooth function of real invariant polynomials. Using this fact, we show that the trivializing homeomorphism defined in Table 2 in the previous section is a diffeomorphism on the complement of the exceptional orbits.

Let $G$ be $\text{SU}(2)$ or $\text{SO}(3)$ and let $(M,\omega,\Phi)$ be a six dimensional Hamiltonian $G$-manifold with a tall zero fiber. First we list all the exceptional orbits in the local models of an orbit in the zero fiber:

| Local model $Y$ | Trivializing homeomorphism $F_Y$ |
|-----------------|----------------------------------|
| $\text{SO}(3) \times S_1 \mathbb{R}^2 \times \mathbb{C}$ | $F_Y(g,\mu,|z| = (||\mu||^2, z)$ |
| $\text{SO}(3) \times N(S_1) \times \mathbb{R}^2 \times \mathbb{C}$ | $F_Y(g,\mu,|z| = (||\mu||^2, z)$ |
| $\text{SO}(3) \times N(S_1) (\mathbb{R}^2 \times \mathbb{C})$ | $F_Y(g,\mu,|z| = (||\mu||^2, z^2)$ |
| $\mathbb{C}^2 \setminus \{q + \sqrt{-1}p\}$ | $F_Y(q,p) = (||q||^2, ||p||^2, 2\langle q,p \rangle)$ |
| $\mathbb{C}^2 \times \mathbb{C}$ | $F_Y(w,|z| = (\frac{1}{2}||w||^2, z)$ |

Table 2. Trivializing homeomorphisms.

| Local model $Y$ | Exceptional orbits |
|-----------------|--------------------|
| $\text{SO}(3) \times S_1 \mathbb{R}^2 \times \mathbb{C}$ | none |
| $\text{SO}(3) \times N(S_1) \times \mathbb{R}^2 \times \mathbb{C}$ | none |
| $\text{SO}(3) \times N(S_1) (\mathbb{R}^2 \times \mathbb{C})$ | $\{[g,\mu,0]\}$ |
| $\mathbb{C}^2 \times \mathbb{C}$ | none |

Table 3. Exceptional Orbits.

Lemma 6.1. Let $G$ be $\text{SU}(2)$ or $\text{SO}(3)$ and $(M,\omega,\Phi)$ be a six dimensional Hamiltonian $G$-manifold such that $0 \in \Phi(M)$. Assume that the zero fiber is tall. Let $Y$ be a local model of an orbit in the zero fiber and $F_Y$ its trivializing homeomorphism defined in Table 2. Then on the complement of the exceptional orbits, $F_Y$ is a diffeomorphism.
Corollary 6.2. Let $G$ be SU(2) or SO(3) and $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold such that $0 \in \Phi(M)$. Assume that the zero fiber is tall. Let $Y$ be a local model of an orbit in the zero fiber and $F_Y$ its trivializing homeomorphism defined in Table 2. Then the restriction of $F_Y$ to the reduced space on the complement of the exceptional orbits is also a diffeomorphism.

Proof of Lemma 6.1. This is an application of Schwarz [Sch1] and a result of direct computation. We will prove for the local model $C^3$ as an example.

As in the proof of Lemma 5.2, for $Y = C^3$, the trivializing homeomorphism $F_Y$ is the composition $F_2 \circ F_1$, where $F_1: C^3/SO(3) \to \mathbb{R}^3$ is given by $F_1(p, q) = (|p|^2, |q|^2, \langle p, q \rangle)$ and $F_2: \mathbb{R}^3 \to \mathbb{R}_+ \times C$ is given by $F_2(\alpha, \beta, \gamma) = (\alpha \beta - \gamma^2, \alpha - \beta, 2\gamma)$. By Schwarz, $F_1$ pulls back smooth functions to smooth invariant functions. The inverse $F_2^{-1}$ is given by $F_2^{-1}(a, b, c) = (\frac{1}{2}(b + \sqrt{4a + b^2 + c^2}), \frac{1}{2}(-b + \sqrt{4a + b^2 + c^2}, \frac{2}{a})$). By direct computation, $F_2^{-1}$ is smooth except when $4a + b^2 + c^2 = 0$, i.e., when $a = b = c = 0$. So $F_2$ is a diffeomorphism except when $a = \beta = \gamma = 0$. So $F_Y$ is a diffeomorphism except at $q = p = 0$, the exceptional orbit.

\[\square\]

7. The SU(2) case

Let $G$ be SU(2) and $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold with $0 \in \Phi(M)$. Assume that the zero fiber $\Phi^{-1}(0)$ is tall. By Table 1, the principal stabilizer of $\Phi^{-1}(0)$ is SU(2) and the local model of every orbit in the zero fiber is $Y = C^2 \times C$. By Table 3, there is no exceptional orbits. Sections 5 and 6 show that the trivializing homeomorphism is a diffeomorphism. Near the zero fiber, the quotient $M/G$ is a smooth manifold with corners since $Y/G$ is diffeomorphic to $|\Phi(M)|^2 \times \mathbb{C}$.

Restricting to a smaller neighborhood $V$ of $0 \in \mathfrak{g}^*$, the norm square of the moment map $\overline{\Phi} = |\Phi|^2$ is a proper submersion; there is a diffeomorphism from $\Phi^{-1}(V)/G$ to $(|\Phi(M)|^2 \cap V) \times (\Phi^{-1}(0)/G)$. In particular, $|\Phi(M)|^2 \cap V$ is an interval and $\Phi^{-1}(V)$ satisfies Condition 3.1.

The reduced space $\Phi^{-1}(0)/G$ is a Riemann surface, and is determined by its genus.

Therefore, the genus of the zero fiber determines $\Phi^{-1}(V)$ up to $\Phi$-diffeomorphisms.

Shrink $V$ if necessary, by Propositions 4.11 and 3.3, we have the following version of Theorem A for SU(2):

Proposition 7.1. Let $G$ be SU(2). Let $(M, \omega, \Phi)$, and $(M', \omega', \Phi')$ be compact connected six dimensional Hamiltonian $G$-manifolds such that $0 \in \Phi(M) = \Phi(M')$. Assume that the zero fibers are tall. Then there exists an invariant neighborhood $V$ of $0$ in $\mathfrak{g}^*$ over which the Hamiltonian $G$-manifolds are isomorphic if and only if

- their Duistermaat-Heckman functions coincide on $V$, and
- their genus at $0$ are the same.

8. Grommets

Let $G$ be SO(3) and $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold with $0 \in \Phi(M)$. Assume that the zero fiber $\Phi^{-1}(0)$ is tall. According to Table 1, there are more than one possible local model for an orbit in the zero fiber. In particular, some local models have exceptional orbits.
The arguments in the previous section fail near the exceptional orbits. Sections 8-11 are dedicated to deal with this problem. The techniques used here are adapted from [KT1].

We start by defining charts at every exceptional orbits so that we can fix the smooth structure later.

**Definition 8.1.** Let $G$ be a compact Lie group and let $(M, \omega, \Phi)$ be a complexity one Hamiltonian $G$-manifold with $0 \in \Phi(M)$. A **grommet over 0** is a $\Phi$-G-diffeomorphism $\psi: D \to M$ where $D$ is a $G$-invariant open subset of a local model $Y = G \times_H (h^0 \times V)$ of an orbit in the zero fiber.

**Definition 8.2.** Let $G$ be SO(3) or SU(2), and let $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold with $0 \in \Phi(M)$. Let $Y$ be a local model of an orbit in the zero fiber. The **exceptional sheet** in $Y$ is the subset

$$S = \{ [g, \mu, z] \in G \times_H (h^0 \times V) \mid P_Y([g, \mu, z]) = 0 \}$$

where $P_Y$ is the component function of the trivializing homeomorphism defined in Table 2.

**Remark 8.3.** The exceptional orbits are always in the exceptional sheets. However, the exceptional sheets might include nonexceptional orbits. For example, the exceptional sheet of the local model $Y = \mathbb{C}^3$ has a nonexceptional orbit in every $\Phi_Y^{-1}(G \cdot \alpha)$ for $\alpha \neq 0 \in \Phi_Y(Y)$.

**Definition 8.4.** Let $G$ be SO(3) or SU(2), and let $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold with $0 \in \Phi(M) = U$. Let $\psi: D \to M$ be a grommet over 0 where $D$ is a $G$-invariant open subset of a local model $Y = G \times_H (h^0 \times V)$ of an orbit in the zero fiber. The grommet $\psi$ is **wide** if $D$ contains its part of the exceptional sheet. That is, $\Phi_Y^{-1}(U) \cap S \subset D$.

Wide grommets with pairwise disjoint closures ensure that the exceptional sheets in different local models are separated. The following lemma states that we can always find such grommets over 0:

**Lemma 8.5.** Let $G$ be SO(3) or SU(2), and let $(N, \omega, \Phi)$ be a neighborhood of the zero fiber in a six dimensional Hamiltonian $G$-manifold with $0 \in \Phi(N)$. Assume that the zero fiber is tall. Let $\{E_j\}$ denote the exceptional orbits in $\Phi^{-1}(0)$. After replacing $N$ by the preimage of some $G$-invariant neighborhood of 0 in $\Phi(N)$, there exist wide grommets $\psi_j: D_j \to N$ over 0 such that $\psi_j([g, 0, 0]) = E_j$ and $\psi_j(D_j)$ have pairwise disjoint closures.

**Proof.** By Local Normal Form Theorem 2.1, there is a local model $Y_j$ for each exceptional orbit $E_j \in \Phi^{-1}(0)$ with a moment map $\Phi_j: Y_j \to g^*$. We can choose grommets $\psi_j: D_j \subset Y_j \to N$ such that $\psi_j([g, 0, 0]) = E_j$.

By Lemma 5.2 and Definition 8.2, the restriction of the norm square of the moment map $\Phi_j$ to $S_j/G$ is a homeomorphism onto its image. So there exists a $G$-invariant neighborhood $W_j$ of 0 in $g^*$ such that $S_j \cap D_j = S_j \cap \Phi_j^{-1}(W_j)$. Let $W = \bigcap W_j$, and replace $N$ by $N \cap \Phi^{-1}(W)$ and $D_j$ by $D_j \cap \Phi^{-1}(W)$. Then the grommets are wide.

Since each $\psi_j(S_j \cap D_j)$ is closed, $\psi_j(S_j \cap D_j) \cap \psi_j(S_j \cap D_j)$ is closed in $N$. It does not intersect $\Phi^{-1}(0)$ since the exceptional orbits in the zero fiber are isolated. Because the moment map is proper, we can find a $G$-invariant neighborhood $V \subset W$ of 0 such that $\Phi^{-1}(V)$ does not intersect any of these intersections.
Replace $N$ by $N \cap \Phi^{-1}(V)$ and $D_j$ by $D_j \cap \Phi^{-1}(V)$. The grommets $\psi$ are still wide and $\psi(S_i \cap D_i) \cap \psi(S_j \cap D_j)$ is empty. Shrinking each $D_j$ to a smaller neighborhood of $S_j \cap D_j$, we obtain wide grommets with pairwise disjoint closures.

9. Flattening the Quotient

Let $G$ be SO(3) and $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold such that $0 \in \Phi(M)$. In this section, we show that the quotient $M/G$ near the zero fiber is topologically a surface bundle. We find a homeomorphism between $\Phi^{-1}(V)$ and $(|\Phi(M)|^2 \cap V) \times (\Phi^{-1}(0)/G)$ for some $G$-invariant neighborhood $V$ of $0$ in $g^*$. Away from the exceptional sheets (see Definition 8.2), this homeomorphism is a diffeomorphism. Near the exceptional sheets, it is determined by the grommets.

**Definition 9.1.** Let $G$ be SO(3) and $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold such that $0 \in \Phi(M)$. Assume that the zero fiber is tall. Let $Y$ denote the local model $G \times_H (g^0 \times V)$ of an orbit in the zero fiber. A **standard flattening** of $Y$ is the map $\delta: Y/G \to (\text{image } \Phi_Y) \times (\Phi_Y^{-1}(0)/G)$ given by

$$\delta = (\overline{\Phi}_Y, \overline{P}_0^{-1} \circ \overline{P}_Y)$$

where $\overline{\Phi}_Y$ is induced by the moment map, and $\overline{P}_Y: Y/G \to \mathbb{C}$ and $\overline{P}_0: \Phi_Y^{-1}(0)/G \to \mathbb{C}$ are the component functions of the trivializing homeomorphisms defined in Section 5, Table 2.

By Lemma 5.2 and Corollary 5.3, the standard flattening is a homeomorphism. By Lemma 6.1 and Corollary 6.2, it is a diffeomorphism away from the exceptional sheet.

**Definition 9.2.** Let $G$ be SO(3) and $(N, \omega, \Phi)$ be a neighborhood of the zero fiber in a six dimensional Hamiltonian $G$-manifold such that $0 \in \Phi(N)$. Assume that the zero fiber is tall. The **flattening** of $N$ about $0$ consists of

1. a homeomorphism $\delta: N/G \to (\text{image } \Phi_Y) \times (\Phi_Y^{-1}(0)/G)$, and
2. a wide grommet $\psi_j: D_j \to N$ at each exceptional orbit $E_j$ in $\Phi_Y^{-1}(0)$ such that $\psi_j([g,0,0]) = E_j$, where $D_j$ is a $G$-invariant open subset of a local model $G \times_H (h^0 \times V)$ of an orbit in the zero fiber,

such that the following two conditions are satisfied:

1. $\delta$ is a diffeomorphism on the complement of the exceptional sheets; that is,
   $$\delta: N/G \setminus \sqcup \psi_j(S_j \cap D_j)/G \to (\text{image } \Phi_Y) \times (\Phi_Y^{-1}(0) \setminus \sqcup E_j)/G$$
   is a diffeomorphism.
2. Near the exceptional sheets, $\delta$ is the standard flattening of the local models; namely, the following diagram commutes:

$$\begin{array}{ccc}
D_j/G & \xrightarrow{\delta_j} & \mathbb{R}_+ \times (\Phi_Y^{-1}(0) \cap D_j)/G \\
\downarrow{\psi_j} & & \downarrow{(\text{id}, \overline{\psi}_j)} \\
N/G & \xrightarrow{\delta} & \mathbb{R}_+ \times (\Phi_Y^{-1}(0)/G)
\end{array}$$
where $\psi_j: D_j/G \to N/G$ is induced by the grommets, $\Phi_j$ is the moment map on the local model $Y_j \supset D_j$, and $\delta_j: Y_j/G \to (\text{image } \Phi_j) \times (\Phi_j^{-1}(0)/G)$ is the standard flattening of $Y_j$.

The following proposition asserts that a flattening about 0 always exists.

**Proposition 9.3.** Let $G$ be $SO(3)$ and $(M,\omega,\Phi)$ be a six dimensional Hamiltonian $G$-manifold such that $0 \in \Phi(M)$. Assume that the zero fiber is tall. Then there exists a $G$-invariant neighborhood $V$ of 0 in $\Phi(M)$ such that $\Phi^{-1}(V)$ admits a flattening about 0.

**Proof.** Let $\{E_j\}$ denote the exceptional orbits in the zero fiber. By Lemma 8.5, there exists a $G$-invariant neighborhood $V$ of 0 in $\Phi(M)$ and wide grommets $\psi_j: D_j \to \Phi^{-1}(V) = N$ such that $\psi_j(\{[g,0,0]\}) = E_j$ and $\psi_j(D_j)$ have disjoint closures in $N$.

The grommets together with the standard flattening $\delta_j$ of the local models $Y_j$ define a partial flattening $\delta$ such that the following diagram commutes.

\[
\begin{array}{ccc}
D_j/G & \xrightarrow{\delta_j} & \mathbb{R}^+ \times ((\Phi_j^{-1}(0) \cap D_j)/G) \\
\downarrow{\psi_j} & & \downarrow{(\text{id},\psi_j)} \\
\sqcup \psi_j(D_j/G) & \xrightarrow{\delta} & \mathbb{R}^+ \times (\Phi_j^{-1}(0)/G)
\end{array}
\]

By Lemma 6.1, $\Phi: N/G \setminus \sqcup \psi_j(D_j \cap S_j)/G \to (\text{image } \Phi)$ is a submersion on the complement of the exceptional sheets. The partial flattening $\delta$ defined above then determines an Ehresmann connection on the open set $\sqcup \psi_j(D_j \setminus S_j)/G$. This connection can be extended on the entire complement of the exceptional sheets with a partition of unity. Using the parallel transport $\gamma$ associated with the connection, we define $\delta(p) = (\Phi(p), \gamma(p))$ for all $p \in N/G$. $\square$

This has a few immediate corollaries:

**Corollary 9.4.** Let $G$ be $SO(3)$ and $(M,\omega,\Phi)$ be a six dimensional Hamiltonian $G$-manifold with $0 \in \Phi(M)$. Assume that the zero fiber is tall. There exists a $G$-invariant neighborhood $V$ of 0 in $\Phi(M)$ such that $\Phi^{-1}(V)/G$ is a surface bundle.

**Corollary 9.5.** Let $G$ be $SO(3)$ and $(M,\omega,\Phi)$ be a six dimensional Hamiltonian $G$-manifold with $0 \in \Phi(M)$. Assume that the zero fiber is tall. Then all the reduced spaces $\Phi^{-1}(G \cdot \alpha)/G$ have the same genus for $\alpha$ sufficiently close to 0.

**Corollary 9.6.** Let $G$ be $SO(3)$ and $(M,\omega,\Phi)$ be a six dimensional Hamiltonian $G$-manifold with $0 \in \Phi(M)$. Assume that the zero fiber is tall. Then there exists a $G$-invariant neighborhood $V$ of 0 in $\mathfrak{g}^*$ such that the restriction map

\[H^*(\Phi^{-1}(V)/G) \to H^*(\Phi^{-1}(G \cdot \alpha)/G)\]

is an isomorphism for all $\alpha \in V$. In particular, $\Phi^{-1}(V)$ satisfies Condition 3.1.
10. THE ASSOCIATED MARKED SURFACE

Let $G$ be $SO(3)$ and $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold with $0 \in \Phi(M)$. Assume that the zero fiber is tall. Then the reduced space $\Phi^{-1}(0)$ is topologically a surface. We can give it a smooth structure according to the grommets on $M$. We first define a grommet on a surface. This is simply a notion for a coordinate chart at a marked point.

**Definition 10.1.** Let $\Sigma$ denote a closed oriented surface. A grommet at a point $q \in \Sigma$ is a diffeomorphism $\varphi : B \to \Sigma$ where $B$ is a neighborhood of $0$ in $\mathbb{C}$ and $\varphi(0) = q$.

**Definition 10.2.** Let $\Sigma$ and $\Sigma'$ be closed oriented surfaces with labelled marked points $\{q_i\} \subset \Sigma$ and $\{q_i'\} \subset \Sigma'$. Let $\varphi_i$ and $\varphi'_i$ denote the grommets at these marked points. Assume that $q_i$ and $q'_i$ have the same labels for all $i$. An orientation preserving diffeomorphism $g : \Sigma \to \Sigma'$ is **rigid** if for all $i$

1. $g(q_i) = q'_i$;
2. $\varphi'^{-1}_i \circ g \circ \varphi_i$ is a rotation on some neighborhood of $0 \in \mathbb{C}$.

**Lemma 10.3.** Let $\Sigma$ and $\Sigma'$ be closed oriented surfaces with labelled marked points $\{q_i\} \subset \Sigma$ and $\{q_i'\} \subset \Sigma'$. Let $\varphi_i$ and $\varphi'_i$ denote the grommets at these marked points. Assume that $q_i$ and $q'_i$ have the same labels for all $i$ and that $\Sigma$ and $\Sigma'$ have the same genus. Then every bijection from the marked points $\{q_i\}$ to the marked points with the same labels $\{q_i'\}$ extends to a rigid map from $\Sigma$ to $\Sigma'$.

**Proof.** This uses standard techniques in differential topology. For details, see for example [Ko].

**Remark 10.4.** Let $G$ be $SO(3)$ and $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold. Assume that the zero fiber is tall. Let $\psi : D \to M$ be a grommet over $0$ where $D$ is a $G$-invariant open subset of a local model $G \times_H (\mathfrak{h}^0 \times V)$ of an orbit in the zero fiber. Assume that $\psi([g,0,0]) = \mathcal{O} \subset \Phi^{-1}(0)$. Then the grommet $\psi$ induces a coordinate chart on the reduced space $\Phi^{-1}(0)/G$. Explicitly, the map $\overline{\psi} : (D \cap \Phi^{-1}(0))/G \to \mathbb{C}$ given by the trivializing homeomorphism in Table 2 is a homeomorphism onto its image $B$. And $\varphi = \overline{\psi} \circ \overline{\Phi^{-1}} : B \to \Phi^{-1}(0)/G$ is a homeomorphism onto its image such that $\varphi(0) = \mathcal{O}/G$, where $\overline{\psi} : D/G \to M/G$ is induced by $\psi$.

**Definition 10.5.** Let $G$ be $SO(3)$ and $(N, \omega, \Phi)$ be a $G$-invariant neighborhood of the zero fiber in a six dimensional Hamiltonian $G$-manifold. Assume that the zero fiber is tall. For each exceptional orbit $E_j$ in $\Phi^{-1}(0)$, let $\psi_j : D_j \to N$ be the grommet over $0$ such that $\psi_j([g,0,0]) = E_j$. The **associated marked surface** of $N$ consists of the following data:

1. The connected oriented topological surface $\Sigma = \Phi^{-1}(0)/G$.
2. The set of marked points $\{q_j\} \subset \Sigma$ corresponding to the exceptional orbits $\{E_j\} \subset \Phi^{-1}(0)$; i.e. for each $j$, $q_j = E_j/G$.
3. The smooth structure on $\Sigma$ given by the following coordinate charts. For each exceptional orbit $E_j$ in $\Phi^{-1}(0)$, take the given grommet. For each nonexceptional orbit $\mathcal{O}$ in $\Phi^{-1}(0)$, choose an arbitrary grommet such that $\psi([g,0,0]) = \mathcal{O}$. For each grommet, take the induced coordinate chart as described in Remark 10.4.
(4) The grommets on \( \Sigma \) at the marked points \( \{q_j\} \) given by the above coordinate charts.

(5) A label at each marked point \( q_j \) describing the isotropy representation of the corresponding exceptional orbit \( E_j \).

### 11. Diffeomorphism between Quotients

Here we show that a diffeomorphism between the associated marked surfaces that behaves nicely near the marked points induces a \( \Phi \)-diffeomorphism between the quotients.

**Proposition 11.1.** Let \( G = \text{SO}(3) \) and let \((N, \omega, \Phi)\) and \((N', \omega', \Phi')\) be neighborhoods of the zero fibers in six dimensional Hamiltonian \( G \)-manifolds such that \( 0 \in \Phi(N) = \Phi'(N') \). Assume that the zero fibers are tall and have the same principal stabilizer. Assume that \( N \) and \( N' \) admit flattenings about \( 0 \). Let \( \Sigma \) and \( \Sigma' \) denote the associated marked surfaces of \( N \) and \( N' \) respectively. Then any rigid map \( h: \Sigma \to \Sigma' \) extends to a \( \Phi \)-diffeomorphism \( H: N/G \to N'/G \).

**Proof.** If there exists a rigid map \( h: \Sigma \to \Sigma' \), the labels of the marked points on \( \Sigma \) and \( \Sigma' \) are the same. Topologically, \( \Phi^{-1}(0)/G = \Sigma \) and \( \Phi'^{-1}(0)/G = \Sigma' \), so the isotropy data at \( 0 \) are the same for \( N \) and \( N' \). Let \( \Phi \) and \( \Phi' \) be the norm squares of the moment maps. We know that \( \text{image} \ \Phi = \text{image} \ \Phi' \).

Let \( \delta \) and \( \delta' \) denote the homeomorphisms given by the flattenings of \( N \) and \( N' \) about \( 0 \). We define \( H = \delta'^{-1} \circ (\text{id}, h) \circ \delta \) so that the following diagram commutes:

\[
\begin{array}{ccc}
N/G & \xrightarrow{H} & N'/G \\
\downarrow \delta & & \downarrow \delta' \\
(\text{image} \ \Phi) \times \Sigma & \xrightarrow{(\text{id}, h)} & (\text{image} \ \Phi') \times \Sigma'
\end{array}
\]

We claim that \( H \) is a \( \Phi \)-diffeomorphism.

It is orientation preserving since \( h \) is. \( H^* \Phi' = \Phi \) since \( H \) is induced from an identity map between (image \( \Phi \)) and (image \( \Phi' \)). We need to prove that it is a diffeomorphism and it locally lifts in a neighborhood of each exceptional orbit.

Let \( \{E_j\} \) and \( \{E_j'\} \) denote the exceptional orbits in \( \Phi^{-1}(0) \) and \( \Phi'^{-1}(0) \), respectively. The rigid map \( h \) determines an identification between the exceptional orbits \( E_j \) and \( E_j' \) with the same isotropy representation. We can then denote the local model by \( Y_j \) for both \( E_j \in \Phi^{-1}(0) \) and \( E_j' \in \Phi'^{-1}(0) \). Let

\[
\psi_j: D_j \to N \quad \text{and} \quad \psi_j': D_j' \to N'
\]

denote the grommets given by the flattenings of \( N \) and \( N' \) about \( 0 \), where \( D_j \subset Y_j \) and \( D_j' \subset Y_j \) are \( G \)-invariant open subsets. Let \( S_j \) and \( S_j' \) denote the exceptional sheets.

Since the homeomorphisms \( \delta \) and \( \delta' \) given in the flattenings are diffeomorphisms on the complement of the exceptional sheets, and since the smooth structures on the reduced space \( \Phi^{-1}(0)/G \) and the associated surface \( \Sigma \) agree off the exceptional orbits, the restriction of \( H \) to \( (N \setminus \bigcup_j \psi_j(S_j \cap D_j))/G \) is a diffeomorphism. This is easy to see from the following diagram, where

\[
\tilde{h}: \left( \Phi^{-1}(0) \setminus \bigcup_j E_j \right)/G \to \left( \Phi'^{-1}(0) \setminus \bigcup_j E_j' \right)/G
\]
is a diffeomorphism induced by $h: \Sigma \to \Sigma'$:

$$
\begin{array}{c}
N \setminus \sqcup_j \psi_j(S_j \cap D_j) \\
\downarrow \delta \\
\text{(image } \Phi) \times \Phi^{-1}(0) \setminus \sqcup_j E_j \\
\downarrow \Phi_j^{-1}(0) \setminus \sqcup_j E'_j
\end{array}
\xrightarrow{H} 
\begin{array}{c}
N' \setminus \sqcup_j \psi'_j(S'_j \cap D'_j) \\
\downarrow \delta' \\
\text{(image } \Phi') \times \Phi'^{-1}(0) \setminus \sqcup_j E'_j
\end{array}
$$

It remains to show that $H$ is a $\Phi$-diffeomorphism in a neighborhood of each exceptional sheets $\psi_j(S_j \cap D_j)/G$.

Let $\varphi_j: B_j \to \Sigma$ and $\varphi'_j: B'_j \to \Sigma'$ be the grommets of the associated marked surfaces. The fact that $h$ is rigid implies that $\varphi'_j \circ h \circ \varphi_j$ is a rotation by $a_j \in S^1$ on some neighborhood of 0. That is, we have the following diagram:

$$
\begin{array}{c}
B_j \\
\downarrow \varphi_j \\
\Sigma \\
\downarrow h \\
\Sigma'
\end{array}
\xrightarrow{H} 
\begin{array}{c}
B'_j \\
\downarrow \varphi'_j \\
\Sigma' \\
\downarrow h \\

\end{array}
$$

For each local model $Y_j$ of an exceptional orbit, we define a map $f_j: Y_j \to Y_j$ as follows:

**Case 1:** $Y_j = SO(3) \times_{N(S^1)} (\mathbb{R}^2 \times \mathbb{C})$, $f_j([g, \mu, z]) = [g, \mu, \pm a_j^{1/2} z]$.

**Case 2:** $Y_j = \mathbb{C}^3$, $f_j(x, y, z) = \pm (a_j^{1/2} x, a_j^{1/2} y, a_j^{1/2} z)$.

This map $f_j$ is an equivariant symplectomorphism which respects the moment maps. It induces a $\Phi$-diffeomorphism $\tilde{f}_j: Y_j/G \to Y_j/G$ on the quotient. In some neighborhood of $\psi_j(S_j \cap D_j)$, if we identify $Y_j/G$ with $(\text{image } \Phi_j) \times \mathbb{C}$ by the trivializing homeomorphism, $\tilde{f}_j$ sends $(\alpha, w)$ to $(\alpha, a_j w)$, and therefore agrees with $H$. □

12. PROOF OF THE LOCAL UNIQUENESS THEOREM OVER ZERO

We now arrive at the stage to prove our main local theorem.

**Theorem A** (Local Uniqueness over 0). Let $G$ be SU(2) or SO(3). Let $(M, \omega, \Phi)$ and $(M', \omega', \Phi')$ be compact connected six dimensional Hamiltonian $G$-manifolds such that $0 \in \Phi(M) = \Phi'(M')$. There exists a $G$-invariant neighborhood $V$ of 0 in $\mathfrak{g}^*$ over which the Hamiltonian $G$-manifolds are isomorphic if and only if

- their Duistermaat-Heckman functions coincide on $V$,
- their isotropy data and genus at 0 are the same,
- their principal stabilizers of the zero fibers are the same,
- if the zero fibers are tall with principal stabilizer $S^1$, the first Stiefel-Whitney classes of $\Phi^{-1}(0)$ and $\Phi'^{-1}(0)$ in $H^1(M_0^{reg}, \mathbb{Z}_2)$ and $H^1(M'_0^{reg}, \mathbb{Z}_2)$ are equal (under an identification of $M_0$ and $M'_0$ that respects the isotropy data).

**Proof.** It is clear that these conditions are necessary conditions when applicable. We show that they are also sufficient conditions.

If the zero fibers are short, this is Proposition 2.2.
If $G$ is SU(2) and the zero fibers are tall, this is Proposition 7.1.

Assume that $G$ is SO(3) and that the zero fibers are tall. By Proposition 9.3, there exists a $G$-invariant neighborhood $V$ of 0 in $\Phi(M) = \Phi'(M')$ such that $\Phi^{-1}(V)$ and $\Phi'^{-1}(V)$ admit flattenings about 0. By assumption, their genus and isotropy data are the same at 0. Then the associated marked surfaces $\Sigma$ and $\Sigma'$ have the same genus and labels. By Lemma 10.3, there exists a rigid map $h: \Sigma \to \Sigma'$. By Proposition 11.1, it extends to a $\Phi$-diffeomorphism $\psi: \Phi^{-1}(V)/G \to \Phi'^{-1}(V)/G$. Now Condition 3.1 is satisfied because of the flattenings. By assumption, the Duistermaat-Heckman functions of $\Phi^{-1}(V)$ and $\Phi'^{-1}(V)$ are the same and the zero fibers have the same principal stabilizer and the first Stiefel-Whitney class. Proposition 4.11 implies that there exists a $\Phi$-$G$-diffeomorphism from $\Phi^{-1}(V)$ to $\Phi'^{-1}(V)$, and Proposition 3.3 implies that there exists an equivariant symplectomorphism from $\Phi^{-1}(V)$ to $\Phi'^{-1}(V)$ which respects the moment maps. □

13. SYMPLECTIC CROSS-SECTION

In this section we begin to study the preimage of a neighborhood away from zero. First, we factor out the coadjoint orbit directions in the sense of the symplectic cross-section theorem (cf. Theorem 26.7 in [GS2]).

**Theorem 13.1.** Let $G$ be a compact Lie group and let $(M, \omega, \Phi)$ be a Hamiltonian $G$-manifold. Suppose that $S$ is a submanifold of $g^*$ passing through a point $\alpha \in g^*$ satisfying $T_\alpha S \oplus T_\alpha (G \cdot \alpha) = g^*$ and suppose that $S$ is $G_\alpha$-invariant. Then for a small enough $G_\alpha$-invariant neighborhood $B$ of $\alpha$ in $S$ the preimage $\Phi^{-1}(B)$ is a symplectic submanifold of $(M, \omega)$ and the action of $G_\alpha$ on $\Phi^{-1}(B)$ is Hamiltonian. Its moment map is the restriction of $\Phi$ on $\Phi^{-1}(B)$ followed by the projection onto $T_\alpha S \simeq g_\alpha^*$, the dual of the Lie algebra of the stabilizer of $\alpha$.

The submanifold $X = \Phi^{-1}(B)$ is called a **symplectic cross-section**. It is proved in [GLS] that if $G$ is compact we can choose the manifold $S$ and $B$ be so large that it is all the interior of the Weyl chamber. The set $\Phi^{-1}(G \cdot B) = G \cdot \Phi^{-1}(B) = G \cdot X$ is an open subset of $M$, which is $G$-equivariantly isomorphic to the associated bundle $G \times_{G_\alpha} X$, and the map $\pi: G \times_{G_\alpha} X \to G \cdot \alpha$ given by $[g,x] \mapsto g \cdot \alpha$ is a symplectic fibration. The symplectic connection on the bundle $G \times_{G_\alpha} X \to G \cdot \alpha$ is the same as the connection determined by the splitting $T_\alpha S \oplus T_\alpha (G \cdot \alpha) = g^*$. While the symplectic form on $G \times_{G_\alpha} X$ comes naturally from its identification with $G \cdot X \to M$, it can be reconstructed from its restriction to the fiber $X$ and the connection corresponding to the splitting.

Let $G$ be SU(2) or SO(3), and let $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold. For all $\alpha \neq 0 \in g^*$, we have $G_\alpha = S^1$ and $G \cdot \alpha = S^2$. The symplectic cross-section $X$ is the preimage of an open ray $\mathbb{R}_{>0} = \mathbb{R}_+ \setminus \{0\}$. It is symplectic, connected, four dimensional and it has a Hamiltonian circle action. Its moment map $\Phi_X: X \to \mathbb{R}_{>0}$ is given by

$$g \cdot \Phi_X(x) = \Phi([g,x])$$

where $\Phi$ is the moment map of the $G$ action on $G \times_{G_\alpha} X$. Let $A$ denote the connection one-form. The symplectic form on $G \times_{G_\alpha} X$ can be reconstructed by (13.1)

$$\omega = \omega_X - d(\Phi_X, A) + \pi^* \omega_{S^2}$$

where $\omega_X$ is a symplectic form on the symplectic cross-section $X$, and $\pi^* \omega_{S^2}$ is the pull-back of the natural symplectic form on the coadjoint orbit $G \cdot \alpha = S^2$. 
A special case of Theorem B follows from the symplectic cross-section theorem:

**Proposition 13.2.** Let $G$ be SU(2) or SO(3). Let $(M,\omega,\Phi)$ and $(M',\omega',\Phi')$ be compact connected six dimensional Hamiltonian $G$-manifolds such that 0 $\notin \Phi(M) = \Phi'(M)$. Then $M$ and $M'$ are isomorphic if and only if they have the same principal stabilizer, the same Duistermaat-Heckman function, the same genus, and the same isotropy skeleton.

**Proof.** Since $M$ and $M'$ have the same principal stabilizer $\mathbb{Z}_n \simeq \mathbb{Z}/n\mathbb{Z}$, their symplectic cross-sections $X$ and $X'$ inherit actions of $S^1/\mathbb{Z}_n \simeq S^1$ from the actions of $G$ on $M$ and $M'$. The symplectic cross-sections are then four dimensional Hamiltonian $S^1$-manifolds on which $\mathbb{Z}_n$ acts trivially. By [K1] or [KT2], these are determined by the Duistermaat-Heckman function, the genus, and the isotropy skeleton. Hence $X$ and $X'$ are isomorphic and $M \simeq G \times S^1 X$ and $M' \simeq G \times S^1 X'$ are isomorphic. $\square$

We now examine some previous definitions and properties on a Hamiltonian $G$-manifold in terms of its symplectic cross-section.

**Lemma 13.3.** Let $G$ denote SU(2) or SO(3). Let $(M,\omega,\Phi)$ and $(M',\omega',\Phi')$ be Hamiltonian $G$-manifolds and let $X$ and $X'$ be their symplectic cross-sections $\Phi^{-1}(\mathbb{R}_{>0})$ and $\Phi'^{-1}(\mathbb{R}_{>0})$ respectively. Identify $M \setminus \Phi^{-1}(0)$ and $M' \setminus \Phi'^{-1}(0)$ with $G \times S^1 X$ and $G \times S^1 X'$. There exists a one-to-one correspondence between $\Phi$-diffeomorphisms $F: G \times S^1 X \rightarrow G \times S^1 X'$ and $\Phi^{-1}-$diffeomorphisms $f: X \rightarrow X'$.

**Proof.** One direction is easy. Assume that there exists a $\Phi_X^{-1}$-diffeomorphism $f: X \rightarrow X'$. The map $F: G \times S^1 X \rightarrow G \times S^1 X'$ defined by $F([g,x]) = [g,f(x)]$ is a $\Phi$-diffeomorphism.

Now assume there exists a $\Phi$-diffeomorphism $F: G \times H X \rightarrow G \times H X'$. The fact that $F$ is equivariant implies that $F([g,x]) = [gf_1(x),f_2(x)]$ for some smooth functions $f_1: X \rightarrow G$ and $f_2: X \rightarrow X'$. Since $F$ respects the moment maps, $g \cdot \Phi_X(x) = gf_1(x) \cdot \Phi'_X(f_2(x))$. Note $\Phi_X$ and $\Phi'_X$ map into $\mathbb{R}_{>0}$. We derive that $f_1(x) \in S^1$ and $\Phi_X(x) = \Phi'_X(f_2(x))$. We rewrite $F([g,x]) = [gf_1(x),f_2(x)] = [g,f_1(x)f_2(x)] = [g,f(x)]$.

To show that $f$ is a $\Phi_X^{-1}$-diffeomorphism, we need to show that $f$ is an orientation preserving diffeomorphism which respects the moment maps. Since $F$ is well-defined under the $S^1$ action on $G \times X$, we have $F([g,a,x]) = [g,f(ax)] = F([ga^{-1},ax]) = [ga^{-1},f(ax)]$ for all $a \in S^1$. So $f(ax) = af(x)$ for all $a \in S^1$; namely, $f$ is $S^1$-equivariant. We know that this function $f: X \rightarrow X'$ also respects the moment map since $\Phi_X'$ is $S^1$-invariant and $\Phi_X'(f(x)) = \Phi_X'(f_1(x)f_2(x)) = \Phi_X'(f_2(x)) = \Phi_X(x)$. Finally, the fact that $F$ preserves the orientation on $G \times S^1 X$ implies that $f$ preserves the orientation on $X$. $\square$

**Definition 13.4.** Let $\mathbb{R}$ denote the dual of the Lie algebra of $S^1$. Let $M$ and $M'$ be oriented manifolds with $S^1$ actions and $S^1$-equivariant maps $\Phi: M \rightarrow \mathbb{R}$ and $\Phi': M' \rightarrow \mathbb{R}$. A $\Phi$-**diffeomorphism** from $M/S^1$ to $M'/S^1$ is an orientation preserving diffeomorphism $\psi: M/S^1 \rightarrow M'/S^1$ such that $\psi^*\overline{\Phi} = \overline{\Phi'}$ and such that $\psi$ and $\psi^{-1}$ lift to $\Phi^{-1}$-diffeomorphisms in a neighborhood of each exceptional orbit. Here $\overline{\Phi}$ and $\overline{\Phi'}$ are induced by the moment maps as in (4.1).

**Lemma 13.5.** Let $G$ denote SU(2) or SO(3). Let $(M,\omega,\Phi)$ and $(M',\omega',\Phi')$ be Hamiltonian $G$-manifolds and let $X$ and $X'$ be their symplectic cross-sections $\Phi^{-1}(\mathbb{R}_{>0})$ and $\Phi'^{-1}(\mathbb{R}_{>0})$ respectively. Identify $M \setminus \Phi^{-1}(0)$ and $M' \setminus \Phi'^{-1}(0)$
with $G \times S^1 X$ and $G \times S^1 X'$. There exists a one-to-one correspondence between $\Phi$-diffeomorphisms $F$: $(G \times S^1 X)/G \to (G \times S^1 X')/G$ and $\Phi_X$-diffeomorphisms $f$: $X/S^1 \to X'/S^1$.

Proof. Let $H$ denote $S^1$. Consider the map $\tilde{\gamma}: X/H \to (G \times H X)/G$ induced by the inclusion $i: X \to G \times H X$ such that $i(x) = [e, x]$. It is easy to check that $\tilde{\gamma}$ is a homeomorphism. We have the commutative diagram that gives the correspondence between a $\Phi$-diffeomorphism $F$ and a $\Phi_X$-diffeomorphism $f$.

$$
\begin{array}{ccc}
X/H & \xrightarrow{f} & X'/H \\
\downarrow \tilde{\gamma} & & \downarrow \tilde{\gamma} \\
(G \times H X)/G & \xrightarrow{F} & (G \times H X')/G \\
\downarrow \Phi_X & & \downarrow \Phi_X \\
\mathbb{R}_{>0} & & \mathbb{R}_{>0}
\end{array}
$$

Let $H_x$ denote the stabilizer of a point $x \in X$ with respect to the $H$ action on $X$. The stabilizer of $[g, x] \in G \times H X$ with respect to the $G$ action on $G \times H X$ is given by $G_{[g,x]} = g H_x g^{-1}$. Since $\Phi([g, x]) = g \cdot \Phi_X(x)$, there is a one-to-one correspondence between exceptional $S^1$-orbits in $\Phi_X^{-1}(\alpha)$ and exceptional $G$-orbits in $\Phi^{-1}(G \cdot \alpha)$.

By Lemma 13.3, if the $\Phi_X$-diffeomorphism $f$ lifts to a $\Phi_X$-$S^1$-diffeomorphism in a neighborhood of an exceptional $S^1$-orbit, the induced $\Phi$-diffeomorphism $F = \tilde{\gamma} \circ f \circ \tilde{\gamma}^{-1}$ lifts to a $\Phi$-$G$-diffeomorphism in a neighborhood of the corresponding exceptional $G$-orbit. \hfill \Box

**Lemma 13.6.** Following the notations in Lemma 13.3 and 13.5, a $\Phi$-diffeomorphism $F$ lifts to a $\Phi$-$G$-diffeomorphism $\tilde{F}$ if and only if the corresponding $\Phi_X$-diffeomorphism $f$ lifts to the corresponding $\Phi_X$-$S^1$-diffeomorphism $\tilde{f}$.

Proof. Let $[,]$ denote the $S^1$ equivalence class and let $\{,\}$ denote the $G$ equivalence class. A $\Phi$-diffeomorphism $F$: $(G \times S^1 X)/G \to (G \times S^1 X')/G$ lifts to a $\Phi$-$G$-diffeomorphism $\tilde{F}: G \times S^1 X \to G \times S^1 X'$ if and only if

$$
\tag{13.2}
\tilde{F}([g, x]) = F([g, x]),
$$

and a $\Phi_X$-diffeomorphism $f$: $X/S^1 \to X'/S^1$ lifts to a $\Phi_X$-$S^1$-diffeomorphism $\tilde{f}$: $X \to X'$ if and only if

$$
\tag{13.3}
[\tilde{f}(x)] = f([x]).
$$

By Lemma 13.3,

$$
\tag{13.4}
\tilde{F}([g, x]) = [g, \tilde{f}(x)].
$$

By Lemma 13.5,

$$
\tag{13.5}
F([e, x]) = \langle [e, f([x])].
$$

We only need to show that (13.3), (13.4), and (13.5) together imply (13.2), but this is easy.

$$
\langle \tilde{F}([g, x]) \rangle = \langle [g, \tilde{f}(x)] \rangle = \langle [e, \tilde{f}(x)] \rangle = \langle [gh^{-1}, h \tilde{f}(x)] \rangle
$$

where $h$ is some element in $S^1$. \hfill \Box
The moment image of a Hamiltonian $S^1$-manifold can be translated by a constant $\alpha \in \mathfrak{a}^* \simeq \mathbb{R}$. Hence, the Local Normal Form Theorem 2.1 applies for any orbit in any $\alpha$ fiber. It follows from [KT1] that trivializing homeomorphisms exist, and that grommets, exceptional sheets, wide grommets, flattenings, and associated marked surfaces described in Sections 8, 9, and 10 are defined over any tall $\alpha$ fiber for the symplectic cross-section $X$ of a six dimensional Hamiltonian $G$-manifold where $G = SU(2)$ or $SO(3)
$.

Let $Y$ be a local model for an $S^1$-orbit $O$ in $\Phi_X^{-1}(\alpha)$ for $\alpha \in \mathbb{R}_{>0}$. The associated bundle $G \times_{S^1} Y$ is a local model for the $G$-orbit $G \cdot O$ in $\Phi^{-1}(G \cdot \alpha)$. A trivializing homeomorphism on $Y$ determines a trivializing homeomorphism on $G \times_{S^1} Y$. We can then define grommets, exceptional sheets, wide grommets, flattenings, and associated marked surfaces of $M$ away from 0 in a similar fashion.

The following propositions and lemmas are derived from [KT1] based on the properties of the symplectic cross-section.

**Proposition 13.7.** Let $G$ be $SU(2)$ or $SO(3)$ and let $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold. Assume that $\Phi^{-1}(\alpha)$ is tall for $\alpha \in \Phi(M) \cap \mathbb{R}_{>0}$. Then there exists a neighborhood $I$ of $\alpha$ in $\Phi(M) \cap \mathbb{R}_{>0}$ such that the preimage $\Phi^{-1}(G \cdot I)$ admits a flattening about $\alpha$.

**Proposition 13.8.** Let $G$ be $SU(2)$ or $SO(3)$. Let $(M, \omega, \Phi)$ and $(M', \omega', \Phi')$ be six dimensional Hamiltonian $G$-manifolds equipped with flattenings about $\alpha \in \Phi(M) = \Phi(M')$. Let $\Sigma$ and $\Sigma'$ be the associated marked surfaces of $M$ and $M'$, respectively. Then any rigid map $h: \Sigma \to \Sigma'$ extends to a $\Phi$-diffeomorphism $g: M/G \to M'/G$.

**Proposition 13.9.** Let $G$ be $SU(2)$ or $SO(3)$. Let $(M, \omega, \Phi)$ and $(M', \omega', \Phi')$ be compact connected six dimensional Hamiltonian $G$-manifolds. Assume that their $\alpha$ fibers are tall for $\alpha \neq 0$ in $\Phi(M) = \Phi(M')$. Then there exists a $G$-invariant neighborhood $V$ of $\alpha$ such that $\Phi^{-1}(V)$ and $\Phi'^{-1}(V)$ are $\Phi$-diffeomorphic if and only if the reduced spaces $\Phi^{-1}(G \cdot \alpha)/G$ and $\Phi'^{-1}(G \cdot \alpha)/G$ have the same isotropy data and genus.

**Proof.** Without loss of generality, we can assume $\alpha \in \mathbb{R}_{>0}$. By Proposition 13.7, there exists a neighborhood $I$ of $\alpha$ in $\mathbb{R}_{>0}$ such that $\Phi^{-1}(G \cdot I)$ and $\Phi'^{-1}(G \cdot I)$ admit flattenings about $\alpha$. Since the isotropy data and genus of the reduced spaces $M_\alpha$ and $M'_\alpha$ are the same, the associated marked surfaces $\Sigma$ and $\Sigma'$ have the same genus and labels. By Lemma 10.3, there exists a rigid map $h: \Sigma \to \Sigma'$. By Proposition 13.8, there exists a $\Phi$-diffeomorphism $g: \Phi^{-1}(G \cdot I)/G \to \Phi'^{-1}(G \cdot I)/G$. $\square$

14. **Global structure of the orbit space $M/G$**

Let $G$ be $SU(2)$ or $SO(3)$ and let $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold. Assume that every moment fiber is tall. Corollary 9.4 states that there exists a $G$-invariant neighborhood $V$ of $\alpha$ such that $\Phi^{-1}(V)/G$ is topologically a trivial surface bundle. Most importantly, we have shown that the restriction map $H^i(\Phi^{-1}(V)/G; \mathbb{Z}) \to H^i(\Phi^{-1}(G \cdot \alpha)/G; \mathbb{Z})$ is one-to-one for $i = 1, 2$ and $\alpha \in V$.

Assume $\alpha \neq 0$ and $\alpha \in \Phi(M)$. There also exists a $G$-invariant neighborhood $V$ of $\alpha$ such that the restriction map $H^i(\Phi^{-1}(V)/G; \mathbb{Z}) \to H^i(\Phi^{-1}(G \cdot \beta)/G; \mathbb{Z})$ is one-to-one for $i = 1, 2$, and $\beta \in V$. This essentially follows from [KT1] since $\Phi^{-1}(G \cdot \beta)/G = \Phi_X^{-1}(\beta)/S^1$, and $\Phi^{-1}(G \cdot I)/G = (G \times S^1, \Phi_X^{-1}(I))/G = \Phi_X^{-1}(I)/S^1$. 


for any interval $I \subset \mathbb{R}_{>0}$. Here $X$ denotes the symplectic cross-section $\Phi^{-1}(\mathbb{R}_{>0})$ and $\Phi_X$ is the corresponding moment map for the Hamiltonian circle action on $X$.

In this section, we will show that the injectivity of the restriction map holds not only locally but also for the manifold $M$.

**Proposition 14.1.** Let $G = SU(2)$ or $SO(3)$ and let $(M, \omega, \Phi)$ be a connected six dimensional Hamiltonian $G$-manifold. Then the restriction map $H^i(M/G; \mathbb{Z}) \rightarrow H^i(\Phi^{-1}(G \cdot \alpha)/G; \mathbb{Z})$ is one-to-one for any $\alpha \in \Phi(M)$ and $i = 1, 2$.

We need several observations to carry out the proof.

**Lemma 14.2.** Let $G$ be $SU(2)$ or $SO(3)$ and let $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold with a moment map $\Phi: M \rightarrow \mathbb{R}^3$. Assume $\Phi^{-1}(G \cdot \alpha)$ for $\alpha \in \mathbb{R}_+$ consists of a single $G$-orbit. Then every neighborhood of $\alpha$ in $\mathbb{R}_+$ contains a smaller neighborhood $V$ such that the quotient $\Phi^{-1}(G \cdot \alpha)/G$ is contractible.

Moreover, any reduced space $\Phi^{-1}(G \cdot \beta)/G$ of complexity one is homeomorphic to a 2-sphere for $\beta \in V$ and $\beta \neq \alpha$.

**Proof.** Assume $\alpha = 0$, and $\Phi^{-1}(0)$ consists of a single orbit. The stabilizer of this single orbit is either $S^1$, or a finite subgroup $\Gamma \subset G$. The local model of the orbit is $G \times_{S^1} (\mathbb{R}^2 \times \mathbb{C})$ or $G \times_{\Gamma} \mathbb{R}^3$. The quotients $(G \times_{S^1} (\mathbb{R}^2 \times \mathbb{C}))/G$ and $(G \times_{\Gamma} \mathbb{R}^3)/G$ are homeomorphic to $(\mathbb{R}^2 \times \mathbb{C})/S^1$ and $\mathbb{R}^3/\Gamma$ respectively. They are both contractible. For $\beta \neq 0$ in the local model, direct computation shows that $\Phi^{-1}(G \cdot \beta)/G = \Phi^{-1}(\beta)/S^1$ is homeomorphic to $S^2$.

For $\alpha \neq 0$, it suffices to show that the statements hold on the symplectic cross-section $X$, which is a complexity one Hamiltonian $S^1$-manifold. It is immediate from [KT1].

**Lemma 14.3.** Let $G$ be $SU(2)$ or $SO(3)$ and let $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold with a moment map $\Phi: M \rightarrow \mathbb{R}^3$. Denote by $M^\circ$ the union of all the tall moment fibers in $M$ and let $I^\circ = \Phi(M^\circ) \cap \mathbb{R}_+$. Then for every $\alpha \in I^\circ$, there exists a neighborhood $V \subset \mathbb{R}_+$ such that $\Phi^{-1}(G \cdot \alpha)$ is homeomorphic to $\Sigma \times (V \cap I^\circ)$, where $\Sigma$ is a surface. In other words, $\overline{\Phi}: M^\circ/G \rightarrow I^\circ$ is topologically a surface bundle where the map $\overline{\Phi}$ is induced by the moment map.

**Proof.** If $\alpha = 0$, this is Corollary 9.4. If $\alpha \neq 0$, we deduce from Proposition 13.7.

In particular, the set $I^\circ$ of points in $\mathbb{R}_+$ whose moment fiber is tall is connected. We have the following result:

**Corollary 14.4.** Let $G$ be $SU(2)$ or $SO(3)$ and let $(M, \omega, \Phi)$ be a six dimensional Hamiltonian $G$-manifold with a moment map $\Phi: M \rightarrow \mathbb{R}^3$. Then all the tall symplectic quotients have the same genus.

Define the genus of a point to be zero. Then the genus is a well-defined notion for the complexity one Hamiltonian $G$-manifold $M$.

**proof of proposition 14.1.** We use Leray-Serre spectral sequence for the induced map $\overline{\Phi}: M/G \rightarrow I = \Phi(M) \cap \mathbb{R}_+$. Cover $I$ with open sets $\{U_i\} = U$, then $\overline{\Phi}^{-1}U$ is a cover for $M/G$. There is a spectral sequence converging to $H^*(M/G)$ with $E_2$ term

$$E_2^{p,q} = H^p(U, \mathcal{H}^q)$$

where $\mathcal{H}^q(U) = H^q(\overline{\Phi}^{-1}(U))$ is the presheaf on $I$. 
Assume first that all the moment fibers in \( M \) are tall. Then the quotient \( M/G \) by Lemma 14.3 is topologically a surface bundle over \( I \). Then \( H^q \) is locally constant on \( U \) and the groups \( H^q(U) = H^q(\Phi^{-1}(U)) = H^q(\Sigma) \) are constant on contractible open sets \( U \). So \( E_{2}^{0,q} = H^q(\Sigma) \), and \( E_{2}^{p,q} = 0 \) for \( p \neq 0 \). The restriction map \( H^i(M/G;\mathbb{Z}) \to H^i(\Sigma;\mathbb{Z}) \) is an isomorphism.

If there exists a short moment fiber, by Lemma 14.2, the genus must be 0. Since the moment fiber is tall for all interior points of \( I \), the short fiber takes place at the end point of \( I \). We cover \( I = [\alpha, \beta] \) with three connected open sets such that \( U_0 \) covers \( \alpha \), \( U_2 \) covers \( \beta \), and \( U_{01}, U_{12} \) are connected and \( U_{02} \) is empty. Then \( E_{1}^{0,0} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, E_{1}^{1,0} = \mathbb{Z} \oplus \mathbb{Z}, E_{1}^{1,2} = \mathbb{Z} \oplus \mathbb{Z}, E_{1}^{2,0} = F_{\alpha} \oplus \mathbb{Z} \oplus F_{\beta} \) where \( F_{\alpha} = \mathbb{Z} \) if \( \Phi^{-1}(\alpha) \) is tall, and \( F_{\alpha} = 0 \) if \( \Phi^{-1}(\alpha) \) is short, and all other \( E_{1}^{p,q} \) vanish. So we have \( E_{2}^{0,0} = \mathbb{Z}, E_{2}^{p,0} = 0 \) for \( p \geq 1 \), \( E_{2}^{p,1} = 0 \) for all \( p \), and \( E_{2}^{2,2} = 0 \). We see \( E_{2}^{0,2} = 0 = E_{2}^{2,1} = E_{2}^{2,0} = 0, \) and \( E_{2}^{0,1} = E_{2}^{1,0} = 0 \). So \( H^i(M/G;\mathbb{Z}) = 0 \) for \( i = 1,2 \). So the restriction to \( H^i(\Sigma;\mathbb{Z}) \) is one-to-one for \( i = 1,2 \). \( \square \)

15. PASSING TO \( M/G \)

**Proposition 15.1.** Let \( G \) be \( SU(2) \) or \( SO(3) \) and let \((M,\omega,\Phi)\) and \((M',\omega',\Phi')\) be compact connected six dimensional Hamiltonian \( G \)-manifolds such that \( \Phi(M) = \Phi'(M') \). There exists an equivariant symplectomorphism from \( M \) to \( M' \) that respects the moment maps if and only if they have the same Duistermaat-Heckman function and there exists a \( \Phi \)-symplectomorphism from \( M \) to \( M' \).

**Proof.** Assume \( f: M \to M' \) is a \( \Phi \)-symplectomorphism. By Proposition 14.1, the restriction map \( H^2(M/G;\mathbb{Z}) \to H^2(\Phi^{-1}(G \cdot \alpha)/G;\mathbb{Z}) \) is injective. The same arguments as in Section 3 apply here; we only need to show that the 2-form \( \omega_i = (1-t)\omega + tf^*\omega' \) is nondegenerate everywhere for \( 0 \leq t \leq 1 \).

Over zero, this is true by Lemma 3.5. Away from zero, we reconstruct the two-form \( \omega_i \) by (13.1). Lemma 3.6 in [KT1] guarantees the nondegeneracy on the symplectic cross-section and therefore on the manifold. \( \square \)

**Proposition 15.2.** Let \( G \) be \( SU(2) \) or \( SO(3) \). Let \((M,\omega,\Phi)\) and \((M',\omega',\Phi')\) be compact connected six dimensional Hamiltonian \( G \)-manifolds such that \( \Phi(M) = \Phi'(M') \). Assume that \( M \) and \( M' \) have the same Duistermaat-Heckman function, that their principal stabilizers of the zero fibers are the same, and that their first Stiefel-Whitney classes of the zero fibers are the same when applicable. There exists a \( \Phi \)-symplectomorphism from \( M \) to \( M' \) if and only if there exists a \( \Phi \)-symplectomorphism from \( M/G \) to \( M'/G \).

**Proof.** Assume there exists a \( \Phi \)-symplectomorphism \( \Psi \) from \( M/G \) to \( M'/G \). Let \( X \) denote the symplectic cross-section \( \Phi^{-1}(\mathbb{R}_{>0}) \) and \( \Phi_X \) denote its moment map. By Lemma 13.5, there exists a \( \Phi_X \)-symplectomorphism \( \psi \) on \( X \). By [KT1], \( \psi \) locally lifts to \( \Phi_X \)-symplectomorphisms.

Pick an open \( S^1 \)-invariant cover \( \{U_i\} \) on \( \Phi^{-1}(\mathbb{R}_+) \) such that \( \Psi \) lifts to a \( \Phi \)-symplectomorphism \( \Psi_1 \) on \( U_1 \) and such that \( U_i \cap \Phi^{-1}(0) = \emptyset \) for all \( i \neq 1 \). We can further refine \( \mathcal{U} \) so that \( (U_i \cap U_j)/S^1 \) is simply connected for all \( i \neq j \) and on each \( U_i, i \neq 1 \), there exists a \( \Phi_X \)-symplectomorphism \( \psi_i: U_i \to M' \) that is a lift of \( \psi \). By Lemma 13.6, \( \Psi_1 \) induces a \( \Phi_X \)-symplectomorphism \( \psi_1 \) on \( U_1 \cap \Phi^{-1}(0) \) and \( \psi_1 \) is a lift of \( \psi \).

By Theorem 15.3 below, there exist smooth \( S^1 \)-invariant functions \( g_{ij}: U_i \cap U_j \to S^1 \) such that \( \psi_j^{-1} \circ \psi_i(m) = g_{ij}(m) \cdot m \) for all \( m \in U_i \cap U_j \). These functions form a
Čech cocycle \( g \in C^1(U, S^1) \). If there exists \( g_i : U_i \to S^1 \) such that \( g_{ij} = g_i^{-1} \cdot g_j \), then \( g_i \cdot \psi_i = g_j \cdot \psi_j \) on \( U_i \cap U_j \). Namely, \( \{ g_i \cdot \psi_i \} \) form a global \( \Phi_X \)-\( S^1 \)-diffeomorphism.

Consider the sheaves of \( S^1 \)-invariant functions in \( \mathbb{R}, S^1, \) and \( \mathbb{Z} \). Since \( (U_i \cap U_j)/S^1 \) are simply connected, \( H^1(U_i \cap U_j; \mathbb{Z}) = 0 \), and the exponential map

\[
\exp : \mathbb{R}(U_i \cap U_j) \to S^1(U_i \cap U_j)
\]

is surjective. The short exact sequence \( 0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1 \) induces an exact sequence \( H^1(U; \mathbb{R}) \to H^1(U; S^1) \to H^2(U; \mathbb{Z}) \to H^2(U; \mathbb{R}) \). Since there exists a partition of unity on \( U \), the cohomology \( H^1(U; \mathbb{R}) = H^2(U; \mathbb{R}) = 0 \) and hence the Bockstein operator \( \delta : H^1(U; S^1) \to H^2(U; \mathbb{Z}) \) is an isomorphism.

Now consider the image \( [c] \) of \( [g] \) in \( H^2(U; \mathbb{Z}) \). Since \( (U_i \cap U_j)/S^1 \) is simply connected, we choose any particular branch of the logarithm, and obtain that \( c_{ijk} = \log g_{jk} - \log g_{ik} + \log g_{ij} \). Since \( M \) and \( M' \) have the same Duistermaat-Heckman function, \([KT1]\) asserts that there exist \( b_{ij} \) such that \( c_{ijk} = b_{jk} - b_{ik} + b_{ij} \). We can choose a different branch of the logarithm \( \log^\text{new} g_{ij} = \log^\text{old} g_{ij} - b_{ij} \) so that \( c_{ijk} = \log g_{jk} - \log g_{ik} + \log g_{ij} = 0 \). Take a partition of unity \( \lambda_i \) subordinate to \( \{ U_i \} \), and define

\[
g_i = \exp\left( - \sum_k \lambda_k \log g_{ik} \right).
\]

Then \( g_i^{-1} \cdot g_j = \exp(\sum_k \lambda_k \log g_{ik} - \sum_k \lambda_k \log g_{jk}) = \exp(\sum \lambda_k \log g_{ij}) = g_{ij} \) on \( U_i \cap U_j \). Note \( g_i = \exp(- \sum \lambda_k \log g_{ik}) = 1 \) in \( U_1 \setminus U_k \) for \( k \neq 1 \). In particular, \( g_1 = 1 \) near \( \Phi^{-1}(0) \). By Lemma 13.3, we extend \( g_i \cdot \psi_i \) from the symplectic cross-section to the entire manifold to obtain a global \( \Phi \)-\( G \)-diffeomorphism and \( g_1 \cdot \Psi_1 = \Psi_1 \) in a neighborhood of \( \Phi^{-1}(0) \).

**Theorem 15.3 ([HS]).** Let \( S^1 \) act on a manifold \( M \). Let \( f : M \to M \) be an equivariant diffeomorphism that preserves the orbits. There exists a smooth invariant function \( h : M \to S^1 \) such that \( f(m) = h(m) \cdot m \) for all \( m \in M \).

**16. Global Uniqueness**

Let \( G \) be \( SU(2) \) or \( SO(3) \) and let \( (M, \omega, \Phi) \) be a six dimensional Hamiltonian \( G \)-manifold. Let \( E \) denote the set of exceptional orbits in \( M \). We consider the projections \( M \to M/G \) and \( g^* \to g^*/G \) and the map \( \Phi \) induced by the moment map \( \Phi \). The **isotropy skeleton** is the space \( E/G \) where each point is labeled by its isotropy representation, together with the map \( \Phi : E/G \to g^*/G \). Two isotropy skeletons are considered the same if there exists a homeomorphism \( f : E/G \to E'/G \) that sends each point to a point with the same isotropy representation and such that \( \Phi = \Phi \circ f \).

We have the following global uniqueness theorem:

**Theorem B.** Let \( G \) be \( SU(2) \) or \( SO(3) \). Let \( (M, \omega, \Phi) \) and \( (M', \omega', \Phi') \) be compact connected six dimensional Hamiltonian \( G \)-manifolds such that \( \Phi(M) = \Phi'(M) \). Then \( M \) and \( M' \) are isomorphic if and only if they have the same Duistermaat-Heckman function, the same genus, the same isotropy skeleton, the same principal stabilizers of the manifolds and of the zero fibers, and the same first Stiefel-Whitney class of the zero fibers when applicable.

We now introduce the final ingredient in the proof of the above theorem:
Lemma 16.1. Let $G$ be SU(2) or SO(3) and $(M, \omega, \Phi)$ and $(M', \omega', \Phi')$ be six dimensional Hamiltonian $G$-manifolds. Let $I_1$ and $I_2$ be open intervals in $\mathbb{R}_{>0} \subset \mathfrak{g}^*$ such that $I_1 \cap I_2 \neq \emptyset$. Let $U_i = \Phi^{-1}(G \cdot I_i)$ and $U'_i = \Phi'^{-1}(G \cdot I_i)$ for $i = 1, 2$. Assume that $U_i$ and $U'_i$ admit flattenings and that $U_1 \cap U_2$ contains only orbits with finite stabilizers. Let $g_1: U_1/G \to U'_1/G$ and $g_2: U_2/G \to U'_2/G$ be $\Phi$-diffeomorphisms. Then there exists a $\Phi$-diffeomorphism $\tilde{g}: (U_1 \cup U_2)/G \to (U'_1 \cup U'_2)/G$ such that $\tilde{g} = g_1$ on $U_1 \setminus U_2$.

Proof. Without loss of generality, let $I_1 = (a, b)$, $I_2 = (a, \beta)$, and $I_1 \cap I_2 = (a, b)$. First assume that $U_1 \cap U_2$ has no exceptional orbits. Hence $(U_1 \cap U_2)/G$ is diffeomorphic to the product surface bundle $\Sigma \times (a, b)$ where $\Sigma$ is the reduced space $\Phi^{-1}(G \cdot \mu)/G$ at any $\mu \in \Phi(U_1 \cap U_2)$. Similarly $(U'_1 \cap U'_2)/G \simeq \Sigma' \times (a, \beta)$ where $\Sigma'$ is the reduced space at any $\nu \in \Phi'(U'_1 \cap U'_2)$. Denote $\Sigma \times \{t\}$ by $\Sigma_t$ and $\Sigma' \times \{t\}$ by $\Sigma'_t$. We then set $(g_t)_t = g_t|_{\Sigma_t}$ so that $(g_t)_t$ is a diffeomorphism from $\Sigma_t$ to $\Sigma'_t$ for $t \in (a, b)$. We can choose connections on these two surface bundles so that the flows $f^s: \Sigma_t \to \Sigma_{t+s}$ and $f'^s: \Sigma'_t \to \Sigma'_{t+s}$ of the horizontal lift of the vector field $\partial_s$ on $(a, b)$ satisfy $f'^s \circ (g_t)_t = (g_{t+s})_t \circ f^s$ for $t, t + s \in (a, a + \epsilon)$ and $f'^s \circ (g_{2t})_t = (g_{2t})_{t+\epsilon} \circ f^s$ for $t, t + s \in (b - \epsilon, b)$ for some small $\epsilon > 0$. Moreover, we may assume that these are also defined over $a$ and $b$. We define $\gamma_t: \Sigma_t \to \Sigma'_t$ for $t \in (a, b)$ by

$$
\gamma_t = f^{t-a} \circ (g_t)_a \circ f^{-(t-a)}.
$$

(16.1)

It’s easy to see that $\gamma_t = (g_t)_t$ for $t \in (a, a + \epsilon)$ and

$$
\gamma_t = (g_{2t})_t \circ h_t \quad \text{ for } \quad t \in (b - \epsilon, b)
$$

(16.2)

where $h_t: \Sigma_t \to \Sigma_t$ is a diffeomorphism determined by $(g_1)_a$, $(g_2)_b$, and the flows $f^s$ and $f'^s$.

With the help of a smooth function $\rho: (b - \epsilon, b) \to (b - \epsilon, b)$ such that $\rho(t) = t$ near $b - \epsilon$ and $\rho(b) = 1$ near $b$, we can reparametrize $h_t$ and $\gamma_t$ for $t \in (b - \epsilon, b)$ so that $\gamma_t$ in (16.2) becomes $(g_{2t})_t \circ h$ for $t \in (c, b)$, where $h$ denotes the diffeomorphism $h_b$ and $c$ is some value in $(b - \epsilon, b)$ near $b$.

The flattening of $U_2$ determines a new trivialization $\Phi^{-1}(G \cdot (c, b))/G \simeq \Sigma \times (c, b)$. The exceptional sheets defined by this trivialization correspond to a set of points $\{x_i\}$ on $\Sigma$. There exists a rigid diffeomorphism $\lambda: \Sigma \to \Sigma$ such that $\lambda(h(x_i)) = x_i$. In fact, there exists an isotopy $\lambda_t: \Sigma_t \to \Sigma_t$ such that $\lambda_t \equiv \text{id}$ near $c$ and $\lambda_t = \lambda$ near $b$. So we can construct a new map $g: (U_1 \cup U_2)/G \to (U'_1 \cup U'_2)/G$ such that

$$
g|_{\Sigma_t} = \begin{cases} 
(g_1)_t, & a < t < a + \epsilon \\
\gamma_t, & a + \epsilon \leq t \leq b - \epsilon \\
(g_2)_t \circ H_t, & b - \epsilon \leq t \leq b - b - \delta \\
(g_2)_t \circ H_t, & b - \delta < t < b 
\end{cases}
$$

(16.3)

where $\epsilon > \delta > 0$ are small and $H: \Sigma \to \Sigma$ is a rigid diffeomorphism which sends the points on $\Sigma$ that correspond to the exceptional sheets on $U_2$ to themselves.

By Proposition 13.8, the rigid map $H$ extends to a $\Phi$-diffeomorphism $\tilde{H}$ from $(U_2 \setminus U_1)/G$ to $(U'_2 \setminus U'_1)/G$. And the new $\Phi$-diffeomorphism $g: (U_1 \cup U_2)/G \to (U'_1 \cup U'_2)/G$ can be defined as $g_1$ on $(U_1 \cup U_2)/G$, $g$ on $(U_1 \cup U_2)/G$, and $g_2 \circ \tilde{H}$ on $(U_2 \setminus U_1)/G$.

Now assume there exist exceptional orbits in $U_1 \cap U_2$. By assumption, every orbit in $U_1 \cap U_2$ has a finite stabilizer. The moment map $\Phi: U_1 \cap U_2 \to \mathbb{R}^3$ is then a proper
submersion. Since $0 \notin \Phi(U_1 \cap U_2)$, the norm square of the moment map $|\Phi|^2$ from $U_1 \cap U_2$ to $I_1 \cap I_2$ is also a proper submersion. Therefore $U_1 \cap U_2$ is $G$-equivariant diffeomorphic to a trivial bundle $Z \times (a,b)$ where $Z/G$ is homeomorphic to the reduced space $\Phi^{-1}(G \cdot \mu)/G \simeq \Sigma$.

Let $g_i : U_i/G \to U'_i/G$ lift to $\Phi$-$G$-diffeomorphisms $\tilde{g}_i : U_i \to U'_i$. We proceed as before, but take $G$-equivariant connections consistent with $\tilde{g}_i$. Again we use the smooth function $\rho$ to reparametrize so that we obtain a new map $\tilde{g} : U_1 \cap U_2 \to U'_1 \cap U'_2$ such that

$$
\tilde{g}|_{Z_t} = \begin{cases}
(\tilde{g}_1)_t, & a < t < a + \varepsilon \\
(\tilde{g}_2)_t \circ \tilde{h}_t, & a + \varepsilon \leq t \leq b - \varepsilon \\
(\tilde{g}_2)_t \circ \tilde{h}_t, & b - \delta < t < b
\end{cases}
$$

We then return to the $\Phi$-diffeomorphism level. Since a $\Phi$-diffeomorphism induced from a $\Phi$-$G$-diffeomorphism takes each exceptional orbit to an exceptional orbit with the same stabilizer, we only need to construct a rigid diffeomorphism that sends the exceptional sheets defined by the flattening of $U_2$ to themselves. This and the rest follow the same arguments as before. \qed

We also need the following lemma: (See [K1] Appendix B, and [S].)

**Lemma 16.2.** Let $k : S^2 \to S^2$ be a diffeomorphism such that $k$ is a rotation on a neighborhood of the north pole and on a neighborhood of the south pole. Then there exists an isotopy $k_t : S^2 \to S^2$, with $k_0 = k$ and $k_1 = \text{id}$, such that each $k_t$ is a rotation on a neighborhood of each pole.

**Proof of Theorem B.** If $0 \notin \Phi(M)$, this is Proposition 13.2. Assume the moment image contains zero. By the assumption on the genus and the isotropy skeletons, there exist a $G$-invariant neighborhood $V$ of $\alpha$ such that $\Phi^{-1}(V)/G$ and $\Phi^{-1}(V)/G$ are $\Phi$-diffeomorphic for each $\alpha \in \frak{g}^*$. We can cover $\Phi(M) \cap \frak{r}_+$ by connected open intervals such that $g_j : \Phi^{-1}(G \cdot I_j)/G \to \Phi^{-1}(G \cdot I_j)/G$ are $\Phi$-diffeomorphisms. Taking refinements if necessary, we can assume that $\{I_j\}$ have no triple intersections, that $\min I_j < \min I_{j+1}$, that $U_j = \Phi^{-1}(G \cdot I_j)$ have flattenings if they contain only tall fibers, and that $U_j \cap U_{j+1}$ contain only orbits with finite stabilizers.

Replace $U_j$ by $U_j \setminus \Phi^{-1}(0)$. We can use induction and Lemma 16.1 to modify $g_1, \ldots, g_{n-1}$ so that we obtain a new $\Phi$-diffeomorphism $\tilde{g}_1 : (U_1 \cup \cdots \cup U_{n-1})/G \to (U'_1 \cup \cdots \cup U'_{n-1})/G$ with $\tilde{g}_1 = g_1$ on $U_1 \setminus U_2$.

If the reduced space $\Phi^{-1}(G \cdot \alpha)/G$ is tall for all $\alpha \in I_n$, we can use induction again on $U_n$ so that we obtain a new $\Phi$-diffeomorphism $\tilde{g} : (U_1 \cup \cdots \cup U_n)/G \to (U'_1 \cup \cdots \cup U'_n)/G$ with $\tilde{g} = g_1$ on $U_1 \setminus U_2$. So $\tilde{g}$ naturally extends to $\Phi^{-1}(0)/G$ and we obtain a global $\Phi$-diffeomorphism $\tilde{g} : M/G \to M'/G$.

By Lemmas 14.2 and 14.3, if the reduced space $\Phi^{-1}(G \cdot \beta)/G$ is short for $\beta = \max I_n$, the reduced space $\Phi^{-1}(G \cdot \alpha)/G$ is homeomorphic to $S^2$ for any $\alpha \neq \beta$ in $I_n$. By [K1], there exist at most two exceptional orbits in $\Phi^{-1}(G \cdot \alpha)$.

Replace $I_n$ by $I_n \setminus \{\beta\}$ and $U_n$ by $U_n \setminus \Phi^{-1}(G \cdot \beta)$. Denote $I_{n-1} \cap I_n$ by $(a,b)$. We construct a map $g : (U_{n-1} \cap U_n)/G \to (U'_{n-1} \cap U'_n)/G$ as in the proof of Lemma 16.1 up to the form in (16.3). By Lemma 16.2, there exists a rigid isotopy $k_1 : S^2 \to S^2$ such that $k_0 = H$ and $k_1 = \text{id}$. So we obtain a new map...
g: \((U_{n-1} \cap U_n)/G \to (U_{n-1}' \cap U_n')/G\) such that
\[
g|_{\Sigma_t} = \begin{cases} 
(\tilde{g}_1)_t, & a < t < a + \epsilon \\
\gamma_t, & a + \epsilon \leq t \leq b - \epsilon \\
(\gamma_n)_t \circ H_t, & b - \epsilon \leq t \leq b - \delta \\
(\gamma_n)_t, & b - \delta < t < b
\end{cases}
\]
where \(\epsilon > \delta > 0\) are small numbers. Therefore the new \(\Phi\)-diffeomorphism \(\tilde{g}\) given by \(\tilde{g}_1\) on \(((U_1 \cup \cdots \cup U_{n-1}) \setminus U_n)/G\), \(g\) on \(((U_1 \cup \cdots \cup U_{n-1}) \cap U_n)/G\), and \(g_n\) on \((U_n \setminus (U_1 \cup \cdots \cup U_{n-1}))/G\) is well-defined on \(U_1 \cup \cdots \cup U_n\) such that \(\tilde{g} = g_1\) on \(U_1 \setminus U_2\) and \(\tilde{g} = g_n\) on \(U_n \setminus U_{n-1}\). Hence \(\tilde{g}\) naturally extends to \(\Phi^{-1}(0)/G\) as well as \(\Phi^{-1}(G \cdot \alpha)/G\), where \(\alpha = \max I_n\). The map \(\tilde{g}: M/G \to M'/G\) is a global \(\Phi\)-diffeomorphism.

By Proposition 15.2, there exists a \(\Phi\)-diffeomorphism from \(M\) to \(M'\). By Proposition 15.1, there exists an equivariant symplectomorphism from \(M\) to \(M'\).

\[\square\]

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