On pulse broadening for optical solitons

H.J.S. Dorren and J.J.B. van den Heuvel
Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Abstract

Pulse broadening for optical solitons due to birefringence is investigated. We present an analytical solution which describes the propagation of solitons in birefringent optical fibers. The special solutions consist of a combination of purely solitonic terms propagating along the principal birefringence axes and soliton-soliton interaction terms. The solitonic part of the solutions indicates that the decay of initially localized pulses could be due to different propagation velocities along the birefringence axes. We show that the disintegration of solitonic pulses in birefringent optical fibers can be caused by two effects. The first effect is similar as in linear birefringence and is related to the unequal propagation velocities of the modes along the birefringence axes. The second effect is related to the nonlinear soliton-soliton interaction between the modes, which makes the solitonic pulse-shape blurred.

E-mail: H.J.S.Dorren@ele.tue.nl
I. Introduction

The demand for long-distance high-bandwidth data transfer in the field of optical telecommunications has lead to research about the use of optical envelope solitons as information carriers. Solitons are ideal for optical telecommunications because the dispersion of the optical fiber is exactly counterbalanced by the nonlinearity. As a result of this, the information carrier can maintain its pulse-shape over long distances. In an ideal optical fiber, the nonlinear Schrödinger equation describes the electromagnetic field envelope in a single polarization case. Actual optical fibers however, have the property that there is a difference in the propagation velocity for the two different polarization states of the electromagnetic field. This property is called birefringence. The effects of birefringence are almost never steady, but they vary randomly, in both magnitude and orientation. The random birefringence will ensure that an initially localized pulse will eventually disintegrate. This effect is called Polarization Mode Dispersion (PMD). PMD is important in situations where high-bit-rate optical signals have to be transported over long distances such as submarine intercontinental optical connections. This is the reason that the topic has already had considerable attention in both theoretical and experimental research in the field of optical telecommunications [1]-[13].

The dynamics of nonlinear waves in birefringent optical fibers is described in the literature by the following set of coupled nonlinear differential equations, which were originally introduced by Berkhoer and Zakharov [12]:

\[ i u_{1x} + i \delta u_{1t} + \frac{1}{2} u_{1tt} = -\left( |u_1|^2 + \gamma |u_2|^2 \right) u_1, \]
\[ i u_{2x} - i \delta u_{2t} + \frac{1}{2} u_{2tt} = -\left( |u_2|^2 + \gamma |u_1|^2 \right) u_2. \]

A detailed derivation of Eq.(1), which are from now on called the birefringence equations, can be found in the book by Hasegawa [14]. In Eq.(1), the (scaled) electromagnetic field envelopes of the different polarization states (modes) are described by \( u_1 \) and \( u_2 \) respectively. The parameter \( \gamma \) describes the strength of the cross-phase modulation. For single mode optical fibers we can use \( \gamma = \frac{2}{3} \). The parameter \( \delta \) describes the group velocity birefringence. In the limit \( \delta \to 0 \), the well known Manakov equation is retained. Ueda and Kath have argued that the parameter \( \delta \) is actually a random parameter with zero mean:

\[ \langle \delta \rangle = 0. \]

As a result of this, Eq.(1) is a stochastic differential equation. In the following, we treat the parameter \( \delta \) as the half-difference of the group velocity between the local principal birefringence axes. If we are able to solve Eq.(1) analytically however, we can obtain analytical expressions for the propagation properties of an initially localized soliton in a birefringent medium.

The set of coupled equations (1) have also been investigated by others. In Ref.[15] the behavior of solutions of Eq.(1) is investigated numerically. One of the conclusions of Ref.[15] is that for solitonic initial conditions the two partial pulses (modes) always move together. A complicated pattern of soliton-soliton interactions between the modes is presented. Another important result is presented in Ref.[16]. Firstly, it is shown in this publication that by applying the transformation:

\[ u_1 = \tilde{u}_1 \exp \left( \frac{i \delta^2}{2} x - i \delta t \right), \]
\[ u_2 = \tilde{u}_2 \exp \left( \frac{i \delta^2}{2} x + i \delta t \right), \]

the set of equations Eq.(1) transform into:

\[ i \tilde{u}_{1x} + \frac{1}{2} \tilde{u}_{1tt} = -\left( |\tilde{u}_1|^2 + \gamma |\tilde{u}_2|^2 \right) \tilde{u}_1, \]
\[ i \tilde{u}_{2x} + \frac{1}{2} \tilde{u}_{2tt} = -\left( |\tilde{u}_2|^2 + \gamma |\tilde{u}_1|^2 \right) \tilde{u}_2. \]
If $\gamma = 1$, the set of equations (4) are called the Manakov equations. The Manakov equations are well-studied and solutions can be obtained in several ways (17, 18, 19, 20, 21, 22, 23). It has to be remarked that the transformation (3) leads to special solutions of Eq.(1), in which the effect of the birefringence appears in the phase factor. Since under experimental conditions usually the optical power is measured (i.e. $|u_i|^2$), the transformation (3) leads to special solutions of Eq.(1), which can not explain the decay of optical pulses due to birefringence (since the birefringence is in the phase). On the other hand in Ref.[16] solutions are reported which show that Eq.(4) has modes which propagate with a different velocity. The existence of such a solution might be important for explaining the decay of solitonic pulses in birefringent optical fibers. For linear birefringence it is well known that the solitonic part of the modes propagate with a different velocity. The second effect, which is absent in linear birefringence, is the nonlinear soliton-soliton interaction between the modes. The paper is concluded with a discussion.

In Sect.II, we apply concepts derived in earlier publications to find special solutions of the birefringence problem. We derive solutions consisting of of two unperturbed solitons propagating with a different velocity along the principal birefringence axes. Moreover the solutions contain interactions terms between the pure solitonic solutions. It is concluded that two effects play a role in the decay of solitonic pulses. The effect is similar as in linear birefringence and is due to the unequal propagation velocity of the modes. The second effect, which is absent in linear birefringence is the nonlinear soliton-soliton interaction between the modes. The paper is concluded with a discussion.

II. Special solutions of the birefringence equations

For solving Eq.(4), we try a starting point the following free wave solutions:

$$u_1(x,t) = Ae^{i(k_1 t - \omega_1 x)},$$
$$u_2(x,t) = Be^{i(k_2 t - \omega_2 x)}.$$  \tag{5}

If we substitute Eq.(5) into Eq.(4), we obtain:

$$\begin{align*}
\left[\omega_1 - \delta k_1 - \frac{1}{2}k_1^2\right]Ae^{i(k_1 t - \omega_1 x)} &= -A|A|^2e^{2i(k_1 t - \omega_1 x)}e^{-i(k_2 t - \omega_2 x)} - \gamma A|B|^2e^{i(k_1 t - \omega_1 x)}e^{i(k_1 - k_2 t - [\omega_2 - \omega_1] x)}, \\
\left[\omega_2 + \delta k_2 - \frac{1}{2}k_2^2\right]Be^{i(k_2 t - \omega_2 x)} &= -B|B|^2e^{2i(k_2 t - \omega_2 x)}e^{-i(k_2 t - \omega_2 x)} - \gamma B|A|^2e^{i(k_2 t - \omega_2 x)}e^{i(k_1 - k_2 t - [\omega_2 - \omega_1] x)}.
\end{align*}$$ \tag{6}

In Eq.(6), it is assumed that $k_1$ and $\omega_1$ are both complex (the bars indicate the complex conjugates). We require that the following dispersion relationships are valid:

$$\omega_1 = \delta k_1 + \frac{1}{2}k_1^2,$$
$$\omega_2 = -\delta k_2 + \frac{1}{2}k_2^2.$$ \tag{7}

We can conclude that for nonzero coefficients $A$ and $B$, the try-solution (5) does not satisfy Eq.(4) and hence the birefringence equations have no free wave solutions. Before deriving special solutions of Eq.(4) the following simplifications are imposed to simplify the computations which will follow:

$$I: \quad k_1 = k_2 = k; \quad k = a + ib,$$
$$II: \quad a = 0; \quad k = ib.$$ \tag{8}
Condition I is related to the fact that we strive for a situation in which both the linear part as the nonlinear part of Eq. (1) are expanded in the same basis functions. In experiments, this situation can be obtained by optimizing the “launching conditions”. Condition II is introduced by realizing that in fiber optics only envelope solutions can be measured. In this paper, we search for modes $u_1$ and $u_2$ propagating with unequal velocity. We therefore substitute the following solutions into Eq. (1):

$$u_1(x, t) = e^{\frac{i}{2}b^2 x} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{A}_{2n+1,m} e^{-[2n+1]b_1} e^{-2mb_2},$$

$$u_2(x, t) = e^{\frac{i}{2}b^2 x} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{B}_{2n+1,m} e^{-2mb_1} e^{-[2n+1]b_2}. \quad (9)$$

In Eq. (9) it is used that $z_1 = t - \delta x$ and $z_2 = t + \delta x$. For the solutions (9) the left-hand side of Eq. (1) is equal to:

$$iu_{1x} + i\delta u_{1t} + \frac{1}{2} u_{1tt} = e^{\frac{i}{2}b^2 x} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ -\frac{1}{2} b^2 - 4mib\delta + \frac{1}{2} (2n + 2m + 1)^2 b^2 \right] \hat{A}_{2n+1,m} e^{-[2n+1]b_1} e^{-2mb_2},$$

$$iu_{2x} - i\delta u_{2t} + \frac{1}{2} u_{2tt} = e^{\frac{i}{2}b^2 x} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ -\frac{1}{2} b^2 + 4mib\delta + \frac{1}{2} (2n + 2m + 1)^2 b^2 \right] \hat{B}_{2n+1,m} e^{-[2n+1]b_2} e^{-2mb_1}. \quad (10)$$

We can conclude that the linear part of Eq. (1) is not modifying the structure of the exponential functions (9). This appears also to be the case for the nonlinear part. If we substitute Eq. (9) into the right-hand side of Eq. (1) we obtain:

$$|u_1|^2 + \gamma |u_2|^2 = e^{\frac{i}{2}b^2 x} \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} \hat{A}_{2q+1,w} \hat{A}_{2p-2q+1,v-w} \hat{A}_{2n-2p-1,m-v} e^{-[2n+1]b_1} e^{-2mb_2},$$

$$+ \gamma e^{\frac{i}{2}b^2 x} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} \hat{B}_{2m-2n-1,n-p} \hat{B}_{2w-1,q} \hat{A}_{2p-2q+1,v-w} e^{-[2n+1]b_1} e^{-2mb_2},$$

$$|u_2|^2 + \gamma |u_1|^2 = e^{\frac{i}{2}b^2 x} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} \hat{B}_{2q+1,w} \hat{B}_{2p-2q+1,v-w} \hat{B}_{2n-2p-1,m-v} e^{-[2n+1]b_2} e^{-2mb_1},$$

$$+ \gamma e^{\frac{i}{2}b^2 x} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{\infty} \hat{A}_{2m-2n-1,n-p} \hat{A}_{2w-1,q} \hat{B}_{2p-2q+1,v-w} e^{-[2n+1]b_2} e^{-2mb_1}. \quad (11)$$

It can be concluded by comparing Eq. (10) and Eq. (11) that solutions of the type (9) ensure that both the linear part and the nonlinear part of Eq. (1) can be expanded in the same basis functions. Given non-zero coefficients $\hat{A}_{1,0}$ and $\hat{B}_{1,0} (A, B \in R)$, all the other coefficients $\hat{A}_{2n+1,m}$ and $\hat{B}_{2n+1,m}$ of the solution (9) are determined by the following
iteration series:

\[ A_{2n+1,m} = - \sum_{p=0}^{n-1} \sum_{q=0}^{p} \sum_{v=0}^{m} \sum_{\nu=0}^{v} \hat{A}_{2q+1,w} \hat{A}_{2p-2q+1,v-\nu} \hat{A}_{2n-2p-1,m-\nu} - \frac{\gamma}{2} b^2 - 4 m b^2 + \frac{1}{2} (2n + 2m + 1)^2 b^2 \]

\[ B_{2n+1,m} = - \sum_{p=0}^{n-1} \sum_{q=0}^{p} \sum_{v=0}^{m} \sum_{\nu=0}^{v} \hat{B}_{2q+1,w} \hat{B}_{2p-2q+1,v-\nu} \hat{B}_{2n-2p-1,m-\nu} - \frac{\gamma}{2} b^2 + 4 m b^2 + \frac{1}{2} (2n + 2m + 1)^2 b^2 \]

(12)

where in the second terms on the right-hand side \( m > 0 \).

The solution (8) of Eq.(12) is purely formal and does not give adequate insight of the behavior of solitonic solutions in birefringent media. In order to obtain this insight, it is convenient to separate the solutions \( u_1(x,t) \) and \( u_2(x,t) \) into two parts:

\[ u_1(x,t) = e^{\frac{\theta}{2} b^2 x} \sum_{n=0}^{\infty} \hat{A}_{2n+1,0} e^{-[2n+1]b_2 x} + e^{\frac{\theta}{2} b^2 x} \sum_{n=0}^{\infty} \hat{A}_{2n+1,m} e^{-[2n+1]b_2 x} e^{-2m b z}, \]

\[ u_2(x,t) = e^{\frac{\theta}{2} b^2 x} \sum_{n=0}^{\infty} \hat{B}_{2n+1,0} e^{-[2n+1]b_2 x} + e^{\frac{\theta}{2} b^2 x} \sum_{n=0}^{\infty} \hat{B}_{2n+1,m} e^{-[2n+1]b_2 x} e^{-2m b z}. \]

(13)

In this special case, we find that given the coefficients \( \hat{A}_{2n+1,0} \) and \( \hat{B}_{2n+1,0} \) are given by:

\[ \hat{A}_{2n+1,0} = - \sum_{p=0}^{n-1} \sum_{q=0}^{p} \frac{\hat{A}_{2q+1,w} \hat{A}_{2p-2q+1,v} \hat{A}_{2n-2p-1,w}}{- \frac{\gamma}{2} b^2 + \frac{1}{2} (2n + 1)^2 b^2} \]

\[ \hat{B}_{2n+1,0} = - \sum_{p=0}^{n-1} \sum_{q=0}^{p} \frac{\hat{B}_{2q+1,w} \hat{B}_{2p-2q+1,v} \hat{B}_{2n-2p-1,w}}{- \frac{\gamma}{2} b^2 + \frac{1}{2} (2n + 1)^2 b^2} \]

(14)

We firstly analyze the first part the right-hand side of Eq.(13). By assuming that \( \hat{A}_{1,0} = A \) and \( \hat{B}_{1,0} = B \) all the other coefficients \( \hat{A}_{2n+1,0} \) and \( \hat{B}_{2n+1,0} \) follow from Eq.(14). This implies that Eq.(13) can be reformulated in the following form [23]:

\[ u_1(x,t) = \frac{1}{2} A e^{\frac{\theta}{2} b^2 x} e^{i \xi_0} \text{Sech}(b |t - \delta x| + \xi_0) + e^{\frac{\theta}{2} b^2 t} \sum_{n=0}^{\infty} \hat{A}_{2n+1,m} e^{-[2n+1]b_2 x} e^{-2m b z}, \]

\[ \xi_0 = - \frac{1}{2} \log \left( \frac{A^2}{4 b^2} \right); \]

\[ u_2(x,t) = \frac{1}{2} B e^{\frac{\theta}{2} b^2 x} e^{i \xi_0} \text{Sech}(b |t + \delta x| + \xi_0) + e^{\frac{\theta}{2} b^2 t} \sum_{n=0}^{\infty} \hat{B}_{2n+1,m} e^{-[2n+1]b_2 x} e^{-2m b z}, \]

\[ \xi_0 = - \frac{1}{2} \log \left( \frac{B^2}{4 b^2} \right); \]

(15)

The first term on the right-hand side of Eq.(13) represents unperturbed solitons. The propagation velocity of these solitons is determined by the birefringence coefficients \( \delta \). It follows from Eq.(13) that the two solitons \( u_1(x,t) \) and \( u_2(x,t) \) propagate with a relative velocity \( 2b \delta \). This makes that even in an “ideal situation”, in which the non-solitonic
terms in Eq. (15) are negligible an initially localized pulse is still unstable due do the unequal propagation velocity of the solitonic solutions along the principal birefringence axes.

The second effect, which leads to instabilities, is the interaction between the unperturbed solitonic solutions. The asymptotic expansion of these instabilities is given by the sum of terms on the right-hand side of Eq. (15). In principle, by using the recursion relationship (12), we can compute all the expansion coefficients \( \hat{A}_{2n+1,m} \) and \( \hat{B}_{2n+1,m} \). Only in special cases it is possible to carry out the summation explicitly to obtain an analytical expression for the interaction.

In Appendix A it is indicated that the summation in Eq. (15) converges if \( \delta < b \). The interaction terms are proportional to a factor depending on \( b, \delta \) and \( \gamma \). From this result the conclusion can be drawn that the interaction terms are small if (Appendix A):

\[
\left| \frac{b(1 + \gamma)}{2\delta - 6b} \right| \ll 1. \tag{16}
\]

If the condition (16) is satisfied, the interaction between the modes is small compared to the solitonic part, and hence, the only that causes initially localized pulses to disintegrate is the unequal propagation velocity of the modes.

III. Discussion

We have shown that the equations (1) have special solutions which consist of solitonic pulses each propagating with a different velocity along the principal birefringence axes. The purely solitonic pulses are subject to soliton-soliton interactions between the modes. This indicates that there are two mechanisms in birefringent optical fibers which can be responsible for the decay of optical solitons in birefringent optical fibers.

The first mechanism is exactly equal as for linear optical pulses. The solitonic optical pulses can decay because of a different propagation velocity of the two modes along the principal birefringence axes. It is no surprise that also solitonic pulses are subject to linear birefringence since the propagation velocity of the solitonic part of the solutions (15) is determined by the linear part (dispersion relations) of the differential equation (1). The nonlinearity in the differential equation (1) has only influence on the shape of the solutions and keeps the propagation velocity unaffected. If we consider for instance the propagation properties of solitons in birefringent optical fibers, we can conclude that for nonlinear birefringence similar criteria as for linear birefringence apply. If we remember that Eq. (1) is defined in a coordinate frame moving with velocity \( v_0 \), we find that solutions \( u_1(x, t) \) and \( u_2(x, t) \) propagate with velocities \( v_0 - (\delta v_0) \) and \( v_0 + (\delta v_0) \). In Ref. [13] it is shown by using a mathematical analysis based upon Ref. [27] that in this case the PMD in long birefringent fibers increases as the square root of the length.

The second effect, which can lead to decay of solitonic optical pulses is the nonlinear soliton-soliton interaction between the modes. This effect has been reported in Refs. [15, 16]. In the numerical results presented in Ref. [15] it is concluded that only soliton-soliton interactions played a role in the decay of solitons. It is not observed that two initially solitonic solutions propagate with a different velocity along the birefringence axes. We think that this result can be explained by the fact that in Ref. [15] perfect solitonic pulse-shapes are taken as the initial condition. If we solve Eq. (4), it follows from Eq. (7) that in the case that \( \delta = 0 \), both modes can be chosen to have equal dispersion relations. Hence the solutions of Eq. (4) can be expanded in the following form:

\[
\tilde{u}_1(x, t) = e^{\frac{1}{2} b^2 x} \sum_{n=1}^{\infty} \hat{A}_n e^{-nb t},
\]

\[
\tilde{u}_2(x, t) = e^{\frac{1}{2} b^2 x} \sum_{n=1}^{\infty} \hat{B}_n e^{-nb t}.
\]

For nonzero coefficients \( \hat{A}_1 \) and \( \hat{B}_2 \), all the other coefficients can be determined by a similar iteration series as presented in [27]. This implies that the Manakov equations (4) have solitonic solutions. It might be possible that by choosing
purely solitonic initial conditions, the numerical simulations in Ref. \cite{15} lead to the special solutions \cite{13}. In realistic (experimental) optical systems it is difficult to make pure solitons and a certain amount of noise will always be present. The appearance of noise can cause that the modes propagate with a different velocity and that the actual situation is more complicated as presented in \cite{15}.

We want to conclude by remarking that there is another way to understand the result obtained in Eq.\((15)\). We therefore firstly implement the following Galileian transformations:

\[
z_1 = t - \delta x \\
z_2 = t + \delta x
\]  

(18)

As a result of this transformation we find that for \(\epsilon = 1\), Eq.\((1)\) can be reformulated as:

\[
i\hat{u}_{1x} + \frac{1}{2}i\hat{u}_{1z_1z_1} + |\hat{u}_1|^2\hat{u}_1 = -\epsilon\gamma |\hat{u}_2|^2\hat{u}_1, \\
i\hat{u}_{2x} + \frac{1}{2}i\hat{u}_{2z_2z_2} + |\hat{u}_2|^2\hat{u}_2 = -\epsilon\gamma |\hat{u}_1|^2\hat{u}_2.
\]  

(19)

Eq.\((19)\) describes a perturbed nonlinear Schrödinger equation for both the modes \(\hat{u}_1\) and \(\hat{u}_2\). It should be remarked that both the modes \(\hat{u}_1\) and \(\hat{u}_2\) have a different time-evolution due to the rescaled time. If we try to solve Eq.\((19)\) by implementing a perturbation series:

\[
\hat{u}_1 = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdots, \\
\hat{u}_2 = g_0 + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3 + \cdots,
\]  

(20)

we obtain for the lowest order:

\[
i f_{0x} + \frac{1}{2} f_{0z_1z_1} + |f_0|^2 f_0 = 0, \\
i g_{0x} + \frac{1}{2} g_{0z_2z_2} + |g_0|^2 g_0 = 0.
\]  

(21)

\((19)\) is the nonlinear Schrödinger equation. The solutions for \(f_0\) and \(g_0\) are therefore nonlinear Schrödinger solitons which are defined with respect to the transformed time \(z_1 = t - \delta\) and \(z_2 = t + \delta\) respectively. Furthermore, \(f_0\) and \(g_0\) are contaminated with the higher order perturbations \(f_n\) and \(g_n\). This is exactly what we find in Eq.\((15)\) where the asymptotic behavior of \(f_n\) and \(g_n\) is presented. As expected, the functions \(f_n\) and \(g_n\) have a time-evolution which is a combination of (exponential) functions which are defined with respect to the transformed time \(z_1\) and \(z_2\).

**Acknowledgments**

This research was supported by the Netherlands Organization for Scientific Research (N.W.O.) through the “N.R.C. Photonics” grant.
Appendix A

Convergence of Eq. (15)

In this Appendix the convergence of the summation in Eq. (15) is discussed. We estimate the coefficients \( \hat{A}_{2n+1,m} \) and \( \hat{B}_{2n+1,m} \) with respect to \( \hat{A}_{2n+1,0} \) and \( \hat{B}_{2n+1,0} \).

\[
\left| \hat{A}_{2n+1,m} \right| = \left| \sum_{p=0}^{n-1} \sum_{q=0}^{p} \sum_{v=0}^{m} \sum_{w=0}^{v} \frac{\hat{A}_{2q+1,w} \hat{A}_{2p-2q+1,v-w} \hat{A}_{2n-2p-1,m-v}}{\frac{1}{2} b^2 - 4mb\delta + \frac{1}{2} (2n + 2m + 1)^2 b^2} \right|

+ \gamma \left| \sum_{p=0}^{n} \sum_{q=0}^{p} \sum_{v=0}^{m-1} \sum_{w=0}^{v} \frac{\hat{B}_{2n-2q-1,m-p} \hat{B}_{2w-1,q} \hat{A}_{2p-2q+1,v-w}}{\frac{1}{2} b^2 - 4mb\delta + \frac{1}{2} (2n + 2m + 1)^2 b^2} \right|

\leq (1 + \gamma) \left( \sum_{p=0}^{n-1} \sum_{q=0}^{p} \sum_{v=0}^{m} \sum_{w=0}^{v} \hat{C}_{2q+1,w} \hat{C}_{2p-2q+1,v-w} \hat{C}_{2n-2p-1,m-v} \left( \frac{1}{2} b^2 + \frac{1}{2} (2n + 2m + 1)^2 b^2 \right) \right)

\leq (1 + \gamma) \left( \sum_{p=0}^{n-1} \sum_{q=0}^{p} \sum_{v=0}^{m} \sum_{w=0}^{v} \hat{C}_{2q+1,w} \hat{C}_{2p-2q+1,v-w} \hat{C}_{2n-2p-1,m-v} \left( \frac{1}{2} b^2 + \frac{1}{2} (2n + 2m + 1)^2 b^2 \right) \right)

\leq (1 + \gamma) \left( \sum_{p=0}^{n} \sum_{q=0}^{p} \sum_{v=0}^{m} \sum_{w=0}^{v} \hat{C}_{2q+1,w} \hat{C}_{2p-2q+1,v-w} \hat{C}_{2n-2p-1,m-v} \left( \frac{1}{2} b^2 + \frac{1}{2} (2n + 2m + 1)^2 b^2 \right) \right)

\leq \frac{b(1 + \gamma)}{2\delta - 6b} \left( \sum_{p=0}^{n} \sum_{q=0}^{p} \hat{C}_{2q+1,0} \hat{C}_{2p-2q+1,0} \hat{C}_{2n-2p-1,0} \right)

= \frac{b(1 + \gamma)}{2\delta - 6b} \left( \hat{C}_{2n+1,0} \right)

\hat{C}_{2n+1,0} = \max \left\{ \hat{A}_{2n+1,0}, \hat{B}_{2n+1,0} \right\}

The coefficients \( \hat{B}_{2n+1,m} \) can be estimated in a similar way as the coefficients \( \hat{B}_{2n+1,m} \). In Step 1, we replace either the coefficients \( \hat{A}_{n,m} \) or \( \hat{B}_{n,m} \) by \( \hat{C}_{n,m} = \max \{ \hat{A}_{n,m}, \hat{B}_{n,m} \} \). This replacement is justified because:

\[
|u_1|^2 u_1 + |u_2|^2 u_2 \leq (1 + \gamma)|u|^2 v \quad v = \max \{ u_1, u_2 \}
\]

In Step 2, the triangle inequality is applied to the complex denominator. In Step 3, the quotient presented in in Eq. (A-1) reaches an upper-limit by putting \( m = 0 \) in the denominator. Finally, in Step 4 it is used that:

\[
\frac{1}{2} b^2 + \frac{1}{2} (2n + 2m + 1)^2 b^2 > 4mb\delta
\]
Eq. (A-3) is valid if $\delta \leq b$ for all $n, m > 0$. Furthermore, the summation over $m$ is estimated to be smaller than $m^2$ times the summation over coefficients for which $m = 0$. From Eq. (A-3), it follows that:

$$m^2 < \frac{b}{2\delta - 6b}$$ (A-4)

As a result of Eq. (A-4) we can estimate an upper limit of the double sum in Eq. (13). Because the coefficients $|\hat{A}_{2n+1,m}|$ and $|\hat{B}_{2n+1,m}|$ are proportional to $|\hat{A}_{2n+1,0}|$ and $|\hat{B}_{2n+1,0}|$, we can estimate the summation over $n$ in Eq. (13) is proportional to the unperturbed soliton. The summation over $m$ can be carried out independently, and remains finite because it forms a geometric series. Finally we obtain:

$$u_1(x, t) \leq \frac{1}{2} A e^{\frac{b}{4} b^2 x} e^{i \xi_0} \text{sech}(b[t - \delta x] + \xi_0) + \frac{1}{2} C \frac{b(1 + \gamma)}{2\delta - 6b} \bigg|^{C=\max\{A, B\}}$$

$$u_2(x, t) = \frac{1}{2} B e^{\frac{b}{4} b^2 x} e^{i \hat{\xi}_0} \text{sech}(b[t + \delta x] + \hat{\xi}_0) + \frac{1}{2} C \frac{b(1 + \gamma)}{2\delta - 6b} \bigg|^{C=\max\{A, B\}}.$$ (A-5)
References

[1] M. Fontaine, B. Wu, V.P. Tzolov, W.J. Bock, W. Urbanczyk, *Theoretical and experimental analysis of thermal stress effects on modal polarization properties of highly birefringent optical fibers*, IEEE J. Lightwave Technol 14, 585-591 (1996).

[2] M. Midrio, P. Franco, M. Crivellari, M. Romagnoli, F. Matera, *Polarization shift keying for highbit-rate multilevel soliton transmissions*, J. Opt. Soc. Am. B 13, 1526-1535 (1996).

[3] T. Ueda and W.L. Kath, *Dynamics of optical pulses in randomly birefringent fibers*, Physica D 55, 166-181 (1992).

[4] P.K.A. Wai and C.R. Menyuk, *Polarization mode dispersion, decorrelation and diffusion in optical fibers with randomly varying birefringence*, IEEE J. Lightwave Technol 14, 148-157 (1996).

[5] R.E. Schlueh, J.G. Ellison, A.S. Siddiqui, D.H.O Bebbington, *Polarization OTDR measurements and theoretical analysis on fibers with twist and their implication for estimation of PMD*, Electronics Letters 32, 387-388 (1996).

[6] O. Aso and H. Nakamura, *Principal states of polarization: time domain measurements with coherent optical pulse*, Electronics Letters 32, 578-579 (1996).

[7] B. Perny, C. Zimmer, F. Prieto, N. Gisin, *Polarization mode dispersion: Large scale comparison of Jones matrix eigenanalysis against interferometric measurements techniques*, Electronics Letters 32, 680-681 (1996).

[8] A. Galtarossa, G. Gianello, C.G. Someda, M. Schiano, *In-Field comparison among polarization mode dispersion measurement techniques*, IEEE J. Lightwave Technol 14, 42-48 (1996).

[9] E. Iannone, A. Mecozi, F. Matera, M. Settembre *Single-channel transmissions over very long optical links with zero average chromatic dispersion adopting NRZ intensity and polarization modulation*, Optics Communications 123, 89-93 (1996).

[10] B.L. Heffner, *Influence of optical source characteristics on the measurements of polarization mode dispersion of highly mode coupled fibers*, Optics Letters 21, 113-115 (1996).

[11] C.R. Menyuk, Opt. Lett. 12, 614 (1987); J. Opt. Soc. Am. B 5, 392 (1988).

[12] A.L. Berkhoer and V.E. Zakharov, *Self excitation of waves with different polarizations in nonlinear media*, Sov. Phys. JETP 31, 486-490, 1970.

[13] N. Gisin, J.P. Von der Weid and J.P. Pellaux, *Polarization Mode Dispersion of short and long single mode fibers*, J. Lightwave Technol. 9, 821, 1991.

[14] A. Hasegawa and Y.Kodama, *Solitons in optical communications*, Clarendon Press, Oxford, 1995.

[15] C.R. Menyuk, *Stability of solitons in birefringent optical fibers II. Arbitrary Amplitudes*, J. Opt. Soc. Am. B, 5, pp. 392-402, 1988.

[16] T. Ueda and W.L. Kath, *Dynamics of coupled solitons in nonlinear optical fibers*, Phys. Rev. A. 42 pp. 563-571, 1990.

[17] S.V. Manakov, *On the theory of two-dimensional stationary self-focusing of electromagnetic waves*, Sov. Phys. JETP 38, 248-252, 1974.

[18] V.E. Zakharov and E.I. Schulman, *Degenerative dispersion laws, motion invariants and kinetic equations*, Physica 1D, 192-202, 1980.
[19] R. Radhakrishnan and M. Lakshmanan, *Suppression and enhancement of soliton switching during interaction in periodically twisted birefringent fiber*, solv-int/9904006.

[20] V. Shchesnovich, *Polarization scattering by soliton-soliton collisions*, solv-int/9712020.

[21] F.P. Zen, H.I. Elim, *Soliton Solution of the Integrable Coupled Nonlinear Schrodinger Equation of Manakov Type*, hep-th/9812215.

[22] F.P. Zen, H.I. Elim, *Multi-soliton Solution of the Integrable Coupled Nonlinear Schrodinger Equation of Manakov Type*, solv-int/9901010.

[23] F.P. Zen, H.I. Elim, *Lax Pair Formulation and Multi-soliton Solution of the Integrable Vector Nonlinear Schrodinger Equation*, solv-int/9902010.

[24] H.J.S. Dorren, *A linearization method for the Korteweg-de Vries equation; generalizations to higher dimensional S-integrable differential equations*, J. Math. Phys. 39, 3711-3729, 1998.

[25] G.P. Agrawal, *Nonlinear fiber optics, 2nd ed.*, Academic Press, London, 1995.

[26] H.J.S. Dorren, *On the integrability of nonlinear partial differential equations*, J. Math. Phys 40, 1966-1976, 1999 (solv-int/9807007).

[27] M. Kac, *A stochastic model related to the telegrapher’s equation*, Rocky Mountain J. Math. 4, 497, 1974.