Distinct Trivial Phases Protected by a Point-Group Symmetry in Quantum Spin Chains

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The ground state of the $S=1$ antiferromagnetic Heisenberg chain belongs to the Haldane phase—a well known example of symmetry-protected topological phase. However, an applied staggered field breaks all the symmetries protecting the Haldane phase and reduces it to a trivial phase, i.e., the phase is smoothly connected to an antiferromagnetic product state. Nevertheless, as long as the symmetry under site-centered inversion combined with a spin rotation is preserved, the phase is still distinct from another trivial phase. We demonstrate the existence of such distinct symmetry-protected trivial phases using a field-theoretical approach and numerical calculations. Furthermore, a general proof and a non-local order parameter are given in terms of an matrix-product state formulation.

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Introduction.—While symmetry broken phases can be completely classified using the Landau theory, there still exists no exhaustive understanding of topological quantum phases. A very general scheme for the classification of quantum phases is based on local unitary transformations (LUT) [1–3]. In this scheme, two gapped ground states of local Hamiltonians belong to the same phase if and only if they can be transformed into each other by LUTs. Equivalently, two gapped ground states belong to the same phase if and only if the one is connected adiabatically to the other by a continuous parameter in the Hamiltonian. An LUT can only change short-range entanglement (SRE) and thus a ground state with intrinsic topological order cannot be connected to a product state by an LUT as it contains long-range entanglement. In contrast, any SRE state can be changed by LUTs to a product state, and thus all SRE states belong to the same “trivial phase”. There thus exists only one SRE phase in this sense.

Once symmetries are imposed, a rich variety of phases appears. That is, if we restrict LUTs to those respecting the imposed symmetry, certain SRE states cannot be connected to another SRE state. These are either states with spontaneous symmetry breaking or belong to the class of symmetry-protected topological (SPT) phase [2–8]. Examples of SPT phases include topological insulators [9], which are protected by time reversal symmetry, and the Haldane phase [10, 11] in one dimension, which is protected by either one of the time reversal, bond-centered inversion, or the dihedral group of the spin rotations [5]. Although SPT phases are often characterized by the existence of gapless edge excitations, these are in fact absent in an SPT phase in some circumstances. For example, when bond-centered inversion is the only symmetry which protects the Haldane phase, there is no physical edge state and thus the ground state on an open chain is not necessarily degenerate. In fact, it was shown that a very general characterization of the Haldane phase is the double degeneracy of the entire entanglement spectrum [5].

While the notion of the SPT phases is now established and widely recognized, in this work we demonstrate that site-centered inversion symmetry allows to distinguish different trivial one-dimensional phases. That is, there are multiple “symmetry-protected trivial” (SPt) phases, i.e., symmetric phases connected to product states by LUTs, which are distinct in the presence of the imposed symmetry. We note that, the word “symmetry-protected trivial phase” is sometimes used in place of the standard terminology of SPT (symmetry-protected topological) phase, because the entanglement in such a phase is short-ranged and is removable by an LUT if the symmetry is disregarded. In contrast, in what is called an SPT phase in this work, the entanglement can be completely removed by an LUT which respects the imposed symmetry, to reduce the state to a product state. However, it can be still distinct from another trivial phase.

The model.—In order to make the discussion concrete, let us first consider the following simple model of $S = 1$ chain:

$$H = \sum_i \left[ \vec{S}_i \cdot \vec{S}_{i+1} + D_z \left( S^{z^2}_i \right)^2 - h_z (-1)^i S^z_i \right]. \quad (1)$$

The first term is the standard Heisenberg model with antiferromagnetic exchange interactions which stabilize the celebrated Haldane gap [10]. The $D_z$ term is the uniaxial single-ion anisotropy, which is commonly present in magnetic ions with $S=1$ such as Ni$^{2+}$. The model with uniaxial anisotropy $D_z$ and $h_z = 0$ is well understood [12–14]. For small $D_z \geq 0$, the system is in the Haldane phase and undergoes a quantum phase transition into the “large-$D$” phase at $D_z \approx 1$. Both phases are gapped and have the full symmetry of the Hamiltonian. The Haldane phase is a well-known example of SPT...
phases [5, 15] as discussed in the introduction. The large-$D$ phase is a trivial phase which is adiabatically connected to the product state $|D⟩ = |⋯ + 000⋯⟩$, where $|0⟩$ represents the local spin state with $S^z = 0$. This state is the exact ground state of the Hamiltonian (1) in the limit $D_z → ∞$. The $h_z$ term represents a staggered field, which occurs in many quasi-one-dimensional materials, including Haldane gap systems, with an alternating crystal structure under an applied (uniform) field. For simplicity, we only include the staggered field term without the uniform one. In the limit $h_z → ∞$, the spins are fully polarized along the staggered field, and the ground state is reduced to another trivial product state $|N⟩ = |⋯ + − + −⋯⟩$, where $+$ and $−$ represent the local spin states with $S^z = +1$ and $S^z = −1$, respectively. It was recognized earlier that there is no phase transition for $0 < h_z < ∞$ (for $D_z = 0$) [16]. That is, the Haldane phase is adiabatically connected to the Neel state $|N⟩$ with imposed antiferromagnetic (AF) order, and thus there is no distinction between the Haldane phase and the imposed AF phase. In the SPT framework, this is naturally understood since the staggered field breaks all the symmetries that protect the Haldane phase as an SPT phase. That is, the “Haldane phase” is reduced to a trivial phase in the absence of the symmetries. Now let us discuss the model with both $D_z$ and $h_z$. In fact, this model has been studied in Ref. [17] by a field theory and numerical methods where a quantum phase transition between the large-$D$ phase and the imposed AF phase was found. This is rather surprising, since both phases are trivial and are adiabatically connected to product states $|D⟩$ and $|N⟩$ that have the full symmetry of the Hamiltonian. It is perhaps even more surprising in the light of the recent concept of the SPT phases, where the existing classification scheme [3, 6, 7] would not distinguish them. While the details of the phase transition were studied in Ref. [17], why (and when) these two trivial phases are distinguished was not completely clarified. In the remainder of this paper, we demonstrate that this is an example of distinct SPT phases and identify the symmetry which protects them.

**Bosonization.**—While there are several symmetries in Hamiltonian (1) which can protect the two SPT phases, we focus here on the combined operation $I' = I_S × R_z$, where $I_S$ is the site-centered inversion and $R_z$ is the global $π$-rotation of spins about $z$ axis. The standard bosonization procedure of $S = 1$ chains starts from two coupled $S = 1/2$ chains, and the low-energy effective field theory for $H$ is given by the Hamiltonian [18–20]

$$H_{\text{eff}} = \frac{v}{2\pi} \int dx \left[ K(\partial_x \theta)^2 + \frac{1}{K}(\partial_x \phi)^2 \right] + g_{\text{eff}} \int dx \cos(2\phi),$$

(2)

where $\phi$ and $\theta$ are dual field of each other, satisfying $[\phi(x), \theta(x')] = i(\pi/2)[\text{sgn}(x - x') + 1]$. The Hamiltonian (2) represents the so-called sine-Gordon field theory, which is ubiquitous in many problems in $1 + 1$ dimensions. Its properties essentially depend on the coupling constant $K$. When $K > 2$, the coupling $g_{\text{eff}}$ is irrelevant under the renormalization group (RG), and the system is renormalized in the low-energy limit into the free boson theory with $g_{\text{eff}} = 0$, which is nothing but a gapless Tomonaga-Luttinger liquid (TLL). On the other hand, if $K < 2$ and $g_{\text{eff}}$ is non-vanishing, the interaction is RG-relevant, and the system acquires an excitation gap. In the absence of the staggered field, the Haldane and the large-$D$ phases correspond to $g_{\text{eff}} > 0$ and $g_{\text{eff}} < 0$ respectively, both with $K < 2$. The quantum phase transition between the two phases is identified with $g_{\text{eff}} = 0$, where the system is a gapless TLL. It is easy to see that, within the effective Hamiltonian (2), the two phases with $g_{\text{eff}} > 0$ and $g_{\text{eff}} < 0$ are always separated by the critical point $g_{\text{eff}} = 0$. This actually comes from the fact that $\cos(2\phi)$ is the only interaction compatible with the symmetry and the compactification, $\phi \sim \phi + \pi$ and $\theta \sim \theta + 2\pi$, up to subleading terms $\cos(2n\phi)$ ($n \geq 2$). In general, the effective theory can also have the $\sin(2\phi)$ term, in addition to $\cos(2\phi)$. The two terms can be combined as $\cos(2\phi + \alpha)$ with a phase shift $\alpha$. It is clear that, by changing $\alpha$ from $0$ to $\pi$, the two phases with $g_{\text{eff}} > 0$ and $g_{\text{eff}} < 0$ are adiabatically connected without closing the gap [19]. Thus, for the two phases to be distinct, $\sin(2\phi)$ has to be forbidden by some symmetry.

In fact, in the framework of bosonization, this is how a symmetry protects the Haldane phase as an SPT phase which is distinct from the trivial large-$D$ phase. Any of the three symmetries, which are known to protect the Haldane phase, forbids the $\sin(2\phi)$ interaction [21]. Here, for brevity, among these three symmetries, we only show the representation of the bond-centered inversion $I_b$ in terms of the bosonic field $\phi$, $\theta$ in Table I. The action of $I_b$, $\phi(x) → −\phi(−x)$, forbids $\sin(2\phi)$, and leads to the distinction of the two phases with $g_{\text{eff}} > 0$ and $g_{\text{eff}} < 0$. On the other hand, Table I also shows that the site-centered inversion $I_s$ also has the same action on $\phi$, and thus it also does not allow $\sin(2\phi)$. However, only $I_s$ is not sufficient to protect the transition. Another hidden assumption in the derivation of Eq. (2) was that the vertex operators of the dual field $\theta$, such as $\cos(n\theta)$, do not appear in the Hamiltonian. These terms are forbidden by the $U(1)$ symmetry under global spin rotation about $z$ axis by an arbitrary angle. Therefore, the combination of the $U(1)$ spin-rotational symmetry and $I_s$ is sufficient to protect the transition. A more careful analysis [21] shows that, it suffices to forbid the leading vertex operators $e^{±i\theta}$, but not necessarily $e^{in\theta}$ with $n \geq 2$, to protect the transition. According to Table I, it is then sufficient to have a symmetry under the global $\pi$-rotation about $z$ axis, $\mathcal{R}_z$. In fact, it is sufficient to have the invariance under the combined operation $\mathcal{I}' = I_s × \mathcal{R}_z$, instead of the two individual symmetries.
TABLE I. Symmetry transformations for the original spins and the bosonic fields.

| Symmetry operation          | Symbol | Transformation for spins | Transformation for fields ($\phi, \theta$) |
|-----------------------------|--------|--------------------------|------------------------------------------|
| Bond-centered inversion     | $I_b$  | $S^z_i \rightarrow S^z_{i-1}$ | $\phi(x) \rightarrow -\phi(-x), \theta(x) \rightarrow \theta(-x) + \pi$ |
| Site-centered inversion     | $I_s$  | $S^z_i \rightarrow S^z_{i-1}$ | $\phi(x) \rightarrow -\phi(-x) + \pi, \theta(x) \rightarrow \theta(-x)$ |
| $\pi$ rotation about $z$ axis | $R_z$  | $S^x_i \rightarrow -S^x_{i+1}, S^y_i \rightarrow S^y_{i+1}$ | $\phi \rightarrow \phi, \theta \rightarrow \theta + \pi$ |

FIG. 1. (Color online) Correlation lengths calculated for the spin-1 chain (3) are plotted against $D_z$. The parameters are varied as (a) $h_z = 0.1$, $d_x = 0$, and (b) $h_z = d_x = 0.1$. Each color and symbol denotes the different number of kept states $\chi$ from 50 to 200. Insets show the nonlocal order parameters $O_i(L)/\text{Tr} \Lambda^2$ for $L = 100$ and 200 (see text).

**Numerical results.**—The above bosonization analysis suggests that we can introduce microscopic models with less symmetries than Eq. (1), but with the symmetry under $T'$, to maintain the two distinct phases. As an example, we consider the following Hamiltonian:

$$\mathcal{H}' = \mathcal{H} + \sum_i d_x \left(S^x_i S^z_{i+1} - S^y_i S^y_{i+1}\right)$$  \hspace{1cm} (3)

The new term $d_x$ represents the (uniform) Dzyaloshinskii-Moriya (DM) interaction with the DM vector parallel to $x$ axis. This term breaks not only the U(1) spin-rotational symmetry about $z$ axis, but also both $I_b$ and $R_z$ as individual symmetries. However, $\mathcal{H}'$ with a non-vanishing $d_x$ still preserves the symmetry $T'$ under the composite operation.

We numerically study Hamiltonian (3) using infinite density-matrix renormalization group (iDMRG) [22–24]. The correlation lengths as functions of $D_z$ are plotted in Fig. 1 for different numbers $\chi$ of kept states and parameters of the model. A divergent correlation length with increasing $\chi$ indicates a critical point. For $h_z = 0.1$ and $d_x = 0$, we find that the Haldane phase and the Néel state ($D_z \rightarrow -\infty$) are adiabatically connected since all of the three symmetries protecting the Haldane phase are broken. However, as found in Refs. [17, 25], the transition at $D_z \sim 1$ still exists (see Fig. 1 (a)). This indicates that there is a phase transition between two trivial phases connected to $|N\rangle$ and $|D\rangle$. To confirm that this transition is protected by $T'$ alone, we further introduce $d_x$ in Fig. 1 (b). A single transition in Fig. 1 (a) is now split into two transitions, but the two phases are still separated by (two) transitions and thus are distinct. In the intermediate phase, an AF order along $x$ axis occurs and thus $T'$ is spontaneously broken. Further details about this calculation are shown in [26].

Once an explicit dimerization in introduced, e.g., by adding a term $\delta \sum_i (-1)^i S^z_i \cdot S^z_{i+1}$ with $\delta \neq 0$, $T'$ is broken without affecting any other symmetries in $\mathcal{H}'$, and there is only one trivial phase. Numerically, we observe that the correlation length remains finite for all values of $D_z$ when $d_x = 0$ [26]. In fact, this can also be shown analytically by considering the limit of $\delta = 1$ with $d_x = 0$. Here, the entire chain is decomposed into isolated dimers. In particular, at $D_z = h_z = 0$, the ground state is simply given by a product of spin singlet states on each dimer. It can be shown, by solving the two-spin problem explicitly [26], that this dimerized state is connected adiabatically to both $D_z \rightarrow \infty$ and $h_z \rightarrow \infty$ limits. Thus, the two trivial product states $|D\rangle$ and $|N\rangle$ can be adiabatically connected through the dimerized limit, and belong to a single phase, in the presence of $\delta$. This fact rules out a possibility that the two trivial phases are distinct under the two-site translation invariance and some on-site symmetry, as indicated in Refs. [6, 7].

**Matrix-product state formulation.**—Matrix-product states (MPS) can represent gapped ground states of local Hamiltonians in one dimension faithfully. Thus, the classification of gapped phases in one dimension, including the SPt phase proposed in the present work, can be proven rigorously within the MPS formalism. Let us begin with the general MPS [27, 28], without assuming any translation invariance:

$$|\psi\rangle = \sum_{\{m_n\}} \cdots \Gamma^{[n+1]} \Lambda^{[n+\frac{1}{2}]} \Gamma^{[n]} \Lambda^{[n-\frac{1}{2}]} \cdots \Gamma^{[1]} \Lambda^{[\frac{1}{2}]} \Gamma^{[0]} \cdots |\psi\rangle,$$  \hspace{1cm} (4)

where $\Lambda^{[n]}$ is a $\chi_a \times \chi_a$ positive diagonal matrix, $\Gamma^{[n]}$ is a
\(\chi_{n-1/2} \times \chi_{n+1/2}\) matrix, and \(m_n\) represents the physical degrees of freedom on site \(n\). An MPS representation is not unique for a given state but we can always choose the canonical MPS \([29]\) satisfying \(\text{Tr}\left[\left(\Lambda^{|a|}\right)^2\right] = 1\), and

\[
\mathcal{E}^{|a|}(\mathbb{I}_{\chi_{n+1/2}}) = \mathbb{I}_{\chi_{n-1/2}}, \quad \mathcal{E}^{|a|}(\mathbb{I}_{\chi_{n-1/2}}) = \mathbb{I}_{\chi_{n+1/2}};
\]

where \(\mathbb{I}_\chi\) is the \(\chi \times \chi\) identity matrix, and \(\mathcal{E}^{|a|}\) and \(\mathcal{E}^{\dagger |a|}\) are completely positive maps defined by

\[
\mathcal{E}^{|a|}(X)(Y) \equiv \sum_m \left(\Gamma_m^{|a|}\right)^\dagger \Lambda_{[n-\frac{1}{2}]} X \Lambda_{[n+\frac{1}{2}]} \left(\Gamma_m^{|a|}\right),
\]

(6)

By introducing the metric \(|X|^2 \equiv \text{Tr}[X(\Lambda^{|a|})^2 X^\dagger]\) in the vector space of \(\chi_a \times \chi_a\) matrices, we can introduce a singular value decomposition of \(\mathcal{E}^{|a|}\) and \(\mathcal{E}^{\dagger |a|}\). The canonical condition Eq. (5) means that the identity matrices are left/right “eigenvectors” of \(\mathcal{E}^{|a|}\) and \(\mathcal{E}^{\dagger |a|}\) belonging to the largest singular value 1. Furthermore, we require the MPS to be pure, that is the largest singular value 1 is non-degenerate. The pureness of a translation-invariant MPS is equivalent to the uniqueness of the ground state (or absence of any long-range order). In a non-translation-invariant MPS, the pureness is a sufficient but not a necessary condition for the uniqueness of the ground state \([6]\). Nevertheless, we can group several sites into a supersite and consider the canonical MPS based on the supersites. If the MPS is not pure (i.e. has a degeneracy in the largest singular value 1) for an arbitrary large supersite, the ground state is not unique. Therefore the physical requirement of uniqueness of the ground state is equivalent to the pureness of the MPS in the end, if a sufficiently large supersite is constructed. Thus, in the pureness of the MPS is assumed with this proviso in mind. In order to consider the symmetry \(I'\), we define \(n \in \mathbb{Z}\) so that \(I_n\) can be identified with \(n \rightarrow -n\) with the inversion center at site \(n = 0\). Following Refs. \([5, 30]\), if \(|\psi\rangle\) is invariant under the combined symmetry \(I'\), it satisfies

\[
\sum_{m,n'} u_{mm'} \left(\Gamma_m^{|a|}\right)^\dagger = e^{i\theta_{I'}^{|a|}} \left(U_{I'}^{[-n-\frac{1}{2}]}\right)^\dagger \Gamma_{m'}^{|a|} U_{I'}^{[-n+n+\frac{1}{2}]},
\]

(7)

where \(u_{mm'}\) is the representation of \(R_z\) acting on the physical Hilbert space of each site, \(\theta_{I'}^{|a|}\) is a phase, and \(U_{I'}^{|a|}\) is a \(\chi_a \times \chi_a\) unitary matrix commuting with \(\Lambda^{|a|}\). \(I'\) also implies that \(\Lambda^{|a|} = \Lambda^{[-|a|]}\) and \(\chi_a = \chi_{-a}\). Using the above relation twice, we obtain

\[
\mathcal{E}^{|a|}(A^{[n-\frac{1}{2}]}) = e^{-i(\theta_{I'}^{|a|} + \theta_{I'}^{[-|a|]})} A^{[n+\frac{1}{2}]},
\]

(8)

where \(A^{|a|} \equiv \left(U_{I'}^{[-|a|]}\right)^\dagger \left(U_{I'}^{[|a|]}\right)^\dagger\). Since \(|A^{|a|}|^2 = 1\), Eq. (8) implies \(A^{[n+1/2]}\) are left/right eigenvectors of \(\mathcal{E}^{|a|}\) belonging to the singular value 1. The assumption of the pure MPS, namely non-degeneracy of the singular value 1 of \(\mathcal{E}^{|a|}(X)\), implies

\[
A^{|a|} = e^{i\phi_{I'}^{|a|}} \mathbb{I}_{\chi_a},
\]

(9)

where

\[
\phi_{I'}^{|a|} + \phi_{I'}^{[-|a|]} = 0 \mod 2\pi.
\]

(10)

Combining Eqs. (8), (9), and the canonical condition, we obtain \(\theta_{I'}^{|a|} + \theta_{I'}^{[-|a|]} = 0 \mod 2\pi\). In particular, for \(n = 0\), together with Eq. (10), we find

\[
2(\theta_{I'}^{|0|} - \phi_{I'}^{[1/2]}) = 0 \mod 2\pi.
\]

(11)

As a consequence, \(\theta_{I'}^{|0|} - \phi_{I'}^{[1/2]}\) is quantized to either 0 or \(\pi\); it cannot change unless the system undergoes a quantum phase transition. This implies that, in the presence of the \(I'\) symmetry, there are two distinct phases corresponding to \(\phi_{I'}^{|0|} = 0\) and \(\pi\). Let us now consider the limits of the trivial product states \(|D\rangle\) and \(|N\rangle\). Here, all the matrices \(\Gamma_m^{|a|}\), \(\Lambda_m^{|a|}\), and \(U_{I'}^{|a|}\) are reduced to scalars (1 \(\times 1\) matrices) and thus commute with each other. Then the fundamental relation (7) for \(n = 0\) reads \(\theta_{I'}^{|0|} - \phi_{I'}^{[1/2]} = 0\) for \(|D\rangle\) and \(\theta_{I'}^{|0|} - \phi_{I'}^{[1/2]} = \pi\) for \(|N\rangle\). This establishes that, under the \(I'\) symmetry, the two product states \(|D\rangle\) and \(|N\rangle\) indeed belong to distinct phases, which are always separated by a quantum phase transition.

As in the case of SPT phases, no local order parameter can distinguish SPT phases. However, using the MPS framework, we can directly derive non-local order parameters \([31]\) which are sensitive to the phase factor \(\theta_{I'}^{|0|} - \phi_{I'}^{[1/2]}\). In particular, we can define an operator \(I'_s(2L + 1)\) which inverts a block of \(2L + 1\) consecutive sites. For \(L\) much larger than the correlation length, we find that

\[
O_s(L) = \langle \psi|I'_s(2L + 1)|\psi\rangle \approx \text{Tr} \Lambda^4 e^{i(\theta_{I'}^{|0|} - \phi_{I'}^{[1/2]})}.
\]

(12)

The insets in Fig. 1 show that the different SPT phases are indeed distinguished by \(O_s(L)/\text{Tr} \Lambda^4 = \pm 1\) while \(O_s(L) = 0\) when \(I'\) is broken.

**Conclusion and discussion.**— We demonstrated that there exists two distinct SPT phases in the presence of the symmetry under the site-centered inversion combined with a spin rotation. We showed the existence of such phases by field-theoretical arguments based on bosonization and presented a general proof based on the MPS formalism. While it is known that distinct trivial phases can exist in translation-invariant systems \([6, 7]\), it is surprising that only point-group symmetries can stabilize distinct trivial phases in simple 1D systems. Our finding
implies that more studies are needed for complete classification of quantum phases in one dimension under symmetries. The notion of the SPt phases is presumably not restricted in one dimension. In fact, what appear as examples of SPt phases in two dimensions were discussed in Ref. [32]. The lack of universal theoretical description of quantum many-body systems in higher dimensions makes a systematic analysis of SPt phases more difficult than in one dimension. Nevertheless, it would be certainly an interesting direction for the future.

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DETAILS FOR NUMERICAL CALCULATION

In this supplemental material, we present several numerical details about the spin-1 chain,

\[ H = \sum_i \left[ (1 + \delta (-1)^i) \vec{S}_i \cdot \vec{S}_{i+1} + D_z (S_i^z)^2 + h_z (-1)^i S_i^z + d_x (S_i^y S_{i+1}^y - S_i^z S_{i+1}^z) \right]. \]  

(S1)

Central charge, magnetization, and entanglement spectrum

For \( h_z = 0.1 \) and \( d_x = \delta = 0 \), from the divergent behavior of the correlation length in Fig.1(a) of the main text, we expect a Gaussian transition at \( D_z \sim 1 \). To confirm this, we calculate the von Neumann entanglement entropy \( S \) for a bipartition of the system into two half chains as a function of the correlation length \( \xi \). From conformal field theory, the entanglement entropy is known to scale as \([S1, S2]\)

\[ S = \frac{c}{6} \log(\xi/a) + c', \]

where \( c \) is the central charge, \( a \) is the lattice spacing (we set \( a = 1 \)), and \( c' \) is a nonuniversal constant. As shown in Fig. S1 (a), the entanglement entropy \( S \) at \( D_z = 1.00 \) is well fitted by a linear function of \( \log(\xi) \), and the central charge is estimated as \( c \approx 1.02 \). This is close to the expected value \( c = 1 \) at the Gaussian transition.

For \( h_z = d_x = 0.1 \) and \( \delta = 0 \), a Gaussian transition at \( d_x = 0 \) splits into two Ising transitions, and we have an intermediate phase between these transitions. In fact, between the two peaks in Fig.1(b) of the main text, the \( \mathbb{Z}_2 \) spin reversal symmetry in \( x \) axis \((S_i^x \rightarrow -S_i^x)\) is spontaneously broken. Then the staggered magnetizations along \( x \) axis, \( M_x \), take finite expectation values in the intermediate phase, as shown in Fig. S1 (b). Such an intermediate Néel phase between two distinct gapped symmetric phases is also observed in a two-leg spin-1/2 ladder \([S3]\) when the \( U(1) \) symmetry is explicitly broken.

To see a clear signature of the topologically trivial phases, we check the degeneracy in the entanglement spectra. For \( h_z = d_x = \delta = 0 \), the Haldane phase is protected by time reversal, bond-centered inversion, and dihedral group of the spin rotations, as observed in Ref. \([S4]\). From Fig. S2 (a), the whole entanglement spectrum is two-fold degenerate in a region \(-0.3 \lesssim D_z \lesssim 1 \). On the other hand, as shown in Fig. S2 (b), once we introduce a finite staggered magnetic field \( h_z \), this two-fold degeneracy is lifted, and the Haldane phase is merged with the imposed AF ordered phase which is topologically trivial.

Adiabatic connection between \( |D\rangle \) and \( |N\rangle \)

A nonzero dimerization \( \delta \) breaks the site-centered inversion symmetry but still preserves the two-site translational invariance. In the main text, we discuss an adiabatic continuity between the two trivial states \( |D\rangle \) and \( |N\rangle \) in the

![Diagram](image_url)

FIG. S1. (a) von Neumann entanglement entropy as a function of the correlation length \( \xi \) for \( D_z = 1, h_z = 0.1, \) and \( d_x = \delta = 0 \). The solid line is a logarithmic fitting function \( S = 0.170 \log \xi + 0.792 \). (b) Staggered magnetization in \( x \) axis as a function of \( D_z \) for \( \chi = 200, h_z = d_x = 0.1, \) and \( \delta = 0 \). The correlation length (divided by 200) is again shown for comparison.
FIG. S2. Low-lying entanglement spectra as functions of $D_z$ for (a) $h_z = d_z = \delta = 0$ and (b) $h_z = 0.1$, $d_z = \delta = 0$. Both data are obtained with $\chi = 200$.

FIG. S3. Correlation length as a function of $D_z$ for the spin-1 chain with $h_z = \delta = 0.1$ and $d_x = 0$. Each color and symbol denotes the different number of $\chi$ varied from 50 to 200.

absence of the combined symmetry, $\mathcal{I}' = \mathcal{I}_x \times R_z$. We first show the correlation length as a function of $D_z$ for $h_z = \delta = 0.1$ and $d_x = 0$ in Fig. S3. Compared with Fig. 1(a) in the main text, the correlation length around $D_z \simeq 1$ becomes shorter and exhibits a saturating behavior by increasing $\chi$. This indicates the absence of the phase transition due to the dimerization which breaks the combined symmetry $\mathcal{I}'$. In fact, this is easily and rigorously seen from the perfectly dimerized limit $\delta = 1$. In this limit, the Hamiltonian with $d_x = 0$ is reduced to the sum of independent

FIG. S4. Energy spectra of the two-site Hamiltonian (S3). $D_z$ are varied and $h_z = 0$ on the left panel, while $h_z$ are varied and $D_z = 0$ on the right panel.
two-site Hamiltonians,
\[ H_{\text{two-site}} = 2\vec{S}_1 \cdot \vec{S}_2 + D_z[(S^z_1)^2 + (S^z_2)^2] + h_z(S^z_1 - S^z_2). \]  
(S3)

Again we can obtain the two trivial states \(|D\rangle\) and \(|N\rangle\) in the limits \(D_z \to \infty\) and \(h_z \to \infty\), respectively. Therefore, the continuity between \(|D\rangle\) and \(|N\rangle\) is confirmed by finding a path on which no level crossing occurs in the lowest energy spectrum between these limits. In Fig. S4, we plot energy spectra of the two-site Hamiltonian. The singlet state at \(D_z = h_z = 0\) is adiabatically connected to both the states \(|00\rangle\) and \(|-+\rangle\). From our numerical data we thus conclude that, by breaking the site-centered inversion symmetry, the two states \(|D\rangle\) and \(|N\rangle\) are adiabatically connected and thus no longer distinguished.

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