ORBITAL STABILITY FOR THE MASS-CRITICAL AND SUPERCritical PSEUDO-RELAtivistic nonlinear SCHRöDINGER EQUATION

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Abstract. For the one-dimensional mass-critical/supercritical pseudo-relativistic nonlinear Schrödinger equation, a stationary solution can be constructed as an energy minimizer under an additional kinetic energy constraint and the set of energy minimizers is orbitally stable. In this study, we proved the local uniqueness and established the orbital stability of the solitary wave by improving that of the energy minimizer set. A key aspect thereof is the reformulation of the variational problem in the non-relativistic regime, which we consider to be more natural because the proof extensively relies on the subcritical nature of the limiting model. Thus, the role of the additional constraint is clarified, a more suitable Gagliardo-Nirenberg inequality is introduced, and the non-relativistic limit is proved. Subsequently, this limit is employed to derive the local uniqueness and orbital stability.

1. Introduction

We consider the pseudo-relativistic nonlinear Schrödinger equation (NLS)

\[ i \partial_t u = \left( \sqrt{m^2c^4 - c^2 \partial_x^2} - mc^2 \right) u - |u|^{p-1} u, \tag{1.1} \]

where \( u = u(t, x) : I(\subset \mathbb{R}) \times \mathbb{R} \to \mathbb{C} \). The operator \( \sqrt{m^2c^4 - c^2 \partial_x^2} - mc^2 \) is defined as a Fourier multiplier. The constant \( c \) represents the speed of light, and \( m \) is the particle mass. This operator is called pseudo-relativistic or semi-relativistic because it describes the intermediate dynamics between the non-relativistic regime \( c \gg 1 \) and the ultra-relativistic regime \( 0 < c \ll 1 \). When the power-type nonlinearity is replaced by Hartree nonlinearity, the equation is referred to as the boson star equation \([9, 12]\).

We suppose that \( 1 < p < 5 \). Then, the Cauchy problem for (1.1) is locally well-posed in \( H^{1/2}(\mathbb{R}) \) (see \([3, 10, 14, 18]\)), and the solutions preserve the mass

\[ \mathcal{M}(u) = \int_\mathbb{R} |u(x)|^2 dx \]

and the energy

\[ \mathcal{E}_{m,c}(u) = \frac{1}{2} \int_\mathbb{R} \left( \sqrt{m^2c^4 - c^2 \partial_x^2} - mc^2 \right) u(x) \overline{u(x)} dx - \frac{1}{p+1} \int_\mathbb{R} |u(x)|^{p+1} dx. \]

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\(^1\text{The higher-dimensional cases are considered in [1], but the radially symmetric assumption is imposed.}\)
When the nonlinearity is mass-subcritical, that is, $1 < p < 3$, the equation (1.1) is globally well-posed in $H^{\frac{1}{2}}(\mathbb{R})$, and the corresponding energy minimization problem

$$J(M) := \inf \left\{ E_{m,c}(u) : u \in H^{\frac{1}{2}}(\mathbb{R}) \text{ and } M(u) = M \right\}$$

(1.2)

yields an orbitally stable ground state (see [1, 6]). In contrast, in the mass-critical or supercritical case $p \in [3, 5)$, a blow-up is expected to occur (see [4, 14]), and the variational problem (1.2) is not appropriately formulated because $J(M) = -\infty$.

The meaning of the mass-critical nonlinearity is however ambiguous because of its semi-relativistic nature. In the non-relativistic regime $c \gg 1$, equation (1.1) is formally approximated by the mass-subcritical non-relativistic NLS

$$i \partial_t u = -\frac{1}{2m} \partial_x^2 u - |u|^{p-1}u.$$ 

Based thereupon, in the non-relativistic regime, an orbitally stable state can be constructed from a modified energy minimization problem, which is one of the main observations by Bellazzini, Georgiev, and Visciglia [1] (see also [2] for orbitally stable ground states to NLS with a partial confinement). Precisely, the energy minimization problem with an additional constraint

$$\inf \left\{ E_{1,1}(u) : \|u\|_{H^{\frac{1}{2}}(\mathbb{R})} \leq \frac{1}{2} \text{ and } M(u) = M \right\}.$$ 

(1.3)

Furthermore, the existence of an energy minimizer is established, provided that the mass $M$ is sufficiently small. Then, using the standard argument [6, 10], the orbital stability of the set of local energy minimizers follows.

In this study, we revisit the variational problem (1.3), but we rescale it to clarify the connection to the non-relativistic problem, as in the work by Lenzmann [13]. Hereafter, we take $m = \frac{1}{2}$ for numerical simplicity and denote

$$\mathcal{H}_c := \sqrt{-c^2 \partial_x^2 + \frac{c^4}{4} - \frac{c^2}{2}} \quad \text{and} \quad \mathcal{E}_c(u) := \mathcal{E}_{\frac{1}{2},c}(u) = \frac{1}{2} \|\sqrt{\mathcal{H}_c} u\|_{L^2(\mathbb{R})}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R})}^{p+1}.$$ 

Then, instead of fixing $c = 1$, we choose $M > 0$ (where $M$ is not necessarily small) and consider the energy minimization problem

$$J_c(M) := \inf \left\{ \mathcal{E}_c(u) : \|\sqrt{\mathcal{H}_c} u\|_{L^2(\mathbb{R})} \leq c^{\frac{p+1}{2(p+3)}} \text{ and } M(u) = M \right\},$$

(1.4)

which is equivalent to (1.3) by scaling (see Section 2.1). Using the formal series expansion

$$\sqrt{-c^2 \partial_x^2 + \frac{c^4}{4} - \frac{c^2}{2}} = -\partial_x^2 - \frac{c^2}{2}(\partial_x^2)^2 + \cdots,$$

via the non-relativistic limit $c \to \infty$, the variational problem (1.4) is closely related to the mass-subcritical non-relativistic energy minimization problem

$$J_{\infty}(M) := \inf \left\{ \mathcal{E}_{\infty}(u) : u \in H^1(\mathbb{R}) \text{ and } M(u) = M \right\},$$

(1.5)

where

$$\mathcal{E}_{\infty}(u) := \frac{1}{2} \|\partial_x u\|_{L^2(\mathbb{R})}^2 - \frac{1}{p+1} \|u\|_{L^{p+1}(\mathbb{R})}^{p+1}.$$ 

(1.6)

Although this connection was previously employed in [1], it is rather implicit.
Motivated by the formal convergence, we state the existence of a minimizer for the variational problem (1.4) as follows.

**Theorem 1.1 (Minimizer for \( J_c(M) \)).** Let \( p \in [3, 5) \), and suppose that
\[
c \geq \max \{ (\alpha M)^\frac{p-1}{5-p}, (\alpha^\frac{p}{p+3} M)^\frac{p-1}{5-p} \},
\]
where \( \alpha \) is given by (2.2). Then, the minimization problem \( J_c(M) \) possesses a minimizer, which must be of the following form:
\[
e^{i\theta} Q_c(\cdot - x_0) \quad \text{with } x_0 \in \mathbb{R}, \theta \in \mathbb{R},
\]
where \( Q_c \) is non-negative symmetric, and it solves the elliptic equation
\[
\mathcal{H}_c Q_c - Q_c^p = -\mu_c Q_c.
\]
Moreover, if \( 3 < p < 5 \) and \( c \geq \max \{ (\alpha M)^\frac{p-1}{5-p}, (\alpha^\frac{p}{p+3} M)^\frac{p-1}{5-p}, (\frac{M}{p-3})^\frac{p-1}{5-p} \} \), then \( Q_c \) is a ground state on \( N \) in the sense that under the mass constraint \( M(v) = M \), it occupies the least energy among solutions to the nonlinear elliptic equation \( \mathcal{H}_c u - u^p = -\lambda u \) for some \( \lambda > 0 \).

**Remark 1.2.** (1) Theorem 1.1 was essentially proved in [1]. The difference is that the assumption of a small mass \( M \ll 1 \) [1] is transferred to the non-relativistic regime assumption \( c \gg 1 \). Indeed, it could be regarded as simply being a matter of scaling, but it is helpful to clarify the setup of the problem and to improve our intuition. This leads us to modify the Gagliardo-Nirenberg inequality, which is a more appropriate fit for the pseudo-relativistic operator (Proposition 2.1), but the constraint in the minimization problem is also refined (Lemma 3.2). Consequently, we can obtain several useful properties of the minimizer, which can be employed to prove our main result.

(2) A similar existence theorem was established for the mass-critical/supercritical pseudo-relativistic Hartree-type equation [17].

Our main result provides the uniqueness and the orbital stability of the ground state.

**Theorem 1.3.** Let \( p \in [3, 5) \). For any sufficiently large \( c \geq 1 \), let \( Q_c \) be an energy minimizer for \( J_c(M) \) constructed in Theorem 1.1.

(i) \( Q_c \) is unique up to translation and phase shift.

(ii) For \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if
\[
\inf_{x_0, \theta_0 \in \mathbb{R}} \| \sqrt{1 + \mathcal{H}_c(e^{i\theta_0} u_0(x - x_0) - Q_c(x))} \|_{L^2(\mathbb{R})} \leq \delta,
\]
then the global solution \( u(t) \) to
\[
i \partial_t u = \left( \sqrt{\frac{c^4}{4} - c^2 \partial_x^2 - \frac{c^2}{2}} \right) u - |u|^{p-1} u \tag{1.8}
\]

\[\text{Precisely, } \mathcal{E}_c(Q_c) = \inf \{ \mathcal{E}_c(u) : \mathcal{E}_c|_{N}(u) = 0 \}, \text{ where } N = \{ v \in H^\frac{1}{2}(\mathbb{R}) : M(v) = M \}.
\]
with the initial data $u_0$ satisfies
\[
\inf_{x_1, \theta_1 \in \mathbb{R}} \| \sqrt{1 + \mathcal{H}_c(e^{i\theta_1}u(t, x - x_1) - Q_c(x))} \|_{L^2(\mathbb{R})} \leq \epsilon \quad \text{for all } t \in \mathbb{R}.
\]

**Remark 1.4.** Theorem 1.3 (i) can be extended to the multidimensional case $d \geq 2$ with power-type nonlinearity $-|u|^{p-1}u$ with $1 + \frac{2}{d} < p < 1 + \frac{4}{d}$. However, because the well-posedness of the time-dependent problem (1.1) is known under the radial assumption, only the orbital stability against symmetric perturbations can be obtained (see [1]).

**Remark 1.5.** By using scaling in Section 2.1 for small $M > 0$, one can find a minimizer $u_M$ for (1.3). Then, our main result can be reformulated as follows: for $\epsilon > 0$, there exists $\delta > 0$ such that if
\[
\inf_{x_0, \theta_0 \in \mathbb{R}} \| e^{i\theta_0}v_0(x - x_0) - u_M(x) \|_{H^\frac{1}{2}(\mathbb{R})} \leq \delta,
\]
then the global solution $v(t)$ to
\[
i\partial_t v = \sqrt{1 - \partial_x^2} v - |v|^{p-1} v
\]
with the initial data $v_0$ satisfies
\[
\inf_{x_1, \theta_1 \in \mathbb{R}} \| e^{i\theta_1}v(t, x - x_1) - u_M(x) \|_{H^\frac{1}{2}(\mathbb{R})} \leq \epsilon \quad \text{for all } t \in \mathbb{R}.
\]

Note that, compared with the previous result [1], the possibility of transforming into a different low-energy state is eliminated.

To prove the main theorem, we employ a connection to the non-relativistic variational problem $\mathcal{J}_\infty(M)$. Let $Q_\infty$ be a symmetric, positive decreasing ground state for $\mathcal{J}_\infty(M)$. This minimizer is known to be unique up to translation and phase shift and solves the nonlinear elliptic equation
\[
-\partial_x^2 Q_\infty - Q_\infty^p = -\mu_\infty Q_\infty,
\]
where $\mu_\infty$ is the Lagrange multiplier. A key step in our analysis is to show the convergence from relativistic to non-relativistic minimizers.

**Proposition 1.6** (Non-relativistic limit). Let $p \in [3, 5]$. For a sufficiently large $c \geq 1$, let $Q_c$ be the ground state constructed in Theorem 1.1. Then, $Q_c \to Q_\infty$ in $H^1(\mathbb{R})$ and $\mu_c \to \mu_\infty$ as $c \to \infty$.

By the convergence $Q_c \to Q_\infty$, we obtain a certain coercivity estimate for the linearized operator at $Q_c$ (see Lemma 5.2). Then, using this estimate, we deduce a contradiction by comparing the energies, provided that there are two minimizers (see Section 5).

**Remark 1.7.** In the mass-critical/supercritical case, a radially symmetric solution to (1.7) is constructed near $Q_\infty$ in [7] via the contraction mapping argument. The convergence $Q_c \to Q_\infty$ and the local uniqueness show that it is identified with the solution obtained by the variational method.
1.1. Organization of this paper. The remainder of this paper is organized as follows. In Section 2, we provide preliminaries for the proof of the main results. This section describes a scaling property, a modified Gagliardo-Nirenberg inequality, well-posedness of the pseudo-relativistic NLS, and non-relativistic minimization problem. In Section 3, we prove Theorem 1.1 using the modified Gagliardo-Nirenberg inequality and the results for the non-relativistic minimization problem. In Section 4, we present the non-relativistic limit of a minimizer \( Q_c \) for \( \mathcal{J}_c(M) \) (Proposition 1.6). In Section 5, we establish the uniqueness of minimizer \( Q_c \) for \( \mathcal{J}_c(M) \) (Theorem 1.3). Finally, in Appendix A, for completeness of the paper, we present that a minimizer \( Q_c \) is a ground state solution as stated in Theorem 1.1.

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2. Preliminaries

2.1. Scaling. By a simple change of the variable, the parameter \( c \) in the variational problem (1.4) is scaled as follows. Let \( u(x) = c^{\frac{2}{p-1}} v(cx) \). Then, we have

\[
\mathcal{M}(u) = c^{\frac{5}{p-1}} \mathcal{M}(v),
\]

\[
\mathcal{E}_{m,c}(u) = c^{\frac{p+3}{p-1}} \mathcal{E}_{m,1}(v),
\]

\[
\| \sqrt{\mathcal{H}_{m,c} u} \|_{L^2(\mathbb{R})} = c^{\frac{p+3}{2(p-1)}} \| \sqrt{\mathcal{H}_{m,1} v} \|_{L^2(\mathbb{R})},
\]

where \( \mathcal{H}_{m,c} = \sqrt{m^2 c^4 - c^2 \partial_x^2 - mc^2} \). Thus, the minimization problem

\[
\inf \left\{ \mathcal{E}_{m,c}(u) : \| \sqrt{\mathcal{H}_{m,c} u} \|_{L^2(\mathbb{R})} \leq c^{\frac{p+3}{2(p-1)}} \text{ and } \mathcal{M}(u) = M \right\}
\]

is equivalent to

\[
\inf \left\{ \mathcal{E}_{m,1}(v) : \| \sqrt{\mathcal{H}_{m,1} v} \|_{L^2(\mathbb{R})} \leq 1 \text{ and } \mathcal{M}(v) = c^{-\frac{5}{p-1}} M \right\}.
\]

2.2. Modified Gagliardo-Nirenberg inequality. The Gagliardo-Nirenberg inequality states that, for \( p \geq 1 \) with \( \frac{1}{p+1} > \frac{1-2s}{2} \),

\[
\| u \|_{L^{p+1}(\mathbb{R})} \leq C_{s,p+1} \| u \|_{L^2(\mathbb{R})}^{\frac{(2s-1)p+(2s+1)}{2s}} \| \partial_x \|^{\frac{p-1}{2}} L^2(\mathbb{R}),
\]

where \( u \in L^2(\mathbb{R}) \), \( |\partial_x|^s u \in L^2(\mathbb{R}) \) and \( C_{s,p+1} \) is a sharp constant. For the pseudo-relativistic NLS and its non-relativistic limit, this inequality is modified considering that the pseudo-relativistic operator acts differently on low and high frequencies.

**Proposition 2.1** (Modified Gagliardo-Nirenberg inequality). For \( p > 1 \), let

\[
C_{GN} = 2^{\frac{3p-1}{2}} \max \{ C_{1,p+1}, C_{2,p+1} \},
\]
where \( C_{s,p+1} \) is the sharp constant for the Gagliardo-Nirenberg inequality (2.1). Then, for any \( c \geq 1 \) and \( \delta \in (0,1) \), we have

\[
\|u\|_{L^{p+1}(\mathbb{R})}^{p+1} \leq C_{\text{GN}} \left\{ \|u\|_{L^2(\mathbb{R})} \sqrt{\mathcal{H}} P_{\leq c\delta} u \|_{L^2(\mathbb{R})}^{p+1} + \frac{1}{(c\delta)^{\alpha}} \|u\|_{L^2(\mathbb{R})} \sqrt{\mathcal{H}} P_{> c\delta} u \|_{L^2(\mathbb{R})}^{p+1} \right\},
\]

where \( P_{\leq c\delta} u = (1 - \chi_{c\delta}) u \) and \( P_{> c\delta} u = u - P_{\leq c\delta} u \).

The following bounds for the symbol are useful.

**Lemma 2.2** (Symbol of the pseudo-relativistic operator). For \( \delta \in (0,1) \), we have

\[
\sqrt{c^2|\xi|^2 + \frac{c^4}{4} - \frac{c^2}{2}} \geq \begin{cases} \frac{1}{2} |\xi|^2 & \text{if } |\xi| \leq c\delta, \\ \frac{c}{2}|\xi| & \text{if } |\xi| \geq c\delta. \end{cases}
\]

and

\[
\sqrt{c^2|\xi|^2 + \frac{c^4}{4} - \frac{c^2}{2}} \leq |\xi|^2 \quad \text{for all } \xi.
\]

**Proof.** Because \( \sqrt{c^2|\xi|^2 + \frac{c^4}{4} - \frac{c^2}{2}} = \frac{c}{2} f(\frac{4|\xi|^2}{c^2}) \), where \( f(t) = \sqrt{1 + t} - 1 \), it suffices to show the appropriate bounds for \( f(t) \). Evidently, \( f(t) \leq \frac{t}{2} \). Moreover, if \( 0 \leq t \leq 3\delta^2 \), then by the mean-value theorem, there exists \( t_* \in [0,3\delta^2] \) such that \( f(t) = \frac{1}{2\sqrt{t_*+1}} t \geq \frac{1}{4} t \). On the other hand, if \( t \geq 3\delta^2 \), then \( f(t) = \frac{t}{2\sqrt{t_*+1}} \geq \frac{\delta}{2\sqrt{\delta^2}} \) because \( \frac{t}{\sqrt{t_*+1}} \) increases. \( \square \)

**Proof of Proposition 2.1.** Decomposing the high and low frequencies and then applying the Gagliardo-Nirenberg inequality, we obtain

\[
\|u\|_{L^{p+1}(\mathbb{R})}^{p+1} \leq 2^p \left\{ \|P_{\leq c\delta} u\|_{L^{p+1}(\mathbb{R})}^{p+1} + \|P_{> c\delta} u\|_{L^{p+1}(\mathbb{R})}^{p+1} \right\}
\]

\[
\leq 2^p \left\{ C_{1,p+1} \|u\|_{L^2(\mathbb{R})} \|\partial_x |P_{\leq c\delta} u\|_{L^2(\mathbb{R})}^{p+1} + C_{\frac{1}{2},p+1} \|u\|_{L^2(\mathbb{R})} \|\partial_x |P_{> c\delta} u\|_{L^2(\mathbb{R})}^{p+1} \right\}
\]

Hence, using the lower bound in Lemma 2.2, we complete the proof. \( \square \)

**Corollary 2.3.** Let \( p \in [3,5) \). Then, \( J_c(M) > -\infty \) for all \( c \geq 1 \).

Throughout this paper, we let

\[
\alpha = \frac{4C_{\text{GN}}}{p+1} = \frac{2^{\frac{3(p+1)}{p+1}}}{p+1} \max \left\{ C_{1,p+1}, C_{\frac{1}{2},p+1} \right\},
\]

where \( C_{s,p+1} \) is the sharp constant for the Gagliardo-Nirenberg inequality (2.1).

Another important consequence of Proposition 2.1 is the separation of the negative energy function set for the pseudo-relativistic NLS.

**Corollary 2.4** (Separation of the negative energy function space). Let \( p \in [3,5) \) and \( c \geq \max\{ (\alpha M)^{\frac{3}{5-p}}, (\alpha^{p+1} M)^{\frac{3}{5-p}} \} \), where \( \alpha \) is given by (2.2). Suppose that \( M(u) = M \) and \( E_c(u) < 0 \). Then, either \( \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})} > e^{-\frac{p+1}{p-1}} \) or \( \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})} \leq \alpha^{\frac{2}{5-p}} M^{\frac{p+1}{5-p}} \) holds.
Remark 2.5. In other words, if $\mathcal{M}(u) = M$, $\mathcal{E}_c(u) < 0$ and $\|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})} \leq c^{\frac{p+3}{2(p-1)}}$, then $\|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})} \leq \alpha^{\frac{2}{p-1}} M^{\frac{p+3}{2(p-1)}}$.

Proof of Corollary 2.4. Note that, because $c \geq (\alpha M^{\frac{4}{p+3}})^{\frac{p-1}{p}}$, we have $c^{\frac{p+3}{2(p-1)}} \geq \alpha^{\frac{2}{p-1}} M^{\frac{p+3}{2(p-1)}}$.

By Proposition 2.1 with $\delta = 1$, we write

$$\mathcal{E}_c(u) \geq \frac{1}{2} \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^2 - \alpha \left\{ \|u\|_{L^2(\mathbb{R})}^{\frac{p+1}{2}} \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^{\frac{p-1}{2}} + c^{-\frac{p-1}{2}} \|u\|_{L^2(\mathbb{R})}^{\frac{p-1}{2}} \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^{\frac{p-1}{2}} \right\}.$$  

If $\|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})} \leq c^{\frac{p+3}{2(p-1)}}$, then by the assumptions, it follows that

$$\mathcal{E}_c(u) \geq \frac{1}{2} \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^2 - \alpha \left\{ M \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^{\frac{p+1}{2}} + c^{-\frac{p+3}{2}} \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^{\frac{p-1}{2}} \right\}$$

$$= \left( \frac{1}{2} - \alpha \frac{c^{-\frac{p+3}{2}}}{4} \right) \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^{\frac{p+1}{2}} \left\{ \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^{\frac{p-1}{2}} - \frac{\alpha M^{\frac{p+3}{2}}}{4 \left( \frac{1}{2} - \alpha \frac{c^{-\frac{p+3}{2}}}{4} \right)} \right\}$$

$$\geq \frac{1}{4} \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^{\frac{p-1}{2}} \left\{ \|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})}^{\frac{p-1}{2}} - \alpha M^{\frac{p+3}{2}} \right\},$$

where $c \geq (\alpha M)^{\frac{4}{p+3}}$ in the previous step. Hence, because $\mathcal{E}_c(u) < 0$, we conclude that $\|\sqrt{\mathcal{H}} c u\|_{L^2(\mathbb{R})} \leq \alpha^{\frac{2}{p-1}} M^{\frac{p+3}{2(p-1)}}$. \hfill $\square$

2.3. Well-posedness of the pseudo-relativistic NLS. Even in the mass-(super)critical case, if $c \geq 1$ is sufficiently large, a negative energy solution exists globally in time, and its kinetic energy is not extremely large.

Proposition 2.6 (Global well-posedness). Let $p \in [3, 5)$ and

$$c \geq \max\{ (\alpha M)^{\frac{p-1}{2(p-1)}}, (\alpha M)^{\frac{1}{p-1}} \},$$

where $\alpha$ is given by (2.2). Suppose that $\mathcal{M}(u_0) = M$, $\mathcal{E}_c(u_0) < 0$, and $\|\sqrt{\mathcal{H}} c u_0\|_{L^2(\mathbb{R})} \leq c^{\frac{p+3}{2(p-1)}}$, and let $u(t)$ be the local solution to the pseudo-relativistic NLS (1.1) with initial data $u_0$. Then, $u(t)$ exists globally in time, and

$$\sup_{t \in \mathbb{R}} \|\sqrt{\mathcal{H}} c u(t)\|_{L^2(\mathbb{R})} \leq \alpha^{\frac{2}{p-1}} M^{\frac{p+3}{2(p-1)}}.$$  

Proof. Because $\|\sqrt{\mathcal{H}} c u_0\|_{L^2(\mathbb{R})} \leq c^{\frac{p+3}{2(p-1)}}$ and $u_0$ has a negative energy, it follows from Remark 2.5 that $\|\sqrt{\mathcal{H}} c u_0\|_{L^2(\mathbb{R})} \leq \alpha^{\frac{2}{p-1}} M^{\frac{p+3}{2(p-1)}}$. Hence, by continuity, $\|\sqrt{\mathcal{H}} c u(t)\|_{L^2(\mathbb{R})} \leq \alpha^{\frac{2}{p-1}} M^{\frac{p+3}{2(p-1)}}$ for all sufficiently small $|t|$, but from Remark 2.5 the solution obeys $\|\sqrt{\mathcal{H}} c u(t)\|_{L^2(\mathbb{R})} \leq \alpha^{\frac{2}{p-1}} M^{\frac{p+3}{2(p-1)}}$ for all small $|t|$. Repeating this analysis, we conclude that the same bound holds for all $t$. \hfill $\square$
2.4. Non-relativistic minimization problem. We summarize the known results for the limit case \((1.5)\). The variational problem \((1.5)\) attains a positive minimizer \(Q_\infty\). This minimizer is unique up to translation and phase shift, and it solves the Euler-Lagrange equation \(-\partial_x^2 Q_\infty - Q_\infty^{p} = -\mu_\infty Q_\infty\) for some Lagrange multiplier \(\mu_\infty > 0\). Moreover, it is symmetric, smooth, and exponentially decreasing (see [5,16]). Using the following identities,

\begin{align*}
0 &= \|\partial_x Q_\infty\|_{L^2(\mathbb{R})}^2 - \|Q_\infty\|_{L^{p+1}(\mathbb{R})}^{p+1} + \mu_\infty \|Q_\infty\|_{L^2(\mathbb{R})}^2
\end{align*}

and

\begin{align*}
0 &= 2 \int_{\mathbb{R}} x \partial_x Q_\infty (-\partial_x^2 Q_\infty - Q_\infty^p + \mu_\infty Q_\infty) \, dx \\
&= \|\partial_x Q_\infty\|_{L^2(\mathbb{R})}^2 + \frac{2}{p+1} \|Q_\infty\|_{L^{p+1}(\mathbb{R})}^{p+1} - \mu_\infty \|Q_\infty\|_{L^2(\mathbb{R})}^2,
\end{align*}

we find that

\begin{align}
\|\partial_x Q_\infty\|_{L^2(\mathbb{R})}^2 &= \frac{p-1}{p+3} \mu_\infty M, \\
\|Q_\infty\|_{L^{p+1}(\mathbb{R})}^{p+1} &= \frac{2(p+1)}{p+3} \mu_\infty M, 
\end{align}

(2.4)

where \(M = \mathcal{M}(Q_\infty)\). In [20], it was proved that \(Q_\infty\) provides a sharp constant for the Gagliardo-Nirenberg inequality (see (2.1))

\[ C_{1,p+1} = \frac{\|Q_\infty\|_{L^{p+1}(\mathbb{R})}^{p+1}}{\|Q_\infty\|_{L^2(\mathbb{R})}^2} = \frac{2(p+1)}{p+3} \frac{\mu_\infty^{5-p}}{4 \sqrt{(p+1)}} M^{\frac{p+3}{2}}, \]

(2.6)

where (2.4) are used in the final step. Then, inserting \(\mu_\infty = (p+3)\left(\frac{p+1}{4(p+1)^4} M^{4(p+1)}\right)^{\frac{5-p}{2}}\) into (2.4), we obtain

\[ \|\partial_x Q_\infty\|_{L^2(\mathbb{R})}^2 = \left[\frac{(p+1)C_{1,p+1}}{2(p+1)}\right]^{\frac{4}{5-p}} M^{\frac{p+3}{2}} \]

(2.5)

and

\[ J_\infty(M) = -\frac{5-p}{2(p+3)} \mu_\infty M = -\frac{5-p}{2} \left[\frac{(p+1)C_{1,p+1}}{2(p+1)}\right]^{\frac{4}{5-p}} M^{\frac{p+3}{2}} < 0. \]

(2.6)

3. Existence of a minimizer: Proof of Theorem 1.1

We consider the variational problem \(J_c(M)\) (see (1.4)) and prove the existence of a minimizer and its basic properties (Theorem 1.1). As a first step, we show that \(J_c(M)\) has a negative upper bound.

**Lemma 3.1** (Comparison between \(J_c(M)\) and \(J_\infty(M)\)). Assume that \(p \in [3,5)\) and \(c \geq (\alpha^{\frac{1}{p+3}} M)^{\frac{p-1}{5-p}}\). Then, \(Q_\infty\) is admissible for \(J_c(M)\), and

\[ J_c(M) \leq E_c(Q_\infty) \leq E_\infty(Q_\infty) = J_\infty(M) < 0. \]

*Proof*. By (2.2), (2.5), Lemma 2.2 and the assumptions \(p \in [3,5)\) and \(c \geq (\alpha^{\frac{1}{p+3}} M)^{\frac{p-1}{5-p}}\), we obtain

\[ \|\sqrt{\mathcal{H}} Q_\infty\|_{L^2(\mathbb{R})} \leq \|\partial_x Q_\infty\|_{L^2(\mathbb{R})} = \left[\frac{(p+1)C_{1,p+1}}{2(p+1)}\right]^{\frac{2}{5-p}} M^{\frac{p+3}{2(5-p)}} \leq \alpha^{\frac{2}{5-p}} M^{\frac{p+3}{2(5-p)}} \leq c^{\frac{p+3}{2(5-p)}}. \]
Moreover, by (2.6) and the fact that \( \|\sqrt{\mathcal{H}} c \|_{L^2(\mathbb{R})} \leq \|\partial_x Q_\infty\|_{L^2(\mathbb{R})} \), we obtain \( \mathcal{E}_c(Q_\infty) \leq \mathcal{E}_\infty(Q_\infty) = J_\infty(M) < 0 \). This proves the lemma by the definition of \( J_c(M) \).

Next, we observe from Corollary 2.4 that the constraint in \( J_c(M) \) can be refined.

**Lemma 3.2** (Refined constraint minimization). Assume that \( p \in [3, 5) \). Then, for any \( c \geq \max\{ (\alpha M)^{\frac{1}{p+1}}, (\frac{4}{p+4} M)^{\frac{1}{p+1}} \} \), we have

\[
J_c(M) = \inf \left\{ \mathcal{E}_c(u) : \|\sqrt{\mathcal{H}} u\|_{L^2(\mathbb{R})} \leq \alpha \frac{2}{p-2} M^{\frac{p+3}{2(p-2)}} \text{ and } \|u\|_{L^2(\mathbb{R})}^2 = M \right\}.
\]

**Proof.** Let \( \{u_n\}_{n=1}^\infty \) be a minimizing sequence for \( J_c(M) \). For a sufficiently large \( n \geq 1 \), by Lemma 3.1 the set of admissible functions is not empty and \( \mathcal{E}_c(u_n) < 0 \). Hence, Corollary 2.4 implies that \( \|\sqrt{\mathcal{H}} u_n\|_{L^2(\mathbb{R})} \leq \alpha \frac{2}{p-2} M^{\frac{p+3}{2(p-2)}} \) holds.

Now, we can show that a minimizer exists for (1.4).

**Proof of Theorem 1.1.** Let \( \{u_n\}_{n=1}^\infty \) be a minimizing sequence for \( J_c(M) \). Then, by Lemma 3.2 it is uniformly bounded in \( H^{1/2}(\mathbb{R}) \), and thus, \( u_n \rightharpoonup \tilde{u} \) in \( H^{1/2}(\mathbb{R}) \) up to a subsequence. However, we have

\[
\liminf_{n \to \infty} \|u_n\|_{L^{p+1}(\mathbb{R})} > 0,
\]

because, by Lemma 3.1 \( 0 > J_c(M) = \mathcal{E}_c(u_n) - o_n(1) \geq - \frac{1}{p+1} \|u_n\|_{L^{p+1}(\mathbb{R})}^{p+1} - o_n(1) \). Here, \( o_n(1) \) means that \( o_n(1) = 0 \) as \( n \to \infty \), where \( a_n \in \mathbb{R} \). Hence, \( \tilde{u} \neq 0 \) in \( L^{p+1}(\mathbb{R}) \).

We claim that \( u_n \to \tilde{u} \) in \( L^2(\mathbb{R}) \). If the claim is not true, then passing to a subsequence,

\[
\|\tilde{u}\|_{L^2(\mathbb{R})}^2 = M', \quad \|u_n - \tilde{u}\|_{L^2(\mathbb{R})}^2 \to M - M' \in (0, M).
\]

and thus,

\[
\mathcal{E}_c(u_n) = \mathcal{E}_c(\tilde{u}) + \mathcal{E}_c(u_n - \tilde{u}) + o_n(1) \geq J_c(M') + J_c(M - M') - o_n(1).
\]

Let \( \{v_n\}_{n=1}^\infty \) be a minimizing sequence for \( J_c(M') \). Then, as observed above (see 3.2), passing to a subsequence, \( \liminf_{n \to \infty} \|v_n\|_{L^{p+1}(\mathbb{R})}^{p+1} \geq \delta_0 > 0 \) for a small \( \delta_0 > 0 \). Furthermore, \( \sqrt{M/M'} v_n \) is admissible for \( J_c(M) \), because Lemma 3.2 implies that \( \|\sqrt{\mathcal{H}} c (\sqrt{M/M'} v_n)\|_{L^2(\mathbb{R})} \leq \frac{M}{M'} \alpha \frac{4}{p+4} M^{\frac{p+1}{p+4}} \) and \( \|\sqrt{M/M'} v_n\|_{L^2(\mathbb{R})}^2 = M \). Therefore, it follows that

\[
J_c(M) \leq \mathcal{E}_c(\sqrt{M/M'} v_n) - \frac{1}{p + 1} \frac{M}{M'} \left[ \left( \frac{M}{M'} \right)^{\frac{p+1}{2}} - 1 \right] \|v_n\|_{L^{p+1}(\mathbb{R})}^{p+1}
\]

\[
\leq \frac{M}{M'} J_c(M') - \frac{1}{p + 1} \frac{M}{M'} \left[ \left( \frac{M}{M'} \right)^{\frac{p+1}{2}} - 1 \right] \delta_0 - o_n(1),
\]

and thus,

\[
J_c(M') \geq \frac{M'}{M'} J_c(M) + \frac{1}{p + 1} \left[ \left( \frac{M}{M'} \right)^{\frac{p+1}{2}} - 1 \right] \delta_0 - o_n(1).
\]
Moreover, by switching the roles of $M'$ and $M - M'$, we have
\[ \mathcal{J}_c(M - M') \geq \frac{M - M'}{M} \mathcal{J}_c(M) + \frac{1}{p + 1} \left[ \left( \frac{M}{M - M'} \right)^{\frac{p-1}{2}} - 1 \right] \delta_0 - o_n(1) \]
for sufficiently small $\delta_0 > 0$. In combination, we obtain
\[ \mathcal{J}_c(M) \geq \mathcal{J}_c(M) + \frac{1}{p + 1} \left[ \left( \frac{M}{M'} \right)^{\frac{p-1}{2}} + \left( \frac{M}{M - M'} \right)^{\frac{p-1}{2}} - 2 \right] \delta_0 - o_n(1), \]
which deduces a contradiction.

We now show that $u_n \to \tilde{u}$ in $H^{1/2}(\mathbb{R})$, and the limit $\tilde{u}$ is a minimizer for $\mathcal{J}_c(M)$. Indeed, $\tilde{u}$ is admissible because $u_n \to \tilde{u}$ in $L^2(\mathbb{R})$. Moreover, by applying the Gagliardo-Nirenberg inequality (Proposition 2.1) to $(u_n - \tilde{u})$, we can show that $u_n \to \tilde{u}$ in $L^{p+1}(\mathbb{R})$. Thus, owing to the weak lower semi-continuity of the kinetic energy, it follows that $\mathcal{J}_c(M) = \mathcal{E}_c(u_n) + o_n(1) \geq \mathcal{E}_c(\tilde{u}) + \frac{1}{2} \parallel \sqrt{\mathcal{H}_c}(u_n - \tilde{u}) \parallel_{L^2(\mathbb{R})} + o_n(1)$. By minimality, it is shown that $\mathcal{E}_c(\tilde{u}) = \mathcal{J}_c(M)$ and $u_n \to \tilde{u}$ in $H^{1/2}(\mathbb{R})$.

It remains to show that $\tilde{u}$ is non-negative up to the phase shift, decreasing, and symmetric. Let $\tilde{u}^*$ be the symmetric rearrangement of $\tilde{u}$. Then, $\parallel \sqrt{\mathcal{H}_c}\tilde{u}^* \parallel_{L^2(\mathbb{R})} \leq \parallel \sqrt{\mathcal{H}_c}\tilde{u} \parallel_{L^2(\mathbb{R})}$, where the equality holds only if $\tilde{u} = e^{i\theta}\tilde{u}^*(-x_0)$ for some $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ (see [15] Appendix A or the proof of [19] Proposition 2.1 for details). Moreover, we have $\parallel \tilde{u}^* \parallel_{L^{p+1}(\mathbb{R})} = \parallel \tilde{u} \parallel_{L^{p+1}(\mathbb{R})}$ and $\parallel \tilde{u}^* \parallel_{L^2(\mathbb{R})}^2 = \parallel \tilde{u} \parallel_{L^2(\mathbb{R})}^2 = M$. Thus, symmetrization strictly decreases the energy, and a minimizer must be the symmetrization of itself. \hfill \Box

4. Non-relativistic limit: proof of Proposition 1.6

In this section, we prove the non-relativistic limit. First, we show that relativistic minimizers are uniformly more regular according to a standard argument.

**Lemma 4.1** (Elliptic regularity). Let $p \in [3, 5)$. Then, there exists $c_0 \geq 1$ such that $\sup_{c \geq c_0} \parallel Q_c \parallel_{H^1(\mathbb{R})} < \infty$, where $Q_c$ is a minimizer constructed in Theorem 1.4.

**Proof.** Let $c \geq c_0$ be a sufficiently large $c_0 \geq 1$. The implicit constants below are independent of $c \geq c_0$. Note that $\mu_c = -\frac{1}{2} \frac{\mathcal{J}_c(M)}{\mu_c}$, where $\mu_c$ is the Lagrange multiplier for $\mathcal{H}_c Q_c = Q_c^p = -\mu_c Q_c$, since by Lemma 3.1
\[ \mathcal{J}_c(M) \geq \mathcal{E}_c(Q_c) = \frac{1}{2} \parallel \sqrt{\mathcal{H}_c} Q_c \parallel_{L^2(\mathbb{R})}^2 - \frac{1}{p + 1} \parallel Q_c \parallel_{L^{p+1}(\mathbb{R})}^{p+1} \geq \frac{1}{p + 1} \int_{\mathbb{R}} (\mathcal{H}_c Q_c - Q_c^p) Q_c dx = -\frac{\mu_c}{p + 1} \parallel Q_c \parallel_{L^2(\mathbb{R})}^2 = -\frac{M}{p + 1} \mu_c. \] (4.1)

Hence, by Lemma 2.2 and the claim, $\parallel (\mathcal{H}_c + \mu_c)^{-1} f \parallel_{L^2(\mathbb{R})} \lesssim \parallel f \parallel_{H^{-1}(\mathbb{R})}$. Thus, by Lemma 2.2 and Lemma 3.2 we obtain
\[ \parallel Q_c \parallel_{H^1(\mathbb{R})} = \parallel (\mathcal{H}_c + \mu_c)^{-1} Q_c^p \parallel_{H^1(\mathbb{R})} \lesssim \parallel Q_c^p \parallel_{L^2(\mathbb{R})} \]
\[ \lesssim \parallel Q_c \parallel_{H^{\frac{1}{2}}(\mathbb{R})}^p \lesssim (\parallel \sqrt{\mathcal{H}_c} Q_c \parallel_{L^2(\mathbb{R})} + \parallel Q_c \parallel_{L^2(\mathbb{R})})^p \leq (\alpha \frac{2}{p} M \frac{p+1}{p+3} + \sqrt{M})^p, \]
and hence
\[ \|Q_c\|_{L^2(\mathbb{R})} = \|(H_c + \mu_c)^{-1}Q_c\|_{L^2(\mathbb{R})} \lesssim \|Q_c\|_{L^2(\mathbb{R})} \lesssim \|Q_c\|_{L^2(\mathbb{R})} \lesssim \|Q_c\|_{H^1(\mathbb{R})}. \]

Proof of Proposition 4.6. To prove the non-relativistic limit \( Q_c \to Q_\infty \) in \( H^1(\mathbb{R}) \), by the concentration-compactness property of the well-known variational problem \( \mathcal{J}_\infty(M) \) (see [16]) and by the uniqueness of the minimizer \( Q_\infty \) (see [11]), it is sufficient to show that \( \{Q_c\}_{c \geq c_0} \) is a minimizing sequence for \( \mathcal{J}_\infty(M) \), where \( c_0 \geq 1 \) is a large number chosen in Lemma 4.1. Indeed, we have

\[ \mathcal{E}_\infty(Q_c) = \mathcal{E}_c(Q_c) + \frac{1}{2} \left\{ \|\partial_x Q_c\|_{L^2(\mathbb{R})}^2 - \|\sqrt{H_c} Q_c\|_{L^2(\mathbb{R})}^2 \right\}, \]

but by the Plancherel theorem,

\[
\|\partial_x Q_c\|_{L^2(\mathbb{R})}^2 - \|\sqrt{H_c} Q_c\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ |\xi|^2 - \left( \sqrt{c^2|\xi|^2 + \frac{c^4}{4}} - \frac{c^2}{2} \right) \right\} |\hat{Q}_c(\xi)|^2 d\xi
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{|\xi|^4}{\sqrt{c^2|\xi|^2 + \frac{c^4}{4}} + \frac{c^2}{2} + |\xi|^2} |\hat{Q}_c(\xi)|^2 d\xi
\]

\[
\leq \frac{1}{2\pi c^2} \|\xi^2 \hat{Q}_c\|_{L^2(\mathbb{R})}^2 = \frac{1}{c^2} \|Q_c\|_{L^2(\mathbb{R})}^2. \tag{4.2}
\]

Thus, by Lemma 4.1, \( \mathcal{J}_\infty(M) \leq \mathcal{E}_\infty(Q_c) = \mathcal{E}_c(Q_c) + o_c(1) = \mathcal{J}_c(M) + o_c(1) \), and Lemma 3.1 proves that \( \{Q_c\}_{c \geq c_0} \) is a minimizing sequence for \( \mathcal{J}_\infty(M) \). Here, \( o_c(1) \) means that \( o_c(1) = a_c \to 0 \) as \( c \to \infty \), where \( a_c \in \mathbb{R} \).

Finally, by equation (4.2) and the non-relativistic limit \( Q_c \to Q_\infty \) in \( H^1(\mathbb{R}) \), we prove that

\[
(\mu_c - \mu_\infty)M = \mu_c\|Q_c\|_{L^2(\mathbb{R})}^2 - \mu_\infty\|Q_\infty\|_{L^2(\mathbb{R})}^2
\]

\[
= -\|\sqrt{H_c} Q_c\|_{L^2(\mathbb{R})}^2 + \|Q_c\|_{L^{p+1}(\mathbb{R})}^{p+1} + \|\partial_x Q_\infty\|_{L^2(\mathbb{R})}^2 - \|Q_\infty\|_{L^{p+1}(\mathbb{R})}^{p+1} \to 0.
\]

\[ \square \]

5. Uniqueness: Proof of Theorem 1.3

We establish the uniqueness of a pseudo-relativistic minimizer, provided that \( c \geq 1 \) is sufficiently large. A key component is the property of the linearized operator

\[ \mathcal{L}_\infty = -\partial_x^2 - pQ_\infty^{p-1} + \mu_\infty, \]

where \( Q_\infty \) is a unique positive, symmetric, decreasing minimizer for \( \mathcal{J}_\infty(M) \). We denote the set of collections of symmetric \( H^s(\mathbb{R}) \) functions by \( H^s_{rad}(\mathbb{R}) \).

Lemma 5.1 (Weinstein [20]). For \( p \in (1, 5) \), there exists \( C_0 > 0 \) such that

\[ \langle \mathcal{L}_\infty v, v \rangle_{L^2(\mathbb{R})} \geq C_0 \|v\|_{H^1(\mathbb{R})}^2 \]

for all \( v \in H^1_{rad}(\mathbb{R}) \) such that \( \langle v, Q_\infty \rangle_{L^2(\mathbb{R})} = 0 \).
We prove that a similar lower bound holds for the linearized operator
\[ \mathcal{L}_c = -\mathcal{H}_c - p Q_c^{p-1} + \mu_c, \]
where \( Q_c \) is a minimizer for \( J_c(M) \).

**Lemma 5.2.** If \( 3 \leq p < 5 \), then there exists \( C > 0 \), independent of sufficiently large \( c \geq 1 \), such that
\[ \langle \mathcal{L}_c v, v \rangle_{L^2(\mathbb{R})} \geq C \| v \|_{H^{1/2}(\mathbb{R})}^2 \]
for all \( v \in H^{1/2}_{rad}(\mathbb{R}) \) such that \( \langle v, Q_c \rangle_{L^2(\mathbb{R})} = 0 \).

**Proof.** It is sufficient to show that
\[ \langle \mathcal{L}_c v, v \rangle_{L^2(\mathbb{R})} \geq \tilde{C} \| v \|_{L^2(\mathbb{R})}^2 \]  \( (5.1) \)
for all \( v \in H^{1/2}_{rad}(\mathbb{R}) \), such that \( \langle v, Q_c \rangle_{L^2(\mathbb{R})} = 0 \), where \( \tilde{C} > 0 \) is a constant independent of sufficiently large \( c \geq 1 \). Indeed, if it is true but there is \( \{ v_c \}_{c \geq c_0} \subset H^{1/2}_{rad}(\mathbb{R}) \) such that \( \langle v_c, Q_c \rangle_{L^2(\mathbb{R})} = 0 \), \( \| v_c \|_{H^{1/2}(\mathbb{R})} = 1 \) but \( \langle \mathcal{L}_c v_c, v_c \rangle_{L^2(\mathbb{R})} \to 0 \), then by \( (5.1) \), \( \| v_c \|_{L^2(\mathbb{R})} = \tilde{C}^{-1} \langle \mathcal{L}_c v_c, v_c \rangle_{L^2(\mathbb{R})} \to 0 \), and thus \( v_c \to 0 \) in \( H^{1/2}(\mathbb{R}) \). Thus, we have
\[ \alpha_c(1) = \langle \mathcal{L}_c v_c, v_c \rangle_{L^2(\mathbb{R})} = \langle (\mathcal{H}_c + \mu_c) v_c, v_c \rangle_{L^2(\mathbb{R})} - p \langle Q_c^{p-1} v_c, v_c \rangle_{L^2(\mathbb{R})} \geq \| v_c \|_{H^{1/2}(\mathbb{R})}^2 \alpha_c(1), \]
which deduces a contradiction.

We suppose that \( c \geq 1 \) is sufficiently large and define
\[ \lambda_c := \inf \left\{ \langle \mathcal{L}_c v, v \rangle_{L^2(\mathbb{R})} : v \in H^{1/2}_{rad}(\mathbb{R}), \| v \|_{L^2(\mathbb{R})} = 1, \langle v, Q_c \rangle_{L^2(\mathbb{R})} = 0 \right\}. \]  \( (5.2) \)
Let \( C_0 \) be a constant in Lemma \( 5.1 \) and let \( \mu_\infty \) be the Lagrange multiplier for the elliptic equation for the non-relativistic minimizer \( Q_\infty \). If \( \lambda_c \geq \frac{1}{2} \min \{ C_0, \mu_\infty \} \), then \( (5.1) \) follows. We assume that \( \lambda_c < \frac{1}{2} \min \{ C_0, \mu_\infty \} \). Let \( \{ v_{c,n} \}_{n=1}^\infty \) be a minimizing sequence for \( (5.2) \). Then, it is uniformly bounded in \( H^{1/2}(\mathbb{R}) \). Hence, \( v_{c,n} \to \tilde{v}_c \) in \( H^{1/2}(\mathbb{R}) \).

If \( \tilde{v}_c \equiv 0 \), then
\[ \lambda_c + \alpha_c(1) = \langle \mathcal{L}_c v_{c,n}, v_{c,n} \rangle_{L^2(\mathbb{R})} \]
\[ \geq \mu_c \| v_{c,n} \|^2_{L^2(\mathbb{R})} - p \langle Q_c^{p-1} v_{c,n}, v_{c,n} \rangle_{L^2(\mathbb{R})} \]
\[ \geq \mu_c - \alpha_c(1) = \mu_\infty - \alpha_c(1) - \alpha_c(1), \]
which contradicts \( \lambda_c < \frac{\mu_\infty}{2} \).

Suppose that \( \tilde{v}_c \neq 0 \). Then, by replacing \( \tilde{v}_c \) with its normalization \( \frac{\tilde{v}_c}{\| \tilde{v}_c \|_{L^2(\mathbb{R})}} \), but still denoted by \( \tilde{v}_c \), we obtain a minimizer \( \tilde{v}_c \) for \( (5.2) \). Then, because \( (5.2) \) is a two-constraint minimization problem, \( \tilde{v}_c \) must obey the linear elliptic equation
\[ \mathcal{L}_c \tilde{v}_c = \lambda_c \tilde{v}_c + \tilde{\lambda}_c Q_c \]
for some Lagrange multipliers, $\lambda_c$ and $\tilde{\lambda}_c$. Thus, by applying the standard elliptic regularity argument in the proof of Lemma 4.1, we can show that $\sup_{c \geq 1} \| \tilde{v}_c \|_{L^2(\mathbb{R})} \leq C'$, where $C'$ is independent of $c \geq 1$. Now, we let

$$V_c = \tilde{v}_c - \frac{\langle \tilde{v}_c, Q_\infty \rangle_{L^2(\mathbb{R})}}{\|Q_\infty\|_{L^2(\mathbb{R})}^2} Q_\infty$$

$$= \tilde{v}_c - \frac{\langle \tilde{v}_c, Q_c \rangle_{L^2(\mathbb{R})}}{M} Q_c - \frac{\langle \tilde{v}_c, Q_\infty - Q_c \rangle_{L^2(\mathbb{R})}}{M} Q_c - \frac{\langle \tilde{v}_c, Q_\infty \rangle_{L^2(\mathbb{R})}}{M} (Q_\infty - Q_c),$$

because $\langle \tilde{v}_c, Q_c \rangle_{L^2(\mathbb{R})} = 0$ and $Q_c \to Q_\infty$ in $H^1(\mathbb{R})$ by the non-relativistic limit (Proposition 1.6). Here, $R_c$ denotes a function that converges to zero in $H^1(\mathbb{R})$ as $c \to \infty$. Then by (4.2), Proposition 1.6, Lemma 5.1 and the facts that $\langle V_c, Q_\infty \rangle_{L^2(\mathbb{R})} = 0$ and $\sup_{c \geq 1} \| \tilde{v}_c \|_{H^2(\mathbb{R})} \leq C'$, we prove that

$$C_0 \leq \frac{\langle \mathcal{L}_\infty V_c, V_c \rangle_{L^2(\mathbb{R})}}{\|V_c\|_{L^2(\mathbb{R})}^2} = \frac{\langle \mathcal{L}_c \tilde{v}_c, \tilde{v}_c \rangle_{L^2(\mathbb{R})} + o_c(1)}{\|\tilde{v}_c\|_{L^2(\mathbb{R})}^2 + o_c(1)} = \lambda_c + o_c(1).$$

□

Proof of Theorem 1.3. For a contradiction, we assume that the variational problem $J_c(M)$ has two different minimizers $Q_c$ and $\tilde{Q}_c$, which are non-negative, symmetric and decreasing, and that $Q_c$ solves the Euler-Lagrange equation $\mathcal{H}_c Q_c - Q_c^p = -\mu_c Q_c$ for some Lagrange multiplier $\mu_c > 0$. We decompose

$$\tilde{Q}_c = \sqrt{1 - \delta_c^2} Q_c + \epsilon_c, \quad \epsilon_c \neq 0,$$

where $\langle \epsilon_c, Q_c \rangle_{L^2(\mathbb{R})} = 0$. Note that

$$\delta_c = \frac{1}{\sqrt{M}} \| \epsilon_c \|_{L^2(\mathbb{R})} \to 0,$$

because $M = \mathcal{M}(\tilde{Q}_c) = (1 - \delta_c^2) M + \| \epsilon_c \|_{L^2(\mathbb{R})}^2$ and $(1 - \sqrt{1 - \delta_c^2})^2 M + \| \epsilon_c \|_{L^2(\mathbb{R})}^2 = \| \tilde{Q}_c - Q_c \|_{L^2(\mathbb{R})}^2 \to 0$ by the non-relativistic limit.

We introduce the functional

$$\mathcal{I}_c(u) = \mathcal{E}_c(u) + \frac{\mu_c}{2} \mathcal{M}(u).$$
Clearly, we have \( I_c(Q_c) = I_c(\tilde{Q}_c) = J_c(M) + \frac{\mu}{2} M. \) Furthermore, by the uniform boundedness of \( Q_c \) and \( \tilde{Q}_c, \) we have
\[
I_c(\tilde{Q}_c) = \frac{1}{2} \left\| \sqrt{\mathcal{H}_c + \mu_c(\sqrt{1 - \delta_c^2} Q_c + \varepsilon_c)} \right\|_{L^2(\mathbb{R})}^2 - \frac{1}{p+1} \left\| \sqrt{1 - \delta_c^2} Q_c + \varepsilon_c \right\|_{L^{p+1}(\mathbb{R})}^{p+1} \]
\[
= \left\{ \frac{1 - \delta_c^2}{2} \left\| \sqrt{\mathcal{H}_c + \mu_c Q_c} \right\|_{L^2(\mathbb{R})}^2 - \frac{(1 - \delta_c^2)\frac{p+1}{2}}{p+1} \left\| Q_c \right\|_{L^{p+1}(\mathbb{R})}^{p+1} \right\}
+ \langle \sqrt{1 - \delta_c^2} (\mathcal{H}_c + \mu_c) Q_c - (1 - \delta_c^2)\frac{p}{2} Q_c^{p-1} + \mu_c \varepsilon_c, \varepsilon_c \rangle_{L^2(\mathbb{R})}
+ \frac{1}{2} \langle \mathcal{H}_c \varepsilon_c - p(1 - \delta_c^2)\frac{p-1}{2} Q_c^{p-1} \varepsilon_c + \mu_c \varepsilon_c, \varepsilon_c \rangle_{L^2(\mathbb{R})} + o_c(1) \left\| \varepsilon_c \right\|_{H^{1/2}(\mathbb{R})}^2.
\]
Then, by applying the equation \((\mathcal{H}_c + \mu_c) Q_c - Q_c^p = 0\) to the first two lines, we write
\[
I_c(\tilde{Q}_c) = \frac{1}{2} \left\| \sqrt{\mathcal{H}_c + \mu_c Q_c} \right\|_{L^2(\mathbb{R})}^2 - \frac{(1 - \delta_c^2)\frac{p+1}{2}}{p+1} \left\| Q_c \right\|_{L^{p+1}(\mathbb{R})}^{p+1} \]
\[
+ \langle \langle \sqrt{1 - \delta_c^2} - (1 - \delta_c^2)\frac{p}{2} Q_c^{p-1} + \mu_c \varepsilon_c, \varepsilon_c \rangle_{L^2(\mathbb{R})}
+ \frac{1}{2} \langle \mathcal{H}_c \varepsilon_c - p(1 - \delta_c^2)\frac{p-1}{2} Q_c^{p-1} \varepsilon_c + \mu_c \varepsilon_c, \varepsilon_c \rangle_{L^2(\mathbb{R})} + o_c(1) \left\| \varepsilon_c \right\|_{H^{1/2}(\mathbb{R})}^2.
\]
Thus, it follows from (5.3) that
\[
I_c(\tilde{Q}_c) = I_c(Q_c) + \frac{1}{2} \langle \mathcal{L}_c \varepsilon_c + \mu_c \varepsilon_c, \varepsilon_c \rangle_{L^2(\mathbb{R})} + o_c(1) \left\| \varepsilon_c \right\|_{H^{1/2}(\mathbb{R})}^2.
\]
Then, the non-degeneracy of \( \mathcal{L}_c \) yields \( 0 \geq \frac{C_0}{2} \left\| \varepsilon_c \right\|_{H^{1/2}(\mathbb{R})}^2 + o_c(1) \left\| \varepsilon_c \right\|_{H^{1/2}(\mathbb{R})}^2, \) which contradicts the fact that \( \varepsilon_c \neq 0. \)

**APPENDIX A. ENERGY MINIMIZER AS A GROUND STATE**

In this appendix, we show that although the energy minimizer \( Q_c \) constructed in Theorem 1.1 is not a global minimizer, it can be still called a *ground state* in the sense that
\[
\mathcal{E}_c(Q_c) = \inf \left\{ \mathcal{E}_c(u) : \mathcal{E}_c|_{\mathcal{N}(u)} = 0 \right\},
\]
where \( \mathcal{N} = \{ v \in H^{1, \frac{2}{3}}(\mathbb{R}) : \mathcal{M}(v) = M \}. \) Indeed, this was shown in [1], but it is sketched here again for the sake of completeness.

We start with the following Pohozaev identity (see [8]).

**Lemma A.1** (Pohozaev identity). If \( u \in H^{1, \frac{2}{3}}(\mathbb{R}) \) solves
\[
\mathcal{H}_c u = |u|^{p-1} u = -\mu u \tag{A.1}
\]
for some \( \mu > 0, \) then
\[
\frac{1}{2} \left\| \sqrt{\mathcal{H}_c} u \right\|_{L^2(\mathbb{R})}^2 - \frac{p-1}{2(p+1)} \left\| u \right\|_{L^{p+1}(\mathbb{R})}^{p+1} + \frac{1}{4} \int_{\mathbb{R}} \frac{\sqrt{c^2|\xi|^2 + c^2 + \varepsilon_c^2}}{\sqrt{\frac{1}{4} + \varepsilon_c^2}} |\hat{u}(\xi)|^2 d\xi = 0.
\]
Proof. The proof of the result was proved in [8], but we briefly sketch of the proof for the convenience of the reader. We multiply $u$ to the equation (A.1) and then integration by parts shows

$$\| \sqrt{\mathcal{H}_c} u \|^2_{L^2(\mathbb{R})} + \mu \| u \|^2_{L^2(\mathbb{R})} - \| u \|^{p+1}_{L^{p+1}} = 0.$$  \hspace{1cm} (A.2)

On the other hand, from the result of [8, Proposition 1.1], we see that $u \in H^2(\mathbb{R})$ and $|u| + |x \partial_x u| \leq C(1 + |x|^2)^{-1}$. Then, by the Plancherel theorem and elementary calculations, we observe that

$$\left\langle \left( -\frac{c^2}{2} + \mu \right) u - u^p, x \partial_x u \right\rangle = \int_{\mathbb{R}} \left( -\frac{c^2}{2} + \mu \right) x \partial_x (u^2) - \frac{1}{p+1} x \partial_x (u^{p+1}) dx$$

$$= \frac{1}{2} \left( \frac{c^2}{2} - \mu \right) \| u \|^2_{L^2(\mathbb{R})} + \frac{1}{p+1} \| u \|^{p+1}_{L^{p+1}}.$$  

Hence, combining these with $\frac{c^4}{4\sqrt{c^2|\xi|^2 + \frac{c^4}{4}}} = \sqrt{c^2|\xi|^2 + \frac{c^4}{4}} - \frac{c^2|\xi|^2}{\sqrt{c^2|\xi|^2 + \frac{c^4}{4}}}$ and using the equation (A.1), we obtain

$$-\frac{1}{2} \| \sqrt{\mathcal{H}_c} u \|^2_{L^2(\mathbb{R})} - \frac{\mu}{2} \| u \|^2_{L^2(\mathbb{R})} + \frac{1}{p+1} \| u \|^{p+1}_{L^{p+1}} + \frac{c^2}{2} \int_{\mathbb{R}} \frac{|\xi|^2|\hat{u}(\xi)|^2}{\sqrt{c^2|\xi|^2 + \frac{c^4}{4}}} d\xi = 0.$$ \hspace{1cm} (A.3)

Thus, by (A.2) and (A.3), we prove the result. \hfill \Box

Next, we prove that the local minimizer $Q_c$ constructed in Theorem 1.1 is a ground state.

**Proposition A.2.** Assume that $p \in (3, 5)$ and $c \geq \max \{ (\alpha M)^{\frac{p-1}{p-3}}, (\alpha \frac{4}{p-3} M)^{\frac{p-1}{p-3}}, (\frac{M}{p-3})^{\frac{p-1}{p-3}} \}$. Then we have

$$\mathcal{J}_c(M) = \inf \left\{ \mathcal{E}_c(u) : \mathcal{E}_c|_{\mathcal{N}}(u) = 0 \right\},$$

where $\mathcal{N} = \{ v \in H^\frac{1}{2}(\mathbb{R}) : \mathcal{M}(v) = M \}$.

**Proof.** We assume on the contrary that there exists a critical point $u$ for $\mathcal{E}_c$ on $\mathcal{N}$ with $\mathcal{E}_c(u) < \mathcal{J}_c(M)$. Then $u \in H^\frac{1}{2}(\mathbb{R})$ is a solution of

$$\mathcal{H}_c u - |u|^{p-1} u = -\mu u.$$
for some $\mu \in \mathbb{R}$. We observe that by (4.1) Lemma 3.1 $\mu > 0$. Then by Lemma 3.1 and Lemma A.1, we get

$$0 > J_c(M) > \mathcal{E}_c(u)$$

$$= \frac{1}{2} \left\| \sqrt{\mathcal{H}_c} u \right\|^2_{L^2(\mathbb{R})} - \frac{1}{p+1} \left\| u \right\|^{p+1}_{L^{p+1}(\mathbb{R})}$$

$$= \frac{1}{2} \left( 1 - \frac{2}{p-1} \right) \left\| \sqrt{\mathcal{H}_c} u \right\|^2_{L^2(\mathbb{R})} - \frac{c^2}{2(p-1)} \left\| u \right\|^2_{L^2(\mathbb{R})} + \frac{c^4}{4(p-1)} \int_{\mathbb{R}} \frac{\left| \hat{u}(\xi) \right|^2}{c^2 |\xi|^2 + c^4} d\xi$$

$$\geq \frac{p-3}{2(p-1)} \left\| \sqrt{\mathcal{H}_c} u \right\|^2_{L^2(\mathbb{R})} - \frac{c^2}{2(p-1)} \left\| u \right\|^2_{L^2(\mathbb{R})},$$

From this and the assumption $c \geq (\frac{M}{p-3})^{\frac{p-1}{p-3}}$, we have $\left\| \sqrt{\mathcal{H}_c} u \right\|_{L^2(\mathbb{R})} \leq \sqrt{\frac{M}{p-3}} c \leq c^{\frac{p-3}{p-1}},$

which implies that $J_c(M) \leq \mathcal{E}_c(u).$  \hfill \Box

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