Gauss Law Constraints in Chern-Simons Theory From BRST
Quantization

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Abstract

The physical state condition in the BRST quantization of Chern-Simons field theory is used to derive Gauss law constraints in the presence of Wilson loops, which play an important role in explicitly establishing the connection of Chern-Simons field theory with 2-dimensional conformal field theory.

When we discuss knot invariants in terms of Chern-Simons theory\textsuperscript{1} and the relationship between Chern-Simons field theory and conformal field theory\textsuperscript{2,3}, an important relation is Gauss law constraint in presence of Wilson line, which was first given in ref.\textsuperscript{4}. This relation plays an important role in proving that states of Chern-Simons theory satisfy the Knizhnik-Zamolodchikov equation\textsuperscript{4,5}. In this letter we intend to derive the Gauss law constraints from BRST quantization of Chern-Simons field theory. We think this investigation is significant since in some sense BRST quantization formulation is defined better than a formal manipulation without gauge fixing\textsuperscript{6}. We will show that when Wilson lines exist the physical state condition in BRST quantization will lead to Gauss law constraints

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with source terms just as those given in ref. [4]. The procedure we will adopt is similar to the one in ref. [15], where the equivalence between Dirac’s first-class constraints and BRST treatment for Yang-Mills theory is formally proved.

Let us first write down the BRST quantization of Chern-Simons theory. The action of Chern-Simons field theory takes the following form

\[
S_{CS} = \frac{k}{4\pi} \int_M \text{Tr} \left[ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right],
\]

where \( A = A_\mu dx^\mu = A^a_\mu T^a dx^\mu \) with \( T^a \) being the generators in some representation of gauge group \( G \). The parameter \( k \) must be chosen to be an integer in order to make the theory gauge invariant under large gauge transformations. Without loss of generality, we choose \( G = SU(N) \) and the normalization \( \text{Tr}(T^a T^b) = \frac{1}{2} \).

Choosing the Lorentz gauge \( \partial_\mu A_\mu = 0 \) and performing BRST gauge fixing, we obtain the following effective action

\[
S_{\text{eff}} = \int d^3x \ L_{\text{eff}} = S_{CS} + \int \text{Tr} \delta [\bar{c} (\partial_\mu A^\mu + B)]
\]

\[
= \int d^3x \left\{ \frac{k}{16\pi} \epsilon^{\mu\nu\rho} \left[ A^\mu_\rho (\partial_\nu A^a_\rho - \partial_\rho A^a_\nu) + \frac{2}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho \right] - \frac{ik}{8\pi} A^{\mu a} \partial_\mu B^a + \frac{ik}{8\pi} (B^a)^2 - \frac{1}{2} \partial_\mu \bar{c}^a D^\mu c^a \right\},
\]

where \( B = B^a T^a \) is the auxiliary field and \( A_\mu = A^a_\mu T^a, c = c^a T^a, \bar{c} = \bar{c}^a T^a \). The BRST transformations of the fields are as follows

\[
\delta A^a_\mu = D_\mu c^a, \quad \delta B^a = 0,
\]

\[
\delta c^a = -\frac{1}{2} f^{abc} c^b c^c, \quad \delta \bar{c}^a = \frac{ik}{4\pi} B^a.
\]

These transformations are nilpotent, i.e., \( \delta^2 = 0 \). Now, obviously the classical configuration space is enlarged by the introduction of new fields—ghost fields \( c^a \), anti-ghost fields \( \bar{c}^a \) and multiplier fields \( B^a \). The canonically conjugate momenta can be well defined by \( \Pi_\Phi = \frac{\partial L}{\partial \dot{\Phi}} \), with \( \Phi = \{A_1, B, c, \bar{c}\} \):

\[
\Pi^a_{A_1} = \frac{k}{8\pi} A^a_2, \quad \Pi^a_B = -i \frac{k}{8\pi} A^a_0,
\]

\[
\Pi^a_c = -\frac{1}{2} D^0 c^a, \quad \Pi^a_{\bar{c}} = \frac{1}{2} \bar{c}^a.
\]
These fields and their canonically conjugate momenta satisfy the Poisson brackets (for bosonic fields) or anti-brackets (for fermionic fields):

\[
\{\Pi^i_\Phi(x,t) , \Phi^j(y,t)\}_{\pm\text{PB}} = -i\delta^i_j\delta^{(2)}(x-y),
\]

\[
\{\Pi^i_\Phi(x,t), \Pi^j_\Phi(y,t)\}_{\pm\text{PB}} = \{\Phi^i(x,t), \Phi^j(y,t)\}_{\pm\text{PB}} = 0.
\] (5)

The BRST charge can be obtained by the Noether theorem

\[
Q = \int d^2x \left[ \frac{k}{8\pi} \epsilon^{ij} D_i c^a A_j^a - \frac{1}{4} f^{abc} c^a c^b c^c - \frac{i k}{8\pi} B^a D^0 c^a \right]
\]

\[
= \int d^2x \left[ -\frac{k}{8\pi} c^a F_{12}^a - \frac{1}{2} f^{abc} \Pi^a c^b c^c + \frac{i k}{4\pi} B^a \Pi^a c^a \right].
\] (6)

It is easy to show that

\[
\delta\Phi = \{Q, \Phi\}_{\pm\text{PB}}, \{Q, Q\}_{\pm\text{PB}} = Q^2 = 0.
\] (7)

When we perform quantization, the classical observables are replaced by operators, and (anti-) Poisson brackets by (anti-) commutative Lie brackets. With the present polarization choice, the Hilbert space are composed of square integrable functionals of \(\Phi\). The quantum BRST charge operator \(\hat{Q}\) is nilpotent

\[
\frac{1}{2}\{\hat{Q}, \hat{Q}\} = \hat{Q}^2 = 0,
\] (8)

where a hat ”\(^{\hat{}}\)” denotes an operator. It is well known that the state space here possesses indefinite metric. According to the general principle of BRST quantization, physical states satisfy the so-called “BRST-closed” condition

\[
\hat{Q}|\text{phys}\rangle = 0.
\] (9)

Notice that above condition (9) determines a physical state up to “BRST-exact states”, i.e.

\[
|\text{phys}\rangle \sim |\text{phys}\rangle + |\chi\rangle, \ |\chi\rangle = \hat{Q}|\text{any states}\rangle.
\] (10)

Obviously these states \(|\chi\rangle\) are normal to all physical states including themselves,

\[
\langle\chi|\text{phys}\rangle = \langle\chi_1|\chi_2\rangle = 0.
\] (11)
Thus they are zero norm states and make no contribution to the physical observables. Now we define the physical operator $\hat{\Phi}$ to be an operator that generates a physical state from vacuum. It is easy to show that the physical operator $\hat{\Phi}$ must satisfy the condition

$$[\hat{\Phi}, \hat{Q}]_\pm = f[\hat{\Phi}] \hat{Q}$$

(12)
due to Eq.(9). Furthermore, the operators can be divided into two classes. According to ref. [15], we call them as the A-type and the B-type. An A-type operator transforms a physical state into another one

$$\hat{A}|\text{phys}\rangle = |\text{phys}^\prime\rangle.$$  

(13)

A B-type operator transforms a physical state into a BRST exact state and has the form

$$\hat{B} = [*, \hat{Q}]_\pm,$$

(14)

where $*$ represents some operator. Eq.(14) implies that a B-type operator can be regarded as the generator of a kind of gauge transformation since it does not affect physical observables. The (anti-)commutators of B-type operators with an arbitrary physical operator $\hat{\Phi}$ have the form

$$[\hat{B}_i, \hat{\Phi}]_\pm = g[\hat{\Phi}]_{ij} \hat{B}_j,$$

(15)

which means that B-type operators form an ideal in the operator algebra

$$[\hat{A}, \hat{A}]_\pm \subset \{\hat{A} \& \hat{B}\},$$

(16)

$$[\hat{A}, \hat{B}]_\pm \subset \{\hat{B}\}, \ [\hat{B}, \hat{B}]_\pm \subset \{\hat{B}\}.$$ 

The product of an arbitrary operator $\hat{K}$ (physical or nonphysical) with a B-type operator can also be regarded as the generator of gauge transformations due to the fact that

$$[(\hat{K}\hat{B})_i, \hat{\Phi}]_\pm = h[\hat{\Phi}]_{ij} (\hat{K}\hat{B})_j.$$  

(17)

In addition, $\hat{K}\hat{B}$ operators also form a closed algebra.
\[(\hat{K}\hat{B})_i, \hat{K}\hat{B})_j \pm = U_{ij}^k (\hat{K}\hat{B})_k. \quad (18)\]

Notice that the $\hat{K}\hat{B}$ operator transforms a physical state out of the genuine physical state space \[13\]. We can see in the following that the properties of $\hat{B}$ or $\hat{K}\hat{B}$ operators play a crucial role in our derivation.

From the BRST charge given in Eq.(6) we can show that
\[
\hat{B}_1^a \equiv [\hat{Q}, \hat{\Pi}_c^a] = -\frac{k}{8\pi} \hat{F}_{12}^a - \frac{1}{2} f^{abc} \hat{\Pi}_c^b \hat{\Pi}_c^c,
\]
\[
\hat{B}_2^a \equiv [\hat{Q}, \hat{\Pi}_B^a] = \frac{k}{4\pi} \hat{\Pi}_c^a,
\]
\[
\hat{B}_3^a \equiv [\hat{Q}, \partial^\mu \hat{A}_\mu^a] = M_{ab} \hat{c}^b,
\]
where $M_{ab} = \frac{k}{8\pi} [\hat{F}_{12}^a, \partial^\mu \hat{A}_\mu^b]$. Furthermore, we know that the matrix $(M_{ab})$ is non-singular from the fact that $\hat{F}_{12}$ and $\partial^\mu \hat{A}_\mu$ constitute a pair of second-class constraints \[13,18\]. Note that in deriving the Eqs.(19) we have used the $B$-field equation of motion (on-shell condition). The non-singularity of $M$ ensures that ghost field operators can be written as $\hat{c}^a = (M^{-1})^{ab} \hat{B}_3^b$ and belong to the B-type. Hence they are indeed the generators of gauge transformations.

From Eqs.(6), (19) and the above arguments, one can see that the three terms composed of BRST charge $\hat{Q}$ are all gauge transformation generators. However the second and third terms are $\hat{B}$- or $\hat{K}\hat{B}$-type operators. Thus when $\hat{Q}$ acts on physical states, the second and the third terms transform the physical state to non-physical state. Since there exists no coupling between the nonphysical gauge transformation generators $\hat{c}^a$ and the physical ones, after the action of BRST charge, the transformed state $| \rangle$ can be written as
\[
| \rangle = |\text{non-phys}\rangle \oplus |\text{phys}\rangle. \quad (20)
\]
So the physical state condition $\hat{Q}|\text{phys}\rangle = 0$ reduces to
\[
\hat{F}_{12}^a|\text{phys}\rangle = 0, \quad (21)
\]
when no Wilson loop exists. Now we turn to the case in the presence of Wilson loops. Let us take the manifold $M = \Sigma \times R$ as in ref. \[1\], where $R$ is the time variable space and $\Sigma$
is the spatial surface. The physical state at some time \( t \) in the presence of a Wilson loop can be represented by a punctured surface \( \Sigma \), the puncture points being produced by the intersections of the surface \( \Sigma \) with the links where the Wilson loop operators are defined. This has been given in ref. \[16\]

\[
|\text{phys}\rangle = \prod_{n=1}^{N} e^{i \sum_{i=1,2}^{2} A_i^{(n)}(\mathbf{x})dx^i} |0\rangle ,
\]

where \( n \) denotes the \( n \)th component of links, \( \Gamma_n \) is the projection on \( \Sigma \) of links located in the three dimensional space-time region less than time \( t \) and \( P_n, Q_n \) are the endpoints of \( \Gamma_n \). In the polarization chosen above, the state functional can be written down explicitly in a path integral form \[16\]

\[
\Psi_{\text{phys}}[\Phi] = \langle \hat{\Phi} | \text{phys} \rangle = \left( \int D\Xi \{ e^{i \int_{-\infty}^{t} dt' \int_{\Sigma} d^2x L_{\text{eff}} - \frac{k}{2\pi} \int_{\Sigma} d^2x \sum_{i=1}^{2} A_i^{\ast a} A_i^{a} \} \times \prod_{n=1}^{N} e^{i \sum_{i=1}^{2} A_i^{(n)}(\mathbf{x})dx^i} |\Psi_0\rangle \} [\Phi] ,
\]

\( \Phi = (A_1, B, c, \bar{c}), \quad \Xi = (A_\mu, B, c, \bar{c}), \quad \mu = 0, 1, 2 \)

where \( \Psi_0 \) is the vacuum state functional at time \( t = -\infty \) and it is determined by Eq.(21). Eq.(23) is in fact the gauge-fixed version of the state functional given in ref. \[10\]. Therefore, we have that

\[
\hat{Q}|\text{phys}\rangle = \left( \int d^2x \left[ \frac{k}{8\pi} i^{ij} D_i \hat{c}^a A_j^a + \frac{f^{abc}}{4\pi} \hat{c}^a \hat{c}^b \hat{c}^c + \frac{i k}{8\pi} \hat{B}^a D^a \hat{c}^a \right] \times \prod_{n=1}^{N} e^{i \sum_{i=1,2}^{2} A_i^{(n)}(\mathbf{x})dx^i} \right) |0\rangle
\]

\[
= \left\{ \int d^2x \left[ -\hat{c}^a \left[ \frac{k}{8\pi} \hat{F}_{12}^a - \sum_{n=1}^{N} T_{(a)}(\mathbf{x} - \mathbf{x}_{P_n}) - \delta^{(2)}(\mathbf{x} - \mathbf{x}_{Q_n}) \right] + \frac{k}{4\pi} \hat{B}^a \hat{\Gamma}_c^{a} - \frac{1}{2} f^{abc} \hat{\Gamma}_c^{a} \hat{c}^b \hat{c}^c \right] \right\} |\text{phys}\rangle
\]

\[
= \left\{ \int d^2x \left[ -\hat{c}^a \left[ \frac{k}{8\pi} \hat{F}_{12}^a - \sum_{n=1}^{N} T_{(a)}(\mathbf{x} - \mathbf{x}_{P_n}) - \delta^{(2)}(\mathbf{x} - \mathbf{x}_{Q_n}) \right] \right] \right\} |\text{phys}\rangle
\]

\( \oplus|\chi_1\rangle > \oplus|\chi_2\rangle >= 0 ,
\]

where Eq.(3) and the following operator equations have been used:
\[
\hat{A}_2^a = \frac{8\pi}{k} \hat{I}_{A_1}^a - \frac{8i\pi}{k} \frac{\delta}{\delta \hat{A}_1^a},
\]

\[
\left[ \frac{\partial}{\partial P_n^a} \frac{\delta}{\delta \hat{A}_1^a(x_{P_n})}, \exp \left[ i \int_{P_n(\Gamma_n)} \sum_{i=1,2} \hat{A}_i^{(n)}(x) dx^i \right] \right] = -i T_{(n)}^a \delta^{(2)}(x - x_{P_n})
\]

\[
\times \exp \left[ i \int_{P_n(\Gamma_n)} \sum_{i=1,2} \hat{A}_i^{(n)}(x) dx^i \right],
\]

\[
\left[ \frac{\partial}{\partial Q_n^a} \frac{\delta}{\delta \hat{A}_1^a(x_{Q_n})}, \exp \left[ i \int_{P_n(\Gamma_n)} \sum_{i=1,2} \hat{A}_i^{(n)}(x) dx^i \right] \right] = i T_{(n)}^a \delta^{(2)}(x - x_{Q_n})
\]

\[
\times \exp \left[ i \int_{P_n(\Gamma_n)} \sum_{i=1,2} \hat{A}_i^{(n)}(x) dx^i \right].
\]

Thus, the physical state condition \( \hat{Q}|\text{phys}\rangle = 0 \), can be reduced to the form

\[
\left[ \frac{k}{8\pi} \hat{F}_{12}^a - \sum_{n=1}^N T_{(n)}^a \left( \delta^{(2)}(x - x_{P_n}) - \delta^{(2)}(x - x_{Q_n}) \right) \right]|\text{phys}\rangle = 0.
\]

These are exactly the Gauss law constraints given by Witten in the case that Wilson loop operators are present.

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