Modeling nonlinearity and hysteresis due to critical-state flux penetration in hard superconductors
T. Dasgupta, Durga P. Choudhury and S. Sridhar
Department of Physics, Northeastern University, Boston, MA 02115

Abstract—The nonlinear surface impedance of a thin superconducting strip carrying a microwave current has been calculated numerically from first principles based upon flux penetration due to a current-induced critical state. The calculations approach analytic results in the appropriate limit. The technique is useful for design of passive superconducting microwave circuits.

Keywords—Critical state, Hysteresis, Nonlinearity, Microwave Transmission Lines

I. INTRODUCTION

Since its discovery, the new breed of oxide superconductors with transition temperatures above the boiling point of nitrogen have shown great potential as a replacement for metals for passive microwave structures owing to their very low surface resistance[1], [2], [3]. However, a key feature of the HTSC is the inherently nonlinear response at microwave frequencies, which is present over a wide range of input powers. This nonlinear responses also poses a major challenge for numerical device design using these materials. Most existing computer aided design software do not take into account the inherent nonlinearity of superconductors. The few analytical models that do exist[4] are valid only for highly idealized geometries and need to be incorporated into practical device design.

There are several physical models of nonlinear surface resistance, each valid in a different regime of temperature and microwave power[5]. The most pursued ones are pinning induced critical state[4], [6], [7], transport current induced self-heating [8], and depairing [8]. The current-driven critical state model [4] has proven to be particularly useful not only because it fits experimental measurement of $R_S$ very accurately over a wide range of temperature and power levels [9], but also because its usefulness extends to modeling hysteretic losses in superconducting transmission lines.

The (field-induced) critical state model was first proposed by Bean[10] to explain the unusual magnetization behavior of hard superconductors and was first solved for the geometry of a semi-infinite slab where the field was applied parallel to its face. The geometry where the field is applied perpendicular to a thin strip was solved relatively recently by several groups and the field and current distribution[11], [12] and the surface impedance from hysteretic loss[4] was obtained analytically.

Though very elegant, the analytic results were restricted to certain “ideal” geometries only where symmetry could be used to simplify the problems. However, these idealizations are not always satisfied in real world applications of superconducting strips in microwave circuits. In the wake of the fact that the model does describe experimental data accurately [14], it is imperative to be able to implement a numerical paradigm without the constraints of the idealized geometry required by the analytic approach.

In this paper, we demonstrate a numerical approach to incorporate the nonlinear response of superconducting materials using assumptions equivalent to those of the current induced critical state.

II. THE CRITICAL STATE MODEL

Let us consider a stripline with the ground plane far away from the center conductor so that it carries a negligible current and hence does not effect the magnetic field distribution inside the center conductor. It has been shown in literature that such an assumption is reasonable[13] (We will discuss the effects of ground planes somewhere else.) The geometry of the center conductor and our choice of axes is shown in of Fig. 1.

Starting from the most generic physical assumptions that retain the essential ideas of critical states:
1. $|j(r)| \leq j_c$, where $j$ is the current density.
2. The magnetic induction $B(x, y) = 0$ for $|y| \leq b \leq a$

The first assumption highlights the fact that flux lines are pinned until the Lorenz force on them exceeds a certain critical value. The second assumption follows from the fact that the vortices start penetrating from the edges. Hence
for any given current there would be a flux free region in the center of the strip. Note that the depinning critical current is at least an order of magnitude smaller than the depairing critical current where superconductivity would be quenched by the self field.

In the following we show that for a strip with finite thickness, this model can be solved numerically and that it leads to unique field and current distribution within the strip. We further show that (1) and (2) leads to hysteresis irreversibility, this model can be solved numerically and that it leads to a quenching of the critical current where superconductivity would be quenched by the self field.

The above analytical expression for the specific strip geometry under consideration. With respect to the geometry and study the hysteresis laws for any given current there would be a flux free region in the center of the strip. Note that the depinning critical current is at least an order of magnitude smaller than the depairing critical current where superconductivity would be quenched by the self field.

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III. Computational Details

A semi-numerical approach was used to determine a current distribution which satisfies the two assumptions presented above. Assuming an uniform current distribution along the x-direction one can show that

\[ B(x = 0, y) = \frac{\mu_0}{\pi} \int_{-a}^{a} j(y') \tan^{-1}\left(\frac{\delta}{2(y - y')}ight) dy' \]

(1)

The above analytical expression for \( B(x = 0, y) \) in Eq(1) was used to save computational time.

In the model which requires that \( j(y : b \leq |y| \leq a) \equiv J_c \) and \( B(x = 0, |y| \leq b < a) = 0 \), one needs to solve for the current distribution in the field free region \( |y| \leq b < a \)

\[ \int_{-a}^{a} j(y') \tan^{-1}\left(\frac{\delta}{2(y - y')}ight) dy' = 0 \text{ for } |y| \leq b < a \]

(2)

Numerical solutions of integral equations such as eq(2) are rather complicated. We found a simpler iterative approach to find the current density in the field free region of the superconducting strip. Here we give a brief description of the algorithm that was used.

Suppose that the thickness of the strip is infinite (\( \delta \to \infty \)). Then for any given initial current distribution \( j(y) \), a field free region \( B(|y| \leq b) \equiv 0 \) can be obtained by replacing the current density in that region by

\[ j(|y| < b) \Rightarrow j(|y|) + \frac{\partial}{\partial y} B_x \]

(3)

Also note that the magnetic field strength on the line \( x = 0 \) for a slab (\( \delta \to \infty \)) is mostly affected by the current on and around the line \( x = 0 \). Therefore, for the case of a finite \( \delta \), the transformation in eq(3) will bring the field strength in the region \( |y| < b \) a little closer to zero. Hence an iterative implementation of the above transformation of \( J(y) \) in eq(3) will finally converge bringing the magnetic field strength in the region \( |y| < b \) to zero, irrespective of the size of \( \delta \).

In a nutshell, we start with a flat current distribution, \( j(y) = j_c \) in an one dimensional grid of size 4000 and we iteratively make \( B(x = 0, |y| < b) \) vanish by replacing the current density by \( j_y^{n+1} = j_y^n + (B_y^{n+1} - B_y^n) / \Delta y \).

IV. Results

A. Penetration Law

We calculate the flux penetration depth \( d/a \) for a strip of aspect ratio of 1% as a function of the applied current. In Fig.2 we display \( (1 - d/a) = b/a \) as a function of the applied current \( y/a \) and compare that with the analytical results for a strip of zero thickness. Fig.3 and Fig.4 shows the corresponding current and field distributions. One can see from Fig.2 and Fig.3 that the penetration law get’s more linear and the current density in the field free region decreases as we go from zero thickness to a finite thickness. These results are expected since, in the limit \( \delta \to \infty \) we expect a linear penetration law and \( J(x = 0, |y| < b < a) = 0 \). These results can then determine the current critical states and the corresponding magnetic field distribution for AC currents with a peak current of \( I_0 \).

V. Hysteresis

The hysteresis law can be derived by studying the current density distribution for an oscillating current over a full cycle. If the current density distribution for a total current \( I \) is \( j(y, I, j_c \to \infty) \) then in absence of pinning the current...
density will be $Nj(y, I, j_c \to \infty)$ for a total current of $NI$. In other words

$$j(y, N I, j_c \to \infty) = Nj(y, I, j_c \to \infty)$$  \hspace{1cm} (4)

However, since $j(y) \leq j_c$ everywhere, the current density distribution depends on the total current. That is

$$j(y, N I, j_c) \neq Nj(y, I, j_c).$$  \hspace{1cm} (5)

When the current is reversed, after reaching the peak $I_0$, current density starts to decrease from the edges with an effective critical current density in the opposite direction of $2J_c$. This is because at the edges the current density is $j_c$ in the positive $x$-direction.

$$j_{cL} = 2J_c \uparrow$$  \hspace{1cm} (6)

Thus the hysteresis law on the downward cycle is given by $[12]$,

$$j_L(y, I, j_c) = \max(y, I_0, j_c) - \max(y, I_0 - I, 2j_c)$$  \hspace{1cm} (7)

Fig. 5 displays the magnetic field distribution when the input current is reduced from the maximum current $2J_c\delta a$ to $-2J_c\delta a$. As the decreases through 0 one does not see any field free region inside the strip, that is there are locked in fields from field distribution when the total current was $2J_c\delta a$: something that is unique to hysteresis.

### VI. SURFACE IMPEDANCE

Assuming that the strip is excited with a current of amplitude $I_0$ and frequency $\omega_0$, the electric field $E(y, t)$ is be given by

$$E(z, y, t) = \frac{d}{dt} \int_{-\infty}^{\infty} dy' B_x(y', t)$$  \hspace{1cm} (8)

and the energy lost in a half cycle is

$$U_{H.C.} = 2 \int_{0}^{\infty} dy \int_{-T/4}^{T/4} dt j(y, t) E(y, t)$$  \hspace{1cm} (9)

Substituting eq(8) and integrating by parts we get

$$U_{H.C.} = 2 \int_{0}^{\infty} dy \int_{-T/4}^{T/4} dt j_c \frac{d}{dt} \int_{b(I_0)}^{a} dy' B_x(y', t)$$

$$= 4\mu_0 J_c \int_{b(I_0)}^{a} dy \int_{b(I_0)}^{a} dy' B_x(y', t)$$  \hspace{1cm} (10)

This definition of $U_{H.C.}$ was also used by Brandt in ref. $[12]$. This defines the surface resistance as

$$R_s = 2\frac{4\mu_0 J_c}{I_0^2} \int_{b(I_0)}^{a} dy \int_{b(I_0)}^{a} dy' B_x(y', t)$$  \hspace{1cm} (11)

Fig. 6 displays the surface resistance $R_s$ as a function of the peak current, $I_0$. The surface reactance $X_s$ per unit length can be calculated using the average total electrostatic energy per unit length.

$$X_s = \frac{1}{\mu_0 I_0^2} \int_{0}^{T} \left( B^2(x, y) \right) dx dy$$  \hspace{1cm} (12)

We carry out the space integral until $B^2(x, y)$ falls down by 99% of it’s peak value. Fig. 7, displays the surface reactance $X_s$ as a function of the peak current, $I_0$. Our results are close to the analytical results in ref.[8].

### VII. CONCLUSION

A first principle numerical calculation on a strip of finite thickness using assumptions similar to the critical state model yields results that approach the analytic results in the appropriate limits ($\delta \to 0$) but without the unrealistic idealizations of the later. Further a numerical approach gives one the flexibility of accommodating higher complexities like effects due to the ground planes and more accurate calculations of surface impedance and harmonics. This topic will be addressed in future work. This approach can be implemented into a computer aided design framework for the design of passive superconducting microwave components.
Fig. 6. Surface resistance $R_s$ as a function of the current $I/I_{\text{max}}$.

Fig. 7. Surface reactance $X_s$ as a function of the current $I/I_{\text{max}}$.

VIII. Acknowledgments

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