PROPAGATION AT THE VERTEX OF A SECTOR

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Abstract. We discuss holomorphic extension across a boundary point in terms of sector property. The point is of infinite type and the sector is accordingly “cusped” at the vertex.

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1. Introduction

This paper looks at the holomorphic extension across a smooth real hypersurface, the boundary $b\Omega$ of a domain of the complex space, about a point that we fix as the origin 0. Our program is to give the analogous of the result of Baouendi and Treves [1] and Baracco, Zaitsev and Zampieri [3] on holomorphic extension across the boundary in terms of the “sector property”. In our setting the boundary may have infinite type and the sector, described by the inverse function of the type, is accordingly singular at its vertex 0. We also prefer to use the language of propagation instead of forced extension; however, the two points of view do not differ substantially. In coordinates $(z, z', w) \in \mathbb{C} \times \mathbb{C}^{n-2} \times \mathbb{C}$, let $b\Omega$ be defined by $s=h$ for $h(0)=0$, $dh(0)=0$ and, for a holomorphic function $F(z)$, $z \in \mathbb{C}$, suppose that $h$ satisfies $h|_{z'=0} = O(F(|z|))$ (with some minor additional requirements such as (2.3), (2.4) and (2.7) below). The sector property of $b\Omega$ over $S_\alpha$, parametrized by $F_\alpha^*: = F^*(\epsilon z^\alpha)$, $z = 1 - \tau \in 1 - \Delta$ (where $F^*$ is the inverse to $F$ and $\Delta$ is the standard disc) consists in $h|_{S_\alpha} \leq 0$ for $\alpha > 1$. Our result is that,

(i) If the sector property holds, then the disc $S_\alpha$ attached to $b\Omega$ over $S_\alpha$ is a propagator of holomorphic extendibility from $\Omega$ to $s < h$ at the vertex 0.

(ii) Under some additional condition (cf. (3.2) below), the sector property is necessary for holomorphic extendibility. If it is not satisfied, there is a fundamental system of neighborhoods of $\Omega \cup S_\alpha$ which are pseudoconvex. Thus propagation cannot occur.

Our result is best understood in the model situation in which $n=2$. We first treat the finite type.
Example 1.1. (Finite type.) We consider the domain $\Omega \subset \mathbb{C}^2$ defined by $s > h$ for $h = |z|^{2m} + c|z|^{2m-2p}(\text{Re} \, z)^{2p}$ and discuss the choice of $c$ which yields the sector property. For this, we set $F^* := z^\frac{1}{2m}$ and $S_\alpha = F_\alpha^*(1 - \Delta)$. Then $h < 0$ in a sector $S_\alpha$ for $\alpha > 1$ if and only if $c > \cos^{-1} (\frac{2\pi}{4m})$ (cf. [4] Proposition 4.1). Under this assumption, $S_\alpha \times \{0\}$ is contained in $\bar{\Omega}$ and thus it is a propagator by Theorem 2.2 and Remark 2.3 below (or also by [1]). When $c \leq \cos^{-1} (\frac{2\pi}{4m})$ (and in case $p$ is a divisor of $m$) there are holomorphic functions on $\Omega$ which do not extend down at 0 (cf. [4] Corollary 4.3) since, in new holomorphic coordinates, it is contained in the half space $s > 0$. Notice that the condition on $c$ for the pseudoconvexity of $\Omega$ is different; for instance for $m = 2$ and $p = 1$, this is $c \leq \frac{4}{3}$ whereas the condition for the sector property is $c \leq \cos^{-1} (\frac{\pi}{4}) = \sqrt{2}$. This means that for the intermediate values $\frac{4}{3} < c < \sqrt{2}$ we have holomorphic extension at points arbitrarily close to 0 but not at 0.

We pass to discuss the infinite type. We consider pairs of functions such as $(F, F^*) = (e^{-\frac{1}{z}}, \frac{1}{-\log z})$ or $(F, F^*) = (e^{-e^{z}}, \frac{1}{\log(-\log z)})$.

Lemma 1.2. The sector $S_\alpha$ parametrized by $F_\alpha^*(\epsilon(1-\tau)^\alpha)$ in the two respective cases listed above, satisfies

\[ S_\alpha \sim \{ z \in \mathbb{C}^+ : |y| < \alpha x^{a+1} \}, \]
\[ (\text{resp. } S_\alpha \sim \{ z \in \mathbb{C}^+ : |y| < c_\alpha x^{a+1} e^{-\frac{1}{2\pi} x \epsilon} e^{-\frac{1}{2\pi} |\alpha\tau|} \} \text{ for } c_\alpha, x = 1 + \frac{\log \alpha}{\log(-\log x)}). \]

Proof. In proving (1.1) we neglect the factor $\epsilon$ which is irrelevant at the vertex. We start from the first of (1.1) and observe that $S_\alpha$ at $z = 0$ is the $\frac{1}{\alpha^\pi}$-homotetic set of $S_1 = \{ \frac{1}{(-\log(1-\tau))^{\frac{1}{\pi}}} : \tau \in \Delta \}$ at $\tau = 1$. Now, to describe the points $z = x + iy \in bS_1$, we write $\tau = e^{i\theta}$, for $|\theta|$ small, and get

\[
x + iy \sim \frac{1}{\log(|1 - e^{i\theta}|) \frac{1}{\pi}} + i \frac{\arg(1 - e^{i\theta})}{(-\log(|1 - e^{i\theta}|) \frac{1}{\pi} + 1)}
\]
\[
\sim \frac{1}{\log \theta} \frac{1}{\frac{1}{\pi} + 1} + i \frac{\arctan \left( \frac{\pi}{\theta} \right)}{-\log \theta} \frac{1}{\frac{1}{\pi} + 1}
\]
\[
\sim \frac{1}{\log \theta} \frac{1}{\frac{1}{\pi} + 1} \pm i \frac{1}{(-\log \theta) \frac{1}{\pi} + 1}.
\]

Thus $z = x + iy \in bS_\alpha$ if and only if $|y| \sim x^{a+1}$. Taking the $\frac{1}{\alpha^\pi}$-homotetic set, the inequality changes into $|y| \sim \alpha x^{a+1}$.
To prove the second of (1.1), an easy calculation shows that 

$$S_{\alpha}$$ is approximately the $\frac{1}{c_{\alpha,x}}$-homotetic of $S_1$ for $c_{\alpha,x} = 1 + \frac{\log \alpha}{\log(-\log x)}$. We observe that, since $-\log(1 - e^{i\theta}) \sim -\log \theta^2 + i\theta$, then $\arg(-\log(1 - e^{i\theta})) \sim \frac{\theta}{-\log \theta^2}$. Thus, the points $z = x + iy \in bS_1$ are described by

$$x + iy = \frac{1}{(\log(-\log |\theta|))^{\frac{1}{a}}} + i \frac{\arg(-\log(1 - e^{i\theta})}{(\log(-\log |\theta|))^{\frac{1}{a}+1}}.$$

Thus $S_1$ is described by $|y| < x^{a+1} e^{-\frac{1}{|\alpha|^a}} e^{-\frac{1}{|\alpha|^a}}$ and its $\frac{1}{c_{\alpha,x}}$-homotetic $S_{\alpha}$ by $|y| < c_{\alpha,x} x^{a+1} e^{-\frac{1}{|\alpha|^a}} e^{-\frac{1}{|\alpha|^a}}$. This completes the proof of (1.1).

\[\square\]

**Example 1.3.** (Infinite type.) If $S_{\alpha}$ is the sector parametrized by $F_\alpha^*$ for $\alpha > 1$ (in the two respective cases), we introduce the cut-off $\chi = \chi(y_2 \alpha x^{a+1})$ (resp. $\chi = \chi(y_2 \alpha x^{a+1})$) which has support in $S_{2\alpha}$ and is 1 on $S_{\alpha}$. We choose $b$ such that $a < b < a + 1$ and consider the domain $\Omega$ of infinite type defined by $s > h$ for the two choices

$$\begin{cases}
h = e^{-\frac{1}{|\alpha|^a}} - \chi(y_2 \alpha x^{a+1}) \Re e^{-\frac{s}{|\alpha|^a}}; \\
(\text{resp. } h = e^{-\frac{1}{|\alpha|^a}} - \chi(y_2 \alpha x^{a+1}) \Re e^{-\frac{s}{|\alpha|^a}}).
\end{cases}$$

We observe that $\Re e^{-\frac{s}{|\alpha|^a}}$ is positive on $\text{supp } \chi$ and the same is true for the double exponential. Moreover, because of $b < a + 1$, we have for any $\alpha > 1$ and in a suitable neighborhood of 0

$$\begin{cases}
\Re e^{-\frac{s}{|\alpha|^a}} > e^{-\frac{1}{|\alpha|^a}} & \text{on } S_{\alpha}, \\
(\text{resp. } \Re e^{-\frac{s}{|\alpha|^a}} > e^{-\frac{1}{|\alpha|^a}} & \text{on } S_{\alpha}).
\end{cases}$$

Thus $\Omega \supset S_{\alpha} \times \{0\}$. Also, because of $b > a$, we have satisfied (2.1) and (2.3) (whereas (2.4) and (2.7) are obvious). Hence the hypotheses of Theorem 2.2 are all fulfilled and $S_{\alpha} \times \{0\}$ is a propagator of holomorphic extendibility at 0; in fact, a stronger property holds, that is, forced extension.

Note that, when $a \geq 1$, it has been proved in [2] that the half line $\mathbb{R}^+$ itself is a propagator. We can also observe that $e^{-\frac{1}{c^{|\alpha|^a}}} < e^{-\frac{1}{|\alpha|^a}}$ for any $c$ and thus in particular for $c > 1$; thus $\mathbb{R}^+$ is also a propagator.
for the domain \( s > e^{-\frac{1}{|y|}} \) regardless who is \( a \). However, this requires different tools from the present context.

**Example 1.4.** (Failure of the sector property in the exponentially degenerate type.) Let us consider the “tube” domain defined by \( s > e^{-\frac{1}{|y|}} \) for \( a < 1 \); is \( \mathbb{R}^+ \) still a propagator? To decide it, we take \( b \) such that \((b+1)a < b\), consider the holomorphic function \( e^{-\frac{1}{|y|}} \), its inverse \( \frac{1}{(-\log|y|)} \) and the related sector \( \{ z \in \mathbb{C}^+ : |y| < x_{b+1} \} \). (Here the use of \( \alpha \) has become irrelevant.) Note that the choice of the holomorphic function is no longer related to the vanishing order of the boundary. We introduce the modified domain \( \Omega \) defined by \( s > h \) for \( h = e^{-\frac{1}{|y|}} |y|_{a} \) and the related sector \( \{ z \in \mathbb{C}^+ : |y| < x_{b+1} \} \).\( \chi \left( \frac{y}{2x_{b+1}} \right) \) is 1 for \( y < x_{b+1} \) and has support in \( y < 2x_{b+1} \). We have \( \partial_z \partial_{\bar{z}} h > \frac{\partial_z e^{-\frac{1}{|y|}}}{|e^{-\frac{1}{|y|}}|} \) on \( \text{supp} \chi \) and therefore, by Proposition 3.1 below, \( h \) is plurisubharmonic, \( \Omega \) is pseudoconvex, and \( S_a \times \{0\} \) is not a propagator for \( \Omega \) since we can subtract to \( h \) a further term \( \chi \text{Re} e^{-\frac{1}{|z|}} \) without destroying its pseudoconvexity. In particular, \( \mathbb{R}^+ \) is not a propagator of extendibility for holomorphic functions on the initial tube domain. (To prove directly the plurisubharmonicity of \( h \), it suffices to notice that, over \( \text{supp} \chi \),

\[
\partial_z \partial_{\bar{z}} e^{-\frac{1}{|y|}} \sim e^{-\frac{1}{x_{a(b+1)}}} \sim \frac{1}{x_{b+1}(2a+2)} \sim \frac{1}{a^{b+1} e^{-\frac{c}{x^b}}} \quad \text{for} \quad c < 1
\]

since \( a(b+1) < b \)

\[
\sim \partial_z \partial_{\bar{z}} \left( \chi \left( \frac{y}{2x_{b+1}} \right) \text{Re} e^{-\frac{1}{|z|}} \right).
\]

The conclusion does not contradict Theorem 2.2, because, in the present case, the sector is not calibrated on the type of \( b\Omega \) by our choice of \( b > a \). If, instead, \( a < b < (a+1)a \), the sector \( \{ z \in \mathbb{C}^+ : y < x_{b+1} \} \) is a propagator for the domain defined as in Example 1.3; more precisely, we have forced extension.

2. Propagation of holomorphic extendibility

Let \( F^* \) be a holomorphic, injective function defined on the points \( \varepsilon z^a \), for \( z \in \mathbb{C}^+ := \{ z \in \mathbb{C} : \text{Re} z > 0 \} \) whose range contains a neighborhood of 0 in the half line \( \mathbb{R}^+ \). For positive \( \alpha \) and for \( F^*_\alpha := F^*(\varepsilon z^a) \), let \( S = S_\alpha \) be the sector \( \{ z = F^*_\alpha(1 - \tau), \tau \in \Delta \} \) parametrized by \( F^*_\alpha \) over the standard disc \( \Delta \); this is possibly singular at \( \tau = 1 \). We denote
by $F$ the inverse to $F^*$. The pairs of functions $(F^*, F)$ that we have in mind are $(z \frac{1}{\sqrt{\log z}}, z^{2m})$ or else $(\frac{1}{-\log z}, e^{-\frac{x}{2}})$. Let $\Omega$ be a domain in $\mathbb{C}^n$ defined by $s > h$ for $h \leq cF(|z|)$ when $z$ moves in a complex curve. We take coordinates $(z, z', w) \in \mathbb{C} \times \mathbb{C}^{n-2} \times \mathbb{C}$, $z = x + iy$, $z' = x' + iy'$, $w = r + is$, assume that $b\Omega$ is defined by $s = h(z, z', r)$ for $h(0) = 0$, $dh(0) = 0$ and that the complex curve is the $z$-axis. With the notation $h_r := h(*, 0, r)$, we are assuming that

$$h_r|_{z' = 0} = O(F(|z|)) \text{ uniformly in } r. \quad (2.1)$$

In particular,

$$|h_r||_{S_n \times \{0\}} \leq c|\theta|^\alpha, \text{ for } e^{i\theta} \in \partial \Delta \text{ uniformly in } r. \quad (2.2)$$

Let $D^1$ and $D^2$ denote various first and second order derivatives; we assume, together with (2.1) and uniformly in $r$

$$D^1h_r|_{S_n \times \{0\}} = O(|\partial_z F|), \quad D^2h_r|_{S_n \times \{0\}} = O(|\partial^2_z F|). \quad (2.3)$$

We also assume

$$|\partial^2_z F^*| \lesssim \frac{|\partial_z F^*|}{|z|}, \quad |\partial^2_z F^*| \lesssim \frac{|\partial_z F^*|}{|z|}. \quad (2.4)$$

Again, (2.1), (2.3) and (2.4) hold for our two main models. We have

**Lemma 2.1.** Let $1 < \alpha < 2$ and set $\beta := \alpha - 1$; then $h_r(F^*_\alpha(1 - e^{i\theta})) \in C^{1, \beta}$.

**Proof.** Since $F^*_\alpha(1 - e^{i\theta})$ is singular only at $\theta = 0$, by the Hardy-Littlewood lemma it suffices to prove that $\frac{d}{d\theta}h_r(F^*_\alpha) = O(\theta^{\alpha-1})$ and $\frac{d^2}{d\theta^2}h_r(F^*_\alpha) = O(\theta^{\alpha-2})$ at $\theta = 0$. Now, for $z = 1 - e^{i\theta}$, we have

$$\left| \frac{d}{d\theta}(h_r(F^*_\alpha)) \right| \lesssim \left| D^1h_r(F^*_\alpha)D^1F^*_\alpha \right| \lesssim \left| \partial_z F(F^*_\alpha) \right| (\partial_z F^*_\alpha)_{\alpha \theta^{\alpha-1}} \lesssim O(|\theta|^{-1+\alpha}),$$

$$\left| \frac{d^2}{d\theta^2}(h_r(F^*_\alpha)) \right| \lesssim \left| D^1h_r(F^*_\alpha)D^2F^*_\alpha \right| + \left| D^2h_r(F^*_\alpha)(D^1F^*_\alpha)^2 \right| \lesssim \left| \partial_z F(F^*_\alpha) \right| (\partial^2_z F^*_\alpha)_{\alpha \theta^{2\alpha-2}} + \left| (\partial_z F^*_\alpha)_{\alpha \theta^{\alpha-2}} \right| + \left| \partial^2_z F(F^*_\alpha) \right| (\partial_z F^*_\alpha)_{\alpha \theta^{2\alpha-2}} \lesssim O(|\theta|^{-2+\alpha}).$$

We consider the (Bishop) equation

$$u - T_1h(F^*_\alpha, 0, u) = 0, \quad (2.5)$$
where $T_1$ is the Hilbert transform normalized by the value 0 at $\tau = 1$. Recall that $T_1$ is continuous on $C^{1,\beta}(b\Delta)$; thus the mapping $(\epsilon, u) \mapsto u - T_1 h(F_\alpha^*, 0, u)$, $\mathbb{R} \times C^{1,\beta} \to C^{1,\beta}$ (for $F_\alpha^* = F^*(\epsilon z^\alpha)$) is differentiable and its differential in $u$ is $1 - T_1 \partial_\tau h \sim 1$. By the implicit function theorem, we have that, for $\epsilon$ small, there is a unique solution $u = u(\tau)$ to (2.5). Moreover, if we consider a 1-parameter family of deformations $h_\eta$ of $h$ so that $\eta \mapsto h_\eta(F_\alpha^*, 0, r)$, $\mathbb{R} \to C^{1,\beta}(b\Delta)$ is $C^k$ uniformly with respect to $r$, then $\eta \mapsto u_\eta$ is also $C^k$. We put $v|_{b\Delta} : = T_1 u|_{b\Delta}$ and use the same notation $u, v$ for the harmonic extensions from $b\Delta$ to $\Delta$. We define $S = S_\alpha$ by

\begin{equation}
S := \{(F_\alpha^*(1 - \tau), 0, u(\tau) + iv(\tau)) : \tau \in \Delta\}.
\end{equation}

This is the holomorphic disc attached to $b\Omega$ over the sector $S \times \{0\}$. We assume $F(x)$, $F^*(x)$ increasing, and $\partial_x F^*(x)$ decreasing. We make an additional assumption. For this, we write $z = \rho e^{i\psi}$ or else $z = \sigma(1 - e^{i\theta})$ for $(\rho, \psi) \in (0, \epsilon) \times (\pi/2, \pi/2)$ or $(\sigma, \theta) \in (1 - \epsilon, 1) \times ((\epsilon, -\epsilon) \setminus \{0\})$. They are related by the change

\begin{equation}
\begin{aligned}
\rho &= 2\sigma \sin \frac{\theta}{2} \\
\psi &= \pi - \theta.
\end{aligned}
\end{equation}

In particular

$$
\partial_\sigma = 2 \sin \left(\frac{\pi}{2} - \psi\right) \partial_\rho.
$$

With these preliminaries we suppose that $\partial_\sigma \Re F_\alpha^*(\sigma(1 - e^{i\theta}))$ and $\partial_\sigma \Im F_\alpha^*(\sigma(1 - e^{i\theta}))$ have the properties that

\begin{equation}
\begin{aligned}
(i) & \quad \text{they keep the same sign for fixed } \theta \\
(ii) & \quad \text{their absolute value is decreasing with respect to } \sigma.
\end{aligned}
\end{equation}

It is readily seen that $F^* = z^{1/\sigma}$ and $F^* = \frac{1}{(-\log z)^2}$ satisfy (2.7).

**Theorem 2.2.** In the above situation, in particular under (2.1) and (2.7) and for $\alpha > 1$, we further assume that $S$ is tangent to $b\Omega$ at the vertex, that is,

\begin{equation}
\partial_\tau v = 0 \quad \text{at } \tau = 1.
\end{equation}

Then there is propagation of holomorphic extendibility across $b\Omega$ from any point of $bS$ to the vertex 0.

**Proof.** Let holomorphic extendibility occur in the $\eta_0$-neighborhood $V_{\eta_0}$ of $\tau = -1$; we show that it also occurs at $\tau = 1$. Let $\chi = \chi(1 - \tau)$ be a cut-off in $V_{\eta_0}$ and let $u = u_\eta$ be the harmonic extension of the solution of the equation

$$
u - T_1(h(F_\alpha^*, 0, u) - \eta \chi) = 0.$$
Then, \( \eta \mapsto u_\eta + iv_\eta, \mathbb{R} \to C^{1,\beta} \) is \( C^\infty \) since \( \eta \mapsto h(F^*\alpha) - \chi \eta \) is also \( C^\infty \).

In the coordinate \( \tau = te^i\theta \in \Delta \) we have therefore the Taylor expansion

\[
\partial_t v_\eta = \partial_t v \bigg|_{\eta=0} + \eta \partial_\eta \partial_t v_\eta \bigg|_{\eta=0} + o(\eta).
\]

Now, the first term in the right side is 0 by (2.8) at \( \tau = 1 \). On the other hand, the radial derivative of \( \partial_\eta v_\eta \) at \( \tau = 1 \) can be calculated by means of the convergent integral

\[
\partial_t \partial_\eta v_\eta \sim \partial_t \int \partial_\eta v_\eta \cdot \frac{1 - t^2}{1 + t - 2 \cos \theta} d\theta \\
\sim -\partial_t \int \chi \cdot \frac{1 - t^2}{1 + t - 2 \cos \theta} d\theta \\
\sim + \int \frac{\chi}{1 - \cos \theta} d\theta = c, \text{ at } \tau = 1.
\]

Thus, for \( \eta \) small, \( \partial_\eta v_\eta > 0 \) at \( \tau = 1 \); we pick up such \( v = v_\eta \). After reparametrization \( z = F^*\alpha(1 - \tau) \), and with \( F \) denoting as always the inverse to \( F^* \), our temporary conclusion is that \( v_\eta \) satisfies

\[
(2.9) \quad v_\eta(t) \leq -cF(t) \quad t \in (1 - \epsilon, 1).
\]

We are now tempted to move the vertex of the sector \( S \) to \(-\epsilon\) and to attach to \( b\Omega \) an \( \epsilon \)-parameter family of discs over the sectors \(-\epsilon + S\). But, over the new vertex \(-\epsilon\), we do not have any more the condition (2.1) in which the singularity of \( F^* \) is balanced by the vanishing order of the defining function \( h \) of \( b\Omega \). Instead, we use approximation by the smooth sectors \( S_\nu \) parametrized by

\[
F^*_{\alpha,\nu} = F^*_{\alpha}(1 - \tau + \frac{1}{\nu}) - F^*_{\alpha}(\frac{1}{\nu}), \quad \tau \in \Delta.
\]

We have

\[
h\left(F^*_{\alpha}(1 - e^{i\theta}), r\right) \leq |\theta|^\alpha.
\]

\[\boxed{(2.1)}\]
We also have
\begin{equation}
|F^*_\alpha(1 - e^{i\theta} + \frac{1}{\nu}) - F^*_\alpha(\frac{1}{\nu})| = \left| \int_0^1 \partial_\sigma F^*_\alpha(\sigma(1 - e^{i\theta}) + \frac{1}{\nu}) d\sigma \right|
\end{equation}
\begin{align}
&\leq (i) \int_0^1 \left| \partial_\sigma F^*_\alpha(\sigma(1 - e^{i\theta}) + \frac{1}{\nu}) \right| d\sigma \\
&\leq (ii) \int_0^1 \left| \partial_\sigma F^*_\alpha(\sigma e^{i\theta}) \right| d\sigma \\
&\leq (i) \int_0^1 \left| \partial_\sigma F^*_\alpha(\sigma(1 - e^{i\theta})) \right| d\sigma \\
&= |F^*_\alpha(1 - e^{i\theta})|.
\end{align}

It follows
\begin{equation}
|h \left( \left( F^*_\alpha(1 - e^{i\theta} + \frac{1}{\nu}) - F^*_\alpha(\frac{1}{\nu}) \right), 0, r \right)| = O(F(|F^*_\alpha(1 - e^{i\theta})|)) \sim \theta^\alpha.
\end{equation}

Thus \( \frac{|h(F^*_\alpha, 0, r)|}{1 - \cos \theta} \leq |\theta|^{-2+\alpha} = |\theta|^{-1+\beta}, \beta > 0 \); we can therefore apply the dominated convergence theorem to the sequence \( \{ \partial_t v_\nu \}_\nu \) and conclude that \( \partial_t v_\nu|_{t=1} \to \partial_t v|_{t=1} \). It follows that for \( \nu \) large, the disc over \( F^*_\alpha(1 - \Delta) \) is transversal to \( b\Omega \), that is
\[ \partial_t v_\nu > 0. \]

Thus the disc \( S_\nu \) attached to \( b\Omega \) over the sector \( F^*_\alpha(1 - \Delta) \) “points down” at \( \tau = 1 \) for \( \nu \) large. By the aid of the discs attached to \( b\Omega \) over a family of translations of the sector \( F^*_\alpha(1 - \Delta) \), we sweep out a full neighborhood of 0 in the complement of \( \Omega \) and thus get the extension at 0 of a holomorphic function (if this extends at \( F^*_\alpha(2) \)).

\[ \square \]

**Remark 2.3.** If \( h|_{S_\alpha \times \{0\} \times (-\varepsilon, \varepsilon)} \leq 0 \), that is, \( S_\alpha \times \{0\} \times (-\varepsilon, \varepsilon) \subset \bar{\Omega} \), then the component \( v \) of the disc \( S_\alpha \) satisfies \( \partial_t v \geq 0 \); if, moreover \( h < 0 \) at some point of \( bS_\alpha \), then \( \partial_t v > 0 \). Hence we have in hands from the beginning a disc satisfying \( (2.9) \). This simplifies the proof of Theorem (2.2) and also dispense from taking the bumping \( h - \eta\chi \). In other words, we have forced extension at 0 of a holomorphic function \( f \) on \( \Omega \); in order that a holomorphic function \( f \) extends at 0, it needs not to extend at some other point of \( bS \).
3. Pseudoconvex bumps

Let \( F^* \) be a holomorphic, injective, function of the \( \epsilon \)-neighborhood of \( 0 \) in a sector of \( \mathbb{C} \) with axis \( \mathbb{R}^+ \) and angle \( > \pi \) whose range contains \( \mathbb{R}^+ \) at 0, denote by \( F \) the inverse to \( F^* \), and let \( \Omega \) be a domain in \( \mathbb{C}^n \) defined by \( s > h(z,z',r) \) with \( h = O(F) \). We also make the assumptions (2.2) and (2.3) and (2.4) on \( h \) and \( F^* \) which yield \( h_r(F_\alpha^*) \in C^{1,\beta} \) for \( h_r = h(\cdot,0,r) \) according to Lemma 2.1. We set \( F^*_\alpha(z) := F^*(\epsilon z^\alpha) \) and consider the sector \( S_\alpha = \{ z : z = F^*_\alpha(1 - \tau), \tau \in \Delta \} \) and the disc \( S_\alpha \) attached to \( b\Omega \) over \( S_\alpha \times \{ 0 \} \), that is, \( S_\alpha = \{ (F^*_\alpha(1 - \tau), 0, u(\tau) + iv(\tau)) : \tau \in \Delta \} \) where \( u \) and \( v \) are the harmonic extension of the functions satisfying \( v - h(F^*_\alpha(1 - \tau), 0, u)|_{b\Delta} = 0 \) and \( v|_{b\Delta} = T_1 u|_{b\Delta} \). We will suppose, all through this section, that \( \alpha < 1 \). We take \( \alpha_1 \) with \( \alpha < \alpha_1 < 1 \); the crucial and elementary remark which underlies this part of the discussion is that, since \( F \circ F^* \) is the identity of \( \mathbb{C}^+ \), then \( \Re F > 0 \) on \( \mathbb{C}^+ \) and moreover, since \( \Re z \sim |z| \) for \( z = \epsilon(1 - \tau)^{\alpha_1} \), then

\[
\Re F \sim |F| \quad \text{on } S_{\alpha_1}.
\]

**Proposition 3.1.** Let \( h = h(z) \) with \( h(0) = 0 \) be plurisubharmonic, let \( F \) be holomorphic and assume, uniformly on \( z' \) and \( r \)

\[
\partial \bar{\partial} h \sim \frac{\partial z}{{\left| {F} \right|}^2} \quad \text{on } S_{\alpha_1} \setminus S_\alpha \text{ for } \alpha < \alpha_1 < 1.
\]

Then, there is \( \tilde{h} \) with \( \tilde{h}(0) = 0 \), which is subharmonic and satisfies

\[
\begin{cases}
\tilde{h} \leq h, \\
\tilde{h} < h \quad \text{on } S_\alpha.
\end{cases}
\]

**Proof.** We define

\[
\tilde{h} = h - \eta \chi_{S_{\alpha_1} \setminus S_\alpha} \Re F,
\]

where the cut-off \( \chi_{S_{\alpha_1} \setminus S_\alpha} \) is the pull-back under \( F \) of the conical cut-off which is 0 for \( |\arg z| \geq \frac{\alpha_1 \pi}{2} \) and 1 for \( |\arg z| \leq \frac{\alpha_1 \pi}{2} \), that is, \( \chi_{S_{\alpha_1} \setminus S_\alpha} = \chi_{\left( \frac{F - F}{F + F} \right)} \). Thus

\[
\partial z \chi_{\left( \frac{F - F}{F + F} \right)} = \frac{F \partial_z F}{(F + F)^2} < \frac{|\partial_z F|}{|F|},
\]
\[ \left| \partial_z \partial_{\bar{z}} \left( \frac{F - \bar{F}}{F + \bar{F}} \right) \right| \lesssim \frac{\left| \partial_z F \right|^2 (F + \bar{F})^2 - \left| \partial_z F \right|^2 \bar{F}(F + \bar{F})^2}{(F + \bar{F})^4} \]

It then follows
\[ \partial_z \partial_{\bar{z}} (\chi_{S_{\alpha}} \setminus S_{\alpha} \Re F) = \partial_z \partial_{\bar{z}} (\chi_{S_{\alpha}} \setminus S_{\alpha} \Re F) - 2\Re (\partial_z \chi_{S_{\alpha}} \setminus S_{\alpha}) \partial_{\bar{z}} \Re F \]

\[ \lesssim \frac{\left| \partial_z F \right|^2}{|F|^2}. \]

\[ (3.6) \]

From (3.6) we readily conclude that \( \partial_z \partial_{\bar{z}} \tilde{h} \geq 0 \); thus \( \tilde{h} \) is subharmonic. It also satisfies the other requirements of the statement.

As an immediate consequence of Proposition 3.1 we have the proof of

**Theorem 3.2.** Let \( \Omega \) be a pseudoconvex domain defined by \( s = h(z, z', r) \) with \( h \) satisfying (3.2), let \( \tilde{h} \) be obtained by the technique of Proposition 3.1, and let \( \tilde{\Omega} \) be defined by \( s > \tilde{h}(z, z', r) \). Then \( \tilde{\Omega} \) has the following properties

\[ \begin{aligned} \tilde{\Omega} & \text{ is pseudoconvex,} \\
\tilde{\Omega} & \supset \Omega \cup (b\Omega \cap \pi^{-1}(S_{\alpha} \times \{0\} \times \{0\})) \quad \text{where} \ \pi \text{ is the projection to} \ w = 0, \\
0 & \notin \tilde{\Omega}. \end{aligned} \]

In particular, there is a function which is holomorphic in \( \Omega \), extends holomorphically across \( b\Omega \) at any point of the boundary of the attached disc \( bS_{\alpha} \setminus \{0\} \), but is singular at 0; thus the sector \( S_{\alpha} \) attached to \( b\Omega \) over \( S_{\alpha} \times \{0\} \times \{0\} \), even in case is contained in \( \Omega \) or is tangent to \( b\Omega \) at 0, nonetheless is not a propagator of holomorphic extendibility at the vertex 0.

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