ANOSOV REPRESENTATIONS AND PROPER ACTIONS

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Abstract. We give several new characterizations of Anosov representations of word hyperbolic groups into real reductive Lie groups, in terms of a Cartan projection of the Lie group. Using a properness criterion of Benoist and Kobayashi, we derive applications to proper actions on homogeneous spaces of reductive groups.

1. Introduction

Anosov representations of word hyperbolic groups $\Gamma$ into real semisimple (or more generally reductive) Lie groups $G$ were first introduced by Labouerie [Lab06]. They provide an interesting class of discrete subgroups of Lie groups, with a rich structure theory. In many respects, they generalize convex cocompact subgroups of rank-one Lie groups to the setting of Lie groups of higher real rank [GW12, KLPa, KLPb]. They also play an important role in the context of higher Teichmüller spaces.

The definition of Anosov representations involves the flow space of a word hyperbolic group, whose construction is not completely straightforward. In this paper, we establish several new characterizations of Anosov representations that do not involve the flow space. A central role in these characterizations is played by the Cartan projection of $G$ (associated with a fixed Cartan decomposition), which measures dynamical properties of diverging sequences in $G$. We apply our new characterizations to the study of proper actions on homogeneous spaces by establishing a direct link between the properties, for a representation $\rho : \Gamma \to G$ to be Anosov, and for $\Gamma$ to act properly discontinuously via $\rho$ on certain homogeneous spaces of $G$.

We now describe our results in more detail.

1.1. Existence of continuous boundary maps. Since the foundational work of Furstenberg and the celebrated rigidity theorems of Mostow and Margulis, the existence of boundary maps has been playing an important role in the study of discrete subgroups of Lie groups. Given a representation

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\( \rho : \Gamma \to G \) of a finitely generated group \( \Gamma \) into a reductive (e.g. semisimple) Lie group \( G \), measurable \( \rho \)-equivariant boundary maps from a Poisson boundary of \( \Gamma \) to a boundary of \( G \) exist under rather weak assumptions, e.g. Zariski density of \( \rho(\Gamma) \) in \( G \). However, if \( \Gamma \) comes with some geometric or topological boundary, obtaining a continuous \( \rho \)-equivariant boundary map is in general difficult.

Anosov representations of word hyperbolic groups come, by definition, with a pair of continuous equivariant boundary maps. More precisely, let \( \rho : \Gamma \to G \) be a \( P_\theta \)-Anosov representation. (We always assume \( G \) to be noncompact and linear, and use the standard notation \( P_\theta, P_\theta^- \) for its parabolic subgroups with the convention that \( P_\theta = G \), see Section 2.2.) Then there exist two \( \rho \)-equivariant boundary maps \( \xi^+ : \partial_\infty \Gamma \to G/P_\theta \) and \( \xi^- : \partial_\infty \Gamma \to G/P_\theta^- \) that are continuous. These boundary maps have additional remarkable properties: they are transverse, i.e. for any distinct points \( \eta, \eta' \in \partial_\infty \Gamma \) the images \( \xi^+ (\eta) \in G/P_\theta \) and \( \xi^- (\eta') \in G/P_\theta^- \) are in general position, and they are dynamics-preserving, which means that for any \( \gamma \in \Gamma \) of infinite order with attracting fixed point \( \eta^+ \in \partial_\infty \Gamma \), the point \( \xi^+ (\eta^+) \) (resp. \( \xi^- (\eta^+) \)) is an attracting fixed point for the action of \( \rho(\gamma) \) on \( G/P_\theta \) (resp. \( G/P_\theta^- \)). Furthermore, these maps satisfy an exponential contraction property involving certain bundles over the flow space of \( \Gamma \) (see Section 2.4).

In this paper, given a word hyperbolic group \( \Gamma \) and a representation \( \rho : \Gamma \to G \), we construct, under some growth assumption for the Cartan projection of \( G \) on \( \rho(\Gamma) \) (Theorem 1.1.(1)), an explicit pair \((\xi^+, \xi^-)\) of continuous, \( \rho \)-equivariant, dynamics-preserving boundary maps. (Recall that the Cartan projection \( \mu : G \to \mathfrak{p}_+ \) of \( G \), defined from a Cartan decomposition \( G = K(\exp \mathfrak{t}_+)K \), is a continuous, proper, surjective map; see Section 2.3.1, and Example 2.14 for \( G = \text{GL}_d(\mathbb{R}) \).) Under an additional assumption on the growth of \( \mu \) along geodesic rays in \( \Gamma \) (Theorem 1.1.(2)), we prove that the pair of maps \((\xi^+, \xi^-)\) is also transverse and that \( \rho \) is Anosov. This assumption involves the following notion: we say that a sequence \((x_n) \in (\mathbb{R}_+)^\mathbb{N}\) is CLI (i.e. has coarsely linear increments) if there exist \( \kappa, \kappa', \kappa'', \kappa''' > 0 \) such that for all \( n, m \in \mathbb{N}, \)

\[
\kappa m - \kappa' \leq x_{n+m} - x_n \leq \kappa'' m + \kappa'''.
\]

In other words, \( n \mapsto x_n \) is a quasi-isometric embedding of \( \mathbb{N} \) into \( \mathbb{R}_+ \). In this case we say that \((x_n) \in (\mathbb{R}_+)^\mathbb{N}\) is \((\kappa, \kappa')\)-lower CLI and \((\kappa'', \kappa''')\)-upper CLI.

**Theorem 1.1.** Let \( \Gamma \) be a word hyperbolic group and \( | \cdot |_\Gamma : \Gamma \to \mathbb{N} \) its word length function with respect to some fixed finite generating subset of \( \Gamma \). Let \( G \) be a real reductive Lie group and \( \rho : \Gamma \to G \) a representation. Fix a nonempty subset \( \theta \subset \Delta \) of the simple restricted roots of \( G \) (see Section 2.2.2), and let \( \Sigma_\theta^+ \) be the set of positive roots that do not belong to the span of \( \Delta \setminus \theta \).

(1) Suppose that for any \( \alpha \in \theta, \)

\[
< \alpha, \mu(\rho(\gamma)) > - 2 \log |\gamma|_\Gamma \xrightarrow[\gamma \to \infty]{} +\infty.
\]

Then there exist continuous, \( \rho \)-equivariant, dynamics-preserving boundary maps \( \xi^+ : \partial_\infty \Gamma \to G/P_\theta \) and \( \xi^- : \partial_\infty \Gamma \to G/P_\theta^- \).
(2) Suppose moreover that for any $\alpha \in \Sigma^+_g$ and any geodesic ray $(\gamma_n)_{n \in \mathbb{N}}$ in the Cayley graph of $\Gamma$, the sequence $(\langle \alpha, \mu(\rho(\gamma_n)) \rangle)_{n \in \mathbb{N}}$ is CLI. Then $\xi^+$ and $\xi^-$ are transverse and $\rho$ is $P_\theta$-Anosov.

By $\gamma \to \infty$ we mean that $\gamma$ leaves every finite subset of $\Gamma$.

Let us briefly discuss the meaning of the assumptions of Theorem 1.1. Let $\| \cdot \|$ be a Euclidean norm on $\mathfrak{a}$ which is invariant under the restricted Weyl group of $\mathfrak{a}$ in $G$. For any $\alpha \in \Delta$, the function $\langle \alpha, \cdot \rangle : \mathfrak{a}^+ \to \mathbb{R}_+$ is proportional to the distance function to the wall $\text{Ker} \alpha$, with respect to $\| \cdot \|$. Thus the assumption of Theorem 1.1.(1) means that the set $\mu(\rho(\Gamma))$ “drifts away at infinity” from the union of walls $\bigcup_{\alpha \in \theta} \text{Ker} \alpha$ in $\mathfrak{a}^+$, faster than logarithmically. The CLI assumption in Theorem 1.1.(2) means that the image under $\mu \circ \rho$ of any geodesic ray in the Cayley graph of $\Gamma$ drifts away “forever linearly” from the union of walls $\bigcup_{\alpha \in \theta} \text{Ker} \alpha$ (see Section 2.3.1). Inside the Riemannian symmetric space $G/K$ of $G$, the function $\| \mu(\cdot) \| : G \to \mathbb{R}_+$ gives the distance between the basepoint $x_0 = eK \in G/K$ and its image by an element of $G$ (see (2.5)). The functions $\langle \alpha, \mu(\cdot) \rangle$ can be thought of as refinements of $\| \mu(\cdot) \|$. The CLI assumption in Theorem 1.1.(2) means that $\langle \alpha, \mu \circ \rho(\cdot) \rangle : \Gamma \to \mathbb{R}_+$ restricts to a quasi-isometric embedding on any geodesic ray in the Cayley graph of $\Gamma$, for $\alpha \in \Sigma^+_g$.

Remarks 1.2. (a) Theorem 1.1.(1) extends a result of Floyd [Flo80] to the setting of higher rank.

(b) In Theorem 5.2 we give a stronger result than Theorem 1.1. In particular, we provide a sufficient condition for the existence of continuous, $\rho$-equivariant boundary maps that are not necessarily dynamics-preserving. The subtleties in dropping one assumption among continuity, dynamics-preservation, or transversality are illustrated by Examples 5.4 and A.5. The methods of the proof of Theorem 5.2 might be interesting in broader contexts where $\Gamma$ is not necessarily word hyperbolic.

(c) The CLI assumption in Theorem 1.1.(2) can be restricted to the set of geodesic rays starting at the identity element $e \in \Gamma$. In fact, we only need the CLI assumption for one quasi-geodesic representative per point in the boundary at infinity $\partial_\alpha \Gamma$ (see Proposition 5.10). If $\theta = \Delta$ (i.e. $P_\theta$ is a minimal parabolic subgroup of $G$), then the CLI assumption for all $\alpha \in \Sigma^+_g$ is equivalent to the CLI assumption for all $\alpha \in \theta$.

(d) If one assumes the CLI constants to be uniform, then Theorem 1.1.(2) can also be deduced from [KLPb, KLPc]: see Remark 1.4.(b). However, we do not require the CLI constants to be uniform here.

(e) For a quasi-geodesic ray $(\gamma_n)_{n \in \mathbb{N}}$, the sequence $(\langle \alpha, \mu(\rho(\gamma_n)) \rangle)_{n \in \mathbb{N}}$ is always upper CLI: see (2.8) and Fact 2.16.

1.2. New characterizations of Anosov representations. Theorem 1.1 provides sufficient conditions for a representation $\rho : \Gamma \to G$ to be Anosov in terms of the Cartan projection $\mu$. Conversely, we prove that any Anosov representation satisfies these conditions. This yields the following new characterizations of Anosov representations.

**Theorem 1.3.** Let $\Gamma$ be a word hyperbolic group, $G$ a real reductive Lie group, and $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of $G$. For any representation $\rho : \Gamma \to G$, the following statements are equivalent:
(1) \( \rho \) is \( P_\theta \)-Anosov;

(2) There exist two continuous, \( \rho \)-equivariant, dynamics-preserving, and transverse maps \( \xi^+ : \partial_\infty \Gamma \to G/P_\theta \) and \( \xi^- : \partial_\infty \Gamma \to G/P_\theta^- \), and for any \( \alpha \in \theta \) we have \( \langle \alpha, \mu(\rho(\gamma)) \rangle \to +\infty \) as \( \gamma \to +\infty \) in \( \Gamma \);

(3) There exist two continuous, \( \rho \)-equivariant, dynamics-preserving, and transverse maps \( \xi^+ : \partial_\infty \Gamma \to G/P_\theta \) and \( \xi^- : \partial_\infty \Gamma \to G/P_\theta^- \), and constants \( c, C > 0 \) such that \( \langle \alpha, \mu(\rho(\gamma)) \rangle \geq c|\gamma| - C \) for all \( \alpha \in \theta \) and \( \gamma \in \Gamma \);

(4) There exist \( \kappa, \kappa' > 0 \) such that for any \( \alpha \in \Sigma^+_\theta \) and any geodesic ray \( (\gamma_n)_{n \in \mathbb{N}} \) with \( \gamma_0 = e \) in the Cayley graph of \( \Gamma \), the sequence \((\langle \alpha, \mu(\rho(\gamma_n)) \rangle)_{n \in \mathbb{N}} \) is \((\kappa, \kappa')\)-lower CLI.

Remarks 1.4. (a) From Theorem 1.3.(3) we recover the fact [Lab06], [GW12, Th. 5.3] that any Anosov representation is a quasi-isometric embedding. For a semisimple Lie group \( G \) of real rank one, being a quasi-isometric embedding is equivalent to being Anosov with respect to a proper parabolic subgroup of \( G \) (see Remark 2.7 and Example (a) in Section 2.4.3), and in particular is an open property. In higher real rank this is not true: being a quasi-isometric embedding is not an open property (see Appendix A), whereas being Anosov is. In higher real rank it is more difficult to find natural constraints on quasi-isometric embeddings \( \rho : \Gamma \to G \) that define an open subset of Hom\((\Gamma, G)\). Characterization (4) in Theorem 1.3 gives one answer to this problem.

(b) Kapovich, Leeb, and Porti [KLPa, KLPb] have been developing a coarse geometric approach to Anosov representations by characterizing their images as RCA (regular, conical, antipodal) subgroups of the groups of isometries of Riemannian symmetric spaces. The equivalence \( (1) \iff (2) \) in Theorem 1.3 also follows from their characterizations. We shared and discussed our results with Kapovich and Porti in the summer 2014, in particular our equivalence \( (1) \iff (4) \) in Theorem 1.3. In their recent preprint [KLPc], they prove a result similar to that equivalence; in fact they prove an even stronger characterization of the images of Anosov representations as \( \tau_{mod} \)-regular undistorted subgroups, which implies a positive answer to the following question that we had asked:

**Question 1.5.** For \( \rho : \Gamma \to G \) to be \( P_\theta \)-Anosov, is it sufficient that there exist \( c, C > 0 \) such that for any \( \alpha \in \theta \) and \( \gamma \in \Gamma \),

\[
\langle \alpha, \mu(\rho(\gamma)) \rangle \geq c|\gamma| - C.
\]

By [KLPc], if there exist such \( c, C > 0 \), then \( \Gamma \) is automatically word hyperbolic (assuming it is finitely generated). The proofs in [KLPa, KLPb, KLPc] involve some fine geometry of higher-rank Riemannian symmetric spaces and use asymptotic cones, while our approach is based on an explicit understanding of actions of diverging sequences of elements of \( G \) on flag manifolds, which should have applications to nonhyperbolic groups as well.

The characterization of Anosov representations given by Theorem 1.3.(4) does not involve the boundary \( \partial_\infty \Gamma \), but only the behavior of the Cartan projection along geodesic rays. Here are some consequences.
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Remarks 1.6. (1) Theorem 1.3.(4) provides a notion of Anosov representations of word hyperbolic groups into $p$-adic Lie groups. Indeed, the Cartan projection $\mu$, with values in a convex cone inside some Euclidean space, is also well defined, with similar properties, when $G$ is a reductive group over $\mathbb{Q}_p$ (or more generally a non-Archimedean local field). See also [KLPc] for the notion of Morse actions on Euclidean buildings.

(2) Theorem 1.3.(4) can be used to define new classes of representations into real Lie groups. For instance, for a free group $\Gamma$, requiring Condition (4) only for certain “primitive” geodesic rays gives rise to a notion of primitive stable representations into higher-rank Lie groups (generalizing the notion introduced by Minsky [Min13] for $G = \operatorname{PSL}_2(\mathbb{C})$).

In Section 3, we prove other characterizations of Anosov representations, analogous to Theorem 1.3 but involving the Lyapunov projection $\lambda : g \mapsto \lim_n \mu(g^n)/n$ associated with the Jordan decomposition of $G$ (Theorem 3.2).

A consequence of our proof is the following statement, which is of independent interest (see Section 2.4.4 for a definition of the semisimplification).

**Corollary 1.7.** Let $\Gamma$ be a word hyperbolic group, $G$ a real reductive Lie group, and $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of $G$. Let $\rho : \Gamma \to G$ be a representation and $\rho^{\operatorname{ss}}$ its semisimplification. Then $\rho$ is $P_{\theta}$-Anosov $\iff$ $\rho^{\operatorname{ss}}$ is $P_{\theta}$-Anosov.

Since the character variety of $\Gamma$ in $G$ can be viewed as the quotient of $\operatorname{Hom}(\Gamma, G)$ by the relation “having the same semisimplification”, Corollary 1.7 means that the notion of being $P_{\theta}$-Anosov is well defined in the character variety.

1.3. Anosov representations and proper actions. The first applications of Anosov representations to proper actions on homogeneous spaces were investigated in [GW12] through constructions of domains of discontinuity. Theorem 1.3 now provides a more direct link via the following properness criterion of Benoist and Kobayashi.

**Properness criterion** [Ben96, Kob96]: Let $G$ be a reductive Lie group and $H, \Gamma$ two closed subgroups of $G$. Then $\Gamma$ acts properly on $G/H$ if and only if for any compact subset $C$ of $a$ the intersection $(\mu(\Gamma) + C) \cap \mu(H) \subset a$ is compact.

In other words, $\Gamma$ acts properly on $G/H$ if and only if the set $\mu(\Gamma)$ drifts away at infinity from $\mu(H)$. The quotient manifolds $\Gamma \backslash G/H$ are sometimes called Clifford–Klein forms of $G/H$.

Based on the properness criterion, a strengthening of the notion of proper discontinuity was introduced in [KK]: a discrete subgroup $\Gamma < G$ is said to act sharply (or strongly properly discontinuously) on $G/H$ if the set $\mu(\Gamma)$ drifts away from $\mu(H)$ at infinity “with a nonzero angle”, i.e. there are constants $c, C > 0$ such that for all $\gamma \in \Gamma$,

$$d_a(\mu(\gamma), \mu(H)) \geq c \|\mu(\gamma)\| - C,$$
where \( d_a \) denotes the metric on \( a \) induced by \( \| \cdot \\| \). The quotient \( \Gamma \backslash G/H \) is said to be a sharp Clifford–Klein form. Many (but not all) properly discontinuous actions are sharp; the sharpness constants \((c, C)\) give a way to quantify this proper discontinuity. Sharp actions are interesting for several reasons. Firstly, in all known examples, sharp actions on reductive homogeneous spaces are stable under small deformations; this is not true for properly discontinuous actions that are not sharp. Secondly, there are applications to spectral theory in the setting of affine symmetric spaces \( G/H \) by [KK], if the discrete spectrum of the Laplacian on \( G/H \) is nonempty (which is equivalent to the rank condition \( \text{rank} \ G/H = \text{rank} \ K/(K \cap H) \)), then the discrete spectrum of the Laplacian is infinite on any sharp Clifford–Klein form \( \Gamma \backslash G/H \).

Here is an immediate consequence of the implication \((1) \Rightarrow (3)\) of Theorem 1.3 and of \((2.8)\) below (which expresses the subadditivity of \( \| \mu \| \)).

**Corollary 1.8.** Let \( \Gamma \) be a word hyperbolic group, \( G \) a real reductive Lie group, and \( \theta \subset \Delta \) a nonempty subset of the simple restricted roots of \( G \). For any \( P_\theta \)-Anosov representation \( \rho : \Gamma \to G \), the group \( \rho(\Gamma) \) acts sharply (in particular, properly discontinuously) on \( G/H \) for any closed subgroup \( H \) of \( G \) such that \( \mu(H) \subset \bigcup_{\alpha \in \theta} \ker \alpha \). The set of such representations is open in \( \text{Hom}(\Gamma, G) \).

This corollary applies for instance to Hitchin representations of surface groups (which are Anosov with respect to a minimal parabolic subgroup, i.e. \( \theta = \Delta \)) and to maximal representations (which are Anosov with respect to a specific maximal proper parabolic subgroup, i.e. \( |\theta| = 1 \)). We refer to [Lab06, BIW10, GW12] for definitions.

**Corollary 1.9.** Let \( (G, H) \) be a pair in Table 1. For any Hitchin representation \( \rho : \pi_1(\Sigma) \to G \), the group \( \pi_1(\Sigma) \) acts sharply on \( G/H \).

| \( G \) | \( H \) | Conditions |
|---|---|---|
| \( \text{SL}_d(\mathbb{R}) \) | \( \text{GL}_k(\mathbb{R}) \) | \( k < d - 1 \) |
| \( \text{SL}_d(\mathbb{R}) \) | \( \text{SO}(d-k,k) \) | \( |d-2k| > 1 \) |
| \( \text{SL}_{2d}(\mathbb{R}) \) | \( \text{SL}_d(\mathbb{C}) \) | \( |k| > 1 \) |
| \( \text{SO}(d,d) \) | \( \text{SO}(k,\ell) \times \text{SO}(d-k,d-\ell) \) | \( |k - \ell| > 1 \) |
| \( \text{SO}(d,d+1) \) | \( \text{SO}(k,\ell) \times \text{SO}(d-k,d+1-\ell) \) | \( \ell \notin \{k, k+1\} \) |
| \( \text{SO}(d,d) \) | \( \text{GL}_k(\mathbb{R}) \) | \( k < d - 1 \) |
| \( \text{SO}(d,d+1) \) | \( \text{U}(d,d) \) | \( k < d \) |
| \( \text{SO}(2d,2d) \) | \( \text{U}(d-k,k) \) | \( \text{Sp}(2k,\mathbb{R}) \) |
| \( \text{Sp}(2d,\mathbb{R}) \) | \( \text{Sp}(2d,\mathbb{C}) \) | \( k < d \) |

Table 1. In these examples, \( 0 \leq k, \ell \leq d \) are any integers with \( d \geq 2 \), satisfying the specified conditions.

**Corollary 1.10.** Let \( (G, H) \) be a pair in Table 2. For any maximal representation \( \rho : \pi_1(\Sigma) \to G \), the group \( \pi_1(\Sigma) \) acts sharply on \( G/H \).
The lists of pairs $(G, H)$ in Tables 1 and 2 are taken from Okuda’s classification [Oku13] of semisimple symmetric spaces admitting proper actions by surface groups, and from some extension to the nonsymmetric case by Bocheński–Jastrzębski–Talle [BJT]. Applying Corollary 1.8, it is easy to find many other examples with similar properties.

Conversely to Corollary 1.8, we can prove that certain properly discontinuous actions give rise to Anosov representations, using the implication (4) ⇒ (1) of Theorem 1.3. This works well, for instance, in the so-called standard case, namely when the discrete group $\Gamma$ lies inside some Lie subgroup $G_1$ of $G$ that itself acts properly on $G/H$; in this case the action of $\Gamma$ is automatically properly discontinuous, and even sharp (see [KK, Ex. 4.10] or Section 5.6).

**Corollary 1.11.** Let $G$ be a real reductive Lie group, $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of $G$, and $H$ a closed subgroup of $G$ such that $\mu(H) \supset \left( \bigcup_{\alpha \in \theta} \text{Ker} \alpha \right) \cap \overline{\alpha}^\perp$. Let $G_1$ be a reductive subgroup of $G$, of real rank 1, acting properly on $G/H$. Then for any convex cocompact subgroup $\Gamma$ of $G_1$, the inclusion of $\Gamma$ into $G$ is $P_\theta$-Anosov.

Recall that $\Gamma$ being convex cocompact in $G_1$ means that there is a non-empty, $\Gamma$-invariant, closed, convex subset $C$ of the Riemannian symmetric space of $G_1$ such that $\Gamma \backslash C$ is compact. Since $G_1$ has real rank 1, this is equivalent to $\Gamma$ being quasi-isometrically embedded in $G_1$ (see Remark 2.7).

Corollary 1.11 applies in particular to the examples in Table 3 below. For more examples, including exceptional groups, see [KY05]. In examples (i) to (iv) when $d = 2k$, in example (v) when $d = 4\ell$, and in example (vii), the group $G_1$ acts cocompactly on $G/H$, hence compact Clifford–Klein forms of $G/H$ can be obtained by taking $\Gamma$ to be a uniform lattice in $G_1$. We refer to [Oku13, BJT] for many more examples to which Corollary 1.11 applies, where $G_1$ is locally isomorphic to $\text{SL}_2(\mathbb{R})$. 

| $G$               | $H$               | Conditions                      |
|------------------|------------------|---------------------------------|
| $\text{SO}(2, d)$ | $\text{SO}(1, d)$|                                 |
| $\text{Sp}(2d, \mathbb{R})$ | $U(d - k, k)$     |                                 |
| $\text{Sp}(2d, \mathbb{R})$ | $\text{Sp}(2k, \mathbb{R})$ | $k < d$                        |
| $\text{Sp}(4d, \mathbb{R})$ | $\text{Sp}(2d - 2, \mathbb{C})$ |                                 |
| $\text{SU}(2d + 1, 2d + 1)$ | $\text{SO}^*(4d + 2)$ |                                 |
| $\text{SU}(p, q)$  | $\text{SU}(k, \ell) \times \text{SU}(p - k, q - \ell)$ | $(p - q)(k - \ell) < 0$         |
| $\text{SO}^*(2d)$  | $\text{U}(d - k, k)$ |                                 |
| $\text{SO}^*(4d)$  | $\text{SO}^*(4d - 2)$ |                                 |
| $E_6(-14)$        | $F_4(-20)$        |                                 |
| $E_7(-25)$        | $E_6(-14)$        |                                 |
| $E_7(-25)$        | $\text{SU}(6, 2)$ |                                 |

Table 2. In these examples, $0 \leq k, \ell \leq d, p, q$ are any integers with $d \geq 2$, satisfying the specified conditions.
In these examples, $d, k, \ell$ are any integers with $0 < k \leq d/2$ and $0 < \ell \leq d/4$. We denote by $\alpha_0$ the simple root of $G$ such that $P_{\alpha_0}$ is the stabilizer of an isotropic line, and by $\alpha_1$ the simple root of $G$ such that $P_{\alpha_1}$ is the stabilizer of a maximal isotropic subspace.

The geometric construction of domains of discontinuity from [GW12] can be applied in many of the cases of Table 3 to furthermore obtain compactifications of the corresponding Clifford–Klein forms $\Gamma \backslash G/H$. These compactifications generalize for instance the conformal compactifications of Fuchsian and quasi-Fuchsian groups. This is the object of Section 7.

Remarks 1.12. (a) The properness criterion of Benoist and Kobayashi also applies when $G$ is a reductive group over a non-Archimedean local field (e.g. $\mathbb{Q}_p$): see [Ben96]. Corollaries 1.8 and 1.11 also hold in this setting (see Remark 1.4.(1)).

(b) From Corollaries 1.8 and 1.11, we recover the main result of [Kas12]: in the setting of Corollary 1.11 (over $\mathbb{R}$ or $\mathbb{Q}_p$), there is a neighborhood $U \subset \text{Hom}(\Gamma, G)$ of the natural inclusion such that for any $\varphi \in U$ the group $\varphi(\Gamma)$ is discrete in $G$ and acts properly discontinuously on $G/H$.

1.4. **Proper actions on group manifolds.** For a Lie group $G$, let $\text{Diag}(G)$ be the diagonal of $G \times G$. The homogeneous space $(G \times G)/\text{Diag}(G)$ identifies with $G$ endowed with the transitive action of $G \times G$ by left and right translation, and is thus called a group manifold. Using the full equivalence (1) $\iff$ (3) of Theorem 1.3, we obtain a particularly satisfying characterization of quasi-isometrically embedded groups acting properly on $(G \times G)/\text{Diag}(G)$ when $G$ is semisimple of real rank 1. This covers in particular the cases of anti-de Sitter 3-manifolds ($G = \text{PSL}_2(\mathbb{R})$) and of Riemannian holomorphic complex 3-manifolds with constant nonzero sectional curvature ($G = \text{PSL}_2(\mathbb{C})$).

**Theorem 1.13.** Let $G$ be a semisimple Lie group of real rank 1 and $\Gamma$ a finitely generated subgroup of $G \times G$. Then the following are equivalent:

1. $\Gamma$ acts properly discontinuously on $(G \times G)/\text{Diag}(G)$ and the inclusion $\Gamma \hookrightarrow G \times G$ is a quasi-isometric embedding,
2. $\Gamma$ acts sharply on $(G \times G)/\text{Diag}(G)$ and the inclusion $\Gamma \hookrightarrow G \times G$ is a quasi-isometric embedding,
3. $\Gamma$ is word hyperbolic, of the form $\Gamma = \{ (\rho_L(\gamma), \rho_R(\gamma)) \mid \gamma \in \Gamma_0 \}$,
where \( \rho_L, \rho_R : \Gamma_0 \to G \) are representations and, up to switching the two factors of \( G \times G \), the representation \( \rho_L \) has finite kernel and convex cocompact image and \( \rho_L \) uniformly dominates \( \rho_R \).

Here we say that the representation \( \rho_L \) uniformly dominates \( \rho_R \) if there exists \( c < 1 \) such that for all \( \gamma \in \Gamma \),

\[
\lambda(\rho_R(\gamma)) \leq c \lambda(\rho_L(\gamma)),
\]

where \( \lambda : G \to \mathbb{R}_+ \) is the translation length function in the Riemannian symmetric space \( G/K \) of \( G \), given by \( \lambda(g) = \inf_{x \in G/K} d(x, g \cdot x) \) for all \( g \in G \).

Remark 1.14. Theorem 1.13, together with Corollary 1.17 below, was first established in [Kas09] for \( G = \text{PSL}_2(\mathbb{R}) \simeq \text{SO}(1, 2)_0 \), then in [GK] for \( G = \text{SO}(1, d) \) with \( d \geq 2 \). (For the \( p \)-adic version, with \( G \) of relative rank 1 over a non-Archimedean local field, see [Kas10].) The fact that for a general Lie group \( G \) of real rank 1, any discrete subgroup of \( G \times G \) acting properly discontinuously on \( (G \times G)/\text{Diag}(G) \) is of the form \((\rho_L, \rho_R)(\Gamma_0)\) where \( \rho_L \) or \( \rho_R \) is discrete with finite kernel, was proved in [Kas08]; see Theorem 6.12 for a precise statement.

To prove Theorem 1.13, we relate conditions (1), (2), (3) to the fact that \( \Gamma \) is word hyperbolic and its natural inclusion inside some larger group containing \( G \times G \) is Anosov. Such a relation exists even when \( G \) has higher real rank: see Theorem 6.3 for a general and precise statement.

Here is what we prove in the special case that \( G = \text{Aut}_K(b) \) is the group of automorphisms of a vector space over \( K = \mathbb{R} \) preserving a nondegenerate bilinear (symmetric or symplectic) form \( b \), or the group of automorphisms of a vector space over \( K = \mathbb{C} \) or \( \mathbb{H} \) (quaternions) preserving a nondegenerate Hermitian or anti-Hermitian form \( b \). This situation includes all classical simple groups of real rank 1, namely \( \text{SO}(1, d) \), \( \text{SU}(1, d) \), \( \text{Sp}(1, d) \) (Example 6.4).

Theorem 1.15. For \( K = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H} \), let \( V \) be a \( K \)-vector space and \( b : V \otimes_V V \to K \) an \( R \)-bilinear form which is symmetric, antisymmetric, Hermitian, or anti-Hermitian over \( K \). Let \( Q_0(b \oplus b) \) be the stabilizer in \( \text{Aut}_K(b \oplus b) \) of a \((b \oplus b)\)-isotropic line in \( V \oplus V \), and similarly for \( \oplus (-b) \). For a discrete subgroup \( \Gamma \) of \( G \times G \), the following two conditions are equivalent:

1. \( \Gamma \) is word hyperbolic and the natural inclusion
   \( \Gamma \hookrightarrow G 	imes G = \text{Aut}(b) \times \text{Aut}(b) \hookrightarrow \text{Aut}(b \oplus b) \)
   is \( Q_0(b \oplus b) \)-Anosov,

2. \( \Gamma \) is word hyperbolic and the natural inclusion
   \( \Gamma \hookrightarrow G 	imes G = \text{Aut}(b) \times \text{Aut}(-b) \hookrightarrow \text{Aut}(b \oplus (-b)) \)
   is \( Q_0(b \oplus (-b)) \)-Anosov.

If (4) or (5) holds, then (1), (2), (3) of Theorem 1.13 hold (where, in case \( G \) has higher real rank, convex cocompact is replaced by \( Q_0(b) \)-Anosov and uniform domination by \( Q_0(b) \)-uniform domination, see Definition 6.1). The converse is true if and only if \( G \) has real rank 1.

We refer to Remark 6.5 for an explanation of why (2) does not imply (4) when \( G \) has higher real rank.
Even though $\text{Aut}(b) = \text{Aut}(-b)$, the embeddings in (4) and (5) are in general quite different. For instance, for $\text{Aut}(b) = O(1, d)$, these embeddings are $O(1, d) \times O(1, d) \rightarrow O(2, 2d)$ and $O(1, d) \times O(1, d) \rightarrow O(d + 1, d + 1)$.

The following corollaries are immediate consequences of Theorems 1.13 and 1.15; the second one uses the fact that being Anosov is an open property.

**Corollary 1.16.** Let $G$ be a semisimple Lie group of real rank 1 and $\Gamma$ a discrete subgroup of $G \times G$. If the action of $\Gamma$ on $(G \times G) / \text{Diag}(G)$ is properly discontinuous and cocompact, then it is in fact sharp.

**Corollary 1.17.** Let $G$ be a semisimple Lie group of real rank 1 and $\Gamma$ a finitely generated quasi-isometrically embedded subgroup of $G \times G$. If $\Gamma$ acts properly discontinuously on $(G \times G) / \text{Diag}(G)$, then there is a neighborhood $U \subset \text{Hom}(\Gamma, G \times G)$ of the natural inclusion such that for any $\rho \in U$, the group $\rho(\Gamma)$ is quasi-isometrically embedded in $G \times G$, and acts properly discontinuously on $(G \times G) / \text{Diag}(G)$. If moreover the action of $\Gamma$ on $(G \times G) / \text{Diag}(G)$ is cocompact, then $\rho(\Gamma)$ also acts cocompactly on $(G \times G) / \text{Diag}(G)$.

**Remark 1.18.** For $G$ semisimple of real rank 1, Corollary 1.17 together with [Tho, Th. 3] implies that the space of complete $(G \times G, (G \times G) / \text{Diag}(G))$-structures on a compact manifold $M$ is a union of connected components of the space of $(G \times G, (G \times G) / \text{Diag}(G))$-structures on $M$.

**Conventions.** In the whole paper, we assume the reductive group $G$ to be noncompact, equal to a finite union of connected components (for the real topology) of $G(R)$ for some algebraic group $G$. We set $R_+ := [0, +\infty)$, as well as $N := Z \cap R_+$ and $N^* := N \setminus \{0\}$.

1.5. **Organization of the paper.** In Section 2 we review some background material. In Section 3 we prove the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) of Theorem 1.3 (characterizations of Anosov representations involving boundary maps). In Section 4 we summarize some facts about linear representations of reductive Lie groups, which are later used in Sections 5, 6, and 7. Section 5 contains several key results: we prove Theorem 1.1 (construction of the boundary maps, and transversality under the CLI assumption) and establish the equivalence (1) $\Leftrightarrow$ (4) of Theorem 1.3; we also provide a short proof of Corollary 1.11 in Section 5.6. The link between Anosov representations and proper actions on group manifolds is established in Section 6, where we prove Theorems 1.13 and 1.15. Finally, in Section 7 we describe some geometric features and compactifications of the Clifford–Klein forms coming from Anosov representations.

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2. Preliminaries

In this section we set up notation and recall some useful facts about word hyperbolic groups, real reductive Lie groups, and Anosov representations.

2.1. Word hyperbolic groups and their boundary at infinity. Recall that a finitely generated group $\Gamma$, with finite generating set $S \subset \Gamma$, is said to be word hyperbolic if its Cayley graph $\mathcal{C}(\Gamma, S)$, equipped with the natural graph metric, is Gromov hyperbolic.

2.1.1. The boundary at infinity. If $X$ and $X'$ are geodesic metric spaces and if $f : X \rightarrow X'$ is a quasi-isometry, then $X$ is Gromov hyperbolic if and only if $X'$ is, and in this case $f$ induces a homeomorphism $\partial_{\infty} f : \partial_{\infty} X \rightarrow \partial_{\infty} X'$ between the visual boundaries [CDP90, Ch. III, Th. 2.2]. This fundamental fact has the following consequences:

(i) the notion of word hyperbolic group does not depend on the choice of finite generating set,
(ii) the boundary at infinity $\partial_{\infty} \Gamma = \partial_{\infty} \mathcal{C}(\Gamma, S)$ is well defined and the group $\Gamma$ acts on it by homeomorphisms.

It is known that $\Gamma$ acts on $\partial_{\infty} \Gamma$ as a uniform convergence group (see e.g. [Bow98]), which means that it acts properly discontinuously and cocompactly on the set $\partial_{\infty} \Gamma^{3*}$ of triples of pairwise distinct elements of $\partial_{\infty} \Gamma$. As a consequence, it satisfies the following dynamical properties:

(1) For any sequence $(\gamma_n) \in \Gamma^N$ going to infinity, there exist $\eta, \eta' \in \partial_{\infty} \Gamma$ and a subsequence $(\gamma_{\phi(n)})$ such that $\gamma_{\phi(n)}|_{\partial_{\infty} \Gamma \setminus \{\eta'\}}$ converges, in the compact-open topology, to the constant map with image $\{\eta\}$. 

(2) For any $\gamma \in \Gamma$ of infinite order, there exist $\eta^+ \neq \eta^- \in \partial_{\infty} \Gamma$ such that $\lim_{n \rightarrow +\infty} \gamma^n \cdot \eta = \eta^+$ for all $\eta \neq \eta^-$ and $\lim_{n \rightarrow -\infty} \gamma^{-n} \cdot \eta = \eta^-$ for all $\eta \neq \eta^+$.

Moreover, the pairs $(\eta^+, \eta^-)$ of attracting and repelling fixed points of elements $\gamma \in \Gamma$ of infinite order form a dense subset of $\partial_{\infty} \Gamma \times \partial_{\infty} \Gamma \setminus \text{Diag}(\partial_{\infty} \Gamma)$. When $\Gamma$ is nonelementary, i.e. when $\#\partial_{\infty} \Gamma \geq 3$ (or equivalently when $\Gamma$ is not virtually cyclic), the action of $\Gamma$ on its boundary is minimal (i.e. every nonempty $\Gamma$-invariant subset is dense).

2.1.2. Word length, stable length, and translation length. Associated with the word length function $|\cdot| : \Gamma \rightarrow \mathbb{N}$ is the stable length function $|\cdot|_{\infty} : \Gamma \rightarrow \mathbb{N}$, given by

$$|\gamma|_{\infty} = \lim_{n \rightarrow +\infty} \frac{1}{n} |\gamma^n|$$

for all $\gamma \in \Gamma$. It is easily seen to be invariant under conjugation: $|\beta \gamma \beta^{-1}|_{\infty} = |\gamma|_{\infty}$ for all $\beta, \gamma \in \Gamma$. Moreover, it is related to the translation length function on the Cayley graph

$$\gamma \mapsto \ell_{\Gamma}(\gamma) = \inf_{\beta \in \Gamma} |\beta \gamma \beta^{-1}|$$
via the following proposition.

**Proposition 2.1 ([CDP90, Ch. X, Prop. 6.4]).** If the group $\Gamma$ is $\delta$-hyperbolic, then $\ell_\Gamma(\gamma) - 16\delta \leq |\gamma|_\infty \leq \ell_\Gamma(\gamma)$ for all $\gamma \in \Gamma$.

For $c, C > 0$, we shall say that a sequence $(\gamma_n) \in \Gamma^\mathbb{N}$ defines a $(c, C)$-quasi-geodesic ray in the Cayley graph of $\Gamma$ if for all $n, m \in \mathbb{N}$,

$$c^{-1}|n - m| - C \leq |\gamma_n^{-1}\gamma_m| \leq c|n - m| + C.$$

**2.1.3. The flow space.** An important object for the definition of an Anosov representation below and for some proofs in this paper is the flow space $G_\Gamma$ of the word hyperbolic group $\Gamma$. It is a proper metric space $G_\Gamma$ with the following properties:

1. $G_\Gamma$ is Gromov hyperbolic.
2. $G_\Gamma$ is equipped with a properly discontinuous and cocompact action of $\Gamma$ by isometries. In particular, any orbit map $\gamma \mapsto \gamma \cdot v$ from $\Gamma$ to $G_\Gamma$ is a quasi-isometry, and so $\partial_\infty \Gamma$ is equivariantly homeomorphic to $\partial_\infty G_\Gamma$.
3. $G_\Gamma$ is equipped with a flow $\{\varphi_t\}_{t \in \mathbb{R}}$ (i.e. a continuous $\mathbb{R}$-action) that commutes with the $\Gamma$-action and for which there exist $c, C > 0$ such that any $\mathbb{R}$-orbit $\mathbb{R} \to G_\Gamma$ is a $(c, C)$-quasi-isometric embedding. This implies the existence of two continuous maps

$$\varphi_{\pm \infty} : G_\Gamma \to \partial_\infty \Gamma, \quad v \mapsto \lim_{t \to \pm \infty} \varphi_t \cdot v$$

associating to $v \in G_\Gamma$ the endpoints of its orbit.
4. $G_\Gamma$ is equipped with an isometric $\mathbb{Z}/2\mathbb{Z}$-action commuting with $\Gamma$ and anticommuting with $\mathbb{R}$.
5. The natural map

$$(\varphi_{-\infty}, \varphi_{+\infty}) : \mathbb{R} \times G_\Gamma \to \partial_\infty \Gamma^2 := \{(\eta, \eta') \in \partial_\infty \Gamma^2 \mid \eta \neq \eta'\}$$

is a homeomorphism.

The flow space was constructed by Gromov [Gro87, Th. 8.3.C], and more details were provided by Champetier [Cha94, § 4]. Mineyev [Min05] introduced a different construction of the flow space of a hyperbolic graph with bounded valency. It is based on the existence of a hyperbolic metric $\hat{d}$ on the Cayley graph satisfying some subtle properties (see [Min05, Th. 26] and [MY02, Th. 17]). In Mineyev’s version the $\mathbb{R}$-orbits are geodesics and not only quasi-geodesics. There is also a uniqueness statement for the flow space $G_\Gamma$ as a $\Gamma \times (\mathbb{R} \times \mathbb{Z}/2\mathbb{Z})$-space up to quasi-isometry and up to reparameterization of the $\mathbb{R}$-orbits, but we shall not need it.

**Remark 2.2.** It follows from Section 2.1.1 that the union of periodic geodesics of the flow space $G_\Gamma$ is dense in $G_\Gamma$.

**2.1.4. Geodesics in $\Gamma$ and in its flow space.** We make the following definition.

**Definition 2.3.** A sequence $(x_n) \in \mathbb{R}^\mathbb{N}$ is CLI (i.e. has coarsely linear increments) if $n \mapsto x_n$ is a quasi-isometric embedding of $\mathbb{N}$ into $[a, +\infty)$ for some $a \in \mathbb{R}$, i.e. there exist $\kappa, \kappa', \kappa'', \kappa''' > 0$ such that for all $n, m \in \mathbb{N}$,

$$\kappa m - \kappa' \leq x_{n+m} - x_n \leq \kappa'' m + \kappa'''.$$
In this case we say that \((x_n)_{n \in \mathbb{N}}\) is \((\kappa, \kappa')\)-lower CLI.

The following result will be used several times throughout the paper.

**Proposition 2.4.** Let \(\Gamma\) be a word hyperbolic group with flow space \(\mathcal{G}_\Gamma\). For any \(c, C > 0\), there exist a compact subset \(\mathcal{D}\) of \(\mathcal{G}_\Gamma\) and constants \(\kappa, \kappa' > 0\) with the following property: for any \((c, C)\)-quasi-geodesic ray \((\gamma_n)_{n \in \mathbb{N}}\) with \(\gamma_0 = e\) in the Cayley graph of \(\Gamma\), there exist \(v \in \mathcal{D}\) and a \((\kappa, \kappa')\)-lower CLI sequence \((t_n) \in \mathbb{R}^\mathbb{N}\) such that \(\varphi_{t_n} \cdot v \in \gamma_n \cdot \mathcal{D}\) for all \(n \in \mathbb{N}\).

**Proof.** Any \((c, C)\)-quasi-geodesic ray \((\gamma_n)_{n \in \mathbb{N}}\) with \(\gamma_0 = e\) can be extended to a full uniform quasi-geodesic \((\gamma_n)_{n \in \mathbb{Z}}\) in the Cayley graph of \(\Gamma\). Let \(\psi : \Gamma \to \mathcal{G}_\Gamma\) be an orbital application; it is a quasi-isometry. By hyperbolicity, \((\psi(\gamma_n))_{n \in \mathbb{Z}}\) lies at uniformly bounded Hausdorff distance \(R > 0\) from the \(\mathbb{R}\)-orbit in \(\mathcal{G}_\Gamma\) with the same endpoints at infinity. Let us write this \(\mathbb{R}\)-orbit as \((\varphi_t \cdot v)_{t \in \mathbb{R}}\) where \(v\) lies at distance \(\leq R\) from \(\psi(e)\). For any \(n \in \mathbb{N}\), the point \(\psi(\gamma_n) \in \mathcal{G}_\Gamma\) lies at distance \(\leq R\) from \(\varphi_{t_n} \cdot v\) for some \(t_n \in \mathbb{R}\), and the sequence \((t_n)_{n \in \mathbb{N}}\) is CLI because \(\psi\) is a quasi-isometry. The lower CLI constants of \((t_n)_{n \in \mathbb{N}}\) depend only on \((c, C)\) and on the quasi-isometry constants of \(\psi\). \(\Box\)

**Corollary 2.5.** Let \(\Gamma\) be a word hyperbolic group. Then there exists a compact set \(\mathcal{D} \subset \mathcal{G}_\Gamma\) and constants \(c_1, c_2 > 0\) with the following property: for any \(\gamma \in \Gamma\) there exist \(v \in \mathcal{D}\) and \(t \geq 0\) such that \(t \geq c_1|\gamma|_{\Gamma} - c_2\) and \(\varphi_t \cdot v \in \gamma \cdot \mathcal{D}\).

**Proof.** Any element \(\gamma\) belongs to a uniform quasi-geodesic \((\gamma_n)_{n \in \mathbb{N}}\) with \(\gamma_0 = e\) and \(|\gamma|_{\Gamma} = \gamma\). We conclude using Proposition 2.4. \(\Box\)

2.1.5. **Negatively curved Riemannian manifolds.** Suppose \(X\) is a simply connected Riemannian manifold with sectional curvature bounded above by \(-a^2\), for some \(a \neq 0\), and \(\Gamma\) is a convex cocompact subgroup of the group of isometries \(\text{Isom}(X)\), i.e. there exists a nonempty convex subset \(\mathcal{C} \subset X\) on which \(\Gamma\) acts properly and cocompactly by isometries. Then the boundary at infinity of \(\Gamma\) is homeomorphic to the limit set \(\Lambda_\Gamma \subset \partial_\infty X\). Moreover, the flow space of \(\Gamma\) is a subflow of the unit tangent bundle \(T^1(X)\):

**Fact 2.6.** Let \(X\) be a simply connected Riemannian manifold with sectional curvature bounded above by \(-a^2\), for some \(a \neq 0\), and \(\Gamma\) a convex cocompact subgroup of \(\text{Isom}(X)\). Then a flow space of \(\Gamma\) is

\[
\mathcal{G}_\Gamma = \{v \in T^1(X) \mid (\varphi_{+\infty}, \varphi_{-\infty})(v) \in \Lambda_\Gamma \times \Lambda_\Gamma\}
= \{v \in T^1(X) \mid \forall t \in \mathbb{R}, \ \pi(\varphi_t \cdot v) \in \mathcal{C}\},
\]

with its natural \(\Gamma \times (\mathbb{R} \ltimes \mathbb{Z}/2\mathbb{Z})\)-action.

This example illustrates the nonuniqueness of the flow space as a metric space, since a given convex cocompact subgroup can have nontrivial deformations.

**Remark 2.7.** If \(G\) has real rank 1 and \(\rho : \Gamma \to G\) is a quasi-isometric embedding, then \(\rho\) has finite kernel and the group \(\rho(\Gamma)\) is convex cocompact in \(G\) (see [Bou95, Bow93]). Thus, for many applications in this paper, the reader can ignore the general theory of flow spaces and work with the description of Fact 2.6.
2.2. Parabolic subgroups of reductive Lie groups. Let $G$ be a non-compact real reductive Lie group. We assume that $G$ is a finite union of connected components (for the real topology) of $G(\mathbb{R})$ for some algebraic group $G$. Recall that $G$ is the almost product of $Z(G)_0$ and $G_s$, where $Z(G)_0$ is the identity component (for the real topology) of the center $Z(G)$ of $G$, and $G_s = D(G)$ is the derived subgroup of $G$, which is semisimple.

2.2.1. Parabolic subgroups. By definition, a parabolic subgroup of $G$ is a subgroup of the form $P = G \cap P(\mathbb{R})$ for some algebraic subgroup $P$ of $G$ with $G(\mathbb{R})/P(\mathbb{R})$ compact.

Definition 2.8. Two parabolic subgroups $P$ and $Q$ are said to be

- transverse (or opposite, or in generic position) if their intersection is a reductive subgroup;
- compatible (or in singular position) if their intersection is a parabolic subgroup.

When $G$ has real rank 1, two parabolic subgroups are either transverse (i.e. distinct) or compatible (i.e. equal), but when $G$ has higher real rank there are other cases between these two extremes.

Remark 2.9. Any parabolic subgroup $P$ is its own normalizer in $G$, hence $G/P$ identifies (as a $G$-set) with the set of conjugates of $P$ in $G$. In the sequel, we shall make no distinction between elements of $G/P$ and parabolic subgroups. In particular, the terminology transverse and compatible will be used for elements of $G/P \times G/Q$.

Remark 2.10. Let $X = G/K$ be the Riemannian symmetric space of $G$; it has nonpositive curvature and its visual boundary $\partial_\infty X$ is a sphere. Geometrically, a proper parabolic subgroup of $G$ is the stabilizer in $G$ of a (not necessarily unique) point $\xi \in \partial_\infty X$. Two proper parabolic subgroups $P$ and $Q$ are transverse if and only if there is a bi-infinite geodesic $c : \mathbb{R} \to X$ such that $P = \text{Stab}_G(\lim_{\to \infty} c)$ and $Q = \text{Stab}_G(\lim_{\leftarrow \infty} c)$.

Example 2.11. Let $K$ be $\mathbb{R}$, $\mathbb{C}$, or the ring $\mathbb{H}$ of quaternions, and let $G = \text{GL}_K(V)$ for some (right) $K$-vector space $V$. Any parabolic subgroup of $G$ is the stabilizer of a flag in $V$. Two parabolic subgroups are transverse if and only if the flags are transverse.

2.2.2. Lie algebra decompositions. Let $\mathfrak{g}$ (resp. $\mathfrak{z}(\mathfrak{g})$, resp. $\mathfrak{g}_s$) be the Lie algebra of $G$ (resp. of the center $Z(G)$, resp. of the derived group $G_s$). Then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_s$, and this decomposition is orthogonal with respect to the Killing form of $\mathfrak{g}$, whose restriction to $\mathfrak{z}(\mathfrak{g})$ (resp. $\mathfrak{g}_s$) is zero (resp. nondegenerate). Here are some algebraic and combinatorial objects needed to give a more comprehensive description of the parabolic subgroups of $G$:

- $K$: a maximal compact subgroup of $G$, with Lie algebra $\mathfrak{k}$;
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$: the orthogonal decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$, for the Killing form;
- $\mathfrak{a} \subset \mathfrak{k}^\perp$: a Cartan subspace of $\mathfrak{g}$, i.e. a maximal abelian subspace of $\mathfrak{k}^\perp$, which we fix; it is the direct sum of $\mathfrak{k}^\perp \cap \mathfrak{z}(\mathfrak{g})$ and of a maximal abelian subspace $\mathfrak{a}_s$ of $\mathfrak{k}^\perp \cap \mathfrak{g}_s$ (unique up to the $\text{Ad}(K)$-action);
We can take $\Delta = \bigoplus_{a \in \Sigma} g_a$: the decomposition of $g$ into $\text{ad}(a)$-eigenspaces: by definition,

$$(\text{ad} \, Y)(Y') = \alpha(Y)Y'$$

for all $Y \in a$ and $Y' \in g_a$. The eigenspace $g_0$ is the centralizer of $a$ in $g_1$; it is the direct sum of $\mathfrak{z}(g)$ and of the centralizer of $a$ in $g_a$. The set $\Sigma \subset \mathfrak{a}^*$ projects to a (possibly nonreduced) root system of $\mathfrak{a}_s^*$, and each $\alpha \in \Sigma$ is called a (restricted) root of $a$ in $g$.

- $\Delta \subset \Sigma$: a simple system (see [Kna02, §II.6, p. 164]), i.e. any root is expressed uniquely as a linear combination of elements of $\Delta$ with coefficients all of the same sign; elements of $\Delta$ are called simple (or indecomposable) roots;
- $\Sigma^+ \subset \Sigma$: the set of positive roots, i.e. roots that are nonnegative linear combinations of elements of $\Delta$; then $\Sigma = \Sigma^+ \cup (-\Sigma^+)$.

Note that $\Delta$ projects to a basis of the vector space $\mathfrak{a}_s^*$. The real rank of $G$ is by definition the dimension of $a$. Let

$$\Pi^+ := \{Y \in a \mid \alpha(Y) \geq 0, \forall \alpha \in \Sigma^+\} = \{Y \in a \mid \alpha(Y) \geq 0, \forall \alpha \in \Delta\}$$

be the closed positive Weyl chamber of $a$ associated with $\Sigma^+$.

Given a subset $\theta \subset \Delta$, we define $P_\theta$ (resp. $P_{\theta}^-$) to be the normalizer in $G$ of the Lie algebra

$$u_\theta = \bigoplus_{\alpha \in \Sigma^+} g_\alpha \quad \text{(resp. } u_{\theta}^- = \bigoplus_{\alpha \in \Sigma^+} g_{-\alpha})$$

where $\Sigma^+_{\theta} = \Sigma^+ \setminus \text{span}(\Delta \setminus \theta)$ is the set of positive roots that do not belong to the span of $\Delta \setminus \theta$. The group $P_\theta$ (resp. $P_{\theta}^-$) is a parabolic subgroup of $G$, equal to the semidirect product of its unipotent radical $U_\theta := \exp(u_\theta)$ (resp. $\exp(u_{\theta}^-)$) and of the Levi subgroup

$$(2.2) \quad L_\theta := P_\theta \cap P_{\theta}^-.$$ 

Explicitly,

$$\text{Lie}(P_\theta) = g_0 \oplus \bigoplus_{\alpha \in \Sigma^+} g_\alpha \oplus \bigoplus_{\alpha \in \Sigma^+ \cap \text{span}(\Delta \setminus \theta)} g_{-\alpha}.$$ 

In particular, $P_\theta = G$ and $P_{\Delta}$ is a minimal parabolic subgroup of $G$.

Example 2.12. Let $K$ be $R$, $C$, or the ring $H$ of quaternions and let $G = \text{GL}_d(K)$, seen as a real Lie group. Its derived group is $G_s = D(G) = \text{SL}_d(K)$. If $K = R$ (resp. $C$, resp. $H$), then we can take $K$ to be $O(d)$ (resp. $U(d)$, resp. $\text{Sp}(d)$), and in all cases we can take $a \subset \mathfrak{gl}_d(K)$ to be the set of real diagonal matrices of size $d \times d$. For $1 \leq i \leq d$, let $\varepsilon_i \in \mathfrak{a}^*$ be the evaluation of the $i$-th diagonal entry. Then $a = \mathfrak{z}(g) + a_s$, where $\mathfrak{z}(g) = \sum_{1 \leq i < j \leq d} \text{Ker}(\varepsilon_i - \varepsilon_j)$ is the set of real scalar matrices and $a_s = \text{Ker}(\varepsilon_1 + \cdots + \varepsilon_d)$ the set of traceless real diagonal matrices. The set of restricted roots of $a$ in $G$ is

$$\Sigma = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq d\}.$$ 

We can take $\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq d - 1\}$, so that

$$\Sigma^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq d\}$$

and $\Sigma$ is the set of the elements of $a$ whose entries are in nonincreasing order. For $\theta = \{\varepsilon_{n_1} - \varepsilon_{n_1+1}, \ldots, \varepsilon_{nm} - \varepsilon_{nm+1}\}$ with $1 \leq n_1 < \cdots < n_m \leq d - 1$, the
The following classical fact will be used in Section 4.

Fact 2.13 (see e.g. [Kna02, Prop. 7.76]). Any Lie subalgebra of \( \mathfrak{g} \) containing \( \text{Lie}(\mathfrak{p}_\Delta) \) is of the form \( \text{Lie}(\mathfrak{p}_\theta) \) for a unique subset \( \theta \) of \( \Delta \).

2.2.3. Conjugacy classes of parabolic subgroups and invariant distributions on \( G/L_\theta \). Recall that any parabolic subgroup is conjugate to \( P_\theta \) for some unique \( \theta \subset \Delta \); any pair of opposite parabolic subgroups \( (P,Q) \) is conjugate to a pair \( (P_\theta,P^-_\theta) \) for a unique \( \theta \subset \Delta \) (see [BT65, § 5]). Since the stabilizer in \( G \) of \( (P_\theta,P^-_\theta) \) is \( L_\theta = P_\theta \cap P^-_\theta \), the set of pairs \( (P,Q) \) of transverse parabolic subgroups of \( G \) identifies, as a \( G \)-set, with the disjoint union of the \( G/L_\theta \) for \( \theta \subset \Delta \). More precisely, with the identification of Remark 2.9,

\[
\{(P,Q) \in G/P_\theta \times G/P^-_\theta \mid P,Q \text{ transverse}\} \simeq G/L_\theta,
\]

and \( G/L_\theta \) is the unique open \( G \)-orbit in \( G/P_\theta \times G/P^-_\theta \). From this the tangent bundle \( T(G/L_\theta) \) inherits a decomposition

\[
T(G/L_\theta) = E^+ \oplus E^-.
\]

This decomposition is \( G \)-invariant, and so for any bundle with fiber \( G/L_\theta \) there is a corresponding decomposition of the vertical tangent space, for which we shall use again the notation \( E^+ \) and \( E^- \).

2.3. The Cartan and Lyapunov projections. A central role in this paper is played by the Cartan projection \( \mu \), which can be used to measure dynamical properties of diverging sequences in \( G \).

2.3.1. The Cartan projection. Recall that, with the notation of Section 2.2.2, the cartan decomposition \( G = K(\exp \mathfrak{t}^+)K \) holds: any \( g \in G \) may be written \( g = k(\exp \mu(g))k' \) for some \( k,k' \in K \) and a unique \( \mu(g) \in \mathfrak{t}^+ \) (see [Hel01, Ch. IX, Th. 1.1]). This defines a map

\[
\mu : G \rightarrow \mathfrak{t}^+
\]

\[
g \rightarrow \mu(g),
\]

called the Cartan projection, inducing a homeomorphism \( K\backslash G/K \simeq \mathfrak{t}^+ \).

Example 2.14. For \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), let \( G = \text{GL}_d(\mathbb{K}) \), and let \( K \subset G \) and \( \mathfrak{t}^+ \) be as in Example 2.12. Then the diagonal entries of \( \mu(g) \) are the logarithms of the singular values of \( g \) (i.e. of the square roots of the eigenvalues of \( ^t\bar{g}gg \), where \( \bar{g} \) is the complex conjugate of \( g \)), in nonincreasing order.

The (restricted) Weyl group of \( \mathfrak{a} \) in \( \mathfrak{g} \) is the group \( W = N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \), where \( N_K(\mathfrak{a}) \) (resp. \( Z_K(\mathfrak{a}) \)) is the normalizer (resp. centralizer) of \( \mathfrak{a} \) in \( K \). We now fix a \( W \)-invariant scalar product \( (\cdot,\cdot) \) on \( \mathfrak{a} \) and denote by \( \| \cdot \| \) the corresponding Euclidean norm. By a little abuse of notation, we shall use the same symbols for the induced scalar product and norm on the dual space \( \mathfrak{a}^* \).

If \( G \) is simple, then \( (\cdot,\cdot) \) is unique up to scale: it is the restriction to \( \mathfrak{a} \) of the Killing form of \( \mathfrak{g} \). In general, \( (\cdot,\cdot) \) is not unique, but any choice will do.
This choice determines the Riemannian metric $d_{G/K}$ on the symmetric space $G/K$, and for any $g \in G$ we have
\begin{equation}
\|\mu(g)\| = d_{G/K}(x_0, g \cdot x_0),
\end{equation}
where $x_0 := eK \in G/K$.

Seen as a subgroup of $\text{GL}(a)$, the Weyl group $W$ is a finite Coxeter group generated by the orthogonal reflections $s_\alpha$ in the hyperplanes $\text{Ker}(\alpha) \subset a$, for $\alpha \in \Delta$. The group $W$ acts simply transitively on the set of connected components of $a \setminus \bigcup_{\alpha \in \Sigma} \text{Ker}(\alpha)$ (open Weyl chambers). Therefore there is a unique element $w_0 \in W$ such that $w_0 \cdot (-\overline{\alpha}^+) = \overline{\alpha}^+$; it is the longest element with respect to the generating set $\{s_\alpha \mid \alpha \in \Delta\}$. The involution of $a$ defined by $Y \mapsto -w_0 \cdot Y$ is called the opposition involution\footnote{This involution is nontrivial only if the restricted root system $\Sigma$ is of type $A_n$, $D_{2n+1}$, or $E_6$, where $n \geq 2$.}; it sends $\mu(g)$ to $\mu(g^{-1})$ for any $g \in G$. The corresponding dual linear map preserves $\Sigma$. We shall denote it by
\begin{equation}
a^* \longrightarrow a^*
\end{equation}
\begin{equation}
\alpha \longrightarrow \alpha^* = -w_0 \cdot \alpha.
\end{equation}

By definition, for any $\alpha \in \Sigma$ and any $g \in G$,
\begin{equation}
\langle \alpha, \mu(g) \rangle = \langle \alpha^*, \mu(g^{-1}) \rangle.
\end{equation}

**Example 2.15.** Take $G = \text{GL}_d(K)$ with $K$, $K^+$, and $\overline{\alpha}^+$ as in Example 2.12. The Weyl group $W$ is the symmetric group $\Sigma_d$, which permutes the diagonal entries of the elements of $a$. The longest element $w_0$ of $W$ is the permutation $(1, \ldots, d) \mapsto (d, \ldots, 1)$. For $\alpha = \varepsilon_i - \varepsilon_{i+1} \in \Delta$, we have $\alpha^* = \varepsilon_{d-i} - \varepsilon_{d-i+1}$.

Here are some useful properties (see for instance [Kas08, Lem.2.3]), expressing that the map $\mu$ is “strongly subadditive”.

**Fact 2.16.** For any $g, g_1, g_2, g_3 \in G$,
\begin{enumerate}
\item $\|\mu(g)\| = \|\mu(g^{-1})\|$;
\item $\|\mu(g_1 g_2) - \mu(g_1)\| \leq \|\mu(g_2)\|$;
\item in particular, $\|\mu(g_1 g_2 g_3) - \mu(g_2)\| \leq \|\mu(g_1)\| \|\mu(g_3)\|$.
\end{enumerate}

As a consequence, for any representation $\rho : \Gamma \rightarrow G$, there exists $k > 0$ such that for any $\gamma \in \Gamma$,
\begin{equation}
\|\mu(\rho(\gamma))\| \leq k |\gamma|_\Gamma.
\end{equation}

Indeed, we can take $k := \max_{s \in S} \|\mu \circ \rho(s)\|$ where $S$ is the finite generating set of $\Gamma$ defining the word length $|\cdot|_\Gamma$.

A crucial point is that the map $\mu : G \rightarrow \overline{\alpha}^+$ is proper, by compactness of $K$. This implies the following.

**Remark 2.17.** Let $\Gamma$ be a finitely generated discrete group and $\rho : \Gamma \rightarrow G$ a representation. The representation $\rho$ has finite kernel in $\Gamma$ and discrete image in $G$ if and only if there is a function $f : \mathbb{N} \rightarrow \mathbb{R}$ with $\lim_{+\infty} f = +\infty$ such that for all $\gamma \in \Gamma$,
\begin{equation}
\|\mu(\rho(\gamma))\| \geq f(|\gamma|_\Gamma).
\end{equation}

The map $\rho$ is a quasi-isometric embedding if and only if $f$ can be taken to be affine.
In particular, if \( \rho \) is a quasi-isometric embedding, then for any positive root \( \alpha \in \Sigma^+ \) the following two conditions are equivalent:

(i) There exist \( c, C > 0 \) such that \( \alpha, \mu(\rho(\gamma)) \geq c\|\mu(\rho(\gamma))\| - C \) for all \( \gamma \in \Gamma \).

(ii) There exist \( c, C > 0 \) such that \( \alpha, \mu(\rho(\gamma)) \geq c|\gamma|_\Gamma - C \) for all \( \gamma \in \Gamma \).

Condition (i) means that the set \( \mu(\rho(\Gamma)) \) avoids some translate \( C' \) of the cone \( C := \{ x \in a \mid \langle \alpha, x \rangle < c\|x\| \} \) in the Euclidean space \( a \) (see Figure 1, left panel).

In Theorem 1.1.(2) we consider the following slightly different condition: for any geodesic ray \( \mathcal{R} = \langle \gamma_n \rangle_{n \in \mathbb{N}} \) in the Cayley graph of \( \Gamma \), the sequence \( (\alpha, \mu(\rho(\gamma_n)))_{n \in \mathbb{N}} \in \mathbb{R}^N \) is CLI, i.e. there exist \( c_R, C_R > 0 \) such that for all \( n, m \in \mathbb{N} \),

\[
\langle \alpha, \mu(\rho(\gamma_{n+m})) - \mu(\rho(\gamma_n)) \rangle \geq c_R m - C_R.
\]

This means that there is a translate \( C'_R \) in \( a \) of the cone

\[
C_R := \{ x \in a \mid \langle \alpha, x \rangle < c_R\|x\| \}
\]

such that for any \( n \in \mathbb{N} \), the sequence \( (\mu(\rho(\gamma_{n+m}))_{n \in \mathbb{N}} \) avoids \( \mu(\rho(\gamma_n)) + C'_R \) (see Figure 1, right panel). This “nested cone” property is what we mean when we say (in the introduction) that the sequence \( (\mu(\rho(\gamma_n)))_{n \in \mathbb{N}} \in (\mathbb{R}^+)^N \) drifts away “forever linearly” from Ker \( \alpha \).

![Figure 1](image)

**Figure 1.** Here \( G = \text{SL}_2(\mathbb{R}) \) and \( \theta = \{ \varepsilon_2 - \varepsilon_3 \} \). Left panel: Condition (i) above. Right panel: The CLI condition of Theorem 1.1.(2) for a geodesic ray \( \mathcal{R} = \langle \gamma_n \rangle_{n \in \mathbb{N}} \).

If \( G \) has real rank 1, then Proposition 2.4 implies the following strengthening of Remark 2.17 (which yields the implication (1) \( \Rightarrow \) (4) of Theorem 1.3 in that case):

**Corollary 2.18.** Let \( \Gamma \) be a discrete group and \( G \) a semisimple Lie group of real rank 1. If \( \rho : \Gamma \to G \) is a quasi-isometric embedding, then for any geodesic ray \( \langle \gamma_n \rangle_{n \in \mathbb{N}} \) in the Cayley graph of \( \Gamma \), the sequence \( (\|\mu(\rho(\gamma_n))\|)_{n \in \mathbb{N}} \) is CLI.

**Proof.** The property that \( (\|\mu(\rho(\gamma_n))\|)_{n \in \mathbb{N}} \) be CLI for any geodesic ray \( \langle \gamma_n \rangle \) is invariant under passing to a finite-index subgroup, by subadditivity of \( \|\mu\| \) (Fact 2.16). Therefore, we may assume that \( \Gamma \) is torsion-free.

Let \( X = G/K \) be the Riemannian symmetric space of \( G \). If \( \rho : \Gamma \to G \) is a quasi-isometric embedding, then \( \Gamma \) is word hyperbolic and admits a flow space \( \mathcal{G}_\Gamma \) which isometrically and \( \rho \)-equivariantly embeds into the tangent
space $T^1(X)$, with the flow of $\mathcal{G}_T$ corresponding to the geodesic flow on $T^1(X)$ (see Fact 2.6 and Remark 2.7 above). We conclude using Proposition 2.12 and Remark 2.7.

2.3.2. The Jordan decomposition and the Lyapunov projection. The natural projection induced by the Jordan decomposition is called the Lyapunov projection; we denote it by $\lambda : G \to \overline{\alpha}^+$. Explicitly, any $g \in G$ can be written uniquely as the commuting product $g = g_hg_\alpha g_u$ of a hyperbolic, an elliptic, and a unipotent element (see e.g. [Ebe96, Th. 2.19.24]). The conjugacy class of $g_h$ intersects $\exp(\overline{\alpha}^+)$ in a unique element $\exp(\lambda(g))$. This projection can also be defined as a limit: for any $g \in G$,

$$\lambda(g) = \lim_{n \to +\infty} \frac{1}{n} \mu(g^n).$$

Example 2.19. For $K = \mathbb{R}$ or $\mathbb{C}$, let $G = GL_d(K)$, and let $K \subset G$ and $\overline{\alpha}^+$ be as in Example 2.12. Then the diagonal entries of $\lambda(g)$ are the logarithms of the moduli of the complex eigenvalues of $g$, in nonincreasing order.

Note that by definition of the opposition involution, and similarly to (2.7),

$$\langle \alpha, \lambda(g) \rangle = \langle \alpha^*, \lambda(g^{-1}) \rangle$$

for all $\alpha \in \Sigma$ and $g \in G$.

2.3.3. The Cartan projection for $L_\theta$ and the $\theta$-coset distance map. Let $\theta \subset \Delta$ be a nonempty set of simple roots. Recall the Levi subgroup $L_\theta$ of $F_\theta$ from (2.2). The group $K_\theta := K \cap L_\theta$ is a maximal compact subgroup of $L_\theta$, and $L_\theta$ admits the Cartan decomposition $L_\theta = K_\theta(\exp(\overline{\alpha}^+_\theta))K_\theta$ where

$$\overline{\alpha}^+_\theta := \{ Y \in \mathfrak{a} \mid \alpha(Y) \geq 0 \ \forall \alpha \in \Delta \setminus \theta \}.$$ 

We denote by

$$(2.12) \quad \mu_\theta : L_\theta \longrightarrow \overline{\alpha}^+_\theta$$

the corresponding Cartan projection of $L_\theta$. As in Section 2.3.1, the map $\mu_\theta$ induces a homeomorphism $K_\theta \setminus L_\theta / K_\theta \simeq \overline{\alpha}^+_\theta$.

The Weyl chamber $\overline{\alpha}^+_\theta$ for $L_\theta$ is a convex union of finitely many $W$-iterates of the Weyl chamber $\overline{\alpha}^+$ for $G$.

Example 2.20. Take $G = GL_d(K)$ with $K$, $K$ and $\overline{\alpha}^+$ as in Example 2.12. For $\theta = \{ \varepsilon_i - \varepsilon_{i+1} \}$, the set $\overline{\alpha}^+_\theta$ consists of elements $\text{diag}(t_1, \ldots, t_d) \in \mathfrak{a}$ with $t_j \geq t_{j+1}$ for all $j \in \{1, \ldots, d-1\} \setminus \{i\}$.

The Cartan projection $\mu_\theta$ induces a Weyl-chamber-valued metric on the Riemannian symmetric space of $L_\theta$:

$$d_{\mu_\theta} : L_\theta/K_\theta \times L_\theta/K_\theta \longrightarrow \overline{\alpha}^+_\theta \quad (gK_\theta, hK_\theta) \longmapsto \mu_\theta(g^{-1}h).$$

This extends to a coset distance map on the set of pairs of elements of $G/K_\theta$ projecting to the same element in $G/L_\theta$.

Definition 2.21. The $\theta$-coset distance map is the map

$$d_{\mu_\theta} : \{(gK_\theta, hK_\theta) \in G/K_\theta \times G/K_\theta \mid gL_\theta = hL_\theta \} \longrightarrow \overline{\alpha}^+_\theta \quad (gK_\theta, hK_\theta) \longmapsto \mu_\theta(g^{-1}h).$$

This map was introduced in the study of Anosov representations in [GW12]; it will play a crucial role in Section 3.
2.4. Anosov representations. In this section we recall the definition and some properties of Anosov representations. For more details and proofs we refer to [GW12].

Remark 2.22. In this paper our convention for the notation of parabolic subgroups is different from the one adopted in [GW12]: definitions and statements involving $\theta \subset \Delta$ should be changed to $\Delta \setminus \theta$ when compared with their versions in [GW12].

2.4.1. The dynamical definition. Let $\Gamma$ be a word hyperbolic group with flow space $\mathcal{G}_\Gamma$, let $G$ be a reductive group, let $\theta \subset \Delta$ be a nonempty subset of the simple restricted roots of $G$, and let $\rho : \Gamma \to G$ be a representation. The space

$$\mathcal{E}(\rho) := \Gamma\backslash(G \times G/\theta)$$

is a $G/\theta$-bundle over $\Gamma\backslash \mathcal{G}_\Gamma$. The subbundles $E^+, E^-$ of $T(G/\theta)$ defined in (2.4) induce vector bundles (still denoted by $E^+, E^-$) over $\mathcal{E}(\rho)$. In fact, $E^+$ and $E^-$ are subbundles of the vertical tangent bundle

$$vT\mathcal{E}(\rho) := \Gamma\backslash(G \times T(G/\theta)).$$

The geodesic flow $\{\varphi_t\}_{t \in \mathbb{R}}$ naturally acts on the products $\mathcal{G}_\Gamma \times G/\theta$ and $\mathcal{G}_\Gamma \times T(G/\theta)$ (leaving the second coordinate unchanged), hence on their quotients $\mathcal{E}(\rho)$ and $vT\mathcal{E}(\rho)$; the subbundles $E^\pm$ are flow-invariant.

Definition 2.23. The representation $\rho$ is called $P_\theta$-Anosov if there is a continuous section $\sigma : \Gamma\backslash \mathcal{G}_\Gamma \to \mathcal{E}(\rho)$ with the following properties:

(i) $\sigma$ is flow-equivariant (i.e. its image $F = \sigma(\Gamma\backslash \mathcal{G}_\Gamma)$ is flow-invariant),
(ii) the action of the flow on the vector bundle $E^+|_F \subset vT\mathcal{E}(\rho)$ is dilating,
(iii) the action of the flow on the vector bundle $E^-|_F \subset vT\mathcal{E}(\rho)$ is contracting.

It is known that (ii) implies (iii) [GW12, Prop. 16] and that the section $\sigma$ is unique. This definition is useful to determine certain properties of Anosov representations, such as openness, or to define $\Gamma$-invariant metrics on spaces of Anosov representations [BCLS].

2.4.2. Another equivalent definition. The following notions will be needed for various characterizations of Anosov representations.

Definition 2.24. Two maps $\xi^+ : \partial_\infty \Gamma \to G/P_\theta$ and $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$ are said to be

- transverse if for any $\eta \neq \eta'$ in $\partial_\infty \Gamma$, the points $\xi^+(\eta)$ and $\xi^-(\eta')$ are transverse in the sense of Definition 2.8;
- compatible if for any $\eta \in \partial_\infty \Gamma$, the points $\xi^+(\eta)$ and $\xi^-(\eta)$ are compatible in the sense of Definition 2.8;
- dynamics-preserving for a representation $\rho : \Gamma \to G$ if for any $\gamma \in \Gamma$ of infinite order with attracting fixed point $\eta_\gamma^+ \in \partial_\infty \Gamma$, the point $\xi^+(\eta_\gamma^+)$ (resp. $\xi^-(\eta_\gamma^-)$) is an attracting fixed point for the action of $\rho(\gamma)$ on $G/P_\theta$ (resp. $G/P_\theta^-$); in this case this attracting fixed point is unique.

Remark 2.25. Given a representation $\rho : \Gamma \to G$, a pair $(\xi^+, \xi^-)$ of continuous, dynamics-preserving boundary maps for $\rho$, if it exists, is unique.
Such a pair is necessary $\rho$-equivariant and compatible, since the attracting fixed points $\xi^+(\eta^\gamma_\theta)$ and $\xi^-(\eta^\gamma_\theta)$ are always compatible for any $\gamma$ and the set $\{\eta^\gamma_\theta\}_{\gamma\in\Gamma}$ is dense in $\partial_\infty\Gamma$.

A section $\sigma$ as in Definition 2.23 is equivalent to a $\rho$-equivariant map $\tilde{\sigma} : \mathcal{G}_\Gamma \rightarrow G/L_\theta$. Working out the properties of $\tilde{\sigma}$, one obtains the following equivalent definition of Anosov representations (see [GW12, Def. 2.10]), which still makes use of the flow space $\mathcal{G}_\Gamma$ of $\Gamma$ but avoids the language of bundles.

**Definition 2.26.** A representation $\rho : \Gamma \rightarrow G$ is $P_\theta$-Anosov if there exist continuous, $\rho$-equivariant maps $\xi^+ : \partial_\infty\Gamma \rightarrow G/P_\theta$ and $\xi^- : \partial_\infty\Gamma \rightarrow G/P_\theta$ with the following properties:

(i) $\xi^+$ and $\xi^-$ are transverse. This implies that $\xi^+$ and $\xi^-$ combine, via (2.3), to a continuous, $\Gamma$-equivariant, flow-invariant map

$$\tilde{\sigma} : \mathcal{G}_\Gamma \rightarrow G/L_\theta$$

$$v \mapsto (\xi^+ \circ \varphi_{+\infty}(v), \xi^- \circ \varphi_{-\infty}(v)).$$

Such a map $\tilde{\sigma}$ always admits a continuous, $\Gamma$-equivariant lift

$$\tilde{\beta} : \mathcal{G}_\Gamma \rightarrow G/K_\theta,$$

(this follows from the contractibility of $L_\theta/K_\theta$), i.e. $\text{pr} \circ \tilde{\beta} = \tilde{\sigma}$.

(ii) There exist $c, C > 0$ such that for all $\alpha \in \theta$, all $v \in \mathcal{G}_\Gamma$, and all $t \in \mathbb{R}$,

$$\langle \alpha, d_{\mu_\theta}(\tilde{\beta}(v), \tilde{\beta}(\varphi_t(v))) \rangle \geq ct - C.$$

This last inequality expresses the exponential contraction in Definition 2.23. Note that the left-hand side of the inequality is nonnegative for $\alpha \in \Delta \setminus \theta$ as well, by Definition 2.21 of $d_{\mu_\theta}$ and definition (2.11) of $\overline{\pi}_0^\theta$.

The maps $\xi^+, \xi^-$ of an Anosov representation are always dynamics-preserving (see [GW12, Lem. 3.1]); in particular, they are unique (Remark 2.25).

**Remark 2.27.** When $\Gamma$ is elementary (i.e. virtually cyclic), $\partial_\infty\Gamma$ consists of two points (assuming $\Gamma$ is infinite), and a representation $\rho : \Gamma \rightarrow G$ is $P_\theta$-Anosov if and only if there exist transverse maps $\xi^+ : \partial_\infty\Gamma \rightarrow G/P_\theta$ and $\xi^- : \partial_\infty\Gamma \rightarrow G/P_\theta$ that are dynamics-preserving for $\rho$.

Let $\theta^* \subset \Delta$ be the image of $\theta$ under the opposition involution (2.6). By definition, the group $P_\theta$ is conjugate to $P_{\theta^*}$. It is sometimes useful to reduce to the case that $\theta = \theta^*$: this is always possible by the following fact.

**Fact 2.28.** [GW12, Lem. 3.18] A representation $\rho : \Gamma \rightarrow G$ is $P_\theta$-Anosov if and only if it is $P_{\theta^*, \theta^*}$-Anosov.

**Remark 2.29.** If $\theta = \theta^*$, then the maps $\xi^+$ and $\xi^-$ associated to an Anosov representation are equal (after identifying $G/P_\theta$ and $G/P_{\theta^*}$ with the set of parabolic subgroups of $G$ conjugate to $P_\theta$, see Remark 2.9).

### 2.4.3. Examples and properties

Examples of Anosov representations include:

(a) Convex cocompact subgroups in semisimple Lie groups $G$ of real rank 1 [Lab06, GW12] (here $|\Delta| = 1$, and so $P_\Delta$ is the only proper parabolic subgroup of $G$ up to conjugacy);

(b) Surface group representations in the Hitchin components, when $G$ is a split real form [Lab06, GW12, § 6.3];
(c) Maximal representations of surface groups, when the Riemannian symmetric space of $G$ is Hermitian [BILW05, BIW, GW12];
(d) The inclusion of quasi-Fuchsian subgroups in $\text{SO}(2,d)$ [Bar, BM12];
(e) Holonomies of compact convex $\mathbb{RP}^n$-manifolds whose fundamental group is word hyperbolic [Ben04].

Here are some basic properties of Anosov representations [Lab06, GW12]:

1. The section $\sigma : \Gamma \backslash \mathcal{G}_\Gamma \to \mathcal{E}(\rho)$ and the $\rho$-equivariant maps $(\xi_+^\rho, \xi_-^\rho) =: (\xi_+^\Gamma, \xi_-^\Gamma)$ depend only on the conjugacy class of $P_\theta$ and on $\rho$.

2. The set of $P_\theta$-Anosov representations is open in $\text{Hom}(\Gamma, G)$ and the maps $\rho \mapsto \xi_+^\rho$ and $\rho \mapsto \xi_-^\rho$ are continuous.

3. A $P_\theta$-Anosov representation $\rho : \Gamma \to G$ is a quasi-isometric embedding; by Remark 2.17, this means that there exist $c, C > 0$ such that for all $\gamma \in \Gamma$,

$$\|\mu(\rho(\gamma))\| \geq c|\gamma|_\Gamma - C,$$

where $| \cdot |_\Gamma$ is the word length with respect to the fixed finite generating set $S$. (The upper bound for $\|\mu(\rho(\gamma))\|$ is always satisfied by (2.8).)

4. Let $\Gamma'$ be a finite-index subgroup of $\Gamma$. A representation $\rho : \Gamma \to G$ is $P_\theta$-Anosov if and only if its restriction to $\Gamma'$ is.

Remark 2.30. To check the Anosov property, it is always possible to restrict to a semisimple Lie group instead of a reductive one. Indeed, if $G$ is reductive with center $Z(G)$, then the group $G' := G/Z(G)$ is semisimple, and $\Delta$ identifies with a basis of simple roots of $G'$. For $\theta \subset \Delta$, let $P'_\theta$ be the corresponding parabolic subgroup of $G'$. Then $\Gamma/P_\theta'$ identifies with $G'/P'_\theta$. A representation $\rho : \Gamma \to G$ is $P_\theta$-Anosov if and only if the induced representation $\rho' : \Gamma \to G'$ is $P'_\theta$-Anosov. Similarly, $\rho$ admits equivariant (resp. continuous, resp. transverse, resp. dynamics-preserving) boundary maps if and only if $\rho'$ does.

2.4.4. Semisimple representations of discrete groups. Let $\rho : \Gamma \to G$ be a representation of the discrete group $\Gamma$ into the reductive Lie group $G$. We shall say that $\rho$ is semisimple if the Zariski closure of $\rho(\Gamma)$ in $G$ is reductive. This terminology comes from the fact that $\rho$ is semisimple if and only if for any linear representation $\tau : G \to \text{GL}(V)$ we can write $V$ as a direct sum of irreducible $(\tau \circ \rho)(\Gamma)$-modules.

In general, we define the semisimplification of $\rho$ as follows. Let $H$ be the Zariski closure of $\rho(\Gamma)$ in $G$. Choose a Levi decomposition $H = L \rtimes R_u(H)$, where $R_u(H)$ is the unipotent radical of $H$. The composition of $\rho$ with the projection onto $L$ does not depend, up to conjugation by $R_u(H)$, on the choice of the Levi factor $L$. We shall call this representation the semisimplification of $\rho$, denoted by $\rho^{ss}$. The $G$-orbit of $\rho^{ss}$ (for the action of $G$ by conjugation at the target) is the unique closed orbit in the closure of the $G$-orbit of $\rho$.

As a consequence of the openness of the set of Anosov representations, a representation is Anosov as soon as its semisimplification is. The converse is also true, and will be proved in Section 4.5 (Corollary 1.7).

**Proposition 2.31.** Let $\Gamma$ be a word hyperbolic group, $G$ a reductive Lie group, $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of $G$, and $\rho : \Gamma \to G$ a representation. If a representation $\rho'$ belonging to the closure of
the $G$-orbit of $\rho$ in $\text{Hom}(\Gamma, G)$ is $P_0$-Anosov, then the representation $\rho$ itself is $P_0$-Anosov. In particular, if the semisimplification of $\rho$ is $P_0$-Anosov, then $\rho$ is $P_0$-Anosov.

**Proof.** Recall that being $P_0$-Anosov is an open property which is invariant under the action of $G$ on $\text{Hom}(\Gamma, G)$. Let $\rho' \in \text{Hom}(\Gamma, G)$ be $P_0$-Anosov. There is a neighborhood $\mathcal{U} \subset \text{Hom}(\Gamma, G)$ of $\rho'$ consisting of $P_0$-Anosov representations. If $\rho'$ belongs to the closure of the $G$-orbit of $\rho$, then $\rho$ admits a conjugate in $\mathcal{U}$, hence $\rho$ is $P_0$-Anosov. □

Finally, semisimplification does not change the values taken by the Lyapunov projection $\lambda$ of Section 2.3.2:

**Lemma 2.32.** Let $\rho : \Gamma \to G$ be a homomorphism from a group $\Gamma$ into a reductive Lie group $G$, and let $\lambda : \Gamma \to \mathfrak{p}^+$ be a Lyapunov projection for $G$. Then the semisimplification $\rho^{ss}$ of $\rho$ satisfies, for all $\gamma \in \Gamma$, $\lambda(\rho^{ss}(\gamma)) = \lambda(\rho(\gamma))$.

**Proof.** There is a sequence $(\rho_n)_{n \in \mathbb{N}}$ of conjugates of $\rho$ converging to $\rho^{ss}$. The map $\lambda$ is invariant under conjugation and continuous. □

2.5. **Approaching the Lyapunov projection by the Cartan projection.** The following result on semisimple representations will be used in Section 3.5. It was established by Benoist [Ben97] as a consequence of a theorem of Abels–Margulis–Soifer [AMS95]. Since the result is not explicitly stated as such in [Ben97], we recall a sketch of the proof.

**Theorem 2.33 ([Ben97]).** Let $\Gamma$ be a discrete group, $G$ a real reductive Lie group, and $\rho : \Gamma \to G$ a semisimple representation. Then there exist a finite set $F \subset \Gamma$ and a constant $C_\rho > 0$ such that for any $\gamma \in \Gamma$ we can find $f \in F$ with

$$\|\lambda(\rho(\gamma f)) - \mu(\rho(\gamma))\| \leq C_\rho.$$ 

**Proof.** By subadditivity of $\mu$ (Fact 2.16), it is sufficient to prove the existence of a finite set $F \subset \Gamma$ and a constant $C > 0$ such that for any $\gamma \in \Gamma$ we can find $f \in F$ with $\|\lambda(\rho(\gamma f)) - \mu(\rho(\gamma f))\| \leq C$. Since $\rho$ is semisimple by assumption, i.e. the Zariski closure of $\rho(\Gamma)$ in $G$ is reductive, we may assume without loss of generality that $\rho(\Gamma)$ is actually Zariski-dense in $G$. It is then known that the set $\alpha(\mu(\rho(\Gamma))) \subset \mathbb{R}_+$ is unbounded for all $\alpha \in \Delta$ (see [GM89, GR89, BL93, Pra94]). By Lemma 4.5 below, for any $\alpha \in \Delta$ there is an irreducible linear representation $\tau_\alpha$ of $G$ into a finite-dimensional vector space $V_\alpha$ whose highest weight $\chi_\tau_\alpha$ is a multiple of the fundamental weight $\omega_\alpha$ associated with $\alpha$ (see Section 4.1), and such that the highest-weight space $(V_\alpha)_{\chi_\tau_\alpha}$ is a line. By [AMS95, Th. 5.17], there are a finite set $F \subset \Gamma$ and a constant $\varepsilon > 0$ with the following property: for any $\gamma \in \Gamma$ we can find $f \in F$ such that for any $\alpha \in \Delta$, the element $\tau_\alpha \circ \rho(\gamma f) \in \text{GL}(V_\alpha)$ is $\varepsilon$-proximal in $\text{P}(V_\alpha)$ in the sense of [Ben97, § 2.2]. We conclude as in [Ben97, Lem. 4.5]: on the one hand, for any $\alpha \in \Delta$, there is a constant $C_\alpha > 0$ such that for any $g \in G$, if $\tau_\alpha(g)$ is $\varepsilon$-proximal in $\text{P}(V_\alpha)$, then $|\langle \omega_\alpha, \mu(g) - \lambda(g) \rangle| \leq C_\alpha$ [Ben97, Lem. 2.2.5]; on the other hand, the fundamental weights $\omega_\alpha$, for $\alpha \in \Delta$, form a basis of $\mathfrak{a}^*$, and for any $g \in G$ the elements $\lambda(g)$ and $\mu(g)$ of $\mathfrak{a}$ have the same projection to $\mathfrak{g}$. □
3. Anosov representations in terms of boundary maps and Cartan or Lyapunov projections

In this section we prove the equivalences (1) ⇔ (2) ⇔ (3) of Theorem 1.3. (The equivalence (1) ⇔ (4) will be the object of Section 5.) More precisely, we prove the following.

**Theorem 3.1.** Let $\Gamma$ be a word hyperbolic group, $G$ a real reductive Lie group, and $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of $G$. Let $\rho : \Gamma \to G$ be a representation, and suppose there exist two continuous, $\rho$-equivariant, and transverse maps $\xi^+ : \partial_\infty \Gamma \to G/P_\theta$ and $\xi^- : \partial_\infty \Gamma \to G/P^-_\theta$. Then the following statements are equivalent:

1. The maps $\xi^+, \xi^-$ lift to a map $\tilde{\beta} : \mathcal{G}_\Gamma \to G/K_\theta$ satisfying the contraction property (ii) of Definition 2.26, i.e., $\rho$ is $P_\theta$-Anosov;
2. The maps $\xi^+, \xi^-$ are dynamics-preserving for $\rho$ and
   $$\exists c, C > 0, \forall \alpha \in \theta, \forall \gamma \in \Gamma, \quad \langle \alpha, \mu(\rho(\gamma)) \rangle \geq c |\gamma| - C;$$
3. The maps $\xi^+, \xi^-$ are dynamics-preserving for $\rho$ and
   $$\exists c, C > 0, \forall \alpha \in \theta, \forall \gamma \in \Gamma, \quad \langle \alpha, \mu(\rho(\gamma)) \rangle \geq c \|\mu(\rho(\gamma))\| - C;$$
4. The maps $\xi^+, \xi^-$ are dynamics-preserving for $\rho$ and
   $$\forall \alpha \in \theta, \quad \lim_{\gamma \to \infty} \langle \alpha, \mu(\rho(\gamma)) \rangle = +\infty.$$

We also give the following similar characterizations of Anosov representations in terms of the Lyapunov projection $\lambda : G \to \mathfrak{a}^\perp$ of (2.9). Recall the notation $\ell_\Gamma(\cdot)$ for the translation length function (2.1) on the Cayley graph of $\Gamma$.

**Theorem 3.2.** Let $\Gamma$, $G$, $\rho$, $\theta$ and $\xi^\pm$ be as in Theorem 3.1. Then the following statements are equivalent:

1. $\rho$ is $P_\theta$-Anosov;
2. The maps $\xi^+, \xi^-$ are dynamics-preserving for $\rho$ and
   $$\exists c, C > 0, \forall \alpha \in \theta, \forall \gamma \in \Gamma, \quad \langle \alpha, \lambda(\rho(\gamma)) \rangle \geq c \ell_\Gamma(\gamma) - C;$$
3. The maps $\xi^+, \xi^-$ are dynamics-preserving for $\rho$ and
   $$\exists c, C > 0, \forall \alpha \in \theta, \forall \gamma \in \Gamma, \quad \langle \alpha, \lambda(\rho(\gamma)) \rangle \geq c \|\lambda(\rho(\gamma))\| - C;$$
4. The maps $\xi^+, \xi^-$ are dynamics-preserving for $\rho$ and
   $$\forall \alpha \in \theta, \quad \lim_{\ell_\Gamma(\gamma) \to +\infty} \langle \alpha, \lambda(\rho(\gamma)) \rangle = +\infty.$$

**Remarks 3.3.**

(a) Conditions (2), (3), (4) of Theorem 3.1 use the Cartan projection $\mu$, whereas (1) uses the coset distance map $d_{\mu_\theta}$ as in Definition 2.26.(ii) of the Anosov property. Recall that the range of $d_{\mu_\theta}$ is a finite union of copies of the range of $\mu$ (see Section 2.3.3).

(b) The existence of continuous, $\rho$-equivariant, compatible, and transverse maps $\xi^+, \xi^-$ already implies that $\rho$ has finite kernel and discrete image. In certain cases the existence of such a boundary map is actually already enough to ensure the Anosov property [GW12, Th.4.11].

(c) By Fact 2.28, as well as (2.7) and (2.10), all conditions of Theorems 3.1 and 3.2 are invariant under replacing $\theta$ with $\theta^*$ or with $\theta \cup \theta^*$. Thus, without loss of generality, we may assume that $\theta$ is symmetric (i.e. $\theta = \theta^*$).
The implication (2) ⇒ (3) of Theorem 3.1 follows from the subadditivity of \( ||\mu|| \) (see (2.8)). The implication (3) ⇒ (4) is immediate. We start by proving the implications (1) ⇒ (2) and (4) ⇒ (1) of Theorem 3.1 in Sections 3.2 and 3.4, respectively. Then Theorem 3.2 is proved in Sections 3.5 and 4.5.

3.1. A preliminary result. In the setting of Theorem 3.1, the maps \( \xi^+, \xi^- \) always combine as in Definition 2.26.(i) into a continuous, \( \rho \)-equivariant, flow-invariant map \( \bar{\sigma} : \mathcal{G}_\Gamma \to G/L_\theta \), of which we choose a continuous lift \( \bar{\beta} : \mathcal{G}_\Gamma \to G/K_\theta \). We further choose a \( \rho \)-equivariant set-theoretic lift \( \tilde{\beta} : \mathcal{G}_\Gamma \to G \) of \( \bar{\beta} \). Then for any \( (t, v) \in \mathbb{R} \times \mathcal{G}_\Gamma \), since \( \bar{\beta}(\varphi_t \cdot v)L_\theta = \bar{\sigma}(\varphi_t \cdot v) = \bar{\sigma}(v) = \bar{\beta}(v)L_\theta \), we have

\[
\bar{\beta}(\varphi_t \cdot v) = \tilde{\beta}(v) \cdot t_v
\]

for some \( t_v \in L_\theta \) satisfying the \( \Gamma \)-invariance property

(3.1) \[ t_{\gamma \cdot v} = \bar{\beta}(\gamma \cdot v)^{-1} \bar{\beta}(\varphi_t \cdot \gamma \cdot v) = \tilde{\beta}(v)^{-1} \rho(\gamma)^{-1} \rho(\gamma) \bar{\beta}(\varphi_t \cdot v) = t_v \]

for all \( \gamma \in \Gamma \). By definition, \( \mu_\theta(t_v) = d_{\mu_\theta}(\tilde{\beta}(v), \tilde{\beta}(\varphi_t \cdot v)) \) with the notation of Definition 2.21, and the following cocycle condition is satisfied:

(3.2) \[ t_{t+s,v} = t_{t,v} t_{s,v} \]

for all \( t, s \in \mathbb{R} \) and \( v \in \mathcal{G}_\Gamma \). We fix this map \((t, v) \mapsto t_v\) for the remainder of Section 3.

The following lemma will be used in Sections 3.2 and 3.4 below to prove the implications (1) ⇒ (2) and (4) ⇒ (1) of Theorem 3.1.

**Lemma 3.4.** For any compact subset \( \mathcal{D} \) of \( \mathcal{G}_\Gamma \), the set \( \tilde{\beta}(\mathcal{D}) \) is relatively compact in \( G \); in particular, \( K := 2 \sup_{v \in \mathcal{D}} ||\mu(\tilde{\beta}(v))|| \) is finite. Moreover, for any \( v \in \mathcal{D} \), any \( t \geq 0 \), and any \( \gamma \in \Gamma \) such that \( \varphi_t \cdot v \in \gamma \cdot \mathcal{D} \),

\[
||\mu(t_v) - \mu(\rho(\gamma))|| \leq K.
\]

**Proof.** The set \( \tilde{\beta}(\mathcal{D}) \) is mapped onto the compact set \( \tilde{\beta}(\mathcal{D}) \) by the proper map \( G \to G/K_\theta \), hence it is relatively compact in \( G \). In particular, the continuous function \( ||\mu|| \) is bounded on \( \tilde{\beta}(\mathcal{D}) \), i.e. \( K < +\infty \). Consider \( v \in \mathcal{D} \), \( t \geq 0 \), and \( \gamma \in \Gamma \) such that \( \varphi_t \cdot v \in \gamma \cdot \mathcal{D} \). Then

\[
\tilde{\beta}(v) t_v = \tilde{\beta}(\varphi_t \cdot v) = \rho(\gamma) \tilde{\beta}(\gamma^{-1} \cdot \varphi_t \cdot v)
\]

and \( \gamma^{-1} \cdot \varphi_t \cdot v \in \mathcal{D} \). Therefore, \( \rho(\gamma) = g_1 t_v g_2^{-1} \) for some \( g_1, g_2 \in \tilde{\beta}(\mathcal{D}) \). The bound follows by Fact 2.16. \( \square \)

3.2. The Cartan projection on an Anosov representation. We first prove the implication (1) ⇒ (2) of Theorem 3.1.

Let \( \Gamma \) be a word hyperbolic group, \( G \) a real reductive Lie group, \( \theta \subset \Delta \) a nonempty subset of the simple restricted roots of \( G \), and \( \rho : \Gamma \to G \) a representation. Suppose condition (1) of Theorem 3.1 is satisfied, i.e. \( \rho \) is \( P_\theta \)-Anosov. Then the exponential contraction property (ii) in Definition 2.26 is satisfied: there are constants \( c, C > 0 \) such that for all \( \alpha \in \theta \), all \( v \in \mathcal{G}_\Gamma \), and all \( t \in \mathbb{R} \),

(3.3) \[ \langle \alpha, d_{\mu_\theta}(\bar{\beta}(v), \tilde{\beta}(\varphi_t \cdot v)) \rangle \geq ct - C. \]
For any $t \in \mathbb{R}$ and $v \in G_\Gamma$, let $l_{t,v} \in L_\theta$ be defined as in Section 3.1. Then (3.3) can be rephrased as $\langle \alpha, \mu_\theta(l_{t,v}) \rangle \geq ct - C$ for all $\alpha \in \theta$, all $v \in G_\Gamma$, and all $t \in \mathbb{R}$. We claim that if $t \geq C/c$, then

$$\mu(l_{t,v}) = \mu_\theta(l_{t,v}).$$

Indeed, we always have $\langle \alpha, \mu_\theta(l_{t,v}) \rangle \geq 0$ for $\alpha \in \Delta \setminus \theta$ by definition (2.11) of the range $\overline{\mathcal{T}_\theta}$ of $\mu_\theta$, and if $t \geq C/c$ then $\langle \alpha, \mu_\theta(l_{t,v}) \rangle \geq 0$ for $\alpha \in \theta$ by (3.3). Therefore, if $t \geq C/c$, then $\mu_\theta(l_{t,v}) \in \overline{\mathcal{T}}^+$ for all $v \in G_\Gamma$, i.e. $\mu(l_{t,v}) = \mu_\theta(l_{t,v}).$

By Corollary 2.5, there are a compact set $\mathcal{D} \subset G_\Gamma$ and constants $c_1, c_2 > 0$ such that for any $\gamma \in \Gamma$ there exists $(t, v) \in \mathbb{R} \times \mathcal{D}$ with $t \geq c_1|\gamma|\Gamma - c_2$ and $\varphi_t \cdot v \in \gamma \cdot \mathcal{D}$. In particular, if $|\gamma|\Gamma \geq n_0 := (C + c_2)/c_1$, then $t \geq C/c$ and so (3.4) holds; applying Lemma 3.4 to the compact set $\mathcal{D}$, we obtain

$$\langle \alpha, \mu(\rho(\gamma)) \rangle \geq \langle \alpha, \mu(l_{t,v}) \rangle - \mathcal{K}\|\alpha\|$$

$$= \langle \alpha, \mu_\theta(l_{t,v}) \rangle - \mathcal{K}\|\alpha\|$$

$$\geq cc_1|\gamma|\Gamma - C',$$

where $\mathcal{K} > 0$ is given by Lemma 3.4 and $C' = cc_2 + C + \mathcal{K}\|\alpha\|$. Up to adjusting the additive constant $C'$, the same inequality also holds for the (finitely many) $\gamma \in \Gamma$ such that $|\gamma|\Gamma < n_0$, hence condition (2) of Theorem 3.1 holds.

This completes the proof of the implication $(1) \Rightarrow (2)$ of Theorem 3.1.

### 3.3. Weak contraction and Anosov representations

In the course of proving $(4) \Rightarrow (1)$ of Theorem 3.1, we will need the following characterization of Anosov representations.

**Proposition 3.5.** Let $\Gamma$ be a word hyperbolic group, $G$ a reductive Lie group, $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of $G$, and $\rho : \Gamma \to G$ a representation. Suppose there is a pair of continuous, $\rho$-equivariant, transverse maps $\xi^+ : \partial_\infty \Gamma \to G/P_\theta$ and $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$ and let $\tilde{\sigma} : G_\Gamma \to G/L_\theta$ be the induced equivariant map and $\tilde{\beta} : G_\Gamma \to G/K_\theta$ a lift, as in Definition 2.26(i).

Suppose also that for any $\alpha \in \theta$,

$$\lim_{t \to +\infty} \inf_{v \in G_\Gamma} \langle \alpha, d_{\mu_\theta}(\tilde{\beta}(v), \tilde{\beta}(\varphi_t \cdot v)) \rangle = +\infty.$$

Then the representation is $P_\theta$-Anosov (Definition 2.26): there exist $c, C > 0$ such that

$$\forall \alpha \in \theta, \forall t \in \mathbb{R}, \forall v \in G_\Gamma, \quad \langle \alpha, d_{\mu_\theta}(\tilde{\beta}(v), \tilde{\beta}(\varphi_t \cdot v)) \rangle \geq ct - C.$$

**Remark 3.6.** For fixed $t \in \mathbb{R}$, by $\Gamma$-invariance (3.1) of $l_{t,v}$,

$$\inf_{v \in G_\Gamma} \langle \alpha, d_{\mu_\theta}(\tilde{\beta}(v), \tilde{\beta}(\varphi_t \cdot v)) \rangle = \inf_{v \in D} \langle \alpha, d_{\mu_\theta}(\tilde{\beta}(v), \tilde{\beta}(\varphi_t \cdot v)) \rangle$$

for any $\mathcal{D} \subset G_\Gamma$ such that $\Gamma \cdot \mathcal{D} = G_\Gamma$. Choosing a compact set $\mathcal{D}$ in the above equality shows that the infimum is in fact a minimum.

For the proof it will be useful to relate the contraction on $G/P_\theta$ with a lower bound for the roots $\alpha \in \theta$.

**Lemma 3.7.** Let $G$ be a reductive Lie group and $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of $G$. Fix a $K_\theta$-invariant norm on the tangent space $TP_\theta(G/P_\theta) \simeq \mathfrak{u}_\theta^-$. Then there is a constant $C > 0$ such that

$$\langle \alpha, d_{\mu_\theta}(\tilde{\beta}(v), \tilde{\beta}(\varphi_t \cdot v)) \rangle \geq C - \mathcal{K}_t$$

for some $\mathcal{K}_t$. Then $P_\theta$-contractive (Definition 2.26).
(i) For any \( t \in \mathbb{R} \), if the left multiplication action of \( l \in L_\theta \) on \( T_p(G/P_\theta) \) is \( e^{-t} \)-Lipschitz, then \( \langle \alpha, \mu_\theta(l) \rangle \geq t \) for all \( \alpha \in \theta \).

(ii) For any \( t \in \mathbb{R} \) and \( l \in L_\theta \), if \( \langle \alpha, \mu_\theta(l) \rangle \geq t \) for all \( \alpha \in \theta \), then the action of \( l \) on \( T_p(G/P_\theta) \) is \( Ce^{-t} \)-Lipschitz.

**Proof of Lemma 3.7.** The action of \( l \in L_\theta \) on \( T_p(G/P_\theta) \) identifies with the adjoint action of \( l \) on \( u_\theta = \bigoplus_{\alpha \in \Sigma_\theta^+} g_{-\alpha} \). We write \( l \in K_\theta(\exp \mu_\theta(l))K_\theta \), where \( \mu_\theta(l) \in \mathfrak{a} \). By definition, any \( a \in \exp(\mathfrak{a}) \) acts on \( g_{-\alpha} \) by multiplication by the scalar \( e^{-\langle \alpha, \log a \rangle} \). The constant \( C \) comes from the choice of the \( K_\theta \)-invariant norm on \( T_p(G/P_\theta) \approx u_\theta \). \( \square \)

**Proof of Proposition 3.5.** As in Section 3.1, we choose a \( \rho \)-equivariant set-theoretic lift \( \tilde{\beta} : \mathcal{G}_\Gamma \rightarrow G \) of \( \beta \), and for any \( (t, v) \in \mathbb{R} \times \mathcal{G}_\Gamma \) we set \( l_{t,v} := \tilde{\beta}(v)^{-1}\tilde{\beta}(\varphi \cdot v) \in L_\theta \), so that \( d_{\mu_\theta}(\tilde{\beta}(v), \tilde{\beta}(\varphi \cdot v)) = \mu_\theta(l_{t,v}) \).

Suppose (3.5) holds for all \( \alpha \in \theta \), i.e.

\[
\inf_{\alpha \in \theta, \ \tau \in \mathcal{G}_\Gamma} \langle \alpha, \mu_\theta(l_{t,v}) \rangle \longrightarrow_{t \rightarrow +\infty} +\infty.
\]

By Lemma 3.7(ii), there exists \( \tau > 0 \) such that \( l_{t,v} \) is \( e^{-1} \)-contracting on \( T_p(G/P_\theta) \) for all \( v \in \mathcal{G}_\Gamma \). Let \( M \geq 1 \) be a bound for the Lipschitz constant of \( l_{s,w} \) acting on \( T_p(G/P_\theta) \) for all \( s \in [0, \tau] \) and \( w \in \mathcal{G}_\Gamma \). By (3.2), for any \( t \geq 0 \) and \( v \in \mathcal{G}_\Gamma \),

\[
l_{t,v} = l_{t-n_\tau, v} l_{t, (n-1)_\tau, v} \cdots l_{\tau, v}
\]

where \( n = \left\lfloor \frac{t}{\tau} \right\rfloor \), hence \( l_{t,v} \) is \( Me^{-n} \)-Lipschitz on \( T_p(G/P_\theta) \). By Lemma 3.7(i), for any \( \alpha \in \theta \),

\[
\langle \alpha, \mu_\theta(l_{t,v}) \rangle \geq \left\lfloor \frac{t}{\tau} \right\rfloor - \log M \geq \frac{1}{\tau}t - \log M - 1.
\]

\( \square \)

### 3.4. When the Cartan projection drifts away from the \( \theta \)-walls.

We now address the implication (4) \( \Rightarrow \) (1) of Theorem 3.1. We first fix an arbitrary representation \( \rho : \Gamma \rightarrow G \) of the word hyperbolic group \( \Gamma \) into the reductive group \( G \) and establish the following.

**Proposition 3.8.** If condition (4) of Theorem 3.1 holds, then there exists \( T \geq 0 \) such that for all \( t \geq T \) and \( v \in \mathcal{G}_\Gamma \),

\[
\mu_\theta(l_{t,v}) = \mu(l_{t,v}).
\]

In order to prove Proposition 3.8, we use the following observation.

**Remark 3.9.** For \( l \in L_\theta \), if \( \mu_\theta(l) \in \mathfrak{a}^+ \), then \( \mu_\theta(l) = \mu(l) \).

Recall that condition (4) of Theorem 3.1 states that there exist two continuous, \( \rho \)-equivariant, transverse, dynamics-preserving maps \( \xi^+ : \partial_\infty \Gamma \rightarrow G/P_\theta \) and \( \xi^- : \partial_\infty \Gamma \rightarrow G/P_\theta^- \), such that for any \( \alpha \in \theta \),

\[
\lim_{\gamma \rightarrow \infty} \langle \alpha, \mu(\rho(\gamma)) \rangle = +\infty.
\]

Consider maps \( \tilde{\alpha}, \tilde{\beta}, \tilde{\beta} \) and \( l_{t,v} \) as in Section 3.1 and let

\[ K := 2 \sup_{v \in D} \| \mu(\tilde{\beta}(v)) \| < +\infty \]

(see Lemma 3.4). By (3.7), there is a finite subset \( F \) of \( \Gamma \) such that for any \( \alpha \in \theta \) and \( \gamma \in \Gamma \setminus F \),

\[
\langle \alpha, \mu(\rho(\gamma)) \rangle \geq 1 + K.
\]
Applying Corollary 2.5 and Lemma 3.4, we deduce the existence of $T > 0$ such that

$$(3.8) \quad \forall t \geq T, \{v \in \mathcal{G}_T | \forall \alpha \in \theta, \langle \alpha, \mu(l_{t,v}) \rangle \geq 1 \} = \mathcal{G}_T.$$ 

Using the Weyl group $W$, we define

$$
\mathcal{s}_\theta := \{Y \in \mathbb{R}^+ | \langle \alpha, Y \rangle \geq 1 \text{ for all } \alpha \in \theta\},
$$

$$
\mathcal{s}_\theta' := \mathbb{R}^+ \cap (W \cdot \mathcal{s}_\theta).
$$

**Lemma 3.10.** The set $\mathcal{s}_\theta$ is a connected component of $\mathcal{s}_\theta'$.

**Proof.** The set $\mathcal{s}_\theta$ is clearly connected (in fact, convex) and contained in $\mathcal{s}_\theta'$. Since $\mathcal{s}_\theta'$ is a union of $W$-translates of $\mathcal{s}_\theta$, it is sufficient to prove that for any nontrivial $W \in W$, if $w \cdot \mathcal{s}_\theta \subseteq \mathcal{s}_\theta'$, then $w \cdot \mathcal{s}_\theta \cap \mathcal{s}_\theta = \emptyset$ (see Figure 2). For this, it is sufficient to see that for any nontrivial $w \in W$, if $w \cdot \mathcal{s}_\theta \cap \mathcal{s}_\theta = \emptyset$, then $w \cdot \mathcal{s}_\theta \not\subseteq \mathcal{s}_\theta'$. Note that for any nontrivial $w \in W$ there exists $\alpha \in \Delta$ such that $\langle \alpha, Y \rangle \leq 0$ for all $Y \in w \cdot \mathcal{R}^+$. If $\mathcal{s}_\theta \cap w \cdot \mathcal{s}_\theta = \emptyset$, then any such $\alpha$ belongs to $\Delta \setminus \theta$ by definition of $\mathcal{s}_\theta$, and so $w \cdot \mathcal{s}_\theta \not\subseteq \mathcal{s}_\theta'$ by definition (2.11) of $\mathcal{R}_n^+$.

Let $\gamma \in \Gamma$ be an element of infinite order. The invariant axis $\mathcal{A}_\gamma \subseteq \mathcal{G}_\Gamma$ of $\gamma$ is the set of $v \in \mathcal{G}_\Gamma$ such that $(\varphi_{-\infty}, \varphi_{-\infty})(v) = (\eta_\gamma^+ \cdot v, \eta_\gamma^-)$, where $\eta_\gamma^+$ and $\eta_\gamma^-$ are respectively the attracting and the repelling fixed point of $\gamma$ in $\partial_\infty \Gamma$. There is a constant $T_\gamma > 0$ (the *period of $\gamma$*) such that $\gamma \cdot v = \varphi_{T_\gamma} \cdot v$ for all $v \in \mathcal{A}_\gamma$. The assumption from Theorem 3.1 that the maps $\xi^+, \xi^-$ are dynamics-preserving for $\rho$ implies the following:

**Lemma 3.11.** Let $\gamma \in \Gamma$ be an element of infinite order, with invariant axis $\mathcal{A}_\gamma \subseteq \mathcal{G}_\Gamma$. Then for any $v \in \mathcal{A}_\gamma$,

$$
\mu_\theta(l_{T_\gamma,v}) \in \mathcal{R}_n^+,
$$

i.e. $\langle \alpha, \mu_\theta(l_{T_\gamma,v}) \rangle \geq 0$ for all $\alpha \in \theta$ (for $\alpha \in \Delta \setminus \theta$ the same inequality is true a priori by definition (2.12) of $\mu_\theta$).

**Proof.** Let $v \in \mathcal{A}_\gamma$. Then $\tilde{\beta}(v) = \tilde{\beta}(v)K_\theta$, and so $\tilde{\sigma}(v) = \tilde{\beta}(v)L_\theta$ and $\xi^+(\varphi_{-\infty}(v)) = \xi^+(\eta_\gamma^+ \cdot v) \equiv \tilde{\beta}(v)P_\theta$. Also,

$$
\tilde{\beta}(v)l_{T_\gamma,v} = \tilde{\beta}(\varphi_{T_\gamma} \cdot v) = \tilde{\beta}(\gamma v) = \rho(\gamma)\tilde{\beta}(v)
$$

(from the construction of $l_{t,v}$), hence $l_{T_\gamma,v}$ and $\rho(\gamma)$ are conjugate by $\tilde{\beta}(v)$. In particular, since the attracting fixed point of $\rho(\gamma)$ in $G/P_\theta$ is $\tilde{\beta}(v)P_\theta$, the
attracting fixed point for $l_{T,v}$ is $P_0$. By Lemma 3.7,
\[ \langle \alpha, \mu_\theta(l_{T,v}) \rangle > 0 \]
for all $\alpha \in \theta$, hence $\mu_\theta(l_{T,v}) \in \mathfrak{a}^\gamma$.

Recall the constant $T$ from (3.8).

Lemma 3.12. Let $\gamma \in \Gamma$ be an element of infinite order with invariant axis $A_\gamma \subset \mathcal{G}_\Gamma$. For any $v \in A_\gamma$ and any $t \geq T$,
\[ \mu_\theta(l_{t,v}) \in \mathfrak{s}_\theta. \]

Proof. We can rephrase (3.8) by saying that the map $\psi : [T, +\infty) \to \mathfrak{s}_\theta'$ sending $t$ to $\mu_\theta(l_{t,v})$ is well defined. It is continuous since $(t, v) \mapsto K_\theta l_{t,v} K_\theta$ is continuous and so is the Cartan projection $\mu_\theta$ (which is bi-$K_\theta$-invariant). Applying Lemma 3.11 to an appropriate power of $\gamma$, we see that $\psi(t) \in \mathfrak{s}_\theta$ for some $t \geq T$. Lemma 3.10 then implies $\psi(t) \in \mathfrak{s}_\theta$ for all $t \geq T$.

Proof of Proposition 3.8. If $\Gamma$ is virtually the cyclic group generated by some element $\gamma$ of infinite order, then $\mathcal{G}_\Gamma = A_\gamma \cup A_{\gamma^{-1}}$ and the proposition follows from Lemma 3.12 and Remark 3.9. We now assume that $\Gamma$ is nonelementary. Using Lemma 3.12, we see that for any $t \geq T$ the closed set
\[ \{ v \in \mathcal{G}_\Gamma \mid \mu_\theta(l_{t,v}) \in \mathfrak{s}_\theta \} \]
contains the invariant axis $A_\gamma$ of any element $\gamma \in \Gamma$ of infinite order. By density of the union of those axes (Remark 2.2), this set is all of $\mathcal{G}_\Gamma$. Using again Remark 3.9, we see that $\mu_\theta(l_{t,v}) = \mu(l_{t,v})$ for all $t \geq T$ and all $v \in \mathcal{G}_\Gamma$.

Proof of the implication $(4) \Rightarrow (1)$ of Theorem 3.1. Suppose that (4) holds, i.e. there exist two continuous, $\rho$-equivariant, transverse, dynamics-preserving maps $\xi^+, \xi^-$ and (3.7) holds. By Proposition 3.5, in order to show that $\rho$ is $P_0$-Anosov, it is enough to prove that
\[ (3.9) \quad \forall \alpha \in \theta, \lim_{t \to +\infty, v \in \mathcal{G}_\Gamma} \inf\langle \alpha, \mu_\theta(l_{t,v}) \rangle = +\infty. \]

By Remark 3.6, the infimum needs to be taken only on a compact set $\mathcal{D}$ such that $\Gamma \cdot \mathcal{D} = \mathcal{G}_\Gamma$. By Lemma 3.4 and Proposition 3.8, there is a constant $K > 0$ such that
\[ \| \mu_\theta(l_{t,v}) - \mu(\rho(\gamma)) \| \leq K \]
for all $v \in \mathcal{D}$, all $t \geq T$, and all $\gamma \in \Gamma$ such that $\varphi_t \cdot v \in \gamma \cdot \mathcal{D}$. Since $\Gamma \to \mathcal{G}_\Gamma$ is a quasi-isometry (and such $\gamma$ exist), we obtain $\gamma \to \infty$ as $t \to +\infty$, and so the assumption (3.7) readily implies (3.9).

3.5. Characterizations in terms of Lyapunov projections. The implication $(3) \Rightarrow (4)$ of Theorem 3.2 is immediate.

Let us check $(1) \Rightarrow (2)$ of Theorem 3.2. Suppose that (1) holds, i.e. $\rho$ is a $P_0$-Anosov representation. By Theorem 3.1.(2), there are constants $c, C > 0$ such that for any $\alpha \in \theta$ and $\gamma \in \Gamma$,
\[ \langle \alpha, \mu(\rho(\gamma)) \rangle \geq c |\gamma|_\Gamma - C. \]
Using (2.9), we find
\[
\langle \alpha, \lambda(\rho(\gamma)) \rangle = \lim_{n \to \infty} \frac{1}{n} \langle \alpha, \mu(\rho(\gamma^n)) \rangle \\
\geq \lim_{n \to \infty} \frac{1}{n} (c |\gamma^n|_\Gamma - C) = c |\gamma|_\infty
\]
for all \(\alpha \in \theta\) and \(\gamma \in \Gamma\). Then (2) holds by Proposition 2.1.

For the implication (2) \(\Rightarrow\) (3) of Theorem 3.2, we note that by (2.8) and (2.9), there exists \(k > 0\) such that for any \(\gamma \in \Gamma\),
\[
\|\lambda(\rho(\gamma))\| = \lim_{n \to +\infty} \frac{1}{n} \|\mu(\rho(\gamma^n))\| \leq k \lim_{n \to +\infty} \frac{1}{n} |\gamma^n|_\Gamma = k |\gamma|_\infty.
\]
We conclude using Proposition 2.1.

We now prove the implication (4) \(\Rightarrow\) (1) of Theorem 3.2 under the additional assumption that \(\rho\) is semisimple in the sense of Section 2.4.4. We postpone the proof of the general case to Section 4.5, since it involves linear representations of \(G\) to be introduced in the next section.

**Proof of the implication (4) \(\Rightarrow\) (1) of Theorem 3.2 for semisimple \(\rho\).** Let \(F\) be the finite subset of \(\Gamma\) and \(C_\rho > 0\) the constant given by Theorem 2.33. Let \((\gamma_n) \in \Gamma^N\) be a sequence of elements of \(\Gamma\) going to infinity. The existence of \(\rho\)-equivariant, transverse maps \(\xi^+, \xi^-\) implies that \(\rho\) has finite kernel and discrete image (Remark 3.3.(b)), and so the sequence \((\mu(\rho(\gamma_n)))_{n \in N}\) goes to infinity in \(\overline{\Gamma}^+\) by properness of the map \(\mu\). By Theorem 2.33, for any \(n \in N\) there exists \(f_n \in F\) such that \(\|\lambda(\rho(\gamma_n f_n)) - \mu(\rho(\gamma_n))\| \leq C_\rho\). Thus \(\lambda(\rho(\gamma_n f_n)) \to +\infty\) and the sequence of conjugacy classes of \(\gamma_n f_n\) goes to infinity. The hypothesis (4) says that \(\langle \alpha, \lambda(\rho(\gamma_n f_n)) \rangle \to +\infty\) for any \(\alpha \in \theta\), hence \(\langle \alpha, \mu(\rho(\gamma_n)) \rangle \to +\infty\). Thus condition (4) of Theorem 3.1 is satisfied, which means that \(\rho\) is \(P_\theta\)-Anosov. \(\square\)

4. **Linear representations of reductive Lie groups**

Let \(G\) be a reductive Lie group and \(\theta \subset \Delta\) a nonempty subset of the simple restricted roots of \(G\). The goal of this section is to show that there exist (many) finite-dimensional linear representations \((\tau, V)\) of \(G\) such that a homomorphism \(\rho : \Gamma \to G\) is \(P_\theta\)-Anosov if and only if the composed homomorphism \(\tau \circ \rho : \Gamma \to \text{GL}(V)\) is Anosov with respect to the stabilizer of a line (Lemma 4.5 and Proposition 4.8). This will be used with real vector spaces \(V\) in the proofs of Section 5: it will make computations simpler by reducing them to the group \(\text{GL}(V)\). Certain technical lemmas, and the possibility of working with \(K\)-vector spaces (\(K = \mathbb{R}, \mathbb{C},\) or \(\mathbb{H}\)), will also be used in Section 6.

In Section 4.1 we recall the notion of (restricted) weight for linear representations of \(G\) and introduce some notation. In Section 4.2 we recall the notion of proximality for elements \(g \in G\) acting on \(G/P_\theta\) and characterize it in terms of the growth of \(\langle \alpha, \mu(g^n) \rangle\) as \(n \to +\infty\) (Lemma 4.2). In Section 4.3 we introduce the notion of \(\theta\)-proximal linear representation of \(G\) and show that irreducible such representations have the good behavior mentioned above with respect to Anosov representations of \(\Gamma\); a technical proof is postponed to Section 4.4. Finally, in Section 4.5 we use this machinery
to address the implication $(4) \Rightarrow (1)$ of Theorem 3.2 in the general situation where $\rho$ is not necessarily semisimple.

All linear representations in this paper are understood to be finite-dimensional.

4.1. Reminders on irreducible linear representations of $G$ and their weights. Let $G$ be a real reductive Lie group as in Section 2.2. We use the notation of Section 2. In particular, $(\cdot, \cdot)$ denotes a $W$-invariant scalar product on $\mathfrak{a}$ and $\| \cdot \|$ the induced Euclidean norm on $\mathfrak{a}$; we use the same symbols for the induced scalar product and norm on $\mathfrak{a}^*$. Any irreducible linear representation $(\tau, V)$ of $G$ decomposes under the action of $\text{exp}(\mathfrak{a})$; the joint eigenvalues (elements of $\mathfrak{a}^*$) are called the restricted weights of $(\tau, V)$. The union of the restricted weights of all irreducible linear representations of $G$ is the set

$$\Phi = \left\{ \alpha \in \mathfrak{a}^* \mid 2 \frac{(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}, \forall \beta \in \Sigma \right\}$$

which projects to a lattice of $\mathfrak{a}_s^*$; the set $\Phi$ is discrete if and only if $G$ is semisimple. Let $\mathfrak{g}(\mathfrak{g})^\perp$ be the subspace of $\mathfrak{a}^*$ consisting of linear forms vanishing on $\mathfrak{g}(\mathfrak{g})$; it identifies with the dual $\mathfrak{a}_s^*$ of $\mathfrak{a}_s$. Similarly $\mathfrak{a}_s^\perp$ is the space of linear forms vanishing on $\mathfrak{a}_s$. For any $\alpha \in \Delta$, let $\omega_\alpha \in \mathfrak{g}(\mathfrak{g})^\perp \subset \mathfrak{a}^*$ be the fundamental weight associated with $\alpha$, defined by

$$(4.1) \quad 2 \frac{(\omega_\alpha, \beta)}{(\beta, \beta)} = \delta_{\alpha, \beta} \quad \text{for all } \beta \in \Delta$$

where $\delta_\cdot$ is the Kronecker symbol. Then

$$\Phi = \mathfrak{a}_s^\perp + \sum_{\alpha \in \Delta} \mathbb{Z} \omega_\alpha,$$

and $(\omega_\alpha)_{\alpha \in \Delta}$ projects to a basis of $\mathfrak{a}_s^*$. The set of dominant weights is the semigroup $\Phi^+ = \mathfrak{a}_s^\perp + \sum_{\alpha \in \Delta} N \omega_\alpha$. The cone generated by the positive roots (or, equivalently, by the simple roots) determines a partial ordering on $\mathfrak{a}^*$, given by

$$\nu \leq \nu' \iff \nu' - \nu \in \sum_{\alpha \in \Sigma^+} \mathbb{R}_+ \alpha = \sum_{\alpha \in \Delta} \mathbb{R}_+ \alpha.$$ 

Given an irreducible linear representation $(\tau, V)$ of $G$, the set of restricted weights of $\tau$ admits, for that ordering, a unique maximal element $\chi_\tau$ (see e.g. [GW09, Cor.3.2.3]); it is a dominant weight called the highest weight of $\tau$.

4.2. Proximality in $G/P_\theta$. Let $V$ be a finite-dimensional $K$-vector space. Recall that an element $g \in \text{GL}_K(V)$ is said to be proximal in $P_K(V) = (V - \{0\})/K^*$ if it has a unique eigenvalue of maximal modulus and if the corresponding eigenspace is one-dimensional. This eigenspace gives rise to a unique attracting fixed point $\xi_g^+ \in P_K(V)$ for the action of $g$ on $P_K(V)$. There is a unique complementary hyperplane $H^-_g$ stable under $g$, and $\lim_{n \to +\infty} g^n \cdot x = \xi_g^+$ for all $x \in P_K(V) \setminus P_K(H^-_g)$.

Let $\theta \subset \Delta$ be a nonempty subset of the simple restricted roots of $G$. We say that an element $g$ of the reductive group $G$ is proximal in $G/P_\theta$ if it has two fixed points $\xi_g^+ \in G/P_\theta$ and $\xi_g^- \in G/P_{\theta^-}$ with the following dynamical
behavior: \( \lim_{n \to +\infty} g^n \cdot x = \xi^+_g \) for all \( \xi \in G/P_\theta \) transverse to \( \xi^-_g \). The fixed points \( \xi^+_g \) and \( \xi^-_g \) are always transverse.

**Remark 4.1.** If \( \rho : \Gamma \to G \) is a \( P_\theta \)-Anosov representation, then the boundary map \( \xi^+ : \partial_{\infty} \Gamma \to G/P_\theta \) is dynamics-preserving (see Section 2.4.2), and so \( \rho(\gamma) \in G \) is proximal in \( G/P_\theta \) for any \( \gamma \in \Gamma \) of infinite order.

**Lemma 4.2.** For any element \( g \in G \), the following are equivalent:

(i) \( g \) is proximal in \( G/P_\theta \),

(ii) \( \langle \alpha, \lambda(g) \rangle > 0 \) for all \( \alpha \in \theta \),

(iii) \( \langle \alpha, \mu(g^n) \rangle \to 2 \log n \to +\infty \) as \( n \to +\infty \), for any \( \alpha \in \theta \).

Lemma 4.2 is based on the following,

**Claim 4.3.** For any unipotent element \( u \in G \), there is a constant \( C_u > 0 \) such that for all \( \alpha \in \Delta \) and \( n \in \mathbb{N}^* \),

\[ \langle \alpha, \mu(u^n) \rangle \leq 2 \log n + C_u. \]

**Proof of Claim 4.3.** Consider the following two standard elements of \( \mathfrak{sl}_2(\mathbb{R}) \):

\[ x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

Let \( \mu_{\mathfrak{sl}_2}(R) : \mathfrak{sl}_2(\mathbb{R}) \to \mathbb{R}_+^\times x \) be the Cartan projection of \( \mathfrak{sl}_2(\mathbb{R}) \) with respect to the Cartan decomposition \( \mathfrak{sl}_2(\mathbb{R}) = \mathbb{SO}(2)/(\exp R_+ x)\mathbb{SO}(2) \). For any \( n \in \mathbb{N} \), an elementary computation shows that \( \mu_{\mathfrak{sl}_2}(R)(\exp n e_+) = t_n x \) for \( t_n = \text{argsinh}(\frac{n}{2}) \in [0, \log(n + 1)] \).

Let \( u \in G \) be unipotent. By the Jacobson-Morozov theorem (see e.g. [Bou75, Ch. VIII, § 11, Prop. 2]), there is a homomorphism \( \tau : \mathfrak{sl}_2(\mathbb{R}) \to G \) such that \( \tau(\exp e_+) = u \). Up to conjugating in \( G \) (which only changes \( \mu \) by a bounded additive amount, see Fact 2.16), we may assume that \( d_\tau(x) \in \mathfrak{a}^+ \) and that \( \tau(\mathbb{SO}(2)) \subset K \). Then \( \mu(u^n) = t_n d_\tau(x) \) and so

\[ \langle \alpha, \mu(u^n) \rangle = t_n \langle \alpha, d_\tau(x) \rangle \leq \log(n + 1) \langle \alpha, d_\tau(x) \rangle \]

for all \( \alpha \in \Delta \) and \( n \in \mathbb{N} \). We conclude using the fact [Kos59, Lem. 5.1] that \( \langle \alpha, d_\tau(x) \rangle \in \{0, 1, 2\} \) for all \( \alpha \in \Delta \). \( \square \)

**Proof of Lemma 4.2.** For (i) \( \iff \) (ii), see [Ben97, Rem. 2.5.2]. The implication (ii) \( \Rightarrow \) (iii) follows from (2.9). To prove (iii) \( \Rightarrow \) (ii), let

\[ \theta' := \{ \alpha \in \Delta \mid \langle \alpha, \lambda(g) \rangle > 0 \}. \]

Let us prove the existence of a constant \( C > 0 \) such that for all \( \beta \in \Delta \setminus \theta' \) and \( n \in \mathbb{N} \),

\[ \langle \beta, \mu(g^n) \rangle \leq 2 \log n + C. \]

Consider the Jordan decomposition \( g = g_h g_u g_a \) of \( g \) (see Section 2.3.2). By Fact 2.16, \( \| \mu(g^n) - \mu(g_h^n g_u^n) \| \leq \| \mu(g_a^n) \| \) is bounded, and so we may assume \( g_a = 1 \). Up to conjugation (which only changes \( \mu \) by a bounded additive amount, by Fact 2.16 again), we may assume that \( g_h = \exp(\lambda(g)) \in \mathfrak{a}^+ \) and that \( g_u \in \exp(\bigoplus_{\beta \in \Delta^+ \cap \text{span}(\Delta \setminus \theta')} g_\beta) \). Then \( g_h \) belongs to the center of the group \( L_{\theta'} \) of (2.2); the element \( g_u \) commutes with \( g_h \), hence belongs to

\[ \exp \left( \bigoplus_{\beta \in \Delta^+ \cap \text{span}(\Delta \setminus \theta')} g_\beta \right) \subset L_{\theta'}; \]
we have \( \mu_\varphi(g^n) = n \lambda(g) + \mu_\varphi(g^\theta_n) \) for all \( n \in \mathbb{N} \). By Fact 2.16, for any \( \alpha \in \theta' \) and \( n \in \mathbb{N} \),
\[
|\langle \alpha, \mu_\varphi(g^n) - \mu_\varphi(g^\theta_n) \rangle| \leq \|\alpha\| \|\mu_\varphi(g^n) - \mu_\varphi(g^\theta_n)\| \leq \|\alpha\| \|\mu_\varphi(g^\theta_n)\|.
\]
By Claim 4.3 applied to \( g_u \in L_{\theta'} \), the right-hand side grows logarithmically with \( n \), while \( \langle \alpha, \mu_\varphi(g^\theta_n) \rangle = n (\alpha, \lambda(g)) \) grows linearly; therefore, \( \langle \alpha, \mu_\varphi(g^n) \rangle \geq 0 \) for all large enough \( n \in \mathbb{N} \). This holds for all \( \alpha \in \theta' \), and so for all large enough \( n \in \mathbb{N} \) we have \( \mu_\varphi(g^n) \in \mathfrak{p}^\vee \), or equivalently \( \mu(g^n) = \mu_\varphi(g^n) \). In particular, for any \( \beta \in \Delta \setminus \theta' \) and large enough \( n \in \mathbb{N} \),
\[
\langle \beta, \mu(g^n) \rangle = \langle \beta, \mu_\varphi(g^n) \rangle = \langle \beta, n \lambda(g) + \mu_\varphi(g^\theta_n) \rangle = \langle \beta, \mu_\varphi(g^\theta_n) \rangle.
\]
By Claim 4.3 again, there is a constant \( C_{g_u} > 0 \) such that the right-hand side is \( \leq 2 \log n + C_{g_u} \) for all \( n \in \mathbb{N} \).

If \( g \) satisfies condition (iii), then by the above equation (4.2) \( \theta \) and \( \Delta \setminus \theta' \) cannot intersect. Hence \( \theta \subset \theta' \) and \( g \) satisfies condition (ii) by the definition of \( \theta' \).

\qed

4.3. Compatible and proximal linear representations of \( G \). Recall that a representation \( \tau : G \to \text{GL}_K(V) \) of the reductive group \( G \) is said to be \textit{proximal} if the group \( \tau(G) \subset \text{GL}_K(V) \) contains an element which is proximal in \( P_K(V) \). For irreducible \( \tau \), this is equivalent to the highest-weight space \( V^\chi_\tau \) being a line.

We introduce the following notions.

**Definition 4.4.** Let \( \theta \subset \Delta \) be a nonempty subset of the simple restricted roots \( G \). An irreducible representation \( \tau : G \to \text{GL}(V) \) with highest weight \( \chi_\tau \) is called

1. \( \theta \)-\textit{compatible} if 
   \[
   \{ \alpha \in \Delta \mid (\chi_\tau, \alpha) > 0 \} = \theta,
   \]
2. \( \theta \)-\textit{proximal} if it is proximal and \( \theta \)-compatible.

Recall from Section 4.1 that the highest weight \( \chi_\tau \) of any irreducible representation \( (\tau, V) \) belongs to \( \Phi_+ = a^+_\theta + \sum_{\alpha \in \Delta} N \omega_\alpha \). Therefore, \( (\tau, V) \) is \( \theta \)-compatible if and only if
\[
\chi_\tau \in a^+_\theta + \sum_{\alpha \in \theta} N^* \omega_\alpha.
\]

We shall use the following fact.

**Lemma 4.5.** For any reductive Lie group \( G \), there is an integer \( N \geq 1 \) such that any \( \chi \in N \sum_{\alpha \in \Delta} N \omega_\alpha \) is the highest weight of some irreducible proximal representation \( \tau \) of \( G \). By definition, such a representation \( \tau \) is \( \theta \)-compatible (for some nonempty subset \( \theta \) of \( \Delta \)) if and only if \( \chi \in \sum_{\alpha \in \theta} N^* \omega_\alpha \).

**Proof.** The image \( H := \text{Ad}(G_0) \subset \text{GL}(g) \) of the identity component \( G_0 \) of \( G \) under the adjoint representation is a connected, semisimple, linear Lie group whose Lie algebra \( h \) is isomorphic to \( g_\theta \). Any weight \( \chi \) for the group \( G \) induces a weight for the group \( H \). By results of Abels–Margulis–Soifer [AMS95, Th. 6.3] and Helgason [Hel84, Ch. V, Th. 4.1] (see [Ben00, §2.3]), any weight \( \chi_1 \in 2 \sum_{\alpha \in \Delta} N \omega_\alpha \) is the highest weight of an irreducible representation \( (\tau_1, V_1) \) of \( H \), hence of \( G_0 \), and this representation is proximal. The induced
representation $V_2 = \text{Ind}_{G_0}^G V_1$ is an irreducible representation of $G$; its highest weight is again $\chi_1$ but the weight space $V_2^{\chi_1}$ is now $p$-dimensional, where $p := [G : G_0]$ is the number of connected components of $G$. The irreducible factor $(\tau, V)$ in $\Lambda^p(V_2)$ containing $\Lambda^p(V_2^{\chi_1})$ is then an irreducible proximal representation of $G$ with highest weight $\chi = p\chi_1$. Thus $N = 2p$ has the desired property. □

The relevance of the notion of $\theta$-proximality lies in the following two propositions proved in Section 4.4 just below.

**Proposition 4.6.** Let $(\tau, V)$ be an irreducible, $\theta$-proximal representation of $G$ with highest weight $\chi_\tau$. Let $V^{\chi_\tau}$ be the weight space corresponding to $\chi_\tau$ in $V$ and $V_{<\chi_\tau}$ the sum of all other weight spaces.

(a) The stabilizer in $G$ of $V^{\chi_\tau}$ (resp. $V_{<\chi_\tau}$) is the parabolic subgroup $P_\theta$ (resp. $P_{\theta}^-$).

(b) The maps $g \mapsto \tau(g) V^{\chi_\tau}$ and $g \mapsto \tau(g) V_{<\chi_\tau}$ induce $\tau$-equivariant embeddings

$$\iota^+: G/P_\theta \to \mathbf{P}_K(V) \quad \text{and} \quad \iota^-: G/P_{\theta}^- \to \mathbf{P}_K(V^*) .$$

Two parabolic subgroups $P \in G/P_\theta$ and $Q \in G/P_{\theta}^-$ of $G$ are transverse if and only if $\iota^+(P)$ and $\iota^-(Q)$ are transverse.

(c) An element $g \in G$ is proximal in $G/P_\theta$ if and only if $\tau(g)$ is proximal in $\mathbf{P}_K(V)$. In this case, the attracting fixed point $\xi^+_{\tau(g)} \in \mathbf{P}_K(V)$ of $\tau(g)$ is the image of $\xi^+_g \in G/P_\theta$ by $\iota^+$, i.e. $\xi^+_{\tau(g)} = \iota^+(\xi^+_g)$. Similarly, $\xi^-_{\tau(g)} = \iota^-(\xi^-_g)$.

Here we use the identification of Remark 2.9. Recall also Example 2.11 characterizing transversality in $\mathbf{P}_K(V)$.

**Remark 4.7.** For $G = \text{GL}_d(\mathbf{R})$, for $\theta = \{\epsilon_i - \epsilon_{i+1}\}$, and for $V = \Lambda^i \mathbf{R}^d$, the space $G/P_{\theta}$ is the Grassmannian of $i$-dimensional planes of $\mathbf{R}^d$ and the map $\iota^+$ of Proposition 4.6 is the Plücker embedding.

**Proposition 4.8.** Let $(\tau, V)$ be an irreducible, $\theta$-proximal, linear representation of $G$ over $K = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. Then a representation $\rho : \Gamma \to G$ is $P_{\theta}$-Anosov if and only if $\tau \circ \rho : \Gamma \to \text{GL}(V)$ is Anosov with respect to the stabilizer of a line, i.e. it is $P_{\epsilon_1 - \epsilon_2}$-Anosov (see Example 2.12).

This was proved in [GW12, § 4] over $K = \mathbf{R}$; we now provide a proof in the general case.

### 4.4. Embeddings into projective spaces.

This subsection is devoted to the proof of Propositions 4.6 and 4.8. As in Theorem 1.1, let $\Sigma^\theta_d$ be the set of positive roots that do \textit{not} belong to the span of $\Delta \setminus \theta$.

**Remark 4.9.** For a representation $(\tau, V)$ of $G$, we will always choose a Cartan decomposition of $\text{GL}_K(V)$ compatible with that of $G$. This means that the basis $(\epsilon_1, \ldots, \epsilon_d)$ of $V$ providing the isomorphism $\text{GL}_K(V) \simeq \text{GL}_d(K)$ is a basis of eigenvectors of $\mathfrak{a}$. The group $\tau(K)$ is included in $O(d)$ or $U(d)$ or $\text{Sp}(d)$, depending on whether $K = \mathbf{R}$ or $\mathbf{C}$ or $\mathbf{H}$. We shall always assume that $\epsilon_1 \in V^{\chi_\tau}$, so that $\langle \chi_\tau, Y \rangle = \langle \epsilon_1, d_\tau(Y) \rangle$ for all $Y \in \mathfrak{a}$. The Cartan projection for $\text{GL}_K(V)$ will be denoted by $\mu_{\text{GL}_K(V)}$ and the Lyapunov projection by $\lambda_{\text{GL}_K(V)}$.  

Proposition 4.6 relies on the following fact, which will also be used in Section 6.

**Lemma 4.10.** Suppose \( \tau : G \rightarrow \text{GL}(V) \) is \( \theta \)-compatible. Then

1. for any weight \( \chi \) of \( \tau \), we have \( \chi - \chi = \sum_{\alpha \in \Sigma^+} n_{\alpha} \chi \);
2. for any \( \alpha \in \theta \), the element \( \chi - \alpha \in a^* \) is a weight of \( \tau \);
3. If \( \tau \) is \( \theta \)-proximal then, for any \( g \in G \), \( \langle \epsilon_1 - \epsilon_2, \mu_{\text{GL}(V)}(\tau(g)) \rangle = \min_{\alpha \in \theta}(\alpha, \mu(g)) \) and \( \langle \epsilon_1 - \epsilon_2, \lambda_{\text{GL}(V)}(\tau(g)) \rangle = \min_{\alpha \in \theta}(\alpha, \lambda(g)) \).

**Proof.** Let \( \Phi := \sum_{\alpha \in \Delta} Z_{\alpha} \subset \Phi \) be the root lattice of \( G \). The set of weights of \( \tau \) is the intersection of \( \chi_{\tau} + \Phi \) with the convex hull of the \( W \)-orbit of \( \chi_{\tau} \) in \( a^* \) (see [GW09, Prop. 3.2.10]). Therefore, in order to prove (1), it is sufficient to prove that \( \chi_{\tau} - w \cdot \chi_{\tau} = \sum_{\alpha \in \Sigma^+} n_{\alpha} \chi \) for all \( w \in W \), or in other words that \( \chi_{\tau} - w \cdot \chi_{\tau} \notin \text{span}(\Delta \setminus \theta) \) for all \( w \in W \) with \( w \cdot \chi_{\tau} \neq \chi_{\tau} \). By definition of \( \theta \)-compatibility, \( \chi_{\tau} \) belongs to the orthogonal of \( \text{span}(\Delta \setminus \theta) \). Therefore, for any \( w \in W \), if \( \chi_{\tau} - w \cdot \chi_{\tau} \in \text{span}(\Delta \setminus \theta) \), then

\[
\| w \cdot \chi_{\tau} \|^2 = \| \chi_{\tau} \|^2 + \| \chi_{\tau} - w \cdot \chi_{\tau} \|^2.
\]

But this is also equal to \( \| \chi_{\tau} \|^2 \) by \( W \)-invariance of the norm, and so \( \chi_{\tau} = w \cdot \chi_{\tau} \). This proves (1). For any \( \alpha \in \Delta \), the orthogonal reflection \( s_{\alpha} \in W \) in the hyperplane \( \text{Ker}(\alpha) \) satisfies

\[
s_{\alpha} \cdot \chi_{\tau} = \chi_{\tau} - 2 \frac{\langle \chi_{\tau}, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha,
\]

and \( s_{\alpha} \cdot \chi_{\tau} \) is a weight of \( \tau \). Therefore the intersection of \( \chi_{\tau} + Z_{\alpha} \) with the segment \( [\chi_{\tau}, \chi_{\tau} - 2 \frac{\langle \chi_{\tau}, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha] \) consists of weights of \( \tau \). In particular, \( \chi_{\tau} - \alpha \) is a weight of \( \tau \) since \( 2 \frac{\langle \chi_{\tau}, \alpha \rangle}{\langle \alpha, \alpha \rangle} \geq 1 \) and \( \alpha \in \Phi \). This proves (2).

Point (3) follows from (1) and (2). \( \square \)

**Proof of Proposition 4.6.** (a). We prove that the stabilizer in \( G \) of \( V^{\chi_{\tau}} \) is \( P_{\theta} \). We first examine the stabilizer of \( V^{\chi_{\tau}} \) in \( g \). To simplify notation, we abusively write \( \tau \) for the derivative \( d_{\tau} : g \rightarrow \text{gl}(V) \). The fact that this derivative is a Lie algebra homomorphism implies that for any root \( \alpha \in \Sigma \) and any weight \( \chi \) of \( \tau \),

\[
(4.3) \quad \tau(g_{\alpha}) V^{\chi} \subset V^{\chi + \alpha}.
\]

In particular, \( \tau(g_0) V^{\chi_{\tau}} \subset V^{\chi_{\tau}} \), and \( \tau(g_{\alpha}) V^{\chi_{\tau}} = \{0\} \) for all \( \alpha \in \Sigma^+ \) by (4.3) and maximality of \( \chi_{\tau} \) for the partial order of Section 4.1. This means that the stabilizer of \( V^{\chi_{\tau}} \) in \( g \) contains the Lie algebra \( \text{Lie}(P_{\Delta}) \), hence it is of the form \( \text{Lie}(P_{\theta}) \) for a unique subset \( \theta \) of \( \Delta \) (see Fact 2.13). By (4.3) and Lemma 4.10, for any \( \alpha \in \Delta \setminus \theta \) we have \( \tau(g_{-\alpha}) V^{\chi_{\tau}} = \{0\} \subset V^{\chi_{\tau}} \); moreover, for any \( \alpha \in \theta \) we have \( \{0\} \neq \tau(g_{-\alpha}) V^{\chi_{\tau}} \subset V^{\chi_{\tau} - \alpha} \) (see e.g. [GW09, proof of Lem. 3.2.9]), hence \( \tau(g_{-\alpha}) V^{\chi_{\tau}} \neq V^{\chi_{\tau}} \). Therefore, \( \theta = \theta \), i.e. the stabilizer of \( V^{\chi_{\tau}} \) in \( g \) is \( \text{Lie}(P_{\theta}) \). In particular, the stabilizer of \( V^{\chi_{\tau}} \) in \( G \) is contained in \( P_{\theta} \). For the reverse inclusion, note that for any \( g \in P_{\theta} \) the line \( \tau(g) V^{\chi_{\tau}} \) is stable under \( \text{Ad}(g) \text{Lie}(P_{\theta}) = \text{Lie}(P_{\theta}) \), hence in particular under \( \text{Lie}(P_{\Delta}) \). But \( V^{\chi_{\tau}} \) is the only \( \text{Lie}(P_{\Delta}) \)-invariant line in \( V \), by arguments similar to the above (any \( \text{Lie}(P_{\Delta}) \)-invariant line \( L \) is \( \alpha \)-invariant, hence contained in a weight space \( V^\chi \), and for \( \chi \neq \chi_{\tau} \) there always exists \( \alpha \in \Delta \) with \( \{0\} \neq \tau(g_{\alpha}) L \subset V^{\chi + \alpha} \). This
proves that the stabilizer of $V^x\tau$ in $G$ is exactly $P_\theta$. Similarly, the stabilizer in $G$ of the hyperplane $V_{<\chi_{\tau}}$ is $P_\theta^-$.

(b) By (a), the map $\iota^+: G/P_\theta \to \mathbf{P}(V)$ is well defined and injective; it is clearly a $\tau$-equivariant embedding. Similarly, $\iota^-: G/P_\theta^- \to \mathbf{P}(V^*)$ is a $\tau$-equivariant embedding.

Let us show that two parabolic subgroups $P \in G/P_\theta$ and $Q \in G/P_\theta^-$ of $G$ are transverse if and only if $\iota^+(P)$ and $\iota^-(Q)$ are transverse. Note that transversality is invariant under the $G$-action, both in $G/P_\theta \times G/P_\theta^-$ and in $\mathbf{P}(V) \times \mathbf{P}(V^*)$ (by $\tau$-equivariance of $\iota^+$ and $\iota^-$). We can write $P = gp_\theta P_\theta$ and $Q = hP_\theta^- h^{-1}$ where $g, h \in G$. By the Bruhat decomposition (see [BT65, Th. 5.15]), there exist $(p, p') \in P_\theta^\perp \times P_\theta$ and $w \in W = N_K(a)/Z_K(a)$ such that $h^{-1}g = wpwp'$. Up to conjugating both $P$ and $Q$ by $p^{-1}h^{-1}$, we may assume that $(P, Q) = (wP_\theta w^{-1}, P_\theta^-)$, so that $\iota^+(P) = V^{w\chi_{\tau}}$ and $\iota^-(Q) = V^{<\chi_{\tau}}$.

Then $\iota^+(P)$ and $\iota^-(Q)$ are transverse if and only if $w \cdot \chi_{\tau} = \chi_{\tau}$. By [Bou68, Ch. V, §3, Prop. 1], this happens if and only if $w$ belongs to the subgroup $W_{\Delta, \theta}$ of $W$ generated by the reflections $s_\alpha$ for $\alpha \in \Delta \setminus \theta$. Therefore, it is sufficient to prove that $P = wP_\theta w^{-1}$ and $Q = P_\theta^-$ are transverse if and only if $w \in W_{\Delta, \theta}$. If $w \in W_{\Delta, \theta}$, it is not difficult to see that

$$\text{Ad}(w)\text{Lie}(P) = \text{Lie}(P_\theta),$$

and so $P$ and $Q$ are transverse. Conversely, suppose $w \notin W_{\Delta, \theta}$; let us prove that $P = wP_\theta w^{-1}$ and $Q = P_\theta^-$ are not transverse. Let $s_{\alpha_1} \cdots s_{\alpha_q}$ be a reduced expression of $w$ and $i \in \{1, \ldots, q\}$ the smallest index such that $\alpha_i$ belongs to $\theta$. By [Bou68, Ch. VI, §1, Cor. 2], the root $\beta = s_{\alpha_q} \cdots s_{\alpha_{i+1}}(\alpha_i)$ is positive, hence $g_\beta \subset \text{Lie}(P_\theta)$. Its image under $w$ is $-\beta' = -s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$, which satisfies

$$g_{-\beta'} = \text{Ad}(w)g_\beta \subset \text{Ad}(w)\text{Lie}(P_\theta) = \text{Lie}(wP_\theta w^{-1}).$$

Since $\omega_{\alpha_i}$ is invariant under the reflections $s_\alpha$ for $\alpha \neq \alpha_i$ and since the scalar product $(\cdot, \cdot)$ is $W$-invariant,

$$(\beta', \omega_{\alpha_i}) = (s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i), s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\omega_{\alpha_i})) = (\alpha_i, \omega_{\alpha_i}) > 0.$$

This forces $\beta'$ to belong to $\Sigma_\theta^+$; this means that $g_{-\beta'} \in u_\theta^+$ and $\text{Lie}(wP_\theta w^{-1})$ intersects nontrivially the unipotent radical of $\text{Lie}(P_\theta^-)$. Thus $P = wP_\theta w^{-1}$ and $Q = P_\theta^-$ are not transverse.

(c). By Lemma 4.2 applied to $g \in G$ and to $\tau(g) \in \text{GL}_K(V)$, we have

(i) $g$ is proximal in $G/P_\theta$ if and only if $\min_{\alpha \in \theta} \langle \alpha, \chi(g) \rangle > 0$.
(ii) $\tau(g)$ is proximal in $\text{P}_K(V)$ if and only if $\langle \varepsilon_1 - \varepsilon_2, \chi_{\text{GL}_K(V)}(g) \rangle > 0$.

So the equivalence follows from Lemma 4.10.(3). For such an element $g$, let $g = gh_\tau g_\tau$ be its Jordan decomposition in $G$ so that $\tau(g_\tau) \tau(g_\tau)$ is the Jordan decomposition of $\tau(g)$ in $\text{GL}_K(V)$. Since $\xi_{\tau}^\perp + = \xi_{\tau}^+ + \xi_{\tau}^+$ and $\xi_{\tau(g_\tau)}^+ = \xi_{\tau(g_\tau)}^+$, we may assume that $g = \exp(Y)$ with $Y \in \mathfrak{X}^+$ satisfying $\min_{\alpha \in \theta} \langle \alpha, Y \rangle > 0$. In that case $\xi_{\tau}^+ = P_\theta$ and $\iota^+(\xi_{\tau}^+) = V^{\chi_{\tau}}$ and $\tau(\exp(Y))$ is a diagonal matrix in $\text{GL}_K(V)$ whose eigenvector $e_1$ is associated with the eigenvalue of largest modulus. Hence $\xi_{\tau(g)}^+ = V^{\chi_{\tau}}$ and the equality $\xi_{\tau(g)}^+ = \iota^+(\xi_{\tau}^+) = \iota^+(\xi_{\tau}^+)$ holds. □
Proof of Proposition 4.8. Let $\rho : \Gamma \to G$ be $P_\theta$-Anosov and let $\xi^+ : \partial_\infty \Gamma \to G/P_\theta$, $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$ be its continuous, equivariant, dynamics-preserving boundary maps. Then $\iota^+ \circ \xi^+$ and $\iota^- \circ \xi^-$ are continuous and $(\tau \circ \rho)$-equivariant, they are transverse by Proposition 4.6.(b) and dynamics-preserving by Proposition 4.6.(c). From the equality (Lemma 4.10.(3))

$$\langle \varepsilon_1 - \varepsilon_2, \mu_{\GL_K(V)}(\tau(g)) \rangle = \min_{\alpha \in \theta} \langle \alpha, \mu(g) \rangle,$$

one gets, applying Theorem 3.1.(4) to $\rho : \Gamma \to G$, that

$$\langle \varepsilon_1 - \varepsilon_2, \mu_{\GL_K(V)}(\tau \circ \rho(g)) \rangle \xrightarrow{\gamma \to \infty} +\infty.$$

Thus, applying this time Theorem 3.1 to $\tau \circ \rho$, we deduce that $\tau \circ \rho$ is $P_{\varepsilon_1 - \varepsilon_2}$-Anosov.

Conversely, suppose that $\tau \circ \rho$ is Anosov with boundary maps $\xi^+_V$, $\xi^-_V$. For any $\gamma \in \Gamma$ of infinite order, $\tau \circ \rho(\gamma)$ is proximal in $\text{P}_K(V)$ with attracting fixed point $\xi^+_V(\eta^+_\gamma) (\eta^+_\gamma \in \partial_\infty \Gamma$ being the attracting fixed point of $\gamma$). By Proposition 4.6.(c), $\rho(\gamma)$ is proximal in $G/P_\theta$ and $\xi^+_V(\eta^+_\gamma) = \iota^+(\xi^+_{\rho(\gamma)}).$ Thus the set

$$\{ \eta \in \partial_\infty \Gamma \mid \xi^+_V(\eta) \in \iota^+(G/P_\theta) \}$$

is closed and contains the dense set $\{ \eta^+_\gamma \mid \gamma \in \Gamma \text{ of infinite order} \}$, hence it is equal to $\partial_\infty \Gamma$. Therefore there is a unique map $\xi^+ : \partial_\infty \Gamma \to G/P_\theta$ such that $\xi^+_V = \iota^+ \circ \xi^+.$ Similarly there is $\xi^- \gamma$ such that $\xi^- = \iota^- \circ \xi^-.$

The maps $\xi^+$ and $\xi^-$ are continuous and $\rho$-equivariant, they are transverse by Proposition 4.6.(b) and dynamics-preserving by Proposition 4.6.(c). Using one more time Lemma 4.10.(3), for any $\alpha \in \theta$

$$\langle \alpha, \mu(\rho(\gamma)) \rangle \xrightarrow{\gamma \to \infty} +\infty.$$

By Theorem 3.1.(4), $\rho$ is $P_\theta$-Anosov. \hfill $\square$

4.5. Anosov representations and the Lyapunov projection. Using the results of Section 4.3, we can now address the implication (4) $\Rightarrow$ (1) of Theorem 3.2 in the general situation where $\rho$ is not necessarily semisimple. (The semisimple case has already been treated in Section 3.5.)

Proof of (4) $\Rightarrow$ (1) in Theorem 3.2. Suppose there exist two continuous, $\rho$-equivariant, and transverse maps $\xi^+ : \partial_\infty \Gamma \to G/P_\theta$ and $\xi^- : \partial_\infty \Gamma \to G/P_\theta^-$ which are dynamics-preserving for $\rho$, and that for any $\alpha \in \theta$

$$\lim_{\ell(\gamma) \to +\infty} \langle \alpha, \lambda(\rho(\gamma)) \rangle = +\infty.$$

Let us prove that $\rho$ is $P_\theta$-Anosov. If $\Gamma$ is elementary (i.e. virtually cyclic), this follows from Remark 2.27. We now assume that $\Gamma$ is not elementary.

Consider an irreducible, $\theta$-proximal representation $(\tau, V)$ of $G$ (Lemma 4.5). By Proposition 4.8, it is sufficient to prove that $\tau \circ \rho : \Gamma \to \GL(V)$ is Anosov with respect to the stabilizer of a line. We shall prove that the semisimplification $(\tau \circ \rho)''$ is Anosov with respect to the stabilizer of a line, and conclude using the semisimple case treated in Section 3.5.
By Proposition \ref{prop:semisimplification}, there are \(\tau\)-equivariant embeddings \(\iota^+ : G/P_0 \to \mathbf{P}(V)\) and \(\iota^- : G/P_0 \to \mathbf{P}(V^*)\) such that the maps

\[
\begin{cases}
\xi^+_V := \iota^+ \circ \xi^+ : \partial_\infty \Gamma \to \mathbf{P}(V), \\
\xi^-_V := \iota^- \circ \xi^- : \partial_\infty \Gamma \to \mathbf{P}(V^*)
\end{cases}
\]

are continuous, transverse, and dynamics-preserving for \(\tau \circ \rho\). Viewing \(\mathbf{P}(V)\) and \(\mathbf{P}(V^*)\) as the sets of lines and hyperplanes of \(V\), respectively, we define the linear subspaces

\[
U := \sum_{\eta \in \partial_\infty \Gamma} \xi^+_V(\eta) \quad \text{and} \quad U' := \bigcap_{\eta \in \partial_\infty \Gamma} \xi^-_V(\eta)
\]

of \(V\); they are \(\tau \circ \rho(\Gamma)\)-invariant. Let \(V_1 := U \cap U'\), let \(V_2\) be a complementary subspace of \(V_1\) in \(U'\), and let \(V_3\) be a complementary subspace of \(U'\) in \(V\). Then

\[
V = U' + V_3 = V_1 + V_2 + V_3.
\]

Under this decomposition of \(V\), the representation \(\tau \circ \rho\) is upper triangular: there are representations \(\rho_i : \Gamma \to \text{GL}(V_i)\) and maps \(a_{i,j} : \Gamma \to \text{Hom}(V_i, V_j)\), for \(1 \leq i, j \leq 3\), such that for any \(\gamma \in \Gamma\),

\[
\tau \circ \rho(\gamma) = \begin{pmatrix}
\rho_1(\gamma) & a_{1,2}(\gamma) & a_{1,3}(\gamma) \\
a_{2,1}(\gamma) & \rho_2(\gamma) & a_{2,3}(\gamma) \\
a_{3,1}(\gamma) & a_{3,2}(\gamma) & \rho_3(\gamma)
\end{pmatrix}.
\]

The semisimplification of \(\tau \circ \rho\) (see Section \ref{subsec:semisimplification}) is given by

\[
(\tau \circ \rho)^{ss}(\gamma) = \begin{pmatrix}
\rho_1^{ss}(\gamma) \\
\rho_2^{ss}(\gamma) \\
\rho_3^{ss}(\gamma)
\end{pmatrix}
\]

for all \(\gamma \in \Gamma\), where \(\rho_i^{ss}\) is the semisimplification of \(\rho_i\). We now construct continuous, equivariant, transverse, and dynamics-preserving boundary maps \(\xi^+_V : \partial_\infty \Gamma \to \mathbf{P}(V)\) and \(\xi^-_V : \partial_\infty \Gamma \to \mathbf{P}(V^*)\) for \((\tau \circ \rho)^{ss}\).

We first note that \(\xi^+_V\) takes values in \((V_1 + V_2) \setminus V_1\), and \(\xi^-_V\) takes values in \(V_1^+ \setminus (V_1 + V_2)^\perp\), where we denote by \(V_1^+\) the subspace of \(V^*\) consisting of linear forms vanishing on \(V_1\), and similarly for \((V_1 + V_2)^\perp\). Let

\[
p : U' = V_1 + V_2 \to V_2
\]

be the linear projection onto \(V_2\) with kernel \(V_1\) and \(p : \mathbf{P}(U' \setminus V_1) \to \mathbf{P}(V_2)\) the induced map. Similarly, let

\[
p^* : V_1^+ = (V_1 + V_2)^\perp \to (V_1 + V_3)^\perp
\]

be the linear projection onto \((V_1 + V_3)^\perp\) with kernel \((V_1 + V_2)^\perp\) and \(p^* : \mathbf{P}(V_1^+ \setminus (V_1 + V_2)^\perp) \to \mathbf{P}((V_1 + V_3)^\perp)\) the induced map. We set

\[
\begin{cases}
\xi^+_V := p \circ \xi^+_V : \partial_\infty \Gamma \to \mathbf{P}(V_2) \subset \mathbf{P}(V), \\
\xi^-_V := p^* \circ \xi^-_V : \partial_\infty \Gamma \to \mathbf{P}((V_1 + V_3)^\perp) \subset \mathbf{P}(V^*)
\end{cases}
\]

These maps are continuous and transverse by construction, as well as dynamics-preserving for \(\rho_1^{ss} + \rho_2^{ss} + \rho_3^{ss} : \Gamma \to \text{GL}(V_1 + V_2 + V_3)\). To see that they are dynamics-preserving for \((\tau \circ \rho)^{ss} : \Gamma \to \text{GL}(V)\), it is sufficient to prove that \(\rho_2 = \rho_2^{ss}\).

We claim that in fact \(\rho_2 : \Gamma \to \text{GL}(V_2)\) is irreducible. Indeed, let \(R\) be a \(\rho_2(\Gamma)\)-invariant subspace of \(V_2\) and let \(\gamma \in \Gamma\) be an element of infinite order.
with attracting (resp. repelling) fixed point \( \eta^+ \) (resp. \( \eta^- \)) in \( \partial_\infty \Gamma \). Since \( \xi^+_V \) and \( \xi^-_V \) are dynamics-preserving for \( \rho_2 \), for any \( x \in \text{P}(V) \setminus \xi^-_V(\eta^-) \) we have \( \rho_2(\gamma^n) \cdot x \to \xi^+_V(\eta^+) \) as \( n \to +\infty \). Since \( R \) is closed and \( \rho_2(G) \)-invariant, we obtain that either \( \text{P}(R) \subset \xi^-_V(\eta^-) \) or \( \xi^+_V(\eta^+) \in \text{P}(R) \). Thus, one of the closed \( \Gamma \)-invariant subsets \( \{ \eta \in \partial_\infty \Gamma \mid \text{P}(R) \subset \xi^-_V(\eta) \} \) or \( \{ \eta \in \partial_\infty \Gamma \mid \xi^+_V(\eta) \in \text{P}(R) \} \) is nonempty, hence equal to \( \partial_\infty \Gamma \) by minimality of the action of the nonelementary group \( \Gamma \) on \( \partial_\infty \Gamma \). This shows that \( R = \{ 0 \} \) or \( R = V_2 \).

Thus \( \rho_2 \) is irreducible, and in particular \( \rho_2 = \rho^{ss}_2 \).

In conclusion, the semisimplification \( (\tau \circ \rho)^{ss} : \Gamma \to \text{GL}(V) \) admits continuous, equivariant, transverse, and dynamics-preserving boundary maps \( \xi^+_V : \partial_\infty \Gamma \to \text{P}(V) \) and \( \xi^-_V : \partial_\infty \Gamma \to \text{P}(V^*) \). By Lemma 2.32 it satisfies condition (4) of Theorem 3.2, hence by Section 3.5 it is Anosov with respect to the stabilizer of a line. By Proposition 2.31, the representation \( \tau \circ \rho : \Gamma \to \text{GL}(V) \) is Anosov as well with respect to the stabilizer of a line, and so \( \rho \) is \( P_\theta \)-Anosov by Proposition 4.8.

Corollary 1.7 is a consequence of this proof.

**Proof of Corollary 1.7.** Suppose \( \rho : \Gamma \to G \) is \( P_\theta \)-Anosov and let \( (\tau, V) \) be an irreducible, \( \theta \)-proximal representation of \( G \) (Lemma 4.5). By the proof just above, the semisimplification \( (\tau \circ \rho)^{ss} \) is Anosov with respect to the stabilizer of a line. Since the representation \( \tau \circ \rho^{ss} \) is semisimple and lies in the closure of the \( \text{GL}(V) \)-orbit of \( \tau \circ \rho \), we have \( \tau \circ \rho^{ss} = (\tau \circ \rho)^{ss} \) (up to conjugacy), and so \( \tau \circ \rho^{ss} \) is Anosov with respect to the stabilizer of a line. Consequently, \( \rho^{ss} \) is \( P_\theta \)-Anosov (Proposition 4.8). The converse implication holds by Proposition 2.31. \( \square \)

### 5. Construction of the boundary maps

In this section we prove Theorem 1.1 (which implies in particular (4) \( \Rightarrow \) (1) in Theorem 1.3) by establishing an explicit version of it, namely Theorem 5.2. We also complete the proof of Theorem 1.3 by establishing its implication (1) \( \Rightarrow \) (4).

Let \( G \) be a reductive Lie group with Cartan decomposition \( K(\exp \overline{a}^+)K \) and corresponding Cartan projection \( \mu : G \to \overline{a}^+ \), and let \( \theta \subset \Delta \) be a nonempty subset of the simple restricted roots of \( \mathfrak{a} \) in \( G \); we use the notation of Section 2.2. We define two maps \( \Xi^+_\theta : G \to G/P_\theta \) and \( \Xi^-_\theta : G \to G/P^-_\theta \) as follows: for any \( g \in G \), we choose \( k_g, k'_g \in K \) such that \( g = k_g(\exp \mu(g))k'_g \), and set

\[
\begin{align*}
\Xi^+_\theta(g) &= k_g \cdot P_\theta \in G/P_\theta, \\
\Xi^-_\theta(g) &= k_g \cdot P^-_\theta \in G/P^-_\theta.
\end{align*}
\]

This does not depend on the choice of \( k_g, k'_g \) as soon as \( \langle \alpha, \mu(g) \rangle > 0 \) for all \( \alpha \in \theta \) (see [Hel01, Ch. IX, Cor. 1.2]). It is not difficult to see that if \( g \) is proximal in \( G/P_\theta \) (in the sense of Section 4.2), then the sequence \( (\Xi^*_\theta(g^n))_{n \in \mathbb{N}} \) converges to the attracting fixed point of \( g \) in \( G/P_\theta \) (Lemma 5.9), and similarly for \( G/P^-_\theta \) and \( (\Xi^-_\theta(g^n))_{n \in \mathbb{N}} \). More generally, we prove that for a word hyperbolic group \( \Gamma \) and a representation \( \rho : \Gamma \to G \), the maps \( \Xi^*_\theta \) and \( \Xi^-_\theta \) induce \( \rho \)-equivariant maps on the boundary \( \partial_\infty \Gamma \) as soon as \( \rho \) has the following property.
Definition 5.1. A representation \( \rho : \Gamma \to G \) has the gap summation property with respect to \( \theta \) if for any \( \alpha \in \theta \) and any geodesic ray \( (\gamma_n)_{n \in \mathbb{N}} \) in the Cayley graph of \( \Gamma \),

\[
\sum_{n \in \mathbb{N}} e^{-(\alpha, \mu(\rho(\gamma_n)))} < +\infty.
\]

The representation \( \rho \) has the uniform gap summation property with respect to \( \theta \) if this series converges uniformly for all geodesic rays \( (\gamma_n)_{n \in \mathbb{N}} \) with \( \gamma_0 = e \), i.e. if for any \( \alpha \in \theta \),

\[
\sup_{\{\gamma_n\}_{n \in \mathbb{N}} \text{geodesic with } \gamma_0 = e} \sum_{n \geq n_0} e^{-(\alpha, \mu(\rho(\gamma_n)))} \longrightarrow 0.
\]

Theorem 5.2. Let \( \Gamma \) be a word hyperbolic group, \( G \) a real reductive Lie group, \( \theta \subset \Delta \) a nonempty subset of the simple restricted roots of \( G \), and \( \rho : \Gamma \to G \) a representation.

1. If \( \rho \) has the gap summation property with respect to \( \theta \), then the maps \( \Xi^+_{\rho} : \rho : \Gamma \to G/P_\theta \) and \( \Xi^-_{\rho} : \rho : \Gamma \to G/P^-_\theta \) induce \( \rho \)-equivariant boundary maps \( \xi^+ : \partial_\infty \Gamma \to G/P_\theta \) and \( \xi^- : \partial_\infty \Gamma \to G/P^-_\theta \). The maps \( \xi^+ , \xi^- \) are independent of the choices involved in the definition of \( \Xi^+_{\rho} , \Xi^-_{\rho} \).

2. If moreover for any \( \alpha \in \theta \) and any \( \gamma \in \Gamma \) of infinite order,

\[
(\alpha, \mu(\rho(\gamma^n))) - 2 \log n \longrightarrow +\infty,
\]

then \( \xi^+ \) and \( \xi^- \) are dynamics-preserving for \( \rho \).

3. If \( \rho \) has the uniform gap summation property with respect to \( \theta \), then \( \xi^+ \) and \( \xi^- \) are continuous.

4. If moreover for any \( \alpha \in \Sigma^+_{\rho} \) and any geodesic ray \( (\gamma_n)_{n \in \mathbb{N}} \) the sequence \( ((\alpha, \mu(\rho(\gamma_n)))_{n \in \mathbb{N}} \) is CLI (Definition 2.3), then \( \xi^+ \) and \( \xi^- \) are transverse and \( \rho \) is \( P_\theta \)-Anosov.

Remark 5.3. Let \( \gamma \in \Gamma \) be an element of infinite order, with attracting fixed point \( \eta^+_\gamma \in \partial_\infty \Gamma \). The image of \( \eta^+_\gamma \) by \( \xi^+ \) (or by any \( \rho \)-equivariant map \( \partial_\infty \Gamma \to G/P_\theta \)) is always a fixed point of \( \rho(\gamma) \) in \( G/P_\theta \); however, this fixed point \( \xi^+(\eta^+_\gamma) \) is attracting only if \( (\alpha, \mu(\rho(\gamma^n))) \) grows faster than \( 2 \log(n) \) for every \( \alpha \) in \( \theta \) (Lemma 4.2). This shows that the growth assumption of (2) above is optimal. For example, here is a case where the assumptions of (1) and (3) are satisfied but the conclusion of (2) fails:

Example 5.4. Let \( G = \text{SL}_2(\mathbb{R}) \), with Cartan projection \( \mu : G \to \mathbb{R}_+ \) obtained by identifying \( \mathfrak{a}^+ \) with \( \mathbb{R}_+ \). Let \( \Gamma \) be a finitely generated Schottky subgroup of \( G \) containing a parabolic element \( u \). There is a constant \( C > 0 \) such that \( \mu(\rho(\gamma)) \geq 2 \log |\gamma| r - C \) for all \( \gamma \in \Gamma \); in particular, \( \rho \) satisfies the uniform gap summation property (5.3). However, this growth rate cannot be improved since \( \mu(u^n) = 2 \log n + O(1) \). The continuous, equivariant boundary map \( \xi : \partial_\infty \Gamma \to \partial_\infty \mathbb{H}^2 \) given by Theorem 5.2.(3) is not dynamics-preserving since the fixed point of \( u \) is neither attracting nor repelling in \( G/P_\theta = \partial_\infty \mathbb{H}^2 \); thus the conclusion of Theorem 5.2.(2) fails. The map \( \xi \) is also not transverse (i.e. not injective, i.e. (4) fails) since \( \xi(\lim_{+\infty} u^n) = \xi(\lim_{+\infty} u^{-n}) \).

We first discuss the gap summation property (Section 5.1), then establish some estimates for the maps \( \Xi^+_{\rho} \) and \( \Xi^-_{\rho} \) (Section 5.2), which are useful in the
proof of Theorem 5.2.1–(2)–(3) (Section 5.3). The proof of Theorem 5.2.(4) is more delicate, and is the object of Section 5.4. Finally, in Sections 5.5 and 5.6, we establish the implication (1) \(\Rightarrow\) (4) of Theorem 1.3 and give a short proof of Corollary 1.11.

5.1. The gap summation property. The following observation will be useful in the proof of Theorem 5.2.

Lemma 5.5. For any \(c, C > 0\), there exists \(C_0 > 0\) such that for any \((c, C)\)-quasi-geodesic rays \((\gamma_n)_{n \in \mathbb{N}}\) and \((\gamma'_n)_{n \in \mathbb{N}}\) in the Cayley graph of \(\Gamma\), with the same initial point \(\gamma_0 = \gamma'_0\) and the same endpoint in \(\partial_\infty \Gamma\), and for any \(\alpha \in \Delta\),

\[
\sum_{n \in \mathbb{N}} e^{-\langle \alpha, \mu(\gamma'_n) \rangle} \leq C_0 \sum_{n \in \mathbb{N}} e^{-\langle \alpha, \mu(\gamma_n) \rangle}.
\]

Therefore, in the definition 5.1 of the gap summation property, it is equivalent to ask for the convergence of the series (5.2) for all quasi-geodesic rays.

Proof. Since \((\gamma_n)_{n \in \mathbb{N}}\) and \((\gamma'_n)_{n \in \mathbb{N}}\) have the same endpoint at infinity in the word hyperbolic group \(\Gamma\), there is a \((c', C')\)-quasi-isometry \(\phi : \mathbb{N} \to \mathbb{N}\) such that the sequence \((\gamma'_n \gamma_{\phi(n)})_{n \in \mathbb{N}}\) is contained in the \(R\)-ball \(B_e(R)\) centered at \(e\) in \(\Gamma\). The constants \(c', C',\) and \(R\) depend only on \((c, C)\) (and on the hyperbolicity constant of \(\Gamma\)). Let \(M\) be a real number such that \(\|\mu(\gamma)\| \leq M\) for all \(\gamma \in B_e(R)\). By subadditivity of \(\mu\) (Fact 2.16.(3)), for any \(n \in \mathbb{N}\),

\[
|\langle \alpha, \mu(\gamma_{\phi(n)}) \rangle - \mu(\gamma_n)| \leq \|\mu(\gamma_{\phi(n)}) - \mu(\gamma_n)\| \leq \|\mu(\gamma_{\phi(n)})\| \leq M.
\]

Moreover, for any \(p \in \mathbb{N}\), the set \(\{n \in \mathbb{N} \mid \phi(n) = p\}\) has at most \(c'C' + 1\) elements. Thus, for any \(n \in \mathbb{N}\),

\[
\sum_{n \in \mathbb{N}} e^{-\langle \alpha, \mu(\gamma'_n) \rangle} \leq e^M \sum_{n \in \mathbb{N}} e^{-\langle \alpha, \mu(\gamma_{\phi(n)}) \rangle} \leq e^M (c'C' + 1) \sum_{p \in \mathbb{N}} e^{-\langle \alpha, \mu(\gamma_p) \rangle}.
\]

For any \(g \in G\), we set

\[
T_\theta(g) := \min_{\alpha \in \theta} \langle \alpha, \mu(g) \rangle \geq 0.
\]

If the gap summation property holds with respect to \(\theta\), then the series \(\sum_n e^{-T_\theta(\rho(\gamma_n))}\) converges for every quasi-geodesic ray \((\gamma_n)_{n \in \mathbb{N}}\). Here is an immediate consequence of Lemma 5.5.

Corollary 5.6. If the uniform gap summation property holds with respect to \(\theta\), then for any \(c, C > 0\),

\[
\sup_{(\gamma_n)_{n \in \mathbb{N}} \text{ \(\text{(c,C)-quasi-geodesic} \)} \sum_{n \geq n_0} e^{-T_\theta(\rho(\gamma_n))} \overset{n_0 \to +\infty}{\longrightarrow} 0.
\]

5.2. Metric estimates for the map \(\Xi^+_\theta\). Consider a finite-dimensional, irreducible, \(\theta\)-proximal representation \((\tau, V)\) of \(G\) (Definition 4.4); such a representation exists by Lemma 4.5. Let \(\|\cdot\|_V\) be a \(K\)-invariant Euclidean
norm on $V$ for which the weight spaces of $\tau$ are orthogonal. It defines a $K$-invariant metric $d_{P(V)}$ on $P(V)$, given by

$$d_{P(V)}([v],[v']) = |\sin \angle(v,v')|$$

for all nonzero $v,v' \in V$. By Proposition 4.6, the space $G/P_\theta$ embeds into $P(V)$ as the closed $G$-orbit of $x_\tau := P(V^{x_\tau})$, and inherits from this a $K$-invariant metric $d_{G/P_\theta}$: for any $g,g' \in G$,

$$d_{G/P_\theta}(g \cdot P_\theta, g' \cdot P_\theta) = d_{P(V)}(g \cdot x_\tau, g' \cdot x_\tau).$$

Here, and sometimes in the rest of the section, to simplify notation, we omit the $\tau$-action of $G$ on both $V$ and $P(V)$. The goal of this subsection is to establish the following useful estimates. (Similar estimates hold for $\Xi_\theta$, with the same proof.)

**Lemma 5.7.** For any compact subset $\mathcal{M}$ of $G$, there is a constant $C_\mathcal{M} > 0$ such that for any $g \in G$ and $m \in \mathcal{M}$,

1. $d_{G/P_\theta}(\Xi_\theta^+(gm), \Xi_\theta^+(g)) \leq C_\mathcal{M} e^{-T_\theta(g)}$,
2. $d_{G/P_\theta}(\Xi_\theta^-(mg), m \cdot \Xi_\theta^+(g)) \leq C_\mathcal{M} e^{-T_\theta(g)}$.

By Lemma 4.10, the quantity $T_\theta(g)$ of (5.5) is the logarithm of the ratio of the two largest singular values of $\tau(g)$.

In order to prove Lemma 5.7, we make the following observation, where for $g \in G$ we denote by

$$R(g) := \langle \chi_\tau, \mu(g) \rangle = \max_{v \in V \setminus \{0\}} \log \frac{\|g \cdot v\|_V}{\|v\|_V} \geq 0$$

the logarithm of the largest singular value of $\tau(g)$.

**Observation 5.8.** Let $v \in V$ be a highest-weight vector for $\tau$ with $\|v\|_V = 1$. For any $h \in G$ and $a,b \in \exp \mathfrak{p}^+$,

$$d_{G/P_\theta}(P_\theta, h \cdot P_\theta) \leq e^{-T_\theta(a)} e^{R(a)-R(b)+R(h^{-1})} \|a^{-1}hb \cdot v\|_V.$$

**Proof of Observation 5.8.** Write $h \cdot v = tv + w$ where $t \in \mathbb{R}$ and $w \in v^\perp$. Then $\|v\|_V = \|h \cdot v\|_V \sin \angle(v,h \cdot v) = \|h \cdot v\|_V d_{G/P_\theta}(P_\theta, h \cdot P_\theta)$, and so

$$\|a^{-1}h \cdot v\|_V \geq \|a^{-1} \cdot w\|_V \geq e^{-R(a)+T_\theta(a)} \|h \cdot v\|_V d_{G/P_\theta}(P_\theta, h \cdot P_\theta).$$

To conclude, note that $1 = \|h^{-1}h \cdot v\|_V \leq e^{R(h^{-1})}\|h \cdot v\|_V$ and $b \cdot v = e^{R(b)}v$. 

**Proof of Lemma 5.7.** Let $\mathcal{M}$ be a compact subset of $G$. By continuity of $\mu$, there is a constant $\delta > 0$ such that $\|m \cdot v\|_V \leq e^\delta \|v\|_V$ and $\|\mu(m)\| \leq \delta$ for all $m \in \mathcal{M}$ and $v' \in V$, where $\|\cdot\|$ is the $W$-invariant Euclidean norm on $V$. Let $v \in V$ be a highest-weight vector for $\tau$ with $\|v\|_V = 1$. Recall the elements $k_g,k'_g \in K$ defined before (5.1). For any $g \in G$ and $m \in \mathcal{M}$,

$$\|\exp(\mu(g))^{-1}(k_g^{-1}gm) \cdot v\|_V = \|k'_gmk'_g^{-1} \cdot v\|_V \leq e^\delta.$$

By applying Observation 5.8 to $(a,b,h) = (\exp(\mu(g)), \exp(\mu(gm)), k_g^{-1}gm)$ we obtain

$$d_{G/P_\theta}(\Xi_\theta^+(g), \Xi_\theta^+(gm)) = d_{G/P_\theta}(k_g \cdot P_\theta, k_g^{-1}gm \cdot P_\theta) \leq e^{-T_\theta(g)} e^{R(g)-R(gm)} e^\delta.$$
By strong subadditivity of \( \mu \) (Fact 2.16),
\[
|R(g) - R(gm)| = |(\chi_\tau, \mu(g) - \mu(gm))| \leq \|\mu(g) - \mu(gm)\| \|\chi_\tau\|
\leq \|\mu(m)\| \|\chi_\tau\| \leq \delta \|\chi_\tau\|.
\]
Therefore,
\[
d_{G/P}(\Xi^+_\vartheta(g), \Xi^+_\vartheta(gm)) \leq e^{\delta(1+\|\chi_\tau\|)}e^{-T_\theta(g)}
\]
i.e. (i) holds with \( C_M = e^{\delta(1+\|\chi_\tau\|)} \). Similarly, we have
\[
\|\exp(\mu(mg))^{-1}(k_{mg}^{-1}mk_g) \exp(\mu(g)) \cdot v\|_V = \|k_{mg}^{-1}mk_g^{-1} \cdot v\|_V \leq e^{\delta}.
\]
By applying Observation 5.8 to \((a, b, h) = (\exp(\mu(mg)), \exp(\mu(g)), k_{mg}^{-1}mk_g)\) we obtain
\[
d_{G/P}(\Xi^+_\vartheta(mg), m \cdot \Xi^+_\vartheta(g)) = d_{G/P}(k_{mg} \cdot P_\theta, mk_g \cdot P_\theta)
\leq e^{-T_\theta(g)}e^{R(mg) - R(g) + R(m^{-1})}e^{\delta}.
\]
As above, \(|R(mg) - R(g)| \leq \|\mu(m)\| \|\chi_\tau\| \leq \delta \|\chi_\tau\|, \text{ and}
\]
\[
R(m^{-1}) \leq \|\mu(m^{-1})\| \|\chi_\tau\| \leq \delta \|\chi_\tau\|,
\]

hence
\[
d_{G/P}(\Xi^+_\vartheta(mg), m \cdot \Xi^+_\vartheta(g)) \leq e^{\delta(1+\|\chi_\tau\|)}e^{-T_\theta(g)},
\]
i.e. (ii) holds with \( C_M' = e^{\delta(1+\|\chi_\tau\|)}) \). \(\square\)

5.3. Existence, equivariance, continuity, and dynamics-preserving property for the boundary maps. We now give a proof of statements (1), (2), (3) of Theorem 5.2.

Proof of Theorem 5.2 (1). Let \((\gamma_n)_{n \in \mathbb{N}}\) be a quasi-geodesic ray in the Cayley graph of \( \Gamma \), with endpoint \( \eta \in \partial_\infty \Gamma \). The set \( \{\gamma^{-1}_n \gamma_{n+1} \mid n \in \mathbb{N}\} \) is bounded: if \((\gamma_n)_{n \in \mathbb{N}}\) is \((c, C)\)-quasi-geodesic, then it is contained in the ball \( B_c(c + C) \) of radius \( c + C \) centered at \( e \in \Gamma \). Applying Lemma 5.7. (i) to \( \rho(\gamma_n) \) and to \( \mathcal{M} := \rho(B_c(c + C)) \), we obtain
\[
d_{G/P}(\Xi^+_\vartheta \circ \rho(\gamma_n), \Xi^+_\vartheta \circ \rho(\gamma_{n+1})) \leq C_M e^{-T_\theta(\rho(\gamma_n))}
\]
for all \( n \in \mathbb{N} \). The gap summation property and Lemma 5.5 imply
\[
\sum_{n \in \mathbb{N}} e^{-T_\theta(\rho(\gamma_n))} < +\infty.
\]
Thus \((\Xi^+_\vartheta \circ \rho(\gamma_n))_{n \in \mathbb{N}}\) is a Cauchy sequence in \( G/P_\theta \), and so its limit \( \xi^+(\eta) \in G/P_\theta \) exists. More precisely, for any \( n_0 \in \mathbb{N} \),
\[
d_{G/P}(\Xi^+_\vartheta \circ \rho(\gamma_{n_0}), \xi^+(\eta)) \leq C_M \sum_{n \geq n_0} e^{-T_\theta(\rho(\gamma_n))}.
\]
(5.6)
This limit \( \xi^+(\eta) \) does not depend on the choice of the quasi-geodesic ray \((\gamma_n)_{n \in \mathbb{N}}\), because any other quasi-geodesic ray \((\gamma'_n)_{n \in \mathbb{N}}\) lies within some distance \( R \in \mathbb{R}_+ \) from \((\gamma_n)\), and so we can apply Lemma 5.7. (i) again, taking for \( \mathcal{M} \) the image under \( \rho \) of the Cayley \( R \)-ball centered at \( e \).

To see that \( \xi^+ \) is \( \rho \)-equivariant, consider a quasi-geodesic ray \((\gamma_n)_{n \in \mathbb{N}}\) with endpoint \( \eta \in \partial_\infty \Gamma \) and an element \( \gamma \in \Gamma \). Then \((\gamma_n)_{n \in \mathbb{N}}\) is a quasi-geodesic ray with endpoint \( \gamma \cdot \eta \in \partial_\infty \Gamma \). By Lemma 5.7. (ii),
\[
d_{G/P}(\Xi^+_\vartheta (\rho(\gamma \gamma_n)), \rho(\gamma) \cdot \Xi^+_\vartheta (\rho(\gamma_n))) \to 0,
\]
\(n \to +\infty\),
hence \( \xi^+(\gamma \cdot \eta) = \rho(\gamma) \cdot \xi^+(\eta) \) by passing to the limit.

The gap summation property also imposes that \( \langle \alpha, \mu(\rho(\gamma_n)) \rangle > 0 \) for all \( \alpha \in \theta \) and all large enough \( n \in \mathbb{N} \), in which case \( \Xi_\theta^+(\rho(\gamma_n)) = k_{\rho(\gamma_n)} \cdot P_\theta \) is independent of the choice of \( k_{\rho(\gamma_n)} \). We deduce that the limit \( \xi^+(\eta) = \lim_n \Xi_\theta^+(\rho(\gamma_n)) \) is independent of the choices involved in the definition of \( \Xi_\theta^+ \).

We can argue similarly for \( \xi^- \).

\( \square \)

Theorem 5.2.(2) is based on the following observation.

**Lemma 5.9.** For any \( g \in G \) which is proximal in \( G/P_\theta \),

\[
\Xi_\theta^+(g^n) \xrightarrow{n \to +\infty} \xi_g^+,
\]

where \( \xi_g^+ \) is the attracting fixed point of \( g \) in \( G/P_\theta \).

**Proof.** The element \( g \) admits a Jordan decomposition \( g = g_h g_e g_u \) where \( g_h \) is hyperbolic, \( g_e \) is elliptic, and \( g_u \) is unipotent, and \( g_h, g_e, g_u \) commute. Since \( \lambda(g_e g_u) = 0 \) by definition of \( \lambda \), we have \( \|\mu(g^n g_u^n)\| = o(n) \) as \( n \to +\infty \), by (2.9). Let us write \( g_h = m z m^{-1} \) where \( m \in G \) and \( z \in \exp(\bar{a}^+) \). Then \( \xi_g^+ = m \cdot P_\theta \). Let \( (\tau, V) \) and \( \| \cdot \|_V \) as in Section 5.2, and let \( v \in V \) be a highest-weight vector for \( \tau \) with \( \|v\|_V = 1 \). For any \( n \in \mathbb{N} \),

\[
\|z^{-n} m^{-1} k_{g^n} \exp(\mu(g^n)) \cdot v\|_V = \|m^{-1} g_h^n g^n k_{g^n}^{-1} \cdot v\|_V = \|m^{-1} g_h^n g^n k_{g^n}^{-1} \cdot v\|_V.
\]

By Observation 5.8 applied to \( (a, b, h) = (z^n, \exp(\mu(g^n)), m^{-1} k_{g^n}) \), we have

\[
d_{G/P_\theta}(s_g^+, \Xi_\theta^+(g^n)) = d_{G/P_\theta}(m \cdot P_\theta, k_{g^n} \cdot P_\theta) \leq C d_{G/P_\theta}(P_\theta, m^{-1} k_{g^n} \cdot P_\theta).
\]

Let \( C \geq 0 \) be a Lipschitz constant for the action of \( m \) on \( G/P_\theta \). By strong subadditivity of \( \mu \) (Fact 2.16),

\[
|R(z^n) - R(g^n)| = |\langle \chi_{\tau}, \mu(z^n) - \mu(m z m^{-1} g_h^n g_u^n) \rangle| < \|\chi_{\tau}\| \|\mu(m^{-1})\| + \|\mu(g_h^n g_u^n)\| = o(n).
\]

Similarly,

\[
\|m^{-1} g_h^n g_u^n k_{g^n}^{-1} \cdot v\|_V \leq e^{\langle \chi_{\tau}, \mu(m^{-1} g_h^n g_u^n) \rangle} \leq e^{\|\chi_{\tau}\| \|\mu(m^{-1})\| + \|\mu(g_h^n g_u^n)\|} = e^{o(n)}.
\]

Therefore, \( d_{G/P_\theta}(s_g^+, \Xi_\theta^+(g^n)) = e^{-n T_\theta(z) + o(n)} \to 0 \) as \( n \to +\infty \). \( \square \)

**Proof of Theorem 5.2.(2).** Let \( \gamma \in \Gamma \) be an element of infinite order with attracting fixed point \( \eta_\gamma^+ \) in \( \partial_\infty \Gamma \). Suppose that for any \( \alpha \in \theta \) we have \( \langle \alpha, \mu(\rho(\gamma_n)) \rangle - 2 \log n \to +\infty \) as \( n \to +\infty \). By Lemma 4.2, the element \( \rho(\gamma) \in G \) is proximal in \( G/P_\theta \). By Lemma 5.9, the sequence \( (\Xi_\theta^+ \circ \rho(\gamma_n))_{n \in \mathbb{N}} \) converges to the attracting fixed point of \( \rho(\gamma) \) in \( G/P_\theta \). On the other hand, this sequence converges to \( \xi^+(\eta_\gamma^+) \) by construction of \( \xi^+ \). Thus \( \xi^+ \) is dynamics-preserving for \( \rho \). We argue similarly for \( \xi^- \).

Under the uniform gap summation property, we prove the continuity of the maps \( \xi^+ \) and \( \xi^- \).

**Proof of Theorem 5.2.(3).** There exist \( c, C > 0 \) with the following property: for any point \( \eta \in \partial_\infty \Gamma \) which is the endpoint of a geodesic \( (\gamma_n)_{n \in \mathbb{N}} \) with \( \gamma_0 = e \), a basis of neighborhoods of \( \eta \) in \( \partial_\infty \Gamma \) is given by the family \( (Y_{n_0})_{n_0 \in \mathbb{N}} \).
where we set
\[ V_{n_0} = \{ \eta' = \lim_{n} \gamma'_{n} \mid (\gamma'_{n})_{n \in \mathbb{N}} \text{ (c, C)-quasi-geodesic with } \gamma'_{k} = \gamma_{k} \forall k \leq n_0 \}. \]
By Corollary 5.6,
\[ \varepsilon(n_0) := \sup_{(\gamma'_{n})_{n \in \mathbb{N}}} \sum_{n \geq n_0} e^{-T_\theta(\rho(\gamma'_{n}))} \]
tends to 0 as \( n_0 \to +\infty \). By (5.6), for any \( n_0 \in \mathbb{N} \) and any \( \eta' \in V_{n_0} \),
\[ d_{G/P} (\xi^+(\eta'), \Xi_p(\rho(\gamma_{n_0}))) \leq C_{\theta} \varepsilon(n_0). \]
Since this holds also for \( \eta \), the continuity of \( \xi^+ \) follows from the uniform gap summation property assumption \( \lim_{n_0 \to +\infty} \varepsilon(n_0) = 0 \). \( \square \)

5.4. Transversality of the boundary maps. In order to prove the transversality of the boundary maps under the assumption that the sequence \( (\alpha, \mu(\rho(\gamma_n)))_{n \in \mathbb{N}} \) is CLI for any \( \alpha \in \Sigma^+ \) and any geodesic ray \( (\gamma_n)_{n \in \mathbb{N}} \) (Theorem 5.2.1), we first consider the special case where \( G = \text{GL}_d(\mathbb{R}) \) and \( P_\theta \) is the stabilizer of a line, i.e. \( G/P_\theta = \mathbb{P}^{d-1}(\mathbb{R}) \). The general case is treated in Section 5.4.2: we reduce to this special case using the results of Section 4.

5.4.1. Transversality in \( \text{GL}_d(\mathbb{R}) \). Let \( G = \text{GL}_d(\mathbb{R}) \). As in Example 2.12, we take \( K \) to be \( O(d) \) and \( \mathbb{P}^+ \) to be the set of real diagonal matrices of size \( d \times d \) with entries in nonincreasing order; for \( 1 \leq i \leq d \) we denote by \( \varepsilon_i \in a^* \) the evaluation of the \( i \)-th diagonal entry.

**Proposition 5.10.** Let \( G = \text{GL}_d(\mathbb{R}) \) and \( \theta = \{ \varepsilon_1 - \varepsilon_2 \} \). Let \( \Gamma \) be a word hyperbolic group and \( \rho : \Gamma \to G \) a representation. Suppose the maps \( \xi^+ \), \( \xi^- \) of Theorem 5.2.1 are well defined, continuous, \( \rho \)-equivariant, and dynamics-preserving. Let \( (\gamma_n)_{n \in \mathbb{N}} \) be a quasi-geodesic ray in the Cayley graph of \( \Gamma \), with endpoint \( \eta \in \partial_\infty \Gamma \). If the sequences \( (\varepsilon_1 - \varepsilon_i, \mu(\rho(\gamma_n)))_{n \in \mathbb{N}} \) are CLI (Definition 2.3) for all \( 2 \leq i \leq d \), then \( \xi^+(\eta) \notin \xi^-(\eta') \) for all \( \eta' \in \partial_\infty \Gamma \setminus \{\eta\} \).

Before proving Proposition 5.10, let us fix some notation in that setting. We see \( G/P_\theta \) as \( \mathbb{P}^{d-1}(\mathbb{R}) \) and \( G/P_\theta^- \) as the space of projective hyperplanes in \( \mathbb{P}^{d-1}(\mathbb{R}) \). Let \( (e_1, e_2, \ldots, e_d) \) be the canonical basis of \( \mathbb{R}^d \) and \( w_0 \) the longest element of the Weyl group \( W \simeq \mathbb{S}_d \) (see Example 2.15). The point \( P_\theta \in G/P_\theta \) corresponds to \( x_0 := [e_1] \in \mathbb{P}^{d-1}(\mathbb{R}) \). The points \( P_\theta^- \) and \( w_0 \cdot P_\theta^- \) of \( G/P_\theta^- \) correspond to the projective hyperplanes
\[ X_0 := \mathbb{P}(\text{span}(e_1, \ldots, e_{d-1})) \quad \text{and} \quad Y_0 := \mathbb{P}(\text{span}(e_2, \ldots, e_d)), \]
respectively. By construction, for any \( \eta \in \partial_\infty \Gamma \), the point
\[ \xi^+(\eta) = \lim_{n \to +\infty} k_n \cdot x_0 \in \mathbb{P}^{d-1}(\mathbb{R}) \]
belongs to the projective hyperplane
\[ \xi^- (\eta) = \lim_{n \to +\infty} k_n \cdot X_0 \subset \mathbb{P}^{d-1}(\mathbb{R}). \]
For any \( n \in \mathbb{N} \), we set \( g_n := \rho(\gamma_n) \) and \( a_n := \exp(\mu(g_n)) \) (this is a diagonal matrix with positive entries, in nonincreasing order), and choose elements...
\( k_n, k_n' \in K \) such that
\( g_n = k_n a_n k_n' \).

The crux of the proof of Proposition 5.10 will be to control the elements
\[
H_n^m := k_n^{-1} k_{n+m} \in K.
\]

More precisely, we shall establish the following two lemmas, from which Proposition 5.10 easily follows. We use the \( K \)-invariant metric \( d_{G/P_0} \) of Section 5.2 on \( G/P_0 = \mathbb{R}^{d-1} \); being defined as the sine of an angle, \( d_{G/P_0} \) is valued in \([0,1]\).

**Lemma 5.11.** In this setting, if there exists \( 0 < \delta \leq 1 \) such that for infinitely many \( n \in \mathbb{N} \),
\[
\limsup_{m \to +\infty} d_{G/P_0} (x_0, (a_n^{-1} H_n^m a_n) \cdot x_0) \leq 1 - \delta,
\]
then \( \xi^+ (\eta) \notin \xi^- (\eta') \) for all \( \eta' \in \partial_\infty \Gamma \setminus \{\eta\} \).

**Lemma 5.12.** If for every \( 2 \leq i \leq d \) the sequence \( (\varepsilon_i - \varepsilon_i, \mu(g_n))_{n \in \mathbb{N}} \) is CLI, then for every \( 2 \leq i \leq d \) the absolute value of the \((i,1)\)-th entry of the matrix \( a_n^{-1} H_n^m a_n \) is uniformly bounded for \((n,m) \in \mathbb{N}^2 \).

**Proof of Proposition 5.10 assuming Lemmas 5.11 and 5.12.** By Lemma 5.11, in order to prove that \( \xi^+ (\eta) \notin \xi^- (\eta') \) for all \( \eta' \in \partial_\infty \Gamma \setminus \{\eta\} \), it is enough to prove the existence of \( \delta > 0 \) such that (5.8) holds for infinitely many \( n \in \mathbb{N} \). To prove this, it is sufficient to see that

(i) the first column of the matrix \( a_n^{-1} H_n^m a_n \) is uniformly bounded,
(ii) the \((1,1)\)-th entry of the matrix \( a_n^{-1} H_n^m a_n \), which is also the \((1,1)\)-th entry of \( H_n^m \), is uniformly bounded from below for \((n,m) \in \mathbb{N}^2 \).

(Here all bounds are meant for the absolute values of the entries.) Suppose that the sequence \( (\varepsilon_1 - \varepsilon_i, \mu(g_n))_{n \in \mathbb{N}} \) is CLI for every \( 2 \leq i \leq d \). By Lemma 5.12, for every \( 2 \leq i \leq d \) the \((i,1)\)-th entry of \( a_n^{-1} H_n^m a_n \) is uniformly bounded for \((n,m) \in \mathbb{N}^2 \). The \((1,1)\)-th entry is also bounded since \( H_n^m \in K = O(d) \), hence (i) holds. On the other hand, Lemma 5.12 implies that for every \( 2 \leq i \leq d \) the \((i,1)\)-th entry of \( H_n^m \) is uniformly bounded by a constant multiple of \( e^{-\varepsilon_i - \varepsilon_i, \mu(g_n)} \), which goes to 0 as \( n \to +\infty \). Since \( H_n^m \in K = O(d) \), this in turn implies that the \((1,1)\)-th entry of \( H_n^m \) is close to 1 for all large enough \( n \) and all \( m \). Thus (ii) holds.

We now prove Lemmas 5.11 and 5.12.

**Proof of Lemma 5.11.** Fix \( \eta' \in \partial_\infty \Gamma \setminus \{\eta\} \). We will show that for infinitely many \( n \in \mathbb{N} \),
\[
d_{G/P_0} (g_n^{-1} \cdot \xi^+ (\eta), g_n^{-1} \cdot \xi^- (\eta')) \geq \frac{\delta}{2}.
\]

This inequality for only one \( n \) already shows that the point \( \xi^+ (\eta) \) does not belong to the hyperplane \( \xi^- (\eta') \).

Let \( (\gamma_n)_{n \in \mathbb{N}} \) be a subsequence of \( (\gamma_n)_{n \in \mathbb{N}} \) such that for any \( n \in \mathbb{N} \),
\[
\limsup_{m \to +\infty} d_{G/P_0} (x_0, a_{\phi(n)}^{-1} H_{\phi(n)}^m a_{\phi(n)} \cdot x_0) \leq 1 - \delta.
\]

Applying property (1) from Section 2.1.1, we may assume that the sequence \( (\gamma_n^{-1} t)_{n \in \mathbb{N}} \) converges to some point \( \eta'' \in \partial_\infty \Gamma \) and that \( \lim \gamma_n^{-1} \cdot t = \eta'' \) for all
\( t \in \partial_\infty \Gamma \setminus \{ \eta \} \). Let \( \tilde{\omega}_0 \in O(d) \) be the permutation matrix sending \((e_1, \ldots, e_d)\) to \((e_d, \ldots, e_1)\), representing the longest element \( w_0 \) of the Weyl group \( W = \mathfrak{S}_d \). For any \( n \), the decomposition \( g_{\phi(n)} = k_{\phi(n)} a_{\phi(n)} k'_{\phi(n)} \in K(\exp \mathfrak{a}^+) K \) induces the decomposition \( g_{\phi(n)}^{-1} = l_{\phi(n)} b_{\phi(n)} l'_{\phi(n)} \in K(\exp \mathfrak{a}^+) K \) with
\[
(l_{\phi(n)}, b_{\phi(n)}, l'_{\phi(n)}) = (k'_{\phi(n)}^{-1} w_0, w_0^{-1} a_{\phi(n)}^{-1} w_0, w_0^{-1} k_{\phi(n)}).
\]
In particular, by construction of \( \xi^- \) we have
\[
\xi^-(\eta'') = \lim_{n \to +\infty} l_{\phi(n)} \cdot X_0 = \lim_{n \to +\infty} k'_{\phi(n)}^{-1} \cdot Y_0.
\]
By \( \rho \)-equivariance and continuity of \( \xi^- \), we deduce
\[
\lim_n g_{\phi(n)}^{-1} \cdot \xi^-(\eta') = \xi^-(\lim_n \gamma_{\phi(n)}^{-1} \cdot \eta') = \xi^-(\eta'') = \lim_n k'_{\phi(n)}^{-1} \cdot Y_0.
\]
Thus the desired inequality (5.9) will hold if for any \( n \in \mathbb{N} \),
\[
d_{CP\theta} (g_{\phi(n)}^{-1} \cdot \xi^+(\eta), k'_{\phi(n)}^{-1} \cdot Y_0) \geq \delta,
\]
or equivalently
\[
d_{CP\theta} (a_{\phi(n)}^{-1} k'_{\phi(n)}^{-1} \cdot \xi^+(\eta), Y_0) \geq \delta
\]
(5.10) (using the fact that the metric \( d_{CP\theta} \) is \( K \)-invariant). By construction of \( \xi^+ \) and definition (5.7) of \( H_{\phi(n)}^m \), for any \( n \in \mathbb{N} \),
\[
k_{\phi(n)} H_{\phi(n)}^m \cdot x_0 = k_{\phi(n)+m} \cdot x_0 \xrightarrow{m \to +\infty} \xi^+(\eta).
\]
Therefore (5.10) is equivalent to
\[
\liminf_{m \to +\infty} d_{CP\theta} (a_{\phi(n)}^{-1} H_{\phi(n)}^m \cdot x_0, Y_0) \geq \delta.
\]
By definition of \( d_{CP\theta} \) (see Section 5.2), for any \( x \in \mathbb{P}^{d-1}(\mathbb{R}) \),
\[
d_{CP\theta}(x, Y_0)^2 + d_{CP\theta}(x, x_0)^2 = 1.
\]
Since \( x_0 = a_{\phi(n)} \cdot x_0 \), we see that (5.10) is equivalent to
\[
\limsup_{m \to +\infty} d_{CP\theta} (a_{\phi(n)}^{-1} H_{\phi(n)}^m a_{\phi(n)} \cdot x_0, x_0) \leq \sqrt{1 - \delta^2},
\]
which is satisfied by assumption and by the inequality \( 1 - \delta \leq \sqrt{1 - \delta^2} \) (valid as soon as \( \delta \leq 1 \)). \( \square \)

For any \( n \in \mathbb{N} \), let \( h_n := k_n^{-1} k_{n+1} \in K \), so that \( H_n = h_n h_{n+1} \cdots h_{n+m-1} \) for all \( n, m \in \mathbb{N} \). In order to establish Lemma 5.12, we first observe that we have a control on the entries of the matrices \( h_n \). For \( 1 \leq i, j \leq d \), the \((i, j)\)-th entry of a matrix \( h \in \text{GL}_d(\mathbb{R}) \) is denoted by \( h(i, j) \).

**Lemma 5.13.** There is a constant \( r \geq 0 \) such that for any \( 1 \leq i, j \leq d \) and any \( n \in \mathbb{N} \),
\[
|h_n(i, j)| \leq e^{r - |(\varepsilon_i - \varepsilon_j, \mu(g_n))|}.
\]

**Proof of Lemma 5.13.** Since \( (\gamma_n)_{n \in \mathbb{N}} \) is a geodesic ray in the Cayley graph of \( \Gamma \), the sequence \( \gamma_n^{-1} \gamma_{n+1} \) is bounded (it takes only finitely many values). Therefore the sequence \( \rho(\gamma_n^{-1} \gamma_{n+1}) = g_n^{-1} g_{n+1} = k_{n+1}^{-1} a_{n+1}^{-1} h_n a_{n+1} k_{n+1}' \) is bounded, as well as the sequence \( \mu(g_n) - \mu(g_{n+1}) = \log(a_{n+1}' a_n) \) by Fact 2.16. Thus the sequence \( a_{n+1}' a_n = (a_n^{-1} h_n a_{n+1}) (a_{n+1}' a_n) \) is bounded in \( \text{GL}_d(\mathbb{R}) \),
as well as its transposed inverse \( a_n h_n a_n^{-1} \). We can take for \( e^r \) a bound for the absolute values of the entries of these matrices \( (a_n^{\pm 1} h_n a_n^{\mp 1})_{n \in \mathbb{N}} \), using the fact that \( a_n = \exp(\mu(g_n)) \).

**Proof of Lemma 5.12.** The strategy is to write the entries of \( H_n^m \) in terms of the matrices \( (h_p)_{n \leq p < n+m} \) and to use Lemma 5.13 to bound the terms appearing. The good news is that most of those terms are bounded by \( e^{-\kappa m} \) and the bad news is that their number is exponential in \( m \). The work is to reorganize the sum in order to have a polynomial number of terms.

Let us first introduce some notation. For any integer \( 1 \leq i \leq d \) and any matrix \( h \in \text{GL}_d(\mathbb{R}) \), we denote by \( h[i] \in \text{GL}_{d-1}(\mathbb{R}) \) the matrix obtained by crossing out the \((i-1)\) topmost rows and the \((i-1)\) leftmost columns of \( h \). When \( h \) is orthogonal, \( h[i] \) is the matrix of a 1-Lipschitz linear transformation.

Similarly to \( H_n^m \), we define

\[
i H_n^m := h[i] h_{n+1}[i] \cdots h_{n+m-1}[i] \in \text{GL}_{d+1-i}(\mathbb{R}).
\]

Then \( i H_n^m \) is the matrix of a 1-Lipschitz transformation and its entries are bounded by 1 in absolute value. As above, the \((k, \ell)\)-th entry of a matrix \( h \in \text{GL}_d(\mathbb{R}) \) is denoted by \( h(k, \ell) \). To make things more natural, we index the entries of \( i H_n^m \) by pairs \((k, \ell) \in \{1, \ldots, d\}^2\).

Using this notation, we now compute the \((i, 1)\)-th entry of \( H_n^m \):

\[
H_n^m(i, 1) = \sum_u \mathcal{H}_u,
\]

where the sum is over all \((m+1)\)-tuples \( u = (u_0, u_1, \ldots, u_m) \in \{1, \ldots, d\}^{m+1} \) with \( u_0 = i \) and \( u_m = 1 \) and we set

\[
\mathcal{H}_u := h_n(u_0, u_1) h_{n+1}(u_1, u_2) \cdots h_{n+m-1}(u_{m-1}, u_m).
\]

This sum admits the following decomposition:

\begin{equation}
H_n^m(i, 1) = \sum_{s \in \{0, \ldots, i-1\}} \sum_{i, j, m} \mathcal{H}_{i, j, m},
\end{equation}

where, for fixed \( s \in \{0, \ldots, i-1\} \), the sum is over all \( i, j, m \) such that

- \( i = (i_0, i_1, \ldots, i_s) \in \mathbb{N}^{s+1} \) satisfies \( i_0 = i \) and \( i_0 > i_1 > \cdots > i_s = 1 \),
- \( j = (j_0, j_1, \ldots, j_{s-1}) \in \mathbb{N}^s \) satisfies \( j_k \geq i_k \) for all \( 0 \leq k \leq s-1 \),
- \( m = (m_0, m_1, \ldots, m_s) \in \mathbb{N}^{s+1} \) satisfies \( m_0 = 1 \leq m_1 < m_2 < \cdots < m_s \leq m \),

and the summand is:

\[
\mathcal{H}_{i, j, m} = i_0 H_{n-m_0}^{m_0-m_0}(i_0, j_0) h_{n+m_0-1}(j_0, i_1) \cdots i_s H_{n+m_s-1}^{m_s-m_s-1}(i_s, j_{s-1}) h_{n+m_{s-1}}(j_{s-1}, i_s) H_{n+m_{s-1}}^{m_{s-1}}(i_s, 1).
\]

Indeed, \( \mathcal{H}_{i, j, m} \) is the sum of the \( \mathcal{H}_u \) over all \( u = (u_0, u_1, \ldots, u_m) \in \{1, \ldots, d\}^{m+1} \) with \( u_0 = i \) and \( u_m = 1 \) such that:

- the smallest index \( k \) with \( u_k = u_0 = i \) is \( m_1 \),
- the smallest index \( k \) with \( u_k = u_{m_0} \) is \( m_2 \),
- \( \cdots \)
- the smallest index \( k \) with \( u_k = u_{m_{s-1}} \) is \( m_{s-1} \),
- the smallest index \( k \) with \( u_k = u_m = 1 \) is \( m_s \),
and such that for any \( k \in \{0, 1, \ldots, s\} \) we have \( u_m = i_k \) and \( u_{m+1} = j_k \).

From this we see that the subsums \( H_{i,j,m} \) for varying \( s \) and \( i,j,m \), form a partition of all the \( H_u \).

Since the entries of \( H_{i,j,m} \) are bounded by 1 in absolute value, we have
\[
|H_{i,j,m}| \leq |h_{n+m_1}(j_0,i_1)| |h_{n+m_2}(j_1,i_2)| \cdots |h_{n+m_s}(j_s,i_s)|.
\]
We take the convention that \( \log(0) = -\infty \).

Using the CLI assumption, we deduce
\[
\sum_k \langle \varepsilon_{i_k}, \mu(g_{n+m_k}) \rangle = \sum_k \langle \varepsilon_{j_k}, \mu(g_{n+m_k}) \rangle.
\]

Going back to the formula (5.11) for \( H_{n}^m(i_1,1) \), we obtain
\[
|a_n^{-1}H_n^m(a_n)(i,1)| = e^{\langle \varepsilon_{i_1}, \mu(g_{n+k}) \rangle} |H_n^m(i,1)|
\]
\[
\leq \sum_{s=0}^{i-1} e^{(r+k') + \kappa} \sum_{i,j,m} e^{-\kappa m_s}.
\]

Observe now that for a fixed integer \( q \) there is at most a polynomial number \( P(q) \) of possible \( i,j,m \) with \( m_s = q \), hence the above sum is bounded by a multiple of the converging series \( \sum_q P(q) e^{-2q} \). The real numbers \( (a_n^{-1}H_n^m a_n)(i,1) \) are therefore uniformly bounded for \( (n,m) \in \mathbb{N}^2 \).

5.4.2 Transversality in general. We now prove Theorem 5.2.(4) in full generality.

**Proposition 5.14.** Let \( \Gamma \) be a word hyperbolic group, \( G \) a reductive Lie group, \( \theta \subset \Delta \) a nonempty subset of the simple restricted roots of \( G \), and \( \rho : \Gamma \to G \) a representation. Suppose that the maps \( \xi^+, \xi^- \) of Theorem 5.2.(1) are well defined, continuous, \( \rho \)-equivariant, and dynamics-preserving. Let \( (\gamma_n)_{n \in \mathbb{N}} \) be a geodesic ray in the Cayley graph of \( \Gamma \), with endpoint \( \eta \in \partial_\Gamma \). If the sequences \((\alpha, \mu(\rho(\gamma_n)))_{n \in \mathbb{N}}\) for \( \alpha \in \Sigma^+ \) are CLI, then \( \xi^+(\eta') \) and \( \xi^-(\eta) \) are transverse for all \( \eta' \in \partial_\Gamma \setminus \{\eta\} \).

To prove Proposition 5.14 (and also later Proposition 5.17), we shall use the following lemma.
Lemma 5.15. For $1 \leq i \leq D$, let $(x_n^{(i)})_{n \in \mathbb{N}}$ and $(y_n^{(i)})_{n \in \mathbb{N}}$ be sequences of real numbers. Suppose that the $D$ sequences $(x_n^{(i)})_{n \in \mathbb{N}}$, for $1 \leq i \leq D$, are all $(\kappa, \kappa')$-lower CLI, and that for any $n \in \mathbb{N}$ there exists $\sigma_n \in \mathcal{G}_D$ such that $y_n^{(i)} = x_n^{(\sigma_n^{(i)})}$ for all $1 \leq i \leq D$. Suppose also that we are in one of the following two cases:

\begin{itemize}
  \item[(*)] $y_n^{(1)} \geq \cdots \geq y_n^{(D)}$ for all $n \in \mathbb{N}$,
  \item[(**)] there exists $M > 0$ such that $|y_{n+1}^{(i)} - y_n^{(i)}| \leq M$ for all $n \in \mathbb{N}$ and $1 \leq i \leq D$.
\end{itemize}

Then the $D$ sequences $(y_n^{(i)})_{n \in \mathbb{N}}$, for $1 \leq i \leq D$, are all $(\kappa, \kappa')$-lower CLI, where we set

$$\kappa' := \begin{cases} 
  \kappa' & \text{in case (*)}, \\
  \kappa' + D(\kappa + \kappa' + M) & \text{in case (**}).
\end{cases}$$

Remark 5.16. An analogous statement holds for the upper CLI property of sequences $(x_n^{(i)})_{n \in \mathbb{N}}$ and $(y_n^{(i)})_{n \in \mathbb{N}}$: this follows from Lemma 5.15 applied to the sequences $(R_n - x_n^{(i)})$ and $(R_n - y_n^{(i)})$, where $R > \kappa$ is any large enough positive real.

Proof of Lemma 5.15. Suppose we are in case (*). Then for any $n \in \mathbb{N}$ and $1 \leq i \leq D$ the number $y_n^{(i)}$ can be defined as the largest real number $y$ such that at least $i$ of the $D$ numbers $x_n^{(1)}, \ldots, x_n^{(D)}$ are $\geq y$. By the CLI hypothesis, for any $m \geq 0$, at least $i$ of the $D$ numbers $x_{n+m}^{(1)}, \ldots, x_{n+m}^{(D)}$ are $\geq y + \kappa m - \kappa'$, and so $y_{n+m}^{(i)} \geq y + \kappa m - \kappa'$. This proves that $(y_n^{(i)})_{n \in \mathbb{N}}$ is lower CLI with constants $(\kappa, \kappa')$.

We now treat case (**). By case (*), up to reordering the $x_n^{(i)}$ for each $n \in \mathbb{N}$, we may assume that $x_n^{(1)} \geq \cdots \geq x_n^{(D)}$ for all $n \in \mathbb{N}$. Fix an integer $1 \leq i \leq D$ and an integer $n \in \mathbb{N}$, and focus on the sequence $(y_n^{(i)})_{m \geq 0}$. For any $m \geq 0$, let $r_m := \sigma_{n+m}(i) \in [1, D]$, so that $y_{n+m}^{(i)} = x_{n+m}^{(r_m)}$ is the $r_m$-th largest number in the family $(y_n^{(1)}, \ldots, y_n^{(D)})$. There exist an integer $s \leq D$ and a finite maximal list $0 = m_0 < m_1 < \cdots < m_{s+1} = +\infty$ such that

$$r_m < r_{m_1} < \cdots < r_{m_s} \leq D \quad \text{and} \quad r_m \leq r_{m_j} \quad \text{for all} \quad m < m_{j+1}.$$ 

For any $m_j \leq m < m_{j+1}$,

$$y_{n+m}^{(i)} - y_{n+m_j}^{(i)} = x_{n+m}^{(r_m)} - x_{n+m_j}^{(r_{m_j})} \geq x_{n+m}^{(r_{m_j})} - x_{n+m_j}^{(r_{m_j})} \geq \kappa(m - m_j) - \kappa',$$

where the first inequality comes from the assumption $x_{n+m}^{(1)} \geq \cdots \geq x_{n+m}^{(D)}$ and the second inequality from the CLI hypothesis. Then for any $m_j \leq m <
m_{j+1}, using the CLI hypothesis again as well as (**), we can bound

\[ y^{(i)}_{n+m} - y^{(i)}_n = y^{(i)}_{n+m} - y^{(i)}_{n+m_j} + \sum_{k=1}^{j} \left( (y^{(i)}_{n+m_k} - y^{(i)}_{n+m_{k-1}}) + (y^{(i)}_{n+m_{k-1}} - y^{(i)}_{n+m_{k-1}}) \right) \]

\[ \geq \kappa(m - m_j) - \kappa' + \sum_{k=1}^{j} \left( -M + \kappa(k - 1 - m_{k-1}) - \kappa' \right) \]

\[ = \kappa m - j(M + \kappa) - (j + 1)\kappa'. \]

Since \( j < s \leq D \), this produces the desired bounds. \( \square \)

**Proof of Proposition 5.14.** Let \((\tau, V)\) be a finite-dimensional, irreducible, proximal, \(\theta\)-compatible representation of \(G\) (Lemma 4.5). By Proposition 4.6, it induces embeddings \(\iota^+: G/P_0 \rightarrow \mathbf{P}(V)\) and \(\iota^-: G/P_0 \rightarrow \mathbf{P}(V^*)\). We identify \(\text{GL}(V)\) with \(\text{GL}_d(\mathbf{R})\) for some \(d \in \mathbf{N}\), and denote by \(\mathfrak{a}_{\text{GL}_d}\) the set of diagonal matrices in \(\mathfrak{gl}_d(\mathbf{R})\) and \(\pi^{\mathfrak{gl}_d}_{\text{GL}_d}\) its subset with entries in non-increasing order. Up to conjugating \(\tau\), we may assume that the Cartan decomposition \(G = K(\exp \mathfrak{a}^+)K\) of \(G\) is compatible with the Cartan decomposition \(\text{GL}_d(\mathbf{R}) = O(d)(\exp \pi^{\mathfrak{gl}_d}_{\text{GL}_d})O(d)\) of \(\text{GL}_d(\mathbf{R})\) (Example 2.12), in the sense that \(\tau(K) \subset O(d)\) and \(d_{\iota}(\mathfrak{a}) \subset \mathfrak{a}_{\text{GL}_d}\). We distinguish the Cartan projections \(\mu: G \rightarrow \mathfrak{a}^+\) and \(\mu_{\text{GL}_d}: \text{GL}_d(\mathbf{R}) \rightarrow \pi^{\mathfrak{gl}_d}_{\text{GL}_d}\) corresponding to these two decompositions. For any \(g \in G\), the matrix \(d_{\iota}\tau(\mu(g))\) is diagonal with entries \(\langle \chi, \mu(g) \rangle\), where \(\chi \in \mathfrak{a}^*\) ranges through the weights of \(\tau\). The matrix \(\mu_{\text{GL}_d}(\tau(g))\) is the diagonal matrix with the previous entries ordered: \(\langle \varepsilon_1, \mu_{\text{GL}_d}(\tau(g)) \rangle = \langle \chi_{\tau}, \mu(g) \rangle\), and for any \(2 \leq i \leq d\) there is a weight \(\chi \neq \chi_{\tau}\) (depending on \(g\) and \(i\)) such that \(\langle \varepsilon_i, \mu_{\text{GL}_d}(\tau(g)) \rangle = \langle \chi, \mu(g) \rangle\).

By Lemma 4.10, for any weight \(\chi \neq \chi_{\tau}\) of \(\tau\), we can write \(\chi_{\tau} - \chi = \sum_{\alpha \in \Sigma_{\theta}^+} m_{\alpha} \alpha\) where \(m_{\alpha} \geq 0\) for all \(\alpha\). Using that sums of CLI sequences are again CLI and applying case (*) of Lemma 5.15 to the sequences \(x^\chi = (\langle \chi_{\tau} - \chi, \mu(\rho(\gamma_n)) \rangle)_{n \in \mathbf{N}}\) for \(\chi\) ranging through the set of weights of \(\tau\) different from \(\chi_{\tau}\) and

\[ y^{(i)} = (\langle \varepsilon_1 - \varepsilon_i, \mu_{\text{GL}_d}(\tau \circ \rho(\gamma_n)) \rangle)_{n \in \mathbf{N}} \]

for \(i\) ranging through \(\{2, \ldots, d\}\), we see that if the sequences \(\langle \alpha, \mu(\rho(\gamma_n)) \rangle_{n \in \mathbf{N}}\) for \(\alpha \in \Sigma_{\theta}^+\) are CLI, then so are the sequences \(\langle \varepsilon_1 - \varepsilon_i, \mu_{\text{GL}_d}(\rho(\gamma_n)) \rangle_{n \in \mathbf{N}}\) for all \(2 \leq i \leq d\). By Proposition 5.10, the maps \(\iota^+ \circ \xi^+ : \partial_{\infty} \Gamma \rightarrow \mathbf{P}(V)\) and \(\iota^- \circ \xi^- : \partial_{\infty} \Gamma \rightarrow \mathbf{P}(V^*)\) are transverse. Therefore, \(\xi^+\) and \(\xi^-\) are transverse as well by Proposition 4.6. \( \square \)

This concludes the proof of Theorem 1.1, and hence also of the implication (4) \(\Rightarrow\) (1) of Theorem 1.3.

**5.5. Contraction properties for Anosov representations.** We now establish the implication (1) \(\Rightarrow\) (4) in Theorem 1.3.

**Proposition 5.17.** Let \(\Gamma\) be a word hyperbolic group, \(G\) a real reductive Lie group, \(\theta \subset \Delta\) a nonempty subset of the simple restricted roots of \(G\). Let \(\rho: \Gamma \rightarrow G\) be a \(P_0\)-Anosov representation. Then there exist \(\kappa, \kappa' > 0\) such that for any \(\alpha \in \Sigma_{\theta}^+\) and any geodesic ray \((\gamma_n)_{n \in \mathbf{N}}\) with \(\gamma_0 = e\) in the Cayley graph of \(\Gamma\), the sequence \((\langle \alpha, \mu(\rho(\gamma_n)) \rangle)_{n \in \mathbf{N}}\) is \((\kappa, \kappa')\)-lower CLI.
To prove Proposition 5.17, we shall use the following lemma.

**Lemma 5.18.** If an ellipsoid $B$ is contained in an ellipsoid $B'$ in $\mathbb{R}^d$, then for any $1 \leq i \leq d$, the $i$-th principal axis of $B$ is at most as long as the $i$-th principal axis of $B'$.

**Proof.** It suffices to show that the length $r_i$ of the $i$-th largest axis of $B$ is the diameter of the largest possible copy of the Euclidean ball of an $i$-dimensional subspace (or $i$-ball) of $\mathbb{R}^d$ that fits into $B$. Clearly, an $i$-ball of diameter $r_i$ fits into $B$, for instance inside the $i$-dimensional vector space $V_i$ spanned by the $i$ largest axes. Since $V_i^{\perp}$ intersects $B$ along a $(d - i + 1)$-dimensional ellipsoid whose axes are all of length $\leq r_i$, it is also clear that no larger $i$-ball can fit into $B$, since it would have a diameter in $V_i^{\perp}$ by a simple dimension count. \hfill $\square$

**Proof of Proposition 5.17.** By Fact 2.28, we may assume without loss of generality that $\theta = \theta^*$, so that $\Sigma^+_\theta = (\Sigma^+_\theta)^*$. Let $\tilde{\sigma} : \mathcal{G}_\Gamma \to G/L_\theta$ be the section associated with the $P_{\theta}$-Anosov representation $\rho$. As in Section 3.1, we choose a $\rho$-equivariant continuous lift $\tilde{\beta} : \mathcal{G}_\Gamma \to G/K_\theta$ of $\tilde{\sigma}$ and a $\rho$-equivariant set-theoretic lift $\tilde{\beta} : \mathcal{G}_\Gamma \to G$ of $\tilde{\beta}$. For any $(t, v) \in \mathbb{R} \times \mathcal{G}_\Gamma$ there exists $l_{t,v} \in L_\theta$ such that

$$\tilde{\beta}(\varphi_t \cdot v) = \tilde{\beta}(v) l_{t,v}.$$ 

By Proposition 2.4, and Proposition 3.8, there are a compact subset $D$ of $\mathcal{G}_\Gamma$ and constants $K, n_0, \kappa_0, \kappa'_0 > 0$ with the following property: for any geodesic ray $(\gamma_n)_{n \in \mathbb{N}}$ with $\gamma_0 = e$ in the Cayley graph of $\Gamma$, there exist $v \in D$ and a $(\kappa_0, \kappa'_0)$-lower CLI sequence $(t_n) \in \mathbb{R}^N$ such that for all $n \geq n_0$,

$$\| \mu(\rho(\gamma_n)) - \mu_\theta(l_{t_n,v}) \| \leq K.$$

Therefore it is enough to prove the existence of $\kappa, \kappa' > 0$ such that for any $\alpha \in \Sigma^+_\theta$, any $t, s \in \mathbb{R}$, and any $v \in D$,

$$(5.13) \quad \langle \alpha, \mu_\theta(l_{t+s,v}) - \mu_\theta(l_{t,v}) \rangle \geq \kappa s - \kappa'.$$

In fact, we only need to establish (5.13) for $t, s \in \mathbb{N}$. Indeed, $l_{t,v}$ is uniformly close to $\{ l_{t,v} \}$ for $t \in \mathbb{R}$ and $v \in D$, by the cocycle property (3.2) of the map $(t, v) \mapsto l_{t,v}$, the $\Gamma$-invariance property (3.1), and the fact that the set $\{ l_{s,v} \mid s \in [0, 1], v \in D \}$ is bounded in $G$ since $\{ \tilde{\beta}(\varphi_s \cdot v) \mid s \in [0, 1], v \in D \}$ is by Lemma 3.4.

Recall the subbundle $E^+_{\theta}$ of $T(G/L_\theta)$ from (2.4). Since $\tilde{\beta} : \mathcal{G}_\Gamma \to G$ is a lift of $\tilde{\sigma} : \mathcal{G}_\Gamma \to G/L_\theta$, we have $E^+_{\tilde{\beta}(v)} = \tilde{\beta}(v) \cdot T_{\beta_0}(G/P_\beta)$ for all $v \in \mathcal{G}_\Gamma$, where we write $\tilde{\beta}(v)$ for the derivative of the action of $\tilde{\beta}(v)$ by left translation. Fix a $K_\theta$-invariant Euclidean norm $\| \cdot \|_0$ on $T_{\beta_0}(G/P_\beta) \simeq u^-_\beta$. For any $v \in \mathcal{G}_\Gamma$,

$$(5.14) \quad w \mapsto \| w \|_v := \| \tilde{\beta}(v)^{-1} \cdot w \|_0$$

defines a Euclidean norm on $E^+_{\tilde{\beta}(v)}$. The family $(\| \cdot \|_v)_{v \in \mathcal{G}_\Gamma}$ is continuous and $\rho$-equivariant. By Definition 2.23.(ii) of a $P_{\theta}$-Anosov representation, there exist $\kappa, \kappa' > 0$ such that

$$\| w \|_v \leq e^ {-\kappa s + \kappa'} \| w \|_{\varphi_s v}.$$
for all $v \in \mathcal{G}_\Gamma$ and all $w \in E^+_{\sigma(v)}$. In other words, for any $v \in \mathcal{G}_\Gamma$, the unit ball $B_{t+s,v}$ of $\| \cdot \|_{\varphi,v}$ in $E^+_{\sigma(v)}$ is contained in $e^{-\kappa s + \kappa'}$ times the unit ball $B_{t,v}$ of $\| \cdot \|_{\varphi,v}$, for all $t, s \geq 0$. We now apply Lemma 5.18 to $E^+_{\sigma(v)}$ endowed with the Euclidean norm $\| \cdot \|$: for any $1 \leq i \leq \dim G/P_\theta$, the length of the $i$-th principal axis of $B_{t+s,v}$ is at most $e^{-\kappa s + \kappa'}$ times that of the $i$-th principal axis of $B_{t,v}$. By construction, the lengths of the principal axes of $B_{t,v}$ are the $e^{-\langle \alpha, \mu_\sigma(l_{i,v}) \rangle}$ for $\alpha$ ranging through $\Sigma^+_{\theta}$, or in other words the $e^{-\langle \alpha, \mu_\sigma(l_{i,v}) \rangle}$ for $\alpha$ ranging through $(\Sigma^+_{\theta})^* = \Sigma^+_{\theta}$. Since $\|\mu_\theta(l_{i+1,v}) - \mu_\theta(l_{i,v})\|$ is bounded, we may apply case (**) of Lemma 5.15: there exists $\tilde{r}' > 0$ such that for any $\alpha \in \Sigma^+_{\theta}$, any $t, s \in \mathbb{N}$, and any $v \in \mathcal{D}$,

$$\langle \alpha, \mu_\theta(l_{i+1,v}) - \mu_\theta(l_{i,v}) \rangle \geq \kappa s - \tilde{r}'$$

Thus (5.13) holds, which completes the proof. \hfill \Box

5.6. Proper actions of rank-1 reductive groups yield Anosov representations. In this section we prove Corollary 1.11.

Let $G$ be a real reductive Lie group, $\theta \subset \Delta$ a nonempty subset of the simple roots of $\mathfrak{a}$ in $G$, and $H$ a closed subgroup of $G$ such that $\mu(H) \supset (\bigcup_{\alpha \in \theta} \Ker \alpha) \cap \mathfrak{a}^\perp$. Let $G_1$ be a reductive subgroup of $G$, of real rank 1, acting properly on $G/H$. Up to conjugation (which only modifies $\mu$ by a compact set, see Fact 2.16), we may assume that $G_1$ admits the Cartan decomposition

$$G_1 = (K \cap G_1)(\exp(\mathfrak{a}) \cap G_1)(K \cap G_1).$$

The set $\mu(G_1) = \mu(\exp(\mathfrak{a}) \cap G_1) = \mathfrak{a}^\perp \cap W \cdot (\mathfrak{a} \cap G_1)$ is then a union of two (possibly equal) rays $\mathcal{L}_1, \mathcal{L}_2$ starting at 0. Since $\mu(H) \supset (\bigcup_{\alpha \in \theta} \Ker \alpha) \cap \mathfrak{a}^\perp$, the properness criterion of Benoist and Kobayashi (see also the earlier paper [Kob89]) implies that for any $i \in \{1, 2\}$ and $\alpha \in \theta$ we have $\mathcal{L}_i \cap \Ker \alpha = \{0\}$; thus there is a constant $C_{\alpha,i} > 0$ such that $\langle \alpha, Y \rangle = C_{\alpha,i} \| Y \|$ for all $Y \in \mathcal{L}_i$. In particular, any discrete subgroup $\Gamma$ of $G_1$ acts sharply on $G/H$. In fact, such a constant $C_{\alpha,i} > 0$ exists for any $\alpha \in \Sigma^+_{\theta}$, since $\mathcal{L}_i \cap \Ker \alpha = \{0\}$.

Let $\Gamma$ be a convex cocompact subgroup of $G_1$. Let us prove that the natural inclusion of $\Gamma$ in $G$ is $P_\theta$-Anosov.

We first assume that $G_1$ is semisimple of real rank 1, so that there is only one ray $\mathcal{L}_1 = \mathcal{L}_2$. Let $\mu_{G_1}$ be a Cartan projection for $G_1$, associated with the decomposition (5.15). Since $G_1$ has real rank 1, we may see $\mu_{G_1}$ as a map from $G_1$ to $\mathbb{R}_+$, and there is a constant $C > 0$ such that $\|\mu(g)\| = C \mu_{G_1}(g)$ for all $g \in G_1$. Consider a geodesic ray $(\gamma_n)_{n \in \mathbb{N}}$ in the Cayley graph of $\Gamma$. For all $\alpha \in \Sigma^+_{\theta}$ and all $n \in \mathbb{N}$,

$$\langle \alpha, \mu(\gamma_n) \rangle = C_{\alpha,1} C \mu_{G_1}(\gamma_n).$$

Since $\Gamma$ is convex cocompact in $G_1$, its inclusion in $G_1$ is Anosov with respect to a minimal parabolic subgroup of $G_1$, and so the sequence $(\mu_{G_1}(\gamma_n))_{n \in \mathbb{N}}$ is CL1 by Theorem 1.3 (or Corollary 2.18). By (5.16), the sequence $((\alpha, \mu(\gamma_n)))_{n \in \mathbb{N}}$ is CL1 as well for $\alpha \in \Sigma^+_{\theta}$, and so the inclusion of $\Gamma$ in $G$ is $P_\theta$-Anosov by Theorem 1.3.

We now assume that $G_1$ is not semisimple. It is then a central extension of the compact group $K \cap G_1$ by the one-parameter group $\exp(\mathfrak{a}) \cap G_1$, and
Γ is virtually the cyclic group generated by some element γ in G1 \ K. Let μG1 be a Cartan projection for G1, associated with the decomposition (5.15). We may see μG1 as a map from G1 to \( \mathbb{R} \), and there is a constant C > 0 such that \( ||\mu(\gamma^n)|| = C |n| ||\mu_G(\gamma)|| \) for all \( n \in \mathbb{Z} \). For any \( \alpha \in \Sigma^0_+ \), the estimates \( \langle \alpha, Y \rangle = C_{\alpha,i} |Y||_Y \) for \( Y \in \mathcal{L}_1 \) imply that the sequences \((\langle \alpha, \mu(a^n) \rangle)_{n \in \mathbb{N}} \) and \((\langle \alpha, \mu(a^{-n}) \rangle)_{n \in \mathbb{N}} \) are CLI, and so the inclusion of Γ in G is \( P_\theta \)-Anosov by Theorem 1.3.

6. PROPER ACTIONS ON GROUP MANIFOLDS

In view of the properness criterion of Benoist and Kobayashi stated in Section 1.3, our characterizations of Anosov representations \( \rho : \Gamma \to G \) in terms of the Cartan projection \( \mu \) (Theorem 1.3) provide a direct link with the properness of the action of \( \Gamma \) (via \( \rho \)) on homogeneous spaces of \( G \).

In this section, we consider properly discontinuous actions on group manifolds and deduce Theorems 1.13 and 1.15 and Corollaries 1.16 and 1.17 from Theorems 1.3 and 3.2.

6.1. Proper actions on group manifolds, uniform domination, and Anosov representations. Before stating our main Theorem 6.3, we introduce some useful notation and terminology.

6.1.1. Uniform \( P \)-domination. Let \( \Gamma \) be a discrete group, \( G \) a real reductive Lie group, and \( \theta \subset \Delta \) a nonempty subset of the simple restricted roots of \( a \) in \( G \). Recall that \( \omega_\alpha \) denotes the corresponding fundamental weight (4.1) of a simple root \( \alpha \in \Delta \). We adopt the following terminology.

**Definition 6.1.** A representation \( \rho_L \in \text{Hom}(\Gamma, G) \) uniformly \( P_\theta \)-dominates a representation \( \rho_R \in \text{Hom}(\Gamma, G) \) if there exists \( c < 1 \) such that for all \( \alpha \in \theta \) and \( \gamma \in \Gamma \),

\[
\langle \omega_\alpha, \lambda(\rho_R(\gamma)) \rangle \leq c \langle \omega_\alpha, \lambda(\rho_L(\gamma)) \rangle.
\]

If \( G \) has real rank 1, then \( \theta = \Delta \) is a singleton and we say simply that \( \rho_L \) uniformly dominates \( \rho_R \).

Note that the notion of uniform \( P \)-domination depends only on the conjugacy class of the parabolic subgroup \( P \). Uniform \( P_\theta \)-domination is equivalent to \( P_{\theta,\rho_R} \)-domination by (2.10).

We shall use uniform \( P \)-domination both in \( G \) and in the following setting.

6.1.2. Automorphism groups of bilinear forms. Let \( K \) be \( \mathbb{R} \), \( \mathbb{C} \), or the ring \( \mathbb{H} \) of quaternions. Let \( V \) be a \( K \)-vector space (where \( K \) acts on the right in the case of \( \mathbb{H} \)), and let \( b : V \otimes_{\mathbb{R}} V \to K \) be an \( R \)-bilinear form which is symmetric or antisymmetric (if \( K = \mathbb{R} \) or \( \mathbb{C} \)), or Hermitian or anti-Hermitian (if \( K = \mathbb{C} \) or \( \mathbb{H} \)). Let \( \text{Aut}_K(b) \) be the subgroup of \( \text{GL}_K(V) \) preserving \( b \). Table 4 gives the list of all possible examples.

We denote by \( Q_0(b) \subset \text{Aut}_K(b) \) the stabilizer of a \( b \)-isotropic line in \( V \). It is a maximal proper parabolic subgroup, and \( \mathcal{F}_0(b) := \text{Aut}_K(b)/Q_0(b) \) identifies with the set of \( b \)-isotropic \( K \)-lines of \( V \) inside the projective space \( \mathbb{P}_K(V) = (V \setminus \{0\})/K^* \). As above we say that a representation \( \tau : G \to \text{Aut}_K(b) \) is proximal if some element of \( \tau(G) \) has an attracting fixed point in \( \mathbb{P}_K(V) \).
6.1.3. A useful normalization. We shall use the following normalization to avoid having to switch $\rho_L$ and $\rho_R$ in Theorem 6.3.

Remark 6.2. Let $\tau : G \to \text{Aut}_K(b)$ be an irreducible representation of $G$ with highest weight $\chi$. For any $\rho_L, \rho_R \in \text{Hom}(\Gamma, G)$, the following always holds up to switching $\rho_L$ and $\rho_R$:

$$\sup_{\gamma \in \Gamma} \langle \chi, \lambda(\rho_L(\gamma)) - \lambda(\rho_R(\gamma)) \rangle \geq 0.$$ (6.1)

If $G$ is semisimple of real rank 1, then $\theta = \Delta$ is a singleton $\{\alpha\}$ and (assuming $\tau$ to be nonzero) the inequality (6.1) is equivalent to

$$\sup_{\gamma \in \Gamma} \langle \omega, \lambda(\rho_L(\gamma)) - \lambda(\rho_R(\gamma)) \rangle \geq 0.$$

For $G$ of arbitrary real rank, (6.1) always holds when $\rho_L$ uniformly $P_0$-dominates $\rho_R$ and $\tau$ is $\theta$-compatible (Definition 4.4).

6.1.4. The main theorem. Theorems 1.13 and 1.15 and Corollary 1.16 are immediate consequences of the following result (which holds for $G$ of arbitrary real rank) and of the fact from [Kas08] (see Theorem 6.12 below) that if $G$ has real rank 1, then any finitely generated quasi-isometrically embedded subgroup of $G \times G$ acting properly discontinuously on $(G \times G)/\text{Diag}(G)$ is word hyperbolic and has one quasi-isometric projection to $G$.

**Theorem 6.3.** Let $\Gamma$ be a discrete group, $G$ a reductive Lie group, and $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of $G$, with $\theta = \theta^*$. For $K = R, C,$ or $H$, let $V$ be a $K$-vector space and $\tau : G \to \text{GL}_K(V)$ an irreducible, $\theta$-proximal representation preserving an $R$-bilinear form $b : V \otimes_R V \to K$ which is symmetric, antisymmetric, Hermitian, or anti-Hermitian over $K$. Let $b'$ be a nonzero real multiple of $b$. For a pair $(\rho_L, \rho_R) \in \text{Hom}(\Gamma, G)^2$ with the normalization (6.1) (see Remark 6.2), consider the following conditions:

1. $\Gamma$ is word hyperbolic and $\rho_L$ is $P_0$-Anosov and uniformly $P_0$-dominates $\rho_R$;
2. $\Gamma$ is word hyperbolic, $\rho_L$ is $P_0$-Anosov, and $\tau \circ \rho_L : \Gamma \to \text{Aut}_K(b)$ uniformly $Q_0(b)$-dominates $\tau \circ \rho_R$;
3. $\Gamma$ is word hyperbolic and $\tau \circ \rho_L : \Gamma \to \text{Aut}_K(b)$ is $Q_0(b)$-Anosov and uniformly $Q_0(b)$-dominates $\tau \circ \rho_R$;
4. $\Gamma$ is word hyperbolic and $\tau \circ \rho_L \oplus \tau \circ \rho_R : \Gamma \to \text{Aut}_K(b \oplus b')$ is $Q_0(b \oplus b')$-Anosov;

| $\text{Aut}_K(b)$ | $K$ | $\text{dim}_K(V)$ | Description of $b$ |
|------------------|-----|-----------------|--------------------|
| $O(p,q)$         | $R$ | $p + q$         | symmetric          |
| $U(p,q)$         | $C$ | $p + q$         | Hermitian          |
| $\text{Sp}(p,q)$ | $H$ | $p + q$         | Hermitian          |
| $O(d,C)$         | $C$ | $d$            | symmetric          |
| $\text{Sp}(2d,R)$ | $R$ | $2d$            | antisymmetric      |
| $\text{Sp}(2d,C)$ | $C$ | $2d$            | antisymmetric      |
| $O^*(2d)$        | $H$ | $2d$            | anti-Hermitian     |

Table 4. In these examples, $p, q, d$ are any integers $\geq 1$. 

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(5) \( \rho_L \) is a quasi-isometric embedding and the action of \( \Gamma \) on \( (G \times G)/\text{Diag}(G) \) via \((\rho_L, \rho_R)\) is sharp;
(6) \( \rho_L \) is a quasi-isometric embedding and the action of \( \Gamma \) on \( (G \times G)/\text{Diag}(G) \) via \((\rho_L, \rho_R)\) is properly discontinuous;
(7) \((\rho_L, \rho_R) : \Gamma \to G \times G\) is a quasi-isometric embedding and the action of \( \Gamma \) on \( (G \times G)/\text{Diag}(G) \) via \((\rho_L, \rho_R)\) is properly discontinuous.

The following implications always hold:

\[(1) \implies (2) \iff (3) \iff (4) \implies (5) \implies (6) \implies (7)\]

If \( \theta \) is a singleton, then \((2) \implies (1)\) holds as well. If \( G \) has real rank 1, then all conditions are equivalent.

We note that an irreducible, \( \theta \)-proximal representation \( \tau : G \to \text{Aut}_K(b) \) always exists when \( \theta = \theta^* \): see Proposition 6.7 below. The condition \( \theta = \theta^* \) is not restrictive: see Fact 2.28.

In Theorem 6.3 there are essentially two choices for \( b' \): either \( b' = b \) or \( b' = -b \). The groups \( \text{Aut}_K(b \oplus b) \) and \( \text{Aut}_K(b \oplus (-b)) \) both sit inside \( \text{GL}_K(V \oplus V) \) and their intersection is \( \text{Aut}_K(b) \times \text{Aut}_K(b) \). These groups are not isomorphic in general: see Example 6.4. Even when they are isomorphic, the corresponding two embeddings of \( \Gamma \) via \((\rho_L, \rho_R)\) tend to be of a quite different nature: this is the case for instance when \( b \) is a real symplectic form.

Example 6.4. For \( G = \text{SO}(1,d) \), we may take \( \text{Aut}_K(b) \) to be \( G \) and \( \tau : G \to \text{Aut}_K(b) \) to be the identity map, with \( \theta \) consisting of the unique simple restricted root of \( G \). Theorem 6.3 then states the equivalence of the following conditions, for a discrete group \( \Gamma \) and a pair \((\rho_L, \rho_R) \in \text{Hom}(\Gamma, \text{SO}(1,d))^2\) of representations:

\begin{enumerate}
  \item \( \Gamma \) is word hyperbolic and one of the representations \( \rho_L \) or \( \rho_R \) is convex cocompact and uniformly dominates the other;
  \item \((\rho_L, \rho_R) : \Gamma \to \text{SO}(2,2d) \) is Anosov with respect to the stabilizer of an isotropic line in \( \mathbb{R}^{2,2d} \);
  \item \( \Gamma \) is word hyperbolic and \( \rho_L \oplus \rho_R : \Gamma \to \text{SO}(d+1,d+1) \) is Anosov with respect to the stabilizer of an isotropic line in \( \mathbb{R}^{d+1,d+1} \);
  \item one of the representations \( \rho_L \) or \( \rho_R \) is a quasi-isometric embedding and the action of \( \Gamma \) on \((\text{SO}(1,d) \times \text{SO}(1,d))/\text{Diag}(\text{SO}(1,d))\) via \((\rho_L, \rho_R)\) is sharp;
  \item one of the representations \( \rho_L \) or \( \rho_R \) is a quasi-isometric embedding and the action of \( \Gamma \) on \((\text{SO}(1,d) \times \text{SO}(1,d))/\text{Diag}(\text{SO}(1,d))\) via \((\rho_L, \rho_R)\) is properly discontinuous;
  \item \((\rho_L, \rho_R) : \Gamma \to \text{SO}(1,d) \times \text{SO}(1,d) \) is a quasi-isometric embedding and the action of \( \Gamma \) on \((\text{SO}(1,d) \times \text{SO}(1,d))/\text{Diag}(\text{SO}(1,d))\) via \((\rho_L, \rho_R)\) is properly discontinuous.
\end{enumerate}

Here we see \( \text{SO}(2,2d) \) (resp. \( \text{SO}(d+1,d+1) \)) as the stabilizer in \( \text{SL}_{2d+2}(\mathbb{R}) \) of the quadratic form \( x_0^2 - x_1^2 - \cdots - x_d^2 + y_0^2 - y_1^2 - \cdots - y_d^2 \) (resp. \( x_0^2 - x_1^2 - \cdots - x_d^2 - y_0^2 + y_1^2 + \cdots + y_d^2 \)). Similar equivalences are true after replacing \( (\text{SO}(1,d), \text{SO}(2,2d), \text{SO}(d+1,d+1), \mathbb{R}^{2d+2}) \) with \((\text{SU}(1,d), \text{SU}(2,2d), \text{SU}(d+1,d+1), \mathbb{C}^{2d+2})\) or with \((\text{Sp}(1,d), \text{Sp}(2,2d), \text{Sp}(d+1,d+1), \mathbb{H}^{2d+2})\), or after taking compact extensions of these groups.
We refer to [Gol85, Ghy95, Kob98, Sal00, GK, GKW, DT,Tho, DGK] for examples of discrete subgroups of \( SO(1,d) \times SO(1,d) \) satisfying the equivalent conditions of Example 6.4.

Remark 6.5. When \( G \) has higher real rank, the implication (5) \( \Rightarrow \) (4) of Theorem 6.3 is false. Indeed, if \( \Gamma \) is quasi-isometrically embedded in \( G \times G \) and acts sharply on \( (G \times G)/\text{Diag}(G) \), it does not need to be word hyperbolic: for instance, for \( G = SO(2,2d) \), any discrete subgroup of \( SO(1,2d) \times U(1,d) \subset G \times G \) acts sharply on \( (G \times G)/\text{Diag}(G) \). The implication is actually false even if we assume \( \Gamma \) to be word hyperbolic: for instance, take \( \rho_L \) to be any quasi-isometric embedding which is not \( P_0 \)-Anosov (see e.g. Appendix A) and \( \rho_R \) to be the constant representation.

6.1.5. A complement to the main theorem. In Theorem 6.3, we may replace the notion of Anosov representation into \( \text{Aut}_K(b) \) with that of Anosov representation into \( \text{GL}_K(V) \), as follows.

Proposition 6.6. In the setting of Theorem 6.3, let \( P_{\varepsilon_1-\varepsilon_2}(V) \) be the stabilizer in \( \text{GL}_K(V) \) of a line of \( V \) and \( P_{\varepsilon_1-\varepsilon_2}(V \oplus V) \) the stabilizer in \( \text{GL}_K(V \oplus V) \) of a line of \( V \oplus V \). Condition (3) of Theorem 6.3 is equivalent to

\( (3') \Gamma \) is word hyperbolic and \( \tau \circ \rho_R : \Gamma \rightarrow \text{GL}_K(V) \) is \( P_{\varepsilon_1-\varepsilon_2}(V) \)-Anosov and uniformly \( P_{\varepsilon_1-\varepsilon_2}(V) \)-dominates \( \tau \circ \rho_R \).

Condition (4) of Theorem 6.3 is equivalent to

\( (4') \Gamma \) is word hyperbolic and \( \tau \circ \rho_L \oplus \tau \circ \rho_R : \Gamma \rightarrow \text{GL}_K(V \oplus V) \) is \( P_{\varepsilon_1-\varepsilon_2}(V \oplus V) \)-Anosov.

6.2. Linear representations into automorphism groups of bilinear forms. Before proving Theorem 6.3 and Proposition 6.6, we make a few useful observations and fix some notation.

6.2.1. Existence of representations. The following proposition justifies the assumptions in Theorem 6.3.

Proposition 6.7. Let \( G \) be a noncompact reductive Lie group and \( \theta \subset \Delta \) a nonempty subset of the simple restricted roots of \( G \). For \( K = R, C, \) or \( H \), there exists an irreducible, \( \theta \)-proximal representation \( \tau : G \rightarrow \text{GL}_K(V) \) preserving a bilinear form \( b : V \otimes_R V \rightarrow K \) if and only if \( \theta = \theta^* \).

Note that the group \( \text{Aut}_K(b) \) is necessarily noncompact since it contains an element which is proximal in \( P_K(V) \).

One implication of Proposition 6.7 is given by the following observation.

Lemma 6.8. For \( K = R, C, \) or \( H \), let \( \tau : G \rightarrow \text{GL}_K(V) \) be an irreducible representation with highest weight \( \chi_\tau \). If the group \( \tau(G) \) preserves a nonzero bilinear form \( b : V \otimes_R V \rightarrow K \), then \( \chi_\tau = \chi_\tau^* \) and the bilinear form is unique up to scale. When \( K = R \) and \( \tau \) is proximal, the converse also holds.

Proof of Lemma 6.8. The dual representation \( (\tau^*, V^* = \text{Hom}_K(V,K)) \) has highest weight \( \chi_\tau^* \). Therefore, if there exists a \((\tau, \tau^*)\)-equivariant isomorphism \( \psi : V \rightarrow V^* \) then \( \chi_\tau = \chi_\tau^* \); in this case \( \psi \) is unique up to scale by the Schur lemma. We then note that the space of nonzero \( \tau(G) \)-invariant bilinear forms \( b : V \otimes_K V \rightarrow K \) identifies with the space of \((\tau, \tau^*)\)-equivariant isomorphisms
$V \to V^*$ by sending $b$ to the isomorphism $v \mapsto b(v, \cdot)$. This treats the case of a symmetric or antisymmetric form.

For the case of Hermitian and anti-Hermitian forms (where $K = C$ or $H$), we observe that the (real vector) space of forms $\text{Sym}^r V$ and $\text{Ant}^r V$ isomorphic to $GL(V) \to K$ that are $K$-linear in the second variable and anti-linear in the first variable identifies with the space of $(\tau, \bar{\tau}^*)$-equivariant morphisms $V \to V^*$ where $\bar{\tau}^*$ is the representation of $G$ on the space $V^*$ of antilinear forms $V \to K$. The highest weight of $\bar{V}^*$ is also $\chi^*_\tau$.

When $K = R$, the equality $\chi_\tau = \chi^*_\tau$ implies the existence of an equivariant isomorphism $V \to V^*$, hence of an invariant nonzero bilinear form.

**Proof of Proposition 6.7.** Suppose there exists an irreducible, $\theta$-proximal representation $\tau : G \to GL_K(V)$ preserving a bilinear form $b : V \otimes_R V \to K$. By Lemma 6.8, the highest weight $\chi_\tau$ of $\tau$ satisfies $\chi_\tau = \chi^*_\tau$. By definition of $\theta$-compatibility, $\theta$ is the set of $\alpha \in \Delta$ such that $(\chi_\tau, \alpha) > 0$. Since the $W$-invariant scalar product $(\cdot, \cdot)$ on $\mathfrak{a}^*$ is invariant under $\alpha \mapsto \alpha^*$, we conclude that $\theta = \theta^*$.

Conversely, suppose $\theta = \theta^*$. By Lemma 4.5, we can find an irreducible proximal real representation $\tau$ of $G$ with highest weight $\chi_\tau \in \sum_{\alpha \in \theta} N^\ast \omega_\alpha$ satisfying $\chi_\tau = \chi^*_\tau$; it is $\theta$-compatible by definition. By Lemma 6.8, the group $\tau(G)$ preserves a nondegenerate real bilinear form. Tensoring with $K$ gives an irreducible, $\theta$-proximal $K$-representation $V$ together with an invariant bilinear form $b : V \otimes_R V \to K$.

**Remark 6.9.** For $K = R$ or $C$, we can always assume $b$ to be symmetric up to replacing $V$ with the irreducible representation of highest weight $2\chi_\tau$, which is a subrepresentation of $\text{Sym}^2(V)$.

### 6.2.2. Cartan and Lyapunov projections for $G$, $\text{Aut}_K(b)$, and $GL_K(V)$

As in Theorem 6.3, let $b : V \otimes_R V \to K$ be a bilinear form on a $K$-vector space $V$ and $\tau : G \to \text{Aut}_K(b) \subset GL_K(V)$ an irreducible, $\theta$-proximal representation. We see $\text{Aut}_K(b)$ as a subgroup of $GL_K(V)$. We identify $GL_K(V)$ with $GL_d(K)$ where $d$ is the dimension of $V$ over $K$, and use the notation of Example 2.12. Up to conjugating, we may assume that the real Lie groups $G$, $\text{Aut}_K(b)$, and $GL_K(V)$ have compatible Cartan decompositions, in the sense of inclusion of the corresponding maximal compact subgroups and inclusion of the corresponding Cartan subspaces (see Remark 4.9). We denote the corresponding Cartan projections by $\mu$, $\mu_b$, $\mu_{GL_K(V)}$, and the corresponding Lyapunov projections by $\lambda$, $\lambda_b$, $\lambda_{GL_K(V)}$. Let $\alpha_0(b)$ be a simple restricted root of $\text{Aut}_K(b)$ for our chosen Cartan subspace of $\text{Lie}(\text{Aut}_K(b))$, determining the parabolic subgroup $Q_0(b)$ of Section 6.1.2; then $\alpha_0(b) = \alpha_0(b)^*$. Let $\omega_{\alpha_0(b)}$ be the corresponding fundamental weight. We use similar notation for $b \oplus b'$ on $V \oplus V$. Then the following equalities hold.

**Lemma 6.10.** Let $\nu$ be either the Cartan projection $\mu$ or the Lyapunov projection $\lambda$. For any $g \in G$,

1. $\langle \omega_{\alpha_0(b)}, \nu_\theta(\tau(g)) \rangle = \langle \varepsilon_1, \nu_{GL_K(V)}(\tau(g)) \rangle = \langle \chi_\tau, \nu(g) \rangle$,
2. $\langle \alpha_0(b), \nu_\theta(\tau(g)) \rangle = \langle \varepsilon_1 - \varepsilon_2, \nu_{GL_K(V)}(\tau(g)) \rangle$.

For any $g, g' \in G$ with $\langle \chi_\tau, \nu(g) \rangle \geq \langle \chi_\tau, \nu(g') \rangle$,
The embedding \( \chi \) yields

\[
\langle \alpha_0(b \oplus b'), \nu_{b \oplus b'}(\tau(g) \oplus \tau(g')) \rangle = \langle \varepsilon_1 - \varepsilon_2, \nu_{\text{GL}(V \oplus V)}(\tau(g) \oplus \tau(g')) \rangle = \min \left\{ \langle \alpha_0(b), \nu_b(\tau(g)) \rangle, \langle \omega_{\alpha_0(b)}, \nu_b(\tau(g)) \rangle - \nu_b(\tau(g')) \right\}.
\]

The space \( \mathcal{F}_0(b) = \text{Aut}_K(b)/Q_0(b) \) identifies with the subset of \( \mathbf{P}_K(V) \) consisting of \( b \)-isotropic lines, and similarly for \( \mathcal{F}_0(b \oplus b') \) inside \( \mathbf{P}_K(V \oplus V) \). The embedding \( V \simeq V \oplus \{0\} \hookrightarrow V \oplus V \) induces natural embeddings \( \mathcal{F}_0(b) \hookrightarrow \mathcal{F}_0(b \oplus b) \).

Similarly to Lemma 6.10.(3) for \( \lambda \), the following holds:

Remark 6.11. Let \( g, g' \in G \) satisfy \( \langle \chi_{\tau}, \lambda(g) - \lambda(g') \rangle > 0 \). Then the element \( \tau(g) \oplus \tau(g') \in \text{Aut}_K(b \oplus b') \) is proximal in \( \mathcal{F}_0(b \oplus b') \) if and only if \( \tau(g) \in \text{Aut}_K(b) \) is proximal in \( \mathcal{F}_0(b) \). In this case the attracting fixed point of \( \tau(g) \oplus \tau(g') \) is the image of the attracting fixed point of \( \tau(g) \) under the natural embedding \( \mathcal{F}_0(b) \hookrightarrow \mathcal{F}_0(b \oplus b') \), and the same holds with \( (g^{-1}, g'^{-1}) \) instead of \( (g, g') \), by (2.10) and the fact that \( \chi_{\tau} = \chi_{\tau}^+ \) (Lemma 6.8).

6.2.3. The properness criterion of Benoist and Kobayashi for group manifolds. If \( \mu : G \to \mathfrak{a}^+ \) is a Cartan projection for \( G \) as above, then

\[
\mu \times \mu : G \times G \to \mathfrak{a}^+ \times \mathfrak{a}^+)
\]

is a Cartan projection for \( G \times G \). It sends \( \text{Diag}(G) \) to the diagonal of \( \mathfrak{a}^+ \times \mathfrak{a}^+ \). Let \( \| \cdot \| \) be a \( W \)-invariant Euclidean norm on \( \mathfrak{a} \) as in Section 2.3.1. In this setting the properness criterion of Benoist and Kobayashi (see Section 1.3) can be expressed as follows.

**Properness criterion for group manifolds** [Ben96, Kob96]: A discrete subgroup \( \Gamma' \) of \( G \times G \) acts properly discontinuously on \( (G \times G)/\text{Diag}(G) \) if and only if

\[
\| \mu(\gamma'_1) - \mu(\gamma'_2) \| \xrightarrow{\gamma' = (\gamma'_1, \gamma'_2) \to \infty} +\infty,
\]

where \( \gamma' \to \infty \) means that \( \gamma' \) exits every finite subset of \( \Gamma' \).

The action of \( \Gamma' \) on \( (G \times G)/\text{Diag}(G) \) is sharp, in the sense of (1.2), if and only if there exist \( c, C > 0 \) such that for any \( \gamma' = (\gamma'_1, \gamma'_2) \in \Gamma' \),

\[
\| \mu(\gamma'_1) - \mu(\gamma'_2) \| \geq c \left( \| \mu(\gamma'_1) \| + \| \mu(\gamma'_2) \| \right) - C.
\]

6.3. Proof of Theorem 6.3 and Corollary 1.17. In the theorem, the implication (5) \( \Rightarrow \) (6) is immediate from the definition (1.2) of sharpness and the properness criterion of Benoist and Kobayashi. Remark 2.17 yields the implication (6) \( \Rightarrow \) (7).

We now prove the other implications in Theorem 6.3, using the notation of Section 6.2. Note that our proofs of (3) \( \Leftrightarrow \) (4), as well as (6) \( \Rightarrow \) (4) for \( G \) of real rank 1, rely on our characterization of Anosov representations given by Theorem 1.3.(2), whereas (4) \( \Rightarrow \) (5) relies on Theorem 1.3.(3).

**Proof of (1) \( \Rightarrow \) (2) in Theorem 6.3.** By definition of \( \theta \)-compatibility, we can write \( \chi_{\tau} = \sum_{\alpha \in \theta} n_{\alpha} \omega_{\alpha} \) where \( n_{\alpha} > 0 \) for all \( \alpha \in \theta \). Lemma 6.10.(1) for the
Lyapunov projection $\lambda$ then yields that for any $\gamma \in \Gamma$,
\[
\langle \omega_{\alpha_0(b)}, \lambda_b(\tau \circ \rho_L(\gamma)) \rangle = \sum_{\alpha \in \theta} n_{\alpha} \langle \omega_{\alpha}, \lambda(\rho_L(\gamma)) \rangle,
\]
\[
\langle \omega_{\alpha_0(b)}, \lambda_b(\tau \circ \rho_R(\gamma)) \rangle = \sum_{\alpha \in \theta} n_{\alpha} \langle \omega_{\alpha}, \lambda(\rho_R(\gamma)) \rangle.
\]
Therefore, if $\rho_L$ uniformly $P_\theta$-dominates $\rho_R$, then $\tau \circ \rho_L$ uniformly $Q_0(b)$-dominates $\tau \circ \rho_R$. □

If $\theta$ is a singleton (e.g. if $G$ is semisimple of real rank 1), then the previous proof shows that the uniform $P_\theta$-dominance of $\rho_R$ by $\rho_L$ is equivalent to the uniform $Q_0(b)$-dominance of $\tau \circ \rho_R$ by $\tau \circ \rho_L$, i.e. (2) $\Rightarrow$ (1) holds as well.

**Proof of (2) $\Leftrightarrow$ (3′) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (4′) in Theorem 6.3 and Proposition 6.6.** Suppose $\Gamma$ is word hyperbolic. The natural inclusion $\text{Aut}_K(b) \hookrightarrow \text{GL}_K(V)$ is $\alpha_0(b)$-proximal, hence Proposition 4.8 applies: a representation $\tau \circ \rho_L : \Gamma \to \text{Aut}_K(b)$ is $Q_0(b)$-Anosov if and only if $\tau \circ \rho_L : \Gamma \to \text{GL}_K(V)$ is $P_{-\varepsilon_2}(V)$-Anosov. Moreover, Lemma 6.10.(1) for the Lyapunov projection $\lambda$ yields that $\tau \circ \rho_L$ uniformly $Q_0(b)$-dominates $\tau \circ \rho_R$ as representations into $\text{Aut}_K(b)$ if and only if $\tau \circ \rho_L$ uniformly $P_{-\varepsilon_2}(V)$-dominates $\tau \circ \rho_R$ as representations into $\text{GL}_K(V)$. Thus (3) $\Leftrightarrow$ (3′) holds.

The equivalence (4) $\Leftrightarrow$ (4′) follows from the same argument, using $b \oplus b'$ on $V \oplus V$ instead of $b$ on $V$.

Similarly, $\rho_L$ is $P_\theta$-Anosov if and only if $\tau \circ \rho_L$ is $P_{-\varepsilon_2}(V)$-Anosov by Proposition 4.8; thus (2) $\Leftrightarrow$ (3′) holds. □

**Proof of (3) $\Leftrightarrow$ (4) in Theorem 6.3.** Suppose condition (3) of Theorem 6.3 holds, i.e. $\Gamma$ is word hyperbolic and $\tau \circ \rho_L$ is $Q_0(b)$-Anosov and uniformly $Q_0(b)$-dominates $\tau \circ \rho_R$. Since $\alpha_0(b) = \alpha_0(b)^*$, the two boundary maps of $\tau \circ \rho_L$ identify (Remark 2.29); we denote them by $\xi : \partial_\infty \Gamma \to F_0(b)$. By Remark 4.1, for any $\gamma \in \Gamma$ of infinite order, $\tau \circ \rho_L(\gamma)$ is proximal in $F_0(b)$.

Since $\tau \circ \rho_L$ uniformly $Q_0(b)$-dominates $\tau \circ \rho_R$, Remark 6.11 implies that $(\tau \circ \rho_L \oplus \tau \circ \rho_R)(\gamma)$ is proximal in $F_0(b \oplus b')$ and $\xi$ sends the attracting fixed point of $\gamma$ in $\partial_\infty \Gamma$ to the attracting fixed point of $(\tau \circ \rho_L \oplus \tau \circ \rho_R)(\gamma)$ in $F_0(b \oplus b')$, after embedding $F_0(b)$ into $F_0(b \oplus b')$. Thus the continuous, transverse map $\xi : \partial_\infty \Gamma \to F_0(b) \hookrightarrow F_0(b \oplus b')$ is dynamics-preserving for $\tau \circ \rho_L \oplus \tau \circ \rho_R$. On the other hand, since $\tau \circ \rho_L$ is $Q_0(b)$-Anosov, Theorem 1.3.(2) implies that
\[
\langle \alpha_0(b), \lambda(\tau \circ \rho_L(\gamma)) \rangle \xrightarrow{t_{\gamma}(\gamma) \to +\infty} +\infty.
\]
Moreover, uniform $Q_0(b)$-domination implies that
\[
\langle \omega_{\alpha_0(b)}, \lambda_b(\tau \circ \rho_L(\gamma)) - \lambda_b(\tau \circ \rho_R(\gamma)) \rangle \xrightarrow{t_{\gamma}(\gamma) \to +\infty} +\infty.
\]

By Lemma 6.10.(3) for the Lyapunov projection $\lambda$, we have
\[
\langle \alpha_0(b \oplus b'), \lambda(b \oplus b') \left( (\tau \circ \rho_L \oplus \tau \circ \rho_R)(\gamma) \right) \rangle \xrightarrow{t_{\gamma}(\gamma) \to +\infty} +\infty.
\]
Therefore $\tau \circ \rho_L \oplus \tau \circ \rho_R$ is $Q_0(b \oplus b')$-Anosov by Theorem 3.2.(4), i.e. condition (4) of Theorem 6.3 holds.

Conversely, suppose condition (4) of Theorem 6.3 holds, i.e. $\Gamma$ is word hyperbolic and $\tau \circ \rho_L \oplus \tau \circ \rho_R : \Gamma \to \text{Aut}_K(b \oplus b')$ is $Q_0(b \oplus b')$-Anosov. Let
\( \xi : \partial_\infty \Gamma \to \mathcal{F}_0(b \oplus b') \) be the boundary map of \( \tau \circ \rho_L \oplus \tau \circ \rho_R \). By Remark 4.1, for any \( \gamma \in \Gamma \) of infinite order, \((\tau \circ \rho_L \oplus \tau \circ \rho_R)(\gamma)\) is proximal in \( \mathcal{F}_0(b \oplus b') \).

By the normalization (6.1) and Remark 6.11, there exists \( \gamma \in \Gamma \) of infinite order for which \( \tau \circ \rho_L(\gamma) \) is proximal in \( \mathcal{F}_0(b) \) and the attracting fixed point of \((\tau \circ \rho_L \oplus \tau \circ \rho_R)(\gamma)\) in \( \mathcal{F}_0(b \oplus b') \) is the image of the attracting fixed point of \( \tau(\rho_L(\gamma)) \) under the natural embedding \( \mathcal{F}_0(b) \hookrightarrow \mathcal{F}_0(b \oplus b') \), and the same holds with \( \gamma^{-1} \) instead of \( \gamma \). In particular, the closed \( \Gamma \)-invariant set

\[
\{ \eta \in \partial_\infty \Gamma \mid \xi(\eta) \in \mathcal{F}_0(b) \subset \mathcal{F}_0(b \oplus b') \}
\]

contains the attracting fixed points \( \eta^+_\gamma, \eta^-_{\gamma^{-1}} \) of \( \gamma \) and \( \gamma^{-1} \), hence is nonempty. This set is actually equal to \( \partial_\infty \Gamma \), by minimality of the action of \( \Gamma \) on \( \partial_\infty \Gamma \) if \( \Gamma \) is nonelementary, and by the fact that \( \partial_\infty \Gamma = \{ \eta^+_\gamma, \eta^-_{\gamma^{-1}} \} \) if \( \Gamma \) is elementary. Therefore, \( \xi \) defines a continuous map from \( \partial_\infty \Gamma \) to \( \mathcal{F}_0(b) \) which is equivariant and dynamics-preserving for \( \tau \circ \rho_L \).

Moreover, since \( \tau \circ \rho_L \oplus \tau \circ \rho_R \) is \( \mathcal{Q}_0(b \oplus b') \)-Anosov, Theorem 1.3.(2) implies that

\[
\langle \alpha_0(b \oplus b'), \mu_{b \oplus b'}(\tau \circ \rho_L(\gamma)) \rangle \xrightarrow{\gamma \to \infty} +\infty.
\]

By Lemma 6.10.(3) for the Cartan projection \( \mu \), we have

\[
\langle \alpha_0(b), \mu_\gamma(\tau \circ \rho_L(\gamma)) \rangle \xrightarrow{\gamma \to \infty} +\infty.
\]

Therefore \( \tau \circ \rho_L \) is \( \mathcal{Q}_0(b) \)-Anosov by Theorem 1.3.(2). On the other hand, using Theorem 3.2.(2) and Lemma 6.10.(3) for the Lyapunov projection \( \lambda \), we see that there exist \( c, C > 0 \) such that for any \( \gamma \in \Gamma \),

\[
\langle \omega_{\alpha_0(b)}, \lambda_\gamma(\tau \circ \rho_L(\gamma)) \rangle - \lambda_\gamma(\tau \circ \rho_R(\gamma)) \geq c \langle \omega_{\alpha_0(b)}, \lambda_\gamma(\tau \circ \rho_L(\gamma)) \rangle - C.
\]

Applying this to \( \gamma^n \), dividing by \( n \), and taking the limit, we obtain

\[
\langle \omega_{\alpha_0(b)}, \lambda_\gamma(\tau \circ \rho_R(\gamma)) \rangle \leq (1 - c) \langle \omega_{\alpha_0(b)}, \lambda_\gamma(\tau \circ \rho_L(\gamma)) \rangle.
\]

Thus \( \tau \circ \rho_L(\gamma) \) uniformly \( \mathcal{Q}_0(b) \)-dominates \( \tau \circ \rho_R(\gamma) \), i.e. condition (3) of Theorem 6.3 holds.

Proof of (2), (4) \( \Rightarrow \) (5) in Theorem 6.3. Suppose that (2) and (4) hold (we have seen that they are equivalent). Since \( \rho_L \) is \( \mathcal{F}_b \)-Anosov, it is a quasi-isometric embedding (see Section 2.4.3). Since \( \tau \circ \rho_L \oplus \tau \circ \rho_R \) is \( \mathcal{Q}_0(b \oplus b') \)-Anosov, Theorem 1.3.(3), Lemma 6.10.(1)–(3) for the Cartan projection \( \mu \), and the normalization (6.1) show that there exist \( c, C > 0 \) such that for any \( \gamma \in \Gamma \),

\[
\langle \chi_\gamma, \mu(\rho_L(\gamma)) \rangle - \mu(\rho_R(\gamma)) \rangle \geq c |\gamma| - C.
\]

Using (2.8), we see that there exist \( c', C' > 0 \) such that for any \( \gamma \in \Gamma \),

\[
||\mu(\rho_L(\gamma)) - \mu(\rho_R(\gamma))|| \geq c' (||\mu(\rho_L(\gamma))|| + ||\rho_R(\gamma)||) - C',
\]

where \( || \cdot || \) is the \( W \)-invariant Euclidean norm on \( \alpha \) from Section 2.3.1. Thus the action of \( \Gamma \) on \( (G \times G)/\text{Diag}(G) \) via \( (\rho_L, \rho_R) \) is sharp (see Section 6.2.3).

Note that any subgroup of \( G \times G \) is always of the form

\[
\Gamma^{\rho_L, \rho_R} = \{ (\rho_L(\gamma), \rho_R(\gamma)) \mid \gamma \in \Gamma \},
\]

where \( \Gamma \) is a group and \( \rho_L, \rho_R \in \text{Hom}(\Gamma, G) \) two representations, corresponding to the two projections of \( G \times G \) onto \( G \).
In the case that \( G \) is semisimple of real rank 1, the implication (7) \( \Rightarrow \) (6) of Theorem 6.3 is an immediate consequence of the following result. (Since \( G \) has real rank 1, we identify \( \mathfrak{T}^+ \) with \( \mathbb{R}_+ \) and see \( \lambda \) as a function \( G \to \mathbb{R}_+ \).)

**Theorem 6.12 ([Kas08]).** Let \( G \) be a semisimple Lie group of real rank 1. Then any discrete subgroup of \( G \times G \) acting properly discontinuously on \( (G \times G)/\text{Diag}(G) \) is of the form \( \Gamma^{\text{pl},\rho_R} \) where \( \Gamma \) is a discrete group and \( \rho_L, \rho_R \in \text{Hom}(\Gamma, G) \) two representations and, up to switching the two factors of \( G \times G \), the representation \( \rho_L \) has finite kernel and discrete image and

\[ (6.2) \quad \lambda(\rho_R(\gamma)) < \lambda(\rho_L(\gamma)) \quad \text{for all } \gamma \in \Gamma \text{ of infinite order}. \]

In particular, if \( \Gamma^{\text{pl},\rho_R} \) is finitely generated and quasi-isometrically embedded in \( G \times G \), then it is word hyperbolic.

We now use this result to prove the implication (6) \( \Rightarrow \) (4) of Theorem 6.3.

**Proof of (6) \( \Rightarrow \) (4) in Theorem 6.3 for \( G \) semisimple of real rank 1.** Suppose \( G \) is semisimple of real rank 1 and (6) holds. Then \( \Gamma \) is word hyperbolic by Theorem 6.12. Since \( G \) has real rank 1, the quasi-isometric embedding \( \rho_L \) is \( P^1\)-Anosov: see [GW12] or Theorem 1.3. The boundary map of \( \rho_L \) induces a boundary map \( \xi : \partial_\infty \Gamma \to \mathcal{F}_0(b \oplus b') \) that is continuous, \( (\tau \circ \rho_L) \oplus (\tau \circ \rho_R) \)-equivariant, and transverse. The normalization (6.1) implies the normalization (6.2) in Theorem 6.12, and so \( \xi \) is dynamics-preserving. By the properness criterion of Benoist and Kobayashi, we have

\[ \|\mu(\rho_L(\gamma)) - \mu(\rho_R(\gamma))\| \xrightarrow{\gamma \to \infty} +\infty. \]

Since \( G \) is semisimple of real rank 1, this also holds if we replace the norm \( \| \cdot \| \) on \( a \) with \( \langle \cdot , \cdot \rangle \). Using Lemma 6.10.(3) for the Cartan projection \( \mu \), as well as the fact that \( \rho_L \) is a quasi-isometric embedding and Remark 2.17, we deduce that

\[ \langle \alpha_0(b \oplus b'), \mu_{b \oplus b'}((\tau \circ \rho_L) \oplus (\tau \circ \rho_R)(\gamma)) \rangle \xrightarrow{\gamma \to \infty} +\infty. \]

Therefore, \( \tau \circ \rho_L \oplus \tau \circ \rho_R \) is \( \Gamma_0(b \oplus b') \)-Anosov by Theorem 1.3.(2). \( \Box \)

**Proof of Corollary 1.17.** The first statement (openness) follows from Theorem 1.13 and from the fact that being Anosov is an open property [Lab06, GW12]. The second statement (compactness) follows from a classical cohomological dimension argument (see for instance [GK, § 7.7]). \( \Box \)

7. Compactifications of Clifford–Klein forms

In this section we recall the construction of domains of discontinuity developed in [GW12], and show how it can be used to give another proof of the implication (4) \( \Rightarrow \) (6) of Theorem 6.3. This leads to the description of some additional geometric features of the quotient manifolds \( \Gamma \setminus (G \times G)/\text{Diag}(G) \).

As in Section 6.1.2, let \( K \) be \( \mathbb{R} \), \( \mathbb{C} \), or the ring \( \mathbb{H} \) of quaternions. Let \( V \) be a \( K \)-vector space (where \( K \) acts on the right in the case of \( \mathbb{H} \)), and let \( b : V \otimes_{\mathbb{R}} V \to K \) be an \( \mathbb{R} \)-bilinear form which is symmetric or antisymmetric (if \( K = \mathbb{R} \) or \( \mathbb{C} \)), or Hermitian or anti-Hermitian (if \( K = \mathbb{C} \) or \( \mathbb{H} \)). Let \( \text{Aut}_K(b) \) the subgroup of \( GL_K(V) \) preserving \( b \), and \( r \in \mathbb{N} \) its real rank. Let \( Q_0(b) \) (resp. \( Q_1(b) \)) be the stabilizer in \( \text{Aut}_K(b) \) of a \( b \)-isotropic line.
(resp. a maximal $b$-isotropic subspace) of $V$; it is a maximal proper parabolic subgroup of $\text{Aut}_K(b)$. Similarly to Section 6, the space of $b$-isotropic $K$-lines of $V$

$$\mathcal{F}_0(b) := \{L \in \mathcal{P}_K(V) \mid b|_{L \times L} = 0\} \simeq \text{Aut}_K(b)/Q_0(b).$$

The space of maximal $b$-isotropic $K$-subspaces of $V$ is

$$\mathcal{F}_1(b) := \{W \in \text{Gr}_r(V) \mid b|_{W \times W} = 0\} \simeq \text{Aut}_K(b)/Q_1(b),$$

where $\text{Gr}_r(V)$ is the Grassmannian of $r$-dimensional $K$-planes of $V$. As we saw in Section 6, the parabolic subgroup $Q_0(b)$ is conjugate to any of its opposite parabolic subgroups, and so Remark 2.29 applies: the two boundary maps $\xi^+, \xi^-$ of a $Q_0(b)$-Anosov representation coincide with a unique map $\xi$ with values in $\mathcal{F}_0(b)$. We shall use the following result.

**Theorem 7.1** ([GW12, Th. 8.6]). In the setting above, let $\Gamma$ be a word hyperbolic group and $\rho : \Gamma \to \text{Aut}_K(b)$ a $Q_0(b)$-Anosov representation, with boundary map $\xi : \partial_{\infty}\Gamma \to \mathcal{F}_0(b)$. Let $\Lambda_\rho = \xi(\partial_{\infty}\Gamma) \subset \mathcal{F}_0(b)$ be the limit set of $\rho(\Gamma)$ and let

$$K_\rho := \{W \in \mathcal{F}_1(b) \mid \exists L \in \Lambda_\rho, \ L \subset W\}.$$ 

Then the action of $\Gamma$ on the $\rho(\Gamma)$-invariant open set

$$(7.1) \quad \Omega_\rho := \mathcal{F}_1(b) \setminus K_\rho$$

is properly discontinuous and cocompact.

We now show how to use this result to construct properly discontinuous actions on reductive homogeneous spaces with explicit compactifications of the corresponding Clifford–Klein forms. More details, as well as further examples, will be given in [GGKW].

7.1. **Properly discontinuous actions on group manifolds.** We use the notation above, not only for $b$ on $V$, but also for $b \oplus (-b)$ on $V \oplus V$. Consider the following open subset of $\mathcal{F}_1(b \oplus (-b))$:

$$(7.2) \quad \mathcal{U} := \{W \in \mathcal{F}_1(b \oplus (-b)) \mid W \cap (V \oplus \{0\}) = \{0\}\}.$$ 

It is clearly invariant under $\text{Aut}_K(b) \times \text{Aut}_K(b) = \text{Aut}_K(b) \times \text{Aut}_K(-b)$, seen as a subgroup of $\text{Aut}_K(b \oplus (-b))$.

**Lemma 7.2.** As an $(\text{Aut}_K(b) \times \text{Aut}_K(b))$-space, $\mathcal{U}$ is isomorphic to $$(\text{Aut}_K(b) \times \text{Aut}_K(b))/\text{Diag}(\text{Aut}_K(b)).$$

**Proof.** We first check that the action of $\text{Aut}_K(b) \times \text{Aut}_K(b)$ on $\mathcal{U}$ is transitive. Let $W' := \{(v,v) \mid v \in V\}$. By definition, any $W \in \mathcal{U}$ satisfies $W \cap (V \oplus \{0\}) = \{0\}$, hence is the graph $\{(g(v),v) \mid v \in V\}$ of some endomorphism $g$ of $V$. Since $W \in \mathcal{F}_1(b \oplus (-b))$, we have $(b \oplus (-b))(g(v),v) = 0$, i.e. $b(g(v),g(v')) = b(v,v')$, for all $v, v' \in V$, and so $g \in \text{Aut}_K(b)$. Thus $W = (g,e) \cdot W'$ lies in the $(\text{Aut}_K(b) \times \text{Aut}_K(b))$-orbit of $W'$, proving transitivity.

We then check that the stabilizer of $W'$ in $\text{Aut}_K(b) \times \text{Aut}_K(b)$ is $\text{Diag}(\text{Aut}_K(b))$. For any $(g_1,g_2) \in \text{Aut}_K(b) \times \text{Aut}_K(b)$,

$$(g_1,g_2) \cdot W' = \{(g_1(v),g_2(v)) \mid v \in V\} = \{(g_2^{-1}g_1(v),v) \mid v \in V\},$$

where $W' := \{(v,v) \mid v \in V\}$ and $\mathcal{F}_1(b \oplus (-b))$.

[GGKW]
and so \((g_1, g_2) \cdot W = W'\) if and only if \(g_1 = g_2\).

The proof shows that we also have
\[
\mathcal{U} := \{ W \in \mathcal{F}_1(b \oplus (-b)) \mid W \cap (\{0\} \oplus V) = \{0\} \}.
\]
The complement of \(\mathcal{U}\) in \(\mathcal{F}_1(b \oplus (-b))\) is a finite union of subvarieties of lower dimension. Applying Lemma 7.2, we see that \(\mathcal{F}_1(b \oplus (-b))\) provides a compactification of \((\text{Aut}_K(b) \times \text{Aut}_K(b))/\text{Diag}(\text{Aut}_K(b))\).

We now consider quotients by discrete groups \(\Gamma\).

Lemma 7.3. Let \(\Gamma\) be a word hyperbolic group and \(\rho_L, \rho_R : \Gamma \to \text{Aut}_K(b)\) two representations. If \(\rho_L \oplus \rho_R : \Gamma \to \text{Aut}_K(b \oplus (-b))\) is \(Q_0(b \oplus (-b))\)-Anosov, then the set \(\Omega_{\rho_L \oplus \rho_R}\) of (7.1) contains \(\mathcal{U}\).

Proof. Taking \(G\) to be \(\text{Aut}_K(b)\) and \(\tau : G \to \text{Aut}_K(b)\) to be the identity map, it follows from the implication (4) \(\Rightarrow\) (3) of Theorem 6.3 and its proof in Section 6.3, together with Remark 6.2, that up to switching \(\rho_L\) and \(\rho_R\) the representation \(\rho_L\) is \(Q_0(b)\)-Anosov and the boundary map \(\xi : \partial_\infty \Gamma \to \mathcal{F}_0(b \oplus (-b))\) of \(\rho_L \oplus \rho_R\) is the composition of the boundary map \(\xi_L : \partial_\infty \Gamma \to \mathcal{F}_0(b)\) of \(\rho_L\) with the natural embedding \(\mathcal{F}_0(b) \hookrightarrow \mathcal{F}_0(b \oplus (-b))\). In particular, if we see \(\mathcal{F}_0(b \oplus (-b))\) as a subset of the projective space \(P_K(V \oplus V)\), then \(\xi\) takes values in \(P_K(V \oplus \{0\}) \subset P_K(V \oplus V)\). This immediately implies that the set \(\mathcal{U}\) of (7.2) is contained in
\[
\Omega_{\rho_L \oplus \rho_R} = \{ W \in \mathcal{F}_1(b \oplus (-b)) \mid \xi(\eta) \not\subset W \quad \forall \eta \in \partial_\infty \Gamma \}.
\]

In the setting of Lemma 7.3, Theorem 7.1 shows that the action of \(\Gamma\) on \(\Omega_{\rho_L \oplus \rho_R}\) via \(\rho_L \oplus \rho_R\) is properly discontinuous and cocompact. If moreover \(\Gamma\) is torsion-free, then the quotient \(\Gamma \backslash \Omega_{\rho_L \oplus \rho_R}\) is a compact manifold. By Lemma 7.3, the action of \(\Gamma\) on \(\mathcal{U}\) via \((\rho_L, \rho_R)\) is properly discontinuous as well (but not necessarily cocompact). By Lemma 7.2, this action identifies with the action of \(\Gamma\) on \((\text{Aut}_K(b) \times \text{Aut}_K(b))/\text{Diag}(\text{Aut}_K(b))\) via \((\rho_L, \rho_R)\). We thus obtain the following.

Proposition 7.4. Let \(\Gamma\) be a torsion-free word hyperbolic group and \(\rho_L, \rho_R : \Gamma \to \text{Aut}_K(b)\) two representations. Suppose that the representation \(\rho_L \oplus \rho_R\) is \(Q_0(b \oplus (-b))\)-Anosov. Then \(\Gamma \backslash \mathcal{U}\) is an open and dense submanifold of the compact manifold \(\Gamma \backslash \Omega_{\rho_L \oplus \rho_R}\). In particular, \(\Gamma \backslash \Omega_{\rho_L \oplus \rho_R}\) provides a natural compactification of
\[
\Gamma \backslash (\text{Aut}_K(b) \times \text{Aut}_K(b))/\text{Diag}(\text{Aut}_K(b)) \simeq \Gamma \backslash \mathcal{U}.
\]

Proof. By construction, we have \(\mathcal{U} \subset \Omega_{\rho_L \oplus \rho_R} \subset \mathcal{F}_1(b \oplus (-b))\). Since \(\mathcal{U}\) is open and dense in \(\mathcal{F}_1(b \oplus (-b))\), it is also open and dense in \(\Omega_{\rho_L \oplus \rho_R}\). Taking the quotient by \(\Gamma\), we obtain that \(\Gamma \backslash \mathcal{U}\) is an open and dense submanifold of \(\Gamma \backslash \Omega_{\rho_L \oplus \rho_R}\). It identifies with \(\Gamma \backslash (\text{Aut}_K(b) \times \text{Aut}_K(b))/\text{Diag}(\text{Aut}_K(b))\) by Lemma 7.2.

Composing a pair of representations of \(\Gamma\) into a reductive Lie group \(G\) with an irreducible, proximal representation \(\tau : G \to \text{Aut}_K(b)\), we obtain the following.

Corollary 7.5. Let \(\Gamma\) be a word hyperbolic group, \(G\) a reductive Lie group, and \(\rho_L, \rho_R : \Gamma \to G\) two representations of \(\Gamma\). Let \(\tau : G \to \text{Aut}_K(b)\) be an
irreducible, proximal representation of $G$. Suppose that
\[ \tau \circ \rho_L \oplus \tau \circ \rho_R : \Gamma \rightarrow \text{Aut}_K(b \oplus (-b)) \]
is $Q_0(b \oplus (-b))$-Anosov. Then the action of $\Gamma$ on $(G \times G)/\text{Diag}(G)$ via $(\rho_L, \rho_R) : \Gamma \rightarrow G \times G$ is properly discontinuous.

Proof. As we just saw, the action of $\Gamma$ on $(\text{Aut}_K(b) \times \text{Aut}_K(b))/\text{Diag}(\text{Aut}_K(b))$ via $\tau \circ \rho_L \oplus \tau \circ \rho_R$ is properly discontinuous. Since $(\tau(G) \times \tau(G))/\text{Diag}(\tau(G))$ embeds into $(\text{Aut}_K(b) \times \text{Aut}_K(b))/\text{Diag}(\text{Aut}_K(b))$ as the $(\tau(G) \times \tau(G))$-orbit of $(e,e)$, the action of $\Gamma$ on $(\tau(G) \times \tau(G))/\text{Diag}(\tau(G))$ via $\tau \circ \rho_L \oplus \tau \circ \rho_R$ is also properly discontinuous. Thus the action of $\Gamma$ on $(G \times G)/\text{Diag}(G)$ via $(\rho_L, \rho_R)$ is properly discontinuous. \qed

Corollary 7.6. In the setting of Corollary 7.5, a compactification of
\[ \Gamma \backslash (\tau(G) \times \tau(G))/\text{Diag}(\tau(G)) \]
(where $\Gamma$ acts via $\tau \circ \rho_L \oplus \tau \circ \rho_R$) is obtained by taking its closure in $\Gamma \backslash \Omega_{\tau \circ \rho_L \oplus \tau \circ \rho_R}$.
If $\tau : G \rightarrow \text{Aut}_K(b)$ is injective, this provides a compactification of $\Gamma \backslash (G \times G)/\text{Diag}(G)$ (where $\Gamma$ acts via $(\rho_L, \rho_R)$).

7.2. Properly discontinuous actions on other homogeneous spaces.
A construction similar to Section 7.1 can be applied in other situations where the properness of the action of $\Gamma$ on $G/H$ is linked to the Anosov property, possibly inside a larger group $G_2$ containing $G$ and relative to some parabolic subgroup $P$ of $G_2$. Table 5 lists several examples where the homogeneous space $G/H$ embeds as a $G$-orbit into a generalized flag variety $G_2/Q$. More precisely, in these examples, given a representation $\rho : \Gamma \rightarrow G \subset G_2$ which is $P$-Anosov as a representation into $G_2$, the construction of [GW12] gives a domain in $G_2/Q$ on which $\rho(\Gamma)$ acts properly discontinuously and cocompactly, and which contains $G/H$, sometimes even as an open and/or dense subset (see Proposition 7.7). The quotient of the intersection of the closure of $G/H$ in $G_2/Q$ with the domain of discontinuity then gives rise to a compactification of the corresponding Clifford–Klein forms $\Gamma \backslash G/H$. Examples of noncompact Clifford–Klein forms that become compactified in this way are representations $\rho : \Gamma \rightarrow G_1 \subset G \subset G_2$, where $G_1$ is a semisimple Lie group of real rank 1 acting properly on $G/H$ (see Table 3) and $\Gamma$ is a convex cocompact subgroup of $G_1$. Small deformations of such $\rho$ into $G$, or more generally representations $\rho : \Gamma \rightarrow G \subset G_2$ which are $P$-Anosov as representations into $G_2$, provide further examples. This compactification process generalizes for example the usual, conformal compactification of convex cocompact Fuchsian and quasi-Fuchsian groups acting on $\mathbb{H}^3$.

Proposition 7.7. Let $\Gamma$ be a hyperbolic group, let $(G,H,G_2,P,Q)$ be as in Table 5, and let $\rho : \Gamma \rightarrow G \subset G_2$ be a representation which is $P$-Anosov as a representation into $G_2$. Then the domain of discontinuity $\Omega_\rho$ in $G_2/Q$ constructed in [GW12] contains a $G$-orbit diffeomorphic to $G/H$. The quotient of the intersection of the closure of this $G$-orbit with $\Omega_\rho$ gives a compactification of the noncompact Clifford–Klein form $\Gamma \backslash G/H$.

Remarks 7.8. (1) In all examples in Table 5 except (vi) and (vii), the space $G/H$ is an affine symmetric space. Of particular interest are
Thus an $\mathbf{H}$-Hermitian form is completely determined by its real part.

Proof. Case (i) was treated in [GW12, Prop. 13.1]; we recall it for the reader’s convenience. If $p : \Gamma \to G = \text{O}(p+1,q)$ is $Q_1(b_{p+1,q})$-Anosov, with boundary map $\xi : \partial_\infty \Gamma \to \mathcal{F}_1(b_{p+1,q})$, then the composed representation $\rho_2 : \Gamma \to \text{GL}(2,q)$ is $\text{GL}(2,q)$-Anosov, with boundary map $\xi_2 : \partial_\infty \Gamma \to \mathcal{F}_2(\rho_2)$. We describe in detail how $\rho_2$ determines $\xi_2$ in the next section.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$G$ & $H$ & $G_2$ & $P$ & $Q$
\hline
(i) & $\text{O}(p+1,q)$ & $\text{O}(p,q)$ & $\text{O}(p+1,q+1)$ & $\text{Stab}(W)$ & $\text{Stab}(L)$
\hline
(ii) & $\text{U}(p+1,q)$ & $\text{U}(p,q)$ & $\text{O}(p+1,q+1)$ & $\text{Stab}(W)$ & $\text{Stab}(L)$
\hline
(iii) & $\text{Sp}(p+1,q)$ & $\text{Sp}(p,q)$ & $\text{Sp}(p+1,q+1)$ & $\text{Stab}(W)$ & $\text{Stab}(L)$
\hline
(iv) & $\text{O}(2p,2q)$ & $\text{U}(p,q)$ & $\text{O}(2p+2q,\mathbf{C})$ & $\text{Stab}(L)$ & $\text{Stab}(W)$
\hline
(v) & $\text{U}(2p,2q)$ & $\text{Sp}(p,q)$ & $\text{Sp}(p+q,p+q)$ & $\text{Stab}(L)$ & $\text{Stab}(W)$
\hline
(vi) & $\text{O}(4p,4q)$ & $\text{Sp}(2p,2q)$ & $\text{Sp}(2p+2q,2p+2q)$ & $\text{Stab}(L)$ & $\text{Stab}(W') \subset W$
\hline
(vii) & $\text{Sp}(4n,\mathbf{R})$ & $\text{O}^*(2n)$ & $\text{O}^*(8n)$ & $\text{Stab}(L)$ & $\text{Stab}(W') \subset W$
\hline
(viii) & $\text{Sp}(2n,\mathbf{R})$ & $\text{U}(p,n-p)$ & $\text{Sp}(2n,\mathbf{C})$ & $\text{Stab}(L)$ & $\text{Stab}(W)$
\hline
(ix) & $\text{U}(n,n)$ & $\text{U}(n) \times U(n)$ & $\text{GL}(2n,\mathbf{C})$ & $\text{Stab}(L)$ & $\text{Stab}(H)$
\hline
(x) & $\text{O}^*(4n)$ & $\text{U}(2n)$ & $\text{GL}(2n,\mathbf{H})$ & $\text{Stab}(L)$ & $\text{Stab}(H)$
\hline
\end{tabular}
\caption{Here, $n,p,q$ are any integers with $p+1 \leq q$. We denote by $L$ an isotropic line and by $W$ a maximal isotropic subspace, over $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. We denote by $H$ an $n$-dimensional subspace of $\mathbf{C}^{2n}$ or $\mathbf{H}^{2n}$, and by $W' \subset W$ a partial flag of two isotropic subspaces with $W$ maximal and $\dim(W) = 2\dim(W')$.}
\end{table}

Examples (viii) through (x), which provide examples of compactifications of Hermitian locally symmetric spaces.

(2) The cocompact domain of discontinuity in $G_2/Q$ lifts to a cocompact domain of discontinuity in $G_2/Q'$ for any parabolic subgroup $Q'$ of $G_2$ contained in $Q$. The compactifications we describe for the quotients of $G/H$ thus also give rise to compactifications of the quotients of the $G$-orbits in $G_2/Q'$. We use this lifting in cases (vi) and (vii) below.

In order to prove Proposition 7.7, let us introduce some notation. For $p,q \in \mathbf{N}$, we denote by $\mathbf{R}^{p,q}$ the vector space $\mathbf{R}^{p+q}$ endowed with the symmetric bilinear form

$$b_{p,q} : (v,v') \mapsto v_1v'_1 + \cdots + v_pv'_p - v_{p+1}v'_{p+1} - \cdots - v_{p+q}v'_{p+q},$$

and by $\mathbf{C}^{p,q}$ the vector space $\mathbf{C}^{p+q}$ endowed with the Hermitian form

$$h_{p,q} : (z,z') \mapsto \overline{z}_1z'_1 + \cdots + \overline{z}_pz'_p - \overline{z}_{p+1}z'_{p+1} - \cdots - \overline{z}_{p+q}z'_{p+q},$$

so that $\text{O}(p,q) = \text{Aut}_\mathbf{R}(b_{p,q})$ and $\text{U}(p,q) = \text{Aut}_\mathbf{C}(h_{p,q})$. We use $h^\mathbf{C}_{p,q}$ for the complex symmetric bilinear form extending $h_{p,q}$ on $\mathbf{R}^{p+q} \otimes \mathbf{R} \mathbf{C}$. Recall that a Hermitian form $h$ on a complex vector space $V$ is completely determined by its real part $b$: for any $v,v' \in V$,

$$h(v,v') = b(v,v') - \sqrt{-1}b(v,\sqrt{-1}v').$$

Similarly, an $\mathbf{H}$-Hermitian form $h_\mathbf{H}$ on an $\mathbf{H}$-vector space $V$ is completely determined by its complex part $h$: for any $v,v' \in V$,

$$h_\mathbf{H}(v,v') = h(v,v') - h(v,v' j) j.$$
$G \hookrightarrow G_2 = O(p+1,q+1)$ is $Q_1(b_{p+1,q+1})$-Anosov and its boundary map $\xi_{p^2} : \partial_\infty \Gamma \to F_1(b_{p+1,q+1})$ is the composition of $\xi_\rho$ with the natural inclusion $F_1(b_{p+1,q}) \hookrightarrow F_1(b_{p+1,q+1})$. By Theorem 7.1, a domain of discontinuity $\Omega_{p^2}$ with compact quotient is given by the complement in $F_0(b_{p+1,q+1})$ of

$$K_{p^2} = \{ L \in F_0(b_{p,q+1}) | \exists \eta \in \partial_\infty \Gamma, \ L \subset \xi_{p^2}(\eta) \}.$$ 

Note that $K_{p^2} \subset F_0(b_{p+1,q})$ by construction of $\xi_{p^2}$, hence $\Omega_{p^2}$ contains the $G$-invariant open set

$$U := F_0(b_{p+1,q+1}) \setminus F_0(b_{p+1,q})$$

and $\Gamma$ acts properly discontinuously on $U$. One checks that $U$ is the $G$-orbit of the line $L_0 := \mathbb{R}(v_{1,0} + v_{0,1})$ where $v_{1,0} \in \mathbb{R}^{1,0} \oplus \{0\} \subset \mathbb{R}^{1,0} \oplus \mathbb{R}^{p,q+1} = \mathbb{R}^{p+1,q+1}$ satisfies $b_1(v_{1,0}) = 1$, and $v_{0,1} \in \{0\} \oplus \mathbb{R}^{0,1} \subset \mathbb{R}^{p+1,q} \oplus \mathbb{R}^{0,1} = \mathbb{R}^{p+1,q+1}$ satisfies $b_0(v_{0,1}) = -1$. It identifies with $G/H$ since the stabilizer in $G$ of $L_0$ is the stabilizer in $G$ of the vector $v_{1,0}$, namely $H = O(p,q)$.

Cases (ii) and (iii) are treated similarly, using complex or quaternionic Hermitian forms instead of real quadratic forms.

For case (iv), note that the subcase $p = 1$ was treated in [GW12, Th. 13.3]; we now write the proof for general $p \geq 1$. We identify $C^{p+q}$ with $\mathbb{R}^{2p+2q}$ and see $H = U(p,q)$ as the subgroup of $G = O(2p,2q)$ commuting with the multiplication by $\sqrt{-1}$, which we denote by $I \in G$. If $\rho : \Gamma \to G$ is $Q_0(b_{2p,2q})$-Anosov, with boundary map $\xi_\rho : \partial_\infty \Gamma \to F_0(b_{2p,2q})$, then the composed representation $\rho_2 : \Gamma \to G \hookrightarrow G_2 = O(2p+2q,\mathbb{C}) = \text{Aut}(b_{2p,2q}^\mathbb{C})$ is $Q_0(b_{2p,2q}^\mathbb{C})$-Anosov and its boundary map $\xi_{p^2} : \partial_\infty \Gamma \to F_0(b_{2p,2q}^\mathbb{C})$ is the composition of $\xi_\rho$ with the natural inclusion $F_0(b_{2p,2q}) \hookrightarrow F_0(b_{2p,2q}^\mathbb{C})$. By Theorem 7.1, a domain of discontinuity $\Omega_{p^2}$ with compact quotient is given by the complement in $F_1(b_{2p,2q}^\mathbb{C})$ of

$$K_{p^2} = \{ W \in F_1(b_{2p,2q}^\mathbb{C}) | \exists \eta \in \partial_\infty \Gamma, \ \xi_{p^2}(\eta) \subset W \}.$$ 

Note that $K_{p^2} \subset \{ W \in F_1(b_{2p,2q}) | W \cap \mathbb{R}^{2p+2q} \neq \{0\} \}$, hence $\Omega_{p^2}$ contains the $G$-invariant open set

$$U := \{ W \in F_1(b_{2p,2q}^\mathbb{C}) | W \cap \mathbb{R}^{2p+2q} = \{0\} \}$$

and $\Gamma$ acts properly discontinuously on $U$. One checks that $U$ is the $G$-orbit of

$$W_0 := \{ x + \sqrt{-1}I(x) | x \in \mathbb{R}^{2p+2q} \} \in F_1(b_{2p,2q}^\mathbb{C}).$$

It identifies with $G/H$ since the stabilizer of $W_0$ in $G$ is the set of elements $g \in O(2p,2q)$ commuting with $I$, namely $H = U(p,q)$.

We now address case (v). Let us write $H = \mathbb{C} + j \mathbb{C}$ where $j^2 = -1$. We identify $\mathbb{H}^{2p+2q}$ with $\mathbb{C}^{2p+2q}$ and see $H = \text{Sp}(p,q)$ as the subgroup of $G = U(2p,2q)$ commuting with the right multiplication by $j$, which we denote by $J \in G$. The tensor product $\mathbb{C}^{2p+2q} \otimes \mathbb{C} H$ can be realized as the set of “formal” sums

$$\mathbb{C}^{2p+2q} \otimes \mathbb{C} H = \{ v_1 + j v_2 | v_1, v_2 \in \mathbb{C}^{2p+2q} \},$$

on which $H$ acts by right multiplication: $(v_1 + j v_2) \cdot z = z v_1 + j \bar{z} v_2$ and $(v_1 + j v_2) \cdot j = -v_2 + v_1 j$ for $z \in \mathbb{C}$. Consider the $\mathbb{C}$-Hermitian form $h$ on
We identify \(I,J\) and then this orbit identifies with \(\mathcal{O}(24)\). The embedding of \(G = U(2p, 2q) = AutC(h_{2p, 2q})\) into \(G_2\) induces a natural embedding of \(F_0(h_{2p, 2q})\) into \(F_0(h_{12})\), given explicitly by

\[L \mapsto \{v_1 + v_2 j | v_1, v_2 \in L\}\]

for any \(h_{2p, 2q}\)-isotropic line \(L\) of \(C^{2p+2q}\). If a representation \(\rho : \Gamma \to G\) is \(Q_0(h_{2p, 2q})\)-Anosov, with boundary map \(\xi_\rho : \partial_\infty \Gamma \to F_0(h_{2p, 2q})\), then the composed representation \(\rho_2 : \Gamma \to G_2\) is \(Q_0(h_{12})\)-Anosov and its boundary map \(\xi_{\rho_2} : \partial_\infty \Gamma \to F_0(h_{12})\) is the composition of \(\xi_\rho\) with the natural inclusion \(F_0(h_{2p, 2q}) \to F_0(h_{12})\). By Theorem 7.1, a domain of discontinuity \(\Omega_{\rho_2}\) with compact quotient is given by the complement in \(F_1(h_{12})\) of

\[K_{\rho_2} = \{W \in F_1(h_{12}) | \exists \eta \in \partial_\infty \Gamma, \xi_{\rho_2}(\eta) \in W\}\]

Note that \(K_{\rho_2} \subset \{W \in F_1(h_{12}) | W \cap C^{2p+2q} \neq \{0\}\}\), hence \(\Omega_{\rho_2}\) contains the \(G\)-invariant open set

\[U : = \{W \in F_1(h_{12}) | W \cap C^{2p+2q} = \{0\}\}\]

which itself contains the element

\[W_0 : = \{v + Jv j | v \in C^{2p, 2q}\}\]

In particular, the action of \(G\) on the \(G\)-orbit of \(W_0\) is properly discontinuous. This orbit identifies with \(G/H\) since the stabilizer of \(W_0\) in \(G\) is the set of elements \(g \in U(2p, 2q)\) commuting with \(J\), namely \(H = Sp(p, q)\). For each \(\langle i, j \rangle\), we write \(H = R + R i + R j + R k\) where \(i = \sqrt{-1}\) and \(i j = k\). We identify \(H_{2p+q}\) with \(R^{4p+4q}\), and see \(H = Sp(p, q)\) as the subgroup of \(G = O(4p, 4q)\) commuting with the right multiplications by \(i\) and by \(j\), which we denote respectively by \(I, J \in G\). The tensor product \(R^{4p+4q} \otimes R H\) can be realized as the set of “formal” sums

\[R^{4p+4q} \otimes R H = \{v_1 + v_2 i + v_3 j + v_4 k | v_1, v_2, v_3, v_4 \in R^{4p+4q}\}\]

Consider the real bilinear form \(b\) on \(R^{4p+4q} \otimes R H\) given by

\[b(v_H, v_H') = b_{4p, 4q}(v_1, v_1') - b_{4p, 4q}(v_2, v_2') + b_{4p, 4q}(v_3, v_3') - b_{4p, 4q}(v_4, v_4')\]

for any \(v_H = v_1 + v_2 i + v_3 j + v_4 k\) and \(v_H' = v_1' + v_2' i + v_3' j + v_4' k\) in \(R^{4p+4q} \otimes R H\), and let \(b_H\) be the \(H\)-Hermitian form on \(R^{4p+4q} \otimes R H\) with real form \(b\). Then \(G_2 = Sp(2p + 2q) = Sp(2p, 2q)\) identifies with \(Aut_H(b_H)\), and the natural embedding of \(G = O(4p, 4q) = Aut_R(b_{4p, 4q})\) into \(G_2\) induces a natural embedding of \(F_0(b_{4p, 4q})\) into \(F_0(h_{12})\). If a representation \(\rho : \Gamma \to G\) is \(Q_0(b_{4p, 4q})\)-Anosov, with boundary map \(\xi_\rho : \partial_\infty \Gamma \to F_0(b_{4p, 4q})\), then the composed representation \(\rho_2 : \Gamma \to G_2\) is \(Q_0(h_{12})\)-Anosov and its boundary map \(\xi_{\rho_2} : \partial_\infty \Gamma \to F_0(h_{12})\) is the composition of \(\xi_\rho\) with the natural inclusion \(F_0(b_{4p, 4q}) \to F_0(h_{12})\). By Theorem 7.1, a domain of discontinuity \(\Omega_{\rho_2}\) with compact quotient is given by the complement in \(F_1(h_{12})\) of

\[K_{\rho_2} = \{W \in F_1(h_{12}) | \exists \eta \in \partial_\infty \Gamma, \xi_{\rho_2}(\eta) \in W\}\]
Let $F_2(b_H)$ be the space of partial flags $W' \subset W$ of $\mathbb{R}^{4p+4q} \otimes \mathbb{R} H$ with $W \in F_1(b_H)$ and $\dim_H(W) = 2n + 2 = 2 \dim_H(W')$. It fibers $G_2$-equivariantly over $F_1(b_H)$ with compact fiber, hence $\Gamma$ acts properly discontinuously via $\rho_2$ on the preimage $\tilde{\Omega}_{\rho_2}$ of $\Omega_{\rho_2}$ in $F_2(b_H)$. One checks that $\tilde{\Omega}_{\rho_2}$ contains the $G$-invariant open set
\[\{(W' \subset W) \in F_2(b_H) \mid W \cap \mathbb{R}^{4p+4q} = \{0\}\},\]
which itself contains the $G$-orbit of the flag $W_0' \subset W_0$ where
\[W_0' := \{v + Iv_i + Jv_j + Kv_k \mid v \in \mathbb{R}^{4p+4q}\},\]
\[W_0 := \{v + Iv_i + Jv'_j + K\overline{v}'_k \mid v, v' \in \mathbb{R}^{4p+4q}\}.\]
This orbit identifies with $G/H$ since the stabilizer of $W_0' \subset W_0$ in $G$ is the set of elements $g \in O(4p, 4q)$ commuting with $I$ and $J$, namely $H = \text{Sp}(p, q)$.

Case (vii) is entirely similar to case (vi), where $b_{4p, 4q}$ is replaced by a symplectic form $\omega$ on $\mathbb{R}^{4n}$, and $b$ is now the symplectic form $\omega(v_1, v'_1) - \omega(v_2, v'_2) + \omega(v_3, v'_3) - \omega(v_4, v'_4)$ on $\mathbb{R}^{4n} \otimes \mathbb{R} H$. The subgroup of $G = \text{Sp}(4n, \mathbb{R}) = \text{Aut}_R(\omega)$ of elements that commute with $I$ and $J$ is precisely $H = O(2n)$.

Case (viii) for $p = 0$ is discussed in [GW12, §12]. Let $b$ (resp. $b_C$) be the standard symplectic form on $\mathbb{R}^{2n}$ (resp. $\mathbb{C}^{2n}$), so that $G = \text{Sp}(2n, \mathbb{R}) = \text{Aut}_R(b)$ and $G_2 = \text{Sp}(2n, \mathbb{C}) = \text{Aut}_C(b_C)$. If $\rho : \Gamma \to G$ is $Q_0(b)$-Anosov, with boundary map $\xi : \partial_\infty \Gamma \to F_0(b) \simeq \mathbb{RP}^{2n-1}$, then the composed representation $\rho_2 : \Gamma \to G \to G_2$ is $Q_0(b_C)$-Anosov and its boundary map $\xi_{\rho_2} : \partial_\infty \Gamma \to F_0(b) \simeq \text{CP}^{2n-1}$ is the composition of $\xi$, with the natural inclusion $\mathbb{RP}^{2n-1} \to \text{CP}^{2n-1}$. By Theorem 7.1, a domain of discontinuity $\tilde{\Omega}_{\rho_2}$ with compact quotient is given by the complement in the space $F_1(b_C)$ of Lagrangians of $\mathbb{C}^{2n}$ of $K_{\rho_2} = \{W \in F_1(b_C) \mid \exists t \in \partial_\infty \Gamma, \xi_{\rho_2}(t) \subset W\}$.

In particular, $\Omega_{\rho_C}$ contains the $G$-invariant open set
\[\mathcal{U} = \{W \in F_1(b_C) \mid W \cap \mathbb{R}^{2n} = \{0\}\}.
\]
For any $W \in \mathcal{U}$, the restriction of $h$ to $W \times W$ is nondegenerate. If it has signature $(p, n - p)$, then the $\text{Sp}(2n, \mathbb{R})$-orbit of $W$ is isomorphic to $\text{Sp}(2n, \mathbb{R})/U(p, n - p)$.

For case (ix), note that any $Q_0(h_{n,n})$-Anosov representation $\rho : \Gamma \to G = U(n, n)$ gives rise to a $P_{x_1-x_2}$-Anosov representation $\rho_2 : \Gamma \to G \to G_2 = \text{GL}(2n, \mathbb{C})$. Let $\xi : \partial_\infty \Gamma \to \text{CP}^{2n-1}$ be the boundary map. Then the domain of discontinuity $\tilde{\Omega}_{\rho_2}$ with compact quotient constructed in [GW12, §10.2.5] in the space of Grassmannian $n$-planes $\text{Gr}_n(\mathbb{C}^{2n})$ is the complement of $K_{\rho_2} = \{H \in \text{Gr}_n(\mathbb{C}^{2n}) \mid \exists t \in \partial_\infty \Gamma, \xi(t) \subset H\}$.

Since $\xi(t)$ is an $h$-isotropic line, $\Omega_{\rho_2}$ contains in particular $\mathcal{U}_+ \cup \mathcal{U}_-$, where
\[\mathcal{U}_\pm = \{H \in \text{Gr}_n(\mathbb{C}^{2n}) \mid \pm h|_H \times H \text{ is positive definite}\}.
\]
Note that $\mathcal{U}_\pm \cong U(n, n)/U(n) \times U(n)$.

The case (x) follows by the same argument, considering $H^{2n}$ with an anti-Hermitian form and replacing $\mathbb{C}$ with $\mathbb{H}$. □
Appendix A. An unstable quasi-isometrically embedded subgroup

In this appendix we give an example of a quasi-isometric embedding $\rho_0$ of a free group $\Gamma$ into a semisimple Lie group of higher real rank which is not stable under small deformations; in particular $\rho_0$ is not an Anosov representation of $\Gamma$. This example was first described in [Gui].

**Proposition A.1.** Let $\Gamma$ be a free group on two generators. Then there is a continuous family $\{\rho_t\}_{t \in [0,1]}$ of representations $\Gamma \to \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ such that

- $\rho_0$ is a quasi-isometric embedding;
- for any $t \notin \mathbb{Q}$, the group $\rho_t(\Gamma)$ is dense in $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ (for the real topology).

In order to prove Proposition A.1, we consider a free generating subset $\{a, b\}$ of $\Gamma$. For any $\gamma = a^{m_1}b^{n_1} \cdots a^{m_N}b^{n_N} \in \Gamma$ with $m_i \neq 0$ for all $i > 1$ and $n_i \neq 0$ for all $i < N$, we set

$$
\begin{align*}
|\gamma|_a &= |m_1| + \cdots + |m_N|, \\
|\gamma|_b &= |n_1| + \cdots + |n_N|.
\end{align*}
$$

Then the word length function $|\cdot| : \Gamma \to \mathbb{N}$ with respect to $\{a, b\}$ satisfies

$$
|\gamma| = |\gamma|_a + |\gamma|_b
$$

for all $\gamma \in \Gamma$. We identify the Weyl chamber $\overline{\mathfrak{a}}^+$ with $\mathbb{R}_+$, so that the Cartan projection $\mu : \text{SL}_2(\mathbb{R}) \to \overline{\mathfrak{a}}^+$ of Section 2.3.1 takes values in $\mathbb{R}_+$. With this notation, Proposition A.1 is an easy consequence of the following.

**Proposition A.2.** Let $\Gamma$ be a free group on two generators $a, b$ and let $K > 0$. Then there are a continuous family $\{\rho_{\alpha, t}\}_{t \in [0,1]}$ of representations $\Gamma \to \text{SL}_2(\mathbb{R})$ such that

- $\mu(\rho_{\alpha, 0}(\gamma)) \geq K|\gamma|_a$ for all $\gamma \in \Gamma$;
- for any $t \notin \mathbb{Q}$, the group $\rho_{\alpha, t}(\Gamma)$ is dense in $\text{SL}_2(\mathbb{R})$ (for the real topology) and the element $\rho_{\alpha, t}(a)$ (resp. $\rho_{\alpha, t}(b)$) is hyperbolic (resp. elliptic).

**Proof of Proposition A.1 using Proposition A.2.** Let $\{\rho_{\alpha, t}\}_{t \in [0,1]}$ be the continuous family of representations $\Gamma \to \text{SL}_2(\mathbb{R})$ given by Proposition A.2 (for $K = 1$ say), and $\{\rho_{\beta, t}\}_{t \in [0,1]}$ be defined by $\rho_{\beta, t} = \rho_{\alpha, t} \circ \varsigma$ where $\varsigma$ is the automorphism of $\Gamma$ switching $a$ and $b$. For any $t \in [0,1]$, consider the representation

$$
\rho_t = (\rho_{\alpha, t}, \rho_{\beta, t}) : \Gamma \to \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}).
$$

Then $\rho_0$ is a quasi-isometric embedding because

$$
\mu(\rho_{\alpha, 0}(\gamma)) + \mu(\rho_{\beta, 0}(\gamma)) \geq c(|\gamma|_a + |\gamma|_b) = c|\gamma|_r
$$

for all $\gamma \in \Gamma$. Let $G_t$ be the closure of $\rho_t(\Gamma)$ in $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. If $t \notin \mathbb{Q}$, then the two projections of $G_t$ to $\text{SL}_2(\mathbb{R})$ are equal to the full group $\text{SL}_2(\mathbb{R})$. In that case, by Goursat’s lemma (see e.g. [Pet09]), either $G_t = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ or $G_t = \{h, g h g^{-1} \mid h \in \text{SL}_2(\mathbb{R})\}$ for some $g$ in $\text{GL}_2(\mathbb{R})$. The second case cannot occur since it would imply that the hyperbolic element $\rho_{\alpha, t}(a)$ is conjugate to the elliptic element $\rho_{\beta, t}(a)$. □
Proof of Proposition A.2. We define \( \rho_{a,t}(a) \) to be a hyperbolic element \( A \in \text{SL}_2(\mathbb{R}) \), independent of \( t \), whose properties will be specified in a moment. For \( t > 0 \), we also define
\[
\rho_{a,t}(b) = \begin{pmatrix} \cos \pi t & \frac{1}{\pi t} \sin \pi t \\ \pi t \sin \pi t & \cos \pi t \end{pmatrix},
\]
and extend this by continuity to \( \rho_{a,0}(b) = B := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

For \( t \) in \([0, 1] \setminus \mathbb{Q}\), the element \( \rho_{a,t}(b) \) is conjugate to an irrational rotation, and so the closure of the group spanned by \( \rho_{a,t}(b) \) is a conjugate of \( \text{SO}(2) \); since \( \text{SL}_2(\mathbb{R}) \) is generated by any hyperbolic element and \( \text{SO}(2) \), we conclude that \( \rho_{a,t}(\Gamma) \) is dense in \( \text{SL}_2(\mathbb{R}) \).

We now show that, for some appropriate choice of the hyperbolic element \( A \), we have \( \mu(\rho_{a,0}(\gamma)) \geq K |\gamma|_a \) for all \( \gamma \in \Gamma \). Endow \( \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\} \) with the round metric centered at \( \sqrt{-1} \in \mathbb{H}^2 \). The parabolic element \( B = \rho_{a,0}(b) \) fixes the point \( \infty \in \mathbb{P}^1(\mathbb{R}) \) and there are two compact intervals \( V_- \) and \( V_+ \) of \( \mathbb{P}^1(\mathbb{R}) \) (intersecting only at the point \( \infty \)) such that
\[
\bullet \quad B(\mathbb{P}^1(\mathbb{R}) \setminus V_-) \subset V_+ \quad \text{and} \quad B|_{\mathbb{P}^1(\mathbb{R}) \setminus V_-} \text{ is 1-Lipschitz};
\]
\[
\bullet \quad B^{-1}(\mathbb{P}^1(\mathbb{R}) \setminus V_+) \subset V_- \quad \text{and} \quad B^{-1}|_{\mathbb{P}^1(\mathbb{R}) \setminus V_+} \text{ is 1-Lipschitz}.
\]
Choose the hyperbolic element \( A = \rho_{a,0}(a) \) so that there are disjoint compact intervals \( U_- \) and \( U_+ \) in \( \mathbb{P}^1(\mathbb{R}) \setminus (V_- \cup V_+) \) with the following properties:
\[
\bullet \quad A(\mathbb{P}^1(\mathbb{R}) \setminus U_-) \subset U_+ \quad \text{and} \quad A|_{\mathbb{P}^1(\mathbb{R}) \setminus U_-} \text{ is } e^{-K_1} \text{-contracting};
\]
\[
\bullet \quad A^{-1}(\mathbb{P}^1(\mathbb{R}) \setminus U_+) \subset U_- \quad \text{and} \quad A^{-1}|_{\mathbb{P}^1(\mathbb{R}) \setminus U_+} \text{ is } e^{-K_1} \text{-contracting}.
\]
Then for any \( \gamma \in \Gamma \), the element \( \rho_{a,0}(\gamma) \) is \( e^{-K_1} \)-contracting at every point in \( \mathbb{P}^1(\mathbb{R}) \setminus (V_- \cup V_+ \cup U_- \cup U_+) \). We obtain that \( \mu(\rho_{a,0}(\gamma)) \geq K |\gamma|_a \) for all \( \gamma \in \Gamma \) from the following lemma. \( \square \)

Lemma A.3. For any \( g \in \text{SL}_2(\mathbb{R}) \) and \( t \geq 0 \), if \( g \) is \( e^{-t} \)-contracting at some point of \( \mathbb{P}^1(\mathbb{R}) \), then \( \mu(g) \geq t \).

Proof. The assumptions do not change if we multiply \( g \) by elements of \( \text{SO}(2) \). Thus we can suppose that \( g = \left( \begin{smallmatrix} e^{-s/2} & e^{s/2} \\ e^{-s/2} & e^{s/2} \end{smallmatrix} \right) \) with \( s \geq 0 \) (so that \( \mu(g) = s \)). Let \( x_0 \in \mathbb{P}^1(\mathbb{R}) \) be a point where the differential \( (dg) \) is \( e^{-t} \)-contracting. Then \( x_0 \neq \infty \). Let \( \partial / \partial t \) be the translation-invariant vector field on \( \mathbb{R} \). Denoting by \( || \cdot ||_x \) the Riemannian norm of tangent vectors at a point \( x \in \mathbb{P}^1(\mathbb{R}) \), the norm \( ||\partial / \partial t||_x \) is a decreasing function of \( |x| \), and the contraction of \( dg \) at \( x_0 \) is
\[
\frac{\|dg \cdot \partial / \partial t\|_{g \cdot x_0}}{\|\partial / \partial t\|_{x_0}} = e^{-s} \frac{\|\partial / \partial t\|_{g \cdot x_0}}{\|\partial / \partial t\|_{x_0}} \leq e^{-t}.
\]
Since \( 1 \leq \frac{\|\partial / \partial t\|_{x_0}}{\|\partial / \partial t\|_{g \cdot x_0}} \), the conclusion \( \mu(g) = s \geq t \) follows. \( \square \)

Remark A.4. For \( t \in \mathbb{Q} \setminus \{0\} \), the representation \( \rho_t \) has a non-trivial kernel: a power \( b^n \) of \( b \) is in the kernel of \( \rho_{a,t} \) and \( a^n \) is in the kernel of \( \rho_{b,t} \) and therefore the commutator \( a^nb^n a^{-n}b^{-n} \) is in the kernel of \( \rho_t \). Thus \( \rho_0 \) is the endpoint of a continuous family of representations, all of them being nondiscrete or nonfaithful.

We can use the representation \( \rho_0 \) of Proposition A.1 to construct a representation \( \rho : \Gamma \to \text{SL}_d(\mathbb{R}) \) with the following properties:
• $\rho$ admits continuous, $\rho$-equivariant, transverse boundary maps $\xi^+ : \partial_\infty \Gamma \to P(R^6)$ and $\xi^- : \partial_\infty \Gamma \to P((R^6)^*)$.

• these boundary maps are not dynamics-preserving and $\rho$ is not Anosov with respect to the stabilizer of a line in $R^6$.

**Example A.5.** Let $\Gamma$ be a free group on two generators and $\rho' : \Gamma \to SL_2(R)$ a convex cocompact representation. We see $SL_2(R)$ as a subgroup of $G = SL_6(R)$ by embedding it into the upper left corner of $G$, and use the notation of Example 2.12 for $G$. By (2.8), there exists $k > 0$ such that

$$\langle \epsilon_1, \mu(\rho'(\gamma)) \rangle \leq k|\gamma|\Gamma$$

for all $\gamma \in \Gamma$. On the other hand, if we choose $K > k$ in Proposition A.2, then the representation $\rho_0 : \Gamma \to SL_2(R) \times SL_2(R)$ constructed in Proposition A.1, seen as a representation into $G = SL_6(R)$ by embedding $SL_2(R) \times SL_2(R)$ into the lower right corner of $G$, satisfies

$$\langle \epsilon_1, \mu(\rho_0(\gamma)) \rangle \geq K|\gamma|\Gamma$$

for all $\gamma \in \Gamma$. In particular, using (2.9), we see that $\rho_0$ uniformly $P_{\epsilon_1-\epsilon_2}$-dominates $\rho'$ as representations into $G$ (Definition 6.1). Consider the representation

$$\rho := (\rho_0, \rho') : \Gamma \longrightarrow (SL_2(R) \times SL_2(R)) \times SL_2(R) \longrightarrow G.$$

It admits two continuous, $\rho$-equivariant, transverse boundary maps $\xi^+ : \partial_\infty \Gamma \to P(R^6)$ and $\xi^- : \partial_\infty \Gamma \to P((R^6)^*)$, obtained by composing the boundary maps of the Anosov representation $\rho' : \Gamma \to SL_2(R)$ (Example 2.4.3.(a)) with the inclusion of $P(R^2) \simeq P(R^2 \times \{0\})$ into $P(R^6)$. These boundary maps cannot be dynamics-preserving: indeed, since the representation $\rho_0$ uniformly $P_{\epsilon_1-\epsilon_2}$-dominates $\rho'$, the attracting fixed point of $\rho(\gamma)$ is contained in $P(\{0\} \times R^4) \subset P(R^6)$ for any $\gamma \in \Gamma$ of infinite order. Furthermore, the representation $\rho$ is not $P_{\epsilon_1-\epsilon_2}$-Anosov, since otherwise the representation $\rho_0$ would have to be Anosov, contradicting Proposition A.1.
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[BIW10] Surface group representations with maximal Toledo invariant, Ann. of Math. (2) 172 (2010), 517–566.

[BJT] Maciej Boček, Piotr Jastrzębski, and Aleksy Tralle, Proper $\text{SL}(2, \mathbb{R})$-actions on homogeneous spaces, preprint, arXiv:1405.4167.

[BL93] Yves Benoist and François Labourie, Sur les difféomorphismes d’Anosov affines à feuilletages stable et instable différentiables, Invent. Math. 111 (1993), 285–308.

[BM12] Thierry Barbot and Quentin Mérigot, Anosov AdS representations are quasi-Fuchsian, Groups Geom. Dyn. 6 (2012), 441–483.

[Bou68] Nicolas Bourbaki, Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV : Groupes de Coxeter et systèmes de Tits. Chapitre V : Groupes engendrés par des réflexions. Chapitre VI : Systèmes de racines, Actualités Scientifiques et Industrielles 1337, Hermann, Paris, 1968.

[Bou75] Éléments de mathématique. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII : Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII : Algèbres de Lie semi-simples déployées, Actualités Scientifiques et Industrielles 1364, Hermann, Paris, 1975.

[Bou95] Marc Bourdon, Structure conforme au bord et flot géodésique d’un CAT(−1)-espace, Enseign. Math. (2) 41 (1995), 63–102.

[Bow95] Brian H. Bowditch, Geometrical finiteness with variable negative curvature, Duke Math. J. 77 (1995), 229–274.

[Bow98] A topological characterisation of hyperbolic groups, J. Amer. Math. Soc. 11 (1998), 643–667.

[BT65] Armand Borel and Jacques Tits, Groupes réductifs, Publ. Math. Inst. Hautes Études Sci. 27 (1965), 55–150.

[CDP90] Michel Coornaert, Thomas Delzant, and Athanase Papadopoulos, Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], with an English summary, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, Berlin, 1990, Géométrie et théorie des groupes.

[Cha94] Christophe Champetier, Petite simplification dans les groupes hyperboliques, Ann. Fac. Sci. Toulouse Math. 3 (1994), 161–221.

[DGK] Jeffrey Danciger, François Guéritaud, and Fanny Kassel, Margulis spacetimes via the arc complex, preprint, arXiv:1407.5422.

[DT] Bertrand Deroin and Nicolas Tholozan, Dominating surface group representations by Fuchsian ones, preprint, arXiv:1311.2919.

[Ebe96] Patrick B. Eberlein, Geometry of nonpositively curved manifolds, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996.

[Flo80] William J. Floyd, Group completions and limit sets of Kleinian groups, Invent. Math. 57 (1980), 205–218.

[GGKW] François Guéritaud, Olivier Guichard, Fanny Kassel, and Anna Wienhard, Compactification of Clifford–Klein forms, in preparation.

[Ghy95] Étienne Ghys, Déformations des structures complexes sur les espaces homogènes de $\text{SL}(2, \mathbb{C})$, J. Reine Angew. Math. 468 (1995), 113–138.

[GK] François Guéritaud and Fanny Kassel, Maximally stretched laminations on geometrically finite hyperbolic manifolds, Geom. Topol., to appear.

[GW] François Guéritaud, Fanny Kassel, and Maxime Wolff, Compact anti-de Sitter $3$-manifolds and folded hyperbolic structures on surfaces, Pacific J. Math., to appear.

[GM89] Ilya Ya. Gol’dsheid and Grigory A. Margulis, Lyapunov exponents of a product of random matrices, Russian Math. Surveys 44 (1989), 11–71.

[Gold85] William M. Goldman, Nonstandard Lorentz space forms, J. Differential Geom. 21 (1985), 301–308.

[GR89] Yves Guivarc’h and G. Raugi, Propriétés de contraction d’un semi-groupe de matrices inversibles, Israel J. Math. 65 (1989), 165–196.

[Gro87] Mikhail Gromov, Hyperbolic groups, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263.
[Tho] Nicolas Tholozan, Sur la complétude de certaines variétés pseudo-riemann niennes localement symétriques, Ann. Inst. Fourier, to appear.

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