**A_p-A_∞ estimates for general multilinear sparse operators**

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**Abstract.** In this paper, we study the $A_p-A_∞$ estimates for a class of multilinear dyadic positive operators. As applications, the $A_p-A_∞$ estimates for different operators e.g. multilinear square functions and multilinear Fourier multipliers can be deduced very easily.

1. Introduction

The weighted norm inequality is a hot topic in harmonic analysis. In 1980s, Buckley [1] studied the quantitative relation between the weighted bound of Hardy-Littlewood maximal function and the $A_p$ constant. Specifically, he showed that

$$\|M\|_{L^p(w)} \leq c[w]_{A_p}^\frac{1}{p-1},$$

where recall that

$$[w]_{A_p} := \sup_Q \langle w \rangle_Q \langle w^1-p' \rangle_Q^{p-1}.$$ 

Here and through out, $\langle \cdot \rangle_Q$ denotes the average over $Q$.

Since then, the sharp weighted estimates for Calderón-Zygmund operators has attracted many authors’ interest, which was referred to as the famous $A_2$ conjecture. The $A_2$ conjecture (now theorem) asserts that

$$\|T\|_{L^p(w)} \leq c[w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}.$$ 

It was finally proved by Hytönen [11]. The interested readers can consult [12] for a survey on the history of the different proofs given for $A_2$ theorem. Moreover, Hytönen and Lacey [13] extends the $A_2$ theorem to the so-called $A_p-A_∞$ type estimates, i.e.,

$$\|T(\sigma)\|_{L^p(\sigma) \to L^p(w)} \leq c[w, \sigma]_{A_p}^{\frac{1}{p}} ([w]_{A_∞}^\frac{1}{p} + [\sigma]_{A_∞}^\frac{1}{p}),$$

where

$$[w, \sigma]_{A_p} := \sup_Q \langle w \rangle_Q \langle \sigma \rangle_Q^{p-1}, \quad [w]_{A_∞} := \sup_Q \frac{1}{w(Q)} \int_Q M(1_Q w) dx$$

and $\sigma$ needn’t to be the dual weight of $w$, i.e., we don’t require that $\sigma = w^{1-p'}$.
Now the story goes to the multilinear case. First we need to extend the $A_p$ weights to the multilinear case. Let $1 < p_1, \ldots, p_m < \infty$ and $p$ be numbers such that $1/p = 1/p_1 + \cdots + 1/p_m$ and denote $\bar{p} = (p_1, \ldots, p_m)$. Now we define $[w, \sigma]_{A_{\bar{p}}}$ constant:

$$[w, \sigma]_{A_{\bar{p}}} = \sup_Q \langle w \rangle_Q \prod_{i=1}^m \langle \sigma_i \rangle_{p_i}.$$ 

In the one weight case, i.e., $\sigma_i = w_i^{1-\frac{1}{p_i}}$ and $w = \prod_{i=1}^m w_i^{\frac{p}{p_i}}$, we say that $\bar{w}$ satisfies the $A_{\bar{p}}$ condition if $[w, \sigma]_{A_{\bar{p}}} < \infty$, see [22]. For the Buckley type estimate, the second author, Moen and Sun [23] studied the sharp weighted estimates for multilinear maximal Calderón-Zygmund operators when $p > 1$. The corresponding $A_{p^0}A_\infty$ estimate was obtained in [7] and [24], respectively. Specifically, the result for multilinear maximal operators reads as

$$\|M(\bar{\sigma})\|_{Lp_1(\sigma_1) \times \cdots \times Lp_m(\sigma_m) \to Lp(w)} \leq [w, \sigma]_{A_{\bar{p}}} \prod_{i=1}^m [\sigma_i]_{A_{\infty}}.$$ 

As to the multilinear Calderón-Zygmund operators, if $p > 1$, then

$$\|T(\bar{\sigma})\|_{Lp_1(\sigma_1) \times \cdots \times Lp_m(\sigma_m) \to Lp(w)} \leq [w, \sigma]_{A_{\bar{p}}} \left( \prod_{i=1}^m [\sigma_i]_{A_{\infty}} + [w]_{A_{\infty}} \prod_{j=1}^m \prod_{i \neq j} [\sigma_i]_{A_{\infty}} \right).$$

The spirit of the above results is reducing the problem to consider the so-called sparse operators. Recall that given a dyadic grid $D$, we say a collection $S \subset D$ is sparse if

$$\left| \bigcup_{Q' \subseteq Q} Q' \right| \leq \frac{1}{2}|Q|,$$

and we denote $E_Q := Q \setminus \bigcup_{Q' \subseteq S, Q' \not\subseteq Q} Q'$. Now given a sparse family $S$ over a dyadic grid $D$ and $\gamma \geq 1$, a general multilinear sparse operator is an averaging operator over $S$ of the form

$$T_{p_0, \gamma, S}(\bar{f})(x) = \left( \sum_{Q \in S} \left[ \prod_{i=1}^m \langle f_i \rangle_{Q, p_0} \right]^{\gamma} \chi_Q(x) \right)^{1/\gamma},$$

where $p_0 \in [1, \infty)$ and for any cube $Q$,

$$\langle f \rangle_{Q, p_0} := \left( \frac{1}{|Q|} \int_Q |f(x)|^{p_0} \, dx \right)^{1/p_0}.$$ 

It was proved in [5] that the multilinear Calderón-Zygmund operators are dominated pointwisely by $T_{1,1,S}$. In [4], Bui and the first author also showed that the multilinear square functions are dominated pointwisely by $T_{1,2,S}$, and therefore, they obtained the Buckley type estimate for multilinear square functions. For $\gamma = 1$ and general $p_0$, it was shown in [22] that $T_{p_0,1,S}$ can dominates a large class of operators with rough kernels (which include multilinear Fourier multipliers) as well. Therefore, everything are reduced to study $T_{p_0,\gamma,S}$. Our main result states as follows.
Theorem 1.1. Let $\gamma > 0$. Suppose that $p_0 < p_1, \ldots, p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w$ and $\sigma$ be weights satisfying that $[w, \sigma]_{A_{p/p_0}} < \infty$ and $w, \sigma_i \in A_\infty$ for $i = 1, \ldots, m$. If $\gamma \geq p_0$, then
\[
\left\| T_{p_0, \gamma, \mathcal{S}}(f) \right\|_{L^p(w)} \lesssim [w, \sigma]_{A_{p/p_0}}^\frac{1}{p} \left( \prod_{i=1}^{m} [\sigma_i]_{A_\infty}^\frac{1}{p_i} + [w]_{A_\infty}^{\left( \frac{1}{p_0} - \frac{1}{p} \right)} \right) \sum_{j=1}^{m} \prod_{i \neq j} [\sigma_i]_{A_\infty} \times \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)},
\]
where $w_i = \sigma_i^{\frac{1}{p_0} - \frac{1}{p_i}}$, $i = 1, \cdots, m$ and
\[
\left( \frac{1}{\gamma} - \frac{1}{p} \right)_+ := \max \left\{ \frac{1}{\gamma} - \frac{1}{p}, 0 \right\}.
\]
If $\gamma < p_0$, then the above result still holds for all $p > \gamma$.

The proof of Theorem 1.1 is quite technical. In the literature, the $A_p - A_\infty$ estimates usually follows from testing condition. Our technique provide a way to obtain $A_p - A_\infty$ estimates without testing conditions. The idea follows from a recent paper by Lacey and the second author [15], where they studied the $A_p - A_\infty$ estimates for square functions in the linear case. We generalize their method to suit for the multilinear case with general parameters $\gamma$ and $p_0$.

2. PROOF OF THEOREM 1.1

Let us first observe that it suffices to prove Theorem 1.1 for $p_0 = 1$. Indeed, suppose Theorem 1.1 holds for $p_0 = 1$. Consider the two weight norm inequality
\[
\|T_{p_0, \gamma, \mathcal{S}}(f, g)\|_{L^p(w)} \leq \mathcal{N} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)},
\]
where we use $\mathcal{N}$ to denote the best constant such that (2.1) holds. Rewrite (2.1) as
\[
\|T_{p_0, \gamma, \mathcal{S}}(f^{1/p_0}, g^{1/p_0})\|_{L^p(w)}^{p_0} \leq \mathcal{N}^{p_0} \|f^{1/p_0}\|_{L^{p_1}(w_1)} \|g^{1/p_0}\|_{L^{p_2}(w_2)}^{p_0},
\]
which is equivalent to the following
\[
\|T_{1/p_0, \gamma, \mathcal{S}}(f, g)\|_{L^{p_0}(w)} \leq \mathcal{N}^{p_0} \|f\|_{L^{p_1}(w_1)} \|g\|_{L^{p_2}(w_2)}^{p_0}.
\]
Then by our assumption, we have
\[
\mathcal{N} \lesssim [w, \sigma]_{A_{p_0/p_0}}^{\frac{1}{p_0}} \left( [\sigma_1]_{A_\infty}^{\frac{1}{p_1}} [\sigma_2]_{A_\infty}^{\frac{1}{p_2}} + [w]_{A_\infty}^{\left( \frac{1}{p_0} - \frac{1}{p} \right)_+} \left( [\sigma_1]_{A_\infty}^{\frac{1}{p_1}} + [\sigma_2]_{A_\infty}^{\frac{1}{p_2}} \right) \right).
\]
So we concentrate on the case $p_0 = 1$. As in [24], we begin with $m = 2$, that is we deal with the dyadic bilinear operators:
\[
T(f, g) := \left( \sum_{Q \in \mathcal{S}} \langle f \rangle_Q^\gamma \langle g \rangle_Q^\gamma 1_Q \right)^\gamma
\]
and we shall give the corresponding $A_p - A_\infty$ estimate.

Without loss of generality, we can assume that all cubes in $\mathcal{S}$ are contained in some root cube. As usual we only work on a subfamily $\mathcal{S}_\alpha$, which is defined by the following
\[
\mathcal{S}_\alpha := \{ Q \in \mathcal{S} : 2^a < \langle w \rangle_Q \langle \sigma_1 \rangle_Q^{\frac{1}{p_1}} \langle \sigma_2 \rangle_Q^{\frac{1}{p_2}} \leq 2^{a+1} \}.
\]
Now we can define the principal cubes $\mathcal{F}$ for $(f, \sigma_1)$ and $\mathcal{G}$ for $(g, \sigma_2)$. Namely,

$$
\mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_k, \quad \mathcal{F}_0 := \{ \text{maximal cubes in } S_a \}
$$

$$
\mathcal{F}_{k+1} := \bigcup_{F \in \mathcal{F}_k} \text{ch}_F(F), \quad \text{ch}_F(F) := \{ Q \subseteq F \text{ maximal s.t. } (f)_{Q}^{p_1} > 2(f)_{F}^{p_1} \},
$$

and analogously for $\mathcal{G}$. We use $\pi_F(Q)$ to denote the minimal cube in $\mathcal{F}$ which contains $Q$ and $\pi(Q) = (F, G)$ means that $\pi_F(Q) = F$ and $\pi_G(Q) = G$. By our construction, it is easy to see that

$$
\sum_{F \in \mathcal{F}} ((f)_{F}^{p_1})^{p_1} \sigma_1(F) \lesssim \| f \|_{L^p_{\mathcal{F}}(\sigma_1)}^p.
$$

We are going to prove that if $w, \sigma_1, \sigma_2$ be weights satisfying that $[w, \bar{\sigma}]_{A_\rho} < \infty$ and $w, \sigma_1, \sigma_2 \in A_\infty$. Then

$$
\| T(f \sigma_1, g \sigma_2) \|_{L^p(w)} \lesssim [w, \bar{\sigma}]_{A_\rho} \left( \left[ \sigma_1 \right]_{A_\infty}^{\frac{1}{p_1}} \left[ \sigma_2 \right]_{A_\infty}^{\frac{1}{p_2}} \right) + \left[ w \right]_{A_\infty}^{\frac{1}{p}} \left( \left[ \sigma_1 \right]_{A_\infty}^{\frac{1}{p_1}} + \left[ \sigma_2 \right]_{A_\infty}^{\frac{1}{p_2}} \right) \| f \|_{L^p_\mathcal{F}(\sigma_1)} \| g \|_{L^p_\mathcal{G}(\sigma_2)}.
$$

First, we consider the case $p \leq \gamma$ with $\gamma \geq 1$. In this case, we have

$$
\left\| \left( \sum_{Q \in \mathcal{S}_a} \langle f \sigma_1 \rangle_{Q}^{\gamma} \langle g \sigma_2 \rangle_{Q}^{\gamma} 1_Q \right)^{\frac{1}{\gamma}} \right\|_{L^p(w)}
$$

$$
= \left\| \left( \sum_{Q \in \mathcal{S}_a} ((f)_{Q}^{p_1})^{\gamma} \langle g \rangle_{Q}^{p_2} \langle \sigma_1 \rangle_{Q}^{\gamma} \langle \sigma_2 \rangle_{Q}^{\gamma} 1_Q \right)^{\frac{1}{\gamma}} \right\|_{L^p(w)}
$$

$$
\lesssim \left\| \left( \sum_{F \in \mathcal{F}} ((f)_{F}^{p_1})^{\gamma} \sum_{G \in \mathcal{G}} \langle g \rangle_{G}^{p_2} \sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_{Q}^{\gamma} \langle \sigma_2 \rangle_{Q}^{\gamma} 1_Q \right)^{\frac{1}{\gamma}} \right\|_{L^p(w)}
$$

$$
\leq \left( \sum_{F \in \mathcal{F}} ((f)_{F}^{p_1})^{p} \sum_{G \in \mathcal{G}} \langle g \rangle_{G}^{p_2} \left\| \sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_{Q}^{\gamma} \langle \sigma_2 \rangle_{Q}^{\gamma} 1_Q \right\|_{L^p(w)}^{p} \right)^{\frac{1}{p}}
$$

$$
\lesssim \left( \sum_{F \in \mathcal{F}} ((f)_{F}^{p_1})^{p} \sum_{G \in \mathcal{G}} \langle g \rangle_{G}^{p_2} \left\| \sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_{Q}^{\gamma} \langle \sigma_2 \rangle_{Q}^{\gamma} 1_Q \right\|_{L^p(w)}^{p} \right)^{\frac{1}{p}}
$$

$$
+ \left( \sum_{G \in \mathcal{G}} \langle g \rangle_{G}^{p_2} \sum_{F \in \mathcal{F}} ((f)_{F}^{p_1})^{p} \left\| \sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_{Q} \langle \sigma_2 \rangle_{Q} 1_Q \right\|_{L^p(w)}^{p} \right)^{\frac{1}{p}}
$$

$$
:= I + II.
$$
By symmetry we only focus on estimating $I$. From [21], we already know that

\begin{equation}
\| \sum_{Q \in S_n} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q 1_Q \|_{L^p(u)} \lesssim 2^p \left( \sum_{Q \in S_n} \sigma_1(Q) \right)^{\frac{p}{p_1}} \left( \sum_{Q \in S_n} \sigma_2(Q) \right)^{\frac{p}{p_2}}.
\end{equation}

We also recall a fact that, for $\sigma \in A_\infty$ and $S$ a sparse family, we have

$$\sum_{Q \in S} \sigma(Q) \leq 2 \sum_{Q \in S} \langle \sigma \rangle_Q |E_Q| \leq 2 \int_R M(1_R \sigma) dx \lesssim 2[\sigma]_{A_\infty} \sigma(R).$$

Therefore,

$$I \lesssim 2^p \left( \sum_{F \in F} ((f)_F^1)^p \sum_{G \in G_F} ((g)_G^2)^p \left( \sum_{Q \in S_n} \sigma_1(Q) \right)^{\frac{p}{p_1}} \left( \sum_{Q \in S_n} \sigma_2(Q) \right)^{\frac{p}{p_2}} \right)^{\frac{1}{p}}$$

$$\lesssim 2^p \left[ \sigma_2 \right]_{A_\infty} \left( \sum_{F \in F} ((f)_F^1)^p \left( \sum_{G \in G_F} ((g)_G^2)^p \left( \sum_{Q \in S_n} \sigma_1(Q) \right)^{\frac{p}{p_1}} \right)^{\frac{p}{p_2}} \right)^{\frac{1}{p}}$$

$$\leq 2^p \left[ \sigma_2 \right]_{A_\infty} \left( \sum_{F \in F} ((f)_F^1)^p \sum_{G \in G_F \pi_F(G)=F} \sum_{Q \in S_n} \sigma_1(Q) \right)^{\frac{1}{p_1}}$$

$$\times \left( \sum_{F \in F} \sum_{G \in G_{\pi_F(G)=F}} ((g)_G^2)^p \sigma_2(G) \right)^{\frac{1}{p_2}}$$

$$\lesssim 2^p \left[ \sigma_1 \right]_{A_\infty} \left[ \sigma_2 \right]_{A_\infty} \| f \|_{L^{p_1}(\sigma_1)} \| g \|_{L^{p_2}(\sigma_2)},$$

where (2.2) is used in the last step.

2.2. The case $p > \gamma$ with $p_1 = \max \{p_1, p_2, q'\}$. Here $q = p/\gamma$. By duality, we have

$$\left\| \left( \sum_{Q \in S_n} ((f)_Q^1) \gamma ((g)_Q^2) \gamma \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q 1_Q \right)^{\frac{1}{\gamma}} \right\|_{L^p(u)}$$

$$= \sup_{\| h \|_{L^{q'}(u)}} \sum_{Q \in S_n} ((f)_Q^1) \gamma ((g)_Q^2) \gamma \langle h \rangle_Q \gamma \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \gamma \langle \sigma \rangle_Q Q w(Q).$$

Now we suppress the supremum, and denote by $\mathcal{H}$ the principal cubes associated to $(h, w)$. Similarly, $\pi(Q) = (F, G, H)$ means that $\pi_F(Q) = F$, $\pi_G(Q) = G$ and $\pi_H(Q) = H$. We
First we estimate $I$. We have

$$I \lesssim \sum_{F \in F} \langle f \rangle_F^\gamma \sum_{G \in G} \langle g \rangle_G^\gamma \sum_{H \in H} \langle h \rangle_H^w \sum_{Q \in S_a} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q w(Q)$$

$$\leq \sum_{F \in F} \langle f \rangle_F^\gamma \sum_{G \in G} \langle g \rangle_G^\gamma \int \left( \sum_{H \in H} \sum_{Q \in S_a} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \langle h \rangle_H^w \langle 1 \rangle_Q \right) \left( \sup_{H' \in H} \frac{\langle h \rangle_{H'}}{\langle 1 \rangle_{H'}} \right) \, dw$$

$$\leq \sum_{F \in F} \langle f \rangle_F^\gamma \sum_{G \in G} \langle g \rangle_G^\gamma \left\| \sum_{H \in H} \sum_{Q \in S_a} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \langle h \rangle_H^w \langle 1 \rangle_Q \right\|_{L^q(w)} \left( \sup_{H' \in H} \langle h \rangle_{H'}^w \langle 1 \rangle_{H'} \right)_{L^{q'}(w)}$$

$$\leq 2 \sum_{F \in F} \langle f \rangle_F^\gamma \sum_{G \in G} \langle g \rangle_G^\gamma \left( \sum_{H \in H} \sum_{Q \in S_a} \langle \sigma_1 \rangle_Q \right) \left( \sum_{H \in H} \sum_{Q \in S_a} \langle h \rangle_H^w \langle 1 \rangle_Q \right)_{L^q(w)}$$

$$\times \left( \sum_{Q \in S_a} \langle \sigma_2 \rangle_Q \right)^{\frac{\gamma'}{\gamma}} \left( \sum_{H \in H} \langle h \rangle_H^q \langle 1 \rangle_H \right)^{\frac{1}{q'}}.$$
Since $\frac{\gamma}{p_1} + \frac{\gamma}{p_2} + \frac{1}{q'} = 1$, by using Hölder’s inequality twice we have

$$I \lesssim 2^{\frac{\gamma}{p_1}} \left( \sum_{F \in \mathcal{F}} \left( \langle f \rangle_{p_1}^{\sigma} \right)^{p_1} \sum_{G \in \mathcal{G}} \sum_{H \in \mathcal{H}} \sum_{Q \in \mathcal{S}_a} \sigma_1(Q) \right)^{\frac{q}{p_1}}$$

$$\times \left( \sum_{F \in \mathcal{F}} \sum_{G \in \mathcal{G}} \sum_{H \in \mathcal{H}} \sum_{Q \in \mathcal{S}_a} \langle f \rangle_{p_1}^{\sigma_2} \langle g \rangle_{p_2} \langle h \rangle_{q'} \right)^{\frac{1}{p'}}.$$ 

It is obvious that $I'$ can be estimated similarly. Next we estimate $I_{II}$. We have

$$\sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_Q^{\gamma} \langle \sigma_2 \rangle_Q^{\gamma} w(Q)$$

$$= \sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_Q^{\gamma} \langle \sigma_2 \rangle_Q^{\gamma} \langle w \rangle_Q |Q|$$

$$= \sum_{Q \in \mathcal{S}_a} \langle \sigma_2 \rangle_Q^{\gamma-\frac{p_2}{p}} \langle w \rangle_Q^{1-\frac{p_1}{p}} |Q|$$

$$\lesssim 2^{\frac{\gamma p_1}{p}} \sum_{Q \in \mathcal{S}_a} \langle \sigma_2 \rangle_Q^{\gamma-\frac{p_2}{p}} \langle w \rangle_Q^{1-\frac{p_1}{p}} |Q|.$$ 

Since $p_1 = \max\{p_1, p_2, q'\}$ and $p > \gamma$, it is easy to check that

$$0 \leq \gamma - \frac{\gamma p_1}{p_2} < 1, \quad 0 \leq 1 - \frac{\gamma p_1}{p} < 1,$$

and

$$\frac{1}{r} := \gamma - \frac{\gamma p_1}{p_2} + 1 - \frac{\gamma p_1}{p} < 1.$$ 

Set

$$\frac{1}{s} := \gamma - \frac{\gamma p_1}{p_2} + \frac{1 - \frac{1}{r}}{2}.$$ 

Then

$$\frac{1}{s'} = 1 - \frac{\gamma p_1}{p} + \frac{1 - \frac{1}{r}}{2},$$

and therefore,

$$\sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_Q^{\gamma} \langle \sigma_2 \rangle_Q^{\gamma} w(Q) \lesssim 2^{\frac{\gamma p_1}{p}} \int_G M(\sigma_2 \mathbf{1}_G)^{\gamma-\frac{p_2}{p}} M(w \mathbf{1}_G)^{1-\frac{p_1}{p}} dx$$
\[ \leq 2 \frac{\gamma p a}{p} \left( \int_G M(\sigma_2 1_G)^s(\gamma - \frac{\gamma p}{p}) \, dx \right)^{\frac{1}{s}} \left( \int_G M(w 1_G)^s(1 - \frac{\gamma p}{p}) \, dx \right)^{\frac{1}{s}} \]

Before we give further estimate, we introduce the Kolmogorov’s inequality (see for example [22]): Let \(0 < p < q < \infty\), then there exists a constant \(C = C_{p,q}\) such that for any locally integrable function \(f\),
\[
\|f\|_{L^p(Q, \frac{df}{|df|})} \leq C \|f\|_{L^q(\frac{df}{|df|})}.
\]
With this inequality in hand, we have
\[
\frac{1}{|G|} \int_G M(w 1_G)^s(1 - \frac{\gamma p}{p}) \, dx \leq \|M(w 1_G)^s(1 - \frac{\gamma p}{p})\|_{L^1(\frac{df}{|df|})} \leq \langle w \rangle_G^{s(1 - \frac{\gamma p}{p})},
\]
and
\[
\left( \frac{1}{|G|} \int_G M(\sigma_2 1_G)^s(\gamma - \frac{\gamma p}{p}) \, dx \right) \leq \|M(\sigma_2 1_G)^s(\gamma - \frac{\gamma p}{p})\|_{L^1(\frac{df}{|df|})} \leq \langle \sigma_2 \rangle_G^{s(\gamma - \frac{\gamma p}{p})}.
\]
Thus we get
\[
\sum_{Q \in S} \langle \sigma_1 \rangle_Q^\gamma \langle \sigma_2 \rangle_Q^\gamma w(Q) \lesssim 2 \frac{\gamma p a}{p} \langle \sigma_2 \rangle_G^{\gamma - \frac{\gamma p}{p}} \langle w \rangle_G^{1 - \frac{\gamma p}{p}} |G| \lesssim 2 \frac{\gamma p a}{p} w(G)^{1 - \frac{\gamma}{p}} \sigma_1(G)^{\frac{\gamma}{p}} \sigma_2(G)^{\frac{\gamma}{p2}}.
\]
It follows that
\[
II \lesssim \sum_{F \in \mathcal{F}} \sum_{H \in \mathcal{H}} \sum_{G \in \mathcal{G}} \sum_{Q \in S} \langle \sigma_1 \rangle_Q^\gamma \langle \sigma_2 \rangle_Q^\gamma w(Q)
\]
\[
\lesssim \sum_{F \in \mathcal{F}} \sum_{H \in \mathcal{H}} \sum_{G \in \mathcal{G}} \sum_{Q \in S} \langle \sigma_1 \rangle_Q^\gamma \langle \sigma_2 \rangle_Q^\gamma \langle g \rangle_G^\gamma \langle h \rangle_H^\gamma w(Q) \lesssim 2 \frac{\gamma p a}{p} \langle \sigma_2 \rangle_G^{\gamma - \frac{\gamma p}{p}} \langle w \rangle_G^{1 - \frac{\gamma p}{p}} \sigma_1(G)^{\frac{\gamma}{p}} \sigma_2(G)^{\frac{\gamma}{p2}}
\]
\[
\lesssim 2 \frac{\gamma p a}{p} |w|_{A_{p2}}^{1 - \frac{\gamma}{p}} \langle \sigma_1 \rangle_{A_{p2}}^\gamma \langle g \rangle_{L^p(p, \sigma_1)}^\gamma \langle h \rangle_{L^p(p, \sigma_2)}^\gamma \langle w \rangle_{L^p(q)}^\gamma \langle \sigma_1 \rangle_{A_{p2}}^\gamma \langle \sigma_2 \rangle_{A_{p2}}^\gamma \langle g \rangle_{L^p(p, \sigma_1)}^\gamma \langle h \rangle_{L^p(q)}^\gamma \langle w \rangle_{L^p(q)}^\gamma,
\]
where again, the Hölder’s inequality and [22,22] are used in the last step.

Now we estimate \(II'\). By similar arguments as that in the above, we have
\[
\sum_{Q \in S} \langle \sigma_1 \rangle_Q^\gamma \langle \sigma_2 \rangle_Q^\gamma w(Q) \lesssim 2 \frac{\gamma p a}{p} \langle \sigma_2 \rangle_G^{\gamma - \frac{\gamma p}{p}} \langle w \rangle_G^{1 - \frac{\gamma p}{p}} \sigma_1(G)^{\frac{\gamma}{p}} \sigma_2(G)^{\frac{\gamma}{p2}}.
\]
Then it follows that
\[
II' \lesssim \sum_{G \in \mathcal{G}} \sum_{H \in \mathcal{H}} \sum_{F \in \mathcal{F}} \sum_{Q \in S} \langle \sigma_1 \rangle_Q^\gamma \langle g \rangle_G^\gamma \langle h \rangle_H^\gamma w(Q)
\]
\[
\leq 2 \frac{\gamma p a}{p} |w|_{A_{p2}}^{1 - \frac{\gamma}{p}} \langle \sigma_2 \rangle_{A_{p2}}^\gamma \langle g \rangle_{L^p(p, \sigma_1)}^\gamma \langle h \rangle_{L^p(q)}^\gamma \langle w \rangle_{L^p(q)}^\gamma \langle \sigma_1 \rangle_{A_{p2}}^\gamma \langle \sigma_2 \rangle_{A_{p2}}^\gamma \langle g \rangle_{L^p(p, \sigma_1)}^\gamma \langle h \rangle_{L^p(q)}^\gamma \langle w \rangle_{L^p(q)}^\gamma.
\]

\(III\) and \(III'\) can also be estimated similarly.

2.3. **The case \(p > \gamma\) with \(p_2 = \max\{p_1, p_2, q\}\).** By symmetry, this case can be estimated similarly as that in the previous subsection.
2.4. **The case \( p > \gamma \) with \( q' = \max\{p_1, p_2, q'\} \).** Again, we can decompose the summation to \( I + I' + II + II' + III + III' \). The estimates of \( I \) and \( I' \) have no differences with the previous case. Now we consider \( II \). We have

\[
\sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_Q^\gamma \langle \sigma_2 \rangle_Q^\gamma w(Q) = \sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_Q^\gamma \langle \sigma_2 \rangle_Q^\gamma |w(Q)|
\]

\[
= 2^{a} \sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_Q^\gamma \langle \sigma_2 \rangle_Q^\gamma |w(Q)|
\]

Since \( p > \gamma \) and \( q' \geq \max\{p_1, p_2\} \), we have

\[
\gamma - \frac{p}{p_1} \geq 0, \quad \gamma - \frac{p}{p_2} \geq 0,
\]

and

\[
\gamma - \frac{p}{p_1} + \gamma - \frac{p}{p_2} = 2\gamma + 2p - 1 < 1.
\]

Then follow the same arguments as that in the above, we get

\[
\sum_{Q \in \mathcal{S}_a} \langle \sigma_1 \rangle_Q^\gamma \langle \sigma_2 \rangle_Q^\gamma w(Q) \lesssim 2^{a} \langle \sigma_1 \rangle_G^{\gamma - \frac{p}{p_1}} \langle \sigma_2 \rangle_G^{\gamma - \frac{p}{p_2}} |G|
\]

\[
\lesssim 2^{a} \left( \langle w \rangle_G^{\gamma - \frac{p}{p_1}} \langle \sigma_1 \rangle_G^{\gamma - \frac{p}{p_1}} \right) \left( \langle w \rangle_G^{\gamma - \frac{p}{p_2}} \langle \sigma_2 \rangle_G^{\gamma - \frac{p}{p_2}} \right) |G|
\]

\[
= 2^{a} w(G)^{\gamma - \frac{p}{p_1}} \sigma_1(G)^{\gamma - \frac{p}{p_1}} \sigma_2(G)^{\gamma - \frac{p}{p_2}}.
\]

Then follow the same arguments as the previous subsection we can get the desired conclusion. The estimates of \( II' \), \( III \) and \( III' \) can also be estimated similarly.

3. **Applications**

Theorem 1.1 has some new applications. It is obvious that if an operator reduced to \( T_{p_0, \gamma, S} \) for some \( p_0 \) and \( \gamma \), then it is enough to apply Theorem 1.1 for those particular \( p_0 \) and \( \gamma \). Thus, to find out the \( A_p-A_\infty \) estimates for *Multilinear square functions* (which were introduced and investigated in [6, 28, 29]), considering Proposition 4.2. of [4], it is enough to apply Theorem 1.1 for \( T_{1,2,S} \).

To observe the other application, we first recall the class of multilinear integral operator which is bounded on certain products of Lebesgue spaces on \( \mathbb{R}^n \) where associated kernel satisfies some mild regularity condition which is weaker than the usual Hölder continuity of those in the class of multilinear Calderón-Zygmund singular integral operators. This class of the operators motivated from the recent works [3, 10, 14, 22, 25, 26, 27] and weighted bounds for such operators studied in [2] very recently. The main example of such operators is *Multilinear Fourier multipliers*. Now, to deduce the \( A_p-A_\infty \) estimates for the operators of such class, it is enough to apply Theorem 1.1 for \( T_{p_0,1,S} \) applying the main theorems of [2]. It is worth-mentioning that the \( A_p-A_\infty \) estimates for linear Fourier multipliers was unknown as well as other noted multilinear operators.
References

[1] S. Buckley, Estimates for operator norms on weighted spaces and reverse Jensen inequalities. Trans. Amer. Math. Soc. 340 (1993), 253–272.
[2] T. A. Bui, J. M. Conde-Alonso, X. T. Duong, and M. Hormozi, Weighted bounds for multilinear operators with non-smooth kernels, Preprint. 2015. [arXiv:1506.07752]
[3] T. A. Bui and X. T. Duong, Weighted norm inequalities for multilinear operators and applications to multilinear Fourier multipliers, Bull. Sci. Math. 137 (2013), no. 1, 63–75.
[4] T. A. Bui and M. Hormozi, Weighted bounds for multilinear square functions, Preprint. 2015. [arXiv:1502.05490]
[5] J. Conde-Alonso and G. Rey, A pointwise estimate for positive dyadic shifts and some applications, Preprint. 2014. [arXiv:1409.4351]
[6] X. Chen, Q. Xue and K. Yabuta, On multilinear Littlewood-Paley operators, Nonlinear Anal 115 (2015), 25–40.
[7] W. Dahmen, A. Lerner and C. Pérez, Sharp weighted bounds for multilinear maximal functions and Calderón-Zygmund operators, J. Fourier Anal. Appl., 21(2015), 161–181.
[8] L. Grafakos, Modern Fourier Analysis, 3rd Edition, GTM 250, Springer, New York, 2014.
[9] L. Grafakos and Z. Si, The Hörmander type multiplier theorem for multilinear operators, J. Reine Angew. Math. 668 (2012), 133–147.
[10] L. Grafakos and R.H. Torres, Multilinear Calderón–Zygmund theory, Adv. Math. 165 (2002), 124–164.
[11] T. Hytönen, The sharp weighted bound for general Calderón–Zygmund operators, Ann. of Math. (2) 175 (2012), no. 3, 1473–1506.
[12] T. Hytönen, The $A_2$ theorem: Remarks and complements, Contemp. Math., 612, Amer. Math. Soc., Providence, RI (2014), 91–106.
[13] T. Hytönen and M. Lacey, The $A_p^r-A_{q}^s$ inequality for general Calderón-Zygmund operators, Indiana Univ. Math. J. 61(2012), 2041–2052.
[14] D. S. Kurtz and R. L. Wheeden, Results on weighted norm inequalities for multipliers, Trans. Amer. Math. Soc. 255 (1979), 343–362.
[15] M. Lacey and K. Li, On $A_p^r-A_{q}^s$ type estimates for square functions, available at http://arxiv.org/abs/1505.00195
[16] A.K. Lerner, A pointwise estimate for the local sharp maximal function with applications to singular integrals, Bull. London Math. Soc. 42 (2010), no. 5, 843–856.
[17] A.K. Lerner, On an estimate of Calderón-Zygmund operators by dyadic positive operators, J. Anal. Math. 121 (2013), 141–161.
[18] A.K. Lerner, A simple proof of the $A_2$ conjecture, Int. Math. Res. Not. 14 (2013), 3159–3170.
[19] A.K. Lerner, On sharp aperture-weighted estimates for square functions, J. Fourier Anal. Appl. 20 (2014), no. 4, 784–800.
[20] A.K. Lerner, Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals, Adv. Math. 226 (2011), 3912–3926.
[21] A. K. Lerner and F. Nazarov, Intuitive dyadic calculus. Available at [http://www.math.kent.edu/~znvar/Lerner Nazarov Book.pdf.]
[22] A.K. Lerner, S. Ombrosi, C. Pérez, R.H. Torres and R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, Advances in Math. 220, 1222-1264 (2009).
[23] K. Li, K. Moen and W. Sun, The sharp weighted bound for multilinear maximal functions and Calderón-Zygmund operators, J. Fourier Anal. Appl., 20 (2014) 751–765.
[24] K. Li and W. Sun, Weak and strong type weighted estimates for multilinear Calderón-Zygmund operators, Advances in Mathematics, 254(2014), 736–771.
[25] M. Lorente, J. M. Martell, C. Pérez and M. S. Riveros, Generalized Hörmander’s conditions and weighted endpoint estimates, Studia Math. 195 (2009), no. 2, 157–192.
[26] M. Lorente, J. M. Martell, M. S. Riveros and A. de la Torre, Generalized Hörmander’s condition, commutators and weights, J. Math. Anal. Appl. 342 (2008), 1399–1425.
[27] M. Lorente, M. S. Riveros and A. de la Torre, Weighted estimates for singular integral operators satisfying Hörmander’s conditions of Young type, J. Fourier Anal. Appl. 11 (2005), 497–509.

[28] S. Shi, Q. Xue, Qingying and K. Yabuta, On the boundedness of multilinear Littlewood–Paley $g^*$ function, J. Math. Pures Appl. 101 (2014), 394–413.

[29] Q. Xue and J. Yan, On multilinear square function and its applications to multilinear Littlewood–Paley operators with non-convolution type kernels, J. Math. Anal. Appl. 422 (2015), 1342-1362.

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