Generalized Dirichlet-process-means for $f$-separable distortion measures

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Abstract DP-means clustering was obtained as an extension of $K$-means clustering. While it is implemented with a simple and efficient algorithm, it can estimate the number of clusters simultaneously. However, DP-means is specifically designed for the average distortion measure. Therefore, it is vulnerable to outliers in data, and it can cause large maximum distortion in clusters. In this work, we extend the objective function of the DP-means to $f$-separable distortion measures and propose a unified learning algorithm to overcome the above problems by the selection of the function $f$. Furthermore, the influence function of the estimated cluster center is analyzed to evaluate the robustness against outliers. We show the effectiveness of the generalized method by numerical experiments using real datasets.

Keywords Clustering · Dirichlet-process-means · $f$-separable distortion measures · Bregman divergence · Influence function · Maximum distortion

1 Introduction

$K$-means is one of the most popular clustering methods. This is because its algorithm is simple and can be executed at high speed in linear time with respect to the number of data. However, it is necessary to specify the number of clusters in advance. Therefore, it is necessary to apply some heuristics or examine results with multiple cluster numbers. To estimate the number of clusters, a learning method of Gaussian mixtures by the nonparametric Bayes approach was proposed (Gershman and Blei, 2012). This led to an extension of $K$-means capable of estimating the number of clusters in the limit where the variance of the Gaussian com-
ponent approaches 0. This method is called Dirichlet-Process-means (DP-means) (Kulis and Jordan, 2012).

DP-means enables the estimation of the number of clusters from data. Because it retains the advantages of $K$-means, it can be executed in linear time with respect to the number of data and it is easy to apply to large scale data. As an attempt to further speed up DP-means, parallelization has been applied (Pan et al., 2013). Another attempt considerably reduced the computational time at the expense of precision (Bachem et al., 2013). DP-means specialized for application to large scale genetic data has been devised, and it has been shown that it is superior to the state-of-the-art from both aspects of accuracy and efficiency (Jiang et al., 2017). To improve the accuracy of clustering, there are studies to avoid local minimum solutions (Odashima et al., 2016). An extension using Bregman divergence was also given, which introduces an appropriate distance measure when data has a special type such as binary or non-negative integer value (Banerjee et al., 2005; Jiang et al., 2012).

From the viewpoint of information theory, the algorithm of DP-means monotonically decreases the average distortion of the training data, whereas the penalty parameter, which controls the number of clusters, has been interpreted as the maximum distortion of the data (Kobayashi and Watanabe, 2017). This motivates us to consider the modification of DP-means where the maximum distortion is minimized instead of the average distortion. In fact, the maximum distortion minimization is an important problem. It is also known as the $K$-center problem (Gonzalez, 1985) and the smallest enclosing ball problem when the number of clusters is one (Bádoiu and Clarkson, 2003). Also, a method to calculate the smallest enclosing Bregman ball with the radius of the smallest enclosing ball measured by Bregman divergence has been studied (Nock and Nielsen, 2005).

DP-means has a problem that it is prone to the influence of outliers because of the nature of the objective function, the average distortion. Therefore, we extended the objective function of DP-means and invented two objective functions, either of which bridges maximum distortion and robust distortion measures, and constructed the algorithms to minimize them (Kobayashi and Watanabe, 2018). However, the degree of the robustness against outliers induced by these objective functions has yet to be clarified.

On the other hand, in order to extend the linear distortion measure as the average distortion to nonlinear distortion measures with respect to the distortion of each data point, $f$-separable distortion measures using $f$-mean has been proposed. In particular, for this distortion measure, the rate-distortion function showing the limit of lossy compression was elucidated (Shkel and Verdú, 2018).

In this paper, we further generalize the above previous work and extend the objective function of DP-means to $f$-separable distortion measure using a monotonically increasing function $f$. We derive a cluster center update rule for a sufficiently wide class of the function $f$, and show that the objective function monotonically decreases if $f$ is concave. As concrete examples of the function $f$, we show that two kinds of functions $f$ including the parameter $\beta$ can unify the minimization of robust distortion measures and the minimization of the maximum distortion by adjusting the parameter $\beta$. Furthermore, we derive the influence function and evaluate the robustness against outliers. Experiments using real datasets demonstrate that DP-means generalized by the function $f$ improves the performance of original DP-means.
The paper is organized as follows. Section 2 introduces DP-means. Section 3 generalizes the objective function of DP-means to $f$-separable distortion measures and explains the behavior of the objective function corresponding to the selection of the function $f$. In Section 4, the generalized DP-means algorithm is constructed based on the generalized objective function. In Section 5, we derive the influence function and evaluate robustness against outliers. In Section 6, we present the results of numerical experiments using real datasets to demonstrate the effectiveness of the generalized DP-means. In Section 7, we discuss further modification of the objective function in terms of the pseudo-distance. Finally, Section 8 concludes this paper.

2 DP-means

DP-means requires data $x^n = \{x_1, \ldots, x_n\}$ and penalty parameter $\lambda$ as inputs. Suppose that each data point is $L$-dimensional, $x_i = (x_i^{(1)}, \ldots, x_i^{(L)})^T \in \mathbb{R}^L$. The algorithm of DP-means is basically the same as $K$-means. Let $\{\theta_1, \ldots, \theta_K\}$ be cluster centers. DP-means executes calculation of cluster centers and the assignments of the data points to clusters until the following objective function converges:

$$L((\theta_k)_{k=1}^K, \{c(i)\}_{i=1}^n) = \sum_{i=1}^n d_\phi(x_i, \theta_{c(i)}) + \lambda K.$$  \hfill (1)

Here, $c(i) \triangleq \arg\min_k d_\phi(x_i, \theta_k)$ denotes the cluster label of the data point $x_i$. However, the following two points are different from $K$-means. DP-means is initialized with one cluster, $K = 1$. When the pseudo-distance between a data point and its nearest cluster center is greater than the penalty parameter $\lambda$, a new cluster is generated. In other words, a new cluster is generated when the following is satisfied,

$$d_\phi(x_i, \theta_{c(i)}) > \lambda.$$  \hfill (2)

Also, the cluster center $\theta_k$ is calculated as the average of data points assigned to the $k$-th cluster,

$$\theta_k = \frac{\sum_{i=1}^n r_{ik} x_i}{\sum_{j=1}^K r_{jk}}.$$  

Here, $r_{ik}$ is given by

$$r_{ik} = \begin{cases} 1 & (c(i) = k), \\ 0 & (c(i) \neq k). \end{cases}$$

In this paper, we assume the Bregman divergence as the pseudo-distance, which generalizes the squared distance. Specifically, the Bregman divergence is the pseudo-distance determined from a differentiable strictly convex function $\phi$ as

$$d_\phi(x, \theta) \triangleq \phi(x) - \phi(\theta) - \langle x - \theta, \nabla \phi(\theta) \rangle,$$  \hfill (3)

where $\nabla \phi$ represents the gradient vector of $\phi$ and $\langle \cdot, \cdot \rangle$ is the inner product. The Bregman divergence satisfies non-negativity and identity of indiscernibles in distance axioms. Moreover, the Bregman divergences are related to the probability distributions in the exponential family bijectively. For example, if data are
of a particular type such as binary or non-negative integer, the Bregman divergences corresponding to the Bernoulli and Poisson distributions are used as the more suitable distance measures than the usual squared distance for real numbers (Banerjee et al., 2005; Jiang et al., 2012).

The average distortion and the maximum distortion are defined by

\[
\frac{1}{n} \sum_{i=1}^{n} d_\phi \left( x_i, \theta_{c(i)} \right),
\]

\[
\max_{1 \leq i \leq n} d_\phi \left( x_i, \theta_{c(i)} \right),
\]

respectively. Note that the minimization of the objective function (1) with respect to \( \theta_k \) is equivalent to that of the average distortion. On the other hand, the penalty parameter \( \lambda \), which determines the number of clusters, can be interpreted as the maximum distortion (Kobayashi and Watanabe, 2017). Therefore, we can consider the maximum distortion as the measure for cluster increment.

3 Generalized objective function

3.1 Generalization with \( f \)-separable distortion measures

In this paper, we propose the objective function that generalizes the objective function of DP-means to \( f \)-separable distortion measures as follows:

\[
L_f \left( \{ \theta_k \}_{k=1}^K, \{ c(i) \}_{i=1}^n \right) = \sum_{i=1}^{n} f \left( d_\phi \left( x_i, \theta_{c(i)} \right) \right) + f \left( \lambda K \right). \tag{4}
\]

As we will discuss in Section 4.2, this objective function is guaranteed to decrease monotonically with respect to \( \{ \theta_k \}_{k=1}^K \) and \( \{ c(i) \}_{i=1}^n \). In this paper, we assume that the function \( f \) is differentiable and continuous monotonically increasing. Its domain is \( z \geq 0 \) and argument is given by the Bregman divergence or the penalty parameter \( \lambda \). In particular, we consider the following three types:

\[ \nearrow \] : linear,

\[ \searrow \] : concave,

\[ \swarrow \] : convex.

If the function \( f(z) \) is a differentiable and strictly monotonically increasing function, an inverse function \( f^{-1}(z) \) exists, and (4) can be normalized to the distortion measure in the form of \( f \)-mean (Tikhomirov, 1991). The \( f \)-separable distortion measures using this inverse function \( f^{-1}(z) \) corresponds to that of the literature (Shkel and Verdú, 2018). The generalized objective function (4) can be monotonically transformed with the inverse function \( f^{-1} \). Therefore, minimizing (4) is equivalent to minimize the \( f \)-mean,

\[
f^{-1} \left( \frac{1}{n + K} \left\{ \sum_{i=1}^{n} f \left( d_\phi \left( x_i, \theta_{c(i)} \right) \right) + f \left( \lambda K \right) \right\} \right).
\]

Table I summarizes the behavior of the objective function, its monotonic improvement property, and the calculation order required to execute the learning.
Table 1: Algorithm behavior corresponding to function $f$.

| $f(z)$ | $f^{-1}(z)$ behavior | monotonic decrease | calculation order |
|--------|----------------------|-------------------|------------------|
| $\nearrow$ | average distortion minimization | yes | $O(\ln n)$ |
| $\searrow$ | robustness against outliers | yes | $O(\ln n)$ |
| $\searrow$ | approaches the maximum distortion minimization | gradient descent optimization: requires/depends |

algorithm. If $f$ is the liner type ($\nearrow$), it becomes the original objective function, which corresponds to the average distortion in the original DP-means. If $f$ is the convex type ($\swarrow$), distortion with a larger pseudo-distance value tends to be minimized. In particular, the faster the function $f$ diverges to infinity, the more the objective function approaches the maximum distortion. Conversely, If $f$ is the concave type ($\searrow$), distortion with a smaller pseudo-distance value will be prioritized to be minimized. That is, the influence of data points far from other data points, such as outliers is weakened. In other words, there is a trade-off between the maximum distortion, which is the maximum radius of the cluster, and the robustness against outliers. Robustness against outliers is explained more in detail in Section 3.

Eguchi and Kano (2001) generalized the likelihood of a probabilistic model to $\Psi$-likelihood using a convex function like (4), and devised a $\Psi$-estimator as a robust estimator against outliers. The $\Psi$-estimator focuses on the following two points. The first point is to obtain robustness against outliers. The second point is to guarantee the unbiasedness of the estimation equation. Therefore, the objective function includes a bias correction term whose calculation is complicated in general. On the other hand, we consider a wider class of functions, and not only the robustness against outliers but also the maximum distortion minimization is within our focus. $\Psi$-likelihood assumes a probabilistic model, whereas in this study, only the Bregman divergence is assumed. As we will discuss, the update rule of the cluster center derived from the combination of the function $f$ and Bregman divergence enables to execute the learning algorithm in the linear order on the number of data as the original DP-means.

3.2 Examples of function $f$

In this section, we show two concrete examples of the functions with a parameter $\beta$. When the parameter $\beta$ is changed, the generalized objective function changes its behavior as average distortion, maximum distortion, and robust distortion measures.

3.2.1 Power mean objective

For the function

$$f(z) = \frac{1}{\beta} \left\{ (z + a)^\beta - 1 \right\}$$

(5)
the corresponding $f$-mean is given by
\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} (z_i + a)^\beta \right\}^{\frac{1}{\beta}} - a.
\] 
(6)

The first term of (6) is called power mean. Here, parameters are $\beta \in \mathbb{R}$, $a \geq 0$. The parameter $\beta$ determines the effect of the objective function and the parameter $a$ is introduced to avoid an algorithmic disadvantage. Table 2 shows the characteristics of the objective function (6) and the corresponding function $f$ for different choices of $\beta$.

As shown in Table 2, the behavior of the objective function varies around $\beta = 1$, it shows robust characteristics when $\beta < 1$, and the smaller the $\beta$, the smaller the influence of outliers. When $\beta > 1$, the larger the value of $\beta$, the more the objective function approaches the maximum distortion. In particular, $\beta \to \infty$ implies maximum distortion minimization. When the Bregman divergence is the squared distance, the objective function using (6) with $\beta > 0$ and $a = 0$ corresponds to the objective function derived in the framework of MAP-based Asymptotic Derivations (MAD-Bayes) (Broderick et al., 2013) when the generalized Gaussian distribution is assumed as the component of the nonparametric mixture model. The proof of this fact is put in Appendix A.1. Similarly, assuming a deformed $t$-distribution as the component of a nonparametric mixture model, the same objective function when $\beta = 0$ and $a > 0$ is obtained. The proof of this fact is put in Appendix A.2.

### 3.2.2 Log-sum-exp objective

For the function
\[
f(z) = \frac{1}{\beta - 1} \{ \exp ((\beta - 1)z) - 1 \},
\] 
(7)
the corresponding $f$-mean is given by
\[
\frac{1}{\beta - 1} \ln \left\{ \frac{1}{n} \sum_{i=1}^{n} \exp ((\beta - 1)z_i) \right\}.
\] 
(8)
Equation (8) is known as a differentiable approximation of the maximum value function when $\beta = 2$, and is called the log-sum-exp function (Boyd and Vandenberghe, 2004). As in the case of the power mean, $\beta \in \mathbb{R}$ determines the characteristics of the objective function as a parameter. Table 3 shows the characteristics of the objective function using (8) and the corresponding function $f$ for different choices of $\beta$.

As shown in Table 3, the objective function behaves differently around $\beta = 1$ as in the case of the power mean. It becomes robust when $\beta < 1$, and approaches the maximum distortion when $\beta > 1$. In particular, when $\beta \to \infty$, its limit is the maximum distortion. In addition, (8) corresponds to the objective function for the estimation of mixture models in (Watanabe and Ikeda, 2015) when the variance of each component approaches 0.

1 It can also be expressed as $\ln_{1-\beta}(z + a) = \frac{1}{\beta}((z + a)^\beta - 1)$ by using Tsallis $q$-function $\ln_q(z) \equiv \frac{z^{1-q} - 1}{1-q}$, for which $\ln(z) = \lim_{q \to 1} \ln_q(z)$ (Tsallis, 2009).

2 It can also be expressed as $\ln_{2-\beta}(\exp(z)) = \frac{\exp((\beta-1)z) - 1}{\beta-1}$.
Table 2: Behavior of the power mean objective and corresponding function $f$.

| $\beta$            | $f^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} f(z_i) \right)$ | $f(z)$                       | $f'(z)$ | behavior                        |
|---------------------|---------------------------------------------------------------|------------------------------|----------|---------------------------------|
| $\beta = 1$        | $\frac{1}{n} \sum_{i=1}^{n} z_i$                             | $z + a - 1$                  | $z$      | average distortion minimization |
| $\beta = 0$        | $(\prod_{i=1}^{n} (z_i + a))^{\frac{1}{\beta}} - a$        | $\ln(z + a)$                | $\frac{1}{\beta}$ | robustness against outliers     |
| $-\infty < \beta < 0$ | $(\frac{1}{n} \sum_{i=1}^{n} (z_i + a)^{\beta})^{\frac{1}{\beta}} - a$ | $\frac{1}{\beta} \{(z + a)^{\beta} - 1 \}$ | $(z + a)^{\beta - 1}$ | approaches the maximum distortion minimization |
| $0 < \beta < 1$    |                                                                 |                              |          |                                 |
| $1 < \beta < \infty$ | max$_{1 \leq i \leq n} z_i$                                  |                              |          | maximum distortion minimization |
| $\beta \rightarrow \infty$ |                                                             |                              |          |                                 |

Table 3: Behavior of the log-sum-exp objective and corresponding function $f$.

| $\beta$            | $f^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} f(z_i) \right)$ | $f(z)$                        | $f'(z)$ | behavior                        |
|---------------------|---------------------------------------------------------------|------------------------------|----------|---------------------------------|
| $\beta = 1$        | $\frac{1}{n} \sum_{i=1}^{n} z_i$                             | $z$                          | $z$      | average distortion minimization |
| $-\infty < \beta < 1$ | $\frac{1}{\beta} \ln \left( \frac{1}{n} \sum_{i=1}^{n} \exp \left( (\beta - 1) z_i \right) \right)$ | $\exp(\frac{(\beta - 1) z}{\beta - 1}) - 1$ | $\exp((\beta - 1) z)$ | robustness against outliers     |
| $1 < \beta < \infty$ |                                                                 |                              |          |                                 |
| $\beta \rightarrow \infty$ |                                                             |                              |          | maximum distortion minimization |
| $\beta \rightarrow \infty$ |                                                             |                              |          |                                 |
4 Construction of generalized algorithm

In this section, we construct a generalized DP-means algorithm based on the objective function proposed in Section 3. First, we derive the update rule of the cluster center in Section 4.1. Second, we show that the objective function decreases monotonically with the derived update rule in Section 4.2. Finally, we explain small problems in the execution of the algorithm and the solutions to them in Section 4.3. The generalized algorithms are constructed from the original DP-means by replacing the update rule of cluster centers and the objective function used for convergence diagnosis (Algorithm 1). This algorithm differs only in the update rule of the cluster center, and the computation time required for execution is of linear order with respect to the number of data as the original algorithm.

4.1 Derivation of update rules

The updating equations of cluster centers are derived from the stationary conditions that the gradient of the cluster center $\theta_k$ of the generalized objective function (4) is 0. Here, $f'$ represents the derivative of the function $f$. That is, the update rule of the cluster center is given as

$$\theta_k = \frac{\sum_{i=1}^{n} w_{ik} x_i}{\sum_{j=1}^{n} w_{jk}},$$

(9)

$$w_{ik} = r_{ik} f'(d_\phi(x_i, \theta_k)).$$

(10)

which is a weighted mean of $x_i$ weighted by $f'$ with Bregman divergence as its argument. However, the objective function monotonically decreases by this update rule (9) only when the function $f$ is linear or concave.

4.2 Guarantee of monotonic decreasing property

The DP-means algorithm iterates the cluster center updating step and the assignments of data points to clusters. Because of the monotonic decreasing property of the objective function in each step, the algorithm converges within finite iterations. Even in the generalized objective function (4), the cluster assignment step is same as the original DP-means. Therefore, only the monotonic decreasing property of the objective function in the cluster center updating step is considered. In the following, we explain the two cases, namely, concave and convex cases, separately.

The next theorem applies to the case where the function $f$ is concave.

Theorem 1 If the function $f$ is concave, the updating of the cluster center using (9) monotonically decreases the objective function for general Bregman divergence.

Proof We show that the objective function (4) monotonically decreases when the $k$-th cluster center $\theta_k$ is newly updated to $\tilde{\theta}_k$ by (9). More specifically, we prove
Algorithm 1: Generalized DP-means for \( f \)-separable distortion measures

1. **Input:** \( x^n = \{x_1, \ldots, x_n\} \), \( \lambda \), generic function \( f \)
2. **Output:** \( \{\theta_k\}_{k=1}^K \), \( \{c(i)\}_{i=1}^n \), \( K \)
3. \( K = 1 \), \( \theta_1 = \frac{1}{n} \sum_{i=1}^n x_i \), \( c(i) = 1 \) \((i = 1, \ldots, n)\)
4. **repeat**
   5. calculate \( w_{ik} \) by (10) \((i = 1, \ldots, n)\)
   6. update \( \theta_1 \) by (9)
5. **until** (4) converges with respect to \( \theta_1 \)
6. **repeat**
   7. for \( i = 1 \) to \( n \) do
   8. \( d_k = \min_{k'} d_{\phi}(x_i, \theta_{k'}) \) \((k' = 1, \ldots, K)\)
   9. \( k = \arg \min_{k'} d_{\phi}(x_i, \theta_{k'}) \) \((k' = 1, \ldots, K)\)
   10. if \( d_k > \lambda \) then
   11. \( K = K + 1 \), \( c(i) = K \), \( \theta_K = x_i \)
   12. else
   13. \( c(i) = k \)
   14. for \( k = 1 \) to \( K \) do
   15. **repeat**
   16. calculate \( w_{ik} \) by (10) \((i = 1, \ldots, n)\)
   17. update \( \theta_k \) by (9)
   18. **until** (4) converges with respect to \( \theta_k \)
5. **until** (4) convergences

that, \( L(\theta_k) \geq L(\tilde{\theta}_k) \), where \( L(\theta_k) \) is the sum of the terms related to \( \theta_k \) in (4), that is,

\[
L(\theta_k) = \sum_{i=1}^n r_{ik} f(d_{\phi}(x_i, \theta_k)).
\]

To prove this, we focus on the relationship between the function \( f \) and its tangent line. The tangent line \( y \) of \( f(z) \) at the point \( a \) is expressed by the following equation:

\[
y = f'(a)(z - a) + f(a).
\]

Furthermore, since it is \( y \geq f(z) \) from the concavity of the function \( f \), the following inequality holds:

\[
f(a) - f(z) \geq f'(a)(a - z).
\]

From this inequality, the following holds:

\[
L(\theta_k) - L(\tilde{\theta}_k)
\]

\[
= \sum_{i=1}^n r_{ik} \left\{ f(d_{\phi}(x_i, \theta_k)) - f(d_{\phi}(x_i, \tilde{\theta}_k)) \right\}
\]

\[
\geq \sum_{i=1}^n r_{ik} f'(d_{\phi}(x_i, \theta_k)) \left\{ d_{\phi}(x_i, \theta_k) - d_{\phi}(x_i, \tilde{\theta}_k) \right\}
\]

\[
= d_{\phi}(\tilde{\theta}_k, \theta_k) \sum_{i=1}^n r_{ik} f'(d_{\phi}(x_i, \theta_k)) \geq 0,
\]
where (9) and (10) were used in the last equality.

The following corollary immediately follows from Theorem 1.

**Corollary 1** When the objective function is constructed by the power mean (6) or the log-sum-exp function (8), the following holds. When $\beta \leq 1$, the updating of the cluster center using (9) monotonically decreases the objective function for general Bregman divergence.

The proof of Theorem 1 is a generalization of the monotonic decreasing property of ei-means ($\varepsilon = 0$) proposed in (Watanabe, 2015), which corresponds to the case where $f(z) = \sqrt{z}$ and $d_{\phi}(x, \theta) = \|x - \theta\|^2$.

Next, we turn to the case where the function $f$ is convex. When the objective function (4) exhibits a characteristic close to the maximum distortion minimization, that is, when the function is convex, if the cluster center is updated by (9), the value of the objective function can oscillate and may not decrease monotonically. Therefore, it is necessary to apply a gradient descent optimization such as the steepest descent method or the Newton’s method. In the case where the pseudo-distance is the general Bregman divergence, the problem of calculating the cluster centers is not generally a convex optimization problem, so the gradient is not necessarily in the descent direction. However, monotonic decreasing property is also guaranteed for the Bregman divergence in general by using the algorithm that updates to the descending direction of the gradient like the modified Newton’s method. In particular, when symmetry is satisfied among distance axioms such as the squared distance, the problem of calculating the cluster centers is reduced to a convex optimization problem, so the gradient direction is always in the descent direction (Nock and Nielsen, 2005; Boyd and Vandenberghe, 2004, Section 3.2).

When the function $f$ is convex, the calculation time in the case of the steepest descent method is $O(Ln)$, and the Newton’s method with the Cholesky decomposition is $O(L^2n)$. In this way, there is no change in the linear order with respect to the number of data.

### 4.3 Problem and solution

When the objective function shows robustness against outliers, that is, when the function $f$ is a concave, if a cluster center overlaps a data point, subsequent updates are not performed. In the assignment step of DP-means, a new cluster center is generated exactly on the data point satisfying (2). Thus the situation where a data point and a cluster center overlap can occur frequently. Therefore, we show the condition that cluster center updating does not take place and give a solution.

When the function $f$ is a concave and satisfies

$$\lim_{z \to 0} f'(z) = \infty,$$

if a cluster center overlaps a data point, updating of the cluster center is not performed. It is assumed that one point in $x^* = \{x_1, \ldots, x_n\}$ overlaps the cluster center $\theta_k$. That is, $x_i^* = \theta_k \iff d_{\phi}(x_i^*, \theta_k) = 0$. Here, if the function $f$ satisfies (12),

$$f'(d_{\phi}(x_i^*, \theta_k)) = \begin{cases} 1 & (i = i^*), \\ 0 & (i \neq i^*), \end{cases}$$

$$f'(d_{\phi}(x_i^*, \theta_k)) \neq \begin{cases} 1 & (i = i^*), \\ 0 & (i \neq i^*), \end{cases}$$
holds. Therefore, we have

$$\theta_k = \sum_{i=1}^{n} \frac{w_{ik}}{f(d_\phi(x_i^*, \theta_k))} x_i = x_i^*,$$

which means that the cluster center does not move from the data point.

If (12) is satisfied, when a new data point other than the data point overlapping the cluster center is newly assigned to the cluster, updating can be performed by giving an average value of data points included at that time as an initial value of the cluster center.

In the case of the power mean, when \( a = 0 \), since (12) is satisfied, updating of the cluster center is not performed. Therefore, it is better to set \( a > 0 \). However, \( a \) must be smaller than the pseudo-distance value. In the case of the log-sum-exp function, cluster center updating is performed because (12) is not satisfied.

5 Analysis of influence function

The influence function is one of the indicators of the robustness against outliers and it shows how much the estimator is influenced by the contamination of a small number of outliers. In this section, we derive the influence function and evaluate to what extent the generalized objective function is robust to outliers.

5.1 Influence function for general function \( f \)

Derivation of the influence function in this section is based on the derivation of the influence function of the total Bregman divergence (Vemuri et al., 2011; Amari, 2016) (see also Section 7). We consider the influence function when an outlier is mixed in the \( k \)-th cluster. However, since the influence of mixed outliers is independent for each cluster, in the derivation of subsequent influence functions, the subscript of \( \theta_k \) is omitted and expressed as \( \theta \). Suppose that the \( k \)-th cluster contains \( m \) samples of data \( x^m = \{x_1, \ldots, x_m\} \), and the cluster center estimated from \( x^m \) is \( \theta \). When an outlier \( x^* \) is mixed into the data \( x^{m+1} = \{x_1, \ldots, x_m, x^*\} \). Let \( \delta \eta \) be the difference between the estimator \( \tilde{\theta} \) including outliers and the estimator \( \theta \) without outliers,

$$\tilde{\theta} - \theta = \delta \eta.$$

The influence function is defined by

$$\text{IF}(x^*) = m \delta \eta. \quad (13)$$

The influence function (13) defined by a finite sample is also specifically called a sensitivity curve in the field of robust statistics (Hampel et al., 2005). Then, the new cluster center \( \tilde{\theta} \) minimizes

$$\frac{1}{m+1} \sum_{i=1}^{m} f\left(d_\phi(x_i, \tilde{\theta})\right) + \frac{1}{m+1} f\left(d_\phi(x^*, \tilde{\theta})\right). \quad (14)$$
Now, we compute the first three terms of the Taylor expansion around the old cluster center $\theta$, which minimizes $\sum_{i=1}^{m} f(d_\phi(x_i, \theta))$. Therefore, the first derivative with respect to $\delta \eta$ of the first term of (14) becomes

$$\sum_{i=1}^{m} \nabla f(d_\phi(x_i, \theta)) = 0.$$  

The second derivative with respect to $\delta \eta$ of the second term of (14) becomes very small when $m$ is large because it is $O(m^{-3})$. Here, $\nabla$ expresses the gradient with respect to $\theta$. From the above arguments, the term related to $\delta \eta$ is given as follows:

$$\frac{1}{m+1} \left\{ \frac{1}{2} \delta \eta^T G \delta \eta + \nabla f(d_\phi(x^*, \theta)) \delta \eta \right\},$$

where

$$G = \sum_{i=1}^{m} \nabla^2 f(d_\phi(x_i, \theta)).$$

The $\delta \eta$ that minimizes (15) is solved by completing the square with respect to $\delta \eta$. Then, the influence function in (13) is given by

$$IF(x^*) = -m G^{-1} \nabla f(d_\phi(x^*, \theta)).$$

Essentially, (16) is same as the influence function in M-estimation [Hampel et al., 2005]. Since the matrix $G$ does not depend on the outlier, in order to investigate the boundedness of the influence function, we evaluate the following term:

$$- \nabla f(d_\phi(x^*, \theta)) = -f'(d_\phi(x^*, \theta)) \nabla d_\phi(x^*, \theta)$$

$$= f'(d_\phi(x^*, \theta)) \left( \nabla^2 \phi(\theta)(x^*-\theta) \right).$$

Therefore, the boundedness of the influence function depends on the function $f$ and the strictly convex function $\phi$ constituting the Bregman divergence. If (17) is bounded, the estimated cluster center is robust against outliers.

5.2 Necessary condition for boundedness

**Theorem 2** The following condition on the function $f$ is necessary for its influence function to be bounded for all $x^*$:

$$\lim_{z \to \infty} f'(z) = 0.$$  

**Proof** As $\|\theta\| < \infty$, when $\|x^*\|$ is large, $d_\phi(x^*, \theta)$ is large. When the function $f(z)$ is linear ($f'$), or convex ($f''$), $f'(z)$ is a constant or monotonically increasing with respect to $z$. As $\|x^*\|$ is increased, the norm of (17) becomes as large as possible, and the influence function is not bounded. In order to reduce the norm of (17) when $\|x^*\|$ is large, $f'(z)$ must be monotonically decreasing. The type of the function $f(z)$ that satisfies this condition is only concave ($f''$). When $f'(z)$ does not satisfy (18), the norm of (17) is divergent at $\|x^*\| \to \infty$, and hence the influence function is not bounded. Therefore, the function $f(z)$ must satisfy (18). □

From the proof of this theorem, we know that $f(z)$ is restricted to a subclass of concave functions in order to obtain the robustness.
**Remark 1** In some cases, the factor \( f'(d_\phi(x^*, \theta)) \) in (17) can diverge to infinity around \( x^* = \theta \). Even in such a case, if we consider the region of \( x^* \) satisfying \( d_\phi(x^*, \theta) > \delta \) for a constant \( \delta \), the condition of Theorem 2 provides a necessary condition for the boundedness of the influence function on the region.

Under the condition of Theorem 2, the norm of (17) is bounded. In the following, we consider if the norm of the influence function vanishes as \( \|x^*\| \to \infty \). In particular, if

\[
\lim_{\|x^*\| \to \infty} \|IF(x^*)\| = 0 \tag{19}
\]

holds, the influence function is said to be redescending, and an outlier that is too large is automatically ignored.

The following Assumption 1 is assumed in the following discussion.

**Assumption 1** The input dimension, the norm of the cluster center \( \theta \), and that of \( \phi \) at \( \theta \) are finite, that is, \( L < \infty \), \( \|\theta\| < \infty \), \( \|\nabla \phi(\theta)\| < \infty \). The Bregman divergence \( d_\phi \) satisfies the followings:

\[
\|x^*\| \to \infty \iff d_\phi(x^*, \theta) \to \infty,
\]

\[
\exists l, \left\| x^{(l)} \right\| \to \infty \Rightarrow d_\phi(x^*, \theta) \to \infty, \tag{20}
\]

\[
\tilde{x} = (\theta^{(1)}, \ldots, \theta^{(l-1)}, x^{(l)}, \theta^{(l+1)}, \ldots, \theta^{(L)})^T \Rightarrow d_\phi(\tilde{x}, \theta) \leq d_\phi(x^*, \theta). \tag{21}
\]

Equations (20) and (21) hold true if the Bregman divergence \( d_\phi(x, \theta) \) is defined additively with respect to \( L \)-dimensions. Below, we discuss the situation where \( \|x^*\| \to \infty \) holds under these assumptions. In some cases, such as the Bregman divergence corresponding to the binomial distribution, \( \|x^*\| \to \infty \) cannot occur. However we can investigate the behavior of the influence function for finite \( x^* \). We illustrate the behavior of (17) for such a case in Appendix E.

### 5.3 Power mean

From (17) and (5), to evaluate the influence function in the case of the power mean, we evaluate the following term:

\[
\lim_{\|x^*\| \to \infty} \left\| \nabla \nabla \phi(\theta)(x^* - \theta) \right\| \left\{ d_\phi(x^*, \theta) + a \right\}^{1-\beta}. \]

**Theorem 3** For the function \( f \) of the power mean (5) with \( \beta < 0 \), the influence function is redescending for general Bregman divergences.

The proof of Theorem 3 is put in Appendix B.2.1. However, when \( 0 \leq \beta < 1 \), the redescending or boundedness property of the influence function depends on the Bregman divergence. In the following, we investigate the influence functions for concrete examples of the Bregman divergences.

The \( \alpha \)-divergence is a subclass of the Bregman divergences, including Itakura Saito divergence (\( \alpha = 0 \)), generalized KL divergence (\( \alpha = 1 \)), and the squared
distance ($\alpha = 2$) as special cases (Hennequin et al., 2011). The $\alpha$-divergence and corresponding convex functions are given by

$$d_\alpha(x, \theta) = \begin{cases} \frac{x - \ln \left( \frac{x}{\theta} \right) - 1}{\theta} & (\alpha = 0), \\ x \ln \left( \frac{x}{\theta} \right) - (x - \theta) & (\alpha = 1), \\ \frac{x^{\alpha} + (\alpha - 1)\theta^{\alpha} - a_\alpha^{\alpha - 1}}{\alpha(\alpha - 1)} & \text{(otherwise)} \end{cases}$$

$$\phi_\alpha(x) = \begin{cases} -\ln x + x - 1 & (\alpha = 0), \\ x \ln x - x + 1 & (\alpha = 1), \\ \frac{x}{a_\alpha^{\alpha - 1}} - \frac{1}{\alpha} & \text{(otherwise)} \end{cases}$$

respectively. In the convex function, when the parameter $\alpha$ is a positive even number other than 0, its domain is defined as $\mathbb{R}$, otherwise it is $\mathbb{R}^{++}$. If the data is multidimensional, the Bregman divergence and the corresponding convex function are defined additively with respect to dimensions as follows:

$$d_\phi(x, \theta) = \sum_{l=1}^{L} d_\alpha(x^{(l)}, \theta^{(l)}),$$

$$\phi(x) = \sum_{l=1}^{L} \phi_\alpha(x^{(l)}).$$

We calculated the influence function for the $\alpha$-divergence and found that it can be classified into divergent, bounded, and redescending types with respect to $\alpha$ and $\beta$ according to the following conditions (The proof is in Appendix B.2.2):

$$\alpha < 1 : \begin{cases} \beta > 0 \Rightarrow \|\vec{IF}(x^*)\| \text{ is divergent,} \\ \beta = 0 \Rightarrow \|\vec{IF}(x^*)\| \text{ is bounded,} \\ \beta < 0 \Rightarrow \|\vec{IF}(x^*)\| \text{ is redescending,} \end{cases}$$

$$\alpha = 1 : \begin{cases} \beta > 0 \Rightarrow \|\vec{IF}(x^*)\| \text{ is divergent,} \\ \beta < 0 \Rightarrow \|\vec{IF}(x^*)\| \text{ is redescending,} \end{cases}$$

$$\alpha > 1 : \begin{cases} \beta > 0 \frac{1}{1 - \frac{\alpha}{\alpha - 1}} \Rightarrow \|\vec{IF}(x^*)\| \text{ is divergent,} \\ \beta = 0 \frac{1}{1 - \frac{\alpha}{\alpha - 1}} \Rightarrow \|\vec{IF}(x^*)\| \text{ is bounded,} \\ \beta < 0 \frac{1}{1 - \frac{\alpha}{\alpha - 1}} \Rightarrow \|\vec{IF}(x^*)\| \text{ is redescending.} \end{cases}$$

Specifically, Figure 1 shows the regions of ($\alpha, \beta$) corresponding to the three types. As shown in Figure 1, the boundedness property holds at the boundary line between the divergent and the redescending properties of the influence function. This boundary line is continuous except for the point of $\alpha = 1$, and gradually approaches $\beta = 0$ when $\alpha \to \infty$. The case of another divergence, the exp-loss is shown in Appendix B.2.3.

---

3 The $\alpha$-divergence here is usually termed as the $\beta$-divergence with the parameter $\beta$. We denote the parameter by $\alpha$ in order not to be confused with the $\beta$ in (5).
Generalized Dirichlet-process-means for $f$-separable distortion measures

### 5.4 Log-sum-exp

From (17) and (7), to evaluate the influence function in the case of the log-sum-exp function, we evaluate the following term:

$$
\lim_{\|x^*\| \to \infty} \left\| \frac{\nabla \phi(x^* - \theta)}{\exp \left\{ (1 - \beta) d_{\phi}(x^*, \theta) \right\}} \right\|
$$

**Theorem 4** For the function $f$ of the log-sum-exp (7) with $\beta < 1$, the influence function is redescending for general Bregman divergences.

The proof of Theorem 4 is put in Appendix B.3. In other words, Theorem 4 shows that when the estimated cluster center is robust against outliers, the redescending property always holds for any Bregman divergences.

From the above examples, it can be seen that the robustness property of $f$-mean strongly depends on the function $f$.

### 6 Experiments

#### 6.1 UCI experiment

In this section, we report the results of experiments with benchmark datasets in UCI Machine Learning Repository\(^4\) to demonstrate the effectiveness of the objective function generalized by the monotonically increasing function $f$.

---

\(^4\) https://archive.ics.uci.edu/ml/datasets.html
Table 4: UCI datasets.

| dataset                  | n        | K | L |
|--------------------------|----------|---|---|
| Breast Cancer Wisconsin\(^5\) | 683      | 2 | 9 |
| Heart\(^6\)              | 297      | 5 | 13|
| HeartK2                  | 297      | 2 | 13|
| HTRU2                    | 17898    | 2 | 8 |
| Iris                     | 150      | 3 | 4 |
| Mice Protein Expression  | 552      | 8 | 77|
| Pima                     | 768      | 2 | 8 |
| Seeds                    | 210      | 3 | 7 |
| Thyroid\(^7\)            | 215      | 3 | 5 |
| Wine                     | 178      | 3 | 13|
| Yeast                    | 1484     | 10| 8 |

6.1.1 Dataset

We summarize the datasets used for the experiment in Table 4 where \(n\), \(K\), and \(L\) denote the number of data, the number of clusters, and the number of dimensions, respectively. In Table 4, links to the specific datasets are given as footnotes when there are multiple datasets. We use datasets for classification problems by assuming classes as the true clusters. HeartK2 dataset is made of Heart dataset by coarsening the cluster labels. Heart dataset consists of data on heart disease with five clusters. Four clusters represent heart disease and one cluster represents no heart disease. HeartK2 dataset is made by the two clusters with and without heart disease. We deleted data points with missing values beforehand.

6.1.2 Evaluation criteria

We used normalized mutual information (NMI), entropy, purity, F-measure, maximum distortion as the evaluation criteria. NMI, entropy, purity, and F-measure are criteria for evaluating the clustering result, and take values from 0 to 1. The closer the entropy is to 0, the better the result. As for the other evaluation criteria than entropy, the closer the criterion is to 1, the better the result. NMI is defined by the following equation:

\[
\text{NMI}(C, A) = \frac{I(C, A)}{\sqrt{H(C)H(A)}}
\]

for the label set \(C\) of the clustering result and the label set \(A\) of the correct cluster. Here \(I(\cdot, \cdot)\) and \(H(\cdot)\) represent mutual information and entropy, respectively.

\(^5\) http://archive.ics.uci.edu/ml/machine-learning-databases/thyroid-disease/new-thyroid.data

\(^6\) https://archive.ics.uci.edu/ml/machine-learning-databases/heart-disease/processed.cleveland.data

\(^7\) https://archive.ics.uci.edu/ml/machine-learning-databases/breast-cancer-wisconsin/breast-cancer-wisconsin.data
order to confirm the behavior of the objective function, we examined the behavior of the maximum distortion against the change of $\beta$.

### 6.1.3 Method

For preprocessing of clustering, we standardized data so that the each dimension is transformed as $x_i^{(l)} \leftarrow \frac{x_i^{(l)}}{\sqrt{\sum_{n=1}^{n} (x_i^{(l)})^2}}$. We chose the squared distance $d_\phi(x, \theta) = \frac{1}{L} \|x - \theta\|^2$ as the distortion measure, which is averaged with respect to the dimensions.

In the experiment, we investigated the change of each evaluation criteria when changing the parameter $\beta$ for each case of the power mean $\phi$ with $a = 0$ ("pow") and log-sum-exp function $\phi$ ("LSE"). The range of $\beta$ examined was $-2 \sim 5$. DP-means returns a local minimum solution depending on the order of data. Hence, the order of data was shuffled 100 times, and for each evaluation criteria, the average value on the number of shuffles was calculated, and this was used as the result. Since the execution result depends on the penalty parameter $\lambda$, we changed $\lambda$ to obtain results with different numbers of clusters, and the average value of each evaluation criterion was calculated for the result with the correct number of clusters. The details on the experimental procedure are described in Appendix C.

### 6.1.4 Result

Table 5 shows the optimal $\beta$ and the optimal value of each evaluation criterion with the results of the original DP-means. We performed paired t-tests between evaluation criteria for each $\beta$ and those for $\beta = 1$ (original DP-means, "origin"), and let the one with the optimal value and the significance level less than 5% be the optimal $\beta$. If there were multiple optimal $\beta$, we chose the value close to $\beta = 1$ as the optimal $\beta$. Table 5 shows that the optimal value of $\beta$ that minimizes the maximum distortion is $\beta > 1$ in many cases, and we can see the tendency that the maximum distortion decreases when increasing $\beta$. Theoretically, the larger the $\beta$, the closer the generalized DP-means to the maximum distortion minimization. The above result supports this fact. As for other evaluation criteria, in many cases, the performance is improved compared to the case of $\beta = 1$. The optimal $\beta$ varies depending on the dataset and the evaluation criteria. However, it can be seen that it is better to set $\beta < 1$ as a tendency. In addition, the behavior of each evaluation criterion with respect to the change of $\beta$ showed a similar tendency in these two generalizations.
Table 5: Result: optimal $\beta$ and optimal value of each criterion.

| dataset   | type | max distortion | NMI     | entropy | purity | F-measure |
|-----------|------|----------------|---------|---------|--------|-----------|
|           |      | $\beta$ value  | $\beta$ value | $\beta$ value | $\beta$ value | $\beta$ value |
| BCW       | origin | 1.797401 | 0.709719 | 0.278678 | 0.951684 | 0.951302 |
|           | pow   | 1.262827 | 0.784083 | 0.202992 | 0.967039 | 0.966688 |
|           | LSE   | 1.293603 | -0.837505 | -0.837505 | -0.837505 | -0.837505 |
| Heart     | origin | 0.687251 | 0.186732 | 0.667591 | 0.567311 | 0.528103 |
|           | pow   | 0.594202 | -0.247655 | -0.247655 | -0.247655 | -0.247655 |
|           | LSE   | 0.611663 | -0.836005 | -0.836005 | -0.836005 | -0.836005 |
| HeartK2   | origin | 1.053749 | 0.254393 | 0.746796 | 0.78543 | 0.782754 |
|           | pow   | 0.837753 | 0.254393 | 0.746796 | 0.78543 | 0.782754 |
|           | LSE   | 0.883888 | 0.259732 | 0.741662 | 0.787946 | 0.78533 |
| HTRU2     | origin | 0.563794 | 0.529009 | 0.251357 | 0.957872 | 0.95242 |
|           | pow   | 0.523274 | 0.551319 | 0.225665 | 0.962979 | 0.959666 |
|           | LSE   | 4.272006 | 0.560864 | 0.217081 | 0.964465 | 0.962264 |
| Iris      | origin | 0.045814 | 0.864186 | 0.136024 | 0.96 | 0.959984 |
|           | pow   | 0.036784 | 0.864186 | 0.136024 | 0.96 | 0.959984 |
|           | LSE   | 0.045814 | 0.864186 | 0.136024 | 0.96 | 0.959984 |
| MPE       | origin | 0.082285 | 0.377545 | 0.680344 | 0.372967 | 0.344446 |
|           | pow   | 0.080255 | 0.399729 | 0.642607 | 0.409622 | 0.415579 |
|           | LSE   | 0.082285 | 0.327545 | 0.706314 | 0.87257 | 0.842849 |
| Pima      | origin | 1.093682 | 0.017221 | 0.917006 | 0.651042 | 0.612956 |
|           | pow   | 1.37352 | 0.03817 | 0.89914 | 0.673177 | 0.692306 |
|           | LSE   | 1.478223 | 0.04042 | 0.897921 | 0.671875 | 0.684285 |
| Seeds     | origin | 0.102133 | 0.518463 | 0.489484 | 0.899524 | 0.8128 |
|           | pow   | 0.04198 | 0.518463 | 0.489484 | 0.899524 | 0.8128 |
|           | LSE   | 0.056606 | 0.536118 | 0.46851 | 0.809524 | 0.813394 |
| Thyroid   | origin | 3.083999 | 0.299619 | 0.619662 | 0.762862 | 0.744757 |
|           | pow   | 1.67311 | 0.358383 | 0.583573 | 0.775411 | 0.734067 |
|           | LSE   | 1.81135 | -1.478343 | 0.50713 | 0.804651 | 0.777106 |
| Wine      | origin | 0.207957 | 0.763574 | 0.232191 | 0.92714 | 0.926552 |
|           | pow   | 0.181908 | 0.784429 | 0.212726 | 0.92714 | 0.926552 |
|           | LSE   | 0.207957 | -0.378166 | -0.212479 | 0.927392 | 0.926768 |
| Yeast     | origin | 0.304851 | 0.198939 | 0.602141 | 0.495828 | 0.430757 |
|           | pow   | 0.212187 | 0.216864 | 0.602141 | 0.495828 | 0.430757 |
|           | LSE   | 0.213919 | 0.198775 | -1.4060995 | -0.454005 | 0.41852 |
6.2 Image compression task

Here, we experimentally demonstrate that the generalized DP-means is more effective than the original DP-means ($\beta = 1$) through the application of vector quantization to an image compression task. We used the generalized DP-means with power mean $f(\alpha) (\alpha = 0)$. In particular, we consider the case where the minimization measure approaches the maximum distortion minimization, that is, $\beta$ is sufficiently large. Although in this experiment, image compression is handled, the purpose is to examine the performance of the generalized DP-means. A previous work compared the maximum distortion minimization and the average distortion minimization in an image compression task by using a clustering method called the kernel vector quantization [Tipping and Schölkopf, 2001]. In this experiment, the same comparison is carried out using the same image (Figure 2). This image is a color image of the size $384 \times 256$. We obtained data points by dividing it into block images of $8 \times 8$. Each data point consists of $8 \times 8 \times 3 = 192$ dimensions from the block size and the color information. The uncompressed image is represented by the dataset of $384 \times 256 / 64 = 1536$ points. Image compression is performed while increasing the penalty parameter and carrying out clustering using Algorithm 1 with $f$ given by (5) and $\beta = 1$ as the average distortion minimization and $\beta = 200$ as the approximation of the maximum distortion minimization. As the penalty parameter increases, the number of clusters decreases. The preprocessing for clustering is the same as the previous experiment. We chose the squared distance, $d_\phi(x, \theta) = \|x - \theta\|^2$ as the distance measure. Newton’s method was used for gradient descent optimization to calculate cluster centers because the convergence speed is second order. The parameter $\beta = 200$ is a relatively large value among $\beta$ which do not cause any divergence in the calculation. Hereafter, $\beta = 200$ is regarded as the maximum distortion minimization. In this experiment, we focused on whether or not the letter string of the license plate in Figure 2 is recognizable [Tipping and Schölkopf, 2001]. Figure 3 shows an image compressed to the limit at which the license plate letter string can be read for each of the average distortion minimization and the maximum distortion minimization. Furthermore, when the letter string of the license plate can only be read partly, comparison under the same compression ratio is shown in Figure 4.

Figure 3 shows that, the maximum distortion minimization achieved the better compression ratio than the average distortion minimization when all the letter string can be read. In Figure 4, it is possible to read several characters of the li-

Fig. 2: No compression, compression ratio:100%, number of clusters 1536 [Tipping and Schölkopf, 2001].
Fig. 3: Limited compression that can read the license plate.

Fig. 4: Compression below limit, compression rate: 3.91%, number of clusters 60.

cense plate in the case of the maximum distortion minimization, while it is almost impossible to read in the case of the average distortion minimization. In the image compression task focusing on the letter string in the image, it is suggested that better performance is obtained by using the generalized DP-means with a large value of $\beta$ which approaches the maximum distortion minimization than the original DP-means. The reason why the generalized objective function with a large $\beta$ was effective may be discussed as follows. In the average distortion minimization, the license plate consisting of a small number of patterns in the entire image tends to have large distortion from the cluster center to which it belongs, whereas in the case of the maximum distortion minimization, this leads to the reduction in the distortion of the blocks from the license plate.

7 Discussion: total Bregman divergence

So far we have discussed with the Bregman divergence as a prerequisite. The same discussion can be made when the total Bregman divergence is used as the pseudo-distance. The total Bregman divergence is invariant to the rotation of the coordinate axes, and the cluster center obtained by minimizing the average distortion is shown to be robust to outliers [Vemuri et al., 2011]. The Bregman divergence is known to have a bijective relationship with the exponential family, whereas the total Bregman divergence corresponds to the lifted exponential family.
The total Bregman divergence is defined by
\[ tBD(x, \theta) \triangleq \frac{d_\phi(\theta, x)}{\sqrt{1 + c^2\|\nabla \phi(x_i)\|^2}}, \]
where \( c > 0 \) (Vemuri et al., 2011; Amari, 2016). Note here that the arguments of \( d_\phi \) in the numerator is reversed compared to the \( d_\phi(x, \theta) \) in (3). In the case of \( c = 1 \), it coincides with the definition in (Vemuri et al., 2011), and when \( c = 0 \), it coincides with the case of the reversed Bregman divergence. When the total Bregman divergence is used for the pseudo-distance, as in Section 4.1, the update rule of the cluster center can be obtained as
\[ \theta_k = \left(\nabla \phi\right)^{-1} \left(\sum_{i=1}^{n} w_{ik} \nabla \phi(x_i) \right) \left(\sum_{j=1}^{n} w_{jk} \right), \]  
where
\[ w_{ik} = \frac{r_{ik} f' \left(tBD(x_i, \theta_k)\right)}{\sqrt{1 + c^2\|\nabla \phi(x_i)\|^2}}. \]  
Here, \( \left(\nabla \phi\right)^{-1} \) denotes the inverse function of \( \nabla \phi \). As in Section 4.2, when the function \( f \) is concave, Theorem 6 claiming the monotonic decreasing property of the objective function holds (Appendix D.1). When the function \( f \) is convex, since the problem of updating the cluster center is a convex optimization problem, the gradient direction always becomes the descent direction by the gradient descent optimization. The influence function is derived in the same flow as in Section 5.1. As the theorem on the boundedness of the influence function, the following holds.

**Theorem 5** The following Condition 1 on the function \( f \) is a necessary and sufficient condition for its influence function to be bounded for all \( x^* \) and Condition 2 provides a necessary and sufficient condition for it to be redescending:
1. \( f'(z) \) is monotonically decreasing function \( \iff \) \( f(z) \) is a concave function or linear function,
2. \[ \lim_{z \to \infty} f'(z) = 0. \]  

The proof of this theorem is put in Appendix D.3. See Remark 1 for the discussion when \( f' \left(tBD(x^*, \theta)\right) \to \infty \) around \( x^* = \theta \). Recall that Condition 2 is a necessary condition for the boundedness of the influence function when the standard Bregman divergence \( d_\phi(x, \theta) \) is used as the pseudo-distance (Theorem 2).

**8 Conclusion**

In this paper, we generalized the average distortion of DP-means to \( f \)-separable distortion measures by using a monotonically increasing function \( f \). If the function \( f \) has an inverse function \( f^{-1} \), the \( f \)-separable distortion measure can be expressed by \( f \)-mean. We classified the function \( f \) into three types, namely, linear, convex and concave. These three types correspond to the original average distortion, distortion measures approaching the maximum distortion, and those with robustness...
against outliers, respectively. We showcased two kinds of functions including the parameter $\beta$. The objective function constituted by these functions can change the characteristics according to the value of the parameter $\beta$. Furthermore, based on this generalized objective function, an algorithm with guaranteed convergence was constructed. Like original DP-means, this algorithm has the computational complexity of the linear order of the number of data. In order to evaluate the robustness against outliers, we derived the influence function on the general form of the function $f$ and showed the necessary condition for the influence function to be bounded. For each concrete example of the function $f$, we examined the condition under which the boundedness of the influence function holds. We proved that the log-sum-exp function shows the robustness against outliers regardless of the Bregman divergence. Although the above discussion assumes the Bregman divergence as pseudo-distance, we also discussed that the same argument holds true for the total Bregman divergence. In addition, experiments using real datasets demonstrated that the generalized DP-means improves the performance of the original DP-means. Our future directions include analyzing the generalization error consisting of the bias and variance in the estimation of the cluster centers. This will lead to a principled design of the combination of the function $f$ and the Bregman divergence (or pseudo-distance) by investigating the trade-off between the generalization error and the robustness.

A Derivation of objective functions through MAD-Bayes

A.1 Generalized Gaussian distribution

We focus on the objective function with the function $f$ in (5) ($\beta > 0, a = 0$) and the Bregman divergence as the squared distance $\|x - \theta\|^2$. We prove that the objective function of this case is derived in the framework of MAD-Bayes (Broderick et al., 2013) when the generalized Gaussian distribution is assumed as a component. The generalized Gaussian distribution is given by

$$p(x|\theta, \alpha, \beta) = \frac{1}{C} \exp \left( -\left( \frac{\|x - \theta\|^2}{a} \right)^{\beta} \right),$$

where the normalization constant is

$$C = \frac{\beta \Gamma \left( \frac{L}{\beta} \right)}{2\pi \Gamma \left( \frac{L}{\beta} \right) a \Gamma \left( \frac{\tau}{\beta} \right)},$$

and $\Gamma(\cdot)$ is the gamma function. The parameters are $\alpha > 0$ and $\beta > 0$. It includes the Laplace distribution ($\beta = \frac{1}{2}$), the Gaussian distribution ($\beta = 1$), and the uniform distribution ($\beta \to \infty$) as special cases. The likelihood is given by

$$p(x^n|r, \{\theta_k\}_{k=1}^K) = \prod_{k=1}^K \prod_{i:r_i=k} p(x_i|\theta_k, \alpha, \beta).$$

The Chinese restaurant process, which is an instance of Dirichlet processes, is given by

$$p(r) = \tau^{K-1} \frac{\Gamma(\tau + 1)}{\Gamma(\tau + n)} \prod_{k=1}^K (S_{n,k} - 1)!,$$
where \( S_{n,k} = \sum_{j=1}^{n} r_{ijk} \) and \( \tau > 0 \) is the hyperparameter \((\text{Gershman and Blei 2012})\). When an arbitrary distribution that creates the cluster center is defined as \( p(\theta_k) \), the simultaneous distribution is expressed by \( p(\{\theta_k\}_{k=1}^{K}) \). The simultaneous distribution of data, cluster assignments and cluster centers are expressed by the following equation:

\[
p(x^n, r, \{\theta_k\}_{k=1}^{K}) = \prod_{k=1}^{K} \prod_{i, r_{ik}=1} \frac{1}{C} \exp \left(-\frac{\|x_i - \theta_k\|^2}{\alpha}\right)^{\beta} \cdot \tau^{K-1} \frac{\Gamma(\tau + 1)}{\Gamma(\tau + n)} \prod_{k=1}^{K} (S_{n,k} - 1)! \cdot \prod_{k=1}^{K} p(\theta_k).
\]

Then, setting \( \tau = \exp\left(-\frac{\lambda^2}{\alpha}\right) \), we consider the limit \( \alpha \to 0 \). We have

\[
-\ln p(x^n, r, \{\theta_k\}_{k=1}^{K}) = \sum_{k=1}^{K} \sum_{i, r_{ik}=1} \{O(\ln(\alpha)) + \left(\frac{\|x_i - \theta_k\|^2}{\alpha^3}\right)\} + (K - 1)\lambda^2 + O(1).
\]

It follows that

\[
-\alpha^2 \ln p(x^n, r, \{\theta_k\}_{k=1}^{K}) = \sum_{k=1}^{K} \sum_{i, r_{ik}=1} \|x_i - \theta_k\|^2 + (K - 1)\lambda^2.
\]

This objective function is equivalent to the objective function \((4)\) with the function \( f \) in \((5)\) \((\beta \neq 0, a = 0)\) as follows:

\[
\sum_{i=1}^{n} \|x_i - \theta_{a(1)}\|^2 + \lambda^2 K + O(1).
\]

Note, however, that \( \beta \) must be positive in the generalized Gaussian distribution.

### A.2 Deformed \( t \)-distribution

We consider the same objective function as that of Appendix A.1 except that \( \beta = 0 \) and \( a > 0 \) instead of \( \beta \neq 0 \). We prove that the objective function of this case is derived in the framework of MAD-Bayes when the deformed \( t \)-distribution is assumed as a component. The \( t \)-distribution is given by

\[
p(\theta | \nu, \sigma^2) = \frac{\Gamma\left(\frac{\nu + L}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(\nu\pi\sigma^2\right)^{\frac{L}{2}}} \left(1 + \frac{\|x - \theta\|^2}{\nu}\right)^{-\frac{\nu + L}{2}},
\]

where \( \nu > 0 \) is the degree of freedom, \( L \) is the dimension of data. It includes the Cauchy distribution \((\nu = 1)\) and the Gaussian distribution \((\nu \to \infty)\) as special cases. Here, we use the following distribution obtained by transforming this \( t \)-distribution:

\[
p(\theta | \nu) = \frac{1}{C} \left[1 + \frac{\|x - \theta\|^2}{\nu}\right]^{-\frac{\nu + L}{2}},
\]

where the normalization constant is

\[
C = \frac{\Gamma\left(\frac{\nu + L}{2} - \frac{L}{2}\right)(\nu\pi)^{\frac{L}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}.
\]
The likelihood is given by

\[ p(x^n | r, \{ \theta_k \}_{k=1}^K) = \prod_{k=1}^K \prod_{i \neq r_k} p(x_i | \theta_k, \nu, \sigma^2). \]

As in Appendix A.1, the simultaneous distribution of data, cluster assignments, and cluster centers are expressed by the following equation:

\[
p(x^n, r, \{ \theta_k \}_{k=1}^K) = p(x^n | r, \{ \theta_k \}_{k=1}^K) p(r) p(\{ \theta_k \}_{k=1}^K)
= \prod_{k=1}^K \prod_{i \neq r_k} p(x_i | \theta_k, \nu, \sigma^2) \cdot \tau^{K-1} \frac{\Gamma(\tau + 1)}{\Gamma(\tau + n)} \prod_{k=1}^K (S_{n,k} - 1)! \cdot \prod_{k=1}^K p(\theta_k).
\]

Then, setting \( \tau = (\nu + \lambda)^{-\frac{\nu + \lambda}{\nu + 2\sigma^2}} \), we consider the limit \( \sigma^2 \to 0 \). We have

\[
- \ln p(x^n, r, \{ \theta_k \}_{k=1}^K) = \sum_{k=1}^K \sum_{i \neq r_k} \left\{ \ln C + \frac{\nu + L}{2\sigma^2} \ln \left( 1 + \frac{\| x_i - \theta_k \|^2}{\nu} \right) \right\} + \frac{\nu + L}{2\sigma^2} (K - 1) \ln (\nu + \lambda) + O(1).
\]

It follows that

\[
- \frac{2\sigma^2}{\nu + L} \ln p(x^n, r, \{ \theta_k \}_{k=1}^K)
= \sum_{k=1}^K \sum_{i \neq r_k} \left\{ \frac{2\sigma^2}{\nu + L} \ln C + \ln \left( 1 + \frac{\| x_i - \theta_k \|^2}{\nu} \right) \right\} + (K - 1) \ln (\nu + \lambda)
= \sum_{k=1}^K \sum_{i \neq r_k} \left\{ \frac{2\sigma^2}{\nu + L} \ln C + \ln \frac{1}{\nu} + \ln (\nu + \| x_i - \theta_k \|^2) \right\} + (K - 1) \ln (\nu + \lambda).
\]

Because \( \sigma^2 \ln C \to \text{constant} \) as \( \sigma^2 \to 0 \), we obtain the objective function as follows:

\[
\sum_{k=1}^K \sum_{i \neq r_k} \ln (\nu + \| x_i - \theta_k \|^2) + (K - 1) \ln (\nu + \lambda).
\]

We put \( \nu = a \). This objective function is equivalent to the objective function \( 3 \) with the function \( f \) in \( 9 \) \( (\beta = 0, a \geq 0) \) as follows:

\[
\sum_{i=1}^n \ln (\| x_i - \theta_{d(i)} \|^2 + a) + \ln (\lambda + a) K.
\]

Note, however, that \( a \) must not be equal to 0 in the deformed \( t \)-distribution.

**B Proof of bounded influence function**

**B.1 Proof of Lemma 1**

Under Assumption 1, the following lemma holds.

**Lemma 1** For \( \mathbf{x} = (\theta^{(1)}, \ldots, \theta^{(t-1)}, x^{(t)}, \theta^{(t+1)}, \ldots, \theta^{(t)})^T \), let

\[
\bar{F}_1 = \left| \lim_{|x^{(t)}| \to \infty} f^\prime (d_\phi (\mathbf{x}, \theta)) (x^{(t)} - \theta^{(t)}) \right|,
\]

(25)
be the influence function of the \( l \)-th dimension. Then, it holds that

\[
\lim_{\|x^*\| \to \infty} \|\text{IF}(x^*)\| = \begin{cases} 
\infty & \text{if } \exists l, \text{ IF}_l \text{ is divergent,} \\
\text{constant} & \text{if } \forall l, \text{ IF}_l \text{ is bounded,} \\
0 & \text{if } \forall l, \text{ IF}_l \text{ is 0.}
\end{cases}
\]

Proof Let the \( ij \)-th component of the matrix \( \nabla \nabla \phi (\theta) \) be \( b_{ij} \), and \( \nabla \nabla \phi (\theta) (x^* - \theta) = u = (u(1), \ldots, u(L))^T \in \mathbb{R}^L \). It follows that

\[
\left( \begin{array}{c} b_{11} \ldots b_{1L} \\ \vdots \\ b_{L1} \ldots b_{LL} \end{array} \right) \left( \begin{array}{c} x^{* (1)} - \theta^{(1)} \\ \vdots \\ x^{* (L)} - \theta^{(L)} \end{array} \right) = \left( \begin{array}{c} u(1) \\ \vdots \\ u(L) \end{array} \right),
\]

where

\[
u^{(j)} = \sum_{l=1}^{L} b_{lj} (x^{* (l)} - \theta^{(l)}). \tag{26}\]

If the norm of \( \text{IF} \) is bounded, the norm of the influence function is bounded. Here, the norm of \( \text{IF} \) is

\[
\| f' (d_{\phi} (x^*, \theta)) \nabla \nabla \phi (\theta) (x^* - \theta) \| = \sqrt{\sum_{l=1}^{L} |f' (d_{\phi} (x^*, \theta)) u^{(j)}|^2}
\]

\[
= \sqrt{\sum_{j=1}^{L} \sum_{l=1}^{L} b_{lj} f' (d_{\phi} (x^*, \theta)) (x^{* (l)} - \theta^{(l)})^2}.
\]

Hence, we have

\[
\lim_{\|x^*\| \to \infty} \| f' (d_{\phi} (x^*, \theta)) \nabla \nabla \phi (\theta) (x^* - \theta) \| = \sqrt{\sum_{j=1}^{L} \sum_{l=1}^{L} b_{lj} \lim_{\|x^*\| \to \infty} f' (d_{\phi} (x^*, \theta)) (x^{* (l)} - \theta^{(l)})^2}.
\]

when \( \|x^*\| \to \infty \). If the function \( f \) satisfies \( \text{IF} \) of Theorem 2, which implies the concavity of \( f \), the following holds:

\[
|f' (d_{\phi} (x^*, \theta)) (x^{* (l)} - \theta^{(l)})| \geq |f' (d_{\phi} (x^*, \theta)) (x^{* (l)} - \theta^{(l)})|.
\]

Thus, the bounded and descending properties of the left hand side as \( |x^{* (l)}| \to \infty \) imply those of the right hand side as \( \|x^*\| \to \infty \), respectively. This means that if \( \text{IF}_l \) is bounded or converging to 0 for all \( l \), so is \( \lim_{\|x^*\| \to \infty} \|\text{IF}(x^*)\| \). If \( \text{IF}_l = \infty \) for some \( l \), putting \( x^* = x \) and taking the limit \( |x^{* (l)}| \to \infty \), we have \( \lim_{\|x^*\| \to \infty} \|\text{IF}(x^*)\| = \infty \). \( \square \)
B.2 Power mean

\textbf{B.2.1 Proof of Theorem 3: redescending property for power mean}

Evaluate the following expression (25) of Lemma 1 for the function \( f(x, \theta) \) in (B) \((\beta < 0)\). It follows from l'Hopital's rule that

\[
I^F_I = \lim_{|x^*(t)| \to \infty} \frac{x^*(t) - \theta(t)}{(d_\alpha(x^*(t), \theta(t)) + a)^{1-\beta}}
\]

\[
= \lim_{|x^*(t)| \to \infty} \frac{1}{1 - \beta} \left\{ \frac{d_\alpha(x^*(t), \theta(t)) - \beta \left( \frac{\partial \phi(x, \theta)}{\partial x^{(t)}} \right)}{(d_\alpha(x^*(t), \theta(t)) + a)^{1-\beta}} \right\} = 0.
\]

Therefore, from Lemma 1 the redescending property holds.

\textbf{B.2.2 \(\alpha\)-divergence}

Here, since the \(\alpha\)-divergence is additively defined, it holds that \(d_\alpha(x, \theta) = d_\alpha(x(t)^*, \theta(t))\). Then, the \(\alpha\)-divergence is expressed as

\[
d_\alpha(x(t), \theta(t)) = \begin{cases} 
  x(t) + O(x(t)^\alpha) & \alpha < 1 \\
  x(t) \ln x(t) + O(x(t)) & \alpha = 1 \\
  x(t)^\alpha + O(x(t)) & \alpha > 1.
\end{cases}
\]

Evaluate the following expression (26) of Lemma 1 for the function \( f(x, \theta) \) in (B):

\[
I^F_I = \lim_{|x^*(t)| \to \infty} \frac{x^*(t) - \theta(t)}{d_\alpha(x^*(t), \theta(t)) + a}^{1-\beta}
\]

\[
= \lim_{|x^*(t)| \to \infty} \frac{1}{1 - \beta} \left\{ \frac{x^*(t) - \theta(t)}{d_\alpha(x^*(t), \theta(t)) + a} \right\}^{1-\beta}.
\]

1. \(\alpha < 1\)

It follows from (27) that

\[
I^F_I = \left\{ \lim_{|x^*(t)| \to \infty} \frac{\alpha(\alpha - 1)}{x(t)} \right\}^{1-\beta}
\]

\[
= \left\{ \alpha(\alpha - 1) \lim_{|x^*(t)| \to \infty} \frac{x^*(t) - \theta(t)}{x(t)} \right\}^{1-\beta}
\]

\[
= \left\{ \alpha(\alpha - 1) \lim_{|x^*(t)| \to \infty} \frac{x^*(t) - \theta(t)}{x(t)^{(1-\beta)}} \right\}^{1-\beta}.
\]
Therefore, it holds that
\[
\begin{align*}
1 > 1 - \beta & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is divergent}, \\
1 = 1 - \beta & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is bounded}, \\
1 > 1 - \beta & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is redescending}, \\
\end{align*}
\]
\[
\Leftrightarrow \begin{align*}
\beta > 0 & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is divergent}, \\
\beta = 0 & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is bounded}, \\
\beta < 0 & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is redescending}. \\
\end{align*}
\]

2. \( \alpha = 1 \) (generalized Kullback-Leibler divergence)

It follows from \((27)\) that
\[
\tilde{\mathbf{I}}_{\alpha} = \left\{ \lim_{|x^*(t)| \to \infty} \frac{x^*(t) - g(t)}{x^*(t)^\alpha \ln(x^*(t)) + O(x^*(t))} \right\}^{1-\beta}
\]
\[
= \left\{ \lim_{|x^*(t)| \to \infty} \frac{x^*(t) - g(t)}{x^*(t)^\alpha \ln(x^*(t))} \right\}^{1-\beta}
\]
\[
= \left\{ \lim_{|x^*(t)| \to \infty} \frac{x^*(t) - g(t)}{x^*(t)^{\alpha-1} \ln(x^*(t))} \right\}^{1-\beta}.
\]

Therefore, it holds that
\[
\begin{align*}
\beta > 0 & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is divergent}, \\
\beta \leq 0 & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is redescending}. \\
\end{align*}
\]

3. \( \alpha > 1 \)

It follows from \((27)\) that
\[
\tilde{\mathbf{I}}_{\alpha} = \left\{ \lim_{|x^*(t)| \to \infty} \frac{\alpha(x^*(t) - g(t))}{x^*(t)^\alpha + O(x^*(t))} \right\}^{1-\beta}
\]
\[
= \left\{ \lim_{|x^*(t)| \to \infty} \frac{x^*(t) - g(t)}{x^*(t)^\alpha} \right\}^{1-\beta}
\]
\[
= \left\{ \lim_{|x^*(t)| \to \infty} \frac{x^*(t) - g(t)}{x^*(t)^{\alpha-1}} \right\}^{1-\beta}.
\]

Therefore, it holds that
\[
\begin{align*}
\alpha(1 - \beta) > 1 & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is divergent}, \\
\alpha(1 - \beta) = 1 & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is bounded}, \\
\alpha(1 - \beta) > 1 & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is redescending}, \\
\end{align*}
\]
\[
\Leftrightarrow \begin{align*}
\beta > 1 - \frac{1}{\alpha} & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is divergent}, \\
\beta = 1 - \frac{1}{\alpha} & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is bounded}, \\
\beta < 1 - \frac{1}{\alpha} & \Rightarrow \| \mathbf{I}\!\!\!\!\!F(x^*) \| \text{ is redescending}. \\
\end{align*}
\]
B.2.3 Exp-loss

When the function $f$ is given by (5) and the convex function constituting the Bregman divergence is given by the exponential function, we investigate the boundedness of the influence function. For the convex function

$$\phi(x) = \exp(x),$$

the corresponding Bregman divergence is given by (Banerjee et al., 2005),

$$d_\phi (x, \theta) = \exp(x) - \exp(\theta) - \exp(\theta)(x - \theta).$$

For multidimensional data, we additively define the divergence as follows:

$$d_\phi (x, \theta) = \sum_{l=1}^{L} d_\phi (x^{(l)}, \theta^{(l)}), \quad (28)$$

$$\phi(x) = \sum_{l=1}^{L} \phi(x^{(l)}).$$

Here, since (28) is additively defined, it holds that

$$d_\phi (\tilde{x}, \theta) = d_\phi (x^{(l)}, \theta^{(l)}).$$

Evaluate the following expression (25) of Lemma 1 for the function $f$ in (5) as follows:

$$\tilde{IF} = \lim_{|x^*| \to \infty} \frac{x^*(l) - \theta(l)}{\{d_\phi (\tilde{x}, \theta) + a\}^{1-\beta}} \quad (29)$$

$$\tilde{IF} = \lim_{|x^*| \to \infty} \frac{1}{(1 - \beta) \{d_\phi (\tilde{x}, \theta) + a\}^{1-\beta}} \left\{ \begin{array}{ll} \frac{\beta}{\exp(x^*(l)) - \exp(\theta(l))} & \text{if } x^*(l) \to \infty, \\ \frac{\beta}{\exp(\theta(l))} & \text{if } x^*(l) \to -\infty. \end{array} \right. \quad (30)$$

Here, when $\beta < 1$ and $\beta \neq 0$, (30) is 0, from Lemma 1 the redescending property holds. When $\beta = 0$, it follows from (29) that:

$$\frac{1}{\exp(x^*(l)) - \exp(\theta(l))} = \begin{cases} 0 & \text{if } x^*(l) \to \infty, \\ \exp(-\theta(l)) & \text{if } x^*(l) \to -\infty. \end{cases}$$

That is, when $\beta = 0$, at least one of the elements of $x^*$ satisfies $x^*(l) \to -\infty$, the influence function is bounded, however it is not redescending. If there is no such an element, the influence function satisfies the redescending property. The results are summarized as follows:

$$\begin{array}{l} \beta = 0 \quad \text{redescending if } \exists l, x^*(l) \to -\infty, \\
0 < \beta < 1 \quad \text{bounded if } \exists l, x^*(l) \to -\infty, \\
\beta < 0 \quad \text{redescending,} \\
\beta < 0 \quad \text{redescending.} \end{array}$$
B.3 Proof of Theorem 4: redescending property for log-sum-exp

Evaluate the following expression (25) of Lemma 1 for the function \( f \) in (7) \( (\beta < 1) \). It follows from l’Hôpital’s rule that

\[
\lim_{|x^*(l)| \to \infty} \frac{|x^*(l)| - \theta(l)}{(1 - \beta)d_\varphi(\tilde{x}, \theta)} \exp \left\{ (1 - \beta)d_\varphi(\tilde{x}, \theta) \right\} = 0.
\]

Therefore, from Lemma 1, the redescending property holds.

C Details of experimental procedure

The experiment of Section 6.1 was carried out by the following procedure.

(i) Randomly rearrange the sequence of data.
(ii) Change \( \beta \) from \(-2\) to \( 5 \) in \( 0.1 \) increments.
(ii.a) Change the penalty parameter \( \lambda \) for each \( \beta \) and execute the algorithm.
(ii.b) For each result with the number of clusters estimated correctly, an average value is calculated for each evaluation criterion.
(iii) Repeat the above steps (i) and (ii) \( 100 \) times, find the average value of each evaluation criterion obtained in (ii) for each \( \beta \), and use it as the result.

Note that when the Bregman divergence is defined as the average with respect to the dimension \( L \), in the case of the log-sum-exp function, the effective value of the parameter \( \beta \) depends on \( L \). When the parameter to be given is \( \beta^* \), the effective value of the parameter is \( \beta = \frac{\beta^*}{L} + 1 \). Thus, the range of \( \beta \) is \( 1/L \sim 4/L \) and the step size is \( \frac{0.1}{L} \).

D Properties of total Bregman divergence

D.1 Guarantee of monotonic decreasing property

Theorem 6 If the function \( f \) is concave, the updating of the cluster center using (22) monotonically decreases the objective function for general total Bregman divergence.

Proof The flow of the proof is almost the same as the proof of Theorem 1. We show that the objective function (4) monotonically decreases when the \( k \)-th cluster center \( \theta_k \) is newly updated to \( \tilde{\theta}_k \) by (22). More specifically, we prove that, \( L(\theta_k) \geq L(\tilde{\theta}_k) \), where \( L(\theta_k) \) is the sum of the terms related to \( \theta_k \) in (4), that is,

\[
L(\theta_k) = \sum_{i=1}^{n} r_{ik} f(\text{tBD}(x_i, \theta_k)).
\]

From the inequality in (11), the following holds:

\[
L(\theta_k) - L(\tilde{\theta}_k) = \sum_{i=1}^{n} r_{ik} \left\{ f(\text{tBD}(x_i, \theta_k)) - f(\text{tBD}(x_i, \tilde{\theta}_k)) \right\} \\
\geq \sum_{i=1}^{n} r_{ik} f'(\text{tBD}(x_i, \theta_k)) \left\{ \text{tBD}(x_i, \theta_k) - \text{tBD}(x_i, \tilde{\theta}_k) \right\} \\
= d_\varphi(\theta_k, \tilde{\theta}_k) \sum_{i=1}^{n} r_{ik} f'(\text{tBD}(x_i, \theta_k)) \frac{1}{\sqrt{1 + c^2 \| \nabla \phi(x_i) \|^2}} \geq 0,
\]

where (24) and (26) were used in the last equality. \( \Box \)
The following corollary immediately follows from Theorem 6.

**Corollary 2** When the objective function is constructed by the power mean or the log-sum-exp function, the following holds. When \( \beta \leq 1 \), the updating of the cluster center using monotonically decreases the objective function for general total Bregman divergence.

### D.2 Influence function

When we derive the influence function as in Section 5.1, it is given by

\[
\text{IF}(x^*) = -mG^{-1}\nabla f(t\text{BD}(x^*, \theta)),
\]

\[
G = \sum_{i=1}^{m} \nabla f(t\text{BD}(x_i, \theta)).
\]

Because the matrix \( G \) does not depend on \( x^* \), the robustness against outliers is evaluated by

\[
-\nabla f(t\text{BD}(x^*, \theta)) = -f'(t\text{BD}(x^*, \theta)) \nabla \phi(x^*)
\]

\[
= f'(t\text{BD}(x^*, \theta)) \frac{\nabla \phi(x^*) - \nabla \phi(\theta)}{\sqrt{1 + c^2\|\nabla \phi(x^*)\|^2}}.
\]

### D.3 Proof of Theorem 5

Evaluate the following expression, which is the norm of (31):

\[
f'(t\text{BD}(x^*, \theta)) \frac{\|\nabla \phi(x^*) - \nabla \phi(\theta)\|}{\sqrt{1 + c^2\|\nabla \phi(x^*)\|^2}}.
\]

Even if \( \|x^*\| \) has any value, \( \frac{\|\nabla \phi(x^*) - \nabla \phi(\theta)\|}{\sqrt{1 + c^2\|\nabla \phi(x^*)\|^2}} \) is bounded \cite{Vemuri et al., 2011}. Therefore, it is a necessary and sufficient condition for the influence function to be bounded that \( f'(z) \) is bounded for all \( z \). As \( \|\theta\| < \infty \), when \( \|x^*\| \) is large, \( t\text{BD}(x^*, \theta) \) is large. When the function \( f(z) \) is convex, \( f'(z) \) is a monotonically increasing function. As \( \|x^*\| \) is increased, \( f'(z) \) grows unboundedly, and the influence function is not bounded. In order to reduce \( f'(z) \) when \( \|x^*\| \) is large, \( f'(z) \) must be a monotonically decreasing function. The function \( f(z) \) that satisfies this condition is concave or linear \( (\text{Condition 1}) \). As \( \|\theta\| < \infty \), when \( \|x^*\| \to \infty \), \( \frac{\|\nabla \phi(x^*) - \nabla \phi(\theta)\|}{\sqrt{1 + c^2\|\nabla \phi(x^*)\|^2}} \) does not become 0. Therefore, the necessary and sufficient condition for satisfying the re-descending property \( (\text{Condition 2}) \) is (24).

### E Plots of influence functions

The Bregman divergence corresponding to the binomial distribution is given by

\[
d_\phi(x; \theta) = x \ln \left( \frac{x}{\theta} \right) + (N - x) \ln \left( \frac{N - x}{N - \theta} \right).
\]

where \( N \) is a non-negative integer value and \( x \in \{0, 1, \ldots, N\} \) in \cite{Banerjee et al., 2005}. In the following, \ref{plots} in the one-dimensional case is illustrated as a function of \( x^* \) for each of Bregman divergence for the power mean and the log-sum-exp. It is 0 at \( x^* = \theta \). We can see the tendencies of the influence functions as discussed in Section 5.3 and Section 5.4 for different \( f \) and Bregman divergences.
Fig. 5: Power mean and $f$.

Fig. 6: Power mean and squared distance.

Fig. 7: Power mean and generalized KL divergence.
Fig. 8: Power mean and Itakura Saito divergence.

(a) \( \theta = 1000, \alpha = 1 \)
(b) \( \theta = 1000, \alpha = 0.1 \)

Fig. 9: Power mean and exp-loss.

(a) \( \theta = 0, \alpha = 1 \)
(b) \( \theta = 0, \alpha = 0 \)
Generalized Dirichlet-process-means for $f$-separable distortion measures

(a) KL, $\theta = 50$, $N = 100$

(b) Squared distance, $\theta = 0$

(c) Generalized KL divergence, $\theta = 100$

(d) Itakua Saito divergence, $\theta = 1000$

(e) Exp-loss, $\theta = 0$

Fig. 10: Log-sum-exp
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