The fractional Fisher information and the central limit theorem for stable laws

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Abstract. A new information-theoretic approach to the central limit theorem for stable laws is presented. The main novelty is the concept of relative fractional Fisher information, which shares most of the properties of the classical one, included Blachman-Stam type inequalities. These inequalities relate the fractional Fisher information of the sum of \( n \) independent random variables to the information contained in sums over subsets containing \( n - 1 \) of the random variables. As a consequence, a simple proof of the monotonicity of the relative fractional Fisher information in central limit theorems for stable law is obtained, together with an explicit decay rate.

Keywords. Central limit theorem, Fractional calculus, Fisher information, Information inequalities, Stable laws.

1 Introduction

The entropy functional (or Shannon’s entropy) of a real valued random variable \( X \) with density \( f \) is defined as

\[
H(X) = H(f) = - \int_{\mathbb{R}} f(x) \log f(x) \, dx. \tag{1}
\]

provided that the integral makes sense. Among random variables with the same variance \( \sigma \) the standard Gaussian \( Z \) with variance \( \sigma \) has the largest entropy. If the \( X_j \)’s are independent copies of a centered random variable \( X \) with variance 1, then the (classical) central limit theorem implies that the law of the normalized sums

\[
S_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j
\]

converges weakly to the law of the centered standard Gaussian \( Z \), as \( n \) tends to infinity.

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A direct consequence of the entropy power inequality, postulated by Shannon [40] in the fourthies, and subsequently proven by Stam [42] (cf. also Blachman [7]), implies that
\[ H(S_2) \geq H(S_1). \]
The entropy of the normalized sum of two independent copies of a random variable is larger than that of the original. A shorter proof was obtained later by Lieb [31] (cf. also [3, 28, 29] for exhaustive presentation of the subject). While inductively expected that the entire sequence \( H(S_n) \) should increase with \( n \), as conjectured by Lieb in 1978 [31], a rigorous proof of this result was found only 25 years later by Artstein, Ball, Barthe and A. Naor [1, 2].

More recently, simpler proofs of the monotonicity of the sequence \( H(S_n) \) have been obtained by Madiman and Barron [36, 37] and Tulino and Verdú [44]. Madiman and Barron [36], by means of a detailed analysis of variance projection properties, derived new entropy power inequalities for sums of independent random variables, and, as a consequence, the monotonicity of entropy in central limit theorems for independent and identically distributed random variables. Tulino and Verdú [44] obtained analogous results by taking advantage of projection through minimum mean-squared error interpretation. As observed in [37], the proofs of the main result in both [37, 44] share essential similarities.

As suggested by Stam’s proof of the entropy power inequality [7, 42], most of the results about monotonicity benefit from the reduction from entropy to another information-theoretic notion, the Fisher information of a random variable. For sufficiently regular densities, the Fisher information can be written as
\[
I(X) = I(f) = \int_{\{f>0\}} \frac{|f'(x)|^2}{f(x)} \, dx. \tag{2}
\]
Among random variables with the same variance \( \sigma \), the Gaussian \( Z \) has smallest Fisher information \( 1/\sigma \). Fisher information and entropy are related each other by the so-called de Bruijn relation [3, 42]. If \( u(x, t) = u_t(x) \) denotes the solution to the initial value problem for the heat equation in the whole space \( \mathbb{R} \),
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{3}
\]
leaving from an initial probability density function \( f(x) \),
\[
I(f) = \frac{d}{dt} H(u_t)|_{t=0}.
\]
A particularly clear explanation of this link is given in the article of Carlen and Soffer [13] (cf. also Barron [3] and Brown [11]).

It is noticeable that the connection between Fisher information and the central limit theorem was noticed at the time of Stam’s proof of entropy power inequality by Linnik [34], who first used Fisher information in a proof of the central limit theorem.

Recently, the role of Fisher information in limit theorems has been considered also in situations different from the classical central limit theorem. In [10] (cf. also [8, 9]) Bobkov, Chistyakov and Götze enlightened the possibility to make use of the relative Fisher information to study convergence towards a stable law [15, 20, 29]. If the \( X_j \)’s are
independent copies of a centered random variable $X$ which lies in the domain of normal attraction of a random variable $Z_\lambda$ with Lévy symmetric stable distribution, the central limit theorem for stable laws implies that the law of the normalized sums

$$T_n = \frac{1}{n^{1/\lambda}} \sum_{j=1}^{n} X_j$$

converges weakly to the law of the centered stable $Z_\lambda$, as $n$ tends to infinity. Given a random variable $X$, with probability distribution $G(x)$, the Fourier transform of $G(x)$ will be denoted by

$$\mathcal{F}G(\xi) = \hat{G}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} dG(x), \quad \xi \in \mathbb{R}.$$ 

In case $X$ possesses a probability density $g(x) = G'(x)$, we will still denote the Fourier transform of $g(x)$ by

$$\mathcal{F}g(\xi) = \hat{g}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} g(x) dx, \quad \xi \in \mathbb{R}.$$ 

A Lévy symmetric stable distribution $L_\lambda(x)$ of order $\lambda$ is defined in terms of its Fourier transform by

$$\hat{L}_\lambda(\xi) = e^{-|\xi|^\lambda}. \quad (5)$$

While the Gaussian density is related to the linear diffusion equation (3), Lévy distributions are deeply related to linear fractional diffusion equations

$$\frac{\partial u(x, t)}{\partial t} = D_{2\alpha} u(x, t). \quad (6)$$

For the classical diffusion case described by (3), $\alpha = 1$ and the diffusion operator models a Brownian diffusion process. For fractional diffusion, where $1/2 < \alpha < 1$, the $D_{2\alpha}$ operator in (6) is commonly referred to as anomalous diffusion, and the underlying stochastic process is a Lévy stable flight.

Indeed, the fractional diffusion equation (6) can be fruitfully described in terms of Fourier variables, where it takes the form

$$\frac{\partial \hat{u}(\xi, t)}{\partial t} = -|\xi|^{2\alpha} \hat{u}(\xi, t). \quad (7)$$

Equation (7) can be easily solved. Its solution

$$\hat{u}(\xi, t) = \hat{u}_0(\xi) e^{-|\xi|^{2\alpha} t}. \quad (8)$$

shows that the fractional diffusion equation admits a fundamental solution, given by a scaled in time Lévy distribution of order $\lambda = 2\alpha$.

Equation (8) enhances a strong analogy between the solution of the heat equation (3) and the solution to the fractional diffusion equation (6), and suggests that information techniques used for the former could be fruitfully used, by suitably adapting these techniques to the new situation, to the latter [19].
At difference with the analysis of Bobkov, Chistyakov and Götze [10], in this note we will show that this analogy can be reasonably stated at the level of Fisher information by resorting to a new definition, which in our opinion is better adapted to the anomalous diffusion case. In Section 2 we introduce a generalization of (relative) Fisher information, the relative fractional Fisher information, that is constructed to vanish on the Lévy symmetric stable distribution, and shares most of the properties of the classical one defined by [2]. By using this new definition, we will present in Section 3 an information-theoretic proof of monotonicity of the fractional Fisher information of the normalized sums (4) by adapting the ideas developed in [37]. At difference with the classical case, it will be showed in Section 3 that the monotonicity also implies convergence in fractional Fisher information at a precise rate, which depends on the value of the exponent that characterizes the Lévy distribution.

Fractional in space diffusion equations appear in many contexts. Among others, the review paper by Klafter et al. [25] provides numerous references to physical phenomena in which these anomalous diffusions occur (cf. [6, 15, 21, 38, 41] and the reference therein for various details on both mathematical and physical aspects). Also, fractional diffusion equations in the nonlinear setting have been intensively studied in the last years by Caffarelli, Vazquez et al. [12, 14, 45, 46]. The reading of these papers was essential in moving my interest towards the present topic.

2 Scores and fractional Fisher information

In the rest of this paper, if not explicitly quoted, and without loss of generality, we will always assume that any random variable $X$ we will consider is centered, i.e. $E(X) = 0$, whereas usual $E(\cdot)$ denotes mathematical expectation.

In this section we briefly summarize the mathematical notations and the meaning of the fractional diffusion operator $D_{2\alpha}$ which appears in equation (4). For $0 < \alpha < 1$ we let $R_{\alpha}$ be the one-dimensional normalized Riesz potential operator defined for locally integrable functions by [39, 43]

$$R_{\alpha}(f)(x) = S(\alpha) \int_{\mathbb{R}} \frac{f(y) \, dy}{|x - y|^{1 - \alpha}}.$$

The constant $S(\alpha)$ is chosen to have

$$\hat{R}_{\alpha}(f)(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

(9)

Since for $0 < \alpha < 1$ it holds [32]

$$\mathcal{F}|x|^{\alpha - 1} = |\xi|^{-\alpha} \pi^{1/2} \Gamma \left( \frac{1 - \alpha}{2} \right) \Gamma \left( \frac{\alpha}{2} \right),$$

(10)

where, as usual $\Gamma(\cdot)$ denotes the Gamma function, the value of $S(\alpha)$ is given by

$$S(\alpha) = \left[ \pi^{1/2} \Gamma \left( \frac{1 - \alpha}{2} \right) \Gamma \left( \frac{\alpha}{2} \right) \right]^{-1}.$$
Note that $S(\alpha) = S(1 - \alpha)$.

We then define the fractional derivative of order $\alpha$ of a real function $f$ as $(0 < \alpha < 1)$

$$
\frac{d^{\alpha}f(x)}{dx^{\alpha}} = D_{\alpha}f(x) = \frac{d}{dx}R_{1-\alpha}(f)(x).
$$

(11)

Thanks to (9), in Fourier variables

$$
\hat{D}_{\alpha}f(\xi) = i \frac{\xi}{|\xi|} |\xi|^\alpha \hat{f}(\xi).
$$

(12)

In theoretical statistics, the score or efficient score \[16, 37\] is the derivative, with respect to some parameter $\theta$, of the logarithm of the likelihood function (the log-likelihood). If the observation is $X$ and its likelihood is $L(\theta; X)$, then the score $\rho_L(X)$ can be found through the chain rule

$$
\rho_L(\theta, X) = \frac{1}{L(\theta; X)} \frac{\partial L(\theta; X)}{\partial \theta}.
$$

(13)

Thus the score indicates the sensitivity of $L(\theta; X)$ (its derivative normalized by its value).

In older literature, the term linear score refers to the score with respect to an infinitesimal translation of a given density. In this case, the likelihood of an observation is given by a density of the form $L(\theta; X) = f(X + \theta)$. According to this definition, given a random variable $X$ in $\mathbb{R}$ distributed with a differentiable probability density function $f(x)$, its linear score $\rho$ (at $\theta = 0$) is given by

$$
\rho(X) = \frac{f'(X)}{f(X)}.
$$

(14)

The linear score has zero mean, and its variance is just the Fisher information \[2\] of $X$.

Also, the notion of relative score has been recently considered in information theory \[22\] (cf. also \[10\]). For every pair of random variables $X$ and $Y$ with differentiable density functions $f$ (respectively $g$), the score function of the pair relative to $X$ is represented by

$$
\tilde{\rho}(X) = \frac{f'(X)}{f(X)} - \frac{g'(X)}{g(X)}.
$$

(15)

In this case, the relative to $X$ Fisher information between $X$ and $Y$ is just the variance of $\tilde{\rho}(X)$. This notion is satisfying because it represents the variance of some error due to the mismatch between the prior distribution $f$ supplied to the estimator and the actual distribution $g$. Obviously, whenever $f$ and $g$ are identical, then the relative Fisher information is equal to zero.

Let $z_\sigma(x)$ denote the Gaussian density in $\mathbb{R}$ with zero mean and variance $\sigma$

$$
z_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left( -\frac{|x|^2}{2\sigma} \right).
$$

(16)

Then a Gaussian random variable of density $z_\sigma$ is uniquely defined by the score function

$$
\rho(Z_\sigma) = -Z_\sigma / \sigma.
$$
Also, the relative (to $X$) score function of $X$ and $Z_\sigma$ takes the simple expression
\begin{equation}
\tilde{\rho}(X) = \frac{f'(X)}{f(X)} + \frac{X}{\sigma},
\end{equation}
which induces a (relative to the Gaussian) Fisher information
\begin{equation}
\tilde{I}(X) = \tilde{I}(f) = \int_{\{f>0\}} \left( \frac{f'(x)}{f(x)} + \frac{x}{\sigma} \right)^2 f(x) \, dx.
\end{equation}
Clearly, $\tilde{I}(X) \geq 0$, while $\tilde{I}(X) = 0$ if $X$ is a centered Gaussian variable of variance $\sigma$.

The concept of linear score can be naturally extended to cover fractional derivatives. Given a random variable $X$ in $\mathbb{R}$ distributed with a probability density function $f(x)$ that has a well-defined fractional derivative of order $\alpha$, with $0 < \alpha < 1$, its linear fractional score, denoted by $\rho_{\alpha+1}$ is given by
\begin{equation}
\rho_{\alpha+1}(X) = \frac{D_\alpha f(X)}{f(X)}.
\end{equation}
Thus the linear fractional score indicates the non local (fractional) sensitivity of $f(X + \theta)$ at $\theta = 0$ (its fractional derivative normalized by its value). Differently from the classical case, the fractional score of $X$ is linear in $X$ if and only if $X$ is a Lévy distribution of order $\alpha + 1$. Indeed, for a given positive constant $C$, the identity
\begin{equation}
\rho_{\alpha+1}(X) = -CX,
\end{equation}
is verified if and only if, on the set $\{f > 0\}$
\begin{equation}
D_\alpha f(x) = -Cxf(x).
\end{equation}
Passing to Fourier transform, this identity yields
\begin{equation}
i\xi|\xi|^{\alpha-1} \hat{f}(\xi) = -iC \frac{\partial \hat{f}(\xi)}{\partial \xi},
\end{equation}
and from this follows
\begin{equation}
\hat{f}(\xi) = \hat{f}(0) \exp \left\{ -\frac{|\xi|^{\alpha+1}}{C(\alpha + 1)} \right\}.
\end{equation}
Finally, by choosing $C = (\alpha+1)^{-1}$, and imposing that $f(x)$ is a probability density function (i.e. by fixing $\hat{f}(\xi = 0) = 1$), we obtain that the Lévy stable law of order $\alpha + 1$ is the unique probability density solving (20).

It is important to remark that, unlike in the case of the linear score, the variance of the fractional score is in general unbounded. One can easily realize this by looking at the variance of the fractional score in the case of a Lévy variable. For a Lévy variable, in fact, the variance of the fractional score coincides with a multiple of its variance, which is unbounded [20,30]. For this reason, a consistent definition in this case is represented by
the relative fractional score. In reason of (21), a Lévy random variable of density $z_\lambda$, with $1 < \lambda < 2$ is uniquely defined by a linear fractional score function

$$\rho_\lambda(Z_\lambda) = -\frac{Z_\lambda}{\lambda},$$

the relative (to $X$) fractional score function of $X$ and $Z_\lambda$ assumes the simple expression

$$\tilde{\rho}_\lambda(X) = \frac{D\lambda-1f(X)}{f(X)} + \frac{X}{\lambda},$$

which induces a (relative to the Lévy) fractional Fisher information (in short $\lambda$-Fisher relative information)

$$I_{\lambda}(X) = I_{\lambda}(f) = \int_{\{f>0\}} \left( \frac{D\lambda-1f(x)}{f(x)} + \frac{x}{\lambda} \right)^2 f(x) \, dx.$$  

(23)

The fractional Fisher information is always greater or equal than zero, and it is equal to zero if and only if $X$ is a Lévy symmetric stable distribution of order $\lambda$. At difference with the relative standard relative Fisher information, $I_{\lambda}$ is well-defined any time that the the random variable $X$ has a probability density function which is suitably closed to the Lévy stable law (typically lies in a subset of the domain of attraction). We will define by $\mathcal{P}_\lambda$ the set of probability density functions such that $I_{\lambda}(f) < +\infty$, and we will say that a random variable $X$ lies in the domain of attraction of the $\lambda$-Fisher information if $I_{\lambda}(X) < +\infty$. More in general, for a given positive constant $\nu$, we will consider other relative fractional score functions given by

$$\tilde{\rho}_{\lambda,\nu}(X) = \frac{D\lambda-1f(X)}{f(X)} + \frac{X}{\lambda \nu}.$$  

(24)

This leads to the relative fractional Fisher information

$$I_{\lambda,\nu}(X) = I_{\lambda,\nu}(f) = \int_{\{f>0\}} \left( \frac{D\lambda-1f(x)}{f(x)} + \frac{x}{\lambda \nu} \right)^2 f(x) \, dx.$$  

(25)

Clearly, $I_{\lambda} = I_{\lambda,1}$. Analogously, we will define by $\mathcal{P}_{\lambda,\nu}$ the set of probability density functions such that $I_{\lambda,\nu}(f) < +\infty$, and we will say that a random variable $X$ lies in the domain of attraction if $I_{\lambda,\nu}(X) < +\infty$.

Remark 1. The characterization of the functions which belong to the domain of attraction of the relative fractional Fisher information is not an obvious task. However, it can be seen that this set of functions is not empty. We will present an explicit example in the Appendix.

3 Monotonicity of the fractional Fisher information

We now proceed to recover some useful properties of the relative fractional score function $\tilde{\rho}_\lambda(X)$ defined in (22). Most of these properties are easy generalizations of analogous ones proven in [37] for the standard linear score function. The main difference here is that we need to resort to the relative one. The following Lemma holds
Therefore, to conclude the proof, it is enough to remark that the following identity holds true
\[
\hat{\rho}_\lambda(x) = E \left[ \delta \hat{\rho}_\lambda^{(1)}(X_1) + (1 - \delta) \hat{\rho}_\lambda^{(2)}(X_2) | X_1 + X_2 = x \right].
\] (26)

Proof. Let \( f_j, j = 1, 2 \) and \( f \) be the densities of \( X_j, j = 1, 2 \) and \( X_1 + X_2 \). Then, the density of \( X_1 + X_2 \) is given by the convolution product of \( f_1 \) and \( f_2 \), \( f = f_1 * f_2 \). To start with, we remark that the fractional derivatives, as given by (11) and (12), share the same behavior, with respect to convolutions, of the usual derivatives. Indeed, thanks to (12), it follows that
\[
D_{\lambda-1}f(x) = (D_{\lambda-1}f_1) * f_2(x) = f_1 * (D_{\lambda-1}f_2)(x).
\]

The previous identity can be rewritten in terms of expectations as (27)
\[
D_{\lambda-1}f(x) = D_{\lambda-1}E[f_1(x - X_2)] = E[D_{\lambda-1}f_1(x - X_2)] = E[\rho_\lambda^{(1)}(x - X_2)f_1(x - X_2)].
\]

Therefore
\[
\rho_\lambda(x) = \frac{D_{\lambda-1}f(x)}{f(x)} = E\left[ \rho_\lambda^{(1)}(x - X_2)f_1(x - X_2) \right] = E\left[ \rho_\lambda^{(1)}(X_1) | X_1 + X_2 = x \right].
\]

As usual, given the random variables \( X \) and \( Y \), we denoted by \( E[X|Y] \) the conditional expectation of \( X \) given \( Y \). Exchanging the indexes, we obtain an identical expression which relates the fractional score of the sum to the second variable \( X_2 \). Hence, for each positive constant \( \delta \) with \( 0 < \delta < 1 \) we have
\[
\rho_\lambda(x) = \delta E\left[ \rho_\lambda^{(1)}(X_1) | X_1 + X_2 = x \right] + (1 - \delta)E\left[ \rho_\lambda^{(2)}(X_2) | X_1 + X_2 = x \right].
\] (27)

To conclude the proof, it is enough to remark that the following identity holds true
\[
x = \delta E\left[ \frac{X_1}{\delta} | X_1 + X_2 = x \right] + (1 - \delta)E\left[ \frac{X_2}{1 - \delta} | X_1 + X_2 = x \right].
\]

Lemma 1 has several interesting consequences. Since the norm of the relative fractional score is not less than that of its projection (i.e. by the Cauchy–Schwarz inequality), we obtain
\[
I_\lambda(X_1 + X_2) = E \left[ \hat{\rho}_\lambda^2(X_1 + X_2) \right] \leq E \left[ \left( \delta \hat{\rho}_\lambda^{(1)}(X_1) + (1 - \delta) \hat{\rho}_\lambda^{(2)}(X_2) \right)^2 \right] = \delta^2 I_{\lambda, \delta}(X_1) + (1 - \delta)^2 I_{\lambda, 1-\delta}(X_2).
\] (28)
Inequality (28) is the analogous of the Blachman–Stam inequality [7, 42], and allows to bound the relative fractional Fisher information of the sum of independent variables in terms of the relative fractional Fisher information of its addends. Inequality (28) can be reduced to a normal form by making use of scaling arguments. Indeed, for any given random variable \( X \) such that one of the two sides is bounded, and positive constant \( \upsilon \), the following identity holds
\[
I_{\lambda, \upsilon}(\upsilon^{1/\lambda}X) = \upsilon^{-2(1-1/\lambda)}I_{\lambda}(X).
\] (29)

Using identity (29) into inequality (28) we can write it in the form
\[
I_{\lambda}(\delta^{1/\lambda}X_1 + (1 - \delta)^{1/\lambda}X_2) \leq \delta^{2/\lambda}I_{\lambda}(X_1) + (1 - \delta)^{2/\lambda}I_{\lambda}(X_2).
\] (30)

Note that equality into (30) holds if and only if both \( X_1 \) and \( X_2 \) are Lévy variables of order \( \lambda \). Indeed, equality into (28) holds if and only if the relative score functions satisfy
\[
\tilde{\rho}^{(1)}_{\lambda}(x) = c_1; \quad \tilde{\rho}^{(2)}_{\lambda}(x) = c_2.
\] (31)

In fact, if this is the case, the Cauchy–Schwarz inequality we used to obtain (28) is satisfied with the equality sign. On the other hand, proceeding as in the derivation of (21), we conclude that (31) implies that the densities of the \( X_j \)'s, \( j = 1, 2 \) have Fourier transforms
\[
\hat{f}_j(\xi) = \exp \left\{-|\xi|^\lambda + ic_j\xi\right\}.
\] (32)

We proved

**Theorem 2.** Let \( X_j, j_1, 2 \) be independent random variables such that their relative fractional Fisher information functions \( I_{\lambda}(X_j), j = 1, 2 \) are bounded for some \( \lambda \), with \( 1 < \lambda < 2 \). Then, for each constant \( \delta \) with \( 0 < \delta < 1 \), \( I_{\lambda}(\delta^{1/\lambda}X_1 + (1 - \delta)^{1/\lambda}X_2) \) is bounded, and inequality (30) holds. Moreover, there is equality in (30) if and only if, up to translation, both \( X_j, j = 1, 2 \) are Lévy variables of exponent \( \lambda \).

A posteriori, we can use the result of Theorem 2 to avoid inessential difficulties in proofs by means of a smoothing argument. Indeed, since we are interested in inequalities for convolutions of densities, and a Lévy symmetric stable law is stable with respect to the operation of convolution, we can consider in the following text densities suitably smoothed by convolution with a Lévy symmetric stable law. In fact, if \( Z_1 \) and \( Z_2 \) denote two independent copies of a symmetric Lévy stable law \( Z \) of order \( \lambda \), for any given positive constants \( \epsilon_j, j = 1, 2 \) the random variable \( \epsilon_1^{1/\lambda}Z_1 + \epsilon_2^{1/\lambda}Z_2 \) is symmetric Lévy stable with the law of \((\epsilon_1 + \epsilon_2)^{1/\lambda}Z\). Therefore, for any given positive constant \( \epsilon < 1 \), and random variable \( X \) with density function \( f \) we will denote by \( f_\epsilon \) the density of the random variable \( X_\epsilon \) given by \((1 - \epsilon)^{1/\lambda}X + \epsilon^{1/\lambda}Z\), where the symmetric stable Lévy variable \( Z \) of order \( \lambda \) is independent of \( X \).

By virtue of Theorem 2
\[
I_{\lambda}(X_\epsilon) \leq (1 - \epsilon)^{2/\lambda}I_{\lambda}(X).
\] (33)
Hence, the relative fractional Fisher information of the smoothed version $X_\varepsilon$ of the random variable is always smaller than the relative fractional Fisher information of $X$. Moreover,

$$\lim_{\varepsilon \to 0} I_\lambda(X_\varepsilon) = I_\lambda(X).$$

This statement follows from (33) and Fatou’s lemma. Indeed, in view of (33), it only remains to show that $\liminf_{\varepsilon \to 0} I_\lambda(X_\varepsilon) \geq I_\lambda(X)$. On $\{f > 0\}$, $(D_{\lambda-1}f)^2/f_\varepsilon$ converges a.e. to $(D_{\lambda-1}f)^2/f$. This is enough to imply by Fatou’s lemma the desired inequality.

The next ingredient in the proof of monotonicity deals with the so-called variance drop inequality [37]. The idea goes back at least to the pioneering work of Hoeffding on $U$ statistics [23]. Let $[n]$ denote the index set $\{1, 2, \ldots, n\}$, and, for any $s \subset [n]$, let $X_s$ stand for the collection of random variables $(X_i : i \in s)$, with the indices taken in their natural increasing order. Then we have

**Theorem 3.** Let the function $\Phi : \mathbb{R}^m \to \mathbb{R}$, with $1 \leq m \in \mathbb{N}$, be symmetric in its arguments, and suppose that $E[\Phi(X_1, X_2, \ldots, X_m)] = 0$. Define

$$U(X_1, X_2, \ldots, X_n) = \frac{m!(n-m)!}{n!} \sum_{\{s \subset [n] : |s| = m\}} \Phi(X_s). \tag{34}$$

Then

$$E[U^2] \leq \frac{m}{n} E[\Phi^2]. \tag{35}$$

In theoretical statistical, $U$ defined in (34) is called a $U$-statistic of degree $m$ with symmetric, mean zero kernel $\Phi$ that is applied to data of sample size $n$. Thus, the essence of inequality (35) is to give a quantitative estimate of the reduction of the variance of a $U$-statistic when the sample size $n$ increases. It is remarkable that, as soon as $m > 1$ the functions $\Phi(X_s)$ are no longer independent. Nevertheless, the variance of the $U$-statistic drops by a factor $m/n$.

With the essential help of Theorem 3 we prove

**Theorem 4.** Let $T_n$ denote the sum (4), where the random variables $X_j$ are independent copies of a centered random variable $X$ with bounded relative $\lambda$-Fisher information, $1 < \lambda < 2$. Then, for each $n > 1$, the relative $\lambda$-Fisher information of $T_n$ is decreasing in $n$, and the following bound holds

$$I_\lambda(T_n) \leq \left(\frac{n-1}{n}\right)^{(2-\lambda)/\lambda} I_\lambda(T_{n-1}). \tag{36}$$

**Proof.** In what follows, for $n > 1$, $S_n = \sum_{j \in [n]} Y_j$ will denote the (unnormalized) sum of the independent and identically distributed random variables $Y_j$. Likewise, $S^{(k)}_n = \sum_{j \neq k} Y_j$ will denote the leave-one-out sum leaving out $Y_k$. Since $S_n = S^{(k)}_n + Y_k$, for each $k \in [n]$ we can write

$$x = E[Y_k|S_n = x] + E[S^{(k)}_n|S_n = x].$$

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On the other hand, since the $Y_j$ are independent and identically distributed, we have the identity
\[ E[S_n^{(k)}|S_n = x] = (n - 1)E[Y_k|S_n = x], \]
which implies that we can write
\[ x = \frac{n}{n-1}E[S_n^{(k)}|S_n = x]. \]
Hence, for the relative fractional score of $S_n$ we conclude that, if $\upsilon_n = (n - 1)/n$,
\[ \tilde{\rho}_\lambda(S_n) = E \left[ \tilde{\rho}_{\lambda, \upsilon_n}(S_{n-1}^{(k)})|S_n \right], \]
for all $k \in [n]$, and hence
\[ \tilde{\rho}_\lambda(S_n) = E \left[ \frac{1}{n} \sum_{k \in [n]} \tilde{\rho}_{\lambda, \upsilon_n}(S_{n-1}^{(k)})|S_n \right]. \]

Proceeding as in Lemma 1, namely by using the fact that the norm of the fractional relative score is not less than that of its projection, we obtain
\[ I_\lambda(S_n) = E \left[ \tilde{\rho}_\lambda^2(S_n) \right] \leq E \left[ \left( \frac{1}{n} \sum_{k \in [n]} \tilde{\rho}_{\lambda, \upsilon_n}(S_{n-1}^{(k)}) \right)^2 \right]. \]

To this point, Theorem 3 yields
\[ E \left[ \left( \frac{1}{n} \sum_{k \in [n]} \tilde{\rho}_{\lambda, \upsilon_n}(S_{n-1}^{(k)}) \right)^2 \right] \leq (n - 1) \sum_{k \in [n]} \frac{1}{n^2} E \left[ \tilde{\rho}_{\lambda, \upsilon_n}^2(S_{n-1}^{(k)}) \right] = \frac{n-1}{n} I_{\lambda, \upsilon_n}(S_{n-1}). \]

If we suppose that the right-hand side in the previous inequality is bounded, we obtained, for $n > 1$ the bound
\[ I_\lambda(S_n) \leq \frac{n-1}{n} I_{\lambda, \upsilon_n}(S_{n-1}). \quad (37) \]

To end up, let us choose $Y_k = X_k/n^{1/\lambda}$. In this case, $S_n = T_n$, where $T_n$ is the sum 4. Moreover
\[ S_{n-1} = \frac{X_1 + X_2 + \cdots + X_{n-1}}{n^{1/\lambda}} = \left( \frac{n-1}{n} \right)^{1/\lambda} T_{n-1} = v_n^{1/\lambda} T_{n-1}. \]

On the other hand, thanks to formula 11,
\[ I_{\lambda, \upsilon_n}(S_{n-1}) = I_{\lambda, \upsilon_n}(v_n^{1/\lambda} T_{n-1}) \leq v_n^{-2(1-1/\lambda)} I_\lambda(T_{n-1}). \]

Substituting into (37) gives the result.
Remark 2. Surprisingly enough, at difference with the case of the standard central limit theorem, where $\lambda = 2$ and the monotonicity result of the classical relative Fisher information reads $I(S_n) \leq I(S_{n-1})$, in the case of the central limit theorem for stable laws, the monotonicity of the relative $\lambda$-Fisher information also gives a rate of decay. Indeed, formula (36) of Theorem 4 shows that, for all $n > 1$

$$I_\lambda(T_n) \leq \left(\frac{1}{n}\right)^{(2-\lambda)/\lambda} I_\lambda(X),$$

namely convergence in relative $\lambda$-Fisher information sense at rate $1/n^{(2-\lambda)/\lambda}$.

Remark 3. This result allows to enlighten, at the level of Fisher information, a strong difference between the classical central limit theorem and the central limit theorem for stable laws. In the former case, the domain of attraction is very large and contains all random variables with finite variance, while the attraction in terms of relative Fisher information is, without additional assumptions, very low (only monotonicity is guaranteed). In the latter, the domain of attraction is very restricted and contains only random variables with distribution which has the same tails at infinity of the Lévy stable law. However, in this case the attraction in terms of the relative fractional fisher information is very strong, and it is inversely proportional to the exponent $\lambda$ which characterizes the Lévy stable law.

4 The relative $\lambda$-entropy

The previous results show that theoretical information techniques can be fruitfully employed to obtain a new approach to the central limit theorem for stable laws. In particular, the new concept of relative fractional Fisher information seems nicely adapted to the subject, since most of the classical properties can be easily extended to this case. In particular, subadditivity of the classical Fisher information with respect to weighted convolution is shown to hold also for the relative fractional Fisher information.

However, up to now, Fisher information has been considered as a useful instrument to obtain results for Shannon’s entropy functional defined in (1). The main finding in this context was in fact the proof of the monotonicity of Shannon’s entropy on the weighted sums in the central limit theorem. If stable laws are concerned, at difference with Fisher information, it seems difficult to find an explicit expression of the (non-local) corresponding of Shannon’s entropy, say the relative fractional entropy.

One possible way to obtain a consistent definition would be to establish between the relative fractional entropy and the relative fractional Fisher information the same link which connects Shannon entropy to Fisher information through the solution to the heat equation (4). In addition to de Bruijn relation, further connections between entropy and Fisher information have been established by Barron [3] and Carlen and Soffer [13]. Given two random variables $X$ and $Y$ of densities $f(x)$ and, respectively, $g(x)$, the relative entropy $H(X|Y)$ of $X$ and $Y$ is defined as

$$H(X|Y) = H(f|g) = \int_{\mathbb{R}} f(x) \log \frac{f(x)}{g(x)} \, dx.$$
If $X$ is a random variable with a density $f(x)$ and arbitrary finite variance, and $f_t(x) = f(x, t)$ denotes the solution to the heat equation (3) such that $f(x, t = 0) = f(x)$ then it holds \[ H(X|Z) = \int_0^\infty I(f_t|z_{1+t}) \, dt, \]
where $Z$ denotes as usual the Gaussian density of variance 1, and $I(f_t|z_{1+t})$ is the relative (to the Gaussian of variance $1 + t$) Fisher information defined in (18). In analogous way, one can define the relative (to the stable law) relative entropy.

Let $X$ be a random variable with density $f(x)$, and let $f_t(x) = f(x, t)$ denote the solution to the fractional diffusion equation (6) such that $f(x, t = 0) = f(x)$. Then, it appears natural to define the fractional relative entropy of order $\lambda$ as

\[ H_\lambda(X) = \int_0^\infty I_{\lambda,1+t}(f_t) \, dt, \tag{39} \]

where $I_{\lambda,1+t}(f_t)$ denotes the relative fractional Fisher information defined as in (25).

The relative fractional entropy given by (39) is well-defined. In fact, if the random variable $X$ has a density $f(x)$, the solution to the fractional diffusion equation (6) has the density $f(x, t)$ of the sum $X_t = X + t^{1/\lambda} Z$. By inequality (29)

\[ I_{\lambda,1+t}(f_t) = I_{\lambda,1+t}(f_t) \leq (1 + t)^{-2(1-1/\lambda)} I_{\lambda}(X (1 + t)^{-1/\lambda}). \]

In addition, since

\[ X_t(1 + t)^{-1/\lambda} = \left( \frac{1}{1 + t} \right)^{1/\lambda} X + \left( \frac{t}{1 + t} \right)^{1/\lambda} X, \]

thanks to inequality (33) it holds

\[ I_{\lambda}(X_t(1 + t)^{-1/\lambda}) \leq I_{\lambda}(X). \]

Finally,

\[ I_{\lambda,1+t}(f_t) \leq (1 + t)^{-2(1-1/\lambda)} I_{\lambda}(X), \]

so that, integrating both sides from 0 to $\infty$, we obtain

\[ H_{\lambda}(X) \leq \frac{\lambda}{2 - \lambda} I_{\lambda}(X). \tag{40} \]

Analogously to the classical case, inequality (40) implies that the domain of attraction of the relative $\lambda$-Fisher information is a subset of the domain of attraction of the relative $\lambda$-entropy.

Despite its complicated structure, as it happens in the classical situation, all inequalities satisfied by the relative fractional Fisher information also hold for the relative fractional entropy. By formula (39) we can easily show that Theorem 4 also is valid for the fractional relative entropy of order $\lambda$. In the classical case, however, Csiszar–Kullback inequality (17) allows to pass from convergence in relative entropy to convergence in $L^1(\mathbb{R})$. It would be interesting to show that a similar result still holds in the case of the relative fractional entropy, but at present this remains an open question.
5 Conclusions

Starting from the pioneering work of Linnik [34], the role of Fisher information to obtain alternative proofs of the central limit theorem has been enlightened by a number of papers (cf. [1, 2, 3, 10, 13, 28, 29, 36, 44] and the references therein). Only recently, Bobkov, Chistyakov and Götze [10] used of the relative Fisher information to study convergence to stable laws. At difference with the previous existing literature, we studied the role of Fisher-like functionals in the central limit theorem for stable laws by resorting to the new concept of relative fractional Fisher information. This nonlocal functional relies on the consideration of a linear fractional score function. As the linear score function of a random variable X identifies Gaussian variables as the unique random variables for which the score is linear, Lévy symmetric stable laws are here identified as the unique random variables for which the fractional linear score is linear. This analogy is pushed further to show that the relative fractional Fisher information, defined as the variance of the relative score, satisfies almost all properties of the classical relative Fisher information. While the fractional Fisher information represents in our opinion a powerful instrument to study convergence towards symmetric stable laws, the role of the analogous of Shannon’s entropy in this context at present remains obscure, and will deserve further investigations.

6 Appendix

To clarify that the domain of attraction of the fractional relative Fisher information constitute a notion that could be fruitfully used, we retain of paramount important to prove that this domain is not an empty subset of the classical domain of attraction of the stable law. To this aim, we will provide in this appendix an explicit example of a density which belongs both to the domain of attraction of the stable law, and to the domain of attraction of the relative fractional Fisher information.

To start with, let us briefly recall some information about the domain of attraction of a stable law. More details can be found in the book [24] or, among others, in the papers [1, 3]. A centered distribution F belongs to the domain of normal attraction of the λ-stable law [3] with distribution function \( L_\lambda(x) \) if and only if \( F \) satisfies \( |x|^\lambda F(x) \to c \) as \( x \to -\infty \) and \( x^\lambda(1-F(x)) \to c \) as \( x \to +\infty \) i.e.

\[
F(-x) = \frac{c}{|x|^\lambda} + S_1(-x) \quad \text{and} \quad 1 - F(x) = \frac{c}{x^\lambda} + S_2(x) \quad (x > 0)
\]

\[
S_i(x) = o(|x|^{-\lambda}) \quad \text{as} \quad |x| \to +\infty, \quad i = 1, 2
\]

(41)

where \( c = \frac{\Gamma(\lambda)}{\pi} \sin \left( \frac{\pi \lambda}{2} \right) \).

If the distribution function \( F \) belongs to the domain of normal attraction of the \( \lambda \)-stable law, for any \( \nu \) such that \( 0 < \nu < \lambda \)

\[
\int_\mathbb{R} |x|^\nu dF(x) < +\infty.
\]

(42)

The behavior of \( F \) in the physical space [11] leads to a characterization of the domain of normal attraction of the \( \lambda \)-stable law [4] in terms of Fourier transform. Indeed, if \( \hat{f} \) is the
Fourier transform of the distribution function $F$ satisfying (41), then

$$1 - \hat{f}(\xi) = (1 - R(\xi))|\xi|^\lambda,$$

(43)

where

$$R(\xi) \in L^\infty(\mathbb{R}), \quad \text{and} \quad |R(\xi)| = o(1) \quad \xi \to 0.$$  

Let us consider a probability density $f(x)$ that belongs to the domain of attraction of $L_\lambda$, with $\lambda > 1$. Then, we have enough regularity to reckon the Fourier transform of

$$\Psi_\lambda(x) = D_\alpha f(x) + \frac{x}{\lambda} f(x),$$

and, thanks to (43) we obtain

$$\hat{\Psi}_\lambda(x) = i\xi|\xi|^\lambda \hat{f}(\xi) + i \frac{1}{\lambda} \frac{\partial \hat{f}(\xi)}{\partial \xi} = -i\xi|\xi|^{2\lambda-2} + V(\xi).$$

It is known that the leading small $\xi$-behavior of the singular component of the Fourier transform (50) will reflect an algebraic tail of decay of the distribution function (cf. for example Wong [47]). In our case, since $\hat{\Psi}_\lambda(\xi)$ contains the term $\xi|\xi|^{2\lambda-1}$, $\Psi(x)$ should decay at infinity as $|x|^{-2\lambda+1}$.

The leading example of a function which belongs to the domain of attraction of the $\lambda$-stable law is the so-called Linnik distribution [33, 35], expressed in Fourier variable by

$$\hat{\rho}_\lambda(\xi) = \frac{1}{1 + |\xi|^\lambda}.$$  

(44)

For all $0 < \lambda \leq 2$, the function (44) is the characteristic function of a symmetric probability distribution. In addition, when $\lambda > 1$, $\hat{\rho}_\lambda \in L^1(\mathbb{R})$, which, by applying the inversion formula, shows that $\rho_\lambda$ is a probability density function.

The main properties of Linnik’s distributions can be extracted from its representation as a mixture (cf. Kotz and Ostrovskii [26]). For any given pair of positive constants $a$ and $b$, with $0 < a < b \leq 2$ let $g(s,a,b)$ denote the probability density

$$g(s,a,b) = \left(\frac{b}{\pi} \sin \frac{\pi a}{b}\right) \frac{s^{a-1}}{1 + s^{2a} + 2s^a \cos \frac{\pi a}{b}}, \quad 0 < s < \infty.$$  

Then, the following equality holds [26]

$$\hat{\rho}_a(\xi) = \int_0^\infty \hat{\rho}_b(\xi/s) g(s,a,b) \, ds,$$

(45)

or, equivalently

$$\rho_a(x) = \int_0^\infty \rho_b(sx) g(s,a,b) \, ds.$$  

This representation allows us to generate Linnik distributions of different parameters starting from a convenient base, typically from the Laplace distribution (corresponding to
In this case, since \( \hat{p}_2(\xi) = 1/(1 + |\xi|^2) \) (alternatively \( p_2(x) = e^{-|x|/2} \) in the physical space), for any \( \lambda \) with \( 1 < \lambda < 2 \) we obtain the explicit representation

\[
\hat{p}_\lambda(\xi) = \int_0^\infty \frac{s^2}{s^2 + |\xi|^2} g(s, \lambda, 2) \, ds,
\]

or, in the physical space

\[
p_\lambda(x) = \int_0^\infty \frac{s}{2} e^{-s|x|} g(s, \lambda, 2) \, ds.
\]

Owing to (47) we obtain easily that, for \( 1 < \lambda < 2 \), Linnik’s probability density is a symmetric and bounded function, non-increasing and convex for \( x > 0 \). Moreover, since Linnik’s distribution belongs to the domain of attraction of the stable law of order \( \lambda \), \( p_\lambda(x) \) decays to zero like \( |x|^{1+\lambda} \) as \( |x| \to \infty \). These properties insure that there exist positive constants \( A_\lambda \) and \( B_\lambda \) such that

\[
p_\lambda^{-1}(x) \leq A_\lambda + B_\lambda|x|^{1+\lambda}.
\]

In reason of (48), we obtain the bound

\[
I_\lambda(p_\lambda) = \int_\mathbb{R} \left( \frac{D_{\lambda-1} p_\lambda(x)}{p_\lambda(x)} + \frac{x}{\lambda} \right)^2 p_\lambda(x) \, dx = \int_\mathbb{R} \left( D_{\lambda-1} p_\lambda(x) + \frac{x}{\lambda} p_\lambda(x) \right)^2 p_\lambda^{-1}(x) \, dx \leq \int_\mathbb{R} g_\lambda^2(x)(A_\lambda + B_\lambda|x|^{1+\lambda}) \, dx.
\]

In (49) we defined

\[
g_\lambda(x) = D_{\lambda-1} p_\lambda(x) + \frac{x}{\lambda} p_\lambda(x).
\]

Explicit computations give

\[
\hat{g}_\lambda(\xi) = i\xi |\xi|^{\lambda-2} \frac{d^2 p_\lambda(\xi)}{d\xi}.
\]

As discussed before, the algebraic tail of decay of a function is given by the leading small \( \xi \)-behavior of the singular component of its Fourier transform (50). Hence, while the leading singular component in the Linnik distribution (44) is \( |\xi|^\lambda \), which induces condition (42), by (50) the leading singular component in \( \hat{g}_\lambda^2 \) is \( |\xi|^{4\lambda-2} \), which would imply that the right-hand side of (49) is bounded, in reason of the fact that \( 4\lambda - 2 > \lambda + 1 \) for \( \lambda > 1 \). This suggests that the relative fractional Fisher information of Linnik’s distribution is bounded.

An explicit proof of this property follows owing to representation (45). To this extent, consider that the function \( \hat{g}_\lambda(\xi) \) defined in (50) can be also written in the form

\[
\hat{g}_\lambda(\xi) = -i|\xi|^{\lambda} \frac{d p_\lambda(\xi)}{d\xi}.
\]

Therefore, differentiating under the integral sign in (46) we obtain the equivalent expression

\[
\hat{g}_\lambda(\xi) = \int_0^\infty \hat{h}_\lambda(\xi/s)s^{\lambda-1} \, g(s, \lambda, 2) \, ds,
\]
where
\[ \hat{h}_\lambda(\xi) = \frac{2i}{\lambda} \frac{\xi |\xi|^\lambda}{(1 + |\xi|^2)^2}. \]  

(52)

Since \(1 < \lambda < 2\), it follows that both \(\hat{h}_\lambda\) and \(\hat{h}_\lambda''\) belong to \(L^2(\mathbb{R})\). Hence, by Plancherel’s identity we obtain
\[
\int_{\mathbb{R}} |x|^4 h_\lambda^2(x) \, dx = \int_{\mathbb{R}} |x|^2 h_\lambda(x) |x|^2 \, dx = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{h}_\lambda''(\xi)|^2 \, d\xi < +\infty.
\]  

(53)

Thus, for any given \(R > 0\)
\[
\int_{\mathbb{R}} |x|^{1+\lambda} h_\lambda^2(x) \, dx \leq \int_{\{x| \leq R\}} |x|^{1+\lambda} h_\lambda^2(x) \, dx + \frac{1}{R^{3-\lambda}} \int_{\{x| > R\}} |x|^4 h_\lambda^2(x) \, dx \leq \quad \leq (2R)^{1+\lambda} \int_{\mathbb{R}} h_\lambda^2(x) \, dx + \frac{1}{R^{3-\lambda}} \int_{\mathbb{R}} |x|^4 h_\lambda^2(x) \, dx.
\]

Optimizing over \(R\) we obtain
\[
\int_{\mathbb{R}} |x|^{1+\lambda} h_\lambda^2(x) \, dx \leq C_\lambda \left( \int_{\mathbb{R}} h_\lambda^2(x) \, dx \right)^{(3-\lambda)/4} \left( \int_{\mathbb{R}} |x|^4 h_\lambda^2(x) \, dx \right)^{(1+\lambda)/4},
\]

where \(C_\lambda\) is an explicitly computable constant. Note that, in view of (53), the right-hand side of the previous inequality is bounded. Finally, by Jensen’s inequality we have
\[
\int_{\mathbb{R}} |x|^{1+\lambda} g_\lambda^2(x) \, dx = \int_{\mathbb{R}} |x|^{1+\lambda} \left[ \int_{0}^{\infty} s h_\lambda(sx) s^{\lambda-1} g(s, \lambda, 2) \, ds \right]^2 \, dx \leq \quad \leq \int_{\mathbb{R}} |x|^{1+\lambda} \left[ \int_{0}^{\infty} (h_\lambda(sx) s^\lambda)^2 g(s, \lambda, 2) \, ds \right] \, dx = \quad = \int_{\mathbb{R}} |x|^{1+\lambda} h_\lambda^2(x) \, dx \int_{0}^{\infty} s^{\lambda-2} g(s, \lambda, 2) \, ds.
\]

By definition, the probability density \(g(s, \lambda, 2) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\). This implies that, for \(1 < \lambda < 2\)
\[
\int_{0}^{\infty} s^{\lambda-2} g(s, \lambda, 2) \, ds < +\infty.
\]

Hence the relative fractional Fisher information of the Linnik’s density is bounded.

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