Nucleating Black Holes via Non-Orientable Instantons

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Abstract. We extend the analysis of black hole pair creation to include non-orientable instantons. We classify these instantons in terms of their fundamental symmetries and orientations. Many of these instantons admit the pin structure which corresponds to the fermions actually observed in nature, and so the natural objection that these manifolds do not admit spin structure may not be relevant. Furthermore, we analyse the thermodynamical properties of non-orientable black holes and find that in the non-extreme case, there are interesting modifications of the usual formulae for temperature and entropy.

Introduction

Recently, there has been considerable interest in the application of semi-classical Euclidean quantum gravity techniques to the study of black hole pair creation. Here, the analogy is with ordinary electron-positron pair production, where one computes the probability for the process by looking at the action of the ‘Wick rotated’ solution, which is an electron moving on a circle in a uniform field on Euclidean space. Likewise, in Euclidean quantum gravity, one models generic tunnelling phenomena by first finding an instanton (a manifold $M$, Riemannian metric $g$, and matter fields $\{\phi\}$ which solve
the relevant field equations) and then matching the instanton to a Lorentzian solution across a three-surface $\Sigma$ of vanishing extrinsic curvature. The amplitude for such a decay process is then given by $e^{-S}$, where $S$ is the Euclidean action, i.e.,

$$S = -\frac{1}{16\pi G} \int_M (R - 2\Lambda) \sqrt{g} \, d^4x - \int_M \mathcal{L}_m \sqrt{g} \, d^4x - \frac{1}{8\pi G} \int_{\partial M} (K - K^0) \sqrt{h} \, d^3x$$

where $G$ is Newton’s constant, $g$ is the determinant of the four-metric, $h$ is the determinant of the three-metric on $\partial M$ (which we assume is positive definite), $\mathcal{L}_m$ is the Lagrangian of any matter fields, $R$ is the scalar curvature of the four-manifold $M$, $\Lambda$ is the cosmological constant, $K$ is the trace of the second fundamental form of the boundary (relative to the metric $g_{ab}$ on $M$) and $K^0$ is the trace of the second fundamental form of the boundary imbedded in flat space.

Instantons (with or without matter fields) which correspond to a black hole moving on a loop in Euclidean space have the topology $S^2 \times S^2$. In the simplest case (without matter fields), one takes the product metric, $g_R$, given by the direct sum of the two round metrics on each of the $S^2$ factors. The nucleation surface is then the $\Sigma \simeq S^1 \times S^2$, with vanishing extrinsic curvature. Thus, this metric (known as the Nariai instanton) ‘nucleates’ a wormhole $S^1 \times S^2$. Given the presence of horizons in the Lorentzian section, one can think of this instanton as modelling black hole pair production in a De Sitter background, as was first noted in [3], [4].

It has been argued [1] that the only instantons which are of any real interest, to black hole pair creation, are those which are simply connected and admit a spin structure. However, this restriction may be physically too severe.

As an example, take the case of the Nariai instanton, and identify the ‘spacelike’ $S^2$ under the antipodal map, to obtain an instanton with topology $S^2 \times \mathbb{R}\mathbb{P}^2$ and nucleation surface $\Sigma \simeq S^1 \times \mathbb{R}\mathbb{P}^2$. The Lorentzian section of this solution contains ‘black holes’ (which are now non-orientable), and so it corresponds to the birth of a non-orientable wormhole. Because $w_2(\mathbb{R}\mathbb{P}^2) \neq 0$, the instanton does not admit a spin structure. As we shall show, however, it does admit the pin structure observed in nature (more precisely, $S^2 \times \mathbb{R}\mathbb{P}^2$ admits the pin structure which particle physicists customarily use to construct the discrete maps $P$ and $T$ on the Hilbert space of solutions to the Dirac equation [5]). Thus, we can see no reason to exclude this creation
process from consideration. On the contrary, the thesis of this work is that if one accepts the semi-classical approach as a valid approximation to ordinary black hole pair creation, then one must also accept the pair creation of non-orientable black holes.

Historically ([6], [7]), the subject of non-orientable black holes has been rather neglected since of course a non-orientable hole would never form in a realistic astrophysical scenario involving gravitational collapse. Now that we have a physical mechanism for creating such objects, there will hopefully be a renewal of interest. On a more philosophical note, we feel that these issues are important because they focus attention on more ‘exotic’ manifolds which ordinarily are overlooked by those studying the quantum foam. After all, a truly robust implementation of Feynman’s ideas to gravity would require that we sum over all manifolds first, and determine afterwards which contributions may vanish, for example because of a vanishing fermionic determinant factor on an infinite bosonic determinant factor.

I. Schwarzschild and the Elliptic Interpretation

In this section, we recall the properties of ‘classical’ non-orientable black holes, as described previously in [7]. To this end, let \((\mathcal{M}, g)\) denote the Schwarzschild spacetime. We are interested in identifying \(\mathcal{M}\) under the action of certain discrete involutive isometries. In particular, we are interested in the actions of time and space inversion. Since we wish to consider the actions of these inversions on the maximally extended spacetime, it is most natural to use Kruskal coordinates [12], which cover the entire manifold. For reasons which will become apparent, we feel it is useful to first review the relation of these coordinates to the usual Schwarzschild coordinates (which cover only part of the maximal extension).

As usual, let \((t, r, \theta, \phi)\) denote the Schwarzschild coordinates so that the metric reads

\[
\begin{align*}
    ds^2 &= -(1 - \frac{2m}{r})
    \left\{dt^2 + \frac{dr^2}{(1 - \frac{2m}{r})} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\right\}.
\end{align*}
\]
Next introduce null coordinates $u$ and $v$ such that

$$du = dt - \frac{dr}{(1 - \frac{2m}{r})},$$

$$dv = dt + \frac{dr}{(1 - \frac{2m}{r})},$$

or, integrating

$$u = t - r - 2m \log(r - 2m),$$

$$v = t + r + 2m \log(r - 2m).$$

Now form the coordinates $U$ and $V$ by exponentiating

$$U = -e^{\frac{u}{4m}},$$

$$V = e^{\frac{v}{4m}},$$

Then one finds that the coordinates $T$ and $Z$ defined by

$$T = \sinh \left(\frac{t}{4m}\right) e^{\frac{r}{4m} \sqrt{r - 2m}}$$

$$Z = \cosh \left(\frac{t}{4m}\right) e^{\frac{r}{4m} \sqrt{r - 2m}}$$

satisfy the simple algebraic relations

$$U = T + Z,$$

$$V = T - Z,$$

that is, $U$ and $V$ are advanced and retarded null coordinates relative to $T$ and $Z$. One checks that in these coordinates the metric assumes the form

$$ds^2 = e^{\frac{r}{r}} \left(\frac{16m^2}{r}\right) \left(-dT^2 + dZ^2\right) + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

Using the coordinates $(T, Z, \theta, \phi)$, we define total time inversion by the map

$$R_T : (T, Z, \theta, \phi) \rightarrow (-T, Z, \theta, \phi),$$
and likewise space inversion is given by

\[ R_z : (T, Z, \theta, \phi) \rightarrow (T, -Z, \theta, \phi). \]

Of course, neither of these involutions acts freely (they both have fixed points). To obtain a free action, we need to take a product with some other map which is freely acting. Such a map, which we denote as ‘P’, is given as follows:

\[ P : (T, Z, \theta, \phi) \rightarrow (T, Z, \pi - \theta, \phi + \pi) \]

Thus, we can construct the following four freely acting involutions on \( M \): \( P, PR_T, PR_Z \) and \( PR_Z R_T \). We claim that all of these involutions extend to the corresponding Euclidean instanton (the ‘cigar’). Before addressing the Riemannian issue, however, we need to first define and interpret the basic properties of the spacetime obtained when we identify \( M \) under the action of one of these involutions. To this end, let \( J \) be any one of the above involutions. We want to consider the quotient manifold

\[ M_J = M/J \]

Depending on which choice we make for \( J \), \( M_J \) may or may not be asymptotically flat and it may or may not be orientable. However, a little thought establishes the following table:
Thus, we see that the only ‘nice’ quotient manifold (i.e., the only one which is both asymptotically flat and orientable) is $M_{ZP} = M/J$, with $J = R_t R_Z P$. (Note: We will employ this notation from here on, i.e., $M_{TZP}$ denotes $M/J$ with $J = R_t R_Z P$, $M_{TP}$ denotes $M/J$ with $J = R_t P$, and so on). Of course, our point of view is that we should, to begin with at least, consider all of these spacetimes on an equal footing, and not let lack of an orientation dissuade us from studying them (although as we will point out later, a lack of time orientation would seem to be a problem when one introduces quantum mechanics). To this end, consider the spacetime $M_P$.

Although this quotient manifold is not asymptotically flat (the antipodal identification forces the spacelike slices to have the wrong topology, i.e., $\mathbb{R}P^2 \times [0,1)$, at large spatial distances), we consider it anyway since in the context of cosmological pair creation, the background in which the black holes are produced is not asymptotically flat.

Another natural question about $M_P$ is whether or not one can ‘tell’ that it is non-orientable. What effect would a lack of space-orientation have? For example, could we use an object such as the black hole described by

| $J$ | $M_J$ asymptotically flat? | $M_J$ time orientable? | $M_J$ space orientable? |
|-----|--------------------------|------------------------|-------------------------|
| Case I: $J = R_t R_Z P$ | yes                      | no                     | yes                     |
| Case II: $J = R_t P$    | no                       | no                     | no                      |
| Case III: $J = R_Z P$   | yes                      | yes                    | yes                     |
| Case IV: $J = P$        | no                       | yes                    | no                      |

Table 1
\(\mathcal{M}_P\) to turn right-handed people into left-handed people (and vice versa)?

Clearly, the answer to this question is yes, since by simply moving around
the perimeter of the hole an odd number of times we traverse a non-trivial
generator of \(\pi_1(\mathbb{RP}^2)\), and such a curve is by definition a space-orientation
reversing curve in \(\mathcal{M}_P\) (although the space-time curve corresponding to such
causal movement is not closed, it is homotopic to a closed curve in the spatial
factor).

Interesting questions can also arise when one considers the inclusion of
quantum effects on \(\mathcal{M}_P\). For example, the area of the event horizon in \(\mathcal{M}_P\)
(which has topology \(\mathbb{RP}^2\)) is \(\frac{1}{2}\) times the area of the horizon in \(\mathcal{M}\) (with
topology \(S^2\)), the reason being that one way to calculate the area of a non-
orientable surface is to calculate the area of its double-cover and divide by
two. Will this discrepancy in areas affect the temperature? In order to
answer this question properly, we need to look carefully at the corresponding
Riemannian instanton, the ‘identified cigar’. To this end, and also to see
how the freely acting involutive isometries on the Lorentzian section are
related to freely acting involutive isometries on the Riemannian section, let
us write complexified Schwarzschild as an algebraic variety in \(\mathbb{C}^7\) as usual.

More explicitly, let \(\{Z^i | i = 1, \ldots 7\}\) be coordinates on \(\mathbb{C}^7\), so that in terms
of Schwarzschild coordinates (which cover only a subset of the variety), we
have [8]

\[
\begin{align*}
Z^1 &= r \sin \theta \cos \phi, \\
Z^2 &= r \sin \theta \sin \phi, \\
Z^3 &= r \cos \theta, \\
Z^4 &= -2M \sqrt{\frac{2M}{r}} + 4M \sqrt{\frac{r}{2M}}, \\
Z^5 &= 2M \sqrt{3} \sqrt{\frac{2M}{r}}, \\
Z^6 &= 4M \sqrt{1 - \frac{2M}{r \cosh \left(\frac{t}{4M}\right)}}
\end{align*}
\]
\[ Z^7 = 4M \sqrt{1 - \frac{2M}{r}} \sinh \left( \frac{t}{4M} \right). \]

With the coordinates as in (1), it turns out that complexified Schwarzschild (\( \mathcal{M}_C \)) is given as the algebraic variety determined by the three polynomials

\[ (Z^6)^2 - (Z^7)^2 + \frac{4}{3}(Z^5)^2 = 16M^2, \]
\[ \left( (Z^1)^2 + (Z^2)^2 + (Z^3)^2 \right) (Z^5)^4 = 576M^6, \]
\[ \sqrt{3}Z^4Z^5 + (Z^5)^2 = 24M^2. \]

The Lorentzian section (\( \mathcal{M} = \mathcal{M}^L \)) and the Riemannian section (\( \mathcal{M}^R \)) are then specified by finding certain anti-holomorphic involutions acting on the above variety which stabilise either \( \mathcal{M}^L \) or \( \mathcal{M}^R \); that is, we find maps

\[ J_L : \mathcal{M}_C \rightarrow \mathcal{M}_C, \]
\[ J_R : \mathcal{M}_C \rightarrow \mathcal{M}_C, \]

such that \( J_L \) leaves \( \mathcal{M}^L \subset \mathcal{M}_C \) invariant:

\[ J_L(\mathcal{M}^L) = \mathcal{M}^L, \]

and such that \( J_R \) leaves \( \mathcal{M}^R \subset \mathcal{M}_C \) invariant:

\[ J_R(\mathcal{M}^R) = \mathcal{M}^R. \]

As described in [8] \( J_L \) restricted to \( \mathcal{M}^L \) is an anti-holomorphic version of time reversal. \( J_R \) is the map given by reflection through the \( \tau = 0 \) (where \( \tau = it \)) three-surface in the ‘cigar’ instanton (i.e., \( \tau = 0 \) is the ‘Einstein Rosen bridge’ three-surface \( \Sigma \), with topology \( S^2 \times \mathbb{R} \)). Since the surfaces \( t = 0 \) and \( \tau = it = 0 \) correspond to the surface \( Z^7 = 0 \), we see that \( \mathcal{M}^L \) and \( \mathcal{M}^R \) intersect precisely along this Einstein Rosen bridge. Explicitly, we can realise the two maps \( J_L \) and \( J_R \) as follows:

\[ J_L : (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7) \rightarrow (\bar{Z}^1, \bar{Z}^2, \bar{Z}^3, \bar{Z}^4, \bar{Z}^5, \bar{Z}^6, \bar{Z}^7), \]
\[ J_R : (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7) \rightarrow (\bar{Z}^1, \bar{Z}^2, \bar{Z}^3, \bar{Z}^4, \bar{Z}^5, \bar{Z}^6, -\bar{Z}^7). \]
Comparing these explicit formulae for $J_L$ and $J_R$ with the coordinates in (1), we see that $J_R$ is thus obtained from $J_L$ by the transformation $t \rightarrow \tau = it$.

What we want to do now is show how the maps $R_T, R_Z$ and $P$ acting on $\mathcal{M}_L$, and likewise their Euclidean counterparts acting on $\mathcal{M}_R$, are actually just the restrictions to $\mathcal{M}_L$ and $\mathcal{M}_R$ of certain holomorphic involutions acting on $\mathcal{M}_C$. Of course, once we notice that our complex coordinates $Z^6$ and $Z^7$ are (up to a scaling) actually our Kruskal coordinates $Z$ and $T$, it is easy to see that the ‘big’ involutions, $R_Z$ and $R_T$ (which restrict to $R_Z$ and $R_T$ on $\mathcal{M}_L$) are given by

$$R_Z : (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7) \rightarrow (Z^1, Z^2, Z^3, Z^4, Z^5, -Z^6, Z^7),$$

$$R_T : (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7) \rightarrow (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, -Z^7).$$

Clearly, these maps are holomorphic, and since they commute with both $J_L$ and $J_R$, they restrict to well-defined involutions on $\mathcal{M}_L$ and $\mathcal{M}_R$. Thus, $R_Z|_{\mathcal{M}_L} = R_Z$ and $R_T|_{\mathcal{M}_L} = R_T$. For the maps restricted to the Riemannian section, we shall write

$$R_Z|_{\mathcal{M}_R} = \bar{R}_Z : \mathcal{M}_R \rightarrow \mathcal{M}_R$$

$$R_T|_{\mathcal{M}_R} = \bar{R}_T : \mathcal{M}_R \rightarrow \mathcal{M}_R$$

In terms of local coordinates on $\mathcal{M}_R$, these reflections take the form

$$\bar{R}_Z : \tau \rightarrow -\tau + 4\pi m$$

$$\bar{R}_T : \tau \rightarrow -\tau$$

($r, \theta$, and $\phi$ are left invariant by both these maps). Thus, we see that $\bar{R}_T$ is reflection in imaginary time whereas $\bar{R}_Z$ corresponds to rotating through half a period in imaginary time.

Finally, we obtain the involution $\bar{P}$ on $\mathcal{M}_R$ by restricting to $\mathcal{M}_R$ the following map on $\mathcal{M}_C$:

$$\mathcal{P} : (Z^1, Z^2, Z^3, Z^4, Z^5, Z^6, Z^7) \rightarrow (-Z^1, -Z^2, -Z^3, Z^4, Z^5, Z^6, Z^7)$$

Now that we have made sense of how to extend our discrete isometries $R_Z, R_T$ and $P$ from $\mathcal{M}$ to $\mathcal{M}_R$, we can return to the problem of examining
the thermodynamical properties of $\mathcal{M}_J = \mathcal{M}^L/J$ by looking at the instanton $\mathcal{M}^R_\bar{J} = \mathcal{M}^R/\bar{J}$. In particular, let us focus again on $\mathcal{M}_p$.

First of all, since we have only identified under the action of parity inversion, we have not affected the period of the imaginary time coordinate $\tau$. Naïvely, we might therefore expect that the temperature of the hole in $\mathcal{M}_p$ would be the same as that in $\mathcal{M}$, given the thermodynamical principle [20] that the temperature $T$ is inversely related to the period $\beta$:

$$T = \beta^{-1}$$

Indeed, this reasoning is correct and the temperature of $\mathcal{M}$ is in fact equal to the temperature of $\mathcal{M}_p$; however, there are many subtleties which now arise and one finds that in order to maintain this relation between temperature and period, one has to alter the standard formulae which express the relations between temperature, mass and area.

To see how this works, recall first that $\mathcal{M}_p$ is not asymptotically flat since at large radial distances spacelike slices have the topology $\mathbb{RP}^2 \times [0, 1)$. Let $\mathcal{M}_p$ be obtained from $\mathcal{M}$ under the action of $P$, so that $\mathcal{M}$ is the double cover of $\mathcal{M}_p$. Then the horizon in $\mathcal{M}$ is an $S^2$ which is the double cover of the horizon in $\mathcal{M}_p$, which is an $\mathbb{RP}^2$. It follows that the area, $A$, of the horizon in $\mathcal{M}$ is twice the area, $A_p$, of the horizon in $\mathcal{M}_p$:

$$A = 2A_p$$

In a similar way, we can calculate the relationship of the ADM mass, $m_p$, of the hole in $\mathcal{M}_p$ to the ADM mass, $m$, of the hole in $\mathcal{M}$. Of course, one might well wonder how we expect to define mass given a lack of asymptotic flatness as it is usually understood. We posit that it still makes sense to define the mass as a surface integral of some flux density over, a two-surface ‘at infinity’, even if the two-surface has the topology of $\mathbb{RP}^2$, so that the calculation of the mass reduces to calculating the mass of the cover and dividing by 2:

$$m = 2m_p$$

Using $dm = \kappa dA$ we see that consistency requires that $\kappa = \kappa_p$ if and only if $T = T_p$.

However, now recall the formula relating the area $A$ and ADM mass $m$ in $\mathcal{M}$:

$$A = 16\pi m^2$$
Substituting the above formulae for \( m \) and \( A \) into this expression, we obtain

\[
2A_p = 16\pi(4m_p^2)
\]

hence

\[
A_p = 32\pi m_p^2
\]

and so the usual relationship between horizon area and ADM mass is slightly changed in \( \mathcal{M}_P \).

What about temperature and entropy? Well, as we have seen above, the temperature, \( T_p \), of \( \mathcal{M}_P \) must equal the temperature, \( T \), of \( \mathcal{M} \) since the periods are the same:

\[
T = T_p
\]

But \( T = \frac{1}{8\pi m} \), and so the usual relationship between \( T \) and \( m \) on \( \mathcal{M} \) is modified on \( \mathcal{M}_P \) to

\[
T_p = \frac{1}{16\pi m_p}
\]

II. Non-orientability and Wormholes

In this section, we want to point out that if the nucleation surface \( \Sigma \) is closed and non-orientable, then \( b_1(\Sigma) \), the first Betti number, cannot vanish. This means that the fundamental group \( \pi_1(\Sigma) \) must contain elements of infinite order, or put more colloquially, \( \Sigma \) must contain Wheeler wormholes. This is clearly the case for the Nariai solution for which \( \Sigma \cong S^1 \times \mathbb{R}P^2 \). The point we wish to make is that this is always so. Note however that the element of infinite order whose existence is ensured does not necessarily reverse orientation. That is, the wormhole we must always have is not necessarily an orientation reversing wormhole.

The proof of this result is given in [24] and amounts to the observation that the Euler characteristic of any three-manifold, orientable or not, vanishes, thus

\[
\chi(\Sigma) = b_0(\Sigma) - b_1(\Sigma) + b_2(\Sigma) - b_3(\Sigma) = 0
\]

If \( \Sigma \) is connected, then \( b_0(\Sigma) = 1 \) and if \( \Sigma \) is not orientable, then \( b_3(\Sigma) = 0 \). Thus

\[
b_1(\Sigma) = 1 + b_2(\Sigma) \geq 1.
\]
Note that the result is false if $\Sigma$ has dimension greater than 3.

III. Fermions on Non-Orientable Spacetimes

It is often stated in the literature that since it is impossible to define a spin structure on a non-orientable spacetime, it is impossible to define fermions on such spacetimes. We will now show that it is often possible to have fermions regardless of whether or not there exists a spin structure. We would also like to emphasize now that these are ordinary fermions, i.e., particles acted upon by the full inhomogeneous Lorentz group. We will not consider enlarging the group of symmetries by coupling to some internal gauge group, as is done when one passes to a $Spin_c$ structure, since as has been pointed out elsewhere [15] such an enlargement would not correspond to the observed couplings between gauge bosons and fermions. For related reading we refer the reader to ([13], [14]).

Just to be concrete, let us begin by considering the flat space Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0.$$  \hspace{1cm} (3)

Everything we are about to say will go through for the curved space version of eq. (3). As is well-known, Dirac derived (3) by taking the square root of the standard relativistic energy-momentum relation, and making the canonical substitutions of momenta for differential operators: $p_\mu \rightarrow i \partial_\mu$. Dirac found that the equation could only be satisfied if the $\gamma^\mu$s were actually $4 \times 4$ matrices satisfying precisely the Clifford algebra relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$

where $g^{\mu\nu}$ was (for Dirac) the flat Minkowski space metric. Thus, the actual wavefunction $\psi$ representing the electron is a four-component object and we are led naturally to the concept of antiparticles.

Once we form the set of solutions to equation (3) (and put an inner product structure on that space so that it becomes a ‘Hilbert space’, denoted $\mathcal{H}$), it is natural to consider the representation of discrete geometrical transformations on $\mathcal{H}$. Of paramount importance are the representations of $P$ (parity inversion) and $T$ (time reversal) which we must have if we are to
construct a theory of elementary particles transforming under the action of the full inhomogeneous Lorentz group.

The best way to illustrate what we are talking about is with an explicit example. Let us therefore recall how the operators \(C\) (charge conjugation), \(P\) (parity inversion) and \(T\) (time reversal) are represented in the standard particle physics literature [5]: Let \(\mathcal{H}\) be the set of solutions of the Dirac equation on four-dimensional Minkowski space; then \(C\), \(P\), and \(T\) are linear operators on \(\mathcal{H}\) given by the explicit formulae:

\[
C : \psi(x, t) \longrightarrow i\gamma^2 J\psi(x, t),
\]

\[
P : \psi(x, t) \longrightarrow \gamma^0 \psi(-x, t), \quad (4)
\]

\[
T : \psi(x, t) \longrightarrow \gamma^1 \gamma^3 J\psi(x, -t),
\]

where \(\psi\) is any solution and \(J\) denotes the operation of complex conjugation.

We remind the reader that a host of physical considerations goes into the choices made in equations (4). For example, the operator \(J\) is included in the construction of \(T\) in order to ensure that \(T\) takes positive energy states to positive energy states. A number of other choices are possible, the key point being that the other choices are mathematically inequivalent (in a way to be made precise presently).

Now, one of the first things we can notice about the operators \(P\) and \(T\) defined in (4) is that they do not give a Cliffordian representation of the action of space and time inversion. That is, \(P\) and \(T\) do not anti-commute, since in fact they commute:

\[
PT \sim \gamma^0 \gamma^1 \gamma^3 = \gamma^1 \gamma^3 \gamma^0 \sim TP.
\]

Therefore, the operators \(P\) and \(T\) defined in (4) correspond to a non-Cliffordian representation of \(O(3, 1)\) with non-Cliffordian action.

This situation can be contrasted with the case where the representation is Cliffordian. For example, a Cliffordian action can be recovered by the following operator assignment:

\[
P : \psi(x, t) \longrightarrow \gamma^1 \psi(-x, t), \quad (5)
\]

\[
T : \psi(x, t) \longrightarrow \gamma^0 \psi(x, -t).
\]
Clearly, the choices in (5) anti-commute.

Of course, in each of the above examples, the underlying group structure is identical. More precisely, in the operator assignments made in (4), we used the group of elements $\gamma^\mu$ satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ to construct operators $P$ and $T$ whose action on $\mathcal{H}$ is non-Cliffordian, whereas in (5) we used the same group of Cliffordian elements to construct operators $P$ and $T$ with Cliffordian action. It is absolutely essential that we make this distinction between the different actions on a Hilbert space which can be constructed from a given group, and genuinely different groups. This is because we are sympathetic to the philosophy of Wigner [16] who put forward the idea that the irreducible representations of whatever group of symmetries is present in nature should form the basis for any theory of elementary particles. Indeed, Wigner completely classified the set of irreducible representations of the inhomogeneous Lorentz group, $O(3,1)$, on the Hilbert space of solutions to the Dirac equation with $m \neq 0$. He showed that once one ‘fixes’ the sign of the square of parity inversion $P^2$ (fixing this sign corresponds to choosing a signature for spacetime, basically) then there are four inequivalent (non-isomorphic) cases. The first case is the standard particle physics choice made in (4) above. In the remaining three cases, we encounter the phenomenon known as ‘parity doubling’.

Basically, then, there are eight different ways of representing the actions of the operators $P$ and $T$ on the space of fermionic states. Of course, this should not surprise us too much since there are in fact eight distinct non-isomorphic double covers of the inhomogeneous Lorentz group $O(p,q)$ when $p$ and $q$ are both non-zero. Following Dabrowski, we will write these covers as

$$h^{a,b,c} : \text{Pin}^{a,b,c}(p,q) \longrightarrow O(p,q),$$

with $a, b, c \in \{+, -\}$ given as $a = P^2$, $b = T^2$, and $c = (PT)^2$. Thus, a given double cover of $O(p,q)$ is completely characterized by the signs of the squares of parity inversion, time reversal, and the combination of the two. These different double covers are called the ‘pin’ groups, and although our conventions for defining $a$, $b$, and $c$ differ from Dabrowski’s (he takes $a = -(P)^2$), we feel that our notation (which is the notation used in [9]) makes the obstruction theory more transparent.

With this in mind, we can answer the obvious question: Which pin group corresponds to the actions of $P$ and $T$ defined in equations (4) above? To
see the answer, we simply compute:

\[ P^2 = (\gamma^0)^2 = -1, \]
\[ T^2 = (\gamma^1\gamma^3)^2 = -1, \]
\[ (PT)^2 = (\gamma^0\gamma^1\gamma^3)^2 = +1. \]

Thus, the pin group customarily used in particle physics is seen to be \( \text{Pin}^{-,-,+,+}(3,1) \).

This choice, we should point out, cannot be made flippantly since as was pointed out in [18], a different choice for the representations of \( P \) and \( T \) corresponds to a different superselection sector of fermions, i.e., a completely different species of particle.

For example, let us consider a simple scattering experiment, where there is a bubble of non-orientable foam. Furthermore, let us assume that we obtain the bubble by some cut-and-paste construction on Minkowski space, so that we can ignore any curvature effects and so that the '\( S \) matrix' describing scattering off the bubble is given simply as parity inversion:

\[ S = P \]

For example, we could simply decree that any causal path which intersects the spacelike surface \( \{(x, t) : |x| < 1, t = 0\} \) is parity reversing.

Thus, the operator representing \( P \) now appears in the Hilbert space, since by definition the final state \( \psi_f \) is given in terms of the initial state \( \psi_i \) as

\[ \psi_f = P \psi_i \]

(\( P \) is of course a unitary operator in the standard case, e.g. (4) above). It follows that the solutions of the Dirac equation on this parity-reversing bubble are actually sections of a \( \text{pin} \) bundle, i.e., a bundle whose fibres are isomorphic to one of the above eight pin groups. Once we choose which representations of \( P \) and \( T \) we will employ, we determine the fibre group completely. Denote the chosen pin bundle ‘\( \mathcal{B} \)’. Then the only way in which we can sensibly operate on a section of \( \mathcal{B} \) is by using the fibres of \( \mathcal{B} \), i.e., it is mathematically vacuous to say that we want to consider the action on sections of \( \mathcal{B} \) of a group which is not isomorphic to the fibres of \( \mathcal{B} \). Of course,
there may exist other ‘types’ of fermions, given by different choices for $P$ and $T$, but they will not interact with the fermions which lie in $\mathcal{B}$.

Let us now briefly return to our basic examples of non-orientable black holes, the quotient manifolds $\mathcal{M}_J$ constructed above. Which of these manifolds admit Pin$^{- - +}(3,1)$ structure? By the results of [9], we know that the obstruction to Pin$^{- - +}(3,1)$ structure is that the following obstruction vanish on all two-cycles in $\mathcal{M}$:

$$w_2 + w_1^+ \sim w_1^+ + w_1^- \sim w_1^-$$

where $w_1^+$ is the obstruction to space-orientability, $w_1^-$ is the obstruction to time-orientability, $w_2$ is the second Stiefel-Whitney class, and $\sim$ denotes the cup product, as outlined in [9]. With these definitions in mind, consider $\mathcal{M}_P$.

$\mathcal{M}_P$ contains a single non-trivial two-cycle $c$, which is a spacelike $\mathbb{R}P^2$. On this two-cycle, we therefore have

$$w_2[c] = 1$$

$$w_1^+ \sim w_1^+[c] = 1$$

$$w_1^- \sim w_1^-[c] = 0$$

where we are working in additive $\mathbb{Z}_2$. Thus, the above obstruction vanishes mod 2 and so $\mathcal{M}_P$ admits Pin$^{- - +}(3,1)$ structure. Proceeding in this vein for the other examples, we establish the following table:
Table 2

| $J$                  | $\mathcal{M}_J$ admits Pin${}^{-:}{}_{+:}(3, 1)$ structure? |
|----------------------|----------------------------------------------------------|
| $J = R_TR_ZP$        | no                                                       |
| $J = R_TP$           | yes                                                      |
| $J = R_ZP$           | yes                                                      |
| $J = P$              | yes                                                      |

Of course, we are interested in matching these Lorentzian sections to their Riemannian counterparts. We therefore want to know how to ‘Wick rotate’ pinors. With this in mind, let us first review the definition of pinors in Euclidean signature.

Really, the situation in Euclidean signature is much simpler: Given the orthogonal group (inhomogeneous) in $n$-dimensions, $O(n)$, there are just two double-covers of $O(n)$, usually denoted Pin$^+(n)$ and Pin$^-(n)$, where the $\{\pm\}$ denotes the sign of the square of the element in Pin$^\pm(n)$ which covers reflection in $O(n)$ [19]. The obstructions to these structures are similar to the obstructions in Lorentzian signature, and can be summarised as follows:

(1) There exists Pin$^+(n)$ structure iff $w_2(M) = 0$, i.e., iff the manifold is spin.

(2) There exists Pin$^-(n)$ structure iff $w_2(M) + w_1 \sim w_1 = 0$, where $w_1 \sim w_1$ is the cup product of the first Stiefel-Whitney class of $M$ with itself.
Consider now the ‘identified cigar’ instantons, the $\mathcal{M}_j^R$ constructed above. As usual, let us start with $\mathcal{M}_P^R$. Originally, $\mathcal{M}^R$ is topologically $\mathbb{R}^2 \times S^2$. The identification under $\bar{P}$ corresponds to antipodal identification of the $S^2$ factor so that $\mathcal{M}_P^R$ is topologically

$$\mathcal{M}_P^R \cong \mathbb{R}^2 \times \mathbb{R}P^2$$

$\mathcal{M}_P^R$ is not spin, and so it does not admit Pin$^+(4)$ structure. On the other hand,

$$w_1 \sim w_1[\mathbb{R}P^2] = 1$$

and therefore $\mathcal{M}_P^R$ does admit Pin$^-(4)$ structure. This is good, since if we want to ‘match’ the Pin$^-(4)$ structure on $\mathcal{M}_P^R$ to the Pin$^+-+(3,1)$ structure on $\mathcal{M}_P$ (across the Einstein Rosen bridge $\Sigma$, in a way to be made precise in a moment), then we would want the signs of the squares of inversion to match as well.

We then work out the obstructions to Pin$^-(4)$ on $\mathcal{M}_J$, for the other values of $\bar{J}$, and obtain the table:

| $\bar{J}$         | $\mathcal{M}_J^R$ admits Pin$^-(4)$ structure? |
|-------------------|-----------------------------------------------|
| $\bar{J} = \bar{R}_T \bar{R}_Z \bar{P}$ | no                                           |
| $\bar{J} = \bar{R}_T \bar{P}$           | yes                                          |
| $\bar{J} = \bar{R}_Z \bar{P}$           | yes                                          |
| $J = P$                                      | yes                                          |

Table 3
Note how well Table 2 agrees with Table 3. Given this nice correspondence, we can now describe how to ‘match’ the pinors on $\mathcal{M}_J^R$ to the pinors on $\mathcal{M}_J$ across $\Sigma$, so that the data induced on $\Sigma$ by the Euclidean pinors agrees with the data induced on $\Sigma$ by the Lorentzian pinors.

To see how this works, first recall that the structure group of complexified Schwarzschild $\mathcal{M}_C$ is $SO(4, \mathbb{C})$ and that this group splits naturally into two copies of $SL(2, \mathbb{C})$:

$$SO(4, \mathbb{C}) \cong SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$$

(Since the inclusion of inversions, i.e., passing from spin to pin, just involves forming the semi-direct product of these groups with some finite discrete groups, we really only need to show how to match the spinors on ordinary Schwarzschild $\mathcal{M}_L$ across to the spinors on $\mathcal{M}_R$, and then our above discussion (Tables 2 and 3) will take care of the matching when one includes the involutions $J$).

Thus, we want to know how to match the Spin(4) structure on $\mathcal{M}_R$ to the $SL(2, \mathbb{C})$ structure on $\mathcal{M}_L$ across $\Sigma$. But that is easy. After all,

$$\text{Spin}(4) \cong SU(2) \times SU(2)$$

$$\cap \quad \cap$$

$$SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$$

and the Spin(4) structure will induce $SU(2)$ spinors on the Einstein Rosen bridge. We want to think of these spinors as ‘initial data’ for the Lorentzian section. But the spin structure on $\mathcal{M}_L$ is one of the $SL(2, \mathbb{C})$ factors in the above diagram. Therefore, in order to ensure that the two spin structures match, we simply have to make sure that the structure group of $\mathcal{M}_L$ is the $SL(2, \mathbb{C})$ factor which contains the $SU(2)$ factor induced by the Spin(4) structure on $\mathcal{M}_R$.

We can make these matching conditions more explicit by introducing local coordinates for all of the spin structures involved. More precisely, at each point $p \in \mathcal{M}_C$, the tangent space is just

$$T_p(\mathcal{M}_C) \cong \mathbb{C}^4$$

As usual, we can do the ‘twistorial’ thing [25] and rewrite $\mathbb{C}^4$ in terms of $2 \times 2$ complex matrices, i.e., $\mathbb{C}^4 \cong \mathbb{C}^{2 \times 2}$ and the isometry is just given by
the map

\[ \mathbb{C}^4 \ni (z^0, z^1, z^2, z^3) \rightarrow \begin{pmatrix} z^0 + z^3 & z^1 - iz^2 \\ z^1 + iz^2 & z^0 + z^3 \end{pmatrix} = (z^{ij}) \in \mathbb{C}^{2 \times 2} \]

The fact that SO(4, \mathbb{C}) splits into a ‘left’ and a ‘right’ part (via SO(4, \mathbb{C}) \simeq SL(2, \mathbb{C})_L \times SL(2, \mathbb{C})_R) simply means that the action of an element \((S_L, S_R) \in SO(4, \mathbb{C})\) on some \((z^{ij}) \in \mathbb{C}^{2 \times 2}\) can be written

\[ (z^{ij}) \rightarrow S_L (z^{ij}) S_R^{-1} \]

Intuitively, what is going on is that at each point of \(\mathbb{C}^4\) there are two orthogonal two-dimensional complex planes, each of them acted on by an \(SL(2, \mathbb{C})\) (in a way reminiscent of the way \(\mathbb{R}^4\) splits at each point into orthogonal \(\mathbb{R}^2\) factors, each associated with an \(SU(2)\)). \(\mathcal{M}^L \subset \mathcal{M}_C\) has the property that at each \(p\), \(T_p(\mathcal{M}^L) \simeq \mathbb{C}^2\) is acted upon by one of the \(SL(2, \mathbb{C})\) factors which we take to be \(SL(2, \mathbb{C})_L\), without loss of generality. Thus, at each \(T_p(\mathcal{M}^L)\) the above isometry becomes

\[ \mathbb{R}^{3,1} \ni (t, x, y, z) \rightarrow \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix} \in H(2) \]

where \(H(2) = \{2 \times 2 \text{ Hermitian matrices}\}\). We can choose local Minkowskian coordinates about a point in the Einstein-Rosen bridge so that \(t = 0\) corresponds to the intersection \(\mathcal{M}^L \cap \mathcal{M}^R = \Sigma \neq \emptyset\), and so the above Hermitian matrix reduces to

\[ \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \]

which is an element of \(SU(2)\), the spin group for \(SO(3)\), i.e., the spatial rotation group of \(SL(2, \mathbb{C})_L\) is the \(SU(2)\) induced on \(\Sigma\) by the Lorentzian section.

We have now seen how to define fermions on the ‘classical’ non-orientable black holes, and we have touched on the thermodynamical properties of these objects. It is time we turned to the problem of creating these sorts of objects
using the instanton approximation.

**IV. Non-Orientable Instantons**

We begin with the Schwarzschild-de Sitter solution, which may be written in the following form:

\[
\begin{align*}
  ds^2 &= - \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 \\
  &\quad + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \\
\end{align*}
\]

(6)

As usual, we interpret this solution as a black hole immersed in de Sitter space. Upon Wick rotating this metric, one finds that there are apparent singularities in the Riemannian section unless the cosmological constant \( \Lambda \) and the mass \( m \) of the hole are related by

\[
\sqrt{\Lambda} = \frac{1}{3m}
\]

This equality corresponds to the limit in which the black hole and cosmological event horizons merge. Using this equality, the metric on the Riemannian section becomes

\[
\begin{align*}
  ds^2 &= \left(1 - \Lambda \rho^2\right) d\tau^2 + \frac{d\rho^2}{1 - \Lambda \rho^2} + \frac{1}{\Lambda} \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \\
\end{align*}
\]

To see that this metric lies on \( S^2 \times S^2 \), one introduces the coordinate \( \eta \) via

\[
\rho \sqrt{\Lambda} = \cos \eta
\]

whence the metric becomes

\[
\begin{align*}
  ds^2 &= \frac{1}{\Lambda} \left(d\eta^2 + \sin^2 \eta d\tau^2 + d\theta^2 + \sin^2 \theta d\phi^2\right) \\
\end{align*}
\]

(7)

The Euclidean action is calculated to be

\[
S = \frac{-2\pi}{\Lambda}
\]

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The metric on (7) is manifestly the standard product metric on $S^2 \times S^2$, where each $S$ has radius $\frac{1}{\sqrt{\Lambda}}$.

We now wish to form the quotient of this instanton by some discrete involutive freely acting isometries. There will be a number of different possibilities for the set of maps under which we can identify. Let us begin with the Lorentzian section (6).

First, we transform to coordinates $(t, \chi, \theta, \phi)$, where the metric takes the form

$$ds^2 = \frac{1}{\Lambda} \left( -dt^2 + \left( \cosh (\sqrt{\Lambda} t) \right)^2 d\chi^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

i.e., the coordinates $t$ and $\chi$ are (respectively) timelike and spacelike coordinates on two-dimensional de Sitter ($-\infty < t < \infty$, $0 \leq \chi \leq 2\pi$) and $(\theta, \phi)$ are the usual coordinates on $S^2$ ($0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$). In terms of these coordinates, there are then several involutions which we will make use of.

First of all, there is time reversal:

$$T : (t, \chi, \theta, \phi) \rightarrow (-t, \chi, \theta, \phi)$$

This obviously has fixed points ($t = 0$).

Next, there is inversion in the spacelike coordinate:

$$I : (t, \chi, \theta, \phi) \rightarrow (t, \chi + \pi, \theta, \phi)$$

Intriguingly, this involution has no fixed points!

Finally, there is the usual freely acting involution of parity:

$$P : (t, \chi, \theta, \phi) \rightarrow (t, \chi, \pi - \theta, \phi + \pi)$$

Therefore, on the Lorentzian section at least, we now have two freely acting isometries, and thus a much richer range of possibilities. Let $M$ denote the Schwarzschild-de Sitter solution (without identifications). Then we shall follow the notation of Section 2 above when we form quotient spaces, i.e.,

$$M_P = M/P, M_I = M/I, \text{ etc.}$$

As an amusing aside, consider the spacelike $M_{TI}$. This is a non-time orientable spacetime with a single boundary component homeomorphic to $S^1 \times S^2$, and in fact the manifold is basically two-dimensional antipodally
identified de Sitter crossed with a two-sphere (see [13], [14] for more on antipodally identified de Sitter). Thus, this spacetime can be thought of as a Lorentzian path corresponding to the birth of a universe (with a black hole in it) from nothing. We have more about the interplay between Lorentzian path integrals and Euclidean instantons in another recent paper [21].

Now, in analogy with what we did above in Sec. I, let us consider the Riemannian section $M^R \cong S^2 \times S^2$, and let’s construct the maps $T$, $I$ and $P$ which correspond to the maps $T$, $I$ and $P$ on the Lorentzian section. As was done above, we could embed both the Lorentzian and Riemannian sections in some higher dimensional complex space (the Riemannian section would then just be a product of complex projective lines), and we could find ‘big maps’ on the higher dimensional complex manifold which yielded the desired involutions when restricted to the real or imaginary time sections. However, the geometry in this situation is so simple that we can just write down the involutions on the Riemannian instanton by inspection. In terms of the coordinates used in equation (8) above, and remembering to change to imaginary time $\tau = it$, these involutions are given as follows:

$$T: (\tau, \chi, \theta, \phi) \rightarrow (-\tau, \chi, \theta, \phi)$$

($\tau$ now has range $-\pi/2 < \tau < \pi/2$).

$$\bar{I}: (\tau, \chi, \theta, \phi) \rightarrow (\tau, \chi + \pi, \theta, \phi)$$

And as usual, parity:

$$\bar{P}: (t, \chi, \theta, \phi) \rightarrow (t, \chi, \pi - \theta, \phi + \pi)$$

Thus, we see that the transition from the Lorentzian solution to the Riemannian instanton is rather simple in this example. Now notice, however, that $\bar{I}$ is not freely acting on the $(\tau, \chi)$ sphere (it has fixed points at the north and south poles) and so we cannot use $\bar{I}$ to construct freely acting involutions on the instanton. There are thus four instantons which describe the birth from nothing of a pair of non-orientable black holes in a de Sitter background (or, equivalently, one could use these instantons to calculate the rate of decay of de Sitter space into such a black hole pair). These instantons are (using the usual notation): $M^R_P$, $M^R_{IP}$, $M^R_{IP}$, and $M^R_{TIP}$. As usual, we summarize the properties of the Lorentzian solutions which these instantons correspond to in a table:
Table 4

| $J$ | $M_J$ admits $Pin^{-,-,+}(3,1)$ structure? | $M_J$ time orientable? | $M_J$ space orientable? |
|-----|---------------------------------|------------------------|-------------------------|
| Case I: $J = TIP$ | no | no | yes |
| Case II: $J = TP$ | yes | no | no |
| Case III: $J = IP$ | yes | yes | yes |
| Case IV: $J = P$ | yes | yes | no |

As discussed above, the probability for de Sitter space to decay into one of these non-orientable Lorentzian solutions is given by the square of the amplitude, where the amplitude is given in the semiclassical approximation by $e^{-S}$ with $S$ the Euclidean action. (Actually, we have to first divide the amplitude to create a universe with black holes by the amplitude to create ordinary de Sitter in order to obtain the rate of decay of de Sitter, but we will overlook that subtlety here). Consider the instanton

$$M^R_P \cong S^2 \times \mathbb{R}P^2$$

We would like to know whether de Sitter is more likely to decay into $M$ or $M_P$. As we noted above, the action for $M^R$ is given as $S = \frac{-2\pi}{\Lambda}$. On the other hand, the Euler number of $M^R_P$ is 2 (half that of $M^R$) and so the action $S_P$ must be $S_P = \frac{\pi}{\Lambda}$. Thus, recalling that we actually only need the action for ‘half’ the instanton and that the probability is the square of the resulting amplitude, it would seem that the probabilities for these two decay processes differ by a factor of $e^{\pi/\Lambda}$, which measures the suppression of the
rate of non-orientable hole production relative to the rate of orientable hole production.

Of course, the specific results which we have outlined here will go through in general for any instanton and corresponding classical solution which admit at least one discrete, freely acting isometry. For example, the Mellor-Moss instanton [22] (describing the nucleation of charged black holes in a de Sitter background) admits a freely acting isometry as do all of the instantons obtained from the many solutions, derived from the C-metric, which describe the production of charged black holes in background fields [23]. And we have also recently pointed out [21] that non-orientable black holes will be created in the presence of vacuum domain walls. It therefore seems that whenever one has an energy source which can contribute to tunneling phenomena corresponding to the birth of ordinary black holes, that same energy source will also contribute to non-orientable black hole pair production. The only caveat is that the rate of non-orientable black hole production will be suppressed relative to the rate of production of ordinary holes, since the identified instantons will generically have less volume (and hence less action) than the original orientable instantons.

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