Quantum discord of SU(2) invariant states

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Abstract
We have analytically calculated the quantum discord for a system composed of spin-\( j \) and spin-1/2 subsystems possessing SU(2) symmetry. We have compared our results with the quantum discord of states having similar symmetries and seen that in our case the amount of quantum discord is much higher. Moreover, using the well-known entanglement properties of these states, we have also compared their quantum discord with entanglement. Although the system under consideration is almost separable throughout its parameter space as \( j \) increases, we have seen that the discord content remains significantly large. Investigating the quantum discord in SU(2) invariant states may find application in quantum computation protocols that utilize quantum discord as a resource since they arise in many real physical systems.

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1. Introduction

Multipartite quantum states contain different kinds of correlations which either can or cannot be of classical origin. Entanglement has been recognized as the first indicator of non-classical correlations and lies at the heart of quantum information science [1]. It has been considered as the defining resource for almost all protocols in quantum computing. However, recent research on quantum correlations has shown that entanglement is not the only kind of useful quantum correlation. Quantum discord (QD), which is defined as the discrepancy between two classically equal descriptions of quantum mutual information, has also proven to be utilizable in quantum computing protocols [2, 3]. Moreover, QD is more general than entanglement in the sense that it can be present in separable mixed quantum states as well. Following this discovery, much effort has been put into investigating the properties and behavior of QD in various systems ranging from quantum spin chains to open quantum systems [4]. Nevertheless, since evaluation of QD requires a very complex optimization procedure, the significant part of the development in the field is numeric and analytical results are present only for some very restricted sets of states. In general, these restrictions are introduced by forcing certain symmetries and limiting the size and dimension of the system under consideration. A short list of analytical results would include the progress made in X-shaped states of different
dimensions \([5–9]\), \(2 \otimes d\)-dimensional two-parameter classes of states \([10]\), \(d \otimes d\)-dimensional Werner and pseudo-pure states \([11]\), general real density matrices displaying \(Z_2\) symmetry \([12]\), two-mode Gaussian states \([13]\), and \(2 \otimes d\)-dimensional mixed states of rank-2 \([14–17]\) where \(d\) denotes the Hilbert space dimension of the system under consideration. QD witnesses have also been introduced for \(2 \otimes d\) systems \([18]\). Following QD, many other quantum and total correlation quantifiers have been introduced \([19–24]\).

Bipartite \(SU(2)\) invariant states are defined by their invariance under rotation of both spins, \(U_{1(2)} \otimes U_2 \rho U_{1(2)}^\dagger \otimes U_2^\dagger = \rho\), where \(U_{1(2)} = \exp(i\vec{\alpha} \cdot \vec{S}_{1(2)})\) is the usual rotation operator and the length of \(\vec{\alpha}\) is chosen according to the spin length \(|\vec{S}|\) \([25, 26]\). In other words, these states commute with every component of the total spin operator \(\vec{J} = \vec{S}_1 + \vec{S}_2\). In real physical systems, \(SU(2)\) invariant density matrices arise, for example, when considering the reduced state of two spins described by \(SU(2)\) invariant Hamiltonian or multi-photon states generated by parametric down-conversion and then undergoing photon loss \([27]\). The entanglement structure of states under certain symmetries has been vastly explored in the literature \([28–30]\). For \(SU(2)\) invariant states, which are central to this work, negativity has shown to be a necessary and sufficient criterion for separability \([25, 26]\), and the relative entropy of entanglement has been analytically calculated \([31]\) for \((2j + 1) \otimes 2\)- and \((2j + 1) \otimes 3\)-dimensional systems. Furthermore, the entanglement of formation (EoF), a measure which also involves a complex optimization procedure, has been analytically evaluated in the case of mixed \((2j + 1) \otimes 2\)-dimensional systems \([32]\). Recently, a very closely related article arose, which evaluates different correlation measures that are more general than entanglement, in \(O \otimes O\)-invariant states \([33]\). However, to the best of our knowledge, this is the first attempt to explore QD in \((2j + 1) \otimes 2\)-dimensional states.

In this work, we have analytically calculated the QD of an \(SU(2)\) invariant \((2j + 1) \otimes 2\)-dimensional system. We have compared our results with the entanglement properties of these states and other analytical calculations of QD in systems with similar symmetries. We have observed that while the entanglement content decreases as \(j\) increases, the amount of QD remains significantly larger, with its maximum value also following a decreasing trend.

2. Quantum discord

In this section, we shall review the concept of QD. We have very briefly mentioned that QD is the difference between the quantum extensions of the classical mutual information. First and direct generalization of classical mutual information is obtained by replacing the Shannon entropy with its quantum analog, the von Neumann entropy

\[
I(\rho^{ab}) = S(\rho^a) + S(\rho^b) - S(\rho^{ab}).
\]

Here, \(\rho^a\) and \(\rho^b\) are the reduced density matrices of the subsystems and \(S(\rho) = -\text{tr}\ \rho \log_2 \rho\) is the von Neumann entropy. On the other hand, in classical information theory, the mutual information can also be written in terms of the conditional probability. However, generalization of the conditional probability to the quantum case is not straightforward since the uncertainty in a measurement performed by one party depends on the choice of measurement. Therefore, one has to optimize over the set of measurements made on a system \([2–4]\)

\[
C(\rho^{ab}) = S(\rho^a) - \min_{\{\Pi_k\}} \sum_k p_k S(\rho_k^b),
\]

where, in this work, \(\{\Pi_k\}\) is always understood to be the complete set of one-dimensional projective measurements performed on subsystem \(b\) and \(\rho_k^b = (I \otimes \Pi_k^b)\rho^{ab}(I \otimes \Pi_k^b)/p_k\) are the post-measurement states of subsystem \(a\) after obtaining the outcome \(k\) with probability...
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property of entanglement. J. Phys. A: Math. Theor. quantity is also referred to as classical correlations contained in a state [3, 4]. Since classical
versions of the aforementioned expressions for quantum mutual information are the same, one can define a measure for quantum correlations, namely the quantum discord as

$$D(\rho^{ab}) = I(\rho^{ab}) - C(\rho^{ab}).$$

The main challenge in the calculation of QD is the evaluation of classical correlations, since it requires a complex optimization over all measurements on the system. The reason that there are no general analytical results on QD except for very few special cases, is due to this difficulty. It is important to note that QD is dependent on which subsystem the measurements are performed on. Since making the measurements on the spin-1 subsystem will make the optimization procedure even harder, in this work, all measurements are made on the spin-1/2 subsystem. Furthermore, QD can increase or decrease under local operations and classical communication (LOCC) if the LOCC is performed on the measured part of the system [34–37]. This is a rather peculiar behavior since invariance under LOCC is the defining property of entanglement.

3. Results and discussion

The bipartite state under consideration is composed of a spin-j and a spin-1/2 subsystem. $SU(2)$ invariant states are parameterized by a single parameter which will be denoted by $F$ throughout this paper. The density matrix for our system in the total spin basis is given as [25]

$$\rho^{ab} = \frac{F}{2j} \sum_{m=-j+1/2}^{j-1/2} |j-1/2, m\rangle \langle j-1/2, m| + \frac{1-F}{2(j+1)} \sum_{m=-j-1/2}^{j+1/2} |j+1/2, m\rangle \langle j+1/2, m|.$$  

(4)

We shall start by calculating the quantum mutual information. The bipartite density matrix has two eigenvalues $\lambda_1 = F/2j$ and $\lambda_2 = (1-F)/(2j+2)$ with degeneracies $2j$ and $2j+2$, respectively. On the other hand, the reduced density matrices of the subsystems can be found as $\rho^a = I_{2j+1}/(2j+1)$ and $\rho^b = I_2/2$ where $I_2$ are the identity matrices in the dimension of the Hilbert space for the spin-j and spin-1/2 particle, respectively. Note that both $\rho^a$ and $\rho^b$ are maximally mixed independent of $j$. Thus the mutual information of our system is

$$I(\rho) = S(\rho^a) + S(\rho^b) - S(\rho^{ab}) = 1 + \log_2(2j+1) + F \log_2 F - (1-F) \log_2 \frac{1-F}{2j+2}.$$  

(5)

We now turn our attention to the calculation of the classical correlations. We will perform projective measurements on the spin-1/2 part of the density matrix. In order to do so, we first need to write the density matrix in the product basis. By using the Clebsh–Gordan coefficients for coupling a spin-j to a spin-1/2, the density matrix in the product basis can be written as

$$\rho^{ab} = \frac{F}{2j} \sum_{m=-j+1/2}^{j-1/2} a_+^m |m - 1/2\rangle \langle m - 1/2| \otimes |1/2\rangle \langle 1/2| + a_- b_- |m - 1/2\rangle \langle m + 1/2| \otimes |1/2\rangle \langle 1/2| + a_- b_+ |m + 1/2\rangle \langle m - 1/2| \otimes |1/2\rangle \langle 1/2|.$$

3
We are now ready to use the transformation properties of Pauli matrices as given in [5]. Following [5], we can write any von Neumann measurement on $\rho^a$ as

$$B_k = V \Pi_k V^\dagger : k = 0, 1,$$

where $\{\Pi_k = |k\rangle \langle k| : k = 0, 1\}$ and $V = t I + i \vec{y} \cdot \vec{\sigma}$, any unitary matrix in SU(2). Here, both $t$ and $\vec{y}$ are real and $t^2 + y_1^2 + y_2^2 + y_3^2 = 1$. After the measurements are performed, $\rho^{ab}$ will transform into an ensemble of post-measurement states with their corresponding probabilities $\{p_k, p_1\}$. In order to calculate the possible post-measurement states $\rho_k$ and their corresponding probabilities $p_k$, we write

$$p_k, p_1 = (I \otimes B_k) \rho^{ab} (I \otimes B_k) \rho^{ab} (I \otimes V \Pi_k V^\dagger) \rho^{ab} (I \otimes V \Pi_k V^\dagger)$$

$$= (I \otimes V) (I \otimes \Pi_k) (I \otimes V^\dagger) \rho^{ab} (I \otimes V) (I \otimes \Pi_k) (I \otimes V^\dagger).$$

Since transformation of the usual Pauli matrices under $V$ and $\Pi_k$ is known [5], it is easier to calculate the post-measurement states when the spin-1/2 part of the density matrix is written in terms of them. In order to do so, we will use the following identities

$$|1/2\rangle\langle 1/2| = \frac{1}{2} [I + \sigma_3]$$

$$|1/2\rangle\langle -1/2| = \frac{1}{2} [\sigma_1 + i \sigma_2]$$

$$|-1/2\rangle\langle 1/2| = \frac{1}{2} [\sigma_1 - i \sigma_2]$$

$$|-1/2\rangle\langle -1/2| = \frac{1}{2} [I - \sigma_3].$$

We are now ready to use the transformation properties of Pauli matrices as given in [5]

$$V^\dagger \sigma_1 V = (t^2 + y_1^2 - y_2^2 - y_3^2) \sigma_1 + 2 (ty_3 + y_1 y_2) \sigma_2 + 2 (-ty_2 + y_1 y_3) \sigma_3,$$

$$V^\dagger \sigma_2 V = 2 (-ty_3 + y_1 y_2) \sigma_1 + (t^2 + y_2^2 - y_1^2 - y_3^2) \sigma_2 + 2 (-ty_1 + y_2 y_3) \sigma_3,$$

$$V^\dagger \sigma_3 V = 2 (ty_2 + y_1 y_3) \sigma_1 + 2 (-ty_1 + y_2 y_3) \sigma_2 + (t^2 + y_3^2 - y_1^2 - y_2^2) \sigma_3,$$

and $\Pi_0 \sigma_3 \Pi_0 = \Pi_0, \Pi_1 \sigma_1 \Pi_1 = - \Pi_1, \Pi_1 \sigma_2 \Pi_1 = 0$ for $j = 0, 1, k = 1, 2$. We have calculated the probabilities of obtaining two possible post-measurement states as $p_0 = p_1 = 1/2$ and the corresponding post-measurement states themselves as

$$\rho_0 = \left\{ \frac{1}{2j + 1} \left[ \frac{m(2F + F - j)}{\sqrt{J(j + 1) - m(m + 1)(2F - j)}} |m\rangle \langle m| - (z_1 + iz_2) \frac{\sqrt{J(j + 1) - m(m + 1)(2F + F - j)}}{2j + 1} |m\rangle \langle m + 1| \right] \otimes V \Pi_0 V^\dagger \right\}$$

$$\rho_1 = \left\{ \frac{1}{2j + 1} \left[ \frac{m(2F + F - j)}{\sqrt{J(j + 1) - m(m + 1)(2F - j)}} |m\rangle \langle m+1| - (z_1 - iz_2) \frac{\sqrt{J(j + 1) - m(m + 1)(2F + F - j)}}{2j + 1} |m+1\rangle \langle m| \right] \otimes V \Pi_0 V^\dagger \right\}$$

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and

\[
\rho_1 = \left\{ \sum_{m=-j}^{j} \frac{1}{2j+1} + z_3 \frac{m(2Fj + F - j)}{j(2j+1)(2j+1)} |m\rangle\langle m| \right. \\
\left. + (z_1 + iz_2) \frac{\sqrt{j(j+1) - m(m+1)(2Fj + F - j)}}{2j(2j+1)} |m\rangle\langle m+1| \right. \\
\left. + (z_1 - iz_2) \frac{\sqrt{j(j+1) - m(m+1)(2Fj + F - j)}}{2j(2j+1)} |m+1\rangle\langle m| \right\} \otimes V\Pi_1 V^\dagger,
\]  
(14)

where \( z_1 = 2(-ty_2 + y_1 y_3) \), \( z_2 = 2(ty_1 + y_2 y_3) \), \( z_3 = t^2 + y_3^2 - y_1^2 - y_2^2 \) with \( z_1^2 + z_2^2 + z_3^2 = 1 \). The eigenvalues of the post-measurement states are the same and by inspection, they can be found as

\[
\lambda_n^\pm = \frac{1}{2j+1} \pm \frac{j-n}{j(j+1)(2j+1)} |(F(2j+1) - j)|,
\]  
(15)

where \( n = 0, \ldots, [j] \) for half-integer \( j \) with \([\cdot]\) being the floor function and \( n = 0, \ldots, j \) for integer \( j \).

In our calculation of the post-measurement states, we have followed the method introduced in [5]. Considering the symmetry of the states considered in this work, an alternative and more direct way to obtain the eigenvalues of the post-measurement states is present. Continuing directly from (8)

\[
p_k \rho_k = (I \otimes V\Pi_k V^\dagger) \rho^{ab} (I \otimes V\Pi_k V^\dagger) \\
= (I \otimes V\Pi_k V^\dagger) (V \otimes V) \rho^{ab} (V^\dagger \otimes V^\dagger) (I \otimes V\Pi_k V^\dagger) \\
= (I \otimes V\Pi_k) (V \otimes I) \rho^{ab} (V^\dagger \otimes I) (I \otimes V\Pi_k V^\dagger) \\
= (V \otimes V) (I \otimes \Pi_k) \rho^{ab} (I \otimes \Pi_k V) (V^\dagger \otimes V^\dagger). 
\]  
(16)

Applying the projection operators to the spin-1/2 part of the density matrix, one can get the post-measurement states as

\[
p_0 \rho_0 = \frac{F}{2j} \sum_{m=-j+1/2}^{j+1/2} a^2 |m-1\rangle\langle m-1| + \frac{1 - F}{2(j+1)} \sum_{m=-j-1/2}^{j-1/2} a^2 |m-1\rangle\langle m-1| 
\]  
(17)

and

\[
p_1 \rho_1 = \frac{F}{2j} \sum_{m=-j+1/2}^{j+1/2} b^2 |m-1\rangle\langle m-1| + \frac{1 - F}{2(j+1)} \sum_{m=-j-1/2}^{j-1/2} b^2 |m-1\rangle\langle m-1|.
\]  
(18)

Since both of these matrices are diagonal and free of measurement parameters, it is straightforward to calculate the eigenvalues and eventually, the QD of these states. The eigenvalues obtained from these post-measurement states are equivalent to the ones presented in (15).

It can be clearly seen that that the eigenvalues do not depend on the measurement parameters. Therefore, calculation of the classical correlations do not require any optimization over the projective measurements. Then, the classical correlations can be written as

\[
C(\rho^{ab}) = S(\rho^a) - \sum_k p_k S(\rho_k^a) = \log_2 (2j + 1) + \sum_{n=0}^{j} \lambda_n^\pm \log_2 (\lambda_n^\pm).
\]  
(19)
Combining the above equation with (5), we have obtained an analytical expression for QD in the system under consideration

\[
D(\rho^{ab}) = 1 + F \log_2 \frac{F}{2j} + (1 - F) \log_2 \frac{1 - F}{2j + 2} - \sum_{n=0}^{j} \lambda_n^\pm \log_2 (\lambda_n^\pm),
\]

(20)

where \(\lambda_n^\pm\) is given at (15).

In figure 1, we present our results on QD and \(C(\rho^{ab})\) as a function of our system parameter \(F\) for different dimensions. We recover the results obtained in [5, 38] in the special case of two spin-1/2 systems. We know that for \(\rho^{ab}\), the boundary between separable and entangled states is at \(F_s = 2j/(2j + 1)\) [25], which is half of the value at which both QD and \(C(\rho^{ab})\) vanish, \(F_d = j/(2j + 1)\). One can observe that as the dimension of the system increases, both QD and \(C(\rho^{ab})\) increase in the region \(F < F_s\) and decrease in the region \(F > F_s\). Eventually, in the infinite \(j\) limit, both of them become symmetric around the point \(F = 1/2\) where they are exactly zero. The symmetry around \(F = 1/2\) clearly starts to manifest itself at system dimensions as low as \(j = 9/2\) \((d = 10)\). The maximum value of QD is attained for \(F = 1\) for all system dimensions, which corresponds to the state that is the projector onto the spin-(\(j - 1/2\)) subspace. It is important to note that as \(j \to \infty\), our system becomes completely separable while QD remains finite except at a certain point, with its maximum value following a decreasing trend. This behavior can also be seen explicitly if we look at the large \(j\) limit of (20) as

\[
D(\rho^{ab}) = 1 + F \log_2 F + (1 - F) \log_2 (1 - F) - \log_2 (2j + 1) - \sum_{n=0}^{j} A_n^\pm \log_2 A_n^\pm,
\]

(21)

where \(A_n^\pm = 1/(2j \pm (j - n))(2F - 1)/(2j^2)\). The symmetry point \(F = 1/2\) is apparent in the above equation and the decreasing trend of the maximum value of QD can also be seen analytically as a function of \(j\). In the same limit for \(d \otimes d\) Werner states \(F_s = F_d = 1/2\) and QD is again symmetric around this point. Therefore, for QD \(< 1\), it is possible to find an entangled and a separable state possessing the same amount of QD [11]. From the right panel of figure 1, it is clear that classical correlations decay in the limit \(j \to \infty\). However, its maximum settles to a fairly high value as compared to \(d \otimes d\) Werner states.

We will now compare the amount of QD and entanglement possessed in our system. The EoF for a spin-1/2 and a spin-\(j\) SU(2) invariant state is given by [32]

\[
\text{EoF} = \begin{cases} 
0, & F \in [0, 2j/(2j + 1)] \\
H \left( \frac{1}{j+1} (\sqrt{2F - \sqrt{2j(1 - F)}})^2 \right), & F \in [2j/(2j + 1), 1],
\end{cases}
\]
where $H(x) = -x \log x - (1-x) \log(1-x)$ is the binary entropy. In contrast to $d \otimes d$ Werner states, the point in the parameter space at which the EoF becomes non-zero is dependent on $j$. In [11], it was shown that the EoF becomes a general upper bound for QD in $d \otimes d$ Werner states. However, in figure 2 we can see that except in the $j = 1/2$ case, QD always remains larger than the EoF for all $F$ and the difference between these quantities increases as $j \to \infty$. Note that the region in which the EoF remains zero covers the whole parameter space in the same limit.

4. Conclusion

We have analytically calculated the QD of an $SU(2)$ invariant system, consisting of a spin-$j$ and a spin-1/2 subsystem. We have compared our results with the entanglement structure of these systems and the QD of states having similar symmetries. It is known that a very small subset of the set of states addressed in this work possess entanglement as the dimension of the spin-$j$ particle becomes larger. We have shown that in the large $j$ limit, QD remains significantly larger than the entanglement. On the other hand, we have seen that maximum value of QD decreases with the increasing system size. The observation of $SU(2)$ invariant states in many real physical systems makes them a good candidate for utilization in quantum computing protocols that rely on QD.

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Figure 2. QD (solid line) and EoF (dashed line) versus $F$ for $j = 1/2 (d = 2)$ (left panel) and for $j = 9/2 (d = 10)$ (right panel).
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