Two-dimensional superfluidity of exciton-polaritons requires strong anisotropy

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Fluids of exciton-polaritons, excitations of two dimensional quantum wells in optical cavities, show collective phenomena akin to Bose condensation. However, a fundamental difference from standard condensates stems from the finite life-time of these excitations, which necessitate continuous driving to maintain a steady state. A basic question is whether a two dimensional condensate with long range algebraic correlations can exist under these non-equilibrium conditions. Here we show that such driven two-dimensional Bose systems cannot exhibit algebraic superfluid order except in low-symmetry, strongly anisotropic systems. Our result implies, in particular, that recent apparent evidence for Bose condensation of exciton-polaritons must be an intermediate scale crossover phenomenon, while the true long distance correlations fall off exponentially. We obtain these results through a mapping of the long-wavelength condensate dynamics onto the anisotropic Kardar-Parisi-Zhang equation.

One of the most striking discoveries to emerge from the study of non-equilibrium systems is that they sometimes exhibit ordered states that are impossible in their equilibrium counterparts. For example, it has been shown [1] that a two-dimensional “flock” - that is, a collection of moving, self-propelled entities - can develop long-ranged orientational order in the presence of finite noise (the non-equilibrium analog of temperature), and in the absence of both rotational symmetry breaking fields and long ranged interactions. In contrast, a two-dimensional equilibrium system with short-ranged interactions (e.g., a two-dimensional ferromagnet) cannot order at finite temperature; this is the Mermin-Wagner theorem [2].

In this paper, we report an example of the opposite phenomenon: A driven, two-dimensional Bose system, such as a gas of polariton excitations in a two-dimensional isotropic quantum well [3], cannot exhibit off-diagonal algebraic correlations (i.e., two-dimensional superfluidity). In the polariton gas, the departure from thermal equilibrium is due to the incoherent pumping needed to counteract the intrinsic losses and maintain a constant excitation density.

The critical properties of related driven quantum systems have been the subject of numerous theoretical studies; in certain cases it can be shown that the low frequency correlation functions induced by driving are identical to those in equilibrium systems at an effective temperature set by the driving [4–8]. Such emergent equilibrium behavior occurs in three dimensional bosonic systems, although non-equilibrium effects can change the dynamical critical behavior [8, 9]. Here, we show that the non-equilibrium conditions imposed by the driving have a much more dramatic effect on two-dimensional Bose systems: effective equilibrium is never established in the generic isotropic case; instead, the non-equilibrium nature of the fluctuations inevitably destroys the condensate at long scales.

This conclusion follows from the known [10–13] connection between the Complex Ginzburg-Landau equation (CGL) (which describes the long wavelength dynamics of a driven condensate) and the Kardar-Parisi-Zhang (KPZ) equation [14], or, in the anisotropic case, the anisotropic KPZ equation [15], which were originally formulated to describe randomly growing interfaces. The non-equilibrium fluctuations generated by the drive translate into the non-linear terms of the KPZ equation.

Our results suggest that recent experiments [16–19] done with isotropic semiconductor quantum wells purporting to show evidence for the long sought [20] Bose condensation of polariton excitations are in fact observing an intermediate length scale crossover phenomenon, and not the true long-distance behavior of correlations.
We remark that earlier work, which predicted long range algebraic order in two-dimensional driven condensates [21], relied on a linear (Bogoliubov) theory, which may appear on intermediate scales but, as our analysis shows, is invalidated at long distances due to the relevant non-linearity.

On the other hand, performing the same mapping on the anisotropic CGL leads, as noted by Grinstein et al. [12, 13], to the anisotropic KPZ equation. This suggests, as also noted by those authors, that algebraic order can prevail if the system is anisotropic. Then the transition to the disordered phase occurs by a standard equilibrium-like Kosterlitz-Thouless transition. This requires very strong anisotropy, which may seem unnatural in the case of exciton-polaritons in two dimensional quantum wells. However, mapping a realistic model of such a system to the anisotropic KPZ equation shows that the anisotropy of the KPZ non-linearities is a function of the driving laser power. Surprisingly, we find that even if the intrinsic anisotropy of the system is moderate, the effective anisotropy increases with pump power and eventually passes the threshold allowing for an effective equilibrium description. Then, not only does an algebraically ordered phase occur, but it does so in a reentrant manner: the phase is entered, and then left, as the driving laser power is increased.

Model – The dynamics of a driven-dissipative system like a polariton condensate, described by a complex scalar order parameter field \( \psi \), is determined by a set of equations for coherent and dissipative processes respectively. A model of the condensate dynamics that incorporates these processes is

\[
\partial_t \psi(x, t) = \frac{\delta H_d}{\delta \psi^*} + i \frac{\delta H_c}{\delta \psi^*} + \zeta(x, t). \tag{1}
\]

Here, the effective Hamiltonians \( H_\ell (\ell = c, d) \) that generate the coherent and dissipative dynamics respectively read

\[
H_\ell = \int_{x,y} \left[ r_\ell |\psi|^2 + K^x_\ell |\partial_x \psi|^2 + K^y_\ell |\partial_y \psi|^2 + \frac{1}{2} u_\ell |\psi|^4 \right]. \tag{2}
\]

The last term \( \zeta(x, t) \) in Eq. (1) is a zero mean Gaussian white noise with short-ranged spatiotemporal correlations: \( \langle \zeta^*(x, t) \zeta(x', t') \rangle = 2\sigma \delta^d(x - x') \delta(t - t') \), \( \langle \zeta(x, t) \zeta(x', t') \rangle = 0 \).

Eq. (1) is widely known as the complex Ginzburg-Landau equation [22, 23], or in the context of polariton condensates, as the dissipative Gross-Pitaevskii equation [24, 25], although usually only the isotropic (i.e., \( K^x = K^y \)), noise free (\( \zeta = 0 \)) case is considered (but see [26]). Modifications of this equation, e.g., including higher powers of \( \psi \) and \( \zeta \), higher derivatives, or combinations of the two, can readily be shown to be irrelevant in the Renormalization Group (RG) sense: they have no effect on the long-distance, long-time scaling properties of either the ordered phase, or the transition into it [27].

Each of the parameters appearing in the model has a clear physical origin, as we now review. The coefficient \( r_d \) is the single particle loss rate \( \gamma_l \) (spontaneous decay) offset by the pump rate \( \gamma_p \), that is, \( r_d = \gamma_l - \gamma_p \). In contrast, \( r_c \) is an effective chemical potential, which is completely arbitrary. Indeed, it can be adjusted by a temporally local gauge transformation \( \psi(x, t) = \psi'(x, t)e^{i\omega t} \), such that \( r'_c = r_c - \omega \). In the following, we choose \( r_c \) so that, in the absence of noise, the equation of motion has a stationary, spatially uniform solution.

The term proportional to \( u_c \) is the pseudo-potential which describes the elastic scattering of two polaritons, whereas \( u_d \) is the non-linear loss or, alternatively, a reduction of the pump rate with density, that ensures saturation of particle number. The coefficients \( K_{x,y} = \hbar^2/(2m_{x,y}) \), where \( m_{x,y} \) are the eigenvalues of the effective polariton mass tensor, with principal axes \( x, y \). Under typical circumstances, the diffusion-like term \( K_d \) is expected to be small, but is allowed by symmetry, and so will always be generated [28, 29]. Finally, the noise is given by the total rate of particles entering and leaving the system. In polariton condensates, where \( u_d \) reflects a non-linear reduction of the pumping rate rather than an additional loss mechanism (see supplementary information), the noise level at steady state is simply set by the single particle loss, i.e. \( 2\sigma = 2\gamma_l \). [30]

Before proceeding, it is important to clarify under what conditions Eq. (1) describes an effective thermal equilibrium at all wavelengths. Imposing the additional condition that the field follows a thermal Gibbs distribution at steady state translates to the simple requirement \( H_d = RH_c \), where \( R \) is a multiplicative constant [26, 31–33]. This condition can also be seen as a symmetry of the dynamics, which ensures detailed balance [8, 9]. In a driven system, the relation \( H_d = RH_c \) is not satisfied in general, because the dissipative and coherent parts of the dynamics are generated by independent processes. This relation can, however, arise as an emergent symmetry at low frequencies and long wavelengths. This was shown to be the case for a three-dimensional driven condensate [8, 9]. Below we shall derive the hydrodynamic long-wavelength description of a two-dimensional driven condensate and determine if it flows to effective thermal equilibrium.

Mapping to a KPZ equation – In the long-wavelength limit, Eq. (1) reduces to a KPZ equation [14] for the phase variable [12]. As in equilibrium, in a hydrodynamic description of the condensate the order-parameter field is written in the amplitude-phase representation as \( \psi(x, t) = (M_0 + \chi(x, t))e^{i\theta(x,t)} \). Integrating out the gapped amplitude mode, and keeping only terms which are not irrelevant in the sense of the renormalization group, we obtain a closed equation for \( \theta \) (see Method
\[
\partial_t \theta = D_x \partial_x^2 \theta + D_y \partial_y^2 \theta + \frac{\lambda_x}{2} (\partial_x \theta)^2 + \frac{\lambda_y}{2} (\partial_y \theta)^2 + \zeta(x, t),
\]
with \((\alpha = x, y)\):
\[
D_\alpha = K_\alpha^x \left[1 + \frac{K_\alpha^x u_c}{K_\alpha^y u_d}\right],
\]
\[
\lambda_\alpha = 2K_\alpha^x \left[\frac{K_\alpha^x u_c}{K_\alpha^y u_d} - 1\right],
\]
and noise level (replacing \(\sigma\) in the noise correlations above)
\[
\Delta = \frac{(a_x^2 + a_y^2)\gamma}{2u_d(\gamma_p - \gamma)}.
\]

Eq. (3) is the anisotropic KPZ equation, originally formulated to describe the roughness of a growing surface due to random deposition of particles on it \([14, 15]\), in which case \(\theta\) is the height of the interface. It reduces to the isotropic KPZ equation when \(D_x = D_y\) and \(\lambda_x = \lambda_y\). This reduction can also be achieved by a trivial rescaling of lengths if \(\Gamma \equiv \frac{\lambda_x D_x}{\lambda_y D_y} = 1\). Thus, when \(\Gamma \neq 1\), the system is anisotropic.

Crucially, the presence of the non-linearity directly reflects non-equilibrium conditions \([34]\). Indeed, the coefficients \(\lambda_x, \lambda_y\) that measure the deviation from thermal equilibrium vanish identically when the conditions
\[
K_x^x/K_x^y = K_y^x/K_y^y = u_c/u_d,
\]
which follow from the equilibrium requirement that \(H_d = RH_c\), are met.

It is furthermore important to note that our KPZ model differs from that formulated for a description of randomly growing interfaces \([14]\) in that the analog of the interface height variable in our model is actually a compact phase; hence, topological defects in this field are possible. This difference with the conventional KPZ equation also arises in “Active Smectics” \([35]\).

Analysis of Eq. (3) in absence of vortices is the analogue of the low temperature spin-wave (linear phase fluctuation) theory of the equilibrium XY model. Indeed, without the non-linear terms, the KPZ equation reduces to linear diffusion, which would bring the field to an effective thermal equilibrium with power-law off-diagonal correlations (in \(d = 2\)). A transition to the disordered phase in this equilibrium situation can occur only as a Kosterlitz-Thouless (KT) transition through proliferation of topological defects in the phase field.

In a driven condensate, the non-linear terms are in general present, and in two dimensions have the same canonical scaling dimension as the linear terms. A more careful RG analysis is therefore required to determine how the system behaves at long scales even without defect proliferation. Such an analysis has been done in Refs. \([15, 35]\) for the anisotropic KPZ equation. In this case, the flow is closed in the two parameter space of scaled non-linearity \(g \equiv \frac{\lambda_x^2 \Delta}{D_x^2 \sqrt{D_x D_y}}\) and scaled anisotropy \(\Gamma \equiv \frac{\lambda_x D_x}{\lambda_y D_y}\), and is given, to leading order in \(g\), by:
\[
\frac{dg}{dl} = \frac{g^2}{32\pi} (\Gamma^2 + 4\Gamma - 1),
\]
\[
\frac{d\Gamma}{dl} = \frac{g}{32\pi} (1 - \Gamma^2).
\]

These flows are illustrated in Fig. 1. We see that in an isotropic system, \(\Gamma = 1\), and the nonlinear coupling \(g\), which embodies the non-equilibrium fluctuations, is relevant. Moreover, for a wide range of anisotropies (namely, all \(\Gamma > 0\)) the flow is attracted to the isotropic line: the system flows to strong coupling, with emergent rotational symmetry. On the other hand, if the anisotropy is sufficiently strong, so that \(\Gamma < 0\), the non-linearity becomes irrelevant and the system can flow to an effective equilibrium state at long scales.

We will now discuss the physics of these two regimes, starting with the isotropic case, which is most relevant to current experiments with polariton condensates.

**Isotropic systems** – As noted above, rotational symmetry is emergent at long scales if the anisotropy is not too strong at the outset. This is also the regime in which current experimental quantum well polaritons lie. We therefore consider this case first.

On the line \(\Gamma = 1\), the scaling of the non-linear coupling \(dg/dl = g^2/8\pi\) drives \(g \rightarrow \infty\); in the growing surface problem the system goes to the “rough” state, with height fluctuations scaling algebraically with length. The analogous behavior in the phase field \(\theta\) would lead to stretched exponentially decaying order parameter corre-
FIG. 2. Dependence of the emergent KPZ length scale $L_*$ (in units of the microscopic healing length) on the tuning parameter $x = \gamma_p/\gamma_l - 1$. This curve was obtained by inserting the expression Eq. (9) for the bare coupling $g_0$ into Eq. (8) for $L_*$. While $L_*$ is exponentially large when $x > \bar{u}^2\pi/2\pi$, it goes to a microscopic value $\xi_0$ at the mean field transition $x = 0^+$. The shaded region marks the scales at which a system would exhibit algebraic correlations. Upon decreasing the tuning parameter $x$, a finite system will lose its algebraic order in one of two ways: (1) When $L_*$ falls below system size, as in the case of system $L_1$ shown, or (2) in a KT transition before $L_*$ falls below system size, as in the case of system $L_2$. Here we have used $\bar{\gamma} = 0.7$, $\bar{u} = 0.5$.

lations. However, the fact that the phase field is compact implies that topological defects (vortices) in this field exist. Our expectation, based on analogy with equilibrium physics (which admittedly may be an untrustworthy analogy), is that vortices will unbind at the strong coupling fixed point of the KPZ equation. If this happens, it will lead to simple exponential correlations. Testing this expectation will be the object of future work.

We have thus established that the non-linearity, no matter how weak, destroys the condensate at long distances, leading to either stretched or simple exponential decay of correlations throughout the isotropic regime. However, the effects of the nonlinearity only become apparent when $g$ gets to be of order one. Solving the scaling equation we see that this occurs at the characteristic “RG time” $l_\ast = 8\pi/g_0$; the corresponding length scale is:

$$L_* = \xi_0 e^{l_\ast} = \xi_0 e^{8\pi/g_0},$$

where $\xi_0$ is the mean field healing length of the condensate. If the bare value of $g_0$ is small, then the scale $L_*$ can be huge. On length scales smaller than $L_*$, the system is governed by the linearized isotropic KPZ equation, which, as noted earlier, is the same as an equilibrium XY model. Thus, all of the equilibrium physics associated with two-dimensional BEC, including power law correlations and a Kosterlitz-Thouless defect unbinding transition, can appear in a sufficiently small system.

As parameters, such as the pump power, are changed, the system can lose its apparent algebraic order in one of two ways: (i) the KPZ length $L_*$ is gradually reduced below the system size, or (ii) $L_*$ remains large while the correlations within the system size $L$ are destroyed by unbinding of vortex anti-vortex pairs at the scale $L$. The latter type of crossover would appear as a KT transition broadened by the finite size. Of course, for any given set of system parameters, a sufficiently large system ($L > L_*$) will always be disordered.

We shall now discuss how the system parameters determine what type of crossover, if any, will be seen in an experiment. We assume that the main tuning parameter is the pump power $\gamma_p$, and it will be convenient to track the behavior as a function of a dimensionless tuning parameter $x = \gamma_p/\gamma_l - 1$, and set $K_d = 0$ since this parameter is thought to be small in current experimental realizations. In the supplementary material, we derive the parameters of the KPZ equation for a realistic model of a polariton condensate, described by a two fluid model which includes an upper polariton band, into which polaritons are pumped, and a lower band, where the condensate forms. In particular, we obtain an expression for the bare dimensionless coupling constant $g_0$ in this model, which measures the bare deviation from equilibrium:

$$g_0 = \frac{\Delta A^2}{D^3} = 2\bar{u}^2\bar{\gamma}^2 \left( \frac{\bar{\gamma}^2 + (1 + x)^2}{x(1 + x)^3} \right).$$

Here $\bar{u} \equiv u_c/K_c$ is the dimensionless interaction constant, and $\bar{\gamma} \sim \gamma_l$ the dimensionless loss rate (see supplementary material). Note that $g_0$ diverges as we approach the mean field transition at $x \to 0^+$, while it decays as $1/x^2$ as $x \to \infty$ at very high pump power.

Hence, in the latter regime the KPZ length scale $L_* = \xi_0 \exp(8\pi/g_0)$ is certainly much larger than any reasonable system size. As the pump power is decreased, and the system approaches the mean field transition at $x = 0$, $L_*$ drops sharply to a microscopic healing length $L^\ast \approx \xi_0$. $L_*$ drops below the system size when $x \lesssim x_\ast$, where

$$x_\ast(1 + x_\ast)^3 \approx \frac{\bar{u}^2\bar{\gamma}^2}{4\pi} \ln(L/\xi_0).$$

For pump powers corresponding to $x > x_\ast$, the system will appear to be at effective equilibrium, and, hence, may sustain power law order within its confines, whereas for pump power $x < x_\ast$, the non-equilibrium fluctuations become effective and destroy the algebraic correlations at the scale of the system size. However, it is possible that this crossover at $x_\ast$ is preceded by unbinding of vortices at values of $x = x_{KT} > x_\ast$, while the finite size is still at effective equilibrium.

To determine which crossover occurs in a particular system, let us estimate the value of the tuning parameter $x_{KT}$ at which the putative Kosterlitz-Thouless transition would occur if the non-linear term $\lambda$ vanished, or was
vortex unbinding is controlled by the (bare) parameters equivalent to an equilibrium to flow to the linear regime, which, as noted earlier, is fact that they were derived neglecting vortices. recursion relations Eq. (7) for our problem, despite the to nearly zero). If this is the case, then we can use the crossover controlled by vortex unbinding through the KT mechanism, i.e., \( x_{KT} > x_\ast \), if the system size is \( L < \xi_0 \exp(2/\gamma^2) \). For larger system size the crossover will be controlled by the nonlinearities of the KPZ equation. This crossover behavior is summarized in Fig. 2.

**Strong anisotropy** – If the bare value of the anisotropy parameter is negative \( \Gamma < 0 \), then the RG equations (7) lead to a fixed point at \( g = 0 \). Because the non-linear \( \lambda_{x,y} \) terms in (3) are irrelevant in this region of parameter space, the linear (and, hence, equilibrium) version of the theory applies. Hence it is possible, for \( \Gamma < 0 \), to obtain both a power law phase and a KT defect unbinding transition out of it.

To estimate the extent of this phase, we can utilize the RG flow of the anisotropic KPZ equation for \( \Gamma < 0 \) analyzed in Ref. [35]. In principle, we should add to these recursion relations terms coming from the vortices. Instead, we will follow reference [35] and assume that the vortex density is low enough that vortices only become important on length scales far longer than those at which the nonlinear effects have become unimportant (i.e., those at which the scaled non-linearity \( g \) has flowed to nearly zero). If this is the case, then we can use the recursion relations Eq. (7) for our problem, despite the fact that they were derived neglecting vortices.

Our strategy is then to use those recursion relations to flow to the linear regime, which, as noted earlier, is equivalent to an equilibrium XY model. In this regime, vortex unbinding is controlled by the (bare) parameters \( \kappa_0 \equiv \Delta/\sqrt{D_x D_y} \) (cf. Eq. (5), or (24) in the supplementary material for the two-band model), giving the scaled noise level and replacing the temperature of the equilibrium problem as above, as well as the scaled anisotropy \( \Gamma_0 \). Following [35], the phase boundary in the \( \kappa_0 - \Gamma_0 \) plane is then a locus in the plane of bare scaled noise and anisotropy parameters given by [35]

\[
\kappa_0 = \frac{4\pi \Gamma_0}{(1 - \Gamma_0)^2},
\]

see the supplementary materials for more details.

There is a broad range of parameters for which a system enters a regime \( \Gamma_0 = (D_x \lambda_y)/(D_y \lambda_x) < 0 \), in which true power-law order and a KT transition exist. Within the “two-band polariton model” (see supplementary material), as a function of microscopic parameters we obtain

\[
\Gamma_0 = \frac{[\nu_y (1 + x) - \gamma] [\nu_x \gamma + 1 + x]}{[\nu_x (1 + x) - \gamma] [\nu_y \gamma + 1 + x]},
\]

\[
\kappa_0 = \frac{\bar{\nu}}{2x} \frac{[\gamma^2 + (1 + x)^2]}{\sqrt{[\nu_y \gamma + (1 + x)] [\nu_x \gamma + (1 + x)]}}.
\]

with the ratios of the dissipative to coherent phase stiffnesses along the two directions, \( \nu_\alpha = K_\alpha^D/K_\alpha^c \). Now consider gradually increasing the pump power, and hence \( x \), from the mean field threshold \( x = 0 \). For system parameters \( \gamma > \nu_y > \nu_x \), \( \Gamma_0(0) \) starts out positive at \( x = 0 \), is reduced to negative values as \( x \) is increased past \( x = \frac{\gamma^2}{\nu_y^2} - 1 \), and eventually runs off to \( \Gamma_0 = -\infty \) at a finite value of \( x \) (namely, \( x = \frac{\gamma^2}{\nu_y^2} - 1 \)). If at the same time \( \bar{\nu} \) is sufficiently small, then the experimental trajectory in the \( \kappa_0 - \Gamma_0 \) plane is guaranteed to cross the dome marking the condensate (algebraic order) phase as determined in Eq. (29). The condition on \( \bar{\nu} \) for this crossing to occur is

\[
\bar{\nu} < 2\pi \frac{(\gamma - \nu_y)}{\gamma \nu_y (1 + \nu_y^2)}.
\]

Thus, we not only naturally achieve the ordered phase in this anisotropic system by varying the driving, but we do so in a reentrant manner: we enter the phase, and then leave it again, as the driving is increased. The analysis for \( \gamma > \nu_y > \nu_x \) is the same if we take \( \Gamma \to 1/\Gamma \).

**Outlook** – Our analysis can be extended to three dimensions. There, for weak deviations from equilibrium, i.e.
a small bare value of the KPZ non-linearity, it predicts a true Bose condensate which may be established through the dynamical phase transition described in [8]. However, beyond a critical strength of the equilibrium deviation, one may also encounter a different, non-equilibrium transition controlled by a strong coupling fixed point of the three dimensional KPZ equation [36]. This opens up the possibility for a new non-equilibrium phase of matter with short-ranged order, distinct from the usual uncondensed state in that vortex loops do not proliferate. This will be explored in future work.

**METHODS**

Here we review the mapping [12], in the long-wavelength limit, between the model (1) and an anisotropic KPZ equation [14, 15]. We work in the amplitude-phase representation \( \psi(x, t) = (M_0 + \chi(x, t)) e^{i\theta(x, t)} \), with \( M_0, \chi \), and \( \theta \) all real. Here \( M_0 \) is determined by requiring that \( \chi = 0, \theta = 0 \) is a static uniform solution of Eq. (1) in the absence of fluctuations \( \langle \zeta(x, t) \rangle = 0 \). The real and imaginary parts of Eq. (1) then give \( M_0^2 = -r_d/u_d \) and \( c_c = -u_c M_0^2 \), respectively. We can satisfy the second condition by exploiting the freedom to choose \( r_c \) mentioned in the main text. As explained there, by varying the strength of the pump laser, one can experimentally control \( r_d \), which determines the amplitude \( M_0 \). The mean field transition occurs at the point \( r_d = 0 \) (i.e., when \( \gamma_p = \gamma_l \)), where the amplitude \( M_0 \) vanishes. For later convenience we define the dimensionless tuning parameter \( x = \gamma_l/\gamma_l - 1 \).

Plugging the amplitude-phase representation of \( \psi \) into Eq. (1), and linearizing in the amplitude fluctuations \( \chi \), we obtain the pair of equations

\[
\partial_t \chi = -2u_d M_0^2 \chi - K_x^2 M_0 \partial_x^2 \theta - K_y^2 M_0 \partial_y^2 \theta - K_d M_0 (\partial_x \theta)^2 - K_y M_0 (\partial_y \theta)^2 + \text{Re} \zeta, \\
M_0 \partial_t \theta = -2u_c M_0^2 \chi + K_x^2 M_0 \partial_x^2 \theta + K_y^2 M_0 \partial_y^2 \theta - K_x^2 M_0 (\partial_x \theta)^2 - K_y^2 M_0 (\partial_y \theta)^2 + \text{Im} \zeta,
\]

where we have used the freedom discussed earlier to choose \( r_c = -u_c M_0^2 \) to simplify this expression.

Note that if we have no dissipation \( (H_d = 0) \), so that \( u_d = 0 \), both \( \chi \) and \( \theta \) are “slow” variables, in the sense of evolving at rates that vanish as the wavevector goes to zero. In this case we can substitute Eq. (16) into the time derivative of Eq. (17) to obtain a wave equation for \( \theta \) supplemented by irrelevant non-linear corrections. This gives the linear dispersion of the undamped Goldstone modes characteristic of a massless condensate with exact particle number conservation. In contrast, without particle number conservation (i.e., in the presence of loss and drive), \( u_d \neq 0 \), and we can therefore neglect the \( \partial_t \chi \) term (which vanishes as frequency \( \omega \to 0 \)) on the left hand side of Eq. (16) relative to the \( 2u_d M_0^2 \chi \) on the right hand side for any “hydrodynamic mode” (i.e., in the low frequency limit). Doing so turns Eq. (16) into a simple linear algebraic equation relating \( \chi \) to spatial derivatives of \( \theta \). Substituting the solution for \( \chi \) of this equation into Eq. (17) gives Eq. (3), a closed equation for \( \theta \). The noise variable in that equation is related to the original noise through \( \zeta = (\text{Im} \zeta - u_c \text{Re} \zeta)/u_d \), and hence \( \langle \zeta(x, t) \zeta(x', t') \rangle = 2\Delta \delta^2(x-x') \delta(t-t') \) with \( \Delta \) given in Eq. (5). The stochastic equation for \( \theta \) includes all terms that are marginal and relevant by canonical power-counting, while neglecting irrelevant terms like \( \partial_t^2 \theta, \partial_t \nabla \theta, \) and \( \delta_t (\nabla \theta)^2 \).

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Supplementary Material

Polariton condensate model with reservoir

In the main text we worked with a Ginzburg-Landau model including only the lower polariton band. Such a model clearly gives the correct universal physics. However, in order to find how the parameters of the anisotropic KPZ equation change as actual experimental parameters are varied requires to start from a more microscopic model of the polariton degrees of freedom.

The standard model for describing these systems is a “two fluid” model which includes the particles in the “upper-polariton band” acting as a reservoir with local density \( n_R \) for the condensate which forms in the “lower polariton band” \([3]\). Here we generalize the model slightly in order to include dissipative mass terms and anisotropy:

\[
\partial_t \psi = \sum_{\alpha=x,y} (iK_c^\alpha + K_d^\alpha) \partial^2_{\alpha} - ir_c - \gamma_l - iu_c|\psi|^2 + Rn_R \psi + \zeta, \\
\partial_t n_R = P - Rn_R|\psi|^2 - \gamma_R n_R, 
\]

where \( \langle \zeta^*(x,t)\zeta(x',t') \rangle = 2\sigma\delta(x - x')\delta(t - t') \). It is usually assumed that the reservoir relaxation time \( \gamma_R \) is faster than all other scales. Hence we may solve the reservoir density independently assuming it is time independent

\[
\partial_t \psi = \sum_{\alpha} (iK_c^\alpha + K_d^\alpha) \partial^2_{\alpha} - ir_c - \gamma_l - iu_c|\psi|^2 + \frac{P}{\eta + |\psi|^2} \psi + \zeta, 
\]

where we have eliminated \( R \) and \( \gamma_R \) for the single parameter \( \eta = \gamma_R/R \). We note that the amplitude of the white noise is given by the total loss rate (\( \gamma_l \)) and gain, and since in steady state the loss and gain must be equal we simply have \( \sigma = \gamma_l \) in this case.

In the following, as in the main text we work in the phase-amplitude representation \( \psi(x,t) = (M_0 + \chi(x,t))e^{i\theta(x,t)} \) and expand around the homogeneous mean field solution. Let us therefore first solve for the mean field steady state. The real part of the equation gives \( \gamma_l = P/(\eta + M_0^2) \) from which we can deduce the condensate density \( M_0^2 = P/\gamma_l - \eta \). The imaginary part of the equation is \( r_c = -u_c M_0^2 \). It is also worth noting that loss comes only from the term \( \gamma_l \), since there is no two-particle loss term in this model (instead saturation is reached due to the non-linear reduction of the pump term). Hence in steady state, when loss is equal to gain, the noise term is simply \( \sigma = \gamma_l \).

We now proceed to write the equations of motion for \( \chi \) and \( \theta \) to linear order in \( \chi \). This gives

\[
M_0^{-1}\partial_t \chi = -2\gamma_l^2 P^{-1}M_0 \chi - K_c^c \partial^2_{\alpha}\theta - K_d^\alpha (\partial_{\alpha}\theta)^2 + M_0^{-1}\text{Re}\zeta, \\
\partial_t \theta = -2u_c \chi + K_d^\alpha \partial^2_{\alpha}\theta - K_c^\alpha (\partial_{\alpha}\theta)^2 - M_0^{-1}\text{Im}\zeta. 
\]

Now as in the main text we can eliminate \( \chi \) to obtain the KPZ equation for \( \theta \), where \( \alpha = x, y \) is summed over and

\[
\partial_t \theta = D^\alpha \partial^2_{\alpha}\theta + \frac{1}{2} \lambda^\alpha (\partial_{\alpha}\theta)^2 + \tilde{\zeta}, 
\]

where

\[
\tilde{\zeta} = M_0^{-1} \left( \text{Re}\zeta - \frac{u_c P}{\gamma_l^2} \text{Im}\zeta \right). 
\]

The noise parameter in \( \langle \tilde{\zeta}^*(x,t)\tilde{\zeta}(x',t') \rangle = 2\Delta\delta(x -
The main text corresponds to this special case and the dimensionless interaction strength $\bar{\nu}$ and the ratios $\nu = K_c(1+x)/2\bar{\gamma}$ and $\lambda = -K_c$. The dimensionless coupling constant $g$ is then given by

$$g = \frac{\Delta \lambda^2}{D^3} = 2\bar{u}\gamma^2 \left( \frac{\bar{\gamma}^2 + (1+x)^2}{x(1+x)^3} \right). \quad (27)$$

From the expression for $g$ we can extract the dependence of the KPZ length on the tuning parameter:

$$\log(L_*/\xi_0) = \left( \frac{4\pi}{\bar{u}\gamma^2} \right) \frac{x(1+x)^3}{\bar{\gamma}^2 + (1+x)^2}. \quad (28)$$

### Crossover scales in isotropic polariton condensates

We will now use the results just presented for the isotropic case without dissipative mass terms; i.e., $\nu_x = \nu_y = 0$ (for the anisotropic case, see main text). This implies $D = K_c(1+x)/2\bar{\gamma}$ and $\lambda = -K_c$. The dimensionless coupling constant $g$ is then given by

$$D_x = K_c \left( \frac{K_c^3 + u_x P}{K_c^2 \gamma_i^2} \right) = K_c \left( \nu^0 + \frac{1+x}{\gamma} \right), \quad (25)$$

$$\lambda = 2K_c \left( \frac{K_c^2 u_x P}{K_c^2 \gamma_i^2} - 1 \right) = 2K_c \left( \nu^0 \frac{1+x}{\gamma} - 1 \right).$$

In order to make contact to the main text, we note that the expressions for the diffusion constants $D_x$ and non-linear coefficients $\lambda$ can be obtained from the predictions Eq. (25) for the Ginzburg-Landau model Eq. (1), if we make the replacement

$$u_d = \frac{\gamma_i^2}{P} = \frac{u_c \bar{\gamma}}{1+x}. \quad (26)$$

The parameter $K_d$ is thought to be small in isotropic two-dimensional quantum wells. If we take $K_d = 0$, then $D = K_c u_c/u_d = K_c(1+x)/\bar{\gamma}$, and $\lambda = -2K_c$. Eq. (9) in the main text corresponds to this special case $K_d = 0$.

### Critical locus in the anisotropic case.

The critical point for vortex unbinding can be estimated by solving for the renormalized scaled noise $\nu_\ell = \nu_0 = 0$ as a function of the bare value using the RG equations of the non-compact KPZ equation; this involves additional recursion relations for $D_x$ and $\Delta$ as well as Eq. (7); for details see reference [35]. This analysis gives $\kappa(\infty) = -\kappa_0(1-\Gamma_0)^2/(4\Gamma_0)$. The KT transition occurs at the point where this renormalized value $\kappa(\infty)$ of $\kappa$ reaches $\pi$. Hence the phase boundary in the $\kappa_0$-$\Gamma_0$ plane is then a locus in the plane of bare scaled noise and anisotropy parameters given by [35]

$$\kappa_0 = \frac{-4\pi \Gamma_0}{(1-\Gamma_0)^2}. \quad (29)$$