A CO-VARIANT APPROACH TO ASHTEKAR’S
CANONICAL GRAVITY

Brian P. Dolan and Kevin P. Haugh

Department of Mathematical Physics, St. Patrick’s College
Maynooth, Ireland

e-mail: bdolan@thphys.may.ie, khaugh@thphys.may.ie

ABSTRACT
A Lorentz and general co-ordinate co-variant form of canonical gravity, using Ashtekar’s
variables, is investigated. A co-variant treatment due to Crnkovic and Witten is used,
in which a point in phase space represents a solution of the equations of motion and a
symplectic functional two form is constructed which is Lorentz and general co-ordinate
invariant. The subtleties and difficulties due to the complex nature of Ashtekar’s variables
are addressed and resolved.

PACS Nos. 04.20.Cv 04.20.Fy
1. Introduction

In 1986, Abhay Ashtekar [1] discovered a set of canonical variables for the gravitational field as described by the general theory of relativity. Ashtekar found that they led to a considerable simplification of the constraints associated with the Hamiltonian formulation of Einstein’s theory. Indeed, Ashtekar’s constraints are polynomials in the canonical variables. Ashtekar’s canonical gravity is definite progress in the direction of a quantum theory of gravity since it gives rise to a closed constraint algebra [2].

Hamiltonian models of physical phenomena have always distinguished between time and space. The Hamiltonian of a dynamical system generates time translations, that is to say it determines the time evolution of the dynamical variables. Relativity regards time and space as being components of a single entity: space-time. An equation, describing the way a physical quantity changes with time, does not look the same to all relativistic observers. In other words, an equation of this kind is not co-variant. It is usual to develop the Hamilton mechanics of a relativistic field by specifying a space-time foliated by space-like hyper-surfaces of constant time, and a Hamiltonian functional on this space-time. However, this approach spoils co-variance from the beginning because a time co-ordinate must be singled out, in order for the required foliation to make sense [3].

One way of viewing the role of canonical variables is that their initial values determine a solution of the Hamilton equations. In other words, there is a one-to-one correspondence between the canonical variables at any time, t, and the canonical variables initially [4]. Thus we can describe the phase space as the set of solutions of the Hamilton equations of motion. For a field theory, a knowledge of the initial canonical variables requires a knowledge of the field configuration and its time derivatives on a space-like hyper-surface, and a point in phase space is a solution of the Hamilton equations at a given time. The object of this paper is to describe Ashtekar’s gravity in a manifestly co-variant way. One possible way of achieving this goal is to use a simple construction due to Crnkovic and Witten.

The essence of the Crnkovic-Witten construction is the observation that a co-variant theory must have an invariant symplectic form, and that each point in phase space rep-
resents a solution of the equations of motion. One can thus dispense with the Hamiltonian, and focus on the symplectic structure and the points of phase space as providing a co-variant description of the dynamics in phase space. This idea has been successfully applied by Crnkovic and Witten [5] to the Yang-Mills field and to general relativity (using the 3-metric and the extrinsic curvature as canonical variables) where there is an additional complication due to gauge invariance.

It is not immediately obvious how to implement the Crnkovic-Witten construction in the framework of Ashtekar’s canonical gravity. In particular, the complex nature of the canonical variables leads to difficulties which will be addressed here. It will be shown that these difficulties can be overcome, and the Crnkovic-Witten construction can be applied successfully to give a co-variant version of Ashtekar’s theory.

2. Ashtekar’s Canonical Gravity

In this section, we shall review Ashtekar’s Hamiltonian formulation with a view to establishing our notation and conventions. Ashtekar’s canonical variables are the inverse densitized triads, $E^{ai}$, and the Ashtekar connection, $A_{ai}$, defined on a space-like hypersurface, $\Sigma_t$, of constant time, $t$. (Ashtekar’s canonical variables can also be defined on a null hyper-surface [6].) Here $a$ and $i$ are orthonormal and co-ordinate indices respectively, ranging from 1 to 3. The metric signature is $-+++$, and the completely anti-symmetric Levi-Civita tensor is taken to be $\varepsilon_{0123} = 1$. For a space-like foliation, a set of orthonormal 1-forms is given by

$$
e^0 = N \, dt, \quad e^a = h^a_i N^i \, dt + h^a_i \, dx^i,$$

(1)

where $N$ and $N^i$ are the lapse and shift functions respectively. The dual basis vectors are

$$\beta_0 = \frac{1}{N} \left( \frac{\partial}{\partial t} - N^i \frac{\partial}{\partial x^i} \right), \quad \beta_a = (h^{-1})^i_a \frac{\partial}{\partial x^i}.$$

(2)

Each $\beta_a$ is space-like, and the normal $\beta_0$ is time-like. The densitized triads are defined by

$$(E^{-1})_{ai} = \frac{1}{h} h_{ai},$$

(3)
where \( h \) is the determinant of the matrix, \([h_{ai}]\). The densitized triads are real-valued on any co-ordinate patch provided that \( h^0_{\,\,i} = 0 \) [7]. This is the *time gauge* condition, which can be relaxed by allowing the densitized triads to become complex-valued (see the appendix). The local group of local tangent space rotations, that preserves the time gauge condition, is the rotation group, \( SO(3) \). The *inverse densitized triads*, \( E^{ai} \), satisfy

\[
E^{ai}(E^{-1})_{aj} = \delta^i_j. \tag{4}
\]

Let \( E \) be the determinant of the matrix, \([E^{ai}]\). Now

\[
E^{ai} = h(h^{-1})^{ai}, \quad E = h^2. \tag{5}
\]

We record the useful relations:

\[
h_{ai} = \sqrt{E}(E^{-1})_{ai}, \quad (h^{-1})^{ai} = \frac{E^{ai}}{\sqrt{E}}. \tag{6}
\]

The torsion-free, metric-compatible connection 1-forms \( \omega_{AB} \) are given by

\[
\omega_{AB} = \frac{1}{2} \left[ (i_A i_B d e_C)e^C - i_A d e_B + i_B d e_A \right], \tag{7}
\]

where \( A, B, \ldots \) range from 0 to 3, \( i_A \) is the interior derivative along \( \beta_A \), and \( d \) is the exterior derivative. The *Ashtekar connection*, \( A_{ai} \), can then be defined in terms of the components of the connection 1-forms, \( \omega_{AB} \):

\[
A_{ai} = \omega_{0ai} - \frac{i}{2} \varepsilon_{abc} \omega^{bc}_i. \tag{8}
\]

The curvature 2-forms, \( R_{AB} \), are given by

\[
R_{AB} = d\omega_{AB} + \omega_{AC} \wedge \omega^C_B, \quad R_{BA} = -R_{AB}. \tag{9}
\]

They satisfy the Hodge duality relations:

\[
* R_{AB} = \frac{1}{2} \varepsilon_{ABCD} R^{CD}, \quad ** R_{AB} = -R_{AB}. \tag{10}
\]

The self-dual curvature 2-forms, \( ^+ R_{AB} \), are then given by
\[ +R_{AB} = R_{AB} - i \star R_{AB}, \quad \star +R_{AB} = i +R_{AB}. \quad (11) \]

In the absence of torsion, we have the identity:
\[ R_{AB} \wedge e^B = 0. \quad (12) \]

Thus, for a vacuum gravitational field, the action density 4-form can be written:
\[ \mathcal{L}d^4x = \frac{1}{2} R_{AB} \wedge \star e^{AB} \]
\[ = \frac{1}{2} \star R_{AB} \wedge e^{AB} + \frac{i}{2} R_{AB} \wedge e^{AB} \]
\[ = \frac{i}{2} +R_{AB} \wedge e^{AB}, \quad (13) \]

where \( e^{AB} \) stands for \( e^A \wedge e^B \). Now (11) tells us
\[ +R^{bc} = i \varepsilon^{abc} + R_{0a}, \quad (14) \]

and therefore,
\[ \mathcal{L}d^4x = i +R_{0a} \wedge \left( e^{0a} + \frac{i}{2} \varepsilon^{abc} e_{bc} \right). \quad (15) \]

Writing
\[ F_a = +R_{0a}, \quad \Lambda^a = e^{0a} + \frac{i}{2} \varepsilon^{abc} e_{bc}; \quad (16) \]

we have
\[ \mathcal{L}d^4x = i F_a \wedge \Lambda^a. \quad (17) \]

It is straightforward to obtain the important relations:
\[ F_a = \frac{1}{2} F_{a\mu} dx^\mu \wedge dx^\nu \]
\[ = \left( \dot{A}_{ai} - \partial_i A^0_a - i \varepsilon_{abc} A^b_0 A^c_i \right) dt \wedge dx^i \]
\[ + \frac{1}{2} \left( \partial_i A_{aj} - \partial_j A_{ai} - i \varepsilon_{abc} A^b_i A^c_j \right) dx^i \wedge dx^j, \quad (18) \]
\[ \Lambda^a = \frac{1}{2} \Lambda^a_{\mu\nu} dx^\mu \wedge dx^\nu \]
with the help of (1), (6), (8), and (11). Here, we have used the symbol \( N \) for \( N/\sqrt{E} \), and the Greek letters \( \mu, \nu, \ldots \) for co-ordinate indices, ranging from 0 to 3. It follows that

\[
\mathcal{L} d^4 x = \left[ A_{a_0} \left( \partial_i E^{a_0} - i \varepsilon^{abc} A_{b_0} E_c^i \right) \right.
\]

\[
+ \left. \frac{i}{2} N \varepsilon^{abc} F_{aij} E_b^i E_c^j + N F_{aij} E^{aj} \right] dt \wedge d^3 x.
\]

Thus we see that the Ashtekar connection, \( A_{a_0} \), are the momenta conjugate to the inverse densitized triads, \( E^{ai} \), and that Ashtekar’s Hamiltonian is

\[
H = \int_{\Sigma_t} d^3 x \left[ A_{a_0} \left( \partial_i E^{a_0} - i \varepsilon^{abc} A_{b_0} E_c^i \right) - \frac{i}{2} N \varepsilon^{abc} F_{aij} E_b^i E_c^j - N F_{aij} E^{aj} \right],
\]

for a space-like hyper-surface, \( \Sigma_t \). This is the form of the Hamiltonian given in [8].

The general theory of relativity has a phase space structure analogous to that of the \( SU(2) \) Yang-Mills field, where local \( SO(3) \) tangent space rotations, or more generally, Lorentz transformations play the role of gauge transformations in Yang-Mills theory. General co-ordinate invariance and Lorentz invariance require the introduction of redundant canonical variables. This leads to constraints expressing the resulting interdependence of the canonical variables.

In Ashtekar’s formulation, the constraints take the polynomial form:

\[
\frac{\delta H}{\delta A_{a_0}} = \partial_i E^{a_0} - i \varepsilon^{abc} A_{b_0} E_c^i = 0,
\]

\[
\frac{\delta H}{\delta N} = -i \varepsilon^{abc} F_{aij} E_b^i E_c^j = 0,
\]

\[
\frac{\delta H}{\delta N^i} = -F_{aij} E^{aj} = 0,
\]

everywhere on the hyper-surface, \( \Sigma_t \).

Ashtekar [9] has shown that these secondary constraints are first-class. We see that the Hamiltonian is a linear combination of the constraints. It is therefore first-class and
weakly zero. It is important to recall that the Yang-Mills Hamiltonian is not weakly zero in general. This reflects a dynamical difference between the Yang-Mills field and the gravitational field.

3. Crnkovic-Witten Theory

Let us review the Crnkovic-Witten construction in the case of the scalar field. We begin with the action of the scalar field in flat space-time:

\[ S = \int_M d^4x \ L, \]
\[ L = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - V(\phi)). \]  
(23)

Crnkovic and Witten’s idea involves the introduction of a *symplectic current* at each space-time point, \( x \):

\[ J_\mu(x) = \delta \left( \frac{\delta L}{\delta (\partial_\mu \phi)} \right) \wedge \delta \phi(x) = \delta \partial_\mu \phi(x) \wedge \delta \phi(x), \]  
(24)

where \( \delta \) stands for the functional exterior derivative of forms on the phase space of the scalar field [5]. Now

\[ \delta J_\mu(x) = \delta (\delta \partial_\mu \phi(x)) \wedge \delta \phi(x) - \delta \partial_\mu \phi(x) \wedge \delta (\delta \phi(x)) \]
\[ = \partial_\mu \delta (\delta \phi(x)) = 0. \]  
(25)

This means that \( J_\mu \) is closed as a functional 2-form. Further,

\[ \partial^\mu J_\mu(x) = \delta (\partial^\mu \partial_\mu \phi(x)) \wedge \delta \phi(x) + \delta \partial_\mu \phi(x) \wedge \delta \partial^\mu \phi(x) \]
\[ = -V''(\phi) \delta \phi(x) \wedge \delta \phi(x) + \delta \partial_\mu \phi(x) \wedge \delta \partial^\mu \phi(x) \]
\[ = -\delta \partial_\mu \phi(x) \wedge \delta \partial^\mu \phi(x) = 0, \]  
(26)

with the help of the equation of motion

\[ \partial_\mu \partial^\mu \phi + V'(\phi) = 0. \]  
(27)
Stokes’ theorem then implies that

\[ \int_N d^4x \, \partial^\mu J_\mu(x) = \int_{\partial N} d\sigma^\mu(x) J_\mu(x) = 0, \tag{28} \]

where \( N \) is a sub-manifold of \( M \) with boundary, \( \partial N \). Suppose \( \partial N = \Sigma_{t_1} \cup \Sigma_{t_2} \cup \Sigma \), where \( \Sigma_{t_1}, \Sigma_{t_2} \) are space-like hyper-surfaces of constant time, and \( d\sigma^\mu J_\mu \) vanishes everywhere on the hyper-surface, \( \Sigma \). Then

\[ \int_{\Sigma_{t_1}} d\sigma^\mu(x) J_\mu(x) = \int_{\Sigma_{t_2}} d\sigma^\mu(x) J_\mu(x), \tag{29} \]

where \( d\sigma^\mu \) is chosen to point in the same temporal direction on both \( \Sigma_{t_1} \) and \( \Sigma_{t_2} \). This means that the closed functional 2-form

\[ \Omega = \int_{\Sigma_t} d\sigma^\mu(x) J_\mu(x) \]

\[ = \int_{\Sigma_t} d\sigma^\mu(x) \delta \partial_\mu \phi(x) \wedge \delta \phi(x) \tag{30} \]

is independent of the choice of \( \Sigma_t \). When we perform a Lorentz transformation \( \Sigma_t \rightarrow \Sigma_{t'} \) and \( \Omega \rightarrow \Omega' \), where

\[ \Omega' = \int_{\Sigma_{t'}} d\sigma^\mu(x') J_\mu(x') = \int_{\Sigma_t} d\sigma^\mu(x) J_\mu(x) = \Omega. \tag{31} \]

We conclude that \( \Omega \) is a Lorentz invariant symplectic form on the phase space of the scalar field, and that it is possible to formulate the Hamiltonian theory of the scalar field in a manifestly co-variant way. The Lorentz invariance of the symplectic form allows us to choose a space-like hyper-surface, \( \Sigma_t \), such that

\[ d\sigma^0(x) = d^3x, \quad d\sigma^i(x) = 0 \tag{32} \]

for all \( x \in \Sigma_t \), and hence, we obtain the standard symplectic form

\[ \Omega = \int_{\Sigma_t} d^3x \, \delta \dot{\phi}(x) \wedge \delta \phi(x) = \int_{\Sigma_t} d^3x \, \delta \left( \frac{\delta L}{\delta \dot{\phi}(x)} \right) \wedge \delta \phi(x), \tag{33} \]
where \( \dot{\phi} \) is the momentum canonically conjugate to \( \phi \).

Next we consider the construction of a Lorentz invariant and gauge invariant symplectic form on the phase space of the \( SU(2) \) Yang-Mills field, \( A_\mu \), in flat space-time. In this case, the action is

\[
S = -\frac{1}{4} \int_M d^4x \ tr(F_{\mu\nu}F^{\mu\nu}),
\]

where

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{34}
\]

The symplectic current is taken to be

\[
J_\mu = tr \left( \delta \left( \frac{\delta L}{\delta (\partial_\mu A_\nu)} \right) \right) \wedge \delta A^\nu = tr \ \delta F_{\mu\nu} \wedge \delta A^\nu, \tag{35}
\]

where \( \delta \) is the functional exterior derivative of forms on the Yang-Mills phase space [5]. This symplectic current is closed, since

\[
\delta(\delta F_{\mu\nu}) = 0, \quad \delta(\delta A^\nu) = 0, \tag{36}
\]

and therefore,

\[
\delta J_\mu = tr \ \delta(\delta F_{\mu\nu}) \wedge \delta A^\nu - tr \ \delta F_{\mu\nu} \wedge \delta(\delta A^\nu) = 0. \tag{37}
\]

On introducing a basis, say \( \{ T^a \} \), for the \( SU(2) \) Lie algebra, we have

\[
\nabla_\mu A_{a\nu} = \partial_\mu A_{a\nu} + [A_\mu, A_\nu]_a = \partial_\mu A_{a\nu} - \epsilon_{abc} A^b_\mu A^c_\nu. \tag{38}
\]

Thus

\[
\delta F_{a\mu\nu} = \nabla_\mu \delta A_{a\nu} - \nabla_\nu \delta A_{a\mu},
\]

or

\[
\delta F_{\mu\nu} = \nabla_\mu \delta A_\nu - \nabla_\nu \delta A_\mu. \tag{39}
\]
As a result of a gauge transformation, \( A_{a\mu} \rightarrow A'_{a\mu} \) with

\[
A'_{a\mu} = A_{a\mu} + \partial_\mu \lambda_a + [A_{\mu}, \lambda]_a
\]

(40)

for some infinitesimal real-valued function \( \lambda \) on space-time. We have

\[
\delta A'_{a\mu} = \delta A_{a\mu} + [\delta A_{\mu}, \lambda]_a,
\]

(41)

\[
\delta F'_{a\mu\nu} = \delta F_{a\mu\nu} + [\delta F_{\mu\nu}, \lambda]_a.
\]

(42)

The symplectic current transforms according to

\[
J'_\mu = \delta F'_{a\mu\nu} \wedge \delta A'^{a\nu}
\]

\[
= J_\mu - \varepsilon_{abc} \lambda^c (\delta F^b_{\mu\nu} \wedge \delta A^{a\nu} + \delta F^a_{\mu\nu} \wedge \delta A^{b\nu}) + O(\lambda^2)
\]

\[
= J_\mu + O(\lambda^2).
\]

(43)

Thus the symplectic current is an \( SU(2) \)-singlet. This allows us to write

\[
\partial^\mu J_\mu = \nabla^\mu J_\mu
\]

\[
= tr \nabla^\mu \delta F_{\mu\nu} \wedge \delta A^{\nu} + tr \delta F_{\mu\nu} \wedge \nabla^\mu \delta A^{\nu}.
\]

(44)

The equations of motion

\[
\nabla^\mu F_{\mu\nu} = \partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0,
\]

(45)

imply that

\[
\nabla^\mu \delta F_{\mu\nu} = -[\delta A^\mu, F_{\mu\nu}].
\]

(46)

Also

\[
tr \nabla^\mu \delta F_{\mu\nu} \wedge \delta A^{\nu} = \varepsilon_{abc} F^c_{\mu\nu} \delta A^{a\nu} \wedge \delta A^{b\mu}
\]

\[
= 0.
\]

(47)

Next we consider
\[
tr \delta F_{\mu\nu} \wedge \nabla^\mu \delta A^\nu = \frac{1}{2} \tr \delta F_{\mu\nu} \wedge (\nabla^\mu \delta A^\nu - \nabla^\nu \delta A^\mu)
\]
\[
= \frac{1}{2} \tr \delta F_{\mu\nu} \wedge \delta F^{\mu\nu} = 0. \quad (48)
\]
Combining (47) and (48), we see that
\[
\partial^\mu J_\mu = 0 \quad (49)
\]
by (44). It follows that the closed functional 2-form, \(\Omega\), given by
\[
\Omega = \int_{\Sigma_t} d\sigma^\mu J_\mu = \int_{\Sigma_t} d\sigma^\mu \tr \delta F_{\mu\nu} \wedge \delta A^\nu,
\]
is Lorentz invariant. Thus we have constructed a Lorentz invariant and gauge invariant symplectic form, \(\Omega\), on the \(SU(2)\) Yang-Mills phase space.

We can obtain the standard \(SU(2)\) Yang-Mills symplectic form by a suitable choice of the space-like hyper-surface, \(\Sigma_t\):
\[
\Omega = \int_{\Sigma_t} d^3x \tr \delta E_i \wedge \delta A^i, \quad (50)
\]
where \(E_i = F_{0i}\) is the momentum canonically conjugate to \(A^i\).

4. A Symplectic Form For Ashtekar’s Canonical Gravity

The inverse densitized triads and the Ashtekar connection act as symplectic coordinates in the phase space of Ashtekar’s canonical gravity. We wish to put a symplectic form on Ashtekar’s phase space in a manner consistent with the Crnkovic-Witten construction. An extra difficulty here, over and above the problem of gauge invariance, is the complex nature of Ashtekar’s canonical variables. Denoting the functional exterior derivative of forms on Ashtekar’s phase space by \(\delta\), we have
\[
\delta A_{ai} = \frac{\delta \omega_{0ai}}{\delta E^bj} \delta E^b_j + \frac{\delta \omega_{0ai}}{\delta \dot{E}^b_j} \delta \dot{E}^b_j - \frac{i}{2} \varepsilon_{acd} \frac{\delta \omega^{cd}_i}{\delta E^{bj}} \delta E^b_j,
\]
where the shorthand notation \(\frac{\delta \omega_{0ai}(x)}{\delta E^b_j(y)} d^4 y\) is understood.
We require the symplectic form,

\[ \Omega = \int_{\Sigma_t} d^3x \ (\delta E^{ai} \wedge \delta A_{ai}) \]

\[ = -\int_{\Sigma_t} d^3x \left( \frac{\delta A_{ai}}{\delta E^{bj}} \delta E^{bj} \wedge \delta E^{ai} + \frac{\delta A_{ai}}{\delta E^{bj}} \delta \dot{E}^{bj} \wedge \delta E^{ai} \right), \tag{53} \]

to be real-valued in order to have a unique symplectic structure on Ashtekar’s phase space. A complex-valued symplectic form would give rise to two real symplectic structures. Moreover, a real-valued symplectic form produces real-valued Poisson brackets.

Working in the time gauge and using (6) and (7), it is straightforward to show that

\[ \int_{\Sigma_t} d^3x \frac{\delta \omega_0^{ai}}{\delta E^{bj}} \delta E^{bj} \wedge \delta E^{ai} = \frac{1}{2N} \left[ \left((E^{-1})_{bi}(E^{-1})_{ak} + (E^{-1})_{ci}(E^{-1})_{ck}\delta_{ab}\right)\partial_j N^k 

- ((E^{-1})_{bi}(E^{-1})^c_j (E^{-1})_{ck} + (E^{-1})_{bk}(E^{-1})^c_i (E^{-1})_{cj}) \left( \dot{E}_a^k + E_a^\ell \partial_\ell N^k - N^\ell \partial_\ell E_a^k \right) 

- (E^{-1})_{ai}(E^{-1})_{cj}(E^{-1})_{bk} \left( N^\ell \partial_\ell E^{ck} - \dot{E}^{ck} \right) \right] \delta E^{bj} \wedge \delta E^{ai}, \tag{54} \]

\[ \int_{\Sigma_t} d^3x \frac{\delta \omega_0^{ai}}{\delta \dot{E}^{bj}} \delta \dot{E}^{bj} \wedge \delta E^{ai} = \]

\[ \frac{1}{2N} \left[ (E^{-1})^c_i (E^{-1})_{cj} \eta_{ab} + (E^{-1})_{bi}(E^{-1})_{aj} - (E^{-1})_{ai}(E^{-1})_{bj} \right] \delta \dot{E}^{bj} \wedge \delta E^{ai}, \tag{55} \]

Following Henneaux et al [7], we can write

\[ \varepsilon_{acd} \omega_{di}^{cd} = \frac{\delta}{\delta E^{ai}} \int_{\Sigma_t} d^3x \ G, \]

where

\[ G = \varepsilon^{jk\ell} h_{bj} \partial_k h_{\ell}. \tag{56} \]

Then

\[ \int_{\Sigma_t} d^3x \ \varepsilon_{acd} \frac{\delta \omega_{di}^{cd}}{\delta E^{bj}} \delta E^{bj} \wedge \delta E^{ai} = \frac{\delta}{\delta E^{bj}} \left( \frac{\delta}{\delta E^{ai}} \int_{\Sigma_t} d^3x \ G \right) \delta E^{bj} \wedge \delta E^{ai} \tag{57} \]

\[ = - \frac{\delta}{\delta E^{bj}} \left( \frac{\delta}{\delta E^{ai}} \int_{\Sigma_t} d^3x \ G \right) \delta E^{bj} \wedge \delta E^{ai} = 0. \]

Thus the complex part of the symplectic form, \( \Omega \), is zero in the time gauge.
Now we must show that the symplectic form is real-valued for all other gauges, apart from the time gauge. When we go from a time gauge hyper-surface to a more general hyper-surface, the group of local symmetries enlarges from the rotation group, \( SO(3) \), to the Lorentz group, \( SO(3,1) \) (see the appendix).

Ashtekar’s action can be written:

\[
S = \int_M \mathcal{L} d^4x,
\]

where

\[
\mathcal{L} d^4x = iF_a \wedge A^a,
\]

as in (17) and (20). The symplectic form (53) can be then be written:

\[
\Omega = \int_{\Sigma_t} d^3x \, \delta \left( \frac{\delta L}{\delta A_{ai}} \right) \wedge \delta A^{ai}.
\]

The analogy with the Hamiltonian formulation of the \( SU(2) \) Yang-Mills field suggests that we ought to postulate a functional 2-form on Ashtekar’s phase space, with the vector density on space-time:

\[
J_\mu = \delta \left[ \frac{\delta L}{\delta (\partial_\mu A_{ai})} \right] \wedge \delta A^{ai} = \delta \left[ \frac{\delta L}{\delta (\partial_\mu A_{a\nu})} \right] \wedge \delta A^{a\nu}
\]

\[
= i \, \text{tr} \, \delta \Lambda_{\mu\nu} \wedge \delta A^\nu,
\]

as a symplectic current. Here, the trace \( \text{tr} \) relates to a representation of \( sl(2,\mathbb{C}) \), the Lie algebra of \( SO(3,1) \). We can associate an \( SO(3,1) \) co-variant derivative, \( D_\mu \), with the connection, \( A_\mu \), such that

\[
\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu.
\]

Under a local \( SO(3,1) \) transformation, \( A_\mu \to A_\mu + \partial_\mu \lambda + [A_\mu, \lambda] \), where \( \lambda \) is a real-valued function on space-time. It is found that \( \delta A_\mu \to \delta A_\mu + [\delta A_\mu, \lambda] \) and \( \delta \Lambda_{\mu\nu} \to \delta \Lambda_{\mu\nu} + [\delta \Lambda_{\mu\nu}, \lambda] \). It follows that the symplectic current is an \( SO(3,1) \)-singlet. Thus we can write:

\[
\partial^\mu J_\mu = D^\mu J_\mu = i \, \text{tr} \, D^\mu \delta \Lambda_{\mu\nu} \wedge \delta A^\nu + i \, \text{tr} \, \delta \Lambda_{\mu\nu} \wedge D^\mu \delta A^\nu.
\]
Since
\[ D^\mu \Lambda_{\mu\nu} = 0, \]  
we have
\[ D^\mu \delta \Lambda_{\mu\nu} = -[\delta A^\mu, \Lambda_{\mu\nu}] \]
and
\[
i \text{tr} \ D^\mu \delta \Lambda_{\mu\nu} \wedge \delta A^\nu = -i \text{tr} \ [\delta A^\mu, \Lambda_{\mu\nu}] \wedge \delta A^\nu
= \frac{i}{2} \text{tr} \ [\delta A^\mu, \delta A^\nu] \wedge \Lambda_{\mu\nu} = 0. \tag{65} \]

When we vary the action with respect to the orthonormal 1-forms, while keeping the connection 1-forms fixed, the equations of motion imply:
\[ F^{\mu\nu} \delta \Lambda_{\mu\nu} = 0. \tag{66} \]

Consequently,
\[
i \text{tr} \ \delta \Lambda_{\mu\nu} \wedge D^\nu \delta A^\mu = \frac{i}{2} \text{tr} \ \delta \Lambda_{\mu\nu} \wedge \delta F^{\mu\nu} = -i \frac{i}{2} \text{tr} \ \delta (F^{\mu\nu} \delta \Lambda_{\mu\nu}) = 0. \tag{67} \]

It is clear from (65) and (67) that the divergence of the symplectic current in (63) vanishes, and it follows that the closed functional 2-form
\[
\Omega = \int_{\Sigma_t} d\sigma^\mu J_\mu = i \int_{\Sigma_t} d\sigma^\mu \text{tr} \ \delta \Lambda_{\mu\nu} \wedge \delta A^\nu, \tag{68} \]
is a Lorentz invariant and general co-ordinate invariant symplectic form on Ashtekar’s phase space. In particular, since the imaginary part of \( \Omega \) vanishes in the time gauge, and \( \Omega \) is Lorentz invariant, then \( \Omega \) is real-valued in any local Lorentz frame, even one in which the inverse densitized triads are complex-valued.
5. Conclusions

We have described Ashtekar’s canonical gravity in a manifestly co-variant way by using a construction due to Crnkovic and Witten. This construction had worked for the ADM formulation of general relativity, so we hoped it might work for Ashtekar’s formulation, at least in the time gauge. The only obstacles to be overcome were gauge invariance, and the complex nature of Ashtekar’s canonical variables.

Gauge invariance was incorporated into the symplectic form we put on Ashtekar’s phase space, along with Lorentz invariance, as in the work of Crnkovic and Witten. Using a result in a paper by Henneaux et al, we showed that the symplectic form is real-valued in the time gauge, thereby giving rise to a unique symplectic structure on Ashtekar’s phase space, as well as real-valued Poisson brackets.

It remained to show that the symplectic form is real-valued for all other gauges, in addition to the time gauge, when the canonical variables are all \( sl(2, \mathbb{C}) \)-valued. This was accomplished using the analogy with the Hamiltonian formulation of the \( SU(2) \) Yang-Mills field. As a result, we know that the Crnkovic-Witten construction can be applied to Ashtekar’s canonical gravity.

Appendix

A general orthonormal basis can be obtained from one adapted to a space-like hypersurface, \( \Sigma_t \), as follows. Denoting 4-dimensional orthonormal and co-ordinate indices by \( A, B, \ldots \) and \( \mu, \nu, \ldots \) respectively, let

\[
h^A_{\mu} = \begin{bmatrix} N & 0 \\ h^a_j N^j & h^a_i \end{bmatrix}
\]  
(69)

be a tetrad with \( h^0 = N dx^0 \) normal to \( \Sigma_t \). This choice of tetrad is compatible with the time gauge condition. Here \( N \) is the lapse function, \( N^i \) are the shift functions, and \( h^a_i \) is an orthonormal triad on \( \Sigma_t \) satisfying

\[
h^a_i h_{aj} = g_{ij}, \quad (h^{-1})_a^i h^a_j = \delta^i_j,
\]  
(70)
where $g_{ij}$ is the 3-dimensional metric on $\Sigma_t$. An arbitrary Lorentz boost, tangent to $\Sigma_t$, with 3-velocity, $v^a$, is given by

$$L(v)^A_B = \begin{bmatrix} \gamma & -\gamma v_b \\ -\gamma v^a & \delta^a_b + \frac{\gamma^2}{1+\gamma} v^a v_b \end{bmatrix},$$

where

$$\gamma = (1 - v^a v_a)^{-\frac{1}{2}}. \tag{71}$$

So an arbitrary tetrad is of the form:

$$e^A_\mu = L(v)^A_B h^B_\mu. \tag{72}$$

Note, however, that

$$e^a_i = h^a_i + \frac{\gamma^2}{1+\gamma} v^a v_b h^b_i \tag{73}$$

are not an orthonormal triad because $e^a_i e_{aj} \neq g_{ij}$. Let $e$ be the determinant of the matrix, $[e_{ai}]$. Defining the inverse densitized triads, $E^{ai}$, by

$$E^{ai} = e(e^{-1})^a_i - i\epsilon^{ijk} e^a_j e^b_k v^b, \tag{74}$$

Ashtekar’s Lagrangian takes the form:

$$L = \int_{\Sigma_t} d^3x \left[ A_{ai} \dot{E}^{ai} - A_{a0} \left( \partial_i E^{ai} - i\varepsilon^{abc} A_{b0} E^{ci} \right) \right. \left. + \frac{i}{2} \mathcal{N} \varepsilon^{abc} F_{a_1j} E^a_i E^b_j E^c_j + N^i F_{aij} E^{aj} \right], \tag{75}$$

where $\mathcal{N} = \gamma N/e$. This shows that $E^{ai}$ and $A_{ai}$ are canonically conjugate, with $\mathcal{N}, N^i$, and $A_{a0}$ behaving as Lagrange multipliers for the secondary constraints. It is straightforward to verify that the complex inverse densitized triads satisfy

$$E^{ai} E^{aj}_a = gg^{ij}, \quad E^{ai} E^{aj}_b g_{ij} = g\delta^a_b, \tag{76}$$

where $g$ is the determinant of the matrix, $[g_{ij}]$, and so they can be regarded, in a sense, as a complex orthonormal triad density. The effect of an infinitesimal Lorentz boost on the
canonical variables is easily calculated. Using

$$\delta L^A_{\ B}(0) = \begin{bmatrix} 0 & -\delta v^a \\ -\delta v^a & 0 \end{bmatrix}, \quad (77)$$

we find

$$\delta v A^a_i = -\left(\partial_i \delta v^a - i \varepsilon^{abc} A_{bi} \delta v_c\right), \quad (78)$$

$$\delta v E^{ai} = i \varepsilon^{abc} E^b \delta v_c, \quad (79)$$

which is to be compared with the effect of an infinitesimal tangent space rotation on $\Sigma_t$, parameterised by $\delta \theta^a$,

$$\delta \theta A^a_i = i \left(\partial_i \delta \theta^a - i \varepsilon^{abc} A_{bi} \delta \theta_c\right), \quad (80)$$

$$\delta \theta E^{ai} = \varepsilon^{abc} E^b \delta \theta_c. \quad (81)$$

As an extra check that these variables are canonically conjugate, it is instructive to prove that Lorentz transformations leave the Poisson brackets unchanged. It is easy to verify that an infinitesimal boost leaves the Poisson bracket unchanged as, of course, do infinitesimal rotations. As boosts and rotations form a group, we can simply exponentiate and deduce that finite Lorentz transformations also leave the Poisson bracket invariant. Hence,

$$\{E^{ai}, A_{bj}\} = \delta^a_b \delta^i_j \quad (82)$$

must hold for the complex $E^{ai}$ with $v^a \neq 0$. In conclusion, it has been shown that it is not necessary to match the choice of an orthonormal frame to the foliation of space-time in Ashtekar’s canonical gravity. The inverse densitized triads are now complex-valued, but there are still conditions on them, since the imaginary part only has three degrees of freedom, $v^a$, rather than the nine which would be necessary for a complex $3 \times 3$ matrix. The Ashtekar connection, $A_{ai}$, become $sl(2, \mathbb{C})$-valued. The infinitesimal $sl(2, \mathbb{C})$ gauge transformations are given above in (78) and (79). Finally, this appendix is equivalent to the work of Ashtekar et al in [10], where the results are formulated in spinor notation.


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