A Novel Augmented Lagrangian Approach for Inequalities and Convergent Any-Time Non-Central Updates

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Abstract. Motivated by robotic trajectory optimization problems we consider the Augmented Lagrangian approach to constrained optimization. We first propose an alternative augmentation of the Lagrangian to handle the inequality case (not based on slack variables) and a corresponding “central” update of the dual parameters. We prove certain properties of this update: roughly, in the case of LPs and when the “constraint activity” does not change between iterations, the KKT conditions hold after just one iteration. This gives essential insight on when the method is efficient in practice. We then present our main contribution, which are consistent any-time (non-central) updates of the dual parameters (i.e., updating the dual parameters when we are not currently at an extremum of the Lagrangian). Similar to the primal-dual Newton method, this leads to an algorithm that parallelly updates the primal and dual solutions, not distinguishing between an outer loop to adapt the dual parameters and an inner loop to minimize the Lagrangian. We again prove certain properties of this any-time update: roughly, in the case of LPs and when constraint activities would not change, the dual solution converges after one iteration. Again, this gives essential insight in the caveats of the method: if constraint activities change the method may destabilize. We propose simple smoothing, step-size adaptation and regularization mechanisms to counteract this effect and guarantee monotone convergence. Finally, we evaluate the proposed method on random LPs as well as on standard robot trajectory optimization problems, confirming our motivation and intuition that our approach performs well if the problem structure implies moderate stability of constraint activity.

1 Introduction

To motivate this work we first mention some empirical findings. We tested standard interior point and Augmented Lagrangian methods on random LPs and QPs as well as on non-linear constrained robot trajectory optimization problems. For random LPs and QPs, we found Augmented Lagrangian methods less efficient as plain log-barrier. However, for our trajectory optimization problems Augmented Lagrangian methods performed extremely well, only by a small factor slower than unconstrained non-linear trajectory optimization—

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in constrast to less efficient log-barrier methods. We believe a reason for this is that in the
trajectory optimization case the constraints play a “simpler” role than in random LPs: the
problem is dominated by the non-linear cost function $f(x)$, the number of constraints is
smaller than the primal problem dimensionality, and empirically we find that constraint
activity is rather stable, i.e., does not vary much over optimization iterations.

These views motivate us to investigate in Augmented Lagrangian methods, extending
them to deal efficiently also with inequality constraints particularly in cases where the con-
straint activity is rather stable. We will propose an alternative augmentation to deal with
inequality, analyze it and generalize it towards an any-time primal dual update. This anal-
ysis gives insight into why this Augmented Lagrangian might be particularly appropriate
when constraint activity is rather stable. We will detail the contributions after introducing
related work.

2 Related work

In Section 17.4, Nocedal and Wright (1999) propose an “unconstrained formulation” of the
Augmented Lagrangian in the case of inequalities. The specific dual parameter update
(their Eq. (17.63)) is the same as the “central” update we consider below. However, their
specific augmentation (17.64) is different to the Augmented Lagrangian we will propose—
only includes a squared penalty $g^2$ if $2\mu g + \lambda \leq 0$ (translated to our notation). We will
explicitly address the difference when discussing the implications of our choice. Nocedal
and Wright (1999) state that their proposition has not been practically evaluated and we are
not aware of evaluations of their approach. Further, they do not extend towards any-time
primal-dual updates.

LANCELOT is the most popular software using the Augmented Lagrangian for globally
convergent non-linear optimization (Conn et al., 1991, 2010). Inequalities are handled with
slack variables $\xi$, which implies that the dimensionality of the optimization problem is
increased and the state space will be subject to bound constraints ($\xi \geq 0$) (Nocedal and
Wright, 1999), prohibiting straight-forward Newton methods. Both of these aspects makes
the approach less attractive in the high-dimensional trajectory optimization domain. Fur-
ther, we are not aware of any-time updates used within such approaches.

Another approach is to consider shifted barriers (e.g., log-barriers) as augmentation in the
inequality case (Conn et al., 1997; Noll et al., 2004; Noll, 2007). We find these approaches
very interesting and at first sight very different to our $[\lambda > 0 \lor g > 0]g^2$ augmentation we
will discuss below. Again, we are not aware of any-time updates having been proposed
for such types of augmentations. We believe our approach to any-time updates could be
generalized also to the case of shifted barrier augmentations.

We would also like to point to a very interesting historical discussion of interior point meth-
ods by Forsgren et al. (2002), where the authors nicely clarify the original motivation for
Augmented Lagrangian methods: Log-barrier (and squared penalty) methods lead to an
ill-conditioning of the Hessian in the limit of $\mu \to 0$ (strict barriers). This was considered a
problem and motivation for the Augmented Lagrangian, which happens to beautifully not
modify the conditioning of the Hessian at all. However, in the late 80ies it was thoroughly
understood that the log-barrier’s ill-conditioning of the Hessian is, surprisingly, not a prob-
lem (confirming the practical success), which lead to the rise of interior point methods and
efficient primal-dual formulations, diminishing the interest in the Augmented Lagrangian. As mentioned in the introduction we feel that it very much depends on the concrete structure of the problem whether interior point or Augmented Lagrangian methods might be more efficient.

Our contributions over this previous work are:

1. We propose an alternative augmentation for the inequality case. We analyze the properties of a centered update (also proposed in (Nocedal and Wright, 1999, Eq. (17.63))) with this augmentation, giving sufficient conditions for when the update yields the dual solution. This result gives essential insights on when the approach is promising in practice.

2. Based on these results we reason about which any-time (non-centered) dual update (i.e., an update of dual parameters while not being at an extremum of the Lagrangian) would also yield the correct dual solution (under similar sufficient conditions). We propose such an any-time update and provide these sufficient conditions, generalizing the result of the first part.

3. Finally we consider a straightforward extension to account for the local Hessian of \( f(x) \), leading to a 2nd order any-time update.

### 3 Alternative augmentation and centered update

Let \( x \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^n \to \mathbb{R}^m, h : \mathbb{R}^n \to \mathbb{R}^l \). We consider

\[
\min_x f(x) \quad \text{s.t.} \quad g(x) \leq 0, \ h(x) = 0.
\]

We denote the dual variables as \( \lambda \in \mathbb{R}^m, \kappa \in \mathbb{R}^l \). The KKT conditions are

\[
\begin{align*}
v f(x) + \lambda^T v g(x) + \kappa^T v h(x) &= 0 \quad \text{(stationarity)} \quad (2) \\
g(x) \leq 0 \land h(x) = 0 \quad \text{(primal feasibility)} \quad (3) \\
\lambda \geq 0 \quad \text{(dual feasibility)} \quad (4) \\
\lambda \cdot g_i(x) &= 0 \quad \text{(complementary)} \quad (5)
\end{align*}
\]

Let use introduce some notation we use throughout. By primal-dual state we refer to an arbitrary tuple \((x, \lambda, \kappa)\). In any state \((x, \lambda, \kappa)\) we call the \( i \)th constraint active iff \( \lambda_i > 0 \lor g_i(x) > 0 \). For two vectors \( v \) and \( w \), \( (v; w) = (v^T, w^T)^T \) denotes their “stacking” (analogously for matrices).

We consider the following Augmented Lagrangian, which includes for any active constraint a squared penalty \( g_i(x)^2 \) pulling \( g_i \) to zero.

**Definition 1.** We define our Augmented Lagrangian as

\[
L(x, \lambda, \kappa) = f(x) + \mu \sum_{i=1}^m [\lambda_i > 0 \lor g_i(x) > 0] g_i(x)^2 + \lambda^T g(x) \\
+ \nu \sum_{i=1}^l h_i(x)^2 + \kappa^T h(x) \quad (6) \\
= f(x) + [\mu I_\lambda(x) g(x) + \lambda^T g(x) + \nu h(x) + \kappa^T h(x)], \quad (7)
\]

where \( I_\lambda(x) := \text{diag}([g_i(x) \geq 0 \lor \lambda_i > 0]) \).
Its gradient is
\[ \nabla L(x, \lambda, \kappa) = \nabla f(x) + [2\mu I(x)g(x) + \lambda]^{\top}\nabla g(x) + [2\nu h(x) + \kappa]^{\top}\nabla h(x), \] (8)

**Definition 2.** For any state \((x, \lambda, \kappa)\) we define the centered update \(U^{cen}\) as
\[
U^{cen}(x, \lambda, \kappa) = (\lambda', \kappa') \quad \text{with}
\]
\[
\lambda' = \max\{0, \lambda + 2\mu g(x)\} \quad \text{(9)}
\]
\[
\kappa' = \kappa + 2\nu h(x) \quad \text{(10)}
\]
where the \(\max\) operator is interpreted element-wise.

This update is also introduced in (Nocedal and Wright, 1999, Eq. (17.63)). The centered update is meant to be applied at a minimum \(x' = \arg\min_x L(x, \lambda, \kappa)\). The standard nested loop approach uses an inner loop to converge to \(x' = \arg\min_x L(x, \lambda, \kappa)\) for given dual parameters and an outer loop to update \((x', \lambda', \kappa')\). An intuition behind the update, related to the following theorem, is the following: Assuming initially \(\lambda = \kappa = 0\), \(x' = \arg\min_x L(x, 0, 0)\) will violate constraints. The squared penalties counteract these violations by generating the gradients \(2\mu g(x')^{\top}I(x')\nabla g(x') + 2\nu h(x')^{\top}\nabla h(x')\) at \(x'\). The centered update will generate exactly these gradients in the next iteration. In other terms, the dual parameters are chosen such that \(\lambda^{\top}g(x) + \kappa^{\top}h(x)\) will generated the gradients that have previously been generated by the squared penalties. This is made more rigorous in the following result.

**Theorem 1.** For any \((\lambda, \kappa)\), let
\[
x' = \arg\min_x L(x, \lambda, \kappa), \quad (\lambda', \kappa') = U^{cen}(x', \lambda, \kappa), \quad x^* = \arg\min_x L(x, \lambda', \kappa'). \] (12)

For any Linear Program \((f, g \text{ and } h \text{ linear})\), if all active constraints are linearly independent at \(x^*\) (non-zero rows of \(I_\lambda(x^*)\) \(\nabla g(x^*)\) are linearly independent), then
\[
[\forall i : \lambda_i > 0 \Rightarrow \lambda'_i > 0] \Rightarrow \text{KKT hold at } x^* \] (13)

**Proof.** Note that (element-wise)
\[
[\lambda > 0 \Rightarrow \lambda' > 0] \iff [\lambda = 0 \lor \lambda' > 0] \iff [\max\{0, \lambda + 2\mu g(x')\} = \lambda + 2\mu I(x')g(x')] \text{.} \] (14)

This is obvious for \(\lambda = 0\). In the case \(\lambda > 0 \land \lambda' > 0\) we have \(I_\lambda(x') = 1\) and \(\lambda + 2\mu g(x') > 0\), from which the RHS follows.

We consider the gradient at \(x'\),
\[
0 = \nabla L(x', \lambda, \kappa) = \nabla f + [2\mu I(x')g(x') + \lambda]^{\top}\nabla g + [2\nu h(x') + \kappa]^{\top}\nabla h, \] (16)

where \(\nabla f, \nabla g, \nabla h\) are independent of \(x'\), and compare it to the gradient at \(x^*\),
\[
0 = \nabla L(x^*, \lambda', \kappa') \quad \text{(17)}
\]
\[
= \nabla f + [2\mu I(x^*)g(x^*) + \max\{0, \lambda + 2\mu g(x')\}]^{\top}\nabla g + [2\nu h(x^*) + \kappa + 2\nu h(x')]^{\top}\nabla h \quad \text{(18)}
\]
\[
= \nabla f + [2\mu I(x^*)g(x^*) + \lambda + 2\mu I(x')g(x')]^{\top}\nabla g + [2\nu h(x^*) + \kappa + 2\nu h(x')]^{\top}\nabla h, \] (19)
\[ = 2\mu g(x^*)^\top J_x(x^*) \nabla g + 2\nu h(x^*)^\top \nabla h, \tag{20} \]

where in the 3rd line we use the implication of \([\forall i : \lambda_i > 0 \Rightarrow \lambda_i' > 0]\), and the last line inserted (16). If all non-zero rows of \(I_x(x^*) \nabla g\) are linearly independent the gradient \(\nabla L(x^*, \lambda', \kappa')\) is zero at \(x^*\) iff

\[ I_x(x^*) g(x^*) = 0, \quad h(x^*) = 0. \tag{21} \]

Note that (21) implies primal feasibility, complementarity, as well as \(\mathcal{U}(x^*, \lambda', \kappa') = (\lambda', \kappa')\). With \(\nabla L(x^*, \lambda', \kappa') = 0\) and (21) the stationarity holds. Dual feasibility holds by construction.

The theorem states that, for a linear program, the updated dual parameters \((\lambda', \kappa')\) are optimal under two conditions: 1) \([\lambda > 0 \Rightarrow \lambda' > 0]\), that is, none of the constraints becomes inactive when it was previously active. And 2), all active constraints (non-zero rows of \(I_x(x^*) \nabla g\) are linearly independent. The discussion of these two conditions is interesting and gives insight into our choice of the augmentation itself.

Let us first discuss the case \([\lambda > 0 \Rightarrow \lambda' > 0]\), where in some iteration \(\lambda > 0\), then \(x' = \arg\min_x L(x, \lambda, \kappa)\) pushes far outside the constraint \((2\mu g(x^*) < \lambda < 0)\) such that the subsequent update chooses \(\lambda' = 0\). In this case, even in the locally linearized view, the next optimization does not reach a KKT point. This is intuitive as the inner loop optimization of \(x'\) considered the constraint to be strictly active and therefore included a penalty \(g(x)^2\) even when \(g(x) < 0\). It pulled towards the constraint \(g(x) = 0\) even though \(g(x) < 0\). This explains that the update failed to lead to a KKT point directly: In the equations we see that (18) becomes unequal to (19) when the \(\max\) selects \(\lambda' = 0\) while \(\lambda > 0\), and therefore the \(\lambda'\) does not generate the necessary gradients to achieve stationarity in the next centering. A trivial solution seems to initialize \(\lambda = 0\) in the first iteration, which avoids \([\lambda > 0 \Rightarrow \lambda' > 0]\); however, here the second conditions gets into play.

Whether non-zero rows of \(I_x(x^*) \nabla g\) are linear independent typically depends on \(\sum_i [\lambda_i' > 0 \lor g_i(x^*) > 0] \leq n\), that is, how many constraints are “active”. Note that \(\lambda'\) has been computed at \(x'\) while \(g(x^*)\) is evaluated at \(x^*\). Therefore, \(I_x(x^*)\) includes constraints that have been active at \(x'\) or \(x^*\), which can well be more than \(n\).

If both conditions are fulfilled, the Theorem shows it is effective to include the penalty \(g(x)^2\) even when \(g(x) < 0\), because it leads to a ‘correct’ retuning of the active dual parameter \(\lambda' > 0\)—under the given assumptions. It penalizes the inequality just like an equality, assuming that \(\lambda\) might remain active when it was active before.

### 4 Any-time, non-centered update

**Definition 3.** We define the any-time update as

\[ \mathcal{U}^{\text{any}}(x, \lambda, \kappa) = (\lambda', \kappa') \quad \text{with} \]

\[ \frac{\lambda'}{\kappa'} = \arg\min_{(\lambda', \kappa') : \lambda' \geq 0} \left\| \begin{pmatrix} \lambda' \\ \kappa' \\ 0 \end{pmatrix} - \begin{pmatrix} \lambda + 2\mu I_x(x) g(x) \\ \kappa + 2\nu h(x) \end{pmatrix} \right\| \left( \nabla g(x) + \nabla h(x) \right) + \nabla L(x, \lambda, \kappa) \right\|^2 \tag{23} \]

This update is a bounded quadratic program, aiming to minimize the difference between the gradients \([\lambda + 2\mu I_x(x) g(x)] \nabla g + [\kappa + 2\nu h(x)] \nabla h - \nabla L(x', \lambda, \kappa)\) before the update, and \(\lambda' \nabla g + \kappa' \nabla h\) after the update.
Theorem 2. Let \((x', \lambda, \kappa)\) be arbitrary and
\[
(\lambda', \kappa') = \mathcal{U}^{\text{arg}}(x', \lambda, \kappa), \quad x^* = \underset{z}{\text{argmin}} L(x, \lambda', \kappa').
\] (24)

For any Linear Program \((f, g \text{ and } h \text{ linear}), \) if all active constraints are linearly independent at \(x^*\) (non-zero rows of \(I_x(x^*) \nabla g(x^*)\) are linearly independent), and if the \(\text{argmin} \) in the update \((23)\) reaches zero, then KKT hold at \(x^*\).

Proof. We have
\[
\begin{align*}
\nabla L(x, \lambda', \kappa') &= \nabla f + [2\mu I_x(x)g(x) + \lambda']^\top \nabla g + [2\nu h(x) + \kappa']^\top \nabla h \\
&= \nabla f + [2\mu I_x(x)g(x) + \lambda + 2\mu I_x(x')g(x')]^\top \nabla g \\
&\quad + [2\nu h(x) + \kappa + 2\nu h(x')]^\top \nabla h - \nabla L(x', \lambda, \kappa) \\
&= 2\mu g(x)^\top I_x(x) \nabla g + 2\nu h(x)^\top \nabla h,
\end{align*}
\] (25)

The rest of the proof is as previously.

The above theorem makes a statement under the strong assumption that we can minimize the \(\text{argmin} \) in \((23)\) to zero. A particular complication here is the bound constraint \(\lambda' \geq 0\) of the minimization, which in the centered update translated to the \(\lambda' \leftarrow \min\{..., 0\}, \) which in turn was related to the assumption \(\lambda_i > 0 \Rightarrow \lambda_i' > 0\) we made in Theorem 1.

To avoid the bounded optimization problem \((23)\) we first consider even stronger assumption which leads an analytical solution:

Corollary 3. Let \(\forall i : \lambda_i > 0 \Leftrightarrow \lambda_i' > 0 \Leftrightarrow g_i(x') \geq 0, \) then the minimum of \((23)\) is given analytically as
\[
\begin{bmatrix} \lambda' \\ \kappa' \end{bmatrix} = y - (AA^\top)^{-1}A \nabla L(x, \lambda, \kappa), \quad y = \begin{bmatrix} \lambda + 2\mu g(x) \\ \kappa + 2\nu h(x) \end{bmatrix}_{>0}, \quad A = \begin{bmatrix} \nabla g(x) \\ \nabla h(x) \end{bmatrix}_{>0},
\] (28)

where the notation \((\_\_\_ > 0)\) refers to rows for which \(\lambda_i > 0\) only.

Proof. Under the strong assumption, all inactive constraints drop out of the minimization \((23)\) (as when and (as \(\lambda_i' > 0\) for the active ones) \((23)\) becomes and unconstrained minimization that can be solve analytically. We have
\[
\begin{bmatrix} \lambda' \\ \kappa' \end{bmatrix} = \underset{(\lambda, \kappa)}{\text{argmin}} \left\| \begin{bmatrix} \lambda \\ \kappa \end{bmatrix} - \begin{bmatrix} \lambda + 2\mu g(x) \\ \kappa + 2\nu h(x) \end{bmatrix} \right\|^2_y \\
= y - (AA^\top)^{-1}A \nabla L(x, \lambda, \kappa),
\] (29)

which gives the minimum via a pseudo-inverse of the active constraint matrix \(A.\)

In our evaluations we employed an approximation to \((23),\) where we analytically solve the unconstrained problem \((30)\) and then impose the bound \(\lambda' \geq 0\) by clipping values.

Corollary 4.
\[
\nabla L(x, \lambda, \kappa) = 0 \quad \Rightarrow \quad \mathcal{U}^{\text{arg}}(x, \lambda, \kappa) = \mathcal{U}^{\text{en}}(x, \lambda, \kappa)
\]

That is, when \(\nabla L(x, \lambda, \kappa) = 0\) the any-time update coincides with the centered update—as the \(\text{argmin} \) reaches zero when for \((\lambda', \kappa') = \mathcal{U}^{\text{en}}(x, \lambda, \kappa).\)
Algorithm 1 Newton with adaptive step size and Levenberg-Marquardt parameter

**Input:** start point \( x \), tolerance \( \delta \), functions \( x \mapsto (f(x), \nabla f(x), \nabla^2 f(x)) \), parameters (defaults: \( \alpha^0 = \beta^0 = 1, \alpha^+ = 2, \alpha^- = 0.1, \beta^+ = \beta^- = 1, \varrho = 0.01 \))

**Output:** converged point \( x^1 \)

1: initialize \( \alpha = \alpha^0, \beta = \beta^0 \)

2: compute \( f, \nabla f, \nabla^2 f \) at \( x \)

3: repeat

4: compute \( \Delta \) to solve \((\nabla^2 f + \beta I) \Delta = -\nabla f \) \hspace{1cm} // backtracking line search (for \( \beta^+ = 1 \))

5: repeat

6: \( x' \leftarrow x + \alpha \Delta \) \hspace{1cm} // computing \( \nabla f, \nabla^2 f \) can be postponed

7: if \( f' \leq f + g_0 \nabla f(x)^\top \Delta \) then \hspace{1cm} // step is accepted (Wolfe condition)

8: \( x \leftarrow x', \quad (f', \nabla f', \nabla^2 f') \leftarrow (f', \nabla f', \nabla^2 f') \)

9: \( \beta \leftarrow \beta^- \beta, \quad \alpha \leftarrow \min\{\alpha^+ \alpha, 1\} \) \hspace{1cm} // adapt \( \alpha \) towards 1

10: else \hspace{1cm} // step is rejected

11: if \( |\Delta|_\infty \ll \delta \) or evaluations exceed then abort with failure \hspace{1cm} // gradient seems incorrect

12: \( \beta \leftarrow \beta^+ \beta, \quad \alpha \leftarrow \alpha^- \alpha \)

13: end if

14: until step accepted or \( \beta^+ \neq 1 \) \hspace{1cm} // change of \( \beta \) requires recomputing \( \Delta \)

15: until \( \beta \leq 1 \land |\Delta|_\infty < \delta \) or evaluations exceed

**Heuristic update.** Solving the bound constraint problem (23) becomes yet another constrained optimization problem. A heuristic is to update with (30) and then truncate \( \lambda_i' \leftarrow \max\{0, \lambda_i\} \). Again, for \( \nabla L = 0 \) this coincides with the centered update. For \( \nabla L \neq 0 \) this is clearly a suboptimal update. Empirical studies need to evaluate the benefit of the any-time update.

5 Experiments

5.1 Algorithmic details

In our experiments we use a basic Newton method for solving the unconstrained problem \( x' = \text{argmin}_x L(x, \lambda, \kappa) \) up to a stopping criterion. The method includes adaptive stepsize and Levenberg-Marquardt damping, see Algorithm 1. In the case of the any-time update we increase the tolerance \( \delta \) by a factor 2 in each Newton step, leading to an early stopping such that \( x' \approx \text{argmin}_x L(x, \lambda, \kappa) \) only crudely approximates the Lagrangian minimum. This is then alternated with the any-time update. To ensure that the monotonicity check (line 8) remains sensible, the any-time update also needs to update the stored values of \((L, \nabla L, \nabla^2 L)\) (stored in line 9) consistently.

5.2 Random LPs

We first compare the performance on random \( n \)-dimensional LPs of the form

\[
\min_x \sum_{i=1}^n x_i \quad \text{s.t.} \quad G\left(\begin{array}{c} 1 \\ x \end{array}\right) \leq 0
\]
Figure 1: Number of evaluations until convergence (tolerance $10^{-4}$) averaged over 10 random LPs for different dimensions $n = \dim(x)$. Errorbars indicate the deviation of the mean estimator.

where the constraint-defining matrix $G \in \mathbb{R}^{m \times n+1}$ was randomly generated as follows: First, each $G_{ij} \sim N(0, 1)$; second, if $G_{i1} > 0$ : $G_{i1} \leftarrow -G_{i1}$, which ensures that $x = 0$ is feasible; third, $G_{i1} \leftarrow G_{i1} - 1$ to increase the constraint distance from $x = 0$. Figure 5.2 compares the novel methods AugLag and AnyAugLag with standard LogBarrier and SqrPenalty. All methods reliably find the same optimum with very small constraint violation $\sum_i [g_i(x)]_+$. Interestingly, the any-time augmented lagrangian methods performs extremely well for moderate problem sizes, but clearly looses its benefits for larger sizes. We inspected its behavior and found qualitatively that the declined performance coincides with significant non-stationarity of the constraint activity also in the later stage of the random LP optimization. As anticipated by our discussion and motivation of the proposed method, for random LPs we should not expect stationarity of constraint activity during optimization—in fact, finding the set of active constraints is the main problem for LPs and if we knew this set early the remaining optimization would be trivial. The severe non-stationarity of constraints works against the implicit assumptions made in the AnyAula update, explaining its declining performance for many constraints.

### 5.3 Robotic Trajectory Optimization

We also tested the performance on standard robotic trajectory optimization problems, as illustrated in Figure 5.3. Before discussing the results, we would like to characterize such problems and the methods we seek for: Feasible path finding, let alone finding globally optimal paths, are in general hard computational problems (NP-complete when discretizing the configuration space). Therefore the approaches can roughly be separated in two categories: path finding methods that are globally (probabilistically) complete and local trajectory optimization methods that aim to converge robustly and fast to a local feasible optimum. Note that locality here is meant in the trajectory space, not configuration space. Therefore, depending on the specifics of the cost function and whether the optimization method allows to temporarily traverse infeasible regions, local optimization can very well solve problems that are in other contexts (potential fields) considered as local deadlocks. In this view, here we aim for optimization methods that robustly and aggressively move towards local optima and, in the vicinity of such local optima, efficiently minimize the local
Figure 2: Start and (optimized) goal configuration of a typical trajectory optimization problem. The configuration space is 25-dimensional, the trajectory composed of 200 time slices, making this a 5000-dimensional problem over $x \in \mathbb{R}^{25 \times 200}$. The computational cost is fully dominated by the number of evaluations of $f(x)$ and $g(x)$, which implies computing potential collisions.

| method       | $\lambda$ or $\mu$-updates | $f$ evaluations | suboptimality |
|--------------|-----------------------------|-----------------|--------------|
| AnyAula      | 20.25±2.3                   | 48.25±4.93      | 0.05±0.03    |
| Aula         | 22.8±1.3                    | 64.2±2.03       | 0.14±0.12    |
| LogBarrier   | 11±0                        | 60.2±4.0        | 72337±3325   |
| SqrP         | 11±0                        | 42.6±3.2        | 4.45±0.91    |

Table 1: Performance on the robot trajectory optimization problem. Averages are taken over randomization of the initial configuration. Suboptimality denotes the difference in the found minimum $f(x^*)$ to the best found by all methods.

non-linear convex problem.

Table 5.3 displays the performance of the various methods. Both, AnyAula and Aula converge to the same optimum (modulo stopping criterion tolerance), whereas LogBarrier fails to find any reasonable solution. In the light of the above discussion this can be explained as follows: Finding a fully feasible path from the initial trajectory $x$ to the final optimal trajectory $x^*$, where none of the intermediate trajectories violates constraints, is very hard. Aula and AnyAula implicitly relax the constraints in early iterations, leading to much better convergence to a local optimum.

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