Existence Analysis for the One-Dimensional Nonlinear Telegraph Equation with Operator Forcing and Constraint

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Abstract

The one-dimensional nonlinear telegraph equation is studied within the framework of semigroups. This accommodates operator forcing, allowing for the incorporation of concurrent physical processes. The solution is constrained, above a threshold value zero. The initial state is assumed quiescent, and homogeneous boundary values are assumed. It is shown that, for every value above the threshold, there is a time interval $J$ such that the system moves from the initial state to a corresponding admissible state. The forcing is assumed operator continuous, not operator Lipschitz continuous, which imposes technical issues for the application of fixed point theorems.

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1 Introduction.

The nonlinear telegraph equation, also characterized as the damped wave equation, has assumed renewed importance in recent years because of its connection to unresolved special cases in MEMS modeling [14] (see Table 1, p. 472, \( \gamma > 0, \beta = 0 \)). In the cited application, the forcing is the result of a complex physical process involving elastic/electrostatic interactions, and is best described via operator composition. Also, the wave motion tracked by the model is constrained to avoid ‘touchdown’. The present article does not resolve the special case cited, but uses it as a benchmark example of a more general interpretation of the telegraph equation, which has been extensively studied (cf. [4, 6, 8, 10, 15]), often in tandem with related equations such as the Klein-Gordon equation. For a pedagogical introduction to the telegraph equation, cf. [17]. Its derivation dates to the 1880s, when it was derived by Heaviside to describe attenuated electrical transmission.

In this article, we consider the following constrained initial/boundary-value problem. \( I = (-1,1) \) represents the spatial domain. The system considered is given as follows, where subscripts represent (partial) derivatives with respect to the indicated variables.

\[
\begin{align*}
  u_{tt} &= -\nu u_t + \kappa u_{xx} + F(u), \\
  u(-1,t) &= u(1,t) = 0, \quad t > 0, \\
  u(0,x) &= u_t(0,x) = 0, \quad x \in I, \\
  \inf_{x \in I} G(u) &> 0.
\end{align*}
\]

Here, \( \nu \) and \( \kappa \) are positive physical constants, \( u \) depends on the physical context, and may represent membrane deflection, voltage, etc. The hypotheses for the operator \( F(u) \) and the constraint \( \inf_{x \in I} G(u) > 0 \) are discussed in Definition 3.1. \( F \) and \( G \) are defined on \( H^2(I) \cap H^1_0(I) \), and act continuously, with range in \( H^1_0(I) \) and \( C(\bar{I}) \), resp.

We interpret the solution of (1) as follows. Any solution of this system, for which \( \inf_{x \in I} G(u) > 0 \), represents an admissible state. The main result of this article (Theorem 4.1) is the demonstration of the existence of weak solutions, which attain given super-threshold values. The time interval depends upon the given value. The proof identifies an infinitesimal generator \( U \) and a continuous semigroup \( T(t) \). Because \( F \) is assumed operator continuous, and not Lipschitz continuous, we employ a modal approximation procedure, using approximate solutions in the domain of \( U \). The limiting procedure gives rise to a weak solution limit.

2 Preliminaries

We discuss an explicit construction of the semigroup associated with the linear part of the equation in (1). Although this equation is well understood, we will develop special features necessary for the analysis given here. The semigroup
Constrained nonlinear telegraph equation

$T(t)$ will initially be defined on a Hilbert space $H$. Much is known about the norm of $T(t)$ on this space. For example, the decay of the semigroup norm in this case is derived in a more general setting in the celebrated paper [18], and is currently under study by many mathematicians. We are particularly interested here in solutions contained in the domain $D(U)$ of the semigroup generator. Instrumental to this is the study of the norm of $T(t)$ when restricted to this smoother space. This distinction assumes especial importance in the work of Kato [12, Ch. 6] in his construction of the evolution operator.

After a standard definition, we will concisely state and derive the required results. We will make extensive use of parts (2,3) of Proposition 2.1. Define $v = u_t$, and

$$U = \begin{bmatrix} 0 & \mathcal{I} \\ \kappa \frac{d^2}{dx^2} & -\nu \mathcal{I} \end{bmatrix}. \quad (2)$$

Here $\mathcal{I}$ represents the identity. Then the equation,

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = U \begin{bmatrix} u \\ v \end{bmatrix},$$

is a standard equivalent representation of $u_{tt} = -\nu u_t + \kappa u_{xx}$. In order to position this in an operator framework, we retain the notation $I = (-1, 1)$, and formulate the following definitions.

**Definition 2.1.** Define the function space, $\mathcal{H} = H^1_0(I) \times L^2(I)$, and the domain $D(U)$ of $U$, $D(U) = (H^2(I) \cap H^1_0(I)) \times H^1_0(I)$. We shall employ an equivalent norm on $H^1_0(I)$ by utilizing the inner product:

$$(u, w)_{H^1_0} = \kappa(u', w')_{L^2}. \quad (3)$$

For elements $f_j = \begin{bmatrix} u_j \\ v_j \end{bmatrix}$ in $D(U)$, $j = 1, 2$, we employ the $D(U)$ inner product given by the sum,

$$(f_1, f_2)_{D(U)} = (u''_1, u''_2)_{L^2} + (u_1, u_2)_{H^1_0} + (v_1, v_2)_{H^1_0}. \quad (4)$$

We employ the notation $D(U)$ for the first components of the elements of $D(U)$. The corresponding inner product is defined by the truncation of (4):

$$(u_1, u_2)_{D(U)} = (u''_1, u''_2)_{L^2} + (u_1, u_2)_{H^1_0}. \quad (5)$$

For clarity, throughout section two we will use vector notation to represent the components in the Cartesian products which represent $\mathcal{H}$ and $D(U)$. Throughout the remainder of the article, for $X$ a Banach space, and $J$ a closed time interval, $C(J, X)$ is the usual Banach space with norm $\|u\|_{C(J, X)} := \max_{t \in J} \|u(t)\|_X$.

The following proposition contains the results needed for the subsequent sections.

**Proposition 2.1.** $U$ is a closed linear operator in $\mathcal{H}$. The following properties hold.
1. The resolvent set of $U$ satisfies $\rho(U) \supset \{ \lambda : \text{Re}\lambda > \theta \}$, where, for $\theta_n = 4n^2\pi^2\kappa - \nu^2$,

$$\theta = \begin{cases} -\nu/2, & \text{if } \theta_n \geq 0, \forall n \geq 1. \\ \max_{n: \theta_n < 0} \{-\nu/2 + \sqrt{\nu^2 - 4n^2\pi^2\kappa/2}\}, & \text{otherwise.} \end{cases}$$

(5)

In particular, the imaginary axis is contained in $\rho(U)$.

2. $U$ is the generator of a contraction semigroup $T(t)$. Suppose $T_0 > 0$ is arbitrary. If $F_0 \in D(U)$, and $F_1 \in C(J, D(U))$, where $J = [0, T_0]$, then

$$V(t) = T(t)F_0 + \int_0^t T(t-s)F_1(s) \, ds,$$

(7)

is in $C(J, D(U))$ and satisfies the initial value problem,

$$V_t = UV(t) + F_1(t), \quad V(0) = F_0, \quad 0 < t \leq T_0.$$

(8)

3. $\|T(t)\|_{D(U)}$, defined with respect to $D(U)$, decays to zero as $t \to \infty$. In particular, there is a number $\omega$ such that this norm satisfies $\|T(t)\|_{D(U)} \leq \omega, t \geq 0$.

Proof. The property that $U$ is closed follows routinely from the definitions. If $\begin{bmatrix} u_n \\ v_n \end{bmatrix} \to \begin{bmatrix} u \\ v \end{bmatrix}$, $U \begin{bmatrix} u_n \\ v_n \end{bmatrix} \to \begin{bmatrix} w \\ z \end{bmatrix}$, in $\mathcal{H}$, $\begin{bmatrix} u_n \\ v_n \end{bmatrix} \in D(U)$,

then we conclude directly that $v = w$ and the $L^2$ limit of $\kappa u_n''$ is $z + \nu w$. From this relation, we conclude that $u'$ has an $L^2$ derivative, so that

$$\begin{bmatrix} u \\ v \end{bmatrix} \in D(U), \text{ with image } \begin{bmatrix} w \\ z \end{bmatrix}.$$ 

It follows that $U$ is closed.

To prove statement (1), suppose that Re $\lambda > \theta$, and consider the formal system,

$$(\lambda I - U) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} w \\ z \end{bmatrix}.$$ 

Since $\lambda$ is permitted to be complex, for this part of the proof we interpret $\mathcal{H}$ and $D(U)$ as complex Hilbert spaces. We show that $\lambda I - U$ is an algebraic isomorphism from $D(U)$ to $\mathcal{H}$, i.e., the formal system is uniquely solvable. By using the definition of $U$, we see that $v = \lambda u - w$, provided $u$ is determined. This means that the following differential equation, with homogeneous boundary values, must determine a unique solution $u$ in $H^2(I) \cap H_0^1(I)$: $-\kappa u'' + \lambda(\lambda + \nu)u = z + (\lambda + \nu)w$, for this pair $w, z$. Now the spectrum of the self-adjoint operator $-\kappa((d/dx)^2)$ consists of the eigenvalues $\kappa n^2\pi^2, n = 1, 2, \ldots$, so that the product $-\lambda(\lambda + \nu)$ must exclude these numbers. A calculation, based upon proof by
contradiction, shows that this is achieved for \( \text{Re } \lambda > \theta \). We conclude that \( \lambda \mathcal{L} - U \) is surjective and injective from \( D(U) \) to \( \mathcal{H} \). It is, by the definition of norms, a bounded linear operator between these spaces. Its algebraic inverse, \( R(\lambda, U) \), is a bounded linear operator, as follows from the open mapping theorem. Indeed, a bound on the \( D(U) \) norm, which dominates the \( \mathcal{H} \) norm, is obtained.

We now prove part (2), and operate within real Hilbert spaces. We verify that the resolvent of \( U \), \( R(\lambda, U) \), satisfies the norm inequality in \( \mathcal{H} \),

\[
\|R(\lambda, U)\| \leq \frac{1}{\lambda}, \ \lambda > 0. \tag{9}
\]

The Hille-Yosida theorem \cite{7} then implies that \( U \) generates a contraction semigroup. To establish \( \| \), it is sufficient to show that the inner products \( (Uf, f) = (\mathcal{H}_n, f) \) are nonpositive, as is seen by expanding the square of the \( \mathcal{H} \)-norms of both sides of \( \lambda f - Uf = g \). Now we compute, for \( f = \begin{bmatrix} u \\ v \end{bmatrix} \),

\[(Uf, f)_{\mathcal{H}} = (v, u)_{H^1_0} + (\kappa u'' - \nu v, v)_{L^2}.
\]

After integration by parts, the rhs reduces to \(-\nu(v, v)_{L^2}\), which is nonpositive. Note that the cancelation involved here depends on the equivalent norm introduced in (3).

The statement that \( \| \) is well defined and provides a solution of \( \mathcal{H} \) depends fundamentally on the characterization of \( D(U) \) in terms of the limit of difference quotients of \( T(t) \). The result follows by direct computation. The details of the computation may be found in [12, Section 6.4], in the general case of the evolution operator \( \mathcal{U}(t, s) \). For the current result, \( \mathcal{U}(t, s) = T(t - s) \).

To establish part (3), we use the lemma cited at the conclusion of the proof, in conjunction with part (1). This concludes the proof.

The following lemma is quoted from [20, Theorem 3].

**Lemma 2.1.** A bounded \( C_0 \)-semigroup \( e^{At} \) on a Banach space \( X \) with (unbounded) generator \( A \) satisfies

\[\|e^{At}(A - \lambda \mathcal{L})^{-1}\| \to 0, \text{ as } t \to \infty, \lambda \in \rho(A),\]

if and only if the imaginary axis is in the resolvent set of \( A \): \( i\mathbb{R} \subset \rho(A) \).

An equivalent form of this result was obtained earlier in [3]. The special case of this result for \( \lambda = 0 \) was obtained in [1,2] and [21].

### 3 Fourier Projections: Approximate Problems

Recall the notation \( I = (-1, 1) \).

**Definition 3.1.** We retain the meaning of \( D(U) \), introduced in Definition 2.7. Suppose that \( F: D(U) \to H^1_0(I) \) is a given continuous operator, with \( F(0) = 0 \). We make the following additional assumptions.
• \( F \) is locally bounded: \( \forall C > 0, \) and \( \mathcal{B} = \{ u \in \mathcal{D}(U) : \| u \|_{\mathcal{D}(U)} \leq C \} \), there is a constant \( c = c(C) \) such that
\[
\| F(u) \|_{H^1_0(I)} \leq c, \; \forall u \in \mathcal{B}.
\]

• \( F \) satisfies a closure property: For any sequence \( \{ u_n \} \subset \mathcal{D}(U) \):
\[
\text{If } u_n \to u, \text{ in } H^1_0(I), \text{ and } F(u_n) \to f \text{ in } L^2(I),
\]
then \( f \in H^1_0(I) \) and \( f = F(u) \).

Also, we assume that \( G : \mathcal{D}(U) \mapsto C(\bar{I}) \) is continuous, with \( \inf_I G(0) > 0 \).

The reduced assumption of continuity on \( F \) seems to exclude the application of standard fixed point theorems in the space \( C(J, H^1) \). Ultimately, we will look for strong solutions of approximate problems, defined via Fourier projection operators.

**Definition 3.2.** Consider the system \( \{ \phi_n(x) := \sin(n\pi x) \}_{n \geq 1} \), which is a complete orthogonal system in \( L^2(I) \) and \( H^1_0(I) \), and orthonormal in \( L^2(I) \). Finally, denote by \( P_n \) the (closed) linear subspace of \( C(J, H^1_0(I)) \) defined by
\[
P_n = \left\{ \sum_{k=1}^{n} \alpha_k(t)\phi_k(x) : \alpha_k \in C(J), k = 1, \ldots, n \right\},
\]
and write \( P_n \) for the projection in \( C(J, H^1_0(I)) \) onto \( P_n \).

A direct implementation of the definition shows that \( Q_n u \) has the same formal representation as the \( L^2(I) \) projection. Furthermore, as verified in [14], \( P_n \) is closed, \( P_n = P^2_n \) is a standard projection, and \( P_n \) can be interpreted, for each fixed \( t \in J \), as the orthogonal projection \( Q_n \) onto \( \mathcal{M}_n \).

### 3.1 The unconstrained equation with projection

For a given operator \( F = F(u) \), satisfying Definition 3.1, consider the system,
\[
\begin{align*}
\frac{u_{tt}}{} &= -\nu u_t + \kappa u_{xx} + P_n F(u), \\
u(-1, t) &= u(1, t) = 0, \; t > 0, \\
\frac{u(0, x)}{u_t(0, x)} &= 0, \; x \in I,
\end{align*}
\]
where \( P_n \) is defined in Definition 3.2.

**Definition 3.3.** Suppose that \( F \) is described in Definition 3.1 and suppose \( C > 0 \) is fixed. For \( \omega \) defined in Proposition 2.1, we define the terminal time \( T_0 \) as
\[
T_0 = \frac{C}{\omega c},
\]
where \( c = c(C) \) is the constant defined in Definition 3.1.
The mapping $K$ is invariant on $B_n$ if $T_0$ is the time given by (11).

Proof. Given $z$, and $F_0, F_1$ as in Definition 3.3 we use (7) in the form,

$$V(t) = \int_0^t T(t - s)F_1(s) \, ds, \quad t \leq T_0,$$  \hfill (12)

which is estimated in the norm of $C(J, D(U))$ by the product $T_0 \omega_c$. Note that we have made use of Proposition 2.1. It follows that the first component $u(t)$ of $V(t)$ satisfies $\|u\|_{C(J, D(U))} \leq C$, so that for $v = P_n F(u)$ we have $\|v\|_{C(J, H^1_0)} \leq c$. By the definition of $K$, we conclude that $K$ acts invariantly on $B_n$.  \hfill \Box

We now establish the properties required to verify that $K$ is a compact mapping.

Lemma 3.2. The range of $K$ is relatively compact in $C(J, H^1_0(I))$.

Proof. According to [19, Th. 2.3.14], [16, Th. 6.1], it suffices to verify the following properties.

1. There is a fixed set $C \subset H^1_0(I)$, with compact closure, such that, for each $t \in J$, $\{Kz(t) : z \in B_n\} \subset C$.

2. The set $KB_n$ is equicontinuous in $C(J, H^1_0(I))$. This means that, given $\epsilon > 0$, there is $\delta = \delta(\epsilon, C, n)$, such that, $\forall t_1, t_2 \in J$, and $z \in B_n$,

$$\|Kz(t_1) - Kz(t_2)\|_{H^1_0(I)} \leq \epsilon \text{ if } |t_1 - t_2| \leq \delta.$$ 

The verification of property (1) is immediate; take $C$ to be the closed ball in $\mathcal{M}_n$ of radius $c$.

The verification of property (2) requires the preliminary observation that the family $u(t)$, occurring in the definition of $K$, is equicontinuous. This follows immediately from (12). Moreover, this family has range in $\mathcal{M}_n$, for each $t$. This can be verified by a standard separation of variables argument in conjunction with (12); we provide this at the conclusion of the proof. It follows that the family $F(u(t))$ is equicontinuous, since $F$ is uniformly continuous on an appropriate compact subset of $\mathcal{M}_n$. Composition with $P_n$ preserves the stated equicontinuity.

We now consider the range of $u(t)$, which is constructed from the integral in (12), where $F_1 = (0, z), z \in B_n$. Linearity of the semigroup $T$ allows us to examine the action of $T(t)$ on $(0, \phi_n)$. We observe that the separated solutions of $u_{tt} = -nu_t + \kappa u_{xx}$ fall into three categories, depending on the expression $4n^2 \pi^2 \kappa - \nu^2$. 

\footnote{Constrained nonlinear telegraph equation}
Constrained nonlinear telegraph equation

1. $4n^2\pi^2\kappa - \nu^2 > 0$. In this case, the separated solutions are of the form, 
   \[(aC_n(t) + bS_n(t))\phi_n(x),\] where 
   \[C_n(t) := \exp(-\nu/2)t \cos(\omega_n t), \quad S_n(t) := \exp(-\nu/2)t \sin(\omega_n t),\] 
   \[\omega_n = \frac{\sqrt{4n^2\pi^2\kappa - \nu^2}}{2}.\]

2. $4n^2\pi^2\kappa - \nu^2 = 0$. In this case, the separated solutions are constructed by replacing $C_n$ and $S_n$ by $c_n$ and $s_n$, where 
   \[c_n(t) := \exp(-\nu/2)t, \quad s_n(t) := t \exp(-\nu/2)t.\]

3. $4n^2\pi^2\kappa - \nu^2 < 0$. Here, the replacements are $D_n$ and $E_n$, where 
   \[D_n(t) := \exp((-\nu/2 + \rho_n)t), \quad E_n(t) := \exp((-\nu/2 - \rho_n)t).\] 
   Here, $\rho_n = \sqrt{\nu^2 - 4n^2\pi^2\kappa}/2$.

We can construct the action of $T(t)$ on $(0, \phi_n)$ by using the properties $T(0) = \mathcal{I}$, $(d/dt)T(t) = UT(t)$.

1. $4n^2\pi^2\kappa - \nu^2 > 0$. We have $T(t)(0, \phi_n) = (u_*, v_*)$, where 
   \[u_*(x, t) = (1/\omega_n)S_n(t)\phi_n(x), \quad v_*(x, t) = (C_n(t) - \nu/(2\omega_n)S_n(t))\phi_n(x).\]

2. $4n^2\pi^2\kappa - \nu^2 = 0$. We have $T(t)(0, \phi_n) = (u_*, v_*)$, where 
   \[u_*(x, t) = s_n(t)\phi_n(x), \quad v_*(x, t) = (c_n(t) - (\nu/2)s_n(t))\phi_n(x).\]

3. $4n^2\pi^2\kappa - \nu^2 < 0$. We have $T(t)(0, \phi_n) = (u_*, v_*)$, where 
   \[u_*(x, t) = 1/(2\rho_n)(D_n(t) - E_n(t))\phi_n(x), \quad v_*(x, t) = (1/2)[(1 - \nu/(2\rho_n))D_n(t) + (1 + \nu/(2\rho_n))E_n(t)]\phi_n(x).\]

In all three cases, since formula (12) involves integration in the variable $t$, we conclude that the first component $u$ is a function from $J$ to $\mathcal{M}_n$.

\[\square\]

We will use the Schauder fixed point theorem in the following form [3].

**Lemma 3.3.** Let $K$ be a closed convex subset of a Banach space $\mathcal{X}$, and let $K$ be a continuous mapping of $K$ into itself, such that the image is relatively compact, i.e., has compact closure. Then $K$ has a fixed point.

**Corollary 3.1.** Suppose that the function $F$ satisfies Definition [3]. There is a (local) strong solution to the unconstrained system [10]. More precisely, if $J = [0, T_0]$, with $T_0$ defined by [11], there is a function $u \in C(J, D(U))$, with $u_{tt} \in C(J, L^2(J))$, such that [10] holds.
Proof. The hypotheses of the Schauder fixed point theorem are satisfied regarding the mapping $K : B_n \mapsto B_n$. Specifically, we have shown that $K$ is compact. The continuity of $K$ follows from (12), combined with the continuity of $F$. In particular, there is a fixed point for $K$, which coincides with a solution of (10).

The final result of this section justifies the use of the Fourier projection.

Lemma 3.4. Suppose that $\{h_n\}$ is a sequence which is bounded in $C(J, H^1_0(I))$, and convergent in $C(J, L^2(I))$ to a function $h$. Then $P_n h_n \to h$, $n \to \infty$.

Proof. Suppose that $b$ is a bound for $h_n$ in $C(J, H^1_0(I))$. For each fixed $t$, consider the self-adjoint operator $R = -(\kappa/b^2)(d/dx)^2$ on $D(U)$, and the ellipsoid in $L^2(I)$ defined by

$$ R(t) = \{ u \in D(U) : (Ru, u)_{L^2} \leq 1 \}. $$

Then the $L^2(I)$ $n$-width of $R(t)$ is attained by the subspace $M_n$ for each fixed $t$ [11], so that, by closure, this holds true for the widths $d_n$ of the closed $H^1_0(I)$-ball of radius $b$. The latter are directly computable [11] in terms of the eigenvalues of $R$: $d_n = \lambda_{n+1}^{-1/2} = \frac{b}{(n+1)\pi\sqrt{\kappa}}$. This implies that, in $L^2(I)$, uniformly in $t$, $Q_n h_n - h_n \to 0$, $n \to \infty$. Equivalently,

$$ P_n h_n - h_n \to 0, \ n \to \infty. $$

The result follows from the triangle inequality.

3.2 The constrained equation with projection

We now add the constraint to the system [13]. We have:

Theorem 3.1. Suppose that $G$ is given as in Definition 3.1. Suppose, by the assumed continuity, that $C > 0$ is selected so that $\inf_I G(u) \geq \alpha > 0, u \in B$. Then the solution identified in Corollary 3.1 satisfies [10] and the additional condition that $\inf_I G(u) \geq \alpha > 0, t \leq T_0$.

Proof. The result follows from the selection of $C$ and the corollary. Note that $C$ does not depend on $n$. 

4 Main Result

4.1 The local weak solution for the model

Definition 4.1. Suppose there exist $T_0$ and $u$ such that:

1. $(u, u_t)$ is weakly continuous from $J$ to $D(U)$, continuous from $J$ to $H$, with $u_{tt}$ essentially bounded from $J$ to $L^2(I)$.
2. The initial conditions are satisfied: \( u(0, x) = u_t(0, x) = 0, \ x \in I \).

3. \( \forall \phi \in D(U), \forall 0 < t \leq T_0, \)

\[
(u_{tt}, \phi)_{L^2} = -\nu(u_t, \phi)_{L^2} - \kappa(u_x, \phi_x)_{L^2} + (F(u), \phi)_{L^2},
\]

\[
\inf_I G(u) > 0.
\]

Then we say that there exists a weak local solution for the system \((1)\).

**Theorem 4.1.** There exists a weak local solution satisfying Definition [4.1].

**Proof.** We consider the family \( \{u_n\} \) of solutions of the approximate problems described in Theorem 3.1. The sequence constructed from this family and its time derivative is bounded in \( D(U) \) and satisfies the hypotheses of [5, Prop. 1.1.2] where \( X \) of [5] is identified with \( D(U) \) and \( Y \) of [5] is identified with the Hilbert space \( \mathcal{H} \). The assumed equicontinuity of the sequence relative to the metric in \( \mathcal{H} \) follows directly from the fundamental theorem of calculus applied to the integral representation of the \( D(U) \)-bounded sequence. We may conclude the existence of a pair \((u, u_t)\) satisfying (1) of Definition 4.1. Furthermore, this pair is the weak limit in \( D(U) \), uniformly in time, of a subsequence, say, \( \{u_{n_k}\} \) and the corresponding time derivatives, and the strong limit in \( \mathcal{H} \). We may select the subsequence so that \( \{F(u_{n_k})\} \) is convergent in \( L^2(I) \). Consider the weak relations satisfied by the approximate solutions, which follow directly from the strong solution definition: \( \forall \phi \in D(U), \forall 0 < t \leq T_0, \)

\[
(u_{tt}, \phi)_{L^2} = -\nu(u_t, \phi)_{L^2} - \kappa(u_x, \phi_x)_{L^2} + (F(u), \phi)_{L^2},
\]

\[
\inf_I G(u_{n_k}) \geq \alpha > 0.
\]

By taking limits in (13), and using Lemma 3.4 together with the closure property of \( F \), we obtain

\[
(u_{tt}, \phi)_{L^2} = -\nu(u_t, \phi)_{L^2} - \kappa(u_x, \phi_x)_{L^2} + (F(u), \phi)_{L^2}, \ \inf_I G(u) \geq \alpha > 0.
\]

The initial conditions follow from the continuity properties.

\[\square\]

## 5 Concluding Remarks

We have analyzed an operator forcing for the nonlinear telegraph equation, which reflects recent studies of this equation, allowing for an intermediate or concurrent process. The forcing is continuous and bounded from the domain of the generator of the linear part into \( H^1_0 \). Although some of the results of section two can be extended to the multidimensional case, we have retained the one-dimensional framework throughout.
The assumed continuity, rather than uniform or Lipschitz continuity, imposes a technical hurdle, which is overcome by modal analysis. The semigroup is invariant on the individual modes, which is maintained by the convolution formula. This permits continuity to be extended to uniform continuity, and thus allows us to establish the equicontinuity, necessary for operator fixed point compactness.

The semigroup format fits nicely with the constraint imposed on the model. The initial quiescent state satisfies the constraint, and the system evolves according to a time selected to maintain an admissible state.

Although the operator case for forcing has been emphasized, the standard example of a smooth real-valued function, vanishing at the origin, is included. In particular, arbitrary monomial functions are included. Even rapid growth functions, such as \( \sinh(u) \), are included. However, the principal application envisioned is the case where \( u \) is the input to a model, such as a boundary value problem, and \( F(u) \) is the output. This is the case in [14]; however, in this application, \( F \) is continuous only into \( H^\sigma(I) \), for \( 0 \leq \sigma < 1/2 \). Existence and related questions remain open in this case. For this case, the constraint is readily handled by the choice \( G(u) = 1 + u \).

In this article, uniqueness is not established, either for the approximate solutions or for the weak solutions. However, as required in the first part of Definition 4.1, the regularity class for the weak solution and its time derivative is the same as for the strong solution.

References

[1] C.J.K. Batty, Tauberian theorems for the Laplace-Stieltjes transform, Trans. Amer. Math. Soc. 322 (1990), 783-804.

[2] C.J.K. Batty, Asymptotic behavior of semigroups of operators. Functional analysis and operator theory (Warsaw, 1992). 30, pp. 35-52. Banach Center Publ., Polish Acad. Sci. Warsaw (1994).

[3] C.J.K. Batty and T. Duyckaerts, Non-uniform stability for bounded semigroups on Banach spaces. J. Evol. Equ. 8 (2008), 765-780.

[4] C. Bereau, Periodic solutions of the nonlinear telegraph equations with bounded nonlinearities. J. Math. Anal. Appl. 343 (2008), 758-762.

[5] T. Cazenave, Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics 10. New York University, Courant Institute of Mathematical Sciences. American Mathematical Society, Providence R. I., 2003.

[6] M.S. El-Azab and M. El-Gamel, A numerical algorithm for the solution of telegraph equations. Appl. Math. Comp. 190 (2007), 757-764.

[7] K.-J. Engel and R. Nagel, One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics, vol. 194. Springer-Verlag, New York, 2000.
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[8] S. Fucik and J. Mawhin, Generalized periodic solutions of nonlinear telegraph equations. Nonlinear Anal. 2 (1978), 609-617.

[9] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order. Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[10] T.S. Jang, A new solution procedure for the nonlinear telegraph equation. Commun. Nonlinear Sci. Numer. Simul. 29 (2015), 307-326.

[11] J.W. Jerome, On the $L_2$ n-width of certain classes of functions of several variables. J. Math. Anal. Appl. 20 (1967), 110-123.

[12] J.W. Jerome, Approximation of Nonlinear Evolution Equations. Academic Press, 1983.

[13] J.W. Jerome, A tight nonlinear approximation theory for time dependent closed quantum systems. J. Numer. Math., to appear (arXiv: 1709.09063).

[14] P. Laurencot and C. Walker, Some singular equations modeling MEMS. Bull. Amer. Math. Soc. 54 (2017), no. 3, 437-479.

[15] J. Mawhin, R. Ortega, and A.M. Robles-Pérez, A maximum principle for bounded solutions of the telegraph equations and applications to nonlinear forcings. J. Math. Anal. Appl. 251 (2000), 695-709.

[16] J.R. Munkres, Topology, A First Course. Prentice-Hall, 1975.

[17] M.A. Pinsky, Partial Differential Equations and Boundary-Value Problems with Applications, Reprint of the third edition. Pure and Appl. Undergraduate Texts, 15. Amer. Math. Soc., Providence, R. I., 2011.

[18] J. Rauch and M. Taylor, Exponential decay of solutions to hyperbolic equations on bounded domains. Indiana Univ. Math. J. 24 (1974), 79-86.

[19] B. Simon, Real analysis, A Comprehensive Course in Analysis, Part 1, Amer. Math. Soc., Providence, R. I., 2015.

[20] G. Sklyar, On the decay of bounded semigroup on the domain of its generator, Vietnam J. Math. 43 (2015), no. 2, 207-213.

[21] Q.-P. Vu, Theorems of Katznelson-Tzafrirri type for semigroups of operators. J. Funct. Anal. 103 (1992), 74-84.