BRST COHOMOLOGY FOR 2D GRAVITY

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Abstract

The BRST cohomology group in the space of local functionals of the fields for the two-dimensional conformally invariant gravity is calculated. All classical local actions (ghost number equal to zero) and all candidate anomalies are given and discussed for our model.
1 Introduction

Gauge fields play a very important role in all theories which describe the fundamental interactions [1]. The most efficient way to study the quantization and the renormalization of all gauge (local) theories is given by the BRST transformations and by the so-called BRST cohomology [2, 3, 4]. Despite the fact that gravity could be introduced as a gauge theory associated with local Lorentz invariance [5], its action has a different structure and it is difficult to connect it to a special form of the Yang-Mills theory, known as the topological quantum field theory [6]. However, in the BRST quantization framework the structure of the invariant action, the anomalies and the Schwinger terms can be obtained in a purely algebraic way by solving the BRST consistency conditions in the space of the integrated local field polynomials [7, 8]. This fact has been known since the work of Wess and Zumino [9] and for a general gauge theory the general form of these can be elegantly formulated in the BV formalism [3]. In this framework, the Wess-Zumino consistency condition [7, 8] can be written as:

\[ sA = (S, A) = 0 \] (1.1)

where \( S \) is the proper solution of the master equation, \( A \) are integrated local functionals and \( s \) is the BRST nilpotent differential [7, 8, 10]. The solutions of (1.1) modulo the exact forms \( A = sB \) can be organized in an abelian group \( H(s) \), the BRST cohomology group. We can introduce a graduation in \( H(s) \) by the ghost number \((gh = g)\) and we can decompose it in a direct sum of subspaces with a definite ghost number:

\[ H(s) = \oplus H^g(s) \] (1.2)

Anomalies are represented by cohomology classes of \( H^1(s) \), but it is often useful to compute \( H^g(s) \) for other values of \( g \) as well since \( H^0 \) contains the BRST invariant action and \( H^2(s) \) the Schwinger terms. Besides, the whole \( H(s) \) could play an important role in solving and understanding the descent equations [7, 8] and in the study of structure of the field configurations.

In this paper we are going to investigate the structure of \( H(s) \) for a class of two dimensional models which are conformally invariant at the classical level. We will not characterize these models by specific conformally invariant classical actions but we rather specify the field content
and the gauge invariances of the theory. We will try to obtain \( H(s) \) for a general framework independent on the local classical action \( S_0 \). Thus we are not going to introduce the antifields \( \mathbb{F}^\mathbb{F} \) which have the BRST transformations dependent on \( S_0 \).

The paper is organized as follows: in Sect.2 we recall the field content and the gauge symmetries of our model. In Sect.3 we give the equations which have to be solved in order to find out invariant Lagrangians, anomalies and Schwinger terms. Sect.4 deals with the analysis of the algebra of all fields and their derivatives. We shall show that \( A \) can be splitted in its contractive part \( C \) and its minimal subalgebra \( M \). In the minimal algebra we introduce, following \([11]\) a very convenient basis where the solution of (1.1) are very simple. At the end we give the structure of all nontrivial sectors of \( H(s) \) and in Sect.5 we give some comments of our results.

## 2 Two-dimensional conformal gravity

Our main aim is to compute the BRST cohomology group for a class of two-dimensional models which are conformally invariant at the classical level. We are not going to characterize these models by the local classical action \( S_0 \), but we will provide only the field contents and the gauge invariance of the classical theory. Then we can define the BRST differential \( s \) and we can compute the BRST cohomology group \( H^*(s) \) in the space of local functionals of the fields. Thus, for ghost number zero, this group provides the most general local classical action \( S_0 \). For ghost number one, \( H^1(s) \) gives us the most general local anomalies. In this paper we are not going to take into account the antifields and we do not need the concrete form of the classical action \( S_0 \) which enters the BRST transformations.

The fields of our theory are the components of the 2D metric \( g_{\mu\nu} = g_{\nu\mu} \) and the set of bosonic scalar matter fields \( X = X^A, A = 1, \ldots, D \). It is convenient to replace the metric \( g_{\mu\nu} \) by the zweibein fields \( e^a_\mu \) such that:

\[
g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \tag{2.3}
\]

where \( \mu, \nu = 0, 1 \), \( a, b = 0, 1 \) and \( \eta_{ab} = (+, -) \) or by the moving frame

\[
e^a = e^a_\mu dx^\mu \tag{2.4}
\]
The conformal properties of the two-dimensional spacetime are most conveniently described in terms of light-cone coordinates:

\[ x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^1) \]

and the differential operators

\[ \frac{\partial}{\partial x^\pm} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right) \tag{2.5} \]

For the moving frame one defines the one-forms:

\[ e^\pm = \frac{1}{\sqrt{2}} (e^0 \pm e^1) \tag{2.6} \]

with the coefficients

\[ e^+ = (dx^+ + h_{--} dx^-) e^+_+ \tag{2.7} \]
\[ e^- = (dx^- + h_{++} dx^+) e^-^- \tag{2.8} \]

Here

\[ h_{++} = \frac{e^+_+}{e^+_+}, \quad h_{--} = \frac{e^-^-}{e^-^-} \tag{2.9} \]

are the gauge fields which occur for the nonchiral Virasoro algebra. They can be expressed in terms of the components of the metric \( g_{\mu\nu} \) by using the definitions \( (2.3) \) and \( (2.4) \). The metric becomes:

\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{2.10} \]

where \( \alpha, \beta = +, - \). The components of the metric in light-cone frame read out as:

\[ g_{\pm\pm} = \frac{1}{2} (g_{00} \pm 2g_{01} + g_{11}) \]
\[ g_{+-} = \frac{1}{2} (g_{00} - g_{11}) \tag{2.11} \]
\[ g_{-+} = g_{+-} \]

and employing the zweibein

\[ ds^2 = \eta_{ab} e^a e^b \tag{2.12} \]

Thus, we can write the gauge fields as follows:

\[ h_{\pm\pm} = \frac{g_{\pm\pm}}{g_{+-} + \sqrt{g}} \tag{2.13} \]
with \( g = |\text{det}(g_{\mu\nu})| \).

The fields \( h_{\pm\pm} \) are inert under local Lorentz (structure) group transformations and change under diffeomorphisms (i.e., from one chart to another) in the following way:

\[
h_{++} = \frac{(\partial x'^+ / \partial x^-) + (\partial x'^- / \partial x^-) h'_{++}(x')}{(\partial x'^+ / \partial x^-) + (\partial x'^- / \partial x^-) h'_{++}(x')}
\]

\[
h_{--} = \frac{(\partial x'^- / \partial x^+)}{(\partial x'^- / \partial x^+) + (\partial x'^+ / \partial x^-) h'_{--}(x')}\]

(2.14)

(2.15)

Besides, the fields \( h_{\pm\pm} \) remain invariant under the Weyl transformation \( g_{\alpha\beta} \rightarrow e^\sigma g_{\alpha\beta} \). The conformal factors \( e^+_- \) and \( e^-_- \) carry entirely the local Lorentz group transformations and the Weyl transformation. Thus, if we want to define a theory that is invariant under the local Weyl transformations of the metric \( g_{\alpha\beta} \) we have to work only with the fields \( h_{\pm\pm} \) instead of the whole metric \( g_{\alpha\beta} \).

In the parametrization (2.7) and (2.8) of the moving frame, the structure group and Weyl transformations are carried entirely by the conformal factors \( \epsilon^\pm_\pm \). Besides, upon general coordinate transformations they change as:

\[
e^+_+(x) = \left[ \frac{\partial x'^+}{\partial x^+} + \frac{\partial x'^-}{\partial x^-} h'_{++}(x) \right] e^+_+(x')
\]

(2.16)

\[
e^-_-(x) = \left[ \frac{\partial x'^-}{\partial x^-} + \frac{\partial x'^+}{\partial x^+} h'_{--}(x) \right] e^-_-(x')
\]

(2.17)

Under a Weyl transformation, the fields \( h_{\pm\pm} \) and the matter fields \( X \) remain invariant and the conformal factors \( e^+_\pm \) and \( e^-_\pm \) change as:

\[
e^+_+(x) = e^{-\tilde{\sigma}_+} e^+_+(x)
\]

(2.18)

\[
e^-_-(x) = e^{-\tilde{\sigma}_-} e^-_-(x)
\]

Upon a general coordinate transformation they change as:

\[
e^+_+(x) = \left[ \frac{\partial x'^+}{\partial x^+} + \frac{\partial x'^-}{\partial x^-} h'_{++}(x) \right] e^+_+(x)
\]

(2.19)

\[
e^-_-(x) = \left[ \frac{\partial x'^-}{\partial x^-} + \frac{\partial x'^+}{\partial x^+} h'_{--}(x) \right] e^-_-(x)
\]

(2.20)
The geometric description, developed so far, provides the base for the description of the two-dimensional theories which have local reparametrization (diffeomorphism) as well as Weyl and Lorentz invariance. Upon quantization, the whole theory must undergo the standard BRST treatment (or quantization) to control the degeneracies due to the local gauge transformations. This will be done in the next section.

The matter field $X = \{X^\mu\}$ are scalar under the diffeomorphism, i.e., they have the following transformation under a general coordinate transformation:

$$X(x) = X'(x').$$

We will suppose that $X(x)$ have the Weyl weight zero, i.e., they are invariant under the Weyl transformation. In fact, in D dimensions the Weyl weight for $X(x)$ is

$$w(X) = -\frac{D-2}{2}$$

which is zero for $D = 2$.

3  **BRST transformations of the model**

The $BRST$ transformations and the $BRST$ differential can be obtained in general by simply replacing the infinitesimal gauge parameters $\varepsilon^\alpha$ which occur in the gauge transformation:

$$\delta_{\varepsilon} \varphi^i = R^i_\alpha \varepsilon^\alpha$$

by the ghosts $c^\alpha(x)$:

$$s \varphi^i = R^i_\alpha c^\alpha$$

The $BRST$ transformations of the ghosts are defined by demanding that $s$ should be nilpotent.

For an infinitesimal diffeomorphism we have

$$x'^\mu = x^\mu + \varepsilon^\mu(x)$$

with $\varepsilon^0$ and $\varepsilon^1$ infinitesimal arbitrary functions of $x$. Under the transformation \([3.1]\) $h_{\pm\pm}$ change as:

$$\delta h_{++} = h_{++}(\partial_- \varepsilon^- + h_{++} \partial_+ \varepsilon^+) - (\partial_+ \varepsilon^- + h_{++} \partial_+ \varepsilon^+) - (\varepsilon^+ \partial_+ + \varepsilon^- \partial_-)h_{++}$$

$$\delta h_{--} = h_{--}(\partial_- \varepsilon^+ + h_{--} \partial_+ \varepsilon^-) - (\partial_- \varepsilon^+ + h_{--} \partial_+ \varepsilon^-) - (\varepsilon^- \partial_- + \varepsilon^+ \partial_+)h_{--}$$
where $\partial_+ = \partial/\partial x^+$ and $\partial_- = \partial/\partial x^-$. 

The matter field $X$ has the following transformation under the diffeomorphism (3.1):

$$
\delta X = (\varepsilon^+ \partial_+ + \varepsilon^- \partial_-)X
$$

(3.3)

From (3.1), (3.3), (3.3) we get the BRST transformations by simply replacing the infinitesimal gauge parameters $\varepsilon^\pm$ by the diffeomorphism ghost $\xi^\pm$. These transformations can be simplified if new ghost fields $c^\pm$ are introduced following the line of [11, 17, 19, 20]:

$$
c^\pm = \xi^\pm + h^\pm \xi^\mp
$$

(3.4)

Thus, the BRST transformations of $h_{++}$, $h_{--}$, $X$ are:

$$
sh_{++} = \partial_+ c^- - h_{++} \partial_- c^- + c^- \partial_- h_{++}
$$

$$
sh_{--} = \partial_- c^+ - h_{--} \partial_+ c^+ + c^+ \partial_+ h_{--}
$$

(3.5)

$$
sX = c^+ D_+ X + c^- D_- X
$$

where:

$$
D_\pm X = \frac{1}{1 - y}(\partial_\pm - h_{\pm \pm} \partial_\mp)X = \frac{1}{1 - y} \nabla_\pm X
$$

(3.6)

The BRST transformations of the ghosts $c^\pm$ can be obtained from the nilpotency of $s$ and the fact that it commutes with $\partial_\pm$. In this way we conclude that

$$
s c^\pm = c^\pm \partial_\pm c^\pm
$$

(3.7)

Under the Weyl transformation $h_{++}$ and $h_{--}$ remain invariant and only the conformal factors $e^+_+$ and $e^-_-$ change.

In the framework of the BRST transformations, the search for invariant Lagrangian anomalies and Schwinger terms can be done in a purely algebraic way, by solving the BRST consistency condition in the space of integrated local polynomials [1, 12, 13]. This amounts to study the non-trivial solutions of the equation

$$
s \mathcal{A} = 0
$$

(3.8)
where $\mathcal{A}$ is an integrated local functional $\mathcal{A} = \int d^2f$. The condition (3.8) translates into the local descent equations [14]:

$$
\begin{align*}
\mathbf{s}\omega_2 + d\omega_1 &= 0 \\
\mathbf{s}\omega_1 + d\omega_0 &= 0 \\
\mathbf{s}\omega_0 &= 0
\end{align*}
$$

(3.9)

where $\omega_2$ is a 2-form with $\mathcal{A} = \int \omega_2$ and $\omega_1, \omega_0$ are local 1-forms, respectively 0-forms. It is well known [12, 13, 16] that the descent equations terminate in the bosonic string or the superstring in Beltrami or super-Beltrami parametrization always with a nontrivial 0-form $\omega_0$ and that their integration is trivial:

$$
\begin{align*}
\omega_1 &= \mathbf{\delta}\omega_0; \\
\omega_2 &= \frac{1}{2} \mathbf{\delta}^2 \omega_0
\end{align*}
$$

(3.10)

where $\mathbf{\delta}$ is a linear operator, which allows to express the exterior derivative $d$ as a BRST commutator

$$
d = -[s, \mathbf{\delta}]
$$

(3.11)

This operator was introduced by Sorella for the Yang-Mills theory [14] and it was used for solving the descent equations (3.10) for the bosonic string [12] and superstring [16] in the Beltrami and super-Beltrami parametrization. It is easy to see that, once the last equation in the tower (3.10), i.e.,

$$
\mathbf{s}\omega_0 = 0
$$

(3.12)

is solved, the rest of the equations from (3.10) can be solved with the help of the operator $\mathbf{\delta}$ with the solutions (3.10). Thus, due to the operator $\mathbf{\delta}$, the study of the cohomology of $\mathcal{A} \mod d$ is essentially reduced to the study of the local cohomology of $s$ which, in turn, can be systematically analyzed by using different powerful techniques from the algebraic topology as the Sullivan and Künneth theorems, the spectral sequences [8], etc. Actually, as proven in [15] for the Yang-Mills theory, the solutions obtained by utilizing the decomposition (3.10) turn out to be completely equivalent to that based on the Russian formula [7].

The main purpose of our paper is to solve the descent equations (3.10) in the algebra of the local polynomials of all fields and their derivatives $\mathcal{A}$. A basis of this algebra can be chosen to
consist of:
\[
\{\partial^p_\pm \partial^n_\pm \psi, \partial^p_\pm \partial^n_\pm c^\pm\}
\]

where \(\psi = (X, h_{++}, h_{--})\) and \(p, q = 0, 1, 2, \ldots\). However, the BRST transformations of this basis are quite complicated and contain many terms which can be eliminated in \(H(s)\). In the next section we shall eliminate a part of this basis and introduce a new basis in which the action of \(s\) is quite simple. *****

4 The structure of the fields algebra

The calculation of \(H(s)\) can be considerably simplified if we take into account that \(\mathcal{A}\) as a free differential algebra and make use of a very strong theorem due to Sullivan. A free differential algebra is an algebra generated by a basis, endowed with a differential. The Sullivan’s theorem tells us the following:

**The most general free differential algebra \(\mathcal{A}\) is a tensor product of a contractible algebra and a minimal one**

A minimal differential algebra \(\mathcal{M}\) with the differential \(s\) is one for which \(s\mathcal{M} \subset \mathcal{M}^+, \mathcal{M}^+\) being the part of \(\mathcal{M}\) in positive degree, i.e. \(\mathcal{M} = \mathcal{C} \oplus \mathcal{M}^+\) and a contractible differential algebra \(\mathcal{C}\) is one isomorphic to a tensor product of algebras of the form \(\Lambda(x, sx)\).

On the other hand, due to Künneth’s theorem the cohomology of \(\mathcal{A}\) is given by the cohomology of its minimal part and we can say that the contractible part \(\mathcal{C}\) can be neglected in the calculation of \(H(s)\).

For our algebra, the construction of \(\mathcal{C}\) and \(\mathcal{M}\) is straightforward and we do not need any general method to accomplish that. In fact, it is easy to see from the BRST transformations of \(h_{\pm \pm}, X, c^\pm\) that the generators
\[
\partial^p_\pm \partial^n_\pm c^+, \partial^p_\pm \partial^n_\pm c^-
\]
with \(p = 0, 1, 2, \ldots\) and \(n = 1, 2, \ldots\) can be replaced by:
\[
\{\partial^p_\pm \partial^n_\pm X, \partial^p_\pm \partial^n_\pm \phi, s\partial^p_\pm \partial^n_\pm \phi, \partial^p_\pm c^\pm\}
\]
The Sullivan decomposition can be easily obtained from (4.2). Indeed, the contractible sub-
algebra is generated by:
\[ \{ \partial^p_- \partial^p_+ h_{\pm \pm}, s(\partial^p_- \partial^p_+ h_{\pm \pm}) \} \] (4.3)
and the minimal subalgebra \( \mathcal{M} \) by:
\[ \{ \partial^p_- \partial^p_+ X, \partial^p_\pm c^\pm \} \] (4.4)

Now, to calculate the cohomology \( H(s) \) we take into account only the basis (4.4). Never-
thless, this basis is not convenient since the differential \( s \) has a complicated action on it. The
investigation of the BRST cohomology, i.e., the solution of (3.12) is considerable simplified
by using an appropriate new basis substituting the fields \((X, c^\pm)\) and their derivatives. The con-
struction of this new basis is a crucial step towards the calculation of \( H(s) \). This new basis has
been proposed by Brandt, Troost and Van Proeyen [11] and it is basically intended to substitute
one by one the elements of the basis (4.4) by:
\[ \Delta^p_- \Delta^q_+ X = X^{p,q} \] (4.5)
\[ \frac{1}{(p + 1)!} \Delta^{p+1}_\pm c^\pm = c^p_\pm \] (4.6)
where
\[ \Delta_\pm = \left\{ s, \partial \frac{\partial}{\partial c^\pm} \right\} = s \frac{\partial}{\partial c^\pm} + \frac{\partial}{\partial c^\pm} s. \] (4.7)

It is crucial to remark that the linear operators \( \Delta^\pm \) act on the algebra \( \mathcal{A} \) as derivatives, i.e.,
they obey the Leibnitz rule:
\[ \Delta_\pm (ab) = \Delta_\pm a \cdot b + a \Delta_\pm b \] (4.8)
The action of \( s \) on the elements (4.3) which form a new basis, can be calculated directly
\[ sX^{p,q} = \Delta^p_- \Delta^q_+ (sX) = \sum_{k=0}^{\infty} \left[ \left( \begin{array}{c} p \\ k \end{array} \right) (\Delta^k_+ c^+) X^{k+1,q} + \left( \begin{array}{c} q \\ k \end{array} \right) (\Delta^k_- c^-) X^{p,k+1} \right] \] (4.9)
\[ sc^\pm_n = \frac{1}{(n + 1)!} \Delta^{n+1}_\pm (c^\pm \Delta_\pm c^\pm) \] (4.10)
since \( s \) commutes with \( \Delta_\pm \) and
\[ sX = c^+ X^{1,0} + c^- X^{0,1} \]
\[ \Delta_{\pm} c_{\pm} = 0. \]

The remarkable property of this new basis is the fact that its BRST transformation is given by the Virasoro algebra with associated ghosts just \( c_{\pm} \). Indeed, the last two equations (4.9), (4.10) can be rewritten as:

\[
s X^{p,q} = \sum_{k \geq -1} \left( c_{\pm}^{k} L_{k}^{\pm} \right) X^{p,q} \quad (4.11)
\]

\[
s c_{\pm}^{k} = \frac{1}{2} t_{mn}^{c_{\pm}^{m} c_{\pm}^{n}} \quad (4.12)
\]

where \( L_{k}^{\pm} \) and \( L_{-k}^{-} \) are given by the following equations:

\[
L_{k}^{\pm} X^{p,q} = A_{k}^{p} X^{p-k,q} \quad (4.13)
\]

\[
L_{-k}^{-} X^{p,q} = A_{k}^{p} X^{p,q-k} \quad (4.14)
\]

and

\[
A_{k}^{p} = \frac{p!}{(p-k-1)!} \quad (4.15)
\]

The BRST transformations of the ghosts \( c_{\pm}^{p} \) can easily be written as:

\[
s c_{\pm}^{p} = \frac{1}{(n+1)!} \Delta_{\pm}^{n+1} (c_{\pm}^{p} \partial_{\pm} c_{\pm}^{p}) = \sum_{k=-1}^{p} (p-k) c_{\pm}^{k} c_{\pm}^{p-k} = \frac{1}{2} \sum_{m,n \geq 0} f_{m,n}^{p} c_{\pm}^{m} c_{\pm}^{n} \quad (4.16)
\]

where

\[
f_{m,n}^{p} = (m-n) \delta_{m+n}^{p} \quad (4.17)
\]

are the structure constants of the Virasoro algebra.

Now, it is easy to see that \( L_{n}^{\pm} \) represent, on the basis \( X^{p,q} \), the Virasoro algebra according to:

\[
[L_{m}^{\pm}, L_{n}^{\pm}] = f_{m,n}^{k} L_{k}^{\pm} \quad [L_{m}^{\pm}, L_{n}^{\pm}] = 0 \quad (4.18)
\]

From (4.11) we can give another representation for the generators \( L_{n}^{\pm} \) on the algebra spanned by \( X^{p,q} \). They have the form:

\[
L_{n}^{\pm} = \left\{ s, \frac{\partial}{\partial c_{\pm}^{n}} \right\} \quad (4.19)
\]

and they can be extended to \( c_{\pm}^{n} \).

It is worthwhile remarking that on the basis \( c_{\pm}^{p}, X^{p,q} \) the BRST differential \( s \) can be written in
\[ s = \sum_{k=1}^{n} (c^+_k L^+_k + c^-_k L^-_k) + \frac{1}{2} \sum_{m,n,k} f^k_{m,n} (c^+_m c^+_n + \partial \frac{\partial}{\partial c^+_k} + c^-_m c^-_n \frac{\partial}{\partial c^-_k}) \] (4.20)

The generators \( L^\pm_0 \) are **diagonal** on all elements of the new basis. Indeed, one has:

\[ L^+_0 X^p,q = pX^p,q \quad L^-_0 X^p,q = qX^p,q \] (4.21)

\[ L^\pm_0 c^p_\pm = p c^p_\pm \] (4.22)

\[ L^\pm_0 c^p_\pm = 0 \] (4.23)

Thus, any product of elements of this basis is an eigenvector of the \( L^\pm_0 \).

Due to the fact that \( L^\pm_0 \) have the form:

\[ L^\pm_0 = s \frac{\partial}{\partial c^\pm} + \frac{\partial}{\partial c^0_\pm} \] (4.24)

we can conclude that the solutions of \( s \omega_0 = 0 \) must have the total weight \((0,0)\), all other contributions to \( \omega_0 \) being trivial.

All monomials with weight \((0,0)\) are tabulated below:

| Ghost | Monomial | \( s(\text{Monomial}) \) |
|-------|----------|--------------------------|
| 0     | \( F \)  | \((c^+ X^{1,0} + c^- X^{0,1})\partial F\) |
| 1     | \( c^0_+ F \)  | \(2c^+ c^1_+ F + c^0_+ \partial F(c^+ X^{1,0} + c^- X^{0,1})\) |
|       | \( c^0_- F \)  | \(2c^- c^1_- F + c^0_- \partial F(c^+ X^{1,0} + c^- X^{0,1})\) |
|       | \( c^+ X^{1,0} F \)  | \(-c^+ c^- X^{1,0} X^{0,1} \partial F + c^+ c^- X^{1,1} F\) |
|       | \( c^- X^{0,1} F \)  | \(-c^+ c^- X^{1,0} X^{0,1} \partial F - c^+ c^- X^{1,1} F\) |
| 2     | \( c^+ c^1_+ F \)  | \(-c^+ c^- c^1_+ X^{0,1} \partial F\) |
|       | \( c^- c^1_- F \)  | \(c^+ c^- c^1_- X^{1,0} \partial F\) |
| Ghost | Monomial | $s(\text{Monomial})$ |
|-------|----------|------------------|
| 2     | $c^0_+ c^0_- F$ | $c^0_+ c^0_- (c^+ X^{1,0} + c^- X^{0,1}) \partial F + 2(-c^0_+ c^1_- + c^+ c^1_+ c^0_-) F$ |
|       | $c^+ c^- X^{1,1} F$ | 0 |
|       | $c^+ c^- X^{1,0} X^{0,1} F$ | 0 |
|       | $c^0_+ X^{1,0} F$ | $-c^+ c^- c^0_+ X^{1,1} F + c^+ c^- c^0_+ X^{1,0} X^{0,1} \partial F$ |
|       | $c^- c^0_+ X^{0,1} F$ | $c^- c^0_+ c^0_+ X^{0,1} F - 2c^- c^+ c^1_+ X^{1,0} F$ |
|       | $c^0_+ c^- X^{0,1} F$ | $c^- c^0_+ c^0_+ X^{1,0} X^{0,1} \partial F$ |
|       | $c^+ c^0_+ X^{1,0} F$ | $-c^- c^- c^0_+ X^{1,0} X^{0,1} \partial F - c^+ c^- c^0_+ X^{1,1} F$ |
|       | $c^+ c^1_+ c^0_+ F$ | $-2c^- c^- c^1_+ X^{1,0} F - c^+ c^- c^0_+ X^{1,0} X^{0,1} \partial F + c^+ c^- c^0_+ X^{1,1} F$ |
| 3     | $c^0_+ c^0_+ c^0_- F$ | $c^+ c^0_+ c^0_+ c^- X^{0,1} \partial F$ |
|       | $c^- c^0_+ X^{1,0} F$ | $-c^- c^0_+ c^- + c^0_+ X^{1,0} \partial F$ |
|       | $c^+ c^1_+ c^0_+ F$ | $-c+ c^1_+ c^0_+ c^- X^{0,1} \partial F + 2c^+ c^1_+ c^- c^1_+ F$ |
|       | $c^- c^1_+ c^0_+ F$ | $-c^- c^1_+ c^0_+ c^+ X^{1,0} \partial F + 2c^- c^- c^1_+ c^1_+ F$ |
|       | $c^0_+ c^0_+ c^+ X^{1,0} F$ | $-c^0_+ c^+ c^- (c^0_+ X^{1,0} X^{0,1} \partial F + c^0_+ X^{1,1} F + 2c^1_+ X^{1,0} F)$ |
|       | $c^0_+ c^0_+ c^- X^{0,1} F$ | $-c^0_+ c^- c^- (c^0_+ X^{1,0} X^{0,1} \partial F + c^0_+ X^{1,1} F + 2c^1_+ X^{1,0} F)$ |
|       | $c^+ c^0_+ c^- X^{0,1} F$ | 0 |
|       | $c^- c^0_+ c^- X^{0,1} F$ | 0 |
|       | $c^0_+ c^- X^{1,0} X^{0,1} F$ | 0 |
|       | $c^+ c^- X^{1,0} X^{0,1} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,1} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,1} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{0,1} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
|       | $c^+ c^- c^0_+ X^{1,0} F$ | 0 |
TABLE 1

| Ghost | Monomial                                      | s(Monomial) |
|-------|-----------------------------------------------|-------------|
| 5     | $c^0_+ c^0_- c^1_+ c^- X^{0,1} F$             | 0           |
|       | $c^0_+ c^0_- c^1_+ c^+ X^{0,1} F$             | 0           |
|       | $c^0_+ c^1_+ c^- c^1_- F$                     | 0           |
|       | $c^0_+ c^1_+ c^- c^1_- F$                     | 0           |
| 6     | $c^+ c^1_+ c^- c^1_- ^0_+ c^0_- ^0$           | 0           |

TABLE 2

where $F = F(X)$ is an arbitrary smooth function of the matter fields $X$ and $\partial F = \partial F/\partial X$.

From this table we can calculate the cohomology group $H^g(s)$ by finding out which of the combinations of the above monomials are non trivial solutions of the equation $s \omega_0 = 0$.

The independent non trivial solutions are given in Table 2.
REMARKS.

1. In the Table 2 we have included only independent solutions of Eq. (3.12) and we have not given the solutions which differ by those from Table 2 by an s-exact term. For instance for ghost $g = 2$ there are two additional solutions of Eq. (3.12) given by

$$\eta_1 = c^+ c^- c_0^0 X^{1,0} F$$ (4.25)
$$\eta_2 = c^+ c^- c_0^0 X^{1,1} F$$ (4.26)

but it is easy to verify that they are s-dependent on the ones given in Table 2. Indeed a little algebra shows that

$$\eta_1 = c^+ c^- c_0^0 X^{1,0} X^{0,1} \partial F + s \left[ c^+ c_0^0 X^{1,0} F \right]$$

and

$$\eta_2 = c^+ c^- c_0^0 X^{1,0} X^{0,1} \partial F + s \left[ -c^- c_0^0 X^{0,1} F \right].$$

2. For $gh = 4$ we have only three independent solutions, the rest of five being or s-exact or a linear combination of these three solutions and some s-exact terms. For instance one can write

$$c^+ c^- c_0^0 c_0^0 X^{1,1} F = \frac{1}{2} s(c_+^0 c_-^0 X^{1,0} F - c_-^0 c_0^0 X^{0,1} F) - c^+ c^- c_0^0 c_0^1 X^{1,0} F - c^+ c^- c_0^1 c^1_0 X^{0,1} F$$

and

$$c^+ c^- c_0^0 c_0^1 X^{0,1} \partial F = s(c^+ c_0^1 c^1_0 F).$$

3. For $gh = 5$ we have four solutions but only two of them are independent. In this case the non-independent solutions are

$$c^+ c^- c_0^0 c_0^1 c_1^1 F = \frac{1}{2} s(c^+ c_0^0 c_0^1 c_1^1 F + c^+ c^- c_0^0 c_0^1 X^{0,1} F)$$
$$c^+ c^- c_0^1 c^1_0 c_1^0 F = \frac{1}{2} s(c_+^0 c_0^0 c_0^1 F + c^+ c^- c_0^1 c_1^0 X^{1,0} F)$$
We must point out that all elements of $H(s)$ given in Table 2 are solutions in the space of local functions, i.e., in the space of 0-forms. If we want to compute the BRST cohomology in the space of local functionals $\omega = \int d^3 xf$ which fulfil the equation $s\omega = 0$ we have to solve the descent equation. As we have been pointing out, these equations can be solved by using the operator $\delta$ defined by

$$\delta = dx^\alpha \frac{\partial}{\partial \xi^\alpha}$$

(4.27)

on this way we can write the integrand of $\omega$, $\omega_2 = d^2 x f$ in the following form

$$\omega_2 = \frac{1}{2} \delta \delta \omega_0$$

(4.28)

and we can compute all terms from the BRST cohomology in the space of local functionals.

The operator $\delta$ can be defined directly on the basis used by us as

$$\delta c^\pm = dx^\pm + h_{\pm \pm} dx^\mp$$

$$d \phi = 0 \quad \text{for} \quad \phi = \{h_{\pm \pm}, X\}$$

(4.29)

and in addition

$$[\delta, \partial_{\pm}] = 0.$$

Now it is easy to see that $\delta$ is of degree zero and obeys the following relations

$$d = -[s, \delta] \quad \quad \quad \quad [d, \delta] = 0.$$

(4.30)

In order to solve the tower (3.10) we shall make use of the following identity

$$e^\delta s = (s + d)e^\delta$$

(4.31)

which is a direct consequence of (4.30) (see [13]). Thanks to this identity we can obtain the higher cocycles $\omega_1$ and $\omega_2$ once a non-trivial solution $\omega_0$ is known. Indeed, if one apply the identity (4.31) to $\omega_0$ we get

$$(s + d)[e^\delta \omega_0] = 0.$$  

(4.32)

On the other hand one can see from eq. (4.27) that the operator $\delta$ acts as a translation on the ghosts $\xi^\pm$ with the amount $dx^\pm$ and eq. (4.32) can be rewritten as

$$(s + d)\omega_0(\xi^\pm + dx^\pm, X) = 0.$$  

(4.33)
Thus the expansion of the zero form cocycle $\omega_0(\xi^\pm, dx^\pm, X)$ in power of the one-form $dx^\pm$ yields all the cocycles $\omega_1$ and $\omega_2$.

In this way we can compute all terms from the BRST cohomology in the space of local functionals.

Our results are given in Table 3.

| Ghost | Monomial | $\delta^2(\text{Monomial})/dx^+ \wedge dx^-$ |
|-------|----------|------------------------------------------|
| 0     | -        | -                                        |
| 1     | -        | -                                        |
| 2     | $c^+c^-X^{1.0}X^{0.1}F$ | $2(1-y)X^{1.0}X^{0.1}F$ |
| 3     | $c^+c^-c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ | $(1-y)c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ |
|       | $c^+c^-c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ | $(1-y)c_+^0X^{1.0}X^{0.1}F$ |
|       | $c^+c^-c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ | $(1-y)c_+^0X^{1.0}X^{0.1}F$ |
|       | $c^+c^-c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ | $(1-y)c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ |
| 4     | $c^+c^-c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ | $(1-y)c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ |
| 5     | $c^+c^-c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ | $(1-y)c_+^0c_-c_+^0X^{1.0}X^{0.1}F$ |
| 6     | $c^+c^-c_+^0c_-c_+^0c_-c_+^1F$ | $(1-y)c_+^0c_-c_+^0c_-c_+^1F$ |

**TABLE 3**

In Table 3 we made use of the following notations

$$
\xi_+^0 = \partial_+\xi^+ + h_-\partial_+\xi^-
$$

$$
\xi_-^0 = \partial_-\xi^- + h_+\partial_-\xi^+
$$

$$
\xi_+^1 = \partial_+^2\xi^+ + 2\partial_+h_-\partial_+\xi^- + h_-\partial_-^2\xi^-
$$

$$
\xi_-^1 = \partial_-^2\xi^- + 2\partial_-h_+\partial_-\xi^+ + h_+\partial_+^2\xi^+
$$

All solutions of the descent equations (3.10) are discussed in the last section where we will connect our results with the previous ones.
5 Discussions and Conclusions

We have determined the complete BRST cohomology group in the space of local fields and
local functionals for a 2D gravitational theory invariant under diffeomorphisms and local Weyl
transformations.

The elements of the BRST group in the space of the local functionals, i.e., the term \( \omega_2 \) in the
descent equations are particularly interesting because they represent classical actions, anomalies
and Schwinger terms. For \( \text{ghost} = 0 \) there is only one element of \( H^2(s) \) and it corresponds
to the unique classical action. This action has the form of the \( \sigma \) - model with a torsion term.
For \( \text{ghost} = 1 \) there are four non-trivial elements of \( H^3(s) \) which can be grouped in two types.

Representatives of the first type can be chosen to be independent of the matter fields. There are
two independent terms of this type:

\[
\int d^2 x c^\pm \partial^3_{\pm \mp} h_{\mp \mp} \tag{5.38}
\]

and they represent the candidates for the anomalies \[17\]. Representatives of the second type
depends nontrivially on the matter fields and have the form:

\[
\int d^2 x (1 - y)(\partial_{\pm} \xi^\pm + h_{\pm \pm} \partial_{\pm} \xi^{\mp})(D_+ X)(D_- X) \tag{5.39}
\]

where

\[
\xi^\pm = \frac{1}{1 - y}(c^\pm - h_{\pm \pm} c^{\mp}) \tag{5.40}
\]

are the diffeomorphism ghosts. In fact the anomalies of the second type cannot occur in the
perturbative calculations since the classical action \( S_0 \) does not contain a self-interactive term
in the matter fields. Thus, the numerical coefficient of the corresponding Feynman diagrams
automatically vanishes.

All these solutions have been obtained by Werneck de Oliveira, Schweda and Sorella \[12\] and
by Brandt, Troost and Van Proeyen \[11\]. The remarkable point here is the fact that these are
the only possible solutions with ghost number less than two and for the ghost number zero we
have a unique solution.

The solutions of the descent equations \((3.10)\) with ghost number bigger than one do not have
any direct physical significance. Nevertheless we will give all of them for the sake of completeness.
and for the future use.

For \( gh=2 \) we have the following solution of the form:

\[
\mathcal{A}^2 = \int d^2 \left[ \xi_0^0 \xi_1^1 \nabla_+ X^\mu f_\mu^1 (X) + \xi_0^1 \xi_1^0 \nabla_- X^\mu f_\mu^2 (X) + \frac{1}{1-\gamma} \xi_0^0 \xi_0^1 \nabla_+ X^\mu \nabla_- X^\nu f_{\mu\nu} \right] \tag{5.41}
\]

where \( f_{\mu}^{1,2}(X), \ f_{\mu\nu}(X) \) are some arbitrary functions of \( X \). In this case the solutions of eqs. (3.10) depend only on the diffeomorphism ghosts \( \xi^\pm \).

For \( gh=3 \) we have also two independent solutions of the form

\[
\mathcal{A}^3 = \int d^2 x \xi_0^0 \xi_0^0 \left[ \xi_1^1 (\nabla_- X^\mu f_\mu^1 (X) + \xi_1^1 (\nabla_+ X^\mu) f_{\mu}^2 (X) \right] . \tag{5.42}
\]

Again in \( \mathcal{A}^3 \) occur only the ghosts \( \xi^\pm \).

In the last possible case \( gh=4 \) we have obtained a unique solution of the form

\[
\mathcal{A}^4 = \int d^2 x (1-\gamma) \xi_0^0 \xi_0^0 \xi_0^1 \xi_1^1 F(X) \tag{5.43}
\]

with \( F(X) \) an arbitrary scalar function of \( X \).

All solutions with ghost number bigger than one are new and as far as we are aware of this is the first place where they are done.

Finally we want to mention that a similar calculation with the antifields included can be done \([11]\), but in this case the results strongly depends on the form of the classical action one starts with.
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