On the role of MMSE estimation in approaching the information-theoretic limits of linear Gaussian channels: Shannon meets Wiener

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Abstract

We discuss why MMSE estimation arises in lattice-based schemes for approaching the capacity of linear Gaussian channels, and comment on its properties.

1 Introduction

Recently, Erez and Zamir [8, 22] have cracked the long-standing problem of achieving the capacity of additive white Gaussian noise (AWGN) channels using lattice codes and lattice decoding. Their method uses Voronoi codes (nested lattice codes), dither, and an MMSE estimation factor $\alpha$ that had previously been introduced in more complex multiterminal scenarios, such as Costa’s “dirty-paper channel” [5]. However, they give no fundamental explanation for why an MMSE estimator, which is seemingly an artifact from the world of analog communications, plays such a key role in the digital communications problem of achieving channel capacity.

The principal purpose of this paper is to provide such an explanation, in the lattice-based context of a mod-$\Lambda$ AWGN channel model. We discuss various properties of MMSE-based schemes in this application, some of which are unexpected.

MMSE estimators also appear as part of capacity-achieving solutions for more general linear Gaussian channel scenarios; e.g., in MMSE-DFE structures (including precoding) for ISI channels [9, 2], and generalized MMSE-DFE structures for vector and multi-user channels [3, 20]. Some of the explanation for the “canonicality” of MMSE-DFE structures in these more general scenarios is no doubt information-theoretic [18, 13]. The observations of this paper complement these results by showing why lattice-type codes combine so well with MMSE equalization structures, as shown previously in [14, 22].

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2 Lattice-based coding for the AWGN channel

Consider the real discrete-time AWGN channel \( Y = X + N \), where \( E[X^2] \leq S_x \) and \( N \) is independent\(^1\) zero-mean Gaussian noise with variance \( S_n \). The capacity is \( C = \frac{1}{2} \log_2(1 + \text{SNR}) \) bits per dimension (b/d), where \( \text{SNR} = S_x / S_n \). Following Erez and Zamir [8, 22], we will show how lattice-based transmission systems can approach the capacity of this channel at all SNRs.

2.1 Lattices and spheres

Geometrically, an \( N \)-dimensional lattice \( \Lambda \) is a regular infinite array of points in \( \mathbb{R}^N \). Algebraically, \( \Lambda \) is a discrete subgroup of \( \mathbb{R}^N \) which spans \( \mathbb{R}^N \). A Voronoi region \( R_V(\Lambda) \) of \( \Lambda \) represents the quotient group \( \mathbb{R}^N / \Lambda \) by a set of minimum-energy coset representatives for the cosets of \( \Lambda \) in \( \mathbb{R}^N \). For any \( x \in \mathbb{R}^N \), “\( x \mod \Lambda \)” denotes the unique element of \( R_V(\Lambda) \) in the coset \( \Lambda + x \). Geometrically, \( \mathbb{R}^N \) is the disjoint union of the translated Voronoi regions \( \{ R_V(\Lambda) + \lambda, \lambda \in \Lambda \} \). The volume \( V(\Lambda) \) of \( R_V(\Lambda) \) is therefore the volume of \( \mathbb{R}^N \) associated with each point of \( \Lambda \).

As \( N \to \infty \), the Voronoi regions of some \( N \)-dimensional lattices can become more or less spherical, in various senses. As \( N \to \infty \), an \( N \)-sphere (ball) of squared radius \( N \rho^2 \) has normalized volume (per two dimensions)

\[
V_\bigodot(N\rho^2)^{2/N} \xrightarrow{N \to \infty} 2\pi e \rho^2.
\]

The average energy per dimension of a uniform probability distribution over such an \( N \)-sphere goes to \( P_\bigodot(N\rho^2) = \rho^2 \). The probability that an iid Gaussian random \( N \)-tuple with zero mean and symbol variance \( S_n \) falls outside the \( N \)-sphere becomes arbitrarily small for any \( S_n < \rho^2 \).

It is known that there exist high-dimensional lattices whose Voronoi regions are quasi-spherical in the following second moment sense. The normalized second moment of a compact region \( \mathcal{R} \subset \mathbb{R}^N \) of volume \( V(\mathcal{R}) \) is defined as

\[
G(\mathcal{R}) = \frac{P(\mathcal{R})}{V(\mathcal{R})^{2/N}},
\]

where \( P(\mathcal{R}) \) is the average energy per dimension of a uniform probability distribution over \( \mathcal{R} \). The normalized second moment of \( \mathcal{R} \) exceeds that of an \( N \)-sphere. The normalized second moment of an \( N \)-sphere decreases monotonically with \( N \) and approaches \( \frac{1}{2\pi e} \) as \( N \to \infty \). Poltyrev (reported in Feder-Zamir [21]) showed that there exist lattices \( \Lambda \) such that \( \log 2\pi e G(\Lambda) \) is arbitrarily small, where \( G(\Lambda) \) denotes the normalized second moment of \( R_V(\Lambda) \). Such lattices are said to be “good for quantization,” or “good for shaping.”

Poltyrev [17] also showed that there exist high-dimensional lattices whose Voronoi regions are quasi-spherical in the sense that the probability that an iid Gaussian noise \( N \)-tuple with symbol variance \( S_n \) falls outside the Voronoi region \( R_V(\Lambda) \) is arbitrarily small as long as

\[
S_n < \frac{V(\Lambda)^{2/N}}{2\pi e}.
\]

Such lattices are said to be “good for AWGN channel coding,” or “sphere-bound-achieving” [12].

\(^1\)Note that without the independence of \( N \), the “additive” property is vacuous, since for any real-input, real-output channel we may define \( N = Y - X \), and then express \( Y \) as \( Y = X + N \). We exploit this idea later.
2.2 Mod-lattice transmission and capacity

We now show that the mod-Λ transmission system shown in Figure 1 can approach the channel capacity \( C = \frac{1}{2} \log_2 (1 + \text{SNR}) \) b/d arbitrarily closely, provided that \( G(\Lambda) \approx \frac{1}{2\pi e} \) and \( f(Y) \) is a MMSE estimator of \( X \).

![Diagram](https://via.placeholder.com/150)

Figure 1. Mod-Λ transmission system over an AWGN channel.

This system is based on an \( N \)-dimensional lattice \( \Lambda \) whose Voronoi region \( \mathcal{R}_V(\Lambda) \) has volume \( V(\Lambda) \), average energy per dimension \( P(\Lambda) = S_x \) under a uniform probability distribution over \( \mathcal{R}_V(\Lambda) \), and thus normalized second moment \( G(\Lambda) = P(\Lambda)/V(\Lambda)^{2/N} \).

The \( N \)-dimensional input vector \( X \) is restricted to the Voronoi region \( \mathcal{R}_V(\Lambda) \). The output vector \( Y \) is mapped by some function \( f \) to another vector \( f(Y) \in \mathbb{R}^N \), which is then mapped modulo \( \Lambda \) to \( Y' = f(Y) \mod \Lambda \), also in the Voronoi region \( \mathcal{R}_V(\Lambda) \).

Our main result is that capacity can be approached in the system of Figure 1 if and only if the lattice \( \Lambda \) is “good for shaping” and the function \( f(Y) \) is an MMSE estimator.

(The sufficiency of these conditions was shown in [8, 22].)

As a first step, we derive a lower bound:

**Theorem 1 (Mod-Λ channel capacity)** The capacity \( C(\Lambda, f) \) of the mod-Λ transmission system of Figure 1 is lowerbounded by

\[
C(\Lambda, f) \geq C - \frac{1}{2} \log_2 2\pi e G(\Lambda) - \frac{1}{2} \log_2 \frac{S_{e,f}}{S_e} \text{ b/d,}
\]

where \( C = \frac{1}{2} \log_2 (1 + \text{SNR}) \) b/d is the capacity of the underlying AWGN channel, \( G(\Lambda) \) is the normalized second moment of \( \mathcal{R}_V(\Lambda) \), and \( S_{e,f} \) and \( S_e \) are the average energies per dimension of \( E_f = f(Y) - X \) and of \( E = \hat{X}(Y) - X \), respectively, where \( \hat{X}(Y) \) is the linear MMSE estimator of \( X \) given \( Y \).

The key to the proof of this theorem is the introduction of a dither variable \( U \) that is known to both transmitter and receiver, and whose probability distribution is uniform over the Voronoi region \( \mathcal{R}_V(\Lambda) \), as in [8, 22]. Given a data vector \( V \in \mathcal{R}_V(\Lambda) \), the channel input is taken as

\[
X = V + U \mod \Lambda.
\]

This makes \( X \) a uniform random variable over \( \mathcal{R}_V(\Lambda) \), statistically independent of \( V \). This property follows from the following lemma:\(^2\)

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\(^2\)We call this the crypto lemma because if we take \( X \) as plaintext, \( N \) as a cryptographic key, and \( Y = X + N \) as the encrypted message, then the encrypted message is independent of the plaintext provided that the key is uniform, so no information can be obtained about the plaintext from the encrypted message without the key. On the other hand, given the key, the plaintext may be easily recovered from the encrypted message via \( X = Y - N \). This is the principle of the one-time pad, which, as Shannon showed, is essentially the only way to achieve perfect secrecy in a cryptographic system.
Lemma 2 (Crypto lemma) Let $G$ be a compact abelian group with group operation $+$, and let $Y = X + N$, where $X$ and $N$ are random variables over $G$ and $N$ is independent of $X$ and uniform over $G$. Then $Y$ is independent of $X$ and uniform over $G$.

Proof. Since $y - x$ runs through $G$ as $y$ runs through $G$ and $p_N(n)$ is constant over $n \in G$, the distribution $p_{Y|X}(y|x) = p_N(y - x)$ is constant over $y \in G$ for any $x \in G$. □

One effect of the dither $U$ is thus to ensure that the channel input $X = V + U \mod \Lambda$ is uniform over $\mathcal{R}_V(\Lambda)$ and thus has average energy per dimension $P(\Lambda) = S_x$. A second and more important effect is to make $X$ and thus also $Y = X + N$ independent of $V$.

The dither may be subtracted out at the output of the channel, mod $\Lambda$, to give

$$Z = f(Y) - U \mod \Lambda.$$ 

The end-to-end channel is illustrated in Figure 2.

Now let us regard $f(Y)$ as an estimator of $X$, and define the estimation error as $E_f = f(Y) - X$. Since $Y$ and $X$ are independent of $V$, so is $E_f$. Then

$$Z = X + E_f - U = V + E_f \mod \Lambda.$$ 

In short, we have created a mod-$\Lambda$ additive noise channel $Z = V + E_f \mod \Lambda$, where $E_f$ is independent of $V$. This equivalent channel is illustrated in Figure 3.

As is well known, the capacity of an additive-noise channel $Z = V + E_f \mod \Lambda$ is achieved when the input distribution is uniform over $\mathcal{R}_V(\Lambda)$, in which case the output distribution is uniform as well, by the crypto lemma. The capacity is equal to

$$C(\Lambda, f) = \frac{1}{N} (h(Z) - h(Z \mid V)) = \frac{1}{N} (\log_2 V(\Lambda) - h(E'_f)) \text{ b/d},$$ 

where $h(Z) = \log_2 V(\Lambda)$ is the differential entropy of a uniform distribution over a region of volume $V(\Lambda)$, and $h(E'_f)$ is the differential entropy of the $\Lambda$-aliased additive noise $E'_f = E_f \mod \Lambda$. Now since $E'_f$ is the result of applying the many-to-one mod-$\Lambda$ map to $E_f$, we have

$$h(E'_f) \leq h(E_f).$$

The group $G$ is required to be compact so that its Haar (translation-invariant) measure $\mu(G)$ is finite and thus normalizable to a uniform probability distribution over $G$. However, $G$ need not be abelian.
Moreover, if \( E_f \) has average energy per dimension \( S_{e,f} \), then we have

\[
h(E_f) \leq \frac{N}{2} \log_2 2\pi e S_{e,f},
\]

the differential entropy of an iid zero-mean Gaussian distribution with the same average energy. Combining these results, using \( V(\Lambda)^{2/N} = P(\Lambda)/G(\Lambda) \) and \( P(\Lambda) = S_x \), we have

\[
C(\Lambda, f) \geq \frac{1}{N} \log_2 V(\Lambda) - \frac{1}{2} \log_2 2\pi e S_{e,f} = \frac{1}{2} \log_2 \frac{S_x}{S_{e,f}} - \frac{1}{2} \log_2 2\pi e G(\Lambda) \quad \text{b/d}.
\]

The linear MMSE estimator \( \hat{X}(Y) \) of \( X \) is \( \hat{X}(Y) = \alpha Y \), where

\[
\alpha = \frac{S_x}{S_x + S_n} = \frac{\text{SNR}}{1 + \text{SNR}}.
\]

By the orthogonality principle of MMSE estimation theory, the linear MMSE estimation error \( E = X - \alpha Y = (1 - \alpha)X - \alpha N \) is then uncorrelated with \( Y \). \(^4\) The average energy of the estimation error per dimension becomes

\[
S_e = (1 - \alpha)^2 S_x + \alpha^2 S_n = \frac{S_x S_n}{S_x + S_n} = \alpha S_n
\]

(see footnote). Finally, since \( S_x/S_e = S_y/S_n = 1 + \text{SNR} \), we have

\[
C(\Lambda, f) \geq \frac{1}{2} \log_2 \frac{S_x}{S_e} + \frac{1}{2} \log_2 \frac{S_e}{S_{e,f}} - \frac{1}{2} \log_2 2\pi e G(\Lambda) = C - \frac{1}{2} \log_2 2\pi e G(\Lambda) - \frac{1}{2} \log_2 \frac{S_{e,f}}{S_e} \quad \text{b/d}.
\]

This completes the proof of Theorem 1. \( \square \)

**Remark 1** (dither is unnecessary). Evidently a channel \( Z = V + u + E_f \mod \Lambda \) with a fixed dither vector \( u \in \mathcal{R}_V(\Lambda) \) has the same capacity \( C(\Lambda, f) \). Therefore introducing the random dither variable \( U \) is just a tactic to prove Theorem 1; dither is not actually

\(^4\)These relations are illustrated by the “Pythagorean” right triangle shown in Figure 4 below, which follows from interpreting covariances as inner products of vectors in a two-dimensional Hilbert space. Since \( E[X,N] = 0 \), the two vectors corresponding to \( X \) and \( N \) are orthogonal. Their squared lengths are given by \( E[X^2] = S_x \) and \( E[N^2] = S_n \). The hypotenuse corresponds to the sum \( Y = X + N \), and has squared length \( S_y = S_x + S_n \). Since \( E[XY] = S_x \), the projection of \( Y \) onto \( X \) is \( \hat{Y}(X) = (E[XY]/E[X^2])X = X \), and the projection of \( X \) onto \( Y \) is \( \hat{X}(Y) = (E[XY]/E[Y^2])Y = \alpha Y \). Then \( E = X - \hat{X}(Y) = X - \alpha Y \) is orthogonal to \( Y \). The inner right triangle in Figure 4 with sides \( (\hat{X}(Y), E, X) \) is similar, so since \( S_x = \alpha S_y \) the squared lengths of its sides are \( (S_x = \alpha S_x, S_e = \alpha S_n, S_x = \alpha S_y) \), respectively.

![Figure 4. “Pythagorean” right triangle with sides \((X, N, Y)\), with similar inner right triangle with sides \((\hat{X}(Y) = \alpha Y, E = X - \hat{X}(Y), X)\).](image-url)
needed to achieve $C(\Lambda, f)$. However, dither is key to decoupling the Shannon and the Wiener problems.\footnote{This is analogous to the tactic used by Elias \cite{Elias:1962} to prove that binary linear block codes can achieve the capacity of a binary input-symmetric channel, namely the introduction of a random translate $C + U$ of a binary linear block code $C$ of length $N$, where $U$ is a random uniform binary $N$-tuple in $(\mathbb{F}_2)^N$.}

**Remark 2** (MMSE estimation and bias). Notice that the signal-to-noise ratio of the channel $Z = V + E \mod \Lambda$ is $S_x/S_e = S_y/S_n = 1 + \text{SNR}$, and moreover this channel has no bias. Thus the MMSE factor $\alpha$ and random dither increase the effective signal-to-noise ratio from $1 + \text{SNR}$ without introducing bias. This is evidently a different way of approaching capacity than that given in \cite{Erez:2006}, where the apparent $\text{SNR}_{\text{MMSE-DFE}}$ was discounted to $\text{SNR}_{\text{MMSE-DFE,UE}} = \text{SNR}_{\text{MMSE-DFE}} - 1$ to account for bias.\footnote{Zamir and Feder \cite{Zamir:2005} show that if $X_n$ is uniform over an $N(n)$-dimensional region $R_n$ of average energy $S_x$ and $G(R_n) \to 1/(2\pi e)$, then the normalized divergence $\frac{1}{N(n)}D(X_n||N_n) \to 0$, where $N_n$ is an iid Gaussian random vector with zero mean and variance $S_x$. They go on to show that this implies that any finite-dimensional projection of $X_n$ converges in distribution to an iid Gaussian vector.}

**Remark 3** ("dirty-paper" capacity). This approach easily extends to give a constructive proof of Costa’s result \cite{Costa:1983} that channel interference known to the transmitter does not reduce capacity; see, e.g., \cite{Zamir:2005}. Let the channel model be $Y = X + N + S$, where $S$ is an arbitrary interference vector known to the transmitter. Then let the channel input be $X = V + U - \alpha S \mod \Lambda$. The channel input is still uniform and independent of $V$, by the crypto lemma, while the effect of the interference $S$ is entirely cancelled in $Z = \alpha Y - U = V + E_f \mod \Lambda$. Thus the receiver needs to know nothing about the interference, the equivalent channel model is the same, and $C(\Lambda, f)$ is unaffected.\footnote{Since $E_f = E + (f(Y) - \bar{X}(Y))$ and $Y$ and $E$ are independent, $S_{e,f} = S_e + \frac{1}{2}E[|f(Y) - \bar{X}(Y)|^2]$. Thus $f(Y)$ is an MMSE estimator if and only if $E[|f(Y) - \bar{X}(Y)|^2] = 0$.}

Theorem 1 implies that the capacity $C = \frac{1}{2} \log_2(1 + \text{SNR})$ can be approached arbitrarily closely by the mod-$\Lambda$ channel of Figure 1 if $\log 2\pi e G(\Lambda) \to 0$ and $f(Y)$ is the linear MMSE estimator $\bar{X}(Y) = \alpha Y$, which is the main result of Erez and Zamir \cite{Erez:2006}.

We now show that the conditions $\log_2 2\pi e G(\Lambda) \to 0$ and $S_{e,f} = S_e$ are not only sufficient but also necessary to reach capacity. Briefly, the arguments are as follows:

1. The differential entropy per dimension of $X$ and $Z$, namely

$$\frac{1}{N} h(X) = \frac{1}{N} h(Z) = \frac{1}{2} \log_2 V(\Lambda)^{2/N} = \frac{1}{2} \log_2 2\pi e S_x - \frac{1}{2} \log_2 2\pi e G(\Lambda)$$

creates $\frac{1}{2} \log_2 2\pi e S_x$ if and only if $\log_2 2\pi e G(\Lambda) \to 0$. This condition is necessary because the capacity of an AWGN channel with input power constraint $S_x$ can be approached arbitrarily closely only if $h(X)/N$ approaches $\frac{1}{2} \log_2 2\pi e S_x$.

2. The channel output $Y = X + N$ is then also Gaussian in the limit, so the linear MMSE estimator $\bar{X}(Y) = \alpha Y$ becomes a true MMSE estimator in the limit. The MMSE estimation error $E = -(1-\alpha)X + \alpha N$ becomes Gaussian in the limit with symbol variance $S_e = \alpha S_n$, and becomes independent of $Y$. In order that $C(\Lambda, f) \to C$, it is then necessary that $S_{e,f} = S_e$, which by definition implies that $f(Y)$ is an MMSE estimator.\footnote{Since $E_f = E + (f(Y) - \bar{X}(Y))$ and $Y$ and $E$ are independent, $S_{e,f} = S_e + \frac{1}{2}E[|f(Y) - \bar{X}(Y)|^2]$. Thus $f(Y)$ is an MMSE estimator if and only if $E[|f(Y) - \bar{X}(Y)|^2] = 0$.}
In summary, these two conditions are necessary as well as sufficient:

**Theorem 3 (Necessary conditions to approach \(C\))** The capacity of the mod-\(\Lambda\) channel of Figure 1 approaches \(C\) if and only if \(\log 2 \pi e G(\Lambda) \to 0\) and \(f(Y)\) is an MMSE estimator of \(X\) given \(Y\).

** Remark 5 (MMSE estimation and lattice decoding).** One interpretation of the Erez-Zamir result is that the scaling introduced by the MMSE estimator is somehow essential for lattice decoding of a fine-grained coding lattice \(\Lambda_c\). Theorem 3 shows however that in the mod-\(\Lambda\) channel an MMSE estimator is necessary to achieve capacity, quite apart from any particular coding and decoding scheme.

**Remark 6 (aliasing becomes negligible).** Under these conditions, Theorem 1 says that \(C(\Lambda, f) \geq C\). Since \(C(\Lambda, f)\) cannot exceed \(C\), this implies that all inequalities in the proof of Theorem 1 must tend to equality, and in particular that

\[
\frac{h(E')}{N} \to \frac{h(E)}{N} \to \frac{1}{2} \log 2 \pi e S_e,
\]

where \(E' = E\) mod \(\Lambda\) is the \(\Lambda\)-aliased version of the estimation error \(E\). So not only must \(E\) become Gaussian in the limit, i.e., \(h(E)/N \to \frac{1}{2} \log 2 \pi e S_e\), but also \(E'\) must tend to \(E\), which means that the effect of the mod-\(\Lambda\) aliasing must become negligible. This is as expected, since \(E\) is Gaussian in the limit with symbol variance \(S_e\) and \(\mathcal{R}_V(\Lambda)\) is quasi-spherical with average energy per dimension \(S_x > S_e\).

### 2.3 Voronoi codes

A **Voronoi code** \(C((\Lambda_c + u)/\Lambda) = (\Lambda_c + u) \cap \mathcal{R}_V(\Lambda)\) is the set of points in a translate \(\Lambda_c + u\) of an \(N\)-dimensional “coding lattice” \(\Lambda_c\) that lie in the Voronoi region \(\mathcal{R}_V(\Lambda)\) of a “shaping” sublattice \(\Lambda \subset \Lambda_c\). (Such codes were called “Voronoi codes” in [1], “Voronoi constellations” in [11], and “nested lattice codes” in [8, 22, 1]. Here we will use the original term.)

A Voronoi code has \(|\Lambda_c/\Lambda| = V(\Lambda)/V(\Lambda_c)|\) code points, and thus rate

\[
R(\Lambda_c/\Lambda) = \frac{1}{N} \log_2 \frac{V(\Lambda)}{V(\Lambda_c)} = \frac{1}{2} \left( \log_2 \frac{V(\Lambda)^{2/N}}{2 \pi e} - \log_2 \frac{V(\Lambda_c)^{2/N}}{2 \pi e} \right) \text{ b/d.}
\]

Erez and Zamir [8, 22] have shown rigorously (not employing the Gaussian approximation principle) that there exists a random ensemble \(C((\Lambda_c + U)/\Lambda)\) of dithered Voronoi codes that can approach the capacity \(C(\Lambda)\) of the mod-\(\Lambda\) transmission system of Figure 1 arbitrarily closely, if \(f(Y) = \hat{X}(Y) = \alpha Y\). The decoder may be the usual minimum-Euclidean-distance decoder, even though the effective noise \(E = -(1 - \alpha) X + \alpha N\) is not Gaussian.

If \(C(\Lambda) \approx C\) and \(P(\Lambda) = S_x\), this implies that \(2 \pi e G(\Lambda) \approx 1\); i.e., \(\Lambda\) is “good for shaping.” Furthermore, since the effective noise has variance \(S_e\), if the error probability is arbitrarily small and \(R(\Lambda_c/\Lambda) \approx C = \frac{1}{2} \log_2 S_x/S_e\), then

\[
\log_2 S_e \approx \log_2 \frac{V(\Lambda_c)^{2/N}}{2 \pi e} ;
\]

i.e., \(\Lambda_c\) is “good for AWGN channel coding,” or “sphere-bound-achieving.”
The ensemble $\mathcal{C}((\Lambda_c + U)/\Lambda)$ is an ensemble of fixed-dither Voronoi codes $\mathcal{C}((\Lambda_c + u)/\Lambda)$. The average probability of decoding error $\Pr_U(E) = E_U[\Pr_u(E)]$ is arbitrarily small over this ensemble, using a decoder that is appropriate for random dither (i.e., minimum-distance decoding). This implies not only that there exists at least one fixed-dither code $\mathcal{C}((\Lambda_c + u)/\Lambda)$ such that $\Pr_u(E) \leq \Pr_U(E)$, using the same decoder, but also that at least a fraction $1 - \varepsilon$ of the fixed-dither codes have $\Pr_u(E) \leq \frac{1}{\varepsilon} \Pr_U(E)$; i.e., almost all fixed-dither codes have low $\Pr_u(E)$.

This result is somewhat counterintuitive, since for fixed dither $u$, $X$ is not independent of $V$; indeed, there is a one-to-one correspondence given by $X = V + u \mod \Lambda$. Therefore, the error

$$\mathbf{E} = -(1 - \alpha)\mathbf{X} + \alpha\mathbf{N} = -(1 - \alpha)(\mathbf{V} + u \mod \Lambda) + \alpha\mathbf{N}$$

is not independent of $V$; i.e., there is bias in the equivalent channel output $Z = V + \mathbf{E} \mod \Lambda$. Even so, we see that capacity can be achieved by a suboptimum decoder which ignores bias.

Since almost all fixed-dither codes achieve capacity, we may as well use the code $\mathcal{C}((\Lambda_c + u)/\Lambda)$ that has minimum average energy $S_{\text{min}} \leq P(\Lambda) = S_x$ per dimension. But if $S_{\text{min}} < S_x$, then we could achieve a rate greater than the capacity of an AWGN channel with signal-to-noise ratio $S_{\text{min}}/S_n < S_x/S_n$. We conclude that the average energy per dimension of $\mathcal{C}((\Lambda_c + u)/\Lambda)$ cannot be materially less than $S_x = P(\Lambda)$ for any $u$, and thus must be approximately $S_x$ for almost all values of the dither $u$, in order for the average over $U$ to be $S_x$. In summary:

**Theorem 4 (Average energy of Voronoi codes)** If $\mathcal{C}((\Lambda_c + u)/\Lambda)$ is a capacity-achieving Voronoi code, then $\Lambda_c$ is good for AWGN channel coding, $\Lambda$ is good for shaping, the decoder may ignore bias, and the average energy per dimension of $\mathcal{C}((\Lambda_c + u)/\Lambda)$ is $\approx P(\Lambda)$.

**Remark 7** (Average energy of Voronoi codes). Theorem 4 shows that the hope of [12] that one could find particular Voronoi codes with average energy $S_x - S_c$ was misguided. For Voronoi codes, the original “continuous approximation” of [10] holds, not the “improved continuous approximation” of [12].

**Remark 8** (observations on output scaling). It is surprising that a decoder for Voronoi codes which first scales the received signal by $\alpha$ and then does lattice decoding should perform better than one that just does lattice decoding. Optimum (ML) decoding on this channel is minimum-distance (MD) decoding, and ordinary lattice decoding is equivalent to minimum-distance decoding except on the boundary of the support region.

Scaling by $\alpha$ seems excessive. Scaling the output by $\alpha$ reduces the received variance to $S_x = \alpha^2 S_y = \alpha S_x$, less than the input variance. This means that the scaled output $\alpha Y$ is almost surely going to lie in a spherical shell of average energy per dimension $\approx \alpha S_x$, whereas the code vectors in the Voronoi code $\mathcal{C}(\Lambda_c/\Lambda)$ almost all lie on a spherical shell of average energy $\approx S_x$. Yet the subsequent lattice decoding to $\mathcal{C}((\Lambda_c + u)/\Lambda)$ works, even though it seems that the decoder should decode to $\alpha \mathcal{C}((\Lambda_c + u)/\Lambda)$.

These questions about scaling may be resolved if as $N \to \infty$ it suffices to decode Voronoi codes based on angles, ignoring magnitudes. Then whether the decoder uses $Y, \alpha Y$ or $\sqrt{\alpha} Y$ as input, the optimum minimum-angle decoder would be the same. Indeed, Urbanke and Rimoldi [19], following Linder et al. [15], have shown that as $N \to \infty$ a suboptimum decoder for spherical lattice codes that does minimum-angle decoding to the subset of codewords in a spherical shell of average energy $\approx S_x$ suffices to approach capacity.
Of course, lattice decoding does depend on scale, so it seems that scaling the lattice decoder is just a trick to analyze the optimal minimum-angle decoder performance, as well as to show that lattice decoding of Voronoi codes suffices to reach capacity.

Finally, note that with a fixed code and scaling by $\alpha$, as $N \to \infty$ the output $\alpha Y$ almost surely lies in a sphere of average energy $\approx \alpha S_x < S_x$, inside $R_V(\Lambda)$, so the mod-$\Lambda$ operation in the receiver has negligible effect and may be omitted. \hfill \square

Remark 9 (Shannon codes, spherical lattice codes, and Voronoi codes). In Shannon’s random code ensemble for the AWGN channel, the code point $X$ asymptotically lies almost surely in a spherical shell of average energy $\approx S_x$, the received vector $Y$ lies almost surely in a spherical shell of average energy $\approx S_y$, and the noise vector $N$ lies almost surely in a spherical shell of average energy $\approx S_n$. Thus we obtain a geometrical picture in which a “output sphere” of average energy $\approx S_y$ is partitioned into $\approx (S_y/S_n)^{N/2}$ probabilistically disjoint “noise spheres” of squared radius $\approx S_n$. Curiously, the centers of the noise spheres are at average energy $\approx S_x$, even though practically all of the volumes of the noise spheres are at average energy $\approx S_y$.

Urbanke and Rimoldi \[19\] have shown that spherical lattice codes (the set of all points in a lattice $\Lambda_c$ that lie within a sphere of average energy $S_x$) can achieve the channel capacity $C = \frac{1}{2} \log_2 S_y/S_n$ b/d with minimum-distance decoding. Since again $Y$ and $N$ must lie almost surely in spheres of average energy $S_y$ and $S_n$, respectively, we again have a picture in which the output sphere must be partitioned into $\approx (S_y/S_n)^{N/2}$ effectively disjoint noise spheres whose centers are the points in the spherical lattice code, which have average energy $\approx S_x$.

Voronoi codes evidently work differently. The Voronoi region $R_V(\Lambda)$ has average energy $S_x$, and so does any good Voronoi code $C((\Lambda_c + u)/\Lambda)$. Moreover, $R_V(\Lambda)$ is the disjoint union (mod $\Lambda$) of $\Lambda_c$ and small Voronoi regions, whose centers are the points in $C((\Lambda_c + u)/\Lambda)$. So the centers have the same average energy as the bounding region, in contrast to the spherical case.

By the sphere bound \[12\ \[17\] $\log_2 V(\Lambda_c)^{2/N} / (2\pi e) \geq \log_2 S_c$, where $S_c$ is the channel noise variance, so the capacity of the mod-$\Lambda$ channel is limited to $\frac{1}{2} \log_2 S_x/S_c$. If the channel noise has variance $S_c = S_n$, then the capacity is limited to $\frac{1}{2} \log_2 S_x/S_c = \frac{1}{2} \log_2 \text{SNR}$, which is the best that de Buda and others \[6\ \[16\] were able to achieve with Voronoi codes prior to [8]. However, the MMSE estimator reduces the effective channel noise variance to $S_c = S_c = \alpha S_y$, which allows the capacity to approach $C = \frac{1}{2} \log_2 S_x/S_c = \frac{1}{2} \log_2 (1 + \text{SNR})$. So in the mod-$\Lambda$ setting the MMSE estimator is the crucial element that precisely compensates for the Voronoi code capacity loss from $C$ to the “lattice capacity” $C$.

Finally, consider a “backward-channel” view of the Shannon ensemble. The jointly Gaussian pair $(X,Y)$ is equally well modeled by the forward-channel model $Y = X + N$ or the backward-channel model $X = \alpha Y + E$. From the latter perspective, the transmitted codeword $X$ lies almost surely in a spherical shell of average energy $\approx S_c$ about the scaled received word $\alpha Y$, which lies almost surely in a spherical shell of average energy $\approx \alpha^2 S_y = \alpha S_x$. Thus we obtain a geometrical picture in which an “input sphere” of average energy $\approx S_x$ is partitioned into $\approx (S_x/S_c)^{N/2}$ probabilistically disjoint “decision spheres” of squared radius $\approx S_c$. The centers of the decision spheres are codewords of average energy $\approx S_x$.

Capacity-achieving Voronoi codes thus appear to be designed according to the backward-channel view of the Shannon ensemble, whereas capacity-achieving spherical lattice codes appear to be designed according to the forward-channel view.
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