3-Point Functions in $\mathcal{N} = 4$ Yang-Mills

P.S. Howe$^a$, E. Sokatchev$^b$ and P.C. West$^a$

$^a$ Department of Mathematics, King’s College, Strand, London WC2R 2LS, United Kingdom

$^b$ Laboratoire d’Annecy-le-Vieux de Physique Théorique, LAPTH, Chemin de Bellevue, B.P. 110, F-74941 Annecy-le-Vieux, France

Abstract

Three-point functions of analytic (chiral primary) operators in $\mathcal{N} = 4$ Yang-Mills theory in four dimensions are calculated using the harmonic superspace formulation of this theory. In the case of the energy-momentum tensor multiplet anomaly considerations determine the coefficient. Analyticity in $\mathcal{N} = 2$ harmonic superspace is explicitly checked in a two-loop calculation.
Although there was no known example of a four dimensional conformally invariant quantum field theory in the 1960’s and 1970’s, the properties of such theories were investigated. It was realised that conformal invariance could be used to determine the two- and three-point Green’s functions up to constants in any dimension and the space-time dependence of many such correlators were found. With the discovery of supersymmetry, examples of conformally invariant quantum field theory were found. The first such theory to be found was the $\mathcal{N} = 4$ Yang-Mills theory. Conformal invariance was very successfully exploited in two dimensions to determine Green’s functions for higher-point functions in certain theories. However, these developments in two dimensions relied on the infinite nature of the two dimensional conformal group and the existence of null vectors in certain representations of this algebra. In four dimensions, the conformal group is only a finite dimensional group and it appeared that, unlike in two dimensions, one would not be able to exploit conformal invariance to solve for higher-point Green’s functions. One indication to the contrary concerned the Green’s functions that involved $\mathcal{N} = 2$ chiral superfields of the same chirality in the two dimensional $\mathcal{N} = 2$ minimal model series. Since these correlators belong to a minimal conformal field theory, it was to be expected that one could solve for these Green’s functions explicitly, but in it was shown that one could do this using only the globally defined superconformal group and chirality. As it is the globally defined part of the two dimensional conformal group that generalises to higher dimensions, this work lead to the hope that the constrained nature of the superfields that describe supersymmetric theories when combined with conformal invariance might be sufficient, even in higher dimensions, to solve for more than just the two- and three-point Green’s functions. A related example of this phenomenon is the simple relation between the anomalous weight of a chiral superfield and its R weight in any superconformal theory, which in many cases allows one to deduce the anomalous weight of the chiral superfield.

In fact all four dimensional supersymmetric theories of interest are described by constrained superfields. The Wess-Zumino model and the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ Yang-Mills field strengths are described by chiral superfields. The remaining theories of extended rigid supersymmetry are the $\mathcal{N} = 2$ matter and the $\mathcal{N} = 4$ Yang-Mills theory. The $\mathcal{N} = 2$ matter is best described by a harmonic superspace formulation in which it is represented by a single component superfield $q^+$ which satisfies an analyticity condition. Here, analytic means that $q^+$ is both Grassmann analytic, i.e. it depends on only half of the fermionic coordinates in a similar manner to a chiral superfield, and analytic on the internal (bosonic) space, a compact, complex manifold which is used to extend standard ($\mathcal{N} = 2$) Minkowski superspace to the harmonic superspace of interest. The $\mathcal{N} = 4$ Yang-Mills theory also has a succinct description when formulated on an appropriate harmonic superspace. Explicitly, the Yang-Mills field strength multiplet is described by a single-component analytic superfield $W$ (taking its values in the Lie algebra of $SU(N)$). Although, in the non-Abelian case, $W$ is covariantly analytic (with respect to the gauge group), the gauge invariant operators, $A_q$, defined by

$$A_q = \text{tr}(W^q)$$

are analytic fields in the strict sense.

In references the constraints due to superconformal invariance on four dimensional Green’s functions involving chiral or harmonic superfields were found. It was clear that these constraints were very strong and it was suggested that they were sufficiently powerful to determine, up to constants, a class of these Green’s functions. These included all the Green’s functions of operators composed of $\mathcal{N} = 2$ matter of sufficiently low dimension in the $\mathcal{N} = 2$ su-
persymmetric theories and operators composed of gauge invariant polynomials of the harmonic superfield $W$, also of sufficiently low dimension, in the $\mathcal{N} = 4$ Yang-Mills theory. In the latter case these are just Green’s functions of the above operators $A_q = \text{tr}(W^q)$ for sufficiently small $q$. The calculations required to establish this result are complicated, but have been successfully completed for the four-point Green’s functions involving $\mathcal{N} = 2$ matter [13]. This result encourages us to believe that some four-point functions in the $\mathcal{N} = 4$ theory may be amenable to a similar analysis. Some other four-point calculations from a different viewpoint have appeared more recently [14, 15].

Recently there has been considerable interest in the Maldacena conjecture which relates string theory on $AdS$ backgrounds to conformal field theory on the boundary [16]. In the most studied example it is conjectured that classical IIB supergravity on $AdS_5 \times S^5$ is equivalent to the large $N$ limit of $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory on the boundary which in this case is four-dimensional Minkowski spacetime. A key ingredient in this conjecture is the fact that the symmetry groups of the supergravity background and the conformal field theory are the same, namely $SU(2,2\vert 4)$. Although not all gauge-invariant operators in the $\mathcal{N} = 4$ theory are of the type given in equation (1) it turns out that it is precisely this set of operators that is relevant to the Maldacena conjecture in its simplest form. The spectrum of IIB supergravity on $AdS_5 \times S^5$ consists of the gauged $D = 5, \mathcal{N} = 8$ supergravity multiplet together with the massive Kaluza-Klein multiplets. These all fall into short representations of $SU(2,2\vert 4)$ with maximum spin 2 and are in one-to-one correspondence with the superfields $A_q$ introduced above [17].

An important example of this type of operator is the supercurrent $T = A_2 = \text{tr}(W^2)$. This multiplet has $128 + 128$ components and contains amongst them the traceless, conserved energy-momentum tensor, four gamma-traceless, conserved supersymmetry currents and fifteen conserved currents corresponding to the internal $SU(4)$ symmetry of the theory.

In the present paper we shall focus on the two- and three-point functions of the $\mathcal{N} = 4$ theory. On general grounds it is to be expected that one should be able to solve for the two- and three-point functions of an arbitrary conformal field theory in any dimension [1]. However, the advantage of our formalism is that it allows us to solve for the complete superfield correlation functions in a very simple way, not least because the operators we are interested in all have only one component. This means that the tensor structures which arise in a component approach are dealt with automatically. In addition, for those three-point functions which have non-zero leading terms in a $\theta$ expansion, it is easy to show that the solutions we obtain are unique. In fact, the form of the two- and three-point function can be found as a special case of the general formula given in ref. [3] for any Green’s function for which there are no corresponding superconformal invariants. The procedure to determine certain of the higher-point Green’s functions works in essentially the same way, but the details are very much more complicated.

Analytic fields on harmonic superspace are most simply described in the setting of complex spacetime [8, 10]. In this setting they are holomorphic fields on a complex superspace with 8 even and 8 odd coordinates. These coordinates may be assembled into a supermatrix $X$ as follows:

$$X = \begin{pmatrix}
x^{\alpha \dot{\alpha}} & \lambda^{\alpha \alpha'} \\
\pi^{\alpha \dot{\alpha}} & y^{\alpha \alpha'}
\end{pmatrix}.$$ (2)

Here the indices $\alpha, \dot{\alpha}, a$ and $a'$ each take on two values. The underlying body of this superspace is, locally, a product of complex Minkowski space and an internal space which also has four
complex dimensions and which is coordinatised by the \( y' \)'s. Locally, the internal bosonic space is the same as complex Minkowski space but globally this is not so, however, since one is usually interested in non-compact Minkowski space, whereas the internal space is always compact; in this instance it is the Grassmanian of 2-planes in \( \mathbb{C}^4 \). From a computational point of view the \( a \) and \( a' \) indices behave in exactly the same way as the two-component spacetime spinor indices \( \alpha \) and \( \dot{\alpha} \).

The action of an infinitesimal superconformal transformation on \( X \) is given by

\[
\delta X = B + AX + XD + XCX
\]

where \( \delta g \in \mathfrak{sl}(4|4) \) (the complexified superconformal algebra) is given by

\[
\delta g = \begin{pmatrix} -A & B \\ -C & D \end{pmatrix}
\]

and where the matrices \( A, B, C \) and \( D \) are now \((2|2) \times (2|2)\) supermatrices. We note that there are 8 odd coordinates \( \lambda^{a\dot{a}} \) and \( \pi^{a\dot{a}} \) whereas complexified \( N = 4 \) super Minkowski space has 16. This means that fields defined on analytic superspace depend on only half of the usual odd coordinates and are therefore to be thought of as chiral in a generalized sense. The fields we shall consider are also analytic in the internal \( y \) coordinates; since the internal part of the space is a compact complex manifold this means that their dependence on these coordinates is severely restricted.

These fields are in fact holomorphic in all the coordinates and are characterized by a positive integer \( q \); under superconformal transformations they transform as

\[
\delta A_q = \mathcal{V}A_q + q\Delta A_q
\]

where \( \mathcal{V} \) is the vector field generating the transformation (3) and where

\[
\Delta = \text{str}(A + XC) .
\]

In this language the (free) field strength tensor \( W \) is such a field with charge \( q = 1 \); in the non-Abelian case \( W \) is not actually defined on this superspace (rather it is covariantly analytic) but gauge-invariant operators of the form

\[
A_q = \text{tr}(W^q)
\]

are analytic operators and transform as in equation (3). We observe that these superfields define different short representations of \( SL(4|4) \) depending on the value of \( q \) which must be integral. Assuming that quantum effects do not disturb this representation structure \( q \) must remain unchanged, and so these fields will not have any anomalous dimensions because the dimension of \( A_q \), which is fixed by the above transformation law, is also given by \( q \). This is similar to the situation for chiral superfields where the dimensions are determined by the \( R \)-charges.

We now consider the two-point functions of such operators. The basic building block is the two-point function for the free field strength \( W \); it is
\[ <W(1)W(2)> \propto g_{12} \]  

where

\[ g_{12} = (\text{sdet} X_{12})^{-1} = \frac{\hat{y}_{12}^2}{x_{12}^2}. \]

Here \( X_{12} = X_1 - X_2 \), etc, and the hatted \( y \) variable is defined by

\[ \hat{y}_{12} = y_{12} - \pi_{12} x_{12}^{-1} \lambda_{12}. \]

This variable is invariant under \( S \)-supersymmetry transformations. The function \( g_{12} \) can also be expressed in terms of a hatted \( x \) variable which is \( Q \)-supersymmetric as

\[ g_{12} = \frac{y_{12}^2}{x_{12}} \]

where

\[ \hat{x}_{12} = x_{12} - \lambda_{12} y_{12}^{-1} \pi_{12}. \]

For the operators \( A_q \) we find

\[ <A_{q_1}(1)A_{q_2}(2)> \propto \delta_{q_1,q_2}(g_{12})^{q_1}. \]

In the case of the two-point function of two energy-momentum tensors \( T = A_2 \) the constant of proportionality can be determined by anomaly considerations as we discussed below. It is essentially the central charge of the theory.

We now turn to the three-point functions. They are expressions of the form

\[ <A_{q_1}(X_1)A_{q_2}(X_2)A_{q_3}(X_3)> \equiv G_{q_1,q_2,q_3}(X_1,X_2,X_3). \]

On the assumption that analyticity is preserved in the quantum theory\(^2\) the Ward Identity is

\[ \sum_{i=1}^{3} (V_i + q_i \Delta_i) G_{q_1q_2q_3} = 0 \]

where \( V_i \) is the vector field generating a superconformal transformation at the \( i^{th} \) point. If the sum of the charges is even, the solution to (15), up to an overall constant, is

\[ G_{q_1q_2q_3} = (g_{12})^{k_1} (g_{23})^{k_2} (g_{31})^{k_3} \]

where

\(^2\)An explicit two-loop calculation confirming this assumption is presented in the second half of the paper.
\[ k_1 = \frac{1}{2}(q_1 + q_2 - q_3) \]
\[ k_2 = \frac{1}{2}(q_2 + q_3 - q_1) \]
\[ k_3 = \frac{1}{2}(q_3 + q_1 - q_2) \]  
(17)

The restriction to the sum of the \( q \)'s being even ensures that the \( k \)'s are positive integers so that the solution is regular in the \( y \)'s. This must be the case since each field can be expanded as a polynomial in \( y \) with coefficients which are fields defined on ordinary superspace. That (14) solves (13) is easy to demonstrate. From (8) we have

\[(\mathcal{V}_1 + \mathcal{V}_2 + \Delta_1 + \Delta_2)g_{12} = 0 \]  
(18)

Using this and the values for the \( k \)'s given in (17) we find immediately that the Ward Identity (13) is satisfied.

Furthermore, this solution is unique. Suppose there was another solution, \( \mathcal{G}' \) say, then this could be written as \( \mathcal{G} \times \mathcal{G}' \) where \( \mathcal{G} \) is the above solution. But the ratio of the two solutions would be an invariant under all superconformal transformations of three points, and there are no such objects [11].

If the sum of the charges is odd, it does not necessarily mean that the corresponding correlation function should vanish because the charges are also carried by the odd coordinates. Hence, in this case, the three-point functions would be nilpotent. Furthermore, it is not so easy to establish uniqueness for this type of correlation function since one cannot divide by nilpotent quantities. Moreover, one might be able to multiply a given solution by a function which is invariant up to terms that are annihilated by the nilpotent leading term in the correlator.

We now compare our results for three-point functions with some of the partial component results that have been given in the literature. Expressions for the leading terms, i.e. the correlation functions of the Lorentz scalar fields which form the leading components of each of the fields \( A_q \), were given in early work on conformal invariance [1] and discussed in this context in [18]. In our formalism this Green’s function is simply obtained from the fully supersymmetric answer by dropping the hats on the \( y \)'s. If we denote the leading scalars by \( a_q(x, y) \) then we have the universal formula

\[ < a_{q_1}a_{q_2}a_{q_3} > = (g_{12}^0)^{k_11}(g_{23}^0)^{k_22}(g_{31}^0)^{k_33} \]  
(19)

where

\[ g_{12}^0 = \frac{y_{12}^2}{x_{12}^2} \]  
(20)

The structure of the \( x \)-factors in the denominator agrees with that expected from reference [1]. The role of the \( y \)'s is to take care of the group theory; when one expresses any field \( a(x, y) \) explicitly in terms of \( y \)'s and a scalar field with \( SL(4) \) indices, one finds that the numerators of three-point functions of fields of the latter type are given by combinations of \( SL(4) \) invariant tensors.
The next examples we shall consider concern Green’s functions that contain the $SL(4)$ (complexification of $SU(4)$) current. The $SL(4)$ currents $\mathcal{J}_{\mu}^\lambda$ appear at order $\lambda \pi$ in the expansion of the superfield $T$. Unfortunately, the spacetime derivatives of the leading scalars also appear at this level, so that one has to separate out their contribution. Explicitly, one has

$$
T \sim \lambda \pi \alpha' \pi' \lambda' \left( \tilde{J}_{\alpha' \lambda' \pi' \pi} + \frac{1}{2} \partial_{\alpha} \partial_{\alpha'} T_0 \right)
$$

(21)

where $T_0(x, y)$ is the leading component of $T$. The notation here is that an $SL(4)$ superscript $i$ is replaced by a subscript $a$ and a superscript $a'$ while an $SL(4)$ subscript $i$ is replaced by a superscript $a$ and a subscript $a'$. The $a$ and $a'$ indices are raised and lowered using the epsilon tensor as usual. Thus we have

$$
J_\lambda \rightarrow \left( \begin{array}{cc} J_a & J^a' \\ J'_a & J'_a \end{array} \right),
$$

(22)

with $J_a + J'^a = 0$. The field $\tilde{J}^{aa'}$ is then given by

$$
\tilde{J}^{aa'} = J^{aa'} + y^{ab'} J^b_a \alpha' - J^{a'b} o - y^{ab'} J^b \alpha'.
$$

(23)

We now extract from the $< TTT >$ the Greens function, the component that has one $SL(4)$ current and two $T_0$ operators. This occurs in the coefficient of $< TTT >$ that has one factor of $\lambda_1 \pi_1$, however we must take into account the occurrence of the space-time derivatives of $T_0$ given into equation [21]. One finds that

$$
< \tilde{J}_{\alpha' \lambda' \pi'}(x_1, y_1) T_0(x_2, y_2) T_0(x_3, y_3) \sim y_{23} \frac{y_{23}^2 (y_{31}^2 + y_{21}^2 (y_{31}^2) y_{x^2}^2 - (x_{12}^2)_{x^2}^2 - (x_{31}^2)_{x^2}^2)}{x_{12}^2 x_{23}^2 x_{31}^2} (24)
$$

The $y$ factors just encode the correct $SL(4)$ group theory while the space-time dependence agrees with that expected on grounds of just conformal invariance for the amplitude which has one vector and two scalar operators [1] and discussed in this context in [19].

Finally, we consider the component of $< TTT >$ which contains three $SL(4)$ currents. This will occur in the part of the $< TTT >$ that has the factor $\Pi_{i=1}^4 \lambda_4 \pi_4$. However, in order to extract it, we must as before subtract out the contributions of the scalar operator $T_0$. It is straightforward to do this, but somewhat lengthy, so we shall focus on showing that the anomaly term in $< JJJ >$ is indeed present. One finds a term of the form

$$
< J_{\alpha' \lambda' \pi'}(x_1) \mathcal{J}_{\beta \gamma} \mathcal{J}_{\delta \epsilon} \mathcal{J}_{\delta' \epsilon'} \mathcal{J}(x_3) > \sim \left\{ \begin{array}{c} \left( \delta_{\lambda_1} \delta_{\lambda_2} \delta_{\lambda_3} \delta_{\lambda_4} \delta_{\lambda_5} \delta_{\lambda_6} \delta_{\lambda_7} \delta_{\lambda_8} \right) \\
- \frac{1}{2} \left( \delta_{\lambda_1} \delta_{\lambda_2} \delta_{\lambda_3} \delta_{\lambda_4} \delta_{\lambda_5} \delta_{\lambda_6} \delta_{\lambda_7} \delta_{\lambda_8} \right) \\
+ \frac{1}{4} \delta_{\lambda_1} \delta_{\lambda_2} \delta_{\lambda_3} \delta_{\lambda_4} \delta_{\lambda_5} \delta_{\lambda_6} \delta_{\lambda_7} \delta_{\lambda_8} \left( x_1, x_2, x_3 \right) \end{array} \right\}
$$

(25)

where the function $f$ is given by
\[
f_{\alpha\beta\gamma\delta}(x_1, x_2, x_3) = \left( (x_{12})^4 (x_{23})^4 (x_{31})^4 \right)^{-1} \left( (x_{12})_{\alpha\delta}(x_{23})_{\gamma\beta}(x_{31})_{\alpha\gamma} - (x_{12})_{\alpha\beta}(x_{23})_{\gamma\delta}(x_{31})_{\alpha\gamma} \right).
\]

The combination of \( SL(4) \) delta’s in (25) is simply the \( SL(4) \) \( d \)-tensor expressed in these indices while the function \( f \) is the anomaly triangle graph expression in two-component form. Explicitly, the combination of \( x \)'s appearing in the numerator of \( f \) is

\[
(x_{12})_{\alpha\beta}(x_{23})_{\gamma\delta}(x_{31})_{\alpha\gamma} - (x_{12})_{\alpha\delta}(x_{23})_{\beta\gamma}(x_{31})_{\alpha\gamma} = (\sigma_{\mu})_{\alpha\alpha}(\sigma_{\nu})_{\beta\beta}(\sigma_{\rho})_{\gamma\gamma} \times
\]

\[
\text{tr}(\gamma_5 \gamma_{\mu} (\gamma \cdot x_{12}) \gamma_{\nu} (\gamma \cdot x_{23}) \gamma_{\rho} (\gamma \cdot x_{31}))
\]

Substituting (26) into (25) and using (27) we find an expression that agrees with that expected from conformal invariance alone [1] and agrees with that given in [19]. Although this procedure gives the functional forms for these correlation functions it does not determine the overall coefficients in terms of the coupling constant.

For supercurrent correlators it turns out that the coefficients of two and three-point functions are determined by their one-loop (free-field) values. An argument demonstrating this non-renormalization theorem for two and three point correlators involving internal currents and the energy momentum tensor was given in [19] using the results of [20]. We now give an alternative argument for the one-loop nature of the coefficients of the two- and three-point functions of \( T \). We are interested in \( \mathcal{N} = 4 \) Yang-Mills theory with the operators \( T = \text{tr}(W^2) \) as operator insertions. The latter couples to the \( \mathcal{N} = 4 \) conformal supergravity multiplet. Hence if we consider the quantum theory of \( \mathcal{N} = 4 \) Yang-Mills theory coupled to a classical \( \mathcal{N} = 4 \) conformal supergravity background the correlation functions of \( T \) will automatically be included. Although the \( \mathcal{N} = 4 \) Yang-Mills theory is finite and free of anomalies in flat space-time [2], it is not in the presence of \( \mathcal{N} = 4 \) conformal supergravity [21]. However, it is known that in perturbation theory this coupled theory is finite above one loop [22], [23]. As a result, the anomaly in the superconformal symmetry has only a one-loop contribution. The three point function \( \langle TTT \rangle \) contains, for example, the three-point correlation function of the \( SU(4) \) currents \( J \). By taking the divergence of the latter we find that this component of the three-point function is related to the axial anomaly. Since the \( \langle TTT \rangle \) correlation function has only one unknown coefficient, this coefficient must be the anomaly coefficient, up to a numerical factor. However, as we have just discussed, the anomaly only has a one-loop contribution and as a result the overall coefficient in \( \langle TTT \rangle \) is determined by this one-loop contribution.

For correlators involving fields with higher values of \( q \) it is not immediately apparent that the above argument is applicable, but recent calculations seem to suggest that non-renormalisation theorems may be true for some three-point correlation functions of analytic operators [24], [18]. A possible explanation for this is that the fields \( A_q \) couple to the boundary fields of the bulk AdS Kaluza-Klein supergravity multiplets. However, from a ten-dimensional point of view all of these multiplets combine into the ten-dimensional IIB supergravity multiplet so that one might suspect that this common ten-dimensional origin could have some implications for the conformal fields on the boundary.

In the rest of the paper we shall present an explicit calculation of the two-loop contribution to the three-point function \( \langle +2 + 3 + 3 \rangle \) in an \( \mathcal{N} = 2 \) theory consisting of complex (Fayet-
Sohnius) hypermultiplets coupled to Yang-Mills (as is well-known, if the matter is in the adjoint representation, such a model describes $N = 4$ Yang-Mills in terms of $N = 2$ superfields). The main purpose of this example is to show that the assumption of harmonic analyticity made earlier is indeed justified. We also show that the two-loop contribution to this class of correlators actually vanishes, in line with the results of ref. [24].

The $N = 2$ matter and Yang-Mills multiplets will be described in a way which maintains the $SU(2)$ symmetry manifest [8]. For this purpose we introduce the Grassmann-analytic (G-analytic) harmonic superspace with coordinates

$$x^\mu_A, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u^\pm_i.$$ \hfill (28)

Here $u^\pm_i$ are the harmonic variables parametrising the coset space $SU(2)/U(1) \sim S^2$, i.e. one is to regard $u^+_i$ as the two columns of an $SU(2)$ matrix; the index $i$ transforms under the (right) $SU(2)$ and $\pm$ are its harmonic (left) $U(1)$ projections. As a consequence, they have the defining properties:

$$u^-_i = (u^+_i)^*, \quad u^+_i u^-_i = 1.$$ \hfill (29)

The Grassmann variables $\theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}$ are $U(1)$ harmonic projections of the odd coordinates of $N = 2$ superspace,

$$\theta^{+\alpha, \dot{\alpha}} = u^+_i \theta^{+\alpha}_i, \bar{\theta}^{+\dot{\alpha}}.$$ \hfill (30)

The G-analytic space-time coordinate $x^\mu_A$ is obtained by shifting $x^\mu$:

$$x^\mu_A = x^\mu - 2i \theta^{+\alpha}(\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{+\dot{\alpha}})u^+_i u^-_j.$$ \hfill (31)

Under $Q$-supersymmetry it transforms into the $+$ projections $\theta^+$ and not their complex conjugates $\theta^-$. This is the reason why the superspace (28) is called G-analytic. In what follows we shall always work in the G-analytic superspace, therefore we shall drop the index $A$ of $x^\mu_A$.

Given two points in $x$ space, $x_{1,2}$, one can define the $Q$-supersymmetry invariant difference

$$\hat{x}_{12} = x_{12} + \frac{2i}{(12)}[(1^- 2^+ \theta^+ \bar{\theta}^+ - (1^- 2^-)(\theta^+_2 \bar{\theta}^+_2 + \theta^-_2 \bar{\theta}^-_2 + \theta^+_1 \bar{\theta}^-_2)].$$ \hfill (32)

Here $(12), (1^- 2), (12^-)$ are short-hand notations for contractions of harmonic variables:

$$(12) \equiv u^+_1 u^+_2, \quad (1^- 2) \equiv u^-_1 u^+_2, \quad (12^-) \equiv u^+_1 u^-_2.$$ \hfill (12)

In fact, $(12)$ and $\hat{x}_{12}$ are the $SU(2)$ covariant counterparts of the variables $y_{12}$ and $\hat{x}_{12}$ from eq.(12).

The matter and Yang-Mills multiplets are described by the analytic superfields $q^+_r(x, \theta^+, \bar{\theta}^+, u)$ and $V^{a+}_r(x, \theta^+, \bar{\theta}^+, u)$, with $r$ and $a$ being indices of the matter and adjoint representations of the gauge group, respectively. The details can be found in [8], here we only give a brief summary of the Feynman rules [25]. The matter propagator $\Pi$, the gluon propagator $P$ in the Fermi-Feynman gauge and the only vertex relevant to our calculation are indicated below:
The expression of the matter propagator is

\[(\Pi_{12})^r_s = \langle \bar{q}^{\dagger+}(1) | q^+_s(2) \rangle = \frac{(12)}{x_{12}^2} \delta^r_s. \quad (33)\]

Here we make use of the \(Q\)-invariant variable (32). Eq. (33) is in fact the \(SU(2)\) covariant counterpart of eq. (11). The G-analyticity of \(\Pi_{12}\) is manifest, since only the + projections of the Grassmann variables appear. Note that the original form of the matter propagator given in [25] is different, but the equivalent form (33) is best suited for our purposes in this paper. For the gluon propagator we shall use the standard form from [25]:

\[(P_{12})_{ab} = \langle V^{++}_a(1) | V^{++}_b(2) \rangle = (D^+_1)^4 \left( \frac{\delta^8(\theta_1 - \theta_2)}{x_{12}^2} \right) \delta^{(-2,2)}(u_1, u_2) \delta_{ab}. \quad (34)\]

Its G-analyticity with respect to the first argument is manifest, since it contains the maximal number four of plus-projected spinor derivatives \(D^+_a = u^+_a D^i\) (just like the chiral matter propagators in \(N = 1\) supersymmetry). G-analyticity with respect to the second argument is assured by the presence of the Grassmann and harmonic delta functions which allow us to transfer the spinor derivatives from point 1 to point 2.

Finally, the vertex describing the coupling of the gauge superfield to the hypermultiplet is shown in Figure 1. It involves a G-analytic superspace integral. Note that the harmonic integral must always be done after the Grassmann one, since the analytic Grassmann measure \(d^4\theta^+\) carries a harmonic charge. The full Yang-Mills Feynman rules involve gluon vertices of arbitrary order, as well as ghosts, but none of them show up at the two-loop level.

The three-point function we want to compute involves gauge invariant composite operators of harmonic charges \((+2 + 3 + 3)\). The simpler case \((+2 + 2 + 2)\) turns out trivial, since the matter propagator (33) obey fermion type rules and a Furry-like theorem. The first relevant graph is shown in Figure 2:
It should be remembered that this is a graph in $x$ space, therefore the true loops are those involving the internal line 4-5, as opposed to the lines 2-3 which are just free propagators. Having this in mind and applying the Feynman rules above, we find the corresponding expression (the gauge group indices and factors are not shown):

$$I_1 = (\Pi_{23})^2 \int d^4x_{4,5} d^4u_{4,5} d^4\theta^+_4 \Pi_{1442}\Pi_{3551} (D^+_4)^4 \left( \frac{\delta^8(\theta_4 - \theta_5)}{x_{45}^2} \right) \delta^{(-2,2)}(u_4, u_5). \quad (35)$$

The first step in evaluating this graph consists in using the four spinor derivatives $(D^+_4)^4$ from the gluon propagator to restore the full Grassmann integral $\int d^4\theta^+_4 (D^+_4)^4 = \int d^8\theta_4$. This is made possible by the explicit G-analyticity of the matter propagators $\Pi_{14}$ and $\Pi_{42}$. Then the Grassmann $\delta^8(\theta_4 - \theta_5)$ and harmonic $\delta^{(-2,2)}(u_4, u_5)$ delta functions can be used to do the integrals $\int du_4 d^8\theta_4$, thus identifying the Grassmann and harmonic points 4 and 5. In order to simplify the calculation, we shall evaluate the graph with all the external Grassmann variables put to zero, $\theta_1 = \theta_2 = \theta_3 = 0$. This corresponds to taking the lowest-order term in the $\theta$ expansion of the amplitude. This step allows us to easily deal with the hats $\hat{x}$ in the matter propagators (see (32)). For the propagators $\Pi_{23}$ the choice $\theta_{1,2,3} = 0$ amounts to just removing the hat, but for those involved in the vertex integrals, e.g. $\Pi_{14}$, there is still the shift due to the integration variable $\theta_5$. Now, since the points 4 and 5 have been identified, the hats in all the propagators involve the same Grassmann structure $\theta^+_5 \bar{\theta}^+_5$ but different harmonic ones. All this allows us to rewrite the amplitude (35) as follows:

$$I_1(\theta_{1,2,3} = 0) = \frac{(23)^2}{x_{23}} \int du_5 (15)^2 (25)(35) \int d^4\theta_5^+ \exp \left\{ 2i\theta_5^+ \bar{\theta}_5^+ \left[ \frac{(15^-)}{(15)} \partial_1 + \frac{(25^-)}{(25)} \partial_2 + \frac{(35^-)}{(35)} \partial_3 \right] \right\} f(1, 2, 1, 3). \quad (36)$$

Here $f(1, 2, 1, 3)$ denotes the two-loop $x$-space integral

$$f(1, 2, 1, 3) = \int d^4x_4 d^4x_5 \frac{d^4x_1 d^4x_2 d^4x_3}{x_{14}^2 x_{24}^2 x_{15}^2 x_{35}^2 x_{45}^2}. \quad (37)$$

Using the translational invariance of $f(1, 2, 1, 3)$ we can substitute $\partial_1 f = -(\partial_2 + \partial_3) f$ in (36). Then we use harmonic cyclic identities of the type $(25^-)(15) - (15^-)(25) = (12)$ (see the defining property (29)). Next we expand the exponential and do the Grassmann integral, after which (36) is reduced to (up to an overall factor)
\[ I_1(0) = \frac{(23)^2}{x_{23}} \int d\mu_5 \left[ \frac{(12)^2(35)}{(25)} \partial_2 \cdot \partial_2 + \frac{(13)^2(25)}{(35)} \partial_3 \cdot \partial_3 + (12)(13) 2\partial_2 \cdot \partial_3 \right] f(1,2,1,3). \]  

The harmonic integral of the third term in (38) is trivially done \((f d\mu_5 \equiv 1)\), and the first two ones are computed as follows (see [25] for details):

\[ \int d\mu_5 \frac{(35)(25)}{(25)} = \int d\mu_5 \frac{\partial_5^{++}(35^-)}{(25)} = \int d\mu_5 (35^-)\delta^{(-1,1)}(u_2, u_5) = (32^-). \]

Further, the box operators in (38) reduce the two-loop integral \(f\) to a one-loop one, e.g.

\[ \partial_2 \cdot \partial_2 \int \frac{d^4 x_4 d^4 x_5}{x_{14}^2 x_{24}^2 x_{35}^2 x_{45}^2} = \frac{4\pi^2 i}{x_{12}^2} \int \frac{d^4 x_5}{x_{15}^2 x_{25}^2 x_{35}^2} \equiv \frac{g(1,2,3)}{x_{12}^2}. \]

The end result of all this is:

\[ I_1(0) = \frac{(23)^2}{x_{23}^4} \left[ (12)^2(32^-) \frac{g(1,2,3)}{x_{12}^2} + (13)^2(23^-) \frac{g(1,2,3)}{x_{13}^2} + (12)(13) 2\partial_2 \cdot \partial_3 f(1,2,1,3) \right]. \]  

We immediately remark the presence of negative-charged harmonics in (39) which means that the contribution of the graph in Figure 2 is not harmonic analytic. However, the situation changes when we take into account the two other graphs of similar type shown in Figure 3:

Figure 3

They can be computed in exactly the same way as the one in Figure 2. Putting all three contributions together and using harmonic cyclic identities we find that the non-analytic negative-charged harmonics disappear and we obtain the harmonic analytic result:

\[ I(0) = I_1 + I_2 + I_3 = \frac{(12)(13)(23)^2}{x_{12}^2 x_{13}^2 x_{23}^4} g(1,2,3) \]

where
\[ a(1, 2, 3) = x_{12}^2 g(1, 2, 3) + x_{12}^2 x_{13}^2 2 \partial_2 \cdot \partial_3 f(1, 2, 1, 3) + \text{cycle}. \]  

(41)

In (40) we observe a product of four matter propagators multiplied by the coefficient function \( a(1, 2, 3) \). The latter, according to the general theory, must be a conformally invariant function of three points and hence can only be a constant. In fact, it can be shown to vanish. A simple argument using the Lorentz, translational and scaling properties of the integrals involved in (41) leads to the identity

\[ 2 \partial_2 \cdot \partial_3 f(1, 2, 1, 3) = \frac{x_{23}^2 - x_{12}^2 - x_{13}^2}{x_{12}^2 x_{13}^2} g(1, 2, 3). \]

Substituting this in (41) gives \( a(1, 2, 3) = 0 \), so

\[ I(\theta_1 = \theta_2 = \theta_3 = 0) = 0. \]  

(42)

In other words, the lowest-order \((\theta = 0)\) term in the amplitude is zero, and this can then be generalised to the entire amplitude.

The example of a three-point correlator presented above is not unique. It is easy to construct three-point functions with higher \( U(1) \) charges by simply attaching more matter propagators to the external points. This does not affect the loop structure of the graphs and thus leads to the same result as above.

In conclusion we should mention that there exist further two-loop graphs involving the following insertions:

They need not be considered because both of them vanish (strictly speaking, the second one is proportional to \( \delta(x_{12}) \), but in our analysis we always keep the external points of an \( n \)-point function apart).

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