Continuous dynamical decoupling of spin chains: modulating the spin-environment and spin-spin interactions

Sharoon Austin,1 Muhammad Qasim Khan,1 Maryam Mudassar,1 and Adam Zaman Chaudhry1,∗

1School of Science & Engineering, Lahore University of Management Sciences (LUMS), Opposite Sector U, D.H.A, Lahore 54792, Pakistan

For spin chains to be useful for quantum information processing tasks, the interaction between the spin chain and its environment generally needs to be suppressed. In this paper, we propose the use of static and oscillating control fields in order to effectively remove the spin-chain-environment interaction. We find that our control fields can also effectively transform the spin chain Hamiltonian. In particular, interaction terms which are absent in the original spin chain Hamiltonian appear in the time-averaged effective Hamiltonian once the control fields are applied, implying that spin-spin interactions can be engineered via the application of static and oscillating control fields. This transformation of the spin chain can then potentially be used to improve the performance of the spin chain for quantum information processing tasks. For example, our control fields can be used to achieve almost perfect quantum state transfer across a spin chain even in the presence of noise. As another example, we show how the use of static and oscillating control fields not only suppresses the effect of the environment, but can also improve the generation of two-spin entanglement in the spin chain.

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I. INTRODUCTION

Spin chains have been a subject of constant study for many years now in diverse areas. For example, spin chains have been used to study phase transitions [1, 2], quantum chaos [3], high-temperature superconductivity [4], and Anderson localization [5]. On the experimental front, the physical realization of spin chains ranges from trapped ions [6] to optical lattices [7], solid state setups [8], and photonic systems [9]. In the context of quantum information and computation, spin chains have been, for example, extensively studied to achieve perfect quantum state transfer from one site to another [10–12], and to generate and distribute entanglement [13–17]. However, one of the major hurdles towards the use of spin chains in such quantum information tasks is the inevitable coupling of the spin chain to its environment [18–20], which results in the rapid decoherence of the fragile, generally many-body entangled, quantum spin chain state. As such, it is worthwhile studying ways in which the quantum spin chain can be effectively protected from its environment.

One promising method of protecting the quantum spin chain is to use dynamical decoupling [20–24]. In dynamical decoupling, control fields are applied rapidly on the quantum system that needs to be protected. The usual approach is to consider different pulse sequences applied to the system [20, 22, 30–32] which effectively modulate the system-environment interaction, thereby greatly extending the decoherence timescale. However, one can envisage applying instead strong static and oscillating control fields to dynamically decouple the spin chain, as has been done for a single qubit [33, 34], two qubits [35, 36], and an effective large spin system [38]. This scheme has the advantage that one need not worry about the timing of the different fields; one simply turns on the required fields to achieve effective decoupling of the system from its environment. However, at the same time, the spins in the spin chain are also interacting, and, as a result of the control fields applied, this interaction is also modulated. Consequently, the interactions between the spins are also changed due to the static and oscillating control fields. Instead of considering the change of the spin-spin interactions as a nuisance, we can think about using this change to our advantage. That is, can we apply simple static and oscillating fields to the spin chain such that not only is the spin chain effectively decoupled from its environment (at least to lowest order), but the spin chain Hamiltonian is also changed in such a way that, for example, state transfer fidelity improves or the performance of the spin chain in generating entanglement increases? This is the question that we intend to answer in this paper. We note that control fields, in the form of pulse sequences, have been used to engineer spin chain Hamiltonians [32–34]; however, these pulse sequences can be rather complicated. Our static and oscillating control fields can be used in conjunction with schemes based on pulses, thereby realizing hybrid Hamiltonian engineering techniques.

We start by considering the Hamiltonian of a general one-dimensional spin chain which is an anisotropic version of the usual XYZ Hamiltonian [43]. We assume that each spin in this spin chain is coupled ‘locally’ to its environment [35, 36]. In such a situation, we first find suitable static and oscillating control fields that, when applied to the spin chain, are able to dynamically decouple the spin chain, at least to lowest order. The nice feature of these control fields is that the same field needs to
be applied to each spin. We then proceed to investigate how the spin chain interactions are modulated by these continuous dynamical decoupling control fields. We find that the spin-spin interactions fundamentally change depending on the control fields, and the effective spin chain Hamiltonian contains interactions that are not present in the original spin chain Hamiltonian. Interestingly, for a special set of control fields, even more additional interaction terms, similar to those in the Dzyaloshinskii-Moriya interaction [44–48], can be generated. Our aim then is to analyze the spin chain with these control fields. We first look at the possibility of achieving perfect quantum state transfer by removing the effect of the environment and, at the same time, suitably engineering the spin chain Hamiltonian. We then investigate the entanglement generated between two spins of the spin chain via the spin chain interactions. To this end, we present numerical simulations that first show that the control fields are able to effectively dynamically decouple the spin chain. Second, the simulations show that the dynamics of the spin chain in the presence of the control fields is captured very well by the effective time-averaged Hamiltonian which, in general, contains additional interaction terms. Third, we show that for special control fields, the generation of entanglement can be enhanced even more due to the additional interaction terms. After these numerical simulations, we subsequently endeavor to analytically solve the dynamics of the time-averaged effective spin chain Hamiltonian. We show that if we impose a condition on the coupling coefficients in the spin chain, we can transform our problem to a system of non-interacting fermions via the Jordan-Wigner transformation [43]. With this approach, we are able to significantly reduce the computational complexity of the problem. We then demonstrate that our special control fields are able to enhance the entanglement generation, even for larger spin chains.

This paper is organized as follows. In Sec. II, we present the static and oscillating control fields we use to dynamically decouple the spin chain from its environment, and derive the effective spin chain Hamiltonian in the presence of these control fields. The use of these control fields towards obtaining perfect quantum state transfer is investigated in Sec. III. The performance of the control fields in entanglement generation is then numerically analyzed in Sec. IV. In Sec. V, we diagonalize the effective Hamiltonian via the Jordan-Wigner transformation, and thereby extend our study of entanglement generation to relatively larger spin chains. Finally, we conclude in Sec. VI.

II. THE FORMALISM

We start by considering the usual XYZ Hamiltonian which describes a one-dimensional spin chain. Considering only nearest-neighbor coupling, the Hamiltonian, with zero magnetic field, can be written as (we take $\hbar = 1$ throughout)

$$H_0 = \sum_{j=1}^{N-1} \sum_{k=1}^{3} \lambda_{jk} \sigma^j_k \sigma^{(j+1)}_k. \quad (1)$$

Here $\lambda_{jk}$ are the coupling strengths between the spins, $j$ labels the sites, and $k = 1, 2, 3$ denotes $x$, $y$ and $z$ respectively. As usual, $[\sigma^j_p, \sigma^q_m] = 2i\delta_{pq}\delta_{lm}\sigma^l_n$, and note that we are not using cyclic boundary conditions.

We want to dynamically decouple the spin chain from its environment. To this end, we first need to model the spin chain-environment interaction. We assume that each spin interacts ‘locally’ with the environment so that the interaction between the spin chain and its environment is given by

$$H_{SB} = \sum_{j=1}^{N-1} B^{(j)}_x \sigma^j_x + B^{(j)}_y \sigma^j_y + B^{(j)}_z \sigma^j_z. \quad (2)$$

Here $B^{(j)}_k$ are arbitrary environment operators (or randomly fluctuating noise terms for a classical bath). Our basic strategy is to apply periodic control fields to the spin chain to modulate the interaction between the spin chain and its environment in such a way that the spin chain becomes effectively decoupled from its environment, at least to lowest order. Corresponding to these continuous control fields, there is a unitary operator $U_c(t)$ such that $\frac{\partial U_c(t)}{\partial t} = H_c(t)U_c(t)$, where $H_c(t)$ is the Hamiltonian describing the action of the control fields on the spin chain. Since we are considering periodic control fields, $U_c(t+t_c) = U_c(t)$. Furthermore, in order to decouple the spin chain from the environment to lowest order, we have the condition [33 34 36 37]

$$\int_0^{t_c} dt U^\dagger_c(t)H_{SB}U_c(t) = 0. \quad (3)$$

For completeness, the reasoning behind this condition is shown in Appendix A. Keeping the form of $H_{SB}$ in mind, we guess that

$$U_c(t) = \prod_{i=1}^N e^{i\omega n_x \sigma^i_x t} e^{i\omega n_y \sigma^i_y t}, \quad (4)$$

where $\omega = 2\pi/t_c$ and $n_x$ and $n_y$ are integers, is one possible choice that can dynamically decouple the spin chain from its environment. Our task then is to check that this is indeed the case. It is trivial to check that $U_c(t+t_c) = U_c(t)$. Then, using $U_c^{(j)}(t)\sigma^{(j)}_x U_c^{(j)}(t) = \cos(2\omega n_y t)\sigma^j_x - \sin(2\omega n_y t)\sigma^j_z$, and

$$U_c^{(j)}(t)\sigma^{(j)}_y U_c^{(j)}(t) = \sin(2\omega n_x t)\sin(2\omega n_y t)\sigma^j_x + \cos(2\omega n_x t)\sigma^j_y + \sin(2\omega n_x t)\cos(2\omega n_y t)\sigma^j_z,$$
We now define
\[ U_c^{(j)}(t) = \cos(2\omega n_x t) \sin(2\omega n_y t) \sigma_x^{(j)} - \sin(2\omega n_x t) \sigma_y^{(j)} + \cos(2\omega n_x t) \cos(2\omega n_y t) \sigma_z^{(j)}, \]

it is straightforward to see that as long as \( n_x \neq n_y \), we meet the condition given by Eq. (3). The corresponding control field Hamiltonian is given by

\[ H_c(t) = \sum_{i=1}^{N} \left\{ \omega n_y [\sin(2\omega n_x t) \sigma_z^{(i)} - \cos(2\omega n_x t) \sigma_y^{(i)}] \right\}, \]

with \( n_x \neq n_y \). Thus, provided that \( \omega \) is large enough, we are able to dynamically decouple the spin chain from its environment, at least to lowest order, by using two oscillating fields in the \( y \) and \( z \) directions and a static field in the \( x \) direction.

The key observation now is that the control fields not only serve to dynamically decouple the spin chain, but they also modify the spin chain Hamiltonian itself. Provided that the control fields are strong enough and oscillating fast enough, the effective spin chain Hamiltonian in the presence of the control fields is

\[ \bar{H} = \frac{1}{t_c} \int_{0}^{t_c} dt U_c^{(j)}(t) H_0 U_c^{(-j)}(t). \]

For completeness, this relation is also derived in Appendix A. In our case, the effective Hamiltonian becomes

\[ \bar{H} = \frac{1}{t_c} \sum_{j=1}^{N-1} \int_{0}^{t_c} dt \sum_{k=1}^{3} \lambda_{jk} h_{j,k}(t) h_{j+1,k}(t), \]

with

\[ h_{j,k}(t) = U_c^{(j)}(t) \sigma_k^{(j)} U_c^{(-j)}(t). \]

Defining \( X = -\omega n_x t \) and \( Y = -\omega n_y t \) for brevity, we can write

\[ h_{j,1}(t) = \cos(2Y) \sigma_x^{(j)} - \sin(2Y) \sigma_z^{(j)}, \]
\[ h_{j,2}(t) = \sin(2X) \sin(2Y) \sigma_z^{(j)} + \cos(2X) \sigma_y^{(j)} + \sin(2X) \cos(2Y) \sigma_z^{(j)} \]
\[ + \sin(2X) [2 \sin^2(Y) + \cos(2Y)] \sigma_y^{(j)}. \]

We now define

\[ I_1^{(j)} = \frac{1}{t_c} \int_{0}^{t_c} h_{j,1}(t) h_{j+1,1}(t) dt, \]
\[ I_2^{(j)} = \frac{1}{t_c} \int_{0}^{t_c} h_{j,2}(t) h_{j+1,2}(t) dt, \]
\[ I_3^{(j)} = \frac{1}{t_c} \int_{0}^{t_c} h_{j,3}(t) h_{j+1,3}(t) dt. \]

With these, the effective Hamiltonian becomes

\[ \bar{H} = \sum_{j=1}^{N-1} \left[ \lambda_{j1} I_1^{(j)} + \lambda_{j2} I_2^{(j)} + \lambda_{j3} I_3^{(j)} \right]. \]

The remaining task is to evaluate the integrals. Recalling that \( n_x \) and \( n_y \) are integers with \( n_x \neq n_y \) (since we want to dynamically decouple the spin chain from its environment), we find that, if \( n_y \neq 2n_x \),

\[ I_1^{(j)} = \frac{1}{4} [\sigma_x^{(j)} \sigma_x^{(j+1)} + \sigma_z^{(j)} \sigma_z^{(j+1)}], \]
\[ I_2^{(j)} = \frac{1}{4} [\sigma_x^{(j)} \sigma_x^{(j+1)} + 2\sigma_y^{(j)} \sigma_y^{(j+1)} + \sigma_z^{(j)} \sigma_z^{(j+1)}], \]
\[ I_3^{(j)} = I_2^{(j)}. \]

This leads to

\[ \bar{H}_1 = \sum_{j=1}^{N-1} \left\{ \lambda_{j1} \left[ \frac{1}{2} (\sigma_x^{(j)} \sigma_x^{(j+1)} + \sigma_z^{(j)} \sigma_z^{(j+1)}) \right] + (\lambda_{j2} + \lambda_{j3}) \left[ \frac{1}{4} (\sigma_x^{(j)} \sigma_x^{(j+1)} + 2\sigma_y^{(j)} \sigma_y^{(j+1)} + \sigma_z^{(j)} \sigma_z^{(j+1)}) \right] \right\}, \]

for \( n_x \neq n_y \) and \( n_y \neq 2n_x \). Notice that the spin chain Hamiltonian has been fundamentally transformed. For example, the original spin chain may be the quantum Ising spin chain such that \( \lambda_{j1} \) is non-zero while \( \lambda_{j2} = \lambda_{j3} = 0 \). In this case, \( \bar{H}_1 \) becomes essentially the quantum XX spin chain with zero magnetic field. Thus, by applying local control fields, one can not only decouple spin chains from their environment, but also transform the interactions between the spins themselves. In fact, we can go further along these lines if we consider instead the case \( n_x \neq n_y \), but now \( n_y = 2n_x \). In this case, \( I_1^{(j)} \) is the same as before, but now

\[ I_2^{(j)} = \frac{1}{4} [\sigma_x^{(j)} \sigma_x^{(j+1)} + 2\sigma_y^{(j)} \sigma_y^{(j+1)} + \sigma_z^{(j)} \sigma_z^{(j+1)} + \sigma_x^{(j)} \sigma_y^{(j+1)} + \sigma_y^{(j)} \sigma_z^{(j+1)}], \]
\[ I_3^{(j)} = \frac{1}{4} [\sigma_x^{(j)} \sigma_x^{(j+1)} + 2\sigma_y^{(j)} \sigma_y^{(j+1)} + \sigma_z^{(j)} \sigma_z^{(j+1)} + \sigma_y^{(j)} \sigma_z^{(j+1)}]. \]

In this case, we can then write the effective Hamiltonian as

\[ \bar{H}_2 = \sum_{j=1}^{N-1} \left[ \frac{\lambda_{j1}}{2} \left[ \sigma_x^{(j)} \sigma_x^{(j+1)} + \sigma_z^{(j)} \sigma_z^{(j+1)} \right] + \frac{\lambda_{j2}}{4} \left[ \sigma_x^{(j)} \sigma_x^{(j+1)} \right] \right. \]
\[ + 2\sigma_y^{(j)} \sigma_y^{(j+1)} + \sigma_y^{(j)} \sigma_y^{(j+1)} + \sigma_z^{(j)} \sigma_z^{(j+1)} + \sigma_z^{(j)} \sigma_z^{(j+1)} \]
\[ \left. + \frac{\lambda_{j3}}{4} \left[ \sigma_y^{(j)} \sigma_z^{(j+1)} + 2\sigma_y^{(j)} \sigma_y^{(j+1)} - \sigma_y^{(j)} \sigma_y^{(j+1)} \right] \sigma_z^{(j)} \sigma_z^{(j+1)} \right]. \]
This case is even more interesting due to the additional presence of the ‘cross-interactions’ such as $\sigma_x^{(j)} \sigma_y^{(j+1)}$. Such ‘cross-interactions’ arise in spin chains when one studies Dzyaloshinskii-Moriya interactions in spin chains (although the signs of our additional terms differ). However, in our case, these interactions are simply an effective result of applying control fields to each spin.

III. QUANTUM STATE TRANSFER

As a first example of our formalism, we study the transfer of a quantum state from one end of a quantum spin chain to the other \[11, 49\]. The most commonly studied scenario involves the quantum XX model

$$H_{XX} = \sum_{j=1}^{N-1} \lambda_j \left[ \sigma_x^{(j)} \sigma_x^{(j+1)} + \sigma_y^{(j)} \sigma_y^{(j+1)} \right] + \sum_{j=1}^{N} B_j \sigma_z^{(j)}.$$  

The idea is that an arbitrary state for the spin chain can be transferred to the other end. Writing the eigenstates of the $\sigma_z$ operator as $|0\rangle$ and $|1\rangle$, it has been found that if the initial state of the spin chain is $|\Psi(0)\rangle = (a|0\rangle + b|1\rangle) \otimes |0\rangle \otimes \ldots \otimes |0\rangle$, the state of the spin chain after some time $T$ is $|\Psi(T)\rangle = |0\rangle \otimes \ldots \otimes |0\rangle \otimes (a|0\rangle + e^{i\phi}b|1\rangle)$, where $\phi$ is known so that the phase can be corrected at the end of the state transfer. Since our decoupling fields will remove the influence of the term $\sum_{j=1}^{N} B_j \sigma_z^{(j)}$, we look to achieve this for the case where all $B_j = 0$. One way to then achieve perfect state transfer is that we set $\lambda_j = \sqrt{j(N-j)} \ [50, 51]$, which is optimal in terms of the transfer time $[52]$. Perfect state transfer is then achieved after time $T = \pi/2$, with the phase factor $e^{i\phi} = (-i)^{N-1}$ that can be removed $[52]$. Practically speaking, however, such perfect state transfer is difficult due to the unwanted influences of the environment. To remove this detrimental effect, pulse sequences $[54]$ have been considered and the direct modulation of the spin-spin coupling has also been investigated $[55]$. With our scheme, as we have discussed, local noise terms can be eliminated to lowest order by applying a static as well as oscillating control fields. These control fields also modify the spin chain Hamiltonian. In particular, it is clear that the quantum XX spin chain does not remain the quantum XX spin chain in the presence of the control fields. To get around this, we note that if we originally have the quantum Ising model (with zero magnetic field),

$$H_0 = \sum_{j=1}^{N-1} \lambda_j \sigma_x^{(j)} \sigma_x^{(j+1)},$$  

the corresponding time-averaged effective Hamiltonian in the presence of the control fields is

$$\tilde{H}_1 = \sum_{j=1}^{N-1} \frac{\lambda_j}{2} \left[ \sigma_x^{(j)} \sigma_x^{(j+1)} + \sigma_y^{(j)} \sigma_y^{(j+1)} \right].$$  

In this case, it turns out that $\tilde{H}_2 = \tilde{H}_1 = \tilde{H}$, so $n_x$ being equal to $2n_z$ or not makes no difference. Notice that the effective Hamiltonian contains $\sigma_z^{(j)} \sigma_z^{(j+1)}$ interactions which are absent in the original Ising chain. In particular, the effective Hamiltonian is simply a rotated version of the XX model. Defining the rotation operator $U_R = \Pi_j e^{-i\sigma_z^{(j)} \phi / 4}$, we find that $H_R = U_R H U_R^\dagger = H_{XX}$ with $B_j = 0$. In other words, if we start from the Ising spin chain, and apply the control fields given by

$$H_c(t) = \sum_{n=1}^{N} \left\{ \omega n_x [\sin(2\omega n_x t) \sigma_y^{(n)} + \cos(2\omega n_x t) \sigma_z^{(n)}] - \omega n_x \sigma_z^{(n)} \right\},$$  

where $n_x$ and $n_z$ are integers with $n_x \neq n_z$, we can achieve excellent state transfer $(a|0\rangle + b|1\rangle) \otimes |0\rangle \otimes \ldots \otimes |0\rangle \rightarrow |0\rangle \otimes \ldots \otimes |0\rangle \otimes (a|0\rangle + b|1\rangle)$ even in the presence of the local noise terms.

![FIG. 1. Dynamics of the fidelity of the state transfer starting from the quantum Ising model. Here we have $N = 4$. We have used $\lambda_j = \sqrt{j(N-j)}$, and $n_x = 2$, $n_y = 1$. The dot-dashed, magenta curve shows the fidelity with the full Hamiltonian $H_0 + H_{SB}$ numerically solved, while the solid, black curve shows the dynamics using the time-averaged effective Hamiltonian $H_R$. The dashed blue curve shows the fidelity if we simply use only the XX spin chain $H_{XX}$ in the presence of noise. We are working in dimensionless units with $\hbar$ set equal to one and $t_c = 0.01$.

We now numerically check our claims. Since the Hamiltonian $H_R$ preserves the number of excitations, we restrict ourselves to studying the transfer $|\psi_i\rangle = |1\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \rightarrow |\psi_f\rangle = |0\rangle \otimes \ldots \otimes |1\rangle$ for simplicity. To quantify the quality of the state transfer, we use the fidelity $F(t) = \langle \psi_f | \rho(t) | \psi_f \rangle$ where $\rho(t)$ is the spin state density matrix at time $t$. We model the effect of the environment on the spins via classical noise fields acting on each spin. For simplicity, we assume that $B_k^{(j)}$ is the same for every $j$. $B_k^{(j)}$, $B_x^{(j)}$, and $B_{yy}^{(j)}$ are then generated via independent Ornstein-Uhlenbeck processes, each with long-term mean zero, reversion speed ten, and volatility equal to 0.5.

In Fig. 1, we illustrate state transfer for $N = 4$. First, the fidelity is captured very well by the effective Hamilto-
nian since the solid black curve essentially overlaps with the dot-dashed magenta curve. Second, the effect of the noise is effectively removed. Third, if we use the XX model with no control fields and noise present, the fidelity of the state transfer is significantly lower as shown by the dashed blue curve. Similar results are obtained for larger spin chains as illustrated in Fig. 2. Thus, we have demonstrated that if we use the quantum Ising model to begin with, we can obtain excellent quantum state transfer simply by the use of static and oscillating control fields even in the presence of noise.

IV. NUMERICAL RESULTS FOR ENTANGLEMENT GENERATION

We now quantitatively analyze the results of applying the control fields for entanglement generation. To do this, we look at the concurrence [56] between two spins in the spin chain. Our strategy is simple. We consider the spin chain in the presence of local noise fields. To begin, we do not apply any control fields, and examine the behavior of the concurrence between two spins as a function of time. Thereafter, we apply our strong and rapidly oscillating control fields. We find the concurrence between two spins in the presence of the noise fields by solving the Schrödinger equation, thereby showing the effectiveness of the control fields in dynamically decoupling the spin chain. We also show that the dynamical behavior is captured very well by the time-averaged effective Hamiltonian approach. Finally, we compare the performance of the control fields with $n_y \neq 2n_x$ and $n_y = 2n_x$.

A. The quantum Ising model

In this case, the spin chain Hamiltonian (with zero magnetic field) is given by Eq. [3], while the corresponding effective Hamiltonian in the presence of the control fields is shown in Eq. [4]. From now on, for simplicity, we will be assuming that the coupling strengths are the same throughout the chain, that is, $\lambda_{jk}$ is independent of $j$. As outlined before, we aim to find the concurrence between two spins in the spin chain as a function of time. We find the density matrix as a function of time, and then take the partial trace over all the spins other than the two spins whose concurrence we are interested in. Having found this two-spin density matrix $\rho_2(t)$ as a function of time, we find the concurrence [56] by first finding\[
R = \sqrt{\bar{\rho}_2^{-1}} \bar{\rho}_2 \bar{\rho}_2^{-1},\]
with\[
\bar{\rho}_2 = (\sigma_y \otimes \sigma_y) \rho_2^* (\sigma_y \otimes \sigma_y).
\]
The concurrence $C$ is then given by\[
C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)
\]
where $\lambda_i$ are the eigenvalues of $R$ in descending order.

Let us now present our results for the quantum Ising model. In Fig. 3 we illustrate three points. First, as can be seen by comparing the dashed, black curve with the dot-dashed, magenta curve, the time-averaged Hamiltonian reproduces the exact numerical results very well. Second, the concurrence in the presence of the noise is very low - the solid, blue curve is nearly on top of the horizontal axis. Third, the control fields are able to average out the effect of the noise fields. However, since there is no difference in $H_1$ and $H_2$, we now investigate the XY model.

![Image](image.png)

FIG. 2. Same as Fig. 1 except that we now have $N = 6$.

![Image](image.png)

FIG. 3. Dynamics of the concurrence between spins 1 and 4 for the quantum Ising model with $N = 4$. We have used $\lambda_1 = 0.25$, and $n_y = 3$, $n_x = 1$. The dynamics for the concurrence without any control fields is shown by the solid blue curve, while the dot-dashed, magenta curve shows the dynamics with the full Hamiltonian $H_0 + H_{SB} + H_\nu(t)$ numerically solved. The dashed, black curve shows the dynamics using the time-averaged effective Hamiltonian. We are working in dimensionless units with $\hbar$ set equal to one and $t_0 = 0.1$ from now on. Furthermore, the initial state of the spin chain used throughout is the state with all spins up in the $z$ direction. As before, we model the effect of the environment on the spins via classical noise fields acting on each spin. We assume that $B^{(j)}_k$ is the same for every $j$. $B^{(j)}_x$, $B^{(j)}_y$, and $B^{(j)}_z$ are then generated via independent Ornstein-Uhlenbeck processes, each with long-term mean zero, reversion speed ten, and volatility equal to 0.1 from now on.
B. XY model

For the XY model, the spin chain Hamiltonian, with zero magnetic field, is
\[ H_0 = \sum_{j=1}^{N-1} \left[ \lambda_1 \sigma_x^{(j)} \sigma_x^{(j+1)} + \lambda_2 \sigma_y^{(j)} \sigma_y^{(j+1)} \right]. \]

In the presence of the control fields, the effective Hamiltonian is
\[
\begin{align*}
\bar{H}_1 &= \sum_{j=1}^{N-1} \left\{ \frac{\lambda_1}{2} \left[ \sigma_x^{(j)} \sigma_x^{(j+1)} + \sigma_y^{(j)} \sigma_y^{(j+1)} \right] \\
&\quad + \frac{\lambda_2}{4} \left[ \sigma_x^{(j)} \sigma_x^{(j+1)} + 2\sigma_y^{(j)} \sigma_y^{(j+1)} + \sigma_y^{(j)} \sigma_y^{(j+1)} \right] \right\},
\end{align*}
\]
(10)
if \( n_y \neq 2n_x \). Once again, note the presence of the additional spin-spin interactions. On the other hand, if \( n_y = 2n_x \), the effective Hamiltonian is
\[
\begin{align*}
\bar{H}_2 &= \sum_{j=1}^{N-1} \left\{ \frac{\lambda_1}{2} \left[ \sigma_x^{(j)} \sigma_x^{(j+1)} + \sigma_y^{(j)} \sigma_y^{(j+1)} \right] \\
&\quad + \frac{\lambda_2}{4} \left[ \sigma_x^{(j)} \sigma_x^{(j+1)} + 2\sigma_y^{(j)} \sigma_y^{(j+1)} + \sigma_y^{(j)} \sigma_y^{(j+1)} \right] \\
&\quad + \sigma_y^{(j)} \sigma_y^{(j+1)} + \sigma_y^{(j)} \sigma_y^{(j+1)} \right\} \right\},
\end{align*}
\]
There are now even more additional spin-spin interactions; \( \bar{H}_2 \) contains ‘cross-interaction’ terms \( \sigma_x^{(j)} \sigma_y^{(j+1)} \) and \( \sigma_y^{(j)} \sigma_y^{(j+1)} \) absent in \( \bar{H}_1 \).

We now present numerical simulations illustrating the effect of these additional terms. First, Fig. 4 shows the concurrence between the first and last spins of the spin chain with \( N = 4 \) where we are using control fields such that \( n_y \neq 2n_x \). The blue curve (which is on top of the horizontal axis) illustrates that, in the absence of the control fields, entanglement generation is negligible. However, in the presence of the control fields, significant entanglement can be generated, as evidenced by the dot-dashed, magenta curve. Moreover, the dashed black curve, which lies essentially on top of the magenta curve, shows that the dynamics are captured very well by the time-averaged effective Hamiltonian \( \bar{H}_1 \). Moving on, in Fig. 5 we have again shown the dynamics of the entanglement between spins 1 and 4, but we have now used the special control fields with \( n_y = 2n_x \). As before, considerable entanglement is generated in the presence of the control fields, and the dynamics is captured very well by the effective Hamiltonian (which is \( \bar{H}_2 \) now). Moreover, comparing Figures 4 and 5, we see that the special choice \( n_y = 2n_x \) is able to generate more entanglement (at least between spins 1 and 4). If we look instead at spins 2 and 3 of the spin chain [see Figures 6 and 7], we reach a similar conclusion. Thus, a judicious choice of the control fields can impact the spin-spin interactions in such a way that generation of a valuable quantum resource such as quantum entanglement is improved.

**FIG. 4.** Plot of the concurrence between spins 1 and 4 for the quantum XY model with \( N = 4 \). Here we have \( n_y \neq 2n_x \) (\( n_y = 3, n_x = 1 \)), and \( \lambda_1 = \lambda_2 = 1 \). We have shown the dynamics without any control fields (solid blue), with control fields using the total Hamiltonian \( H_0 + H_{SB} + H_c(t) \) (dot-dashed, magenta curve), and using the time-averaged Hamiltonian \( \bar{H}_1 \) (dashed, black curve).

**FIG. 5.** Same as Fig. 4 except that we are now using control fields with \( n_y = 2n_x \) (\( n_y = 2, n_x = 1 \)).

**FIG. 6.** Same as Fig. 4 except that we are now showing the concurrence between spins 2 and 3 of the spin chain.
C. XYZ model

We now look at the full XYZ spin chain Hamiltonian, given by

\[ H_0 = \sum_{k=1}^{3} \lambda_k \sum_{j=1}^{N-1} \sigma_k^{(j)} \sigma_k^{(j+1)}. \]

One important comment is in order. If the spin chain is fully isotropic, that is, \( \lambda_1 = \lambda_2 = \lambda_3 \), then the control fields cannot alter the spin chain Hamiltonian. The reason is simple - for the fully isotropic case, the Hamiltonian can be written as an inner product of a vector consisting of the Pauli matrices, and this inner product is of course invariant under unitary operations. Therefore, we will focus on the anisotropic case where the coupling strengths are not all equal to each other. The time-averaged Hamiltonian is given by \( \bar{H}_1 \) if \( n_y \neq 2n_x \) [see Eq. (6)] and by \( \bar{H}_2 \) [see Eq. (7)] if \( n_y = 2n_x \). Having previously shown that the effect of the control fields is well approximated by the effective Hamiltonians, we show in Fig. 7 the dynamics of the concurrence between the first and last spin for \( N = 4 \) using the effective Hamiltonians only. As shown by the dot-dashed magenta curve, it is clear that the control fields with \( n_y = 2n_x \) are clearly better at generating entanglement for the given values of the interaction strengths. Once again, the choice of the control fields can affect the interactions, and thereby the generation of a quantum resource such as entanglement, to a very large degree.

V. COMPARING THE EFFECTIVE HAMILTONIANS \( \bar{H}_1 \) AND \( \bar{H}_2 \) FOR LARGER \( N \)

Having now demonstrated that if we apply sufficiently strong and rapidly oscillating control fields, the spin chain Hamiltonian can be approximated by \( \bar{H}_1 \) if \( n_y \neq 2n_x \) and by \( \bar{H}_2 \) if \( n_y = 2n_x \), we now aim to cast \( \bar{H}_1 \) and \( \bar{H}_2 \) in diagonal form so that the concurrence for larger spin chains can be worked out easily. A commonly used tool in such a calculation is the Jordan-Wigner transformation which allows one to transform the problem of interacting spins to a problem of spinless fermions. Unfortunately, the presence of the \( \sigma_x^{(j)} \sigma_x^{(j+1)} \) interactions in \( \bar{H}_1 \) and \( \bar{H}_2 \) means that the spinless fermions are interacting. However, if the original spin chain is such that \( 2\lambda_1 + \lambda_2 + \lambda_3 = 0 \), then the Jordan-Wigner transformation allows us to transform both \( \bar{H}_1 \) and \( \bar{H}_2 \) to non-interacting fermions, thereby making the problem tractable and greatly reducing the computational complexity. We now present the details.

With the condition \( \lambda_3 = -2\lambda_1 - \lambda_2 \) to suppress the interaction terms \( \sigma_x^{(i)} \sigma_x^{(i+1)} \), we find that the effective Hamiltonians are

\[ \bar{H}_1 = -\lambda_1 \sum_{j=1}^{N-1} \sigma_y^{(j)} \sigma_y^{(j+1)}, \]

while

\[ \bar{H}_2 = \sum_{j=1}^{N-1} \left\{ \frac{(\lambda_1 + \lambda_2)}{2} \left[ \sigma_x^{(j)} \sigma_y^{(j+1)} + \sigma_y^{(j)} \sigma_x^{(j+1)} \right] - \lambda_1 \left[ \sigma_y^{(j)} \sigma_y^{(j+1)} \right] \right\}. \]

We now perform a Jordan-Wigner transformation. First, we introduce the raising and lowering operators \( a_i^\dagger = \frac{1}{2} \sigma_x^{(i)} + i \sigma_y^{(i)} \) and \( a_i = \frac{1}{2} \sigma_x^{(i)} - i \sigma_y^{(i)} \). Thereafter, the
fermionic operators $c_1$ and $c_1^\dagger$ are defined as
\[ c_i = \exp \left[ \pi i \sum_{j=1}^{i-1} a_j^\dagger a_j \right] a_i, \]
\[ c_i^\dagger = a_i^\dagger \exp \left[ -\pi i \sum_{j=1}^{i-1} a_j a_j^\dagger \right]. \]

Using the Jordan-Wigner transformation for $H_2$, we get
\[ \bar{H}_2 = \lambda_1 \sum_{i,j} \left[ c_i^\dagger J_{ij} c_j + \frac{1}{2} (e^{-i\phi} c_i^\dagger K_{ij} c_j + \text{h.c.}) \right], \tag{11} \]
where
\[ J = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \]
\[ K = \gamma \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \]
and
\[ \gamma = \frac{\sqrt{\lambda_1^2 + (\lambda_1 + \lambda_2)^2}}{\lambda_1}, \]
\[ \phi = \arctan \left[ (\lambda_1 + \lambda_2)/\lambda_1 \right]. \]
The form of $\bar{H}_1$ after the Jordan-Wigner transformation found to be the same as that in Eq. (11), except that now $\phi = 0$ and $\gamma = 1$.

Following Ref. 57], we look for a linear transformation of the form
\[ \eta_k = \sum_i [g_{ki} c_i + h_{ki} c_i^\dagger], \tag{12} \]
\[ \eta_k^\dagger = \sum_i [g_{ki} c_i^\dagger + h_{ki}^* c_i], \tag{13} \]
such that $\bar{H}_2$ becomes
\[ \bar{H}_2 = \sum_k \Lambda_k \eta_k^\dagger \eta_k + \text{constant} \tag{14} \]
The constant term is inconsequential. Our central task in finding the dynamics is to find the eigenvalues $\Lambda_k$. To do this, we note that if Eq. (14) is true, then
\[ [\eta_k, H] - \Lambda_k \eta_k = 0 \tag{15} \]
Substituting Eq. (12) in Eq. (15), we get
\[ \sum_m [g_{km} J_{mj} - e^{i\phi} h_{km} K_{mj}] = \frac{\Lambda_k}{\lambda_1} g_{kj}, \tag{16} \]
\[ \sum_m [g_{km} K_{mj} e^{-i\phi} - h_{km} J_{mj}] = \frac{\Lambda_k}{\lambda_1} h_{kj}. \tag{17} \]
These are further simplified by introducing the linear combinations
\[ \Phi_{ki} = g_{ki} + e^{i\phi} h_{ki}, \]
\[ \Psi_{ki} = g_{ki} - e^{i\phi} h_{ki}. \]
Eqs. (16) and (17) can be cast in the form of matrix equations as
\[ \Phi_k (J - K) = \frac{\Lambda_k}{\lambda_1} \Psi_k, \tag{18} \]
\[ \Psi_k (J + K) = \frac{\Lambda_k}{\lambda_1} \Phi_k, \tag{19} \]
in where $\Psi_k$ and $\Phi_k$ denote the $k$th row of matrices $\Phi$ (whose matrix elements are given by $\Phi_{ki}$) and $\Psi$ (whose matrix elements are given by $\Psi_{ki}$) respectively. Eliminating $\Psi_k$, we get
\[ \Phi_k (J - K)(J + K) = \left( \frac{\Lambda_k}{\lambda_1} \right)^2 \Phi_k. \tag{20} \]
We then view Eq. (20) as an eigenvalue problem to solve for $\Lambda_k$. However, for $H_1$, it turns out that $\Lambda_k$ can be zero, therefore $\Phi_k$ and $\Psi_k$ are solved using Eqs. (16) and (17) as a null space problem.

We now aim to find the concurrence for any two spins in our spin chain. Since concurrence is a two-particle measure, we need to work with a reduced $4 \times 4$ density matrix which is obtained by tracing out the state of the spin chain over all the other spins. Since the Pauli matrices form a basis, we can write the two-spin state as
\[ \rho^{(ij)}(t) = \text{Tr}_{ij} [\rho_{\text{tot}}(t)] = \frac{1}{4} \sum_{\alpha\beta} \Theta_{\alpha\beta}(t) \sigma^{(i)}_{\alpha} \otimes \sigma^{(j)}_{\beta}, \]
where we have introduced the correlation functions
\[ \Theta_{\alpha\beta} = \langle \sigma^{(i)}_{\alpha} \sigma^{(j)}_{\beta} \rangle = \text{Tr}_{ij} [\rho^{(ij)}(t) \sigma^{(i)}_{\alpha} \sigma^{(j)}_{\beta}], \]
and $\rho_{\text{tot}}(t)$ is the density matrix for the complete spin chain. Once we can figure out these correlation functions, we have the
relevant two-spin density matrix, and thereby the concurrence. As an example of finding the correlation functions, we explain how to find \( \sigma_x^{(1)} \sigma_x^{(N)} \). First, we write this in terms of the raising and lowering operators as

\[
\langle \sigma_x^{(1)} \sigma_x^{(N)} \rangle = \langle a_1^\dagger a_N + a_1 a_N + a_1^\dagger a_N + a_1 a_N \rangle.
\]

To evaluate each of these terms, we take as an example

\[
\langle a_1^\dagger a_N \rangle = \langle c_1^\dagger L_{1,N}c_N \rangle,
\]

where

\[
L_{l,m} = e^{i \pi \sum_{i=m}^l c_i^\dagger c_i}.
\]

Further simplification leads to

\[
\langle a_1^\dagger a_N \rangle = \langle c_1^\dagger \langle c_N^\dagger + c_N \rangle \langle c_N^\dagger - c_N \rangle L_{1,N} \rangle = \text{Tr}[\rho(t) c_1^\dagger \langle c_N^\dagger + c_N \rangle \langle c_N^\dagger - c_N \rangle L_{1,N}].
\]

Now, it can be shown that (see Appendix B)

\[
[L_{1,N}, H_2] = 0,
\]

allowing us to write, using the anti-commutation relations,

\[
\langle a_1^\dagger a_N \rangle = \langle c_1^\dagger(t) c_N^\dagger(t) L_{1,N} \rangle,
\]

where the time-dependence is carried by the operators now, that is,

\[
c_i(t) = e^{i H_2 t} c_i e^{-i H_2 t},
\]

\[
c_i^\dagger(t) = e^{i H_2 t} c_i^\dagger e^{-i H_2 t}.
\]

To find these these time-evolving operators, the strategy is to first transform the operators \( c_i \) and \( c_i^\dagger \) to the operators \( \eta_k \) and \( \eta_k^* \). These operators can be easily evolved as the Hamiltonian is diagonal in terms of these operators. Then, we transform back to the operators \( c_i \) and \( c_i^\dagger \). The result is that we can write

\[
c_i(t) = \sum_l [A_{lt}(t)c_l + B_{lt}(t)c_l^\dagger] \]

and

\[
c_i^\dagger(t) = \sum_l [A_l^*(t)c_l^\dagger + B_l^*(t)c_l],
\]

where the matrices \( A \) and \( B \) are defined as

\[
A = g'(t)g + h'(t)h^*,
\]

\[
B = g'(t)h + h'(t)g^*.
\]

Here \( g'(t) = g'e^{-itF} \) and \( h'(t) = h'e^{itF} \) where

\[
F = \lambda_1 \begin{pmatrix} A_1 & \cdots & 0 \\ 0 & A_2 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & A_{N-1} \\ 0 & 0 & \cdots & A_N \end{pmatrix}.
\]

The matrices \( g \) and \( h \) are the transformation matrices given in Eqs. (12) and (13), while \( g' \) and \( h' \) are the inverse transformation matrices, that is,

\[
c_i = \sum_j [g'_{ij} \eta_j + h'_{ij} \eta_j^*],
\]

\[
c_i^\dagger = \sum_j [g'_{ij}^* \eta_j^\dagger + h'_{ij}^* \eta_j].
\]

With the matrices \( A \) and \( B \) in hand, we find that

\[
\langle \sigma_x^{(1)} \sigma_x^{(N)} \rangle = \sum_l [A_{lt}^*(t)B^*(t)_{NI} + B_l(t)A^*(t)_{NI} - A_l^* A_{NI}(t) - B_l(t)A(t)_{NI}].
\]

This recipe can be repeated for the the other correlation coefficients. Once we have these, we have the two-spin density matrix, and thus the concurrence.

**FIG. 9.** Dynamics of entanglement between spins 1 and 4 for the XYZ model with \( N = 4 \). We have used \( \lambda_1 = 0.5, \lambda_2 = 1, \lambda_3 = -2 \). The dot-dashed magenta curve shows the entanglement with \( H_2 \), while the dashed black curve is for \( H_1 \). The black curve overlaps with the horizontal axis.

Using the approach presented above, we have checked that for small spin chains, the results produced are the same as those obtained numerically. For example, the results shown in Fig. 9 were reproduced using the formalism presented above. Interestingly, in this case, we can achieve almost perfect entanglement between the first and last spins with \( H_2 \), while \( H_1 \) generates no entanglement at all. We then used this formalism to obtain
FIG. 10. Dynamics of the entanglement between spins 1 and 12 for the XYZ model with $N = 12$. We have used $\lambda_1 = 0.25$, $\lambda_2 = 0.5$, and $\lambda_3 = -1$. The magenta curve shows the concurrence with the time-averaged Hamiltonian $\bar{H}_2$, while the dashed black curve shows the dynamics with $\bar{H}_1$. The black curve again overlaps with the horizontal axis.

FIG. 11. Same as Fig. 10 except that we are now looking at the concurrence between spins 6 and 7.

VI. CONCLUSION

In summary, we have shown that applying suitable control fields to a spin chain can, at least to a large extent, eliminate the interaction between the spin chain and its environment. Moreover, we have also shown that the application of the control fields modulates the spin-spin interaction in ways that can effectively generate spin-spin interactions that are absent in the original spin chain Hamiltonian. As an example of the constructive use of this modification, we have shown how, starting from the quantum Ising chain, perfect quantum state transfer can be achieved provided that control fields of sufficient strength and frequency are applied. We have also presented numerical simulations that show that two-spin entanglement generation in the spin chain can be improved by using particular control fields. Moreover, we have also diagonalized the effective spin chain Hamiltonian, at least for special coupling strengths, to show how the effect of the control fields can be analyzed for larger spin chains. Due to the great theoretical interest in spin chains, as well as their experimental realizations, our results should be interesting not only from the perspective of effectively isolating spin chains from their environment, but also for engineering spin-spin interactions via simple static and oscillating control fields.

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Appendix A: Eliminating the spin chain-environment interaction and transformation of the spin chain Hamiltonian

Following Refs. [34, 36], let us write the Hamiltonian for the spin chain in the presence of the control fields as

$$H_{\text{tot}} = H_0 + H_c(t) + H_B + H_{SB} = H' + H_c(t).$$

Here $H_c(t)$ is the control field Hamiltonian (acting on the spin chain), $H_B$ is the Hamiltonian of the environment, and $H_{SB}$ is the interaction between the spin chain and its environment. It is interesting to note that the environment of a quantum system itself has been modeled as a spin chain (see, for instance, Refs. [59–61] and references therein). For future convenience, let us define

$$H' = H_0 + H_B + H_{SB}. \tag{A1}$$

Our goal is to see how a state evolves under the action of the total Hamiltonian. To this end, let us first rotate the basis by $U_c(t)$. In this frame, the unitary time-evolution

the concurrence for larger spin chains. As an example, Fig. 11 we have shown the dynamics of the entanglement between spins 1 and 12 of a spin chain with $N = 12$ with both $H_1$ and $H_2$. It is clear that the entanglement generated if the special condition ($n_y = 2n_x$) is used is considerable, while no entanglement is generated when $n_y \neq 2n_x$. It should also be kept in mind that we are now dealing with a larger spin chain; as the number of spins is increased, we should consider a rescaled concurrence that takes into account the number of spins. For example, in Ref. [58], the rescaled concurrence has been defined as $C_R = (N - 1)C$. It is then clear that the entanglement generated by the spin chain dynamics is very significant if control fields are used. Moreover, if we look instead at spins 6 and 7 of the spin chain, we again observe that $H_2$ is better at generating entanglement as compared to $H_1$. 

$\text{VI. CONCLUSION}$

In summary, we have shown that applying suitable control fields to a spin chain can, at least to a large extent, eliminate the interaction between the spin chain and its environment. Moreover, we have also shown that the application of the control fields modulates the spin-spin interaction in ways that can effectively generate spin-spin interactions that are absent in the original spin chain Hamiltonian. As an example of the constructive use of this modification, we have shown how, starting from the quantum Ising chain, perfect quantum state transfer can be achieved provided that control fields of sufficient strength and frequency are applied. We have also presented numerical simulations that show that two-spin entanglement generation in the spin chain can be improved by using particular control fields. Moreover, we have also diagonalized the effective spin chain Hamiltonian, at least for special coupling strengths, to show how the effect of the control fields can be analyzed for larger spin chains. Due to the great theoretical interest in spin chains, as well as their experimental realizations, our results should be interesting not only from the perspective of effectively isolating spin chains from their environment, but also for engineering spin-spin interactions via simple static and oscillating control fields.

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$$H' = H_0 + H_B + H_{SB}. \tag{A1}$$

Our goal is to see how a state evolves under the action of the total Hamiltonian. To this end, let us first rotate the basis by $U_c(t)$. In this frame, the unitary time-evolution
operator for the spin chain and the environment as a whole is

$$
\hat{U}_{\text{tot}}(t) = \mathcal{T}\exp \left[ -i \int_0^t \hat{H}'(s) \, ds \right],
$$

where $\hat{H}' = \hat{U}_c^\dagger(s) \hat{H}' \hat{U}_c(s)$. At time $t = N t_c$ ($N$ is a positive integer), due to the periodicity of the control fields,

$$
\hat{U}_{\text{tot}}(t) = \left[ \hat{U}_{\text{tot}}(t_c) \right]^N,
$$

and

$$
\hat{U}_{\text{tot}}(t_c) = \mathcal{T}\exp \left[ -i \int_0^{t_c} \hat{H}'(s) \, ds \right].
$$

Now comes the key step. We use the Magnus expansion to write

$$
\hat{U}_{\text{tot}}(t) \approx \exp[ -i t_c (\hat{H}^{(0)} + \hat{H}^{(1)} + \cdots)].
$$

Imposing the condition

$$
\int_0^{t_c} \hat{U}_c(t) H_{SB} \hat{U}_c(t) \, dt = 0,
$$

we find that

$$
\hat{U}_{\text{tot}}(t_c) \approx e^{-i t_c \hat{H}^{(0)}} e^{i t_c \hat{H} + \cdots} = e^{-it\hat{H}} e^{-it\hat{H}_B}.
$$

Transforming it back to the original frame, we find that the unitary evolution operator is

$$
\hat{U}_{\text{tot}}(t) \approx \hat{U}_c(t) e^{-it\hat{H}} e^{-it\hat{H}_B}. \quad (A2)
$$

Thus, the spin chain and its environment have been effectively decoupled, at least to lowest order. Moreover, we have found how the spin chain Hamiltonian effectively changes in the presence of the control fields. Since $U_c(0) = 1$, and the fields are rapidly oscillating, we can simply replace $H_0$ by $\hat{H}$.

**Appendix B: Proof of $[L_{1,N}, \hat{H}_2]$**

We begin by writing $L_{1,N} = e^Q$, where $Q = i\pi \sum_{m=1}^N c_m^\dagger c_m$. We also write

$$
\hat{H}_2 = \lambda_1 (H^{(1)} + H^{(2)}),
$$

with

$$
H^{(1)} = \sum_{ij} J_{ij} c_i^\dagger c_j,
$$

and

$$
H^{(2)} = \frac{1}{2} \sum_{ij} \left( e^{-i\phi} K_{ij} c_i^\dagger c_j + \text{h.c.} \right).
$$

Now, it is easy to see that $[H^{(1)}, e^Q] = 0$, which immediately implies $[H^{(1)}, e^Q] = 0$. Moving on, we find that

$$
[Q, H^{(2)}] = i\pi \sum_{ij} \left( e^{-i\phi} K_{ij} c_i^\dagger c_j - e^{i\phi} K_{ij} c_j c_i \right),
$$

and

$$
[Q, [Q, H^{(2)}]] = (2i\pi)^2 H^{(2)}.
$$

Using the Baker-Campbell-Hausdorff lemma,

$$
e^Q H^{(2)} e^{-Q} = \hat{H}_2 + [Q, H^{(2)}] + \frac{1}{2!} [Q, [Q, H^{(2)}]] + \ldots,
$$

we find that

$$
e^Q H^{(2)} e^{-Q} = \cos(2\pi) H^{(2)} + [Q, H^{(2)}] \frac{\sin(2\pi)}{2\pi} = H^{(2)}.
$$

Therefore, we have shown that $[L_{1,N}, H^{(2)}] = 0$. Since $[L_{1,N}, H^{(1)}] = 0$ as well, $[L_{1,N}, \hat{H}_2] = 0$.

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[1] P. Pfeuty, Ann. Phys., NY 57, 79 (1970).
[2] A. C. M. Carollo and J. K. Pachos, Phys. Rev. Lett. 95, 157203 (2005).
[3] A. Gubin and L. F. Santos, Am. J. Phys. 80, 246 (2012).
[4] T. Vuletić, B. Korin-Hamzić, T. Ivec, S. Tomić, B. Gornushinov, M. Dressel, and J. Akimitsu, Phys. Rep. 428, 169 (2006).
[5] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[6] R. Islam, E. E. Edwards, K. Kim, S. Korenblit, C. Noh, H. Carmichael, G. D. Lin, L. M. Duan, C. C. Joseph Wang, J. K. Freericks, and C. Monroe, Nat. Commun. 2, 377 (2011).
[7] J. K. Pachos and P. L. Knight, Phys. Rev. Lett. 91, 107902 (2003).
[8] J. Majer, J. M. Chow, J. M. Gambetta, J. Koch, B. R. Johnson, J. A. Schreier, L. Frunzio, D. I. Schuster, A. A. Houck, A. Wallraff, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Nature 449, 443 (2007).
[9] M. Graefe, R. Heilmann, A. Perez-Leija, R. Keil, F. Dreisow, M. Heinrich, H. Moya-Cessa, S. Nolte, D. N.
