A NEW VISTA TO THE $SU(N)$ GAUGE FIELD CONFINEMENT

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Abstract

Based on our exact effective spin model, the zero temperature $SU(N)$ gauge field confinement has been studied by the Wilson line, rather than the quark antiquark potential. The procedure may allow the confinement be studied in the true continuum limit. For the $(1+1)$ $SU(2)$ gauge field, we have shown that the confinement (here we mean that the Wilson line vanishes.) corresponds to a linear zero temperature Polyakov quark antiquark potential. Also, for the first time, the Polyakov and Wilson potentials are shown to be identical.

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A substantial amount of numerical evidences have been accumulated in the past decade which indicate that the zero temperature (3 + 1) dimensional SU(2) and SU(3) gauge fields could indeed be permanently confined [1]. Still, one may argue that such a proof is not quite within the realm of the true continuum world [2], and therefore a direct study of the confinement in the continuum world is still highly desirable. In this paper, we shall introduce a new approach which appears to have this feasibility.

Let us first consider the confinement of the finite temperature SU(N) gauge field. It was speculated as early as the late seventies that in this case there is a transition from the color confined to the color deconfined phase with the Wilson line as its order parameter. Although this suggests that it is natural to use the Wilson line to study the zero temperature SU(N) confinement by checking whether it is zero or not, it was not carried out as far as we are aware of [3]. Recently, the exact form of the effective spin model of the SU(N) gauge field was derived [4]. For the zero temperature lattice gauge theory (LGT), we found that one can in principle compute the coupling constant of the effective spin model because the effective action for the space-like link can be obtained exactly. Hence by computing the coupling constant of the effective spin model in the continuum limit, one can determine whether or not the gauge field confines since the 3 dimensional effective spin model can be easily carried out numerically. This thus opens up a possibility to study the zero temperature confinement by the Wilson line. In this paper, we will show how to use our effective spin model to study the zero temperature confinement in general and will implement such an approach for the (1 + 1) SU(2) gauge field.

We shall first briefly introduce the effective spin model for the general d + 1 dimensional finite temperature SU(N) LGT. The finite temperature behavior of the LGT can be studied by the following partition function on a hypercubic lattice of size $N_d^s \times N_r$,

$$Q = \sum e^S$$  \hspace{1cm} (1)

where $S$ is the Wilson action

$$S = \beta E \sum_n \sum_{\mu < \nu} \frac{1}{N} Re Tr(U_n^\mu U_{n+\mu} U_{n+\nu} U_n^{\nu \dagger})$$  \hspace{1cm} (2)

In eq.(2), $\beta E$ is the coupling constant, $n$, $\mu$ and $\nu$ the space-time coordinate and directions respectively and $U$ the gauge field link variables. Finite tem-
perature can be introduced to the system by imposing a periodic boundary condition with $N_\tau$ period in the time direction. The temperature $T$ is then $\frac{1}{N_\tau a}$, where $a$ is the lattice spacing. Naturally the zero temperature LGT can be obtained by simply letting $T \to 0$.

To manifest the Wilson line variable in the action, we will introduce a thermal gauge fixing for the gauge field.

$$U_{n,n_\tau} = 1 \quad (1 \leq n_\tau \leq N_\tau - 1),$$  \hspace{1cm} (3)

and the link $U_{n,N_\tau}$ remains unchanged. Noticed that we have relabelled the space-time coordinate $n$ as $\mathbf{n}$ the space vector and $n_\tau$ the time coordinate.

With the thermal gauge, the trace of $U_{n,n_\tau}$ becomes a Wilson line (labelled here as $W_n$) and is manifested in the action $S$. We shall rewrite the action $S$ as a sum of $S^g$ and $S^\tau$.

$$S = S^g + S^\tau$$  \hspace{1cm} (4)

where

$$S^g = \beta_E \sum_{\mathbf{n},i<j} \sum_{n_\tau=1}^{N_\tau} \frac{1}{N} \text{Re} Tr(U_{\mathbf{n},n_\tau}^{i,j} U_{\mathbf{n}+i,n_\tau}^{j,i} U_{\mathbf{n}+j,n_\tau}^{i,j} U_{\mathbf{n},n_\tau}^{j,i})$$

$$+ \beta_E \sum_{\mathbf{n},i} \sum_{n_\tau=1}^{N_\tau-1} \frac{1}{N} \text{Re} Tr(U_{\mathbf{n},n_\tau}^{i} U_{\mathbf{n},n_\tau+1}^{i})$$  \hspace{1cm} (5)

and

$$S^\tau = \beta_E \sum_{\mathbf{n},i} \frac{1}{N} \text{Re} Tr(U_{\mathbf{n},n_\tau}^{i} W_{\mathbf{n}+i}^{i} W_{\mathbf{n}}^{i})$$  \hspace{1cm} (6)

Using eqs. (3) and (6), we can construct the effective action in which the field variables are the space-like link and the Wilson line.

To derive the effective spin model, we will decouple the action into two parts: one depends on the Wilson line and the other the space-like field:

$$S_{\text{eff}} = S_{\text{eff}}^W(\{W_n\}) + S_{\text{eff}}^U(\{U^{i}_{\mathbf{n},n_\tau}\}),$$  \hspace{1cm} (7)

In eq. (7), $S_{\text{eff}}^W$ and $S_{\text{eff}}^U$ are

$$S_{\text{eff}}^W = \ln \left( \frac{\sum_{\mathbf{n}} \exp(S^g + S^\tau)}{\sum_{\mathbf{n}} \exp(S_{\text{eff}}^U)} \right)$$  \hspace{1cm} (8)

$$S_{\text{eff}}^U = \ln \left( \frac{\sum_{W} \exp(S^g + S^\tau)}{\sum_{W} \exp(S_{\text{eff}}^W)} \right)$$  \hspace{1cm} (9)
In this way, we can keep the partition function of the action $S$ the same as $S_{\text{eff}}$. Clearly, $S_{\text{eff}}^{U}$ and $S_{\text{eff}}^{W}$ will describe exactly the behavior of $U$ and $W$ respectively.

Of course, $S_{\text{eff}}$ should preserve the symmetry of $S$ given by eqs. (3) and (6). Since there is a local gauge invariance for $W$ in $S$ (eqs. (5) and (6)),

$$W_{n} \rightarrow g_{n}W_{n}g_{n}^{-1}, \quad U_{n,n_{r}}^{i} \rightarrow g_{n,n_{r}}U_{n,n_{r}}^{i}g_{n+i,n_{r}}^{-1}.$$  

where $g_{n}$ and $g_{n,n_{r}}$ are the $SU(N)$ matrices, $S_{\text{eff}}^{W}$ must depend only on $Tr(W^{m})$ (where $m$ is an integer) and its conjugate (from now on written as a function of $Tr(W^{m})$).

$$S_{\text{eff}}^{W}({\{W_{n}\}}) \equiv S_{\text{eff}}^{W}({\{\frac{1}{N}Tr(W_{n}^{m})\}})$$  

We then use the variational principle to derive the exact form of $S_{\text{eff}}^{W}$ in Ref.[4]

$$S_{\text{eff}}^{W} = \alpha_{E} \sum_{n,i} Re(\frac{1}{N}TrW_{n+i}\frac{1}{N}TrW_{n}^{+}),$$  

where

$$\alpha = Re((\frac{1}{N}Tr(U_{n,n_{r}}^{i}U_{n,n_{r}}^{d})_{U}).$$

The symbol $\langle \cdots \rangle_{U}$ represents an average in $S_{\text{eff}}^{U}$. To facilitate subsequent discussions, we shall redefine

$$\beta \equiv \alpha \beta_{E}.$$  

To study $S_{\text{eff}}^{W}$ quantitatively, $\beta$ must be computed. This means that we need the exact form of $S_{\text{eff}}^{U}$. Physically, this also means that we should find $S_{\text{eff}}^{U}$ which can describe the $U$ link field behavior exactly. To this end, we found that $S^{g}$ will describe the exact zero temperature $U$ field behavior, and thus the important result that $S_{\text{eff}}^{U}$ is simply $S^{g}$.

$$S_{\text{eff}}^{U}(T = 0) = S^{g}$$

Once $S_{\text{eff}}^{U}$ is known, $\beta$ can be computed using eq. (12), and we can now study the effective spin model.
With the effective spin model in eq. (11), the zero temperature $SU(N)$ confinement can now be studied. We shall take two steps to carry out such a task. First, we shall calculate the nearest-neighbor coupling $\beta$ using eqs. (12) and (13). Second, the $d$ dimensional effective spin model of eq. (11) will be examined numerically or analytically in order to obtain the deconfinement-confinement phase diagram with respect to the coupling constant $\beta$. To see the phase diagram clearly, let us define the effective spin model critical coupling constant of the deconfinement-confinement phase transition as $\beta_c$. We then plot the phase diagram in Fig. 1. The comparison of the coupling constant $\beta$ in the continuum limit with $\beta_c$ in Fig. 1 can be used as a criteria to determine whether the system confines or not.

Obviously, computing the continuum limit $\beta$ is the heart of the matter. If it is possible, then we can determine whether the gauge field is confined or not by ”simply” placing $\beta$ in Fig. 1. For situations in which $\beta$ cannot be computed exactly, two facts may still render the study tractable. First it is clear that $\beta$ is a local space parameter for the effective spin model (see eqs. (11), (12) and (13)). Therefore, it may be obtained perturbatively. Second, from Fig. 1, we see that if the calculated upper limit of $\beta$ in the continuum limit is less than $\beta_c$, then the non-Abelian gauge field confinement is proved. These two points imply the feasibility of this approach.

We will now implement our idea on the $(1+1)$ dimensional $SU(2)$ gauge theory. First we shall calculate $\beta$ by studying the $S_{eff}^U$. In the fundamental representation, the parametrized link variable $U$ is

\[ U = S^0 + i\vec{S} \cdot \vec{\sigma} \]  

where $S^0$ is a real number, $\vec{S}$ a real vector and $\vec{\sigma}$ a Pauli matrix. Since $U$ is unitary, $S^0$ and $\vec{S}$ must satisfy the following equation

\[ |S^0|^2 + |\vec{S}|^2 = 1. \]  

We see that $S_{eff}^U$ is a one dimensional vector model with four internal degrees of freedom, i.e.

\[ S_{eff}^U = \beta_E \sum_{i=1}^{N_x} S_i \cdot S_{i+1} \]  

where $S_i$ is a unit vector of four components, with $S_i^0$ and $\vec{S}_i$ as components. We find that this action can be exactly solved \[ 3 \] and from which one can
Figure 1: The $\langle W \rangle_{SW}$ axis is the Wilson line. $\beta$ axis is the nearest-neighbor coupling of the effective spin model. The thickline represents the $SU(N)$ confinement region and $\beta > \beta_c$ the deconfinement region.

obtain $\alpha$ as

$$\alpha = \left( \frac{I_2(\beta E)}{I_1(\beta E)} \right)^{N_\tau - 1}$$

(18)

where $I_1(\beta E)$ and $I_2(\beta E)$ are the modified Bessel functions, and thus the nearest-neighbor coupling $\beta$ of the effective spin model.

$$\beta = \beta_E \left( \frac{I_2(\beta E)}{I_1(\beta E)} \right)^{N_\tau - 1}$$

(19)

Since $I_2(\beta E) < I_1(\beta E)$ and $N_\tau$ approach infinity, $\beta$ must be an infinitesimally small number. For $\beta_E \to \infty$ which is the continuum limit, we still have $\beta$ as

$$\beta = \beta_E \left( 1 - \frac{15}{8\beta_E} + O\left( \frac{1}{\beta_E^2} \right) \right)^{N_\tau - 1}$$

(20)

From the above one sees that $\beta$ is still infinitesimally small. We therefore conclude that $\beta$ is infinitesimally small for all $\beta_E$.

Once $\beta$ is known, we can then study the effective spin model. To this end, let us parametrize $W$ as

$$W = P^0 + i \vec{P} \cdot \vec{\sigma}$$

(21)
where $P^0$ is real number and $\vec{P}$ a real vector. Then the 1 dimensional effective spin model is given by

$$S_{\text{eff}}^W = \beta \sum_i P_i^0 P_{i+1}^0$$

(22)

Obviously, for an infinitesimally small $\beta$, $\langle TrW \rangle_{S_{\text{eff}}^W}$ vanishes. This is of course not surprising because the $(1+1)$ dimensional $SU(2)$ gauge field is known to be permanently confined.

To show the consistency of our method with others, we shall use this model to calculate the Polyakov static quark potential which is defined by

$$V_{q\bar{q}}^P(R) = -\frac{1}{N_t} \ln \frac{\langle \frac{1}{2} Tr W R \frac{1}{2} Tr W_0^\dagger \rangle_{S_{\text{eff}}^W}}{\langle \frac{1}{2} Tr W_0 \frac{1}{2} Tr W_0^\dagger \rangle_{S_{\text{eff}}^W}}$$

(23)

In the above, $R$ is the lattice distance between a quark $q$ and an antiquark $\bar{q}$ and $\langle \cdots \rangle_{S_{\text{eff}}^W}$ represents an average in the action $S_{\text{eff}}^W$. The $SU(2)$ group measure is

$$[dW] = \frac{2}{\pi} \sqrt{1 - (P^0)^2} dP^0$$

(24)

Since $\beta$ is infinitesimally small, this one dimensional effective spin model can be solved by the strong coupling expansion. The results are

$$\langle \frac{1}{2} Tr W R \frac{1}{2} Tr W_0^\dagger \rangle_{S_{\text{eff}}^W} = \beta^R b^{R+1} (1 + O(\beta^2) + \cdots)$$

(25)

and

$$\langle \frac{1}{2} Tr W_0 \frac{1}{2} Tr W_0^\dagger \rangle_{S_{\text{eff}}^W} = b(1 + O(\beta^2) + \cdots)$$

(26)

where $b$ is

$$b = \int_{-1}^{1} dP^0 (P^0)^2 \frac{2}{\pi} \sqrt{1 - (P^0)^2}$$

(27)

From this one can readily show that the Polyakov static quark antiquark potential is a linear one

$$V_{q\bar{q}}^P(R) = R \ln \left( \frac{I_1(\beta_E)}{I_2(\beta_E)} \right)$$

(28)

Therefore we have shown that the confinement (here we mean that the Wilson line vanishes.) corresponds to the linear Polyakov quark antiquark potential.
Next we will also calculate the Wilson potential by studying the $R \times L$ Wilson loop in Fig. 2. Here $L$ and $R$ are the lattice distance in the time and space directions respectively. The quark antiquark potential in the action $S^g$ can be defined as

$$V_W^{q\bar{q}}(R) = - \lim_{L \to \infty} \frac{1}{L} \ln \langle W_{RL} \rangle_{S^g}$$  \hspace{1cm} (29)$$

Here $W_{RL}$ is the Wilson loop, $V_W^{q\bar{q}}(R)$ is referred to as the Wilson potential and $\langle \cdots \rangle_{S^g}$ represents an average in the action $S^g$. We can then simply obtain $W_{RL}$ in the $S^g$ as

$$\langle W_{RL} \rangle_{S^g} = \left( \frac{I_2(\beta_E)}{I_1(\beta_E)} \right)^{RL}_R,$$  \hspace{1cm} (30)$$

from which the Wilson potential

$$V_W^{q\bar{q}}(R) = R ln \left( \frac{I_1(\beta_E)}{I_2(\beta_E)} \right)$$  \hspace{1cm} (31)$$

is obtained. Clearly, the zero temperature Polyakov potential $V_{q\bar{q}}^P(R)$ is identical to the Wilson potential $V_{q\bar{q}}^W(R)$.

In summary, a new vista to study the zero temperature $SU(N)$ confinement has been introduced. We have solved the effective spin model of the $(1 + 1) SU(2)$ gauge field exactly, and from which its confinement and linear Polyakov quark antiquark potential are shown. Also for the first time the Polyakov potential is shown to be identical to the Wilson potential. For $(3+1) SU(3)$ gauge field, the study will certainly be more complicated. However,
since our analysis have shown that the effective spin model coupling constant is a local space parameter and thus can be computed perturbatively, it is hoped that this approach can lead to a resolution of the important question of the \((3 + 1) \, SU(3)\) confinement in the continuum limit.

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