A positivity property of the dimer entropy of graphs

P. Butera,\textsuperscript{1} P. Federbush,\textsuperscript{2} and M. Pernici\textsuperscript{3,}\textsuperscript{†}

\textsuperscript{1}Dipartimento di Fisica Universit\'a di Milano-Bicocca and
Istituto Nazionale di Fisica Nucleare
Sezione di Milano-Bicocca
3 Piazza della Scienza, 20126 Milano, Italy
\textsuperscript{2}Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1043, USA
\textsuperscript{3}Istituto Nazionale di Fisica Nucleare
Sezione di Milano
16 Via Celoria, 20133 Milano, Italy

(Dated: November 26, 2014)

The entropy of a monomer-dimer system on an infinite bipartite lattice can be written as a
mean-field part plus a series expansion in the dimer density. In a previous paper it has been
conjectured that all coefficients of this series are positive. Analogously on a connected regular
graph with $v$ vertices, the “entropy” of the graph $\ln N(i)/v$, where $N(i)$ is the number of ways of setting
down $i$ dimers on the graph, can be written as a part depending only on the number of the dimer
configurations over the completed graph plus a Newton series in the dimer density on the graph.
In this paper, we investigate for which connected regular graphs all the coefficients of the Newton
series are positive (for short, these graphs will be called positive). In the class of connected regular
bipartite graphs, up to $v = 20$, the only non positive graphs have vertices of degree 3. From $v = 14$
to $v = 30$, the frequency of the positivity violations in the 3-regular graphs decreases with increasing
$v$. In the case of connected 4-regular bipartite graphs, the first violations occur in two out of the
2806490 graphs with $v = 22$. We conjecture that for each degree $r$ the frequency of the violations,
in the class of the $r$–regular bipartite graphs, goes to zero as $v$ tends to infinity.

This graph-positivity property can be extended to non-regular or non-bipartite graphs.

We have examined a large number of rectangular grids of size $N_x \times N_y$ both with open and
periodic boundary conditions. We have observed positivity violations only for $\min(N_x, N_y) = 3$ or
4.

PACS numbers: 05.50.+q, 64.60.De, 75.10.Hk, 64.70.F-, 64.10.+h
Keywords: Dimer problem

I. INTRODUCTION

For periodic cubical $d$-dimensional lattices with even length sides, the number of dimer configurations covering a
fraction $p$ of the vertices (called “matchings” of the lattice graph in the language of graph theory) has long been studied.
As the number of vertices $v$ tends to infinity, the number of distinct dimer configurations grows\textsuperscript{1} as $\exp(v\lambda_d(p))$, where $\lambda_d(p)$ is called the dimer entropy and $p$ is the dimer density. In \cite{2}, the entropy has been written as the sum of
a mean field term and of a series expansion in the dimer density, computed through order 6; in Refs.\textsuperscript{3,4} this series
has been extended at least through the order 20. More precisely, for lattices of dimension $2 \leq d \leq 4$ the first 24
series coefficients were derived\textsuperscript{3,4}, while 20 coefficients have been obtained for all $d > 4$ (in $d = 5,6$ respectively
22, 21 coefficients have been computed). In the case of $d = 1$, all coefficients are positive and have long been known.
It is striking that, for any integer $d$, all the series coefficients that we could compute, are positive. This led us to
conjecture\textsuperscript{4} that: \textit{for all (hyper)-cubical lattices the series coefficients are all positive}. Moreover, after examining the
series coefficients for various non-square bipartite lattices, we were led to extend the conjecture to all bipartite lattices
(i.e. to the lattices in which the sites can be divided into even and odd, such that no edge connects two vertices with
the same parity), while the positivity does not hold in the case of non-bipartite lattices.

We observe that, in analogy with the case of infinite bipartite lattices, on a connected simple undirected regular
bipartite graph $G$ with set of vertices $V$ and set of edges $E$, the “graph dimer entropy” $\ln N(i)/v$ - where $N(i)$ is the
number of ways of setting down $i$ dimers on the graph and $v = |V|$ is the number of vertices of the graph - can be
written as a part depending on the dimer configurations counting over the completion of the graph and a Newton
expansion in the dimer density on the graph. If all the coefficients of the Newton series are positive, we shall say that
the graph satisfies the \textit{graph-positivity} property. For not too large number of vertices $v$, we have generated all regular
(bi)connected bipartite graphs (RBB graphs from now on) using the \textit{Nauty} program\textsuperscript{5} via the \textit{Sage} interface\textsuperscript{6}. In
examining all RBB graphs with vertices of degree 3 for $v \leq 30$, we have observed no violations of the graph positivity

""
for $v < 14$ vertices. A single violation is observed for $v = 14$. For $v \geq 14$, as the number $v$ of vertices increases, the frequency of violations decreases. No positivity violation occurs in all RBB graphs with vertices of degree greater than 3, up to $v = 20$. For $v = 22$ and degree 4 there are two violations. For $r-$regular graphs with $r > 4$ the absence of violations could be checked up to $v = 20$. Although we found no positivity violation for RBB graphs with vertex degree larger than 4 in the large sample studied, we can point out examples of positivity violation occurring for larger values of $v$, in some $r-$regular graphs with $r = 5, 6, 7, 8$, which were defined in [7]. We conjecture that, for RBB graphs of any degree the frequency of positivity violations becomes vanishingly small as $v$ becomes large.

An interesting observation emerging from the systematic study of RBB graphs with $v \leq 30$ and degree 3, is that the average order of the automorphism group of the graphs not satisfying the graph-positivity property is larger by an order of magnitude than the average order of the automorphism group for all RBB graphs.

In addition to this systematic survey of RBB graphs, we have focused the attention on the class of square and hexagonal lattices with periodic boundary conditions (bc), which are also RBB graphs. In the case of rectangular grids of sizes $N_x \times N_y$ with periodic bc, we examined the cases $N_x = N_y$ up to sizes $N_y = 12$. For $N_x \geq N_y$ we found positivity violations only for $N_y = 4$ and $N_x \geq 432$.

For periodic hexagonal lattices of sizes $N_x \times N_y$ in the brick-wall representation, positivity is valid when $N_x \geq N_y$, apart from the case $N_x = N_y = 4$, while violations occur for $N_x < N_y$. We have tested this class of grids up to the size $14 \times 14$.

This positivity property can also be studied for graphs which are not RBB. Since we originally proposed the positivity conjecture for infinite square lattices, it is natural to consider also rectangular grids with open b. c. In the case of rectangular grids $N_x \times N_y$ with open bc, in the cases $N_x = N_y$ we considered up to the case $N_y = 19$; the graph positivity holds except for the case of $N_x \times 3$ graphs. We have also examined the case of rectangular grids $N_x \times N_y$ with bc periodic only in the $y$ direction. Again, we found positivity violations only for narrow grids, as in the case of rectangular grids with periodic and with open bc.

We considered also several cubic grids of sizes $N_x \times N_y \times N_z$ with open bc, up to the $5 \times 5 \times 4$ case: no positivity violation was observed, except in the $N_x \times 2 \times 2$ case, which is isomorphic to the $N_y = 4$ rectangular grid case periodic in the $y$ direction.

In the case of non-bipartite graphs, the positivity property is not common. However there are exceptions for specific classes of graphs: for example, we have singled out a sequence of “nanotubes” $C_{40+20N}$, with $N$ the number of hexagonal strips, and have tested graph positivity for $1 \leq N \leq 300$. All the nanotubes with $N > 7$ are graph-positive.

To perform our study of graphs, it was necessary to compute the graph matching generating polynomial

$$M(t) = \sum_{i=0}^{[v/2]} N(i)t^i$$  \hspace{2cm} (1)

where $N(i)$ is the number of configurations of $i$ dimers on the graph. We used the algorithm discussed in [8] to perform the computation.

The paper is organized as follows. In the second Section, we formulate the graph positivity property. The third Section summarizes the results of the graph tests for a variety of graphs and lattices. The fourth Section contains our conclusions. In Appendix A, the graph positivity is proven for polygons, for complete bipartite graphs and for an approximate average distribution of graphs. In Appendix B the positivity property is examined for a sequence of nanotubes. In Appendix C we give a few examples of RBB graphs violating positivity.

II. THE POSITIVITY PROPERTY FOR GRAPHS

To formulate the graph-positivity properly, let us turn now to a cubic lattice graph $G$ with $v$ vertices of degree $2d$ and periodic bc. Let $N(i)$ be the number of ways of setting down $i$ dimers on the edges of $G$ (with no overlap). If we consider the sequence of larger and larger periodic lattices used to compute the entropy $\lambda_d(p)$ on the infinite lattice, one must have

$$\hat{\lambda}_d(p_i) = \frac{1}{v} \ln N(i) \rightarrow \lim_{v \rightarrow \infty} \frac{1}{v} \ln(\exp v \lambda_d) = \lambda_d(p)$$  \hspace{2cm} (2)

where the arrow indicates convergence as $v \rightarrow \infty$. Here $i$ is related to $p$ by

$$p_i = \frac{2i}{v} \approx p$$  \hspace{2cm} (3)
where the integer $i$ is chosen to make this approximation best. We refer now to the notation in [2] observing that from Eq.(5.9) of this

$$\lambda_d(p) = \frac{p}{2} \ln(2d) + \lim_{v \to \infty} \frac{1}{v} \ln Z$$

(4)

Observe now that $Z$ can be factored into a “mean field” term $Z_0$ times the rest $Z^*$ as in Eq.(5.12) of [2]

$$\lambda_d(p) = \frac{p}{2} \ln(2d) + \lim_{v \to \infty} \frac{1}{v} \ln(Z_0) + \lim_{v \to \infty} \frac{1}{v} \ln(Z^*)$$.

(5)

From Eq.(2) and from Eq.(5.11) and (7.1) of [2], we have

$$\frac{1}{v} \ln \frac{N(i)}{(2d)^i} - \lim_{v \to \infty} \frac{1}{v} \ln(Z_0) \to \sum_{k=2} a_k(d)p^k$$

(6)

with

$$\lim_{v \to \infty} \frac{1}{v} \ln(Z_0) = -\frac{p}{2} \ln(p) - (1 - p) \ln(1 - p) - \frac{p}{2}.$$

(7)

It is an easy computation to show that for the graph $\bar{G}$, defined as the completion of $G$ (namely the non-bipartite graph constructed by connecting the vertices of $G$ in all possible ways), if we set $\bar{N}(i)$ as the number of ways of setting down $i$ dimers on $\bar{G}$ (without overlap),

$$\bar{N}(i) = \frac{v!}{(v - 2i)!i!2^i}$$

(8)

then

$$\frac{1}{v} \ln \frac{\bar{N}(i)}{(v-1)^i} \to \lim_{v \to \infty} \frac{1}{v} \ln(Z_0)$$

(9)

We now define

$$d(i) \equiv \ln \frac{N(i)}{(2d)^i} - \ln \frac{\bar{N}(i)}{(v-1)^i}$$

(10)

and see that

$$\frac{1}{v} d(i) \to \sum_{k=2} a_k(d)p^k$$

(11)

The rhs of Eq.(11) is the series part of the dimer entropy in Eq.(6) in [4], where it is conjectured that the $a_k$ are all positive. We now observe that as $v \to \infty$, the finite differences at $i = 0$ on the lhs of Eq. (11) become the derivatives on the rhs.

We write $\Delta$ for the forward finite difference with respect to $i$. One has

$$\frac{v}{2} \Delta \approx \frac{d}{dp}$$

(12)

so differentiating Eq.(11) once, one gets

$$\frac{1}{2}(d(i+1) - d(i)) \to \frac{d}{dp} \sum_{k=2} a_k(d)p^k.$$

(13)

Writing $d(i) = \hat{d}(\hat{p})$ with $\hat{p} = \frac{2i}{v}$ and $h = \frac{2}{v}$, one has the Newton expansion

$$\hat{d}(\hat{p}) = \sum_{k=0}^{v/2} \frac{\Delta_k^{(i)}(0)}{k!} (\hat{p})^k$$

(14)
where \((\hat{p})_k = \hat{p}(\hat{p} - h)...(\hat{p} - (k - 1)h) = (2^k)(i)_k\), \((i)_k\) is the falling factorial and \(\Delta_h\) is the finite difference with step \(h\), \(\Delta_h \hat{d}(\hat{p}) = \frac{\hat{d}(\hat{p} + h) - \hat{d}(\hat{p})}{h}\).

We can now state the definition of graph positivity in greater generality. Let \(G\) be a simple regular biconnected graph, with \(v\) vertices of degree \(r\). Let \(\bar{G}\) be the completion of \(G\). Let \(N(i)\) be the number of ways of laying down \(i\) dimers on \(G\) (without overlap) and \(\bar{N}(i)\) the same quantity for \(\bar{G}\). The equations introduced in this section for the hypercubic lattices generalize to regular biconnected graphs replacing \(2d\) with the degree \(r\).

Define

\[
\hat{d}(\hat{p}) = d(i) = \ln \frac{N(i)}{r^i} - \ln \frac{\bar{N}(i)}{(v - 1)^i},
\]

(15)

and

\[
\hat{\lambda}(\hat{p}_k) = R(i) = \frac{\hbar}{2} \sum_{k=0}^{v/2} \frac{\Delta_k \hat{d}(0)}{k!} \hat{p}_k
\]

(16)

where

\[
R(i) = \frac{1}{v} \ln \left( \frac{r^i}{(v - 1)^i} \bar{N}(i) \right)
\]

(17)

Using the Stirling formula for large \(v\)

\[
R(i) \approx \frac{1}{2} (p \ln(v) - p \ln(p) - 2(1 - p) \ln(1 - p) - p)
\]

(18)

Then \(G\) satisfies graph positivity, if all the \(d(i)\) are non-negative, as are all terms derived by any number of non-trivial finite differences i.e. if all terms in the sequence \(d(0), d(1), d(2), d(3),...\) are non-negative as are all terms in the sequence \(d(1) - d(0), d(2) - d(1), d(3) - d(2),...,\), and the sequence \(d(2) - 2d(1) + d(0), d(3) - 2d(2) + d(1), d(4) - 2d(3) + d(2),...\), etc., that is

\[
\Delta^k d(i) \geq 0
\]

(19)

for \(k = 0, ..., \nu\) and \(i = 0, ..., \nu - k\). In fact, it is sufficient that the first term \(\Delta^k d(0)\) in each of these sequences is non-negative, non-negativity of the other terms would then follow \((\Delta^k d(0) = 0\) for \(k = 0, 1\ since \(d(0) = d(1) = 0\)). Equivalently, positivity means that all the coefficients in the Newton series of \(\hat{d}(\hat{p})\) are non-negative.

In all the tests we made, when a graph is positive it has \(\Delta^k d(i)\) strictly positive apart from the trivial case \(k = 0, i = 0, 1\) and \(k = 1, i = 0\).

For infinite regular lattices, in the bipartite models \(a_k > 0\) while it is not true for non-bipartite ones, so we can expect that regular connected bipartite (hence “biconnected”) graphs (RBB graphs) tend to be positive, while non-bipartite graphs not.

Notice that using \(N(1) = \nu^2\) one can rewrite \(d(i)\) in Eq.\((15)\), with no explicit dependence on degree of the vertices

\[
d(i) = \ln \frac{N(i)}{N(1)^i} - \ln \frac{\bar{N}(i)}{N(1)^i}
\]

(20)

for \(i = 1, ..., \nu\), where \(\nu\) is the maximum value of \(i\) with \(N(i)\) not zero, and \(\bar{N}(i)\) is the number of configurations of \(i\) dimers on a complete graph with \(\bar{v} = 2\nu\) vertices (if \(\bar{v} = v\) the graph has perfect matchings). This suggests that it is worthwhile to investigate also when the graph-positivity holds for non-regular graphs.

Let us comment on the use of the graph completion in [2]. If one studies the definition of \(Z_0\) in [2], the expression on the lhs of [2] is seen to arise naturally. But we get no theoretical understanding of why the completion of a graph, rather than the bipartite completion, is taken. One could have in [2] used a similar expression with the bipartite completion, but one can show that there are more violations of the bounds Eq.\((19)\) for \(k \geq 2\) than with the definition used; in particular the RBB graphs with \(v \leq 18\) would all violate positivity, apart from the complete bipartite graphs, which have trivially \(d(i) = 0\).
III. TESTS OF GRAPH POSITIVITY

As a first step, we have tested systematically the validity of the positivity property on RBB graphs, exploring them in ascending number of vertices, namely we have counted the configurations of $i$ dimers on these graphs and checked the inequalities for the higher finite differences. To generate systematically the RBB graphs, we have used the \textit{geng} program in the \textit{Nauty} package\cite{1}, via the \textit{Sage} interface\cite{2}. We have used an ordinary desktop personal computer based on a processor Intel i7 860 and a RAM of 8 GB. The first result of this extensive graph survey is that, while most RBB graphs share the graph-positivity property, a few of them do not, see Table I for the graphs with degree 3.

There are no violations for $v < 14$. The positivity test for graphs with 30 vertices was completed in about 20 days. Most of the computing time was spent in enumerating graphs with degree greater than 4, we can exhibit cases of positivity-violating graphs with degree up to 8, belonging to the class $\text{Aut}(G)$ of all the RBB graphs with the same number of vertices. In particular, the positivity violating graph with $v = 14$ is the most symmetrical of all $v = 14$ RBB graphs with degree 3.

For degrees greater than 3, there is no violation of positivity for RBB graphs with number of vertices up to $v = 20$. For $v = 22$, there are 2806490 RBB graphs with vertices of degree 4, and two violations, given in Appendix C. The average order of the automorphism group of all the RBB graphs with the same number of vertices. In particular, the positivity violating graph with $v = 14$ is the most symmetrical of all $v = 14$ RBB graphs with degree 3.

For 3-regular graphs, the frequency of violations decreases with $v$, for $v \geq 14$.

It is interesting to observe that, in the cases considered in Table I, the average order of the automorphism group of the positivity-violating graphs is roughly an order of magnitude greater than the average order of the automorphism group for graphs violating the positivity property. The average order of the automorphism group of all the RBB graphs with number of vertices of degree 3 is 1080, the average order of the automorphism group for graphs violating the positivity property is roughly an order of magnitude greater than the average order of the automorphism group for graphs violating the positivity property.

For degrees greater than 3, there is no violation of positivity for RBB graphs with number of vertices up to $v = 20$. For $v = 22$, there are 2806490 RBB graphs with vertices of degree 4, and two violations, given in Appendix C. The average order of the automorphism group of all the RBB graphs with $v = 22$ and degree equal to 4 is 1.5.

Although in this systematic examination of graphs with low order we found no positivity-violating graphs with degree greater than 4, we can exhibit cases of positivity-violating graphs with degree up to 8, belonging to the class of $k$-regular graphs $G_{k,f}$ considered in Example 1 of \cite{3}. The $G_{k,f}$ graphs, with $f \geq 2$ and $k \geq 3$ are RBB graphs with degree $k$ and $v = 2n$ vertices where $n = (k^2f + 1)$. The vertex classes of $G_{k,f}$ are

$$U = \{s\} \cup \{u_{a,b,c} : 1 \leq a \leq k; 1 \leq b \leq k; 1 \leq c \leq f\}$$

and

$$V = \{t\} \cup \{v_{a,b,c} : 1 \leq a \leq k; 1 \leq b \leq k; 1 \leq c \leq f\}$$

The edges are $E = E_1 \cup E_2 \cup E_3$ with

$$E_1 = \{(u_{a,b,c}, v_{a,\beta,c}) : 1 \leq a, b, \beta \leq k; (b, \beta) \neq (1, 1); 1 \leq c \leq f\}$$

$$E_2 = \{(u_{a,1,1}, v_{a,1,c+1}) : 1 \leq a \leq k; 1 \leq c \leq f - 1\}$$

$$E_3 = \{(s, v_{a,1,1}) : 1 \leq a \leq k\} \cup \{(u_{a,1,f}, t) : 1 \leq a \leq k\}$$

and dimer countings for almost complete and complete matching

$$N(n - 1) = \frac{((k - 1)f)!kf}{(k - 2)^2}((k + 2)^2(k - 1)^{k^2f+2} + ((k - 2)^2kf - 3k^2 + 4)k^2(k - 1)^{(k-1)f} + 2k^3(k - 1)^{(k-2)f+1})$$

TABLE I: For RBB graphs with $14 \leq v \leq 30$ vertices of degree 3, we have listed the number of graphs in this class, the number of violations of the graph positivity, the average order $ng$ of the automorphism group of graphs, the average order $ngv$ of this group for graphs violating the positivity property.

| $v$ | number of graphs | violations | $ng$ | $ngv$ |
|-----|------------------|------------|------|-------|
| 14  | 13               | 1          | 44.5 | 1336  |
| 16  | 38               | 2          | 18.6 | 112   |
| 18  | 149              | 2          | 15.1 | 176   |
| 20  | 703              | 8          | 8.7  | 118   |
| 22  | 4132             | 17         | 4.5  | 72    |
| 24  | 29579            | 49         | 3.3  | 92    |
| 26  | 245627           | 115        | 2.3  | 66    |
| 28  | 2291589          | 514        | 1.9  | 55    |
| 30  | 23466857         | 3949       | 1.7  | 37    |
The graph $G_{5,2}$ with 102 vertices of degree 5 and the graph $G_{6,2}$ with 146 vertices of degree 6 show a first violation in the case of $\Delta^2 d(i)$ for some $i > 0$. In the finite differences at $i = 0$, a first violation occurs for $\Delta^3 d(0)$. The graphs $G_{7,2}$ and $G_{8,2}$ have $\Delta^2 d(0) < 0$. The symmetry groups of the graphs $G_{6,2}$ with $k = 5, \ldots, 8$ are larger than $10^{20}$.

While all the examples of violations $\Delta^2 d(i) < 0$ mentioned so far have $j \geq 2$, there are violations for $j = 1$. $\Delta d(i) \geq 0$ is equivalent to

$$
N(n) = k((k - 1)!^2((k - 1)(k - 1)f
(25)
$$

where $v = 2n$ and $r$ is the degree. This implies $N(n - 1)/N(n) \leq n(2n - 1)/r$. From Eqs. (24, 25) it follows that the graph $G_{3,6}$ with 110 vertices fails to satisfy this inequality.

The 0th order inequality $d(i) \geq 0$ gives

$$
N(i) \geq \frac{v! r^i}{(v - 2i)!2^i!(v - 1)^i}
(27)
$$

where $r$ is the degree of the vertices. This inequality should be compared to the “Lower Matching Conjecture” [9] recently proven in [10]. In fact Csikvari proves a stronger bound with the right side of (28) divided by $n^i(i/n)^{i(1 - i/n)(n - i)}$. We thank the referee for informing us that he can prove (27) is true provided either $r \leq 5$ or $v \geq 2r^2$.

Gurvits has proved the following slightly weaker lower bound, see Eq.(51) of [11].

$$
N(i) \geq \left(\frac{n}{i}\right)^2 \left(\frac{nr - i}{nr}\right)^{nr - i} \left(\frac{iv}{n}\right)^i.
(28)
$$

We do not know whether there are violations of the inequality Eq. (27).

Let us now consider a sequence of increasingly larger finite rectangular lattices with bc periodic in the $x$ and $y$ directions, as are used in the limiting procedure to extract $\lambda_d(p)$ for $d = 2$. For this class of graphs, we have found no violation of graph-positivity for grids of sizes $(N_x \times N_y)$, with $N_x$ even, such that

- $4 \leq N_x \leq 430$, $N_y = 4$
- $6 \leq N_x \leq 800$, $N_y = 6$
- $8 \leq N_x \leq 100$, $N_y = 8$

and for the grids of sizes $N_x \times N_y = 10 \times 10, 12 \times 10, 12 \times 12$.

However all graphs are not positive for $430 < N_x \leq 5000$, $N_y = 4$.

We have examined also the hexagonal lattices of size $N_x \times N_y$ in the brick-wall representation. We have examined the following cases with $N_x \leq N_y$, with $N_x$ even,

- $4 \leq N_x \leq 800$, $N_y = 4$
- $6 \leq N_x \leq 270$, $N_y = 6$
- $8 \leq N_x \leq 90$, $N_y = 8$
- $10 \leq N_x \leq 14$, $N_y = 10$

and $(N_x, N_y) = (12, 12), (14, 12), (14, 14)$. The only violation occurs in the case $(N_x, N_y) = (4, 4)$, which represents one of the two non positive 3-regular RBB graphs with 16 vertices.

For $N_x < N_y$, we have considered the following cases

- $N_x = 4, 6 \leq N_y \leq 100$, in which all the graphs are non positive.
- $N_x = 6, 8 \leq N_y \leq 100$, in which all the graphs $N_y \geq 26$ are non positive.

The lattice $4 \times 6$, a graph with 24 vertices, is one of the 49 cases of violations observed in the systematic survey of the 3-regular RBB graphs with 24 vertices.

Let us now consider what happens in the case of non-RBB graphs, as mentioned at the end of Section II. In the case of disconnected graphs, the matching generating polynomial is the product of the matching generating polynomials of the connected components. Many violations of the graph positivity property are observed. For instance, the graph formed by $n$ hexagons, with $n \leq 100$, violates positivity for $n \geq 3$. 

$\triangleleft$
For non-regular bipartite connected graphs there are many violations; for $v$ even up to 12 the frequency of violations increases with $v$; for $v = 12$ more than one third of the graphs violate positivity.

For non-regular bipartite biconnected graphs there are in general many violations; for instance enumerating the bipartite biconnected graphs with minimum vertex degree 2 and maximum degree 3, the frequency of violations increases with $v$ for $v = 10, \ldots, 18$.

These violations are due mostly to the vertices of degree 2. We examined a couple of classes of cases excluding vertices of degree 2: bipartite biconnected graphs with minimum degree 3 and maximum degree 4 have a similar pattern as RBB graphs, see Table II; the frequency of violations decreases with $v$ and, on the average, the non-positive graphs have significantly larger symmetries; bipartite biconnected graphs with minimum degree 4 and maximum degree 5 have no violations for $v < 16$; there is one violation out of 189853 graphs for $v = 16$, no violations for $v = 17$ out of 597235 graphs.

**TABLE II:** For connected bipartite graphs with $v$ vertices and with minimum degree 3 and maximum degree 4, we have listed the number of graphs, the number of violations of the graph positivity, the average order $ng$ of the automorphism group of the graphs, the average order $ngv$ of this group for graphs violating the positivity property.

| $v$ | number of graphs | violations | $ng$ | $ngv$ |
|-----|------------------|------------|------|-------|
| 12  | 240              | 1          | 14.1 | 192   |
| 14  | 5183             | 10         | 4.3  | 143   |
| 16  | 190378           | 134        | 2.1  | 41.6  |
| 18  | 9816658          | 6291       | 1.5  | 14.8  |

Let us now consider some non-regular bipartite lattice graphs. In the case of rectangular grids of size $N_x \times N_y$, we found few violations, occurring in the case of narrow grids.

In the case of rectangular grids with bc periodic only in the $y$ direction, we considered the cases in table III.

**TABLE III:** Sizes of the rectangular grids with bc periodic only in the $y$ direction, that have been tested. We have tested the positivity of the grids for all values of $N_x$ in the range indicated in the first row.

| $N_x$ | $N_y$ | 2-5600 | 2-2500 | 2-1500 | 2-1000 | 2-300 | 2-120 | 2-50 | 2-23 |
|-------|-------|--------|--------|--------|--------|-------|-------|------|------|
| 2    | 4     | 6      | 8      | 10     | 12     | 14    | 16    | 18   |

The only positivity violations occur for $N_x \geq 421, N_y = 4$ and $N_x = 3, N_y = 18$.

In the case of rectangular grids with open bc we have considered the cases listed in the table IV.

**TABLE IV:** Tested rectangular grids $N_x \times N_y$ with open bc.

| $N_x$ $\leq$ 5000, $N_y = 2$ | $N_x$ $\leq$ 500, $N_y = 11$ | $N_x$ $\leq$ 1800, $N_y = 3$ | $N_x$ $\leq$ 500, $N_y = 12$ | $N_x$ $\leq$ 3700, $N_y = 4$ | $N_x$ $\leq$ 180, $N_y = 13$ | $N_x$ $\leq$ 3800, $N_y = 5$ | $N_x$ $\leq$ 100, $N_y = 14$ | $N_x$ $\leq$ 2700, $N_y = 6$ | $N_x$ $\leq$ 70, $N_y = 15$ | $N_x$ $\leq$ 2700, $N_y = 7$ | $N_x$ $\leq$ 45, $N_y = 16$ | $N_x$ $\leq$ 1700, $N_y = 8$ | $N_x$ $\leq$ 50, $N_y = 17$ | $N_x$ $\leq$ 1400, $N_y = 9$ | $N_x$ $\leq$ 35, $N_y = 18$ | $N_x$ $\leq$ 700, $N_y = 10$ | $N_x$ $\leq$ 23, $N_y = 19$ |

We have found positivity violations only for $N_y = 3$.

Summarizing, in the case of rectangular grids with any bc, we have found no positivity violations for $N_y > 4$, $y$ being the shortest direction.

In the case of cubic grids of sizes $N_x \times N_y \times N_z$ with open bc, we have tested the positivity in the cases listed in the table V; there are no positivity violations except in the case $N_x \geq 421, N_y = 2, N_z = 2$ which is isomorphic to the rectangular grid $N_x \times 4$ periodic in the $y$-direction considered in table III.

As a final comment concerning biconnected non-bipartite graphs, it appears that the majority of these graphs do not have the graph-positivity property. We have checked that this is true for even $v, v \leq 8$. For odd $v, v \leq 9$ the frequency of violations increases rapidly and it is roughly $1/3$ for $v = 9$. This is consistent with the fact that infinite non-bipartite lattices have been found to be not graph-positive [4].
TABLE V: Sizes of the cubic grids that have been tested for positivity. We have tested the grids for all values of $N_x$ in the range indicated in the first row.

| $N_x$ | 2-5600 | 3-3500 | 4-2000 | 5-800 | 3-1600 | 4-500 | 5-150 | 4-85 | 5-13 |
|-------|--------|--------|--------|-------|--------|-------|-------|-------|-------|
| $N_y$ | 2      | 3      | 4      | 5     | 3      | 4     | 5     | 4     | 5     |
| $N_z$ | 2      | 2      | 2      | 2     | 3      | 3     | 4     | 4     | 4     |

It would be interesting to investigate whether graphs which are not bipartite due to a few defects have the tendency to be positive.

We have examined the positivity of a sequence of nanotubes, $C_{40+20N}$ in Appendix B for $1 \leq N \leq 300$. All these graphs are positive for $N > 7$. These graphs are formed by a nanotube composed of hexagons, and by two caps containing 6 pentagons each, and some more hexagons. Roughly speaking, the nanotube’s hexagon lattice tube tends to make the graph positive, the pentagons negative, so if the nanotube is sufficiently long, one can expect the graph to be positive. However it does not seem to be generally true that a graph formed by a subgraph which is positive and by a small non-bipartite part is positive. For instance consider a $(N_x, N_y)$ rectangular grid with open bc, with $N_y = 5$. We saw that these graphs are positive (checked up to $N_y = 3800$). Let us create defects in the third and fourth vertical strips, by adding $SW-NE$ diagonals to its squares. We have considered these graphs up to $N_x = 4000$: for $N_x \geq 10$, all these graphs violate positivity.

IV. CONCLUSIONS

In a previous paper we noticed that the dimer entropy of infinite bipartite lattices seems to satisfy a positivity property. We have shown that this positivity property can be generalized to finite graphs, as the positivity of the coefficients of the Newton expansion for the "dimer entropy" of graphs.

We considered two natural classes of finite graphs related to infinite bipartite lattices: the connected regular bipartite (RBB) graphs and finite lattices with various bc.

In the first class, we have tested exhaustively a very large sample of RBB graphs. The results of our survey lead us to conjecture that for all degrees $r$ in $r$-regular graphs, the frequency of the positivity violations tends to zero as $v \to \infty$. We have also observed that the graphs showing positivity violations have an average order of the symmetry of the automorphism group sizably larger than the average order for graphs with the same number of vertices.

In the second class, we have examined a large number of rectangular grids $N_x \times N_y$ with various bc. We found no positivity violation for $\min(N_x, N_y) > 4$.

In the three-dimensional case we could only test the positivity for relatively small cubic grids (with open bc). These results give some support to the conjecture that for the infinite hypercubic lattices in any dimension, with open bc, all the coefficients of the series for the dimer entropy are positive.

In the case of hexagonal lattices, in the brick-wall representation $N_x \times N_y$, we found no violations for $N_x = N_y$, except for $N_x = N_y = 4$, while there are several violations for $N_x < N_y$. This may be related to the fact that the entropy of the hexagonal lattice depends on the bc. Also in this case these computations suggest that when taking the limit $N \to \infty$ for the $N \times N$ cases, the coefficients of the Newton expansion for the dimer entropy of the infinite hexagonal lattice are all positive.

Acknowledgments We thank K. Markström for discussions.

V. APPENDIX A: PROOF THAT $C_{2n}$ AND $K_{v,v}$ ARE GRAPH-POSITIVE

A polygon $C_v$ with $v$ vertices has $N(i) = \frac{v}{v-1} \binom{v}{i}$. From (20) one gets for $v$ even

$$d(i) = \ln \left( \frac{(v-i-1)!(v-1)^i}{(v-1)!} \right)$$

from which

$$\Delta d(i) = -\ln \left( 1 - \frac{i}{v-1} \right) = \sum_{j=1}^{\infty} \frac{i^j}{j(v-1)^j}$$
which is positive; and thus from \( d(1) = 0 \) one sees that \( d(2), d(3), \ldots \) are positive.

\[
\Delta^{(k+1)} d(i) = \sum_{j=1}^{\Delta^k (i^j)} \Delta^k (i^j)
\]  

(31)

Observing that \( i^j = \sum_{h=0}^{j} S_{j,h}(i)_h \), with \( S_{j,h} \) the Stirling numbers (of the second kind), and that \( \Delta(i)_m = m(i)_m - 1 \), from the positivity of the Stirling numbers and of the falling factorials, it follows that \( \Delta^k d(i) \) is positive for \( k \geq 1 \), so that \( C_v \) is graph-positive for \( v \) even.

A complete bipartite graph \( K_{n,n} \) has \( N(i) = \binom{n}{i}^2 i! \).

\[
d(i) = \ln \left( \frac{n!}{(n-i)!(2n-1)(2n-3)\ldots(2n-2i+1)} \left( \frac{2n-1}{n} \right)^i \right)
\]  

(32)

One has

\[
d(i + 1) = d(i) + \ln \left( \frac{(n-i)(2n-1)}{(2n-2i-1)n} \right) = d(i) + \ln \left( 1 + \frac{i}{(2n-2i-1)n} \right) > d(i)
\]  

(33)

for \( i < n \); since \( d(1) = 0 \) it follows that \( d(i) \geq 0 \) for \( i < n \). One has

\[
\Delta d(i) = \ln \left( \frac{(2n-1)(n-i)}{n(2n-2i-1)} \right) = \ln \left( 1 + \frac{i}{n(2n-2i-1)} \right)
\]  

(34)

so that \( \Delta d(i) > 0 \) for \( i < n \). From

\[
\Delta d(i) = \ln \frac{2n-1}{2n} - \ln \left( 1 - \frac{1}{2(n-i)} \right) = \ln \frac{2n-1}{2n} + \sum_{j=1}^{i} \frac{1}{j2^j (n-i)^j}
\]  

(35)

it follows that for \( m \geq 1 \)

\[
\Delta^{m+1} d(i) = \sum_{j=1}^{\Delta^m (i^j)} \frac{1}{j2^j} \Delta^m \left( \frac{1}{(n-i)^j} \right)
\]  

(36)

defined for \( i \leq n - m - 1 \). It is not difficult to show that these quantities are non-negative, observing that

\[
\Delta \prod_{j} \frac{1}{(n-i-n_j)}
\]

where \( 0 \leq n_j < k \), is the sum of terms of the form

\[
\prod_{j} \frac{1}{(n-i-n_j)}
\]

where \( 0 \leq n_j < k + 1 \). Therefore \( K_{n,n} \) is graph-positive.

Friedland, Krop and Markström[14] have computed an approximation for the average value of \( N(i) \) over all \( 0 - 1 \) matrices of size \( n \times n \) with row and column sums equal to \( r \)

\[
N^{Av}(i) = \frac{(n!)^2 r 2^i i! (rn-i)!}{(rn)!}
\]  

(37)

One can show by the techniques of this section that the sequence of \( N^{Av}(i) \) leads to \( d(i) \) satisfying graph positivity. This is consistent with our conjecture that as \( n \to \infty \) the frequency of violations of graph positivity goes to zero.
VI. APPENDIX B: A SEQUENCE OF NANOTUBES

A fullerene can be drawn on a sphere; it has 12 pentagonal faces, while the other faces are hexagons.

A nanotube \( C_{40+20N} \) can be formed by cutting a \( C_{40} \) fullerene into a North cap and a South cap, joined by a tube formed by hexagons, belonging to an hexagonal lattice, formed by \( N \) hexagonal strips with 20 nodes characterized by a "chiral vector" \((m, n)\) which describes the periodicity conditions on the tube. In the case we have considered, \( m = n = 5 \). (See Fig. 1)

In Tables VI, VII, VIII the edges must be read from left to right, from up to down. The edges are thus ordered in such a way that the maximum number of 'active nodes' is 11.
TABLE VI: North cap

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 0-2 | 2-8 | 6-8 | 3-6 | 0-3 | 0-1 | 6-7 |
| 8-9 | 2-5 | 4-18 | 17-18 | 1-17 | 1-14 | 16-17 |
| 4-21 | 5-13 | 12-13 | 13-14 | 5-10 | 20-21 | 19-20 |
| 7-22 | 7-25 | 22-23 | 23-24 | 24-25 | 25-26 | 9-26 |
| 26-27 | 10-29 | 28-29 | 27-28 | 10-11 | 11-12 | 14-15 |

|   | 2-8 | 6-8 | 3-6 | 0-3 | 0-1 | 6-7 |
|---|---|---|---|---|---|---|
| 11-14 | 12-17 | 15-18 | 17A-18 | 16A-17 | 18A-19 | 19-20 |
| 19-22 | 16-21 | 21A-22 | 20A-21 | 18A-19 | 19-20 | 21-22 |
| 24A-25 | 23A-24 | 26A-27 | 24-29 | 27-10 | 25A-26 | 25-26 |
| 10A-29 | 28A-29 | 27A-28 | 28-13 | 13A-14 | 14A-15 | 15A-16 |
| 10A-11 | 11A-12 | 12A-13 | 14A-15 | 15A-16 |

TABLE VII: Strips: the first index in an edge $a - b$ stands for $a + 20i$ or for $a + 20(i + 1)$ if it is followed by an $A$; the second index $b$ stands for $a + 20(i + 1)$; 20 is the number of nodes added by a strip; $i = 0, ..., N - 1$ where $N$ is the number of strips.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 11-14 | 12-17 | 15-18 | 17A-18 | 16A-17 | 18A-19 | 19-20 |
| 19-22 | 16-21 | 21A-22 | 20A-21 | 18A-19 | 19-20 | 21-22 |
| 24A-25 | 23A-24 | 26A-27 | 24-29 | 27-10 | 25A-26 | 25-26 |
| 10A-29 | 28A-29 | 27A-28 | 28-13 | 13A-14 | 14A-15 | 15A-16 |

In the algorithm for computing the matching generating polynomial, the ordering of the edges is important for performance.

We considered $1 \leq N \leq 300$; graph positivity holds for $N > 7$.

VII. APPENDIX C: EXAMPLES OF NON-POSITIVE RBB GRAPHS

The non-positive RBB graph with $v = 14$ and degree 3 is the Heawood graph. Its matching generating polynomial coefficients $N(0), ..., N(v/2)$ are

$N = [1, 21, 168, 644, 1218, 1050, 336, 24]$. The coefficients of $\bar{N}(0), ..., \bar{N}(v/2)$ (Eq.8) are $\bar{N} = [1, 91, 3003, 45045, 315315, 945945, 945945, 135135]$. From these and Eq. (20) one computes $d(0), ..., d(v/2)$. The first negative finite difference is $\Delta^4 d(0) = -0.000558$.

To compute numerically the finite differences in Sage we used $RealIntervalField(\text{prec})$ with $\text{prec} = 500$ for the RBB graphs, where $\text{prec}$ is the number of bits of precision used; for long rectangular grids higher precisions were required to have enough significant digits.

There are two non-positive RBB graphs with $v = 16$ and degree 3; the first one has reduced adjacency matrix

$[1111000000, 110100000, 100011000, 010011001, 000011011, 000011101, 010100110, 010011000]$ and $N = [1, 24, 228, 1096, 2830, 3848, 2516, 632, 33]$; the first negative finite difference is $\Delta^4 d(0) = -0.00029$.

The second one has reduced adjacency matrix

$[1111000000, 100110000, 010100110, 010010100, 000011010, 001000110]$ and $N = [1, 24, 228, 1096, 2826, 3816, 2444, 600, 33]$; The first negative finite difference is $\Delta^4 d(0) = -0.000999$.

There are two non-positive RBB graphs with $v = 22$ and degree 4; the first one has reduced adjacency matrix

$[11111000000, 11101000000, 11100100000, 11100010000, 00011101000, 00011101010, 00001101100, 00001100011, 0000000111, 0000000000]$ and $N = [1, 44, 814, 8272, 50675, 193516, 460870, 666512, 552265, 234972, 40968, 1584]$; the first negative finite difference is $\Delta^3 d(0) = -0.00031$.

The second one has reduced adjacency matrix

$[11111000000, 11101000000, 11101000000, 11100010000, 00011000111, 00011100011, 00000001111, 0000000000]$ and $N = [1, 44, 814, 8272, 50678, 193576, 461295, 667872, 554256, 236160, 41076, 1584]$; the first negative finite difference is $\Delta^3 d(0) = -0.000098$. 
TABLE VIII: South cap; to the indices of the edges one must add 20N

| Edge Indices | 11-37 | 28-35 | 35-37 | 33-35 | 34-37 | 27-36 | 24-36 | 36-38 | 33-38 | 38-39 | 23-39 | 20-39 | 32-33 | 19-32 | 31-32 | 16-31 | 31-34 | 30-34 | 12-30 | 15-30 |
|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|              |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
1. J. M. Hammersley, “Existence theorems and Monte Carlo methods for the monomer-dimer problem” in: “Research papers in statistics: Festschrift for J. Neyman”, edited by F.N. David. (Wiley, London 1966), pag 125.

2. P. Federbush and S. Friedland, “An Asymptotic Expansion and Recursive Inequalities for the Monomer-Dimer Problem”, J. Stat. Phys. 143(2011) 306.
DOI= 10.1007/s10955-011-0170-6

3. P. Butera and M. Pernici, “Yang-Lee edge singularities from extended activity expansions of the dimer density for bipartite lattices of dimensionality $2 \leq d \leq 7$”, Phys. Rev. E 86(2012) 011104.
DOI = 10.1103/PhysRevE.86.011104

4. P. Butera, P. Federbush and M. Pernici, “Higher-order expansions for the entropy of a dimer or a monomer-dimer system on $d$-dimensional lattices”, Phys. Rev. E 87(2013) 062113.
DOI= 10.1103/PhysRevE.87.062113

5. B.D. McKay, Congr. Numer. 30(1981) 45-87; 10th Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, 1980) [http://cs.anu.edu.au/bdm/nauty/PGI].
DOI= 10.1006/jagm.1997.0898

6. William A. Stein et al., Sage Mathematics Software (Version 5.7), to be freely downloaded at the [http://www.sagemath.org].

7. I.M. Wanless, “The Holens-Djokovic Conjecture on permanents fails!”, Linear Algebra and Appl. 286(1999) 273-285.
DOI= 10.1016/S0024-3795(98)10177-5

8. P. Butera and M. Pernici, “Sums of permanental minors using Grassmann algebra”, arXiv:1406.5337

9. S. Friedland, E. Krop, P.H. Lundow and K. Markström, “On the Validations of the Asymptotic Matching Conjectures”, J. Stat. Phys. 133(2008) 513-533.
DOI= 10.1007/s10955-008-9550-y

10. P. Csinkvari, “Lower Matching Conjecture, and a New Proof of Schrijver’s and Gurvits’s Theorems”, arXiv:1406.0766 [math.co]

11. L. Gurvits, “Unleashing the power of Schrijver’s permanental inequality with the help of the Bethe Approximation”, Proc. Electron. Colloq. Comput. Complexity 2011 and arXiv:1106.2844v11.

12. C.D. Godsil, “Algebraic Combinatorics”, Chapman and Hall, London, 1993.

13. V. Elser, J. Phys. A: Math. Gen. “Solution of the dimer problem on a hexagonal lattice with boundary” 17(1984) 1509-1513.
DOI= 10.1088/0305-4470/17/7/018

14. S. Friedland, E. Krop and K. Markström, “On the Number of Matchings in Regular Graphs”, Electron. J. Combin. 15(2008) R110.