On the annihilator ideal in the \( bt \)-algebra of tensor space

Steen Ryom-Hansen*

Abstract. We study the representation theory of the braids and ties algebra, or the \( bt \)-algebra, \( \mathcal{E}_n(q) \). Using the cellular basis \( \{m_{n\mathbb{Z}}\} \) for \( \mathcal{E}_n(q) \) obtained in previous joint work with J. Espinoza we introduce two kinds of permutation modules \( M(\lambda) \) and \( \mathcal{M}(\Lambda) \) for \( \mathcal{E}_n(q) \). We show that the tensor product module \( V \otimes_n \) for \( \mathcal{E}_n(q) \) is a direct sum of \( M(\lambda) \)’s. We introduce the dual cellular basis \( \{n_{n\mathbb{Z}}\} \) for \( \mathcal{E}_n(q) \) and study its action on \( M(\lambda) \) and \( \mathcal{M}(\Lambda) \). We show that the annihilator ideal \( \mathcal{I} \) in \( \mathcal{E}_n(q) \) of \( V \otimes_n \) enjoys a nice compatibility property with respect to \( \{n_{n\mathbb{Z}}\} \). We finally study the quotient algebra \( \mathcal{E}_n(q)/\mathcal{I} \), showing in particular that it is a simultaneous generalization of Härterich’s ‘generalized Temperley-Lieb algebra’ and Juyumaya’s ‘partition Temperley-Lieb algebra’.

Keywords: Hecke algebra, cellular algebra, \( bt \)-algebra.

MCS2010: 33D80.

1. Introduction

In this paper we study the representation theory of the braids and ties algebra \( \mathcal{E}_n(q) \), or the \( bt \)-algebra for short. It was introduced by Aicardi and Juyumaya in [Aicar-Juyu1], via an abstraction of the presentation for the Yokonuma-Hecke algebra \( \mathcal{Y}_{r,n}(q) \) of type \( A_{n-1} \). The \( bt \)-algebra carries a diagrammatics consisting of braids and ties, as illustrated below where braids are black and ties are dashed red.

![Diagram of braids and ties]

(1)

The study of \( \mathcal{E}_n(q) \) has attracted much attention recently, both from knot theorists and from representation theorists, see for example [Arcis-Juyu], [Aicar-Juyu2], [Aicar-Juyu3], [Aicar-Juyu4], [Ban], [ChlouPou], [EsphLy], [Flo], [JacoAn], [Juyumaya], [Mar], [Ry] and the references therein. Interesting enough, even though \( \mathcal{E}_n(q) \) is derived from \( \mathcal{Y}_{r,n}(q) \), both the knot theory and the representation theory of \( \mathcal{E}_n(q) \) are almost completely independent from those of \( \mathcal{Y}_{r,n}(q) \), in the sense that almost no statement in the literature about \( \mathcal{E}_n(q) \) is obtained directly from a corresponding statement about \( \mathcal{Y}_{r,n}(q) \).

In our paper [Ry] we constructed a tensor space module \( V \otimes_n \) for \( \mathcal{E}_n(q) \), extending Jimbo’s classical tensor space module for the Iwahori-Hecke algebra \( \mathcal{H}_n(q) \) of type \( A_{n-1} \), and showed that it is faithful when the dimension of \( V \) is large enough, although not in general. Using this we obtained a basis for \( \mathcal{E}_n(q) \) on the form \( \{E_A g_w\} \) where \( A \)

*Supported in part by FONDECYT grant 1171379
runs over set partitions on \( n := \{1, \ldots, n\} \) and \( w \) over the symmetric group \( \Sigma_n \). In particular the dimension of \( E_n(q) \) is \( b_n n! \), where \( b_n \) is the \( n \)'th Bell number, that is the number of set partitions on \( n \).

The \( \{E_{Agw}\}\)-basis plays an important role in the applications of \( E_n(q) \) to knot theory as a key ingredient in the construction of a Markov trace on \( E_n(q) \), see for example [Aicar-Juyu2], [Aicar-Juyu3] and [Aicar-Juyu4]. On knots, the resulting invariant is equivalent to the HOMPLYPT-polynomial, but on links it is stronger than the HOMPLYPT-polynomial, see the discussion in the Addendum to [Aicar-Juyu2].

In the present paper we develop methods that allow us to treat the \( E_n(q)\)-structure on \( V^\otimes n \) when \( V^\otimes n \) is not a faithful, and in particular to determine the annihilator ideal \( I \) for the \( E_n(q)\)-action on \( V^\otimes n \) in all cases. Our methods rely on cellular algebra theory, in particular on the cellular basis \( \{m_{st}\} \) for \( E_n(q) \) that was introduced in [EspRy], in joint work with J. Espinoza, and hence they are independent of the ground ring. We acknowledge that the construction of the \( \{m_{st}\}\)-basis for \( E_n(q) \) is combinatorially much more involved than the construction of the \( \{E_{Agw}\}\)-basis, or other similar non-cellular bases, but throughout our arguments rely crucially on the compatibility between the multiplicative structure and the order relation on \( E_n(q) \), given by the cell datum. In fact, it appears to be very difficult to obtain the results of our paper without using an appropriate order relation, even for the ground field \( \mathbb{C} \) and generic \( q \).

The basis \( \{m_{st}\} \) is a generalization of Murphy’s standard basis \( \{x_{st}\} \) for \( H_n(q) \). In the classical representation theory of \( H_n(q) \), see for example [DipperJames] and [DipperJamMathas], an important application of \( \{x_{st}\} \) is to introduce permutation modules \( M(\lambda) \) for \( H_n(q) \), where \( \lambda \) runs over integer partitions of \( n \), and to realize Jimbo’s tensor space module \( V^\otimes n \) as a direct sum of these \( M(\lambda) \)'s. The \( M(\lambda) \)'s have many important properties, but for us it is of special relevance that they are endowed with canonical symmetric bilinear forms \( (\cdot, \cdot)_\lambda \), that satisfy a certain non-vanishing property with respect to the dual basis \( \{y_{st}\} \) of \( \{x_{st}\} \).

In the paper we generalize the \( M(\lambda) \)'s to \( E_n(q)\)-permutation modules \( M(\lambda) \), for \( \lambda \) running over multipartitions, and show in our Theorem 17 that the \( E_n(q)\)-tensor module \( V^\otimes n \) from [Ry] is a direct sum of these \( M(\lambda) \)'s in analogy with the \( H_n(q) \)-case. The \( M(\lambda) \)'s are endowed with bilinear forms, in analogy with the \( M(\lambda) \)'s, but the parametrizing poset \( \mathcal{L}_n \) for \( E_n(q) \) is a combinatorial object which is more complicated than multipartitions, and so the \( M(\lambda) \)'s are not sufficient for our determination of the annihilator ideal \( I \).

To solve this problem we introduce, via \( \{m_{st}\} \), another kind of permutation modules that we denote \( M(\lambda) \). We show in a series of Lemmas, culminating with Lemma 14, that the canonical bilinear form \( (\cdot, \cdot)_\lambda \) on \( M(\lambda) \) indeed satisfies the nonvanishing property with respect to the dual basis \( \{y_{st}\} \).

The above results constitute the bulk of work of our paper. The situation is not completely analogous to the situation for \( H_n(q) \), but in general there is an inclusion \( M(\lambda) \subseteq M(\lambda) \) and this is in fact enough to execute Härterich’s wonderful compatibility argument, see [Härterich], for the determination of the annihilator ideal of the action of \( H_n(q) \) in Jimbo’s tensor space \( V^\otimes n \), but in the \( E_n(q) \)-setting. We do so in Theorem 18.

When working over a ground field \( k \), Theorem 18 gives a \( k \)-vector space basis for \( I \) in terms of an explicit subset of \( \{m_{st}\} \), thus generalizing Härterich’s compatibility result. Note that Härterich’s compatibility result is the starting point for the author’s joint work with D. Plaza on KLR-gradings on the Temperley-Lieb algebras of type \( A \) and \( B \), see [PlaRy]. Note also that other compatibility results for \( \{m_{st}\} \) were obtained in [EspRy].

In the final paragraphs of the paper we introduce and study the quotient algebra \( ETL_n(q) := E_n(q)/I \). It is a simultaneous generalization of Härterich’s ‘generalized Temperley-Lieb’ algebra and of Juyumaya’s ‘partition Temperley-Lieb algebra
\( \mathcal{PT}\mathcal{L}_n(q) \)'s, that themselves are generalizations of the original Temperley-Lieb algebra. Our Theorem 18 allows us to determine the dimension of \( \mathcal{E}\mathcal{T}\mathcal{L}_{n,N}(q) \) as the cardinality of an explicit subset of \( \{n\in\mathbb{Z}\} \). This shows in particular that Juyumaya’s conjectural basis for \( \mathcal{PT}\mathcal{L}_n(q) \) is wrong, but gives a correct basis that may be useful for constructing a Markov trace on \( \mathcal{PT}\mathcal{L}_n(q) \).

Let us now indicate the layout of the paper. In the next section we introduce the basic notation that shall be used throughout the paper. It is mostly concerned with combinatorial notions related to the symmetric group \( \mathfrak{S}_n \), that is partitions, Young diagrams, Young tableaux, etc. Throughout the paper, \( \mathfrak{S}_n \) is viewed as a Coxeter group and we only consider subgroups of \( \mathfrak{S}_n \) that are parabolic subgroups, in the sense of Coxeter groups. It is well known that parabolic subgroups of Coxeter groups give rise to distinguished coset representatives. Throughout we need these distinguished coset representatives and their associated decompositions of the group elements, and we therefore briefly explain why they exist, but from an algorithmic point of view which is possibly less known. In the final paragraphs of this section we recall the definition of \( \mathcal{E}_n(q) \), together with its \( \{E_{\lambda\mu}\} \) basis.

In section 3 we first recall from [EspRy] the ideal decomposition

\[
\mathcal{E}_n(q) = \bigoplus_{a \in \mathcal{P}\mathcal{A}_n} \mathcal{E}_n^a(q)
\]

that reduces the study of \( \mathcal{E}_n(q) \) to the study of its summands \( \mathcal{E}_n^a(q) \). We next recall from [EspRy] the construction of the cellular basis \( \{m_{e\in\mathbb{Z}}\} \) for \( \mathcal{E}_n^a(q) \) and at the same time we introduce the new dual cellular basis \( \{n_{e\in\mathbb{Z}}\} \) by making the appropriate adaptations. It is a generalization of Murphy’s dual standard basis \( \{y_{\alpha}\} \) for \( \mathcal{H}_n(q) \). The cell poset \( \mathcal{L}_n(a) \) for both bases consists of pairs \( (\Lambda, \mu) \) of multipartitions, satisfying certain conditions.

In section 4 we first introduce the permutation module \( M(\Lambda) \). In Lemma 10 we describe a basis \( \{x_\alpha\} \) for \( M(\Lambda) \) together with its \( \mathcal{E}_n^a(q) \)-action. We next introduce, for \( \Lambda \in \mathcal{L}_n(a) \), the permutation module \( M(\Lambda) \). We describe in the Lemmas 11 and 12 a basis \( \{m_\alpha\} \) together with the \( \mathcal{E}_n^a(q) \)-action on it. We then introduce the bilinear form \( (\cdot, \cdot)_\Lambda \) on \( M(\Lambda) \) and show that it is symmetric and invariant. Finally, in Lemma 14 we prove the following crucial property, alluded to above

\[
(m_{e_n} n_{e'_{n'}} m_{e_0})_\Lambda \neq 0.
\]

In section 5 we obtain the main results of our paper. We first give the decomposition of \( V^*n \) corresponding to (2) and then the decomposition in terms of the \( M(\Lambda) \)'s. Finally, in the main Theorem 18, we give the description of \( I \), proving that a subset of \( \{n_{e\in\mathbb{Z}}\} \) induces a basis for it.

It is a great pleasure to thank J. Espinoza for many useful discussions related to the contents of the paper, for sending us comments on preliminary versions of it, and for sharing with us his formula for \( \dim \mathcal{PT}\mathcal{L}_n(q) \). We also thank J. Juyumaya for communicating to us his and P. Papi’s MAGMA calculations and for pointing out an error in a preliminary version of the paper. We thank the referee for useful comments that helped us improve the text. Finally, it is a special pleasure to thank M. Härterich for introducing us to Murphy’s standard basis and all its deep properties.

2. Notation and Basic Concepts

We fix the ground ring \( S := \mathbb{Z}[q, q^{-1}] \), where \( q \) is an indeterminate.

Let \( \mathfrak{S}_n \) be the symmetric group consisting of bijections of \( n := \{1, 2, \ldots, n\} \). Throughout we consider the natural action of \( \mathfrak{S}_n \) on \( n \) as a right action. As is well known, \( \mathfrak{S}_n \) is a Coxeter group on \( \Sigma_n := \{s_1, \ldots, s_{n-1}\} \) with relations

\[
s_is_j = s_js_i \quad \text{for} \quad |i - j| > 1
\]

\[
s_is_{i+1}s_i = s_{i+1}s_iss_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, n - 2
\]

\[
s_i^2 = 1 \quad \text{for} \quad i = 1, 2, \ldots, n - 1
\]
where \( s_i := (i, i + 1) \). We denote by \( < \) the Bruhat-Chevalley order and by \( \ell(\cdot) \) the length function on \( \mathfrak{S}_n \).

Let \( \mathbb{N} \) denote the set of natural numbers and set \( \mathbb{N}^\emptyset := \mathbb{N} \cup \{0\} \). For the notions of compositions and partitions of \( \mathbb{N}^\emptyset \) we shall adhere to the notation introduced in section 2 of [EspRy], except that the Young diagram of a composition or partition \( \mu \) shall be denoted \( \mathcal{Y}(\mu) \). We set \( \text{Comp} := \bigcup_n \text{Comp}_n \) and \( \mathcal{P} \text{ar} := \bigcup_n \mathcal{P}\text{ar}_n \).

Suppose that \( \mu \models n \). Then a \( \mu \)-tableau is a bijection \( t : \mathbf{n} \rightarrow \mathcal{Y}(\mu) \). If \( t \) is a \( \mu \)-tableau we identify it with a labelling of the nodes of \( \mathcal{Y}(\mu) \), using the elements of \( \mathbf{n} \). For example, if \( \mu = (4, 4, 4, 1) \) then

\[
\begin{bmatrix}
1 & 4 & 3 & 13 \\
2 & 6 & 10 & 8 \\
5 & 7 & 11 & 9 \\
12
\end{bmatrix}
\]

(7)

is the \( \mu \)-tableau \( t \), that satisfies \( t(1) = (1, 1), t(4) = (1, 2), t(3) = (1, 3), \) etc. If \( t \) is a \( \mu \)-tableau we define \( \text{Shape}(t) := \mu \). The set of \( \mu \)-tableaux is denoted \( \text{Tab}(\mu) \). A \( \mu \)-tableau \( t \) is \textit{row standard} if the entries in \( t \) increase from left to right in each row and it is \textit{standard} if the entries also increase from top to bottom in each column. The set of row standard \( \lambda \)-tableaux is denoted \( \text{RStd}(\lambda) \) and the set of standard \( \lambda \)-tableaux is denoted \( \text{Std}(\lambda) \). For \( \mu \models n \) we denote by \( t^\mu \) (resp. \( t_\mu \)) the standard tableau in which the integers \( 1, 2, \ldots, n \) are entered in increasing order from left to right along the rows (resp. columns) of \( \mathcal{Y}(\mu) \). For example, if \( \mu = (4, 3) \) then \( t^\mu = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & \end{bmatrix} \) (resp. \( t_\mu = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & \end{bmatrix} \)). If \( \lambda \in \mathcal{P}\text{ar}_n \) and \( t \) is a \( \lambda \)-tableau we denote by \( t^\prime \) the \textit{conjugate tableau}. It is the \( \lambda \)-tableau defined via \( t'(i, j) := t(j, i) \).

\( \mathfrak{S}_n \) acts naturally on the right on \( \text{Tab}(\lambda) \), via composition of a bijection \( t \in \text{Tab}(\lambda) \) with an element \( \sigma \in \mathfrak{S}_n \) viewed as a bijection of \( \mathbf{n} \). For \( t \in \text{Tab}(\lambda) \), we denote by \( d(t) \) the unique element of \( \mathfrak{S}_n \) such that \( t = t^\lambda d(t) \). We set \( w_\lambda := d(t_\lambda) \in \mathfrak{S}_n \). Then for any \( t \in \text{Std}(\lambda) \) we have that

\[
d(t)d(t^\prime)^{-1} = w_\lambda \text{ and } \ell(d(t)) + \ell(d(t^\prime)) = \ell(w_\lambda)
\]

(8)

see for example the proof of Lemma 2.2 of [Murphy92]. The \textit{Young subgroup} \( \mathfrak{S}_\lambda \) associated with \( \lambda \) is the row stabilizer of \( t^\lambda \). It is a parabolic subgroup of \( \mathfrak{S}_n \) in the sense of Coxeter groups. The set \( \{d(t) \mid t \in \text{RStd}(\lambda)\} \) is a set of \textit{distinguished right coset representatives} for \( \mathfrak{S}_\lambda \) in \( \mathfrak{S}_n \), that is \( \ell(w_\lambda d(t)) = \ell(w_\lambda) + \ell(d(t)) \) for \( w_\lambda \in \mathfrak{S}_\lambda \). For any \( w \in \mathfrak{S}_n \) there is a unique decomposition

\[
w = w_0 d(t) \text{ where } w_0 \in \mathfrak{S}_\lambda \text{ and } t \in \text{RStd}(\lambda)
\]

(9)

such that \( \ell(w) = \ell(w_0) + \ell(d(t)) \), see Proposition 3.3 of [Mathas] and chapter 1.10 of [Hump]. The determination of \( w_0 \) and \( t \) can be realized in an algorithmic way that we now explain. Define first \( s \in \text{Tab}(\lambda) \) via \( w = d(s) \). Let next \( s_{j_1}, s_{j_2}, \ldots, s_{j_k} \) be a sequence of elements in \( S \) such that when setting \( s_0 := 5, s_1 := s_0 s_{j_1}, s_2 := s_1 s_{j_2}, \ldots, s_k := s_{k-1} s_{j_k} \) we have that \( j_i \) appears strictly above \( j_{i+1} \) in \( s_j \) and that \( s_k \) is equal to \( t^\lambda \) modulo a permutation of the row elements. Such a sequence always exists. Then \( w_0 = d(s_k) \) and \( t = t^\lambda s_{j_k} s_{j_k-1} \ldots s_{j_1} \), that is \( d(t) = s_{j_k} s_{j_k-1} \ldots s_{j_1} \). Note that although \( w_0 \) and \( d(t) \) are unique, the sequence \( s_{j_k}, s_{j_k-1}, \ldots, s_{j_1} \) is in general not unique. Here is an example of this algorithm, using \( \lambda = (2, 3, 2) \) and \( w = d(s) \) given by

\[
s = \begin{bmatrix} 2 & 6 \\ 3 & 5 & 1 \\ 7 & 4 \end{bmatrix}
\]

(10)

Then a possible sequence of \( s_j \)'s is given as follows

\[
\begin{array}{cccccccc}
2 & 6 & \overset{s_1}{\Rightarrow} & 1 & 6 & \overset{s_2}{\Rightarrow} & 1 & 4 \\
3 & 5 & 1 & 7 & 4 & 3 & 4 & 2 \\
7 & 6 & 7 & 5 & 3 & 5 & 2 & 7
\end{array}
\]

(11)
and so $d(t) = s_2s_3s_4s_5s_1$ whereas $u_0 = s_4s_5s_6$ as can be seen from the last tableau of (11). Note that

$$
t = t^4s_2s_3s_4s_5s_1 = \begin{bmatrix} 2 & 6 \\ 1 & 3 & 5 \\ 4 & 7 \end{bmatrix}
$$

which is also the tableau obtained from $s$ by ordering the rows. This holds in general.

We write $\mu \prec \nu$ for the usual dominance order on $Comp$, see for example section 2 of [EspRy] for the precise definition. For $\lambda \in Comp$ there is a natural extension of $\prec$ to a dominance order on $RStd(\lambda)$, that we shall write the same way $\prec$, see once again [EspRy] for the definition. We have that $t^A$ (resp. $t_A$) is the unique maximal (resp. minimal) row standard $\lambda$-tableau under the dominance order.

An $r$-multicomposition, or just a multicomposition if confusion is not possible, of $n$ is an $r$-tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ of (possibly empty) compositions $\lambda^{(k)}$ such that $\sum_{i=1}^{r'} |\lambda^{(i)}| = n$. We call $\lambda^{(k)}$ the $k$‘th component of $\lambda$. An $r$-multipartition is an $r$-multicomposition whose components are all partitions. The nodes $\mathcal{Y}(\lambda)$ of an $r$-multicomposition $\lambda$ are labelled by triples $(x, y, p)$ with $p$ giving the component number and $(x, y)$ the node of that component. This is the Young diagram for $\lambda$ and is represented graphically as the $r$-tuple of Young diagrams of the components. For example, the Young diagram of $\lambda = ((3,3),(3,1),(1,1,1))$ is

$$
\begin{bmatrix}
\begin{array}{|c|c|c|}
\hline
& & \\
\hline
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{|c|}
\hline
\end{array}
\end{bmatrix}
\end{bmatrix}
$$

The set of $r$-multicompositions of $n$ is denoted by $Comp_{r,n}$ and the subset of $r$-multipartitions of $n$ is denoted by $Par_{r,n}$. If $\lambda \in Comp_{r,n}$, then a $\lambda$-multitableau $t$ is a bijection $t : n \rightarrow \mathcal{Y}(\lambda)$; it is identified with a labelling of $\mathcal{Y}(\lambda)$ using the elements of $n$. The restriction of $t$ to $\lambda^{(i)}$ is the $i$‘th component $t^{(i)}$ of $t$ and we write $t = (t^{(1)}, t^{(2)}, \ldots, t^{(r)})$. We say that $t$ is row standard if all its components are row standard, and standard if they are all standard. If $t$ is a $\lambda$-multitableau we write $Shape(t) = \lambda$. The set of all $\lambda$-tableaux is denoted by $Tab(\lambda)$, the set of all standard $\lambda$-tableaux is denoted by $RStd(\lambda)$ and the set of all standard $\lambda$-multitableaux by $Std(\lambda)$. In the following examples

$$
t = \begin{bmatrix}
\begin{array}{|c|c|c|}
1 & 2 & 3 \\
4 & 5 & \\
\end{array}
\end{bmatrix},
\begin{bmatrix}
\begin{array}{|c|c|}
2 & 7 & 8 \\
1 & 4 & 6 \\
3 & \\
\end{array}
\end{bmatrix}
\end{bmatrix}
$$

$t$ is a standard multitableau whereas $s$ is only a row standard multitableau. We denote by $t^A$ (resp. $t_A$) the $\lambda$-multitableau in which $1, 2, \ldots, n$ appear in order along the rows (resp. columns) of the first component, then along the rows of the second component, and so on. For example, in (13) we have $t = t^4$ with $\lambda = ((3,2),(1,1,1))$. For each multicomposition $\lambda$ we define the Young subgroup $\mathfrak{S}_\lambda$ as the row stabilizer of $t^A$. It is a parabolic subgroup in the sense of Coxeter groups. For $s$ a row standard $\lambda$-multitableau, we denote by $d(s)$ the unique element of $\mathfrak{S}_\lambda$ such that $s = t^Ad(s)$. The set $\{d(s) | s \in RStd(\lambda)\}$ is a set of distinguished right coset representatives for $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$, that is $\ell(w_0d(s)) = \ell(w_0) + \ell(d(s))$ for $w \in \mathfrak{S}_\lambda$. Suppose that $\lambda := (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in Par_{r,n}$ and that $t := (t^{(1)}, \ldots, t^{(r)}) \in Tab(\lambda)$. Then we define the conjugation of $\lambda$ and $t$ componentwise, that is

$$
\lambda' := (\lambda^{(1)}{''}, \ldots, \lambda^{(r)}{''}) \text{ and } t' := (t^{(1)}{''}, \ldots, t^{(r)}{''}).
$$

Note that this definition of conjugation differs from the definition that is often used in the literature, for example in [HuMat].

Similarly to (9), there is for any $w \in \mathfrak{S}_n$ a unique decomposition

$$
w = w_0d(t) \text{ where } w_0 \in \mathfrak{S}_\lambda \text{ and } t \in RStd(\lambda).
$$

Suppose that $w = d(s)$. Then applying the algorithm in (9) on the tableau $s$ obtained from $s = (s^{(1)}, s^{(2)}, \ldots, s^{(r)})$ by concatenating all its components, with $s^{(1)}$ on top followed by $s^{(2)}$ just below $s^{(1)}$ and so on, we obtain $w_0$ and $t$ in (15).
Suppose that $\lambda \in \text{Comp}_{t,n}$ and that $t \in \text{Tab}(\lambda)$ is row standard. Then for $j = 1, \ldots, n$ we set $p_\lambda(j) := k$ if $j$ appears in $t(k)$. We call $p_\lambda(j) \text{ the position of } t$. When $t = t^d$, we write $p_\lambda(j)$ for $p_{t^d}(j)$. We say that $t$ is of the initial kind if $p_\lambda(j) = p_\lambda(k)$ for all $j = 1, \ldots, n$.

Suppose that $\lambda, \mu \in \text{Comp}_{t,n}$. We write $\lambda \preceq \mu$ if $\lambda(i) \preceq \mu(i)$ for all $i = 1, \ldots, n$, this is the dominance order on $\text{Comp}_{t,n}$. If $s$ and $t$ are row standard multitableaux and $m = 1, \ldots, n$ we define $s_{m,s_m}$ and $t_{m,s_m}$ as for usual tableaux and write $s \preceq t$ if $\text{Shape}(s_{m,s_m}) \preceq \text{Shape}(t_{m,s_m})$ for all $m$. Note that our dominance order $\preceq$ is different from the dominance order on multicompositions and multitableaux that is sometimes used in the literature, for example in [DipperRamMathas].

To an $r$-multicomposition $\lambda = (\lambda^1, \ldots, \lambda^r)$ we associate a composition $\|\lambda\|$ of length $r$ in the following way

$$\|\lambda\| := \left(\|\lambda^1\|, \ldots, \|\lambda^r\|\right). \quad (16)$$

Let $\mathcal{S}_\|\lambda\|$ be the associated Young subgroup. Then $w \in \mathcal{S}_\|\lambda\|$ iff $t^d w$ is of the initial kind. For $t = (t^1, \ldots, t^r) \in \text{Tab}(\lambda)$ we define $\|t\| \in \text{Tab}(\|\lambda\|)$ via

$$\|t\| = (\|t^1\|, \ldots, \|t^r\|) \quad (17)$$

where $\|t^i\|$ is the row reading of $t^i$. For the multitableaux $t$ and $s$ in (13) we have for example

$$\|t\| = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}, \quad \|s\| = \begin{pmatrix} 2 & 7 & 8 & 1 & 4 \\ 5 & 6 & 3 & 9 \end{pmatrix}. \quad (18)$$

$\mathcal{S}_\|\lambda\|$ is yet another parabolic subgroup of $\mathcal{S}_n$, with corresponding distinguished right coset representatives $\{t \in \text{Tab}(\lambda) \mid \|t\| \in \text{RStd}(\|\lambda\|)\}$. Hence, for any $w \in \mathcal{S}_n$ there is a unique decomposition

$$w = w_0 d(t) \text{ where } w_0 \in \mathcal{S}_\|\lambda\| \text{ and } t \in \text{Tab}(\lambda) \text{ with } \|t\| \in \text{RStd}(\|\lambda\|). \quad (19)$$

We define $s_0 \in \text{Tab}(\lambda)$ via $w_0 = t^d d(s_0)$; it is of the initial kind. We obtain $s_0$ and $t$ concretely by applying the algorithm given in (11) with respect to $w$ and $\|\lambda\|$. For example, if $w = d(s)$ where

$$s = \begin{pmatrix} 2 & 3 & 7 \\ 6 & 5 & 4 \end{pmatrix} \quad (20)$$

then the calculations in (11) give us

$$s_0 = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 3 & 6 \end{pmatrix}, \quad t = \begin{pmatrix} 2 & 1 & 4 \\ 6 & 3 & 7 \end{pmatrix} \quad (21)$$

where we used $w_0 = s_5 s_3 s_5$ to determine $s_0$. In general, it follows from the algorithm that $s_0$ is row standard if $s$ is row standard.

Suppose that $s \in \text{RStd}(\lambda)$ and $s_1 \in \text{RStd}(\lambda_1)$ with $\|\lambda\| = \|\lambda_1\|$. Let $w = d(s)$, $w_1 = d(s_1)$ and let $w = w_0 d(t)$, $w_1 = (w_1)_0 d(t_1)$ be the decompositions of $w$ and $w_1$ as in (19). Suppose furthermore that $d(t) = d(t_1)$. Then we have that

$$s \preceq s_1 \text{ if and only if } s_0 \preceq (s_1)_0. \quad (22)$$

For $\lambda$ any $r$-multipartition we define $w_\lambda \in \mathcal{S}_n$ as $w_\lambda := d(t_\lambda)$. Suppose now that $s \in \text{Tab}(\lambda)$ and let $s'$ be the conjugate multitableau. Set $w = d(s)$ and $w' = d(s')$ with decompositions $w = w_0 d(t)$ and $w' = (w')_0 d(t')$ as in (19). Then it follows from the algorithm described above that $d(t) = d(t')$ and that $(w')_0 = d(t_0')$ where $w_0 = d(t_0)$. Moreover, since $b_0$ is of the initial kind we get via (8) that

$$d(s) d(s')^{-1} = w_\lambda \text{ and } \ell(d(t_0)) + \ell(d(t'_0)) = \ell(w_\lambda). \quad (23)$$

Note that we cannot here replace $d(t_0)$ and $d(t'_0)$ by $d(s)$ and $d(s')$.

Let us now recall the $brt$-algebra of the title of the entry. It was originally introduced by Aicardi and Juyumaya in [Aicardi-Juyu1].
Definition 1 Let \( n \) be a positive integer. The braids and ties algebra \( \mathcal{E}_n = \mathcal{E}_n(q) \), or the \( bt \)-algebra, is the associative \( S \)-algebra generated by the elements \( g_1, \ldots, g_{n-1} \) and \( e_1, \ldots, e_{n-1} \), subject to the following relations:

\[
\begin{align*}
g_i g_j &= g_j g_i & &\text{for } |i - j| > 1 \\
g_i e_i &= e_i g_i & &\text{for all } i \\
g_i g_j g_i &= g_j g_i g_i & &\text{for } |i - j| = 1 \\
e_i g_j g_i &= g_j g_i e_j & &\text{for } |i - j| = 1 \\
e_i e_j &= e_j e_i & &\text{for all } i, j \\
e_i e_j &= e_j g_i & &\text{for } |i - j| > 1 \\
e_i^2 &= e_i & &\text{for all } i \\
g_i^2 &= 1 + (q - q^{-1}) e_i g_i & &\text{for all } i.
\end{align*}
\]

There is a diagrammatic interpretation of these relations that explains the name of \( \mathcal{E}_n(q) \) and that we now briefly indicate even though it only plays a minor role in the present paper. It uses \( n \) strands connecting a northern and a southern border. Under the interpretation, the \( g_i \)'s correspond to usual simple braids involving the \( i \)'th and the \( i + 1 \)'st strands and the \( e_i \)'s correspond to ties involving the \( i \)'th and \( i + 1 \)'st strands. The relations (27) and (28) are for example visualized as follows.

\[
\begin{align*}
\begin{array}{c}
\includegraphics{braids_ties.png}
\end{array}
\end{align*}
\]

If \( K \) is a commutative ring containing an invertible element, that we also denote \( q \), then we define the specialized algebra \( \mathcal{E}_n^K(q) \) via \( \mathcal{E}_n^K(q) := \mathcal{E}_n(q) \otimes_S K \) where \( K \) is made into an \( S \)-algebra by mapping \( q \in S \) to \( q \in K \).

The concept of set partitions is important in the study of \( \mathcal{E}_n(q) \). Recall that a set partition \( A = \{I_1, I_2, \ldots, I_r\} \) of \( n \) is a set of nonempty and disjoint subsets of \( n \) whose union is \( n \), in order words, \( A \) is the set of classes of an equivalence relation on \( n \). We refer to the \( I_i \)'s as the blocks of \( A \). We denote by \( SP_n \) the set of all set partitions of \( n \). There is a canonical poset structure on \( SP_n \) defined as follows. Suppose that \( A = \{I_1, I_2, \ldots, I_k\}, B = \{J_1, J_2, \ldots, J_l\} \in SP_n \). Then we say that \( A \subseteq B \) if each \( J_j \) is a union of some of the \( I_i \)'s.

Returning to \( \mathcal{E}_n(q) \), for \( 1 \leq i < j \leq n \) we define \( E_{ij} \in \mathcal{E}_n(q) \) by \( E_{ij} = e_i \) if \( i = j - 1 \), and recursively downwards on \( i \) via

\[
E_{ij} := g_i E_{i+1,j} g_i^{-1}.
\]

Note that \( g_i \) is invertible with inverse \( g_i^{-1} = g_i + (q^{-1} - q)e_i \), as follows from (32), so that (35) makes sense. For \( A \in SP_n \) we define \( E_A \in \mathcal{E}_n(q) \) as

\[
E_A := \prod_{i,j} E_{ij}
\]

where the product runs over pairs \( (i, j) \), such that \( i < j \) and such that \( i \) and \( j \) belong to the same block of \( A \). One checks that the \( E_{ij} \)'s commute and so the product is independent of the order in which it is taken. For \( w = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_n \) a reduced expression...
for \( w \) we define \( g_w := g_{t_1} \cdots g_{t_k} \in \mathcal{E}_n(q) \). By the relations it is independent of the choice of reduced expression. We have the following relations in \( \mathcal{E}_n(q) \)

\[
E_{AB}w = g_wE_{Aw} \quad \text{and} \quad E_AE_B = E_C \quad \text{for} \quad w \in \mathfrak{S}_n, A, B \in \mathcal{S}\mathcal{P}_n
\]

(37)

where \( C \in \mathcal{S}\mathcal{P}_n \) is minimal with respect to \( A \subseteq C, B \subseteq C \). Moreover, it was shown in [Ry] that \( \mathcal{E}_n(q) \) has an \( S \)-basis of the form \( \{E_{AB}w | A \in \mathcal{S}\mathcal{P}_n, w \in \mathfrak{S}_n \} \) and so, in particular, the dimension of \( \mathcal{E}_n(q) \) is \( n!b_n \) where \( b_n \) is the Bell number, that is the cardinality of \( \mathcal{S}\mathcal{P}_n \).

3. **Cellular basis for \( \mathcal{E}_n(q) \).**

In the paper [EspRy], a cellular basis \( \{m_{\mu, \pi}\} \) for \( \mathcal{E}_n(q) \) was constructed. Moreover, it was shown that \( \{m_{\mu, \pi}\} \) induces cellular bases for interesting subalgebras of \( \mathcal{E}_n(q) \) and via this an isomorphism between \( \mathcal{E}_n(q) \) and a direct sum of matrix algebras over certain wreath products of Hecke algebras was established.

In this section we first recall the construction of \( \{m_{\mu, \pi}\} \) and at the same time we introduce a new, *dual*, cellular basis for \( \mathcal{E}_n(q) \), that we denote \( \{n_{\mu, \pi}\} \). We next prove, in a series of Lemmas, certain compatibility properties between the dominance order associated with \( \mathcal{E}_n(q) \) and the multiplication of elements from the two bases. These compatibility properties are known in the Hecke algebra setting, and our proofs are essentially reductions, although not completely trivial, to that setting.

Let us first recall the formal definition of cellular algebras, as introduced in [GraLeh].

**Definition 2** Let \( \mathcal{R} \) be an integral domain. Suppose that \( A \) is an \( \mathcal{R} \)-algebra. Suppose that \( (\mathcal{P}, \leq) \) is a poset and that for each \( \lambda \in \mathcal{P} \) there is a finite indexing set \( T(\lambda) \) (the '\( \lambda \)-tableaux') and elements \( c^\lambda_{st} \in A \) such that

\[
C = \left\{ c^\lambda_{st} \mid \lambda \in \mathcal{P} \text{ and } s, t \in T(\lambda) \right\}
\]

(38)

is an \( \mathcal{R} \)-basis of \( A \). The pair \( (C, \mathcal{P}) \) is a *cellular basis* for \( A \) if

(i) The \( \mathcal{R} \)-linear map \( \ast : A \to A \) determined by \( (c^\lambda_{st})^\ast = c^\lambda_{ts} \) for all \( \lambda \in \mathcal{P} \) and all \( s, t \in T(\lambda) \) is an algebra anti-automorphism of \( A \).

(ii) For any \( \lambda \in \mathcal{P}, t \in T(\lambda) \) and \( a \in A \) there exist elements \( r_0 \in \mathcal{R} \) such that for all \( s \in T(\lambda) \)

\[
c^\lambda_{st}a \equiv \sum_{v \in T(\lambda)} r_0 c^\lambda_{sv} \mod A^A
\]

where \( A^A \) is the \( \mathcal{R} \)-submodule of \( A \) with basis \( \left\{ c^\mu_{uv} \mid \mu \in \mathcal{P}, \mu > \lambda \text{ and } u, v \in T(\mu) \right\} \).

If \( A \) has a cellular basis we say that \( A \) is a *cellular algebra* and \( (\mathcal{P}, T, C, \ast) \) is called the *cell datum* for \( A \).

We now recall the, somewhat lengthy, construction of the cell datum for \( \mathcal{E}_n(q) \). For more details on this construction, and for all unexplained notions, the reader should consult [EspRy].

We first consider a certain ideal decomposition of \( \mathcal{E}_n(q) \). Let \( \mu_{\mathcal{S}\mathcal{P}_n} \) be the Möbius function for the lattice \( (\mathcal{S}\mathcal{P}_n, \subseteq) \), see for example section 6.1 of [EspRy] for precise formulas and examples. Using it, we define for \( A \in \mathcal{S}\mathcal{P}_n \) orthogonal idempotents \( E_A \in \mathcal{E}_n(q) \) via

\[
E_A := \sum_{B : A \leq B} \mu(A, B)E_B.
\]

(39)

The following is Proposition 39 in [EspRy].

**Proposition 3** The following properties hold.

(1) \( |E_A| A \in SP_n \) is a set of orthogonal idempotents of \( E_n(q) \).
(2) For all \( w \in \mathfrak{S}_n \) and \( A \in SP_n \) we have \( E_A g_w = g_w E_A w \).
(3) For all \( A \in SP_n \) we have \( E_A E_B = \begin{cases} E_A & \text{if } B \subseteq A \\ 0 & \text{if } B \not\subseteq A. \end{cases} \)

Let \( a \in Par_n \). A set partition \( A = \{I_1, \ldots, I_k\} \) of \( n \) is said to be of type \( a \) if there exists a permutation \( \sigma \) such that \( (|I_{1i}|, \ldots, |I_{ki}|) = a \). We write \( |A| = a \) if \( A \in SP_n \) is of type \( a \) and we let \( SP_n^a \) be the set of set partitions of type \( a \). For \( a \in Par_n \) we define the following element \( E_a \) of \( E_n(q) \)

\[
E_a := \sum_{A \in SP_n, |A| = a} E_A.
\]

Then the \( E_a \)'s form a family of central orthogonal idempotents in \( E_n(q) \) such that \( \sum_{a \in Par_n} E_a = 1 \). As a consequence we have the following decomposition of \( E_n(q) \) into a direct sum of two-sided ideals

\[
E_n(q) = \bigoplus_{a \in Par_n} E_n^a(q)
\]

where \( E_n^a(q) := E_a E_n(q) \). Each ideal \( E_n^a(q) \) is an \( S \)-algebra with identity \( E_a \) and the set

\[
\{E_A g_w \mid w \in \mathfrak{S}_n, |A| = a\}
\]

is an \( S \)-basis for \( E_n^a(q) \). For \( K \) an arbitrary field containing a nonzero element \( q \) we have a specialized version of \( E_n^a(q) \)

\[
E_n^{K,a}(q) := E_n^a(q) \otimes_S K.
\]

In view of the decomposition in (41), in order to give the cell datum for \( E_n(q) \) it is enough to give the cell datum for each \( E_n^a(q) \).

We now explain the poset denoted \( P \) in Definition 2. We first describe \( P \) as a set. For \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in Comp_{r,n} \) we define sets \( I_1 \) via \( I_1 := \{1, 2, \ldots, |\lambda^{(1)}|\} \), \( I_2 := \{|\lambda^{(1)}| + 1, |\lambda^{(1)}| + 2, \ldots, |\lambda^{(1)}| + |\lambda^{(2)}|\} \), and so on. Leaving out the empty \( I_i \)'s we obtain a set partition in \( SP_n \) that we denote \( A_\lambda \). For \( a \in Par_n \) we say that \( \lambda \) is of type \( a \) if \( A_\lambda \) is of type \( a \).

An \( r \)-multipartition \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) is said to be increasing if \( \lambda^{(i)} \prec \lambda^{(j)} \) if \( i < j \), where \( \prec \) is any fixed extension of the dominance order \( \prec \) to a total order.

For \( a \in Par_n \), we now define \( L_n(a) \) to be the set of pairs \( \Lambda = (\lambda \mid \mu) \) where \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) is an increasing \( r \)-multipartition of \( n \) of type \( a \). Let \( \lambda^{(i)} \prec \cdots \prec \lambda^{(r)} \) be the distinct \( \lambda^{(i)} \)'s and let \( m_j \) be the multiplicity of \( \lambda^{(j)} \) in \( (\lambda^{(1)}, \ldots, \lambda^{(r)}) \). Then we require that \( \mu \) be an \( s \)-multipartition \( \mu = (\mu^{(1)}, \ldots, \mu^{(s)}) \) of \( r \) where \( \mu^{(j)} \vdash m_j \). For \( P \) we choose \( L_n(a) \).

We next describe the poset structure on \( L_n(a) \). Suppose that \( \Lambda = (\lambda \mid \mu) \) and \( \Lambda' = (\lambda' \mid \mu' ) = \bar{\Lambda} \) are elements of \( L_n(a) \), such that \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) and \( \lambda' = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \). Then we write \( \lambda \prec \lambda' \) if there exists a permutation \( \sigma \) such that \( (\lambda^{(1)}\sigma, \ldots, \lambda^{(r)}\sigma) \prec (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) where \( \prec \) is the dominance order on \( r \)-multipartitions, introduced above. We then write of \( \Lambda \prec \Lambda' \) if \( \Lambda \prec \Lambda' \) or if \( \Lambda = \Lambda ' \) and \( \mu \prec \mu' \). As usual we set \( \Lambda \leq \Lambda ' \) if \( \Lambda \prec \Lambda ' \) or if \( \Lambda = \Lambda ' \). This is the description of \( L_n(a) \) as a poset.

For \( \Lambda = (\lambda \mid \mu) \in L_n(a) \) as above, we now define the concept of \( \Lambda \)-tableaux. Suppose that \( t \) is a pair \( t = (t \mid u) \). Then \( t \) is called a \( \Lambda \)-tableau if \( t \) is a \( \Lambda \)-multitableau and \( u \) is a \( \mu \)-multitableau of the initial kind. If \( t \) is a \( \Lambda \)-tableau we define \( \text{Shape}(t) := \Lambda \). We let \( \text{Tab}(\Lambda) \) denote the set of \( \Lambda \)-tableaux. We say that \( t = (t \mid u) \in \text{Tab}(\Lambda) \) is row standard if its ingredients are row standard multitableaux in the sense of the previous section and if moreover \( t \) is an increasing multitableau. By increasing we here mean that whenever \( \lambda^{(i)} = \lambda^{(j)} \) we have that \( i < j \) if and only if \( \min(t^{(i)}) < \min(t^{(j)}) \) where
min(t) is the function that reads off the minimal entry of the tableau t. We say that
t = (t ∥ u) ∈ Tab(Λ) is standard if it is row standard and if its ingredients t and u are
standard multitableaux and we define Std(Λ) to be the set of all standard Λ-tableaux.

Note that our notation is here deviating slightly from the one used in [EspRy]
where the condition on t to be increasing was required for standardness of t, but not
for row standardness of t. In the following examples

\[
s := \left( \begin{array}{ccc}
1 & 5 & 6 \\
9 & 1 & 3 \\
7 & 2 & 4 \\
\end{array} \right) \quad \quad \quad \quad t := \left( \begin{array}{ccc}
1 & 5 & 6 \\
9 & 1 & 3 \\
7 & 2 & 4 \\
\end{array} \right)
\]

s is not row standard but t is.

We fix the following combinatorial notation. Suppose that \( Λ = (λ ∥ μ) ∈ L_n(α) \) with
\( λ = (λ^{(1)}, ..., λ^{(r)}) \) and \( μ = (μ^{(1)}, ..., μ^{(s)}) \). By construction, the numbers \( m_i := |μ^{(i)}| \)
are the multiplicities of equal \( λ^{(i)} \)'s. Let \( Σ_Λ ≤ Σ_n \) be the stabilizer subgroup of the set part-
tition \( A_Λ = \{I_1, I_2, ..., I_m \} \). Then the multiplicities give rise to the subgroup \( Σ^m_Λ \) of \( Σ_Λ \)
consisting of the order preserving permutations of the blocks of \( A_Λ \) that correspond
to equal \( λ^{(i)} \)'s. We have \( Σ^m_Λ ≤ Σ_Λ \), and as an abstract group

\[
Σ^m_Λ = Σ_{m_1} × ... × Σ_{m_s}.
\]

For example, if \( Λ = ((1^2), (1^2), (2), (2), (1^3), (1^3), (2, 1)) \) then

\[
t^λ = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \\
\end{pmatrix}
\]

and so \( Σ^m_Λ = Σ_3 × Σ_2 × Σ_3 × Σ_1 \). For \( I_1 \) and \( I_{i+1} \) two consecutive blocks of \( A_Λ \) such
that \( λ^{(i)} = λ^{(i+1)} \) we let \( B_i ∈ Σ_n \) be the element of minimal length, in the sense of
Coxeter groups, that interchanges \( I_i \) and \( I_{i+1} \). For example in the above example we
have \( B_1 = (1, 3)(2, 4), B_2 = (3, 5)(4, 6) \), and so on. \( Σ_Λ \) is generated by the \( B_i \)'s such
that \( λ^{(i)} = λ^{(i+1)} \), in fact it is a Coxeter groups on these \( B_i \)'s.

It is important that the group algebra \( SΣ^m_Λ \) can be realized as a subalgebra of
\( E_n(q) \). Let \( B_j ∈ E_n(q) \) be the element introduced in equation (113) of [EspRy].
Then the following is (1) of Lemma 46 of [EspRy].

**Lemma 4** Suppose that \( Λ = (λ ∥ μ) ∈ L_n(α) \). Then we have an S-algebra embedding
t : \( SΣ^m_Λ \rightarrow E_n(q) \) given by \( B_I \rightarrow B_I \) where \( λ^{(i)} = λ^{(i+1)} \).

We are now ready to recall the construction of the basis elements \( \{m_{k∈ξ} \} \) of the cell
datum for \( E_n(q) \), and at the same time we introduce the new basis elements \( \{n_{kξ} \} \) for
\( E_n(q) \). For each \( Λ ∈ L_n(α) \), we first define elements \( m_Λ \) and \( n_Λ \) that act as starting
points for the bases. Suppose that \( Λ = (λ ∥ μ) \) is as above with \( λ = (λ^{(1)}, ..., λ^{(r)}) \) and
\( μ = (μ^{(1)}, ..., μ^{(s)}) \). We then define \( m_Λ ∈ E_n(q) \) and \( n_Λ ∈ E_n(q) \) as follows

\[
m_Λ := E_Λ x_Λ b_μ, \quad n_Λ := E_Λ y_Λ c_μ.
\]

Let us explain the factors of these products. The idempotent \( E_Λ \) has already been
introduced. The factors \( x_Λ ∈ E_n(q) \) and \( y_Λ ∈ E_n(q) \) are defined as

\[
x_Λ := \sum_{w ∈ Σ_Λ} q^{ℓ(w)} g_w \quad \text{and} \quad y_Λ := \sum_{w ∈ Σ_Λ} (-q)^{-ℓ(w)} g_w.
\]

Note that \( E_Λ \) commutes with \( x_Λ \) and \( y_Λ \) and that for \( w ∈ Σ_Λ \), we have that

\[
E_Λ x_Λ g_w = g_w E_Λ x_Λ = q^{ℓ(w)} E_Λ x_Λ \quad \text{and} \quad E_Λ y_Λ g_w = g_w E_Λ y_Λ = (-q)^{-ℓ(w)} E_Λ y_Λ.
\]
In the Hecke algebra case, corresponding to \( r = 1 \), the elements defined in (47) are the elements denoted \( x_{\lambda} \) and \( y_{\lambda} \) in [Murphy95].

In order to define the factors \( b_{\mu} \) and \( c_{\mu} \) we recall from (45) the decomposition \( \mathfrak{S}_{\lambda}^{m} = \mathfrak{S}_{m_{1}} \times \cdots \times \mathfrak{S}_{m_{i}} \). Let \( x_{\mu}(1) \in \mathfrak{S}_{\lambda}^{m} \) and \( y_{\mu}(1) \in \mathfrak{S}_{\lambda}^{m} \) be the \( q = 1 \) specializations of the elements given in (49) corresponding to the multipartition \( \mu \), that is

\[
x_{\mu}(1) := \sum_{w \in \mathfrak{S}_{\mu}} g_{w} \quad \text{and} \quad y_{\mu}(1) := \sum_{w \in \mathfrak{S}_{\mu}} (-1)^{\ell(w)} g_{w}. \tag{49}
\]

Then \( b_{\mu} \in \mathcal{E}_{n}^{q}(q) \) and \( c_{\mu} \in \mathcal{E}_{n}^{q}(q) \) are defined via

\[
b_{\mu} := i(x_{\mu}(1)) \quad \text{and} \quad c_{\mu} := i(y_{\mu}(1)), \tag{50}
\]

where \( i : \mathfrak{S}_{\lambda}^{m} \rightarrow \mathcal{E}_{n}^{q}(q) \) is the embedding from Lemma 4.

Let \( t^{\Lambda} \) (resp. \( t_{\Lambda} \)) be the \( \Lambda \)-tableau given by \( t^{\Lambda} := (t^{\lambda} \mid t^{\mu}) \) (resp. \( t_{\Lambda} := (t_{\lambda} \mid t_{\mu}) \)). Let now \( s = (s \mid u) \) be an arbitrary \( \Lambda \)-tableau. Then we set \( d(s) := (d(s) \mid i(d(u))) \) where \( d(s) \in \mathfrak{S}_{\Lambda} \) satisfies \( t^{\lambda}d(u) = s \) and \( d(u) \in \mathfrak{S}_{\Lambda}^{m} \) satisfies \( t^{\mu}d(u) = u \). For simplicity, we write \( (d(s) \mid d(u)) \) instead of \( (d(s) \mid i(d(u))) \). Note that since \( u = (u_{1}, \ldots, u_{s}) \) is of the initial kind, we have a decomposition \( d(u) = (d(u_{1}), \ldots, d(u_{s})) \), according to the decomposition in (45) and so

\[
\mathbb{B}_{d(u)} = \mathbb{B}_{d(u_{1})} \cdots \mathbb{B}_{d(u_{s})}. \tag{51}
\]

We can now finally recall the construction of the cellular basis \( m_{s\Lambda} \) from [EspRy] and at the same time introduce the new dual cellular basis \( n_{s\Lambda} \) for \( \mathcal{E}_{n}^{q}(q) \). For \( s = (s \mid u), t = (t \mid v) \) row standard \( \Lambda \)-tableaux we define

\[
m_{s\Lambda} := E_{\Lambda}^{*} g_{d(s)}^{*} x_{\Lambda}^{*} B_{d(v)}^{*} b_{\mu} B_{d(v)} g_{d(t)} \tag{52}
\]

\[
n_{s\Lambda} := E_{\Lambda}^{*} g_{d(s)}^{*} y_{\Lambda}^{*} B_{d(v)}^{*} c_{\mu} B_{d(v)} g_{d(t)}. \tag{53}
\]

We set \( b_{uv} := B_{d(u)}^{*} b_{\mu} B_{d(v)} \) and \( c_{uv} := B_{d(u)}^{*} c_{\mu} B_{d(v)} \) and have then also

\[
m_{s\Lambda} = E_{\Lambda}^{*} g_{d(s)}^{*} x_{\Lambda}^{*} b_{uv} g_{d(t)} \tag{54}
\]

\[
n_{s\Lambda} = E_{\Lambda}^{*} g_{d(s)}^{*} y_{\Lambda}^{*} c_{uv} g_{d(t)}. \tag{55}
\]

In the case where \( s = t^{\Lambda} \) we write for simplicity

\[
m_{s} := m_{s\Lambda} \quad \text{and} \quad n_{s} := n_{s\Lambda}. \tag{56}
\]

In [EspRy] the following Theorem was proved.

**Theorem 5** \( \mathcal{E}_{n}^{q}(q) \) is a cellular basis with cell datum \((\mathcal{L}_{n}(\alpha), \text{Std}(\Lambda), m_{s\Lambda}, *)\).

Making the straightforward adaptations of the proofs given in [EspRy], we also have the following Theorem.

**Theorem 6** \( \mathcal{E}_{n}^{q}(q) \) is a cellular basis with cell datum \((\mathcal{L}_{n}(\alpha), \text{Std}(\Lambda), n_{s\Lambda}, *)\).

By general cellular algebra theory, specialization induces a cell datum on \( \mathcal{E}_{n}^{K,\alpha}(q) \) as well. We shall denote it the same way; in particular \( m_{s\Lambda} \) and \( n_{s\Lambda} \) may refer to elements of both \( \mathcal{E}_{n}^{q}(q) \) and \( \mathcal{E}_{n}^{K,\alpha}(q) \).

The definition of cellular basis does not stipulate a partial order on the tableaux, but in our setting there is a natural such dominance order \( < \) on tableaux that we shall need. It was already introduced in [EspRy], see the paragraph below Remark 43 of loc. cit. for the precise definition. As usual we set \( t \leq t' \) if \( t < t' \) or \( t = t' \).

We now introduce the concept of conjugation of the elements of \( \mathcal{L}_{n}(\alpha) \) and of their tableaux; this concept is also not present in general cellular algebras. Note that it was not formally defined in [EspRy]. Suppose that \( \Lambda = (\Lambda \mid \mu) \in \mathcal{L}_{n}(\alpha) \), with \( \Lambda = (\lambda^{(1)}, \ldots, \lambda^{(\beta)}) \) and \( \mu = (\mu^{(1)}, \ldots, \mu^{(\beta)}) \). Then we define the conjugate of \( \Lambda \) via \( \Lambda' := (\lambda' \mid \)
for tableau $\tau'$ in $\text{Tab}(\Lambda')$ by conjugating all the components of $\tau$, mimicking what we did for $\Lambda$. We have the following compatibility between the dominance orders and conjugation: $\Lambda \leq \Lambda'$ iff $\Lambda' \leq \Lambda$ and $s \leq t$ iff $t' \leq s'$.

The following simple Lemma shall be used repeatedly.

**Lemma 7** Let $\alpha \in \mathcal{P}ar_n$. Suppose that $A \in \mathcal{S}\mathcal{P}_n$ is of type $\alpha$ and that $i$ and $i+1$ are in distinct blocks of $A$. Then in $\mathcal{E}_n^\alpha(q)$ and $\mathcal{E}_n^{K,\alpha}(q)$ the generator $g_i$ verifies the symmetric group quadratic relation when acting on $E_A$, that is

\[ E_A g_i^2 = E_A. \]  

**Proof.** By relation (32) we have that

\[ E_A g_i^2 = E_A(1 + (q - q^{-1})e_i g_i) = E_A + (q - q^{-1})E_A e_i g_i = E_A \]  

where we used (3) of Proposition 3 for the last equality. \hfill $\Box$

For $\Lambda = (\lambda \mid \mu) \in \mathcal{L}_n(\alpha)$, and $\Lambda$-tableaux $s = (s \mid u)$ and $t = (t \mid v)$, we define elements of $\mathcal{E}_n^\alpha(q)$ or $\mathcal{E}_n^{K,\alpha}(q)$

\[ x_{st} := g_d^* \lambda^E \lambda^d(t), \quad y_{st} := g_d^* \lambda^E \lambda^d(t). \]  

(59)

We also need the following Lemma.

**Lemma 8** Suppose that $\Lambda = (\lambda \mid \mu) \in \mathcal{L}_n(\alpha)$, that $\Lambda_1 = (\lambda_1 \mid \mu_1) \in \mathcal{L}_n(\alpha)$ and that $s, t \in \text{Std}(\lambda)$ and $s, t_1 \in \text{Std}(\lambda_1)$. Suppose moreover that $x_{st}y_{st_1} \neq 0$. Then $t \leq s'_{t_1}$.

**Proof.** For usual integer partitions this result is well-known, and as we shall see, the proof can be reduced to that case. Without loss of generality we may assume that $s = t^A$ and $t_1 = t^{A_1}$ and so we have that

\[ x_{st}y_{st_1} = x_{st_1}y_{st_1}. \]  

(60)

For the last step we used that $x_{A^E} E_A = E_A x_{A^E}$, together with part (2) of Proposition 3 and the fact that $E_A = E_{A_1}$ since $\Lambda$ and $\Lambda_1$ are both of type $\alpha$. Now by hypothesis, (60) is nonzero and so $E_{A_1} d(t_1 | d(s_1))^{-1} E_A \neq 0$. Hence using the orthogonality of the $E_A$’s we get that

\[ A_1 d(t) = A_1 d(s_1). \]  

(61)

We now choose decompositions with respect to $G_{||\Lambda||} = G_\alpha$ as in (19)

\[ d(t) = d(t_0) u_t \quad \text{and} \quad d(s_1) = d((s_1)_0) u_{s_1}. \]  

(62)

Clearly $G_n$ acts transitively on $\mathcal{S}\mathcal{P}_n^\alpha$ with $G_\alpha$ as the stabilizer group of $A_\alpha$ and so we have an identification $\mathcal{S}\mathcal{P}_n^\alpha = G_\alpha \backslash \mathcal{S}\mathcal{P}_n$. From (61) we have that $G_\alpha u_t = G_\alpha u_{s_1}$ and so $u_t = u_{s_1}$. Hence (60) becomes

\[ E_A x^L_\lambda g_{d(t_0)} g_{u_t} g_{w_t} g_{d((t_0)_0)} \lambda_1 = x_{t^A t_0} E_A g_{u_t} g_{w_t} g_{d((s_1)_0)} \lambda_1. \]  

(63)

We can now use Lemma 7 repeatedly on the simple transpositions that appear in a reduced expression for $u_t$ and hence (63) becomes

\[ E_A x^L_\lambda S_{(s_1)_0} \lambda_1. \]  

(64)

In view of (22) and the fact that $t_0$ and $(s_1)_0$ are both of the initial kind, the Lemma now follows from the similar result in the Hecke algebra setting. \hfill $\Box$

We wish to generalize the Lemma to a statement involving $m_{\text{reg}}$ and $n_{\text{reg}}$. For this we need to recall from [EspRy] some of the commutation relations between the various factors of $m_{\text{reg}}$ and $n_{\text{reg}}$, as defined in (52) and (53).
The commutation relations involving $E_{\lambda}$ and $x_{\lambda}$ (resp. $y_{\lambda}$) are easy to describe, since $E_{\lambda}$ and $x_{\lambda}$ (resp. $y_{\lambda}$) commute with all the factors of (52) (resp. (53)), except the ones on the extreme left and right. The commutation relations between $B_{\alpha}$ of the initial kind, that is $E_{\lambda}$, since $s, t \in \text{Std}(\Lambda)$ and $s_1, t_1 \in \text{Std}(\Lambda_1)$. If in $E_{\lambda}$ or $E_{\lambda}^\alpha$ we have $m_{ss} n_{tt} \neq 0$ then $ttt \leq s'_{1}$. 

\textbf{Lemma 9} Suppose that $\Lambda = (\Lambda \mid \mu)$ and that $\Lambda_1 = (\Lambda_1 \mid \mu_1) \in \mathcal{L}_\mu(\alpha)$ and that $s, t \in \text{Std}(\Lambda)$ and $s_1, t_1 \in \text{Std}(\Lambda_1)$. If in $E_{\lambda}$ or $E_{\lambda}^\alpha$ we have $m_{ss} n_{tt} \neq 0$ then $ttt \leq s'_{1}$.

\textbf{Proof.} Let $s = (u \mid u)$, $t = (v \mid v)$, $s_1 = (u_1 \mid u_1)$ and $t_1 = (v_1 \mid v_1)$. As in the proof of Lemma 8, we can without loss of generality assume that $s = t^\Lambda$ and $t_1 = t^\Lambda_1$ and so we have that

$$m_{ss} n_{tt} = \mathbb{E}_{A_{\lambda}} x_{\lambda} b_{\mu} B_{\mu} E_{\lambda} d(t) E_{\lambda} d(s_1) B_{\mu}^* E_{\lambda}^\alpha c_{\mu}, y_{\lambda_1}, Y_{\lambda_1}. \tag{66}$$

We now argue largely as in Lemma 8. We first observe that $\mathbb{E}_{A_{\lambda}} = \mathbb{E}_{\lambda}$ since both $\Lambda$ and $\Lambda_1$ are of type $\alpha$. Using the commutation rules involving $E_{\lambda}$, see the paragraph prior to (65) and part (3) of Proposition 3, we get that (66) is equal to

$$x_{\lambda} b_{\mu} B_{\mu} E_{\lambda} d(t) E_{\lambda} d(s_1) B_{\mu}^* E_{\lambda} d(t)d(s_1) B_{\mu}^* c_{\mu}, y_{\lambda_1}, Y_{\lambda_1}. \tag{67}$$

By hypothesis, (66) is nonzero and so $\mathbb{E}_{A_{\lambda}} d(t) d(s_1) = \mathbb{E}_{\lambda}$ which implies that $A_{\lambda} d(t) = A_{\lambda} d(s_1)$. As in Lemma 8 we have decompositions $d(t) = d(t_0) u_1$ and $d(s_1) = d(s_1_0) u_1$ and so $A_{\lambda} u_1 = A_{\lambda} u_1$ and $u_1 = u_1$. Hence (67) becomes

$$x_{\lambda} b_{\mu} B_{\mu} E_{\lambda} d(t_0) E_{\lambda} d(s_1) E_{\lambda} d(t_0) E_{\lambda} d(s_1) B_{\mu}^* c_{\mu}, y_{\lambda_1}, Y_{\lambda_1} =$$

$$x_{\lambda} b_{\mu} B_{\mu} E_{\lambda} d(t_0) E_{\lambda} d(s_1) E_{\lambda} d(t_0) E_{\lambda} d(s_1) B_{\mu}^* c_{\mu}, y_{\lambda_1}, Y_{\lambda_1} =$$

$$x_{\lambda} b_{\mu} B_{\mu} E_{\lambda} d(t_0) E_{\lambda} d(s_1) E_{\lambda} d(t_0) E_{\lambda} d(s_1) B_{\mu}^* c_{\mu}, y_{\lambda_1}, Y_{\lambda_1}.$$ 

Using Lemma 7 repeatedly on a reduced expression for $u_1$ we may cancel $E_{\lambda} d(t_0) E_{\lambda} d(s_1)$ and so (68) becomes

$$x_{\lambda} b_{\mu} B_{\mu} E_{\lambda} d(t_0) E_{\lambda} d(s_1) E_{\lambda} d(t_0) E_{\lambda} d(s_1) B_{\mu}^* c_{\mu}, y_{\lambda_1}, Y_{\lambda_1} =$$

$$x_{\lambda} b_{\mu} B_{\mu} E_{\lambda} d(t_0) E_{\lambda} d(s_1) E_{\lambda} d(t_0) E_{\lambda} d(s_1) B_{\mu}^* c_{\mu}, y_{\lambda_1}, Y_{\lambda_1} =$$

where we used that $y_{\lambda_1}$ is of the initial kind. We now argue once again as in Lemma 8. We know that (69) is nonzero. Since $t_0$ and $s_1_0$ are of the initial kind we deduce from Lemma 8 that $ttt \leq s'_{1}$. But since (69) is nonzero we also get that $b_{\mu} B_{\mu} E_{\lambda} d(t_0) E_{\lambda} d(s_1) B_{\mu}^* c_{\mu}$ is nonzero. Moreover $v$ and $u_1$ are of the initial kind as well, this time by construction, and so we conclude that $ttt \leq u'_{1}$. This proves the Lemma.  

\section{Permutation Modules}

Suppose that $\lambda \in \text{Comp}_{\mu, n}$ is of type $\alpha$. Then we define the $c_{\alpha}^{\alpha}(q)$-permutation module $M(\lambda)$ as the following right ideal of $E_{\mu}^{\alpha}(q)$

$$M(\lambda) := \mathbb{E}_{A_{\lambda}} x_{\lambda} c_{\alpha}^{\alpha}(q) \subseteq c_{\alpha}^{\alpha}(q). \tag{70}$$

We also have a specialized version of the permutation module that we denote the same way

$$M(\lambda) := \mathbb{E}_{A_{\lambda}} x_{\lambda} c_{\alpha}^{\alpha}(q) \subseteq c_{\alpha}^{\alpha}(q). \tag{71}$$

$M(\lambda)$ is a generalization of the permutation modules that appear in the representation theory of the symmetric group, the Hecke algebra, the cyclotomic Hecke algebra, and so on, see [DipperJamMathas] and the references therein. For any $s \in \text{Std}(\lambda)$, we introduce $x_{s_{\lambda}} := \mathbb{E}_{A_{\lambda}} x_{\lambda} E_{\lambda d(s_{\lambda})} \in M(\lambda)$ and define $r_{i = r_{i}}$, $E_{i = E_{i}}$, $p_{i = p_{i}}$ via $s(r_{i}, c_{i}, p_{i}) = i$.

We then have the following Lemma which describes the $c_{\alpha}^{\alpha}(q)$-module (or $c_{\alpha}^{\alpha}(q)$-module) structure on $M(\lambda)$. 

13
Lemma 10  
(1) The set \( \{ x_s | s \in \text{RStd}(\Lambda) \} \) is a basis for \( M(\Lambda) \), over \( S \) or \( K \).

(2) Let \( s \in \text{RStd}(\Lambda) \). Then

\[
x_s g_i = \begin{cases} 
x_{s_{j_i}} & \text{if } p_i \neq p_{i+1} \\
q x_s & \text{if } p_i = p_{i+1} \text{ and } r_i = r_{i+1} \\
x_{s_{j_i}} & \text{if } p_i = p_{i+1} \text{ and } r_i < r_{i+1} \\
(q^{-1} - q)x_s + x_{s_{j_i}} & \text{if } p_i = p_{i+1} \text{ and } r_i > r_{i+1}
\end{cases}
\]

(72)

\[
x_s e_i = \begin{cases} 
x_s & \text{if } p_i = p_{i+1} + 1 \\
0 & \text{otherwise}
\end{cases}
\]

PROOF: Using (42), we get that the set \( \{ E_\Lambda x_A g_w | w \in \mathcal{S}_n \} \) generates \( M(\Lambda) \) over \( S \). On the other hand, for \( w \in \mathcal{S}_n \) we have the decomposition \( w = w_0 d(s) \) with \( w_0 \in \mathcal{S}_A \) and \( s \in \text{RStd}(\Lambda) \), and from this it follows via (48) that

\[
E_\Lambda x_A g_w = q^{f(w_0)} E_\Lambda x_A g_s = q^{f(w_0)} x_s.
\]

(73)

Hence in fact \( \{ x_s | s \in \text{RStd}(\Lambda) \} \) generates \( M(\Lambda) \) over \( S \). In the expansion of \( x_s \) in terms of \( E_\Lambda g_w \)'s we know that \( E_\Lambda g_d(s) \) appears exactly once, since the \( d(s) \)'s are distinguished coset representatives for \( \mathcal{S}_A \) in \( \mathcal{S}_n \), whereas the other terms are of the form \( E_\Lambda g_d(s) g_w \) for \( w \in \mathcal{S}_A \). These elements all belong to the basis for \( E^n_n(q) \) given in (42), and are all distinct, and so the \( x_s \)'s are linearly independent, proving (1).

We next prove (2), where we first consider \( x_s e_i \). Here we have that

\[
x_s e_i = E_\Lambda g_d(s) e_i = g_d(s) E_\Lambda d(s) e_i
\]

(74)

and so the formula for \( x_s e_i \) follows from Proposition 3. Let us then consider \( x_s g_i \) when \( p_i = p_{i+1} + 1 \). If \( r_i = r_{i+1} + 1 \) then we have \( d(s) s_i = s_j d(s) \) for some \( s_j \in \mathcal{S}_A \), and so \( x_s g_i = q x_s \) via (73). If \( r_i < r_{i+1} + 1 \) then \( d(s) s_i > d(s) \) and so \( x_s g_i = x_s s_i \) from the definition of \( x_s s_i \). If \( r_i > r_{i+1} + 1 \) then \( d(s) s_i < d(s) \). We can then choose a reduced expression for \( d(s) \) ending in \( s_i \) and so

\[
x_s g_i = E_\Lambda g_d(s) g_i = E_\Lambda g_d(s) g_i^2 = E_\Lambda g_d(s) g_i^2 \left( 1 + (q^{-1} - q) g_i \right) = x_s s_i + (q^{-1} - q) x_s
\]

(75)

where we used the quadratic relation (32) for the third equality, and the formula for \( x_s e_i \) for the last equality. Finally, if \( p_i \neq p_{i+1} + 1 \) then either \( d(s) s_i > d(s) \) or \( d(s) s_i < d(s) \) and we can repeat the previous arguments, with the only difference that the second term in (75) disappears. These arguments work over \( K \) as well and so the Lemma is proved.

There is another kind of permutation module that we shall need. Suppose that \( \Lambda = (\Lambda | \mu) \in \mathcal{L}_n(\alpha) \). Then we define the permutation module \( M(\Lambda) \) as the following right ideal of \( E^n_n(q) \)

\[
M(\Lambda) := E_\Lambda x_\Lambda b_\mu e^n_n(q) \subseteq E^n_n(q).
\]

(76)

Once again there is a specialized version of \( M(\Lambda) \) obtained by replacing \( E^n_n(q) \) with \( E^{n,k}(q) \). The only difference between \( M(\Lambda) \) and \( M(\Lambda) \) is the factor \( b_\mu \) and so we have \( M(\Lambda) \subseteq M(\Lambda) \), with the inclusion being strict in general. For \( M(\Lambda) \) we have the following Lemma.

Lemma 11  
Let \( \Lambda \in \mathcal{L}_n(\alpha) \). Let \( m_\varepsilon \) be as in (56). Then the set \( \{ m_\varepsilon | t \in \text{RStd}(\Lambda) \} \) is an \( S \)-basis (or a \( K \)-basis) for \( M(\Lambda) \).

PROOF. For \( w \in \mathcal{S}_n \) we have the decomposition \( w = w_0 d(t) \) with \( w_0 \in \mathcal{S}_A \) and \( t \in \text{RStd}(\Lambda) \) and hence, arguing as in (73), we get

\[
E_\Lambda x_\Lambda b_\mu g_w = E_\Lambda x_\Lambda b_\mu g_{w_0} g_t = E_\Lambda x_\Lambda g_{w_0} b_\mu g_t = q^{f(w_0)} E_\Lambda x_\Lambda b_\mu g_t.
\]

(77)
Hence \( \{ \mathcal{E}_A x_A b_{\mu} \mathfrak{B} t | t \in \mathbb{RStd}(\Lambda) \} \) generates \( M(\Lambda) \) over \( S \), but the appearing \( t \) may not be increasing in the sense of the definition of \( \mathbb{RStd}(\Lambda) \). On the other hand, if \( t \) is not increasing we can find \( B_u \in \mathfrak{S}_m \) such that the multitableau \( u \cdot t \in \mathbb{RStd}(\Lambda) \) defined by

\[
d(u \cdot t) = B_u d(t)
\]

(78)
becomes increasing: here \( \mathfrak{S}_m = \mathfrak{S}_{m_1} \times \ldots \times \mathfrak{S}_{m_s} \) is the subgroup of \( \mathfrak{S}_n \) introduced in (45). We then get

\[
\mathcal{E}_A x_A b_{\mu} \mathfrak{B} u \mathfrak{G} u \cdot t = \mathcal{E}_A x_A b_{\mu} \mathfrak{B} u \mathfrak{G} u d(t) = \mathcal{E}_A x_A b_{\mu} \mathfrak{B} u \mathfrak{G} B_u d(t) = \mathcal{E}_A x_A b_{\mu} \mathfrak{B} u \mathfrak{G} u t
\]

(79)
where we for the second equality used Lemma 54 of [EspRy]. Suppose \( \mu = (\mu^{(1)}, \ldots, \mu^{(s)}) \) and denote by \( \mathfrak{S}_{\Lambda, \mu} := \mathfrak{S}_{\mu^{(1)}} \times \ldots \times \mathfrak{S}_{\mu^{(s)}} \) the corresponding diagonal Young subgroup of \( \mathfrak{S}_n \). It gives rise to a diagonal decomposition of \( B_u^{-1} \) that is \( B_u^{-1} = (B_u)_0 d(u) \) where \( (B_u)_0 \in \mathfrak{S}_{\Lambda, \mu} \) and \( u \) is a row standard \( \mu \)-multitableau of the initial kind. Setting \( t = (u \cdot t | u) \in \mathcal{L}_n(\alpha) \), we can now rewrite (79) as

\[
\mathcal{E}_A x_A b_{\mu} \mathfrak{B} u \mathfrak{G} u \cdot t = \mathcal{E}_A x_A b_{\mu} \mathfrak{B} u \mathfrak{G} u d(t) = m_{n_{\mu}}.
\]

(80)
Thus we have proved that the set \( \{ m_{n_{\mu}} \} \) generates \( M(\Lambda) \). To show that it is linearly independent we argue as follows. Suppose that \( s = (s | u) \), that is

\[
m_{n_{\mu}} = \mathcal{E}_A x_A b_{\mu} \mathfrak{B} u \mathfrak{G} d(s).
\]

(81)
We know that \( u \) is row standard of the initial kind and therefore \( b_{\mu} \mathfrak{B} d(u) \) expands as a sum of terms of the form \( \mathfrak{B} u_0 d(u) \) where \( u_0 \in \mathfrak{S}_{\mu^{(1)}} \). But using Lemma 54 of [EspRy] we get that

\[
\mathfrak{B} u_0 d(u) \mathfrak{G} d(s) = \mathfrak{G} B_{u_0} d(u | d(s)).
\]

(82)
We now note that the multitableau \( s \) corresponding to \( B_{u_0} d(u | d(s)) \), in other words \( s \) given by \( d(s) = B_{u_0} d(u | d(s)) \), is a row standard \( \lambda \)-tableau since it is obtained from \( t \) by permuting components with equal \( \lambda^{(i)} \) and so \( m_{n_{\mu}} \) expands as a sum of terms of the form \( \mathcal{E}_A x_A \mathfrak{B} u_0 d(u_0 | d(s)) \) where \( u_0 \in \mathfrak{S}_{\mu^{(1)}} \) and \( x \in \mathfrak{S}_{\lambda} \). These expansions are distinct for distinct pairs \( (s | u) \) and so the set \( \{ m_{n_{\mu}} \} \) is linearly independent, as claimed. Hence it is an \( S \)-basis, and therefore also a \( \Lambda \)-basis, for \( M(\Lambda) \).

\[\Box\]

We next give a description of the action of \( \mathcal{E}_n^a(q) \) on \( \{ m_{n_{\mu}} \} \). We first introduce the following useful notation. Let \( s \in \mathbb{RStd}(\Lambda) \) and let let \( s_i \in \Sigma_n \subseteq \mathfrak{S}_n \) be a simple transposition. Then we define \( s \cdot s_i \) via

\[
s \cdot s_i := \begin{cases} s & \text{if } r_i = r_{i+1} \\ s s_i & \text{if } r_i \neq r_{i+1}. \end{cases}
\]

(83)
This extends to an action of \( \mathfrak{S}_n \) on \( \mathbb{RStd}(\Lambda) \) that we denote \( (s, w) \rightarrow s \cdot w \). We extend this further to \( \mathcal{L}_n(\alpha) \) as follows. Let \( s = (s | u) \in \mathcal{L}_n(\alpha) \) and suppose that \( Shape(t^{(p_i)}) = Shape(t^{(p_{i+1})}) \). Then we define a transposition \( t_i \in \mathfrak{S}_{\Lambda} \) via \( t_i = (p_i, p_{i+1}) \). For \( s \in \mathbb{RStd}(\Lambda) \) we set \( m_i := \min(Shape(t^{(p_i)})) \). Using the notation \( t_i \cdot s \) introduced in (78) we then define

\[
s \cdot s_i := \begin{cases} (s \cdot s_i | u) & \text{if } p_i = p_{i+1} \text{ or otherwise} \\ (s \cdot s_i | u) & \text{if } Shape(t^{(p_i)}) \neq Shape(t^{(p_{i+1})}) \\ (s \cdot s_i | u) & \text{if } Shape(t^{(p_i)}) = Shape(t^{(p_{i+1})}) \text{ and } \{ m_i, m_{i+1} \} \neq \{ i, i+1 \} \\ (t_i \cdot (s \cdot s_i) | u \cdot t_i) & \text{if } Shape(t^{(p_i)}) = Shape(t^{(p_{i+1})}) \text{ and } \{ m_i, m_{i+1} \} = \{ i, i+1 \}. \end{cases}
\]

(84)
One checks that these formulas extend to an action of $\mathfrak{S}_n$ on $\text{RStd}(\Lambda)$. They have their origin in the straightening procedure that carries $s_i$ to an increasing row standard tableaux. Here are two examples, corresponding to the last case in (84):

\[
\begin{pmatrix}
(1,5,3,2) \\
(1,3,4,2)
\end{pmatrix}
\cdot s_1 = \begin{pmatrix}
(1,3,5,2) \\
(1,2,4,3)
\end{pmatrix}
\]

\[
\begin{pmatrix}
(1,5,3,2) \\
(1,3,4,2)
\end{pmatrix}
\cdot s_2 = \begin{pmatrix}
(1,3,5,2) \\
(1,2,4,3)
\end{pmatrix}
\]

Lemma 12 Suppose that $s = (s \mid u) \in \text{RStd}(\Lambda)$. Then

\[
(1) \quad m_{s_i} g_i = \begin{cases}
qm_{s_i} & \text{if } p_i = p_{i+1} \text{ and } r_i = r_{i+1} \\
qm_{s_i} + m_{s_i} & \text{if } p_i = p_{i+1} \text{ and } r_i < r_{i+1} \\
m_{s_i} & \text{if } p_i = p_{i+1} \text{ and } r_i > r_{i+1} \\
0 & \text{if } p_i \neq p_{i+1}.
\end{cases}
\]

(2) \quad m_{s_i} e_i = \begin{cases}
m_{s_i} & \text{if } p_i = p_{i+1} \\
0 & \text{if } p_i \neq p_{i+1}.
\end{cases}

PROOF. The proof of (2) is analogous to the corresponding statement for $x_s e_i$ in Lemma 10, in detail

\[
m_{s_i} e_i = \sum \lambda \chi \mu b_{\mu} \mu u g_{d(s)} e_i = x_{s_i} b_{\mu} \mu u g_{d(s)} e_i
\]

and we conclude via Proposition 3.

We therefore consider (1) where we first focus on the case $p_i = p_{i+1}$. If $r_i = r_{i+1}$ we have $d(s) s_i = s_j d(s)$ for some $s_j \in \mathfrak{S}_\Lambda$, and so

\[
m_{s_i} g_i = \sum \lambda \chi \mu b_{\mu} \mu u g_{d(s)} g_i = \sum \lambda \chi \mu b_{\mu} \mu u g_{d(s)}.
\]

But $s_j$ is of the form $s_j = d(\tau)$ where $\tau$ is a multitableau of the initial kind and so (88) becomes

\[
x_{s_i} g_i b_{\mu} \mu u g_{d(s)} = q x_{s_i} b_{\mu} \mu u g_{d(s)} = q m_{s_i},
\]

as claimed. The two other cases when $p_i = p_{i+1}$, that is $r_i < r_{i+1}$ or $r_i < r_{i+1}$, are proved the same way as in Lemma 10.

We then finally consider the case $p_i < p_{i+1}$. Arguing as in Lemma 10 we have here

\[
m_{s_i} g_i = \sum \lambda \chi \mu b_{\mu} \mu u g_{d(s_i)} g_i = \sum \lambda \chi \mu b_{\mu} \mu u g_{d(s_i)}
\]

which is equal to $m_{s_i}$ in all cases except $[m_i, m_{i+1}] = [i, i+1]$ where $\tau$ is not increasing. But in that case, arguing as we did for (79) and (80) we can rewrite (90) as follows

\[
\sum \lambda \chi \mu b_{\mu} \mu u g_{d(s_i)} = \sum \lambda \chi \mu b_{\mu} \mu u g_{d(s_i)} = \sum \lambda \chi \mu b_{\mu} \mu u g_{d(s_i)} = m_{s_i}.
\]

Given the basis for $M(\Lambda)$, we now introduce a bilinear form $(\cdot, \cdot)_{\Lambda}$ on $M(\Lambda)$ as follows

\[
(m_s, m_t)_{\Lambda} := \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise.}
\end{cases}
\]

It is a generalization of the bilinear forms on the classical permutation modules, see for example [DipperFramMathas].

Lemma 13 The following statements hold.

(1) $(\cdot, \cdot)_{\Lambda}$ is symmetric and nondegenerate.

(2) $(\cdot, \cdot)_{\Lambda}$ is invariant in the sense that for all $m, m_1 \in M(\Lambda)$ and $a \in \mathcal{E}^\alpha_n(q)$ or $a \in \mathcal{E}^\alpha_n(q)$ we have that $(m a, m_1)_{\Lambda} = (m, m_1 a^*)_{\Lambda}$.
The number of cases is reduced to the half by the symmetry of $(\cdot, \cdot)_\Lambda$. Moreover if $p^i_1 = p^i_{i+1}$ and $p^i_1 \neq p^i_{i+1}$ then it follows immediately from Lemma 12 that $(m_\Lambda e_i, m_\Lambda) = (m_\Lambda, m_\Lambda)_{i+1} = 0$ and that $(m_\Lambda g_i, m_\Lambda) = (m_\Lambda, m_\Lambda g_i)_{i+1} = 0$ and so we only need consider the cases where $p^i_1 = p^i_{i+1}$ and $p^i_1 = p^i_{i+1}$ or $p^i_1 \neq p^i_{i+1}$ and $p^i_1 \neq p^i_{i+1}$. In the rest of the proof we shall use the formulas of Lemma 12 repeatedly.

Let us consider $a = g_i$ and suppose that $p^i_1 = p^i_{i+1}$ and $p^i_1 = p^i_{i+1}$. If $p^i_1 < p^i_{i+1}$ and $p^i_1 < p^i_{i+1}$ we have $(m_\Lambda g_i, m_\Lambda) = (m_\Lambda, m_\Lambda)_{i+1}$ whereas $(m_\Lambda, m_\Lambda g_i)_{i+1} = (m_\Lambda, m_\Lambda g_i)_{i+1}$, where $\delta$ is the Kronecker delta. If $p^i_1 < p^i_{i+1}$ and $p^i_1 > p^i_{i+1}$ we have $(m_\Lambda g_i, m_\Lambda) = (m_\Lambda, m_\Lambda)_{i}$ whereas $(m_\Lambda, m_\Lambda g_i)_{i} = (m_\Lambda, m_\Lambda g_i)_{i}$.

Again, the two sides coincide. If $p^i_1 > p^i_{i+1}$ and $p^i_1 > p^i_{i+1}$ we have $(m_\Lambda g_i, m_\Lambda) = (m_\Lambda, m_\Lambda)_{i} + (q - q^{-1})m_\Lambda$ and similarly $(m_\Lambda, m_\Lambda g_i)_{i} = (m_\Lambda, m_\Lambda g_i)_{i}$, and so the two sides coincide.

Finally we consider the case $a = e_i$ and must check that $(m_\Lambda e_i, m_\Lambda) = (m_\Lambda, m_\Lambda e_i)_{i}$. If $p^i_1 = p^i_{i+1}$ and $p^i_1 \neq p^i_{i+1}$ we have $m_\Lambda e_i = m_\Lambda$ and $m_\Lambda e_i = m_\Lambda$, and otherwise, if $p^i_1 \neq p^i_{i+1}$ and $p^i_1 \neq p^i_{i+1}$, we have $m_\Lambda e_i = 0$ and $m_\Lambda e_i = 0$. Hence the two sides also coincide in this case. This finishes the proof of the Lemma.

We have the following crucial lemma.

**Lemma 14** Suppose that $\Lambda = (\Lambda | \mu)$ and that $s, t \in \text{Std}(\Lambda)$. Then we have

$$(m_\Lambda, n_\Lambda)_{g_i} \neq 0 \text{ or equivalently } (m_\Lambda, m_\Lambda n_\Lambda)_{g_i} \neq 0.$$  

(93)

In particular $m_\Lambda n_\Lambda_{g_i} \neq 0$ and $m_\Lambda n_\Lambda_{g_i} \neq 0$.

**Proof.** Suppose that $s = (u | v)$ and $t = (u | v)$. Then the same chain of equalities that took us from (66) to (69) in Lemma 9, but postmultiplied with $d(u) d(v) \cdot d(u) \cdot d(v)$, gives us

$$m_\Lambda n_\Lambda_{g_i} = E^\Lambda \cdot A_{\Lambda} y^{\mu} \mu b_{\mu} \cdot d(v) \cdot d(u) \cdot d(v) \cdot d(u) \cdot d(v) =$$

$$E^\Lambda \cdot A_{\Lambda} \cdot d(v) \cdot d(u) \cdot d(v) \cdot d(u) \cdot d(v)$$

where we used (8) for the last equality: note that $v_0$ and $v$ are both of the initial kind. Using the invariance property of the previous Lemma 13 we get from this that

$$(m_\Lambda, n_\Lambda)_{g_i} = (m_\Lambda, n_\Lambda)_{g_i} = (m_\Lambda, n_\Lambda)_{g_i} = (m_\Lambda, n_\Lambda)_{g_i} =$$

$$(m_\Lambda, n_\Lambda)_{g_i} = (m_\Lambda, n_\Lambda)_{g_i} = (m_\Lambda, n_\Lambda)_{g_i} = (m_\Lambda, n_\Lambda)_{g_i} =$$

$$(m_\Lambda, n_\Lambda)_{g_i} = (m_\Lambda, n_\Lambda)_{g_i} = (m_\Lambda, n_\Lambda)_{g_i} = (m_\Lambda, n_\Lambda)_{g_i} =$$

(95)

In view of (23), and the remarks preceding (23), we have decompositions $d(u') = d(u') d(t)$ and $d(u') = d(u') d(t)$ and hence we can use Lemma 7 repeatedly on a reduced expression of $d(u)$ to rewrite $E^\Lambda \cdot d(u') \cdot d(u') = E^\Lambda \cdot d(u') \cdot d(u')$ and (95) becomes

$$E^\Lambda \cdot A_{\Lambda} \cdot d(u) \cdot d(u) = E^\Lambda \cdot A_{\Lambda} \cdot d(u) \cdot d(u) = E^\Lambda \cdot d(u) \cdot d(u)$$

(96)

Note that (96) does not depend on $s$ and $t$, only on $\Lambda$. The expressions in (96) only involve multitableaux of the initial kind and so the Lemma is reduced to the corresponding Hecke algebra statement. To be precise, for $\Lambda \in \text{Par}_\mu$, each element $w$ of the double coset $G_\Lambda \cdot w \cdot G_\Lambda$ has a unique decomposition $w = w_1 w_2 w_2$ where $w_1 \in G_\Lambda$ and $w_2 \in G_\Lambda$, and moreover $\ell(w) = \ell(w_1) + \ell(w_2) + \ell(w_2)$, see Lemma 1.6 of [DipperJames].
Applying this on $\mathcal{S}_A$ and $\mathcal{S}_\mu$ we conclude that $m_{s,\alpha}$ occurs exactly once in the expansion of $m_{s,\alpha}y_A^c\mu^r$, corresponding to the '1'-term in both $y_A^c$ and $\mu^r$, and so (96) is equal to 1. The Lemma is proved.

5. The tensor space $V^\otimes n$ module for $E_n(q)$

In this section we realize the tensor space $V^\otimes n$ module for $E_n(q)$, introduced in [Ry], as a sum of the permutation modules $\mathcal{M}(\Lambda)$. When the dimension of $V$ is sufficiently large, it is known that $V^\otimes n$ is a faithful $E_n(q)$-module, but here we are interested in the general case where $V^\otimes n$ may not be faithful. Using the results from the previous section we determine the annihilator ideal in $E_n(q)$ of $V^\otimes n$. In turns out that is has a nice description in terms of the dual cellular basis $\{n_{s\alpha}\}$. It is a main point of our constructions and proofs that they work for arbitrary $\mathcal{K}$.

Let $V$ be the free $S$-module with basis

$$B := \{v_i^j | 1 \leq i \leq N, 1 \leq s \leq r\}$$

(97)

that is $V$ is of dimension $rN$. We then define linear maps $E, G \in \text{End}_S(V^{\otimes 2})$ via

$$(v_i^j \otimes v_j^i)E := \begin{cases} v_i^j \otimes v_j^i & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$$

(98)

and

$$(v_i^j \otimes v_j^i)G := \begin{cases} v_i^j \otimes v_j^i & \text{if } s \neq t \\ qv_i^j \otimes v_j^i & \text{if } s = t, i = j \\ v_i^j \otimes v_i^j & \text{if } s = t, i < j \\ (q - q^{-1})v_i^j \otimes v_j^i + v_j^i \otimes v_i^j & \text{if } s = t, i > j. \end{cases}$$

(99)

We extend them to linear maps $E_i$ and $G_i$ acting in the tensor space $V^\otimes n$ by letting $E$ and $G$ act in the $i$'th and $i + 1$'st factors. The following is Theorem 1 of [Ry]

**Theorem 15** The rules $e_i \mapsto E_i$ and $g_i \mapsto G_i$ endow $V^\otimes n$ with the structure of an $E_n(q)$-module.

In the case of $E_n^C(q)$ it was proved in [Ry] that the specialized tensor module $V^{C, \otimes n}$ is faithful when $r, N \geq n$. In [Espby] this faithfulness statement was generalized to $V^\otimes n$ itself, but still only for $r, N \geq n$; in fact it does not hold otherwise, as we shall see.

Let $\text{seq}_N$ be the set of sequences $i = (i_1, \ldots, i_n)$ of integers where each $i_j$ belongs to $\{1, \ldots, N\}$, and let similarly $\text{seq}_r$ be the set of sequences $\underline{s} = (s_1, \ldots, s_n)$ where each $s_j$ belongs to $\{1, \ldots, r\}$. For such $i$ and $\underline{s}$ we define

$$v_i^{\underline{s}} := v_{i_1}^{s_1} \otimes \cdots \otimes v_{i_n}^{s_n} \in V^\otimes n.$$  

(100)

Then the set $\{v_i^{\underline{s}} | i \in \text{seq}_N, \underline{s} \in \text{seq}_r\}$ is a basis for $V^\otimes n$.

To $\underline{s} \in \text{seq}_r$ we associate sets $I_j := \{i | s_i = j\}$, for $j = 1, \ldots, r$. The $I_j$'s may be empty, but leaving out the empty $I_j$'s we obtain a set partition in $\mathcal{SP}_n$, that we denote $A_{\underline{s}}$, and we say that $\underline{s}$ is of type $\alpha$ if $A_{\underline{s}}$ is of type $\alpha$. For example, if $r = 4, n = 13$ and $\underline{s} = (1, 2, 2, 1, 2, 2, 2, 1, 2, 4, 1, 2, 2)$ then $A_{\underline{s}} = \{(1, 4, 8, 11), (2, 3, 5, 6, 7, 9, 12, 13), (10)\}$, and so $\underline{s}$ is of type $(8, 4, 1)$.

Recall now the idempotent decompositions $\sum_{A \in \mathcal{SP}_n} E_A = 1$ and $\sum_{a \in \mathcal{P}_n} E_a = 1$. They give rise to decompositions of $V^\otimes n$

$$V^\otimes n = \bigoplus_{A \in \mathcal{SP}_n} E_AV^\otimes n, \quad V^\otimes n = \bigoplus_{a \in \mathcal{P}_n} E_a V^\otimes n.$$  

(101)

Note that $E_A V^\otimes n$ is canonically an $E_n^C(q)$-module.

We have the following Lemma which gives precise descriptions of $E_AV^\otimes n$ and $E_a V^\otimes n$. 

18
Lemma 16  With the above notations the following statements hold

\begin{align*}
(1) \quad E_A v^j_s &= \begin{cases} 
v^j_s & \text{if } A \subseteq A_s \\
0 & \text{otherwise}
\end{cases} \\
(2) \quad E_A V^\otimes_n &= \text{span}_S \{v^j_s | A_s = A \} \\
(3) \quad E_{\alpha} V^\otimes_n &= \text{span}_S \{v^j_s | \underline{s} \text{ is of type } \alpha \}.
\end{align*}

Proof. Statement (1) is immediate from the definition of $E_A$ given in (36), so let us prove (2). Suppose that $A = \{i_1, i_2, \ldots, i_k\}$. Then by construction, the vector $v^j_s$ belongs to the right hand side of (2) exactly when any two terms $s_i$ and $s_j$ of $\underline{s}$ coincide iff $i$ and $j$ are in the same block $I_j$ of $A$. For example, if $r = 3$ and $A = [(1,2), (3,4), (5,6,7)]$ then the $\underline{s}$ satisfying $A_s = A$ are the following ones

\[
(1,1,2,2,3,3,3), (1,1,2,1,3,3,3), (3,3,1,1,2,2,2), (1,1,2,2,3,3,3), \\
(1,1,3,2,2,2), (2,2,3,3,1,1,1).
\]

(102)

Hence, in view of (39), that is

\[
E_A = \sum_{A \subseteq B} \mu(A, B) E_B,
\]

(103)

together with $\mu(A, A) = 1$, we conclude from (1) that $E_A v^j_s = v^j_s$ whenever $v^j_s$ belongs to the right hand side of (2), which proves the inclusion $\subseteq$ of (2). To prove $\supseteq$ we assume that $v \in E_A V^\otimes_n$ and consider its expansion $v = \sum a^j_i v^j_s$ with coefficients $a^j_i \in S$. By (1) and (103) we have that $A \subseteq A_s$ whenever $a^j_i \neq 0$ and hence if $v$ does not belong to the right hand side of (2), there are $i_0 \in \text{seq}_N$ and $s_0 \in \text{seq}_r$ such that $a^j_i \neq 0$ and $A \subseteq A_s$. We have that $E_A v^j_s = 0$ and hence $E_A v = 0$. By (1) once again. On the other hand, by part (3) of Proposition 3 we also have that $E_A v = 0$, and this gives the desired contradiction. Finally, (3) follows from (2) and the definitions.

For $N$ any natural number we define $\text{Comp}_{r,n,\leq N} \subseteq \text{Comp}_{r,n}$ (resp. $\text{Par}_{r,n,\leq N} \subseteq \text{Par}_{r,n}$) as the set of multicompositions (resp. multipartitions) $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ such that each $\lambda^{(i)}$ is of length less than $N$.

Suppose now that $\underline{\lambda} \in \text{Comp}_{r,n,\leq N}$ and that $\underline{t} \in \text{RStd}(\lambda)$. Then we define $\underline{i}^t = (i_1, \ldots, i_n) \in \text{seq}_N$ by letting $i_j$ be the row number of the node in $\underline{t}$ containing $j$, and similarly we define $\underline{s}^t = (s_1, \ldots, s_n) \in \text{seq}_r$ by letting $s_j$ be the component number of the node in $\underline{t}$ containing $j$. For example if

\[
\underline{t} = \begin{pmatrix} 
3 & 1 & 1 & 7 & 1 & 2 & 12 & 13 & 18 & 21 & 23 & 27 \\
1 & 16 & 19 & 10 & 11 & 14 & 15 & 16 & 19 & 23 & 25 & 31
\end{pmatrix}
\]

(104)

then $\underline{i}^t = (1, 2, 1, 2, 1, 1, 2, 1, 1, 2, 2, 1, 2, 3, 1, 2, 3, 1, 1, 2, 2, 2, 3, 1, 1, 2, 2, 2)$ and $\underline{s}^t = (2, 10, 1, 1, 10, 3, 4, 2, 3, 4, 5, 5, 5, 6, 6, 9, 7, 7, 7, 7, 8, 8, 8, 9)$. For $\underline{t} \in \text{RStd}(\lambda)$ we write $v_\underline{t}$ for $v^\underline{t}_\underline{s}$.

The following Theorem relates the tensor space module $V^\otimes_n$ with the permutation modules $M(\underline{\lambda})$.

Theorem 17  Let $V$ be the free $S$-module (or $K$-vector space) of dimension $rN$ with basis as in (97). Suppose that $\underline{\lambda} \in \text{Comp}_{r,n,\leq N}$ is of type $\alpha \in \text{Par}_r$ and that $\alpha$ is of length $r$.

(1) The $S$-linear map $\iota_\lambda : M(\underline{\lambda}) \rightarrow V^\otimes_n$ given by $\iota_\lambda(x_\underline{t}) = v_\underline{t}$ for $\underline{t} \in \text{RStd}(\lambda)$, is an embedding of $E_n^\alpha(q)$-modules.
(2) Identifying $M(\lambda)$ with $\varphi(\mathcal{S}(\lambda))$ we have $M(\lambda) \subseteq \mathbb{E}_a V^{\otimes n}$ and a direct sum decomposition of $E_n^a(q)$-modules

$$ E_n^a V^{\otimes n} \cong \bigoplus_{\lambda \in \text{Comp}_{r,n,S N}} M(\lambda). \quad (105) $$

**Proof.** (1) follows from Lemma 10 together with (98) and (99) and the definitions and (2) follows from (1) and Lemma 16. \hfill \Box

For any natural number $N$ we define $L_{r,N}(\alpha)$ as the subset of $L_r(\alpha)$ consisting of the pairs $(\lambda | \mu)$ such that the components $\lambda^{(k)}$ of $\lambda$ all have less than $N$ columns, and we set $L_{r,N}(\alpha) := L_r(\alpha) \setminus L_{r,N}(\alpha)$. We can now give the main Theorem of our paper. In the Hecke algebra case the argument of the proof was discovered in [Härlrich], although our presentation of the argument, even in the Hecke algebra case, differs slightly from the one used by Härlrich.

**Theorem 18** Let $V$ be as in (97) with basis $B$ and suppose that $a \in Par_r$ is of length $r$. Let $I \subseteq E_n^a(q)$ be the annihilator ideal of the action of $E_n^a(q)$ in $E_n^a V^{\otimes n}$. Then $I$ is free over $S$ with basis

$$ \left\{ n \in S | s,t \in \text{Std}(\Lambda), \Lambda \in L_{r,N}(\alpha) \right\}. \quad \text{(106)} $$

A similar statement holds over $K$.

**Proof.** Let us focus on the $S$-case, since the $K$-case is done the same way. Let $I_1 \subseteq E_n^a(q)$ be the $S$-span of $\left\{ n \in S | s,t \in \text{Std}(\Lambda), \Lambda \in L_{r,N}(\alpha) \right\}$. We first observe that $I_1$ is a two-sided ideal in $E_n^a(q)$, as one sees from the $n_{s,t}$-version of Lemma 56 of [EspRy]: the number of columns of the components never decreases under the straightening procedure of that Lemma. We first prove that $I_1 \subseteq I$, that is for a basis element $n_{s,t}$ of $I_1$ we prove that $M(\Lambda)n_{s,t} = 0$ for any $M(\Lambda)$ appearing in (105), or equivalently that $M(\Lambda)n_{s,t} = 0$. But since $I_1$ is an ideal, in order to show $M(\Lambda)n_{s,t} = 0$ it is enough to show $\lambda n_{s,t} = 0$. Note that $\lambda$ may not be a multipartition, only a multicomposition, but in any case $\lambda^1$ is of the initial kind and so we can use Lemma 4.4 of [Murphy95] to rewrite $\lambda^1$ as a linear combination of $s_{\lambda^1}$, where $s_{\lambda^1}$ and $s_{\lambda^1}$ are $\lambda^1$-multitableaux of the initial kind, for $\lambda_1$, an $r$-multipartition such that $\lambda_1 \supseteq \lambda$. In general, for any $\alpha \in S_r$ there is an $E_n^a(q)$-isomorphism $M(\alpha) \cong M(\lambda)$ where $\lambda$ is the $r$-multipartition obtained from $\lambda$ by permuting the components, as one checks using Lemma 10, and so we may assume that $\lambda_1$ is increasing. Moreover, since $\lambda_1 \supseteq \lambda$ and $\lambda \in \text{Comp}_{r,N}(\alpha)$ we have $\lambda_1 \in Par_{r,N}$. We now observe that $s_{\lambda^1}$ is also of the form $m_{s_{\lambda^1}}$, where $s_{\lambda^1} = \{s_{\lambda_1} | u_{\lambda_1}\}$ and $t_{\lambda_1} = \{t_{\lambda_1} | v_{\lambda_1}\}$ and where the components of $\text{Shape}(u_{\lambda_1})$ and $\text{Shape}(v_{\lambda_1})$ are all one-column partitions, in particular $d(u_{\lambda_1}) = d(v_{\lambda_1}) = 1$. But from Lemma 9 we know that $m_{s_{\lambda^1}} n_{s,t} \neq 0$ implies $s_{\lambda^1} \subseteq s'$ which is impossible since all components of $\lambda_1$ have less then $N$ rows, whereas at least one component of $\text{Shape}(s')$ has more then $N$ rows; here $s = \{s | u\}$ for some $u$. Hence $m_{s_{\lambda^1}} n_{s,t} = 0$ and so the inclusion $I_1 \subseteq I$ is proved.

Suppose now that the inclusion $I_1 \subseteq I$ is strict. Then there exists $n \in I \setminus I_1$. Our plan is then to construct a $\lambda \in \text{Comp}_{r,N}(\alpha)$ such that $|A_\lambda| = a$ and $M(\Lambda)n = 0$, which is a contradiction since $M(\Lambda)$ is a direct summand of $E_n^a V^{\otimes n}$ by Lemma 17. We shall do so by showing that there exists a $\Lambda := (\lambda | \mu) \in L_r(\alpha)$ with $\lambda \in \text{Comp}_{r,N}(\alpha)$, such that $M(\Lambda)n \neq 0$. This gives the desired contradiction since $M(\Lambda) \subseteq M(\alpha)$.

In order to do so we consider the expansion $n = \sum_{s,t \in \text{Std}(\Lambda), \Lambda \in L_r(\alpha)} \lambda_{s,t} n_{s,t}$. We may assume that $\text{Shape}(s) \in L_{r,N}(\alpha)$ for all $(s,t)$ occurring in the expansion of $n$: otherwise if $\text{Shape}(s) \in L_{r,N}(\alpha)$ for all $(s,t)$ occurring in the expansion of $n$ and still get an element in $I_1 \subseteq I$, by the first part of the proof. We now choose $(s_{\min}, t_{\min})$ with $\lambda_{s_{\min}, t_{\min}} \neq 0$ and minimal in the sense that if $s \leq s_{\min}, t \leq t_{\min}$ and $\lambda_{s,t} \neq 0$ then $(s,t) = (s_{\min}, t_{\min})$. Let $s_{\min} = \{s_{\min} | u_{\min}\}$ and let $\Lambda_{\min} = \text{Shape}(s_{\min}) = (\lambda_{\min} |
\( \mu_{min} \). We then have \( \Lambda'_{min} \in \text{Comp}_{\pi,n,N} \) of type \( \alpha \) and so we may consider the permutation module \( M(\Lambda'_{min}) \). We are interested in the elements \( m_{\mu_{min}} \) and \( m_{\nu_{min}} \) of \( M(\Lambda''_{min}) \). Using the minimality of \( (\mu_{min}, \nu_{min}) \), together with Lemma 9 and the invariance of the bilinear form, we get

\[
(m_{\mu'_{min}}, n_{\mu'_{min}}, m_{\nu'_{min}})' = (m_{\mu'_{min}}, n_{\mu'_{min}}, \mu_{min}, m_{\nu'_{min}})' \Lambda'
\]

which is nonzero by Lemma 14. Hence \( M(\Lambda''_{min}) \neq 0 \), as needed. This proves the Theorem.

In view of the above Theorem, it is natural to consider the following quotient algebra

\[
\mathcal{E}T L_{n,N}(q) := \mathcal{E}n(q)/I
\]

where \( I \) is the ideal from (106). It is a generalization of the ‘generalized Temperley-Lieb’ algebras introduced in [Härterich]. For \( K \) a field, there is also a specialized version of \( \mathcal{E}T L_{n,N}(q) \) that we denote \( \mathcal{E}T L_{n,N}(q) \). By construction \( \mathcal{E}T L_{n,N}(q) \) is the largest quotient of \( \mathcal{E}n(q) \) acting faithfully on \( E_{n}V^{\otimes n} \). We have the following Corollary to Theorem 18.

**Corollary 19** \( \mathcal{E}T L_{n,N}(q) \) is free over \( S \) with basis \( \{ n_{\mu_{\pi}} | s, \tau \in \text{Std}(\Lambda), \Lambda \in \mathcal{L}_{n,N}(\alpha) \} \). A similar statement holds for \( \mathcal{E}T L_{n,N}(q) \).

The following Corollary shows that \( \mathcal{E}T L_{n,N}(q) \) is also a generalization of the partition Temperley-Lieb algebra \( \mathcal{P}T L_{n,N}(q) \) that was introduced in [Juyumaya].

**Corollary 20** Let \( St_{i} \in \mathcal{E}n(q) \) be the \( i \)th Steinberg element given by

\[
St_{i} := -q^{-3} g_{i} g_{i+1} g_{i} + q^{-2} g_{i} g_{i+1} + q^{-2} g_{i} g_{i+1} - q^{-1} g_{i} - q^{-1} g_{i+1} + 1
\]

and define \( \mathcal{P}T L_{n,N}(q) \) as the quotient algebra \( \mathcal{P}T L_{n,N}(q) := \mathcal{E}n(q)/J_{I} \) where \( J_{I} \) is the two-sided ideal of \( \mathcal{E}n(q) \) generated by \( e_{i}e_{i+1}St_{i} \), for some \( i \). Then \( \mathcal{P}T L_{n,N}(q) \) does not depend on the choice of \( i \) and, moreover, \( \mathcal{P}T L_{n,N}(q) = \mathcal{E}T L_{n,N}(q) \). Similar statements hold over \( K \).

**Proof.** In Proposition 4.5 of [Juyumaya] it is shown that the \( e_{i}e_{i+1}St_{i} \)'s are all conjugate to each other in \( \mathcal{E}n(q) \) and so the \( J_{I} \)'s are equal ideals, which proves the first statement. For simplicity we write \( J = J_{I} \). We must then show that \( J = I \) where \( I \) is the ideal described in Theorem 18. The inclusion \( J \subseteq I \) follows from the fact that \( e_{n-1}e_{n}St_{n-1} = n_{\pi_{\mu_{\pi}}} \lambda \) where \( \lambda = (\alpha | \mu) \) is chosen such that all components of \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) are one-column partitions, except \( \lambda^{(r)} \) which is of the form \( (3,1^{s}) \) for some \( s \), and such that all components of \( \mu \) are one-column partitions. For the other inclusion \( J \supseteq I \) we consider a basis element \( n_{\pi_{\mu}} \) for \( I \), as given in (106). Letting \( \text{Shape}(s) = (\alpha | \mu) \), there is a component of \( \alpha \) with more than three columns and so there is an \( i \) such that \( e_{i}e_{i+1}St_{i} \) is a factor of \( n_{\pi_{\mu}} \). This shows \( J \supseteq I \) and concludes the proof of the Corollary.

**Remark.** Using Corollary 19 we can determine the dimension of \( \mathcal{E}T L_{n,N}(q) \), and hence also of \( \mathcal{P}T L_{n,N}(q) := \bigoplus_{\alpha \in \mathcal{P}ar_{n}} \mathcal{P}T L_{n,N}(q) \), since it is the cardinality of

\[
\bigcup_{\alpha \in \mathcal{P}ar_{n}} \{ n_{\mu_{\pi}} | s, \tau \in \text{Std}(\Lambda), \Lambda \in \mathcal{L}_{n,\pi_{2}}(\alpha) \}.
\]

We have for example

\[
\dim \mathcal{P}T L_{3}(q) = 29, \quad \dim \mathcal{P}T L_{4}(q) = 334, \quad \dim \mathcal{P}T L_{5}(q) = 5512.
\]

These values are confirmed by Juyumaya and Papi’s MAGMA calculations and by Espinoza’s preprint, [Esp], that contains a closed general formula for \( \dim \mathcal{P}T L_{n}(q) \).
Remark. In Conjecture 6.5 of [Juyumaya] there is a conjectural basis for $PTL_n(q)$ which would imply that $\dim PTL_n(q) = b_n c_n$ where $c_n = \frac{1}{n+1} \binom{2n}{n}$ is the Catalan number. It follows from (110) that Juyumaya’s conjectural basis is wrong, as his and Papi’s calculations had in fact already shown.

Remark. It would be interesting to use (110) to study Conjecture 7.3 from [Juyumaya] on the existence of a Markov trace on $PTL_n(q)$. One would expect a diagrammatic calculus associated with $PTL_n(q)$ that might be of relevance for this.

REFERENCES

[Arcis-Juyu] D. Arcis, J. Juyumaya, Tied Monoids, Semigroup Forum, 103(2) (2021), 356-394.

[Aicar-Juyu1] F. Aicardi, J. Juyumaya, An algebra involving braids and ties, Preprint ICTP IC/2000/179, Trieste.

[Aicar-Juyu2] F. Aicardi, J. Juyumaya, Markov trace on the algebra of braids and ties, Moscow Mathematical Journal 16(3) (2016), 397-431.

[Aicar-Juyu3] F. Aicardi, J. Juyumaya, Two parameters bt-algebra and invariants for links and tied links, Arnold Math. Journal 6(1) (2020), 131-148.

[Aicar-Juyu4] F. Aicardi, J. Juyumaya, Tied links and invariants for singular links, Advances in Mathematics 381 (2021), 40 pages.

[Ban] E. O. Banjo, The Generic Representation Theory of the Juyumaya Algebra of Braids and Ties, Algebras and Representation Theory 16(5) (2013), 1385-1395.

[Flo] M. Flores, A braids and ties algebra of type B, Journal of Pure and Applied Algebra 224(1) (2020), 1-32.

[FloJuyuLam] M. Flores, J. Juyumaya, S. Lambropoulou, A Framization of the Hecke algebra of Type B, Journal of Pure and Applied Algebra 222(1) (2018), 778-806.

[ChlouPou] M. Chlouveraki, G. Pouchin, Representation theory and an isomorphism theorem for the Framisation of the Temperley-Lieb algebra, Math. Zeitschrift 285(3-4) (2017), 1357-1380.

[DipperJames] R. Dipper, G. James Representations of Hecke algebras of general linear groups, Proc. London Math. Soc. 52(3) (1986), 20-52.

[DipperJamMathas] R. Dipper, G. James, A. Mathas, Cyclotomic q-Schur algebras, Math. Zeitschrift 229 (1998), 385-416.

[Esp] J. Espinoza, An isomorphism Theorem for the partition Temperley-Lieb algebra, preprint.

[EspRy] J. Espinoza, S. Ryom-Hansen, Cell structures for the Yokonuma-Hecke algebra and the algebra of braids and ties, J. Pure Appl. Algebra 222 (2018), 3675-3720.

[GraLeh] J. J. Graham, G. I. Lehrer, Cellular algebras, Inventiones Mathematicae 123 (1996), 1-34.

[Härterich] M. Härterich, Murphy bases of generalized Temperley-Lieb algebras, Archiv der Mathematik 72(5) (1999), 337-345.

[Hump] J. E. Humphreys, Reflection groups and Coxeter groups, volume 29 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.
[JacoAn] N. Jacon, L. Poulain d'Andecy, Clifford theory for Yokonuma-Hecke algebras and deformation of complex reflection groups, J. London Math. Soc., 96(3) (2017), 501-523

[HuMat] J. Hu, A. Mathas, Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A Adv. Math. 225(2) (2010), 598-642

[Juyumaya] J. Juyumaya, A Partition Temperley-Lieb Algebra, arXiv:1304.5158.

[Mar] I. Marin, Artin groups and Yokonuma-Hecke algebras, Int. Math. Res. Not. 13 (2018), 4022-4062

[Martin] P. Martin, Representation theory of a small ramified partition algebra. In Okado Masato Boris Feigin, Jimbo Michio. New Trends in Quantum Integrable Systems: Proceedings of the Infinite Analysis 09. World Scientific Publishing Co. Pte. Ltd. (2010).

[Mathas] A. Mathas, Hecke algebras and Schur algebras of the symmetric group, Univ. Lecture Notes, 15, A.M.S., Providence, R.I., 1999.

[Murphy92] G. E. Murphy, On the Representation Theory of the Symmetric Groups and Associated Hecke Algebras, J. of Algebra 152 (1992), 492–513.

[Murphy95] E. G. Murphy, The representations of Hecke algebras of type $A_n$, J. of Algebra 173 (1995), 97-121.

[PlaRy] D. Plaza, S. Ryom-Hansen Graded cellular bases for Temperley-Lieb algebras of type A and B. J. Algebr. Comb. 40(1) (2014), 137-177.

[Ry] S. Ryom-Hansen, On the Representation Theory of an Algebra of Braids and Ties, J. Algebra Comb. 33 (2011), 57-79.