THE BRILL-NOETHER CURVE OF A STABLE VECTOR BUNDLE ON A GENUS TWO CURVE.

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Abstract. Let $\mathcal{U}_r$ be the moduli space of rank $r$ vector bundles with trivial determinant on a smooth curve of genus $2$. The map $\theta_r: \mathcal{U}_r \to |r\Theta|$, which associates to a general bundle its theta divisor, is generically finite. In this paper we give a geometric interpretation of the generic fibre of $\theta_r$.

1. Introduction.

In this note we deal with the moduli space $\mathcal{U}_r$ of semistable vector bundle of rank $r$ and degree $r(g-1)$ over a smooth, irreducible complex projective curve of genus $g \geq 2$. $\mathcal{U}_r$ is endowed with the Brill-Noether locus

$$\Theta_r := \{ [E] \in \mathcal{U}_r \mid h^0(E) \geq 1 \}$$

which is an integral Cartier divisor and it is known as the generalized theta divisor of $\mathcal{U}_r$, see [8], [5]. Moreover the tensor product defines a morphism

$$f: \mathcal{U}_r \times \text{Pic}^0(C) \to \mathcal{U}_r$$

and we can consider the pull-back $f^*\Theta_r$ of $\Theta_r$. Let $[E] \in \mathcal{U}_r$ be the moduli point of the vector bundle $E$ and let $\det E \cong M^{\otimes r}$, it is well known that then

$$\mathcal{O}_{[E] \times \text{Pic}^0(C)}(f^*\Theta_r) \cong \mathcal{O}_{\text{Pic}^0(C)}(r\Theta_M)$$

where

$$\Theta_M := \{ N \in \text{Pic}^0(C) \mid h^0(M \otimes N) \geq 1 \}.$$

Note that $M$ is a line bundle of degree $g-1$ and that $\Theta_M$ is a theta divisor on $\text{Pic}^0(C)$. We define

$$\Theta_E := f^*\Theta_r \cdot [E] \times \text{Pic}^0(C)$$

if the intersection is proper. In this case we will say that $\Theta_E$ is the theta divisor of $E$. The construction of $\Theta_E$ allows us to define a rational map as follows. Consider

$$\tau_r := \bigcup_{M \in \text{Pic}^{g-1}(C)} |r\Theta_M|.$$
it is a standard fact that $\mathcal{T}_r$ has a natural structure of projective bundle over $\text{Pic}^{r(g-1)}(C)$. So we omit its construction, we only mention that the corresponding projection

$$p : \mathcal{T}_r \to \text{Pic}^{r(g-1)}(C)$$

is defined as follows: $p(D) = M^\otimes$ iff $D \in |r\Theta_M|$. Notice that the only elements of multiplicity $r$ in $|r\Theta_M|$ are exactly the divisors $r\Theta_M \otimes \eta$, where $\eta$ varies in the set of the elements of order $r$ in $\text{Pic}^0(C)$. Therefore the map $p$ is well defined. In the following we will study the rational map

$$\theta_r : \mathcal{U}_r \to \mathcal{T}_r$$

which associates to a general $[E] \in \mathcal{U}_r$ the corresponding theta divisor $\Theta_E \in \mathcal{T}_r$. Let

$$\det : \mathcal{U}_r \to \text{Pic}^{r(g-1)}(C)$$

be the determinant map, it is well known that $\mathcal{T}_r$ is the projectivization of $\det_* \mathcal{O}_{\mathcal{U}_r}(\Theta_r)^*$ and that $\theta_r$ is the induced tautological map. In particular it follows that $p \cdot \theta_r = \det$.

We will say that $\theta_r$ is the theta map.

Too many questions are still unsettled about the theta map, excepted for the case $r \leq 2$: see e.g. [4] for a general survey. This situation is probably related to the fact that the next basic question is still mostly unsolved.

**QUESTION** Is $\theta_r$ generically finite onto its image?

Actually the main difficulty here is that $\theta_r$ is not a morphism in most of the cases [16]. Thus, in spite of the ampleness of $\Theta_r$, it is not a priori granted that $\theta_r$ is generically finite onto its image.

In this paper we give a natural geometric interpretation of the fibres of the map $\theta_r$ for a curve $C$ of genus two. A very special feature of this case is that

$$\dim \mathcal{U}_r = \dim \mathcal{T}_r = r^2 + 1,$$

so the generic finiteness of $\theta_r$ is even more expected. Applying our description of the fibres we prove the generic finiteness of $\theta_r$.

Such a result is not new: Beauville recently proved it using a different, relatively simple method, see [3]. We believe that our description has some interest in itself and we hope to use it for further applications, in particular to compute the degree of $\theta_r$.

Our approach relies on Brill-Noether theory for curves contained in a genus two Jacobian. Let $D \in \theta_r(\mathcal{U}_r)$ be a sufficiently general element, then $D$ is a smooth curve of genus $r^2 + 1$ in $\text{Pic}^0(C)$: see section 2. Consider the Brill-Noether locus

$$W_{r^2}^{r-1}(D) = \{L \in \text{Pic}^r(D) \mid h^0(L) \geq r\}$$
and observe that its expected dimension is one, in other words the Brill-Noether number \( \rho(r-1, r^2, r^2 + 1) \) is one. Our main result can be summarized as follows:

**THEOREM** Each point \([E] \in \theta_{r^{-1}}(D)\) defines an irreducible component

\[
C_E \subset W_{r^2}^{r^{-1}}(D)
\]

biregular to \( C \). Let \( Z \) be the set of all irreducible components of \( W_{r^2}^{r^{-1}}(D) \) and let

\[
i_D : \theta_{r^{-1}}(D) \to Z
\]

be the map sending \([E]\) to \( C_E \). Then \( i_D \) is injective.

The statement clearly implies that \( \theta_r \) is generically finite. We define \( C_E \) as the Brill-Noether curve of \( E \). Fixing appropriately a Poincaré bundle \( P \) on \( D \times \text{Pic}^{r^2}(D) \) it turns out that \( E \) is the restriction of \( \nu_* P \) to \( C_E \), where \( \nu \) is the projection onto \( \text{Pic}^{r^2}(D) \). In particular the family of the fibres of \( E \) is just the family of the spaces \( H^0(L), L \in C_E \).

Notice that the choice of \( P \), hence of \( \det E \), depends on the embedding \( D \subset \text{Pic}^0(C) \) and it is essentially explained in the final part of this note.

To have a typical example of what happens, the reader can consider the case \( r = 2 \). In this case \( D \) is a curve of genus 5 endowed with a fixed point free involution which is induced by the \(-1\) multiplication of \( \text{Pic}^0(C) \). Since \( r = 2 \) the Brill-Noether locus \( W_4^1(D) \) is exactly the singular locus of the theta divisor of \( \text{Pic}^4(D) \). It follows from the theory of Prym varieties that \( W_4^1(D) \) is the union of two irreducible curves: one of them has genus 4, the other one is just a copy of \( C \), (see also [17]). This is the Brill-Noether curve of a stable rank two vector bundle \( E \) such that \( \theta_2([E]) = D \). In higher rank the general theory of Prym-Tjurin varieties can certainly provide further information on \( W_{r^2}^{r^{-1}}(D) \) and hence on the fibres of \( \theta_r \). However, in order to get them, a very explicit description is needed for the Prym-Tjurin realizations of a genus two Jacobian.

On Jacobians of higher genus several extensions of the above constructions are possible and perhaps deserve to be considered in the study of the theta maps. We hope to have underlined with this note the multiplicity of the links between moduli of vector bundles on a curve \( C \), Prym-Tjurin realizations of its Jacobian \( JC \) and Brill-Noether theory for curves in \( JC \).

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2. Notations and preliminary results.

From now on $C$ is a smooth, irreducible, complex projective curve of genus 2. Let $C^{(2)}$ be the 2-symmetric product of $C$, a point of such a surface is a divisor $x + y$ with $x, y \in C$. We consider the map

$$a : C^{(2)} \to \text{Pic}^0(C)$$

sending $x + y \in C^{(2)}$ to $\omega_C(-x - y)$. Of course $a$ is the composition of the Abel map defined by $\omega_C$ with $-1$ multiplication on $\text{Pic}^0(C)$. Therefore $a = -\sigma$, where $\sigma : C^{(2)} \to \text{Pic}^0(C)$ is the blowing up of the zero point. For each fibre $|r\Theta_M|$ of the projective bundle $T_r$ we have the linear isomorphism $a^* : |r\Theta_M| \to |a^* r\Theta_M|$ defined by the pull-back. Let $\Theta_E$ be the theta divisor of $[E]$, we will keep the following notation

$$D_E := a^* \Theta_E.$$ 

$D_E$ is an effective divisor in $C^{(2)}$ which is supported on the set

$$\{ x + y \in C^{(2)} \mid h^0(E \otimes \omega_C(-x - y)) \geq 1 \}.$$ 

$D_E$ is biregular to $\Theta_E$ if the zero point is not in $\Theta_E$, otherwise $D_E$ is the union of the projective line $|\omega_C|$ and of a curve birational to $\Theta_E$. For our convenience we are more interested to $D_E$ than to $\Theta_E$.

At first we want to prove that a general $D_E$ is smooth, to do this we need Laszlo’s singularity theorem, see [11]:

**Theorem 2.1.** The multiplicity of $\Theta_r$ at its stable point $[E]$ is $h^0(E)$.

**Proposition 2.2.** Let $[E]$ be a general stable point of $U_r$ then

$$h^0(E \otimes \omega_C(-x - y)) \leq 1, \forall x + y \in C^{(2)}.$$ 

**Proof.** By induction on $r$. Let $r = 1$ then $D_E = C$ and $E$ is a general line bundle of degree 1, in particular $|E \otimes \omega_C|$ is a base-point-free pencil and this implies the statement. Let $r \geq 2$, we can assume by induction that there exist general $[B] \in U_{r-1}$ and $[A] \in U_1 = \text{Pic}^1(C)$ satisfying the statement. We consider the exact sequence

$$0 \to B \to E \to A \to 0$$

defined by the vector $e \in \text{Ext}^1(A, B)$. Tensoring such a sequence by $\omega_C(-x - y)$ and passing to the long exact sequence we obtain the coboundary map

$$e_{x+y} : H^0(\omega_C \otimes A(-x - y)) \to H^1(\omega_C \otimes B(-x - y)).$$

**Claim** The statement holds for $E$ iff $e_{x+y}$ has maximal rank for every $x + y$. 

Proof: We have $h^0(\omega_C \otimes A(-x - y)) \leq 1$ and
\[ h^0(\omega_C \otimes B(-x - y)) = h^1(\omega_C \otimes B(-x - y)) \leq 1. \]
Then the statement follows from the above mentioned long exact sequence.

Finally it is obvious that $e_{x+y}$ has maximal rank except possibly for points $x + y$ with $h^0(\omega_C \otimes A(-x - y)) = h^0(\omega_C \otimes B(-x - y)) = 1$. The set of these points is $D_A \cap D_B$. Since $A$ is general we can assume that $D_A \cap D_B$ is finite. Let $x + y \in D_A \cap D_B$ then $e_{x+y}$ has not maximal rank iff it is the zero map. It is a standard property that, in the present case, the locus
\[ H_{x+y} = \{ e \in \Ext^1(A, B) \mid e_{x+y} \text{ is the zero map} \} \]
is a hyperplane. Let $H := \bigcup H_{x+y}, x + y \in D_A \cap D_B$, then a general $e \in \Ext^1(A, B) - H$ defines a semistable $E$ satisfying the condition of the statement. Since this condition is open on $\mathcal{U}_r$ the result follows. \qed

Corollary 2.3. Let $[E]$ be a general stable point of $\mathcal{U}_r$, then $D_E$ is smooth.

Proof. Let $f : \mathcal{U}_r \times \text{Pic}^0(C) \to \mathcal{U}_r$ be the map defined via tensor product, recall that
\[ \Theta_E = f^*\Theta_r \cdot [E] \times \text{Pic}^0(C) \subset \mathcal{U}_r \times \text{Pic}^0(C). \]
Therefore $\Theta_E$ is the fibre of the projection $q : f^*\Theta_r \to \mathcal{U}_r$. Then, by generic smoothness, a general $\Theta_E$ is smooth if $\Theta_E \cap \Sing f^*\Theta_r = \emptyset$. On the other hand $f$ is smooth, with fibres biregular to $\text{Pic}^0(C)$. The smoothness of $f$ implies that $\Sing f^*\Theta_r = f^*\Sing \Theta_r$. Therefore, by Laszlo’s singularity theorem and the definition of $f$, we have
\[ \Sing f^*\Theta_r = \{ ([E], \xi) \in \mathcal{U}_r \times \text{Pic}^0(C) \mid h^0(E(\xi)) \geq 2 \}. \]
But the previous proposition implies that $h^0(E(\xi)) \leq 1$, for all $\xi \in \text{Pic}^0(C)$. Then it follows that $\Theta_E \cap \Sing f^*\Theta_r = \emptyset$ and hence a general $\Theta_E$ is smooth. The same holds for $D_E$. \qed

3. The tautological model $P_E$

Now we want to see that the above curve $D_E$ appears as the singular locus of some natural tautological model of $\mathbb{P}E^*$ in $\mathbb{P}^{2r-1}$.

Proposition 3.1. Let $E$ be any semistable point of $\mathcal{U}_r$, then
1) $h^1(\omega_C \otimes E) = 0$ and $h^0(\omega_C \otimes E) = 2r$.
2) $\omega_C \otimes E$ is globally generated unless $E$ is not stable and $\text{Hom}(E, \mathcal{O}_C(x))$ is non zero for some point $x \in C$.

Proof. 1) By Serre duality $h^1(\omega_C \otimes E) = h^0(E^*)$. Since $E^*$ is semistable of slope $-1$ it follows $h^0(E^*) = 0$. Then we have $h^0(\omega_C \otimes E) = 2r$ by Riemann-Roch.
1) By 1) $E$ is globally generated iff $h^0(\omega_C \otimes E(-x)) = r$, $\forall \ x \in C$. By Serre duality this is equivalent to $\text{Hom}(E, \mathcal{O}_C(x)) = 0$, $\forall \ x \in C$. Notice also that $\text{Hom}(E, \mathcal{O}_C(x)) \neq 0$ implies that $E$ is not stable. This completes the proof. □

In this section we assume that $[E] \in \mathcal{U}_r$ has the following properties (satisfied by a general $[E]$):
- $\omega_C \otimes E$ is globally generated,
- $D_E$ exists i.e. $[E]$ is not in the indeterminacy locus of $\theta_r : \mathcal{U}_r \to \mathcal{T}_r$,
- $D_E$ is smooth.

To simplify our notations we put $F := \omega_C \otimes E$ and $P_E := PF^*$.

**Lemma 3.2.** Let $F$ be general and let $\overline{F}$ be defined by the standard exact sequence
$$0 \to F^* \to H^0(F)^* \otimes \mathcal{O}_C \to \overline{F} \to 0$$
induced by the evaluation map. Then $\overline{F}$ is stable. In particular the map
$$j : \mathcal{U}_r \to \mathcal{U}_r$$
sending $[\omega_C^{-1} \otimes F]$ to $[\omega_C^{-1} \otimes \overline{F}]$ is a birational involution.

**Proof.** First of all we claim that $\overline{F}$ is semistable for $F$ general enough. Let $F_o = L^r$, where $L \in \text{Pic}^3(C)$ is globally generated, then $h^0(F_o) = 2r$ and $\overline{F_o} = F_o$. Up to a base change there exists an integral variety $T$ and a vector bundle $F$ over $T \times C$ such that the family of vector bundles $\{F_t := F \otimes \mathcal{O}_{t \times C}, \ t \in T\}$ dominates $\mathcal{U}_r$ and contains $F_o$.

By semicontinuity we can assume, up to replacing $T$ by a non empty open subset, that $h^0(F_t) = h^0(F_o) = 2r$ and $F_t$ is globally generated. So it is standard to construct from $F$ a vector bundle $\overline{F}$ on $T \times C$ with the following property: $\overline{F} \otimes \mathcal{O}_{t \times C} = F_t$, for each $t \in T$. Since $\overline{F} \otimes \mathcal{O}_{o \times C} = L^r$ is semistable, the same holds for a general vector bundle $\overline{F} \otimes \mathcal{O}_{t \times C}$. Hence the claim follows. Let $F$ be a general stable bundle: $\overline{F}$ is semistable, by lemma (3.1) $h^0(\overline{F}) = 2r$, moreover since $h^0(F^*) = 0$, we have $H^0(F)^* \simeq H^0(\overline{F})$ and $\overline{F}$ is globally generated. So $j$ is defined at $\overline{F}$, actually $j(\overline{F}) = F$. This implies that $j$ is a birational involution and $\overline{F}$ is stable too. □

Since $F$ is globally generated the map defined by $\mathcal{O}_{P_E}(1)$ is a morphism
$$u_E : P_E \to \mathbb{P}^{2r-1} := \mathbb{P} H^0(F)^*.$$ In particular the restriction of $u_E$ to any fibre $P_{E,x}$ of $P_E$ is a linear embedding
$$u_{E,x} : P_{E,x} \to \mathbb{P}^{2r-1}.$$ **Definition 3.3.** The image of $u_E$, (of $u_{E,x}$), will be denoted $P_E$, $(P_{E,x})$. 
For any \(d \in C(2)\), \(F_d := F \otimes \mathcal{O}_d\) can be naturally seen as a rank \(r\) vector bundle over \(d\). Note that its projectivization is \(p^*d\), where \(p : \mathbb{P}_E \to C\) is the projection map. In particular the evaluation map \(e_d : H^0(F_d) \otimes \mathcal{O}_d \to F_d\) defines an embedding
\[
i_d : p^*d \to \mathbb{P}H^0(F_d)^*.
\]
We have \(\mathbb{P}H^0(F_d)^* = \mathbb{P}^{2r-1}\) and moreover \(i_d(p^*d)\) is the union of two disjoint linear spaces of dimension \(r-1\) if \(d\) is smooth. The next lemma is therefore elementary.

**Lemma 3.4.** Let \(o \in \mathbb{P}H^0(F_d)^*\) be a point not in \(i_d(p^*d)\), then there exists exactly one line \(L\) containing \(o\) and such that \(Z := i_\lambda L\) is a 0-dimensional scheme of length two. Moreover let \(\lambda\) be the linear projection of centre \(o\), then \(Z\) is the unique 0-dimensional scheme of length two on which \(\lambda \cdot i_d\) is not an embedding.

The central arrow in the long exact sequence
\[
(1) \quad 0 \to H^0(F(-d)) \to H^0(F) \to H^0(F_d) \to H^1(F(-d)) \to 0
\]
defines a linear map
\[
\lambda_d : \mathbb{P}H^0(F_d)^* \to \mathbb{P}H^0(F)^* = \mathbb{P}^{2r-1},
\]
and from the construction clearly \(\lambda_d \cdot i_d\) is the map
\[
u_E|_{p^*d} : p^*d \to \mathbb{P}^{2r-1}.
\]

**Proposition 3.5.** \(u_E|_{p^*d}\) is not an embedding if and only if \(d \in D_{E}\).

**Proof.** From the previous remarks and lemma 3.4 it follows that \(u_E|_{p^*d}\) is not an embedding if \(\lambda_d\) is not an isomorphism. By the long exact sequence (1) \(\lambda_d\) is not an isomorphism iff \(h^0(E(-d)) \geq 1\), that is if \(d \in D_{E}\).

We want to use the previous results to study the singular locus of \(P_E\).
Let \(\text{Hilb}_2(\mathbb{P}_E)\) be the Hilbert scheme of 0-dimensional schemes \(Z \subset \mathbb{P}_E\) of length two, we simply consider its closed subset
\[
\Delta = \{Z \in \text{Hilb}_2(\mathbb{P}_E) \mid u_E|_Z\text{ is not an embedding}\}.
\]
Let \(Z \in \Delta\) then \(Z \subset p^*d\), where \(d := p_*Z\) belongs to \(D_E\). So we have a morphism
\[
p_* : \Delta \to D_E
\]
sending \(Z\) to \(d\).

**Proposition 3.6.** Let \(E\) be general then \(p_* : \Delta \to D_E\) is biregular.

**Proof.** Let \(Z \in \Delta\) and let \(p_*Z = d\), then \(Z\) is embedded in \(p^*d\). Since \(d \in D_E\) we have \(h^0(F(-d)) = 1\), see prop. 2.2. This implies that the linear map
\[
\lambda_d : \mathbb{P}H^0(F_d)^* \to \mathbb{P}^{2r-1}
\]
is the projection of centre a point $o$ with image a hyperplane in $\mathbb{P}^{2r-1}$. Then, by lemma 3.4, $Z$ is the unique element of $\Delta$ which is contained in $p^*d$. Hence $p_*$ is injective. Conversely let $d \in D_E$, then $\lambda_d \cdot i_d$ is not an embedding on exactly one 0-dimensional scheme $Z \subset p^*d$ of length two. Since $\lambda_d \cdot i_d = u_E|_{p^*d}$, it follows that $Z$ is in $\Delta$ and that $p_*$ is surjective. Since $D_E$ is a smooth curve, $p_*$ is biregular. □

**Proposition 3.7.** Assume $r \geq 2$ and $E$ general, then $u_E : \mathbb{P}_E \to P_E$ is the normalization map and Sing $P_E$ is an irreducible curve.

**Proof.** Let $\tilde{D} \subset \mathbb{P}_E$ be the image of the curve

$$\Delta = \{(Z,q) \in \Delta \times \mathbb{P}_E \mid q \in \text{Supp } Z\}$$

under the projection $\Delta \times \mathbb{P}_E \to \mathbb{P}_E$. The set $\tilde{D}$ is the locus of points where $u_E$ is not an embedding: it is a proper closed set as soon as $r \geq 2$. Hence $u_E : \mathbb{P}_E \to P_E$ is a morphism of degree one if $r \geq 2$. Since $\mathbb{P}_E$ is smooth, $u_E$ is the normalization map if each of its fibres is finite. Assume $u_E$ contracts an irreducible curve $B$ to a point $o$. $B$ cannot be is a fibre $\mathbb{P}_{E,p}$: otherwise $D_E$ would contain the curve $\{z + p, z \in C\}$ and would be reducible. Hence $o \in \cap P_{E,x}$, $x \in C$. Let $x + y \in C^{(2)}$ with $x \neq y$, then $P_{E,x} \cup P_{E,y}$ is contained in a hyperplane and hence $h^0(E \otimes \omega_C(-x-y)) \geq 1$. This implies $D_E = C^{(2)}$: a contradiction. It remains to show that Sing $P_E$ is an irreducible curve: this is clear because Sing $P_E = u_E(\tilde{D})$ □

4. **The line bundle $H_E$**

We will keep the generality assumptions and the notations of the previous section. Recall that $d \in D_E$ uniquely defines a 0-dimensional scheme $Z_d \subset p^*d$ of length two such that $u_E|_{Z_d}$ is not an embedding, in particular $u_E(Z_d)$ is a point.

**Definition 4.1.** $h_E : D_E \to \mathbb{P}^{2r-1}$ is the morphism sending $d$ to $u_E(Z_d)$, moreover

$$H_E := h_E^* \mathcal{O}_{\mathbb{P}^{2r-1}}(1).$$

**Remark 4.2.** 1. Let $F := \omega_C \otimes E$ and let $q_1, q_2 : C \times C \to C$ be the projections. Note that $q_1^*F \oplus q_2^*F$ descends to a vector bundle $F^{(2)}$ on $C^{(2)}$ via the quotient map $C \times C \to C^{(2)}$. Moreover the evaluation $H^0(F) \to F_x \oplus F_y$ induces a natural map

$$e : H^0(F) \otimes \mathcal{O}_{C^{(2)}} \to F^{(2)}.$$  

Then $D_E$ is the degeneracy locus of $e$ and $H_E$ is its cokernel. This implies that the sheaf $H_E$ can be defined for every curve $D_E$ and that $H_E$ is a line bundle if and only if $h^0(\omega_C \otimes E(-x-y)) = 1$ for each $x + y \in D_E$. 2. A very simple geometric definition of $h_E$ can be given as follows: let $d = x + y \in D_E$ with $x \neq y$ then

$$h_E(d) = P_{E,x} \cap P_{E,y} = u_E(Z_d).$$
The Brill-Noether curve of a stable vector bundle on a genus two curve.

Proposition 4.3. $h_E : D_E \to \mathbb{P}^{2r-1}$ is generically injective if $E$ is general and $r \geq 2$.

Proof. Let $U = \{x + y \in D_E \mid x \neq y\}$, assume that $d_1, d_2 \in U$ and $h_E(d_1) = h_E(d_2) = o$: then $o \in \bigcap_{i=1}^{4} P_{E,x_i}$, where $\Sigma x_i = d_1 + d_2$. We consider the standard exact sequence

$$0 \to F^* \to H^0(F)^* \otimes \mathcal{O}_C \to \mathcal{F} \to 0,$$

where $F = \omega_C \otimes E$. The long exact sequence identifies $H^0(F)^*$ to a subspace of $\mathcal{H}^0(\mathcal{F})$. Hence $o$ is a 1-dimensional space generated by some $s \in H^0(\mathcal{F})$. It is standard to verify that $o \in \bigcap_{i=1}^{4} P_{E,x_i}$ iff $s$ is zero on $d_1 + d_2 - d$, where $d$ is the M.C.D. of $d_1, d_2$. By Lemma 3.2 $\mathcal{F}$ is stable, hence we must have $\deg d \geq 2$ that is $d_1 = d_2 = d$. □

Proposition 4.4. $H_E$ has degree $r^2 + 2r$.

Proof. Set theoretically we have $h_E(D_E) = \text{Sing } P_E$, hence $\text{Sing } P_E$ is an irreducible curve. Let $o = h_E(x + y)$ be general then $x \neq y$ and moreover $u_E^*(o)$ is supported on two closed points $o'$ and $o''$: this follows because $h_E$ is generically injective.

Claim: The tangent map $du_E$ is injective at $o'$ and $o''$.

Let $\widetilde{D(u_E)}$ be the double point scheme of $u_E$, defined as in [9] p. 166]. $\widetilde{D(u_E)}$ is contained in $\mathbb{P}_E \times \mathbb{P}_E$, where $\pi : \mathbb{P}_E \times \mathbb{P}_E \to \mathbb{P}_E \times \mathbb{P}_E$ is the blowing up of the diagonal $\Delta$. A point of $\widetilde{D(u_E)}$ is either the inverse image by $\pi$ of a pair $(o', o'')$ in $\mathbb{P}_E \times \mathbb{P}_E - \Delta$ such that $u_E(o') = u_E(o'')$ or it is a point in $\pi^{-1}(\Delta)$ parametrizing a 1 dimensional space of tangent vectors to $\mathbb{P}_E$ on which $du_E$ is zero. In the former case we have also $p(o') \neq p(o'')$ because $u_E$ is injective on each fibre of $p$. On the other hand it is clear that, in our situation,

$$q^{-1}(D_E) = (p \times p) \cdot \pi(\widetilde{D(u_E)})$$

where $q : C \times C \to C^{(2)}$ is the quotient map. Thus $du_E$ is not injective at most along fibres $\mathbb{P}_{E,x}$ such that $2\pi \in D_E$. $D_E$ cannot be the diagonal of $C^{(2)}$ because $D_E^2 = 2r^2$. Hence $D_E$ contains finitely many points $2\pi$ and we can choose the above point $o = h_E(x + y)$ so that $2x$ and $2y$ are not in $D_E$. This implies our claim.

Let $T$ be the tangent space to $P_E$ at $o$ and let $T', T'' \subset T$ be the images of $du_E$ at $o', o''$. Since $du_E$ is injective at $o', o''$ and $u_E^{-1}(o) = \{o', o''\}$, it follows that $T' \cup T''$ spans $T$ and that $T' \cap T''$ is the tangent space to $\text{Sing } P_E$ at $o$. We have dim $T' \cap T'' \geq 1$ because $\text{Sing } P_E$ is a curve. On the other hand $P_{E,x} \cap P_{E,y} = o$ implies dim $T \geq 2r - 1$. Since dim $T' = \dim T'' = r$, we deduce that dim $T' \cap T'' = 1$. Hence, as a scheme defined by the Jacobian ideal of $P_E$, $\text{Sing } P_E$ is reduced. Finally the degree of $\text{Sing } P_E$ can be obtained via double point formula, see [9] 9.3], as follows:

$$V \cdot V - c_{r-1}(N_{V|\mathbb{P}^{2r-1}}) = 2 \deg(\text{Sing } P_E)$$
where $V \subset \mathbb{P}^{2r-2}$ is a general hyperplane section of $P_E$, corresponding to a global section $\sigma \in H^0(\mathcal{O}_{P_E}(1)) \simeq H^0(\omega_C \otimes E)$, $c_{r-1}$ denotes the $(r-1)$-Chern class of the normal bundle $N_{V|\mathbb{P}^{2r-2}}$. Note that $V = P_{E'}$, with $E'$, vector bundle of rank $r-1$ defined by the section $\sigma$ as follows:

$$0 \to \mathcal{O}_C \to E \otimes \omega_C \to E' \otimes \omega_C \to 0,$$

and $\mathcal{O}_{P_{E'}}(1) = \mathcal{O}_{P_E}(1)|_{\mathbb{P}_{E'}}$. Let $H = [\mathcal{O}_{P_{E'}}(1)]$ and $f$ be the class of a fibre of $\mathbb{P}_{E'}$, then by computing the total Chern class of the normal bundle, we find

$$c_{r-1}(N_{V|\mathbb{P}^{2r-2}}) = rH^{r-1} + fH^{r-2}[4r^2 - 4r] = 7r^2 - 4r.$$

Finally, we have $\deg \text{Sing } P_E = \deg H_E = r^2 + 2r$.

The genus of $D_E$ is $r^2 + 1$ and $H_E$ has degree $r^2 + 2r$, hence $h^0(H_E) \geq 2r$

**Proposition 4.5.** For a general $E$ the line bundle $H_E$ is non special that is

$$h^0(H_E) = 2r.\]

**Proof.** By induction on $r$. Let $r = 1$ then $A := \omega_C \otimes E$ is a general line bundle of degree 3, $\mathbb{P}_E = C'$ and $u_E : \mathbb{P}_E \to \mathbb{P}^1$ is the triple covering defined by $A$. Moreover $D_E$ is the family of divisors $x + y$ which are contained in a fibre of $u_E$. It is easy to see that $D_E$ is a copy of $C$ and that $h_E = u_E$. Then $H_E = A$ and hence $h^0(H_E) = 2$.

Let $r \geq 2$ and let $[E_{r-1}] \in \mathcal{U}_{r-1}$ and $E_1 \in \text{Pic}^1(C)$ be general points satisfying the statement, then their corresponding curves $D_{r-1}$ and $D_1$ are smooth and transversal. Taking a general semistable extension

$$(2) \quad 0 \to E_{r-1} \to E \to E_1 \to 0$$

we have $h^0(E \otimes \omega_C(-x - y)) \leq 1$ for any $x + y$, (see 2.2 and its proof).

Observe also that $D_E = D_1 \cup D_{r-1}$ and that $h_E$ is a morphism. The restrictions of $h_E$ to $D_1$ and $D_{r-1}$ can be described as follows:

(a) Let $F := \omega_C \otimes E$ and let $A := \omega_C \otimes E_1$: tensoring 2 by $\omega_C$ and passing to the long exact sequence, we obtain a surjective map $H^0(F) \to H^0(A)$. Its dual is a linear embedding $i : \mathbb{P}H^0(A)^* \to \mathbb{P}^{2r-1}$. On the other hand we already know that $h_{E_1}$ is the triple cover of $\mathbb{P}H^0(A)^* \otimes \omega_C$ of centre $\ell$ in $\mathbb{P}^{2r-1}$ which is triple for $h_{E_1}(D_E)$.

(b) Let $B := \omega_C \otimes E_{r-1}$: tensoring 2 by $\omega_C$ and passing to the long exact sequence we get an injection $H^0(B) \to H^0(F)$. Its dual induces a projection $p : \mathbb{P}^{2r-1} \to \mathbb{P}H^0(B)^*$ of centre $\ell$. It is again easy to conclude that $p \cdot h_{E_1}|_{D_{r-1}} = h_{E_{r-1}}$. It follows from the remarks in (b) that

$$H_E \otimes \mathcal{O}_{D_{r-1}} = H_{E_{r-1}}(a),$$
where $a := (h_E|_{D_{r-1}})^* \ell = D_1 \cdot D_{r-1}$. On the other hand (a) implies that

$$H_E \otimes \mathcal{O}_{D_1} = H_{E_1} = A.$$ 

Finally, tensoring by $H_E$ the Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_{D_E} \to \mathcal{O}_{D_{r-1}} \oplus \mathcal{O}_{D_1} \to \mathcal{O}_a \to 0,$$

we obtain

$$0 \to H_E \to H_{E_{r-1}}(a) \oplus A \to \mathcal{O}_a \otimes H_E \to 0.$$ 

By induction $h^1(H_{E_{r-1}}) = 0$, hence $h^1(H_{E_{r-1}}(a)) = 0$. Moreover $h^1(A) = 0$. Passing to the long exact sequence, the vanishing of $H^1(H_E)$ follows if the restriction

$$\rho : H^0(H_{E_{r-1}}(a)) \to \mathcal{O}_a(a) \otimes H_{E_{r-1}}$$

is surjective. Since $h^1(H_{E_{r-1}}) = 0$, this follows from the long exact sequence of

$$0 \to H_{E_{r-1}} \to H_{E_{r-1}}(a) \to \mathcal{O}_a(a) \otimes H_{E_{r-1}} \to 0.$$ 

The vanishing of $h^1(H_E)$ extends by semicontinuity to a general point of $\mathcal{U}_r$. 

5. The Brill-Noether curve of $E$

In the following we will set for simplicity: $D := D_E$. $D$ is an abstract curve endowed with an embedding $D \subset C^{(2)}$. These data are in general not sufficient to reconstruct the vector bundle $E$. As we will see the additional datum of $H_E$ makes possible such a reconstruction.

The embedding $D \subset C^{(2)}$ uniquely defines the family of divisors

$$b_x := C_x \cdot D$$

where $x \in C$ and $C_x := \{x + y \mid y \in C\}$. $b_x$ fits in the standard exact sequence

$$0 \to H^0(\omega_C \otimes E(-x)) \otimes \mathcal{O}_C \to \omega_C \otimes E(-x) \to \mathcal{O}_{b_x} \to 0$$

and its degree is $2r$. The determinant of $E$ can be reconstructed from the family $\{b_x \mid x \in C\}$. Indeed let $x + x' \in |\omega_C|$, then the previous exact sequence implies

$$\det E \cong \mathcal{O}_C(b_x - rx').$$

Let $t + \Theta_E$ be the translate of $\Theta_E$ by $t \in \text{Div}^0(C)$: $D_{E(-t)} = a^*(t + \Theta_E)$. Thus, up to replacing $E$ by $E(-t)$, $D$ is transversal to $C_x$ and $b_x$ is smooth for a general $x$. Mainly we will consider $b_x$ as a divisor on $D$. Let $d \in D$, it is clear that: $d \in \text{Supp } b_x \iff d = x + y \iff h_E(d) \in P_{E,x}$. This implies that

$$b_x = h_E^* P_{E,x}$$
for each $x \in C$. The line bundles $H_E(-b_x)$ have degree $r^2$. Since $D$
has genus $r^2 + 1$ they define a family of points in the theta divisor of
$\text{Pic}^{r^2}(D)$. We can say more:

**Proposition 5.1.** Let $E$ be general then $h^0(H_E(-b_x)) = r$, for each
$x \in C$.

**Proof.** We know from Prop. 4.5 that $h^0(H_E) = 2r$, we also know that
$b_x = h_E^*(P_{E,x})$. Since the space $P_{E,x}$ has dimension $r - 1$, it follows
$h^0(H_E(-b_x)) \geq r$. Moreover the equality holds if the set $h_E(\text{Supp } b_x)$
spans $P_{E,x}$. We prove this by induction on $r$. Let $r = 1$ then $P_{E,x}$ is a
point: since $h_E$ is a morphism $h_E(\text{Supp } b_x) = P_{E,x}$. Let $r \geq 2$, as in
the proof of 4.5 we consider a general extension

$$0 \to E_{r-1} \to E \to E_1 \to 0$$

with $[E_{r-1}] \in \mathcal{U}_{r-1}$ and $E_1 \in \text{Pic}^1(C)$ general. We fix the same
assumptions and notations of the proof of 4.5 which is similar. In particular
the curves $D_{E_{r-1}}$ and $D_{E_1}$ are smooth and transversal, moreover the
exact sequence

$$0 \to B \to F \to A \to 0$$

just denotes the above exact sequence (3) tensored by $\omega_C$. Such a
sequence induces a linear embedding $i : \mathbb{P}H^0(A)^* \to \mathbb{P}H^0(F)^*$. The
image of $i$ is the line $\ell$ considered in 4.5 and it holds the equality
proved there: $h_E|_{D_{E_1}} = i \cdot u_{E_1}$. Then it turns out that

$$\ell \cap P_{E,x} = h_E(C_x \cap D_{E_1}) = \text{one point } o_x$$

for each $x \in C$. On the other hand let $p : P_E \to \mathbb{P}H^0(B)^*$ be the
projection of centre $\ell$, then the latter exact sequence implies that
$p(P_{E,x}) = P_{E_{r-1},x}$. Moreover we also know from the proof of 4.5 that
$h_{E_{r-1}} = p \cdot h_E$. By induction $P_{E_{r-1},x}$ is spanned by

$$h_{E_{r-1}}(C_x \cap D_{E_{r-1}}) = p(h_E(C_x \cap D_{E_{r-1}})).$$

Hence the linear span of $h_E(C_x \cap D_{E_{r-1}})$ is a space $L \subset P_{E,x}$ of
dimension $\geq r - 2$ and such that $p(L) = P_{E_{r-1},x}$. If $o_x \in P_{E,x} - L$ then $P_{E,x}$
is spanned by $h_E(C_x \cap D)$. If $o_x \in L$ then $\dim L = r - 1$ and $L = P_{E,x}$.
In both cases the statement follows. \qed

For each $\ell \in \text{Pic}^1(C)$ we consider the curve

$$B_\ell := \{x + y \in |\ell(z)|, \ z \in C\}.$$

$B_\ell$ is biregular to $C$ unless $\ell = \mathcal{O}_C(x)$ for some $x \in C$. In the latter
case $B_\ell$ is $C_x \cup |\omega_C|$. We define

$$b_\ell := D \cdot B_\ell.$$

Note that $b_\ell = b_x$ if $B_\ell = C_x \cup |\omega_C|$. The reason is that we are assuming
$E$ general, then $h^0(E) = 0$ and hence $D \cap |\omega_C| = \emptyset$. 

Definition 5.3. The crucial curve for doing this can be now defined:

\[ W_{r^2-1}(D) := \{ L \in \Pic^r(D) | h^0(L) \geq r \}. \]

The Brill-Noether number \( \rho(r-1, r^2, r^2+1) \) yields the expected dimension of \( W_{r^2-1}(D) \). We have \( \rho(r-1, r^2, r^2+1) = 1 \) for each \( r \), so we expect that \( C_E \) is an irreducible component of \( W_{r^2-1}(D) \), see [1] ch. V. Of course \( D \) is not a general curve of genus \( r^2+1 \), so the latter property is not a priori granted.

Lemma 5.2. The morphism \( b : \Pic^1(C) \to \Pic^{2r}(D) \) sending \( \ell \) to \( b_\ell \) is an embedding. We will denote its image as \( J_D \):

\[ J_D := \{ \mathcal{O}_D(b_\ell) | \ell \in \Pic^1(C) \}, \]

in particular \( J_D \) contains the canonical theta divisor

\[ C_D := \{ \mathcal{O}_D(b_x) | x \in C \}. \]

Proof. Up to shifting the degrees, \( b \) is a morphism between the complex tori \( \Pic^0(C) \) and \( \Pic^0(D) \). Hence it is an isogeny up to translations, so \( b \) is an embedding if it is injective. Let \( \ell_1, \ell_2 \in \Pic^1(C) \) and set \( L : = \mathcal{O}_D(b_{\ell_1} - b_{\ell_2}) \). \( L \) is defined by the standard exact sequence:

\[ 0 \to \mathcal{O}_{C(\ell)}(-D + B_{\ell_1} - B_{\ell_2}) \to \mathcal{O}_{C(\ell)}(B_{\ell_1} - B_{\ell_2}) \to L \to 0. \]

\( D + B_{\ell_2} - B_{\ell_1} \) is the pull-back by the Abel map \( a : C(\ell) \to \Pic^0(C) \) of a divisor homologous to \( r\Theta \), where \( \Theta \) is a theta divisor in \( \Pic^0(C) \). Since \( r\Theta \) is ample, it follows: \( h^0(-D + B_{\ell_1} - B_{\ell_2}) = h^1(-D + B_{\ell_1} - B_{\ell_2}) = 0 \).

So the associated long exact sequence gives:

\[ h^0(L) = h^0(\mathcal{O}_{C(\ell)}(B_{\ell_1} - B_{\ell_2})). \]

Moreover, it’s easy to see that if \( \ell_1 \neq \ell_2 \), then \( h^0(\mathcal{O}_{C(\ell)}(B_{\ell_1} - B_{\ell_2})) = 0 \) hence \( b_{\ell_1} \) and \( b_{\ell_2} \) are not linearly equivalent. \( \square \)

As an immediate consequence of the lemma, the following map

\[ b_0 : \Pic^0(C) \to \Pic^0(D) \]

sending \( \mathcal{O}_C(x-y) \) to \( \mathcal{O}_D(x-y) \) is an embedding too.

As we already pointed out \( C_D \) is not sufficient to reconstruct \( E \), the crucial curve for doing this can be now defined:

Definition 5.3. The Brill-Noether curve of \( E \) is the curve

\[ C_E := \{ H_E(-b_x), x \in C \}. \]

\( C_E \) is a copy of \( C \) embedded in \( \Pic^{r^2}(D) \). Since \( h^0(H_E(-b_x)) = r \), each point of \( C_E \) is a point of multiplicity \( r \) for the theta divisor

\[ \Theta_D := \{ L \in \Pic^{r^2}(D) | h^0(L) \geq 1 \}. \]

In particular \( C_E \) is contained in the Brill-Noether locus

\[ W_{r^2-1}(D) := \{ L \in \Pic^{r^2}(D) | h^0(L) \geq r \}. \]
Remark 5.4. \( E \) is uniquely reconstructed from the pair \((C_D, H_E)\) as follows:

\[ B = \{ (x, y + z) \in C \times D \mid x \in \{y, z\} \}, \]

with \( B \cdot (x \times D) = b_x \). Let \( p_1 : C \times D \to C \) and \( p_2 : C \times D \to D \) be the projection maps, then we apply the functor \( p_{1*} \) to the exact sequence

\[ 0 \to p_2^* H_E(-B) \to p_2^* H_E \to p_2^* H_E \otimes O_B \to 0. \]

This yields the exact sequence

\[ 0 \to \overline{F}^* \to H^0(H_E) \otimes O_C \to p_{1*} O_B \otimes p_2^* H_E \to R^1 p_{1*} p_2^* H_E(-B) \to 0, \]

where

\[ \overline{F}^* := p_{1*} p_2^* H_E(-B). \]

Let \( F = \omega_C \otimes E \), we have the natural identities

\[ \overline{F}_x^* = H^0(H_E(-b_x)) = H^0(F(-x)). \]

The left one is immediate. Let \( \mathcal{I} \) be the Ideal of \( P_{E,x} \), then we have \( H^0(H_E(-b_x)) = H^0(\mathcal{I}(1)) \) by prop. 5.1. Hence the right equality follows from the identity \( H^0(\mathcal{I}(1)) = H^0(F(-x)) \). The above identities, together with \( H^0(H_E) = H^0(F) \), imply that

\[ F = H^0(H_E) \otimes O_C / \overline{F}^*. \]

As an immediate consequence of the above construction we have:

**Proposition 5.5.** Let \([E_1], [E_2]\) be general points of \( \mathcal{U}_r \). Assume that:

\[ \theta_*([E_1]) = \theta_*([E_2]) = D \text{ and } H_{E_1} = H_{E_2}. \]

Then \([E_1] = [E_2]\).

**Remark 5.6.** The previous construction also defines the vector bundles

\[ \overline{E} : = \overline{F} \otimes \omega_C^{-1}, \quad \overline{E} : = \omega_C^{-1} \otimes (R^1 p_{1*} p_2^* H_E(-B)). \]

We already know from 3.2 that the assignment \([E] \to [\overline{E}]\) defines a birational involution \( j : \mathcal{U}_r \to \mathcal{U}_r \). Notice also that \( \overline{E} \) is semistable for \( E \) general: to prove this it suffices to produce one semistable \( E_o \) such that \( \overline{E}_o \) is semistable. The existence of \( E_o \) follows by induction on \( r \): this is obvious for \( r = 1 \). Let \( r \geq 2 \) and let \( E_o \) be defined by a semistable extension \( e \in \text{Ext}^1(E_1, E_{r-1}) \) where \([U]_{r-1} \in \mathcal{U}_{r-1} \) and \( E_1 \in \mathcal{U}_1 \). It is easy to show that \( \overline{E}_o \) is defined by some \( \overline{e} \in \text{Ext}^1(\overline{E}_1, \overline{E}_{r-1}) \): we leave the details to the reader. Hence \( \overline{E}_o \) is semistable. Due to this property we can define a rational map

\[ \kappa : \mathcal{U}_r \to \mathcal{U}_r \]

sending \([E] \) to \([\overline{E}]\). In addition we have:

**Proposition 5.7.** \( \kappa : \mathcal{U}_r \to \mathcal{U}_r \) is birational, in particular its inverse is \( j \cdot \kappa \cdot j \).
Proof. Let \( T := \mathcal{O}_D(b_x + b_{i(x)}) \), where \( i : C \rightarrow C \) is the hyperelliptic involution. \( T \) does not depend on \( x \) because the family of divisors \( \{b_x + b_{i(x)}, \ x + i(x) \in |\omega_C| \} \) is rational. Then we define the line bundle of degree \( r^2 + 2r \)
\[
\tilde{H}_E := \omega_D \otimes T \otimes H_E^{-1}.
\]
First, we note that \( h^1(\tilde{H}_E) = 0 \) for \( E \) general. Indeed \( \omega_D \otimes \tilde{H}_E^{-1} \) is \( H_E(-b_x - b_{i(x)}) \) and hence \( h^1(\tilde{H}_E) = h^0(H_E(-b_x - b_{i(x)}) \) by Serre duality. Since \( h^0(H_E(-b_x - b_{i(x)}) = h^0(\omega_C \otimes E(-x - i(x)) = h^0(E), \) it follows \( h^1(\tilde{H}_E) = 0 \) for each \( [E] \in U_r - \Theta_r \). Secondly we note that, with the previous notations, Serre duality yields a natural identification
\[
R^1p_1p_2H_E(-B)x = H^0(\tilde{H}_E(-b_{i(x)}))^*, \ \forall x \in C.
\]
It is then easy to deduce that
\[
\omega_C \otimes \tilde{E} = R^1p_1p_2H_E(-B) \cong p_1p_2\tilde{H}_E(-B)^*.
\]
Starting from \( \tilde{H}_E \) it is clear that one obtains \( H_E \) and with the same construction \( \tilde{E} = \omega_C^{-1} \otimes p_1p_2H_E \). Notice also that \( \tilde{H}_E \) is the line bundle \( H_{\tilde{E}} \) defined by the vector bundle \( \tilde{E} \). This implies that \( \kappa^{-1} = j \cdot \kappa \cdot j \) and hence that \( \kappa \) is birational. \( \square \)

6. THE FIBRES OF THE THETA MAP

We want to see that \( E \) is also uniquely reconstructed from the pair \((D, C_E)\). For this we consider more in general any smooth curve \( D \subset C^{(2)} \) such that \( a_xD \in T_r \):

**Definition 6.1.** A Brill-Noether curve of \( D \) is a copy
\[
C' \subset \text{Pic}^{r^2}(D)
\]
of \( C \) satisfying the following property: there exists \( H \in \text{Pic}^{r^2+2r}(D) \) such that
\[
C' = \{ H(-b_y), y \in C \},
\]
moreover \( H \) is non special and \( h^0(H(-b_x)) = r \) for every point \( x \in C \). The set of the Brill-Noether curves of \( D \) will be denoted by \( S_D \).

Let \( D = \theta_r([E]) \) then \( C_E \) is a Brill-Noether curve of \( D \). Let \( r = 1 \) then \( D = C \) and the canonical theta divisor of \( \text{Pic}^1(C) \) is the unique Brill-Noether curve of \( D \).

**Lemma 6.2.** Let \( H \) be as in the previous definition then \( H \) is unique.

**Proof.** Assume that \( C' = \{ H'(b_x), x \in C \} \) for a second \( H' \). Then there exists an automorphism \( u : C \rightarrow C \) which is so defined \( u(x) = y \) iff \( H'(b_x) = H(-b_y) \). Let \( \gamma : \text{Pic}^0(C) \times \text{Pic}^0(C) \rightarrow \text{Pic}^0(D) \) be the map sending \((x, y) \) to \( H' \otimes H^{-1}(b_x - b_y) \); we will show that the image of \( \gamma \) is a copy of \( \text{Pic}^0(C) \) and that \( \gamma : C \times C \rightarrow \text{Pic}^0(C) \) is the difference map. To see this
recall that \( C_D = \{ b_x, x \in C \} \) is the theta divisor of \( J_D \), see (2). The map \( \tilde{\gamma} : C \times C \to \text{Pic}^0(D) \) sending \( (x, y) \to \mathcal{O}_D(b_x - b_y) \) factors through the isomorphism \( t : C \times C \to C_D \times C_D \), sending \( (x, y) \to (b_x, b_y) \), and the difference map. Moreover, we have the following commutative diagram

\[
\begin{array}{ccc}
C \times C & \xrightarrow{t} & C_D \times C_D \\
\downarrow & & \downarrow \\
\text{Pic}^0(C) & \xrightarrow{b_0} & \text{Pic}^0(D)
\end{array}
\]

where the vertical arrows are difference maps and \( b_0 : \text{Pic}^0(C) \to \text{Pic}^0(D) \) sending \( \mathcal{O}_C(x - y) \to \mathcal{O}_D(b_x - b_y) \) is an embedding, see (5.2). This implies that \( \tilde{\gamma} \) is a difference map and \( \gamma \) too. The graph of \( u \) is obviously contracted by \( \gamma \), on the other hand the only curve contracted by the difference map is the diagonal of \( C \times C \). Then \( u \) is the identity and \( H(-b_x) = H'(-b_x) \) for each \( x \in C \). Hence \( H = H' \).

\[\Box\]

**Proposition 6.3.** Let \( [E_1], [E_2] \) be general points in \( U_r \), assume that \( C_{E_1} = C_{E_2} \) and that \( \theta_r([E_1]) = \theta_r([E_2]) = D \). Then \( [E_1] = [E_2] \).

**Proof.** By the previous lemma \( H_{E_1} = H_{E_2} \) and this implies \( [E_1] = [E_2] \), by (5.3).

**Theorem 6.4.** The theta map \( \theta_r : \mathcal{U}_r \to \mathcal{T}_r \) is generically finite.

**Proof.** It suffices to show that \( \theta_r|_U : U \to \theta_r(U) \) is generically finite for a suitable dense open set \( U \). Hence we can assume that \( D \in \theta_r(U) \) is a smooth curve and that the points of \( (\theta_r|_U)^{-1}(D) = \theta_r^{-1}(D) \cap U \) are sufficiently general in \( \mathcal{U}_r \). Let

\[i : (\theta_r|_U)^{-1}(D) \to S_D\]

be the map sending \( [E] \) to \( C_E \). By (6.3) \( E \) is uniquely reconstructed from \( (D, C_E) \), hence it follows that \( i \) is injective. On the other hand recall that \( C_E \) is contained in the Brill-Noether locus \( W_r^{-1}(D) \). Since the Brill-Noether number \( \rho(r - 1, r^2, r^2 + 1) \) is one, each irreducible component of \( W_r^{-1}(D) \) has dimension \( \geq 1 \). This implies that \( \theta_r|_U \) is finite if \( C_E \) is an irreducible component of \( W_r^{-1}(D) \). This property is proved in the next theorem. Hence the statement follows.

\[\Box\]

**Lemma 6.5.** Let \( D = \theta_r([E]) \) for a general \( [E] \in \mathcal{U}_r \) and let \( a = D \cdot D_1 \) for a general \( D_1 \in \mathcal{T}_1 \), then the line bundle \( H_E(a - b_x) \) is non special.

**Proof.** Let \( D_1 = \theta_1([E_1]) \subset C(2) \) with \( E_1 = \mathcal{O}_C(x) \), then we have: \( D_1 = x \times C \cup |\omega_C| \subset C(2) \). Note that \( a = D \cdot D_1 = b_x \) if \( E \) is general. Hence \( H_E(a - b_x) = H_E \) is non special. By semicontinuity, the same is true for a general \( D_1 \).

\[\Box\]

**Theorem 6.6.** For a general \( [E] \in \mathcal{U}_r \) the Brill-Noether curve \( C_E \) is an irreducible component of \( W_r^{-1}(D) \), \( D = \theta_r([E]) \).
Proof. Let $H := H_E$, it is sufficient to show the injectivity of the Petri map
\[ \mu : H^0(H(-b_x)) \otimes H^0(\omega_D \otimes H^{-1}(b_x)) \to H^0(\omega_D) \]
for a general $x \in C$. This implies that the tangent space to $W_{r-1}^r(D)$ at its point $H(-b_x)$ is 1-dimensional, see [1] Ch. V. We proceed by induction on $r$. Let $r = 1$, then $D = C$ and $C_E = \{O_C(x), x \in C\}$. Hence the injectivity of $\mu$ is immediate. For $r \geq 2$ we borrow once more the notations and the method from the proof of proposition 4.5. So we specialize $E$ to the semistable vector bundle defined by the exact sequence
\[ 0 \to E_{r-1} \to E \to E_1 \to 0. \]
Then $D$ is the transversal union of the curves $D_{r-1} = \theta_{r-1}([E_{r-1}])$ and $D_1 = \theta_1([E_1])$, $h_E$ is a morphism and $H$ is the line bundle $h_E^*O_{\mathbb{P}^{2r-1}}(1)$. Let $a = D_{r-1} \cdot D_1$: from [4.5] we have $D_1 = C$ and $H_{E_1} = \omega_C \otimes E_1$ and moreover
\[ H \otimes O_{D_{r-1}} = H_{E_{r-1}}(a) \quad \text{and} \quad H \otimes O_{D_1} = H_{E_1}. \]
Since $x$ is general we can assume $\text{Supp } b_x \cap \text{Sing } D = \emptyset$ so that $O_D(b_x)$ is a line bundle. Let $\mathcal{I}$ be the ideal sheaf of $D_1$ in $D$: at first we show that $\mu|I \otimes W$ is injective, where
\[ I := H^0(\mathcal{I} \otimes H(-b_x)) \quad \text{and} \quad W := H^0(\omega_D \otimes H^{-1}(b_x)). \]
Since $\mathcal{I} \otimes O_{D_{r-1}} = O_{D_{r-1}}(-a)$ and $\omega_D \otimes O_{D_{r-1}} = \omega_{D_{r-1}}(a)$, we have the restriction maps
\[ \rho_I : I \to H^0(H_{E_{r-1}}(-b_{x,r-1})) \quad \text{and} \quad \rho_W : W \to H^0(\omega_{D_{r-1}} \otimes H_{E_{r-1}}^{-1}(b_{x,r-1})) \]
with $b_{x,r-1} = b_x \cdot D_{r-1}$.

Claim : $\rho_I$ is an isomorphism and $\rho_W$ is surjective.

Let’s assume the claim, then $\rho := \rho_I \otimes \rho_W$ is surjective and defines the exact sequence
\[ 0 \to \ker \rho \to I \otimes W \to H^0(H_{E_{r-1}}(-b_{x,r-1})) \otimes H^0(\omega_{D_{r-1}} \otimes H_{E_{r-1}}^{-1}(b_{x,r-1})) \to 0. \]
In particular it follows $\dim \ker \rho = r - 1$. By induction on $r$ the Petri map on the tensor product at the right side is injective. Therefore
\[ \mu|I \otimes W \text{ is injective iff } \mu|\ker \rho \text{ is injective. But our claim implies } \dim \ker \rho_W = 1 \text{ and } \ker \rho = I \otimes \langle w \rangle, \]
where $w$ generates $\ker \rho_W$. Hence $\mu|\ker \rho$ is injective as well as $\mu|I \otimes W$.

Let $V := H^0(H(-b_x))$, now we consider the exact sequence
\[ 0 \to I \otimes W \to V \otimes W \to (V/I) \otimes W \to 0. \]
The map $\mu$ induces a multiplication
\[ \nu : (V/I) \otimes W \to H^0(\omega_D)/\mu(I \otimes W). \]
The injectivity of $\mu|I \otimes W$ implies that $\mu$ is injective iff $\nu$ is injective. On the other hand, $\rho_I$ is an isomorphism, hence $\dim I = r - 1$ and $\dim V/I = 1$. Let $v \in V - I$ then: $\nu$ is injective iff $vW \cap \mu(I \otimes W) = (0)$.
iff no \( w \in W - (0) \) vanishes on \( D_1 \). This is equivalent to the injectivity of the restriction map

\[ u : H^0(\omega_D \otimes H^{-1}(b_x)) \to H^0(\omega_C(a - x)); \]

in fact \( W = H^0(\omega_D \otimes H^{-1}(b_x)) \) and \( \omega_D \otimes H^{-1}(b_x) \otimes \mathcal{O}_{D_1} = \omega_C(a - x) \).

To prove that \( u \) is injective consider the Mayer-Vietoris long exact sequence

\[ 0 \to W \to H^0(\omega_{D_{r-1}} \otimes H^{-1}_{E_{r-1}}(b_{x,r-1})) \oplus H^0(\omega_C(a - x)) \to \mathcal{O}_a \to \ldots \]

The left non zero arrow followed by the projection onto \( H^0(\omega_C(a - x)) \) is exactly \( u \). This implies that \( \ker u \), via the restriction map, injects in \( H^0(\omega_{D_{r-1}} \otimes H^{-1}_{E_{r-1}}(b_{x,r-1} - a)) \). So \( u \) is injective if the latter space is zero that is if \( H^1(H_{E_{r-1}}(a - b_{x,r-1})) = 0 \); this has been shown in lemma \( \ref{lemma:injectivity} \). Hence \( \mu \) is injective. Then, by semicontinuity, the same property is true for a general \( [E] \in \mathcal{U}_r \) and the statement follows.

To complete the proof we show the above claim.
- Let \( h : D \to \mathbb{P}^{2r-1} \) be the map defined by \( H \). As in \( \ref{subsection:linear_pencil} \) \( h(D_1) \) is a line \( \ell \) and \( P_{E,x} \cap \ell \) is a point. Moreover \( P_{E,x} \) is spanned by \( h(b_x) \). Hence we have \( I = H^0(J(1)) \) and dim \( I = r - 1 \), \( J \) being the ideal of \( P_{E,x} \cup \ell \).
  - In particular \( \rho_\ell \) is the pull-back \( (h|_{D_{r-1}})^* \) restricted to a space of linear forms vanishing on \( h(D_1) \). Since \( h(D) \) is non degenerate \( \rho_\ell \) is injective. Then, for dimension reasons, \( \rho_\ell \) is an isomorphism.
  - As in \( \ref{subsection:isomorphism} \) the projection \( p : P_{E,x} \to P_{E_{r-1},x} \) from \( P_{E,x} \cap \ell \) is surjective. Equivalently the restriction \( H^0(E \otimes \omega_C(-x)) \to H^0(E_{r-1} \otimes \omega_C(-x)) \) is surjective. So this property holds for general \( E, E_{r-1} \). Let \( \tilde{E}_r, \tilde{E}_{r-1} \) be defined from \( E_r, E_{r-1} \) as in Remark \( \ref{remark:bracket} \). By \( \ref{remark:general} \) they are general. Hence the restriction \( H^0(\tilde{E} \otimes \omega_C(-x)) \to H^0(\tilde{E}_{r-1} \otimes \omega_C(-x)) \) is surjective: this map is just \( \rho_W \).

We can summarize as follows our partial geometric description of the theta map:

**Theorem 6.7.** Let \( D \in \mathcal{T}_r \) be general and smooth, then there exists a natural injective map \( i_D \) between the fibre of \( \theta_r \) at \( D \) and the set of the Brill-Noether curves of \( D \). Namely the map

\[ i_D : \theta_r^{-1}(D) \to S_D \]

associates to \( [E] \in \theta_r^{-1}(D) \) its Brill-Noether curve \( C_E \in S_D \).

**Proof.** Since \( \theta_r \) is generically finite, each point \( [E] \in \theta_r^{-1}(D) \) is sufficiently general in \( \mathcal{U}_r \). The injectivity then follows from corollary \( \ref{corollary:injectivity} \). \( \square \)

**Remark 6.8.** Each Brill-Noether curve \( C \in S_D \) uniquely defines a vector bundle \( E_C \) of rank \( r \) and degree \( r \): to construct \( E_C \) it suffices to
take the line bundle $H$ appearing in the definition \([6.2]\) of Brill-Noether curve. Applying to the pair $(C, H)$ the reconstruction produced in remark \([5.4]\) we finally obtain such a vector bundle $E_C$. If $E_C$ is semistable it turns out that $\theta_r([E_C]) = D$ and that $i_D([E_C]) = C$. In particular $i_D$ is bijective if each $C \in S_D$ defines a semistable $E_C$. This property seems very plausible for a general $D$, however we do not have a rigorous proof of it.

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