The lattice scale at large $\beta$ in quenched QCD

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Abstract

In this paper we extend the estimate of the value of the lattice spacing $a$ in units of the $r_0$ scale at values of the bare coupling larger than those available. By using results from the computation of the renormalised coupling in the Schrödinger functional formalism we find that from $\beta \simeq 7$ onward the behaviour predicted by asymptotic freedom at tree loop describes very well the data if a value of $r_0 \Lambda$ slightly lower than the latest available is used. We also show that, by sticking to the current value, an effective four loop term can describe as well the data. The systematic relative error on the lattice spacing induced by the choice of the procedure is between 1% and 3% for $6.92 < \beta < 8.5$. 
1 Introduction

In order to obtain physical predictions from a lattice QCD simulation, one has to spend as many “experimental” input as are the free parameters of the theory. This situation is not peculiar of lattice regularisation, but is a common fact to every regularisation/renormalisation procedure. In the pure Yang–Mills case the only free parameter of the theory is the bare coupling $g_0$, and the only dimensionful quantity in a simulation, in the idealized case of a lattice of infinite extent, is the lattice spacing $a$. Asymptotic freedom in the high energy regime predicts the functional dependence of $a$ from $g_0$ once $\Lambda_L$ is known. If the value of the bare coupling gets too large to rely on a perturbative expansion, one has to resort to a non–perturbative definition of the physical scale of the simulation.

One way to perform this task, in the case of a lattice extension large enough to safely account the dynamics of the light quarks, is to use the experimental inputs coming from the spectroscopy of the strange mesons, as explained in refs. [1, 2, 3]. Another way to set the scale, widely used since its proposal in [4], is based on the calculation of the force between static colour sources; the physical input in this case is a length scale, $r_0 \simeq 0.5$ fm.

In [5] and then in [6] the quantity $a/r_0$ has been computed, in the quenched QCD framework, for a range of $\beta \equiv 6/g_0^2$ going from $\beta_{\min} = 5.7$ to $\beta_{\max} = 6.92$ and a parametric description of $\ln(a/r_0)$ as a function of $g_0^2$ is provided that describes the data with an accuracy better than 1% in the whole range. In the quenched approximation the absolute value of the scale is affected by a systematic error, induced by the quenching, of the order of 10%; nevertheless the precise result of ref. [6] has been widely used in literature, at least to set the ratio between different scales and to perform continuum extrapolations. The use of the parametrization of ref. [6] outside its validity range ($5.7 \leq \beta \leq 6.92$) is highly questionable; on the other hand, one can easily encounter situations (see for example ref. [8]) where a precise knowledge of $a/r_0$ for values of $\beta$ greater than 6.92 is required.
In this paper we will show how to obtain an accurate estimate of $a/r_0$ for $\beta > 6.92$, using existing numerical simulations and the behaviour expected from asymptotic freedom: we use the non-perturbative results coming from the computation of the Schrödinger functional renormalised coupling [10, 11] and the parametrization dictated by Renormalisation Group improved perturbation theory.

2 Non-perturbative approach: the running coupling

In a series of publications (see for example refs. [9, 10, 11, 12]) it has been shown how to define and compute a renormalized coupling, whose running with the energy scale can be followed, in the continuum limit, over a very large range, making able to connect the high energy perturbative region with the low energy non-perturbative one.

The calculations has been done using a recursive finite-size technique in the Schrödinger functional framework where the energy scale is given by $\mu = 1/L$. The size of $L_{\text{max}}$, defined implicitly by the relation $\mathcal{F}_{\text{SF}} (L_{\text{max}}) = 3.48$, has been computed in ref. [6] in terms of $r_0$, with the result

$$L_{\text{max}}/r_0 = 0.738(16); \quad (2.1)$$

in the same paper, combining this result with those of ref. [11], the authors quote

$$\Lambda_{\text{MS}}^{(0)} = 0.586(48)/r_0, \quad (2.2)$$

where the superscript (0) remind us that we are dealing with the quenched approximation. Equation (2.1), together with the tab. (6) of ref. [11], will be the handles to extend our non-perturbative knowledge of $a/r_0$ in a wider range of $\beta$ values.

We start by considering a length scale $L_u$, implicitly defined by the relation $\mathcal{F}^2 (L_u) = u$: by construction, this scale is smaller than $L_{\text{max}}$ by a known factor $s_u$. We can use this fact in order to express the scale $L_u$ in terms of $r_0$: 
\[ \frac{L_u}{r_0} \equiv x ; \quad (2.3) \]

if we consider a particular discretisation of \( L_u \), i.e. \( L_u/a = N \), we can write

\[ a/r_0 = x/N . \quad (2.4) \]

From table (6) of ref. [11] we can read off a whole bunch of pairs \( (L_u/a, \beta) \) and, using eq. (2.4), we can compute \( a/r_0 \) corresponding to all these values of \( \beta \).

Given the finite extent used in these calculations we must, of course, assume that lattice artifacts do not play a prominent rôle. Whether this is a good assumption can be checked by using the \( \beta \)-value corresponding to the coupling \( u = 2.77 \) in tab. (6) of ref. [11], that lies within the range in which \( a/r_0 \) has been directly computed. In this case the scale factor \( s_u \) is almost exactly equal to \( 3/2 \).

The results can be seen in table (1): three of the values of \( a/r_0 \) computed through eq. (2.4), including the points computed at \( u = 3.48 \), are at \( \beta < 6.92 \), and nicely fall, within the error, which comes entirely from the error on \( L_{\text{max}}/r_0 \), over the parametrization of ref. [6], even if the points correspond to lattices with a relative poor discretisation, \( N = 8 \) and \( N = 12 \).

Indeed, the results of ref. [11] for the continuum limit of the step scaling function also indicate that lattice artifacts are very small already at \( N = 8 \) and \( N = 12 \), strongly supporting our assumption.

| \( \beta \)     | eq.(2.6) ref.[6] | eq. (2.4) u | \( L_u/a \) |
|----------------|------------------|-------------|-------------|
| 6.4527         | -2.345           | -2.383(22)  | 3.48        | 8           |
| 6.7750         | -2.758           | -2.790(22)  | 3.48        | 12          |
| 6.7860         | -2.773           | -2.789(22)  | 2.77        | 8           |

*Table 1:* Check of eq. (2.4) against non-perturbative result of ref. [6]. The accuracy on the parametrization of ref. [6], in this \( \beta \) range, is about 1%. The error on the results of eq. (2.4) comes entirely from the error on \( L_{\text{max}}/r_0 \).
3 Perturbative approach

In ref. [13] it is argued that the three–loop perturbative expansion in the $q\bar{q}$–scheme may be trusted up to $\alpha_{q\bar{q}} \simeq 0.3$. We test how perturbation theory can help us in safely extending the region where $a/r_0$ has to be considered as known with good precision, contrary to the general belief that lattice bare perturbation theory is well-behaving only in the extremely huge cutoff region.

We recall some well known results. In a generic renormalisation scheme, $S$, the perturbative expansion of the $\beta$–function reads:

$$\mu \frac{d}{d\mu} \bar{g}_S \equiv \beta(\bar{g}_S) = -\bar{g}_S^3 \{ b_0 + b_1 \bar{g}_S^2 + b_2^{(S)} \bar{g}_S^4 + \ldots \}$$

(3.1)

where $b_0$ and $b_1$ are the well–known universal terms,

$$b_0 = \frac{1}{(4\pi)^2} (11 - \frac{2}{3} N_f), \quad b_1 = \frac{1}{(4\pi)^4} (102 - \frac{38}{3} N_f)$$

(3.2)

while, from $b_2^{(S)}$ on, the coefficients of the $\beta$–function start to be scheme–dependent. The $b_2^{(S)}$ coefficient is known in the $\overline{\text{MS}}$ scheme [17]:

$$b_2^{(\overline{\text{MS}})} = \frac{1}{(4\pi)^6} \left( \frac{77139}{54} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2 \right).$$

(3.3)

Since the relation between $\bar{g}_{\overline{\text{MS}}}$ and the bare lattice coupling $g_0$ is known up to two loops [14, 15], some straightforward algebra leads us to the $b_2^L$ coefficient of the lattice $\beta$–function, that we quote in the $N_f = 0$ case only:

$$b_2^L = b_2^{\overline{\text{MS}}} + b_1 d_1(1)/(4\pi) + (d_1^2(1) - d_2(1) b_0)/(4\pi)^2 = -0.0015998323314(3)$$

(3.4)

where $d_1$ and $d_2$ can be found in ref. [14].

In order to integrate the renormalisation group equations, we need the integration constant, usually defined trough the renormalisation QCD scale $\Lambda_S$, which can be related to the running coupling $\bar{g}_S(\mu)$ through the exact relation

$$\Lambda_S = \mu (b_0 \bar{g}_S^2)^{-b_1/(2b_2^L)} e^{-1/(2b_0 \bar{g}_S^2)} \exp \left\{ - \int_0^{\bar{g}_S} dx \left[ \frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\}.$$
where the subscript $S$ indicates a generic scheme. The previous expression obviously becomes a perturbative one upon the insertion of the perturbative formula (3.1) for the $\beta$–function. In the case of lattice bare perturbation theory, we can recast the previous equation in the following form:

$$\ln\left(\frac{a}{r_0}\right) = -\ln(\Lambda_L r_0) - \frac{b_1}{2b_0} \ln(b_0 g_0^2) - \frac{1}{2b_0 g_0^2} - I(g_0),$$

(3.6)

where $I(g_0)$ is the integral appearing in the exponential of eq. (3.5). By using eq. (2.2) and the result from ref. [16]

$$\frac{\Lambda^{(0)}_{\overline{\text{MS}}}}{\Lambda_L} = 28.80934(1)$$

(3.7)

we get

$$\Lambda_L r_0 = 0.0203(17)$$

(3.8)

and, with this input, eq. (3.6) becomes a perturbative tool to set the scale $a$ in terms of $r_0$.

As can be seen in fig. (1), the three–loops formula describes quite well the behaviour of non–perturbative data at high $\beta$ and nicely superimpose to the points computed through eq. (2.4) if the value of $\Lambda$ is lowered by about $1\sigma$ with respect to the value quoted in ref. [14] (the upper bound of the three–loops band in the figure corresponds to $\Lambda^{(0)}_{\overline{\text{MS}}} = 0.538/r_0$).

If we do not stick to the value $\Lambda^{(0)}_{\overline{\text{MS}}} = 0.586/r_0$ but let the three-loops fit decide for its value, we obtain the results listed in table 2. The value of $\Lambda^{(0)}_{\overline{\text{MS}}}$ obtained through this procedure is remarkably constant against the variation of the number of points included in the fit. Whether this fact is really suggestive of a lower value of the $\Lambda$ parameter in quenched approximation could be only clarified with further non–perturbative investigations at smaller values of the bare coupling.

Fig. (1) indicates that, at $\beta \simeq 7$, three-loops perturbation theory is not far from an accurate description of non–perturbative data even when we keep $\Lambda^{(0)}_{\overline{\text{MS}}} = 0.586/r_0$. We can thus try to determine an effective four–loops term by means of a fit. A two parameters fit ($\Lambda$ and $b_3$) gives a value for $\Lambda^{(0)}_{\overline{\text{MS}}} r_0$ totally compatible, in the error, with 0.586, and this make us confident that
Figure 1: $\ln(a/r_0)$ as a function of $g_0^2$. 
reducing the number of parameters by sticking to the known value of $\Lambda_{\text{MS}}^{(0)}r_0$ will give a reliable estimate of $b_3$.

In table (3) we show the results obtained with a one–parameter fit, namely $b_3^{\text{eff}}$ itself, by keeping fixed $\Lambda_{\text{MS}}^{(0)} = 0.586/r_0$. We note that absolute value of $b_3^{\text{eff}}$ agrees with an approximate geometric growth of the coefficients $(4\pi)^{2(n+1)}b_n$.

| $n$ | $\Lambda_{\text{MS}}^{(0)}r_0$ | $\ln(a/r_0)(\beta = 7.5)$ | $\ln(a/r_0)(\beta = 8.5)$ |
|-----|----------------------|----------------------|----------------------|
| 3   | 0.547                | -3.640               | -4.801               |
| 4   | 0.546                | -3.638               | -4.800               |
| 5   | 0.540                | -3.627               | -4.789               |
| 6   | 0.535                | -3.618               | -4.779               |

Table 2: Results for the 3–loop fit to non–perturbative data at large $\beta$, including the last four points obtained through eq. (2.4); $n$ is the number of points included in the fit (most perturbative ones).

| $n$ | $b_3^{\text{eff}}$ | $\ln(a/r_0)(\beta = 7.5)$ | $\ln(a/r_0)(\beta = 8.5)$ |
|-----|-------------------|----------------------|----------------------|
| 3   | -0.0022           | -3.643               | -4.819               |
| 4   | -0.0022           | -3.643               | -4.818               |
| 5   | -0.0023           | -3.638               | -4.815               |
| 6   | -0.0025           | -3.631               | -4.810               |

Table 3: Results for the 4–loop fit to non–perturbative data at large $\beta$, including the last four points obtained through eq. (2.4), keeping fixed $\Lambda_{\text{MS}}^{(0)}r_0 = 0.586$; $n$ is the number of points included in the fit (most perturbative ones).

Summarizing, it is safe to assume that beyond $\beta = 6.92$ the quantity $\ln(a/r_0)$ is well described by an effective four–loops perturbative curve, if not already by the three-loops expression. We give as a final result the value obtained with $n = 6$ and $\Lambda_{\text{MS}}^{(0)}r_0$ fixed to the value 0.586 ($b_3^{\text{eff}} = -0.0025(3)$: the $\chi^2$/n.d.f. of the fit is 0.9), that we chose in order to obtain the lattice scale $a$ in terms of $r_0$ for $\beta > 6.92$: the fit is shown in fig. (1). The accuracy of
this description can be estimated by considering the variation of this quantity when $\Lambda_{\text{MS}}^{(0)}$ changes by 1σ. Of course the value of $b_{3}^{\text{eff}}$ will be readjusted by the fit. We take as a reference points the values of $\ln(a/r_0)$ at $\beta = 7.5$ and $\beta = 8.5$, where we obtain respectively (assuming $r_0 = 0.5$ fm)

\[
\begin{align*}
\beta &= 7.5, & \ln(a/r_0) &= -3.63(2) & a &= 0.0133(3) \text{ fm}, \\
\beta &= 8.5, & \ln(a/r_0) &= -4.81(3) & a &= 0.0041(1) \text{ fm} \quad (3.9)
\end{align*}
\]

We can therefore state that, no matter which solution we adopt, lower $\Lambda$ and 3–loops or standard $\Lambda$ and 4–loops, the uncertainty on $a/r_0$ for $6.92 < \beta < 7.5$ can be bounded by 2%, raising to 3% at $\beta = 8.5$ and asymptotically reaching the value of the uncertainty on $\Lambda$ of 8%.

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