How to Deal With Ratio Metrics When Accounting for Intra-User Correlation in A/B Testing

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ABSTRACT

We consider the A/B testing problem at the presence of correlation among observations coming from the same user. Furthermore, users may come from various segments where levels of correlation are different. A uniformly minimum-variance unbiased estimator of the population mean, called correlation-adjusted mean, is proposed to account for such correlation structure. It is proved theoretically and numerically better than the usual na"ive mean estimator and normalized mean estimator (taking average within users and then across users). The correlation-adjusted mean method is still unbiased but has reduced variance so it gains additional power. Several simulation studies are designed to show the estimation accuracy of the correlation structure, effectiveness in reducing variance, and capability of obtaining more power. An application to the eBay data is conducted to conclude this paper.

KEYWORDS
A/B Testing, repeated measures, uniformly minimum-variance unbiased estimator, stratification, sensitivity, variance reduction

1 INTRODUCTION

The A/B testing is an online randomized controlled experiment that compares the performance of a new design B (treatment) of web service with the current design A (control). It randomly assigns traffic/experiment units (users, browse id/guid, XID and so on) into one of the two groups, collects metrics of interests, conducts hypothesis testing to claim significant treatment effect and estimates the average treatment effect/lift (ATE) over the whole targeting group. Nowadays, such experiment is no longer restricted to evaluate and compare variants of websites but also applied to many other similar business problems. Statistically, the simple A/B test is a two-sample t-test but there are many situations that the data structure is no longer simply two independent groups of observations. For example, a user may visit a website multiple times so those visits can be considered as repeated measures of the same user. There should be a certain level of correlation among those visits. Such intra-user correlation has big impacts when dealing with ratio metrics. Here ratio metrics involve the analysis units having smaller granular level comparing with experiment units, such as click through rate (CTR), average selling price (ASP), search result page to view item (SRP to VI), search result page exit rate (SRP Exit Rate) and so on. At the same time, there can be types of users such as cell phone users, laptop users, and desktop users. In this paper, we call a type of users as a segment. Those users in the same segment shall share some level of behavioral similarity. In this paper, we focus on ratio metrics and further assume the intra-user correlation across the repeated visits is the same for each user. The main purpose of this article is to shed light on how to adjust the A/B test on ratio metrics at the presence of repeated measures of users as well as multiple segments of users.

Many researchers have considered the same kind of data structure [1, 3]. In practice, it is common to use the naive sample mean difference as an unbiased estimator of ATE. The supporters [1, 3] argue that it naturally matches the definition of the metrics, like CTR as summation of all clicks divided by summation of all impressions, taking the expected number of repeated measures as part of the metrics. On the other hand, we notice that there is another way of computing this ratio metrics called normalized mean: compute the ratio metrics for each user/experiment-unit and then take the average of this user level normalized metrics. Critics [3] reported that the later (normalized mean) is not a proper estimator of ATE, at least under point of view from metrics definition. Researchers [4] also observed that the two ratio metrics could lead to contradicted conclusions via two sample z-test.

This article provides another view of this problem and proposes a new way to account for the intra-user correlation for user data with repeated measures. Given which segment each user belongs to, we further assume the correlation among repeated measures for each user is the same. Under this framework, we stratify the data by segments and propose new estimating methods for the correlation. With the accurately estimated correlation, we are able to construct a uniformly minimum-variance unbiased estimator (UMVUE) of the population mean called "correlation-adjusted mean" for the data with repeated measures. Hence, we shall have improved power in the A/B test. We formulate the problem in details in Section 2, deploy the proposed method in Section 3, and support it with numerical analysis in Section 4.

2 PROBLEM FORMULATION

Suppose we are conducting a randomized controlled experiment and denote each experiment unit by $i$. We observe repeated measurements of metrics $(X_{i1}, \ldots, X_{in_i})$ where $n_i$ denotes the number of repetitions for unit $i$. This data structure is common in many fields. In the context of A/B testing for website comparison, an experiment unit is a user while repeated measures refer to the multiple website visits from the same user. We are interested in testing...
We utilize a simplified example with Simpson’s paradox to illustrate where the sub-indices \( \text{ctr} \) varies significantly. The contradiction information in A/B testing further discuss the similarities and differences between the two example explains the contradictory of the two ratio metrics potentially accounts for Simpson’s paradox. At the following, we will further discuss the similarities and differences between the two ratio metrics in a randomized controlled experiment with repeated measures.

### Contradicted Directional Conclusion Explained by Simpson’s Paradox

We utilize a simplified example with Simpson’s paradox to illustrate the contradictory conclusions found in [4] can even be possible using these two ratio metrics. A randomized controlled experiment is conducted in which two observations are repeatedly measured multiple times. Table 1 summarizes the experiment results. We calculate the two ratio metrics:

\[
\hat{R}^A = \frac{\sum_{i=1}^{N} \sum_{j=1}^{m_i} X_{ij}}{\sum_{i=1}^{N} n_i}, \quad \hat{R}^B = N^{-1} \sum_{i=1}^{N} \frac{\sum_{j=1}^{m_i} X_{ij}}{n_i}.
\]

Both ratios can be used to estimate the population mean but which one is better? What are the differences between the two? From many people, \( \hat{R}^A \) is more natural comparing with \( \hat{R}^B \) and many papers from LinkedIn, Microsoft [3], Uber [8], Yandex [1], [4] follow this definition. Yandex [4] reported that these two metrics can have further discussion the similarities and differences between the two ratio metrics in a randomized controlled experiment with repeated measures.

### Remarks.

Usually Simpson’s paradox happens when there are heterogeneous treatment (and control) effects among all subgroups (strata) and the subgroup proportion between treatment and control varies significantly. Currently, evaluating the treatment information in A/B testing provided by \( \hat{R}_{\text{trt}} - \hat{R}_{\text{ctr}} \) and \( \hat{R}_{\text{trt}} - \hat{R}_{\text{ctr}} \) can be a sign of Simpson’s paradox.

### Table 1: Simpson’s Paradox

|          | Control | Treatment |
|----------|---------|-----------|
| \( \sum_{j=1}^{m_i} X_{ij} \) | \( n_i \) | \( \sum_{j=1}^{m_i} X_{ij} \) | \( m_i \) |
| Obs. 1   | 200     | 300       | 30       | 40 |
| Obs. 2   | 10      | 30        | 100      | 200 |

In practice, continuously monitoring strata distribution between treatment and control will help us avoid such situation and the proposed estimator provides less bias for ATE. For situation like unbalanced strata distribution between treatment and control, we suggest to use the proposed estimator (in the following) to estimate \( \mu \) at each strata, then weighted average these estimators with weights proportional to strata population share over the whole population.

### 3 APPROACH

We consider the problem of choosing between these two ratios in a randomized controlled experiment with repeated measures under special designs as follows. The \( X_{ij} \) are identical distributed with correlation \( \rho_i \) and follow the same distribution \( F \) for each unit/user \( i \). Let \( E(X_{ij}) = \mu \) and \( Var(X_{ij}) = \sigma^2 \). It is easy to show that \( r_i = \frac{\sum_{j=1}^{N} X_{ij}/n_i}{} \) is an unbiased estimator of \( \mu \) with variances \( \sigma_i^2 = \sigma^2(1 - \rho_i)/n_i + \rho_i \) (please see Appendix A for proof).

### Obs. 1

| \( n_i \) | \( \rho_i \) | \( \sigma_i \) |
|----------|--------------|--------------|
| A        | 200          | 300          | 30          |
| B        | 10           | 30           | 100         | 200         |

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\( \tau_j \sim N(0, \sigma_j^2) \) representing random deviation due to repeated measures, and \( e_{ij} \sim N(0, \sigma_i^2) \) representing error. With such formulation, \( \text{corr}(Y_{ij}, Y_{ij'}) = \sigma_j^2 / (\sigma_j^2 + \sigma_i^2) \) is a parameter independent of \( i \) or \( j \). This correlation is usually called intra-class correlation. Although our data design does not necessarily follow this formulation, we argue that it is reasonable to assume \( \text{corr}(Y_{ij}, Y_{ij'}) \) is a constant for the problem of repeated measures among users. Hence, we simplify Equation 1 by setting \( \rho_i = \rho \).

**Theorem 1.** With the model setting above, we define

\[
S_1 = \frac{1}{\sum n_i} \sum n_i \sum n_j (X_{ij} - \bar{R}^A)^2,
\]

\[
S_2 = \frac{1}{\sum n_i} \sum n_i \sum n_j (X_{ij} - \bar{R}^B)^2,
\]

\[
S_3 = \frac{1}{\sum (n_i - 1)} \sum n_i \sum (X_{ij} - r_i)^2,
\]

\[
S_4 = \frac{1}{\sum (n_i - 1)} \sum n_i \sum n_j (X_{ij} - r_i)(X_{ij'} - r_i),
\]

\[
S_5 = \frac{1}{\sum n_i} \sum n_i \sum n_j (X_{ij} - \bar{R}^A)(X_{ij'} - \bar{R}^A),
\]

\[
S_6 = \frac{1}{\sum n_i} \sum n_i \sum n_j (X_{ij} - \bar{R}^B)(X_{ij'} - \bar{R}^B).
\]

Then we have

\[
E(S_1) = \sigma^2, \quad E(S_2) = \sigma^2, \quad E(S_3) = (1 - \rho)\sigma^2, \quad E(S_4) = (\rho - 1)\sigma^2, \quad E(S_5) = \rho\sigma^2, \quad E(S_6) = \rho\sigma^2.
\]

The proof of Theorem 1 can be found in the Appendix D. We can get the estimates of \( \rho \) and \( \sigma^2 \) from above and plug it into our optimal formula (1). We shall use simulation and real data to show this is a better solution.

**Remarks.** (1) Intuitively speaking, \( \bar{R}^A \) adopts weights proportional to \( n_i \) (number of repeated measures/segments) by expecting repeated measures within a unit are completely independent (\( \rho = 0 \)); while \( \bar{R}^B \) adopts weights proportional to number of users (\( \rho = 1 \)) by assuming repeated measures are duplicate copies (\( \rho = 1 \)). When \( \rho \) is small, repeated measures within each unit/user are most likely independent, so weighted by sessions (\( \bar{R}^A \), naive mean) is expected to have smaller variance than weighted by users (\( \bar{R}^B \), normalized mean); and vice versa, when \( \rho \) is large, \( \bar{R}^B \) shall have smaller variance than \( \bar{R}^A \). As always, \( \bar{R}^\rho \) shall have the smallest variance at all.

(2) Improving metric sensitivity (aka. variance reduction) is one of primary goals in our large scale trustworthy experimentation platform. The two options proposed follow this guideline: we always prefer metric with smaller variance.

(3) Although equal correlation is assumed between repeated measures within the same observation/user, we argue the the proposed adjustment method may also work when the correlation structure is more general. We will show it through a simulation example in Section 4 below.

### 3.1 Generalization to multi-segment user cases

The proposed correlation-adjusted mean approach only provides a way to obtain a UMVUE of population mean \( \mu \) when expected response (\( \mu_i \)) in repeated measures within each unit \( i \) is the same, aka. the common mean problem. However, in practice, such assumption may not be the case since there may be different types of users which can lead to unequal means. In this article, we call each type of users as a segment. Considering this, we propose to generalize the above estimator in the following way.

In the context of website comparison, most of time, important features of experiment units, i.e. users, are given in advance. A common feature that can define the type of users can be the type of devices that are being used during the experiment. For instance, desktop users may be more or less coherent in each visit to the website than mobile users. Therefore, it is reasonable to assume that the segment information is given in the A/B testing problem considered in this article. Provided which segment/type each user belongs to, we just need to stratify users by segment and compute the correlation-adjusted mean using (1) separately. To integrate the estimators from multiple segments, we can use the weight sum of these estimators with the weight proportional to either the sample size of each segment (weight on users) or the population size of each segment (weight on sessions). The choice of weights is beyond the scope of this article but we refer to [5–7] for more discussion on it.

**Remarks.** Similar to remarks in section 2.1, the large deviation (inconsistency) between \( \bar{R}^A \) and \( \bar{R}^B \) within treatment or control group is a sign of heterogeneous effects among users presence. We need to further categorize each user into segments and integrate the estimators from multiple segments with proper weights. If strata weight within sample is introduced to estimate the population strata weight, we can show that naive mean (\( \bar{R}^A \)) of whole users is the same as integrate multiple naive means within each segments with weights on number of sessions of each segment; normalized mean (\( \bar{R}^B \)) of whole users is the same as integrate multiple normalized means within each segments with weights on number of users of each segment.

### 4 NUMERICAL ANALYSIS

We illustrate the effectiveness of the proposed method with several simulation studies and real data analysis. In the simulation studies, we present four examples to demonstrate the following points.

First, we simulate all users coming from the same segment. With Theorem 1, we have multiple ways to estimate \( \rho \) and we want to evaluate the accuracy for each of them. Second, we simulate another example similar to the previous one. We estimate \( \rho \) from Theorem 1 and plug it in Equation (1) to obtain the correlation-adjusted mean. We empirically calculate the true standard deviation of \( \bar{R}^\rho \), \( \bar{R}^A \), and \( \bar{R}^B \) with bootstrapping (1000 iterations). We will show that the true standard deviation of \( \bar{R}^\rho \) is smaller than that of \( \bar{R}^A \) and \( \bar{R}^B \). We also investigate how the standard errors of \( \bar{R}^A \) and \( \bar{R}^B \) vary with \( \rho \). Third, we simulate an additional example that mimics the real A/B testing where treatment group and control
We demonstrate that more power can be obtained in the A/B test within one experiment group (treatment or control). We find that as we explained above, we design four simulation examples 1000 repetitions. We see from the Table 2 that the last four methods estimate the success of an A/B test, and it could save up to 6.6% to 40.3% sample sizes depending on ρ.

### 4.1 Simulation Studies

As we explained above that we design four simulation examples for four different purposes. The first example only contains one segment of users and we want to show the estimation accuracy of ρ as below.

**Example 1.** We simulate 1000 users in the following manner. For the i-th user, we generate the number of observations ni from Poisson(10)+1. Denote each observation by Xi,j with j ∈ {1, · · · , n1}. Let Xi,j ∼ Bernoulli(pi) with pi ∼ N(0.3, 0.05). Note that observations belonging to the same user may not necessarily independent of each other. We set corr(Xi,j,Xi′j) = ρ for j ̸= j′. There are a number of ways to introduce correlation structure for Bernoulli distribution. We choose to firstly generate multivariate normal observations X̄i = (X̄i1, · · · , X̄in1)⊤ ∼ N(0, Σ(ρ)) with diagonal elements of Σ(ρ) being 1 and off-diagonal elements being ρ. We then dichotomize X̄ij to {0, 1} at Φ −1(1 − pi) where Φ(·) is the cumulative density function of a standard normal distribution. In other words, we let X̄ij = 1 if X̄ij > Φ −1(1 − pi) and X̄ij = 0 otherwise. We consider ρ ∈ {0, 0.2, 0.4, 0.6, 0.8} and present the estimation results in columns 3-5 of Table 2. The column 3 presents the true correlation between dichotomized X̄i,j and X̄i′j. Note that the new correlation (ρ̂) is different from their correlation (ρ̂) before dichotomization. The experiments are repeated 1000 times.

The estimation accuracy of ρ for Example 1 is presented in Table 2. The S3/S1, S3/S2, S4/S1, S4/S2, S5/S1, and S6/S2 refer to the estimation method of ρ by calculating the sample version of 1−E(S3)/E(S1), 1−E(S3)/E(S2), E(S4)/E(S1)+1, E(S4)/E(S2)+1, E(S5)/E(S1), and E(S6)/E(S2), respectively. The columns ρ̂ and SD(ρ̂) denotes the mean and standard deviation of estimated values based on 1000 repetitions. We see from the Table 2 that the last four methods estimate ρ very accurately since ρ̂ is fairly close to the true value ρ.

**Example 2.** The capability of accurately estimating ρ is shown in Example 1. We now proceed to show that the proposed estimator of μ, R̂μ, will have a reduced variance compared to the naive mean estimator and the normalized naive mean estimator. We want to show it for more values of ρ and show the robustness of our method under various settings. Therefore, we simulate a separate example here. Following the notation in Example 1, we generate 1000 users with n1 ∼ Poisson(2)+1, p1 ∼ N(0.3, 0.04), and ρ ∈ {0.1, 0.2, · · · , 0.9}.

| ρ | 0.0022 | 0.1015 | 0.0046 |
|---|---|---|---|
| ρ̂ | 0.1015 | 0.0046 |
| SD(ρ̂) | 0.0046 |

Table 2: Estimation Accuracy of ρ in Example 1

We then present the results of Example 2 in Table 3. There are four estimators that have high accuracy in estimating ρ. We only choose S4/S1 for illustration purpose in Example 2. Results should be similar if the other three estimators are used. In Table 3, column ˚y gives the true value of μ approximated by the mean or normalized mean of all observations; column ˚y gives the estimated value of μ; column σ ˚y gives the true population standard deviation of ˚y calculated based on values of 1000 repetitions; column σ ˚y gives the estimated standard deviation for each method. Three methods, naive mean (R̂μ), normalized mean (R̂μ), and the proposed method, weighted mean (R̂μ) are included in the table. It is clear that the correlation-adjusted mean (R̂μ) has the smallest standard deviation across all values of ρ. In the meanwhile, the standard error of R̂μ is smaller than R̂μ if ρ < 0.3, and the standard error of R̂μ is bigger than R̂μ if ρ > 0.3.

**Example 3.** We further illustrate that additional power can be gained if using the proposed weight mean method. We design an example where users in the treatment and control groups come from different segments. Let us focus on the control group first. Consider...
Table 3: Variance reduction compared to naive and normalized mean in Example 2

|          | $\hat{\bar{y}}$ (truth) | $\hat{\bar{y}}$ (estimate) | $\sigma_{\hat{\bar{y}}}$ (truth) | $\sigma_{\hat{\bar{y}}}$ (estimate) | $\rho$ |
|----------|--------------------------|----------------------------|----------------------------------|----------------------------------|-------|
| Naive    | 0.29971                  | 0.29971                    | 0.00912                          | 0.00947                          | 0.1   |
| Normalized| 0.29953                  | 0.29953                    | 0.00978                          | 0.01016                          | 0.1   |
| Corr. Adj.| 0.29972                  | 0.29972                    | 0.00907                          | 0.00942                          | 0.1   |
| Naive    | 0.30006                  | 0.30006                    | 0.01074                          | 0.01041                          | 0.2   |
| Normalized| 0.29997                  | 0.29997                    | 0.01112                          | 0.01074                          | 0.2   |
| Corr. Adj.| 0.30007                  | 0.30007                    | 0.01061                          | 0.01026                          | 0.2   |
| Naive    | 0.29968                  | 0.29968                    | 0.01162                          | 0.01126                          | 0.3   |
| Normalized| 0.29947                  | 0.29947                    | 0.01145                          | 0.01126                          | 0.3   |
| Corr. Adj.| 0.29966                  | 0.29966                    | 0.01127                          | 0.01100                          | 0.3   |
| Naive    | 0.30015                  | 0.30015                    | 0.01215                          | 0.01206                          | 0.4   |
| Normalized| 0.30009                  | 0.30009                    | 0.01185                          | 0.01177                          | 0.4   |
| Corr. Adj.| 0.30016                  | 0.30016                    | 0.01171                          | 0.01160                          | 0.4   |
| Naive    | 0.30020                  | 0.30020                    | 0.01336                          | 0.01280                          | 0.5   |
| Normalized| 0.30005                  | 0.30005                    | 0.01277                          | 0.01226                          | 0.5   |
| Corr. Adj.| 0.30015                  | 0.30015                    | 0.01272                          | 0.01219                          | 0.5   |
| Naive    | 0.29998                  | 0.29998                    | 0.01374                          | 0.01349                          | 0.6   |
| Normalized| 0.29965                  | 0.29965                    | 0.01303                          | 0.01274                          | 0.6   |
| Corr. Adj.| 0.29982                  | 0.29982                    | 0.01297                          | 0.01270                          | 0.6   |
| Naive    | 0.30064                  | 0.30064                    | 0.01404                          | 0.01417                          | 0.7   |
| Normalized| 0.30060                  | 0.30060                    | 0.01313                          | 0.01321                          | 0.7   |
| Corr. Adj.| 0.30062                  | 0.30062                    | 0.01309                          | 0.01321                          | 0.7   |
| Naive    | 0.29952                  | 0.29952                    | 0.01476                          | 0.01479                          | 0.8   |
| Normalized| 0.29953                  | 0.29953                    | 0.01370                          | 0.01364                          | 0.8   |
| Corr. Adj.| 0.29955                  | 0.29955                    | 0.01384                          | 0.01362                          | 0.8   |
| Naive    | 0.29999                  | 0.29999                    | 0.01541                          | 0.01540                          | 0.9   |
| Normalized| 0.29974                  | 0.29974                    | 0.01412                          | 0.01406                          | 0.9   |
| Corr. Adj.| 0.29977                  | 0.29977                    | 0.01411                          | 0.01407                          | 0.9   |

Table 4: Variance reduction compared to naive and normalized mean in Example 4

|          | $\hat{\bar{y}}$ (truth) | $\hat{\bar{y}}$ (estimate) | $\sigma_{\hat{\bar{y}}}$ (truth) | $\sigma_{\hat{\bar{y}}}$ (estimate) |
|----------|--------------------------|----------------------------|----------------------------------|----------------------------------|
| Naive    | 0.28557                  | 0.28557                    | 0.01403                          | 0.01394                          |
| Normalized| 0.29205                  | 0.29205                    | 0.01376                          | 0.01355                          |
| Corr. Adj.| 0.29074                  | 0.29074                    | 0.01363                          | 0.01342                          |

three segments of subjects/users following the multinomial distribution $(C_1, C_2, C_3)^T \sim \text{Multinomial}(1, 0.5, 0.5, 0.5)$. For segment 1, $n_1 \sim \text{Poisson}(0.5) + 1$ and $X_{ij} \sim \text{Bernoulli}(p_{ij})$ with $p_{ij} \sim N(0.3, 0.04)$. For segment 2, $n_2 \sim \text{Poisson}(0.5) + 1$ and $X_{ij} \sim \text{Bernoulli}(p_{ij})$ with $p_{ij} \sim N(0.5, 0.08)$. For segment 3, $n_3 \sim \text{Poisson}(0.5) + 1$ and $X_{ij} \sim \text{Bernoulli}(p_{ij})$ with $p_{ij} \sim N(0.7, 0.04)$. In any of the three segments, we set $\text{corr}(X_{ij}, X_{ij'}) = 0.3$. Then we simulate the data for the treatment group in a very similar way but the only difference is $d = \phi_{ij}$ follows $N(0.3 + d, 0.04)$, $N(0.5 + d, 0.08)$, and $N(0.7 + d, 0.04)$, respectively. We consider $d \in \{0.01, 0.02, \ldots, 0.08\}$. The experiment is repeated 1000 times for each $d$. Figure 1 shows the comparison of power in this example.

Example 4. We explore the effectiveness of the proposed method in handling unequal correlation structure. To be more specific, we borrow the setting in example 2 but the correlation between $X_{ij}$ and $X_{ij'}$ is $\rho_{ij-j'}$ for any $j \neq j'$. We set $\rho = 0.9$ since in practice, it is common to have high correlation for repeated measures next to each other but low correlation when they are far from each other.

The results for this example is presented in Table 4. It is clear that both true and estimated standard deviations are smaller based on our correlation adjusted method. Note that the true correlation structure is no longer equal between observations but in a more general autoregressive pattern.

4.2 Application to eBay Data

For the purpose of checking how significant of $\rho$ is and how much improvement can be archived in real data, we randomly sampled search activities from users with each size of 500,000 users on eBay global site with primary metrics as "Ratio $R_1$" (alike Exit Rate) for UK and "Ratio $R_2$" (alike CTR) for US in some treatment groups. We further repeated 5 replicas of previous process with equal size 500,000 and showed the average estimation results at table 5. The reported metric estimations were added a constant to hide the real information, but it should not change our conclusion in follows. To properly evaluate the estimator standard errors, we use bootstrap (1000 iterations) with re-sampling id being the experiment unit id to be an educated guess at each replica.

Our results in Table 5 compared the standard errors of naive mean ($\hat{R}^1$), normalized mean ($\hat{R}^2$) and correlation-adjusted mean ($\hat{R}^3$). The method S4/S1 is used to accurately estimate intra-user correlation $\rho$. It clearly revealed that the intra-user correlation coefficient $\rho$ did present in our data set, and varied from 9.6% to 56% for "Ratio $R_1$" at UK site and "Ratio $R_2$" at US site respectively. The root cause of different intra-user correlated behaviors ($\rho$) between UK and US is actually due to a search feature "Multi Aspect Guidance" launched in US site only. This feature provides guided aspects selections for US search users further navigating/retrieving search result information. It ends up with multiple search impressions with same search intention for US site users. The dependence among repeated measures within experiment unit could not be ignored and should be carefully addressed to pick the right estimation method. For small $\rho$ (9.6%), $\hat{R}^A$ showed 3% improvement on standard error comparing with $\hat{R}^B$, which in hence saved 56%.
improvement on sample sizes comparing with $R^A$. For both cases, $R^B$ saved up to 15.2% and 40.3% sample sizes correspondingly.

We noticed the bias between naive mean and normalized mean for "Ratio $R^2\)" could be a sign of heterogeneous user effects presence. A proper segments classification algorithm is needed to further analysis this heterogeneous effect, which is beyond the scope of this paper.

5 CONCLUSION AND RESTRICTION

In this paper, we studied how to estimate ratio metrics in a randomized controlled experiment with repeated measures within each experiment unit. As the intra-unit correlation coefficient $\rho$ could not be ignored, we established a way to accurately measure the severity of intra-unit correlation coefficients $\rho$. We showed the naive mean ($R^A$) and normalized mean ($R^B$) were both weighted user means with different weights. We further proposed correlation-adjusted mean ($R^C$), which adopted the optimal weights depending on $\rho$. Our simulation and real data empirical analysis validated that we can accurately estimate $\rho$ and built more sensitive ratio metric estimators based on $\rho$. Due to the variance reduction technique, we shall improve the power as well as requiring less sample size for A/B testing, which improves experiment efficiency.

The main restriction of the proposed method is on its assumption of common correlation for different users within the same segment. It is possible that there is variation of such correlation for the same type of users (from the same segment). Such observation motivates us to consider more flexible settings where the correlation for a user within the same segment may follow a random distribution, and the random distribution can be different across segments. However, this type of setting is the subject of our future work.

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A VARIANCE FORMULA OF SAMPLE MEAN WITH CORRELATED MEASURE

Claim: The $X_i$ are identical distributed with correlation $\rho$ and follow the same distribution $F$. Let $E(X_j) = \mu$ and $Var(X_j) = \sigma^2$. Then:

- $r = \sum_{j=1}^{n} X_j/n$ is an unbiased estimator of $\mu$.
- The variance of $r$ is $\sigma_r^2 = \sigma^2 / (n(1-\rho/n + \rho))$.

**Proof.** Since

$$E(r) = E\left(\frac{\sum_{j=1}^{n} X_j}{n}\right) = \frac{\sum_{j=1}^{n} E(X_j)}{n} = \frac{n\mu}{n} = \mu,$$

then $r = \sum_{j=1}^{n} X_j/n$ is an unbiased estimator of $\mu$.

The variance of $r$ is

$$\sigma_r^2 = E((r - E(r))^2)$$

$$= E((r - \mu)^2)$$

$$= E\left(\sum_{j=1}^{n} (X_j - \mu)^2\right)$$

$$= \frac{1}{n^2} E\left(\sum_{j=1}^{n} (X_j^2 - \mu^2)\right)$$

$$= \frac{1}{n^2} \left(\sum_{j=1}^{n} (\mu^2 + \sum_{k=1}^{n} \sum_{m=1, m \neq k} E(X_k)(X_m - \mu))\right)$$

$$= \frac{1}{n^2} \left(\sum_{j=1}^{n} \sigma_j^2 + \sum_{k=1}^{n} \sum_{m=1, m \neq k} \sigma_k^2\rho\right)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \sigma_j^2 + \frac{n-1}{n} \sigma^2 \rho$$

$$= \sigma^2 \{(1 - \rho)/(n + \rho)\}.$$

\[\square\]

B PROOF OF UMVUE

Claim: Suppose $r_i$ with $i = 1, \ldots, N$ are independent with each other and $E(r_i) = \mu$, $Var(r_i) = \sigma_i^2$. Then

- for an arbitrary weight vector $W = (w_1, \ldots, w_N)^T$ with $\sum_{i=1}^{N} w_i = 1$, the following quantity is also an unbiased estimator of $\mu$: $\tilde{R}^W = \sum_{i=1}^{N} w_i r_i$.
- the UMVUE of $\mu$ is by setting: $w_i = \frac{1}{\sigma_i^2}$.

**Proof.**

$$E(\tilde{R}^W) = E\left(\sum_{i=1}^{N} w_i r_i\right) = \sum_{i=1}^{N} w_i E(r_i) = \sum_{i=1}^{N} w_i \mu = \mu.$$

Thus, $\tilde{R}^W$ is an unbiased estimator of $\mu$. 

\[\square\]
To find the minimum value of $\text{Var}(\hat{R}^W) = \sum_{i=1}^{N} w_i^2 \sigma_i^2$ under restriction $\sum_{i=1}^{N} w_i = 1$, we use method of Lagrange multiplier: 

$$L = \sum_{i=1}^{N} w_i^2 \sigma_i^2 - \lambda(\sum_{i=1}^{N} w_i - 1).$$

Since $\frac{\partial L}{\partial w_i} = 2\sigma_i^2 - \lambda = 0$ for $i = 1, \ldots, N$, 

$$\frac{\partial L}{\partial w_i} = 2\sigma_i^2 - \lambda w_i = 0$$

and it achieves at:

$$\frac{\partial L}{\partial w_i} = 2\sigma_i^2 - \lambda w_i = 0$$

By solving equations above we have $w_i = \frac{\sigma_i^2}{\sum_{i=1}^{N} \sigma_i^2} \propto \frac{1}{\sigma_i^2}$. \hfill \Box

C \hspace{1em} \text{APPROXIMATION}

Claim: $\frac{n}{1+(n-1)\rho} = n^{1-\rho}$.

Proof.

From Taylor series expansion, the 1st degree polynomial approximation of function $f(n) = n^\rho$ at $n = 1$ is $n^\rho \approx 1 + (n - 1)\rho$. Therefore,

$$n^{1-\rho} \approx \frac{n}{1 + (n - 1)\rho}.$$ \hfill \Box

D \hspace{1em} \text{PROOF OF THEOREM 1}

Without loss of generality, we set $\mu = 0$. It is obvious that:

$$E(r_i^2) = \left(\frac{1}{n_i} + \frac{n_i - 1}{n_i} \rho\right)\sigma_i^2,$$

$$E(\hat{R}^2) = \left(\frac{1}{\sum_{i=1}^{N} n_i} \right)^2 \sum_{i=1}^{N} n_i^2 E(r_i^2) = \left(\frac{1}{\sum_{i=1}^{N} n_i} \right)^2 \sum_{i=1}^{N} n_i \left(1 + (n_i - 1)\rho\right)\sigma_i^2$$

$$= \left(\frac{1}{\sum_{i=1}^{N} n_i} \right)^2 \sum_{i=1}^{N} n_i \left(1 + (n_i - 1)\rho\right)\sigma_i^2.$$

We also have the following for any $i, j$:

$$E(r_i X_{ij}) = \left(\frac{1}{n_i} + \frac{n_i - 1}{n_i} \rho\right)^2 \sigma_i^2$$

$$E(\hat{R}^4 X_{ij}) = \frac{n_i E(r_i X_{ij})}{\sum_{i=1}^{N} n_i} = \frac{n_i \left(1 + (n_i - 1)\rho\right)\sigma_i^2}{\sum_{i=1}^{N} n_i}$$

$$E(\hat{R}^4 N_{ij}) = \frac{E(r_i X_{ij})}{n_i} = \frac{1 + (n_i - 1)\rho}{N n_i} \sigma_i^2.$$

Thus,

$$E((X_{ij} - r_i)^2) = E(X_{ij}^2 - 2r_i X_{ij} + r_i^2)$$

$$= \sigma_i^2 - 2\left(\frac{1}{n_i} + \frac{n_i - 1}{n_i} \rho\right)\sigma_i^2 + \left(\frac{1}{n_i} + \frac{n_i - 1}{n_i} \rho\right)\sigma_i^2$$

$$= \frac{n_i - 1}{n_i} \left(1 - \rho\right)\sigma_i^2.$$

$$E((X_{ij} - \hat{R}^4)^2) = E(X_{ij}^2 - 2\hat{R}^4 X_{ij} + \hat{R}^8)$$

$$= \sigma_i^2 - 2\left(1 + (n_i - 1)\rho\right)\sigma_i^2 + \left(\frac{1}{\sum_{i=1}^{N} n_i} + \frac{\sum_{i=1}^{N} n_i (n_i - 1)}{(\sum_{i=1}^{N} n_i)^2}\rho\right)\sigma_i^2,$$

$$= \left(\frac{1}{\sum_{i=1}^{N} n_i} + \frac{\sum_{i=1}^{N} n_i (n_i - 1)}{(\sum_{i=1}^{N} n_i)^2}\rho\right)\sigma_i^2.$$

Therefore, we have

$$E(S_i) = n_{\sum_{i=1}^{N} n_i - 1} \sum_{i=1}^{N} n_i \left(1 + (n_i - 1)\rho\right)\sigma_i^2 + \left(\frac{1}{\sum_{i=1}^{N} n_i} + \frac{\sum_{i=1}^{N} n_i (n_i - 1)}{(\sum_{i=1}^{N} n_i)^2}\rho\right)\sigma_i^2$$

$$= \sigma_i^2 - \frac{2 + 2(n_i - 1)\rho}{\sum_{i=1}^{N} n_i} + \frac{\sum_{i=1}^{N} n_i (n_i - 1)\rho}{(\sum_{i=1}^{N} n_i)^2}.$$
\[ E(S_k) = \frac{1}{\sum_{i=1}^{N} n_i(n_i - 1)} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \sum_{f \neq f'} \left\{ \rho \sigma^2 - \frac{1 + (n_i - 1)\rho}{\sum_{i=1}^{N} n_i} + \frac{\sum_{i=1}^{N} n_i^2}{(\sum_{i=1}^{N} n_i)^2} \rho \sigma^2 \right\} \]
\[ = \sigma^2 \left\{ \rho + \frac{1}{\sum_{i=1}^{N} n_i} - \frac{2 \sum_{i=1}^{N} n_i^2(n_i - 1)}{(\sum_{i=1}^{N} n_i) \sum_{i=1}^{N} n_i(n_i - 1)} \rho + \frac{\sum_{i=1}^{N} n_i^2}{(\sum_{i=1}^{N} n_i)^2} \rho \right\} \]
\[ = \rho \sigma^2. \]

\[ E(S_k) = \frac{1}{\sum_{i=1}^{N} n_i(n_i - 1)} \sum_{i=1}^{N} \sum_{j=1}^{n_i} \sum_{f \neq f'} \left\{ \rho \sigma^2 - \frac{1 + (n_i - 1)\rho}{N n_i} + \frac{1}{N} \sum_{i=1}^{N} n_i \left\{ 1 + (n_i - 1)\rho \right\} \right\} \]
\[ = \sigma^2 \left\{ \rho - \frac{\sum_{i=1}^{N} n_i(n_i - 1) + (n_i - 1)^2 \rho}{N \sum_{i=1}^{N} n_i(n_i - 1)} + \left\lceil \frac{1}{N} \sum_{i=1}^{N} n_i \left\{ 1 + (n_i - 1)\rho \right\} \right\rceil \right\} \]
\[ = \sigma^2 \left\{ \rho - \frac{\sum_{i=1}^{N} n_i(n_i - 1)}{N \sum_{i=1}^{N} n_i(n_i - 1)} - (1 - \rho) + \left\lceil \frac{1}{N} \sum_{i=1}^{N} n_i \left\{ 1 - \rho \right\} - \frac{\rho}{N} \right\rceil \right\} \]
\[ = \rho \sigma^2. \]