BIFURCATION FROM INFINITY WITH APPLICATIONS TO REACTION-DIFFUSION SYSTEMS

CHIHIRO AIDA
Nakano Junior and Senior High School Attached to Meiji University
3-3-4 Higashi-Nakano, Nakano-ku, 164-0003, Japan

CHAO-NIEN CHEN
Department of Mathematics
National Tsing Hua University
Hsinchu, Taiwan

KOUSUKE KUTO*
Department of Applied Mathematics
Waseda University
3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555, Japan

HIROKAZU NINOMIYA
School of Interdisciplinary Mathematical Sciences
Meiji University
4-21-1 Nakano, Nakano-ku, Tokyo, 164-8525, Japan

Dedicated to Professor Wei-Ming Ni on the occasion of his 70th birthday

Abstract. The bifurcation method is one of powerful tools to study the existence of a continuous branch of solutions. However without further analysis, the local theory only ensures the existence of solutions within a small neighborhood of bifurcation point. In this paper we extend the theory of bifurcation from infinity, initiated by Rabinowitz [11] and Stuart [13], to find solutions of elliptic partial differential equations with large amplitude. For the applications to the reaction-diffusion systems, we are able to relax the conditions to obtain the bifurcation from infinity for the following nonlinear terms; (i) nonlinear terms satisfying conditions similar to [11] (all directions), (ii) nonlinear terms satisfying similar conditions only on the strip domain along the direction determined by the eigenfunction, (iii) $p$-homogeneous nonlinear terms with degenerate conditions.

2010 Mathematics Subject Classification. Primary: 35B32, 35B36; Secondary: 35B44.

Key words and phrases. Bifurcation from infinity, reaction-diffusion systems, $p$-homogeneity.

The second author was supported in part by the Ministry of Science and Technology of Taiwan, grant 105-2115-M-007-009-MY3. The third author was partially supported by JSPS KAKENHI Grant-in-Aid Grant Numbers 15K04948 and 19K03581. The fourth author would like to thank the Mathematics Division of NCTS (Taipei Office) for the warm hospitality and the support of the fourth author’s visit to Taiwan. The fourth author was partially supported by JSPS KAKENHI Grant Numbers JP26287024, JP15K04963, JP16K13778 and JP16KT0022.

* Corresponding author: Kousuke Kuto.
1. **Introduction.** Since in 1952 Turing [14] discovered a diffusion mechanism to generate periodic patterns, many interesting patterns have successfully been found out following from this ingenious idea [7]. This phenomenon is often referred to as *diffusion-induced instability*; that is, stable homogeneous states are destabilized by changing parameters and inhomogeneous periodic patterns emerge. In many situations, local bifurcation theory only ensures the existence of solutions near the bifurcation point, since it is difficult to trace the bifurcation branch globally unless further analysis can be made [1, 2, 3, 10] to obtain enough information on the eigenvalues for the linearization. To construct the solution with large amplitude, the problem-based-studies are often required.

In 1973 Rabinowitz [11] and Stuart [13] studied the bifurcation from infinity. They considered the abstract problem including

\[-(pu')' + qu = \lambda a(x)u + f(x, u, u', \lambda) \quad 0 < x < \pi, \quad (1.1)\]

with various boundary conditions. Here \(q, a\) are continuous in \((0, \pi)\), \(p\) is positive and continuously differentiable, and

\[f(x, u, \xi, \lambda) = o(\sqrt{u^2 + \xi^2}) \quad \text{as} \quad u^2 + \xi^2 \to \infty.\]

The linear Strum-Liouville eigenvalue problem

\[-(pu')' + qu = \mu a(x)u\]

together with suitable boundary conditions possesses an increasing sequence of eigenvalues \(\mu_n\) \((n \in \mathbb{N})\) with \(\mu_n \to \infty\). They showed that a branch of solutions bifurcate from \((\lambda, u) = (\mu_n, \infty)\), provided that \(\mu_n\) is a simple eigenvalue.

The first purpose of this paper is to extend the above mentioned results, ([11], [13]) for a single equation like (1.1), to two-component reaction-diffusion system (1.2). In Section 2.1, seeking the bifurcation from infinity, we give an extension of the above mentioned results, ([11], [13]) for a single equation like (1.1), to two-component reaction-diffusion system (1.2).

The second purpose of this paper is to relax the sufficient condition for the bifurcation from infinity, especially in dealing with Dirichlet boundary conditions. We weaken assumption (1.3) to certain strip domains which are depending on the eigenspaces, and call it a *bifurcation directionally from infinity*. More precisely, the following result will be established.

**Theorem 1.1.** Let \(\sigma\) be the simple eigenvalue of \(-\Delta\) with homogeneous Dirichlet boundary condition on \(\partial \Omega\) and let \(\phi(x)\) be the corresponding eigenfunction with \(\max_{x \in \Omega} \phi(x) = 1\). Assume that there exist \(d_2 > 0\) and \((a_1, a_2) \in S^1\) such that \((U, V) = (a_1, a_2)\phi\) is a solution of

\[d_1 \Delta U + h_{11} U + h_{12} V = 0, \quad d_2 \Delta V + h_{21} U + h_{22} V = 0 \quad \text{in} \quad \Omega\]
BIFURCATION FROM INFINITY

and $U = V = 0$ on $\partial \Omega$. Furthermore, assume that
\[ d_1 \sigma - h_{11} \neq 0, \quad \frac{d_2^2}{d_1^2} (d_1 \sigma - h_{11}) + d_2^2 \sigma - h_{22} \neq 0. \]

If there exist a positive constant $k_f$ and a non-negative continuous function $k(t)$ such that $\lim_{t \to 0} k(t) = 0$ and
\[
\left| f_i \left( \frac{a_1}{t}, \frac{a_2}{t} \right) \right| \leq k_f,
\left| \frac{\partial f_i}{\partial u} \left( \frac{a_1}{t} + z_1, \frac{a_2}{t} + z_2 \right) \right| \leq k(t),
\left| \frac{\partial f_i}{\partial v} \left( \frac{a_1}{t} + z_1, \frac{a_2}{t} + z_2 \right) \right| \leq k(t)
\]
for any $i = 1, 2$, $t \neq 0$ and $z_1, z_2 \in \mathbb{R}$, then, emanated from $(d_2, \infty)$, there exists a bifurcation branch of $\{(d_2, u) \mid (u(x; d_2))\}$ of (1.2) subject to the homogeneous Dirichlet boundary conditions.

Theorem 1.1 enables us to employ the method of bifurcation from infinity to study the stationary solutions in wider classes of reaction-diffusion systems. For example, in some prey-predator models of Holling type II or III, when $f_i(u, v)$ do not satisfy (1.3) but the hypotheses of Theorem 1.1. See Sections 2.2 and 2.3 for more details. We remark that different types of bifurcation from infinity can be found in [12].

The third purpose of this paper is to give a plausible reason for the blow-up induced by diffusion, as demonstrated by Mizoguchi, Ninomiya and Yanagida [5]. In such a reaction-diffusion system
\[
\begin{align*}
    u_t &= d_1 \Delta u + f_1(u, v), \\
    v_t &= d_2 \Delta v + f_2(u, v),
\end{align*}
\]
there exist solutions to blow up, while in the absence of diffusion all the solutions for the system of ordinary differential equations
\[
\begin{align*}
    u_t &= f_1(u, v) \\
    v_t &= f_2(u, v)
\end{align*}
\]
actually converge to the origin as $t$ tends to infinity (see also [4, 6]). A natural question we encountered is:

**What kind of equations do not inherit the global existence of solutions by adding diffusion?**

To clarify this question, Ninomiya and Weinberger [9] studied the influence of the linear perturbation on the global existence of solutions to the homogeneous nonlinearity (see also [8]). The authors shown that the linear term can result in blow-up phenomenon when the solutions, induced from the homogeneous nonlinearity, go to infinity as $t$ tends to infinity. They also pointed out that in this case there are stationary solutions which bifurcate from infinity. We extend this investigation to the reaction-diffusion system with $p$-homogeneous nonlinearity. Under a highly degenerate condition, we show that the bifurcation from infinity leads to the existence of stationary solutions (see Sections 3.1 and 3.2). Along the bifurcation branch, if the stationary solutions are unstable, it is plausible that the unstable manifold of such stationary solutions will reach to infinity which results in the time dependent solution to blow up in finite time. Therefore if a system of ordinary differential equations possesses a global attractor, it seems plausible to create blow-up phenomenon induced by diffusion when the bifurcation from infinity takes place.
2. Turing’s instability and bifurcation from infinity.

2.1. Bifurcation from infinity. In this subsection we apply the theorem for the bifurcation from infinity as in [11, 13] to the reaction diffusion system. Let us consider the semilinear system

\begin{align}
\frac{d}{dt} \Delta u + f_1(u, v) + h_1(u, v) &= 0, \\
\frac{d}{dt} \Delta v + f_2(u, v) + h_2(u, v) &= 0
\end{align}

(2.1)
in a bounded domain $\Omega \subset \mathbb{R}^N$ with homogeneous Neumann boundary conditions

\[ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega \]
or homogeneous Dirichlet boundary conditions

\[ u = v = 0 \text{ on } \partial \Omega. \]

In this section, we assume that the nonlinear terms $f_1(u, v)$ and $f_2(u, v)$ satisfy (1.3), whereas $h_1(u, v)$ and $h_2(u, v)$ are linear terms defined by

\[ h_1(u, v) = h_{11}u + h_{12}v, \quad h_2(u, v) = h_{21}u + h_{22}v \]

with some constants $h_{ij}$ ($i, j = 1, 2$). In what follows, we often regard $d_2$ as a bifurcation parameter. For use of the bifurcation theorem, we assume that

\[ \mathcal{L}^{d_2} := \begin{pmatrix}
\frac{d_1}{d_2} + \frac{1}{d_1} & h_{12} \\
\frac{1}{d_1} & \frac{d_2}{d_1} + h_{22}
\end{pmatrix} \]

has a zero eigenvalue of odd multiplicity at $d_2 = d_2^*$. More precisely, we assume that, if $d_2$ lies in a neighborhood of $\lambda^*_2$, then there exists an eigenvalue $\lambda(d_2)$ of odd multiplicity such that

\[ \mathcal{L}^{d_2} \varphi^{d_2} = \lambda(d_2) \varphi^{d_2}, \quad \lambda(d_2^*) = 0, \]

where $\varphi^{d_2} = (\varphi_1^{d_2}, \varphi_2^{d_2})^T$ represents the corresponding eigenfunction.

In order to apply the bifurcation theorems by Rabinowitz [10, 11] to (2.1), it is convenient to rewrite (2.1) into

\[ \begin{cases}
\frac{d}{dt} \Delta u + f_1(u, v) + h_1(u, v) + u = 0, \\
\frac{d}{dt} \Delta v + f_2(u, v) + h_2(u, v) + v = 0.
\end{cases} \]

(2.2)

The linear and nonlinear parts of the right-hand side of (2.2) will be denoted by

\[ L := (-d_1 \Delta + 1)^{-1} \begin{pmatrix}
\frac{h_{11}}{d_1} + 1 & \frac{h_{12}}{d_1} \\
\frac{h_{21}}{d_1} & \frac{h_{22}}{d_1} + 1
\end{pmatrix}, \]

and

\[ K(u, v) := (-d_1 \Delta + 1)^{-1} \begin{pmatrix}
f_1(u, v) \\
\frac{d_1}{d_2} f_2(u, v)
\end{pmatrix}, \]

respectively. Then (2.2) is expressed as

\[ u = Lu + K(u) \]

(2.3)

where

\[ u := \begin{pmatrix} u \\ v \end{pmatrix}, \quad K(u) := K(u, v). \]
Here we remark that $L = L^{d_2}$ and $K = K^{d_2}$ can be regarded as compact mappings from

$$E = \left\{ u = \begin{pmatrix} u \\ v \end{pmatrix} \mid u, v \in C(\Omega) \right\}$$

equipped with the norm

$$\|u\|_E = \max_{x \in \Omega} |u(x)| + \max_{x \in \Omega} |v(x)|.$$

to $E$. For simplicity of notation we may also use $f_1(u), h_1(u)$ and so on. Then we can obtain the following result similar to [11, Theorem 1.6] or [13] concerning the bifurcation from infinity.

**Theorem 2.1.** If $L^{d_2}$ has a zero eigenvalue of odd multiplicity when $d_2 = d^{\ast}_2$, then there is a bifurcation branch $\{(d_2, u) \mid u(x) = (u(x; d_2), v(x; d_2))\}$ which meets $(d^{\ast}_2, \infty)$.

Setting

$$w := \frac{u}{\|u\|^2},$$

we see that

$$\|u\| = \frac{1}{\|w\|}, \quad u = \frac{1}{\|w\|^2}w.$$

Then (2.3) is rewritten as

$$w = Lw + H(w),$$

where

$$H(w) := \begin{cases} \|w\|^2 K \left( \frac{w}{\|w\|^2} \right), & \text{if } w \neq 0, \\
0, & \text{if } w = 0. \end{cases}$$

**Proof.** To use [10, Lemma 1.2] or [11, Lemma 1.2], it suffices to confirm

(i) $L$ is compact,

(ii) $H(w) = o(\|w\|)$ as $\|w\| \to 0$,

(iii) $H$ is compact,

(iv) $1$ is an eigenvalue of $L$ of odd multiplicity when $d_2 = d^{\ast}_2$.

The first statement (i) is obvious. The statement (ii) follows from (1.3). The statement (iii) is a direct consequence of (i), (ii) and the definition of $H$. The last statement follows from the assumption of the eigenvalue of $L$.

As a consequence of [10, Lemma 1.2], if $1$ is an eigenvalue of $L$ of odd multiplicity, then the unbound component of the bifurcation branch meets $(d^{\ast}_2, \infty)$. □

**Remark 1.** We can use the other parameters such as $h_{ij}$ as a bifurcation parameter. We note that $K(0) = 0$ is not needed.

Let $\lambda$ and $\mu$ be any eigenvalue of $L$ and $L$, respectively. Furthermore, let $\sigma$ be any eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition or Dirichlet boundary condition on $\partial \Omega$. The following transversality condition for the simple zero eigenvalue will be used in the next section.

**Lemma 2.2.** If $L^{d_2}$ (resp. $L^{d_2}$) has a simple eigenvalue $\lambda(d_2) = 0$ (resp. $\mu(d_2) = 1$) with some $d_2 > 0$, then $L^{d_2}$ (resp. $L^{d_2}$) has a simple eigenvalue $\mu(d_2) = 1$ (resp. $\lambda(d_2) = 0$). In addition, if $(d_1 \sigma - h_{11}) \sigma \neq 0$ and $\frac{d_2}{d_1}(d_1 \sigma - h_{11}) + d_2 \sigma - h_{22} \neq 0$, then $\lambda_{d_2} d_2 \neq 0$ and $\mu_{d_2} d_2 \neq 0$. 
Proof. It is easy to check that the characteristic polynomials of $L$ and $L'$ are given by
\[
D(d_2, \lambda) := (\lambda + d_1 \sigma - h_{11})(\lambda + d_2 \sigma - h_{22}) - h_{12} h_{21},
\]
\[
D(d_2, \mu) := \{(1 + d_1 \sigma)\mu - h_{11} - 1\}\left\{\left(\frac{d_2}{d_1} + d_2 \sigma\right)\mu - h_{22} - \frac{d_2}{d_1}\right\} - h_{12} h_{21},
\]
respectively. Namely, $\lambda$ (resp. $\mu$) is an eigenvalue of $L$ (resp. $L'$) if and only if $D(d_2, \lambda) = 0$ (resp $D(d_2, \mu) = 0$). Since the equation $D(d_2, \mu) = 0$ can be rewritten as
\[
\{(1 + d_1 \sigma)(\mu - 1) + d_1 \sigma - h_{11}\}\left\{\frac{d_2}{d_1}\left(1 + d_1 \sigma\right)(\mu - 1) + d_2 \sigma - h_{22}\right\}
\]
\[-h_{12} h_{21} = 0,
\]
the eigenvalue $\lambda = 0$ corresponds to $\mu = 1$. Furthermore, it is possible to check that the eigenspace of $L^d\sigma$ coincides with that of $L'^d\sigma$ when $\lambda(d^*\sigma) = 0$, equivalently, $\mu(d^*\sigma) = 1$.

More precisely, considering the roots $\Lambda^\pm(d_2; \alpha)$ of
\[
(\Lambda + d_1 \sigma - h_{11})(\alpha \Lambda + d_2 \sigma - h_{22}) - h_{12} h_{21} = 0,
\]
we have
\[
\lambda(d_2) = \Lambda^+(d_2; 1), \quad \mu(d_2) = 1 + \frac{1}{1 + d_1 \sigma} \Lambda^\pm\left(\frac{d_2}{d_1}\right).
\]
Differentiating the quadratic equation with respect to $d_2$ and $\alpha$, we get
\[
A_{d_2}\{2\alpha\Lambda + (\alpha d_1 + d_2)\sigma - \alpha h_{11} - h_{22}\} + (\Lambda + d_1 \sigma - h_{11})\sigma = 0
\]
and
\[
A_{\alpha}\{2\alpha\Lambda + (\alpha d_1 + d_2)\sigma - \alpha h_{11} - h_{22}\} + (\Lambda + d_1 \sigma - h_{11})\Lambda = 0,
\]
respectively. Therefore, if $\frac{d_2}{d_1}(d_1 \sigma - h_{11}) + d_2 \sigma - h_{22} \neq 0$, invoking (2.5) and the first equation of (2.4), we obtain
\[
\lambda_{d_2}(d_2^*) = \Lambda^\pm_{d_2}(d_2^*; 1) = -\frac{(d_1 \sigma - h_{11})\sigma}{\frac{d_2}{d_1}(d_1 \sigma - h_{11}) + d_2 \sigma - h_{22}},
\]
whereas (2.5), (2.6) and the second equation of (2.4) lead to
\[
\mu_{d_2}(d_2^*) = \frac{1}{1 + d_1 \sigma}\left\{\Lambda^\pm_{d_2}\left(\frac{d_2^*}{d_1}\right) + \frac{1}{d_1} \Lambda^\pm_{\alpha}\left(\frac{d_2^*}{d_1}\right)\right\}
= \frac{(d_1 \sigma - h_{11})\sigma}{(1 + d_1 \sigma)(\frac{d_2}{d_1}(d_1 \sigma - h_{11}) + d_2 \sigma - h_{22})}.
\]
Then the proof of Lemma 2.2 is complete. 

2.2. Bifurcation directionally from infinity. The condition (1.3) requires that $f_i(u, v)$ ($i = 1, 2$) remain bounded along any directions when $|(u, v)| \to \infty$. Such a uniform boundedness on $(f_1, f_2)$ results in the restriction on the application of Theorem 2.1 in considering certain realistic models. To relax the assumption (1.3), we propose a new mathematical framework for the bifurcation from infinity in the next theorem.
Proof. In order to construct a branch of solutions for (2.3), we decompose the nonlinearity (2.21) will be introduced in the next sub-section. Then we substitute this orthogonal decomposition into equation (2.3), and decompose the equation itself to obtain

\[ \| DK \left( \frac{1}{s} \varphi^{d_2} + w \right) \| \leq k_0(s), \]

\[ \left\| K \left( \frac{1}{s} \varphi^{d_2} \right) \right\| \leq C_K \min \left\{ 1, \frac{1}{\| (I-L)^{-1} Q \|} \right\} \]  \hspace{1cm} (2.7)

for any $0 < |s| \leq \alpha$, $|d_2 - d_2| \leq \delta$ and $w \in Q^{d_2} E$ with $\| w \| \leq M$, where $D$ denotes the Fréchet derivative. Then there is a bifurcation branch $\{ (d_2, u) \mid u(x) = (u(x; d_2), v(x; d_2)) \}$ of (2.3) which meets $(d_2^*, \infty)$.

Remark 2. As a typical reaction-diffusion system (2.1) such as $f_i(u, v)$ ($i = 1, 2$) clear (2.7) and (2.8) but miss (1.3), a prey-predator model with Holling-type III nonlinearity (2.21) will be introduced in the next sub-section.

Proof. In order to construct a branch of solutions for (2.3), we decompose the unknown function $u$ into the $PE$ component and the $QE$ component as follows:

\[ u = \frac{1}{s} \varphi^{d_2} + w, \hspace{1cm} w = Q^{d_2} u. \]

As the Lyapunov-Schmidt reduction procedure, we first look for the solution $w$ of (2.10) for any fixed $s \neq 0$. For the use of the contraction mapping theorem, we introduce the closed subset $\mathcal{M} = \mathcal{M}^{d_2}$ of $E$ as

\[ \mathcal{M} := \{ w \in Q^{d_2} E \mid \| w \|_E \leq M \} \]

and define the mapping $\Phi(s) = \Phi^{d_2}(s)$ with $s \neq 0$ by

\[ \Phi(s)[w] := (I - L)^{-1} Q^{d_2} K \left( \frac{1}{s} \varphi^{d_2} + w \right) \]

for any $w \in \mathcal{M}$. Here $C_K$ is a positive constant in (2.8). Then it can be shown that $\Phi(s)$ is the mapping from $\mathcal{M}$ to itself if $|s| > 0$ is sufficiently small. Actually, (2.7) ensures

\[ \left\| K \left( \frac{1}{s} \varphi^{d_2} + w \right) - K \left( \frac{1}{s} \varphi^{d_2} \right) \right\|_E = \left\| \int_0^1 DK \left( \frac{1}{s} \varphi^{d_2} + \theta w \right) w d\theta \right\|_E \]

\[ \leq k_0(s) \| w \|_E \]  \hspace{1cm} (2.11)
for \( \|w\|_E \leq M \). Then it follows from the above inequality and (2.8) that

\[
\left\| K \left( \frac{1}{s} \psi d_2 + w \right) \right\|_E \leq k_0(s) \|w\|_E + \left\| K \left( \frac{1}{s} \varphi d_2 \right) \right\|_E \\
\leq k_0(s)M + C_K \min \left\{ 1, \frac{1}{\| (I - L)^{-1} Q \|} \right\}.
\]

(2.12)

By the assumption \( \lim_{s \to 0} k_0(s) = 0 \), we may take a constant \( \alpha_1 \in (0, \alpha] \) to satisfy

\[
C_L \sup_{|s| \leq \alpha_1} k_0(s) < \frac{1}{2},
\]

(2.13)

where \( C_L := \|(I - L)^{-1} Q d_2\| \). It follows from (2.12) and (2.13) that

\[
\| \Phi(s)[w] \| \leq \|(I - L)^{-1} Q d_2\| \left\| K \left( \frac{1}{s} \psi d_2 + w \right) \right\| \\
\leq \frac{M}{2} + C_K \leq M.
\]

Therefore, we know that \( \Phi \) is a mapping from \( \mathcal{M} \) to itself if \( 0 < |s| \leq \alpha_1 \). Furthermore, we know from (2.7) that

\[
\| \Phi(s)[w_1] - \Phi(s)[w_2] \| \leq C_L \left\| K \left( \frac{1}{s} \psi d_2 + w_1 \right) - K \left( \frac{1}{s} \psi d_2 + w_2 \right) \right\| \\
\leq C_L k_0(s) \|w_1 - w_2\|.
\]

By (2.13), we have

\[
\| \Phi(s)[w_1] - \Phi(s)[w_2] \| < \frac{1}{2} \|w_1 - w_2\|
\]

and thereby \( \Phi(s) \) is a contraction mapping from \( \mathcal{M} \) to itself if \( 0 < |s| \leq \alpha_1 \). The contraction mapping theorem gives a unique fixed point \( w^*(s) \in \mathcal{M}(s) \) which solves (2.10) as follows

\[
w^*(s) = (I - L^{d_2})^{-1} Q d_2 K \left( \frac{1}{s} \psi d_2 + w^*(s) \right)
\]

(2.14)

for any \( 0 < |s| \leq \alpha_1 \). Since (2.10) implies

\[
\|w^*(s_1) - w^*(s_2)\|_E \\
\leq \|(I - L)^{-1} Q d_2\| \left\| K \left( \frac{1}{s_1} \psi d_2 + w^*(s_1) \right) - K \left( \frac{1}{s_2} \psi d_2 + w^*(s_2) \right) \right\|_E \\
\leq C_L \sup_{0 < |s| \leq \alpha_1} k_0(s) \left| \frac{1}{s_1} - \frac{1}{s_2} \right| \|\varphi d_2\|_E + \|w^*(s_1) - w^*(s_2)\|_E
\]

for any \( s_i \) with \( 0 < |s_i| \leq \alpha_1 \) \( (i = 1, 2) \), we see that

\[
\|w^*(s_1) - w^*(s_2)\|_E \leq 2C_L \sup_{0 < |s| \leq \alpha_1} k_0(s) \left| \frac{1}{s_1} - \frac{1}{s_2} \right| \|\varphi d_2\|_E
\]

for \( 0 < |s_i| \leq \alpha_1 \) \( (i = 1, 2) \).

Hereafter, in the proof, we regard \( d_2 \) as a positive unknown number and construct a curve \( d_2 = d_2(s) \) which solves the equation (2.9). For simplicity of notation, we introduce the projection \( \bar{P}d_2 \) onto \( \mathbb{R} \) instead of \( Pd_2 \) by

\[
P^{d_2}u = \left( \bar{P}^{d_2}u \right) \varphi^{d_2}.
\]
The function $H(s, d_2)$ is continuous in $|s| < \alpha_1$ and $d_2$ close to $d_2^*$. Indeed, it follows from (2.8) and (2.11) that

$$|H(s, d_2) - H(0, d_2)| = |H(s, d_2)|$$

$$\leq |s| \left( \| K \left( \frac{1}{s} \varphi^{d_2} + w^*(s) \right) - K \left( \frac{1}{s} \varphi^{d_2} \right) \|_E + \| K \left( \frac{1}{s} \varphi^{d_2} \right) \|_E \right)$$

$$\leq |s| (k_0(s) \| w^*(s) \|_E + C_K \min\{1, 1/C_L\}).$$

Thus $H(s, d_2)$ is continuous in $s$ for $|s| < \alpha_1$. For use of the contraction mapping theorem, we set

$$G(s, d_2) := 1 - \mu^{d_2} - H(s, d_2), \quad G_0(d_2) := 1 - \mu^{d_2}.$$  

By the assumption that $\mu^{d_2}$ passes 1 at $d_2^*$ transversally, it can be verified that $G(0, d_2^*) = G_0(d_2^*) = 0$ and $G_0, d_2(d_2^*) \neq 0$. Moreover, there is a positive constant $C_0$ satisfying

$$\| (I - L^{d_2})^{-1} Q^{d_2} - (I - L^{d_2})^{-1} Q^{d_2^*} \| \leq C_0 |d_2 - d_2^*|,$$

$$\| \varphi^{d_2} - \varphi^{d_2^*} \|_E \leq C_0 |d_2 - d_2^*|,$$

$$| \tilde{P}^{d_2} - \tilde{P}^{d_2^*} | \leq C_0 |d_2 - d_2^*|$$

for any $d_2, d_2^*$ in the neighborhood of $d_2^*$.

Next we show the Lipschitz continuity of $w^*$ with respect to $d_2$. To clarify the dependency of $d_2$, we write $w^*(s; d_2)$ instead of $w^*(s)$. It follows from (2.7) and (2.14) that

$$\| w^*(s; d_2) - w^*(s; d_2') \|_E$$

$$= \left\| (I - L^{d_2})^{-1} Q^{d_2} K \left( \frac{1}{s} \varphi^{d_2} + w^*(s; d_2) \right) - (I - L^{d_2})^{-1} Q^{d_2} K \left( \frac{1}{s} \varphi^{d_2} + w^*(s; d_2') \right) \right\|_E$$

$$\leq C_0 |d_2 - d_2'| + C_L \left\| K \left( \frac{1}{s} \varphi^{d_2} + w^*(s; d_2) \right) - K \left( \frac{1}{s} \varphi^{d_2} + w^*(s; d_2') \right) \right\|_E$$

$$\leq C_0 |d_2 - d_2'| + C_L k_0(s) \left\| \frac{1}{s} \left( \varphi^{d_2} - \varphi^{d_2'} \right) \right\|_E$$

$$+ C_L k_0(s) \| w^*(s; d_2) - w^*(s; d_2') \|_E$$

$$\leq 2 \left( C_0 + \frac{C_L C_0 k_0(s)}{|s|} \right) |d_2 - d_2'|$$

for any $0 < |s| \leq \alpha_1$. This immediately implies that

$$\| w^*(s; d_2) - w^*(s; d_2') \|_E \leq 2 \left( C_0 + \frac{C_L C_0 k_0(s)}{|s|} \right) |d_2 - d_2'|$$
for $0 < |s| \leq \alpha_1$. We also have
\[
|H(s, d_2) - H(s, d_2')| = \left| s \tilde{d} K \left( \frac{1}{s} \varphi^{d_2} + w^* (s; d_2) \right) - s \tilde{d} K \left( \frac{1}{s} \varphi^{d_2'} + w^* (s; d_2') \right) \right|
\leq C_0 |s| |d_2 - d_2'| + C_1 |s| k_0(s) \left( \left\| \frac{1}{s} (\varphi^{d_2} - \varphi^{d_2'}) \right\|_E + \left\| w^* (s; d_2) - w^* (s; d_2') \right\|_E \right)
\leq C_0 |s| |d_2 - d_2'| + C_1 |s| k_0(s) \left( C_0 |d_2 - d_2'| + 2 (C_0 |s| + C_L C_0 k_0(s)) |d_2 - d_2'| \right)
\leq (C_2 |s| + C_3 k_0(s)) |d_2 - d_2'| \tag{2.17}
\]
where $C_1, C_2$ and $C_3$ are positive constants independent of $d_2, d_2'$ and $s$. Namely, the Lipschitz constant of $H$ with respect to $d_2$ is sufficiently small near $(s, d_2) = (0, d_2^*)$.

In order to construct the curve $d_2 = d_2(s)$ of solutions to (2.15), we define $\eta(\tilde{d}) \in C[-\alpha_1, \alpha_1]$ by
\[
\eta(\tilde{d})(s) := \tilde{d}(s) - \frac{1}{G_{0, d_2}(d_2^*)} \left( G_0(\tilde{d}(s)) - H(s, \tilde{d}(s)) \right)
\]
for $\tilde{d} \in C[-\alpha_1, \alpha_1]$. By the transversality of the eigenvalue $\mu^{d_2}$ at $d_2^*$, (2.16) and (2.17) imply that there exists a constant $\theta \in (0, 1)$ satisfying
\[
|\eta(\tilde{d})(s) - \eta(\hat{d})(s)|
\leq \left| \tilde{d}(s) - \hat{d}(s) - \frac{G_0(\tilde{d}(s)) - G_0(\hat{d}(s))}{G_{0, d_2}(d_2^*)} \right| + \left| \frac{H(s, \tilde{d}(s)) - H(s, \hat{d}(s))}{G_{0, d_2}(d_2^*)} \right|
\leq \theta |\tilde{d}(s) - \hat{d}(s)|
\]
for $\tilde{d}, \hat{d} \in C[-\alpha_1, \alpha_1]$ and $|s| \leq \alpha_1$, which is retaken smaller if necessary. Since
\[
|\eta(\tilde{d})(s) - d_2^*| \leq \left| \tilde{d}(s) - d_2^* - \frac{1}{G_{0, d_2}(d_2^*)} \left( G_0(\tilde{d}(s)) - H(s, \tilde{d}(s)) \right) \right|
\leq \left| \tilde{d}(s) - d_2^* - \frac{1}{G_{0, \mu}(d_2^*)} G_0(\tilde{d}(s)) \right| + \left| \frac{|s| (2k_0(s) C_K + k_P)}{|G_{0, \mu}(d_2^*)|} \right|
\]
we can choose $\delta$ and $\alpha_1$ so small that $\eta$ is a contraction mapping onto
\[
\{ \tilde{d} \in C[-\alpha_1, \alpha_1] \mid \max_{|s| \leq \alpha_1} |\tilde{d}(s) - d_2^*| \leq \delta \}
\]
for any $x \in \Omega$. Applying the contraction mapping theorem, we obtain a unique curve $d_2 = d_2(s)$ as the fixed point of $\eta$ such that
\[
G(x, d_2(s)) = 1 - \mu^{d_2(s)} - H(s, d_2(s)) = 0
\]
for any $x \in \Omega$ and $|s| < \alpha_1$. This implies the existence of a solution $u = \varphi^{d_2(s)} / s + w(s)$ to (2.3) when $d_2$ is close to $d_2^*$. Namely, the solution bifurcates from $(d_2^*, \infty)$.

By taking $\alpha_1$ smaller if necessary, $u = \varphi^{d_2} / s + w(s)$ is positive in $\Omega$ when both components of $\varphi^{d_2}$ are positive. If one of the components of $\varphi^{d_2}$ vanishes, then we need to check case by case.

Concerning (2.1) with homogeneous Dirichlet boundary conditions, the following corollary gives a sufficient condition on $f_1$ and $f_2$ for the bifurcation from infinity to occur. In what follows in this section, $\sigma$ represents the simple eigenvalue of $-\Delta$.
with the Dirichlet boundary condition and \( \phi(x) \) denotes the corresponding positive eigenfunction with \( L^\infty \) normalization, that is,

\[
-\Delta \phi = \sigma \phi, \quad \phi > 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial \Omega, \quad \max_{x \in \Omega} \phi(x) = 1.
\]

**Corollary 1.** Assume that \( \mathcal{L}^{d_2} \) has a simple eigenvalue \( \lambda(d_2) \) satisfying \( \lambda(d_2^+) = 0 \) when \( d_2 \) is near \( d_2^+ \), and

\[
d_1 \sigma - h_{11} \neq 0, \quad \frac{d_2^*}{d_1^*}(d_1 \sigma - h_{11}) + d_2^* \sigma - h_{22} \neq 0.
\]

Let \( \varphi^{d_2} = (a_1, a_2) \phi(x) \) be the eigenfunction of \( \mathcal{L}^{d_2} \) for some \((a_1, a_2) \in \mathbb{S}^1\), that is, \( \mathcal{L}^{d_2} \varphi^{d_2} = \lambda(d_2) \varphi^{d_2} \). Suppose there exist a positive constant \( k_f \) and a non-negative continuous function \( k(t) \) such that \( \lim_{t \to 0} k(t) = 0 \) and

\[
\left| f_i \left( \frac{a_1}{t}, \frac{a_2}{t} \right) \right| \leq k_f, \quad \left| \frac{\partial f_i}{\partial u} \left( \frac{a_1}{t} + z_1, \frac{a_2}{t} + z_2 \right) \right| \leq k(t), \quad \left| \frac{\partial f_i}{\partial v} \left( \frac{a_1}{t} + z_1, \frac{a_2}{t} + z_2 \right) \right| \leq k(t)
\]

for \( t \neq 0, z_1, z_2 \in \mathbb{R}, i = 1, 2 \). Then for (2.1) with homogeneous Dirichlet boundary conditions, there exists a bifurcation branch of solutions \( \{(d_2, u) \mid u(x) = (u(x; d_2), v(x; d_2))\} \) which meets \((d_2^+, \infty)\).

**Proof.** It suffices to show (2.7) and (2.8). Set \( \varphi(x) = (a_1, a_2) \phi(x) \) and \( w = (w_1, w_2) \). From (2.19), it follows that

\[
\left| \frac{\partial f_i}{\partial u} \left( \frac{a_1}{s} \phi(x) + w_1(x), \frac{a_2}{s} \phi(x) + w_2(x) \right) \right| \leq k \left( \frac{s}{\phi(x)} \right),
\]

for any \( x \in \Omega \). By the definition of \( K \), we use the green function \( g \) corresponding to \((-d_1 \Delta + 1)^{-1}\) with Dirichlet boundary condition. Then it is sufficient to consider

\[
(-d_1 \Delta + 1)^{-1} \frac{\partial f_i}{\partial u} \left( \frac{\varphi}{s} + w \right) = \int_{\Omega} g(x, y) \frac{\partial f_i}{\partial u} \left( \frac{\varphi(y)}{s} + w(y) \right) dy,
\]

where we simply write \( f_i(a_1 \phi/s + w_1, a_2 \phi/s + w_2) \) as \( f_i(\varphi/s + w) \) for \( j = 1, 2 \). The Hopf Lemma implies that \( \partial \phi/\partial n \neq 0 \) at \( \partial \Omega \). The first inequality of (2.19) and the above fact imply that

\[
\left\| (-d_1 \Delta + 1)^{-1} \frac{\partial f_i}{\partial u} \left( \frac{\varphi}{s} + w \right) \right\| \leq \left\| \int_{\Omega \cap \{ |\varphi(y)| \leq |s|^{1/2} \}} g(x, y) \left| \frac{\partial f_i}{\partial u} \left( \frac{\varphi(y)}{s} + w(y) \right) \right| dy \right\| + \left\| \int_{\Omega \cap \{ |\varphi(y)| \geq |s|^{1/2} \}} g(x, y) \left| \frac{\partial f_i}{\partial u} \left( \frac{\varphi(y)}{s} + w(y) \right) \right| dy \right\| \leq C(\sqrt{s} + \sqrt{s})
\]

for all \( s \geq 0 \). Using the second inequality of (2.20) yields a similar estimate for \((-d_1 \Delta + 1)^{-1} \partial f_i/\partial v(\varphi/s + w) \). Thus (2.7) is verified.
From the third inequality of (2.19), we see that
\[ \left\| (-d_1 \Delta + 1)^{-1} f_i \left( \frac{\varphi}{s} \right) \right\| \leq \left\| \int_{\Omega} g(x, y) \left| f_i \left( \frac{\varphi(y)}{s} \right) \right| \, dy \right\| \leq Ck_f. \]
Thus (2.8) holds and the proof is complete. \qed

2.3. Examples. In this section, we use Corollary 1 to show the existence of bifurcation branches from infinity in a couple of the prey-predator models.

2.3.1. Prey-predator model with Holling Type III nonlinearity. The first example is one of the prey-predator models, so called Holling Type III:

\[
\begin{align*}
    u_t &= d_1 \Delta u + u - \frac{u^2v}{1+u^2}, \\
    v_t &= d_2 \Delta v + v \left( \frac{au^2}{1+u^2} - 1 \right)
\end{align*}
\]

(2.21)

with homogeneous Dirichlet boundary conditions. Usually this system includes the saturation term, but it is neglected here. To express (2.21) in the form of (2.1), we take

\[ f_1(u, v) = \frac{v}{1+u^2}, \quad f_2(u, v) = -\frac{av}{1+u^2}, \quad h_1(u, v) = u - v, \quad h_2(u, v) = (a - 1)v. \]

Hence \( \lim_{u \to \infty} f_1(u, v) = \lim_{u \to \infty} f_2(u, v) = 0 \) for every \( v \in \mathbb{R} \), but (1.3) does not hold. When \( d_2^2 \sigma = a - 1 \) and \( d_1 \sigma \neq 1 \), the operator of \( L \) has a zero eigenvalue at \( d_2 = d_2^* \). In this situation, (2.18) is also satisfied since \( h_{11} = 1 \) and \( h_{22} = a - 1 \).

The eigenfunction corresponding to a zero eigenvalue of \( L \) is \( \varphi^{d_2} := (a_1, a_2)^T \phi \) where \( \phi \) is a eigenfunction of a Laplace operator and

\[ a_1 = \frac{1}{\sqrt{1 + (-d_1 \sigma + 1)^2}}, \quad a_2 = \frac{-d_1 \sigma + 1}{\sqrt{1 + (-d_1 \sigma + 1)^2}}. \]

when \( d_2 = d_2^* \). Here we note that \( a_2 \neq 0 \) since \( d_1 \sigma \neq 1 \).

We now verify (2.19) to confirm the criteria for the bifurcation from infinity as stated in Theorem 2.3 and Corollary 1. For any \( t \neq 0 \),

\[ |f_1 \left( \frac{a_1}{t}, \frac{a_2}{t} \right) | = \frac{|a_2/t|}{1 + a_1^2/t^2} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0 \]

since \( a_1 \neq 0 \). Since \( f_2 = -af_1 \), the first condition of (2.19) is fulfilled.

Since \( \partial f_1 / \partial u = -2av(1 + u^2)^{-2} \), then for any \( t \neq 0 \) and \( z_i \in \mathbb{R} \),

\[ \left| \frac{\partial f_1}{\partial u} \left( \frac{a_1}{t} + z_1, \frac{a_2}{t} + z_2 \right) \right| = 2 \left| \frac{a_1}{t} + z_1 \right| \left| \frac{a_2}{t} + z_2 \right| \left\{ 1 + \left( \frac{a_1}{t} + z_1 \right)^2 \right\}^{-2} \]

\[ = \frac{2|a_1 t + z_1 t^2| |a_2 t + z_2 t^2|}{(t^2 + (a_1 + z_1 t)^2)^2} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0. \]

Consequently the second condition of (2.19) is fulfilled. Since \( f_2 = -a f_1 \), the third condition is also verified. Applying Corollary 1 deduces that the stationary solutions for (2.21) with Dirichlet boundary conditions bifurcate from infinity at \( d_2 = d_2^* = (a - 1)/\sigma \).
2.3.2. Prey-predator model with Holling Type II nonlinearity. Let us consider the prey-predator model with Holling Type II nonlinearity:

\[
\begin{align*}
  u_t &= d_1 \Delta u + u - \frac{uv}{1 + u}, \\
  v_t &= d_2 \Delta v + v \left( \frac{au}{1 + u} - 1 \right)
\end{align*}
\]

with homogeneous Dirichlet boundary conditions. Namely, we take

\[
\begin{align*}
  f_1(u, v) &= \frac{v}{1 + u}, & f_2(u, v) &= -\frac{av}{1 + u}, & h_1(u, v) &= u - v, & h_2(u, v) &= (a - 1)v
\end{align*}
\]

and \( \varphi = (a_1, a_2)^T \phi \) as above. To confirm (2.19) in Corollary 1, we see that

\[
\begin{align*}
  \left| \frac{\partial f_1}{\partial u} \left( \frac{a_1}{t}, \frac{a_2}{t} \right) \right| &= \frac{|a_2/t|}{1 + a_1/t} = \frac{|a_2|}{t + a_1} \to \frac{|a_2|}{a_1} \quad \text{as } t \to 0.
\end{align*}
\]

Since \( f_2 = -af_1 \), the first condition of (2.19) is fulfilled.

Since \( \partial f_1 / \partial u = -v(1 + u)^{-2} \), then for any \( t \neq 0 \) and \( z_i \in \mathbb{R} \),

\[
\begin{align*}
  \left| \frac{\partial f_1}{\partial u} \left( \frac{a_1}{t} + z_1, \frac{a_2}{t} + z_2 \right) \right| &= \frac{|a_2|}{t + a_1} \to \frac{|a_2|}{1 + a_1 t + z_1} \to 0 \quad \text{as } t \to 0.
\end{align*}
\]

Thus the second condition of (2.19) is fulfilled. The verification of the third condition is similar. Then Corollary 1 deduces that the stationary solutions for (2.21) with Dirichlet boundary conditions bifurcate from infinity at \( d_2 = d_2^* = (a - 1)/\sigma \).

Next we consider the positivity of the solution. By the property of the Green function, we have

\[
|w| \leq C\phi.
\]

If \( d_1 \sigma < 1 \), then \( a_1 > 0 \) and \( a_2 > 0 \). Thus, by taking \( s \) so small, \( u = a_1 \phi/s + w_1 \) and \( v = a_2 \phi/s + w_2 \) are positive in \( \Omega \). Since \( a_1 = a_2 = 1/\sqrt{2} \) at \( d_2 = d_2^* \), \( u \) and \( v \) are positive near the bifurcation point.

3. The case of \( p \)-homogeneous nonlinearities. In the previous section, we consider the case where nonlinear term converges to 0 near infinity; there it is assumed that \( f_1(u, v) \) and \( f_2(u, v) \) converge to 0 as \( u \to \infty \) (or as \( u \to \infty \) as in (2.21)). Instead if we assume that both \( f_1(u, v) \) and \( f_2(u, v) \) vanish in one direction, then nonlinear terms are degenerate and in this case we expect that the lower order terms such as linear ones dominate the dynamics.

First we define \( p \)-homogeneous nonlinearity \( f \) by requiring

\[
\begin{align*}
  f(tu) &= t^p f(u)
\end{align*}
\]

for any \( u \in \mathbb{R}^2 \) and \( t \in \mathbb{R} \) where \( p \) is a positive integer. Let \( f_1 \) and \( f_2 \) be \( p \)-homogeneous nonlinearities. Consider the following reaction-diffusion system with \( p \)-homogeneous nonlinear terms:

\[
\begin{align*}
  u_t &= d_1 \Delta u + f_1(u, v), \\
  v_t &= d_2 \Delta v + f_2(u, v)
\end{align*}
\]
in $\Omega = (-1,1)$ with Dirichlet or Neumann boundary conditions. We assume that there is a constant $k_*$ satisfying

$$f_1(1,k_*) = f_2(1,k_*) = 0. \quad (3.2)$$

Then there exist a family of stationary solutions $\{(s,sk_*)\mid s \in \mathbb{R}\}$. To find stationary solutions, we introduce a new variable $k$ defined by

$$v(x) = ku(x).$$

Next we consider the stationary solution depending on $x$. Substituting $v(x) = ku(x)$ into the second equation of (3.1) and removing $u$ from the first equation, we derive the following condition:

$$d_1 \Delta u + f_1(1,k)u^p = 0, \quad (3.3)$$

$$f_2(1,k) - \frac{d_2 k}{d_1} f_1(1,k) = 0, \quad (3.4)$$

where $k$ is independent of $x$. For the homogeneous Dirichlet boundary conditions, there is a positive solution of (3.3) when $f_1(1,k) > 0$. If the algebraic equation (3.4) possesses a root $k = k^#$ with $f_1(1,k^#) > 0$, there is a positive stationary solution of (3.1).

3.1. $p$-homogeneous systems. In this subsection we show the existence of stationary solutions of (3.1) that bifurcate from infinity.

**Theorem 3.1.** Let

$$\delta := d_2 - d_1 \mu_*.$$ 

Assume that

(i) $p$ is odd and $p > 1$,

(ii) there is a constant $k_*$ satisfying (3.2),

(iii) $\mu_* := \lim_{k \to k_*} \frac{f_2(1,k)}{k f_1(1,k)}$ and $\nu_* := \lim_{k \to k_*} \frac{d}{dk} \frac{f_2(1,k)}{f_1(1,k)}$ exist and $\mu_* \neq \nu_*$,

(iv) there is a positive constant $\delta_0$ such that $f_1\left(1,k_* + \frac{k_* \delta}{d_1(\nu_* - \mu_*)}\right) > 0$ for all $\delta \in (0,\delta_0)$ or for all $\delta \in (-\delta_0,0)$.

Then there exist an interval $V$, a constant $\bar{k} = \bar{k}(\delta)$ and a function $\bar{u} = \bar{u}(x;\delta)$ such that

(i) $0 \in V$,

(ii) $f_1\left(1,k_* + \frac{k_* \delta}{d_1(\nu_* - \mu_*)}\right) > 0$ for all $\delta \in V$,

(iii) $(\bar{u}(x;\delta),\bar{k}(\delta)\bar{u}(x;\delta))$ is a stationary solution of (3.1) for $\delta \in V$,

(iv) $\lim_{\delta \to 0} \bar{k}(\delta) = k_*$,

(v) $\lim_{\delta \to 0} \max_{x \in \Omega} \bar{u}(x;\delta) = \infty$.

Note that assumption (iv) implies that $k_* \neq 0$.

**Proof.** We only consider the case where

$$f_1\left(1,k_* + \frac{k_* \delta}{d_1(\nu_* - \mu_*)}\right) > 0$$

for all $\delta \in (0,\delta_0)$, since the other case can be treated similarly. Define

$$H(\delta,k) := \frac{f_2(1,k)}{f_1(1,k)} - \frac{(d_1 \mu_* + \delta)k}{d_1}.$$
From the assumption,
\[ \lim_{k \to k_*} H(0, k) = 0, \quad \lim_{k \to k_*} H_k(0, k) = \nu_* - \mu_* \neq 0. \]

By the implicit function theorem, there is a function \( \bar{k}(\delta) \) near \( \delta = 0 \) satisfying
\[ \bar{k}(0) = k_*, \quad H(\delta, \bar{k}(\delta)) = 0 \]
when \( k_* \neq 0 \). Since \( p \) is odd, we look for a solution of (3.3) with
\[ \bar{u}(x; \delta) = \rho(\delta)U(x), \]
where \( U \) is a solution of
\[ \begin{cases} U_{xx} + U^p = 0, & x \in (0, L), \\ U(0) = U_x(L) = 0. \end{cases} \tag{3.5} \]
Since
\[ \frac{1}{2} U_x^2 + \frac{1}{p+1} U^{p+1} = \frac{1}{p+1} U(L)^{p+1}, \]
we have
\[ L = \sqrt{\frac{p+1}{2} \int_0^{U(L)} \frac{dU}{U(L)^{p+1} - U^{p+1}}} = \sqrt{\frac{p+1}{2} U(L)^{(1-p)/2} \int_0^1 \frac{ds}{\sqrt{1 - s^{p+1}}}}. \]
It is clear that \( L \to 0 \) as \( U(L) \to \infty \) and there exists a positive solution of (3.5) for any \( L > 0 \).

Substituting \( \bar{u} = \rho U \) into the first equation of (3.1), we have
\[ \rho(\delta) := \left( \frac{f_1(1, k(\delta))}{d_1} \right)^{-1/(p-1)} \]
if \( f_1(1, k(\delta)) \geq 0 \). Since
\[ k(\delta) = k_* + \frac{k_* \delta}{d_1(\nu_* - \mu_*)} + O(\delta^2), \]
\( \rho(\delta) \) and \( U(x; \delta) \) exist if \( \delta \in (0, \delta_1) \) for some \( \delta_1 > 0 \). Since \( k \to k_* \) and \( \rho \to \infty \) as \( \delta \to 0 \), \( \bar{u}(x; \delta) \) goes to infinity as \( \delta \) tends to 0. \( \square \)

Note that \( U \) is not necessarily positive. We also see that infinitely many nodal solutions of (3.5) bifurcate from infinity at the same time.

3.2. Perturbed systems. Hereafter we consider the following reaction-diffusion system:
\[ \begin{align*}
    u_t &= d_1 \Delta u + f_1(u, v) - h_1(u, v), \\
    v_t &= d_2 \Delta v + f_2(u, v) - h_2(u, v), \tag{3.6}
\end{align*} \]
in a domain \( \Omega = (0, 1) \) with the homogeneous Dirichlet or Neumann boundary condition, where \( f_1, f_2 \) are \( p \)-homogeneous and
\[ h_1(u, v) := h_{11} u, \quad h_2(u, v) = h_{21} u + h_{22} v. \]

Suppose the stationary solution of (3.6) is of the form \( (u, ku) \) then
\[ \begin{align*}
    d_1 \Delta u + u^p f_1(1, k) - h_{11} u &= 0, \tag{3.7} \\
    d_2 \Delta (ku) + u^p f_2(1, k) - h_2(1, k) u &= 0. \tag{3.8}
\end{align*} \]
Since $\Delta (ku) = u \Delta k + 2 \nabla k \cdot \nabla u + k \Delta u = \nabla (u^2 \nabla k)/u + k \Delta u$, the second equation is transformed into

$$d_2 \nabla (u^2 \nabla k) + \frac{d_1 f_2(1,k) - d_2 k f_1(1,k)}{d_1} u_{p+1} - \frac{d_1 f_1(1,k) - d_2 k h_1(1,k)}{d_1} u^2 = 0.$$ 

Assume that $k$ is a constant as in the previous section. If $(u, ku)$ is a solution, the following conditions hold:

$$d_1 u_{xx} + f_1(1,k) u - h_1 u = 0,$$

$$\frac{d_1 f_2(1,k) - d_2 k f_1(1,k)}{d_1} = 0,$$

$$\frac{d_1 f_1(1,k) - d_2 k h_1(1,k)}{d_1} = 0. \quad (3.9)$$

Next consider an auxiliary equation

$$d_1 \Delta U + U^p - h_{11} U = 0, \quad \text{in } \Omega, \quad (3.10)$$

with Dirichlet or Neumann boundary condition. It is known that the Dirichlet problem has a positive solution for any $d_1 > 0$, while the Neumann problem has a positive solution for any small $d_1 > 0$.

**Theorem 3.2.** Assume that the hypotheses of Theorem 3.1 are satisfied with the same notation $V$ and $k(\delta)$. Then there are parameters $h_{ij} = h_{ij}(\delta)$ and a bifurcation branch $\{(u^\delta, v^\delta) \mid \delta \in V\}$ such that

(i) $u = \eta(\delta) U$, where

$$\eta(\delta) := \left(\frac{1}{f_1(1,k(\delta))}\right)^{1/(p-1)} \quad (3.11)$$

and $U$ is given by (3.10).

(ii) the following condition holds:

$$h_{21} + \bar{k}(\delta) h_{22} = \left(\mu_* + \frac{\delta}{d_1}\right) h_{11} \bar{k}(\delta). \quad (3.12)$$

**Proof.** As in the proof of Theorem 3.1, there is a $k = \bar{k}(\delta)$ satisfying (3.9). Substituting $u = \eta U(x)$ into (3.7) yields

$$d_1 \Delta U + \eta^{(p-1)} f_1(1,k) U^p - h_{11} U = 0.$$ 

Thus the above equation holds by (3.11) and (3.10); that is,

$$(u^\delta(x), v^\delta(x)) = (\eta(\delta) U(x), \eta(\delta) \bar{k}(\delta) U(x))$$

is a solution of (3.7) and (3.8). Since $\eta(\delta) \to \infty$ as $\bar{k}(\delta) \to k^*$, this family of solutions forms a bifurcation branch emanating from infinity. \qed

When

$$h_{2}(u,v) = \frac{d_2 h_{11}}{d_1} v,$$

(3.12) holds. Namely, $h_{12} = h_{21} = 0, h_{11} > 0, h_{22} = d_2 h_{11}/d_1$, which means that $h_1, h_2$ are independent of $\bar{k}$. The condition (3.12) can be relaxed to

$$d_1 h_2(1, \bar{k}(\delta)) - d_2 h_1(1, \bar{k}(\delta)) = 0$$
3.3. **Inhomogeneous case.** In this subsection, we put $\varepsilon$ in front of the linear term:

\[
\begin{align*}
  u_t &= d_1 u_{xx} + f_1(u,v) - \varepsilon h_1(u,v) \\
  v_t &= d_2 v_{xx} + f_2(u,v) - \varepsilon h_2(u,v)
\end{align*}
\]

(3.13)

in an interval $\Omega = (-1,1)$ with the Neumann boundary condition, where $f_1$, $f_2$ are $p$-homogeneous and $h_1$, $h_2$ are linear. By assuming the odd symmetry of $u$ and $v$, we impose the Dirichlet boundary condition

\[
u(0,t) = v(0,t) = 0.
\]

Thus we consider the problem (3.13) on the interval $\Omega = (0,1)$.

**Theorem 3.3.** Let $d_0 := (d_2 - d_1)/d_1d_2$. Under the assumptions of Theorem 3.1, for any $\delta \in V$ and $|d_0| \leq \delta$, there is an $\varepsilon_0 > 0$ such that the inhomogeneous stationary solution of (3.13) exists if $|\varepsilon| \leq \varepsilon_0$.

First we prepare some lemmas to treat the perturbed system (3.13). Consider the following auxiliary equation:

\[
-(a(x)\psi')' + b(x)\psi = 0
\]

(3.14)

where $a, b$ are analytic functions satisfying

\[
a(x) = a_0 x^2 + o(x^2), \quad b(x) = b_0 x^{p+1} + o(x^{p+1}), \quad a(x), b(x) > 0 \quad (0 < x \leq 1).
\]

(3.15)

By the Frobenius theorem there are two solutions $\psi_1$ and $\psi_2$ of (3.14) with

\[
\begin{align*}
  \psi_1(x) &= O(1), \quad \psi_2(x) = O(|x|^{-1}) \quad \text{as} \ x \to 0, \\
  \psi_1'(x) &= O(1), \quad \psi_2'(x) = O(|x|^{-2}) \quad \text{as} \ x \to 0, \\
  \psi_1''(x) &= O(1), \quad \psi_2''(x) = O(|x|^{-3}) \quad \text{as} \ x \to 0.
\end{align*}
\]

Thus we can assume that

\[
\psi_1(0) = 1, \quad \psi_2(1) = 1, \quad \psi_2'(1) = 0.
\]

Then $\psi_1$ and $\psi_2$ are uniquely determined. Denoted by the Wronskian $W = a\psi_1'\psi_2 - a\psi_1\psi_2'$, which is known to be constant.

**Lemma 3.4.** Suppose that $a$ and $b$ satisfy (3.15). If $a_0$ and $b_0$ are positive, then $W \neq 0$.

**Proof.** Assume that $v$ is a bounded solution with $v'(1) = 0$, then

\[
v' = \frac{1}{a(x)} \int_0^x b(y)v(y)dy.
\]

Without loss of generality, we may assume $v(0) \geq 0$. Then $v'$ is positive near $x = 0$. Then $v$ is increasing, which implies $v > 0$ on $[0,1]$. However $v'$ is positive, which violates the boundary condition. This implies that $\psi_1'(1) > 0$. Thus we have shown that $W = a(1)\psi_1'(1)\psi_2(1) \neq 0$. \qed

**Lemma 3.5.** The solution of

\[-(a(x)v')' + b(x)v = f(x)
\]

with

\[
v'(0) = 0, \quad v'(1) = 0
\]
satisfies
\[ v(x) = \int_0^1 K(x, y)f(y)dy \]
where
\[ K(x, y) := \begin{cases} 
\frac{1}{W} \psi_1(x)\psi_2(y) & 0 < x < y, \\
\frac{1}{W} \psi_1(y)\psi_2(x) & y < x < 1.
\end{cases} \]

We omit the proof, since it is easy to check.

**Lemma 3.6.** The linearized operator
\[ L_1\psi := -\psi_{xx} - pU^{p-1}\psi \] (3.16)
with \( \psi(0) = 0, \psi_x(1) = 0 \) is invertible.

**Proof.** Since \( L_1 \) is a self-adjoint differential operator defined on \((0, 1)\), the spectrum of \( L_1 \) consists of real eigenvalues. Assume that 0 is an eigenvalue of \( L_1 \). We denote the eigenfunction corresponding to the eigenvalue 0 by \( \psi \). Differentiating (3.5) with respect to \( x \) yields
\[ \hat{\psi}_{xx} + pU^{p-1}\hat{\psi} = 0. \]
where \( \hat{\psi} := U_x \). Note that \( \hat{\psi}(1) = 0 \) and \( \hat{\psi} < 0 \) in \((0, 1)\). Note that \( \psi(0) = \psi(1) = 0 \).
By the Strum theorem, \( \psi \) is positive in \((0, 1)\). Multiplying (3.5) by \( \psi \) and integrating over \((0, 1)\), we obtain
\[ 0 = \int_0^1 (U_{xx} + U^p) \psi dx = \int_0^1 (U\psi_{xx} + U^p\psi) dx = \int_0^1 (1 - p)U^p\psi dx. \]
The right hand side of the above equality is not zero, which leads to a contradiction. Thus, zero is not an eigenvalue and \( L_1 \) is invertible.

Next we consider the perturbed system (3.13) with \( \varepsilon > 0 \). Here we seek for stationary solutions in the form \((u, k(x)u)\), which is different from the previous two subsections since \( k \) is inhomogeneous. It follows that
\[
\begin{align*}
d_1u_{xx} + u^pf_1(1,k) - \varepsilon h_1(u, ku) &= 0, \\
d_2(ku)_{xx} + u^pf_2(1,k) - \varepsilon h_2(u, ku) &= 0.
\end{align*}
\]
Since \((ku)_{xx} = k_{xx}u + 2k_xu_x + ku_{xx} = (u^2k_x)_x/u + ku_{xx}\), the second equation is transformed into
\[
\frac{d_2(u^2k_x)_x}{d_1} + \frac{d_1f_2(1,k) - d_2k f_1(1,k)}{d_1} u^{p+1} - \frac{d_1h_2(u, ku) - d_2kh_1(u, ku)}{d_1} \varepsilon u = 0.
\]
To seek a solution near \((\bar{u}, \bar{k}\bar{u})\), we set \((u,k) = (\bar{u} + \rho w, \bar{k} + z) = (\rho(U + w), \bar{k} + z)\).
\[
\begin{align*}
d_1(U + w)_{xx} + \rho^{p-1}(U + w)^pf_1(1,\bar{k} + z) - \varepsilon h_1(U + w, (\bar{k} + z)(U + w)) &= 0, \\
d_2((U + w)^2z)_{xx} + \frac{d_1f_2(1,\bar{k} + z) - d_2(\bar{k} + z)f_1(1,\bar{k} + z)}{d_1} \rho^{p-1}(U + w)^{p+1} \\
- \frac{d_1h_2(U + w, (\bar{k} + z)(U + w)) - d_2(\bar{k} + z)h_1(U + w, (\bar{k} + z)(U + w))}{d_1} \\
\times \varepsilon(U + w) &= 0.
\end{align*}
\]
Set

\[ q(z; \delta) := \frac{d_1 f_2(1, \bar{k} + z) - d_2(\bar{k} + z) f_1(1, \bar{k} + z)}{d_2 f_1(1, \bar{k})}, \]

\[ r(w, z; \delta) := \frac{d_1 h_2(U + w, (\bar{k} + z)(U + w)) - d_2(\bar{k} + z) h_1(U + w, (\bar{k} + z)(U + w))}{d_1 d_2} \]

for simplicity in notation. Since \( \rho^{p-1} = d_1 / f_1(1, \bar{k}) \), the system can be rewritten as

\[
\begin{cases}
    w_{xx} + \frac{f_1(1, \bar{k} + z)}{f_1(1, k)}(U + w)^p - U^p - \frac{\varepsilon}{d_1} h_1(U + w, (\bar{k} + z)(U + w)) = 0, \\
    (U^2 z_x)_x + ((2U w + w^2)z_x)_x + q(z; \delta)(U + w)^{p+1} - \varepsilon r(w, z; \delta)(U + w) = 0.
\end{cases}
\]

Set

\[ K_1(w, z) := \frac{f_1(1, \bar{k} + z)}{f_1(1, k)}(U + w)^p - U^p - pU^{p-1}w, \]

\[ K_2(w, z) := -\frac{\varepsilon}{d_1} h_1(U + w, (\bar{k} + z)(U + w)), \]

\[ K_3(w, z) := q(z; \delta)(U + w)^{p+1} - q_2(0; \delta)z U^{p+1}, \]

\[ K_4(w, z) := -\varepsilon(U + w)r(w, z; \delta), \]

\[ K_5(w, z) := ((2U w + w^2)z_x)_x, \]

\[ F_1(w, z) := K_1(w, z) + K_2(w, z), \]

\[ F_2(w, z) := K_3(w, z) + K_4(w, z) + K_5(w, z). \]

Therefore

\[
\begin{cases}
    L_1 w = F_1(w, z) = K_1(w, z) + K_2(w, z), \\
    L_2 z = F_2(w, z) = K_3(w, z) + K_4(w, z) + K_5(w, z).
\end{cases}
\]

where \( L_1 \) is given by (3.16) and

\[ L_2 z := -(U^2 z_x)_x - q_2(0; \delta)U^{p+1}z \]

with \( z_x(0) = z_x(1) = 0 \). Since \( U(0) = 0 \), then \(-((U^2 z_x)_x - q_2(0; \delta)U^{p+1}z = 0 \) has a regular singular point. We recall that the Green functions for \( L_1 \) and \( L_2 \) are denoted by \( G_1 \) and \( G_2 \) respectively and that the fundamental solutions for \( L_1 \psi = 0 \) (resp. \( L_2 \)) are denoted by \( \psi_{11}, \psi_{12} \) (resp. \( \psi_{21}, \psi_{22} \)). Namely,

\[ L_1 \psi_{1k} = 0, \quad \psi_{11}(0) = 0, \quad \psi'_{11}(0) = 1, \quad \psi_{12}(1) = 1, \quad \psi'_{12}(1) = 0, \]

\[ L_2 \psi_{2k} = 0, \quad \psi_{21}(0) = 1, \quad \psi'_{21}(0) = 0, \quad \psi_{22}(1) = 1, \quad \psi'_{22}(1) = 0 \]

for \( k = 1, 2 \).

Lemma 3.7. The followings hold:

\[
\begin{align*}
    \left\| \frac{1}{x} \int_0^1 G_1(x, y)f(y)dy \right\|_{C^0} & \leq C \left\| f(x) \right\|_{C^0}, \\
    \left\| \int_0^1 G_1(x, y)f(y)dy \right\|_{C^0} & \leq C \left\| f(x) \right\|_{C^0}, \quad (3.17) \\
    \left\| \frac{1}{x} \right\|_{C^1} \left\| f(x) \right\|_{C^1} \leq C \left\| f(x) \right\|_{C^1}, \\
    \left\| \int_0^1 G_2(x, y)f(y)dy \right\|_{C^1} & \leq C \left\| f(x) \right\|_{C^1}. \quad (3.18)
\end{align*}
\]

Note that the norm of the left hand side in the second inequality is \( C^1 \), not \( C^0 \).
Proof. Observe that
\[ G_1(x, y) := \begin{cases} \psi_{11}(x)\psi_{12}(y) & \text{if } 0 \leq x \leq y, \\ \psi_{11}(y)\psi_{12}(x) & \text{if } y < x \leq 1, \end{cases} \]
\[ G_2(x, y) := \begin{cases} \psi_{21}(x)\psi_{22}(y) & \text{if } 0 \leq x \leq y, \\ \psi_{21}(y)\psi_{22}(x) & \text{if } y < x \leq 1. \end{cases} \]

Then
\[
\left| \frac{1}{x} \int_0^1 G_1(x, y)f(y)dy \right| \leq \frac{\psi_{12}(x)}{x} \int_0^x \psi_{11}(y)f(y)dy + \frac{\psi_{11}(x)}{x} \int_x^1 \psi_{12}(y)f(y)dy \\
\leq \frac{\psi_{12}(x)}{x} \left\| \psi_{11} \right\|_{C^0} \left\| f \right\|_{C^0} |x| + \frac{\psi_{11}(x)}{x} \left\| \psi_{12} \right\|_{C^0} \left\| f \right\|_{C^0},
\]
which implies
\[
\left| \frac{1}{x} \int_0^1 G_1(x, y)f(y)dy \right| \leq C \left\| f \right\|_{C^0}.
\]

Similarly,
\[
\left| \int_0^1 G_{1x}(x, y)f(y)dy \right| \leq \left| \psi_{12}'(x) \int_0^x \psi_{11}(y)f(y)dy \right| + \left| \psi_{11}'(x) \int_x^1 \psi_{12}(y)f(y)dy \right| \\
\leq C \left\| f \right\|_{C^0}.
\]
Thus (3.17) follows.

By direct calculation
\[
\left| \int_0^1 G_{2xy}(x, y)f(y)dy \right| \leq \left| \psi_{22}'(x) \int_0^x \psi_{21}'(y)f(y)dy \right| + \left| \psi_{21}'(x) \int_x^1 \psi_{22}'(y)f(y)dy \right| \\
\leq \frac{\|x^2\psi_{22}'\|_{C^0}}{x^2} \left\| \psi_{21}' \right\|_{C^0} \left\| \frac{f}{x^2} \right\|_{C^0} \frac{x^3}{3} + \left| \psi_{21}'(x) \right| \left\| x^2 \psi_{22}' \right\|_{C^0} \left\| \frac{f}{x^2} \right\|_{C^0} \leq C \left\| \frac{f}{x^2} \right\|_{C^0}.
\]
Since \( \left\| \int_0^1 G_2(x, y)f(y)dy \right\|_{C^1} \) can be treated similarly, we omit the detail and thus (3.18) is established. \( \square \)

Consider the following metric space:
\[ \mathcal{M} := \left\{ (w, z) \in C^1(0, 1)^2 \mid w(0) = 0, \ w_x(1) = 0, \ z_x(0) = 0, \ z_x(1) = 0, \ |w/x| + |w_x| \leq M_1 \varepsilon, \ |z| + |z_x| \leq M_2 \delta \varepsilon \right\}, \]
\[ \|(w, z)\|_{\mathcal{M}} := \frac{\|w\|_{C^0} + \|w_x\|_{C^0} + \eta(\|z\|_{C^0} + \|z_x\|_{C^0})}{\|x\|}, \]
with some positive constant \( \eta \). Using the integration by parts yields
\[
L_2^{-1}K_5 = \int_0^1 G_2(x, y)K_5(w(y), z(y))dy \\
= -\int_0^1 G_{2y}(x, y)(2U(y)w(y) + w(y)^2)z_x(y)dy
\]
for \((w, z) \in \mathcal{M}\). It is also denoted by \( \tilde{K}_5 \), namely,
\[ \tilde{K}_5(w, z) := -\int_0^1 G_{2y}(x, y)(2U(y)w(y) + w(y)^2)z_x(y)dy. \]
We take a positive constant $\alpha_0$, which is independent of $\delta$ and $\varepsilon$, to satisfy
\[
\|U\|_{C^0} + \left\| \frac{U}{x} \right\|_{C^0} \leq \alpha_0.
\]
By the assumption of the theorem there are positive constants $\alpha_j$ ($j = 1, \ldots, 6$) depending only on $f_1, f_2, d_1, d_2$ and $p$ such that for any $|z| \leq 1$, $|z| \leq \min\{|\bar{k} - k_*|, 1\}$,
\[
\left| \frac{f_1(1, \bar{k} + z)}{f_1(1, k)} \right| \leq \alpha_1, \quad \left| \frac{f_2(1, \bar{k} + z)}{f_1(1, k)} \right| \leq \alpha_2,
\]
\[
\left| \frac{f_{1,v}(1, \bar{k} + z)}{f_1(1, k)} \right| \leq \frac{\alpha_3}{\delta}, \quad \left| \frac{f_{2,v}(1, \bar{k} + z)}{f_1(1, k)} \right| \leq \frac{\alpha_4}{\delta},
\]
\[
\left| \frac{f_{1,vv}(1, \bar{k} + z)}{f_1(1, k)} \right| \leq \frac{\alpha_5}{\delta^2}, \quad \left| \frac{f_{2,vv}(1, \bar{k} + z)}{f_1(1, k)} \right| \leq \frac{\alpha_6}{\delta^2}.
\]

Lemma 3.8. The followings hold:
\[
\lim_{\delta \to 0} q_\varepsilon(0; \delta) = \frac{\nu_* - \mu_*}{\mu_*},
\]
\[
|q(z; \delta)| \leq \alpha_7 |z|, \quad |q_{zz}(z; \delta)| \leq \frac{\alpha_8}{\delta^2},
\]
where $\alpha_7, \alpha_8$ are a positive constant independent of $\delta$.

Proof. The definition of $q$ implies that
\[
q_\varepsilon(z; \delta) = \frac{d_1 f_{2,v}(1, \bar{k} + z) - d_2 f_1(1, \bar{k} + z) - d_2(\bar{k} + z)f_{1,v}(1, \bar{k} + z)}{d_2 f_1(1, k)}. \tag{3.19}
\]
Then
\[
q_\varepsilon(0; \delta) = \frac{d_1 f_{2,v}(1, \bar{k}) - d_2 f_1(1, \bar{k}) - d_2 \bar{k} f_{1,v}(1, \bar{k})}{d_2 f_1(1, k)}
= \frac{d_1 f_{2,v}(1, \bar{k}) - d_2 \bar{k} f_{1,v}(1, \bar{k})}{d_2 f_1(1, k)} - 1.
\]
Using
\[
d_2 \bar{k} = \frac{d_1 f_2(1, \bar{k})}{f_1(1, k)} \tag{3.20}
\]
gives
\[
q_\varepsilon(0; \delta) = \frac{d_1 f_{2,v}(1, \bar{k}) f_1(1, k) - f_{1,v}(1, \bar{k}) f_2(1, \bar{k})}{d_2 f_1(1, k)^2} - 1.
\]
Letting $\delta \to 0$ yields
\[
\lim_{\delta \to 0} q_\varepsilon(0; \delta) = \frac{\nu_*}{\mu_*} - 1.
\]
Thus the second statement immediately follows. By (3.19) and (3.20),
\[
q_{zz}(z; \delta) = \frac{d_1 f_{2,v}(1, \bar{k} + z) - 2 d_2 f_{1,v}(1, \bar{k} + z) - d_2(\bar{k} + z)f_{1,vv}(1, \bar{k} + z)}{d_2 f_1(1, k)}.
\]
This implies that
\[
|q_{zz}(z; \delta)| \leq \frac{d_1 \alpha_6 + 2 d_2 \alpha_3 \delta + d_2(\bar{k} + |z|) \alpha_5}{d_2 \delta^2}.
\]
The proof of the lemma is been complete. □

Lemma 3.9. There are positive constants $C_j$ ($j = 1, \ldots, 4$) independent of $M_1$ and $M_2$ such that for $(w_j, z_j) \in \mathcal{M}$ ($j = 1, 2$),

$$
\frac{|K_1(w_1, z_1) - K_1(w_2, z_2)|}{|x|} \leq \frac{C_1}{\delta} |z_1 - z_2| + C_2(M_1 + M_2)\varepsilon \frac{|w_1 - w_2|}{|x|},
$$

(3.21)

$$
\frac{|K_2(w_1, z_1) - K_2(w_2, z_2)|}{|x|} \leq C_3\varepsilon |z_1 - z_2| + C_3\varepsilon \frac{|w_1 - w_2|}{|x|},
$$

(3.22)

$$
|K_3(w_1, z_1) - K_3(w_2, z_2)| \leq \frac{C_4M_2^2\varepsilon}{\delta} |z_1 - z_2| + C_5M_1\varepsilon |w_1 - w_2|,
$$

(3.23)

$$
|K_4(w_1, z_1) - K_4(w_2, z_2)| \leq C_6\varepsilon |z_1 - z_2| + C_6\varepsilon |w_1 - w_2|,
$$

(3.24)

$$
\|\tilde{K}_5(w_1, z_1) - \tilde{K}_5(w_2, z_2)\|_{C^1} \leq C_7M_1\varepsilon \|z_1 - z_2\|_{C^0} + C_8M_2\varepsilon \frac{|w_1 - w_2|}{|x|} + C_9 \varepsilon.
$$

(3.25)

Proof. By the definition of $K_1$,

$$
\frac{|K_1(w_1, z_1) - K_1(w_2, z_2)|}{|x|} = \left| \frac{f_1(1, \tilde{k} + z_1)}{f_1(1, k)} (U + w_1)^p - \frac{f_1(1, \tilde{k} + z_2)}{f_1(1, k)} (U + w_2)^p - pU^{p-1}(w_1 - w_2) \right|
$$

$$
\leq \int_0^1 \left| \frac{f_1(1, \tilde{k} + \theta z_1 + (1 - \theta)z_2)}{f_1(1, k)} \right| d\theta |U + w_1|^p |z_1 - z_2|
$$

$$
+ \left| \frac{f_1(1, \tilde{k} + z_2)}{f_1(1, k)} \right| \left| (U + w_1)^p - (U + w_2)^p - pU^{p-1}(w_1 - w_2) \right|
$$

$$
+ \left| \frac{f_1(1, \tilde{k} + z_2) - f_1(1, \tilde{k})}{f_1(1, k)} \right| pU^{p-1}|w_1 - w_2|
$$

$$
\leq \alpha_3 \alpha_0 \delta \frac{p^p}{\delta^p} |x|^p |z_1 - z_2| + \alpha_1 p(p - 1)(\alpha_0 + 1)^{p-2} |x|^{p-2} |w_1 - w_2|^2
$$

$$
+ \frac{\beta_0^{p-1} \alpha_3}{\delta} |x|^{p-1} |w_1 - w_2|.
$$

Thus (3.21) holds.

Next consider (3.22). Since $h_1$ is linear, we have

$$
\frac{|K_2(w_1, z_1) - K_2(w_2, z_2)|}{|x|} \leq \frac{\varepsilon}{d_1} \left| h_1(U + w_1, (\tilde{k} + z_1)(U + w_1)) - h_1(U + w_2, (\tilde{k} + z_2)(U + w_2)) \right|
$$

$$
\leq \frac{\varepsilon \beta_2}{d_1} \left( |w_1 - w_2| + (|\tilde{k}| + |z_1|)|w_1 - w_2| + (|U| + |w_1|)|z_1 - z_2| \right),
$$

which implies (3.22).
Proof of Theorem 3.3. Consider a mapping
\[ T(x, y) = \int_0^1 G_1(x, y) \left( K_1(y) + K_2(y) \right) dy, \]
\[ T_2(x, y) = \int_0^1 G_2(x, y) \left( K_3(y) + K_4(y) \right) dy + \tilde{K}_5(y). \]
Since
\[ \begin{align*}
K_1(0, 0) &= 0, \\
K_2(0, 0) &= -\frac{\varepsilon}{d_1} h_1(U, \bar{k}U), \\
K_3(0, 0) &= 0, \\
K_4(0, 0) &= -\varepsilon U \frac{d_1 h_2(U, \bar{k}U) - d_2 \bar{k}h_3(U, \bar{k}U)}{d_3 d_2}, \\
\tilde{K}_5(0, 0) &= 0,
\end{align*} \]
For (3.23), we have
\[ |K_3(w_1, z_1) - K_3(w_2, z_2)| = \left| q(z_1; \delta)(U + w_1)^{p+1} - q(z_2; \delta)(U + w_2)^{p+1} - q_1(0; \delta) U^{p+1} (z_1 - z_2) \right| \]
\[ \leq |q(z_1; \delta) - q(z_2; \delta) - q_1(0; \delta)(z_1 - z_2)||U + w_1|^{p+1} \]
\[ + |q_1(0; \delta)||U + w_2|^{p+1} - U^{p+1}||z_1 - z_2| \]
\[ \leq \frac{\alpha \varepsilon}{d^2} |x|^{p+1} |z_1 - z_2|^2 + C \varepsilon |x|^p |z_2||w_1 - w_2| + C|\varepsilon|^{-2/3}|w_2||z_1 - z_2|. \]
Next (3.24) follows from
\[ |K_4(w_1, z_1) - K_4(w_2, z_2)| = \left| d_1 h_2(U + w_1, (\bar{k} + z_1)(U + w_1)) - d_2 (\bar{k} + z_1) h_3(U + w_1, (\bar{k} + z_1)(U + w_1)) \right| \]
\[ \times \varepsilon (U + w_1) \]
\[ = d_1 h_2(U + w_2, (\bar{k} + z_2)(U + w_2)) - d_2 (\bar{k} + z_2) h_3(U + w_2, (\bar{k} + z_2)(U + w_2)) \]
\[ \times \varepsilon (U + w_2) \leq C \varepsilon (|w_1 - w_2| + |w_1 - w_2|). \]
Lastly we turn to (3.25). Notice that
\[ \tilde{K}_5(w_1, z_1) - \tilde{K}_5(w_2, z_2) \]
\[ = \int_0^1 G_2(x, y) \left\{ (2U(y)w_1(y) + w_1(y)^2) z'_1(y) - (2U(y)w_2(y) + w_2(y)^2) z'_2(y) \right\} dy. \]
This leads to
\[ \| \tilde{K}_5(w_1, z_1) - \tilde{K}_5(w_2, z_2) \|_{C^1} \leq C M_1 \varepsilon \| z_1 - z_2 \|_{C^1} + C M_2 \varepsilon \| \frac{w_1 - w_2}{|x|} \|_{C^0}. \]
Now the proof is complete. \hfill \Box

Proof of Theorem 3.3. Consider a mapping \( T = (T_1, T_2) \) on \( M \) defined by
\[ T_1(w, z) = \int_0^1 G_1(x, y) \left( K_1(y) + K_2(y) \right) dy, \]
\[ T_2(w, z) = \int_0^1 G_2(x, y) \left( K_3(y) + K_4(y) \right) dy + \tilde{K}_5(y). \]
we see that $T$ is a contraction mapping on $M$ with appropriate constants $M_1, M_2, \varepsilon$, for any $\delta > 0$. Thus the existence of non-constant stationary solutions through the bifurcation from infinity has been shown.

As an application of Theorem 3.3, we will see that adding an arbitrarily small linear perturbation into a $p$-homogeneous system could change bounded solutions to be blow-up.

**Example 3.10.** Consider the system
\[
\begin{align*}
  u_t &= (v - u)(u^2 + 2uv - v^2) + \alpha(1 - \varepsilon)u \\
  v_t &= (v - u)(-u^2 + 2uv + v^2) + \alpha(1 + \varepsilon)v,
\end{align*}
\]
which can be expressed in the polar form
\[
\begin{align*}
  \frac{d \ln r}{dt} &= r^2(\sin \theta - \cos \theta)(\sin \theta + \cos \theta) + \alpha(1 - \epsilon [\cos^2 \theta - \sin^2 \theta]), \\
  \frac{d \theta}{dt} &= r^2(\sin \theta - \cos \theta) + 2\alpha \epsilon \sin \theta \cos \theta.
\end{align*}
\]
When $\alpha = 0$, one obtains the equation
\[
\frac{d \ln r}{d \theta} = \frac{\sin \theta + \cos \theta}{\sin \theta - \cos \theta}.
\]
Integration shows that if $\sin \theta_0 - \cos \theta_0 \neq 0$, then
\[
r = \frac{r_0}{\sin \theta_0 - \cos \theta_0} (\sin \theta - \cos \theta).
\]
Thus the orbit through $(r_0 \cos \theta_0, r_0 \sin \theta_0)$ is a circle centred on the line $u + v = 0$ and tangent to the line $u = v = 0$ at the origin. Since the points of the line $u - v = 0$ are all equilibria, we see that all solutions of the 3-homogeneous system which is obtained by setting $\alpha = 0$ in (3.26) are bounded.

On the other hand, we observe that $h(\pi/4) = \ell(\pi/4) = 0$, that $h'$ and $\ell$ are positive in the interval $(\pi/4, 3\pi/4)$, that $Q(\pi/4) = -\alpha \epsilon$, and that $h' = 2\ell$, so that $\nu_* = 1/2$. Thus Theorem 6.1 of [9] shows that if $\alpha \epsilon < 0$, the system (3.26) has solutions which to be blow-up in a finite time. In particular, we have shown that for any fixed $\varepsilon > 0$, all solutions of the system (3.26) with $\alpha = 0$ are bounded, while if $\alpha < 0$, some solutions turn out to blow up in finite time. Thus even adding an arbitrarily small linear perturbation into a $p$-homogeneous system results in blow-up phenomenon.

Mizoguchi, Ninomiya, and Yanagida [5] have shown that the Neumann problem
\[
\begin{align*}
  u_t &= (1 - \epsilon)\Delta u + (u - v)^3 - u, \\
  v_t &= (1 + \epsilon)\Delta v + (u - v)^3 - v, \\
  \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} &= 0 \text{ on } \partial \Omega
\end{align*}
\]
on a bounded domain $\Omega$ has the property that all solutions converge to the origin when $\epsilon = 0$, but that there are solutions which blow up in a finite time when $0 < \epsilon < 1$.

If we think of $\bar{u}(t)$ as the value at $x = 0$ of a functions which is defined to be $0$ at all the other integer points $x = k \neq 0$, then the the second difference of this grid function at $x = 0$ is $\delta^2 \bar{u}(t, 0) = -2\bar{u}(t, 0)$. Thus the system in the following example may be thought of as a finite difference analog of the system (3.27)
of Mizoguchi, Ninomiya, and Yanagida, but with Dirichlet rather than Neumann boundary conditions.

**Example 3.11.** Consider the system

\[
\begin{align*}
    u_t &= d_1u_{xx} + (u^2 - v^2)v - \varepsilon u, \\
    v_t &= d_2v_{xx} + (u^2 - v^2)u - \varepsilon v.
\end{align*}
\]

(3.28)

In this example, we see that

\[ k^* = 1. \]

Moreover, it is easily confirmed that

\[ \mu^* = \lim_{k \to 1} \frac{f_2(1,k)}{f_1(1,k)} = 1, \quad \nu^* = \lim_{k \to 1} \frac{d}{dk} \left( \frac{f_2(1,k)}{f_1(1,k)} \right) = -1 \neq \mu^* \]

as well as the other conditions stated in Theorem 3.1. It follows from Theorem 3.3 that the inhomogeneous stationary solution resulted from bifurcation from infinity as \( d_2 \) is close to \( d_1 \) and \( d_2 > d_1 \).

**REFERENCES**

[1] C.-N. Chen, Uniqueness and bifurcation for solutions of nonlinear Sturm-Liouville eigenvalue problems, *Arch. Rational. Mech. Anal.*, 111 (1990), 51–85.

[2] C.-N. Chen, Some existence and bifurcation results for solutions of nonlinear Sturm-Liouville eigenvalue problems, *Math. Zeitschrift*, 208 (1991), 177–192.

[3] C.-N. Chen, A survey of nonlinear Sturm-Liouville equations, *Sturm-Liouville Theory*, Birkhäuser, Basel, (2005), 201–216.

[4] M. Fila and K. Ninomiya, Reaction versus diffusion: Blow-up induced and inhibited by diffusivity, *Russian Mathematical Surveys*, 60 (2005), 1217–1235.

[5] N. Mizoguchi, H. Ninomiya and E. Yanagida, Diffusion-induced blowup in a nonlinear parabolic system, J. Dynam. Differential Equations, 10 (1998), 619–638.

[6] J. Morgan, On a question of blow-up for semilinear parabolic systems, *Differential Integral Equations*, 3 (1990), 973–978.

[7] J. D. Murray, *Mathematical Biology II: Spatial Models and Biomedical Applications*, Third edition, Interdisciplinary Applied Mathematics, 18. Springer-Verlag, New York, 2003.

[8] H. Ninomiya and H. F. Weinberger, Pest control may make the pest population explode, *Z. Angew. Math. Phys.*, 54 (2003), 869–873.

[9] H. Ninomiya and H. F. Weinberger, On \( p \)-homogeneous systems of differential equations and their linear perturbations, *Applicable Analysis*, 85 (2006), 225–247.

[10] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *Journal of functional analysis*, 7 (1971), 487–513.

[11] P. H. Rabinowitz, On bifurcation from infinity, *J. Differential Equations*, 14 (1973), 462–475.

[12] S. Rosenblat and S. H. Davis, Bifurcation from infinity, *SIAM Journal on Applied Mathematics*, 37 (1979), 1–19.

[13] C. A. Stuart, Solutions of large norm for non-linear Sturm-Liouville problems, *Quarterly Journal of Mathematics*, 24 (1973), 129–139.

[14] A. M. Turing, The chemical basis of morphogenesis, *Phil. Trans. R. Soc. Lond. Ser. B*, 237 (1952), 37–72.

Received April 2019; revised May 2019.

E-mail address: chihirokamo@gmail.com
E-mail address: chen@math.nthu.edu.tw
E-mail address: kuto@waseda.jp
E-mail address: hirokazu.ninomiya@gmail.com