Counting Classical Nodes in Quantum Networks

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(Dated: March 20, 2019)

Quantum networks\cite{1,2} illustrate the use of connected nodes of individual quantum systems as the backbone of distributed quantum information processing. When the network nodes are entangled in graph states\cite{3}, such quantum platforms are indispensable to almost all the existing distributed quantum tasks\cite{4,11}. Unfortunately, real networks unavoidably suffer from noise and undesirable technical restrictions\cite{12–14}, making nodes transit from quantum to classical at worst. Here, we introduce a figure of merit in terms of the number of classical nodes for quantum networks in arbitrary graph states. This property of a network is evaluated by exploiting genuine multi-subsystem Einstein-Podolsky-Rosen steering. Experimentally, we demonstrate photonic quantum networks of \(n_q\) quantum nodes and \(n_c\) classical nodes with \(n_q\) up to 6 and \(n_c\) up to 18 using spontaneous parametric down-conversion entanglement sources. We show that the proposed method is faithful in quantifying the classical defects in prepared multiphoton quantum networks. Our results provide novel characterization and identification of both generic quantum network architecture\cite{12–16} and multipartite non-classical correlations in graph states\cite{17,18}.

Quantum mechanics enables non-classical correlations to exist across the whole of network via connecting individual quantum nodes, forming a joint quantum many-body system\cite{12}. Quantum networks, therefore, have far greater capacity of state space than the classical ones and serve as well-advanced transmitters of quantum information for all the distant network participants. Such utilities encourage important applications to distributed quantum information processing, from quantum secret sharing\cite{4,5} to distributed sensing\cite{6}, from distributed quantum computation\cite{7,9} to quantum conference key agreement and distribution\cite{10,11}.

The physical realization of these distributed quantum tasks requires suitable connectivities between nodes and network topologies to initialize the nodes in the joint states possessing special multipartite entanglement, known as graph states\cite{3} (shown in Fig. 1a). In a generic network architecture available for preparing graph states with tailored topology, photonic quantum information demands to be sent, received, stored and exchanged between stationary quantum nodes via photonic channels in general\cite{12–16}. Any inevitable imperfections of network nodes, such as the intrinsic fragility of quantum systems and errors present in actual implementations, can cause quantum nodes to become classical systems that obey the laws of classical physics, therefore leading to the failure of state preparation or decay of quantum networks\cite{11,2,12–16}. Moreover, when network participants only have limited knowledge about the node imperfections, the network nodes then become untrusted to the participants as untrusted nodes.

The characterization and identification of quantum networks is a major challenge to distributed quantum information processing. A conventional way to detect entanglement is entanglement witness (EW), which employs deduction from the predictions of quantum theory\cite{19,20}. However, EW is not strict in characterizing quantum correlation in networks as the measurement apparatus in nodes are not always trusted. Two major approaches are currently being pursued to solve this problem. One is device-independent (DI) procedure, which is based on imposing what can be thought of as a classical constraint on network connections between nodes, e.g., Bell-like inequalities\cite{21,22}. An alternative solution is so-called measure-device-independent entanglement witness (MDIEW)\cite{23,24}, in which with proper chosen ancillary states, multipartite entanglement can be witnessed without trusting measurement devices.

Counting defects has played a crucial role in engineering physical systems and processes for a variety of applications, either in evaluating reliability for software development\cite{25}, or in probing exotic properties for solids\cite{26}. However, a method capable of quantifying the defects, such as classical nodes, in quantum networks has remained elusive. The act of counting classical nodes can thus be expected to not only characterize the classical defects in quantum networks, but also quantitatively assess the primitive operations required to realize quantum networks. Nevertheless, neither DI verification nor MDIEW is capable of revealing the number of classical nodes in quantum networks. Moreover, it is not yet

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clear how a created network can be evaluated in terms of the number of classical nodes. Here, we present a solution to fill this gap by revealing novel characteristics of graph states, together with the experimental realization of the defect quantification in photonic quantum networks. This brings us to a new regime of quantum network identification.

An essential difference between quantum and classical nodes is that physical properties of quantum nodes might not have definite values. In contrast, variables in classical physics are in existing states independent of observation. This is known as the assumption of realism [22]. In our framework for network characterization, a node is defined as being classical if, for any physical properties of interest, it is classical realistic.

We utilize the network fidelity function, which measures the closeness of created networks and target graph states, as the basis for counting classical nodes. This makes our framework capable of being used in a wide variety of circumstances and applications based on the fidelity measure. We introduce the following network fidelity function for arbitrary target graph states $|G\rangle$ of $N$ nodes

$$F(N) = \sum_{\vec{m}} h_{\vec{m}} \langle R_{m_1}...R_{m_N} \rangle$$

(1)

where $\vec{m} = (m_1, ..., m_N)$ and $R_{m_k}$ is the outcome of the $m_k$th measurement on the $k$th node. The coefficients $h_{\vec{m}}$ and the measurements on nodes are determined by the decomposition of the graph states into $N$-node tensor products of positive operators; together with the sorts of the mean value of the product $R_{m_1}...R_{m_N}$, denoted as $\langle R_{m_1}...R_{m_N} \rangle$, the fidelity function for a given graph state therefore is not unique (see Methods). When a network is ideally prepared in a target state, we in principle obtain $F = 1$, regardless of what fidelity function is chosen.

As the fidelity functions are characterized in the orthonormal sets of observables from state decomposition,
Expected Value  Expected Value

and described by the equation

$$F_{n_c} = \frac{1}{4}(1 + 2\sqrt{2} + \sqrt{4 + 2n_c})$$

(2)

where $1 \leq n_c \leq N - 1$, independent of the topology of target graph states (see Methods for the derivation of Eq. 2). The threshold fidelities $F_{n_c}$ are strictly decreasing with the number of classical nodes $n_c$. It turns out that there exists a one-to-one correspondence between the number of classical nodes and the relevant maximum fidelity values. For instance, $F_1 \simeq 0.6830$, $F_2 \simeq 0.6036$, and $\lim_{n_c \to \infty} F_{n_c} \simeq 0.5000$. The hybrids of quantum and classical nodes are then comparable in fidelity to the networks composed entirely of quantum nodes with $F \leq F_1$. This implies that the collection $\{F_{n_c}\} \equiv \{F_{n_c} | n_c = 1, 2, \ldots, N - 1\}$ of the threshold fidelities can serve as a set of graduations to indicate the degree of network imperfection. That is, if the measured fidelity $F$ is found to be $F_{n_c} < F \leq F_{n_c}'$, then the created network is evaluated by the number of classical nodes, $n_c$.

Indeed, the collection $\{F_{n_c}\}$ quantitatively describes how the non-classical correlations among nodes of the graph states differ from the quantum-classical hybrids. If $F > F_{n_c}$ for a created network, then it is impossible to simulate the correlations between nodes using any networks mixed with classical defects of the minimum classical nodes, $n_c$. Such quantum characteristic can be interpreted as the genuine multi-subsystem EPR steering (see Methods), a new type of genuine multipartite Einstein-Podolsky-Rosen (EPR) steerability of graph states[27]. Notably, the new-found criterion $F > F_{n_c}$ is stricter than the seminal criterion $F > 1/2$ for genuine multipartite entanglement[19] of networks. A created network containing classical nodes can mimic the networks with genuine multipartite entanglement to show $1/2 < F < F_1$. This can cause flaws in using the verification of genuine multipartite entanglement as an essential criterion for distributed quantum tasks.

We experimentally demonstrate our protocol on multipartite graph states in star graph $|G^{\text{star}}_N\rangle$, which is equivalent to Greenberg-Horne-Zeilinger (GHZ) state $|GHZ\rangle_N = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ via local operation and classical communication (LOCC). The experimental setup to generate a six-photon GHZ state $|GHZ\rangle_6 = \frac{1}{\sqrt{2}}(|H\rangle^{\otimes 6} + |V\rangle^{\otimes 6})$ with $H$ the horizontal polarization and $V$ the vertical polarization is shown in Fig. 1a. An experimental state, denoted as $\rho_6^{GHZ}$, is generated by employing the typical spontaneous parametric down-conversion entangled photon source and photonic interferometry technologies (see Supplementary Information for more details). The network fidelity $F(6)$ of the generated state $\rho_6^{GHZ}$ is measured by the device shown in Fig. 1b, which is consisted of a quarter-wave plate (QWP), half-wave plate (HWP), a polarization beam splitter (PBS) and two detectors. By properly choosing the angle of QWP and HWP, the expected value of $I$, $X$, $Y$ and $Z$ can be readout (see Supplementary Information for more details). The experimental results of measured $F(6)$ are shown in Fig. 2, from which we calculate that $F(6) = 0.792 \pm 0.006$ (shown with blue bar in Fig. 2). Then $F(6)$ exceeds the threshold fidelity $F_1 = 0.683$ more than 18 standard deviation, which indicates there is no classical node in the tested network.

We then consider the case where $n_c$ classical nodes exist in the $N$-node network, and show that $F(N)$ is bounded by the threshold fidelity $F_{n_c}$ even with the optimal “cheating strategy” (OCS). The OCS of one untrusted node in bipartite quantum correlation has been

FIG. 2. Experimental results of network fidelity $F(6)$ of six-node network with classical node number $n_c = 0$ and $n_c = 1$ respectively. a, the experimental results of network fidelity in network with $n_c = 0$. b, the experimental results of network fidelity measurement in network with $n_c = 1$. c, the calculated $F(6)$ from the results in a and b. The black dash line is the threshold of EW, and the red dash line is the threshold fidelity $F_1$. The hybrids of quantum and classical nodes are then comparable in fidelity to the networks composed entirely of quantum nodes with $F \leq F_1$. This implies that the collection $\{F_{n_c}\} \equiv \{F_{n_c} | n_c = 1, 2, \ldots, N - 1\}$ of the threshold fidelities can serve as a set of graduations to indicate the degree of network imperfection. That is, if the measured fidelity $F$ is found to be $F_{n_c} < F \leq F_{n_c}'$, then the created network is evaluated by the number of classical nodes, $n_c$. Indeed, the collection $\{F_{n_c}\}$ quantitatively describes how the non-classical correlations among nodes of the graph states differ from the quantum-classical hybrids. If $F > F_{n_c}$ for a created network, then it is impossible to simulate the correlations between nodes using any networks mixed with classical defects of the minimum classical nodes, $n_c$. Such quantum characteristic can be interpreted as the genuine multi-subsystem EPR steering (see Methods), a new type of genuine multipartite Einstein-Podolsky-Rosen (EPR) steerability of graph states[27]. Notably, the new-found criterion $F > F_{n_c}$ is stricter than the seminal criterion $F > 1/2$ for genuine multipartite entanglement[19] of networks. A created network containing classical nodes can mimic the networks with genuine multipartite entanglement to show $1/2 < F < F_1$. This can cause flaws in using the verification of genuine multipartite entanglement as an essential criterion for distributed quantum tasks. We experimentally demonstrate our protocol on multipartite graph states in star graph $|G^{\text{star}}_N\rangle$, which is equivalent to Greenberg-Horne-Zeilinger (GHZ) state $|GHZ\rangle_N = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ via local operation and classical communication (LOCC). The experimental setup to generate a six-photon GHZ state $|GHZ\rangle_6 = \frac{1}{\sqrt{2}}(|H\rangle^{\otimes 6} + |V\rangle^{\otimes 6})$ with $H$ the horizontal polarization and $V$ the vertical polarization is shown in Fig. 1a. An experimental state, denoted as $\rho_6^{GHZ}$, is generated by employing the typical spontaneous parametric down-conversion entangled photon source and photonic interferometry technologies (see Supplementary Information for more details). The network fidelity $F(6)$ of the generated state $\rho_6^{GHZ}$ is measured by the device shown in Fig. 1b, which is consisted of a quarter-wave plate (QWP), half-wave plate (HWP), a polarization beam splitter (PBS) and two detectors. By properly choosing the angle of QWP and HWP, the expected value of $I$, $X$, $Y$ and $Z$ can be readout (see Supplementary Information for more details). The experimental results of measured $F(6)$ are shown in Fig. 2, from which we calculate that $F(6) = 0.792 \pm 0.006$ (shown with blue bar in Fig. 2). Then $F(6)$ exceeds the threshold fidelity $F_1 = 0.683$ more than 18 standard deviation, which indicates there is no classical node in the tested network. We then consider the case where $n_c$ classical nodes exist in the $N$-node network, and show that $F(N)$ is bounded by the threshold fidelity $F_{n_c}$ even with the optimal “cheating strategy” (OCS). The OCS of one untrusted node in bipartite quantum correlation has been
well discussed. We generalize the OCS of untrusted (classical) nodes with a lack of N-partite entanglement to cheat the other trusted participants in the N-node network as: the nodes first prepare the entangled state \( |\xi\rangle_{n_q} \) for trusted (quantum) nodes based on their knowledge of the N-node network, where \( n_q = N - n_c \). Then, according to the measurement setting for the network fidelity function \( F(N) \) (Methods for details), the experimental results of \( F(6) \) under OCS are shown in Fig. 2, from which we calculate \( F(6) = 0.538 \pm 0.007 \) (shown with red bar in Fig. 2). As shown in Fig. 2b, we can see that \( F(6) \) either with \( n_c = 0 \) or \( n_c = 1 \) exceeds the EW threshold, which is a strong evidence that EW is no longer reliable in quantum network identification. However, with our criteria, the measured \( F(6) \) does not exceed the threshold fidelity \( F_1 \), which indicates there are classical nodes in the measured network.

We also experimentally prepare various N-node quantum networks with \( n_q \) up to 6 and \( n_c \) up to 18, where \( N = n_q + n_c \) (see Supplementary Information for more experimental details). For each network, \( n_c \) nodes employ OCS to achieve maximal network fidelity \( F(N) \). The measured \( F(N) \) are shown in Fig. 3. It is clear that \( F(N) \) with \( n_c \) classical nodes are bounded by the threshold fidelity \( F_{n_c} \). One may notice that \( F(N) \) decreases much faster than \( F_{n_c} \) as \( n_q \) increased. This is mainly caused by the imperfections in the state preparation, in which more imperfections are introduced when coherently manipulating more photons. When \( n_q \geq 3 \), \( F(N) \) decreases below EW threshold (0.5) quickly as \( n_c \) increased (\( n_c \geq 4 \)). We investigate \( F(N) \) of networks with \( n_q = 1 \) and \( n_q = 2 \) for large \( n_c \) as the prepared one-photon and two-photon states are with near unity fidelities. The results of \( F(N) \) for \( n_q = 1 \) and \( n_q = 2 \) are particularly shown in Fig. 3, from which we can see that \( F(N) \) fits \( F_{n_c} \) very well. We analyze the standard deviation \( \mathcal{E} \) of \( F(N) \) in verifying entanglement and evaluating \( n_c \), which is also related to statistical significance \( S(30) \). The significance of \( F(N) \) in EW is defined as \( S(\mathcal{E}) = (\mathcal{S}(N) - 0.5)/\mathcal{E} \), where 0.5 is the threshold of EW and \( \mathcal{E} \) is the statistical error of \( F(N) \) in our experiment. Similarly, the significance of \( F(N) \) in evaluating \( n_c \) is defined as \( S(n_c) = (\mathcal{S}(N) - F_{n_c-1})/\mathcal{E} \). The larger the statistical significance is, the higher confidence interval we can obtain regarding the conclusion. When the statistical significance goes to subtle values or negative values, it means the confidence interval of conclusion is low or it fails to get the conclusion. The significance of \( F(N) \) in networks with \( n_q = 1 \) and \( n_q = 2 \) are shown in Fig. 3. As shown in Figs. 3a and b, the value of significance of \( F(N) \) in EW is greater than 3 and with maximal value of 68.5 when \( n_c \leq 6 \), which represents a high confidence interval in EW. Again, it indicates EW is not reliable in identification of quantum network. With our criteria (shown in Figs. 3a and d), \( F(N) \) can be employed as an confidential indicator to evaluating \( n_c \) when \( n_c \leq 6 \). Although \( F(N) \) could exhibit graduation when \( n_c \geq 7 \) (insets in Figs. 3a and d), it can not be employed as a confidential indicator to evaluating \( n_c \) as \( S(n_c) \) goes.

FIG. 3. Experimental results of network fidelity \( F(N) \) in network with variant quantum and classical node number \( n_q \) and \( n_c \), respectively. a, the bars edged with black line represent threshold of network fidelity \( F_{n_c} \). The filled color bars represent the experimental measured network fidelity \( F(N) \) in its corresponding network. b, the results of \( F(N) \) in networks with \( n_q = 1 \) and \( n_q = 2 \). The bars represent \( F_{n_c} \), and red (blue) dots represent the measured \( F(N) \) in the network with \( n_q = 1 \) (\( n_q = 2 \)).

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**F(N)**

![Image](image-url)
to subtle values negative values. This is caused by the statistic error in our experiment, but not the limitation of our criteria.

Our findings, to the best of our knowledge, is the first method capable of counting the number of classical nodes in quantum networks. Moreover, the proposed method reveals that the quantum-classical hybrid networks with OCS can surpass the seminal EW threshold of $F > 1/2$, which causes serious flaws in using the verification of EW in quantum networks\cite{20}. Our proof-of-principle photonic networking experiments, with $n_q = 2$. The cyan bars represent positive deviation and magenta bars represent negative deviation. The insets are $S(EW)$ and $S(n_c)$ in quantum networks with $n_q = 2$. The cyan bars represent positive deviation and magenta bars represent negative deviation.

**METHODS**

**Graph states.** Suppose that each quantum node is a quantum two-dimensional system (qubit). An edge, say $(i, j) \in E$, corresponds to a two-qubit conditional transformation among the two qubits (vertices) $i$ and $j$ by $U_{i,j} = \sum_{v_i} |v_i\rangle \langle v_i| \otimes (Z_j)^{n_i}$, where $\{v_i\}$ is an orthonormal basis of the $i$th qubit and $Z_j = \sum_{k_j=0}^{1} (-1)^{k_j} |k_j\rangle \langle k_j|$. The state vector of a graph state is determined by the target graph $G(V, E)$, where $|V| = N$ indicates the total node number and $E$ tells us how the nodes are connected together to show the network topology; that is,

$$|G\rangle = \prod_{(i,j) \in E} U_{i,j} |f_0\rangle,$$

where $|f_0\rangle = [(|0\rangle + |1\rangle)/\sqrt{2}]^N$ is the initial state of the network.

**Fidelity function $F(N)$.** The value of the fidelity function \( \text{tr}\) under given experimental results shows the fidelity of the created network and the target graph state. The way the graph state $|G\rangle$ is decomposed decides the construction of the fidelity function, which can be understood by the following explicit general decomposition for arbitrary graph states

$$|G\rangle = \sum_{m} h_{m} \bigotimes_{k=1}^{N} \hat{R}_{m_k},$$

where $\hat{R}_{m_k}$ represents the $m_k$th observable of the $k$th qubit. For instance, suppose that the created network is in a $N$-qubit state described by a density operator $\rho_{\text{expt}}$, the fidelity of $\rho_{\text{expt}}$ and $|G\rangle$ satisfies the relation

$$F(N) = \text{tr}(\rho_{\text{expt}} |G\rangle \langle G|) = \sum_{m} h_{m} \langle R_{m_1}... R_{m_N} \rangle,$$

where $\langle R_{m_1}... R_{m_N} \rangle = \text{tr}(\rho_{\text{expt}} \bigotimes_{k=1}^{N} \hat{R}_{m_k})$.

When we assume that the measurements on each qubit are performed with the observables in the Pauli matrices, $\{\hat{R}_{m_k} |m_k = 0, 1, 2, 3\}$, where $\hat{R}_0 = I$, $\hat{R}_1 = X$, $\hat{R}_2 = Y$, and $\hat{R}_3 = Z$. The spectral decomposition of the Pauli matrices: $\hat{R}_0 = \sum_{v_{m_k}} \hat{R}_{m_k} |v_{m_k}\rangle \langle v_{m_k}|$, and $\hat{R}_{m_k} = \sum_{v_{m_k}} \hat{R}_{m_k} |v_{m_k}\rangle \langle v_{m_k}|$ for $m_k = 1, 2, 3$, reminds us the relation between the measurement outcome $\hat{R}_0 = 1$ and $\hat{R}_{m_k} = v_{m_k}$ and the post-measurement state of the qubit $|v_{m_k}\rangle_{m_k}$. Therefore the states $|G\rangle$ can be specified by the decomposition (4) using the orthonormal set of matrices, $\{\bigotimes_{k=1}^{N} \hat{R}_{m_k}/\sqrt{2}\}$, from which the fidelity function $F$ is then constructed. In this case the constituent $2^N$ matrices with $h_{m} = 2^{-N}$ consist of the stabilizer of the graph state\cite{3}. The fidelity function can be constructed in the same manner when the observables for state decomposition are not orthonormal.
Set of graduations \( \{ F_{n_c} \} \). Suppose a given \( N \)-node network with the desired target graph state decays into a hybrid of quantum and classical nodes. The index set of the network nodes, \( V \), can then be divided into the quantum-node subset, \( V_Q \), and the classical-node subset, \( V_C \), accordingly. We assume that there are \( n_c \) classical nodes in the hybrid network, i.e., \( |V_C| = n_c \), and \( |V_Q| + |V_C| = N \). To examine the minimum deviation from the target graph state, we evaluate the maximum fidelity by performing the following task

\[
F_{n_c} = \max_{V_C, \{ R_{mk} \}} F,
\]

where the maximization is over all vertex sets \( V_C \) with \( |V_C| = n_c \) and all outcomes from measurements on the classical and quantum nodes, \( \{ R_{mk} \} \). First, since the classical nodes possess physical properties that exist independent of observation, the state of each classical node can be specified by a fixed set of measurement outcomes, \( v_k \equiv \{ \hat{R}_{mk} = v_{mk} | m_k = 0, 1, 2, 3 \} \). This gives a complete description of the total state of the \( n_c \) classical nodes in terms of the outcome set, involving all the measurements: \( \{ v_k | k \in V_C \} \). Whereas the measurement outcome derived from the quantum nodes can go beyond this classical realistic assumption.

Second, before the state decay, when explicitly considering Eq. \( [5] \), we note that, the state vector of the target graph state can always be represented in the Schmidt form of rank \( r \): \( |G\rangle = 1/\sqrt{r} \sum_{a=0}^{r-1} |v_a \rangle |v_{ac} \rangle \), for \( r \geq 2 \), where \( \{ |v_a \rangle \}_{a=Q} \) and \( \{ |v_{ac} \rangle \}_{ac} \) are the Schmidt bases for the nodes in the vertex sets \( V_Q \) and \( V_C \), respectively. This representation shows us the state decomposition

\[
|G\rangle = 1/r \sum_{v_v', \tilde{m}_c} h_{m_c v_v'}^{v_v'} |v\rangle |Q\rangle |Q_{ac} \rangle \otimes \hat{R}_{mk},
\]

where \( |v\rangle |Q\rangle |Q_{ac} \rangle = \sum_{\tilde{m}_c} h_{m_c v_v'}^{v_v'} \otimes_{k \in V_C} \hat{R}_{mk} \) and \( \tilde{m}_c \equiv \{ m_k | k \in V_C \} \), by which the fidelity function can be rephrased as the following explicit form

\[
F = \frac{1}{r} \sum_{v_v', \tilde{m}_c} h_{m_c v_v'}^{v_v'} \langle v | Q_{ac} | Q_v \rangle \langle v_v' | Q_{ac} | v_v' \rangle \prod_{k \in V_C} | \hat{R}_{mk} | .
\]

Finally, through this expression for the fidelity function, the maximization task \( [8] \) becomes

\[
F_{n_c} = \max_{\{ v_k | k \in V_C \}} E[f_{00}, f_{01}, f_{10}, f_{11}],
\]

where \( f_{v_v' v_v} = 1/2 \sum_{\tilde{m}_c} h_{m_c v_v'}^{v_v'} \langle \prod_{k \in V_C} | \hat{R}_{mk} \rangle \) and \( E[\cdot] \) denotes the largest eigenvalue of the matrix. Here \( r = 2 \) is shown to be necessary for the maximum of \( F \). One always can find at least one bipartite splitting of the network nodes in the target graph to have such Schmidt rank of state decomposition under the condition \( |V_C| = n_c \). When the fidelity functions are specified in the orthonormal sets of Pauli matrices and the classical nodes are described under the assumption of realism to show \( f_{00} = 1/2, f_{01} = (1/2)^{n_c+1} (1+i)^{n_c}, f_{10} = (1/2)^{n_c+1} (1-i)^{n_c} \), and \( f_{11} = 0 \), we arrive the result of Eq. \( [2] \).

Genuine multi-subsystem EPR steering. Satisfying the criterion

\[
F > F_{n_c},
\]

confirms that, the correlation between the network nodes of a created state is stronger than all the correlations that can be created by the quantum-node subsets, \( V_Q \), and the classical-node subsets, \( V_C \) with \( |V_C| = n_c \), for all possible bipartitions of the hybrid of quantum and classical nodes. This concretely describes the steering effects between two subsystems with \( n_c \) and \( n_q \) nodes, respectively. Moreover, since all possible configurations of splitting \( N \) nodes into two subsystems are considered in the criterion, we call such steerability the genuine multi-subsystem EPR steering. This description generalizes the concept of genuine multipartite EPR steering \( [27, 28] \), where only the extreme value of \( F_{n_c} \) is involved. Such steerability is shown if the created network with a fidelity that goes beyond the threshold

\[
\max_{n_c, |V_C| \in \{ R_{mk} \}} F = \max_{n_c} F_{n_c}.
\]

From the result of \( F_{n_c} \), \( [2] \), it is clear that \( F > F_1 \) is a fidelity criterion for genuine multipartite EPR steering.

Optimal Cheating Strategy (OCS) and its experimental realization. In the \( N \)-node network with \( n_c \) untrusted (classical) nodes and \( n_q \) trusted (quantum) nodes, the \( n_c \) untrusted nodes prepare the entangled state \( |\xi\rangle |n_q \) based on their knowledge of the network fidelity function \( \{ |\xi\rangle \}_{n_q} \). Combining with the measurement results of prepared \( |\xi\rangle |n_q \) the \( n_c \) untrusted nodes broadcast their results in \( \{ +1, -1 \} \), according to the measurement setting for the network fidelity function \( \{ |\xi\rangle \}_{n_q} \), to achieve the maximal \( F(N) \).

For instance, in the case of \( N = 6 \) with \( n_c = 1 \) and \( n_q = 5 \), we experimentally realize OCS by projecting one photon from \( GHZ \) of \( |\xi\rangle = \cos \theta |H\rangle + \sin \theta e^{-i\phi} |V\rangle \) (see Fig. \( [3] \)), which leaves five trusted nodes sharing a five-photon state \( |\xi\rangle_5 = \cos \theta |H\rangle_5 + \sin \theta e^{i\phi} |V\rangle_5 \). The choice of \( \theta \) and \( \phi \) in OCS is not unique so that they constitute an ensemble denoted as \( \{ |\xi\rangle \}_{3} \). In our experiment, we select \( \sin \theta = \frac{4}{3+\sqrt{3}} \) and \( \phi = -\pi/4 \).

The trusted nodes use the experimental setup shown in Fig. \( [4] \) to measure the expected value of \( \hat{R}_{mk} \), in which the transmitted photons are projected on eigenstate \( |+1 \rangle \) of \( \hat{R}_{mk} \) with eigenvalue \( +1 \), while the reflected photons are projected on \( |-1 \rangle \) of \( \hat{R}_{mk} \) with eigenvalue \( -1 \). The angle setting of QWP and HWP to perform measurements of \( X, Y \) and \( Z \) are shown in Table. \( [4] \). Thus, the expected value of \( \langle \hat{R}_{mk} \rangle \) can be obtained by \( \frac{N_+ - N_-}{N_+ + N_-} \),
where \( N_{+1(-1)} \) is the recorded counts on detector behind transmitted (reflected) photon. For \( R_{mk} = I \), the angle setting of QWP and HWP can be arbitrarily chosen, and \( (I) = \frac{N_{+1} + N_{-1}}{N_{+1} + N_{-1}} = 1 \). Then, combining the results from untrusted node \( \in \{+1, -1\} \) according to the measurement setting for the network fidelity function \( F(6) \), we obtain the network fidelity \( F(6) = 0.538 \pm 0.007 \), shown with red bar in Fig. [2]. See Supplementary Information for experimental details of the other cases.

| Observable | QWP | HWP | Expected value |
|------------|-----|-----|----------------|
| \( X \)    | 45° | 22.5° | \( \frac{N_{+1} - N_{-1}}{N_{+1} + N_{-1}} \) |
| \( Y \)    | 0°  | 22.5° |               |
| \( Z \)    | 0°  | 0°   |               |

TABLE I. Angle setting of QWP and HWP in \( X, Y, Z \) detection.

[1] H. J. Kimble, *Nature* **453**, 1023 (2008).
[2] S. Wehner, D. Elkouss, and R. Hanson, *Science* **362**, eaan9288 (2018).
[3] M. Hein, J. Eisert, and H. J. Briegel, *Phys. Rev. A* **69**, 062311 (2004).
[4] Y.-A. Chen, A.-N. Zhang, Z. Zhao, X.-Q. Zhou, C.-Y. Lu, C.-Z. Peng, T. Yang, and J.-W. Pan, *Phys. Rev. Lett.* **95**, 200502 (2005).
[5] H. Lu, Z. Zhang, L.-K. Chen, Z.-D. Li, C. Liu, L. Li, N.-L. Liu, X. Ma, Y.-A. Chen, and J.-W. Pan, *Phys. Rev. Lett.* **117**, 030501 (2016).
[6] P. Kőmőr, E. M. Kessler, M. Bischof, L. Jiang, A. S. Sørensen, J. Ye, and M. D. Lukin, *Nat. Phys.* **10**, 582 (2014).
[7] R. Raussendorf and H. J. Briegel, *Phys. Rev. Lett.* **86**, 5188 (2001).
[8] A. Broadbent, J. Fitzsimons, and E. Kashefi, in *Proceedings of the 2009 50th Annual IEEE Symposium on Foundations of Computer Science, FOCS* ’09 (IEEE Computer Society, Washington, DC, USA, 2009) pp. 517–526.
[9] S. Barz, E. Kashefi, A. Broadbent, J. F. Fitzsimons, A. Zeilinger, and P. Walther, *Science* **335**, 303 (2012).
[10] K. Chen and H.-K. Lo, *Quantum Info. Comput.* **7**, 689 (2007).
[11] M. Epping, H. Kampermann, C. macchiavello, and D. Bruß, *New J. Phys.* **19**, 093012 (2017).
[12] S. Ritter, C. Nölleke, C. Hahn, A. Reiserer, A. Neuzner, M. Uphoff, M. Müller, E. Figueroa, J. Bochmann, and G. Rempe, *Nature* **484**, 195 (2012).
[13] A. Sipahigil, R. E. Evans, D. D. Sukachev, M. J. Burek, J. Borregaard, M. K. Bhaskar, C. T. Nguyen, J. L. Pacheco, H. A. Atikian, C. Meuwly, R. M. Camacho, F. Jelezko, E. Bielejec, H. Park, M. Lončar, and M. D. Lukin, *Science* **354**, 847 (2016).
[14] B. Jing, X.-J. Wang, Y. Yu, P.-F. Sun, Y. Jiang, S.-J. Yang, W.-H. Jiang, X.-Y. Luo, J. Zhang, X. Jiang, X.-H. Bao, and J.-W. Pan, *Nat. Photon.* **13**, 210 (2019).
[15] A. Pirker, J. Wallhöfer, and W. Dür, *New J. Phys.* **20**, 053054 (2018).
[16] K. S. Chou, J. Z. Blumoff, C. S. Wang, P. C. Reinhold, C. J. Axline, Y. Y. Gao, L. Frunzio, M. H. Devoret, L. Jiang, and R. J. Schoelkopf, *Nature* **561**, 368 (2018).
[17] T. Monz, P. Schindler, J. T. Barreiro, M. Chwalla, D. Nigg, W. A. Coish, M. Harlander, W. Hänsel, M. Heinrich, and R. Blatt, *Phys. Rev. Lett.* **106**, 130506 (2011).
[18] L. Aolita, F. de Meo, and L. Davidovich, *Rep. Prog. Phys.* **78**, 042001 (2015).
[19] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
[20] W. McCutcheon, A. Pappa, B. A. Bell, A. McMillan, A. Chailloux, T. Lawson, M. Mafu, D. Markham, E. Diamanti, I. Kerenidis, J. G. Rarity, and M. S. Tame, *Nat. Commun.* **7**, 13251 (2016).
[21] O. Gühne, G. Tóth, P. Hyllus, and H. J. Briegel, *Phys. Rev. Lett.* **95**, 120405 (2005).
[22] N. Brunner, D. Cavalcanti, S. Pirionio, V. Scarani, and S. Wehner, *Rev. Mod. Phys.* **86**, 419 (2014).
[23] C. Branciard, D. Rosset, Y.-C. Liang, and N. Gisin, *Phys. Rev. Lett.* **110**, 060405 (2013).
[24] P. Xu, X. Yuan, L.-K. Chen, H. Lu, X.-C. Yao, X. Ma, Y.-A. Chen, and J.-W. Pan, *Phys. Rev. Lett.* **112**, 140506 (2014).
[25] N. Fenton and J. Bieman, *Software metrics: a rigorous and practical approach* (CRC press, 2014).
[26] J. Feng, H. Deschout, S. Caneva, S. Hofmann, I. Lončarić, P. Lazić, and A. Radenovic, *Nano letters* **18**, 1739 (2018).
[27] Q. Y. He and M. D. Reid, *Phys. Rev. Lett.* **111**, 250403 (2013).
[28] C.-M. Li, K. Chen, Y.-N. Chen, Q. Zhang, Y.-A. Chen, and J.-W. Pan, *Phys. Rev. Lett.* **115**, 010402 (2015).
[29] A. J. Bennett, D. A. Evans, D. J. Saunders, C. Branciard, E. G. Cavalcanti, H. M. Wiseman, and G. J. Pryde, *Phys. Rev. X* **2**, 031003 (2012).
[30] B. Jungeitsch, S. Niekamp, M. Kleinmann, O. Gühne, H. Lu, W.-B. Gao, Y.-A. Chen, Z.-B. Chen, and J.-W. Pan, *Phys. Rev. Lett.* **104**, 210401 (2010).
**SUPPLEMENTARY INFORMATION**

Experimental setup to generate \( N = n_c + n_q \) networks

In this section, we detail the experimental state preparation of quantum networks with \( N = n_c + n_q \) nodes.

1. \( n_c = 0 \)

As we mentioned in the main text, we test our procedure on multiphoton graph state in star graph, which is equivalent to N-photon Greenberger-Horne-Zeilinger (GHZ) state via local operation and classical communication (LOCC). The building block to generate a N-photon GHZ state is entangled photon pair from spontaneous parametric down conversion (SPDC). Experimentally, we use an ultraviolet pulse (with central wavelength of 390nm, pulse duration of 140fs) to pump a 2mm Beta Barium Borate (BBO) crystal (as shown in Fig. 5). The generated twin photons are entangled in polarization degree of freedom, and separated with opening angle of 6°. To compensate the time and spatial walk-off, an half-wave plate set at 45° and a 1mm BBO crystal are inserted in each path. With this setup, the maximally entangled photon pair can be obtained \((GHZ)_{2}\). The generated photons are highly frequency-correlated, which is not suitable for multiphoton experiments. We recombine the generated photons on a polarization beam splitter (PBS) to decrease frequency correlation (similar to the setup shown in Fig. 5b), but remove polarizer (POL), quarter-wave plate (QWP) and one HWP). By properly choosing narrow bandpass filters (full-width half-maximum (FWHM) of 3nm and 8nm in our experiment), we can obtain row bandpass filters (full-width half-maximum (FWHM) of 3nm and 8nm in our experiment), we can obtain (as shown Fig. 5a and Fig. 1b in the main text).

2. \( n_c = 1 \)

The quantum network with \( n_c = 1 \) under optimal “cheating strategy” (OCS) is realized by projecting one photon from \(|GHZ\rangle_N\) on \(|\xi^\prime\rangle = \cos \theta |H\rangle + \sin \theta e^{-i\phi} |V\rangle\). The choice of \(\theta\) and \(\phi\) is dependent on \(n_c\), which is equivalent to the network with \(n_q + n_c\) nodes.

3. \( n_c \geq 2 \)

To generate \(|GHZ\rangle_3\), the ultraviolet pulse goes to shine another 2mm BBO crystal. A heralded single-photon source is obtained by triggering one photon of the second entangled photon pair. The heralded single-photon is rotated to \(|+\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)\) and then superposed with one photon from the first \(|GHZ\rangle_2\) on a PBS. When two photons arrive at PBS simultaneously and come out from two output ports, \(|GHZ\rangle_2\) can be obtained. The setup is shown in Fig. 5c.

\(|GHZ\rangle_4\) is generated without triggering one photon from the second SPDC (as shown in Fig. 5d). Similarly, by successively shining a third BBO crystal and proper operations, \(|GHZ\rangle_5\) and \(|GHZ\rangle_6\) can be obtained (as shown Fig. 5a and Fig. 1b in the main text).

\(|GHZ\rangle_{2^n}\) is generated without triggering one photon from the second SPDC (as shown in Fig. 5d). Similarly, by successively shining a third BBO crystal and proper operations, \(|GHZ\rangle_{2^n}\) and \(|GHZ\rangle_{2^{n+1}}\) can be obtained (as shown Fig. 5a and Fig. 1b in the main text).

The experimental setups to simulate networks with \(n_q + n_c\) nodes are shown in Fig. 5. For network with \(n_q = 1\), its equivalent quantum state is a single-photon state. As shown in Fig. 5a, triggering one photon of \(|GHZ\rangle_2\) on \(|H\rangle\) leaves the other photon on state \(|H\rangle\). Then, by applying a QWP and HWP on it, arbitrary \(|\xi^\prime\rangle\) can be realized.

We do not directly demonstrate OCS on \(|GHZ\rangle_N\) when \(n_c \geq 2\), but simulate the networks with \(n_c \geq 2\) instead. Recall the general OCS for arbitrary \(n_c\): the \(n_c\) untrusted nodes prepare the entangled state \(|\xi\rangle_{n_q}\) for \(n_q\) trusted nodes based on their knowledge of the network fidelity function (1) in the main text. We prepare state \(|\xi\rangle_{n_q} = \cos \theta |H\rangle^{\otimes n_q} + \sin \theta e^{i\phi} |V\rangle^{\otimes n_q}\) with chosen \(\theta\) and \(\phi\).
FIG. 5. Experimental setups for realizing quantum networks with $n_q + n_c$ nodes. a-e, setup to implement networks of $n_q = 1$, $n_q = 2$, $n_q = 3$, $n_q = 4$ and $n_q = 5$, respectively.