MODULAR BOURGAIN-TZA FRIRI RESTRICTED INVERTIBILITY
CONJECTURES AND JOHNSON-LINDENSTRAUSS FLATTENING CONJECTURE

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Abstract: We recently formulated important Modular Bourgain-Tzafriri Restricted Invertibility Conjectures and Modular Johnson-Lindenstrauss Flattening Conjecture in the Appendix of [arXiv: 2207.12799.v1]. For the sake of wide accessibility we give a self-contained treatment of them.

Keywords: C*-algebra, Bourgain-Tzafriri Restricted Invertibility Theorem, Manin Matrix, Johnson-Lindenstrauss Flattening Lemma.

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Contents

1. Modular Bourgain-Tzafriri Restricted Invertibility Conjectures
2. Modular Johnson-Lindenstrauss Flattening Conjecture
References

1. Modular Bourgain-Tzafriri Restricted Invertibility Conjectures

Let $d \in \mathbb{N}$, $K = \mathbb{C}$ or $\mathbb{R}$ and $K^d$ be the standard $d$-dimensional Hilbert space with canonical orthonormal basis $\{e_j\}_{j=1}^d$. Following result obtained by Prof. Bourgain and Prof. Tzafriri in 1987 is one of the highly useful result of 20 century Mathematics which combines algebraic property of a matrix (invertibility) with the analytic property (norm) [12].

Theorem 1.1. [12, 13] (Bourgain-Tzafriri Restricted Invertibility Theorem) There are universal constants $A > 0$, $c > 0$ satisfying the following property. If $d \in \mathbb{N}$, and $T : K^d \to K^d$ is a linear operator with $\|Te_j\| = 1$, $\forall 1 \leq j \leq d$, then there exists a subset $\sigma \subseteq \{1, \ldots, d\}$ of cardinality

$$\text{Card}(\sigma) \geq \frac{cd}{\|T\|^2}$$

such that

$$\left\| \sum_{j \in \sigma} a_j Te_j \right\|^2 \geq A \sum_{j \in \sigma} |a_j|^2, \; \forall a_j \in K, \forall j \in \sigma.$$

In 2009 Tropp gave a polynomial time algorithm for the proof Theorem 1.1 [56]. It came as a surprise in 2009 (arXiv version) when Spielman and Srivastava gave a simple proof of Theorem 1.1 by proving a generalization of Theorem 1.1 [53] due to Vershynin [58]. In 2012 Casazza and Pfander proved infinite dimensional version of Theorem 1.1 [15]. For beautiful descriptions of Bourgain-Tzafriri restricted invertibility theorem and generalizations we refer [14, 16, 19, 28, 45, 47, 48, 51, 54, 55, 58, 60].
We formulate Conjectures 1.2 and 1.3 which are based on Theorem 1.1. Let $d \in \mathbb{N}$, $A$ be a unital C*-algebra and $A^d$ be the left module over $A$ w.r.t. natural operations. Modular $A$-inner product on $A^d$ is defined as

$$
((x_j)_{j=1}^d, (y_j)_{j=1}^d) := \sum_{j=1}^d x_j y_j^*, \quad \forall (x_j)_{j=1}^d, (y_j)_{j=1}^d \in A^d.
$$

Hence the norm on $A^d$ becomes

$$
\| (x_j)_{j=1}^d \| := \left\| \sum_{j=1}^d x_j x_j^* \right\|^{\frac{1}{2}}, \quad \forall (x_j)_{j=1}^d \in A^d.
$$

In this way $A^d$ becomes standard Hilbert C*-module $[37, 50, 52]$.

**Conjecture 1.2.** [(Commutative) Modular Bourgain-Tzafriri Restricted Invertibility Conjecture] Let $A$ be a unital commutative C*-algebra and $\mathcal{I}(A)$ be the set of all invertible elements of $A$. For $d \in \mathbb{N}$, let $\mathbb{M}_{d \times d}(A)$ be the set of all $d$ by $d$ matrices over $A$. For $M \in \mathbb{M}_{d \times d}(A)$, let $\det(M)$ be the determinant of $M$. Let $A^d$ be the standard (left) Hilbert C*-module over $A$ and $\{e_j\}_{j=1}^d$ be the canonical orthonormal basis for $A^d$. There are universal real constants $A > 0$, $c > 0$ ($A$ and $c$ may depend upon C*-algebra $A$) satisfying the following property. If $d \in \mathbb{N}$ and $M \in \mathbb{M}_{d \times d}(A)$ with $\langle Me_j, Me_j \rangle = 1$, $\forall 1 \leq j \leq d$ and $\det(M) \in \mathcal{I}(A) \cup \{0\}$, then there exists a subset $\sigma \subseteq \{1, \ldots, d\}$ of cardinality

$$
\text{Card}(\sigma) \geq \frac{cd}{\|M\|^2}
$$

such that

$$
\sum_{j \in \sigma} \sum_{k \in \sigma} a_j \langle Me_j, Me_k \rangle a_k^* = \left\langle \sum_{j \in \sigma} a_j Me_j, \sum_{k \in \sigma} a_k Me_k \right\rangle \geq A \sum_{j \in \sigma} a_j a_j^*, \quad \forall a_j \in A, \forall j \in \sigma,
$$

where $\|M\|$ is the norm of the Hilbert C*-module homomorphism defined by $M$ as $M : A^d \ni x \mapsto Mx \in A^d$.

We next formulate Conjecture 1.2 for unital C*-algebras which need not be commutative. In the statement we use the notion of Manin matrices. We refer [17, 18] for the basics of Manin matrices.

**Conjecture 1.3.** [(Noncommutative) Modular Bourgain-Tzafriri Restricted Invertibility Conjecture] Let $A$ be a unital C*-algebra and $\mathcal{I}(A)$ be the set of all invertible elements of $A$. For $d \in \mathbb{N}$, let $\mathbb{M}_d(M_d(A))$ be the set of all $d$ by $d$ Manin matrices over $A$. For $M \in \mathbb{M}_d(M_d(A))$, let $\det\_\text{column}(M)$ be the Manin determinant of $M$ by column expansion. Let $A^d$ be the standard (left) Hilbert C*-module over $A$ and $\{e_j\}_{j=1}^d$ be the canonical orthonormal basis for $A^d$. There are universal real constants $A > 0$, $c > 0$ ($A$ and $c$ may depend upon C*-algebra $A$) satisfying the following property. If $d \in \mathbb{N}$ and $M \in \mathbb{M}_d(M_d(A))$ with $\langle Me_j, Me_j \rangle = 1$, $\forall 1 \leq j \leq d$ and $\det\_\text{column}(M) \in \mathcal{I}(A) \cup \{0\}$, then there exists a subset $\sigma \subseteq \{1, \ldots, d\}$ of cardinality

$$
\text{Card}(\sigma) \geq \frac{cd}{\|M\|^2}
$$

such that

$$
\sum_{j \in \sigma} \sum_{k \in \sigma} a_j \langle Me_j, Me_k \rangle a_k^* = \left\langle \sum_{j \in \sigma} a_j Me_j, \sum_{k \in \sigma} a_k Me_k \right\rangle \geq A \sum_{j \in \sigma} a_j a_j^*, \quad \forall a_j \in A, \forall j \in \sigma,
$$

K. MAHESH KRISHNA
where $\|M\|$ is the norm of the Hilbert $C^*$-module homomorphism defined by $M$ as $M : \mathcal{A}^d \ni x \mapsto Mx \in \mathcal{A}^d$.

Remark 1.4. (i) We can surely formulate Conjecture 1.2 by removing the condition $\det(M) \in \mathcal{I}(A) \cup \{0\}$ and Conjecture 1.3 by removing the condition Manin matrices and $\det^{\text{column}}(M) \in \mathcal{I}(A) \cup \{0\}$. But we strongly believe that Conjectures 1.2 and 1.3 fail with this much of generality.

(ii) If Conjecture 1.2 holds but Conjecture 1.3 fails, then we can try Conjecture 1.3 for $W^*$-algebras or $C^*$-algebras with invariant basis number (IBN) property. We refer [27] for the IBN properties of $C^*$-algebras.

2. Modular Johnson-Lindenstrauss Flattening Conjecture

Everything starts from the following surprising result of Prof. Johnson and Prof. Lindenstrauss from 1984 [34].

Theorem 2.1. [34, 46] (Johnson-Lindenstrauss Flattening Lemma - original form) Let $M, N \in \mathbb{N}$ and $x_1, x_2, \ldots, x_M \in \mathbb{R}^N$. For each $0 < \varepsilon < 1$, there exists a Lipschitz map $f : \mathbb{R}^N \to \mathbb{R}^m$ and a real $r > 0$ such that

$$r(1 - \varepsilon)\|x_j - x_k\| \leq \|f(x_j) - f(x_k)\| \leq r(1 + \varepsilon)\|x_j - x_k\|, \quad \forall 1 \leq j, k \leq M,$$

where

$$m = O\left(\frac{\log M}{\varepsilon^2}\right).$$

Over the time, several improvements of Theorem 2.1 were obtained. We recall them.

Theorem 2.2. [25, 34] (Johnson-Lindenstrauss Flattening Lemma - Frankl-Maehara form) Let $0 < \varepsilon < 1$ and $M \in \mathbb{N}$. Define

$$m(\varepsilon, M) := \left\lceil \frac{9}{\varepsilon^2 - \frac{2}{3}} \log M \right\rceil + 1.$$

If $M > m(\varepsilon, M)$, then for any $x_1, x_2, \ldots, x_M \in \mathbb{R}^M$, there exists a map $f : \{x_j\}_{j=1}^M \to \mathbb{R}^m$ such that

$$(1 - \varepsilon)\|x_j - x_k\|^2 \leq \|f(x_j) - f(x_k)\|^2 \leq (1 + \varepsilon)\|x_j - x_k\|^2, \quad \forall 1 \leq j, k \leq M.$$

Theorem 2.3. [22, 34] (Johnson-Lindenstrauss Flattening Lemma - Dasgupta-Gupta form) Let $M, N \in \mathbb{N}$ and $x_1, x_2, \ldots, x_M \in \mathbb{R}^N$. Let $0 < \varepsilon < 1$. Choose any natural number $m$ such that

$$m > 4 - \frac{1}{\varepsilon^2 - \frac{4}{\varepsilon}} \log M.$$

Then there exists a map $f : \mathbb{R}^N \to \mathbb{R}^m$ such that

$$(1 - \varepsilon)\|x_j - x_k\|^2 \leq \|f(x_j) - f(x_k)\|^2 \leq (1 + \varepsilon)\|x_j - x_k\|^2, \quad \forall 1 \leq j, k \leq M.$$

The map $f$ can be found in randomized polynomial time.

Theorem 2.4. [24, 34] (Johnson-Lindenstrauss Flattening Lemma - matrix form) There is a universal constant $C > 0$ satisfying the following. Let $0 < \varepsilon < 1$, $M, N \in \mathbb{N}$ and $x_1, x_2, \ldots, x_M \in \mathbb{R}^N$. Let $0 < \varepsilon < 1$. Choose any natural number $m$ such that

$$m > 4 - \frac{1}{\varepsilon^2 - \frac{4}{\varepsilon}} \log M.$$
For each natural number

\[ m > C_{\varepsilon} \log M, \]

there exists a matrix \( M \in \mathbb{M}_{m \times N}(\mathbb{R}) \) such that

\[(1 - \varepsilon)\|x_j - x_k\|^2 \leq \|M(x_j - x_k)\|^2 \leq (1 + \varepsilon)\|x_j - x_k\|^2, \quad \forall 1 \leq j, k \leq M.\]

In 2016 Larsen and Nelson [41, 42] showed that bound in Johnson-Lindenstrauss flattening lemma is optimal. For a comprehensive look on Johnson-Lindenstrauss flattening lemma from the historical point of view and its various applications we refer important papers and books [1–11, 20, 21, 23, 26, 29–33, 35, 36, 38, 39, 43, 44, 49, 57].

We now formulate the following interesting problem.

**Problem 2.5.** Let \( A \) be the set of all unital C*-algebras. What is the best function \( \phi : A \times (0, 1) \times \mathbb{N} \to (0, \infty) \) satisfying the following. Let \( A \) be a unital C*-algebra. There is a universal constant \( C > 0 \) (which may depend upon \( A \)) satisfying the following. Let \( 0 < \varepsilon < 1, M, N \in \mathbb{N} \) and \( x_1, x_2, \ldots, x_M \in A^N \). For each natural number

\[ m > C\phi(A, \varepsilon, M) \]

there exists a matrix \( M \in \mathbb{M}_{m \times N}(A) \) such that

\[(1 - \varepsilon)\langle x_j - x_k, x_j - x_k \rangle \leq \langle M(x_j - x_k), M(x_j - x_k) \rangle \leq (1 + \varepsilon)\langle x_j - x_k, x_j - x_k \rangle, \quad \forall 1 \leq j, k \leq M.\]

A particular case of Problem 2.5 is the following conjecture.

**Conjecture 2.6.** (Modular Johnson-Lindenstrauss Flattening Conjecture) Let \( A \) be a unital C*-algebra. There is a universal constant \( C > 0 \) (which may depend upon \( A \)) satisfying the following. Let \( 0 < \varepsilon < 1, M, N \in \mathbb{N} \) and \( x_1, x_2, \ldots, x_M \in A^N \). For each natural number

\[ m > C_{\varepsilon} \log M, \]

there exists a matrix \( M \in \mathbb{M}_{m \times N}(A) \) such that

\[(1 - \varepsilon)\langle x_j - x_k, x_j - x_k \rangle \leq \langle M(x_j - x_k), M(x_j - x_k) \rangle \leq (1 + \varepsilon)\langle x_j - x_k, x_j - x_k \rangle, \quad \forall 1 \leq j, k \leq M.\]

**Remark 2.7.**

(i) We believe that Conjecture 2.6 holds at least for W*-algebras (von Neumann algebras) or C*-algebras with IBN property.

(ii) Modular Welch bounds are derived in [40] and various problems including modular Zauner’s conjecture are stated there.

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