A new method using the Forward Backward technique with Contra Harmonic mean formula

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Abstract. We introduce a new method for solving Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs), by making a mixing between the Forward (Predictor) – Backward (Corrector) technique and used it in the Contra Harmonic mean formula, this new method give us a parallelism in numerical calculations and it is more accurate than the old classical Runge – Kutta formula of the same order.

1. Introduction

As it known, the famous Runge – Kutta methods are regard one of the best methods that solving (ODEs), and searching for better new formulas are continuous over time.

Here, we present new method for solving (IVPs) combining between previously techniques and obtain a new formula suitable for parallel computations. In 1967 Miranker [1], present the concern of computation front “the imaginary straight line that separate the values are next to be computed (by some numerical algorithms) from all previously computing value problems", which is not give full advantage of multi processers computers, because it too narrow. By predictor – corrector (PC) method we made an expansion to the computation front.

In 1993 Evans [2] presented the Contra Harmonic mean formula. In 2020 Jasim, M.D [3,4] introduced new methods by using more than one mean with the (PC) method. Our new method namely (PPCCoM1).

2. The Contra Harmonic mean formula.

The Contra Harmonic mean formula has the form,

$$w_{n+1} - w_n = \frac{h}{n} \left( \sum_{i=1}^{n} \frac{t_i^2 + t_{i+1}^2}{t_i + t_{i+1}} \right)$$

(1)

Where,

$$t_i = \Phi(u_n + \beta_i h_i, w_n + h \sum_{j=1}^{i-1} \sigma_i t_j) \quad i = 1, 2, 3, ..., n$$

(2)
is the independent variable, \( w_n \) is the dependent variable since \( t = \Phi(u,w) \), \( h \) is the step size, \( \beta_i = \sum_{j=1}^{i-1} \sigma j \)

we regarded \( \Phi \) a function only of \( w \).

Thus, (2) becomes,

\[
t_i = \Phi\left( w_n + h \sum_{j=1}^{i-1} a_{ij} t_j \right) \quad i = 1, 2, 3, ..., n \quad [5].
\]

3. New method namely PPCCoM1:

The new method PPCCoM1 calculates \( w_n^{p+1} \) depending on \( w_n^{c} \) and the computation of \( w_n^{c} \) depending on \( w_n^{c-1} \) and \( w_n^{p} \), which has form,

\[
w_{n+1}^{p} - w_n^{c -1} = 2h \left( \frac{t_1^2 + t_2^2}{t_1 + t_2} \right)
\]

(4)

\[
t_1 = \Phi(u_{n-1} , w_n^{c -1} ) , \quad t_2 = \Phi(u_{n-1} + 2h , w_n^{c -1} + 2ht_1 )
\]

(5)

And,

\[
w_n^c - w_{n-1}^c = h \left( \frac{v_1^2 + v_2^2}{v_1 + v_2} \right)
\]

(6)

\[
v_1 = \Phi(u_{n-1} , w_n^{p} ) , \quad v_2 = \Phi(u_{n-1} + h , w_n^{p} + hv_1 )
\]

(7)

Illustrated the computation process of PPCCoM1 mode in figure 1 below,

![computation front](image)

Figure 1: information of PPCCoM1- mode.

4. Derivation of PPCCoM1- method.

Predictor Contra Harmonic mean method of two stage second order has form,

\[
w_{n+1}^{p} - w_n^{p -1} = 2h \left( \frac{at_1^2 + bt_2^2}{ct_1 + dt_2} \right)
\]

(8)

\[
t_1 = \Phi(w_{n-1}^{c} ) , \quad t_2 = \Phi(w_{n-1}^{c -1} + ah_t_1 )
\]

(9)

To derive our Forward (predictor) PPCCoM1, expansion of \( t_1 \) and \( t_2 \) gives

\[
t_1 = \Phi(w_{n-1}^{c} ) = \Phi
\]
\[ t_2 = \Phi (w_{n-1}^c + aht_1) = \Phi + aht_1\Phi w_{n-1}^c + o(h^2) \]

\[ t_2 = \Phi + ah\Phi \Phi w_{n-1}^c + o(h^2) \]

Where, \( o(h) \) is the local truncation error.

Now,

\[ at_1^2 = a\Phi^2 \]

\[ bt_2^2 = b(\Phi + ah\Phi \Phi w_{n-1}^c + o(h^2))^2 \]

\[ at_1^2 + bt_2^2 = (a + b)\Phi^2 + bah\Phi^2 \Phi w_{n-1}^c + o(h^2) \]

\[ ct_1 + dt_2 = c\Phi + d(\Phi + ah\Phi \Phi w_{n-1}^c) + o(h^2) \]

\[ \frac{at_1^2 + bt_2^2}{ct_1 + dt_2} = \frac{(a+b)\Phi^2 + 2bah\Phi^2 \Phi w_{n-1}^c + o(h^2)}{(c+d)\Phi + dah\Phi \Phi w_{n-1}^c + o(h^2)} \]

Equation (10) has two different chain in numerator (of order 3) and denominator (of order 2), this means we cannot allow direct chain substitution. So eq.(11) may be written as,

\[ \frac{w_{n+1}^p - w_{n-1}^p}{(c+d)\Phi + dah\Phi \Phi w_{n-1}^c + o(h^2)} \]

\[ w_{n+1}^p = W_{n-1}^p + \frac{UPPER}{LOWER} \]

Where,

\[ UPPER = (a + b)h\Phi^2 + 2bah\Phi^2 \Phi w_{n-1}^c + o(h^3) \]

\[ LOWER = (c + d)\Phi + dah\Phi \Phi w_{n-1}^c + o(h^2) \]

Taylor series of \( w_{n+1}^p \) given as,

\[ TAYLOR = 2h\Phi + 2h^2\Phi \Phi w_{n-1}^c + o(h^3) \]

Written the difference between equation (12) and (13) as,

\[ \frac{upper}{LOWER} = TAYLOR - \text{error} \]

\[ \text{lower} \ast \text{error} = \text{lower} \ast TAYLOR - \text{upper} \]

Compare coefficients of the same terms in (14) until the term in \( o(h^3) \) we get 2 equations.

\[ 2(c + d) = (a + b) \quad , \quad a(b - d) = (c + d) = 0 \]
Which are 2 equations with 5 parameters, that means 3 freedom degree, by choosing \( a = \alpha = b = 2 \), then \( c = d = 1 \).

We get,

\[
 w_{n+1}^p - w_{n-1}^c = 2 \hbar \left( \frac{t_1^2 + t_2^2}{t_1 + t_2} \right) 
\]

(13)

\[
 t_1 = \Phi(u_{n-1}, w_{n-1}^c) , \quad t_2 = \Phi(u_{n-1} + 2\hbar, w_{n-1}^c + 2\hbar t_1) 
\]

(14)

To get the corrector part, from the backward formula, "replace \( h \) by \(-h\)" (see [6]).

\[
 w_{n-1}^c - w_n^c = -h \left( \frac{av_1 + bv_2}{cv_1 + dv_2} \right) 
\]

(15)

\[
 v_1 = \Phi\left( w_n^p \right) \quad , \quad v_2 = \Phi\left( w_n^p - ahv_1 \right) 
\]

(16)

Expansion of \( v_1 \) and \( v_2 \) in (16) gives,

\[
 v_1 = \Phi\left( w_n^p \right) = \Phi \\
 v_2 = \Phi\left( w_n^p - ahv_1 \right) = \Phi - ah\Phi \Phi_{w_n} + o(h^2) \\
 av_1^2 = a\Phi^2 \\
 bv_2^2 = b\Phi^2 - 2ahb\Phi^2\Phi_{w_n} + o(h^2) \\
 cv_1 + dv_2 = (c + d)\Phi - dah\Phi\Phi_{w_n} + o(h^2) \\
 \frac{av_1^2 + bv_2^2}{cv_1 + dv_2} = \frac{(a+b)\Phi^2 - 2ahb\Phi^2\Phi_{w_n} + o(h^2)}{(c+d)\Phi - dah\Phi\Phi_{w_n} + o(h^2)} 
\]

(17)

Substitute (17) in (15) we get,

\[
 w_{n-1}^c - w_n^c = \frac{-\left( a+b \right)\Phi^2 h + 2abh^2\Phi^2\Phi_{w_n} + o(h^3)}{(c+d)\Phi - dah\Phi\Phi_{w_n} + o(h^2)} 
\]

(18)

We have the same problem in equation (11), so equation (18) written as,

\[
 w_{n-1}^c = w_n^c + \frac{\text{UPPER}}{\text{LOWER}} 
\]

(19)

Where,

\[
 \text{UPPER} = -\left( a + b \right)\Phi^2 h + 2abh^2\Phi^2\Phi_{w_n} + o(h^3) \\
 \text{LOWER} = (c + d)\Phi - dah\Phi\Phi_{w_n} + o(h^2) 
\]
Taylor series of \( w_{n-1} \) [5] given as,

\[
TAYLOR = -h\phi + \frac{h^2}{2} \phi \phi w_{n-1} + o(h^3)
\]  

(20)

Written the difference between equation (19) and (20) as,

\[
\frac{upper}{lower} = Taylor - \text{error}
\]

(21)

Compare coefficients of the same terms in (21) until the term in \( o(h^3) \) we get 2 equations

\[
(c + d) = (a + b) , 2a(d - 2b) + (c + d) = 0
\]

Which is two equations with five parameters, so we get three freedom degree.

Choosing \( a = a = b = 1 , c = 1 \), then we have \( d = c = 1 \).

Our backward formula has form,

\[
w_n^c - w_{n-1}^c = h \left( \frac{v_1^2 + v_2^2}{v_1 + v_2} \right)
\]

(22)

\[
v_1 = \phi\left(u_n, w_n^p\right) , \ v_2 = \phi\left(u_n - h, w_n^p - hw_1\right)
\]

(23)

Eq.'s (13) , (14) , (22) and (23) represent the new PPCCoM1.

Where, \( w_{n+1}^p \) represent the forward form, \( w_n^c \) is the backward form.

5. The stability region of PPCCoM1

AS it know, the Runge – Kutta methods are stable, and it have a good quite of stability, when we choose a small step size.

To test the stability of PPCCoM1, we use the test equation \( \dot{w} = \lambda w \), \( \lambda = \partial \phi / \partial w \) is constant [7,8].

Examine the stability of the forward part of PPCCoM1,

\[
w_{n+1}^p - w_{n-1}^c = 2h \left( \frac{t_1^2 + t_2^2}{t_1 + t_2} \right)
\]

\[
t_1 = \phi\left(u_{n-1}, w_{n-1}^c\right) , \ t_2 = \phi\left(u_{n-1} + 2h, w_{n-1}^c + 2ht_1\right)
\]

To find the interval of absolute stability, we used the test equation \( \dot{w} = \lambda w \) [9,10],

\[
t_1 = \phi\left(u_{n-1}, w_{n-1}^c\right) = \lambda w_{n-1}^c
\]

(24)

\[
t_2 = \phi\left(u_{n-1} + 2h, w_{n-1}^c + 2ht_1\right) = \lambda\left( w_{n-1}^c + 2h\lambda w_{n-1}^c\right)
\]

(25)

Substitute equations (24) and (25) in (13) we get,
\[ w_{n+1}^P = w_{n-1}^c + 2h \left( \frac{\lambda (w_{n-1}^c)(1+2h\lambda+2h^2\lambda^2)}{1+h\lambda} \right) \]  

(26)

Dividing (26) by \( w_{n-1}^c \) and putting \( Z = h\lambda \) we get,

\[ \tau_1 = \frac{w_{n+1}^P}{w_{n-1}^c} = 1 + \frac{2\pi (1+Z+2Z^2)}{1+Z} \]  

(27)

The last equation (27) satisfies the absolute stability condition if \( |\tau_1| < 1 \) where \( Z \) identify the condition when \( Z \in (-0.6478, 0) \) which represents the stability region of this method.

6. Numerical Example:

**Table 1:** PPCCoM1 results applied on the example \( \dot{w}=-uw^2, w(0)=2 \) \( h=0.05 \)

| Value of \( u \) | Exact solution \( w \) | The forward solution \( w^P \) | The backward solution \( w^c \) | \(|w - w^P|\) | \(|w - w^c|\) |
|---|---|---|---|---|---|
| 0.00 | 2 | 2 | 0 | 0 |
| 0.05 | 1.995012468 | 1.96000000 | 1.99000000 | 0.0350124688 | 0.0050124688 |
| 0.10 | 1.980198019 | 1.94192571 | 1.973971218 | 0.0386723033 | 0.00622680104 |
| 0.15 | 1.959090220 | 1.91179795 | 1.949309898 | 0.0446104653 | 0.0066803220 |
| 0.20 | 1.923076923 | 1.87244089 | 1.916312390 | 0.0506360230 | 0.0076453233 |
| 0.25 | 1.882352941 | 1.82634089 | 1.875729515 | 0.0560120445 | 0.0066298981 |
| 0.30 | 1.834862385 | 1.77438473 | 1.828511800 | 0.0604776543 | 0.0063508497 |
| 0.35 | 1.781737193 | 1.71779031 | 1.77546732 | 0.0639468790 | 0.0059904608 |
| 0.40 | 1.724137931 | 1.65771721 | 1.718552466 | 0.0664207151 | 0.0055854628 |
| 0.45 | 1.663201663 | 1.59525053 | 1.658038731 | 0.0679511320 | 0.0051629138 |
| 0.50 | 1.60000000 | 1.53137880 | 1.59525813 | 0.0686211951 | 0.0047414861 |
| 0.55 | 1.535508637 | 1.46697685 | 1.531175472 | 0.0685317831 | 0.0043316493 |
| 0.60 | 1.470588235 | 1.40279631 | 1.466643046 | 0.0677919200 | 0.0039451889 |
| 0.65 | 1.405975395 | 1.33946378 | 1.402393971 | 0.0661161426 | 0.0035814233 |
| 0.70 | 1.342281879 | 1.27748520 | 1.339038362 | 0.0647966778 | 0.0032435167 |
| 0.75 | 1.280000000 | 1.21725481 | 1.277068274 | 0.0627451809 | 0.0029317252 |
| 0.80 | 1.219512195 | 1.15906693 | 1.216866712 | 0.0604452584 | 0.0026458295 |
| 0.85 | 1.16103047 | 1.10312904 | 1.158719287 | 0.0579740032 | 0.0023876005 |
| 0.90 | 1.104972375 | 1.04957157 | 1.102827094 | 0.0539719954 | 0.0021452811 |
| 0.95 | 1.051248357 | 0.99847868 | 1.049319707 | 0.0527696707 | 0.0019286497 |
| 1.00 | 1 | 0.94986395 | 0.998267581 | 0.0501360411 | 0.0017324184 |

7. Discussion of the results:

As we see in table 1, Comparison of the exact solution (in column 2) with our PPCCoM1 (in column 4) shows that, the PPCCoM1 is stable in the interval of integration and has a good accuracy in results, comparing with the exact solution. The conclusion is clearly observed in the last column 6 (the absolute difference between the exact solution and the backward solution of PPCCoM1) in table 1.
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8. References

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