Some properties of $h$-$MN$-convexity and Jensen’s type inequalities

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Abstract

In this work, we introduce the class of $h$-$MN$-convex functions by generalizing the concept of MN-convexity and combining it with $h$-convexity. Namely, Let $I$, $J$ be two intervals subset of $(0, \infty)$ such that $(0, 1) \subseteq J$ and $[a, b] \subseteq I$. Consider a non-negative function $h : (0, \infty) \rightarrow (0, \infty)$ and let $M : [0, 1] \rightarrow [a, b]$ be a Mean function given by $M(t) = M(h(t); a, b)$; where by $M(h(t); a, b)$ we mean one of the following functions: $A_{\pm}(a, b) := \frac{h(1-t)}{h(0)}a + \frac{h(t)}{h(1)}b$, $G_{\pm}(a, b) = \frac{h(1-t)}{h(0)}a + \frac{h(t)}{h(1)}b$, and $H_{\pm}(a, b) := \frac{1}{h(0) + h(1-t)}b$; with the property that $M(h(0); a, b)$ and $M(h(1); a, b) = b$.

A function $f : I \rightarrow (0, \infty)$ is said to be $h$-$MN$-convex (concave) if the inequality

$$f(M(t; x, y)) \leq (\geq) N(h(t); f(x), f(y)),$$

holds for all $x, y \in I$ and $t \in [0, 1]$, where $M$ and $N$ are two mean functions. In this way, nine classes of $h$-$MN$-convex functions are established and some of their analytic properties are explored and investigated. Characterizations of each type are given. Various Jensen’s type inequalities and their converses are proved.

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1. Introduction

Throughout this work, $I$ and $J$ are two intervals subset of $(0, \infty)$ such that $(0, 1) \subseteq J$ and $[a, b] \subseteq I$ with $0 < a < b$. A function $f : I \rightarrow \mathbb{R}$ is called convex iff

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for all points $\alpha, \beta \in I$ and all $t \in [0, 1]$. If $-f$ is convex then we say that $f$ is concave. Moreover, if $f$ is both convex and concave, then $f$ is said to be affine.

In 1978, Breckner [7] introduced the class of $s$-convex functions (in the second sense), as follows:

**Definition 1**: Let $I \subseteq [0, \infty)$ and $s \in (0, 1]$, a function $f : I \to [0, \infty)$ is $s$-convex function or that $f$ belongs to the class $K_s(I)$ if for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y).$$

In [8], Breckner proved that every $s$-convex function satisfies the Hölder condition of order $s$. Another proof of this fact was given in [28]. For more properties regarding $s$-convexity see [9] and [16].

In 1985, E. K. Godnova and V. I. Levin (see [?] or [21], pp. 410-433) introduced the following class of functions:

**Definition 2**: We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that $f$ belongs to the class $Q(I)$ if for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y).$$

In the same work, the authors proved that all nonnegative monotonic and nonnegative convex functions belong to this class. For related works see [14] and [20].

In 1999, Pearce and Rubinov [26], established a new type of convex functions which is called $P$-functions.

**Definition 3**: We say that $f : I \to \mathbb{R}$ is $P$-function or that $f$ belongs to the class $P(I)$ if for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Indeed, $Q(I) \supseteq P(I)$ and for applications it is important to note that $P(I)$ also consists only of nonnegative monotonic, convex and quasi-convex functions. A related work was considered in [14] and [31].

In 2007, Varošanec [32] introduced the class of $h$-convex functions which generalize convex, $s$-convex, Godunova-Levin functions and $P$-functions. Namely, the $h$-convex function is defined as a non-negative function $f : I \to \mathbb{R}$ which satisfies

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$
where \( h \) is a non-negative function, \( t \in (0, 1) \subseteq \mathcal{I} \) and \( x, y \in \mathcal{I} \), where \( \mathcal{I} \) and \( \mathcal{J} \) are real intervals such that \((0, 1) \subseteq \mathcal{J}\). Accordingly, some properties of \( h \)-convex functions were discussed in the same work of Varošanec. For more results; generalization, counterparts and inequalities regarding \( h \)-convexity see [1],[5],[6], [10]—[12],[15],[17],[18], [24], [30] and [35].

While he studying \( h \)-convex functions, Alomari [2] proposed a rational geometric and analytic meaning of \( h \)-convexity by introducing the concept of \( h \)-cord as follows:

**Definition 4 : ([2])** The \( h \)-cord joining any two points \((x, f(x))\) and \((y, f(y))\) on the graph of \( f \) is defined to be

\[
L(t;h) := (f(y) - f(x))h\left(\frac{t-x}{y-x}\right) + f(x),
\]

for all \( t \in [x, y] \subseteq \mathcal{I} \) \((x < y)\). In particular, if \( h(t) = t \) then we obtain the well known form of chord, which is

\[
L(t;t) := \frac{f(y) - f(x)}{y-x}(t-x) + f(x).
\]

It’s worth to mention that, if \( h(0) = 0 \) and \( h(1) = 1 \), then \( L(x; h) = f(x) \) and \( L(y; h) = f(y) \), so that the \( h \)-cord \( L \) agrees with \( f \) at endpoints \( x, y \), and this true for all such \( x, y \in \mathcal{I} \).

The \( h \)-convexity of a function \( f : I \rightarrow \mathbb{R} \) means geometrically that the points of the graph of \( f \) are on or below the \( h \)-chord joining the endpoints \((x, f(x))\) and \((y, f(y))\) for all \( x, y \in \mathcal{I}, \ x < y \). In symbols, we write

\[
f(t) \leq \left[ f(y) - f(x) \right]h\left(\frac{t-x}{y-x}\right) + f(x) = L(t;h),
\]

for any \( x \leq t \leq y \) and \( x, y \in \mathcal{I} \). Given any three non-collinear points \( P, Q \) and \( R \) on the graph of \( f \) with \( Q \) between \( P \) and \( R \) (say \( P < Q < R \)). Let \( h \) is super(sub)multiplicative and \( h(a) \geq (\leq) a \), for \( a \in (0, 1) \subseteq \mathcal{J} \). A function \( f \) is \( h \)-convex (concave) if \( Q \) is on or below (above) the \( h \)-chord \( PR \) (see Figure 1).

**Caution:** In special case, for \( h(t) = t^k, \ t \in (0, 1) \) the proposed geometric interpretation is valid for \( k \in (-1,0) \cup (0,\infty) \). In the case that \( k \leq -1 \) or \( k = 0 \) the geometric meaning is inconclusive so we exclude this case (and (and similar cases) from our proposal above.
Definition 5: Let $h : J \to (0, \infty)$ be a non-negative function. Let $f : I \to \mathbb{R}$ be any function. We say $f$ is $h$-midconvex ($h$-midconcave) if

$$
\frac{f(x) + f(y)}{2} \leq (\geq) h\left(\frac{1}{2}\right)[f(x) + f(y)]
$$

for all $x, y \in I$.

In particular, $f$ is locally $h$-midconvex if and only if

$$
h\left(\frac{1}{2}\right)[f(x + p) + f(x - p)] - f(x) \geq 0,
$$

for all $x \in (x - p, x + p)$, $p > 0$.

Generalization of the well known Jensen convexity, could be stated as follows:

Theorem 1: Let $h : J \to (0, \infty)$ be a non-negative function such that $h(a) \geq a$, for all $a \in (0, 1)$. Let $f : I \to \mathbb{R}_+$ be a nonnegative continuous function. $f$ is $h$-convex if and only if it is $h$-midconvex; i.e., the inequality

$$
f\left(\frac{x + y}{2}\right) \leq h\left(\frac{1}{2}\right)[f(x) + f(y)],
$$

holds for all $x, y \in I$.

It’s well known that every convex function is Lipschitz continuous. Moreover, Breckner [8] (see also [3], [9] and [28]) proved that every
s-convex functions is Hölder continuous of order \( s \in (0, 1] \). Recently, Alomari [2] used the concept of control functions in numberical analysis to extend these facts in terms of \( h \)-convex functions. Recall that a function \( h : I \subseteq [0, \infty) \rightarrow [0, \infty] \) is called a control function if

1. \( h \) is nondecreasing,
2. \( \inf_{\delta>0} h(\delta) = 0. \)

A function \( f : I \rightarrow \mathbb{R} \) is \( h \)-continuous at \( x_0 \) if

\[
| f(x) - f(x_0) | \leq h(| x - x_0 |),
\]

for all \( x \in I \). Furthermore, a function is continuous in \( x_0 \) if it is \( h \)-continuous for some control function \( h \).

This approach leads us to refining the notion of continuity by restricting the set of admissible control functions.

For a given set of control functions \( C \) a function is \( C \)-continuous if it is \( h \)-continuous for all \( h \in C \). For example the Hölder continuous functions of order \( \alpha \in (0, 1] \) are defined by the set of control functions

\[
C_{\alpha-\text{Hölder}} = \{ h \mid h(\delta) = H | \delta |^\alpha, H > 0 \}
\]

In case \( \alpha = 1 \), the set \( C_{1-\text{Hölder}} \) contains all functions satisfying the Lipschitz condition.

In [2], Alomari proved the following theorem.

**Theorem 2** : Let \((0, 1) \subseteq J, h : J \rightarrow (0, \infty) \) be a control function which is supermultiplicative such that \( h(\alpha) \geq \alpha \) for each \( \alpha \in (0, 1) \). Let \( I \) be a real interval, \( a, b \in \mathbb{R} (a < b) \) with \( a, b \) in \( I^\circ \) (the interior of \( I \)). If \( f : I \rightarrow \mathbb{R} \) is non-negative \( h \)-convex function on \([a, b]\), then \( f \) is \( h \)-continuous on \([a, b]\).

We recall that, a function \( M : (0, \infty) \rightarrow (0, \infty) \) is called a Mean function if

1. **Symmetry** : \( M(x, y) = M(y, x) \).
2. **Reflexivity** : \( M(x, x) = x \).
3. **Monotonicity** : \( \min\{x, y\} \leq M(x, y) \leq \max\{x, y\} \).
4. **Homogeneity** : \( M(\lambda x, \lambda y) = \lambda M(x, y) \), for any positive scalar \( \lambda \).

The most famous and old known mathematical means are listed as follows:

1. The arithmetic mean :
\[ A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \alpha, \beta \in \mathbb{R}_+. \]

(2) The geometric mean:
\[ G := G(\alpha, \beta) = \sqrt[\alpha\beta], \alpha, \beta \in \mathbb{R}_+. \]

(3) The harmonic mean:
\[ H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}, \alpha, \beta \in \mathbb{R}_+ - \{0\}. \]

In particular, we have the famous inequality \( H \leq G \leq A. \)

In 2007, Anderson et al. in [4] developed a systematic study to the classical theory of continuous and midconvex functions, by replacing a given mean instead of the arithmetic mean.

**Definition 6**: Let \( f : I \to [0, \infty) \) be a continuous function where \( I \subseteq (0, \infty) \). Let \( M \) and \( N \) be any two Mean functions. We say \( f \) is \( MN \)-convex (concave) if
\[ f(M(x, y)) \leq (\geq)N(f(x), f(y)), \quad (1.4) \]
for all \( x, y \in I \) and \( t \in [0, 1] \).

In fact, the authors in [4] discussed the midconvexity of positive continuous real functions according to some Means. Hence, the usual midconvexity is a special case when both mean values are arithmetic means. Also, they studied the dependence of MN-convexity on \( M \) and \( N \) and give sufficient conditions for MN-convexity of functions defined by Maclaurin series. For other works regarding MN-convexity see [22] and [23].

The aim of this work, is to study the main properties of \( h \)-MN-convex functions, such as; addition, product, compositions and some functional type inequalities for some classes. Jensen inequality and its consequences with their converses play significant roles in (almost) all areas of Mathematics and Physics. For example, Jensen inequality used to prove some important inequalities such as AM, GM, HM inequalities and their consequences, moreover it can be used to generate some more ramified inequalities. All this happens using the classical concept of convex set and convex functions, but what happen when we replace these terms by another convexity terms such as \( h \)-MN-convexity? In fact, discovering
new Jensen type inequalities will help us to find, refine, and generate new important inequalities e.g., AM-GM-HM type inequalities which have a wide range of applications.

In this work, the class of $h$-MN-convex functions is introduced. Generalizing and extending some classes of convex functions are given. Some analytic properties for each class of functions are explored and investigated. Characterizations of each type of convexity are established. Some related Jensen’s type inequalities and their converses are proved.

2. The $h$-MN-convexity

Let $h: J \to (0, \infty)$ be a non-negative function. Define the function $M : [0, 1] \to [a, b]$ given by $M(t) = M(h(t); a, b)$; where by $M(t; a, b)$ we mean one of the following functions:

1. $A_h(a, b) := h(t)a + h(1-t)b$; The generalized Arithmetic Mean.
2. $G_h(a, b) = a^{h(0)}b^{h(1)}$; The generalized Geometric Mean.
3. $H_h(a, b) = \frac{ab}{h(1-t)a + h(t)b} = \frac{1}{A_h\left(\frac{1}{a}, \frac{1}{b}\right)}$; The generalized Harmonic Mean.

Note that $M(h(0); a, b) = a$ and $M(h(1); a, b) = b$. Clearly, for $h(t) = t$ with $t = \frac{1}{2}$, the means $A_{\frac{1}{2}}, G_{\frac{1}{2}}$ and $H_{\frac{1}{2}}$, respectively; represents the midpoint of the $A_{\frac{1}{2}}, G_{\frac{1}{2}}$ and $H_{\frac{1}{2}}$ respectively; which was discussed in [4] in viewing of Definition 6.

For $h(t) = t$, we note that the above means are related with celebrated AM-GM-HM inequality

$$H_{t}(a, b) \leq G_{t}(a, b) \leq A_{t}(a, b), \quad \forall t \in [0, 1].$$

Indeed, one can easily prove more general form of the above inequality; that is if $h$ is positive increasing on $[0, 1]$ then the generalized AM-GM-HM inequality is given by

$$H_{h}(a, b) \leq G_{h}(a, b) \leq A_{h}(a, b), \quad \forall t \in [0, 1] \text{ and } a, b > 0. \quad (2.1)$$
2.1 Basic properties of $h$-MN-convex functions

The Definition 6 can be extended according to the defined mean $M(t; a, b)$, as follows: Let $f : I \to (0, \infty)$ be any function. Let $M$ and $N$ be any two Mean functions. We say $f$ is MN-convex (concave) if

$$f(M(t; x, y)) \leq (\geq) N(t; f(x), f(y)),$$

for all $x, y \in I$ and $t \in [0, 1]$.

Next, we introduce the class of $M, N_h$-convex functions by generalizing the concept of $M, N_t$-convexity and combining it with $h$-convexity.

**Definition 7**: Let $h : J \to (0, \infty)$ be a non-negative function. Let $f : I \to (0, \infty)$ be any function. Let $M : [0, 1] \to [a, b]$ and $N : (0, \infty) \to (0, \infty)$ be any two Mean functions. We say $f$ is $h$-MN-convex (-concave) or that $f$ belongs to the class $\mathcal{MN}(h, I)$ if

$$f(M(t; x, y)) \leq (\geq) N(h(t); f(x), f(y)),$$

for all $x, y \in I$ and $t \in [0, 1]$.

Clearly, if $M(t; x, y) = A_t(x, y) = N(t; x, y)$, then Definition 7 reduces to the original concept of $h$-convexity. Also, if we assume $f$ is continuous, $h(t) = t$ and $t = \frac{1}{2}$ in (2.2), then the Definition 7 reduces to the Definition 6.

The cases of $h$-MN-convexity are given with respect to a certain mean, as follow:

1. $f$ is $A, G_h$-convex iff

$$f(t\alpha + (1-t)\beta) \leq [f(\alpha)]^{h(t)} [f(\beta)]^{h(1-t)}, \quad 0 \leq t \leq 1,$$

2. $f$ is $A, H_h$-convex iff

$$f(t\alpha + (1-t)\beta) \leq \frac{f(\alpha) f(\beta)}{h(1-t) f(\alpha) + h(t) f(\beta)}, \quad 0 \leq t \leq 1$$

3. $f$ is $G, A_h$-convex iff

$$f(\alpha' \beta^{1-t}) \leq h(t) f(\alpha) + h(1-t) f(\beta), \quad 0 \leq t \leq 1.$$

4. $f$ is $G, G_h$-convex iff

$$f(\alpha' \beta^{1-t}) \leq [f(\alpha)]^{h(t)} [f(\beta)]^{h(1-t)}, \quad 0 \leq t \leq 1.$$

5. $f$ is $G, H_h$-convex iff
SOME PROPERTIES OF H-MN-CONVEXITY

(6) \( f \) is \( H_{A_h} \)-convex iff
\[
\frac{f(\alpha^t \beta^{-t})}{h(1-t)f(\alpha) + h(t)f(\beta)} \leq f(\alpha)f(\beta), \quad 0 \leq t \leq 1. \tag{2.7}
\]

(7) \( f \) is \( H_{G_h} \)-convex iff
\[
f\left(\frac{\alpha \beta}{(1-t)\alpha + t \beta}\right) \leq h(1-t)f(\alpha) + h(t)f(\beta), \quad 0 \leq t \leq 1. \tag{2.8}
\]

(8) \( f \) is \( H_{H_h} \)-convex iff
\[
f\left(\frac{\alpha \beta}{(1-t)\alpha + t \beta}\right) \leq \frac{f(\alpha)f(\beta)}{h(t)f(\alpha) + h(1-t)f(\beta)}, \quad 0 \leq t \leq 1. \tag{2.10}
\]

Remark 1: In all previous cases, \( h(t) \) and \( h(1-t) \) are not equal to zero at the same time. Therefore, if \( h(0) = 0 \) and \( h(1) = 0 \), then the Mean function \( N \) satisfying the conditions \( N(h(0), f(x), f(y)) = f(x) \) and \( N(h(1), f(x), f(y)) = f(y) \).

Remark 2: According to the Definition 7, we may extend the classes \( Q(I) \), \( P(I) \) and \( K_{2S}^2 \) by replacing the arithmetic mean by another given one. Let \( M : [0, 1] \to [a, b] \) and \( N : (0, \infty) \to (0, \infty) \) be any two Mean functions.

(1) Let \( s \in (0, 1] \), a function \( f : I \to (0, \infty) \) is \( M_{N_s} \)-convex function or that \( f \) belongs to the class \( K_{2s}^2 (I; M_{r} N_{1/r}) \) if for all \( x, y \in I \) and \( t \in [0, 1] \) we have
\[
f(M(t;x,y)) \leq N(t^s; f(x), f(y)). \tag{2.11}
\]

(2) We say that \( f : I \to (0, \infty) \) is an extended Godunova-Levin function or that \( f \) belongs to the class \( Q(I; M_{r}, N_{1/r}) \) if for all \( x, y \in I \) and ** we have
\[
f(M(t;x,y)) \leq N\left(\frac{1}{t}; f(x), f(y)\right). \tag{2.12}
\]

(3) We say that \( f : I \to (0, \infty) \) is \( P_{M_{1/r}} N_{t^{-1}} \)-function or that \( f \) belongs to the class \( P(I; M_r N_i) \) if for all \( x, y \in I \) and \( t \in [0, 1] \) we have
\[ f(M(t;x,y)) \leq N(1; f(x), f(y)). \] (2.13)

In (2.11) – (2.13), setting \( M(t;x,y) = A_t(x,y) = N(t; x,y) \), we then refer to the original definitions of these classes of convexities (see Definitions 1–3).

**Remark 3**: Let \( h \) be a non-negative function such that \( h(t) \geq t \) for \( t \in (0, 1) \). For instance \( h_t(t) = t^r, t \in (0, 1) \) has that property. In particular, for \( r \leq 1 \), if \( f \) is a non-negative \( M_t N_t \)-convex function on \( I \), then for \( x, y \in I, t \in (0, 1) \) we have

\[
\begin{align*}
f(M(t;x,y)) & \leq N(t; f(x), f(y)) \leq N(t'; f(x), f(y)) \\
& = N(h(t); f(x), f(y)),
\end{align*}
\]

for all \( r \leq 1 \) and \( t \in (0, 1) \). So that \( f \) is \( M_t N_t \)-convex. Similarly, if the function satisfies the property \( h(t) \leq t \) for \( t \in (0, 1) \), then \( f \) is a non-negative \( M_t N_t \)-concave. In particular, for \( r \geq 1 \), the function \( h_t(t) \) has that property for \( t \in (0, 1) \). So that if \( f \) is a non-negative \( M_t N_t \)-concave function on \( I \), then for \( x, y \in I, t \in (0, 1) \) we have

\[
\begin{align*}
f(M(t;x,y)) & \geq N(t; f(x), f(y)) \geq N(t'; f(x), f(y)) = N(h(t); f(x), f(y)),
\end{align*}
\]

for all \( r \geq 1 \) and \( t \in (0, 1) \), which means that \( f \) is \( M_t N_t \)-concave.

**Remark 4**: There exists an \( h \)-MN-convex function which is MN-convex. As shown by Varošanec (see Examples 6 and 7 in [32]), one can generate \( h \)-MN-convex functions but not MN-convex.

Next, we give an extended generalization of Theorem 2.4 in [4]. This simply can help to illustrate the concept of \( h \)-MN-convex functions.

**Theorem 3**: Let \( h : J \to (0, \infty) \) be a positive function. \( f : I \to (0, \infty) \) be any function. In parts (4)–(9), let \( I = (0, \tau), 0 < \tau < \infty \).

1. \( f \) is \( A_t A_{\tau} \)-convex (concave) if and only if \( f \) is \( h \)-convex (concave).
2. \( f \) is \( A_t G_{\tau} \)-convex (concave) if and only if \( \log f \) is \( h \)-convex (concave).
3. \( f \) is \( A_t H_{\tau} \)-convex (concave) if and only if \( \frac{1}{f(x)} \) is \( h \)-concave (convex).
4. \( f \) is \( G_t A_{\tau} \)-convex (concave) on \( I \) if and only if \( f(\tau x) \) is \( h \)-convex (concave).
5. \( f \) is \( G_t G_{\tau} \)-convex (concave) if and only if \( \log f(\tau x) \) is \( h \)-convex (concave) on \( (0, \infty) \).
(6) \( f \) is \( G, H \)-convex (-concave) if and only if \( \frac{1}{f(r e^{-t})} \) is \( h \)-concave (-convex) on \((0, \infty)\).

(7) \( f \) is \( H, A \)-convex (-concave) if and only if \( f \left( \frac{1}{x} \right) \) is \( h \)-convex (-concave) on \( \left( \frac{1}{r}, \infty \right) \).

(8) \( f \) is \( H, H \)-convex (-concave) if and only if \( \log f \left( \frac{1}{x} \right) \) is \( h \)-convex (-concave) on \( \left( \frac{1}{r}, \infty \right) \).

(9) \( f \) is \( A, A \)-convex (-concave) if and only if \( f \left( \frac{1}{x} \right) \) is \( h \)-concave (-convex) on \( \left( \frac{1}{r}, \infty \right) \).

**Proof:**

(1) Follows by definition.

(2) Employing (2.3) in the Definition 7, we have

\[
\frac{f(A_t(a,b)) \leq (\geq) G(h(t); f(a), f(b))}{f((1-t)a + tb) \leq (\geq) [f(a)]^{h(1-t)}[f(b)]^{h(t)}}
\]

which proves the result.

(3) Employing (2.4) in the Definition 7, we have

\[
\frac{f(A_t(a,b)) \leq (\geq) H(h(t); f(a), f(b))}{f((1-t)a + tb) \leq (\geq) \frac{f(a)f(b)}{h(t)f(a) + h(1-t)f(b)}}
\]

which proves the result.

(4) Employing (2.5) in the Definition 7 and substituting \( a = r e^{-t} \) and \( b = s e^{-t} \), we have

\[
f(G_t(a,b)) \leq (\geq) A(h(t); f(a), f(b))
\]

which proves the result.

(5) Employing (2.6) in the Definition 7 and substituting \( a = r e^{-t} \) and \( b = s e^{-t} \), we have
Employing (2.7) in the Definition 7 and substituting \( a = e^{-r} \) and \( b = e^{-s} \), we have, we have

\[
f(G_1(a, b)) \leq (\geq)G(h(t); f(a), f(b))
\]

\[
\iff f(a^{1-t}b) \leq (\geq)[f(a)]^{h(1-t)}[f(b)]^{h(t)}
\]

\[
\iff \log f(e^{-[r(1-t)+st]}) \leq (\geq)h(1-t)\log f(e^{-r}) + h(t)\log f(e^{-s}),
\]

which proves the result.

(7) Let \( g(x) = f\left(\frac{1}{x}\right) \) and let \( a, b \in \left[\frac{1}{\tau}, \infty\right) \) with \( a < b \) and \( \tau \geq 1 \). Then \( f \) is \( H_A^{\text{H}} \)-convex (=concave) on \((0, \tau)\) if and only if

\[
f\left(\frac{1}{H_1(a, b)}\right) \leq (\geq)A\left(h(t), \frac{1}{f(a)}, \frac{1}{f(b)}\right)
\]

\[
\iff f\left(\frac{1}{abta + (1-t)b}\right) \leq (\geq)h(t)f\left(\frac{1}{b}\right) + h(1-t)f\left(\frac{1}{a}\right)
\]

\[
\iff g\left(\frac{ab}{ta + (1-t)b}\right) \leq (\geq)h(1-t)g(a) + h(t)g(b),
\]

which proves the result.

(8) Let \( g(x) = \log f\left(\frac{1}{x}\right) \) and let \( a, b \in \left[\frac{1}{\tau}, \infty\right) \) with \( a < b \) and \( \tau \geq 1 \). Then \( f \) is \( H_G^{\text{H}} \)-convex (=concave) on \((0, \tau)\) if and only if

\[
f\left(\frac{1}{H_1(a, b)}\right) \leq (\geq)G(h(t); f(a), f(b))
\]

\[
\iff f\left(\frac{t}{b} + \frac{(1-t)}{a}\right) \leq (\geq)\left[f\left(\frac{1}{b}\right)\right]^{h(t)}\left[f\left(\frac{1}{a}\right)\right]^{h(1-t)}
\]
which proves the result.

(9) Let \( g(x) = \frac{1}{f\left(\frac{1}{x}\right)} \) and let \( a, b \in \left(\frac{1}{\tau}, \infty\right) \) with \( a < b \) and \( \tau \geq 1 \). Then \( f \) is \( H_\tau \), \( H_h \)-convex (-concave) on \((0, \tau)\) if and only if

\[
\log f\left(\frac{t}{b} + \frac{(1-t)}{a}\right) \leq (\geq) h(t)\log f\left(\frac{1}{b}\right) + h(1-t)\log f\left(\frac{1}{a}\right)
\]

\[
\Leftrightarrow \quad \log f\left(\frac{ab}{ta + (1-t)b}\right) \leq (\geq) h(t)g(b) + h(1-t)g(a),
\]

which proves the result.

\[\square\]

**Proposition 1:** Let \( h : J \to (0, \infty) \) be a non-negative function. Then

- if \( f \) is \( A_t H_\tau \)-convex, then \( f \) is \( A_tG_h \)-convex;
- if \( f \) is \( G_\tau H_h \)-convex, then \( f \) is \( G_\tau G_h \)-convex;
- if \( f \) is \( H_\tau H_h \)-convex, then \( f \) is \( H_\tau G_h \)-convex;
- if \( f \) is \( A_t A_h \)-convex, then \( f \) is \( A_t G_h \)-convex.

\( \downarrow f \nearrow \quad \quad \downarrow f \nearrow \quad \quad \downarrow f \nearrow \)

\( \downarrow f \nearrow \quad \quad \downarrow f \nearrow \quad \quad \downarrow f \nearrow \)

\( \downarrow f \nearrow \quad \quad \downarrow f \nearrow \quad \quad \downarrow f \nearrow \)
By \( f \nearrow \) we mean that \( f \) is increasing and by \( f \nwarrow \) we mean that \( f \) is decreasing. For \( h \)-concavity and decreasing monotonicity, the implications are reversed.

**Proof:** The proof of each statement follows from Definition 7 and by noting that for an increasing function \( h \) we have \( H_t(a, b) \leq G_t(a, b) \leq A_t(a, b) \), for all \( t \in [0, 1] \). Furthermore, and for instance we note that if \( f \) is \( A_tH_h \)-convex, therefore we have

\[
f(A_\alpha(x, y)) = f(ax + (1-\alpha)y) \leq \frac{f(x)f(y)}{h(1-\alpha)f(x) + h(\alpha)f(y)}
\]

\[
= \frac{1}{h(1-\alpha) + h(\alpha)}
\]

\[
= H(h(\alpha), f(x), f(y)),
\]

which is employing for \( g(t) = \frac{1}{f(t)} \), i.e.,

\[
g(A_\alpha(x, y)) = g(ax + (1-\alpha)y) = \frac{1}{f(ax + (1-\alpha)y)}
\]

\[
\geq \frac{h(1-\alpha) + h(\alpha)}{f(y) + f(x)} = h(1-\alpha)g(y) + h(\alpha)g(x) = A(h(\alpha), g(x), g(y)),
\]

and this shows that \( g \) is \( A_tA_h \)-concave. \( \square \)

Thus, one can see the implications in Theorem 3 are strict, as shown by the following example:

**Example 1:** Let \( h \) be a non-negative function such that \( h(t) \geq t \) for all \( t \in (0, 1) \). In particular, let \( h(t) = h_k(t) = t^k \), \( k \leq 1 \) and \( t \in (0, 1) \). The functions

1. \( f(x) = \cosh(x) \) is \( A_tG_h \)-convex, hence \( G_tG_h \)-convex and \( H_tG_h \)-convex, on \((0, \infty) \). But it is not \( A_tH_h \)-convex, nor \( G_tH_h \)-convex, nor \( H_tG_h \)-convex.
2. \( f(x) = \arcsin(x) \) is \( A_tA_h \)-convex but it is \( A_tG_h \)-concave for all \( 0 \leq x \leq 1 \).
3. \( f(x) = e^x \) is \( G_tG_h \)-convex and \( H_tG_h \)-convex, but neither \( G_tH_h \)-convex nor \( H_tH_h \)-convex, for all \( x > 0 \).
4. \( f(x) = \log(1 + x) \) is \( G_tA_h \)-convex but \( G_tG_h \)-concave for all \( 0 < x < 1 \).
(5) $f(x) = e^{-x}$ is $G_tA_{h}$-convex for $k \leq \frac{1}{2}$ but not $H_tG_{h}$-convex for all $0 < x < 1$.

Also, $f$ is $H_tA_{h}$-convex but not $H_tG_{h}$-convex for all $x > 1$.

**Proposition 2**: Let $h_1, h_2 : J \to (0, \infty)$ be two positive positive functions with the property that $h_2(t) \leq h_1(t)$ for all $t \in (0, 1)$. If $f$ is $M_tN_h$-convex then $M_tN_{h_1}$-convex and if $f$ is $M_tN_{h_1}$-concave then $M_tN_{h_2}$-concave.

**Proof**: From Definition 7 we have

$$f(M(t; x, y)) \leq (\geq)N(h_2(t); f(x), f(y)) \leq (\geq)N(h_1(t); f(x), f(y)),$$

which is required. □

**Proposition 3**: If $f$ and $g$ are two $M_tN_h$-convex and $\lambda > 0$, then $f + g$, $\lambda f$ and $\max\{f, g\}$.

**Proof**: The proof follows by Definition 7. □

**Proposition 4**: Let $f$ and $g$ be a similarly ordered functions. If $f$ is $A_tA_{h_1}$-convex ($G_tA_{h_1}$-convex, $H_tA_{h_1}$-convex), $g$ is $A_tA_{h_2}$-convex ($G_tA_{h_2}$-convex, $H_tA_{h_2}$-convex), respectively; and $h(t) + h(1 - t) \leq c$, where $h(t) := [h_1(t), h_2(t)]$ and $c$ is a fixed positive real number. Then the product $(fg)$ is $A_tA_{c,h}$-convex ($G_tA_{c,h}$-convex, $H_tA_{c,h}$-convex), respectively.

**Proof**: Since $f$ and $g$ are similarly ordered functions we have

$$f(x)g(x) + f(y)g(y) \geq f(x)g(y) + g(x)f(y).$$

Let $t$ and $s$ be positive numbers such that $t + s = 1$. Then we obtain

$$(fg)(A_t(x, y))$$

$$= (fg)(sx + ty)$$

$$\leq [h_1(s)f(x) + h_1(t)f(y)][h_2(s)g(x) + h_2(t)g(y)]$$

$$\leq h^2(s)f(x)g(x) + h^2(t)f(y)g(y)$$

$$\leq h^2(s)f(x)g(x) + h^2(t)h(s)[f(x)g(x) + f(y)g(y)] + h^2(t)f(y)g(y)$$

$$= (h(s) + h(t))(h(s)(fg)(x) + h(t)(fg)(y))$$

$$= c \cdot h(s)(fg)(x) + c \cdot h(t)(fg)(y)$$

$$= A(c \cdot h(t); (fg)(x), (fg)(y)),$$

which shows that $(fg)$ is $A_tA_{c,h}$-convex. The cases when $fg$ is $G_tA_{c,h}$-convex or $H_tA_{c,h}$-convex, are follow in similar manner. □
**Corollary 1:** Let \( f \) and \( g \) be an oppositely ordered functions. If \( f \) is \( A_{t_{1}} - \text{concave} \) (\( G_{t_{1}} - \text{concave}, H_{t_{1}} - \text{concave} \)), \( g \) is \( A_{t_{2}} - \text{concave} \) (\( G_{t_{2}} - \text{concave}, H_{t_{2}} - \text{concave} \)), respectively; and \( h(t) = \min[h_{1}(t), h_{2}(t)] \) and \( c \) is a fixed positive real number. Then the product \( (fg) \) is \( (c-h)A_{t_{1}} - \text{concave} \) (\( G_{t_{1}} - \text{concave}, H_{t_{1}} - \text{concave} \)), respectively.

**Proposition 5:** If \( f \) is \( A_{t_{1}} - \text{convex} \) (\( G_{t_{1}} - \text{convex}, H_{t_{1}} - \text{convex} \)) and \( g \) is \( A_{t_{2}} - \text{convex} \) (\( G_{t_{2}} - \text{convex}, H_{t_{2}} - \text{convex} \)), respectively; and \( (1) : \max\{h(t), h(1-t)\} \), where \( h(t) + h(1-t) \leq c \). Then the product \( (fg) \) is \( A_{t_{1}} - \text{convex} \) (\( G_{t_{1}} - \text{convex}, H_{t_{1}} - \text{convex} \)), respectively.

**Proof:** Let \( t \in (0,1) \subseteq J \), then
\[
(fg)(A(t,x,y)) = (fg)((1-t)x+ty) \\
\leq \left\{ \left[ f(x) \right]^{h_{t}(1-t)} \left[ f(y) \right]^{h_{t}(1-t)} \cdot \left[ g(x) \right]^{h_{t}(1-t)} \left[ g(y) \right]^{h_{t}(1-t)} \right\} \\
= \left[ f(x) \right]^{h_{1}(1-t)} \left[ g(x) \right]^{h_{2}(1-t)} \cdot \left[ f(y) \right]^{h_{1}(1-t)} \left[ g(y) \right]^{h_{2}(1-t)} \\
\leq \left[ (fg)(x) \right]^{h_{t}(1-t)} \cdot \left[ (fg)(y) \right]^{h_{t}(1-t)} \\
= G(h(t), (fg)(x), (fg)(y)),
\]
which shows that \( (fg) \) is \( A_{t_{1}} - \text{convex} \). The cases when \( fg \) is \( G_{t_{1}} - \text{convex} \) or \( H_{t_{1}} - \text{convex} \), are follow in similar manner. \( \square \)

**Corollary 2:** If \( f \) is \( A_{t_{1}} - \text{concave} \) (\( G_{t_{1}} - \text{concave}, H_{t_{1}} - \text{concave} \)) and \( g \) is \( A_{t_{2}} - \text{convex} \) (\( G_{t_{2}} - \text{convex}, H_{t_{2}} - \text{convex} \)), respectively; and \( h(t) = \max[h_{1}(t), h_{2}(t)] \), where \( h(t) + h(1-t) \geq c \). Then the product \( (fg) \) is \( A_{t_{1}} - \text{concave} \) (\( G_{t_{1}} - \text{concave}, H_{t_{1}} - \text{concave} \)), respectively.

**Proposition 6:** Let \( f \) and \( g \) be an oppositely ordered functions. If \( f \) is \( A_{h_{1}} - \text{convex} \) (\( G_{h_{1}} - \text{convex}, H_{h_{1}} - \text{convex} \)), \( g \) is \( A_{h_{2}} - \text{convex} \) (\( h_{2} - \text{concave}, H_{h_{2}} - \text{convex} \)), respectively; and \( h(t) = \min[h_{1}(t), h_{2}(t)] \) and \( c \) is a fixed positive real number. Then the product \( (fg) \) is \( A_{t_{1}} - \text{convex} \) (\( G_{t_{1}} - \text{convex}, H_{t_{1}} - \text{convex} \)), respectively.

**Proof:** Since \( f \) and \( g \) are oppositely ordered functions
\[
f(x)g(x) + f(y)g(y) \leq f(x)g(y) + g(x)f(y).
\]
Let \( t \) and \( s \) be positive numbers such that \( t + s = 1 \). Then we obtain

\[
\]
\((f \circ g)(A_{i}(x, y))\) = \((f \circ g)(sx + ty)\) \\
\leq \frac{f(x)f(y)}{h_1(t)f(x) + h_1(s)f(y)} \cdot \frac{g(x)g(y)}{h_2(t)g(x) + h_2(s)g(y)} \\\n\leq \frac{(f \circ g)(x)(f \circ g)(y)}{h_1(t)h_2(t)f(x)g(x) + h_1(s)h_2(t)f(y)g(x) + h_1(t)h_2(s)f(x)g(y) + h_1(s)h_2(s)f(y)g(y)} \\\n\leq \frac{c \cdot h(t)(f \circ g)(x) + c \cdot h(s)(f \circ g)(y)}{c \cdot h(t)(f \circ g)(x) + c \cdot h(s)(f \circ g)(y)} = \frac{c \cdot h(t)(f \circ g)(x) + c \cdot h(s)(f \circ g)(y)}{c \cdot h(t)(f \circ g)(x) + c \cdot h(s)(f \circ g)(y)} = H_c(t; (f \circ g)(x), (f \circ g)(y))

which shows that \((f \circ g)\) is \(A_{i}H_{c}\)-convex. The cases when \(fg\) is \(G_{i}H_{c}\)-convex or \(H_{t}H_{c}\)-convex, are follow in similar manner. \(\square\)

**Corollary 3**: Let \(f\) and \(g\) be similarly ordered functions. If \(f\) is \(A_{i}A_{n_{1}}\)-concave (\(G_{i}H_{n_{1}}\)-concave, \(H_{t}H_{n_{1}}\)-concave), \(g\) is \(A_{i}H_{n_{1}}\)-concave (\(h_2G_{i}H_{n_{2}}\)-concave, \(H_{t}H_{n_{2}}\)-concave), respectively; and \(h(t) + h(1-t) \leq c\), where \(h(t) = \max\{h_{1}(t), h_{2}(t)\}\) and \(c\) is a fixed positive real number. Then the product \((f \circ g)\) is \(A_{i}H_{c}\)-concave (\(G_{i}H_{c}\)-concave, \(H_{t}H_{c}\)-concave), respectively.

Sometimes we often use functional inequalities to describe and characterize all real functions that satisfy specific functional inequality. In [32], Varošanec proved a result regarding \(A_{i}A_{n}\)-convex functions, following a similar approach; we next present some results of this type.

**Theorem 4**: Let \(I \subset \mathbb{R}\) with \(0 \in I\). Let \(h\) be a non-negative function on \(I\).

1. Let \(f\) be \(A_{i}G_{i}\)-convex and \(f(0) = 1\). If \(h\) is supermultiplicative, then the inequality

\[
f(\alpha x + \beta y) \leq \left[ f(x) \right]^{h(\alpha)} \left[ f(y) \right]^{h(\beta)}, \tag{2.14}
\]
holds for all $x, y \in I$ and all $a, \beta > 0$ such that $\alpha + \beta \leq 1$.

(2) Assume that $h(\alpha) < \frac{1}{2}$ for some $\alpha \in \left(0, \frac{1}{2}\right)$. If $f$ is a non-negative function such that inequality (2.14) holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \leq 1$, then $f(0) = 1$.

(3) Let $f$ be $A_{G_h}$-concave and $f(0) = 1$. If $h$ is submultiplicative, then the inequality

$$f(\alpha x + \beta y) \geq \left[f(x)\right]^h[(\alpha)] \left[f(y)\right]^h[\beta],$$

holds for all $x, y \in I$ and all $a, \beta > 0$ such that $a + \beta \leq 1$.

(4) Assume that $h(\alpha) > \frac{1}{2}$ for some $\alpha \in \left(0, \frac{1}{2}\right)$. If $f$ is a non-negative function such that inequality (2.15) holds for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta \leq 1$, then $f(0) = 1$.

Proof: Let $\alpha, \beta > 0$ be positive real numbers such that $\alpha + \beta = \lambda \leq 1$.

(1) Define numbers $a$ and $b$ such as $a = \frac{\alpha}{\lambda}$ and $b = \frac{\beta}{\lambda}$. Then $a + b = 1$ and we have the following:

$$f(\alpha x + \beta y) = f(\lambda ax + \lambda by)$$

$$\leq \left[f(\lambda x)\right]^h[\alpha] \left[f(\lambda y)\right]^h[\beta]$$

$$= \left[f(\lambda x + (1 - \lambda) \cdot 0)\right]^h[\alpha] \left[f(\lambda y + (1 - \lambda) \cdot 0)\right]^h[\beta]$$

$$\leq \left[f(x)\right]^h[\lambda] \left[f(0)\right]^h[\lambda(1 - \lambda)] \left[f(x)\right]^h[\lambda] \left[f(0)\right]^h[\lambda(1 - \lambda)]$$

$$= \left[f(x)\right]^h[\alpha] \left[f(\lambda x)\right]^h[\beta(1 - \lambda)]$$

$$= \left[f(x)\right]^h[\alpha] \left[f(y)\right]^h[\beta(\lambda)]$$

$$= \left[f(x)\right]^h[\alpha] \left[f(y)\right]^h[\beta],$$

where we use that $f$ is $A_{G_h}$, $f(0) = 1$ and $h$ is supermultiplicative, respectively.

(2) Suppose that $f(0) \neq 1$. Putting $x = y = 0$ in (2.14) we get

$$f(0) \leq \left[f(0)\right]^h[\alpha + \beta],$$

for all $\alpha, \beta > 0, \alpha + \beta \leq 1$.

Setting $\beta = \alpha, \alpha \in \left(0, \frac{1}{2}\right)$, then $0 \leq (2h(\alpha) - 1)\log f(0)$, it follows that $h(\alpha) \geq \frac{1}{2}$, since $f(0) \neq 1$, which contradicts the assumption of theorem. So that $f(0) = 1$. 
The proofs for cases (3) and (4) are similar to the previous. Hence, the proof is completely established.

**Theorem 5**: Let \( a, b \in \left[ \frac{1}{\tau}, \infty \right) \) with \( a < b \) and \( \tau \geq 1 \). Let \( h \) be a non-negative function on \( J \).

1. Let \( f \) be \( G_{\tau A_h} \)-convex and \( f(1) = 0 \). If \( h \) is supermultiplicative, then the inequality
   \[
   f\left(x^\alpha y^\beta\right) \leq h(\alpha)f(x) + h(\beta)f(y),
   \]
   holds for all \( x, y \in I \) and all \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \).

2. Assume that \( h(\alpha) < \frac{1}{2} \) for some \( \alpha \in \left(0, \frac{1}{2}\right) \). If \( f \) is a non-negative function such that inequality (2.16) holds for all \( x, y \in I \) and all \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \), then \( f(1) = 0 \).

3. Let \( f \) be \( G_{\tau A_h} \)-concave and \( f(1) = 0 \). If \( h \) is submultiplicative, then the inequality
   \[
   f\left(x^\alpha y^\beta\right) \geq h(\alpha)f(x) + h(\beta)f(y),
   \]
   holds for all \( x, y \in I \) and all \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \).

4. Assume that \( h(\alpha) > \frac{1}{2} \) for some \( \alpha \in \left(0, \frac{1}{2}\right) \). If \( f \) is a non-negative function such that inequality (2.17) holds for all \( x, y \in I \) and all \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \), then \( f(1) = 0 \).

**Proof**: Let \( \alpha, \beta > 0 \) be positive real numbers such that \( \alpha + \beta = \lambda \leq 1 \).

1. Define numbers \( a \) and \( b \) such as \( a = \frac{\alpha}{\lambda} \) and \( b = \frac{\beta}{\lambda} \). Then \( a + b = 1 \) and we have the following:
   \[
   f\left(x^\alpha y^\beta\right) = f\left(x^{\frac{\alpha}{\lambda}} y^{\frac{\beta}{\lambda}}\right)
   \leq h(a)f\left(x^{\frac{\alpha}{\lambda}}\right) + h(b)f\left(y^{\frac{\beta}{\lambda}}\right)
   = h(a)f\left(x^{\frac{\alpha}{\lambda}} \cdot 1^{1-\frac{\alpha}{\lambda}}\right) + h(b)f\left(y^{\frac{\beta}{\lambda}} \cdot 1^{1-\frac{\beta}{\lambda}}\right)
   \leq h(a)\left[h(\lambda)f(x) + h(1-\lambda)f(1)\right]
   + h(b)\left[h(\lambda)f(y) + h(1-\lambda)f(1)\right]
   = h(a)h(\lambda)f(x) + h(b)h(\lambda)f(y)
   \leq h(\alpha)f(x) + h(\beta)f(y),
   \]
   where we use that \( f \) is \( G_{\tau A_h} \), \( f(1) = 0 \) and \( h \) is supermultiplicative, respectively.
Suppose that \( f(1) \neq 0 \), since \( f \) is non-negative then \( f(1) > 0 \). Putting \( x = y = 1 \) in (2.16) we get
\[
f(1) \leq h(\alpha) f(1) + h(\beta) f(1), \quad \text{for all } \alpha, \beta > 0, \alpha + \beta \leq 1.
\]
Setting \( \beta = \alpha, \quad \alpha \in \left(0, \frac{1}{2}\right) \), then \( 0 \leq (2h(\alpha) - 1)f(1) \), it follows that \( h(\alpha) \geq \frac{1}{2} \), which contradicts the assumption of theorem. So that \( f(1) = 0 \).

The proofs for cases (3) and (4) are similar to the previous. Hence, the proof is completely established.

**Theorem 6 :** Let \( a, b \in \left(\frac{1}{\tau}, \infty\right) \) with \( a < b \) and \( \tau \geq 1 \). Let \( h \) be a non-negative function on \( I \).

1. Let \( f \) be \( G_{\tau} \)-convex and \( f(1) = 1 \). If \( h \) is supermultiplicative, then the inequality
\[
f\left(x^\alpha y^\beta\right) \leq \left[f(x)\right]^{h(\alpha)}\left[f(y)\right]^{h(\beta)},
\]
holds for all \( x, y \in I \) and all \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \).

2. Assume that \( h(\alpha) < \frac{1}{2} \) for some \( \alpha \in \left(0, \frac{1}{2}\right) \). If \( f \) is a non-negative function such that inequality (2.18) holds for all \( x, y \in I \) and all \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \), then \( f(1) = 1 \).

3. Let \( f \) be \( h\)-\( G_{\tau} \)-concave and \( f(1) = 1 \). If \( h \) is submultiplicative, then the inequality
\[
f\left(x^\alpha y^\beta\right) \geq \left[f(x)\right]^{h(\alpha)}\left[f(y)\right]^{h(\beta)},
\]
holds for all \( x, y \in I \) and all \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \).

4. Assume that \( h(\alpha) > \frac{1}{2} \) for some \( \alpha \in \left(0, \frac{1}{2}\right) \). If \( f \) is a non-negative function such that inequality (2.19) holds for all \( x, y \in I \) and all \( \alpha, \beta > 0 \) such that \( \alpha + \beta \leq 1 \), then \( f(1) = 1 \).

**Proof :** Let \( \alpha, \beta > 0 \) be positive real numbers such that \( \alpha + \beta = \lambda \leq 1 \).

1. Define numbers \( a \) and \( b \) such as \( a = \frac{\alpha}{\lambda} \) and \( b = \frac{\beta}{\lambda} \). Then \( a + b = 1 \) and we have the following:
\[
f(x^\alpha y^\beta) = f(x^{\beta a} y^\beta)
\leq \left[ f(x^\beta) \right]^{\beta(a)} \left[ f(y^\beta) \right]^{\beta(b)}
\leq \left[ f(x^{\beta \cdot 1^{1-\beta}}) \right]^{\beta(a)} \left[ f(y^{\beta \cdot 1^{1-\beta}}) \right]^{\beta(b)}
\leq \left\{ \left[ f(x)^{\beta(a)} \left[ f(1)^{\beta(1-\beta)} \right] \right]^\beta(a) \left\{ \left[ f(y)^{\beta(1-\beta)} \left[ f(1)^{\beta(1-\beta)} \right] \right]^\beta(b) \right\}
\leq \left[ f(x) \right]^{\beta(a)\beta(\beta)} \left[ f(y) \right]^{\beta(b)\beta(\beta)}
\leq \left[ f(x) \right]^{\beta(a)} \left[ f(y) \right]^{\beta(b)}
\]

where we use that \( f \) is \( G_t \cdot G_h \), \( f(1) = 1 \) and \( h \) is supermultiplicative, respectively.

(2) Suppose that \( f(1) \neq 1 \). Putting \( x = y = 1 \) in (2.18) we get
\[
f(1) \leq \left[ f(1) \right]^{\beta(a)} \left[ f(1) \right]^{\beta(b)}
\text{for all } \alpha, \beta > 0, \alpha + \beta \leq 1.
\]
Setting \( \beta = \alpha \), \( \alpha \in \left( 0, \frac{1}{2} \right) \), then \( 1 \leq \left[ f(1) \right]^{2\alpha(a)-1} \), it follows that \( h(\alpha) \geq \frac{1}{2} \), which contradicts the assumption of theorem. So that \( f(1) = 1 \).

The proofs for cases (3) and (4) are similar to the previous. Hence, the proof is completely established. \( \square \)

2.2 Composition of \( h \)-MN-convex functions

In the next three results, we assume the \( g \cdot h_i : J_i \to (0, \infty), i = 1, 2 \), \( h_1(J_2) \subseteq J_1 \) are non-negative functions such that \( h_2(\alpha) + h_2(1 - \alpha) \leq 1 \), for \( \alpha(0,1) \subseteq J_2 \), let \( f : J_1 \to [0, \infty) \), \( g : J_2 \to [0, \infty) \), be functions with \( g(I_2) \subseteq I_1 \).

**Theorem 7**: Let \( f(1) = 0 \). If \( h_i \) is a supermultiplicative function, \( f \) is \( G_t A_{h_1} \)-convex and increasing (decreasing) on \( I_1 \), while \( g \) is \( A_{h_2} \)-convex (-concave) on \( I_2 \), then the composition \( f \circ g \) is \( A_{h_1} A_{h_2} \)-convex on \( I_2 \).

If \( h_i \) is a submultiplicative function, \( f \) is \( G_t A_{h_1} \)-concave and increasing (decreasing) on \( I_1 \), while \( g \) is \( A_{h_2} \)-convex (-convex) on \( I_2 \), then the composition \( f \circ g \) is \( A_{h_1} A_{h_2} \)-concave on \( I_2 \).
Proof: If \( g \) is \( A_t G_{h_2} \)-convex on \( I_2 \) and \( f \) increasing then
\[
f \circ g \left( ax + (1 - \alpha) y \right) \leq f \left( \left[ g(x) \right]^{h_2(\alpha)} \left[ g(y) \right]^{h_2(1-\alpha)} \right),
\]
for all \( x, y \in I_2 \) and \( \alpha \in (0, 1) \). Using Theorem 5(1), we obtain that
\[
f \left( \left[ g(x) \right]^{h_2(\alpha)} \left[ g(y) \right]^{h_2(1-\alpha)} \right) \leq h_1 \left( h_2(\alpha) \right) f \left( g(x) \right) + h_1 \left( h_2(1-\alpha) \right) f \left( g(y) \right)
\]
\[= \left( h_1 \circ h_2 \right) \left( \alpha \right) \left( f \circ g \right) (x) + \left( h_1 \circ h_2 \right) \left( 1-\alpha \right) \left( f \circ g \right) (y),
\]
which means that \( f \circ g \) is \( A_t A_{b_2+b_1} \)-convex on \( I_2 \).

Theorem 8: Let \( 0 \in I_1 \) and \( f(0) = 1 \). If \( h_i \) is a supermultiplicative function, \( f \) is \( A_t G_{b_1} \)-convex and increasing (decreasing) on \( I_1 \), while \( g \) is \( G_t A_{b_2} \)-convex (-concave) on \( I_2 \), then the composition \( f \circ g \) is \( G_t G_{b_1+b_2} \)-convex on \( I_2 \).

If \( h_i \) is a submultiplicative function, \( f \) is \( A_t A_{b_1} \)-concave and increasing (decreasing) on \( I_1 \), while \( g \) is \( G_t G_{b_2} \)-convex (-convex) on \( I_2 \), then the composition \( f \circ g \) is \( G_t G_{b_1+b_2} \)-concave on \( I_2 \).

Proof: The proof is similar to the proof of Theorem 7 and using Theorem 4(1).

Theorem 9: Let \( f(1) = 1 \). If \( h_i \) is a supermultiplicative function, \( f \) is \( G_t G_{b_1} \)-convex and increasing (decreasing) on \( I_1 \), while \( g \) is \( G_t A_{b_2} \)-convex (-concave) on \( I_2 \), then the composition \( f \circ g \) is \( G_t G_{b_1+b_2} \)-convex on \( I_2 \).

If \( h_i \) is a submultiplicative function, \( f \) is \( G_t G_{b_1} \)-concave and increasing (decreasing) on \( I_1 \), while \( g \) is \( G_t A_{b_2} \)-convex (-convex) on \( I_2 \), then the composition \( f \circ g \) is \( G_t G_{b_1+b_2} \)-concave on \( I_2 \).

Proof: The proof is similar to the proof of Theorem 7 and using Theorem 6(1).

Next, we examine functions compositions, one of them is of type \( M_t K_{h_1} \)-convex while the other is \( K_t N_{h_2} \)-convex.

Theorem 10: Let \( M, N \) and \( K \) be three mean functions. Let \( h_1 : J_1 \rightarrow (0, \infty) \) and \( h_1 : J_2 \rightarrow (0, 1) \), \( h_2(0) \in (0, 1) \subseteq J_2 \) are non-negative functions for \( \alpha \in (0, 1) \subseteq J_2 \) and \( h_2(\alpha) \in (0, 1) \subseteq J_1 \)., let \( f : I_1 \rightarrow [0, \infty) \), \( g : I_2 \rightarrow [0, \infty) \), be functions with \( g(I_2) \subseteq I_1 \). If \( f \) is \( K_t N_{b_2} \)-convex and increasing (decreasing) on \( I_1 \), while \( g \) is
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M_{K_{h_2}}-convex (-concave) on I, then the composition $f \circ g$ is M_{N_{h_1}-h_2}-convex on I. Namely, we explore this corollary in the table below.

**Proof:** We select to prove one of the mentioned cases and the others follow in similar fashion. For example, if $g$ is H_{A_{h_2}}-convex on $I_2$ and $f$ is increasing then

$$f \circ g \left( \frac{xy}{\alpha x+(1-\alpha)y} \right) \leq f \left( h_2(1-\alpha)g(x)+h_2(\alpha)g(y) \right),$$

for all $x, y \in I_2$ and $\alpha \in (0, 1)$. Using Definition 7, we obtain that

$$f \left( h_2(1-\alpha)g(x)+h_2(\alpha)g(y) \right) \leq \frac{f(g(x))f(g(y))}{h_1(h_2(\alpha))f(g(x))+h_1(h_2(1-\alpha))f(g(y))} \frac{f \circ g(x)(f \circ g(y))}{(h_1 \circ h_2)(\alpha)(f \circ g)(x)+(h_1 \circ h_2)(1-\alpha)(f \circ g)(y)},$$

for $h_2(\alpha) \in (0,1)$, which shows that $f \circ g$ is H_{H_{h_1}-h_2}-convex on $I_2$. \hfill \square

3. Characterization of $h$-M$_N$-convexity

Let $h : J \rightarrow [0, \infty)$ be a non-negative function and let $f : I \rightarrow \mathbb{R}$ be a function. For all points $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$ such that $x_2 - x_1$, $x_3 - x_2$ and $x_3 - x_1$ in $J$. In [32], Varošanec proved that if $h$ is supermultiplicative, and $f$ is A_{A_{h}}-convex function, then the inequality

$$h(x_3 - x_2)f(x_1) + h(x_2 - x_1)f(x_3) \geq h(x_3 - x_1)f(x_2),$$

holds. Also, if $h$ is submultiplicative, and $f$ is A_{A_{h}}-convex function, then the above inequality is reversed. In what follows, similar results for M_{N_{h}}-convex functions are proved.

**Theorem 11:** Let $h : J \rightarrow [0, \infty)$ be a non-negative function and let $f : I \rightarrow \mathbb{R}$ be a function. For all points $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$ such that $x_2 - x_1$, $x_3 - x_2$ and $x_3 - x_1$ in $J$,

1. If $h$ is supermultiplicative, and $f$ is A_{G_{h}}-convex function, then the following inequality hold:

$$\left[ f(x_1) \right]^{h(x_3 - x_2)} \left[ f(x_3) \right]^{h(x_2 - x_1)} \geq \left[ f(x_2) \right]^{h(x_3 - x_1)},$$
|          |          |          |
|----------|----------|----------|
| $f$      | $g$      | $f \circ g$ |
| $A_r A_{h_1}$-convex | $A A_{h_2}$-convex | $A_r A_{h_1} A_{h_2}$-convex |
| $G_r A_{h_1}$-convex | $A G_{h_2}$-convex | $A_r G_{h_1} A_{h_2}$-convex |
| $H_r A_{h_1}$-convex | $A H_{h_2}$-convex |                    |
| $A_r G_{h_1}$-convex | $A A_{h_2}$-convex | $A_r G_{h_1} A_{h_2}$-convex |
| $G_r G_{h_1}$-convex | $A G_{h_2}$-convex | $G_r G_{h_1} A_{h_2}$-convex |
| $H_r G_{h_1}$-convex | $A H_{h_2}$-convex |                    |
| $A_r H_{h_1}$-convex | $A A_{h_2}$-convex | $A_r H_{h_1} A_{h_2}$-convex |
| $G_r H_{h_1}$-convex | $A G_{h_2}$-convex | $G_r H_{h_1} A_{h_2}$-convex |
| $H_r H_{h_1}$-convex | $A H_{h_2}$-convex | $H_r H_{h_1} A_{h_2}$-convex |
| $A_r A_{h_1}$-convex | $G A_{h_2}$-convex | $A_r A_{h_1} G_{h_2}$-convex |
| $G_r A_{h_1}$-convex | $G G_{h_2}$-convex | $G_r A_{h_1} G_{h_2}$-convex |
| $H_r A_{h_1}$-convex | $G H_{h_2}$-convex |                    |
| $G_r G_{h_1}$-convex | $G G_{h_2}$-convex | $G_r G_{h_1} G_{h_2}$-convex |
| $A_r G_{h_1}$-convex | $G A_{h_2}$-convex | $A_r G_{h_1} G_{h_2}$-convex |
| $H_r G_{h_1}$-convex | $G H_{h_2}$-convex |                    |
| $A_r H_{h_1}$-convex | $G A_{h_2}$-convex | $A_r H_{h_1} G_{h_2}$-convex |
| $G_r H_{h_1}$-convex | $G G_{h_2}$-convex | $G_r H_{h_1} G_{h_2}$-convex |
| $H_r H_{h_1}$-convex | $G H_{h_2}$-convex |                    |
| $A_r A_{h_1}$-convex | $H A_{h_2}$-convex | $A_r A_{h_1} H_{h_2}$-convex |
| $G_r A_{h_1}$-convex | $H G_{h_2}$-convex | $A_r A_{h_1} H_{h_2}$-convex |
| $H_r A_{h_1}$-convex | $H H_{h_2}$-convex |                    |
| $A_r G_{h_1}$-convex | $H A_{h_2}$-convex | $A_r G_{h_1} H_{h_2}$-convex |
| $G_r G_{h_1}$-convex | $H G_{h_2}$-convex | $A_r G_{h_1} H_{h_2}$-convex |
| $H_r G_{h_1}$-convex | $H H_{h_2}$-convex |                    |
| $A_r H_{h_1}$-convex | $H A_{h_2}$-convex | $A_r H_{h_1} H_{h_2}$-convex |
| $G_r H_{h_1}$-convex | $H G_{h_2}$-convex | $A_r H_{h_1} H_{h_2}$-convex |
| $H_r H_{h_1}$-convex | $H H_{h_2}$-convex |                    |
(2) If \( h \) is submultiplicative, and \( f \) is \( A_t \)-convex function, then the following inequality hold:
\[
h(x_3 - x_1) f(x_1) f(x_3) \geq h(x_2 - x_1) f(x_1) f(x_2) + h(x_3 - x_2) f(x_3) f(x_2).
\]

In case of \( A_t \)-concavity the inequalities are reversed.

**Proof**: Let \( x_1, x_2, x_3 \in I \) with \( x_1 < x_2 < x_3 \), such that \( x_2 - x_1, x_3 - x_2 \) and \( x_3 - x_1 \) in \( J \). Consequently, \( \frac{x_2 - x_1}{x_3 - x_1}, \frac{x_3 - x_2}{x_3 - x_1} \in (0,1) \subseteq J \) and \( \frac{x_2 - x_1}{x_3 - x_1}, \frac{x_3 - x_2}{x_3 - x_1} = 1 \).

Also, since \( h \) is super(sub)multiplicative then for all \( p, q \in I \) we have
\[
h(p) = h\left( \frac{p}{q} \cdot q \right) \geq (\leq) h\left( \frac{p}{q} \right) h(q),
\]
and this yield that
\[
\frac{h(p)}{h(q)} \geq (\leq) h\left( \frac{p}{q} \right).
\]

Setting \( t = \frac{x_3 - x_2}{x_3 - x_1}, \alpha = x_1, \beta = x_3 \), therefore we have the following cases:

(1) For \( x_2 = t\alpha + (1-t)\beta \) and since \( f \) is \( A_t \)-convex, then by (2.3)
\[
f(x_2) \leq \left[ f(x_1) \right]^{h(x_2-x_1)} \left[ f(x_3) \right]^{h(x_2-x_3)} \leq \left[ f(x_1) \right]^{h(x_3-x_1)} \left[ f(x_3) \right]^{h(x_2-x_3)}, \tag{3.1}
\]
since \( f \) is positive, then the above inequality equivalent to
\[
h(x_3 - x_1) \log f(x_2) \leq h(x_3 - x_2) \log f(x_1) + h(x_2 - x_1) \log f(x_3).
\]

Rearranging the terms again we get
\[
\left[ f(x_1) \right]^{h(x_3-x_2)} \left[ f(x_3) \right]^{h(x_2-x_1)} \geq \left[ f(x_2) \right]^{h(x_3-x_1)},
\]
as desired.

(2) For \( x_2 = t\alpha + (1-t)\beta \) and since \( f \) is \( A_t \)-convex then by (2.4)
\[
f(x_2) \leq \frac{f(x_1) f(x_3)}{h\left( \frac{x_2 - x_1}{x_3 - x_1} \right) f(x_1) + h\left( \frac{x_3 - x_2}{x_3 - x_1} \right) f(x_3)}
\]
\[
\frac{h(x_3 - x_1) f(x_1) f(x_3)}{h(x_2 - x_1) f(x_1) + h(x_3 - x_2) f(x_3)},
\]  
(3.2)

and this is equivalent to write

\[
h(x_3 - x_1) f(x_1) f(x_3) \geq h(x_2 - x_1) f(x_1) f(x_2) + h(x_3 - x_2) f(x_3) f(x_2),
\]

as desired.

Thus, the proof is completely established.

\[\square\]

**Corollary 4**: Let \( h : (0,1) \rightarrow [0,\infty) \) be a non-negative function and let \( f : (0,1) \rightarrow (0,\infty) \) be a function. For all points \( x_1, x_2, x_3 \in (0,1) \), \( x_1 < x_2 < x_3 \) such that \( x_2 - x_1, x_3 - x_2 \) and \( x_3 - x_1 \) in \( (0,1) \). Let \( h(t) = t^r, r \in (-\infty, -1] \cup [0,1) \).

1. If \( f \) is \( A_tG_h \)-convex function, then the following inequality hold:

\[
\begin{pmatrix}
  f(x_1) \\
  f(x_3)
\end{pmatrix}^{(x_3-x_2)r} \geq
\begin{pmatrix}
  f(x_2)
\end{pmatrix}^{(x_3-x_1)r}.
\]

Furthermore, if \( f(x) = x^\lambda (\lambda < 0) \) we get several Schur type inequalities.

2. If \( f \) is \( A_tH_h \)-convex function, then the following inequality hold:

\[
(x_3 - x_1)^\lambda f(x_1) f(x_3) \geq (x_2 - x_1)^\lambda f(x_1) f(x_2) + (x_3 - x_2)^\lambda f(x_3) f(x_2).
\]

Furthermore, if \( f(x) = x^\lambda (-1 < \lambda < 0) \) we get several Schur type inequalities.

In case of \( A_tN_h \)-concavity the inequalities are reversed.

**Theorem 12**: Let \( h \) : \( I \rightarrow [0, \infty) \) be a non-negative function and let \( f : I \rightarrow \mathbb{R} \) be a function. For all points \( x_1, x_2, x_3 \in I \), \( x_1 < x_2 < x_3 \) such that \( \ln(x_3x_2), \ln(x_2x_1) \) and \( \ln(x_3x_1) \) in \( I \).

1. If \( h \) is supermultiplicative, and \( f \) is \( G_tA_h \)-convex function, then the following inequality hold:

\[
h\left( \ln\left( \frac{x_3}{x_2} \right) \right) \cdot f(x_1) + h\left( \ln\left( \frac{x_2}{x_1} \right) \right) \cdot f(x_3) \geq h\left( \ln\left( \frac{x_3}{x_1} \right) \right) f(x_2).
\]

2. If \( h \) is supermultiplicative, and \( f \) is \( G_tG_h \)-convex function, then the following inequality hold:
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\[ f(x_i) \ln \left( \frac{x_3}{x_i} \right) f(x_3) \geq f(x_2) \ln \left( \frac{x_3}{x_2} \right) f(x_2). \]

(3) If \( h \) is submultiplicative, and \( f \) is \( G_t H_h \)-convex function, then the following inequality hold:

\[
h \left( \ln \left( \frac{x_3}{x_1} \right) \right) f(x_1) f(x_3) + h \left( \ln \left( \frac{x_2}{x_1} \right) \right) f(x_1) f(x_2) + h \left( \ln \left( \frac{x_3}{x_2} \right) \right) f(x_3) f(x_2).
\]

In case of \( G_t N_h \)-concavity the inequalities are reversed.

**Proof:** Let \( x_1, x_2, x_3 \in I \) with \( x_1 < x_2 < x_3 \), such that \( \ln(x_3 x_2), \ln(x_2 x_1) \) and \( \ln(x_3 x_1) \) in \( J \). Consequently, \( \frac{\ln x_2 - \ln x_1}{\ln x_3 - \ln x_1}, \frac{\ln x_3 - \ln x_2}{\ln x_3 - \ln x_1} \in (0, 1) \subseteq I \) and \( \frac{\ln x_2 - \ln x_1}{\ln x_3 - \ln x_1}, \frac{\ln x_3 - \ln x_2}{\ln x_3 - \ln x_1} = 1 \). Setting \( t = \frac{\ln x_3 - \ln x_2}{\ln x_3 - \ln x_1}, \alpha = x_1, \beta = x_3 \), therefore we have the following cases:

1. For \( x_2 = \alpha t^{1-i} \) and since \( f \) is \( G_t A_h \)-convex then by (2.5)

\[
f(x_2) \leq h \left( \frac{\ln(x_3) - \ln(x_2)}{\ln(x_3) - \ln(x_1)} \right) f(x_1) + h \left( \frac{\ln(x_2) - \ln(x_1)}{\ln(x_3) - \ln(x_1)} \right) f(x_3)
\]

and this is equivalent to write

\[
h \left( \ln \left( \frac{x_3}{x_2} \right) \right) f(x_1) + h \left( \ln \left( \frac{x_2}{x_1} \right) \right) f(x_3) \geq h \left( \ln \left( \frac{x_3}{x_1} \right) \right) f(x_2),
\]

as desired.

2. For \( x_2 = \alpha t^{1-i} \) and since \( f \) is \( G_t G_h \)-convex then by (3.4)

\[
f(x_2) \leq \left[ f(x_1) \right]^{h \ln(x_2) - \ln(x_1)} \left[ f(x_3) \right]^{h \ln(x_2) - \ln(x_1)}
\]

\[
\leq \left[ f(x_1) \right]^{h \ln(x_3) - \ln(x_1)} \left[ f(x_3) \right]^{h \ln(x_3) - \ln(x_1)},
\]

(3.4)
since \( f \) is positive therefore

\[
\begin{align*}
\ln\left(\frac{x_3}{x_1}\right) + \ln\left(\frac{x_3}{x_1}\right) \leq \ln\left(\frac{x_3}{x_1}\right) + \ln\left(\frac{x_3}{x_1}\right) \leq \ln\left(\frac{x_3}{x_1}\right) + \ln\left(\frac{x_3}{x_1}\right),
\end{align*}
\]

and this equivalent to write

\[
\begin{align*}
\ln\left(\frac{x_3}{x_1}\right) \ln\left(\frac{x_3}{x_1}\right) & \ln\left(\frac{x_3}{x_1}\right) \ln\left(\frac{x_3}{x_1}\right) \\
\ln\left(\frac{x_3}{x_1}\right) & \ln\left(\frac{x_3}{x_1}\right) \ln\left(\frac{x_3}{x_1}\right) \\
\ln\left(\frac{x_3}{x_1}\right) & \ln\left(\frac{x_3}{x_1}\right) \ln\left(\frac{x_3}{x_1}\right)
\end{align*}
\]

as desired.

(3) For \( x_2 = \alpha \beta^{-t} \) and since \( f \) is \( G_t A_h \)-convex then by (2.7)

\[
\begin{align*}
f(x_2) & \leq \frac{f(x_1)f(x_3)}{h\left(\ln \frac{x_2}{x_1} - \ln \frac{x_3}{x_1}\right) + h\left(\ln \frac{x_2}{x_1} - \ln \frac{x_3}{x_1}\right)} + h\left(\ln \frac{x_2}{x_1} - \ln \frac{x_3}{x_1}\right)f(x_3) \\
& \leq \frac{h\left(\ln \frac{x_2}{x_1} - \ln \frac{x_3}{x_1}\right)f(x_1)f(x_3)}{h\left(\ln x_2 - \ln x_1\right)f(x_1) + h\left(\ln x_2 - \ln x_1\right)f(x_3)}, \quad (3.5)
\end{align*}
\]

which is equivalent to write

\[
\begin{align*}
h\left(\ln \frac{x_3}{x_1}\right)f(x_1)f(x_3) - h\left(\ln \frac{x_3}{x_1}\right)f(x_1)f(x_2) & \geq h\left(\ln \frac{x_3}{x_1}\right)f(x_3)f(x_2),
\end{align*}
\]

as desired.

Thus, the proof is completely established. \( \square \)

**Corollary 5:** Let \( h : (0, 1) \to [0, \infty) \) be a non-negative function and let \( f : (0, 1) \to \mathbb{R} \) be a function. For all points \( x_1, x_2, x_3 \in (0, 1), x_1 < x_2 < x_3 \) such that \( \ln \left(\frac{x_3}{x_2}\right), \ln \left(\frac{x_2}{x_1}\right) \) and \( \ln \left(\frac{x_3}{x_1}\right) \) in \((0, 1)\). For \( h, t = t' \), \( r \in (-\infty, -1] \cup [0, 1] \).

(1) If \( f \) is \( G_t A_h \)-convex function, then the following inequality hold:

\[
\begin{align*}
\ln\left(\frac{x_3}{x_1}\right) + \ln\left(\frac{x_3}{x_1}\right) & \ln\left(\frac{x_3}{x_1}\right) \\
\ln\left(\frac{x_3}{x_1}\right) & \ln\left(\frac{x_3}{x_1}\right) \ln\left(\frac{x_3}{x_1}\right)
\end{align*}
\]

Furthermore, if \( f(x) = x^\lambda \) \((\lambda \in \mathbb{R})\) we get several Schur type inequalities.

(2) If \( f \) is \( G_t G_h \)-convex function, then the following inequality hold:
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\[ \left[ f(x_1) \right]^{\ln \left( \frac{x_3}{x_1} \right)} \left[ f(x_3) \right]^{\ln \left( \frac{x_2}{x_1} \right)} \geq f(x_2) \left[ \frac{x_3}{x_1} \right]^{\ln \left( \frac{x_3}{x_1} \right)} . \]

(3) If \( f \) is \( G_tG_h \)-convex function, then the following inequality hold:

\[ \left( \ln \left( \frac{x_2}{x_1} \right) \right)^{f(x_1)} f(x_3) + \left( \ln \left( \frac{x_2}{x_1} \right) \right)^{f(x_2)} f(x_3) \geq f(x_3) f(x_2). \]

In case of \( G,N_h \)-concavity the inequalities are reversed.

**Theorem 13:** Let \( h : J \to [0, \infty) \) be a non-negative function and let \( f : I \to \mathbb{R} \) be a function. For all points \( x_1, x_2, x_3 \in I, x_1 < x_2 < x_3 \) such that \( x_1(x_3 - x_2), x_3(x_2 - x_1) \) and \( x_2(x_3 - x_1) \) in \( J \),

1. If \( h \) is supermultiplicative, and \( f \) is \( H_A h \)-convex function, then the following inequality hold:

\[ h(x_1(x_3 - x_2)) f(x_1) + h(x_3(x_2 - x_1)) f(x_3) \geq h(x_2(x_3 - x_1)) f(x_2), \]

2. If \( h \) is supermultiplicative, and \( f \) is \( H_G h \)-convex function, then the following inequality hold:

\[ \left( \ln \left( \frac{x_1}{x_1} \right) \right)^{f(x_1)} f(x_3) \geq \left( \ln \left( \frac{x_2}{x_1} \right) \right)^{f(x_2)} f(x_3). \]

3. If \( h \) is submultiplicative, and \( f \) is \( H_H h \)-convex function, then the following inequality hold:

\[ h(x_3(x_2 - x_1)) f(x_1) + h(x_1(x_3 - x_2)) f(x_2) \leq f(x_3) f(x_1), \]

In case of \( H,N_h \)-concavity the inequalities are reversed.

**Proof:** Let \( x_1, x_2, x_3 \in I \) with \( x_1 < x_2 < x_3 \), such that \( x_1(x_3 - x_2), x_3(x_2 - x_1) \),

\[ x_2(x_3 - x_1) \in J. \]

And \( \frac{x_1(x_3 - x_2)}{x_2(x_3 - x_1)} \in (0, 1) \subseteq I \), so that \( \frac{x_1(x_3 - x_2)}{x_2(x_3 - x_1)}, \frac{x_3(x_2 - x_1)}{x_2(x_3 - x_1)} = 1 \). Setting \( t = \frac{x_3(x_2 - x_1)}{x_2(x_3 - x_1)}, a = x_1, \beta = x_3 \), therefore we have the following cases:

1. For \( x_2 = \frac{\alpha \beta}{\alpha + (1-t) \beta} \) and since \( f \) is \( H_A h \)-convex then by (2.8)

\[ f(x_2) \leq h \left( \frac{x_1(x_3 - x_2)}{x_2(x_3 - x_1)} \right) f(x_1) + h \left( \frac{x_3(x_2 - x_1)}{x_2(x_3 - x_1)} \right) f(x_3). \]
which is equivalent to write
\[ h(x_1(x_3-x_2))f(x_1) + h(x_3(x_2-x_1))f(x_3) \geq h(x_2(x_3-x_1))f(x_2), \]
as desired.

(2) For \( x_2 = \frac{\alpha\beta}{t\alpha+(1-t)\beta} \) and since \( f \) is \( H GHC \)-convex then by (2.9)
\[
df(x_2) \leq \left[ f(x_1) \right]^{h(x_1(x_3-x_2))} \left[ f(x_3) \right]^{h(x_3(x_2-x_1))} \leq \left[ f(x_2) \right]^{h(x_2(x_3-x_1))}, \tag{3.7}
\]
and this equivalent to write
\[
\left[ f(x_1) \right]^{h(x_1(x_3-x_2))} \cdot \left[ f(x_3) \right]^{h(x_3(x_2-x_1))} \geq \left[ f(x_2) \right]^{h(x_2(x_3-x_1))},
\]
as desired.

(3) For \( x_2 = \frac{\alpha\beta}{t\alpha+(1-t)\beta} \) and since \( f \) is \( H h HTC \)-convex then by (2.10)
\[
f(x_2) \leq \frac{f(x_1)f(x_3)}{h \left( \frac{x_3(x_2-x_1)}{x_2(x_3-x_1)} \right)f(x_1) + h \left( \frac{x_1(x_3-x_2)}{x_2(x_3-x_1)} \right)f(x_3)} \leq \frac{f(x_1)f(x_3)}{h(x_2(x_3-x_1))f(x_1) + h(x_1(x_3-x_2))f(x_3)}, \tag{3.8}
\]
and this equivalent to write
\[
h(x_3(x_2-x_1))f(x_1)f(x_2) + h(x_1(x_3-x_2))f(x_2)f(x_3) \leq h(x_2(x_3-x_1))f(x_1)f(x_3),
\]
as desired.

Thus, the proof is completely established. \( \square \)

**Corollary 6** : Let \( h: (0,1) \to [0,\infty) \) be a non-negative function and let \( f: (0,1) \to \mathbb{R} \) be a function. For all points \( x_1, x_2, x_3 \in (0,1), x_1 < x_2 < x_3 \)
such that $x_1(x_3 - x_2), x_3(x_2 - x_1)$ and $x_2(x_3 - x_1)$ in $(0, 1)$. For $h_r(t) = t^r$, $r \in (-\infty, -1] \cup [0, 1]$. 

1. If $f$ is $H_{A_h}$-convex function, then the following inequality hold:

$$
\left( x_1 \left( x_3 - x_2 \right) \right)^r f(x_1) + \left( x_3 \left( x_2 - x_1 \right) \right)^r f(x_3) \geq \left( x_2 \left( x_3 - x_1 \right) \right)^r f(x_2).
$$

Furthermore, if $f(x) = x^\lambda$ ($\lambda > 0$) we get several Schur type inequalities.

2. If $f$ is $h_{A_h}$-convex function, then the following inequality hold:

$$
\left[ f(x_1) \right]^{\left( x_1 \left( x_3 - x_2 \right) \right)^r} \left[ f(x_3) \right]^{\left( x_3 \left( x_2 - x_1 \right) \right)^r} \geq \left[ f(x_2) \right]^{\left( x_2 \left( x_3 - x_1 \right) \right)^r}.
$$

3. If $f$ is $H_{A_h}$-convex function, then the following inequality hold:

$$
\left( x_3 \left( x_2 - x_1 \right) \right)^r f(x_1) f(x_2) + \left( x_1 \left( x_3 - x_2 \right) \right)^r f(x_2) f(x_3) \\
\leq \left( x_2 \left( x_3 - x_1 \right) \right)^r f(x_1) f(x_3).
$$

Furthermore, if $f(x) = x^\lambda$ ($1 > \lambda > 0$) we get several Schur type inequalities.

In case of $H_{A_h}$-concavity the inequalities are reversed.

**Remark 5**: In [20], Mitrinović and Pečarić proved the validity of the inequality

$$
\left( x_1 - x_2 \right) \left( x_1 - x_3 \right) f(x_1) + \left( x_2 - x_1 \right) \left( x_2 - x_3 \right) f(x_2) \\
+ \left( x_3 - x_1 \right) \left( x_3 - x_2 \right) f(x_3) \geq 0
$$

for all $x_1, x_2, x_3 \in (0, 1)$ and $f \in Q(I)$. Moreover, if $f(x) = x^\lambda$ ($\lambda \in \mathbb{R}$), then the inequality is of Schur type, see ([21], p.117). A similar inequality for monotone convex functions was proved by Wright in [33]. A generalization to $h$-convex type functions was also presented in [32].

In Corollaries 4–6, if we choose $r = -1$, i.e., $h(x) = x^{-1}$, then several inequalities for $M_{A_h}$-convex functions can be deduced. For inequalities of Schur type choose $f(x) = x^\lambda$ ($\lambda \in \mathbb{R}$), taking into account that some additional assumption on $\lambda$ have to be made to guarantee the $M_{A_h}$-convexity of $f$.

4. **Jensen’s type inequalities**

The weighted Arithmetic, Geometric, and Harmonic Means for $n$-points $x_1, x_2, \cdots, x_n$ ($n \geq 2$) are defined respectively, to be
\[ A(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} t_k x_k \]

\[ G(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} (x_k)^{1/k} \]

\[ H(x_1, x_2, \ldots, x_n) = \frac{1}{A_k \left( \frac{t_1}{x_1}, \frac{t_2}{x_2}, \ldots, \frac{t_n}{x_n} \right)} = \frac{1}{\sum_{k=1}^{n} \frac{t_k}{x_k}} \]

where \( t_k \in [0, 1] \) such that \( \sum_{k=1}^{n} t_k = 1 \) and \((x_1, x_2, \ldots, x_n) \in (0, \infty)^n\). The weighted form of the HM–GM–AM inequality is known as ([23], p. 11):

\[ H(x_1, x_2, \ldots, x_n) \leq G(x_1, x_2, \ldots, x_n) \leq A(x_1, x_2, \ldots, x_n). \]

Let \( w_1, w_2, \ldots, w_n \) be positive real numbers \((n \geq 2)\) and \( h : \mathbb{J} \to \mathbb{R} \) be a non-negative supermultiplicative function. In [32], Varošanec discussed the case that, if \( f \) is a non-negative \( A_+ A_h \)-convex on \( \mathbb{I} \), then for \( x_1, x_2, \ldots, x_n \in \mathbb{I} \) the following inequality holds

\[ f \left( \frac{1}{W_n} \sum_{k=1}^{n} w_k x_k \right) \leq \sum_{k=1}^{n} h \left( \frac{w_k}{W_n} \right) f \left( x_k \right), \]

where \( W_n = \sum_{k=1}^{n} w_k \). If \( h \) is submultiplicative function and \( f \) is an \( h \)-\( A_+ A_h \)-concave then inequality is reversed. A converse result was also given in [32]. For more new results see [12], [13], [19], [25], [27] and [34].

In what follows, Jensen’s type inequalities for \( M_N h \)-convex functions are introduced.

**Theorem 14:** Let \( w_1, w_2, \ldots, w_n \) be positive real numbers \((n \geq 2)\), and \( W_n = \sum_{k=1}^{n} w_k \).

1. If \( w_1, w_2, \ldots, w_n \) is a non-negative supermultiplicative function and \( f \) is a non-negative \( A_+ G_h \)-convex on \( \mathbb{I} \), then for \( x_1, x_2, \ldots, x_n \in \mathbb{I} \) the following inequality holds

\[ f \left( \frac{1}{W_n} \sum_{k=1}^{n} w_k x_k \right) \leq \prod_{k=1}^{n} \left[ f \left( x_k \right) \right]^{h \left( \frac{w_k}{W_n} \right)}. \]  \( \quad \text{(4.1)} \)

If \( h \) is submultiplicative function and \( f \) is an \( A_+ G_h \)-concave then inequality is reversed.

2. If \( h \) is a non-negative submultiplicative function and \( f \) is a non-negative \( A_+ H_h \)-convex on \( \mathbb{I} \), then for \( x_1, x_2, \ldots, x_n \in \mathbb{I} \) the following inequality holds
\[
 f\left( \frac{1}{W_n} \sum_{k=1}^{n} w_k x_k \right) \leq \left\{ \sum_{k=1}^{n} h\left( \frac{w_k}{W_n} \right) f\left( x_k \right) \right\}^{-1}.
\]

(4.2)

If \( h \) is supermultiplicative function and \( f \) is an \( A_nH_n \)-concave then inequality is reversed.

**Proof:** Our proof carries by induction. In case \( n = 2 \), the both results hold.

(1) Assume (4.1) holds for \( n - 1 \) and we are going to prove it for \( n \).

\[
 f\left( \frac{1}{W_n} \sum_{k=1}^{n} w_k x_k \right) = f\left( \frac{w_n}{W_n} x_n + \sum_{k=1}^{n-1} \frac{w_k}{W_n} x_k \right) \\
 = f\left( \frac{w_n}{W_n} x_n + \frac{W_{n-1}}{W_n} \sum_{k=1}^{n-1} \frac{w_k}{W_{n-1}} x_k \right) \\
 \leq \left[ f\left( x_n \right)^{w_n/W_n} \right] f\left( \sum_{k=1}^{n-1} \frac{w_k}{W_{n-1}} x_k \right) \\
 \leq \left[ f\left( x_n \right)^{w_n/W_n} \right] \prod_{k=1}^{n} \left[ f\left( x_k \right)^{W_{n-1}/W_n} \right] \\
 = \prod_{k=1}^{n} \left[ f\left( x_k \right)^{w_k/W_n} \right] \\
\]

and this proves the desired result in (4.1).

(2) Assume (4.2) holds for \( n - 1 \) and we are going to prove it for \( n \).

\[
 f\left( \frac{1}{W_n} \sum_{k=1}^{n} w_k x_k \right) = f\left( \frac{w_n}{W_n} x_n + \sum_{k=1}^{n-1} \frac{w_k}{W_n} x_k \right) = f\left( \frac{w_n}{W_n} x_n + \frac{W_{n-1}}{W_n} \sum_{k=1}^{n-1} \frac{w_k}{W_{n-1}} x_k \right) \\
 \leq \frac{1}{h\left( \frac{w_n}{W_n} \right) + h\left( \frac{W_{n-1}}{W_n} \right) f\left( x_n \right) + \sum_{k=1}^{n-1} \frac{w_k}{W_{n-1}} x_k} \\
 \leq \frac{1}{\frac{h\left( \frac{w_n}{W_n} \right)}{f\left( x_n \right)} + h\left( \frac{W_{n-1}}{W_n} \right) \sum_{k=1}^{n-1} \frac{w_k}{W_{n-1}} f\left( x_k \right)}
\]
\[ \leq \frac{1}{h \left( \frac{w_n}{W_n} \right)} + \sum_{k=1}^{n-1} \frac{h \left( \frac{w_k}{W_n} \right)}{f(x_k)} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{h \left( \frac{w_k}{W_n} \right)}{f(x_k)} , \]

which proves the desired result in (4.2).

Hence, by Mathematical Induction both statements are hold for all \( n \geq 2 \), and therefore the proof is completely established. \( \square \)

The corresponding converse versions of Jensen inequality for \( A_G h \)-convex and \( A_H h \)-convex are incorporated in the following theorem.

**Theorem 15:** Let \( w_1, w_2, \ldots, w_n \) be positive real numbers \( (n \geq 2) \), and \((m, M) \subseteq I\).

1. If \( h : (0, \infty) \to (0, \infty) \) is a non-negative supermultiplicative function and \( f \) is positive \( A_G h \)-convex, then for every finite sequence of points \( x_1, \ldots, x_n \in (m, M) \subseteq I \) we have

\[ \prod_{k=1}^{n} f(x_k)^{h \left( \frac{w_k}{W_n} \right)} \leq \prod_{k=1}^{n} \left\{ f(m)^{\frac{M-x_k}{M-M}} w_k^{\frac{M-x_k}{M-M}} \right\} \cdot f(M)^{\frac{x_k}{M-M}} w_k^{\frac{x_k}{M-M}} , \quad (4.3) \]

If \( h \) is submultiplicative function and \( f \) is an \( A_G h \)-concave then inequality is reversed.

2. If \( h : (0, \infty) \to (0, \infty) \) is a non-negative submultiplicative function and \( f \) is positive \( A_H h \)-convex, then for every finite sequence of points \( x_1, \ldots, x_n \in (m, M) \subseteq I \) we have

\[ \left( \sum_{k=1}^{n} h \left( \frac{w_k}{W_n} \right) \right)^{-1} \leq \left( \sum_{k=1}^{n} \frac{h \left( \frac{x_k-m}{M-M} \right) f(m) + h \left( \frac{M-x_k}{M-M} \right) f(M)}{f(m) f(M)} \right) \left( \sum_{k=1}^{n} h \left( \frac{w_k}{W_n} \right) \right)^{-1} , \quad (4.4) \]

If \( h \) is supermultiplicative function and \( f \) is an \( A_H h \)-concave then inequality is reversed.

**Proof:**

1. In (2.3), setting \( m = x_1, x_2 = x_k \) and \( x_3 = M \) we get
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Since \( f \) is positive therefore we have

\[
[f(x_k)]^{\frac{w_k}{W_n}} \leq \left[ f(m) \right]^{\frac{M-x_k}{M-m}} \left[ f(M) \right]^{\frac{x_k-m}{M-m}} h \left( \frac{w_k}{W_n} \right) h \left( \frac{M-x_k}{M-m} \right) h \left( \frac{w_k}{W_n} \right).
\]

Multiplying the above inequality up to \( n \) we get the required results (4.3).

(2) Setting \( m = x_1, x_2 = x_k \) and \( x_3 = M \) in the reverse of (2.4) we get

\[
f(x_k) \leq \frac{f(m) f(M)}{h \left( \frac{x_k-m}{M-m} \right) f(m) + h \left( \frac{M-x_k}{M-m} \right) f(M)}.
\]

Reversing the inequality and then multiplying the above inequality by \( h \left( \frac{w_k}{W_n} \right) \) we get

\[
\frac{h \left( \frac{w_k}{W_n} \right)}{f(x_k)} \geq \frac{h \left( \frac{x_k-m}{M-m} \right) f(m) + h \left( \frac{M-x_k}{M-m} \right) f(M)}{f(m) f(M)} h \left( \frac{w_k}{W_n} \right).
\]

Summing up to \( n \) and then reverse the above inequality, we get the required result in (4.4).

**Theorem 16:** Let \( w_1, w_2, \ldots, w_n \) be positive real numbers \( (n \geq 2) \), and \( W_n = \sum_{k=1}^{n} w_k \).

1. If \( h \) is a non-negative supermultiplicative function and \( f \) is positive \( G_{A_n} \)-convex on \( I \), then for \( x_1, x_2, \ldots, x_n \in I \) the following inequality holds

\[
f \left( \prod_{k=1}^{n} \left( x_k \frac{w_k}{W_n} \right) \right) \leq \sum_{k=1}^{n} h \left( \frac{w_k}{W_n} \right) f(x_k).
\]

If \( h \) is submultiplicative function and \( f \) is an \( G_{A_n} \)-concave then inequality is reversed.
(2) If $h$ is a non-negative supermultiplicative function and $f$ is positive $G_t G_h$-convex on $I$, then for $x_1, x_2, \ldots, x_n \in I$ the following inequality holds

$$f \left( \prod_{k=1}^{n} (x_k)^{w_k/W_n} \right) \leq \prod_{k=1}^{n} \left[ f \left( x_k \right) \right]^{w_k/W_n}. \quad (4.6)$$

If $h$ is submultiplicative function and $f$ is an $G_t G_h$-concave then inequality is reversed.

(3) If $h$ is a non-negative submultiplicative function and $f$ is positive $G_t G_h$-convex on $I$, then for $x_1, x_2, \ldots, x_n \in I$ the following inequality holds

$$f \left( \prod_{k=1}^{n} (x_k)^{w_k/W_n} \right) \leq \frac{1}{\sum_{k=1}^{n} f(x_k)} \cdot \left( \sum_{k=1}^{n} \frac{w_k}{W_n} \right). \quad (4.7)$$

If $h$ is supermultiplicative function and $f$ is an $G_t H_h$-concave then inequality is reversed.

**Proof**: Our proof carries by induction. In case $n = 2$, the results hold by definition.

(1) Assume (4.5) holds for $n - 1$ and we are going to prove it for $n$.

$$f \left( \prod_{k=1}^{n} (x_k)^{w_k/W_n} \right) = f \left( \frac{x_n^{w_n}}{W_n} \cdot \prod_{k=1}^{n-1} (x_k)^{w_k/W_n} \right)$$

$$= f \left( \frac{x_n^{w_n}}{W_n} \cdot \prod_{k=1}^{n-1} \left( x_k^{w_n} \cdot W_n^{-1} \right)^{w_k/W_n} \right)$$

$$\leq h \left( \frac{w_n}{W_n} \right) f(x_n) + h \left( \frac{W_n-1}{W_n} \right) f \left( \sum_{k=1}^{n-1} \frac{w_k}{W_n-1} x_k \right)$$

$$\leq h \left( \frac{w_n}{W_n} \right) f(x_n) + h \left( \frac{W_n-1}{W_n} \right) \sum_{k=1}^{n-1} h \left( \frac{w_k}{W_n-1} \right) f(x_k)$$

$$\leq h \left( \frac{w_n}{W_n} \right) f(x_n) + \sum_{k=1}^{n-1} \left( \frac{w_k}{W_n} \right) f(x_k) = \sum_{k=1}^{n} \left( \frac{w_k}{W_n} \right) f(x_k),$$

which proves the desired result in (4.5).
(2) Assume (4.6) holds for \( n - 1 \) and we are going to prove it for \( n \).

\[
f\left( \prod_{k=1}^{n} \left( x_k \frac{w_k}{w_n} \right) \right) \leq \left[ f\left( x_n \right) \right]^h \left( \frac{w_n}{w_{n-1}} \right) \left[ f\left( \prod_{k=1}^{n-1} \frac{w_k}{w_{n-1}} x_k \right) \right]^h \left( \frac{w_{n-1}}{w_n} \right)
\]

\[
\leq \left[ f\left( x_n \right) \right]^h \left( \frac{w_n}{w_{n-1}} \right) \prod_{k=1}^{n-1} \left( f\left( x_k \right) \right)^h \left( \frac{w_k}{w_{n-1}} \right) \left( \frac{w_{n-1}}{w_n} \right)
\]

\[
\leq \left[ f\left( x_n \right) \right]^h \left( \frac{w_n}{w_{n-1}} \right) \prod_{k=1}^{n-1} \left( f\left( x_k \right) \right)^h \left( \frac{w_k}{w_{n-1}} \right) = \prod_{k=1}^{n} \left[ f\left( x_k \right) \right]^h \left( \frac{w_k}{w_n} \right),
\]

which proves the desired result in (4.6).

(3) Assume (4.7) holds for \( n - 1 \) and we are going to prove it for \( n \).

\[
f\left( \prod_{k=1}^{n} \left( x_k \frac{w_k}{w_n} \right) \right) \leq \frac{1}{h \left( \frac{w_n}{w_{n-1}} \right) f\left( x_n \right) + h \left( \frac{w_{n-1}}{w_n} \right) \sum_{k=1}^{n-1} \frac{w_k}{f\left( x_k \right)}}\]

\[
= \frac{1}{h \left( \frac{w_n}{w_{n-1}} \right) f\left( x_n \right) + h \left( \frac{w_{n-1}}{w_n} \right) \sum_{k=1}^{n} \frac{w_k}{f\left( x_k \right)}}\]

\[
= \frac{1}{h \left( \frac{w_n}{w_{n-1}} \right) f\left( x_n \right) + \sum_{k=1}^{n-1} \frac{h \left( \frac{w_k}{w_{n-1}} \right) \sum_{k=1}^{n-1} \frac{w_k}{f\left( x_k \right)}}{f\left( x_n \right) + \sum_{k=1}^{n} \frac{w_k}{f\left( x_k \right)}} 
\]

which proves the desired result in (4.7).

Hence, by Mathematical Induction both statements are hold for all \( n \geq 2 \), and therefore the proof is completely established.

The corresponding converse versions of Jensen inequality for \( G_tA_n \)-convex, \( G_tG_n \)-convex and \( G_tH_n \)-convex are incorporated in the following theorem.
Theorem 17: Let \( w_1, w_2, \ldots, w_n \) be positive real numbers \( (n \geq 2) \), and \((m, M) \subseteq I\).

1. If \( h : (m, M) \to [m, M] \) is a non-negative supermultiplicative function and \( f \) is positive \( G_A \)-convex, then for every finite sequence of points \( x_1, \ldots, x_n \in (m, M) \) \( (x_k < x_{k+1}) \) we have

\[
\sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f(x_k) \leq \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) \left[ h\left(\frac{\ln(M) - \ln(x_k)}{\ln(M) - \ln(m)}\right) \cdot f(m) + h\left(\frac{\ln(x_k) - \ln(m)}{\ln(M) - \ln(m)}\right) \cdot f(M) \right]. \tag{4.8}
\]

If \( h \) is submultiplicative function and \( f \) is an \( G_A \)-concave then inequality is reversed.

2. If \( h : (0, \infty) \to (0, \infty) \) is a non-negative supermultiplicative function and \( f \) is positive \( G_A \)-convex, then for every finite sequence of points \( x_1, \ldots, x_n \in (m, M) \subseteq I \) we have

\[
\prod_{k=1}^{n} \left[ f(x_k) \right]^{h\left(\frac{w_k}{W_n}\right)} \leq \prod_{k=1}^{n} \left[ f(m) \right]^{h\left(\frac{\ln(M) - \ln(x_k)}{\ln(M) - \ln(m)}\right)} \left[ f(M) \right]^{h\left(\frac{\ln(x_k) - \ln(m)}{\ln(M) - \ln(m)}\right)} . \tag{4.9}
\]

If \( h \) is submultiplicative function and \( f \) is an \( G_A \)-concave then inequality is reversed.

3. If \( h : (0, \infty) \to (0, \infty) \) is a non-negative submultiplicative function and \( f \) is positive \( G_A \)-convex, then for every finite sequence of points \( x_1, \ldots, x_n \in (m, M) \subseteq I \) we have

\[
\left( \sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f(x_k) \right)^{-1} \leq \left( \sum_{k=1}^{n} h\left(\frac{\ln(x_k) - \ln(m)}{\ln(M) - \ln(m)}\right) \cdot f(M) \right) \left( \sum_{k=1}^{n} h\left(\frac{\ln(m) - \ln(x_k)}{\ln(M) - \ln(m)}\right) \cdot f(m) \right)^{-1} . \tag{4.10}
\]
If $h$ is supermultiplicative function and $f$ is a $G_{tH_{h}}$-concave then inequality is reversed.

**Proof:**

1. In (2.5), setting $m = x_{1}, x_{2} = x_{k}$ and $x_{3} = M$ we get

$$f(x_{k}) \leq h\left(\frac{\ln(M) - \ln(x_{k})}{\ln(M) - \ln(m)}\right) \cdot f(m) + h\left(\frac{\ln(x_{k}) - \ln(m)}{\ln(M) - \ln(m)}\right) \cdot f(M)$$

Multiplying the above inequality by $h\left(\frac{\omega_{k}}{W_{n}}\right)$ and summing up to $n$ we get the required results in (4.8).

2. Setting $m = x_{1}, x_{2} = x_{k}$ and $x_{3} = M$ in (2.6) we get

$$f(x_{k}) \leq \left[f(m)\right]^{\frac{\ln(M) - \ln(x_{k})}{\ln(M) - \ln(m)}} \cdot \left[f(M)\right]^{\frac{\ln(x_{k}) - \ln(m)}{\ln(M) - \ln(m)}} \cdot f(m) \cdot f(M)$$

Since $f$ is positive, the above inequality implies that

$$\left[f(x_{k})\right]^{\frac{\omega_{k}}{W_{n}}} \leq \left[f(m)\right]^{\frac{\ln(M) - \ln(x_{k})}{\ln(M) - \ln(m)}} \cdot \left[f(M)\right]^{\frac{\ln(x_{k}) - \ln(m)}{\ln(M) - \ln(m)}} \cdot \left[f(m)\right]^{\frac{\omega_{k}}{W_{n}}} \cdot f(m) \cdot f(M)$$

Multiplying the above inequality up to $n$ we get the required result in (4.9).

3. Since $f$ is $G_{tH_{h}}$-convex, then (2.7) holds.

$$f(x_{k}) \leq h\left(\frac{\ln x_{k} - \ln m}{\ln M - \ln m}\right) f(m) + h\left(\frac{\ln M - \ln x_{k}}{\ln M - \ln m}\right) f(M)$$

Reversing the order in the inequality we get

$$\frac{1}{f(x_{k})} \geq \frac{h\left(\frac{\ln x_{k} - \ln m}{\ln M - \ln m}\right) f(m) + h\left(\frac{\ln M - \ln x_{k}}{\ln M - \ln m}\right) f(M)}{f(m) f(M)}$$

Multiplying both sides by $h\left(\frac{\omega_{k}}{W_{n}}\right)$ and summing up to $n$ we get

$$\sum_{k=1}^{n} \frac{h\left(\frac{\omega_{k}}{W_{n}}\right)}{f(x_{k})} \geq \sum_{k=1}^{n} \frac{h\left(\frac{\ln x_{k} - \ln m}{\ln M - \ln m}\right) f(m) + h\left(\frac{\ln M - \ln x_{k}}{\ln M - \ln m}\right) f(M) \cdot h\left(\frac{\omega_{k}}{W_{n}}\right)}{f(m) f(M)}$$
Reversing the order in the inequality again we get the required result in (4.10).

**Theorem 18:** Let \( w_1, w_2, \ldots, w_n \) be positive real numbers \((n \geq 2)\), and \( W_n = \sum_{k=1}^{n} w_k \).

1. If \( h \) is a non-negative supermultiplicative function and \( f \) is positive \( H_tA_h \)-convex on \( I \), then for \( x_1, x_2, \ldots, x_n \in I \) the following inequality holds
   \[
   f\left(\frac{1}{W_n} \sum_{k=1}^{n} \frac{w_k}{x_k}\right)^{-1} \leq \sum_{k=1}^{n} h\left(\frac{W_k}{W_n}\right)f(x_k).
   \] (4.11)

2. If \( h \) is submultiplicative function and \( f \) is an \( H_tA_h \)-concave then inequality is reversed.

3. If \( h \) is a non-negative submultiplicative function and \( f \) is positive \( H_tH_h \)-convex on \( I \), then for \( x_1, x_2, \ldots, x_n \in I \) the following inequality holds
   \[
   f\left(\frac{1}{W_n} \sum_{k=1}^{n} \frac{w_k}{x_k}\right)^{-1} \leq \prod_{k=1}^{n} h\left(\frac{W_k}{W_n}\right)^{f(x_k)}.
   \] (4.12)

4. If \( h \) is supermultiplicative function and \( f \) is an \( H_tH_h \)-concave then inequality is reversed.

**Proof:** Our proof carries by induction. In case \( n = 2 \), both results hold.

1. Assume (2.8) holds for \( n - 1 \) and we are going to prove it for \( n \).
Some properties of H-MN-convexity

\[
\begin{align*}
&f\left(\frac{1}{\sum_{k=1}^{n} \frac{w_k}{W_n x_k}}\right) = f\left(\frac{1}{\frac{w_n}{W_n x_n} + \sum_{k=1}^{n-1} \frac{w_k}{W_n x_k}}\right) \\
&= f\left(\frac{w_n}{W_n x_n} + \frac{1}{\sum_{k=1}^{n-1} \frac{w_k}{W_n x_k}}\right) \\
&\leq h\left(\frac{w_n}{W_n}\right) f\left(x_n\right) + h\left(\frac{W_{n-1}}{W_n}\right) f\left(\sum_{k=1}^{n-1} \frac{w_k}{W_n} x_k\right) \\
&\leq h\left(\frac{w_n}{W_n}\right) f\left(x_n\right) + h\left(\frac{W_{n-1}}{W_n}\right) \sum_{k=1}^{n-1} \left(\frac{w_k}{W_n}\right) f\left(x_k\right) \\
&\leq h\left(\frac{w_n}{W_n}\right) f\left(x_n\right) + \sum_{k=1}^{n-1} \left(\frac{w_k}{W_n}\right) f\left(x_k\right),
\end{align*}
\]

which proves the desired result in (4.11).

(2) Assume (2.9) holds for \(n-1\) and we are going to prove it for \(n\).

\[
\begin{align*}
&f\left(\frac{1}{\sum_{k=1}^{n} \frac{w_k}{W_n x_k}}\right) \leq \left[ f\left(x_n\right) \right]^{\frac{w_n}{W_n}} \left[ \prod_{k=1}^{n-1} \left( f\left(x_k\right) \right) \right]^{\frac{w_k}{W_n}} \left\{ \frac{W_{n-1}}{W_n} \right\}^{\frac{w_k}{W_n}} \\
&\leq \left[ f\left(x_n\right) \right]^{\frac{w_n}{W_n}} \left[ \prod_{k=1}^{n-1} \left( f\left(x_k\right) \right) \right]^{\frac{w_k}{W_n}} \left\{ \frac{W_{n-1}}{W_n} \right\}^{\frac{w_k}{W_n}} \\
&\leq \left[ f\left(x_n\right) \right]^{\frac{w_n}{W_n}} \left[ \prod_{k=1}^{n-1} \left( f\left(x_k\right) \right) \right]^{\frac{w_k}{W_n}} \left\{ \frac{W_{n-1}}{W_n} \right\}^{\frac{w_k}{W_n}} \\
&\leq \prod_{k=1}^{n} \left[ f\left(x_k\right) \right]^{\frac{w_k}{W_n}},
\end{align*}
\]

which proves the desired result in (4.12).
(3) Assume (2.10) holds for \( n - 1 \) and we are going to prove it for \( n \).

\[
\frac{1}{\sum_{k=1}^{n} w_k x_k} \leq \frac{1}{h\left(\frac{w_n}{W_n}\right) + \frac{1}{h\left(\frac{W_{n-1}}{W_n}\right)} f(x_n) + \sum_{k=1}^{n-1} \frac{w_k x_k}{W_{n-1}}}
\]

which proves the desired result in (4.13).

Hence, by Mathematical Induction the three statements are hold for all \( n \geq 2 \), and therefore the proof is completely established.

The corresponding converse versions of Jensen inequality for \( H_t A_h \)-convex, \( H_t G_h \)-convex and \( H_t H_h \)-convex are incorporated in the following theorem.

**Theorem 19**: Let \( w_1, w_2, \ldots, w_n \) be positive real numbers \( (n \geq 2) \), and \( (m, M) \subseteq I \).

1. If \( h : (0, \infty) \to (0, \infty) \) is a non-negative supermultiplicative function and \( f \) is positive \( H_t A_h \)-convex, then for every finite sequence of points \( x_1, \ldots, x_n \in (m, M) \subseteq I \) we have

\[
\sum_{k=1}^{n} h\left(\frac{w_k}{W_n}\right) f(x_k) \leq \sum_{k=1}^{n} h\left(\frac{m(M-x_k)}{x_k(M-M)}\right) f(m) + h\left(\frac{M(x_k-m)}{x_k(M-M)}\right) f(M) h\left(\frac{w_k}{W_n}\right).
\]

(4.14)

If \( h \) is submultiplicative function and \( f \) is an \( H_t A_h \)-concave then inequality is reversed.
(2) If \( h : (0, \infty) \to (0, \infty) \) is a non-negative supermultiplicative function and \( f \) is positive \( H_{1}\mathbb{G}_{n} \)-convex, then for every finite sequence of points \( x_{1}, \ldots, x_{n} \in (m, M) \subseteq I \) we have

\[
\prod_{k=1}^{n} \left[ f(x_{k}) \right]^{h \left( \frac{w_{k}}{W_{n}} \right)} \leq \prod_{k=1}^{n} \left[ f(m) \right]^{h \left( \frac{m(M-x_{k})}{x_{k}(M-m)} \right)} \left[ f(M) \right]^{h \left( \frac{M(x_{k}-m)}{x_{k}(M-m)} \right)} \left( \frac{w_{k}}{W_{n}} \right) \right].
\]

(4.15)

If \( h \) is submultiplicative function and \( f \) is an \( H_{1}\mathbb{G}_{n} \)-concave then inequality is reversed.

(3) If \( h : (0, \infty) \to (0, \infty) \) is a non-negative submultiplicative function and \( f \) is positive \( H_{1}H_{n} \)-convex, then for every finite sequence of points \( x_{1}, \ldots, x_{n} \in (m, M) \subseteq I \) we have

\[
\left\{ \sum_{k=1}^{n} \frac{h \left( \frac{w_{k}}{W_{n}} \right)}{f(x_{k})} \right\}^{-1} \leq \left\{ \sum_{k=1}^{n} \frac{h \left( \frac{M(x_{k}-m)}{x_{k}(M-m)} \right) f(m) + h \left( \frac{m(M-x_{k})}{x_{k}(M-m)} \right) f(M)}{f(m) f(M)} \right\}^{-1}.
\]

(4.16)

If \( h \) is supermultiplicative function and \( f \) is an \( H_{1}H_{n} \)-concave then inequality is reversed.

Proof:

(1) In (28), setting \( m = x_{1}, x_{2} = x_{k} \) and \( x_{3} = M \) we get

\[
f(x_{k}) \leq h \left( \frac{m(M-x_{k})}{x_{k}(M-m)} \right) f(m) + h \left( \frac{M(x_{k}-m)}{x_{k}(M-m)} \right) f(M)
\]

Multiplying the above inequality by \( h \left( \frac{w_{k}}{W_{n}} \right) \) and summing up to \( n \) we get the required results in (4.14).

(2) Setting \( m = x_{1}, x_{2} = x_{k} \) and \( x_{3} = M \) in (2.9) we get

\[
f(x_{k}) \leq \left[ f(m) \right]^{h \left( \frac{m(M-x_{k})}{x_{k}(M-m)} \right)} \left[ f(M) \right]^{h \left( \frac{M(x_{k}-m)}{x_{k}(M-m)} \right)}.
\]

Since \( f \) is positive, the above inequality implies that

\[
\left[ f(x_{k}) \right]^{h \left( \frac{w_{k}}{W_{n}} \right)} \leq \left[ f(m) \right]^{h \left( \frac{m(M-x_{k})}{x_{k}(M-m)} \right)} \left( \frac{w_{k}}{W_{n}} \right) \left[ f(M) \right]^{h \left( \frac{M(x_{k}-m)}{x_{k}(M-m)} \right)} \left( \frac{w_{k}}{W_{n}} \right).
\]
Multiplying the above inequality up to \( n \) we get the required result in (4.15).

(3) Setting \( m = x_1, x_2 = x_3 \) and \( x_3 = M \) in (2.10) we get
\[
\frac{f(x_k)}{f(x_1)f(x_3)} \leq \frac{f(x_1)f(x_3)}{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)f(x_1) + h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)f(x_3)}.
\]

Reversing the order in the inequality we get
\[
\frac{1}{f(x_k)} \geq \frac{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)f(x_1) + h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)f(x_3)}{f(x_1)f(x_3)}.
\]

Multiplying both sides by \( h\left(\frac{w_k}{W_n}\right) \) and summing up to \( n \) we get
\[
\sum_{k=1}^{n} \frac{h\left(\frac{w_k}{W_n}\right)}{f(x_k)} \geq \sum_{k=1}^{n} \frac{h\left(\frac{M(x_k-m)}{x_k(M-m)}\right)f(m) + h\left(\frac{m(M-x_k)}{x_k(M-m)}\right)f(M)}{f(m)f(M)} h\left(\frac{w_k}{W_n}\right).
\]

Reversing the order in the inequality again we get the required result in (4.16).

\[\square\]

**Remark 6:** Theorem 22 and Corollary 23 in [32], can be extended to \( M,N_h \)-convexity in similar manner, we omit the details.

**Remark 7:** We note that, in this work, all results are valid for

1. the class \( \mathcal{M}N(h,I) \), whenever \( h(t) = t, t \in [0, 1] \)
2. the class \( Q(I; M, M_0) \), whenever \( h(t) = \frac{1}{t}, t \in (0, 1) \)
3. the class \( P(I; M_0, N_0) \), whenever \( h(t) = 1, t \in (0, 1) \)
4. the class \( K^2(I; M, N_0) \), whenever \( h(t) = t, s \in (0, 1] \) and \( t \in [0, 1] \).

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