HOMOLOGY OF THE STEINBERG VARIETY
AND
WEYL GROUP COINVARIANTS

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Abstract. Let $G$ be a complex, connected, reductive algebraic group with Weyl group $W$ and Steinberg variety $Z$. We show that the graded Borel-Moore homology of $Z$ is isomorphic to the smash product of the coinvariant algebra of $W$ and the group algebra of $W$.

1. Introduction

Suppose $G$ is a complex, reductive algebraic group, $B$ is the variety of Borel subgroups of $G$. Let $g$ be the Lie algebra of $G$ and $N$ the cone of nilpotent elements in $g$. Let $T^*B$ denote the cotangent bundle of $B$. Then there is a moment map, $\mu_0: T^*B \to N$. The Steinberg variety of $G$ is the fibered product $T^*B \times_N T^*B$ which we will identify with the closed subvariety

$$Z = \{ (x, B', B'') \in N \times B \times B \mid x \in \text{Lie}(B') \cap \text{Lie}(B'') \}$$

of $N \times B \times B$. Set $n = \dim B$. Then $Z$ is a $2n$-dimensional, complex algebraic variety.

If $V = \bigoplus_{i \geq 0} V_i$ is a graded vector space, we will frequently denote $V$ by $V_*$. Similarly, if $X$ is a topological space, then $H_i(X)$ denotes the $i$th rational Borel-Moore homology of $X$ and $H_*(X) = \bigoplus_{i \geq 0} H_i(X)$ denotes the total Borel-Moore homology of $X$.

Fix a maximal torus, $T$, of $G$, with Lie algebra $t$, and let $W = N_G(T)/T$ be the Weyl group of $(G, T)$. In [6] Kazhdan and Lusztig defined an action of $W \times W$ on $H_*(Z)$ and they showed that the representation of $W \times W$ on the top-dimensional homology of $Z$, $H_{4n}(Z)$, is equivalent to the two-sided regular representation of $W$. Tanisaki [11] and, more recently, Chriss and Ginzburg [3] have strengthened the connection between $H_*(Z)$ and $W$ by defining a $Q$-algebra structure on $H_*(Z)$ so that $H_i(Z) \ast H_j(Z) \subseteq H_{i+j-4n}(Z)$. Chriss and Ginzburg [3, §3.4] have also given an elementary construction of an isomorphism between $H_{4n}(Z)$ and the group algebra $QW$.

Let $Z_1$ denote the “diagonal” in $Z$:

$$Z_1 = \{ (x, B', B') \in N \times B \times B \mid x \in \text{Lie}(B') \}.$$ 

In this paper we extend the results of Chriss and Ginzburg [3, §3.4] and show in Theorem 2.3 that for any $i$, the convolution product defines an isomorphism $H_i(Z_1) \otimes H_{4n}(Z) \cong H_i(Z)$. It then follows easily that with the convolution product, $H_*(Z)$ is isomorphic to the smash product of the coinvariant algebra of $W$ and the group algebra of $W$.

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Precisely, for $0 \leq i \leq n$ let $\text{Coinv}_2(W)$ denote the degree $i$ subspace of the rational coinvariant algebra of $W$, so $\text{Coinv}_2(W)$ may be identified with the space of degree $i$, $W$-harmonic polynomials on $t$. If $j$ is odd, define $\text{Coinv}_j(W) = 0$. Recall that the smash product, $\text{Coinv}(W)\#\mathbb{Q}W$, is the $\mathbb{Q}$-algebra whose underlying vector space is $\text{Coinv}(W) \otimes_{\mathbb{Q}} \mathbb{Q}W$ with multiplication satisfying $(f_1 \otimes \phi_1) \cdot (f_2 \otimes \phi_2) = f_1\phi_1(f_2) \otimes \phi_1\phi_2$ where $f_1$ and $f_2$ are in $\text{Coinv}(W)$ and $\phi_1$ and $\phi_2$ are in $\mathbb{Q}W$, and $\mathbb{Q}W$ acts on $\text{Coinv}(W)$ in the usual way. The algebra $\text{Coinv}(W)\#\mathbb{Q}W$ is graded by $(\text{Coinv}(W)\#\mathbb{Q}W)_i = \text{Coinv}_i(W)\#\mathbb{Q}W$ and we will denote this graded algebra by $\text{Coinv}_*(W)\#\mathbb{Q}W$. In Theorem 2.5 we construct an explicit isomorphism of graded algebras $H_{4n-\bullet}(Z) \cong \text{Coinv}_*(W)\#\mathbb{Q}W$.

This paper was motivated by the observation, pointed out to the first author by Catharina Stroppel, that the argument in [3, 8.1.5] can be used to show that $H_\bullet(Z)$ is isomorphic to the smash product of $\mathbb{Q}W$ and $\text{Coinv}_*(W)$. The details of such an argument have been carried out in a recent preprint of Namhee Kwon [8]. This argument relies on some deep and technical results: the localization theorem proved by Thomason [12], the bivariant Riemann-Roch Theorem [3, 5.11.11], and the Kazhdan-Lusztig isomorphism between the equivariant $K$-theory of $Z$ and the extended, affine, Hecke algebra [7]. In contrast, and also in the spirit of Kazhdan and Lusztig's original analysis of $H_{4n}(Z)$, and the analysis of $H_{4n}(Z)$ in [3, 3.4], our argument uses more elementary notions and is accessible to readers who are not experts in equivariant $K$-theory and to readers who are not experts in the representation theory of reductive, algebraic groups.

Another approach to the Borel-Moore homology of the Steinberg variety uses intersection homology. Let $\mu : Z \to \mathcal{N}$ be projection on the first factor. Then, as in [3, §8.6], $H_\bullet(Z) \cong \text{Ext}_{D(\mathcal{N})}^{4n-\bullet}(R\mu_*, \mathbb{Q}_\mathcal{N}, R\mu_*, \mathbb{Q}_\mathcal{N})$. The Decomposition Theorem of Beilinson, Bernstein, and Deligne can be used to decompose $R\mu_*, \mathbb{Q}_\mathcal{N}$ into a direct sum of simple perverse sheaves $R\mu_*, \mathbb{Q}_\mathcal{N} \cong \bigoplus_{x,\phi} \text{IC}_{x,\phi}^n$ where $x$ runs over a set of orbit representatives in $\mathcal{N}$, for each $x$, $\phi$ runs over a set of irreducible representations of the component group of $Z_G(x)$, and $\text{IC}_{x,\phi}$ denotes an intersection complex (see [2] or [9, §4.5]). Chriss and Ginzburg have used this construction to describe an isomorphism $H_{4n}(Z) \cong \mathbb{Q}W$ and to in addition give a description of the projective, indecomposable $H_\bullet(Z)$-modules.

It follows from Theorem 2.3 that $H_i(Z) \cong \text{Coinv}_{4n-i}(W) \otimes H_{4n}(Z)$ and so

\begin{equation}
\text{Coinv}_i(W) \otimes H_{4n}(Z) \cong \text{Ext}_{D(\mathcal{N})}^{4n-i}(R\mu_*, \mathbb{Q}_\mathcal{N}, R\mu_*, \mathbb{Q}_\mathcal{N}) \cong \bigoplus_{x,\phi} \bigoplus_{y,\psi} \text{Ext}_{D(\mathcal{N})}^{4n-i}(\text{IC}_{x,\phi}^n, \text{IC}_{y,\psi}^n).
\end{equation}

In the special case when $i = 0$ we have that

\begin{equation}
\text{Coinv}_0(W) \otimes H_{4n}(Z) \cong \text{End}_{D(\mathcal{N})}(R\mu_*, \mathbb{Q}_\mathcal{N}) \cong \bigoplus_{x,\phi} \text{End}_{D(\mathcal{N})}(\text{IC}_{x,\phi}^n).
\end{equation}

The image of the one-dimensional vector space $\text{Coinv}_0(W)$ in $\text{End}_{D(\mathcal{N})}(R\mu_*, \mathbb{Q}_\mathcal{N})$ is the line through the identity endomorphism and $\mathbb{Q}W \cong H_{4n}(Z) \cong \bigoplus_{x,\phi} \text{End}_{D(\mathcal{N})}(\text{IC}_{x,\phi}^n)$ is the Wedderburn decomposition of $\mathbb{Q}W$ as a direct sum of minimal two-sided ideals. For $i < 4n$ we have not been able to find a nice description of the image of $\text{Coinv}_i(W)$ in the right-hand side of (1.1).

The rest of this paper is organized as follows: in §2 we set up our notation and state the main results; in §3 we construct an isomorphism of graded vector spaces between $\text{Coinv}_*(W) \otimes \mathbb{Q}W$ and $H_{4n-\bullet}(Z)$; and in §4 we complete the proof that this isomorphism is
construction used by Chriss and Ginzburg in [3, §X] of $H$ of the main theorems are proved in an appendix. Some very general results about graphs and convolution that we need for the proofs of the main theorems are proved in an appendix.

In this paper $\otimes = \otimes_\mathbb{Q}$, if $X$ is a set, then $\delta_X$, or just $\delta$, will denote the diagonal embedding of $X$ in $X \times X$, and for $g$ in $G$ and $x$ in $\mathfrak{g}$, $g \cdot x$ denotes the adjoint action of $g$ on $x$.

2. Preliminaries and Statement of Results

Fix a Borel subgroup, $B$, of $G$ with $T \subseteq B$ and define $U$ to be the unipotent radical of $B$. We will denote the Lie algebras of $B$ and $U$ by $\mathfrak{b}$ and $\mathfrak{u}$ respectively.

Our proof that $H_\bullet(Z)$ is isomorphic to $\text{Coinv}_\bullet(W)\#\mathbb{Q}W$ makes use of the specialization construction used by Chriss and Ginzburg in [3, §3.4] to establish the isomorphism between $H_{4n}(Z)$ and $\mathbb{Q}W$. We begin by reviewing their construction.

The group $G$ acts diagonally on $B \times B$. Let $O_w$ denote the orbit containing $(B, wBw^{-1})$. Then the rule $w \mapsto O_w$ defines a bijection between $W$ and the set of $G$-orbits in $B \times B$.

Let $\pi_Z: Z \to B \times B$ denote the projection on the second and third factors and for $w$ in $W$ define $Z_w = \pi_Z^{-1}(O_w)$. For $w$ in $W$ we also set $u_w = u \cap w \cdot u$. The following facts are well-known (see [10] and [9, §1.1]):

- $Z_w \cong G \times^{B_{w^{-1}}B} u_w$.
- $\dim Z_w = 2n$.
- The set $\{Z_w \mid w \in W\}$ is the set of irreducible components of $Z$.

Define

$$\tilde{\mathfrak{g}} = \{ (x, B') \in \mathfrak{g} \times B \mid x \in \text{Lie}(B') \},$$

$$\tilde{\mathcal{N}} = \{ (x, B') \in \mathcal{N} \times B \mid x \in \text{Lie}(B') \},$$

and

$$\tilde{\mathcal{Z}} = \{ (x, B', B'') \in \mathfrak{g} \times B \times B \mid x \in \text{Lie}(B') \cap \text{Lie}(B'') \},$$

and let $\mu: \tilde{\mathfrak{g}} \to \mathfrak{g}$ denote the projection on the first factor. Then $\tilde{\mathcal{N}} \cong T^*B$, $\mu(\tilde{\mathcal{N}}) = \mathcal{N}$, $Z \cong \tilde{\mathcal{N}} \times \tilde{\mathcal{N}}$, and $\tilde{\mathcal{Z}} \cong \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$. Let $\pi: \tilde{\mathcal{Z}} \to B \times B$ denote the projection on the second and third factors and for $w$ in $W$ define $\tilde{Z}_w = \pi^{-1}(O_w)$. Then it is well-known that $\dim \tilde{Z}_w = \dim \mathfrak{g}$ and that the closures of the $\tilde{Z}_w$'s for $w$ in $W$ are the irreducible components of $\tilde{\mathcal{Z}}$ (see [9, §1.1]).

Next, for $(x, gBg^{-1})$ in $\tilde{\mathfrak{g}}$, define $\nu(x, gBg^{-1})$ to be the projection of $g^{-1} \cdot x$ in $\mathfrak{t}$. Then $\mu$ and $\nu$ are two of the maps in Grothendieck's simultaneous resolution:

$$\begin{array}{c}
\tilde{g} \\
\mu \\
\nu \\
t \\
t/W
\end{array} \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow
\end{array} \begin{array}{c}
\mathfrak{g} \\
\mathfrak{g} \\
\mathfrak{g}/W
\end{array}$$

It is easily seen that if $\tilde{\mu}: \tilde{\mathcal{Z}} \to \mathfrak{g}$ is the projection on the first factor, then the square

$$\begin{array}{c}
\tilde{\mathcal{Z}} \\
\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}
\end{array} \begin{array}{c}
\tilde{\mu} \\
\mu \times \mu
\end{array} \begin{array}{c}
\rightarrow \\
\rightarrow
\end{array} \begin{array}{c}
\mathfrak{g} \\
\mathfrak{g} \times \mathfrak{g}
\end{array}$$
is cartesian, where the vertical map on the left is given by \((x, B', B'') \mapsto ((x, B'), (x, B''))\). We will frequently identify \(\hat{Z}\) with the subvariety of \(\mathfrak{g} \times \mathfrak{g}\) consisting of all pairs \(((x, B'), (x, B''))\) with \(x\) in \(\text{Lie}(B') \cap \text{Lie}(B'')\).

For \(w\) in \(W\), let \(\Gamma_{w^{-1}} = \{(h, w^{-1} \cdot h) \mid h \in t\} \subseteq t \times t\) denote the graph of the action of \(w^{-1}\) on \(t\) and define

\[
\Lambda_w = \hat{Z} \cap (\nu \times \nu)^{-1}(\Gamma_{w^{-1}}) = \{(x, B', B'') \in \hat{Z} \mid \nu(x, B'') = w^{-1}\nu(x, B')\}.
\]

In the special case when \(w\) is the identity element in \(W\), we will denote \(\Lambda_w\) by \(\Lambda_1\).

The spaces we have defined so far fit into a commutative diagram with cartesian squares:

\[
\begin{array}{ccc}
\Lambda_w & \longrightarrow & \hat{Z} \\
\downarrow & & \downarrow \bar{\mu} \\
(\nu \times \nu)^{-1}(\Gamma_{w^{-1}}) & \longrightarrow & \mathfrak{g} \times \mathfrak{g} \\
\downarrow & & \downarrow \delta_{\mathfrak{g}} \\
\Gamma_{w^{-1}} & \longrightarrow & t \times t
\end{array}
\]

Let \(\nu_w : \Lambda_w \to \Gamma_{w^{-1}}\) denote the composition of the leftmost vertical maps in (2.1), so \(\nu_w\) is the restriction of \(\nu \times \nu\) to \(\Lambda_w\).

For the specialization construction, we consider subsets of \(\hat{Z}\) of the form \(\nu_w^{-1}(S')\) for \(S' \subseteq \Gamma_{w^{-1}}\). Thus, for \(h\) in \(t\) we define \(\Lambda_w^h = \nu_w^{-1}(h, w^{-1}h)\). Notice in particular that \(\Lambda_w^0 = Z\). More generally, for a subset \(S\) of \(t\) we define \(\Lambda_w^S = \bigsqcup_{h \in S} \Lambda_w^h\). Then, \(\Lambda_w^S = \nu_w^{-1}(S')\), where \(S'\) is the graph of \(w^{-1}\) restricted to \(S\).

Let \(t_{\text{reg}}\) denote the set of regular elements in \(t\).

Fix a one-dimensional subspace, \(\ell\), of \(t\) so that \(\ell \cap t_{\text{reg}} = \ell \setminus \{0\}\) and set \(\ell^* = \ell \setminus \{0\}\). Then \(\Lambda_w^\ell = \Lambda_w^{\ell^*} \bigsqcup \Lambda_w^0 = \Lambda_w^\ell \bigsqcup Z\). We will see in Corollary 3.6 that the restriction of \(\nu_w\) to \(\Lambda_w^\ell\) is a locally trivial fibration with fibre \(G/T\). Thus, using a construction due to Fulton and MacPherson ([4, §3.4], [3, §2.6.30]), there is a specialization map

\[
\lim : H_{\bullet+2}(\Lambda_w^\ell) \longrightarrow H_\bullet(Z).
\]

Since \(\Lambda_w^\ell\) is an irreducible, \((2n + 1)\)-dimensional variety, if \([\Lambda_w^\ell]\) denotes the fundamental class of \(\Lambda_w^\ell\), then \(H_{4n+2}(\Lambda_w^\ell)\) is one-dimensional with basis \([\Lambda_w^\ell]\). Define \(\lambda_w = \lim([\Lambda_w^\ell])\) in \(H_{4n}(Z)\). Chriss and Ginzburg [3, §3.4] have proved the following theorem.

**Theorem 2.2.** Consider \(H_\bullet(Z)\) endowed with the convolution product.

(a) For \(0 \leq i, j \leq 4n\), \(H_i(Z) * H_j(Z) \subseteq H_{i+j-4n}(Z)\). In particular, \(H_{4n}(Z)\) is a subalgebra of \(H_\bullet(Z)\).

(b) The element \(\lambda_w\) in \(H_{4n}(Z)\) does not depend on the choice of \(\ell\).

(c) The assignment \(w \mapsto \lambda_w\) extends to an algebra isomorphism \(\alpha : \mathbb{Q}W \xrightarrow{\cong} H_{4n}(Z)\).

Now consider

\[
Z_1 = \{(x, B', B') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \text{Lie}(B')\}.
\]

Then \(Z_1\) may be identified with the diagonal in \(\hat{N} \times \hat{N}\). It follows that \(Z_1\) is closed in \(Z\) and isomorphic to \(\hat{N}\).
Since $\tilde{N} \cong T^*B$, it follows from the Thom isomorphism in Borel-Moore homology [3, §2.6] that $H_{i+2n}(Z_1) \cong H_i(B)$ for all $i$. Since $B$ is smooth and compact, $H_i(B) \cong H^{2n-i}(B)$ by Poincaré duality. Therefore, $H_{4n-i}(Z_1) \cong H^i(B)$ for all $i$.

The cohomology of $B$ is well-understood: there is an isomorphism of graded algebras, $H^\bullet(B) \cong \text{Coinv}_\bullet(W)$. It follows that $H_j(Z) = 0$ if $j$ is odd and $H_{4n-2i}(Z_1) \cong \text{Coinv}_{2i}(W)$ for $0 \leq i \leq n$.

In §3 below we will prove the following theorem.

**Theorem 2.3.** Consider the Borel-Moore homology of the variety $Z_1$.

(a) There is a convolution product on $H_\bullet(Z_1)$. With this product, $H_\bullet(Z_1)$ is a commutative $\mathbb{Q}$-algebra and there is an isomorphism of graded $\mathbb{Q}$-algebras

$$\beta: \text{Coinv}_\bullet(W) \xrightarrow{\cong} H_{4n-\bullet}(Z_1).$$

(b) If $r: Z_1 \rightarrow Z$ denotes the inclusion, then the direct image map in Borel-Moore homology, $r_*: H_\bullet(Z_1) \rightarrow H_\bullet(Z)$, is an injective ring homomorphism.

(c) If we identify $H_\bullet(Z_1)$ with its image in $H_\bullet(Z)$ as in (b), then the linear transformation given by the convolution product

$$H_i(Z_1) \otimes H_{4n}(Z) \xrightarrow{\ast} H_i(Z)$$

is an isomorphism of vector spaces for $0 \leq i \leq 4n$.

The algebra $\text{Coinv}_\bullet(W)$ has a natural action of $W$ by algebra automorphisms, and the isomorphism $\beta$ in Theorem 2.3(a) is in fact an isomorphism of $W$-algebras. The $W$-algebra structure on $H_\bullet(Z_1)$ is described in the next theorem, which will be proved in §4.

**Theorem 2.4.** If $w$ is in $W$ and $H_\bullet(Z_1)$ is identified with its image in $H_\bullet(Z)$, then

$$\lambda_w \ast H_i(Z_1) \ast \lambda_{w^{-1}} = H_i(Z_1).$$

Thus, conjugation by $\lambda_w$ defines a $W$-algebra structure on $H_\bullet(Z_1)$. With this $W$-algebra structure, the isomorphism $\beta: \text{Coinv}_\bullet(W) \xrightarrow{\cong} H_{4n-\bullet}(Z_1)$ in Theorem 2.3(a) is an isomorphism of $W$-algebras.

Recall that $\text{Coinv}(W) \# \mathbb{Q}W$ is graded by $(\text{Coinv}(W) \# \mathbb{Q}W)_i = \text{Coinv}_i(W) \otimes \mathbb{Q}W$. Then combining Theorem 2.2(c), Theorem 2.3(c), and Theorem 2.4 we get our main result.

**Theorem 2.5.** The composition

$$\text{Coinv}_\bullet(W) \# \mathbb{Q}W \xrightarrow{\beta \otimes \alpha} H_{4n-\bullet}(Z_1) \otimes H_{4n}(Z) \xrightarrow{\ast} H_{4n-\bullet}(Z)$$

is an isomorphism of graded $\mathbb{Q}$-algebras.

3. Factorization of $H_\bullet(Z)$

**Proof of Theorem 2.3(a).** We need to prove that $H_\bullet(Z_1)$ is a commutative $\mathbb{Q}$-algebra and that $\text{Coinv}_\bullet(W) \cong H_{4n-\bullet}(Z_1)$.

Let $\pi: \tilde{N} \rightarrow B$ by $\pi(x, B') = B'$. Then $\pi$ may be identified with the vector bundle projection $T^*B \rightarrow B$ and so the induced map in cohomology $\pi^*: H^i(B) \rightarrow H^i(\tilde{N})$ is an
isomorphism. The projection \( \pi \) determines an isomorphism in Borel-Moore homology that we will also denote by \( \pi^* \) (see [3, \$2.6.42]). We have \( \pi^*: H^i(\mathcal{B}) \xrightarrow{\cong} H^i_{2n}(\hat{\mathcal{N}}) \).

For a smooth \( m \)-dimensional variety \( X \), let \( \text{pd}: H^i(X) \to H^i_{2m-i}(X) \) denote the Poincaré duality isomorphism. Then the composition

\[
H^i_{2n-i}(\mathcal{B}) \xrightarrow{\text{pd}^{-1}} H^i(\mathcal{B}) \xrightarrow{\pi^*} H^i(\hat{\mathcal{N}}) \xrightarrow{\text{pd}} H^i_{4n-i}(\hat{\mathcal{N}})
\]

is an isomorphism. It follows from the uniqueness construction in [3, \$2.6.26] that

\[
\text{pd} \circ \pi^* \circ \text{pd}^{-1} = \pi^*: H^i_{2n-i}(\mathcal{B}) \to H^i_{4n-i}(\hat{\mathcal{N}})
\]

and so \( \pi^* \circ \text{pd} = \text{pd} \circ \pi^*: H^i(\mathcal{B}) \to H^i_{4n-i}(\hat{\mathcal{N}}) \).

Recall that \( \text{Coinv}_j(W) = 0 \) if \( j \) is odd and \( \text{Coinv}_2(W) \) is the degree \( i \) subspace of the coinvariant algebra of \( W \). Let \( b_i: \text{Coinv}^i(W) \to H^i(\mathcal{B}) \) be the Borel isomorphism (see [1, §1.5] or [5]). Then with the cup product, \( H^*\mathcal{B}) \) is a graded algebra and \( b_i \) is an isomorphism of graded algebras.

Define \( \beta: \text{Coinv}^i(W) \to H^i_{4n-i}(Z_1) \) to be the composition

\[
\text{Coinv}^i(W) \xrightarrow{b_i} H^i(\mathcal{B}) \xrightarrow{\pi^*} H^i(\hat{\mathcal{N}}) \xrightarrow{\text{pd}} H^i_{4n-i}(\hat{\mathcal{N}}) \xrightarrow{\delta_i} H^i_{4n-i}(Z_1)
\]

where \( \delta = \delta_N \). Then \( \beta \) is an isomorphism of graded vector spaces and

\[
\beta = \delta \circ \text{pd} \circ \pi^* \circ b_i = \delta \circ \pi^* \circ \text{pd} \circ b_i.
\]

The algebra structure of \( H^*\mathcal{B}) \) and \( H^i(\hat{\mathcal{N}}) \) is given by the cup product, and \( \pi^*: H^*\mathcal{B}) \to H^*\mathcal{N}) \) is an isomorphism of graded algebras. Since \( \hat{\mathcal{N}} \) is smooth, as in [3, \$2.6.15], there is an intersection product defined on \( H^*\mathcal{N}) \) using Poincaré duality and the cup product on \( H^*\mathcal{N}) \). Thus, \( \text{pd}: H^*\mathcal{N}) \to H^*_{4n-*}(\hat{\mathcal{N}}) \) is an algebra isomorphism. Finally, it is observed in [3, \$2.7.10] that \( \delta_i: H^*\mathcal{N}) \to H^*_{(Z_1)} \) is a ring homomorphism and hence an algebra isomorphism. This shows that \( \beta \) is an isomorphism of graded algebras and proves Theorem 2.3(a).

**Proof of Theorem 2.3(b).** To prove the remaining parts of Theorem 2.3, we need a linear order on \( W \). Suppose \( |W| = N \). Fix a linear order on \( W \) that extends the Bruhat order. Say \( W = \{w_1, \ldots, w_N\} \), where \( w_1 = 1 \) and \( w_N \) is the longest element in \( W \).

For \( 1 \leq j \leq N \), define \( Z_j = \bigsqcup_{w_j = 1} Z_{w_j} \). Then, for each \( j \), \( Z_j \) is closed in \( Z \), \( Z_{w_j} \) is open in \( Z_j \), and \( Z_j = Z_{j-1} \bigsqcup Z_{w_j} \). Notice that \( Z_N = Z \) and \( Z_1 = Z_{w_1} \).

Similarly, define \( \hat{Z}_j = \bigsqcup_{w_j = 1} \hat{Z}_{w_j} \). Then each \( \hat{Z}_j \) is closed in \( \hat{Z} \), \( \hat{Z}_{w_j} \) is open in \( \hat{Z}_j \), and \( \hat{Z}_j = \hat{Z}_{j-1} \bigsqcup \hat{Z}_{w_j} \).

We need to show that \( r_*: H^*_{(Z_1)} \to H^*_{(Z)} \) is an injective ring homomorphism.

Let \( \text{res}_j: H^*_i(Z_j) \to H^*_i(Z_{w_j}) \) denote the restriction map in Borel-Moore homology induced by the open embedding \( Z_{w_j} \subseteq Z_j \) and let \( r_j: H^*_i(Z_{j-1}) \to H^*_i(Z_j) \) denote the direct image map in Borel-Moore homology induced by the closed embedding \( Z_{j-1} \subseteq Z_j \). Then there is a long exact sequence in homology

\[
\ldots \to H^*_i(Z_{j-1}) \xrightarrow{r_j} H^*_i(Z_j) \xrightarrow{\text{res}_j} H^*_i(Z_{w_j}) \xrightarrow{\partial} H^*_i-1(Z_{j-1}) \to \ldots
\]

It is shown in [3, §6.2] that \( \partial = 0 \) and so the sequence

\[
0 \to H^*_i(Z_{j-1}) \xrightarrow{r_j} H^*_i(Z_j) \xrightarrow{\text{res}_j} H^*_i(Z_{w_j}) \to 0
\]
is exact for every i and j. Therefore, if \( r: Z_j \to Z \) denotes the inclusion, then the direct image \( r_*: H_i(Z_j) \to H_i(Z) \) is an injection for all i. (The fact that r depends on j should not lead to any confusion.)

We will frequently identify \( H_i(Z_j) \) with its image in \( H_i(Z) \) and consider \( H_i(Z_j) \) as a subset of \( H_i(Z) \). Thus, we have a flag of subspaces \( 0 \subseteq H_i(Z_1) \subseteq \cdots \subseteq H_i(Z_{N-1}) \subseteq H_i(Z) \).

In particular, \( r_*: H_i(Z_1) \to H_i(Z) \) is an injection for all i. It follows from [3, Lemma 5.2.23] that \( r_* \) is a ring homomorphism. This proves part (b) of Theorem 2.3.

**Proof of Theorem 2.3(c).** We need to show that the linear transformation given by the convolution product \( H_i(Z_1) \otimes H_{4n}(Z) \to H_i(Z) \) is an isomorphism of vector spaces for \( 0 \leq i \leq 4n \).

The proof is a consequence of the following lemma.

**Lemma 3.2.** The image of the convolution map \( *: H_i(Z_1) \otimes H_{4n}(Z) \to H_i(Z) \) is precisely \( H_i(Z_j) \) for \( 0 \leq i \leq 4n \) and \( 1 \leq j \leq N \).

Assuming that the lemma has been proved, taking \( j = N \), we conclude that the convolution product in \( H_\bullet(Z) \) induces a surjection \( H_i(Z_1) \otimes H_{4n}(Z) \to H_i(Z) \). It is shown in [3, §6.2] that \( \dim H_i(Z) = |W|^2 \) and so \( \dim H_\bullet(Z_1) \otimes H_{4n}(Z) = |W|^2 = \dim H_\bullet(Z) \). Thus, the convolution product induces an isomorphism \( H_i(Z_1) \otimes H_{4n}(Z) \cong H_i(Z) \).

The rest of this section is devoted to the proof of Lemma 3.2.

To prove Lemma 3.2 we need to analyze the specialization map, \( \lim: H_{\bullet + 2}(\Lambda^*_w) \to H_\bullet(Z) \), beginning with the subvarieties \( \Lambda^\ell_w \) and \( \Lambda^\ell_*w \) of \( \Lambda_w \).

**Subvarieties of \( \Lambda_w \).** Suppose that \( \ell \) is a one-dimensional subspace of \( t \) with \( \ell^* = \ell \cap \{0\} = \ell \cap t_{\text{reg}} \). Recall that \( u_w = u \cap w \cdot u \) for \( w \) in \( W \).

**Lemma 3.3.** The variety \( \Lambda^\ell_w \cap \hat{Z}_w \) is the G-saturation in \( \hat{Z} \) of \( \{(h + n, B, wBw^{-1}) \mid h \in \ell, n \in u_w\} \).

**Proof.** By definition,

\[
\Lambda^\ell_w = \Lambda^\ell_*w \coprod \Lambda^0_w = \{(x, B', B'') \in \hat{Z} \mid \nu(x, B'') = w^{-1}\nu(x, B') \in w^{-1}(\ell)\}.
\]

Suppose that \( h \in t_{\text{reg}} \) and \( (x, g_1Bg_1^{-1}, g_2Bg_2^{-1}) \) is in \( \Lambda^\ell_w \). Then \( g_1^{-1} \cdot x = h + n_1 \) and \( g_2^{-1} \cdot x = w^{-1}h + n_2 \) for some \( n_1 \) and \( n_2 \) in \( u_w \). Since \( h \) is regular, there are elements \( u_1 \) and \( u_2 \) in \( U \) so that \( u_1^{-1}g_1^{-1} \cdot h = h \) and \( u_2^{-1}g_2^{-1} \cdot h = w^{-1}h \). Then \( x = g_1u_1 \cdot h = g_2u_2w^{-1} \cdot h \) and so \( g_1u_1 = g_2u_2w^{-1}t \) for some \( t \) in \( T \). Therefore, \( (x, g_1Bg_1^{-1}, g_2Bg_2^{-1}) = g_1u_1 \cdot (h, B, wBw^{-1}) \).

Thus, \( \Lambda^\ell_w \) is contained in the \( G \)-orbit of \( (h, B, wBw^{-1}) \). Since \( \nu \) is \( G \)-equivariant, it follows that \( \Lambda^\ell_w \) is \( G \)-stable and so \( \Lambda^\ell_w \) is the full \( G \)-orbit of \( (h, B, wBw^{-1}) \). Therefore, \( \Lambda^\ell_w \) is the \( G \)-saturation of \( \{(h + n, B, wBw^{-1}) \mid h \in \ell^*, n \in u_w\} \) and \( \Lambda^\ell_w \subset \hat{Z}_w \) for \( h \in \ell^* \).

We have already observed that \( \Lambda^0_w = Z \) and so

\[
\Lambda^\ell_w \cap \hat{Z}_w = \left( \Lambda^\ell_*w \cap \hat{Z}_w \right) \coprod \left( \Lambda^0_w \cap \hat{Z}_w \right) = \Lambda^\ell_*w \coprod \hat{Z}_w.
\]

It is easy to see that \( Z_w \) is the \( G \)-saturation of \( \{(n, B, wBw^{-1}) \mid n \in u_w\} \) in \( Z \). This proves the lemma.

**Corollary 3.4.** The variety \( \Lambda^\ell_w \cap \hat{Z}_w \) is a locally trivial, affine space bundle over \( O_w \) with fibre isomorphic to \( \ell + u_w \), and so there is an isomorphism \( \Lambda^\ell_w \cap \hat{Z}_w \cong G \times^{B\cap wB} (\ell + u_w) \).
Proof. It follows from Lemma 3.3 that the map given by projection on the second and third factors is a $G$-equivariant morphism from $\Lambda^t_w$ onto $O_w$ and that the fibre over $(B, wBw^{-1})$ is $\{ (h + n, B, wBw^{-1}) \mid h \in \ell, n \in u_w \}$. Therefore, $\Lambda^t_w \cong G \times B \times B (\ell + u_w)$. □

Let $g_{rs}$ denote the set of regular semisimple elements in $g$ and define $\tilde{g}_{rs} = \{ (x, B') \in \tilde{g} \mid x \in g_{rs} \}$. For an arbitrary subset $S$ of $t$, define $\tilde{g}^S = \nu^{-1}(S) = \{ (x, B') \in \tilde{g} \mid \nu(x, B') \in S \}$.

For $w$ in $W$, define $\tilde{w}: G/T \times t_{\text{reg}} \rightarrow G/T \times t_{\text{reg}}$ by $\tilde{w}(gT, h) = (gT, w^{-1}h)$. The rule $(gT, h) \mapsto (g \cdot h, gB)$ defines an isomorphism of varieties $f: G/T \times t_{\text{reg}} \rightarrow \tilde{g}_{rs}$ and we will denote the automorphism $f \circ \tilde{w} \circ f^{-1}$ of $\tilde{g}_{rs}$ also by $\tilde{w}$. Notice that if $h$ is in $t_{\text{reg}}$ and $g$ is in $G$, then $\tilde{w}(g \cdot h, gB) = (g \cdot h, gwBw^{-1}g^{-1})$.

**Lemma 3.5.** The variety $\Lambda^e_w$ is the graph of $\tilde{w}|_{G^e}: \tilde{g}^e \rightarrow \tilde{g}^{w^{-1}(e)}$.

**Proof.** It follows from Lemma 3.3 that

$$
\Lambda^e_w = \{ (g \cdot h, gBw^{-1}g^{-1}) \in \tilde{g}_{rs} \times B \times B \mid h \in \ell^e, g \in G \}
= \{ ((g \cdot h, gB^{-1}), (g \cdot h, gwB^{-1}g^{-1})) \in \tilde{g} \times \tilde{g} \mid h \in \ell^e, g \in G \}.
$$

The argument in the proof of Lemma 3.3 shows that $\tilde{g}^e = \{ (g \cdot h, gB^{-1}) \mid h \in \ell^e, g \in G \}$ and by definition $\tilde{w}(g \cdot h, gB) = (g \cdot h, gwB^{-1}g^{-1})$. Therefore, $\Lambda^e_w$ is the graph of $\tilde{w}|_{G^e}$. □

**Corollary 3.6.** The map $\nu_w: \Lambda^e_w \rightarrow \ell^e$ is a locally trivial fibration with fibre isomorphic to $G/T$.

**Proof.** This follows immediately from the lemma and the fact that $\tilde{g}^e \cong G/T \times \ell^e$. □

**The specialization map.** Suppose that $w$ is in $W$ and that $\ell$ is a one-dimensional subspace of $t$ with $\ell^e = \ell \setminus \{0\} = \ell \cap t_{\text{reg}}$. As in [4] and [3, §2.6.30, lim: $H_{i+2}(\Lambda^e_w) \rightarrow H_i(Z)$ is the composition of three maps, defined as follows.

As a vector space over $\mathbb{R}$, $\ell$ is two-dimensional. Fix an $\mathbb{R}$-basis of $\ell$, say $\{v_1, v_2\}$. Define $P$ to be the open half plane $\mathbb{R}_{>0}v_1 \oplus \mathbb{R}v_2$, define $I_{>0}$ to be the ray $\mathbb{R}_{>0}v_1$, and define $I$ to be the closure of $I_{>0}$, so $I = \mathbb{R}_{>0}v_1$.

Since $P$ is an open subset of $\ell^e$, $\Lambda^e_w$ is an open subset of $\Lambda^e_w$ and so there is a restriction map in Borel-Moore homology $\text{res}: H_{i+2}(\Lambda^e_w) \rightarrow H_{i+2}(\Lambda^e_w)$.

The specialization map from $P$ to $I_{>0}$ determines an isomorphism in Borel-Moore homology

$$
\psi: H_{i+2}(\Lambda^e_w) \rightarrow H_{i+1}(\Lambda^e_{\geq 0}).
$$

Since $I = I_{>0} \bigsqcup \{0\}$, we have $\Lambda^I_w = \Lambda^I_{\geq 0} \bigsqcup \Lambda^0_w = \Lambda^I_{\geq 0} \bigsqcup Z$, where $Z$ is closed in $\Lambda^I_w$. The connecting homomorphism of the long exact sequence in Borel-Moore homology arising from the partition $\Lambda^I_w = \Lambda^I_{\geq 0} \bigsqcup Z$ is a map $\partial: H_{i+1}(\Lambda^I_{\geq 0}) \rightarrow H_i(Z)$.

By definition, $\lim = \partial \circ \psi \circ \text{res}$.

Now fix $j$ with $1 \leq j \leq N$ and set $w = w_j$.

Consider the intersection $\Lambda^I_w \cap Z_j = (\Lambda^I_{\geq 0} \cap Z_j) \bigsqcup (Z \cap Z_j)$. Then $Z \cap Z_j$ is closed in $\Lambda^I_w \cap Z_j$ and by construction, $\Lambda^I_{\geq 0} \subset Z_j$ and $Z \cap Z_j = Z_j$. Thus, $\Lambda^I_w \cap Z_j = \Lambda^I_{\geq 0} \bigsqcup Z_j$. Let $\partial_j: H_{i+1}(\Lambda^I_{\geq 0}) \rightarrow H_i(Z_j)$ be the connecting homomorphism of the long exact sequence in Borel-Moore homology arising from this partition. Because the long exact sequence in
Borel-Moore homology is natural, we have a commutative square:

\[
\begin{array}{ccc}
H_{i+1}(\Lambda_{w}^{I>0}) & \xrightarrow{\partial} & H_i(Z) \\
\downarrow & & \uparrow r_* \\
H_{i+1}(\Lambda_{w}^{I=0}) & \xrightarrow{\partial_j} & H_i(Z_j)
\end{array}
\]

This proves the following lemma.

**Lemma 3.7.** Fix \( j \) with \( 1 \leq j \leq N \) and set \( w = w_j \). Then \( \partial : H_{i+1}(\Lambda_w^{I=0}) \rightarrow H_i(Z) \) factors as \( r_* \circ \partial_j \) where \( \partial_j : H_{i+1}(\Lambda_w^{I>0}) \rightarrow H_i(Z_j) \) is the connecting homomorphism of the long exact sequence arising from the partition \( \Lambda_w^{I} \cap \hat{Z} = \Lambda_w^{I>0} \coprod Z_j \).

It follows from the lemma that \( \lim_j H_{i+2}(\Lambda_w^c) \rightarrow H_i(Z) \) factors as

\[
(3.8) \quad H_{i+2}(\Lambda_w^c) \xrightarrow{\text{res}} H_{i+2}(\Lambda_w^P) \xrightarrow{\psi} H_{i+1}(\Lambda_w^{I>0}) \xrightarrow{\partial_j} H_i(Z_j) \xrightarrow{\text{res}_j} H_i(Z).
\]

Define \( \lim_j H_{i+2}(\Lambda_w^c) \rightarrow H_i(Z_j) \) by \( \lim_j = \partial_j \circ \psi \circ \text{res} \).

**Specialization and restriction.** As above, fix \( j \) with \( 1 \leq j \leq N \) and a one-dimensional subspace \( \ell \) of \( t \) with \( \ell^* = \ell \setminus \{0\} = \ell \cap t_{\text{reg}} \). Set \( w = w_j \).

Recall the restriction map \( \text{res}_j : H_i(Z_j) \rightarrow H_i(Z_w) \) from (3.1).

**Lemma 3.9.** The composition \( \text{res}_j \circ \lim_j : H_{i+2}(\Lambda_w^c) \rightarrow H_i(Z_w) \) is surjective for \( 0 \leq i \leq 4n \).

**Proof.** Using (3.8), \( \text{res}_j \circ \lim_j \) factors as

\[
H_{i+2}(\Lambda_w^c) \xrightarrow{\text{res}} H_{i+2}(\Lambda_w^P) \xrightarrow{\psi} H_{i+1}(\Lambda_w^{I>0}) \xrightarrow{\partial_j} H_i(Z_j) \xrightarrow{\text{res}_j} H_i(Z_w).
\]

Lemma 3.11 below shows that \( \text{res} \) is always surjective and the map \( \psi \) is an isomorphism, so we need to show that the composition \( \text{res}_j \circ \partial_j \) is surjective.

Consider \( \Lambda_w^c \cap \hat{Z}_w = (\Lambda_w^I \cap \hat{Z}_j) \cap \hat{Z}_w = \Lambda_w^{I>0} \coprod Z_w \). Then \( \Lambda_w^{I>0} \) is open in \( \Lambda_w^I \cap \hat{Z}_w \) and we have a commutative diagram of long exact sequences

\[
\begin{array}{cccc}
\cdots & \xrightarrow{\partial_w} & H_{i+1}(\Lambda_w^I \cap \hat{Z}_w) & \xrightarrow{\partial_j} & H_i(Z) & \xrightarrow{\text{res}_j} & H_i(Z_w) & \xrightarrow{\partial_w} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \xrightarrow{\partial_j} & H_{i+1}(\Lambda_w^{I>0}) & \xrightarrow{\partial_j} & H_i(Z) & \xrightarrow{\text{res}_j} & H_i(Z_w) & \xrightarrow{\partial_w} & \cdots 
\end{array}
\]

where \( \partial_w \) is the connecting homomorphism of the long exact sequence arising from the partition \( \Lambda_w^I \cap \hat{Z}_w = \Lambda_w^{I>0} \coprod Z_w \). We have seen at the beginning of this section that \( \text{res}_j \) is surjective and so it is enough to show that \( \partial_w \) is surjective.

Recall that \( \{v_1, v_2\} \) is an \( \mathbb{R} \)-basis of \( \ell \) and \( I = \mathbb{R}_{\geq 0}v_1 \). Define

\[
E_I = G \times_{\mathbb{R}_{\geq 0}v_1} \mathbb{R}_{v_1 + u_w}, \\
E_{I>0} = G \times_{\mathbb{R}_{v_1 + u_w}} \mathbb{R}_{I>0}, \text{ and} \\
E_0 = G \times_{\mathbb{R}_{I>0}} u_w.
\]
It follows from Corollary 3.4 that \( E_I \cong \Lambda^I_w \), \( E_{I>0} \cong \Lambda^{I>0}_w \), and \( E_0 \cong Z_w \), so the long exact sequence arising from the partition \( \Lambda^I_w \cap Z_w = \Lambda^{I>0}_w \coprod Z_w \) may be identified with the long exact sequence arising from the partition \( E_I = E_{I>0} \coprod E_0 \):

\[
\cdots \xrightarrow{} H_{i+1}(E_I) \xrightarrow{\partial_E} H_i(E_0) \xrightarrow{} \cdots
\]

Therefore, it is enough to show that \( \partial_E \) is surjective. In fact, we show that \( H_\bullet(E_I) = 0 \) and so \( \partial_E \) is an isomorphism.

Define \( E_\mathbb{R} = G \times B^{w,B} (\mathbb{R} v_1 + u_w) \). Then \( E_\mathbb{R} \) is a smooth, real vector bundle over \( G/B^{w,B} \) and so \( E_\mathbb{R} \) is a smooth manifold containing \( E_I \) as a closed subset. We may apply [3, 2.6.1] and conclude that \( H_i(E_I) \cong H^{4n+1-i}(E_\mathbb{R}, E_\mathbb{R} \setminus E_I) \).

Consider the cohomology long exact sequence of the pair \( (E_\mathbb{R}, E_\mathbb{R} \setminus E_I) \). Since \( E_\mathbb{R} \) is a vector bundle over \( G/B^{w,B} \), it is homotopy equivalent to \( G/B^{w,B} \). Similarly, \( E_\mathbb{R} \setminus E_I \cong G \times (B^{w,B}) (\mathbb{R} v_1 + u_w) \) and so is also homotopy equivalent to \( G/B^{w,B} \). Therefore, \( H^i(E_\mathbb{R}) \cong H^i(E_\mathbb{R} \setminus E_I) \) and it follows that the relative cohomology group \( H^i(E_\mathbb{R}, E_\mathbb{R} \setminus E_I) \) is trivial for every \( i \). Therefore, \( H_\bullet(E_I) = 0 \), as claimed.

This completes the proof of the lemma. \( \square \)

**Corollary 3.10.** The specialization map \( \lim_1 : H_{i+2}(\Lambda^I_1) \longrightarrow H_i(Z_1) \) is surjective for \( 0 \leq i \leq 4n \).

**Proof.** This follows from Lemma 3.9, because \( Z_1 = Z_{w_1} \) and so \( \text{res}_1 \) is the identity map. \( \square \)

The next lemma is true for any specialization map.

**Lemma 3.11.** The restriction map \( \text{res} : H_{i+2}(\Lambda^I_w) \longrightarrow H_{i+2}(\Lambda^P_w) \) is surjective for every \( w \) in \( W \) and every \( i \geq 0 \).

**Proof.** There are homeomorphisms \( \Lambda^I_w \cong G/T \times \ell^I \) and \( \Lambda^P_w \cong G/T \times P \). By definition, \( P \) is an open subset of \( \ell^I \) and so there is a restriction map \( \text{res} : H_2(\ell^I) \rightarrow H_2(P) \). This map is a non-zero linear transformation between one-dimensional \( \mathbb{Q} \)-vector spaces so it is an isomorphism.

Using the Künneth formula we get a commutative square where the horizontal maps are isomorphisms and the right-hand vertical map is surjective:

\[
\begin{array}{ccc}
H_{i+2}(\Lambda^I_w) & \cong & H_i(G/T) \otimes H_2(\ell^I) + H_{i+1}(G/T) \otimes H_1(\ell^I) \\
\downarrow \text{res} & & \downarrow \text{id} \otimes \text{res} + 0 \\
H_{i+2}(\Lambda^P_w) & \cong & H_i(G/T) \otimes H_2(P)
\end{array}
\]

It follows that \( \text{res} : H_{i+2}(\Lambda^I_w) \rightarrow H_{i+2}(\Lambda^P_w) \) is surjective. \( \square \)

**Proof of Lemma 3.2.** Fix \( i \) with \( 0 \leq i \leq 4n \). We show that the image of the convolution map \( * : H_i(Z_j) \otimes H_{4n}(Z_j) \longrightarrow H_i(Z) \) is precisely \( H_i(Z_j) \) for \( 1 \leq j \leq N \) using induction on \( j \).

For \( j = 1 \), \( H_{4n}(Z_1) \) is one-dimensional with basis \( \{ \lambda_1 \} \). It follows from Theorem 2.2(c) that \( \lambda_1 \) is the identity in \( H_\bullet(Z) \) and so clearly the image of the convolution map \( H_i(Z_1) \otimes H_{4n}(Z_1) \longrightarrow H_i(Z) \) is precisely \( H_i(Z_1) \).
Assume that \( j > 1 \) and set \( w = w_j \). We will complete the proof using a commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_i(Z_1) \otimes H_{4n}(Z_{j-1}) & \stackrel{id \otimes (r_{1j})}{\rightarrow} & H_i(Z_1) \otimes H_{4n}(Z_j) & \stackrel{id \otimes \text{res}_j}{\rightarrow} & H_i(Z_1) \otimes H_{4n}(Z_w) & \rightarrow & 0 \\
& & \downarrow {\ast} & & \downarrow {\ast} & & \downarrow {\ast} & & \\
0 & \rightarrow & H_i(Z_{j-1}) & \rightarrow & H_i(Z_j) & \rightarrow & H_i(Z_w) & \rightarrow & 0
\end{array}
\]

and the Five Lemma. We saw in (3.1) that the bottom row is exact and it follows that the top row is also exact. By induction, the convolution product in \( H_\bullet(Z) \) determines a surjective map \( \ast : H_i(Z_1) \otimes H_{4n}(Z) \rightarrow H_i(Z_{j-1}) \). To conclude from the Five Lemma that the middle vertical map is a surjection, it remains to define the other vertical maps so that the diagram commutes and to show that the right-hand vertical map is a surjection.

First we show that the image of the map \( H_i(Z_1) \otimes H_{4n}(Z_j) \rightarrow H_i(Z_j) \) determined by the convolution product in \( H_\bullet(Z) \) is contained in \( H_i(Z_j) \). It then follows that the middle vertical map in (3.12) is defined and so by exactness there is an induced map from \( H_i(Z_1) \otimes H_{4n}(Z_w) \) to \( H_i(Z_w) \) so that the diagram (3.12) commutes. Second we show that the right-hand vertical map is a surjection.

By Lemma 3.5, \( \Lambda^\ell_1 \) is the graph of the identity map of \( \tilde{g}^\ell \), and \( \Lambda^\ell_w \) is the graph of \( \bar{w}|_{\tilde{g}^\ell} \). Therefore, \( \Lambda^\ell_1 \circ \Lambda^\ell_w = \Lambda^\ell_w \) and there is a convolution product

\[
H_{i+2}(\Lambda^\ell_1) \otimes H_{4n+2}(\Lambda^\ell_w) \rightarrow H_{i+2}(\Lambda^\ell_w).
\]

Suppose \( a \) is in \( H_i(Z_1) \). Then by Corollary 3.10, \( a = \lim_1(a_1) \) for some \( a_1 \) in \( H_{i+2}(\Lambda^\ell_1) \). It is shown in [3, Proposition 2.7.23] that specialization commutes with convolution, so \( \lim(a_1 \ast [\Lambda^\ell_w]) = \lim(a_1) \ast \lim([\Lambda^\ell_w]) = a \ast \lambda_w \). Also, \( a_1 \ast [\Lambda^\ell_w] \) is in \( H_{i+2}(\Lambda^\ell_w) \) and \( \lim = r_* \circ \lim_j \) and so \( a \ast \lambda_w = r_* \circ \lim_j (a_1 \ast [\Lambda^\ell_w]) = \text{in } H_i(Z_j) \). By induction, if \( k < j \), then \( a \ast \lambda_{w_k} \) is in \( H_i(Z_k) \) and so \( a \ast \lambda_{w_j} \) is in \( H_i(Z_j) \). Since the set \( \{ \lambda_{w_k} \mid 1 \leq k \leq j \} \) is a basis of \( H_{4n}(Z_j) \), it follows that \( a \ast H_{4n}(Z_j) \subseteq H_{i+2}(\Lambda^\ell_w) \). Therefore, the image of the convolution map \( H_i(Z_1) \otimes H_{4n}(Z_j) \rightarrow H_{i+2}(\Lambda^\ell_w) \) is contained in \( H_i(Z_j) \).

To complete the proof of Lemma 3.2, we need to show that the induced map from \( H_i(Z_1) \otimes H_{4n}(Z_w) \) to \( H_i(Z_w) \) is surjective.

Consider the following diagram:

\[
\begin{array}{ccccccccc}
H_{i+2}(\Lambda^\ell_1) \otimes H_{4n+2}(\Lambda^\ell_w) & \rightarrow & H_{i+2}(\Lambda^\ell_w) \\
\downarrow {\lim_1 \otimes \lim_j} & & \downarrow {\lim_j} \\
H_i(Z_1) \otimes H_{4n}(Z_j) & \rightarrow & H_i(Z_j) \\
\downarrow {id \otimes \text{res}_j} & & \downarrow \text{res}_j \\
H_i(Z_1) \otimes H_{4n}(Z_w) & \rightarrow & H_i(Z_w)
\end{array}
\]

We have seen that the bottom square is commutative. It follows from the fact that specialization commutes with convolution that the top square is also commutative. It is shown in Proposition A.2 that the convolution product \( H_{i+2}(\Lambda^\ell_1) \otimes H_{4n+2}(\Lambda^\ell_w) \rightarrow H_{i+2}(\Lambda^\ell_w) \) is an injection. Since \( H_{i+2}(\Lambda^\ell_1) \) is finite-dimensional and \( H_{4n+2}(\Lambda^\ell_w) \) is one-dimensional, it follows that this convolution mapping is an isomorphism. Also, we saw in Lemma 3.9 that \( \text{res}_j \circ \lim_j \) is surjective. Therefore, the composition \( \text{res}_j \circ \lim_j \circ \ast \) is surjective and it follows that the
bottom convolution map $H_i(Z_1) \otimes H_{4n}(Z_w) \rightarrow H_i(Z_w)$ is also surjective. This completes the proof of Lemma 3.2.

4. Smash Product Structure

In this section we prove Theorem 2.4. We need to show that $\lambda_w \ast H_i(Z_1) \ast \lambda_{w^{-1}} = H_i(Z_1)$ and that $\beta$: Coinv.$\bullet(W) \xrightarrow{\cong} H_{4n-\bullet}(Z_1)$ is an isomorphism of $W$-algebras.

Suppose that $\ell$ is a one-dimensional subspace of $t$ so that $\ell^* = \ell \setminus \{0\} = \ell \cap t_{\text{reg}}$. Recall that for $S \subseteq t$, $\tilde{g}^S = \nu^{-1}(S)$. By Lemma 3.5, if $w$ is in $W$, then $\Lambda_w^\ell$ is the graph of the restriction of $\tilde{w}$ to $\tilde{g}^\ell$. It follows that there is a convolution product

$$H_{4n+2}(\Lambda_w^\ell) \otimes H_{i+2}(\Lambda_1^{w^{-1}(\ell^*)}) \otimes H_{4n+2}(\Lambda_1^{w^{-1}(\ell^*)}) \rightarrow H_{i+2}(\Lambda_1^w).$$

Because specialization commutes with convolution, the diagram

$$
\begin{array}{c}
H_{4n+2}(\Lambda_w^\ell) \otimes H_{i+2}(\Lambda_1^{w^{-1}(\ell^*)}) \otimes H_{4n+2}(\Lambda_1^{w^{-1}(\ell^*)}) \\
\downarrow \text{lim} \otimes \downarrow \text{lim} \\
H_{4n}(Z) \otimes H_i(Z_1) \otimes H_{4n}(Z) \rightarrow H_i(Z)
\end{array}
$$

commutes.

We saw in Corollary 3.10 that $\text{lim}_1: H_{i+2}(\Lambda_1^\ell) \rightarrow H_i(Z_1)$ is surjective. Thus, if $c$ is in $H_i(Z_1)$, then $c = \text{lim}(c_1)$ for some $c_1$ in $H_{i+2}(\Lambda_1^{w^{-1}(\ell^*)})$. Therefore,

$$\lambda_w \ast c \ast \lambda_{w^{-1}} = \text{lim}([\Lambda_w^\ell]) \ast \text{lim}(c_1) \ast \text{lim}([\Lambda_1^{w^{-1}(\ell^*)}]) = \text{lim} \left( [\Lambda_w^\ell] \ast c_1 \ast [\Lambda_1^{w^{-1}(\ell^*)}] \right).$$

Since $\Lambda_w^\ell$ and $\Lambda_1^{w^{-1}(\ell^*)}$ are the graphs of $\tilde{w}$ and $\tilde{w}^{-1}$ respectively, and $\Lambda_1^{w^{-1}(\ell^*)}$ is the graph of the identity function, it follows that $[\Lambda_w^\ell] \ast c_1 \ast [\Lambda_1^{w^{-1}(\ell^*)}]$ is in $H_{i+2}(\Lambda_1^{w^{-1}(\ell^*)})$ and so by (3.8), $\lambda_w \ast c \ast \lambda_{w^{-1}}$ is in $H_i(Z_1)$. This shows that $\lambda_w \ast H_i(Z_1) \ast \lambda_{w^{-1}} = H_i(Z_1)$ for all $i$.

To complete the proof of Theorem 2.4 we need to show that if $w$ is in $W$ and $f$ is in Coinv$_i(W)$, then $\beta(w \cdot f) = \lambda_w \ast \beta(f) \ast \lambda_{w^{-1}}$ where $w \cdot f$ denotes the natural action of $w$ on $f$. To do this, we need some preliminary results.

First, since $\Lambda_1^\ell$ is the diagonal in $\tilde{g}^\ell \times \tilde{g}^\ell$, it is obvious that

$$\delta \circ \tilde{w}^{-1} = (\tilde{w}^{-1} \times \tilde{w}^{-1}) \circ \delta: \tilde{g}^{w^{-1}(\ell^*)} \rightarrow \Lambda_1^\ell.$$

Therefore,

$$\delta_i \circ \tilde{w}_i^{-1} = (\tilde{w}_i^{-1} \times \tilde{w}_i^{-1})_i \circ \delta_i: \tilde{H}_i(\tilde{g}^{w^{-1}(\ell^*)}) \rightarrow H_i(\Lambda_1^\ell)$$

for all $i$. (The first $\delta$ in (4.1) is the diagonal embedding $\tilde{g}^\ell \cong \Lambda_1^\ell$ and the second $\delta$ is the diagonal embedding $\tilde{g}^{w^{-1}(\ell^*)} \cong \Lambda_1^{w^{-1}(\ell^*)}$.)

Next, with $\ell \subseteq t$ as above, $\tilde{g}^\ell = \tilde{g}^\ell \sqcup \nu^{-1}(0) = \tilde{g}^\ell \sqcup \tilde{N}$ and the restriction of $\nu: \tilde{g}^\ell \rightarrow \ell$ to $\tilde{g}^\ell$ is a locally trivial fibration. Therefore, there is a specialization map $\text{lim}_i: H_{i+2}(\tilde{g}^\ell) \rightarrow H_i(\tilde{N})$. Since $\delta_i: H_{i+2}(\tilde{g}^\ell) \rightarrow H_{i+2}(\Lambda_1^\ell)$ and $\delta_i: H_i(Z) \rightarrow H_i(Z_1)$ are isomorphisms, the next lemma is obvious.
Lemma 4.2. Suppose that $\ell$ is a one-dimensional subspace of $t$ so that $\ell^* = \ell \setminus \{0\} \subseteq t_{\text{reg}}$. Then the diagram

$$
\begin{array}{c}
H_{i+2}(\mathfrak{g}^*) \quad \xrightarrow{\delta_*} \quad H_{i+2}(A_1^*) \\
\lim_{t_0} \quad \downarrow \quad \lim_{t_1} \\
H_i(Z) \quad \xrightarrow{\delta_*} \quad H_i(Z_1)
\end{array}
$$

commutes.

Finally, $\tilde{N} \times_{\mathcal{N}} \mathcal{N} = \tilde{N}$ and so $Z \circ \tilde{N} = (\tilde{N} \times_{\mathcal{N}} \mathcal{N}) \circ (\tilde{N} \times_{\mathcal{N}} \mathcal{N}) = \tilde{N} \times_{\mathcal{N}} \mathcal{N}$. Thus, there is a convolution action, $H_{4n}(Z) \otimes H_i(\tilde{N}) \rightarrow H_i(\tilde{N})$, of $H_{4n}(Z)$ on $H_i(\tilde{N})$.

Suppose that $w$ is in $W$ and $z$ is in $H^i(\mathcal{B})$. Then $\pi^* \circ \text{pd}(z)$ is in $H_{4n-i}(\tilde{N})$ and so $\lambda_w \ast (\pi^* \circ \text{pd}(z))$ is in $H_{4n-i}(\tilde{N})$. It is shown in [3, Proposition 7.3.31] that for $y$ in $H_\bullet(\mathcal{B})$, $\lambda_w \ast \pi^*(y) = \epsilon_w \pi^*(w \cdot y)$ where $\epsilon_w$ is the sign of $w$ and $w \cdot y$ denotes the action of $W$ on $H_\bullet(\mathcal{B})$ coming from the action of $W$ on $G/T$ and the homotopy equivalence $G/T \simeq \mathcal{B}$. It is also shown in [3, Proposition 7.3.31] that $\text{pd}(w \cdot z) = \epsilon_w w \cdot \text{pd}(z)$. Therefore,

$$
\lambda_w \ast (\pi^* \circ \text{pd}(z)) = \epsilon_w \pi^*(w \cdot \text{pd}(z)) = \epsilon_w \epsilon_w \pi^* \circ \text{pd}(w \cdot z) = \pi^* \circ \text{pd}(w \cdot z).
$$

This proves the next lemma.

Lemma 4.3. If $w$ is in $W$ and $z$ is in $H_i(\mathcal{B})$, then $\lambda_w \ast (\pi^* \circ \text{pd}(z)) = \pi^* \circ \text{pd}(w \cdot z)$.

Proof of Theorem 2.4. Fix $w$ in $W$ and $f$ in $\text{Coinv}_i(W)$. Using the fact that $\beta = \delta_* \circ \pi^* \circ \text{pd} \circ \text{bi}$ we compute

$$
\lambda_w \ast \beta(f) \ast \lambda_{w^{-1}} = \lim_{t_0} \left( [\Lambda_1^\ell] \ast \lim_{t_1}^{-1}(\beta(f)) \ast [\Lambda_{w^{-1}}^{w^{-1}}(\ell^*)] \right)
$$

$[3, 2.7.23]$

$$
= \lim_{t_1} \circ (\tilde{\omega}_1^{-1} \times \tilde{\omega}_1^{-1}) \circ \lim_{t_1}^{-1} \circ \beta(f)
$$

$\text{Proposition A.3}$

$$
= \lim_{t_1} \circ \delta_* \circ \tilde{\omega}_1^{-1} \circ \delta_{w^{-1}}^{-1} \circ \lim_{t_1}^{-1} \circ \beta(f)
$$

$\text{(4.1)}$

$$
= \delta_* \circ \lim_{t_0} \circ \tilde{\omega}_1^{-1} \circ \delta_{w^{-1}}^{-1} \circ \lim_{t_1}^{-1} \circ \delta_* \circ \delta_{w^{-1}}^{-1} \circ \beta(f)
$$

$\text{Lemma 4.2}$

$$
= \delta_* \circ \lim_{t_0} \circ \tilde{\omega}_1^{-1} \circ \lim_{t_0}^{-1} \circ \delta_{w^{-1}}^{-1} \circ \beta(f)
$$

$\text{Lemma 4.2}$

$$
= \delta_* \circ \lim_{t_0} \circ \tilde{\omega}_1^{-1} \circ \lim_{t_0}^{-1} \circ \pi^* \circ \text{pd} \circ \text{bi}(f)
$$

$$
= \delta_* \circ \lim_{t_0} \left( (\lim_{t_0}^{-1} \circ \pi^* \circ \text{pd} \circ \text{bi}(f)) \ast [\Lambda_{w^{-1}}^{w^{-1}}(\ell^*)] \right)
$$

$[3, 2.7.11]$

$$
= \delta_* ((\pi^* \circ \text{pd} \circ \text{bi}(f)) \ast \lambda_{w^{-1}})
$$

$[3, 2.7.23]$

$$
= \delta_* (\lambda_w \ast (\pi^* \circ \text{pd} \circ \text{bi}(f)))
$$

$\text{Lemma A.1 and [3, 3.6.11]}$

$$
= \delta_* \circ \pi^* \circ \text{pd}(w \cdot \text{bi}(f))
$$

$\text{Lemma 4.3}$

$$
= \delta_* \circ \pi^* \circ \text{pd} \circ \text{bi}(w \cdot f)
$$

$\text{bi is } W\text{-equivariant}$

$$
= \beta(w \cdot f).
$$

This completes the proof of Theorem 2.4.

Appendix A. Convolution and Graphs

In this appendix we prove some general properties of convolution and graphs.
Suppose $M_1$, $M_2$, and $M_3$ are smooth varieties, $\dim M_2 = d$, and that $Z_{1,2} \subseteq M_1 \times M_2$ and $Z_{2,3} \subseteq M_2 \times M_3$ are two closed subvarieties so that the convolution product,

$$H_i(Z_{1,2}) \otimes H_j(Z_{2,3}) \rightarrow H_{i+j-2d}(Z_{1,2} \circ Z_{2,3}),$$

in $[3, \S 2.7.5]$ is defined. For $1 \leq i,j \leq 3$, let $\tau_{i,j}: M_i \times M_j \rightarrow M_j \times M_i$ be the map that switches the factors. Define $Z_{2,1} = \tau_{1,2}(Z_{1,2}) \subseteq M_2 \times M_1$ and $Z_{3,2} = \tau_{2,3}(Z_{2,3}) \subseteq M_3 \times M_2$. Then the convolution product

$$H_j(Z_{3,2}) \otimes H_i(Z_{2,1}) \rightarrow H_{i+j-2d}(Z_{3,2} \circ Z_{2,1})$$

is defined. We omit the easy proof of the following lemma.

**Lemma A.1.** If $c$ is in $H_i(Z_{1,2})$ and $d$ is in $H_j(Z_{2,3})$, then $(\tau_{1,3})_*(c \ast d) = (\tau_{2,3})_* (d \ast (\tau_{1,2})_* (c))$.

Now suppose $X$ is an irreducible, smooth, $m$-dimensional variety, $Y$ is a smooth variety, and $f: X \rightarrow Y$ is a morphism. Then if $\Gamma_X$ and $\Gamma_f$ denote the graphs of $id_X$ and $f$ respectively, using the notation in $[3, \S 2.7]$, we have $\Gamma_X \circ \Gamma_f = \Gamma_f$ and there is a convolution product $*: H_i(\Gamma_X) \otimes H_{2m}(\Gamma_f) \rightarrow H_i(\Gamma_f)$.

**Proposition A.2.** The convolution product $*: H_i(\Gamma_X) \otimes H_{2m}(\Gamma_f) \rightarrow H_i(\Gamma_f)$ is an injection.

*Proof.* For $i, j = 1, 2, 3$, let $p_{i,j}$ denote the projection of $X \times X \times Y$ on the $i^{th}$ and $j^{th}$ factors. Then the restriction of $p_{1,3}$ to $(\Gamma_X \times Y) \cap (X \times \Gamma_f)$ is the map that sends $(x, x, f(x))$ to $(x, f(x))$. Thus, the restriction of $p_{1,3}$ to $(\Gamma_X \times Y) \cap (X \times \Gamma_f)$ is an isomorphism onto $\Gamma_f$ and hence is proper. Therefore, the convolution product in homology is defined.

Since $X$ is irreducible, so is $\Gamma_f$ and so $H_{2m}(\Gamma_f)$ is one-dimensional with basis $[\Gamma_f]$. Suppose that $c$ is in $H_i(\Gamma_X)$. We need to show that if $c \ast [\Gamma_f] = 0$, then $c = 0$.

Fix $c$ in $H_i(\Gamma_X)$. Notice that the restriction of $p_{1,3}$ to $(\Gamma_X \times Y) \cap (X \times \Gamma_f)$ is the same as the restriction of $p_{2,3}$ to $(\Gamma_X \times Y) \cap (X \times \Gamma_f)$. Thus, using the projection formula, we have

$$c \ast [\Gamma_f] = (p_{1,3})_*(p_{1,2}^*c \cap p_{2,3}^*[\Gamma_f]) = (p_{2,3})_*((p_{2,3})^*p_{1,2}^*c \cap p_{2,3}^*[\Gamma_f]) = ([p_{2,3}^*p_{1,2}^*c] \cap [\Gamma_f]),$$

where the intersection product in the last line is from the cartesian square:

\[
\begin{array}{ccc}
\Gamma_f & \cong & \Gamma_f \\
\downarrow & & \downarrow \\
X \times Y & \cong & X \times Y \\
\end{array}
\]

Let $p: X \times Y \rightarrow X$ and $q: \Gamma_X \rightarrow X$ be the first and second projections, respectively. Then the square

\[
\begin{array}{ccc}
\Gamma_X \times Y & \stackrel{p_{2,3}}{\longrightarrow} & X \times Y \\
p_{1,2} & \downarrow & \downarrow p \\
\Gamma_X & \stackrel{q}{\longrightarrow} & X \\
\end{array}
\]

is cartesian. Thus,

$$p_* (c \ast [\Gamma_f]) = p_* \left( ([p_{2,3}]^*p_{1,2}^*c] \cap [\Gamma_f]) \right),$$

and so $c \ast [\Gamma_f] = 0$, we have $c = 0$. \hfill $\Box$
Thus, if \( c \) is an isomorphism, so in particular it is proper.

where we have used the projection formula and the fact that \((p|_{\Gamma_f})_*[\Gamma_f] = [X]\).

Now if \( c \ast [\Gamma_f] = 0 \), then \( q_*c = 0 \) and so \( c = 0 \), because \( q \) is an isomorphism. \( \square \)

Let \( \Gamma_Y \) denote the graph of the identity functions \( id_Y \). Then the following compositions and convolution products in Borel-Moore homology are defined:

- \( \Gamma_f \circ \Gamma_X = \Gamma_f \) and so there is a convolution product \( H_i(\Gamma_f) \otimes H_j(\Gamma_X) \to H_{i+j-m}(\Gamma_f) \).
- \( \Gamma_Y \circ \Gamma_f^{-1} = \Gamma_{f^{-1}} \) and so there is a convolution product \( H_i(\Gamma_X) \otimes H_j(\Gamma_{f^{-1}}) \to H_{i+j-m}(\Gamma_{f^{-1}}) \).
- \( \Gamma_f \circ \Gamma_{f^{-1}} = \Gamma_X \) and so there is a convolution product \( H_i(\Gamma_f) \otimes H_j(\Gamma_{f^{-1}}) \to H_{i+j-m}(\Gamma_X) \).

Thus, if \( c \) is in \( H_i(\Gamma_Y) \), then \([\Gamma_f] \ast c \ast [\Gamma_{f^{-1}}]\) is in \( H_i(\Gamma_X) \). Notice that \( f^{-1} \times f^{-1} : \Gamma_Y \to \Gamma_X \) is an isomorphism, so in particular it is proper.

**Proposition A.3.** If \( c \) is in \( H_i(\Gamma_Y) \), then \([\Gamma_f] \ast c \ast [\Gamma_{f^{-1}}] = (f^{-1} \times f^{-1})_* (c)\).

**Proof.** We compute \(([\Gamma_f] \ast c) \ast [\Gamma_{f^{-1}}]\), starting with \([\Gamma_f] \ast c\).

For \( 1 \leq i, j \leq 3 \) let \( q_{i,j} \) be the projection of the subset

\[
\Gamma_f \times Y \cap X \times Y = \{ (x, f(x), f(x)) \mid x \in X \}
\]

of \( X \times Y \times Y \) onto the \( i, j \)-factors. Then \( q_{1,3} = q_{1,2} \). Therefore, using the projection formula, we see that

\[
[\Gamma_f] \ast c = (q_{1,3})_* (q_{1,2}^*[\Gamma_f] \cap q_{2,3}^*c)
= (q_{1,2})_* (q_{1,2}^*[\Gamma_f] \cap q_{2,3}^*c)
= [\Gamma_f] \cap (q_{1,2})_* q_{2,3}^*c
= (q_{1,2})_* q_{2,3}^*c.
\]

Next, for \( 1 \leq i, j \leq 3 \) let \( p_{i,j} \) be the projection of the subset

\[
\Gamma_f \times X \cap X \times \Gamma_{f^{-1}} = \{ (x, f(x), x) \mid x \in X \}
\]

of \( X \times Y \times X \) onto the \( i, j \)-factors. Then \( p_{1,3} = (f^{-1} \times id) \circ p_{2,3} \). Therefore, using the fact that \([\Gamma_f] \ast c = (q_{1,2})_* q_{2,3}^*c \) and the projection formula, we have

\[
([\Gamma_f] \ast c) \ast [\Gamma_{f^{-1}}] = (p_{1,3})_* (p_{1,2}^*((q_{1,2})_* q_{2,3}^*c) \cap p_{2,3}^*[\Gamma_{f^{-1}}])
= (f^{-1} \times id)_* (p_{2,3})_* (p_{1,2}^*((q_{1,2})_* q_{2,3}^*c) \cap p_{2,3}^*[\Gamma_{f^{-1}}])
= (f^{-1} \times id)_* ((p_{2,3})_* p_{1,2}^* (q_{1,2})_* q_{2,3}^*c \cap [\Gamma_{f^{-1}}])
= (f^{-1} \times id)_* (p_{2,3})_* p_{1,2}^* (q_{1,2})_* q_{2,3}^*c.
\]
The commutative square

\[
\begin{array}{ccc}
\Gamma_f \times X \cap X \times \Gamma_f & \xrightarrow{id \times id \times f} & \Gamma_f \times Y \cap X \times \Gamma_Y \\
\downarrow & & \downarrow q_{1,2} \\
\Gamma_f \times X \cap X \times \Gamma_f & \xrightarrow{p_{1,2}} & \Gamma_f
\end{array}
\]

is cartesian, so \( p_{1,2}^* (q_{1,2})_* = (id \times id \times f)^* \).

Also, the commutative square

\[
\begin{array}{ccc}
\Gamma_f \times X \cap X \times \Gamma_f & \xrightarrow{q_{2,3} \circ (id \times id \times f)} & \Gamma_Y \\
(f^{-1} \times id) \circ p_{2,3} & \downarrow & \downarrow f^{-1} \times f^{-1} \\
\Gamma_X & \xrightarrow{id} & \Gamma_X
\end{array}
\]

is cartesian, so \((f^{-1} \times id)_* (p_{2,3})_* (id \times id \times f)^* q_{2,3}^* = (f^{-1} \times f^{-1})_*\).

Therefore,

\[
([\Gamma_f] * c) * [\Gamma_f^{-1}] = (f^{-1} \times id)_* (p_{2,3})_* p_{1,2}^* (q_{1,2})_* q_{2,3}^* c \\
= (f^{-1} \times id)_* (p_{2,3})_* (id \times id \times f)^* q_{2,3}^* c \\
= (f^{-1} \times f^{-1})_* c.
\]

This completes the proof of the proposition. \(\square\)

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