Brouwer’s fan theorem

Josef Berger

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Abstract

Brouwer’s fan theorem states that every bar is a uniform bar. We give an overview of the status of this axiom in Bishop’s constructive mathematics. In particular, we describe the relationship between the fan theorem, the weak König lemma, and the uniform continuity theorem.

Keywords: Brouwer’s fan theorem, the weak König lemma, the uniform continuity theorem

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1 Introduction

We analyse the relationship between Brouwer’s fan theorem for detachable bars, the weak König lemma, and the uniform continuity theorem:

FAN Every detachable bar is a uniform bar.

WKL Every infinite tree has a path.

UC Every pointwise continuous function $F : \{0,1\}^\mathbb{N} \to \mathbb{N}$ is uniformly continuous.

Extending the well-known basic picture

\[ \text{WKL} \Rightarrow \text{UC} \Rightarrow \text{FAN}, \]
we prove the implications of the following diagram:

\[
\begin{array}{c}
\text{WKL} \Rightarrow \text{FAN} \land \text{DEFU} \iff \text{c-FAN} \Rightarrow \text{FAN} \\
\Downarrow & \Downarrow \\
\text{FAN} \land \text{DECO} \iff \text{UC}
\end{array}
\]

The axiom c-FAN was introduced in [2] as a version of the fan theorem which is equivalent to UC. The axioms DEFU and DECO are auxiliary statements clarifying the gap between FAN and c-FAN, UC respectively.

Most results mentioned in this article have been published before, see

• [9, 10] for results about WKL and FAN,

• [4, 5] for results about unique existence, and

• [1, 2, 8] for results about UC.

2 Notation

Let \( \mathbb{N} = \{0, 1, 2, \ldots \} \) denote the set of natural numbers. We use the variables \( k, l, m, n, N, i, j \) for natural numbers. Let \( \{0, 1\}^* \) denote the set of all finite binary sequences, typically denoted by variables like \( u, v, w \). The symbol \( \varnothing \) denotes the empty sequence. Note that \( \varnothing \in \{0, 1\}^* \). Let \( \{0, 1\}^\mathbb{N} \) denote the set of all infinite binary sequences. We use the variables \( \alpha, \beta, \gamma \) for elements of \( \{0, 1\}^\mathbb{N} \). The length of \( u \) is denoted by \( |u| \). We define

\[\mathcal{N} = \{u \mid |u| > 0 \land \forall i < |u| (u_i = 0)\}\]

and

\[\mathcal{E} = \{u \mid |u| > 0 \land \forall i < |u| (u_i = 1)\} .\]

Let \( \{0, 1\}^n = \{u \mid |u| = n\} \) denote the set of all binary sequences of length \( n \). The concatenation of \( u \) and \( v \) is denoted by \( u \ast v \) and defined by

\[(u_0, \ldots, u_{n-1}) \ast (v_0, \ldots, v_{m-1}) = (u_0, \ldots, u_{n-1}, v_0, \ldots, v_{m-1}).\]

The concatenation of \( u \) and \( \alpha \) is denoted by \( u \ast \alpha \) and defined by

\[(u_0, \ldots, u_{n-1}) \ast \alpha = (u_0, \ldots, u_{n-1}, \alpha_0, \alpha_1, \alpha_2, \ldots).\]
If \( n \leq |w| \), we denote the restriction of \( w \) to the first \( n \) elements by \( \overline{wn} \). Note that \( \overline{w}0 = \emptyset \). We denote the restriction of \( \alpha \) to the first \( n \) elements by \( \overline{\alpha n} \). We write \( u < v \) for

\[
|u| = |v| \land \exists k < |u| \ (\overline{uk} = \overline{vk} \land u_k = 0 \land v_k = 1).
\]

Fix a subset \( A \) of \( \{0, 1\}^* \). We define

- \( A^o = \{u \mid \forall w \ (u * w \in A)\} \),
- \( \overline{A} = \{u * w \mid u \in A \land w \in \{0, 1\}^*\} \), and
- for any \( u \), \( A_u = \{w \mid u * w \in A\} \).

A path of \( A \) is a sequence \( \alpha \) such that \( \forall n (\overline{\alpha n} \in A) \). The set \( A \) hat at most one path if

\[
\forall \alpha, \beta \ (\exists n (\alpha_n \neq \beta_n) \Rightarrow \exists n (\overline{\alpha n} \notin A \lor \overline{\beta n} \notin A)).
\]

A longest path of \( A \) is a sequence \( \alpha \) such that \( \forall u \in A (\overline{|u|} \in A) \). Note that each path of \( A \) is a longest path of \( A \) but not necessarily vice versa.

The set \( A \) is said to be

- detachable (from \( \{0, 1\}^* \)) if
  \[
  \forall u (u \in A \lor u \notin A);
  \]

- a c-set if there exists a detachable set \( D \subseteq \{0, 1\}^* \) such that \( A = D^o \);

- closed under extension if
  \[
  \forall u \in A \forall w (u * w \in A);
  \]

- closed under restriction if
  \[
  \forall u, w (u * w \in A \Rightarrow u \in A);
  \]

- a tree if it is detachable and closed under restriction;

- infinite if
  \[
  \forall n \exists u (|u| = n \land u \in A);
  \]
The weak König lemma is the following axiom:

**WKL** Every infinite tree has a path.

**Lemma 1.** Let $T$ be a tree. The following are equivalent:

(a) $T$ is an infinite tree.

(b) Every longest path of $T$ is a path of $T$.

The notion of longest path is commonly used in graph theory. In the context of constructive mathematics, it was introduced in [10], along with the **longest path lemma**:

**LPL** Every tree has a longest path.

When working with trees and longest paths, it is convenient to have at hand an algorithm which transforms an arbitrary tree $T$ into an infinite tree $T'$, where $T'$ is as closely related to $T$ as possible. Given a tree $T$, we define subsets $L_T$ and $T'$ of $\{0, 1\}^*$ by

$$ u \in L_T \iff (u \in T \land \forall w (|u| < |w| \lor u < w) \Rightarrow w \notin T) \lor (\emptyset \notin T \land u = \emptyset) $$

and

$$ u \in T' \iff u \in T \cup \{\emptyset\} \lor \exists v \in L_T \exists w \in N (u = v * w). $$

Note that
• if \( T = \emptyset \) then \( L_T = \{\emptyset\} \) and \( T' = \mathcal{N} \cup \{\emptyset\} \);  
• if \( T \) is infinite, then \( L_T = \emptyset \) and \( T' = T \);  
• if \( \emptyset \in T \) and the set \( \{|u| \mid u \in T\} \) is bounded, then \( L_T = \{u\} \), where \( u \) is an element of \( T \) of maximal length.

**Proposition 1.** For every tree \( T \) there is an infinite tree \( T' \) such that  
\((a)\) \( T \subseteq T' \);  
\((b)\) if \( T \) is infinite, then \( T = T' \);  
\((c)\) if \( T \) is convex, then \( T' \) is convex as well;  
\((d)\) if \( T \) is finite, then \( T' \) has at most one path;  
\((e)\) every path of \( T' \) is a longest path of \( T \);  
\((f)\) if \( T \) has at most one path, then \( T' \) has at most one path.

The lesser limited principle of omniscience is the following axiom:

**LLPO** For all \( \alpha \),
\[
\forall n, m \left( \alpha_n = \alpha_m = 1 \Rightarrow n = m \right) \Rightarrow \forall n \left( \alpha_{2n} = 0 \right) \vee \forall n \left( \alpha_{2n+1} = 0 \right).
\]

**Proposition 2.** LLPO \( \Rightarrow \) WKL

*Proof.* Let \( T \) be an infinite tree. Let \( \zeta_T(u, m) \) be defined by
\[
\zeta_T(u, m) \iff \exists w \in \{0, 1\}^m (u \ast w \in T).
\]
Moreover, set
\[
Z_T = \{u \mid \forall m \zeta_T(u, m)\}.
\]
Define \( \beta \) by
\[
\beta_{2n} = 0 \iff \zeta_T((0), n)
\]
and
\[
\beta_{2n+1} = 0 \iff \zeta_T((1), n)
\]
and define $\alpha$ by
\[ \alpha_n = 1 \Leftrightarrow \beta_n = 1 \land \forall m < n \left( \beta_m = 0 \right). \]

By LLPO, either
\[ \forall n \left( \alpha_{2n} = 0 \right) \quad (1) \]
or else
\[ \forall n \left( \alpha_{2n+1} = 0 \right). \quad (2) \]

If (1) holds, we can conclude that
\[ \forall n \left( \beta_{2n} = 0 \right), \]
which implies that $0 \in Z_T$.

Analogously, we can show that (2) implies that $1 \in Z_T$.

Iterating this procedure yields a path $\gamma$ of $Z_T$. Since $Z_T \subseteq T$, the sequence $\gamma$ is also a path of $T$.

\begin{proof}
\begin{enumerate}
\item Proposition 3. The following are equivalent:
\begin{enumerate}
\item (a) WKL
\item (b) LPL
\item (c) Every convex tree has a longest path.
\item (d) Every infinite convex tree has a path.
\item (e) LLPO
\end{enumerate}
\end{enumerate}

Proof. “(a) \Rightarrow (b)” This follows from Proposition 1.

“(c) \Rightarrow (d)” Let $T$ be an infinite convex tree. By (c), the tree $T$ has a longest path $\alpha$. By Lemma 1, the sequence $\alpha$ is a path of $T$.

“(d) \Rightarrow (e)” Fix a sequence $\alpha$ with
\[ \forall n, m \left( \alpha_n = \alpha_m = 1 \Rightarrow n = m \right). \]
We define an infinite convex tree $S$ by

$$S = \emptyset, (0), (1) \cup S_0 \cup S_1,$$

where

$$S_0 = \{(0) \ast w \mid w \in \mathcal{E} \land \forall i \leq |w| (\alpha_{2i} = 0)\},$$

$$S_1 = \{(1) \ast w \mid w \in \mathcal{N} \land \forall i \leq |w| (\alpha_{2i+1} = 0)\}.$$

Let $\beta$ be a path of $S$. If $\beta_1 = 0$, then $\forall i (\alpha_{2i} = 0)$. If $\beta_1 = 1$, then $\forall i (\alpha_{2i+1} = 0)$.

“(e) $\Rightarrow$ (a)” This is Proposition 2.

Proposition 2 and Proposition 3 (without the parts about convexity) were proved in [9] and [10].

4 The fan theorem

Brouwer’s fan theorem is the following axiom:

FAN Every detachable bar is a uniform bar.

Lemma 2. If $B \subseteq \{0, 1\}^*$ is closed under extension, the following are equivalent:

(a) $B$ is a uniform bar.

(b) $\exists n (\{0, 1\}^n \subseteq B)$

The next lemma enables us to switch freely between a set $B$ itself and $\overline{B}$, when discussing bar-related properties. This technique was introduced in [10].

Lemma 3. Let $B$ be a subset $\{0, 1\}^*$. Then

(a) $\overline{B}$ is closed under extension,

(b) $B \subseteq \overline{B}$,

(c) $B$ is detachable $\Rightarrow$ $\overline{B}$ is detachable,
(d) $B$ is a bar $\Rightarrow \overline{B}$ is a bar,

(e) $\overline{B}$ is a uniform bar $\Rightarrow B$ is a uniform bar.

Proposition 4. The following are equivalent:

(a) WKL

(b) Each tree with at most one path has a longest path.

(c) Each detachable bar which is closed under extension is a uniform bar.

(d) FAN

Proof. “(a) $\Rightarrow$ (b)” This follows from Proposition 1.

“(b) $\Rightarrow$ (c)” Let $B$ be a detachable bar which is closed under extension. Note that the set

$$T = \{u \mid u \notin B\}$$

is a tree. Since $B$ is a bar, the tree $T$ has at most one path. Let $\alpha$ be a longest path of $T$. There exists $n$ such that $\overline{\alpha}n \in B$. This implies that $T \cap \{0, 1\}^n = \emptyset$ and therefore $\{0, 1\}^n \subseteq B$. Thus, by Lemma 2, $B$ is a uniform bar.

“(c) $\Rightarrow$ (d)” This follows from Lemma 3.

“(d) $\Rightarrow$ (a)” Fix an infinite tree $T$ with at most one path. Define a set $B$ by

$$u \in B :\iff (0) * u \notin T \lor \forall v \in \{0, 1\}^{\lvert u \rvert}((1) * v \notin T).$$

We show that the $B$ is a bar. Fix a sequence $\alpha$. We define $B_\alpha \subseteq \{0, 1\}^*$ by

$$v \in B_\alpha :\iff ((0) * \overline{\alpha}v \notin T \lor (1) * v \notin T).$$

Note that $B_\alpha$ is closed under extension. Since $T$ has a most one path, $B_\alpha$ is a bar and therefore a uniform bar. Thus, there exists $m$ such that $\{0, 1\}^m \subseteq B_\alpha$. Thus, $\overline{\alpha}m \in B$. This concludes the proof that $B$ is a bar.

Since $B$ is is a uniform bar and closed under extension, there exists an $n$ such that $\{0, 1\}^n \subseteq B$. We show that either

(i) $\forall u \in \{0, 1\}^n ((0) * u \notin T)$ or else
(ii) \( \forall u \in \{0, 1\}^n \ ((1) * u \notin T) \).

If (i) fails to hold, there exists \( u \in \{0, 1\}^n \) such that \( (0) * u \in T \). Since \( u \in B \), this implies (ii).

Let \( Z_T \) be defined as in the proof of Proposition 2. If (i) holds, then \( (1) \in Z_T \). If (ii) holds, then \( (0) \in Z_T \). Iterating this construction yields a path \( \gamma \) of \( Z_T \) and therefore of \( T \).

The equivalence of FAN and WKL! was proved in [5].

**Corollary 1.** \( \text{WKL} \Rightarrow \text{FAN} \)

Corollary 1 was originally published in [10].

**Lemma 4.** If \( A \subseteq \{0, 1\}^* \) is co-convex, then \( \overline{A} \) is also co-convex.

**Proof.** Fix \( u, v, w \) such that \( u < v < w \) and \( u, w \notin \overline{A} \). We have to show that \( v \notin \overline{A} \). To this end, fix \( m \leq |v| \). Then \( \overline{m} \notin A \) and \( \overline{w} \notin A \). The co-convexity of \( A \) yields \( \overline{v} \notin A \). Since \( m \leq |v| \) was chosen arbitrarily, this implies that \( v \notin \overline{A} \).

**Proposition 5.**

(a) Every infinite convex tree with at most one path has a path.

(b) Every convex tree with at most one path has a longest path.

(c) Every detachable co-convex bar is a uniform bar.

**Proof.** (a) Let \( T \) be an infinite convex tree with at most one path. Set

\[ \alpha = (0, 1, 1, 1, \ldots) \]

and

\[ \beta = (1, 0, 0, 0, \ldots) \, . \]

Since \( T \) has at most one path, there exists \( n \) such that

\[ \overline{\alpha}(n + 1) \notin T \lor \overline{\beta}(n + 1) \notin T. \]

Since \( T \) is convex, we can conclude that either
(i) \( \forall u \in \{0, 1\}^n ((0) \ast u \notin T) \) or else

(ii) \( \forall u \in \{0, 1\}^n ((1) \ast u \notin T) \).

In the first case, choose \( \alpha_0 = 1 \) and in the second case, choose \( \alpha_0 = 0 \). Iterating this construction leads to a path \( \alpha \) of \( T \).

(b) follows from (a) and Proposition 1.

(c) Let \( B \) be a detachable, co-convex bar. By Lemma 4, we can assume that \( B \) is closed under extension. The set

\[ T = \{ u \mid u \notin B \} \]

is a convex tree. Since \( B \) is a bar, the tree \( T \) has at most one path. Let \( \alpha \) be a longest path of \( T \). There exists \( n \) such that \( \bar{\alpha}n \in B \). Then \( T \cap \{0, 1\}^n = \emptyset \). Thus, \( \{0, 1\}^n \subseteq B \). This implies that \( B \) is a uniform bar.

\[ \square \]

The notion of co-convex bars was introduced in [7]. In [12], the fan theorem for co-convex bars—every detachable co-convex bar is a uniform bar—is used for a case study in program extraction from proofs. For a treatment of convex trees in constructive reverse mathematics see [6].

5 The uniform continuity theorem

Fix a function \( F : \{0, 1\}^\mathbb{N} \to \mathbb{N} \) and \( u \in \{0, 1\}^* \). We define

\[ F(u) = F(u \ast 0), \]

where \( 0 = (0, 0, 0, \ldots) \). Moreover, we define a new function \( F_u : \{0, 1\}^\mathbb{N} \to \mathbb{N} \) by

\[ F_u(\alpha) = F(u \ast \alpha). \]

For \( n \in \mathbb{N} \), we define

\[ F \equiv n \iff \forall \alpha \left( F(\alpha) = n \right). \]

A function \( F : \{0, 1\}^\mathbb{N} \to \mathbb{N} \) is

- pointwise continuous if

\[ \forall \alpha \exists n \forall \beta \left( \bar{\alpha}n = \bar{\beta}n \Rightarrow F(\alpha) = F(\beta) \right); \]
• uniformly continuous if
  \[ \exists n \forall \alpha, \beta (\alpha n = \beta n \Rightarrow F(\alpha) = F(\beta)) ; \]

• bounded if
  \[ \exists N \forall \alpha (F(\alpha) \leq N) ; \]

• constant if
  \[ \exists n (F \equiv n) . \]

**Lemma 5.** If \( F : \{0, 1\}^\mathbb{N} \to \mathbb{N} \) is uniformly continuous,

\[ F \equiv 0 \lor \neg (F \equiv 0) . \]

**Lemma 6.** Fix \( F : \{0, 1\}^\mathbb{N} \to \mathbb{N} . \)

(a) \( F \) is pointwise continuous \( \iff \forall \alpha \exists n (F_{\alpha n} \text{ is constant}) \)

(b) \( F \) is uniformly continuous \( \iff \exists N \forall u \in \{0, 1\}^N (F_u \text{ is constant}) \)

**Lemma 7.** Suppose that \( F : \{0, 1\}^\mathbb{N} \to \mathbb{N} \) is pointwise continuous. Then the set

\[ \{ u \mid F(u) \leq |u| \} \]

is a bar.

**Proof.** Fix \( \alpha \). There exists \( N \) such that

\[ \forall \beta (\alpha \beta N = \beta \alpha N \Rightarrow F(\alpha) = F(\beta)) . \]

We may assume that \( F(\alpha) \leq N \). Note that

\[ F(\alpha \alpha N) = F(\alpha) \leq N . \]

This implies that

\[ \alpha \alpha N \in \{ u \mid F(u) \leq |u| \} . \]

\[ \square \]

**Lemma 8.** Every uniformly continuous function \( F : \{0, 1\}^\mathbb{N} \to \mathbb{N} \) is bounded.

We now consider the uniform continuity theorem:
Every pointwise continuous function \( F : \{0, 1\}^\mathbb{N} \to \mathbb{N} \) is uniformly continuous.

**Proposition 6. UC \Rightarrow FAN**

**Proof.** Let \( B \) be a detachable bar. The function
\[
F(\alpha) = \min \{ n \mid \overline{\pi}n \in B \}
\]
is pointwise continuous. In view of UC, the function \( F \) is uniformly continuous and therefore, by Lemma 8, bounded. Thus, \( B \) is a uniform bar. \( \square \)

Fix functions \( F, M : \{0, 1\}^\mathbb{N} \to \mathbb{N} \). The function \( M \) is a modulus for \( F \) if
\[(a) \ M \text{ is pointwise continuous};\]
\[(b) \ \forall \alpha, \beta, (\overline{\pi}M(\alpha) = \overline{\beta}M(\alpha) \Rightarrow F(\alpha) = F(\beta)).\]

**MUC** Fix \( F, M : \{0, 1\}^\mathbb{N} \to \mathbb{N} \). Assume that \( M \) is a pointwise continuous modulus of \( F \). Then \( F \) is uniformly continuous.

**Proposition 7. MUC \Leftrightarrow FAN**

**Proof.** “\( \Rightarrow \)” Let \( B \) be a detachable bar. Then the function \( F \), given by (3), is a pointwise continuous modulus of itself, which implies that \( F \) is uniformly continuous and therefore bounded. Thus, \( B \) is a uniform bar.

“\( \Leftarrow \)” Fix \( F, M : \{0, 1\}^\mathbb{N} \to \mathbb{N} \) and assume that \( M \) is a pointwise continuous modulus of \( F \). By Lemma 7, the set
\[
B = \{ u \mid M(u) \leq |u| \}
\]
is a bar and therefore, in view of FAN, a uniform bar. Thus, there exists \( N \) such that
\[
\forall \alpha \exists n \leq N (M(\overline{\pi}n) \leq n).
\]
This implies
\[
\forall \alpha (F_{\overline{\pi}N} \text{ is constant}).
\]
By Lemma 6, we can conclude that \( F \) is uniformly continuous. \( \square \)

The axiom MUC and Proposition 7 are taken from [1]. For more about uniform continuity and FAN see [8].
6 The fan theorem for c-sets

In the previous section, we have matched uniform continuity with the fan theorem by replacing UC with its weaker version MUC. Another option is to strengthen FAN by relaxing the requirement that the bar be detachable. This leads to the axiom c-FAN, which was introduced in [2].

**Lemma 9.** For all subsets $A$ of $\{0,1\}^*$ an all $u$ we have

(a) $u \in A^\circ$ $\iff$ $A_u = \{0,1\}^*$ and

(b) $(A_u)^\circ = (A^\circ)_u$.

**Proof.** In order to prove (b), fix $u, v$ and $w$ and note that

$$w \in (A_u)^\circ \iff \forall v (w * v \in A_u) \iff \forall v (u * (w * v) \in A) \iff \forall v ((u * w) * v \in A) \iff u * w \in A^\circ \iff w \in (A^\circ)_u.$$  

\[ \square \]

c-FAN Every bar which is a c-set is a uniform bar.

We introduce the auxiliary axioms DEFU (decidability whether certain bars are full) and DECO (decidability whether certain functions are constant).

**DEFU** Let $D$ be a detachable subset of $\{0,1\}^*$ such that $D^\circ$ is a bar. Then,

$$\exists u (u \notin D) \lor \neg \exists u (u \notin D).$$

**DECO** Let $F : \{0,1\}^\mathbb{N} \to \mathbb{N}$ be pointwise continuous. Then,

$$\exists \alpha, \beta (F(\alpha) \neq F(\beta)) \lor \neg \exists \alpha, \beta (F(\alpha) \neq F(\beta)).$$

**Lemma 10.** DEFU $\iff$ DECO
Proof. “⇒” Fix a pointwise continuous function $F : \{0, 1\}^\mathbb{N} \to \mathbb{N}$. Set

$$D = \{ u \mid F(u) = F(u^* (1)) \}.$$ 

By Lemma 6, $D^\circ$ is a bar. Note that

$$\exists u (u \notin D) \iff \exists \alpha, \beta (F(\alpha) \neq F(\beta)).$$

“⇐” Fix $D \subseteq \{0, 1\}^*$ and suppose that $D^\circ$ is a bar. Define

$$F(\alpha) = \min \{ n \mid \forall m \geq n \ (\alpha^m \in D) \}$$

and note that

$$\exists u (u \notin D) \iff \exists \alpha (F(\alpha) > 0)). \quad (4)$$

In presence of DECO, the right hand side of (4) is decidable, which implies that the left hand side of (4) is decidable. \hfill \Box

The diagram in the next proposition sheds some light on the relationship between bars and functions. Its core part is the equivalence between c-FAN and UC.

**Proposition 8.**

$$\text{WKL} \Rightarrow \text{FAN} \land \text{DEFU} \iff \text{c-FAN} \Rightarrow \text{FAN}$$

$$\Updownarrow \quad \Updownarrow$$

$$\text{FAN} \land \text{DECO} \iff \text{UC}$$

**Proof.** “WKL ⇒ DEFU” Let $D$ be a detachable subset $D$ of $\{0, 1\}^*$ such that $D^\circ$ is a bar. Define an infinite tree $T$ by

$$u \in T \iff \forall w (|w| \leq |u| \land w \notin D \Rightarrow \exists k \leq |w| (\bar{w}k \notin D)).$$

For each $u \in T$ of length $n$ we have

$$\{ \bar{w}k \mid k \leq n \} \subseteq D \iff \{ w \mid |w| \leq n \} \subseteq D.$$ 

Let $\alpha$ be a path of $T$. The following are equivalent:

(i) $\exists u (u \notin D)$
(ii) \( \exists n (\overline{n} \notin D) \)

There exists \( N \) such that \( \overline{N} \in D^\circ \). Thus, (ii) is decidable, which yields the decidability of (i).

“FAN \& DEFU \Rightarrow c-FAN” Fix a detachable subset \( D \) of \( \{0, 1\}^\* \) such that \( D^\circ \) is a bar. We show that \( D^\circ \) is detachable. To this end, fix some \( u \). Note that \( (D^\circ)_u \) is a bar and therefore \( (D_u)^\circ \) is a bar. We can conclude that

\[
D_u = \{0, 1\}^\* \lor \exists w (w \notin D_u),
\]

which implies

\[
u \in D^\circ \lor u \notin D^\circ.
\]

“FAN \& DECO \Rightarrow UC” Let \( F : \{0, 1\}^\mathbb{N} \to \mathbb{N} \) be pointwise continuous. Set

\[
B = \{u \mid F_u \text{ is constant}\}.
\]

In view of DECO, the set \( B \) is detachable. Since \( F \) is pointwise continuous, \( B \) is a bar. Thus, \( B \) is a uniform bar, which implies the uniform continuity of \( F \).

“UC \Rightarrow FAN \& DECO” The implication UC \Rightarrow FAN holds by Proposition 6. The implication UC \Rightarrow DECO follows from Lemma 6.

“c-FAN \Rightarrow FAN” Let \( B \) be a detachable bar which is closed under extension. Then \( B = B^\circ \), which implies that \( B \) is a c-set and therefore, in presence of c-FAN, a uniform bar.

\[\square\]

Another equivalent of c-FAN is the so-called anti-Specker theorem; see [3]. For more on the fan theorem, see [11] and [13].

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