Light scattering from periodic arrays of noble-metal disks

Xiaowei Ji¹, Yu Zhou¹

¹ Liupanshui Normal College, Guizhou, China.
E-mail: jchxhwy@163.com

Abstract. Numerical solution is presented for light scattering from free-standing periodic arrays of disks made of noble-metal. Using the generalized boundary conditions of the surface impedance type, we formulate the boundary value problem into a set of integral equations for unknown electric and magnetic current densities defined over the circular area. Employment of the method of moments allows us to solve the integral equations and give the expansion coefficients of the current densities, from which we can find reflected, transmitted, and absorbed powers. Dependence of the powers on the array parameters and wavelength is discussed in detail from the viewpoint of grating resonance.

1. Introduction

With growing interests in recent technology of controlling lightwaves, it has become necessary to accurately evaluate the optical properties of surface plasmon polaritons [1, 2]. One effective means of facilitating plasmon resonance and enhancing near electromagnetic field is to put multiple nanometer particles close to each other or to compile them into periodic structures. For infinitely long noble-metal scatterers having one-dimensional periodicity, fruitful results have been reported upon circular cylinder [3] and flat strips [4, 5], in which the observed resonances are analytically classified into the local and mutual (due to periodicity) types. These considerations are so instructive that the extension of the conception into two-dimensional geometries related with spheres or disks would be worth a lot. It is known that one of such structures, that is, a metallic plate periodically perforated with cylindrical holes, exhibits so called the extraordinary optical transmission [6, 7] even though the size of aperture cross section is in the order of sub-wavelength. This originates from excitation of surface waves on both sides of the plate. Then analytical approach to understanding its mechanism becomes also an interesting subject.

In view of the above background, the present paper deals with free-standing arrays of disks and apertures as fundamental periodic plane structures. The problem of plane wave scattering is formulated by applying the generalized boundary conditions [8] which replace the effect of electric constants and thickness of the nanometal with both electric resistance and magnetic conductance if the metal thickness is much smaller than the wavelength. Contrary to the use of only one type of surface impedance [9], the incorporation of these two constants is essential in coping with long and short range surface plasmons (LRSP and SRSP), and therefore this set of conditions has successively been applied to the strip problems [4, 5, 10]. The electromagnetic field is expanded by Floquet modes, which were frequently employed in the analysis of frequency selective surfaces [11] and waveguide phased arrays [12] in the microwave region. Combination of the above conditions and expansions leads us to a set of integral equations for unknown electric and magnetic current densities defined on the circular area. Using the method of moments, we obtain the numerical solution of the integral equations and give the expansion coefficients of the current densities, from which we can find power distributions. The behavior of the transmission, reflection, and absorption are examined in detail from
the viewpoint of resonances. We try to reveal the mechanism of power variations by constructing quasi-static solutions of the equivalent admittances of the arrays. The time factor $e^{i\omega t}$ will be omitted throughout.

2. Derivation of the integral equations

As illustrated in Figure 1(a) (b), an infinite number of disks with radius $a$ is allocated in the $xy$ plane with the period $d$ in both $x$ and $y$ directions. The disk array is stacked in the $z$ direction with the period $c$, and the arrays be included in the planes $z = z_l = (l - 1)c$, ($l = 1, 2, ..., L$). The disk array is made of noble-metal with thickness $b(|z| < b/2)$ and relative permittivity $\varepsilon_r$. They are free-standing in the air with electric constants $\varepsilon_0$ and $\mu_0$.

A plane wave $(E^i, H^i)$ is incident with the polar angle $\theta^i$ and the azimuth angle $\phi^i$, as shown in Figure 1(c). The wavenumber vector, propagation constants and wave impedance are given by

$$\begin{align*}
\kappa^i &= k_x \alpha^0 + k_y \beta^0 + k_z \gamma^0 \\
\alpha^0 &= \kappa_0 \sin \theta^i \cos \phi^i, \quad \beta^0 = \kappa_0 \sin \theta^i \sin \phi^i, \quad \gamma^0 = \kappa_0 \cos \theta^i
\end{align*}$$

where $\lambda$ is wavelength, $k_x, k_y$ and $k_z$ are the unit vectors concerning respective coordinate variables.

The periodicity of the system allows us to consider only over one unit cell, that is, $|x| < d/2$ and $|y| < d/2$.

Let us decompose the total electromagnetic fields as $(E, H) = (E^p, H^p) + \sum_{l=1}^{L}(E^{s(l)}, H^{s(l)})$, where the superscripts $p$ and $s$ concern the primary and scattered fields, respectively.

If the metal is electrically very thin ($b \ll \lambda$), and gives a high contrast to the air ($|\varepsilon_r| \gg 1$), we may impose the generalized boundary conditions [8] on the reduced surfaces $z = \pm z_l$ as

$$\begin{align*}
\{ (1/2)[E_T(x, y, z_l + 0) + E_T(x, y, z_l - 0)] &= R_{z_l} \times [H_T(x, y, z_l + 0) - H_T(x, y, z_l - 0)] \\
(1/2)[H_T(x, y, z_l + 0) + H_T(x, y, z_l - 0)] &= -Q_{z_l} \times [E_T(x, y, z_l + 0) - E_T(x, y, z_l - 0)]
\end{align*}$$

where the subscript $T$ means the transverse $(xy)$ components. The left-hand side is the mean tangential field on the two sides of the material, and the right side includes the electric or magnetic current density. Their proportional constants, that is, the electric resistance and magnetic conductance are given by

$$R = (\zeta_0/2\sqrt{\varepsilon_r}) \cot(k_0 b \sqrt{\varepsilon_r}/2), \quad Q = (\sqrt{\varepsilon_r}/4\zeta_0) \cot(k_0 b \sqrt{\varepsilon_r}/4)$$

For perfect conductors, they are reduced to $R = 0$ and $|Q| \rightarrow \infty$. The incident, transmitted, and reflected plane waves are written as

Figure 1. Geometry of the problem. (a) periodic array of disks in the $xy$ plane; (b) periodic array of disks in the $xz$ plane. (c) Incident plane wave.
and the scattered field is expressed in the form of a superposition of the Floquet modes as [12-18].

$$E^{s(l)}(\zeta_0 H^s) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \begin{pmatrix} \Psi_{1pq}(x, y) \\ \Psi_{2pq}(x, y) \\ \Psi_{zpq}(x, y) \end{pmatrix} \begin{pmatrix} A_{1pq}^{(l)} \\ A_{2pq}^{(l)} \end{pmatrix} e^{-i\gamma_{pq}(x-z_1) (z > z_l)} + e^{-i\gamma_{pq}(x-z_1) (z < z_l)}$$

(6)

where the vector mode functions are

$$\begin{pmatrix} \Psi_{1pq}(x, y) \\ \Psi_{2pq}(x, y) \\ \Psi_{zpq}(x, y) \end{pmatrix} = \begin{pmatrix} \beta_0 x + \alpha_0 y \\ \alpha_0 x + \beta_0 y \\ (\alpha_0^2 + \beta_0^2) l_z/k \end{pmatrix} e^{-i(\alpha_p x + \beta_p y)}$$

(7)

with the propagation constants and mode admittances

$$\begin{cases} \alpha_p = \alpha_0 + 2\pi n/d, \beta_p = \beta_0 + 2q\pi/d, \gamma_{pq} = (k_0^2 - \alpha_p^2 - \beta_p^2)^{1/2} \\ \eta_{1pq} = \gamma_{pq}/\zeta_0 k_0, \eta_{2pq} = \kappa_0/\zeta_0 \gamma_{pq} \end{cases}$$

(8)

The constants $A_{spq}$ and $B_{spq}$ are the unknown transmission and reflection coefficients of the $pq$-th mode, respectively. The mode functions satisfy the periodicity condition $\Psi(x + pd, y + qd) = \psi(x, y)e^{-i(p\alpha_0 + q\beta_0)d}$ as well as the orthonormal property

$$\int_{-d/2}^{d/2} \int_{-d/2}^{d/2} \Psi_{spq}(x, y) \Psi_{spq}^*(x, y) dx dy = \delta_{ss} \delta_{pp} \delta_{qq}$$

(9)

where the asterisk denotes complex conjugate and $\delta_{ss}$ is Kronecker’s delta.

Unknown functions are the surface electric and magnetic current densities induced on the disk which are defined by

$$J^{(l)}(x, y) = i \times \left[ H^{s(l)}_T(x, y, z_l + 0) - H^{s(l)}_T(x, y, z_l - 0) \right]$$

$$M^{(l)}(x, y) = -i \times \left[ E^{s(l)}_T(x, y, z_l + 0) - E^{s(l)}_T(x, y, z_l - 0) \right]$$

(10)

Since the incident field is continuous everywhere, we are allowed to replace $(E, H)$ in equation (10) with $(E^s, H^s)$. We substitute the scattered field equation (6) into equation (10), take the dot product with $\psi_{spq}^*(x, y)$, and integrate it over the unit cell. The orthogonality equation (9) permits us to leave only one term out of the infinite sums. Due to the field continuity on the $xy$ plane except the disk part that

$$J^{(l)}(x, y) = M^{(l)}(x, y) = 0 \quad (|x| < d/2, |y| < d/2, \sqrt{x^2 + y^2} > a; l = 1, 2, ..., L)$$

(11)

One can reduce the integration range from the square unit cell to the circular disk surface. Consequently, we arrive at the integral representation of the modal coefficients

$$\begin{pmatrix} A_{1pq}^{(l)} \\ B_{1pq}^{(l)} \end{pmatrix} = \frac{1}{2} \int \int_{\sqrt{x^2 + y^2} < a} \left[ -\frac{1}{\eta_{1pq}} J^{(l)}(x, y) \cdot \Psi_{1pq}^*(x, y) + M^{(l)}(x, y) \cdot \Psi_{2pq}^*(x, y) \right] dx dy$$

(12)
\[
\begin{align*}
\left( \begin{array}{c}
A_{2pq}^{(l)} \\
B_{2pq}^{(l)}
\end{array} \right) &= \frac{1}{2} \iint \frac{1}{\sqrt{x^2+y^2-a^2}} \left[ -\frac{1}{\eta_{2pq}} f^{(l)}(x,y) \cdot \Psi_{spq}^{*}(x,y) \right] dx \ dy \\
(\mathbf{12})
\end{align*}
\]

Let us derive the integral equations by expressing the boundary conditions equation (2) in terms of surface current densities \( J \) and \( M \). The above procedure leads us to the set of integral equations

\[
\begin{align*}
\left( \begin{array}{c}
R f^{(l)}(x,y) \\
\zeta_0 Q M^{(l)}(x,y)
\end{array} \right) + \frac{1}{2} \sum_{s=1}^{2} \sum_{l=1}^{\infty} \sum_{q=-\infty}^{\infty} \frac{1}{2\eta_{spq}} \Psi_{spq}(x,y) \\
\times \iint_{|x^2+y^2-a^2|} \left[ \frac{1}{\zeta_0\eta_{spq}} \left( \begin{array}{c}
\zeta_0 f^{(l)}(x',y') \\
M^{(l)}(x',y')
\end{array} \right) \right] \cdot \Psi_{spq}(x',y') \\
- \text{sgn}(l-\ell)(-1)^{\ell} \left( \begin{array}{c}
\zeta_0 f^{(l)}(x',y') \\
M^{(l)}(x',y')
\end{array} \right) \cdot \Psi_{spq}(x',y') \\
\times dx \ dy e^{-i\gamma_{pq}|x_1-z_1|}
\end{align*}
\]

\[
= \left( \begin{array}{c}
\Psi_{100}(x,y) \\
\Psi_{200}(x,y) / \zeta_0 \eta_{100}
\end{array} \right) \left( \begin{array}{c}
V_1 \\
V_2
\end{array} \right) e^{-i\gamma_{ao}z_1}
\]

\[
(\sqrt{x^2+y^2} < a; l = 1,2, ... , L) \quad (\mathbf{13})
\]

Let us solve the integral equations (13) numerically by means of the method of moments [19]. We approximate the unknown functions by finite sums as

\[
\begin{align*}
\left( \begin{array}{c}
\zeta_0 f^{(l)}(x,y) \\
M^{(l)}(x,y)
\end{array} \right) &\approx \sum_{s=1}^{2} \sum_{m=-M}^{M} \sum_{n=1}^{N} \left( \begin{array}{c}
F_{nm}^{(l)}(r) \\
F_{mn}^{(l)}(r)
\end{array} \right) \Phi_{tnm}(r, \phi) \quad (0 < r < a, -\pi < \phi < \pi; l = 1, 2, ... , L) \quad (\mathbf{14})
\end{align*}
\]

where the basis function is the transverse magnetic field of the \( mn \)-th mode in a circular waveguide with the cylindrical coordinate introduced by \( x = r \cos \phi, y = r \sin \phi \).

\[
\begin{align*}
\Phi_{1mn}(r, \phi) &= (f_{mn}/a) [m(\xi_{mn}r/a) \iota_r - (ima/\xi_{mn}r) f_m(\xi_{mn}r/a) / \iota_{\phi}] e^{im\phi} \\
\Phi_{2mn}(r, \phi) &= (f_{mn}/a) [ma / \xi_{mn}r] f_m(\xi_{mn}r/a) / \iota_{r} + f_m(\xi_{mn}r/a) / \iota_{\phi} e^{im\phi}
\end{align*}
\]

\[
\left( \begin{array}{c}
f_{1mn} = \xi_{mn} / \sqrt{\pi \left( \xi_{mn}^2 - m^2 \right)} f_m(\xi_{mn}) \\
f_{2mn} = 1 / \sqrt{\pi f'_m(\xi_{mn})}
\end{array} \right) \quad \left( \begin{array}{c}
f_{1mn} = \xi_{mn} / \sqrt{\pi \left( \xi_{mn}^2 - m^2 \right)} f_m(\xi_{mn}) \\
f_{2mn} = 1 / \sqrt{\pi f'_m(\xi_{mn})}
\end{array} \right) \quad (\mathbf{15})
\]

The symbols \( \xi_{mn} \) and \( \xi_{mn} \) are the \( n \)-th zero of the Bessel function \( f_m(x) \) and its derivative \( f'_m(x) = df_m(x)/dx \), respectively. Note that the combination of equation (14) and equation (15) exhibits proper edge behavior of the current that \( (J_r, M_r) \to 0 \) and \( (J_\phi, M_\phi) < \infty \) as \( r \to a \), if not exact up to the order [20]. Owing to the normalization constant \( f_{nm} \) (\( t = 1,2 \)), we have the orthonormal property

\[
\int_{-\pi}^{\pi} \int_{0}^{a} \Phi_{tnm}(r, \phi) \cdot \Phi_{tnm}^*(r, \phi) r dr d\phi = \delta_{tt} \delta_{mn} \delta_{nm} \quad (\mathbf{16})
\]

Then, we substitute equation (14) into equation (13), take the dot product with the testing function \( \Phi_{tnm}^*(r, \phi) \) \( (t = 1, 2; m = -M, -M + 1, ... , M; n = 1, 2, ... , N) \), and integrate it over the disk surface \( r < a \). Employment of equation (16) leads us to the set of linear equations

\[
\begin{align*}
\left( \begin{array}{c}
R / \zeta_0 F_{tnm}^{(l)} \\
Q / \zeta_0 F_{tnm}^{(l)}
\end{array} \right) &+ \sum_{l'=1}^{L} \sum_{t'=1}^{2} \sum_{m=-M}^{M} \sum_{n=1}^{N} \left( \begin{array}{c}
k^{(l')}_{tnm,t-m-n} \quad -k^{(l')}_{tnm,t-m-n} \\
k^{(l')}_{tnm,t-m-n} \quad k^{(l')}_{tnm,t-m-n}
\end{array} \right) \left( \begin{array}{c}
F_{t'm'n}^{(l')} \\
F_{t'm'n}^{(l')}
\end{array} \right)
\end{align*}
\]

\[
(t = 1, 2; m = -M, -M + 1, ... , M; n = 1, 2, ... , N; l = 1, 2, ... , L) \quad (\mathbf{17})
\]

where the system and driving elements are written as
IOP Conf. Series: Earth and Environmental Science 541 (2020) 012014    doi:10.1088/1755-1315/541/1/012014

3. Numerical Results

For simplicity, we only consider how the reflected, transmitted and absorbed powers of silver arrays of disks vary with wavelength and the period \( d \) as the number of layers increased. Figure 2 shows the normalized reflected powers for different number layers silver arrays of disks at incidence \( \theta_i = 45^\circ \) as a function of wavelength \( \lambda \), period \( d \). The size is given by \( a=100\text{nm}, c=500\text{nm}, \) and \( b=20\text{nm} \). As the number of layers increasing, the reflected power increases significantly in the range of wavelength between 600nm and 800nm.

As shown in Figure 3, the normalized transmitted powers for silver arrays of disks at incidence \( \theta_i = 45^\circ \) as a function of wavelength \( \lambda \), period \( d \) is very high throughout the region, except for the region of wavelength between 500nm and 650nm. And at the wavelength from 650nm to 800nm, the normalized transmitted power decreases as the number of layers increases. Figure 4 shows the normalized absorbed powers for silver arrays of disks at incidence \( \theta_i = 45^\circ \) as a function of wavelength \( \lambda \), period \( d \). we can find that, as the wavelength is from 200nm to 320nm, the absorbed power increases as the number of layers increases. In the region of 450 < \( \lambda \) < 600 nm, 340<d<420 nm, in a narrow area, the absorbed power increases significantly.
Figure 2. Normalized reflected powers for silver arrays of disks at incidence $\theta_i = 45^\circ$ as a function of wavelength $\lambda$, period $d$. The size is given by $a=100\text{nm}$, $c=500\text{nm}$, and $b=20\text{nm}$. The layers of disks are (a) single layer; (b) two layers; (c) three layers.
Figure 3. Normalized transmitted powers for silver arrays of disks at incidence $\theta_i = 45^\circ$ as a function of wavelength $\lambda$, period $d$. The size is given by $a=100\text{nm}$, $c=500\text{nm}$, and $b=20\text{nm}$. The layers of disks are (a) single layer; (b) two layers; (c) three layers.
Figure 4. Normalized absorbed powers for silver arrays of disks at incidence $\theta^i = 45^\circ$ as a function of wavelength $\lambda$, period $d$. The size is given by $a=100\text{nm}$, $c=500\text{nm}$, and $b=20\text{nm}$. The layers of disks are (a) single layer; (b) two layers; (c) three layers.
Figure 5. Normalized powers for silver arrays of disks at incidence $\theta_i = 45^\circ$ as a function of wavelength $\lambda$, period $c$. The size is given by $a=100\text{nm}$, $d=500\text{nm}$, $b=20\text{nm}$, and $l = 3$. (a) reflected power, (b) transmitted power, (c) absorbed power.

In figure 5, we discuss the reflected, transmitted, and absorbed power of three-layer silver disk arrays as a function of wavelength and period $c$ along $z$ axis. The size is given by $a=100\text{nm}$, $d=500\text{nm}$, and $b=20\text{nm}$. From these figures, we can analyse the influence of the distance between the array layers on the reflected, transmitted, and absorbed power.

4. Conclusion
Numerical solution has been developed to investigate the power distributions of three-dimensional periodic arrays of disks with regard to thin noble-metals. Enforcing the generalized boundary conditions leads us to the set of integral equations, which is treated by means of the method of moments. The computations are carried out to find power distributions as a function of wavelength and array parameters. Though interesting results have been obtained, the present work has limited to treating free-standing structures and showing transmitted and reflected powers which are related with far field. In this case, we need to compute the distribution of near evanescent field to demonstrate the enhancement of electric field density. These problems deserve further attention.

5. References
[1] Maier S A, Plasmonics: Fundamentals and Applications, Springer, 2007.
[2] Kim K Y, Plasmonics: Principles and Applications, InTech, 2012.
[3] Natarov D M, Byelobrov V O, Sauleau R, “Periodicity induced effects in the scattering and absorption of light by infinite and finite gratings of circular silver nanowires,” Optics Express, Vol. 19, No. 22, 22176-190, 2011.
[4] Shapoval O V, Sauleau R, and Nosich A I, “Modeling of plasmon resonances of multiple flat noble-metal nanostrips with a median-line integral equation technique,” IEEE Trans. Nanotechnology, Vol. 12, No. 3, 442-9, 2013.
[5] Shapoval O V, Nosich A I, “Resonance effects in the optical antennas shaped as finite comblike gratings of noble-metal nanostrips,” SPIE Proc. 8781 (Integrated Optics: Physics and Simulations), No. 87810U, 1-8, 2013.

[6] Ebbesen T W, Lezec H J, Ghaemi H F, “Extraordinary optical transmission through subwavelength hole arrays,” Nature, Vol. 391, 667-9, 1998.

[7] Glushko O, Bruner R, Meisels R, “Extraordinary transmission in metal hole array-photonic crystal hybrid structures,” Optics Express, Vol. 20, No. 15, 17174-82, 2012.

[8] Bleszynski E, Bleszyski M, and Jaroszewicz T, “Surface-integral equations for electromagnetic scattering from impenetrable and penetrable sheets,” IEEE Antennas Propag. Mag., Vol. 35, No. 6, 14-25, 1993.

[9] Senior T B, and Volakis J L, Approximate Boundary Conditions in Electromagnetics, IEE, London, 1995.

[10] Shapoval O V, Sauleau R, and Nosich A I, “Scattering and absorption of waves by flat material strips analysed using generalized boundary conditions and Nystrom-type algorithm,” IEEE Trans. Antennas Propagat., Vol. Ap-59, No. 9, 3339-46, 2011.

[11] Munk B A, Frequency Selective Surfaces: Theory and Design, John Wiley & Sons, 2000.

[12] Amitay N, Galindo V, “The analysis of circular waveguide phased arrays,” Bell Syst. Tech. J., Vol. 47, No. 9, 1903-32, 1968.

[13] Chen C C, “Diffraction of electromagnetic waves by a conducting screen perforated periodically with circular holes,” IEEE Trans. Microwave Theory Tech., Vol. MTT-19, No. 5, 475-81, 1971.

[14] Chen C C, “Transmission of microwave through perforated flat plates of finite thickness,” IEEE Trans. Microwave Theory Tech., Vol. MTT-21, No. 1, 1-6, 1973.

[15] Koledintseva M Y, Huang J, Drewniak J L, “Modeling of metasheets embedded in dielectric layers,” Progress In Electromagnetics Research B, Vol. 44, 89-116, 2012.

[16] Hamdi B, Aguii T, Baudrand H, “Floquet modal analysis to modelize and study 2-D planar almost periodic structures in finite and infinite extend with coupled motifs,” Progress In Electromagnetics Research B, Vol. 62, 63-86, 2015.

[17] Matsushima A, Zinenko T L, Nishimori H, “Plane wave scattering from perpendicularly crossed multi-layered strip gratings,” Progress In Electromagnetics Research, Vol. 28, 185-203, 2000.

[18] Matsushima A, Momoka Y, Ohtsu M, “Efficient numerical approach to electromagnetic scattering from three-dimensional periodic array of dielectric spheres using sequential accumulation,” Progress In Electromagnetics Research, Vol. 69, 305-22, 2007.

[19] Harrington R F, Field Computation by Moment Methods, Macmillan, New York, 1968.

[20] Braver I M, Fridberg P S, Garb K L, “The behaviour of the electromagnetic field near the edge of resistive half-plane,” IEEE Trans. Antennas Propagat., Vol. AP-36, No. 12, 1760-8, 1988.

Acknowledgments
Financial support for this work was provided by Research Initiation Fund of Liupanshui Normal University: Study on Near Field Enhancement by Means of Metal-dielectric Composite Nanometer Films (LPSSYKYJJ201814) and the Guizhou Provincial Education Department Foundation Project (No. Qianjiaohe KY Zi [2018]373).