An abstract characterization of
Thompson’s group $F$

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Abstract

We show that Thompson’s group $F$ is the symmetry group of the ‘generic idempotent’. That is, take the monoidal category freely generated by an object $A$ and an isomorphism $A \otimes A \to A$; then $F$ is the group of automorphisms of $A$.

1 Introduction

Our purpose in this paper is to clarify an idea concerning Richard Thompson’s group $F$: that it is, in a suitable sense, the automorphism group of some object known only to be isomorphic to a combination of two copies of itself. This general idea has been known for some years, but it does not seem to have been observed until now that it can be formalized very succinctly. We prove that $F$ can be defined as follows. Take the monoidal category freely generated by an object $A$ and an isomorphism $A \otimes A \to A$; then $F$ is the group of automorphisms of $A$. This result first appeared in our 2005 preprint [FL].

Our characterization is distinct from some superficially similar older characterizations. In particular, it is distinct from Higman’s characterization of Thompson’s group $V$ as the automorphism group of a certain free algebra, and of $F$ as the subgroup consisting of the ‘order-preserving’ automorphisms [Hig, Bro, CFP]. It is also distinct from Freyd and Heller’s characterization of $F$ via conjugacy idempotents [FH]. We do not know of any direct way to deduce our characterization from these older ones, or vice versa.

Intuitively, our result means the following. Suppose that we are handed a mathematical object and told only that it is isomorphic to two copies of itself glued together. We do not know what kind of object it is, nor do we know what ‘gluing’ means except that it is some kind of associative operation. On the basis of this information, what automorphisms does our object have? Our result gives the answer: the elements of $F$.

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Our description of $F$ is not only conceptually simple, but is also a member of a well-established family: many entities of interest can be described via free categories with structure. For example, the braided monoidal category freely generated by one object is the sequence $(B_n)_{n \geq 0}$ of Artin braid groups [JS, Mac]. The monoidal category freely generated by a monoid consists of the finite ordinals; in other words, it is the augmented simplex category [Lawv, Mac]. The symmetric monoidal category freely generated by a commutative monoid consists of the finite cardinals. The symmetric monoidal category freely generated by a commutative Frobenius algebra consists of 1-dimensional smooth oriented manifolds and diffeomorphism classes of 2-dimensional cobordisms. (This last example is a strong form of the equivalence between commutative Frobenius algebras and 2-dimensional topological quantum field theories [Dij]; see [Kock], for instance.) In this vein, our result can be expressed as follows: the monoidal category freely generated by an object $A$ and an isomorphism $A \otimes A \cong A$ is equivalent to the groupoid $1 \coprod F$, where $1$ is the trivial group and $\coprod$ is coproduct of groupoids.

Our result is this:

Theorem 1.1 Let $\mathcal{A}$ be the monoidal category freely generated by an idempotent object $(A, \alpha)$. Then $\text{Aut}_{\mathcal{A}}(A)$ is isomorphic to Thompson’s group $F$.

To make this paper accessible to as wide a readership as possible, we give the definition of Thompson’s group and explain the categorical language used in the statement of this theorem (§2). (The only new piece of terminology is ‘idempotent object’, which means an object $A$ together with an isomorphism $\alpha : A \otimes A \cong A$.) But first we discuss earlier characterizations of Thompson’s group.

Related work

Almost as soon as Thompson introduced the group now called $F$, it began to be understood that $F$ was in some sense the automorphism group of an object known only to be isomorphic to a combination of two copies of itself. This intuition is so crucial that it has been formalized in several ways, of which ours is one.

An early such formalization, due to Thompson and Higman, was as follows. A Jónsson–Tarski algebra [JT], or Cantor algebra, is a set $A$ equipped with a bijection $A \times A \rightarrow A$. Thompson’s group $V$ is the automorphism group of the free Jónsson–Tarski algebra on one generator [Hig, CFP]. Thompson’s group $F$ is the subgroup consisting of those automorphisms that are ‘order-preserving’ in a suitable sense [Bro, CFP]. There is a clear resemblance between these descriptions and ours. However, we know of no direct or simple way to deduce our description of $F$ from the earlier one (or indeed the converse).

There is a sense in which our description of $F$ is more direct. Whenever one works with sets and their cartesian products, one automatically introduces a symmetry in the form of the natural isomorphism $X \times Y \cong Y \times X$ for sets $X$ and $Y$. In particular, for sets $X$, there is a nontrivial natural automorphism of
In Thompson and Higman’s description, symmetry is first created (by working with sets) and then destroyed (by restricting to the order-preserving automorphisms). In our approach symmetry is avoided entirely, by working from the start not with sets, but with objects of a (non-symmetric) monoidal category.

This is also what makes it possible to characterize $F$ as the full automorphism group of some algebraic structure, rather than just a subgroup. As far as we know, this is the first such characterization.

Among all the results related to ours, the closest is probably a theorem of Guba and Sapir [GS]. Given any presentation of a monoid, they define what they call its Squier complex, a 2-dimensional complex whose connected-components are the elements of the monoid. Every element of the monoid therefore gives rise to a ‘diagram group’, the fundamental group of the corresponding component. They show that the diagram group of the presentation $\langle x \mid x^2 = x \rangle$ at the element $x$ is $F$. The connection between their result and ours can be summarized as follows. First, the Squier complex of this presentation is (up to homotopy) the 2-skeleton of the classifying space of the monoidal category freely generated by an idempotent object $(A, \alpha)$. (For explanation of the latter phrase, see [2] for classifying spaces of categories, see [Seg], for instance.) Then, the generator $x$ determines a point of the Squier complex, the object $A$ determines a point of the classifying space, and these two points correspond to one another under the homotopy equivalence. Hence the fundamental group at $x$ is the automorphism group of $A$. In this way, their result can be deduced from ours and vice versa.

Some more distant relatives are the results of Brin [Brin2], Delormoy [De1, De2], and, ultimately, McKenzie and Thompson [MT]. In the context of semigroup theory, our work has connections with recent work of Lawson [Laws1, Laws2].

All of these results express how $F$ arises naturally from two very primitive notions: binary operation and associativity. An advantage of our approach is that it makes this idea precise using only standard categorical language, where other approaches have used language invented more or less specifically for the occasion.

A further advantage is that Thompson’s group $V$, and even higher-dimensional versions of it, have similar characterizations: for $V$, just replace ‘monoidal category’ by ‘symmetric monoidal category’, or equally ‘finite-product category’. We do not know whether there is such a characterization of Thompson’s group $T$; using braided monoidal categories gives not $T$, but the braided version of $V$ defined in [Brin1]. Also, given any $n \geq 2$, if we take the monoidal category freely generated by an object $A$ and an isomorphism $A^\otimes n \rightarrow A$ then the automorphism group of $A^\otimes r$ is canonically isomorphic to the generalized Thompson group $F_{n,r}$ of Brown [Bro].

Freyd and Heller also gave a short categorical definition of $F$, different from ours: it is the initial object in the category of groups equipped with a conjugacy-idempotent endomorphism [FH]. Again, there is a striking resemblance between this description and ours; but again, no one (to our knowledge) has been able to find a direct deduction of one from the other.
The category of forests and the free groupoid on it, which appear in §3 below, have been considered independently by Belk [Belk].

We work throughout with strict monoidal categories. (See below for definitions.) However, the non-strict monoidal category freely generated by an idempotent object \((A', \alpha')\) is monoidally equivalent to the strict one, and in particular, the automorphism group of \(A'\) is \(F\). So, for instance, there is an induced homomorphism from \(F\) to the automorphism group of the free Jónsson–Tarski algebra on one generator. We conjecture that this homomorphism is injective and that its image consists of the order-preserving automorphisms.

In §2 we explain all of the terminology used in the statement of Theorem 1.1. The theorem is proved in §3. Our proof involves almost no calculation, but does use some further concepts from category theory, reviewed in the Appendix. (‘Il faut triompher par la pensée et non par le calcul’—Poincaré.)

Some readers may feel that the language used in the statement of the theorem represents quite enough category theory for their taste, even without the further categorical concepts used in the proof. For them we sketch, at the end of §2 an alternative proof, favouring explicit calculation over conceptual argument.

The novelty of this work lies almost entirely in §3 and in the way in which the categorical and algebraic structures are brought together. In particular, the categorical language explained in §2 is absolutely standard; and while not everything in the Appendix is quite so well known, none of it is by any means new.

2 Terminology

Here we explain the terminology in the statement of Theorem 1.1. Further information on Thompson’s group can be found in [CFP]; for more on the categorical language, see [Mac]. We then sketch a calculational proof of Theorem 1.1 requiring no further categorical concepts.

Thompson’s group \(F\)

In the 1960s Richard Thompson discovered three groups, now called \(F\), \(T\) and \(V\), with remarkable properties. The group \(F\), in particular, is one of those mathematical objects that appears in many diverse contexts and has been rediscovered repeatedly. One definition of \(F\) is that it consists of all bijections \(f : [0, 1] \rightarrow [0, 1]\) satisfying

i. \(f\) is piecewise linear (with only finitely many pieces)

ii. the slope (gradient) of each piece is an integer power of 2

\(^1\)One must prevail by thought, not by calculation.
For example, the 3-piece linear function \( f \) satisfying \( f(0) = 0, f(1/4) = 1/2, f(1/2) = 3/4 \) and \( f(1) = 1 \) is an element of \( F \). In a sense that will be made precise, every element of \( F \) can be built from copies of the halving isomorphism \( \alpha : [0,2] \to [0,1] \) and its inverse; this is shown for our example \( f \) in Figure 1.

So if all we knew about \([0,1]\) was that it was isomorphic to two copies of itself glued together, \( F \) would be the group of all automorphisms of \([0,1]\). This is the spirit of our result.

For the proof we will need an alternative, more combinatorial definition of \( F \). In what follows, tree will mean finite, rooted, planar tree, and a tree is binary if precisely two branches grow out of each vertex. Figure 2 shows a pair of binary trees. Except where mentioned, ‘tree’ will mean ‘binary tree’.

For \( n \in \mathbb{N} = \{0,1,2,\ldots\} \), write \( \text{Tr}_n \) for the set of \( n \)-leafed trees. There are no 0-leafed trees, and there is just one 1-leafed tree: the trivial tree \( l \) with no vertices at all. A non-trivial tree consists of two smaller trees joined at the root, so the sets \( \text{Tr}_n \) can be defined inductively by

\[
\text{Tr}_0 = \emptyset, \quad \text{Tr}_1 = \{l\}, \quad \text{Tr}_n = \bigsqcup_{k+m=n} \text{Tr}_k \times \text{Tr}_m \quad (n \geq 2).
\]

By a subtree of a tree we mean a subtree sharing the same root. For example, the tree \( Y \) has exactly three subtrees: itself, the unique two-leafed tree \( Y \), and the one-leafed tree \( l \).

Given \( n \geq i \geq 1 \), we can join to the \( i \)th leaf of any \( n \)-leafed tree \( \tau \) a copy of the two-leafed tree \( Y \), thus forming an \((n+1)\)-leafed tree \( \omega^n_i(\tau) \). This defines a map \( \omega^n_i : \text{Tr}_n \to \text{Tr}_{n+1} \). Whenever \( \tau \) is a subtree of a tree \( \rho \), there is a finite sequence \( \omega^{n_1}_{i_1}, \ldots, \omega^{n_r}_{i_r} \) of maps such that

\[
\rho = \omega^{n_r}_{i_r} \cdots \omega^{n_1}_{i_1}(\tau).
\]

Moreover, for any two trees \( \sigma \) and \( \tau \), there is a smallest tree containing both as subtrees. This can be obtained by superimposing the pictures of \( \sigma \) and \( \tau \).

The following alternative definition of \( F \) is given in [CFP, §2] and in [Belk, 1.2]. Elements of \( F \) are equivalence classes of pairs \((\tau, \tau')\) of trees with the same number of leaves, where the equivalence relation is generated by identifying \((\tau, \tau')\) with \((\omega^n_i(\tau), \omega^n_i(\tau'))\) whenever \( \tau, \tau' \in \text{Tr}_n \) and \( 1 \leq i \leq n \). Write \([\tau, \tau']\) for the equivalence class of a pair \((\tau, \tau')\).

Under this definition, the element of \( F \) shown in Figure 1 is the same as the element \([\tau, \tau']\) shown in Figure 2. In general, \([\tau, \tau']\) can be read as ‘expand

 iii. the coordinates of the endpoints of each piece are dyadic rationals.
according to $\tau'$ then contract according to $\tau'$. (The order is reversed to agree with the convention of writing maps on the left.)

With this in mind, it is clear what the product (composite) $[\tau, \tau'] [\sigma, \sigma']$ must be when $\tau' = \sigma$: simply $[\tau, \sigma']$. In general, there is a tree containing both $\tau'$ and $\sigma$ as subtrees, so there are maps $\omega_{i_1}^{n_1} \cdots \omega_{i_r}^{n_r}(\tau') = \omega_{j_1}^{m_1} \cdots \omega_{j_s}^{m_s}(\sigma)$, and then—inevitably—

$$[\tau, \tau'] [\sigma, \sigma'] = [\omega_{i_1}^{n_1} \cdots \omega_{i_1}^{n_1}(\tau), \omega_{j_1}^{m_1} \cdots \omega_{j_1}^{m_1}(\sigma)].$$

**Monoidal categories**

A monoid is a set $S$ equipped with a function $S \times S \to S$ and an element $1 \in S$ obeying associativity and unit laws. Similarly, a **monoidal category** is a category $\mathcal{M}$ equipped with a functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and an object $I \in \mathcal{M}$ obeying associativity and unit laws. Explicitly, this means that to each pair $(M, N)$ of objects of $\mathcal{M}$ there is assigned an object $M \otimes N$, and to each pair $$(M \xrightarrow{\phi} M', N \xrightarrow{\psi} N')$$ of maps in $\mathcal{M}$ there is assigned a map $M \otimes N \xrightarrow{\phi \otimes \psi} M' \otimes N'$. Functoriality amounts to the equations

$$(\phi' \circ \phi) \otimes (\psi' \circ \psi) = (\phi' \otimes \psi') \circ (\phi \otimes \psi), \quad 1_M \otimes 1_N = 1_{M \otimes N},$$

and the associativity and unit laws apply to maps as well as objects: $(\phi \otimes \psi) \otimes \chi = \phi \otimes (\psi \otimes \chi)$, etc. A **monoidal functor** is a functor $G$ between monoidal categories that preserves the tensor and unit: $G(M \otimes N) = G(M) \otimes G(N)$, etc.

For example, a monoidal category in which the only maps are identities is simply a monoid. The monoidal category $\text{FinOrd}$ of finite ordinals has as objects the natural numbers; a map $m \to n$ is an order-preserving function $\{0, \ldots, m-1\} \to \{0, \ldots, n-1\}$; the tensor product is given on objects by addition and on maps by juxtaposition; the unit object is 0.

The monoidal categories and functors considered in this paper are properly called **strict** monoidal. The more general notion of monoidal category includes such examples as the category of abelian groups, in which the tensor product is only associative and unital up to (suitably coherent) isomorphism.
Freely generated

We defined \( \mathcal{A} \) as the ‘monoidal category freely generated by an idempotent object \((A, \alpha)\)’. Such use of language is standard in category theory, and extends the familiar notion of free structure in algebra. We now explain what it means.

Informally, it means that \( A \) is constructed by starting with an object \( A \) and an isomorphism \( \alpha : A \otimes A \to A \), then adjoining whatever other objects and maps must be present in order for \( \mathcal{A} \) to be a monoidal category. The only equations that hold are those that are forced to hold by the axioms for a monoidal category. Thus, \( A \) has an object \( A \), so it also has an object \( A^\otimes_0 = A \otimes \cdots \otimes A \) for each \( n \geq 0 \) (with \( A^\otimes_0 = I \)). The maps are built up from \( \alpha \) by taking composites, identities, inverses and tensor products: for instance, there is a map \( A \to A \) given as the composite

\[
A \xrightarrow{\alpha^{-1}} A \otimes A \xrightarrow{\alpha^{-1} \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \alpha} A \otimes A \xrightarrow{\alpha} A.
\]

(Compare Figures [1] and [2].)

Precisely, an idempotent object in a monoidal category \( \mathcal{M} \) is an object \( M \in \mathcal{M} \) together with an isomorphism \( \mu : M \otimes M \to M \). (For example, an idempotent object in the monoidal category of sets, where \( \otimes \) is cartesian product, is a Jónsson–Tarski algebra.) A monoidal category freely generated by an idempotent object is a monoidal category \( \mathcal{A} \) together with an idempotent object \((A, \alpha)\) in \( \mathcal{A} \), satisfying the following universal property:

for any monoidal category \( \mathcal{M} \) and idempotent object \((M, \mu)\) in \( \mathcal{M} \), there is a unique monoidal functor \( G : \mathcal{A} \to \mathcal{M} \) such that \( G(A) = M \) and \( G(\alpha) = \mu \).

The universal property determines \((\mathcal{A}, A, \alpha)\) uniquely, up to isomorphism. That such an \((\mathcal{A}, A, \alpha)\) exists at all is true for quite general categorical reasons, although in fact we will construct it explicitly. We call \((A, \alpha)\) the generic idempotent object.

Specifying a monoidal category in this fashion is closely analogous to what one does in algebra when specifying a group, monoid, etc. by a presentation. Suppose, say, that we define a monoid \( E \) by the presentation \( E = \langle e \mid e^2 = e \rangle \).

Informally, this means that \( E \) is constructed by starting with an element \( e \), then adjoining whatever other elements must be present in order for \( E \) to be a monoid, then imposing only those equations that are forced to hold by \( e^2 = e \) and the axioms for a monoid. (Of course, for this particular presentation it is very easy to describe \( E \) explicitly, but for other presentations it is not.) Precisely, it means that \( E \) is a monoid equipped with an idempotent element \( e \) and satisfying the following universal property:

for any monoid \( X \) and idempotent element \( x \in X \), there is a unique monoid homomorphism \( g : E \to X \) such that \( g(e) = x \).

We might call \( E \) the ‘monoid freely generated by an idempotent element’, and \( e \) the ‘generic idempotent element’, since it is idempotent and satisfies no further equations.
Our definition of \( \mathcal{A} \) can be regarded as a categorification of the definition of \( E \). Monoids have become monoidal categories, elements have become objects, monoid homomorphisms have become monoidal functors, and equations (such as \( e^2 = e \)) have become isomorphisms (such as \( \alpha : A \otimes A \to A \)).

For any monoid \( X \), there is a natural one-to-one correspondence between idempotent elements of \( X \) and homomorphisms \( E \to X \). Similarly, for any monoidal category \( \mathcal{M} \), there is a natural one-to-one correspondence between idempotent objects in \( \mathcal{M} \) and monoidal functors \( \mathcal{A} \to \mathcal{M} \).

**Automorphism group**

Any object \( X \) of any category \( \mathcal{X} \) has an automorphism group \( \text{Aut}_\mathcal{X}(X) \). Its elements are the automorphisms of \( X \), that is, the isomorphisms \( X \to X \) in \( \mathcal{X} \). The group structure is given by composition.

This completes the explanation of the language used in Theorem 1.1. We are now in a position to sketch a proof of the theorem based on explicit calculation, which we do for the reasons stated at the end of the Introduction.

Let \( \mathcal{A} \) be the category whose objects are the natural numbers and whose maps \( m \to n \) are the bijections \( f : [0, m] \to [0, n] \) satisfying conditions (i)–(iii) in the definition of \( F \). Then \( \mathcal{A} \) has a monoidal structure given on objects by addition and on maps by juxtaposition, and there is an isomorphism \( \alpha : 1 \otimes 1 = 2 \to 1 \) given by division by 2. We have \( F = \text{Aut}_\mathcal{A}(1) \) by definition, so our task is to show that \( (\mathcal{A}, 1, \alpha) \) has the universal property stated above.

To do this, first consider trees (binary, as usual). Take a monoidal category \( \mathcal{M} \) and an idempotent object \( (\mathcal{M}, \mu) \) in \( \mathcal{M} \). Then any \( n \)-leafed tree \( \tau \) gives rise to an isomorphism \( \mu_\tau : M^\otimes n \to M \); for instance, if \( \tau = Y \) then \( \mu_\tau = \mu \), and if \( \tau = Y \) then \( \mu_\tau \) is the composite

\[
M \otimes M \otimes M \xrightarrow{\mu \otimes 1} M \otimes M \xrightarrow{\mu} M.
\]

More generally, define a forest to be a finite sequence \( (\tau_1, \ldots, \tau_k) \) of trees \((k \geq 0)\), and let us say that this forest has \( n \) leaves and \( k \) roots, where \( n \) is the sum of the numbers of leaves of \( \tau_1, \ldots, \tau_k \). Any forest \( T = (\tau_1, \ldots, \tau_k) \) with \( n \) leaves and \( k \) roots induces an isomorphism

\[
\mu_T = \mu_{\tau_1} \otimes \cdots \otimes \mu_{\tau_k} : M^\otimes n \to M^\otimes k.
\]

Now, it can be shown that any map \( \phi : m \to n \) in \( \mathcal{A} \) factorizes as

\[
\phi = \left( m \xrightarrow{\alpha_{S}^{-1}} p \xrightarrow{\alpha_T} n \right)
\]

for some \( p \in \mathbb{N} \) and forests \( S \) and \( T \). (The method is given in [CFP §2].) It can also be shown that any monoidal functor \( G : \mathcal{A} \to \mathcal{M} \) satisfying \( G(1) = M \)
Figure 3: Steps in the proof

and $G(\alpha) = \mu$ must also satisfy

$$G(\phi) = \left( M^\otimes \mu_2 \rightarrow M^\otimes \mu_T \rightarrow M^\otimes \right).$$

Although $\phi$ may have many factorizations of the form (1), further calculations show that the right-hand side of (2) is independent of the factorization chosen. Further calculations still show that the $G$ thus defined is a functor, and monoidal. The result follows.

3 Proof of the Theorem

In this section we give a conceptual proof of Theorem 1.1.

To do this, we construct the monoidal category $\mathcal{A}$ freely generated by an idempotent object $(A, \alpha)$. The strategy is to start with a very simple object $B$ and apply several left adjoints in succession (Figure 3). On the one hand, this abstract construction makes the universal property of $\mathcal{A}$ automatic. On the other, each step of the construction can be described explicitly, so it will be transparent that $\text{Aut}_\mathcal{A}(A) \cong F$.

On the left of Figure 3 we have the category $\text{Set}^N$ of 'signatures' and the categories of operads, multicategories, monoidal categories and monoidal groupoids, all non-symmetric. The functors $R_i$ are the evident forgetful functors; they have adjoints $L_i$ and $S_3$ as shown. Definitions, and descriptions of these adjoint functors, are given in the Appendix. On the right of Figure 3 the signature $\mathcal{B}$ consists of a single binary operation: $|\mathcal{B}_2| = 1$ and $|\mathcal{B}_n| = 0$ for $n \neq 2$. Then $C = L_1(\mathcal{B})$, etc.; thus, the monoidal category $\mathcal{A}$ is defined by

$$\mathcal{A} = R_3 L_3 L_2 L_1(\mathcal{B}).$$

The main insight of the proof is that a pair of trees as in Figure 2 can be regarded as a span in the category of forests, and multiplication of such pairs in
the Thompson group is nothing more than the usual composition of spans (by pullback). The only significant work in the proof is to establish the latter fact.

The universal property of \( \mathcal{A} \) is immediate:

**Proposition 3.1** \( \mathcal{A} \) is the monoidal category freely generated by an idempotent object.

**Proof** To lighten the notation, write \( R_i(X) \) as \( X \). Then for any monoidal category \( \mathcal{M} \),

\[
\text{MonCat}(\mathcal{A}, \mathcal{M}) \cong \text{MonGpd}(L_3L_2L_1(\mathcal{B}), S_3(\mathcal{M})) \quad (3)
\]

\[
\cong \text{MonCat}(L_2L_1(\mathcal{B}), S_3(\mathcal{M})) \quad (4)
\]

\[
\cong \text{Multicat}(L_1(\mathcal{B}), S_3(\mathcal{M})) \quad (5)
\]

\[
\cong \{ (M, G) \mid M \in \mathcal{M}, G \in \text{Operad}(L_1(\mathcal{B}), \text{End}_{S_3(\mathcal{M})}(M)) \} \quad (6)
\]

\[
\cong \{ (M, \mu) \mid M \in \mathcal{M}, \mu \in \text{Set}^{\mathcal{B}}(\text{End}_{S_3(\mathcal{M})}(M)) \} \quad (7)
\]

\[
\cong \{ \text{idempotent objects in } \mathcal{M} \} \quad (8)
\]

naturally in \( \mathcal{M} \). Most of these isomorphisms are by adjointness; (4) is from the final observation in the section on multicategories in the Appendix; (9) is the fact that a map from \( \mathcal{B} \) to another signature \( \mathcal{B}' \) just picks out an element of \( \mathcal{B}' \), which in this case is the set of maps \( M \otimes 2 \rightarrow M \) in the groupoid \( S_3(\mathcal{M}) \).

Hence \( \mathcal{A} \) represents the functor \( J : \text{MonCat} \rightarrow \text{Set} \) mapping a monoidal category to the set of idempotent objects in it. The generic idempotent object \( (A, \alpha) \in J(\mathcal{A}) \) is obtained by tracing the element \( 1_{\mathcal{A}} \) through the isomorphisms (3)–(9); then \( (\mathcal{A}, A, \alpha) \) has the universal property required. \( \square \)

To obtain an explicit description of \( (\mathcal{A}, A, \alpha) \), and in particular of the automorphism group of \( A \), we go through each step of the construction using the descriptions of the adjoint functors given in the Appendix.

First step: the free operad \( \mathcal{C} = L_1(\mathcal{B}) \) is the operad of (unlabelled, binary) trees; thus, \( \mathcal{C}_n = \text{Tr}_n \) and composition in \( \mathcal{C} \) is by gluing roots to leaves.

Second step: \( \mathcal{D} = L_2(\mathcal{C}) \) is the monoidal category in which objects are natural numbers and maps \( n \rightarrow k \) are forests with \( n \) leaves and \( k \) roots (as defined in (2)). Composition is by gluing; tensor of objects is addition; tensor of maps is juxtaposition.

**Lemma 3.2** The forest category \( \mathcal{D} \) has pullbacks.

**Proof** Any map \( T : n \rightarrow k \) in \( \mathcal{D} \) decomposes uniquely as a tensor product \( T = T_1 \otimes \cdots \otimes T_k \) with \( T_i : n_i \rightarrow 1 \), so it suffices to prove that every diagram of the form

\[
\begin{array}{ccc}
1 & \overset{m}{\underset{\tau}{\searrow}} & m' \\
\phantom{m} & \downarrow & \\
\phantom{m'} & \downarrow & \\
\phantom{m} & \downarrow \tau' & \\
\end{array}
\]

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has a pullback (where $\tau$ and $\tau'$ are trees with $m$ and $m'$ leaves respectively). Indeed, let $\rho$ be the smallest tree containing both $\tau$ and $\tau'$ as subtrees. Then

$$(\tau) \circ (\sigma_1, \ldots, \sigma_m) = \rho = (\tau') \circ (\sigma'_1, \ldots, \sigma'_{m'})$$

for unique $\sigma_i$ and $\sigma'_i$. Writing $p$ for the number of leaves of $\rho$, the square

```
\begin{tikzcd}
(\sigma_1, \ldots, \sigma_m) \arrow{dr}{p} \arrow{ur}{(\sigma'_1, \ldots, \sigma'_{m'})} & \\
m \arrow{d}{(\tau)} & m' \arrow{d}{(\tau')}
\end{tikzcd}
```

is a pullback. \[\square\]

Third step: $\mathcal{E} = L_3(\mathcal{D})$ is the monoidal groupoid in which objects are natural numbers and maps $k' \rightarrow k$ are equivalence classes of spans

```
\begin{tikzcd}
(\tau'_1, \ldots, \tau'_{k'}) \arrow{dr}{n} \arrow{ur}{(\tau_1, \ldots, \tau_k)} & \\
k' \arrow{d}{k'} & k \arrow{d}{k}
\end{tikzcd}
```

in $\mathcal{D}$. Equivalence is generated by declaring this span to be equivalent to

```
\begin{tikzcd}
(\tau'_1, \ldots, \tau'_{k'}) \circ (\rho_1, \ldots, \rho_n) \arrow{dr}{p} & (\tau_1, \ldots, \tau_k) \circ (\rho_1, \ldots, \rho_n) \arrow{ur}
\end{tikzcd}
```

for any forest $(\rho_1, \ldots, \rho_n)$ with $n$ roots (writing $p$ for its number of leaves), and it makes no difference if we insist that all but one of the $\rho_i$s is trivial and the remaining one is the 2-leaved tree $Y$.

Final step: $\mathcal{A}$ is the underlying monoidal category of $\mathcal{E}$. Under the isomorphisms (3)–(9), the identity $1_{\mathcal{A}}$ corresponds to the idempotent object $(1, \alpha)$ in $\mathcal{A}$, where $\alpha$ is the equivalence class of the span

```
\begin{tikzcd}
2 \arrow{dr}{\text{id}} & 2 \arrow{ur}{(Y)}
\end{tikzcd}
```

So to prove Theorem \[\text{1.1}\] we have to show that $\operatorname{Aut}_{\mathcal{A}}(1) \cong F$.

Since $\mathcal{A}$ is a groupoid, $\operatorname{Aut}_{\mathcal{A}}(1)$ consists of all maps $1 \rightarrow 1$ in $\mathcal{A}$. We have just seen that such a map is an equivalence class of pairs $(\tau, \tau')$ of trees with the same number of leaves, where equivalence is generated by $[\tau, \tau'] = [\omega^n_{\tau}(\tau), \omega^n_{\tau'}(\tau')]$ whenever $\tau, \tau' \in \text{Tr}_n$ and $1 \leq i \leq n$. To compose maps

```
\begin{tikzcd}
1 \arrow{dr}{[\sigma, \sigma']} & 1 \arrow{ur}{[\tau, \tau']}
\end{tikzcd}
```

11
form the diagram

\[
\begin{array}{ccc}
\chi_1, \ldots, \chi_m & \xrightarrow{p} & \zeta_1, \ldots, \zeta_n \\
\downarrow m & \quad & \downarrow n \\
\sigma', \zeta_1, \ldots, \zeta_n & \xrightarrow{\sigma} & \sigma', \chi_1, \ldots, \chi_m \\
1 & \quad & 1
\end{array}
\]

in which the square is a pullback; then

\[ [\tau, \tau'] \circ [\sigma, \sigma'] = [\tau \circ (\zeta_1, \ldots, \zeta_n), \sigma' \circ (\chi_1, \ldots, \chi_m)]. \]

There exist \( i_1, \ldots, i_r, n_1, \ldots, n_r \) with the property that for all \( \pi \in \text{Tr}_n, \)

\[ \pi \circ (\zeta_1, \ldots, \zeta_n) = \omega_i^{n_r} \cdots \omega_i^{n_1}(\pi), \]

and similarly \( j_1, \ldots, j_s, m_1, \ldots, m_s \) for \( (\chi_1, \ldots, \chi_m) \). Hence

\[ \omega_i^{n_r} \cdots \omega_i^{n_1}(\tau') = \omega_j^{m_r} \cdots \omega_j^{m_1}(\sigma) \]

and

\[ [\tau, \tau'] \circ [\sigma, \sigma'] = [\omega_i^{n_r} \cdots \omega_i^{n_1}(\tau), \omega_j^{m_r} \cdots \omega_j^{m_1}(\sigma')]. \]

But this description of \( \text{Aut}_{\sigma}(1) \) is exactly the description of \( F \) in \[2\]. Hence \( \text{Aut}_{\sigma}(1) \cong F \), proving Theorem \[1\].

Finally, we remark that the proof can be recast slightly so that the diagram in Figure 3 becomes a chain of adjunctions

\[
\text{Set}^N \rightleftarrows \text{Operad} \rightleftarrows \text{Multicat} \rightleftarrows \text{MonCat} \rightleftarrows \text{MonGpd}^*.
\]

Here \( \text{Multicat}^* \) denotes the category of multicategories equipped with a distinguished object, and similarly \( \text{MonCat}^* \), and \( \text{MonGpd}^* \). The content of the argument is the same.

### A Appendix: Some categorical structures

Here we review some categorical structures used in the proof of Theorem \[1\]: signatures, operads, multicategories, groupoids, and monoidal groupoids. We also review the basic relationships between these structures.

**Signatures**

We use the category \( \text{Set}^N \), the product of \( \mathbb{N} \) copies of the category of sets. Its objects are sequences \((\mathcal{B}_n)_{n \in \mathbb{N}}\) of sets, which can be regarded as signatures for finitary, single-sorted algebraic theories; \( \mathcal{B}_n \) is thought of as the set of \( n \)-ary operations.
Operads

In this section and the next (on multicategories), trees will not be assumed to be binary: any natural number of branches, including 0, may grow out of each vertex.

If \( D \) is an object of a monoidal category \( \mathcal{D} \) then the sequence \( (\mathcal{D}(D^\otimes n, D))_{n \in \mathbb{N}} \) of hom-sets admits certain algebraic operations. This is the archetypal example of an operad. Formally, an operad consists of a sequence \((\mathcal{C}_n)_{n \in \mathbb{N}}\) of sets together with a composition function

\[
\mathcal{C}_n \times \mathcal{C}_{k_1} \times \cdots \times \mathcal{C}_{k_n} \to \mathcal{C}_{k_1 + \cdots + k_n}
\]

for each \( n, k_1, \ldots, k_n \in \mathbb{N} \), and a unit element \( 1 \in \mathcal{C}_1 \), satisfying associativity and unit axioms. An element of \( \mathcal{C}_n \) can be thought of as an \( n \)-ary operation; then composition is as shown in Figure 4 (ignoring the labels \( C, C_i, C_j \)). The associativity and unit axioms imply that every tree of operations has a well-defined composite.

When \( D \) is an object of a monoidal category \( \mathcal{D} \), the ‘archetypal’ operad mentioned has composition

\[
\theta \circ (\theta_1, \ldots, \theta_n) = \theta \circ (\theta_1 \otimes \cdots \otimes \theta_n).
\]

(See [MSS], [May], or [Lei] for more on operads. Monoidal categories, operads and multicategories—see below—each come in symmetric and non-symmetric versions. We use the non-symmetric versions of everything.)

There is an obvious notion of map of operads, giving a category Operad. Any operad has an underlying signature, giving a forgetful functor Operad \( \rightarrow \) Set\(^{\mathbb{N}}\). This has a left adjoint \( L \); in other words, we may form the free operad \( L(\mathcal{B}) \) on any signature \( \mathcal{B} \). The elements of \( (L(\mathcal{B}))_n \) are the \( n \)-leafed trees in which each vertex is labelled by an element of \( \mathcal{B}_k \), where \( k \) is the number of branches growing out of the vertex. Composition in \( L(\mathcal{B}) \) is
given by gluing roots to leaves, and the unit is the trivial tree. For the proof and further details, see [Lei, 2.3].

**Multicategories**

A **multicategory** resembles a category in that it consists of objects, arrows between objects, and a unique composite for every composable diagram of arrows. The only difference is the shape of the arrows, which in a multicategory are of the form

\[
C_1, \ldots, C_n \xrightarrow{\theta} C
\]

where \( n \in \mathbb{N} \) and \( C_1, \ldots, C_n, C \) are objects. Composition is as shown in Figure 1 to each object \( C \) there is assigned an identity arrow \( 1_C : C \xrightarrow{} C \); and associativity and identity axioms hold, so that every tree of arrows has a well-defined composite. The details can be found in [Lam] or [Lei].

A typical example of a multicategory has vector spaces as objects and multilinear maps as arrows. Composition is given by (10). Similarly, any monoidal category has an **underlying multicategory**: the objects are the same, and an arrow (11) in the multicategory is an arrow \( C_1 \otimes \cdots \otimes C_n \xrightarrow{} C \) in the monoidal category.

Let \( \textbf{MonCat} \) be the category of small monoidal categories and monoidal functors (all strict, as usual). Let \( \textbf{Multicat} \) be the category of small multicategories and maps between them (defined in the obvious way). We have just defined a forgetful functor \( \textbf{MonCat} \xrightarrow{} \textbf{Multicat} \). It has a left adjoint \( L \) given a multicategory \( \mathcal{C} \), an object of \( L(\mathcal{C}) \) is a finite sequence \( (C_1, \ldots, C_n) \) of objects of \( \mathcal{C} \), and an arrow in \( L(\mathcal{C}) \) is a finite sequence of arrows in \( \mathcal{C} \). Thus, arrows

\[
C_1^{k_1}, \ldots, C_n^{k_n} \xrightarrow{\theta_1} C_1, \ldots, C_n^{k_n} \xrightarrow{\theta_n} C_n
\]

in \( \mathcal{C} \) give rise to an arrow

\[
(C_1^{k_1}, \ldots, C_n^{k_n}) \xrightarrow{(\theta_1, \ldots, \theta_n)} (C_1, \ldots, C_n)
\]

in \( L(\mathcal{C}) \). Tensor product in \( L(\mathcal{C}) \) is concatenation of sequences.

For example, let 1 be the terminal multicategory, which has one object and one \( n \)-ary arrow for each \( n \in \mathbb{N} \). Then \( L(1) \) is \( \textbf{FinOrd} \), the monoidal category of finite ordinals and order-preserving maps.

A multicategory \( \mathcal{C} \) with only one object is just as an operad: if we call the object \( C \) and write \( \mathcal{C}^n \) for the set of arrows

\[
\underbrace{C, \ldots, C}_n \xrightarrow{} C
\]

then the multicategory structure on \( \mathcal{C} \) is exactly an operad structure on \( (\mathcal{C}^n)_{n \in \mathbb{N}} \). We write this operad as \( \mathcal{C} \), too.

More generally, every object \( C \) of a multicategory \( \mathcal{C} \) has an **endomorphism operad** \( \text{End}_\mathcal{C}(C) \), whose \( n \)-ary operations are the maps

\[
\underbrace{C, \ldots, C}_n \xrightarrow{} C
\]

in \( \mathcal{C} \).
There is a full and faithful inclusion functor \( \text{Operad} \hookrightarrow \text{Multicat} \). If \( \mathcal{C} \) is an operad and \( \mathcal{C}' \) a multicategory then a map \( \mathcal{C}' \rightarrow \mathcal{C} \) of multicategories amounts to an object \( C \in \mathcal{C} \) together with a map \( \mathcal{C}' \rightarrow \text{End}_\mathcal{C}(C) \) of operads.

**Groupoids**

A **groupoid** is a category in which every map is an isomorphism. The inclusion \( \text{Gpd} \hookrightarrow \text{Cat} \) from small groupoids into small categories has a left adjoint, ‘free groupoid’. It is slightly tricky to describe the free groupoid on an arbitrary category [Pa2], but it is straightforward when the category has pullbacks [Pa1]. This case is all that we will need.

We use the notion of bicategory [Bén]. Given a category \( \mathcal{D} \) with pullbacks, first form the bicategory \( \text{Span}(\mathcal{D}) \) of spans in \( \mathcal{D} \) [Bén 2.6]. Then form the groupoid \( \mathcal{E} \) whose objects are those of \( \mathcal{D} \), whose maps \( D' \rightarrow D \) are the connected-components of the hom-category \( (\text{Span}(\mathcal{D}))(D',D) \), and whose composition is inherited from \( \mathcal{D} \). (This is made possible by the fact that the connected-components functor \( \text{Cat} \rightarrow \text{Set} \) preserves finite products.)

The groupoid \( \mathcal{E} \) can be described explicitly. A **span** from \( D' \) to \( D \) is a diagram in \( \mathcal{D} \) of the form

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & D' \\
\phi' \downarrow & & \downarrow \phi \\
D' & \xrightarrow{\phi} & D,
\end{array}
\]

written \( (\phi, \phi') \); note the reversal. Call two such spans \( (\phi, \phi'), (\psi, \psi') \) **equivalent** if there exists a commutative diagram of the form

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & D' \\
\phi' \downarrow & & \downarrow \phi \\
D' & \xrightarrow{\phi} & D, \\
\psi' \downarrow & & \downarrow \psi \\
Y & \xrightarrow{\psi} & D,
\end{array}
\]

and write \( [\phi, \phi'] \) for the equivalence class of a span \( (\phi, \phi') \). Then the objects of the groupoid \( \mathcal{E} \) are those of \( \mathcal{D} \), the maps from \( D' \) to \( D \) in \( \mathcal{E} \) are the equivalence classes of spans from \( D' \) to \( D \) in \( \mathcal{D} \), composition is by pullback, and \( [\phi, \phi']^{-1} = [\phi', \phi] \).

There is a functor \( \eta_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{E} \) defined on objects as the identity and on maps by \( \left( D' \xrightarrow{\theta} D \right) \mapsto [\theta, 1_{D'}] \).

It is straightforward to check that every functor from \( \mathcal{D} \) to a groupoid factors uniquely through \( \eta_{\mathcal{D}} \); hence \( \mathcal{E} \) is the free groupoid on \( \mathcal{D} \).
Let \( \mathbf{Cat}_{pb} \) be the full subcategory of \( \mathbf{Cat} \) consisting of the categories with pullbacks. Since every groupoid has pullbacks, there is a forgetful functor \( R : \mathbf{Gpd} \to \mathbf{Cat}_{pb} \), and we have just constructed its left adjoint \( L \). It is clear from the construction that \( L \), as well as \( R \), preserves finite products.

**Monoidal groupoids**

Let \( \mathbf{MonGpd} \) be the full subcategory of \( \mathbf{MonCat} \) consisting of the monoidal groupoids. The inclusion \( \mathbf{MonGpd} \to \mathbf{MonCat} \) also has a left adjoint, which again is easily described in the presence of pullbacks.

Let \( \mathbf{MonCat}_{pb} \) be the full subcategory of \( \mathbf{MonCat} \) consisting of the monoidal categories with pullbacks. Taking internal monoids throughout the adjunction \( \mathbf{Gpd} \xleftarrow{L} \mathbf{Cat}_{pb} \) gives an adjunction \( \mathbf{MonGpd} \xleftarrow{L} \mathbf{MonCat}_{pb} \). This new \( R \) is the evident forgetful functor. If \( \mathcal{D} \in \mathbf{MonCat}_{pb} \), then \( L(\mathcal{D}) \) is the free groupoid on \( \mathcal{D} \) as constructed above, with monoidal structure inherited from \( \mathcal{D} \) in the obvious way.

The forgetful functor \( \mathbf{MonGpd} \to \mathbf{MonCat} \) also has a right adjoint, sending a monoidal category to its underlying groupoid (the subcategory consisting of all the objects and all the isomorphisms).

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