A REMARK ON THE UENO-CAMPANA’S THREEFOLD

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Dedicated to Fabrizio Catanese on his 65th birthday

Abstract. We show that the Ueno-Campana’s threefold cannot be obtained as the blow-up of any smooth threefold along a smooth centre, answering negatively a question raised by Oguiso and Truong.

1. Introduction

Let $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau)$ be the complex elliptic curve of period $\tau$. There exist exactly two elliptic curves with automorphism group bigger than $\{\pm 1\}$: these are defined respectively by the periods $\sqrt{-1}$ and the cubic root of unity $\omega := (-1 + \sqrt{-3})/2$.

We consider the diagonal action of the cyclic group generated by $\sqrt{-1}$ (resp. $-\omega$) on the product $E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}}$ (resp. $E_\omega \times E_\omega \times E_\omega$) and we denote by $X_4$ (resp. $X_6$) the minimal resolution of their quotients:

$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}}/(\sqrt{-1})$ (resp. $E_\omega \times E_\omega \times E_\omega/(-\omega)$).

The minimal resolutions are obtained by a single blow-up at the maximal ideal of each singular point of the quotients above.

The threefolds $X_4$ and $X_6$ have been extensively studied in the past. In particular, they admit an automorphism of positive entropy (e.g. see \cite{Ogu15} for more details). The variety $X_4$ is now referred as the...
Ueno-Campana’s threefold. It has been recently shown that $X_4$ and $X_6$ are rational. Indeed, Oguiso and Truong [OT15] showed the rationality of $X_6$, and Colliot-Thélène [CT15] showed the rationality of $X_4$, after the work of Catanese, Oguiso and Truong [COT14]. The unirationality of $X_4$ was conjectured by Ueno [Uen75], whilst Campana asked about the rationality of $X_4$ in [Cam11].

The aim of this note is to give a negative answer to the following question raised by Oguiso and Truong (see [Ogu15][Question 5.11] and [Tru15][Question 2]).

**Question 1.1.** Can $X_4$ or $X_6$ be obtained as the blow-up of $\mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along smooth centres?

Our main result is the following:

**Theorem 1.2.** Let $A$ be an abelian variety of dimension three and let $G$ be a finite group acting on $A$ such that the quotient map

$$\rho : A \to Z = A/G$$

is étale in codimension 2.

Assume that there exists a resolution $f : X \to Z$ given by the blow-up of the singular points of $Z$ and such that the exceptional divisor at each singular point of $Z$ is irreducible.

Then $X$ cannot be obtained as the blow-up of a smooth threefold along a smooth centre.

Note that Theorem 1.2 provides a negative answer to Question 1.1. Very recently, Lesieutre [Les15] announced that Question 1.1 admits a negative answer, using different methods.

**2. Preliminary results**

We use some of the methods introduced in [CT14]. Let $X$ be a normal projective threefold with isolated quotient singularities. Given a basis $\gamma_1, \ldots, \gamma_m$ of $H^2(X, \mathbb{C})$, the cubic form associated to $X$ is the homogeneous polynomial of degree 3 defined by:

$$F_X(x_1, \ldots, x_m) = (x_1\gamma_1 + \cdots + x_m\gamma_m)^3 \in \mathbb{C}[x_1, \ldots, x_m].$$

Note that, modulo the natural action of $\text{GL}(m, \mathbb{C})$, the cubic $F_X$ does not depend on the choice of the base and it is a topological invariant of the underlying manifold $X$ (see [OVdV95] for more details). In particular, if

$$\mathcal{H}_{F_X} = (\partial_{x_i}\partial_{x_j} F_X)_{i,j=1, \ldots, m}$$

denotes the Hessian matrix associated to $F_X$ and $p \in H^2(X, \mathbb{C})$, then the rank of $\mathcal{H}_{F_X}$ at $p$ is well-defined.
The following basic tool was used in [CT14] in a more general context. We provide a proof for the reader’s convenience.

**Lemma 2.1.** Let $Y$ be a normal projective threefold with isolated quotient singularities and let $f: X \to Y$ be the blow-up of $Y$ along a point $q \in Y$ (resp. a curve $C \subseteq Y$). Assume that the exceptional divisor of $f$ is irreducible and let $E$ be its class in $H^2(X, \mathbb{C})$.

Then the rank of the Hessian matrix $\mathcal{H}_{F_X}$ of $F_X$ at $E$ is one (resp. at most two).

Note that by [CT14][Lemma 2.7 and Lemma 2.12] the rank of $\mathcal{H}_{F_X}$ is never zero.

**Proof.** We have $H^2(X, \mathbb{C}) = \langle E, f^*(\gamma_1), \ldots, f^*(\gamma_m) \rangle$ where $\gamma_1, \ldots, \gamma_m$ is a basis of $H^2(Y, \mathbb{C})$.

Consider the cubic form $F_X$ associated to $X$ with respect to this basis:

$$F_X(x_0, \ldots, x_m) = (x_0 E + \sum_{i=1}^{m} x_i f^*(\gamma_i))^3.$$ 

Since $f^*(\gamma_i) \cdot f^*(\gamma_j) \cdot E = 0$ for all $i, j = 1, \ldots, m$, we have

$$F_X(x_0, \ldots, x_m) = x_0^3 E^3 + 3 \sum_{i=1}^{m} x_0 x_i E^2 f^*(\gamma_i) + \left( \sum_{i=1}^{m} x_i f^*(\gamma_i) \right)^3.$$ 

Let $a = E^3$ and let $b_i = E^2 f^*(\gamma_i)$ for $i = 1, \ldots, m$. Note that if $f$ is the blow-up of a point $q \in Y$ then $b_1 = \ldots = b_m = 0$.

Thus, we have

$$F_X(x_0, \ldots, x_m) = a x_0^3 + 3 \sum_{i=1}^{m} b_i x_0^2 x_i + G(x_1, \ldots, x_m),$$

where $G$ is a homogeneous cubic polynomial in the variables $x_1, \ldots, x_m$, i.e. it does not depend on $x_0$. Let $p = y_0 E + \sum_{i=1}^{m} y_i f^*(\gamma_i) \in H^2(X, \mathbb{C})$, for some $y_0, \ldots, y_m \in \mathbb{C}$ and let $p' = (y_1, \ldots, y_m)$. After removing the first row and the first column, the Hessian matrix $\mathcal{H}_{F_X}(p)$ of $F_X$ at $p$, coincides with the Hessian matrix $\mathcal{H}_G(p')$ of $G$ at $p'$.

In particular, if $p = E$, then $p' = (0, \ldots, 0)$ and $\mathcal{H}_G(p')$ is the zero matrix. Thus, the rank of the Hessian of $F_X$ at $p$ is at most two. In addition, if $b_1 = \ldots = b_m = 0$, then the rank of $\mathcal{H}_F$ at $p$ is exactly one. □

3. **Proofs**

**Lemma 3.1.** Let $A$ be an abelian variety of dimension 3 and let $G$ be a finite group acting on $A$ such that the quotient map $\rho: A \to Z = A/G$
is étale in codimension 2. Let $F_Z$ be the cubic form associated to $Z$ and let $p \in H^2(Z, \mathbb{C})$ such that $\text{rk} \mathcal{H}_{F_Z}(p) \leq 1$.

Then $p = 0$.

**Proof.** The morphism $\rho$ induces an immersion of vector spaces

$$\rho^*: H^2(Z, \mathbb{C}) \to H^2(A, \mathbb{C}).$$

Thus, there exists a basis of $H^2(A, \mathbb{C})$ such that if $F_A$ is the cubic associated to $A$ with respect to this basis and $d$ is the degree of $\rho$, then

$$F_Z(x_1, \ldots, x_m) = d \cdot F_A(x_1, \ldots, x_m, 0, \ldots, 0).$$

It is enough to show that if $q \in H^2(A, \mathbb{C})$ is such that the rank of $\mathcal{H}_{F_A}$ at $q$ is not greater than one, then $q = 0$.

Write $A = \mathbb{C}^3/\Gamma$ and consider $z_1, z_2, z_3$ coordinates on $\mathbb{C}^3$. Then a basis of $H^2(A, \mathbb{C})$ is given by

$$z_{ij} = dz_i \wedge dz_j \quad 1 \leq i < j \leq 3,$$

$$z_{ij} = dz_i \wedge d\bar{z}_j \quad i, j \in \{1, 2, 3\},$$

$$z_{ij} = d\bar{z}_i \wedge d\bar{z}_j \quad 1 \leq i < j \leq 3.$$

For any $x \in H^2(A, \mathbb{C})$, let $x_{ij}, x_{ij}$ and $x_{ij}$ be the coordinates of $x$ with respect to the basis above and let $F_A'$ be the cubic associated to this basis. It is enough to show that if $q \in H^2(A, \mathbb{C})$ is such that the rank of $\mathcal{H}_{F_A}$ at $q$ is not greater than one, then $q = 0$.

Let $q_{ij}, q_{ij}$ and $q_{ij}$ be the coordinates of $q$.

The $(2 \times 2)$-minor of $\mathcal{H}_{F_A'}$ at $x$ defined by the rows corresponding to $x_{12}$ and $x_{13}$ and the columns corresponding to $x_{21}$ and $x_{31}$ is given by

$$\begin{pmatrix} 0 & 6x_{23} \\ 6x_{23} & 0 \end{pmatrix}.$$ 

It follows that $q_{23} = 0$. By choosing suitable $(2 \times 2)$-minors, it follows easily that each coordinate of $q$ is zero. Thus, the claim follows. \qed

**Proof of Theorem 1.2.** Suppose not. Then there exists a smooth projective threefold $Y$ such that $X$ can be obtained as the blow-up $g: X \to Y$ at a smooth centre. Let $E$ be the exceptional divisor of $g$. Let $k$ be the number of singular points of $Z$ and let $E_1, \ldots, E_k$ be the exceptional divisors on $X$ corresponding to the singular points of $Z$.

We want to prove that $E = E_i$ for some $i = 1, \ldots, k$. Denote by $p$ the class of $E$ in $H^2(X, \mathbb{C})$. Lemma 2.1 implies that the rank of $\mathcal{H}_{F_X}$ at $p$ is not greater than two.
Let $\gamma_1, \ldots, \gamma_m \in H^2(Z, \mathbb{C})$ be a basis and let $F_Z$ be the associated cubic form. Then $f^*\gamma_1, \ldots, f^*\gamma_m, [E_1], \ldots, [E_k]$ is a basis of $H^2(X, \mathbb{C})$ and if $F_X$ denotes the associated cubic form, we have

$$F_X(x_1, \ldots, x_m, y_1, \ldots, y_k) = F_Z(x_1, \ldots, x_m) + \sum_{i=1}^k a_i y_i^3,$$

where $a_i = E_i^3$ is a non-zero integer, for $i = 1, \ldots, k$.

Thus, the Hessian matrix of $F_X$ is composed by two blocks: one is the Hessian matrix of $F_Z$ and the other one is a diagonal matrix, whose only non-zero entries are $6a_i$ for $i = 1, \ldots, k$. We may write $p = (p^0, p^1) = (p_1^0, \ldots, p_m^0, p_1^1, \ldots, p_k^1)$. We have $\text{rk} \mathcal{H}_F(p^0) \leq 2$.

We distinguish two cases. If $\text{rk} \mathcal{H}_F(p^0) = 2$, then $p^1 = (0, \ldots, 0)$ and in particular $E$ is numerically equivalent to $f^*D$, for some pseudo-effective Cartier divisor $D$ on $Z$. Since $A$ is abelian, it follows that $\rho^*D$ is a nef divisor. Thus $E$ is nef, a contradiction.

If $\text{rk} \mathcal{H}_F(p^0) \leq 1$, then Lemma 3.1 implies that $p^0 = 0$. Thus,

$$E \equiv c_s E_s + c_t E_t$$

for some distinct $s, t \in \{1, \ldots, k\}$ and $c_s, c_t$ rational numbers. Since $E$ is effective non-trivial, at least one of the $c_i$ is positive. By symmetry, we may assume $c_s > 0$. By the negativity lemma, the divisor $E_s$ is covered by rational curves $C$ such that $E_s \cdot C < 0$. Since $E_s$ and $E_t$ are disjoint, it follows that $E \cdot C < 0$, which implies that $C$ is contained in $E$. Thus $E_s$ is contained in $E$. Since $E$ is prime, it follows that $E = E_s$ and $c_t = 0$.

Finally, note that $g$ contracts $E = E_s$ to a point, as otherwise there exists a small contraction $\eta: Y \to Z$ and in particular $Z$ is not $\mathbb{Q}$-factorial, a contradiction. Thus, $\rho: X \to Y$ is the contraction of $E_s$ to the corresponding singular point on $Z$, which is again a contradiction. The claim follows.

**Remark 3.2.** As K. Oguiso kindly pointed out to us, the same proof shows that if $f: X \to Z$ is as in Theorem 1.2 and $g$ is an automorphism on $X$ then the set of exceptional divisors of $f$ is invariant with respect to $g$. Thus, there exists a positive integer $m$ such that the power $g^m$ descends to an automorphism on $Z$.

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