LYAPUNOV–TYPE INEQUALITIES FOR FRACTIONAL DIFFERENTIAL EQUATIONS UNDER MULTI–POINT BOUNDARY CONDITIONS

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Abstract. In this work, we establish new Lyapunov-type inequalities for fractional differential equations under multi-point boundary conditions.

1. Introduction

The well-known result of Lyapunov [9] states that if \( u(t) \) is a nontrivial solution of the differential system

\[
\begin{align*}
\frac{d^2u}{dt^2}(t) + r(t)u(t) &= 0, & t \in (a, b), \\
u(a) &= u(b) = 0,
\end{align*}
\]

where \( r(t) \) is a continuous function defined in \( [a, b] \), then

\[
\int_a^b |r(t)|\,dt > \frac{4}{b-a},
\]

and the constant 4 cannot be replaced by a larger number.

Lyapunov inequality (1.2) is a useful tool in various branches of mathematics including disconjugacy, oscillation theory, and eigenvalue problems. Many improvements and generalizations of the inequality (1.2) have appeared in the literature. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey articles by Cheng [3], Brown and Hinton [1] and Tiryaki [12].

The study of Lyapunov-type inequalities for the differential equation depends on a fractional differential operator was initiated by Rui A. C. Ferreira [4]. He first obtained a Lyapunov-type inequality when the differential equation depends on the Riemann-Liouville fractional derivative, the main result is as follows.

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THEOREM 1.1. If the following fractional boundary value problem
\begin{equation}
(D_{a^+}^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2, \tag{1.3}
\end{equation}
\begin{equation}
u(a) = 0 = u(b), \tag{1.4}
\end{equation}
has a nontrivial solution, where $q$ is a real and continuous function, then
\begin{equation}
\int_a^b |q(s)|ds > \Gamma(\alpha) \left( \frac{4}{b-a} \right)^{\alpha-1}. \tag{1.5}
\end{equation}

Recently, some Lyapunov-type inequalities were obtained for different fractional boundary value problems. In this direction, we refer to Ferreira [5], Jleli and Samet [6,7], O’Regan and Samet [10], Rong and Bai [11], Wang, Liang and Xia [13] and Cabrera, Sadarangani, and Samet [2].

For example, Cabrera, Sadarangani, and Samet [2] obtain some Lyapunov-type inequalities for a higher-order nonlocal fractional boundary value problem, they give the following Lyapunov inequalities.

THEOREM 1.2. If the fractional boundary value problem
\begin{equation}
(D_{a^+}^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 2 < \alpha \leq 3, \tag{1.6}
\end{equation}
\begin{equation}
u(a) = u'(a) = 0, \quad u'(b) = \beta u(\xi), \tag{1.7}
\end{equation}
has a nontrivial solution, where $q$ is a real and continuous function, $a < \xi < b, 0 \leq \beta(\xi-a)^{\alpha-1} < (\alpha-1)(b-a)^{\alpha-2}$, then
\begin{equation}
\int_a^b (b-s)^{\alpha-2}(s-a)|q(s)|ds \geq \left( 1 + \frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}} \right)^{-1} \Gamma(\alpha). \tag{1.8}
\end{equation}

THEOREM 1.3. If the fractional boundary value problem
\begin{equation}
(D_{a^+}^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 2 < \alpha \leq 3, \tag{1.9}
\end{equation}
\begin{equation}
u(a) = u'(a) = 0, \quad u'(b) = \beta u(\xi), \tag{1.10}
\end{equation}
has a nontrivial solution, where $q$ is a real and continuous function, $a < \xi < b, 0 \leq \beta(\xi-a)^{\alpha-1} < (\alpha-1)(b-a)^{\alpha-2}$, then
\begin{equation}
\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}} \left( 1 + \frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}} \right)^{-1}. \tag{1.11}
\end{equation}

Motivated by [2], in this paper, we study the problem of finding some Lyapunov-type inequalities for the fractional differential equations with multi-point boundary conditions.
\begin{equation}
(D_{a^+}^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2, \tag{1.12}
\end{equation}
\begin{equation}u(a) = 0, \quad (D_{a^+}^\beta u)(b) = \sum_{i=1}^{m-2} b_i (D_{a^+}^\beta u)(\xi_i), \tag{1.13}
\end{equation}
where \( D_{a^+}^{\alpha} \) denotes the standard Riemann-Liouville fractional derivative of order \( \alpha \), \( \alpha > \beta + 1, 0 \leq \beta < 1 \), \( a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b \), \( b_i \geq 0 (i = 1, 2, \cdots, m-2), 0 \leq \sum_{i=1}^{m-2} b_i (\xi_i - a)^{\alpha - \beta - 1} < (b-a)^{\alpha - \beta - 1} \) and \( q : [a,b] \to \mathbb{R} \) is a continuous function.

2. Preliminaries

In this section, we recall the concepts of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative of order \( \alpha \geq 0 \).

**Definition 2.1.** [8] Let \( \alpha \geq 0 \) and \( f \) be a real function defined on \([a,b]\). The Riemann-Liouville fractional integral of order \( \alpha \) is defined by \((I_{a^+}^{\alpha} f)(t) \equiv f\) and

\[
(I_{a^+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \ t \in [a,b].
\]

**Definition 2.2.** [8] The Riemann-Liouville fractional derivative of order \( \alpha \geq 0 \) is defined by \((D_{a^+}^{\alpha} f) \equiv f\) and

\[
(D_{a^+}^{\alpha} f)(t) = (D_{a^+}^{m}(I_{a^+}^{\alpha-m} f))(t) = \frac{1}{\Gamma(m-\alpha)} \left( \frac{d}{dt} \right)^{m} \int_{a}^{t} (t-s)^{m-\alpha-1} f(s)ds,
\]

for \( \alpha > 0 \), where \( m \) is the smallest integer greater or equal to \( \alpha \).

**Lemma 2.3.** [8] Assume that \( u \in C(a,b) \cap L(a,b) \) with a fractional derivative of order \( \alpha > 0 \) that belongs to \( C(a,b) \cap L(a,b) \). Then

\[
I_{a^+}^{\alpha}(D_{a^+}^{\alpha} u)(t) = u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n},
\]

where \( c_i \in \mathbb{R}, i = 1, 2, \cdots, n \), and \( n = [\alpha] + 1 \).

**Lemma 2.4.** For \( 1 < \alpha \leq 2, 0 \leq \beta < 1 \), we have

\[
(D_{a^+}^{\beta} (s-a)^{\alpha-1})(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta)} (t-a)^{\alpha - \beta - 1}.
\]

3. Main results

We begin by writing problems (1.12)-(1.13) in its equivalent integral form.

**Lemma 3.1.** We have that \( u \in C[a,b] \) is a solution to the boundary value problem (1.12)-(1.13) if and only if \( u \) satisfies the integral equation

\[
u(t) = \int_{a}^{b} G(t,s)q(s)u(s)ds + T(t) \int_{a}^{b} \left( \sum_{i=1}^{m-2} b_i G(\xi_i,s)q(s)u(s) \right) ds, \quad (3.1)
\]
where Green’s function \( G(t,s) \) is defined by

\[
G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} 
\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\
\frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a \leq t \leq s \leq b.
\end{cases}
\]

\[
T(t) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1} - \sum_{i=1}^{m-2} b_i (\xi_i - a)^{\alpha-1}}, & a \leq t \leq b.
\]

**Proof.** From Lemma 2.3, \( u \in C[a,b] \) is a solution to the boundary value problem (1.12)-(1.13) if and only if

\[
u(t) = c_1 (t-a)^{\alpha-1} + c_2 (t-a)^{\alpha-2} - (I_a^\alpha qu)(t),
\]

for some real constants \( c_1, c_2 \). Using the boundary condition \( u(a) = 0 \), we obtain \( c_2 = 0 \). Therefore

\[
u(t) = c_1 (t-a)^{\alpha-1} - (I_a^\alpha qu)(t).
\]

We apply the operator \( D_a^\beta \) to both side of above equation, we obtain

\[
(D_a^\beta u)(t) = c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1} - (I_a^\alpha qu)(t)
\]

\[
= c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1} - \frac{1}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{\alpha-\beta-1} q(s)u(s)ds,
\]

the boundary condition \( (D_a^\beta u)(b) = \sum_{i=1}^{m-2} b_i (D_a^\beta u)(\xi_i) \) imply that

\[
c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(b-a)^{\alpha-\beta-1} - \frac{1}{\Gamma(\alpha-\beta)} \int_a^b (b-s)^{\alpha-\beta-1} q(s)u(s)ds
\]

\[
= \sum_{i=1}^{m-2} b_i \left[ c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\xi_i - a)^{\alpha-\beta-1} - \frac{1}{\Gamma(\alpha-\beta)} \int_a^{\xi_i} (\xi_i - s)^{\alpha-\beta-1} q(s)u(s)ds \right],
\]

thus

\[
c_1 = \frac{1}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i (\xi_i - a)^{\alpha-\beta-1}]\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-\beta-1} q(s)u(s)ds
\]

\[
- \frac{1}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i (\xi_i - a)^{\alpha-\beta-1}]\Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \int_a^{\xi_i} (\xi_i - s)^{\alpha-\beta-1} q(s)u(s)ds.
\]

By the relation

\[
\frac{1}{(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i (\xi_i - a)^{\alpha-\beta-1}} = \frac{1}{(b-a)^{\alpha-\beta-1}} \sum_{i=1}^{m-2} b_i (\xi_i - a)^{\alpha-\beta-1}
\]

\[
= \frac{1}{(b-a)^{\alpha-\beta-1}} + \frac{\sum_{i=1}^{m-2} b_i (\xi_i - a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i (\xi_i - a)^{\alpha-\beta-1}]}.
\]
we obtain
\[
c_1 = \frac{1}{\Gamma(\alpha)} \int_a^b \frac{(b-s)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} q(s) u(s) ds + \frac{\sum_{i=1}^{m-2} b_i \int_a^b \frac{(\xi_i-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} q(s) u(s) ds}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i (\xi_i-a)^{\alpha-\beta-1}] \Gamma(\alpha)}
\]
therefore
\[
u(t) = c_1(t-a)^{\alpha-1} - (I_{a^+}^\alpha qu)(t)
\]
\[
= \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b \frac{(b-s)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} q(s) u(s) ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s) u(s) ds + \frac{(t-a)^{\alpha-1} \sum_{i=1}^{m-2} b_i \int_a^b \frac{(\xi_i-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} q(s) u(s) ds}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i (\xi_i-a)^{\alpha-\beta-1}] \Gamma(\alpha)}
\]
\[
- \frac{(t-a)^{\alpha-1} \sum_{i=1}^{m-2} b_i \int_a^b \frac{(\xi_i-s)^{\alpha-1}}{(b-a)^{\alpha-\beta-1}} q(s) u(s) ds}{[(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i (\xi_i-a)^{\alpha-\beta-1}] \Gamma(\alpha)}
\]
\[
= \int_a^b G(t,s) q(s) u(s) ds + T(t) \int_a^b \left( \sum_{i=1}^{m-2} b_i G(\xi_i,s) q(s) u(s) \right) ds,
\]
which concludes the proof. \(\square\)

**Lemma 3.2.** The Green’s function \(G\) defined in Lemma 3.1 satisfies the following properties:

(i) \(0 \leq G(t,s) \leq G(s,s) = \frac{(s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1} \Gamma(\alpha)},\)

(ii) For any \(s \in [a,b],\)

\[\max_{s \in [a,b]} G(s,s) = G(s^*,s^*) = (\alpha - \beta - 1)^{\alpha-\beta-1} \frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}{(2\alpha-\beta-2)^{2\alpha-\beta-2} \Gamma(\alpha)},\]

where \(s^*= \frac{\alpha-\beta-1}{2\alpha-\beta-2}a + \frac{\alpha-1}{2\alpha-\beta-2}b.\)

**Proof.** (i) Let us define two functions
\[
g_1(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} - (t-s)^{\alpha-1}, \quad a \leq s \leq t \leq b,
\]
\[
g_2(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}}, \quad a \leq t \leq s \leq b.
\]
Obviously, \(g_2(t,s)\) is an increasing function in \(t\) and \(0 \leq g_2(t,s) \leq g_2(s,s).\) Now we turn our attention to the function \(g_1(t,s).\) By the relation \(\alpha > \beta + 1, 2 - \alpha \geq 0,\) we
have $0 < \left( \frac{b-s}{b-a} \right)^{\alpha-\beta-1} \leq 1$, $0 < \frac{1}{(t-a)^{\alpha-\beta}} \leq \frac{1}{(t-s)^{\alpha-\beta}}$, so we obtain

$$\frac{\partial g_1(t,s)}{\partial t} = (\alpha - 1) \left[ \left( \frac{b-s}{b-a} \right)^{\alpha-\beta-1} \frac{1}{(t-a)^{2-\alpha}} - \frac{1}{(t-s)^{2-\alpha}} \right] \leq 0.$$  

Hence, for a given $s \in [a,b]$, $g_1(t,s)$ is an non-increasing function of $t \in [s,b]$. Therefore, we have

$$g_1(b,s) \leq g_1(t,s) \leq g_1(s,s).$$  

As

$$g_1(b,s) = \frac{(b-a)^{1-\alpha}(b-s)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1}} - (b-s)^{\alpha-1}$$

$$= (b-a)^{1-\alpha}(b-s)^{\alpha-\beta-1} - (b-s)^{\alpha-1}$$

$$= (b-a)^{\alpha-1}(b-s)^{\alpha-\beta} - \frac{1}{(b-a)^{\alpha-1}}$$

$$\geq 0,$$

so we get

$$0 \leq g_1(t,s) \leq g_1(s,s),$$

thus

$$0 \leq G(t,s) \leq G(s,s).$$

(ii) Let $\varphi(s) = (s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}$, $s \in [a,b]$, then

$$\varphi'(s) = (s-a)^{\alpha-2}(b-s)^{\alpha-\beta-2}[(\alpha-1)(b-s) - (\alpha-\beta-1)(s-a)]$$

moreover,

$$\varphi'(s) = 0, s \in (a,b) \iff s = s^* = \frac{\alpha-\beta-1}{2\alpha-\beta-2} a + \frac{\alpha-1}{2\alpha-\beta-2} b.$$  

It is easy to check that $\varphi''(s) < 0, s \in (a,b)$, therefore,

$$\max_{s \in [a,b]} \varphi(s) = \varphi(s^*) = (\alpha - \beta - 1)^{\alpha-\beta-1}(\frac{\alpha-1}{2\alpha-\beta-2}a)^{\alpha-\beta-1}(\frac{\alpha-1}{2\alpha-\beta-2}a)^{\alpha-1},$$

hence

$$\max_{s \in [a,b]} G(s,s) = G(s^*,s^*) = (\alpha - \beta - 1)^{\alpha-\beta-1}(\frac{\alpha-1}{2\alpha-\beta-2})^{\alpha-1}(\frac{\alpha-1}{2\alpha-\beta-2})^{\alpha-1}(2\alpha-\beta-2)^{-\alpha-\beta-2} \Gamma(\alpha).$$

Now, we are ready to prove our first Lyapunov-type inequality.
Theorem 3.3. If a nontrivial continuous solution of the fractional boundary value problem
\[(D^\alpha_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2,\]
\[u(a) = 0, \quad (D^\beta_{a^+}u)(b) = \sum_{i=1}^{m-2} b_i(D^\beta_{a^+}u)(\xi_i),\]
exists, then
\[
\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}|q(s)|ds
\]
\[\geq (b-a)^{\alpha-\beta-1} \frac{(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1} + (b-a)^{\alpha-1} \sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1}} \Gamma(\alpha).
\]

Proof. Let \(B = C[a,b]\) be the Banach space endowed with norm \(\|u\| = \sup_{t \in [a,b]} |u(t)|\).
It follows from Lemma 3.1 that a solution \(u\) to the boundary value problem satisfies the integral equation
\[u(t) = \int_a^b G(t,s)q(s)u(s)ds + T(t)\int_a^b \left( \sum_{i=1}^{m-2} b_i G(\xi_i, s)q(s)u(s) \right) ds.
\]
Now, using Lemma 3.2 (i), we obtain
\[\|u\| \leq \|u\| \int_a^b |G(s,s)||q(s)|ds + \|u\| \sum_{i=1}^{m-2} b_i T(b) \int_a^b |G(s,s)||q(s)|ds,
\]
which yields
\[\|u\| \leq \|u\| \int_a^b \left( 1 + \sum_{i=1}^{m-2} b_i T(b) \right) G(s,s)|q(s)|ds.
\]
Therefore, if \(u\) is a nontrivial continuous solution to (1.12)-(1.13), we have
\[
\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-\beta-1}|q(s)|ds \geq \frac{(b-a)^{\alpha-\beta-1}\Gamma(\alpha)}{1 + \sum_{i=1}^{m-2} b_i T(b)}
\]
\[= (b-a)^{\alpha-\beta-1} \frac{(b-a)^{\alpha-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1}}{(b-a)^{\alpha-\beta-1} + (b-a)^{\alpha-1} \sum_{i=1}^{m-2} b_i - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1}} \Gamma(\alpha). \quad \Box
\]

Now, from Theorem 3.3 and Lemma 3.2 (ii), if problem (1.12)-(1.13) has a nontrivial continuous solution, then we have the following result.

Corollary 3.4. If a nontrivial continuous solution of the fractional boundary value problem
\[(D^\alpha_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2,\]
\[u(a) = 0, \quad (D^\beta_{a^+}u)(b) = \sum_{i=1}^{m-2} b_i(D^\beta_{a^+}u)(\xi_i),\]
exists, then

$$
\int_{a}^{b} |q(s)| ds \\
\geq \frac{\Gamma(\alpha)}{[(\alpha - 1)(b - a)]^{\alpha - 1}} \frac{(2\alpha - \beta - 2)^{2\alpha - \beta - 2}}{(\alpha - \beta - 1)^{\alpha - \beta - 1}} \frac{(b - a)^{\alpha - \beta - 1} - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha - \beta - 1}}{(b - a)^{\alpha - \beta - 1} \Gamma(\alpha) + (b - a)^{\alpha - \beta - 1} \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha - \beta - 1}}.
$$

Let $\beta = 0$ in Theorem 3.3, we obtain

**Corollary 3.5.** If a nontrivial continuous solution of the fractional boundary value problem

$$(D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2,$$

$$u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

exists, then

$$
\int_{a}^{b} |q(s)| ds \\
\geq (b - a)^{\alpha - 1} \frac{(b - a)^{\alpha - 1} - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha - 1}}{(b - a)^{\alpha - 1} \Gamma(\alpha) + (b - a)^{\alpha - 1} \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha - 1}} \Gamma(\alpha).
$$

Let $\beta = 0$ in Corollary 3.4, we have the following result.

**Corollary 3.6.** If a nontrivial continuous solution of the fractional boundary value problem

$$(D_{a+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2,$$

$$u(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

exists, then

$$
\int_{a}^{b} |q(s)| ds \geq \left( \frac{4}{b - a} \right)^{\alpha - 1} \frac{(b - a)^{\alpha - 1} - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha - 1}}{(b - a)^{\alpha - 1} \Gamma(\alpha) + (b - a)^{\alpha - 1} \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha - 1}} \Gamma(\alpha).
$$

**Remark 3.7.** Let $b_1 = b_2 = \cdots = b_{m-2} = 0$ in Corollary 3.6, then we obtain (1.5).
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