\(\mathcal{PT}\) symmetry of a square-wave modulated two-level system

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We study a non-Hermitian two-level system with square-wave modulated dissipation and coupling. Based on the Floquet theory, we achieve an effective Hamiltonian from which the boundaries of the \(\mathcal{PT}\) phase diagram are captured exactly. Two kinds of \(\mathcal{PT}\) symmetry broken phases are found whose effective Hamiltonians differ by a constant \(\omega/2\). For the time-periodic dissipation, a vanishingly small dissipation strength can lead to the \(\mathcal{PT}\) symmetry breaking in the \((2k - 1)\)-photon resonance \((\Delta = (2k - 1)\omega), \) with \(k = 1, 2, 3 \ldots \) It is worth noting that such a phenomenon can also happen in \(2k\)-photon resonance \((\Delta = 2k\omega), \) as long as the dissipation strengths or the driving times are imbalanced, namely \(\gamma_0 \neq -\gamma_1 \) or \(T_0 \neq T_1\). For the time-periodic coupling, the weak dissipation induced \(\mathcal{PT}\) symmetry breaking occurs at \(\Delta_{\text{eff}} = \omega/k, \) where \(\Delta_{\text{eff}} = (\Delta_0 T_0 + \Delta_1 T_1) / T\). In the high frequency limit, the phase boundary is given by a simple relation \(\gamma_{\text{eff}} = \pm \Delta_{\text{eff}}\).

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I. INTRODUCTION

A non-Hermitian Hamiltonian is a natural extension of the conventional Hermitian one to describe the open quantum system. The discovery of the real spectra in non-Hermitian Hamiltonians by Bender and Boettcher \(^1\) has stimulated enormous interests in the systems with parity-time (\(\mathcal{PT}\)) symmetry. \(^2\)\(^,\)\(^3\) Early theoretical and experimental explorations of the non-Hermitian systems with \(\mathcal{PT}\) symmetry mainly focus on the optics and photonics \(^4\)\(^-\)\(^17\). Feng et al. realized the nonreciprocal light propagation in a Silicon photonic circuit which provides a way to chip-scale optical isolators for optical communications and computing. \(^12\) Hodaei et al. stabilized single-longitudinal mode operation in a system of coupled mirroring lasers by harnessing notions from \(\mathcal{PT}\) symmetry, which provides the possibilities to develop optical devices with enhanced functionality. \(^14\) Xiao et al. achieved the first experimental characterization of critical phenomena in \(\mathcal{PT}\)-symmetric nonunitary quantum dynamics. \(^17\) Recent experiments have realized the non-Hermitian Magnon-polaritons systems, and higher-order exceptional points were observed which can be used to measuring the output spectrum of the cavity. \(^18\)\(^-\)\(^20\) The anomalous edge state in a non-Hermitian lattice \(^21\) has intrigued persistent attention to the combination of the non-Hermiticity and the topological phase. \(^22\)\(^-\)\(^32\) The non-Bloch band theory has been developed to describe the non-Hermitian lattice systems. \(^22\)\(^-\)\(^29\) Kawabata et al. established a fundamental symmetry principle in non-Hermitian physics which paved the way towards a unified framework for non-equilibrium topological phase. \(^25\)\(^,\)\(^26\) Yao et al. studied the bulk-boundary correspondence in the non-Hermitian systems and found the non-Hermitian skin effect \(^29\)\(^,\)\(^30\). Xiao et al. observed the topological edge states in \(\mathcal{PT}\)-symmetric quantum walks. \(^31\)

Recently, Joglekar et al. investigated a two-level system coupled to a sinusoidally varying gain-loss potential, namely, the non-Hermitian Rabi model with time-periodic dissipation. \(^38\) They found that there existed multiple frequency windows where \(\mathcal{PT}\) symmetry was broken and restored. The non-Hermitian Rabi model has drawn growing attention due to its especially rich phenomena which are absent in the static counterparts. \(^37\)\(^-\)\(^47\) Lee et al. found the \(\mathcal{PT}\) symmetry breaking at the \((2k - 1)\)-photon resonance and derived the boundaries of the \(\mathcal{PT}\) phase diagram by doing perturbation theory beyond rotating-wave approximation. \(^37\) Gong et al. found that a periodic driving could stabilize the dynamics despite the loss and gain in the non-Hermitian system. \(^40\)\(^,\)\(^41\) Xie et al. studied a non-Hermitian Rabi model with time-periodic coupling and found exact analytical results for certain exceptional points. \(^42\) A synchronous modulation which combined the time-periodic dissipation and coupling was studied in Ref. \(^43\), which provided an additional possibility for pulse manipulation and coherent control of the \(\mathcal{PT}\)-symmetric two-level systems. Experimental approach of Floquet \(\mathcal{PT}\)-symmetric system has been proposed with two coupled high frequency oscillators. \(^44\) A \(\mathcal{PT}\) symmetry breaking transition by engineering time-periodic dissipation and coupling has been realized through state-dependent atom loss in an optical dipole trap of ultracold \(^6\)Li atoms. \(^45\) They confirmed that a weak time-periodic dissipation could lead to \(\mathcal{PT}\)-symmetry breaking in \((2k - 1)\)-photon resonance. It should be noted that the \(\mathcal{PT}\)-symmetry breaking can occur in a finite non-Hermitian system, which is quite different from the quantum phase transition in the Hermitian system where the thermodynamic limit is needed. \(^45\)\(^,\)\(^46\)

In this paper, we study the \(\mathcal{PT}\) symmetry of a two-level
system with time-periodic dissipation and coupling. Instead of the widely used sinusoidal modulation \cite{36, 37, 40-42}, we consider a square-wave one, which is easier to implement in the ultracold atoms experiment \cite{43} and has analytical exact solutions based on the Floquet theory \cite{48, 49}. The square-wave modulation has a broad range of applications in the Hermitian system. It has been used to suppress the quantum dissipation in spin chains \cite{54}, to generate many Majorana modes in a one-dimensional p-wave superconductor system \cite{51}, to generate large-Chern-number topological phases \cite{52}, and so on. The square-wave modulation has also been realized in the non-Hermitian systems \cite{45}. This paper is organized as follows. In section II, we describe the non-Hermitian Hamiltonian of the driving two-level system. In section III, we briefly introduce the Floquet theory and derive the effective static Hamiltonian. In section IV, we achieve the \(\mathcal{PT}\) phase diagram and analyze the influence of multiphoton resonance. An equivalent Hamiltonian is obtained in the high frequency limit. The last section contains some concluding remarks.

II. HAMILTONIAN

We consider a periodically driving two-level system \(H(t) = H(t + T)\), with

\[
H(t) = \frac{\Delta(t)}{2} \sigma_x + i \frac{\gamma(t)}{2} \sigma_z,
\]

where \(\sigma_{x,z}\) are the Pauli matrices, \(T = T_0 + T_1\) is the driving period, \(\omega = 2\pi/T\) is the driving frequency, \(\Delta(t)\) is the time-periodic coupling strength, and \(\gamma(t)\) is the dissipation strength which leads to the periodic gain and loss. Lee et al. \cite{37} studied the \(\mathcal{PT}\) phase diagram of the non-Hermitian two-level system by doing the perturbation theory, which corresponds to \(\Delta(t) = \Delta\) and \(\gamma(t) = 4\lambda \cos(\omega t)\). Xie et al. \cite{12} found the exact analytical results for certain exceptional points of the two-level system with time-periodic coupling, which corresponds to \(\Delta(t) = v_0 + v_1 \cos(\omega t)\) and \(\gamma(t) = \gamma\). Luo et al. \cite{43} studied the analytical results of the non-Hermitian two-level systems with sinusoidal modulations of both \(\Delta(t)\) and \(\gamma(t)\). In order to get the exact analytical results without using perturbation theory, we consider a synchronous square-wave modulation of both dissipation and coupling. The corresponding time-periodic parameters are

\[
f(t) = \begin{cases} 
  f_0, & \text{if } mT - \frac{T}{2} \leq t < mT + \frac{T}{2}, \\
  f_1, & \text{if } mT + \frac{T}{2} \leq t < (m + 1)T - \frac{T}{2},
\end{cases}
\]

with \(f = \Delta, \gamma\) and \(m = \ldots, -1, 0, 1, \ldots\) It’s easy to confirm that the non-Hermitian Hamiltonian has a \(\mathcal{PT}\) symmetry, namely \(\mathcal{P} H(t) \mathcal{P} = H(t)\), where \(H(t)\) is the Hermitian conjugate of \(H(t)\) and \(\mathcal{P} = \mathcal{P}^{-1} = \sigma_x\) is the parity operator \cite{2, 5}. This non-Hermitian system has been realized by Li et al. in the ultracold atoms experiments \cite{44}. However, they focused on a special case with only one time-periodic parameter (either dissipation or coupling), and \(f_0 = f, f_1 = 0, T_0 = T_1 = T/2\). We consider a more general case which relieves those constraints. Two time-independent Hamiltonians \(H_0\) and \(H_1\) appear alternately, with

\[
H_j = \frac{\Delta_j}{2} \sigma_x + i \frac{\gamma_j}{2} \sigma_z, \quad j = 0, 1,
\]

and the corresponding eigenenergies are \(E_j^\pm = \pm h_j\) where

\[
h_j = \sqrt{\Delta_j^2 - \gamma_j^2}/2.
\]

\(H_j\) is one of the simplest non-Hermitian systems with \(\mathcal{PT}\) symmetry \cite{3}. When \(|\Delta_j| > |\gamma_j|\), the eigenenergy is real and it corresponds to the \(\mathcal{PT}\)-symmetric phase. When \(|\Delta_j| < |\gamma_j|\), the eigenenergy is imaginary and the \(\mathcal{PT}\) symmetry is broken. When \(|\Delta_j| = |\gamma_j|\), there exists an exceptional point (EP). The dynamics at each time domain is governed by the time evolution operator

\[
U_j(T_j) = \exp (-i H_j T_j) = \exp \left( -i \int_{-T_j}^{T_j} dt H(t) \right).
\]

III. FLOQUET THEORY

According to the Floquet theory \cite{48, 49}, we can define an effective Hamiltonian \(H_{\text{eff}}\) which satisfies the condition,

\[
U_{\text{eff}}(T) = \exp (-i H_{\text{eff}} T) = \mathcal{T} \exp \left[ -i \int_{-T}^{T} \frac{dt}{T} dt H(t) \right].
\]

The eigenenergies of the effective Hamiltonian correspond to the Floquet quasi-energies. Due to the simplicity of the square-wave modulation, the time evolution operator in a period can be written as

\[
\mathcal{T} \exp \left[ -i \int_{-T}^{T} \frac{dt}{T} dt H(t) \right] = \exp (-i H_1 T_1) \exp (-i H_0 T_0),
\]

Therefore,

\[
U_{\text{eff}}(T) = U_1(T_1) U_0(T_0).
\]

From Eq. (5) and (8), we achieve the effective time evolution operator
\[ U_{\text{eff}}(T) = \left( \cos(h_1 T_1) \cos(h_0 T_0) + \frac{1}{4}(\gamma_1 \gamma_0 - \Delta_1 \Delta_0) T_1 T_0 \sinh(h_1 T_1) \sinh(h_0 T_0) \right) I \]
\[ - \frac{1}{2}(\Delta_0 T_0 \cos(h_1 T_1) \sinh(h_0 T_0) + \Delta_1 T_1 \cos(h_0 T_0) \sinh(h_1 T_1)) \sigma_x \]
\[ + \frac{1}{4}(\Delta_0 \gamma_1 - \Delta_1 \gamma_0) T_1 T_0 \sinh(h_1 T_1) \sinh(h_0 T_0) \sigma_y \]
\[ + \frac{1}{2}\gamma_0 T_0 \cos(h_1 T_1) \sinh(h_0 T_0) + \gamma_1 T_1 \cos(h_0 T_0) \sinh(h_1 T_1)) \sigma_z. \quad (9) \]

Since \( h_j \) can be either a pure real number in the \( \mathcal{PT} \)-symmetric phase or a pure imaginary one in the \( \mathcal{PT} \) symmetry broken phase, both \( \cos(h_j T_j) \) and \( \sin(h_j T_j) \) must be real numbers. Accordingly, the coefficients before \( I, \sigma_y \), and \( \sigma_z \) must be real, while those before \( \sigma_x \) must be imaginary. It’s easy to confirm that the effective Hamiltonian can only be the following form,
\[ H_{\text{eff}} = J \frac{1}{2} \sigma_x + i \left( \frac{\Gamma_y}{2} \sigma_y + \frac{\Gamma_z}{2} \sigma_z \right) + \frac{n \omega}{2} I, \quad (10) \]
with \( n = 0, 1 \). The eigenenergies of \( H_{\text{eff}} \), or the Floquet quasi-energies of \( H(t) \) would be \( E^\pm = \pm h + \frac{n \omega}{2} \), where
\[ h = \sqrt{\frac{J^2 - \Gamma_y^2 - \Gamma_z^2}{2}}. \quad (11) \]

The effective time evolution operator can be rewritten as
\[ U_{\text{eff}}(T) = \exp(-iH_{\text{eff}}T) = (-1)^n \cos(hT)I \]
\[ + \frac{(-1)^n}{2} \frac{\Gamma_y}{\sinh(hT)} J \sigma_x \]
\[ + \frac{(-1)^n}{2} \frac{\Gamma_z}{\sinh(hT)} (\Gamma_y \sigma_y + \Gamma_z \sigma_z). \]

By comparing the coefficients before \( I, \sigma_x, \sigma_y, \) and \( \sigma_z \) in Eq. (9) and Eq. (12), we can directly obtain that
\[ (-1)^{n} \cos(hT) = \cos(h_1 T_1) \cos(h_0 T_0) + \frac{1}{4}(\gamma_1 \gamma_0 - \Delta_1 \Delta_0) T_0 T_1 \sinh(h_1 T_1) \sinh(h_0 T_0), \quad (13) \]
\[ J = \frac{(-1)^n}{T \sinh(hT)} \left( \Delta_0 T_0 \cos(h_1 T_1) \sinh(h_0 T_0) + \Delta_1 T_1 \cos(h_0 T_0) \sinh(h_1 T_1) \right), \quad (14) \]
\[ \Gamma_y = \frac{(-1)^n}{2 T \sinh(hT)} \left( \Delta_0 \gamma_1 - \Delta_1 \gamma_0 \right) T_0 T_1 \sinh(h_1 T_1) \sinh(h_0 T_0), \quad (15) \]
\[ \Gamma_z = \frac{(-1)^n}{T \sinh(hT)} \left( \gamma_0 T_0 \cos(h_1 T_1) \sinh(h_0 T_0) + \gamma_1 T_1 \cos(h_0 T_0) \sinh(h_1 T_1) \right). \quad (16) \]

Once we get \( J, \Gamma_y, \Gamma_z \) and \( n \), the effective Hamiltonian \[U_{\text{eff}}\] is finally determined.

IV. RESULTS AND DISCUSSIONS

The major differences of the effective Hamiltonian and the original one are the dissipation \( \Gamma_y \) in \( y \)-axis and the additional constant \( \omega/2 \). We will show later that the additional constant is closely related with the \( \mathcal{PT} \) symmetry broken phases and the exceptional points. One can easily confirm that the effective Hamiltonian has a \( \mathcal{PT} \) symmetry, namely \( \mathcal{P}H_{\text{eff}}\mathcal{P} = H_{\text{eff}} \), since \( \mathcal{P}x = x \), \( \mathcal{P}\gamma_y \gamma_y = -\gamma_y \) and \( \mathcal{P}\gamma_z \gamma_z = -\gamma_z \). When \( |\cos(hT)| < 1 \), \( h \) must be a real number and the \( \mathcal{PT} \) symmetry is preserved. For the \( \mathcal{PT} \)-symmetric phase, we suppose that the eigenenergies are \( E^\pm_x = \pm h^{(n)} + \frac{n \omega}{2} \). From Eq. (13), we can get that \( \cos(h^{(0)}T) = -\cos(h^{(1)}T) \). Then, \( h^{(1)}T = h^{(0)}T + \pi \), which leads to \( h^{(1)} = h^{(0)} + \frac{\pi}{\omega} \). Finally, \( E^0_+ = E^{(1)}_+ + \omega \) and \( E^0_- = E^{(-1)}_- \). As is well-known, the Floquet quasi-energies are periodic with period \( \omega \), and the total quasi-energies should be \( E^{(l)}_\pm = E^{(l)}_\pm + \omega \) with \( l = 0, \pm 1, \pm 2, \ldots \). Therefore, \( E^{(0)}_+ \) and \( E^{(1)}_- \) are equivalent. From now on, we only consider \( n = 0 \) in the \( \mathcal{PT} \)-symmetric phase.
When \( h \) is an imaginary number, \( \cos(hT) > 1 \), it corresponds to the \( \mathcal{PT} \) symmetry spontaneous breaking. There are two kinds of \( \mathcal{PT} \) symmetry broken phases, and their effective Hamiltonians differ by a constant. For simplicity, we assign the right-hand side of Eq. (13) to \( \Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1) \), namely,

\[
\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1) = \cos(h_1 T_1) \cos(h_0 T_0) + \frac{1}{4} (\gamma_1 \gamma_0 - \Delta_1 \Delta_0) T_0 T_1 \sinh(h_1 T_1) \sin(h_0 T_0).
\]

(17)

If \( \Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1) \) is greater than 1, then \( n = 0 \). If \( \Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1) \) is less than \(-1\), then \( n = 1 \). The exceptional points correspond to \( h = 0 \). From Eq. (13), we can easily find that the exceptional points occur when \( \Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1) = \pm 1 \), where \(+\) \((-\)) corresponds to \( n = 0 \) \((-1\)). Unlike the static Hamiltonians \( H_j \) whose eigenenergies can only be 0 in the exceptional points, the quasi-energies of the driven two-level system can be either 0 for \( n = 0 \) or \( \omega/2 \) for \( n = 1 \). Once the parameters \( \Delta_j, \gamma_j, T_j \) of the driving two-level systems are obtained, we can calculate \( \Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1) \), from which one can determine whether the \( \mathcal{PT} \) symmetry is broken or not.

### A. Multiphoton resonance

For the two-level system with square-wave modulated dissipation and time-independent coupling, the multiphoton resonance refers to the case when the coupling strength \( \Delta \) of the two-level system is an integral multiple of the driving frequency \( \omega \). A vanishingly small dissipation strength can lead to the \( \mathcal{PT} \) symmetry spontaneous breaking in the \((2k-1)\)-photon resonance case \((k = 1, 2, \ldots)\), which has been found in the two-level system with sinusoidal \(^{[37]}\) and square-wave \(^{[45]}\) modulated dissipations.

For the two-level system with a square-wave modulated coupling, one might naively think that the necessary condition for the weak dissipation induced \( \mathcal{PT} \) symmetry breaking is that both \( \Delta_0 \) and \( \Delta_1 \) are integral multiples of \( \omega \). However, it is not the case. The \( \mathcal{PT} \) phase transition induced by the weak dissipation in the multiphoton resonance indicates that \( \Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1) \) deviates from \( \pm 1 \) once the dissipation occurs. We expect that the necessary condition is \( \Pi(\Delta_0, \Delta_1, \gamma_0 = 0, \gamma_1 = 0, T_0, T_1) = \pm 1 \). From Eq. (17), we can obtain that

\[
\Pi(\Delta_0, \Delta_1, \gamma_0 = 0, \gamma_1 = 0, T_0, T_1) \\
= \cos \left( \frac{\Delta_1 T_1}{2} \right) \cos \left( \frac{\Delta_0 T_0}{2} \right) - \sin \left( \frac{\Delta_1 T_1}{2} \right) \sin \left( \frac{\Delta_0 T_0}{2} \right) \\
= \cos \left( \frac{\Delta_0 T_0 + \Delta_1 T_1}{2} \right) \\
= \cos \left( \frac{\Delta_{\text{eff}} T}{2} \right),
\]

where

\[
\Delta_{\text{eff}} = \frac{\Delta_0 T_0 + \Delta_1 T_1}{T}.
\]

(18)

Therefore, the necessary condition for the \( \mathcal{PT} \) phase transition induced by the weak dissipation should be \( \Delta_{\text{eff}} = k \omega \). In another word, the driving frequency should resonate with the effective coupling strength \( \Delta_{\text{eff}} \), rather than \( \Delta_0 \) or \( \Delta_1 \). When \( k \) is an even number, \( \Pi(\Delta_0, \Delta_1, \gamma_0 = 0, \gamma_1 = 0, T_0, T_1) = 1 \). A weak dissipation can lead to \( \Pi > 1 \) which corresponds to the \( \mathcal{PT} \) symmetry broken phase with \( n = 0 \), or \( \Pi < 1 \) which corresponds to the \( \mathcal{PT} \)-symmetric phase. Similarly, when \( k \) is an odd number, a weak dissipation can lead to \( \Pi < -1 \) which corresponds to the \( \mathcal{PT} \) symmetry broken phase with \( n = 1 \), or \( \Pi > -1 \) which corresponds to the \( \mathcal{PT} \)-symmetric phase.

#### 1. Time-periodic dissipation

We firstly consider the two-level system with only square-wave modulated dissipation. The coupling strength is time-independent, namely \( \Delta_0 = \Delta_1 = \Delta \), which leads to \( \Delta_{\text{eff}} = \Delta \). According to the former analysis, we expect that the \( \mathcal{PT} \) phase transition at weak dissipation occurs when \( \Delta = k \omega \). However, Li et al. only showed the \( \mathcal{PT} \)-symmetry breaking in \((2k-1)\)-photon resonance \(^{[45]}\), namely \( \Delta = (2k - 1) \omega \). In Fig. 1(a), we recover the \( \mathcal{PT} \) phase diagram near the one-photon resonance in Ref. \(^{[45]}\), by setting \( T_0 = T_1 = T/2, \gamma_0 = \gamma \) and \( \gamma_1 = 0 \). The boundary of the phase diagram can be determined by either \( \Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1) \) (Fig. 1(b)), or the imaginary part of the quasi-energies (Fig. 1(c)). Near the one-photon resonance region, \( \Pi \) is less than \(-1 \) and the imaginary part of the quasi-energies is nonzero, which indicates that it corresponds to a \( \mathcal{PT} \) symmetry broken phase with \( n = 1 \).

When we further decrease the driving frequency \( \omega \) to the two-photon resonance region, we find that a weak dissipation can also lead to the \( \mathcal{PT} \) symmetry breaking, which is not observed in Ref. \(^{[45]}\). As depicted in Fig. 2(a), the \( \mathcal{PT} \) symmetry broken region is much narrower than that in the one-photon resonance case. Besides, the driving frequency \( \omega \) at the phase boundary tends to decrease with increasing \( \gamma \). Therefore, the \( \mathcal{PT} \) symmetry breaking occurs at the region where \( \omega \) is a bit less than
FIG. 1: (a) $\mathcal{PT}$ phase diagram near the one-photon resonance, showing $\mathcal{PT}$-symmetric phase (grey), and $\mathcal{PT}$ symmetry broken phases with $n = 1$ (black). (b) $\Pi$ (Eq. 17) as a function of $\omega/\Delta$ at $\gamma/\Delta = 0$. The dash line represents $\Pi = -1$, below which corresponds to $\mathcal{PT}$ symmetry broken phase with $n = 1$. (c) Real (black lines) and imaginary (red lines) parts of the quasi-energies as a function of $\omega/\Delta$ at $\gamma/\Delta = 0$. The other parameters are $\Delta_0 = \Delta_1 = \Delta = 1$, $T_0 = T_1 = T/2$, $\gamma_0 = \gamma$ and $\gamma_1 = 0$.

$\Delta/2$. Near the two-photon resonance, $\Pi$ is greater than 1 and the imaginary part of the quasi-energies is nonzero, which indicates that it corresponds to a $\mathcal{PT}$ symmetry broken phase with $n = 0$.

Fig. 3 (a) is a generalization of Figs. 1 (a) and 2 (a), which extends the range of $\omega$. The driving two-level system has a much richer phase diagram than the static one. Clearly, a vanishingly small dissipation strength can lead to the $\mathcal{PT}$ symmetry spontaneous breaking in both $(2k-1)$- and $2k$-photon resonances, which is consistent with our criteria $\Delta = k\omega$. To explain the behavior of the $\mathcal{PT}$ symmetry breaking near the $2k$-photon resonance, we reexamine $\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1)$ in Eq. (17) in more detail. We suppose that $\gamma_0 = \gamma \ll \Delta$, $\gamma_1 = \lambda\gamma$, $T_0 = T_1 = T/2$, and $\Delta \approx 2k\omega$. When $\gamma$ tends to zero, $h_i T_i$ tends to $k\pi$. The first term in the right-hand side of Eq. (17) tends to one while the second term tends to zero. If the second term is greater than zero, it can lead to $\Pi > 1$ and the $\mathcal{PT}$ symmetry broken phase with $n = 0$. Since $(\gamma_0 \gamma_1 - \Delta^2) T_0 T_1/4$ in the second term is less than zero, one need that $\sin (h_1 T_1) \sin (h_0 T_0) < 0$, or $\sin (h_1 T_1) \sin (h_0 T_0) < 0$. Then, the condition for the occurrence of $\mathcal{PT}$ symmetry breaking is that one of $h_i T_i$ should be less than $k\pi$, while the other one should...
FIG. 3: $\mathcal{PT}$ phase diagram for time-periodic dissipation near the multiphoton resonance, showing $\mathcal{PT}$-symmetric phase (grey), and $\mathcal{PT}$ symmetry broken phases with $n = 0$ (white) and $n = 1$ (black). (a) $\gamma_0 = \gamma = \gamma_1 = 0$, $T_0 = T_1$. (b) $\gamma_0 = -\gamma_1 = \gamma$, $T_0 = T_1$. (c) $\gamma_0 = -\gamma_1 = \gamma$, $T_0 = 0.55T$, $T_1 = 0.45T$.

be greater than $k\pi$. If $\Delta$ is a bit less than $2k\omega$, a finite $\gamma$ will always decrease $h_i$, which leads to that both $h_i T_i < \Delta T_i / 2 < k\pi$ and $\Pi < 1$. Therefore, no $\mathcal{PT}$ symmetry breaking occurs when $\Delta < 2k\omega$. If $\Delta$ is a bit larger than $2k\omega$, one can always find certain $\gamma$ which satisfies the condition for the occurrence of $\mathcal{PT}$ symmetry breaking, as long as $\lambda \neq \pm 1$. Fig. 3 (a) corresponds to $\lambda = 0$. Therefore, a finite $\gamma$ can lead to the $\mathcal{PT}$ symmetry breaking near $2k$-photon resonance.

When $\lambda = +1$, namely $\gamma_0 = \gamma_1$, the Hamiltonian (1) becomes time-independent, which is trivial. When $\lambda = -1$, namely $\gamma_0 = -\gamma_1 = \gamma$, $h_0$ equals to $h_1$, $h_1 T_1 = h_0 T_0$ if $T_0 = T_1$, which leads to the $\mathcal{PT}$-symmetric phase with $\Pi < 1$ near the $2k$-photon resonance, as shown in Fig. 3 (b). Following the above analysis, we can easily prove that an imbalanced driving time $T_0 \neq T_1$ can lead to the $\mathcal{PT}$ symmetry breaking when $\gamma_0 = -\gamma_1$, as depicted in Fig. 3 (c). The $\mathcal{PT}$ symmetry breaking near $2k$-photon resonance induced by the imbalanced driving time $T_0 \neq T_1$ is more obvious than that induced by $\gamma_0 \neq -\gamma_1$, when the dissipation strength is very weak. Therefore, the imbalanced driving time $T_0 \neq T_1$ is a more efficient method to access the $\mathcal{PT}$ symmetry breaking near $2k$-photon resonance in the experiments. Figs. 3 (a) and (c) verify our conclusion that the $\mathcal{PT}$ symmetry breaking induced by weak dissipation generally occurs at both $2k$- and $(2k - 1)$-photon resonances, namely $\Delta = k\omega$. The $\mathcal{PT}$ symmetry breaking at $2k$-photon resonance disappears only if $\gamma_0 = -\gamma_1$ and $T_0 = T_1$, as shown in Fig. 3 (c).

2. Time-periodic coupling

FIG. 4: $\mathcal{PT}$ phase diagram for time-periodic coupling near the multiphoton resonance, showing $\mathcal{PT}$-symmetric phase (grey), and $\mathcal{PT}$ symmetry broken phases with $n = 0$ (white) and $n = 1$ (black). (a) $\Delta_0 = 1$, $\Delta_1 = 0$, $T_0 = 0.5T$. (b) $\Delta_0 = 1$, $\Delta_1 = -0.2$, $T_0 = 0.55T$.

For the two-level system with only square-wave modulated coupling, the dissipation strength is time-independent, namely $\gamma_0 = \gamma_1 = \gamma$. Fig. 4 shows the $\mathcal{PT}$ phase diagram for time-periodic coupling near the multiphoton resonance. Li et al. studied the influence of the time-periodic coupling on the non-Hermitian two-level system based on a simpler model with $\Delta_0 = \Delta$, $\Delta_1 = 0$ and $T_0 = T_1 = T/2 \ [45]$, which corresponds to Fig. 4 (a). They concluded that the $\mathcal{PT}$ phase transition induced by the weak dissipation occurs at $\Delta = 2k\omega$, which is consistent with our results $\Delta_{\text{eff}} = k\omega$ due to $\Delta_{\text{eff}} = \Delta / 2$. Fig. 4 (b) introduces a nonzero $\Delta_1$ and
imbalanced driving time $T_0 \neq T_1$, which cannot be explained by Ref. [43]. However, $\Delta_{\text{eff}} = k\omega$ can still provide the right condition at which the $\mathcal{PT}$ phase transitions occur. The $\mathcal{PT}$ symmetry broken phase with $n = 0$ (1) occurs when $k$ is even (odd), which is also consistent with our former analysis.

**B. High frequency limit: $T \to 0$**

![Figure 5: $\mathcal{PT}$ phase diagram, showing $\mathcal{PT}$-symmetric phase (grey), and $\mathcal{PT}$ symmetry broken phase with $n = 0$ (white) at $\Delta_0 = \Delta_1 = \Delta = 1$, $\omega = 3$, $T_0 = 0.4T$, $T_1 = 0.6T$. The red dash lines refer to the analytical results in the high frequency limit.](image)

If the driving frequency is very large, namely $\omega \gg \Delta_j, \gamma_j$, the period $T$ tends to zero. We suppose that $T_0$ and $T_1$ are of same order as $T$. Expanding $\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1)$ to the second order of $T$, we obtain

$$\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1) \approx 1 - \frac{h_0^2 T_0^2}{2} \left(1 - \frac{h_1^2 T_0^2}{2}\right) + \frac{1}{4} (\gamma_1 \gamma_0 - \Delta_1 \Delta_0) T_1 T_0$$

$$\approx 1 + \frac{1}{4} \left[(\gamma_1 \gamma_0 - \Delta_1 \Delta_0) T_1 T_0 - 2h_0^2 T_0^2 - 2h_1^2 T_1^2\right]$$

$$= 1 + \frac{1}{8} \left[(\gamma_0 T_0 + \gamma_1 T_1)^2 - (\Delta_0 T_0 + \Delta_1 T_1)^2\right]$$

$$= 1 + \frac{1}{8} (\gamma_{\text{eff}}^2 - \Delta_{\text{eff}}^2) T^2, \quad (19)$$

where

$$\gamma_{\text{eff}} = \frac{\gamma_0 T_0 + \gamma_1 T_1}{T}. \quad (20)$$

Therefore, the exceptional points, as well as the $\mathcal{PT}$ phase boundary, are located at $\gamma_{\text{eff}} = \pm \Delta_{\text{eff}}$. If $|\gamma_{\text{eff}}| < |\Delta_{\text{eff}}|$, it corresponds to the $\mathcal{PT}$-symmetric phase. Otherwise, the $\mathcal{PT}$ symmetry is broken with $n = 0$. Alternatively, if we expand Eqs. (14)-(16) to the lowest order of $T$, we find that

$$J \approx \Delta_{\text{eff}}, \quad \Gamma_y \approx 0, \quad \Gamma_z \approx \gamma_{\text{eff}}, \quad (21)$$

which give rise to the following effective Hamiltonian,

$$H_{\text{eff}} \approx \frac{\Delta_{\text{eff}}}{2} \sigma_x + i \frac{\gamma_{\text{eff}}}{2} \sigma_z. \quad (22)$$

It leads to the same $\mathcal{PT}$ phase boundary. In a word, we find that when the driving frequency is very large, the Floquet effective Hamiltonian is equivalent to a static one with time-averaged coupling and dissipation strength. When $\Delta_0 \approx -\Delta_1, \Delta_{\text{eff}}$ tends to zero and one can easily achieve the $\mathcal{PT}$ symmetry broken phase no matter how large $\Delta_j$ is. When $\gamma_0 \approx -\gamma_1, \gamma_{\text{eff}}$ tends to zero and one can easily preserve the $\mathcal{PT}$ symmetry no matter how large $\gamma_j$ is.

Fig. 5 shows the $\mathcal{PT}$ phase diagram at $\Delta_0 = \Delta_1 = \Delta$, $\omega/\Delta = 3$ and $T_0/T_1 = 2/3$. The phase boundary $\gamma_{\text{eff}} = \pm \Delta_{\text{eff}}$ fits well with the exact results.

**V. CONCLUSIONS**

We study a non-Hermitian two-level system with square-wave modulated dissipation and coupling. Two time-independent Hamiltonians $H_0$ and $H_1$ appear alternately. Comparing with the formerly well-known sinusoidal modulation, the square-wave modulation has three advantages: Firstly, exact analytical solutions can be achieved by employing the Floquet theory. Secondly, the $\mathcal{PT}$ phase diagram becomes richer. Thirdly, the square-wave modulation has been realized in the ultracold atoms experiment [43].

Based on the Floquet theory, we achieve an effective Hamiltonian with $\mathcal{PT}$ symmetry. We define a parameter $\Pi(\Delta_0, \Delta_1, \gamma_0, \gamma_1, T_0, T_1)$, from which one can derive the boundaries of the $\mathcal{PT}$ phase diagram exactly. The driving two-level system has a much richer phase diagram than the static one. Two kinds of $\mathcal{PT}$ symmetry broken phases are found whose effective Hamiltonians differ by a constant $\omega/2$. When $\Pi > 1$, the $\mathcal{PT}$ symmetry broken phase with $n = 0$ occurs. When $\Pi < -1$, the $\mathcal{PT}$ symmetry broken phase with $n = 1$ occurs. When $-1 < \Pi < 1$, the $\mathcal{PT}$ symmetry is preserved.

With the help of $\Pi$, we firstly study the $\mathcal{PT}$ phase transition with only square-wave modulated dissipation near multiphoton resonance. The coupling strength is time-independent with $\Delta_0 = \Delta_1 = \Delta$. A weak dissipation can lead to the $\mathcal{PT}$ symmetry breaking near the $(2k-1)$-photon resonance $(\Delta = (2k-1)\omega)$, which has been observed in the ultracold atoms experiment [43]. We predict that the $\mathcal{PT}$ symmetry breaking near the $2k$-photon resonance $(\Delta = 2k\omega)$, can also happen as long as the dissipation strengths or the driving times are imbalanced,
with $\gamma_0 \neq -\gamma_1$ or $T_0 \neq T_1$. Our studies pave a way to access the $\mathcal{PT}$ symmetry broken phase near the 2k-photon resonance in the experiments. For the $\mathcal{PT}$ phase transition with square-wave modulated coupling, we define an effective coupling strength $\Delta_{\text{eff}} = (\Delta_0 T_0 + \Delta_1 T_1)/T$. The weak dissipation induced $\mathcal{PT}$ symmetry breaking can occur only if $\Delta_{\text{eff}} = k\omega$. In the high frequency limit, we achieve a simple relation $\gamma_{\text{eff}} = \pm \Delta_{\text{eff}}$, which gives the $\mathcal{PT}$ phase boundary. When $\Delta_0 \simeq -\Delta_1$, one can easily achieve the $\mathcal{PT}$ symmetry broken phase no matter how large the coupling strength $|\Delta_j|$ is. When $\gamma_0 \simeq -\gamma_1$, one can easily preserve the $\mathcal{PT}$ symmetry no matter how large the dissipation strength $|\gamma_j|$ is.

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