DYNAMICS OF POLYNOMIAL SEMIGROUPS: MEASURES, POTENTIALS, AND EXTERNAL FIELDS

By

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Abstract. In this paper, we give a description of a natural invariant measure associated with a finitely generated polynomial semigroup (which we shall call the Dinh–Sibony measure) in terms of potential theory. This requires the theory of logarithmic potentials in the presence of an external field, which, in our case, is explicitly determined by the choice of a set of generators. Along the way, we establish the continuity of the logarithmic potential for the Dinh–Sibony measure, which might be of independent interest. We then use the $F$-functional of Mhaskar and Saff to discuss bounds on the capacity and diameter of the Julia sets of such semigroups.

1 Introduction and statement of main results

A rational semigroup is a subsemigroup of $\text{Hol}(\hat{\mathbb{C}}; \hat{\mathbb{C}})$—the semigroup with respect to composition of holomorphic self-maps of $\hat{\mathbb{C}}$—containing no constant maps (where $\hat{\mathbb{C}}$ denotes the Riemann sphere). The investigation of such semigroups was initiated by Hinkkanen and Martin in [10]. Given a finitely generated rational semigroup $S$ containing at least one element of degree at least 2, and a set of generators $\mathcal{G}$, there happens to be a dynamically meaningful probability measure $\mu_{\mathcal{G}}$ associated with the pair $(S, \mathcal{G})$. This paper is dedicated to the following question: can one describe $\mu_{\mathcal{G}}$, which is constructed purely dynamically, in terms of the theory of logarithmic potentials? The motivation for this is that potential theory in $\mathbb{C}$ is such a well-developed and deeply explored field that identifying $\mu_{\mathcal{G}}$ in potential-theoretic terms would reveal new information about the various invariant sets associated with $S$.

The measure $\mu_{\mathcal{G}}$ is a measure that is preserved, in an appropriate sense, by a holomorphic correspondence on $\hat{\mathbb{C}}$ associated with $(S, \mathcal{G})$. To make precise what this means, we begin with

Definition 1.1. Let $X_1$ and $X_2$ be two compact, connected complex manifolds of dimension $k$. A holomorphic correspondence from $X_1$ to $X_2$ is a formal
linear combination of the form
\[(1.1) \quad \Gamma = \sum_{1 \leq i \leq N} m_i \Gamma_i,\]
where the \(m_i\)’s are positive integers and \(\Gamma_1, \Gamma_2, \ldots, \Gamma_N\) are distinct irreducible complex-analytic subvarieties of \(X_1 \times X_2\) of pure dimension \(k\) that satisfy the following conditions:

(i) for each \(\Gamma_i\) in (1.1), \(\pi_1|_{\Gamma_i}\) and \(\pi_2|_{\Gamma_i}\) are surjective;
(ii) for each \(x \in X_1\) and \(y \in X_2\), \((\pi_1^{-1}\{x\} \cap \Gamma_i)\) and \((\pi_2^{-1}\{y\} \cap \Gamma_i)\) are finite sets for each \(i\)
(where \(\pi_j\) is the projection onto \(X_j\), \(j = 1, 2\)).

A holomorphic correspondence \(\Gamma\) induces a set-valued function, which we denote by \(F_{\Gamma}\),
\[
X_1 \ni A \mapsto \bigcup_{1 \leq i \leq N} \pi_2(\pi_1^{-1}(A) \cap \Gamma_i).
\]
We shall denote \(F_{\Gamma}(\{x\})\) by \(F_{\Gamma}(x)\). If \(X_1 = X_2 = X\) in the above definition then we say that \(\Gamma\) is a **holomorphic correspondence on** \(X\). Two holomorphic correspondences on \(X\) can be composed with each other (see Section 5.2). This, and the map \(F_{\Gamma}\), introduce the dynamical element in the study of holomorphic correspondences.

Definition 1.1 and the discussion immediately prior to it suggest the following natural

**Definition 1.2.** Let \(S\) be a finitely generated rational semigroup and let \(\mathcal{G} = \{g_1, g_2, \ldots, g_N\}\) be a set of generators of \(S\), i.e., \(S = \langle g_1, g_2, \ldots, g_N \rangle\). We call the following holomorphic correspondence
\[(1.2) \quad \Gamma_{\mathcal{G}} := \sum_{1 \leq i \leq N} \text{graph}(g_i)
\]
the **holomorphic correspondence** associated with \((S, \mathcal{G})\).

Now, \(\mu_{\mathcal{G}}\) arises from a very general construction by Dinh and Sibony [8] applied to the holomorphic correspondence \(\Gamma_{\mathcal{G}}\). We provide a little background. Let \(X\) be a compact Riemann surface and \(\Gamma\) a holomorphic correspondence on \(X\). Let \(d_1(\Gamma)\) be the generic number of inverse images and \(d_0(\Gamma)\) the generic number of forward images under \(\Gamma\), both counted according to multiplicity (see Section 5.2). Dinh–Sibony show that regular Borel measures can be pulled back under \(\Gamma\); for such a measure \(\mu\), let \(F_{\Gamma}^* \mu\) denote its pull-back (see [8, Section 2.4] for details).
The main results of [8] applied to the latter set-up imply that when \( d_1(\Gamma) > d_0(\Gamma) \), there exist a polar set \( E \subsetneq X \) and a regular Borel probability measure \( \mu_\Gamma \) such that
\[
\frac{1}{d_1(\Gamma)^n} F^*_{\Gamma\rightarrow}(\delta_a) \overset{\text{weak}^*}{\longrightarrow} \mu_\Gamma \quad \text{as} \; n \to \infty, \; \forall a \in X \setminus E.
\]
The measure \( \mu_G \) is the limit measure given by (1.3) taking \( \Gamma = \Gamma_G \).

As \( \mu_G \) is a special case of a construction in [8], we shall call it the Dinh–Sibony measure associated with \((S, G)\). We must mention that, under a further constraint on \( S \)—as above—which is reflected in his choice of generating set \( G \) of \( S \), the measure \( \mu_G \) was discovered by Boyd [3]. Also see [17] by Sumi for another approach to associating dynamically interesting measures with rational semigroups. Boyd’s construction is not based on the formalism of correspondences. Our main theorems, however, do not rely principally on Boyd’s construction, nor do they rely on his methods. We shall not dwell on the reasons for this, but the interested reader is referred to [2, Remark 4.1] and to the fact that the semigroups that we shall consider are allowed to have degree-one elements. The semigroups we shall consider are described by the following

**Definition 1.3.** A rational semigroup \( S \) is called a **polynomial semigroup** if all its elements are polynomials, any degree-one element of \( S \) has an attracting fixed point at \( \infty \), and \( S \) contains at least one element of degree at least 2.

**Remark 1.4.** Unlike what its name suggests, a polynomial semigroup cannot contain arbitrary degree-one elements. Yet, we choose the latter name for the semigroups considered here because we want the terminology to evoke Brolin’s result [5, Theorem 16.1] on an invariant measure associated with a polynomial \( P \) of degree at least 2. That \( P \) has an attracting fixed point at \( \infty \) is a crucial part of Brolin’s proof. This is what motivates our condition on degree-one elements in Definition 1.3. In fact, we shall see that our Theorem 1.6 subsumes Brolin’s theorem.

We first show that the conditions defining a finitely generated polynomial semigroup \( S \) imply something interesting about its generators. A set of generators \( G = \{ g_1, g_2, \ldots, g_N \} \) of \( S \) is called **minimal** if no function \( g_i \) can be expressed as a composition involving the remaining generators. The existence of such sets is clear, but more can be said:

**Proposition 1.5.** Every finitely generated polynomial semigroup has a unique minimal generating set.

This proposition is important because, given a finitely generated polynomial semigroup \( S \), it identifies a set of generators of \( S \) that is, in a precise sense, **canonical**. We shall denote the unique minimal generating set of \( S \) as \( G_S \).
In [5, Theorem 16.1], Brolin constructed an invariant measure associated with a polynomial $P$ of degree at least 2 (which turns out to be precisely the Dinh–Sibony measure associated with $(S, G = \{ P \})$) and showed it to be the equilibrium measure of the Julia set of $P$. This result cannot extend naively to finitely generated polynomial semigroups $S$ with more than one generator because:

(i) It is easy to construct finitely generated polynomial semigroups $S$ whose Julia sets $J(S)$ (see Section 2 for a definition) have non-empty interiors. See, for instance, [10, Example 1].

(ii) There exist finitely generated polynomial semigroups $S$ as in (i) and a choice of generating set $G$ such that $\text{supp}(\mu_G) = J(S)$. See [3, Theorem 1].

Now, if for a semigroup of the above kind, and a choice of generating set as in (ii), the measure $\mu_G$ were the equilibrium measure of $J(S)$, then it would have to be supported on the exterior boundary of $J(S)$, which would contradict (ii). This is the fundamental problem one must understand in order to answer the question posed at the beginning of this section.

The latter problem is solved by turning to the theory of logarithmic potentials in the presence of an external field. Roughly speaking, an equilibrium measure associated with an external field gives the distribution of a unit charge on a conductor, in the presence of an external electrostatic potential, that minimizes energy (the classical equilibrium measure gives the latter distribution in the absence of any external field). To make mathematical sense, this electrostatic potential must satisfy certain admissibility conditions and, in the mathematical literature, is called an external field: see Section 4 for definitions and [15, Chapter I] by Saff and Totik for details. Our first theorem says, in essence, that, given a finitely generated polynomial semigroup $S$ and a finite set of generators $G$, the measure $\mu_G$ is the equilibrium measure associated with an external field that is described explicitly by $G$. Now we introduce this external field. For $(S, G)$ as above, we define the dynamical Green’s function associated with $(S, G)$ to be the upper semicontinuous regularization of

$$G_{G}(z) := \limsup_{n \to \infty} \frac{1}{d_1(G_G)^n} \log \left( \prod_{l_G=n} |g(z) - a| \right),$$

where $a$ is arbitrary element outside a certain polar set (see Section 2 for the meaning of the above product). Let us denote the upper semicontinuous regularization of $G_{G}$—see Section 6 for a definition—by $G^{*}_{G}$. That $G^{*}_{G}$ does not depend on $a$ as above is a consequence of (1.3) with $\Gamma = \Gamma_{G}$—we shall say more about this; see Remark 6.6. The external field that is relevant to our problem is $G^{*}_{G}$ restricted to $J(S)$. 
A point \( z \in \hat{\mathbb{C}} \) is called **exceptional** if the set 

\[
O^-(z) := \{ x \in \hat{\mathbb{C}} : g(x) = z \text{ for some } g \in S \}
\]

is finite. We denote the set of exceptional points by \( \mathcal{E}(S) \). It is well known that, for a rational semigroup \( S \), \( \#(\mathcal{E}(S)) \leq 2 \). Note that for a polynomial semigroup \( S \), \( \infty \in \mathcal{E}(S) \), and hence it has at most one exceptional point in \( \mathbb{C} \). Now we are in the position to state

**Theorem 1.6.** Let \( S \) be a finitely generated polynomial semigroup. Define

\[
C^*[S] := \{ c \in \mathbb{C} : g'(c) = 0 \text{ for some } g \in G_S \}, \\
C[S] := \{ c \in \mathcal{J}(S) : g'(c) = 0 \text{ for some } g \in G_S \}.
\]

Suppose \( S \) has the property that if \( \#C^*[S] = 1 \) then \( C[S] \cap \mathcal{E}(S) = \emptyset \). Then, for any finite set of generators \( \mathcal{G} \) of \( S \), the Dinh–Sibony measure \( \mu_{\mathcal{G}} \) is the equilibrium measure associated with the external field \( G_{\mathcal{G}}^* |_{\mathcal{J}(S)} \).

**Remark 1.7.** The condition stated in Theorem 1.6 is very mild. Finitely generated polynomial semigroups that do not satisfy the condition stated are very exceptional—see Section 3.1 for a precise discussion. If, for a finitely generated polynomial semigroup \( S \), \( \#C^*[S] = 1 \) and \( C[S] \cap \mathcal{E}(S) \neq \emptyset \), then it is unclear whether the logarithmic potential of the measures \( \mu_{\mathcal{G}} \) (for any generating set \( \mathcal{G} \) of \( S \)) is continuous. That being said, the behaviour of these potentials is not intractable. But since a completely different analysis would be required to study these exceptional semigroups, we focus here on the semigroups addressed by Theorem 1.6.

If \( P \) is a polynomial of degree at least 2 and we consider the iterative dynamics of \( P \) (i.e., \( \mathcal{G} = \{ P \} \) in our notation), it follows from [5] that \( G_{\mathcal{G}} \) restricted to the unbounded component of the Fatou set of \( P \) is the Green’s function of the latter with pole at \( \infty \), and the external field is identically 0 in this case. This is the framework of classical potential theory. From our remark above on the equilibrium measure (see Section 4 for a precise statement), and as \( \mu_{\mathcal{G}} \) here is the equilibrium measure of the Julia set, we see that Brolin’s theorem is subsumed by Theorem 1.6.

We provide a very short sketch of the proof of Theorem 1.6 to point out some features of it that are novel (in what follows, \( D(z, r) \) denotes the open disc with centre \( z \in \mathbb{C} \) and radius \( r \)):

- We establish that the logarithmic potential of \( \mu_{\mathcal{G}} \) (let us denote it by \( U^{\mu_{\mathcal{G}}} \)) is continuous. This is Theorem 6.3—which may be of independent interest.
• We show that there is a constant $\alpha > 0$ such that for each $z \in \mathbb{C}$,

$$\mu_g(D(z, r)) \lesssim r^\alpha$$

when $r > 0$ is sufficiently small. The continuity of $U^{\mu_g}$ follows from this using a characterization of the continuity of the logarithmic potential by Arsove [1, Theorem 1].

• Using the continuity of $U^{\mu_g}$ we are able to show a very strong relation between $U^{\mu_g}$ and the external field in Theorem 1.6, from which this result follows.

We should mention that the proof of the power bound on $\mu_g(D(z, r))$ is inspired by an argument by Bharali and Sridharan [2, Section 5]. However, their argument addresses only the case $\mathcal{C}[S] = \emptyset$. We make a careful analysis of the local orders at critical points to show that the power bound on $\mu_g(D(z, r))$ can be obtained even when $\mathcal{C}[S] \neq \emptyset$. This is our Proposition 6.1.

With Theorem 1.6 at hand, one expects that a growing understanding of the external field would lead to new information about dynamically interesting objects associated with $S$ as in Theorem 1.6. In this work, we focus on the (logarithmic) capacity of $J(S)$. Our next result shows why it is fundamentally hard to compute the capacity of $J(S)$ for $S$ with more than one generator. To be specific: for the latter $S$, the analogue of the Robin constant for $J(S = \langle P \rangle)$, $P$ a polynomial, is the modified Robin constant (see Section 4), $F_g$, for the external field given by Theorem 1.6. Just like the Robin constant for $J(S = \langle P \rangle)$, $F_g$ is not hard to compute. But the relationship between $F_g$ and capacity is badly vitiated by the external field $G^*_g|_{J(S)} =: Q_g$.

One may ask: when is $Q_g \neq 0$? Theorem 1.8 below provides a sufficient condition under which $Q_g \neq 0$. Actually, we conjecture that the condition (ii) in Theorem 1.8-(a) is superfluous and that condition (i) is necessary and sufficient for $Q_g \neq 0$: refer to Conjecture 1.9 for a precise statement. We shall prove that:

• The condition (i) in Theorem 1.8-(a) is a necessary condition for $Q_g \neq 0$ for some finite generating set $\mathcal{G}$, which is part of our evidence for Conjecture 1.9.

• Under the conditions on $S$ alluded to, $F_g$ is always greater than the Robin constant of $J(S)$ (for any finite generating set $\mathcal{G}$), which gives a strict lower bound on the capacity of $J(S)$.

Some notation: $\Lambda(g)$ will denote the coefficient of the highest-degree term of a polynomial $g$. 
**Theorem 1.8.** Let $S$ be a finitely generated polynomial semigroup having the properties stated in Theorem 1.6. For a set of generators $\mathcal{G} = \{g_1, g_2, \ldots, g_N\}$ of $S$, let $Q_{\mathcal{G}}$ denote the external field associated with $(S, \mathcal{G})$ given by Theorem 1.6.

(a) Assume that for some $z_0 \in J(S)$: (i) its orbit under $S$, $O(z_0)$, is unbounded, and (ii) $O(z_0)$ is not dense in $\mathbb{C}$. Then $Q_{\mathcal{G}} \not\equiv 0$ for any finite set of generators $\mathcal{G}$ of $S$.

(b) If $Q_{\mathcal{G}} \not\equiv 0$ for some finite set of generators $\mathcal{G}$ then there exists a point $z_0 \in J(S)$ such that $O(z_0)$ is unbounded.

Moreover, if $S$ satisfies conditions (i) and (ii) and each element of $S$ is of degree at least 2 then

$$\text{cap}(J(S)) > \exp(-F_{\mathcal{G}}),$$

for any set of generators $\mathcal{G}$ as above. Here $F_{\mathcal{G}}$ is the modified Robin constant for $Q_{\mathcal{G}}$, and equals $(D - N)^{-1} \log |\Lambda(g_1)\Lambda(g_2) \cdots \Lambda(g_N)|$, where

$$D := \deg(g_1) + \deg(g_2) + \cdots + \deg(g_N).$$

As discussed above, we have some evidence to propose the following

**Conjecture 1.9.** Let $S$ be a finitely generated polynomial semigroup. For a finite set of generators $\mathcal{G}$ of $S$, let $Q_{\mathcal{G}}$ denote the external field associated with $(S, \mathcal{G})$ given by Theorem 1.6. Then, the following are equivalent:

(a) For some point $z_0 \in J(S)$, its orbit under $S$ is unbounded.

(b) $Q_{\mathcal{G}} \not\equiv 0$ for some finite set of generators $\mathcal{G}$ of $S$.

(c) $Q_{\mathcal{G}} \not\equiv 0$ for every finite set of generators $\mathcal{G}$ of $S$.

We conclude this section with another indication of future research. To this end, we mention a recent work by Dinh et al. [6], with which this work shares some common features. In [6], the authors revisit the problem in random matrix theory of determining the asymptotic behaviour of the random products $s_n \cdots s_1$, $n = 1, 2, 3, \ldots$, where $s_1, s_2, s_3, \ldots$ are sampled independently and identically from $\text{SL}_2(\mathbb{C})$ relative to a non-elementary probability measure $\mu$ on $\text{SL}_2(\mathbb{C})$. The approach in [6] involves interpreting each $s_n \in \text{SL}_2(\mathbb{C})$ as an element in $\text{Aut}(\hat{\mathbb{C}})$. The problem considered leads to the study of the dynamical properties of a “generalized correspondence”—in the sense that if supp($\mu$) were a finite set then the latter object would be a correspondence of the form (1.1). This suggests a class of problems on $S$—as in Theorems 1.6 and 1.8—wherein one considers random compositions in $S$ by endowing $S$ with a probability measure $\mu$ analogous to those considered in [6]. The analytical methods in [6] suggest, for instance, a way to describe the asymptotic behaviour of random compositions in the higher-degree setting.
2 Classical notions on rational semigroups

For a rational semigroup $S$, the Fatou set of $S$—which we denote by $F(S)$—is the largest open subset of $\hat{\mathbb{C}}$ on which functions in $S$ form a normal family. The Julia set of $S$, $\mathcal{J}(S)$, is the complement of $F(S)$: i.e., $\mathcal{J}(S) := \hat{\mathbb{C}} \setminus F(S)$. If $S$ is generated by a set $\mathcal{G}$ then we write $S = \langle f : f \in \mathcal{G} \rangle$. If $S$ is generated by a single rational function $f$ then we abbreviate $F(\langle f \rangle)$ and $\mathcal{J}(\langle f \rangle)$ to $F(f)$ and $\mathcal{J}(f)$, respectively.

If a rational semigroup $S$ contains a function of degree at least 2 then we can say more about $\mathcal{J}(S)$.

**Result 2.1** (Hinkkanen–Martin, [10]). Let $S$ be a rational semigroup containing at least one function of degree at least 2. Then the following hold:

(a) The set of all repelling fixed points of all the elements of $S$ is dense in $\mathcal{J}(S)$.
(b) The Julia set of $S$ satisfies $\mathcal{J}(S) = \bigcup_{f \in S} \mathcal{J}(f)$.

Let $S$ be a rational semigroup and let $\mathcal{G}$ be a set of generators (or a generating set) of $S$. A **word** will refer to any composition $f_{i_n} \circ \cdots \circ f_{i_1}$, $n \in \mathbb{Z}_+$, where $f_{i_1}, \ldots, f_{i_n} \in \mathcal{G}$. We shall call $n$ the **length** of the word $f_{i_n} \circ \cdots \circ f_{i_1}$. For $f \in S$, the expression $l(f) = n$ is the shorthand for the following implication:

$$l(f) = n \implies \exists f_{i_1}, \ldots, f_{i_n} \in \mathcal{G} \text{ such that } f = f_{i_n} \circ \cdots \circ f_{i_1}.$$ 

In the above expression, the word $f_{i_n} \circ \cdots \circ f_{i_1}$ is called as a representation of $f$.

**Definition 2.2.** Let $S$ be a polynomial semigroup. The filled-in Julia set of $S$, denoted by $\mathcal{K}(S)$, is the set

$$\mathcal{K}(S) := \{ z \in \mathbb{C} : \text{O}(z) \text{ has finite limit point} \},$$

where $\text{O}(z) := \{ f(z) : f \in S \}$ denotes the orbit of $z \in \mathbb{C}$. We shall call the complement of $\mathcal{K}(S)$, denoted by $\mathcal{A}(S) = \hat{\mathbb{C}} \setminus \mathcal{K}(S)$, the **basin of attraction** of $\infty$ for $S$.

A couple of observations are in order. First: we recall that the expression $\text{O}(z)$ has finite limit point means that there exists a sequence $\{ h_n \}$ in $S$ consisting of distinct elements of $S$ such that $\{ h_n(z) \}$ converges to some point in $\mathbb{C}$. Second: it is not immediate from the above definition why $\mathcal{A}(S)$ is called a “basin of attraction”. The terminology is made clearer by Lemma 3.2 below, which provides an alternative description of $\mathcal{A}(S)$.
Julia sets of finitely generated rational semigroups have an interesting property that we will need in our proof of Theorem 1.8. We first state this property and then the pertinent result.

**Definition 2.3.** Let $\Sigma$ be a closed subset of $\hat{\mathbb{C}}$. We say that $\Sigma$ is uniformly perfect if $\Sigma$ contains at least two points and there exists a number $M \in (0, \infty)$ such that for any conformal annulus $A \subset \hat{\mathbb{C}}$ that separates $\Sigma$ (which means:

- $A \cap \Sigma = \emptyset$, and
- $\Sigma$ intersects both the connected components of $\hat{\mathbb{C}} \setminus A$)

the modulus of $A$ is at most $M$.

**Remark 2.4.** Let $\Sigma$ be a non-empty closed subset of $\mathbb{C}$. It is a classical fact—which follows from the work of Pommerenke [13]—that uniform perfectness is equivalent to the following property: there exists a number $c \in (0, 1)$ such that for each $z \in \Sigma$ and each $r \in (0, \text{diam}(\Sigma))$,

$$\Sigma \cap \{ w \in \mathbb{C} : cr \leq |w - z| \leq r \} \neq \emptyset.$$ 

Here, $\text{diam}(\Sigma)$ denotes the diameter of $\Sigma$, and takes values in $(0, \infty]$.

**Result 2.5** (paraphrasing of [16, Theorem 3.1]). Let $S$ be a finitely generated rational semigroup such that $J(S)$ has at least three points. Then $J(S)$ is uniformly perfect.

### 3 Structural lemmas for polynomial semigroups

In this section, we present certain lemmas about polynomial semigroups that will be of use in later sections in this work. Recall the definition of a polynomial semigroup from Section 1: such a semigroup contains at least one polynomial of degree at least 2. For a polynomial semigroup $S$, if $G$ is a set of generators, then $G$ must contain at least one element of degree at least 2. This fact will be used implicitly, with no further comment, in the rest of this paper.

If $g(z) = az + b$ is a polynomial such that $|a| > 1$, then $J(g) = \{ b/(1 - a) \}$. If $p$ is a polynomial of degree at least 2, then $\infty$ is a superattracting fixed point and $\infty \in F(g)$. This information is not enough for concluding whether $\infty \in F(S)$. But if we have a finitely generated polynomial semigroup then $\infty \in F(S)$. To prove this result, we need a lemma, which is interesting in its own right.
**Lemma 3.1.** Let $S$ be a finitely generated polynomial semigroup and let $\mathcal{G} = \{g_1, g_2, \ldots, g_N\}$ be a set of generators of $S$. For every $g \in S$, there are at most finitely many representations of $g$ in terms of elements of $\mathcal{G}$.

**Proof.** Fix $g \in S$. There is a finite number, say $m$, such that in any representation of $g$, the number of elements of $\mathcal{G}$ with degree at least 2 does not exceed $m$. Thus, if every element of $\mathcal{G}$ is of degree at least 2 then the result follows. We now consider the case when there are degree one elements in $\mathcal{G}$. Then, let

$$\Lambda_1 := \min\{ |\Lambda(h) : h \in \mathcal{G}, \deg(h) = 1\},$$

$$\Lambda_2 := \min\{ |\Lambda(h) : h \in \mathcal{G}, \deg(h) \geq 2\},$$

$$d := \max\{ \deg(h) : h \in \mathcal{G} \}.$$

Note that $\Lambda_1 > 1$ and $\Lambda_2 > 0$.

Consider a word $g_{i_n} \circ \cdots \circ g_{i_1}$ with at most $m$ elements with degree at least 2 and $n > m$. Note that

$$\Lambda(g_{i_n} \circ \cdots \circ g_{i_1}) = \Lambda(g_{i_n})\Lambda(g_{i_{n-1}})^{d_{i_{n-1}}} \cdots \Lambda(g_{i_1})^{d_{i_1} \cdots d_{i_n}},$$

where $d_{i_j} = \deg(g_{i_j})$. It is easy to see that if $\Lambda_2 < 1$ then

$$|\Lambda(g_{i_n} \circ \cdots \circ g_{i_1})| \geq \Lambda_1^{n-m} \Lambda_2^{d_{i_1} + \cdots + d_{i_{n-1}}}.$$

Similarly, if $\Lambda_2 \geq 1$ then

$$|\Lambda(g_{i_n} \circ \cdots \circ g_{i_1})| \geq \Lambda_1^{n-m}.$$

In both cases, the right-hand sides of the above inequalities approach $\infty$ as $n \to \infty$. So there exists $n_0 \in \mathbb{Z}_+$ such that (here $g \in S$ is as fixed above)

$$(3.1) \quad |\Lambda(g_{i_n} \circ \cdots \circ g_{i_1})| > |\Lambda(g)| \quad \forall n \geq n_0.$$  

If $g_{i_n} \circ \cdots \circ g_{i_1}$ is a representation of $g$ then $\Lambda(g_{i_n} \circ \cdots \circ g_{i_1}) = \Lambda(g)$. Therefore, by (3.1), $n < n_0$. Since $\mathcal{G}$ is finite, there are at most finitely many words with length $\leq n_0$. Thus there are at most finitely many representations of $g$ in terms of elements of the generating set $\mathcal{G}$. □

Let $x \in \mathcal{A}(S)$. By definition, either $x = \infty$ or $O(x)$ does not have finite limit points in the sense of the explanation following Definition 2.2. Thus, given $r > 0$, $\{g \in S : |g(x)| \leq r\}$ is a finite set. Owing to Lemma 3.1, for $g \in S$, there at most finitely many representations. Thus there exists $n_r(x) \in \mathbb{Z}_+$ such that if $l(g) \geq n_r(x)$ then $|g(x)| > r$.  

Lemma 3.2. Let $S$ be a finitely generated polynomial semigroup and let $\mathcal{G} = \{g_1, g_2, \ldots, g_N\}$ be a set of generators of $S$. Then:

(a) There exist constants $M > 1$ and $R > 0$ such that, for each $i \in \{1, \ldots, N\}$, $|g_i(z)| > M|z|$ whenever $|z| > R$.

(b) If we set $\mathcal{U} := \{|z| > R\} \cup \{\infty\}$ and define

$$A_\infty(\mathcal{G}) := \bigcup_{n=1}^{\infty} \left( \bigcap_{l(g)=n} g^{-1}(\mathcal{U}) \right),$$

then $A_\infty(\mathcal{G})$ does not depend on the choice $R > 0$ in the definition of $\mathcal{U}$ for any $R > 0$ and $M > 1$ for which the conclusion of (a) holds true.

(c) $A(S) = A_\infty(\mathcal{G})$ and, therefore, $\mathcal{K}(S)$ is a compact subset of $\mathbb{C}$.

The proof of this lemma is routine. Therefore, we shall not provide a proof but, instead, make a few explanatory remarks. The conclusion of (a) is a consequence of the fact that, by definition, $\infty$ is an attracting fixed point for each $g_i$, $i \in \{1, \ldots, N\}$. The equality of $A(S)$ and $A_\infty(\mathcal{G})$ relies on the fact—observed just prior to the statement of Lemma 3.2—that whenever $x \in A(S)$, there exists $n_R(x) \in \mathbb{Z}_+$ such that if $l(g) \geq n_R(x)$ then $g(x) \in \mathcal{U}$.

Since, by definition of $\mathcal{K}(S)$, $J(g) \subset \mathcal{K}(S)$ for every $g \in S$, it follows from Result 2.1 that $J(S) \subset \mathcal{K}(S)$. Thus $J(S)$ is also compact in $\mathbb{C}$ and thus $\infty \in F(S)$. In general, if $S$ is a polynomial semigroup (not necessarily finitely generated), then the above lemma is not true—just consider $S = \langle z^2/n : n \in \mathbb{Z}_+ \rangle$.

If $g$ is a polynomial of degree at least 2 then it is known that $J(g) = \partial \mathcal{K}(g)$. For a polynomial $g(z) = az + b$ with $|a| > 1$, we can see that $J(g) = \mathcal{K}(g) = \partial \mathcal{K}(g)$. It is now easy to see that

$$J(S) \subset \bigcup_{g \in S} \mathcal{K}(g) \subset \mathcal{K}(S).$$

In the above expression, inclusions can be strict but if every element in $S$ is of degree at least 2 then we have

Result 3.3 (Boyd, [4]). Let $S$ be a finitely generated polynomial semigroup where each element of $S$ has degree at least 2. Then the unbounded components of $A(S)$ and $F(S)$ are the same.

Remark 3.4. In the version of the above result that Boyd establishes, see [4, Theorem 4.1], he considers a class of semigroups that he calls polynomial semigroups of finite type—see [4, Definition 3.1]. It follows from Lemma 3.2 that a finitely generated polynomial semigroup where each element of $S$ has degree at least 2 is of finite type, which gives us Result 3.3.
We now consider two results that have been referenced in Section 1. First, we provide

**The proof of Proposition 1.5.** Let $S$ be a finitely generated polynomial semigroup. Consider a finite generating set for $S$. Now remove one-by-one the generators that can be expressed as compositions of the other generators. Eventually we will end up having a minimal generating set. So, $S$ has at least one minimal generating set. Let

$$S = \langle g_1, g_2, \ldots, g_N \rangle = \langle h_1, h_2, \ldots, h_{N'} \rangle,$$

where each generating set is minimal. Now, if such sets are not unique then we may assume $\{g_1, g_2, \ldots, g_N\} \neq \{h_1, h_2, \ldots, h_{N'}\}$. Without loss of generality we may assume that $g_1 \not\in \{h_1, h_2, \ldots, h_{N'}\}$. But as $\{h_1, h_2, \ldots, h_{N'}\}$ is a generating set, $g_1 = h_{i_r} \circ \cdots \circ h_{i_1}$ for some $i_1, \ldots, i_r \in \{1, \ldots, N\}$ and $r \geq 2$. Now, since $h_{i_j} \in S = \langle g_1, g_2, \ldots, g_N \rangle$, for every $1 \leq j \leq r$, we get

\begin{equation}
(3.2) \quad g_1 = g_{i_n} \circ \cdots \circ g_{i_1}
\end{equation}

for some $i_1, \ldots, i_n \in \{1, \ldots, N\}$ and $n \geq 2$. If $g_1 \neq g_{i_j}$ for every $1 \leq j \leq n$ then we would have a contradiction of the minimality of the generating set $\{g_1, g_2, \ldots, g_N\}$. Hence, $g_1 = g_{i_{j^*}}$ for some $1 \leq j^* \leq n$.

If $\deg(g_1) = 1$ then, by (3.2), $\deg(g_{i_1}) = \cdots = \deg(g_{i_n}) = 1$. Since $|\Lambda(g)| > 1$ for every $g \in S$ such that $\deg(g) = 1$, $|\Lambda(g_{i_n} \circ \cdots \circ g_{i_1})| = |\Lambda(g_{i_n}) \cdots \Lambda(g_{i_1})| > |\Lambda(g_{i_{j^*}})| = |\Lambda(g_1)|,$ which contradicts (3.2). Now, if $\deg(g_1) \geq 2$ then by (3.2),

$$g_1 = p_2 \circ g_{i_{j^*}} \circ p_1,$$

where $p_1 = g_{i_{j^*+1}} \circ \cdots \circ g_{i_1}$ and $p_2 = g_{i_1} \circ \cdots \circ g_{i_{j^*+1}}$. Therefore, $\deg(p_1) = \deg(p_2) = 1$ and $|\Lambda(p_1)| = |\Lambda(p_2)| > 1$. We compute:

$$|\Lambda(g_{i_n} \circ \cdots \circ g_{i_1})| = |\Lambda(p_2 \circ g_{i_{j^*}} \circ p_1)| = |\Lambda(p_2) \Lambda(g_{i_{j^*}}) \Lambda(p_1)^{\deg(g_{i_{j^*}})}| > |\Lambda(g_{i_{j^*}})| = |\Lambda(g_1)|,$$

which again contradicts (3.2). Thus $S$ has a unique minimal generating set. □
We now state and prove a simple lemma that is essential to the proof of Theorem 1.6. To do so, we need some notation. If $A$ is a finite set (respectively, a finite list) whose elements (respectively, terms) are the non-constant polynomials $g_1, g_2, \ldots, g_N$, then we set

\begin{align*}
(3.3) & \quad C^*(A) := \{ c \in \mathbb{C} : g_i'(c) = 0 \text{ for some } i \in \{1, \ldots, N\} \}, \\
(3.4) & \quad C(A) := \{ c \in J(S) : g_i'(c) = 0 \text{ for some } i \in \{1, \ldots, N\} \}.
\end{align*}

With this notation, we have:

**Lemma 3.5.** Let $S$ be a finitely generated polynomial semigroup. The condition

\[ \# C^*(G_S) = 1 \Rightarrow C(G_S) \cap E(S) = \emptyset \]

holds true if and only if the condition

\[ \# C^*(G) = 1 \Rightarrow C(G) \cap E(S) = \emptyset \]

holds true for any set of generators $G$ of $S$.

**Proof.** The “if” part of the above assertion is obvious. Now suppose that the condition ($\# C^*(G_S) = 1 \Rightarrow C(G_S) \cap E(S) = \emptyset$) holds true. If, for some set of generators $G$, we have $\# C^*(G) = 1$, then $C^*(G_S) = C^*(G)$. This is because $G_S \subseteq G$ and $C^*(G_S) \neq \emptyset$. Thus $C(G_S) = C(G)$, whence ($\# C^*(G) = 1 \Rightarrow C(G) \cap E(S) = \emptyset$) holds true. \(\square\)

We conclude this section with a discussion on the type of polynomial semigroups excluded by the condition in Theorem 1.6.

**3.1 On the exceptional semigroups of Remark 1.7.** Let $S$ be a finitely generated polynomial semigroup that does not satisfy the condition in Theorem 1.6. Let us write $G_S = \{ g_1, g_2, \ldots, g_N \}$. As $\# C^*[S] = 1$ and $C[S] \cap E(S) \neq \emptyset$, there exists a point $a \in \mathbb{C}$ such that $C^*[S] = C[S] = \{a\}$. From this, and the fact—evident from the definition—that $g^{-1}(E(S)) \subseteq E(S)$ for every $g \in S$, we see that

\[ g_j(z) = B_j(z - a)^{n_j} + a \quad \text{whenever deg}(g_j) \geq 2, \]

for some constant $B_j \in \mathbb{C} \setminus \{0\}$ and some $n_j \in \mathbb{Z}_+ \setminus \{1\}$. At this stage, we record the following

**Fact.** If $\mathcal{S}$ is a finitely generated rational semigroup each of whose elements has degree at least 2, then $E(\mathcal{S}) \subset F(\mathcal{S})$. 


The proof is routine. Since $C[S] \cap \mathcal{E}(S) \neq \emptyset$, it follows from the above fact that $\mathcal{S}_S$ must contain degree-one elements. Once again, as $g^{-1}(\mathcal{E}(S)) \subseteq \mathcal{E}(S)$ for every $g \in S$, we get
\[ g_j(z) = B_j(z - a) + a \quad \text{whenever } \deg(g_j) = 1, \]
for some constant $B_j \in \mathbb{C}$ with $|B_j| > 1$. From this discussion it follows that:

- Every element of $S$ is of the form $B(z - a)^m + a$, where $B \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{Z}_+$.
- $S$ has degree-one elements, for all of which $a$ is a repelling fixed point.

4 Essential definitions and results in potential theory

This section is devoted to presenting a number of essential definitions in potential theory that we had deferred in Section 1. Additionally, we collect here several important results that we shall require for our proofs.

**Definition 4.1.** Let $\sigma$ be a Borel probability measure on $\mathbb{C}$ with compact support. Its **logarithmic potential** is the function $U^\sigma : \mathbb{C} \to (-\infty, \infty]$ defined by
\[ U^\sigma(z) = \int_{\mathbb{C}} \log \frac{1}{|z - t|} d\sigma(t) \]
and its **logarithmic energy** is given by
\[ I(\sigma) := \int_{\mathbb{C}} \int_{\mathbb{C}} \log \frac{1}{|z - t|} d\sigma(z) d\sigma(t) = \int_{\mathbb{C}} U^\sigma(z) d\sigma(z). \]

The potential $U^\sigma$ is superharmonic in $\mathbb{C}$ and harmonic outside the support of $\sigma$. As noted by Frostman [9], the potential $U^\sigma$ is finite at $z_0$ (i.e., does not take the value $+\infty$) if for some $\epsilon > 0$ the integral
\[ \int_0^\epsilon \frac{\sigma(D(z_0, r))}{r} dr \]
exists and is finite. However, one can say much more.

**Result 4.2 (Arsove, [1]).** For the potential $U^\sigma$ of a Borel probability measure $\sigma$ to be continuous at $z_0$, it is necessary and sufficient that
\[ \lim_{\epsilon \to 0} \left\{ \limsup_{z \to z_0} \int_0^\epsilon \frac{\sigma(D(z, r))}{r} dr \right\} = 0. \]
Moreover, if $\text{supp}(\sigma)$ lies in a closed set $\Sigma$, then the approach of $z$ to $z_0$ in the above limit can be restricted to points $z$ of $\Sigma$. 
Remark 4.3. It follows from Result 4.2 that the potential $U_\sigma$ will be finite and continuous at $z_0$ if $\sigma$ satisfies a condition of the form

$$\sigma(D(z, r)) \leq Cr^\alpha \quad \forall r \in (0, r_0),$$

where $|z - z_0| < \delta$ and $C, \alpha, r_0, \delta$ are positive constants depending only on $\sigma$ and $z_0$.

If $E \subset \mathbb{C}$ is a Borel set, then $\mathcal{M}(E)$ will always denote the collection of all Borel probability measures $\sigma$ with $\text{supp}(\sigma) \subset E$. Let $K \subset \mathbb{C}$ be a compact subset of the complex plane. Define

$$V := \inf \{ I(\sigma) : \sigma \in \mathcal{M}(K) \}.$$ 

Then $V$ turns out to be finite or $+\infty$. The quantity

$$\text{cap}(K) := \exp(-V)$$

is called the logarithmic capacity (or simply capacity) of $K$. The capacity of an arbitrary Borel set $E$ is defined as

$$\text{cap}(E) := \sup \{ \text{cap}(K) : K \subset E, \ K \text{ compact} \}$$

and every set (not necessarily a Borel set) that is contained in a Borel set of zero capacity is considered to have zero capacity. A property is said to hold quasi-everywhere (which we shall often abbreviate to q.e.) on a set $E$ if the set of points in $E$ at which this property does not hold is of logarithmic capacity zero.

The results and definitions that follow are from the book [15] by Saff and Totik. First, we state a couple of results which describe the behaviour of the logarithmic potential with respect to a convergent sequence of measures in the weak* topology.

Result 4.4 (Principle of Descent and Lower Envelope Theorem). Let $\sigma_n, n = 1, 2, \ldots$, be a sequence of Borel probability measures all having support in a fixed compact subset of $\mathbb{C}$. If $\sigma_n \to \sigma$ in the weak* topology then

$$\liminf_{n \to \infty} U^\sigma_n(z) \geq U^\sigma(z) \quad \text{for every } z \in \mathbb{C},$$

$$\liminf_{n \to \infty} U^\sigma_n(z) = U^\sigma(z) \quad \text{for q.e. } z \in \mathbb{C}.$$ 

We will now introduce some basic definitions and results from weighted potential theory.

Let $\Sigma \subset \mathbb{C}$ be a closed set and $\omega : \Sigma \to [0, \infty)$. We call such a function a weight function on $\Sigma$. 
**Definition 4.5.** A weight function $w$ on $\Sigma$ is said to be **admissible** if it satisfies the following three conditions:

(i) $w$ is upper semi-continuous;
(ii) $\Sigma_0 := \{ z \in \Sigma : w(z) > 0 \}$ has positive capacity;
(iii) if $\Sigma$ is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty$, $z \in \Sigma$.

Consider an admissible weight function on $\Sigma$, and define $Q \equiv Q_w$ by

$$w(z) = \exp(-Q(z)).$$

Then $Q : \Sigma \to (-\infty, \infty]$ is lower semi-continuous, $Q(z) < \infty$ on a set of positive capacity and if $\Sigma$ is unbounded, then

$$\lim_{|z| \to \infty, z \in \Sigma} \{ Q(z) - \log |z| \} = \infty.$$

The function $Q$ is called an **external field**.

Let $\Sigma \subset \mathbb{C}$ be a closed set. For any $\sigma \in \mathcal{M}(\Sigma)$, and $w$ an admissible weight function on $\Sigma$, we define the **weighted energy integral**

$$I_w(\sigma) := \int_\Sigma \int_\Sigma \log \frac{1}{|z - t|} w(z) w(t) \, d\sigma(z) d\sigma(t)$$

$$= \int_\Sigma \int_\Sigma \log \frac{1}{|z - t|} \, d\sigma(z) d\sigma(t) + 2 \int_\Sigma Q \, d\sigma,$$

where the last representation is valid whenever both integrals exist and are finite. It follows from the definition of an admissible weight that the first integral is well defined.

**Definition 4.6.** Let $w$ be an admissible weight on the closed set $\Sigma$ and let

$$V_w := \inf\{ I_w(\sigma) : \sigma \in \mathcal{M}(\Sigma) \}.$$  

Then a measure $\sigma$ is called an **equilibrium measure** associated with $w$ (or, equivalently in view of (4.1), an equilibrium measure associated with $Q$) if

$$I_w(\sigma) = V_w.$$  

**Remark 4.7.** In view of the relation (4.1), we shall use the phrases “admissible weight” and “external field” interchangeably, both of which are standard in the literature. We used the term “external field” in Section 1 because it has a well-understood meaning in electrostatics.

Now we are ready to state the fundamental theorem of the theory, which gives existence and uniqueness of equilibrium measures associated with $w$. 
Result 4.8. Let $w$ be an admissible weight on the closed set $\Sigma$. Then $V_w$ is finite and there exists a unique equilibrium measure $\sigma_w \in \mathcal{M}(\Sigma)$ associated with $w$. Moreover, $\sigma_w$ has finite logarithmic energy.

Remark 4.9. If $w \equiv 1$ (i.e., $Q \equiv 0$) then $I_w(\sigma) = I(\sigma)$. In this case, if $\Sigma$ is a compact subset of $\mathbb{C}$ (of positive capacity) then we recover the classical theory of logarithmic potentials. The unique equilibrium measure associated with the weight $w \equiv 1$ on the compact set $\Sigma$ is called the equilibrium measure of $\Sigma$. It is denoted by $\sigma_\Sigma$.

With $w$ as above, define

$$S_w := \text{supp}(\sigma_w),$$
$$F_w := V_w - \int_\mathbb{C} Q d\sigma_w.$$

The constant $F_w$ is called the modified Robin constant for $w$.

Result 4.10. Let $w$ be an admissible weight on the closed set $\Sigma$. If $\sigma \in \mathcal{M}(\Sigma)$ has compact support and finite logarithmic energy, and $U^q(z) + Q(z)$ coincides with a constant $F$ for quasi-every $z$ in $\text{supp}(\sigma)$, and $U^q + Q \geq F$ quasi-everywhere on $\Sigma$, then $\sigma = \sigma_w$ and $F = F_w$.

Let $K$ be a compact subset of $\Sigma$ of positive capacity, and define

$$F(K) := \log \text{cap}(K) - \int_\Sigma Q d\sigma_K,$$

where $\sigma_K$ denotes the equilibrium measure of the compact set $K$. This is the so-called $F$-functional of Mhaskar and Saff, which is one of the most powerful tools in finding $\sigma_w$ and $S_w$. We will use the $F$-functional to estimate the logarithmic capacity of the Julia set of a finitely generated polynomial semigroup.

Result 4.11. Let $w$ be an admissible weight on the closed set $\Sigma$. Then the following hold:

(a) For every compact set $K \subset \Sigma$ of positive capacity, $F(K) \leq F(S_w)$.
(b) $F(S_w) = -F_w$, where $F_w$ is the modified Robin constant for $w$.

5 Complex-analytic preliminaries

This section gathers together a number of results in complex analysis, along with some consequences thereof, that we shall need in our proofs in Sections 6 and 7.
5.1 Orders and degrees. Let $f$ be a non-constant holomorphic $\hat{\mathbb{C}}$-valued map defined in a neighbourhood of $a \in \hat{\mathbb{C}}$. Let $(U, \phi)$ and $(V, \psi)$ be holomorphic charts at $a$ and $f(a)$ respectively such that $\tilde{f} := \psi \circ f \circ \phi^{-1}$ is defined. Suppose the Taylor expansion of $\tilde{f}$ at $\phi(a)$ has the form

$$\tilde{f}(z) = b_0 + b_m(z - \phi(a))^m + b_{m+1}(z - \phi(a))^{m+1} + \cdots,$$

where $b_m \neq 0$. Recall that the (unique) integer $m$ does not depend on the choice of the charts $(U, \phi)$ or $(V, \psi)$, and is called the order of $f$ at $a$ and is denoted by $\text{ord}_a(f)$.

**Result 5.1** (Riemann–Hurwitz Formula). For any non-constant rational map $f$

$$\sum_{z \in \hat{\mathbb{C}}} (\text{ord}_z(f) - 1) = 2 \deg(f) - 2.$$

For a non-constant polynomial $g$, we have $\text{ord}_{\infty}(g) = \deg(g)$. Thus, in this case, Result 5.1 becomes

$$\sum_{z \in \mathbb{C}} (\text{ord}_z(g) - 1) = \deg(g) - 1.$$

Observe that the general term in the sum is positive only when $z$ is a critical point of $g$.

Let the non-constant polynomials $g_1, g_2, \ldots, g_N$ be the entries of the list $A$ (whence they are not necessarily distinct). Recall the definition of $C^*(A)$: see (3.3). In Section 6, we will need to explicitly refer to these polynomials. To this end, we define $C^*(g_1, g_2, \ldots, g_N) := C^*(A)$.

The next lemma is needed in the proof of Proposition 6.1. Recall, from the discussion in Section 1, that Proposition 6.1 establishes for the semigroups of our interest a result analogous to that in [2]—but which allows elements of $\mathcal{J}$ to have critical points in $\mathcal{J}(S)$. The following lemma is the key to dealing with the latter situation.

**Lemma 5.2.** Let $g_1, g_2, \ldots, g_N$ be a collection of non-constant polynomials such that $\deg(g_i) \geq 2$ for some $i \in \{1, \ldots, N\}$. Set $D := \sum_{i=1}^N \deg(g_i)$. If $\nabla(C^*(g_1, g_2, \ldots, g_N)) > 1$ then there exists $\kappa \in \mathbb{Z}_+$ such that

$$\sum_{i : g_i'(x) \neq 0} \left( \frac{D}{N} \right)^{\frac{1}{2}} + \sum_{i : g_i'(x) = 0} \text{ord}_x(g_i) \leq D - \frac{1}{2} \quad \forall x \in C^*(g_1, g_2, \ldots, g_N).$$
**Proof.** For simplicity, we shall denote the set $C^*(g_1, g_2, \ldots, g_N)$ as $C^*$. Let $x \in C^*$. Suppose first that there exists $i^* \in \{1, \ldots, N\}$ such that $\deg(g_{i^*}) \geq 2$ and $g'_{i^*}(x) \neq 0$. Then, since $\text{ord}_x(g_{i^*}) = 1$, $\text{ord}_x(g_{i^*}) \leq \deg(g_{i^*}) - 1$. On the other hand, if $g'_{i^*}(x) = 0$ for every $i^* \in \{1, \ldots, N\}$, then, since $\#C^* > 1$, there exists $x' (\neq x) \in C^*$ such that $g'_{i^*}(x') = 0$ for some $i' \in \{1, \ldots, N\}$. Thus $\text{ord}_{x'}(g_{i'}) \geq 2$.

Now, by (5.1), $\text{ord}_x(g_{i'}) \leq \deg(g_{i'}) - 1$. To summarize: for each $x \in C^*$ there exists $i \in \{1, \ldots, N\}$ such that $\text{ord}_x(g_i) \leq \deg(g_i) - 1$. Of course, in general, we have $\text{ord}_x(g_i) \leq \deg(g_i)$ for all $i \in \{1, \ldots, N\}$. So summing over all $i \in \{1, \ldots, N\}$ gives

$$ \sum_{i=1}^{N} \text{ord}_x(g_i) \leq \left( \sum_{i=1}^{N} \deg(g_i) \right) - 1 = D - 1. $$

It is easy to see that for each $x \in C^*$,

$$ \sum_{i: g'_i(x) \neq 0} \left( \frac{D}{N} \right) + \sum_{i: g'_i(x) = 0} \text{ord}_x(g_i) \longrightarrow \sum_{i=1}^{N} \text{ord}_x(g_i) \quad \text{as} \quad k \to \infty. $$

Thus, in view of the last inequality, we can choose $\kappa \in \mathbb{Z}_+$ such that for all $x \in C^*$,

$$ \sum_{i: g'_i(x) \neq 0} \left( \frac{D}{N} \right) + \sum_{i: g'_i(x) = 0} \text{ord}_x(g_i) \leq D - \frac{1}{2}. $$

Thus we have the proof. \[\square\]

### 5.2 The Dinh–Sibony measure.

We now provide a brief discussion of the formalism and the results underlying the convergence statement (1.3). The first basic observation is that with $X$ as in Section 1, any two holomorphic correspondences on $X$ can be composed with each other. Since compositions of correspondences of the most general kind are not relevant to the proofs in this paper, we shall just make the following observations on the subject of composing two correspondences. (For readers who are more comfortable with complex analysis in one dimension, we refer to [2] for a more detailed discussion.) They are:

(i) The **topological degree** of a holomorphic correspondence $\Gamma$ is the generic number of preimages of a point counted according to multiplicity. To elaborate: representing $\Gamma$ as in (1.1), it is classical that there is a Zariski-open set $W \subset X_2$ and $v_i \in \mathbb{Z}_+$ such that $(\pi_2^{-1}(W) \cap \Gamma, W, \pi_2)$ is a $v_i$-sheeted covering. The topological degree of $\Gamma$ is defined as

$$ \text{deg}_{\text{top}}(\Gamma) := \sum_{1 \leq i \leq N} m_i v_i. $$
In the 1-dimensional case, we abbreviate \( \text{deg}_{\text{top}}(\Gamma) \) to \( d_1(\Gamma) \), as introduced in Section 1. In what follows, \( \Gamma^\dagger \) will denote the **adjoint** of \( \Gamma \). In the notation of (1.1),

\[
\Gamma^\dagger := \sum_{1 \leq i \leq N} m_i \Gamma_i^\dagger
\]

where \( \Gamma_i^\dagger := \{(y, x) \in X_2 \times X_1 : (x, y) \in \Gamma_i\} \). In the 1-dimensional case, \( d_0(\Gamma) := d_1(\Gamma^\dagger) \).

(ii) A holomorphic correspondence \( \Gamma \) determines a relation from \( X_1 \) to \( X_2 \) given, in the notation of (1.1), by \( \bigcup_{1 \leq i \leq N} \Gamma_i^j \). Thus, given two holomorphic correspondences on \( X \), their composition is, in essence, the composition of the underlying relations with a little care taken to account for the multiplicities (the integers \( m_1, m_2, \ldots, m_N \) in the notation of (1.1)). The observation concerning this “accounting” that is relevant to us is that if \( X \) is a compact Riemann surface and \( \Gamma^1 \) and \( \Gamma^2 \) are holomorphic correspondences on \( X \) then \( d_j(\Gamma^2 \circ \Gamma^1) = d_j(\Gamma^2) d_j(\Gamma^1), j = 0, 1 \). We shall denote the \( n \)-fold iterated composition of \( \Gamma \) by \( \Gamma^{\circ n} \).

Also of immediate relevance is the following formula: given the following collections of non-constant rational maps \( g_1, g_2, \ldots, g_N \) and \( f_1, f_2, \ldots, f_M \), not necessarily distinct, and

\[
\Gamma^1 := \sum_{1 \leq i \leq N} \text{graph}(g_i) \quad \text{and} \quad \Gamma^2 := \sum_{1 \leq j \leq M} \text{graph}(f_j),
\]

it turns out that

\[
\Gamma^2 \circ \Gamma^1 = \sum_{1 \leq i \leq N} \left( \sum_{1 \leq j \leq M} \text{graph}(f_j \circ g_i) \right).
\]

Deferring for the moment the discussion of pull-backs of currents by holomorphic correspondences, we fix the following notation: with \( X \) as in Section 1, \( \Gamma \) a holomorphic correspondence on \( X \) and \( T \) a current that can be pulled back by \( \Gamma \), we will denote the pull-back of \( T \) by \( F_\Gamma^* T \). With this, we can state the two results from which (1.3) follows. These are results by Dinh–Sibony. The specific results cited actually address much more general (including multi-dimensional) situations than ours. In the form in which they appear, they are heavily paraphrased in two ways:

- they are stated merely for holomorphic correspondences on \( \hat{\mathbb{C}} \); and
- the convergence stated below actually holds on a larger class of test functions (which were introduced in [8]; also see [7]), but weak* convergence suffices for our purposes.

With these words, the results needed are:
**Result 5.3** (Théorème 5.1 of [8] paraphrased for \(\hat{\mathbb{C}}\)). Let \(\Gamma_n, n \in \mathbb{Z}_+\), be holomorphic correspondences on \(\hat{\mathbb{C}}\). Suppose that the series
\[
\sum_{n \in \mathbb{Z}_+} \left( \frac{d_0(\Gamma_1)}{d_1(\Gamma_1)} \cdots \frac{d_0(\Gamma_n)}{d_1(\Gamma_n)} \right)
\]
calculates. Then, there exists a regular Borel probability measure \(\mu\) such that
\[
d_1(\Gamma_1)^{-1} \cdots d_1(\Gamma_n)^{-1} F_{\Gamma_1 \cdots \Gamma_1}^* (\omega_{FS}) \overset{\text{weak}^*}{\longrightarrow} \mu \text{ as measures, as } n \to \infty.
\]
The measure \(\mu\) places no mass on polar sets.

**Result 5.4** (Théorème 1.1 of [8] paraphrased for \(\hat{\mathbb{C}}\)). Let \(\Gamma_n, n \in \mathbb{Z}_+\), be holomorphic correspondences on \(\hat{\mathbb{C}}\). Suppose
\[
\sum_{n \in \mathbb{Z}_+} \frac{d_0(\Gamma_n)}{d_1(\Gamma_n)}
\]
calculates. Then, there exists a Borel polar set \(E \subset \hat{\mathbb{C}}\) such that for any \(a \in \hat{\mathbb{C}} \setminus E\),
\[
d_1(\Gamma_n)^{-1} \left( F_{\Gamma_n}^* (\omega_{FS}) - F_{\Gamma_n}^* (\delta_a) \right) \overset{\text{weak}^*}{\longrightarrow} 0 \text{ as } n \to \infty.
\]

In both these results, \(\omega_{FS}\) stands for the Fubini–Study form on \(\hat{\mathbb{C}}\). As this is a volume form on \(\hat{\mathbb{C}}\), it and its pull-backs are treated as measures in Result 5.3. Both results involve the notion of the pull-back of a measure by a correspondence. A measure—as discussed in [8, Sections 2–3]—is an example of a current that can be pulled back by a holomorphic correspondence, which is the general framework for the results in [8]. Since there is a fairly detailed discussion of the definition and computation of pull-backs of measures in the one-dimensional setting in [2, Section 4.1], we refer the reader to it.

To conclude this section, we present the following pull-back formula (the details of whose computation are presented in the last reference). Let \(g_1, g_2, \ldots, g_N\) be a collection of non-constant polynomials, not necessarily distinct. Call this collection \(\mathcal{C}\) and write
\[
\Gamma_{\mathcal{C}} := \sum_{1 \leq i \leq N} \text{graph}(g_i).
\]
Let \(a \in \hat{\mathbb{C}}\). For simplicity of notation, we shall abbreviate here, and in the sections that follow, the pull-back \(F_{\Gamma_{\mathcal{C}}}^* \delta_a\) as \(F_{\mathcal{C}}^* \delta_a\). Then,
\[
(F_{\mathcal{C}}^* \delta_a) = \sum_{1 \leq i \leq N} \sum_{x \in g_i^{-1}\{a\}} \delta_x.
\]
Here, the notation \(x \in g_i^{-1}\{a\}\) signifies that \(x\) is repeated according to multiplicity as it varies through \(g_i^{-1}\{a\}\). Also, let us abbreviate \(F_{\Gamma_{\mathcal{C}}}^* = (F_{\mathcal{C}}^*)^n \delta_a\). Then, from Results 5.3 and 5.4, we conclude that there exist a Borel polar set \(E \subset \hat{\mathbb{C}}\) and a measure \(\mu_{\mathcal{C}}\) having the properties stated in Result 5.3 such that
\[
d_1(\Gamma_{\mathcal{C}})^{-n} (F_{\mathcal{C}}^*)^n \delta_a \overset{\text{weak}^*}{\longrightarrow} \mu_{\mathcal{C}} \text{ as } n \to \infty, \forall a \in \hat{\mathbb{C}} \setminus E.
\]
6 Theorem 1.6 and associated results

This section is devoted to proving several results—including the theorem alluded to in Section 1 in the discussion following the statement of Theorem 1.6—that are closely tied to the proof of the latter theorem. To do so, we need to fix certain notations. Let $S$ be a finitely generated polynomial semigroup. Let $g_1, g_2, \ldots, g_N$ (not necessarily distinct) be polynomials such that $S = \langle g_1, g_2, \ldots, g_N \rangle$, and define

$$M := \max \{|g'_i(z)| : z \in J(S), \ i \in \{1, \ldots, N\}\},$$

$$R := \frac{D}{N} \quad \text{and} \quad \lambda := \frac{\log R}{\log M},$$

where $D := \sum_{i=1}^N \deg(g_i)$. Thus $M > R^\frac{\lambda}{2}$. Note that $R > 1$ and by Result 2.1, repelling fixed points of all elements of $S$ are dense in $J(S)$. Thus $M > 1$. Also define

$$C(g_1, g_2, \ldots, g_N) := \{ c \in J(S) : g'_i(c) = 0 \ \text{for some} \ i \in \{1, \ldots, N\}\}.$$

In what follows, a collection denoted by $A^\bullet$ will represent a list; the objects in $A^\bullet$ will be repeated according to multiplicity. Also $A$ will denote the set underlying $A^\bullet$. The notation $\sharp A^\bullet$ will denote the number of objects in $A^\bullet$, counted according to multiplicity. All other notations will be as introduced in Sections 1 and 5.

We are now ready to state the following

**Proposition 6.1.** Let $S$ be a finitely generated polynomial semigroup and let $g_1, g_2, \ldots, g_N$ (not necessarily distinct) be polynomials such that

$$S = \langle g_1, g_2, \ldots, g_N \rangle.$$

Assume

$$\sharp(C^\bullet(g_1, g_2, \ldots, g_N)) > 1.$$

Consider the correspondence

$$\Gamma := \sum_{1 \leq i \leq N} \text{graph}(g_i),$$

and abbreviate $(F^n)^\dagger := F_{(\Gamma)^n}$. Then there exist $r_0 > 0$ and $\kappa \in \mathbb{Z}_+$ such that for any $r \in (0, r_0]$ and $y \in J(S)$, we have

$$\sharp((F^n)^\dagger(y) \cap D(z, r)) \leq \max \left(D^{n-\frac{\nu}{\lambda}+1}N^{\frac{\nu}{\lambda}-1}, \left(D - \frac{1}{2}\right)^n\right)$$

for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$, where $\nu \in \mathbb{Z}_+$ is the unique integer such that

$$r \in I(\nu) := \left(r_0R^{-\frac{2\nu}{\lambda}}, r_0R^{-\frac{2\nu-1}{\lambda}}\right].$$
Proof. For $\varepsilon > 0$, let us write
\[ J^\varepsilon := \bigcup_{\xi \in \mathcal{J}(S)} D(\xi, \varepsilon) \quad \text{and} \quad \bar{J}^\varepsilon := \overline{J^\varepsilon}. \]

In this proof we will abbreviate $\mathcal{C}(g_1, g_2, \ldots, g_N)$ to $\mathcal{C}$. If $\mathcal{C} \neq \emptyset$ then denote this (finite) set of points by $\{c_1, c_2, \ldots, c_q\}$. Note that (5.2) holds for all $c \in \mathcal{C}$.

Consider the following quantities:

- If $\mathcal{C} \neq \emptyset$ then let $\delta_1 > 0$ be so small that:
  1. $D(c_j, 2\delta_1)$ are pairwise disjoint for $j = 1, 2, \ldots, q$,
  2. if for some $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, q\}$, $g_i'(c_j) = 0$ then $|g_i'(z)| < 1$ for every $z \in D(c_j, 2\delta_1)$ and $g_i$ maps at most $\text{ord}_c(c_j)$ points of $D(c_j, 2\delta_1)$ to a single point of $\mathbb{C}$,
  3. if for some $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, q\}$, $g_i'(c_j) \neq 0$ then $|g_i'(z)| \neq 0$ for every $z \in D(c_j, 2\delta_1)$.

If $\mathcal{C} = \emptyset$, then we just set $\delta_1 := 1$.

- Let $\delta_2 > 0$ be such that $g_i'(z) \neq 0$ for every $z \in \overline{J^{2\delta_1}} \setminus \mathcal{J}(S)$ and $i = 1, \ldots, N$.
- Let $\delta_3 > 0$ be such that $|g_i'(z)| < R^2$ for every $z \in \overline{J^{\delta_2}}$ and $i = 1, \ldots, N$.

Next, we introduce the following open covers:

(i) Define
\[ \emptyset_0 := \begin{cases} 
\{D(\xi, r(\xi)) : \xi \in \overline{J^{2\delta_1}} \setminus \bigcup_{j=1}^q D(c_j, \delta_1)\}, & \text{if } \mathcal{C} \neq \emptyset, \\
\{D(\xi, r(\xi)) : \xi \in \overline{J^{2\delta_1}}\}, & \text{if } \mathcal{C} = \emptyset,
\end{cases} \]

where $r(\xi) > 0$ is such that $g_i|_{D(\xi, r(\xi))}$ is injective for $i = 1, 2, \ldots, N$.

(ii) If $\mathcal{C} \neq \emptyset$, then for each $j = 1, 2, \ldots, q$, $\emptyset_j := \{D(\xi, r(\xi)) : \xi \in \overline{D(c_j, \delta_1)}\}$, where $r(\xi) > 0$ is such that if $g_i'(c_j) \neq 0$ then $g_i|_{D(\xi, r(\xi))}$ is injective.

Finally, we set $\delta_4 > 0$ to be the minimum of the Lebesgue numbers of all the covers introduced above, and write:
\[ r_0 := \frac{\min\{\delta_1, \delta_2, \delta_3, \delta_4\}}{4}. \]

If $\mathcal{C} = \emptyset$ then set $\kappa = 1$, else fix $\kappa \in \mathbb{Z}_+$ such that (5.2) holds true for $g_1, g_2, \ldots, g_N$ as above. With this choice of $r_0$ and $\kappa$, we will prove our result by induction on $n$. To this end, we must mention that, by definition, $(F^0)^\dagger(y) := \{y\}$ for any $y \in \mathcal{C}$.

Fix an arbitrary $y \in \mathcal{J}(S)$. If $n = 0$, then the second term on the right-hand side of (6.1) is 1 and the left-hand side of (6.1) is by definition $\leq 1$ for any $z \in \mathbb{C}$ and $r \in (0, r_0]$. Hence, the inequality (6.1) follows for these $r$ and $z$. Now assume that (6.1) holds for $n = m$, where $m \geq 0$, for every $z \in \mathbb{C}$ and $r \in (0, r_0]$. We will
study what this implies for \( n = m + 1 \). Consider an arbitrary \( z \in \mathbb{C} \) and \( r \in (0, r_0] \).

If \((z, r)\) is such that \( D(z, r) \cap J(S) = \emptyset \) then, as \((F^n)^\dagger(y) \subset J(S)\), the left-hand side of (6.1) is zero and the latter inequality trivially follows for all \( n \in \mathbb{N} \). Hence, assume that \( D(z, r) \cap J(S) \neq \emptyset \).

If, for the \( r \) chosen, \( \nu = 1 \), then the first term on the right-hand side of (6.1) is

\[
D^{m+1} \left( \frac{D}{N} \right)^{1-\frac{1}{\nu}} \geq D^{m+1},
\]

whence, by definition, the left-hand side of (6.1) \( \leq D^{m+1} \). Thus (6.1) (for \( \nu = 1 \)) follows. We therefore consider \( \nu > 1 \).

**Case 1.** Either \( C = \emptyset \) or \((C \neq \emptyset \text{ and } z \notin \bigcup_{j=1}^{\nu} D(c_j, \delta_1))\).

Observe that \( D(z, r) \cap J(S) \neq \emptyset \) implies that \( z \in J' \). Let us write

\[
\mathcal{B} := \begin{cases} \bigcup_{j=1}^{\nu} D(c_j, \delta_1), & \text{if } C \neq \emptyset, \\ \emptyset, & \text{if } C = \emptyset. \end{cases}
\]

Since \( r \leq \delta_2/4 \), we get that \( z \in J_0^{\delta_2} \setminus \mathcal{B} \). Also, since \( r \leq \delta_4/4 \), there exists a disc \( D(\xi, r(\xi)) \in \mathcal{B}_0 \) such that \( D(\xi, r(\xi)) \supset D(z, r) \). Thus, by the choice of \( r(\xi) \) in defining the cover \( \mathcal{B}_0 \), \( g_i|_{D(z, r)} \) is an injective map for \( i = 1, \ldots, N \), and we get

\[
\sharp((F^{m+1})^\dagger(y) \cap D(z, r))^\bullet = \sum_{i=1}^{N} \sharp((F^{m})^\dagger(y) \cap g_i(D(z, r)))^\bullet.
\]

Now, \( r \leq \delta_3/4 \) and \( z \in J' \) implies that \( D(z, r) \subset J_0^{\delta_3} \). Thus \( |g_i'(\xi)| < R_2/2 \) for all \( \xi \in D(z, r) \) and \( i = 1, \ldots, N \). Therefore, by an application of the Mean Value Inequality, we get

\[
g_i(D(z, r)) \subset D(g_i(z), rR_2^{2/\nu}).
\]

Thus

\[
\sum_{i=1}^{N} \sharp((F^m)^\dagger(y) \cap g_i(D(z, r)))^\bullet \leq \sum_{i=1}^{N} \sharp((F^m)^\dagger(y) \cap D(g_i(z), rR_2^{2/\nu}))^\bullet.
\]

Observe that \( rR_2^{2/\nu} \in I(\nu - 1) \); thus, by the induction hypothesis and the above observations,

\[
\sharp((F^{m+1})^\dagger(y) \cap D(z, r))^\bullet \leq N \max \left( D^m - \frac{1}{\nu} + 1, N^{-\frac{1}{\nu} - 1} \left( D - \frac{1}{2} \right)^m \right)
\]

(6.2)

\[
= \max \left( D^{m+1} - \frac{x}{\nu} + 1, N^{\frac{1}{\nu} - 1} \left( \frac{N}{D} \right)^{1-\frac{1}{\nu}}, N^{\frac{1}{\nu} - 1} N^{\frac{1}{\nu} - 1} \left( D - \frac{1}{2} \right)^m \right).
\]
As $N < D$, we see that
\[
N \leq D - \frac{1}{2} \quad \text{and} \quad \left( \frac{N}{D} \right)^{\frac{1}{2}} \leq 1.
\]
Thus, from (6.2) and the above, we have the desired claim for $n = m + 1$:
\[
\#((F^{m+1})^\dagger(y) \cap D(z, r))^\bullet \leq \max \left( D^{m+1-\frac{1}{2}+1}N^{\frac{1}{2}-1}, \left( D - \frac{1}{2} \right)^{m+1} \right).
\]

When $C \neq \emptyset$, we must also consider

**Case 2.** $z \in \bigcup_{j=1}^{q} D(c_j, \delta_i)$.

By our choice of $\delta_i > 0$, there is a unique $j^0 \in \{1, \ldots, q\}$ such that $z \in D(c_{j^0}, \delta_i)$. For simplicity, we shall write $c := c_{j^0}$. If $i \in \{1, \ldots, N\}$ is such that $g_i^p(c) \neq 0$ then the description in (ii) of the cover for $D(c, \delta_i)$ implies that $g_i|_{D(z, r)}$ is an injective map. Note that $D(z, r) \subset D(c, 2\delta_i)$. Thus, from our discussion on the choice of $\delta_i > 0$, if $i \in \{1, \ldots, N\}$ is such that $g_i^p(c) = 0$ then $g_i$ maps at most $\ord_c(g_i)$ points of $D(z, r)$ to a single point of $C$. Thus we have
\[
\#((F^{m+1})^\dagger(y) \cap D(z, r))^\bullet \leq \sum_{1 \leq i \leq N} \#((F^{m})^\dagger(y) \cap g_i(D(z, r)))^\bullet
\]
\[
+ \sum_{1 \leq i \leq N} \ord_c(g_i) \#((F^{m})^\dagger(y) \cap g_i(D(z, r)))^\bullet.
\]

In the above inequality, the primed sum denotes the sum over only those indices $i$ such that $g_i^p(c) \neq 0$ while the starred sum denotes the sum over only those indices $i$ such that $g_i^p(c) = 0$. These will have the same meaning in the expressions below.

If $g_i^p(c) \neq 0$ then $|g_i^p(\zeta)| \leq R^{2/\lambda}$ for all $\zeta \in D(z, r) \subset J^{j^0}$. Now, by the Mean Value Inequality, $g_i(D(z, r)) \subset D(g_i(z), rR^{2/\lambda})$. Similarly, if $g_i^p(c) = 0$ then $|g_i^p(\zeta)| \leq 1$ for all $\zeta \in D(z, r) \subset D(c, 2\delta_i)$, and we get
\[
g_i(D(z, r)) \subset D(g_i(z), r).
\]

Thus:
\[
\#((F^{m+1})^\dagger(y) \cap D(z, r))^\bullet \leq \sum_{1 \leq i \leq N} \#((F^{m})^\dagger(y) \cap D(g_i(z), rR^{2/\lambda}))^\bullet
\]
\[
+ \sum_{1 \leq i \leq N} \ord_c(g_i) \#((F^{m})^\dagger(y) \cap D(g_i(z), r))^\bullet.
\]
Now, by the induction hypothesis and noting that $r R^{2/\lambda} \in I(\nu - 1)$,

$$\#((F^{m+1})^\dagger(y) \cap D(z, r))^* \leq \sum_{1 \leq i \leq N} \max \left( D^{m-\frac{\nu}{2}+1} N^{\frac{\nu}{2}-1}, (D - \frac{1}{2})^m \right)$$

$$+ \sum_{1 \leq i \leq N} \operatorname{ord}_c(g_i) \max \left( D^{m-\frac{\nu}{2}+1} N^{\frac{\nu}{2}-1}, (D - \frac{1}{2})^m \right)$$

$$\leq \left( \sum_{1 \leq i \leq N} \left( \frac{D}{N} \right)^\frac{\nu}{2} + \sum_{1 \leq i \leq N} \operatorname{ord}_c(g_i) \right)$$

$$\times \max \left( D^{m-\frac{\nu}{2}+1} N^{\frac{\nu}{2}-1}, (D - \frac{1}{2})^m \right).$$

Now, by our choice of $\kappa$ and Lemma 5.2, we get

$$\#((F^{m+1})^\dagger(y) \cap D(z, r))^* \leq \left( D - \frac{1}{2} \right) \max \left( D^{m-\frac{\nu}{2}+1} N^{\frac{\nu}{2}-1}, (D - \frac{1}{2})^m \right)$$

$$\leq \max \left( D^{m+1-\frac{\nu}{2}+1} N^{\frac{\nu}{2}-1}, (D - \frac{1}{2})^{m+1} \right).$$

Thus we have the desired claim for $n = m + 1$ in this case too.

From Cases 1 and 2, (6.1) is true for $n = m + 1$. By induction, (6.1) is true for all $n \in \mathbb{N}$. Since $y \in J(S)$ was arbitrary, the proof is complete.

**Remark 6.2.** We saw in the above proof that in case, for $g_1, g_2, \ldots, g_N$ as in Proposition 6.1, $C(g_1, g_2, \ldots, g_N) = \emptyset$, then the situation discussed in Case 2 does not even arise. Observe that, in this circumstance, the condition $\#(C^*(g_1, g_2, \ldots, g_N)) > 1$ is irrelevant. In short: if $C(g_1, g_2, \ldots, g_N) = \emptyset$ then the conclusion of Proposition 6.1 holds true with no conditions on $C^*(g_1, g_2, \ldots, g_N)$. This follows from the argument presented under Case 1. That, in essence, is the argument in [2].

We now present a result that, apart from being central to the proof of Theorem 1.6, might be of independent interest.

**Theorem 6.3.** Let $S$ be a finitely generated polynomial semigroup. Suppose $S$ satisfies the property that if $\#C^*[S] = 1$ then $C[S] \cap E(S) = \emptyset$. Then, for any finite set of generators $\mathcal{G}$ of $S$, the potential $U_{\mu^G}$ is finite and continuous on $\mathbb{C}$.

**Proof.** We fix a set of generators $\mathcal{G} = \{g_1, g_2, \ldots, g_N\}$ of $S$. Let $E(\mathcal{G})$ denote the Borel polar set associated with $\mathcal{G}$ described just prior to (5.4). It is a classical fact that for any polynomial $g$ with $\deg(g) \geq 2$,

$$\operatorname{cap}(J(g)) = |\Lambda(g)|^{\frac{1}{\deg(g)}} > 0.$$
Since for any $g \in S$ we have $J(g) \subset J(S)$, it follows that $\text{cap}(J(S)) > 0$. Therefore, $J(S) \setminus \text{E}(S) \neq \emptyset$. Hence, we can pick a $a \in J(S) \setminus \text{E}(S)$, which we shall fix for the remainder of this proof. Write $\mu_n := \mu_0^a := D^{-n}(F_0^a)^n(\delta_a)$ (recall the notation introduced in Section 5.2). Then, by (5.4),

$$\mu_n \xrightarrow{\text{weak}^*} \mu_0 \quad \text{as} \ n \to \infty.$$  

To begin with, we consider the case when $\sharp(C^*(\mathcal{G})) > 1$. We apply Proposition 6.1 to $g_1, g_2, \ldots, g_N$ that we have fixed above. Then, by this proposition, there exists $r_0 > 0$ such that if $r \in (0, r_0]$ then

$$\sharp((F^n)^{(a)} \cap D(z, r))^* \leq \max \left( D^{n-\frac{\nu}{\lambda}+1}N^{\frac{\nu}{\lambda}-1}, \left( D - \frac{1}{2} \right)^n \right)$$

(using the abbreviated notation in Proposition 6.1) for any $z \in \mathbb{C}$. Then, in view of the formula (5.3) and (6.4), we have for $n$ sufficiently large:

$$\mu_n(D(z, r)) = \frac{1}{D^n} \sharp((F^n)^{(a)} \cap D(z, r))^* \leq \left( \frac{D}{N} \right)^{1-\frac{\nu}{\lambda}},$$

where $\nu$ is the unique integer such that $r \in I(\nu)$—the latter as introduced in the statement of Proposition 6.1. Thus $r > r_0 R^{-\nu/\lambda}$. Recalling that $R := D/N$, the last two inequalities give

$$\mu_n(D(z, r)) \leq \left( \frac{R}{r_0^{\nu/2\kappa}} \right)^{r_0^{\nu/2\kappa}} = C_1 r^\alpha$$

for all $n$ sufficiently large, where $C_1 := R/(r_0^{\nu/2\kappa}) > 0$, $\alpha := \lambda/2\kappa > 0$, $r \in (0, r_0]$ and $z \in \mathbb{C}$.

From (6.3) and (6.5), we see that for every $r \in (0, r_0]$,

$$\mu_0(D(z, r)) \leq C_1 r^\alpha.$$  

Invoking Remark 4.3, $U^{\mu_0}$ is finite and continuous on $\mathbb{C}$.

Now consider the case when $\sharp(C^*(\mathcal{G})) = 1$. In this case, by Lemma 3.5, $C(\mathcal{G}) \cap \text{E}(S) = \emptyset$. If $C(\mathcal{G}) = \emptyset$ then, by Remark 6.2, the conclusion of Proposition 6.1 still holds true. Thus we get (6.5) (with $\kappa = 1$ this time). Consequently, arguing as before, $U^{\mu_0}$ is finite and continuous.

It remains to consider the case when $\sharp(C^*(\mathcal{G})) = 1$, $C(\mathcal{G}) \cap \text{E}(S) = \emptyset$ and $C(\mathcal{G}) \neq \emptyset$. Since $\sharp(C^*(\mathcal{G})) = 1$ and $C(\mathcal{G}) \neq \emptyset$, we get that

$$C(\mathcal{G}) = C^*(\mathcal{G}).$$

Consider the holomorphic correspondence associated with the list of polynomials $\mathcal{G}^2 := \{ g_i \circ g_j : 1 \leq i, j \leq N \}^*$, i.e.,

$$\Gamma_{\mathcal{G}^2} := \sum_{1 \leq i, j \leq N} \text{graph}(g_i \circ g_j).$$

Also, let $S' := \{ g_i \circ g_j : 1 \leq i, j \leq N \}$.  

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It is easy to see that for $g \circ g \in S'$ for any $g \in S$. From this and the classical fact that $J(g) = J(g \circ g)$, we deduce—in view of Result 2.1—that $J(S') = J(S)$. Thus the $a$ that we had fixed above belongs to $J(S')$. Write $\mu'_n := D^{-2n}(F_{g_2}^*)^n(\delta_a)$. It is easy to see that $\mu'_n = \mu_{2n}$ for all $n \in \mathbb{Z}_+$. Thus, by (6.3):

\[
\mu'_n \xrightarrow{\text{weak}^*} \mu_g \quad \text{as } n \to \infty.
\]

Now, let

\[
C^*(\mathcal{G}^2) := \{ c \in \mathbb{C} : (g_i \circ g_j)'(c) = 0 \text{ for some } i, j \in \{1, \ldots, N\}\}.
\]

Observe that for the list $\mathcal{G}^2$,

\[
C^*(\mathcal{G}^2) = C^*(\mathcal{G}) \cup \left[ \bigcup_{i=1}^N g_i^{-1}(C^*(\mathcal{G})) \right] = C(\mathcal{G}) \cup \left[ \bigcup_{i=1}^N g_i^{-1}(C(\mathcal{G})) \right],
\]

where the second equality is a consequence of (6.6). We now argue that $\sharp(C^*(\mathcal{G}^2)) > 1$. Since $C(\mathcal{G}) \cap E(S) = \emptyset$ and $\sharp(C(\mathcal{G})) = 1$, there exists $x$ such that $x \notin C(\mathcal{G})$ but $g_i(x) \in C(\mathcal{G})$ for some $i \in \{1, \ldots, N\}$, i.e.,

\[
x \in \bigcup_{i=1}^N g_i^{-1}(C(\mathcal{G})).
\]

Consequently, by (6.8), $\sharp(C^*(\mathcal{G}^2)) > 1$. Thus Proposition 6.1 can be applied to the correspondence $\Gamma_{\mathcal{G}^2}$ and, as the $a$ we had fixed lies in $J(S')$, its conclusion applies to this $a$. By a computation analogous to the one in the second paragraph of this proof—with $\mu'_n$ replacing $\mu_n$ and using (6.7)—we deduce that there exists $r_0 > 0$ such that if $r \in (0, r_0]$ then

\[
\mu_g(D(z, r)) \leq C_2 r^\beta
\]

for some positive constants $C_2$ and $\beta$. Once again, by Remark 4.3, $U^{\mu_g}$ is finite and continuous on $\mathbb{C}$ in this final case as well.

Recall that for a polynomial $g$, $\Lambda(g)$ denotes the coefficient of the highest degree term of the polynomial $g$.

**Lemma 6.4.** Let $S$ be a finitely generated polynomial semigroup and let $\mathcal{G} = \{g_1, g_2, \ldots, g_N\}$ be a set of generators of $S$. Then

\[
\lim_{n \to \infty} \frac{1}{D^n} \log \left( \prod_{l(\mathcal{G}) = n} |\Lambda(g)| \right)
\]

exists and equals $(D - N)^{-1} \log |\Lambda(g_1)\Lambda(g_2) \ldots \Lambda(g_N)|$. 

Recall that for a polynomial $g$, $\Lambda(g)$ denotes the coefficient of the highest degree term of the polynomial $g$. 

**Lemma 6.4.** Let $S$ be a finitely generated polynomial semigroup and let $\mathcal{G} = \{g_1, g_2, \ldots, g_N\}$ be a set of generators of $S$. Then
Proof. Fix \( n \in \mathbb{Z}_+ \). Recall that if \( g = g_{i_n} \circ \cdots \circ g_{i_1} \) then
\[
\Lambda(g) = \Lambda(g_{i_n}) \Lambda(g_{i_{n-1}})^{d_{i_n}} \cdots \Lambda(g_{i_1})^{d_{i_n} \cdots d_{i_1}},
\]
where \( d_{i_j} = \deg(g_{i_j}) \). Hence,
\[
(6.9) \prod_{l(g) = n} |\Lambda(g)| = \prod_{(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n} |\Lambda(g_{i_n}) \Lambda(g_{i_{n-1}})^{d_{i_n}} \cdots \Lambda(g_{i_1})^{d_{i_n} \cdots d_{i_1}}|.
\]
If we fix a word \( (g_{i_n} \circ \cdots \circ g_{i_{k+1}}) \), \( 1 \leq k \leq n \), and an \( i \in \{1, \ldots, N\} \), then the number of words of the form \( g_{i_n} \circ \cdots \circ g_{i_{k+1}} \circ g_i \circ f \) that contribute a factor of \( \Lambda(g_i)^{d_{i_k+1} \cdots d_{i_n}} \) to the right-hand side of (6.9) is \( N^{k-1} \); where
\[
g_{i_n} \circ \cdots \circ g_{i_{k+1}} := \text{id}_C \text{ if } k = n, \quad \text{and}\]
\[
f = \begin{cases} 
\text{a word with } l(f) = (k - 1), & \text{if } k \geq 2, \\
\text{id}_C, & \text{if } k = 1.
\end{cases}
\]
Here, we shall set \( d_{i_{k+1}} \cdots d_{i_n} := 1 \text{ if } k = n \). Thus the right-hand side of (6.9) can be reorganized as follows:
\[
\prod_{l(g) = n} |\Lambda(g)| = \prod_{i=1}^{N} \left( |\Lambda(g_i)|^{N^{i-1}} \times \prod_{k=1}^{n-1} \left( \prod_{(i_{k+1}, \ldots, i_n) \in \{1, \ldots, N\}^{n-k}} (|\Lambda(g_i)|^{d_{i_{k+1}} \cdots d_{i_n}})^{N^{k-1}} \right) \right)
= \prod_{i=1}^{N} \prod_{k=1}^{n} (|\Lambda(g_i)|^{D^{i-1+k}})^{N^{k-1}}
= \prod_{i=1}^{N} |\Lambda(g_i)|^{(D^{i-1} + ND^{i-2} + \cdots + N^{m-2} + N^{m-1})},
\]
since \( d_1 + d_2 + \cdots + d_N = D \). For simplicity of notation, let us write
\[
A := |\Lambda(g_1) \Lambda(g_2) \cdots \Lambda(g_N)|.
\]
Then, it follows from above that
\[
\frac{1}{D^n} \log \left( \prod_{l(g) = n} |\Lambda(g)| \right) = \frac{1}{D^n} \log(A^{(D^{i-1} + ND^{i-2} + \cdots + N^{m-2} + N^{m-1})})
= \frac{1 - (N/D)^n}{D - N} \log A.
\]
Since \( N < D \) by assumption, \( (N/D)^n \to 0 \text{ as } n \to \infty \), from which the result follows.

Before proving Theorem 1.6, we formally define a term that was used in Section 1.
**Definition 6.5.** Let $X$ be a topological space, and let $u : X \to [-\infty, \infty)$ be a function that is locally bounded above on $X$. Its **upper semicontinuous regularization** $u^* : X \to [-\infty, \infty)$ is defined by

$$u^*(x) := \lim_{y \to x} \sup_{y \in \mathcal{N}} u(y) = \inf_{y \in \mathcal{N}} (\sup u(y)) \quad \forall x \in X,$$

the infimum being taken over all neighbourhoods $\mathcal{N}$ of $x$.

It is easily checked that $u^*$ is an upper semicontinuous function on $X$ such that $u^* \geq u$, and also that it is the least upper semicontinuous function that dominates $u$.

**The proof of Theorem 1.6.** Note that if $g$ is a polynomial then

$$g(z) - a = \Lambda(g) \prod_{j=1}^{\deg(g)} (z - x_j),$$

where $x_1, \ldots, x_{\deg(g)}$ are the solutions of $g(z) = a$ repeated according to multiplicity. Now fix a set of generators $\mathcal{G} = \{g_1, g_2, \ldots, g_N\}$. Fix $a \in \mathbb{C} \setminus E(\mathcal{G})$, where $E(\mathcal{G})$ is as in the proof of Theorem 6.3, and let $\mu_n^a$ be as defined in the first paragraph of that proof. Set $\mu_n := \mu_n^a$. Then, by the definition of the logarithmic potential,

$$U_{\mu_n}^a(z) = \frac{1}{D^n} \sum_{l(g)=n} \left( \sum_{x \in g^{-1} \{a\}^*} \log \left( \frac{1}{|z - x|} \right) \right)$$

$$= \frac{1}{D^n} \log \left( \prod_{l(g)=n} \frac{|\Lambda(g)|}{|g(z) - a|} \right).$$

Therefore, by Lemma 6.4,

$$\liminf_{n \to \infty} U_{\mu_n}^a(z) = \liminf_{n \to \infty} \frac{1}{D^n} \log \left( \prod_{l(g)=n} \frac{|\Lambda(g)|}{|g(z) - a|} \right)$$

$$= \liminf_{n \to \infty} \frac{1}{D^n} \log \left( \prod_{l(g)=n} \frac{1}{|g(z) - a|} \right) + \lim_{n \to \infty} \frac{1}{D^n} \log \left( \prod_{l(g)=n} |\Lambda(g)| \right)$$

$$= -G_\mathcal{G}(z) + \log A \frac{D}{D - N},$$

where $A = |\Lambda(g_1)\Lambda(g_2) \ldots \Lambda(g_N)|$.

Since, by (5.4), $\mu_n \to \mu_\mathcal{G}$ in the weak* topology, Result 4.4 implies:

$$U_{\mu_\mathcal{G}}^a(z) \leq -G_\mathcal{G}(z) + \log A \frac{D}{D - N} \quad \text{for every } z \in \mathbb{C},$$

$$U_{\mu_\mathcal{G}}^a(z) = -G_\mathcal{G}(z) + \log A \frac{D}{D - N} \quad \text{for q.e. } z \in \mathbb{C}.$$
By Theorem 6.3, $U^\mu G$ is continuous. Thus $G^*_G$ is locally bounded above. Hence, $G^*_G$ (the upper semicontinuous regularization of $G^*_G$) exists. Since $G^*_G$ is the least upper semicontinuous function that dominates $G^*_G$ and $U^\mu G$ is continuous, we get

\begin{align}
U^\mu G(z) &\leq -G^*_G(z) + \frac{\log A}{D - N} \quad \text{for every } z \in \mathbb{C}, \quad (6.10) \\
U^\mu G(z) &= -G^*_G(z) + \frac{\log A}{D - N} \quad \text{for q.e. } z \in \mathbb{C}. \quad (6.11)
\end{align}

Now, if $U^\mu G$ is continuous at some $z_0 \in \mathbb{C}$ and

$$U^\mu G(z_0) < -G^*_G(z_0) + \frac{\log A}{D - N},$$

then by lower semicontinuity of $-G^*_G$, we can find an open disc $\Delta$ with centre $z_0$ such that for every $z \in \Delta$,

$$U^\mu G(z) < -G^*_G(z) + \frac{\log A}{D - N}.$$

Since $\text{cap}(\Delta) > 0$, the last inequality contradicts (6.11). Thus

$$U^\mu G(z_0) = -G^*_G(z_0) + \frac{\log A}{D - N}.$$ 

As $U^\mu G$ is continuous, the above argument implies that

$$U^\mu G(z) = -G^*_G(z) + \frac{\log A}{D - N} \quad \forall z \in \mathbb{C}. \quad (6.12)$$

Therefore, $G^*_G$ is continuous on $\mathbb{C}$. For the remainder of our argument we shall draw upon results in Section 4. We introduce some notation in order to identify objects featuring in our proof with those in Section 4. We define $Q_G := G^*_G|_{J(S)}$. Then

$$U^\mu G(z) + Q_G(z) = \frac{\log A}{D - N} \quad \forall z \in J(S). \quad (6.13)$$

In the notation of Section 4, consider the closed set $\Sigma := J(S)$. Now observe:

(i) $Q_G$ is continuous on $\Sigma$;
(ii) $Q_G(z)$ is finite for every $z \in \Sigma$.

As seen earlier, since $S$ contains a polynomial of degree at least 2,

$$\text{cap}(\Sigma) = \text{cap}(J(S)) > 0.$$

By observation (ii) above and since $\Sigma$ is of positive capacity, the set on which $Q_G < \infty$ is of positive capacity. Since $Q_G$ is continuous on $\Sigma$ and $J(S)$ is compact,
it follows that \( Q_\mathcal{G} \) is an external field (or equivalently, \( w_\mathcal{G}(z) := \exp(-Q_\mathcal{G}(z)) \) is an admissible weight).

Since \( \text{supp}(\mu_\mathcal{G}) \subset J(S) \), \( \mu_\mathcal{G} \) is compactly supported in \( \mathbb{C} \). It is easy to see that, since \( U^{\mu_\mathcal{G}} \) is continuous on \( \mathbb{C} \), \( \mu_\mathcal{G} \) has finite logarithmic energy. By (6.13) and Result 4.10, \( \mu_\mathcal{G} \) is the weighted equilibrium measure associated with the external field \( Q_\mathcal{G} \) and

\[
F_\mathcal{G} := F_{w_\mathcal{G}} = \frac{1}{D-N} \log |\Lambda(g_1)\Lambda(g_2) \ldots \Lambda(g_N)|
\]
is the modified Robin constant for \( Q_\mathcal{G} \).

\[
\text{Remark 6.6.} \text{ We point out that the last proof reveals that the function } G_\mathcal{G}^* \text{ does not depend on the choice of } a \in \mathbb{C}, \text{ provided } a \notin E_\mathcal{G}. \text{ With this exception, we know from (5.4) that } \mu_\mathcal{G}, \text{ and hence } U^{\mu_\mathcal{G}}, \text{ is independent of } a. \text{ Thus, in view of (6.12), the stated independence of the choice of } a \text{ follows. We shall exploit this fact in Section 7, where we shall work with } G_\mathcal{G}^* \text{ and } Q_\mathcal{G}. \]

7 The proof of Theorem 1.8

Before giving a proof of Theorem 1.8, we state couple of results that we will need.

\[
\text{Result 7.1 (Pommerenke, [12]). Let } \Sigma \text{ be a non-empty closed subset of } \mathbb{C}. \text{ Then, } \Sigma \text{ is uniformly perfect if and only if there is a constant } \delta > 0 \text{ such that } \text{cap}(\Sigma \cap \overline{D(z, r)}) \geq \delta r \text{ for all } z \in \Sigma \text{ and } 0 < r < \text{diam}(\Sigma) \text{ (where } \text{diam}(\Sigma) \text{ denotes the diameter of } \Sigma).}
\]

\[
\text{Remark 7.2. The above result was stated in [12] for unbounded closed sets in } \mathbb{C}. \text{ However, the only place where the unboundedness of } \Sigma \text{ is needed in its proof is in showing } \Sigma \text{ to have the property stated in Remark 2.4 (which is almost immediate), which in [12] is stated with } \Sigma \text{ being unbounded.}
\]

The measure \( \mu_\mathcal{G} \) in the next result is as in the previous sections.

\[
\text{Result 7.3 (Boyd, [3]). Let } S \text{ be a finitely generated rational semigroup where each element of } S \text{ has degree at least 2. Let } \mathcal{G} \text{ be a finite set of generators. Then } \text{supp}(\mu_\mathcal{G}) = J(S).
\]

The exterior boundary of \( J(S) \) will be denoted by \( \partial_e J(S) \). Also, we recall that owing to Theorem 6.3, (6.12) tells us that \( G_\mathcal{G}^* \), and hence \( Q_\mathcal{G} \), are continuous. We shall use this without any further comment in

\[
\text{The proof of Theorem 1.8. We begin with the proof of part (a). Fix a finite set of generators } \mathcal{G}. \text{ By hypothesis, the orbit of } z_0, O(z_0), \text{ is not dense}
\]
in $\mathbb{C}$. Thus there exist $p \in \mathbb{C}$ and $\varepsilon > 0$ such that $O(z_0) \cap D(p, 2\varepsilon) = \emptyset$. As $\text{cap}(D(p, \varepsilon)) > 0$, there is a point $a \in D(p, \varepsilon)$ such that $a \notin E(\mathcal{G})$. Recall that $G^*_\mathcal{G}$ is the upper semicontinuous regularization of the function

$$G^*_\mathcal{G}(z) = \lim_{n \to \infty} \frac{1}{D^n} \log \left( \prod_{l(g) = n} |g(z) - a| \right)$$

and the external field $Q^*_\mathcal{G} = G^*_\mathcal{G}|_{\mathcal{J}(S)}$. Note that, by Remark 6.6, $Q^*_\mathcal{G}$ does not depend on the choice of $a$, where $a \in \mathbb{C} \setminus E(\mathcal{G})$. Let $\rho_1 > 0$ be such that

$$|g_i(z)| > |z|, \quad |g_i(z) - a| > \frac{|\Lambda(g_i)|}{2}|z|^{\deg(g_i)} \quad \forall z : |z| > \rho_1,$$

for $i = 1, \ldots, N$ and $a = 0, a$. Then, owing to Lemma 6.4 and the above choice of $\rho_1$, we get

$$G^*_\mathcal{G}(z) \geq \log |z| + \frac{1}{D - N}(\log |\Lambda(g_1)\Lambda(g_2)\ldots\Lambda(g_N)| - N \log 2), \quad \forall z : |z| > \rho_1.$$ 

Hence, there exists a $\rho_2 \geq \rho_1$ such that $G^*_\mathcal{G}(z) > 0$ whenever $|z| > \rho_2$. Since $O(z_0)$ is unbounded, there exists a word $h$ such that $|h(z_0)| > \rho_2$, whence $G^*_\mathcal{G}(h(z_0)) > 0$. Let $l(h) = M$. Observe that by the choice of $a$, for any $g \in S$, $|g(z_0) - a| > \varepsilon$. Thus, for $n \geq M + 1$,

$$\log \left( \prod_{l(g) = n} |g(z_0) - a| \right) > \sum_{l(g) = n - M} \log |g(h(z_0)) - a| + (N^n - N^{n-M}) \log \varepsilon.$$ 

Divide both sides above by $D^n$. Then, it follows from the definition of $G^*_\mathcal{G}$, and as $N/D < 1$, that $G^*_\mathcal{G}(z_0) \geq D^{-M}G^*_\mathcal{G}(h(z_0))$. Recall that $G^*_\mathcal{G}(h(z_0)) > 0$. It follows that $Q^*_\mathcal{G}(z_0) = G^*_\mathcal{G}(z_0) \geq G^*_\mathcal{G}(z_0) > 0$. Since the choice of $\mathcal{G}$ was arbitrary, this establishes (a).

We shall prove part (b) by establishing its contrapositive. Assume that for every $z \in \mathcal{J}(S)$, $O(z)$ is bounded. Now fix a finite set of generators $\mathcal{G}$. By Lemma 3.2, there exists $R > 0$ (which depends on $\mathcal{G}$) such that $O(z) \subset D(0, R)$ for every $z \in \mathcal{J}(S)$. Choose $a \in \mathbb{C}$ such that $|a| > 2R$ and $a \notin E(\mathcal{G})$. Observe:

$$N^n \log R \leq \log \left( \prod_{l(g) = n} |g(z) - a| \right) \leq N^n \log(R + |a|) \quad \forall z \in \mathcal{J}(S).$$ 

Since $N/D < 1$, it follows that $G^*_\mathcal{G}(z) = 0$ for every $z \in \mathcal{J}(S)$. By (6.11), $G^*_\mathcal{G} = G^*_\mathcal{G}$ quasi-everywhere on $\mathbb{C}$. In particular, $Q^*_\mathcal{G} = 0$ quasi-everywhere on $\mathcal{J}(S)$. Suppose there is some $\zeta \in \mathcal{J}(S)$ such that $Q^*_\mathcal{G}(\zeta) \neq 0$. Then, as $Q^*_\mathcal{G}$ is continuous, there exists a disc $\Delta$ with centre $\zeta$ such that $Q^*_\mathcal{G} \neq 0$ on $\mathcal{J}(S) \cap \Delta$. In view of Results 2.5 and 7.1, $\mathcal{J}(S) \cap \Delta$ must have positive capacity: a contradiction. Thus $Q^*_\mathcal{G} \equiv 0$. Since this is true for any choice of $\mathcal{G}$, (b) follows.
It now remains to prove the capacity estimate in (1.4). To do so, we use the $F$-functional of Mhaskar and Saff with external field $Q_g$ given by Theorem 1.6. We introduce some notation in order to identify objects pertinent to our analysis with those in Section 4. Abbreviate $\delta_g := \delta_{w_g}$; the latter is as introduced just prior to Result 4.10. By Theorem 1.6, $\delta_g = \text{supp}(\mu_g)$. With this notation, Result 4.11 gives

$$\log \text{cap}(\delta_g) = -F_g + \int_{J(S)} Q_g d\sigma_{\delta_g}.$$ 

By assumption, each element of $S$ is of degree at least 2. Thus, by Result 7.3 and since $\delta_g = \text{supp}(\mu_g)$, we have $\delta_g = J(S)$. Also note that, by (6.14), the modified Robin constant is $F_g = (D - N)^{-1} \log |\Lambda(g_1)\Lambda(g_2)\ldots\Lambda(g_N)|$. Consequently, we get

$$(7.1) \quad \log \text{cap}(J(S)) = \frac{1}{N - D} \log |\Lambda(g_1)\Lambda(g_2)\ldots\Lambda(g_N)| + \int_{J(S)} Q_g d\sigma_{J(S)}.$$ 

Let $V$ denote the unbounded component of $F(S)$, whence $\partial_e J(S) := \partial V$. With $z_0$ as in our hypothesis, the proof of part (a) tells us that $G_g^*(z_0) > 0$. We claim that there exists $z_1 \in \partial_e J(S)$ such that $Q_g(z_1) > 0$. If $z_0 \notin \partial_e J(S)$, take $z_1 := z_0$ and we are done. If $z_0 \notin \partial_e J(S)$, then $\Omega := \mathbb{C} \setminus (\partial_e J(S) \cup V)$ is a non-empty open set. Recall that, by (6.12), $G_g^*$ is a continuous subharmonic function on $\mathbb{C}$. Since $z_0 \in \Omega$ and $G_g^*(z_0) > 0$, by the maximum principle applied to $G_g^*|_\Omega$, the desired claim follows.

Let $x \in V \setminus \{\infty\}$. By Result 3.3, $x$ also belongs to the unbounded component of $A(S)$. Thus, as argued just prior to Lemma 3.2, given $r > 0$, there exists $n_r(x) \in \mathbb{Z}_+$ such that if $l(g) \geq n_r(x)$ then $|g(x)| > r$. It follows by taking $r > 0$ large enough that $G_g(x) \geq 0$. Since $G_g^* \geq G_g$, and $x \in V$ was arbitrary, $G_g^* \geq 0$ on $V$. So, as each point in $\partial_e J(S)$ is a limit point of $V$, it follows from the continuity of $G_g^*$ that $G_g^* \geq 0$ on $\partial_e J(S)$. Thus $Q_g \geq 0$ on $\partial_e J(S)$.

We now appeal to potential theory to prove $\text{supp}(\sigma_{J(S)}) = \partial_e J(S)$. It is well known that $\text{supp}(\sigma_{J(S)}) \subseteq \partial_e J(S)$. Let $\zeta \in \partial_e J(S)$ and assume that $\zeta \notin \text{supp}(\sigma_{J(S)})$. Then, there exists an open disc $\Delta$ with centre $\zeta$ on which $U^{\sigma_{J(S)}}$ is harmonic. By Result 2.5, $J(S)$ is uniformly perfect. Thus, by Result 7.1, it follows that $V$ is regular—see, for instance [18, Corollary 2 to Theorem III-62]. As $\zeta$ is a regular boundary point of $V$, and $\infty \in V$, it is a classical fact that

$$U^{\sigma_{J(S)}}(\zeta) = -\log \text{cap}(J(S)).$$

But by Frostman’s theorem, $U^{\sigma_{J(S)}} \leq -\log \text{cap}(J(S))$ on $\mathbb{C}$. Thus, applying the maximum principle to $U^{\sigma_{J(S)}}|_{\Delta}$, we have $U^{\sigma_{J(S)}}|_{\Delta} \equiv -\log \text{cap}(J(S))$. As $U^{\sigma_{J(S)}}$ is harmonic on $V \setminus \{\infty\}$, the identity principle for harmonic functions implies
that \( U^{\sigma_{J(S)}} \equiv -\log \text{cap}(J(S)) \) on \( V \setminus \{\infty\} \). This contradicts the fact that
\[
U^{\sigma_{J(S)}}(z) = -\log |z| + o(1)
\]
as \( z \to \infty \). Hence, \( \text{supp}(\sigma_{J(S)}) = \partial_e J(S) \). Thus, by the continuity of \( Q_{\mathcal{S}} \), there is a \( J(S) \)-open neighbourhood \( \mathcal{N} \) of \( z_1 \) with \( \sigma_{J(S)}(\mathcal{N}) > 0 \) such that \( Q_{\mathcal{S}} > 0 \) on \( \mathcal{N} \). This, together with the conclusions of the last two paragraphs, gives
\[
\int_{J(S)} Q_{\mathcal{S}} d\sigma_{J(S)} > 0.
\]
By (7.1), we get the desired inequality. \( \square \)

**Corollary 7.4.** Let \( S \) be a finitely generated polynomial semigroup as in Theorem 1.8 and let \( \mathcal{S} = \{g_1, g_2, \ldots, g_N\} \) be a set of generators of \( S \). If each element of \( S \) is of degree at least 2 then
\[
diam(J(S)) > 2|\Lambda(g_1)\Lambda(g_2)\cdots\Lambda(g_N)|^{\frac{1}{n-1}},
\]
where \( \text{diam}(J(S)) \) denotes the diameter of \( J(S) \) with respect to the Euclidean metric on \( \mathbb{C} \).

**Proof.** The estimate is a consequence of the following relation between logarithmic capacity and diameter: if \( K \) is a compact subset of \( \mathbb{C} \) then
\[
\text{cap}(K) \leq \frac{\text{diam}(K)}{2};
\]
see, for instance, [14, Theorem 5.3.4]. The result now follows from (1.4).

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