CONSTRUCTION OF SOLUTIONS FOR SOME LOCALIZED NONLINEAR SCHröDINGER EQUATIONS

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(Communicated by Juan Diego Davila)

Abstract. For an $N$-body system of linear Schrödinger equation with space dependent interaction between particles, one would expect that the corresponding one body equation, arising as a mean field approximation, would have a space dependent nonlinearity. With such motivation we consider the following model of a nonlinear reduced Schrödinger equation with space dependent nonlinearity

$$-\varphi'' + V(x)h'(|\varphi|^2)\varphi = \lambda \varphi,$$

where $V(x) = -\chi_{(-1,1)}(x)$ is minus the characteristic function of the interval $[-1,1]$ and where $h'$ is any continuous strictly increasing function. In this article, for any negative value of $\lambda$ we present the construction and analysis of the infinitely many solutions of this equation, which are localized in space and hence correspond to bound-states of the associated time-dependent version of the equation.

1. Introduction. Autonomous Nonlinear Schrödinger (NLS) equations appear in many fields of theoretical physics, in particular as effective models in the study of many body quantum systems [24], with the standard form:

$$i\partial_t u = -\Delta u + f(x, u, Du), \quad x \in \mathbb{R}^d,$$

where $f$ is a suitable real-valued function and $Du$ denotes the spatial derivatives of the complex-valued solution $u$. The Gross-Pitaevskii equation is the archetype of such evolution equations with $f$ given by $f(x, u, Du) = V(x)u - |u|^2u$, where the potential $V$ is a real-valued function. The specification of the function $f$ leads to various models of NLS equations, some of which have been extensively studied in the literature like the semilinear case $f(x, u, Du) = V(x)u - g(|u|)u$, where $V$ (the linear potential) and $g$ are real-valued. For an overview of the subject we refer to e.g. [1, 4, 5, 10, 33, 39, 40].

In the case of semilinear NLS equations, as in the linear case, the bound states (or standing waves) are defined as solutions of the form $e^{-i\lambda t}\varphi(x)$, where $\lambda$ is a real
number and $\varphi$ is a non trivial solution of the corresponding stationary non-linear equation,

$$-\Delta \varphi + (V(x) - \lambda)\varphi = g(|\varphi|)\varphi,$$

which belongs to $H^1(\mathbb{R}^d)$ and vanishes at infinity. The existence and properties of bound states in the semilinear case have been widely studied with a quickly growing literature. These studies have mainly tackled the existence problem, in perturbative regimes (see [43] and references therein), in semiclassical regimes, see e.g. [2, 7, 13, 20, 30, 44] and for asymptotically linear perturbations, see e.g. [15, 27, 28]. Some of these approaches also include nonlinear terms with some spatial modulations and some additional regular linear potential. In absence of potentials, see also [12]. These results have also been extended to include singular potentials [21, 17], asymptotically periodic settings [16, 33, 38] and magnetic fields [3, 14, 19]. In general, the construction of such bound states is not very explicit and there are few results on the control of the multiplicity of the associated “eigenvalue”. For the dynamical consequences related to the existence of bound states, see e.g. [37, 41, 42, 43]. An important effort has also been placed in the specific study of non-negative solutions (ground states), for results on existence, non-existence and regularity of ground states see e.g. [22, 34] and references therein.

In this paper we consider the existence problem of bound states for semilinear NLS equations with a compactly supported non-linearity. Such a model may arise as a mean field approximation of a many body quantum system, with space dependent interaction between particles. To our knowledge, this situation has not been studied systematically and provides interesting and useful insights for more general cases, see e.g. [8, 31]. By studying a one dimensional basic model we provide a construction of infinitely many bound states for any negative energy of the system (see Theorems 3.3 and 3.4). These bound states will be characterized by their oscillation properties. The situation contrasts abruptly with the linear case, where the spectrum is known to be bounded from below and it is discrete in any compact interval of negative energies (Weyl Theorem). Our approach relies on very precise manipulations of ODE techniques and adequately constructed barrier functions.

The paper is structured as follows. The model and basic setting is presented in section 2, the main results are in section 3. In section 4, we study some consequences of our results, considering the limiting case of a singular non-linearity (like in [8, 31]) and using the solution of our stationary equation to describe soliton-type solutions for some non-autonomous NLS equations. The proofs of the main results are presented in section 5, which is divided in three subsections. Subsection 5.1 develops the technical results in ODEs that are required as the main tools, that are used in subsection 5.2, to present the proofs of the main theorems in subsection 3.1. While subsection 5.3 contains some technical observations.

2. Setting of the problem and notation. The following notation and hypotheses will be used throughout sections 3 and 4, and will be omitted in the statements of the results. Let

$$V(x) = \begin{cases} -1 & |x| \leq 1, \\ 0 & |x| > 1. \end{cases}$$

(1)

Let $h' : \mathbb{R}_+ \to \mathbb{R}$ be a continuous strictly increasing function satisfying $h'(0) \geq 0$ and $\lim_{x \to +\infty} h'(x) = \infty.$
We consider the equation
\[
\begin{aligned}
-\varphi'' + V(x)h'(|\varphi|^2)\varphi &= \lambda \varphi, \quad x \in \mathbb{R}, \\
\varphi(x) &\to 0 \text{ when } x \to \pm \infty.
\end{aligned}
\]
(2)

By solutions of equation (2) we mean $H^2(\mathbb{R})$ solutions, which in this setting ends up being equivalent to $C^1(\mathbb{R})$ solutions.

As shown in subsection 5.3 we only need to consider $\lambda \in \mathbb{R}$ and $\varphi$ real valued. Furthermore, theorem 3.1 will imply $\lambda < 0$.

By letting $h(x) = \int_0^x h'(s)ds$, $h$ satisfies $h(0) = 0$ and the hypotheses on $h'$ are equivalent to assume $h \in C^1(\mathbb{R}_+)$ is strictly increasing and strictly convex with unbounded derivative.

3. Main results.

3.1. Main theorems. In this subsection we present the main theorems describing the solutions of equation (2). The proofs of these results are presented later in subsection 5.2.

The potential $V(x)$ vanishes outside the interval $[-1, 1]$ and therefore the non-linear part of equation (2) is restricted to the interval $[-1, 1]$. The first step in the analysis is the study of equation (2) in each component of $\mathbb{R} \setminus \{-1, 1\}$, obtaining the following characterization.

**Theorem 3.1.** If $\varphi$ is a non-trivial solution of equation (2) then $\lambda < 0$ and there exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that
\[
\varphi(x) = \begin{cases} 
\alpha e^{\sqrt{-\lambda}(x+1)} & x \leq -1, \\
\pm \alpha e^{\sqrt{-\lambda}(1-x)} & x \geq 1,
\end{cases}
\]
(3)
\[
\frac{|\varphi'|}{\sqrt{-\lambda \varphi^2 - h(\varphi^2) + h(\alpha^2)}}, \text{ for } |x| < 1, \\
\varphi(-1) = \alpha = \pm \varphi(1),
\]
(4)

and $\varphi' = 0$ only at isolated points.

Theorem 3.1 directs our attention to solutions of equation (4) that only vanish at isolated points. The analysis of equation (4), of the form $|u'| = \sqrt{g(u)}$, requires some technical considerations due to the right hand side not necessarily Lipschitz at $g(u) = 0$. All the technical aspects of this equation are addressed in subsection 5.1 and a summary of such results is the following.

**Theorem 3.2.** Let $\lambda < 0$. There exists $m_0 \in \mathbb{N} \cup \{0\}$ such that $\forall m \geq m_0, \exists \alpha = \alpha(m) > 0$ for which there is a $C^1([-1, 1])$ solution of
\[
\begin{cases} 
|\varphi'| = \sqrt{-\lambda \varphi^2 - h(\varphi^2) + h(\alpha^2)}, \text{ for } |x| < 1, \\
\varphi(-1) = \alpha = (-1)^m \varphi(1), \\
\varphi'(-1) > 0 \text{ and } (-1)^{m+1} \varphi'(1) > 0, \\
\varphi \text{ has exactly } m \text{ zeroes}, \\
\varphi' = 0 \text{ only at isolated points}.
\end{cases}
\]
(5)
Such solution is unique and \( \varphi \in C^2([-1, 1]) \). Additionally, if \( h'(0) < -\lambda \) then \( m_0 = 0 \).

Theorem 3.1 divides the original problem into two parts, one of decaying exponentials outside \([-1, 1]\) and a difficult non-linear problem inside \([-1, 1]\). Theorem 3.2 gives a definite answer for the non-linear problem inside \([-1, 1]\). And by pasting the solutions obtained in theorem 3.2 inside \([-1, 1]\) to the decaying exponential outside we obtain all the possible solutions to the original equation (2). Hence, Theorem 3.3.

For any \( \lambda < 0 \), \( \exists m_0 \in \mathbb{N} \cup \{0\} \) such that \( \forall m \geq m_0 \) there is exactly one \( H^2(\mathbb{R}) \) non-trivial solution, up to a change of sign, of equation (2) with exactly \( m \) zeroes. If \( h'(0) < -\lambda \) then \( m_0 = 0 \).

3.2. Non-linear Schrödinger equation in the torus. The techniques developed to analyze equation (2) can also be adapted to study non-linear Schrödinger equations on the one dimensional torus (related aspects can be found in [32]). Namely, consider the equation

\[
\begin{cases}
-\varphi'' - h'(\|\varphi\|^2)\varphi = \lambda \varphi, x \in \mathbb{R}, \\
\varphi(x) = \varphi(x+1), x \in \mathbb{R}.
\end{cases}
\]

(6)

In this setting, we have that:

Theorem 3.4. For any \( \lambda < 0 \), \( \exists m_0 \in \mathbb{N} \cup \{0\} \) such that \( \forall m \geq m_0 \) there exist \( H^2(\mathbb{R}) \) non-trivial solution of equation (6) with exactly \( 2m \) zeroes. If \( h'(0) < -\lambda \) then \( m_0 = 0 \).

The proof of this result is provided in subsection 5.2.

3.3. Examples. Theorem 3.3 applies to a number of non-linearities of interest, in here we provide a few examples.

a) If \( h'(x) = x \) then equation (2) becomes

\[
\begin{cases}
-\varphi'' + V(x)|\varphi|^2\varphi = \lambda \varphi, x \in \mathbb{R}, \\
\varphi(x) \to 0 \text{ when } x \to \pm \infty.
\end{cases}
\]

As mentioned before, cubic NLS equations are effective models in the analysis of many body quantum systems involving two-body interactions in the limit of large particles numbers, see e.g. [25, 26] and references therein.

b) If \( h'(x) = x^p \) then equation (2) becomes

\[
\begin{cases}
-\varphi'' + V(x)|\varphi|^{2p}\varphi = \lambda \varphi, x \in \mathbb{R}, \\
\varphi(x) \to 0 \text{ when } x \to \pm \infty.
\end{cases}
\]

Higher order NLS equations also appear as effective models in the study of many body quantum systems with more complex interactions, see e.g. [11].

c) If \( h'(x) = e^x \) then equation (2) becomes

\[
\begin{cases}
-\varphi'' + V(x)\exp(|\varphi|^2)\varphi = \lambda \varphi, x \in \mathbb{R}, \\
\varphi(x) \to 0 \text{ when } x \to \pm \infty.
\end{cases}
\]

Observe that in this case \( h'(0) = 1 \) hence in theorem 3.3, \( m_0 = 0 \) only for \( \lambda < -1 \). This kind of nonlinearity appear in plasma physics [23]. We refer to [18] and discussion therein for related problems on bounded domains of \( \mathbb{R}^d \).
4. Related consequences.

4.1. Convergence of the potential to a delta function. The results in sections 3 and 5 provide a very complete characterization of the solutions of equation (2). In this subsection we use those results to study in detail what happens to the positive solution of equation (2) when the potential \( V(x) \) approaches a delta function. Namely, for \( \mu > 0 \) let

\[
W_\mu(x) = \begin{cases} 
-\mu & |x| < 1/2\mu, \\
0 & |x| > 1/2\mu,
\end{cases}
\]

which satisfies that \( \lim_{\mu \to \infty} W_\mu(x) = -\delta(x) \) in the sense of distributions. Let \( \lambda < 0 \) and consider the equation

\[
\begin{aligned}
-v'' + W_\mu h'(v^2)v &= \lambda v, & x &\in \mathbb{R},
\end{aligned}
\]

\[
\begin{aligned}
v(x) &\to 0 \text{ as } x \to \pm \infty.
\end{aligned}
\]

(7)

As \( \mu \to \infty \) equation (7) resembles the equation

\[
\begin{aligned}
-v'' - \delta(x) h'(v^2)v &= \lambda v, & x &\in \mathbb{R},
\end{aligned}
\]

\[
\begin{aligned}
v(x) &\to 0 \text{ as } x \to \pm \infty,
\end{aligned}
\]

(8)

and we will establish (assuming \( h'(0) = 0 \)) that the positive solution of equation (7) converges to the positive solution of equation (8). Let us start by describing the solution of equation (8).

Lemma 4.1. Let \( \lambda < 0 \) and assume that \( h'(0) < \min\{-\lambda, 2\sqrt{-\lambda}\} \). Let \( s > 0 \) be the unique number such that \( h'(s^2) = 2\sqrt{-\lambda} \). Then the function \( v_\infty(x) = se^{-|x|\sqrt{-\lambda}} \) is the only \( H^1(\mathbb{R}) \) (weak) positive solution of the equation (8).

Proof. It is immediate that equation (8), in \((−\infty, 0)\) and in \((0, \infty)\), implies that any positive solution \( v(x) \) has the form

\[
v(x) = \begin{cases} 
\alpha e^{-x\sqrt{-\lambda}} & x < 0 \\
\beta e^{-x\sqrt{-\lambda}} & x > 0
\end{cases}
\]

for some \( \alpha, \beta > 0 \). Continuity of \( v(x) \) at \( x = 0 \) implies \( \alpha = \beta \), hence \( v(x) = \alpha e^{-x\sqrt{-\lambda}} \) for some \( \alpha > 0 \). Taking derivatives in a weak sense we obtain \( v'' = -\delta(x)(2\sqrt{-\lambda})v - \lambda v \). Since \( \delta(x) \) is supported in \( \{0\} \) we conclude that \( v \) is solution of equation (8) if and only if \( (2\sqrt{-\lambda} = h'(v^2(0)) = h'(s^2)) \). Hence \( \alpha = s \) and the lemma is proved. \( \square \)

And now let us establish the convergence of the solutions.

Theorem 4.2. Let \( \lambda < 0 \) and assume that \( h'(0) = 0 \). For \( \mu > 0 \) let \( v_\mu \) be the unique positive solution of equation (7) and let \( v_\infty \) be the unique positive solution of equation (8) (in lemma 4.1). Then \( v_\mu(x) \to v_\infty(x) \) in \( H^1(\mathbb{R}) \) as \( \mu \to \infty \).

Proof. Let \( \mu > 0 \). Since \( h'(0) = 0 \) then \( \mu h'(0) < -\lambda \), and it is not difficult to see that the results in subsection 3.1 and the proofs in subsection 5.2 can be slightly modified to show that there exists a unique \( v = v_\mu \in H^2(\mathbb{R}) \) positive solution (without zeroes) of equation (7). This solution is characterized as follows: there exists a unique \( \alpha = \alpha(\mu) > 0 \) such that,

\[
v(x) = \begin{cases} 
\alpha e^{(x+\frac{\lambda}{\mu})\sqrt{-\lambda}} & x < -1/2\mu, \\
\alpha e^{(x-\frac{\lambda}{\mu})\sqrt{-\lambda}} & x > 1/2\mu.
\end{cases}
\]
and

\[
\begin{align*}
v' &= \sqrt{-\lambda v^2 - \mu h(v^2) + \mu h(\alpha^2)}, \quad x \in [-1/2\mu, 0], \\
v' &= -\sqrt{-\lambda v^2 - \mu h(v^2) + \mu h(\alpha^2)}, \quad x \in [0, 1/2\mu], \\
v'(0) &= 0, \\
v\left(\frac{1}{2\mu}\right) &= \alpha.
\end{align*}
\] (9)

Define \( c = c(\mu) = v_\mu(0) \), which is the only number \( c > 0 \) such that \( \mu h(c^2) + \lambda c^2 = \mu h(\alpha^2) \), and observe that \( \alpha < c \).

Given the description of \( v_\mu \) in \((-\infty, -\frac{1}{2\mu}) \cup (\frac{1}{2\mu}, \infty)\), and given the bounds \(|v_\mu'(x)| \leq c\) and \(|v_\mu''(x)| \leq c\sqrt{-\lambda} \) for \( x \in \mathbb{R} \), to prove that \( \lim_{\mu \to \infty} v_\mu = v_\infty \) in \( H^1(\mathbb{R}) \), it is enough to show that \( \lim_{\mu \to \infty} \alpha(\mu) = \lim_{\mu \to \infty} c(\mu) = s \), where \( s \), prescribed in lemma 4.1, is the only number \( s \geq 0 \) satisfying \( h'(s^2) = 2\sqrt{-\lambda} \). In other words, it is enough to prove that \( \lim_{\mu \to \infty} h'(\alpha^2) = \lim_{\mu \to \infty} h'(c^2) = 2\sqrt{-\lambda} \).

In order to do this, we will use the bounds in lemma 5.2 applied to equation (9). Namely, let \( g(y) = -\lambda y^2 - \mu h(y^2) + \mu h(\alpha^2) \), then \( g : [\alpha, c] \to \mathbb{R}_+ \) and \( v_\mu \) satisfy the equation

\[
\begin{align*}
v' &= \sqrt{g(v)}, \quad x \in [-\frac{1}{2\mu}, 0], \\
v\left(\frac{1}{2\mu}\right) &= \alpha \quad \text{and} \quad v'(0) = 0, \\
v'(x) &= 0, \quad \forall x \in [-\frac{1}{2\mu}, 0].
\end{align*}
\]

If \( \kappa_1, \kappa_2 > 0 \) are such that \( \kappa_1(c - y) \leq g(y) \leq \kappa_2(c - y) \) for \( y \in [\alpha,c] \), then lemma 5.2 implies

\[
2\sqrt{\frac{c - \alpha}{\kappa_2}} \leq \frac{1}{2\mu} \leq 2\sqrt{\frac{c - \alpha}{\kappa_1}},
\]

but since \( h \) is convex and \( \mu h(c^2) + \lambda c^2 = \mu h(\alpha^2) \) then the inequalities above become

\[
\begin{align*}
\kappa_1 \leq 16\mu^2(c - \alpha) &\leq \kappa_2, \\
\Rightarrow \kappa_1 \frac{h(c^2) - h(\alpha^2)}{c - \alpha} &\leq -\lambda c^2 16\mu \leq \kappa_2 \frac{h(c^2) - h(\alpha^2)}{c - \alpha}, \\
\Rightarrow \kappa_1 \frac{c + \alpha}{c^2} h'(\alpha^2) &\leq -\lambda 16\mu \leq 2\kappa_2 \frac{h'(c^2)}{c}.
\end{align*}
\] (10)

(this last equation is also true for \( \kappa_1 \leq 0 \). Now, observe that

\[
g(y) = \mu h(c^2) + \lambda c^2 - \mu h(y^2) - \lambda y^2 = \left[ \mu \frac{h(c^2) - h(\alpha^2)}{c^2 - y^2} + \lambda \right] (c + y)(c - y),
\]

and since \( h \) convex, and \( y \in [\alpha,c] \), we can choose \( \kappa_1 = (c + \alpha)[\mu h'(\alpha^2) + \lambda] \) and \( \kappa_2 = 2c[\mu h'(c^2) + \lambda] \), and then equation (10) becomes

\[
\left( \frac{c + \alpha}{c} \right)^2 \left[ h'(\alpha^2) + \frac{\lambda}{\mu} \right] h'(c^2) \leq -\lambda 16 \leq 4 \left[ h'(c^2) + \frac{\lambda}{\mu} \right] h'(c^2).
\] (11)

(remembering that \( \alpha = \alpha(\mu), c = c(\mu) \) while \( \lambda \) is fixed and independent of \( \mu \)). Since \( \lim_{x \to +\infty} h'(x) = +\infty \) and \( h'(0) = 0 \), equation (11) immediately implies that \( \alpha(\mu) \) remains bounded above, while \( c(\mu) \) remains bounded below away from
zero, as \( \mu \to \infty \). Therefore \( h(c^2)/c^2 \) remains bounded below away from zero, but \( h(c^2)/c^2 \leq (h(c^2) - h(\alpha^2))/(c^2 - \alpha^2) \) and
\[
\left(1 - \frac{\alpha^2}{c^2}\right) \frac{h(c^2) - h(\alpha^2)}{c^2 - \alpha^2} = \frac{h(c^2) - h(\alpha^2)}{c^2} = \frac{-\lambda}{\mu} \to 0 \quad \text{as} \quad \mu \to \infty,
\]
hence \( \lim_{\mu \to \infty} \alpha/c = 1 \). Then both \( \alpha(\mu) \) and \( c(\mu) \) remain bounded above and away from zero and they both have the same cluster points. Using this in equation (11) we conclude that
\[
4 \left( \lim_{\mu \to \infty} h'(\alpha^2) \right)^2 = -\lambda 16 = 4 \left( \lim_{\mu \to \infty} h'(c^2) \right)^2,
\]
this gives \( \lim_{\mu \to \infty} h'(\alpha^2) = \lim_{\mu \to \infty} h'(c^2) = 2\sqrt{-\lambda} \),
completing the proof. \( \square \)

4.2. Description of soliton solutions for the dynamic equation. Let us consider the dynamic nonlinear Schrödinger equation
\[
i \frac{\partial \psi}{\partial t} = -\Delta \psi + \nabla(x, t)|\psi|^2 \psi. \tag{12}
\]
As mentioned in the introduction, such models may arise as a mean field approximation of a many body quantum system. More specifically, in Bose-Einstein condensates, the system is studied using a nonlinear equation with interaction between particles summarized by a nonlinear potential of the form \( \frac{1}{2}(|\psi|^2 * w)\psi \), where \( * \) denotes the convolution and \( w \) is the pair potential, i.e. the model for interaction between particles (see e.g. [24]). When the interaction between particles in the system is a very short range interaction (contact interaction) that happens all over the space, the pair potential \( w \) is proportional to the delta function and the Gross-Pitaevskii equation is obtained. If the interaction between particles in the system is a short range interaction that additionally has a space and time dependency, one might expect the corresponding mean field equation to be of the form (12).

For some specific cases of \( \nabla(x, t) \), an adequate ansatz of solution allows equation (12) to be reduced to the equation
\[
-\Delta \varphi + V(x)|\varphi|^2 \varphi = \lambda \varphi. \tag{13}
\]
Some examples of potentials \( \nabla(x, t) \) where such reduction is possible are the following cases.

**Lemma 4.3.** Assume that \( \lambda \in \mathbb{R} \) and assume that \( \varphi(x) \) is a solution of equation (13). Then, in the following cases, equation (12) admits the corresponding \( \psi(x, t) \) as a solution:

- **a)** \( \psi(x, t) = \varphi(x)e^{-i\lambda t} \) for \( \nabla(x, t) = V(x) \).
- **b)** \( \psi(x, t) = \varphi(x - ct)e^{-i(\lambda x + |c|^2/4)t - c|x|^2/2)} \) for \( \nabla(x, t) = V(x - ct) \), where \( c \in \mathbb{R}^n \).
- **c)** \( \psi(x, t) = \varphi \left( \frac{x}{t^{n/2}} \right) e^{i(x/\sqrt{t})^2/4t} \) for \( \nabla(x, t) = t^{n/2}V(x/t) \).

After a change of variable, this results also tackles potentials of the form \( \nabla(x, t) = V(ax - b, |a|^2 t + \tau) \), \( a \in \mathbb{R} \setminus \{0\} \), \( b \in \mathbb{R}^n \), \( \tau \in \mathbb{R} \), for one of the potentials \( \nabla(x, t) \) above.
Proof. It is obtained by direct calculations. To exemplify the calculations we prove case $c)$. Let $\psi(x,t) = \varphi(\frac{x}{t}) e^{i(\lambda t + |x|^2/4t)}$, then
\[
\frac{\partial \psi}{\partial t} = \left[ -\frac{x}{t^2} \cdot [\nabla \varphi](\frac{x}{t}) + \varphi(\frac{x}{t}) \left( -\frac{n}{2t} - \frac{\lambda}{t^2} - \frac{|x|^2}{4t^2} \right) \right] e^{i(\lambda t + |x|^2/4t)} t^{n/2} \frac{e^{i(\lambda t + |x|^2/4t)}}{t^{n/2}}, \quad \text{and}
\]
\[
\Delta \psi = \left[ \frac{1}{t^2} [\Delta \varphi](\frac{x}{t}) + \frac{x}{t^2} \cdot [\nabla \varphi](\frac{x}{t}) + \left( \frac{n}{2t} - \frac{|x|^2}{4t^2} \right) \varphi(\frac{x}{t}) \right] e^{i(\lambda t + |x|^2/4t)} t^{n/2} \frac{e^{i(\lambda t + |x|^2/4t)}}{t^{n/2}}.
\]
Using these in equation (12), and since $V(x,t) = t^{n-2}V(x/t)$, we obtain,
\[
i \frac{\partial \psi}{\partial t} = -\Delta \psi + V(x,t)|\psi|^2 \psi,
\]
\[
\Leftrightarrow \left[ -i \frac{x}{t^2} \cdot [\nabla \varphi](\frac{x}{t}) + \varphi(\frac{x}{t}) \left( -\frac{n}{2t} + \frac{\lambda}{t^2} + \frac{|x|^2}{4t^2} \right) \right] e^{i(\lambda t + |x|^2/4t)} t^{n/2} \frac{e^{i(\lambda t + |x|^2/4t)}}{t^{n/2}}
\]
\[
= \left[ -\frac{1}{t^2} [\Delta \varphi](\frac{x}{t}) - \frac{x}{t^2} \cdot [\nabla \varphi](\frac{x}{t}) + \left( \frac{n}{2t} + \frac{|x|^2}{4t^2} \right) \varphi(\frac{x}{t}) \right] e^{i(\lambda t + |x|^2/4t)}
\]
\[
+ t^{n-2}V(\frac{x}{t})|\varphi(\frac{x}{t})|^2 \left( \frac{x}{t} \right) \frac{e^{i(\lambda t + |x|^2/4t)}}{t^{n/2}} \left( \frac{x}{t} \right) \right] e^{i(\lambda t + |x|^2/4t)} t^{n/2} \frac{e^{i(\lambda t + |x|^2/4t)}}{t^{n/2}}.
\]
\[
\Leftrightarrow \lambda \varphi(\frac{x}{t}) = -[\Delta \varphi](\frac{x}{t}) + V(\frac{x}{t})|\varphi|^2 \varphi(\frac{x}{t}).
\]
which reduces equation (12) into equation (13) as prescribed. \hfill \square

Corollary 4.4. If $\varphi(x)$ is a solution of equation (2), provided in theorem 3.3, then $\psi(x,t)$ constructed in lemma 4.3 is a soliton solution of equation (12) in the one dimensional case for the corresponding potential $V(x,t)$.

5. Technical results and main proofs.

5.1. Some technical ODE results. This subsection can be read independently from the rest of the article. In particular, in here we are not assuming the hypotheses of section 2.

In this subsection we study the technical aspects of autonomous equations of the form $|u'| = \sqrt{g(u)}$ and some particular cases of it. Such equations can be analyzed with standard ODE results when $g(u) \neq 0$ and difficulties arise only when $g(u) = 0$. Given our final setting, we will restrict our attention to solutions satisfying $u' = 0$ only at isolated points, hence we will proceed with the following strategy: construct local solutions of $|u'| = \sqrt{g(u)}$ satisfying $g(u) \neq 0$ except at the end points, then paste these local solutions in an appropriate way.

We start this subsection with results that analyze these local solutions or building blocks. Let us recall a basic comparison result of ODEs that will be used repeatedly throughout the subsection.

Since we are working with autonomous equations, for simplicity and without loss of generality we will write the results with initial conditions at $x = 0$.

Lemma 5.1. Let $\alpha \geq 1$ and let $y_0, y_1 \in \mathbb{R}$ such that $\alpha y_0 \leq y_1$, let $d > y_1$. Let $g : [y_0, d/\alpha) \rightarrow (0, \infty)$ and $h : [y_1, d) \rightarrow (0, \infty)$ be continuous functions such that
\[
\alpha g(y/\alpha) \leq h(y), \forall y \in [y_1, d). \tag{14}
\]
Let \( u, v \in C^1([0, b]) \) be solutions of

\[
\begin{cases}
  u' = g(u), x \in [0, b), \\
  u(0) = y_0, \\
  v' = h(v), x \in [0, b), \\
  v(0) = y_1.
\end{cases}
\]

Then \( au(x) \leq v(x), \forall x \in [0, b] \), with strict inequality for \( x \in (0, b] \) if (14) holds with strict inequality.

Proof. Since \( g(y), h(y) > 0 \) then \( u, v \) are strictly increasing on \([0, b]\), hence injective. Fix \( s \in [0, b] \) and let \( t = u(s) \). Let \( \tau = \min\{\alpha t, v(b)\} \) and define \( \delta = \int_{\tau}^{\alpha t} (1/h(y)) dy \). Notice that

\[
\tau < \alpha t \iff [v^{-1}(\tau) = b \text{ and } \delta > 0] \quad \text{and} \quad \tau = \alpha t \iff \delta = 0.
\]

We observe that

\[
s = \int_0^s dx = \int_0^s \frac{u'(x)}{g(u(x))} dx = \int_{y_0}^{\alpha t} \frac{1}{g(y)} dy
\]

\[
= \int_{\delta}^{\alpha t} \frac{1}{\alpha g(y/\alpha)} dy \geq \int_{\delta}^{\alpha t} \frac{1}{\alpha g(y/\alpha)} dy \geq \int_{\delta}^{\alpha t} \frac{1}{h(y)} dy
\]

\[
= \int_{\delta}^{\alpha t} \frac{1}{h(y)} dy + \delta = \int_{0}^{\tau} \frac{v'(x)}{h(v(x))} dx + \delta = v^{-1}(\tau) + \delta.
\]

If \( \tau < \alpha t \) then \( s \geq b + \delta > b \), contradicting \( s \in [0, b] \). Therefore \( \tau = \alpha t \) and \( s \geq v^{-1}(\alpha t) \). Since \( v \) is increasing we conclude \( v(s) \geq \alpha t = au(s) \) where \( s \in [0, b] \) was arbitrary.

This comparison result will be used frequently with \( \alpha = 1 \) and allows us to establish the existence of the building block solutions described at the beginning of the subsection, under adequate conditions.

**Lemma 5.2.** Let \( 0 < \kappa_1 \) and \( y_0 < c \). Let \( g(y) \) be a Lipschitz continuous function satisfying

\[
g : [y_0, c] \to \mathbb{R}^+, g(c) = 0, \quad \text{and} \quad 0 < \kappa_1(c - y) \leq g(y), \forall y \in [y_0, c),
\]

Then there exists a unique number \( a \in (0, \infty) \) and a unique \( u \in C^1([0, a]) \) solution of

\[
\begin{cases}
  u' = \sqrt{g(u)}, x \in [0, a], \\
  u(0) = y_0.
\end{cases}
\]

satisfying

\[
u'(x) > 0 \quad \text{and} \quad u(x) \in [y_0, c), \forall x \in [0, a),
\]

\[
u(a) = c \quad \text{and} \quad u'(a) = 0.
\]

The condition \( \kappa_1(c - y) \leq g(y) \) also implies \( a \leq 2 \sqrt{(c - y_0)/\kappa_1} \) and if \( \exists \kappa_2 > 0 \) such that \( g(y) \leq \kappa_2(c - y), \forall y \in [y_0, c], \) then \( a \geq 2 \sqrt{(c - y_0)/\kappa_2} \).

Proof. At any \( y \in [y_0, c) \) we have \( g(y) > 0 \). So \( \sqrt{g} \) is locally Lipschitz for \( y \in [y_0, c) \). Standard ODE theory (Picard-Lindelöf Theorem) implies the existence of a maximal \( a > 0 \) such that equation (15) admits a unique solution \( u \in C^1([0, a]) \). Since
Lemma 5.3. \(\sqrt{g(y)} > 0\) then \(u(x)\) is strictly increasing and \(y_0 \leq u(x) < c, \forall x \in [0,a)\). We now prove that \(a < \infty\), in which case it follows that \(\lim_{x \to a^-} u(x) = c\) (otherwise the solution could be extended beyond \(a\), contradicting the maximality of \(a\)).

Let \(b = 2\sqrt{(c - y_0)/\kappa_1}\) and let
\[
v(x) = \begin{cases} c - \left(\sqrt{c - y_0 - x\sqrt{\kappa_1}/2}\right)^2, & x \in [0, b], \\ c, & x \geq b. \end{cases}
\]
By direct calculation, \(v \in C^1(\mathbb{R}_+)\) and
\[
v' = \sqrt{\kappa_1(c - v)}, \forall x \geq 0, \quad v(0) = y_0.
\]
Since \(\sqrt{g(y)} \geq \sqrt{\kappa_1(c - y)}\) for all \(y \in [y_0, c]\) lemma 5.1 yields that \(u(x) \geq v(x), \forall x \in [0, \min\{a, b\}]\). Since \(v(b) = c\) we conclude that \(a \leq b = 2\sqrt{(c - y_0)/\kappa_1} < \infty\), hence proving that \(\lim_{x \to a^-} u(x) = c\) and showing that \(\lim_{x \to a^-} u'(x) = \sqrt{g(c)} = 0\).

If additionally \(\kappa_2 > 0\) is such that \(g(y) \leq \kappa_2(c - y), \forall y \in [y_0, c]\), lemma 5.1 implies the bound \(a \geq 2\sqrt{(c - y_0)/\kappa_2}\) in a similar way.

We now establish the continuous dependence of the solution in lemma 5.2 with respect to the parameters of the equation. There are two main difficulties that we need to address in this result and they lie at the right end of the domain interval. They are: 1) the right end point moves when changing the parameters of the equation, therefore changing the domain of the solution; and 2) at the right end point the ODE stops being locally Lipschitz.

Lemma 5.3. Let \(\kappa_1 > 0\) and let \(y_0, y_1 < c \leq d\). Let \(g(x)\) and \(h(x)\) be Lipschitz continuous functions satisfying
\[
g : [y_0, c) \to \mathbb{R}_+, g(c) = 0, \text{ and } 0 < \kappa_1(c - y) \leq g(y), \forall y \in [y_0, c),
\]
\[
h : [\min\{y_0, y_1\}, d) \to \mathbb{R}_+, h(d) = 0, \text{ and } 0 < \kappa_1(d - y) \leq h(y), \forall y \in [y_1, d).
\]
Let \(a > 0, u \in C^1([0, a])\) and let \(b > 0, v \in C^1([0, b]\) be the solutions (as in lemma 5.2) of
\[
\begin{align*}
\begin{cases}
u' = \sqrt{g(u)}, x \in [0, a], \\
u(0) = y_0 \text{ and } u'(a) = 0, \\
u'(x) > 0, \forall x \in [0, a],
\end{cases} & \quad 
\begin{cases}
u' = \sqrt{h(v)}, x \in [0, b], \\
u(0) = y_1, \text{ and } v'(b) = 0, \\
u'(x) > 0, \forall x \in [0, b],
\end{cases}
\end{align*}
\]
Then there exists \(\delta > 0\) that only depends on \(\kappa_1, (c - y_0), (d - y_1)\) and the Lipschitz constant of \(h(x)\), such that for any \(0 < \epsilon < 1\) small enough,
\[
|y_1 - y_0| + |d - c| + \sup_{[y_0, c]} |g(y) - h(y)| \leq \epsilon,
\]
implies \(|b - a|^2 + \sup_{[0, \min\{a, b\}]} |u(x) - v(x)| \leq (-\delta/\ln(\epsilon))^2\).

Proof. Our first step is the comparison of the solutions \(u(x)\) and \(v(x)\) for \(x \in [0, \min\{a, b\}]\). To do this we split the interval \([0, \min\{a, b\}]\) into \([0, \min\{a, b\} - \epsilon]\) and \([\min\{a, b\}, \min\{a, b\}]\) and choose \(\epsilon\) appropriately such that: a) in \([0, \min\{a, b\} - \epsilon]\) we can use standard Lipschitz estimates, and such that; b) in \([\min\{a, b\}, \min\{a, b\}]\) the solutions are almost constant.
The appropriate criteria for the choice of $x$ is the following:

$$x = \sup\{x \in [0, \min\{a, b\}] :$$

$$\max\{\sqrt{g(u(x))}, \sqrt{h(v(x))}\} \geq -\delta/\ln(\epsilon), \forall x \in [0, \tilde{x}]\}$$

where $\delta > 0$ is a constant that will be prescribed later.

Let us proceed to a): the interval where the equations are Lipschitz. For $x \in [0, \tilde{x})$ we have

$$|u' - v'| = |\sqrt{g(u)} - \sqrt{h(v)}|$$

$$= \frac{|g(u) - h(v)|}{\sqrt{g(u)} + \sqrt{h(v)}}$$

$$\leq \frac{|g(u) - h(v)|}{\ln(\epsilon)} \frac{1}{\delta}$$

$$\leq \frac{|g(u) - h(u)| + |h(u) - h(v)|}{\ln(\epsilon)} \frac{1}{\delta}$$

$$\leq \frac{(\epsilon + L|u - v|)}{\ln(\epsilon)} \frac{1}{\delta},$$

where $L$ is the Lipschitz constant of $h$. Since the solution of $w' = (\epsilon/L + w)k$ is $w(x) = ((w(0) + \epsilon/L)e^{hx} - \epsilon/L)$, comparison of solutions in lemma 5.1 implies that for $x \in [0, \tilde{x})$

$$|u(x) - v(x)| \leq ([u(0) - v(0)] + \epsilon/L) \exp[-\ln(\epsilon) Lx/\delta] - \epsilon/L.$$ 

Since $|u(0) - v(0)| \leq \epsilon$, choosing $\delta$ large enough such that $\delta > 2L \min\{a, b\} \geq 2L\tilde{x}$, we conclude,

$$\sup_{x \in [0, \tilde{x})} |u(x) - v(x)| \leq (\epsilon + \epsilon/L) \frac{1}{\sqrt{\epsilon}} = \sqrt{\epsilon} \left(1 + \frac{1}{L}\right),$$

i.e. we control the difference between $u$ and $v$ in $[0, \tilde{x})$ as desired.

This estimate implies in particular that $\tilde{x} < \min\{a, b\}$ for $\epsilon$ small enough. Indeed, assume that $\tilde{x} = a$ (hence $\tilde{x} > 0$ and $a \leq b$). Since $g(a) = 0$, the definition of $\tilde{x}$ implies $h(a) \geq (-\delta/\ln(\epsilon))^2$. On the other hand, the estimate above implies $|u(a) - v(a)| \leq \sqrt{\epsilon}(1 + 1/L)$, but $u(a) = c$ and $|c - d| < \epsilon$, hence $|v(a) - d| < \epsilon + \sqrt{\epsilon}(1+1/L)$. Since $h$ is Lipschitz, $h(a) = (h(a) - h(d)) < L(\epsilon + \sqrt{\epsilon}(1+1/L))$. In summary $(-\delta/\ln(\epsilon))^2 \leq h(a) < (L\epsilon + \sqrt{\epsilon}(L + 1))$ which is a contradiction for $\epsilon > 0$ small enough. Assuming $\tilde{x} = b$ (hence $\tilde{x} > 0$ and $b \leq a$) the analysis is similar. On one hand $g(b) = c$ and $|c - d| < \epsilon$, hence $|v(b) - d| < \epsilon + \sqrt{\epsilon}(1+1/L)$. Since $h$ is Lipschitz then $h(b) < L(\epsilon + \sqrt{\epsilon}(L + 1))$ and since $\sup_{[y, \tilde{x}]} |g(y) - h(y)| < \epsilon$ we conclude $g(b) < \epsilon + \sqrt{\epsilon}(L + 1)$. In summary, $(-\delta/\ln(\epsilon))^2 \leq g(b) < (\epsilon + \sqrt{\epsilon}(L + 1))$ which is a contradiction for $\epsilon > 0$ small enough.

Since $\tilde{x} < \min\{a, b\}$, the definition of $\tilde{x}$ and the continuity of the functions imply that $\max\{\sqrt{g(u(x))}, \sqrt{h(v(x))}\} \leq -\delta/\ln(\epsilon)$. With this we now proceed to b), the analysis in the interval $[x, \min\{a, b\}]$, where the functions are almost constant and close to their limiting values $c$ and $d$ respectively. The equation for $u$ implies

$$u(x) \leq u(x) \leq c, \forall x \in [x, a],$$

the condition on $g$ implies

$$\kappa_1(c - u(x)) \leq g(u(x)),$$
while the choice of $x$ gives

$$g(u(x)) \leq (-\delta / \ln(\epsilon))^2.$$  

These three conditions together imply

$$c - \frac{1}{\kappa_1} \left( -\delta \frac{1}{\ln(\epsilon)} \right)^2 \leq u(x) \leq u(x) \leq c, \quad \forall x \in [x, a].$$

Similarly for $v$, we have

$$d - \frac{1}{\kappa_1} \left( -\delta \frac{1}{\ln(\epsilon)} \right)^2 \leq v(x) \leq v(z) \leq d, \quad \forall z \in [x, b].$$

Since $|c - d| \leq \epsilon$ we conclude

$$|u(x) - v(z)| \leq \epsilon + \frac{1}{\kappa_1} \left( -\delta \frac{1}{\ln(\epsilon)} \right)^2, \quad \forall x \in [x, a], \forall z \in [x, b]. \quad (17)$$

In particular

$$\sup_{x \in [x, \min\{a, b\}]} |u(x) - v(x)| \leq \epsilon + \frac{1}{\kappa_1} \left( -\delta \frac{1}{\ln(\epsilon)} \right)^2.$$  

In summary, for $\delta$ large enough and for $\epsilon > 0$ small enough, the estimates in $[0, x]$ and $[x, \min\{a, b\}]$ imply that

$$\sup_{x \in [0, \min\{a, b\}]} |u(x) - v(x)| \leq \epsilon + \sqrt{\epsilon} (1 + 1/L) + \frac{1}{\kappa_1} \left( -\delta \frac{1}{\ln(\epsilon)} \right)^2.$$  

Our second step is to estimate $|a - b|$. Observe that $u$ and $v$ solve

$$\begin{cases} u' = \sqrt{g(u)}, x \in [x, a], \\
        u(x)|_{x = a} = u(a), \\
        u'(a) = 0, \\
    \end{cases} \quad \text{and} \quad \begin{cases} v' = \sqrt{h(v)}, x \in [x, b], \\
        v(x)|_{x = a} = v(a), \\
        v'(b) = 0, \\
    \end{cases}$$

Lemma 5.2 implies $0 \leq a - x \leq 2 \sqrt{(c - u(x))/\kappa_1}$. Since we also have $\kappa_1(c - u(x)) \leq g(u(x)) \leq (-\delta / \ln(\epsilon))^2$, we conclude

$$0 \leq a - x \leq -\frac{2\delta}{\kappa_1 \ln(\epsilon)}.$$  

Similarly, we obtain

$$0 \leq b - x \leq -\frac{2\delta}{\kappa_1 \ln(\epsilon)}.$$  

And these bounds imply

$$|b - a| \leq -\frac{2\delta}{\kappa_1 \ln(\epsilon)}.$$  

To finish the proof we review the dependence of $\delta$ on the parameters of the equation. The argument above requires $\delta \geq 2L \min\{a, b\}$, but lemma 5.2 implies $\min\{a, b\} \leq 2 \sqrt{(d - y_1)/\kappa_1}$, hence $\delta$ can be chosen as only depending on $\kappa_1, (d - y_1)$ and $L$.

The estimates obtained are the same as the one described in the statement of the lemma after absorbing some constants into $\delta$. \qed
Lemma 5.2 and lemma 5.3 study solutions \( u(x) \) of equations of the form \( u' = \sqrt{g(u)} \) on intervals where \( u'(x) = 0 \) only at the right end of the interval. We now study how to use these solutions as building blocks, pasting them at points where \( u' = 0 \) and constructing solutions of the equation \( |u'| = \sqrt{g(u)} \) for larger domains. As we will see below, by allowing \( u' = 0 \) only at isolated points this construction also solves the equation \( u'' = g'(u)/2 \) if \( g \) is \( C^1 \).

**Lemma 5.4.** Let \( \kappa_1 > 0 \) and \( c > 0 \). Let \( g(y) \) be a Lipschitz continuous function satisfying

\[
g : [0, c] \to \mathbb{R}_+, g(c) = 0, \quad \text{and} \quad 0 < \kappa_1(c - y) \leq g(y), \forall y \in [0, c).
\]

Let \( a > 0 \) and \( u \in C^1([0, a]) \) be the solution in lemma 5.2 of the equation

\[
\begin{cases}
u' = \sqrt{g(u)} , & x \in [0, a],
\nu(0) = 0, \\
u'(x) > 0, & x \in [0, a) \text{ and } u'(a) = 0.
\end{cases}
\]

Define

\[
v(x) = \begin{cases} u(x), & x \in [0, a], \\
u(2a - x), & x \in [a, 2a].
\end{cases}
\]

Then \( v(x) \) is the unique \( C^1([0, 2a]) \) solution of

\[
\begin{cases}
|v'| = \sqrt{g(v)}, & x \in [0, 2a], \\
v(0) = 0 \text{ and } v'(0) > 0, \\
v'(x) = 0 \quad \text{only at isolated points in } [0, 2a],
\end{cases}
\]

and \( a \) and the solution \( v \) depend continuously on the parameters of the equation.

Moreover, if \( g \in C^1([0, c]) \), then \( v \in C^2([0, 2a]) \) and it solves the equation

\[
v'' = g'(v)/2, \quad \forall x \in [0, 2a],
\]

**Proof.** Our first step is to verify that \( v(x) \) is a solution of the equation (18). From its definition it is clear that \( v(x) \) is \( C^1 \) in \([0, a) \cup (a, 2a] \). Also

\[
v'(x) = \begin{cases} u'(x) = \sqrt{g(v(x))}, & x \in [0, a), \\
u'(2a - x) = -\sqrt{g(v(x))}, & x \in (a, 2a].
\end{cases}
\]

Since \( v(x) \) is continuous at \( a \) and \( \lim_{x \to a^-} v'(x) = 0 \), then \( v'(a) \) exists and \( v'(a) = 0 = \sqrt{g(v(a))} \); i.e. \( v(x) \in C^1([0, 2a]) \) and \( |v'| = \sqrt{g(v)} \) in \([0, 2a] \). The other conditions in equation (18) are readily verified since \( v(0) = u(0) = 0, v'(0) = \sqrt{g(u(0))} > 0 \) and \( v'(x) = 0 \) only at \( x = a \).

Our second step is to establish the uniqueness of \( v \), which is a direct consequence of the uniqueness of solutions in lemma 5.2. Namely, if \( v \) is a solution of equation (18), whenever \( |v'| = v' \neq 0 \) the uniqueness in lemma 5.2 implies that \( v(x) \) has to be of the form \( u(x_0 + x) \) for some \( x_0 \), and whenever \( |v'| = -v' \neq 0 \) then \( v(x) \) has to be of the form \( u(x_1 - x) \). Since \( v' \) is continuous and \( v' \neq 0 \) only at isolated points, the conditions \( v(0) = 0 \) and \( v'(0) > 0 \) forces \( x_0 = 0 \) and then \( x_1 = 2a \), i.e. any solution \( v(x) \) has the prescribed form of the lemma.
Theorem 5.5. Let $\kappa_1 > 0$ and let $c > 0$. Let $f$ be a continuously differentiable function satisfying
\[ f : [0, c^2] \to \mathbb{R}_+, f(c^2) = 0, \quad \text{and} \quad 0 < \kappa_1(c^2 - y) \leq f(y), \forall y \in [0, c^2). \]
Let $a > 0$ and let $w \in C^1([0, a])$ be the solution in lemma 5.2 of
\[
\begin{cases}
w' = \sqrt{f(w^2)}, x \in [0, a], \\
w(0) = 0, \\
w'(x) > 0, \forall x \in [0, a] \text{ and } w'(a) = 0.
\end{cases}
\]
Let $v$ be the odd extension of $w$ to the interval $[-a, a]$, i.e.
\[ v(x) = \begin{cases} w(x) & x \in [0, a] \\
-w(-x) & x \in [-a, 0]. \end{cases} \]
Let $u(x) = (-1)^n v(x - 2an)$ for $x \in 2an + [-a, a]$, $n \in \mathbb{Z}$. Then $u \in C^2(\mathbb{R})$ is the unique solution of
\[
\begin{cases}
|u'| = \sqrt{f(u^2)}, \forall x \in \mathbb{R}, \\
u(0) = 0, \\
u'(0) > 0, \text{ and } u' = 0 \text{ only at isolated points } \mathbb{R}.
\end{cases}
\]
This $u$ also solves the equation $u'' = uf'(u^2)$ and it depends continuously on the parameters of the equation in any bounded interval.
Moreover, the function $u$ is $(4\kappa_1)$-periodic with $a \leq 2/\sqrt{\kappa_1}$ and $a$ depending continuously on the parameters of the equation. And if $\exists \kappa_2 > 0$ such that $f(z) \leq \kappa_2(c^2 - z)$, $z \in [0, c^2)$, then also $a \geq \sqrt{2/\kappa_2}$.

Proof. Let $g : [0, c] \to \mathbb{R}_+$ be defined as $g(y) = f(y^2)$. Then
\[
\begin{align*}
[f(c^2) = 0] & \Rightarrow [g(c) = 0], \\
0 < \kappa_1(c^2 - y^2) \leq f(y^2) & \Rightarrow [c \kappa_1(c - y) \leq g(y)], y \in [0, c], \\
f(y^2) \leq \kappa_2(c^2 - y^2) & \Rightarrow [g(y) \leq 2c \kappa_2(c - y)], y \in [0, c].
\end{align*}
\]
Hence we are in the setting of the previous results and \( w \) solves \( w' = \sqrt{f(w^2)} \) in \([0, a]\) as described in lemma 5.2. Because of the symmetries, \( v' = \sqrt{f(v^2)} \) in \([-a, a]\) and we can apply lemma 5.4 on each end of the intervals \( 2an + [-a, a] \), concluding that the prescribed \( u \) is indeed the only solution of equation (20). The stated properties of \( u \) are direct consequence of lemma 5.2 and lemma 5.4 and \( u \) is 4\(a\)-periodic by construction.

In theorem 5.5 we have a very complete description of the solutions of \( |u'| = \sqrt{f(u^2)} \) satisfying \( u' = 0 \) at isolated points. These solutions are periodic and the last part of this appendix is devoted to study how the period of these solutions depend on some very specific parameters of the equation when \( \beta - f \) is a convex function for some constant \( \beta > 0 \). The family of convex functions providing the structure for the analysis is the following.

Definition 5.6. We define the family of functions \( H \) as

\[
H = \{ h \in C^1(\mathbb{R}_+) \mid h(0) = 0; \lim_{z \to \infty} h'(z) = \infty; h \text{ strictly convex} \}.
\]

The next lemma remarks on some useful basic properties for \( h \in H \) and introduce the definition of some quantities that will be used later in the parameters of the equations or in the estimates.

Lemma 5.7. Let \( h \in H \), then

1. \( \lim_{z \to -\infty} h(z) = \infty \) and \( \forall \beta > 0, \exists \! c = c(\beta) > 0 \) such that \( h(c^2) = \beta \). Such \( c(\beta) \) is continuous strictly increasing on \( \beta \) and satisfies \( \lim_{\beta \to -\infty} c = \infty \).
2. Let \( r, \rho, t, \tau \in \mathbb{R}_+ \) be such that \( r \leq \rho \) and \( t \leq \tau \), one of the inequalities strict, then

\[
h'(r) < \frac{h(t) - h(r)}{t - r} < \frac{h(\tau) - h(\rho)}{\tau - \rho} \leq h'(\tau).
\]
3. Let \( \kappa_1 = \kappa_1(\beta) = \beta/c^2 \) and \( \kappa_2 = \kappa_2(\beta) = h'(c^2) \), then \( 0 < \kappa_1 < \kappa_2 \) and \( \lim_{\beta \to -\infty} \kappa_1 = \infty \).
4. \( \beta - h(z) : [0, c^2] \to \mathbb{R}_+ \) is such that

\[
\kappa_1(c^2 - z) \leq (\beta - h(z)) \leq \kappa_2(c^2 - z), \forall z \in [0, c^2].
\]

Proof. The conditions defining \( H \) imply that \( h \) restricted to \( h^{-1}((0, \infty)) \) is a strictly increasing homeomorphism onto \((0, \infty)\), hence property (1) follows. Property (2) is the convexity of \( h \). Property (3) follows directly from the fact that a differentiable convex function satisfies \( \lim_{z \to -\infty} h(z)/z = \lim_{z \to -\infty} h'(z) \). And property (4) is property (2) with \( \tau = t = c^2, \rho = 0 \) and \( \rho = z \).

Lemma 5.7 implies that if \( h \in H \) and \( \beta > 0 \) then theorem 5.5 applies to the equation \( |u'| = \sqrt{\beta - h(u^2)} \). The next theorem studies the the behavior of 4\(a\), the period of the solution \( u \), with respect to the parameter \( \beta \).

Theorem 5.8. Let \( h \in H \). For \( \beta > 0 \) define \( c = c(\beta) > 0 \) as in lemma 5.7, so \( h(c^2) = \beta \). Let \( a = a(\beta) > 0 \) and \( u = u \in C^1([0, a]) \) be the solution in lemma 5.2 of

\[
\begin{align*}
u' &= \sqrt{\beta - h(u^2)}, & x &\in [0, a], \\
u(0) &= 0, & \\
u'(x) &> 0 \text{ for } x \in [0, a) \text{ and } u(a) = c.
\end{align*}
\]

Then \( a(\beta) \) is continuous strictly decreasing on \( \beta \) and \( \lim_{\beta \to -\infty} a = 0 \). If additionally \( h'(0) < 0 \) then \( \lim_{\beta \to 0^+} a = +\infty \).
Proof. Let $0 < \beta < \gamma$. Let $a = a(\beta), c = c(\beta), b = b(\gamma), d = d(\gamma)$ and let $u \in C^1([0, a]), v \in C^1([0, b])$ be such that

$$
\begin{align*}
    u' &= \sqrt{\beta - h(u^2)}, \quad x \in [0, a], \\
    u(0) &= 0, \\
    u'(x) &= 0, \quad x \in [0, a], \\
    u(a) &= c \text{ and } u'(a) = 0,
\end{align*}
\begin{align*}
    v' &= \sqrt{\gamma - h(v^2)}, \quad x \in [0, b], \\
    v(0) &= 0, \\
    v'(x) &= 0, \quad x \in [0, b], \\
    v(b) &= d \text{ and } v'(b) = 0.
\end{align*}
$$

Lemma 5.7 implies $0 < c < d$ and then another part of lemma 5.7 implies

$$
\frac{h(c^2) - h(\frac{c^2}{d} y^2)}{c^2 - \frac{c^2}{d} y^2} < \frac{h(d^2) - h(y^2)}{d^2 - y^2}, \quad \forall y \in [0, d),
$$

$$
\Rightarrow \frac{d^2}{c^2} \left( \beta - h(\frac{c^2}{d} y^2) \right) < \left( \gamma - h(y^2) \right), \quad \forall y \in [0, d),
$$

$$
\Rightarrow \frac{d}{c} \left( \sqrt{\beta - h(\frac{c^2}{d} y^2)} \right) < \sqrt{\gamma - h(y^2)}, \quad \forall y \in [0, d).
$$

Lemma 5.1 with $y_0 = y_1 = 0$ and $\alpha = d/c$ implies $\frac{d}{c} u(x) < v(x), \forall x \in (0, \min(a, b)]$. Since $u(a) = c, v(b) = d$ and $v$ is strictly increasing we conclude $b < a$. Hence $a(\beta)$ is strictly decreasing. And $a(\beta)$ is continuous on $\beta$ directly from theorem 5.5.

Also, theorem 5.5 and lemma 5.7 imply $a \leq \frac{2}{\sqrt{\kappa_1}}$ for $\kappa_1 = \beta/c^2$. Since lemma 5.7 implies $\lim_{\beta \to \infty} \kappa_1 = \infty$ then $\lim_{\beta \to \infty} a = 0$.

And finally, if $h'(0) < 0$ then $c_0 = \lim_{\beta \to 0^+} c(\beta)$ satisfies $c(\beta) > c_0 > 0, \forall \beta > 0$. Assume $\beta \in (0, c_0^2)$ then $\exists! x_0 \in (0, a)$ such that $u(x_0) = \sqrt{\beta}$, since $h$ is convex and $u$ is positive and strictly increasing

$$
u' = \frac{\beta - h(u^2)}{u^2 - h'(0)u^2} \leq u' \leq u \sqrt{1 - h'(0)}, \quad x \in [x_0, a].
$$

Lemma 5.1 then implies $u(x) \leq \sqrt{\beta} \exp \left( \frac{1}{\sqrt{1 - h'(0)}} \right), \quad x \in [x_0, a]$. Evaluating in $x = a$ and some manipulation imply

$$
a > a - x_0 \geq \frac{1}{\sqrt{1 - h'(0)}} \ln \left( \frac{c_0}{\sqrt{\beta}} \right).
$$

We conclude that $\lim_{\beta \to 0^+} a = \infty$. \qed

Theorem 5.8 will be applied with a slightly different notation in the main part of the article. For simplicity we rewrite theorem 5.8 in such notation as a corollary.

**Corollary 5.9.** Let $h \in H$ and let $\lambda < 0$. For $\beta > 0$ define $c = c(\beta) > 0$ such that $h(c^2) + \lambda c^2 = \beta$. Let $a = a(\beta) > 0$ and $u \in C^1([0, a])$ be the solution in lemma 5.2 of

$$
\begin{align*}
    u' &= \sqrt{-\lambda u^2 - h(u^2) + \beta}, \quad x \in [0, a], \\
    u(0) &= 0, \\
    u'(x) &= 0 \quad \text{for } x \in [0, a) \text{ and } u(a) = c.
\end{align*}
$$

Then $c(\beta)$ is continuous strictly increasing on $\beta$, $a(\beta)$ is continuous strictly decreasing on $\beta$, $\lim_{\beta \to \infty} c = \infty$ and $\lim_{\beta \to \infty} a = 0$. If additionally $h'(0) < -\lambda$ then $\lim_{\beta \to 0^+} a = +\infty$.

**Proof.** If $h \in H$ then $(\lambda + h) \in H$ for any $\lambda \in \mathbb{R}$, hence the corollary is exactly theorem 5.8 and some parts of lemma 5.7. \qed
And the last result in this appendix is a version of corollary 5.9 considering a specific initial condition.

**Lemma 5.10.** Let $h \in H$ and let $\lambda < 0$. For $\beta > 0$ define $c = c(\beta) > 0$ such that $\lambda c^2 + h(c^2) = \beta$ and define $r = r(\beta) > 0$ such that $h(r^2) = \beta$. Let $a_0 = a_0(\beta) > 0$ and $u \in C^1([0, a_0])$ be the unique solution of

\[
\begin{cases}
u' = \sqrt{-\lambda u^2 - h(u^2) + \beta}, & x \in [0, a_0], \\
u(0) = r, \\
u'(x) > 0 \text{ for } x \in [0, a_0), \\

u(a_0) = c \text{ and } u'(a_0) = 0.
\end{cases}
\]

Then $r(\beta), c(\beta)$ are well defined, continuous and strictly increasing on $\beta$. Also $a_0(\beta)$ is continuous strictly decreasing on $\beta$. Additionally, if $h'(0) \in [0, -\lambda)$ then $\lim_{\beta \to 0^+} a_0 = \infty$.

**Proof.** The facts that $r, c$ are well defined and that they are continuous and strictly increasing on $\beta$ are direct from lemma 5.7. The fact that $a_0$ is continuous on $\beta$ is direct from lemma 5.3. To establish the monotone behavior of $a_0$ consider $0 < \beta < \gamma$, analogously to the proof of theorem 5.8, we will use lemma 5.1 to conclude that $a_0(\gamma) < a_0(\beta)$. Let $0 < r(\beta) < s(\gamma)$ and $0 < c(\beta) < d(\gamma)$ be such that $h(r^2) = \beta, \lambda c^2 + h(c^2) = \beta, h(s^2) = \gamma$ and $\lambda d^2 + h(d^2) = \gamma$.

In the proof of theorem 5.8 we already obtained

\[
\frac{d}{c} \sqrt{\beta - h\left(\frac{y^2}{d^2}\right)} < \sqrt{\gamma - h\left(y^2\right)}, \quad \forall y \in [0, d),
\]

hence all that we need to apply lemma 5.1 is that $y_0 = r$ and $y_1 = s$ satisfy $\frac{d}{c} y_0 \leq y_1$. From the definitions,

\[
\begin{align*}
h(d^2) - h(s^2) &= -\lambda d^2 > 0, \\
h(c^2) - h(r^2) &= -\lambda c^2 > 0,
\end{align*}
\]

and from lemma 5.7

\[
\frac{h(c^2) - h(r^2)}{c^2 - r^2} < \frac{h(d^2) - h(s^2)}{d^2 - s^2},
\]

hence

\[
\frac{-\lambda c^2}{c^2 - r^2} < \frac{-\lambda d^2}{d^2 - s^2},
\]

\[
\Rightarrow \frac{d}{c} r < s.
\]

Applying lemma 5.1 as in the proof of theorem 5.8 implies $a_0(\gamma) < a_0(\beta)$.

Let us study what happens with $a_0(\beta)$ as $\beta \to 0^+$. On one hand, we have that $u' = \sqrt{-\lambda u^2 - h(u^2) + \beta} \leq \sqrt{-\lambda u^2} = u \sqrt{-\lambda}$ and lemma 5.1 implies $u(x) \leq r e^{x \sqrt{-\lambda}}$, $x \in [0, a_0]$. Evaluating in $x = a_0$ and using that $c = u(a_0)$ we conclude $a_0(\beta) \geq \frac{1}{\sqrt{-\lambda}} \ln \left(\frac{c(\beta)}{r(\beta)}\right)$. On the other hand, since $h \in H$ and $h'(0) \geq 0$ we get $\lim_{\beta \to 0^+} r(\beta) = 0^+$. Similarly, $h \in H$ and $h'(0) < -\lambda$ implies $c(\beta) \geq c(0) > 0$ for all $\beta > 0$. Hence $\lim_{\beta \to 0^+} \frac{c(\beta)}{r(\beta)} = +\infty$ and therefore $\lim_{\beta \to 0^+} a_0 = +\infty$. \qed
5.2. Proofs of the main theorems. Now we proceed to use the results and tools from subsection 5.1 to present the proofs of the main theorems of this paper, which were introduced in section 3.

Proof of Theorem 3.1. As mentioned in section 2 we consider $\varphi$ real valued and $\lambda \in \mathbb{R}$. First, observe that equation (2) can be written as
\[
\begin{align*}
-\varphi'' &= \lambda \varphi, \quad \text{for } x \in (-\infty, -1) \text{ and } \lim_{x \to -\infty} \varphi(x) = 0, \\
-\varphi'' &= \lambda \varphi, \quad \text{for } x \in (1, \infty) \text{ and } \lim_{x \to \infty} \varphi(x) = 0, \\
-\varphi'' - h'(|\varphi|^2)\varphi &= \lambda \varphi, \quad \text{for } x \in (-1, 1),
\end{align*}
\]
(21)
plus continuity of $\varphi$ and $\varphi'$ at $\{-1, 1\}$. On the one hand, this immediately implies that $\lambda < 0$ and
\[
\begin{align*}
\varphi(x) &= \varphi(-1)e^{(x+1)\sqrt{-\lambda}}, \quad \text{for } x \in (-\infty, -1), \\
\varphi(x) &= \varphi(1)e^{(1-x)\sqrt{-\lambda}}, \quad \text{for } x \in (1, \infty).
\end{align*}
\]
On the other hand, multiplying equation (21) by $\varphi'$ and integrating we obtain
\[
\begin{align*}
(\varphi')^2 &= -\lambda \varphi'^2, \quad \text{for } x \in (-\infty, -1), \\
(\varphi')^2 &= -\lambda \varphi'^2, \quad \text{for } x \in (1, \infty), \\
(\varphi')^2 &= -h(\varphi^2' - \lambda \varphi^2 + \beta, \quad \text{for } x \in (-1, 1),
\end{align*}
\]
for some constant $\beta$. The continuity of $\varphi$ and $\varphi'$ at $\{-1, 1\}$ implies that $\beta = h(\varphi^2(1)) = h(\varphi^2(-1))$, and the conditions on $h$ imply that $\beta \geq 0$ (moreover $\beta > 0$ if $\alpha \neq 0$) and that $|\varphi(1)| = |\varphi(-1)|$. Letting $\alpha = \varphi(-1) \in \mathbb{R}$ we obtain formulas (3) and (4) of theorem 3.1. Let us verify that $\alpha \neq 0$. If $\alpha = 0$ then equation (3) implies $\varphi \equiv 0$ for $|x| > 1$. While for $|x| < 1$ equation (22) becomes $|\varphi'| = \sqrt{-\lambda \varphi^2 - h(\varphi^2)} \leq c|\varphi|$ for some constant $c > 0$ for $|\varphi|$ small. Since $\varphi(-1) = 0$ then Grönwall’s inequality implies $\varphi \equiv 0$ in $[1, 1]$ and $\varphi$ is a trivial solution. I.e. if $\varphi$ is a non-trivial solution then $\alpha \neq 0$.

Finally, let us prove that $\varphi' = 0$ only at isolated points. In equation (3) this is straightforward at $|x| \geq 1$ since $\alpha \neq 0$. For $|x| < 1$, multiply equation (21) by $\varphi$ and add equation (22) to obtain, in $(-1, 1)$,
\[
(\varphi')^2 = \varphi''\varphi = \beta + h'(\varphi^2)\varphi^2 - h(\varphi^2).
\]
(23)
The hypotheses on $h$ imply $h(z)/z < h'(z), \forall z > 0$, hence $h'(\varphi^2)\varphi^2 - h(\varphi^2) \geq 0$. Since $\beta > 0$ the right hand side in equation (23) is strictly positive, hence $\varphi', \varphi''$ cannot both vanish at the same time, which implies that $\varphi' = 0$ only at isolated points.

Proof of Theorem 3.2. For each $\alpha > 0$ consider the following quantities and equations. Let $a_0 = a_0(\alpha) > 0$ and let $w \in C^1([0, a_0])$ be such that
\[
\begin{align*}
w' &= \sqrt{-\lambda w^2 - h(w^2) + h(\alpha^2)}, \quad x \in [0, a_0], \\
w(0) &= \alpha \text{ and } w'(a_0) = 0, \\
w'(x) &> 0 \text{ for } x \in [0, a_0).
\end{align*}
\]
Let $a = a(\alpha) > 0$ and let $v \in C^2(\mathbb{R})$ be $4\alpha$-periodic such that
\[
\begin{align*}
|v'| &= \sqrt{-\lambda v^2 - h(v^2) + h(\alpha^2)}, \quad x \in \mathbb{R}, \\
v(0) &= 0 \text{ and } v'(a) = v'(-a) = 0, \\
v'(0) &> 0 \text{ and } v' = 0 \text{ only at isolated points.}
\end{align*}
\]
Such quantities and solutions exist and are unique due to lemma 5.2 and 5.5. For \( m \in \mathbb{N} \cup \{0\} \) define \( u : [0, 2a_0 + 2ma] \to \mathbb{R} \) as

\[
\begin{cases}
  u(x) = w(x), & x \in [0, a_0], \\
  u(x + a_0) = v(x + a), & x \in [0, 2ma], \\
  u(x + a_0 + 2ma) = (-1)^mw(a_0 - x), & x \in [0, a_0].
\end{cases}
\]

As in theorem 5.5, lemma 5.4 implies that \( u \in C^2([0, 2a_0 + 2ma]) \) is the unique solution of

\[
\begin{cases}
|u'| = \sqrt{-\lambda u^2 - h(u^2)} + \hat{h}(\alpha^2), & x \in [0, 2a_0 + 2ma], \\
 u(0) = \alpha = (-1)^mu(2a_0 + 2ma), \\
 u'(0) > 0 \text{ and } u' = 0 \text{ only at isolated points},
\end{cases}
\]

if \( u \) has exactly \( m \) zeroes.

To conclude all we need is to show that there is a unique \( \alpha > 0 \) such that \( 2a_0 + 2ma = 2 \) (then \( u(x + 1) \) is the solution in Theorem 3.2). Corollary 5.9 and lemma 5.10 imply \( a_0, a > 0 \) and that \( 2a_0 + 2ma \) is continuous, strictly decreasing on \( \alpha \) and \( \lim_{\alpha \to \infty} 2a_0 + 2ma = 0 \). Therefore, if \( m_0 \geq 0 \) is such that \( m_0 > \inf\{ (1 - a_0) / a : \alpha > 0 \} \), then for any \( m \geq m_0 \), \( \exists \alpha = \alpha(m) > 0 \) satisfying \( 2a_0 + 2ma = 2 \). Additionally, if \( h''(0) \in [0, -\lambda) \) then \( \lim_{\alpha \to 0^+} a_0 = \infty \), hence we can choose \( m_0 = 0 \).

**Proof of Theorem 3.3.** Let \( \lambda < 0 \). Let \( m_0 \) be the one provided by theorem 3.2 and let \( m \geq m_0 \). Let \( \alpha = \alpha(m) > 0 \) and \( \varphi(x), x \in [1, 1] \), be the ones prescribed by theorem 3.2. By extending \( \varphi(x) \) to \( x \in (-\infty, -1) \cup (1, \infty) \) as

\[
\varphi(x) = \begin{cases}
  \alpha^e(x+1)\sqrt{-\lambda} & x \leq -1, \\
  (-1)^m \alpha^e(1-x)\sqrt{-\lambda} & x > 1,
\end{cases}
\]

it is not difficult to see that \( \varphi \in H^2(\mathbb{R}) \) and it is a solution of equation (2). Uniqueness in theorem 3.1 and theorem 3.2 imply that \( \varphi \) is the only \( H^2(\mathbb{R}) \) solution to equation (2) with \( m \) zeroes and \( \varphi(-1) > 0 \). Also, \( \hat{\varphi} = -\varphi \) will be the only solution with \( m \) zeroes and \( \hat{\varphi}(-1) < 0 \).

**Proof of Theorem 3.4.** For each \( \alpha > 0 \) let \( a = a(\alpha) > 0 \) and let \( v \in C^2(\mathbb{R}) \) be the \( 4\alpha \)-periodic solution of

\[
\begin{cases}
  |v'| = \sqrt{-\lambda v^2 - h(v^2)} + \hat{h}(\alpha^2), & x \in \mathbb{R}, \\
  v(0) = 0 \text{ and } v'(a) = v'(-a) = 0, \\
  v'(0) > 0 \text{ and } v' = 0 \text{ only at isolated points},
\end{cases}
\]

which exists and is unique due to theorem 5.5. Corollary 5.9 implies that \( a \) is continuous strictly decreasing on \( \alpha \) and that \( \lim_{\alpha \to \infty} a = 0 \). Therefore, if \( m_0 \geq 0 \) is such that \( m_0 > \inf\{ 1/4a : \alpha > 0 \} \), then for any \( m \geq m_0 \), \( \exists \alpha > 0 \) satisfying \( 4ma = 1 \). For such \( \alpha \), theorem 5.5 implies that \( v \) has exactly \( 2m \) zeroes and solves equation (6). Additionally, if \( h'(0) < -\lambda \) then two things happen. First, \( \lim_{\alpha \to 0^+} a = \infty \), hence we can choose \( m_0 = 1 \). Second, since \( \lim_{x \to +\infty} h'(x) = +\infty \) there exists \( p_0 > 0 \) such that \( h'(p_0) = -\lambda \), in which case \( \varphi \equiv p_0 \) is a solution of (6) without zeroes and we can actually choose \( m_0 = 0 \).
5.3. **Real valued parameters.** In section 2 we mentioned that without loss of generality we can restrict our attention to \( \lambda \in \mathbb{R} \) and solutions \( \varphi \) that were real-valued. In this section we justify such consideration. We let \( V(x) = -1 \) for \( |x| \leq 1 \) and \( V(x) = 0 \) for \( |x| > 1 \). We assume \( h' : \mathbb{R}_+ \to \mathbb{R} \) is continuous strictly increasing, satisfying \( h'(0) \geq 0 \) and \( \lim_{z \to \infty} h'(z) = \infty \). Let us study the equation

\[
\begin{align*}
-\varphi'' + V(x)h'(|\varphi|^2)\varphi &= \lambda \varphi, \quad x \in \mathbb{R}, \\
\varphi(x) &\to 0 \text{ when } x \to \pm \infty,
\end{align*}
\]

(24)

**Lemma 5.11.** If there is \( \varphi : \mathbb{R} \to \mathbb{C} \), a non-trivial \( H^2(\mathbb{R}) \) solution of equation (24), then \( \lambda \in \mathbb{R} \).

**Proof.** Multiplying equation (24) by \( \overline{\varphi} \), integrating over \( \mathbb{R} \) and integrating by parts we obtain

\[
\int_{\mathbb{R}} (|\varphi'|^2 + V(x)h'(|\varphi|^2)|\varphi|^2)dx = \lambda \int_{\mathbb{R}} |\varphi|^2dx.
\]

Since \( \int |\varphi|^2 \neq 0 \) then \( \lambda \in \mathbb{R} \).

**Lemma 5.12.** If \( \varphi : \mathbb{R} \to \mathbb{C} \) is a non-trivial \( H^2(\mathbb{R}) \) solution of equation (24), then \( \alpha \varphi : \mathbb{R} \to \mathbb{R} \) for some constant \( \alpha \in \mathbb{C} \setminus \{0\} \).

**Proof.** Let us write \( \varphi(x) = \rho(x)e^{i\phi(x)} \), where \( \rho, \phi \) are real valued, they have the same regularity as \( \varphi \) whenever \( \varphi \neq 0 \) and \( \rho \in C^1(\mathbb{R}) \) (this last part is possible by imposing that \( \rho \) changes sign whenever \( \varphi = 0 \) and \( \varphi' \neq 0 \)). On any interval where \( \rho \neq 0 \) we have

\[
\varphi'' = \left( |\rho''| - (\phi')^2 \rho + i(2\phi' \rho' + \phi'' \rho) \right)e^{i\phi}.
\]

Since \( V(x), h(x), \lambda \in \mathbb{R} \), multiplying equation (24) by \( e^{-i\phi} \) and computing the imaginary part gives \( (2\phi' \rho' + \phi'' \rho) = 0 \), which implies that \( \rho^2 \phi' \) is constant on each interval where \( \rho \neq 0 \). If \( x < -1 \), equation (24) implies \( \varphi(x) = \gamma e^{-x\sqrt{-\lambda}} \) for some constant \( \gamma \in \mathbb{C} \), hence \( \phi'(x) = 0 \) for \( x < -1 \). Similarly, \( \phi'(x) = 0 \) for \( x > 1 \). For \( |x| \leq 1 \), after multiplying by \( \overline{\varphi'} \), taking real part and integrating, equation (24) gives that for some \( \beta \geq 0 \)

\[
|\varphi'|^2 = -\lambda|\varphi|^2 - h(|\varphi|^2) + \beta, \quad x \in [-1, 1],
\]

since \( \varphi = \rho e^{i\phi} \) this can be written as,

\[
(\rho')^2 + (\phi')^2 \rho^2 = -\lambda \rho^2 - h(\rho^2) + \beta, \quad x \in [-1, 1].
\]

(25)

Assume there is \( k \neq 0 \) such that \( I = \{ \rho^2 \phi' = k \} \) is non-empty. Since \( \rho^2 \phi' \) is constant in intervals where \( \rho \neq 0 \), then \( I \) is open and \( \rho(x) = 0 \) at the boundary of \( I \). Also \( I \subset [-1, 1] \) because \( \phi' = 0 \) for \( |x| > 1 \). In \( I \) equation (25) can be written as

\[
(\rho')^2 = -k^2/\rho^2 - \lambda \rho^2 - h(\rho^2) + \beta, \quad x \in I,
\]

which means that \( \rho'(x) \to \infty \) at the boundary of \( I \) (\( \rho(x) = 0 \) at the boundary of \( I \)). But \( \rho \in C^1(\mathbb{R}) \), therefore this cannot happen and we conclude that \( \rho^2 \phi' \equiv 0 \) in \( \mathbb{R} \). Equation (25) then becomes

\[
(\rho')^2 = -\lambda \rho^2 - h(\rho^2) + \beta, \quad x \in [-1, 1].
\]

We also have \( \rho \in C^1(\mathbb{R}) \) and

\[
\rho'' = -\lambda \rho, \quad |x| > 1.
\]
The solution $\rho$ of this equation is exactly the one constructed in theorem 3.3 and therefore $\rho = 0$ only at isolated points. Since $\rho^2\varphi' = 0$ in $\mathbb{R}$ then $\phi$ is piecewise constant. In particular $\varphi' = \rho e^{i\phi}$ in $\mathbb{R}\setminus\{\rho = 0\}$, but $\varphi \in C^1(\mathbb{R})$ and $\rho$ satisfies $\rho(x) \neq 0$ in $\{\rho = 0\}$ (see e.g. the proof of theorem 3.1, when proving that $\alpha \neq 0$ to conclude that $\beta > 0$), hence $e^{i\phi}$ has to be continuous across points where $\rho = 0$ and $e^{i\phi}$ has to be constant in $\mathbb{R}$. In summary, there exists a constant, namely $\alpha = e^{i\phi} \in \mathbb{C}\setminus\{0\}$, such that $\overline{\varphi} = \rho : \mathbb{R} \to \mathbb{R}$. 

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Received January 2018; revised July 2018.

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