VERY AMPLENESS AND PROJECTIVE NORMALITY ON CERTAIN CALABI-YAU
AND HYPERKÄHLER VARIETIES

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Abstract. In this article we produce new results on effective very ampleness and projective normality on certain $K_X$ trivial varieties. In the first part we produce an effective projective normality result on ample line bundles on regular fourfolds with trivial canonical bundle. In the second part we emphasize on the projective normality of powers of ample and globally generated line bundles on two classes of known examples (upto deformation) of projective hyperkähler varieties.

Introduction

Part 1: Regular, $K_X$ Trivial Varieties. Geometry of linear series on $K_X$ trivial varieties is a topic that has motivated a lot of research. The question of what multiple of an ample bundle is very ample was extensively studied by many mathematicians including Gallego, Oguiso, Peternell, Purnaprajna, Saint-Donat (see [9], [26], [27]). Saint-Donat proves the following theorem on a K3 surface (see [27]) which is defined as a smooth projective surface $S$ with $K_S = 0$ and $H^1(\mathcal{O}_S) = 0$.

Theorem A. Let $S$ be a smooth projective K3 surface and let $B$ be an ample line bundle on $S$. Then $B^\otimes n$ is very ample (in fact projectively normal) for $n \geq 3$.

Gallego and Purnaprajna proved the following generalization of Saint-Donat’s result on projective normality for smooth, projective, regular ($H^1(\mathcal{O}_X) = 0$) threefold with trivial canonical bundle (see [9]).

Theorem B. Let $X$ be a smooth, projective threefold with $K_X = 0$ and $H^1(\mathcal{O}_X) = 0$. Suppose $B$ be an ample line bundle on $X$. Then $B^\otimes n$ is projectively normal for $n \geq 8$. If $B^3 > 1$ then $A^\otimes n$ is projectively normal if $n \geq 6$.

In order to prove the theorem above, Gallego and Purnaprajna gave a classification theorem for a regular, $K_X$ trivial threefold that maps onto a variety of minimal degree by a complete linear series of an ample and globally generated line bundle. Varieties that appear as covers of varieties of minimal degree play an important role in the geometry of algebraic varieties. They are extremal cases in a variety of geometric situations from algebraic curves to higher dimensional varieties (see [9], [10], [11], [14]). In this article, we prove the following classification theorem where we study the situation when a smooth regular $K_X$ trivial fourfold $X$ maps to a variety of minimal degree by the complete linear system of an ample and base point free line bundle $B$.

Theorem 1. (See Theorem 2.3) Let $X$ be a smooth regular $K_X$ trivial fourfold. Let $\pi$ be the morphism induced by an ample and base point free line bundle $B$ on $X$ with $h^0(B) = r + 1$ and let $n$ be the degree of $\pi$. If $\pi$ maps $X$ to a variety of minimal degree $Y$ then $n \leq \frac{24(r - 1)}{r - 3}$ and one of the following happens.
(1) $Y = \mathbb{P}^4$.
(2) $Y$ is a smooth quadric hypersurface in $\mathbb{P}^5$.
(3) $Y$ is a smooth rational normal scroll of dimension 4 in $\mathbb{P}^6$ or $\mathbb{P}^7$ and $X$ is fibered over $\mathbb{P}^1$ and the general fibre is a smooth threefold $G$ with $K_G = 0$.
(4) $Y$ is a smooth rational normal scroll in $\mathbb{P}^r$ for $r \geq 8$ and $X$ is fibered over $\mathbb{P}^1$ and the general fibre is a three-fold $G$ with $K_G = 0$ and the degree $n$ of $\pi$ satisfies $2 \leq n \leq 18$.
(5) $Y$ is a singular four-fold which is either a triple cone over a rational normal curve or a double cone over the Veronese surface in $\mathbb{P}^5$.

This result can be thought of as an analogue to the classification theorem obtained by Gallego and Purnaprajna that we mentioned before. As a consequence of the above theorem and Fujita’s conjecture in dimension four that has been proved by Kawamata (see [16]), we are able to give an effective projective normality result and hence very ampleness result on smooth $K_X$ trivial fourfolds which can be thought of as a generalization of Theorem A and Theorem B.

**Theorem 2.** (See Theorem 2.4) Let $X$ be a smooth fourfold with trivial canonical bundle and let $A$ be an ample line bundle on $X$ then

(i) $nA$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 16$.

(ii) If $H^1(\mathcal{O}_X) = 0$ then $nA$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 15$.

We note that standard methods of Castelnuovo-Mumford regularity (see Lemma 1.1.5) and Theorem 1.3, [9] yields in the situation above the following; $nA$ satisfies projective normality for $n \geq 21$.

**Part 2: Hyperkähler Varieties.** Note that the definition of $K3$ surface is equivalent to having a holomorphic symplectic form on $S$. However in higher dimensions these two notions do not coincide which is clear from the fact that existence of a holomorphic symplectic form on a Kähler manifold demands that its dimension is even whereas there are examples of smooth projective algebraic varieties in odd dimensions with trivial canonical bundle and $H^1(\mathcal{O}_X) = 0$, for example smooth hypersurfaces of degree $n + 1$ in $\mathbb{P}^n$. So essentially we can have two different kinds of generalizations of a $K3$ surface. At this point we make a few definitions before stating a theorem of Beauville and Bogomolov that summarizes the importance of the study of the classes of varieties we just discussed.

**Definition 0.1.** A compact Kähler manifold $M$ of dimension $n \geq 3$ is called Calabi-Yau if it has trivial canonical bundle and the hodge numbers $h^{p,0}(M)$ vanish for all $0 < p < n$.

**Definition 0.2.** A compact Kähler manifold $M$ is called hyperkähler if it is simply connected and its space of global holomorphic two forms is spanned by a symplectic form.

The following theorem is due to Beauville-Bogomolov (see [1], [2]).

**Theorem C.** Every smooth projective variety with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ admits a finite cover isomorphic to a product of Abelian varieties, Calabi-Yau varieties and hyperkähler varieties.

Hence one can see that Calabi-Yau and hyperkähler varieties can be thought of as “building blocks” of varieties with trivial canonical bundle.

In the paper mentioned before (see [27]) Saint-Donat proves the following theorem for ample and base point free line bundles on $K3$ surfaces.
**Theorem D.** Let $S$ be a smooth projective $K3$ surface and let $B$ be an ample and base point free line bundle on $S$. Then

(i) $B^\otimes 2$ is very ample and $|B^\otimes 2|$ embeds $S$ as a projectively normal variety unless the morphism given by the complete linear system $|B|$ maps $S$, $2 : 1$ onto $\mathbb{P}^2$.

(ii) $B$ is very ample and $|B|$ embeds $S$ as a projectively normal variety unless the morphism $m$ given by the complete linear system $|B|$ maps $S$, $2 : 1$ onto $\mathbb{P}^2$ or to a variety of minimal degree.

Gallego and Purnaprajna proved the following generalization of Saint-Donat’s result (see [9]).

**Theorem E.** Let $X$ be a smooth, projective, regular, $K_X$ trivial threefold and let $B$ be an ample and base point free line bundle on $X$. Then

(i) $B^\otimes 3$ is very ample and $|B^\otimes 3|$ embeds $X$ as a projectively normal variety unless the morphism given by the complete linear system $|B|$ maps $X$, $2 : 1$ onto $\mathbb{P}^3$.

(ii) $B^\otimes 2$ is very ample and $|B^\otimes 2|$ embeds $X$ as a projectively normal variety unless the morphism given by the complete linear system $|B|$ maps $X$, $2 : 1$ onto $\mathbb{P}^3$ or to a variety of minimal degree.

They also proved that $4B$ is projectively normal on smooth, projective, regular, $K_X$ trivial fourfolds when the morphism induced by the complete linear series of an ample and globally generated line bundle $B$ is birational onto its image and $h^0(B) \geq 7$ (see Theorem 1.11, [9]). Niu proved an analogue of Theorem E in dimension four. In fact he proved a general result for smooth, projective, regular, $K_X$ trivial varieties in all dimensions (see [23]) with an additional assumption of $H^2(O_X) = 0$.

We see that it is a natural question to ask whether and to what extent these theorems generalize to the other class of higher dimensional analogues of $K3$ surfaces, namely hyperkähler varieties. Before stating our results we briefly recall the known examples of hyperkähler varieties.

There are many families of examples for Calabi-Yau varieties but only few classes of examples for hyperkähler varieties are known. Beauville first produced examples of two distinct deformation classes of compact hyperkähler manifolds in all even dimensions greater than or equal to 2 (see [1]). The first example is the Hilbert scheme $S^{[n]}$ of length $n$ subschemes on a $K3$ surface $S$. The second one is the generalized Kummer variety $K^n(T)$ which is the fibre over the 0 of an abelian variety $T$ under the morphism $\phi \circ \psi$ (see the diagram below)

$$
T^{[n+1]} \overset{\psi}{\longrightarrow} T^{(n+1)} \overset{\phi}{\longrightarrow} T
$$

where $T^{[n+1]}$ Hilbert scheme of length $n + 1$ subschemes on the abelian variety $T$, $T^{(n+1)}$ is the symmetric product, $\psi$ is the Hilbert chow morphism and $\phi$ is the addition on $T$. Two other distinct deformation classes of hyperkähler manifolds are given by O’Grady in dimensions 6 and 10 which appear as desingularizations of certain moduli spaces of sheaves over symplectic surfaces (see [24], [25]). All other known examples are deformation equivalent to one of these.

We prove structure theorems for two known deformation classes of polarized hyperkähler four, six, eight and tenfolds $(X, L)$ where $L$ maps to variety of minimal degree. We use the results to prove new results on very ampleness and projective normality for ample and globally generated line bundles. These are analogues of Saint-Donat’s result on $K3$ surfaces. We state below the results for dimension four in detail. Table 2 at the end of Section 3 gives the results for higher dimensions.
Our first result on hyperkähler varieties deformation equivalent to a Hilbert scheme of two points on a $K3$ surface is precisely as follows.

**Theorem 3.** (See Theorem 3.1.1) Suppose $L$ be a base point free line bundle on a projective hyperkähler manifold $X$ which is deformation equivalent to the Hilbert scheme of 2 points on a $K3$ surface $S$. Assume that the morphism given by $L$ is generically finite onto its image. Then
(i) The degree $d$ of the generically finite morphism given by $L$ is bounded above by 23.
(ii) If the morphism maps to a variety of minimal degree then the one of the following happens:
(a) $X$ maps $6 : 1$ to a quadric in $\mathbb{P}^5$.
(b) $X$ maps $8 : 1$ to a singular rational normal scroll of degree 6 in $\mathbb{P}^9$ which is the triple cone over the rational normal curve in $\mathbb{P}^6$.

The following is an analogous result for hyperkähler varieties deformation equivalent to a generalized Kummer variety.

**Theorem 4.** (See Theorem 3.1.4) Suppose $L$ be a base point free line bundle on a projective hyperkähler manifold $X$ which is deformation equivalent to a generalized Kummer variety of dimension four. Assume that the morphism given by $L$ is generically finite onto its image Then
(i) The degree $d$ of the generically finite morphism given by $L$ is bounded above by 23.
(ii) The morphism never maps $X$ to a variety of minimal degree.

With these classification theorems we proceed to prove generalizations of Saint-Donat’s theorem on the above classes of hyperkähler manifolds. We first state the result for hyperkähler varieties deformation equivalent to $K3^{[2]}$.

**Corollary 5.** (See Corollary 3.1.3) Let $X$ be a projective hyperkähler fourfold deformation equivalent to Hilbert scheme of two points on a $K3$ surface. Let $B$ be an ample and base point free line bundle on $X$. Then
(1) $B^\otimes n$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 4$.
(2) $B^\otimes n$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 3$ unless the complete linear series of $|B|$ maps $X$ to a variety of minimal degree, i.e, unless one of the two cases in Theorem 3 happens.

The result for generalized Kummer varieties is the following.

**Corollary 6.** (See Corollary 3.1.5) Let $X$ be a projective hyperkähler fourfold deformation equivalent to a generalized Kummer variety. Let $B$ be an ample and base point free line bundle on $X$. Then $B^\otimes n$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 3$.

At this point we recall the Fujita’s very ampleness conjecture which states that for a smooth projective fourfold $X$ with canonical bundle $K_X$, we have that $K_X + 6B$ is very ample for any ample line bundle $B$. Hence for $K_X$ trivial varieties we get that according to the conjecture, $6B$ is very ample. Here we prove that $3B$ is very ample if we take $B$ to be ample and globally generated.

There are examples of $X = S^{[2]}$, where $S$ is a $K3$ surface, that maps onto a variety of minimal degree by the complete linear series of an ample and globally generated line bundle where the degree of the morphism is 6 (see Example 3.1.6) and we believe that under the same hypothesis it can not map 8:1 onto a variety of minimal degree. The above theorems crucially use two key characteristics of a hyperkähler variety which are the existence of a primitive integral quadratic
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1. Preliminaries and Notations

Throughout this article, $X$ will always denote a smooth, projective variety over $\mathbb{C}$. $K$ or $K_X$ will denote its canonical bundle. We will use the multiplicative and the additive notation of line bundles interchangeably. Thus, for a line bundle $L$, $L^\otimes r$ and $rL$ are the same. We have used the notation $L^{-r}$ for $(L^*)^{\otimes r}$. We will use $L^r$ to denote the intersection product.

1.1. Background on projective normality. For a globally generated line bundle $L$ on a smooth projective variety $X$, we have the following short exact sequence.

$$0 \longrightarrow M_L \longrightarrow H^0(L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0 \quad (*)$$

We have the following necessary and sufficient condition for the $N_p$ property of an ample and base point free line bundle on $X$.

**Theorem 1.1.1.** Let $L$ be an ample, globally generated line bundle on $X$. If the group $H^1(\bigwedge^{p'+1} M_L \otimes L^{\otimes k})$ vanishes for all $0 \leq p' \leq p$ and for all $k \geq 1$, the $L$ satisfies the property $N_p$. If in addition $H^1(L^{\otimes r}) = 0$ for all $r \geq 1$, then the above is a necessary and sufficient condition for $L$ to satisfy $N_p$.

Since we are working over a field of characteristic zero, we will always show that $H^1(M_L^{\otimes p'+1} \otimes L^{\otimes k}) = 0$ in order to prove that $L$ satisfy the $N_p$ property.

We have made use of the following observation of Gallego and Purnaprajna (see for instance [9]) to show projective normality.

**Observation 1.1.2.** Let $E$ and $L_1, L_2, \ldots, L_r$ be coherent sheaves on a variety $X$. Consider the map $H^0(E) \otimes H^0(L_1 \otimes L_2 \otimes \ldots \otimes L_r) \xrightarrow{\psi} H^0(E \otimes L_1 \otimes \ldots \otimes L_r)$ and the following maps

$\begin{align*}
H^0(E) \otimes H^0(L_1) &\xrightarrow{\alpha_1} H^0(E \otimes L_1), \\
H^0(E \otimes L_1) \otimes H^0(L_2) &\xrightarrow{\alpha_2} H^0(E \otimes L_1 \otimes L_2), \\
&\vdots \\
H^0(E \otimes L_1 \otimes \ldots \otimes L_{r-1}) \otimes H^0(L_r) &\xrightarrow{\alpha_r} H^0(E \otimes L_1 \otimes \ldots \otimes L_r).
\end{align*}$

If $\alpha_1, \alpha_2, \ldots, \alpha_r$ are surjective then $\psi$ is also surjective.

The technique we use to show projective normality of an ample and globally generated line bundle on a variety is to use Koszul resolution to restrict the bundle on a smooth curve section and then showing the surjectivity of an appropriate multiplication map. It is worth mentioning that Koszul resolution is the special case of a particular complex, known as Skoda complex which we define below.
Definition 1.1.3. Let $X$ be a smooth projective variety of dimension $n \geq 2$. Let $B$ be a globally generated and ample line bundle on $X$.

(1) Take $n - 1$ general sections $s_1, \ldots, s_{n-1}$ of $H^0(B)$ so the intersection of the divisor of zeroes $B_i = (s_i)_0$ is a nonsingular projective curve $C$, that is $C = B_1 \cap \ldots \cap B_{n-1}$.

(2) Let $\mathcal{I}$ be the ideal sheaf of $C$ and let $W = \text{span}\{s_1, \ldots, s_{n-1}\} \subseteq H^0(B)$ be the subspace spanned by $s_i$. Note that $W \subseteq H^0(B \otimes \mathcal{I})$. For $i \geq 1$, define the Skoda complex $I_i$ as

$$
0 \rightarrow W \otimes B^{-(n-1)} \otimes \mathcal{I}^{-(n-1)} \rightarrow \cdots \rightarrow W \otimes B^{-1} \otimes \mathcal{I}^{i-1} \rightarrow \mathcal{I}^i \rightarrow 0
$$

where $\mathcal{I}^k$ stands for $\mathcal{I} \otimes k$, we have used the convention that $\mathcal{I}^0 = \mathcal{O}_X$ for $k \leq 0$.

In this article we have only used $I_1$ which is the following

$$
0 \rightarrow W \otimes B^{-(n-1)} \rightarrow \cdots \rightarrow W \otimes B^{-2} \rightarrow W \otimes B^{-1} \rightarrow \mathcal{I} \rightarrow 0.
$$

and it is just the Koszul resolution of $\mathcal{I}$. In fact, Lazarsfeld showed that the complex $I_i$ is exact for any $i \geq 1$ (see [18]).

Once we boil down our problem to a problem on curve, we use the following two results. The first one is a result of Green (see [11]).

Lemma 1.1.4. Let $C$ be a smooth, irreducible curve. Let $L$ and $M$ be line bundles on $C$. Let $W$ be a base point free linear subsystem of $H^0(C, L)$. Then the multiplication map $W \otimes H^0(M) \rightarrow H^0(L \otimes M)$ is surjective if $h^1(M \otimes L^{-1}) \leq \dim(W) - 2$.

The second one is known as Castelnuovo-Mumford lemma (see [22]).

Lemma 1.1.5. Let $L$ be a base point free line bundle on a variety $X$ and let $\mathcal{F}$ be a coherent sheaf on $X$. If $H^i(\mathcal{F} \otimes L^{-i}) = 0$ for all $i \geq 1$ then the multiplication map $H^0(\mathcal{F} \otimes L^0) \otimes H^0(L) \rightarrow H^0(\mathcal{F} \otimes L^{0+i})$ surjects for all $i \geq 0$.

Now we give some background on hyperkähler varieties.

1.2. Background on hyperkähler varieties. For the definition of a hyperkähler manifold see Definition 0.2. We start by the following theorem of Beauville and Fujiki (see [1] and [8]) which we have crucially used in our proofs.

Theorem 1.2.1. Let $X$ be a hyperkähler variety of dimension $2n$. There exists a quadratic form $q_X : H^2(X, \mathbb{C}) \rightarrow \mathbb{C}$ and a positive constant $c_X \in \mathbb{Q}_+$ such that for all $\alpha$ in $H^2(X, \mathbb{C})$, $\int_X \alpha^{2n} = c_X q_X(\alpha)^n$.

The above equation determines $c_X$ and $q_X$ uniquely if one assumes the following two conditions.

(I) $q_X$ is a primitive integral quadratic form on $H^2(X, \mathbb{Z})$;

(II) $q_X(\sigma, \bar{\sigma}) > 0$ for all $0 \neq \sigma \in H^{2,0}(X)$

Here $q_X$ and $c_X$ are called the Beauville form and Fujiki constant respectively.

Remark 1.2.2. The Beauville form and Fujiki constants are fundamental invariants of a hyperkähler variety. The following table gives the list of Beauville forms, Fujiki constants and the lattice structure induced on $H^2(X, \mathbb{Z})$ by the Beauville form on the first two classes of known examples of hyperkähler varieties, namely when $X = S^{[n]}$ (Hilbert scheme of $n$ points on $S$) where $S$ is a $K3$ surface and when $X = K^n(T)$ (generalized Kummer variety of dimension $2n$) where $T$ is an
Abelian variety.

| $X$      | $\dim(X)$ | $b_2(X)$ | $c_X$ | $(H^2(X, \mathbb{Z}), c_X)$            |
|----------|-----------|----------|-------|----------------------------------------|
| $S^{[n]}$| $2n$      | 23       | $\frac{(2n)!}{n!^2}$ | $H^2 \oplus -E_8 \oplus (-2(n-1))$ |
| $K^n(T)$ | $2n$      | 7        | $\frac{(2n)!}{n!^2(n+1)}$ | $H^2 \oplus (-2(n-1))$        |

Table 1

In this table the lattice $H$ is the standard hyperbolic plane, the lattice $-E_8$ is the unique negative definite even unimodular lattice of rank eight and $(i)$ is the rank 1 lattice generated by an element whose square is $i$.

Let us recall the following theorem which helps to find the explicit form of the Riemann-Roch theorem on hyperkähler varieties.

**Theorem 1.2.3.** (See [8], [12]) Let $X$ be a hyperkähler variety of dimension $2n$. Assume that $\alpha \in H^4_j(X, \mathbb{C})$ of type $(2j, 2j)$ on all small deformations of $X$. Then there exists a constant $C(\alpha) \in \mathbb{C}$ depending on $\alpha$ such that $\int_X \alpha \beta^{2n-2j} = C(\alpha) q_X(\beta)^{n-j}$ for all $\beta \in H^2(X, \mathbb{C})$.

**Remark 1.2.4.** As a consequence of the theorem above, we get the following form of the Riemann-Roch formula for an line bundle $L$ on a hyperkähler variety of dimension $2n$ (see [15]).

$$\chi(X, L) = \sum_{i=0}^{n} \frac{a_i}{(2i)!} q_X(c_1(L))^i$$

where $a_i = C(td_{2n-2i}(X))$. Here $a_i$’s are constants depending only on the topology of $X$.

**Remark 1.2.5.** Elingsrad-Gottsche-Lehn computes the rational constants of the Riemann roch expression for hyperkähler manifolds of deformation type $K^3[n]$ (See [7]) and Britze-Nieper computes the same for generalized Kummer varieties of dimension $2n$ (see [3]).

If $X$ is of $K^3[n]$ type we have that

$$\chi(L) = \binom{\frac{1}{2}q(L) + n + 1}{n}.$$

If $X$ is a generalized Kummer variety of dimension $2n$ we have that

$$\chi(L) = (n+1) \binom{\frac{1}{2}q(L) + n}{n}.$$

Now we are ready to give the proofs of our theorems.

2. **Proof of the Main Result on Regular Fourfolds with Trivial Canonical Bundle**

The main aim of this section is to prove results on effective very ampleness and projective normality on a four dimensional variety with trivial canonical bundle. We start with a general statement on projective normality and normal presentation.
Corollary 2.1. Let $X$ be a $n$-fold with trivial canonical sheaf. Let $B$ be an ample and base point free line bundle on $X$. Let $h^0(B) \geq n + 2$. Then $lB$ satisfies the property $N_0$ for all $l \geq n$. Moreover, if $X$ is Calabi-Yau, then $lB$ satisfies the property $N_1$ for all $l \geq n$.

Proof. Follows immediately from Theorem 2.3 and Theorem 3.4 of [21].

Now we want to find out what multiple of an ample line bundle is very ample on a four dimensional variety with trivial canonical bundle. We will use the Fujita freeness on four folds that has been proved by Kawamata in [16]. We begin with a lemma.

Lemma 2.2. Let $X$ be a fourfold with trivial canonical bundle. Let $A$ be an ample line bundle and let $B = nA$ for $n \geq 5$. Then the multiplication map $H^0(3B + kA) \otimes H^0(B) \rightarrow H^0(4B + kA)$ is surjective for $k \geq 1$.

Proof. We already know that $B$ is base point free by Kawamata’s proof of Fujita’s base point freeness theorem on fourfolds (see [16]). We prove the statement for $k = 1$. For $k > 1$ the proof is exactly the same.

Let $C$ be a smooth and irreducible curve section of the linear system $|B|$ and let $\mathcal{I}$ be the ideal sheaf of $C$ in $X$. We have the following commutative diagram with the two horizontal rows exact.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^0(B \otimes \mathcal{I}) \otimes H^0(3B + A) & \rightarrow & H^0(B) \otimes H^0(3B + A) & \rightarrow & V \otimes H^0(3B + A) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^0((4B + A) \otimes \mathcal{I}) & \rightarrow & H^0((4B + A)) & \rightarrow & H^0(4B + A|_C) & \rightarrow & 0
\end{array}
\]

Now we claim that the leftmost vertical map is surjective. Consider the following exact sequence (see Definition 1.1.3).

\[0 \rightarrow \bigwedge^3 W \otimes B^{-3} \rightarrow \bigwedge^2 W \otimes B^{-2} \rightarrow \bigwedge W \otimes B^{-1} \rightarrow \mathcal{I} \rightarrow 0\]

Tensor it with $4B + A$ to get the following exact sequence.

\[0 \rightarrow \bigwedge^3 W \otimes (B + A) \rightarrow \bigwedge^2 W \otimes (2B + A) \rightarrow \bigwedge W \otimes (3B + A) \rightarrow (4B + A) \otimes \mathcal{I} \rightarrow 0\]

That gives us two short exact sequences.

\[0 \rightarrow \text{Ker}(f_1) \rightarrow W \otimes (3B + A) \rightarrow (4B + A) \otimes \mathcal{I} \rightarrow 0\]

\[0 \rightarrow \bigwedge^3 W \otimes (B + A) \rightarrow \bigwedge^2 W \otimes (2B + A) \rightarrow \text{Ker}(f_1) \rightarrow 0\]

Taking long exact sequence of cohomology in the second sequence we get the following.

\[\bigwedge^2 W \otimes H^1(2B + A) \rightarrow H^1(\text{Ker}(f_1)) \rightarrow \bigwedge^3 W \otimes H^2(B + A)\]

Hence $H^1(\text{Ker}(f_1)) = 0$ since the other terms of the exact sequence vanish by Kodaira Vanishing. The long exact sequence of cohomology associated to the first sequence is the following.

\[W \otimes H^0(3B + A) \rightarrow H^0((4B + A) \otimes \mathcal{I}) \rightarrow H^1(\text{Ker}(f_1))\]
We showed that the last term is zero and hence we have the surjection of the map \( W \otimes H^0(3B + A) \to H^0(((4B + A) \otimes \mathcal{I})) \).

Since \( W \subseteq H^0(B \otimes \mathcal{I}) \) we have the surjection of the multiplication map \( H^0(B \otimes \mathcal{I}) \otimes H^0(3B + A) \to H^0((4B + A) \otimes \mathcal{I}) \).

In order to prove the lemma we are left to show that \( V \otimes H^0(3B + A) \to H^0((4B + A)|_c) \) surjects. Since we have the surjection of \( H^0(3B + A) \to H^0((3B + A)|_c) \), it is enough to show the surjection of \( V \otimes H^0(3B + A|_c) \to H^0((4B + A)|_c) \).

Using Lemma 1.1.4 it is enough to prove the inequality

\[ h^1(2B + A|_c) \leq \dim V - 2. \]

Now consider the exact sequence

\[
0 \longrightarrow 3 \wedge W \otimes B^{-2} \xrightarrow{f_3} 2 \wedge W \otimes B^* \xrightarrow{f_2} W \otimes \mathcal{O}_X \xrightarrow{f_1} B \otimes \mathcal{I} \longrightarrow 0.
\]

So we get the following two exact sequences.

\[
0 \longrightarrow \text{Ker}(f_1) \longrightarrow W \otimes \mathcal{O}_X \xrightarrow{f_1} B \otimes \mathcal{I} \longrightarrow 0
\]

\[
0 \longrightarrow 3 \wedge W \otimes B^{-2} \xrightarrow{f_3} 2 \wedge W \otimes B^* \xrightarrow{f_2} \text{Ker}(f_1) \longrightarrow 0
\]

The long exact sequence of cohomology associated to the second sequence gives

\[
2 \wedge W \otimes H^0(B^*) \longrightarrow H^0(\text{Ker}(f_1)) \longrightarrow 3 \wedge W \otimes H^1(B^{-2}).
\]

Hence \( H^0(\text{Ker}(f_1)) = 0 \) since \( H^1(B^{-2}) = 0 \) by Kodaira vanishing and \( H^0(B^*) = 0 \) since \( B^* \) is negative of an ample divisor.

Taking cohomology once more we have the following exact sequence

\[
2 \wedge W \otimes H^1(B^*) \longrightarrow H^1(\text{Ker}(f_1)) \longrightarrow 3 \wedge W \otimes H^2(B^{-2}).
\]

Hence \( H^1(\text{Ker}(f_1)) = 0 \) since the other terms of the exact sequence vanish by Kodaira Vanishing.

The long exact sequence of cohomology associated to the first sequence is the following.

\[
H^0(\text{Ker}(f_1)) \longrightarrow W \otimes H^0(\mathcal{O}_X) \longrightarrow H^0(B \otimes \mathcal{I}) \longrightarrow H^1(\text{Ker}(f_1))
\]

But the first and last terms are zero by Kodaira Vanishing and hence \( h^0(B \otimes \mathcal{I}) = \dim W \leq 3 \). Hence \( \dim V - 2 \geq h^0(B) - 5 \).

On the other hand the canonical bundle of \( C \) is given by \( 3B|_c \). Applying Serre-Duality it is enough to prove that \( h^0(B - A) \leq h^0(B) - 5 \) i.e. \( h^0(n - 1)A) \leq h^0(nA) - 5 \).

Applying Riemann Roch theorem we get that

\[
h^0(nA) = \frac{n^4}{24} A^4 + \frac{n^2}{24} A^2.c_2 + 2 - 2h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X).
\]

Similarly, \( h^0((n - 1)A) = \frac{(n - 1)^4}{24} A^4 + \frac{(n - 1)^2}{24} A^2.c_2 + 2 - 2h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X). \)

Subtracting we get that \( h^0(nA) - h^0((n - 1)A) = \frac{n^4 - (n - 1)^4}{24} A^4 + \frac{n^2 - (n - 1)^2}{24} A^2.c_2. \)
Now using a result of Miyaoka (see [20]), we have that \( A^2.c_2 \geq 0 \) which gives
\[
h^0(nA) - h^0((n - 1)A) \geq \frac{n^4 - (n - 1)^4}{24} A^4 \geq 5 \text{ if } n \geq 5 \text{ and hence we are done.}
\]

Now we give a classification theorem in which we classify the varieties which come as an image of a regular fourfold with trivial canonical bundle by an ample, globally generated line bundle with an additional property of being a variety of minimal degree.

**Theorem 2.3.** Let \( X \) be a regular four-fold with trivial canonical bundle. Let \( \pi \) be the morphism induced by an ample and base point free line bundle \( B \) on \( X \) with \( h^0(B) = r + 1 \) and let \( n \) be the degree of \( \pi \). If \( \pi \) maps \( X \) to a variety of minimal degree \( Y \) then \( n \leq \frac{24(r - 1)}{r - 3} \) and one of the following happens.

1. \( Y = \mathbb{P}^4 \).
2. \( Y \) is a smooth quadric hypersurface in \( \mathbb{P}^5 \).
3. \( Y \) is a smooth rational normal scroll of dimension 4 in \( \mathbb{P}^6 \) or \( \mathbb{P}^7 \) and \( X \) is fibered over \( \mathbb{P}^1 \) and the general fibre is a smooth threefold \( G \) with \( K_G = 0 \) and the degree \( n \) satisfies
   \[
   2 \leq n \leq \min\left\{6h^0(B|_G), \frac{24(r - 1)}{r - 3}\right\}.
   \]
   If in addition \( G \) is regular we have the following
   \[
   2h^0(B|_G) - 6 \leq n \leq \min\left\{6(h^0(B|_G) - 1), \frac{24(r - 3)}{r - 3}\right\}, \text{ if } n \text{ is even and}
   \]
   \[
   2h^0(B|_G) - 5 \leq n \leq \min\left\{6(h^0(B|_G) - 1), \frac{24(r - 3)}{r - 3}\right\}, \text{ if } n \text{ is odd.}
   \]
4. \( Y \) is a smooth rational normal scroll in \( \mathbb{P}^r \) for \( r \geq 8 \) and \( X \) is fibered over \( \mathbb{P}^1 \) and the general fibre is a three-fold \( G \) with \( K_G = 0 \) and the degree \( n \) of \( \pi \) satisfies \( 2 \leq n \leq 18 \).
5. \( Y \) is a singular four-fold which is either a triple cone over a rational normal curve or a double cone over the Veronese surface in \( \mathbb{P}^5 \).

**Proof.** We first prove the inequality. Using Riemann-Roch we can see that
\[
h^0(B) = \frac{1}{24} B^4 + \frac{1}{24} B^2.c_2 + 2
\]
and we also have that \( B^4 = n(r - 3) \) since \( Y \) is a variety of minimal degree. By Miyaoka’s result (see [20]) we have that \( B^2.c_2 \geq 0 \) and hence we have the inequality \( n \leq \frac{24(r - 1)}{r - 3} \).

We now describe the cases when \( Y \) is a smooth variety of minimal degree. We have that \( r \geq 4 \).

**Case 1.** If \( r = 4 \), we have that \( Y = \mathbb{P}^4 \).

**Case 2.** If \( r = 5 \), we have that codimension of \( Y \) is one and degree is 2 which implies that \( Y \) is a smooth quadric hypersurface.

**Case 3.** If \( r \geq 6 \), we have that \( Y \) is a smooth rational normal scroll and is hence fibered over \( \mathbb{P}^1 \). Let this map from \( Y \) to \( \mathbb{P}^1 \) be \( \phi \). Composing this with \( \pi \) we get a map \( \phi \circ \pi : X \to \mathbb{P}^1 \). Hence \( X \) is fibered over \( \mathbb{P}^1 \) and we have that the general fibre is a smooth threefold \( G \) with \( K_G = 0 \) by adjunction.

We first settle the case for \( r \geq 8 \). Let the general fibre of \( Y \) be denoted by \( F \) and that of \( X \) is denoted
by $G$. We have the following exact sequence of cohomology of line bundles on $X$.

$$0 \longrightarrow H^0(B(-G)) \longrightarrow H^0(B) \longrightarrow H^0(B \otimes O_G) \longrightarrow H^1(B(-G))$$

Now we claim that $H - F$ is a nef and big divisor in $Y$ where $H$ is a hyperplane section in $Y$. We have that $Y$ is $S(a_0, a_1, a_2, a_3)$ i.e., $Y$ is the image of $\mathbb{P}(E)$ where $E$ is the vector bundle $O(a_0) \oplus O(a_1) \oplus O(a_2) \oplus O(a_3)$, mapped to the projective space by $|O_{\mathbb{P}(E)}(1)|$. We have that $H - F$ is nef (in fact base point free). Now we compute $(H - F)^4 = H^4 - 4H^3 F$. We have $H_i = \sum_{i=0}^3 a_i H_i^3 F$ and $r = \sum_{i=0}^3 a_i + 3$.

So, $r \geq 8$ gives $\sum_{i=0}^3 a_i \geq 5$ which gives $(H - F)^4 > 0$ as $H$ is ample. Hence, $H - F$ is nef and big and consequently $B - G$ is nef and big as well.

So, by Kawamata-Viehwag vanishing, we have that $H^1(B - G) = 0$. Hence $\pi|_G$ is given by the complete linear system $|B|_G$.

Since $G$ maps to $F = \mathbb{P}^3$ we have that $h^0(B|_G) = 4$.

Now, the degree of $\pi$ is also the degree of $\pi|_G$ for a general fibre $G$. Hence by a result of Gallego and Purnaprajna (see [9], Theorem 1.6) we have that $2 \leq n \leq 18$.

Now for the cases $r = 5$ or $r = 6$ we again use the fact that degree of $\pi$ is equal to the degree of $\pi|_G$ and then use Riemann-Roch theorem on the threefold $G$ noticing the fact that $K_G = 0$ and that $B|_G$ is ample and base point free. This gives the upper bound $6h^0(B|_G)$ since we have that $B|_{G.c2} \geq 0$ (see [20]). The lower bound 2 is due to the fact that $G$ cannot be birational to $\mathbb{P}^3$. Now assuming $G$ is regular and hence Calabi-Yau we have that $h^0(B|_G) \geq 1/6 (B|_G)^3 + 1$ and hence $n \leq 6(h^0(B|_G) - 1)$.

The lower bound is obtained by Proposition 2.2, part (1) of [17].

Case 4. Suppose the image $Y$ of $X$ under the morphism defined by $|B|$ is a singular variety. If $Y$ is a cone over a smooth 3 dimensional scroll or a double cone over a smooth 2 dimensional scroll then the codimension of the singular locus of $Y$ is $> 2$. Then by [13] Proposition 2.1, part 2, the corresponding projective bundle $Y'$ gives a small resolution of singularities of $Y$. Hence it follows that there exist a birational morphism from a variety (the fibre product $X$ and $Y'$ over $Y$) to $X$ with the exceptional locus having no divisorial component which contradicts the factoriality of $X$. Hence $Y$ can be either a triple cone over a rational normal curve or a double cone over the Veronese surface in $\mathbb{P}^5$.

Now we prove our main result of this section using the first part of the previous theorem. We notice that since part (ii) requires a regular fourfold with trivial canonical bundle, we see that according to our definition, it holds for both hyperkähler and Calabi-Yau fourfolds in dimension four.

**Theorem 2.4.** Let $X$ be a four fold with trivial canonical bundle and let $A$ be an ample line bundle on $X$ then

(i) $nA$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 16$.

(ii) If $H^1(\mathcal{O}_X) = 0$ then $nA$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 15$.

**Proof of (i).** By the result of Kawamata (see [16]) we have that on a fourfold with trivial canonical bundle if $A$ is ample then $nA$ is base point free for $n \geq 5$. Now using CM lemma (Lemma 1.1.5) we can easily prove that $nA$ satisfies the property $N_0$ for $n \geq 21$.

If we set $B = 5A$ then $20A = 4B$ and it satisfies the property $N_0$ by Corollary 2.1.

Using Lemma 2.2, CM Lemma (Lemma 1.1.5) and Observation 1.1.2, we can see that $H^0(nkA) \otimes
Lemma 2.2
Lemma 1.1.5
and CM Lemma (Lemma 1.1.5).
Observation 1.1.2

Theorem 2.3

we need to show that

Using

.G
the rational normal curve in

P
3.1.
Hyperkähler fourfolds. Let

B
Let

3

Hyperkähler fourfolds.
and

G
we are done.
□

8 : 1

(b) \( X \) maps

which also follows from CM lemma (Lemma 1.1.5).

Therefore, by Observation 1.1.2 we need to show that \( H^0(26A) \otimes H^0(6A) \to H^0(32A) \) is surjective which also follows from CM lemma (Lemma 1.1.5).

Proof of (ii). Suppose \( H^1(\mathcal{O}_X) = 0 \). We just need to show that \( 15A \) satisfies the property \( N_0 \).

Let \( B = 5A \) which is ample and base point free (see [16]). Now By the result of Green (see [10]),

3B is projectively normal unless the image of the morphism induced by \( B \) is a variety of minimal degree. So, we have to show that the image of the morphism induced by \( 5A \) is not a variety of minimal degree.

Applying Riemann-Roch we get that \( h^0(5A) = \frac{625}{24} A^4 + \frac{25}{24} A^2 c_2 + 2 \geq 28 \).

Now suppose that the image is a variety of minimal degree. However, since the codimension of the image is \( \geq 24 \), we have that the image cannot be a quadric hypersurface or a cone over the veronese embedding of \( \mathbb{P}^2 \) in \( \mathbb{P}^5 \). Hence the image is a rational normal scroll (which might be singular). Let \( h^0(B) = r + 1 \). Hence the degree of the image is \( r - 3 \). Also, let the degree of the finite morphism given by \( |B| \) be \( n \).

We know that \( n \leq \frac{24(r - 1)}{r - 3} \) by Theorem 2.3. Using \( h^0(B) \geq 28 \) we have that \( r \geq 27 \) and hence \( n \leq 26 \).

Since the image of the morphism is a rational normal scroll of dimension 4 we can choose a general \( \mathbb{P}^3 = F \) and then take the pullback of the divisor \( F \) under the morphism induced by \( |B| \) and call it \( G \). The degree of the morphism restricted to \( G \) is again \( n \).

Since the degree of \( F \) in the image is 1 we have that \( n = B^3 . G = 125A^2 . G \geq 125 \) (since \( A \) is ample and \( G \) is effective) contradicting \( n \leq 26 \). Hence the image cannot be variety of minimal degree and we are done. \( \square \)

3. Proof of the Main Results on Hyperkähler Varieties

3.1. Hyperkähler fourfolds. First we prove the following theorem that studies the situation when a projective hyperkähler manifold of \( K3^{2l} \) type maps to a variety of minimal degree. This theorem will help us to get results on effective very ampleness.

Theorem 3.1.1. Suppose \( L \) be an base point free line bundle on a projective hyperkähler manifold \( X \) which is deformation equivalent to the Hilbert scheme of 2 points on a smooth, projective \( K3 \) surface \( S \). Assume the morphism given by the complete linear series \( |L| \), say \( \phi \), is generically finite. Then

(1) The degree \( d \) of \( \phi \) is bounded above by 23.
(2) If \( \phi \) maps to a variety of minimal degree then the one of the following happens.
(a) \( X \) maps \( 6 : 1 \) to a quadric in \( \mathbb{P}^5 \).
(b) \( X \) maps \( 8 : 1 \) to a singular rational normal scroll of degree 6 in \( \mathbb{P}^9 \) which is the triple cone over the rational normal curve in \( \mathbb{P}^6 \).
Proof. (1) Let $Y$ be the image of $\phi$. Then we have $L^4 = d \cdot \deg(Y)$. We have the following
\[ \deg(Y) \geq 1 + \text{codim}(Y) = 1 + h^0(L) - 1 = h^0(L) - 4. \]
Hence if $q_L(x) = x$, using the Riemann-Roch for $X$ (see Remark 1.2.5) and noting that all the higher cohomologies of $L$ vanish we have that $24x^2 \geq d(x^2 + 10x - 8) \implies (24 - d)x^2 - 10dx + 8d \geq 0$. We have that for $L$ is nef and big and therefore $q_L(x) = x > 0$. Hence we have that $d < 24$.

(2) Now consider $\phi$ maps to a variety of minimal degree. Then we have the following equation
\[ (24 - d)x^2 - 10dx + 8d = 0. \]
Considering that we have an even integer solution of $x$ for this equation and $0 < d < 24$ we have that the only possible choices are $d = 6, x = 2, h^0(L) = 6$ and $d = 8, x = 4, h^0(L) = 10$. Now the image is either a quadric (possibly singular) or a smooth rational normal scroll or a cone over a smooth rational normal scroll by Eisenbud, Harris’s classification of varieties of minimal degree (see [6]).

Suppose the image is not a quadric. We first claim that the image is not a smooth rational normal scroll. Suppose on the contrary the image is a smooth rational normal scroll. Since a smooth scroll admits a morphism to $\mathbb{P}^1$ we have a composed morphism from $X$ to $\mathbb{P}^1$. Now take the stein factorization of this morphism which has connected fibres and notice that since $X$ is smooth this further factors through a normalization. So we get a morphism from $X$ to a normal base of dimension 1 (hence smooth in this case) with connected fibres which contradicts Matsushita’s result on the fibre space structure of a holomorphic symplectic manifold (see [19]). Hence our claim is proved.

Now singular varieties of minimal degree are obtained by taking cones over smooth scrolls or over a Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. Now if the scroll is a single cone or a double cone over a smooth scroll then the codimension of singular locus is $> 2$. Hence the corresponding projective bundle produces a small resolution (see [13], Proposition 2.1) which contradicts the factoriality of $X$. Also the double cone over the veronese embedding is in $\mathbb{P}^7$ which again contradicts the fact that the image is non-degenerate in either $\mathbb{P}^5$ or $\mathbb{P}^9$. Hence the theorem is proved. \hfill \Box

The upper bound to the degree of a generically finite morphism from $X = K3^{[2]}$ by a complete linear series of a base point free line bundle has an interesting corollary on the secant lines of an embedding of a $K3$ surface which we state and prove next.

**Corollary 3.1.2.** Let $(S, L)$ be a polarized $K3$ surface such that $L$ is very ample. Consider the projective embedding of $S$ in $\mathbb{P}^{h^0(L)-1}$ and the closed subvariety of $Gr(2, h^0(L))$ consisting of lines that intersect the $K3$ surface at a subscheme of length at least 2. Then a general such line intersects $S$ at a subscheme of length $\leq 7$.

**Proof.** Given the above conditions we construct a generically finite morphism $f$ from $X =: \text{hilbert scheme of two points on } S$ to $Gr(2, h^0(L))$. Given a point on $X$ we take the length 2 subscheme it defines on $S$ and send it to the linear span of the length two subscheme inside $\mathbb{P}^{h^0(L)-1}$. Since a general such line doesn’t lie on the $S$, it intersects $S$ at finitely many points. So a general point in the image of the morphism $f$ has got finite fibers. Hence $f$ is a generically finite morphism. Also this morphism is given by the complete linear series $L - 2\delta$ where $2\delta$ is the class of the divisor in $S^{[2]}$ that parametrizes non-reduced subschemes of length 2 on the $K3$ surface $S$. By the above
theorem $\deg(f) \leq 23$. Now if a line intersects $S$ at $k$ points then the line has $\binom{k}{2}$ preimages under the morphism $f$. Hence for a general line $\binom{k}{2} \leq 23$ and hence $k \leq 7$. \hfill \square

Now we are ready to state our results on very ampleness and projective normality.

**Corollary 3.1.3.** Let $X$ be a projective hyperkähler fourfold deformation equivalent to Hilbert scheme of two points on a smooth, projective $K3$ surface. Let $B$ be an ample and base point free line bundle on $X$. Then

1. $B^\otimes n$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 4$.
2. $B^\otimes n$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 3$ unless the complete linear series of $|B|$ maps $X$ to a variety of minimal degree, i.e., unless one of the two cases in Theorem 3.1.1 happens.

**Proof.** (1) We notice that from the Riemann-Roch formula on ample line bundles on $X$ that $h^0(B) \geq 6$ and hence the proof follows by Theorem 2.3, [21].

(2) We show the surjectivity of the multiplication map $H^0(3B) \otimes H^0(B) \rightarrow H^0(4B)$. Choose a smooth threefold section $T$ of the ample and base point free line bundle $B$. We have the following commutative diagram.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^0(3B) \otimes H^0(O_X) & \rightarrow & H^0(3B) \otimes H^0(B) & \rightarrow & H^0(3B) \otimes H^0(K_T) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^0(3B) & \rightarrow & H^0(4B) & \rightarrow & H^0(4K_T) & \rightarrow & 0
\end{array}
\]

Hence it is enough to prove that right most vertical map surjects. However since we have the surjection from $H^0(3B)$ to $H^0(3B|T) = H^0(3K_T)$, it is enough to show the surjection of $H^0(3K_T) \otimes H^0(K_T) \rightarrow H^0(4K_T)$.

Now we appeal to [10], p. 1089, (3) to see that the required map surjects unless $X$ is mapped to a variety of minimal degree by $|B|$. \hfill \square

Now we give analogous theorems for a hyperkähler variety deformation equivalent to a generalized Kummer variety of dimension four.

**Theorem 3.1.4.** Let $X$ be a projective hyperkähler fourfold deformation equivalent to a generalized Kummer variety. Let $L$ be a base point free line bundle on $X$. Assume the morphism given by the complete linear system $|L|$ is, say $\phi$ is generically finite. Then

1. The degree $d$ of $\phi$ is bounded above by 23.
2. $\phi$ will never map to a variety of minimal degree.

**Proof.** As in the proof of Theorem 3.1.1, we get $(72 - 3d)x^2 - 18dx + 8d \geq 0$ using Riemann-Roch (see Remark 1.2.5) where $q_X(L) = x$. Part (1) follows from the fact that this inequality forces $d \leq 23$ since $x$ is a positive even integer. Part (2) follows from the fact that $(72 - 3d)x^2 - 18dx + 8d = 0$ has no even integer solution for any $1 \leq d \leq 23$. \hfill \square

As before, the theorem above gives the following corollary.
Corollary 3.1.5. Let $X$ be a projective hyperkähler fourfold deformation equivalent to a generalized Kummer variety. Let $B$ be an ample and base point free line bundle on $X$. Then $B^{\otimes n}$ is very ample and embeds $X$ as a projectively normal variety for $n \geq 3$.

Proof. The proof follows along the same lines as Corollary 3.1.2 with the observation that a generalized Kummer variety does not map to a variety of minimal degree by an ample and base point free complete linear system. □

The following example is taken from the survey article of Debarre (see [5]). It gives an example of an ample and globally generated line bundle on a hyperkähler variety of the form $K3^{[2]}$ that maps it 6:1 onto a variety of minimal degree.

Example 3.1.6 Let $(S, L)$ be a polarized $K3$ surface with $\text{Pic}(S) = \mathbb{Z}L$ and $L^2 = 4$. Then $L$ is very ample and consequently we get a morphism $\phi : S^{[2]} \rightarrow G(2, 4)$ to the Grassmannian. Moreover, the pullback of the Plücker line bundle on the Grassmannian has class $L_2 - \delta$. Therefore, if $(S, L)$ is general then it contains no line and consequently $\phi$ will be finite of degree $6$. This gives an example of a hyperkähler variety of the form $K3^{[2]}$ getting mapped 6:1 onto a variety of minimal degree by the complete linear series of an ample and globally generated line bundle.

3.2. Higher dimensional calculations. In this section, we carry out similar computations on hyperkähler six, eight and tenfolds which are deformation equivalent to $K3^{[n]}$ or a generalized Kummer variety. More precisely, we want to figure out whether $B^{\otimes 2n-1}$ is very ample for an ample and globally generated line bundle $B$. As before, first we will find out $q_X(B)$ and the degree of the morphism induced by the complete linear system of $|B|$ if it maps to a variety of minimal degree. Of course we will get an affirmative answer if it never maps to a variety of minimal degree.

Let $X$ be a hyperkähler variety deformation equivalent to $K3^{[n]}$. Recall that we proved in Theorem 3.1.1 that for an ample and basepoint free line bundle $B$ on a hyperkähler variety deformation equivalent to $K3^{[2]}$, $3B$ is very ample unless the morphism induced by the complete linear series of $B$ maps it 6:1 or 8:1 onto $S(0, 0, 0, 2)$ or $S(0, 0, 0, 6)$ respectively where $S(0, 0, ..., 0, r)$ is the variety obtained by taking cones over a rational normal curve of degree $r$.

Calculations similar to the proof of that theorem shows that if the complete linear system of an ample and globally generated line bundle $B$ maps onto a variety of minimal degree then we have the following equation

$$c_X.x^n = d\left(\left(\frac{x+n+1}{n}\right) - 2n\right)$$

where $x = q_X(B)$, $d$ is the degree of the morphism and $c_X = \frac{(2n)!}{n!2^n}$. As before, the degree of the morphism $d$ satisfies $1 \leq d \leq (2n)! - 1$. We are trying to find out the positive even integer solutions of $x$, $B^{\otimes 2n-1}$ will be very ample if there is none.

Similarly, if $X$ is a hyperkähler variety deformation equivalent to a generalized Kummer variety $K^n(T)$ then we have to work with

$$c_X.x^n = d\left((n+1)\left(\frac{x+n}{n}\right) - 2n\right).$$
\[ c_X = (n + 1) \frac{(2n)!}{n! 2^n} \] in this case.

Similar arguments as in the proof of the last part of Theorem 3.1.1 shows that if the morphism induced by the ample and globally generated line bundle on an hyperkähler variety of one of the two above types maps to a variety of minimal degree, then the embedded variety will be obtained by taking cones over a rational normal curve.

We ran a computer program using Python to find those solutions for \( X \) which is a hyperkähler six, eight or tenfold deformation equivalent to \( K^n[2] \) or a generalized Kummer variety \( K^n(T) \). The following table is the summary of the results we have obtained. Here \( 2n \) is the dimension of the variety, \( d \) is the degree of morphism induced by the complete linear series of an ample and globally generated line bundle \( B \), \( x = q_X(B) \), \( r \) is the degree of the embedded variety. If all the entries of \( d \), \( x \) and \( r \) are ‘—’ in a row, that means it will never map to a variety of minimal degree.

| Deformation Type | \( d \) | \( x \) | \( r \) | Is \((2n-1)B\) projectively normal? |
|------------------|-------|-------|-------|-------------------------------|
| \( K^n[2] \)     | 6     | 2     | 2     | Yes unless it maps 6:1 onto a quadric in \( \mathbb{P}^5 \) |
|                  | 8     | 4     | 6     | Yes unless it maps 8:1 onto \( S(0, 0, 0, 6) \) |
| \( K^n[3] \)     | 30    | 2     | 4     | Yes unless it maps 30:1 onto \( S(0, 0, 0, 0, 4) \) |
| \( K^n[4] \)     | 240   | 2     | 7     | Yes unless it maps 240:1 onto \( S(0, 0, 0, 0, 0, 0, 7) \) |
| \( K^n[3](T) \)  | —     | —     | —     | Yes |
| \( K^n[5](T) \)  | —     | —     | —     | Yes |
| \( K^n[6](T) \)  | —     | —     | —     | Yes |
| \( K^n[7](T) \)  | —     | —     | —     | Yes |

Table 2

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