Axionic symmetry gaugings in $\mathcal{N} = 4$ supergravities and their higher-dimensional origin

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ABSTRACT: We study the class of four-dimensional $\mathcal{N} = 4$ supergravities obtained by gauging the axionic shift and axionic rescaling symmetries. We formulate these theories using the machinery of embedding tensors, characterize the full gauge algebras and discuss several specific features of this family of gauged supergravities. We exhibit in particular a generalized duality between massive vectors and massive two-forms in four dimensions, inherited from the gauging of the shift symmetry. We show that these theories can be deduced from higher dimensions by a Scherk–Schwarz reduction, where a twist with respect to a non-compact symmetry is required. The four-dimensional generalized duality plays a crucial role in identifying the higher-dimensional ascendent.

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1. Introduction

The effective field theories describing the low-energy dynamics of typical string and M-theory compactifications are plagued with massless scalar fields that hamper any attempt to contact four-dimensional phenomenology. One way of eliminating some of these unwanted massless scalars is the introduction of fluxes in the internal compactification space (see [1] for a review). From the effective field theory perspective, turning on fluxes corresponds to a “gauging” of the original theory obtained without fluxes. The term gauging refers to the fact that a subgroup of the global duality symmetry of the original theory is promoted to a local gauge symmetry. Simultaneously, part of the original Abelian gauge symmetry is promoted to a non-Abelian one and various fields, including the scalars, acquire minimal couplings to the gauge fields. The connection with the moduli stabilization issue stems from the fact that for extended supergravity theories, the only way to generate a potential is through the gauging. The resulting theories are known as gauged supergravities (see [2] for a concise review).

One can envisage a bottom-up approach to the problem of moduli stabilization where instead of looking for a specific higher-dimensional background whose corresponding low-energy effective theory is phenomenologically viable, one first constructs a gauged supergravity theory with the required phenomenological properties and then attempts to engineer it from higher dimensions. Such a programme was initiated in [3] and models were proposed there exhibiting full moduli stabilization.
Evidently such a programme depends crucially on having a good picture of the “landscape” of possible gauged supergravity theories. Therefore, a problem of paramount importance is to classify and describe all possible gaugings of supergravity in diverse spacetime dimensions and with various amounts of supersymmetry. This problem was tackled in a series of publications [4, 5, 6, 7, 8] where the general method of embedding tensors was developed.

Ultimately, however, one would like to know that the effective theory constructed in the bottom-up approach is consistent, i.e. can be embedded in a certain string or M-theory setup. Although for many classes of gauged supergravities their higher-dimensional realization is known, for the generic gauged supergravity there is no recipe – not even guaranteed existence of a higher-dimensional origin. Presumably such an endeavor would require a better understanding of the classes of string backgrounds dubbed non-geometric, whose significance has recently been investigated in the framework of flux compactifications and supergravity theories.

Here, our aim is to contribute in this direction by analyzing a family of four-dimensional \( \mathcal{N} = 4 \) gauged supergravities and explain how they can be obtained from string theory or higher-dimensional supergravity theories. Gauged supergravity with \( \mathcal{N} = 4 \) supersymmetry was the arena of [3] since string theories with 16 supercharges are the starting point of a variety of realistic string constructions. Recently the most general theory of this type was explicitly constructed in [9] using the formalism of embedding tensors. We will use this formalism to study in detail the specific class of \( \mathcal{N} = 4 \) theories obtained by gauging the axionic symmetries, namely the axionic shifts and rescalings. Consistency requires that several \( SO(6,6) \) directions are also gauged. The general structure of the gauge algebra is systematically worked out and it exhibits the following characteristic property: it is non-flat contrary to what happens in more conventional gaugings.

In order to uncover the higher-dimensional origin of these gaugings one needs to perform a Scherk–Schwarz reduction of heterotic supergravity (or of the common sector in general) on a torus. The crucial ingredient here is a twist by a non-compact duality symmetry of ten-dimensional supergravity.

The identification of the theory obtained from the reduction with the gauged supergravity under consideration is not at all straightforward. It relies heavily on a duality between massive vectors and massive two-forms in four dimensions. This duality is a necessary extension of the more standard duality between a massless two-form and an axion scalar field in four dimensions: it incorporates the Stückelberg-like terms that are generated by the axionic gauging.

The organization of this paper is as follows. Section 2 is devoted to a reminder of gauged [10, 11, 12] \( \mathcal{N} = 4 \) supergravity in \( D = 4 \) [13, 14, 15]. We emphasize the approach of the embedding tensor following Refs. [8, 9, 16]. In Sec. 3 we specialize to the so-called electric gaugings and we analyze a particular class of algebras for which we elaborate on the corresponding gauged supergravity. These theories admit two equivalent formulations related by a duality between massive vectors and massive two-forms in four dimensions. In Sec. 4 we move on to higher dimensions. Our aim is to analyze the ten-dimensional origin of the four-dimensional theory constructed in Sec. 3. We show that it can be obtained
using a generalized Scherk–Schwarz reduction of heterotic $\mathcal{N} = 1$ $D = 10$ supergravity. Furthermore, we comment on the higher-dimensional origin of other classes of $\mathcal{N} = 4$ gaugings. In Sec. 5 we present our conclusions and discuss some open problems.

2. Reminder on $\mathcal{N} = 4$ gauged supergravities in $D = 4$

In this section we review $\mathcal{N} = 4$ gauged supergravity following Ref. [9]. After some general remarks on the possible gauge algebras and the constraints on the gauging parameters we present for reference the most general bosonic Lagrangian.

2.1 Gauge algebras and the embedding tensor

Four-dimensional $\mathcal{N} = 4$ supergravity has a very restricted structure. It contains generically the gravity multiplet and $n$ vector multiplets. The bosonic sector of the theory consists of the graviton, $n + 6$ vectors and $2 + 6n$ scalars. In the ungauged version, the gauge group is Abelian, $U(1)^{6+n}$, and there is no potential for the scalars. Interactions are induced upon elimination of the auxiliary fields. They affect the scalars whose non-linearities are captured by a universal coset manifold [17, 18]

\[ \mathcal{M} = \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, n)}{SO(6) \times SO(n)}. \]  

All fields are non-minimally coupled to the Abelian vectors. The gauge kinetic terms have scalar-field-dependent coefficients, whereas the action is at most quadratic in the gauge-field strengths with no explicit dependence on the gauge potentials. For this reason, the $SL(2, \mathbb{R}) \times SO(6, n) \subset Sp(12+2n, \mathbb{R})$ symmetry of the scalar manifold is globally realized as a U-duality symmetry. Although the scalar manifold survives any deformation of the plain theory triggered by gaugings, the U-duality is broken as a consequence of the introduction of non-Abelian field strengths and minimal couplings, all of which depend explicitly on the gauge potentials.

The duality group acts as a symmetry of the field equations and the Bianchi identities of the gauge fields. In the standard formulation of supergravity only a subgroup of it is realized off-shell as a genuine symmetry of the Lagrangian. This includes the $SO(6, n)$ plus a two-dimensional non-semi-simple subalgebra of $SL(2, \mathbb{R})$ generated by axionic shifts and axionic rescalings. The third transformation in $SL(2, \mathbb{R})$, corresponding to the truly electric-magnetic duality, is an on-shell symmetry which relates different Lagrangians associated to different choices of symplectic frames.

The only known deformation of $\mathcal{N} = 4$ supergravity compatible with supersymmetry is the gauging. This consists in transmuting part of the $U(1)^{6+n}$ local symmetry into an non-Abelian gauge symmetry, or equivalently in promoting part of the global U-duality symmetry to a local symmetry, using some of the available vectors. This operation should not alter the total number of propagating degrees of freedom, as required e.g. by supersymmetry.

It is possible to parameterize all gaugings of $\mathcal{N} = 4$ supergravity by using the so-called embedding tensor. The latter describes how the gauge algebra is realized in terms of the
U-duality generators. It captures all possible situations, including those where some of the gauge fields are the magnetic duals of the vectors originally present in the Lagrangian, as well as the option of gauging the duality rotation between electric and magnetic vectors, which, as already stressed, appears only as an on-shell symmetry. In this procedure, one doubles the number of vector degrees of freedom, keeping however unchanged the number of propagating ones thanks to appropriate auxiliary fields. One therefore avoids any choice of symplectic frame until a specific gauging is performed. We will not elaborate on these general properties but summarize the structure of the embedding tensor that we will use in the ensuing. We refer the reader to [8, 9, 16] for further information on this subject.

The $n+6$ electric vector fields $A^{M+}$, $M = 1, \ldots, 6+n$ belong to the fundamental vector representation of $SO(6, n)$. Their magnetic duals $M^{-}$ form also a vector of $SO(6, n)$, but carry opposite charge with respect to the $SO(1, 1) \subset SL(2, \mathbb{R})$ that generates the axionic rescalings.

The $SL(2, \mathbb{R})$ algebra is generated by $S_{\alpha\beta} = S_{\beta\alpha}$, $\alpha, \beta \in \{+,-\}$, which obey the following commutation relations:

$$[S_{\alpha\beta}, S_{\gamma\delta}] = -\epsilon_{\alpha\gamma} S_{\beta\delta} - \epsilon_{\beta\delta} S_{\alpha\gamma} - \epsilon_{\alpha\delta} S_{\beta\gamma} - \epsilon_{\beta\gamma} S_{\alpha\delta}$$

(2.2)

with $\epsilon^{+-} = 1 = \epsilon_{+-}$. In the vector representation $(S_{\gamma\delta})^\beta = \epsilon_{\gamma\delta} \delta_\beta^\alpha + \epsilon_{\delta\alpha} \delta_\gamma^\beta$ and they explicitly read

$$S_{++} = S^{--} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad S_{+-} = -S^{+-} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_{--} = S^{++} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}.$$  

(2.3)

The axion $a$ and the dilaton $\phi$ form a complex scalar $\tau = a + i e^{-2\phi}$, which parameterizes the $SL(2, \mathbb{R})/U(1)$ coset. We define

$$M_{\alpha\beta} = \frac{1}{\text{Im} \tau} \begin{pmatrix} |\tau|^2 \text{Re} \tau \\ \text{Re} \tau \end{pmatrix}$$

(2.4)

and we denote by $M^{\alpha\beta}$ its inverse. The action of

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

(2.5)

is linear on $M$: $M \rightarrow gMg^T$ while it acts on $\tau$ as a M"obius transformation: $\tau \rightarrow a \tau + b / c \tau + d$. Therefore, $S^{++}$ generates the axionic shifts, $S^{+-}$ the axionic rescalings, whereas the electric-magnetic duality is generated by $S^{--}$. In this basis, $(\{A^{M+}\}, \{A^{M-}\})$ form a doublet of $SL(2, \mathbb{R})$ with diagonal $S^{+-}$ and correspondingly the rescaling charges are $+1$ and $-1$.

The $SO(6, n)$ is generated by $T_{MN} = -T_{NM}$, $M, N \in \{1, \ldots, 6+n\}$ obeying

$$[T_{KL}, T_{JM}] = \eta_{LJ} T_{KM} + \eta_{KM} T_{LJ} - \eta_{KJ} T_{LM} - \eta_{LM} T_{KJ}.$$  

(2.6)

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1Since with the present conventions for $\epsilon_{\alpha\beta}, \epsilon_{\alpha\alpha} \epsilon^{\beta\beta} = \delta_\alpha^\beta$, we can raise and lower $\alpha$-indices unambiguously as follows: $A_\alpha = A^\beta \epsilon_{\beta\alpha}$ and $B^{\alpha} = \epsilon_{\alpha\beta} B_\beta$. This leads to $A_+ = -A_-$ and $A_- = A^+$. In particular, $S_{++} = S^{--}$, $S_{+-} = -S^{+-}$ and $S_{--} = S^{++}$.
with $\eta_{LI}$ being the $SO(6, n)$ metric. In the fundamental representation the generators read: $(T_{KL})^J_I = \eta_{KI}\delta_J^L - \eta_{LI}\delta_J^K$.

The $6n$ scalars coming from the $n$ vector multiplets live on the $SO(6, n)/(SO(6) \times SO(n))$ coset and can be parameterized by a symmetric matrix $M$ of elements $M_{MN}$. Introducing vielbeins $V = (V_M^a, V_N^a)$ with $m = 1, \ldots, 6$ and $a = 1, \ldots, n$ we can write $M = VV^T$ and define the fully antisymmetric tensor

$$M_{MNPQRS} = \epsilon_{mnpqrs}V^m_MV^p_NV^q_PV^r_QV^s_R$$

(2.7)

that appears in the scalar potential. We will denote by $M^{MN}$ the inverse of $M_{MN}$.

The gauging of $\mathcal{N} = 4$ supergravity proceeds by selecting a subalgebra of $SL(2, \mathbb{R}) \times SO(6, n)$ generated by some linear combination of $\{S_{\alpha\beta}, T_{KL}\}$. The coefficients of this combination are the components of the embedding tensor and are subject to various constraints discussed in detail in the aforementioned references. In summary, this tensor belongs a priori to the $(2 \times 3, \text{Vec}) + (2, \text{Vec} \times \text{Adj})$ of $SL(2, \mathbb{R}) \times SO(6, n)$. However, there are linear constraints resulting from the requirement that the commutator provides an adjoint action and that supersymmetry is preserved. These reduce the representation content of the embedding tensor to $(2, \text{Vec})+(2, \text{Ant})^3$. Furthermore, there are quadratic constraints guaranteeing the closure of the gauge algebra – Jacobi identity. Putting everything together, one finds that the admissible generators of the gauge algebra are of the form

$$\Xi_{\alpha L} = \frac{1}{2} \left( f_{\alpha LMN} T_{MN} - \eta_{PQ} \xi_{\alpha P} T_{QL} + \epsilon^{\gamma\beta} \xi_{\alpha L} S_{\gamma\alpha} \right)$$

(2.8)

where $f_{\alpha LMN}$ are fully antisymmetric in $L, M, N$ and $\xi_{\alpha L}$ satisfy

$$(i) \quad \eta_{MN} \xi_{\alpha M} \xi_{\beta N} = 0 \quad \forall \alpha, \beta.$$  

(2.9)

These parameters characterize completely the gauging.

The set of consistency conditions is completed as follows:

$$(ii) \quad \eta^{MN} \xi_{(\alpha M} f_{\beta)NIJ} = 0,$$

(2.10)

$$(iii) \quad \epsilon^{\alpha\beta} (\xi_{\alpha I} \xi_{\beta J} + \eta^{MN} \xi_{\alpha M} f_{\beta NIJ}) = 0$$

(2.11)

and

$$\begin{align*}
(iv) \quad \eta^{MN} f_{\alpha M[I} f_{J\beta KL]} &= \frac{1}{2} \xi_{\alpha[I} f_{J\beta KL]\gamma} + \frac{1}{6} \epsilon_{\alpha\beta} \epsilon^{\gamma\delta} \xi_{\gamma I} f_{\delta JKL} \\
&\quad - \frac{1}{2} \eta^{MN} \xi_{\alpha M} f_{\beta N[JK} \eta_{L]\gamma] I} - \frac{1}{6} f_{\alpha M[I} f_{\beta JKL} \xi_{\gamma I},

(2.12)
\end{align*}$$

where $[]$ and $(.)$ stand for antisymmetrization and symmetrization with respect to different indices belonging to the same family (e.g. $[L\beta N] = 1/2(L\beta N - N\beta L)$).

$^2$Indices $M, N, \ldots$ are lowered and raised with $\eta_{IJ}$ and $\eta^{KM}$ (inverse matrix).

$^3$This is actually the minimal set of constraints that one can consistently impose and they also guarantee that a Lagrangian exists propagating the correct number of degrees of freedom, as one learns from general studies on gaugings of maximal supergravities [4, 5, 6, 7, 8].
Several comments are in order here. A general gauging is manifestly expressed in terms of $2 \times (6 + n) + 2 \times (6 + n)(5 + n)(4 + n)/6$ parameters subject to four conditions (i, ii, iii, iv). The gauge algebra, as defined by this set of parameters, is characterized by the commutation relations of the subset of independent $\Xi_{\alpha L}$’s. These generators are indeed constrained (they satisfy e.g. $\Xi_{(\alpha L} \xi_{\beta])M} \eta^{LM} = 0$ as a consequence of their definition (2.8) and Eqs. (2.9), (2.10), (2.11)) as they should since no more that $6 + n$ vectors can propagate. The structure constants of the gauge algebra are not directly read off from the $f_{\alpha LMN}$’s, which are not necessarily structure constants of some algebra. They can, however, be expressed in terms of the $f_{\alpha LMN}$’s and $\xi_{\alpha M}$’s and describe a variety of situations which capture simple, semi-simple or even non-semi-simple examples. For all of these, $\eta_{MN}$ always provides an invariant metric although the Cartan–Killing metric of the corresponding gauge algebra can be degenerate.

The duality phases of de Roo and Wagemans [19, 20, 21] are also captured by the present formalism when the $\xi_{\alpha M}$’s are absent \footnote{For vanishing $\xi_{\alpha M}$’s, Eq. (2.12) is the only constraint; it is an ordinary Jacobi identity when $\alpha = \beta$.}, as relative orientations of $f^+_{LMN}$ with respect to $f^-_{LMN}$ for each simple component [9]. We will not elaborate any longer on the general aspects of the embedding tensor and the variety of physical possibilities and for further details we will refer the reader to the already quoted literature.

2.2 Lagrangian formulation

The bosonic Lagrangian corresponding to the most general $N = 4$ gauging was presented in [9]. For the sake of completeness we reproduce this result here and provide several comments. The Lagrangian consists of a kinetic term, a topological term, and a potential for the scalars:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{top}} + \mathcal{L}_{\text{pot}}.$$ (2.13)

The kinetic term reads

$$e^{-1} \mathcal{L}_{\text{kin}} = \frac{1}{2} R + \frac{1}{16} D_{\mu} M_{MN} D^{\mu} M^{MN} - \frac{1}{4(4\pi)^2} D_{\mu} \tau D^{\mu} \tau - \frac{1}{4} e^{-2\phi} M_{M} H_{\mu\nu}^M H_{\kappa\lambda}^N + 8 a \eta_{MN} \epsilon^{\mu\nu\kappa\lambda} H_{\mu}^{M} H_{\nu}^{N}.$$ (2.14)

with the covariant derivatives defined as

$$D_{\mu} M_{MN} = \partial_{\mu} M_{MN} + 2g A_{\mu}^{P} \Theta_{\alpha P \{M}^{Q} M_{N\}}^{Q},$$

$$D_{\mu} \tau = \partial_{\mu} \tau + g A_{\mu}^{M} \xi_{M} + g (A_{\mu}^{M} - A_{\mu}^{M} - \xi_{M}) \tau - ig A_{\mu}^{M} \xi_{M} \tau^2.$$ (2.15)

and the generalized gauge-field strengths being

$$H_{\mu\nu}^{M} = 2 \partial_{\mu} A_{\nu}^{M} - g f_{\alpha N \mu} A_{\alpha}^{N \nu} + \frac{g}{2} \Theta_{\alpha N \mu} C_{\mu\nu}^{NP} + \frac{g}{2} \epsilon^{M} C_{\mu\nu}^{+} + \xi_{M}^{+} C_{\mu\nu}^{-}.$$ (2.16)

The combinations $\Theta_{\alpha MNP}$ and $f_{\alpha MNP}$ are defined in terms of the gauging parameters $f_{\alpha MNP}$ and $\xi_{\alpha N}$ as

$$\Theta_{\alpha MNP} = f_{\alpha MNP} - \xi_{\alpha [N} \eta_{\mu]} M, \quad \Theta_{\alpha MNP} = f_{\alpha MNP} - \xi_{\alpha [M} \eta_{\mu]} N - \frac{3}{2} \xi_{\alpha N} \eta_{MP}.$$ (2.17)
The tensor gauge fields $C_{\mu \nu}^{MN} = C_{[\mu \nu]}^{[MN]}$ and $C_{\mu \nu}^{\alpha \beta} = C_{(\mu \nu)}^{(\alpha \beta)}$ are auxiliary and their elimination ensures that the correct number of gauge field degrees of freedom are propagated. For that purpose one needs to introduce a topological term in the Lagrangian

\[
e^{-1} \mathcal{L}_{\text{top}} = -\frac{g}{2} \epsilon^{\mu \nu \rho \lambda} \left( \xi_{+M} \eta_{NP} A_{\mu}^{M-N} A_{\nu}^{N+} \partial_{\rho} A_{\lambda}^{P+} - \left( f_{-MNP} + 2 \xi_{-N} \eta_{MP} \right) A_{\mu}^{M} A_{\nu}^{N} \partial_{\rho} A_{\lambda}^{P} \right)
\]

\[
-\frac{9}{4} f_{\alpha MNR} f_{\beta PQ} R^{\alpha \beta} A_{\mu}^{M} A_{\nu}^{N} A_{\rho}^{P} A_{\lambda}^{Q} + \frac{g}{16} \Theta_{+MNP} \Theta_{-MQR} C_{\mu \nu}^{NP} C_{\rho \lambda}^{QR} \quad (2.20)
\]

Finally, there is a potential for the scalar fields that takes the form

\[
e^{-1} \mathcal{L}_{\text{pot}} = -\frac{g^{2}}{16} \left( f_{\alpha MNP} f_{\beta QRS} M^{\alpha \beta} \left( \frac{1}{3} M^{MP} M^{QR} M^{PS} + \left( \frac{2}{3} \eta_{MP} - M^{MP} \right) \eta^{QR} \eta^{PS} \right) \right.
\]

\[
-\frac{4}{9} f_{\alpha MNP} f_{\beta QRS} \epsilon^{\alpha \beta} M^{MNPQRS} + 3 \xi_{\alpha}^{M} \xi_{\beta}^{N} M^{\alpha \beta \mu \nu} \right).
\]

The basic feature of this Lagrangian is that it depends explicitly on both the electric gauge potentials $A_{\mu}^{M+}$ and their magnetic duals $A_{\mu}^{M-}$. Therefore it allows the gauging of any subgroup of the full duality group $SL(2, \mathbb{R}) \times SO(6, n)$, where in principle both electric and magnetic potentials can participate. The field equations derived from this Lagrangian and the gauge transformations can be found in [9].

### 3. Gauging the $\mathcal{N} = 4$ axionic symmetries

Here we present a class of $\mathcal{N} = 4$ gauged supergravities obtained by making local the axionic shifts and axionic rescalings. This is a subclass of the electric gaugings and we will refer to them as non-unimodular gaugings. Their existence was pointed out in [9] but here we analyze in detail and full generality the properties of the corresponding gauge algebra and discuss some interesting features of their Lagrangian description, such as a duality between massive vectors and massive two-forms.

#### 3.1 Electric-magnetic duality and electric gaugings

The most general $\mathcal{N} = 4$ gauging is described in terms of two tensors $f_{\alpha LMN}$ and $\xi_{\beta I}$, $\alpha, \beta \in \{+,-\}$ and $I, L, M, N \in \{1, \ldots, 6 + n\}$, satisfying four quadratic conditions (i, ii, iii, iv) displayed in Eqs. (2.9)–(2.12). This general formalism, defined in an arbitrary symplectic frame, captures in particular the gauging of the electric-magnetic duality symmetry generated by $S_{+}$ (see Sec. 2.1).

Gaugings with pure $f_{\alpha LMN}$’s have been studied extensively in the literature. They correspond to switching on gauge algebras entirely embedded in $SO(6, n)$, as shown by (2.8). On the other hand, turning on the $\xi_{\beta I}$’s allows one to gauge both $SL(2, \mathbb{R})$ and $SO(6, n)$. This situation has not attracted much attention and only a few examples of the corresponding gauge algebras have been analyzed (see e.g. [9, 35]). Our aim is to study
systematically a class of such gaugings and show that they correspond to a specific pattern of higher-dimensional reduction which generalizes the Scherk–Schwarz mechanism.

We will focus here on electric gaugings, namely gaugings that do not involve the $S_{++}$ generator of $SL(2, \mathbb{R})$. Axionic shifts $S_{--}$ or axionic rescalings $S_{+-}$ will however be gauged, accompanied by the appropriate $SO(6, n)$ generators. Hence, this class of gaugings is defined by setting

$$\xi_{-I} = 0.$$  \hspace{1cm} (3.1)

We will also set

$$f_{-LMN} = 0,$$  \hspace{1cm} (3.2)

although this is not compulsory for electric gaugings, while keeping $\xi_{+I} \equiv \xi_I \neq 0$ and $f_{+LMN} \equiv f_{LMN} \neq 0$. Furthermore, we will focus on the case $n = 6$, which is related to pure gravity in ten dimensions, and adopt the off-block-diagonal 6 + 6 metric:

$$\eta = \begin{pmatrix} 0 & \mathbb{I}_6 \\ \mathbb{I}_6 & 0 \end{pmatrix}.$$  \hspace{1cm} (3.3)

The quadratic constraint (iii), Eq. (2.11), is now automatically satisfied while the constraints (i, ii, iv) – Eqs. (2.9), (2.10), (2.12) – reduce to

$$\eta^{MN} \xi_M \xi_N = 0, \quad \eta^{MN} \xi_M f_{N[IJ]} = 0,$$

$$\eta^{MN} f_{MI[N]f_{KL]}N} = \frac{2}{3} f_{[IJK] \xi_L}.$$  \hspace{1cm} (3.4)

3.2 Non-unimodular gaugings

The fundamental representation $12$ of $SO(6, 6)$ decomposes into $6_{+1} + 6_{-1}$ under the diagonal $GL(6) = U(1) \times SL(6)$ subgroup and correspondingly the $I$-indices decompose into $(i, i')$ with both $i$ and $i'$ ranging from 1 to 6. Then, the 6 + 6 metric can be written in the following way: $\eta_{ij} = \eta_{i'j'} = 0$, whereas $\eta_{ii'} = \eta_{i'i} = \delta_{ii'}$. In this basis the $SO(6, 6)$-invariant inner product takes the form $A_M B^M = A_m B^m + A_{m'} B^{m'}$ and we can write $A^m \equiv A_{m'}$, $A^{m'} \equiv A_m$.

A specific solution with $f$'s and $\xi$'s. Now, a non-trivial solution to Eqs. (3.4) and (3.5) is

$$\xi_i = \lambda_i \quad \xi_{i'} = 0 \quad \text{for all} \quad i' \equiv$$

$$f_{i'ij} = f_{ij'i} = f_{ji'i} = -\delta_i \delta_{j'} \quad \text{all others vanishing},$$  \hspace{1cm} (3.6)

with $\lambda_i$ arbitrary real numbers. The existence of the gauging described in Eqs. (3.6) and (3.7) was pointed out in Ref. [9]. Actually, there exist a whole class of gaugings of this type with more components of the tensor $f_{IJK}$ turned on. As discussed in [9], besides Eqs. (3.6) and (3.7) one can turn on $f_{ijk}$. Then, the quadratic constraints reduce to a single equation

$$f_{[ijk] \lambda_l] = 0.$$  \hspace{1cm} (3.8)

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What is meant by electric gauging is not universally set in the literature. In [9], for example, electric gaugings are defined as those with $\xi_{3M} = f_{-LMN} = 0$, which is somewhat too restrictive.
Since the novel feature of this class of gaugings is the presence of a non-zero $\xi$ parameter, we will restrict ourselves to the simplest example, namely the gauging with $f_{ijk} = 0$.

The gauging under consideration will be called “non-unimodular” for reasons that will become clear at the end of Sec. 4.2, or “traceful” since

$$f_{ij}^j = -\frac{5}{2} \lambda_i.$$  

(3.9)

This is slightly misleading, however, since the gauge algebra is traceless as a consequence of the full antisymmetry of its genuine structure constants. The latter are not $f_{ijk}$ but specific combinations of $f_{IJK}$ and $\xi_I$ read off from the commutation relations of generators (2.8).

**The gauge algebra.** In the rest of this section, we will characterize the gauge algebra, which will be further studied from a higher-dimensional perspective in the next chapters.

Using Eqs. (3.6) and (3.7) in expression (2.8), we obtain:

$$\Xi_{-i} = -\frac{\lambda_i}{2} S_{-} \equiv \lambda_i \Upsilon,$$

(3.10)

$$\Xi_{-i'} = 0,$$

(3.11)

$$\Xi_{+i} = -\frac{\lambda_i}{2} \left( T^j_j + S_{-} \right) \equiv \lambda_i \Xi,$$

(3.12)

$$\Xi_{+i'} = -\lambda_j T^j_i' \equiv \Xi_i'.$$

(3.13)

The gauge algebra at hand has at most 18 non-vanishing generators but only 7 are independent: $\Upsilon, \Xi$, plus 5 of the $\Xi_{i'}$ due to the constraint $\lambda^i \Xi_i = 0$.

Their commutation relations follow from Eqs. (2.2) and (2.6):

$$[\Xi_{i'}, \Xi_{j'}] = 0,$$

(3.14)

$$[\Upsilon, \Xi_{j'}] = 0,$$

(3.15)

$$[\Xi_{i'}, \Xi] = \Xi_i'.$$

(3.16)

$$[\Upsilon, \Xi] = -\Upsilon.$$

(3.17)

The set $\{ \Upsilon, \Xi_{i'} \}$ spans an Abelian Lie subalgebra. Furthermore, the algebra generated by $\{ \Upsilon, \Xi \}$ is a non-compact $A_{2,2}$ subalgebra of $SL(2, \mathbb{R}) \times SO(6,6)$.

The above algebra is non-flat\(^6\), in contrast to the algebras obtained by standard Scherk–Schwarz reductions. As we will see in later, the gaugings at hand are related to twisted versions of these reductions, which relax therefore the flatness of the gauge algebras.

The particular case $\lambda_i = \lambda \delta_{ii}$ appears when compactifying from five to four dimensions (see [35]) with a non-compact twist generated by the five-dimensional rescaling. The full algebra (3.10)–(3.17) will also emerge (see Sec. 4) as a ten-dimensional heterotic reduction with twist. Richer non-Abelian extensions are possible in this case, that eventually lead to non-vanishing $f_{ijk}$ as already advertised, and which are not possible when compactifying from five dimensions. These issues will be extensively analyzed in Sec. 4.2.

\(^6\)An algebra is called flat when it is generated by a set $\{ Q, X_i \}$ satisfying the commutation relations $[Q, X_i] = M_i^j X_j, [X_i, X_j] = 0$ with $M_i^j = -M_i^j$. Then the Levi–Civita connection on the corresponding group manifold has zero curvature, therefore justifying the name flat.
3.3 Lagrangian description

It is straightforward to derive the bosonic Lagrangian for the gaugings we have just described by using the general formulas of the previous section. As a first step, we will implement (3.1) and (3.2), and later set (3.6) and (3.7). Finally, we will dualize a vector, which acquires a St"uckelberg-like mass via the gauging at hand, into a massive two-form field potential.

Electric gaugings $\xi_I - f_{-KLM} = 0$. The kinetic terms read

$$e^{-1}L_{\text{kin}} = \frac{1}{2} R + \frac{1}{16} D_{\mu} M_{MN} D^{\mu} M^{MN} - \frac{1}{4} e^{0} D_{\mu} a D^{\mu} a - D_{\mu} \phi D^{\mu} \phi$$

$$- \frac{1}{4} e^{-2\phi} M_{MN} H_{\mu\nu}^M H_{\kappa\lambda}^N + \frac{1}{8} a \eta_{MN} e^{\mu \nu \kappa \lambda} H_{\mu \nu}^M H_{\kappa \lambda}^N,$$

(3.18)

where now

$$D_{\mu} M_{MN} = \partial_{\mu} M_{MN} + 2g A_{\mu}^P \Theta_{P(M} Q_{N)Q}$$

(3.19)

and

$$H_{\mu\nu}^M = 2\partial_{[\mu} A_{\nu]}^M - g \hat{f}_{NP}^M A_{[\mu}^N A_{\nu]}^P + \frac{g}{2} \epsilon^M C_{\mu\nu}$$

(3.20)

with $C_{\mu\nu} := C_{\mu\nu}^{++}$.

Since $f_{-MNP}$ and $\xi_{-M}$ are zero and hence $\Theta_{-MNP}$ and $\hat{f}_{-MNP}$ are zero as well, we have omitted the “+” index from all coefficients. We will use the notation $A_{\mu}^M \equiv A_{\mu}^{M+}, A_{\mu}^{-M} \equiv X_{\mu}^M$ for the gauge potentials in order to avoid cluttering of the formulas with indices. Furthermore, we define the linear combinations

$$X_{\mu} \equiv \xi_{M} X_{\mu}^{M}, \quad A_{\mu} \equiv \xi_{M} A_{\mu}^{M}.$$  

(3.21)

The gauge covariant derivatives of the axion and dilaton take the form\footnote{From now on we set the coupling constant $g$ equal to 1 for simplicity.}

$$D_{\mu} a = \partial_{\mu} a + X_{\mu} a + A_{\mu} a,$$

(3.22)

$$D_{\mu} \phi = \partial_{\mu} \phi - \frac{1}{2} A_{\mu}.$$  

(3.23)

By turning on the parameters $\xi$ we have gauged a non-Abelian two-dimensional subgroup of the $SL(2, \mathbb{R})$ global axion-dilaton symmetry. The “magnetic” potential $X_{\mu}$ corresponds to the gauging of the shift symmetry of the axion $a \rightarrow a + c$ and acts as a St"uckelberg field, while $A_{\mu}$ gauges the dilatation symmetry $a \rightarrow e^{-2\lambda} a, \phi \rightarrow \phi + \lambda$. In terms of the notation of the previous subsection, the corresponding gauge algebra is the one spanned by $\Xi$ and $\Upsilon$ with commutation relation (3.17).

The topological term in the Lagrangian for the class of gaugings under consideration becomes

$$e^{-1}L_{\text{top}} = -\frac{1}{2} \epsilon^{\mu \nu \rho \lambda} \left( \xi_{M} \eta_{NP} X_{\mu}^{M} A_{\nu}^{N} A_{\rho}^{P} \partial_{\lambda} a_{\lambda} - \frac{1}{4} \xi_{M} C_{\mu\nu} C_{\rho\lambda}^{M} \right.$$  

$$- \frac{1}{4} \hat{f}_{MNP} \hat{f}_{PQR} A_{\mu}^{M} A_{\nu}^{N} A_{\rho}^{P} X_{\lambda}^{Q} \right).$$

(3.24)
We have defined the following gauge field strengths

\[ F_{\mu\nu}^M = 2\partial_{[\mu} A_{\nu]}^M - \tilde{f}_{NP}^M A_{\mu}^N A_{\nu}^P, \]  
\[ G_{\mu\nu}^M = 2\partial_{[\kappa} X_{\lambda]}^M - \tilde{f}_{QR}^M A_{\mu}^Q X_{\nu}^R, \]  
(3.25)
(3.26)
in terms of which one can write \( H_{\mu\nu}^M = F_{\mu\nu}^M + \frac{1}{2} \xi^M C_{\mu\nu}. \)

Finally, the potential terms are

\[ e^{-1} L_{\text{pot}} = -\frac{1}{16} M^{++} \left( 3 \xi_M \xi_N M^{MN} + f_{MNP} f_{QRS} \left( \frac{1}{3} M^{MQ} M^{NR} M^{PS} + \left( \frac{2}{3} \eta^{MQ} - M^{MQ} \right) \eta^{NR} \eta^{PS} \right) \right) \]  
(3.27)
with \( M^{++} = e^{2\phi}. \)

**Non-unimodular gaugings.** We will now specify the gauge parameters by setting (3.6) and (3.7). With this gauging, the indices \( i \) and \( i' \) corresponding to the \( 6_{+1} \) and \( 6_{-1} \) of \( U(1) \times SL(6) \subset SO(6,6) \) are treated differently. We first notice that the magnetic field strengths \( G_{\mu\nu}^M \) do not appear in the Lagrangian. The gauge field strengths that appear are

\[ F_{\mu\nu}^m = 2\partial_{[\mu} A_{\nu]}^m, \]  
\[ F_{\mu\nu}^{m'} = 2\partial_{[\mu} A_{\nu]}^{m'} + \left( 2\lambda_j A_{\mu}^{m'} A_{\nu}^j - \lambda_m A_{\mu}^{m'} A_{\nu}^j \right), \]  
\[ G_{\kappa\lambda}^m = 2\partial_{[\kappa} X_{\lambda]}^m + \left( \lambda_i A_{\kappa}^i X_{\lambda}^m + \lambda_i A_{[\kappa}^i X_{\lambda]}^m \right), \]  
(3.28)
(3.29)
(3.30)
where now

\[ X_{\mu} = \xi_M X_{\mu}^M = \lambda_m X_{\mu}^m, \quad A_{\mu} = \xi_M A_{\mu}^M = \lambda_m A_{\mu}^m. \]  
(3.31)

One further notices that the Lagrangian actually depends only on the linear combination \( \lambda_m G_{\mu\nu}^m. \) This combination can be written in terms of \( A_{\mu} \) and \( X_{\mu} \) as

\[ G_{\mu\nu} = 2\partial_{[\mu} X_{\nu]} + 2A_{[\mu} X_{\nu]}. \]  
(3.32)

Similarly we introduce \( F_{\mu\nu} = \lambda_m F_{\mu\nu}^m = 2\partial_{[\mu} A_{\nu]}. \)

The natural prescription is to integrate out the auxiliary two-form \( C_{\mu\nu} \) in order to obtain the final Lagrangian. If we do so, starting from (3.18), (3.24) and (3.27), we obtain

\[ e^{-1} L = \frac{1}{2} R + \frac{1}{16} D_{\mu} M_{MN} D^\mu M^{MN} - \frac{1}{4} e^{2\phi} D_{\mu} a D^\mu a - D_{\mu} \phi D^\mu \phi \]  
\[ - \frac{1}{4} e^{-2\phi} M_{MN} F_{\mu\nu}^M F_{N\rho\sigma}^N + \frac{1}{4} \eta_{MN} F_{\mu\nu}^M \tilde{F}^{N\mu\nu} - \frac{1}{4\lambda^2} e^{2\phi} Z^{\mu\nu} Z_{\mu\nu} \]  
\[ - \frac{1}{12} \epsilon_{\mu\nu\rho\lambda} X_{\mu} \omega_{\rho\lambda\nu} + e^{-1} L_{\text{pot}}. \]  
(3.33)

In this expression \( Z^{\mu\nu} = a F^{\mu\nu} + G^{\mu\nu} + e^{-2\phi} \lambda^m M^{m'N} \tilde{F}^{N\mu\nu}, \) where \( \tilde{F}^{N\mu\nu} \) is the Hodge–Poincaré dual of \( F_{\mu\nu}^N; \) we have also defined \( \lambda^2 = \lambda_m M^{m\lambda} \lambda_n \) and we have introduced the Chern–Simons form:

\[ \omega_{\rho\lambda\nu} = A_{\nu}^n F_{\rho\lambda n} + A_{\nu} M_{\rho\lambda} A_{\mu}^n + \frac{1}{2} A_{\nu} A_{\lambda}^m A_{\mu}^n + \frac{1}{2} A_{\nu} A_{\lambda}^m A_{\mu}^n + \text{cyclic}. \]  
(3.34)
The local gauge invariance under axionic shifts enables us to gauge away the axion \( a = 0 \). Then, the magnetic potential \( X_\mu \) acquires a mass through its Stückelberg coupling to the axion. The final expression for the gauge-fixed Lagrangian is therefore

\[
e^{-1} \mathcal{L} = \frac{1}{2} R + \frac{1}{16} D_\mu M_{MN} D^\mu M^{MN} - D_\mu \phi D^\mu \phi - \frac{1}{4} e^{-2\phi} M_{MN} F^M_{\mu \nu} F^{N\mu \nu} - \frac{1}{4 \Lambda^2} \left( C^{\mu \nu} + e^{-2\phi} \chi m' M_{m'N} \tilde{F}^{N\mu \nu} \right) \left( C_{\mu \nu} + e^{-2\phi} \chi m' M_{m'N} \tilde{F}^{N}_{\mu \nu} \right) - \frac{1}{12} \epsilon^{\mu \nu \rho \lambda} X_\mu \omega_{\rho \lambda \nu} + e^{-1} \mathcal{L}_{\text{pot}}.
\]

(3.35)

It captures all the relevant information carried by the axionic-symmetry gauging.

**Stückelberg mass and dualization.** There is a different formulation of the theory, which is actually more suggestive of a higher-dimensional origin. Instead of integrating out the auxiliary antisymmetric tensor \( C_{\mu \nu} \), one can promote \( C_{\mu \nu} \) to a propagating field and integrate out \( X_\mu \). Using as previously (3.18), (3.24) and (3.27), this procedure yields the following Lagrangian:

\[
e^{-1} \tilde{\mathcal{L}} = \frac{1}{2} R + \frac{1}{16} D_\mu M_{MN} D^\mu M^{MN} - D_\mu \phi D^\mu \phi - \frac{1}{4} e^{-2\phi} M_{MN} \left( F^M_{\mu \nu} + \frac{1}{2} \xi^M C_{\mu \nu} \right) \left( F^{N\mu \nu} + \frac{1}{2} \xi^N C_{\mu \nu} \right) - \frac{3}{8} e^{-4\phi} \left( \partial_{[\kappa} C_{\mu \rho \nu]} - A_{[\kappa} C_{\mu \rho \nu]} - \frac{1}{3} \omega_{\kappa \mu \nu} \right)^2 + e^{-1} \mathcal{L}_{\text{pot}},
\]

(3.36)

where the two-form \( C_{\mu \nu} \) acquires now a scalar-field-dependent mass.

The key observation is that the theory described by (3.36) is related to that described by (3.35) by an interesting generalized duality. Recall that a massless two-form potential in four dimensions carries one propagating degree of freedom and can be dualized into a scalar. In the case where the two-form comes from the reduction of the ten-dimensional NS-NS two-form in the gravity multiplet, the dual scalar is the axion. If instead, as it happens here, the two-form is massive, it carries three degrees of freedom and the dual potential is a massive vector. This duality is a particular instance of a generalized duality between massive \( p \)-forms and massive \( (D - p - 1) \)-forms in \( D \) dimensions \([22, 23]\).

For the benefit of the reader we will present schematically how this generalized duality works, leaving as an exercise its precise implementation between (3.35) and (3.36). Start from a massive gauge field \( X_\mu \) with Lagrangian

\[
\mathcal{L} = -\frac{1}{4g^2} (\partial_\mu X_\nu - \partial_\nu X_\mu) (\partial^\mu X^\nu - \partial^\nu X^\mu) + \frac{1}{2} m^2 X_\mu X^\mu.
\]

(3.37)

This Lagrangian can be obtained from

\[
\tilde{\mathcal{L}} = 2 C_{\mu \nu} \partial^\mu X^\nu + g^2 C_{\mu \nu} C^{\mu \nu} + \frac{1}{2} m^2 X_\mu X^\mu
\]

(3.38)

by integrating \( C_{\mu \nu} \), which is an auxiliary non-propagating antisymmetric tensor. Instead, we can integrate by parts \( \tilde{\mathcal{L}} \) so that \( X_\mu \) becomes non-dynamical, while \( C_{\mu \nu} \) acquires a
dynamics. Integrating out finally $X_\mu$ yields an action for a massive two-form,

$$\hat{\mathcal{L}} = -\frac{2}{m^2} \partial_\mu C^\mu \partial_\nu C_{\lambda \nu} + g^2 C_\mu C^\mu,$$

(3.39)

which can be brought to a more familiar form for antisymmetric tensors, by Hodge–Poincaré-dualizing $C_\mu$.

Following a similar pattern, one can replace the massive vector $X_\mu$ in $\mathcal{L}$ (Eq. (3.35)) by a two-form. This yields precisely $\hat{\mathcal{L}}$ (Eq. (3.36)), therefore demonstrating that the Lagrangians $\mathcal{L}$ and $\hat{\mathcal{L}}$ describe equivalent physics.

4. Higher-dimensional origin

In this section we perform a generalized dimensional reduction of heterotic supergravity to four dimensions and show that the resulting effective theory belongs to the class of $\mathcal{N} = 4$ gauged supergravities studied in the previous section. Recall that the usual dimensional reduction of heterotic supergravity, which can be thought of as compactification on a six-torus keeping only the massless modes, results in a four-dimensional theory with 16 supercharges, Abelian gauge group $U(1)^{12+p}$ and a global off-shell symmetry $SO(6,6+p)$ [24]. Six of the Abelian vectors are graviphotons while another six of them come from reducing the NS-NS two-form on the 1-cycles of the torus.

One can also reduce some of the vectors present already in ten dimensions and obtain $p$ additional Abelian gauge fields. Usually this is done for the vectors lying in the Cartan torus of the ten-dimensional gauge group, therefore yielding a theory with $p = 16$. Since our goal here is to make contact with a gauged supergravity with $p = 0$ we will ignore this possibility and consider the reduction of the gravity-multiplet fields only. It is straightforward to extend the reduction to the Yang–Mills sector and obtain generalizations of the gaugings we discussed so far.

4.1 Heterotic reduction with duality twist

Our starting point is the bosonic action of heterotic supergravity in the string frame\(^8\)

$$S = \int_{M_4} dx \int_{K_6} dy \sqrt{-G} e^{-\Phi} \left( R + G^{MN} \partial_M \Phi \partial_N \Phi - \frac{1}{12} G^{MM'} G^{NN'} G^{KK'} H_{MNNKH_{M'N'K'}} \right).$$

(4.1)

We assume a decomposition of the ten-dimensional spacetime in a four-dimensional non-compact part $M_4$ parameterized by coordinates $x^\mu$ and a six-dimensional internal manifold $K_6$ parameterized by $y^i$. The spacetime indices will be decomposed as $M = (\mu, i)$. As usual $\Phi$ is the dilaton and $H = dB$ is the three-form field strength of the NS-NS antisymmetric tensor $B$. As said, we neglect the ten-dimensional Yang–Mills fields.

\(^8\)The ten-dimensional spacetime indices $M, N, \ldots$ of this section should not be confused with the fundamental $SO(6,6)$ indices of the previous sections.
Taking $K_6$ to be a flat six-torus and keeping only the $y$-independent modes yields a theory with 12 Abelian vectors and 38 massless scalars, two of which are the dilaton and the axion. The latter is the dual of the two-form $B_{\mu\nu}$ obtained by reducing $B_{MN}$. One way to obtain a more interesting theory is by introducing fluxes in the torus. The most general reduction of this type, where both NS-NS and geometric fluxes were present, was studied in [25].

The introduction of geometric fluxes has an alternative interpretation as a reduction with a twist for the spacetime fields [26]. This is actually a particular case of a generalized reduction scheme that usually is referred to as “Scherk–Schwarz” reduction [27]. The characteristic property of reductions of this type is that the reduction ansatz can incorporate a dependance on the coordinates of the internal torus. This dependance is not arbitrary however; on a technical level it is dictated by the requirement that the Lagrangian should be independent of the internal coordinates. This is implemented by selecting a profile for the fields whose consistency is guaranteed by some symmetry of the original theory. Such reductions in the context of supergravity have been studied in [28, 29, 30, 31, 32, 33, 34, 35, 36] while the reader is referred to [37] for a general discussion on reductions with duality twists.

A subtle point that is not usually emphasized is the following. These reduction schemes yield an effective theory for a finite set of modes selected out of the infinitude of higher-dimensional modes according to some symmetry principle. Hence, it is not necessarily true that they encompass all low-energy modes and further analysis is required in order to establish that the effective theory obtained through a Scherk–Schwarz reduction is actually a low-energy effective theory. It would be interesting to perform such an analysis for the reduction scheme presented below but this issue lies beyond the scope of this paper.

The symmetry we will employ in the present paper is the $SO(1,1)$ scaling symmetry of (4.1), under which the fields transform as

$$\Phi \to \Phi + 4\lambda, \quad G_{MN} \to e^{\lambda}G_{MN}(x), \quad B_{MN} \to e^{\lambda}B_{MN}(x).$$

This enables us to trade the usual periodic ansatz, which assumes no dependance on the torus coordinates, with the following one

$$\Phi(x,y) = \Phi(x) + 4\lambda_4 y^4, \quad G_{MN}(x,y) = e^{\lambda y^4}G_{MN}(x), \quad B_{MN}(x,y) = e^{\lambda y^4}B_{MN}(x).$$

The parameters $\lambda_i$ are arbitrary real numbers that dictate the twisting of the fields along the six one-cycles of the torus.

The decomposition of the ten-dimensional metric tensor in terms of four-dimensional fields is the usual one

$$G_{\mu\nu} = g_{\mu\nu} + A_{\mu}^i A_{\nu}^i h_{ij}, \quad G_{\mu i} = A_{\mu}^i h_{ij}, \quad G_{ij} = h_{ij}. \quad (4.4)$$

Here $g_{\mu\nu}$ is the metric on $M_4$, $h_{ij}$ are the 21 metric moduli of the six-torus and $A_{\mu}^i$ are the Kaluza–Klein gauge fields. Similarly, the antisymmetric tensor is decomposed as

$$B_{\mu\nu} = b_{\mu\nu}, \quad B_{\mu i} = b_{\mu i}, \quad B_{ij} = b_{ij}. \quad (4.5)$$
We obtain a two-form $b_{\mu\nu}$, which usually is dualized to an axion, six vectors $B^i_{\mu}$ and 15 scalar moduli $b_{ij}$.

The above decompositions hold for the full ten-dimensional fields that are assumed to have a dependence on $y^i$ of the type dictated by the $SO(1, 1)$ scaling symmetry. Consistency implies that

$$g_{\mu\nu}(x, y) = e^{\lambda_k y^k} g_{\mu\nu}(x), \quad b_{\mu\nu}(x, y) = e^{\lambda_k y^k} b_{\mu\nu}(x), \quad h_{ij}(x, y) = e^{\lambda_k y^k} h_{ij}(x),$$

$$A^i_{\mu}(x, y) = A^i_{\mu}(x), \quad b_{\mu}(x, y) = e^{\lambda_k y^k} b_{\mu}(x), \quad (4.6)$$

$$\phi(x, y) = \phi(x) + \lambda_k y^k,$$  

(4.7)

where the four-dimensional dilaton is defined as $\phi = \Phi - \frac{1}{2} \log \det h$. The $y$-independent modes on the right-hand sides are the four-dimensional fields for which we would like to derive the effective action. The ansatz is consistent in the sense that the $y$-dependence is totally eliminated from the action and the integration over $y$ yields an overall multiplicative factor. Notice that the volume of the internal six-torus is encoded in the metric moduli $h_{ij}$.

Let us first reduce the Einstein–Hilbert part of the action along with the dilaton kinetic term. The most efficient way of performing this reduction is the following. We start from the metric

$$G'_{MN}(x, y) = \Omega^2(y) G_{MN}(x),$$

(4.9)

where $\Omega(y) = \exp(\frac{1}{2} y^i \lambda_i)$, use the relation between Ricci scalars for conformally related metrics, and finally apply the usual reduction formulas for $G_{MN}(x)$. After the above redefinition of the dilaton, necessary for absorbing the factor of $\sqrt{h}$, the conformal rescaling $g_{\mu\nu} \rightarrow \frac{\exp \phi}{2} g_{\mu\nu}$, which brings us to the Einstein frame in four dimensions, and a final rescaling $\phi \rightarrow 2\phi$, we obtain

$$S_{\text{gravity}} = \int_{M_4} d^4x \sqrt{-\tilde{g}} \left( \frac{1}{2} R_4 + \frac{1}{8} \left( \partial_{\mu} h_{ij} - A^k_{\mu} \lambda_k h_{ij} \right) \left( \partial^\mu h^{ij} + A^{ij}_{\mu} \lambda^\mu h^{ij} \right) \right)$$

$$- g^{\mu\nu} \left( \partial_{\mu} \phi - \frac{1}{2} A^i_{\mu} \lambda_i \right) \left( \partial_{\nu} \phi - \frac{1}{2} A^i_{\nu} \lambda_i \right)$$

$$- \frac{1}{4} e^{-2\phi} h_{ij} F^i_{\mu\nu} F^{j\mu\nu} - \frac{1}{2} e^{2\phi} \lambda_i h^{ij} \lambda_j, \quad (4.10)$$

where $F^m_{\mu\nu} = \partial_{\mu} A^m_{\nu} - \partial_{\nu} A^m_{\mu}$ and all fields are exclusively $y$-dependent.

There are several observations in order. First, the metric moduli $h_{ij}$ become charged under the Kaluza–Klein gauge fields $A^i_{\mu}$. The charges are given by the vector of twisting coefficients $\lambda_i$. The dilaton is also coupled in a St"uckelberg fashion to $A^i_{\mu}$. This signals the gauging of a shift symmetry as expected from a reduction where the ansatz was twisted by employing such a symmetry. Notice, furthermore, that the twisting does not result in a non-Abelian gauge symmetry for the Kaluza–Klein gauge fields. It does however lead to a potential for the dilaton and the metric moduli.

The reduction of the NS-NS part of the Lagrangian is more easily performed using the tangent-space components of the antisymmetric three-form [24]. Furthermore, some field

\footnotetext{9}{Note that the dilaton is shifted under the $SO(1, 1)$ scaling.}
redefinitions are necessary in order to bring the resulting Lagrangian to a more standard form. One defines vector fields $Y_{\mu n}$ and a two-form $B_{\mu \nu}$ as

\begin{align}
Y_{\mu n} &= b_{\mu n} + b_{nm} A_{\mu}^{m}, \\
B_{\mu \nu} &= b_{\mu \nu} + A_{\mu}^{m} Y_{\nu m} - A_{\mu}^{m} A_{\nu}^{n} b_{mn}.
\end{align}

We get

\begin{align}
S_{\text{NS-NS}} &= \int_{M_4} \left( -\frac{1}{8} h_{mn}^{\mu} D_{\mu} b_{m \ell} D^{\mu} b_{nk} \\
&\quad - \frac{1}{6} e^{-2\phi} \left( 3 \left( \partial_{[\mu} B_{\nu \lambda]} - \lambda_{k} A_{[\mu}^{k} B_{\nu \lambda]} \right) - \frac{1}{2} \Omega^{\mu \nu \lambda} \right) \\
&\quad \times \left( 3 \left( \partial_{[\mu} B_{\nu \lambda]} - \lambda_{\ell} A_{[\mu}^{\ell} B_{\nu \lambda]} \right) - \frac{1}{2} \Omega^{\mu \nu \lambda} \right) \\
&\quad - \frac{1}{4} e^{2\phi} h^{mn} \left( Y_{\mu m} + \lambda_{m} B_{\mu \nu} - b_{m \ell} F_{\mu \nu}^{\ell} \right) \left( Y_{\nu n} + \lambda_{n} B_{\mu \nu} - b_{nk} F_{\mu \nu}^{k} \right) \\
&\quad + \frac{1}{16} e^{2\phi} \lambda_{i} \left( h_{ij} h^{kt} b_{kn} h^{mn} b_{nk} - 2 h^{kt} b_{km} h^{mn} b_{ nr} h^{rj} \right) \lambda_{j} \sqrt{-g} \, dx,
\end{align}

where we have defined

\begin{align}
D_{\mu} b_{n \ell} &\equiv \partial_{\mu} b_{n \ell} + \lambda_{\ell} Y_{\mu n} - \lambda_{n} Y_{\mu \ell} - \lambda_{m} A_{\mu}^{m} b_{n \ell}, \\
Y_{\mu \ell} &\equiv \partial_{\mu} Y_{\nu \ell} - \partial_{\nu} Y_{\mu \ell} + \left( \frac{1}{2} \lambda_{\ell} \delta^{mn} - \lambda_{m} \delta^{\ell n} \right) \left( A_{\mu m} Y_{\nu n} - Y_{\mu m} A_{\nu n} \right),
\end{align}

while the Chern–Simons three-form is

\begin{align}
\Omega_{\mu \nu \lambda} &= Y_{\mu \ell} A_{\lambda}^{\ell} + A_{\mu}^{\ell} Y_{\nu \lambda} - \frac{1}{2} \lambda_{\ell} A_{\mu}^{\ell} A_{\nu}^{m} Y_{\mu m} + \frac{1}{2} \lambda_{\ell} A_{\lambda}^{\ell} A_{\nu}^{m} Y_{\mu m} + \text{cyclic}.
\end{align}

We observe that the NS-NS moduli are charged under the Kaluza–Klein gauge fields but have also St"uckelberg couplings to the NS-NS gauge potentials. The latter couplings are due to the gauging of the shift symmetries of those moduli induced by the duality twist. A crucial difference with the case of the ordinary dimensional reduction is that the four-dimensional two-form $B_{\mu \nu}$ acquires a mass. This prohibits the standard dual formulation in terms of an axion but, according to the discussion of the previous section on dualities between massive fields, suggests that a dual formulation in terms of a massive vector is possible. Let us finally stress that the reduction of the NS-NS sector also contributes to the potential for the $h_{ij}$ and $b_{ij}$ moduli (last line of (4.13)).

4.2 Contact with $N = 4$ gauged supergravity

We will now show that the effective theory described by the sum of actions (4.10) and (4.13) is nothing but the $N = 4$ gauged supergravity worked out in Sec. 3. Using the standard parameterization of the moduli matrix $M^{MN}$

\begin{align}
M^{MN} &= \begin{pmatrix} h_{mn}^{\mu} & -h_{mk}^{\mu} b_{kn} \\
\ b_{mk} h^{kn} & h_{mn} - b_{mk} h^{k\ell} b_{\ell n} \end{pmatrix},
\end{align}
the $\mathcal{N} = 4$ potential (3.27) obtained for the non-unimodular gauging reads:

$$V = \frac{1}{16} e^{2\phi} \lambda_i \left( 8h^{ij} - h^{ij} h^{k\ell} b_{\ell m} h^{m n} b_{nk} + 2h^{ik} b_{k m} h^{m n} b_{nr} h^{r j} \right) \lambda_j. \quad (4.18)$$

This is precisely the potential in the effective theory (4.10) plus (4.13). Notice that this identification clarifies the higher-dimensional interpretation of the gauging parameters $\xi$: they correspond to the parameters used to twist the boundary conditions by $SO(1,1)$ scalings along the six one-cycles of the torus.

It is straightforward to check that the rest of the terms in (4.10) plus (4.13) match exactly those of (3.36) provided we identify the gauge fields $A^m_\mu$, $A^\prime_\mu$ in the gauged-supergravity Lagrangian with the Kaluza–Klein and NS-NS gauge fields $A^m_\mu$, $Y_{m\mu}$ in the heterotic reduction and the antisymmetric tensors as $C_{\mu\nu} \leftrightarrow 2B_{\mu\nu}$. This elucidates the higher-dimensional origin of the gauged supergravity of Sec. 3 and confirms the prominent role of the generalized duality performed in four dimensions for reaching (3.36). It is amusing that the four-dimensional two-form $B_{\mu\nu}$ that comes from the NS-NS antisymmetric tensor in ten dimensions is actually the auxiliary tensor gauge field required for consistency of the gauging in the formalism of [8].

Let us mention at this point that ordinary reductions of the heterotic theory with NS-NS fluxes and geometric fluxes also yield $\mathcal{N} = 4$ gauged supergravities [25]. The correspondence with the embedding-tensor language is as follows: there are no $\xi$’s turned on and the only non-vanishing parameters are the $f_{IJK}$. Under the decomposition of indices $I = (i, i')$, the background NS-NS fluxes $\beta_{ijk}$ and geometric fluxes $\gamma_{ij}^k$ are identified with the components of $f_{IJK}$ as

$$f_{ijk} = -3\beta_{ijk}, \quad f_{ijk'} = 2\gamma_{ij}^k, \quad (4.19)$$

all other components being zero. The remaining non-trivial quadratic constraint is (iv) (Eq. (2.12)) and it corresponds to the Bianchi identities for the NS-NS fluxes and the Jacobi identity for the geometric fluxes. From this we conclude that the more general class of gaugings we mentioned in Sec. 3.2 with non-zero $f_{ijk}$ originates from a ten-dimensional reduction with an $SO(1,1)$ duality twist combined with background NS-NS fluxes. The condition (3.8) found then is a consequence of the Bianchi identity resulting from the ansatz (4.3).

An interesting observation is in order here. The correspondence between the components $f_{ijk'}$ and the geometric fluxes $\gamma_{ij}^k$ provides an alternative perspective on the gauging we have performed in Sec. 3.2 and the subsequent heterotic reduction of Sec. 4.1. Indeed, if we interpret the $y$-dependence of the internal metric (c.f. Eqs. (4.6)) as inducing a geometric flux, this flux is automatically “non-unimodular” since $\gamma_{ij}^i \neq 0$. In ordinary reductions, the unimodularity condition ensures consistency of the truncation of the higher-dimensional Lagrangian [26, 38]. This well-known obstruction is circumvented in our approach thanks to the compensating duality twist\(^{11}\).

\(^{10}\)We use the notation of [25].

\(^{11}\)This statement refers to a reduction performed in the string frame. The metric in the Einstein frame is not affected by the duality twist and the corresponding geometric flux must always be unimodular.
5. Conclusions and open problems

In this paper we studied in detail the class of $\mathcal{N} = 4$ axionic-symmetry gaugings and established that they can be embedded in heterotic theory. More specifically, they arise through a reduction where the boundary conditions for the fields are twisted by an $SO(1,1)$ scaling symmetry. Similar reductions are possible for type II strings yielding $\mathcal{N} = 8$ gauged supergravities or $\mathcal{N} = 4$ upon appropriate orbifolding/orientifolding. For M-theory, instead, there is no scaling symmetry of the action and that implies that the Lagrangian cannot be consistently truncated for fields with boundary conditions of this type. However, one can still perform such reductions at the level of the equations of motion and it is expected that the reduced equations of motion correspond to gaugings of the type we studied here. In passing, we also note that since the dilaton becomes a component of the internal geometry when a type IIA background is lifted to M-theory, reductions with dilaton twists should lift to M-theory reductions with purely geometric twists of the type studied in [39, 40].

Some obvious extensions of the current work include twisted reductions of the heterotic theory taking into account the ten-dimensional gauge fields or similar type II reductions in the presence of branes and orientifolds. This should yield electric $\mathcal{N} = 4$ gaugings where the gauge algebra is a subgroup of $SL(2, \mathbb{R}) \times SO(6, n)$ for $n \geq 6$.

The fact that the formulation of gauged supergravity through the embedding tensor is duality-covariant implies that these theories capture the effective dynamics of backgrounds related by duality transformations. Recently it has become increasingly clear that the majority of these backgrounds are non-geometric and cannot be described using the familiar notions of geometry and ordinary fluxes. From one point of view this demonstrates the power of the effective bottom-up approach, since four-dimensional physics can be derived without the need to delve into the microscopic details of a higher-dimensional setup. On the other hand, one could argue that a better understanding of non-geometric backgrounds may still be obtained through analyzing gauged supergravity.

For instance, the “non-geometric” fluxes $Q$ and $R$ proposed in [41] as T-dual of the familiar NS-NS and geometric fluxes, are automatically captured for heterotic compactifications on a six-torus by the formulation of $\mathcal{N} = 4$ gauging we have been discussing. Using the notation of [41] our gauging parameters $f_{+IJK}$ describe all possible situations through

$$f_{+ijk} \sim H_{ijk}, \quad f_{+ijk'} \sim f_{ijk}^k, \quad f_{+i'j'k} \sim Q^{ij}_{k'}, \quad f_{+i'j'k'} \sim R^{ij}_{k'}. \quad (5.1)$$

Besides this set of $SO(6,6)$-dual fluxes, the most general $\mathcal{N} = 4$ gauging comprises of another set of S-dual fluxes $f_{-IJK}$. It would be extremely interesting to understand the microscopic origin of all those non-geometric fluxes directly in ten dimensions and derive the corresponding gauged supergravities using an appropriate reduction scheme (see [42] for some recent ideas in this direction. Also, the non-geometric fluxes can be interpreted as geometric fluxes in an appropriate generalized geometry [43]). Among others, this should shed some light on the open problem of lifting gauged supergravities with non-trivial duality phases in heterotic string theory.
A related question concerns the higher-dimensional origin of the gauging constraints (i)-(iv). For example, although some of these constraints have a clear origin as Bianchi identities in the internal space, this is not so for the null condition (i) $\boldsymbol{\xi}_M \boldsymbol{\xi}^M = 0$. The reduction we performed depends naturally on six parameters that fill up $\boldsymbol{\xi}_M$ in such a way that it is automatically null. It would be interesting to understand how $\mathcal{N} = 4$ gaugings with more general parameters $\boldsymbol{\xi}_M$ can be obtained from higher dimensions and where the null condition comes from.

We conclude by emphasizing that formulations of string and M-theory of the type presented in [44, 45, 46] as well as the mathematical framework of generalized complex geometry [47] may provide the appropriate tools for resolving the above issues.

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