On the second Dirichlet eigenvalue of some nonlinear anisotropic elliptic operators

Francesco Della Pietra, Nunzia Gavitone, Gianpaolo Piscitelli

Università degli studi di Napoli Federico II, Dipartimento di Matematica e Applicazioni “R. Caccioppoli”
Via Cintia, Monte S. Angelo - 80126 Napoli, Italia.

March 11, 2022

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, $n \geq 2$. In this paper we mainly study some properties of the second Dirichlet eigenvalue $\lambda_2(p, \Omega)$ of the anisotropic $p$-Laplacian

$$-Q_p u := -\text{div} \left( F^{-1}(\nabla u) \nabla (F(\nabla u)) \right),$$

where $F$ is a suitable smooth norm of $\mathbb{R}^n$ and $p \in ]1, +\infty[$. We provide a lower bound of $\lambda_2(p, \Omega)$ among bounded open sets of given measure, showing the validity of a Hong-Krahn-Szego type inequality. Furthermore, we investigate the limit problem as $p \to +\infty$.

MSC 2010: 35P15 - 35P30 - 35J60
Keywords: Nonlinear eigenvalue problems - Hong-Krahn-Szego inequality - Finsler metrics

1 INTRODUCTION

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, $n \geq 2$. The main aim of this paper is to study some properties of the Dirichlet eigenvalues of the anisotropic $p$-Laplacian operator:

$$-Q_p u := -\text{div} \left( F^{-1}(\nabla u) \nabla (F(\nabla u)) \right), \quad (1)$$

where $1 < p < +\infty$, and $F$ is a sufficiently smooth norm on $\mathbb{R}^n$ (see Section 2 for the precise assumptions on $F$), namely the values $\lambda$ such that the problem

$$\begin{cases}
-\nabla^i (\nabla^j F(\nabla^k u)) = \lambda |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases} \quad (2)$$

admits a nontrivial solution in $W^{1,p}_0(\Omega)$.

The operator in (1) reduces to the $p$-Laplacian when $F$ is the Euclidean norm on $\mathbb{R}^n$. For a general norm $F$, $Q_p$ is anisotropic and can be highly nonlinear. In literature, several papers...
are devoted to the study of the smallest eigenvalue of \((2)\), denoted by \(\lambda_1(p, \Omega)\), in bounded domains (see Section 2), which has the variational characterization

\[
\lambda_1(p, \Omega) = \min_{\varphi \in \mathcal{W}_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} F^p(\nabla \varphi) \, dx}{\int_{\Omega} |\varphi|^p \, dx}.
\]

Let \(\Omega\) be a bounded domain. It is known (see [BFK, DG2]) that \(\lambda_1(p, \Omega)\) is simple, the eigenfunctions have constant sign and it is isolated; moreover the only positive eigenfunctions are the first eigenfunctions. Furthermore, the Faber-Krahn inequality holds (see [BFK]):

\[
\lambda_1(p, \Omega) \geq \lambda_1(p, W_R)
\]

where \(W_R\) is the so-called Wulff shape, that is the ball with respect to the dual norm \(F^0\) of \(F\), having the same measure of \(\Omega\) (see Section 2). Many other results are known for \(\lambda_1(p, \Omega)\). The interested reader may refer, for example, to [BFK, BKJ, BGM, DG3, KN, P, WX]. As matter of fact, also different kind of boundary conditions have been considered as, for example, in the papers [DG, DGP] (Neumann case), [DG2] (Robin case).

Among the results contained in the quoted papers, we recall that if \(\Omega\) is a bounded domain, it has been proved in [BKJ] that

\[
\lim_{p \to \infty} \lambda_1(p, \Omega)^{\frac{1}{p}} = \frac{1}{\rho_F(\Omega)},
\]

where \(\rho_F(\Omega)\) is the anisotropic inradius of \(\Omega\) with respect to the dual norm (see Section 2 for its definition), generalizing a well-known result in the Euclidean case contained in [JLM].

Actually, very few results are known for higher eigenvalues in the anisotropic case. In [F] the existence of a infinite sequence of eigenvalues is proved, obtained by means of a min–max characterization. Actually, as in the Euclidean case, it is not known if this sequence exhausts all the set of the eigenvalues. Here we will show that the spectrum of \(-Q_p\) is a closed set, that the eigenfunctions are in \(C^{1,\alpha}(\Omega)\) and that admit a finite number of nodal domains. We recall the reference [L2], where many results for the spectrum of the \(p\)-Laplacian in the Euclidean case have been summarized.

The core of the paper relies in the study of the second eigenvalue \(\lambda_2(p, \Omega)\), \(p \in ]1, +\infty[\), in bounded open sets, defined as

\[
\lambda_2(p, \Omega) := \begin{cases} \min\{\lambda > \lambda_1(p, \Omega) : \lambda \text{ is an eigenvalue}\} & \text{if } \lambda_1(p, \Omega) \text{ is simple} \\ \lambda_1(p, \Omega) & \text{otherwise}, \end{cases}
\]

and in analyzing its behavior when \(p \to \infty\).

First of all, we show that if \(\Omega\) is a domain, then \(\lambda_2(p, \Omega)\) admits exactly two nodal domains. Moreover, for a bounded open set \(\Omega\), we prove a sharp lower bound for \(\lambda_2\), namely the Hong-Krahn-Szego inequality

\[
\lambda_2(p, \Omega) \geq \lambda_2(p, \widetilde{W}),
\]

where \(\widetilde{W}\) is the union of two disjoint Wulff shapes, each one of measure \(\frac{1}{2} |\Omega|\).

In the Euclidean case, such inequality is well-known for \(p = 2\), and it has been recently studied for any \(1 < p < +\infty\) in [BF2].

Finally, we address our attention to the behavior of \(\lambda_2(p, \Omega)\) when \(\Omega\) is a bounded open set and \(p \to +\infty\). In particular, we show that

\[
\lim_{p \to \infty} \lambda_2(p, \Omega)^{\frac{1}{p}} = \frac{1}{\rho_{2,F}(\Omega)},
\]

where \(\rho_{2,F}(\Omega)\) is the anisotropic inradius of \(\Omega\) with respect to the dual norm (see Section 2 for its definition).
where \( \rho_{2, F}(\Omega) \) is the radius of two disjoint Wulff shapes \( W_1, W_2 \) such that \( W_1 \cup W_2 \) is contained in \( \Omega \). Furthermore, the normalized eigenfunctions of \( \lambda_p(2, \Omega) \) converge to a function \( u_\infty \) that is a viscosity solution to a suitable fully nonlinear elliptic problem (see Section 5). In the Euclidean case, this kind of result has been proved for bounded domains in [JL]. We consider both the nonconnected case and general norm \( F \). In a forthcoming paper we will deal with the limit case \( p \to 1 \).

As enhanced before, the aim of the paper is twofold: first, to consider the case of a general Finsler norm \( F \); second, to extend also the results known in the case of domains, to the case of nonconnected sets. We structured the paper as follows. In Section 2 we recall the main definitions as well as some basic fact of convex geometry, and we fix the precise assumptions on \( F \). In Section 3 we state the general eigenvalue problem for \(-Q_p\), recalling some known results, extending them, where it is possible, to the case of nonconnected sets; moreover we provide several properties of the first and of higher eigenvalues and eigenfunctions. In Section 4 the attention will be focused on the second eigenvalue \( \lambda_2(p, \Omega) \) of \(-Q_p\) proving, among the other properties of \( \lambda_2 \), the Hong-Krahn-Szego inequality. Finally, in Section 5 we study the limit case \( p \to \infty \).

2 NOTATION AND PRELIMINARIES

Throughout the paper we will consider a convex even 1-homogeneous function

\[ \xi \in \mathbb{R}^n \mapsto F(\xi) \in [0, +\infty], \]

that is a convex function such that

\[ F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R}, \xi \in \mathbb{R}^n, \] (4)

and such that

\[ a|\xi| \leq F(\xi), \quad \xi \in \mathbb{R}^n, \] (5)

for some constant \( 0 < a \). Under this hypothesis it is easy to see that there exists \( b \geq a \) such that

\[ F(\xi) \leq b|\xi|, \quad \xi \in \mathbb{R}^n. \]

Moreover, throughout the paper we will assume that

\[ \nabla^2_\xi[F^p](\xi) \text{ is positive definite in } \mathbb{R}^n \setminus \{0\}, \] (6)

with \( 1 < p < +\infty \).

The hypothesis (6) on \( F \) assures that the operator

\[ \Omega_p[u] := \text{div} \left( \frac{1}{p} \nabla_\xi[F^p](\nabla u) \right) \]

is elliptic, hence there exists a positive constant \( \gamma \) such that

\[ \frac{1}{p} \sum_{i,j=1}^{n} \nabla^2_{\xi_i \xi_j}[F^p](\eta)\xi_i \xi_j \geq \gamma|\eta|^{p-2}|\xi|^2, \]

for some positive constant \( \gamma \), for any \( \eta \in \mathbb{R}^n \setminus \{0\} \) and for any \( \xi \in \mathbb{R}^n \).
Remark 2.1. We stress that for \( p \geq 2 \) the condition
\[
\nabla^2_w[F^2](\xi) \text{ is positive definite in } \mathbb{R}^n \setminus \{0\},
\]
implies (6).

The polar function \( F^o : \mathbb{R}^n \to [0, +\infty) \) of \( F \) is defined as
\[
F^o(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F(\xi)}.
\]

It is easy to verify that also \( F^o \) is a convex function which satisfies properties (4) and (5). Furthermore,
\[
F(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{F^o(\xi)}.
\]

From the above property it holds that
\[
|\langle \xi, \eta \rangle| \leq F(\xi)F^o(\eta), \quad \forall \xi, \eta \in \mathbb{R}^n.
\]

The set
\[
W = \{ \xi \in \mathbb{R}^n : F^o(\xi) < 1 \}
\]
is the so-called Wulff shape centered at the origin. We put \( \kappa_n = |W| \), where \( |W| \) denotes the Lebesgue measure of \( W \). More generally, we denote with \( W_r(x_0) \) the set \( rW + x_0 \), that is the Wulff shape centered at \( x_0 \) with measure \( \kappa_n r^n \), and \( W_r(0) = W_r \).

The following properties of \( F \) and \( F^o \) hold true (see for example [BP]):
\[
\begin{align*}
\langle \nabla \xi F(\xi), \xi \rangle &= F(\xi), \\
\langle \nabla \xi F^o(\xi), \xi \rangle &= F^o(\xi), \\
F(\nabla \xi F^o(\xi)) &= F^o(\nabla \xi F(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \\
F^o(\xi)\nabla \xi F(\nabla \xi F^o(\xi)) &= F(\xi)\nabla \xi F^o(\nabla \xi F(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.
\end{align*}
\]

Finally, we will recall the following

Definition 1. A domain of \( \mathbb{R}^n \) is a connected open set.

3 THE DIRICHLET EIGENVALUE PROBLEM FOR \(-Q_p\)

Here we state the eigenvalue problem for \( \Omega_p \). Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), \( n \geq 2 \), \( 1 < p < +\infty \), and consider the problem
\[
\begin{aligned}
-\Omega_p u &= \lambda |u|^{p-2}u \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\quad (7)
\]

Definition 2. We say that \( u \in W^{1,p}_0(\Omega) \), \( u \neq 0 \), is an eigenfunction of (7), if
\[
\int_{\Omega} \langle F^{p-1}(\nabla u)\nabla \xi F(\nabla u), \nabla \varphi \rangle \, dx = \lambda \int_{\Omega} |u|^{p-2}u \varphi \, dx \quad (8)
\]
for all \( \varphi \in W^{1,p}_0(\Omega) \). The corresponding real number \( \lambda \) is called an eigenvalue of (7).

Obviously, if \( u \) is an eigenfunction associated to \( \lambda \), then
\[
\lambda = \frac{\int_{\Omega} F^p(\nabla u) \, dx}{\int_{\Omega} |u|^p \, dx} > 0.
\]
3.1 The first eigenvalue

Among the eigenvalues of (7), the smallest one, denoted here by $\lambda_1(p, \Omega)$, has the following well-known variational characterization:

$$\lambda_1(p, \Omega) = \min_{\varphi \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \varphi|^p \, dx}{\int_\Omega |\varphi|^p \, dx}.$$  

(9)

In the following theorems its main properties are recalled.

**Theorem 3.1.** If $\Omega$ is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, there exists a function $u_1 \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ which achieves the minimum in (9), and satisfies the problem (7) with $\lambda = \lambda_1(p, \Omega)$. Moreover, if $\Omega$ is connected, then $\lambda_1(p, \Omega)$ is simple, that is the corresponding eigenfunctions are unique up to a multiplicative constant, and the first eigenfunctions have constant sign in $\Omega$.

**Proof.** The proof can be immediately adapted from the case of $\Omega$ connected and we refer the reader, for example, to [L, BFK].

**Theorem 3.2.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 2$. Let $u \in W_0^{1,p}(\Omega)$ be an eigenfunction of (7) associated to an eigenvalue $\lambda$. If $u$ does not change sign in $\Omega$, then there exists a connected component $\Omega_0$ of $\Omega$ such that $\lambda = \lambda_1(p, \Omega_0)$ and $u$ is a first eigenfunction in $\Omega_0$. In particular, if $\Omega$ is connected then $\lambda = \lambda_1(p, \Omega)$ and a constant sign eigenfunction is a first eigenfunction.

**Proof.** If $\Omega$ is connected, a proof can be found in [L, DG2]. Otherwise, if $\Omega \ni 0$ in $\Omega$ disconnected, by the maximum principle $u$ must be either positive or identically zero in each connected component of $\Omega$. Hence there exists a connected component $\Omega_0$ such that $u$ coincides in $\Omega_0$ with a positive eigenfunction relative to $\lambda$. By the previous case, $\lambda = \lambda_1(p, \Omega_0)$ and the proof is completed.

Here we list some other useful and interesting properties that can be proved in a similar way than the Euclidean case.

**Proposition 3.3.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, the following properties hold.

1. For $t > 0$ it holds $\lambda_1(p, t\Omega) = t^{-p} \lambda_1(p, \Omega)$.

2. If $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, then $\lambda_1(p, \Omega_1) \geq \lambda_1(p, \Omega_2)$.

3. For all $1 < p < s < +\infty$ we have $p[\lambda_1(p, \Omega)]^{1/p} < s[\lambda_1(s, \Omega)]^{1/s}$.

**Proof.** The first two properties are immediate from (9). As regards the third property, the inequality derives from the Hölder inequality, similarly as in [L1]. Indeed, taking $\phi = |\psi|^{s-1} \psi$, $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\psi \geq 0$, we have by (4) that

$$[\lambda_1(p, \Omega)]^{\frac{1}{s}} \leq \frac{s}{p} \left( \frac{\int_\Omega |\psi|^{s-p} F_p(\nabla \psi) \, dx}{\int_\Omega |\psi|^s \, dx} \right)^{\frac{1}{s}} \leq \frac{s}{p} \left( \frac{\int_\Omega F_p(\nabla \psi) \, dx}{\int_\Omega |\psi|^s \, dx} \right)^{\frac{1}{s}}.$$

By minimizing with respect to $\psi$, we get the thesis. □

In addition, the Faber-Krahn inequality for $\lambda_1(p, \Omega)$ holds.
Theorem 3.4. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 2$, then
\[ |\Omega|^{p/N} \lambda_1(p, \Omega) \geq k_n^{p/N} \lambda_1(p, \omega). \] \hspace{1cm} (10)
Moreover, equality sign in (10) holds if $\Omega$ is homothetic to the Wulff shape.

The proof of this inequality, contained in [BFK], is based on a symmetrization technique introduced in [AFLT] (see [ET, FV] for the equality cases).

Using the previous result we can prove the following property of $\lambda_1(p, \omega)$.

Proposition 3.5. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. The first eigenvalue of (7), $\lambda_1(p, \Omega)$, is isolated.

Proof. We argue similarly as in [L2]. For completeness we give the proof. For convenience we write $\lambda_1$ instead of $\lambda_1(p, \Omega)$. Let $\lambda_k \neq \lambda_1$ a sequence of eigenvalues such that
\[ \lim_{k \to +\infty} \lambda_k = \lambda_1 \]

Let $u_k$ be a normalized eigenfunction associated to $\lambda_k$ that is,
\[ \lambda_k = \int_{\Omega} F^p(\nabla u_k) \, dx \quad \text{and} \quad \int_{\Omega} |u_k|^p \, dx = 1 \] \hspace{1cm} (11)

By (11), there exists a function $u \in W_0^{1,p}(\Omega)$ such that, up to a subsequence
\[ u_k \to u \quad \text{in} \quad L^p(\Omega) \quad \nabla u_k \rightharpoonup \nabla u \quad \text{weakly in} \quad L^p(\Omega). \]

By the strong convergence of $u_k$ in $L^p(\Omega)$ and, recalling that $F$ is convex, by weak lower semicontinuity, it follows that
\[ \int_{\Omega} |u|^p \, dx = 1 \quad \text{and} \quad \int_{\Omega} F^p(\nabla u) \, dx \leq \lim_{k \to \infty} \lambda_k = \lambda_1. \]

Hence, $u$ is a first eigenfunction. On the other hand, being $u_k$ not a first eigenfunction, by Theorem 3.2 it has to change sign. Hence, the sets $\Omega_k^+ = \{ u_k > 0 \}$ and $\Omega_k^- = \{ u_k < 0 \}$ are nonempty and, as a consequence of the Faber-Krahn inequality and of Theorem 3.2, it follows that
\[ \lambda_k = \lambda_1(p, \Omega_k^+) \geq \frac{C_{n,p}}{|\Omega_k^+|^{\frac{p}{n}}}, \quad \lambda_k = \lambda_1(p, \Omega_k^-) \geq \frac{C_{n,p}}{|\Omega_k^-|^{\frac{p}{n}}}. \]

This implies that both $|\Omega_k^+|$ and $|\Omega_k^-|$ cannot vanish as $k \to +\infty$ and finally, that $u_k$ converges to a function $u$ which changes sign in $\Omega$. This is in contradiction with the characterization of the first eigenfunctions, and the proof is completed. \hfill \Box

3.2 Higher eigenvalues

First of all, we recall the following result (see [F, Theorem 1.4.1] and the references therein), which assures the existence of infinite eigenvalues of $-\nabla p$. We use the following notation. Let $S^{n-1}$ be the unit Euclidean sphere in $\mathbb{R}^n$, and
\[ M = \{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |u|^p \, dx = 1 \}. \] \hspace{1cm} (12)

Moreover, let $\mathcal{C}_n$ be the class of all odd and continuous mappings from $S^{n-1}$ to $M$. Then, for any fixed $f \in \mathcal{C}_n$, we have $f : \omega \in S^{n-1} \mapsto f_\omega \in M$. 

Proposition 3.6. Let $\Omega$ a bounded open set of $\mathbb{R}^n$, for any $k \in \mathbb{N}$, the value

$$\bar{\lambda}_k(p, \Omega) = \inf_{f \in \mathbb{C}_c} \max_{\omega \in S^{n-1}} \int_{\Omega} F^p(\nabla f_\omega) \, dx$$

is an eigenvalue of $-\bar{Q}_p$. Moreover,

$$0 < \tilde{\lambda}_1(p, \Omega) = \lambda_1(p, \Omega) \leq \tilde{\lambda}_2(p, \Omega) \leq \ldots \leq \tilde{\lambda}_k(p, \Omega) \leq \tilde{\lambda}_{k+1}(p, \Omega) \leq \ldots,$$

and

$$\tilde{\lambda}_k(p, \Omega) \to \infty \text{ as } k \to \infty.$$

Hence, we have at least a sequence of eigenvalues of $-\bar{Q}_p$. Furthermore, the following proposition holds.

Proposition 3.7. Let $\Omega$ a bounded open set of $\mathbb{R}^n$. The spectrum of $-\bar{Q}_p$ is a closed set.

Proof. Let $\lambda_k$ be a sequence of eigenvalues converging to $\mu < +\infty$ and let $u_k$ be the corresponding normalized eigenfunctions, that is such that $\|u_k\|_{L^p(\Omega)} = 1$. We have to show that $\mu$ is an eigenvalue of $-\bar{Q}_p$.

We have that

$$\int_{\Omega} \langle F^{-1}(\nabla u_k)^p \varphi, \nabla F(\nabla u_k) \rangle \, dx = \lambda_k \int_{\Omega} |u_k|^{p-2} u_k \varphi \, dx \quad (13)$$

for any test function $\varphi \in W^{1,p}_0(\Omega)$. Since

$$\lambda_k = \int_{\Omega} F^p(\nabla u_k) \, dx,$$

and being $\lambda_k$ a convergent sequence, up to a subsequence we have that there exists a function $u \in W^{1,p}_0(\Omega)$ such that $u_k \to u$ strongly in $L^p(\Omega)$ and $\nabla u_k \to \nabla u$ weakly in $L^p(\Omega)$. Our aim is to prove that $u$ is an eigenfunction relative to $\lambda$.

Choosing $\varphi = u_k - u$ as test function in the equation solved by $u_k$, we have

$$\int_{\Omega} \langle F^{-1}(\nabla u_k)^p \nabla F(\nabla u_k) - F^{-1}(\nabla u)^p \nabla F(\nabla u), \nabla (u_k - u) \rangle \, dx$$

$$= \lambda_k \int_{\Omega} |u_k|^{p-2} u_k (u_k - u) \, dx - \int_{\Omega} F^{-1}(\nabla u)^p \nabla F(\nabla u), \nabla (u_k - u) \rangle \, dx.$$

By the strong convergence of $u_k$ and the weak one of $\nabla u_k$, the right-hand side of the above identity goes to zero as $k$ diverges. Hence

$$\lim_{k \to \infty} \int_{\Omega} \langle F^{-1}(\nabla u_k)^p \nabla F(\nabla u_k) - F^{-1}(\nabla u)^p \nabla F(\nabla u), \nabla (u_k - u) \rangle \, dx = 0$$

By nowadays standard arguments, this limit implies the strong convergence of the gradient, hence we can pass to the limit under the integral sign in (13) to obtain

$$\int_{\Omega} \langle F^{-1}(\nabla u)^p \nabla F(\nabla u), \nabla \varphi \rangle \, dx = \lambda \int_{\Omega} |u|^{p-2} u \varphi \, dx$$

This shows that $\lambda$ is an eigenvalue and the proof is completed.

Finally, we list some properties of the eigenfunctions, well-known in the Euclidean case (see for example [L2, AFT]). Recall that a nodal domain of an eigenfunction $u$ is a connected component of $\{u > 0\}$ or $\{u < 0\}$. 
Proposition 3.8. Let $p > 1$, and let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Then the following facts hold.

(i) Any eigenfunction of $-\mathcal{Q}_p$ has only a finite number of nodal domains.

(ii) Let $\lambda$ be an eigenvalue of $-\mathcal{Q}_p$, and $u$ be a corresponding eigenfunction. The following estimate holds:

\[
\|u\|_{L^\infty(\Omega)} \leq C_{n,p,F} \lambda^\frac{p}{2} \|u\|_{L^1(\Omega)}
\]  

where $C_{n,p,F}$ is a constant depending only on $n$, $p$ and $F$.

(iii) All the eigenfunctions of (i) are in $C^{1,\alpha}(\Omega)$, for some $\alpha \in (0,1)$.

Proof. Let $\lambda$ be an eigenvalue of $-\mathcal{Q}_p$, and $u$ a corresponding eigenfunction.

In order to prove (i), let us denote by $\Omega^+_j$ a connected component of the set $\Omega^+ := \{u > 0\}$. Being $\lambda = \lambda_1(\Omega^+_j)$, then by (10)

\[
|\Omega^+_j| \geq C_{n,p,F} \lambda^{-\frac{p}{2}}.
\]

Then, the thesis follows observing that

\[
|\Omega| \geq \sum_j |\Omega^+_j| \geq C_{n,p,F} \lambda^{-\frac{p}{2}} \sum_j 1.
\]

In order to prove (ii), let $k > 0$, and choose $\varphi(x) = \max\{u(x) - k, 0\}$ as test function in (8). Then

\[
\int_{A_k} F_p(\nabla u) \ dx = \lambda \int_{A_k} |u|^{p-2} u (u - k) \ dx 
\]

where $A_k = \{x \in \Omega : u(x) > k\}$. Being $k|A_k| \leq \|u\|_{L^1(\Omega)}$, then $|A_k| \to 0$ as $k \to \infty$. By the inequality $a^{p-1} \leq 2^{p-1} (a - k)^{p-1} + 2^{p-1} k^{p-1}$, we have

\[
\int_{A_k} |u|^{p-2} u (u - k) \ dx \leq 2^{p-1} \int_{A_k} (u - k)^p \ dx + 2^{p-1} k^{p-1} \int_{A_k} (u - k) \ dx.
\]

By Poincaré inequality and property (5), then (15) and (16) give that

\[
(1 - \lambda C_{n,p,F} |A_k|^{p/n}) \int_{A_k} (u - k)^p \ dx \leq \lambda |A_k|^{p/n} C_{n,p,F} k^{p-1} \int_{A_k} (u - k) \ dx.
\]

By choosing $k$ sufficiently large, the Hölder inequality implies

\[
\int_{A_k} (u - k) \ dx \leq \tilde{C}_{n,p,F} \lambda^{\frac{1}{p-1}} |A_k|^{\frac{n}{n(p-1)}}.
\]

This estimate allows to apply [LU, Lemma 5.1, p. 71] in order to get the boundedness of $\text{ess sup } u$. Similar argument gives that $\text{ess inf } u$ is bounded.

Since (14) holds, by standard elliptic regularity theory (see e.g. [LU]) the eigenfunction is $C^{1,\alpha}(\Omega)$. \qed

4 The Second Dirichlet Eigenvalue of $-\mathcal{Q}_p$

If $\Omega$ is a bounded domain, Proposition 3.5 assures that the first eigenvalue $\lambda_1(p, \Omega)$ of (1) is isolated. This suggests the following definition.
Definition 3. Let $\Omega$ be a bounded open set of $\mathbb{R}^n$. Then the second eigenvalue of $-\Omega_p$ is

$$\lambda_2(p, \Omega) := \begin{cases} \min\{\lambda > \lambda_1(p, \Omega) : \lambda \text{ is an eigenvalue}\} & \text{if } \lambda_1(p, \Omega) \text{ is simple} \\ \lambda_1(p, \Omega) & \text{otherwise.} \end{cases}$$

Remark 4.1. If $\Omega$ is connected, by theorems 3.1 and 3.2 we deduce the following characterization of the second eigenvalue:

$$\lambda_2(p, \Omega) = \min\{\lambda : \lambda \text{ admits a sign-changing eigenfunction}\}.$$ \hfill (17)

We point out that in [F] it is proved that in a bounded open set it holds

$$\lambda_2(p, \Omega) = \bar{\lambda}_2(p, \Omega) = \inf_{\gamma \in \Gamma_{\Omega}(u_1, v_1, u_2, v_2)} \max_{S \in \gamma(\partial \Omega)} \int_\Omega F^p(\nabla u(x)) \, dx$$ \hfill (18)

where $\bar{\lambda}_2(p, \Omega)$ is given in Proposition 3.6, and

$$\Gamma_{\Omega}(u, v) = \{ \gamma : [0, 1] \to M : \gamma \text{ is continuous and } \gamma(0) = u, \gamma(1) = v\},$$

with $M$ as in (12). As immediate consequence of (18) we get

**Proposition 4.2.** If $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$, then $\lambda_2(p, \Omega_1) \geq \lambda_2(p, \Omega_2)$.

By adapting the method contained in [CDG1, CDG2], it is possible to prove the following result.

**Proposition 4.3.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. The eigenfunctions associated to $\lambda_2(p, \Omega)$ admit exactly two nodal domains.

**Proof.** We will proceed as in the proof of [CDG2, Th. 2.1]. In such a case, $\lambda_2(p, \Omega)$ is characterized as in (17). Then any eigenfunction $u_2$ has to change sign, and it admits at least two nodal domains $\Omega_1 \subset \Omega^+$ and $\Omega_2 \subset \Omega^-$. Let us assume, by contradiction, the existence of a third nodal domain $\Omega_3$ and let us suppose, without loss of generality, that $\Omega_3 \subset \Omega^+$.

**Claim.** There exists a connected open set $\tilde{\Omega}_2$, with $\tilde{\Omega}_2 \subset \Omega_2 \subset \Omega$ such that $\tilde{\Omega}_2 \cap \Omega_1 = \emptyset$ or $\tilde{\Omega}_2 \cap \Omega_3 = \emptyset$.

The proof of the claim follows line by line as in [CDG2, Th. 2.1]. One of the main tools is the Hopf maximum principle, that for the operator $-\Omega_p$ is proved for example in [CT, Th. 2.1].

Now, without loss of generality, we assume that $\tilde{\Omega}_2$ is disjoint of $\Omega_1$ and from this fact a contradiction is derived.

By the fact that $u_2$ does not change sign on the nodal domains and by Proposition 4.2, we have that $\lambda_1(p, \Omega_1) = \lambda_2(p, \Omega)$ and that $\lambda_1(p, \tilde{\Omega}_2) < \lambda_1(p, \Omega_2) = \lambda_2(p, \Omega)$. Now, we may construct the disjoint sets $\tilde{\Omega}_2$ and $\tilde{\Omega}_1$ such that $\Omega_2 \subset \tilde{\Omega}_2 \subset \tilde{\Omega}_2$ and $\Omega_1 \subset \tilde{\Omega}_1$, in order to have

$$\lambda_1(p, \tilde{\Omega}_1) < \lambda_2(p, \Omega), \quad \lambda_1(p, \tilde{\Omega}_2) < \lambda_2(p, \Omega).$$

Now let $v_1$ and $v_2$ be the extension by zero outside $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, respectively, of the positive normalized eigenfunctions associated to $\lambda_1(p, \tilde{\Omega}_1)$ and $\lambda(p, \tilde{\Omega}_2)$. Hence we easily verify that the function $v = v_1 - v_2$ belongs to $W^{1,p}_0(\Omega)$, it changes sign and satisfies

$$\frac{\int_\Omega F^p(\nabla v_+) \, dx}{\int_\Omega v_+^p \, dx} < \lambda_2(p, \Omega), \quad \frac{\int_\Omega F^p(\nabla v_-) \, dx}{\int_\Omega v_-^p \, dx} < \lambda_2(p, \Omega).$$
The final aim is to construct a path $\gamma([0, 1])$ such

$$\max_{u \in \gamma([0, 1])} \int_{\Omega} F_p(\nabla u(x)) \, dx < \lambda_2(p, \Omega),$$

obtaining a contradiction from (18). The construction of this path follows adapting the method contained in [CDG1, CDG2].

\[ \square \]

**Remark 4.4.** In order to better understand the behavior of $\lambda_1(p, \Omega)$ and $\lambda_2(p, \Omega)$ on disconnected sets, an meaningful model is given when

$$\Omega = W_{r_1} \cup W_{r_2},$$

with $r_1, r_2 > 0$ and $W_{r_1} \cap W_{r_2} = \emptyset$.

We distinguish two cases.

**Case** $r_1 < r_2$. We have

$$\lambda_1(p, \Omega) = \lambda_1(p, W_{r_2}).$$

Hence $\lambda_1(p, \Omega)$ is simple, and any eigenfunction is identically zero on $W_1$ and has constant sign in $W_2$. Moreover,

$$\lambda_2(p, \Omega) = \min(\lambda_1(p, W_{r_1}), \lambda_2(p, W_{r_2})).$$

Hence, if $r_1$ is not too small, then the second eigenvalue is $\lambda_1(W_{r_1})$, and the second eigenfunctions of $\Omega$ coincide with the first eigenfunctions of $W_{r_1}$, that do not change sign in $W_{r_1}$, and vanish on $W_{r_2}$.

**Case** $r_1 = r_2$. We have

$$\lambda_1(p, \Omega) = \lambda_1(p, W_{r_1}), \quad i = 1, 2.$$

The first eigenvalue $\lambda_1(p, \Omega)$ is not simple: choosing, for example, the function $U = u_1 x_{W_{r_1}} - u_2 x_{W_{r_2}}$, where $u_i$, $i = 1, 2$, is the first normalized eigenfunction of $\lambda_1(p, W_{r_1})$, and $V = u_1 x_{W_{r_1}}$, then $U$ and $V$ are two nonproportional eigenfunctions relative to $\lambda_1(p, \Omega)$. Hence, in this case, by definition,

$$\lambda_2(p, \Omega) = \lambda_1(p, \Omega) = \lambda_1(p, W_{r_1}).$$

In order to prove the Hong-Krahn-Szego inequality, we need the following key lemma.

**Proposition 4.5.** Let $\Omega$ be an open bounded set of $\mathbb{R}^n$. Then there exists two disjoint domains $\Omega_1, \Omega_2$ of $\Omega$ such that

$$\lambda_2(p, \Omega) = \max(\lambda_1(p, \Omega_1), \lambda_1(p, \Omega_2)).$$

**Proof.** Let $u_2 \in W^{1,p}_0(\Omega)$ be a second normalized eigenfunction. First of all, suppose that $u_2$ changes sign in $\Omega$. Then, consider two nodal domains $\Omega_1 \subseteq \Omega_+$ and $\Omega_2 \subseteq \Omega_-$. By definition, $\Omega_1$ and $\Omega_2$ are connected sets. The restriction of $u_2$ to $\Omega_1$ is, by Theorem 3.2, a first eigenfunction for $\Omega_1$ and hence $\lambda_2(p, \Omega) = \lambda_1(p, \Omega_1)$. Analogously for $\Omega_2$, hence

$$\lambda_2(p, \Omega) = \lambda_1(p, \Omega_1) = \lambda_1(p, \Omega_2),$$

and the proof of the proposition is completed, in the case $u_2$ changes sign.
In the case that $u_2$ has constant sign in $\Omega$, for example $u_2 \geq 0$, then by Theorem 3.2 $\Omega$ must be disconnected. If $\lambda_1(p,\Omega)$ is simple, by definition $\lambda_2(p,\Omega) > \lambda_1(p,\Omega)$. Otherwise, $\lambda_1(p,\Omega) = \lambda_2(p,\Omega)$. Hence in both cases, we can consider a first nonnegative normalized eigenfunction $u_1$ not proportional to $u_2$.

Observe that in any connected component of $\Omega$, by the Harnack inequality, $u_i$, $i = 1, 2$, must be positive or identically zero. Hence we can choose two disjoint connected open sets $\Omega_1$ and $\Omega_2$, contained respectively in $\{x \in \Omega : u_1(x) > 0\}$ and $\{x \in \Omega : u_2(x) > 0\}$. Then, $u_1$ and $u_2$ are first Dirichlet eigenfunctions in $\Omega_1$ and $\Omega_2$, respectively, and

$$\lambda_1(p,\Omega) = \lambda_1(p,\Omega_1) \leq \lambda_2(p,\Omega), \quad \lambda_2(p,\Omega) = \lambda_1(p,\Omega_2),$$

and the proof is completed. \hfill \Box

Now we are in position to prove the Hong-Krahn-Szego inequality for $\lambda_2(p,\Omega)$.

**Theorem 4.6.** Let $\Omega$ be a bounded open set of $\mathbb{R}^n$. Then

$$\lambda_2(p,\Omega) \geq \lambda_2(p,\tilde{W}),$$

where $\tilde{W}$ is the union of two disjoint Wulff shapes, each one of measure $\frac{|\Omega|}{2}$. Moreover equality sign in (19) occurs if $\Omega$ is the disjoint union of two Wulff shapes of the same measure.

**Proof.** Let $\Omega_1$ and $\Omega_2$ given by Proposition 4.5. By the Faber-Krahn inequality we have

$$\lambda_2(p,\Omega) = \max\{\lambda_1(p,\Omega_1), \lambda_1(p,\Omega_2)\} \geq \max\{\lambda_1(p,\Omega_0,\Omega_1), \lambda_1(p,\Omega_0,\Omega_2)\},$$

with $|\Omega_0| = |\Omega|$. By the rescaling property of $\lambda_1(p,\cdot)$, and observing that, being $\Omega_1$ and $\Omega_2$ disjoint subsets of $\Omega$, $|\Omega_1| + |\Omega_2| \leq |\Omega|$, we have that

$$\max\{\lambda_1(p,\Omega_1), \lambda_1(p,\Omega_2)\} \geq \lambda_1(p,\Omega) \kappa\Omega^p \max\left\{|\Omega_1|^{-\frac{p}{n}}, |\Omega_2|^{-\frac{p}{n}}\right\} \geq \lambda_1(p,\Omega) \kappa\Omega^p \left(\frac{|\Omega|}{2}\right)^{-\frac{p}{n}} = \lambda_1(p,\tilde{W}).$$

\hfill \Box

## 5 The Limit Case $p \to \infty$

In this section we derive some information on $\lambda_2(p,\Omega)$ as $p$ goes to infinity. First of all we recall some known result about the limit of the first eigenvalue. Let us consider a bounded open set $\Omega$.

The anisotropic distance of $x \in \overline{\Omega}$ to the boundary of $\Omega$ is the function

$$d_F(x) = \inf_{y \in \partial\Omega} F^o(x - y), \quad x \in \overline{\Omega}.$$

We stress that when $F = |\cdot|$ then $d_F = d_E$, the Euclidean distance function from the boundary.

It is not difficult to prove that $d_F$ is a uniform Lipschitz function in $\overline{\Omega}$ and

$$F(\nabla d_F(x)) = 1 \quad \text{a.e. in } \Omega.$$

Obviously, $d_F \in W^{1,\infty}_0(\Omega)$. Let us consider the quantity

$$\rho_F = \max(d_F(x), x \in \overline{\Omega}).$$

If $\Omega$ is connected, $\rho_F$ is called the anisotropic inradius of $\Omega$. If not, $\rho_F$ is the maximum of the inradii of the connected components of $\Omega$.

For further properties of the anisotropic distance function we refer the reader to [CM].
Remark 5.1. It is easy to prove (see also [JLM, BKJ]) that the distance function satisfies
\begin{equation}
\frac{1}{\rho_F(\Omega)} = \frac{1}{\|d_F\|_{L^\infty(\Omega)}} = \min_{\varphi \in W_0^{1,\infty}(\Omega) \setminus \{0\}} \frac{\|F(\nabla \varphi)\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^\infty(\Omega)}}.
\end{equation}

Indeed it is sufficient to observe that if \( \varphi \in C^1_0(\Omega) \cap C(\overline{\Omega}) \), then \( \varphi \in C^1_0(\Omega_i) \cap C(\overline{\Omega}_i) \), for any connected component \( \Omega_i \) of \( \Omega \). Then for a.e. \( x \in \Omega_i \), for \( y \in \partial \Omega_i \) which achieves \( F^0(x-y) = d_F(x) \), it holds
\[ |\varphi(x)| = |\varphi(x) - \varphi(y)| = |\langle \nabla \varphi(\xi), x-y \rangle| \leq \|F(\nabla \varphi(\xi))\|_{L^\infty(\Omega)} F^0(x-y) \leq \|F(\nabla \varphi)\|_{L^\infty(\Omega)} d_F(x).\]

Passing to the supremum and by density we get (20).

The following result holds (see [BKJ, JLM]).

Theorem 5.2. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), and let \( \lambda_1(p, \Omega) \) be the first eigenvalue of (7). Then
\[ \lim_{p \to \infty} \lambda_1(p, \Omega)^{\frac{1}{p}} = \frac{1}{\rho_F(\Omega)}. \]

Now let us define
\[ \Lambda_1(\infty, \Omega) = \frac{1}{\rho_F(\Omega)}. \]

The value \( \Lambda_1(\infty, \Omega) \) is related to the so-called anisotropic infinity Laplacian operator defined in [BKJ], that is
\[ \Omega_\infty u = \langle \nabla^2 u J(\nabla u), J(\nabla u) \rangle, \]
where \( J(\xi) = \frac{1}{2} \nabla \xi \[F^2(\xi) \] \xi \). Note that we mean, by continuous extension, \( J(0) = 0 \). This is possible being \( F \) 1-homogeneous and \( F(0) = 0 \).

Indeed, in [BKJ] the following result is proved.

Theorem 5.3. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Then, there exists a positive solution \( u_\infty \in W_0^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \) which satisfies, in the viscosity sense, the following problem:
\begin{align}
\begin{cases}
\min\{F(\nabla u) - \Lambda u, -\Omega_\infty u\} = 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{align}

(21)

with \( \Lambda = \Lambda_1(\infty, \Omega) \). Moreover, any positive solution \( v \in W_0^{1,\infty}(\Omega) \) to (21) with \( \Lambda = \Lambda_1(\infty, \Omega) \) satisfies
\[ \frac{\|F(\nabla v)\|_{L^\infty(\Omega)}}{\|v\|_{L^\infty(\Omega)}} = \min_{\varphi \in W_0^{1,\infty}(\Omega) \setminus \{0\}} \frac{\|F(\nabla \varphi)\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^\infty(\Omega)}} = \Lambda_1(\infty, \Omega) = \frac{1}{\rho_F(\Omega)}. \]

Finally, if problem (21) admits a positive viscosity solution in \( \Omega \), then \( \Lambda = \Lambda_1(\infty, \Omega) \).

Proposition 5.4. Theorem 5.2 holds also when \( \Omega \) is a bounded open set of \( \mathbb{R}^n \).

Proof. Suppose that \( \Omega \) is not connected, and consider a connected component \( \Omega_0 \) of \( \Omega \) with anisotropic inradius \( \rho_F(\Omega) \). By the monotonicity property of \( \lambda_1(p, \Omega) \) given in Proposition 3.3, we have
\[ \lambda_1(p, \Omega) \leq \lambda_1(p, \Omega_0). \]
Then up to a subsequence, passing to the limit as \( p \to +\infty \) and using Theorem 5.2 we have

\[
\bar{\Lambda} = \lim_{p_j \to \infty} \lambda_1(p_j, \Omega)^{\frac{1}{p_j}} \leq \frac{1}{\rho_F(\Omega)}.
\]  

(22)

In order to prove that \( \bar{\Lambda} = \rho_F(\Omega)^{-1} \), let \( u_{p_j} \) the first nonnegative normalized eigenfunction associated to \( \lambda_1(p_j, \Omega) \). Reasoning as in [BKJ], the sequence \( u_{p_j} \) converges to a function \( u_\infty \) in \( C^0(\Omega) \) which is a viscosity solution of \((21)\) associated to \( \bar{\Lambda} \). Then by the maximum principle contained in [BB, Lemma 3.2], in each connected component of \( \Omega \), \( u_\infty \) is either positive or identically zero. Denoting by \( \bar{\Omega} \) a connected component of \( \{u_\infty > 0\} \), by the uniform convergence, for \( p_j \) large, also \( u_{p_j} \) is positive in \( \bar{\Omega} \). Then by Theorem 3.2 we have

\[
\lambda_1(p_j, \bar{\Omega}) = \lambda_1(p_j, \Omega), \quad \text{and then} \quad \frac{1}{\rho_F(\bar{\Omega})} = \bar{\Lambda}.
\]

By (22) and by definition of \( \rho_F \), \( \bar{\Lambda} \leq \rho_F(\Omega)^{-1} \leq \rho_F(\bar{\Omega})^{-1} = \bar{\Lambda} \); then necessarily \( \bar{\Lambda} = \rho_F(\Omega)^{-1} \). \( \square \)

In order to define the eigenvalue problem for \( \Omega_\infty \), let us consider the following operator

\[
A_A(s, \xi, X) = \begin{cases} 
\min\{F(\xi) - \Lambda s, -\langle XJ(\xi), J(\xi)\rangle\} & \text{if } s > 0, \\
-\langle XJ(\xi), J(\xi)\rangle & \text{if } s = 0, \\
\max\{-F(\xi) - \Lambda s, -\langle XJ(\xi), J(\xi)\rangle\} & \text{if } s < 0,
\end{cases}
\]

with \((s, \xi, X) \in \mathbb{R} \times \mathbb{R}^n \times S^{n \times n} \), where \( S^{n \times n} \) denotes the space of real, symmetric matrices of order \( n \). Clearly \( A_A \) is not continuous in \( s = 0 \).

For completeness we recall the definition of viscosity solution for the operator \( A_A \).

**Definition 4.** Let \( \Omega \subset \mathbb{R}^n \) a bounded open set. A function \( u \in C(\Omega) \) is a viscosity subsolution (resp. supersolution) of \( A_A(x, u, \nabla u) = 0 \) if

\[
A_A(\phi(x), \nabla \phi(x), \nabla^2 \phi(x)) \leq 0 \quad (\text{resp. } A_A(\phi(x), \nabla \phi(x), \nabla^2 \phi(x)) \geq 0),
\]

for every \( \phi \in C^2(\Omega) \) such that \( u - \phi \) has a local maximum (resp. minimum) zero at \( x \). A function \( u \in C(\Omega) \) is a viscosity solution of \( A_A = 0 \) if it is both a viscosity subsolution and a viscosity supersolution and in this case the number \( \Lambda \) is called an eigenvalue for \( \Omega_\infty \).

**Definition 5.** We say that \( u \in C(\bar{\Omega}) \), \( u|_{\partial \Omega} = 0, u \neq 0 \) is an eigenfunction for the anisotropic \( \infty \)–Laplacian if there exists \( \Lambda \in \mathbb{R} \) such that

\[
A_A(u, \nabla u, \nabla^2 u) = 0 \quad \text{in } \Omega
\]

(23)

in the viscosity sense. Such value \( \Lambda \) will be called an eigenvalue for the anisotropic \( \infty \)–Laplacian.

In order to define the second eigenvalue for \( \Omega_\infty \) we introduce the following number:

\[
\rho_{2, F}(\Omega) = \sup\{\rho > 0: \text{there are two disjoint Wulff shapes } \mathcal{W}_1, \mathcal{W}_2 \subset \Omega \text{ of radius } \rho\},
\]

and let us define

\[
\Lambda_2(\infty, \Omega) = \frac{1}{\rho_{2, F}(\Omega)}.
\]

Clearly

\[
\Lambda_1(\infty, \Omega) \leq \Lambda_2(\infty, \Omega).
\]
Remark 5.5. It is easy to construct open sets $\Omega$ such that $\Lambda_1(\infty, \Omega) = \Lambda_2(\infty, \Omega)$. For example, this holds when $\Omega$ coincides with the union of two disjoint Wulff shapes with same measure, or their convex envelope.

Remark 5.6. A simple example of $\rho_{2,F}(\Omega)$ is given when $\Omega$ is the union of two disjoint Wulff sets, $\Omega = W_{r_1} \cup W_{r_2}$, with $r_2 \leq r_1$. In this case, $\Lambda_1(\infty, \Omega) = \frac{1}{r_1}$ and, if $r_2$ is not too small, then $\Lambda_2(\infty, \Omega) = \frac{1}{r_2}$.

Theorem 5.7. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $\lambda_2(p, \Omega)$ be the second Dirichlet eigenvalue of $-\Omega_p$ in $\Omega$. Then

$$\lim_{p \to \infty} \lambda_2(p, \Omega)^\frac{1}{p} = \lambda_2(\alpha, \Omega) = \frac{1}{\rho_{2,F}(\Omega)}.$$  

Moreover $\lambda_2(\alpha, \Omega)$ is an eigenvalue of $\Omega_{\infty}$, that is $\lambda_2(\alpha, \Omega)$ is an eigenvalue for the anisotropic infinity Laplacian in the sense of Definition 5.

Proof. First we observe that $\lambda_2(p, \Omega)^\frac{1}{p}$ is bounded from above with respect to $p$. More precisely we have

$$\Lambda_1(\infty, \Omega) \leq \limsup_{p \to \infty} \lambda_2(p, \Omega)^\frac{1}{p} \leq \Lambda_2(\infty, \Omega). \tag{24}$$

Indeed if we consider two disjoint Wulff shapes $W_1$ and $W_2$ of radius $\rho_{2,F}(\Omega)$, clearly $W_1 \cup W_2 \subset \Omega$ and then by monotonicity property (Proposition 4.2) of $\lambda_2(p, \Omega)$ we have

$$\lambda_1(p, \Omega)^\frac{1}{p} \leq \lambda_2(p, \Omega)^\frac{1}{p} \leq \lambda_2(p, W_1 \cup W_2)^\frac{1}{p} = \lambda_1(p, W_1)^{\frac{1}{p}},$$

where last equality follows from Remark 4.4. Then passing to the limit as $p \to \infty$ in the right hand side, by Theorem 5.2 we have (24). Hence there exists a sequence $p_j$ such that $p_j \to +\infty$ as $j \to \infty$, and

$$\frac{1}{\rho_{2,F}(\Omega)} = \Lambda_1(\infty, \Omega) \leq \lim_{j \to \infty} \lambda_2(p_j, \Omega)^{\frac{1}{p_j}} = \overline{\lambda} \leq \Lambda_2(\infty, \Omega) = \frac{1}{\rho_{2,F}(\Omega)}. \tag{25}$$

In order to conclude the proof we have to show that $\overline{\lambda}$ is an eigenvalue for $\Omega_{\infty}$ and that $\overline{\lambda} = \Lambda_2(\infty, \Omega)$.

Let us consider $u_j \in W_0^{1,p}(\Omega)$ eigenfunction of $\lambda_2(p_j, \Omega)$ such that $\|u_j\|_{L^p(\Omega)} = 1$. Then by standard arguments $u_j$, converges, up to a subsequence of $p_j$, uniformly to a function $u \in W_0^{1,\infty}(\Omega) \cap C(\overline{\Omega})$. The function $u$ is a viscosity solution of (23) with $\Lambda = \overline{\lambda}$. Indeed, let $x_0 \in \Omega$. If $u(x_0) > 0$, being $u$ continuous, it is positive in a sufficiently small ball centered at $x_0$. Then it is possible to proceed exactly as in [BKJ] in order to obtain that, in the viscosity sense,

$$\min(F(\nabla u(x_0)) - \overline{\lambda} u(x_0), -\Omega_{\infty} u(x_0)) = 0.$$  

Similarly, if $u(x_0) < 0$ then

$$\max(-F(\nabla u(x_0)) - \overline{\lambda} u(x_0), -\Omega_{\infty} u(x_0)) = 0.$$  

It remains to consider the case $u(x_0) = 0$. We will show that $u$ is a subsolution of (23).

Let $\varphi$ a $C^2(\Omega)$ function such that $u - \varphi$ has a strict maximum point at $x_0$. By the definition of $A_{\overline{\lambda}}$, we have to show that $-\Omega_{\infty} \varphi(x_0) \leq 0$.  

For any \( j \), let \( x_j \) be a maximum point of \( u_j - \varphi \), so that \( x_j \to x_0 \) as \( j \to \infty \). Such sequence exists by the uniform convergence of \( u_j \). By [BKJ, Lemma 2.3] \( u_j \) verifies in the viscosity sense

\[
-\Omega_p u_j = \lambda_2(p_j, \Omega)|u_j|^{p_j - 2} u_j.
\]

Then

\[
- \Omega_p \varphi_j(x_j) =
- (p_j - 2)F^{p_j - 4} (\nabla \varphi(x_j))(\nabla^2 \varphi(x_j))(J(\nabla \varphi(x_j)), J(\nabla \varphi(x_j))) +
- F^{p_j - 2} (\nabla \varphi(x_j)) \nabla^2 \varphi(x_j) \otimes \nabla J(\nabla \varphi(x_j)) =
- (p_j - 2)F^{p_j - 4} (\nabla \varphi(x_j)) \Omega_\infty \varphi_j(x_j) - F^{p_j - 2} (\nabla \varphi(x_j)) \Omega_2 \varphi_j(x_j) \leq
\]

\[
\leq \lambda_2(p_j, \Omega)|u_j(x_j)|^{p_j - 2} u_j(x_j);
\]

here \( A \otimes B := \sum_{i,k} A_{ik}B_{ik} \), for two \( n \times n \) matrices \( A, B \). If \( \nabla \varphi(x_0) \neq 0 \), then dividing the above inequality by \( (p_j - 2)F^{p_j - 4}(\nabla \varphi) \) we have

\[
- \Omega_\infty \varphi_j(x_j) \leq \frac{F^2(\nabla \varphi(x_j)) \Omega_2 \varphi_j(x_j)}{p_j - 2} + \left( \frac{\lambda_2(p_j, \Omega)\frac{1}{p_j - 2}|u_j(x_j)|}{F(\nabla \varphi(x_j))} \right)^{p_j - 4} \frac{u_j(x_j)^3}{p_j - 2} =: t_j.
\]

Passing to the limit as \( j \to \infty \), recalling that \( \varphi \in C^2(\Omega) \), \( F \in C^2(\mathbb{R}^n \setminus \{0\}) \), \( \lambda_2(p_j, \Omega) \to \tilde{\Lambda} \), \( \nabla \varphi(x_0) \neq 0 \) and \( u_j(x_j) \to 0 \) we get

\[
- \Omega_\infty \varphi(x_0) \leq 0.
\]

Finally, we note that if \( \nabla \varphi(x_0) = 0 \), the above inequality is trivially true. Hence, we can conclude that \( u \) is a viscosity subsolution.

The proof that \( u \) is also a viscosity supersolution can be done by repeating the same argument than before, considering \(-u\).

Last step of the proof of the Theorem consists in showing that \( \tilde{\Lambda} = \Lambda_2(\infty, \Omega) \). We distinguish two cases.

**Case 1:** The function \( u \) changes sign in \( \Omega \).

Let us consider the following sets

\[
\Omega^+ = \{ x \in \Omega : u(x) > 0 \} \quad \Omega^- = \{ x \in \Omega : u(x) < 0 \}.
\]

Being \( u \in C^0(\Omega) \) then \( \Omega^+, \Omega^- \) are two disjoint open sets of \( \mathbb{R}^n \) and \( |\Omega^+| > 0 \) and \( |\Omega^-| > 0 \).

By Theorem 5.3 we have

\[
\overline{\Lambda} = \Lambda_1(\infty, \Omega^+) \quad \text{and} \quad \overline{\Lambda} = \Lambda_1(\infty, \Omega^-).
\]

Then by definition of \( \rho_{2, F} \) we get

\[
\rho_F(\Omega^+) = \rho_F(\Omega^-) = \frac{1}{\overline{\Lambda}} \leq \rho_{2, F}(\Omega),
\]

that implies, by (25) that

\[
\overline{\Lambda} = \Lambda_2(\infty, \Omega).
\]

**Case 2:** The function \( u \) does not change sign in \( \Omega \).

We first observe that in this case \( \Omega \) cannot be connected. Indeed since \( u_j \) converges to \( u \) in \( C^0(\overline{\Omega}) \), for sufficiently large \( p \) we have that there exist second eigenfunctions relative to \( \lambda_2(p, \Omega) \) with constant sign in \( \Omega \) and this cannot happen if \( \Omega \) is connected.
Then in this case, we have to replace the sequence \( u_j \) (and then the function \( u \)) in order to find two disjoint connected open subsets \( \Omega_1, \Omega_2 \) of \( \Omega \), such that

\[
\Lambda_1(\infty, \Omega) = \Lambda_1(\infty, \Omega_1)
\]

and

\[
\Lambda = \Lambda_1(\infty, \Omega_2).
\]

Once we prove that such subsets exist, by (25) and the definition of \( \rho_{2,F} \) we obtain

\[
\rho_F(\Omega_2) = \frac{1}{\Lambda} \leq \rho_{2,F}(\Omega) \leq \rho_F(\Omega) = \rho_F(\Omega_1),
\]

that implies, again by (25),

\[
\Lambda = \Lambda_2(\infty, \Omega).
\]

In order to prove (26) and (27), we consider \( u_{1,\infty} \), an eigenfunction associated to \( \Lambda_1(\infty, \Omega) \), obtained as limit in \( C^0(\Omega) \) of a sequence \( u_{1,p} \) of first normalized eigenfunctions associated to \( \lambda_1(p, \Omega) \), and consider a connected component of \( \Omega \), say \( \Omega_1 \), where \( u_{1,\infty} > 0 \) and such that \( \Lambda_1(\infty, \Omega) = \Lambda_1(\infty, \Omega_1) \). The argument of the proof of Proposition 5.4 gives that such \( u_{1,\infty} \) and \( \Omega_1 \) exist. Then, let \( u_{2,p} \geq 0 \) be a normalized eigenfunction associated to \( \lambda_2(\mathbb{R}, \Omega) \) such that for any \( p \) sufficiently large, \( \text{spt}(u_{2,p}) \cap \Omega_1 = \emptyset \).

The existence of such a sequence is guaranteed from this three observations:

- if \( u_{2,p} \) changes sign for a divergent sequence of \( p \)'s, then we come back to the case 1;
- by the maximum principle, in each connected component of \( \Omega \) \( u_{2,p} \) is either positive or identically zero;
- the condition \( \text{spt}(u_{2,p}) \cap \Omega_1 = \emptyset \) depends from the fact that \( u_{2,p} \) can be chosen not proportional to \( u_{1,p} \).

Hence, there exists \( \Omega_2 \) connected component of \( \Omega \) disjoint from \( \Omega_1 \), such that \( u_{2,p} \) converges to \( u_{2,\infty} \) (up to a subsequence) in \( C^0(\Omega_2) \), and where \( u_{2,\infty} > 0 \). By Theorem 5.3, (27) holds.

**Theorem 5.8.** Given \( \Omega \) bounded open set of \( \mathbb{R}^n \), let \( \Lambda > \Lambda_1(\infty, \Omega) \) be an eigenvalue for \( \Omega_\infty \). Then \( \Lambda \geq \Lambda_2(\infty, \Omega) \) and \( \Lambda_2(\infty, \Omega) \) is the second eigenvalue of \( \Omega_\infty \), in the sense that there are no eigenvalues of \( \Omega_\infty \) between \( \Lambda_1(\infty, \Omega) \) and \( \Lambda_2(\infty, \Omega) \).

**Proof.** Let \( u_\Lambda \) be an eigenfunction corresponding to \( \Lambda \). We distinguish two cases.

**Case 1:** The function \( u_\Lambda \) changes sign in \( \Omega \).

Let us consider the following sets

\[
\Omega^+ = \{ x \in \Omega : u_\Lambda(x) > 0 \} \quad \Omega^- = \{ x \in \Omega : u_\Lambda(x) < 0 \}.
\]

Being \( u_\Lambda \in C^0(\Omega) \) then \( \Omega^+, \Omega^- \) are two disjoint open sets of \( \mathbb{R}^n \) and \( |\Omega^+| > 0 \) and \( |\Omega^-| > 0 \).

By Theorem 5.3 we have

\[
\Lambda = \Lambda_1(\infty, \Omega^+) \quad \text{and} \quad \Lambda = \Lambda_1(\infty, \Omega^-).
\]

Then by definition of \( \rho_{2,F} \) we get

\[
\rho_F(\Omega^+) = \rho_F(\Omega^-) = \frac{1}{\Lambda} \leq \rho_{2,F}(\Omega),
\]
that implies, by (25) that
\[ \Lambda \geq \Lambda_2(\infty, \Omega). \]

**Case 2:** The function \( u_\Lambda \) does not change sign in \( \Omega \).

By Theorem 5.3 \( \Omega \) cannot be connected being \( \Lambda > \Lambda_1(\infty, \Omega) \).

In this case, again by By Theorem 5.3 we can find two disjoint connected open subsets \( \Omega_1, \Omega_2 \) of \( \Omega \), such that
\[ \Lambda_1(\infty, \Omega) = \Lambda_1(\infty, \Omega_1) \tag{28} \]
and
\[ \Lambda = \Lambda_1(\infty, \Omega_2). \tag{29} \]

Being \( \Lambda > \Lambda_1(\infty, \Omega) \), we obtain
\[ \rho_F(\Omega_2) = \frac{1}{\Lambda} < \rho_F(\Omega) = \rho_F(\Omega_1), \]
that by the definition of \( \rho_{2,F} \) implies,
\[ \Lambda \geq \Lambda_2(\infty, \Omega). \]

**Remark 5.9.** We observe that if \( \Omega \) is a bounded open set and \( \tilde{W} \) is the union of two disjoint Wulff sets with the same measure \( |\Omega|/2 \), it holds that
\[ \rho_{2,F}(\Omega) \leq \rho_{2,F}(\tilde{W}), \]
that is,
\[ \Lambda_2(\infty, \Omega) \geq \Lambda_2(\infty, \tilde{W}), \]
that is the Hong-Krahn-Szego inequality for the second eigenvalue of \( -\Delta_{\infty} \).

**ACKNOWLEDGEMENTS**

This work has been partially supported by the FIRB 2013 project “Geometrical and qualitative aspects of PDE’s” and by GNAMPA of INdAM.

**REFERENCES**

[AFLT] A.Alvino, V.Ferone, P.L. Lions, G. Trombetti. Convex symmetrization and applications. Ann. Inst. H. Poincaré Anal. non linéaire 14 (1997), 275-293.

[AFT] A.Alvino, V.Ferone, G. Trombetti. On the properties of some nonlinear eigenvalues. SIAM J. Math. Anal. 29 (1998), 437-451.

[BB] G. Barles, J. Busca. Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth order term. Comm. PDE 26 (2001): 2323–2337.

[BP] G. Bellettini, M. Paolini. Anisotropic motion by mean curvature in the context of Finsler geometry. Hokkaido Math. J., 25 (1996): 537-566.
