Directed polymer in a random medium - an introduction

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This is a set of introductory lectures on the behaviour of a directed polymer in a random medium. Both the intuitive picture that helps in developing an understanding and systematic approaches for quantitative studies are discussed.

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I. DIRECTED POLYMERS

A physicist’s polymer is a long string like object, devoid of any specific inner structure. It can be treated as a random walk on a lattice or as an elastic string in continuum. The lattice problem is well-defined though in many situations the continuum version is easier to treat. For the lattice problem, the sites visited are the monomers with the steps as bonds. The monomers are indexed by a coordinate $z$ which becomes a continuous variable in the continuum model. See Fig. 1.

A polymer as of (a) can be drawn in $d+1$ dimensions. This is like a path of a quantum particle in nonrelativistic quantum mechanics.

An important quantity for a polymer is its size or the spatial extent as the length $N$ becomes large. For a translationally invariant system with one end ($z = 0$) fixed at origin, the average position at $z = N$ is zero but the size is given by the rms value

$$\langle r_N \rangle = 0, \quad \langle r_N^2 \rangle^{1/2} \sim N^{\nu},$$

with $\nu = 1/2$, for the free case. In general the size exponent $\nu$ defines the polymer universality class and it depends only on a few basic elements of the polymer. In addition to this geometric property, the usual thermodynamic quantities, e.g. free energy (or energy at $T = 0$), are also important, especially if one wants to study phase transitions.

FIG. 1: (a) A random walk in $d$ dimensions with $z$ as the variable along the contour of the polymer i.e. giving the location of the monomers. (b) Directed polymer on a square lattice. A polymer as of (a) can be drawn in $d+1$ dimensions. This is like a path of a quantum particle in nonrelativistic quantum mechanics.

We consider the problem of a polymer where each monomer sees a different, independent, identically distributed random potentials. Geometrically this can be...
achieved if the monomers live in separate spaces. One way to get that is to consider the polymer to be a $d+1$ dimensional string with the monomers in $d$ dimensional planes but connected together in the extra dimension. As shown in Fig. 1, this is a polymer which is directed in one particular direction. Hence the name directed polymer.

For a directed polymer, the size would now refer to the size in the transverse $d$-directions and so Eq. 1 refers to the transverse size as the length in the special $z$-direction increases.

A. Why bother?

The major issue for the random problem is to see if the universality class of the pure polymer changes to a new one in presence of disorder and if any new phase transition can take place.

Because of randomness (or disorder), any quantity of interest has to be a stochastic variable (realization dependent). Therefore, an additional disorder averaging needs to be done over and above the usual thermal averaging for each realization. For thermal fluctuations, there is generally no need to go beyond second moments or cumulants (related to various response functions like the specific heat, susceptibility etc.) but that cannot be said offhand for the disorder averaging. The randomness is put in by hand and is not thermalizable. There are possibilities of rare events for which it is necessary to distinguish between the average value and the typical (i.e. most probable) value. In such situations higher moments of the quantity concerned become important. It is this aspect that makes disorder problems interesting and difficult.

Example: Typical vs. average
Consider a random variable $x$ that takes two values

$$X_1 = e^{\alpha \sqrt{N}} \text{ and } X_2 = e^{\beta N}, \beta > 1,$$  

(2a)

with probabilities

$$p_1 = 1 - e^{-N}, \text{ and } p_2 = e^{-N}.$$  

(2b)

In the limit $N \to \infty$, the average value $[x]_{av} \to e^{(\beta - 1)N}$ while the typical or most probable value is $x = X_1$ with probability 1. On the other hand $[\ln x]_{av} \to \alpha \sqrt{N}$ in the same limit, showing that $[\ln x]_{av}$ is determined by the typical value of the variable while the moments are controlled by the rare events. Note that this peculiarity disappears if $x$ has a smooth probability distribution in the sense of no special or rare events.

The significance of directed polymer lies in the fact that the pure system is very well understood and exactly solvable in all dimensions. Polymers show critical-like behaviour (e.g., power laws for size) without any need of any fine tuning. One hopes that the understanding gained in this topic on the role of disorder may be useful in other situations as well. Wistful sighs...

Background: Familiarity with the basic ideas of renormalization group or critical phenomena (Ref. 1) and polymers (Refs. 2, 3) would be useful.

II. HAMILTONIAN AND RANDOMNESS

A. Pure case

Taking the polymer as an elastic string, one may define a Hamiltonian

$$H_0 = \frac{d}{2} K \int dz \left( \frac{\partial r}{\partial z} \right)^2$$  

(3)

which gives a normalized probability distribution of the position vector $r$ at length $z$

$$P(r, z) = \frac{1}{(2\pi z)^{d/2}} e^{-r^2/2z} \quad (Kd/k_B T = 1).$$  

(4)

For the lattice random walk, there is no “energy” and the elastic Hamiltonian of Eq. 3 just simulates the entropic effect at non-zero temperatures. One needs to look at the lattice problem in case one is interested in low or zero temperature behaviour.

For a polymer of length $N$ the probability distribution gives

$$\langle r \rangle = 0, \quad \langle r^2 \rangle = N, \quad (Kd/k_B T = 1)$$  

(5)

so that the transverse size of the polymer is given by

$$R_0 \sim \langle r^2 \rangle^{1/2} \sim N^{\nu}$$  

with $\nu = 1/2$.  

(6)

The power law growth of the size of a polymer as the length increases in a reflection of the absence of any “length scale” in the Hamiltonian.

Exercise II.1 Put in $d$, $K$ and $T$ in Eqs. 4, 5, 6. Get Eq. 4 from Eq. 3.

B. Let’s put randomness

1. Hamiltonian

Let us now put this polymer in a random medium. In the lattice model of Fig. 1 each site has an independent random energy and the total energy of the lattice polymer is the sum of the energies of the sites visited. In continuum, the Hamiltonian can be written as

$$H = H_0 + \int_0^N dz \eta(r, z) \delta(r(z) - r)$$  

(7)

where $\eta(r, z)$ is an identical, independent gaussian distributed random variable with zero mean and variance $\Delta > 0$,

$$\langle \eta(r, z) \rangle_{av} = 0, \quad \langle \eta(r, z) \eta(r', z') \rangle_{av} = \Delta \delta(r - r')\delta(z - z').$$  

(8)

With this distribution of random energies, we see $[H]_{av} = H_0$ and so the average Hamiltonian is not of much use.
2. Partition function

The partition function for a polymer in a random medium or potential is given by

\[ Z = \int \mathcal{D}r \, e^{-\beta H}. \]  

(9)

This is a symbolic notation to denote sum over all configurations and is better treated as a continuum limit of a well-defined lattice partition function

\[ Z = \sum_{\text{paths}} e^{-\beta \eta(r,z)} \]  

(10)

where the sum is over all possible paths of \( N \) steps starting from \( r = 0 \) at \( z = 0 \). and \( Z_0 \) is the partition function of the free walker. It often helps to define the partition function such that \( Z(\{\eta = 0\}) = 1 \) to avoid problems of going to the continuum limit (see Eq. 4). This is done by dividing (or normalizing) \( Z \) by \( Z_0 = \mu^N \), \( Z_0 \) being the partition function of the free walker with \( \mu \) as the connectivity constant (\( = 2 \) for Fig. 1).

3. Averages

Because of the randomness, there is no unique partition function but rather a probability distribution of the partition function, \( P(Z) \), is needed. Any quantity of interest needs to be averaged over such a distribution. This averaging is to be called *sample averaging*, denoted by \([\ldots]_{\text{av}} \) (as opposed to thermal averaging, denoted by \( \langle \ldots \rangle \)). Consequently, sample to sample fluctuations may become important and would be a new class of quantities which are not meaningful in the pure thermal case. The extreme situation is the possibility of “rare” events which require distinctions between typical and average behaviour.

C. What to look for?

For the pure case (\( \Delta = 0 \)), there is no “energy”, only configurational entropy of the polymer. But with \( \Delta \neq 0 \), there may be one or more lowest energy states. What is the nature of the ground state?

**Exercise II.2** For the lattice problem, the energy is the sum of \( N \rightarrow \infty \) random energies of the sites visited. This is tantalizingly similar to what one needs for applying the central limit theorem. Is it so?

D. Size: disorder and thermal correlation

So far as the geometrical properties are concerned, we first note the lack of translational invariance for a particular realization of disorder and therefore \( \langle bfr \rangle \neq 0 \), but on averaging over randomness, translational invariance will be restored statistically and so \( \langle (bfr)_{\text{av}} \rangle \) has to be zero. One may therefore consider the size of the polymer by the “disorder” correlations

\[ C_{\text{dis}} \equiv \langle (bfr^2)_{\text{av}} \rangle \text{ or } C_{\text{dis}} \equiv \langle (bfr)_{\text{av}}^2 \rangle \text{ with } C_{\text{dis}} \sim N^{2\nu}. \]  

(11)

Furthermore, the usual correlation function is the thermal correlation

\[ C_T \equiv \langle (bfr^2)_{\text{av}} \rangle - \langle (bfr)_{\text{av}} \rangle^2. \]  

(12)

These are defined at all temperatures. Of course, the probability distribution \( P(\langle bfr \rangle) \) is also of importance.

1. Free energy fluctuation

Apart from these geometric properties one would like to know the fluctuations in free energy at any non-zero temperature (or ground-state energy at \( T = 0 \)). The probability distributions for these quantities would be the ideal things to look for. In absence of knowledge of this distribution, one may look for various moments and fluctuations.

Let us define \( F(r, N) = -T \ln Z(r, N) \), a restricted free energy that the end at \( z = N \) is at \( r \). This free energy, being realization dependent, is also a stochastic quantity. We may then define a correlation function

\[ C_F(r, N) = \langle (F(\mathbf{r} + \mathbf{r}_0, N) - F(\mathbf{r}_0, N))^2 \rangle_{\text{av}}. \]  

(13)

which may be assumed to satisfy a scaling form

\[ C_F(r, N) = N^{2\theta} f \left( \frac{r}{N^\nu} \right). \]  

(14)

Note that the exponent \( \theta \) has no meaning in the pure case where \( \theta = 0 \). This is a new quantity for the disordered system, \( P(r, z) \) of Eq. 4, if used for \( Z(r, z) \), satisfies Eq. 11 with \( \theta = 0 \) and \( \nu = 1/2 \).

For \( r \ll N^\nu \), \( C_F \) should be a function of \( r \). Taking \( f(x) \sim x^p \) for \( x \rightarrow 0 \), we need \( p = \theta/\nu \) and get

\[ C_F(r, N) = r^{2\theta/\nu}, \text{ for } r \ll N^\nu \]

\[ = N^{2\theta}, \text{ for } r \geq N^\nu. \]  

(15)

**2. Our Aim**

Our aim is two-fold: to get the quantitative estimates of \( \nu \) and \( \theta \), and to develop an intuitive picture. In fact, there are quantitative changes in both the exponents but, more importantly, those changes are consequences of a drastic change in the qualitative behaviour of the polymers. At the end, however, a simple picture borrowed from pure polymers remain valid.

**On Notation:**
• To avoid proliferation of symbols, we reserve the symbol \( f \) to denote an arbitrary or unspecified function, not necessarily same everywhere.

• Sample averaging is denoted by \([\ldots]_{av}\) while thermal averaging is denoted by \langle \ldots \rangle.

• The Boltzmann constant is set, most often, to one, \( k_B = 1 \).

• “Disorder” and “randomness” will be used interchangeably.

3. Digression: Self-averaging

Our interest is in large systems. A question of building up such a large system arises. In the pure case, we may add blocks systematically (see Fig. 2) to generate a unique way of going to the limit. For a random system, it is not so straightforward. For a given block \( A \), there are many possibilities for each of \( B, C, D \ldots \), well, calling for a probabilistic description. Even the issue of thermodynamic limit has to be probabilistic in nature; we may expect \( \ln Z \) to be extensive for large size with probability one but not necessarily for each and every realization. A quantity of import would be the probability distribution (sample to sample variation) of the free energy itself, \( P(\ln Z) \). If \( P(\ln Z) \) is sharply peaked for large size, then any large sample would show the average behaviour. A quantity with this property is often called self-averaging. This may not be the case if the distribution is broad especially in the sense discussed in Sec. 1.1. A self-averaging quantity has the advantage that one may study one realization of a large enough system without any need of further disorder or sample averaging.

![FIG. 2: Adding blocks to build a larger system.](image)

To be quantitative, let us choose a quantity \( M \) which is extensive meaning \( M = Nm \) where \( m \) is the “density” or per particle value. This is based on the additive property over subsystems \( M = \sum M_i \). For a random system we better write \( M = M(N, \{Q\}) \), with \( \{Q\} \) representing all the random variables. To recover thermodynamics, we want \( [M]_{av} \) to be proportional to \( N \) for \( N \to \infty \). Now, if it so happens that for large \( N \)

\[
M(N, \{Q\}) \to \text{Nm}_d,
\]

with \( m_d \) independent of the explicit random variables, then \( M \) is said to have the self averaging property. Note that no averaging has been done in Eq. 16. One way to guarantee this self-averaging is to have a probability distribution

\[
P(M/N) \xrightarrow{N \to \infty} \delta(m_d).
\]

This is equivalent to the statement that the sum over a large number of subsystems gives the average value, something that would be expected in case the central limit theorem (CLT) is applicable. But for many critical systems CLT may not be applicable.

A more practical procedure for testing self-averaging behaviour of a quantity \( X \) is to study the fluctuations \( \sigma_X^2 = \langle X^2 \rangle_{av} - \langle X \rangle_{av}^2 \) and then check if

\[
R_{X,N} = \frac{\sigma_X^2}{\langle X \rangle_{av}^2} \to 0, \text{ as } N \to \infty.
\]

One may define more general measures like \( R_{X,N}^{(p)} = \sigma_X^{(p)} / \langle X \rangle_{av}^{(p)} \) where \( \sigma_X^{(p)} \) is the \( p \)th cumulant. A quantity is not self-averaging if the corresponding \( R \) does not decay to zero. A very slow decay (“weakly” self averaging) would also signal practical difficulties. A thumb rule would be that an extensive quantity is generally self-averaging, others may not be.

III. Q.: IS DISORDER RELEVANT?

Let us first see if disorder is at all relevant.

A. Annealed average: low temperature problem

The averaged Hamiltonian is of no use but what about the average partition function \( [Z]_{av} \)? This is called an annealed average. With a gaussian distribution for the random energy, from Eq. 14,

\[
[Z]_{av} = \exp(\beta^2 \Delta N/2) \exp(N \ln \mu),
\]

so that \( F/N = -T \ln \mu + \beta^2 \Delta/2 \) The entropy obtained from this partition function \( S = -\beta F/\beta T \) by definition has to be positive. This gives a limit \( T \geq T_A \equiv \sqrt{\lambda / 2 \ln \mu} \).

Annealed averaging will not work at very low temperatures.

B. Moments of \( Z \)

For thermodynamic quantities we know that it is the average of the free energy that is required. This is called quenched averaging.

We may write

\[
\ln [Z]_{av} = \ln \{ [Z]_{av} + (Z - [Z]_{av}) \}_{av}
= \ln [Z]_{av} + \frac{[Z^2]_{av} - [Z]_{av}^2}{2[Z]_{av}^2} + ...
\]

to show the importance of the variance of the partition function. If the variance remains small, in the limit \( N \to \infty \), then the polymer can be described by the average partition function which is more or less like a pure problem. Otherwise not.
### 1. Hamiltonian for moments

Higher order terms involve higher moments \([Z^n]_{av}\) or rather cumulants. These moments can be written as the partition function of an \(n\)-polymer problem with an extra interaction induced by the disorder. Starting from \(H\) as given by Eq. (4), and the gaussian distribution of Eq. \(8\), we have

\[
[Z^n]_{av} = \int \left( \int DR \, e^{-\beta H} \right)^n P(\eta) \, d\eta
\]

\[
= \int DR_1 \, DR_2 \ldots DR_n e^{-\beta \sum H_{0i} \times [e^{-\beta \int \sum \delta(r_i(z) - r_j(z)) \eta(r, z)]_{av}}}
\]

\[
= \int DR_1 \, DR_2 \ldots DR_n e^{-\beta \sum H_{0i} \times \exp (\beta^2 \Delta \sum_{i<j} \delta(r_i(z) - r_j(z)))}
\]

\[
\equiv \int DR_1 \, DR_2 \ldots DR_n e^{-\beta H_n},
\]

where \(H_n = \frac{dK}{2} \int_0^N dz \sum_i \left( \frac{\partial r_i}{\partial z} \right)^2 - \beta \Delta \int_0^N dz \sum_{i<j} \delta(r_i(z) - r_j(z)) \) \(21\)

This particular form can be understood in terms of two polymers. These two polymers start from the same point and do their random walk as they take further steps. If there is a site which is energetically favourable, both the polymers would like to be there. The effect is like an attractive interaction between the two polymers - an interaction induced by the randomness.

### 2. Bound state: two polymer problem

For the second moment, we have a two polymer problem. The analogy with quantum mechanics tells us that for \(d < 2\), any binding potential can form a bound state but a critical strength is required for a bound state for \(d > 2\). In the polymer language, this means that any small disorder will change the behaviour of the free (pure) chain for \(d < 2\) (disorder is always relevant) but for \(d > d_c = 2\), if \(\beta \Delta < (\beta \Delta)_c\), the chain remains pure-like (disorder is irrelevant). Actually in higher dimensions \((d > 2)\) the delta-function potential needs to be regularized appropriately (e.g., by a “spherical” well).

Such cases are better treated by renormalization group (RG) which also helps in making the definitions of relevance/irrelevance more precise. We discuss this below.

In short, the second term (fluctuation in partition function) in the expansion of Eq. \(20\) cannot be ignored if \(d < 2\) or if \(\beta \Delta\) is sufficiently large for \(d > 2\). This signals a disorder dominated phase for all disorders in low dimensions or at low temperatures (weak disorders) in higher dimensions.

### C. RG approach

It is instructive to develop a renormalization group approach for the two polymer problem.

#### 1. Expansion in potential

We do an expansion in the interaction potential and just look at the first contributing term. Full series can of course be treated exactly. On averaging, the first order terms drop out, yielding

\[
[Z^2]_{av} - [Z]_{av}^2 = \int DR_1 \int DR_2 e^{-\beta H_{01}} e^{-\beta H_{02}} \times dz \beta^2 \Delta \delta(r_1(z) - r_2(z)) + ... \quad 22
\]

This is the first order term if Eq. \(21\) is used. A diagrammatic representation is often helpful in book-keeping as shown in Fig. 3 but we avoid the details here.

#### 2. Reunion

For this two polymer problem, the interaction contributes whenever there is a meeting or reunion of the two polymers at a site. At the order we are considering, there is only one reunion but this reunion can take place anywhere along the chain and anywhere in the transverse direction.

\[
\begin{align*}
\bigtriangledown &= \bigtriangledown + \bigtriangleup
\end{align*}
\]

FIG. 3: Renormalization of the interaction. The polymers are represented by the two lines and an intersection represents an interaction.

The probability that two walkers starting from origin would meet at \(r\) at \(z\) is given by \(P^2(r, z)\) (Eq. \(4\)) so that a reunion anywhere is given by a space integral of this probability which gives

\[
R_z = \int dr P^2(r, z) = (4\pi z)^{-\Psi}, \text{ with } \Psi = d/2. \quad 23
\]

This exponent \(\Psi\) will be called the reunion exponent. The occurrence of a power law is again to be noted. The eventual renormalization group approach hinges on this power law behaviour.
3. Divergences

The contribution in Eq. 22 apart from some constants involve an integral over the reunion behaviour given in Eq. 23. This integral in the limit \( N \to \infty \) is

\[
\int_0^N \frac{dz}{z^{d/2}} \sim a^{1-d/2} \quad \text{for } d > 2 \quad \text{(24)}
\]

\[
\sim N^{1-d/2} \quad \text{for } d < 2. \quad \text{(25)}
\]

For a finite cut-off, as is usually the case, the integral is finite for \( d > 2 \), and therefore \( [(Z - [Z]_{\text{av}})^2]_{\text{av}} \approx O(\beta^2 \Delta) \).

This however is not the case for \( d < 2 \) as a borderline case. The problem we face here is ideal for a renormalization group approach.

4. RG flows

Let us introduce an arbitrary length scale \( L \) in the transverse direction and define a dimensionless “running” coupling constant

\[
u(L) = (\beta^2 K \Delta) L^\epsilon, \quad \epsilon = 2 - d \quad \text{(26)}
\]

and study the RG flow of the coupling constant as the scale \( L \) is changed. This takes care of reunions at small scales to define the effective coupling on a longer scale (with rescaling). The flow equation is

\[ L \frac{du}{dL} = (2 - d)u + u^2 \quad \text{(27)} \]

The magnitude of the coefficient of the \( u^2 \) term is not very crucial because at this order, this coefficient can be absorbed in the definition of the \( u \) itself. What matters is the sign of the \( u^2 \) term.

**Exercise III.1** By change of variables in Eq. 27, show that \( u(L) \) as defined in Eq. 26 is the correct dimensionless variable.

5. Relevant, irrelevant and marginal

A coupling that grows (decays) with length scale is called a relevant (irrelevant) term because at long scales the contribution of this quantity cannot (can) be ignored. A coupling that does not change with scale is called a marginal variable.

For example at the pure fixed point (\( u = 0 \)), disorder is relevant (see Eq. 26) for \( d < 2 \) but becomes irrelevant for \( d > 2 \), while it is marginal at \( d = 2 \). Marginal variables are important because the renormalization procedure may add higher order corrections, like the \( u^2 \) term in Eq. 27, which then determines the growth or decay of the term. Here we see that the disorder is a marginally relevant variable, eventually increasing with length (at \( d = 2 \)). A marginally relevant variable introduces a new phase transition (or fixed point) in higher dimensions. The emergence of the new fixed point for \( d > 2 \) signals a phase transition.

6. Fixed points: strong vs weak disorder

Since \( u \) comes from the variance of the distribution, it cannot be negative. We therefore need to concentrate only on the \( u \geq 0 \) part with the initial condition of \( u(L = a) = u_0 \). What we see that for \( d < 2 \), the flow on the positive axis goes to infinity indicating a strong disorder phase for any amount of disorder provided we look at long enough length scale. An estimate of this length scale may be obtained from the nature of divergence for a given \( u_0 \). An integration of the flow equation gives \( L \sim u_0^{1/(2-d)} \) for \( d < 2 \), a crossover length beyond which the effect of the disorder is appreciable.

![Flow diagram](image)

**FIG. 4**: RG Fixed points for \( u \). (a) Based on the second moment of the partition function. (b) Based on the KPZ equation. Arrows show the flow of \( u \). \( \epsilon = 2 - d \).

For \( d > 2 \), there is a fixed point at \( u^* = |\epsilon| \) where \( \epsilon = 2 - d \). For \( u < u^* \), the disorder strength goes to zero and one recovers a “pure”-like behaviour. This is a weak disorder limit. But, if \( u > u^* \) the disorder is relevant.

D. Ans.: Yes, it is!

Based on the fixed point analysis, we conclude, as already mentioned, that disorder can be relevant depending on the dimensions we are in (i.e. the value of \( d \)) and temperature or strength of disorder. In particular, follows:

1. A disorder-dominated or strong disorder phase for all temperatures for \( d \leq 2 \).
2. A disorder dominated or strong disorder phase at low temperatures for \( d > 2 \).

For \( d > 2 \), one sees a phase transition by changing the strength of the disorder or equivalently temperature for a given \( \Delta \). This is an example of a phase transition induced by disorder which cannot exist in a pure case. The flow equation around the fixed point for \( d > 2 \) shows that one may define a “length-scale” associated with the critical point that diverges as \( \xi \sim |u - u^*|^{-\zeta} \) with \( \zeta = 1/[2-d] \). In the weak disorder phase where the disorder is irrelevant,
[ln Z]_{av} \approx ln[Z]_{av}, and therefore one may put a bound on the transition temperature $T_c$ for a lattice model as $T_c \geq T_4$ as defined below Eq. (19).

In absence of any fixed point for the strong disorder phase in this approach, no further quantitative results can be obtained about the phase itself.

IV. QUANTITATIVE RESULTS ON EXPONENTS

A strong disorder phase has characteristic sample fluctuations (described by the exponent $\theta$). Even the size need not be gaussian like (i.e., $\nu \neq 1/2$). We discuss various ways of getting these quantities, some of which are general and some are very particular for the directed polymer problem.

A. Replica

Even in the absence of any knowledge of the probability distribution like e.g. $P(\ln Z)$, various information can be extracted from the moments. This starts from the simple identity that

$$[Z^n]_{av} = [e^{n \ln Z}]_{av} = \exp \left( \sum_{m} \frac{1}{m!} \langle [\ln Z]^m \rangle_{av} \right),$$

(28)

where $\langle [\ln Z]^m \rangle_{av}$ are the cumulants. As an interacting polymer problem, each polymer of length $N$, one expects an extensive term and a correction which depends on $n$ and $N$, as

$$\ln [Z^n]_{av} = n N \epsilon + \mathcal{F}(n, N).$$

(29)

It is the correction term that is more useful and informative. Our interest is generally in small $n$ and $N \to \infty$, as shown in Fig. 5. This also becomes clear from a more familiar form for the free energy. Eq. (28) can be written as

$$\ln [Z]_{av} = \lim_{n \to 0} \frac{[Z^n]_{av} - 1}{n}.$$  

(30)

so that to compute the average free energy we may consider a case of $n$-replicas of the original system or after averaging, an $n$-polymer problem with extra interactions induced by the disorder though an $n \to 0$ limit is to be taken at the end. A few possible paths to take the limit for long chains are shown in Fig. 5.

Nontrivial results are expected if and only if the origin in Fig. 5 is a singular point so that the limits $n \to 0$ and $N \to \infty$ become non-interchangeable. In other words, the $n$ and $N$ dependences should be coupled so that the appropriate path is a scaling path like (c) in the figure.

Exercise IV.1 Path dependence with rare events

(i) Consider the probability distribution of Eqs. (28) and calculate $[\ln x]_{av}$.

(ii) Calculate the moments $[x^n]_{av}$ and use Eq. (30). Go along paths (a) and (c) of Fig. 5 and compare with the result of (i).

(iii) Show that correct average is obtained if a scaling path is chosen with $n\sqrt{N} = constant$.

IV. QUANTITATIVE RESULTS ON EXPONENTS

$$1. \quad n \to 0 \text{ and } N \to \infty$$

Now, $[(\ln Z)^2]_{av}$ is related to the free energy fluctuation and, as per Eq. (14), it is expected to scale as $N^{2\theta}$. If higher order fluctuations (or cumulants) do not require any new exponent, then it is fair to expect $[(\ln Z)^m]_{av} \sim N^{m\theta}$. A natural choice is $\tilde{F}(n, N) = F(nN^\theta)$.

If we demand that $\ln [Z^n]_{av}$ is proportional to $N$ for large $N$, then, for $x = nN^{\theta} \to \infty$, $\mathcal{F}(x) \sim x^{1/\theta}$ so that

$$\ln [Z^n]_{av} = n \epsilon N + an^{1/\theta} N, \quad (N \to \infty)$$

(31)

This is for path (a) of Fig. 5. In contrast, if we take $n \to 0$ for finite $N$, path (c), a Taylor series expansion gives

$$\ln [Z^n]_{av} = n N \epsilon + an^{2N^{2\theta}} + \ldots, \quad (n \to 0)$$

(32)

Eqs. (31) and (32) can be used to calculate $\theta$, the free energy fluctuation exponent.

Exercise IV.2 Prove that $\ln [Z^n]_{av} \propto N$.

The size of a polymer can also be handled in this replica approach by a careful limit as follows,

$$[< r_N^2 >]_{av} = \lim_{n \to 0} \frac{\int dR r_N^2 \exp(-\beta H)}{\int dR \exp(-\beta H)}_{av}$$

$$= \lim_{n \to 0} \left( \int dR \exp(-\beta H) \right)^{-1} \int dR r_N^2 \exp(-\beta H)_{av}.$$  

(33)

Using the identity $\lim_{n \to 0} \left( \int dR \exp(-\beta H) \right)^n_{av} = 1$, one may consider the partition function of the $n$-replica problem, compute the value of average of $r_N^2$ (without normalization) for one of the replicas and then take $n \to 0$. However this procedure is not easy to implement.

B. Bethe ansatz

The replica approach requires an evaluation of $[Z^n]_{av}$. For a gaussian distributed, delta-correlated disorder,
\([Z^n]_{\text{av}}\) corresponds to the partition function of an \(n\)-polymer system with the Hamiltonian given by Eq. (21). Noting the similarity with the quantum Hamiltonian with \(z\) playing the role of imaginary time, finding \(N^{-1}\ln[Z^n]_{\text{av}}\) for \(N \to \infty\) is equivalent to finding the ground state energy \(E\) of a quantum system of \(n\) particles. This problem can be solved exactly only in one dimension \((d = 1)\) using the Bethe ansatz. This gives the ground state energy as

\[
E = -K(n - n^3) \quad (d = 1)
\]

which gives

\[
\theta = \frac{1}{3} \implies \nu = \frac{2}{3} \quad (d = 1) \quad \text{(see below)}
\]

The polymer has swollen far beyond the random walk or gaussian behaviour. What looks surprising in this approach is that there is no “variance” (2nd cumulant) contribution. It is just not possible to have a probability distribution whose variance vanishes identically! This is a conspiracy of the \(N \to \infty\) limit inherent in the quantum mapping and the value of the exponent \(\theta\) that suppressed the second cumulant contribution (see Eq. 31). To repeat, in this quantum ground state approach, \(N \to \infty\) has been taken first and so there are no higher order terms.

**Exercise IV.3** Why cannot the Bethe ansatz be extended to higher dimensions?

1. Flory approach

Using the quantum analogy, we may try to estimate the ground state energy in a simple minded calculation. The elastic energy is like the kinetic energy of quantum particles which try to delocalize the polymers (random walk) while the attractive potential tries to keep the polymer together. For \(n\) polymers there are \(n(n - 1)/2\) interactions. We take the large \(n\) limit so that if the particles are bound in a region of size \(R\), the energy is (using dimensionally correct form with \(R\) as the only length scale)

\[
E = \frac{n}{R^2} - \frac{n^2}{R} \Delta
\]

which on minimization gives \(E \sim n^3\) consistent with the Bethe ansatz solution. At this point we see the problem of the replica approach if the limit is taken too soon. Since our interest is eventually in \(n \to 0\), we could have used in this argument the linear term of the combinatorics. That would have made energy “extensive” with respect to the number of particle and replaced the disorder-induced attraction by a repulsion (note the negative sign). The end result would however have no \(n^3\) dependence. This is a real danger and any replica calculation has to watch out of these pitfalls. Quite strangely we see that the correct answer came by taking \(n \to \infty\) first and then \(n \to 0\), probably better to say, by staying along the “attractive part” of the interaction only.

**Exercise IV.4** Does the Bethe ansatz require \(n \to \infty\)?

**V. CAN WE HANDLE FREE ENERGY?**

So far we have been looking at the moments of the partition function and trying to extract relevant information from that. Is it possible to avoid the replica trick altogether and treat the free energy explicitly? This is something unique to directed polymers that there is a way to study the average free energy and implement RG directly for the free energy bypassing the \(n \to 0\) problem of Fig. 4 completely. This is a very important step because it gives an independent way of checking the results of replica approach.

**A. Free energy and the KPZ equation**

For a polymer, the partition function satisfies a diffusion or Schrodinger-like equation. This equation can be transformed to an equation for the free energy \(F(r, z) = -T \ln Z(r, z)\). This is the free energy of a polymer whose end point at \(z\) is fixed at \(r\). To maintain the distance fixed at \(r\) a force is required which is given by \(g = -\nabla F\). If we want to increase the length of a polymer by one unit, we need to release the constraint at the previous layer (think of a lattice). The change in free energy would then depend on the force at that point, and of course the random energy at the new occupied site. The change \(\partial F(r)/\partial z\) in free energy required then depend on the two scalars \(\nabla \cdot g, g^2\). A direct derivation of the differential equation for the free energy shows that these are the three terms required. The differential equation, now known as the Kardar-Parisi-Zhang equation, is

\[
\frac{\partial F}{\partial z} = \frac{T}{2K} \nabla^2 F - \frac{1}{2K} (\nabla F)^2 + \eta(r, z).
\]

If we can solve this exact equation and average over the random energy \(\eta\), we get all the results we want.

One may also write down the equation for the force in this “fixed distance” ensemble as

\[
\frac{\partial g}{\partial z} = \frac{T}{2K} \nabla^2 g - \frac{1}{2K} g \cdot \nabla g + \nabla \eta(r, z).
\]

This equation is known as the Burgers equation.

**B. Thermal correlation and an exact result**

In the previous section we considered the force required to maintain the distance fixed. We now consider a fixed force ensemble that helps us in determining the response functions.
It is known in statistical mechanics that the response of a system in equilibrium is determined by the fluctuations. If we consider the response of a directed polymer to an applied force, the averaged response function comes from the Hamiltonian

\[ H = \frac{d}{2} K \int dz \left( \frac{\partial r}{\partial z} \right)^2 + \int \eta(r(z), z) - g \cdot \int \frac{\partial r}{\partial z} dz \]

\[ = \frac{d}{2} K \int dz \left( \frac{\partial}{\partial z} (r - \frac{g^2}{dK}) \right)^2 + \int \eta(r(z), z) - \frac{1}{2} \frac{g^2 N}{d.K}, \]

(39)

The disorder is gaussian-distributed as in Eq. \( S \).

The general response function for the force is

\[ C_T|_{ij} = \frac{\partial^2 [\ln Z]|_{av}}{\partial g_i \partial g_j} \bigg|_{g=0} = \langle r_i r_j \rangle - \langle r_i \rangle \langle r_j \rangle |_{av}, \]

(40)

\( i, j \) representing the components.

1. Exact result on response: pure like

By a redefinition of the variables and using the \( \delta \)-correlation of the disorder in the z direction, we have

\[ [\ln Z(g)]_{av} = [\ln Z(g = 0)]_{av} + \frac{g^2 N}{d.K}, \]

(41)

from which it follows that

\[ C_T = \frac{TN}{d.K}, \]

(42)

as one would expect in a pure system, Eq. \( \bar{\text{4}} \). And there are no higher order correlations.

Two things played important roles in getting this pure-like result: (i) The disorder correlation has a statistical translational invariance coming from the delta function in the z-coordinate, and (ii) the quadratic nature of the Hamiltonian. If disorder had any correlation along the length of the polymer, Eq. \( \bar{\text{4}} \) will not be valid.

Exercise V.1 Put a semi-flexibility term \( (\partial^2 r/\partial z^2)^2 \) in the Hamiltonian and then derive the expression for \( C_T \).

Exercise V.2 Is Eq. \( \bar{\text{72}} \) valid for a lattice problem? Note also that the full range \([-\infty, \infty]\) has been used.

Exercise V.3 Prove that

\[ P(F(r_N)) = P(F(0) + \frac{d.Kr^2}{2N}) \]

Hint: Choose \( g = d.Kr/N \).

C. Free energy of extension: pure like

We want to know the free energy cost in pulling a polymer of length \( N \) from origin (where the other end is fixed) to a position \( r \). For the pure case, the free energy follows from Eq. \( \bar{\text{4}} \) (with \( K \) inserted) as

\[ F(r, N) - F(0, N) = \frac{d}{2} K \frac{r^2}{N} \] (pure) \( \bar{\text{44}} \)

For the disordered case, we use Eq. \( \bar{\text{48}} \) to see the surprising result,

\[ |F(r, N) - F(0, N)|_{av} = \frac{d}{2} K \frac{r^2}{N} \] (disorder) \( \bar{\text{45}} \)

Therefore, on the average the stretching of a chain is pure-like (elastic) with the same elastic constant though the fluctuation is anomalous (\( \theta \neq 0 \)). This result has a far reaching consequence that in a renormalization group procedure, the elastic constant must remain an invariant. As we shall see, this invariance condition puts a constraint on \( \nu \) and \( \theta \), making only one independent.

D. RG of the KPZ equation

To analyze the nonlinear KPZ equation, an RG procedure may be adopted. This RG is based on treating the nonlinear term in an iterative manner by starting from the linear equation. This is a bit unusual because here we are not starting with a “gaussian” polymer problem, rather, a formal linear equation that does not satisfy the exponent relation Eq. \( \bar{\text{45}} \). Leaving aside such peculiarity, one may implement the coarse-graining of RG to see how the couplings change with length scale.

1. Scale transformation and an important relation

As a first step, we do the scale transformation

\[ x \rightarrow bx, z \rightarrow b^{1/\nu} z, \text{ and } F \rightarrow b^{\theta/\nu} F, \]

(46)

so that the randomness transforms like

\[ [\eta(r_1, z_1) \eta(r_2, z_2)]_{av} = \Delta \delta(r_1 - r_2) \delta(z_1 - z_2) \]

\[ \rightarrow b^{-d-(1/\nu)} \Delta \delta(r_1 - r_2) \delta(z_1 - z_2) \]

(47)

This transformation done on Eq. \( \bar{\text{48}} \) shows that for \( K \) to be an invariant (no \( b \)-dependence) we must have,

\[ \theta + 1 = 2\nu \]

(48)

This is trivially valid for the gaussian pure polymer problem but gives a relation between the free energy fluctuation and the size of the polymer. This is borne out by the intuitive picture we develop below. This relation gives the size exponent \( \nu = 2/3 \) in \( d = 1 \) (See Eq. \( \bar{\text{45}} \)).
The equation in terms of the transformed variables is then

\[ \frac{\partial F}{\partial z} = \frac{T}{2K} b^{(1-2\nu)/\nu} \nabla^2 F - \frac{1}{2K} b^{(\theta-2\nu+1)/\nu} (\nabla F)^2 + b^{(1-d\nu-2\theta)/\nu} \eta(r, z) \]  

(49)

The \( \theta \)-dependent factors can be absorbed to define new parameters, except for \( K \).

The temperature however gets renormalized. Its flow is described by the flow equation

\[ L \frac{\partial T}{\partial L} = \frac{1-2\nu}{\nu} T \]  

(to leading order)  

(50)

For \( \nu > 1/2 \), \( T(L) \to 0 \). The disorder dominated phase is therefore equivalent to a zero temperature problem. In other words, the fluctuation in the ground state energy and configurations dominate the behaviour at finite temperatures in situations with \( \nu > 1/2 \). It is this renormalization that was missing in Sec. IIIIC.

The nonlinearity (or \( \theta^2 \)) contributes further to the renormalization of the temperature through the appropriate dimensionless variable \( u = (K\Delta/T^3)L^{2-d} \) (same as in Eq. 29). The important flow equation is for this parameter \( u \) (upto constant factors) and it is

\[ L \frac{du}{dL} = (2-d)u + \frac{2d-3}{4d} u^2. \]  

(51)

One immediately sees a major difference with the flow equation Eq. 27 for \( d = 1 \). Because of the change of sign of the quadratic term in Eq. 41 there is now a f.p. for \( d = 1 \) as shown in Fig. 3(b). This flow equation does not behave properly in a range \( d \in [1.5, 2] \) but that is more of a problem of RG than DP as such and so, may be ignored here. Note also that no extra information can be obtained for \( d \geq 2 \) other than what we have obtained so far in Sec. III. A point to stress is that the fixed point at \( d = 1 \) gives back the exact exponents we have derived so far.

VI. VIRTUAL REALITY: INTUITIVE PICTURE

Powered by the quantitative estimates of the free energy fluctuation and size exponents, we now try to generate a physical picture. To do this we exploit a few more exact results.

A. Rare events

How to understand the exact result of Eq. 12? We have shown that there is a low temperature region or in lower dimensions, disorder or randomness results in a new phase but then its response is the same as the pure system?

For the pure case as \( N \to \infty \) the width of \( P(r_N, N) \) increases. Hence the increase of \( C_T \) with \( N \). With randomness, for \( T \to 0 \) we need to look for the minimum energy path. Let us suppose that there is a unique ground state, i.e. \( E(r_N) \) or \( F(r_N) \) is a minimum for a particular path. This tells us that as the temperature is changed \( T \) still low, the polymer explores the nearby region so that the probability distribution gains some width which is determined by the thermal length. Susceptibility would be the width of the distribution and this is independent of \( N \). This cannot satisfy the relation. Since the relation is valid on the average, one way to satisfy it is to assume that most of the samples have a unique ground-state but once in a while (rare samples) there may be more than one ground state which happens to be far away from each other. Suppose there are such rare samples, whose probabilities decay as \( N^{-\kappa} \), where the paths are separated by \( N^\nu \), then the contribution to the fluctuation from these samples would be \( N^{2\nu-\kappa} \). In case \( \kappa = 2\nu-1 \), we get back the exact result. The relation of Eq. 45 tells us \( \kappa = \theta \). The rare events control the free energy fluctuation.

What we see here that though the average behaviour is the same as that of the pure system, the underlying phenomenon is completely different; the average thermal response is determined by the rare samples that have widely separated degenerate ground state and the probability of such states also decay as a power law of the chain length.

In a given sample elastic energy \( \sim r^2/N \sim N^{2\nu-1} \). The pinning energy would also grow with length say as \( N^\bar{\theta} \). We see, \( \bar{\theta} = 2\nu - 1 = \theta \). One way to say this is that the sample to sample fluctuation and the energy scale for a given sample are the same.

FIG. 6: Various paths for various locations of the end point.

These results can now be combined for an image of the minimum energy paths. If the end points at \( z = N \) are separated by \( r \ll N^\nu \), the paths remain separated (each path exploring an independent disordered region) until they join at \( \Delta z \sim r^{1/\nu} \), after which they follow the same path. If the end points are separated by a distance \( r \gg N^\nu \), the two paths explore independent regions and they need not meet. This (Fig 6) picture is often alluded to as the river-basin network.

B. Probability distribution

For a pure polymer, the probability distribution of the end point is gaussian but it need not be so for the disordered case. One way to explore the probability dis-
tration is to study the response of the polymer as we take it out of its optimal or average position, e.g. by applying a force. In a previous section we used the fixed distance ensemble where the end point was kept fixed and we looked at the force $g$ required to maintain that distance (see Eq. 55). Here we consider the conjugate fixed force ensemble.

1. Response to a force

Let us apply a force that tries to pull the end of the polymer beyond the equilibrium value $r \sim R_0$. In equilibrium, the average size $R$ or extension by the force can be expressed in a scaling form

$$R = R_0 f \left( \frac{gR_0}{T} \right).$$  \hspace{1cm} (52)

This is because for zero force one should get back the unperturbed size while the force term may enter only in a dimensionless form in the above equation where the quantities available are the size $R_0$ and the thermal energy. For small $g$, linearity in $g$ is expected. This requirement gives

$$R = R_0 \frac{R_0}{T} g (k_B = 1),$$  \hspace{1cm} (53)

$R$ is not proportional to $N$ if $\nu \neq 1/2$ ($R_0 \sim N^\nu$). The polymer acts as a spring with $T/R_0^2$ as the effective spring constant.

2. Scaling approach

Let us develop a physical picture and the corresponding algebraic description (called a scaling theory). The polymer in absence of any force has some shape of characteristic size $R_0$. The force stretches it in a way that it breaks up into blobs of size $\xi_g = T/g$. For size $< \xi_g$ the polymer looks like a chain without any force and these blobs, connected linearly by geometry, act as a “new” polymer to respond to the force by aligning along it. We now have two scales $R_0$ and $\xi_g = T/g$. A dimensionless form is then

$$R = R_0 f \left( \frac{R_0}{\xi_g} \right) \sim \frac{R_0^2}{T}.$$  \hspace{1cm} (54)

Now each blob is of length $N_g$ so that $\xi_g = N_g^\nu$ and there are $N/N_g$ blobs. We therefore expect

$$R = \frac{N}{N_g} \xi_g = N \xi_g^{1-1/\nu} = N \left( \frac{g}{T} \right)^{(\nu-1)/\nu}.$$  \hspace{1cm} (55)

This gives a susceptibility $\chi = \partial R/\partial g \sim g^{(1-2\nu)/\nu}$.

3. And so the probability distribution is

Let us try to get the susceptibility of Eq. 55 in another way. Let us assume that the probability distribution for large $R$ is

$$P(R) \sim \exp \left( -\frac{r}{R_0} \right).$$  \hspace{1cm} (56)

The entropy is given by $S(r) = -\ln P$. The free energy in presence of a force which stretches the polymer to the tail region is given by

$$F = T \left( \frac{r}{R_0} \right) - gr.$$  \hspace{1cm} (57)

This on minimization gives $g = \frac{T}{T_0} \left( \frac{r}{R_0} \right)^{\delta-1}$. By equating this form with Eq. 55, we get $\delta = 1/(1-\nu)$ and

$$P(R) \sim \exp \left( -\left( \frac{r}{R_0} \right)^{1/(1-\nu)} \right).$$  \hspace{1cm} (58)

For $\nu = 1/2$ we do get back the gaussian distribution.

![FIG. 7: A blob picture of the polymer under a force. Though drawn as sphere, the z-direction is elongated with isotropy in the transverse direction.](image)

The above analysis, done routinely for polymers, relies on the fact that there is only one length scale in the problem, namely, the size of the polymer. If we are entitled to do the same for the disorder problem, namely only one scale, $R_0 \sim N^\nu$, matters, then the blob picture goes through in toto. The chain breaks up into “blobs” and the blobs align as dictated by the force. Each blob is independent and the polymer inside a blob is exploring its environment like a DP pinned at one end. The probability distribution is therefore given by Eq. 55 which for $d = 1$ is

$$P(r) \sim \exp \left( -\left| r \right|^3 / l^2 \right) \quad (d = 1).$$  \hspace{1cm} (59)

If we use the relation $\Delta F \equiv F(x, N) - F(0, N) \sim x^2 / N$, then the above probability distribution can be mapped to the distribution of the free energy as

$$P(F) \sim \exp \left( -\frac{\left| \Delta F \right|^3}{N^{\nu/2}} \right) \quad (d = 1).$$  \hspace{1cm} (60)
C. Confinement energy

Suppose we confine the polymer in a tube of diameter $D$. This is like the localization length/argument used to justify the energy in the quantum formulation. The polymer in the random medium won’t feel the wall until its size is comparable to that, $D \sim N_0^\nu$ which gives the length at which the polymer feels the wall. Elastic energy of a blob is $D^2/N_0$. But because of the tube, the polymer will be stretched in the tube direction. The number of blobs is $N/N_0$, so that the energy is

$$\frac{N}{N_0} \frac{D^2}{N_0} = N \left( \frac{D}{N_0} \right)^2 = N \frac{1}{D^{2(1-\nu)/\nu}}. \quad (61)$$

This gives the known form $1/D^2$ used in the quantum analogy (consistent with dimensional analysis) but for $\nu = 2/3$, this gives $1/D$.

A cross-check of this comes from the energy of a blob. Each blob has the fluctuation energy $N_0^\nu$ and so free energy per unit length $F/N \sim N_0^\nu/N_0 \sim D^{-2(1-\nu)/\nu}$.

VII. OVERLAP

The problem we face in a disordered system is that there is no well defined ground state - the ground state is sample dependent. There is therefore no predefined external field that will force the system into its ground state (e.g., a magnetic field in a ferromagnetic problem). This is a generic problem for any random system.

But, suppose, we put in an extra fictitious (ghost) polymer and let it choose the best path. Now we put in the actual polymer in the same random medium but with a weak attraction $v$ with the ghost. At $T = 0$, this DP will then sit on top of the ghost. In absence of any interaction ($v \to 0$), the DP would go over the ghost in any case if there is a unique ground state, not otherwise. At non-zero temperatures there will be high energy excursions and how close to the ground state we are will be determined by the number of common points of the two polymers. This is called overlap which may be quantitatively defined as

$$q_i = \frac{1}{N} \int \delta(r_1(z) - r_2(z)), \quad (62)$$

for a given sample $i$ and then one has to average over the disorder samples, $q \equiv \langle q_i \rangle_{\text{av}}$.

In case of a repulsive interaction, the situation will be different. If there are more than one ground state, the two chains will occupy two different paths and there is no energetic incentive to collapse on top of each other when the repulsion $v \to 0+$. In such a scenario, the overlap $q(v \to 0+) \neq q(v \to 0-)$. Conversely, a situation like this for the overlap would indicate presence of degenerate ground states. For a unique ground state, the second chain would follow a nearby excited path with certain amount of overlaps with the ground state.

In a replica approach, such occupancy of different ground states may be achieved by “replica symmetry breaking” (i.e., various replicas occupying various states) but the difficulty arises from the $n \to 0$ limit. In the case of directed polymer, we have argued that the degenerate states occur only rarely and therefore the effect of “replica symmetry breaking” if any has to be very small. This is why the Bethe ansatz gave correct results without any breaking of replica symmetry.

The method to compute the overlap was developed by Mukherji. By introducing a repulsive potential $v f dz \delta(r_1(z) - r_2(z))$ for the two polymers in the same random medium, the free energy $F(r_1(z), r_2(z), z, v)$ can be shown to satisfy a modified KPZ type equation

$$\frac{\partial F}{\partial z} = \sum_i \left( \frac{T}{2K} \nabla^2 F - \frac{1}{2K} (\nabla_i F)^2 + 2q(r_i, z) \right) + v \delta(r_1 - r_2). \quad (63)$$

which can be studied by RG. The mutual interaction has no effect on the single chain behaviour but the interaction gets renormalized. The flow equation for the dimensionless parameter $u$ of Eq. 27 remains the same. The exponent relation of Eq. 37 also remains valid. The interaction gets renormalized as

$$L \frac{\partial v}{\partial L} = \left( \frac{1}{\nu} - d + \frac{u}{2} \right) v, \quad (64)$$

where $v$ is in a dimensionless form. For the pure problem ($\theta = 0, \nu = 1/2$) this reduces to the expected flow equation of Eq. 27 for repulsive interaction ($u \to -u$). For overlap one needs only the first order term because we need $v \to 0$.

The overlap can be written in a polymer-type scaling form $q = N^\Sigma f(vN^{-\phi \nu})$, where $\Sigma = \theta - \phi \nu - 1$. The above RG equation for $v$ shows that the exponent $\Sigma = 0$ at the stable fixed point for $u$ of Fig 2(b) at $d = 1$. However, $\Sigma < 0$ at the transition point for $d > 2$. This means that the overlap vanishes at the transition point from the strong disorder side as $q \sim |T - T_c|^{2\Sigma}$. This approach to overlap can be extended to $m$-chain overlaps also, which show a nonlinear dependence on $m$ at the transition point. This suggests that even though the size exponent is $\nu = 1/2$ gaussian like, there is more intricate structure than the pure gaussian chain.

VIII. SUMMARY AND OPEN PROBLEMS

The behaviour of a directed polymer in a random medium in $1+1$ dimensions seems to be well understood. There is a strong disordered phase at all temperatures. For $d > 2$ renormalization group analysis shows a phase transition from a low temperature strong disordered phase to a weak disorder, pure-like phase. There are rare configurations with degenerate widely separated ground states, giving a contribution to “overlap”. However most questions on the higher dimensional strong-disorder phase remain open.
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