ON ZAGIER’S CONJECTURE ABOUT THE INVERSE OF A MATRIX RELATED TO DOUBLE ZETA VALUES

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Abstract. We prove a conjecture of Zagier about the inverse of a \((K-1) \times (K-1)\) matrix \(A = A_K\) using elementary methods. This formula allows one to express the product of single zeta values \(\zeta(2r)\zeta(2K+1-2r), 1 \leq r \leq K-1,\) in terms of the double zeta values \(\zeta(2r, 2K+1-2r), 1 \leq r \leq K-1\) and \(\zeta(2K+1).\)

1. Introduction

This paper addresses a conjecture of Zagier he put forward in the paper [6]. Following [6], for positive integers \(k_1, \ldots, k_n\) with \(k_n \geq 2,\) define the multiple zeta value \(\zeta(k_1, k_2, \ldots, k_n)\) by

\[
\zeta(k_1, k_2, \ldots, k_n) = \sum_{1 \leq m_1 < \ldots < m_n} \frac{1}{m_1^{k_1} \ldots m_n^{k_n}}. \tag{1.1}
\]

\(k = k_1 + k_2 + \ldots + k_n\) is called the weight of this multiple zeta value.

When \(n = 1,\) we have the classical Riemann zeta value

\[
\zeta(k) = \sum_{m=1}^{\infty} \frac{1}{m^k}.
\]

When \(n = 2,\) the double sum

\[
\zeta(k_1, k_2) = \sum_{m=2}^{\infty} \frac{1}{m^{k_2}} \sum_{j=1}^{m-1} \frac{1}{j^{k_1}} \tag{1.2}
\]

has been considered by Euler.

Let \(H(0) = 1\) and define

\[
H(n) = \zeta(2, 2, \ldots, 2) \quad \text{for } n \geq 1.
\]

It is well known that for \(n \geq 0,\)

\[
H(n) = \frac{\pi^{2n}}{(2n+1)!}.
\]

When \(a\) and \(b\) are nonnegative integers, define

\[
H(a, b) = \zeta(\underbrace{2, \ldots, 2}_{a}, 3, 2, \ldots, 2) \underbrace{\ldots}_{b}.
\]

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In \cite{6}, Zagier derived the formula
\[ H(a, b) = 2 \sum_{r=1}^{K} (-1)^r \left[ \frac{2r}{2a+2} - \frac{1}{2^{2r}} \right] \left( \frac{2r}{2b+1} \right) H(K-r) \zeta(2r+1), \] (1.3)
where \( K = a + b + 1 \). Here for a real number \( n \) and a nonnegative integer \( k \), we define the generalized binomial coefficient \( \binom{n}{k} \) by
\[
\binom{n}{k} = \begin{cases} 
  \frac{n(n-1)\ldots(n-k+1)}{k!}, & k \geq 1, \\
  1, & k = 0.
\end{cases}
\]
In particular, if \( n \) is an integer and \( n < k \), \( \binom{n}{k} = 0 \).

The formula (1.3) expresses \( H(a, b) \) as rational linear combinations of \( H(K-r) \zeta(2r+1) \). It can be used to prove that the following two sets
\begin{align*}
\mathcal{B}_1 &= \{ H(a, K-1-a) \mid 0 \leq a \leq K-1 \}, \\
\mathcal{B}_2 &= \{ H(K-r) \zeta(2r+1) \mid 1 \leq r \leq K \}
\end{align*}
span the same \( \mathbb{Q} \)-vector space over \( \mathbb{Q} \). On the other hand, Euler has found that all double zeta values of odd weight can be expressed as rational linear combinations of the Riemann zeta values. In particular, when \( 1 \leq r \leq K-1 \),
\[ \zeta(2r, 2K+1-2r) = \frac{1}{2} \zeta(2K+1) + \sum_{s=1}^{K-1} A_{r,s} \zeta(2s) \zeta(2K+1-2s), \] (1.4)
where
\[ A_{r,s} = \left( \frac{2K-2s}{2r-1} \right) + \left( \frac{2K-2s}{2K-2r} \right). \] (1.5)
In \cite{6}, Zagier used an elementary argument to show that the \( (K-1) \times (K-1) \) matrix \( A = A_K = [A_{r,s}] \) has nonzero determinant, and thus it is invertible. Using the fact that both \( \zeta(2s) \) and \( H(2s) \) are rational multiples of \( \pi^{2s} \), this shows that the set
\[ \mathcal{B}_3 = \{ \zeta(2r, 2K+1-2r) \mid 1 \leq r \leq K-1 \} \cup \{ \zeta(2K+1) \} \]
spans the same \( \mathbb{Q} \)-vector space as the set \( \mathcal{B}_2 \).

In \cite{6}, Zagier formulated three conjectures about the matrix \( A_K \). The first one is about its determinant, the second one is about its \( LU \)-decomposition, and the third one is about its inverse. The main objective of this paper is to prove the third conjecture, which states a pair of conjectural formulas for \( A^{-1} \).

Let \( P \) and \( Q \) be the \( (K-1) \times (K-1) \) matrices with entries
\[
P_{s,r} = \frac{2}{2s-1} \sum_{n=0}^{2K-2s} \binom{2r-1}{2K-2s-n+1} \binom{n+2s-2}{n} B_n, \\
Q_{s,r} = -\frac{2}{2s-1} \sum_{n=0}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n.
\]
Here \( B_n \) are the Bernoulli numbers.
Zagier conjectured that both $P$ and $Q$ are the inverse of $A$. This implies that $P$ and $Q$ must be the same matrix. This conjecture was proved by D. Ma in [4] using generating functions. In this work, we use a totally different approach.

In Section 2, we will prove that $P = Q$. In Section 3, we will prove that they indeed give the inverse of $A$.

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2. The equality of the two conjectural formulas

Recall that the Bernoulli numbers $B_n$ are defined by the generating function [3]:

$$
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.
$$

One can compute $B_n$ recursively by $B_0 = 1$, and

$$
B_n = -n! \sum_{k=0}^{n-1} \frac{B_k}{k!(n-k+1)!}, \quad \text{for } n \geq 1.
$$

It is well known that $B_1 = -1/2$, and for any odd integer $n$ larger than 1, $B_n = 0$.

Let $K$ be an integer greater than or equal to 2, and let $P$ and $Q$ be the $(K - 1) \times (K - 1)$ matrices with entries defined by

$$
P_{s,r} = \frac{2}{2s-1} 2^{K-2s} \sum_{n=0}^{2K-2s} \left( \frac{2r - 1}{2K - 2s - n + 1} \right) \binom{n + 2s - 2}{n} B_n,
$$

$$
Q_{s,r} = -\frac{2}{2s-1} 2^{K-2s} \sum_{n=0}^{2K-2s} \left( \frac{2K - 2r}{2K - 2s - n + 1} \right) \binom{n + 2s - 2}{n} B_n.
$$

(2.1)

Our goal is to prove that $P_{s,r} = Q_{s,r}$ for all $1 \leq r, s \leq K - 1$ using a generating function technique that is totally different from that used in [4]. We begin with the following theorem which is interesting of its own right.

Theorem 2.1. Let $s$ be a positive integer. Define the function $f_s(t)$ by

$$
f_s(t) = \frac{t^{2s-1}}{e^t - 1}.
$$

If $m$ is a positive integer, we have the following relation that relates the derivatives of $f_s$ up to order $m$.

$$
e^t f_s^{(m)}(t) = \sum_{p=0}^{m} (-1)^{m-p} \binom{m}{p} f_s^{(p)}(t)
$$

$$
+ \sum_{p=0}^{\min\{m,2s-1\}} (-1)^{m-p} \binom{m}{p} \frac{(2s-1)!}{(2s-1-p)!} t^{2s-1-p}.
$$

(2.2)

\footnote{In [6], the summations of $n$ in (2.1) are taken to be until the term $n = 2K - 2s + 1$. However, since $s \leq K - 1$, $2K - 2s + 1$ is an odd number greater than 2, and so $B_{2K - 2s + 1} = 0$. Thus the summations can be taken to be until $n = 2K - 2s$ only.}
Proof. By the definition of $f_s(t)$, we have
\[ f_s(t) = e^{-t} f_s(t) + t^{2s-1} e^{-t}. \]
Differentiate both sides $m$ times and apply Leibniz rule, we have
\[ f_s^{(m)}(t) = \sum_{p=0}^{m} \binom{m}{p} f_s^{(p)}(t) \frac{d^{m-p}}{dt^{m-p}} e^{-t} + \sum_{p=0}^{m} \binom{m}{p} \frac{d^p}{dt^p} t^{2s-1} \frac{d^{m-p}}{dt^{m-p}} e^{-t}. \]
Since
\[ \frac{d^p}{dt^p} t^{2s-1} = 0 \quad \text{if } p > 2s - 1, \]
we find that
\[ f_s^{(m)}(t) = \sum_{p=0}^{\min(m,2s-1)} (-1)^{m-p} \binom{m}{p} e^{-t} f_s^{(p)}(t) \]
\[ + \sum_{p=0}^{\min(m,2s-1)} (-1)^{m-p} \binom{m}{p} e^{-t} \frac{(2s-1)!}{(2s-1-p)!} t^{2s-1-p}. \]
Multiply both sides by $e^t$ give (2.2).

Now we can prove the main theorem in this section.

**Theorem 2.2.** If $K$ is a positive integer larger than or equal to 2, $r$ and $s$ are positive integers less than $K$, then
\[ \frac{(2s-2)!}{(2K-2r)!} \sum_{n=0}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n = - \frac{(2s-2)!}{(2K-2r)!} \sum_{n=0}^{2K-2s} \binom{2r-1}{2K-2s-n+1} \binom{n+2s-2}{n} B_n. \]
(2.3)

In particular, this implies that the matrices $P$ and $Q$ defined by (2.1) are equal.

**Proof.** The left hand side of (2.3) can be rewritten as
\[ \sum_{n=\max\{0,2r-2s+1\}}^{2K-2s} \frac{1}{(2K-2s-n+1)!} \frac{(n+2s-2)!}{n!} B_n, \]
(2.4)
and the right hand side of (2.3) can be rewritten as
\[ - \sum_{n=\max\{0,2K-2s-2r+2\}}^{2K-2s} \frac{(2r-1)!}{(2K-2s-n+1)!} \frac{(n+2s-2)!}{(2r+2s-2K+n-2)!} (2K-2r)! \frac{1}{n!} B_n. \]
(2.5)

The proof of (2.3) is by taking $m = 2r - 1$ in the equation (2.2) and comparing the coefficients of $t^{2K-2r}$ on both sides. Namely, we want to compare the coefficients of $t^{2K-2r}$ on both sides of the equation
\[ e^t f_s^{(2r-1)}(t) = - \sum_{p=0}^{2r-1} (-1)^p \binom{2r-1}{p} f_s^{(p)}(t) \]
\[ + \sum_{p=0}^{\min\{2r-1,2s-1\}} (-1)^p \binom{2r-1}{p} \frac{(2s-1)!}{(2s-1-p)!} t^{2s-1-p}. \]
(2.6)
Notice that
\[ f_s(t) = t^{2s-2} \times \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n 2s-2. \]

Therefore,
\[ f_s^{(p)}(t) = \sum_{n=\max\{0,p-2s+2\}}^{\infty} B_n \frac{(n+2s-2)!}{(n+2s-2-p)!} t^{n+2s-2-p}. \]

First we consider the coefficient of \( t^{2K-2r} \) in the left hand side of (2.6), namely, the coefficient of \( t^{2K-2r} \) in
\[ e^t f_s^{(2r-1)}(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=\max\{0,2r-2s+1\}}^{\infty} B_n \frac{(n+2s-2)!}{(n+2s-2-r-1)!} t^{n+2s-2-r-1}. \] (2.7)

It is given by the expression (2.5).

The term on the right hand side of (2.6) can be written as \( T_1 + T_2 \), where
\[ T_1 = - \sum_{p=0}^{2r-1} (-1)^p \binom{2r-1}{p} f_s^{(p)}(t) \]
\[ = - \sum_{p=0}^{2r-1} (-1)^p \frac{(2r-1)!}{p!(2r-1-p)!} \sum_{n=\max\{0,p-2s+2\}}^{\infty} B_n \frac{(n+2s-2)!}{(n+2s-2-p)!} t^{n+2s-2-p} \]
\[ = - \sum_{n=0}^{\infty} \sum_{p=0}^{\min\{2r-1,n+2s-2\}} (-1)^p \frac{(2r-1)!}{p!(2r-1-p)!} B_n \frac{(n+2s-2)!}{(n+2s-2-p)!} t^{n+2s-2-p}, \]
\[ T_2 = - \sum_{p=0}^{\min\{2r-1,2s-1\}} (-1)^p \frac{(2r-1)!}{p!(2r-1-p)!} \frac{(2s-1)!}{(2s-1-p)!} t^{2s-1-p}. \]

\( T_2 \) contains a term in \( t^{2K-2r} \) if and only if \( 2s-1 \geq 2K-2r \), or equivalently, \( r+s \geq K+1 \). In this case the coefficient of \( t^{2K-2r} \) in \( T_2 \) is
\[ \frac{(2r-1)!}{(2r+2s-2K-1)!(2K-2s)!} \frac{(2s-1)!}{(2K-2r)!}. \] (2.8)

For the term \( T_1 \), the coefficient of \( t^{2K-2r} \) is
\[ - \sum_{n=\max\{0,2K-2s-2r+2\}}^{2K-2s} (-1)^n \frac{(2K-2s-n)!}{(2K-2s-n+1)!(2r+2s-2K+n-2)!} \frac{(2r-1)!}{(2K-2r)!} B_n \frac{(n+2s-2)!}{n!}. \] (2.9)

When \( r+s \leq K \), \( 2K-2s-2r+2 \geq 2 \). Hence, the sum over \( n \) in (2.9) does not contain \( n = 1 \) term. Since \( B_n = 0 \) when \( n \) is odd and larger than 2, we find that (2.9) is equal to (2.5). Since there are no contribution from \( T_3 \) to the term \( t^{2K-2r} \) when \( r+s \leq K \), this proves that when \( r+s \leq K \), the coefficient of \( t^{2K-2r} \) in the right hand side of (2.6) is (2.5).
When \( r + s \geq K + 1 \), there is a term with \( n = 1 \) in (2.9). Using the fact that \( B_1 = \frac{1}{2} \), we find that this term is given by
\[
-\frac{1}{2} \frac{(2r - 1)!}{(2r + 2s - 2K - 1)!} \frac{(2s - 1)!}{(2K - 2r)!},
\]
which is \(-1/2\) of the term (2.8). Summing the coefficients of \( t^{2K-2r} \) from \( T_1 \) and \( T_2 \), we find that the sum is equal to (2.5). Therefore, when \( r + s \geq K + 1 \), the coefficient of \( t^{2K-2r} \) in the right hand side of (2.6) is (2.5).

Thus, we have shown that the coefficient of \( t^{2K-2r} \) in the left hand side of (2.6) is (2.4), and the coefficient of \( t^{2K-2r} \) in the right hand side of (2.6) is (2.5), this completes the proof of the theorem.

\[\square\]

As we mentioned before, this theorem has been proved in [4] using a totally different method, with the help of the Carlitz’s Bernoulli number identity [1, 5]. Our proof uses directly the generating function of the Bernoulli numbers.

**Remark 2.3.** Carlitz’s identity says that for any nonnegative integers \( m \) and \( n \),
\[
(-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k}. \tag{2.10}
\]
Prodinger [5] gave a short proof using an exponential generating function of two variables. Here we show that this identity can be derived directly from (2.2) by setting \( s = 1 \). Namely, we consider the generating function of the Bernoulli numbers
\[
f(t) = e^t - 1.
\]
Since (2.8) is symmetric in \( m \) and \( n \), it is sufficient to consider the case \( 0 \leq m < n \). In this case, \( n \geq 1 \). Equation (2.2) says that
\[
e^t f^{(m)}(t) = (-1)^m t + (-1)^{m-1} m + \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f^{(k)}(t).
\]
This gives
\[
\sum_{l=0}^{\infty} \frac{t^l}{l!} \sum_{k=m}^{\infty} \frac{B_k}{(k-m)!} t^{k-m} = (-1)^m t + (-1)^{m-1} m + \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{l=k}^{\infty} \frac{B_l}{l!} t^{l-k}.
\]
Compare the coefficients of \( t^n \) on both sides, we have
\[
\sum_{k=m}^{m+n} \frac{1}{(k-m)!(m+n-k)!} B_k = (-1)^m \delta_{n,1} + \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \frac{B_{n+k}}{n!}.
\]
If \( n = 1 \), the last sum contains the term \((-1)^m B_1\), which can be combined with \((-1)^m \delta_{n,1}\) to yield \((-1)^{m-1} B_1\). Since \( B_k = 0 \) when \( k \) is odd and greater than 2, we find that
\[
\sum_{k=m}^{m+n} \frac{n!}{(k-m)!(m+n-k)!} B_k = \sum_{k=0}^{m} (-1)^{m-n} \binom{m}{k} B_{n+k}.
\]
Multiplying \((-1)^n\) on both sides and shifting the summation variables on the left hand side, we obtain
\[
(-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k},
\]
which is the Carlitz identity.

3. The proof of the conjecture

In this section, we prove that the inverse of the matrix \(A = A_K\) is the matrix \(P\).

**Theorem 3.1** (Zagier’s Conjecture).

If \(K\) is an integer larger than 1, \(A_K\) is the \((K-1) \times (K-1)\) matrix defined by
\[
A_{r,s} = \left( \frac{2K - 2s}{2r - 1} \right) + \left( \frac{2K - 2s}{2K - 2r} \right),
\]
then the inverse of \(A\) is the matrix \(P\) defined by one of the following two formulas that are equal.

\[
P_{s,r} = \frac{2}{2s - 1} \sum_{n=0}^{2K - 2s} \left( \frac{2r - 1}{2K - 2s - n + 1} \right) \binom{n + 2s - 2}{n} B_n,
\]

\[
= - \frac{2}{2s - 1} \sum_{n=0}^{2K - 2s} \left( \frac{2K - 2r}{2K - 2s - n + 1} \right) \binom{n + 2s - 2}{n} B_n.
\]

**Proof.** Define the \((K-1) \times (K-1)\) matrices \(B\) and \(C\) by
\[
B_{r,s} = \left( \frac{2K - 2s}{2r - 1} \right), \quad C_{r,s} = \left( \frac{2K - 2s}{2K - 2r} \right),
\]
so that \(A = B + C\). Notice that \(B_{r,s} = 0\) if \(r + s > K\), and \(C_{r,s} = 0\) if \(r < s\).

The strategy of proof is to show that the matrix \(PA = PB + PC\) is indeed the identity matrix, by showing that if \(s\) and \(s'\) are positive integers less than \(K\), then
\[
(PB)_{s,s'} + (PC)_{s,s'} = \begin{cases} 
1, & s = s', \\
0, & s \neq s'.
\end{cases}
\]

(3.3)

We will use the following two elementary identities of combination numbers. If \(n \geq 1\), then
\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} = 2^{n-1},
\]
\[
\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k + 1} = 2^{n-1}.
\]

(3.4)

First we compute \((PB)_{s,s'}\) using the first formula in (3.2) for \(P_{s,r}\). The cases where \(s \leq s'\) and \(s > s'\) are considered separately.
If $s \leq s'$, then for $r \leq K - s'$, we have $r \leq K - s$. Hence, $2K - 2s - 2r + 2 \geq 2$. Therefore,

\[(PB)_{s,s'} = \frac{2}{2s - 1} \sum_{n=2s'-2s+2}^{K-s'} \left( \frac{2K - 2s'}{2r - 1} \right) \sum_{r=2}^{2K-2s} \left( \frac{2r - 1}{2K - 2s - n + 1} \right) \binom{n + 2s - 2}{n} B_n \]

Notice that the summation over $n$ only contains terms with $n$ even since $2s' - 2s + 2 \geq 2$.

It can be easily verified that

\[
\left( \frac{2K - 2s'}{2r - 1} \right) \left( \frac{2r - 1}{2K - 2s - n + 1} \right) = \left( \frac{2K - 2s'}{2s - 2s' + n - 1} \right) \left( \frac{2s' + n - 1}{2r + 2s - 2K + n - 2} \right).
\]

Therefore,

\[(PB)_{s,s'} = \frac{2}{2s - 1} \sum_{n=2s'-2s+2}^{K-s'} \left( \frac{2K - 2s'}{2s - 2s' + n - 1} \right) \binom{n + 2s - 2}{n} B_n \times \sum_{r=K-s-\frac{n}{2}+1}^{K-s'} \left( \frac{2s - 2s' + n - 1}{2r + 2s - 2K + n - 2} \right) ;
\]

For $n \geq 2s' - 2s + 2$, we have $2s - 2s' + n - 1 \geq 1$. The first formula in (3.4) implies that

\[
\sum_{r=K-s-\frac{n}{2}+1}^{K-s'} \left( \frac{2s - 2s' + n - 1}{2r + 2s - 2K + n - 2} \right) = \sum_{r=0}^{s-s'+n/2-1} \left( \frac{2s - 2s' + n - 1}{2r} \right) = 2^{2s-2s'+n-2}.
\]

This shows that when $s \leq s'$,

\[(PB)_{s,s'} = \frac{2}{2s - 1} \sum_{n=2s'-2s+2}^{K-s'} \left( \frac{2K - 2s'}{2s - 2s' + n - 1} \right) \binom{n + 2s - 2}{n} 2^{2s-2s'+n-2} B_n. \quad (3.6)
\]

When $s > s'$,

\[(PB)_{s,s'} = \frac{2}{2s - 1} \sum_{n=0}^{2K-2s} \sum_{K-s-\frac{n}{2}+1 \leq r \leq K-s'} \left( \frac{2K - 2s'}{2r - 1} \right) \left( \frac{2r - 1}{2K - 2s - n + 1} \right) \binom{n + 2s - 2}{n} B_n.
\]
Splitting out the \( n = 1 \) term, we have

\[
(PB)_{s,s'} = \frac{2}{2s-1} \left\{ \sum_{0 \leq n \leq 2K-2s \atop n \text{ is even}} \binom{2K-2s'}{2s-2s'+n-1} \binom{n+2s-2}{n} B_n \right. \\
\times \left. \sum_{r=K-s-\frac{s'}{2}+1}^{K-s'} \binom{2s-2s'+n-1}{2r+2s-2K+n-2} \right. \\
+ \left. \binom{2K-2s'}{2s-1} B_1 \sum_{r=K-s+1}^{K-s'} \binom{2s-2s'}{2r+2s-2K-1} \right\}.
\]

Notice that the second formula in (3.4) give

\[
\sum_{r=K-s-\frac{s'}{2}+1}^{K-s'} \binom{2s-2s'+n-1}{2r+2s-2K+n-2} = 2^{2s-2s'-1}.
\]

Together with (3.5), we find that when \( s > s' \),

\[
(PB)_{s,s'} = \frac{2}{2s-1} \sum_{n=0}^{2K-2s} \binom{2K-2s'}{2s-2s'+n-1} \binom{n+2s-2}{n} 2^{2s-2s'+n-2} B_n.
\]

Next we compute \((PC)_{s,s'}\). Using the second expression in (3.2) for \( P_{s,r} \), we have

\[
(\text{PC})_{s,s'} = -\frac{2}{2s-1} \sum_{r=s'}^{K-s} \binom{2K-2s'}{2K-2r} \sum_{n=\max\{0,2r-2s+1\}}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n.
\]

Again, it is easy to verify that

\[
\binom{2K-2s'}{2K-2r} \binom{2K-2s'}{2K-2s-n+1} = \binom{2K-2s'}{2s-2s'+n-1} \binom{2s-2s'+n-1}{2r-2s'}.
\]

Now we discuss the cases \( s < s' \), \( s = s' \) and \( s > s' \) separately.

When \( s = s' \),

\[
(\text{PC})_{s,s} = -\frac{2}{2s-1} \sum_{r=s}^{K-s} \binom{2K-2s}{2K-2r} \sum_{n=2r-2s+1}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n.
\]

In this case, we have a \( n = 1 \) term when \( r = s \). When \( r > s \), summation over \( n \geq 2r - 2s + 1 \) is the same as summation over \( 2r - 2s + 2 \). The term with \( r = s \) and \( n = 1 \) contribute the term 1. Therefore,

\[
(\text{PC})_{s,s} = 1 - \frac{2}{2s-1} \sum_{r=s}^{K-s} \binom{2K-2s}{2K-2r} \sum_{n=2r-2s+2}^{2K-2s} \binom{2K-2r}{2K-2s-n+1} \binom{n+2s-2}{n} B_n
\]

\[
= 1 - \frac{2}{2s-1} \sum_{n=2}^{2K-2s+n/2-1} \binom{2K-2s}{n-1} \binom{n+2s-2}{n} B_n
\]

\[
= 1 - \frac{2}{2s-1} \sum_{n=2}^{2K-2s} \binom{2K-2s}{n-1} \binom{n+2s-2}{n} 2^{n-2} B_n.
\]
The last equality follows from the first equation in (3.4). Compare to the \( s = s' \) case in (3.6) show that

\[ (PB + PC)_{s,s} = 1. \]

Next we consider the case \( s < s' \). In this case, if \( r \geq s' \), then \( r > s \) and hence \( 2r - 2s + 1 \geq 3 \). Therefore,

\[ (PC)_{s,s'} = \frac{2}{2s - 1} \sum_{r=s'}^{2K-2s} \sum_{n=2r-2s+2}^{K-1} \left( \frac{2K - 2s'}{2s - 2s' + n - 1} \right) \left( \frac{2s - 2s' + n - 1}{r - 2s'} \right) \left( \frac{n + 2s - 2}{n} \right) B_n \]

\[ = \frac{2}{2s - 1} \sum_{n=2s' - 2s + 2}^{2K-2s} \left( \frac{2K - 2s'}{2s - 2s' + n - 1} \right) \left( \frac{n + 2s - 2}{n} \right) B_n \sum_{r=s'}^{s+n/2-1} \left( \frac{2s - 2s' + n - 1}{2r - 2s'} \right) \]

\[ = \frac{2}{2s - 1} \sum_{n=2s' - 2s + 2}^{2K-2s} \left( \frac{2K - 2s'}{2s - 2s' + n - 1} \right) \left( \frac{n + 2s - 2}{n} \right) 2^{2s-2s'+n-2} B_n \]

\[ = - (PB)_{s,s'}. \]

Finally, we consider the case \( s > s' \). In this case

\[ (PC)_{s,s'} = \frac{2}{2s - 1} \sum_{n=0}^{2K-2s} \sum_{s' \leq r \leq s+n/2-1} \left( \frac{2K - 2s'}{2K - 2r} \right) \left( \frac{2K - 2r}{2K - 2s - n + 1} \right) \left( \frac{n + 2s - 2}{n} \right) B_n. \]

Splitting out the \( n = 1 \) term, we have

\[ (PC)_{s,s'} = \frac{2}{2s - 1} \left\{ \sum_{0 \leq n \leq 2K - 2s \atop n \text{is even}} \left( \frac{2K - 2s'}{2s - 2s' + n - 1} \right) \left( \frac{n + 2s - 2}{n} \right) B_n \sum_{r=s'}^{s+n/2-1} \left( \frac{2s - 2s' + n - 1}{2r - 2s'} \right) \right\} + \frac{2K - 2s'}{2s - 2s'} \left( \frac{2s - 2s'}{2s - 2s'} \right) \left( \frac{2s - 2s'}{2s - 2s'} \right) \left( \frac{2K - 2s'}{2s - 2s'} \right) \left( \frac{2s - 2s'}{2s - 2s'} \right) B_n \sum_{r=s'}^{s+n/2-1} \left( \frac{2s - 2s' + n - 1}{2r - 2s'} \right) \]

\[ = - (PB)_{s,s'}. \]

This completes the proof of (3.3), and so the assertion of the theorem is proved. □

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