On comparison of fractional Laplacians

Alexander I. Nazarov

Abstract. For $s > -1$, $s \notin \mathbb{N}_0$, we compare two natural types of fractional Laplacians $(-\Delta)^s$, namely, the restricted Dirichlet and the spectral Neumann ones. We show that for the quadratic form of their difference taken on the space $\tilde{H}^s(\Omega)$ is positive or negative depending on whether the integer part of $s$ is even or odd. For $s \in (0, 1)$ and convex domains we prove also that the difference of these operators is positivity preserving on $H^s(\Omega)$. This paper complements [10] and [11] where similar statements were proved for the spectral Dirichlet and the restricted Dirichlet fractional Laplacians.

1 Introduction

In recent decades a lot of efforts have been invested in studying nonlocal differential operators and nonlocal variational problems. Model operators here are various fractional Laplacian (FLs for the brevity) $(-\Delta)^s$, mainly for $s \in (0, 1)$.

Recall that the spectral Dirichlet and Neumann FLs are the $s$th powers of conventional Dirichlet and Neumann Laplacian in the sense of spectral theory. In a Lipschitz bounded domain $\Omega$, they can be defined by corresponding quadratic forms

$$Q_s^{\text{DSP}}[u] \equiv ((-\Delta)^s_{\text{DSP}} u, u) := \sum_{j=1}^{\infty} \lambda_j^s |(u, \varphi_j)|^2;$$

$$Q_s^{\text{NSP}}[u] \equiv ((-\Delta)^s_{\text{NSP}} u, u) := \sum_{j=0}^{\infty} \mu_j^s |(u, \psi_j)|^2,$$

*St.Petersburg Department of Steklov Institute, Fontanka, 27, St.Petersburg, 191023, Russia and St.Petersburg State University, Universitetskii pr. 28, St.Petersburg, 198504, Russia. E-mail: al.il.nazarov@gmail.com. Supported by RFBR grant 20-01-00630.
where $\lambda_j$, $\varphi_j$, and $\mu_j$, $\psi_j$ are eigenvalues and (normalized) eigenfunctions of the Dirichlet and Neumann Laplacian in $\Omega$, respectively. Notice that $\mu_0 = 0$ and $\psi_0 \equiv const$.

For $s \in (0,1)$ the domains of these quadratic forms are the classical Sobolev–Slobodetskii spaces (see [15 Ch. 4] or [5])

$$\text{Dom}(Q^\text{DSp}_s) = \tilde{H}^s(\Omega); \quad \text{Dom}(Q^\text{NSp}_s) = H^s(\Omega) \quad (1)$$

(we recall that

$$\tilde{H}^s(\Omega) = H^s(\Omega) \quad \text{if} \quad 0 < s < 1/2; \quad \tilde{H}^s(\Omega) \subsetneq H^s(\Omega) \quad \text{if} \quad s \geq 1/2,$$

see, e.g., [15 4.3.2]).

The first equality in (1) is proved in [10 Lemma 1]; the proof of the second one is quite similar.

For $s > 1$ the domains of spectral quadratic forms are more complicated but the following relations are always true:

$$\tilde{H}^s(\Omega) \subset \text{Dom}(Q^\text{DSp}_s); \quad \tilde{H}^s(\Omega) \subset \text{Dom}(Q^\text{NSp}_s).$$

On the other hand, the quadratic form of restricted Dirichlet FL is defined as follows:

$$Q^\text{DR}_s[u] \equiv ((-\Delta_\Omega)^s_{\text{DR}}u, u) := \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi$$

where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} u(x) \, dx.$$ 

Corresponding domain is $\tilde{H}^s(\Omega)$ for all $s > 0$.

For $s \in (0,1)$ the following relation holds:

$$Q^\text{DR}_s[u] = c_{n,s} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy,$$

where

$$c_{n,s} = 2^{2s-1} \pi^{-n/2} \frac{\Gamma(n+2s)}{|\Gamma(-s)|}.$$
Remark 1 Notice that for \( s \in (0, 1) \) the quadratic form of restricted Neumann (or regional) FL is

\[ Q_{NR}^s[u] \equiv \left( (-\Delta_{\Omega})_{NSp}^s u, u \right) := c_{n,s} \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy. \]

For some other types of fractional Laplacians see, e.g., [12] and references therein.

The operators \((-\Delta_{\Omega})_{DSP}^s\) and \((-\Delta_{\Omega})_{DR}^s\) were compared in the sense of quadratic forms and in the pointwise sense in [10] \((s \in (0, 1))\) and [11] (for partial results see also [4], [6], [7], [13]).

Theorem 1 (Theorem 2 in [10] and Theorem 1 in [11]) Let \( s > -1 \) and \( s \notin \mathbb{N}_0 \). Suppose that \( u \in \tilde{H}^s(\Omega), \ u \neq 0 \). Then the following relation holds:

\[ Q_{DSP}^s[u] > Q_{DR}^s[u], \quad \text{if} \quad 2k < s < 2k + 1, \quad k \in \mathbb{N}_0; \]
\[ Q_{DSP}^s[u] < Q_{DR}^s[u], \quad \text{if} \quad 2k - 1 < s < 2k, \quad k \in \mathbb{N}_0. \]

Theorem 2 1. (Theorem 1 in [10]) Let \( s \in (0, 1) \), and let \( u \in \tilde{H}^s(\Omega), \ u \geq 0, \ u \neq 0 \). Then the following relation holds in the sense of distributions:

\((-\Delta_{\Omega})_{DSP}^s u > (-\Delta_{\Omega})_{DR}^s u.\]

2. (Theorem 3 in [11]) Let \( s \in (-1, 0) \). Suppose that \( u \in \tilde{H}^s(\Omega), \ u \geq 0 \) in the sense of distributions, \( u \neq 0 \). Then the following relation holds:

\((-\Delta_{\Omega})_{DSP}^s u < (-\Delta_{\Omega})_{DR}^s u.\]

In this paper we prove similar results for the operators \((-\Delta_{\Omega})_{DR}^s\) and \((-\Delta_{\Omega})_{NSp}^s\). Since the domains of their quadratic forms are in general different, we consider them on the smaller domain \( \tilde{H}^s(\Omega) \).

Theorem 3 Let \( s > -1 \) and \( s \notin \mathbb{N}_0 \). Suppose that \( u \in \tilde{H}^s(\Omega), \ u \neq 0 \). Then the following relation holds:

\[ Q_{DSP}^s[u] > Q_{NSp}^s[u], \quad \text{if} \quad s \in (2k, 2k + 1), \quad k \in \mathbb{N}_0; \]
\[ Q_{DSP}^s[u] < Q_{NSp}^s[u], \quad \text{if} \quad s \in (2k - 1, 2k), \quad k \in \mathbb{N}_0. \]

\(^1\)For \( n = 1 \) and \( s \leq -\frac{1}{2} \) assume in addition that \( (u, 1) = 0 \).

\(^2\)For \( s < 0 \) assume in addition that \( (u, 1) = 0 \).
Remark 2 Notice that a weaker inequality $Q_s^{DSP}[u] \geq Q_s^{NSP}[u]$ for $u \in \tilde{H}^s(\Omega)$, $s \in (0, 1)$, is a particular case of the well-known Heinz inequality [3]. On the other hand, the inequality $Q_s^{DR}[u] \geq Q_s^{NR}[u]$ for $u \in \tilde{H}^s(\Omega)$, $s \in (0, 1)$, is trivial.

Theorem 4 Suppose that $\Omega$ is convex. Let $s \in (0, 1)$, and let $u \in \tilde{H}^s(\Omega)$, $u \geq 0$, $u \not\equiv 0$. Then the following relation holds in the sense of distributions:

$$\left(-\Delta_{\Omega}\right)^s_{DR} u > \left(-\Delta_{\Omega}\right)^s_{NSP} u \quad \text{in} \quad \Omega.$$  \hspace{1cm} (4)

The structure of our paper is as follows. In Section 2 we recall the basic facts on the generalized harmonic extensions related to fractional Laplacians of orders $\sigma \in (0, 1)$ and $-\sigma \in (-1, 0)$. Theorems 3 and 4 are proved in Section 3. Also we show that the assumption of convexity in Theorem 4 cannot be removed.

2 Fractional Laplacians as D-to-N and N-to-D operators

It is a common knowledge nowaday that some of FLs of order $\sigma \in (0, 1)$ are related to the so-called harmonic extension in $n + 2 - 2\sigma$ dimensions and to the generalized Dirichlet-to-Neumann map (notice that for $\sigma = \frac{1}{2}$ it was known long ago).

Let $u \in H^\sigma(\mathbb{R}^n)$ (in our consideration, we always assume that $u \in \tilde{H}^\sigma(\Omega)$ is extended by zero to $\mathbb{R}^n$). In the pioneering paper [2], it was shown that there exists a unique solution $w^\sigma_{DR}(x, y)$ of the BVP in the half-space

$$-\text{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \quad w|_{y=0} = u,$$

with finite energy (weighted Dirichlet integral)

$$E^\sigma_{DR}(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} |\nabla w(x, y)|^2 \, dx \, dy,$$

and the relation

$$\left(-\Delta_{\Omega}\right)^\sigma_{DR} u(x) = -C_\sigma \cdot \lim_{y \to 0^+} y^{1-2\sigma} \partial_y w^\sigma_{DR}(x, y)$$  \hspace{1cm} (5)
holds in the sense of distributions and pointwise at every point of smoothness of $u$. Here the constant $C_\sigma$ is given by

$$C_\sigma := \frac{4^\sigma \Gamma(1 + \sigma)}{\Gamma(1 - \sigma)}.$$ 

Moreover, the function $w^{\text{DR}}_\sigma(x, y)$ minimizes $\mathcal{E}^{\text{DR}}_\sigma$ over the set

$$\mathcal{W}^{\text{DR}}_\sigma(u) = \left\{ w(x, y) : \mathcal{E}^{\text{DR}}_\sigma(w) < \infty, \ w\mid_{y=0} = u \right\},$$

and the following equality holds:

$$Q^{\text{DR}}_\sigma[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}^{\text{DR}}_\sigma(w^{\text{DR}}_\sigma). \quad (6)$$

In [14] this approach was transferred to quite general situation. In particular, it was shown that for $u \in H^\sigma(\Omega)$ there is a unique solution $w^{\text{NSp}}_\sigma(x, y)$ of the BVP in the half-cylinder

$$-\text{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+; \quad w\mid_{y=0} = u, \quad \partial_n w\mid_{x \in \partial \Omega} = 0$$

(here $n$ is the unit vector of exterior normal to $\partial \Omega$) having finite energy

$$\mathcal{E}^{\text{NSp}}_\sigma(w) = \int_0^\infty \int_\Omega y^{1-2\sigma} |\nabla w(x, y)|^2 \, dx \, dy,$$

and the relation

$$(-\Delta_\Omega)^\sigma_{\text{NSp}} u(x) = -C_\sigma \cdot \lim_{y \to 0^+} y^{1-2\sigma} \partial_y w^{\text{NSp}}_\sigma(x, y). \quad (7)$$

holds in the sense of distributions on $\Omega$ and pointwise at every point of smoothness of $u$.

Moreover, the function $w^{\text{NSp}}_\sigma(x, y)$ minimizes $\mathcal{E}^{\text{NSp}}_\sigma$ over the set

$$\mathcal{W}^{\text{NSp}}_{\sigma, \Omega}(u) = \left\{ w(x, y) : \mathcal{E}^{\text{NSp}}_\sigma(w) < \infty, \ w\mid_{y=0} = u \right\},$$

and the following equality holds:

$$Q^{\text{NSp}}_\sigma[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}^{\text{NSp}}_\sigma(w^{\text{NSp}}_\sigma). \quad (8)$$
In a similar way, one can connect FLs of order \(-\sigma \in (-1, 0)\) with the generalized Neumann-to-Dirichlet map. It was done in [3] for the spectral Dirichlet FL and in [1] for the FL in \(\mathbb{R}^n\) (and therefore for the restricted Dirichlet FL). Variational characterization of these operators was given in [11]. We formulate this result for the operator\(^3\) \((-\Delta_\Omega)^{-\sigma}_{\text{DR}}\).

Let \(u \in \widetilde{H}^{-\sigma}(\Omega)\) (for \(n = 1\) and \(\sigma \geq \frac{1}{2}\) assume in addition that \((u, 1) = 0\)). We consider the problem of minimizing the functional

\[
\mathcal{E}^{\text{DR}}_{-\sigma}(w) = \mathcal{E}^{\text{DR}}_{\sigma}(w) - 2 \left( u, w \right|_{y=0})
\]

over the set \(\mathcal{W}^{\text{DR}}_{-\sigma}\), that is closure of smooth functions on \(\mathbb{R}^n \times \mathbb{R}_+\) with bounded support, with respect to \(\mathcal{E}^{\text{DR}}_{\sigma}(\cdot)\). We notice that by the result of [2] the duality \((u, w)\) is well defined.

If \(n > 2\sigma\) (this is a restriction only for \(n = 1\)) then the minimizer is determined uniquely. Denote it by \(w^{-\sigma}_{\text{DR}}(x, y)\). Then formulae (5) and (6) imply the relations

\[
Q^{\text{DR}}_{-\sigma}[w] = -\frac{2\sigma}{C_{\sigma}} \cdot \mathcal{E}^{\text{DR}}_{-\sigma}(w^{-\sigma}_{\text{DR}}); \quad (-\Delta_\Omega)^{-\sigma} u(x) = \frac{2\sigma}{C_{\sigma}} w^{-\sigma}_{\text{DR}}(x, 0) \quad (9)
\]

(the second relation holds for a.a. \(x \in \Omega\)).

In case \(n = 1 \leq 2\sigma\) the minimizer \(w^{-\sigma}_{\text{DR}}(x, y)\) is defined up to an additive constant. However, by assumption \((u, 1) = 0\) the functional \(\mathcal{E}^{\text{DR}}_{-\sigma}(w^{-\sigma}_{\text{DR}})\) does not depend on the choice of the constant, and the first relation in (9) holds. The second equality in (9) also holds if we choose the constant such that \(w^{-\sigma}_{\text{DR}}(x, 0) \to 0\) as \(|x| \to \infty\).

Notice that the function \(w^{-\sigma}_{\text{DR}}\) solves the Neumann problem in the half-space

\[-\text{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \quad \lim_{y \to 0^+} y^{1-2\sigma} \partial_y w = -u\]

(the boundary condition holds in the sense of distributions). So, we obtain the “dual” Caffarelli–Silvestre characterization of \((-\Delta_\Omega)^{-\sigma}_{\text{DR}}\) as the Neumann-to-Dirichlet map.

Now we introduce the “dual” Stinga–Torrea characterization of \((-\Delta_\Omega)^{-\sigma}_{\text{NSp}}\) in almost the same way as it was done in [11] for \((-\Delta_\Omega)^{-\sigma}_{\text{DSp}}\). Namely, let

\(^3\)We emphasize that \((-\Delta_\Omega)^{-\sigma}_{\text{DR}}\) is not inverse to \((-\Delta_\Omega)^{-\sigma}_{\text{DR}}\).
\( u \in H^{−\sigma}(\Omega) \) and let \((u, 1) = 0\). Then the function \(w_{−\sigma}^{NSp}(x, y)\) minimizing the functional
\[
\tilde{\mathcal{E}}_{−\sigma}^{NSp}(w) = \mathcal{E}_{σ}^{NSp}(w) − 2(u, w|_{y=0})
\]
over the set
\[
\mathcal{W}_{−\sigma;Ω}(u) = \{w(x, y) : \mathcal{E}_{σ}^{NSp}(w) < \infty\},
\]
is defined up to an additive constant. By assumption \((u, 1) = 0\) the functional \(\tilde{\mathcal{E}}_{−\sigma}^{NSp}(w_{−\sigma}^{NSp})\) does not depend on the choice of the constant, and formulae (8) and (7) imply
\[
Q_{−\sigma}^{NSp}[u] = −\frac{2\sigma}{C_σ} \cdot \tilde{\mathcal{E}}_{−\sigma}^{NSp}(w_{−\sigma}^{NSp}); \quad (−\Delta)_Ω^{−\sigma}u(x) = \frac{2\sigma}{C_σ}w_{−\sigma}^{NSp}(x, 0) \quad (10)
\]
(The second equality holds for a.a. \(x \in Ω\) if we choose the constant such that \(w_{−\sigma}^{NSp}(x, y) \rightarrow 0\) as \(y \rightarrow +\infty\)).

Also the function \(w_{−\sigma}^{NSp}\) solves the Neumann problem in the half-cylinder
\[-\text{div}(y^{1−2\sigma}\nabla w) = 0 \quad \text{in} \quad Ω \times \mathbb{R}^+; \quad \lim_{y \to 0^+} y^{1−2\sigma}\partial_y w = −u, \quad \partial_n w|_{x \in \partial Ω} = 0
\]
(the boundary condition on the bottom holds in the sense of distributions).

3 Proof of main results

Proof of Theorem 3. We split the proof in three parts.

1. Let \(s \in (0, 1)\). For any \(w \in \mathcal{W}_s^{DR}(u)\) we have \(w|_{Ω \times \mathbb{R}^+} \in \mathcal{W}_{s;Ω}(u)\). Therefore, relations (6) and (8) provide
\[
Q_s^{NSp}[u] = \frac{C_s}{2s} \cdot \inf_{w \in \mathcal{W}_{s;Ω}(u)} \mathcal{E}_s^{NSp}(w) \leq \frac{C_s}{2s} \mathcal{E}_s^{NSp}(w_s^{DR}) \leq \frac{C_s}{2s} \mathcal{E}_s^{DR}(w_s^{DR}) = Q_s^{DR}[u],
\]
and (2) follows with the large sign.

Finally, the equality in (2) implies \(\nabla w_s^{DR} = 0\) on \((\mathbb{R}^n \setminus Ω) \times \mathbb{R}^+_\). Since any \(x\)-derivative of \(w_s^{DR}\) solves the same equation in the whole half-space \(\mathbb{R}^n \times \mathbb{R}_+\), it should be zero everywhere that is impossible for \(u \not\equiv 0\).

2. Let \(s \in (−1, 0)\). We define \(σ = −s \in (0, 1)\) and construct the extension \(w_{−σ}^{DR}\) as described in Section 2.

7
We again have \( w_{-\sigma}^{DR}|_{\Omega \times \mathbb{R}^+} \in \mathcal{W}^{NSp}_{-\sigma, \Omega}(u) \). Therefore, relations (9) and (10) provide

\[
- Q_s^{NSp}[u] = \frac{2\sigma}{C_\sigma} \cdot \inf_{w \in \mathcal{W}^{NSp}_{-\sigma, \Omega}} \mathcal{E}^{NSp}_{-\sigma}(w) \leq \frac{2\sigma}{C_\sigma} \mathcal{E}^{NSp}_{-\sigma}(w^{DR}_{-\sigma}) \leq \frac{2\sigma}{C_\sigma} \mathcal{E}^{DR}_{-\sigma}(w^{DR}_{-\sigma}) = - Q_s^{DR}[u],
\]

and (3) follows with the large sign. To complete the proof, we repeat the argument of the first part.

3. Now let \( s > 1, s \notin \mathbb{N} \). We put \( k = \left\lfloor \frac{s+1}{2} \right\rfloor \) and define for \( u \in \tilde{H}^s(\Omega) \)

\[
v = (-\Delta)^k u \in \tilde{H}^{s-2k}(\Omega), \quad s - 2k \in (-1, 0) \cup (0, 1).
\]

Note that \( v \neq 0 \) if \( u \neq 0 \), and

\[
(v, 1) = Fv(0) = |\xi|^{2k} Fu(\xi)|_{\xi=0} = 0.
\]

Then we have

\[
Q_s^{DR}[u] = Q_{s-2k}^{DR}[v], \quad Q_s^{NSp}[u] = Q_{s-2k}^{NSp}[v],
\]

and the conclusion follows from cases 1 and 2. \( \square \)

**Proof of Theorem 4.** We recall the representation formulae for \( w_{s}^{DR} \) and \( w_{s}^{NSp} \), see [2] and [14], respectively:

\[
w_{s}^{DR}(x, y) = \text{const} \cdot \int_{\mathbb{R}^n} \frac{y^{2s} u(\xi) \, d\xi}{(|x - \xi|^2 + y^2)^{\frac{n+2s}{2}}};
\]

\[
w_{s}^{NSp}(x, y) = \sum_{j=0}^{\infty} (u, \psi_j)_{L^2(\Omega)} \cdot Q_s(y \sqrt{\mu_j}) \psi_j(x), \quad Q_s(\tau) = \frac{2^{1-s} \tau^s}{\Gamma(s)} K_s(\tau),
\]

where \( K_s(\tau) \) stands for the modified Bessel function of the second kind.

First of all, these formulae imply for \( u \geq 0, u \neq 0 \)

\[
\lim_{y \to +\infty} w_{s}^{DR}(x, y) = 0; \quad \lim_{y \to +\infty} w_{s}^{NSp}(x, y) = (u, \psi_0)_{L^2(\Omega)} \cdot \psi_0(x) > 0;
\]

8
the second relation follows from the asymptotic behavior (see, e.g., [14, (3.7)])

\[ K_s(\tau) \sim \Gamma(s) 2^{s-1} \tau^{-s}, \quad \text{as } \tau \to 0; \]

\[ K_s(\tau) \sim \left( \frac{\pi}{2\tau} \right)^{\frac{s}{2}} e^{-\tau} (1 + O(\tau^{-1})) \quad \text{as } \tau \to +\infty. \]

Next, for \( x \in \partial \Omega \) we derive by convexity of \( \Omega \)

\[ \partial_n w_{DR}^s(x, y) = \text{const} \cdot \int_{\mathbb{R}^n} \frac{y^{2s} \langle (\xi - x), n \rangle u(\xi) \, d\xi}{(|x - \xi|^2 + y^2)^{s+1}} < 0. \]

Thus, the difference \( W(x, y) = w_{NSp}^s(x, y) - w_{DR}^s(x, y) \) has the following properties in the half-cylinder \( \Omega \times \mathbb{R}^+ \):

\[-\text{div}(y^{1-2s} \nabla W) = 0; \quad W|_{y=0} = 0; \quad W|_{y=\infty} > 0; \quad \partial_n W|_{x \in \partial \Omega} > 0. \]

By the strong maximum principle, \( W > 0 \) in \( \Omega \times \mathbb{R}^+ \). Finally, we apply the boundary point principle (the Hopf–Oleinik lemma, see [9]) to the function \( W(x, t^{\frac{1}{2s}}) \) and obtain (cf. [10], Theorem 1)

\[ \liminf_{y \to 0^+} y^{1-2s} \partial_y W(x, y) = \liminf_{y \to 0^+} \frac{W(x, y)}{y^{2s}} = \liminf_{t \to 0^+} \frac{W(x, t^{\frac{1}{2s}})}{t} > 0, \quad x \in \Omega. \]

This completes the proof in view of (5) and (7). \( \square \)

**Remark 3** For non-convex domains the relation (4) does not hold in general. We provide corresponding counterexample.

Put temporarily \( \Omega = \Omega_1 \cup \Omega_2 \) where \( \Omega_1 \cap \Omega_2 = \emptyset \). If \( u \geq 0 \) is a smooth function supported in \( \Omega_1 \) then easily \( (-\Delta^s)^{\Delta^s}_{NSp} u \equiv 0 \) in \( \Omega_2 \). On the other hand, \( w_{DR}^s(x, y) > 0 \) for all \( x \in \mathbb{R}^n \), \( y > 0 \), and the Hopf–Oleinik lemma gives \( (-\Delta^s)^{\Delta^s}_{DR} u < 0 \) in \( \Omega_2 \).

Finally, if we join \( \Omega_1 \) with \( \Omega_2 \) by a small channel then the inequality \( (-\Delta^s)^{\Delta^s}_{DR} u < (-\Delta^s)^{\Delta^s}_{NSp} u \) in \( \Omega_2 \) holds by continuity.

**References**

[1] X. Cabré and Y. Sire, *Nonlinear equations for fractional Laplacians. I: Regularity, maximum principles, and Hamiltonian estimates*, AIHP – AN. 31 (2014), no. 1, 23–53.
[2] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Comm. PDEs. 32 (2007), no. 7-9, 1245–1260.

[3] A. Capella, J. Dávila, L. Dupaigne and Y. Sire, Regularity of radial extremal solutions for some non-local semilinear equations, Comm. PDEs. 36 (2011), no. 8, 1353–1384.

[4] Z.-Q. Chen and R. Song, Two-sided eigenvalue estimates for subordinate processes in domains, J. Funct. Anal. 226 (2005), 90–113.

[5] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math., 136 (2012), no. 5, 521–573.

[6] M.M. Fall, Semilinear elliptic equations for the fractional Laplacian with Hardy potential, Nonlin. Analysis – TMA, 193 (2020), 111311, DOI 10.1016/j.na.2018.07.008. Arxiv preprint 1109.5530v4 (2012).

[7] R. L. Frank and L. Geisinger, Refined semiclassical asymptotics for fractional powers of the Laplace operator, J. reine und angew. Math. (Crelles Journal), 712 (2016), 1–37.

[8] E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung, Math. Ann., 193 (1951), 415–438.

[9] L. I. Kamynin and B. N. Himčenko, Theorems of Giraud type for second order equations with a weakly degenerate non-negative characteristic part, Sib. Math. J. 18 (1977), 76–91.

[10] R. Musina and A.I. Nazarov, On fractional Laplacians, Comm. PDEs, 39 (2014), no. 9, 1780–1790.

[11] R. Musina and A.I. Nazarov, On fractional Laplacians–2, Ann. Inst. Henri Poincaré – An. Nonlin., 33 (2016), no. 6, 1667–1673.

[12] R. Musina and A.I. Nazarov, Strong maximum principles for fractional Laplacians, Proc. Roy. Soc. Edinburgh A. 149 (2019), no. 5, 1223–1240.

[13] R. Servadei and E. Valdinoci, On the spectrum of two different fractional operators, Proc. Roy. Soc. Edinburgh A, 144 (2014), no. 4, 831–855.

[14] P. R. Stinga and J. L. Torrea, Extension problem and Harnack’s inequality for some fractional operators, Comm. PDEs. 35 (2010), no. 11, 2092–2122.

[15] H. Triebel, Interpolation theory, function spaces, differential operators, Deutscher Verlag Wissensch., Berlin, 1978.