On integrable systems outside Nijenhuis and Haantjes geometry

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Abstract

We study non-invariant Killing tensors with non-zero Nijenhuis torsion in the three-dimensional Euclidean space. Generalizing the corresponding integrable systems we construct two new families of superintegrable systems in n-dimensional Euclidean space.

Keywords: Killing tensors, integrable systems, separation of variables

1 Introduction

Let us consider Riemannian or pseudo-Riemannian manifold \( M \), \( \dim M = n \) endowed with coordinates \( q = (q_1, \ldots, q_n) \). Metric \( g(q) \) and potential \( V(q) \) define the Hamilton function on the cotangent bundle \( T^*M \)

\[
H_1 = \sum_{i,j=1}^{n} g^{ij}(q)p_ip_j + V(q).
\]  

(1.1)

A criterion for orthogonal separability of the corresponding Hamilton-Jacobi equation is given by Benenti [1]:

The Hamiltonian system defined by (1.1) is orthogonally separable if and only if there exists a valence-two symmetric Killing tensor \( K \) with

1. pointwise simple and real eigenvalues with respect to metric \( g \);
2. orthogonally integrable eigenvectors with respect to metric \( g \);
3. such that

\[
d(KdV) = 0.
\]  

(1.2)

A Killing tensor satisfying conditions (1-2) is called a characteristic Killing tensor.

Thus, we have a family of the St¨ ackel systems which are completely defined by Hamiltonian \( H_1 \) and yet another integral of motion

\[
H_2 = \sum_{i,j=1}^{n} K^{ij}p_ip_j + U(q)
\]  

(1.3)

involving characteristic Killing tensor. Other \( n - 2 \) independent integrals of motion in the involution \( H_2, \ldots, H_n \) are defined by recurrence relations [12], all these integrals are polynomials of second order in momenta. Similar construction of \( n - 2 \) additional integrals in more generic case including non-potential forces was proposed by Kozlov [13].

The integrability of eigenvectors condition (2) is equivalent to a system of non-linear partial differential equations which can be represented in various forms, see [10] and references within. For instance, a given symmetric Killing tensor \( K \) has integrable eigenvectors if

\[
[N^{\ell}_{\,ijk}g_{ij}]_\ell = 0, \quad [N^{\ell}_{\,ijk}K_{ij}]_\ell = 0, \quad [N^{\ell}_{\,ijk}K_{ij}mK^m]_\ell = 0,
\]  

(1.4)
where the square brackets stand for antisymmetrisation and $N$ denotes the Nijenhuis torsion of (1,1) tensor field $K$, i.e. $N = N_K$.

According to Haantjes, these partial differential equations may be rewritten in the equivalent form

$$H_K = 0$$

(1.5)

where $H_K$ is the Haantjes torsion of (1,1) tensor field $K$. Now Nijenhuis and Haantjes tensors can be found in various parts of mathematics, mathematical physics, and classical mechanics, but the overwhelming majority of applications is related to the vanishing of one of these tensors.

In Eisenhart found eleven characteristic Killing tensors $K$ in the three-dimensional Euclidean space $\mathbb{R}^3$ so that

$$N_K(u,v) \neq 0, \quad \text{but} \quad H_K(u,v) = 0$$

(1.6)

for any vector fields $u$ and $v$, and proposed construction of the completely integrable Stäckel systems associated with this tensor $K$.

Invariant construction of these eleven Killing tensors in $\mathbb{R}^3$ is discussed in [10]. For other existing Killing tensors $K$ in the three-dimensional Euclidean space $\mathbb{R}^3$ we have

$$N_K \neq 0 \quad \text{and} \quad H_K \neq 0.$$

If such tensor $K$ is invariant under action a Killing vector field $X$

$$H_K \neq 0, \quad \text{but} \quad L_X K = 0,$$

where $L$ is a Lie derivative and $X$ belongs to the isometry group $E(3)$, then we also have an integrable system with integrals of motion $H_{1,2}$ and

$$H_3 = \sum_{i=1}^n X_i^i p_i,$$

Thus, thanks to Noether’s theorem, we have an integrable Hamiltonian system outside Nijenhuis and Haantjes geometry.

In [16, 17] we found two completely non-invariant (1,1) Killing tensors $K$ in the three-dimensional Euclidean space $\mathbb{R}^3$ so that

$$H_K \neq 0 \quad \text{and} \quad L_X K \neq 0, \quad \forall X \in \mathbb{R}^3.$$

The corresponding integrable Hamiltonian systems admit two with integrals of motion $H_{1,2}$ and

$$H_3 = \sum_{i,j,k,\ell=1}^3 A^{ijkl} p_ip_jp_kp_\ell + \sum_{i,j=1}^3 S^{ij} p_ip_j + W(q).$$

In this note, we obtain a few generalizations of these Killing tensors in $\mathbb{R}^4$ that allows us to describe two families of superintegrable Hamiltonian systems in $\mathbb{R}^n$ outside of the Nijenhuis and Haantjes geometry. All these systems have $n - 1$ quadratic and one quartic integrals of motion in the involution.

### 1.1 Killing, Nijenhuis and Haantjes tensors

There are a few equivalent definitions of the Killing tensors.

For instance, a Killing tensor $K$ of valence $p$ defined in $(M, g)$ is a symmetric $(m,0)$ tensor satisfying the Killing tensor equation

$$[K,g] = 0$$

(1.7)
where \([\cdot, \cdot]\) denotes the Schouten bracket. When \(m = 1\), vector field \(K = X\) is said to be a Killing vector (infinitesimal isometry) and this equation reads as

\[
\mathcal{L}_X g = 0
\]

(1.8)

where \(L\) denotes the Lie derivative operator.

According to another definition, if \(q_i(t)\) is a geodesic, then \(K\) is a \((m,0)\) Killing tensor if scalar

\[
C = \sum_{i_1, \ldots, i_m} K^{i_1 \ldots i_m} p_{i_1} \cdots p_{i_m}
\]

is constant along a geodesic. Here \(p_i = \dot{q}_i(t)\) is the tangent vector of the geodesic.

At \(m = 2\) we do not separate the motion along geodesics and the motion in a potential field, so we consider Killing tensor \(K\) as a solution of the equation

\[
\{H_1, H_2\} = 0
\]

where

\[
H_1 = \sum_{i,j=1} g^{ij}(q) p_i p_j + V(q) \quad \text{and} \quad H_2 = \sum_{i,j=1} K^{ij} p_i p_j + U(q),
\]

with "non-trivial" potentials \(V(q)\) and \(U(q)\). Below we explain what means "non-trivial" potentials.

Metric establishes an isomorphism between the tangent space and its dual. This identifies co- and contravariant tensor components via lowering or rising indices using the metric. In particular a tensor \(K\) of valency two can be identified with \((0, 2), (2,0),\) or \((1, 1)\) tensor field.

If \(K\) be \((1,1)\) tensor field in \(M\), then Nijenhuis and Haantjes tensors on \(M\) are

\[
N_K(u, v) = K^2 [u, v] + [Ku, Kv] - K ([Ku, v] + [u, Kv])
\]

and

\[
H_K(u, v) = K^2 N_K(u, v) + N_K(Ku, Kv) - K (N_K(Ku, v) + N_K(u, Kv))
\]

where \(u, v\) are arbitrary vector fields and \([\cdot, \cdot]\) denotes the commutator of two vector fields.

On a local coordinate chart \(q = (q_1, \ldots, q_n)\) the alternating \((1, 2)\) Nijenhuis tensor takes the form

\[
(N_K)^i_{jk} = \sum_{\alpha = 1}^n \left( \frac{\partial K^i_j}{\partial q\alpha} K^\alpha_k - \frac{\partial K^i_k}{\partial q\alpha} K^\alpha_j + \left( \frac{\partial K^\alpha_j}{\partial q\alpha} - \frac{\partial K^\alpha_k}{\partial q\alpha} \right) K^i_k \right)
\]

The corresponding Haantjes tensor looks like

\[
(H_K)^i_{jk} = \sum_{\alpha, \beta = 1}^n \left( K^\alpha_{\alpha j} K^\beta_{\beta k} (N_K)^j_{\alpha k} + (N_K)^j_{\alpha k} K^\alpha_{\alpha j} K^\beta_{\beta k} - K^\alpha_i \left( (N_K)^j_{\alpha k} K^\beta_j + (N_K)^j_{\alpha j} K^\beta_k \right) \right)
\]

Properties of these tensors are discussed in [4, 5]. In [16, 17] we used components of these tensors for the description of the non-invariant Killing tensors with \(H_K \neq 0\).

2 Three-dimensional Euclidean space

Let us consider Euclidean space \(\mathbb{R}^3\) with Cartesian coordinates \(q_1, q_2, q_3\) and metric

\[
g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

(2.1)

The canonical basis of the Killing vectors consists of the translational Killing vectors

\[
X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]
and from the rotational Killing vectors

\[ R_1 = \begin{pmatrix} 0 & -q_3 \\ q_2 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} q_3 & 0 \\ 0 & -q_1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} -q_2 & 0 \\ q_1 & 0 \end{pmatrix}. \]

Any Killing tensor of valence two in \( \mathbb{R}^3 \) is represented as a linear combination of symmetric products of the basic Killing vectors

\[ K = \sum_{ij} A^{ij} X_i \otimes X_j + \sum_{ij} B^{ij} X_i \otimes R_j + \sum_{ij} C^{ij} R_i \otimes R_j, \quad (2.2) \]

where matrices

\[ A = \begin{pmatrix} a_1 & a_1 & a_2 \\ a_2 & a_2 & a_3 \\ a_3 & a_3 & a_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & \gamma_1 & \gamma_2 \\ \gamma_1 & c_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & c_3 \end{pmatrix} \]

depend on twenty-one parameters. It is easy to observe that entries of \( K \) involve only the differences of the diagonal coefficients \( b_{11}, b_{22} \) and \( b_{33} \). Defining

\[ \beta_1 = b_{22} - b_{33}, \quad \beta_2 = b_{33} - b_{11}, \quad \beta_3 = b_{11} - b_{22}, \]

yields the constraint \( \beta_1 + \beta_2 + \beta_3 = 0 \), thereby showing that there are only twenty independent parameters. It coincides with the Delong-Takeuchi-Thompson formula

\[ d = \frac{1}{n} \binom{n + m}{m + 1} \binom{n + m - 1}{m} = \frac{1}{3} \binom{3 + 2}{2 + 1} \binom{3 + 2 - 1}{3} = 20, \quad (2.3) \]

for the dimension of vector space of the Killing tensors of valency \( m \) in \( n \)-dimensional Riemannian space, see [10] for details.

In generic case \( K \) is a completely non-invariant tensor with non-vanishing Haantjes torsion

\[ L_X K \neq 0, \quad L_R K \neq 0, \quad \text{and} \quad H_K \neq 0. \]

Thus, we have two integrals of motion \( H_{1,2} \)

\[ H_1 = \sum_{i,j=1}^3 g^{ij}(q)p_ip_j + V(q) \quad \text{and} \quad H_2 = \sum_{i,j=1}^3 K^{ij}p_ip_j + U(q). \quad (2.4) \]

and

- at \( H_K = 0 \) we can construct third independent second-order polynomial in momenta \( H_3 \) using Eisenhart construction [8];
- at \( L_V K = 0 \), where \( V \) is a linear combination of translations \( X_i \) and rotations \( R_i \), we can construct a third independent linear integral of motion \( H_3 \) using Noether’s theorem.

For instance, let us consider symmetric Killing tensor

\[ K = \begin{pmatrix} a_1 + 2b_{13}q_2 + c_3q_3^2 & \alpha_3 - b_{13}q_1 + b_{23}q_2 - c_3q_1q_2 & \alpha_2 - \beta_1q_2 \\ * & a_2 - 2b_{23}q_1 + c_3q_1^2 & \alpha_1 + \beta_1q_1 \\ * & * & a_3 \end{pmatrix} \]

which is invariant under translation

\[ L_{X_i} K = 0. \]
Because $H_K \neq 0$ we cannot apply standard Eisenhart’s construction. Nevertheless, using Noether’s theorem we easily obtain desired integrals of motion in the involution

$$H_1 = \sum_{i,j}^3 g^{ij} p_i p_j + V(q_1, q_2), \quad H_2 = \sum_{i,j}^3 K^{ij} p_i p_j + U(q_1, q_2), \quad H_3 = p_3.$$  

Similarly, we can construct integrable systems associated with rotational and helicoidal symmetries. So, integrable systems outside Nijenhuis and Haantjes geometry exist.

At $H_K = 0$, i.e. inside Nijenhuis and Haantjes geometry, two integrals of motion completely define the integrable system with quadratic integrals of motion, see [1] and more generic case in [13].

At $H_K \neq 0$, i.e. outside Nijenhuis and Haantjes geometry, we can also suppose that two integrals of motion completely determine an integrable system with "non-trivial" potentials. Indeed, let us substitute generic Killing tensor $K$ depending on 20 parameters into the equation

$$\{H_1, H_2\} = 0 \quad \Rightarrow \quad d(KdV) = 0$$

and try to solve the resulting equation imposing the following restrictions on $K$

$$\mathcal{L}_X, K \neq 0, \quad \mathcal{R}_L, K \neq 0, \quad \text{and} \quad H_K \neq 0.$$  

Here we do not use grading by momenta when we have to study the geodesic motion and only then add the suitable potential to each obtained geodesic. We prefer to solve an equation on the potential $V$ depending on 20 parameters which describe all the possible geodesics.

As a result, we obtain completely non-invariant Killing tensor

$$K = \begin{pmatrix} -q_2 & \frac{q_1}{2} & 0 \\ \frac{q_1}{2} & 0 & -a q_3 \\ 0 & -\frac{a q_3}{2} & a q_2 \end{pmatrix}, \quad a = 1, -1/2, -2,$$

and potentials

- first solution: $a = 1$, \quad $V(q) = \alpha \left(q_1^4 + 6q_1^2 q_3^2 + q_3^4 + 12q_2^2 (q_1^2 + q_3^2) + 16q_2^4\right)$, \quad (2.6)

- second solution: $a = -1/2, -2$, \quad $V(q) = \frac{\alpha (q_1^2 + 4q_2^2 + 4q_3^2)}{q_1^2}$, \quad (2.7)

Other solutions are "trivial", i.e. potential is separable in Cartesian coordinates

$$V(q_1, q_2, q_3) = f_1(q_1) + f_2(q_2) + f_3(q_3)$$

and some of these separable potentials may be added to (2.6, 2.7), see [16, 17].

### 2.1 First solution

If $a = 1$, then solution of equation $d(KdV)$ is equal to

$$V(q) = \alpha \left(q_1^4 + 6q_1^2 q_3^2 + q_3^4 + 12q_2^2 (q_1^2 + q_3^2) + 16q_2^4\right),$$
solution of the corresponding equation \( \{ H_1, H_2 \} = 0 \) has the form
\[
U(q) = 2\alpha q_2 (q_1^2 - q_3^2)(q_1^2 + q_2^2 + q_3^2).
\]
The third integral of motion is the polynomial of the fourth-order in momenta
\[
H_3 = p_1^2 p_3^2 + 2\alpha \sum_{i,j} S_{ij}(q)p_ip_j + 4\alpha^2 W(q)
\]
where
\[
S(q) = \begin{pmatrix}
2q_2^2 q_3^2 & -2q_1 q_2 q_3^2 & q_1 q_3 (q_1^2 + 4q_2^2 + q_3^2) \\
-2q_1 q_2 q_3^2 & 2q_1^2 q_3^2 & -2q_2 q_1^2 q_3 \\
q_1 q_3 (q_1^2 + 4q_2^2 + q_3^2) & -2q_2 q_1^2 q_3 & 2q_1^2 q_2^2
\end{pmatrix}
\]
and
\[
W(q) = q_1^2 q_3^2 (q_1^2 + 2q_2^2 + q_3^2)^2.
\]
Thus, we have a non-trivial integrable system with two quadratic and one quartic invariants outside Nijenhuis and Haantjes geometry.

### 2.2 Second solution

If \( \alpha = -1/2, -2 \), then Hamiltonian
\[
H_1 = \sum_{i,j=1}^3 g^{ij} p_i p_j + V(q) = p_1^2 + p_2^2 + p_3^2 + \frac{\alpha (q_1^2 + 4q_2^2 + 4q_3^2)}{q_1^2},
\]
commutes with the following two polynomials of second order in momenta
\[
\begin{align*}
H_2 &= \sum_{i,j=1}^3 A^{ij} p_i p_j + U_2(q) = p_1 J_{12} - 2p_3 J_{23} - \frac{2\alpha q_2 (q_1^2 + 2q_2^2 + 2q_3^2)}{q_1^2}, \\
H_3 &= \sum_{i,j=1}^3 B^{ij} p_i p_j + U_3(q) = p_1 J_{13} + 2p_2 J_{23} - \frac{2\alpha q_3 (q_1^2 + 2q_2^2 + 2q_3^2)}{q_1^2},
\end{align*}
\]
and with a component
\[
J_{23} = q_2 p_3 - q_3 p_2
\]
of the angular momentum operator \( J_{ij} = q_i p_j - q_j p_i \).

Here \( A \) and \( B \) are the Killing tensor of valency two in \( \mathbb{R}^3 \) having non-vanishing Haantjes torsion
\[
H_A \neq 0 \quad \text{and} \quad H_B \neq 0,
\]
thus suggesting non-separability of the corresponding Hamilton-Jacobi equation in orthogonal curvilinear coordinate systems in \( \mathbb{R}^3 \).

Algebra of these integrals of the motion reads as
\[
\{ H_1, H_2 \} = \{ H_1, H_3 \} = \{ H_1, J_{23} \} = 0, \quad \{ J_{23}, H_2 \} = H_3, \quad \{ J_{23}, H_3 \} = -H_2
\]
and
\[
\{ H_2, H_3 \} = -4H_1 J_{23}.
\]
Its second central element is the following polynomial of fourth order in momenta
\[
H_A = 4H_1 J_{2,3}^2 - H_2^2 - H_3^2.
\]
In this case, there are three sets of commuting integrals of motion

\((H_1, H_2, H_4)\), \((H_1, H_3, H_4)\), \((H_1, J_2, H_4)\)

which always involve two quadratic integrals and one quartic integral of motion.

Thus, we have a non-trivial superintegrable system with two quadratic and one quartic invariants outside Nijenhuis and Haantjes geometry.

3 Four-dimensional Euclidean space

Let us consider Euclidean space \(\mathbb{R}^4\) with Cartesian coordinates \(q_1, q_2, q_3, q_4\) and metric

\[
g = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

According to the Delong-Takeuchi-Thompson formula (2.3), the dimension of vector space of the Killing tensors of valency 4 in 4-dimensional Euclidean space is

\[
d = \frac{1}{n} \left( \frac{n + m}{m + 1} \right) \left( \frac{n + m - 1}{m} \right) = \frac{1}{4} \left( \frac{4 + 2}{2 + 1} \right) \left( \frac{4 + 2 - 1}{3} \right) = 50.
\]

Unfortunately, we can not use modern computer algebra systems to calculate the generic solution of a system of partial differential equations in \(\mathbb{R}^4\)

\[
d(KdV) = 0, \quad \mathcal{L}_X K \neq 0, \quad \mathcal{L}_R K \neq 0, \quad \text{and} \quad H_K \neq 0,
\]

depending on 50 parameters.

So, let us study the following deformation of the Killing tensor (2.5)

\[
K = \begin{pmatrix}
-q_2 & \frac{q_1}{2} & 0 & b_{q_2} + b_3 q_3 + b_4 q_4 + d_1 \\
\frac{q_1}{2} & 0 & -\frac{aq_3}{2} & b_{1 q_3} - b_{2 q_1} + b_5 q_4 + d_2 \\
0 & -\frac{aq_3}{2} & aq_2 & -b_{1 q_2} - b_{3 q_1} + b_6 q_4 + d_3 \\
& b_{2 q_2} + b_3 q_3 + b_4 q_4 + d_1 & b_{1 q_3} - b_{2 q_1} + b_5 q_4 + d_2 & -b_{1 q_2} - b_{3 q_1} + b_6 q_4 + d_3
\end{pmatrix},
\]

which is a partial solution of the Killing equation (1.7) depending on eleven parameters \(a, b_1, \ldots, b_6\) and \(d_1, \ldots, d_4\). This deformation is invariant with respect to rotation

\[
\mathcal{L}_{R_{1,4}} K = 0, \quad R_{1,4} = q_1 \frac{\partial}{\partial q_4} - q_4 \frac{\partial}{\partial q_1}.
\]

As a result, we obtain three "nontrivial" solutions of the equations (3.1) associated with the Killing tensor

\[
K = \begin{pmatrix}
-q_2 & \frac{q_1}{2} & 0 & 0 \\
\frac{q_1}{2} & 0 & -\frac{aq_3}{2} & bq_4 \\
0 & -\frac{aq_3}{2} & aq_2 & 0 \\
0 & bq_4 & 0 & -2bq_2
\end{pmatrix}
\]

(3.2)

at the special values of parameters
• \( a = 1 \) and \( b = 1/2 \);
• \( a = -1/2, -2 \) and \( b = 1/2 \);
• \( a = -1/2, -2 \) and \( b = 1 \).

The corresponding integrable systems are discussed below.

### 3.1 First solution

If \( a = 1 \) and \( b = 1/2 \), then nonseparable in Cartesian coordinates solution of (3.3) is

\[
V(q) = \alpha \left( q_1^4 + 12q_1^2q_2^2 + 6q_1^2q_3^2 + 2q_2^2q_3^2 + 16q_1^4 + 12q_2^2q_3^2 + 12q_2^2q_4^2 + q_1^4 + 6q_3^2q_4^2 + q_1^4 \right), \quad (3.3)
\]

solution of the equation \(\{H_1, H_2\} = 0\) has the form

\[
U(q) = 2\alpha q_2(q_1^2 - q_2^2)(q_2^2 + q_3^2 + q_4^2). \quad (3.4)
\]

Other integrals of motion read as

\[
J_{14} = p_1q_4 - p_4q_1
\]

and

\[
H_3 = p_3^2(p_1^2 + p_4^2) + 2\alpha \sum_{i,j=1}^4 S_{ij}(q)p_ip_j + \alpha^2 W(q),
\]

where

\[
S(q) = \begin{pmatrix}
2q_3^2(q_2^2 + q_4^2) & -2q_1q_2q_3^2 & q_1q_3(q_1^2 + 4q_2^2 + q_4^2) & -2q_1q_4q_2^2 \\
-2q_1q_3q_2^3 & 2q_1^2q_3^2 + q_4^2 & -2q_2q_3(q_2^2 + q_4^2) & -2q_2q_4q_3^2 \\
q_1q_3(q_1^2 + 4q_2^2 + q_4^2) & -2q_2q_3(q_2^2 + q_4^2) & -2q_2q_4(4q_2^2 + 4q_2^2 + q_4^2) & q_4q_3(q_1^2 + 4q_3^2 + q_4^2) \\
-2q_4q_1q_2^3 & -2q_4q_2q_3^2 & q_4q_3(q_1^2 + 4q_3^2 + q_4^2) & 2q_3^2(q_1^2 + q_4^2)
\end{pmatrix}
\]

and

\[
W(q) = 4q_3^2(q_1^2 + q_4^2)(q_2^2 + q_3^2 + q_4^2)^2.
\]

We can also add separable in Cartesian coordinates potentials to (3.3) \(U_{ab} = c_1(q_1^2 + 4q_2^2 + q_3^2) / b + c_2q_2 + c_3q_3^2 + c_4q_4 + c_5 / bq_4^2\)

\[
V_{ab} = \frac{c_1(q_1^2 + 4q_2^2 + q_3^2)}{b} + \frac{c_2q_2}{q_1^2} + \frac{c_3q_3^2}{q_1^2 + 4q_2^2} + \frac{c_4q_4}{q_1^2} - \frac{c_5q_2}{q_1^2} - \frac{2c_5q_2}{q_1^2}.
\]

This integrable system is a natural generalization of the 3D system (2.6).

### 3.2 Second and third solution

If \( a = -1/2, -2 \) and \( b = 1/2 \), then Hamiltonian is equal to

\[
H_1 = p_1^2 + p_2^2 + p_3^2 + p_4^2 + \frac{\alpha(q_1^2 + 4q_2^2 + 4q_3^2 + q_4^2)}{(q_1^2 + q_4^2)^3}.
\]

If \( a = -1/2, -2 \) and \( b = 1 \), then Hamiltonian reads as

\[
H_1 = p_1^2 + p_2^2 + p_3^2 + p_4^2 + \frac{\alpha(q_1^2 + 4q_2^2 + 4q_3^2 + q_4^2)}{q_1^2}.
\]

These Hamiltonian have quadratic invariants of the form (2.5), linear invariants which are components of the angular momentum tensor, and quartic invariant which is a central element of the corresponding algebra of invariants.
For instance, at \( a = -2 \) and \( b = 1/2 \) we have quartic invariant

\[
H_4 = (p_1^2 + p_3^2) \left( J_{12}^2 + J_{13}^2 + J_{24}^2 + J_{34}^2 \right) - (p_2^2 + p_3^2) J_{14}^2 + \frac{\alpha}{4(q_1^2 + q_4^2)^3} \sum_{i,j=1}^{4} S_{ij}(q)p_i p_j + \alpha^2 W(q)
\]

where

\[
W(q) = \frac{\alpha(q_3^2 + q_4^2)(q_1^2 + 2q_3^2 + 2q_4^2 + q_2^2)^2}{4(q_1^2 + q_4^2)^6}
\]

and symmetric matrix \( S \) is equal to

\[
S = \begin{pmatrix}
  s_{11} & -2q_1 q_2 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -2q_1 q_3 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -2q_1 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) \\
-2q_2 q_3 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & 4q_1^2 (q_1^2 + q_3^2) & -4q_2 q_3 (q_1^2 + q_4^2) & -2q_2 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) \\
-2q_1 q_3 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -4q_2 q_3 (q_1^2 + q_3^2) & 4q_4^2 (q_1^2 + q_4^2) & -2q_1 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) \\
-2q_1 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -2q_2 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -2q_1 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & 4q_1^2 (q_1^2 + q_3^2 + q_4^2)
\end{pmatrix}
\]

where

\[
s_{11} = 4q_1^2 q_2^2 + 4q_1^2 q_3^2 + q_1^2 q_4^2 + 8q_1^2 + 16q_2^2 q_3^2 + 8q_2^2 q_4^2 + 8q_3^2 + 8q_4^2 + q_4^2
\]

\[
s_{14} = q_1^2 + 4q_1^2 q_2^2 + q_1^2 q_3^2 + q_1^2 q_4^2 + 8q_1^2 + 16q_2^2 q_3^2 + 4q_2^2 q_4^2 + 8q_3^2 + 4q_4^2
\]

At \( a = -2 \) and \( b = 1/2 \) quartic invariant reads as

\[
H_4 = p_1^2 \left( J_{12}^2 + J_{13}^2 + J_{24}^2 + J_{34}^2 \right) + \frac{2\alpha}{q_1^2} \sum_{i,j=1}^{4} S_{ij}(q)p_i p_j + \alpha^2 W(q),
\]

where

\[
W(q) = \frac{(q_3^2 + q_4^2)(q_1^2 + 2q_3^2 + 2q_4^2 + q_2^2)^2}{4q_1^{12}}
\]

and symmetric matrix \( S \) is equal to

\[
S = \begin{pmatrix}
  s_{11} & -2q_1 q_2 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -2q_1 q_3 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -2q_1 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) \\
-2q_2 q_3 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & 2q_1^2 (q_1^2 + q_3^2) & -2q_1 q_3 (q_1^2 + q_4^2) & -2q_1 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) \\
-2q_1 q_3 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -2q_1 q_3 (q_1^2 + q_3^2) & 2q_4^2 (q_1^2 + q_4^2) & -2q_1 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) \\
-2q_1 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -2q_2 q_3 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & -2q_1 q_4 (q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2) & 2q_1^2 (q_1^2 + q_3^2 + q_4^2)
\end{pmatrix}
\]

where

\[
s_{11} = 2(q_2^2 + q_3^2 + q_4^2)(q_1^2 + 2q_2^2 + 2q_3^2 + q_4^2)
\]

These integrable systems are the generalization of the 3D superintegrable system \([2.7]\).

Using the result of the brute force solutions of the equations \([3.1]\) we obtain a generalization of these superintegrable systems in \( \mathbb{R}_3 \) and \( \mathbb{R}^4 \) to \( \mathbb{R}^n \) which will be considered in the next Section.

4 \( n \)-dimensional Euclidean space

Below we change indexes in our previous formulae to get more compact expressions for additional integrals of motion.
4.1 First family of superintegrable systems

Let us consider Hamiltonian

\[ H_1 = \sum_{i,j=1}^{n} g^{ij} p_i p_j + V(q_1, q_2, \rho) = \sum_{i=1}^{n} p_i^2 + \alpha (4q_1^2 + q_2^2 + \rho) + 4\alpha (q_1^2 + q_2^2) \rho + 4\alpha q_1^2 q_2^2, \]

commuting with the second integral of motion

\[ H_2 = \sum_{i,j=1}^{n} K^{ij} p_i p_j + U(q_1, q_2, \rho) = \left( \sum_{i=1}^{n} p_i J_{1,1} \right) - 2p_2 J_{2,1} + 2\alpha q_1 (\rho - q_2^2) \left( 2q_1^2 + q_2^2 + \rho \right), \quad (4.5) \]

Here

\[ \rho = q_3^2 + \cdots + q_n^2, \]

and \( J_{ij} \) are components of the angular momentum operator \( J \) in \( T^*\mathbb{R}^n \)

\[ J = \begin{pmatrix} 0 & J_{1,2} & J_{1,3} & \cdots & J_{1,n} \\ J_{2,1} & 0 & J_{2,3} & \cdots & \vdots \\ J_{3,1} & J_{3,2} & 0 & \ddots & \vdots \\ & \vdots & \ddots & \ddots & \vdots \\ J_{n,1} & \cdots & \cdots & J_{n,n-1} & 0 \end{pmatrix}, \quad J_{i,j} = q_i p_j - q_j p_i. \quad (4.6) \]

Integrals of motion \( H_1 \) and \( H_2 \) are in involution with the following quartic polynomial

\[ H_3 = p_2^2 \sum_{i=3}^{n} p_i^2 + \sum_{i,j=1}^{n} S^{ij}(q_1, q_2, \rho) p_i p_j + W(q_1, q_2, \rho), \quad W = 4\alpha^2 q_2^2 \rho (\rho + 2q_1^2 + q_2^2)^2, \]

where \( S \) is a symmetric matrix with the following entries

\[ S^{2,2} = 4\alpha q_1^2 \rho, \quad S^{i,i} = 4\alpha q_2^2 (\rho + q_1^2 - q_i^2), \quad i \neq 2; \]

\[ S^{1,2} = -4\alpha q_1 q_2 \rho, \quad S^{i,2} = 2\alpha q_2 (\rho + 4q_1^2 + q_2^2), \quad i \neq 1, 2, \]

and in other cases

\[ S^{i,j} = -4\alpha q_i q_j q_2^2. \]

Integrals of motion \( H_1, H_2 \) and \( H_3 \) commute with all the entries of the submatrix \( \hat{J} \) obtained by deleting the first and second row and column in \( J \) [1161]

\[ \hat{J} = \begin{pmatrix} 0 & J_{3,4} & J_{3,5} & \cdots & J_{3,n} \\ J_{4,3} & 0 & J_{4,5} & \cdots & \vdots \\ J_{5,4} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ J_{n,3} & \cdots & \cdots & J_{n,n-1} & 0 \end{pmatrix}, \]

It means that \( H_1, H_2 \) and \( H_3 \) are invariants of action of the group \( O(n-2) \) which is a subgroup of the corresponding isometry group \( \text{E}(n) \).

Although the \( \hat{J}_{ij} \) are in involution with \( H_1, H_2 \) and \( H_3 \), they are not in involution with each other. As usual [1111], we can introduce \( n - 3 \) integrals of motion in involution through

\[ I_k = \sum_{j<k}^{n} \hat{J}_{j,k}^2, \quad k = 3, \ldots, n \quad \text{and} \quad \lambda = \sum_{k=3}^{n} I_k. \]

It provides integrability of at \( n = 3, 4 \) and superintegrability at \( n > 4. \)
4.2 Second family of superintegrable systems

Let us consider Hamiltonian

\[ H_1 = p_1^2 + \cdots + p_n^2 + \frac{\alpha}{r^2} \left( 1 + \frac{4 \rho}{r} \right), \quad r = q_1^2 + \cdots + q_m^2, \quad \rho = q_{m+1}^2 + \cdots + q_n^2, \]

commuting with \( n - 1 \) polynomials of second order in momenta

\[ H_k = \sum_{i=1}^{m} p_i J_{ik} + 2 \sum_{j=m+1}^{n} p_j J_{jk} - \frac{2 \alpha q_k}{r^2} \left( 1 + \frac{2 \rho}{r} \right), \quad k = m + 1 \ldots n, \]

Here \( m \) is arbitrary integer on the interval \( 0 < m < n \).

Because Hamiltonian \( H_1 \) depends only on \( r \) and \( \rho \) it remains invariant under action of the subgroups \( O(m) \) and \( O(n - m - 1) \) in the isometry group \( \mathbb{E}(n) \). As a result, \( H_1 \) commutes with entries of the truncated angular momentum tensor

\[ \hat{J}_{ij} = \begin{pmatrix} J_m & 0 \\ 0 & J_{n-m-1} \end{pmatrix} \]

The central element of the corresponding algebra of integrals of motion is the following polynomial of fourth order in momenta

\[ C = 4H_1 \sum_{j>i}^{n-1} \hat{J}_{ij}^2 - \sum_{k=2}^{n-1} H_k^2, \]

which has to be included in all the sets of \( n \) integrals of motion in the involution. Other central elements are the Hamiltonian and the Casimir element of \( O(m) \).

For instance, at \( n = 5 \) and \( m = 3 \) Hamiltonian is equal to

\[ H_1 = \sum_{i=1}^{5} p_i^2 + \frac{\alpha}{(q_1^2 + q_2^2 + q_3^2)^2} \left( 1 + \frac{4(q_4^2 + q_5^2)}{q_1^2 + q_2^2 + q_3^2} \right). \]

It commutes with integrals of motion \( H_4 \) and \( H_5 \)

\[ H_k = \sum_{i=1}^{5} p_i J_{ik} + 2 \sum_{j=m+1}^{n} p_j J_{jk} - \frac{2 \alpha q_k}{(q_1^2 + q_2^2 + q_3^2)^2} \left( 1 + \frac{2(q_4^2 + q_5^2)}{q_1^2 + q_2^2 + q_3^2} \right), \quad k = 4, 5, \]

and with the following entries of the angular momentum vector \( J_{1,2}, J_{1,3} \) and \( J_{2,3} \). Non-trivial elements of the algebra of integrals of motion are

\[ \{H_4, H_5\} = -4J_{4,5}H_1, \quad \{H_4, J_{45}\} = -H_5, \quad \{H_5, J_{45}\} = H_4, \]

and

\[ \{J_{1,2}, J_{1,3}\} = J_{2,3}, \quad \{J_{1,3}, J_{2,3}\} = J_{1,2}, \quad \{J_{2,3}, J_{1,2}\} = J_{1,3}. \]

The central element of this algebra of integrals of motion has the form

\[ C = 4H_1 \left( J_{1,2}^2 + J_{1,3}^2 + J_{2,3}^2 + J_{45}^2 \right) - H_4^2 - H_5^2, \]

which can be rewritten as

\[ \tilde{C} = 4H_1 J_{45}^2 - H_4^2 - H_5^2 \]

using other central elements \( H_1 \) and \( J_{1,2}^2 + J_{1,3}^2 + J_{2,3}^2 \).
5 Conclusion

Sometimes all the quadratic conservation laws for equations of mathematical physics can be
determined using only two quadratic integrals of motion \[1, 13\]. In this note, we proved that
\( n - 1 \) quadratic and one quartic constant of motion can be also determined using only two
quadratic integrals of motion

\[
H_1 = \sum_{i,j=1}^{3} g^{ij}(q)p_i p_j + V(q) \quad \text{and} \quad H_2 = \sum_{i,j=1}^{3} K^{ij} p_i p_j + U(q),
\]

where \( K \) is the completely non-invariant Killing tensor outside Nijenhuis and Haantjes geometry

\( H_K \neq 0 \).

In three-dimensional Euclidean space, these Killing tensors are characterized by the following
condition on the off-diagonal entries of Haantjes tensor \( H_K \)

\[
(H_K)_{23}^1 = 0, \quad (H_K)_{31}^2 \neq 0, \quad (H_K)_{12}^3 \neq 0, \quad (5.7)
\]

up to permutation of indexes, see \[16, 17\]. This experimental fact allows us to study similar
Killing tensors in another three-dimensional space. For instance, we can prove that in the
conformal Euclidean space with metric

\[
\hat{g} = \frac{1}{1 + \lambda f(q)} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \lambda \in \mathbb{R},
\]

Killing tensors satisfying \( (5.7) \) do not exist at

\[
f(q) = q_1^2 + q_2^2 + q_3^2
\]

and exist at

\[
f(q) = q_1^2 + q_2^2 + 4q_3^2.
\]

In the last case, we can construct at least two new integrable systems with 2 quadratic and 1
quartic invariants in the involution.

Similarly, we can study more complicated deformations of the flat metric. For instance, we
suppose that satisfying equations \( (5.7) \) Killing tensors

- do not exist in the space 3-manifold for Schwarzschild coordinates in the Schwarzschild
  spacetime when the metric is the following deformation of the flat metric \( g \)

\[
\hat{g} = \sigma g, \quad \sigma = \left(1 + \frac{1}{2mr}\right)^4
\]

- exist in the space 3-manifold for Boyer-Lindquist coordinates in the Kerr spacetime, be-
  cause the corresponding metric is more complicated deformation of the flat metric

\[
\hat{g} = \sigma g + s \times s
\]

where \( \sigma \) and \( s \) are respectively a scalar and a one-form depending on all the spherical
coordinates \( r, \phi \) and \( \theta \). \[7\].

Of course, it is more interesting to study non-invariant Killing or Yano-Killing tensors in the
four-dimensional Schwarzschild spacetime and Kerr spacetime, but it is a more complicated
task.

In 4-dimensional and \( n \)-dimensional Euclidean space we also have a few vanishing off-
diagonal entries of Haantjes tensor \( H_K \) whereas other off-diagonal entries do not equal to zero.
Because the number of vanishing off-diagonal entries depends on a choice of local coordinates we need in the geometric description of the non-invariant Killing tensors with $H_K \neq 0$ suitable to construction of quartic invariants.

Similar to transformation from the Nijenhuis conditions to more geometric Haantjes conditions, we have to transform (5.7) to coordinate independent conditions working in any dimension. It is the main open theoretical problem appearing in our mathematical experiments with the non-invariant Killing tensors in Euclidean space.

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