Synchronization transitions through metastable state on structured networks

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Recently, we considered the fully connected competing Kuramoto model with uniform intrinsic frequency distribution \(g(\omega)\). This competing Kuramoto model assigns two opposite-sign coupling constants \(K_1 < 0\) and \(K_2 > 0\) to \(1 - p\) and \(p\) fractions of nodes, respectively. In our previous paper, we briefly reported a rich phase diagram that includes incoherent (IC), \(\pi\), and traveling wave (TW) phases and abnormal properties of a hybrid phase transition that occurs through an intermediate metastable \(\pi\) state. Here, we present the detailed derivation of the self-consistent equations, the phase diagram, and physical properties of the hybrid phase transition from the incoherent to \(\pi\) with the critical exponent \(\beta_p = 2/3\). Next, we extend our study to the case that oscillators locate on structured random networks. Within the heterogeneous mean-field scheme, phase diagram and transition types are overall similar to those on the completely connected networks. However, numerical simulations on the structured networks produce different results. When the mean degree of the structured networks is small, TW state does not appear and transition types are changed, in contrast to the mean-field solutions on the annealed random networks, which predict TW states under appropriate control parameters.

I. INTRODUCTION

The Kuramoto model (KM) \textsuperscript{[1]} is a coupled phase oscillator model which exhibits a synchronization transition from an incoherent to a synchronized state. This model is written as

\[
\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i),
\]

where \(\theta_i\) denotes the phase of oscillator \(i\), \(\omega_i\) is the natural frequency of an oscillator \(i\) which follows a distribution \(g(\omega)\), \(K\) is the coupling constant with positive sign, and \(N\) is the number of oscillators in the system. We note that each oscillator has interactions with all other oscillators, which is conventionally used as a mean-field type model. This KM has served as a prototypical model to understand synchronization phenomena in diverse real-world systems ranging from biological systems such as fireflies flashing to physical systems such as electric power grids \textsuperscript{[2–4]}. In the original KM, \(g(\omega)\) takes a Gaussian distribution, and a continuous synchronization transition occurs as the coupling strength \(K\) is increased.

The coupling constant is not necessarily positive, but can be generalized to a mixture of positive and negative constants when the system contains competing interactions. Here we consider a KM with such competing coupling constants, inspired by neural networks with excitatory and inhibitory couplings between neurons, and by spin glass systems with ferromagnetic and antiferromagnetic interactions. Such competing interactions, for instance, in spin glass systems, often lead to the frustration of magnetic ordering and the emergence of a glass state. Metastable slow dynamics proceeds across the free energy landscape. Accordingly, one may expect similar metastable behaviors to arise in the competing KM (c-KM).

The KM with competing interaction was first considered by Daido \textsuperscript{[5]} a long time ago. He used the Sherrington-Kirkpatrick type random coupling constant \(K_{ij}\) between all pairs of oscillators \((i, j)\), which follows a Gaussian distribution with zero mean and a finite standard deviation. Under such competing interactions, several interesting behaviors were noticed, for instance, quasi-entrainment, diffusive dynamics in phase space and slow power-law decay of coherence. Furthermore, the potential existence of glassy-oscillating state, leading to the so-called volcano transition was suggested \textsuperscript{[5,6]}.

Meanwhile, Hong and Strogatz \textsuperscript{[7–9]} considered a tractable KM with node-based random couplings \(\{K_i\}\), in which \(1 - p\) fraction of nodes have negative coupling constant \(K_1 < 0\), while the remaining nodes have positive coupling constant \(K_2\). The authors adopted a Lorentzian \(g(\omega)\) and used the Ott-Antonsen (OA) ansatz \textsuperscript{[10,11]}, finding that the synchronized state of this c-KM can be characterized by the dynamics of two groups of oscillators roughly separated by an angle \(\pi\) in the phase space. They are either static or traveling.

The OA method has been successful in the bifurcation analysis of several KMs with unimodal \(g(\omega)\) such as the Lorentzian distribution \textsuperscript{[8,13,19]}. That is because a \(N \rightarrow \infty\) continuum of oscillator degrees of freedom are exactly reduced to a few coupled modes. The Watanabe-Strogatz method is also useful to reduce the degrees of freedom of the coupled nonlinear equations of \(N\) identical oscillators \textsuperscript{[7,13,16]}. The order parameter dynamics is exactly calculable in these OA and Watanabe-Strogatz reducible systems. However, for a uniform distribution, reduction is hardly achieved. Instead, one can at best seek for the stationary solution using the complex self-consistency (SC) equation.

In our previous paper \textsuperscript{[17]}, we considered a c-KM model with a uniform \(g(\omega)\) with interval \([-\gamma, \gamma]\), called the c-Winfree-Pazó (c-WP) model. Using the complex
SC equation, we showed briefly that a discontinuous synchronization transition occurs with critical behavior, often called as a hybrid phase transition. We mainly focused on two-step discontinuous transitions that arises above the hybrid transition point: from incoherent (IC) to \( \pi \), and then to traveling wave (TW) state, where the intermediate \( \pi \) state is characterized by a long-lasting metastability.

In this paper, we first recapitulate a general formalism of the complex SC equation which is applicable to general types of \( g(\omega, K) \) \cite{13}. Next we derive explicitly the complex SC equation for the \( c \)-WP model. In this case, \( \pi \) and TW states can emerge in the synchronized regime.

The imaginary part of the SC equation determines the TW solution. A rich phase diagram is obtained and critical behaviors are derived explicitly as a function of the parameter set \( \{p, Q \equiv |K_1|/K_2, \gamma\} \). On the other hand, the complex SC equation does not determine the stability of the solution rigorously. Instead, an empirical linear-stability of each complex SC equation solution can be analyzed \cite{13}. However, we have shown numerically that their linear stability criterion could not discern the metastable states, which arises in the \( c \)-WP model.

Next, we consider the \( c \)-KM on Erdős and Rényi (ER) networks. First, using the heterogeneous mean-field approach, we check analytically whether the synchronization transitions of the all-to-all coupling case occur even on ER networks. We find that the synchronization transitions are overall similar. However, simulation results show significant differences. On ER networks, the mean degree of ER network \( \langle q \rangle \) additionally plays an important role in determining synchronization transition properties.

When \( \langle q \rangle \) is small, TW state can disappear, even though the analytic and numerical solutions on the anneal ER networks predict the existence of the TW state. Moreover, transition type can be changed.

This paper is organized as follows: In Sec. II we introduce the \( c \)-KM and investigate the origin of TW of oscillators. Mean-field solution of the TW is also presented. In Sec. III we derive the complex SC equation by following the original Kuramoto method and the OA ansatz. In Sec. IV we consider the \( c \)-WP model and derive the SC equation explicitly. Using the SC equation, we derive the order parameter \( R \) and \( \Omega \) of the \( c \)-WP model for various \( Q = |K_1|/K_2 \) and the half-width \( \gamma \) of uniform distribution \( g(\omega) \). Thus, diverse phases such as IC, \( \pi \) and TW phases are characterized. Moreover, diverse phase transitions such as continuous, discontinuous, and hybrid phase transitions are observed. Based on such results, we construct phase diagrams for each \( Q < 1 \) and \( Q > 1 \). In Sec. V we consider the \( c \)-KM model on ER networks. Using the heterogeneous mean-field approach, we obtain the SC equation for ER networks. In Sec. VI we investigate the synchronization transitions on ER networks by solving the SC equation numerically and performing simulations. Phase diagrams and the behaviors of the order parameters are obtained as a function of \( p \) for various values of \( \gamma, Q \), and mean degree \( \langle q \rangle \). In Sec. VII we summarize our results.

II. THE COMPETING KURAMOTO MODEL

The \( c \)-KM on the completely connected networks is written as

\[
\dot{\theta}_i = \omega_i + \frac{K_i}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, N, \tag{2}
\]

where each oscillator phase \( \theta_i \) has the intrinsic frequency \( \omega_i \) following a distribution function \( g(\omega) \). In this Section, we do not assume a specific form of \( g(\omega) \) yet. \( K_i \) is a coupling constant of oscillator \( i \). \((1-p)\) fraction of nodes have a coupling constant \( K_1 < 0 \), while the remaining nodes have \( K_2 > 0 \). \( p \) is a control parameter. This \( c \)-KM can be possibly generalized to an arbitrary number of species of coupling constants.

The synchronization transition is characterized by a complex order parameter at time \( t \),

\[
Z(t) \equiv R(t)e^{i\psi(t)} \equiv \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}, \tag{3}
\]

where \( R \) is the coherence of oscillators and \( \psi \) is the average phase. In the long time limit, the system falls into the stationary state, in which \( Z \approx Re^{i\Omega} \). The synchronization order parameter \( R \) distinguishes coherent (C) and IC phases. The C phase is further distinguished into \( \pi \) and traveling wave (TW) phase by the TW order parameter \( \Omega \). Overall, three phases are possible: IC \((R = 0)\), \( \pi \) \((R \neq 0, \Omega = 0)\), and TW \((R \neq 0, \Omega \neq 0)\) phases.

Plugging the definition of the complex order parameter into the Kuramoto equation leads to an effective decoupling of oscillators,

\[
\dot{\theta}_i = \omega_i + K_i R(t) \sin(\psi(t) - \theta_i), \quad i = 1, 2, \ldots, N. \tag{4}
\]

Depending on the sign of coupling \( K_i \), the stability of an oscillator at a velocity-balancing position is reversed. Each oscillator with \( K_2 > 0 \) is attracted towards the average phase \( \psi \) with strength proportional to the coherency \( r \), while each oscillator with \( K_1 < 0 \) is drawn towards the antipod \( \psi + \pi \) (Fig. 1). Therefore, in the synchronized phase the oscillators are clustered into two groups, in which each positive or negative coupling population forms a separate group. The \( K_1 \) group, i.e., the group of oscillators with the coupling constant \( K_1 < 0 \), generally shows a more widespread angle distribution than the \( K_2 \) group, because each \( K_1 \) oscillator is repulsive to all other oscillators while each \( K_2 \) oscillator is attractive to all other oscillators.

In the \( \pi \) phase, the two groups are balanced at a separation angle \( \pi \) and stay fixed on the phase circle. In the TW phase, a separation less than \( \pi \) is maintained, where the attractive \( K_2 \) group tries to catch up with the \( K_1 \) group while the repulsive \( K_1 \) group tries to keep the
The force exerted on the distance with the \( K \) denoted by \( \pi \) asymmetric interaction. We stress that the sign differs; the angular speed \( \Omega \).

can run in the same direction. A nonvanishing net interaction, however, does not necessarily lead to a steady TW propagation. The TW phase can emerge only if \( Q \equiv [K_1]/K_2 < 1 \), when the attraction of the \( K_2 \) group is larger than the repulsion of the \( K_1 \) group, otherwise a separation less than \( \pi \) is not maintained. We construct a mean field theory in the following as the first attempt to understand the emergence of TW.

A. The origin of TW

TW is possible in the competing model due to the asymmetric interaction. We stress that the sign difference in \( K_1 \) and \( K_2 \) is crucial in the formation of TW. The force exerted on \( K_1 \) by \( K_2 \) and the corresponding reaction on \( K_2 \) by \( K_1 \) are no longer in opposite direction, i.e., the action-reaction principle is broken. Therefore both \( K_1 \) and \( K_2 \) can run in the same direction.

A nonvanishing net interaction, however, does not necessarily lead to a steady TW propagation. The TW phase can emerge only if \( Q \equiv [K_1]/K_2 < 1 \), when the attraction of the \( K_2 \) group is larger than the repulsion of the \( K_1 \) group, otherwise a separation less than \( \pi \) is not maintained. We construct a mean field theory in the following as the first attempt to understand the emergence of TW.

B. Mean field theory for the TW phase

In the TW phase, the system collectively rotates at a nonzero speed. The average angular speed of the two groups \( K_1 \) and \( K_2 \) and that of the total system are calculated as follows:

\[
v_1 \equiv \langle \dot{\theta} \rangle_1 = \frac{1}{N_1} \sum_{j \in K_1} \omega_j + \frac{K_1}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j)
= p|K_1|r_1r_2 \sin \Delta
\]

\[
v_2 \equiv \langle \dot{\theta} \rangle_2 = \frac{1}{N_2} \sum_{j \in K_2} \omega_j + \frac{K_2}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j)
= (1 - p)K_2r_1r_2 \sin \Delta
\]

\[
v = pv_2 + (1 - p)v_1
= p(1 - p)(|K_1| + K_2)r_1r_2 \sin \Delta
\]

As a mean field approximation, we let \( \Delta \approx \langle \theta_{12} \rangle \equiv \frac{1}{N_1N_2} \sum_{j \in K_1} \sum_{k \in K_2} (\theta_j - \theta_k) \) and \( \psi_\alpha \approx \langle \theta_{12} \rangle \alpha (1 + \langle \theta_{12}^2 \rangle \alpha - \langle \theta_{12}^2 \rangle \alpha / 2!) \approx \langle \theta_{12} \rangle \alpha \) for each group, i.e., \( r_1 \approx r_2 \approx 1 \). The approximation is valid as long as the distribution angle of each group is small. Let \( \Delta \equiv \pi - \delta \). Assuming that the amplitude parts are stable, the phase dynamics of \( \delta \) is given as

\[
\dot{\delta} \approx -\langle \theta_{12} \rangle \alpha = -(V_1 - V_2)
= -(p - p_u)(|K_1| + K_2)r_1r_2 \sin \delta,
\]

where \( \langle e^{i\theta} \rangle_{1,2} \equiv r_1e^{i\psi_{1,2}} \) and the separation between the two groups is denoted as \( \psi_1 - \psi_2 \equiv \Delta \). \( \langle \omega_j \rangle_2 = 0 \) for \( g(\omega) \) with even symmetry. Also \( \langle \sin \theta_{12} \rangle \equiv \frac{1}{N_1N_2} \sum_{j \in K_1} \sum_{k \in K_2} \sin(\theta_j - \theta_k) = r_1r_2 \sin \psi_1 - \psi_2 = r_1r_2 \sin \Delta \). The summation counts only for inter-group interactions because the intra-group interactions pairwise cancel up to zero. When \( \Delta = \pi, \sin \Delta \) is zero and therefore \( v_1 = v_2 = v = 0 \). However, in the TW phase, \( \Delta \) is less than \( \pi \) and the angular speed \( v \) is nonzero. Now for each group \( \alpha = 1 \) and 2,

\[
\psi_\alpha = \arctan \left( \frac{\sin \theta_{12} \alpha}{\cos \theta_{12} \alpha} \right) = \arctan \left( \frac{\theta_{12} \alpha}{1 - \theta_{12}^2 \alpha / 2 + \cdots} \right)
\]

III. THE COMPLEX SC EQUATION

Here we seek for steady state solutions \( Z \approx Re^{i\Omega t} \). For this purpose, we apply the SC method as follows: In the rotating frame \( \phi_j = \theta_j - \Omega t \), each oscillator follows

\[
\dot{\phi}_j = \omega_j - \Omega - K_j R \sin \phi_j.
\]

Depending on the relative strengths between the frequency disorder \( \omega_j = \Omega \) and the coupling strength \( |K_j| \), each oscillator is either phase-locked or drifting. The order parameter is written as

\[
R = \frac{1}{N} \sum_j e^{i\phi_j(0)},
\]

and time average is taken in the steady state regime for the self-consistency, which is denoted by the overline. Each oscillator with natural frequency satisfying \( |\omega_j - \Omega| \leq |K_j| R \) becomes phase-locked at

\[
\phi_j^* = \arcsin \left( \frac{\omega_j - \Omega}{K_j R} \right),
\]
where $a$ eventually contributes to the total synchronization order. Meanwhile each drifting oscillator satisfying $|\omega_j - \Omega| \geq |K_j| R$ will repeatedly lapse forward or behind the locked population. Their rotation period is calculated as

$$T_j = \int_0^{2\pi} \frac{d\phi_j}{|\phi_j|} = \frac{2\pi}{\sqrt{(\omega_j - \Omega)^2 - (K_j R)^2}}. \quad (11)$$

Therefore the contribution of the drifting oscillators to the order parameter $R$ is zero. Indeed individual drifting oscillator contributes only to the order parameter $\Omega$ as

$$e^{i\phi_j} = \frac{1}{T_j} \int_0^{2\pi} \frac{d\phi_j}{\phi_j} e^{i\phi_j} = \frac{i \text{sgn}(\omega_j - \Omega)}{K_j R} \left[ |\omega_j - \Omega| - \sqrt{(\omega_j - \Omega)^2 - (K_j R)^2} \right]. \quad (12)$$

After some calculations, the complex SC equation is obtained as follows:

$$R = \int_{-\infty}^{\infty} dK d\omega g(K,\omega) e^{i\phi}$$

$$= \int_{-\infty}^{\infty} dK \int_{\Omega - |K| R}^{\Omega + |K| R} d\omega g(K,\omega) \text{sgn}(K) \sqrt{1 - \left( \frac{\omega - \Omega}{KR} \right)^2}$$

$$+ i \int_{-\infty}^{\infty} dK \int_{\text{drifting}} d\omega g(K,\omega) KR \left[ \omega - \Omega - \text{sgn}(\omega - \Omega) \sqrt{(\omega - \Omega)^2 - (KR)^2} \right], \quad (13)$$

where $g(K,\omega)$ is the distribution of disorders of the c-KM, and $\text{sgn}(x) = 1$ and $-1$ for $x > 0$ and $x < 0$, respectively.

The imaginary part on the right hand side of the complex SC equation\[13\] has to vanish, because $R$ in the left hand side is real valued. Usually in the KM, $\Omega$ should be given as the mean intrinsic frequency of the system, i.e. the traveling wave order is absent. Therefore only the real part of the equation is needed to be solved. Indeed, when $g$ is symmetric with respect to $\Omega$, one can easily check that the integrand of the imaginary part becomes odd in $\omega$ and vanishes to zero. To obtain TW solution; however, the real and imaginary parts of the equation\[13\] must be solved simultaneously for $R$ and $\Omega$.

The complex SC equation can also be obtained by applying the OA ansatz to the continuum version of the KM.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta} \left[ \rho(K,\omega,\theta,t) \left( \omega + K \frac{Ze^{-i\theta} - Z^* e^{i\theta}}{2t} \right) \right] = 0. \quad (14)$$

The density function is assumed to have the form,

$$\rho = \frac{g(K,\omega)}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} \left( a_{\omega,\omega,K}(t) e^{in\theta} + \text{c.c.} \right) \right],$$

where $a_{\omega,\omega,K}$ satisfies the equation,

$$\dot{a}_{\omega,\omega,K}(t) = -i \omega a_{\omega,\omega,K} + \frac{K}{2} (Z^* - Z a_{\omega,\omega,K}^2). \quad (15)$$

Then the order parameter is given as

$$Z(t) = \int dK d\omega d\theta e^{i\theta} \rho(K,\omega,\theta,t)$$

$$= \int dK d\omega g(K,\omega) a_{\omega,\omega,K}^*(t). \quad (16)$$

In the steady state, Eqs.\[15\] and \[16\] also lead to the complex SC equation that we derived in Eq.\[13\]\[18]. For the Lorentzian frequency distribution $g(\omega) = \frac{\gamma/\pi}{\omega^2 + \gamma^2}$, the above integration can be carried out in the complex domain and only a single complex mode $a_{\omega,\omega,K}^*(t)$ (for each $K$) is relevant to the order parameter. Such a dimensional reduction does not occur in general for other types of frequency distributions, but Eq.\[16\] remains valid. Therefore exact bifurcation analysis was studied in limited intrinsic frequency distributions, for instance, the Watanabe-Strogatz transform\[14\]\[15\] for identical oscillators or the OA methods for Lorentzian-type intrinsic frequency distributions\[6\]\[8]\[13]\[19]. A general theory for the stability analysis is yet unknown.

It should be mentioned that the SC method does not determine the stability of the solution in general. The stability of each SC solution needs to be checked numerically. An empirical linear stability criterion proposed in Ref\[18\] have succeeded to correctly tell the stability of the self consistently obtained solutions in several cases with different intrinsic frequency distributions. However, we have pointed out in the previous paper that this criterion is incomplete for the metastable states with almost neutral stability, in case of the uniform intrinsic frequency distribution\[17\].
IV. THE c-WP MODEL

Thus far, we developed a general framework within which the c-KM can be explored. Here we study the c-KM with the uniform intrinsic frequency distribution in the range $[-\gamma, \gamma]$ and two types of coupling constants $K_1 < 0$ or $K_2 > 0$ at a ratio of $1 - p$ to $p$. This model is called the competing Winfree-Pazó (c-WP) model. The distribution $g(K, \omega)$ is written as

$$g(K, \omega) = \frac{1}{2\gamma} \Theta(\gamma - |\omega|) [(1-p)\delta(K-K_1) + p\delta(K-K_2)].$$  \hspace{1cm} (17)

In fact, the KM with uniform frequency distribution, called the Winfree-Pazó (WP) model, is known to exhibit a hybrid synchronization transition, which is a discontinuous phase transition 21 accompanying a critical behavior 21. We also observe a hybrid phase transition as $p$ is varied in the c-WP model with a critical exponent $\beta_p = 2/3$ at $p_c^+$, analogous to $\beta_K = 2/3$ as $K$ is varied at $K_+^*$ in the WP model 21. This transition occurs when $Q \equiv |K_1|/K_2 < 1$ (Fig. 2(a)). Furthermore when $Q > 1$, there exists an unstable hybrid transition as shown in Fig. 2(b).

When $\Omega = 0$, i.e., for the IC and $\pi$ states, the imaginary part of Eq. (13) vanishes because the integrand becomes odd in $\omega$. The transition from IC to $\pi$ state is characterized by solving the real part of the SC equation,

$$R = -(1-p) \int_{-\left|K_1\right|R}^{\left|K_1\right|R} d\omega g(\omega) \sqrt{1 - \left(\frac{\omega}{|K_1|R}\right)^2} + p \int_{-\left|K_2\right|R}^{\left|K_2\right|R} d\omega g(\omega) \sqrt{1 - \left(\frac{\omega}{|K_2|R}\right)^2},$$ \hspace{1cm} (18)

where $g(\omega) = \frac{1}{\sqrt{2\gamma}} \Theta(\gamma - |\omega|)$. In the following, analytic solutions of Eq. (18) are obtained for different parameter values.

It should be noticed that the integration is evaluated differently depending on the relative sizes of $|K_1|R, K_2R$ and $\gamma$. i) For $Q \equiv |K_1|/K_2 < 1$, $\gamma$ can fall in one of the three ranges $(0,|K_1|R], (|K_1|R,|K_2|R], (|K_2|R,\infty)$. ii) For $Q > 1$, $\gamma$ can fall in one of the three ranges $(0, K_2R], (K_2R, |K_1|R]$, and $(|K_1|R, \infty)$. 

![Fig. 2. Schematic plots of the hybrid phase transition and the first-order transition with hysteresis: (a) a hybrid phase transition with $\beta_p = 2/3$ occurs for $Q < 1$, while (b) a first-order transition with hysteresis occurs for $Q > 1$, in which an unstable hybrid phase transition with $\beta_p = 2/3$ is hidden. Thick solid and dashed lines denote stable and unstable SC solutions, and the thin dotted vertical lines denote jumps at the transition points.](image)

A. The SC solution for IC and $\pi$

1. Supercritical hybrid phase transition ($Q < 1$)

The integration of Eq. (18) when $Q < 1$ leads to

$$R = \begin{cases} \frac{1-p}{2\gamma} |K_1| R + \frac{p}{2\gamma} \frac{|K_2|}{2} K_2 R & \text{for } R < \frac{\gamma}{|K_2|} < \frac{\gamma}{|K_1|}, \\ \frac{1-p}{2\gamma} \frac{|K_1|}{2} R + p \left( \frac{K_2 R}{2\gamma} \arcsin \frac{\gamma}{K_2 R} + \frac{1}{2} \sqrt{1 - \left(\frac{\omega}{|K_2|R}\right)^2} \right) & \text{for } \frac{\gamma}{|K_2|} < R < \frac{\gamma}{|K_1|}, \\ \frac{1-p}{2\gamma} |K_2| R + p \left( \frac{K_2 R}{2\gamma} \arcsin \frac{\gamma}{|K_2| R} + \frac{1}{2} \sqrt{1 - \left(\frac{\omega}{|K_2| R}\right)^2} \right) + p \left( \frac{K_2 R}{2\gamma} \arcsin \frac{\gamma}{K_2 R} + \frac{1}{2} \sqrt{1 - \left(\frac{\omega}{|K_2|R}\right)^2} \right) & \text{for } \frac{\gamma}{|K_2|} < \frac{\gamma}{|K_1|} < R. \end{cases}$$ \hspace{1cm} (19)
Note the IC ($R = 0$) state is a trivial solution of the above SC equation. The remaining nontrivial solutions correspond to $\pi$ states. After inverting the above equations and solving for $p$, we obtain the inverse function of the order parameter curve $p(R)$ as follows:

$$p = \begin{cases} \frac{|K_1| + 4\gamma/\pi}{|K_1| + K_2} & \text{for } 0 < R < \frac{\gamma}{K_2}, \\ \frac{(\pi|K_1| + 4\gamma)R}{\pi|K_1| + 2\gamma\sqrt{1 - \left(\frac{\gamma}{K_2}\right)^2 + K_2^2R\arcsin\left(\frac{\gamma}{K_2}\right)}} & \text{for } \frac{\gamma}{K_2} < R < \frac{\gamma}{|K_1|}, \\ \frac{\gamma\sqrt{1 - (\gamma/|K_1|)^2 + |K_1|\arcsin\left(\frac{\gamma}{|K_1|}\right)} + 2\gamma R}{\gamma\sqrt{1 - (\gamma/|K_1|)^2 + K_2^2R\arcsin\left(\frac{\gamma}{K_2}\right)}} & \text{for } \frac{\gamma}{K_2} < \frac{\gamma}{|K_1|} < R. \end{cases} \quad (20)$$

Numerical solutions of $R(p)$ from the SC equations are represented as solid and dashed curves in Fig. 3. Solid (dashed) curves are stable (unstable) through the linear stability analysis \[18\], which will be discussed later.

![Fig. 3](image_url)

**FIG. 3.** Diverse types of synchronization transitions (STs) for $Q = 0.5$ and various $\gamma$. Green triangles and red circles denote data points of $R(p)$ and $|\Omega(p)|$ obtained from simulations starting from the IC and C initial states, respectively. Solid (dashed) blue curves are SC solutions representing stable (unstable) states, according to the stability criterion, Eq. (30). In (a), a hybrid phase transition (HPT) occurs with the critical exponent $\beta_p = 2/3$ at $p_c$. A close check of the exponent value is shown in the inset. The black line guides a slope of $2/3$. (b) The TW phase emerges at $\gamma_b$ and exists in the range $[p_c, p_a]$. When $\gamma_a < \gamma < \gamma_b$, IC $\rightarrow$ TW $\rightarrow$ TW $\rightarrow$ IC occur with increasing $p$. (c) At $\gamma = \gamma_a$, $p_c = p_a$; thus, IC $\rightarrow$ TW $\rightarrow$ IC occur. The part of the $\pi$ line (indicated by arrow) that is stable according to the criterion is actually metastable. (d) When $\gamma < \gamma_a$ ($\gamma = 0.05$), $p_c < p_a < p_a$, $R$ jumps from the IC state to the TW state, and a hysteresis occurs between the IC and TW states at $[p_c, p_a]$, where IC $\rightarrow$ TW $\rightarrow$ TW $\rightarrow$ IC occurs. Different types of arrows distinguish the types of phase transitions: continuous ($\rightarrow$), discontinuous ($\leftarrow$), and hybrid ($\leftrightarrow$). A hysteresis behavior appears in (d), while it does not in (b) and (c).

The order parameter curve $R(p)$ shows a discontinuous jump of size $R_c$ at the critical point $p_c$.

$$p_c = \frac{|K_1| + 4\gamma/\pi}{|K_1| + K_2} = \frac{Q + \frac{4\gamma}{|K_2|}}{Q + 1},$$

$$R_c = \gamma/K_2. \quad (21)$$

Expanding the intermediate branch $\frac{\gamma}{K_2} < R < \frac{\gamma}{|K_1|}$ in powers of $\epsilon \equiv (R - R_c)/R_c$ after the jump gives:

$$p = \frac{(\pi|K_1| + 4\gamma)R}{\pi|K_1| + 2\gamma\sqrt{1 - \left(\frac{\gamma}{K_2}\right)^2 + K_2^2R\arcsin\left(\frac{\gamma}{K_2}\right)}} \quad \approx \quad \frac{(\pi|K_1| + 4\gamma)R_c(1 + \epsilon)}{\pi|K_1|R_c(1 + \epsilon) + 2\gamma\left(\sqrt{26} - \frac{3\epsilon^{3/2}}{2\sqrt{2}} + O(\epsilon^{5/2})\right) + 2\gamma(1 + \epsilon)\left(\frac{\pi}{\sqrt{26}} - \frac{5\epsilon^{3/2}}{6\sqrt{2}} + O(\epsilon^{5/2})\right)}$$

$$\approx p_c + \frac{8\sqrt{2}}{3\pi} \frac{p_c}{(Q + 1)} \epsilon^{3/2} + O(\epsilon^{5/2}). \quad (22)$$
Therefore the transition from IC to \( \pi \) state is hybrid as \((R - R_c) \sim (p - p_c)^{6/3}\) with \(\beta_p = 2/3\) for \(Q < 1\). Notice at \(p_c\), the competing system has a mean coupling strength

\[
\langle K \rangle_c = (1 - p_c)K_1 + p_cK_2 = \frac{4\gamma}{\pi},
\]

(23)

which is equivalent to the critical coupling strength \(K_c = 2/\pi g(0) = 4\gamma/\pi\) for IC \(\rightarrow\) C of the WP model \[20, 21\]. Also the coincidence with the critical exponent value \(\beta_K = 2/3\) should be noticed for the WP model. This is rather natural because the WP model corresponds to a particular case of the c-WP model with \(p = 1\). Besides these, a new type of transition additionally occurs in the c-WP model below.

Thus far, the IC and \(\pi\) states and the transitions between the two phases are obtained exactly. TW solutions, however, could not be obtained in a closed form. Instead we solve the complex SC equation numerically. Solving the c-WP model below.

2. Subcritical hybrid phase transition \((Q > 1)\)

\[
\begin{align*}
\text{(a) } & \gamma = 0.2 \quad \text{(b) } \gamma = 0.4 \quad \text{(c) } \gamma = 0.6 \quad \text{(d) } \gamma = 0.785
\end{align*}
\]

\text{FIG. 4.} Plot of the order parameter versus \(p\) for \(Q = 3\) and various \(\gamma\): (a) \(\gamma = 0.2\), (b) \(\gamma = 0.4\), (c) \(\gamma = 0.6\), and (d) \(\gamma = 0.785\). A first-order transition and hysteresis occur between \(p_c\) and \(p_{c,b}\). As \(\gamma\) is increased, \(p_{c,b}\) and \(p_c\) are increased at different rates and hysteresis range is reduced. The unstable solution contains a discontinuous jump and a hidden critical exponent value \(\beta_p = 2/3\). Notice also that in (d) at \(\gamma = \gamma_h = \pi K_2/4 \approx 0.785\), \(p_c^+ = p_c = 1\) and \(R_c = \gamma/K_2\). At \(p = 1\), the c-WP model reduces to the WP model \[20, 21\]. For larger values of \(\gamma\) beyond \(\gamma_h\), the system is incoherent because the value of \(K_2\) is subcritical i.e. \(K_2 < K_c \equiv 4\gamma/\pi\), where \(K_c\) is the critical coupling strength of the WP model.

The integration of Eq. \[18\] when \(Q > 1\) leads to

\[
R = \begin{cases} 
\frac{-1 - p}{2\gamma} \frac{1}{|K_1|} |R| + \frac{p}{2\gamma} \frac{1}{|K_2|} |R| K_2 R & \text{for } R < \frac{\gamma}{|K_1|} < \frac{\gamma}{K_2} \\
-(1 - p) \left( \frac{|K_1|}{2\gamma} \arcsin \frac{\gamma}{|K_1|} + \frac{1}{2} \sqrt{1 - \left( \frac{\omega}{|K_1|} \right)^2} \right) + \frac{p}{2\gamma} \frac{1}{|K_2|} |R| K_2 R & \text{for } \frac{\gamma}{|K_1|} < R < \frac{\gamma}{K_2} \\
-(1 - p) \left( \frac{|K_1|}{2\gamma} \arcsin \frac{\gamma}{|K_1|} + \frac{1}{2} \sqrt{1 - \left( \frac{\omega}{|K_1|} \right)^2} \right) + \left( \frac{K_2 R}{2\gamma} \arcsin \frac{\gamma}{K_2 R} + \frac{1}{2} \sqrt{1 - \left( \frac{\omega}{K_2 R} \right)^2} \right) & \text{for } \frac{\gamma}{K_1} < R < \frac{\gamma}{K_2}. 
\end{cases}
\]

(24)
Taking a similar step, we obtain that

\[ p = \begin{cases} 
\frac{|K_1| + 4\gamma / \pi}{|K_1| + K_2} & \text{for } 0 < R < \frac{\gamma}{|K_1|} \\
\frac{2R\gamma + \gamma\sqrt{1 - (\frac{\gamma}{|K_1|})^2} + |K_1|R \arcsin \left(\frac{\gamma}{|K_1|}\right)}{\pi K_2 R + \gamma\sqrt{1 - (\frac{\gamma}{|K_1|})^2} + |K_1|R \arcsin \left(\frac{\gamma}{|K_1|}\right)} & \text{for } \frac{\gamma}{|K_1|} < R < \frac{\gamma}{K_2} \\
\frac{\gamma\sqrt{1 - (\frac{\gamma}{|K_1|})^2} + |K_1|R \arcsin \left(\frac{\gamma}{|K_1|}\right) + 2\gamma R}{\gamma\sqrt{1 - (\frac{\gamma}{|K_1|})^2} + |K_1|R \arcsin \left(\frac{\gamma}{|K_1|}\right) + \gamma\sqrt{1 - \frac{\gamma}{K_2} |R|^2} + K_2 R \arcsin \left(\frac{\gamma}{K_2 R}\right)} & \text{for } \frac{\gamma}{K_2} < R.
\end{cases} \tag{25}\]
FIG. 5. Phase diagram of synchronization transitions in the 
\((p, \gamma)\) plane for various \(Q\). \(p\) is the fraction of oscillators with 
positive coupling \(K_2\) and \(\gamma\) is the half width of the uniform 
distribution \(g(\omega)\). (a)–(b) When \(Q > 1\), the phase diagrams 
contain IC and \(\pi\) phases, and the hysteresis zone \(H\) of the 
two phases. Both types of dashed lines represent discontinuous 
transitions, but the forward transition from IC to \(\pi\) phase is 
hybrid. At \(Q = 1\), the hysteresis vanishes. The \(p = 1\) line 
corresponds to the phase diagram of the WP model, and the 
symbol \(\bullet\) at \(\gamma_h = \pi K_2/4 \approx 0.78\) denotes the hybrid critical 
synchronization transition point for \(K_2 = 1\) [20] [21]. (c)–(f) 
When \(Q < 1\), some part of \(\pi\) state is replaced by TW phase. 
The hysteresis region reappears between IC and TW. The symbols \(\blacktriangleleft\) and \(\blacksquare\) represent critical points across which different 
types of phases or phase transitions emerge.

bility matrix \(\hat{S}\) is reproduced as follows:

\[
\begin{align*}
\begin{pmatrix}
\delta R \\
\delta \psi
\end{pmatrix} &= A
\begin{pmatrix}
\langle \partial_R F_R \rangle - 1 & \partial_R F_{\Omega} \\
R^{-1} \partial_R F_{\Omega} & R \partial_R F_{\Omega}
\end{pmatrix}
\begin{pmatrix}
\delta R \\
\delta \psi
\end{pmatrix} \\
&= \hat{S} \begin{pmatrix}
\delta R \\
\delta \psi
\end{pmatrix},
\end{align*}
\]

(29)

where \(F_R\) and \(F_{\Omega}\) are the real and imaginary parts of the 
SC complex order parameter. Explicitly they are written as

\[
\begin{align*}
F_R(R, \Omega) &= \int_{\text{locked}} dK d\omega g(\omega, K) \sqrt{1 - (\omega/KR)^2} \\
F_{\Omega}(R, \Omega) &= \int_{\text{drifting}} dK d\omega g(\omega, K) \sqrt{(\omega/KR)^2 - 1}.
\end{align*}
\]

(30)

In the stability matrix \(\hat{S}\), the relative strength between the 
proportional coefficients of \(\delta R\) and \(\delta \Omega\) had been 
determined empirically [18].

The empirical stability of the SC solutions of the c-WP model is represented by the blue solid (stable) and dashed (unstable) curves in Fig. 5. Our numerical result suggests that the linear stability criterion of [18] is partly fulfilled; some portions of the “stable” \(\pi\) curve are not covered by the simulation data points in the long-time limit. Interestingly, however, the order parameter stays for quite a long time at these uncovered parts, before it finally settles down in the stable stationary line occupied by the symbols in Figs. (3c) and (3d). These parts uncovered by simulation data are not stable but metastable. The detailed analysis were presented in Ref. [17].

V. THE SC EQUATION ON ER NETWORKS

Now we generalize the competing KM to structured networks. The Kuramoto equation is written as

\[
\dot{\theta}_i = \omega_i + \frac{K_i}{\langle q \rangle} \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i),
\]

(31)

where the normalization \(\langle q \rangle\) is equivalent to the expected 
number of terms in the summation. \(a_{ij}\) is the adjacent 
matrix. On the ER network, \(a_{ij} = 1\) if two nodes \(i\) and \(j\) are connected, otherwise \(a_{ij} = 0\). In case of all-to-all connections, this equation reduces to Eq. (2). Now we consider an annealed ER network, in which

\[
a_{ij} = \frac{q_i q_j}{\langle q \rangle}.
\]

(32)

\(q_i\) and \(q_j\) are degrees of nodes \(i\) and \(j\), respectively. Each 
degree follows a Poisson distribution,

\[
P_q(q) = \frac{(\langle q \rangle)^q e^{-\langle q \rangle}}{q!},
\]

(33)

where again \(\langle q \rangle\) is the mean degree. With the order pa-
rameter defined as the degree weighted phase coherence

\[
H e^{i\phi} = \sum_j q_j e^{i\phi_j} + \frac{1}{\langle q \rangle} \sum_j q_j e^{i\phi_j},
\]

(34)

the model is written as

\[
\dot{\theta}_i = \omega_i + \frac{K_i q_i H}{\langle q \rangle} \sin(\phi - \theta_i),
\]

(35)

where we take \(\phi = 0\) for simplicity. In the stationary 
limit, the oscillators satisfying \(|\omega_i - \Omega| \leq |K_i q_i H/\langle q \rangle|\) are 
phase locked, where \(\Omega\) denotes the angular velocity of the 
complex order parameter \(H e^{i\phi}\). Each locked oscillator 
contributes a phasor of

\[
e^{i\phi_j} = \sqrt{1 - \left(\frac{\langle q \rangle (\omega_j - \Omega)}{K_j q_j H}\right)^2 + i \frac{\langle q \rangle (\omega_j - \Omega)}{K_j q_j H}},
\]

(36)

while each drifting oscillator rotating with a period

\[
T_j = \int_0^{2\pi} \frac{d\theta_j}{|\dot{\theta}_j|} = \frac{2\pi}{\sqrt{(\omega_j - \Omega)^2 - (K_j q_j H/\langle q \rangle)^2}}
\]

(37)

has the time averaged contribution of
\[ \bar{e}^{i\theta_j} = \frac{1}{T_j} \int_0^{2\pi} d\theta_j e^{i\theta_j} \]
\[ = i \langle q \rangle \text{sgn}(\omega_j - \Omega) \frac{K_j q_j H}{|\omega_j - \Omega| - \sqrt{(\omega_j - \Omega)^2 - (K_j q_j H / \langle q \rangle)^2}}. \tag{38} \]

Thus the self-consistency equation of the competing KM on an ER network is written as
\[
H = \int d\omega dK g(\omega, K) \frac{\sum_q q P_d(q) e^{i\theta_j}}{\sum_q q P_d(q)}
\]
\[ = \int d\omega dK g(\omega, K) \frac{1}{\langle q \rangle} \left[ \sum_{q < |X|} q P_d(q) e^{i\theta_j} + \sum_{q > |X|} q P_d(q) e^{i\theta_j} \right]
\]
\[ = \int d\omega dK g(\omega, K) \frac{1}{\langle q \rangle} \left[ \sum_{q > |X|} P_d(q) \sqrt{q^2 - X^2} \right.
\]
\[ + iX - i \text{sgn}(X) \sum_{q < |X|} P_d(q) \sqrt{X^2 - q^2} \] \tag{39}

where we used a shorthand notation \( X = (q)(\omega - \Omega)/(KH) \).

VI. SYNCHRONIZATION TRANSITIONS OF c-WP MODEL ON ER NETWORKS

SC solutions of the c-WP model on the annealed ER networks were obtained numerically by solving Eq. (39). The phase diagram is obtained in Fig. 6. TW regime extends as the network mean degree \( \langle q \rangle \) is increased. As \( \langle q \rangle \to N \), we confirm that the solution is reduced to the previous solution on the all-to-all network in Fig. 5(c). The stability of these heterogeneous mean field solutions was tested by numerical simulations on the annealed ER network governed by Eq. (35). The black symbols in Fig. 7 represent the stationary states of the simulation, reached after waiting a sufficiently long time. The solid (dotted) blue curves in Fig. 7 denote stable (unstable) solutions of the SC equations under the linear empirical stability criterion, which was calculated similarly as before [17, 18]. We observe that TW and \( \pi \) phases also occur in the annealed ER network. Also, the empirical stability condition works in this heterogeneous mean field solution in most of the regime, however as before, metastability do arise in the solid blue \( \pi \) states uncovered by the simulation data points.

Numerical simulations were performed on ER random networks using the KM with the conventional adjacency matrix [31]. In Fig. 7, the simulation data points initially started from coherent/incoherent state are denoted by green (▲)/red (●) symbols, and some similarity and difference are noticed from the mean field results on the annealed ER networks. TW and \( \pi \) phases also arise in the c-KM model on ER networks, but the mean degree \( \langle q \rangle \) needs to be sufficiently large. Moreover, the order of transition may change depending on \( \langle q \rangle \). In the ER network with low mean degree (a)–(d), TW is not observed and the synchronization transition to \( \pi \) requires the fraction of \( K_2 \) oscillators \( p \) larger than \( p_c \approx 0.66 \), and the transition is continuous; however, the \( p_c \) of the heterogeneous mean field result is smaller and the transition can be discontinuous. For intermediate mean degree (e)–(h), we observe the emergence of TW similarly to the mean-field prediction, but the order of transition depends on the frequency half width \( \gamma \) in contrast. We investigate what happens in the system for the case of transition in Fig. 7(e). In the forward transition, a jump is noticed at \( p_c = 0.55 \). In Fig. 8 we find at this point the \( K_2 \) oscillators begin to phase lock into a semi-circle in the phase circle, while prior to which the oscillators are not locked. This amounts to the mechanism of hybrid jump transition of the Pazó model [21]. In Fig. 7(c) we unexpectedly find a hysteresis behavior for \( p \) in range [0.54, 0.55], which is not seen from the mean field calculations and simulations. We remark in Fig. 7(g) the TW regime exists, although the angular velocity is very small (Fig. 9). For higher mean degree Fig. 7(i)–(l), the hybrid jump transition at \( p_c \) is gradually restored.

Furthermore, metastability is noticed at the transition points of Figs. 7(i) and 7(j), similarly to our previous all-to-all model [17] and the heterogeneous mean field model. A metastable jump trajectory and linearized
flows around the SC solutions are shown in Fig. 10. Starting from an incoherent state, the system quickly jumps to the (meta)stable $\pi$ (red), and spends some time at the metastable basin characterized by the $\pi$ and the TW saddle (green), and finally reaches the stable TW (red). In contrary to our previous paper, the saddle TW (green) is placed close to the metastable $\pi$ and has a small traveling speed $|\Omega|$. However, the qualitative understandings
FIG. 8. The phase distributions of $K_1$ and $K_2$ subpopulations (a) before and (b) after the incoherent to coherent synchronization transition of Fig. 7(e), when $\langle q \rangle = 40, \gamma = 0.04$. The red (+) and the black (×) symbols represent the number of $K_1$ and $K_2$ oscillators at each bin, respectively. The $K_2$ oscillators range $2\pi$ in the incoherent phase ($p = 0.54$) and $\pi$ immediately after the transition point ($p_c = 0.55$). This amounts to the sudden jump in the order parameter and the hybrid jump transition of the Pazó model [21]. Beyond the transition point, the width of range reduces alongside the increase in the coherence $H$. The oscillator phase values at the end of each simulation $T = 10^4$ were histogrammed in the domain [0,2$\pi$] with bin size $b = 0.1$.

on the metastability remains the same.

VII. SUMMARY AND DISCUSSION

We have investigated synchronization transitions in the $c$-KM with uniform intrinsic frequency distribution on completely connected networks and on ER networks. For the former case, we presented the SC solutions and the critical exponent of the IC to $\pi$ hybrid synchronization transition explicitly. Phase diagrams including the IC, $\pi$, and TW phases were obtained analytically for different ratios $Q \equiv |K_1|/K_2$ of coupling constants as a function of $p$ and $\gamma$, the fraction of nodes with positive coupling constant $K_2$ and the half width of the uniform distribution, respectively. Diverse synchronization transition types and critical behaviors were determined analytically or numerically. Linear stability of each SC solution was tested by numerical simulations.

Next, the synchronization transition was investigated on the random network. Some of the synchronization features depend on the level of network mean degree. When mean degree is large, overall behaviors of synchronization transition are similar to those on the completely connected networks. However, when mean degree is small, the TW phase was not observed in numerical simulation, even though the heterogeneous mean field solution on the annealed ER networks predicts the existence of TW state under appropriate other control parameters. Transition features appear differently. We hope that our results are helpful for understanding the synchronization pattern and improving the control strategy of synchronization in brain [23-26].

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FIG. 10. A metastable π state on the way to TW from IC. (a) $H(t)$ shows a long plateau before jumping to the stable TW state of the c-KM model on an ER network with $\langle q \rangle = 160, \gamma = 0.064, Q = 0.5$ and $p = 0.48$. (b)–(d) The linearized flow near each SC solution of the c-KM model on corresponding annealed network with the same mean degree is shown in the $(H, \Omega)$ plane. Two stable points of π and TW are represented by red circles in (b) and (d), and a saddle point of the TW state is shown in green in (c). (e) An actual flow obtained from a simulation. We remark that the stable and saddle points in (c) and (d) are close but slightly separated in position. Also, we remark that the coherence $H$ and TW order parameter $|\Omega|$ of the metastable π and stable TW states of the ER network denoted by blue curves in (a) and (d) are slightly different from the values obtained from the corresponding annealed network.

[1] Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki, Lecture Notes in Physics Vol. 30 (Springer, New York, 1975).
[2] S. H. Strogatz, *Sync: The Emerging Science of Spontaneous Order* (Hyperion, New York, 2003).
[3] G. V. Osipov, J. Kurths, and C. Zhou, *Synchronization in Oscillatory Networks* (Springer, Berlin, 2007).
[4] S. Boccaletti, *The Synchronized Dynamics of Complex Systems*, (Elsevier, Oxford, UK, 2008).
[5] H. Daido, Phys. Rev. Lett. 68, 1073 (1992).
[6] B. Ottino-Löffler and S. H. Strogatz, Phys. Rev. Lett. 120, 264102 (2018).
[7] H. Hong and S. H. Strogatz, Phys. Rev. E. 84, 046202 (2011).
[8] H. Hong and S. H. Strogatz, Phys. Rev. Lett. 106, 054102 (2011).
[9] H. Hong and S. H. Strogatz, Phys. Rev. E. 85, 056210 (2012).
[10] E. Ott and T. M. Antonsen, Chaos 18, 037113 (2008).
[11] E. Ott and T. M. Antonsen, Chaos 19, 023117 (2009).
[12] E. A. Martens, E. Barreto, S. H. Strogatz, E. Ott, P. So, and T. M. Antonsen, Phys. Rev. E 79, 026204 (2009).
[13] I. M. Kloumann, I. M. Lizarraga, and S. H. Strogatz, Phys. Rev. E. 89, 012904 (2014).
[14] S. Watanabe and S. H. Strogatz, Physica D 74, 197 (1994).
[15] S. A. Marvel, R. E. Mirollo, and S. H. Strogatz, Chaos 19, 043104 (2009).
[16] A. Pikovsky and M. Rosenblum, Phys. Rev. Lett. 101, 264103 (2008).
[17] J. Park and B. Kahng, Phys. Rev. E 97, 020203 (2018).
[18] D. Iatsenko, S. Petkoski, P. V. E. McClintock, and A. Stefanovska, Phys. Rev. Lett. 110, 064101 (2013).
[19] S. Petkoski, D. Iatsenko, L. Basnarkov, and A. Stefanovska, Phys. Rev. E 87, 032908 (2013).
[20] A. T. Winfree, *The Geometry of Biological Time* (Springer, Berlin, 1980).
[21] D. Pazó, Phys. Rev. E 72, 046211 (2005).
[22] J. Gómez-Gardeñes, Y. Moreno, and A. Arenas, Phys. Rev. Lett. 98, 034101 (2007).
[23] M. Kim, G. A. Mashour, S.B. Moraes, G. Vanini, V. Tarnai, E. Janke, and U. Lee, Front Comput Neurosci 10, 1 (2016).
[24] J. Y. Moon, J. Kim, T. W. Koh, M. Kim, Y. Iturria-Medina, J. H. Choi, J. Lee, G. A. Mashour, and U. Lee,
Sci. Rep. **7**, 46606 (2017).

[25] U. Lee, M. Kim, K. Lee, C. M. Kaplan, D. J. Clauw, S. Kim, G. A. Mashour, and R. E. Harris, *Sci. Rep.* **8**, 243. (2018)

[26] H. Kim, J-Y Moon, G. A. Mashour, and U. Lee, *PLoS Comput. Biol.* **14**, e1006424 (2018).