MOCHIZUKI-TROOSHIN TYPE THEOREM FOR
STURM-LIOUVILLE PROBLEM ON TIME SCALES

I. ADALAR AND A. S. OZKAN

Abstract. In this paper, we consider an interior inverse Sturm-Liouville problem on time scale $T = [0, a_1] \cup [a_2, l]$ and give a Mochizuki-Trooshin type theorem.

1. Introduction

Time scale theory was introduced by Hilger in order to unify continuous and discrete analysis [22]. This approach has applied quickly to various areas in mathematics. Sturm-Liouville theory on time scales was studied first by Erbe and Hilger [15] in 1993. Some important results on the properties of eigenvalues and eigenfunctions of a Sturm-Liouville problem on time scales were given in various publications (see e.g. [2]-[5], [12]-[14], [16], [19], [20], [22], [23], [26]-[34] and the references therein).

Inverse spectral problems consist in recovering the coefficients of an operator from its spectral characteristics. The first results on inverse theory of classical Sturm-Liouville operator were given by Ambarzumyan and Borg [1], [9]. Inverse Sturm-Liouville problems which appear in mathematical physics, mechanics, electronics, geophysics and other branches of natural sciences have been studied for about ninety years (see [6], [11] and [17]).

Specially, the inverse problem for interior spectral data of the differential operator consists in reconstruction of this operator from the given eigenvalues and some information on eigenfunctions at an internal point. This kind of problems for the Sturm-Liouville operator were studied firstly by Mochizuki and Trooshin [46]. Similar results to Mochizuki and Trooshin have been studied in various papers until today [50]-[52].

Although the literature for inverse Sturm–Liouville problems on a continuous interval is vast, there is only a few studies about this subject on time scales [31], [30] and [47]. Such problems is useful in many applied problems, for example in string theory, in dynamics of population, in spatial networks problems etc.

In this paper, we consider an interior inverse Sturm–Liouville problem on a time scale and give a Mochizuki and Trooshin-type theorem. We hope that our results will contribute to the development of inverse spectral theory on time scales and to obtain stronger results in some applied sciences.

For basic concepts of the time scale theory we refer to the textbooks [7] and [8].

In this paper we give an Mochizuki-Trooshin theorem in [46] on a time scale. Throughout this paper we assume that $T = [0, a_1] \cup [a_2, l]$ is a bounded time scale for $a_1 < a_2$. Let us consider the boundary value problem. We consider following boundary value problem $L$ on $T = [0, a_1] \cup [a_2, l]$:

\begin{align*}
\ell y &:= -y^{\Delta\Delta}(t) + q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in T_{k^2} \\
U(y) &:= y^{\Delta}(0) - hy(0) = 0 \\
V(y) &:= y^{\Delta}(l) + H y(l) = 0
\end{align*}

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where $q(t)$ is real valued continuous function on $\mathbb{T}$, $h, H \in \mathbb{R}$ and $\lambda$ is the spectral parameter.

Together with $L$, we consider a boundary value problem $\tilde{L} = L(\tilde{q}(t), h, H)$ of the same form but with different coefficients $\tilde{q}(t)$. We assume that if a certain symbol $s$ denotes an object related to $L$, then $\tilde{s}$ will denote an analogous object related to $\tilde{L}$.

A function $y$, defined on $\mathbb{T}$, is called a solution of equation (1) if $y \in C^2_{\text{rd}}(\mathbb{T})$ and $y$ satisfies (1) for all $t \in \mathbb{T}$. The values of the $\lambda$ parameter, for which (1)-(3) has nonzero solutions are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions [2]. It is proven in [2] that the problem (1)-(3) has countable many eigenvalues which are real, simple and bounded below, and can be ordered as $-\infty < \lambda_1 < \lambda_2 < ... < \lambda_n < ....$

2. Main Result

Prior to calculations, we need some preliminaries.

Let $\varphi(t, \lambda)$ be the solution of (1) under the initial conditions

\[
\varphi(0, \lambda) = 1, \quad \varphi''(0, \lambda) = h
\]

The zeros of the function $\Delta(\lambda) = \varphi''(l, \lambda) + H\varphi(l, \lambda)$ coincide with the eigenvalues of the problem (1)-(3). It is proven in [29] that the functions $\varphi(t, \lambda), \varphi''(t, \lambda)$ and so $\Delta(\lambda)$ are entire on $\lambda$.

It is clear that $\varphi(t, \lambda)$ satisfies the following integral equation on $(0, a_1)$

\[
\varphi(t, \lambda) = \cos \sqrt{\lambda}t + \frac{h}{\sqrt{\lambda}} \sin \sqrt{\lambda}t + \frac{1}{\sqrt{\lambda}} \int_0^t \sin \sqrt{\lambda}(t - \xi)q(\xi)\varphi(\xi)d\xi.
\]

On the other hands, $\varphi''(t, \lambda)$ is continuous at $a_1$, and so the relation

\[
\alpha \varphi'(a_1 - 0) = \varphi(a_2) - \varphi(a_1)
\]

holds, where $\alpha := a_2 - a_1$. From (5) and (6), we have the next lemma [30].

**Lemma 1.** The following asymptotic formula holds for $|\lambda| \to \infty$;

\[
\varphi(t, \lambda) = \begin{cases} 
\cos \sqrt{\lambda}t + \frac{h}{\sqrt{\lambda}} \sin \sqrt{\lambda}t + O \left( \frac{1}{\sqrt{\lambda}} \exp |\tau| t \right), & t \in [0, a_1] \\
a^2\lambda \sin \sqrt{\lambda}a_1 \sin \sqrt{\lambda}(t - a_2) + O \left( \sqrt{\lambda} \exp |\tau| (t - a_2 + a_1) \right), & t \in [a_2, l]
\end{cases}
\]

where $\tau := \text{Im} \sqrt{\lambda}$.

The zeros of the function

\[
\Delta(\lambda) = \varphi''(l, \lambda) + H\varphi(l, \lambda) = \varphi'(l, \lambda) + H\varphi(l, \lambda)
\]

coincide with the eigenvalues of the problem (1)-(3). From Lemma 1 we have the following asymptotic relation.

\[
\Delta(\lambda) = a^2\lambda^{3/2} \sin \sqrt{\lambda}a_1 \cos \sqrt{\lambda}(l - a_2) + O \left( \lambda \exp |\tau| (l - a_2 + a_1) \right)
\]

If we assume $l - a_2 = a_1$, then we can prove by using well-known Rouche’s theorem that $\{\lambda_n\}_{n \geq 1}$ satisfies the following asymptotic formula for $n \to \infty$:

\[
\sqrt{\lambda_n} = \frac{(n - 1)\pi}{2a_1} + O \left( \frac{1}{n} \right)
\]

**Lemma 2.** The system of functions $\{\cos 2\sqrt{\lambda_n}t\}_{n=1}^{\infty}$ is complete in $L_2(0, a_1)$.
Then the asymptotic equality \( \sqrt{n} = \frac{(n-1)\pi}{a_1} + O\left(\frac{1}{n}\right) \) holds for sufficiently large \( n \). Taking care of (11) we get

\[
\sum_{n=1}^{\infty} \left| 2\sqrt{\lambda_n} - \sqrt{\mu_n} \right|^2 < \infty.
\]

It is known \([35, 36]\) that \( \left\{ \cos \sqrt{\mu_n} t \right\}_{n=1}^{\infty} \) is complete in \( L_2(0, a_1) \). By using Lemma 3.1. in \([36]\), the proof is completed. \( \square \)

We state the main result of this article.

Let \( \Lambda := \{\lambda_n\}_{n \geq 1} \) and \( \tilde{\Lambda} := \{\tilde{\lambda}_n\}_{n \geq 1} \) be the eigenvalues sets of \( L \) and \( \tilde{L} \), \( \varphi(t, \lambda_n) \) and \( \tilde{\varphi}(t, \lambda_n) \) are eigenfunctions related to this eigenvalues, respectively.

**Theorem 1.** If \( \Lambda = \tilde{\Lambda} \), \( a_1 + a_2 = l \) and

\[
\frac{\varphi^\Delta(a_1, \lambda_n)}{\varphi(a_1, \lambda_n)} = \frac{\tilde{\varphi}^\Delta(a_1, \lambda_n)}{\tilde{\varphi}(a_1, \lambda_n)}
\]

then \( q(t) = \tilde{q}(t) \) on \( T \).

**Remark 1.** Let \( x = \begin{cases} t, & t \in [a_1, l] \\ t - a_2 + a_1, & t \in [a_2, l] \end{cases} \). From (5) and (6), the problem (1)-(3) can be replaced by the following boundary value problem

\[
\begin{align*}
-y'' + q(t)y &= \lambda y, \quad t \in (0, a_1) \\
y'(0) - hy(0) &= y'(a_1) + Hy(a_1) = 0, \\
y(a_1 + 0) - y(a_1 - 0) &= a y'(a_1 - 0), \\
y'(a_1 + 0) - y'(a_1 - 0) &= \frac{1}{a} \left[ a^2 \lambda + b + 1 \right] y(a_1 + 0),
\end{align*}
\]

where \( a := a_2 - a_1 > 0 \), \( b := -a^2 q(a_1) - 1 \). Thus the eigenvalue problem considered in Theorem 1 can be transformed into the Sturm-Liouville problem with an interior discontinuity. The interior inverse problems for Sturm-Liouville operators with various discontinuity conditions are discussed in \([37-45]\).

### 3. Proof

Now we are ready to prove our main result.

**Proof of Theorem 1.** Consider the following equalities

\[
\begin{align*}
-\varphi^\Delta(t, \lambda) + q(t)\varphi^\sigma(t, \lambda) &= \lambda \varphi^\sigma(t, \lambda) \\
-\tilde{\varphi}^\Delta(t, \lambda) + \tilde{q}(t)\tilde{\varphi}^\sigma(t, \lambda) &= \lambda \tilde{\varphi}^\sigma(t, \lambda).
\end{align*}
\]

From (11) and (12) one can obtain

\[
\begin{align*}
\left[ \varphi(t, \lambda)\tilde{\varphi}^\Delta(t, \lambda) - \varphi^\Delta(t, \lambda)\tilde{\varphi}(t, \lambda) \right]^{a_1} = \int_{0}^{a_1} \left[ q(t) - \tilde{q}(t) \right] \varphi^\sigma(t, \lambda)\tilde{\varphi}^\sigma(t, \lambda) \Delta t
\end{align*}
\]

By \( \Delta \)-integrating both sides of (13) on \( [0, a_1] \), we get

\[
\begin{align*}
\left[ \varphi(t, \lambda)\tilde{\varphi}^\Delta(t, \lambda) - \varphi^\Delta(t, \lambda)\tilde{\varphi}(t, \lambda) \right]^{a_1} = \int_{0}^{a_1} \left[ q(t) - \tilde{q}(t) \right] \varphi^\sigma(t, \lambda)\tilde{\varphi}^\sigma(t, \lambda) \Delta t
\end{align*}
\]
\( \varphi \) satisfies the equation

\[
\Delta(t, \lambda) - \varphi(t, \lambda) \varphi(t, \lambda) = 0.
\]

From the assumption of theorem and initial conditions (4) we have \( H(\lambda_n) = 0 \) for all \( \lambda_n \in \Lambda \) and so \( \chi(\lambda) := \frac{H(\lambda)}{\Delta(\lambda)} \) is entire on \( \lambda \). Additionally, it follows from (8) and Lemma 1 that

\[
|\chi(\lambda)| \leq C |\lambda|^{-1/2}.
\]

Thus \( \chi(\lambda) = 0 \) for all \( \lambda \). Hence, \( H(\lambda) \equiv 0 \).

From (13), the equality \( \int_0^{a_1} [q(t) - \tilde{q}(t)] \varphi(t, \lambda) \varphi(t, \lambda) dt = 0 \) is valid on the whole \( \lambda \)-plane. On the other hand, the following representation holds on \([0, a_1] \)

\[
\varphi(t, \lambda) \varphi(t, \lambda) = \frac{1}{2} \left[ 1 + \cos 2\sqrt{\lambda t} + \int_0^t V(t, \tau) \cos 2\sqrt{\lambda} \tau d\tau \right],
\]

where \( V(t, x) \) is a continuous function which does not depend on \( \lambda \). This is used to get

\[
\frac{1}{2} \int_0^{a_1} [q(t) - \tilde{q}(t)] \left( 1 + \cos 2\sqrt{\lambda t} \right) dt + \int_0^{a_1} [q(t) - \tilde{q}(t)] \int_0^t V(t, \tau) \cos 2\sqrt{\lambda} \tau d\tau d\tau = 0.
\]

Taking into account (15) we conclude that

\[
\int_0^{a_1} \cos 2\sqrt{\lambda t} [q(t) - \tilde{q}(t)] + \int_0^{a_1} [q(t) - \tilde{q}(t)] V(t, \tau) d\tau d\tau = 0
\]

Therefore, it follows from the completeness of the functions \( \cos 2\sqrt{\lambda t} \) in Lemma 2 that

\[
q(t) - \tilde{q}(t) + \int_0^{a_1} [q(\tau) - \tilde{q}(\tau)] V(t, \tau) d\tau = 0, \quad t \in [0, a_1]
\]

Since the equation (17) is a homogenous Volterra integral equation, it has only trivial solution. Thus \( q(t) = \tilde{q}(t) \) on \([0, a_1] \). To prove that \( q(t) = \tilde{q}(t) \) on \([a_2, l] \), we will consider the supplementary problem \( L_1 \):

\[
\begin{align*}
-y^{\Delta} + q_1(t) y^\rho &= \lambda y^\rho, \quad t \in T, \\
y^{\Delta}(0) - H y(0) &= y^{\Delta}(l) + h y(l) = 0,
\end{align*}
\]

where \( q_1(t) = q(l - t) \). By using chain rule in (10), we have \( \varphi(t, \lambda) = \varphi(l - t, \lambda) \) satisfies the equation \( -\varphi^{\Delta} + q_1(t) \varphi^\rho = \lambda \varphi^\rho \) and the initial conditions \( \varphi(1, \lambda) = 1, \varphi^{\Delta}(t, \lambda) = -h \). Furthermore, the assumption \( \varphi_{\lambda n}^{\Delta}(a_2, \lambda_n) = \varphi_{\lambda n}^\rho(a_2, \lambda_n) \) holds.

If we repeat the above arguments then we replace equation (13) by

\[
\left[ \varphi(t, \lambda) \varphi^\rho(t, \lambda) - \varphi^\rho(t, \lambda) \varphi(t, \lambda) \right]^{\Delta} = [q_1(t) - \tilde{q}_1(t)] \varphi^\rho(t, \lambda) \varphi(t, \lambda).
\]
By integrating (in the sense of $\nabla$-integral) both sides of this equality on $[0, a_2]$, we obtain
\[
\left[ \varphi_1(t, \lambda) \varphi_1^\nabla(t, \lambda) - \varphi_1^\nabla(t, \lambda) \varphi_1(t, \lambda) \right]_{0}^{a_2} = \int_{0}^{a_2} [q_1(t) - \tilde{q}_1(t)] \varphi_1(t, \lambda) \varphi_1^\nabla(t, \lambda) \nabla t
\]
\[
= \int_{0}^{a_1} [q_1(t) - \tilde{q}_1(t)] \varphi_1(t, \lambda) \varphi_1(t, \lambda) dt
\]
\[
+ \int_{a_1}^{a_2} [q_1(t) - \tilde{q}_1(t)] \varphi_2(t, \lambda) \varphi_2^\nabla(t, \lambda) \nabla t
\]
Since $q(a_1) = \tilde{q}(a_1)$, then $q_1(a_2) = \tilde{q}_1(a_2)$ and
\[
\int_{a_1}^{a_2} [q_1(t) - \tilde{q}_1(t)] \varphi_2(t, \lambda) \varphi_2^\nabla(t, \lambda) \nabla t = [q_1(a_2) - \tilde{q}_1(a_2)] \varphi_2(a_2, \lambda) \varphi_2^\nabla(a_2, \lambda)(a_2 - a_1) = 0.
\]
Therefore
\[
\varphi_1^\nabla(a_2, \lambda) \varphi_1(a_2, \lambda) - \varphi_1(a_2, \lambda) \varphi_1^\nabla(a_2, \lambda) = \int_{0}^{a_2} [q_1(t) - \tilde{q}_1(t)] \varphi_1(t, \lambda) \varphi_1(t, \lambda) dt.
\]
Let
\[
K(\lambda) := \int_{0}^{a_1} [q_1(t) - \tilde{q}_1(t)] \varphi_1(t, \lambda) \varphi_1(t, \lambda) dt.
\]
It is obvious that $K(\lambda_n) = 0$ for all $\lambda_n \in \Lambda$ and so $\omega(\lambda) := \frac{K(\lambda)}{\Delta(\lambda)}$ is entire on $\Lambda$. On the other hand, from asymptotics $\varphi_1^\nabla(a_2, \lambda)$ and $\varphi_1(a_2, \lambda)$, $K(\lambda) = O(\lambda \exp 2 |\tau| a_1)$ and $|\Delta(\lambda)| \geq C_\gamma |\lambda|^{3/2} \exp 2 |\tau| a_1$ for sufficiently large $|\lambda|$. Thus
\[
|\omega(\lambda)| \leq C |\lambda|^{-1/2}.
\]
From Liouville’s Theorem $\omega(\lambda) = 0$ for all $\lambda$. Hence, $K(\lambda) \equiv 0$.

By integrating again both sides of the equality (18) on $[0, a_1)$, we get
\[
\varphi_1'(a_1, \lambda) \varphi_1(a_1, \lambda) = \varphi_1(a_1, \lambda) \varphi_1'(a_1, \lambda)
\]
Put $\psi(t, \lambda) := \varphi_1(a_1 - t, \lambda)$. It is clear that $\psi(t, \lambda)$ is the solution of the following initial value problem
\[
-y'' + q_1(a_1 - t) y = \lambda y, \ t \in (0, a_1)
\]
\[
y(a_1) = 1, \ y'(a_1) = -H
\]
It follows from (22) that
\[
\psi'(0, \lambda) \psi(0, \lambda) = \psi(0, \lambda) \psi'(0, \lambda).
\]
Taking into account Theorem 1.4.7. in [17] it is concluded that $q_1(t) = \tilde{q}_1(t)$ on $[0, a_1]$, that is $q(t) = \tilde{q}(t)$ on $[0, a_2]$. This completes the proof. \qed

References

[1] Ambarzumyan, V.A.: Über eine Frage der Eigenwerttheorie. Z. Phys. 53, 690–695 (1929)
[2] Agarwal, R.P., Bohner, M., Wong, P.J.Y.: Sturm-Liouville eigenvalue problems on time scales. Appl. Math. Comput. 99, 153–166 (1999)
[3] Allahverdiev, B.P., Eryilmaz, A., Tuna, H.: Dissipative Sturm-Liouville operators with a spectral parameter in the boundary condition on bounded time scales. Electronic Journal of Differential Equations, 95, 1–13 (2017)

[4] Amster, P., De Na`poli, P., Pinaesco, J.P.: Eigenvalue distribution of second-order dynamic equations on time scales considered as fractals. J. Math. Anal. Appl. 343, 573–584 (2008)

[5] Amster, P., De Na`poli, P., Pinaesco, J.P.: Detailed asymptotic of eigenvalues on time scales, J. Differ. Equ. Appl. 15, 225–231 (2009)

[6] Atkinson, F.: Discrete and Continuous Boundary Problems. Academic Press, New York (1964)

[7] Bohner, M., Peterson, A.: Dynamic Equations on Time Scales. Birkhäuser, Boston, MA (2001)

[8] Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston, MA (2003)

[9] Borg, G., Eine Umkehrung der Sturm–Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte, Acta Math. 78, 1–96 (1946)

[10] Bohner, M., Koyunbakan, H.: Inverse problems for Sturm–Liouville difference equations. Filomat, 30(5), 1297-1304 (2016)

[11] Chadan, K., Colton, D., Paivarinta, L., Rundell, W.: An introduction to inverse scattering and inverse spectral problems. Society for Industrial and Applied Mathematics. (1997)

[12] Davidson, F.A., Rynne, B.P.: Global bifurcation on time scales. J. Math. Anal. Appl. 367, 345–360 (2002)

[13] Davidson, F.A., Rynne, B.P.: Self-adjoint boundary value problems on time scales. Electron. J. Differ. Equ. 175, 1–10 (2007)

[14] Davidson, F.A., Rynne, B.P.: Eigenfunction expansions in L2 spaces for boundary value problems on time-scales. J. Math. Anal. Appl. 335, 1038–1051 (2007)

[15] Erbe, L., Hilger, S.: Sturmian theory on measure chains. Differ. Equ. Dyn. Syst. 1, 223–244 (1993)

[16] Erbe, L., Peterson, A.: Eigenvalue conditions and positive solutions. J. Differ. Equ. Appl. 6, 165–191 (2000)

[17] Freiling, G., Yurko V.A.: Inverse Sturm–Liouville Problems and their Applications, Nova Science, New York (2001)

[18] Gesztesy F., Simon B., Inverse spectral analysis with partial information on the potential. II. The case of discrete spectrum. Trans. Am. Math. Soc. 352(6), 2765-2787 (2000)

[19] Guseinov, G.S.: Eigenfunction expansions for a Sturm-Liouville problem on time scales. Int. J. Differ. Equ. 2, 93–104 (2007)

[20] Guseinov, G.S.: An expansion theorem for a Sturm-Liouville operator on semi-unbounded time scales. Adv. Dyn. Syst. Appl. 3, 147–160 (2008)

[21] Hald, O.H.: Discontinuous inverse eigenvalue problems, Comm. Pure Appl. Math. 37, 539-577 (1984)

[22] Hilger, S.: Analysis on measure chains – a unified approach to continuous and discrete calculus. Results in Math. 18, 18–56. (1990)

[23] Hilscher, R.S., Zemanek, P.: Weyl-Titchmarsh theory for time scale symplectic systems on half line. Abstr. Appl. Anal. Art. ID 738520, 41 pp. (2011)

[24] Hochstadt, H., Lieberman, B.: An inverse Sturm–Liouville problem with mixed given data. SIAM Journal on Applied Mathematics. 34(4), 676-680 (1978)

[25] Horvath, M.: Inverse spectral problems and closed exponential systems, Ann. of Math. 162 885-918 (2005)

[26] Huseynov, A.: Limit point and limit circle cases for dynamic equations on time scales. Hacet. J. Math. Stat. 39, 379–392 (2010)

[27] Huseynov, A., Bairamov, E.: On expansions in eigenfunctions for second order dynamic equations on time scales. Nonlinear Dyn. Syst. Theory 9, 7–88 (2009)

[28] Kong, Q.: Sturm-Liouville problems on time scales with separated boundary conditions. Results Math. 52, 111–121 (2008)

[29] Ozkan, A.S.: Sturm-Liouville operator with parameter-dependent boundary conditions on time scales. Electron. J. Differential Equations. 212, 1-10 (2017)

[30] Ozkan, A. S., Adalar, I. Half-inverse Sturm-Liouville problem on a time scale. Inverse Problems (2019). https://doi.org/10.1088/1361-6420/aab2a1

[31] Ozkan, A. S.: Ambarzumyan-type theorems on a time scale. Journal of Inverse and Ill-posed Problems. 26(5), 633–637 (2018)

[32] Rynne, B.P.: L2 spaces and boundary value problems on time-scales. J. Math. Anal. Appl. 328, 1217–1236 (2007)

[33] Sakhnovich L.: Half inverse problems on the finite interval, Inverse Problems, 17, 527–532 (2001)
[34] Sun, S., Bohner, M., Chen, S.: Weyl-Titchmarsh theory for Hamiltonian dynamic systems. Abstr. Appl. Anal. Art. ID 514760, 18 pp. (2010)
[35] He, X., & Volkmer, H. (2001). Riesz bases of solutions of Sturm-Liouville equations. Journal of Fourier Analysis and Applications, 7(3), 297-307.
[36] Harutyunyan, T., Pailevanyan, A., Srapıonyan, A. (2013). Riesz bases generated by the spectra of Sturm-Liouville problems. Electronic Journal of Differential Equations, 2013(71), 1-8.
[37] Hald, O. H., Discontinuous inverse eigenvalue problems. Comm. Pure Appl. Math., 37, 539–577 (1984)
[38] Ozkan, A. S.: Inverse Sturm–Liouville problems with eigenvalue dependent boundary and discontinuity conditions. Inverse Probl. Sci. Eng., 20(6), 857–868 (2012)
[39] Shieh, C. T., Yurko, V. A.: Inverse nodal and inverse spectral problems for discontinuous boundary value problems. J. Math. Anal. Appl., 347(1), 266–272 (2008)
[40] Yurko, V. A.: Integral transforms connected with discontinuous boundary value problems. Integral Transforms Spec. Funct., 10(2), 141–164 (2000)
[41] Yurko, V. A.: On boundary value problems with jump conditions inside the interval. Diff. Uravn., 36(8), 1139–1140 (2000); English Transl. in Diff. Equations, 8, 1266–1269 (2000)
[42] Amirov RKh. On Sturm-Liouville operators with discontinuity conditions inside an interval. J. Math. Anal. Appl. 2006:317:163–176.
[43] A.McNabb, R.Anderesen, E.Lapwood, Asymptotic behaviour of the eigenvalues of a Sturm–Liouville system with discontinuous coefficients, J.Math.Anal.Appl.54 (1976)741–751.
[44] C.F. Yang, An interior inverse problem for discontinuous boundary-value problems, Integral Equations Operator Theory 65 (2009) 593–604.
[45] Yang, C-F. Uniqueness theorems for differential pencils with eigenparameter boundary conditions and transmission conditions. J. Differ. Equ. 255, 2615-2635 (2013)
[46] Mochizuki, K., Troshin, I., Inverse problem for interior spectral data of Sturm-Liouville operator, J. Inverse Ill-posed Probl. 9 (2001) 425-433.
[47] Yurko, V.A. Inverse Problems for Sturm-Liouville Differential Operators on Closed Sets. Tamkang Journal of Mathematics 50.3 (2019): 199-206.
[48] Atici, F. M., Guseinov, G. Sh. On Green’s functions and positive solutions for boundary value problems on time scales. Journal of Computational and Applied Mathematics 141.1-2 (2002): 75-99.
[49] Bartosiewicz, Z., Piotrowska, E. Lyapunov functions in stability of nonlinear systems on time scales. Journal of Difference Equations and Applications 17.03 (2011): 309-325.
[50] Adalar, I. On Mochizuki-Troshin Theorem for Sturm-Liouville Operators. Cumhuriyet Science Journal 40.1 (2019): 108-116.
[51] Ozkan, A.S., Amirov, R. Kh., An interior inverse problem for the impulsive Dirac operator, Tamkang Journal of Mathematics, 42 (2011) 259-263.
[52] Mochizuki, K., Troshin, I., Inverse problem for interior spectral data of the Dirac operator on a finite interval, Publ. RIMS, Kyoto Univ. 38 (2002) 387-395.

Current address: Zara Veyesel Dursun Colleges of Applied Sciences, Sivas Cumhuriyet University Zara/Sivas, Turkey
E-mail address: iadar@cumhuriyet.edu.tr

Current address: Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University 58140 Sivas, Turkey