MODELLING OF 1/f NOISE BY SEQUENCES OF STOCHASTIC PULSES OF DIFFERENT DURATION

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Abstract

We present and analyze the simple analytically solvable model of 1/f noise, which can be relevant for the understanding of the origin, main properties and parameter dependencies of the flicker noise. In the model, the currents or signals represented as sequences of the random pulses, which recurrence time intervals between transit times of pulses are uncorrelated with the shape of the pulse, are analyzed. It is shown that for the pulses of fixed area with random duration, distributed uniformly in a wide interval, 1/f behavior of the power spectrum of the signal or current in wide range of frequency may be obtained.

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1 Introduction

The origin and omnipresence of 1/f noise is one of the oldest problems of the contemporary physics. Since the first observation of the flicker noise in the currents of electron tubes by Johnson [1], fluctuations of signals and physical variables exhibiting behavior characterized by a power spectral density diverging at low frequencies like 1/f have been observed in a wide variety of systems [2]. The widespread occurrence suggest that some underlying mechanism might exist. However, a fully satisfactory explanation has not yet been found and the general theory of 1/f noise is still an open question.

A simple procedures of integration or differentiation of the convenient (white noise, Brownian motion or so) fluctuating signals do not yield in the signal exhibiting 1/f noise. There are no simple linear, even stochastic, differential equations generating signals with 1/f noise. Therefore, 1/f noise is often modeled as the superposition of Lorentzian spectra
with a wide range of relaxation times [3]. Summation or integration of the Lorentzians with the appropriate weights may yield $1/f$ noise [4].

In many cases the physical processes can be represented by a sequence of random pulses. Recently, considering signals and currents as consisting of pulses we have shown [5–7] that the intrinsic origin of $1/f$ noise may be a Brownian motion of the interevent time of the signal pulses, similar to the Brownian fluctuations of the signal amplitude, resulting in $1/f^2$ noise.

The model, proposed in [5, 6], can be extended taking into account finite duration of the pulse. The spectrum of the signal, consisting of the pulse sequences which belong to the class of Markov process, was investigated in [8, 9].

In this article we present a different model of pulses. We consider a signal consisting of a sequence of uncorrelated pulses. The shape of the pulses is determined by only one random parameter — pulse duration. We will show that by suitably choosing of the distribution of the pulse duration the $1/f$ noise can be obtained.

2 Signal as sequence of pulses

We will investigate a signal consisting from a sequence of pulses. We assume that:

(i) the pulse sequences are stationary and ergodic;

(ii) interevent times and the shapes of different pulses are independent.

The general form of such signal can be written as

$$I(t) = \sum_k A_k(t - t_k)$$  \hspace{1cm} (1)

where functions $A_k(t)$ determine the shape of individual pulses and time moments $t_k$ determine when a pulse occurs. The power spectrum is given by the equation

$$S(f) = \lim_{T \to \infty} \left\langle \frac{2}{T} \left| \int_{t_i}^{t_f} I(t)e^{-i2\pi ft} dt \right|^2 \right\rangle$$  \hspace{1cm} (2)

where $T = t_f - t_i$. Substituting Eq. (1) into Eq. (2) we have

$$S(\omega) = \lim_{T \to \infty} \left\langle \frac{2}{T} \sum_{k,k'} e^{i\omega(t_k - t_{k'})} \int_{t_i-t_k}^{t_f-t_k} du \int_{t_i-t_{k'}}^{t_f-t_{k'}} du' A_k(u)A_{k'}(u')e^{i\omega(u-u')} \right\rangle.$$  \hspace{1cm} (3)

We assume that functions $A_k(u)$ decrease sufficiently fast when $|u| \to \infty$. Since $T \to \infty$, the bounds of the integration in Eq. (3) can be changed to $\pm \infty$. We also assume that time moments $t_k$ are not correlated with the shape of the pulse $A_k$. Then the power spectrum is

$$S(\omega) = \lim_{T \to \infty} \frac{2}{T} \sum_{k,k'} \left\langle e^{i\omega(t_k - t_{k'})} \right\rangle \left\langle \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} du' A_k(u)A_{k'}(u')e^{i\omega(u-u')} \right\rangle.$$
After introducing the functions
\[ \Psi_{k,k'}(\omega) = \left\langle \int_{-\infty}^{+\infty} du A_k(u) e^{i\omega u} \int_{-\infty}^{+\infty} du' A_{k'}(u') e^{-i\omega u'} \right\rangle \] (4)
and
\[ \chi_{k,k'}(\omega) = \left\langle e^{i\omega(t_k-t_{k'})} \right\rangle \] (5)
the spectrum can be written as
\[ S(\omega) = \lim_{T \to \infty} \frac{2}{T} \sum_{k,k'} \chi_{k,k'}(\omega) \Psi_{k,k'}(\omega). \] (6)

2.1 Stationary process

Equation (6) can be further simplified assuming that the process is stationary. In the stationary case all averages can depend only on \( k - k' \). Then
\[ \Psi_{k,k'}(\omega) \equiv \Psi_{k-k'}(\omega). \] (7)
and
\[ \chi_{k,k'}(\omega) \equiv \chi_{k-k'}(\omega). \] (8)
Equation (6) then reads
\[ S(\omega) = \lim_{T \to \infty} \frac{2}{T} \sum_{k,k'} \chi_{k-k'}(\omega) \Psi_{k-k'}(\omega). \]
Changing the variables into \( k \equiv k' \) and \( q \equiv k - k' \) and changing the order of summation we obtain
\[
S(\omega) = \lim_{T \to \infty} \frac{2}{T} \sum_{q=1}^{k_{\text{max}}-k_{\text{min}}} \chi_q(\omega) \Psi_q(\omega)
+ \lim_{T \to \infty} \frac{2}{T} \sum_{q=k_{\text{min}}}^{k_{\text{max}}-1} \sum_{k=k_{\text{min}}}^{k_{\text{max}}-q} \chi_q(\omega) \Psi_q(\omega)
+ \lim_{T \to \infty} \frac{2}{T} \sum_{k=k_{\text{min}}}^{k_{\text{max}}} \Psi_0(\omega).
\]
Introducing \( N = k_{\text{max}} - k_{\text{min}} \) we have
\[
S(\omega) = 2\Psi_0(\omega) \bar{v} + \lim_{T \to \infty} 4 \sum_{q=1}^{N} \left( \bar{v} - \frac{q}{T} \right) \text{Re} \chi_q(\omega) \Psi_q(\omega)
\] (9)
where
\[
\bar{v} = \lim_{T \to \infty} \left\langle \frac{N+1}{T} \right\rangle \] (10)
is the mean number of pulses per unit time.

When the sum $\sum_{q=1}^{N} q \text{Re} \chi_q(\omega)\Psi_q(\omega)$ converges and $T \to \infty$ then the second term in the sum (9) vanishes and the spectrum is

$$S(\omega) = 2\bar{\nu} \Psi_0(\omega) + 4\bar{\nu} \sum_{q=1}^{\infty} \text{Re} \chi_q(\omega)\Psi_q(\omega)$$

$$= 2\bar{\nu} \sum_{q=-\infty}^{\infty} \chi_q(\omega)\Psi_q(\omega).$$

2.2 Fixed shape pulses

When the shape of the pulses is fixed ($k$-independent) then the function $\Psi_{k,k'}(\omega)$ does not depend on $k$ and $k'$ and, therefore, $\Psi_{k,k'}(\omega) = \Psi_{0,0}(\omega)$. Then equation (6) yields the power spectrum

$$S(\omega) = \Psi_{0,0}(\omega) \lim_{T \to \infty} \frac{2}{T} \sum_{k,k'} \chi_{k,k'}(\omega) \equiv \Psi_{0,0}(\omega) S_\delta(\omega).$$

This is the spectrum of one pulse multiplied by the spectrum of the sequence of $\delta$-shaped pulses $S_\delta(\omega)$. It has been shown [5, 6] that the spectrum of such a sequence can exhibit $1/f$-like behaviour in a broad frequency range if the interevent times $\tau_k = t_k - t_{k-1}$ follow an autoregressive process.

2.3 Uncorrelated pulses

When the pulses are uncorrelated and $k \neq k'$ then

$$\Psi_{k-k'}(\omega) = \left\langle \int_{-\infty}^{+\infty} A_k(u)e^{iu\omega}du \right\rangle \left\langle \int_{-\infty}^{+\infty} A_{k'}(u')e^{-iu'\omega}du' \right\rangle$$

$$= \left| \langle F_k(\omega) \rangle \right|^2$$

where

$$F_k(\omega) = \int_{-\infty}^{+\infty} A_k(u)e^{iu\omega}du.$$  \hfill (14)

is the Fourier transform of the pulse $A_k$. When $k = k'$ then

$$\Psi_0(\omega) = \left\langle \left| F_k(\omega) \right|^2 \right\rangle.$$

From Eq. (11) we obtain the spectrum

$$S(\omega) = 2\bar{\nu} \left\langle \left| F_k(\omega) \right|^2 \right\rangle + 4\bar{\nu} \left| \langle F_k(\omega) \rangle \right|^2 \sum_{q=1}^{\infty} \text{Re} \chi_q(\omega).$$  \hfill (15)

When the interevent times $\tau_k = t_k - t_{k-1}$ are random and uncorrelated then

$$\chi_q(\omega) = \left\langle e^{i\omega(t_{k+q}-t_k)} \right\rangle = \left\langle e^{i\omega\tau_k} \right\rangle \equiv \chi_\tau(\omega)^q.$$  \hfill (16)
From Eq. (15) we obtain

\[ S(\omega) = 2\bar{\nu} \left\langle |F_k(\omega)|^2 \right\rangle + 4\bar{\nu} |\langle F_k(\omega)\rangle|^2 \Re \frac{\chi_T(\omega)}{1 - \chi_T(\omega)}. \]  

(17)

Here

\[ \bar{\nu} = \left[-i \frac{d\chi_T(\omega)}{d\omega}\right]_{\omega=0}^{-1}. \]  

(18)

If the occurrence times of the pulses \( t_k \) are distributed according to Poisson process then the interevent time probability distribution is \( \Psi(\tau) = \frac{1}{\bar{\tau}} e^{-\frac{\tau}{\bar{\tau}}} \). The characteristic function obeys the equality \( \Re \frac{\chi_T(\omega)}{1 - \chi_T(\omega)} = 0 \) and the spectrum is

\[ S(\omega) = 2\bar{\nu} \left\langle |F_k(\omega)|^2 \right\rangle. \]  

(19)

We will investigate this case more deeply.

3 Pulses of variable duration

Let the only random parameter of the pulse is the duration. We take the form of the pulse as

\[ A_k(t) = T_k^\beta A \left( \frac{t}{T_k} \right), \]

(20)

where \( T_k \) is the characteristic duration of the pulse. The value \( \beta = 0 \) corresponds to fixed height pulses; \( \beta = -1 \) corresponds to constant area pulses. Differentiating the fixed area pulses we obtain \( \beta = -2 \). The Fourier transform of the pulse (20) is

\[ F_k(\omega) = \int_{-\infty}^{+\infty} T_k^\beta A \left( \frac{t}{T_k} \right) e^{i\omega t} dt = T_k^{\beta+1} \int_{-\infty}^{+\infty} A(u) e^{i\omega T_k u} du \equiv T_k^{\beta+1} F(\omega T_k). \]

From Eq. (19) the power spectrum is

\[ S(\omega) = 2\bar{\nu} \left\langle T_k^{2\beta+2} |F(\omega T_k)|^2 \right\rangle. \]

(21)

Introducing the probability density \( P(T_k) \) of the pulses durations \( T_k \) we can write

\[ S(\omega) = 2\bar{\nu} \int^{\infty}_0 T_k^{2\beta+2} |F(\omega T_k)|^2 P(T_k) dT_k. \]

(22)

If \( P(T_k) \) is a power-law distribution, then the expressions for the spectrum are similar for all \( \beta \).
3.1 Spectrum at small frequencies $\omega$

For small frequencies we expand the Fourier transform of the pulse into Taylor series. The first coefficients are

$$F(0) = a, \quad \frac{dF(0)}{d\omega} = ia\langle t \rangle, \quad \frac{d^2F(0)}{d\omega^2} = -a\langle t^2 \rangle,$$

(23)

where

$$a = \int_{-\infty}^{+\infty} A(t)dt$$

(24)

is the area of the pulse,

$$\langle t \rangle = \frac{1}{a} \int_{-\infty}^{+\infty} tA(t)dt, \quad \langle t^2 \rangle = \frac{1}{a} \int_{-\infty}^{+\infty} t^2A(t)dt.$$

(25)

Then the spectrum from Eq. (22) is

$$S(\omega) \approx 2\bar{\nu} a^2 \int_{0}^{\infty} T_k^{2\beta+2}(1 - \Delta t^2\omega^2 T_k^2)P(T_k)dT_k,$$

where $\Delta t^2 = \langle t^2 \rangle - \langle t \rangle^2$. We obtain

$$S(\omega) = 2\bar{\nu} a^2 \langle T_k^{2\beta+2} \rangle (1 - \Delta t^2\omega^2 \langle T_k^{2\beta+4} \rangle),$$

(26)

where

$$\langle T_k^\xi \rangle = \int_{0}^{\infty} T_k^\xi P(T_k)dT_k.$$

(27)

3.2 Power-law distribution

We take the power-law distribution of pulse durations

$$P(T_k) = \begin{cases} \frac{a+1}{T_{\max}^{\alpha+1} - T_{\min}^{\alpha+1}} T_k^\alpha, & T_{\min} \leq T_k \leq T_{\max}, \\ 0, & \text{otherwise.} \end{cases}$$

(28)

From Eq. (22) we have the spectrum

$$S(\omega) = 2\bar{\nu} \frac{\alpha+1}{T_{\max}^{\alpha+1} - T_{\min}^{\alpha+1}} \int_{0}^{\infty} T_k^{\alpha+2\beta+2} |F(\omega T_k)|^2 dT_k$$

$$= \frac{2\bar{\nu}(\alpha + 1)}{\omega^{\alpha+2\beta+3}(T_{\max}^{\alpha+1} - T_{\min}^{\alpha+1})} \int_{\omega T_{\min}}^{\omega T_{\max}} u^{\alpha+2\beta+2} |F(u)|^2 du.$$

When $\alpha > -1$ and $\frac{1}{T_{\max}} \ll \omega \ll \frac{1}{T_{\min}}$ then the expression for the spectrum can be approximated as

$$S(\omega) \approx \frac{2\bar{\nu}(\alpha + 1)}{\omega^{\alpha+2\beta+3}(T_{\max}^{\alpha+1} - T_{\min}^{\alpha+1})} \int_{0}^{\infty} u^{\alpha+2\beta+2} |F(u)|^2 du.$$

(29)
If \( \alpha + 2\beta + 2 = 0 \) then in the frequency domain \( \frac{1}{T_{\text{max}}} \ll \omega \ll \frac{1}{T_{\text{min}}} \) the spectrum is

\[
S(\omega) \approx \frac{2\bar{\nu}(\alpha + 1)}{\omega(T_{\alpha+1}^{\alpha+1} - T_{\alpha+1}^{\alpha+1})} \int_{0}^{\infty} |F(u)|^2 du .
\]  

(30)

We obtained \( 1/f \) spectrum. The condition \( \alpha + 2\beta + 2 = 0 \) is satisfied, e.g., for the fixed area pulses \( (\beta = -1) \) and uniform distribution of pulse durations or for fixed height pulses \( (\beta = 0) \) and uniform distribution of inverse durations \( \gamma = T_k^{-1} \), i.e. for \( P(T_k) \propto T_k^{-2} \).

If \( \alpha + 2\beta + 4 = 0 \) then in the frequency domain \( \frac{1}{T_{\text{max}}} \ll \omega \ll \frac{1}{T_{\text{min}}} \) the spectrum is

\[
S(\omega) \approx \frac{2\bar{\nu}(\alpha + 1)\omega}{(T_{\alpha+1}^{\alpha+1} - T_{\alpha+1}^{\alpha+1})} \int_{0}^{\infty} |F(u)|^2 du / u^2 .
\]  

(31)

Such a spectrum can be obtained after differentiation of the signal exhibiting \( 1/f \) spectrum.

4 Example

4.1 Rectangular pulses

As an example we will obtain the spectrum of rectangular constant area pulses. The duration of the pulse is \( T_k \). The Fourier transform of the pulse is

\[
F(\omega T_k) = a \int_{0}^{1} du e^{i\omega T_k u} = a e^{i\omega T_k} - 1 = a e^{i\omega T_k} - 1 = a e^{i\omega T_k} - 1 = \frac{2\sin \left( \frac{\omega T_k}{2} \right)}{\omega T_k} .
\]  

(32)

Then the spectrum according to Eqs. (22), (28) and (32) is

\[
S(\omega) = \frac{4\tilde{\nu}a^2(\alpha + 1)(T_{\alpha+1}^{\alpha+1} - T_{\alpha+1}^{\alpha+1}) + 4\tilde{\nu}a^2(\alpha + 1)}{\omega^2(\alpha - 1)(T_{\alpha+1}^{\alpha+1} - T_{\alpha+1}^{\alpha+1}) + \omega^{\alpha+1}(T_{\alpha+1}^{\alpha+1} - T_{\alpha+1}^{\alpha+1})} \times \text{Re} \left\{ i^{1-\alpha} \left( \Gamma(\alpha - 1, i\omega T_{\text{max}}) - \Gamma(\alpha - 1, i\omega T_{\text{min}}) \right) \right\} ,
\]  

(33)

where \( \Gamma(a, z) \) is the incomplete gamma function, \( \Gamma(a, z) = \int_{z}^{\infty} u^{a-1}e^{-u}du \).

When \(-1 < \alpha < 1\) then the term with \( \Gamma(\alpha - 1, i\omega T_{\text{max}}) \) is small and can be neglected. We also assume that \( T_{\text{min}} \ll T_{\text{max}} \) and when \( \alpha < -1 \) we neglect the term \( (T_{\text{min}}/T_{\text{max}})^{\alpha+1} \). Then we have

\[
S(\omega) \approx -4\tilde{\nu}a^2(\alpha + 1) / \omega^{\alpha+1} T_{\alpha+1}^{\alpha+1} \cos \left( \frac{\pi}{2} (\alpha - 1) \right) \Gamma(\alpha - 1) .
\]  

(34)

For \( \alpha = 0 \) we have the uniform distribution of the pulses duration. Using the result of the limit

\[
\lim_{\alpha \to 0} \cos \left( \frac{\pi}{2} (\alpha - 1) \right) \Gamma(\alpha - 1) = -\frac{\pi}{2} ,
\]  

(35)

we obtain \( 1/f \) spectrum

\[
S(\omega) \approx \frac{2\pi\tilde{\nu}a^2}{\omega T_{\alpha+1}^{\alpha+1}} .
\]  

(36)

The spectrum was also obtained from numerical calculations. Typical signal for rectangular fixed area pulses is shown in Fig. 1 and the power spectrum in Fig. 2.
Figure 1: Typical signal consisting from the fixed area rectangular pulses with uniformly distributed durations. The time intervals between the pulses are distributed according to Poisson process with the average $\bar{\tau} = 5$. The used parameters are $T_{\text{min}} = 0.01$, $T_{\text{max}} = 100$.

Figure 2: The spectrum of the signal consisting from the fixed area pulses with uniformly distributed durations. The used parameters are the same as in Fig. 1. The dashed line corresponds to the spectrum obtained according Eq. [36].
Figure 3: The spectrum of differentiated signal consisting from the fixed area pulses. The used parameters are the same as in Fig. 1. The dashed line corresponds to the spectrum obtained according Eq. (40).

4.2 Differentiated rectangular pulses

After differentiation of the signal the shape of the pulse becomes

$$A \left( \frac{t}{T_k} \right) = a \left( \delta \left( \frac{t}{T_k} \right) - \delta \left( \frac{t}{T_k} - 1 \right) \right).$$

(37)

The Fourier transform of the pulse is

$$F(\omega T_k) = a(1 - e^{i\omega T_k}) = -2iae^{i\omega T_k} \sin \left( \frac{\omega T_k}{2} \right).$$

(38)

Then the spectrum is

$$S(\omega) = \frac{4\beta a^2(\alpha + 1)(T_{\text{max}}^\alpha - T_{\text{min}}^\alpha)}{(\alpha - 1)(T_{\text{max}}^{\alpha+1} - T_{\text{min}}^{\alpha+1}) + \omega^{\alpha-1}(T_{\text{max}}^{\alpha+1} - T_{\text{min}}^{\alpha+1})} \times \text{Re} \left( i^{1-\alpha}(\Gamma(\alpha - 1, i\omega T_{\text{max}}) - \Gamma(\alpha - 1, i\omega T_{\text{min}})) \right).$$

(39)

When $\alpha = 0$ we obtain

$$S(\omega) \approx \frac{2 \pi \beta a^2}{T_{\text{max}}} \omega.$$ 

(40)

The spectrum, obtained numerically, is shown in Fig. 3.
5 Conclusion

We investigate signals consisting of a sequence of uncorrelated pulses with random durations. By suitably choosing distribution of the pulse duration the $1/f$ noise can be obtained. Signal of fixed area pulses yields $1/f$ noise when width of the pulse is uniformly distributed in a wide interval. The spectrum is given by Eq. (30). This conclusion does not depend on particular shape of the pulse. For the fixed amplitude pulses $1/f$ spectrum yields when the inverse duration of the pulses $\gamma = T_k^{-1}$ is distributed uniformly or $P(T_k) \propto T_k^{-2}$.

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