\textbf{δ}_k-\textbf{SMALL SETS IN GRAPHS}

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\textbf{Abstract.} Let \( G \) be a simple \( n \)-vertex graph and \( W \subseteq V(G) \). We say that \( W \) is a \( \delta_k \)-small set if
\[
\sqrt{\frac{\sum_{v \in W} d_k(v)}{|W|}} \leq n - |W|.
\]

Let \( \varphi_k(G) \) denote the smallest natural number \( r \) such that \( V(G) \) decomposes into \( r \) \( \delta_k \)-small sets, and let \( \alpha_k(G) \) denote the maximal number of vertices in a \( \delta_k \)-small set of \( G \). In this paper we obtain bounds for \( \alpha_k(G) \) and \( \varphi_k(G) \). Since \( \varphi_k(G) \leq \omega(G) \leq \chi(G) \) and \( \alpha(G) \leq \alpha_k(G) \), we obtain also bounds for the clique number \( \omega(G) \), the chromatic number \( \chi(G) \) and the independence number \( \alpha(G) \).

1. \textbf{Introduction}

We consider only finite, non-oriented graphs without loops and multiple edges. We shall use the following notations:
\( V(G) \) – the vertex set of \( G \);
\( e(G) \) – the number of edges of \( G \);
\( \omega(G) \) – the clique number of \( G \);
\( \chi(G) \) – the chromatic number of \( G \);
\( d(v) \) – the degree of a vertex \( v \);
\( \Delta(G) \) – the maximal degree of \( G \);
\( \delta(G) \) – the minimal degree of \( G \).
All undefined notation are from [8].

\textbf{Definition 1.} Let \( G \) be an \( n \)-vertex graph and \( W \subseteq V(G) \). We say that \( W \) is a \textit{small set} in the graph \( G \) if
\[
d(v) \leq n - |W|, \text{ for all } v \in W.
\]

With \( \varphi(G) \) we denote the smallest natural number \( r \) such that \( V(G) \) decomposes into \( r \) small sets.

\( \varphi(G) \) is defined for the first time in [6]. Some properties of \( \varphi(G) \) are proved in [6] and [2]. Further \( \varphi(G) \) is more thoroughly investigated in [1]. There an effective algorithm for the calculation of \( \varphi(G) \) is given. First of all let us note the following bounds for \( \varphi(G) \).

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Proposition 1.1 (I).
\[
\frac{n}{n - d_1(G)} \leq \varphi(G) \leq \frac{n}{n - \Delta(G)},
\]
where $d_1(G)$ is the average degree of the graph $G$.

Let $G$ be a graph and $W \subseteq V(G)$. We define
\[
d_k(W) = \sqrt[k]{\frac{\sum_{v \in W} d^k(v)}{|W|}}, \quad d_k(G) = d_k(V(G)).
\]

Definition 2. Let $G$ be an $n$-vertex graph and $W \subseteq V(G)$. We say that $W$ is a $\delta_k$-small set of $G$ if
\[
d_k(W) \leq n - |W|.
\]
With $\varphi^{(k)}(G)$ we denote the minimal number of $\delta_k$-sets of $G$ into which $V(G)$ decomposes.

Remark 1. $\delta_1$-small sets are defined in [1] as $\beta$-small sets and $\varphi^{(1)}(G)$ is denoted by $\varphi^{\beta}(G)$. Also in [1] it is proven

Proposition 1.2 (I).
\[
\varphi^{(1)}(G) \geq \frac{n}{n - d_1(G)}.
\]

Further we shall need the following

Proposition 1.3. Let $G$ be an $n$-vertex graph. Then
(i) Every small set of $G$ is a $\delta_k$-small set of $G$ for all natural $k$.
(ii) Every $\delta_{k-1}$-small set of $G$ is a $\delta_k$-small set of $G$.

Proof. Let $W$ be a small set of $G$. Then $d(v) \leq n - |W|, \forall v \in W$. Therefore $d_k(W) \leq n - |W|$, i.e. $W$ is a $\delta_k$-small set.

The statement in (ii) follows from the inequality $d_{k-1}(W) \leq d_k(W)$ (cf. [1], [2]).

Let us note that if $G$ is an $r$-regular graph then $d_k(W) = r$ for all natural $k$. So, in this case, every $\delta_k$-set of $G$ is a small set of $G$.

In this paper we shall prove that for a given graph $G$ and for sufficiently large natural $k$ every $\delta_k$-small set of $G$ is a small set of $G$ (Theorem 2.1).

Proposition 1.4. Let $G$ be a graph. Then
\[
\varphi^{(1)}(G) \leq \varphi^{(2)}(G) \leq \cdots \leq \varphi^{(k)}(G) \leq \cdots \leq \varphi(G) \leq \omega(G) \leq \chi(G).
\]

Proof. The inequality $\chi(G) \geq \omega(G)$ is obvious. The inequality $\varphi(G) \leq \omega(G)$ is proven in [6] (see also [1]). The inequality $\varphi^{(k)}(G) \leq \varphi(G)$ follows from Proposition 1.3 (i) and the inequality $\varphi^{(k-1)}(G) \leq \varphi^{(k)}(G)$ follows from Proposition 1.3 (ii).

According to Proposition 1.4 every lower bound for $\varphi^{(k)}(G)$ is a lower bound for $\varphi(G)$, $\omega(G)$ and $\chi(G)$. In this paper we shall obtain a lower bound for $\varphi^{(k)}(G)$ (Theorem 3.2) from which we shall derive new lower bounds for $\varphi(G)$, $\omega(G)$ and $\chi(G)$. As a corollary we shall get and some results for $\varphi(G)$, $\omega(G)$ and $\chi(G)$ already from [1] and [2].
Proposition 1.5.
\[ \left\lfloor \frac{n}{n - d_1(G)} \right\rfloor \leq \varphi^{(k)}(G) \leq \left\lfloor \frac{n}{n - \Delta(G)} \right\rfloor. \]

Proof. The right inequality follows from Proposition 1.1 and Proposition 1.4. The left inequality follows from Proposition 1.2 and Proposition 1.4. \(\square\)

2. Strengthening Proposition 1.4

Theorem 2.1. Let \(G\) be a graph. There exists a natural \(k_0 = k_0(G)\) such that for all \(k \geq k_0\) we have

(i) Every \(\delta_k\)-small set of \(G\) is a small set of \(G\).
(ii) \(\varphi^{(1)}(G) \leq \cdots \leq \varphi^{(k_0)}(G) = \varphi^{(k_0 + 1)}(G) = \cdots = \varphi(G)\).

Proof. Fix a subset of \(V(G)\), say \(W\), and let \(\Delta(W) = \max\{d(v) \mid v \in W\}\). Then \(d_k(W) \leq \Delta(W)\) and \(\lim_{k \to \infty} d_k(W) = \Delta(W)\) (see [4]).

Therefore, since \(V(G)\) has only finitely many subsets, there exists \(k_0\) such that for arbitrary \(W \subseteq V(G)\)

\[ \Delta(W) - \frac{1}{2} \leq d_k(W), \quad \text{if } k \geq k_0. \]

Let us suppose now that \(W\) is a \(\delta_k\)-small set of \(G\) and \(k \geq k_0\), i.e.

\[ d_k(W) \leq n - |W|. \]

From (2.1) and (2.2) we have that

\[ \Delta(W) - \frac{1}{2} \leq n - |W|. \]

Since \(\Delta(W)\) and \(n - |W|\) are integers, from the last inequality we derive that \(\Delta(W) \leq n - |W|\). From the definition of \(\Delta(W)\) it follows \(d(v) \leq n - |W|\) for all \(v \in W\), i.e. \(W\) is a small set. Therefore (i) is proven. The statement (ii) obviously follows from (i). \(\square\)

3. Lower bounds for \(d_k(G)\) and \(\varphi^{(k)}(G)\)

Lemma 3.1. Let \(\beta_1, \beta_2, \ldots, \beta_r \in [0, 1]\) and \(\beta_1 + \beta_2 + \cdots + \beta_r = r - 1\). Then for all natural \(k \leq r\) is held the inequality

\[ \sum_{i=1}^{r} (1 - \beta_i) \beta_k^i \leq \left( \frac{r - 1}{r} \right)^k. \]

Proof. The case \(k = r\) is proven in [1]. That’s why we suppose that \(k \leq r - 1\). For all natural \(n\) we define

\[ S_n = \beta_1^n + \beta_2^n + \cdots + \beta_r^n. \]

We can rewrite the inequality (3.1) in following way

\[ S_k - S_{k+1} \leq \left( \frac{r - 1}{r} \right)^k. \]

Since

\[ \frac{r - 1}{r} = \frac{S_1}{r} \leq \sqrt[k]{\frac{S_k}{r}} \leq \sqrt[k+1]{\frac{S_{k+1}}{r}} \quad \text{(cf. [2, 5])}, \]
we have
\[(3.3)\quad S_{k+1} \geq \frac{1}{\sqrt{r}}S_k^{\frac{k+1}{k}}\]
and
\[(3.4)\quad S_k \geq \frac{(r-1)^k}{r^{k-1}}.\]
From (3.3) we see that
\[(3.5)\quad S_k - S_{k+1} \leq S_k - \frac{1}{\sqrt{r}}S_k^{\frac{k+1}{k}}.\]
We consider the function
\[f(x) = x - \frac{1}{\sqrt{r}}x^{\frac{k+1}{k}}, \quad x > 0.\]
According to (3.2) and (3.5) it is sufficient to prove that
\[f(S_k) \leq \left(\frac{r-1}{r}\right)^k.\]
From \(f'(x) = 1 - \frac{k+1}{k}\sqrt{r}x^{\frac{k-1}{k}}\), it follows that \(f'(x)\) has unique positive root
\[x_0 = \frac{rk^k}{(k+1)^k}\]
and \(f(x)\) decreases in \([x_0, \infty)\). According to (3.4), \(S_k \geq \frac{(r-1)^k}{r^{k-1}}\). Since \(k \leq r - 1\), \(\frac{(r-1)^k}{r^{k-1}} \geq x_0\). Therefore
\[f(S_k) \leq f\left(\frac{(r-1)^k}{r^{k-1}}\right) = \left(\frac{r-1}{r}\right)^k.\quad \square\]

**Theorem 3.2.** Let \(G\) be an \(n\)-vertex graph and
\[V(G) = V_1 \cup V_2 \cup \cdots \cup V_r, \quad V_i \cap V_j = \emptyset, \quad i \neq j,\]
where \(V_i\) are \(\delta_k\)-small sets. Then for all natural \(k \leq r\) the following inequalities are satisfied

(i) \(d_k(G) \leq \frac{n(r-1)}{r};\)

(ii) \(r \geq \frac{n - d_k(G)}{n};\)

Proof. Let \(n_i = |V_i|, \quad i = 1, 2, \ldots, r.\) Then
\[\sum_{v \in V(G)} d^k(v) = \sum_{i=1}^r \sum_{v \in V_i} d^k(v) \leq \sum_{i=1}^r n_i(n - n_i)^k.\]
Let \(\beta_i = 1 - \frac{n_i}{n}, \quad i = 1, 2, \ldots, r.\) Then
\[\sum_{v \in V(G)} d^k(v) \leq n^{k+1} \sum_{i=1}^r \beta_i(1 - \beta_i)k, \quad k \geq r.\]
The inequality (i) follows from the last inequality and Lemma 3.1. Solving the inequality (i) for \( r \), we derive the inequality (ii). □

4. SOME COROLLARIES FROM THEOREM 3.2

**Corollary 4.1.** Let \( G \) be an \( n \)-vertex graph and let \( k \) and \( s \) be natural numbers such that \( k \leq \varphi^{(s)}(G) \). Then

\[
\begin{align*}
(i) \quad &d_k(G) \leq \frac{(\varphi^{(s)}(G) - 1)n}{\varphi^{(s)}(G)} \leq \frac{(\varphi(G) - 1)n}{\varphi(G)} \leq \frac{(\omega(G) - 1)n}{\omega(G)} \leq \frac{(\chi(G) - 1)n}{\chi(G)}; \\
(ii) \quad &\varphi^{(s)}(G) \geq \frac{n}{n - d_k(G)}.
\end{align*}
\]

**Proof.** Let \( \varphi^{(s)}(G) = r \) and \( V(G) = V_1 \cup V_2 \cup \cdots \cup V_r \), \( V_i \cap V_j = \emptyset \), where \( V_i \) are \( \delta_k \)-small sets. Then the left inequality in (i) follows from Theorem 3.2 (i). The other inequalities in (i) follow from the inequalities \( \varphi^{(s)}(G) \leq \varphi(G) \leq \omega(G) \leq \chi(G) \). The inequality (ii) follows from Theorem 3.2 (ii). □

**Remark 2.** In the case \( k = s = 1 \), Corollary 4.1 is proven in [1] (cf. Theorem 6.3 (i) and Theorem 6.2 (ii)).

**Corollary 4.2.** Let \( G \) be an \( n \)-vertex graph. Then for all natural \( s \geq 2 \),

\[ \varphi^{(s)}(G) \geq \frac{n}{n - d_2(G)}. \]

**Proof.** If \( \varphi^{(2)}(G) = 1 \) then \( E(G) = \emptyset \), i.e. \( G = K_n \) and the inequality is obvious. If \( \varphi^{(2)}(G) \geq 2 \) then \( \varphi^{(s)}(G) \geq 2 \) because \( s \geq 2 \). Therefore Corollary 4.2 follows from Corollary 4.1 (ii). □

**Corollary 4.3 ([2]).** For every \( n \)-vertex graph

\[ \varphi(G) \geq \frac{n}{n - d_2(G)}. \]

**Proof.** This inequality follows from Corollary 4.2 because \( \varphi^{(s)}(G) \leq \varphi(G) \). □

**Corollary 4.4 ([1]).** Let \( G \) be an \( n \)-vertex graph. Then for every natural \( k \leq \varphi(G) \)

\[ \varphi(G) \geq \frac{n}{n - d_k(G)}. \]

**Proof.** According to Theorem 2.1 there exists a natural number \( s \) such that \( \varphi(G) = \varphi^{(s)}(G) \). Since \( k \leq \varphi^{(s)}(G) \) from Corollary 4.1 (ii) we derive

\[ \varphi(G) = \varphi^{(s)}(G) \geq \frac{n}{n - d_k(G)}. \]

□

**Corollary 4.5.** Let \( G \) be an \( n \)-vertex graph. Then for every natural \( s \geq 3 \)

\[ \varphi^{(s)}(G) \geq \frac{n}{n - d_3(G)}. \]
Proof. Since \( s \geq 3 \), \( \varphi(s)(G) \geq \varphi(3)(G) \). Therefore it is sufficient to prove the inequality

\[
\varphi(3)(G) \geq \frac{n}{n - d_3(G)}.
\]

If \( \varphi(3)(G) \geq 3 \) then \((\ref{eq:ineq1})\) follows from Corollary \((\ref{cor:ineq1})\)(ii). If \( \varphi(3)(G) = 1 \) then the inequality \((\ref{eq:ineq1})\) is obvious because \( d_3(G) = 0 \). Let \( \varphi(3)(G) = 2 \) and \( V(G) = V_1 \cup V_2 \), where \( V_i, i = 1, 2 \) are \( \delta_3 \)-small sets. Let \( n_i = |V_i|, i = 1, 2 \).

Then

\[
\sum_{v \in V(G)} d^3(v) = \sum_{v \in V_1} d^3(v) + \sum_{v \in V_2} d^3(v) \leq n_1(n - n_1)^3 + n_2(n - n_2)^3 = n_1n_2(n^2 - 2n_1n_2) \leq \frac{n^4}{8}.
\]

Therefore \( d_3(G) \leq \frac{n}{2} \) and we obtain

\[
\frac{n}{n - d_3(G)} \leq 2 = \varphi(3)(G).
\]

Since \( \varphi(G) \geq \varphi(3)(G) \) from Corollary \((\ref{cor:ineq1})\) we derive

**Corollary 4.6.** \( (\ref{cor:ineq1}) \). For every \( n \)-vertex graph \( G \)

\[
\varphi(G) \geq \frac{n}{n - d_3(G)}.
\]

**Corollary 4.7.** Let \( G \) be an \( n \)-vertex graph and \( \varphi(4)(G) \neq 2 \). Then for every natural \( s \geq 4 \),

\[
\varphi(s)(G) \geq \frac{n}{n - d_4(G)}.
\]

Proof. Since \( \varphi(s)(G) \geq \varphi(4)(G) \) for \( s \geq 4 \), it sufficient to prove the inequality

\[
\varphi(4)(G) \geq \frac{n}{n - d_4(G)}.
\]

If \( \varphi(4)(G) \geq 4 \) the inequality \((\ref{eq:ineq1})\) follows from Corollary \((\ref{cor:ineq1})\)(ii). If \( \varphi(4)(G) = 1 \) the inequality \((\ref{eq:ineq1})\) is obvious because \( d_4(G) = 0 \). It remains to consider the case \( \varphi(4)(G) = 3 \). Let \( V(G) = V_1 \cup V_2 \cup V_3 \), where \( V_i \), are \( \delta_4 \)-small sets and let \( n_i = |V_i|, i = 1, 2, 3 \). Then

\[
\sum_{v \in V(G)} d^4(v) = \sum_{v \in V_1} d^4(v) + \sum_{v \in V_2} d^4(v) + \sum_{v \in V_3} d^4(v) \leq n_1(n - n_1)^4 + n_2(n - n_2)^4 + n_3(n - n_3)^4.
\]

Denoting \( \beta_i = 1 - \frac{n_i}{n}, i = 1, 2, 3 \) we receive

\[
\sum_{v \in V(G)} d^4(v) \leq n^4 \left( \sum_{i=1}^{3} (1 - \beta_i)^4 \right).
\]
Let $\alpha$ and Proposition 5.1. Let $\alpha$. hence Theorem 5.2. For every graph $\{v_1, v_2, \ldots, v_n\}$ with $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$. Then 

$$\alpha^{(k)}(G) = \max \{s \mid d_k(\{v_1, v_2, \ldots, v_s\}) \leq n - s\} = \max \{s \mid \{v_1, v_2, \ldots, v_s\} \text{ is } \delta_k\text{-small set in } G\}.$$ 

Proof. Let $s_0 = \max \{s \mid \{v_1, v_2, \ldots, v_s\} \text{ is } \delta_k\text{-small set in } G\}$. Then $s_0 \leq \alpha^{(k)}(G)$. Let $\alpha^{(k)}(G) = r$ and let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ be a $\delta_k$-small set. Since $d_k(\{v_1, v_2, \ldots, v_r\}) \leq d_k(\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\})$ it follows that $\{v_1, v_2, \ldots, v_r\}$ is $\delta_k$-small set too. Therefore $\alpha^{(k)}(G) = r \leq s_0$. 

Proposition 5.4. For every natural $k$ are held the inequalities 

$$n - \Delta(G) \leq \alpha^{(k)}(G) \leq n - \delta(G).$$ 

Proof. The left inequality follows from the inequality $S(G) \geq n - \Delta(G)$ from [1] and Proposition 5.1. Let $r = \alpha^{(k)}(G)$. According to Proposition 5.3 $\{v_1, v_2, \ldots, v_r\}$ is a $\delta_k$-small set. So 

$$\delta(G) = d(v_1) \leq d_k(\{v_1, v_2, \ldots, v_r\}) \leq n - r = n - \alpha^{(k)}(G),$$

hence $\alpha^{(k)}(G) \leq n - \delta(G)$. 

Remark 5. The inequality $\alpha(G) \geq n - \Delta(G)$ is not always true. For example, $\alpha(C_5) < 5 - \Delta(C_5) = 3$. 

Since $\sum_{i=1}^{3}(1 - \beta_i)\beta_i^4 \leq \frac{2}{3}$ (see the proof of Theorem 5.4 (iii) in [1]) we take 

$$d_k(G) \leq \frac{2}{3} = \frac{\varphi^{(k)}(G) - 1}{\varphi^{(k)}(G)}.$$

Solving the last equation for $\varphi^{(k)}(G)$ we obtain (4.5). 

Corollary 4.8. Let $G$ be an $n$-vertex graph and $\varphi^{(k)}(G) \neq 2$. Then 

$$(4.5) \quad \varphi(G) \geq \frac{n}{n - d_1(G)}.$$ 

Remark 3. In [1] it is proven that the inequality (4.5) is held if $\varphi(G) \neq 2$. 

5. Maximal $\delta_k$-sets 

We denote the maximal number of vertices in a $\delta_k$-set of $G$ by $\alpha^{(k)}(G)$. $S(G)$ is the maximal number of vertices of small sets of $G$. From Proposition 1.3 is easy to see that the next proposition holds.

Proposition 5.1. For every graph $G$ 

$$\alpha^{(1)}(G) \geq \alpha^{(2)}(G) \geq \cdots \geq \alpha^{(k)}(G) \geq \cdots \geq S(G) \geq \alpha(G).$$ 

Remark 4. Note that $\alpha^{(1)}(G)$ is denoted in [1] by $S^{\alpha}(G)$. 

From Theorem 2.1 we have 

Theorem 5.2. For every graph $G$ there exists an unique number $k_0 = k_0(G)$ such that 

$$\alpha^{(1)}(G) \geq \alpha^{(2)}(G) \geq \cdots \geq \alpha^{(k_0)}(G) = \alpha^{(k_0+1)}(G) \cdots = S(G).$$ 

Proposition 5.3. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_n)$. Then 

$$\alpha^{(k)}(G) = \max \{s \mid d_k(\{v_1, v_2, \ldots, v_s\}) \leq n - s\} = \max \{s \mid \{v_1, v_2, \ldots, v_s\} \text{ is } \delta_k\text{-small set in } G\}.$$ 

Proof. Let $s_0 = \max \{s \mid \{v_1, v_2, \ldots, v_s\} \text{ is } \delta_k\text{-small set in } G\}$. Then $s_0 \leq \alpha^{(k)}(G)$. Let $\alpha^{(k)}(G) = r$ and let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ be a $\delta_k$-small set. Since $d_k(\{v_1, v_2, \ldots, v_r\}) \leq d_k(\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\})$ it follows that $\{v_1, v_2, \ldots, v_r\}$ is $\delta_k$-small set too. Therefore $\alpha^{(k)}(G) = r \leq s_0$. 

Proposition 5.4. For every natural $k$ are held the inequalities 

$$n - \Delta(G) \leq \alpha^{(k)}(G) \leq n - \delta(G).$$ 

Proof. The left inequality follows from the inequality $S(G) \geq n - \Delta(G)$ from [1] and Proposition 5.1. Let $r = \alpha^{(k)}(G)$. According to Proposition 5.3 $\{v_1, v_2, \ldots, v_r\}$ is a $\delta_k$-small set. So 

$$\delta(G) = d(v_1) \leq d_k(\{v_1, v_2, \ldots, v_r\}) \leq n - r = n - \alpha^{(k)}(G),$$

hence $\alpha^{(k)}(G) \leq n - \delta(G)$.

Remark 5. The inequality $\alpha(G) \geq n - \Delta(G)$ is not always true. For example, $\alpha(C_5) < 5 - \Delta(C_5) = 3$. 

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Theorem 5.5. Let $A \subseteq V(G)$ be a $\delta_1$-small set of $G$ and $s = d_1(V(G) \setminus A)$. Then

$$|A| \leq \left\lfloor \frac{n-s}{2} + \sqrt{\frac{(n-s)^2}{4} + ns - 2e(G)} \right\rfloor$$

Proof. Let $2e(G) = \sum_{v \in V(G)} d(v) = \sum_{v \in A} d(v) + \sum_{v \in V(G) \setminus A} d(v) \leq |A| (n - |A|) + s(n - |A|)$. Solving the derived quadric inequality for $|A|$ we obtain the inequality (5.1).

Corollary 5.6 ([1]). For every number $k$

$$\alpha^{(k)}(G) \leq \left\lfloor \frac{n - \Delta(G)}{2} + \sqrt{\frac{(n - \Delta(G))^2}{4} + n\Delta(G) - 2e(G)} \right\rfloor \leq \left\lfloor 1 + \sqrt{\frac{1}{4} + n^2 - n - 2e(G)} \right\rfloor.$$

Proof. According to Proposition 5.1 it is sufficient to prove (5.2) only in the case $k = 1$. Let $A$ be a maximal $\delta_1$-small set, i.e. $|A| = \alpha^{(1)}(G)$, and $s = d_1(V(G) \setminus A)$. According to Theorem 5.5 the inequality (5.1) holds. Since the right side of (5.1) is an increasing function for $s$ and $s \leq \Delta(G) \leq n - 1$, the inequalities (5.2) follows from (5.1).

6. $\alpha$-SMALL SETS

Definition 3 ([1]). Let $G$ be an $n$-vertex graph and let $W \subseteq V(G)$. We say that $W$ is an $\alpha$-small set if

$$\sum_{v \in W} \frac{1}{n - d(v)} \leq 1.$$

We denote the smallest natural number $r$ for which $V(G)$ decomposes into $r$ $\alpha$-small sets by $\varphi^\alpha(G)$.

The idea for $\alpha$-small sets is coming from the following Caro-Wey inequality ([3] and [7])

$$\omega(G) \geq \sum_{v \in V(G)} \frac{1}{n - d(v)}.$$

We have the proposition

Proposition 6.1 ([1]).

$$\varphi^{(1)}(G) \leq \varphi^\alpha(G) \leq \varphi(G).$$

The following problem is inspired from Proposition 6.1 and Theorem 2.1

Problem. Is it true that for every graph $G$ there exists natural number $k_0 = k_0(G)$ such that $\varphi^{(\alpha)}(G) = \varphi^{(k_0)}(G)$?
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