APPROXIMATION OF THE AVERAGE OF SOME RANDOM MATRICES

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Abstract. Rudelson’s theorem states that if a set of unit vectors forms a John decomposition of the identity operator on $\mathbb{R}^d$, then a random sample of $C d \log d$ of them yields a decomposition of a matrix close to the identity. First, we observe that the same proof yields a more general statement about the average of positive semidefinite matrices. Second, we show that the $\log d$ factor cannot be removed. Then we present a stability version of the statement which extends to non-symmetric matrices, with applications to the study of the Banach–Mazur distance of convex bodies. We show also that in some cases, one needs to take a sample of the vectors of order $d^2$ to approximate the identity.

1. Introduction

For vectors $u, v \in \mathbb{R}^d$, their tensor product (or, diadic product) is a linear operator on $\mathbb{R}^d$ defined as $(u \otimes v)x = \langle u, x \rangle v$ for every $x \in \mathbb{R}^d$, where $\langle u, x \rangle$ denotes the standard inner product.

A random vector $v$ in $\mathbb{R}^d$ is called isotropic, if $\mathbb{E}v \otimes v = I$, where $\mathbb{E}$ denotes the expectation of a random variable, and $I$ is the identity operator on $\mathbb{R}^d$.

According to Rudelson’s theorem [Rud99], if we take $k$ independent copies $y_1, \ldots, y_k$ of an isotropic random vector $y$ in $\mathbb{R}^d$ for which $|y|^2 \leq \gamma$ almost surely, with

$$k = \left\lceil \frac{c\gamma \ln d}{\varepsilon^2} \right\rceil,$$

then

$$\mathbb{E} \left\| \frac{1}{k} \sum_{i=1}^{k} y_i \otimes y_i - I \right\| \leq \varepsilon.$$

A sequence of unit vectors $u_1, \ldots, u_m$ in $\mathbb{R}^d$ is said to yield a John decomposition of $I$, if $\frac{1}{d} I \in \text{conv}\{u_i \otimes u_i : i \in [m]\}$, that is, if there are scalars $\alpha_1, \ldots, \alpha_m \geq 0$ with $\sum_{i=1}^{m} \alpha_i = 1$ such that

$$\sum_{i=1}^{m} \alpha_i u_i \otimes u_i = \frac{1}{d} I.$$

Rudelson’s result applies in this setting as well. The coefficients $\alpha_i$ define a probability distribution on $[m]$. Let $\sigma = \{i_1, \ldots, i_k\}$ be a multiset obtained by $k$ independent draws from $[m]$ according to this distribution, and consider the following average of matrices

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Theorem 1.2. For any integer \( \sigma \) there is a multi-subset \( P \) in the operator norm, provided that \( I \) is a positive semidefinite matrix with \( a \) states that for every convex body \( K \) in \( \mathbb{R}^d \), there is a unique ellipsoid of maximal volume contained in \( K \), and this ellipsoid is the \( 1 \)-centered Euclidean unit ball \( B_2^d \) if, and only if, there are contact points \( u_1, \ldots, u_m \in \text{bd} (K) \cap \text{bd} (B_2^d) \) such that for some scalars \( \alpha_1, \ldots, \alpha_m > 0 \), we have (1) and \( \sum_{i=1}^m \alpha_i u_i = 0 \).

Gordon, Litvak, Meyer and Pajor [GLMP04] proved (extending similar results from [BR02, GPT01, Lew79], see also [TJ89, Theorem 14.5]) that the maximum volume affine image of any convex body \( K \) contained in \( L \) also yields a decomposition of the identity similar to John’s. In order to state it, we recall some terminology.

The polar of a convex body \( K \) in \( \mathbb{R}^d \) is defined as \( K^o = \{ x : \langle x, y \rangle \leq 1 \text{ for every } y \in K \}. \)
Definition 1.3. Let $K$ and $L$ be convex bodies in $\mathbb{R}^d$. We say that $K$ is in John’s position in $L$ if $K \subseteq L$ and for some scalars $\alpha_1, \ldots, \alpha_m > 0$ with $\sum_{i=1}^m \alpha_i = 1$, we have

\[ \frac{1}{d} I = \sum_{i=1}^m \alpha_i u_i \otimes v_i \]

and

\[ 0 = \sum_{i=1}^m \alpha_i u_i = \sum_{i=1}^m \alpha_i v_i, \]

where $u_1, \ldots, u_m \in \text{bd}(L) \cap \text{bd}(K)$, $v_1, \ldots, v_m \in \text{bd}(L^c) \cap \text{bd}(K^c)$ with $\langle u_i, v_i \rangle = 1$ for all $i \in [m]$.

Note that if $K$ and $L$ are origin-symmetric and (2) is satisfied for a set of vectors then by including the opposites of the vectors too, (3) is also satisfied.

Theorem 3.8 in [GLMP04] generalizes John’s theorem.

Theorem 1.4. Let $K$ and $L$ be two convex bodies in $\mathbb{R}^d$ such that $K \subseteq L$, and among all affine images of $K$ contained in $L$, $K$ has maximum volume. Assume also that $0 \in \text{int} L$. Then there exists $z \in \text{int}(K)$ such that $K - z$ is in John’s position in $L - z$ with $m \leq n^2 + n$.

Definition 1.5. Let $K$ be a convex body in $\mathbb{R}^d$. We denote the Banach–Mazur distance of $K$ to the Euclidean ball by

\[ r(K) = \inf \{ \lambda : E \subset K - a \subset \lambda E, \text{ for some ellipsoid } E \text{ and vector } a \in \mathbb{R}^d \}. \]

By John’s theorem, $r(K) \leq d$ for any convex body $K$ in $\mathbb{R}^d$, and $r(K) \leq \sqrt{d}$ for all centrally-symmetric convex bodies.

Moreover, for the unit balls of $\ell_p$ spaces, we have

\[ r(K) \leq \begin{cases} d^{1/p - 1/2} & \text{for } p \in [1, 2], \\ d^{1/2 - 1/p} & \text{for } p \in [2, \infty). \end{cases} \]

Theorem 1.6. Let $K$ be a convex body in $\mathbb{R}^d$ with $r(K) \leq 2$ such that the ellipsoid $E$ at which the infimum attains in (4) is the Euclidean unit ball. Let $K$ be in John’s position in $L$, and let vectors $u_i$ and $v_i$ for $i \in [m]$ satisfy the conditions of Definition 1.3. Then for any $0 < \varepsilon < 1$ and

\[ k \geq \frac{c d \ln d}{\varepsilon^2} r(K) \left( d \sqrt{r(K)} - 1 + 1 \right) \]

there is a multiset $\sigma \subset [m]$ of size $k$ such that

\[ \left\| \frac{d}{k} \sum_{i \in \sigma} u_i \otimes v_i - I \right\| \leq \varepsilon \]

and

\[ \frac{1}{k} \left\| \sum_{i \in \sigma} u_i \right\| \leq \frac{\varepsilon}{\sqrt{d}}, \quad \text{and} \quad \frac{1}{k} \left\| \sum_{i \in \sigma} v_i \right\| \leq \frac{\varepsilon}{\sqrt{d}}. \]

As an immediate corollary, we obtain a stability version of Rudelson’s result, that is, when $K$ is very close to the Euclidean ball, then we can approximate the identity with diads coming from $O(d \ln d)$ contact pairs.
Corollary 1.7. Let $K$ be a convex body in $\mathbb{R}^d$ with $B_2^d \subseteq K \subseteq (1 + 1/d^2)B_2^d$. Let $K$ be in John’s position in $L$, and let vectors $u_i$ and $v_i$ for $i \in [m]$ satisfy the conditions of Definition 1.3. Then for any $\varepsilon \in (0, 1)$ and for

$$k = \left\lceil \frac{cd \ln d}{\varepsilon^2} \right\rceil$$

there is a multiset $\sigma \subset [m]$ of size $k$ such that (5) and (6) hold.

On the other hand, when $K$ is not so close to the Euclidean ball, approximation of $I$ using only a few vector-pairs cannot be guaranteed.

Theorem 1.8. For a positive integer $d$, $\varepsilon \in (0, 1/2)$ and $\delta \in \left(0, \sqrt{d/4}\right)$, there is an origin-symmetric convex body $K$ contained in the cube $[-1, 1]^d$, the largest volume ellipsoid of which is $B_2^d$, such that there are contact points of $K$ and $[-1, 1]^d$ satisfying (2) with the following property. If $M$ is any subset of the diads appearing in (2) such that some linear combination of elements of $M$ is at distance at most $\varepsilon$ from $I$ in the operator norm, then

$$|M| \geq \left(\frac{\delta}{4\varepsilon}\right)^2 d \geq \left(\frac{r(K)}{4\varepsilon}\right)^2 d.$$  

2. Symmetric matrices

Let $\mathcal{P}^d$ denote the cone of positive semi-definite symmetric matrices in $\mathbb{R}^{d \times d}$. The Schatten $p$-norm of a matrix $A$ in $\mathcal{P}^d$ is defined as

$$\|A\|_{C_p^d} := \left(\sum_{i=1}^{d} (\lambda_i(A))^p\right)^{1/p},$$

where $(\lambda_i(A))_{i=1}^{d}$ is the sequence of eigenvalues of $A$. We recall that $\|A\| \leq \|A\|_{C_p^d}$ for all $p \geq 1$, and we also have

$$(7) \quad \|A\| \leq \|A\|_{C_p^d} \leq e \|A\| \quad \text{for} \quad p = \ln d,$$

where $\ln$ denotes the natural logarithm and $e$ denotes its base.

We state the following inequality due to Lust-Piquard and Pisier [LP86, LPP91], essentially in the form as it appears in the book [Pis98, Theorem 8.4.1].

**Theorem 2.1** (Lust-Piquard). $2 \leq p < \infty$. For any $d$ and any $Q_1, \ldots, Q_k$ square matrices of size $d$ (not necessarily positive definite) we have

$$\left[\mathbb{E}_r \left\| \sum_{j=1}^{k} r_j Q_j \right\|_{C_p^d}^p \right]^{1/p} \leq c \sqrt{p} \max \left\{ \left\| \sum_{j=1}^{k} Q_j Q_j^* \right\|_{C_p^d}^{1/2}, \left\| \sum_{j=1}^{k} Q_j^* Q_j \right\|_{C_p^d}^{1/2} \right\}$$

for a universal constant $c > 0$, where $r_1, \ldots, r_k$ are i.i.d random variables uniformly distributed on $\{1, -1\}$. 

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Note that for any $d \times d$ matrix $Q$, the product $Q^*Q$ is positive semidefinite. Since, by Weyl’s inequality, the Schatten $p$-norm is monotone on the cone of positive semidefinite matrices, we may deduce from the theorem of Lust–Piquard the following inequality

\[(8) \quad \left[ \mathbb{E}_r \left\| \sum_{j=1}^k r_j Q_j \right\|_{C_p^d}^p \right]^{1/p} \leq c \sqrt{p} \left\| \left( \sum_{j=1}^k Q_j Q_j^* + Q_j^* Q_j \right) \right\|_{C_p^d}^{1/2}.
\]

**Lemma 2.2 (Symmetrization by Rademacher variables).** Let $q_1, \ldots, q_k$ be independent random vectors distributed according to (not necessarily identical) probability distributions $\mathcal{P}_1, \ldots, \mathcal{P}_k$ on a normed space $X$ with $\mathbb{E} q_i = q$ for all $i \in [k]$.

Then

\[
\mathbb{E}_{\mathbf{r}} \left| \frac{1}{k} \sum_{\ell=1}^k q_{\ell} - q \right| \leq \frac{2}{k} \mathbb{E}_{\mathbf{q}_1, \ldots, \mathbf{q}_k} \mathbb{E}_{\mathbf{r}} \left| \sum_{\ell=1}^k r_{\ell} q_{\ell} \right|,
\]

where $\mathbf{r} = (r_1, \ldots, r_k)$ are Rademacher variables, that is, random variables uniformly distributed on \{1, −1\} independent of each other and other random variables.

**Proof of Lemma 2.2** Let $\bar{q}_1, \ldots, \bar{q}_k$ be independent random vectors chosen according to $\mathcal{P}_1, \ldots, \mathcal{P}_k$, respectively.

\[
\mathbb{E}_{\mathbf{q}_1, \ldots, \mathbf{q}_k} \mathbb{E}_{\mathbf{r}} \left| \frac{1}{k} \sum_{\ell=1}^k q_{\ell} - q \right| \leq \frac{1}{k} \mathbb{E}_{\mathbf{r}} \mathbb{E}_{\mathbf{q}_1, \ldots, \mathbf{q}_k, \mathbf{\bar{q}}_1, \ldots, \mathbf{\bar{q}}_k} \left| \sum_{\ell=1}^k r_{\ell} (q_{\ell} - \bar{q}_{\ell}) \right| \leq \frac{2}{k} \mathbb{E}_{\mathbf{r}} \mathbb{E}_{\mathbf{q}_1, \ldots, \mathbf{q}_k} \left| \sum_{\ell=1}^k r_{\ell} q_{\ell} \right|.
\]

**Proof of Theorem 1.1** Let $\mathbf{r} = (r_1, \ldots, r_k)$ be a sequence of $k$ random variables uniformly distributed on \{1, −1\} independent of each other and other random variables. Denote by $D = \frac{1}{k} \sum_{i \in [k]} Q_i - A$, and $p = \ln d$.

\[
\mathbb{E}_{\mathbf{Q}_1, \ldots, \mathbf{Q}_k} \left\| D \right\| \leq \frac{1}{k} \mathbb{E}_{\mathbf{Q}_1, \ldots, \mathbf{Q}_k} \left\| D \right\|_{C_p^d} \leq \frac{2}{k} \mathbb{E}_{\mathbf{Q}_1, \ldots, \mathbf{Q}_k} \mathbb{E}_{\mathbf{r}} \left\| \sum_{\ell=1}^k r_{\ell} Q_{\ell} \right\|_{C_p^d} (\text{S}) \leq \frac{c_0 \sqrt{p}}{k} \mathbb{E}_{\mathbf{Q}_1, \ldots, \mathbf{Q}_k} \left\| \left( \sum_{\ell=1}^k Q_{\ell}^2 \right)^{1/2} \right\|_{C_p^d} (\text{L-P}) \leq \frac{c_1 \sqrt{p}}{k} \mathbb{E}_{\mathbf{Q}_1, \ldots, \mathbf{Q}_k} \left[ \max_{\ell \in [k]} \left\| Q_{\ell} \right\|^{1/2} \cdot \left\| \left( \sum_{\ell=1}^k Q_{\ell} \right)^{1/2} \right\|_{C_p^d} \right].
\]

(PSD) \quad \leq \frac{c_1 \sqrt{p}}{k} \mathbb{E}_{\mathbf{Q}_1, \ldots, \mathbf{Q}_k} \left[ \max_{\ell \in [k]} \left\| Q_{\ell} \right\|^{1/2} \cdot \left\| \left( \sum_{\ell=1}^k Q_{\ell} \right)^{1/2} \right\|_{C_p^d} \right].
\[
\left( \frac{c_1 \sqrt{\gamma p}}{k} \right)^{1/2} \leq \frac{c_1 \sqrt{\gamma p}}{\sqrt{k}} \left[ \mathbb{E}_{Q_1, \ldots, Q_k} \left( \left\lVert \sum_{\ell=1}^{k} Q_{\ell} \right\rVert_2 \right) \right]^{1/2} \leq \frac{c_1 \sqrt{\gamma p}}{\sqrt{k}} \left[ \mathbb{E}_{Q_1, \ldots, Q_k} \left( \left\lVert D \right\rVert + \left\lVert A \right\rVert \right) \right]^{1/2},
\]

we assume that \( c_0 \) and \( c_1 \) are positive constants. Here, we use Lemma 2.2 in (S) and the inequality (8) in (L-P). The inequality (PSD) relies on the fact that the matrices \( Q_i \) are positive semidefinite, and (H) follows from Hölder’s inequality.

Thus, we obtain

\[
\mathbb{E} \left\lVert D \right\rVert \leq \frac{c_1 \sqrt{\gamma \ln d}}{\sqrt{k}} \sqrt{\mathbb{E} \left\lVert D \right\rVert} + a.
\]

Denoting by \( \alpha = \left( \frac{c_1 \sqrt{\gamma \ln d}}{\sqrt{k}} \right)^2 \), we have

\[
(\mathbb{E} \left\lVert D \right\rVert)^2 - \alpha \mathbb{E} \left\lVert D \right\rVert - \alpha a \leq 0.
\]

Therefore, we get \( \mathbb{E} \left\lVert D \right\rVert \leq \alpha + \sqrt{\alpha a} \), and thus the inequality

\[
\mathbb{E} \left\lVert D \right\rVert \leq \frac{c_1 \gamma \ln d}{k} + \frac{c_1 \sqrt{\gamma a \ln d}}{\sqrt{k}} \leq \varepsilon
\]

holds for sufficiently large \( c \) (see the definition of \( k \)). Theorem 1.2 is proved.

\[\Box\]

3. Non-symmetric diads – upper bound

We will show that Theorem 1.6 follows from the following more general result.

**Theorem 3.1.** Suppose that \( 0 < \varepsilon < 1 \) is a given number, \( Q_1, \ldots, Q_m \) and \( A \) are square matrices of size \( d \) such that

\[
A = \sum_{i=1}^{m} \alpha_i Q_i,
\]

where \( \alpha_i \geq 0 \) and \( \sum_{i=1}^{m} \alpha_i = 1 \). Let \( \gamma = \max_i \left\lVert Q_i \right\rVert \),

\[
U_i = \frac{1}{2\gamma} \left( Q_i Q_i^* + Q_i^* Q_i \right), \quad B = \sum_{i=1}^{m} \alpha_i U_i
\]

and \( b = \left\lVert B \right\rVert \). Suppose that

\[
k \geq \frac{c \ln d}{\varepsilon^2} \gamma (1 + b)
\]

for some positive constant \( c \). Then there is a multi-subset \( \sigma \subset [m] \) such that

\[
\left\lVert \frac{1}{k} \sum_{i \in \sigma} Q_i - A \right\rVert \leq \varepsilon.
\]
3.1. **Proof of Theorem 3.1.** Let

\[ E_B = \mathbb{E}_\sigma \left\| \frac{1}{k} \sum_{j \in \sigma} U_j - B \right\| . \]

Note that \( \| U_i \| \leq \gamma \). Since the \( U_i \) are positive semidefinite matrices, we can apply Theorem 1.1 and get that \( E_B \leq \varepsilon \).

Setting \( p = \ln d \), we obtain that

\[
\frac{1}{k} \mathbb{E}_\sigma \left\{ \left\| \left( \sum_{j \in \sigma} U_j \right)^{1/2} \right\|_{C^d_p} \right\} = \frac{1}{\sqrt{k}} \mathbb{E}_\sigma \left\{ \left\| \left( \frac{1}{k} \sum_{j \in \sigma} U_j \right)^{1/2} \right\|_{C^d_p} \right\}
\]

\[
\leq \frac{e}{\sqrt{k}} \mathbb{E}_\sigma \left\{ \left\| \left( \frac{1}{k} \sum_{j \in \sigma} U_j \right)^{1/2} \right\| \right\} = \frac{e}{\sqrt{k}} \mathbb{E}_\sigma \left\{ \left\| \frac{1}{k} \sum_{j \in \sigma} U_j \right\|^{1/2} \right\}
\]

\[
(\text{T}) \quad \leq \frac{e}{\sqrt{k}} \mathbb{E}_\sigma \left\{ \left( \left\| \frac{1}{k} \sum_{j \in \sigma} U_j - B \right\| + \| B \| \right) \right\}^{1/2}
\]

\[
(\text{H}) \quad \leq \frac{e}{\sqrt{k}} \mathbb{E}_\sigma \left( \mathbb{E}_\sigma \left( \left\| \frac{1}{k} \sum_{j \in \sigma} U_j - B \right\| + \| B \| \right) \right)^{1/2} = \frac{e}{\sqrt{k}} (E_B + \| B \|)^{1/2} \leq \frac{e \sqrt{1 + b}}{\sqrt{k}}.
\]

Here, (T) and (H) follow from the triangle inequality and Hölder’s inequality respectively.

Denote by \( D_\sigma \) the matrix

\[ \frac{1}{k} \sum_{j \in \sigma} Q_j - A. \]

Our aim is to show that \( \mathbb{E}_\sigma \| D_\sigma \| \leq \varepsilon \).

Let \( r_1, \ldots, r_m \) be random variables uniformly distributed on \( \{1, -1\} \) and independent of each other and of other random variables. Next, we have

\[
\mathbb{E}_\sigma \| D_\sigma \| \leq \mathbb{E}_\sigma \| D_\sigma \|_{C^d_p} \leq \frac{2}{k} \mathbb{E}_\sigma \mathbb{E}_r \left\| \sum_{i \in \sigma} r_i Q_i \right\|_{C^d_p} \leq \frac{2}{k} \mathbb{E}_\sigma \left[ \mathbb{E}_r \left\| \sum_{i \in \sigma} r_i Q_i \right\|_{C^d_p} \right]^{1/p}
\]

\[
(\text{L-P}) \quad \leq \frac{c_1 \sqrt{p} \sqrt{\gamma}}{k} \mathbb{E}_\sigma \left\{ \left\| \left( \sum_{j \in \sigma} [Q_j Q_j^* + Q_j^* Q_j] \right)^{1/2} \right\|_{C^d_p} \right\} = \frac{c_1 \sqrt{p} \sqrt{\gamma}}{k} \mathbb{E}_\sigma \left\{ \left\| \left( \sum_{j \in \sigma} U_j \right)^{1/2} \right\|_{C^d_p} \right\}
\]

\[
(\text{S}) \quad \leq \frac{cc_1 \sqrt{p} \sqrt{\gamma} \sqrt{1 + b}}{\sqrt{k}} \leq \varepsilon,
\]

where \( c_1 \) is some positive constant. Note that (S) and (L-P) follow from Lemma 2.2 and (8) respectively and the last inequality holds for a sufficiently large constant \( c \). This finishes the proof of Theorem 3.1.
3.2. **Proof of Theorem 1.6.** For each \(i \in [m]\), set \(Q_i = du_i \otimes v_i\), and use the notation \((U_i, B, b, \gamma)\) of Theorem 3.1. As \(B_2^d \subset K \subset r(K)B_2^d\), we have \(1 \leq \|u_i\| \leq r(K)\) and \(1/r(K) \leq \|v_i\| \leq 1\). Since \(\|Q_i\| = d\|u_i\|\|v_i\|\), we have

\[
d/r(K) \leq \|Q_i\| \leq r(K)d,
\]

and in particular, we have

\[
d/r(K) \leq \gamma \leq r(K)d. \tag{10}
\]

Note that

\[
\|v_i^2 d^2 \gamma u_i \otimes u_i - du_i \otimes v_i\| = d|u_i| d|v_i^2 u_i - v_i| \\
\leq d|u_i| \left( \left( \frac{d|v_i^2 r(K)}{\gamma} \right)^2 + 1 - 2 \frac{d|v_i^2 r(K)}{\gamma} \right)^{1/2} \\
\leq d|u_i| \left[ r(K)^4 + 1 - 2/r(K)^2 \right]^{1/2} \leq 6d \left[ r(K)^6 - 1 \right]^{1/2} \leq 100d \left[ r(K) - 1 \right]^{1/2}. \tag{11}
\]

Similarly, we have

\[
\|u_i^2 \gamma^2 v_i \otimes v_i - dv_i \otimes u_i\| \leq 100d \left[ r(K) - 1 \right]^{1/2}. \tag{12}
\]

On the other hand,

\[
\sum_{i=1}^m \alpha_i U_i = \sum_{i=1}^m \alpha_i \left[ Q_i Q_i^* - \gamma du_i \otimes v_i \right] + \left[ Q_i^* Q_i - \gamma dv_i \otimes u_i \right] + \sum_{i=1}^m \alpha_i \frac{u_i \otimes v_i + v_i \otimes u_i}{2},
\]

where the last summand is equal to the identity operator \(I\). Since \(Q_i Q_i^* = d^2|u_i|^2 v_i \otimes v_i\), and \(Q_i^* Q_i = d^2|v_i|^2 u_i \otimes u_i\), by \(11\) and \(12\), the formula above yields

\[
b \leq 100d \left[ r(K) - 1 \right]^{1/2} + 1,
\]

which, combined with \(10\) yields \(5\).

To obtain the balancedness bound \(6\), we form new vectors \(a_i\) and \(b_i\) in \(\mathbb{R}^{d+1}\) by concatenating \(v_i\) and \(u_i\) together with \(1/\sqrt{d}\). It is easy to see, that \(\sum_{i \in m} \alpha_i d a_i \otimes b_i\) is the identity in \(\mathbb{R}^{d+1}\), and, that

\[
\|da_i \otimes b_i\| \leq d \sqrt{r^2(K) + 1/d} \leq 2dr(K).
\]

Therefore, \(5\) in \(\mathbb{R}^{d+1}\) yields \(d \sum_{i \in \sigma} a_i \otimes b_i - I \| \leq \varepsilon\). Observing, that

\[
\frac{d}{k} \sum_{i \in \sigma} u_i \otimes v_i, \quad \frac{\sqrt{d}}{k} \sum_{i \in \sigma} u_i^*, \quad \frac{\sqrt{d}}{k} \sum_{i \in \sigma} v_i
\]

are submatrices of \(d/k \sum_{i \in \sigma} a_i \otimes b_i\), we complete the proof of Theorem 1.6.
4. The log factor is needed for symmetric matrices

In this section, we prove Theorem 1.2. First, in Lemma 4.1 we show that in $\ell_1^n$, a point in the convex hull of other points may not be well approximated in terms of the dimension $t$. Then, we use the fact that $\ell_1^n$ embeds isometrically in $\ell_\infty^n$ for $d = 2^t$, which embeds isometrically in the space of matrices of size $d \times d$.

**Lemma 4.1.** Consider the point $a = \frac{1}{12k}(1, \ldots, 1) \in \ell_1^n$, where $k$ and $t$ are positive integers. Denote by $e_1, \ldots, e_t$ the standard basis of $\ell_1^n$ and by $e_{t+1}$ the zero vector.

Then, for any non-empty multiset $\sigma_0 \subseteq [t+1]$ of size $s$, where $s \leq 3k$, we have

$$
\left\| \frac{1}{s} \sum_{i \in \sigma_0} e_i - a \right\|_1 \geq \frac{t}{12k}.
$$

**Remark 1.** In the proof of the theorem, we choose $t = \lceil \log_2 d \rceil$.

**Proof of Lemma 4.1.** Since the $i$-th coordinate $b_i$ of $\frac{1}{s} \sum_{i \in \sigma_0} e_i/2$ is either equal to 0 or at least $\frac{1}{2s} \geq \frac{1}{6k}$, we have $|b_i - \frac{1}{12k}| \geq \frac{1}{12k}$ for every $i \in [t]$, which finishes the proof of the lemma. \(\Box\)

Without loss of generality, we may assume $d = 2^t$, where $t$ is a non-negative integer. Indeed, if we prove Theorem 1.2 for $d = 2^t$, that is, we find proper matrices $Q_1, \ldots, Q_t$, then the matrices $Q'_1, \ldots, Q'_m$ of size $d \times d$ with the following properties satisfy the conditions of the theorem provided that $t = \lceil \log_2 d \rceil$. The matrix $Q_i$ is a diagonal minor of $Q'_i$, the complementing minor of $Q'_i$ is the identity submatrix, and the rest elements of $Q_i$ are zeros.

Note that it is sufficient to consider multisets $\sigma$ such that

$$m/2 < |\sigma| \leq m. $$

Indeed, if the theorem is proved for such multisets, then it holds for a multiset $\sigma'$ with $|\sigma| \leq m/2$: the multiset $\sigma'$ consisting of $2^t$ copies of $\sigma$, where $l = \lceil \log_2 (m/|\sigma|) \rceil$, satisfies (13), and thus the statement of the theorem holds for $\sigma'$. Since $\sigma'$ consists of several copies of $\sigma$, we can easily conclude that Theorem 1.2 is true for $\sigma$.

We identify the space of $d \times d$ real diagonal matrices equipped with the operator norm with $\ell_\infty^d$. Enumerate all $\pm 1$ sequences of length $t$ as $s_1, \ldots, s_d$. Clearly, the linear map

$$
\phi : \ell_1^n \to \ell_\infty^d \text{ such that } \phi(x) = (\langle x, s_1 \rangle, \ldots, \langle x, s_d \rangle)
$$

embeds $\ell_1^n$ isometrically into $\ell_\infty^d$ which we consider as a subspace of $\mathbb{R}^{d \times d}$.

Next, we are going to construct the desired matrices $Q_i$. Let $k$ be an integer such that

$$
(14) \quad \frac{m}{\gamma} = \left\lfloor \frac{t}{96\varepsilon} \right\rfloor,
$$

that is $k \geq 1$. Using the notation introduced in Lemma 4.1, for every $i \in [t+1]$, put

$$
Q_i = \gamma \psi(e_i/2), \quad \text{where } \psi(x) = \phi(x - a) + I.
$$

Note that $\psi$ is an affine isometry from $\ell_1^n$ into $\ell_\infty^d$.

By (14), we have that

$$
\left\| \frac{e_i}{2} - a \right\|_1 \leq \left\| \frac{e_i}{2} \right\|_1 + \left\| a \right\|_1 \leq \frac{1}{2} + \frac{t}{12k} \leq \frac{1}{2} + \frac{96t\varepsilon}{12t} = \frac{1}{2} + 8\varepsilon \leq 1,
$$

as desired.
for every \( i \in [t+1] \), and therefore \( \| Q_i \| \leq \gamma (\| I \| + \| e_i/2 - a \|_1) < 2\gamma \) and the matrix \( Q_i \) is positive definite.

Assuming \( \lambda_i = \frac{1}{12k} \) for \( i \in [t] \) and \( \lambda_{t+1} = 1 - \frac{1}{12k} \), we have

\[
a = \sum_{i=1}^{t+1} \lambda_i e_i.
\]

Since \( \sum_{i=1}^{t+1} \lambda_i = 1 \) and \( \lambda_i \geq 0 \) for every \( i \in [t+1] \), we obtain \( a \in \text{conv} \{ \frac{e_1}{2}, \ldots, \frac{e_{t+1}}{2} \} \). Thus, denoting by \( Q_{t+2} \) the zero matrix, we get

\[
\sum_{i=1}^{t+1} \frac{\lambda_i}{\gamma} Q_i + \left( 1 - \frac{1}{\gamma} \right) Q_{t+2} = \sum_{i=1}^{t+1} \lambda_i (\phi(e_i/2 - a) + I) = \phi \left( \sum_{i=1}^{t+1} \lambda_i e_i/2 \right) - \phi(a) + I = I,
\]

that is,

\[
I \in \text{conv} \{ Q_1, \ldots, Q_{t+2} \}.
\]

To prove the theorem, assume that there is a multiset \( \sigma \) of \([t+2]\) with \( (13) \) such that

\[
\left\| \frac{1}{|\sigma|} \sum_{i \in \sigma} Q_i - I \right\| < \varepsilon.
\]

Next, \( (t+2) \) is an element of \( \sigma \) with multiplicity \( |\sigma| - s \) for some non-negative integer \( s \). Denote by \( \sigma_0 \) the multi-subset of \( \sigma \) which does not contain \( (t+2) \) and \( |\sigma_0| = s \).

Since \( \text{trace}(\phi(y)) = 0 \), we obtain \( \text{trace}(Q_i) = \gamma \text{trace}(I) = \gamma d \) for every \( i \in [t+1] \). Thus, it follows from \( (15) \) and the inequality \( \text{trace}(A) \leq d \| A \| \) for an arbitrary matrix \( A \) of size \( d \) that

\[
(16) \quad \left| \frac{\gamma s}{|\sigma|} - 1 \right| < \varepsilon,
\]

and therefore

\[
\frac{|\sigma|}{\gamma} (1 - \varepsilon) < s < \frac{|\sigma|}{\gamma} (1 + \varepsilon).
\]

By \( (13), (16), \) and \( (14) \), we obtain

\[
(17) \quad s < \frac{|\sigma|}{\gamma} (1 + \varepsilon) \leq \frac{m}{\gamma} (1 + \varepsilon) < 3k
\]

and

\[
(18) \quad s > \frac{|\sigma|}{\gamma} (1 - \varepsilon) > \frac{m}{2\gamma} (1 - \varepsilon) > \frac{k}{4},
\]

Since \( \psi \) is an affine map, we have

\[
\left\| \frac{1}{|\sigma|} \sum_{i \in \sigma} Q_i - I \right\| = \left\| \frac{1}{|\sigma|} \sum_{i \in \sigma_0} \psi \left( \frac{e_i}{2} \right) - I \right\| = \left\| \frac{1}{|\sigma|} \psi \left( \sum_{i \in \sigma_0} \frac{e_i}{2} \right) - I \right\| =
\]

\[
\frac{\gamma}{|\sigma|} \phi \left( \sum_{i \in \sigma_0} \frac{e_i}{2} - sa \right) \quad + \quad \frac{\gamma s}{|\sigma|} I - I \geq
\]

\[
\frac{\gamma s}{|\sigma|} \phi \left( \frac{1}{s} \sum_{i \in \sigma_0} \frac{e_i}{2} - a \right) \quad - \quad \frac{\gamma s}{|\sigma|} - 1 \geq
\]

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\[ \frac{\gamma s}{|\sigma|} \left\| \frac{1}{s} \sum_{i \in \sigma} e_i - a \right\|_1 - \varepsilon, \]

where, in the last inequality, we combine the fact that \( \phi \) is an isometry with (16).

By (17), we may apply Lemma 4.1 to the multiset \( \sigma_0 \) to obtain the inequality

\[ \left\| \frac{1}{|\sigma|} \sum_{i \in \sigma} Q_i - I \right\| > \frac{\gamma s}{|\sigma|} \cdot \frac{t}{12k} - \varepsilon, \]

which, by (15) and (18), yields

\[ 2\varepsilon > \frac{\gamma t}{48|\sigma|}, \]

and thus

\[ m \geq |\sigma| > \frac{\gamma t}{96\varepsilon}, \]

completing the proof of Theorem 1.2.

5. LARGE BANACH-MAZUR DISTANCE TO THE BALL AND SLOW APPROXIMATION

In this section, we prove Theorem 1.8.

We use \( \{e_1, \ldots, e_d\} \) to denote the standard basis in \( \mathbb{R}^d \). For every \( i, j \in [d] \) let \( w_i^j \) be vectors that satisfy the following conditions:

1. \( \langle w_i^j, e_i \rangle = 1 \) for all \( j \in [d] \);
2. \( |w_i^j - e_i| = \delta \) for all \( j \in [d] \) and \( w_i^j \) belongs to the facet \( \{x : x_i = 1\} \) of a cube \([-1, 1]^d\);
3. \( \{w_i^1, \ldots, w_i^d\} \) are the vertices of the \( (d - 1) \)-dimensional regular simplex with the center at \( e_i \). That is, \( \sum_{j=1}^d w_i^j = de_i \).

Clearly, \( I = \frac{1}{d} \sum_{i,j=1}^d w_i^j \otimes e_i \).

Let

\[ K = \text{conv} \left( B_2^d \cup \{\pm w_i^j\}_{i,j \in [d]} \right). \]

Clearly, \( \{\pm w_i^j\}_{i,j \in [d]} \subset \text{bd} (K) \) and \( \{e_1, \ldots, e_d\} \subset \text{bd} (K^c) \). By K. Ball’s refinement of John’s theorem, \( B_2^d \) is the largest volume ellipsoid in \( K \).

Let us show that one needs to use a large number of operators \( dw_i^j \otimes e_i \) to approximate the identity. For simplicity, we use \( Q_{ij} \) to denote \( dw_i^j \otimes e_i \). The set of all operators \( Q_{ij} \), where \( i, j \in [d] \), is denoted by \( Q \).

Let \( A \) be an \( \varepsilon \)-approximation of the identity by elements of \( M \). Assume we use \( \ell \) operators from the set \( \{Q_{11}, \ldots, Q_{1d}\} \). This means that there exists a vector of non-negative coefficients \((c_1, \ldots, c_n)\) with \( \ell \) non-zero components such that \( \sum_{j=1}^d c_j Q_{1j} \) is a summand of \( A \). Since
\[ \|A - I\| \leq \varepsilon, \text{ we have} \]

\[ \varepsilon \geq \| (A - I) e_1 \| = \| (A - I) e_1, e_1 \| = \left\| \sum_{j=1}^{d} c_j d \left\langle w_1^j, e_1 \right\rangle - 1 \right\| = \left\| d \sum_{j=1}^{d} c_j - 1 \right\|. \]

Therefore,

\[ d \sum_{j=1}^{d} c_j \geq \frac{1}{2}. \]

Set

\[ x = \frac{\sum_{j: c_j \neq 0} (w_1^j - e_1)}{\sum_{j: c_j \neq 0} (w_1^j - e_1)} \]

Now, \( x \) is a unit vector with \( \langle x, e_1 \rangle = 0 \) and

\[ \langle (w_1^j - e_1), x \rangle = \frac{\delta}{\sqrt{\ell}} \sqrt{\frac{d - \ell}{d - 1}} \geq \frac{\delta}{2\sqrt{\ell}} \quad \text{for all} \quad j: c_j \neq 0. \]

On the other hand,

\[ \varepsilon \geq \|(A - I) x\| \geq \langle (A - I) x, e_1 \rangle = \left\langle \left( \sum_{j=1}^{d} c_j Q_{1j} \right) x, e_1 \right\rangle = d \sum_{j=1}^{d} c_j \left\langle w_1^j, x \right\rangle = d \sum_{j=1}^{d} c_j \left\langle (w_1^j - e_1), x \right\rangle \geq d \sum_{j=1}^{d} c_j \frac{\delta}{2\sqrt{\ell}} \geq \frac{\delta}{4\sqrt{\ell}} \]

Or, equivalently, \( \ell \geq \left( \frac{\delta}{4\varepsilon} \right)^2 \)

Finally, the total number of operators \( M \) is at least \( \ell \cdot d \).

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