The modal analysis of a double disc flexible rotor-bearing system with isotropic stiffness and damping properties

Abhigyan Bhuyan¹, Rajiv Verma²

¹Department of Mechanical Engineering, NIT Kuruskshetra-136119, India
²Associate professor, Department of Mechanical Engineering, NIT Kuruskshetra-136119, India

abhi2012gyan@gmail.com

Abstract. This paper shows the analysis of the modal characteristics of a flexible rotor system having two non-identical discs mounted on it using the finite element method. It uses the Timoshenko beam elements theory including shear deformation, rotary bending effects and gyroscopic effects. The bearing is considered to be isotropic having translational stiffness and damping values constant in x and y directions. There are no cross-coupling terms. The conduct of the framework is investigated by reference to natural frequency maps and mode shape diagrams. A comparative study has been conducted between this system having isotropic bearings of stiffness and damping properties and the one in which the bearings have only stiffness keeping other parameters same. The gyroscopic effect is seen to be prominent at higher rotating speeds for both the cases which cause the splitting of frequency pairs into forward and backward whirls. The Campbell diagram is computed which helps in finding the critical speeds associated with each pair of diverging natural frequencies. The bearing system represented by stiffness and damping properties shows a comparatively better control over the critical speed range.

1. Introduction

The aspect of rotor dynamics has a great significance in the present day world of science and technology. While designing the rotating structures or rotating machine elements for various applications, the study of the dynamic characteristics of such a system has become a matter of great concern and importance. Nowadays, new computational techniques and higher grounds of research methodologies have taken this domain of importance into new heights. Finite element method (FEM) is one such modern-day computational techniques. One of the very first works on rotor dynamics using FEM could be enlisted by [1] and [2] as they studied a turbo rotor system’s stability and unbalance response using FEM including elastic bending and translational kinetic energy. They concluded that it was necessary to reduce the number of degrees of freedom to accurately model a system. But in [3], it was concluded that FEM can be effectively used for modelling rotor bearing system for determining critical speeds, stability etc. They studied a flexible rotor model supported on bearings of linear stiffness and viscous damping using Rayleigh beam theory. In [4], the authors extended this work by incorporating internal viscous and hysteretic damping in the model. They found that both forms of damping destabilized the rotor bearing system and induced nonsynchronous forward precession.

In the year 1972, authors [5] used the Timoshenko beam elements theory in the dynamic study of a rotor bearing system. They derived the corresponding matrices for the beam elements. In the year 1973,[6] a research work on tapered beams using Timoshenko beam elements theory was presented
where each node had 3 degrees of freedom. In [7], the authors developed the matrices for stiffness, mass and gyroscopic couple of a tapered finite elements beam.

The use of Timoshenko beam elements in rotor dynamics was generalized and the shape functions using transverse shear effects was established in [8]. In the paper [9], the author obtained the consistent mass and gyroscopic matrices for a constant section shaft element considering shear deformation and transversal inertia and showed that systems incorporated with rotational inertia alone, not shear deformation were not accurate. In [10], the authors performed modal analysis of a continuous rotor bearing system to find out the mode shapes, whirl speeds (both forward and backward whirl) of the rotating shaft. In the research works of [11] and [12], the authors computed the natural whirl speeds and the effects of unbalance in a rotor bearing system using both shear deformation and internal damping. In the research work of [13], they presented a new model of Timoshenko beam using FEM in order to tackle its free vibration analysis. The beam being non-uniform was supported on variable two parameter foundation and hence the methodology led our way to deal with common non-uniform structures such as rectangular, circular, tubular etc. In order to study the whirl speeds and stability of a system having flexible rotor mounted on linear stiffness and viscous damping bearings, the work in [14] put forward a Timoshenko beam model of C0 class using FEM incorporating shear deformation and internal viscous and hysteretic damping and proved better convergence and high accuracy with numerical results. In [15], the authors used the Lagrangian approach in their FEM formulation of a rotor bearing system and incorporated phenomena of gyroscopic effects and inertia coupling between bending and torsional vibrations. The authors in [16] considered a multi branched shaft system having rigid coupling and flexible connections as their model for modal synthesis.

In the light of the above literature survey, having observed the absence of emphasis on dynamic analysis of double disc rotor systems, the dynamic analysis on a rotor model having two non-identical discs mounted on a flexible shaft supported by isotropic bearings of linear stiffness and damping properties is performed here. A similar work by the same authors has been published in a conference proceeding [18] on the same model with a difference that the bearings have stiffness properties only. Here, a comparative study has been conducted between the results of both the models i.e. Model 1: a flexible rotor with two non-identical discs mounted on linear stiffness bearings[18] and Model 2: a flexible rotor with two non-identical discs mounted on linear stiffness and viscous damping bearings. With the inclusion of gyroscopic effects, the dynamic analysis is performed using finite element computations of Timoshenko beam elements.

2. Modal analysis of the rotor bearing system

Figure 1 shows the rotor bearing system having two non-identical discs mounted on a flexible rotor which is supported by two isotropic bearings. The bearing type 3 represents isotropic bearing having constant translational stiffness and damping and no cross coupling terms.

Assumptions:

a. The disc is rigid and symmetrical.

Figure 1. Model of the rotor system
b. The spin speed of the rotor is constant.
c. The shaft is flexible and symmetric.
d. The bearings are isotropic in nature. The values of stiffness and damping are constant in both x and y directions.
e. There is no cross couple terms for bearing stiffness and damping.

The shaft is divided into 6 Timoshenko beam elements having 7 nodes in such a way that the two discs fall on two nodes. In the following sub sections the elemental equations and matrices of different components of the system will be presented.

2.1. Elemental matrices for rigid disc
Since the disc is considered to be rigid, the strain energy calculations are neglected. Only the kinetic energy is calculated. The kinetic energy of the disc due to its translation and rotation is given by,

\[ \frac{1}{2} I_d (\dot{\omega}_z^2 + \dot{\omega}_\theta^2) + \frac{1}{2} \rho I \dot{\omega}_z^2 \]

Hence the total kinetic energy of the disc using instantaneous angular velocities w.r.t. a frame of reference that rotates with the disc and neglecting the higher order terms,

\[ T_d = \frac{1}{2} m_d (\dot{u}^2 + \dot{v}^2) + \frac{1}{2} I_d (\dot{\theta}^2 + \dot{\psi}^2) + \frac{1}{2} \rho I (\dot{\omega}_z^2 - 2\dot{\psi}) \]

The last term of the equation accounts for the gyroscopic effect of the disc. In order to get the element matrices, the Lagrange’s equation is applied to equation (2). Thus the element mass matrix of the disc[17] is given by,

\[
M_e = \begin{bmatrix}
m_d & 0 & 0 & 0 \\
0 & m_d & 0 & 0 \\
0 & 0 & I_d & 0 \\
0 & 0 & 0 & I_d \\
\end{bmatrix}
\]

(3)

The element gyroscopic matrix of the disc[17] is given by,

\[
G_e = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & I_p & 0 \\
0 & 0 & -I_p & 0 \\
\end{bmatrix}
\]

(4)

2.2. Elemental matrices for shaft
The rotor shaft is divided into 6 elements using the 6 Timoshenko beam elements theory considering both shear deformation and rotary inertia effects. Each element possesses two nodes and four degrees of freedom per node i.e. transverse displacements in x and y directions and rotations about x and y axes.

[17] The mass matrix is given by

\[
M_e = \frac{\rho_e A e l_e}{840(1 + \nu_e)^2} \begin{bmatrix}
m_1 & 0 & 0 & m_2 & m_3 & 0 & 0 & m_4 \\
0 & m_1 & -m_2 & 0 & 0 & m_3 & -m_4 & 0 \\
0 & -m_2 & m_5 & 0 & 0 & m_6 & m_4 & 0 \\
0 & m_2 & 0 & 0 & m_5 & -m_4 & 0 & 0 & m_6 \\
0 & m_3 & m_4 & 0 & 0 & m_1 & m_2 & 0 \\
0 & -m_4 & m_6 & 0 & 0 & m_2 & m_5 & 0 \\
0 & m_4 & 0 & 0 & m_6 & -m_2 & 0 & 0 & m_5 \\
\end{bmatrix}
\]

\[+ \frac{\rho_e l_e}{30(1 + \nu_e)^2} \begin{bmatrix}
m_7 & 0 & 0 & m_8 & -m_7 & 0 & 0 & m_8 \\
0 & m_7 & -m_8 & 0 & 0 & -m_7 & -m_8 & 0 \\
0 & -m_8 & m_9 & 0 & 0 & m_8 & m_{10} & 0 \\
- m_7 & 0 & 0 & m_8 & -m_7 & m_9 & 0 & 0 & -m_8 \\
0 & -m_7 & m_8 & 0 & 0 & m_7 & m_8 & 0 \\
0 & -m_8 & m_{10} & 0 & 0 & m_8 & m_9 & 0 \\
m_8 & 0 & 0 & m_{10} & -m_8 & 0 & 0 & m_9 \\
\end{bmatrix}
\]

(5)
\[ m_1 = 312 + 588\phi_e + 2800\phi_e^2 : m_2 = (44 + 77\phi_e + 350\phi_e^2)l_e : m_3 = 108 + 252\phi_e + 140\phi_e^2 \\
m_4 = -(26 + 63\phi_e + 35\phi_e^2)l_e : m_5 = (8 + 14\phi_e + 7\phi_e^2)l_e^2 : m_a = -(6 + 14\phi_e + 7\phi_e^2)l_e^2 \\
m_6 = 36 : m_a = (3 - 15\phi_e)l_e : m_9 = (4 + 5\phi_e + 100\phi_e^2)l_e^2 : m_{10} = -(1 - 50\phi_e + 50\phi_e^2)l_e^2 \]

The stiffness matrix is given by,

\[ K_e = \frac{E_e l_e}{(1 + \phi_e)l_e^3} \begin{bmatrix} 12 & 0 & 0 & 6l_e & -12 & 0 & 0 & 6l_e \\
0 & 12 & -6l_e & 0 & 0 & -12 & 6l_e & 0 \\
0 & 0 & l_e^2(4 + \phi_e) & 0 & 0 & 0 & l_e^2(2 - \phi_e) & 0 \\
-12 & 0 & 0 & -6l_e & 12 & 0 & 6l_e & 0 \\
0 & 0 & 0 & 6l_e & -12 & 0 & 6l_e & 0 \\
0 & 0 & 0 & l_e^2(2 - \phi_e) & 0 & 0 & l_e^2(4 + \phi_e) & 0 \\
6l_e & 0 & 0 & l_e^2(2 - \phi_e) & -6l_e & 0 & 0 & l_e^2(4 + \phi_e) \end{bmatrix} \]

The gyroscopic matrix is given by

\[ G_e = \frac{\rho_e l_e}{15(1 + \phi_e)l_e^2} \begin{bmatrix} 0 & g_3 & -g_2 & 0 & 0 & -g_1 & -g_2 & 0 \\
g_2 & 0 & g_3 & g_2 & 0 & -g_1 & -g_2 & 0 \\
g_2 & 0 & 0 & g_3 & -g_2 & 0 & 0 & g_4 \\
g_1 & g_2 & 0 & 0 & g_2 & g_3 & g_2 & 0 \\
g_1 & 0 & g_2 & 0 & -g_1 & 0 & 0 & g_2 \\
g_2 & 0 & 0 & g_3 & g_2 & 0 & g_3 & 0 \\
g_1 & 0 & 0 & g_3 & 0 & g_2 & g_3 & 0 \\
0 & g_2 & -g_4 & 0 & 0 & -g_2 & -g_3 & 0 \end{bmatrix} \]

\[ g_1 = 36 : g_2 = (3 - 15\phi_e)l_e : g_3 = (4 + 5\phi_e + 100\phi_e^2)l_e^2 : g_4 = -(1 - 50\phi_e + 50\phi_e^2)l_e^2 \]

2.3. Elemental matrices for bearings

In order to simplify the dynamics, the load deflection relationship for the bearings is considered to be linear. [17] Thus the relationship between the forces acting on the shaft due to the bearing and the resultant velocities and displacements of the shaft are approximated as,

\[ \begin{bmatrix} f_x \\ f_y \end{bmatrix} = -\begin{bmatrix} k_{uu} & k_{uv} \\ k_{vv} & k_{vv} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} c_{uu} & c_{uv} \\ c_{vv} & c_{vv} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} \]

Here the cross couple terms of stiffness (i.e. \( k_{uv} \) and \( k_{vu} \) ) and damping (i.e. \( c_{uv} \) and \( c_{vu} \)) are taken to be zero.

2.4. Assembly of elemental equations

The general form of the equation after assembling all element level equations for all elements is given by

\[ M\ddot{q} + \Omega G\dot{q} + Kq = 0 \]

3. Results

The values of the system parameters are as follows

\[ L_a = 1.5m; D_a = 0.05m; D_d = 0.28m; D_{ed} = 0.35m; \tau = 0.07m; E = 211 \text{ GN/m}^2; G = 81.2 \text{ GN/m}^2; \rho = 7810 \text{ kg/m}^3; k_{uu} = k_{vv} = 1 \text{ MN/m}^2; c_{uu} = c_{vv} = 3 \text{kN/m} \]

Case 1: When the rotor spin speed is 0 rev/min.

In this case, the eigen values occur in pairs. The roots have a negative real part and an imaginary part which accounts for the fact that the damping ratio is not zero. The eigen values (s), natural frequencies (\( \omega_n \)), damped natural frequencies (\( \omega_d \)) and damping ratios (\( \zeta \)) of this system (i.e. Model 2) at 0 rev/min and 4000 rev/min is shown in table 1.
Table 1: Eigen values (s), natural frequencies (\(\omega_n\)), damped natural frequencies (\(\omega_d\)) and damping ratios (\(\zeta\)) for a rotor bearing system of model 2

| Speed (rev/min) | Root s (rad/sec) | \(\omega_n\) (Hz) | \(\omega_d\) (Hz) | \(\zeta\) |
|-----------------|------------------|-------------------|-------------------|---------|
| 0               | -4.424\pm87.26j  | 13.91             | 13.89             | 0.051   |
|                 | -4.424\pm87.26j  | 13.91             | 13.89             | 0.051   |
|                 | -78.24\pm292.4j  | 48.18             | 46.54             | 0.258   |
|                 | -78.24\pm292.4j  | 48.18             | 46.54             | 0.258   |
|                 | -566.5\pm648.6j  | 137.06            | 103.22            | 0.658   |
|                 | -566.5\pm648.6j  | 137.06            | 103.22            | 0.658   |
|                 | -657.3\pm834.8j  | 169.10            | 132.86            | 0.619   |
|                 | -657.3\pm834.8j  | 169.10            | 132.86            | 0.619   |
| 4000            | -4.083\pm85.97j  | 13.70             | 13.68             | 0.048   |
|                 | -4.742\pm88.41j  | 14.09             | 14.07             | 0.054   |
|                 | -74.10\pm263.8j  | 43.61             | 41.98             | 0.270   |
|                 | -78.81\pm318.2j  | 52.18             | 50.65             | 0.240   |
|                 | -402.6\pm655.0j  | 122.37            | 104.25            | 0.524   |
|                 | -667.2\pm663.9j  | 149.81            | 105.66            | 0.709   |
|                 | -609.5\pm686.5j  | 168.87            | 138.23            | 0.574   |
|                 | -694.7\pm818.2j  | 170.82            | 130.22            | 0.647   |

Case 2: When the rotor spin speed is 4000 rev/min.
Unlike case 1, the eigen values are not identical. The mode shapes occur in non-identical pairs as shown in figure 2 for a speed of 4000rev/min. With increasing spin speed, each damped natural frequency pair diverges. The one that increases is called forward whirl frequency(FW) whereas the one that decreases is called backward whirl frequency(BW).

![Figure 2. Mode shapes of the rotor at 4000 rev/min](image)

Figure 2. Mode shapes of the rotor at 4000 rev/min

Figure 3. The axial view of the mode shapes of the left disc(node 3, solid line) and right disc( node 5, dashed line) for the rotor at 4000 rev/min (Cross denotes the start of the orbit and diamond denotes the end)
Figure 4 shows the Campbell diagram indicating the relation between damped natural frequency (Hz) and rotor spin speed (rev/min). The Campbell diagram also helps finding the critical speeds associated with each pair of diverging (i.e. the forward and backward) natural frequencies as shown in Table 3.

![Campbell diagram of the rotor system of model 2](image)

**Figure 4. Campbell diagram of the rotor system of model 2**

![Campbell diagram of the rotor system of model 1](image)

**Figure 5. Campbell diagram of the rotor system of model 1**

| Mode no. | Critical speed (rev/min) |
|----------|-------------------------|
| Mode 1   | 416                     |
| Mode 2   | 417.2                   |
| Mode 3   | 1351                    |
| Mode 4   | 1443                    |
| Mode 5   | 3130                    |
| Mode 6   | 3144                    |
| Mode 7   | 3908                    |
| Mode 8   | 4154                    |

**Table 3. Critical speed values for model 2**

| Mode no. | Critical speed (rev/min) |
|----------|-------------------------|
| Mode 1   | 413                     |
| Mode 2   | 414                     |
| Mode 3   | 1270                    |
| Mode 4   | 1340                    |
| Mode 5   | 3000                    |
| Mode 6   | 3940                    |
| Mode 7   | >4500                   |
| Mode 8   | >4500                   |

**Table 4. Critical speed values for model 1**

4. **Comparison**

1. At zero rotor speed, the mode shape diagrams for both the models are same i.e. they are identical in xz and yz plane as shown in [18]. Hence the figures for this model in the stationary case is not shown here.

2. With increasing speed, the natural frequency pair diverges (i.e. Forward and backward whirls) in both the cases. For comparison, the eigen values and natural frequencies of the rotor for model 1 are shown in table 2.

3. The Campbell diagram shows that there is a better control over the critical speed range in model 2 as compared to model 1. Figure 5 shows the Campbell diagram and table 4 shows the critical speed range for model 1. The critical speed values for mode 7 and 8 in table 4 is beyond the range of spin speed and hence not mentioned.

5. **Conclusion**

In this paper, the modal characteristics of a double disc flexible rotor system incorporated with gyroscopic effects and supported by isotropic bearings having stiffness and damping properties is studied. Also a comparative study has been conducted between the model described in this paper (model 2) and the one already described in [18] (Model 1). The following conclusions can be drawn.
1. As it is evident from both the models, the dynamic performance is not influenced by gyroscopic effects at 0 rev/min. It is prominent at higher spin speeds only. It is the gyroscopic couple which causes the frequency pairs to split into forward whirl and backward whirl.

2. The mode shape diagrams in both the models show identical behaviour at 0 rev/min as shown in [18]. As the spin speed increases, the mode shapes form circular paths as shown in figure 2 with relative amplitude varying along the shaft. The fact that the orbitals are circular as shown in figure 3 is due to the absence of cross-coupling terms in the stiffness and damping matrices.

3. The critical speed range for model 1 as shown in table no. 4 indicates a wider span as compared to the table no. 3 for model 2. It suggests that bearings represented as both stiffness and damping can be used for better control over critical speed response of the system. As it is quite evident that the rotor system experiences large oscillations during the passage of the critical speeds, designers are to be strictly concerned about the range while designing the rotor system so that the general operation is not bothered in any circumstances.

6. Appendix

| Symbol | Description |
|--------|-------------|
| $c_{uu}$, $c_{vv}$ | Damping at the bearings in x and y directions (kNs/m) |
| $D_s$ | Diameter of the shaft/rotor (m) |
| $D_{dL}$, $D_{dR}$ | Diameter of the left disc(m), Diameter of the right disc (m) |
| $E$, $E_e$ | Young’s modulus of the shaft material(GN/m²), Young’s modulus of the beam elemental (GN/m²) |
| $f_x$, $f_y$ | Dynamic forces in x and y directions (N) |
| $G$ | Rigidity modulus of the shaft material(GN/m²) |
| $I_d$, $I_p$ | Diametral moment of inertia of the disc(m²), Polar moment of inertia of the disc(m⁴) |
| $K$ | Stiffness matrix |
| $k_{uu}$, $k_{vv}$ | Stiffness of spring at the bearings in x and y directions (MN/m) |
| $L_s$ | Length of the shaft/rotor (m) |
| $M$ | Mass matrix |
| $m_d$ | Mass of the disc (kg) |
| $q$ | Displacement and rotations at the nodes |
| $T_d$ | Total kinetic energy of the disc |
| $t$ | Thickness of both the discs (m) |
| $u, v$ | Dynamic displacements of the shaft journal relative to the bearing housing (m) |
| $u_e$ | Lateral displacement(m) |
| $\dot{u}$, $\dot{v}$ | Velocities in x and y directions (m/sec) |
| $\rho$ | Density of both shaft and disc(kg/m³) |
| $\psi_e$ | Angle of the beam cross-section |
| $\omega_x$, $\omega_y$, $\omega_z$ | Instantaneous angular velocities in x, y and z directions(rad/sec) the axes fixed on the disc |
| $\phi_e$ | Constant (dimensionless) |
| $\dot{\theta}$, $\dot{\psi}$ | Instantaneous angular velocities about x and y axes(rad/sec) the axes are fixed in space |
| $\Omega$ | Disc rotational speed (rad/sec) |
| $\rho_e$, $A_e$, $I_e$ | Density(kg/m³), Area(m²), Length(m) of a beam element |
| $\omega_n$ | Natural frequency (Hz) |
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