On the Correlation Functions of the Characteristic Polynomials of Random Matrices with Independent Entries: Interpolation Between Complex and Real Cases

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Abstract

The paper is concerned with the correlation functions of the characteristic polynomials of random matrices with independent complex entries. We investigate how the asymptotic behavior of the correlation functions depends on the second moment of the common probability law of the matrix entries, a sort of “reality measure” of the entries. It is shown that the correlation functions behave like that for the Complex Ginibre Ensemble up to a factor depending only on the second moment and the fourth absolute moment of the common probability law of the matrix entries.

Key words: Random matrix theory, Ginibre ensemble, correlation functions of characteristic polynomials, moments of characteristic polynomials, SUSY.

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1 Introduction

The ensemble of random matrices with independent entries was introduced by Ginibre in 1965 [32]. To be exact, he introduced a partial case when entries of the matrices have Gaussian distribution. Anyway, the ensemble appeared to be significant and has been attracting scientists’ attention since that time.

Random matrices with independent entries are usually considered over three fields: complex numbers, real numbers and quaternions. An asymptotic behavior of the correlation functions of the characteristic polynomials was recently computed in the complex case [2] and in the real case [3]. The goal of the current article is to obtain a similar result in the intermediate case between the complex and the real ones.

Let us proceed to precise definitions. We consider the matrices of the form

\[ M_n = \frac{1}{\sqrt{n}} X = \frac{1}{\sqrt{n}} (x_{jk})^n_{j,k=1}, \]

(1.1)

where \( x_{jk} \) are i.i.d. complex random variables such that

\[ E\{x_{jk}\} = 0, \quad E\{|x_{jk}|^2\} = 1, \quad E\{x_{jk}^2\} =: \kappa_{2,0}. \]

(1.2)

Here and everywhere below \( E \) denotes an expectation with respect to (w.r.t.) all random variables. In the particular case if the entries \( x_{jk} \) are complex or real Gaussian this ensemble
is known as Complex or Real Ginibre Ensemble respectively (Gin(\mathbb{C}) resp. Gin(\mathbb{R})). The parameter \( \kappa_{2,0} \) plays a role of a “reality measure”. Indeed, on the one hand \( \kappa_{2,0} = 0 \) in the complex case. On the other hand \( \kappa_{2,0} = 1 \) in the real case.

Notice that the ensemble (1.1) has various applications in physics, neuroscience, economics, etc. For detailed information see [4] and references therein.

Define the Normalized Counting Measure (NCM) of eigenvalues as

\[
N_n(\Delta) = \#\{\lambda^{(n)}_j \in \Delta, \ j = 1, \ldots, n\}/n,
\]

where \( \Delta \) is an arbitrary Borel set in the complex plane, \( \{\lambda^{(n)}_j\}^n_{j=1} \) are the eigenvalues of \( M_n \).

The NCM is known to converge to the uniform distribution on the unit disc. This distribution is called the circular law. This result has a long and rich history. Mehta was the first who obtained it for \( x_{jk} \) being complex Gaussian in 1967 [42]. The proof strongly relied on the explicit formula for the common probability density of eigenvalues due to Ginibre [32].

Unfortunately, there is no such a formula in the general case. That is why other methods have to be used. The Hermitization approach introduced by Girko [33] appeared to be an effective method. The main idea is to reduce the study of matrices (1.1) to the study of Hermitian matrices using the logarithmic potential of a measure

\[
P_\mu(z) = \int_{\mathbb{C}} \log |z - \zeta| \ d\mu(\zeta).
\]

This approach was successfully developed by Girko in the next series of works [34–37]. The final result in the most general case was established by Tao and Vu [56]. Notice that there are a lot of partial results besides those listed above. The interested reader is directed to [8].

The Central Limit Theorem (CLT) for linear statistics of non-Hermitian random matrices of the form (1.1) was first proven for radial-invariant test functions in the complex case by Forrester [26]. The study was continued in the complex case by Rider and Silverstein [47], Rider and Virag [48], in the real case by O’Rourke and Renfrew [45], in both cases by Tao and Vu [57] and Kopel [40]. The best result for today was obtained by Cipolloni, Erdős and Schröder for the complex case in [14] and for the real case in [15]. They proved CLT for a bit more than twice differentiable test functions assuming that the common distribution of matrix entries has finite moments.

A local regime for matrices (1.1) is worse studied. The asymptotic behavior of the \( k \)-point correlation function for Ginibre ensembles is well-known, see [32,42] for Gin(\mathbb{C}) and [9,22,25] for Gin(\mathbb{R}). A general distribution case was considered in [57]. It was established in both cases that the \( k \)-point correlation function converges in vague topology to that for Ginibre ensemble if \( x_{jk} \) having the first four moments as in the Gaussian case. The condition of matching moments was recently overcome at the edge of the spectrum (i.e. at \( |z| = 1 \)) in [16]. The last result strongly relies on an estimate for the least singular value obtained in [13] using the supersymmetry technique (SUSY).

One can observe that non-Hermitian random matrices are more complicated than their Hermitian counterparts. Indeed, the Hermitian case was successfully dealt with using the Stieltjes transform or the moments method. However, a measure in the plane can not be recovered from its Stieltjes transform or its moments. Thus these approaches to the analysis fail in the non-Hermitian case.

The present article suggests to use the SUSY. It is a rather powerful method which is widely applied at the physical level of rigor (for instance [11,12,27,29,31,44,60]). There are also a lot of rigorous results, which were obtained using SUSY in the recent years, e.g.
Supersymmetry technique is usually used in order to obtain an integral representation for ratios of determinants. Since the main spectral characteristics such as density of states, spectral correlation functions, etc. often can be expressed via ratios of determinants, SUSY allows to get the integral representation for these characteristics too. For detailed discussion on connection between spectral characteristics and ratios of determinants see \cite{10,18,55}. See also \cite{31,46}.

Let us consider the second spectral correlation function \( R_2 \) defined by the equality

\[
E \left\{ 2 \sum_{1 \leq j_1 < j_2 \leq n} \eta \left( \lambda_{(n)}^{(j_1)}, \lambda_{(n)}^{(j_2)} \right) \right\} = \int_{\mathbb{C}^2} \eta(\lambda_1, \lambda_2)R_2(\lambda_1, \lambda_2)d\lambda_1d\lambda_2d\lambda_2d\lambda_2,
\]

where the function \( \eta: \mathbb{C}^2 \to \mathbb{C} \) is bounded, continuous and symmetric in its arguments. Using the logarithmic potential, \( R_2 \) can be represented via ratios of the determinants of \( M_n \) with the most singular term of the form

\[
\int_0^{\varepsilon_0} \int_0^{\varepsilon_0} \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2} E \left\{ \prod_{j=1}^2 \det \left( (M_n - z_j)(M_n - z_j)^* + \delta_j \right) \right\} \bigg|_{\delta = \varepsilon} d\varepsilon_1 d\varepsilon_2
\]

The integral representation for (1.3) obtained by SUSY will contain both commuting and anti-commuting variables. Such type integrals are rather difficult to analyse. That is why one would investigate a more simple but similar integral to shed light on the situation. This integral arises from the study of the correlation functions of the characteristic polynomials. Moreover, the correlation functions of the characteristic polynomials are of independent interest. They were studied for many ensembles of Hermitian and real symmetric matrices, for instance \cite{1,11,12,51,53,54} etc. The other result on the asymptotic behavior of the correlation functions of the characteristic polynomials of non-Hermitian matrices of the form \( H + i\Gamma \), where \( H \) is from Gaussian Unitary Ensemble (GUE) and \( \Gamma \) is a fixed matrix of rank \( M \), was obtained in \cite{28}. The kernel computed there, in the limit of rank \( M \to \infty \) of the perturbation \( \Gamma \) (taken after matrix size \( n \to \infty \)) after appropriate rescaling approaches the form (1.8). It was demonstrated in \cite[Sec. 2.2]{21}.

Let us introduce the \( m^{th} \) correlation function of the characteristic polynomials

\[
f_m(Z) = E \left\{ \prod_{j=1}^m \det (M_n - z_j)(M_n - z_j)^* \right\},
\]

where

\[
Z = \text{diag}\{z_1, \ldots, z_m\}
\]

and \( z_1, \ldots, z_m \) are complex parameters which may depend on \( n \). We are interested in the asymptotic behavior of (1.4), as \( n \to \infty \), for

\[
z_j = z_0 + \frac{\zeta_j}{\sqrt{n}}, \quad j = 1,2,\ldots,m,
\]

where \( z_0 \) is either in the bulk (\( |z_0| < 1 \)) or at the edge (\( |z_0| = 1 \)) of the spectrum and \( \zeta_1, \ldots, \zeta_m \) are \( n \)-independent complex numbers. The functions (1.4) are well-studied for the Complex Ginibre Ensemble, see \cite{5,59}. A general distribution case was considered in \cite{2,3}. It was showed that in the complex case for any \( z_0 \) in the unit disk

\[
\lim_{n \to \infty} n^{-\frac{2m}{2}2m} \frac{f_m(Z)}{f_1(z_1)\cdots f_1(z_m)} = e^{m^2(1-|z_0|^2)^2} \kappa_{2,2,2} \left. \frac{\det(K_C(\zeta_j, \zeta_k))_{j,k=1}^n}{|\triangle(Z)|^2} \right|_{\zeta_j = z_0, \zeta_k \to \infty}.
\]
where $\kappa_{2,2} = E[|x_{11}|^4] - |E[x_{11}^2]|^2$ and

$$K_C(z, w) = e^{-|z|^2/2-|w|^2/2+zw},$$

$$Z = \text{diag}\{\zeta_1, \ldots, \zeta_m\},$$

and $\Delta(Z)$ is a Vandermonde determinant of $\zeta_1, \ldots, \zeta_m$. Whereas in the real case for any $z_0 \in [-1, 1]$,

$$\lim_{n \to \infty} n^{-\frac{1}{2}} \frac{f_2(Z)}{f_1(z_1)f_1(z_2)} = C e^{(1-|z_0|^2)^2\kappa_{2,2}} \frac{\text{Pf}(K_R(\zeta_j, \zeta_k))_{j,k=1}^n}{\Delta(\zeta_1, \zeta_2, \zeta_1, \zeta_2)},$$

where

$$K_R(\zeta_j, \zeta_k) = e^{-\frac{|\zeta_j|^2}{2}} \frac{|\zeta_k|^2}{2} \left(\left(\zeta_j - \zeta_k\right)e^{\zeta_j^*\zeta_k} - \left(\zeta_j - \zeta_k\right)e^{\zeta_j\zeta_k*}\right).$$

In the current paper we extend the results of [2, 3] to the case of arbitrary $\kappa_{2,0}$, $|\kappa_{2,0}| \leq 1$. The main result is

**Theorem 1.** Let an ensemble of real random matrices $M_n$ be defined by (1.1) and (1.2). Let the first $2m$ moments ($m > 1$) of the common distribution of entries of $M_n$ be finite and $z_j, j = 1, \ldots, m$, have the form (1.6). Let also $z_0$ and $\kappa_{2,0}$ satisfy at least one of two following conditions

(i) $|\kappa_{2,0}| < 1$ and $|z_0| < 1$;

(ii) $|\kappa_{2,0}| = 1$ and $|z_0| < 1, z_0 \notin \mathbb{R}$.

Then the $m^{th}$ correlation function of the characteristic polynomials (1.4) satisfies the asymptotic relation

$$\lim_{n \to \infty} n^{-\frac{1}{2} - m} \frac{f_m(Z)}{f_1(z_1) \cdots f_1(z_m)} = C_{m,z_0} e^{d(\kappa_{2,0}, \kappa_{2,2})} \frac{\text{det}(K_C(\zeta_j, \zeta_k))_{j,k=1}^n}{|\Delta(Z)|^2},$$

where $C_{m,z_0}$ is some constant, which does not depend on the common distribution of entries and on $\zeta_1, \ldots, \zeta_m$; $\kappa_{2,2} = E[|x_{11}|^4] - |E[x_{11}^2]|^2$,

$$d(\kappa_{2,0}, \kappa_{2,2}) = -m \log \left\{ \frac{1}{z_0^2} \right\}^2 - |\kappa_{2,0}|^2 (1 - |z_0|^2)^2 \right\},$$

$$+ \frac{m^2 - m}{2} (1 - |z_0|^2)^2 \kappa_{2,2},$$

$\Delta(Z)$ is a Vandermonde determinant of $\zeta_1, \ldots, \zeta_m$, and $K_C(z, w)$ is defined in (1.8).

Note that (1.10) has an additional factor compared with (1.7). This factor shows the dependence of the asymptotics of $f_m$ (here and below we omit $Z$ only if $Z = \text{diag}\{z_1, \ldots, z_m\}$) on $\kappa_{2,0}$.

The paper is organized as follows. Section 2 discusses a suitable integral representation for $f_m$. In Section 3 we apply the steepest descent method to the suitable integral representation and find out the asymptotic behavior of $f_m$. In order to compute it, the Harish-Chandra/Itskovsk–Zuber formula is used. For the reader convenience the latter section is divided into two parts. The first part deals with a simpler partial case whereas the second one treats a general case.

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| Set of matrices | Matrix | Vector | Entry |
|----------------|--------|--------|-------|
| $Q$            | $Q_{p,s}$ | $\phi$ | $\phi_j$ |
| $\theta$      | $\theta_j$ |
| $Y_{p,s}$      | $g_{\alpha\beta}^{(p,s)}$ |
| $U$            | $u_{kj}$ |
| $V$            | $v_{kj}$ |

Table 1.1: Notation correspondence

1.1 Notations

Throughout the article lower-case letters denote scalars, bold lower-case letters denote vectors, upper-case letters denote matrices and bold upper-case letters denote sets of matrices. We use the same letter for a matrix, for its columns and for its entries. Table 1.1 shows the exact correspondence. Besides, for any matrix $A$ we denote by $(A)_j$ its $j$-th column and by $(A)_{kj}$ its entry in the $k$-th row and in the $j$-th column.

The term “Grassmann variable” is a synonym for “anti-commuting variable”. The variables of integration $\phi, \theta$ and $\rho$ are Grassmann variables, all the other variables of integration unspecified by an integration domain are either complex or real. We split all the generators of Grassmann algebra into two equal sets and consider the generators from the second set as “conjugates” of that from the first set. I.e., for Grassmann variable $\upsilon$ we use $\upsilon^*$ to denote its “conjugate”. Furthermore, if $\Upsilon = (\upsilon_{jk})$ means a matrix of Grassmann variables then $\Upsilon^+$ is a matrix $(\upsilon_{kj}^*)$. $d$-dimensional vectors are identified with $d \times 1$ matrices.

Integrals without limits denote either integration over Grassmann variables or integration over the whole space $\mathbb{C}^d$ or $\mathbb{R}^d$. Let also $dt^* dt$ ($t = (t_1, \ldots, t_d)^T \in \mathbb{C}^d$) denote the measure $\prod_{j=1}^d dt_j dt_j$ on the space $\mathbb{C}^d$. Similarly, for vectors with anti-commuting entries $d\tau^* d\tau = \prod_{j=1}^d d\tau_j^* d\tau_j$. Note that the space of matrices is a linear space over $\mathbb{C}$. Thus the same notations are used for matrices as well.

$\langle \cdot, \cdot \rangle$ denotes a standard scalar product on $\mathbb{C}^d$. For matrices $\langle A, B \rangle = \operatorname{tr} B^* A$. For sets of matrices $\langle A, B \rangle = \sum_j \langle A_j, B_j \rangle$. The norm we use is defined by $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$.

$(m \choose p) \times (m \choose s)$ matrices appear in the statement of Proposition 2. It is natural to number rows and columns of such matrices by subsets of a $m$-element set. To this end, set

$$I_{m,p'} = \{ \alpha \in \mathbb{Z}^{p'} \mid 1 \leq \alpha_1 < \ldots < \alpha_{p'} \leq m \}. \quad (1.12)$$

If $p' = 0$ we define $I_{m,0'}$ as $\{ \emptyset \}$.

The cumulants $\kappa_{p,s}$ are defined as follows. Consider the function

$$\psi(t_1, t_2) := \mathbb{E} \{ e^{t_1 x_{11} + t_2 \Upsilon_{11}} \}.$$ 

Then

$$\kappa_{p,s} = \frac{\partial^{p+s}}{\partial t_1^p \partial t_2^s} \log \psi(t_1, t_2) \bigg|_{t_1 = t_2 = 0}. \quad (1.13)$$

In particular, $\kappa_{2,2} = \mathbb{E}\{ |x_{11}|^4 \} - \left| \mathbb{E}\{ x_{11}^2 \} \right|^2 - 2$. 

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Throughout the article $U(m)$ denotes a group of unitary $m \times m$ matrices. $\mu$ denotes a corresponding Haar measure. In addition, $C$, $C_1$ denote various $n$-independent constants which can be different in different formulas.

## 2 Integral representation for $f_m$

The following integral representation is true

**Proposition 2.** Let an ensemble $M_n$ be defined by (1.1) and (1.2). Then the $m^{th}$ correlation function of the characteristic polynomials $f_m$, defined by (1.4) can be represented in the following form

$$f_m = \left( \frac{n}{\pi} \right)^{cm} \int g(Q)e^{(n-cm)f(Q)} dQ,$$

where $c_m = 2^{2m-1}$, $Q = (Q_j)_{j=0}^m$, $Q_j = \{Q_{p,s} \mid p + s = 2j, 0 \leq p, s \leq m\}$, $Q_{p,s}$ is a complex $(m \times (m)$ matrix, $dQ = \prod_{p+s \text{ is even}}^{m} dQ_{p,s} dQ_{p,s}$ and

$$f(Q) = -(Q, Q) + \log h(Q);$$

$$g(Q) = (h(Q)^{cm} + n^{-1/2} p_n(Q)) \exp \{-c_m (Q, Q)\};$$

$$h(Q) = \text{Pf} F(Q_1) + n^{-1/2} \tilde{h}(Q_1, Q_2) + n^{-1} \text{Pc}(Q_1, Q_{>1});$$

$$F(Q_1) = \begin{pmatrix} \sqrt{2} B_{2,0} & 0 & -Z \sqrt{2} B_{2,0}^* & Q_1 \\ Z & \sqrt{2} B_{0,2} & -Q_1^* & -Z^* \\ -Q_1^* & -Q_1 & 0 & \sqrt{2} B_{0,2}^* \\ Z^* & \sqrt{2} B_{0,2}^* & 0 & \sqrt{2} B_{2,0} \end{pmatrix}.$$ (2.4)

$B_{2,0}$ and $B_{0,2}$ are skew-symmetric matrices such that

$$(B_{2,0})_{a_1a_2} = -q_{a_0}^{(2,0)}, \quad (B_{0,2})_{a_1a_2} = -q_{a_0}^{(0,2)}, \quad a \in \mathcal{I}_{m,2}$$

and $\mathcal{I}_{m,2}$ is defined in (1.12). Moreover,

$$\tilde{h}(Q_1, Q_2) = -\int \sum_{p+s=4} \left( \text{tr} Y_{p,s} Q_{p,s} + \text{tr} Q_{p,s} Y_{p,s} \right) e^{-\frac{i}{2} \rho^T F \rho} d\phi^* d\phi d\theta^* d\theta, \quad (2.5)$$

$$\rho = (\phi^* \theta^* \phi^T \theta^T)^T,$$

$$y^{(p,s)}_{\alpha \beta} = \sqrt{p_{\alpha \beta}} (-1)^p \prod_{r=s}^{1} \theta_{\beta r} \prod_{q=p}^{1} \phi_{\alpha q}, \quad (2.6)$$

$$y^{(p,s)}_{\alpha \beta} = \sqrt{p_{\alpha \beta}} \prod_{q=1}^{p} \phi_{\alpha q} \prod_{r=1}^{s} \theta_{\beta r}. \quad (2.7)$$

$k_{p,s}$ are defined in (1.13), $p_n(Q)$ and $p_c(Q_1, Q_{>1})$ are certain polynomials such that $p_c(Q_1, 0) = 0$, and $Q_{>1}$ contains all $Q_j$ except $Q_1$.

**Proof.** Proposition 2 was proved for the case $k_{2,0} = 1$ in [3, Proposition 2.1]. The most part of the provided proof goes in the frames of a general case, and only in the very end $k_{2,0} = 1$ is substituted. Therefore it is easy to understand from [3] that the only distinction of the general case from the partial one is in the presence of $k_{2,0}$ in (2.4).
Remark 3. Let $Q_1 = U \Lambda V^*$ be the singular value decomposition of the matrix $Q_1$, i.e. $\Lambda = \text{diag}(\lambda_j)_{j=1}^m$, $\lambda_j \geq 0$, $U, V \in U(m)$. In order to perform asymptotic analysis let us change the variables $Q_1 = U \Lambda V^*$, $B_{2,0} \rightarrow UB_{2,0}U^T$, $B_{0,2} \rightarrow \nabla B_{0,2}V^*$ in (2.1). Since the Jacobian is $\frac{2^n \pi^{n^2}}{(\prod_{j=1}^{m-1} j!)^2} \Delta^2(\Lambda^2) \prod_{j=1}^m \lambda_j$ (see e.g. [39]) we obtain

$$f_m = C n^{c_m} \int_\mathcal{D} \Delta^2(\Lambda^2) \prod_{j=1}^m \lambda_j \left[ g_0(\Lambda, \hat{Q}) + \frac{1}{\sqrt{n}} g_r(U \Lambda V^*, \hat{Q}) \right]$$

$$\times \exp \left\{ (n - c_m) \left[ f_0(\Lambda, \hat{Q}) + \frac{1}{\sqrt{n}} f_r(U \Lambda V^*, \hat{Q}) \right] \right\}$$

$$\times d\mu(U)d\mu(V)d\lambda d\hat{Q},$$

where $\hat{Q}$ contains all the matrices $Q_{p,s}$ except $Q_1$, $\mathcal{D} = \{(\Lambda, U, V, \hat{Q}) \mid \lambda_j \geq 0, j = 1, \ldots, m, U, V \in U(m)\}$, $\mu$ is a Haar measure, $d\lambda = \prod_{j=1}^m d\lambda_j$ and

$$f_0(Q) = -(Q, Q) + \log h_0(Q_1);$$

$$g_0(Q) = h_0(Q_1)^{c_m} \exp \{-c_m(Q, Q)\} = e^{c_m f_0(Q)};$$

$$h_0(Q_1) = \text{Pf} \tilde{F}(Q_1), \quad \tilde{F}(Q_1) := \begin{pmatrix} B_{2,0} & 0 & -z_0 I_m & \Lambda \\ 0 & B_{0,2}^* & -\Lambda & -z_0 I_m \\ z_0 I_m & \Lambda & B_{2,0}^* & 0 \\ -\Lambda & -z_0 I_m & 0 & B_{0,2} \end{pmatrix};$$

$$f_r(Q) = \sqrt{n}(f(Q) - f_0(Q));$$

$$g_r(Q) = \sqrt{n}(g(Q) - g_0(Q));$$

$B_{2,0} = \sqrt{\kappa_{2,0}} B_{2,0}$ and $B_{0,2} = \sqrt{\kappa_{2,0}} B_{0,2}$. Notice that $f_0(U \Lambda V^*, \hat{Q}) = f_0(\Lambda, \hat{Q})$ and the same for $g_0$.

Remark 4. In the special case $m = 1$ the matrices $B_{2,0}$ and $B_{0,2}$ are zeros and we have

$$f_1(z) = \frac{n}{\pi} \int \exp \left\{ n(-|q|^2 + \log(|z|^2 + |q|^2)) \right\} dq dq.$$

Changing variables to polar coordinates and performing a simple Laplace integration, we obtain

$$f_1(z) = 2n \int_{0}^{+\infty} r \exp \left\{ n(-r^2 + \log(|z|^2 + r^2)) \right\} dr$$

$$= \sqrt{2\pi} n e^{|z|^2}(1 + o(1)).$$

3 Asymptotic analysis

The goal of the section is to investigate the asymptotic behavior of the integral representation (2.8). To this end, the steepest descent method is applied. As usual, the hardest step is to choose stationary points of $f(Q)$ and a $N$-dimensional (real) manifold $M_s \subset \mathbb{C}^N$ such that for any chosen stationary point $Q_\ast \in M_s$

$$\Re f(Q) < \Re f(Q_\ast), \quad \forall Q \in M_s, \ Q \text{ is not chosen.}$$
Note that \( N \) is equal to the number of real variables of the integration, i.e. in our case \( N = 2^{2m} \).

The present proof proceeds by a standard scheme for the case when function \( f(Q) \) has the form

\[
f(Q) = f_0(Q) + n^{-1/2} f_r(Q),
\]

where \( f_0(Q) \) does not depend on \( n \), whereas \( f_r(Q) \) may depend on \( n \). We choose stationary points of \( f_0(Q) \) of the form \( Q_1 = U \lambda_0 V^* \), \( Q = 0 \), where \( \lambda_0 = \lambda_0 I \), \( \lambda_0 = \sqrt{1 - |z_0|^2} \) and \( U, V \) vary in \( U(m) \). The manifold \( M_n \) is \( \mathbb{R}^N \). Then the steepest descent method is applied to the integral over \( \Lambda \) and \( Q \). In the process \( U \) and \( V \) are considered as parameters and all the estimates are uniform in \( U \) and \( V \). As soon as the domain of integration is restricted by a small neighborhood we recall about the integration over \( U \) and \( V \). After several changes of the variables the integral is reduced to the form (1.10).

We start with an analysis of the function \( f_0 \).

**Lemma 5.** Let the function \( f_0 : \mathbb{R}^{2^{2m}} \to \mathbb{C} \) be defined by (2.9). Then the function \( \Re f_0(\Lambda, \hat{Q}) \) attains its global maximum value only at the point

\[
\lambda_1 = \cdots = \lambda_m = \lambda_0, \quad \hat{Q} = 0,
\]

where \( \lambda_0 = \sqrt{1 - |z_0|^2} \).

**Proof.** From (2.9) and (2.10) we get

\[
\Re f_0(\Lambda, \hat{Q}) = - \sum_{j \neq 1} (Q_j, Q_j) - (Q_1, Q_1) + \frac{1}{2} \log |\det \tilde{F}| \leq - (Q_1, Q_1) + \frac{1}{2} \log |\det \tilde{F}|. \tag{3.2}
\]

Hadamard’s inequality yields

\[
\frac{1}{2} \log |\det \tilde{F}| \leq \frac{1}{2} \log \left\{ \prod_{j=1}^{m} \left( |z_0|^2 + \lambda_j^2 + |\kappa_2,0| \sum_{k=1}^{m} |q_{(j,k)\varnothing}^{(2,0)}| \right)^{\frac{1}{2}} \right\}
\]

\[
\times \left( |z_0|^2 + \lambda_j^2 + |\kappa_2,0| \sum_{k=1}^{m} |q_{(j,k)\varnothing}^{(2,0)}| \right)^{\frac{1}{2}} \tag{3.3}
\]

\[
\times \left( |z_0|^2 + \lambda_j^2 + |\kappa_2,0| \sum_{k=1}^{m} |q_{(j,k)\varnothing}^{(2,0)}| \right)^{\frac{1}{2}}.
\]

where \( q_{(j,k)\varnothing}^{(2,0)} = -q_{(k,j)\varnothing}^{(2,0)} \), \( q_{(j,k)\varnothing}^{(0,2)} = -q_{(k,j)\varnothing}^{(0,2)} \) for \( j > k \) and \( q_{(j,j)\varnothing}^{(2,0)} = q_{(j,j)\varnothing}^{(0,2)} = 0 \). Simplifying the r.h.s. of (3.3) and taking into account (3.2) we obtain

\[
\Re f_0(\Lambda, \hat{Q}) \leq - (Q_1, Q_1) + \frac{1}{2} \sum_{j=1}^{m} \log \left\{ \left( |z_0|^2 + \lambda_j^2 + |\kappa_2,0| \sum_{k=1}^{m} |q_{(j,k)\varnothing}^{(2,0)}| \right)^{2} \right\} \tag{3.4}
\]

\[
\times \left( |z_0|^2 + \lambda_j^2 + |\kappa_2,0| \sum_{k=1}^{m} |q_{(j,k)\varnothing}^{(2,0)}| \right)^{\frac{1}{2}} \right\},
\]

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The inequality \( \log x \leq x - 1 \) and (3.4) imply
\[
\Re f_0(\Lambda, Q) \leq -\langle Q_1, Q_1 \rangle + \frac{1}{2} \sum_{j=1}^m \left( \left| z_0 \right|^2 + \lambda_j^2 + \left| \kappa_{2,0} \right| \sum_{k=1}^m \left| q_{(j,k)2}^{(2,0)} \right|^2 \right)
\[
\quad + \left( \left| z_0 \right|^2 + \lambda_j^2 + \left| \kappa_{2,0} \right| \sum_{k=1}^m \left| q_{(j,k)2}^{(0,2)} \right|^2 \right) - 2 \right)
\]
\[
= -\langle Q_1, Q_1 \rangle + m \left| z_0 \right|^2 - m + \sum_{j=1}^m \lambda_j^2 + \left| \kappa_{2,0} \right| \sum_{\alpha \in I_m,2} \left| q_{\alpha,2}^{(0,2)} \right|^2
\]
\[
+ \left| \kappa_{2,0} \right| \sum_{\alpha \in I_m,2} \left| q_{\alpha,2}^{(0,2)} \right|^2.
\]
(3.5)

Finally, since \( \left| \kappa_{2,0} \right| = \left| \mathbf{E}\{x_{11}^2\} \right| \leq \mathbf{E}\{|x_{11}|^2\} = 1 \), we have
\[
\Re f_0(\Lambda, Q) \leq -\langle Q_1, Q_1 \rangle + \sum_{j=1}^m \lambda_j^2 + \sum_{\alpha \in I_m,2} \left\{ \left| q_{\alpha,2}^{(2,0)} \right|^2 + \left| q_{\alpha,2}^{(0,2)} \right|^2 \right\}
\]
\[
+ m \left| z_0 \right|^2 - m = -\langle Q_1, Q_1 \rangle + \langle Q_1, Q_1 \rangle + m \left| z_0 \right|^2 - m = m(\left| z_0 \right|^2 - 1). \quad (3.6)
\]

Therefore, the function \( \Re f_0(\Lambda, Q) \) attains its global maximum value at the point (3.1). It remains to show that there is no other point for which \( \Re f_0(\Lambda, Q) = m(\left| z_0 \right|^2 - 1) \). Indeed, equality in (3.2) is attained if and only if \( q_{21} = 0 \). Moreover, the r.h.s. of (3.6) and (3.5) are equal if and only if \( \left| \kappa_{2,0} \right| = 1 \) or \( q_{\alpha,2}^{(2,0)} = q_{\alpha,2}^{(0,2)} = 0 \). Let us consider the following two cases.

1. \( \left| \kappa_{2,0} \right| < 1 \implies q_{\alpha,2}^{(2,0)} = q_{\alpha,2}^{(0,2)} = 0 \) for all \( \alpha \in I_{m,2} \).

Since the equality \( \log x = x - 1 \) holds if and only if \( x = 1 \), then we obtain from the equality of the r.h.s. of (3.4) and (3.5) that
\[
\left| z_0 \right|^2 + \lambda_j^2 + \left| \kappa_{2,0} \right| \sum_{k=1}^m \left| q_{(j,k)2}^{(2,0)} \right|^2 = 1.
\]

Thus for any \( j \)
\[
\lambda_j = \sqrt{1 - \left| z_0 \right|^2}.
\]

2. \( \left| \kappa_{2,0} \right| = 1 \) and \( z_0 \not\in \mathbb{R} \).

Equality in Hadamard’s inequality is attained if and only if columns of a matrix are orthogonal vectors. Hence, if equality is attained in (3.3), then the columns of the matrix \( \tilde{F} \) are orthogonal. In particular, the orthogonality of the first and the 2m + 2\textsuperscript{nd} yields
\[-(B_{2,0})_{21} \xi_0 + z_0(B_{2,0})_{21} = 0.
\]

Since \( z_0 \neq \xi_0 \), the last identity implies
\[
q_{(1,2)2}^{(2,0)} = \frac{1}{\sqrt{\kappa_{2,0}}} (B_{2,0})_{21} = 0.
\]

Using a similar argument, we get that all \( q_{\alpha,2}^{(2,0)} \) and \( q_{\alpha,2}^{(0,2)} \) are zeros. Next, similarly to the first case we obtain \( \lambda_1 = \cdots = \lambda_m = \sqrt{1 - \left| z_0 \right|^2} \).

Totally, the assertion of the lemma is proven. \( \square \)

To simplify the reading, the remaining steps are first explained in the case when the cumulants \( \kappa_{p,s}, p + s > 2 \) are zeros.
3.1 Case of zero high cumulants

Now we proceed to the integral estimates. In a standard way the integration domain in (2.8) can be restricted as follows

\[
f_m = C n^{c_m} \int_{\Sigma_r} \frac{1}{\Delta^2(\Lambda^2)} \prod_{j=1}^m \lambda_j x g(U\Lambda V^*, Q) e^{(n-c_m)f(UAV^*, Q)} d\mu(U) d\mu(V) d\Lambda dQ
\]

+ O(e^{-nr/2}),

where

\[
\Sigma_r = \left\{ (\Lambda, U, V, Q) \mid \|\Lambda\| + \|Q\| \leq r \right\}.
\]

The next step is to restrict the integration domain by

\[
\Omega_n = \left\{ (\Lambda, U, V, Q) \mid \|\Lambda - \Lambda_0\| + \|Q\| \leq \frac{\log n}{\sqrt{n}} \right\}. \quad (3.7)
\]

To this end we need the estimate of \(\Re f\) given by the following lemmas.

**Lemma 6.** Let \(\hat{\Lambda}\) and \(\hat{Q}\) satisfy the condition \(\|\hat{\Lambda}\| + \|\hat{Q}\| \leq \log n\). Then uniformly in \(U\) and \(V\)

\[
f(U(\Lambda_0 + n^{-1/2}\hat{\Lambda})V^*, n^{-1/2}\hat{Q}) = -m\lambda_0^n + n^{-1/2} tr(\Sigma_0 Z + z_0 Z^*)
\]

\[
- \frac{1}{2n} tr(2\lambda_0 \hat{\Lambda} + \Sigma_0 Z_U + z_0 Z_V^*)^2 + \frac{1}{n} tr Z_U Z_V
\]

\[
- \frac{1}{2n} tr \left[ (1 - |\kappa_{2,0}| z_0^2) \tilde{B}_{2,0}^* \tilde{B}_{2,0} + (1 - |\kappa_{2,0}| z_0^2) \tilde{B}_{0,2}^* \tilde{B}_{0,2}
\]

\[
- |\kappa_{2,0}| \lambda_0^2 \tilde{B}_{0,2} \tilde{B}_{2,0} - |\kappa_{2,0}| \lambda_0^2 \tilde{B}_{2,0}^* \tilde{B}_{0,2}^* \right] - \frac{1}{n} \|Q_{\geq 1}\|^2
\]

+ O(n^{-3/2} \log^3 n), \quad (3.8)

where \(Z_W = W^* Z W\).

**Proof.** If \(Q_1 = U(\Lambda_0 + n^{-1/2}\hat{\Lambda})V^*\), then \(F\) has the form

\[
F = \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & V \end{pmatrix} (F_0 + \frac{1}{\sqrt{n}} F_1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & V^* & 0 & 0 \\ 0 & 0 & U^* & 0 \\ 0 & 0 & 0 & V^* \end{pmatrix},
\]

where

\[
F_0 = \begin{pmatrix} 0 & A_0 \\ -A_0^T & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} B & A \\ -A^T & B^* \end{pmatrix},
\]

\[
A_0 = \begin{pmatrix} -z_0 I_m & \Lambda_0 \\ -\Lambda_0 & -z_0 I_m \end{pmatrix}, \quad A = \begin{pmatrix} -Z_U & \hat{\Lambda} \\ -\hat{\Lambda} & -Z_V^* \end{pmatrix}, \quad B = \begin{pmatrix} \tilde{B}_{2,0} & 0 \\ 0 & \tilde{B}_{0,2} \end{pmatrix}. \quad (3.9)
\]

Taking into account that

\[
\det F_0 = \left[ \det \begin{pmatrix} -z_0 & \lambda_0 \\ -\lambda_0 & -z_0 \end{pmatrix} \right]^m = 1,
\]
one gets
\[ \log \det F = \text{tr} \log(1 + n^{-1/2} F_0^{-1} F_1) \]
\[ = \frac{1}{\sqrt{n}} \text{tr} F_0^{-1} F_1 - \frac{1}{2n} \text{tr}(F_0^{-1} F_1)^2 + O\left(\frac{\log^3 n}{\sqrt{n^3}}\right) \] (3.10)
uniformly in \( U \) and \( V \). Further,
\[ F_0^{-1} F_1 = \begin{pmatrix} (A_0^T)^{-1} A^T & -(A_0^T)^{-1} B^* \\ A_0^{-1} B & A_0^{-1} A \end{pmatrix} \]
(3.11)
and
\[ (F_0^{-1} F_1)^2 = \begin{pmatrix} \left((A_0^{-1} A)^T\right)^2 & -(A_0^T)^{-1} B^* A_0^{-1} B \\ * & -A_0^{-1} B (A_0^T)^{-1} B^* + (A_0^{-1} A)^2 \end{pmatrix}. \] (3.12)
Moreover,
\[ A_0^{-1} A = \begin{pmatrix} z_0 Z_U + \lambda_0 \bar{\Lambda} & -z_0 \bar{\Lambda} + \lambda_0 Z_V^* \\ -\lambda_0 Z_U + z_0 \bar{\Lambda} & \lambda_0 \bar{\Lambda} + z_0 Z_V^* \end{pmatrix}, \]
\[ (A_0^T)^{-1} B^* A_0^{-1} B = \begin{pmatrix} \lambda_0^2 \bar{B}_{2,0}^* \bar{B}_{0,2} + \lambda_0^2 \bar{B}_{0,2} \bar{B}_{2,0} & * \\ * & \lambda_0^2 \bar{B}_{2,0}^* \bar{B}_{0,2} + z_0 \bar{B}_{0,2} \bar{B}_{2,0} \end{pmatrix}. \] (3.13)
Combining (3.10)–(3.13) and (2.2), we get
\[
f(U(\Lambda_0 + n^{-1/2} \bar{\Lambda}) V^*, n^{-1/2} \bar{Q}) = -\text{tr} \left[ \Lambda_0^2 + 2n^{-1/2} \lambda_0 \bar{\Lambda} + n^{-1} \bar{\Lambda}^2 \right] \\
- \frac{1}{2n} \text{tr}[\bar{B}_{2,0}^* \bar{B}_{0,2} + \bar{B}_{0,2} \bar{B}_{2,0}] \left[ 2\lambda_0 \bar{\Lambda} + z_0 Z_U + z_0 Z_V^* \right] \\
- \frac{1}{\sqrt{n}} \left\| \bar{Q} \right\|_1^2 + \frac{1}{n^{1/2}} \text{tr}[2 \lambda_0 \bar{\Lambda} + z_0 Z_U + z_0 Z_V^*] \\
+ \frac{1}{2n} \left| \kappa_{2,0} \right| \text{tr}[\bar{B}_{2,0}^* \bar{B}_{0,2} + \lambda_0^2 \bar{B}_{0,2} \bar{B}_{2,0} + \lambda_0^2 \bar{B}_{2,0} \bar{B}_{0,2} + z_0 \bar{B}_{0,2} \bar{B}_{2,0}] \\
+ O(n^{-3/2} \log^3 n). \]

Hence the last expansion yields (3.8).

**Corollary 7.** Let the function \( f_0 : \mathbb{R}^{2n} \to \mathbb{C} \) be defined by (2.9). Then the following assertions are true:

(i) the point \( Q_\ast = (\Lambda_0, 0) \) is a stationary point of the function \( f_0(\Lambda, \bar{Q}) \);

(ii) the Hessian matrix of the function \( \Re f_0(\Lambda, \bar{Q}) \) (as a function of real argument) at the point \( Q_\ast \) is negative definite.

**Proof.** Let us put \( Z = 0 \) and \( p_c = 0 \). Then
\[ f_0(\Lambda, \bar{Q}) = f(\Lambda, \bar{Q}). \]
Therefore it is possible to consider the expansion (3.8) as Taylor formula for \( f_0(\Lambda, \hat{Q}) \) at the point \((\Lambda_0, 0)\). We obtain

\[
f_0(\Lambda_0 + n^{-\frac{1}{2}}\Lambda, n^{-\frac{1}{2}}\hat{Q}) = -m\lambda_0^2 - n^{-1}2\lambda_0^2 \text{ tr } \Lambda^2 - n^{-1} \|\hat{Q}_{>1}\|^2
- \frac{1}{2n} \text{ tr } \left[ (1 - |\kappa_{2,0}| z_0^2) \hat{B}_{2,0}^* \hat{B}_{2,0} + (1 - |\kappa_{2,0}| z_0^2) \hat{B}_{0,2}^* \hat{B}_{0,2}
- |\kappa_{2,0}| \lambda_0^2 \hat{B}_{2,0}^* \hat{B}_{2,0}
- |\kappa_{2,0}| \lambda_0^2 \hat{B}_{0,2}^* \hat{B}_{0,2} \right] + O(n^{-\frac{3}{2}} \log^3 n).
\]

Thus the gradient of the function \( f_0(\Lambda, \hat{Q}) \) is evidently zero at the point \((\Lambda_0, 0)\). Assertion (i) is proven. Note that

\[
\frac{1}{2n} \text{ tr } \left[ (1 - |\kappa_{2,0}| z_0^2) \hat{B}_{2,0}^* \hat{B}_{2,0} + (1 - |\kappa_{2,0}| z_0^2) \hat{B}_{0,2}^* \hat{B}_{0,2}
- |\kappa_{2,0}| \lambda_0^2 \hat{B}_{2,0}^* \hat{B}_{2,0}
- |\kappa_{2,0}| \lambda_0^2 \hat{B}_{0,2}^* \hat{B}_{0,2} \right] = \frac{1}{n} \sum_{\alpha \in \mathcal{I}_{m,2}} \left[ (1 - |\kappa_{2,0}| z_0^2) \left| q_{\alpha \hat{\omega}} \right|^2
+ (1 - |\kappa_{2,0}| z_0^2) \left| q_{\alpha \hat{\omega}} \right|^2
+ |\kappa_{2,0}| \lambda_0^2 \left( q_{\alpha \hat{\omega}}^4 + \overline{q_{\alpha \hat{\omega}}^4} \right) \right].
\]

(3.14)

Hence, in order to prove assertion (ii) it is enough to show that the following quadratic form of \( x_1 \) and \( x_2 \)

\[
(1 - |\kappa_{2,0}| z_0^2) x_1^2 + (1 - |\kappa_{2,0}| z_0^2) x_2^2 
+ 2 |\kappa_{2,0}| \lambda_0^2 x_1 x_2
\]

is positive definite. A straightforward check yields

\[
1 - |\kappa_{2,0}| z_0^2 > 0;
(1 - |\kappa_{2,0}| z_0^2)^2 - |\kappa_{2,0}|^2 \lambda_0^4 
\geq (1 - |\kappa_{2,0}| z_0^2)^2 - |\kappa_{2,0}|^2 \lambda_0^4 \geq 0.
\]

(3.15)

Besides, if parameters \( \kappa_{2,0} \) and \( z_0 \) are such those in the assertion of Theorem 1 then the
inequality (3.15) is strict.

\[ \square \]

Lemma 8. Let \( \tilde{f}(Q_1, \hat{Q}) = f(Q_1, \hat{Q}) - f(\Lambda_0, 0) \). Then for sufficiently large \( n \)

\[
\max_{\lambda \in \Lambda - \Lambda_0, \|\hat{Q}\| \leq \epsilon} \Re \tilde{f}(UAV^*, \hat{Q}) \leq -C \frac{\log^2 n}{n}
\]

uniformly in \( U \) and \( V \).

Proof. First let us check that the first and the second derivatives of \( f_r \) are bounded in the
\( \delta \)-neighborhood of \( \Lambda_0 \), where \( f_r \) is defined in (2.11) and \( \delta \) is \( n \)-independent. Indeed, since \( h \) and \( h_0 \) are polynomials and \( h \Rightarrow h_0 \) on compacts

\[
\left| \frac{1}{\sqrt{n}} \frac{\partial f_r}{\partial x} \right| \leq \left| \frac{1}{\sqrt{n}} \frac{\partial f_r}{\partial x} \right| = \left| \frac{\partial f_r}{\partial x} \right| = \left| \frac{\partial f - f_0}{\partial x} \right| = \left| \frac{\partial (\log h - \log h_0)}{\partial x} \right|
\leq \left| \frac{1}{h_0} \frac{\partial h_0}{\partial x} - \frac{1}{h} \frac{\partial h}{\partial x} \right| \leq \frac{C}{\sqrt{n}}.
\]
where $x$ is either $\lambda_j$ or an entry of $Q_{p,s}$, $(p, s) \neq (1, 1)$. Let $\Lambda_E$ be a real diagonal matrix of unit norm and let $Q_E, \|Q_E\| = 1$, be a set of matrices which sizes correspond to those of $Q$. Then for any $\Lambda_E$ and $Q_E$ and for $\frac{\log n}{\sqrt{n}} \leq t \leq \delta$ we have

$$\frac{d}{dt} \Re \tilde{f}(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{Q}_E) = \langle \nabla_{\Lambda,Q} \Re f_0(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{Q}_E), v(E) \rangle + n^{-1/2} \langle \nabla_{\Lambda,Q} \Re f(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{Q}_E), v(E) \rangle$$

$$= \langle \nabla_{\Lambda,Q} \Re f_0(\Lambda_0 + t\Lambda_E, t\hat{Q}_E), v(E) \rangle + O(n^{-1/2}),$$

where $v(E)$ denotes a vector with $m$ components are all the real variables of $\Lambda_E$ and $\hat{Q}_E$ and $\langle \cdot, \cdot \rangle$ is a standard real scalar product. Expanding the scalar product by Taylor formula and considering that $\nabla_{\Lambda,Q} f_0(\Lambda_0, 0) = 0$, we obtain

$$\frac{d}{dt} \Re \tilde{f}(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{Q}_E) = t((\Re f_0)''(\Lambda_0, 0)v(E), v(E)) + r_1 + O(n^{-1/2}),$$

where $(\Re f_0)''$ is a matrix of second order derivatives of $\Re f_0$ w.r.t. $\Lambda$, $\Re \hat{Q}$ and $\Im \hat{Q}$ and $|r_1| \leq C t^2$. $(\Re f_0)''(\Lambda_0, 0)$ is negative definite according to Corollary 7. Hence $\frac{d}{dt} \Re \tilde{f}(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{Q}_E)$ is negative and

$$\max_{\frac{\log n}{\sqrt{n}} \leq \|\Lambda - \Lambda_0\| + \|\hat{Q}\| \leq \delta} \Re \tilde{f}(U\Lambda V^*, \hat{Q}) = \max_{\|\Lambda - \Lambda_0\| + \|\hat{Q}\| = \frac{\log n}{\sqrt{n}}} \Re \tilde{f}(U\Lambda V^*, \hat{Q}) \leq \Re f(U\Lambda_0 V^*, 0) - C \frac{\log^2 n}{n} - f(\Lambda_0, 0). \tag{3.16}$$

Notice that $f_r$ is bounded from above uniformly in $n$. This fact and Lemma 5 imply that $\delta$ in (3.16) can be replaced by $r$

$$\max_{\frac{\log n}{\sqrt{n}} \leq \|\Lambda - \Lambda_0\| + \|\hat{Q}\| \leq r} \Re \tilde{f}(U\Lambda V^*, \hat{Q}) \leq \Re f(U\Lambda_0 V^*, 0) - f(\Lambda_0, 0) - C \frac{\log^2 n}{n}. \tag{3.17}$$

It remains to deduce from Lemma 6 that $\Re f(U\Lambda_0 V^*, 0) - f(\Lambda_0, 0) = O(n^{-1})$ uniformly in $U$ and $V$. \hfill \square

Lemma 8 and the formula (2.8) yield

$$f_m = C n^c m \int_{\Omega_n} \triangle^2(\Lambda) \prod_{j=1}^m \lambda_j g(\hat{Q}) e^{-c_m f(U\Lambda V^*, \hat{Q})}$$

$$\times e^{f(U\Lambda V^*, \hat{Q})} d\mu(U) d\mu(V) d\lambda d\hat{Q} + O(e^{-C_1 \log^2 n}),$$

where $\Omega_n$ is defined in (3.7). Changing the variables $\Lambda = \Lambda_0 + \frac{1}{\sqrt{n}} \tilde{\Lambda}$, $\hat{Q} = \frac{1}{\sqrt{n}} \hat{Q}$ and expanding $f$ according to Lemma 6 we obtain

$$f_m = C k_n \int_{\sqrt{m} \Omega_n} \triangle^2(\tilde{\Lambda}) g(\hat{Q}) e^{-c_m f(\Lambda_0, 0)} d\mu(U) d\mu(V) d\lambda d\hat{Q}(1 + o(1))$$

$$\times \exp \left\{ -\frac{1}{2} \text{tr}(2\lambda_0 \tilde{\Lambda} + \tilde{z}_0 \tilde{Z}_U + z_0 \tilde{Z}_V) \right. - \left\{ \text{tr} \left[ (1 - |\kappa_{2,0}| \tilde{z}_0^2) \tilde{B}_{2,0} \right] + (1 - |\kappa_{2,0}| \tilde{z}_0^2) \tilde{B}_{2,0} \right. \right.$$  \hspace{1cm} \left. - |\kappa_{2,0}| \lambda_0^2 \tilde{B}_{0,2} \tilde{B}_{0,2} - |\kappa_{2,0}| \lambda_0^2 \tilde{B}_{0,2} \tilde{B}_{0,2} \right\}, \tag{3.17}$$

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where
\[ k_n = n^{m^2/2} e^{-mn\lambda_0^2 + \sqrt{n} \text{tr}(z_0 Z + z_0 Z^*)}. \] (3.18)

Since (3.14) the integral over \( \tilde{Q} \) can be computed separately over real and imaginary parts of the entries of \( \tilde{Q} \). Because \( g(Q_\ast)e^{-cnf(\lambda_0,0)} = 1 + o(1) \), the integration implies
\[
f_m = C k_n d_1(\kappa_2,0)^{-m} \int \Delta^2(\tilde{\Lambda}) d\mu(U) d\mu(V) d\tilde{\Lambda}(1 + o(1))
\]
\[
\times \exp \left\{ -\frac{1}{2} \text{tr} (2\lambda_0 \tilde{\Lambda} + z_0 Z + z_0 Z^*)^2 + \text{tr} Z_U Z_V^* \right\},
\]
where
\[
d_1(\kappa_2,0) = \left| 1 - |\kappa_2,0| z_0^2 \right|^2 - |\kappa_2,0|^2 \lambda_0^4. \] (3.20)

Let us change the variables \( V = WU \). Taking into account that the Haar measure is invariant w.r.t. shifts we get
\[
f_m = C k_n d_1(\kappa_2,0)^{-m} \int_{\mathbb{R}^m} \int_{U(m)} \int_{U(m)} \Delta^2(\tilde{\Lambda}) d\mu(U) d\mu(W) d\tilde{\Lambda}(1 + o(1))
\]
\[
\times \exp \left\{ -\frac{1}{2} \text{tr} (2\lambda_0 \tilde{\Lambda} + U^* (z_0 Z + z_0 Z_W^*) U)^2 + \text{tr} ZU^* Z^* W \right\}
\]
\[
= C k_n d_1(\kappa_2,0)^{-m} \int_{\mathbb{R}^m} \int_{U(m)} \int_{U(m)} \Delta^2(\tilde{\Lambda}) d\mu(U) d\mu(W) d\tilde{\Lambda}(1 + o(1))
\]
\[
\times \exp \left\{ -\frac{1}{2} \text{tr} (2\lambda_0 U \tilde{\Lambda} U^* + (z_0 Z + z_0 Z_W^*))^2 + \text{tr} ZU^* Z^* W \right\}.
\]

The next step is to change the variables \( H = U \tilde{\Lambda} U^* \). The Jacobian is \( \frac{\prod_{j=1}^{m-1} j!}{(2\pi)^{m(m-1)/2}} \Delta^{-2}(\tilde{\Lambda}) \) (see e.g. [39]). Thus
\[
f_m = C k_n d_1(\kappa_2,0)^{-m} \int_{\mathcal{H}_m} \int_{U(m)} d\mu(W) dH(1 + o(1))
\]
\[
\times \exp \left\{ -\frac{1}{2} \text{tr} (2\lambda_0 H + (z_0 Z + z_0 Z_W^*))^2 + \text{tr} ZU^* Z^* W \right\},
\]
where \( \mathcal{H}_m \) is a space of hermitian \( m \times m \) matrices and
\[
dH = \prod_{j=1}^{m} d(H)_{jj} \prod_{j<k} d\Re(H)_{jk} d\Im(H)_{jk}.
\]

The Gaussian integration over \( H \) implies
\[
f_m = C k_n d_1(\kappa_2,0)^{-m} \int_{U(m)} \exp \{ \text{tr} ZW^* Z^* W \} d\mu(W)(1 + o(1)). \] (3.21)

For computing the integral over the unitary group, the following Harish-Chandra/Itsykson–Zuber formula is used
Proposition 9. Let $A$ and $B$ be normal $d \times d$ matrices with distinct eigenvalues \{a_j\}_{j=1}^d$ and \{b_j\}_{j=1}^d respectively. Then
\[
\int_{U(d)} \exp\{z \operatorname{tr} AU^* BU\} d\mu(U) = \left(\prod_{j=1}^{d-1} j!\right) \frac{\det\{\exp(z a_j b_k)\}_{j,k=1}^d}{z^{(d^2-d)/2} \triangle(A) \triangle(B)},
\]
where $z$ is some constant, $\mu$ is a Haar measure, and $\triangle(A) = \prod_{j>k} (a_j - a_k)$.

For the proof see, e.g., [43, Appendix 5].

Applying the Harish-Chandra/Itsykson–Zuber formula to (3.21) we obtain
\[
f_m = C \kappa_n e^{-m \log d_1(\kappa_2,0)} \frac{\det\{e^{c_{1j}^k} \kappa_1^{m}\}_{j,k=1}^m}{|\triangle(Z)|^2} (1 + o(1)),
\]
which in combination with (2.12) yields the result of Theorem 1.

3.2 General case

In the general case the proof proceeds by the same scheme as in the case of zero high cumulants. In this subsection we focus on the crucial distinctions from the partial case considered above and refine the corresponding assertions from the previous subsection.

At the point we are ready to generalize Lemma 6.

Lemma 10. Let $\|\hat{\lambda}\| + \|\hat{Q}\| \leq \log n$. Then uniformly in $U$ and $V$
\[
f(U(\Lambda_0 + n^{-1/2} \hat{\lambda}) V^*, n^{-1/2} \hat{Q}) = -m \lambda_0^2 + n^{-1/2} \operatorname{tr}(\tau_0 Z + z_0 Z^*) - \frac{1}{2n} \operatorname{tr}(2 \lambda_0 \hat{\lambda} + \tau_0 Z_U + z_0 Z_U^*)^2 + \frac{1}{n} \operatorname{tr} Z_U Z_U^* - \frac{1}{2n} \operatorname{tr} [1 - |\kappa_2,0| \tau_0^2] \hat{B}_{2,0}^* \hat{B}_{2,0} + (1 - |\kappa_2,0| \tau_0^2) \hat{B}_{0,2}^* \hat{B}_{0,2}
\]
\[- |\kappa_2,0| \lambda_0 \hat{B}_{0,2} \hat{B}_{2,0} - |\kappa_2,0| \lambda_0^2 \hat{B}_{0,2}^* \hat{B}_{0,2}] - \frac{1}{n} \left\| \hat{Q}_{>1} \right\|^2 + n^{-1} \lambda_0 \sqrt{n^{-2}} \operatorname{tr} (\langle \wedge^2 V U^* \rangle \hat{Q}_2 + \hat{Q}_2 (\wedge^2 U V^*)) + O(n^{-3/2} \log^3 n),
\]
where we keep the notations of Lemma 6 and $\wedge^2 B$ is the second exterior power of a linear operator $B$ (see [58] for the definition and properties of an exterior power of a linear operator).

Proof. Differently from the previous subsection the function $f$ has an additional term $n^{-1/2} \hat{h}(Q_1, Q_2) + n^{-1} p_c(\hat{Q})$ under the logarithm, where $\hat{h}$ is defined in (2.5) and $p_c$ is a polynomial such that $p_c(0) = 0$. Therefore, the contribution of the term $n^{-1} p_c(n^{-1/2} \hat{Q})$ is $O(n^{-3/2} \log n)$. Hence, it remains to determine the contribution of the term $n^{-1/2} \hat{h}(Q_1, Q_2)$.

\[
n^{-1/2} \hat{h}(n^{-1/2} \hat{Q}_1, n^{-1/2} \hat{Q}_2) = n^{-1} \hat{h}(n^{-1/2} \hat{Q}_1, \hat{Q}_2)
\]
\[
= -\frac{1}{n} \int_{p+s=4} \left( \operatorname{tr} \tilde{Y}_{p,s} \hat{Q}_{p,s} + \operatorname{tr} \hat{Q}_{p,s}^* Y_{p,s} \right) e^{-\frac{1}{2} \rho^T F(\sqrt{n} \hat{Q}_1) \rho} d\phi^+ d\phi^- d\theta^+ d\theta^- \theta, \tag{3.23}
\]
where $\rho$ is defined in (2.6), $F$ is defined in (2.4), $\tilde{Y}_{p,s}$ and $Y_{p,s}$ are defined by (2.7).
Let us change the variables $\hat{\phi} = U^*\phi$, $\hat{\theta}^+ = \phi^+ U$, $\hat{\theta} = V^*\theta$, $\hat{\theta}^+ = \theta^+ V$. We have

$$
\frac{1}{\sqrt{\kappa_{p,s}}} y_{\alpha\beta}^{(p,s)} = \frac{1}{\sqrt{\kappa_{p,s}}} \sum_{q=1}^{p} \phi_{\alpha q} \prod_{r=1}^{s} \theta_{\beta r}^s = \frac{1}{\sqrt{\kappa_{p,s}}} (U \hat{\phi})_{\alpha q} \prod_{r=1}^{s} (\hat{\theta}^+ V^*)_{\beta r}
$$

$$
= \prod_{q=1}^{p} \sum_{\gamma_q=1}^{m} u_{\alpha q \gamma_q} \tilde{\phi}_{\gamma_q} \prod_{r=1}^{s} \tilde{\theta}_{\delta_r}^s v_{\beta \gamma_q \delta_r}.
$$

$$
= \sum_{\gamma \in I_m, \delta \in I_m} a_{\alpha \beta}^{(p,s)} \prod_{q=1}^{p} \tilde{\phi}_{\gamma_q} \prod_{r=1}^{s} \tilde{\theta}_{\delta_r}^s.
$$

(3.24)

where $a_{\alpha \beta \gamma \delta}^{(p,s)}$ just denotes the coefficient at $\prod_{q=1}^{p} \tilde{\phi}_{\gamma_q} \prod_{r=1}^{s} \tilde{\theta}_{\delta_r}^s$. Similarly

$$
\frac{1}{\sqrt{\kappa_{p,s}}} y_{\beta \alpha}^{(p,s)} = \sum_{\gamma \in I_m, \delta \in I_m} a_{\alpha \beta}^{(p,s)} \prod_{q=1}^{p} \tilde{\phi}_{\gamma_q} \prod_{r=1}^{s} \tilde{\theta}_{\delta_r}^s.
$$

(3.25)

Besides,

$$
\rho^T F \rho = \hat{\rho}^T F_0 \hat{\rho} + O(n^{-1/2} \log n),
$$

(3.26)

where $F_0$ is defined in (3.9) and

$$
\hat{\rho} = \begin{pmatrix}
(\hat{\phi}^+)^T \\
(\hat{\theta}^+)^T \\
\hat{\phi}
\end{pmatrix}.
$$

The “measure” changes as follows

$$
d\phi^+ d\phi d\phi^+ d\theta = \det^{-1} U \det^{-1} U^* \det^{-1} V \det^{-1} V^* d\phi^+ d\phi d\phi^+ d\theta
$$

$$
= d\phi^+ d\phi d\phi^+ d\theta.
$$

(3.27)

Eventually, substitution of (3.24)–(3.27) into (3.23) yields

$$
n^{-1} h(n^{-1/2} Q_1, Q_2) = - \frac{1}{n} \int e^{-\frac{1}{2} \hat{\phi}^+ F_0 \hat{\rho} d\phi^+ d\phi d\phi^+ d\theta}
$$

$$
\times \sum_{p+s=4, \alpha, \beta \in I_m, \gamma, \delta \in I_m} \left( \sqrt{\kappa_{p,s}} a_{\alpha \beta}^{(p,s)} \prod_{q=1}^{p} \tilde{\phi}_{\gamma_q} \prod_{r=1}^{s} \tilde{\theta}_{\delta_r}^s \right)
$$

$$
+ \sqrt{\kappa_{p,s}} a_{\alpha \beta}^{(p,s)} \prod_{q=1}^{p} \tilde{\phi}_{\gamma_q} \prod_{r=1}^{s} \tilde{\theta}_{\delta_r}^s
$$

$$
+ O(n^{-3/2} \log^3 n)
$$

(3.28)

uniformly in $U$ and $V$.

The integration in (3.28) can be performed over $\tilde{\phi}_j$, $\tilde{\theta}_j$ separately for every $j$ due to the structure of $F_0$. Thus it remains to compute the integrals of the form

$$
\int \prod_{q=1}^{p} \tilde{\phi}_{\gamma_q} \prod_{r=1}^{s} \tilde{\theta}_{\delta_r}^s \exp \left\{ \sum_{j=1}^{\tilde{\phi}_j, \tilde{\theta}_j} \left( \lambda_0 \tilde{\phi}_j \tilde{\theta}_j^* + \lambda_0 \tilde{\theta}_j \phi_j^* + z_j \phi_j \tilde{\phi}_j^* - \lambda_0 \tilde{\phi}_j \tilde{\theta}_j^* \right) \right\} d\tilde{\phi}_j d\tilde{\theta}_j d\phi_j d\theta_j
$$

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Furthermore, expanding the exponent into series, one can observe that all the integrals are non-zero only if \( p = s = 2 \) and \( \gamma = \delta \). Moreover,

\[
\begin{align*}
&\int \tilde{\phi}_j \tilde{\theta}_j^* e^{-2 \tilde{\phi}_j \tilde{\theta}_j + \lambda_0 \tilde{\phi}_j \tilde{\theta}_j + z_0 \tilde{\phi}_j + z_0 \tilde{\theta}_j - \lambda_0 \tilde{\phi}_j \tilde{\theta}_j} d\tilde{\phi}_j d\tilde{\theta}_j = -\lambda_0, \\
&\int \tilde{\theta}_j \tilde{\phi}_j^* e^{-2 \tilde{\phi}_j \tilde{\theta}_j + \lambda_0 \tilde{\phi}_j \tilde{\theta}_j + z_0 \tilde{\phi}_j + z_0 \tilde{\theta}_j - \lambda_0 \tilde{\phi}_j \tilde{\theta}_j} d\tilde{\phi}_j d\tilde{\theta}_j = \lambda_0, \\
&\int e^{-2 \tilde{\phi}_j \tilde{\theta}_j + \lambda_0 \tilde{\phi}_j \tilde{\theta}_j + z_0 \tilde{\phi}_j + z_0 \tilde{\theta}_j - \lambda_0 \tilde{\phi}_j \tilde{\theta}_j} d\tilde{\phi}_j d\tilde{\theta}_j = 1.
\end{align*}
\]

The last thing we need is values of \( a_{\alpha \beta \gamma \delta}^{(2,2)} \) and \( a_{\beta \alpha \delta \gamma}^{(2,2)} \). The formula (3.24) implies

\[ a_{\alpha \beta \gamma \delta}^{(2,2)} = (\Lambda^2 U)_{\alpha \gamma} (\Lambda^2 V^*)_{\delta \beta}. \]

Similarly

\[ a_{\beta \alpha \delta \gamma}^{(2,2)} = (\Lambda^2 V)_{\beta \gamma} (\Lambda^2 U^*)_{\delta \alpha}. \]

Finally

\[
\begin{align*}
\frac{1}{n} \bar{h}(n^{-1/2} \tilde{Q}_1, \tilde{Q}_2) &= \frac{1}{n} \lambda_0^2 \sqrt{2} \text{tr}(\Lambda^2 U^*) \tilde{Q}_2 (\Lambda^2 V) + \text{tr}(\Lambda^2 V^*) \tilde{Q}_2^* (\Lambda^2 U) + o \left( \frac{1}{n} \right) \\
&= n^{-1} \lambda_0^2 \sqrt{2} \text{tr}(\Lambda^2 V U^*) \tilde{Q}_2 + \text{tr} \tilde{Q}_2^* (\Lambda^2 U V^*) \\
&\quad + O(n^{-3/2} \log^3 n).
\end{align*}
\]

The above relation completes the proof of (3.22).

Lemma 8 is still valid in the general case, despite the proof needs some insignificant changes due to a non-zero term \( n^{-1/2} \bar{h}(Q_1, Q_2) \).

Following the proof in the Gaussian case one can see that (3.17) transforms into

\[
f_m = C k_n \int \frac{\Delta^2 (\tilde{\Lambda}) g(Q_*) e^{-c_m f(\Lambda_0, 0)} d\mu(U) d\mu(V) d\tilde{\Lambda} d\tilde{Q}}{\sqrt{m_n}} \times \exp \left\{ -\frac{1}{2} \text{tr}(2\lambda_0 \tilde{\Lambda} + z_0 Z_U + z_0 Z_V)^2 + \text{tr} Z_U Z_V^* \right\} \\
&\quad - \left( 1 - |\kappa_0| \right) \tilde{B}_{20}^* B_{20} + (1 - |\kappa_2|) \tilde{B}_{20,2}^* B_{20,2} \\
&\quad + \lambda_0^2 \sqrt{2} \text{tr} \bigl[ (\Lambda^2 V U^*) \tilde{Q}_2 + \tilde{Q}_2^* (\Lambda^2 U V^*) \bigr],
\]

where \( k_n \) is defined in (3.18). The Gaussian integration over \( \tilde{Q} \) yields

\[
f_m = C k_n d_1(\kappa_{20})^{-m} \int \frac{\Delta^2 (\tilde{\Lambda}) d\mu(U) d\mu(V) d\tilde{\Lambda}}{\sqrt{m_n}} \times \exp \left\{ -\frac{1}{2} \text{tr}(2\lambda_0 \tilde{\Lambda} + z_0 Z_U + z_0 Z_V)^2 + \text{tr} Z_U Z_V^* \\
&\quad + \lambda_0^4 \kappa_{20} \text{tr} \bigl[ (\Lambda^2 V U^*) (\Lambda^2 U V^*) \bigr] \right\}.
\]
Note that $\wedge^2 VU^*$ and $\wedge^2 UV^*$ are mutually inverse matrices. Therefore,

$$f_m = Ck_n d_1(\kappa_{2,0})^{-m} \exp \left\{ \frac{m^2 - m}{2} \lambda_0^4 \kappa_{2,2} \right\} \int \Delta^2(\tilde{\Lambda}) d\mu(U) d\mu(V) d\tilde{\Lambda}(1 + o(1)) \times \exp \left\{ -\frac{1}{2} \text{tr}(2\lambda_0 \tilde{\Lambda} + z_0 Z_U + z_0 Z_V^*)^2 + \text{tr} Z_U Z_V^* \right\}.$$

The last formula differs from (3.19) only by a factor $\exp \left\{ \frac{m^2 - m}{2} \lambda_0^4 \kappa_{4,4} \right\}$. Hence, there are no differences in further proof up to this factor.

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