A short survey of duality in special functions

Tom H. Koornwinder

Abstract

This is a tutorial on duality properties of special functions, mainly of orthogonal polynomials in the \((q-)\)Askey scheme. It is based on the first part of the 2017 R. P. Agarwal Memorial Lecture delivered by the author.

1 Introduction

Classical orthogonal polynomials \(p_n(x)\) and their generalizations in the Askey and \(q\)-Askey scheme have the property that they are eigenfunctions of some second order operator \(L\) with eigenvalues depending on \(n\), which therefore may be called the spectral variable. Moreover, being orthogonal polynomials, the \(p_n(x)\) satisfy a three-term recurrence relation and are therefore, as functions of \(n\), eigenfunctions of a so-called Jacobi operator with eigenvalues \(x\). This duality phenomenon was also guiding for the author in the companion paper \[17\], where he derived the dual addition formula for continuous \(q\)-ultraspherical polynomials.

This paper gives a brief tutorial type survey of duality, mainly for orthogonal polynomials, but also a little bit for transcendental special functions. This paper is based on the first part of the \(R. \text{ P. Agarwal Memorial Lecture}\), which the author delivered on November 2, 2017 during the conference ICSFA-2017 held in Bikaner, Rajasthan, India. See \[17\] for the paper based on the second part.

With pleasure I remember to have met Prof. Agarwal during the workshop on Special Functions and Differential Equations held at the Institute of Mathematical Sciences in Chennai, January 1997, where he delivered the opening address \[1\]. I cannot resist to quote from it the following wise words, close to the end of the article:

“I think that I have taken enough time and I close my discourse- with a word of caution and advice to the research workers in the area of special functions and also those who use them in physical problems. The corner stones of classical analysis are ‘elegance, simplicity, beauty and perfection.’ Let them not be lost in your work. Any analytical generalization of a special function, only for the sake of a generalization by adding a few terms or parameters here and there, leads us nowhere. All research work should be meaningful and aim at developing a quality technique or have a bearing in some allied discipline.”
Note For definition and notation of \((q-)\)-shifted factorials and \((q-)\)-hypergeometric series see \([9]\) §1.2. In the \(q = 1\) case we will mostly meet terminating hypergeometric series

\[
\begin{align*}
{}_{r}F_{s}\left(-n, a_2, \ldots, a_r; \frac{b_1}{b_1, \ldots, b_s} \right) := \sum_{k=0}^{n} \frac{(-n)_k}{k!} \frac{(a_2, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} \left(\frac{b_1}{b_1, \ldots, b_s}\right)^k.
\end{align*}
\] (1.1)

Here \((b_1, \ldots, b_s)_k := (b_1)_k \cdots (b_s)_k\) and \((b)_k := b(b+1) \cdots (b+k-1)\) is the Pochhammer symbol or shifted factorial. In \([11]\) we even allow that \(b_i = -N\) for some \(i\) with \(N\) integer \(\geq n\). There is no problem because the sum on the right terminates at \(k = n \leq N\).

In the \(q\)-case we will always assume that \(0 < q < 1\). We will only meet terminating \(q\)-hypergeometric series of the form

\[
\sum_{k=0}^{n} \frac{(q^{-n})_k}{(q)_k} \frac{(a_2, \ldots, a_{s+1})_k}{(b_1, \ldots, b_s)_k} \left(\frac{q}{b_1, \ldots, b_s}\right)^k.
\] (1.2)

Here \((b_1, \ldots, b_s)_k := (b_1)_k \cdots (b_s)_k\) and \((b)_k := (1-b)(1-qb) \cdots (1-q^{k-1}b)\) is the \(q\)-Pochhammer symbol or \(q\)-shifted factorial. In \([12]\) we even allow that \(b_i = q^{-N}\) for some \(i\) with \(N\) integer \(\geq n\).

For formulas on orthogonal polynomials in the \((q-)\)-Askey scheme we will often refer to Chapters 9 and 14 in \([13]\). Almost all of these formulas, with different numbering, are available in open access on \url{http://aw.twi.tudelft.nl/~koekoek/askey/}.

2 The notion of duality in special functions

With respect to a (positive) measure \(\mu\) on \(\mathbb{R}\) with support containing infinitely many points we can define orthogonal polynomials (OPs) \(p_n\) \((n = 0, 1, 2, \ldots)\), unique up to nonzero real constant factors, as (real-valued) polynomials \(p_n\) of degree \(n\) such that

\[
\int_{\mathbb{R}} p_m(x) p_n(x) d\mu(x) = 0 \quad (m, n \neq 0).
\]

Then the polynomials \(p_n\) satisfy a three-term recurrence relation

\[
x p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x) \quad (n = 0, 1, 2, \ldots),
\] (2.1)

where the term \(C_n p_{n-1}(x)\) is omitted if \(n = 0\), and where \(A_n, B_n, C_n\) are real and

\[
A_{n-1} C_n > 0 \quad (n = 1, 2, \ldots).
\] (2.2)

By Favard’s theorem \([8]\) we can conversely say that if \(p_0(x)\) is a nonzero real constant, and the \(p_n(x)\) \((n = 0, 1, 2, \ldots)\) are generated by \((2.1)\) for certain real \(A_n, B_n, C_n\) which satisfy \((2.2)\), then the \(p_n\) are OPs with respect to a certain measure \(\mu\) on \(\mathbb{R}\).

With \(A_n, B_n, C_n\) as in \((2.1)\) define a Jacobi operator \(M\), acting on infinite sequences \(\{g(n)\}_n\), by

\[
(M g)(n) = M_n (g(n)) := A_n g(n+1) + B_n g(n) + C_n g(n-1) \quad (n = 0, 1, 2, \ldots),
\]

2
where the term $C_n g(n-1)$ is omitted if $n = 0$. Then (2.1) can be rewritten as the eigenvalue equation

\[ M_n(p_n(x)) = x p_n(x) \quad (n = 0, 1, 2, \ldots). \]

One might say that the study of a system of OPs $p_n$ turns down to the spectral theory and harmonic analysis associated with the operator $M$. From this perspective one can wonder if the polynomials $p_n$ satisfy some dual eigenvalue equation

\[ (Lp_n)(x) = \lambda_n p_n(x) \]

for $n = 0, 1, 2, \ldots$, where $L$ is some linear operator acting on the space of polynomials. We will consider various types of operators $L$ together with the corresponding OPs, first in the Askey scheme and next in the $q$-Askey scheme.

### 2.1 The Askey scheme

**Classical OPs**  
Bochner’s theorem [5] classifies the second order differential operators $L$ together with the OPs $p_n$ such that (2.4) holds for certain eigenvalues $\lambda_n$. The resulting classical orthogonal polynomials are essentially the polynomials listed in the table below. Here $d\mu(x) = w(x)\,dx$ on $(a, b)$ and the closure of that interval is the support of $\mu$. Furthermore, $w_1(x)$ occurs in the formula for $L$ to be given after the table.

| name   | $p_n(x)$          | $w(x)$          | $\frac{w_1(x)}{w(x)}$ | $(a, b)$ | constraints                          | $\lambda_n$ |
|--------|-------------------|-----------------|------------------------|---------|--------------------------------------|-------------|
| Jacobi | $P_n^{\alpha,\beta}(x)$ | $(1 - x)^\alpha (1 + x)^\beta$ | $1 - x^2$ | $(-1, 1)$ | $\alpha, \beta > -1$ | $-n(n + \alpha + \beta + 1)$ |
| Laguerre | $L_n^{(\alpha)}(x)$ | $x^\alpha e^{-x}$ | $x$ | $(0, \infty)$ | $\alpha > -1$ | $-n$ |
| Hermite | $H_n(x)$ | $e^{-x^2}$ | $1$ | $(-\infty, \infty)$ | $\alpha > -1$ | $-2n$ |

Then

\[ (Lf)(x) = w(x)^{-1} \frac{d}{dx} \left( w_1(x) f'(x) \right). \]

For these classical OPs the duality goes much further than the two dual eigenvalue equations (2.3) and (2.4). In particular for Jacobi polynomials it is true to a large extent that every formula or property involving $n$ and $x$ has a dual formula or property where the roles of $n$ and $x$ are interchanged. We call this the **duality principle**. If the partner formula or property is not yet known then it is usually a good open problem to find it (but one should be warned that there are examples where the duality fails).

The Jacobi, Laguerre and Hermite families are connected by limit transitions, as is already suggested by limit transitions for their (rescaled) weight functions:

- Jacobi $\rightarrow$ Laguerre: $x^\alpha (1 - \beta^{-1}x)^\beta \rightarrow x^\alpha e^{-x}$ as $\beta \rightarrow \infty$;

- Jacobi $\rightarrow$ Hermite: $(1 - \alpha^{-1}x^2)^\alpha \rightarrow e^{-x^2}$ as $\alpha \rightarrow \infty$;

- Laguerre $\rightarrow$ Hermite: $e^{\alpha(1 - \log \alpha)}(2\alpha)^{\frac{1}{2}} x + \alpha e^{-\alpha} x^{-\alpha} \rightarrow e^{-x^2}$ as $\alpha \rightarrow \infty$.

Formulas and properties of the three families can be expected to be connected under these limits. Although this is not always the case, this **limit principle** is again a good source of open problems.
Discrete analogues of classical OPs  Let $L$ be a second order difference operator:

$$(Lf)(x) := a(x) f(x + 1) + b(x) f(x) + c(x) f(x - 1). \quad (2.5)$$

Here as solutions of (2.4) we will also allow OPs $\{p_n\}_{n=0}^{N}$ for some finite $N \geq 0$, which will be orthogonal with respect to positive weights $w_k (k = 0, 1, \ldots, N)$ on a finite set of points $x_k (k = 0, 1, \ldots, N)$:

$$\sum_{k=0}^{N} p_m(x_k)p_n(x_k)w_k = 0 \quad (m, n = 0, 1, \ldots, N; m \neq n).$$

If such a finite system of OPs satisfies (2.4) for $n = 0, 1, \ldots, N$ with $L$ of the form (2.5) then the highest $n$ for which the recurrence relation (2.1) holds is $n = N$, where the zeros of $p_{N+1}$ are precisely the points $x_0, x_1, \ldots, x_N$. The classification of OPs satisfying (2.4) with $L$ of the form (2.5) (first done by O. Lancaster, 1941, see [2]) yields the four families of Hahn, Krawtchouk, Meixner and Charlier polynomials, of which Hahn and Krawtchouk are finite systems, and Meixner and Charlier infinite systems with respect to weights on countably infinite sets.

Krawtchouk polynomials [13, (9.11.1)] are given by

$$K_n(x; p, N) := _2F_1\left( \begin{array}{c} -n, -x \\ -N \end{array} ; p^{-1} \right) \quad (n = 0, 1, 2, \ldots, N). \quad (2.6)$$

They satisfy the orthogonality relation

$$\sum_{x=0}^{N} (K_mK_nw)(x; p, N) = \frac{(1-p)^N}{w(n; p, N)} \delta_{m,n}$$

with weights

$$w(x; p, N) := \binom{N}{x} p^x (1-p)^{N-x} \quad (0 < p < 1).$$

By (2.6) they are self-dual:

$$K_n(x; p, N) = K_x(n; p, N) \quad (n, x = 0, 1, \ldots, N).$$

The three-term recurrence relation (2.3) immediately implies a dual equation (2.4) for such OPs.

The four just mentioned families of discrete OPs are also connected by limit relations. Moreover, the classical OPs can be obtained as limit cases of them, but not conversely. For instance, Hahn polynomials [13 (9.5.1)] are given by

$$Q_n(x; \alpha, \beta, N) := _3F_2\left( \begin{array}{c} -n, n + \alpha + \beta + 1, -x \\ \alpha + 1, -N \end{array} ; 1 \right) \quad (n = 0, 1, \ldots, N) \quad (2.7)$$

and they satisfy the orthogonality relation

$$\sum_{x=0}^{N} (Q_mQ_nw)(x; \alpha, \beta, N) = 0 \quad (m, n = 0, 1, \ldots, N; m \neq n; \alpha, \beta > -1)$$
with weights
\[ w(x; \alpha, \beta, N) := \frac{(\alpha + 1)x(\beta + 1)N_x}{x!(N-x)!}. \]

Then by (2.7) (rescaled) Hahn polynomials tend to (shifted) Jacobi polynomials:
\[ \lim_{N \to \infty} Q_n(Nx; \alpha, \beta, N) = {}_2F_1 \left( -n, n + \alpha + \beta + 1 ; \alpha + 1 ; x \right) = \frac{R_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}. \]  

(2.8)

Continuous versions of Hahn and Meixner polynomials
A variant of the difference operator (2.5) is the operator
\[ (Lf)(x) := A(x)f(x+i) + B(x)f(x) + \bar{A}(x)f(x-i) \quad (x \in \mathbb{R}), \]  

(2.9)
where \( B(x) \) is real-valued. Then further OPs satisfying (2.4) are the continuous Hahn polynomials and the Meixner-Pollaczek polynomials [13, Ch. 9].

Insertion of a quadratic argument
For an operator \( \tilde{L} \) and some polynomial \( \sigma \) of degree 2 we can define an operator \( L \) by
\[ (Lf)(\sigma(x)) := \tilde{L}_x(f(\sigma(x))), \]  

(2.10)
Now we look for OPs satisfying (2.4) where \( \tilde{L} \) is of type (2.5) or (2.9). So
\[ \tilde{L}_x(p_n(\sigma(x))) = \lambda_n p_n(\sigma(x)). \]  

(2.11)
The resulting OPs are the Racah polynomials and dual Hahn polynomials for (2.11) with \( \tilde{L} \) of type (2.5), and Wilson polynomials and continuous dual Hahn polynomials for (2.11) with \( \tilde{L} \) of type (2.9), see again [13, Ch. 9].

The OPs satisfying (2.4) in the cases discussed until now form together the Askey scheme, see Figure 1. The arrows denote limit transitions.

In the Askey scheme we emphasize the self-dual families: Racah, Meixner, Krawtchouk and Charlier for the OPs with discrete orthogonality measure, and Wilson and Meixner-Pollaczek for the OPs with non-discrete orthogonality measure. We already met perfect self-duality for the Krawtchouk polynomials, which is also the case for Meixner and Charlier polynomials. For the Racah polynomials the dual OPs are still Racah polynomials, but with different values of the parameters:
\[ R_n(x(x + \delta - N); \alpha, \beta, -N-1, \delta) := {}_4F_3 \left( -n, n + \alpha + \beta + 1, -x, x + \delta - N ; \alpha + 1, \beta + \delta + 1, -N \right; 1 \)
\[ = R_x(n(n + \alpha + \beta + 1); -N-1, \delta, \alpha, \beta) \quad (n, x = 0, 1, \ldots, N). \]

The orthogonality relation for these Racah polynomials involves a weighted sum of terms \( (R_m R_n)(x(x + \delta - N); \alpha, \beta, -N-1, \delta) \) over \( x = 0, 1, \ldots, N \).
For Wilson polynomials we have also self-duality with a change of parameters but the self-duality is not perfect, i.e., not related to the orthogonality relation:

\[
\text{const. } W_n(x^2; a, b, c, d) := {}_4F_3\left(-n, n + a + b + c + d - 1, a + ix, a - ix; 1 \right)_{a + b, a + c, a + d} = \text{const. } W_{-ix-a} \left((i(n + a'))^2; a', b', c', d' \right), \quad (2.12)
\]

where \(a' = \frac{1}{2}(a + b + c + d - 1), \ a' + b' = a + b, \ a' + c' = a + c, \ a' + d' = a + d.\) The duality \((2.12)\) holds for \(-ix-a = 0, 1, 2, \ldots,\) while the orthogonality relation for the Wilson polynomials involves a weighted integral of \((W_mW_n)(x^2; a, b, c, d)\) over \(x \in [0, \infty).\)

As indicated in Figure 1, the dual Hahn polynomials

\[
R_n(x(x + \alpha + \beta + 1); \alpha, \beta, N) := {}_3F_2\left(-n, -x, x + \alpha + \beta + 1; 1 \right)_{\alpha + 1, -N} \quad (n = 0, 1, \ldots, N)
\]

are dual to the Hahn polynomials \((2.1)\):

\[
Q_n(x; \alpha, \beta, N) = R_x\left(n(n + \alpha + \beta + 1); \alpha, \beta, N \right) \quad (n, x = 0, 1, \ldots, N).
\]
The duality is perfect: the dual orthogonality relation for the Hahn polynomials is the orthogonality relation for the dual Hahn polynomials, and conversely. There is a similar, but non-perfect duality between continuous Hahn and continuous dual Hahn.

The classical OPs are in two senses exceptional within the Askey scheme. First, they are the only families which are not self-dual or dual to another family of OPs. Second, they are the only continuous families which are not related by analytic continuation to a discrete family.

With the arrows in the Askey scheme given it can be taken as a leading principle to link also the formulas and properties of the various families in the Askey scheme by these arrows. In particular, if one has some formula or property for a family lower in the Askey scheme, say for Jacobi, then one may look for the corresponding formula or property higher up, and try to find it if it is not yet known. In particular, if one could find the result on the highest Racah or Wilson level, which is self-dual then, going down along the arrows, one might also obtain two mutually dual results in the Jacobi case.

2.2 The $q$-Askey scheme

The families of OPs in the $q$-Askey scheme result from the classification of OPs satisfying (2.4), where $L$ is defined in terms of the operator $\tilde{L}$ and the function $\sigma$ by (2.10), where $\tilde{L}$ is of type (2.5) or (2.9), and where $\sigma(x) = q^{x}$ or equal to a quadratic polynomial in $q^{x}$. This choice of $\sigma(x)$ is the new feature deviating from what we discussed about the Askey scheme. And here $q$ enters, with $0 < q < 1$ always assumed. The $q$-Askey scheme is considerably larger than the Askey scheme, but many features of the Askey scheme return here, in particular it has arrows denoting limit relations. Moreover, the $q$-Askey scheme is quite parallel to the Askey scheme in the sense that OPs in the $q$-Askey scheme, after suitable rescaling, tend to OPs in the Askey scheme as $q \uparrow 1$. Parallel to Wilson and Racah polynomials at the top of the Askey scheme there are Askey–Wilson polynomials and $q$-Racah polynomials at the top of the $q$-Askey scheme. These are again self-dual families, with the self-duality for $q$-Racah being perfect.

The guiding principles discussed before about formulas or properties related by duality or limit transitions now extend to the $q$-Askey scheme: both within the $q$-Askey scheme and in relation to the Askey scheme by letting $q \uparrow 1$. For instance, one can hope to find as many dual pairs of significant formulas and properties of Askey–Wilson polynomials as possible which have mutually dual limit cases for Jacobi polynomials. In fact, we realize this in [17] with the addition and dual addition formula by taking limits from the continuous $q$-ultraspherical polynomials (a self-dual one-parameter subclass of the four-parameter class of Askey–Wilson polynomials) to the ultraspherical polynomials (a one-parameter subclass of the two-parameter class of Jacobi polynomials).

One remarkable aspect of duality in the two schemes concerns the discrete OPs living there. Leonard (1982) classified all systems of OPs $p_{n}(x)$ with respect to weights on a countable set $\{x(m)\}$ for which there is a system of OPs $q_{m}(y)$ on a countable set $\{y(n)\}$ such that

$$p_{n}(x(m)) = q_{m}(y(n)).$$

1See [http://homepage.tudelft.nl/11r49/pictures/large/q-AskeyScheme.jpg](http://homepage.tudelft.nl/11r49/pictures/large/q-AskeyScheme.jpg)
His classification yields the OPs in the $q$-Askey scheme which are orthogonal with respect to weights on a countable set together with their limit cases for $q \uparrow 1$ and $q \downarrow -1$ (where we allow $-1 < q < 1$ in the $q$-Askey scheme). The $q \downarrow -1$ limit case yields the Bannai–Ito polynomials [4].

### 2.3 Duality for non-polynomial special functions

For Bessel functions $J_{\alpha}$ see [23, Ch. 10] and references given there. It is convenient to use a different standardization and notation:

$$J_{\alpha}(x) := \Gamma(\alpha + 1) \left(\frac{2}{x}\right)^{\alpha} J_{\alpha}(x).$$

Then (see [23, (10.16.9)])

$$J_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{4} x^2)^k}{(\alpha + 1)_k k!} = {}_0F_1\left(-\frac{1}{4} x^2; \frac{-\alpha + 1}{\alpha + 1}\right) \quad (\alpha > -1).$$

$J_{\alpha}$ is an even entire analytic function. Some special cases are

$$J_{-1/2}(x) = \cos x, \quad J_{1/2}(x) = \frac{\sin x}{x}. \quad (2.13)$$

The Hankel transform pair [23, §10.22(v)], for $f$ in a suitable function class, is given by

$$\begin{cases}
\hat{f}(\lambda) = \int_0^{\infty} f(x) J_{\alpha}(\lambda x) x^{2\alpha+1} dx, \\
f(x) = \frac{1}{2^{2\alpha+1} \Gamma(\alpha + 1)^2} \int_0^{\infty} \hat{f}(\lambda) J_{\alpha}(\lambda x) \lambda^{2\alpha+1} d\lambda.
\end{cases}$$

In view of (2.13) the Hankel transform contains the Fourier-cosine and Fourier-sine transform as special cases for $\alpha = \pm \frac{1}{2}$.

The functions $x \mapsto J_{\alpha}(\lambda x)$ satisfy the eigenvalue equation [23, (10.13.5)]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x} \frac{\partial}{\partial x}\right) J_{\alpha}(\lambda x) = -\lambda^2 J_{\alpha}(\lambda x). \quad (2.14)$$

Obviously, then also

$$\left(\frac{\partial^2}{\partial \lambda^2} + \frac{2\alpha + 1}{\lambda} \frac{\partial}{\partial \lambda}\right) J_{\alpha}(\lambda x) = -x^2 J_{\alpha}(\lambda x). \quad (2.15)$$

The differential operator in (2.15) involves the spectral variable $\lambda$ of (2.14), while the eigenvalue in (2.15) involves the $x$-variable in the differential operator in (2.14).

The Bessel functions and the Hankel transform are closely related to the Jacobi polynomials (2.8) and their orthogonality relation. Indeed, we have the limit formulas

$$\lim_{n \to \infty} \frac{P_n^{(\alpha,\beta)}(\cos(n^{-1} x))}{P_n^{(\alpha,\beta)}(1)} = J_{\alpha}(x), \quad \lim_{\nu \to \infty} \frac{P_n^{(\alpha,\beta)}(\cos(\nu^{-1} x))}{P_n^{(\alpha,\beta)}(1)} = J_{\alpha}(\lambda x).$$
There are many other examples of non-polynomial special functions being limit cases of OPs in the \((q-)\)Askey scheme, see for instance [16], [14].

In 1986 Duistermaat & Grünbaum [6] posed the question if the pair of eigenvalue equations (2.14), (2.15) could be generalized to a pair

\[
L_x(\phi_\lambda(x)) = -\lambda^2 \phi_\lambda(x), \quad M_\lambda(\phi_\lambda(x)) = \tau(x) \phi_\lambda(x)
\]

for suitable differential operators \(L_x\) in \(x\) and \(M_\lambda\) in \(\lambda\) and suitable functions \(\phi_\lambda(x)\) solving the two equations. Here the functions \(\phi_\lambda(x)\) occur as eigenfunctions in two ways: for the operator \(L_x\) with eigenvalue depending on \(\lambda\) and for the operator \(M_\lambda\) with eigenvalue depending on \(x\). Since the occurring eigenvalues of an operator form its spectrum, a phenomenon as in (2.16) is called **bispectrality**. For the case of a second order differential operator \(L_x\) written in potential form \(L_x = d^2/dx^2 - V(x)\) they classified all possibilities for (2.16). Beside the mentioned Bessel cases and a case with Airy functions (closely related to Bessel functions) they obtained two other families where \(M_\lambda\) is a higher than second order differential operator. These could be obtained by successive **Darboux transformations** applied to \(L_x\) in potential form. A Darboux transformation produces a new potential from a given potential \(V(x)\) by a formula which involves an eigenfunction of \(L_x\) with eigenvalue 0. Their two new families get a start by the application of a Darboux transformation to the Bessel differential equation (2.14), rewritten in potential form

\[
\phi_\lambda''(x) - (\alpha^2 - \frac{1}{4})x^{-2}\phi_\lambda(x) = -\lambda^2 \phi_\lambda(x), \quad \phi_\lambda(x) = (\lambda x)^{\alpha + \frac{1}{2}} J_\alpha(\lambda x).
\]

Here \(\alpha\) should be in \(\mathbb{Z} + \frac{1}{2}\) for a start of the first new family or in \(\mathbb{Z}\) for a start of the second new family. For other values of \(\alpha\) one would not obtain a dual eigenvalue equation with \(M_\lambda\) a finite order differential operator.

Just as higher order differential operators \(M_\lambda\) occur in (2.16), there has been a lot of work on studying OPs satisfying (2.4) with \(L\) a higher order differential operator. See a classification in [20], [19]. All occurring OPs, the so-called **Jacobi type** and **Laguerre type polynomials**, have a Jacobi or Laguerre orthogonality measure with integer values of the parameters, supplemented by mass points at one or both endpoints of the orthogonality interval. Some of the Bessel type functions in the second new class in [6] were obtained in [7] as limit cases of Laguerre type polynomials.

### 2.4 Some further cases of duality

The self-duality property of the family of Askey-Wikson polynomials is reflected in Zhedanov’s **Askey–Wilson algebra** [25]. A larger algebraic structure is the **double affine Hecke algebra** (DAHA), introduced by Cherednik and extended by Sahi. The related special functions are so-called **non-symmetric** special functions. They are functions in several variables and associated with root systems. Again there is a duality, both in the DAHA and for the related special functions. For the (one-variable) case of the non-symmetric Askey–Wilson polynomials this is treated in [22]. In [18] limit cases in the \(q\)-Askey scheme are also considered.
Finally we should mention the manuscript [15]. Here the author extended the duality [17 (4.2)] for continuous $q$-ultraspherical polynomials to Macdonald polynomials and thus obtained the so-called Pieri formula [21 §VI.6] for these polynomials.

Acknowledgement I thank Prof. M. A. Pathan and Prof. S. A. Ali for the invitation to deliver the 2017 R. P. Agarwal Memorial Lecture and for their cordiality during my trip to India on this occasion.

References

[1] R. P. Agarwal, Special functions, associated differential equations—their role, applications and importance, in: Special functions and differential equations, Allied Publishers, New Delhi, 1998, pp. 1–9.

[2] W. A. Al-Salam, Characterization theorems for orthogonal polynomials, in: Orthogonal polynomials: theory and practice, NATO ASI Series C, Vol. 294, Kluwer, 1990, pp. 1–24.

[3] R. Askey and J. A. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Mem. Amer. Math. Soc. 54 (1985), no. 319.

[4] E. Bannai and T. Ito, Algebraic combinatorics. I: Association schemes, Benjamin-Cummings, 1984.

[5] S. Bochner, Allgemeine lineare Differenzengleichungen mit asymptotisch konstanten Koeffizienten, Math. Z. 33 (1931), 426–450.

[6] J. J. Duistermaat and F. A. Grünbaum, Differential equations in the spectral parameter, Comm. Math. Phys. 103 (1986), 177–240.

[7] W. N. Everitt and C. Markett, On a generalization of Bessel functions satisfying higher-order differential equations, J. Comput. Appl. Math. 54 (1994), 325–349.

[8] J. Favard, Sur les polynomes de Tchebicheff, C. R. Acad. Sci. Paris 200 (1935), 2052–2053.

[9] G. Gasper and M. Rahman, Basic hypergeometric series, Cambridge University Press, Second ed., 2004.

[10] F. A. Grünbaum and L. Haine, The $q$-version of a theorem of Bochner, J. Comput. Appl. Math. 68 (1996), 103–114.

[11] W. Hahn, Über Orthogonalpolynome, die $q$-Differenzengleichungen genügen, Math. Nachr. 2 (1949), 4–34.

[12] M. E. H. Ismail, A generalization of a theorem of Bochner, J. Comput. Appl. Math. 159 (2003), 319–324.
[13] R. Koekoek, P. A. Lesky and R. F. Swarttouw, Hypergeometric orthogonal polynomials and their $q$-analogues, Springer-Verlag, 2010.

[14] E. Koelink and J. V. Stokman, The Askey–Wilson function transform scheme, in: Special Functions 2000: Current perspective and future directions, NATO Science Series II, Vol. 30, Kluwer, 2001, pp. 221–241; arXiv:math/9912140.

[15] T. H. Koornwinder, Self-duality for $q$-ultraspherical polynomials associated with root system $A_n$, unpublished manuscript, 1988; https://staff.fnwi.uva.nl/t.h.koornwinder/art/informal/dualmacdonald.pdf.

[16] T. H. Koornwinder, Jacobi functions as limit cases of $q$-ultraspherical polynomials, J. Math. Anal. Appl. 148 (1990), 44–54.

[17] T. H. Koornwinder, Dual addition formulas: the case of continuous $q$-ultraspherical and $q$-Hermite polynomials, arXiv:1803.09636v3 [math.CA], 2021.

[18] T. H. Koornwinder and M. Mazzocco, Dualities in the $q$-Askey scheme and degenerate DAHA, Studies Appl. Math. 141 (2018), 424–473.

[19] K. H. Kwon and D. W. Lee, Characterizations of Bochner–Krall orthogonal polynomials of Jacobi type, Constr. Approx. 19 (2003), 599–619.

[20] K. H. Kwon, L. L. Littlejohn and G. J. Yoon, Orthogonal polynomial solutions of spectral type differential equations: Magnus’ conjecture, J. Approx. Theory 112 (2001), 189–215.

[21] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford University Press, Second Edition, 1995.

[22] M. Noumi and J. V. Stokman, Askey–Wilson polynomials: an affine Hecke algebraic approach, in: Laredo Lectures on Orthogonal Polynomials and Special Functions, Nova Sci. Publ., Hauppauge, NY, 2014, pp. 111–144; arXiv:math/0001033.

[23] F. W. J. Olver et al., NIST Handbook of Mathematical Functions, Cambridge University Press, 2010; http://dlmf.nist.gov.

[24] L. Vinet and A. Zhedanov, Generalized Bochner theorem: characterization of the Askey–Wilson polynomials, J. Comput. Appl. Math. 211 (2008), 45–56.

[25] A. S. Zhedanov, “Hidden symmetry” of Askey–Wilson polynomials, Theoret. and Math. Phys. 89 (1991), 1146–1157.

T. H. Koornwinder, Korteweg-de Vries Institute, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands; email: thkmath@xs4all.nl