JSJ decompositions: definitions, existence, uniqueness.
I: The JSJ deformation space.

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Abstract

We give a simple general definition of JSJ decompositions by means of a universal maximality property. The JSJ decomposition should not be viewed as a tree (which is not uniquely defined) but as a canonical deformation space of trees. We prove that JSJ decompositions of finitely presented groups always exist, without any assumption on edge groups. Many examples are given.

1 Introduction

JSJ decompositions first appeared in 3-dimensional topology with the theory of the characteristic submanifold by Jaco-Shalen and Johannson [JaSh Jo]. For an orientable irreducible closed 3-manifold $M$, this can be described as follows. Let $T \subset M$ be a maximal disjoint union of non-parallel embedded tori such that any immersed torus can be homotoped to be disjoint from them. Then $T$ is unique up to isotopy, and any connected component of $M \setminus T$ is either atoroidal, or a Seifert fibered manifold.

This was carried over to group theory by Kropholler [Kro] for some Poincaré duality groups of dimension at least 3, and by Sela for torsion-free hyperbolic groups [Sel3]. The initials JSJ, standing for Jaco-Shalen and Johannson, were popularized by Sela, and constructions of JSJ decompositions were given in more general settings by many authors [RiSe, Bow, DuSa, FuPa, DuSw, ScSw].

In this group-theoretical context, one has a group $G$ and a class of subgroups $\mathcal{A}$ (such as cyclic groups, abelian groups...), and one tries to understand splittings (i.e. graph of groups decompositions) of $G$ over groups in $\mathcal{A}$ (in [DuSw ScSw], one looks at almost invariant sets rather than splittings, in closer analogy to the 3-manifold situation). The family of tori $T$ of the 3-manifold is replaced by a splitting of $G$ over $\mathcal{A}$. The authors construct a canonical splitting enjoying a long list of properties, rather specific to each case.

These ideas have had a vast influence and range of applications, from the isomorphism problem and the structure of the group of automorphisms of hyperbolic groups, to diophantine geometry over groups [Sel3 Sel1 Sel4].

In this paper, we do not propose a construction of JSJ decompositions, but rather a simple abstract definition stated by means of a universal maximality property, together with general existence and uniqueness statements stated in terms of deformation spaces (see below).

The JSJ decompositions constructed in [RiSe Bow DuSa FuPa] are JSJ decompositions in our sense, and we view these constructions as descriptions of JSJ decompositions and their flexible vertices. The regular neighbourhood of [ScSw] is of a different nature, whose relation with usual JSJ decompositions is explored in [GL5].

Several results of this paper and its sequel [GL6] were announced in [GL3].

A universal property. To explain the definition, let us first consider free decompositions of a group $G$, i.e. decompositions of $G$ as the fundamental group of a graph of
groups with trivial edge groups, or equivalently actions of $G$ on a tree $T$ with trivial edge stabilizers.

If $G = G_1 \ast \cdots \ast G_n$ where each $G_i$ is non-trivial, non-cyclic, and freely indecomposable, the Bass-Serre tree $T_0$ of this decomposition is maximal: if $T$ is associated to any other free decomposition, then $T_0$ dominates $T$ in the sense that there is a $G$-equivariant map $T_0 \to T$. In other words, among free decompositions of $G$, the tree $T_0$ is as far as possible from the trivial tree (a point): its vertex stabilizers are as small as possible (they are conjugates of the $G_i$’s). The maximality condition does not determine $T_0$ uniquely, we will come back to this key fact later.

If now $G = G_1 \ast \cdots \ast G_p \ast F_q$ with $G_i$ as above and $F_q$ a free group, one can take $T_0$ to be the universal covering of one of the graphs of groups pictured on Figure 1. Its vertex stabilizers are precisely the conjugates of the $G_i$’s (if $G$ is free, its action on $T_0$ is free). We call such a tree a Grushko tree.

![Figure 1: Graph of groups decompositions corresponding to two Grushko trees](image)

When more general decompositions are allowed, there may not exist a maximal tree. The fundamental example is the following. Consider a closed surface $\Sigma$, and two simple closed curves $c_1, c_2$ in $\Sigma$ with non-zero intersection number. Let $T_i$ be the Bass-Serre tree of the associated splitting of $\pi_1(\Sigma)$ over $\mathbb{Z} \simeq \pi_1(c_i)$. Since $c_1$ and $c_2$ have positive intersection number, $\pi_1(c_1)$ is hyperbolic in $T_2$ (it does not fix a point) and vice-versa. Using one-endedness of $\pi_1(\Sigma)$, it is an easy exercise to check that there is no splitting of $\pi_1(\Sigma)$ which dominates both $T_1$ and $T_2$. In this case there is no hope of having a maximal splitting over cyclic groups, similar to $T_0$ in the case of free splittings.

To overcome this difficulty, one restricts to universally elliptic splittings, defined as follows. Let $\mathcal{A}$ be a class of subgroups of $G$, stable under taking subgroups and under conjugation. We only consider $G$-trees with edge stabilizers in $\mathcal{A}$, which we call $\mathcal{A}$-trees.

**Definition 1.** An $\mathcal{A}$-tree is universally elliptic if its edge stabilizers are elliptic in every $\mathcal{A}$-tree.

Recall that $H$ is elliptic in $T$ if it fixes a point in $T$. Free decompositions are universally elliptic, but the trees $T_1, T_2$ introduced above are not.

If an $\mathcal{A}$-tree $T$ is universally elliptic, then one can read any $\mathcal{A}$-tree with finitely generated edge stabilizers from $T$ by blowing up vertices, and then performing a finite sequence of collapses and folds on this blow-up (see Remark 3.4).

**Definition 2.** A JSJ decomposition (or JSJ tree) of $G$ over $\mathcal{A}$ is an $\mathcal{A}$-tree $T$ such that

- $T$ is universally elliptic;
- $T$ dominates any other universally elliptic tree.

The second condition is a maximality condition. It means that, if $T'$ is universally elliptic, there is an equivariant map $T \to T'$. Equivalently, vertex stabilizers of $T$ are elliptic in every universally elliptic tree.

If $\mathcal{A}$ consists of all subgroups with a given property (being cyclic, abelian, slender, ...), we refer to, say, cyclic trees, cyclic JSJ decompositions when working over $\mathcal{A}$.
Uniqueness. JSJ trees are not unique. Returning to the example of free decompositions, one obtains trees with the same maximality property as $T_0$ by precomposing the action of $G$ on $T_0$ with any automorphism of $G$. One may also change the topology of the quotient graph $T_0/G$ (see Figure [1]). The canonical object is not a single tree, but the set of all trees with trivial edge stabilizers and non-trivial vertex stabilizers conjugate to the $G_i$'s, a deformation space.

**Definition 3** (Forester). The deformation space of a tree $T$ is the set of trees $T'$ such that $T$ dominates $T'$ and $T'$ dominates $T$. Equivalently, two trees are in the same deformation space $D$ if and only if they have the same elliptic subgroups.

More generally, given a family of subgroups $\tilde{\mathcal{A}} \subset \mathcal{A}$, one considers deformation spaces over $\tilde{\mathcal{A}}$ by restricting to trees in $D$ with edge stabilizers in $\tilde{\mathcal{A}}$.

For instance, Culler-Vogtmann’s outer space (the set of free actions of $F_n$ on trees) is a deformation space. Just like outer space, any deformation space is a complex in a natural way, and it is contractible (see [GL2, Cla1]).

If $T$ is a JSJ tree, a tree $T'$ is a JSJ tree if and only if $T'$ is universally elliptic, $T$ dominates $T'$, and $T'$ dominates $T$. In other words, $T'$ should belong to the deformation space of $T$ over $\mathcal{A}_{ell}$, where $\mathcal{A}_{ell}$ is the family of universally elliptic groups in $\mathcal{A}$.

**Definition 4.** If non-empty, the set of all JSJ decompositions of $G$ is a deformation space over $\mathcal{A}_{ell}$ called the JSJ deformation space (of $G$ over $\mathcal{A}$). We denote it by $D_{JSJ}$.

The canonical object is therefore not a particular JSJ decomposition, but the JSJ deformation space.

It is a general fact that two trees belong to the same deformation space if and only if one can pass from one to the other by applying a finite sequence of moves of certain types (see [For1, GL2, For3, Cla2] and Remark 2.3). The statements about uniqueness of the JSJ up to certain moves which appear in [RiSc, DuSa, FuPa] are special cases of this general fact.

If $\mathcal{A}$ is invariant under the group of automorphisms of $G$ (in particular if $\mathcal{A}$ is defined by restricting the isomorphism type), then so is $D_{JSJ}$, and one can gain information about Aut($G$) and Out($G$) by studying their action on the contractible complex $D_{JSJ}$ [CuVo, McMi, GL1, Cla2].

There are nice situations (for instance, splittings of a one-ended hyperbolic group over two-ended subgroups [Bow]) when one can construct a preferred tree in $D_{JSJ}$ (or in a related deformation space). Such a tree is a fixed point for the action of Out($G$). In [GL4, GL6], we explain how fixed points may sometimes be constructed as trees of cylinders or as maximal decompositions encoding compatibility of splittings (see the discussion at the end of this introduction).

Existence. Existence of the JSJ deformation space over any class $\mathcal{A}$ of subgroups (without any smallness assumption) when $G$ is finitely presented is a simple consequence of Dunwoody’s accessibility.

**Theorem 4.3.** If $G$ is finitely presented, the JSJ deformation space $D_{JSJ}$ of $G$ over $\mathcal{A}$ exists. It contains a tree whose edge and vertex stabilizers are finitely generated.

In the sequel to this paper [GL6] we use acylindrical accessibility to construct and describe JSJ decompositions of certain finitely generated groups, for instance abelian decompositions of CSA groups.
Given $G$ and $A$, all JSJ trees have the same vertex stabilizers, provided we restrict to stabilizers not in $A$. A vertex $v$, or its stabilizer $G_v$, is **rigid** if $G_v$ is universally elliptic (i.e. it fixes a point in every $A$-tree). For instance, all vertices of the Grushko tree $T_0$ studied above are rigid.

But the essential feature of JSJ theory is the description of **flexible** vertices (those which are not rigid), in particular the fact that in many contexts (see Theorem 7.7) flexible vertex stabilizers are extensions of 2-orbifold groups with boundary, attached to the rest of the group in a particular way (quadratically hanging subgroups). In other words, the example of trees $T_1, T_2$ given above using intersecting curves on a surface is often the only source of flexibility.

We prove in [GL6] that this is also the case for flexible subgroups of the JSJ deformation space of a relatively hyperbolic group with small parabolic subgroups over the class of its small subgroups.

However, in certain natural situations, flexible groups are not quadratically hanging subgroups, for instance for JSJ decompositions over abelian groups (see Subsection 8.3). We also consider relative JSJ decompositions, where one imposes that finitely many finitely generated subgroups be elliptic in all splittings considered, and we show that the description of flexible subgroups remains valid in this context (Theorem 8.20).

**Problem.** Describe flexible vertices of the JSJ deformation space of a finitely presented group over small subgroups.

In a sequel to this paper [GL6], we produce a canonical tree instead of a canonical deformation space, by replacing universal ellipticity by the more rigid notion of universal compatibility. If $T$ is universally elliptic, and $T'$ is arbitrary, one can refine $T$ (by blowing up vertices) to a tree $\hat{T}$ which dominates $T'$. We say that $T$ is **universally compatible** if, given $T'$, one can refine $T$ to a tree $\hat{T}$ which refines $T'$. We show that, if $G$ is finitely presented, there exists a maximal deformation space containing a universally compatible tree. Unlike $D_{JSJ}$, this deformation space always contains a preferred tree $T_{co}$, the **compatibility JSJ tree**. This is somewhat similar to Scott and Swarup’s construction who construct a canonical tree $T_{SS}$ which is compatible with any almost invariant set [ScSw]. The tree $T_{co}$ dominates $T_{SS}$, sometimes strictly.

Additionally, we use acylindrical accessibility and trees of cylinders to produce and describe JSJ deformation spaces and compatibility JSJ trees for some classes of finitely generated groups including CSA groups, $\Gamma$-limit groups with $\Gamma$ a hyperbolic group (possibly with torsion), and relatively hyperbolic groups. This also includes relative JSJ decompositions, relative to an arbitrary (possibly infinite) family of subgroups.

Let us now describe the contents of this paper. After preliminary sections, the JSJ deformation space is defined and constructed in Section 4. We also explain there why the constructions of [RiSe] [DuSa] [FuPa] are JSJ decompositions in our sense. Section 5 is devoted to relative JSJ decompositions. In Section 6, we give examples of JSJ decompositions with all vertices rigid (like the Grushko trees $T_0$ in the discussion above, these trees dominate every tree under consideration). This includes splittings over finite groups, parabolic splittings of relatively hyperbolic groups (relative to parabolic subgroups), and locally finite trees with small edge stabilizers (in particular, generalized Baumslag-Solitar groups).

In Section 7 we define quadratically hanging (QH) subgroups, and give some general properties of these groups. We state results from [RiSe] [DuSa] [FuPa] showing that flexible subgroups are QH-subgroups in JSJ decompositions over cyclic groups, or two-ended groups, or slender groups (Theorem 7.7). We also show that, under suitable assumptions, any QH vertex stabilizer of any $A$-tree is elliptic in JSJ trees (Section 7.5).
Finally, in Section 8 we give examples of flexible subgroups over abelian groups which are not QH-subgroups. We also show how a relative JSJ decomposition of a group $G$ may be viewed as an absolute decomposition of a larger group $\hat{G}$, using a filling construction. This allows us to extend the description of flexible subgroups to the relative case (the original proofs probably extend to the relative case, but this is unlikely to ever appear in print).

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2 Preliminaries

In all of this paper, $G$ will be a finitely generated group. Sometimes finite presentation will be needed, for instance to prove existence of JSJ decompositions.
We also fix a family $\mathcal{A}$ of subgroups of $G$ which is stable under conjugation and under taking subgroups. If $H$ is a subgroup, we sometimes write $\mathcal{A}|_H$ for the set of subgroups of $H$ which belong to $\mathcal{A}$.

Trees

A splitting of $G$ is an isomorphism of $G$ with the fundamental group of a graph of groups. A one-edge splitting (a graph of groups with one edge) is a splitting as an amalgam or an HNN extension. Using Bass-Serre theory, we view a splitting as an action of $G$ on a tree $T$ without inversion. The action $G \acts T$ is trivial if $G$ fixes a point, and minimal if there is no proper $G$-invariant subtree.

In this paper, all trees are endowed with a minimal action of $G$ without inversion (we allow the trivial case when $T$ is a point). We identify two trees if there is an equivariant isomorphism between them.

An $\mathcal{A}$-tree is a tree $T$ whose edge stabilizers belong to $\mathcal{A}$. We sometimes say that $T$ is over $\mathcal{A}$. We also say cyclic tree (abelian tree, ...) when $\mathcal{A}$ is the family of cyclic (abelian, ...) subgroups (we consider the trivial group as cyclic).

An element or a subgroup of $G$ is elliptic in $T$ if it fixes a point. If $H_1 \subset H_2 \subset G$ with $H_1$ of finite index in $H_2$, then $H_1$ is elliptic if and only if $H_2$ is. An element which is not elliptic is hyperbolic, it has a unique axis on which it acts by translation. If a finitely generated subgroup of $G$ is not elliptic, it contains a hyperbolic element and has a unique minimal invariant subtree.

A tree $T$ is irreducible if there exist two hyperbolic elements $g, h \in G$ whose axes intersect in a compact set. Equivalently, $T$ is not irreducible if and only if $G$ fixes a point, or an end, or preserves a line of $T$.

We denote by $V(T)$ and $E(T)$ the set of vertices and (non-oriented) edges of $T$ respectively, by $G_v$ or $G_e$ the stabilizer of a vertex $v$ or an edge $e$.

If $v \in V(T)$, its incident edge groups are the stabilizers of edges containing $v$, viewed as subgroups of $G_v$. We denote by $P_v$ the set of incident edge groups. It is a finite union of conjugacy classes of subgroups of $G_v$.

Relative trees

Besides $\mathcal{A}$, we sometimes also fix an arbitrary set $\mathcal{H}$ of subgroups of $G$, and we restrict to $\mathcal{A}$-trees $T$ such that each $H \in \mathcal{H}$ is elliptic in $T$ (in terms of graphs of groups, $H$ is contained in a conjugate of a vertex group). We call such a tree an $(\mathcal{A}, \mathcal{H})$-tree, or a tree over $\mathcal{A}$ relative to $\mathcal{H}$. The set of $(\mathcal{A}, \mathcal{H})$-trees does not change if we enlarge $\mathcal{H}$ by making it invariant under conjugation.

In particular, suppose that $G_v$ is a vertex stabilizer of a tree. We often consider splittings of $G_v$ relative to $P_v$, the set of incident edge groups. Such a splitting extends to a splitting of $G$ (see Remark 3.5).

Morphisms, collapse maps, refinements

Maps between trees will always be $G$-equivariant and linear on edges. We mention two particular classes of maps.

A map $f : T \to T'$ between two trees is a morphism if each edge of $T$ can be written as a finite union of subsegments, each of which is mapped bijectively onto a segment in $T'$. Equivalently, $f$ is a morphism if and only if one may subdivide $T$ and $T'$ so that $f$ maps
each edge onto an edge (in particular, no edge of $T$ is collapsed). Folds are examples of morphisms (see [BeFe]).

A collapse map $f : T \to T'$ is a map obtained by collapsing edges to points (by equivariance, the set of collapsed edges is $G$-invariant). Equivalently, $f$ preserves alignment: the image of any arc $[a, b]$ is a point or the arc $[f(a), f(b)]$. We say that $T'$ is a collapse of $T$, and that $T$ is a refinement of $T'$. In terms of graphs of groups, one obtains $T'/G$ by collapsing edges in $T/G$.

**Domination and deformation spaces**

**Definition 2.1.** A tree $T_1$ dominates $T_2$ if there is an equivariant map from $T_1$ to $T_2$. Equivalently, $T_1$ dominates $T_2$ if every vertex stabilizer of $T_1$ fixes a point in $T_2$: every subgroup which is elliptic in $T_1$ is also elliptic in $T_2$.

In particular, every refinement of $T_1$ dominates $T_1$. Beware that domination is defined by considering ellipticity of subgroups, not just of elements.

**Definition 2.2.** Having the same elliptic subgroups is an equivalence relation on the set of $A$-trees, an equivalence class is called a deformation space over $A$.

In other words, if $T_0$ is an $A$-tree, the deformation space of $T_0$ is the set of all $A$-trees $T$ such that $T$ and $T_0$ dominate each other, or equivalently the set of $A$-trees having the same elliptic subgroups as $T_0$. We say that a deformation space $D$ dominates a space $D'$ if trees in $D$ dominate those of $D'$. The deformation space of the trivial tree is called the trivial deformation space. It is the only deformation space in which $G$ is elliptic.

**Remark 2.3.** The deformation space of $T_0$ is the space of trees that may be obtained from $T_0$ by applying a finite sequence of collapse and expansion moves ([For1], see also [ClFo]). When the deformation space $D$ is non-ascending as defined in [GL2], for instance when all groups in $A$ are finite, any two reduced trees in $D$ may be joined by a finite sequence of slide moves. These facts will not be used in the present paper.

**Slender, small subgroups**

A group $H$ is slender if $H$ and all its subgroups are finitely generated. Examples of slender groups include virtually polycyclic groups. We denote by $VPC_n$ (resp. $VPC_{\leq n}$) the class of virtually polycyclic groups of Hirsch length exactly (resp. at most) $n$. If a slender group acts on a tree, it fixes a point or leaves a line invariant.

Following [GL2, page 177], we say that a subgroup $H \subset G$ (possibly infinitely generated) is small in $A$-trees if, whenever $G$ acts on an $A$-tree, $H$ fixes a point, or an end, or leaves a line invariant. Any small subgroup (in the sense of [BeFe]), in particular any subgroup not containing $F_2$, is small in $A$-trees. Unlike smallness, smallness in $A$-trees is stable under taking subgroups. It is also a commensurability invariant (but it is not invariant under abstract isomorphism). Moreover, $H$ is small in $A$-trees if and only if all its finitely generated subgroups are.

**Accessibility**

Constructions of JSJ decompositions are based on accessibility theorems stating that, given suitable $G$ and $A$, there is an a priori bound for the number of orbits of edges of $A$-trees, under the assumption that there is no redundant vertex: if $v$ has valence 2, it is the unique fixed point of some $g \in G$. This holds in particular:

(1) if $G$ is finitely presented and all groups in $A$ are finite [Dun];
(2) if $G$ is finitely presented, all groups in $\mathcal{A}$ are small, and the trees are reduced in the sense of [BeFe];

(3) if $G$ is finitely generated and all groups in $\mathcal{A}$ are finite with bounded order [Lin];

(4) if $G$ is finitely generated and the trees are $k$-acylindrical for some $k$ [Sc2];

(5) if $G$ is finitely generated and the trees are $(k,C)$-acylindrical: the stabilizer of any segment of length $>k$ has order $\leq C$ [We] (Del for finitely presented groups).

In this paper, we use a version of Dunwoody’s accessibility given in [FuPa] (see Proposition 4.3). In [GL6] we use acylindrical accessibility.

3 Universal ellipticity

Fix $\mathcal{A}$ as above. Let $T_1, T_2$ be two $\mathcal{A}$-trees.

Definition 3.1 (Ellipticity of trees). $T_1$ is elliptic with respect to $T_2$ if every edge stabilizer of $T_1$ is elliptic in $T_2$.

When $T_1$ is elliptic with respect to $T_2$, one can read $T_2$ from $T_1$: there is a refinement $\hat{T}_1$ of $T_1$ which dominates $T_2$. More precisely:

Lemma 3.2. Let $T_1, T_2$ be minimal $\mathcal{A}$-trees. If $T_1$ is elliptic with respect to $T_2$, there is a minimal $\mathcal{A}$-tree $\hat{T}_1$ such that:

1. $\hat{T}_1$ is a refinement of $T_1$ and dominates $T_2$;

2. the stabilizer of any edge of $\hat{T}_1$ fixes an edge in $T_1$ or in $T_2$;

3. every edge stabilizer of $T_2$ contains an edge stabilizer of $\hat{T}_1$;

4. a subgroup of $G$ is elliptic in $\hat{T}_1$ if and only if it is elliptic in both $T_1$ and $T_2$.

Remark 3.3. Conversely, if some refinement $\hat{T}_1$ dominates $T_2$, then $T_1$ is elliptic with respect to $T_2$: edge stabilizers of $T_1$ are elliptic in $\hat{T}_1$, hence also in $T_2$.

Remark 3.4. If edge stabilizers of $T_2$ are finitely generated, then $T_2$ can be obtained from $\hat{T}_1$ by a finite number of collapses and folds [BeFe].

Proof. For each vertex $v \in V(T_1)$, choose any $G_e$-invariant subtree $Y_v$ of $T_2$ (possibly $T_2$ itself). For each edge $e = vw \in E(T_1)$, choose vertices $p_v \in Y_v$ and $p_w \in Y_w$, fixed by $G_e$; this is possible because $G_e$ is elliptic in $T_2$. We make these choices $G$-equivariantly.

We can now define a tree $\hat{T}_1$ by blowing up each vertex $v$ of $T_1$ into $Y_v$, and attaching edges of $T_1$ using the points $p_v$. Formally, we consider the disjoint union $\bigcup_{v \in V(T_1)} Y_v$, and for each edge $vw$ of $T_1$ we attach an edge between $p_v \in Y_v$ and $p_w \in Y_w$.

Clearly, $\hat{T}_1$ is a refinement of $T_1$ and dominates $T_2$: one recovers $T_1$ from $\hat{T}_1$ by collapsing each $Y_v$ to a point, and one constructs a map $f : \hat{T}_1 \to T_2$ by mapping each added edge $vw$ to the segment $[p_v, p_w]$. If $\hat{T}_1$ is not minimal, we replace it by its minimal subtree. We now consider stabilizers.

The stabilizer of any edge of $\hat{T}_1$ fixes an edge in $T_2$ or in $T_1$, depending on whether the edge lies in some $Y_v$ or not. In particular, $\hat{T}_1$ is an $\mathcal{A}$-tree. By minimality, any edge $e$ of $T_2$ is contained in the image of some edge of $\hat{T}_1$, so $G_e$ contains an edge stabilizer of $\hat{T}_1$.

If a subgroup $H \subset G$ is elliptic in $\hat{T}_1$, then it is elliptic in $T_1$ and $T_2$. Conversely, if $H$ is elliptic in $T_1$ and $T_2$, let $v \in T_1$ be fixed by $H$. Since $H$ fixes a point in $Y_v$, it is elliptic in $\hat{T}_1$. \qed
Lemma 3.2. This element is hyperbolic in $T$ the same deformation space. Then some hyperbolic element of $T$ given by Lemma 3.2 does not belong to the same deformation space as the refinement of $\hat{T}$.

Proof. One needs only prove the first assertion when $T_2$ is obtained from $T_1$ by collapsing the orbit of an edge $e = uv$ if $u$ and $v$ are in the same orbit, or if $G_e \neq G_u$ and $G_e \neq G_v$, then some hyperbolic element of $T_1$ becomes elliptic in $T_2$. Otherwise, $T_1$ and $T_2$ are in the same deformation space.

For the second assertion, assume that $T_1$ does not dominate $T_2$. Then the tree $\hat{T}$ given by Lemma 3.2 does not belong to the same deformation space as $\hat{T}_1$. Since it is a refinement of $T_1$, some $g \in G$ is elliptic in $T_1$ and hyperbolic in $\hat{T}_1$. By Assertion (4) of Lemma 3.2, this element is hyperbolic in $T_2$, a contradiction. Thus $T_1$ dominates $T_2$.

Definition 3.7 (Universally elliptic). A subgroup $H \subset G$ is universally elliptic (over $A$) if it is elliptic in every $A$-tree. An $A$-tree $T$ is universally elliptic if its edge stabilizers are universally elliptic, i.e. if $T$ is elliptic with respect to every $A$-tree.

Lemma 3.8. (1) If $T_1, T_2$ are universally elliptic, the refinement $\hat{T}_1$ given by Lemma 3.2 is universally elliptic.

(2) If there is a morphism $f : S \rightarrow T$, and $T$ is universally elliptic, then so is $S$ (see Section 2 for the definition of morphisms).

(3) If $T_1, T_2$ are universally elliptic and have the same elliptic elements, they belong to the same deformation space.

Proof. The first assertion follows from Assertion (2) of Lemma 3.2. The second assertion is clear. The third one follows from the second assertion of Lemma 3.6.

The following lemma will be useful in [GL6].

Lemma 3.9. Let $(T_i)_{i \in I}$ be any family of trees. There exists a countable subset $J \subset I$ such that, if $T$ is elliptic with respect to every $T_i$ ($i \in I$), and $T$ dominates every $T_j$ for $j \in J$, then $T$ dominates $T_i$ for all $i \in I$.

Proof. Since $G$ is countable, we can find a countable $J$ such that, if an element $g \in G$ is hyperbolic in some $T_i$, then it is hyperbolic in some $T_j$ with $j \in J$. If $T$ dominates every $T_j$ for $j \in J$, any $g$ which is elliptic in $T$ is elliptic in every $T_i$. By Lemma 3.6, the tree $T$ dominates every $T_i$.

For many purposes, it is enough to consider one-edge splittings, i.e. $A$-trees with only one orbit of edges.

Lemma 3.10. Let $S$ be an $A$-tree.
(1) \( S \) is universally elliptic if and only if it is elliptic with respect to every one-edge splitting (over \( A \)).

(2) \( S \) dominates every universally elliptic \( A \)-tree if and only if it dominates every universally elliptic one-edge splitting.

**Proof.** By induction on the number of orbits of edges, using the following lemma. \( \square \)

**Lemma 3.11.** Let \( T \) be a tree, and \( H \) a subgroup of \( G \). Let \( E_1 \sqcup E_2 \) be a partition of \( E(T) \) into two \( G \)-invariant sets. Let \( T_1, T_2 \) be the trees obtained from \( T \) by collapsing \( E_1 \) and \( E_2 \) respectively.

(1) If a subgroup \( H \) is elliptic in \( T_1 \) and \( T_2 \), then \( H \) is elliptic in \( T \).

(2) If a tree \( T' \) dominates \( T_1 \) and \( T_2 \), then it dominates \( T \).

**Proof.** Let \( x_1 \in T_1 \) be a vertex fixed by \( H \). Let \( Y \subset T \) be its preimage. It is a subtree because the map \( T \to T_1 \) is a collapse map. Now \( Y \) is \( H \)-invariant and embeds into \( T_2 \). Since \( H \) is elliptic in \( T_2 \), it fixes a point in \( Y \), therefore it is elliptic in \( T \). One shows (2) by applying (1) to vertex stabilizers of \( T' \). \( \square \)

4 The JSJ deformation space

4.1 Definitions

We fix \( A \) as above. We define \( A_{\text{ell}} \subset A \) as the set of groups in \( A \) which are universally elliptic (\( A_{\text{ell}} \) is stable under conjugating and taking subgroups). An \( A \)-tree is universally elliptic if and only if it is an \( A_{\text{ell}} \)-tree.

**Definition 4.1** (JSJ deformation space). A deformation space \( D_{\text{JSJ}} \) of \( A_{\text{ell}} \)-trees which is maximal for domination is called the JSJ deformation space of \( G \) over \( A \) (it is unique if it exists by the first assertion of Lemma 3.8).

Trees in \( D_{\text{JSJ}} \) are called JSJ trees, or JSJ decompositions, of \( G \) over \( A \). They are precisely those \( A \)-trees \( T \) which are universally elliptic, and which dominate every universally elliptic tree.

In general there are many JSJ trees, but they all belong to the same deformation space and therefore have a lot in common (see Section 4 of [GL2]). In particular [GL2] Corollary 4.4), they have the same vertex stabilizers, except possibly for vertex stabilizers in \( A_{\text{ell}} \) (the groups in \( A \) which are universally elliptic).

**Definition 4.2** (Rigid and flexible vertices). Let \( H = G_v \) be a vertex stabilizer of a JSJ tree \( T \) over \( A \). We say that \( H \) is rigid if it is universally elliptic, flexible if it is not. We also say that the vertex \( v \) is rigid (flexible). If \( H \) is flexible, we say that it is a flexible subgroup of \( G \) over \( A \).

The definition of flexible subgroups of \( G \) does not depend on the choice of the JSJ tree \( T \). The heart of JSJ theory is to understand flexible groups. They will be discussed in Sections 7 and 8.

4.2 Existence

**Theorem 4.3.** Assume that \( G \) is finitely presented. Then the JSJ deformation space \( D_{\text{JSJ}} \) of \( G \) over \( A \) exists. It contains a tree whose edge and vertex stabilizers are finitely generated.
There is no hypothesis, such as smallness or finite generation, on the elements of $\mathcal{A}$. Finite presentability of $G$ is used to prove the existence of $D_{JSJ}$, through the following version of Dunwoody’s accessibility whose proof will be given after that of Theorem 4.3.

**Proposition 4.4** (Dunwoody’s accessibility). Let $G$ be finitely presented. Assume that $T_1 \leftarrow \cdots \leftarrow T_k \leftarrow T_{k+1} \leftarrow \cdots$ is a sequence of refinements of $\mathcal{A}$-trees. There exists an $\mathcal{A}$-tree $S$ such that:

1. for $k$ large enough, there is a morphism $S \to T_k$ (in particular, $S$ dominates $T_k$);
2. each edge and vertex stabilizer of $S$ is finitely generated.

Remark. Note that the maps $T_{k+1} \to T_k$ are required to be collapse maps.

One may view the following standard result as a special case (apply Proposition 4.4 to the constant sequence $T \leftarrow T \leftarrow \cdots$).

**Corollary 4.5.** If $G$ is finitely presented, and $T$ is an $\mathcal{A}$-tree, there exists a morphism $f : S \to T$ where $S$ is an $\mathcal{A}$-tree with finitely generated edge and vertex stabilizers (if $T$ is universally elliptic, so is $S$ by Lemma 3.8).

**Proof of Theorem 4.3.** Let $\mathcal{U}$ be the set of minimal universally elliptic $\mathcal{A}$-trees with finitely generated edge and vertex stabilizers, up to equivariant isomorphism. It is non-empty since it contains the trivial tree. An element of $\mathcal{U}$ is described by a finite graph of groups with finitely generated edge and vertex groups. Since $G$ only has countably many finitely generated subgroups, and there are countably many homomorphisms from a given finitely generated group to another, the set $\mathcal{U}$ is countable.

By Corollary 4.5, it suffices to produce a universally elliptic $\mathcal{A}$-tree dominating every $U \in \mathcal{U}$. Choose an enumeration $\mathcal{U} = \{U_1, U_2, \ldots, U_k, \ldots\}$. We define inductively a universally elliptic $\mathcal{A}$-tree $T_k$ which refines $T_{k-1}$ and dominates $U_1, \ldots, U_k$ (it may have infinitely generated edge or vertex stabilizers). We start with $T_1 = U_1$. Given $T_{k-1}$ which dominates $U_1, \ldots, U_{k-1}$, we let $T_k$ be a universally elliptic refinement of $T_{k-1}$ which dominates $U_k$, given by Lemma 3.8. Then $T_k$ also dominates $U_1, \ldots, U_{k-1}$ because $T_{k-1}$ does.

Apply Proposition 4.4 to the sequence $T_k$. The tree $S$ is universally elliptic because there are morphisms $S \to T_k$. Furthermore $S$ dominates every $T_k$, hence every $U_k$. This shows that $S$ is a JSJ tree over $\mathcal{A}$.

**Proof of Proposition 4.4.** This is basically Proposition 5.12 of [FuPa]. We sketch the argument for completeness.

Let $X$ be a Cayley simplicial 2-complex for $G$, i.e. $X$ is a simply connected simplicial 2-complex with a free cocompact action of $G$. By considering preimages of midpoints of edges under a suitable equivariant map $X \to T_k$, one constructs a $G$-invariant pattern $P_k$ on $X$ such that there is a morphism $S_{P_k} \to T_k$, where $S_{P_k}$ is the tree dual to $P_k$. Since $T_{k+1}$ refines $T_k$, one can assume $P_k \subset P_{k+1}$.

By Dunwoody’s theorem [Dim] Theorem 2.2, there is a bound on the number of non-parallel tracks in $X/G$. Thus there exists $k_0$ such that for all $k \geq k_0$ one obtains $S_{P_k}$ from $S := S_{P_{k_0}}$ by subdividing edges, and therefore there is a morphism $S \to T_k$ for $k \geq k_0$. Edge and vertex stabilizers of $S$ are finitely generated since they are images in $G$ of fundamental groups of nice compact subsets of $X/G$.

**Remark 4.6.** When $\mathcal{A}$ consists of finite groups with bounded order, Proposition 4.4 is true if $G$ is only assumed to be finitely generated. This follows from Linnell’s accessibility [Lin]: for $k$ large, $T_{k+1}$ is just a subdivision of $T_k$. The JSJ deformation space therefore always exists in this case.

We record the following simple facts for future reference.
Lemma 4.7. Any $A$-tree $T$ with universally elliptic vertex stabilizers is a JSJ tree.

Proof. This is clear since $T$ dominates every $A$-tree. \hfill \Box

Lemma 4.8. If there is a JSJ tree, then any $A$-tree $T$ with universally elliptic edge stabilizers may be refined to a JSJ tree.

Proof. Let $T_2$ be a JSJ tree. Apply Lemma 3.2 with $T_1 = T$. The refinement $\hat{T}_1$ of $T$ is universally elliptic and dominates $T_2$, so is a JSJ tree. \hfill \Box

4.3 Relation with other constructions

Several authors have constructed JSJ splittings of finitely presented groups in various settings. We explain here why those splittings are JSJ splittings in our sense (results in the literature are often stated only for one-edge splittings, but this is not a restriction by Lemma 3.10).

In [RiSe], Rips and Sela consider cyclic splittings of a one-ended group $G$ (so $A$ consists of all cyclic subgroups of $G$, including the trivial group). Theorem 7.1 in [RiSe] says that the produced JSJ splitting is universally elliptic (this is statement (iv)) and maximal (statement (iii)). The uniqueness up to deformation is statement (v).

In [FuPa], Fujiwara and Papasoglu consider all splittings of a group over the class $A$ of its slender subgroups. Statement (2) in [FuPa, Theorem 5.13] says that the produced JSJ splitting is elliptic with respect to any minimal splitting. By Proposition 3.7 in [FuPa], any splitting is dominated by a minimal splitting, so universal ellipticity holds. Statement (1) of Theorem 5.15 in [FuPa] implies maximality.

In the work of Dunwoody-Sageev [DuSa], the authors consider splittings of a group $G$ over slender subgroups in a class $\mathcal{ZK}$ such that $G$ does not split over finite extensions of infinite index subgroups of $\mathcal{ZK}$ (there are restrictions on the class $\mathcal{ZK}$, but one can typically take $\mathcal{ZK} = VPC_n$, see [DuSa] for details). In our notation, $A$ is the set of subgroups of elements of $\mathcal{ZK}$. Universal ellipticity of the constructed splitting follows from statement (3) in the Main Theorem of [DuSa], and from the fact that any edge group is contained in a white vertex group. Maximality follows from the fact that white vertex groups are universally elliptic (statement (3)) and that black vertex groups either are in $\mathcal{ZK}$ (in which case they are universally elliptic by the non-splitting assumption made on $G$), or are $\mathcal{K}$-by-orbifold groups and hence are necessarily elliptic in any JSJ tree (see Proposition 7.13 below).

5 Relative JSJ decompositions

5.1 Definition and existence

The construction of the JSJ deformation space may be done in a relative setting. In this subsection, we show how to adapt the construction given in Section 4. In Section 8, we will give another approach (valid under similar hypotheses) and study flexible vertices. In [GL6] we will construct relative JSJ decompositions of certain groups using acylindrical accessibility instead of Dunwoody’s accessibility (this does not require finite presentability).

Besides $A$, we fix a set $\mathcal{H}$ of subgroups of $G$ and we only consider $(A, \mathcal{H})$-trees: $A$-trees $T$ such that each $H \in \mathcal{H}$ is elliptic in $T$. Of course, universal ellipticity is defined using only $(A, \mathcal{H})$-trees. We now denote by $A_{\text{ell}}$ the set of elements of $A$ which are elliptic in every $(A, \mathcal{H})$-tree.

The JSJ deformation space of $G$ over $A$ relative to $\mathcal{H}$, if it exists, is the unique deformation space of $(A_{\text{ell}}, \mathcal{H})$-trees which is maximal for domination. Note that the JSJ space does not change if we enlarge $\mathcal{H}$ by making it invariant under conjugation.
Suppose that $\mathcal{H} = \{H_1, \ldots, H_p\}$ is a finite family. One says that $G$ is \textit{finitely presented relative to $\mathcal{H}$} if it is a quotient $G \simeq (F * H_1 * \cdots * H_p)/N$, where $F$ is a finitely generated free group, $N$ is the normal closure of a finite set, and each $H_i$ maps isomorphically onto $H_i$. If we drop the finiteness condition on $N$, we say that $G$ is \textit{finitely generated relative to $\mathcal{H}$}. Note that replacing each $H_i$ by a conjugate does not change the fact that $G$ is relatively finitely presented or generated. A finitely presented group is finitely presented relative to any finite collection of finitely generated subgroups. A relatively hyperbolic group is finitely presented relative to its maximal parabolic subgroups [Osi].

**Theorem 5.1.** Assume that $G$ is finitely presented relative to $\mathcal{H} = \{H_1, \ldots, H_p\}$. Then the JSJ deformation space $D_{JSJ}$ of $G$ over $A$ relative to $\mathcal{H}$ exists.

Moreover, $D_{JSJ}$ contains a tree with finitely generated edge stabilizers (and finitely generated vertex stabilizers if all $H_i$’s are finitely generated).

The theorem is proved as in the non-relative case, using a relative version of Proposition 4.4, except that one has to work with a relative Cayley 2-complex $X$ as follows. Consider connected simplicial complexes $X_1, \ldots, X_p$ (not necessarily compact), with base points $*$, such that $\pi_1(X_i,*) = H_i$. Glue finitely many 2-cells to the wedge of the $X_i$’s and finitely many circles to get a space $X$ with $\pi_1(X) = G$. In the universal cover $\tilde{\pi} : \tilde{X} \to X$, consider $Y = \pi^{-1}(X_1) \sqcup \ldots \sqcup \pi^{-1}(X_p)$. The stabilizer of each connected component of $Y$ is conjugate to some $H_i$, and $\tilde{\pi}(\tilde{X} \setminus Y)$ is relatively compact. If each $H_i$ fixes a point in a tree $T$, there is an equivariant map $f : \tilde{X} \to T$ that maps each component of $Y$ to a vertex of $T$, and preimages of midpoints of edges of $T$ define a $G$-invariant pattern in $\tilde{X} \setminus Y$. The tree dual to this pattern is relative to $\mathcal{H}$ because the pattern does not intersect $Y$. One concludes as in the absolute case using Dunwoody’s bound on the number of non-parallel tracks in $X \setminus (X_1 \sqcup \cdots \sqcup X_p)$.

### 5.2 JSJ decompositions of vertex groups

We now record relations between the JSJ decomposition of a group, and relative splittings of vertex stabilizers. All trees are assumed to have edge stabilizers in $A$. Splittings of a vertex stabilizer $G_v$ are over $A_{G_v}$, the family of subgroups of $G_v$ belonging to $A$, and relative to the incident edge stabilizers.

**Lemma 5.2.** Let $G_v$ be a vertex stabilizer of a universally elliptic tree $T$. Let $\mathcal{P}_v$ be the set of incident edge stabilizers.

1. A subgroup of $G_v$ is universally elliptic (as a subgroup of $G$) if and only if it is elliptic in every splitting of $G_v$ relative to $\mathcal{P}_v$.

2. Assume that $T$ is a JSJ tree. Then $G_v$ has no non-trivial splitting relative to $\mathcal{P}_v$ over a universally elliptic subgroup. The group $G_v$ is universally elliptic if and only if it has no non-trivial splitting relative to $\mathcal{P}_v$.

3. Let $S$ be a JSJ tree of $G$. The action of $G_v$ on its minimal subtree in $S$ is a JSJ tree of $G_v$ relative to $\mathcal{P}_v$. In particular, $G_v$ is elliptic in the JSJ deformation space of $G$ if and only if its JSJ decomposition relative to $\mathcal{P}_v$ is trivial.

**Proof.** If $H \subset G_v$ is not elliptic in some tree with an action of $G$, restricting the action to $G_v$ shows the “if” direction of the first assertion; note that the groups of $\mathcal{P}_v$ are elliptic in the induced splitting because they are universally elliptic. By Remark 3.5, any splitting of $G_v$ relative to $\mathcal{P}_v$ may be used to refine $T$. This shows the other direction.
The second assertion follows similarly, using the maximality of JSJ trees. For the third assertion, we may assume by Lemma 4.8 that $S$ is a refinement of $T$. We then use Assertion (1).

**Lemma 5.3.** Let $T$ be a universally elliptic tree. Then $G$ has a JSJ tree if and only if every vertex stabilizer $G_v$ of $T$ has a JSJ tree relative to $\mathcal{P}_v$.

In this case, one can refine $T$ using these trees so as to get a JSJ tree of $G$.

**Proof.** If $G$ has a JSJ tree, so does $G_v$ by Lemma 5.2 (3). Conversely, assume that $G_v$ has a JSJ tree $T_v$ relative to $\mathcal{P}_v$. Choose such a $T_v$ in a $G$-equivariant way. Let $\hat{T}$ be the corresponding refinement of $T$ as in Remark 3.5. It is universally elliptic by Assertion (1) of Lemma 5.2. Maximality of $\hat{T}$ follows from maximality of $T_v$ as in the proof of Lemma 4.8: given a universally elliptic $T_2$, the refinement of $\hat{T}$ which dominates $T_2$ belongs to the same deformation space as $\hat{T}$ by Assertion (1) of Lemma 5.2.

**Remark 5.4.** Lemmas 5.2 and 5.3 hold if one considers JSJ trees relative to $\mathcal{H}$, as in Subsection 5.1, provided one adds to $\mathcal{P}_v$ all subgroups of $G_v$ of the form $H \cap G_v$ with $H$ conjugate to a group of $\mathcal{H}$. Since $H \cap G_v$ fixes an edge incident to $v$ if $H$ is not contained in $G_v$, it suffices to add to $\mathcal{P}_v$ the subgroups of $G_v$ which are conjugate to a group of $\mathcal{H}$.

6 Rigid examples

In this section, we shall construct $\mathcal{A}$-trees whose vertex stabilizers are universally elliptic; they are JSJ trees by Lemma 4.7 (they dominate every $\mathcal{A}$-tree), and all their vertices are rigid. Unless specified otherwise, we only assume that $G$ is finitely generated, so JSJ trees are not guaranteed to exist.

6.1 Free groups

Let $G = F_n$ be a finitely generated free group, and let $\mathcal{A}$ be arbitrary. Then the JSJ deformation space of $F_n$ over $\mathcal{A}$ is the space of free actions (unprojectivized Culler-Vogtmann’s Outer Space $[\text{CuVo}]$).

More generally, if $G$ is virtually free and $\mathcal{A}$ contains all finite subgroups, then $D_{JSJ}$ is the space of trees with finite vertex stabilizers.

6.2 Free splittings: the Grushko deformation space

Let $\mathcal{A}$ consist only of the trivial subgroup of $G$. Then the JSJ deformation space exists, it is the outer space introduced in $[\text{GL}]$ (see $[\text{McMi}]$ when no free factor of $G$ is $\mathbb{Z}$, $[\text{CuVo}]$ when $G = F_n$). This is the set of Grushko trees $T$ in the following sense: edge stabilizers are trivial and vertex stabilizers are freely indecomposable (we consider $\mathbb{Z}$ as freely decomposable since it splits over the trivial group). Denoting by $G = G_1 \ast \cdots \ast G_p \ast F_q$ a decomposition of $G$ given by Grushko’s theorem (with $G_i$ non-trivial and freely indecomposable, and $F_q$ free), the quotient graph of groups $T/G$ is homotopy equivalent to a wedge of $q$ circles, it has one vertex with group $G_i$ for each $i$, and all other vertex groups are trivial (see Figure 1).

6.3 Splittings over finite groups: the Stallings-Dunwoody deformation space

If $\mathcal{A}$ is the set of finite subgroups of $G$, the JSJ deformation space is the set of trees whose edge groups are finite and whose vertex groups have 0 or 1 end (by Stallings’s theorem, an $\mathcal{A}$-tree is maximal for domination if and only if its vertex stabilizers have at most one
end). By definition, the JSJ deformation space exists if and only if 

$G$ is accessible, in particular if $G$ is finitely presented by Dunwoody’s original accessibility result \cite{Dun}.

As mentioned in Remark \[4.6\], Linnell’s accessibility implies that the JSJ space exists if $G$ is finitely generated and $A$ consists of finite groups with bounded order.

### 6.4 Parabolic splittings

Assume that all groups in $A$ are universally elliptic. Then all vertex stabilizers of a JSJ tree $T$ are rigid by Lemma \[5.2\] (and Remark \[5.4\] in the relative case).

In particular, assume that $G$ is hyperbolic relative to a family of finitely generated subgroups $H = \{H_1, \ldots, H_p\}$. Recall that a subgroup of $G$ is parabolic if it is contained in a conjugate of an $H_i$. We let $P$ be the family of parabolic subgroups, and we consider splittings over $P$ relative to $H$ (equivalently, relative to $P$).

Parabolic subgroups are universally elliptic (relative to $H$!), so the JSJ trees over parabolic subgroups, relative to $H$, do not have flexible vertices (these trees exist because $G$ is finitely presented relative to $H$, see Subsection \[5.1\]).

### 6.5 Splittings of small groups

Recall that $G$ is small in $A$-trees if it has no irreducible action on a $A$-tree (irreducible meaning non-trivial, without any invariant line or end).

**Proposition 6.1.** If $G$ is small in $A$-trees, there is at most one non-trivial deformation space containing a universally elliptic $A$-tree.

**Proof.** If $T$ is a non-trivial universally elliptic tree, every vertex stabilizer contains an edge stabilizer with index at most 2 (the index is 2 if $G$ acts dihedrally on a line), so is universally elliptic. It follows that any two such trees belong to the same deformation space.

If there is a deformation space as in the proposition, it is the JSJ space. Otherwise, the JSJ space is trivial.

Consider for instance cyclic splittings of solvable Baumslag-Solitar groups $BS(1, n)$. If $n = 1$ (so $G = \mathbb{Z}$), there are infinitely many deformation spaces and no non-trivial universally elliptic tree. If $n = -1$ (Klein bottle group), there are exactly two deformation spaces but no non-trivial universally elliptic tree. If $n \neq \pm 1$, the JSJ space is non-trivial, as we shall now see.

### 6.6 Generalized Baumslag-Solitar groups

Let $G$ be a generalized Baumslag-Solitar group, i.e. it acts on a tree $T$ with all vertex and edge stabilizers infinite cyclic. Let $A$ be the set of cyclic subgroups of $G$ (including the trivial subgroup). Unless $G$ is isomorphic to $\mathbb{Z}$, $\mathbb{Z}^2$, or to the Klein bottle group, the deformation space of $T$ is the JSJ deformation space \cite{For2}.

Here is a short proof (the arguments are contained in \cite{For2}). We show that every vertex stabilizer $H$ of $T$ is universally elliptic. Clearly, the commensurator of $H$ is $G$. If $H$ is hyperbolic in an $A$-tree $T'$, its commensurator $G$ preserves its axis, so $T'$ is a line. This implies that $G$ is $\mathbb{Z}$, $\mathbb{Z}^2$, or a Klein bottle group, a contradiction.

### 6.7 Locally finite trees

We generalize the previous example to small splittings (i.e. splittings over small groups).

We suppose that $G$ acts irreducibly on a locally finite tree $T$ with all edge stabilizers small in $A$-trees (local finiteness is equivalent to edge stabilizers having finite index in neighboring vertex stabilizers; in particular, vertex stabilizers are small in $A$-trees).
Lemma 8.5, we proved that all such trees $T$ belong to the same deformation space. This happens to be the JSJ deformation space.

**Lemma 6.2.** Suppose that all groups of $A$ are small in $A$-trees. Then any locally finite irreducible $A$-tree $T$ belongs to the JSJ deformation space over $A$.

**Proof.** We show that every vertex stabilizer $H$ of $T$ is universally elliptic. Assume that $H$ is not elliptic in some $A$-tree $T'$. If $H$ contains a hyperbolic element, then it preserves a unique line or end of $T'$ because it is small in $A$-trees. If it consists of elliptic elements but does not fix a point, then $H$ fixes a unique end of $T'$. Moreover, in both cases, any finite index subgroup of $H$ preserves the same unique line or end.

As in the previous subsection, local finiteness implies that $G$ commensurizes $H$, so it preserves this $H$-invariant line or end of $T'$ (in particular, $T'$ is not irreducible). We now define a normal subgroup $G' \subset G$ which is small in $A$-trees. If $G$ does not act dihedrally on a line, we let $G' = [G, G]$. It is small in $A$-trees: any finitely generated subgroup pointwise fixes a ray of $T'$, so is contained in an edge stabilizer $G_e \in A$. If $G$ acts dihedrally, we let $G'$ be the kernel of the action.

Consider the action of the normal subgroup $G'$ on $T$. If it is elliptic, its fixed point set is $G$-invariant, so by minimality the action of $G$ factors through the action of an abelian or dihedral group; this contradicts the irreducibility of $T$. If $G'$ preserves a unique line or end, this line or end is $G$-invariant because $G'$ is normal, again contradicting the irreducibility of $T$. \[\square\]

7 **QH subgroups**

Flexible subgroups of the JSJ deformation space are most important, as understanding their splittings conditions the understanding of the splittings of $G$. In well-understood cases, flexible subgroups are *quadratically hanging* (QH) subgroups. In this section, after preliminaries about 2-orbifold groups, we define QH-subgroups and we establish some general properties. We then quote several results saying that flexible vertices are QH, and we prove that, under suitable hypotheses, any QH-subgroup is elliptic in the JSJ deformation space.

### 7.1 2-orbifolds

We consider a compact 2-orbifold $\Sigma$ with $\pi_1(\Sigma)$ not virtually abelian. Such an orbifold is hyperbolic, we may view it as the quotient of a compact orientable hyperbolic surface $\Sigma_0$ with geodesic boundary by a finite group of isometries $\Lambda$. If we forget the orbifold structure, $\Sigma$ is homeomorphic to a surface $\Sigma_{\text{top}}$.

The image of $\partial \Sigma_0$ is the boundary $\partial \Sigma$ of $\Sigma$. Each component $C$ of $\partial \Sigma$ is either a component of $\partial \Sigma_{\text{top}}$ (a circle) or an arc contained in $\partial \Sigma_{\text{top}}$. The (orbifold) fundamental group of $C$ is $\mathbb{Z}$ or an infinite dihedral group accordingly. A *boundary subgroup* is a subgroup of $\pi_1(\Sigma)$ which is conjugate to the fundamental group of a component $C$ of $\partial \Sigma$.

The closure of the complement of $\partial \Sigma$ in $\partial \Sigma_{\text{top}}$ is a union of *mirrors*: a mirror is the image of a component of the fixed point set of an orientation-reversing element of $\Lambda$. Each mirror is itself a circle or an arc contained in $\partial \Sigma_{\text{top}}$. Mirrors may be adjacent, whereas boundary components of $\Sigma$ are disjoint.

**Lemma 7.1.** Let $C$ be a boundary component of a compact hyperbolic 2-orbifold $\Sigma$. There exists a non-trivial splitting of $\pi_1(\Sigma)$ over $\{1\}$ or $\mathbb{Z}/2\mathbb{Z}$ relative to the fundamental groups $J_k$ of all boundary components distinct from $C$.

**Proof.** Any arc $\gamma$ properly embedded in $\Sigma_{\text{top}}$ and with endpoints on $C$ defines a free splitting of $\pi_1(\Sigma)$ relative to the groups $J_k$. In most cases one can choose $\gamma$ so that this
splitting is non-trivial. The exceptional cases are when $\Sigma_{\text{top}}$ is a disc or an annulus, and $\Sigma$ has no conical point.

If $\Sigma_{\text{top}}$ is a disc, its boundary circle consists of components of $\partial \Sigma$ and mirrors. Since $\Sigma$ is hyperbolic, there must be a mirror $M$ not adjacent to $C$ (otherwise $\partial \Sigma_{\text{top}}$ would consist of $C$ and one or two mirrors, or two boundary components and two mirrors). An arc $\gamma$ with one endpoint on $C$ and the other on $M$ defines a splitting over $\mathbb{Z}/2\mathbb{Z}$, which is non-trivial because $M$ is not adjacent to $C$.

If $\Sigma_{\text{top}}$ is an annulus, there are two cases. If $C$ is an arc, one can find an arc $\gamma$ from $C$ to $C$ as in the general case. If $C$ is a circle in $\partial \Sigma_{\text{top}}$, the other circle contains a mirror $M$ (otherwise $\Sigma$ would be a regular annulus) and an arc $\gamma$ from $C$ to $M$ yields a splitting over $\mathbb{Z}/2\mathbb{Z}$.

Remark 7.2. If the splitting constructed is over $K = \mathbb{Z}/2\mathbb{Z}$, this $K$ is contained in a 2-ended subgroup (generated by $K$ and a conjugate).

We denote by $\tilde{\Sigma}$ the universal covering of $\Sigma$, a convex subset of $\mathbb{H}^2$ with geodesic boundary. A (bi-infinite) geodesic $\gamma \subset \tilde{\Sigma}$ is closed if its image in $\Sigma$ is compact, and simple if $h\gamma$ and $\gamma$ are equal or disjoint for all $h \in \pi_1(\Sigma)$. If $\gamma \not\subset \partial \tilde{\Sigma}$, we say that its projection $\delta$ is an essential simple closed geodesic in $\Sigma$ (possibly one-sided). We then denote by $H_\gamma$ the 2-ended subgroup of $\pi_1(\Sigma)$ consisting of elements which preserve $\gamma$ and each of the half-spaces bounded by $\gamma$.

There is a non-trivial one-edge splitting of $\pi_1(\Sigma)$ over $H_\gamma$ relative to the boundary subgroups, we say that it is dual to $\delta$ (if $\delta$ is one-sided, this splitting can be viewed as the splitting dual to the boundary of a regular neighbourhood of $\delta$, a connected 2-sided simple 1-suborbifold). More generally, any family of disjoint simple closed geodesics $\delta_i$ gives rise to a dual splitting.

7.2 Quadratically hanging subgroups

Let $Q$ be a vertex stabilizer of an $A$-tree.

Definition 7.3. We say that $Q$ is a QH-subgroup (over $A$) if it is an extension $1 \to F \to Q \to \pi_1(\Sigma) \to 1$, with $\Sigma$ a hyperbolic 2-orbifold as above, and each incident edge group is an extended boundary subgroup: its image in $\pi_1(\Sigma)$ is finite or contained in a boundary subgroup of $\pi_1(\Sigma)$ (in particular, the preimage in $Q$ of the stabilizer of a cone point of $\Sigma$ is an extended boundary subgroup). We call $F$ the fiber.

We say that a boundary component $C$ of $\Sigma$ is used if there exists an incident edge group whose image in $\pi_1(C)$ is contained in $\pi_1(\Sigma)$ with finite index (recall that $\pi_1(C)$ is cyclic or dihedral, so having finite index is the same as being infinite).

Note that the terminology QH is used by Rips-Sela in [RiSe] with a more restrictive meaning ($F = \{1\}$ and $\Sigma$ has no mirror).

7.3 General properties of QH subgroups

Let $\Sigma$ be as above. We first recall that splittings of $\pi_1(\Sigma)$ over small (i.e. virtually cyclic) subgroups are dual to families of simple closed curves.

Lemma 7.4. Let $T$ be a tree with a non-trivial minimal action of $\pi_1(\Sigma)$, without inversion, with small edge stabilizers and with all boundary subgroups elliptic. Then $T$ is equivariantly isomorphic to the tree dual to a family of disjoint simple closed geodesics of $\Sigma$ (possibly one-sided, see Subsection 7.1).

If edge stabilizers are not assumed to be small, $T$ is still dominated by a tree dual to a family of disjoint simple closed geodesics of $\Sigma$. 

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Proof. When $\Sigma$ is an orientable surface, this follows from Theorem II I.2.6 of [McSh], noting that all small subgroups of $\pi_1(\Sigma)$ are cyclic. If $\Sigma$ is a 2-orbifold, we consider a covering surface $\Sigma_0$ as in Subsection [7.1]. The action of $\pi_1(\Sigma_0)$ on $T$ is dual to a family of closed geodesics on $\Sigma_0$. This family is $\Lambda$-invariant and projects to the required family on $\Sigma$. The action of $\pi_1(\Sigma)$ on $T$ is dual to this family, as defined in Subsection [7.1].

The second statement follows from standard arguments (see the proof of [MoSh, Theorem III.2.6]).

The following proposition shows that Lemma 7.4 applies under natural conditions.

**Proposition 7.5.** Let $Q$ be a QH vertex group of an $A$-tree $T$, with fiber $F$ and underlying orbifold $\Sigma$.

1. If $A$ consists of slender groups, and $F$ is slender, then $F$ is universally elliptic.
2. If $F$ is universally elliptic, and if $T'$ is any $A$-tree, there is a $Q$-invariant subtree $T_Q \subset T'$ such that the action of $Q$ on $T_Q$ factors through an action of $\pi_1(\Sigma)$.
3. Assume that $F$, and any subgroup containing $F$ with index 2, belongs to $A$. If $T$ is a JSJ decomposition over $A$, and if $F$ is universally elliptic, then every boundary component of $\Sigma$ is used. Moreover, every extended boundary subgroup of $Q$ is universally elliptic.

Proof. Suppose $F$ is not universally elliptic. Being slender, it would act non-trivially on a line in some $A$-tree with an action of $G$. Since $F$ is normal in $Q$, this line would be $Q$-invariant and $Q$ would act on it with slender edge stabilizers. This is a contradiction since $Q$ is not slender.

For the second assertion, let $T_Q$ be the fixed subtree of $F$. Since $F$ is normal in $Q$, it is $Q$-invariant.

Let $C$ be a boundary component of $\Sigma$. Lemma 7.4 yields a non-trivial splitting of $Q$ over a group containing $F$ with index $\leq 2$, hence in $A$ and universally elliptic. If $C$ is not used, every incident edge group is elliptic in this splitting. By Remark 3.5, one can use this decomposition of $Q$ to produce a universally elliptic splitting of $G$ in which $Q$ is not elliptic. This contradicts the maximality of the JSJ decomposition. Since $F$ and all edge stabilizers of $T$ are universally elliptic, so are extended boundary subgroups.

We now prove that, under natural hypotheses, the only universally elliptic elements in a QH-subgroup $Q$ lie in extended boundary subgroups.

**Proposition 7.6.** Let $Q$ be a QH vertex group of an $A$-tree $T$, with fiber $F$ and underlying orbifold $\Sigma$.

1. If $F$, and extended boundary subgroups of $Q$, are universally elliptic, but $Q$ is not universally elliptic, then $\Sigma$ contains a essential simple closed geodesic (as defined in Subsection [7.1]).
2. Assume that the preimages in $Q$ of all two-ended subgroups of $\pi_1(\Sigma)$ belong to $A$. If $\Sigma$ contains an essential simple closed geodesic, any universally elliptic element of $Q$ lies in an extended boundary subgroup. In particular, $Q$ is not universally elliptic.

Proof. Since $Q$ is not universally elliptic, it acts non-trivially on some $A$-tree $T'$. As in Proposition 7.5, the action of $Q$ on its minimal subtree $T_Q$ factors through $\pi_1(\Sigma)$. By assumption, the boundary subgroups of $\pi_1(\Sigma)$ fix a point in $T'$, so the action is dominated by one which is dual to a system of simple closed essential geodesics (Lemma 7.4). In particular, we can find an essential simple closed geodesic $\gamma$ in $\Sigma \setminus \partial \Sigma$. 

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For (2), let \( g \) be an element of \( Q \) that does not lie in an extended boundary subgroup, and represent the image of \( g \) in \( \pi_1(\Sigma) \) by an immersed geodesic \( \delta \) not parallel to the boundary.

If \( \Sigma \) is an orientable surface, the existence of an essential simple closed geodesic implies that \( \Sigma \) is filled by such geodesics. In particular, \( \delta \) intersects such a simple geodesic \( \gamma \), and \( g \) is hyperbolic in the splitting of \( G \) dual to \( \gamma \). Our assumptions guarantee that this splitting is over \( A \), so \( g \) is not universally elliptic.

If \( \Sigma \) is an orbifold, the argument works the same, but we need to check that simple geodesics fill the orbifold. Starting with a simple geodesic \( \gamma \) as above, Lemma 5.3 of [Gui] ensures that there exists another simple geodesic \( \gamma_1 \) intersecting \( \gamma \) non-trivially. Let \( \Gamma_1 \subset \Sigma \) be the suborbifold with geodesic boundary filled by \( \gamma, \gamma_1 \), constructed by projecting the (necessarily \( \Lambda \)-invariant) subsurface with geodesic boundary filled by the preimage of \( \gamma \cup \gamma_1 \) in the surface \( \Sigma_0 \). Applying again Lemma 5.3 of [Gui] to any component of \( \partial \Sigma \setminus \partial \Gamma_1 \), we get a larger suborbifold \( \Gamma_2 \), and the repetition of this process has to stop with some \( \Gamma_i = \Sigma \), which concludes the proof.

### 7.4 When flexible groups are QH

The following theorem collects results from [RiSe, DuSa, FuPa].

**Theorem 7.7.** Let \( G \) be finitely presented. Let \( Q \) be a non-slender flexible vertex stabilizer of a JSJ tree \( T_A \) over \( A \). In each of the following cases, \( Q \) is a QH-subgroup:

1. \( A \) consists of all finite or cyclic subgroups of \( G \). In this case, \( F \) is trivial, and the underlying orbifold has no mirror.

2. \( A \) consists of all finite and 2-ended subgroups of \( G \). In this case, \( F \) is finite.

   More generally, \( A \) consists of all \( VPC \leq n \) subgroups of \( G \), for some \( n \geq 1 \), and \( G \) does not split over a \( VPC_{n-1} \) subgroup. In this case, \( F \) is \( VPC_{n-1} \).

3. \( A \) consists of all slender subgroups of \( G \). In this case, \( F \) is slender.

These results are usually stated under the assumption that \( G \) is one-ended (in this case, one does not need to include finite subgroups in \( A \)), but they hold in general ([RiSe, p. 107], see Corollary 8.4).

**Remark 7.8.** In [GL6], we will prove that similar results hold for splittings of relatively hyperbolic groups with small parabolic subgroups over small subgroups. In [DG10], one studies the flexible subgroups of the JSJ deformation space of a one-ended hyperbolic group over the class \( Z \) of virtually cyclic groups with infinite center (and their subgroups). Flexible vertices then are QH-subgroups whose underlying orbifold has no mirror. In particular, the fundamental group of an orbifold with mirrors usually has a non-trivial JSJ deformation space over \( Z \). One also studies a (variation of) JSJ splitting over maximal subgroups in \( Z \). Its flexible groups are orbisockets, i.e. QH-subgroups without mirrors, amalgamed to virtually cyclic groups over maximal extended boundary subgroups ([DG10], see also [RiSe]).

Propositions 7.5 and 7.6 immediately imply:

**Corollary 7.9.** Under the assumptions of Theorem 7.7, any flexible subgroup of a JSJ decomposition of \( G \) is QH with universally elliptic slender fiber, and all its boundary components are used. Moreover the underlying orbifold contains an essential simple closed geodesic.

For instance, the underlying orbifold cannot be a pair of pants.
7.5 Quadratically hanging subgroups are elliptic in the JSJ

The goal of this subsection is to prove that, under suitable hypotheses, a QH vertex group of any splitting is elliptic in the JSJ deformation space (if we do not assume existence of the JSJ deformation space, we obtain ellipticity in every universally elliptic tree).

We start with the following fact (see [FuPa] Remark 2.3).

Lemma 7.10. Suppose that $T_1$ is elliptic with respect to $T_2$, but $T_2$ is not elliptic with respect to $T_1$.

Then $G$ splits over a group which has infinite index in an edge stabilizer of $T_2$.

Proof. Let $\hat{T}_1$ be as in Lemma 3.2. Let $G_e$ be an edge stabilizer of $T_2$ which is not elliptic in $T_1$. By Assertion (3) of Lemma 3.2 it contains an edge stabilizer $J$ of $T_1$. Since $J$ is elliptic in $T_1$ and $G_e$ is not, the index of $J$ in $G_e$ is infinite. 

Corollary 7.11. If $G$ splits over a group $K \in A$, but does not split over any infinite index subgroup of $K$, then $K$ is elliptic in the JSJ deformation space over $A$.

Proof. Take $T_1$ to be a JSJ tree, and $T_2$ a one-edge splitting over $K$. \hfill \Box

Remark 7.12. If $K$ is not universally elliptic, it fixes a unique point in any JSJ tree. Also note that being elliptic or universally elliptic is a commensurability invariant, so the same conclusions hold for groups commensurable to $K$.

In [SaB, RiSe, DuSa], it is proved that, if $Q$ is a QH-subgroup in some splitting (in the class considered), then $Q$ is elliptic in the JSJ deformation space.

This is not true in general, even if $A$ is the class of cyclic groups: $F_n$ contains many quadratically hanging subgroups, none of them elliptic in the JSJ deformation space. This happens because $G = F_n$ splits over groups in $A$ having infinite index in each other, something which is prohibited by the hypotheses of the papers mentioned above (in [FuPa], $G$ is allowed to split over a subgroup of infinite index in a group in $A$, but $Q$ has to be the enclosing group of minimal splittings, see Definition 4.5 and Theorem 5.13(3) in [FuPa]).

A different counterexample will be given in Example 8.10 of Subsection 8.3, with $A$ the family of abelian groups. In that example the QH-subgroup only has one abelian splitting, which is universally elliptic, so it is not elliptic in the JSJ space.

These examples explain the hypotheses in the following general result.

Proposition 7.13. Consider an $A$-tree $T$. Let $Q$ be a QH vertex stabilizer of $T$, with fiber $F$ and underlying orbifold $\Sigma$. Assume that, if $\hat{J} \subset Q$ is the preimage of a 2-ended subgroup $J \subset \pi_1(\Sigma)$, then $\hat{J}$ belongs to $A$ and $G$ does not split over a subgroup of infinite index of $J$. If $\partial \Sigma = \emptyset$, assume additionally that $F$ is universally elliptic.

Then $Q$ is elliptic in the JSJ deformation space over $A$.

Remark 7.14. We do not assume that the fiber $F$ is slender.

Proof. We first claim that every boundary component of $\Sigma$ is used by an edge of $T$ (see Definition 7.3). Otherwise, we argue as in the proof of Proposition 7.13. Using Lemma 7.10 we construct a splitting of $Q$ relative to its incident edge groups over a group $F'$ containing $F$ with index 1 or 2. By Remark 7.12 the group $F'$ is contained in the preimage $\hat{J}$ of a 2-ended subgroup $J \subset \pi_1(\Sigma)$. The splitting of $Q$ extends to a splitting of $G$, and this is a contradiction since $F'$ has infinite index in $\hat{J}$.

We deduce that $G$ splits non-trivially over the preimage $\hat{B}$ of every boundary subgroup $B \subset \pi_1(\Sigma)$. Let $T_J$ be a JSJ tree, and assume that $Q$ is not elliptic in $T_J$. Since $\hat{B} \in A$ by assumption, $\hat{B}$ is elliptic in $T_J$ by Corollary 7.11. In particular, $F$ is elliptic in $T_J$ (this follows from our additional assumption if $\partial \Sigma = \emptyset$). As in Proposition 7.3, the action of $Q$ on the fixed point set of $F$ in $T_J$ factors through an action of $\pi_1(\Sigma)$. Note that
every boundary subgroup is elliptic, so Lemma 7.14 implies the existence of a simple closed geodesic γ in \( \Sigma \setminus \partial \Sigma \) (as defined in Subsection 7.1).

Given such a γ, recall that \( H_\gamma \) is the subgroup of \( \pi_1(\Sigma) \) consisting of elements which preserve γ and each of the half-spaces bounded by γ. We let \( \tilde{H}_\gamma \) be the preimage of \( H_\gamma \) in \( Q \). It belongs to \( \mathcal{A} \), and \( G \) has a non-trivial one-edge splitting over \( \tilde{H}_\gamma \). Lemma 5.3 of [Gu1] implies that every essential simple closed geodesic crosses another one. In particular, \( H_\gamma \) is not universally elliptic. By Corollary 7.11, \( \tilde{H}_\gamma \) fixes a point \( c_\gamma \in T_J \). By Remark 7.12, this point is unique.

**Lemma 7.15.** Let \( \gamma, \gamma' \subset \Sigma \setminus \partial \Sigma \) be simple geodesics. If \( \gamma \) and \( \gamma' \) intersect, then \( c_\gamma = c_{\gamma'} \).

**Proof.** Let \( T \) be the Bass-Serre tree of the splitting of \( G \) determined by \( \gamma' \). It contains an edge \( e \) with stabilizer \( \tilde{H}_\gamma \). Since \( \gamma \) and \( \gamma' \) intersect, the group \( \tilde{H}_\gamma \) acts hyperbolically on \( T \), and its minimal subtree \( M \) contains \( e \). Let \( T_1 \) be a refinement of \( T_J \) which dominates \( T \), and let \( M_1 \subset T_1 \) be the minimal subtree of \( \tilde{H}_\gamma \).

The image of \( M_1 \) in \( T_J \) consists of the single point \( c_\gamma \) (because \( T_1 \) is a refinement of \( T_J \)), and its image by any equivariant map \( f : T_1 \to T \) contains \( M \). Let \( e_1 \) be an edge of \( M_1 \) such that \( f(e_1) \) contains \( e \). The stabilizer \( G_{e_1} \) of \( e_1 \) is contained in \( \tilde{H}_{\gamma'} \) with finite index, so \( c_{\gamma'} \) is its unique fixed point. But \( G_{e_1} \) fixes \( c_\gamma \), so \( c_\gamma = c_{\gamma'} \). □

We can now conclude. First suppose that \( \Sigma \) is a surface. Choose a finite set \( \Gamma \) of simple closed geodesics which fill \( \Sigma \). The family \( \tilde{\Gamma} \) of lifts of elements of \( \Gamma \) is then connected. Given \( \gamma, \gamma' \in \tilde{\Gamma} \), we can find simple geodesics \( \gamma = \gamma_0, \gamma_1, \ldots, \gamma_p = \gamma' \) such that \( \gamma_i \) and \( \gamma_{i+1} \) belong to \( \tilde{\Gamma} \) and intersect. The lemma implies that \( c_\gamma \) is independent of \( \gamma \in \tilde{\Gamma} \). It is therefore fixed by \( Q \), a contradiction.

The proof is similar when \( \Sigma \) is an orbifold, taking \( \Gamma \) to be a set of simple geodesics which fill \( \Sigma \) as in the proof of Proposition 7.6. □

**Remark 7.16.** Let \( Q \) be a QH vertex stabilizer as in Proposition 7.13. Assume moreover that all groups in \( \mathcal{A} \) are slender, and that \( G \) does not split over a subgroup of \( Q \) whose image in \( \pi_1(\Sigma) \) is finite. Then \( Q \) fixes a unique point \( v \in T_J \), so \( Q \subset G_v \). We claim that, if the stabilizer \( G_v \) (hence also \( Q \)) is universally elliptic, then \( v \) is a QH-vertex of \( T_J \). This is used in [GL6].

Let \( T \) be an \( \mathcal{A} \)-tree in which \( Q \) is a vertex group \( G_w \). First note that \( G_v \) is elliptic in \( T \), so \( G_v = Q \). We have to show that, if \( e \subset T_J \) contains \( v \), then \( G_e \) is an extended boundary subgroup of \( Q \). Let \( \tilde{T} \) be a refinement of \( T_J \) which dominates \( T \), as in Lemma 3.2. Let \( \tilde{w} \) be the unique point of \( \tilde{T} \) fixed by \( Q \), and let \( f : \tilde{T} \to T \) be an equivariant map. Let \( \tilde{e} \) be the lift of \( e \) to \( \tilde{T} \).

If \( f(\tilde{e}) \neq \{w\} \), then \( G_e \) fixes an edge of \( T \) adjacent to \( w \), so is an extended boundary subgroup of \( Q \). Otherwise, consider a segment \( x\tilde{w} \), with \( f(x) \neq w \), which contains \( \tilde{e} \). Choose such a segment of minimal length, and let \( \varepsilon = xy \neq \tilde{e} \) be its initial edge. We have \( G_e \subset G_y \subset G_w = Q \), and \( G_e \) fixes an edge of \( T \) adjacent to \( w \). Since \( G \) does not split over groups mapping to finite groups in \( \pi_1(\Sigma) \), the image of \( G_e \) in \( \pi_1(\Sigma) \) is a finite index subgroup of a boundary subgroup \( B \subset \pi_1(\Sigma) \). But we also have \( G_e \subset G_w = G_{\tilde{w}} \), so that \( G_e \subset G_{\tilde{e}} = G_e \). Being slender, the image of \( G_e \) in \( \pi_1(\Sigma) \) has to be contained in \( B \).

## 8 JSJ decompositions with flexible vertices

We shall now consider JSJ trees with flexible vertices. By Lemma 7.2, the stabilizer of such a vertex has a non-trivial splitting (relative to the incident edge groups), but not over a universally elliptic subgroup.
8.1 Changing edge groups

First we fix two families of subgroups $A$ and $B$ and we compare the associated JSJ splittings. For example:

- $A$ consists of the trivial group, or the finite subgroups of $G$. This allows us to reduce to the case when $G$ is one-ended when discussing flexible subgroups (see Corollary 8.4).
- $A$ consists of the finitely generated abelian subgroups of $G$, and $B$ consists of the slender subgroups. This will be useful to describe the abelian JSJ in Subsection 8.3.
- $G$ is relatively hyperbolic, $A$ is the family of parabolic subgroups, and $B$ is the family of elementary subgroups.

There are now two notions of universal ellipticity, so we shall distinguish between $A$-universal ellipticity (being elliptic in all $A$-trees) and $B$-universal ellipticity.

Two trees are compatible if they have a common refinement.

**Proposition 8.1.** Assume $A \subseteq B$. Let $T_B$ be a JSJ tree over $B$. If there is a JSJ tree over $A$, there is one which is compatible with $T_B$. It may be obtained by refining $T_B$, and then collapsing all edges whose stabilizer is not in $A$.

**Proof.** Let $T$ be an $A$-universally elliptic $A$-tree. It is $B$-universally elliptic, so is dominated by $T_B$. Consider an edge $e$ of $T$ whose stabilizer is not in $A$. Then $G_e$ fixes a unique point of $T_B$, so any equivariant map from $T_1$ to $T_2$ is constant on $e$. It follows that the tree obtained from $T_B$ by collapsing all edges whose stabilizer is not in $A$ dominates $T_2$. Being $A$-universally elliptic by Assertion (2) of Lemma 3.2, it is a JSJ tree over $A$.

**Proposition 8.2.** Assume that $A \subseteq B$, and every $A$-universally elliptic $A$-tree is $B$-universally elliptic.

1. Let $T_B$ be a JSJ tree over $B$, and let $\pi : T_B \to T_A$ be the map that collapses all edges whose stabilizer is not in $A$. Then:
   
   (a) $T_A$ is a JSJ tree over $A$;
   
   (b) if $v$ is a vertex of $T_A$, with stabilizer $G_v$, then $\pi^{-1}(v)$ contains a JSJ tree $T_v$ of $G_v$ over $B_{|G_v}$ relative to the incident edge groups.

2. Conversely, suppose that $T_A$ is a JSJ tree over $A$, and for every stabilizer $G_v$, there exists a JSJ tree $T_v$ over $B_{G_v}$ relative to the incident edge groups. Then one can refine $T_A$ using these trees so as to get a JSJ tree over $B$.

**Proof.** For the first assertion, let $T$ be an $A$-universally elliptic $A$-tree. It is $B$-universally elliptic, so is dominated by $T_B$. As in the previous proof, one shows that any equivariant map from $T_B$ to $T$ factors through $T_A$, so $T_A$ is a JSJ tree over $A$. Statement (b) follows from Assertion (3) of Lemma 5.2 (applied over $B$).

For the second assertion, we view $T_A$ as a $B$-universally elliptic tree. The proposition then follows from Lemma 5.3 (applied over $B$).

**Remark 8.3.** This proposition remains true in a relative setting, provided that one enlarges $\mathcal{P}_v$ as in Remark 5.4.

The hypothesis of the proposition is satisfied if all groups in $A$ are finite (or, more generally, have Serre’s property FA). In this case $T_v$ is simply a (non-relative) JSJ tree over $B_{G_v}$. In particular, letting $A$ consist of the trivial group (resp. all finite subgroups), we deduce that one may assume one-endedness of $G$ when studying flexible subgroups.

**Corollary 8.4.** Let $B$ be arbitrary.

1. $G$ has a JSJ deformation space over $B$ if and only if each of its non-cyclic free factors does. If so, every flexible subgroup of $G$ is a flexible subgroup of a non-cyclic free factor.
(2) If $B$ contains all finite subgroups, then $G$ has a JSJ deformation space over $B$ if and only if $G$ is accessible (see Subsection 6.3) and every maximal one-ended subgroup has a JSJ space. If so, every flexible subgroup of $G$ is a flexible subgroup of a maximal one-ended subgroup.

8.2 Peripheral structure of quadratically hanging vertices

In this subsection, we study the incident edge groups of a QH vertex stabilizer, and how they depend on the particular JSJ tree chosen.

Let $Q$ be a QH vertex stabilizer of a JSJ tree $T_A$ as in Theorem 7.7. If $T'$ is another JSJ tree, it has a vertex stabilizer equal to $Q$, but possibly with different incident edge groups. For instance, it is always possible to modify $T_A$ within its deformation space so that each incident edge group is a maximal extended boundary subgroup of $Q$ (see Definition 7.3). But one loses information in the process (see Example 8.6 below).

The relevant structure, which does not depend on the choice of a JSJ tree, is the peripheral structure of $Q$, as defined in Section 4 of [GL2]. This is a finite family $M_0$ of conjugacy classes of extended boundary subgroups of $Q$. In the case at hand, one may define $M_0$ as follows (see [GL2] for details): a subgroup of $Q$ represents an element of $M_0$ if and only if it fixes an edge in every JSJ tree, and is maximal for this property.

We have seen (Corollary 7.9) that $M_0$ uses every boundary component of $\Sigma$. Apart from that, the peripheral structure of $Q$ may be fairly arbitrary. We shall now give examples. In particular, even when $G$ is one-ended, it is possible for an incident edge group to meet $F$ trivially, or (in the slender case) to have trivial image in $\pi_1(\Sigma)$.

Construction. Let $Q$ be an extension $1 \to F \to Q \to \pi_1(\Sigma) \to 1$ with $F$ slender and $\Sigma$ a compact orientable surface (with genus $\geq 1$, or with at least 4 boundary components). Let $H_1, \ldots, H_k$ be a finite family of infinite extended boundary subgroups of $Q$ as defined in Definition 7.3 (note that they are slender). For each boundary subgroup $B$ of $\pi_1(\Sigma)$, there should be an $i$ such that a conjugate of $H_i$ maps onto a finite index subgroup of $B$ (i.e. every boundary component is used in the sense of Definition 7.3). Let $R_i$ be a non-slender finitely presented group with Serre’s property FA, for instance $SL(3,\mathbb{Z})$. We define a finitely presented group $G$ by amalgamating $Q$ with $K_i = H_i \times R_i$ over $H_i$ for each $i$; in other words, $G = (((Q *_{H_i} K_1) * \ldots) *_{H_k} K_k)$.

Lemma 8.5. The Bass-Serre tree $T$ of the amalgam defining $G$ is a slender JSJ tree, $Q$ is a flexible subgroup, and $G$ is one-ended. If no $H_i$ is conjugate in $Q$ to a subgroup of $H_j$ for $i \neq j$, the peripheral structure $M_0$ consists of the conjugacy classes of the $H_i$’s.

Proof. We work over the family $A$ consisting of all slender subgroups. Let $T'$ be any tree with an action of $G$. Each $R_i$ fixes a unique point, and this point is also fixed by $H_i$. In particular, $H_i \times R_i$, $H_i$, and $T$, are universally elliptic. To prove that $T$ is a JSJ tree, it suffices to see that $Q$ is elliptic in any universally elliptic $A$-tree $T'$.

By Proposition 7.5, $F$ is universally elliptic. If $Q$ is not elliptic in $T'$, then by Proposition 7.5, the action of $Q$ on its minimal subtree $T_Q \subset T'$ factors through a nontrivial action of $\pi_1(\Sigma)$ with slender (hence cyclic) edge stabilizers. Since $H_i$, hence every boundary subgroup of $\pi_1(\Sigma)$, is elliptic, the action is dual to a system of disjoint essential simple closed geodesics on $\Sigma$ by Lemma 7.4. By Proposition 7.6, no edge stabilizer of $T_Q$ is universally elliptic, contradicting universal ellipticity of $T'$.

Thus, $T$ is a JSJ tree, and $Q$ is flexible because $\Sigma$ was chosen to contain intersecting simple closed curves.

By Proposition 8.2, one obtains a JSJ tree of $G$ over finite groups by collapsing all edges of $T$ with infinite stabilizer. Since each $H_i$ is infinite, this JSJ is trivial, so $G$ is one-ended.

The assertion about $M_0$ follows from the definition of $M_0$ given in [GL2].
Example 8.6. Let \( \Sigma \) be a punctured torus, with fundamental group \( \langle a, b \rangle \). Write \( u = [a, b] \).

Let \( Q = F \times \langle a, b \rangle \), with \( F \) finite and non-trivial. Let \( H_1 = \langle F, u^2 \rangle \) and \( H_2 = \langle u \rangle \). The tree \( T \) is a JSJ tree over 2-ended (and finite) groups. The peripheral structure of \( Q \) consists of two elements, though \( \Sigma \) only has one boundary component. There is a JSJ tree \( T' \) such that incident edge groups are conjugate to \( \langle F, u \rangle \) (the quotient \( T'/G \) is a tripod), but it does not display the peripheral structure of \( Q \).

Example 8.7. Let \( \Sigma, a, b, u \) be as above. Again write \( Q = F \times \langle a, b \rangle \), but now \( F = \langle t \rangle \) is infinite cyclic. Let \( H_1 = \langle u \rangle \) and \( H_2 = \langle t \rangle \). Then \( H_1 \) meets \( F \) trivially, while \( H_2 \) maps trivially into \( \pi_1(\Sigma) \).

8.3 Flexible vertices of abelian JSJ decompositions

We have described flexible subgroups over cyclic groups, 2-ended groups, slender groups (see Theorem 7.7). Things are more complicated over abelian groups.

The basic reason is the following: if a group \( Q \) is an extension \( 1 \to F \to Q \to \pi_1(\Sigma) \to 1 \) with \( F \) a finitely generated abelian group and \( \Sigma \) a surface, a splitting of \( \pi_1(\Sigma) \) along a simple closed curve induces a splitting of \( Q \) over a subgroup which is slender (indeed polycyclic) but not necessarily abelian. In the language of [FuPa], the enclosing graph decomposition of two splittings over abelian groups is not necessarily over abelian groups.

In fact, we shall now construct examples showing:

**Proposition 8.8.** (1) Flexible subgroups of abelian JSJ trees are not always slender-by-orbifold groups.

(2) One cannot always obtain an abelian JSJ tree by collapsing edges in a slender JSJ tree.

By Proposition 8.1 one can obtain an abelian JSJ tree by refining and collapsing a slender JSJ tree. The point here is that collapsing alone is not always sufficient. We will see, however, that things change if \( G \) is assumed to be CSA (see Proposition 8.12).

We use the same construction as in the previous subsection, but now \( \pi_1(\Sigma) \) will act non-trivially on the fiber \( F \).

Example 8.9. In this example \( F \cong \mathbb{Z} \). Let \( \Sigma \) be obtained by gluing a once-punctured torus to one of the boundary components of a pair of pants. Let \( M \) be a circle bundle over \( \Sigma \) which is trivial over the punctured torus but non-trivial over the two boundary components of \( \Sigma \). Let \( Q = \pi_1(M) \), and \( H_1, H_2 \) be the fundamental groups of the components of \( \partial M \) (homeomorphic to Klein bottles). Note that \( H_1, H_2 \) are non-abelian. Construct \( G \) by amalgamation with \( H_i \times R_i \) as above. We claim that the abelian JSJ decomposition of \( G \) is trivial, and \( G \) is flexible (but not slender-by-orbifold).

We argue as in the proof of Lemma 8.5. We know that \( H_1 \times R_1 \) and \( H_2 \times R_2 \) (hence also \( F \)) are universally elliptic. If \( T \) is any tree with abelian edge stabilizers, the action of \( Q \) on its minimal subtree factors through \( \pi_1(\Sigma) \), and the action of \( \pi_1(\Sigma) \) is dual to a system of simple closed curves (Proposition 7.5 (2) and Lemma 7.4). But not all curves give rise to an abelian splitting: they have to be “positive”, in the sense that the bundle is trivial over them.

To prove that none of these splittings is universally elliptic, hence that the abelian JSJ space of \( G \) is trivial, it suffices to see that any positive curve intersects (in an essential way) some other positive curve. This is true for the curve \( \delta \) separating the pair of pants from the punctured torus (one easily constructs a positive curve meeting \( \delta \) in 4 points). It is also true for curves meeting \( \delta \). Curves disjoint from \( \delta \) are contained in the punctured torus, and the result is true for them.

It is clear that \( G \) is flexible. If one performs the construction adding a third group \( H_3 = F \), then \( G \) becomes a flexible vertex group in a group \( \hat{G} \) with non-trivial abelian JSJ.
Example 8.10. Now $F = \mathbb{Z}^2$. Let $\Sigma$ be a surface of genus $\geq 2$ with two boundary components $C_1, C_2$. Let $\gamma$ be a simple closed curve separating $C_1$ and $C_2$. Let $\Sigma'$ be the space obtained from $\Sigma$ by collapsing $\gamma$ to a point. Map $\pi_1(\Sigma)$ to $SL(2, \mathbb{Z}) \subset \text{Aut}(\mathbb{Z}^2)$ by projecting to $\pi_1(\Sigma')$ and embedding the free group $\pi_1(\Sigma')$ into $SL(2, \mathbb{Z})$. Let $Q$ be the associated semi-direct product $\mathbb{Z}^2 \rtimes \pi_1(\Sigma)$, and $H_i = \mathbb{Z}^2 \rtimes \pi_1(C_i)$. Construct $G$ as before.

Abelian splittings of $G$ now come from simple closed curves on $\Sigma$ belonging to the kernel of $\rho : \pi_1(\Sigma) \to SL(2, \mathbb{Z})$. But it is easy to see that $\gamma$ is the only such curve. It follows that the one-edge splitting dual to $\gamma$ is an abelian JSJ decomposition of $G$. It has two rigid vertex groups. It cannot be obtained by collapsing a slender JSJ splitting.

Remark 8.11. In order to obtain an abelian JSJ tree from a slender JSJ tree, one must in general refine the tree and then collapse edges with non-abelian stabilizer (Proposition 8.1). There is some control over how a flexible vertex group $Q$ is refined. Suppose for instance that $Q = \mathbb{Z}^n \rtimes \pi_1(\Sigma)$ with $\Sigma$ an orientable surface. As in the previous example, the refinement uses curves $\gamma$ in the kernel of $\rho : \pi_1(\Sigma) \to \text{Aut}(\mathbb{Z}^n)$ which do not intersect other curves in the kernel. One may check that there is at most one such curve, and it is separating.

We now show that the abelian JSJ is very easy to describe when $G$ is CSA. Recall that $G$ is CSA if the centralizer of any non-trivial element is abelian and malnormal.

**Proposition 8.12.** Let $G$ be a torsion-free, finitely presented CSA group.

1. One obtains an abelian JSJ tree $T$ by collapsing all edges with non-abelian stabilizer in a slender JSJ tree.

2. All non-abelian flexible subgroups of $T$ are QH-subgroups with trivial fiber: they are isomorphic to $\pi_1(\Sigma)$, with $\Sigma$ a surface and all incident edge groups contained in boundary subgroups.

See [GL6] for abelian JSJ decompositions of finitely generated torsion-free CSA groups.

**Proof.** Denote by $A$ and $S$ the families of abelian and slender subgroups of $G$ respectively. If $G$ has infinitely generated abelian subgroups, one does not have $A \subset S$. But any $S$-universally elliptic subgroup is $A$-universally elliptic by Corollary 4.5

Let $T_A, T_S$ be JSJ trees. The key point is to show that $T_S$ dominates $T_A$. This implies Assertion (1) as in the proof of Proposition 8.1. Any map $T_S \to T_A$ sends edges with non-abelian stabilizer to points, so collapsing these edges yields an abelian JSJ tree.

To prove that $T_S$ dominates $T_A$, we consider a vertex stabilizer $Q = G_v$ of $T_S$ which is not $A$-universally elliptic (hence not $S$-universally elliptic), and we show that it is elliptic in $T_A$.

We first show that $Q$ is either an abelian group or a surface group (a QH-subgroup with $F$ trivial). There are two cases. If $Q$ is slender, let $S$ be an $A$-tree on which $Q$ acts non-trivially. By slenderness, $S$ contains a $Q$-invariant line, and $Q$ is abelian by the CSA property as it maps onto $Z$ or an infinite dihedral group with abelian kernel. If $Q$ is not slender, it is a QH-subgroup with slender fiber $F$ by Theorem 7.7. We prove that $F$ is abelian, hence trivial by the CSA property. We have seen that $F$ is $S$-universally elliptic (Proposition 7.3). If $F$ were not abelian, it would fix a unique point in every $A$-tree. This point would be fixed by $Q$ because $F$ is normal, contradicting the fact that $Q$ is not $A$-universally elliptic.

We can now show that $Q = G_v$ is elliptic in $T_A$. Since all its slender subgroups are abelian, the JSJ deformation space of $Q$ relative to the incident edge groups is the same over $A$ as over $S$. Note that these edge groups are abelian. Applying Assertion (3) of Lemma 6.2 with $T$ the tree obtained from $T_S$ by collapsing all edges with non-abelian stabilizer, we see that $Q$ is elliptic in the JSJ space over $A$ since it is elliptic over $S$. This proves the first assertion of the proposition.
We now prove the second assertion. By (1), we can assume that $T_A$ is obtained from $T_S$ by collapsing. No edge adjacent to a QH-vertex of $T_S$ with trivial fiber is collapsed, so such a vertex remains a flexible vertex stabilizer of $T_A$, with the same incident edge groups. We have seen that all other vertex stabilizers of $T_S$ are abelian or $A$-universally elliptic. Thus a vertex stabilizer of $T_A$ which is not an abelian group or a surface group is the fundamental group of a graph of groups with non-abelian edge groups and $A$-universally elliptic vertex groups. Such a group is $A$-universally elliptic. It follows that all non-abelian flexible subgroups of $T_A$ are surface groups. □

8.4 Relative JSJ decompositions

Fix a finitely presented group $G$ and a family $A$. In Subsection 5.1 we have shown the existence of the JSJ deformation space of $G$ relative to a finite set $H = \{H_1, \ldots, H_p\}$ of finitely generated subgroups. We now give an alternative construction, using the (absolute) JSJ space of another group $\hat{G}$ obtained by the filling construction of Subsection 5.2. This will allow us in the next subsection to extend the description of flexible vertices of Theorem 7.7 to the relative case.

Let $\hat{G}$ and $H$ be as above. As in Subsection 8.2, we define a group $\hat{G}$ by amalgamating with $K_i = H_i \times R_i$ over $H_i$, where $R_i$ is a finitely presented group with property FA. The group $\hat{G}$ is the fundamental group of a graph of groups with one central vertex $v$ (with vertex group $G$), and edges $e_i = vv_i$ with $G_{e_i} = H_i$ and $G_{v_i} = H_i \times R_i$. It is finitely presented. We denote by $T$ the Bass-Serre tree of this amalgam.

**Remark 8.13.** This construction can be adapted to the more general setting where $G$ is finitely presented relative to $\{H_1, \ldots, H_p\}$, and each $H_i$ is finitely generated and recursively presented. Since $H_i$ embeds into a finitely presented group $H_i$, one can take for $\hat{G}$ the amalgam of $G$ with $H_i' \times R_i$ over $H_i$, a finitely presented group.

Fix a family $B$ of subgroups of $\hat{G}$ such that $B_{\hat{G}} = A$ and $R_i \notin B$, for instance the family of subgroups of $\hat{G}$ having a conjugate in $A$ (note that two subgroups of $G$ which are conjugate in $\hat{G}$ are also conjugate in $G$ because $H_i$ is central in $K_i$, so this family $B$ is stable under conjugation).

We also define a family $B_H$, by adding to $B$ all subgroups of $\hat{G}$ having a conjugate contained in some $H_i$. Note that $R_i \notin B_H$, since dividing $\hat{G}$ by the normal closure of $G$ kills every $H_i$ but does not kill $R_i$.

**Lemma 8.14.** $H_i \times R_i$ is $B_H$-universally elliptic. A subgroup $J \subset G$ is $(A, H)$-universally elliptic if and only if $J$ (viewed as a subgroup of $\hat{G}$) is $B$-universally elliptic.

**Proof.** Consider any $B_H$-tree. The group $R_i$ fixes a point, which is unique since $R_i \notin B_H$. This point is also fixed by $H_i$ since $H_i$ commutes with $R_i$. Since $T$ is $B$-universally elliptic, the second assertion follows from Lemma 5.2. □

Given an $(A, H)$-tree $T_v$, with an action of $G$, we may use $T$ to extend the action of $G$ on $T_v$ to an action of $\hat{G}$ on a tree $\hat{T}$ containing $T_v$, as in Remark 3.5 (at the graph of groups level, this amounts to gluing edges amalgamating $H_i$ to $H_i \times R_i$ onto $T_v/G$). To get a $B$-tree $\hat{T}$, we collapse the orbit of the edge stabilized by $H_i$ whenever $H_i \notin A$. We say that $\hat{T}$ is an extension of $T_v$ to $\hat{G}$.

**Lemma 8.15.** Some JSJ tree $T_B$ of $\hat{G}$ over $B$ is an extension of a JSJ tree $T_A$ of $G$ over $A$ relative to $H$.

**Proof.** Being finitely presented, $\hat{G}$ has a JSJ deformation space over $B_H$ by Theorem 4.3. The Bass-Serre tree $T$ is $B_H$-universally elliptic, so by Lemma 8.15 it may be refined to a JSJ tree of $\hat{G}$ over $B_H$. By Proposition 8.1 this JSJ tree is compatible with some JSJ tree $T_B$ of $\hat{G}$ over $B$.  

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One obtains $T_B$ from $T$ by refining, and then collapsing all edges with stabilizer not in $B$. By universal ellipticity of $H_i \times R_i$, no refinement takes place over the vertex $v_i$, so only $v$ is refined. The edge $e_i$ is collapsed if and only if $H_i \notin A$. This shows that $T_B$ is an extension of an $(A, H)$-tree $T_A$.

The tree $T_A$ is $(A, H)$-universally elliptic by Lemma 8.14. If it is not maximal, it may be refined as an $(A, H)$-universally elliptic tree. An extension of this refinement is $B$-universally elliptic by Lemma 8.14, and contradicts the maximality of $T_B$ as a $B$-universally elliptic tree. This shows that $T_A$ is a JSJ tree of $G$ relative to $H$.

This gives another proof of Theorem 5.1:

Corollary 8.16. The JSJ deformation space of $G$ over $A$ relative to $H$ exists.

8.5 Relative QH-subgroups

In a relative setting, one needs to slightly modify the definition of a QH-subgroup by taking groups of $H$ into account.

Definition 8.17. Given $G$, $A$, and a family of subgroups $H$, one says that $Q$ is a relative QH-subgroup (over $A$, relative to $H$) if:

1. $Q$ is a vertex stabilizer of an $(A, H)$-tree;
2. $Q$ is an extension $1 \rightarrow F \rightarrow Q \rightarrow \pi_1(\Sigma) \rightarrow 1$, with $\Sigma$ a hyperbolic 2-orbifold;
3. each incident edge group, and each intersection of a conjugate of a group in $H$ with $Q$, is an extended boundary subgroup as in Definition 7.3.

A boundary component $C$ of $\Sigma$ is used if there is an incident edge group, or a conjugate of a group in $H$, whose image in $\pi_1(\Sigma)$ is contained in $\pi_1(C)$ with finite index.

Remark 8.18. As in Remark 5.4, one can replace the assumption on $H$ in (3) by the following one: for any $H \in H$ and any $g \in G$ such that $H^g \subset Q$, then $H^g$ is an extended boundary subgroup of $Q$.

Remark 8.19. Propositions 7.5 and 7.6 apply in a relative setting (the meaning of QH and of used being understood as in Definition 8.17).

The description of flexible vertices of JSJ decompositions over classes of slender groups occurring in Theorem 7.7 can be extended to the relative case.

Theorem 8.20. Let $G$ be finitely presented, let $A$ be a class of subgroups as below, and let $H = \{H_1, \ldots, H_p\}$ be a finite set of finitely generated subgroups. Let $T$ be a JSJ decomposition of $G$ over $A$ relative to $H$.

In each of the following cases, the flexible vertices of $T$ with non-slender stabilizer are relative QH-subgroups, with fiber $F$ as in Theorem 7.7, and every boundary component is used:

1. $A$ consists of all finite or cyclic subgroups of $G$.
2. $A$ consists of all finite and 2-ended subgroups of $G$.

More generally, $A$ consists of all $VPC_{\leq n}$ subgroups of $G$, for some $n \geq 1$, no $H_i$ is $VPC_{\leq n-1}$, and $G$ does not split over a $VPC_{\leq n-1}$ subgroup relative to $H$.
3. $A$ consists of all slender subgroups of $G$. 

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Let $G$ be hyperbolic relative to a family of finitely generated subgroups $\mathcal{H} = \{H_1, \ldots, H_p\}$, as in Subsection 6.4. Recall that $H < G$ is elementary if it is finite, virtually cyclic, or contained in a parabolic subgroup. In [GL6], we will study the elementary JSJ deformation space relative to any family $\mathcal{H}$ containing every $H_i$ which is not small, and we will show that non-elementary flexible subgroups are relative QH-subgroups with finite fiber.

**Proof of Theorem 8.20.** Choose $R_i$ so that it does not embed into a slender-by-orbifold group, for instance $R_i = SL(3, \mathbb{Z})$. Construct $\hat{G}$ as above, and let $\mathcal{B}$ be the natural class of groups extending $A$ on $\hat{G}$ (for instance, the class of slender groups of $\hat{G}$ in [3]). Using Lemma 8.15, consider a JSJ tree $T_A$ of $G$ relative to $\mathcal{H}$ as above, such that some extension $\hat{T}_A = T_B$ is a JSJ tree of $\hat{G}$. Recall that $\hat{T}_A/G$ is obtained by attaching edges to $T_A/G$ amalgamating $H_i$ to $H_i \times R_i$, and that $\hat{T}_A = T_B$ is obtained by collapsing edges of $\hat{T}_A$ with stabilizer not in $\mathcal{B}$ (i.e. not slender in case [3]).

By Theorem 7.7, the non-slender flexible vertex stabilizers of $T_B$ are QH-subgroups (in the $VPC_n$ case, we must check that $\hat{G}$ does not split over a $VPC_{\leq n-1}$ subgroup; this holds because in any tree with $VPC_{\leq n-1}$ edge stabilizers $H_i \times R_i$ is elliptic, so $G$ is elliptic, and all these groups fix the same point because no $H_i$ is $VPC_{\leq n-1}$).

Let $Q = G_v$ be a non-slender flexible vertex stabilizer of $T_A$. We denote by $\hat{v}$ and $\hat{\nu}$ the vertex corresponding to $v$ in $\hat{T}_A$ and $\hat{T}_A$. The stabilizer of $\hat{v}$ is $Q$, and the stabilizer of $\hat{\nu}$ is some $Q \supset H_i$.

First, $Q$ is flexible since any splitting of $G$ over $A$ relative to $\mathcal{H}$ extends to a splitting of $\hat{G}$ over $B$. By Theorem 7.4, $Q$ is an (absolute) QH vertex group of $T_B$. Since $R_i$ does not embed into a slender-by-orbifold group, no edge of $\hat{T}_A$ incident on $\hat{v}$ is collapsed in $\hat{T}_A$. It follows that $\hat{Q} = Q$. Since $\hat{\nu}$ is a QH vertex, edge groups incident on $\hat{\nu}$ are extended boundary subgroups of $Q$. This implies that each conjugate of $H_i$ intersects $Q$ in an extended boundary subgroup of $Q$. Thus, $Q$ is a relative QH vertex group of $T_A$.

The fact that every boundary component is used is a consequence of Proposition 7.5 ([3]), extended to the relative case by Remark 8.19.

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