Noether symmetries, energy-momentum tensors and conformal invariance in classical field theory

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Abstract

In the framework of classical field theory, we first review the Noether theory of symmetries, with simple rederivations of its essential results, with special emphasis given to the Noether identities for gauge theories. Will this baggage on board, we next discuss in detail, for Poincaré invariant theories in flat spacetime, the differences between the Belinfante energy-momentum tensor and a family of Hilbert energy-momentum tensors. All these tensors coincide on shell but they split their duties in the following sense: Belinfante’s tensor is the one to use in order to obtain the generators of Poincaré symmetries and it is a basic ingredient of the generators of other eventual spacetime symmetries which may happen to exist. Instead, Hilbert tensors are the means to test whether a theory contains other spacetime symmetries beyond Poincaré. We discuss at length the case of scale and conformal symmetry, of which we give some examples. We show, for Poincaré invariant Lagrangians, that the realization of scale invariance selects a unique Hilbert tensor which allows for an easy test as to whether conformal invariance is also realized. Finally we make some basic remarks on metric generally covariant theories and classical field theory in a fixed curved background.

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1 Introduction

It might seem an almost impossible task to say something new as regards the theory of Noether symmetries in classical field theory, even more so if the considerations are mostly made in flat space. Noether theory, which has been widely developed and employed in mathematical and theoretical physics since its dawn in 1918 [1] (for more modern expositions of Noether theorems see [2][3]), seems quite complete. There are a certain number of subjects, though, which are scarcely considered in the literature and which are nevertheless relevant in order to extract and take advantage of all the potentialities of the Noether formulation. We mention in this respect the detailed connection between Belinfante’s [4] and Hilbert’s [5] approaches to the energy-momentum tensor for Poincaré invariant theories, first investigated by Rosenfeld [6], and their different role either in giving the generators of the symmetry or in implementing the conditions for the existence of a symmetry. In particular, the role of a specific Hilbert tensor –among a variety of Hilbert tensors– in the implementation of scale and conformal symmetry in flat spacetime. Relevant papers on this case of conformal symmetry are [7, 8], which are mostly devoted to its realization in quantum field theory. As regards the classical setting, valuable contributions to Noether theory and energy-momentum tensors include [9, 10]. Our emphasis, though, is different, particularly regarding the different roles played by the several energy-momentum tensors and the discussion on scale and conformal invariance. We believe that our approach gives an integrated picture, always in the classical setting and in the language common to physics, which completes what it is already in the literature.

The essence of Noether theory is the connexion between continuous symmetries –of a certain type– and conservation laws (currents and charges) for theories whose dynamics is derived from a variational principle. Another aspect of Noether theory concerns the canonical formalism, where the conserved charges associated with a Noether symmetry become, through the Poisson bracket structure, the infinitesimal generators of the symmetry.

In this paper we will show that whereas there is a single Belinfante energy-momentum tensor associated with a theory in flat spacetime –a theory described by a Poincaré invariant Lagrangian–, which may be subject to improvements¹, one does not have a unique Hilbert tensor, but a family of them, all coinciding on shell. An additional and related remark, already pointed out by Rosenfeld, is that Belinfante’s energy-momentum tensor, which is in general only symmetric on shell, only coincides with the Hilbert tensors on shell. These observations could be thought unremarkable were it not for two facts. One is that it is the Belinfante tensor that contains the right infor-

¹An improvement is the addition to the energy-momentum tensor of a functional of the fields with identically vanishing divergence.
formation to construct the Poincaré symmetry generators and, basically, other spacetime symmetries that may exist; the other is that it is a Hilbert tensor that contains the right information to examine the eventual implementation of these other symmetries beyond Poincaré, like scale and conformal invariance. The analysis presented here relies heavily on the Noether identities for gauge theories.

The classical realization of scale invariance takes place when the Lagrangian has no dimensional parameters. In such case we show how to determine a specific Hilbert tensor for the theory. The trace of this tensor is the divergence of a quantity that can be computed. This quantity is then used to set up a simple test as to whether conformal invariance can be classically realized. This test coincides with the one obtained in [7] from a different analysis, not involving the Hilbert tensor.

In the subjects we have dealt with, we have tried to be complete, at the risk of, in some sections, reobtaining well known results. In this case we have tried, however, to produce new, brief, and clean presentations for old subjects, in a manner which we think could be useful for an introduction to Noether symmetries in general and Poincaré symmetries in flat spacetime in particular. Whenever we encounter spacetime transformations, either in flat spacetime or as diffeomorphisms in generally covariant theories, we always take the active view of the transformations: acting on the fields and leaving the coordinates unchanged. We believe that the active view is the most efficient one, and allows the spacetime and internal symmetries to be dealt with on the same footing. In addition, it is worth noticing that, in the canonical formalism, the action of the symmetry generators through the Poisson bracket corresponds to the active view.

In the present paper only the bosonic case is considered. An extension to the spinorial case is left for future work.

The organization of the paper is as follows. In section 2 we explore the basics of continuous symmetries, including Noether symmetries, either rigid or gauge, and conserved currents, in the Lagrangian formulation. Some remarks are made on the existence, in gauge theories, of first class constraints. Section 3 is devoted to theories in flat spacetime with Poincaré invariance, and the relation is given between the Belinfante tensor and a family of Hilbert tensors, and their respective roles are explained. In section 4 we discuss scale and conformal invariance and obtain, out of the trace of a specific Hilbert tensor, an expression to check for a given scale invariant theory whether conformal invariance is also realized. In section 5 we briefly consider generally covariant theories. An appendix complements section 3.

We set the notation used in the paper for the different energy-momentum tensors. \( \hat{T}^{\mu\nu} \) stands for the canonical energy-momentum tensor, \( T_b^{\mu\nu} \) for the Belinfante tensor, and \( T^{\mu\nu} \) for a generic Hilbert tensor. When we need to specify a Hilbert tensor associated with some density weights of the fields, we will write \( T^{(n)\mu\nu} \).
2 Noether symmetries

2.1 Some identities in the variational calculus

The variational calculus exhibits some identities which are very useful in implementing the conditions for a continuous transformation to be a symmetry. Let us consider a field theory, governed by an action principle

$$ S = \int \mathcal{L}, $$

where $\mathcal{L}$ is the Lagrangian density, with depends on the fields and their derivatives, in a finite number. The equations of motion are obtained by demanding extremality under arbitrary variations of the fields,

$$ \delta S = \int \delta \mathcal{L} = \int [\mathcal{L}]_A \delta \phi^A + \text{b.t.} = 0, $$

where $\phi^A$ is a generic field or field component, $[\mathcal{L}]_A$ stands for the Euler-Lagrange (E-L) functional derivative of $\mathcal{L}$ with respect to $\phi^A$ (we use also the notation $\frac{\delta \mathcal{L}}{\delta \phi^A}$), and b.t. represents boundary terms, that is, an integration on the boundary $\partial \mathcal{M}$ of the manifold $\mathcal{M}$ where the integration in (1) takes place. For arbitrary $\delta \phi^A$, except for some restrictions at the boundary that will help to make the b.t. to vanish—and thus to make $S$ a differentiable functional [11]—, the requirement $\delta S = 0$ is equivalent to $[\mathcal{L}]_A = 0$, which are the Euler-Lagrange equations of motion (e.o.m.).

Here we are not interested as much in the dynamics as in some properties of the variations themselves. Let us now perform a second variation so that

$$ \delta(\delta \phi^A) = \frac{\partial \delta \phi^A}{\partial \phi^B} \delta \phi^B + \frac{\partial \delta \phi^A}{\partial \phi^B_\mu} \delta \phi^B_\mu + \frac{\partial \delta \phi^A}{\partial \phi^B_{\mu\nu}} \delta \phi^B_{\mu\nu} + \ldots, $$

where $\phi^B := \partial_\mu \phi^B$ are derivatives with respect to the coordinates of the manifold—in a given patch—, etc. Here and henceforth, dots as in the last equation represent obvious contributions form higher derivatives of the fields. When acting on $\delta S$ we may choose two ways to expand the second variation, either

$$ \delta(\delta S) = \int \delta(\delta \mathcal{L}) = \int [\delta \mathcal{L}]_A \delta \phi^A + \text{b.t.} , $$

(form now on b.t. is generic for any boundary term) or

$$ \delta(\delta S) = \int \delta([\mathcal{L}]_A \delta \phi^A) + \text{b.t.} = \int \delta([\mathcal{L}]_A) \delta \phi^A + \int [\mathcal{L}]_A \delta(\delta \phi^A) + \text{b.t.} $$

$$ = \int \delta([\mathcal{L}]_A) \delta \phi^A + \int [\mathcal{L}]_A \left( \frac{\partial \delta \phi^A}{\partial \phi^B} \delta \phi^B \right) $$
+ \frac{\partial \delta \phi^A}{\partial \phi^B_{,\mu}} \delta \phi^B_{,\mu} + \frac{\partial \delta \phi^A}{\partial \phi^B_{,\mu\nu}} \delta \phi^B_{,\mu\nu} + \ldots ) + \text{b.t.}

= \int \delta ([L]_A) \delta \phi^A + \int \left( [L]_A \frac{\partial \delta \phi^A}{\partial \phi^B} \delta \phi^B 

- \partial_\mu ([L]_A \frac{\partial \delta \phi^A}{\partial \phi^B_{,\mu}}) + \partial_{\mu\nu} ([L]_A \frac{\partial \delta \phi^A}{\partial \phi^B_{,\mu\nu}}) + \ldots ) \right) \delta \phi^B + \text{b.t.} . \quad (3)

In both expressions (2) and (3) the bulk term depends on \( \delta \phi \), which is an arbitrary variation. Subtracting one from the other we get

\[ 0 = \int \left( \delta [L]_A \right) - [\delta L]_A + [L]_B \frac{\partial \delta \phi^B}{\partial \phi^A} 

- \partial_\mu ([L]_B \frac{\partial \delta \phi^B}{\partial \phi^A_{,\mu}}) + \partial_{\mu\nu} ([L]_B \frac{\partial \delta \phi^B}{\partial \phi^A_{,\mu\nu}}) + \ldots ) \right) \delta \phi^A + \text{b.t.} . \quad (4)

Since \( \delta \phi^A \) is arbitrary, the vanishing of the bulk term implies the identities\(^2\)

\[ \delta [L]_A - [\delta L]_A + [L]_B \frac{\partial \delta \phi^B}{\partial \phi^A} - \partial_\mu ([L]_B \frac{\partial \delta \phi^B}{\partial \phi^A_{,\mu}}) + \partial_{\mu\nu} ([L]_B \frac{\partial \delta \phi^B}{\partial \phi^A_{,\mu\nu}}) + \ldots = 0 \quad (5) \]

These identities are valid for an arbitrary variation \( \delta \phi \). A direct check is feasible and straightforward, but cumbersome. What the identities do in essence is to give a computation of the variation of the E-L derivatives, \( \delta [L]_A \), in terms of combinations –including derivatives with respect to the coordinates of the manifold– of the E-L derivatives themselves plus the term \([\delta L]_A\).

### 2.2 Continuous symmetries

An immediate application of (5) is to establish conditions for the existence of continuous symmetries –which we will explore with the infinitesimal variations \( \delta \). Our definition of a symmetry is simple: an invertible map sending solutions of the e.o.m. into solutions. So let us suppose that we have a solution \( \phi_0 \) (that is, \( \phi^A_0, \forall A \)) of the e.o.m.,

\[ [L]_{\phi_0} = 0, \]

\(^2\)To our knowledge, these identities were obtained in the language of mechanics by Kiyoshi Kamimura, in the early eighties, by direct computation, and never published. A particular case of (5), for variations satisfying the Noether condition –see below–, was written in [12], eq (2.7). We thank D. Salisbury for pointing this out to us.
and let $\phi^A_0 \rightarrow \phi^A_0 + \delta \phi^A_0$ be the transformation to the new configuration\(^3\), infinitesimally close to the old one. For it to be another solution of the e.o.m., we need

$$[\mathcal{L}]|_{\phi_0^A + \delta \phi_0^A} = 0,$$

which can also be written, at first order in the infinitesimal parameter hidden in the variation, as

$$[\mathcal{L}]|_{\phi_0^A} + (\delta [\mathcal{L}])|_{\phi_0^A} = 0.$$

The first term vanishes because $\phi^A_0$ is a solution. As for the second, notice that, according to (5),

$$(\delta [\mathcal{L}])|_{\phi_0^A} = [\delta \mathcal{L}]|_{\phi_0^A},$$

and thus we end up with the necessary and sufficient\(^4\) condition of the variation $\delta \phi^A$ to define an infinitesimal symmetry:

$$[\delta \mathcal{L}]|_{\phi_0^A} = 0 \tag{6}$$

for any solution $\phi_0$.

### 2.2.1 Noether Symmetries

One strong way to ensure (6) is the adoption of the Noether setting: require that

$$\delta \mathcal{L} = \partial_\mu F^\mu$$

for some $F^\mu$, which guarantees $[\delta \mathcal{L}]_A = 0, \forall A$, identically. We have thus proved that $\delta \mathcal{L}$ being a divergence is a sufficient condition for the variation $\delta$ to map solutions into solutions. It is worth noticing that, unlike the general condition of symmetry (6), the Noether condition is a direct condition on the Lagrangian and not on the e.o.m. The Noether condition must be satisfied on and off shell.

General relativity (GR) provides with an elementary example of a symmetry which is not Noether. The Einstein-Hilbert lagrangian $\mathcal{L}_{\text{EH}} = \sqrt{|g|} R$ admits a scaling symmetry (rigid Weyl rescaling) $g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}$, under which $\sqrt{|g|} R \rightarrow \lambda^{\frac{d-2}{2}} \sqrt{|g|} R$ where $d$ is the spacetime dimension. For $d > 2$ this is not a Noether symmetry but it is still a symmetry. With $\lambda = 1 + \delta \lambda$ one has $\delta \mathcal{L}_{\text{EH}} = \frac{d-2}{2} \delta \lambda \mathcal{L}_{\text{EH}}$ and thus satisfies (6).

\(^3\)As stated in the introduction, we only consider active variations on the fields. Any transformation of the coordinates—passive transformation—is rewritten as an active one.

\(^4\)This sufficient condition may be further restricted in some cases if some boundary conditions are imposed on the acceptable solutions.
The obtention of a conserved current for a Noether symmetry is straightforward. Condition (7) can also be written as

$$\left[ L \right]_A \delta \phi^A + \partial_\mu J^\mu = 0$$

(8)

for some current density $J^\mu$. This is the on shell conserved current associated with a Noether symmetry. With some additional caveats in the case of gauge theories\(^5\), the spatial integration of $J^0$, when appropriately expressed in phase space, will become the generator, under the Poisson bracket, of the Noether symmetries.

As a matter of notation, whenever a Noether symmetry is realized, we will say that the theory has invariance under such symmetry. This is the meaning we attach to the concept of a Poincaré invariant theory, a diffeomorphism invariant theory, etc. even though the variation of the Lagrangian under the infinitesimal Noether transformation may not vanish, being in general a divergence.

### 3 Noether identities

Consider a field theory described by a first order Lagrangian density—except perhaps for a divergence term—$L$. We also consider the Lagrangian having some symmetries of the type

$$\delta \phi^A = R^A_a \epsilon^a + R^A_{\mu} \partial_\mu \epsilon^a,$$

(9)

where $\epsilon^a$ are the infinitesimal parameters of the symmetries, with the index $a$ running over the number of independent symmetries, and with $R^A_a$, $R^A_{\mu}$ functions of the fields and their derivatives. Up to now everything has been general. Now let us be specific: the case of our interest is when $\epsilon^a$ are arbitrary functions of the coordinates; then the associated symmetries are called gauge transformations. We assume that these gauge transformations are of Noether type, which means that $\delta L$ is a divergence.

The dependence of $\delta \phi^A$ on the arbitrary functions $\epsilon^a$ imposes that $J^\mu$ in (8) must be of the form $J^\mu = C^\mu_a \epsilon^a$ (up to the divergence of an arbitrary antisymmetric tensor density, as we discuss below). Hence we find, using the arbitrariness of the functions $\epsilon^a$, that $C^\mu_a$ can be taken so as to satisfy

$$C^\mu_a = -\left[ L \right]_A R^A_{\mu} \quad \text{and} \quad \partial_\mu C^\mu_a = -\left[ L \right]_A R^A_a.$$

(10)

The general solution of $J^\mu$ in (8) is $J^\mu = C^\mu_a \epsilon^a + \partial_\mu N^{\nu\mu}$, with $N^{\nu\mu}$ an arbitrary antisymmetric tensor density (we work locally and do not consider global issues).

\(^5\)There exists a projectability issue in going from tangent space to phase space in gauge theories, see [13] for the general theory and [14, 15] for its application to generally covariant theories.
particular solution found for $C^\mu_a$ in (10) vanishes on shell, we infer that, up to terms vanishing on shell, the conserved Noether currents associated with gauge symmetries are always trivial and completely undetermined by the formalism. In the standard spacetime setting, $\mu = (0, i)$, the values of conserved charges $\int_{\mathcal{M}_t} d^{d-1}x J^0$ (where $\mathcal{M}_t$ is a spacelike slice of $\mathcal{M}$) will depend on $N^{i0}$ as a boundary integral. The 2-form Hodge dual to $N^{\nu\mu}$ in a standard metric theory is usually called the superpotential.

Notice that elimination of $C^\mu_a$ in (10) yields the Noether identities\textsuperscript{6}

$$[\mathcal{L}]_A R^A_a - \partial_\mu ([\mathcal{L}]_A R^{A\mu}_a) = 0$$

(11)

Let us pause to reobtain these identitites in another way --not essentially different, though. Let us integrate (8) on the manifold, and in addition consider the case when the arbitrary functions $\epsilon^a$ have compact support. This means that all boundary terms depending on the arbitrary functions --and derivatives-- will vanish. Then we have

$$0 = \int [\mathcal{L}]_A \delta \phi^A = \int [\mathcal{L}]_A (R^A_a \epsilon^a + R^{A\mu}_a \partial_\mu \epsilon^a) = \int \left( [\mathcal{L}]_A R^A_a - \partial_\mu ([\mathcal{L}]_A R^{A\mu}_a) \right) \epsilon^a ,$$

(12)

from which (11) is obtained owing to the arbitrariness of $\epsilon^a$. Saturating (11) with $\epsilon^a$, it can be equivalently written as

$$[\mathcal{L}]_A \delta \phi^A - \partial_\mu ([\mathcal{L}]_A R^{A\mu}_a) = 0 ,$$

(13)

which explicitely shows the current obtained above.

Notice that in a more general case of a gauge symmetry, for instance

$$\delta \phi^A = R^A_a \epsilon^a + R^{A\mu}_a \partial_\mu \epsilon^a + R^{A\mu\nu}_a \partial_{\mu\nu} \epsilon^a + \ldots$$

(14)

(where $R^{A\mu\nu}$ can always be taken symmetric in the $\mu \nu$ indices) we would have obtained, for the Noether identity,

$$[\mathcal{L}]_A R^A_a - \partial_\mu ([\mathcal{L}]_A R^{A\mu}_a) + \partial_{\mu\nu} ([\mathcal{L}]_A R^{A\mu\nu}_a) + \ldots = 0$$

(15)

The case with second derivatives $\partial_{\mu\nu} \epsilon^a$ takes place for instance in the Palatini formalism for GR.

### 3.1 First class constraints from Noether identities

Observe that the quantities $C^\mu_a$ in (10) vanish on shell, but this is not a sufficient reason to qualify them as constraints. Here we enter the realm of the Rosenfeld-Dirac-Bergmann\textsuperscript{7} theory of constrained systems [17, 12, 18, 19, 20, 21]. We will not dwell

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\textsuperscript{6}This is the contents of the often called Noether’s second theorem.

\textsuperscript{7}Rosenfeld’s contribution, which has been overlooked for a long time, has recently resurfaced thanks to the work of D. Salisbury and it is discussed in [16].
in this theory\(^8\) but will only borrow a few concepts from it. To distinguish what is a constraint and what is not we must first individuate an evolution parameter in our theory. If we are in Minkowsky spacetime or in a Lorentzian manifold, we typically consider a coordinate \(x^0 = t\) (for ”time”) such that the equal time surfaces are spacelike. The e.o.m. are second order, and initial conditions –positions and velocities– are given on the initial time surface and the e.o.m. are the differential conditions required on a solution –which will satisfy the initial conditions by construction. An on shell vanishing quantity will qualify as a constraint if it does not depend on the second time derivatives. This means that such a quantity places a restriction on the initial value problem.

Now we will prove that, assuming that the functions \(R^A_a\) and \(R^A_{\mu a}\) do not depend on the second time derivatives, the quantities \(C^0_a\) are actually constraints. This is the situation in general covariant, Maxwell and YM theories.

To this effect, consider the second equation in (10) written in the form \((\mu = (0, i))\)
\[
\partial_0 C^0_a = -[\mathcal{L}]_A R^A_a - \partial_i C^i_a.
\]
The rhs depends at most on the second time derivatives, according the our assumptions, but this implies, looking at the lhs, that \(C^0_a\) depends at most on the first time derivatives, which means that it is a constraint. Thus we have proven that the special combination of the e.o.m. given by
\[
C^0_a = -[\mathcal{L}]_A R^A_0
\]
is a constraint.

In addition, since these constraints participate in the 0-component of the current \(J^\mu\), they must be projectable to phase space (\(\int d^3x J^0\) is the generator of the Noether symmetry, and it is always projectable) and first class\(^9\) because they participate in a generator of a symmetry, which necessarily preserves the constraints.

### 3.1.1 A generalization

In the more general case (14) with second derivatives of \(\epsilon^a\), assuming that \(R^A_a\), \(R^A_{\mu a}\) and \(R^A_{\mu\nu a}\) depend on the fields and their first derivatives with respect to the coordinates, we find the particular solution for \(J^\mu\) in (8) as \(J^\mu = C^\mu_a \epsilon_a + \partial_\mu (C^\mu_{\nu a} \epsilon^a)\), with
\[
C^\mu_{\nu a} = -[\mathcal{L}]_A R^A_{\nu a}, \quad C^\mu_a = -[\mathcal{L}]_A R^A_\mu + 2 \partial_\nu ([\mathcal{L}]_A R^A_{\nu \mu}).
\]
(Note that \(C^\mu_{\nu a}\) is symmetric in \(\mu\nu\) because \(R^A_{\mu\nu a}\) is so.) Substitution of these relations into \([\mathcal{L}]_A R^A_a + \partial_\mu C^\mu_a + \partial_\mu C^\mu_{\nu a} = 0\) (which is also consequence of (8)), produces the Noether identity (15).

\(^8\)See for instance [22] and [23]. A brief exposition of its basics can be found in [24].

\(^9\)Dirac introduced the concept of a first class function as a function whose Poisson bracket with the constraints vanishes on the constraints’ surface.
Notice that (16) shows that $C^\mu_a$ and $C^{\mu\nu}_a$ vanish on shell. Let us identify, in analogy with the previous subsection, some projectable constraints out of these objects. We continue to assume that the E-L e.o.m. are of second order, at least in the time derivatives. Examination of the Noether identity (15) as regards the presence of time derivatives shows that the combinations $[L]_A R^A_{\mu 00}$ and $[L]_A R^A_{\mu 0} - 2 \partial_\mu ([L]_A R^A_{\mu 0}) + \partial_0 ([L]_A R^A_{\mu 00})$ must be constraints, that is, they can not contain second order time derivatives, otherwise the Noether identities can not be fulfilled. In terms of the coefficients in $J^\mu$, the constraints are $C^0_{a0}$ and $C^0_a + \partial_0 C^0_{a0}$. Let us see now the logic behind this finding. The conserved quantity associated with the Noether transformation is

$$G = \int d^3x J^0 = \int d^3x \left( C^0_a \dot{\epsilon}^a + \partial_\mu (C^{\mu 0}_a \epsilon^a) \right) = \int d^3x \left( (C^0_a + \partial_0 C^0_{a0}) \epsilon^a + C^0_{a0} \dot{\epsilon}^a \right)$$

(17)

where $\dot{\epsilon}^a$ stands for the time derivative of the arbitrary function $\epsilon^a$. The boundary term is relevant –and indeed essential since the bulk term vanishes on shell– as regards the values of conserved quantities, but the symmetry is generated by the bulk term only. When written in terms of canonical variables –which is always possible because $G$ is projectable to phase space–, the generator $G$ will be expressed in terms of first class constraints and it will contain an additional piece proportional to $\ddot{\epsilon}^a$ –to be able to generate the transformation (14). The coefficients in this new piece will be primary first class constraints, which are invisible in the Lagrangian formalism because their pullback to configuration-velocity space vanishes identically. Thus we have shown that the Lagrangian constraints $C^0_{a0}$ are projectable from configuration-velocity space to the canonical formalism to become secondary first class constraints, and $C^0_a + \partial_0 C^0_{a0}$ to become tertiary first class constraints. Including the primary ones, these constraints will exhaust the first class constraints of the theory, if some regularity conditions are met.

### 3.1.2 An example: first class constraints for general relativity

In GR, the infinitesimal gauge transformations are given by the Lie derivative,

$$\delta g^{\mu\nu} = \epsilon^\lambda \partial_\lambda g^{\mu\nu} - g^{\mu\lambda} \partial_\lambda \epsilon^\nu - g^{\lambda\nu} \partial_\lambda \epsilon^\mu,$$

out of which we get

$$R^{(\mu\nu)\rho}_a = -(g^{\mu\alpha} \delta^\nu_a + g^{\nu\alpha} \delta^\mu_a).$$

With the notation $L_{EH}$ for the Einstein-Hilbert Lagrangian and $[L_{EH}]_{\mu\nu} = G_{\mu\nu}$ for the Einstein tensor density, the constraints are

$$C^0_a = -[L_{EH}]_{\mu\nu} R^{(\mu\nu)0}_a = 2G_{\mu\nu} g^{\mu0}.$$

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Up to a factor, $g^{\mu 0}$ is the vector orthonormal to the equal time surfaces,

$$n^\mu = - \frac{g^{\mu 0}}{\sqrt{|g^{00}|}},$$

and therefore we have deduced that the combinations of the e.o.m.,

$$G_{\mu \sigma} n^\mu,$$

are constraints, i.e., they do not depend on the second time derivatives. These constraints are the Hamiltonian and momentum constraints of GR. Notice that we have obtained them from first principles, without ever examining the e.o.m.. In the phase space picture, these constraints are secondary, the primary ones being the momenta conjugate to the lapse and shift variables.

### 3.2 Examples of Noether identities

#### 3.2.1 The Bianchi identities in GR

Sometimes it is convenient to express (9) with the help of the covariant derivative,

$$\delta \phi^A = U^A_\rho \epsilon^\rho + R^A_\rho \nabla \mu \epsilon^\rho,$$

with $U^A_\rho = R^A_\rho - R^A_\nu \Gamma^\nu_{\mu \rho}$, and the Noether identity (11) becomes

$$[\mathcal{L}_A U^A_\rho - \nabla_\mu ([\mathcal{L}_A R^A_\rho]) = 0.$$  \hspace{1cm} (19)

In the case of the metric field $g^{\mu \nu}$ we have simply

$$\delta g^{\mu \nu} = -\nabla^\mu \epsilon^\nu - \nabla^\nu \epsilon^\mu = -(g^{\mu \rho} \delta^\nu_\sigma + g^{\nu \rho} \delta^\mu_\sigma) \nabla_\rho \epsilon^\sigma,$$

which identifies $U^A_{\rho} = 0$ (now $\{\mu \nu\}$ has the role of the index $A$) and

$$R^{A \{\mu \nu\}}_{\rho} = -(g^{\mu \rho} \delta^\nu_\sigma + g^{\nu \rho} \delta^\mu_\sigma),$$

as before. The Noether identities (19) for GR are then

$$\nabla_\rho ([\mathcal{L}_{EH}]_{\mu \nu} R^{\{\mu \nu\}}_{\rho}) = -2 \nabla_\rho G^\rho_{\sigma} = 0,$$

which are the doubly contracted Bianchi identities.
3.2.2 Noether identities for Yang-Mills coupled to gravity

For $\mathcal{L}_{YM} = -\frac{1}{4} \sqrt{|g|} F^a_{\mu\nu} F^{a\mu\nu}$, in addition to the well known Noether identities associated with the internal gauge symmetry,

$$D_\mu D_\nu (\sqrt{|g|} F^a_{\mu\nu}) = 0,$$

we get as Noether identities for the diffeomorphism invariance,

$$-2D_\mu \frac{\delta \mathcal{L}_{YM}}{\delta g_{\mu\nu}} + \left( D_\rho (\sqrt{|g|} F^a_{\rho\mu}) \right) F^a_{\sigma\mu} g^{\sigma\nu} = 0,$$

where $D_\rho$ is the total covariant derivative -including the Riemmanian connection and the $SU(N)$ connection. Actually, $D_\mu$ in the first term acts only as the Riemmanian covariant derivative because there are no YM indices involved. On the contrary, in the second term, the Riemmanian connection is superfluous because $\sqrt{|g|} F^a_{\rho\mu}$ is an antisymmetric tensor density.

3.2.3 Noether identities for the Palatini formalism of GR

Also for the diffeomorphism invariance, the Palatini formalism of GR exhibits the following Noether identities

$$\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \nabla_\lambda g_{\mu\nu} - 2 \nabla_\mu \left( \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} g_{\nu\lambda} \right) - 2 \left( \nabla_\nu \frac{\delta \mathcal{L}}{\delta R^{\sigma}_{\mu\rho\lambda}} R^\sigma_{\mu\lambda\rho} \right) - 2 \nabla_\rho \nabla_\mu \nabla_\nu \left( \frac{\delta \mathcal{L}}{\delta R^{\lambda}_{\mu\nu\rho}} \right) = 0.$$

4 Poincaré invariant theories

Here we will consider standard flat spacetime field theories with Poincaré invariance satisfying the Noether condition (8). There is a standard route to construct the energy-momentum tensor encompassing the four currents associated with the four translational symmetries in flat space with Cartesian coordinates. An improvement –Belinfante– of this tensor allows for it to provide also for an easy construction of the currents associated with the Lorentz symmetries.

4.1 Belinfante’s energy-momentum tensor

Although the material is standard, here we discuss, for the sake of completeness, Belinfante’s improvement, [4], of the canonical energy-momentum tensor for Poincaré invariant theories in Minkowski spacetime. Throughout this subsection, $\epsilon^\mu$ will stand for the infinitesimal Poincaré coordinate transformations. In Cartesian coordinates,

$$\epsilon^\mu = a^\mu + \omega^\mu_{\nu} x^\nu, \quad \omega^\mu_{\nu} = -\omega^\nu_{\mu}, \quad (20)$$
with $a^\mu$ and $\omega^\mu_\nu$ infinitesimal parameters ($\omega^\mu_\nu = \eta^\mu_\rho \omega^\rho_\nu$). In the active view of the action of Poincaré group, the fields transform as

$$\delta \phi^A = \epsilon^\mu \partial_\mu \phi^A + S^{A\mu\nu}_B \omega^\mu_\nu \phi^B,$$

with $S^{A\mu\nu}_B = S^{A[\mu\nu]}_B$ being a linear representation of the Lorentz group. Let us make contact with the notation (9) applied to the case of diffeomorphism transformations and show how this object $S^{A\mu\nu}_B$ is recovered from it. We will do it in the case of a vector field $A_\mu$, but the method is general for scalar, vector and tensor representations. An infinitesimal diffeomorphism acts as

$$\delta \epsilon A_\mu = \epsilon^\sigma \partial_\sigma A_\mu + A_\sigma \partial_\mu \epsilon^\sigma,$$

thus we extract, from (9),

$$R^{(\mu)}_\sigma = \partial_\sigma A_\mu, \quad R^\rho_\mu = A_\sigma \delta^\rho_\mu,$$

Then, for $\epsilon^\mu$ as in (20),

$$R^{(\mu)}_\sigma \partial_\rho \epsilon^\sigma = \frac{1}{2} \left( R^{(\rho)}_\sigma \eta^{\sigma\nu} - R^{(\sigma)}_\rho \eta^{\rho\nu} \right) \omega^\nu_\rho =: S^{(\lambda)}_{(\mu)} \omega^\nu_\rho \partial_\lambda,$$

with

$$S^{(\lambda)}_{(\mu)} = (\delta^\rho_\mu \eta^{\lambda\nu} - \delta^\nu_\mu \eta^{\lambda\rho})$$

being the matrices of the vector representation of the Lorentz algebra. Keeping (9) in mind, in the general case we will have

$$S^{A\mu\nu}_B \phi^B = \left( R^{A\lambda}_\sigma \eta^{\sigma\mu} - R^{A\mu}_\sigma \eta^{\sigma\nu} \right).$$

All is summarized in the decomposition into antisymmetric and symmetric components

$$R^{A\lambda}_\sigma \eta^{\rho\sigma} = \frac{1}{2} \left( S^{A\mu\nu}_B \phi^B + Q^{A\mu_\nu}_B \phi^B \right),$$

where $S^{A\mu\nu}_B = S^{A[\mu\nu]}_B$ and $Q^{A\mu_\nu}_B = Q^{A(\mu\nu)}_B$, and the observation that only the antisymmetric term contributes to (8) when $\epsilon^\mu$ is as in (20). The matrices $(M^{\mu\nu})^A_B := S^{A\mu\nu}_B$ form a representation of the Lorentz algebra.

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10Note however that we restrict here the covariant behaviour of the fields (scalars, vectors, tensors) for we do not allow them to be densities. For instance, our scalars here transform as $\delta \phi = \epsilon^\mu \partial_\mu \phi$ whereas a scalar density of weight $n$ transforms as $\delta \phi = \epsilon^\mu \partial_\mu \phi + n \phi \partial_\mu \epsilon^\sigma$. This restriction will be lifted in section 4.3.

11We use the standard notation $[ \ ]$ for antisymmetry and $( )$ for symmetry.
Taking into account that for a Poincaré scalar Lagrangian $L_f$ ($f$ is for flat), $\delta L_f = \epsilon^{\mu} \partial_\mu L_f = \partial_\mu (\epsilon^{\mu} L_f)$ ($\epsilon^{\mu}$ is just (20)), we realize the existence of the Noether symmetries, and (8) becomes

$$0 = [L_f]_A \delta \phi^A + \partial_\mu \left( \frac{\partial L_f}{\partial \phi^A_{,\mu}} \delta \phi^A - \delta_\mu \epsilon^\nu L_f \right) = [L_f]_A \delta \phi^A + \partial_\mu \left( \hat{T}^{\mu}_{\rho} \epsilon^\rho + \frac{\partial L_f}{\partial \phi^A_{,\rho}} R^{\Lambda \nu}_{\rho} \partial_\nu \epsilon^\rho \right), \tag{25}$$

where the canonical energy-momentum tensor is defined as

$$\hat{T}^{\mu}_{\nu} = \frac{\partial L_f}{\partial \phi^A_{,\mu}} \delta^A_{,\nu} - \delta^{\mu}_{\nu} L_f \tag{26}$$

It is possible, and highly convenient, to get rid of the first derivatives of $\epsilon^\rho$ within the current in (25). To end observe that, using the antisymmetry of $\omega_{\sigma\nu}$,

$$\frac{\partial L_f}{\partial \phi^A_{,\mu}} R^{\Lambda \nu}_{\rho} \partial_\nu \epsilon^\rho = \frac{1}{2} \frac{\partial L_f}{\partial \phi^A_{,\mu}} S^{A \sigma \nu}_{B} \omega_{\sigma \nu}$$

$$= \frac{1}{2} \left( \frac{\partial L_f}{\partial \phi^A_{,\mu}} S^{A \sigma \nu}_{B} + \frac{\partial L_f}{\partial \phi^A_{,\sigma}} S^{A \mu \nu}_{B} + \frac{\partial L_f}{\partial \phi^A_{,\nu}} S^{A \sigma \mu}_{B} \right) \omega_{\sigma \nu}$$

$$= F^{\nu \mu \sigma} \omega_{\sigma \nu} \tag{27}$$

with $F^{\nu \mu \sigma} = F^{\nu[\mu \sigma]}$. Thus we have, for (25)

$$0 = [L_f]_A \delta \phi^A + \partial_\mu \left( \hat{T}^{\mu}_{\rho} \epsilon^\rho + F^{\nu \mu \sigma} \omega_{\sigma \nu} \right) = [L_f]_A \delta \phi^A + \partial_\mu \left( \hat{T}^{\mu}_{\rho} \epsilon^\rho - F^{\nu \mu \sigma} \eta_{\nu \rho} \partial_\sigma \epsilon^\rho \right)$$

$$= [L_f]_A \delta \phi^A + \partial_\mu \left( \hat{T}^{\mu}_{\rho} + \partial_\sigma F^{\nu \mu \sigma} \eta_{\nu \rho} \right) \epsilon^\rho), \tag{28}$$

where in the last equality we have used that $F^{\nu \mu \sigma}$ is antisymmetric in its last two indices. In the last line of (28) we identify Belinfante’s improvement of the canonical energy-momentum tensor,

$$T^{\mu}_{\rho} := \hat{T}^{\mu}_{\rho} + \partial_\sigma F^{\nu \mu \sigma} \eta_{\nu \rho} \tag{29}$$

Note that (28) can be written as

$$0 = [L_f]_A \delta \phi^A + \partial_\mu \left( T^{\mu}_{\rho} a^\rho + \frac{1}{2} (T^{\mu \nu \rho \sigma} - T^{\mu \sigma \rho \nu} \omega_{\rho \sigma}) \right), \tag{30}$$

which allows for a neat identification of the currents associated with translations and Lorentz transformations. The on shell conservation of these currents guarantees that Belinfante’s energy-momentum tensor is symmetric on shell. In fact, using $\delta \phi^A$ from (21), and the determination, from (30), of $\partial_\mu T^{\mu}_{\rho} = -[L_f]_A \phi^A_{,\rho}$, one can compute, also from (30), the antisymmetric component in $T^{\mu \nu}_{\rho}$. One finds

$$T^{\mu \nu}_{\rho} - T^{\nu \mu}_{\rho} = [L_f]_A S^{A \mu \nu \sigma}_{B} \phi^B.$$
In particular we note that $T^\mu_\nu + \frac{1}{2} [\mathcal{L}_A \phi^A_B] \delta \phi^B$ is symmetric on and off shell. It is also symmetric, noticing (24), the combination

$$T^\mu_\nu + [\mathcal{L}_f] A R^A_\mu \eta^\sigma_\nu .$$

We anticipate that this is indeed a Hilbert energy-momentum tensor, the objects to which we devote the next section.

4.2 Hilbert’s energy-momentum tensor(s)

Here we show a way to bypass the constructions made above by using Hilbert’s prescription of substituting the Minkowski metric $\eta$ by a general metric and extracting the energy-momentum tensor as the functional derivative of the Lagrangian with respect to the new metric, which, after derivation, is set again to be Minkowski. So consider a Poincaré invariant Lagrangian $\mathcal{L}_f(\phi, \partial \phi)$ in Minkowski spacetime. We will assume that it is possible to ”covariantize” it, that is, we assume that there exists a scalar density Lagrangian $\mathcal{L}_g$, with the metric $g$ as a new field such that it becomes the original Lagrangian $\mathcal{L}_f$ when $g \rightarrow \eta$. This is easy to do, by just promoting Poincaré scalars, vectors, tensors to scalars, vectors, tensors under diffeomorphisms and to place the usual square root of the determinant of the metric as a factor in the Lagrangian in order to make it the desired scalar density -notice that fermions are excluded as of now-. Thus we assume that, under variations given by the Lie derivative (that is, infinitesimal diffeomorphisms),

$$\delta \mathcal{L}_g = \partial_\mu (\epsilon^\mu \mathcal{L}_g) .$$

Let us define the tensor density

$$T^\mu_\nu := -2 \frac{\delta \mathcal{L}_g}{\delta g^\mu_\nu} .$$

The Noether identity (13) associated with diffeomorphisms, now written for fields $\phi^A$, $g^\mu_\nu$, becomes

$$[\mathcal{L}_g] \delta \phi^A - \partial_\mu ([\mathcal{L}_g] A R^A_\rho \epsilon^\rho - T^\mu_\sigma g_{\sigma_\rho} \epsilon^\rho) - \frac{1}{2} T^\mu_\nu \delta g^\mu_\nu = 0 .$$

Now take the limit $g \rightarrow \eta$, which implies $[\mathcal{L}_g] A \rightarrow [\mathcal{L}_f] A$, and define the Hilbert tensor $T^\mu_\nu$ as the limit of $T^\mu_\nu$ for $g \rightarrow \eta$. In this limit, $\delta g^\mu_\nu = \nabla^\mu \epsilon^\nu + \nabla^\nu \epsilon^\mu \rightarrow \partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu$, with $\epsilon^\mu$ now simply $\epsilon^\mu = \eta^\mu_\nu \epsilon^\nu$, and these $\epsilon^\nu$ still being the components of an arbitrary vector field. Thus we get, in flat space, the identity

$$[\mathcal{L}_f] \delta \phi^A - \partial_\mu ([\mathcal{L}_f] A R^A_\rho \epsilon^\rho - T^\mu_\sigma \eta^\rho_\sigma \epsilon^\rho) - \frac{1}{2} T^\mu_\nu (\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) = 0 .$$

14
for arbitrary $\epsilon^\nu$. Now we can consider several cases. Let us start with the Poincaré symmetry, sitting with Cartesian coordinates. Let $\epsilon^\mu$ be as in (20), that is, a Poincaré infinitesimal coordinate transformation. The important fact is that with this $\epsilon^\mu$, $\delta \eta_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 0$ (our choice for $\epsilon^\mu$ is the general solution of the equation for Killing vectors in flat space, $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 0$). The last term in the lhs of (33) disappears and we end up with

$$[L_f]_A \delta \phi^A + \partial_\mu \left( (T^\mu_\rho - [L_f]_A R_\rho^{A\mu})(a^\rho + \omega^\rho_\nu x^\nu) \right) = 0,$$

(34)

where now $\delta \phi^A$ are the infinitesimal Poincaré transformations of the fields or field components $\phi^A$. Having (8) in mind, (34) identifies the Noether conserved current associated with Poincaré invariance as

$$J^\mu = (T^\mu_\rho - [L_f]_A R_\rho^{A\mu})(a^\rho + \omega^\rho_\nu x^\nu).$$

(35)

As a matter of fact, $T^\mu_\rho - [L_f]_A R_\rho^{A\mu}$ is exactly the Belinfante improved energy-momentum tensor obtained in section 4.1. This result, first obtained in [6]^12, is proven in detail in the appendix. A possible ambiguity of the covariantization procedure, discussed in the next subsection, will not make any difference on this result.

Before moving on, let us note that it is from the Belinfante energy-momentum tensor in (35) that one gets the generator of the Poincaré symmetries, as the spatial integration of the appropriate expression in phase space of the time component $J^0$ of the current, see also (30). The term $[L_f]_A R_\rho^{A\mu}$, though vanishing on shell, is crucial to produce the right transformation

13.

### 4.3 Covariantization procedures. Variety of Hilbert tensors

There is in principle total freedom in the choice of the density behaviour of the fields. The minimal procedure of covariantization is to promote all the fields to geometric entities without implementing any density behaviour for them. This is what we have done above. But one could also have promoted the Poincaré fields to field densities of arbitrary weights. The only consistency needed is that $L_g$ be a scalar density of weight 1, which is easily met. To every different covariantization there will correspond a different Hilbert tensor.

Let us write $L_g^{(0)}$ for the minimal covariantization of $L_f$, that is, with all fields without density behaviour. There is a specific role of $t := \sqrt{|g|}$ to make $L_g^{(0)}$ a density.
Now let us introduce densitizations. Let $n_A$ be the weight associated with $\phi^A$, so that $\tilde{\phi}^A := t^{-n_A} \phi^A$ (no sum for $A$) is not a density scalar, density tensor... any more, but just a scalar, tensor... . Let $n$ symbolize the full set of densitites $n_A$. One can define a new scalar density Lagrangian

$$\mathcal{L}_g^{(n)} = \mathcal{L}_g^{(0)} \bigg|_{\phi^A \rightarrow \tilde{\phi}^A}.$$ 

We do not claim that this procedure will exhaust the ways of obtaining different covariantized Lagrangians but, looking at the result, it is conceivable that the Hilbert tensors thus obtained are the most general ones, except for improvements\(^{14}\). Noticing that $\frac{\partial \mathcal{L}^{(n)}}{\partial g_{\mu \nu}} = -\frac{1}{2} g_{\mu \nu}$ and $\frac{\partial \mathcal{L}^{(n)}}{\partial g_{\sigma g \mu \nu}} = -\frac{1}{2} \delta^\sigma_\rho g_{\mu \nu}$, we can compute

$$\frac{\delta \mathcal{L}_g^{(n)}}{\delta g^\mu\nu} \bigg|_{g \rightarrow \eta} = \frac{\delta \mathcal{L}_g^{(n)}}{\delta g^\mu\nu} \bigg|_{g \rightarrow \eta} + (\frac{\partial \mathcal{L}_g^{(n)}}{\partial \phi^A} \frac{\partial (t^{-n_A} \phi^A)}{\partial t}) \frac{\partial \mathcal{L}_g^{(n)}}{\delta g^\mu\nu} \bigg|_{g \rightarrow \eta} + (\frac{\partial \mathcal{L}_g^{(n)}}{\partial \phi^A} \frac{\partial (t^{-n_A} \phi^A)}{\partial \sigma}) \frac{\partial \mathcal{L}_g^{(n)}}{\delta g^\mu\nu} \bigg|_{g \rightarrow \eta}$$

$$- \frac{\partial}{\partial \sigma} \left( \frac{\partial \mathcal{L}_g^{(n)}}{\partial \sigma} \frac{\partial (t^{-n_A} \phi^A)}{\partial t} \right) \frac{\partial \mathcal{L}_g^{(n)}}{\delta g^\mu\nu} \bigg|_{g \rightarrow \eta}$$

\begin{equation}
\frac{\delta \mathcal{L}_g^{(n)}}{\delta g^\mu\nu} \bigg|_{g \rightarrow \eta} = \frac{1}{2} T_{\mu\nu}^{(n)} + \frac{1}{2} n_A \eta_{\mu\nu} [\mathcal{L}_f]_A \phi^A,
\end{equation}

we have used $T_{\mu\nu} = 2 \frac{\delta \mathcal{L}_g}{\delta g^\mu\nu} \bigg|_{g \rightarrow \eta}$, and put a superscript $(0)$ as a reminder) and so the new Rosenfend tensor $T_{\mu\nu}^{(n)}$ is

\begin{equation}
T_{\mu\nu}^{(n)} = T_{\mu\nu}^{(0)} + n_A [\mathcal{L}_f]_A \phi^A \eta_{\mu\nu}.
\end{equation}

This result is fully consistent with what should have been expected on the grounds that if $\delta^{(0)} \phi^A$ is the Lie derivative of $\phi^A$ considered without density behaviour, then

\begin{equation}
\delta^{(n)} \phi^A = \delta^{(0)} \phi^A + n_A \phi^A \partial_\rho \epsilon^\rho,
\end{equation}

is the Lie derivative for an object of density weight $n_A$. Notice that $\delta^{(0)} \phi^A \rightarrow \delta^{(n)} \phi^A$ implies a change $R^{(0)}_{\rho}{A}\mu \rightarrow R^{(n)}_{\rho}{A}\mu = R^{(0)}_{\rho}{A}\mu + n_A \phi^A \delta^\mu_\rho$.

Using (37) and (38) and taking into account (9), one can easily pass form (33) with superscripts $(0)$ to the same relation with superscripts $(n)$. Note in particular that the Belinfante tensor

\begin{equation}
T_{\mu\nu}^{(n)} - [\mathcal{L}_f]_A R^{(n)}_{\rho}{A}\mu = T_{\mu\nu}^{(0)} - [\mathcal{L}_f]_A R^{(0)}_{\rho}{A}\mu
\end{equation}

\(14\)Improvements that can also be introduced, see subsection 4.4, by adding to the covariantized Lagrangian terms that vanish in flat space. An example of this procedure is given in [29].
still remains the same no matter which superscripts are used. The final form of (33) is then,
\[
[\mathcal{L}_f]_A \delta \phi^A + \partial_\mu (T^\mu_{\nu} \epsilon^\nu) - T^\mu_{\nu} \partial_\mu \epsilon^\nu = 0
\]
which is an identity in flat spacetime but for arbitrary \( \epsilon^\nu \). Note that both the Belinfante tensor and the Hilbert tensor have a role in this expression (40). A change \( T^{(0)}_{\nu} \rightarrow T^{(n)}_{\nu} \) in (40) is compensated by a change in the transformation \( \delta^{(0)}_{\nu} \phi^A \rightarrow \delta^{(n)}_{\nu} \phi^A \), (38).

As regards Poncaré symmetries, (20), the term \( \partial_\rho \epsilon^\rho \) in (38) vanishes because of the antisymmetry of \( \omega_{\mu\nu} \), and thus \( \delta^{(n)}_{\nu} \phi^A \) coincides with \( \delta^{(0)}_{\nu} \phi^A \), as Poincaré symmetries are concerned. But the tensors \( T^{(n)}_{\mu\nu} \) are different if the weights are different. In particular, for a specific choice \( (n) \) of density weights –which means a choice of the transformations (38) as well– one may have other symmetries, for instance conformal invariance, which will only hold for this particular choice. Although their difference vanishes on shell, \( T^{(n)}_{\mu\nu} \) is not an improvement of \( T^{(0)}_{\mu\nu} \), but a basically different Hilbert tensor, which can be used to check whether a new symmetry is realized.

We conclude that whereas there is a unique –up to improvements– Belinfante energy-momentum tensor associated with a Poincaré invariant theory, one does not have a single Hilbert tensor, but a family of them. One may ask whether, among all these possibilities, there is a preferred choice for the Hilbert tensor. The answer is in the positive and it consist in determining the density weights \( n_A \) out of the physical dimensions of the fields. This issue is discussed in section 5.

### 4.4 Improvements

An improvement of the energy-momentum tensor has been already defined in the introduction as the addition of a functional of the fields with identically vanishing divergence. An example of such operation is the addition of the term \( \partial_\sigma F^{[\mu\sigma]}_{\nu} \) to the canonical energy-momentum tensor in (29) to obtain the Belinfante energy-momentum tensor. If we already have a symmetric energy-momentum tensor, improvements preserving this condition must be [30, 31] of the form \( \partial_{\alpha\beta} Y^{\mu\nu\alpha\beta} \) with \( Y^{\mu\nu\alpha\beta} \) having the symmetries of the Riemman tensor, that is, antisymmetric in \( \mu\alpha \), antisymmetric in \( \nu\beta \), and symmetric under the exchange \( \mu\alpha \leftrightarrow \nu\beta \).

A mechanism to improve symmetric energy-momentum tensors is provided [29] by the covariantization method: one may add to the covariantized Lagrangian a new term which vanishes in the flat space limit. Supose we call \( \mathcal{L}_d(g, \phi) \) this new addition, such that \( \mathcal{L}_d|_{g \rightarrow \eta} = 0 \), which also implies \( [\mathcal{L}_d]_{A|_{g \rightarrow \eta}} = 0 \), \( \forall \phi^A \). The Noether identity (19), in
the flat space limit, becomes
\[ \partial_\mu \left( \frac{\delta \mathcal{L}_d}{\delta g_{\mu\nu} |_{g \rightarrow \eta}} \right) = 0, \] (41)

identically. Thus the addition to an energy-momentum tensor of a term\(^{15}\) proportional to \(\frac{\delta \mathcal{L}_d}{\delta g_{\mu\nu} |_{g \rightarrow \eta}}\) is an improvement that preserves the symmetry condition. An application of this procedure will be seen in subsection 5.3.1.

5 Scale and conformal invariance in classical field theory for \(d > 2\)

5.1 General conditions

In this section we consider Poincaré invariant Lagrangians \(\mathcal{L}_f\) in a \(d\)-dimensional Minkoskwi spacetime, \(d > 2\), with \(c\) and \(\hbar\) as universal constants. The condition for a given spacetime vector field \(\epsilon^\mu\) to generate a Noether symmetry in flat spacetime can be read off from comparison of (8) and (33): it requires that \(T^{\mu\nu}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)\) be a divergence. Now we see that Hilbert tensor is more fundamental than Belinfante’s for examining the conditions to have flat spacetime symmetries beyond Poincaré’s, because it is with the Hilbert tensor that we control the existence of new symmetries.

We will explore this possibility of new symmetries for the case of conformal transformations. Vector fields \(\epsilon^\mu\) generating conformal transformations\(^{16}\) in flat spacetime are the solutions for \(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\) to be proportional to \(\eta_{\mu\nu}\) (conformal Killing vectors of flat spacetime). For dimensions of spacetime \(d > 2\) they are
\[ \epsilon^\mu = ax^\mu + \frac{1}{2}(x \cdot x)b^\mu - (x \cdot b)x^\mu, \] (42)

(a is the infinitesimal parameter for scale transformations and \(b^\mu\) are the infinitesimal parameters for a special conformal transformation).

5.1.1 Scale invariance

Every Poincaré invariant Lagrangian \(\mathcal{L}_f\) not containing dimensional parameters implements scale invariance as a classical Noether symmetry. Indeed, the scale transformation \(\delta_s\) for a field –or field component– \(\phi^A\) is (we drop an infinitesimal constant parameter form the transformation)
\[ \delta_s \phi^A = x^\mu \partial_\mu \phi^A + d_A \phi^A \] (43)

\(^{15}\)Although \(\mathcal{L}_d |_{g \rightarrow \eta} = 0\), the quantities \(\frac{\delta \mathcal{L}_d}{\delta g_{\mu\nu} |_{g \rightarrow \eta}}\) need not vanish.

\(^{16}\)Notice that here we only analyze spacetime conformal transformations, always within the active view. Sometimes the same language is used for Weyl transformations, which are not spacetime transformations. Weyl symmetries, which can be rigid or gauge, are dealt with in [10].
(the first term originates from the active view of spacetime transformations. No sum for \( A \) in the last term), where \( d_A \) is its dimension in units of mass. If, and only if, there are no dimensional parameters, \( \mathcal{L}_f \) transforms as

\[
\delta_s \mathcal{L}_f = x^\mu \partial_\mu \mathcal{L}_f + d \mathcal{L}_f = \partial_\mu (x^\mu \mathcal{L}_f)
\]

and hence it is a Noether symmetry.

Note that the scale transformation (43) can be understood as the flat spacetime limit of a diffeomorphism transformation when \( \epsilon^\mu = x^\mu \) (\( x^\mu \) are Cartesian coordinates, for which the flat metric is expressed as \( \eta_{\mu\nu} \)). To make this connection, one asks for

\[
\delta_s \phi^A = \delta^{(0)}_s \phi^A + n_A \phi^A \partial_\mu \epsilon^\mu
\]

(\( \delta^{(0)}_s \) is the Lie derivative for a field without density behaviour) to agree with (43) when \( \epsilon^\mu = x^\mu \). In fact, for such an \( \epsilon^\mu \), we have, from (9), \( \delta^{(0)}_s \phi^A = x^\mu \partial_\mu \phi^A + R^{(0)A\nu}_\nu \), and, using (24), \( R^{(0)A\nu}_\nu = \frac{1}{2} Q^{(0)A\mu}_B \eta_{\mu\nu} \phi^B \). The computation of this piece, \( Q^{(0)A\mu}_B \eta_{\mu\nu} \), depends on the number of covariant and contravariant indices of the field. For a field \( \phi^A \) of the type \( P^{\mu_1\mu_2...\mu_r}_{\nu_1\nu_2...\nu_s} \), one has \( \frac{1}{2} Q^{(0)A\mu}_B \eta_{\mu\nu} = (s - r) \delta^A_B \) and thus (43) and (45) are the same if one defines the density weight of the field \( P^{\mu_1\mu_2...\mu_r}_{\nu_1\nu_2...\nu_s} \) as

\[
n_p = \frac{d_p + r - s}{d}.
\]

Note that the choice of the density weights as determined by (46) dictates a preferred choice for the Hilbert tensor. We will address this important issue in subsection 5.2.

As examples of the application of (46), the scalar field, of dimension \( d_\phi = \frac{d-2}{2} \) has a density weight \( n_\phi = \frac{d-2}{2d} \), and the Maxwell field, of dimension \( d_A = \frac{d-2}{2} \) has a density weight \( n_A = \frac{d-4}{2d} \).

### 5.1.2 Conformal invariance

Similarly to the scale transformations, special conformal transformations \( \delta_c \) can be read as the flat spacetime limit of diffeomorphisms with \( \epsilon^\mu = \frac{1}{2} (x \cdot b) b^\mu - (x \cdot b) x^\mu \). Using (24), one gets

\[
\delta_c \phi^A = \epsilon^\mu \partial_\mu \phi^A - x_\mu b_\nu S^{A\mu\nu}_B \phi^B - (x \cdot b) d_A \phi^A,
\]

where \( d_A \) has been obtained from (46). Equation (47) appears in [32] – see also [33]– by consistency requirements of the transformations with the algebra of the conformal group. Here it is a consequence of the flat spacetime limit of diffeomorphisms associated with conformal transformations, with the key use of (46).
For conformal Killing vectors (42), one has
\[ \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2(a - (x \cdot b))\eta_{\mu\nu} \]
and thus, to generate a Noether symmetry, we need
\[ (a - (x \cdot b)) T = \text{divergence}, \tag{48} \]
where \( T \) is the trace \( T^\mu_\mu \).

Thus the condition for scale invariance is that \( T \) must be a divergence. This condition can be rephrased as the requirement that the e.o.m. for the “Lagrangian” \( T \) vanish identically, that is, \( [T] = 0 \) for any field configuration.

The condition for special conformal invariance is that \( x^\rho T \) to be a divergence, that is, it requires \( [x^\rho T] = 0 \) identically. Allowing –just for simplicity– dependencies up to second spacetime derivatives, one has
\[ [x^\rho T] = x^\rho [T] - \frac{\partial T}{\partial \phi^\rho} + 2\partial_\mu \left( \frac{\partial T}{\partial \phi^\rho} \right) =: x^\rho G(\phi, \partial\phi, \partial^2\phi) + F(\phi, \partial\phi, \partial^2\phi) = 0 \tag{49} \]
for any field configuration. Considering translations \( x^\rho \rightarrow x^\rho + \lambda^\rho \) we can easily convince ourselves that this last equation implies \( G(\phi, \partial\phi, \partial^2\phi) = 0 \) and \( F(\phi, \partial\phi, \partial^2\phi) = 0 \) separately. In particular this means that \( [T] = 0 \). We conclude that the assumption of special conformal invariance, \( [x^\rho T] = 0 \) identically, already guarantees scale invariance, \( [T] = 0 \) identically, and is thus equivalent to full conformal invariance\(^{17}\).

When scale invariance is realized, \( T = \partial_\mu D^\mu \) for some \( D^\mu \). Plugging this relation into (49) gives \( [D^\mu] = 0 \) which means that \( D^\mu \) must be in its turn a divergence for full conformal invariance to be realized. This is equivalent to the requirement for \( T \) to be a double divergence, that is, there must exist \( C^{\alpha\beta} \) such that \( T = \partial_{\alpha\beta} C^{\alpha\beta} \). Notice that we can always take \( C^{\alpha\beta} \) symmetric, which we do in the following. If these quantities do exist, there is a standard procedure [8] to improve the energy-momentum tensor so that it will explicitly exhibit the tracelessness condition. In \( d > 2 \) spacetime dimensions, one defines
\[ Y^{\mu\alpha\beta} = \frac{1}{d-2} \left( \eta^{\mu\nu} C^{\alpha\beta} + \eta^{\alpha\beta} C^{\mu\nu} - \eta^{\alpha\mu} C^{\beta\nu} - \eta^{\beta\mu} C^{\alpha\nu} \right) - \frac{1}{(d-1)(d-2)} \left( \eta^{\mu\nu} \eta^{\alpha\beta} - \eta^{\alpha\nu} \eta^{\beta\mu} \right) C^\rho_\rho. \tag{50} \]
\( Y^{\mu\alpha\beta} \) is the only structure, made with the Minkowski metric and \( C^{\alpha\beta} \), with the symmetries of the Riemman tensor, and with the property \( \partial_{\alpha\beta}(Y^{\mu\alpha\beta}\eta_{\mu\nu}) = \partial_{\alpha\beta} C^{\alpha\beta} \). Then the improved tensor
\[ T^{\mu\nu}_C := T^{\mu\nu} - \partial_{\alpha\beta} Y^{\mu\alpha\beta} \tag{51} \]
\(^{17}\)This result is also obtained by examining the algebra of the Poincaré plus conformal trasformations.
is traceless. Equation (33) now becomes
\[ [L_f]_A \delta \phi^A - \partial_\mu \left( [L_f]_A R^A_\mu \epsilon^\rho - T^\mu_\nu \eta_{\rho\nu} \epsilon^\rho \right) = 0. \]
with \( \epsilon^\rho \) the vector field for conformal transformations (42). Note that \( T_\nu^\mu = [L_f]_A R^A_\mu \eta^\rho_\nu \) is an improvement of Belinfante’s tensor. In fact the conserved current in (52) is \( J^\mu = (T^\mu_\nu - \partial_\alpha \beta V^{\mu\alpha\beta} \eta_{\nu\rho}) \epsilon^\rho \), out of which, after moving to phase space, one can obtain the canonical generators of scale and conformal transformations.

In conclusion, scale invariance is equivalent to \( T \) –the trace of the Hilbert tensor– being a divergence, \( T = \partial_\mu D^\mu \), whereas special conformal invariance, which already implies scale invariance and thus full conformal invariance, is equivalent to \( T \) being a double divergence, \( T = \partial_\alpha \beta C^{\alpha\beta} \), which is a much more restrictive condition.

### 5.2 Selecting the Hilbert tensor

Consider again (40) and apply it to the scale invariance property of a Poincaré invariant Lagrangian without dimensional parameters. For \( \delta \phi^A \) to become the scale transformation \( \delta \phi^A \) in (43) we need the satisfaction of (46), which means that the Hilbert tensor present in (40) is in fact \( T^{(n)} \) associated with a specific covariantization within the family introduced in 4.3. We infer that The Hilbert tensor becomes completely determined –up to improvements– when scale invariance is implemented. Now we will compute the trace of this tensor for a scale invariant theory. Let us write (40) for \( \epsilon^\mu = x^\mu \),
\[ [L_f]_A \delta \phi^A + \partial_\mu (T^\mu_\nu \eta_{\rho\nu} x^\rho) - T^{(n)} = 0. \]
We can isolate the trace \( T^{(n)} \) from (53) and obtain, using (44),
\[
T^{(n)} = \partial_\mu \left( x^\mu L_f - \frac{\partial L_f}{\partial \phi^A_\mu} \delta \phi^A + T^\mu_\nu \eta_{\rho\nu} x^\rho \right) = \partial_\mu \left( x^\mu (T^\mu_\nu - \tilde{T}^\mu_\nu) \eta_{\rho\nu} - \frac{\partial L_f}{\partial \phi^A_\mu} d_\lambda \phi^A \right)
\]
\[
= \partial_\mu \left( x^\mu (\partial_\sigma F^{\nu\mu\sigma} \eta_{\rho\nu} - \frac{\partial L_f}{\partial \phi^A_\mu} d_\lambda \phi^A) \right) = \partial_\sigma \left( F^{\nu\mu\sigma} \eta_{\rho\nu} - \frac{\partial L_f}{\partial \phi^A_\mu} d_\lambda \phi^A \right)
\]
\[
= \partial_\sigma \left( \frac{\partial L_f}{\partial \phi^A_\mu} (S_B^{A\sigma\nu} \eta_{\mu\nu} \phi^B - \delta^\sigma d_\lambda \phi^A) \right) = \partial_\sigma \left( \frac{\partial L_f}{\partial \phi^A_\mu} \eta_{\mu\nu} (S_B^{A\sigma\nu} - d_\lambda \delta^A_B \eta^{\sigma\nu}) \phi^B \right),
\]
where we have used (44) and the definitions (26), (27) and (29), and the antisymmetry properties of \( F^{\nu\mu\sigma} \). From the first equality in (54) we see that as expected \( T^{(n)} \) is a divergence, which is the condition for scale invariance.

Once this tensor is determined, the fate of conformal invariance is sealed and its possible additional realization can be further investigated. In fact, (54) shows explicitly the new condition for implementing full conformal invariance in the presence
of scale invariance: for $T^{(n)}$ to be a double divergence (which is the requirement found in subsection 5.1.2) we need

$$\frac{\partial \mathcal{L}_f}{\partial \phi} \eta_{\mu\nu} \left( S_B^{A\sigma\nu} - d_A \delta_B^{A\sigma\nu} \right) \phi^B = \text{divergence}. \quad (55)$$

This result can be stated as the following

**Proposition.** A Poincaré invariant Lagrangian $\mathcal{L}_f$ in $d$-dimensional Minkowski spacetime, with $d > 2$, with no dimensional parameters –and thus implementing scale invariance–, is conformal invariant if and only if expression (55) is fulfilled.

This equation (55) was obtained in [7] (see section 5.2 in that paper) on the basis of an analysis of the conditions for $\delta_c$ in (47) to be a Noether transformation. The difference with our approach is that we can specify the Hilbert tensor which contains in its trace the information as to whether a scale invariant theory realizes conformal invariance. It is worth noticing that this Hilbert tensor is not the Belinfante tensor –though they differ in terms vanishing on shell– and it is only this Hilbert tensor that becomes traceless –if conformal invariance holds– after the improvement devised at the end of subsection 5.1.2.

### 5.3 Examples

#### 5.3.1 Massless real scalar field with $\phi^m$ interaction in $d > 2$ dimensions

The Poincaré invariant Lagrangian is $\mathcal{L}_f = \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi \eta^{\mu\nu} + \Lambda \phi^m$. The dimension of the field is $d_\phi = \frac{d_\phi}{2}$. According to subsection 5.1.1, scale invariance is guaranteed as long as $\Lambda$ is dimensionless, which is equivalent to $m = \frac{2d}{d-2}$. Its integer solutions are $(d,m) = (6,3), (4,4), (3,6)$. It is now easy to check that conformal invariance holds at any dimension. Indeed, the kinetic term satisfies (55) –because $S_B^{A\sigma\nu}$ vanishes whereas $\phi \partial_{\mu} \phi$ is a divergence –and so does trivially the selfinteraction term –because since it has no derivatives, it does not contribute to (55). The rhs of (55) is actually $\partial_{\mu} \left( -\frac{d-2}{4} \eta^{\mu\sigma} \phi^2 \right)$.

Constructing the improved, traceless, energy-momentum tensor, is, after (50), straightforward. As explained in subsection 4.4, an alternative procedure to introduce the improvement is the addition to the covariantized Lagrangian of a term proportional to $\sqrt{|g|}R \phi^2$, see [29], taking into account that $\frac{\delta(\sqrt{|g|}R \phi^2)}{\delta g^{\mu\nu}} |_{g \to \eta} = \partial_{\alpha\beta} \left( (\eta_{\mu\nu} \eta^{\alpha\beta} - \delta^\alpha_\mu \delta^\beta_\nu) \phi^2 \right)$.

The extension to $O(N)$ symmetry, with interaction term $\Lambda(\phi^2)^2$, is immediate and one can check that for dimensionless $\Lambda$, both scale and conformal invariance are realized at any spacetime dimension $d > 2$. 

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5.3.2 Pure Maxwell or YM theory

Let us apply (55) to the Maxwell Lagrangian, \( \mathcal{L}_{\text{max}} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \), with \( F_{\mu \nu} := \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \), for which \( d_{A} = \frac{d-2}{2} \). Expression (55) becomes

\[
\frac{\partial \mathcal{L}_{\text{max}}}{\partial A_{\rho, \mu}} \eta_{\mu \nu} (S_{(\rho)}^{(\lambda) \sigma \nu} - \frac{d-2}{2} \delta_{\rho}^{\lambda} \eta_{\sigma \nu}) A_{\lambda} = -F_{\mu \rho} \eta_{\mu \nu} (\delta_{\rho}^{\nu} \eta^{\lambda} - \delta_{\rho}^{\sigma} \eta^{\lambda \nu} - \frac{d-2}{2} \delta_{\rho}^{\lambda} \eta_{\sigma \nu}) A_{\lambda}
\]

\[
= -F_{\mu \rho} (\eta_{\mu \rho} A^{\sigma} - \delta_{\rho}^{\sigma} A_{\mu} - (1 + \frac{d-4}{2}) \delta_{\rho}^{\sigma} A_{\mu}) = \frac{d-4}{2} F_{\sigma \rho} A_{\rho} ,
\]

(where we have used (22)) which is only a divergence –actually it vanishes– for \( d = 4 \). Thus pure Maxwell, which is scale invariant at any spacetime dimension, only becomes conformal invariant at four dimensions.

One can easily check that scalar electrodynamics for \( d = 4 \) also satisfies (55). On the other hand, For pure YM, due to the presence of the selfinteraction terms, scale and/or conformal invariance are only achieved at four dimensions (it is the spacetime dimension for which the coupling constant is dimensionless). The proof repeats the calculations done for the Maxwell case but with a new summation index over the dimension of the gauge group.

5.3.3 \( p \)-forms

Let us first consider the free theory of 2 forms, with gauge field \( B = \frac{1}{2} B_{\mu \nu} dx^{\mu} \wedge dx^{\nu} \), with \( B_{\mu \nu} \) antisymmetric. The field strenght is \( H_{\mu \nu \sigma} = \partial_{\mu} B_{\nu \sigma} + \partial_{\nu} B_{\sigma \mu} + \partial_{\sigma} B_{\mu \nu} \) and the Lagrangian is the kinetic term \( \mathcal{L}_{B} = \frac{1}{6} H_{\mu \nu \sigma} H^{\mu \nu \sigma} \). To compute the lhs of (55) we note that the representation of the Lorentz algebra defined in (23) is now\(^{18} (A = [\mu \nu], \ B = [\alpha \beta]) \)

\[
S_{(\mu \nu)}^{[\alpha \beta] \sigma \rho} = \frac{1}{2} \left( \eta_{\rho \mu} \delta_{\sigma}^{\rho} \delta_{\nu}^{\sigma} - \eta_{\rho \nu} \delta_{\sigma}^{\rho} \delta_{\mu}^{\sigma} + \eta_{\sigma \beta} \delta_{\rho}^{\sigma} \delta_{\mu}^{\rho} - \eta_{\sigma \beta} \delta_{\rho}^{\sigma} \delta_{\nu}^{\rho} - \eta_{\sigma \beta} \delta_{\rho}^{\sigma} \delta_{\mu}^{\rho} - (\mu \leftrightarrow \nu) \right),
\]

and (55) becomes

\[
\frac{\partial \mathcal{L}_{B}}{\partial B_{\mu \nu, \lambda}} \eta_{\lambda \rho} (S_{(\mu \nu)}^{[\alpha \beta] \sigma \rho} - \frac{d-2}{2} \delta_{(\mu \nu)}^{[\alpha \beta]} \eta_{\sigma \rho}) B_{\alpha \beta}
\]

\[
= H_{\lambda \mu \nu} \eta_{\lambda \rho} \left( -\delta_{(\mu}^{\sigma} B_{\lambda \nu} - \delta_{(\nu}^{\sigma} B_{\mu \lambda} - (2 + \frac{d-6}{2}) \delta_{(\lambda}^{\sigma} B_{\mu \nu}) \right)
\]

\[
= \frac{d-6}{2} H^{\sigma \mu \nu} B_{\mu \nu} ,
\]

(where we have used in the second and third equalities the full antisymmetry of \( H_{\lambda \mu \nu} \) which is only a divergence –in fact it vanishes– for \( d = 6 \). It is not difficult to identify

\(^{18}\text{We apply the practical convention that any object with indices that saturate those of } B_{\mu \nu} \text{ must be taken antisymmetric in such indices. This is the reason of the factor } \frac{1}{2}.\)
a pattern in the expressions (56) and (57) which allows for a generalization to $p$-forms. The result for the $p$-form free theory is that conformal invariance is only achieved for $d = 2(p + 1)$ dimensions. Since, according to (46), the density weight of a $p$-form field is $n_p = \frac{d-2(p+1)}{2d}$, one can deduce from the generalization of (57) that the Hilbert tensor $T^{(0)}_{\mu\nu}$ obtained from the minimal covariantization will become tracelessness for $d = 2(p + 1)$.

5.3.4 Interaction term with scale invariance but not conformal invariance

Consider the interaction term of a scalar field with a vector field $L_{int} = \Lambda \eta^{\mu\nu} (\partial_{\mu} A_{\nu}) \phi^m$. It breaks $U(1)$ gauge as well as rigid invariance\(^{19}\) and we can even take the scalar field real. We assume the standard dimensions for the fields $A_{\mu}$ and $\phi$. Scale invariance is guaranteed with dimensionless $\Lambda$, which means $m = \frac{d}{d-2}$, whose integer solutions are $(d, m) = (3, 3), (4, 2)$. As regards conformal invariance, the lhs of (55) becomes in this case $\Lambda^2 A^\sigma \phi^m$, which is not a divergence. Thus the model admits scale invariance but not conformal invariance. This model though does not lead to a unitary quantum field theory. In this sense, the issue [8] as to whether there exists a $d = 4$ unitary quantum field theory realizing scale invariance but not conformal invariance remains unsettled\(^{20}\).

5.3.5 Abelian Chern-Simons theory in $d = 2n + 1$ dimensions

The Lagrangian density is

$$L_{CS}^{2n+1} = \epsilon^{\mu_0\mu_1...\mu_{2n}} A_{\mu_0} F_{\mu_1\mu_2} ... F_{\mu_{2n-1}\mu_{2n}}.$$ (58)

This Lagrangian, which is metric independent, is already diffeomorphism invariant and therefore scale and conformal invariance are trivially realized as part of it. The dimension of the field is $d_{A_{\mu}} = 1$ and the corresponding density weight, from (46), is $n_{A_{\mu}} = 0$, which is consistent with the fact that the Hilbert tensor vanishes identically ($n_{A_{\mu}} \neq 0$ would have suggested a rewriting of the "covariantized" Lagrangian which would have allowed to define along the lines of subsection 4.3, see (37), a nonvanishing Hilbert tensor $T^{(n)}_{\mu\nu}$ even when $T^{(0)}_{\mu\nu}$ was vanishing). One can also check that the lhs of (55) vanishes identically. Equation (40) remains valid but without the last term of the lhs. One directly identifies from this expression the Noether current for diffeomorphism invariance as $J^\mu = T^\mu_{b\nu} \epsilon^\nu$, where, being zero the Hilbert tensor, the Belinfante tensor can be extracted from (39) as $T^\mu_{\rho\nu} = -[L_f] A R^A_{\rho\mu}$, which makes equation (40) to become the Noether identity (13). The Belinfante tensor vanishes on shell but it is capable of providing in the canonical picture for the generators of the gauge symmetry of diffeomorphism invariance.

\(^{19}\)In [7] the case with an interaction term of the type $(\partial_{\mu} A^\nu)^2$ is considered.

\(^{20}\)The assumptions made in [32] exclude our example.
6 Classical field theory with gravity

Consider a matter covariant Lagrangian $\mathcal{L}_m(g, \partial g, \phi, \partial \phi)$ (where $g$ is the metric field and $\phi$ the matter fields). Following Hilbert's prescription, we define the matter energy-momentum tensor (Hilbert) density $T^\mu\nu = -2 \partial^\mu \mathcal{L}_m / \partial g^\mu\nu$. Since $\mathcal{L}_m$ is a scalar density, we can write the Noether identities with respect to diffeomorphism invariance for $\mathcal{L}_m$. We will use the manifestly covariant form of the Noether identities discussed on subsection 3.2.1. We have, for the variations (see (18)),

$$\delta g^\mu\nu = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu,$$

$$\delta \phi^A = U^A_\rho \epsilon^\rho + R^A_\rho \nabla_\mu \epsilon^\rho,$$

and the the Noether identities (19), now for fields $g^\mu\nu$ and $\phi^A$, become

$$\nabla_\mu T^\mu\nu = -[\mathcal{L}_m]_A U^A_\nu + \nabla_\mu \left( [\mathcal{L}_m]_A R^A_\nu \right).$$

(59)

Saturating with $\epsilon^\nu$ we obtain an expression formally identical to (33),

$$[\mathcal{L}_m] \delta \phi^A + \partial_\mu \left( (T^\mu_\nu - [\mathcal{L}_m]_A R^A_\nu) \epsilon^\nu \right) - T^\mu_\nu \nabla_\mu \epsilon^\nu = 0,$$

(60)

but now in curved, dynamical spacetime. More important is to notice that (59) tells that $T^\mu_\nu$ is a covariantly conserved tensor density as long as the matter fields satisfy the e.o.m., and this result has no relation whatsoever with the Einstein equations—we are only considering the matter Lagrangian (had we considered the addition of the purely gravitational Einstein-Hilbert Lagrangian, the Einstein e.o.m. will also imply independently that $T^\mu_\nu$ is covariantly conserved). This point, also made in [9], is basically known since the early years of GR [5] but sometimes gets blurred. Contrary to some popularizations, it is not that the choice for the Einstein tensor density—the gravity side of the Einstein equations—is restricted because one wants to have a covariantly conserved matter energy-momentum tensor, but because one already has it. Of course, when the e.o.m. are derived from a variational principle there is no restriction any more because any purely gravitational metric Lagrangian $\mathcal{L}_{grav}$ will yield, as Noether identities associated with diffeomorphism invariance, the covariant conservation of $\delta \mathcal{L}_{grav} / \delta g^\mu\nu$.

6.1 Field theory in nondynamical curved spacetime

One could decide to freeze the background, let’s name it $\tilde{g}$, and to make it non-dynamical. This describes matter in a fixed background without taking into account the backreaction induced by the matter on it. There is a case, though, where the backreaction is indeed included, which is when, in addition to the satisfaction of the matter e.o.m., $\tilde{g}$ is solution of the e.o.m. for $\mathcal{L}_{grav} + \mathcal{L}_m$.

\[\text{\textsuperscript{21}}\text{it is a tensor density of weight 1 because the Lagrangian is a scalar density.}\]
Now the Lagrangian $L_m$ becomes a functional of the matter fields only, $L_m = L_m|_{g \rightarrow \bar{g}}$. Equations (59) and (60) are still applicable with the only circumstance of sending $g \rightarrow \bar{g}$. Thus for any matter Lagrangian in any fixed background, there is covariant conservation of the Hilbert tensor. Is this conservation related to any Noether symmetry? The answer in general is no, because, except for some special cases, there is no way to extract from this tensor density a conserved vector density –to play the role of the Noether current. The exceptions: first, when the background has Killing vectors. If $\xi^\mu$ is a Killing vector for the background $\bar{g}_{\mu\nu}$ then the last term in (60) vanishes for $\epsilon^\mu \rightarrow \xi^\mu$ and we end up with the Noether conserved current $J^\mu = (T^\mu_\nu - [L_m]_{A} R^A_{\mu\nu}) \xi^\nu$. Another exception is the case when $\epsilon^\mu$ is such that $T^\mu_\nu \nabla^\nu \epsilon^\mu$ happens to be a divergence (see (60)). This is reminiscent of our explorations in subsection 5.1 concerning conformal symmetry in flat spacetime. In fact, for a conformal Killing vector the condition is that $(\nabla^\mu \epsilon^\mu) tr(T)$ must be a divergence, which includes the tracelessness case $tr(T) = 0$, now in curved space.

The tensor $T^\mu_\nu - [L_m]_{A} R^A_{\mu\nu}$ could well receive the name of Belinfante –or Belinfante-Rosenfeld– tensor in nondynamical curved space, because it is the tensor that participates in defining the Noether conserved current and the canonical Noether symmetry generator, if such a symmetry is realized.

7 Conclusions

We have presented a basic review of classical aspects of Noether symmetries, with special emphasis in the use of Noether identities for gauge theories. Next we have discussed in detail the role of different energy-momentum tensors in Poincaré invariant field theories and explored the eventual realization of scale and conformal invariance. The core of this part of paper is devoted to flat spacetime.

The main points worth remarking are

- The introduction and use of some identities of the variational calculus, (5), in order to obtain the conditions for a continuous symmetry.
- The use of the Noether identities to find the first class structure –except for the primary constraints in phase space– of constraints of a gauge theory, subsections 3.1 and 3.1.1.

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22 Since the background is fixed, there is no longer the gauge symmetry of diffeomorphism invariance. A residual, non-gauge, subgroup of diffeomorphisms can still realize Noether symmetries, as we shall see.

23 Conserved or covariantly conserved is the same for a vector density, with the standard Riemmanian connection.
- The observation that whereas the Belinfante energy-momentum tensor is unique and it is directly linked with the Poincaré generators, there is a family of Hilbert tensors, subsection 4.3, which are the means to test whether further spacetime symmetries can be realized. All these tensors only differ in terms vanishing on shell.

- The selection of a unique Hilbert tensor for Poincaré invariant Lagrangians realizing scale invariance, subsection 5.2.

- An example, subsection 5.3.4, of a toy model that exhibits scale invariance but not conformal invariance at the classical level.

- A simple expression, (55), derived from the computation of the trace of the Hilbert tensor, as a means to check for a specific scale invariant theory whether conformal invariance is also realized.

Let us elaborate on this last point, summarizing our results. We start with Poincaré invariant theories in Minkowski spacetime. In the classical setting, scale invariance is identified by the absence of dimensional parameters in the Lagrangian. The correct Hilbert tensor for such a scale invariant theory is determined through (37) by using the relation (46) between the dimensions of the fields and their density weights upon covariantization. The trace of this tensor is a divergence –thus describing scale invariance–, and it is computed in (54). Then the additional requirement of conformal invariance is (55).

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**Appendix. Belinfante - Hilbert**

Here we prove the result advanced in subsection 4.2 concerning the relation between Belinfante tensor and the Hilbert tensors. Let us examine again the consequences of
the Noether condition (7) for the minimally covariantized Lagrangian \( L_g := L_g^{(0)} \) (we eliminate superscripts for simplicity) of the flat spacetime Lagrangian \( L_f \),

\[
\delta L_g = -\frac{1}{2} T_g^{\mu\nu} \delta g_{\mu\nu} + \partial_\sigma (\frac{\partial L_g}{\partial g_{\mu\nu,\sigma}} \delta g_{\mu\nu}) + [L_g]_A \delta \phi^A + \partial_\sigma (\frac{\partial L_g}{\partial \phi^A_\sigma} \delta \phi^A) = \partial_\sigma (\epsilon^\sigma L_g)
\]

where we have used the definition (31).

We can make the limit \( g \to \eta \) in this expression, still keeping \( \epsilon^\sigma \) arbitrary. \( T_g^{\mu\nu} \) becomes the Hilbert tensor \( T^{\mu\nu} \) associated with the covariantization. We get,

\[
-\frac{1}{2} T^{\mu\nu} \delta \eta_{\mu\nu} + \partial_\sigma (\frac{\partial L_g}{\partial g_{\mu\nu,\sigma}} \delta \eta_{\mu\nu}) + [L_f]_A \delta \phi^A + \partial_\sigma (\frac{\partial L_f}{\partial \phi^A_\sigma} \delta \phi^A) = \partial_\sigma (\epsilon^\sigma L_f) \tag{61}
\]

with \( \delta \eta_{\mu\nu} = \eta_{\mu\rho} \epsilon^\rho_\nu + \eta_{\nu\rho} \epsilon^\rho_\mu \) and \( \delta \phi^A = R^A_\rho \epsilon^\rho + R^{A\mu}_\rho \partial_\mu \epsilon^\rho \). Since \( \epsilon^\mu \) is arbitrary, we can get different identities by equalizing the coefficients of \( \epsilon^\mu \), \( \partial_\sigma \epsilon^\rho \) and \( \partial_\sigma \epsilon^\rho \) on both sides of (61). For \( \epsilon^\mu \) we just get the conservation on shell of the canonical energy-momentum tensor \( T^{\mu\nu} \), (26). For \( \partial_\sigma \epsilon^\rho \) we obtain

\[
-T^\rho_\sigma + \dot{T}^\sigma_\rho + [L_f]_A R^A_\rho + \partial_\lambda (2 \frac{\partial L_g}{\partial g_{\mu\sigma,\lambda}} \eta_{\mu\rho} + \frac{\partial L_f}{\partial \phi^A_\sigma} R^A_\rho) = 0 , \tag{62}
\]

and finally, for \( \partial_\sigma \epsilon^\rho \) we obtain that the quantity inside the \( \lambda \)-derivative in (62) must be antisymmetric in the \( \sigma \lambda \) indices, that is, the quantity

\[
G^{\rho\sigma\lambda} := 2 \frac{\partial L_g}{\partial g_{\rho\sigma,\lambda}} \eta_{\mu\rho} + \frac{\partial L_f}{\partial \phi^A_\sigma} R^A_\rho \eta_{\mu\rho} . \tag{63}
\]

satisfies \( G^{\rho\sigma\lambda} = G^{[\rho|\sigma|\lambda]} \). Substituting (24) in the last term of (63), one sees that the structure of (63) is of the type

\[
A^{(\rho\sigma)}_\lambda + B^{[\rho\sigma]}_\lambda = G^{[\rho|\sigma|\lambda]} , \tag{64}
\]

and it is well known that knowledge of the coefficients \( B \) completely determines the coefficients \( A \) and \( G \). Indeed one gets

\[
A^{(\rho\sigma)}_\lambda = B^{[\rho\sigma]}_\lambda + B^{[\lambda\rho]}_\sigma .
\]

In our case, we obtain\(^{24}\)

\[
2 \frac{\partial L_g}{\partial g_{\rho\sigma,\lambda}} \eta_{\mu\rho} = \frac{1}{2} \left( \frac{\partial L_f}{\partial \phi^A_\rho} S^A_\lambda \phi^B + \frac{\partial L_f}{\partial \phi^A_\sigma} S^A_\rho \phi^B - \frac{\partial L_f}{\partial \phi^A_\lambda} Q^A_\rho \phi^B \right) , \tag{65}
\]

\(^{24}\)This equation could in principle have been alternatively obtained by noticing that the dependences of \( L_g \) on derivatives of the metric must come from terms in \( L_f \) depending on the derivatives of the fields, since these derivatives become covariant derivatives upon covariantization.
and equation (62) becomes

\[ T^{\sigma\rho} = \hat{T}^{\sigma\rho} + [\mathcal{L}_f]_A R^A_{\mu\sigma} \eta^{\mu\rho} + \frac{1}{2} \partial_\lambda \left( \frac{\partial \mathcal{L}_f}{\partial \phi^A_\rho} S^A_{B\lambda\sigma} \phi^B + \frac{\partial \mathcal{L}_f}{\partial \phi^A_\sigma} S^A_{B\lambda\rho} \phi^B + \frac{\partial \mathcal{L}_f}{\partial \phi^A_\lambda} S^A_{B\rho\sigma} \phi^B \right) \]

\[ = \hat{T}^{\sigma\rho} + [\mathcal{L}_f]_A R^A_{\mu\sigma} \eta^{\mu\rho} + \partial_\lambda (F^{\rho\sigma\lambda}) \]  

(66)

where \( F^{\rho\sigma\lambda} = F^{[\rho|\sigma]\lambda} = G^{\rho|\sigma]\lambda} \) are the quantities defined in (27). Using the definition of Belinfante’s tensor, (29), we arrive at

\[ T^{\sigma\rho} = T_b^{\sigma\rho} + [\mathcal{L}_f]_A R^A_{\mu\sigma} \eta^{\mu\rho}, \]  

(67)

which shows the connection between Belinfante’s and the minimal –that is, obtained by the minimal covariantization– Hilbert’s tensor, anticipated in section 4.2. If, however, instead of the minimal covariantization, \( \mathcal{L}^{(0)}_g \), we take any covariantization, \( \mathcal{L}^{(n)}_g \), and the respective change for the variations, \( \delta^{(0)} \phi^A \rightarrow \delta^{(n)} \phi^A = \delta^{(0)} \phi^A + n_A \phi^A \partial_\mu \epsilon^\mu \), note that \( S^A_{B\rho\sigma} \phi^B \) in (24) remains unchanged and, as a consequence, the Belinfante tensor in (66),(67), remains unchanged as well.

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