Algebraic Kan extensions in double categories

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Abstract

We study Kan extensions in three weakenings of the Eilenberg-Moore double category associated to a normal pseudo double monad, that was introduced by Grandis and Paré. To be precise, given such a double monad $T$ on a double category $K$, we consider the double categories consisting of pseudo $T$-algebras, ‘weak’ vertical $T$-morphisms, horizontal $T$-morphisms and $T$-cells, where ‘weak’ means either ‘lax’, ‘colax’ or ‘pseudo’. Denoting these double categories by $\text{Alg}_w(T)$, where $w = l, c$ or $\text{ps}$ accordingly, our main result gives, in each of these cases, conditions ensuring that pointwise Kan extensions can be lifted along the forgetful double functor $\text{Alg}_w(T) \to K$. As an application we recover and generalise a result by Getzler, on the lifting of pointwise left Kan extensions along symmetric monoidal enriched functors.

Introduction

When given a symmetric monoidal functor $j: A \to B$ one, instead of considering ‘ordinary’ Kan extensions along $j$, often considers ‘symmetric monoidal’ Kan extensions along $j$. Precisely, while we can consider Kan extensions along $j$ in the 2-category $\text{Cat}$ of categories, functors and transformations, it is often more useful to consider such extensions in the 2-category $\text{sMonCat}$ of symmetric monoidal categories, symmetric monoidal functors and monoidal transformations.

For example consider the notion of an algebra of a ‘PROP’: informally, a PROP $P$ is a certain kind of symmetric monoidal category that describes a type of algebraic structure, on the objects $A$ of any symmetric monoidal category, which involves operations of the form $A^\otimes n \to A^\otimes n$. There is, for instance, a PROP that describes monoids, and one that describes Hopf monoids. An algebra $A$ of $P$, in a symmetric monoidal category $M$, is then an object $A$ in $M$ that is equipped with the algebraic structure described by $P$ — formally, $A$ is simply a symmetric monoidal functor $A: P \to M$. Presenting algebraic structures as symmetric monoidal functors like this has the advantage that, often, freely generated such structures can be constructed as left Kan extensions in $\text{sMonCat}$. For example, the PROP $C$ of cocommutative comonoids embeds into the PROP $\mathbb{H}$ of bicommutative Hopf monoids, and the bicommutative Hopf monoids in $M$ that are freely generated by cocommutative comonoids correspond precisely to left Kan extensions along the embedding $C \hookrightarrow \mathbb{H}$. We will show in §5 that, as a consequence of our main result, all such Kan extensions exist under reasonable conditions on $M$.

As a second example, Getzler shows in [Get09] that ‘operads’ in $M$ — which describe algebraic structures involving operations of the form $A^\otimes n \to A$ — can also be regarded as symmetric monoidal functors $T \to M$, where $T$ is a symmetric monoidal category that describes the algebraic structure of operads. Several generalisations of operads, such as ‘cyclic operads’ and ‘modular operads’ can be similarly presented. The main result of [Get09] gives conditions ensuring that left
Kan extensions can be ‘lifted’ along the forgetful 2-functor \( \text{sMonCat} \to \text{Cat} \) which, as a consequence, gives a coherent way of freely generating many types of generalised operad. To describe this ‘lifting’ of Kan extensions more precisely, let us consider symmetric monoidal functors \( j: A \to B \) and \( d: A \to M \): we say that the ordinary left Kan extension \( l: B \to M \) of \( d \) along \( j \), in \( \text{Cat} \), can be lifted to \( \text{sMonCat} \) whenever \( l \) admits a ‘canonical’ symmetric monoidal structure that makes it into the left Kan extension of \( d \) along \( j \) in \( \text{sMonCat} \).

Using the language of double categories, (the horizontal dual of) the main result of this paper, which is stated below, can be thought of as generalising Getzler’s result, of ‘lifting symmetric monoidal Kan extensions’, to the broader idea of ‘lifting algebraic Kan extensions’. In the remainder of this introduction we will informally explain some of the details of the main result, and at the same time describe the contents of this paper. We will however not explain the condition (p), except for remarking that, to recover Getzler’s result, we apply the main result to the double monad whose algebras are symmetric monoidal categories and, in that case, (p) holds as soon as the tensor product of \( M \) preserves colimits in each variable.

**Theorem 4.7.** Let \( T \) be a normal pseudo double monad on a double category \( K \), and assume that \( T \) is pointwise left exact. Let ‘weak’ mean either ‘colax’, ‘lax’ or ‘pseudo’. Given pseudo \( T \)-algebras \( A, B \) and \( M \), consider the following conditions on a horizontal \( T \)-morphism \( J: A \Rightarrow B \) and a weak vertical \( T \)-morphism \( d: A \to M \):

1. (p) the algebraic structure of \( M \) preserves the pointwise left Kan extension of \( d \) along \( J \);
2. (e) the structure cell of \( J \) is pointwise left \( d \)-exact;
3. (l) the forgetful double functor \( \text{Alg}_w(T) \to K \) lifts the pointwise left Kan extension of \( d \) along \( J \).

The following hold:

1. if ‘weak’ means ‘colax’ then (e) implies (l);
2. if ‘weak’ means ‘lax’ then (p) implies (l);
3. if ‘weak’ means ‘pseudo’ then any two of (p), (e) and (l) imply the third.

We start in §1 by recalling the relevant terminology on double categories. Briefly, the features of double categories that distinguish them from bicategories are that, besides objects, they consist of two types of morphisms, ‘vertical’ ones denoted \( f: A \to B \) and ‘horizontal’ ones denoted \( J: A \Rightarrow B \), and cells that are shaped like squares, which can be composed both vertically and horizontally. Amongst others, we shall recall the notions of ‘restriction’ and ‘extension’ of a horizontal morphism along vertical morphisms, as well as the notion of ‘tabulation’ of a horizontal morphism — the latter can be thought of as generalising the notion of comma object in 2-categories. Following this we recall the notion of ‘left Kan extension’ in any double category \( K \) which, as can be deduced from the way the main result is stated, defines the extension of a vertical morphism \( d: A \to M \) along a horizontal morphism \( J: A \Rightarrow B \); the resulting extension, if it exists, is a vertical morphism of the form \( l: B \to M \) which, like in the case of 2-categories, is defined by a cell satisfying a universal property. In fact, the notion of Kan extension in \( K \) generalises that of Kan extension in the 2-category \( \mathcal{V}(K) \), which is the ‘vertical part’ of \( K \), as soon as \( K \) has all restrictions. Double categories that have all restrictions are called ‘equipments’.

\(^1\)In fact it gives conditions ensuring that left Kan extensions can be lifted along the forgetful 2-functor \( \mathcal{V}: \text{sMonCat} \to \mathcal{V}:\text{Cat} \), from the 2-category of symmetric monoidal categories enriched in a suitable closed symmetric monoidal category \( \mathcal{V} \), to the 2-category of \( \mathcal{V} \)-categories.
We continue by recalling the stronger notion of ‘pointwise’ Kan extension and, to get some feeling for it, study in detail such extensions in the equipment $\mathcal{V}$-$\text{Prof}$, whose objects are categories enriched in a suitable monoidal category $\mathcal{V}$ while its vertical morphisms are $\mathcal{V}$-functors and its horizontal morphisms are $\mathcal{V}$-profunctors. In doing so we are naturally lead to an extension of the notions of ‘weighted limit’ and ‘Kan extension along $\mathcal{V}$-functors’, to the setting in which $\mathcal{V}$ is just a monoidal category, where classically (see e.g. [Kel82]) $\mathcal{V}$ is assumed to be closed symmetric monoidal. Lastly we recall the notion of ‘pointwise left exactness’, which is crucial for the statement of the main result: a cell of a double category is called pointwise left exact if, whenever it is vertically postcomposed with a cell defining a pointwise left Kan extension, the resulting composite defines again such an extension. For example, any $\mathcal{V}$-natural transformation between $\mathcal{V}$-functors that satisfies the ‘left Beck-Chevalley condition’, in the sense of e.g. [Gui80], gives rise to a pointwise left exact cell in $\mathcal{V}$-$\text{Prof}$.

In §2 we recall the description of the 2-category consisting of double categories, so-called ‘normal pseudo’ double functors, and double transformations; by a normal pseudo double monad $T = (T, \mu, \eta)$ on a double category $\mathcal{K}$, as in the main result, we simply mean a monad on $\mathcal{K}$ in this 2-category. We remark that, like all double transformations, the multiplication $\mu$ and the unit $\eta$ consist of a natural family of vertical morphisms, one for each object in $\mathcal{K}$, as well as a natural family of cells, one for each horizontal morphism in $\mathcal{K}$. We call the double monad $T$ pointwise left exact, as is required in the main result, whenever each of these cells is pointwise left exact.

Any normal pseudo double monad $T$ on $\mathcal{K}$ induces a strict 2-monad on the vertical part $V(\mathcal{K})$ of $\mathcal{K}$, so that we can consider pseudo $V(T)$-algebras in $V(\mathcal{K})$, in the usual 2-categorical sense of e.g. [Str74], as well as ‘weak’ $V(T)$-morphisms between them, where we take ‘weak’ to mean either ‘lax’, ‘colax’ or ‘pseudo’. Following this we shall consider a notion of ‘horizontal $T$-morphism’ between pseudo $V(T)$-algebras, which generalises slightly a notion introduced in [GP04], and show that, for each choice of ‘weak’, pseudo $V(T)$-algebras, weak $V(T)$-morphisms, horizontal $T$-morphisms, together with an appropriate notion of ‘$T$-cell’, form a double category $\text{Alg}_w(T)$, where the subscript $w \in \{c, l, ps\}$ according to the choice of weakness. We remark that, without going into details, a horizontal $T$-morphism $J: A \to B$ consists of a horizontal morphism $J: A \Rightarrow B$ in $\mathcal{K}$ that is equipped with a cell $\bar{J}$ defining an algebraic structure on $J$. If it exists, the cell defining the pointwise left Kan extension of a map $d: A \to M$ along $J$, in $\mathcal{K}$, can be vertically postcomposed with $J$ and, in these terms, condition (e) of the main result means that the resulting composite defines again a pointwise left Kan extension.

In §3 we consider Kan extensions in $\text{Alg}_w(T)$. We show that some restrictions (of horizontal morphisms) can be lifted along the double functor $\text{Alg}_w(T) \to \mathcal{K}$, that forgets the algebraic structure, as well as all tabulations. We conclude that, if $\mathcal{K}$ is an equipment that has so-called ‘opcartesian’ tabulations, then pointwise Kan extensions in $\text{Alg}_w(T)$ can be defined in terms of ordinary Kan extensions, in a way that is analogous to Street’s definition of pointwise Kan extension in 2-categories, that was introduced in [Str74].

Finally, §4 is devoted to stating and proving the main result. We also show that Getzler’s result can be recovered and, in §5, that the existence of freely generated bicommutative Hopf monoids can be obtained as an application, as promised.

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1 Kan extensions in double categories

Throughout this paper the terminology and notation of [Kou14] are used, which we recall here.

1.1 Double categories

By a double category we mean a weakly internal category in the 2-category Cat of categories, functors and natural transformations, as follows.

Definition 1.1. A double category $\mathcal{K}$ consists of a diagram of functors

$$
\begin{array}{ccc}
\mathcal{K}_c & \xrightarrow{\pi_1} & \mathcal{K}_v \\
\downarrow{\pi_2} & & \downarrow{L} \\
\mathcal{K}_c & \xrightarrow{\circ} & \mathcal{K}_c \\
\end{array}
$$

(where $\mathcal{K}_c R \times_L \mathcal{K}_c$ is the pullback of $R$ and $L$, with projections $\pi_1$ and $\pi_2$), such that

$$L \circ \circ = L \circ \pi_1, \quad R \circ \circ = R \circ \pi_2 \quad \text{and} \quad L \circ 1 = \text{id} = R \circ 1,$$

together with natural isomorphisms

$$\alpha: (J \circ H) \circ K \cong J \circ (H \circ K), \quad 1: 1_A \circ M \cong M \quad \text{and} \quad \tau: M \circ 1_B \cong M,$$

for all $(J, H, K) \in \mathcal{K}_c R \times_L \mathcal{K}_c R \times_L \mathcal{K}_c$ and $M \in \mathcal{K}_v$ with $LM = A$ and $RM = B$. The natural isomorphisms $\alpha, 1$ and $\tau$ are required to satisfy the usual coherence axioms for a monoidal category or bicategory (see e.g. Section VII.1 of [ML98]), while their images under both $R$ and $L$ must be identities.

The category $\mathcal{K}_v$ consists of the objects and vertical morphisms of $\mathcal{K}$, while the objects and morphisms of $\mathcal{K}_c$ form the horizontal morphisms and cells of $\mathcal{K}$. We denote a horizontal morphism $J \in \mathcal{K}_c$ with $LJ = A$ and $RJ = B$ as a barred arrow $J: A \Rightarrow B$, while a cell $\phi: f \Rightarrow g: A \Rightarrow C$ with $L\phi = f: A \Rightarrow C$ and $R\phi = g: B \Rightarrow D$ will be depicted as

$$
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow{f} & \downarrow{\phi} & \downarrow{g} \\
C & \xrightarrow{K} & D
\end{array}
$$

(1)

and denoted by $\phi: J \Rightarrow K$; we call $J$ and $K$ the horizontal source and target of $\phi$, while $f$ and $g$ are called its vertical source and target. A cell whose vertical source and target are identities is called horizontal.

The compositions of $\mathcal{K}_v$ and $\mathcal{K}_c$, which are associative and unital, define vertical compositions for the vertical morphisms and cells of $\mathcal{K}$, both of which we denote by $\circ$, and whose identities are denoted $\text{id}_A: A \Rightarrow A$ and $\text{id}_J: J \Rightarrow J$ (which is a horizontal cell). The functors $\circ: \mathcal{K}_c R \times_L \mathcal{K}_c \to \mathcal{K}_c$ and $1: \mathcal{K}_v \to \mathcal{K}_c$ define horizontal compositions for the horizontal morphisms and cells of $\mathcal{K}$, which are associative up to invertible horizontal cells $\alpha: (J \circ H) \circ K \cong J \circ (H \circ K)$, that are called associators, and unital up to invertible horizontal cells $1: 1_A \circ M \cong M$ and $\tau: M \circ 1_B \cong M$, called unitors. A cell $\phi$ as in (1), that has units $J = 1_A$ and $K = 1_C$ as horizontal source and target, is called vertical; we will often denote it by the more descriptive $\phi: f \Rightarrow g$.

To make our drawings of cells more readable we will depict both vertical identities and horizontal units simply as $A \Rightarrow A$. Likewise when writing down, or depicting, compositions of cells we will often leave out the associators and unitors.

Every double category $\mathcal{K}$ contains both a vertical 2-category $V(\mathcal{K})$, consisting of its objects, vertical morphisms and vertical cells, as well as a horizontal bicategory $H(\mathcal{K})$.
H(K), consisting of its objects, horizontal morphisms and horizontal cells. For details see Definition 1.7 of [Kou14]. Like 2-categories, any double category K has both a vertical dual K^{op}, that is given by taking (K^{op})_v = (K_v)^{op} and (K^{op})_c = (K_c)^{op}, and a horizontal dual K^{co}, that is obtained by swapping the functors L and R: K_v → K_c; for details see Definition 1.6 of [Kou14].

Example 1.2. The archetypical double category is that of profunctors. Denoted Prof, it has small categories as objects and functors as vertical morphisms, while its horizontal morphisms J: A ⇓ B are profunctors, that is functors of the form J: A^{op} × B → Set. A cell φ in Prof, of the form (1), is a natural transformation φ: J ⇒ K(f, g) where K(f, g) = K ⋙ (f^{op} × g). The horizontal composite J ⊗ H of profunctors J: A ⇓ B and H: B ⇓ E is given by choosing a coequaliser (J ⊗ H)(x, z) for each pair of functions

\[
\prod_{v: y_1 \to y_2 \in B} J(x, y_1) × H(y_2, z) \Rightarrow \prod_{y \in B} J(x, y) × H(y, z),
\]

that are induced by postcomposing the maps in J(x, y_1) with v: y_1 → y_2 and precomposing the maps in H(y_2, z) with v. The unit profunctor 1_A: A ⇓ A is given by the hom-sets 1_A(x_1, x_2) = A(x_1, x_2). We shall describe a V-enriched variant of Prof, where V is a suitable monoidal category, in detail in Example 1.12.

Example 1.3. A span J: A ⇓ B in a category E is a diagram A ⇓_d J ⇓ B in E. Spans can be composed as soon as E has pullbacks; with this composition objects and morphisms of E, together with spans in E and their morphisms, form a double category Span(E). Details can be found in Example 1.4 of [Kou14].

Example 1.4. Given a category V and sets A and B, a V-matrix J: A ⇓ B is simply a family of V-objects J(x, y), one for each pair of objects x ∈ A and y ∈ B. If V is equipped with a monoidal structure (⊗, 1), and has coproducts that are preserved by ⊗ on both sides, then sets, functions between sets and V-matrices form a double category V-Mat as follows. A cell φ in V-Mat, of the form (1), is given by a family of V-maps φ_{x,y}: J(x, y) → K(f x, g y), while the horizontal composite J ⊗ H of J: A ⇓ B and H: B ⇓ E is given by ‘matrix multiplication’:

\[
(J ⊗ H)(x, z) = \prod_{y \in B} J(x, y) \otimes H(y, z).
\]

The unit matrix 1_A: A ⇓ A is given by 1_A(x, x) = 1 and 1_A(x_1, x_2) = ∅, the initial object of V, whenever x_1 ≠ x_2.

1.2 Equipments

Important to the theory of double categories are the notions of cartesian and op-cartesian cells. A cartesian cell defines the restriction of a horizontal morphism along a pair of vertical morphisms and, dually, op-cartesian cells define extensions, as follows.

Definition 1.5. The cell φ on the left below is called cartesian if any cell ψ, as in the middle, factors uniquely through φ as shown.

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \phi \quad \downarrow \psi \\
C \xrightarrow{\phi} D
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{h} Y \\
\downarrow k \\
C \xrightarrow{h}
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{h} Y \\
\downarrow k \\
C \xrightarrow{h}
\end{array}
\quad
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow g \\
C \xrightarrow{g}
\end{array}
\quad
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow g \\
C \xrightarrow{g}
\end{array}
\quad
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow g \\
C \xrightarrow{g}
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{h} Y \\
\downarrow k \\
C \xrightarrow{h}
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{h} Y \\
\downarrow k \\
C \xrightarrow{h}
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{h} Y \\
\downarrow k \\
C \xrightarrow{h}
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{h} Y \\
\downarrow k \\
C \xrightarrow{h}
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{h} Y \\
\downarrow k \\
C \xrightarrow{h}
\end{array}
\]
Vertically dual, the cell \( \phi \) is called \textit{opcartesian} if any cell \( \chi \) as on the right factors uniquely through \( \phi \) as shown.

If a cartesian cell like \( \phi \) exists then we call \( J \) the \textit{restriction of} \( K \) \textit{along} \( f \) \textit{and} \( g \), and write \( K(f,g) = J \); if \( K = 1_C \) then we write \( C(f,g) = 1_C(f,g) \). By their universal property, any two cartesian cells defining the same restriction factor through other as invertible horizontal cells. Moreover, since the vertical composite of two cartesian cells is again cartesian, and since vertical identities \( id_K \) are cartesian, it follows that restrictions are pseudofunctorial, in the sense that \( K(f,g)(h,k) \cong K(f \circ h, g \circ k) \) and \( K(id,id) \cong K \). Dually, if an opcartesian cell like \( \phi \) exists then we call \( K \) the \textit{extension of} \( J \) \textit{along} \( f \) \textit{and} \( g \); like restrictions, extensions are unique up to isomorphism and pseudofunctorial. We shall usually not name cartesian and opcartesian cells, but simply depict them like the two cells below.

For each vertical morphism \( f: A \to C \) the restriction \( f_* = C(id,id): A \Rightarrow C \), if it exists, is called the \textit{companion} of \( f \); it is defined by a cartesian cell as on left below. Dually the extension of \( 1_A \) along \( f \) and \( id_A \), if it exists, is called the \textit{conjoint} of \( f \); it is denoted by \( f^* \) and defined by an opcartesian cell as on the right.

\[
\begin{array}{ccc}
A & \xrightarrow{f_*} & C \\
\downarrow^f & cart & \downarrow^C \\
C & \xrightarrow{\text{cart}} & C
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{\text{opcart}} & A \\
\downarrow^f & \downarrow^{f^*} & \downarrow^C \\
C & \xrightarrow{\text{opcart}} & A
\end{array}
\]

\text{Example 1.6.} In Example 1.2 we have already used the notation \( K(f,g) \) to denote the profunctor \( K \circ (f^{op} \times g): A \Rightarrow B \). It is readily seen that the cell \( \varepsilon: K(f,g) \Rightarrow K \) given by the identity transformation on \( K \circ (f^{op} \times g) \) is cartesian, so that \( K \circ (f^{op} \times g) \) is indeed the restriction of \( K \) along \( f \) and \( g \) in the double category \( \text{Prof} \).

We can take the conjoint \( f^* \) of a functor \( f: A \Rightarrow C \) to be the restriction \( C(id,f) \): the opcartesian cell defining it is given by the actions \( f: (x_1,x_2) \Rightarrow C(fx_1,fx_2) \) of \( f \) on the hom-sets. That \( f^* \cong C(id,f) \) is no coincidence is explained below.

Generalising the situation above, in every double category the conjoint \( f^* \) of a vertical morphism \( f: A \Rightarrow C \) can be equivalently defined as the restriction \( C(id,f) \). This is because the vertical identity cell \( 1_f \) factors uniquely through the opcartesian cell defining \( f^* \) as a cartesian cell that defines \( C(id,f) \), and conversely. Horizontally dual, the same relation exists between the companion \( f_* \) and the extension of \( 1_A \) along \( id_A \) and \( f \).

Thus companions and conjoints are defined by cartesian, or equivalently opcartesian, cells. Conversely the existence of all companions and conjoints implies the existence of all restrictions and extensions: for any \( K: C \Rightarrow D \) the composite on the left below is cartesian while, for any \( J: B \Rightarrow A \), the composite on the right is opcartesian. For details see Theorem 4.1 of [Shu08] or Theorem 2.9 of [Kon14].

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow^f & \downarrow^\text{cart} & \downarrow^{\text{cart} g} \\
C & \xrightarrow{\text{cart}} & K \\
\downarrow^g & \downarrow^K & \downarrow^D \\
D & \xrightarrow{\text{cart} g} & B
\end{array}
\quad \quad \begin{array}{ccc}
B & \xrightarrow{J} & A \\
\downarrow^g & \downarrow^{\text{opcart} f} \\
D & \xrightarrow{\text{opcart} f} & B
\end{array}
\]

(3)

In summary, the following conditions on a double category \( \mathcal{K} \) are equivalent: \( \mathcal{K} \) has all companions and conjoints; \( \mathcal{K} \) has all restrictions; \( \mathcal{K} \) has all extensions.

\textbf{Definition 1.7.} An \textit{equipment} is a double category that satisfies the conditions above.
Example 1.8. Since the double category $\text{Prof}$ has all restrictions, as we saw in Example 1.6, it is an equipment. The double categories $\text{Span}(\mathcal{E})$ (Example 1.3) and $\mathcal{V}\text{-Mat}$ (Example 1.4) are equipments as well. Indeed the extension of a span $A \overset{f}{\to} J \overset{d_0}{\to} J \overset{d_1}{\to} B$ in $\mathcal{E}$, along morphisms $f: A \to C$ and $g: B \to D$, is given by the span $C \overset{\rho \circ f}{\to} J \overset{g \circ d_1}{\to} D$, while the restriction $K(f,g)$ of a $\mathcal{V}$-matrix $K: C \to D$ is given by the family of $\mathcal{V}$-objects $K(f,g)(x,y) = K(fx,gy)$.

1.3 Monoids and bimodules

Next we recall the notions of monoid and bimodule in double categories, following Section 11 of [Shul08]. These notions are useful: for example, as we recall below, monoids and bimodules in $\text{Span}$ and $\mathcal{V}\text{-Mat}$ are $\mathcal{V}$-enriched categories and $\mathcal{V}$-profunctors. These notions are useful: for example, as we recall below, monoids and bimodules in $\text{Span}(\mathcal{E})$ are categories and profunctors internal in $\mathcal{E}$ while monoids and bimodules in $\mathcal{V}\text{-Mat}$ are $\mathcal{V}$-enriched categories and $\mathcal{V}$-profunctors.

Definition 1.9. Let $\mathcal{K}$ be a double category.

- A monoid in $\mathcal{K}$ consists of a quadruple $A = (A_0, A, \mu, \eta)$ where $A: A_0 \to A_0$ is a horizontal morphism in $\mathcal{K}$ and $\mu: A \circ A \Rightarrow A$ and $\eta: 1_{A_0} \Rightarrow A$ are horizontal cells satisfying the usual coherence axioms for monoids.

- Given monoids $A$ and $C$, a morphism of monoids $f: A \to C$ consists of a vertical morphism $f_0: A_0 \to C_0$ and a cell $f$, as on the left below, such that $\mu_C \circ (f \circ f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_C \circ 1_{f_0}$.

- Given monoids $A$ and $B$, an $(A,B)$-bimodule $J: A \Rightarrow B$ consists of a horizontal morphism $J_0: A_0 \Rightarrow B_0$ that is equipped with horizontal cells $\lambda: A \circ J \Rightarrow J$ and $\rho: J \circ B \Rightarrow J$ defining the actions of $A$ and $B$ on $J$, which satisfy the usual coherence axioms for bimodules.

- Given morphisms of monoids $f: A \Rightarrow C$ and $g: B \Rightarrow D$, and bimodules $J: A \Rightarrow B$ and $K: C \Rightarrow D$, a cell $\phi$ as in the middle above is a cell in $\mathcal{K}$ as on the right, such that $\lambda_K \circ (f \circ \phi) = \phi \circ \lambda_J$ and $\rho_K \circ (\phi \circ g) = \phi \circ \rho_J$.

- The horizontal composite $J \odot B$ of bimodules $J: A \Rightarrow B$ and $H: B \Rightarrow E$ is the following reflexive coequaliser in $H(\mathcal{K})(A,E)$, if it exists:

$$J \odot B \odot H \xrightarrow{\rho_J \odot \text{id}} J \odot H \xrightarrow{\text{id} \odot \lambda_H} J \odot B H.$$

The following is Proposition 11.10 of [Shul08]. Recall that each double category $\mathcal{K}$ contains a bicategory $H(\mathcal{K})$ consisting of horizontal morphisms and horizontal cells. We say that $\mathcal{K}$ has local reflexive coequalisers if the categories $H(\mathcal{K})(A,B)$ have reflexive coequalisers that are preserved by horizontal composition on both sides.

Proposition 1.10 (Shulman). If an equipment $\mathcal{K}$ has local reflexive coequalisers then monoids and bimodules in $\mathcal{K}$, together with their morphisms and cells, again form an equipment $\text{Mod}(\mathcal{K})$ that has local reflexive coequalisers, whose horizontal composition is given as above.
Example 1.11. Let \( \mathcal{E} \) be a category with pullbacks. Monoids in \( \text{Span}(\mathcal{E}) \) are internal categories in \( \mathcal{E} \) and morphisms of such monoids are internal functors. Bimodules and their cells in \( \text{Span}(\mathcal{E}) \) are internal profunctors in \( \mathcal{E} \) and their transformations. Internal categories and internal profunctors are described in some detail in Example 1.5 of [Kou14]. If \( \mathcal{E} \) has reflexive coequalisers preserved by pullback then internal categories, functors, profunctors and transformations in \( \mathcal{E} \) form an equipment \( \text{Mod}(\text{Span}(\mathcal{E})) \), which is denoted \( \text{Prof}(\mathcal{E}) \).

For a suitable monoidal category \( \mathcal{V} \) the equipment of bimodules in \( \mathcal{V}\text{-Mat} \) (Example [L3]) is that of \( \mathcal{V} \)-enriched profunctors, which we shall now describe; it will be used as an example throughout. We remark that, in the case that \( \mathcal{V} \) is closed symmetric monoidal—hence enrich over itself—, the classical meaning of a \( \mathcal{V} \)-profunctor \( A \Rightarrow B \), where \( A \) and \( B \) are \( \mathcal{V} \)-categories, is that of a \( \mathcal{V} \)-functor of the form \( A^{op} \otimes B \to \mathcal{V} \). The advantage of defining \( \mathcal{V} \)-profunctors as bimodules in \( \mathcal{V}\text{-Mat} \) is that, while this extends the classical definition, in this way it is not necessary for \( \mathcal{V} \) to be closed symmetric monoidal—\( \mathcal{V} \) being a monoidal category suffices. This allows us, in the next subsection, to extend the classical notions of ‘weighted limit’ and ‘enriched Kan extension’ to settings in which the enriching category \( \mathcal{V} \) is not closed symmetric monoidal.

Example 1.12. Let \( \mathcal{V} = (\mathcal{V} \otimes, 1) \) be a monoidal category that has coproducts which are preserved by \( \otimes \) on both sides. A monoid in \( \mathcal{V}\text{-Mat} \) is a category enriched in \( \mathcal{V} \), while a morphism of such monoids is a \( \mathcal{V} \)-functor, both in the usual sense; see e.g. Section 1.2 of [Kel82]. A bimodule \( J: A \Rightarrow B \) in \( \mathcal{V}\text{-Mat} \) is a \( \mathcal{V} \)-profunctor in the sense of Section 7 of [DS97]; it consists of a family \( J(x, y) \) of \( \mathcal{V} \)-objects, indexed by pairs of objects \( x \in A \) and \( y \in B \), that is equipped with associative and unital actions

\[
\lambda: A(x_1, x_2) \otimes J(x_2, y) \to J(x_1, y) \quad \text{and} \quad \rho: J(x, y_1) \otimes B(y_1, y_2) \to J(x, y_2)
\]

satisfying the usual compatibility axiom for bimodules. Given a map \( f: x_1 \to x_2 \) in \( A \) we write \( \lambda f \) for the composite

\[
J(x_2, y) \xrightarrow{\lambda \otimes \text{id}} A(x_1, x_2) \otimes J(x_2, y) \xrightarrow{\lambda} J(x_1, y);
\]

likewise for \( y: y_1 \to y_2 \) in \( B \) we write \( \rho y = \rho \circ (\text{id} \otimes y): J(x, y_1) \to J(x, y_2) \).

If \( \mathcal{V} \) is closed symmetric monoidal then \( \mathcal{V} \)-profunctors \( J: A \Rightarrow B \) can be identified with \( \mathcal{V} \)-functors of the form \( J: A^{op} \otimes B \to \mathcal{V} \). Indeed the actions of \( J \) correspond, under the adjunctions \( - \otimes J(x, y) \dashv [J(x, y), -] \) that are part of the closed structure on \( \mathcal{V} \), to families of maps

\[
A(x_1, x_2) \to [J(x_2, y), J(x_1, y)] \quad \text{and} \quad B(y_1, y_2) \to [J(x, y_1), J(x, y_2)]
\]

respectively, which define families of partial \( \mathcal{V} \)-functors \( J(-, y): A^{op} \to \mathcal{V} \) and \( J(x, -): B \to \mathcal{V} \). The compatibility axiom for bimodules ensures that the latter correspond to a single \( \mathcal{V} \)-functor \( J: A^{op} \otimes B \to \mathcal{V} \); for details see Section 1.4 of [Kel82].

A cell of \( \mathcal{V} \)-profunctors

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \varphi & & \downarrow \rho \\
C & \xrightarrow{g} & D,
\end{array}
\]

As is the custom, by a map \( f: x_1 \to x_2 \) in a \( \mathcal{V} \)-category \( A \) we mean a \( \mathcal{V} \)-map \( f: 1 \to A(x_1, x_2) \).
called a *transformation*, consists of a family of $\mathcal{V}$-maps $\phi_{(x,y)} : J(x, y) \to K(fx, gy)$ (often simply denoted $\phi$) that are compatible with the actions, in the sense that the identities

$$\lambda \circ (f \otimes \phi) = \phi \circ \lambda : A(x_1, x_2) \otimes J(x_2, y) \to K(fx_1, gy)$$

and

$$\rho \circ (\phi \otimes g) = \phi \circ \rho : J(x, y_1) \otimes B(y_1, y_2) \to K(fx, gy_2)$$

are satisfied, where $f$ and $g$ denote the actions of the $\mathcal{V}$-functors $f$ and $g$ on the hom-objects of $A$ and $B$ respectively. If $\mathcal{V}$ is closed symmetric monoidal then transformations in the above sense can be identified with the usual $\mathcal{V}$-natural transformations between the $\mathcal{V}$-functors $J : A^{op} \otimes B \to \mathcal{V}$ and $K(f, g) = K \circ (f^{op} \otimes g)$. If $\mathcal{V}$ has all colimits, preserved by $\otimes$ on both sides, then $\mathcal{V}$-categories, $\mathcal{V}$-functors, $\mathcal{V}$-profunctors and their transformations form a double category $\text{Mod}(\mathcal{V}-\text{Mat})$, which is denoted $\mathcal{V}$-$\text{Prof}$. Its horizontal composite $J \circ H$, of $\mathcal{V}$-profunctors $J : A \Rightarrow B$ and $H : B \Rightarrow E$, is obtained by choosing coequalisers $(J \circ H)(x, z)$ for the pairs of $\mathcal{V}$-maps

$$\prod_{y_1, y_2 \in B} J(x, y_1) \otimes B(y_1, y_2) \otimes H(y_2, z) \Rightarrow \prod_{y \in B} J(x, y) \otimes H(y, z), \quad (4)$$

that are induced by letting $B(y_1, y_2)$ act on $J(x, y_1)$ and $H(y_2, z)$ respectively; compare the unenriched situation \(^2\). The horizontal composite $\phi \circ \chi$ of the transformations on the left below is given by the family of unique factorisations shown on the right, where the $\mathcal{V}$-maps drawn horizontally are the coequalisers defining $(J \circ H)(x, z)$ and $(K \circ L)(fx, hz)$, and the $\mathcal{V}$-map on the left is induced by the tensorproducts $\phi_{(x, y)} \otimes \chi_{(y, z)}$.

The unit $\mathcal{V}$-profunctor $1_A : A \Rightarrow A$, for a $\mathcal{V}$-category $A$, is given by the hom-objects $1_A(x_1, x_2) = A(x_1, x_2)$; its actions are given by the composition of $A$. Finally, $\mathcal{V}$-$\text{Prof}$ is an equipment in which the restriction $K(f, g)$ of a $\mathcal{V}$-profunctor $K : C \Rightarrow D$ along $\mathcal{V}$-functors $f : A \to C$ and $g : B \to D$ is given by the family of $\mathcal{V}$-objects $K(f, g)(x, y) = K(fx, gy)$, that is equipped with actions induced by those of $K$.

We remark that, for $\mathcal{V}$-functors $f$ and $g : A \to C$, the vertical cells $f \Rightarrow g$ in $\mathcal{V}$-$\text{Prof}$ can be identified with $\mathcal{V}$-natural transformations $f \Rightarrow g$, in the classical sense. Indeed, any vertical cell $\phi : f \Rightarrow g$, given by a family of $\mathcal{V}$-maps $\phi_{(x_1, x_2)} : A(x_1, x_2) \to C(fx_1, gx_2)$ is, because of its compatibility with the actions of $A$ and $C$, completely determined by the composites

$$\phi_x = [1 \xrightarrow{\eta_x} A(x, x) \xrightarrow{\phi_{(x, x)}} C(fx, gx)],$$

which make the diagram below commute; that is, they form a $\mathcal{V}$-natural transformation of $\mathcal{V}$-functors $f \Rightarrow g$, see Section 1.2 of [Kel82].

$$\begin{array}{c}
A(x_1, x_2) \xrightarrow{f \otimes \phi_{x_2}} C(fx_1, fx_2) \otimes C(fx_2, gx_2) \\
\phi_{x_1} \otimes g \downarrow \mu \\
C(fx_1, gx_1) \otimes C(gx_1, gx_2) \xrightarrow{\mu} C(fx_1, gx_2)
\end{array}$$

This identification induces an isomorphism

$$\mathcal{V}(\mathcal{V}$-$\text{Prof}) \cong \mathcal{V}$-$\text{Cat}$$

\(5\)
between the vertical 2-category contained in \( \mathcal{V}\)-Prof and the 2-category \( \mathcal{V}\)-Cat of \( \mathcal{V}\)-categories, \( \mathcal{V}\)-functors and \( \mathcal{V}\)-natural transformations.

1.4 Kan extensions in double categories

Analogous to that in 2-categories there is a notion of Kan extension in double categories, that was introduced in [GP08] and which is recalled below. The stronger notion of pointwise Kan extension, which we also consider, was introduced in [Kou14]. To get some feeling for the latter we shall consider pointwise Kan extensions in the double category \( \mathcal{V}\)-Prof in the next subsection.

**Definition 1.13.** Let \( d: B \to M \) and \( J: A \rightharpoonup B \) be morphisms in a double category. The cell \( \varepsilon \) in the right-hand side below is said to define \( r \) as the right Kan extension of \( d \) along \( J \) if every cell \( \phi \) below factors uniquely through \( \varepsilon \) as shown.

\[
\begin{array}{c}
A \xrightarrow{d} B \\
M \downarrow \phi \downarrow d = \Downarrow \varepsilon \downarrow \Downarrow r \downarrow d
\end{array}
\]

We call the right Kan extension \( r \) above pointwise if cells \( \phi \) of the more general form below also factor uniquely through \( \varepsilon \), as shown.

\[
\begin{array}{c}
C \xrightarrow{H} A \xrightarrow{J} B \\
M \downarrow \phi \downarrow d = \Downarrow \varepsilon \downarrow \Downarrow r \downarrow d
\end{array}
\]

The following, which is a simple consequence of the universal property of op-cartesian cells, shows that the notion of Kan extension in double categories generalises that of Kan extension in 2-categories (for a definition see e.g. Section 3 of [Kou14]). Remember that any double category \( \mathcal{K} \) contains a 2-category \( \mathcal{V}(\mathcal{K}) \) of objects, vertical morphisms and vertical cells of \( \mathcal{K} \).

**Proposition 1.14.** Let \( d: B \to M \), \( j: B \to A \) and \( r: A \to M \) be morphisms in a double category \( \mathcal{K} \), and assume that the conjoint \( j^*: A \rightharpoonup B \) of \( j \) exists. Consider a vertical cell \( \varepsilon \) as on the left below, as well as its factorisation through the op-cartesian cell defining \( j^* \), as shown.

\[
\begin{array}{c}
B \xrightarrow{j} B \\
A \Downarrow \varepsilon \Downarrow d = A \xrightarrow{j^*} B \\
M \downarrow r \downarrow d = M \xrightarrow{j^*} M
\end{array}
\]

The vertical cell \( \varepsilon \) defines \( r \) as the right Kan extension of \( d \) along \( j \) in \( \mathcal{V}(\mathcal{K}) \) precisely if its factorisation \( \varepsilon' \) defines \( r \) as the right Kan extension of \( d \) along \( j^* \) in \( \mathcal{K} \).

The following result for iterated pointwise Kan extensions is a straightforward consequence of Definition 1.13.
Proposition 1.15. Consider horizontally composable cells

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow s & & \downarrow H \\
M & & M
\end{array}
\quad
\begin{array}{ccc}
J & \xrightarrow{r} & J \\
\downarrow & & \downarrow \\
M & & M
\end{array}
\quad
\begin{array}{ccc}
C & \xleftarrow{d} & C \\
\downarrow & & \downarrow \\
M & & M
\end{array}
\] 

in a double category, and suppose that \(\varepsilon\) defines \(r\) as the pointwise right Kan extension of \(d\) along \(H\). Then \(\gamma\) defines \(s\) as the pointwise right Kan extension of \(r\) along \(J\) precisely if \(\gamma \odot \varepsilon\) defines \(s\) as the pointwise right Kan extension of \(d\) along \(J \odot H\).

For completeness we record the definition of pointwise left Kan extension in a double category \(K\), which coincides with that of pointwise right Kan extension in its horizontal dual \(K^{op}\).

Definition 1.16. Let \(d: A \to M\) and \(J: A \to B\) be morphisms in a double category. The cell \(\eta\) in the right-hand side below defines \(l\) as the pointwise left Kan extension of \(d\) along \(J\) if every cell \(\phi\) below factors uniquely through \(\eta\) as shown.

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow d & & \downarrow H \\
M & & M
\end{array}
\quad
\begin{array}{ccc}
J & \xleftarrow{\varepsilon} & J \\
\downarrow & & \downarrow \\
M & & M
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{h} & C \\
\downarrow & & \downarrow \\
M & & M
\end{array}
\] 

1.5 Pointwise Kan extensions in \(V\)-Prof

Here we study pointwise Kan extensions in the double category \(V\)-Prof and show that they can be described in terms of ‘\(V\)-weighted limits’. While such limits are classically defined in the case that \(V\) is a closed symmetric monoidal category, we shall consider a more general definition for which a monoidal structure on \(V\) alone suffices; this enables us to treat all cases of \(V\)-Prof.

For the rest of this subsection we denote by \(V\) a cocomplete monoidal category whose tensor product preserves colimits on both sides, so that \(V\)-categories, \(V\)-functors, \(V\)-profunctors and their transformations form a double category \(V\)-Prof; see Example 1.12. We write 1 for the unit \(V\)-category, that has a single object \(*\) and hom-object \(1(*,*) = 1\). We identify \(V\)-functors \(1 \to A\) with objects in \(A\) and \(V\)-profunctors \(1 \to 1\) with \(V\)-objects; transformations of such profunctors are identified with \(V\)-maps.

By a \(V\)-weight \(J\) on a \(V\)-category \(B\) we mean a \(V\)-profunctor \(J: B \to \mathbb{1}\).

Definition 1.17. Let \(d: B \to M\) be a \(V\)-functor, \(J\) a \(V\)-weight on \(B\) and \(r\) an object of \(M\). A cell \(\varepsilon\) in \(V\)-Prof, as in the right-hand side below, is said to define \(r\) as the \(J\)-weighted limit of \(d\) if every cell \(\phi\) below factors uniquely through \(\varepsilon\) as shown.

\[
\begin{array}{ccc}
1 & \xrightarrow{H} & 1 \\
\downarrow s & & \downarrow J \\
M & & M
\end{array}
\quad
\begin{array}{ccc}
J & \xrightarrow{\varepsilon} & J \\
\downarrow & & \downarrow \\
M & & M
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{r} & C \\
\downarrow & & \downarrow \\
M & & M
\end{array}
\] 

The definition above extends the classical definition of weighted limit as follows. Note that if \(V\) is closed symmetric monoidal then \(V\)-weights on \(B\) can be identified with \(V\)-functors \(J: B \to V\), using the isomorphism \(1^{op} \otimes B = 1 \otimes B \cong B\); see Example 1.12. This recovers the classical definition of \(V\)-weight, see e.g. Section 3.1.
of \texttt{Kel82} where such \(\mathcal{V}\)-functors are called ‘indexing types’. Using this identification, the cell \(\varepsilon\) above can be regarded as a \(\mathcal{V}\)-natural transformation between the \(\mathcal{V}\)-functors \(J: B \to \mathcal{V}\) and \(M(r, d) : B \to \mathcal{V}\).

**Proposition 1.18.** Let \(d, J, r\) and \(\varepsilon\) be as above. If \(\mathcal{V}\) is complete and closed symmetric monoidal then the cell \(\varepsilon\) defines \(r\) as the \(J\)-weighted limit of \(d\), in the above sense, precisely if the pair \((r, \varepsilon)\), where \(\varepsilon : J \Rightarrow M(r, d)\) is regarded as a \(\mathcal{V}\)-natural transformation of \(\mathcal{V}\)-functors \(B \to \mathcal{V}\), forms the ‘limit of \(d\) indexed by \(J\)’ in the sense of Section 3.1 of \texttt{Kel82}.

**Proof.** In terms of \(\mathcal{V}\)-natural transformations \(\mathcal{V}\)-functors \(B \to \mathcal{V}\), the unique factorisations through \(\varepsilon\) above can be restated as the following universal property for the composite \(\mathcal{V}\)-functors

\[
\varepsilon_s = [M(s, r) \otimes J \xrightarrow{id \otimes s} M(s, r) \otimes M(r, d) \xrightarrow{\varepsilon} M(s, d)],
\]

where \(s \in M\): for any \(\mathcal{V}\)-natural transformation \(\phi : H \odot J \Rightarrow M(s, d)\), where \(H \in \mathcal{V}\), there exists a unique \(\mathcal{V}\)-map \(\phi^\prime : H \to M(s, r)\) such that

\[
\phi = [H \odot J \xrightarrow{\phi \odot id} M(s, r) \otimes J \xrightarrow{\varepsilon} M(s, r)].
\]

Now the adjunctions \(- \otimes - \dashv [Y, -]\), that define the closed structure on \(\mathcal{V}\), induce a bijective correspondence between \(\mathcal{V}\)-natural transformations \(H \odot J \Rightarrow M(s, d)\), of \(\mathcal{V}\)-functors \(B \to \mathcal{V}\), and \(\mathcal{V}\)-natural transformations \(H = [J, M(s, d)]\), from the \(\mathcal{V}\)-object \(H\) to the \(\mathcal{V}\)-functor \([J, M(s, d)]\); \(B^{op} \otimes B \to \mathcal{V}\) given on objects by \((y_1, y_2) \mapsto [J y_1, M(s, d y_2)]\); the latter in the sense of Section 2.1 of \texttt{Kel82}. Since this correspondence is natural in \(H\) the universality of the transformations \(\varepsilon_s : M(s, r) \odot J \Rightarrow M(s, d)\) above translates into the universality of the corresponding transformations \(M(s, r) \Rightarrow [J, M(s, d)]\); in other words they define \(M(s, r)\) as the end \(\int_{y \in B} [J y, M(s, d y)]\), for each \(s \in M\). The latter is equivalent to the meaning of \((r, \varepsilon)\) is the limit of \(d\) indexed by \(J\) that is given in Section 3.1 of \texttt{Kel82}; see the sentence therein containing formula (3.3). \(\square\)

Next we describe the pointwise Kan extensions of \(\mathcal{V}\text{-}\text{Prof}\) in terms of weighted limits and, as a consequence, obtain an ‘enriched variant’ of Proposition\texttt{1.14}. Recall that vertical cells in \(\mathcal{V}\text{-}\text{Prof}\) can be identified with \(\mathcal{V}\)-natural transformations of \(\mathcal{V}\)-functors, under the isomorphism \(\mathcal{B}\).

**Proposition 1.19.** Consider \(\mathcal{V}\)-functors \(r : A \to M\) and \(d : B \to M\) as well as a \(\mathcal{V}\text{-}\text{prof}\) \(J : A \Rightarrow B\).

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\xrightarrow{r} & \xrightarrow{x} & \xleftarrow{\text{cart}} \\
M & \xrightarrow{\varepsilon} & d \\
\xrightarrow{r} & \xleftarrow{\text{cart}} & \xrightarrow{d} \\
M & \xrightarrow{M} & M
\end{array}
\]

For a cell \(\varepsilon\) in \(\mathcal{V}\text{-}\text{Prof}\) as on the left above, the following are equivalent:

(a) \(\varepsilon\) defines \(r\) as the pointwise right Kan extension of \(d\) along \(J\);

(b) for each object \(x \in A\) the composite in the middle above defines \(r x\) as the \(J(x, \text{id})\)-weighted limit of \(d\).
In particular, pointwise right Kan extensions along a \( \mathcal{V} \)-weight \( J : 1 \to B \) coincide with \( \mathcal{J} \)-weighted limits.

Finally assume that \( \mathcal{V} \) is complete and closed symmetric monoidal. If \( J = j^* \), for a \( \mathcal{V} \)-functor \( j : B \to A \), and \( \varepsilon \) is the factorisation of a \( \mathcal{V} \)-natural transformation \( \psi : r \to j \) through the opcartesian cell defining \( j^* \), as on the right above, then condition (b) coincides with the meaning of the definition of \( \psi \) exhibits \( r \) as the right Kan extension of \( d \) along \( j' \) that is given in Section 4.1 of [Kel82].

**Proof.** For each \( x \in A \) we denote by \( \varepsilon_x \) the composite in the middle above.

(a) \( \Rightarrow \) (b). By Theorem 1.23 below condition (a) implies that \( \varepsilon_x \) defines \( r x \) as the pointwise right Kan extension of \( d \) along \( J(x, \text{id}) \), for each \( x \in A \). By restricting the universal property of Definition 1.13 for \( \varepsilon_x \) to cells \( \phi \) with \( H : 1 \to 1 \), we obtain the universal property of Definition 1.17 for \( \varepsilon_x \), so that (b) follows.

(b) \( \Rightarrow \) (a). Consider a cell \( \phi \) in \( \mathcal{V} \text{-Prof} \), as in the left-hand side below; we have to show that it factors uniquely as \( \phi = \phi' \circ \varepsilon \).

\[
\phi'_{(z, x)} = \begin{array}{ccc}
1 & \xrightarrow{H(z, x)} & 1 \\
\varepsilon_x & \text{cart} & \varepsilon_x \\
\downarrow & & \downarrow \\
C & \xrightarrow{H(z, x)} & A \\
\phi & \xrightarrow{\phi'_{(z, x)}} & B \\
M & \xrightarrow{d} & M
\end{array} = \begin{array}{ccc}
1 & \xrightarrow{H(z, x)} & 1 \\
\varepsilon_x & \text{cart} & \varepsilon_x \\
\downarrow & & \downarrow \\
C & \xrightarrow{H(z, x)} & A \\
\phi & \xrightarrow{\phi'_{(z, x)}} & B \\
M & \xrightarrow{d} & M
\end{array}
\]

Because the cells \( \varepsilon_x \) define weighted limits, the restrictions \( \phi'_{(z, x)} \) of \( \phi \) above, for each pair \( z \in C \) and \( x \in A \), factor uniquely through \( \varepsilon_x \) as cells \( \phi'_{(z, x)} \) as shown. These factorisations are simply \( \mathcal{V} \)-maps \( \phi'_{(z, x)} : H(z, x) \to M(s z, r x) \), and it remains to show that, as a family, they combine to form a transformation \( \phi' : H \to M(s, r) \). Indeed, the unique factorisations above then imply that \( \phi \) factors uniquely as \( \phi = \phi' \circ \varepsilon \), as needed.

Thus we have to show that the \( \mathcal{V} \)-maps \( \phi'_{(z, x)} \) are compatible with the left and right actions of \( C, A \) and \( M \). In terms of cells in \( \mathcal{V} \text{-Prof} \), the compatibility with the left actions means that the identities hold, where the cell \( \lambda \) is the \( \mathcal{V} \)-map \( C(z_1, z_2) \otimes H(z_2, x) \to H(z_1, x) \) given by the left action of \( C \) on \( H \), and the cell \( s \) is the \( \mathcal{V} \)-map \( C(z_1, z_2) \to M(s z_1, s z_2) \) given by the action of the \( \mathcal{V} \)-functor \( s \) on hom-objects. Because factorisations through \( \varepsilon_x \) are unique we may equivalently show that this identity holds after composing both sides on the right with \( \varepsilon_x \), so that it is shown below where, to save space, only the non-identity cells are depicted while objects and morphisms are left out. The first and last identities here follow from the factorisations (6) while the second identity follows from the compatibility of \( \phi \) with the action of \( C \).

\[
\begin{array}{ccc}
\lambda & \phi'_{(z_1, x)} & \varepsilon_x \\
\phi_{(z_1, x)} & \varepsilon_x \\
\end{array} = \begin{array}{ccc}
\lambda & \phi_{(z_1, x)} & \varepsilon_x \\
\phi_{(z_1, x)} & \varepsilon_x \\
\end{array} = \begin{array}{ccc}
\phi'_{(z_2, x)} & \varepsilon_x \\
\phi'_{(z_2, x)} & \varepsilon_x \\
\end{array}
\]
That the family $\mathcal{V}$-maps $\phi'_{(z,x)}$ is compatible with the right actions too follows similarly, from the following equation of transformations (which are of the form $H(z,x_1) \circ A(x_1, x_2) \circ J(x_2, \text{id}) \Rightarrow 1_M$) and the fact that factorisations through $\varepsilon_{x_2}$ are unique.

$$
\begin{array}{c}
\varepsilon_{x_2} \\
\phi'_{(z,x_2)} \\
\rho
\end{array}
\Rightarrow
\begin{array}{c}
\varepsilon_{x_1} \\
\phi'_{(z,x_1)} \\
\lambda
\end{array}
\Rightarrow
\begin{array}{c}
\epsilon_{x_1} \\
\phi_{(z,x_1)} \\
\lambda
\end{array}
\Rightarrow
\begin{array}{c}
\varepsilon_{x_2} \\
\phi_{(z,x_2)} \\
\rho
\end{array}
\Rightarrow
\begin{array}{c}
\epsilon_{x_2} \\
\phi_{(z,x_2)} \\
\rho
\end{array}
\Rightarrow
\begin{array}{c}
\varepsilon_{x_1} \\
\phi_{(z,x_1)} \\
\lambda
\end{array}
\Rightarrow
\begin{array}{c}
\epsilon_{x_1} \\
\phi_{(z,x_1)} \\
\lambda
\end{array}
\Rightarrow
\begin{array}{c}
\varepsilon_{x_2} \\
\phi'_{(z,x_2)} \\
\rho
\end{array}
$$

The identities here follow from the factorisations (6); the fact that the domain $H \odot J$ of $\phi$ coequalises the actions of $A$ on $H$ and on $J$, see (4); the factorisations (6) again; the compatibility of $\varepsilon$ with the left actions.

To prove the final assertion we assume that $J = j^*$ for a $\mathcal{V}$-functor $j: B \to A$, and that $\varepsilon$ is the unique factorisation of a $\mathcal{V}$-natural transformation $\psi: rj \Rightarrow d$ through the opcartesian cell defining $j^*$. As soon as we consider $\psi$ as a vertical cell of $\mathcal{V}$-$\text{Prof}$, under the isomorphism (5), it is easy to check that $\varepsilon$ must be given by the $\mathcal{V}$-maps

$$
A(x, y) \xrightarrow{\phi y} M(rx, rjy) \xrightarrow{\rho y} M(rx, dy),
$$

for pairs $x \in A$ and $y \in B$. It then follows from Proposition 1.18 that, if $\mathcal{V}$ is complete and closed symmetric monoidal, then condition (b) coincides with condition (ii) of Theorem 4.6 of [Kel82], which lists equivalent meanings of the definition of $\psi$ exhibits $r$ as the right $\text{Kan}$ extension of $d$ along $j^*$.

### 1.6 Pointwise Kan extensions in terms of Kan extensions

The main goal of [Kou14] was to give a condition on equipments $\mathcal{K}$ ensuring that an analogue of Proposition 1.14 holds for pointwise $\text{Kan}$ extensions as well, where by ‘pointwise $\text{Kan}$ extension in the 2-category $\mathcal{V}($\mathcal{K})’, we mean the classical notion given by Street in [Str74]. This condition is given in terms of the notion of tabulation, as follows.

**Definition 1.20.** Given a horizontal morphism $J: A \Rightarrow B$ in a double category $\mathcal{K}$, the tabulation $\langle J \rangle$ of $J$ consists of an object $\langle J \rangle$ equipped with a cell $\pi$ as on the left below, satisfying the following 1-dimensional and 2-dimensional universal properties.

$$
\begin{array}{c}
\langle J \rangle \\
\pi_A \\
\downarrow \pi \\
\pi_B \\
A \\
\xrightarrow{J} B
\end{array}
\qquad
\begin{array}{c}
X \\
\phi_A \\
\downarrow \phi \\
\phi_B \\
A \\
\xrightarrow{J} B
\end{array}
$$

Given any other cell $\phi$ in $\mathcal{K}$ as on the right above, the 1-dimensional property states that there exists a unique vertical morphism $\phi': X \to \langle J \rangle$ such that $\pi \circ 1_{\phi'} = \phi$.

The 2-dimensional property is the following. Suppose we are given a further cell $\psi$ as in the identity below, which factors through $\pi$ as $\psi': Y \Rightarrow \langle J \rangle$, like $\phi$ factors as $\phi'$. Then for any pair of cells $\xi_A$ and $\xi_B$ as below, so that the identity holds, there exists a unique cell $\xi'$ as on the right below such that $1_{\pi_A} \circ \xi' = \xi_A$ and $1_{\pi_B} \circ \xi' = \xi_B$.

$$
\begin{array}{c}
X \xrightarrow{H} Y \\
\phi_A \\
\downarrow \phi \\
\psi_B \\
A \\
\xrightarrow{J} B
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{H} Y \\
\phi_A \\
\downarrow \phi \\
\xi_B \\
A \\
\xrightarrow{J} B
\end{array}
\quad
\begin{array}{c}
X \xrightarrow{H} Y \\
\phi' \\
\downarrow \phi' \\
\psi' \\
\langle J \rangle \\
\xrightarrow{\epsilon} \langle J \rangle
\end{array}
$$

A tabulation is called opcartesian whenever its defining cell $\pi$ is opcartesian.
Example 1.21. The tabulation $⟨ J ⟩$ of a profunctor $J: A^{op} × B → \text{Set}$ is the category that has triples $(x, u, y)$ as objects, where $(x, y) ∈ A × B$ and $u: x → y$ in $J(x, y)$, while a map $(x, u, y) → (x′, u′, y′)$ is a pair $(p, q): (x, y) → (x′, y′)$ in $A × B$ making the diagram

$$
\begin{array}{ccc}
x & u & y \\
p & & q \\
x' & u' & y'
\end{array}
$$

commute in $J$. The functors $π_A$ and $π_B$ are the projections and the natural transformation $π: 1_{⟨ J ⟩} ⇒ J$ maps the pair $(p, q)$ to the diagonal $u′ ◦ p = q ◦ u$. It is easy to check that $π$ satisfies both the 1-dimensional and 2-dimensional universal property above, and that it is opcartesian.

Example 1.22. Generalising the previous example, it was shown in Proposition 5.12 of [Kou14] that the double category $\text{Prof}(E)$, of profunctors internal to $E$, admits all opcartesian tabulations.

The main result, Theorem 5.9, of [Kou14] is the following.

Theorem 1.23. In an equipment $K$ consider the cell $ε$ on the left below.

$$
\begin{array}{ccc}
A & d & B \\
r & \downarrow j & \downarrow d \\
M & = & M
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{J(f, id)} & B \\
f & \downarrow \text{cart} & \downarrow B \\
A & \xrightarrow{j} & B \\
r & \downarrow l & \downarrow d \\
M & = & M
\end{array}
$$

For the following conditions the implications $(a) ⇔ (b) ⇒ (c)$ hold, while $(c) ⇒ (a)$ holds as soon as $K$ has opcartesian tabulations.

(a) The cell $ε$ defines $r$ as the pointwise right Kan extension of $d$ along $j$;

(b) for all $f: C → A$ the composite on the right above defines $r ◦ f$ as the pointwise right Kan extension of $d$ along $J(f, id)$;

(c) for all $f: C → A$ the composite on the right above defines $r ◦ f$ as the right Kan extension of $d$ along $J(f, id)$.

As a consequence the analog of Proposition 1.14 for pointwise Kan extensions below follows, which is Proposition 5.10 of [Kou14].

Proposition 1.24. Let $d: B → M$, $j: B → A$ and $r: A → M$ be morphisms in an equipment $K$ that has opcartesian tabulations. Consider a vertical cell $ε$ as on the left below, as well as its factorisation through the opcartesian cell defining $j^*$, as shown.

$$
\begin{array}{ccc}
B & d & B \\
j & \downarrow j & \downarrow \text{opcart} \\
A & \xrightarrow{d} & A \\
r & \downarrow l & \downarrow d \\
M & = & M
\end{array}
\quad
\begin{array}{ccc}
B & d & B \\
j & \downarrow j^* & \downarrow d \\
A & \xrightarrow{d} & A \\
r & \downarrow l & \downarrow d \\
M & = & M
\end{array}
$$

The vertical cell $ε$ defines $r$ as the pointwise right Kan extension of $d$ along $j$ in $V(K)$, in the sense of [Str74], precisely if its factorisation $ε'$ defines $r$ as the pointwise right Kan extension of $d$ along $j^*$ in $K$. 

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1.7 Exact cells

The final notions that we need to recall are those of ‘exact’ and ‘initial’ cell.

**Definition 1.25.** In a double category consider a cell \( \phi \), as on the left below, and a vertical morphism \( d: D \to M \). We call \( \phi \) (pointwise) right \( d \)-exact if for any cell \( \varepsilon \), as on the right, that defines \( r \) as the (pointwise) right Kan extension of \( d \) along \( K \), the composite \( \varepsilon \circ \phi \) defines \( r \circ f \) as the (pointwise) right Kan extension of \( d \circ g \) along \( J \). If the converse holds as well then we call \( \phi \) (pointwise) \( d \)-initial.

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow \phi & & \downarrow \varepsilon \\
C & \xrightarrow{k} & D
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{K} & D \\
r & \downarrow \varepsilon & \downarrow d \\
M & \xrightarrow{M} & M
\end{array}
\]

If \( \phi \) is (pointwise) right \( d \)-exact for all vertical morphisms \( d: D \to M \), where \( M \) varies, then it is called (pointwise) right exact. Likewise \( \phi \) is called (pointwise) initial whenever it is (pointwise) \( d \)-initial for all \( d: D \to M \).

The uniqueness of Kan extensions implies that the notions of right \( d \)-exactness and \( d \)-initialness do not depend on the choice of the cell \( \varepsilon \) that defines the Kan extension.

**Example 1.26.** In Example 4.4 of [Kou14] it was shown that every ‘initial’ functor \( f: A \to C \) induces an initial cell in \( \text{Prof} \), while in Example 4.7 of the same paper any natural transformation \( \phi: fj \Rightarrow kg \) between composites of functors, that satisfies the ‘right Beck-Chevalley condition’, was shown to give rise to a pointwise right exact cell. We shall return to this condition later, at the end of Section 4.

The following, which combines Proposition 4.2 and Corollary 4.6 of [Kou14], describes classes of initial and right exact cells.

**Proposition 1.27.** In a double category consider a cell \( \phi \) as on the left below, and assume that the companions \( f_*: A \to C \) and \( g_*: B \to D \) of \( f \) and \( g \) exists. It follows that the opcartesian and cartesian cell in the composite on the right exist, see (3), and we write \( \phi_* \) for the unique factorisation of \( \phi \) through these cells, as shown.

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\downarrow \phi & & \downarrow \phi_* \\
C & \xrightarrow{K} & D
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{K(f, \text{id})} & D \\
\downarrow \phi_* & & \downarrow \text{cart} \\
A & \xrightarrow{g_*} & D
\end{array}
\]

The following hold:

(a) if \( f = \text{id}_A \) and \( \phi \) is opcartesian then \( \phi \) is both initial and pointwise initial;

(b) \( \phi \) is pointwise right exact whenever the following equivalent conditions hold: \( \phi_* \circ \text{opcart} \) is opcartesian; \( \text{cart} \circ \phi_* \) is cartesian; \( \phi_* \) is invertible.
2 Eilenberg-Moore double categories

In this section we recall the notion of a ‘normal oplax double monad’ $T$ on a double category and consider several weakenings of the ‘Eilenberg-Moore double category’ associated to $T$, that was introduced in Section 7.1 of [GP04].

More precisely, by a ‘normal oplax double monad’ we shall mean a monad $T$ in the 2-category $\text{Dbl}_{\text{no}}$ of double categories, ‘normal oplax double functors’ and their transformations. Since the assignment $K \mapsto V(K)$, that maps a double category $K$ to its vertical 2-category $V(K)$, extends to a 2-functor $\text{Dbl}_{\text{no}} \to \text{2-Cat}$, any normal oplax double monad $T$ on $K$ induces a strict 2-monad $V(T)$ on $V(K)$. After generalising slightly the notions of ‘horizontal $T$-morphism’ and ‘$T$-cell’, that were introduced in [GP04], we will, for any choice of ‘weak’ $\in \{\text{colax}, \text{lax}, \text{pseudo}\}$, show that lax $V(T)$-algebras, weak $V(T)$-morphisms, horizontal $T$-morphisms and $T$-cells form a double category $\text{Alg}_w(T)$.

2.1 Double functors and their transformations

We start by recalling the notions of double functor and double transformation; references include Sections 7.2 and 7.3 of [GP99] and Section 6 of [Shr08].

Definition 2.1. Let $K$ and $L$ be double categories. A lax double functor $F: K \to L$ consists of a pair of functors $F_v: K_v \to L_v$ and $F_c: K_c \to L_c$ (both will be denoted $F$), such that $LF_c = F_v L$ and $RF_c = F_v R$, together with natural transformations whose components are horizontal cells, that satisfy the usual associativity and unit axioms for monoidal functors, see e.g. Section XI.2 of [ML98]. The transformations $F_\otimes$ and $F_\mathbf{1}$ are called the compositor and unitor of $F$.

Horizontally dual, in the definition of an oplax double functor $F: K \to L$ the directions of the compositor $F_\otimes$ and unitor $F_\mathbf{1}$ are reversed. A pseudo double functor is a lax (or, equivalently, an oplax) double functor whose compositor and unitor are invertible; a lax, oplax, or pseudo double functor whose unitor $F_\mathbf{1}$ is the identity is called normal.

We shall mostly be interested in normal oplax double functors. Unpacking the above, such a double functor $F: K \to L$ maps the objects, vertical and horizontal morphisms, as well as cells of $K$ to those of $L$, in a way that preserves horizontal and vertical sources and targets. Moreover, vertical composition of morphisms and cells is preserved, as well as the horizontal units: $F1_A = 1_{FA}$ and $F1_f = 1_{Ff}$ for each $A$ and $f: A \to C$ in $K$, while horizontal composition is preserved only up to natural coherent cells

$F(1_A \otimes J) \Rightarrow F1_A \otimes FJ = 1_{FA} \otimes FJ \Rightarrow FJ$ (7)

for each composable pair $J: A \Rightarrow B$ and $H: B \Rightarrow E$ in $K$. We will often use the unit axioms for $F$: they state that, for any horizontal morphism $J: A \Rightarrow B$, the composite
coincides with $F\iota$, and similar for $r$.

**Example 2.2.** Any functor $F: \mathcal{D} \to \mathcal{E}$ between categories with pullbacks induces a normal oplax double functor $\text{Span}(F): \text{Span}(\mathcal{D}) \to \text{Span}(\mathcal{E})$, simply by applying $F$ to the spans of $\mathcal{D}$. The compositor of $\text{Span}(F)$ is induced by the universal property of pullbacks; in particular, if $F$ preserves pullbacks then $\text{Span}(F)$ is a normal pseudo double functor.

**Definition 2.3.** A double transformation $\xi: F \Rightarrow G$ between lax double functors $F$ and $G: \mathcal{K} \to \mathcal{L}$ is given by natural transformations $\xi_c: F_c \Rightarrow G_c$ and $\xi_a: F_\iota \Rightarrow G_\iota$ (both will be denoted $\xi$), with $L\xi_c = \xi_cL$ and $R\xi_c = \xi_cR$, such that the following diagrams commute, where $(J, H) \in \mathcal{K}_c \times_{\mathcal{L}} \mathcal{K}_c$ and $A \in \mathcal{K}_v$.

$$
\begin{align*}
FJ \odot FH & \Rightarrow F(J \odot H) \\
\xi_J \odot \xi_H & \Rightarrow \xi_{J \odot H} \\
GJ \odot GH & \Rightarrow G(J \odot H)
\end{align*}
$$

$$
\begin{align*}
1_{FA} & \Rightarrow F1_A \\
1_{\xi_A} & \Rightarrow \xi_{1_A} \\
1_{GA} & \Rightarrow G1_A
\end{align*}
$$

Analogously, in the definition of a double transformation $\xi: F \Rightarrow G$ between oplax double functors $F$ and $G: \mathcal{K} \to \mathcal{L}$ the directions of the coherence cells in the diagrams above are reversed; in both cases we shall call the commuting of these diagrams respectively the composition and unit axiom for $\xi$.

Double categories, lax double functors and double transformations form a 2-category which we denote $\text{Dbl}$; analogously $\text{Dbl}_o$ denotes the 2-category of double categories, oplax double functors and double transformations. We denote by $\text{Dbl}_{lo} \subset \text{Dbl}_l$ and $\text{Dbl}_{no} \subset \text{Dbl}_o$ the sub-2-categories consisting of normal double functors. Notice that the unit axiom for a double transformation $\xi: F \Rightarrow G$ between normal double functors reduces to the identity $\xi_1 = 1_{\xi_1}$, for all objects $A$ of $\mathcal{K}$.

**Example 2.4.** Every natural transformation $\xi: F \Rightarrow G$, between functors $F$ and $G: \mathcal{D} \to \mathcal{E}$ of categories that have pullbacks, induces a double transformation $\text{Span}(\xi): \text{Span}(F) \Rightarrow \text{Span}(G)$ between normal oplax double functors, that is given by $A \mapsto \xi_A$ on objects and $(A \leftarrow J \to B) \mapsto \xi_J$ on spans. We conclude that the assignment $\mathcal{E} \mapsto \text{Span}(\mathcal{E})$, that maps a category $\mathcal{E}$ with pullbacks to the double category $\text{Span}(\mathcal{E})$ of spans in $\mathcal{E}$, extends to a 2-functor $\text{Span}: \text{pbCat} \to \text{Dbl}_{no}$, where $\text{pbCat}$ denotes the 2-category of categories with pullbacks, all functors between such categories and their transformations.

The following propositions record some useful properties of double functors and their transformations. Remember that every pseudo double category $\mathcal{K}$ contains a vertical 2-category $V(\mathcal{K})$; in the proposition below $\mathcal{2}\text{-Cat}$ denotes the 2-category of 2-categories, strict 2-functors and 2-transformations.

**Proposition 2.5.** The assignment $\mathcal{K} \mapsto V(\mathcal{K})$ extends to a strict 2-functor $V: \text{Dbl}_{no} \to \mathcal{2}\text{-Cat}$.

**Sketch of the proof.** The image $V(F): V(\mathcal{K}) \to V(\mathcal{L})$ of a double functor $F: \mathcal{K} \to \mathcal{L}$ is simply the restriction of $F$ to objects, vertical morphisms and vertical cells; that $V(F)$ is a strict 2-functor follows from the unit axiom for $F$. Likewise the vertical part $\xi_v: F_v \Rightarrow G_v$ of any double transformation $\xi: F \Rightarrow G$ forms a 2-transformation $V(\xi): V(F) \Rightarrow V(G)$.

The following is Proposition 6.8 of [Shulman].

**Proposition 2.6 (Shulman).** Any lax double functor between equipments preserves cartesian cells, and any oplax double functor between equipments preserves opcartesian cells.
Having introduced double functors and their transformations we turn to double monads.

**Definition 2.7.** An oplax double monad on a double category $K$ is a monad $T = (T, \mu, \eta)$ on $K$ in the 2-category $Dbl$, consisting of an oplax double endofunctor $T: K \to K$, a multiplication $\mu: T^2 \Rightarrow T$ and a unit $\eta: \text{id}_K \Rightarrow T$, which satisfy the associativity axiom $\mu \circ T \mu = \mu \circ \mu T$ and unit axiom $\mu \circ T \eta = \text{id}_K = \mu \circ \eta T$.

Dually a lax double monad is a monad in $Dbl$. An oplax, or lax, double monad is called normal or pseudo whenever its underlying double endofunctor is a normal or pseudo double functor.

Notice that, by the proposition above, any normal oplax double monad $T$ on $K$ restricts to a strict 2-monad on the vertical 2-category $V(K)$. The following example describes the ‘free monoid’-monad on $V\text{-Mat}$. In Example 2.10 below we will see that it induces the ‘free strict monoidal $V$-category’-monad on $V\text{-Prof}$, which we shall use throughout.

For many other examples of lax double monads we refer to Section 3 of [CS10], where ‘monads on virtual double categories’ are considered. Analogous to the generalisation of the notion of monoidal category to that of ‘multicategory’, a ‘virtual double category’ consists of objects, vertical morphisms and horizontal morphisms, like a double category, but has cells of the form $(J_1, \ldots, J_n) \Rightarrow K$, where $(J_1, \ldots, J_n)$ is a sequence of horizontal morphisms $J_1: A_0 \Rightarrow A_1, \ldots, J_n: A_{n-1} \Rightarrow A_n$. Virtual double categories that have all ‘composites’ and ‘units’, in the sense of Section 5 of [CS10], can be identified with double categories in our sense, and ‘monads’ on such virtual double categories correspond to the lax double monads in our sense.

**Example 2.8.** Let $V = (V, \otimes, 1, \delta)$ be a symmetric monoidal category that has coproducts preserved by $\otimes$ on both sides, so that $V$-matrices form a double category $V\text{-Mat}$; see Example 1.4. The pseudo double endofunctor $T$ underlying the ‘free monoid’-monad on $V\text{-Mat}$ assigns to a set $A$ the free monoid $TA = \bigoplus_{n \geq 0} A^n$ and, accordingly, maps a function $f: A \to C$ to $Tf = \bigoplus_{n \geq 0} f^n$. The elements of $TA$ are (possibly empty) sequences $\underline{x} = (x_1, \ldots, x_n)$ of elements in $A$; we write $|\underline{x}| = n$. The function $Tf$ simply applies $f$ coordinatewise. The assignments $A \mapsto TA$ and $f \mapsto Tf$ extend to a pseudo double endofunctor $T$ on $V\text{-Mat}$ as follows. Its image of a profunctor $J: A \Rightarrow B$ is given by

$$TJ(\underline{x}, \underline{y}) = \begin{cases} \bigotimes_{i=1}^n J(x_i, y_i) & \text{if } |\underline{x}| = n = |\underline{y}|; \\ \emptyset & \text{otherwise.} \end{cases}$$

Likewise the image $T\phi$ of the cell on the left below is given by the family of $V$-maps on the right.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \phi & | \downarrow \phi \bigg| & \downarrow \phi \bigg| \\ C & \xleftarrow{g} & D \end{array}$$

$$(T\phi)(\underline{x}, \underline{y}) = \begin{cases} \bigotimes_{i=1}^n \phi(x_i, y_i) & \text{if } |\underline{x}| = n = |\underline{y}|; \\ \text{id}_\emptyset & \text{otherwise.} \end{cases}$$

Next the compositor $T\circ: TJ \circ TH \cong T(J \circ H)$ is given, on sequences $\underline{x}$ and $\underline{z}$ of equal length $n$, by the canonical isomorphisms

$$(TJ \circ TH)(\underline{x}, \underline{z}) = \prod_{y \in TB} TJ(\underline{x}, y) \circ TH(y, \underline{z}) \cong \prod_{y \in TB} \left( \bigotimes_{i=1}^n J(x_i, y_i) \otimes H(y, z_i) \right) \cong \prod_{y \in B^n} \left( \bigotimes_{i=1}^n J(x_i, y_i) \otimes H(y, z_i) \right) = T(J \circ H)(\underline{x}, \underline{z}),$$
where the first and the third isomorphism follow from the fact that \( \otimes \) preserves coproducts on both sides, while the second isomorphism is given by reordering the factors \( J(x_i, y_i) \) and \( H(y_i, z_i) \); of course \( (T_\otimes)_x = \text{id}_x \) if \( x \neq x' \). On a set \( A \) the unitor \( T_1: 1 \to TA \) is given by the isomorphisms \( 1_{TA}(x, y) = 1 = 1^\otimes n = (T_1)(x, y) \) if \( x = y \) and \( \text{id}_x \) otherwise. This completes the definition of the pseudo double endofunctor \( T \) on \( \mathcal{V}\text{-Mat} \).

The multiplication \( \mu: T^2 \Rightarrow T \) and unit \( \eta: \text{id} \Rightarrow T \), that make \( T \) into a double monad, restrict to sets as concatenation of double sequences and insertion as one-element sequences respectively. On a \( \mathcal{V}\)-matrix \( J: A \to B \) the multiplication \( \mu_J: T^2J \Rightarrow TJ \) is given as follows: if \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) are double sequences of equal length, such that \( |x_i| = m_i = |y_i| \) for all \( i = 1, \ldots, n \), then \( (\mu_J)(x, y) \) is the \( \mathcal{V}\)-isomorphism

\[
(T^2J)(x, y) = \bigotimes_{i=1}^n \left( \bigotimes_{j=1}^{m_i} J(x_{ij}, y_{ij}) \right) \cong \bigotimes_{i=1}^{m_1 + \cdots + m_n} J((\mu_x)_i, (\mu_y)_i) = (TJ)(\mu_x, \mu_y); \quad (8)
\]

in all other cases it is the unique \( \mathcal{V}\)-map \( \emptyset \to (TJ)(\mu_x, \mu_y) \). Finally, the unit \( \eta_J: J \Rightarrow TJ \) is given by the isomorphisms \( J(x, y) \cong (TJ)(\eta x, \eta y) \); this completes the description of the pseudo double monad \( T \).

We remark that \( T \) can be modified into the following lax double monad on \( \mathcal{V}\text{-Mat} \), which we denote \( T_\Sigma \), by introducing ‘twists’ in the images \( T \mathcal{J} \), as follows:

\[
T_\Sigma J(x, y) = \begin{cases} 
\prod_{\sigma \in \Sigma_n} \bigotimes_{i=1}^n J(x_{\sigma i}, y_i) & \text{if } |x| = n = |y|; \\
\emptyset & \text{otherwise,}
\end{cases}
\]

where \( \Sigma_n \) denotes the group of permutations of the set \( \{1, \ldots, n\} \). The double monad \( T_\Sigma \) induces the ‘free symmetric strict monoidal \( \mathcal{V}\text{-category’-monad} on \( \mathcal{V}\text{-Prof} \); see Example 2.10.

Remember that the monoids and bimodules in an equipment \( \mathcal{K} \) form again an equipment \( \mathcal{Mod}(\mathcal{K}) \) whenever \( \mathcal{K} \) has local reflexive coequalisers, that is each category \( H(\mathcal{K})(A, B) \) has reflexive coequalisers which are preserved by \( \otimes \) on both sides; see Proposition 1.10. In the proposition below, which is Proposition 11.12 of [Shul08], \( \mathcal{qEqu}_{\text{ip}} \subset \mathcal{Db}_1 \) and \( \mathcal{qEqu}_{\text{ip}} \subset \mathcal{Db}_1 \) denote the full sub-2-categories consisting of equipments that have local reflexive coequalisers.

**Proposition 2.9** (Shulman). The assignment \( \mathcal{K} \mapsto \mathcal{Mod}(\mathcal{K}) \) extends to a 2-functor \( \mathcal{Mod}: \mathcal{qEqu}_{\text{ip}} \to \mathcal{qEqu}_{\text{ip}} \), whose image of a pseudo double functor \( F: \mathcal{D} \to \mathcal{E} \) is again pseudo whenever \( F \) preserves local reflexive coequalisers.

By applying \( \mathcal{Mod} \) to the ‘free monoid’-monad on \( \mathcal{V}\text{-Mat} \), of Example 2.8, we obtain the ‘free strict monoidal \( \mathcal{V}\text{-category’-monad on \( \mathcal{V}\text{-Prof} \), as follows; this double monad, whose algebras are monoidal \( \mathcal{V}\text{-categories (see Example 2.18), will be our main example.

**Example 2.10.** Let \( \mathcal{V} \) be a cocomplete symmetric monoidal category whose tensor product preserves colimits on both sides. We denote the image \( \text{Mod}(T) \) of the ‘free monoid’-monad on \( \mathcal{V}\text{-Mat} \), of Example 2.8 again by \( T \); it is the ‘free strict monoidal \( \mathcal{V}\text{-category’-monad, which is given as follows. Its image of a \( \mathcal{V}\text{-category } A \) is the free strict monoidal \( \mathcal{V}\text{-category generated by } A, that is } TA = \bigoplus_{n \geq 0} A^\otimes n; \text{ likewise its image of a } \mathcal{V}\text{-functor } f: A \to C \text{ is given by } Tf = \bigoplus_{n \geq 0} f^\otimes n. \text{ In detail, the objects of } TA \text{ are (possibly empty) sequences } x = (x_1, \ldots, x_n) \text{ of objects in } A, \text{ while its hom-objects } (TA)(x, y) \text{ are given by the tensor products } \bigotimes_{i=1}^n A(x_i, y_i) \text{ if } |x| = |y|, \text{ and } \emptyset \text{ otherwise. The functor } TF \text{ simply applies } f \text{ coordinatewise.}
The assignments $A \mapsto TA$ and $f \mapsto Tf$ extend to a double functor $T$ on $\mathcal{V}$-$\mathsf{Prof}$ as follows. Its image of a profunctor $J : A \Rightarrow B$ is given by

$$TJ(x,y) = \begin{cases} \bigotimes_{i=1}^n J(x_i,y_i) & \text{if } |x| = n = |y|; \\ \emptyset & \text{otherwise}, \end{cases}$$

on which the hom-objects of $TA$ and $TB$ act coordinatewise. It is clear that $T$ is normal, that is $T1_A = 1_{TA}$ for all $\mathcal{V}$-categories $A$. The invertible composites $T_\circ : T \circ TH \cong T(J \circ H)$, for composable $\mathcal{V}$-profunctors $J$ and $H$, are induced by those of the ‘free monoid’-monad $\mathsf{Mod}$; that they are again invertible follows, by Proposition 2.9, from the fact that the latter preserves local reflexive coequalisers.

This is a direct consequence from the fact that a tensor product of reflexive $\mathcal{V}$-categories and $\mathcal{V}$-monads, are induced by those of the ‘free monoid’-monad on $\mathcal{V}$-$\mathsf{Prof}$, that was obtained by introducing twists in the images $\mathcal{V}$-$\mathcal{V}$-profunctors form a double category $\mathcal{V}_\Sigma$ on $\mathcal{V}$-$\mathsf{Prof}$, whose algebras are symmetric monoidal $\mathcal{V}$-categories; see Example 2.19. Again denoting it $T\Sigma$, its image on a $\mathcal{V}$-category $A$ is given as follows: the objects of $T\Sigma A$ are (possibly empty) sequences of objects in $A$, while its hom-objects are given by

$$T\Sigma A(x,y) = \begin{cases} \prod_{\sigma \in \Sigma_n} \bigotimes_{i=1}^n A(x_{\sigma_i}, y_{\sigma_i}) & \text{if } |x| = n = |y|; \\ \emptyset & \text{otherwise}. \end{cases}$$

Its action on $\mathcal{V}$-profunctors is given analogously; that $T\Sigma$ is a normal pseudo double monad on $\mathcal{V}$-$\mathsf{Prof}$ is shown in Proposition 3.23 of [Kou13].

In closing this subsection we briefly consider monoidal double categories, and study pointwise Kan extensions along tensor products of horizontal morphisms. The following is Definition 9.1 of [Shu08].

**Definition 2.11.** A monoidal double category is a double category $\mathcal{K}$ equipped with a pseudo double functor $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ and an object $1 \in \mathcal{K}$, together with invertible double transformations $\otimes \circ (\otimes \times \mathsf{id}) \cong \otimes \circ (\mathsf{id} \times \otimes)$ and $1 \otimes \equiv \mathsf{id} \equiv \mathsf{id} \otimes 1$ that satisfy the usual coherence axioms.

We abbreviate by $1 = 1_1 : 1 \Rightarrow 1$ the horizontal unit for $1$. For an unraveling of the above definition we refer to Section 9 of [Shu08]; for us it suffices to notice the ‘interchange’ isomorphisms

$$\otimes \otimes : (J \otimes H) \otimes (K \otimes L) \cong (J \otimes K) \otimes (H \otimes L),$$

which form the invertible composites of $\otimes$.

**Example 2.12.** Let $\mathcal{V}$ be a cocomplete monoidal category whose tensor product preserves colimits on both sides, so that $\mathcal{V}$-$\mathsf{Prof}$ form a double category $\mathcal{V}$-$\mathsf{Prof}$ as we saw in Example 1.12. If $\mathcal{V}$ is symmetric monoidal then we can form tensor products of $\mathcal{V}$-categories and $\mathcal{V}$-functors; see e.g. Section 1.4 of [Kel82]. It is straightforward to extend these tensor products to a monoidal structure on $\mathcal{V}$-$\mathsf{Prof}$, in which the tensor product of $\mathcal{V}$-profunctors $J \otimes H : A \otimes C \Rightarrow B \otimes D$ is given by the $\mathcal{V}$-objects $((x,z) \otimes J(y,w)) = J(x,y) \otimes H(z,w)$, and whose unit is the $\mathcal{V}$-category $1$ consisting of a single object $*$ and hom-object $1(*,*) = 1$.

We consider pointwise Kan extensions along tensor products of horizontal morphisms. The following is a direct consequence of Proposition 1.15.
Proposition 2.13. Consider horizontally composable cells

\[
\begin{array}{c}
A \otimes C \xrightarrow{J \otimes 1_C} B \otimes C \xrightarrow{1_B \otimes H} B \otimes D \\
\downarrow s \quad \downarrow \gamma \quad \downarrow \varepsilon \quad \downarrow d \\
M \quad M \quad M \quad M
\end{array}
\]

in a monoidal double category and suppose that \(\varepsilon\) defines \(r\) as the pointwise right Kan extension of \(d\) along \(1_B \otimes H\). Then \(\gamma\) defines \(s\) as the pointwise right Kan extension of \(r\) along \(J \otimes 1_C\) precisely if the composite

\[
J \otimes H \xrightarrow{\varepsilon^{-1} \otimes \varepsilon^{-1}} (J \otimes 1_B) \otimes (1_C \otimes H) \xrightarrow{\otimes r^{-1}} (J \otimes 1_C) \otimes (1_B \otimes H) \xrightarrow{\gamma \otimes \varepsilon} 1_M
\]
defines \(s\) as the pointwise right Kan extension of \(d\) along \(J \otimes H\).

Lastly we consider Kan extensions along \(\mathcal{V}\)-profunctors of the form \(J \otimes 1_C\), in the monoidal double category \(\mathcal{V}\text{-Prof}\).

\[
\begin{array}{c}
A \otimes C \xrightarrow{J \otimes 1_C} B \otimes C \\
\downarrow s \quad \downarrow \gamma \quad \downarrow d \\
M \quad M \quad M
\end{array}
\quad
\begin{array}{c}
A \xrightarrow{J} B \\
\downarrow s(\cdot, z) \quad \downarrow \gamma_z \quad \downarrow d(\cdot, z) \\
M \quad M \quad M
\end{array}
\]

Given a cell \(\gamma\) in \(\mathcal{V}\text{-Prof}\) as on the left above and an object \(z \in C\), we write \(\gamma_z\) for the cell on the right that is given by the composite below; here we have identified \(z\) with the corresponding \(\mathcal{V}\)-functor \(z: 1 \to C\).

\[J \cong J \otimes 1 \xrightarrow{id \otimes 1} J \otimes 1_C \xrightarrow{\gamma} 1_M.\]

Proposition 2.14. Let \(\mathcal{V}\) be a cocomplete symmetric monoidal category whose tensor product preserves colimits on both sides, so that \(\mathcal{V}\)-profunctors form a monoidal double category \(\mathcal{V}\text{-Prof}\). A cell \(\gamma\), as on the left above, defines \(s\) as the pointwise right Kan extension of \(d\) along \(J \otimes 1_C\) precisely if the cells \(\gamma_z\) on the right, for each \(z \in C\), define \(s(\cdot, z)\) as the pointwise right Kan extension of \(d(\cdot, z)\) along \(J\). An analogous result holds for pointwise right Kan extensions along the \(\mathcal{V}\)-profunctor \(1_C \otimes J\).

Proof. For any \(x \in A\) we write \(\varepsilon_x\) for the cartesian cell defining the restriction \(J(x, id)\). Likewise for any \(z \in C\) we write \(\varepsilon_z\) for the cartesian cell defining the companion \(z: 1 \to C\) of \(z: 1 \to C\); remember from the discussion following Example 1.6 that the unit cell \(1_z: 1 \to 1_C\) factors as \(1_z = \varepsilon_z \circ \eta_z\), where \(\eta_z: 1 \to z\) is an opcartesian cell that defines \(z\) as the extension of \(1: 1 \Rightarrow 1\) along \(id\) and \(z\).

Notice that, by Proposition 2.10, the tensor product \(\varepsilon_x \otimes \varepsilon_z: J(x, id) \otimes z \Rightarrow J \otimes 1_C\) is a cartesian cell defining the restriction of \(J \otimes 1_C\) along \(x \otimes z: 1 \otimes 1 \to A \otimes C\). It follows from Proposition 1.19 that we may equivalently prove that, for each pair \((x, z)\), the composite \(\gamma_{(x,z)} = \gamma \circ (\varepsilon_x \otimes \varepsilon_z)\) defines \(s(x, z)\) as the \((J(x, id) \otimes z, \cdot, \cdot)\)-weighted limit of \(d\) if and only if the composite \(\gamma_z \circ \varepsilon_x\) defines \(s(x, z)\) as the \((J(x, id), \cdot)\)-weighted limit of \(d(\cdot, z)\).

To see this we use that \(id_{J(x, id)} \otimes \eta_z\) is pointwise initial, because tensor products of opcartesian cells are opcartesian (Proposition 2.6) and by Proposition 1.27(a). By regarding weighted limits as pointwise Kan extensions (Proposition 1.19), it follows that each \(\gamma_{(x,z)}\) defines \(s(x, z)\) as the \((J(x, id) \otimes z, \cdot, \cdot)\)-weighted limit of \(d\) precisely if the composite \(\gamma_{(x,z)} \circ (id \otimes \eta_z)\) defines \(s(x, z)\) as the \((J(x, id) \otimes 1, \cdot)\)-weighted limit of \(d(\cdot, z)\). To complete the proof we notice that \(\gamma_{(x,z)} \circ (id \otimes \eta_z) = \gamma \circ (\varepsilon_x \otimes 1_z)\) which, after composition with \(J(x, id) \cong J(x, id) \otimes 1\), coincides with \(\gamma_z \circ \varepsilon_x\). \(\square\)
2.2 Eilenberg-Moore double categories

In this subsection we define, for a normal oplax double monad $T$, several weakenings of the ‘Eilenberg-Moore double category’ associated to $T$, that was introduced by Grandis and Paré in [GP04]. We start by recalling from [Str74] the definitions of lax algebras for strict 2-monads, as well as several notions of weak morphism between such algebras, and the cells between those. The strict 2-monads of interest to us are those of the form $V(T)$, where $T$ is a normal oplax double monad.

Consider a strict 2-monad $T = (T, µ, η)$ on a 2-category $C$, consisting of a strict 2-functor $T: C → C$ and 2-natural transformations $µ: T² → T$ and $η: 1_C → T$ satisfying the usual axioms: $µ ◦ Tµ = µ ◦ µT$ and $µ ◦ Tη = ηT = µ ◦ η$.

**Definition 2.15.** A lax $T$-algebra $A$ is a quadruple $A = (A, a, ˘a, ˆa)$ consisting of an object $A$ equipped with a morphism $a: TA → A$, its structure map, and cells $˘a: a ◦ Ta → a ◦ µA$ and $ˆa: id_A → a ◦ ηA$, its associator and unitor, that satisfy the following three coherence conditions.

\[
\begin{align*}
T²A & → TA \\
T³A & → T²A & A = T³A \\
T²A & → TA
\end{align*}
\]

\[
\begin{align*}
T²A & → TA \\
T³A & → T²A & A = T³A \\
T²A & → TA
\end{align*}
\]

\[
\begin{align*}
T²A & → TA \\
T³A & → T²A & A = T³A \\
T²A & → TA
\end{align*}
\]

Dually, in the notion of a colax $T$-algebra $A = (A, a, ˘a, ˆa)$ the directions of the associator $˘a: a ◦ µA → a ◦ Ta$ and the unitor $ˆa: a ◦ ηA → id_A$ are reversed. A lax or colax $T$-algebra $A$ is called a pseudo $T$-algebra if the cells $˘a$ and $ˆa$ are isomorphisms; if they are identities then $A$ is called strict.

**Definition 2.16.** Given lax $T$-algebras $A = (A, a, ˘a, ˆa)$ and $C = (C, c, ˘c, ˆc)$, a lax $T$-morphism $A → C$ is a morphism $f: A → C$ that is equipped with a structure cell $f: c ◦ Tf → f ◦ a$, which satisfies the following associativity and unit axioms.

\[
\begin{align*}
T²C & → TC \\
T³C & → T²C & C = T³C \\
T²C & → TC
\end{align*}
\]

\[
\begin{align*}
T²C & → TC \\
T³C & → T²C & C = T³C \\
T²C & → TC
\end{align*}
\]

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Dually a colax $T$-morphism $f: A \to C$ is equipped with a cell $\bar{f}: f \circ a \Rightarrow c \circ T f$ such that

$$(\bar{c}.T^2f) \circ (c.Tf) \circ (\bar{f}.Ta) = (\bar{f}.\mu_A) \circ (f.\bar{a}) \quad \text{and} \quad (\bar{f}.\eta_A) \circ (f.\hat{a}) = \hat{c}.f.$$ 

Lax or colax $T$-morphisms whose structure cell $\bar{f}$ is invertible are called pseudo $T$-morphisms; notice that, for an invertible cell $f: c \circ T f \Rightarrow f \circ a$, the pair $(f, \bar{f})$ forms a lax $T$-morphism if and only if $(f, f^{-1})$ forms a colax $T$-morphism. Pseudo $T$-morphisms are called strict whenever $\bar{f}$ is the identity cell.

This leaves the definition of cells between $T$-morphisms.

**Definition 2.17.** Given lax $T$-morphisms $f$ and $g: A \to C$, a $T$-cell between $f$ and $g$ is a cell $\phi: f \Rightarrow g$ satisfying

$$TA \xrightarrow{Tf} TC \quad \text{for} \quad A \xrightarrow{g} C \quad \text{and} \quad TA \xrightarrow{Tf} TC \quad \text{for} \quad A \xrightarrow{f} C.$$ 

Likewise a $T$-cell between colax $T$-morphisms $f$ and $g: A \to C$ is a cell $\phi: f \Rightarrow g$ satisfying $(c.T\phi) \circ \bar{f} = \bar{g} \circ (\phi.a)$.

For any strict 2-monad $T$ lax $T$-algebras, lax $T$-morphisms and $T$-cells form a 2-category $\text{Alg}^l(T)$, and we denote by $\text{Alg}^l(T)$ the sub-2-category of lax $T$-algebras, pseudo $T$-morphisms and all $T$-cells between them. Likewise lax $T$-algebras, colax $T$-morphisms and $T$-cells form a 2-category $\text{Alg}^c(T)$.

**Example 2.18.** Let $T$ be the ‘free strict monoidal $V$-category’-monad described in Example 2.10. A lax $V(T)$-algebra $A = (A, \otimes, a, i)$ is a lax monoidal $V$-category: it consists of a $V$-category $A$ equipped with a $V$-functor $\otimes: TA \to A$ that defines the tensor product $(x_1 \otimes \cdots \otimes x_n) = \otimes(z)$ of each (possibly empty) sequence $z$ in $TA$, as well as $V$-natural maps

$$a: ((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{n1} \otimes \cdots \otimes x_{nm_n})) \to (x_{11} \otimes \cdots \otimes x_{nm_n}),$$

for each double sequence $z \in T^2A$, and $i: x \to (x)$, where $x \in A$. These transformations satisfy certain coherence conditions, see Definition 3.1.1 of [Lei04] for example. A lax monoidal $V$-category $A$ whose structure transformations $a$ and $i$ are invertible is simply called a monoidal $V$-category; if they are identities then $A$ is called a strict monoidal $V$-category. The above notions of monoidal category are often called unbiased, referring to the fact that the tensor products of all arities are part of their structure, in contrast to the classical notion which is biased towards the nullary (unit) and binary tensor products.
Lax $\mathcal{V}(T)$-morphisms are \textit{lax monoidal $\mathcal{V}$-functors}, that is $\mathcal{V}$-functors $f: A \to C$ equipped with $\mathcal{V}$-natural maps
\[ f_\otimes: (fx_1 \otimes \cdots \otimes fx_n) \to f(x_1 \otimes \cdots \otimes x_n), \]
its \textit{compositors}, that are compatible with the coherence transformations of $A$ and $C$; see Definition 3.1.3 of [Lei04]. Dually, in the notion of colax $\mathcal{V}(T)$-morphism the direction of the compositors is reversed; such $\mathcal{V}$-functors are called \textit{colax monoidal $\mathcal{V}$-functors}. Lax or colax monoidal $\mathcal{V}$-functors whose compositors are invertible are simply called \textit{monoidal $\mathcal{V}$-functors}; those with identities as compositors are called \textit{strict} monoidal $\mathcal{V}$-functors.

A $\mathcal{V}(T)$-cell $f \Rightarrow g$ between lax monoidal $\mathcal{V}$-functors $f$ and $g$, called a \textit{monoidal transformation}, is simply a $\mathcal{V}$-natural transformation $\phi: f \Rightarrow g$ that is compatible with the structure transformations $f_\otimes$ and $g_\otimes$: the diagrams below commute for each sequence $x \in TA$. Monoidal transformations between colax monoidal $\mathcal{V}$-functors are defined analogously.

\[
\begin{array}{ccc}
(fx_1 \otimes \cdots \otimes fx_n) & \xrightarrow{f_\otimes} & f(x_1 \otimes \cdots \otimes x_n) \\
(\phi_{x_1} \otimes \cdots \otimes \phi_{x_n}) & & \phi_{(x_1 \otimes \cdots \otimes x_n)} \\
(gx_1 \otimes \cdots \otimes gx_n) & \xrightarrow{g_\otimes} & g(x_1 \otimes \cdots \otimes x_n)
\end{array}
\]

\textbf{Example 2.19.} Consider the ‘free symmetric strict monoidal $\mathcal{V}$-category’-monad $T_\Sigma$, that was briefly considered at the end of Example 2.10. The notions of lax $\mathcal{V}(T_\Sigma)$-algebra and lax $\mathcal{V}(T_\Sigma)$-morphism are that of symmetric lax monoidal category and symmetric lax monoidal functor, as follows. A \textit{symmetric lax monoidal $\mathcal{V}$-category} $A$ is a lax monoidal $\mathcal{V}$-category $A$ that is equipped with $\mathcal{V}$-natural symmetries
\[ s_\sigma: (x_1 \otimes \cdots \otimes x_n) \to (x_{\sigma 1} \otimes \cdots \otimes x_{\sigma n}), \]
for all $\sigma \in \Sigma_n$ and $x \in A^{\otimes n}$, that are functorial in the sense that $s_\tau \circ s_\sigma = s_{\tau \circ \sigma}$ and $s_{id} = id$, and that make the following diagrams commute. Firstly, given permutations $\sigma_i \in \Sigma_{m_i}$, the diagrams
\[
\begin{array}{ccc}
((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{n1} \otimes \cdots \otimes x_{nm_n})) & \xrightarrow{a} & (x_{11} \otimes \cdots \otimes x_{nm_n}) \\
(s_{\sigma_1} \otimes \cdots \otimes s_{\sigma_n}) & & \sigma_{(\sigma_1, \ldots, \sigma_n)} \\
((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{nm_n} \otimes \cdots \otimes x_{n1})) & \xrightarrow{a} & (x_{1m_1} \otimes \cdots \otimes x_{nm_n})
\end{array}
\]

commute where, writing $N = \Sigma_m$, the permutation $(\sigma_1, \ldots, \sigma_n) \in \Sigma_N$ denotes the disjoint union of the $\sigma_i$. Secondly, for each $\tau \in \Sigma_n$, the diagrams
\[
\begin{array}{ccc}
((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{n1} \otimes \cdots \otimes x_{nm_n})) & \xrightarrow{\alpha} & (x_{11} \otimes \cdots \otimes x_{nm_n}) \\
\sigma_{\tau} & & \tau_{(m_1, \ldots, m_n)} \\
((x_{11} \otimes \cdots \otimes x_{1m_1}) \otimes \cdots \otimes (x_{m_1} \otimes \cdots \otimes x_{nm_n})) & \xrightarrow{\alpha} & (x_{m_1} \otimes \cdots \otimes x_{nm_n})
\end{array}
\]

commute, where the ‘block permutation’ $\tau_{(m_1, \ldots, m_n)} \in \Sigma_N$ permutes the $n$ blocks of the $(m_1, \ldots, m_n)$-partition of $\{1, \ldots, N\}$ exactly like $\tau$ permutes the elements of $\{1, \ldots, n\}$, while preserving the ordering of the elements in each block.

A lax monoidal functor $f: A \to C$ between symmetric lax monoidal categories is called \textit{symmetric} whenever the diagrams below commute. Transformations between
symmetric lax monoidal functors are simply monoidal transformations.

\[
\begin{align*}
(f x_1 \otimes \cdots \otimes f x_n) & \xrightarrow{f_\sigma} f (x_1 \otimes \cdots \otimes x_n) \\
(f x_{\sigma 1} \otimes \cdots \otimes f x_{\sigma n}) & \xrightarrow{f_\sigma} f (x_{\sigma 1} \otimes \cdots \otimes x_{\sigma n})
\end{align*}
\] (9)

Having recalled, for a normal oplax double monad \( T \) on \( K \), the notions of lax \( V(T) \)-algebra, weak \( V(T) \)-morphism and \( V(T) \)-cell in the 2-category \( V(K) \), we now introduce the notion of horizontal \( T \)-morphism between lax \( V(T) \)-algebras and that of a cell between such morphisms. Both are straightforward generalisations of notions that were introduced by Grandis and Paré in Section 7.1 of [GP04], where \( V(T) \)-algebras were assumed to be strict. Afterwards, in Proposition 2.24, we will show that lax \( V(T) \)-algebras, weak \( V(T) \)-morphisms, horizontal \( T \)-morphisms and their cells form a double category.

**Definition 2.20.** Let \( T = (T, \mu, \eta) \) be a normal oplax double monad on a double category \( K \). Given lax \( V(T) \)-algebras \( A = (A, a, \bar{a}, \hat{a}) \) and \( B = (B, b, \bar{b}, \hat{b}) \), a horizontal \( T \)-morphism \( A \Rightarrow B \) is a horizontal morphism \( J: A \Rightarrow B \) equipped with a structure cell

\[
\begin{array}{ccc}
TA & T(J) & TB \\
a & J & b \\
\downarrow & \downarrow & \downarrow \\
A & J & B
\end{array}
\]

satisfying the following associativity and unit axioms.

\[
\begin{array}{ccc}
T^2A & T^2J & T^2B \\
\downarrow & \downarrow & \downarrow \\
TA & J & TB \\
\downarrow & \downarrow & \downarrow \\
A & J & B
\end{array}
\]

\[
\begin{array}{ccc}
T^2A & T^2J & T^2B \\
\downarrow & \downarrow & \downarrow \\
TA & J & TB \\
\downarrow & \downarrow & \downarrow \\
A & J & B
\end{array}
\]

**Example 2.21.** Consider the double monad \( T \) of free strict monoidal \( V \)-categories described in Example 2.10; we will call a horizontal \( T \)-morphism \( A \Rightarrow B \) between lax monoidal \( V \)-categories \( A \) and \( B \) a monoidal \( V \)-profunctor. It consists of a \( V \)-profunctor \( J: A \Rightarrow B \) equipped with \( V \)-maps

\[
J_\otimes: J(x_1, y_1) \otimes \cdots \otimes J(x_n, y_n) \to J((x_1 \otimes \cdots \otimes x_n), (y_1 \otimes \cdots \otimes y_n))
\]

which are compatible with the actions of \( TA \) and \( TB \) and satisfy the following.
Definition 2.22. Let $T$ be a normal oplax double monad on a double category $K$. Consider a cell $\phi$ as on the left below, where $f$ and $g$ are lax $V(T)$-morphisms while $J$ and $K$ are horizontal $T$-morphisms. We call $\phi$ a $T$-cell whenever the identity on the right is satisfied.

\[
\begin{align*}
A &\xrightarrow{J} B & TA &\xrightarrow{T J} TB & TB &\xrightarrow{T} TA &\xrightarrow{T J} TB \\
f &\xrightarrow{\phi} & TJ \phi &\xrightarrow{T g} & Tg &\xrightarrow{b} & Tf &\xrightarrow{a} & T J \phi \\
C &\xrightarrow{K} D & TC &\xrightarrow{TK} TD & TD &\xrightarrow{J b} & B &\xrightarrow{f} & J \phi \\
g &\xrightarrow{c} & JK &\xrightarrow{d} & TD &\xrightarrow{J g} & C &\xrightarrow{f} & JK \\
\end{align*}
\]

Similarly, in the case that $f$ and $g$ are colax $V(T)$-morphisms, we call $\phi$ a $T$-cell whenever $f \circ (K \circ T \phi) = (\phi \circ J) \circ g$.

Example 2.23. Let $T$ be the double monad of ‘free strict monoidal categories’ described in Example 2.10 and consider a transformation $\phi$ in $V$-$\text{Prof}$, as on the left above, where $f$ and $g$ are lax monoidal $V$-functors (Example 2.18) while $J$ and $K$ commute, while the unit axiom means that those of the form

\[
\begin{align*}
J(x, y) &\xrightarrow{\mu_i} J(x, (y)) \\
J((x), (y)) &\xrightarrow{\lambda_i} J((x, y))
\end{align*}
\]

commutes. If $A$ and $B$ are symmetric lax monoidal $V$-categories then we call $J$ a symmetric monoidal $V$-profunctor whenever the diagrams below commute.

\[
\begin{align*}
J(x_1, y_1) &\xrightarrow{\sigma} J(x_1, y_1) \\
J((x_1), (y_1)) &\xrightarrow{\rho_{x,y}} J((x, y))
\end{align*}
\]

The following definition introduces general $T$-cells; we will see in Proposition 2.24 that it generalises the vertical $V(T)$-cells of Definition 2.17.

**Definition 2.22.** Let $T$ be a normal oplax double monad on a double category $K$. Consider a cell $\phi$ as on the left below, where $f$ and $g$ are lax $V(T)$-morphisms while $J$ and $K$ are horizontal $T$-morphisms. We call $\phi$ a $T$-cell whenever the identity on the right is satisfied.

\[
\begin{align*}
A &\xrightarrow{J} B & TA &\xrightarrow{T J} TB & TB &\xrightarrow{T} TA &\xrightarrow{T J} TB \\
f &\xrightarrow{\phi} & TJ \phi &\xrightarrow{T g} & Tg &\xrightarrow{b} & Tf &\xrightarrow{a} & T J \phi \\
C &\xrightarrow{K} D & TC &\xrightarrow{TK} TD & TD &\xrightarrow{J b} & B &\xrightarrow{f} & J \phi \\
g &\xrightarrow{c} & JK &\xrightarrow{d} & TD &\xrightarrow{J g} & C &\xrightarrow{f} & JK \\
\end{align*}
\]

Similarly, in the case that $f$ and $g$ are colax $V(T)$-morphisms, we call $\phi$ a $T$-cell whenever $f \circ (K \circ T \phi) = (\phi \circ J) \circ g$. 

Example 2.23. Let $T$ be the double monad of ‘free strict monoidal categories’ described in Example 2.10 and consider a transformation $\phi$ in $V$-$\text{Prof}$, as on the left above, where $f$ and $g$ are lax monoidal $V$-functors (Example 2.18) while $J$ and $K$
are monoidal $\mathcal{V}$-profunctors (Example 2.21). Remember that $\phi$ is given by a family of $\mathcal{V}$-maps $J(x, y) \to K(fx, gy)$ that is compatible with the actions of $A$ and $B$. The transformation $\phi$ is a $T$-cell if the following diagram commutes; in that case we call $\phi$ a monoidal transformation.

\[
\begin{array}{c}
\begin{array}{c}
J(x_1, y_1) \otimes \cdots \otimes J(x_n, y_n) \xrightarrow{\phi} K(fx_1, gy_1) \otimes \cdots \otimes K(fx_n, gy_n) \\
\end{array} \\
\begin{array}{c}
\downarrow J \otimes \phi \\
J((x_1 \otimes \cdots \otimes x_n), (y_1 \otimes \cdots \otimes y_n)) \xrightarrow{\phi} K((fx_1 \otimes \cdots \otimes fx_n), (gy_1 \otimes \cdots \otimes gy_n)) \\
\end{array} \\
\begin{array}{c}
\downarrow \phi \\
K(f(x_1 \otimes \cdots \otimes x_n), g(y_1 \otimes \cdots \otimes y_n)) \xrightarrow{\lambda_T} K((fx_1 \otimes \cdots \otimes fx_n), g(y_1 \otimes \cdots \otimes y_n))
\end{array}
\end{array}
\]

A monoidal transformation $\phi: J \Rightarrow K(f, g)$ where $f$ and $g$ are colax monoidal functors is defined similarly.

Given a normal oplax double monad $T$, we now show that lax $V(T)$-algebras, any choice of weak $V(T)$-morphisms, horizontal $T$-morphisms, and $T$-cells, form a double category $\mathcal{Alg}_w(T)$ such that $V(\mathcal{Alg}_w(T)) = \mathcal{Alg}_w(V(T))$. Consequently we will call lax $V(T)$-algebras simply lax $T$-algebras, while weak $V(T)$-morphisms will be called weak vertical $T$-morphisms.

**Proposition 2.24.** Let $T$ be a normal oplax double monad on a double category $\mathcal{K}$, and let ‘weak’ mean either ‘colax’ or ‘lax’. The double category structure of $\mathcal{K}$ lifts to make lax $V(T)$-algebras, weak $V(T)$-morphisms, horizontal $T$-morphisms and $T$-cells into a double category $\mathcal{Alg}_w(T)$, such that $V(\mathcal{Alg}_w(T)) = \mathcal{Alg}_w(V(T))$.

We denote by $\mathcal{Alg}_{ps}(T)$ the sub-double category of $\mathcal{Alg}_w(T)$ consisting of lax $T$-algebras, pseudo $V(T)$-morphisms, horizontal $T$-morphisms and all $T$-cells whose vertical source and target are pseudo $V(T)$-morphisms. Clearly $V(\mathcal{Alg}_w(T)) = \mathcal{Alg}_w(V(T))$ implies $V(\mathcal{Alg}_{ps}(T)) = \mathcal{Alg}_w(V(T))$.

**Proof.** We shall show that the double category structure of $\mathcal{K}$ lifts to make lax $V(T)$-algebras, lax $V(T)$-morphisms, horizontal $T$-morphisms and $T$-cells into a double category $\mathcal{Alg}_w(T)$; the case of $\mathcal{Alg}_w(T)$ is similar. First notice that the equality $V(\mathcal{Alg}_w(T)) = \mathcal{Alg}_w(V(T))$ determines the structure on the category $\mathcal{Alg}_w(T)$ completely; in particular the algebra structure on the vertical composite $h \circ f$ of lax $V(T)$-morphisms $f: A \to C$ and $h: C \to E$ must be given by the composite $h \circ f = (h \circ 1_B) \circ (1_h \circ f)$.

To also lift the horizontal structure of $\mathcal{K}$ to $\mathcal{Alg}_w(T)$ it remains to define structure cells for each composite $J \circ H$ of horizontal $T$-morphisms $J: A \Rightarrow B$ and $H: B \Rightarrow C$, as well as for the horizontal units $1_A: A \Rightarrow A$, in a way such that any composite of horizontal units forms again a $T$-cell, the horizontal units $1_f$ form $T$-cells, and the associators and unitors of $\mathcal{K}$ form $T$-cells. We define the structure cell of $J \circ H$ by

\[
\begin{array}{c}
TA \xrightarrow{TJ} TB \xrightarrow{TH} TC \\
\downarrow \phi \\
J \circ H = \begin{array}{c}
TA \xrightarrow{TJ} TB \xrightarrow{TH} TC \\
\downarrow \phi \\
A \xrightarrow{J} B \xrightarrow{H} C.
\end{array}
\end{array}
\]

That this composite satisfies the associativity axiom of Definition 2.20 is shown by the following equation of composites where, to save space, only the non-identity cells
are depicted while objects and morphisms are left out. The cells \( \bar{c} \) and \( \bar{a} \) here are the structure cells of \( C \) and \( A \) respectively (see Definition 2.15) and the identities follow from the naturality of \( T \); the associativity axioms of \( H \) and \( J \) and the fact that \( (T^2)_o = T_o \circ TT_o \); the composition axiom of \( \mu \) (see Definition 2.3). The unit axiom for \( J \circ H \) follows similarly from that for \( H \) and \( J \), as well as the composition axiom for \( \eta \).

\[
\begin{array}{c|c|c}
\hline
\text{T} & \text{J} & \text{H} \\
\hline
\text{TT}_o & J & H \\
\text{T(J \circ H)} & \text{c} & \text{c} \\
\hline
\end{array}
\]

The structure cell of a horizontal unit \( 1_A \) we take to be \( T1_A = 1_{T_A} \xrightarrow{1_T} 1_A \), where \( \alpha: TA \to A \) is the structure map of \( A \). The associativity and unit axioms follow from the unit axioms for \( \mu \) and \( \eta \).

It is now easily checked that vertical and horizontal composites of \( T \)-cells form again \( T \)-cells and that, for a lax \( T \)-morphism \( f: A \to C \), the unit cell \( 1_f \) is a \( T \)-cell. It is also readily seen that the components of the associator and unitors of \( K \) form \( T \)-cells, using the associativity and unit axioms for \( T \); we conclude that, with the chosen algebra structures above, the structure of double category \( K \) lifts to form a double category \( \text{Alg}_w(T) \).

Finally notice that the \( T \)-cell axiom for a vertical cell \( \phi: f \Rightarrow g \) coincides with the \( V(T) \)-cell axiom for \( \phi \) (see Definition 2.17). We conclude that \( V(\text{Alg}_w(T)) = \text{Alg}_w(V(T)) \) as asserted. \( \square \)

### 3 Kan extensions in \( \text{Alg}_w(T) \)

In this section we study Kan extensions in the double category \( \text{Alg}_w(T) \). Firstly, in light of Proposition 1.14, we wonder whether \( \text{Alg}_w(T) \) has restrictions, so that Kan extensions in \( \text{Alg}_w(V(T)) = V(\text{Alg}_w(T)) \) correspond to Kan extensions in \( \text{Alg}_w(T) \). Secondly we wonder whether \( \text{Alg}_w(T) \) has opcartesian tabulations, so that we can obtain analogues of Theorem 2.22 and Proposition 1.24 for \( \text{Alg}_w(T) \). As we shall see, restrictions and tabulations can often be ‘lifted’ along the forgetful double functors \( \text{Alg}_w(T) \to K \).

We start with the lifting of restrictions. Given a double functor \( F: K \to L \) and morphisms \( f: A \to C \), \( K: C \to D \) and \( g: B \to D \) in \( K \), we say that \( F \) lifts the restriction \( K(f, g) \) if for any cartesian cell \( \phi \) in \( L \), as on the left below, there exists a unique cell \( \phi' \) in \( K \) such that \( F\phi' = \phi \) and, moreover, \( \phi' \) is cartesian.

\[
\begin{array}{ccc}
FA & \xrightarrow{Ff} & FB \\
FC & \xrightarrow{Fg} & FD
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
C & \xrightarrow{g} & D
\end{array}
\]

**Proposition 3.1.** For a normal oplax double monad \( T \) on a double category \( K \), the following hold for the forgetful double functors \( \text{Alg}_w(T) \to K \), where \( w \in \{c, l, ps\} \): 

(a) \( \text{Alg}_w(T) \to K \) lifts restrictions \( K(f, g) \) where \( g \) is a pseudo \( T \)-morphism;

(b) \( \text{Alg}_l(T) \to K \) lifts restrictions \( K(f, g) \) where \( f \) is a pseudo \( T \)-morphism;

(c) \( \text{Alg}_{ps}(T) \to K \) lifts all restrictions.

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We conclude that if $K$ is an equipment then so is $\text{Alg}_{\text{ps}}(T)$. Moreover, in that case $\text{Alg}_{\text{ps}}(T)$ has all companions while, in general, it has only conjoints of pseudo $T$-morphisms; dually $\text{Alg}_{\text{li}}(T)$ has all conjoints but only companions of pseudo $T$-morphisms.

**Example 3.2.** If $f: A \to C$ is a monoidal $V$-functor (Example 2.18), $K: C \to D$ a monoidal $V$-profunctor (Example 2.21) and $g: B \to D$ a lax monoidal functor then the restriction $K(f,g)$ admits a canonical monoidal structure that is given by the composites

$$K(f_1, g_1) \otimes \cdots \otimes K(f_n, g_n) \xrightarrow{K \otimes \cdots \otimes K} K\left( (f_1 \otimes \cdots \otimes f_n), (g_1 \otimes \cdots \otimes g_n) \right) \xrightarrow{\rho \otimes \cdots \otimes \rho} K\left( f(x_1 \otimes \cdots \otimes x_n), g(y_1 \otimes \cdots \otimes y_n) \right).$$

**Proof of Proposition 3.1.** We will prove (b). The inclusion $\text{Alg}_{\text{ps}}(T) \to \text{Alg}_{\text{li}}(T)$ lifts all restrictions so that (c) follows, while proving (a) is similar to proving (b).

Consider a cartesian cell $\phi$ in $K$ as on the left below, where $f$ is a pseudo $T$-morphism, $g$ is a lax $T$-morphism and $K$ is a horizontal $T$-morphism. We claim that the unique factorisation $\bar{J}$ of the composite on the right below, through the cartesian cell $\phi$, as shown, forms a structure cell for $J$.

We have to show that $\bar{J}$ satisfies the associativity and unit axiom of Definition 2.20. To show that the first holds consider the following equation of composites in $K$, whose identities are described below. Since its left-hand and right-hand side are the left-hand and right-hand side of the associativity axiom for $\bar{J}$ postcomposed with $\phi$, it implies the associativity axiom for $\bar{J}$, because factorisations through cartesian cells are unique.

The identity (i) above follows from the factorisation (11); (ii) follows from the $T$-image of (11), where the $T$-image of the left-hand side is rewritten as $Tf^{-1} \circ T(K \circ T\phi) \circ Tg$ by using the unit axioms for $T$ (see (7)); (iii) follows from the associativity axiom for $g$; (iv) from the associativity axiom for $K$; (v) from the
naturality of $\mu$ and the associativity axiom for $\bar{f}$; (vi) from the factorisation (11). Showing that $\bar{J}$ satisfies the unit axiom is done analogously: it is equivalent to the identity obtained by composing it with $\phi$, which follows from the factorisation (11) and the unit axioms for $\bar{g}$, $\bar{K}$ and $\bar{f}$. This completes the proof that $\bar{J}$ forms a structure cell making $J$ into a horizontal $T$-morphism.

Composing (11) on the left with $\bar{f}$ we see that $\phi$ forms a $T$-cell $(J, \bar{J}) \Rightarrow (K, \bar{K})$, showing that $\phi$ lifts along $\text{Alg}_w(T) \to K$. In fact, because the factorisation (11) is unique, this lift of $\phi$ is unique as well. To complete the proof we have to show that $\phi$ forms a cartesian $T$-cell. Hence consider a $T$-cell $\psi$ as on the left-hand side below, where $h$ and $k$ are lax $T$-morphisms while $H$ is a horizontal $T$-morphism.

Since $\phi$ is cartesian in $K$, we obtain a unique factorisation $\psi'$ of $\psi$ as shown; we have to prove that $\psi'$ is a $T$-cell. The following equation shows that the $T$-cell axiom for $\psi'$ holds after composing it with $\phi$, where the identities follow from (11), the factorisation above, the $T$-cell axiom for $\psi$ and again the factorisation above.

Since factorisations through $\phi$ are unique the $T$-cell axiom for $\psi'$ follows, completing the proof.

We now turn to tabulations, see Definition 1.20

**Proposition 3.3.** Let $T$ be a normal oplax double monad on an equipment $K$. The forgetful functors $\text{Alg}_w(T) \to K$ lift tabulations in the following sense. If the cell $\pi$ below forms the tabulation of $J$ in $K$, where $J$ is a horizontal $T$-morphism, then there exists a unique lax $T$-algebra structure on $\langle J \rangle$ with respect to which $\pi_A$ and $\pi_B$ are strict $T$-morphisms and $\pi$ is a $T$-cell. Thus a $T$-cell, $\pi$ forms the tabulation of $J$ in $\text{Alg}_w(T)$, which is opcartesian as soon as $\pi$ is opcartesian in $K$.

![Diagram for Proposition 3.3](https://via.placeholder.com/150)

**Example 3.4.** The tabulation $\langle J \rangle$ (see Example 1.21) of a monoidal (unenriched) profunctor $J : A \to B$ (see Example 2.21) admits a canonical lax monoidal structure whose tensor product on objects and morphisms is given by applying the tensor products of $A$, $J$ and $B$ coordinatewise:

$$((x_1, u_1, y_1) \otimes \cdots \otimes (x_n, u_n, y_n)) = ((x_1 \otimes \cdots \otimes x_n), (u_1 \otimes \cdots \otimes u_n), (y_1 \otimes \cdots \otimes y_n));$$

$$((p_1, q_1) \otimes \cdots \otimes (p_n, q_n)) = ((p_1 \otimes \cdots \otimes p_n), (q_1 \otimes \cdots \otimes q_n)).$$
Similarly its coherence maps are given by those of \( A \) and \( B \): 
\[
a_{(x, u, y)} = (a_x, a_y)
\]
for any double sequence \( (x, u, y) \) of objects in \( (J) \), and 
\[
i_{(x, u, y)} = (i_x, i_y)
\]
for any object \( (x, u, y) \) in \( (J) \).

For the proof of Proposition 3.3 it is useful to record the following simple property of tabulations.

**Lemma 3.5.** The projections \( \pi_A \) and \( \pi_B \) of a tabulation \( \pi \), as on the left below, are jointly monic in the following sense. Any two cells \( \phi \) and \( \psi \) as on the right below coincide as soon as 
\[
1_{\pi_A} \circ \phi = 1_{\pi_A} \circ \psi \quad \text{and} \quad 1_{\pi_B} \circ \phi = 1_{\pi_B} \circ \psi.
\]

\[
\begin{array}{ccc}
\langle J \rangle & \cong & \langle J \rangle \\
\pi_A & \downarrow & \pi_B \\
A & \xrightarrow{f} & B
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{H} & Y \\
f & \downarrow \phi & g \\
\langle J \rangle & \cong & \langle J \rangle
\end{array}
\]

Proof. Applying the 2-dimensional universal property of \( \pi \) to the identity
\[
\begin{array}{c}
\phi \\
\pi
\end{array}
= 
\begin{array}{c}
\phi \\
\pi
\end{array}
\]
we find that there is exactly one cell \( \chi \) such that 
\[
1_{\pi_A} \circ \chi = 1_{\pi_A} \circ \phi \quad \text{and} \quad 1_{\pi_B} \circ \chi = 1_{\pi_B} \circ \phi.
\]
Since both \( \chi = \phi \) and \( \chi = \psi \) satisfy these identities we must have \( \phi = \psi \).

**Proof of Proposition 3.3** Given a horizontal \( T \)-morphism \( (J, \bar{J}) : A \rightarrow B \), we assume that there exists a cell \( \pi \) in \( K \), as on the left above, that defines the tabulation of its underlying horizontal morphism \( J : A \rightarrow B \). We will construct a lax \( T \)-algebra structure on \( \langle J \rangle \) and check that it is unique such that \( \pi_A \) and \( \pi_B \) become strict \( T \)-morphisms, and that \( \pi \) becomes a \( T \)-cell. Following this we prove that as a \( T \)-cell \( \pi \) satisfies the universal properties of a tabulation, and that it is opcartesian whenever \( \pi \) is opcartesian in \( K \).

**Lax \( T \)-algebra structure on \( \langle J \rangle \).** The universal properties of the tabulation \( \pi \) induce a lax \( T \)-algebra structure on \( \langle J \rangle \) as follows. For the structure map \( w : T\langle J \rangle \rightarrow \langle J \rangle \) consider the composite on the left-hand side below: by the 1-dimensional universal property of tabulations it factors uniquely through \( \pi \) as a vertical morphism \( w : T\langle J \rangle \rightarrow \langle J \rangle \) as shown.

\[
\begin{array}{ccc}
\langle J \rangle & \xrightarrow{T\pi} & \langle J \rangle \\
\downarrow T\pi & & \downarrow w \\
\langle J \rangle & \xrightarrow{\pi} & \langle J \rangle
\end{array}
\]

\[
\begin{array}{ccc}
T\langle J \rangle & \xrightarrow{TJ} & TB \\
\downarrow T\pi & & \downarrow \pi_B \\
\langle J \rangle & \xrightarrow{\pi} & \langle J \rangle
\end{array}
\]

To obtain the vertical associator cell \( \bar{w} : w \circ T \Rightarrow w \circ \mu_{\langle J \rangle} \) we consider the identity below, which is the associativity axiom for \( J \) precomposed with \( T^2 \pi \). Notice that the second column of the left-hand side coincides with \( \pi \circ 1_w \circ 1_{\mu_{\langle J \rangle}} \), and the first of the right-hand side with \( \pi \circ 1_w \circ 1_{T_w} \). Invoking the 2-dimensional universal property of \( \pi \) we obtain a unique cell \( \bar{w} : w \circ T \Rightarrow w \circ \mu_{\langle J \rangle} \) such that
\[ 1_{\pi_A} \circ \bar{w} = a \circ 1_{T^2\pi_A} \quad \text{and} \quad 1_{\pi_B} \circ \bar{w} = b \circ 1_{T^2\pi_B}; \] take \( \bar{w} \) as the associator of \( (J) \).

\[
\begin{array}{ccc}
T^2(J) = T^2(J) & & T^2(J) = T^2(J) \\
T^2\pi_A & \Downarrow T^2\pi & T^2\pi_B \\
T^2A & \Uparrow \mu_A & \Uparrow \mu_B \\
\Downarrow \tau_a & \Downarrow J & TA \Downarrow J \\
\Downarrow a & \Downarrow \pi_a & \Downarrow \pi_B \\
A & \Downarrow J & B \\
\end{array}
\]

To prove the coherence axiom for \( \bar{w} \) (see Definition 2.15) we use the lemma above as follows. It is clear from the definitions of \( \bar{w} \) and \( \bar{w} \) that, by postcomposing either side of the coherence axiom for \( \bar{w} \) with \( 1_{\pi_A} \), we obtain the corresponding side of the coherence axiom for \( \bar{a} \) precomposed with \( 1_{T^3\pi_A} \). Likewise, postcomposing them with \( 1_{\pi_B} \) we obtain the sides of the coherence axiom for \( \bar{b} \), precomposed with \( 1_{T^3\pi_B} \). Since the coherence axioms for \( \bar{a} \) and \( \bar{b} \) hold, and because \( 1_{\pi_A} \) and \( 1_{\pi_B} \) are jointly monic, the coherence axiom for \( \bar{w} \) follows.

The vertical unitor cell \( \hat{w} \): \( \text{id}_{(J)} \Rightarrow w \circ \eta(J) \) is obtained in a similar way, by applying the 2-dimension universal property of \( \pi \) to the unit axiom for \( J \) precomposed with \( \pi \); it is unique such that \( 1_{\pi_A} \circ \hat{w} = a \circ 1_{\pi_A} \) and \( 1_{\pi_B} \circ \hat{w} = b \circ 1_{\pi_B} \). The coherence axioms for \( \hat{w} \) follow from those for \( \bar{a} \) and \( \bar{b} \), as well as the joint monicity of \( 1_{\pi_A} \) and \( 1_{\pi_B} \), analogous to how the coherence axiom for \( \bar{w} \) followed from that for \( \bar{a} \) and \( \bar{b} \). This completes the definition of the lax \( T \)-algebra \( (J) \).

**Uniqueness of the algebra structure on \( (J) \).** Summarising the above, the algebra structure on \( (J) \) is determined as follows: the structure map \( w \) is uniquely determined by the identity \( \textbf{12} \), while the associator \( \bar{w} \) is uniquely determined by the identities \( 1_{\pi_A} \circ \bar{w} = a \circ 1_{\pi_A} \) and \( 1_{\pi_B} \circ \bar{w} = b \circ 1_{\pi_B} \), and the unitor \( \hat{w} \) by the identities \( 1_{\pi_A} \circ \bar{w} = a \circ 1_{\pi_A} \) and \( 1_{\pi_B} \circ \bar{w} = b \circ 1_{\pi_B} \). It follows from \( \textbf{12} \) that \( \pi \) becomes a \( T \)-cell if we take the structure cells of the projections \( \pi_A \) and \( \pi_B \) to be trivial, while the identities determining \( \bar{w} \) and \( \hat{w} \) are precisely the coherence axioms for these trivial structure cells. Conversely, by the uniqueness of \( w, \bar{w}, \text{and} \hat{w} \), any lift of \( \pi \) to a \( T \)-cell with strict vertical \( T \)-morphisms must have \( (w, \bar{w}, \hat{w}) \) as the algebraic structure on \( (J) \).

**Universal properties of \( \pi \) as a \( T \)-cell.** We will show that \( \pi \) forms a tabulation of \( J \) in \( \text{Alg}_T \) and in \( \text{Alg}_{\phi_\pi}(T) \); proving that it forms one in \( \text{Alg}_T \) is similar to the proof for \( \text{Alg}_T \). For the 1-dimensional universal property consider a \( T \)-cell \( \phi \) as on the left below, where \( \phi_A \) and \( \phi_B \) are lax \( T \)-morphisms. Since \( \pi \) is a tabulation in \( K \) the cell \( \phi \) factors uniquely through \( \pi \) as a vertical morphism \( \phi' \) as shown; we have to show that \( \phi' \) can be lifted to a lax \( T \)-morphism.

\[
\begin{array}{ccc}
X & \Downarrow \phi & X \\
\Downarrow \phi_A & \Downarrow \phi_B & \Downarrow \phi' \\
A & \Downarrow J & B \\
\end{array}
\]

To obtain a structure cell for \( \phi' \) we consider the \( T \)-cell axiom for \( \phi \), as on the right above. Notice that \( \hat{J} \circ \hat{T} \phi' = \pi \circ 1_{\hat{T}} \phi' \) and \( \phi \circ 1_{\hat{x}} = \pi \circ 1_{\hat{\phi}} \circ 1_{\hat{x}} \) here, so that by invoking the 2-dimensional universal property of \( \pi \) we obtain a unique vertical cell \( \hat{\phi}' : \hat{w} \circ \hat{T} \phi' \Rightarrow \phi' \circ x \) such that \( 1_{\pi_A} \circ \hat{\phi}' = \hat{\phi}_A \) and \( 1_{\pi_B} \circ \hat{\phi}' = \hat{\phi}_B \). From these...
identities it follows that the coherence axioms for $\phi_A$ and $\phi_B$ coincide with those of $\phi'$ postcomposed with $1_{\pi_A}$ and $1_{\pi_B}$, respectively. Since the latter are jointly monic, the coherence axioms for $\phi'$ follow, and we conclude that $(\phi', \phi')$ forms a unique lift of $\phi'$; this completes the proof of $\pi$ satisfying the 1-dimensional universal property in $\text{Alg}_\text{ps}(T)$.

To see that the same property is satisfied by $\pi$ in $\text{Alg}_\text{ps}(T)$ assume that, in the above, $\phi_A$ and $\phi_B$ are pseudo $T$-morphisms. Then, using the 2-dimensional universal property of $\pi$ again, we can obtain a unique cell $\chi: \phi' \circ x \Rightarrow w \circ T \phi'$ such that $1_{\pi_A} \circ \chi = \phi_A^{-1}$ and $1_{\pi_B} \circ \chi = \phi_B^{-1}$. Using the joint monicity of $\pi_A$ and $\pi_B$ again, it follows that $\chi$ forms the inverse of $\phi'$ and we conclude that the lift $(\phi', \phi')$ of $\phi'$ is a pseudo $T$-morphism.

The 2-dimensional universal property for $\pi$, as a $T$-cell, follows easily from the corresponding property for $\pi$ as a cell in $\mathcal{K}$. Indeed, consider an identity $\xi_A \circ \psi = \phi \circ \xi_B$ of $T$-cells, as in Definition 1.20. By the 2-dimensional universal property of $\pi$ there exists a unique cell $\xi'$ in $\mathcal{K}$ such that $1_{\pi_A} \circ \xi' = \xi_A$ and $1_{\pi_B} \circ \xi' = \xi_B$. It follows that the $T$-cell axiom for $\xi'$, postcomposed with $1_{\pi_A}$ and $1_{\pi_B}$, coincides with the $T$-cell axioms for $\xi_A$ and $\xi_B$ so that, by the joint monicity of $1_{\pi_A}$ and $1_{\pi_B}$, the $T$-cell axiom for $\xi'$ follows. We conclude that $\pi$ as a $T$-cell forms the tabulation of the horizontal $T$-morphism $J$, both in $\text{Alg}_\text{ps}(T)$ and $\text{Alg}_\text{ps}(T)$.

The $T$-cell $\pi$ is opcartesian as soon as $\pi$ is opcartesian in $\mathcal{K}$. If $\pi$ is an opcartesian cell in $\mathcal{K}$ then so is $T\pi$, by Proposition 2.6. To prove that this ensures that $\pi$ is opcartesian as a $T$-cell we consider a $T$-cell $\phi$ as on the left below; in $\mathcal{K}$ it factors through the opcartesian cell $\pi$ as a cell $\phi'$, as shown.

\[
\begin{array}{c c c c}
\langle J \rangle & \langle J \rangle & \langle J \rangle & \langle J \rangle \\
\pi_A & \pi_B & \pi_A & \pi_B \\
A & \phi & B & \Rightarrow A & \pi & B \\
f & \Rightarrow & g & f & \Rightarrow & \psi' & \Rightarrow g \\
C & \Rightarrow & D & C & \Rightarrow & D
\end{array}
\]

To show that $\phi'$ satisfies the $T$-cell axiom consider the following equation of composites in $\mathcal{K}$, where the identities are given by the factorisation above, the $T$-cell axiom for $\phi$, again the factorisation above, and the $T$-cell axiom for $\pi$.

Since the two sides of the equation above are the sides of the $T$-cell axiom for $\phi'$ precomposed with $T\pi$, the axiom itself follows because $T\pi$ is opcartesian, and factorisations through opcartesian cells are unique. This completes the proof.

Combining Proposition 3.1 and Proposition 3.3 we conclude that if $T$ is a normal oplax double monad on an equipment $\mathcal{K}$ that has opcartesian tabulations, then $\text{Alg}_\text{ps}(T)$ is again an equipment that has opcartesian tabulations. It follows that pointwise Kan extensions in $\text{Alg}_\text{ps}(T)$ can be defined in terms of Kan extensions (by Theorem 1.23) so that pointwise right Kan extensions in the 2-category $\text{Alg}_\text{ps}(\mathcal{V}(T))$ correspond to pointwise right Kan extensions along conjoints in $\text{Alg}_\text{ps}(T)$ (by Proposition 1.23).

Even though in the cases $w = c$ and $w = 1$ the double categories $\text{Alg}_\text{ps}(T)$ fail to be equipments in general, pointwise Kan extensions in $\text{Alg}_\text{ps}(T)$ can still be defined in terms of Kan extensions by the following variant of Theorem 1.23.
Remember that, for a pseudo $T$-morphism $f : C \to A$ and a horizontal $T$-morphism $J : A \to B$, the restriction $J(f, \text{id})$ exists in each of the double categories $\text{Alg}_w(T)$, by Proposition 3.1.

**Theorem 3.6.** Let $T$ be a normal oplax double monad on an equipment $K$ and let $w \in \{c, l, ps\}$. Consider a cell $\varepsilon$ in $\text{Alg}_w(T)$ as on the left below.

\[
\begin{array}{ccc}
A & \xrightarrow{J} & B \\
\text{cart} & \downarrow \varepsilon & \downarrow d \\
M & \xrightarrow{r} & M
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{J(f, \text{id})} & B \\
\text{cart} & \downarrow \varepsilon & \downarrow d \\
A & \xrightarrow{J} & B \\
\downarrow r & & \downarrow r \\
M & \xrightarrow{r} & M
\end{array}
\]

For the following conditions the implications $(a) \Leftrightarrow (b) \Rightarrow (c)$ hold, while $(c) \Rightarrow (a)$ holds as soon as $K$ has opcartesian tabulations.

(a) The cell $\varepsilon$ defines $r$ as the pointwise right Kan extension of $d$ along $J$;

(b) for all pseudo $T$-morphisms $f : C \to A$ the composite on the right above defines $r \circ f$ as the pointwise right Kan extension of $d$ along $J(f, \text{id})$;

(c) for all strict $T$-morphisms $f : C \to A$ the composite on the right above defines $r \circ f$ as the right Kan extension of $d$ along $J(f, \text{id})$.

**Proof.** The proof given in [Kou14] for Theorem 5.9, which has been restated in the present paper as Theorem 1.23, applies to the double categories $\text{Alg}_w(T)$ without needing any adjustment. This is because all conjoints and companions used in that proof exist in $\text{Alg}_w(T)$, as all of them are conjoints or companions of pseudo $T$-morphisms, and because, in the proof of the implication $(c) \Rightarrow (a)$, the projections of the lifted tabulations in $\text{Alg}_w(T)$ are strict $T$-morphisms; see Proposition 3.3.

Just like the proof of Theorem 1.23 applies to the double category $\text{Alg}_l(T)$, so does the proof of Proposition 1.24 (which is Proposition 5.10 of [Kou14]). We thus obtain the following result, which describes pointwise right Kan extensions along conjoints in $\text{Alg}_l(T)$ in terms of right Kan extensions in the 2-category $\text{Alg}_l(V(T))$, in a sense that is analogous to Street’s definition of pointwise Kan extension that was given in [Str74].

In the diagram on the right below $f/j$ denotes the ‘comma object’ of $f : C \to A$ and $j : B \to A$ in the 2-category $\text{Alg}_l(V(T))$, see Section 5.1 of [Kou14]. In terms of the double category $\text{Alg}_l(T)$, the comma object $f/j$ can be defined as the tabulation of the restriction $A(f, j)$; in particular $f/j$ exists whenever $f$ is a pseudo $T$-morphism and $K$ is an equipment that has tabulations, by Proposition 3.1 and Proposition 3.3.

**Proposition 3.7.** Let $T$ be a normal oplax double monad on an equipment $K$ that has opcartesian tabulations. Consider a vertical $T$-cell $\varepsilon$ between lax $T$-morphisms, as on the left below, as well as its factorisation through the opcartesian cell defining $j^*$, as shown.

\[
\begin{array}{ccc}
B & \xrightarrow{j} & B \\
\text{opcart} & \downarrow \varepsilon & \downarrow d \\
A & \xrightarrow{r} & M \\
\downarrow r & & \downarrow \varepsilon \\
M & \xrightarrow{r} & M
\end{array}
\quad
\begin{array}{ccc}
f/j & \xrightarrow{\pi_c} & C \\
\text{opcart} & \downarrow \varepsilon' & \downarrow f \\
B & \xrightarrow{j} & A \\
\downarrow d & & \downarrow r \\
M & \xrightarrow{r} & M
\end{array}
\]
The factorisation $\varepsilon'$ defines $r$ as the pointwise right Kan extension of $d$ along $j^*$ in $\text{Alg}_g(T)$ precisely if, for each strict $T$-morphism $f : C \to A$, the composite on the right above defines $r \circ f$ as the right Kan extension of $d \circ \pi_B$ along $\pi_C$ in $\text{Alg}_g(V(T))$.

Example 3.8. The ‘free strict monoidal category’-monad (Example 2.10) on the equipment $\text{Prof}$ of (unenriched) profunctors satisfies the hypotheses of the proposition above. Thus a monoidal transformation $\varepsilon : r \circ j \Rightarrow d$ of lax monoidal functors, as on the left above, defines $r$ as the pointwise Kan extension of $d$ along $j$ if, for every strict monoidal functor $f : C \to A$, the composite on the right above defines $r \circ f$ as the right Kan extension of $d \circ \pi_B$ along $\pi_C$. In this case $f/j$ is the usual ‘comma category’ that has triples $(z,u,y)$ as objects, where $z \in C$, $y \in B$ and $u : fz \to jy$ in $A$. Its monoidal structure, that is induced by that of $C$ and $B$, maps a sequence $(z_1, \cdots, z_n)$ to the triple consisting of the tensor products $(z_1 \otimes \cdots \otimes z_n)$ and $(y_1 \otimes \cdots \otimes y_n)$ as well as the map $f(z_1 \otimes \cdots \otimes z_n) = (fz_1 \otimes \cdots \otimes fz_n) \xrightarrow{(u_1 \otimes \cdots \otimes u_n)} (jy_1 \otimes \cdots \otimes jy_n) \xrightarrow{j} j(y_1 \otimes \cdots \otimes y_n)$.

4 Lifting algebraic Kan extensions

In this section we prove our main result, which gives conditions allowing Kan extensions to be lifted along the forgetful double functors $\text{Alg}_g(T) \to K$. As an application we recover and generalise a result of Getzler, given in [Get09], on the lifting of symmetric monoidal pointwise left Kan extensions.

We start by making the idea of ‘lifting Kan extensions’ precise.

Definition 4.1. Consider a normal lax double functor $F : K \to L$ as well as morphisms $J : A \to B$ and $d : B \to M$ in $L$. We say that $F$ lifts the (pointwise) right Kan extension of $d$ along $J$ if, given any cell $\varepsilon$ in $L$ on the left below, that defines $r$ as the (pointwise) right Kan extension of $Fd$ along $FJ$, there exists a unique cell $\varepsilon'$ in $K$ on the right such that $F\varepsilon' = \varepsilon$ and, moreover, $\varepsilon'$ defines $r'$ as the (pointwise) right Kan extension of $d$ along $J$.

\[
\begin{array}{ccc}
FA & FJ & FB \\
\varepsilon & \down{F} \varepsilon & \down{Fd} \\
FM & FM & FM
\end{array}
\quad
\begin{array}{ccc}
A & J & B \\
\varepsilon' & \down{\varepsilon'} & \down{d} \\
M & M & M
\end{array}
\]

To state the main theorem we need Definitions 4.2 and 4.5.

Definition 4.2. Let $d : B \to M$, $f : M \to N$ and $J : A \to B$ be morphisms in a double category $K$. We say that $f$ preserves the (pointwise) right Kan extension of $d$ along $J$ if, for any cell $\varepsilon$ on the left below that defines $r$ as the (pointwise) right Kan extension of $Fd$ along $FJ$, there exists a unique cell $\varepsilon'$ in $K$ on the right such that $F\varepsilon' = \varepsilon$ and, moreover, $\varepsilon'$ defines $r'$ as the (pointwise) right Kan extension of $d$ along $J$.

\[
\begin{array}{ccc}
A & J & B \\
\varepsilon & \down{J} \varepsilon & \down{d} \\
M & M & M
\end{array}
\quad
\begin{array}{ccc}
A & J & B \\
\varepsilon & \down{\varepsilon} & \down{d} \\
M & M & M
\end{array}
\quad
\begin{array}{ccc}
TA & TJ & TB \\
\varepsilon & \down{\varepsilon} & \down{Td} \\
TM & TM & TM
\end{array}
\quad
\begin{array}{ccc}
\varepsilon & \down{\varepsilon} & \down{d} \\
M & M & M
\end{array}
\quad
\begin{array}{ccc}
\varepsilon & \down{\varepsilon} & \down{d} \\
M & M & M
\end{array}
\quad
\begin{array}{ccc}
\varepsilon & \down{\varepsilon} & \down{d} \\
M & M & M
\end{array}
\]

Next let $T$ be a normal oplax double monad on $K$ and assume that $M = (M,m,\bar{m},\hat{m})$ is a lax $T$-algebra. As a variation on the above, we say that the
algebraic structure of \( M \) preserves the (pointwise) right Kan extension of \( d \) along \( J \) if for any cell \( \varepsilon \) as on the left above, that defines \( r \) as the (pointwise) right Kan extension of \( d \) along \( J \), the composite \( 1_m \circ T \varepsilon \) on the right defines \( m \circ T r \) as the (pointwise) right Kan extension of \( m \circ T d \) along \( T J \). 

Recall from Proposition \([1.19]\) that, in the double category \( \mathcal{V} \text{-Prof} \), pointwise right Kan extensions along \( \mathcal{V} \)-weights \( J : 1 \twoheadrightarrow B \) coincide with \( J \)-weighted limits. Consequently, we say that a \( \mathcal{V} \)-functor \( f : M \to N \) preserves \( J \)-weighted limits if it preserves all pointwise right Kan extensions along \( J \). That this coincides with the classical notion of ‘preserving weighted limits’, see e.g. Section 3.2 in \([\text{Kel}82]\), in the case that \( \mathcal{V} \) is complete and closed symmetric monoidal, follows from Proposition \([1.18]\). Notice that a \( \mathcal{V} \)-functor preserves all pointwise right Kan extensions whenever it preserves weighted limits, again by Proposition \([1.19]\).

**Example 4.3.** In anticipation of Proposition \([4.8]\) we describe monoidal \( \mathcal{V} \)-categories whose monoidal structure preserves pointwise left Kan extensions. Consider a monoidal \( \mathcal{V} \)-category \( M = (M, \otimes, a, i) \) (Example \([2.18]\)), a \( \mathcal{V} \)-functor \( d : A \to M \) and a \( \mathcal{V} \)-profunctor \( J : A \leftrightarrow B \); we claim that the algebraic structure of \( M \) preserves the pointwise left Kan extension of \( d \) along \( J \) whenever the \( n \)-ary tensor products \( \otimes_n = [M \otimes^n \xrightarrow{j_n} TM \otimes M] \) preserve it in every variable, for each \( n \geq 1 \), where \( j_n : (-) \otimes^n \Rightarrow T \) denotes the double transformation that includes the \( n \)th tensor powers into \( T \). Of course, because the components of \( a \) and \( i \) are isomorphisms, it suffices that the tensor product \( \odot_2 \) preserves this pointwise left Kan extension in either variable. Moreover, by the discussion above, it also suffices that \( \odot_2 \) preserves weighted colimits.

To prove the claim we consider a transformation \( \eta \), as on the left below, that defines \( l \) as the pointwise left Kan extension of \( d \) along \( J \) (see Definition \([1.16]\)). We have to show that the composite \( 1_\otimes \circ T \eta \) defines \( \otimes \circ T l \) as a pointwise left Kan extension. By Lemma \([4.4]\) below we may equivalently prove that the composites \( 1_\otimes \circ \eta \otimes^n \) in the middle define \( \otimes \circ l \otimes^n \) as pointwise left Kan extensions, for each \( n \geq 1 \).

\[
\begin{array}{ccc}
A \otimes^n & \xrightarrow{J \otimes^n} & B \otimes^n \\
\downarrow d \otimes^n & \quad & \downarrow j \otimes^n \\
M \otimes^n & \otimes \eta \otimes^n & \otimes M \\
\downarrow \otimes_n & \quad & \downarrow \otimes_n \\
M & \otimes M & M
\end{array}
\]

To see this consider, for each \( 0 \leq i < n \), the transformation on the right above, where \( i' = i + 1 \). That it defines \( \otimes \circ (l \otimes \otimes d \otimes (n-i')) \) as a pointwise left Kan extension follows, by using Proposition \([2.13]\) twice, from the fact that for all objects \( y_1, \ldots, y_i \in B \) and \( x_{i+2}, \ldots, x_n \in A \) the transformations

\[
(l y_1 \otimes \cdots \otimes l y_i \otimes - \otimes d x_{i+2} \otimes \cdots \otimes d x_n) \circ \eta
\]

define pointwise Kan extensions, by the assumption on \( \otimes_n \). Therefore, by repeatedly applying Proposition \([2.13]\) to the horizontal composite of the transformations on the right above, we can conclude that the composite \( 1_\otimes \circ \eta \otimes^n \) defines \( \otimes \circ l \otimes^n \) as a pointwise left Kan extension, as was needed.

Let \( \mathcal{V} \) be a cocomplete symmetric monoidal category whose tensor product preserves colimits in both variables, so that \( \mathcal{V} \)-profunctors form a monoidal double
category $\mathcal{V}$-$\text{Prof}$ (Example 2.12), and let $T$ denote the ‘free strict monoidal $\mathcal{V}$-category’-monad on $\mathcal{V}$-$\text{Prof}$ (Example 2.10). For each $n \geq 0$ we denote by $j_n : (-)^{\otimes n} \Rightarrow T$ the double transformation that includes the $n$th tensor powers into $T$.

**Lemma 4.4.** Let $\mathcal{V}$ and $j_n : (-)^{\otimes n} \Rightarrow T$ be as above. A cell

$$
\begin{array}{ccc}
TA & \xrightarrow{TJ} & TB \\
\downarrow{d} & & \downarrow{\eta} \\
M & \xrightarrow{j_n} & M
\end{array}
$$

defines $l$ as the pointwise left Kan extension of $d$ along $TJ$ if and only if, for each $n \geq 0$, the composite $J^{\otimes n} \xrightarrow{(j_n)_J} TJ \xrightarrow{\eta} 1_M$ defines $l \circ j_n$ as the pointwise left Kan extension of $d \circ j_n$ along $J^{\otimes n}$.

**Proof.** For each $y \in TB$ of length $n$ consider the identity below.

$$
\begin{array}{ccc}
A^{\otimes n} & \xrightarrow{TJ(j_n, y)} & 1 \\
\downarrow{j_n} & & \downarrow{\eta} \\
TA & \xrightarrow{TJ(id, y)} & 1 \\
\downarrow{\text{cart}} & & \downarrow{\text{cart}} \\
M & \xrightarrow{j_n} & M
\end{array}
$$

In light of Proposition 1.19 it suffices to prove that the composite of the bottom two cells on the left-hand side defines $l(y)$ as a pointwise left Kan extension precisely if the full composite on the right-hand side defines $(l \circ j_n)(y)$ as one. It is readily checked that the top cartesian cell on the left-hand side is opcartesian, so that this equivalence follows from (the horizontal dual of) Proposition 1.27(a).

**Definition 4.5.** A normal oplax double monad $T = (T, \mu, \eta)$ is called (pointwise) right exact if the cells $\mu_J$ and $\eta_J$ are (pointwise) right exact for each horizontal morphism $J : A \Rightarrow B$ (see Definition 1.25).

**Example 4.6.** The ‘free strict monoidal $\mathcal{V}$-category’-monad $T$ of Example 2.10 is both pointwise left and right exact. For example, to show that the cell $\mu_J$, where $J : A \Rightarrow B$ is a $\mathcal{V}$-profunctor, is pointwise left exact it suffices, by (the horizontal dual of) Proposition 1.27(b), to show that the factorisation $\mu'_J$ of $\mu_J$ through the cartesian cell defining $TJ(id, \mu_B)$, as in the right-hand side below, is opcartesian.

$$
\begin{array}{ccc}
T^2A & \xrightarrow{T^2J} & T^2B \\
\downarrow{\mu_A} & & \downarrow{\mu'_{\mu_B}} \\
TA & \xrightarrow{\mu} & T^2B \\
\downarrow{f} & & \downarrow{\mu'_{\mu_B}} \\
C & \xrightarrow{\kappa} & D
\end{array}
$$

To see this consider a cell $\chi$ as above, we have to show that it factors uniquely through $\mu'_J$ as shown. Given $x \in TA$ and $y \in T^2B$ notice that, if $|x| = |\mu y|$, then there exists a unique $\overline{x} \in T^2A$ with $\mu \overline{x} = x$, $|\overline{x}| = |y|$ and $|\overline{x}| = |\overline{y}|$ for all
1 \leq i \leq |\mu y|$, in this case we take as the $V$-map $\chi_{(x, y)}: TJ(x, \mu y) \rightarrow K(f x, gy)$ the composite

$$TJ(x, \mu y) \cong T^2 J(x, y) \xrightarrow{\cong} K(f x, gy),$$

where the isomorphism is the inverse of $(\mu_J)_{(x, y)}$; see (8). In the case that $|x| \neq |\mu y|$, we must take $\chi_{(x, y)}$ to be the unique map $\emptyset \rightarrow K(f x, gy)$. It is clear that, defined like this, $\chi'$ forms the unique factorisation of $\chi$ through $\mu'_J$.

We are now ready to state and prove our main result. Its horizontal dual Theorem 4.7 co, which concerns (pointwise) left Kan extensions between colax $T$-algebras, has already been stated in the introduction where, to simplify exposition, it has been restricted to pseudo $T$-algebras and pointwise left Kan extensions.

**Theorem 4.7.** Let $T$ be a right exact, normal pseudo double monad on a double category $K$ and let ‘weak’ mean either ‘colax’, ‘lax’ or ‘pseudo’. Given lax $T$-algebras $A$, $B$ and $M$, consider the following conditions on a horizontal $T$-morphism $J: A \rightarrow B$ and a weak vertical $T$-morphism $\epsilon: B \rightarrow M$:

\begin{enumerate}
  \item[(p)] the algebraic structure of $M$ preserves the right Kan extension of $\epsilon$ along $J$;
  \item[(e)] the structure cell of $J$ is right $\epsilon$-exact;
  \item[(l)] the forgetful double functor $Alg_w(T) \rightarrow K$ lifts the right Kan extension of $\epsilon$ along $J$.
\end{enumerate}

The following hold:

\begin{enumerate}
  \item[(a)] if ‘weak’ means ‘colax’ then (p) implies (l);
  \item[(b)] if ‘weak’ means ‘lax’ then (e) implies (l);
  \item[(c)] if ‘weak’ means ‘pseudo’ then any two of (p), (e) and (l) imply the third.
\end{enumerate}

If $T$ is pointwise right exact then analogous results hold for pointwise Kan extensions, obtained by replacing ‘right Kan extension’ in (p) and (l) by ‘pointwise right Kan extension’, and ‘right $\epsilon$-exact’ in (e) by ‘pointwise right $\epsilon$-exact’.

**Proof.** We treat the ordinary and pointwise case simultaneously. As usual we denote the algebra structure on $A$ by $A = (A, a, \bar{a}, \hat{a})$, where $a: TA \rightarrow A$ is the structure map, $\bar{a}: a \circ Ta \Rightarrow a \circ \mu_A$ is the associator and $\hat{a}: id_A \Rightarrow a \circ \eta_A$ is the unitor; likewise $B = (B, b, \bar{b}, \hat{b})$ and $M = (M, m, \bar{m}, \hat{m})$, while the structure cell $J$ of $J$ is of the form as on the left below.

\[
\begin{array}{c}
TA \xrightarrow{TJ} TB \\
A \xrightarrow{\bar{a}} \bar{b} \xrightarrow{\hat{b}} B \\
A \xrightarrow{J} B
\end{array}
\quad
\begin{array}{c}
A \xrightarrow{J} B \\
M \xrightarrow{\bar{m}} \bar{m} \xrightarrow{\hat{m}} M
\end{array}
\]

To start we consider a cell $\epsilon$ as on the right above and assume that it defines $\epsilon$ either as the ordinary right Kan extension of $\epsilon$ along $J$ or as the pointwise right Kan extension of $\epsilon$ along $J$.

**Part (a).** If ‘weak’ means ‘colax’ then the algebraic structure on $\epsilon$ is given by a cell $\epsilon: \epsilon \circ b \Rightarrow m \circ Td$ as in the composite on the left-hand side below. Assuming that condition (p) holds, the second column in the right-hand side below defines $m \circ Tr$ as the (pointwise) right Kan extension of $m \circ Td$ along $TJ$, see Definition [4,2].
so that in both the ordinary and the pointwise case the composite on the left factors uniquely as a vertical cell \( r \): \( r \circ a \Rightarrow m \circ T r \), as shown.

\[
\begin{array}{c}
TA \xrightarrow{TJ} TB \\
A \xrightarrow{a} B \\
M \xrightarrow{r} M
\end{array} \quad \begin{array}{c}
TB \xrightarrow{Td} TB \\
B \xrightarrow{b} B \\
M \xrightarrow{d} M
\end{array}
\quad \begin{array}{c}
TA \xrightarrow{TJ} TB \\
A \xrightarrow{a} A \\
M \xrightarrow{r} M
\end{array} \quad \begin{array}{c}
TB \xrightarrow{Td} TB \\
B \xrightarrow{b} B \\
M \xrightarrow{d} M
\end{array}
\]

We claim that the cell \( r \) makes \( r \) into a colax \( T \)-morphism, that is it satisfies the associativity and unit axioms of Definition 2.16. To see that it satisfies the former consider the equation of composites below, whose left-hand and right-hand sides are given by the corresponding sides of the associativity axiom for \( r \), composed with \( T^2 \varepsilon \) on the right. The identities follow from the \( T \)-image the of factorisation \( \{13\} \); the factorisation \( \{13\} \) itself; the associativity axiom for \( d \); the associativity axiom for \( J \); again \( \{13\} \); the naturality of \( \mu \). Now notice that the fourth column of the left and right-hand sides equals \( 1_m \circ 1_{\mu_A} \circ T^2 \varepsilon = 1_m \circ T \varepsilon \circ \mu_J \) which, because \( \mu_J \) is assumed to be (pointwise) right exact, defines \( m \circ T r \circ \mu_A \) as the (pointwise) right Kan extension of \( m \circ T d \circ \mu_B \) along \( T^2 J \). We conclude that the associativity axiom for \( r \) holds after it is composed on the right with a cell defining a right Kan extension; since factorisations through such cells are unique the axiom itself follows.

The unit axiom for \( r \) is proved similarly: we consider the equation below, whose left and right-hand sides coincide with those of the unit axiom for \( r \) composed with \( \varepsilon \) on the right. The identities follow from the unit axiom for \( d \), the unit axiom for \( J \), the factorisation \( \{13\} \) and the naturality of \( \eta \). As before, the unit axiom for \( r \) follows from the fact that the last columns in the left and right-hand sides below are given by \( 1_m \circ 1_{\mu_A} \circ \varepsilon = 1_m \circ T \varepsilon \circ \eta_J \), which defines \( m \circ T r \circ \eta_A \) as the (pointwise) right Kan extension of \( m \circ T d \circ \eta_B \) along \( J \), because \( \eta_J \) is (pointwise) right exact.
We conclude that the cell \( \bar{r} \) forms a colax structure cell for \( r: B \to M \). Moreover, notice that with this algebra structure on \( r \) the factorisation (13) forms the \( T \)-cell axiom for \( \varepsilon \). In fact, since this factorisation is unique, this is the only way to lift the \( K \)-cell \( \varepsilon \) to \( \text{Alg}_c(T) \).

Having obtained an algebra structure on \( r \) it remains to show that \( \varepsilon \), as a \( T \)-cell, defines \( r \) as the (pointwise) right Kan extension of \( d \) along \( J \). To see this consider a \( T \)-cell \( \phi \) as on the left below where, in the case of ordinary Kan extensions, we assume \( H: C \to A \) to be the horizontal unit \( 1_A \). Since \( \varepsilon \) defines \( r \) as the (pointwise) right Kan extension in \( K \) we obtain a unique factorisation \( \phi' \) as shown below; we have to prove that \( \phi' \) is a \( T \)-cell.

For general \( H: C \to A \) consider the equation below, consisting of the following identities. The first is given by the \( T \)-image of the factorisation above precomposed with \( T_\circ: TH \circ TJ \Rightarrow T(H \circ J) \), where we have also used the naturality of \( T_\circ \).

The second follows from the \( T \)-cell axiom of \( \phi \); remember that the structure cell of \( H \circ J \) is given by \( \bar{H} \circ \bar{J} = (\bar{H} \circ \bar{J}) \circ T_\circ^{-1} \), see (10). The third follows from the factorisation (13).

In case \( H \) is the horizontal unit \( 1_A \) the same argument can be used, after adjusting the equation above by replacing the second term by the composite \( \bar{s} \circ (1_m \circ T\phi) \) and, in the third and fourth term, the cell \( \bar{H} \) by the horizontal identity \( 1_A \). This finishes the proof of part (a).

**Part (b).** In the case that \( d \) is a lax \( T \)-morphism, with structure cell \( \bar{d}: m \circ Td \Rightarrow d \circ b \), and the structure cell \( \bar{J} \) of \( J \) is (pointwise) right exact, so that the second column in the right-hand side below defines \( r \circ a \) as the (pointwise) right Kan extension of \( d \circ b \) along \( TJ \), then the composite on the left below factors uniquely as a cell \( \bar{r}: m \circ Tr \Rightarrow r \circ a \) as shown.

Analogous to the proof of part (a) above, we can show that the cell \( \bar{r} \) satisfies the associativity and unit axioms, so that it makes \( r: A \to M \) into a lax \( T \)-morphism.

Indeed, the associativity axiom follows from the identity below, which itself follows from the associativity axiom for \( \bar{J} \), the factorisation above, the associativity axiom for \( \bar{d} \) and the naturality of \( \mu \). As before notice that the identity below is the associativity axiom for \( \bar{r} \) composed on the right with the composite \( \varepsilon \circ \bar{J} \circ \mu_J \).
This composite defines a (pointwise) right Kan extension, since both \( \mu_J \) and \( \bar{J} \) are assumed to be (pointwise) right exact, so that the associativity axiom itself follows.

The unit axiom for \( \bar{r} \) follows in the same way from the identity on the left below, where the identities are given by the unit axioms for \( \bar{J} \) and \( \bar{d} \), the factorisation above and the naturality of \( \eta \).

Having given \( r: A \rightarrow M \) a lax \( T \)-morphism structure, we notice that the factorisation \((15)\) forms the \( T \)-cell axiom for \( \varepsilon \). It remains to show that, as a \( T \)-cell, \( \varepsilon \) defines \( r \) as the (pointwise) right Kan extension of \( d \) along \( J \). To do so we consider a \( T \)-cell \( \phi \) as in \((14)\), where now \( s \) and \( d \) are lax \( T \)-morphisms. Because \( \varepsilon \) defines \( r \) as a (pointwise) right Kan extension, \( \phi \) factors uniquely through \( \varepsilon \) as \( \phi' \), as in \((14)\), and we have to show that \( \phi' \) is a \( T \)-cell. That the \( T \)-cell axiom for \( \phi' \) holds when composed with \( \varepsilon \circ J \) on the right, as on the right above, follows from the \( T \)-cell axiom for \( \varepsilon \) \((15)\), the factorisation \( \bar{\phi} = \phi' \circ \varepsilon \) and the \( T \)-cell axiom for \( \phi \). Since \( \varepsilon \circ J \) defines a (pointwise) right Kan extension the \( T \)-cell axiom for \( \phi' \) follows. This concludes the proof of part (b).

Part (c). In this case \( d \) is a pseudo \( T \)-morphism, with invertible structure cell \( \bar{d}: m \circ T\bar{d} \Rightarrow d \circ b \). If two of (p), (e) and (l) hold then one of (p) and (e) must hold. Suppose that (e) holds; we will show that (p) and (l) are equivalent. By part (b) the cell \( \varepsilon \), that defines \( r \) as a right Kan extension, lifts along \( \text{Alg}_M(T) \rightarrow K \) to a \( T \)-cell defining \( (r, \bar{r}) \) as a right Kan extension in \( \text{Alg}_M(T) \), where \( \bar{r}: m \circ T\bar{r} \Rightarrow r \circ a \) is uniquely determined by the identity \((15)\). If (p) holds then, using the fact that \( \bar{d} \) is invertible, the left-hand side of \((15)\) defines the right Kan extension of \( d \circ b \) along \( TJ \). Because the second column in the right hand side of \((15)\) does too, it follows that the factorisation \( \bar{r} \) is invertible, so that the lift \( (r, \bar{r}) \) of \( r \) is a pseudomorphism; we conclude that condition (l) holds.

On the other hand, if (l) holds then \( r \) can be given the structure of a pseudo \( T \)-morphism such that \( \varepsilon \) forms a \( T \)-cell that defines \( r \) as a right Kan extension in \( \text{Alg}_{\text{ps}}(T) \). Since the lift of \( \varepsilon \) that we obtained by applying part (b) is unique, the structure cell \( m \circ T\bar{r} \Rightarrow r \circ a \) of the lift of \( r \) to \( \text{Alg}_{\text{ps}}(T) \) must coincide with the structure cell \( \bar{r} \) that we obtained in \((15)\). Thus \( \bar{r} \) is invertible and hence, by composing \((15)\) on the right with \( \bar{d}^{-1} \), we see that \( 1_m \circ T\varepsilon \) defines \( m \circ Tr \) as a right Kan extension. This means that the algebraic structure of \( M \) preserves the right Kan extension of \( d \) along \( J \), that is (p) holds. We conclude that if (e) holds then (p) and (l) are equivalent. In the same way one can show that the equivalence of (e) and (l) follows from (p), which completes the proof of part (c).

In the proposition below we apply Theorem \([57]^{\circ} \), that was stated in the introduction, to the double monad of free strict monoidal \( V \)-categories. Consider a (symmetric) monoidal \( V \)-profunctor \( J: A \rightarrow B \) (Example \([2.2]^{\circ} \)). Horizontally dual to the factorisation that was considered in Proposition \([1.27]^{\circ} \) its structure transformation \( J_\circ \) factors uniquely as a transformation \((J_\circ)^*: \otimes \circ TJ \Rightarrow J(\text{id}, \otimes)\), of
\( \mathcal{V} \)-profunctors \( A \to TB \), that is induced by the \( \mathcal{V} \)-maps

\[
A(x, (y_1 \otimes \cdots \otimes y_n)) \otimes J(y_1, z_1) \otimes \cdots \otimes J(y_n, z_n) \xrightarrow{id \otimes J_B} A(x, (y_1 \otimes \cdots \otimes y_n)) \otimes J((y_1 \otimes \cdots \otimes y_n), (z_1 \otimes \cdots \otimes z_n)) \xrightarrow{\lambda} J(x, (z_1 \otimes \cdots \otimes z_n))
\]

where \( x \in A, y \in TA \) and \( z \in TB \). We say that \( J \) satisfies the left Beck-Chevalley condition if the transformation \( (J_\otimes)^* \) is an isomorphism.

In the proposition below \( \mathcal{V}-\text{MonProf}_w \) (respectively \( \mathcal{V}-\text{sMonProf}_w \)) denotes the double category of (symmetric) monoidal \( \mathcal{V} \)-categories, (symmetric) weak monoidal \( \mathcal{V} \)-functors, (symmetric) monoidal \( \mathcal{V} \)-profunctors and their transformations. For a double functor \( F : \mathcal{K} \to \mathcal{L} \), a horizontal morphism \( J : A \to B \) and an object \( M \) in \( \mathcal{L} \), we say that \( F \) lifts all pointwise left Kan extensions along \( J \) into \( M \) if, for every vertical morphism \( d : A \to M \) in \( \mathcal{L} \), \( F \) lifts the pointwise left Kan extension of \( d \) along \( J \).

**Proposition 4.8.** Let \( \mathcal{V} \) be a cocomplete symmetric monoidal category whose tensor product preserves colimits on both sides, let \( A, B \) and \( M \) be monoidal \( \mathcal{V} \)-categories and let \( J : A \to B \) be a monoidal \( \mathcal{V} \)-profunctor. Consider the following conditions:

1. (p) the tensor product \( \otimes_2 : M \otimes M \to M \) preserve pointwise left Kan extensions along \( J \) in each variable;
2. (e) \( J \) satisfies the left Beck-Chevalley condition.

If condition (e) holds then the forgetful double functor \( \mathcal{V}-\text{MonProf}_c \to \mathcal{V}-\text{Prof} \) lifts all pointwise left Kan extensions along \( J \) into \( M \). If condition (p) holds then so does \( \mathcal{V}-\text{sMonProf}_c \to \mathcal{V}-\text{Prof} \); in the case that both conditions hold the same follows for \( \mathcal{V}-\text{sMonProf}_w \to \mathcal{V}-\text{Prof} \).

Analogous assertions hold for the double functors \( \mathcal{V}-\text{sMonProf}_w \to \mathcal{V}-\text{Prof} \), for each \( w \in \{c, l, ps\} \), in the case that \( A, B \) and \( M \) are symmetric monoidal \( \mathcal{V} \)-categories and \( J : A \to B \) is a symmetric monoidal \( \mathcal{V} \)-profunctor.

**Proof.** The assertions for the forgetful double functors \( \mathcal{V}-\text{MonProf}_w \to \mathcal{V}-\text{Prof} \) follow immediately from applying Theorem 1.27 to the ‘free strict monoidal \( \mathcal{V} \)-category’-monad \( T \) of Example 2.10. Indeed, \( T \) is pointwise left exact (Example 1.6) and the conditions (p) and (e) above imply the corresponding conditions in Theorem 1.27, by Example 1.23 and Proposition 1.27b).

To see that the assertions remain true in the symmetric case we claim that, for each choice of \( w \in \{c, l, ps\} \), any lift of a pointwise left Kan extension along the forgetful functor \( \mathcal{V}-\text{MonProf}_w \to \mathcal{V}-\text{Prof} \) can be further lifted along the forgetful functor \( \mathcal{V}-\text{sMonProf}_w \to \mathcal{V}-\text{MonProf}_w \). We will prove this claim in the case that \( w = l \), in the other cases it can be proved analogously. Hence consider symmetric monoidal \( \mathcal{V} \)-categories \( A, B \) and \( M \), a symmetric lax monoidal \( \mathcal{V} \)-functor \( d : A \to M \) and a symmetric monoidal \( \mathcal{V} \)-profunctor \( J : A \to B \), and assume that in \( \mathcal{V}-\text{Prof} \) the transformation \( \eta \) on the left below defines a \( \mathcal{V} \)-functor \( l : B \to M \) as the pointwise left Kan extension of \( d \) along \( J \).

\[
\begin{array}{cccccc}
A & \xrightarrow{d} & B & \xrightarrow{\eta} & M & \xrightarrow{l} & M \\
T \downarrow & T \downarrow & T \downarrow & T \downarrow & T \downarrow & T \downarrow & T \downarrow \\
T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 \\
T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 \\
T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 \\
T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 \\
T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 & T \otimes_2 \\
\end{array}
\]

If the condition (p) holds then \( \lambda \circ \eta \) defines \( \otimes \circ Tl \) as a pointwise left Kan extension and, horizontally dual to (13), the lax monoidal structure \( l_\otimes : \otimes \circ Tl \Rightarrow l \circ \otimes \) that
makes \( l \) into a lax monoidal \( \mathcal{V} \)-functor is obtained as the unique factorisation on the right above.

\[
\begin{array}{ccc}
B^\otimes n & \Rightarrow & B^\otimes n & \Rightarrow & B^\otimes n & \Rightarrow & B^\otimes n \\
\downarrow{\otimes_n} & & \downarrow{\otimes_n} & & \downarrow{\otimes_n} & & \downarrow{\otimes_n} \\
M^\otimes n & \Rightarrow & B & \Rightarrow & B & \Rightarrow & B \\
\downarrow{j \otimes_n} & & \downarrow{j \otimes_n} & & \downarrow{j \otimes_n} & & \downarrow{j \otimes_n} \\
\otimes_n & \Rightarrow & B & \Rightarrow & B & \Rightarrow & B \\
\downarrow{l \otimes_n} & & \downarrow{l \otimes_n} & & \downarrow{l \otimes_n} & & \downarrow{l \otimes_n} \\
M & \Rightarrow & M & \Rightarrow & M & \Rightarrow & M
\end{array}
\]

We claim that \( l \otimes \) is compatible with the symmetries on \( B \) and \( M \), in the sense that it makes a diagram like \( \Box \) commute. Rewritten in terms of vertical cells in \( \mathcal{V} \)-\textit{Prof}, this means that the identity above holds; here \( l \otimes_n = l \otimes 1_n \), where \( j_n: (-) \otimes n \Rightarrow T \) denotes the double transformation that includes the \( n \)th tensor powers into \( T \). The following equation, where \( d \otimes_n = d \otimes 1_n \), \( J \otimes_n = J \otimes (j_n)_! \) and whose identities are described below, shows that the two sides above coincide after composition on the left with \( 1 \otimes \eta^{-1} \). The latter defines a pointwise left Kan extension because \( 1 \otimes T \eta \) does, by Lemma \( \ref{lem:pointwise-left-Kan-extension} \) so that the identity above follows.

\[
\begin{array}{ccc}
\eta \otimes_n & \Rightarrow & d \otimes_n & \Rightarrow & J \otimes_n \\
\downarrow{s_n} & & \downarrow{s_n} & & \downarrow{s_n} \\
\sigma_j & \Rightarrow & \eta & \Rightarrow & \eta
\end{array}
\]

\[
\begin{array}{ccc}
\eta \otimes_n & \Rightarrow & d \otimes_n & \Rightarrow & J \otimes_n \\
\downarrow{s_n} & & \downarrow{s_n} & & \downarrow{s_n} \\
\sigma_j & \Rightarrow & \eta & \Rightarrow & \eta
\end{array}
\]

The identities above follow from the composite of \( \Box \) with \( (j_n)_! \); the symmetry axiom for \( J \) (see Example \( \ref{ex:example-of-symmetry-axiom} \)); the symmetry axiom for \( d \); \( \Box \) composed with \( (j_n)_! \) again; the naturality of \( \sigma \).

We conclude that \( l \), as a lax monoidal \( \mathcal{V} \)-functor, has a unique lift along the forgetful double functor \( U: \mathcal{V} \text{-sMonProf}_w \Rightarrow \mathcal{V} \text{-MonProf}_w \). That \( \eta \), as a cell in \( \mathcal{V} \text{-sMonProf}_w \), defines this lift as the pointwise left Kan extension of \( d \) along \( J \) follows readily from the fact that \( U \) restricts to the identity on cells. This completes the proof.

As a corollary to the previous proposition we can now recover Proposition 2.3 of \( \text{[Get09]} \). Given a symmetric monoidal \( \mathcal{V} \)-functor \( j: A \Rightarrow B \) recall that the representable \( \mathcal{V} \)-functor \( j_!: A \Rightarrow B \) is monoidal, by Proposition \( \ref{prop:representable-monoidal} \). It satisfies the left Beck-Chevalley condition if the transformation \( (j_!)_\lor: \otimes \circ Tj_! \Rightarrow B(j_!, \otimes) \) of profunctors \( A \Rightarrow TB \), that is induced by the compositions

\[
A(x, (y_1 \otimes \cdots \otimes y_n)) \otimes B(jy_1, z_1) \otimes \cdots \otimes B(jy_n, z_n)
\]

\[
\begin{array}{c}
\mathbb{id} \otimes \sigma_{z_1} \Rightarrow A(x, (y_1 \otimes \cdots \otimes y_n)) \otimes B((jy_1 \otimes \cdots \otimes jy_n), (z_1 \otimes \cdots \otimes z_n)) \\
j \otimes \lambda_{z_1} = 1 \Rightarrow B(jx, (y_1 \otimes \cdots \otimes y_n)) \otimes B(j(y_1 \otimes \cdots \otimes y_n), (z_1 \otimes \cdots \otimes z_n)) \\
\lambda_\ast B(jx, (z_1 \otimes \cdots \otimes z_n))
\end{array}
\]
is invertible. To see what this means in the case of $\mathcal{V} = \mathbf{Set}$, first notice that the composite profunctor $\otimes^\ast \circ T_{j^*} : A \Rightarrow TB$, see Example 1.2, is given by the sets $(\otimes^\ast \circ T_{j^*})(x, (z_1, \ldots, z_n))$ forming equivalence classes of sequences of morphisms

$$(x, 1 \otimes \cdots \otimes y_n), jy_1 \xrightarrow{u_1} z_1, \ldots, jy_n \xrightarrow{u_n} z_n),$$

where $p \in A$ and $u_i \in B$, under the smallest equivalence relation $\sim$ that relates the pairs $(p, u_1, \ldots, u_n) \sim (p', u'_1, \ldots, u'_n)$ for which there exist maps $r_i : y_i \rightarrow y'_i$ in $A$ that make the diagrams

\[
\begin{array}{ccc}
(x, 1 \otimes \cdots \otimes y_n) & \xrightarrow{p} & (1 \otimes \cdots \otimes r_n) \\
\downarrow & & \downarrow \\
(p', 1 \otimes \cdots \otimes y'_n) & \xrightarrow{p'} & (1 \otimes \cdots \otimes r_n)
\end{array}
\quad \quad \text{and} \quad \quad
\begin{array}{ccc}
(jy_1, \ldots, jy_n) & \xrightarrow{jr_i} & jz_i \\
\downarrow & & \downarrow \\
(jy'_1, \ldots, jy'_n) & \xrightarrow{jr'_i} & jz_i
\end{array}
\]

commute, the latter for each $i = 1, \ldots, n$. In terms of $\sim$, the profunctor $j_*$ satisfies the left Beck-Chevalley condition if every morphism $v : jx \Rightarrow (z_1 \otimes \cdots \otimes z_n)$ in $B$ can be represented as

\[
\begin{array}{ccc}
jx & \xrightarrow{v} & (z_1 \otimes \cdots \otimes z_n) \\
\downarrow & & \downarrow \\
jy_1 \otimes \cdots \otimes jy_n & \xrightarrow{jr'_i} & jz_i
\end{array}
\]

where $p \in A$ and $u_i \in B$, and that any two such representations $(p, u_1, \ldots, u_n)$ and $(p', u'_1, \ldots, u'_n)$ are related under $\sim$.

In the corollary below we have denoted by $\mathcal{V}-\mathbf{sMonCat}$ the 2-category of symmetric monoidal $\mathcal{V}$-categories, symmetric monoidal $\mathcal{V}$-functors and monoidal $\mathcal{V}$-natural transformations.

**Corollary 4.9** (Getzler). Let $\mathcal{V}$ be a closed symmetric monoidal category that is both complete and cocomplete. Consider symmetric monoidal $\mathcal{V}$-categories $A$, $B$ and $M$, as well as a symmetric monoidal $\mathcal{V}$-functor $j : A \Rightarrow B$. If $j_* : A \Rightarrow B$ satisfies the left Beck-Chevalley condition and $M$ is cocomplete, such that its tensor product preserves $\mathcal{V}$-weighted colimits in each variable, then the functor

\[
\lambda_j : \mathcal{V}-\mathbf{sMonCat}(B, M) \rightarrow \mathcal{V}-\mathbf{sMonCat}(A, M)
\]

that is given by precomposition with $j$, admits a left adjoint that is given by pointwise left Kan extension along $j$.

**Proof.** It is well known that, for any morphism $j : A \rightarrow B$ in a 2-category $\mathcal{C}$, the functor $\lambda_j : \mathcal{C}(B, M) \rightarrow \mathcal{C}(A, M)$, given by precomposition with $j$, has a left adjoint precisely if all left Kan extensions along $j$ exist; see e.g. Section X.3 of [ML98] for the case $\mathcal{C} = \mathbf{Cat}$. It is also well known that all pointwise left Kan extensions into the $\mathcal{V}$-category $M$ (in the sense of Proposition 4.19) exist whenever it is cocomplete; see Proposition 4.33 of [Kel82]. By the assumptions on $M$ and $j_*$, we may apply the previous proposition to these pointwise left Kan extensions, which shows that they are lifted to the double category $\mathcal{V}-\mathbf{sMonProf}_{pr}$. Therefore, to conclude the proof, it suffices to recall that (pointwise) left Kan extensions along $j_*$ in $\mathcal{V}-\mathbf{sMonProf}_{pr}$ correspond to left Kan extensions along $j$ in $\mathcal{V}-\mathbf{sMonCat}$, by (the horizontal dual of) Proposition 4.14. \qed
5 Application: free bicommutative Hopf monoids

In this last section we use Proposition 4.9 above to obtain a left adjoint to the forgetful functor

$$\text{ubichopf}(M) \to \text{ucocComon}(M),$$

from the category of ‘unbiased’ bicommutative Hopf monoids, in a suitable symmetric monoidal category $M$, to the category of unbiased cocommutative comonoids in $M$. The idea is to regard these categories as categories of algebras of ‘PROPs’.

In a symmetric monoidal category, and whose operations may have multiple inputs and multiple outputs; the following formal definition is the unbiased variant of the original definition, that was given in Section 24 of [ML65].

**Definition 5.1.** A PROP $\mathcal{P}$ is a symmetric strict monoidal category $\mathcal{P}$ whose monoid of objects equals $\mathbb{N}$, one for each $k \geq 0$, and symmetries $s_\sigma : n \to n$, that are functorial in $\sigma \in \Sigma_n$, satisfying the following axioms. Firstly the tensor products restrict to addition $\left(n_1, \ldots, n_k\right) \mapsto n_1 + \cdots + n_k$ on objects, with 0 as unit, and they are strictly associative and unital in the sense that

$$((\xi_{11} \otimes \cdots \otimes \xi_{i_1m_1}) \otimes \cdots \otimes (\xi_{k1} \otimes \cdots \otimes \xi_{km_k})) = (\xi_{11} \otimes \cdots \otimes \xi_{km_k}),$$

for any double sequence of morphisms $\xi_{ij} : m_{ij} \to n_{ij}$ in $\mathcal{P}$, and $(\xi) = \xi$ for each single morphism $\xi : m \to n$. Secondly $(s_{\sigma_1} \otimes \cdots \otimes s_{\sigma_k}) = s_{(\sigma_1, \ldots, \sigma_k)}$, where $(\sigma_1, \ldots, \sigma_k)$ denotes the disjoint union of the permutations $\sigma_i \in \Sigma_{n_i}$, while the symmetry $s_{\tau(n_{11}, \ldots, n_{mk})} : n_1 + \cdots + n_k \to n_{\tau 1} + \cdots + n_{\tau k}$, for the block permutation $\tau(n_{11}, \ldots, n_{mk})$ that is induced by $\tau \in \Sigma_k$ (see Example 2.19), is required to be natural in each $n_{11}, \ldots, n_{mk}$.

**Example 5.2.** We denote by $\mathcal{F}$ the PROP that is (isomorphic to) the skeletal category of finite sets. More precisely, we think of a morphism $\xi : m \to n$ in $\mathcal{F}$ as being a function $\xi : [m] \to [n]$ between the ordinals $[m] = \{0 < \cdots < m - 1\}$ and $[n] = \{0 < \cdots < n - 1\}$. The tensor product $\left(\xi_{11} \otimes \cdots \otimes \xi_{kn}\right)$ is given by disjoint union:

$$[m_1 + \cdots + m_k] \cong [m_1] \sqcup \cdots \sqcup [m_k] \mapsto \xi_{11} \mid \cdots \mid \xi_{kn} \mid [n_1] \sqcup \cdots \sqcup [n_k] \cong [n_1 + \cdots + n_k];$$

here the bijections are unique such that they preserve order, where the order of $[m_1] \sqcup \cdots \sqcup [m_k]$ is induced by the orders of each $[m_i]$ and by the rule that, for each pair $i < j$, all elements of $[m_i]$ are less than those of $[m_j]$. Finally, the symmetries $s_\sigma : n \to n$ are simply given by the permutations of $[n]$.

**Example 5.3.** We denote by $\mathbb{H}$ the PROP that is (isomorphic to) the skeletal category of free, finitely generated abelian groups. Precisely, a morphism $\xi : m \to n$ in $\mathbb{H}$ is a homomorphism $\xi : \mathbb{Z}^m \to \mathbb{Z}^n$, which we identify with an $n \times m$-matrix with coefficients in $\mathbb{Z}$. The monoidal structure on $\mathbb{H}$ is given by disjoint sum, so that the tensor product of morphisms $\xi_i : m_i \to n_i$ is identified with the block matrix

$$\begin{pmatrix}
  \xi_1 & 0 & \cdots & 0 \\
  0 & \xi_2 & \cdots & \cdots \\
  \vdots & \ddots & \ddots & 0 \\
  0 & \cdots & 0 & \xi_k
\end{pmatrix}.$$

The symmetries $s_\sigma : n \to n$ are given by permuting the generators of $\mathbb{Z}^n$. 

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The opposite $\mathbb{F}^{op}$ of the PROP $\mathbb{F}$, which is again a PROP, can be embedded into $\mathbb{H}$ by mapping each morphism $m \to n$, corresponding to a function $\xi: [n] \to [m]$, to the $n \times m$-matrix $(\xi_{ij})$ that is given by $\xi_{ij} = 1$ if $i = j$ and 0 otherwise. Thus the image of $\mathbb{F}^{op}$ in $\mathbb{H}$ is the subPROP consisting of all matrices that contain precisely one non-zero entry in each row, whose coefficient is 1. This gives a morphism of PROPs $j: \mathbb{F}^{op} \to \mathbb{H}$.

**Definition 5.4.** Let $\mathbb{P}$ be a PROP and $M = (M, \otimes, a, i)$ a symmetric monoidal category. An algebra $A$ of $\mathbb{P}$ in $M$ is a symmetric monoidal functor $A: \mathbb{P} \to M$. A morphism $f: A \to B$ of such algebras is a monoidal transformation $f: A \Rightarrow B$. In other words, the category of algebras of $\mathbb{P}$ in $M$ is defined as

$$\text{Alg}(\mathbb{P}, M) = \text{sMonCat}(\mathbb{P}, M),$$

where $\text{sMonCat}$ denotes the 2-category of symmetric monoidal categories, symmetric monoidal functors and monoidal transformations.

In describing algebras of PROPs it is useful to abbreviate $A \otimes^n = (A \otimes \cdots \otimes A)$.

**Example 5.5.** An algebra of the PROP $\mathbb{F}$ in $M$ is essentially an unbiased commutative monoid; that is, an object $A$ of $M$ equipped with a family $\mu_n$ of morphisms $\mu_n: A \otimes^n \to A$, one for each $n \geq 0$, that make the following diagrams commute

where on the right $\sigma$ is any permutation in $\Sigma_n$.

To be precise, every algebra $A: \mathbb{F} \to M$ induces the structure of an unbiased commutative monoid on $A(1)$, using the maps

$$A(1) \otimes^n \xrightarrow{\mu_{n_1} \otimes \cdots \otimes \mu_{n_k}} A \xrightarrow{id} A \xrightarrow{\mu_k} A \xrightarrow{A \id} A \xrightarrow{A \id} A \xrightarrow{\sigma} A \xrightarrow{\mu_n} A$$

where $!: [n] \to [1]$ denotes the unique map into the terminal set $[1]$. That this family satisfies the axioms above is easy to check. Moreover, it is not hard to show that the assignment $A \xrightarrow{\cdot} A(1)$ extends to an equivalence of categories

$$\text{Alg}(\mathbb{F}, M) \simeq \text{ucMon}(M),$$

where $\text{ucMon}(M)$ denotes the category of unbiased commutative monoids in $M$, in which a morphism $f: (A, \mu_A) \to (B, \mu_B)$ is simply a map $f: A \to B$ in $M$ such that $\mu_B \circ f^{\otimes n} = f \circ \mu_A$. In proving this it is helpful to notice that each morphism $\xi: m \to n$ in $\mathbb{F}$ factors as

$$m \xrightarrow{s_\sigma} m_1 + \cdots + m_n \xrightarrow{! \otimes \cdots \otimes !}$$

where $\sigma \in \Sigma_m$ is unique up to composition with disjoint unions $(\tau_1, \ldots, \tau_k)$ of symmetries $\tau_i \in \Sigma_{m_i}$.

Dual to the above, in the notion of an unbiased cocommutative comonoid $A$ the structure is given by a family of maps $\delta_n: A \to A^{\otimes n}$. Analogous to the equivalence above we have $\text{Alg}(\mathbb{F}^{op}, M) \simeq \text{ucocComon}(M)$.  

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Example 5.6. An unbiased bicommutative bimonoid in $M$ is a triple $A = (A, \mu, \delta)$ where $(A, \mu)$ forms an unbiased commutative monoid and $(A, \delta)$ forms an unbiased cocommutative comonoid, such that the compatibility diagrams on the left below commute. Here the isomorphism moves each copy of $A$ in $(A^{\otimes n})^{\otimes m}$, that is indexed by $(i, j) \in [m] \times [n]$, to the copy in $(A^{\otimes m})^{\otimes n}$ that is indexed by $(j, i) \in [n] \times [m]$.

A morphism $A \to B$ of unbiased bicommutative bimonoids is a map $A \to B$ in $M$ that is simultaneously a map of monoids and comonoids. Furthermore, an unbiased bicommutative bimonoid $A$ is called an unbiased bicommutative Hopf monoid whenever there exists a map $S: A \to A$ that makes the diagram on the right above commute. Such a map, which is necessarily unique (see e.g. Lemma 37 of [Por13]), is called the antipode of $A$.

The algebras of the PROP $H$ are essentially the unbiased bicommutative Hopf monoids in $M$: each algebra $A: H \to M$ induces the structure of such a Hopf monoid on $A(1)$ using the maps

\[
\begin{align*}
\mu_n &= [A(1)^{\otimes n} \xrightarrow{A} A(n) \xrightarrow{\mu(1\ldots 1)} A(1)], \\
\delta_n &= [A(1) \xrightarrow{\mu(1\ldots 1)^t} A(n) \xrightarrow{A^{-1}} A(1)^{\otimes n}] \\
S &= [A(1) \xrightarrow{A^{-1}} A(1)],
\end{align*}
\]

where $(1\ldots 1)^t$ denotes the transpose of the $1 \times n$-matrix $(1\ldots 1)$. That these maps satisfy the bicommutative bimonoid axioms and the Hopf monoid axiom follows directly from various matrix equations. For example, the top of the diagram on the right above is the $A$-image of the matrix equation

\[
\begin{pmatrix}
1 & 1 \\
-1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

One can show that the assignment $A \mapsto A(1)$ extends to an equivalence of categories

\[
\text{Alg}(H, M) \simeq \text{ubicHopf}(M),
\]

where $\text{ubicHopf}(M)$ denotes the category of unbiased bicommutative Hopf monoids in $M$. In fact it is shown in [Wad08] that, in the case that $M$ is symmetric strict monoidal, the category of symmetric strict monoidal functors $H \to M$ is isomorphic to the category of biased bicommutative Hopf monoids; it is straightforward to modify (parts of) the proof given there into a proof for the equivalence above.

As promised, we finish by showing that unbiased bicommutative Hopf monoids that are freely generated by unbiased cocomutative comonoids exist. For a much more detailed and thorough treatment of free, and cofree, non-commutative Hopf monoids we refer the reader to [Por13].

Theorem 5.7. Let $M$ be a cocomplete symmetric monoidal category whose tensor product preserves colimits in each variable. The forgetful functor

\[
\text{ubicHopf}(M) \to \text{ucocComon}(M),
\]

is an equivalence of categories.
of unbiased bicommutative Hopf monoids to unbiased cocommutative comonoids, admits a left adjoint.

Proof. We may as well prove that the composite of the forgetful functor and the equivalences \( \operatorname{Alg}(\mathbb{H}, M) \simeq \operatorname{ubicHopf}(M) \) and \( \operatorname{ucocComon}(M) \simeq \operatorname{Alg}(\mathbb{F}^{\mathbb{P}}, M) \) admits a left adjoint. In light of Definition 5.4 this composite is simply the functor \( \lambda_{2} : \text{sMonCat}(\mathbb{H}, M) \to \text{sMonCat}(\mathbb{F}^{\mathbb{P}}, M) \) that is given by precomposition with the embedding of PROPs \( j : \mathbb{F}^{\mathbb{P}} \to \mathbb{H} \). Hence the proof follows if we can show that we may apply Corollary 5.3 which means that we have to show that the companion \( j_{*} : \mathbb{F}^{\mathbb{P}} \to \mathbb{H} \) satisfies the left Beck-Chevalley condition.

Thus we consider a morphism \( \xi \in \mathbb{H} \) that is of the form \( \xi : m \to n_{1} + \cdots + n_{k} \); following the discussion preceding Corollary 4.9 we have to show that it can be uniquely represented as a composite \( \xi = (\xi_{1} \otimes \cdots \otimes \xi_{k}) \circ \zeta \), where the \( \xi_{i} : l_{i} \to n_{i} \) are maps in \( \mathbb{H} \) and \( \zeta : m \to l_{1} + \cdots + l_{k} \) is a map in \( \mathbb{F}^{\mathbb{P}} \). Recalling the definitions of \( \mathbb{H} \) and \( j : \mathbb{F}^{\mathbb{P}} \to \mathbb{H} \) (see Example 5.3), this means that \( \xi \) is a \( (n_{1} + \cdots + n_{k}) \times m \)-matrix with coefficients in \( \mathbb{Z} \), the \( \xi_{i} \) are such \( n_{i} \times l_{i} \)-matrices, and \( \zeta \) is an \( (l_{1} + \cdots + l_{k}) \times m \)-matrix that contains precisely one non-zero entry in each row, whose coefficient is 1. It is easy to show that such a representation exists: we take \( l_{i} = m \) and let \( \xi_{i} \) be the blocks of \( \xi \) as shown below. As the \( km \times m \)-matrix \( \xi \) we take the block matrix of identity matrices \( I_{m} \) of dimension \( m \), as on the right.

\[
\xi = \begin{pmatrix}
\xi_{1} \\
\xi_{2} \\
\vdots \\
\xi_{k}
\end{pmatrix} = \begin{pmatrix}
\xi_{1} & 0 & \cdots & 0 \\
0 & \xi_{2} & \cdots & 0 \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \xi_{k}
\end{pmatrix} \begin{pmatrix}
I_{m} \\
I_{m} \\
\vdots \\
I_{m}
\end{pmatrix}
\]

It remains to prove that the representation \( \xi = (\xi_{1} \otimes \cdots \otimes \xi_{k}) \circ \zeta \) is unique, in the sense explained before Corollary 4.9. Hence consider another such representation \( \xi = (\xi_{1}' \otimes \cdots \otimes \xi_{k}') \circ \zeta' \), where each \( \xi_{i}' \) is a \( n_{i} \times l_{i}' \)-matrix and \( \zeta' \) is a \( (l_{1}' + \cdots + l_{k}') \times m \)-matrix in \( \mathbb{F}^{\mathbb{P}} \subset \mathbb{H} \); it suffices to show that there exist \( l_{i}' \times m \)-matrices \( \chi_{i} \) in \( \mathbb{F}^{\mathbb{P}} \) satisfying \( \zeta' = (\chi_{1} \otimes \cdots \otimes \chi_{k}) \circ \zeta' \) as well as \( \xi_{i}' = \xi_{i}' \circ \chi_{i} \), for each \( i = 1, \ldots, k \). Using the same trick, we take the \( \chi_{i} \) to be defined by the identity

\[
\zeta' = \begin{pmatrix}
\chi_{1} \\
\chi_{2} \\
\vdots \\
\chi_{k}
\end{pmatrix} = \begin{pmatrix}
\chi_{1} & 0 & \cdots & 0 \\
0 & \chi_{2} & \cdots & 0 \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \chi_{k}
\end{pmatrix} \begin{pmatrix}
I_{m} \\
I_{m} \\
\vdots \\
I_{m}
\end{pmatrix}
\]

Clearly \( \zeta' \in \mathbb{F}^{\mathbb{P}} \) implies that each \( \chi_{i} \in \mathbb{F}^{\mathbb{P}} \), while \( \xi_{i}' = \xi_{i}' \circ \chi_{i} \) follows by postcomposing both sides of the identity above with \( (\xi_{1}' \otimes \cdots \otimes \xi_{k}') \).

\[\square\]

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