Proof of the finite-time thermodynamic uncertainty relation for steady-state currents

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The thermodynamic uncertainty relation offers a universal energetic constraint on the relative magnitude of current fluctuations in nonequilibrium steady states. However, it has only been derived for long observation times.

Here, we prove a recently conjectured finite-time thermodynamic uncertainty relation for steady-state current fluctuations. Our proof is based on a quadratic bound to the large deviation rate function for currents in the limit of a large ensemble of many copies.

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Introduction. The thermodynamic uncertainty relation offers a fundamental bound on the current fluctuations in nonequilibrium steady states [1–4]. Roughly speaking, it states that small fluctuations come at the cost of more dissipation. This relation, and its cousins [5,6], allows one to constrain thermodynamic forces in enzymatic catalysis [7,8], bound the entropy flow and change in system Shannon entropy—along that transition [21].

Now, as we track a stochastic realization of our system evolving over a finite time interval \( t \in [0,T] \), \( x(t) \), there will be a fluctuating instantaneous current counting every time \( t_k \) the system jumps:

\[
\langle J_{yz}(t) \rangle = -\frac{\pi r_{yz} \pi_x}{\pi y_z} = -\frac{\langle x(t_k) \rangle}{\pi y_z},
\]

with \( x(t_k) \) being the state of the system just before and after a jump. Our interest, though, is in integrated generalized currents, which are obtained by weighing each mesoscopic jump by a factor \( d_{ij}(t) = -d_{ji}(t) \) and summing them up:

\[
J_T = \int_0^T ds \sum_{y<z} d_{yz}(s) J_{yz}(s). \tag{4}
\]

For example, the entropy production is a generalized current with \( d_{yz} = F_{yz} \),

\[
\Sigma_T = \int_0^T ds \sum_{y<z} F_{yz} J_{yz}(s), \tag{5}
\]

whose steady-state average

\[
\langle \Sigma_T \rangle = T \sum_{y<z} F_{yz} \pi_y = T \Sigma^\pi \tag{6}
\]

characterizes the irreversibility of the nonequilibrium steady state. Our goal now is to constrain the fluctuations in \( J_T \) by bounding its large deviation rate function using \( \langle \Sigma_T \rangle = T \Sigma^\pi \), which will lead to (1).

Large deviations for large ensembles. Imagine now not just one instance of our system hopping among its states, but an ensemble of \( N \gg 1 \) independent copies—labeled \( x^\alpha(t) \), \( \alpha = 1, \ldots, N \)—with initial conditions sampled from the steady-state distribution \( \pi \). Then in any given moment we could obtain an empirical estimate of the density to be in mesostate \( y \) at time \( t \) by measuring the instantaneous fraction of copies in state \( y \):

\[
\rho_y(t) = \frac{1}{N} \sum_{\alpha=1}^N \delta(x^\alpha(t),y). \tag{7}
\]
We could additionally estimate the current by counting the total net number of jumps along any link as
\[
\phi_{yz}(t) = \frac{1}{N} \sum_{a=1}^{N} j_{yz}^a(t),
\]
with \(j_{yz}^a(t)\) the instantaneous current of copy \(a\) [cf. (3)]. Indeed, the law of the large numbers guarantees that both empirical measures converge to their expected values as \(N \to \infty\). However, we can also quantify their fluctuations through a large deviation principle. As demonstrated in Ref. [22], the probability to see a fluctuation is exponentially suppressed for large \(N\) as
\[
P[\rho(t), \phi(t)] \propto e^{-N I[\rho(t), \phi(t)]},
\]
where \(\propto\) denotes asymptotic logarithmic equivalence [18], and the large deviation rate function is
\[
I[\rho(t), \phi(t)] = \int_0^T ds \mathcal{I}(\rho(s), \phi(s)) - S(\rho(0) || \pi). \tag{10}
\]
The second term is the relative entropy between the initial fluctuating density \(\rho(0)\) and the steady state \(\pi\), \(S(\rho(0) || \pi) = \sum_{\alpha} \rho_{\alpha}(0) \ln (\rho_{\alpha}(0)/\pi_{\alpha})\). The first term can be put in the form [23, 24]
\[
\mathcal{I}(\rho(t), \phi(t)) = \sum_{y < z} \Psi(\phi_{yz}(t), j_{yz}^p(t), \lambda_{yz}(t)) \tag{11}
\]
with
\[
\Psi(j, j', \lambda) = j_\lambda \left(\arcsin \frac{j}{\lambda} - \arcsin \frac{j'}{\lambda}\right) - \frac{1}{2} \left(\sqrt{\lambda^2 + j^2} - \sqrt{\lambda^2 + j'^2}\right). \tag{12}
\]
\(j_{yz}^p(t) = r_{z\gamma} \rho_{\gamma}(t) - r_{z\alpha} \rho_{\alpha}(t)\) the expected current for density \(\rho\), and \(j_{yz}^p(t) = 2 j_\lambda \rho_{\lambda}(t)\). The expression for \(I\) only applies for fluctuations that conserve probability, \(\dot{\rho}_{\gamma}(t) = \sum_{\alpha \neq \gamma} \phi_{\gamma\alpha}(t)\) with a normalized density \(\sum_{\gamma} \rho_{\gamma}(t) = 1\); otherwise, \(I\) is infinity.

Within this framework, the fluctuations in the generalized current are simply due to the sum over the fluctuations of each member:
\[
\Phi_d = \sum_{a=1}^{N} \left( \int_0^T ds \sum_{y < z} d_{yz}(s) j_{yz}^a(s) \right) \tag{13}
\]
\[
= N \int_0^T ds \sum_{y < z} d_{yz}(s) \phi_{yz}(s) \equiv N \Phi_d. \tag{14}
\]
Importantly, the large-\(N\) scaling of the cumulants of \(\Phi_d\) are identical to the cumulants of our generalized current \(J_T\) [cf. (4)] of interest,
\[
\lim_{N \to \infty} \frac{1}{N} \text{Var}(\Phi_d) = \text{Var}(J_T), \tag{15}
\]
\[
\lim_{N \to \infty} \frac{1}{N} \langle \Phi_d \rangle = \langle J_T \rangle,
\]
since our ensemble of copies are independent and identically distributed. Furthermore, they are encoded in the large deviation rate function \(I(\phi_d)\) for the generalized current. Thus, by bounding \(I(\phi_d)\), as we now do, we constrain the generalized-current fluctuations.

**Bounding the large deviation rate function.** Remarkably, \(\mathcal{I}\) in (11) has the exact same functional form as the level-2.5 large deviation rate function for long-time-averaged empirical density and currents [23–25]. As a consequence, we can almost directly import the proof used to derive the long-time thermodynamic uncertainty relation to this situation. As such we proceed in two steps [3]: First, we bound \(\mathcal{I}\), and then exploit the large-deviation contraction principle to obtain an inequality for the rate function \(I(\phi_d)\).

As shown in Refs. [3,4], \(\mathcal{I}\) satisfies a quadratic inequality, which in this situation reads
\[
I[\rho(t), \phi(t)] \leq \int_0^T ds \sum_{y < z} \frac{[\phi_{yz}(s) - j_{yz}^p(s)]^2}{4 j_{yz}^p(s)^2} \sigma_{yz}(s) - S(\rho(0) || \pi). \tag{16}
\]
where \(\sigma_{yz}(s) = j_{yz}^p(s) \ln [r_{y\gamma} \rho_{\gamma}(s)/r_{z\gamma} \rho_{\gamma}(s)]\) is the expected entropy production along jump \(z \to y\) if the density were \(\rho\).

The next step is to contract down to the large deviation rate function for generalized current. Namely, we can obtain the large deviation function for the generalized current through the minimization [18]:
\[
I(\phi_d) = \inf_{\rho(0), \phi(0)} I[\rho(t), \phi(t)], \tag{17}
\]
where the minimization is constrained by \(\phi_d = \int_0^T ds \sum_{y < z} d_{yz}(s) \phi_{yz}(s)\), the conservation of probability \(\dot{\rho}_{\gamma}(t) = \sum_{\alpha \neq \gamma} \phi_{\gamma\alpha}(t)\), and normalization \(\sum_{\gamma} \rho_{\gamma}(t) = 1\). However, an upper bound to such a minimization can be obtained by choosing any pair of \(\rho\) and \(\phi\) consistent with the constraints. We choose the time-independent pair
\[
\rho_{\gamma}(t) = \pi_{\gamma}, \quad \phi_{yz}(t) = \frac{\phi_d}{\langle J_T \rangle} j_{yz}^p. \tag{18}
\]
Substituting into (17), while exploiting (16), we obtain the quadratic bound
\[
I(\phi_d) \leq \frac{(\phi_d - \langle J_T \rangle)^2}{4 \langle J_T \rangle^2} \int_0^T ds \sum_{y < z} \sigma_{yz}(s) \tag{19}
\]
\[
= \frac{(\phi_d - \langle J_T \rangle)^2}{4 \langle J_T \rangle^2} \langle \Sigma_T \rangle \tag{20}
\]
in terms of the time-integrated steady-state entropy production \(\langle \Sigma_T \rangle = T \Sigma = T \sum_{y < z} \sigma_{yz}\).

The finite-time uncertainty relation (1) now follows readily, by observing that the quadratic bound is zero at the typical value, \(I(\langle J_T \rangle) = 0\), and that the second derivative of \(I(\phi_d)\) at its minimum encodes the large-\(N\) scaling of the variance:
\[
\lim_{N \to \infty} \frac{1}{N} \text{Var}(\Phi_d) = \frac{1}{\langle J''(J_T) \rangle} \geq \frac{2 \langle J_T \rangle^2}{\langle \Sigma_T \rangle}, \tag{21}
\]
by (20). Combining this inequality with the independent-identically-distributed nature of the copies (15) leads to the thermodynamic uncertainty relation in (1).

**Discussion.** Remarkably, the finite-time uncertainty relation can be derived in almost the exact same manner as the long-time uncertainty relation using a large deviation theory for an ensemble of many copies. Consequently, this
finite-time uncertainty relation is expected to also hold for diffusion processes, since the large deviation function for diffusions has a quadratic structure identical to (16) [4,26]. Similarly, we expect that tighter-than-quadratic bounds [5,7] will also hold for finite times. Extending these constructions to an uncertainty relation for finite-time first-passage-time fluctuations would be an interesting and useful extension (cf. [17]). However, an extension to a discrete-time process appears untenable [27].

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