Veronese powers of operads and pure homotopy algebras

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Abstract
We define the \( m \)th Veronese power of a weight graded operad \( \mathcal{P} \) to be its suboperad \( \mathcal{P}^{[m]} \) generated by operations of weight \( m \). It turns out that, unlike Veronese powers of associative algebras, homological properties of operads are, in general, not improved by this construction. However, under some technical conditions, Veronese powers of quadratic Koszul operads are meaningful in the context of the Koszul duality theory. Indeed, we show that in many important cases the operads \( \mathcal{P}^{[m]} \) are related by Koszul duality to operads describing strongly homotopy algebras with only one nontrivial operation. Our theory has immediate applications to objects such as Lie \( k \)-algebras and Lie triple systems. In the case of Lie \( k \)-algebras, we also discuss a similarly looking ungraded construction which is frequently used in the literature. We establish that the corresponding operad does not possess good homotopy properties, and that it leads to a very simple example of a non-Koszul quadratic operad for which the Ginzburg–Kapranov power series test is inconclusive.

Keywords Operad · Veronese power · Homological purity · Koszul duality · Koszulness · Zeilberger’s algorithm

Mathematics Subject Classification 18D50 · 18G55 · 33F10 · 55P48

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1 Introduction

Many examples of algebras with \( m \)-ary structure operations are “pure” versions of homotopy algebras in the following sense. Suppose that \( \mathcal{P} \) is a binary quadratic Koszul operad, and \( \mathcal{P}_\infty = \Omega(\mathcal{P}) \) is its minimal model. We use the term \textit{pure} \( \mathcal{P}_\infty \)-algebras for algebras over the quotient of the operad \( \mathcal{P}_\infty \) by the ideal generated by all its generating operations except for those of arity \( m \). (The space of generators of the thus obtained operad is homologically pure, hence the terminology.)

Another (in many ways more classical) type of algebras with \( m \)-ary structure operations is obtained as follows. Let \( \mathcal{P} \), once again, be a binary quadratic operad. We consider the suboperad of \( \mathcal{P} \) generated by all operations of arity \( m \); experts in classical theory of identities in algebras would probably call algebras over this operad \textit{\( m \)-tuple systems of type} \( \mathcal{P} \). (See, for instance, the work of Jacobson [24], who used the term “triple systems” for this construction in the case of Lie and Jordan algebras.) In modern terms, this construction generalises the construction of Veronese powers which is well known in the cases of graded associative algebras (where it is known to improve homological properties of an algebra).

When writing this paper, we pursued three goals. First, we wanted to record the general definition of Veronese powers for an arbitrary operad and explore its consequences. In that exploration we managed to obtain some counterexamples showing that Veronese powers do not necessarily possess the remarkable properties they have in the case of graded associative algebras, to prove results that guarantee, for particular operads, availability of good homotopical properties for their Veronese powers, and to show how various known definitions of algebraic systems fit into our formalism (at times, this allows for short elegant proofs of results that are otherwise proved by elaborate brute force computations).

Second, we demonstrated that pure homotopy algebras and algebras over Veronese powers are, under some assumptions, related by Koszul duality for operads. To make this into a slogan, one would say that “Veronese powers are Koszul dual to purifications of minimal models”.

Finally, we addressed the issue of “ungraded” pure algebras that are occasionally used in the existing literature (it usually happens when the homotopy Lie algebra identities are imposed for an operation of degree zero on an ungraded vector space). In our previous paper [16], we already demonstrated that in the case of homotopy associative algebras it leads to extra relations and worse homotopical properties of the corresponding operad. In this paper, we prove an analogous result for the case of homotopy Lie algebras, which in particular leads to an example of a non-Koszul operad with a generator of arity three for which the inverse of the Poincaré series has nonnegative coefficients; this is a relative of the counterexample exhibited in [16] in the nonsymmetric case.

Organisation of the paper

In Sect. 2, we recall the key relevant definitions of the theory of operads. In Sect. 3, we define and study Veronese powers of weight graded operads. One of the cen-
central results of that section is Theorem 3.13 providing conditions for the existence of quadratic Gröbner bases for Veronese powers of operads and thus, in particular, for their Koszulness. The matters of Koszulness of Veronese powers of operads are much more subtle, as we demonstrate in Proposition 3.14 which compares the case of operads to the case of associative algebras.

In Sect. 4, we relate our work to research of polynomial identities, both classical and recent. We argue that our definition of Veronese powers allows one to provide clear short proofs of some results on triple systems of various kinds, both classical and recent.

In Sect. 5, we define operads for pure homotopy \( \mathcal{P} \)-algebras, and prove Proposition 5.3 which establishes how these operads are related to Veronese powers by Koszul duality. In particular, this, together with results of previous sections, allows us to establish that the operad of Lie triple systems is Koszul and to give a clean description of its Koszul dual; this is one of the most nontrivial Koszulness results for an operad with nonbinary generators that is known to date.

Finally, in Sect. 6, we demonstrate that the ungraded versions of strong homotopy Lie algebras sometimes used in the literature possess worse homotopical properties than the strong homotopy Lie algebras obtained from the minimal model of the Lie operad. Particularly interesting is the operad for ungraded 3-Lie algebras, since it is so far the simplest known example of an operad which is not Koszul, but for which the Ginzburg–Kapranov positivity criterion is not decisive.

2 Recollections

Throughout this paper, we follow the notational conventions set out in the monographs [5,34]. All the results of this paper are valid for vector spaces and chain complexes over an arbitrary field \( k \) of characteristic zero. We use the notation \( X \cong Y \) for isomorphisms, and the notation \( X \approx Y \) for weak equivalences (quasi-isomorphisms).

To handle suspensions of chain complexes, we introduce an element \( s \) of degree 1, and define, for a graded vector space \( V \), its suspension \( sV \) as \( k \otimes V \). The endomorphism operad \( \text{End}_{k, s} \) is denoted by \( S \). For an operad \( \mathcal{P} \), its operadic suspension is the Hadamard tensor product \( S \otimes \mathcal{P} \). (We must warn the reader that in the literature the terms “operadic suspension” and “operadic desuspension” are sometimes used in the opposite way.)

2.1 Three kinds of operads: weight gradings, compositions, free objects

At various points in this paper, we use all three kinds of operads discussed in [5,34], that is nonsymmetric, symmetric, and shuffle operads. The relationship between them can be summarised as follows. Operads that appear “in the real world” (i.e. acting on various algebras of interest) are symmetric: whenever elements of an operad are actual operations that have actual arguments, one has the actions of the respective symmetric groups by permutations of arguments. However, some symmetric operads may be obtained from much smaller nonsymmetric operads using the functor of symmetrisa-
tion; we usually look at operads of that sort when constructing counterexamples, as it allows us to make the arguments much less technical. On the other hand, shuffle operads are useful as a technical tool for working with arbitrary symmetric operads: in the universe of shuffle operads, one can compute operadic Gröbner bases, so it is often beneficial for computational purposes to replace a symmetric operad by the associated shuffle operad. In each of those cases, i.e., the symmetrisation functor from nonsymmetric to symmetric operads and the forgetful functor from symmetric to shuffle operads, we are dealing with monoidal functors which preserve all key operadic constructions utilised throughout this paper; that is the main reason to invoke those functors where necessary.

We require all operads in this paper to be (nonnegatively) weight graded: this means that every component $P(n)$ admits a direct sum decomposition $P(n) = \bigoplus_{k \geq 0} P(n)_k$ into components of weight $k$ for various $k \geq 0$ for which any operad composition of homogeneous elements of certain weights is a homogeneous element whose weight is the sum of the weights. In addition, we assume all operads reduced ($P(0) = 0$) and connected ($P(n)_0 = k$ and $P(n)_{\emptyset} = 0$ for $n > 1$). A connected operad is automatically augmented, and we denote by $\bar{P}$ the augmentation ideal of $P$.

Let us remark that each operad $P$ has one obvious weight grading where

$$P(n)_{(n-1)} = P(n) \quad \text{for } n \geq 1 \quad \text{and} \quad P(n)_k = 0 \quad \text{otherwise.}$$

For most commonly considered operads, those generated by binary operations subject to ternary relations, this grading is the most convenient one to use; in particular, it is the grading for which the relations are quadratic. However, beyond the binary generated operads, other weight gradings are occasionally more appropriate.

Each of the three kinds of operads we consider has its composition products. In general, for

$$\alpha \in P(k), \quad \beta_1 \in P(n_1), \quad \ldots, \quad \beta_k \in P(n_k)$$

and for a set partition $\pi$ of the form $\{1, \ldots, n_1 + \cdots + n_k\} = I^{(1)} \sqcup \cdots \sqcup I^{(k)}$ with $|I^{(j)}| = n_j$, the composition product $\gamma_\pi(\alpha; \beta_1, \ldots, \beta_k)$ is defined. The only difference between the three kinds of operads is in the types of partitions permitted: in the case of nonsymmetric operads, only the partition with

$$I^{(j)} = \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j\}$$

is allowed, in the case of symmetric operads, all partitions are allowed, and in the case of shuffle operads, only the partitions with $\min(I^{(1)}) < \cdots < \min(I^{(k)})$ are allowed. For any kind of operads, in the case of the partition $\pi$ with $I^{(j)} = \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j\}$ we shall suppress $\pi$ from the subscript, as it is completely determined by the numbers $n_j$, which in turn are completely determined by the arguments of $\gamma$: $n_j$ is the arity of $\beta_j$. 

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Recall that for any kind of operads, the *infinitesimal (partial) composition products* denoted by \( \circ_i \) are available, such products correspond to the partitions

\[
\{1, \ldots, n + m - 1\} = \{1\} \sqcup \cdots \sqcup \{i - 1\} \sqcup \{i, \ldots, i + m - 1\} \sqcup \{i + m\} \sqcup \cdots \sqcup \{n + m - 1\}.
\]

In addition, for shuffle operads, we shall need *infinitesimal shuffle products* \( \circ_i, \sigma \); they correspond to the partitions

\[
\{1, \ldots, n + m - 1\} = \{1\} \sqcup \cdots \sqcup \{i - 1\} \sqcup \{i, \sigma(i + 1), \ldots, \sigma(i + m - 1)\} \\
\sqcup \{\sigma(i + m)\} \sqcup \cdots \sqcup \{\sigma(n + m - 1)\},
\]

where \( \sigma : \{i + 1, \ldots, n + m - 1\} \to \{i + 1, \ldots, n + m - 1\} \) is an \((m - 1, n - i)\)-unshuffle, meaning that

\[
\sigma(i + 1) < \cdots < \sigma(i + m - 1) \quad \text{and} \quad \sigma(i + m) < \cdots < \sigma(n + m - 1).
\]

Using this notion, we can define *left comb products* in any shuffle operad; a left comb product of elements \( x_1, \ldots, x_m \) is an element obtained from them by iterated compositions \( \circ_1, \sigma \) where only the first slot of operations is used.

The *free operad* (of each of the three kinds, where the meaning is always clarified by the surrounding context) generated by a collection \( X \) is denoted by \( T(X) \), the *cofree (conilpotent) cooperad* cogenerated by a collection \( X \) is denoted by \( T^c(X) \); the former is spanned by *tree tensors*, and has its composition product given by grafting of trees, and the latter has the same underlying collection but a different structure, a decomposition coproduct. Whenever \( X \) is weight graded, the underlying collection of \( T(X) \) has a weight grading induced from \( X \). In particular, the *standard* weight grading on \( X \) (all elements are of weight 1) induces the *standard* grading on \( T(X) \).

### 2.2 Operadic Gröbner bases

A very useful technical tool for dealing with operads is given by Gröbner bases. We refer the reader to [5, Chapters 3, 5] for a systematic presentation of operadic Gröbner bases, and only recall the basics here.

Similarly to associative algebras, operads can be presented via generators and relations, that is as quotients of free operads. In both the shuffle and the nonsymmetric case, the free operad generated by a given nonsymmetric collection admits a basis of *tree monomials* which can be defined combinatorially; every composition of tree monomials is again a tree monomial. There exist several ways to introduce a total ordering of tree monomials in such a way that the operadic compositions are compatible with that total ordering (the composition \( \gamma_\pi \) as above, viewed as an operation with \( k + 1 \) arguments \( \alpha, \beta_1, \ldots, \beta_k \), is strictly increasing in each argument). There is also a combinatorial definition of divisibility of tree monomials that agrees with the naïve operadic definition: one tree monomial is a *divisor* of another one if and only
if the latter can be obtained from the former by operadic compositions. A particular case of it that we shall need is the notion of a right divisor. A right divisor of a tree monomial $T$ is a divisor $T_1$ for which each leaf is a leaf of $T$; such a divisor really is a right divisor in that there exists a “complementary divisor” $T_0$ of $T$ for which $T = T_0 \circ_i,\sigma T_1$ in the shuffle case and $T = T_0 \circ_i T_1$ in the nonsymmetric case.

A Gröbner basis of an ideal $J$ of the free operad is a system $S$ of generators of $J$ for which the leading tree monomial of every element of the ideal is divisible by one of the leading terms of elements of $S$. In this case, the quotient by $J$ has a basis of normal tree monomials, those not divisible by leading terms of elements of $S$. There exists an algorithmic way to compute a Gröbner basis starting from any given system of generators (“Buchberger’s algorithm for operads”).

Symmetric operads can be, to an extent, forced into the universe where Gröbner bases methods are available. For that, one uses the forgetful functor from symmetric operads to shuffle operads. While this functor literally forgets the symmetric group actions, it does not change the underlying vector spaces, so if one wants to find a linear basis of an operad, or to prove that some vector space (of homological nature) vanishes, this is a very useful method.

A part of the operad theory which provides one of the most useful known tools to study homological and homotopical algebra for algebras over the given operad is the Koszul duality for operads [21]. A weight graded operad is said to be Koszul if the homology of its bar complex is concentrated on the diagonal (where weight is equal to the homological degree). If a weight graded operad is Koszul, it necessarily is quadratic, that is its defining relations are of weight two for the standard weight grading for which generators are of weight one. Proving that a given quadratic operad is Koszul instantly provides a minimal resolution for this operad, gives a description of the homology theory and, in particular, the deformation theory for algebras over that operad etc. There are a few general methods to prove that an operad is Koszul; one of the simplest and widely applicable methods is to show that a given operad has a quadratic Gröbner basis; this provides a sufficient (but not necessary) condition for Koszulness of an operad. In a way, non-Koszul operads can be regarded as more interesting/challenging, since standard methods of deformation theory do not work for them.

3 Veronese powers of operads

3.1 Naïve Veronese powers

Recall the notion of Veronese powers of weight graded associative algebras: if $A = \bigoplus_{k \geq 0} A_k$, the $d$-th Veronese power of $A$, denoted by $A^{[d]}$, is the subalgebra $\bigoplus_{k \geq 0} A_{kd}$. This definition is motivated by algebraic geometry: taking the $d$-th Veronese power of the ring of polynomial functions on a vector space $V$ corresponds, under the Proj construction, to the Veronese embeddings $\mathbb{P}(V) \hookrightarrow \mathbb{P}(S^d V)$ of projective spaces. Veronese powers of an algebra are known to have “better” properties than the algebra itself, see [1,2,19].
There is an obvious generalisation of this notion to the case of operads, which we call “naïve Veronese powers”.

**Definition 3.1**  Let $O$ be a weight graded operad. The naïve $d$-th Veronese power $V_d(O)$ is the weight graded subcollection of $O$ defined by

$$(V_d(O))(n) = \bigoplus_{k \geq 0} O(n)_{(kd)},$$

with the operad structure induced by that of $O$.

We proceed by justifying the adjective “naïve”. Indeed, one fundamental and easily verifiable property that Veronese powers of associative algebras possess is that for an algebra generated by elements of weight 1, its $d$-th Veronese power is generated by elements of weight $d$. It turns out that for operads this property generally fails.

**Proposition 3.2**  Let $T = T(X)$ be the free operad generated by one commutative binary operation $\mu$. Then for all $d \geq 2$ the operad $V_d(T)$ is not generated by $T_{(d)}$.

**Proof**  Let us denote by $\mu^{(j)}$ the iterated $j$-fold first slot composition of $\mu$: $\mu^{(0)} = \text{id}$, $\mu^{(j+1)} = \mu^{(j)} \circ_1 \mu$; note that $\mu^{(j)} \in T_{(j)}$. Then the element

$$v = \mu^{(d+1)} \circ_{d+1} \mu^{(d-1)} \in T_{(2d)}$$

cannot be represented as a partial composition of two elements from $T_{(d)}$, which is easy to see by direct inspection if we note that

$$v = \gamma(\mu^{(2)}; \mu^{(d-1)}, \mu^{(d-1)}, \text{id}).$$

Let us remark that in the case of operads generated by binary operations (which is the case considered most often) the standard weight grading is the “arity minus one” grading mentioned in Sect. 2, and so the naïve Veronese power $V_d(O)$ is the suboperad on the components of $O$ for all arities $n \equiv 1 \pmod{d}$. One can consider such suboperads in general, but they seem to possess absolutely no notable properties (and so are “very naïve Veronese powers”).

Below we shall propose a more meaningful definition of Veronese powers. As a preparation to that general definition, we first recall the three classical notions of “triple systems”, which correspond to the “Three Graces of the operad theory” (an expression coined by Jean-Louis Loday), the associative operad Ass, the commutative associative operad Com, and the operad Lie of Lie algebras.

### 3.2 Classical definitions of triple systems

The philosophy of triple systems, going back to [24], can be expressed by saying that, for a bilinear operation $a_1, a_2 \mapsto a_1 \star a_2$, the natural trilinear operations $a_1, a_2, a_3 \mapsto (a_1 \star a_2) \star a_3$ and $a_1, a_2, a_3 \mapsto a_1 \star (a_2 \star a_3)$ are of their own merit. (In the cases of Ass, Com, and Lie, it is enough to take just one of those, as they can be expressed through one another using the defining relations and the symmetric group action.)
Definition 3.3 ([33]) A ternary ring, or a totally associative triple system of the first type is a vector space $V$ with a trilinear operation

$(-,-,-): V^3 \to V$

satisfying the properties

$$((a_1, a_2, a_3), a_4, a_5) = (a_1, (a_2, a_3, a_4), a_5) = (a_1, a_2, (a_3, a_4, a_5)).$$  \hspace{1cm} (3.1)

The triple product $(a_1, a_2, a_3) = a_1a_2a_3$ in each associative algebra satisfies these identities; it is also easy to show that every multilinear identity satisfied by all those particular examples follows from (3.1).

Definition 3.4 ([33]) A ternary commutative ring, or a totally commutative associative triple system is a vector space $V$ with a trilinear operation

$(-,-,-): V^3 \to V$

satisfying the properties

$$(a_1, a_2, a_3) = (a_2, a_1, a_3) = (a_1, a_3, a_2),$$ \hspace{1cm} (3.2)

$$((a_1, a_2, a_3), a_4, a_5) = (a_1, (a_2, a_3, a_4), a_5) = (a_1, a_2, (a_3, a_4, a_5)).$$ \hspace{1cm} (3.3)

The triple product $(a_1, a_2, a_3) = a_1a_2a_3$ in each commutative associative algebra satisfies these identities; it is also easy to show that every multilinear identity satisfied by all those particular examples follows from (3.2) to (3.3).

Definition 3.5 ([24,25]) A Lie triple system is a vector space $V$ with a trilinear operation

$$[-,-,-]: V^3 \to V$$

possessing the symmetries

$$[a_1, a_2, a_3] = -[a_2, a_1, a_3],$$ \hspace{1cm} (3.4)

$$[a_1, a_2, a_3] + [a_2, a_3, a_1] + [a_3, a_1, a_2] = 0,$$ \hspace{1cm} (3.5)

and satisfying the identity

$$[a_1, a_2, [a_3, a_4, a_5]] = [[a_1, a_2, a_3], a_4, a_5] + [a_3, [a_1, a_2, a_4], a_5] + [a_3, a_4, [a_1, a_2, a_5]].$$ \hspace{1cm} (3.6)

The triple product $[a_1, a_2, a_3] := [[a_1, a_2], a_3]$ in any Lie algebra satisfies these identities. It is also known [25] that every identity satisfied by the operation $[[a_1, a_2], a_3]$ is a consequence of (3.4), (3.5), and (3.6) (note that the list of identities in the paper [24] where Lie triple systems were first defined contains some redundant ones). We shall re-prove this result below using Gröbner bases for operads.
3.3 Veronese powers

To ensure the property of Veronese powers which we observed to fail, the most natural way is to enforce it, which, in a sense, is exactly what the classical definition of triple systems accomplishes.

**Definition 3.6** Let $O$ be a weight graded operad. The $d$-th Veronese power $O^{[d]}$ is the suboperad of $O$ generated by $O_{(d)}$.

To demonstrate that this definition possesses many reasonable properties, we start by computing Veronese powers of free operads.

**Proposition 3.7** Let $T(X)$ be the free operad generated by $X$ which we equip with the standard weight grading with $X$ placed in weight one. Consider also the operad $T(T(X)_{(d)})$ which we equip with the weight grading where the weight of the generators $T(X)_{(d)}$ is equal to $d$. We have an isomorphism of weight graded collections

$$T(X)^{[d]} \cong T(T(X)_{(d)}).$$

In other words, Veronese powers of free operads are freely generated by elements of the lowest positive weight.

**Proof** Let us prove this for the case of shuffle operads. The case of nonsymmetric operads is analogous (since such an operad also has a combinatorial basis of tree monomials), and the case of symmetric operads follows from the shuffle case via the forgetful functor.

The free shuffle operad $T(X)$ has a basis of tree monomials. By definition, the operad $T(X)^{[d]}$ is generated by $T(X)_{(d)}$. To prove the freeness, it is enough to show that if a tree monomial can be decomposed as an iterated composition of tree monomials of weight $d$, this can be done in an essentially unique way.

For a tree monomial $T$, let us denote by $R_d(T) = \{R_1, \ldots, R_s\}$ the (possibly empty) set of all different right divisors of $T$ with $d$ vertices. We claim that the elements in $R_d(T)$ are disjoint. Indeed, if $R_p$ and $R_q$ are not disjoint, they must clearly share a common leaf, say $j$. As there is an obvious one-to-one correspondence between the vertices of $T$ on the unique path connecting $j$ to the root and right divisors containing $j$ as a leaf, we conclude that $R_p = R_q$.

Let us consider a tree monomial $T \in T(X)$. If it belongs to $T(T(X)_{(d)})$, the set $R_d(T)$ is nonempty. Indeed, if this is so, then $T$ is the result of iterated infinitesimal shuffle compositions of some tree monomials $T_1, \ldots, T_u \in T(X)_{(d)}$. The tree monomial $T_u$ is a right divisor of $T$ with $d$ vertices, i.e., one belonging to $R_d(T)$.

For each tree monomial $T \in T(X)$ belonging to $T(T(X)_{(d)})$, its underlying tree $T$ therefore determines uniquely a nonempty set $R_d(T) = \{R_1, \ldots, R_s\}$ of its (disjoint) right divisors with $d$ vertices. We may factor out in an essentially unique (i.e. modulo the axioms of operads) way the tree monomials $R_1, \ldots, R_s$ from $T$, and induction applies. This establishes a bijection between the basis elements of the collections $T(X)^{[d]}$ and $T(T(X)_{(d)})$, thus completing the proof. \(\square\)
Remark 3.8 The idea of the proof of Proposition 3.7 offers an iterative test whether a tree monomial \( T \in \mathcal{T}(\mathcal{X}) \) belongs to \( \mathcal{T}(\mathcal{T}(\mathcal{X})_{(d)}) \) or not. If the tree is trivial (no internal vertices), it of course belongs to the Veronese power. Otherwise, one should inspect the set \( R_d(T) \) of its right divisors with \( d \) vertices. If \( R_d(T) \neq \emptyset \), remove the trees in \( R_d(T) \) from \( T \) and repeat the same procedure for the subtree of \( T \) obtained this way. For instance, for the underlying tree of the tree monomial \( \nu \) used in the proof of Proposition 3.2 one easily sees that \( R_d(T) = \emptyset \).

3.4 A criterion for Veronese powers to coincide with naïve ones

As we already mentioned, the naïve Veronese power differs from the Veronese power as defined above in the case of the free operad. However, in many classical cases we have \( V_d(O) = O[d] \), and we present one useful criterion for that.

Proposition 3.9 Suppose that \( O \) is a weight graded operad generated by operations \( \mu_1, \ldots, \mu_m \) of weight one for which every weight graded component \( O(k) \) is spanned by orbits of left comb products under the actions of the respective symmetric groups. Then \( V_d(O) = O[d] \) for any \( d \).

Proof Indeed,

\[
(\cdots (\mu_{i_1} \circ_1 \mu_{i_2}) \circ_1 \cdots ) \circ_1 \mu_{i_k} \circ_1 ((\cdots (\mu_{i_1} \circ_1 \mu_{i_2}) \circ_1 \cdots ) \circ_1 \mu_{i_d} \circ_1 ((\cdots (\mu_{i_{d+1}} \circ_1 \mu_{i_{d+2}}) \circ_1 \cdots ) \circ_1 \mu_{i_k}),
\]

so induction applies.

Let us mention a class of examples defined in [14] for which the result we just proved allows us to relate Veronese powers of associative algebras to Veronese powers of operads.

Definition 3.10 ([14]) Let \( A \) be a weight graded commutative associative algebra. We define a symmetric collection which we denote by \( O_A \) by putting \( O_A(n) = A_{n-1} \) with the trivial symmetric group action, and define composition maps

\[
\gamma : O_A(k) \otimes O_A(i_1) \otimes \cdots \otimes O_A(i_k) \to O_A(i_1 + \cdots + i_k)
\]

using the product

\[
A_{k-1} \otimes A_{i_1-1} \otimes \cdots \otimes A_{i_k-1} \to A_{(k-1)+(i_1-1)+\cdots+(i_k-1)} = A_{i_1+\cdots+i_k-1}
\]

in the algebra \( A \).

In [14], it is proved that these composition maps of \( O_A \) define an operad.

Proposition 3.11 Suppose that \( A \) is a weight graded commutative associative algebra generated by elements of weight one. Then for each \( d \geq 1 \) we have

\[
O_A[d] = (O_A)[d] = V_d(O_A).
\]
Proof The equality $(O_A)[d] = V_d(O_A)$ follows from Proposition 3.9 since the first slot compositions span $O_A$. The equality $V_d(O_A) = O_A[d]$ is obvious from the definition. □

3.5 Koszulness of Veronese powers

Let us demonstrate that in a large class of examples Veronese powers of operads have good homotopical properties. We start with a result that immediately connects to Proposition 3.11.

Proposition 3.12 Suppose that $A$ is a weight graded commutative associative algebra generated by elements of weight one. Then for all $d$ big enough the operad $V_d(O_A)$ is Koszul.

Proof By [1,19], for any weight graded commutative associative algebra $A$, its Veronese power $A[1]$ is Koszul for all $d \gg 0$. On the other hand, by [15], the operad $O_B$ is Koszul whenever the algebra $B$ is Koszul, so the operad $O_A[d]$ is Koszul. To conclude the proof, we use the isomorphisms of Proposition 3.11. □

Our next result produces another class of examples where Veronese powers are Koszul. It is an operadic analogue of a known statement about associative algebras, see [39, Proposition 4.4.3].

Theorem 3.13 Let $O$ be a weight graded operad generated by elements of weight one. Assume that the ideal of relations of $O$ has a quadratic Gröbner basis $G$ (for a certain choice of basis in the space of generators and a certain monomial ordering), and that all weight $2d$ normal tree monomials with respect to that Gröbner basis are in $T(X)^{<d} \subset T(X)$. Then the Veronese power $O[d]$ admits a quadratic Gröbner basis of relations. In particular the operad $O[d]$ is Koszul.

Proof Let $Y$ be the collection of all weight $d$ normal tree monomials with respect to $G$. As the collection $Y$ is a basis of $O(d)$, we may choose it as the collection of generators of the operad $O[d]$. There is an embedding $T(Y) \hookrightarrow T(X)$ obtained via the embedding $Y = Y(d) \hookrightarrow T(X)_d$, the identification $T(X)^{<d} \cong T(T(X)_d)$ of Proposition 3.7, and the embedding $T(Y)^{<d} \hookrightarrow T(Y)$. In particular, this embedding induces a monomial ordering of $T(Y)$, and this is the ordering we refer to in the rest of the proof.

We claim that the requisite quadratic Gröbner basis of the operad $O[d]$ consists of all elements in $(G) \cap T(Y)^{<2d}$. To prove that, let us consider all elements $S_1, \ldots, S_p$ of $T(Y)^{<2d}$ which are normal tree monomials with respect to $G$ when viewed as elements of $T(X)^{<2d}$ via the embedding we mentioned. By our assumption on the normal monomials with respect to $G$, these elements form a basis of the weight $2d$ components of $O[d]$. The leading terms of $(G) \cap T(Y)^{<2d}$ are precisely the basis elements $T(Y)^{<2d}$ different from all of the $S_i$’s.

Thus, the result would follow if we show that a tree monomial $T \in T(Y)^{(rd)}$ is not a normal tree monomial with respect to $G$ (when viewed as an element of $T(X)$) if and only if it has a divisor of weight $2d$ different from the $S_i$’s (when viewed as an element of $T(Y)$). In $T(X)$, $T$ is not normal with respect to $G$ if and only if it is divisible by
one of the leading terms of $\mathcal{G}$, say $T_0$. Now we recall that $T \in \mathcal{T}(\mathcal{G}_{(rd)})$, and note that $T_0$ cannot be a divisor of one of the elements of $\mathcal{Y}$ since they all are assumed normal. Also, since the weight of $T_0$ is 2, so its occurrence overlaps with exactly two of the elements of $\mathcal{Y}$, which determines a divisor of weight $2d$ different from one of the $S_j$’s, thus completing the proof.

\section*{3.6 Non-properties of Veronese powers of operads}

We conclude this section by showing that most classically known stronger results valid for Veronese powers of associative algebras are not true for operads in general.

**Proposition 3.14** The following results valid for Veronese powers of algebras are, in general, false in the case of operads:

1. “improvement of defining relations for Veronese powers” ([3, Proposition 3]): if all elements of the minimal set of defining relations of $A$ are of weight at most $(k - 1)d + 1$, then all elements of the minimal set of defining relations of $A^{[d]}$ are of weight at most $k$ (for the standard grading of the Veronese power where all generators have weight one);

2. “improvement of the slope of the off-diagonal homology of the bar complexes for Veronese powers” ([1, Theorem 1]): the rate $r(A)$ defined by the formula

$$
    r(A) := \sup \left\{ \frac{j - 1}{i - 1} : \text{Tor}^A_i(j, k, k) \neq 0 \right\}
$$

satisfies $r(A^{[d]}) \leq \lceil r(A)/d \rceil$;

3. “high” Veronese powers of a weight graded algebra $A$ with a finite Gröbner basis admit a quadratic Gröbner basis ([39, Sect. 4.4, Remark]);

4. all Veronese powers of a weight graded algebra $A$ with a quadratic Gröbner basis admit a quadratic Gröbner basis ([39, Proposition 4.4.3]).

**Proof** We shall exhibit two examples each of which justifies that two of the properties mentioned above are not true in general.

**Example 1** Consider the nonsymmetric operad $\mathcal{O}$ with one binary generator $\nu$ subject to the only monomial relation $\omega \circ_2 (\omega \circ_2 \omega) = 0$ of weight 3. Then for every $d \geq 1$ the minimal set of relations presenting the Veronese power $\mathcal{O}^{[d]}$ as a quotient of $\mathcal{T}(\mathcal{O}_{(d)})$ has weight 3. To demonstrate this, note that, in the notation of the proof of Proposition 3.2, in the operad $\mathcal{O}$ we have $\omega^{(d)} \neq 0$ and $\omega^{(d)} \circ_{d+1} \omega^{(d)} \neq 0$, but however $\omega^{(d)} \circ_{d+1} (\omega^{(d)} \circ_{d+1} \omega^{(d)}) = 0$, which easily implies that the latter element belongs to the minimal set of relations of $\mathcal{O}^{[d]}$. As monomial relations always form a Gröbner basis, this operad clearly serves as a counterexample to statements (1) and (3).

**Example 2** Consider the nonsymmetric operad $\mathcal{O}$ with three binary generators $\mu$, $\nu$, and $\rho$ subject to relations

$$
    \mu \circ_1 \nu = \rho \circ_2 \nu, \quad \rho \circ_1 \nu = 0,
$$

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\[
\begin{align*}
\mu \circ_1 \mu &= \mu \circ_2 \mu = \mu \circ_1 \rho = \mu \circ_2 \rho = \rho \circ_1 \rho = \rho \circ_2 \rho = 0, \\
\rho \circ_1 \mu &= \rho \circ_2 \mu = \nu \circ_1 \mu = \nu \circ_2 \mu = \nu \circ_1 \rho = \nu \circ_2 \rho = 0.
\end{align*}
\]

For the ordering \( \mu > \rho > \nu \), these relations form a Gröbner basis of relations of \( \emptyset \) (this is easy to check directly: most S-polynomials are equal to zero, not just have zero as a normal form). In particular, this operad is Koszul, and \( r(\emptyset) = 1 \), as the nonzero homology of the bar construction is concentrated on the diagonal.

Note that the normal monomials of weight two in \( \emptyset \) are \( A = \nu \circ_1 \nu, B = \mu \circ_2 \nu, C = \rho \circ_2 \nu, \) and \( D = \nu \circ_2 \nu \). Most of the compositions of these elements are either zero or normal monomials of weight four with respect to the Gröbner basis of \( \emptyset \); more precisely, by a direct computation, we have

\[
\begin{align*}
A \circ_1 B &= A \circ_2 B = A \circ_3 B = B \circ_1 B = B \circ_2 B = B \circ_3 B = 0, \\
C \circ_1 B &= C \circ_2 B = C \circ_3 B = D \circ_1 B = D \circ_2 B = D \circ_3 B = 0, \\
A \circ_1 C &= A \circ_2 C = A \circ_3 C = B \circ_1 C = B \circ_2 C = B \circ_3 C = 0, \\
C \circ_1 C &= C \circ_2 C = C \circ_3 C = D \circ_1 C = D \circ_2 C = D \circ_3 C = 0, \\
B \circ_1 A &= C \circ_1 A = C \circ_1 D = 0
\end{align*}
\]

and all other weight two compositions of \( A, B, C, \) and \( D \) are normal monomials with respect to the Gröbner basis of \( \emptyset \) with one exception of the element \( B \circ_1 D \). (In this computation, \( B \circ_1 A \) requires two reductions to get to zero, and all other elements require a single reduction.) This latter element is not a normal monomial with respect to the Gröbner basis of \( \emptyset \), and its normal form with respect to that Gröbner basis is the element \((\rho \circ_2 (\nu \circ_2 \nu)) \circ_2 \nu\), which does not belong to \( T(\mu, \nu, \rho)[2] \). It follows that the basis of \( \emptyset[2]_{(4)} \) is given by the composites of \( A, B, C, \) and \( D \) that are normal monomials with respect to the Gröbner basis of \( \emptyset \), and the element \( B \circ_1 D \). Examining elements of weight 6, we discover that there are two relations between \( A, B, C, \) and \( D \) that do not follow from the weight four ones:

\[
B \circ_1 (D \circ_1 A) = 0 = B \circ_1 (D \circ_1 D).
\]

For instance,

\[
(B \circ_1 D) \circ_1 A = ((\rho \circ_2 (\nu \circ_2 \nu)) \circ_2 \nu) \circ_1 (\nu \circ_1 \nu) = ((\rho \circ_1 \nu) \circ_1 \nu) \circ_2 ((\nu \circ_2 \nu) \circ_1 \nu),
\]

and it remains to recall that \( \rho \circ_1 \nu = 0 \). We conclude that the minimal set of relations of the operad \( \emptyset[2] \) is not quadratic, so the bar complex of this operad operad has off-diagonal homology classes and this operad is not Koszul, so its rate of growth of homology is larger than 1 (the rate of any Koszul operad). Thus this operad serves as a counterexample to statements (2) and (4).

\[\square\]

4 Research of polynomial identities from the Veronese viewpoint

In this section, we discuss some work on polynomial identities, both classical and recent, in the language of Veronese powers.
4.1 Naïve Veronese powers and classical triple systems

Proposition 3.9 (or its version where composition in the first slot is replaced by composition in the last slot) is applicable, among other cases, to the “Three Graces” Ass, Com, Lie. For $d = 2$, this recovers the classical definitions of triple systems.

**Proposition 4.1**
- $V_2(\text{Ass}) = \text{Ass}^{[2]}$ is the operad controlling totally associative triple systems;
- $V_2(\text{Com}) = \text{Com}^{[2]}$ is the operad controlling totally commutative associative triple systems;
- $V_2(\text{Lie}) = \text{Lie}^{[2]}$ is the operad LTS controlling Lie triple systems.

**Proof** In each of these statements, the equality follows from Proposition 3.9, and it only remains to check that there are no other relations. In the first two cases, this is obvious, as already the relations we have give a tight upper bound on the size of the Veronese power. For the third one, to show that all the defining relations of $V_2(\text{Lie}) = \text{Lie}^{[2]}$ are quadratic, we use Theorem 3.13. Indeed, it is known [14] that for the reverse path degree-lexicographic order on shuffle trees the defining relation of Lie forms a Gröbner basis, so the left comb products of the generator are precisely the normal tree monomials. Thus, the operad $\text{Lie}^{[2]}$ is Koszul and therefore quadratic. \hfill $\Box$

4.2 Beyond the naïve Veronese powers

Most of our work is focussed on operads covered by Proposition 3.9, which are operads for which Veronese powers coincide with naïve Veronese powers. We shall now discuss two important cases going beyond this framework, a very classical one in the context of polynomial identities, and a very recent one.

One important example not covered by Proposition 3.9 is the operad of Jordan algebras. Those algebras are usually defined as commutative (non-associative) algebras satisfying the identity

$$(ab)(aa) = a(b(aa))$$

which is not multilinear. Since we work in the context of operad theory, we recall an equivalent multilinear version.

**Definition 4.2** The operad $\text{Jord}$ of Jordan algebras is generated by one operation $a_1, a_2 \mapsto a_1a_2$, subject to the relation

$$((a_1a_2)a_3)a_4 + ((a_1a_4)a_3)a_2 + ((a_2a_4)a_3)a_1$$

$$= (a_1a_2)(a_3a_4) + (a_1a_3)(a_2a_4) + (a_1a_4)(a_2a_3).$$

The classical definition of a Jordan triple system [24] as recalled below is intimately related to our notion of Veronese powers, although the way that definition is usually given slightly obscures this fact.
Definition 4.3 A Jordan triple system is a vector space $V$ with a trilinear operation $\{-,-,-\}: V^3 \to V$ possessing the symmetry $\{a_1, a_2, a_3\} = \{a_3, a_2, a_1\}$ and satisfying the identity

\[
\{a_1, a_2, \{a_3, a_4, a_5\}\} = \{\{a_1, a_2, a_3\}, a_4, a_5\} - \{a_3, \{a_2, a_1, a_4\}, a_5\} + \{a_3, a_4, \{a_1, a_2, a_5\}\}.
\]

These identities are satisfied by the operation

\[
\{a_1, a_2, a_3\} = (a_1a_2)a_3 + a_1(a_2a_3) - a_2(a_1a_3)
\]

in any Jordan algebra. One can easily check that this operation generates $\text{Jord}^{[2]}$ as an $S_3$-module, thus the suboperad of $\text{Jord}$ generated by this operation is $\text{Jord}^{[2]}$. An easy computation shows that the relations of the operad $\text{JTS}$ of Jordan triple systems are the only quadratic relations satisfied by this operation, so in particular, we have a natural surjection from the operad $\text{JTS}$ to the operad $\text{Jord}^{[2]}$. To the best of our knowledge, it is not known whether $\text{JTS} \cong \text{Jord}^{[2]}$. We conjecture that it is the case.

Conjecture 4.4 We have $\text{JTS} \cong \text{Jord}^{[2]}$. In other words, all relations satisfied by the operation

\[
(a_1a_2)a_3 + a_1(a_2a_3) - a_2(a_1a_3)
\]

in any Jordan algebra follow from the axioms of Jordan triple systems. In yet other words, the operad $\text{Jord}^{[2]}$ is quadratic.

Of course, it is even less clear what the higher Veronese powers $\text{Jord}^{[k]}$ are. Note that the “Jordan quadruple systems” [8] are defined in a way that makes it not at all clear how they are related to Veronese powers.

Another important example of an operad not covered by Proposition 3.9 is the pre-Lie operad.

Definition 4.5 The operad $\text{PreLie}$ of pre-Lie algebras is generated by one operation $a_1, a_2 \mapsto a_1a_2$, subject to the relation

\[
(a_1a_2)a_3 - a_1(a_2a_3) = (a_1a_3)a_2 - a_1(a_3a_2).
\]

The Veronese square of this operad was essentially studied by Bremner and Madariaga [7]; their computations take place inside the free dendriform algebra, but as the operad $\text{PreLie}$ is well known to be a suboperad of the dendriform operad, this is sufficient to study free pre-Lie algebras and the pre-Lie operad.

Proposition 4.6 ([7, Theorem 4.5]) Consider the following two elements of the operad $\text{PreLie}^{[2]}$:

\[
[a_1, a_2, a_3]_1 = (a_1a_2)a_3, \quad [a_1, a_2, a_3]_2 = a_1(a_2a_3).
\]
All relations of weight at most two between these elements follow from the relations

\[
[a_1, a_2, a_3]_1 - [a_1, a_2, a_3]_2 = [a_1, a_3, a_2]_1 - [a_1, a_3, a_2]_2,
\]
\[
[[a_1, a_2, a_3]_1, a_4, a_5]_2 - [[a_1, a_4, a_5]_2, a_2, a_3]_1 + [a_1, [a_4, a_5, a_2], a_3]_1
- [a_1, [a_2, a_4, a_5], a_3]_1 + [a_1, a_2, [a_4, a_5, a_3]]_1 - [a_1, a_2, [a_3, a_4, a_5]_2]_1 = 0,
\]
\[
[[a_1, a_2, a_3]_1, a_4, a_5]_1 - [[a_1, a_4, a_2]_1, a_3, a_5]_1
+ [[a_1, a_4, a_3]_1, a_2, a_5]_1 - [[a_1, a_3, a_2]_1, a_4, a_5]_1 - [a_1, [a_2, a_3, a_4]_1, a_5]_1
+ [a_1, [a_4, a_2, a_3]_1, a_5]_1 - [a_1, [a_4, a_3, a_2]_1, a_5]_1 + [a_1, [a_3, a_2, a_4]_1, a_5]_1 = 0,
\]
\[
[[a_1, a_2, a_3]_1, a_4, a_5]_2 - [[a_1, a_4, a_5]_2, a_2, a_3]_1
+ [[a_1, a_4, a_5]_2, a_2, a_3]_2 - [[a_1, a_2, a_3]_2, a_4, a_5]_2 + [a_1, [a_4, a_5, a_2]_1, a_3]_1
+ [a_1, [a_4, a_5, a_3]_1, a_2]_1 - [a_1, [a_2, a_3, a_5]_1, a_4]_1 - [a_1, [a_4, a_5, a_3]_1, a_2]_2
+ [a_1, [a_2, a_3, a_5]_1, a_4]_2 - [a_1, [a_2, a_4, a_5]_2, a_3]_1 - [a_1, [a_3, a_4, a_5]_2, a_2]_1
- [a_1, [a_4, a_2, a_3]_2, a_5]_2 + [a_1, [a_2, a_4, a_5]_2, a_3]_2 + [a_1, [a_3, a_4, a_5]_2, a_2]_2
+ [a_1, a_4, [a_2, a_3, a_5]_1]_1 - [a_1, a_4, [a_5, a_2, a_3]_2]_2 = 0.
\]

Bremner and Madariaga prove [7, Theorem 4.5] that the operad PreLie\(^{[2]}\) has no cubic relations that do not follow from the quadratic ones, and make a conjecture which may be re-stated in terms of Veronese powers as follows.

**Conjecture 4.7** ([7, Conjecture 4.8]) The operad PreLie\(^{[2]}\) is quadratic.

### 4.3 Dialgebras and Veronese powers

In some recent papers on polynomial identities, the notion of \( \mathcal{P} \)-dialgebras (for various operads \( \mathcal{P} \)) has been studied in the context of Veronese powers. Let us recall the relevant definition and its historical context. Recall that the operad Perm of “permutative” algebras encodes (non-unital) associative algebras satisfying the extra identity \( abc = acb \).

**Definition 4.8** Let \( \mathcal{P} \) be an operad. The operad \( \mathcal{P} \)-\( \mathcal{P} \) is the Hadamard product \( \mathcal{P} \otimes \text{Perm} \). Algebras over this operad are called \( \mathcal{P} \)-dialgebras.

This definition goes back to Chapoton [12]. Later, explicit algorithms for computing all identities of \( \mathcal{P} \)-dialgebras from all identities of \( \mathcal{P} \)-algebras were suggested by several authors [10,29,40], and these algorithms were proved to give the same result [30].

Let us establish a result on Veronese powers of Hadamard products of operads which we shall then use to relate the definition of \( \mathcal{P} \)-dialgebras to Veronese powers. Let \( \mathcal{O} \) be an operad and

\[
\mu^\mathcal{O} : \mathcal{F}(\mathcal{O}) \to \mathcal{O}
\]

\( \mathcal{O} \) Springer
the natural map given by the structure operations of $\mathcal{O}$. Recall that the arity $n$ component $\mathcal{F}(\mathcal{O})(n)$ of the free operad $\mathcal{F}(\mathcal{O})$ decomposes into the direct sum,

$$\mathcal{F}(\mathcal{O})(n) = \bigoplus_{[T]} \mathcal{F}_T(\mathcal{O}),$$

over isomorphism classes of labelled rooted trees with $n$ leaves. Denote finally by

$$\mu^\mathcal{O}_T : \mathcal{F}_T(\mathcal{O}) \to \mathcal{O}(n),$$

the restriction of $\mu^\mathcal{O}$ to $\mathcal{F}_T(\mathcal{O})$.

**Proposition 4.9** Let $\mathcal{P}$ be a weight graded operad and $\mathcal{Q}$ an operad. Let us equip the Hadamard product $\mathcal{P} \otimes \mathcal{Q}$ with a weight grading by postulating that the weight of $p \otimes q \in \mathcal{P} \otimes \mathcal{Q}$ equals the weight of $p$. Then

$$V_k(\mathcal{P} \otimes \mathcal{Q}) = V_k(\mathcal{P}) \otimes \mathcal{Q} \quad \text{and} \quad (\mathcal{P} \otimes \mathcal{Q})[k] \subset \mathcal{P}[k] \otimes \mathcal{Q}.$$

Suppose moreover that all restrictions $\mu^\mathcal{Q}_T$ as in (4.1) are epimorphisms. Then the inclusion above is actually an equality

$$(\mathcal{P} \otimes \mathcal{Q})[k] = \mathcal{P}[k] \otimes \mathcal{Q}.$$ 

**Proof** The first part of the proposition is elementary. To prove the second one, notice that elements of $\mathcal{P}[k] \otimes \mathcal{Q}$ are linear combinations of elements of the form $\mu^\mathcal{P}_T(x) \otimes q$, where $x \in \mathcal{F}_T(\mathcal{P}(k))$ and $q \in \mathcal{Q}$. By assumption, $q = \mu^\mathcal{Q}_T(y)$ for some $y \in \mathcal{F}_T(\mathcal{Q})$, therefore

$$\mu^\mathcal{P}_T(x) \otimes q = \mu^\mathcal{P}_T(x) \otimes \mu^\mathcal{Q}_T(y) = \mu^\mathcal{P} \otimes \mathcal{Q}_T(x \otimes y) \in (\mathcal{P} \otimes \mathcal{Q})[k],$$

as claimed. $\square$

The assumption required by the second part of the proposition means, in plain language, that given an arbitrary composition scheme, every element of $\mathcal{Q}$ is decomposable with respect to this scheme. If the operad $\mathcal{Q}$ is generated by binary operations, it is sufficient to require that the restrictions $\mu^\mathcal{Q}_T$ are epimorphisms for all binary trees $T$. Such operads were studied in [45], where it was proved that for such operad $\mathcal{Q}$, the Hadamard product $\mathcal{P} \otimes \mathcal{Q}$ coincides with the Manin white product of $\mathcal{P}$ and $\mathcal{Q}$. In particular, this property holds for the operad Perm, see [45, Corollary 16]. We therefore have

**Corollary 4.10** Let $\mathcal{P}$ be a weight graded operad. For such an operad, the operad of dialgebras construction commutes with Veronese powers:

$$V_k(di-\mathcal{P}) = di-V_k(\mathcal{P}) \quad \text{and} \quad (di-\mathcal{P})[k] = di-(\mathcal{P}[k]).$$
In [9], Leibniz triple systems are introduced, taking as a starting point the operad LTS of Lie triple systems. Namely, Leibniz triple systems are defined as LTS-dialgebras. One of the results of [9] can be re-stated in the language of our paper as follows.

Proposition 4.11 ([9, Theorem 23]) The Veronese power $\text{Leib}^{[2]}$ is isomorphic to the operad of Leibniz triple systems.

Let us give a short alternative proof of this result. By [12], $\text{Leib} = \text{di-Lie}$, and by Proposition 4.1, $\text{LTS} \cong \text{Lie}^{[2]}$. Finally, by Corollary 4.10,

$$\text{Leib}^{[2]} = (\text{di-Lie})^{[2]} = \text{di-}(\text{Lie}^{[2]}) \cong \text{di-LTS},$$

as required.

In [6], the authors define and study a version of the previous definition where Lie algebras are replaced by Jordan algebras. Namely, they define a Jordan triple disystem as a JTS-dialgebra. One of the results proved in [6] can be re-stated in the language of our paper as follows.

Proposition 4.12 ([6, Theorem 7.3]) Generating operations of the Veronese square $(\text{di-Jord})^{[2]}$ satisfy the identities of Jordan triple disystems.

Let us give a short alternative proof of this result. Indeed, by Corollary 4.10, we have

$$(\text{di-Jord})^{[2]} = \text{di-}(\text{Jord}^{[2]}),$$

and $\text{Jord}^{[2]}$ is a quotient of JTS, so $\text{di-}(\text{Jord}^{[2]})$ is a quotient of $\text{di-} \text{JTS}$, as required.

5 Operads of pure homotopy algebras and Koszul duality

In this section, we explain how to compute Koszul duals of Veronese powers of quadratic operads. Let us begin with stating a result on Koszul duals of operads $\text{Ass}$ and $\text{Com}$ which can be proved by a direct calculation.

Proposition 5.1

- The Koszul dual operad of $\text{Ass}^{[k]}$ is the operad controlling a particular class of $A_\infty$-algebras, those with only nonzero structure operation being the one of arity $k+1$, or, equivalently, the operad of partially associative $(k+1)$-ary algebras with the structure operation of homological degree $k - 1$.
- The Koszul dual operad of $\text{Com}^{[k]}$ is the operad controlling a particular class of $L_\infty$-algebras, those with only nonzero structure operation being the one of arity $k+1$, or, equivalently, the operadic desuspension of the the operad of Lie $(k+1)$-algebras [23].

The fact that $(\text{Ass}^{[k]})^!$ has to do with $A_\infty$-algebras and $(\text{Com}^{[k]})^!$ has to do with $L_\infty$-algebras makes one expect that a general statement relating Veronese powers to homotopy algebras exists. The goal of this section is to confirm this guess. For that, we introduce a new general notion of pure homotopy $P$-algebras, and then prove that (appropriate quadratic versions of) Veronese powers of operads and operads controlling pure homotopy algebras are exchanged by Koszul duality.
Definition 5.2 Let $k > 0$ be an integer.

- Let $\mathcal{Q}$ be a connected weight graded cooperad. The weight $k$ Koszul dual operad $Q^i_{[k]}$ is the largest quotient dg operad of the cobar complex $\Omega(\mathcal{Q}) = (T(s^{-1}\mathcal{Q}), d_{\Omega(\mathcal{Q})})$ generated (as a non-dg operad) by $s^{-1}\mathcal{Q}(k)$.
- Let $\mathcal{P}$ be a connected weight graded operad. The weight $k$ Koszul dual cooperad $P^i_{[k]}$ is the smallest dg subcooperad of the bar complex $B(\mathcal{P}) = (T(s\mathcal{P}), d_B(\mathcal{P}))$ cogenerated (as a non-dg cooperad) by $s\mathcal{P}(k)$.

The following proposition makes this definition more explicit.

Proposition 5.3

- For a connected weight graded cooperad $\mathcal{Q}$ and an integer $k > 0$, the operad $Q^i_{[k]}$ is the (non-dg) operad that can be presented via generators $s^{-1}\mathcal{Q}(k)$ and relations which are desuspended quadratic co-relations on $\mathcal{Q}(k)$ in $\mathcal{Q}$.
- For a connected weight graded operad $\mathcal{P}$ and an integer $k > 0$, the cooperad $P^i_{[k]}$ is the (non-dg) cooperad that can be presented via co-generators $s\mathcal{P}(k)$ and co-relations which are suspended quadratic relations on $\mathcal{P}(k)$ in $\mathcal{P}$.

Proof To prove the first statement, we note that the underlying operad of $\Omega(\mathcal{Q})$ is free, so to create a quotient generated by $s^{-1}\mathcal{Q}(k)$, we have to quotient out all elements $s^{-1}\mathcal{Q}(l)$, $l \neq k$. Furthermore, any dg ideal of $\Omega(\mathcal{Q})$ containing all elements of $s^{-1}\mathcal{Q}$ of weight different from $k$ must contain all the differentials of these elements, and so the smallest such ideal coincides with the (usual) ideal of $T(s^{-1}\mathcal{Q})$ generated by all elements of $s\mathcal{Q}$ of weight different from $k$ and their differentials. Note that the differential of any homogeneous element of weight different from $2k$ is a combination of tree tensors involving at least one element of weight different from $k$, so modulo the ideal generated by elements of weight different from $k$ such an element is congruent to zero. Also, modulo the same ideal the differential of any homogeneous element of weight $2k$ is congruent to the suspended image of the decomposition map $\Omega(2k) \to \mathcal{T}^c(\mathcal{Q}(k))_2$. These observations establish both the shape of relations and the lack of differential in the quotient. The second statement is proved analogously.

Let us give some examples of operads of pure homotopy algebras which agree with our previous observations.

Proposition 5.4

- Let $\mathcal{Q} = \text{Com}^c$ be the linear dual of $\text{Com}$. Then the operad $Q^i_{[k]}$ is the operadic suspension of the operad of Lie $(k + 1)$-algebras of $[23]$.
- Let $\mathcal{Q} = \text{Ass}^c$ be the linear dual of $\text{Ass}$. Then the operad $Q^i_{[k]}$ is the operad $\text{pAss}^{k+1}_{k+1}$ of $[35]$.
- Suppose that $\mathcal{Q}$ is quadratic cooperad (cogenerated by elements of weight one). Then the operad $Q^i_{[1]}$ is the classical quadratic dual operad $Q^i$. Similarly, if $\mathcal{P}$ is a quadratic operad (generated by elements of weight one), then the cooperad $P^i_{[1]}$ is the classical quadratic dual cooperad $P^i$.

Proof These are obtained from Proposition 5.3 by a direct inspection.

In general, Veronese powers need not be quadratic, so in order to include Veronese powers in the context of Koszul duality, we should modify the definition appropriately.
Definition 5.5 Let $\mathcal{P}$ be a weight graded operad, and let $d > 0$ be an integer. The quadratic Veronese power $q^{[d]}\mathcal{P}$ is the quotient of $\mathcal{I}(\mathcal{P}_d)$ by the ideal of all quadratic relations satisfied by $\mathcal{P}_d$, that is $\mathcal{I} \cap \mathcal{I}(\mathcal{P}_d)/(2)$, where $\mathcal{I} \subset \mathcal{I}(\mathcal{P}_d)$ is the kernel of the evaluation map $\mathcal{I}(\mathcal{P}_d) \to \mathcal{P}$.

In general, there is a surjection $q^{[d]}\mathcal{P} \to \mathcal{P}^{[d]}$, and an inclusion $\mathcal{P}^{[d]} \hookrightarrow \mathcal{V}_d(\mathcal{P})$, so the set-up we are dealing with is reminiscent of the situation with Manin products and Hadamard products, see [45].

We are ready to formulate the main result of this section.

Theorem 5.6 Let $\mathcal{P}$ be a connected weight graded operad, and let $d > 0$ be an integer. We have $(q^{[d]}\mathcal{P})^i \cong \mathcal{P}_i^{[d]}$.

Note that this statement is valid in full generality, that is the operad $\mathcal{P}$ need not be quadratic, its Veronese power need not be quadratic etc. However, it becomes particularly meaningful under some assumptions on $\mathcal{P}$, for example, assuming that $\mathcal{P}$ is quadratic and that $q^{[d]}\mathcal{P} = \mathcal{P}^{[d]}$.

Proof The result is stated in such a way that the proof becomes almost tautological. Indeed, the cooperad $(q^{[d]}\mathcal{P})^i$ is the subcooperad of $\mathcal{I}^c(s\mathcal{P}_d)$ presented by co-relations which are suspended quadratic relations of $\mathcal{P}^{[d]}$. By Proposition 5.3, this cooperad is isomorphic to $\mathcal{P}_i^{[d]}$. $\Box$

The following proposition demonstrates how our results can be applied in particular cases to produce results which are far from obvious.

Theorem 5.7

- The operad $\text{LTS}$ encoding Lie triple systems is Koszul. Its Koszul dual is the operad $\text{Com}_{\infty,3}$ encoding $C_{\infty}$-algebras whose nonzero structure operations are in arity 3.

- More generally, for every $k \geq 1$ we have $V_k(\text{Lie}) = \text{Lie}^{[k]} = q\text{Lie}^{[k]}$. This operad is Koszul, and its Koszul dual is the operad $\text{Com}_{\infty,k+1}$ encoding $C_{\infty}$-algebras whose nonzero structure operations are in arity $k + 1$.

Proof The first statement is the particular case $k = 2$ of the second one. To prove the second one, we note that from the argument of Proposition 4.1 it follows that the operad $\text{Lie}^{[k]}$ is Koszul, and in particular quadratic, so $\text{Lie}^{[k]} = q\text{Lie}^{[k]}$. Also, we already know from Proposition 3.9 that $V_k(\text{Lie}) = \text{Lie}^{[k]}$. Finally, Theorem 5.6 implies that $(q\text{Lie}^{[k]})^i \cong \text{Lie}^{[k]}_i$. The suspended dual of the latter is manifestly $\text{Com}_{\infty,k+1}$, which completes the proof. $\Box$

We conclude this section with an amusing corollary showing how Bernoulli numbers arise in dimension formulas for pure $\text{Com}_{\infty}$-algebras. For the case of pure $\text{Lie}_{\infty}$-algebras, we refer the reader to [47, Example 4.3.5]. Quite remarkably, in that case there is a simple closed dimension formula

$$\dim \text{Lie}_{\infty,3}(2n - 1) = ((2n - 3)!!)^2.$$
Corollary 5.8  For each $n \geq 1$, we have
\[
\dim \text{Com}_{\infty,3}(2n - 1) = \frac{2^{2n}(2^{2n} - 1)|B_{2n}|}{2n},
\]
where $B_{2n}$ is the $2n$-th Bernoulli number.

Proof  By Theorem 5.7 the operad of Lie triple systems is Koszul, and its Koszul dual cooperad is $\text{Lie}^{[2]}$. That cooperad is cogenerated by ternary operations of homological degree 1, thus, its component of arity $2n - 1$ is concentrated in homological degree $n - 1$. Formula (1) of [16, Corollary 5], generalising the functional equation of [21, Theorem 3.3.2] for operads generated by operations that are not binary (and stated using Euler characteristics rather than dimensions), shows that the generating series of the operad LTS is the compositional inverse of the series
\[
\sum_{n \geq 1} \frac{(-1)^{n-1} \dim \text{Com}_{\infty,3}(2n - 1)}{(2n - 1)!} t^{2n-1}.
\]
Equivalently, the series
\[
g(t) = \sum_{n \geq 1} \frac{(-1)^{n-1} \dim \text{LTS}(2n - 1)}{(2n - 1)!} t^{2n-1}
\]
with modified signs is the compositional inverse of the series
\[
\sum_{n \geq 1} \frac{\dim \text{Com}_{\infty,3}(2n - 1)}{(2n - 1)!} t^{2n-1}.
\]
For the operad LTS of Lie triple systems, we have LTS $\cong V_2(\text{Lie})$, so
\[
\dim \text{LTS}(2n - 1) = \dim \text{Lie}(2n - 1) = (2n - 2)!
\]
for each $n \geq 1$. Thus,
\[
g(t) = \sum_{n \geq 1} \frac{(-1)^{n-1}(2n - 2)!}{(2n - 1)!} t^{2n-1} = \sum_{n \geq 1} \frac{(-1)^{n-1}t^{2n-1}}{2n - 1} = \arctan(t),
\]
so from the well-known formula [36, Appendix B]
\[
\tan(t) = \sum_{n \geq 1} \frac{2^{2n}(2^{2n} - 1)|B_{2n}|}{(2n)!} t^{2n-1}
\]
it immediately follows that
\[
\dim \text{Com}_{\infty,3}(2n - 1) = \frac{2^{2n}(2^{2n} - 1)|B_{2n}|}{2n},
\]
6 Pure homotopy Lie algebras and their mock versions

In this section, we shall discuss some versions of the operad of pure homotopy Lie algebras that occasionally appear in the literature. More precisely, numerous sources tend to call a "strongly homotopy Lie \( n \)-algebra" a vector space with a skew-symmetric \( n \)-ary operation of homological degree zero satisfying the same identity that the operation of a pure homotopy Lie algebra would satisfy. In most such cases, existing literature seems to either silently ignore the matter of homological degrees of operations, see e.g. [13,17,18,22,26,27,32] or to express a belief that ignoring homological degrees does not change much, e.g., [11] notes the discrepancy but states “A little care over the sign factors appearing in the constructions would be required. However, this should be very tractable.”. While several available sources mention that these mock \( n \)-algebras are not really related to homotopy Lie algebras, see e.g. [41,42], there is no clear indication as to what the difference between the two notions is. We shall highlight an important difference by demonstrating that in fact unexpected extra relations occur in that operad, forcing it to not be Koszul; thus, homological degrees and related Koszul signs actually do matter a lot. For this reason, we suggest to call such algebras “mock pure homotopy algebras”, where “mock” in this instance stands for “wrong homological degree”.

Let us emphasise that there are many other \( n \)-ary Lie algebras present in the literature (perhaps the most general class including most of them is described in [46], see also [43, Sect. 5] and references therein), whereas we only focus on those easily confused with homotopy Lie algebras; investigating homotopy properties in other cases is an interesting question for further study. Let us give the relevant definitions.

**Definition 6.1**

- The operad \( t\text{Com}^d_n \) of totally associative commutative \( n \)-ary algebras with operation in homological degree \( d \) is generated by one element \( \mu \) of arity \( n \) and of degree \( d \) which is fully symmetric, that is \( \mu . \sigma = \mu \) for all \( \sigma \in S_n \), and

\[
\mu \circ_i \mu = \mu \circ_j \mu
\]

for arbitrary \( 1 \leq i, j \leq n \).

- The operad \( \text{Lie}^d_n \) of mock \( n \)-pure homotopy Lie algebras with operation in homological degree \( d \) is generated by one element \( \ell \) of arity \( n \) and of degree \( d \) which is fully antisymmetric, that is \( \ell . \sigma = \text{sgn}(\sigma) \ell \) for all \( \sigma \in S_n \), and

\[
\sum_\delta \text{sgn}(\delta)(\ell \circ_1 \ell) . \delta = 0,
\]

where the summation is taken over all \((n, n-1)\)-unshuffles \( \delta \), that is permutations in \( S_{2n-1} \) for which \( \delta(1) < \cdots < \delta(n) \) and \( \delta(n+1) < \cdots < \delta(2n-1) \).

For example, the operad \( t\text{Com}^2_0 \) is the operad encoding the usual commutative associative algebras, and the operad \( \text{Lie}^2_0 \) is the operad encoding Lie algebras. In general,
for $n \geq 2$, the operad $\text{Lie}_{n-2}^n$ encodes $L_\infty$-algebras [31] without differential whose only nontrivial operation is in arity $n$.

For the same technical reasons as in [35] we introduce auxiliary operads by means of operadic suspension:

$$\widetilde{\text{tCom}}_d^n = S \otimes \text{tCom}_{d+n-1}^n$$

$$\widetilde{\text{Lie}}_d^n = S \otimes \text{Lie}_{d+n-1}^n.$$  

**Proposition 6.2**

- The operad $\widetilde{\text{tCom}}_d^n$ is generated by a degree $d$ fully antisymmetric operation $\mu$ of arity $n$ satisfying

$$(-1)^{(i+1)} \mu \circ i \mu = (-1)^{(j+1)} \mu \circ j \mu$$

for arbitrary $1 \leq i, j \leq n$.

- The operad $\widetilde{\text{Lie}}_d^n$ is generated by a degree $d$ fully symmetric operation $\ell$ of arity $n$ satisfying

$$\sum_{\delta} (\ell \circ 1 \ell \delta) = 0,$$

where the summation runs over all $(n, n-1)$-unshuffles $\delta$.

**Proof** The claims will follow from the description of the partial compositions $\circ i$ in the operad $S = \text{End}_{kS}$ given below. We use a one-to-one correspondence between degree $1-r$ linear maps $\omega: kS^\otimes r \to kS \in S(r)$, $r \geq 1$, and degree 0 linear maps $\alpha: k \to k$ given by the commutative diagram

$$
\begin{array}{ccc}
kS^\otimes r & \xrightarrow{\omega} & kS \\
\uparrow & & \downarrow
\end{array}
\quad
\begin{array}{ccc}
k^\otimes r & \xrightarrow{\alpha} & k \\
\uparrow & & \downarrow
\end{array}
$$

where $\uparrow: k \to kS$, resp. $\downarrow: kS \to k$ are the canonical degree +1, resp. degree −1, linear isomorphisms. The arity $n$ piece $S(n)$ of the operad $S$ is then linearly generated by a degree $1-n$ generator $e_n$ defined by the commutativity of

$$
\begin{array}{ccc}
kS^\otimes n & \xrightarrow{e_n} & kS \\
\uparrow & & \downarrow
\end{array}
\quad
\begin{array}{ccc}
k^\otimes n & \xrightarrow{id} & k \\
\uparrow & & \downarrow
\end{array}
$$

For $1 \leq i \leq n$ the Koszul sign rule gives

$$
\downarrow (e_n \circ_i e_n) \uparrow^{\otimes 2n-1} = \downarrow e_n (\uparrow^{\otimes i-1} \otimes e_n \otimes \uparrow^{\otimes n-i}) \uparrow^{\otimes 2n-1}
= (-1)^{(i-1)(1-n)} \downarrow e_n (\uparrow^{\otimes i-1} \otimes \downarrow e_n (\uparrow^{\otimes n} \otimes \uparrow^{\otimes n-i})$$
which translates into
\[ e_n \circ_i e_n = (-1)^{(i-1)(1-n)} e_{2n-1}, \]
with \( e_{2n-1} \) as in the diagram

\[
\begin{array}{ccc}
\mathbb{S}^\otimes 2n-1 & \to & \mathbb{S}^\otimes 2n-1 \\
\uparrow_{e_{2n-1}} & & \downarrow \\
\mathbb{S} & \cong & \mathbb{S}
\end{array}
\]

Let \( \mathcal{P} \) be an operad and \( \mu \in \mathcal{P}(n) \) be homogeneous of degree \( t \). Then \((e_n \otimes \mu) \in (\mathbb{S} \otimes \mathcal{P})(n)\) is of degree \( t + 1 - n \) and
\[
(e_n \otimes \mu) \circ_i (e_n \otimes \mu) = (-1)^{(1-n)} (e_n \circ_i e_n) \otimes (\mu \circ_i \mu) = (-1)^{(1-n)+(i-1)(1-n)} e_{2n-1} \otimes (\mu \circ_i \mu)
= (-1)^{(t+i-1)(1-n)} e_{2n-1} \otimes (\mu \circ_i \mu).
\]

It is clear now that if \( \mu \) fulfills the equality
\[
\mu \circ_i \mu = \mu \circ_j \mu, \quad 1 \leq i, j \leq n,
\]
in \( \mathcal{P} \), then \( \bar{\mu} := e_n \otimes \mu \) satisfies
\[
(-1)^{(t+i-1)(1-n)} (\bar{\mu} \circ_i \bar{\mu}) = (-1)^{(t+j-1)(1-n)} (\bar{\mu} \circ_j \bar{\mu})
\]
which, after multiplying both sides by \((-1)^{(t-1)(n-1)}\), changing \( n \) to \(-n\) and writing \( \mu \) instead of \( \bar{\mu} \), becomes
\[
(-1)^{(n+1)} (\mu \circ_i \mu) = (-1)^{(n+1)} (\mu \circ_j \mu),
\]
which is (6.1). The proof of the first part is finished by observing that the Hadamard product with \( \mathbb{S} \) changes fully symmetric operations to fully antisymmetric ones. The above calculations make the second part of the proposition obvious. \( \square \)

**Proposition 6.3** One has the isomorphisms
\[
(\widetilde{\text{Lie}}_d^n)_i \cong (t\text{Com}_{1+d}^n)_c \quad \text{and} \quad (\text{Lie}_d^n)_i \cong (t\text{Com}_{1+d}^n)_c
\]
(Here the superscript \( c \) denotes component-wise dualization of \( \mathcal{P} \) followed by reversing homological degrees.) Equivalently, using the language of Koszul dual operads,
\[
(\text{Lie}_d^n)_i \cong t\text{Com}_{n-2-d}^n \quad \text{and} \quad (\widetilde{\text{Lie}}_d^n)_i \cong t\text{Com}_{n-2-d}^n.
\]
Proof The first isomorphism is established by direct inspection; the others follow from the standard properties of Koszul duality and operadic suspension.

\[ \Box \]

Theorem 6.4 The operad \( \tilde{\text{Lie}}_d^n \) is Koszul if and only if \( d \) is odd; the operad \( \text{Lie}_d^n \) is Koszul if and only if \( n \) and \( d \) have the same parity. Equivalently, the operad \( \text{tCom}_d^n \) is Koszul if and only if \( d \) is even; the operad \( \tilde{\text{tCom}}_d^n \) is Koszul if and only if \( n \) and \( d \) have different parities.

Note that in the view of Proposition 6.3 and properties of operadic suspension, it is enough to consider the operad \( \text{tCom}_d^n \) for various \( n \) and \( d \). Let us begin with describing the underlying collection of that operad.

Proposition 6.5 For even \( d \), the operad \( \text{tCom}_d^n \) has a quadratic Gröbner basis. Its arity \( k \) component \( \text{tCom}_d^n \) is one-dimensional for \( k \equiv 1 \pmod{n - 1} \) and is equal to zero otherwise.

For odd \( d \), the arity \( k \) component of the operad \( \text{tCom}_d^n \) is one-dimensional for \( k = 1, n, \) or \( 2n - 1 \), and is equal to zero otherwise.

Proof To prove the first statement, note that for the purpose of Gröbner basis computation, only the parity of the homological degree matters, so we may assume \( d = 0 \). In this case, we have \( \text{tCom}_d^n = \text{Com}^{[n - 1]} \), and the result follows from Theorem 3.13.

To prove the second statement, we note that a computation similar to that of [15,35] shows that in addition to the existing quadratic relations, the Gröbner basis contains the element \( \mu \circ_n (\mu \circ_n \mu) \), and hence there are no normal tree monomials of arity greater than \( 2k - 1 \), and the first part of the result follows.

It follows that the operad \( \text{tCom}_d^n \) is Koszul for even \( d \), proving the “easy” part of Theorem 6.4.

For \( n = 2 \) and odd \( d \), the operad \( \text{tCom}_d^n \) is not Koszul; for instance it is the case because it has the same Poincaré series as the mock-commutative operad [20], and the latter is shown to fail the positivity criterion, see op. cit., footnote on p. 180. It remains to establish non-Koszulness of the operad \( \text{tCom}_d^n \). It turns out that the positivity criterion is inconclusive for this operad (which we establish in Sect. 7), so another argument is needed. It is more beneficial to pass to the Koszul dual operad and establish that the operad \( \tilde{\text{Lie}}_0^n \) is not Koszul. We shall accomplish that similarly to [16, Theorem 9] by showing that the minimal model of \( \tilde{\text{Lie}}_0^n \) is not isomorphic to the cobar construction of the cooperad \( \tilde{\text{tCom}}_d^n \) as it would have been in the Koszul case.

By Proposition 6.5, the arity \( k \) component of the operad \( \text{tCom}_d^n \) is one-dimensional for \( k = 1, n \) or \( 2n - 1 \), and is equal to zero otherwise. As a consequence, the cobar construction \( \Omega((\text{tCom}_d^n)^\vee) \) is generated by a fully symmetric generator \( \ell \) of arity \( n \) and degree 0, and one fully symmetric generator \( \xi \) of arity \( 2n - 1 \) and degree 1. Elements of the cobar complex can be represented as linear combinations of shuffle tree monomials whose internal vertices are either of arity \( n \) or arity \( 2n - 1 \). In homological degree 0, allowed shuffle tree monomials have only \( n \)-ary internal vertices, and in homological degree 1 all but one internal vertex are \( n \)-ary, and the remaining vertex is of arity \( 2n - 1 \). In this representation, the differential of the cobar construction is the summation of all possible expansions of the vertex of arity \( 2n - 1 \) into two \( n \)-ary vertices.
Fig. 1  The six left combs in \( B_1 \) for the case of the binary operation \( \ell \)

Let us denote by \( \text{LC}_{(n+1)} \) the set of all \((n + 1)\)-fold left comb products of the generator \( \ell \) in the cobar complex \( \Omega((t\text{Com}_1^n)^c) \). The set \( \text{LC}_{(3)} \) is displayed in Fig. 1.

**Theorem 6.6** There exist nonzero integers \( \epsilon_T \in \mathbb{Z} \) given for each shuffle tree \( T \) of homological degree 1 such that for

\[
\nu := \sum_T \epsilon_T T
\]

one has

\[
\partial \nu = n! \sum_{S \in \text{LC}_{(n+1)}} S.
\]

(6.2)

The proof is a straightforward modification of the proof of [16, Theorem 10] to the set-up of shuffle trees. We recall its salient features below. Let us introduce, only for the purposes of this section, the following terminology. A 0-tree will mean a shuffle tree with \( n + 1 \) vertices of arity \( n \). A 1-tree will be a shuffle tree with \( n - 1 \) vertices of arity \( n \) and one vertex of arity \( 2n - 1 \) called the fat one. With a few obvious exceptions, by a tree we mean either a 0-tree or a 1-tree. Elements of \( \text{LC}_{(n+1)} \) are thus particular examples of 0-trees.

As in [16], we prove Theorem 6.6 by explicitly defining the coefficients \( \epsilon_T \). Denote by \( \text{edg}(X) \) the set of internal edges of a tree \( X \). Assume that we are given a rule that divides internal edges of each tree \( X \) into two disjoint subsets, the set \( \text{reg}(X) \) of regular edges and the set \( \text{sgn}(X) \) of singular ones. For a 0-tree \( S \) and its internal edge \( e \in \text{edg}(S) \) denote by \( S/e \) the tree obtained by collapsing \( e \) into a vertex. Suppose that the rule is such that

\[
\text{card}(\text{reg}(S/e)) = \begin{cases} 
\text{card}(\text{reg}(S)) - 1 & \text{if } e \text{ is regular, and} \\
\text{card}(\text{reg}(S)) & \text{if } e \text{ is singular.}
\end{cases}
\]

(6.3)

The proof of the following result is a verbatim copy of the proof of an analogous [16, Lemma 11]. The only feature which is needed for the latter to work is that the differential \( \partial \) acts by contracting internal edges of trees.

**Lemma 6.7** For a 1-tree \( T \) put \( g = g(T) := \text{card}(\text{reg}(T)) \) and define

\[
\epsilon_T := (-1)^{g+n+1} g!(n - g - 1)!
\]
Then the boundary condition

\[ n! (B_1 - (-1)^n B_0) = \partial \left( \sum_T \epsilon_T T \right), \] (6.4)

in which \( B_1 \) (resp. \( B_0 \)) is the sum of all 0-trees with \( \text{sng}(S) = \emptyset \) (resp. with \( \text{reg}(S) = \emptyset \)) is satisfied.

Let us describe a particular rule satisfying (6.3). Given a tree \( X \), we “flatten” it in by pulling the leftmost leaf down to the same level with the root, resulting in a diagram of the form

\[
\begin{array}{c}
R_1 \\
\vdots \\
R_s
\end{array}
\]

where \( R_i \)'s are, for \( 1 \leq i \leq s \), planar rooted trees. We call the result the body of the tree \( X \). The soul of a tree \( X \) is obtained from its body by removing all the external legs; it is a diagram of the form

\[
\begin{array}{c}
T_1 \\
\vdots \\
T_s
\end{array}
\]

where \( T_i \)'s are trees with no external legs. Note that there is a one-to-one correspondence between the set \( \text{edg}(X) \) of internal edges of \( X \) and the set of edges of its soul.

We call an edge of \( X \) singular if it corresponds to the outgoing edge of a non-fat vertex of the soul of \( X \) with no input edge, i.e., when it looks as

\[
\bullet
\]

where \( \bullet \) has no input edges (the horizontal edges of the soul are not counted as the input ones). All remaining edges of \( X \) are regular. It is easy to see that this division of edges into regular and singular fulfils (6.3).

**Proof of Theorem 6.6** By direct inspection, trees in \( \text{LC}_{(n+1)} \) are the only 0-trees with no singular edge, while each 0-tree has at least one regular edge. Thus (6.2) is an immediate consequence of (6.4).

**Remark 6.8** A notable difference is that, while in [16] the set \( B_1 \) consisted of a single planar rooted tree, in our case it is the sum of all shuffle trees with the same underlying planar tree.
To continue in the proof of the non-Koszulness, we consider the two elements

\[ \alpha_n = \sum_{\sigma \in \text{Sh}_1(n^2-1, n-1)} \ell \circ_{1, \sigma} \nu, \]

\[ \beta_n = \sum_{\sigma \in \text{Sh}_1(n-1, n^2-1)} \nu \circ_{1, \sigma} \ell. \]

Note that

\[ \partial \alpha_n = \sum_{\sigma \in \text{Sh}_1(n^2-1, n-1)} \ell \circ_{1, \sigma} \partial \nu \]

\[ = n! \sum_{\sigma \in \text{Sh}_1(n^2-1, n-1)} \ell \circ_{1, \sigma} \left( \sum_{T \in \text{LC}_{(n+1)}} T \right) = n! \sum_{T \in \text{LC}_{(n+2)}} T \]

and

\[ \partial \beta_n = \sum_{\sigma \in \text{Sh}_1(n-1, n^2-1)} \partial \nu \circ_{1, \sigma} \ell \]

\[ = n! \sum_{\sigma \in \text{Sh}_1(n-1, n^2-1)} \partial \nu \circ_{1, \sigma} \ell = n! \sum_{T \in \text{LC}_{(n+2)}} T. \]

Thus, \( \partial (\alpha_n - \beta_n) = 0 \). Let us show that \( \alpha_n - \beta_n \) is not equal to a boundary of anything in the cobar complex.

Consider the following shuffle tree monomial:

\[ \omega_n := \ell \circ_1 \gamma(\xi; \text{id}, \ldots, \text{id}, \ell, \ldots, \ell), \]

in words, this is a three-level tree obtained by substituting \( n - 1 \) copies of \( \ell \) into the last \( n - 1 \) slots of \( \xi \), and then substituting the result in the first slot of \( \ell \).

Note that \( \omega_n \) appears in \( \alpha_n \) with a nonzero coefficient, and does not appear in \( \beta_n \) (since it is not obtained as shuffle substitution of \( \ell \) in the first slot of anything). Thus, it appears with a nonzero coefficient in the cycle \( \alpha_n - \beta_n \). However, this monomial cannot appear in the differential of anything: the differential of \( \ell \) is zero, and the differential of \( \xi \) can only create trees that have an internal edge between two \( n \)-ary vertices, while \( \omega_n \) does not have such edges. This finishes the proof of Theorem 6.4.

**Remark 6.9** The last argument should be compared with the discussion of naïve Veronese powers in Proposition 3.2; indeed, this argument is only possible because the naïve second Veronese power of the free operad is different from the second Veronese power as defined in this paper.
7 The positivity criterion of Koszulness is not decisive for the operad $\tilde{\text{Lie}}_n$

In this section, we consider the possibility of using the positivity criterion of Koszulness for the operad $\tilde{\text{Lie}}_n$. Since the Koszul dual of this operad is a very simple cooperad $(t \text{Com}_n^1)$, it is natural to try to prove non-Koszulness by establishing that the compositional inverse of the Poincaré series of the latter cooperad has negative coefficients. This works for $n = 2$, but it turns out that already for $n = 3$ the inverse series does not have any negative coefficients, which we demonstrate below. The argument is similar to that of [16].

We first recall a classical result on inversion of power series. To state it, we use, for a formal power series $F(t)$, the notation $[t^k]F(t)$ for the coefficient of $t^k$ in $F(t)$, and the notation $F(t)^{-1}$ for the compositional inverse of $F(t)$ (if that inverse exists).

**Proposition 7.1** (Lagrange’s inversion formula [44, Section 5.4]) Let $f(t)$ be a formal power series without a constant term and with a nonzero coefficient of $t$. Then $f(t)$ has a compositional inverse, and

$$[t^k]f(t)^{-1} = \frac{1}{k} [u^{k-1}] \left( \frac{u}{f(u)} \right)^k.$$ 

Let us now prove the main result of this section. Namely, we show that the compositional inverse of the power series $g_{(t \text{Com}_3)^1}(t)$ has nonnegative coefficients, and hence the positivity criterion [16] cannot be used to establish the non-Koszulness of the operad $\tilde{\text{Lie}}_n$.

**Theorem 7.2** The compositional inverse of the power series

$$g_{(t \text{Com}_3)^1}(t) = t - \frac{t^3}{6} + \frac{t^5}{120}$$

is of the form $th(t^2)$, where $h$ is a power series with positive coefficients.

**Proof** First, let us recall the usual argument explaining the form of the inverse series. By Proposition 7.1, we have

$$[t^k] \left( t - \frac{t^3}{6} + \frac{t^5}{120} \right)^{-1} = \frac{1}{k} [u^{k-1}] \left( \frac{u}{u^3/6 + u^5/120} \right)^k,$$

and the coefficients on the right vanish unless $k = 2n + 1$, so the inverse series is of the form $th(t^2)$, where $h$ is some formal power series.

Let us start the asymptotic analysis of the coefficients of the series $h(t)$.

**Lemma 7.3** The radius of convergence of $h(t)$ is equal to $16(3 + \sqrt{3})/75$. 

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The radius of convergence of \((t - t^3/6 + t^5/120)^{(-1)}\) is equal to the maximal \(r\) for which the inverse function of \(t - t^3/6 + t^5/120\) is analytic for the arguments whose modulus is smaller than \(r\). It is obvious that this \(r\) is the value of \(t - t^3/6 + t^5/120\) at the modulus of the smallest zero of

\[
\left( t - \frac{t^3}{6} + \frac{t^5}{120} \right)' = 1 - \frac{t^2}{2} + \frac{t^4}{24}.
\]

The zeros of the latter are

\[
\sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{4}{24}}} = \sqrt{\frac{1}{2} \pm \sqrt{\frac{1}{12}}} = \sqrt{6 \pm 2\sqrt{3}},
\]

so the smallest one is \(\rho = \sqrt{6 - 2\sqrt{3}}\). Thus, the radius of convergence of the inverse series is

\[
\rho \left( 1 - \frac{6 - 2\sqrt{3}}{6} + \frac{(6 - 2\sqrt{3})^2}{120} \right) = \rho \frac{6 + 2\sqrt{3}}{15}.
\]

Now, as \((t - t^3/6 + t^5/120)^{(-1)} = t \, h(t^2)\), the radius of convergence of \(h(t)\) is

\[
\left( \rho \frac{6 + 2\sqrt{3}}{15} \right)^2 = \left( \frac{6 - 2\sqrt{3}}{6} + \frac{6 + 2\sqrt{3}}{120} \right)^2 = \frac{48(3 + \sqrt{3})}{225} = \frac{16(3 + \sqrt{3})}{75}.
\]

**Lemma 7.4** The \(n\)-th coefficient of \(h(t)\) is equal to

\[
a_n = \frac{1}{2n+1} \sum_{k=\lceil n/2 \rceil}^{n} (-1)^{n-k} \binom{2n+k}{k} \binom{k}{n-k} \left( \frac{1}{6} \right)^{2k-n} \left( \frac{1}{120} \right)^{n-k}.
\]

**Proof** Continuing the computation that utilises the Lagrange inversion formula, we see that the \(n\)-th coefficient of \(h\), or equivalently the coefficient of \(t^{2n+1}\) of \((t - t^3/6 + t^5/120)^{(-1)}\), is equal to

\[
\frac{1}{2n+1} \left[ u^{2n} \right] \left( \frac{1}{1 - \frac{u^2}{6} + \frac{u^4}{120}} \right)^{2n+1} = \frac{1}{2n+1} \left[ v^n \right] \left( 1 - \frac{1}{6} + \frac{v^2}{120} \right)^{2n+1},
\]

and expanding the latter using the binomial theorem, we get

\[
\left( \frac{1}{1 - \frac{v}{6} + \frac{v^2}{120}} \right)^{2n+1} = \sum_{k \geq 0} \binom{2n+k}{k} \left( \frac{v}{6} - \frac{v^2}{120} \right)^k.
\]
therefore the coefficient of $t^{2n+1}$ is given by

$$\frac{1}{2n+1} \sum_{k \geq 0} \sum_{2i+k-i=n} \binom{2n+k}{k} \binom{k}{i} \left(\frac{1}{6}\right)^{k-i} \left(-\frac{1}{120}\right)^i.$$  

Clearly, $i = n - k$, so this simplifies to

$$a_n = \frac{1}{2n+1} \sum_{k=[n/2]}^{n} (-1)^{n-k} \binom{2n+k}{k} \binom{k}{n-k} \left(\frac{1}{6}\right)^{2k-n} \left(\frac{1}{120}\right)^{n-k},$$

as required. \(\square\)

The expression $a_n$ is given by a formula which is a sum of “hypergeometric” terms, we see that Zeilberger’s algorithm [37, Chapter 6] applies. We used the interface to it provided by the \texttt{sumrecursion} function of Maple; this function implements Koepf’s version of Zeilberger’s algorithm [28, Chapter 7]. This function instantly informs us that the sequence $\{a_n\}$ is a solution to a rather remarkable three term finite difference equation

$$s_0(n)x_n - s_1(n)x_{n-1} + s_2(n)x_{n-2} = 0, \quad (7.1)$$

where

$$s_0(n) = 128n(n-1)(2n+1)(2n-1)(5n-6),$$

$$s_1(n) = 80(n-1)(2n-1)(5n-1)(15n^2 - 30n + 14),$$

$$s_2(n) = 3(5n-1)(5n-4)(5n-6)(5n-7)(5n-8).$$

The polynomials $s_i(n)$ are of the same degree 5, and so our equation is of the type considered by Poincaré in [38]. Namely, in [38, Sect. 2] linear finite difference equations of order $k$

$$s_0(n)x_n + s_1(n)x_{n-1} + \cdots + s_k(n)x_{n-k} = 0$$

are considered, with the additional assumption that $s_0(n), \ldots, s_k(n)$ are polynomials of the same degree $d$. To such an equation, one associates its characteristic polynomial

$$\chi(t) = \alpha_0 t^k + \alpha_1 t^{k-1} + \cdots + \alpha_k = 0,$$

where $\alpha_i$ is the coefficient of $t^d$ in $s_i(n)$. If the absolute values of the complex roots of $\chi(t)$ are pairwise distinct, then for any solution $\{a_n\}$ to our equation, the limit

$$\lim_{n \to \infty} \frac{a_n}{a_{n-1}},$$
exists and is equal to one of the roots of $\chi(t)$. Usually, that root will be the one which is maximal in absolute value. The particular case when the root is the minimal in absolute value is the hardest both for computations and for the qualitative analysis of the asymptotic behaviour, since in this case the corresponding solution is unique up to proportionality, and so the situation is not stable under small perturbations. In our case the polynomial $\chi(t)$ is

$$2560t^2 - 12000t + 9375,$$

and its roots are

$$\lambda_- = \frac{25(3 - \sqrt{3})}{32} \approx 0.9905853066 \quad \text{and} \quad \lambda_+ = \frac{25(3 + \sqrt{3})}{32} \approx 3.696914693,$$

so the Poincaré theorem applies. In fact, $\lambda_-$ is immediately seen to be the inverse of the radius of convergence of $h(t)$. By the usual ratio formula for the radius of convergence, we see that $\lim_{n \to \infty} \frac{a_n}{a_{n-1}} = \lambda_-$. Let us consider the auxiliary sequence $\{b_n\}$ satisfying the same finite difference equation (7.1) and the initial conditions $b_0 = 1$, $b_1 = 1$.

**Lemma 7.5** All terms of the sequence $\{b_n\}$ are positive for $n > 0$, and we have

$$\lim_{n \to \infty} \frac{b_n}{b_{n-1}} = \lambda_+.$$

**Proof** First, let us show that for all $n \geq 0$ we have $b_n/b_{n-1} \geq 1$. This is easy to see by induction on $n$. First, for $n = 1$, the statement is obvious. Next, if we suppose that it is true for all values less than the given $n$, we have

$$\frac{b_n}{b_{n-1}} = \frac{s_1(n)}{s_0(n)} - \frac{s_2(n)b_{n-2}}{s_0(n)b_{n-1}} \geq \frac{s_1(n)}{s_0(n)} - \frac{s_2(n)}{s_0(n)},$$

and so it suffices to show that

$$\frac{s_1(n)}{s_0(n)} - \frac{s_2(n)}{s_0(n)} \geq 1.$$

The roots of the polynomial $s_0(n)$ are $0, 1, -\frac{1}{2}, \frac{1}{2}, \frac{6}{5}$, so this polynomial assumes positive values in the given range. Thus, the above inequality is equivalent to

$$0 > s_0(n) - s_1(n) + s_2(n)
= -65n^5 - 9982n^4 + 36457n^3 - 45402n^2 + 21832n - 2912.$$

Using computer algebra software, we find that the latter polynomial has the largest root approximately equal to 1.404, so for all $n \geq 2$ it assumes negative values, and the inductive step is proved. Also, by the Poincaré theorem, the limit of the ratio $b_n/b_{n-1}$ as $n \to \infty$ is equal to either $\lambda_-$ or $\lambda_+$. However, $1 > \lambda_-$, so the inequality

$$\frac{b_n}{b_{n-1}} \geq 1.$$
$b_n/b_{n-1} \geq 1$ shows that the first of the two alternatives is impossible. Hence, the limiting value is $\lambda_+$. □

Our results thus far imply that \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \), as

\[
\frac{a_{n+1}}{b_{n+1}} = \frac{a_n}{b_n} \frac{a_{n+1}/a_n}{b_{n+1}/b_n},
\]

and so \( a_{n+1}/b_{n+1} \) is a multiple of \( a_n/b_n \) by a factor close to \( \lambda_-/\lambda_+ < 1 \) for large \( n \), and thus our sequence can be bounded from above in absolute value by a geometric sequence with a zero limit.

Now it is easy to complete the proof. We note that

\[
\frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}} = \frac{s_1(n)a_{n-1} - s_2(n)a_{n-2}}{s_1(n)b_{n-1} - s_2(n)b_{n-2}} - \frac{a_{n-1}}{b_{n-1}} = \frac{(s_1(n)a_{n-1} - s_2(n)a_{n-2})b_{n-1} - (s_1(n)b_{n-1} - s_2(n)b_{n-2})a_{n-1}}{(s_1(n)b_{n-1} - s_2(n)b_{n-2})b_{n-1}} = \frac{s_2(n)(a_{n-1}b_{n-2} - a_{n-2}b_{n-1})}{s_0(n)b_nb_{n-1}} = \frac{s_2(n)b_{n-2}}{s_0(n)b_n} \left( \frac{a_{n-1}}{b_{n-1}} - \frac{a_{n-2}}{b_{n-2}} \right).
\]

The roots of the polynomial \( s_2(n) \) are \( \frac{1}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5} \), so for \( n \geq 2 \) the sign of \( a_n/b_n - a_{n-1}/b_{n-1} \) is the same as the sign of \( a_{n-1}/b_{n-1} - a_{n-2}/b_{n-2} \), and hence the same as the sign of

\[
\frac{a_1}{b_1} - \frac{a_0}{b_0} = -\frac{5}{6} < 0.
\]

Thus, \( \{a_n/b_n\} \) is a strictly decreasing sequence. For a decreasing sequence with limit zero, all terms must be positive, and hence \( a_n \) is positive for all \( n > 0 \). □

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