Periodic points for amenable group actions on uniquely arcwise connected continua

Enhui Shi  
School of Mathematical Sciences  
Soochow University  
Suzhou 215006, P. R. China  
E-mail: ehshi@suda.edu.cn

Xiangdong Ye  
School of Mathematical Sciences  
University of Science and Technology of China  
Hefei 230022, P. R. China  
E-mail: yexd@ustc.edu.cn

Abstract

We show that if $G$ is a countable amenable group acting on a uniquely arcwise connected continuum $X$, then $G$ has either a fixed point or a 2-periodic point in $X$.

1 Introduction

1.1 Basic notions

Let $X$ be a topological space and let Homeo($X$) be the homeomorphism group of $X$. Let $G$ be a group. A group homomorphism $\phi : G \to \text{Homeo}(X)$ is called an action of $G$ on $X$. For brevity, we usually use $gx$ or $g(x)$ instead of $\phi(g)(x)$ for $g \in G$ and $x \in X$. The orbit of $x \in X$ under the action of $G$ is the set $Gx \equiv \{gx : g \in G\}$; $x$ is called a periodic point of $G$ if $Gx$ is finite, and the number of all elements in $Gx$ is called the order of $x$; if $x$ is a periodic point of order $n$ for some positive integer $n$, then $x$ is called an
n-periodic point; if \(x\) is of order 1, then \(x\) is called a fixed point of \(G\), that is \(gx = x\) for all \(g \in G\). A subset \(Y\) of \(X\) is called \(G\)-invariant, if \(g(Y) \subset Y\) for all \(g \in G\). A Borel measure \(\mu\) on \(X\) is called \(G\)-invariant if \(\mu(g(A)) = \mu(A)\) for every Borel set \(A\) in \(X\) and every \(g \in G\).

Amenability was first introduced by von Neumann. Recall that a countable group \(G\) is called an amenable group if there is a sequence of finite sets \(F_i\) \((i = 1, 2, 3, \ldots)\) such that \(\lim_{i \to \infty} \frac{|g F_i \triangle F_i|}{|F_i|} = 0\) for every \(g \in G\), where \(|F_i|\) is the number of elements in \(F_i\); the set \(F_i\) is called a Følner set. It is well known that solvable groups and finite groups are amenable; every subgroup of an amenable group is amenable. It is also known that any group containing a free noncommutative subgroup is not amenable. An important characterization of countable amenable group is that \(G\) is amenable if and only if every action of \(G\) on a compact metric space \(X\) has a \(G\)-invariant Borel probability measure on \(X\). One may consult [3] for a systematic introduction to amenability.

By a continuum, we mean a connected compact metric space. A continuum is nondegenerate if it is not a single point. An arc is a continuum which is homeomorphic to the closed interval \([0, 1]\). A continuum \(X\) is uniquely arcwise connected if for any two points \(x \neq y \in X\) there is a unique arc \([x, y]\) in \(X\), which connects \(x\) and \(y\). A dendrite is a locally connected, uniquely arcwise connected continuum. A tree is a dendrite which is the union of finitely many arcs. Clearly, the class of uniquely arcwise connected continua is strictly larger than that of dendrites. For example, the Warsaw circle is uniquely arcwise connected but not locally connected.

We provide in the following an example of a uniquely arcwise connected continuum contained in the plane, which will be repeatedly mentioned throughout the paper.

**Example 1.1** (see Fig.1). Let \(\mathbb{R}^2\) be the Euclidean plane with a Descartes’ coordinate system. We use \(P(x, y)\) to denote a point \(P \in \mathbb{R}^2\) with coordinate \((x, y)\). For each positive integer \(n\), let \(I_n, I_{-n}, J_n, J_{-n}\) be the segments between \(P(\frac{n+1}{n}, 0)\) and \(P(\frac{n}{n+1}, 1)\), between \(P(-\frac{n-1}{n}, 0)\) and \(P(-\frac{n}{n+1}, 1)\), between \(P(\frac{n}{n+1}, 0)\) and \(P(\frac{n}{n+1}, 1)\), between \(P(-\frac{n-1}{n}, 0)\) and \(P(-\frac{n}{n+1}, 1)\), respectively. Let \(S^- = \bigcup_{n=1}^{\infty} (I_n \cup J_n)\) and \(S^+ = \bigcup_{n=1}^{\infty} (I_n \cup J_n)\). Let \(L, B, R, M\) be the segments between \(P(-1, -1)\) and \(P(-1, 1)\), between \(P(-1, -1)\) and \((1, -1)\), between \(P(1, -1)\) and \(P(1, 1)\), between \(P(0, -1)\) and \(P(0, 0)\), respectively. Let \(X = L \cup B \cup R \cup M \cup S^- \cup S^+\). Then \(X\) is a uniquely arcwise connected continuum which is not locally connected.
1.2 Backgrounds and the main theorem

For an action of a group $G$ on a topological space $X$, an interesting question is whether there exists a fixed point or a periodic point of $G$ in $X$. The answer to this question depends on the topology of $X$ and the algebraic structure of $G$.

In 1975, Mohler proved in [6] that every homeomorphism (i.e., $\mathbb{Z}$-action) on a uniquely arcwise connected continuum has a fixed point, which answered a question proposed by Bing (see [1]). In 2009, this result is generalized to nilpotent group actions by Shi and Sun (see [9]). In 2010, Shi and Zhou further showed that every solvable group action on such continua has either a fixed point or a 2-periodic point (see [11]). In 2016, Shi and Ye proved that every countable amenable group action on a dendrite either has a fixed point or has a 2-periodic point (see [10]). One may consult [2, 4, 5] for some interesting discussions about fixed point theory for mappings on uniquely arcwise connected continua. We also remark that a continuous map on a uniquely arcwise connected continuum may have no fixed points (see [12]).

We get the following theorem in this paper, which generalize all the corresponding results stated above.

**Theorem 1.2.** Let $G$ be a countable amenable group and let $X$ be a uniquely arcwise connected continuum. Suppose $\phi : G \to \text{Homeo}(X)$ is a group action. Then $G$ has either a fixed point or a 2-periodic point in $X$.

In Section 2, we introduce some basic notions and results concerning the structure and mapping properties of uniquely arcwise connected continua. In Section 3, we construct a convex metric on a special class of arcwise connected subsets of uniquely arcwise connected continua and study their completions with respect to this metric. Specially, we establish a connection between the group actions on dendrites and that on uniquely arcwise connected continua. Based on the connection established in Section 3 and the main theorem in [10], we prove Theorem 1.2 in Section 4.
2 Preliminaries

2.1 Convex hulls

Let $X$ be a uniquely arcwise connected continuum. If $S$ is a subset of $X$, we denote by $[S]$ the intersection of all arcwise connected subsets containing $S$, and call it the convex hull of $S$ in $X$. Clearly, $[S]$ is the minimal one among all the arcwise connected subsets which contain $S$. We remark here that $[S]$ need not be compact in general. If $S = \{a, b\}$, we also denote by $[a, b]$ the convex hull of $S$, which is just the unique arc in $X$ connecting $a$ and $b$; denote by $[a, b), (a, b], and (a, b)$ the sets $[a, b) - \{b\}, [a, b) - \{a\}, and [a, b) - \{a, b\}$, respectively.

The following lemma is clear.

**Lemma 2.1.** If $S$ is a finite set in a uniquely arcwise connected continuum $X$, then $[S]$ is a tree.

**Example 2.2.** In Example 1.1, $[S^-] = S^-$ is not compact; If $S$ is the finite set consisting of points $P(-1, 1)$, $P(1, 1)$, and $P(0, 0)$, then $[S] = L \cup B \cup R \cup M$, which is a tree.

2.2 Dendrites

Let $X$ be a dendrite and let $x \in X$. We use ord$(x, X)$ to denote the cardinality of the set of all components of $X - \{x\}$, which is called the order of $x$ in $X$. The point $x$ is a cut point if ord$(x, X) \geq 2$; is a branch point if ord$(x, X) \geq 3$; is an end point if ord$(x, X) = 1$. For a nondegenerate dendrite $X$, there are at most countably many branch points, there are uncountably many cut points, and there always exist end points. One may consult [7] for more properties about dendrites.

**Proposition 2.3.** Let $f$ be a homeomorphism on a nondegenerate dendrite $X$. Suppose $e$ is an endpoint of $X$ such that $f(e) = e$. Then there is $u \in X, u \neq e$, such that either $f([e, u]) \subset [e, u]$ or $f^{-1}([e, u]) \subset [e, u]$.

**Proof.** Fix a point $v \neq e \in X$. Since $e$ is an end point and $f(e) = e$, there is $w \neq e$ such that $[e, w] = [e, v] \cap [e, f(v)]$. Let $u = f^{-1}(w)$. If $[e, u] \subset [e, w]$, then $f^{-1}([e, u]) \subset f^{-1}([e, w]) = [e, u]$; if $[e, w] \subset [e, u]$, then $f([e, u]) = [e, w] \subset [e, u]$. 

The following two corollaries follow immediately from Proposition 2.3.
Corollary 2.4. Let \( f_1, \ldots, f_n \) be homeomorphisms on a nondegenerate dendrite \( X \), for some positive integer \( n \). Suppose \( e \) is an endpoint of \( X \) such that \( f_i(e) = e \) for all \( i = 1, \ldots, n \). Then there are \( u, v \neq e \in X \) such that \( f_i([e, v]) \cup f_i^{-1}([e, v]) \subset [e, u] \) for all \( i = 1, \ldots, n \).

Corollary 2.5. Let \( f \) be a homeomorphism on a nondegenerate dendrite \( X \). Suppose \( e \) is an endpoint of \( X \) such that \( f(e) = e \). Then there is a sequence \( \{u_i\}_{i=1}^\infty \) in \( X \) satisfying the following two conditions simultaneously:

1. \( [u_1, e] \supset [u_2, e] \supset [u_3, e] \supset \cdots \) and \( \cap_{i=1}^\infty [u_i, e] = \{e\} \);
2. either \( f([u_i, e]) \subset [u_i, e] \) for all \( i \), or \( f^{-1}([u_i, e]) \subset [u_i, e] \) for all \( i \).

Let \( X \) and \( Y \) be metric spaces and let \( f : X \to Y \) be continuous. If \( \text{diam}(f^{-1}(f(x))) \leq \epsilon \) for some \( \epsilon > 0 \) and for every \( x \in X \), then \( f \) is called an \( \epsilon \)-map. A continuum \( X \) is tree-like provided that for every \( \epsilon > 0 \) there is an \( \epsilon \)-map \( f_\epsilon \) from \( X \) onto some tree \( Y_\epsilon \).

Theorem 2.6 ([7, 10.50]). A locally connected continuum is tree-like if and only if it is a dendrite.

Theorem 2.7 ([10 Theorem 1.1]). If \( G \) is a countable amenable group acting on a dendrite \( X \), then either \( G \) has a fixed point or has a 2-periodic point in \( X \).

2.3 Rays and lines

Definition 2.8. Let \( X \) be a compact metric space. If \( \phi : [0, +\infty) \to X \) is a continuous injection, then \( \phi \) or its image \( R = \phi([0, +\infty)) \) is called a ray in \( X \); \( R \) or \( \phi \) is called oscillatory (resp. nonoscillatory) if \( \cap_{n=0}^\infty \phi([n, +\infty)) \) contains at least two points (resp. only one point). If \( \psi : (-\infty, +\infty) \to X \) is a continuous injection, then \( \psi \) or its image \( L = \psi((-\infty, +\infty)) \) is called a line in \( X \); \( L \) or \( \psi \) is called oscillatory if either \( \cap_{n=0}^\infty \psi((-\infty, -n]) \) or \( \cap_{n=0}^\infty \psi([n, +\infty)) \) contains at least two points; is called bi-sided-oscillatory if both \( \cap_{n=0}^\infty \psi((-\infty, -n]) \) and \( \cap_{n=0}^\infty \psi([n, +\infty)) \) contain at least two points; is called one-sided-oscillatory if it is oscillatory but not bi-sided-oscillatory; is called nonoscillatory if it is not oscillatory.

We should note that if \( \phi_1 \) and \( \phi_2 \) are two rays with \( \phi_1([0, +\infty)) = \phi_2([0, +\infty)) \), then \( \phi_1 \) and \( \phi_2 \) have the same types of oscillation. The same conclusion is true for lines. One may consult [3] for more information about rays (the notion “ray” is called “quasi-arc” in [3]).
Definition 2.9. Let $X$ be a compact metric space. Let $R$ be a ray in $X$ and let $L$ be a line in $X$. We say that $L$ is an extension of $R$ if there is a continuous injection $\phi : (-\infty, +\infty) \to X$ such that $L = \phi((-\infty, +\infty))$ and $R = \phi([0, +\infty))$.

The following lemma is clear.

Lemma 2.10. Let $X$ be a uniquely arcwise connected continuum. Let $R$ be a ray in $X$ and let $\phi : [0, +\infty) \to X$ be a continuous injection such that $R = \phi([0, +\infty))$. If there is an arc $[a, b]$ in $X$ such that $\phi(0) \in (a, b)$, then $R$ can be extended to a line $L$ in $X$.

Example 2.11. In Example 1.1, $S^-$ and $S^+$ are one-sided oscillatory rays; $S^- \cup S^+$ is a bi-sided-oscillatory line; $L - P(-1, 1)$ is a nonoscillatory ray; $L - \{P(-1, 1), P(-1, -1)\}$ is a nonoscillatory line; the line $S^- \cup S^+$ is an extension of the ray $S^-$.

2.4 Quasi-retractions

Let $X$ be a uniquely arcwise connected continuum. Let $Y$ be either a tree, or a oscillatory ray, or a bi-sided oscillatory line contained in $X$. Then, by the uniquely arcwise connectivity, for every $x \in X$, there is a unique $y \in Y$ such that $[x, y] \cap Y = \{y\}$; we denote $y = r_Y(x)$, and call the map $r_Y : X \to Y, x \mapsto r_Y(x)$ the quasi-retraction from $X$ onto $Y$. We should note that $r_Y$ is not continuous in general.

Lemma 2.12. Let $X$ be a uniquely arcwise connected continuum. Let $Y$ be either a tree, or a oscillatory ray, or a bi-sided oscillatory line contained in $X$. If $Z$ is an arcwise connected subset of $Y$, then $r_Y^{-1}(Z)$ is an arcwise connected Borel measurable subset of $X$.

Proof. From the definition of $r_Y$, we see that if $x \in r_Y^{-1}(Z)$, then $[x, r_Y(x)] \subset r_Y^{-1}(Z)$, that is every point in $r_Y^{-1}(Z)$ is connected to a point in $Z$ by an arc in $r_Y^{-1}(Z)$. Since $Z \subset r_Y^{-1}(Z)$ and $Z$ is arcwise connected, we know $r_Y^{-1}(Z)$ is arcwise connected. For the measurability of $r_Y^{-1}(Z)$, one may consult [6].

Example 2.13. In Example 1.1, if $Y$ is the tree $L \cup B \cup R \cup M$, then $r_Y^{-1}(P(0, 0)) = S^- \cup S^+$; if $Y$ is the line $S^- \cup S^+$, then $r_Y^{-1}(P(0, 0)) = L \cup B \cup R \cup M$; if $Y$ is the ray $S^-$, then $r_Y^{-1}(P(0, 0)) = L \cup B \cup R \cup M \cup S^+$. 
3 Induced actions on dendrites

3.1 Convex metrics and their completions

Let $X$ be a uniquely arcwise connected space (need not be compact). A metric $d$ on $X$ is convex, if for any $u, v, x, y \in X$ with $[u, v] \subset [x, y]$, we have $d(u, v) \leq d(x, y)$. Suppose $T_1 \subset T_2 \subset T_3 \subset \ldots$ is a strictly increasing sequence of trees contained in $X$. Let $T = \bigcup_{i=1}^{\infty} T_i$. Then $T$ is an arcwise connected subset of $X$. Clearly, $T$ is also the union of infinitely many arcs $I_i (i = 1, 2, 3, \ldots)$ with $I_i \cap I_j$ being a point or empty for any $i \neq j$. Without loss of generality, we may suppose that

$$T_n = \bigcup_{i=1}^{n} I_i$$

for each $n$. Fix a homeomorphism $h_i : I_i \to [0, 1]$ for each $i$. If $[a, b] \subset [0, 1]$, we denote by $l([a, b])$ the length of the interval $[a, b]$ under the Euclidean metric on $[0, 1]$, i.e., $l([a, b]) = |a - b|$. For $x, y \in T$, define

$$(3.1) \quad d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} l(h_i([x, y] \cap I_i)).$$

It is direct to check that $d$ is a convex metric on $T$. Let $\overline{T}$ be the completion of $T$ with respect to the metric $d$. We still use $d$ to denote the naturally induced metric on $\overline{T}$.

**Proposition 3.1.** $(\overline{T}, d)$ is a dendrite.

**Proof.** **Claim A.** $(\overline{T}, d)$ is compact. Indeed, for every $\epsilon > 0$, there is some $n$ such that

$$(3.2) \quad \sum_{i=n+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2}.$$ 

Since $T_n$ is a tree, there is a finite set $\{x_1, \ldots, x_m\} \subset T_n$ for some positive integer $m$, such that

$$T_n \subset \bigcup_{i=1}^{m} B_d(x_i, \frac{\epsilon}{2}),$$

here $B_d(x_i, \frac{\epsilon}{2})$ is the open ball with center $x_i$ and radius $\frac{\epsilon}{2}$ under the metric $d$. For $x \in T - T_n$, let $y = r_{T_n}(x)$. By (3.3), there is some $x_i$ with $y \in B_d(x_i, \frac{\epsilon}{2})$. By (3.2) and the definition of $d$, we have

$$d(x, x_i) \leq d(x, y) + d(y, x_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
E. H. Shi and X. D. Ye

Hence $T \subset \bigcup_{i=1}^{m} B_d(x_i, \epsilon)$ and $\overline{T} \subset \bigcup_{i=1}^{m} \overline{B_d(x_i, \epsilon)}$. So $\overline{T}$ is totally bounded, which implies $\overline{T}$ is compact.

**Claim B.** $(\overline{T}, d)$ is locally connected. Indeed, for every $\epsilon > 0$ there is some $n$ such that

$$
\sum_{i=n+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{3},
$$

(3.4)

Noting that $T_n$ is a tree, there are finitely many arcs $C_1, \ldots, C_m$ such that $T_n = \bigcup_{i=1}^{m} C_i$ and

$$
\text{diam}_d(C_i) < \frac{\epsilon}{3}, \quad \text{for each } i = 1, \ldots, m.
$$

(3.5)

Let $D_i = \{ x \in T : r_{T_n}(x) \in C_i \}$. Then $D_i$ is arcwise connected by Lemma 2.12. From (3.4) and (3.5), for all $x, y \in D_i$, we have

$$
d(x, y) \leq d(x, r_{T_n}(x)) + d(r_{T_n}(x), r_{T_n}(y)) + d(r_{T_n}(y), y) < \epsilon,
$$

that is $\text{diam}(D_i) \leq \epsilon$. Since $T = \bigcup_{i=1}^{m} D_i$, we have $\overline{T} = \bigcup_{i=1}^{m} \overline{D_i}$, where each $\overline{D_i}$ is connected and $\text{diam}_d(\overline{D_i}) \leq \epsilon$. It follows from [7, 8.4] that $\overline{T}$ is locally connected.

**Claim C.** $(\overline{T}, d)$ is tree-like. Indeed, for every $\epsilon > 0$, there is $n$ such that

$$
\sum_{i=n+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{5},
$$

(3.6)

Noting that for every $\epsilon' > 0$, by the convexity of $d$, we always have

$$
d(r_{T_n}(x), r_{T_n}(y)) \leq d(x, y) < \epsilon',
$$

whenever $x, y \in T$ with $d(x, y) < \epsilon'$. This shows that $r_{T_n} : (T, d) \to (T_n, d)$ is uniformly continuous. So, $r_{T_n}$ can be extended to a continuous map

$$
r_{\epsilon} : (\overline{T}, d) \to (T_n, d).
$$

(3.7)

We want to show that $r_{\epsilon}$ is an $\epsilon$-map. Otherwise, there are $x, y \in \overline{T}$ with $d(x, y) > \epsilon$ and $r_{\epsilon}(x) = r_{\epsilon}(y)$. Then by the continuity of $r_{\epsilon}$ and the density of $T$ in $\overline{T}$, there are $x', y' \in T$ such that $d(x, x') < \frac{\epsilon}{5}$, $d(y, y') < \frac{\epsilon}{5}$, and $d(r_{\epsilon}(x'), r_{\epsilon}(y')) < \frac{\epsilon}{5}$. So, by (3.6), we have

$$
d(x, y) \leq d(x, x') + d(x', r_{\epsilon}(x')) + d(r_{\epsilon}(x'), r_{\epsilon}(y')) + d(r_{\epsilon}(y'), y') + d(y, y') < \epsilon,
$$

which is a contradiction. By the arbitrariness of $\epsilon$, we get that $(\overline{T}, d)$ is tree-like. It follows from Claim A, Claim B, Claim C, and Theorem 2.6 that $(\overline{T}, d)$ is a dendrite. \qed
Remark 3.2. The topology induced by $d$ on $T$ may not be the subspace topology of $T$ induced from $X$; that is, the inclusion $i : (T, d) \to X$, $x \mapsto x$, may not be an embedding in general.

Example 3.3. In Example 1.1, if we let $T_n = B \cup M \cup \bigcup_{i=1}^{n} (I_i \cup I_{-i} \cup J_i \cup J_{-i})$ and let $T = \bigcup_{n=1}^{\infty} T_n$, then the completion of $T$ with respect to the metric $d$ defined above is homeomorphic to the graph “H”.

3.2 Induced actions

Let $X$ be a uniquely arcwise connected continuum. Let $G$ be a countable amenable group acting on $X$. Suppose $G = \{g_i : i = 1, 2, 3, \ldots \}$. Take a point $p \in X$. For each positive integer $n$, let $S_n = \{g_i(p) : i = 1, \ldots, n\}$ and let $T_n = [S_n]$. Then we get an increasing sequence of trees:

\begin{equation}
T_1 \subset T_2 \subset T_3 \subset \ldots
\end{equation}

Set $T = \bigcup_{n=1}^{\infty} T_n \ (= [Gp])$. Then $T$ is a $G$-invariant uniquely arcwise connected subset of $X$. We assume that $T$ is not a tree. Then by deleting some $T_i$’s in (3.8) and renumbering the remaining $T_i$’s, we can assume that the sequence in (3.8) is strictly increasing. It follows from Proposition 3.1 that the completion $(\overline{T}, d)$ of $T$ with respect to the metric $d$ defined in (3.1) is a dendrite.

Proposition 3.4. The action of $G$ on $(T, d)$ is uniformly continuous with respect to the metric $d$.

Proof. Let $g \in G$. For every $\epsilon > 0$, there is $m$ such that

\begin{equation}
\sum_{i=m+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{3}.
\end{equation}

Take a sufficiently large $n$ so that $S_n \supset S_m \cup g^{-1}S_m$. Then $g(S_n) \cap S_n \supset S_m$ and

\begin{equation}
g(T_n) \cap T_n \supset T_m.
\end{equation}

By the compactness of $T_n$, there is $\delta > 0$ such that

\begin{equation}
d(g(x), g(y)) < \frac{\epsilon}{3},
\end{equation}

whenever $x, y \in T_n$ with $d(x, y) < \delta$. 
For any \( u, v \in T \), let \( u' = r_{T_n}(u) \) and \( v' = r_{T_n}(v) \). Then, by the convexity of \( d \), \( d(u', v') < \delta \) whenever \( d(u, v) < \delta \). Then by (3.9) (3.10) and (3.11) we have

\[
d(g(u), g(v)) \leq d(g(u), g(u')) + d(g(u'), g(v')) + d(g(v'), g(v)) < \epsilon,
\]

provided that \( d(u, v) < \delta \). This completes the proof. \( \square \)

From Proposition 3.4, we know that every \( g \in G \) can be extended to a continuous map \( \overline{g} : (\overline{T}, d) \to (\overline{T}, d) \).

**Proposition 3.5.** For each \( g \in G \), \( \overline{g} : (\overline{T}, d) \to (\overline{T}, d) \) is a homeomorphism.

**Proof.** We need only to show that \( \overline{g} \) is injective. By the definition of \( \overline{T} \), for any \( x \neq y \in \overline{T} \), there are \( x' \neq y' \in T \) such that \([x, y] \supset [x', y']\). So, \([\overline{g}(x), \overline{g}(y)] \supset [\overline{g}(x'), \overline{g}(y')]\). Then \( d(\overline{g}(x), \overline{g}(y)) > d(\overline{g}(x'), \overline{g}(y')) > 0 \) by the convexity of \( d \). This implies that \( \overline{g} \) is injective. \( \square \)

From Proposition 3.3 and Proposition 3.5, we obtain an action of \( G \) on the dendrite \( (\overline{T}, d) \) by homeomorphisms, which is called the *induced action* from the \( G \)-action on \( T \).

**4 Proof of the main theorem**

In this section, we start to prove Theorem 1.2. Let \( X \) be a uniquely arcwise connected continuum and let \( G \) be a countable amenable group. We want to show that every \( G \)-action on \( X \) has either a fixed point or a 2-periodic point in \( X \).

Fix a point \( p \in X \). Let \( T = [Gp] \) be the convex hull of its orbit. Then \( T \) is an arcwise connected \( G \)-invariant subset of \( X \). If \( T \) is a tree, then \( G \) has either a fixed point or a 2-periodic point in \( T \subset X \) by Theorem 2.7. So, we may as well assume that \( T \) is not a tree. Thus by the discussion in Section 3, there is a metric \( d \) on \( T \) such that the completion \( (\overline{T}, d) \) is a dendrite and there is an induced \( G \)-action on \( (\overline{T}, d) \) by homeomorphisms. It follows from Theorem 2.7 that there is a point \( q \in \overline{T} \), which is either a fixed point or a 2-periodic point of \( G \). If \( q \in T \), then the conclusion of Theorem 1.2 holds, since \( q \in X \). So, we may assume that \( q \in \overline{T} - T \), that is \( q \) is an endpoint of \( \overline{T} \). If \( q \) is a 2-periodic point of \( G \), then \( H \equiv \{g \in G : g(q) = q\} \) is a subgroup of \( G \) with index 2. Notice that \( H \) is also amenable and \( q \) is a fixed point of \( H \). In this case, if we can show that \( H \) has a fixed point \( w \in X \),
then \( w \) is either a fixed point or 2-periodic point of \( G \), and the conclusion of Theorem 1.2 holds. So, we need only to prove the following theorem.

**Theorem 4.1.** If the induced \( G \)-action on \( (T, d) \) has a fixed point \( q \in \overline{T} - T \), then \( G \) has a fixed point in \( X \).

**Proof.** We assume first that \( G \) is finitely generated with a generator set \( \{g_1, \ldots, g_n\} \) for some positive integer \( n \). Fix a point \( o \in T \), then \([o, q) \subset T \subset X\). Let \( \phi : [0, +\infty) \to X \) be a continuous injection such that \([o, q) = \phi([0, +\infty)) \). Then \([o, q) \) becomes a ray.

**Case 1.** \( \phi \) is nonoscillatory. Then there is \( z \in X \) such that \( z = \bigcap_{n=1}^{\infty} \phi((n, +\infty)) \). By Corollary 2.5, we see that \( z \in X \).

**Case 2.** \( \phi \) is oscillatory. By Corollary 2.4, there is \( c_1 > c_2 > 0 \) such that, for all \( i = 1, \ldots, n, \)

\[
(4.1) \quad g_i(\phi([c_2, +\infty})) \cup g_i^{-1}(\phi([c_2, +\infty))) \subset \phi([c_1, +\infty)).
\]

Let \( \prec \) be an ordering on \( \phi([0, +\infty)) \) defined by \( \phi(t) \prec \phi(s) \) if and only if \( t < s \), for any \( t, s \in [0, +\infty) \). If \( z = \sup_{\prec} \{g(\phi(c_2)) : g \in G \} \cap \phi([c_1, +\infty)) \prec +\infty \), then \( g_i(z) = z \) for all \( i = 1, \ldots, n \), because each \( g_i \) preserves the ordering \( \prec \), the restriction of \( \prec \) to \( \phi([c_2, +\infty)) \). Thus we get the conclusion.

So, we may assume that

\[
(4.2) \quad \sup_{\prec} \{g(\phi(c_2)) : g \in G \} \cap \phi([c_1, +\infty)) = +\infty.
\]

Consider the set \( M = \bigcup_{g \in G} g(\phi([c_2, +\infty))) \). Then \( M \) is arcwise connected, since for any \( h_1 \neq h_2 \in G \) there is some \( c' > 0 \) such that \( h_1(\phi([c_2, +\infty))) \cap h_2(\phi([c_2, +\infty))) \supset \phi([c', +\infty)) \) by Corollary 2.4. By Lemma 2.10, we can take a line \( \psi : (-\infty, +\infty) \to M \subset X \) such that \( \psi([0, +\infty)) = \phi([c_2, +\infty)) \).

Set \( L = \psi((0, +\infty)) \).

**Subcase 2.1.** \( \psi \) is bi-sided-oscillatory in \( X \). For each positive integer \( n \), let \( L_n = \psi([n, n + 1)) \) and let \( K_n = \{x \in X : r_L(x) \in L_n\} \). By Lemma 2.12, each \( K_n \) is an arcwise connected Borel measurable set in \( X \). Clearly, these \( K_n \) form a partition of \( X \). Since \( G \) is amenable, there is a \( G \)-invariant probability Borel measure \( \mu \) on \( X \). Suppose \( \mu(K_m) > 0 \) for some integer \( m \).

Since \( \psi(m) \in M \), there is some \( g' \in G \) such that \( g'(\psi(m)) \in \phi([c_2, +\infty)) \), which implies \( r_L(g'(K_m)) \subset \phi([c_2, +\infty)) \). Set \( R = \phi([c_2, +\infty)) \).

\[
(4.3) \quad \mu(r_L^{-1}(R)) \geq \mu(g'(K_m)) = \mu(K_m) > 0.
\]

However, by (4.2), we can take a sequence \( s_i \in G \) such that \( s_1(\phi(c_2)) \prec s_2(\phi(c_2)) \prec s_3(\phi(c_2)) \prec \ldots \in R \) and \( s_i(\phi(c_2)) \to +\infty \) as \( i \to +\infty \), with.
respect to the ordering \( \prec \). Then we have

\[
0 = \mu(\emptyset) = \mu(\cap_{i=1}^{\infty}s_i(r_L^{-1}R)) = \lim_{i \to \infty} \mu(s_i(r_L^{-1}R)) = \mu(r_L^{-1}R).
\]

Since (4.3) and (4.4) are contradict to each other, this subcase does not occur.

Subcase 2.2. \( \psi \) is one-sided-oscillatory in \( X \). Since \( \phi \) is oscillatory, there must exist a point \( z \in X \) such that \( z = \cap_{n=1}^{\infty} \psi(\langle -\infty, -n \rangle) \). If \( z \) is a fixed point of \( G \), then the conclusion holds; otherwise, there is some \( \bar{g} \in G \) with \( \bar{g}(z) \neq z \). Let \( r \in (-\infty, +\infty) \) be such that

\[
\psi([r, +\infty)) = \psi(\langle -\infty, +\infty \rangle \cap \bar{g}(\psi(\langle -\infty, +\infty \rangle))).
\]

Denote \( w = \psi(r) \in M \). Take \( a \in (z, w) \) with \( \bar{g}(a) \in (\bar{g}(z), w) \). Let \( t \in (-\infty, +\infty) \) be such that \( \psi(t) = a \). Set \( P_t = \{x \in X : r_L(x) \in [z, a]\} \) and \( Q_t = \{x \in X : r_L(x) \in \psi([t, +\infty))\} \) (See Fig.2.). Then, by Lemma 2.12, \( P_t \) and \( Q_t \) are arcwise connected and Borel measurable, and \( X = P_t \cup Q_t \) (disjoint union). Since \( G \) is amenable, there is a \( G \)-invariant Borel probability measure \( \mu \) on \( X \). Then

\[
1 = \mu(X) = \mu(P_t) + \mu(Q_t).
\]

Noting that \( \bar{g}(P_t) \subset Q_t \), we have

\[
\mu(Q_t) \geq \mu(\bar{g}(P_t)) = \mu(P_t) > 0
\]

provided that \( \mu(P_t) > 0 \). Thus we always have \( \mu(Q_t) > 0 \). Since \( a \in M \), there is some \( g \in G \) such that \( g(a) \in \phi([c_2, +\infty)) \). Then, by an argument similar to that in Subcase 2.1, we get a contradiction.

Altogether, we finish the proof of Theorem 4.1 under the assumption that \( G \) is finitely generated. Now, suppose that \( G \) is not finitely generated. For any finite subset \( F \) of \( G \), let \( \langle F \rangle \) be the subgroup of \( G \), which is generated by \( F \). Define

\[
X_F = \{x \in X : x \text{ is a fixed point of } \langle F \rangle\}.
\]

Then \( X_F \) is a nonempty closed subset of \( X \). If \( F' \) is another finite subset of \( G \), then \( X_F \cap X_{F'} = X_{F \cup F'} \neq \emptyset \). Thus the family of compact sets \( \{X_F : F \text{ is a finite subset of } G\} \) has the finite intersection property. Hence

\[
\cap \{X_F : F \text{ is finite in } G\} \neq \emptyset,
\]

every point of which is a fixed point of \( G \). Thus we complete the proof of Theorem 4.1. \( \square \)
Acknowledgements

The work is supported by NSFC (No. 11271278, No. 11371339, and No. 11431012).

References

[1] R.H. Bing, The elusive fixed point property. Amer. Math. Monthly 76 (1969), 119-132.

[2] W. Holsztyński, Fixed points of arcwise connected spaces. Fund. Math. 69 (1969), 289-312.

[3] J.H. Mai and E.H. Shi, Structures of quasi-graphs and $\omega$-limit sets of quasi-graph maps. (to appear in Tran. Amer. Math. Soc.)

[4] R. Mańka, On uniquely arcwise connected curves. Colloq. Math. 51 (1987), 227-238.

[5] R. Mańka, On spirals and fixed point property. Fund. Math. 144 (1994), 1-9.

[6] L. Mohler, The fixed point property for homeomorphisms of 1-arcwise connected continua. Proc. Amer. Math. Soc. 52 (1975), 451-456.

[7] S.B. Nadler Jr., Continuum Theory, Marcel Dekker, Inc., New York, 1992.

[8] A.L.T. Paterson, Amenability, American Mathematical Society, Providence, RI, 1988.
[9] E.H. Shi and B.Y. Sun, Fixed point properties of nilpotent group actions on 1-arcwise connected continua. Proc. Amer. Math. Soc. 137 (2009), no. 2, 771-775.

[10] E.H. Shi and X.D. Ye, Periodic points for amenable group actions on dendrites. (to appear in Proc. Amer. Math. Soc.)

[11] E.H. Shi and L.Z. Zhou, Periodic points of solvable group actions on 1-arcwise connected continua. Topology Appl. 157 (2010), no. 7, 1163-1167.

[12] G.S. Young, Fixed-point theorems for arcwise connected continua. Proc. Amer. Math. Soc. 11 (1960), 880-884.