ABSTRACT. We study the ground state energy and ground states of systems coupling non-relativistic quantum particles and force-carrying Bose fields, such as radiation, in the quasi-classical approximation. The latter is very useful whenever the force-carrying field has a very large number of excitations, and thus behaves in a semiclassical way, while the non-relativistic particles, on the other hand, retain their microscopic features. We prove that the ground state energy of the fully microscopic model converges to the one of a nonlinear quasi-classical functional depending on both the particles’ wave function and the classical configuration of the field. Equivalently, this energy can be interpreted as the lowest energy of a Pekar-like functional with an effective nonlinear interaction for the particles only. If the particles are confined, the ground state of the microscopic system converges as well, to a probability measure concentrated on the set of minimizers of the quasi-classical energy.

1. Introduction and Main Results

The description and rigorous derivation of effective models for complex quantum systems is a flourishing line of research in modern mathematical physics. Typically, in suitable regimes, the fundamental quantum description can be approximated in terms of some simpler model retaining the salient physical features, but also allowing a more manageable computational or numerical treatment. The questions addressed in this work naturally belong to such a wide class of problems.

We consider indeed a quantum system composed of $N$ non-relativistic particles interacting with a quantized bosonic field, in the quasi-classical regime. We refer to the series of works [CF18, CFO19a, CCFO19, CFO19b] for a detailed discussion of such a regime: in extreme synthesis, we plan to study field configurations with a suitable semiclassical behavior. We require indeed that there is a large number of field excitations, although each one of the latter is carrying a very small amount of energy, in such a way that the field’s degrees of freedom are almost classical. More precisely, we assume that the average number of force carriers $\langle N \rangle$ is of order $\frac{1}{\varepsilon}$, for some $0 < \varepsilon \ll 1$, and thus much larger than the commutator between $a^\dagger$ and $a$, which is of order 1 (we use units in which $\hbar = 1$). Concretely, this can be realized by
rescaling the canonical variables $a^\dagger, a$ by $\sqrt{\varepsilon}$, i.e., setting $a^\dagger \varepsilon := \sqrt{\varepsilon} a^\dagger$, which leads to
\[
[a^\varepsilon(k), a^\dagger_\varepsilon(k')] = \varepsilon \delta(k - k'), \quad \varepsilon \ll 1. \tag{1.1}
\]
On the other hand, the degrees of freedom associated with the particles are not affected by the scaling limit $\varepsilon \to 0$ and the particles remain quantum. Our goal is precisely to set up and rigorously derive an effective quantum model for the lowest energy state of the system in the quasi-classical regime $\varepsilon \to 0$, when the field becomes classical.

Let us now describe in more detail the type of microscopic models we plan to address. The space of states of the full system is
\[
\mathcal{H}_\varepsilon := L^2(\mathbb{R}^{dN}) \otimes G_\varepsilon(\mathfrak{h}), \tag{1.2}
\]
where $d \in \{1, 2, 3\}$, $\mathfrak{h}$ is the single one-excitation space of the field and $G_\varepsilon$ stands for the second quantization map, so that $G_\varepsilon(\mathfrak{h})$ is the bosonic Fock space constructed over $\mathfrak{h}$, with canonical commutation relations
\[
[a^\varepsilon(\xi), a^\dagger_\varepsilon(\eta)] = \varepsilon \langle \xi | \eta \rangle_{\mathfrak{h}}, \tag{1.3}
\]
for any $\xi, \eta \in \mathfrak{h}$.

The energy of the microscopic system and thus its Hamiltonian is given by the non-relativistic energy of the particles, the field energy and the interaction between the particles and the field, in such a way that
\begin{itemize}
  \item the particle and field energies are a priori of the same order $O(1)$;
  \item the interaction is weak, i.e., a priori subleading w.r.t. the unperturbed energies.
\end{itemize}
This is concretely realized by considering Hamiltonians of the form
\[
H_\varepsilon = K_0 \otimes 1 + 1 \otimes dG_\varepsilon(\omega) + H_I, \tag{1.4}
\]
where:
\begin{itemize}
  \item $K_0$ is the ($\varepsilon$-independent) free particle Hamiltonian
    \[
    K_0 = \sum_{j=1}^{N} (-\Delta_j) + W(x_1, \ldots, x_N) \tag{1.5}
    \]
    which is assumed to be self-adjoint and bounded from below (we specify in \S4 the working assumptions on $W$);
  \item $dG_\varepsilon(\omega)$ is the free field energy and is the second quantization of the positive operator $\omega$ on $\mathfrak{h}$, admitting possibly unbounded inverse $\omega^{-1}$;
  \item the interaction $H_I$ is the only non-factorized term of the Hamiltonian, it depends on $\varepsilon$ only through the creation and annihilation operators $a^\dagger_\varepsilon$ and it is a polynomial of such operators of order between one and two.
\end{itemize}
Such requests meet the scaling conditions mentioned above. Indeed, assuming that the average number $\langle N \rangle$ of bare excitation of the field is $O(\varepsilon^{-1})$, the field energy is of order $\varepsilon \langle N \rangle = O(1)$, due to the rescaling of $a^\dagger_\varepsilon$ and $a_\varepsilon$. For the same reason and since the interaction is at least of order one in the creation and annihilation operators, we have that $H_I$ is of order $O(\sqrt{\varepsilon})$, i.e., a priori subleading w.r.t. the rest of $H_\varepsilon$.

\footnote{We do not take into account the spin degrees of freedom nor the symmetry constraints induced by the presence of identical particles, but such features can be included in the discussion without any effort and the results trivially apply to the corresponding models. In fact, we may even allow for a coupling term between the radiation field and the particle spins \cite{CFO19}, as the one often included in the Pauli-Fierz model.}
The specific models we are considering in the following are:

(a) the Nelson model [Nel64]: the coupling in $H_I$ is simply linear, i.e.,

$$H_I = \sum_{j=1}^{N} A_\varepsilon(x_j),$$

where

$$A_\varepsilon(x) := a_\varepsilon^\dagger(\lambda(x)) + a_\varepsilon(\lambda(x))$$

is the field operator and

$$\lambda, \omega^{-1/2} \lambda \in L^\infty(\mathbb{R}^3; \mathfrak{h})$$

(a typical choice is $\mathfrak{h} = L^2(\mathbb{R}^d)$, $\omega$ the multiplication operator by $\omega(k) \geq 0$ and $\lambda(x; k) = \lambda_0(k)e^{-ik\cdot x}$, with $\lambda_0, \omega^{-1/2}\lambda_0 \in \mathfrak{h}$);

(b) the Fröhlic polaron [Fro37]: it is a variant of the Nelson model where $\mathfrak{h} = L^2(\mathbb{R}^d)$, $\omega = 1$ and

$$\lambda(x; k) = \nu e^{-ik\cdot x} \frac{1}{|k|^{d-1/2}},$$

for some $\alpha > 0$;

(c) the Pauli-Fierz model [PF38]: it is the most elaborate model and we consider only its three-dimensional realization, namely $d = 3$; the interaction is provided by the minimal coupling

$$H_\varepsilon = \sum_{j=1}^{N} \frac{1}{2m_j}(-i\nabla_j + eA_{\varepsilon,j}(x_j))^2 + \mathcal{W}(x_1, \ldots, x_N) + 1 \otimes dG_\varepsilon(\omega),$$

where $\omega \geq 0$, $m_j > 0$, $j = 1, \ldots, N$ and $e$ are the particles’ masses and charge, respectively, and the field operators $A_{\varepsilon,j}$, $j = 1, \ldots, N$, have here the same formal expression as in [1.7] but $\lambda_j = (\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3})$, with

$$\lambda_{j,\ell}, \omega^{-1/2}\lambda_{j,\ell} \in L^\infty(\mathbb{R}^3; \mathfrak{h}),$$

is a vector function to account for the electromagnetic polarizations and the charge distributions of the particles (the standard choice is, indeed, $\mathfrak{h} = L^2(\mathbb{R}^3; \mathbb{C}^2)$) and we fix for convenience the gauge to be the Coulomb’s one, i.e., $\nabla_j \cdot \lambda_j = 0$.

The physical meaning of the three models above is quite different and we refer, e.g., to the monograph [Spo04] for a detailed discussion. The Nelson model is the simplest one and can be applied to model nucleons interacting with a meson field or, in first approximation, to model the interaction of particles with radiation fields, although the case of the electromagnetic field is typically described through the Pauli-Fierz model. The polaron, on the other hand, provides an effective description of quantum particles in a phonon field, e.g., generated by the vibrational models of a crystal. Note also that the quasi-classical limit $\varepsilon \to 0$ itself can have different interpretations in each model. For instance, in the framework of the polaron model, it can be reformulated as a strong coupling limit, which has recently attracted a lot of attention (see, e.g., [Gri17, FG19, LS19, LMS20, Mit20] and references therein).

In the Nelson and Pauli-Fierz Hamiltonians, there is an ultraviolet regularization, made apparent in the assumptions on $\lambda$; we do not consider here the renormalization procedure to remove such ultraviolet cut-off, even if for the Nelson model it is possible to perform it rigorously. We plan to address such a problem in a future work. We also skip at this stage the
discussion of the well-posedness of such models (see § 4.1, § 4.2 and § 4.3 for further details),
but we point out that, with the assumptions made, the operator (1.4) is self-adjoint and
bounded from below in each model.

The main problem we study concerns the behavior of the ground state of the microscopic
Hamiltonian \( H_\varepsilon \) in the quasi-classical limit \( \varepsilon \to 0 \) and, more precisely, we investigate the
convergence in the same limit of the bottom of the spectrum
\[
E_\varepsilon := \inf_{\sigma(H_\varepsilon)} \inf_{\Gamma_\varepsilon \in L^1(H_\varepsilon)} \text{tr}(H_\varepsilon \Gamma_\varepsilon)
\]
(1.12)
of \( H_\varepsilon \) as well as the approximation of any corresponding approximate ground state or mini-
mimizing sequence \( \Psi_{\varepsilon,\delta} \in \mathcal{D}(H_\varepsilon) \) satisfying
\[
\langle \Psi_{\varepsilon,\delta} | H_\varepsilon | \Psi_{\varepsilon,\delta} \rangle_{H_\varepsilon} < E_\varepsilon + \delta,
\]
(1.13)
for some small \( \delta > 0 \).

The quasi-classical counterparts of such quantities are determined via the minimization of a
suitable coupled problem, where the particle’s degrees of freedom are driven by a classical field.
Such a problem is described in detail in § 1.1 below. The quasi-classical energy is given by a
functional \( E_{qc}[\psi, z] \) (see (1.19) below), depending on the particle’s wave function \( \psi \in L^2(\mathbb{R}^{dN}) \)
and on the classical field configuration \( z \in \mathfrak{h}_\omega \). Denoting by \( E_{qc} \) and \( (\psi_{qc}, z_{qc}) \in L^2(\mathbb{R}^{dN}) \oplus \mathfrak{h}_\omega \)
the infimum of a such a quasi-classical energy and the relative minimizing configuration (if
any), respectively, our main results are:

i) **Energy convergence.** Both the quantum and the quasi-classical problems are stable,
\( i.e., E_\varepsilon, E_{qc} > -\infty \) and
\[
E_\varepsilon \xrightarrow{\varepsilon \to 0} E_{qc}.
\]
(1.14)
i) **Convergence of ground states and approximate ground states.** Assuming that
the operator \( K_0 \) has compact resolvent, then any limit point of an approximate ground
state \( \Psi_{\varepsilon,\delta} \) in the sense of quasi-classical Wigner measures is an approximate ground
state of the quasi-classical functional \( E_{qc} \), in a sense to be clarified in Theorem 1.8 be-
low. Furthermore, any limit point of the family of approximate ground states \( \Psi_{\varepsilon,\omega(1)} \) is
concentrated on the set of minimizers of the quasi-classical functional \( E_{qc} \); since the set
of limit points is never empty, the latter admits at least one minimizer in \( L^2(\mathbb{R}^{dN}) \oplus \mathfrak{h}_\omega \). If
\( H_\varepsilon \) has a ground state \( \Psi_{gs} \), then any of its limit points in the sense of quasi-classical
Wigner measures is concentrated on the set of minimizers of \( E_{qc} \).

iii) **Generalized convergence of ground states and approximate ground states.** If
the operator \( K_0 \) does not have compact resolvent, then any limit point of \( \Psi_{\varepsilon,\delta} \) is a gen-
eralized quasi-classical Wigner measure, and it is a minimizing sequence for a suitable
generalization \( E_{gqc} \) of the energy \( E_{qc} \). Furthermore, any limit point of \( \Psi_{\varepsilon,\omega(1)} \) is a mini-
mimizer for \( E_{gqc} \). Let us remark that this does not imply the existence of a minimizing
configuration \( (\psi_{qc}, z_{qc}) \in L^2(\mathbb{R}^{dN}) \oplus \mathfrak{h}_\omega \). If \( H_\varepsilon \) has a ground state \( \Psi_{gs} \), then any of its
limit points in the sense of generalized quasi-classical Wigner measures is concentrated
on the set of minimizers of \( E_{gqc} \).

\footnote{The space \( \mathfrak{h}_\omega \) is constructed starting from \( \mathfrak{h} \) and the dispersion relation \( \omega \) of the semiclassical field; see (1.22)
for a precise definition. It is necessary to use \( \mathfrak{h}_\omega \) in place of \( \mathfrak{h} \) as the field’s configuration space whenever the
field is massless, such as in the Pauli-Fierz model or in the massless Nelson model. For massive fields, \( \mathfrak{h}_\omega \subseteq \mathfrak{h} \).}
We state the above results in all details in §1.2 together with a precise definition of the notions of quasi-classical Wigner measure and generalized quasi-classical Wigner measure and the relative topologies. In the next §1.1, we first introduce and discuss the quasi-classical variational problems. In the rest of the paper, we present the proofs of the above results. We stress that the main techniques we are going to use belong to the framework of semiclassical analysis in infinite dimensional spaces, which was introduced in the series of works AN08, AN09, AN11, AN15a and further discussed in Fal18a, Fal18b. Apart from the aforementioned works on quasi-classical analysis, semiclassical techniques have already been used in the study of variational problems, both for systems with creation and annihilation of particles AFL, and for systems with many bosons, using a slightly different approach called quantum de Finetti theorem (see LNR14, LNR15, LNR16, and references therein contained). We also point out that partially classical regimes have already been explored in the literature in GNV06, AN15b, AJN17, ALN17, although in other contexts and with different purposes.

1.1. Quasi-classical variational problems. As discussed in detail in the series of works CF18, CFO19a, CFO19b, each of the microscopic models introduced so far admits a quasi-classical counterpart in the limit $\varepsilon \to 0$. More precisely, both their stationary CF18, CFO19a and dynamical CFO19b properties can be approximated in such a regime in terms of effective models, where the quantum particle system is driven by a classical field, which in turn is the classical counterpart of the quantized field. In extreme synthesis, the quantum field operator gets replaced by a classical field, which is just a function on $\mathbb{R}^d$, and the interaction term $H_I$ in $H_\varepsilon$ gives rise to a potential $V_z$ depending on the classical field configuration $z \in \mathfrak{h}$. Concretely, the quasi-classical effective Hamiltonian reads

$$H_z = K_0 + \sum_{j=1}^N V_z(x_j) + \langle z \mid \omega \mid z \rangle_\mathfrak{h}, \quad (1.15)$$

and it is self-adjoint on some dense $D \subset L^2(\mathbb{R}^{dN})$ for any $z \in \mathfrak{h}$ (see CF18 Thms. 2.1–2.3 and CFO19a Thm. 1.1). In each model the explicit expression of such an effective potential can be identified explicitly:

(a) in the Nelson model, each particle feels a potential of the form

$$V_z(x) = 2 \text{Re} \langle z \mid \lambda(x) \rangle_\mathfrak{h} \in \mathcal{D}(L^2(\mathbb{R}^d)); \quad (1.16)$$

(b) for the polaron, the formal expression of the potential $V_z$ is the same as in (1.16) above, although, since $\Omega_\varepsilon$ does not belong to $L^\infty(\mathbb{R}^d; \mathfrak{h})$, the expression on the r.h.s. must be interpreted in the proper way (see §4.2); in addition, the obtained potential is no longer bounded but it is infinitesimally form-bounded w.r.t. $-\Delta$;

(c) in the Pauli-Fierz model, the effective operator is obtained via the replacement of the field $A_\varepsilon$ by its classical counterpart $a_z(x) = 2 \text{Re} \langle z \mid \lambda(x) \rangle_\mathfrak{h}$, which is continuous and vanishing at $\infty$ (see CFO19a Rmk. 1.5), and thus, in order to recover the expression (1.15), $V_z$ must be the operator

$$V_z(x) = 2 \sum_{j=1}^N \frac{1}{m_j} \left[ -ie \text{Re} \langle z \mid \lambda_j(x) \rangle_\mathfrak{h} \cdot \nabla_j + e^2 \left( \text{Re} \langle z \mid \lambda_j(x) \rangle_\mathfrak{h} \right)^2 \right]. \quad (1.17)$$
Note that in case (c) the effective operator can in fact be simply rewritten as
\[ H_z = \sum_{j=1}^{N} \frac{1}{2m_j} (-i\nabla_j + ca_2(x_j))^2 + \mathcal{W}(\mathbf{X}) + (z|\omega|z)_h. \] (1.18)

We can now define the effective quasi-classical ground state energy in terms of the energy functional
\[ \mathcal{E}_{qc}[\psi, z] := \langle \psi | H_z | \psi \rangle_{L^2(\mathbb{R}^{dN})}, \quad (\psi, z) \in L^2(\mathbb{R}^{dN}) \oplus \mathfrak{h}_\omega; \] (1.19)
as
\[ E_{qc} := \inf_{(\psi, z) \in \mathcal{D}_{qc}} \mathcal{E}[\psi, z], \] (1.20)
where
\[ \mathcal{D}_{qc} := \left\{ (\psi, z) \in L^2(\mathbb{R}^{dN}) \oplus \mathfrak{h}_\omega \mid ||\psi||_2 = 1, |\mathcal{E}_{qc}[\psi, z]| < +\infty \right\}. \] (1.21)
Here, \( \mathfrak{h}_\omega \) is the Hilbert completion of \( \bigcap_{k \in \mathbb{N}} \mathcal{D}(\omega^k) \) with respect to the scalar product \( \langle \cdot | \cdot \rangle_{\mathfrak{h}_\omega} := \langle \cdot | \omega | \cdot \rangle_{\mathfrak{h}_\omega} \), i.e.,
\[ \mathfrak{h}_\omega := \bigcap_{k \in \mathbb{N}} \mathcal{D}(\omega^k)^{\perp}_{\mathfrak{h}_\omega}. \] (1.22)
We denote by \( (\psi_{qc}, z_{qc}) \in \mathcal{D}_{qc} \) a corresponding minimizing configuration (if any), i.e., such that
\[ E_{qc} = \mathcal{E}_{qc}[\psi_{qc}, z_{qc}]. \] (1.23)

Concretely, the functional \( \mathcal{E}_{qc} \) plays the role of the quasi-classical energy of the system under consideration. However, the reader should be careful and be aware that \( H_z \) is not the Hamiltonian energy of the whole system: the complete environment + small system’s evolution is indeed not of Hamiltonian type. For each fixed \( z \in \mathfrak{h}_\omega \), the Hamilton-Jacobi equations of \( \mathcal{E}_{qc}[\psi, z] \), w.r.t. the (complex) \( \psi \) variable, yield the dynamics of the small system; the environment on the other hand is stationary in the problems under consideration in this paper (see [CFO19b] for a detailed analysis of quasi-classical dynamical systems).

The preliminary questions to address towards the derivation of the above quasi-classical effective models are whether such models are stable and, if this is the case, whether a minimizing configuration does exist: explicitly, if
\[ E_{qc} > -\infty \quad \text{(stability),} \] (VP1)
\[ \exists (\psi_{qc}, z_{qc}) \in \mathcal{D}_{qc} \quad \text{(existence of a ground state).} \] (VP2)

Note that any critical point \( (\psi, z) \in \mathcal{D}_{qc} \) of the functional \( \mathcal{E}_{qc}[\psi, z] \) must satisfy the condition
\[ \delta_{(\psi, z)}|\mathcal{E}_{qc}[\psi, z] - \epsilon ||\psi||_2^2| = 0, \]
which yields the Euler-Lagrange equations
\[ \left\{ \begin{array}{l}
H_z \psi = \epsilon \psi, \\
\omega z + \left\langle \psi \left| \partial_z \sum_j v_j(x_j) \right| \psi \right\rangle_{L^2(\mathbb{R}^{dN})} = 0,
\end{array} \right. \] (1.24)
where the Lagrange multiplier \( \epsilon = \langle \psi | H_z | \psi \rangle \in \mathbb{R} \) takes into account the normalization constraint on \( \psi \). We anticipate that a consequence of the convergence of the microscopic ground state, stated in [Corollary 1.10] below, is that, under suitable assumptions on \( K_0 \) (for instance if \( \mathcal{W} \) is trapping), the answer to both questions in (VP1) and (VP2) is positive and, in particular, the set of minimizers is not empty.

3We use the compact notation \( \mathbf{X} := (x_1, \ldots, x_N) \in \mathbb{R}^{dN}. \)
The variational problem above is strictly related to the more general issue of rigorous derivation of effective theories, since, at least for the polaron model, it is known that the minimization of the microscopic energy can be approximated in the limit $\varepsilon \to 0$ in terms of a nonlinear problem on $\psi$ alone. Indeed, focusing on the particle system, one can naturally approach in a different and a priori inequivalent way, i.e., first one gets rid of the classical field by minimizing over $z \in \mathfrak{h}_\omega$ and then investigates the minimization of the remaining functional on $\psi$, which is obviously nonlinear, since the minimizing $z$ depends on $\psi$ itself. As anticipated, this strategy has been already followed in the literature in the case of the polaron in the strong coupling regime, leading to the Pekar functional and the corresponding variational problem \cite{pek55, dv83, l197}. Such a feature is however not exclusive of the polaron and can be observed in all the models mentioned above: we present below a formal derivation of a Pekar-like functional $\mathcal{E}_\text{Pekar}[\psi]$, for both the Nelson and polaron model. The Pauli-Fierz case is also discussed below, let us remark however that in this case such a procedure does not yield an explicit nonlinear functional of $\psi$ (see \cite{1135} below), because it is in general not possible to solve explicitly the variational equation expressing the minimizing $z$ in terms of $\psi$.

The formal procedure goes as follows: solving the critical point condition $\delta \mathcal{E}_\text{qc} = 0$ w.r.t. the variable $z$ for fixed $\psi$, we find some $z_{\psi}$, that we can plug in $\mathcal{E}_\text{qc}$, thus obtaining the Pekar energy $\mathcal{E}_\text{Pekar}[\psi] := \mathcal{E}_\text{qc}[\psi, z_{\psi}]$. Such a scheme can be made to work rigorously for the polaron case (b) with some care, but the variable $z$ is not the right one to consider in cases (a) and (c). Under the assumptions we have made (recall in particular \cite{118} and \cite{111}), it is indeed more natural to set, since $z \in \mathfrak{h}_\omega$,

$$\eta := \omega^{1/2} z,$$

(note however that in case (b) $\eta = z$) and consider the functional $\mathcal{F}_\text{qc}[\psi, \eta] := \mathcal{E}_\text{qc}[\psi, \omega^{-1/2} \eta]$, which in case (a) reads

$$\mathcal{F}_\text{qc}[\psi, \eta] = \left< \psi \bigg| \mathcal{K}_0 + 2\text{Re} \sum_j \left< \eta \omega^{-1/2} \lambda(x_j) \right| \psi \right>_{\mathcal{L}^2(\mathbb{R}^d; \mathfrak{h})} + \| \eta \|_{\mathfrak{h}}^2$$

$$= \left< \psi \bigg| \mathcal{K}_0 \right| \psi \right>_{\mathcal{L}^2(\mathbb{R}^d; \mathfrak{h})} + 2\text{Re} \left< \eta \bigg| \left< \psi \left| \mathcal{A} \right| \psi \right>_{\mathcal{L}^2(\mathbb{R}^d; \mathfrak{h})} \right>_{\mathfrak{h}} + \| \eta \|_{\mathfrak{h}}^2$$

(1.26)

where $\mathcal{A} \in \mathcal{L}_\infty(\mathbb{R}^d, \mathfrak{h})$ is given by $\mathcal{A}(X) := \sum_{j=1}^N (\omega^{-1/2} \lambda)(x_j)$ (recall again the assumption \cite{113} on $\lambda$) and we have exploited the linearity of the scalar product. Taking now the functional derivative w.r.t. to $\eta$, we get the Euler-Lagrange equation for the minimization of the above energy w.r.t. $\eta \in \mathfrak{h}$, i.e.,

$$\eta + \left< \psi \left| \mathcal{A}(\cdot) \right| \psi \right>_{\mathcal{L}^2(\mathbb{R}^d; \mathfrak{h})} = 0,$$

yielding the minimizing $\eta_{\text{Pekar}}$ as

$$\eta_{\text{Pekar}}[\psi] = - \sum_{j=1}^N \int_{\mathbb{R}^d} dx_1 \cdots dx_N \left( \omega^{-1/2} \lambda \right)(x_j) |\psi(x_1, \ldots, x_N)|^2$$

(1.28)

which can be easily seen to belong to $\mathfrak{h}$ under the assumptions made. Plugging $\eta_{\text{Pekar}}$ back into \cite{1126}, we get

$$\mathcal{E}_\text{Pekar}[\psi] := \inf_{\eta \in \mathfrak{h}} \mathcal{F}_\text{qc}[\psi, \eta] = \mathcal{F}_\text{qc}[\psi, \eta_{\text{Pekar}}[\psi]] = \left< \psi \bigg| \mathcal{K}_0 + \mathcal{V}_\text{Pekar} \star |\psi|^2 \right| \psi \right>.$$  

(1.29)

Here we have denoted by $\star$ the action of the integral kernel $\mathcal{V}_\text{Pekar}(X, Y)$ on $|\psi|^2$, i.e.,

$$\left( \mathcal{V}_\text{Pekar} \star |\psi|^2 \right)(X) := \int_{\mathbb{R}^d} dY \mathcal{V}_\text{Pekar}(X, Y) |\psi(Y)|^2$$

(1.30)
and
\[ Y_{\text{Pekar}}(X, Y) = -\text{Re} \sum_{i,j=1}^{N} \left\langle \lambda(x_i) \left| \omega^{-1} \right| \lambda(y_j) \right\rangle_{\mathfrak{h}} \in L^{\infty}(\mathbb{R}^{2dN}). \] (1.31)

Note that in case of identical particles – either fermionic or bosonic –, the above expressions may be conveniently rewritten using the one-particle density \( \rho_{\psi} \in L^{1}(\mathbb{R}^{d}) \) associated with \( \psi \), i.e.,
\[ \rho_{\psi}(x) := N \int_{\mathbb{R}^{d(N-1)}} dx_2 \cdots dx_N |\Psi(x, x_2, \ldots, x_N)|^2. \] (1.32)
Indeed, in this case, (1.28) reads
\[ \eta_{\text{Pekar}}[\psi] = -\left\langle \rho_{\psi} \left| \omega^{-1/2} \lambda \right| (\cdot) \right\rangle_{L^{2}(\mathbb{R}^{d})}, \]
and the Pekar energy becomes
\[ \mathcal{E}_{\text{Pekar}}[\psi] = \langle \psi | \mathcal{K}_{\alpha} | \psi \rangle_{L^{2}(\mathbb{R}^{dN})} + \langle \rho_{\psi} | \mathcal{U} | \rho_{\psi} \rangle_{L^{2}(\mathbb{R}^{d})}, \] (1.33)
where
\[ \mathcal{U} = U(x, y) := \left\langle \lambda(x) \left| \omega^{-1} \right| \lambda(y) \right\rangle_{\mathfrak{h}}, \] (1.34)
which is its typical form in the literature. For instance, in the polaron case, one recovers the self-interacting potential generated by the kernel \( \mathcal{U}(x - y) = -\alpha |x - y|^{-1} \).

The above derivation can be easily seen to be correct under the assumptions made in case (a). In case (b), however, one can not apply such a derivation straightforwardly because \( \lambda \not\in L^{\infty}(\mathbb{R}^{dN}; \mathfrak{h}) \), but a simple well-known trick (see Remark 4.2) allows to split it into two terms, which can be handled separately as above. In case (c) on the other hand the Pekar functional takes the implicit form
\[ \begin{cases} \eta_{\text{Pekar}} + \sum_{j} \frac{1}{m_j} \left\langle \psi \left| e\omega^{-1/2} \lambda_j \cdot (-i \nabla_j) + 2e^2 \omega^{-1/2} \lambda_j \right| \right\rangle_{\mathfrak{h}} = 0, \\ \mathcal{E}_{\text{Pekar}}[\psi] = \langle \psi | \mathcal{H}_{\text{loc}} | \psi \rangle_{L^{2}(\mathbb{R}^{3N})}, \end{cases} \] (1.35)
where \( \mathcal{H}_{\text{loc}} \) is given by (1.18) and we set \( z_{\psi} := \omega^{-1/2} \eta_{\text{Pekar}}[\psi] \) for short. As before, all the terms in the first equation belong to \( \mathfrak{h} \), thanks to the assumptions on \( \lambda_j \) and the fact that any \( (\psi, z) \in \mathcal{D}_{\eta} \) is such that \( \psi \in H^{1}(\mathbb{R}^{3N}) \). Furthermore, the last term can be thought of as the action on \( \eta_{\text{Pekar}} \) of a linear operator \( T \) on \( \mathfrak{h} \) whose norm is bounded by
\[ 2e^2 \sum_{j=1}^{N} \frac{1}{m_j} \| \omega^{-1/2} \lambda_j \|_{\mathfrak{h}}^{2}, \]
which is smaller than 1, if \( e \) is small enough. In this case, \( 1 + T \) is invertible and there exists a unique solution \( \eta_{\text{Pekar}}[\psi] \in \mathfrak{h} \) of the first equation. More in general, existence and uniqueness of \( \eta_{\text{Pekar}}[\psi] \) for any value of \( e \) follows from the strict convexity of the energy in \( \eta \) (see next Remark 1.2 and Lemma 2.4). Note however that unfortunately it is not possible to write explicitly \( \mathcal{E}_{\text{Pekar}} \) as a functional of \( \psi \) alone, since, due to the presence of an operator – the gradient –, one can not exchange the scalar product in \( L^{2}(\mathbb{R}^{3N}) \) with the one in \( \mathfrak{h} \), as it was done in (1.20). In particular, even for identical particles, the second term in the first equation in (1.35) depends on the reduced density matrix, while the last one is a function of the density alone.
We now define
\[ E_{\text{Pekar}} := \inf_{\psi \in \mathcal{D}_{\text{Pekar}}} E_{\text{Pekar}}[\psi], \] with \( \mathcal{D}_{\text{Pekar}} := \{ \psi \in L^2(\mathbb{R}^dN) \mid \| \psi \|_2 = 1, |E_{\text{Pekar}}[\psi]| < +\infty \} \), as the ground state energy of the Pekar functionals \( (1.29) \) and \( (1.35) \), and denote by \( \psi_{\text{Pekar}} \in \mathcal{D}_{\text{Pekar}} \) any corresponding minimizer. It is then natural to wonder whether there is any connection between the questions \( (\text{VP1}) \) and \( (\text{VP2}) \) and the analogous problems for \( E_{\text{Pekar}} \), i.e.,
\[ E_{\text{Pekar}} > -\infty \] (stability),
\[ 2\psi_{\text{Pekar}} \in L^2(\mathbb{R}^dN) \] (existence of a ground state).

This is of particular interest for physical applications, since the minimization of the nonlinear functional \( E_{\text{Pekar}} \) may be easier to address also in numerical experiments. A priori however it is not at all obvious that such a relation exists, but in the next Proposition 1.1 we are going to state that the two variational problems are actually equivalent, which is particularly interesting in case (c) since the explicit form of \( E_{\text{Pekar}} \) is not available.

**Proposition 1.1 (Equivalence of variational problems).**

Under the assumptions made above,
\[ E_{\text{Pekar}} = E_{\text{qc}} > -\infty. \] (1.37)

Furthermore, if \((\psi_{\text{qc}}, z_{\text{qc}}) \in \mathcal{D}_{\text{qc}} \) is a minimizer of \( E_{\text{qc}}[\psi, z] \), then
\[ E_{\text{Pekar}}[\psi_{\text{qc}}] = E_{\text{Pekar}}. \] (1.38)

Conversely, if \( \psi_{\text{Pekar}} \) is a minimizer of \( E_{\text{Pekar}}[\psi] \), then \( \eta_{\text{Pekar}}[\psi_{\text{Pekar}}] \in \mathfrak{h} \) (given by \( 1.27 \) and \( 1.35 \) with \( \psi = \psi_{\text{Pekar}} \), respectively) and
\[ E[\psi_{\text{Pekar}}, \eta_{\text{Pekar}}] = E_{\text{qc}}. \] (1.39)

**Remark 1.2 (Uniqueness of \( \eta_{\text{Pekar}} \)).**

We prove in Lemma 2.4 that the quasi-classical functional \( F_{\text{qc}}[\psi, \eta] \) (or, equivalently, \( E_{\text{qc}}[\psi, z] \)) is strictly convex in \( \eta \in \mathfrak{h} \) for given \( \psi \in L^2(\mathbb{R}^dN) \). Hence, \( \eta_{\text{Pekar}}[\psi] \) is unique (for fixed \( \psi \)). Note however that the functional \( F_{\text{qc}} \) is not jointly convex in \( (|\psi|^2, \eta) \).

1.2. **Ground state in the quasi-classical regime.** We can now state in detail our main results. We first consider the microscopic ground state energy \( E_\varepsilon \) defined in \( (1.12) \) and its quasi-classical limit. Recall the definition of the quasi-classical energy \( E_{\text{qc}} \) in \( (1.20) \).

**Theorem 1.3 (Ground state energy).**

Under the assumptions made above, \( \exists C < +\infty \) such that \( E_\varepsilon > -C \) and
\[ E_\varepsilon \xrightarrow{\varepsilon \to 0} E_{\text{qc}}, \] (1.40)
which in particular implies that \( (\text{VP1}) \) holds true.

**Remark 1.4 (Assumptions).**

The above result requires only a minimal set of assumptions on the microscopic models, those listed in their definitions, which are the weakest ones guaranteeing the self-adjointness and boundedness from below of the microscopic Hamiltonians. In particular, the quantum potential \( W \) may not be trapping, so that there might be no ground state for both the microscopic and the macroscopic problems.
The above Theorem 1.3 completes and extends analogous results proven in [CFT8 Thm. 2.4] and [CFO19a Thm. 1.9], relaxing the assumptions on the microscopic models and taking into account more general settings. We also point out that the proof of the above result provided in § 3 is quite different and much more general than the ones contained in the above references and involves the new mathematical structure of quasi-classical Wigner measures first introduced in [CFO19b]. In fact, the argument in the proofs given in [CF18, CFO19a] is not complete, since it relies on the assumption that one can find a minimizing sequence which can be decomposed into a linear combination of finitely-many product states, whose number is uniformly bounded in $\varepsilon$. This is a posteriori right (as it follows from the proof of Theorem 1.3), up to errors vanishing in the limit $\varepsilon \to 0$, but it is unproven there.

Once the energy convergence has been stated, it is natural to ask whether, in presence of a microscopic approximate ground state $\Psi_{\varepsilon,\delta}$ or ground state $\Psi_{gs}$, one can prove a suitable convergence respectively to quasi-classical minimizing sequences or configurations $(\psi_{qc}, z_{qc}) \in \mathcal{D}_{qc}$. Let us stress that the question of existence of a ground state of the microscopic energy has been widely studied in the literature and there are more restrictive conditions on the models guaranteeing that $E_\varepsilon \in \sigma_{pp}(H_\varepsilon)$ (see § 4.1 to 4.3); our results about approximate ground states apply even if the microscopic ground state do not exist, and whenever it exists we are able to provide its quasi-classical characterization.

In order to properly formulate the convergence, we first need to introduce a key structure in quasi-classical analysis: the quasi-classical Wigner measures and their relative topologies. We preliminarily recall the definition of the space $P(h_\omega; L^2(\mathbb{R}^{dN}))$ of state-valued probability measures (see [CFO19b, Def. 2.1]), given by measures $m$ on $h_\omega$ taking values in $L^1(L^2(\mathbb{R}^{dN}))$ – the space of positive trace class operators on $L^2(\mathbb{R}^{dN})$ – such that $m(\emptyset) = 0$, the measure is unconditionally $\sigma-$additive in the trace class norm and $\|m(h_\omega)\|_{L^2} = 1$. Starting from such a notion, it is possible to construct a theory of integration of functions with values in the space of bounded operators on $L^2(\mathbb{R}^{dN})$ w.r.t. state-valued measures, so that, for any measurable $B(z) \in \mathcal{B}(L^2(\mathbb{R}^{dN}))$,

$$\int_{h_\omega} dm(z) B(z) \in L^1(L^2(\mathbb{R}^{dN})). \quad (1.41)$$

We refer to Appendix A or to the existing literature (e.g., [Bal85, Ger91, GMS91, FG02, Teu03]) for further details. In particular, we point out that any such state-valued measure $m$ admits a Radon-Nikodým decomposition, i.e., there exist a scalar Borel measure $\mu_m$ and a $\mu_m$-integrable function $\gamma_m(z) \in L^1_+(L^2(\mathbb{R}^{dN}))$ defined a.e. and with values in normalized density matrices, such that

$$dm(z) = \gamma_m(z)d\mu_m(z). \quad (1.42)$$

Hence, (1.41) can be rewritten

$$\int_{h_\omega} dm(z) B(z) = \int_{h_\omega} d\mu_m(z) \gamma_m(z)B(z). \quad (1.43)$$

Finally, let us denote by $W_\varepsilon(z)$, $z \in h$ the Weyl operator constructed over the creation and annihilation operators $a_z^\dagger, a_z$, i.e.,

$$W_\varepsilon(z) := e^{i(a_z^\dagger(z)+a_z(z))}. \quad (1.44)$$
Definition 1.5 (Quasi-classical Wigner measures).
For any family of normalized microscopic states \( \{ \Psi_\epsilon \}_{\epsilon \in (0,1)} \subset \mathcal{H} \), the associated set of quasi-classical Wigner measures \( \mathcal{W}(\Psi_\epsilon, \epsilon \in (0,1)) \subset \mathcal{P}(\mathfrak{h}_\omega; L^1_q(L^2(\mathbb{R}^d) )) \) is the subset of all probability measures \( m \), such that
\[
\Psi_{\epsilon_n} \xrightarrow{\text{qc}} m, \tag{1.45}
\]
where the above convergence means that, for all \( \eta \in \mathcal{D}(\omega^{-1/2}) \) and all compact operators \( K \in \mathcal{L}\infty(L^2(\mathbb{R}^{dN})) \),
\[
\lim_{n \to +\infty} \langle \Psi_{\epsilon_n} | K \otimes \omega_{\epsilon_n}(\eta) \Psi_{\epsilon_n} \rangle_{\mathcal{H}_{\epsilon_n}} = \int_{\mathfrak{h}_\omega} d\mu_m(z) e^{2i\text{Re}(\langle \eta \rangle_\omega)z} \text{tr}_{L^2(\mathbb{R}^{dN})}[\gamma_m(z)K] \\
= \int_{\mathfrak{h}_\omega} d\mu_m(z) e^{2i\text{Re}(\omega^{-1/2}\eta|\omega^{1/2}z)} \text{tr}_{L^2(\mathbb{R}^{dN})}[\gamma_m(z)K]. \tag{1.46}
\]

Remark 1.6 (Measures on \( \mathfrak{h}_\omega \) and test functions).
A reader familiar with infinite dimensional semiclassical analysis or quasi-classical analysis will find the definition of Wigner measures given here slightly different to the usual one \([\text{AN08}, \text{CFO19}]\). Typically, one considers microscopic states that satisfy a number operator estimate, namely for which the expectation of \( dG_\epsilon(1) \) is \( \epsilon \)-uniformly bounded for some \( c > 0 \). The corresponding Wigner measures are concentrated on \( \mathfrak{h} \) \([\text{AN08}]\), and it is natural to test the convergence with Weyl operators having arguments \( \eta \in \mathfrak{h} \). However, in studying variational problems the number operator estimate may not always be available, in particular whenever the field is massless, such as in electromagnetism (Pauli-Fierz model). In that case, only energy estimates, i.e., involving \( dG_\epsilon(\omega) \), are available. The Wigner measures of states satisfying such an energy estimate are concentrated in \( \mathfrak{h}_\omega \), and it is natural to test convergence with Weyl operators having arguments \( \eta \in \mathcal{D}(\omega^{-1/2}) \) belonging to a dense subset of the continuous dual space \([\text{Fal18a}]\). If both the number estimate and the free energy estimate are available, then the measure is concentrated in \( \mathfrak{h} \cap \mathfrak{h}_\omega \); this happens for massive fields, where in addition \( \mathfrak{h} \cap \mathfrak{h}_\omega = \mathfrak{h}_\omega \). Finally, let us remark as well that in all concrete applications \( \mathfrak{h}_\omega \) is in fact the natural domain of definition of the quasi-classical energy \( \mathcal{E}_{\text{qc}} \).

The above notion of quasi-classical convergence, defined in (1.46), is however not the only meaningful topology one can consider for sequences of microscopic states. More precisely, the test in (1.46) may be extended to bounded operators, which means that one is considering the weak-* topology on \( \mathcal{B}(L^2(\mathbb{R}^{dN}) ) \), instead of \( \mathcal{L}^1_q(L^2(\mathbb{R}^{dN}) ) = \mathcal{L}\infty(L^2(\mathbb{R}^{dN}) ) \). In this case, the cluster points belong to a larger space than \( \mathcal{P}(\mathfrak{h}_\omega; L^1_q(L^2(\mathbb{R}^{dN}) ) \), namely the space of generalized state-valued measures (see \([\text{Fal18a}]\) for a detailed and more general discussion).
We thus introduce the set of positive states \( \mathcal{L}^1_q(L^2(\mathbb{R}^{dN}) ) \) in the closure w.r.t. the weak-* topology of the space of trace class operators on \( L^2(\mathbb{R}^{dN}) \): we denote the action of a functional \( F \in \mathcal{L}^1_q(L^2(\mathbb{R}^{dN}) ) \) on a bounded operator \( B \in \mathcal{B}(L^2(\mathbb{R}^{dN}) ) \) as \( F[B] \in \mathbb{C} \) and its norm as
\[
\| F \|_{\mathcal{F}} := \sup_{B \in \mathcal{B}(L^2(\mathbb{R}^{dN}) ), \| B \| = 1} |F[B]|. \tag{1.47}
\]

Definition 1.7 (Generalized quasi-classical Wigner measures).
For any family of normalized microscopic states \( \{ \Psi_\epsilon \}_{\epsilon \in (0,1)} \subset L^2(\mathbb{R}^{dN})_\epsilon \), the associated set of generalized quasi-classical Wigner measures \( \mathcal{W}(\Psi_\epsilon, \epsilon \in (0,1)) \subset \mathcal{P}(\mathfrak{h}_\omega; \mathcal{L}^1_q(L^2(\mathbb{R}^{dN}) ) \)
is the subset of all probability measures \( n \), such that
\[
\Psi_{\epsilon_n} \xrightarrow{qc_{\epsilon_n \to 0}} n,
\]
where the above convergence means that, for all \( \eta \in \mathcal{D}(\omega^{-1/2}) \) and all bounded operators \( B \in \mathcal{B}(L^2(\mathbb{R}^{dN})) \),
\[
\lim_{n \to +\infty} \langle \Psi_{\epsilon_n} | B \otimes W_{\epsilon_n}(\eta) \Psi_{\epsilon_n} \rangle_{\mathcal{H}_{\epsilon_n}} = \int_{\mathcal{H}_\omega} d\mu(z) |B| e^{2i\text{Re}(\omega^{-1/2}\eta)\omega^{1/2}z}_b .
\]

We can now formulate the results about the convergence of microscopic minimizing sequences \( \Psi_{\epsilon,\delta} \) and microscopic minimizers \( \Psi_{gs} \). We start by stating a stronger result with some additional assumptions on the microscopic models. Without such assumptions we are still able to prove a weaker case, but it requires to introduce a generalized variational problem.

**Theorem 1.8** (Convergence of approximate ground states (I)).

If \( K_0 \) has compact resolvent, then, for any \( \delta > 0 \) and for any family of approximate ground states \( \Psi_{\epsilon,\delta} \) satisfying (1.13), \( \mathcal{W}(\Psi_{\epsilon,\delta}, \epsilon \in (0,1)) \neq \emptyset \). Moreover, any family of quasi-classical Wigner measures \( \{m_{\delta}\}_{\delta \in (0,1)} \) is such that, for all \( \delta > 0 \),
\[
\text{tr}_{L^2(\mathbb{R}^{dN})} m_{\delta}(h_\omega) = 1 \quad \text{and it is an approximate ground state of } \mathcal{E}_{qc}[\psi, z], \text{ i.e.},
\]
\[
\int_{\mathcal{H}_\omega} d\mu_{m_{\delta}}(z) \text{tr}_{L^2(\mathbb{R}^{dN})}(\gamma_{m_{\delta}}(z)H_z) < E_{qc} + \delta .
\]
Consequently, there is small \( m_{\delta} \)-probability that \( \mathcal{E}_{qc}(\psi_{\delta}, z_\delta) \) is larger than \( E_{qc} + \delta \): for all \( k \in \mathbb{N} \),
\[
\mathbb{P}_{m_{\delta}} \left\{ \mathcal{E}_{qc}(\psi_{\delta}, z_\delta) \geq E_{qc} + k\delta \right\} < \frac{1}{k} .
\]

**Corollary 1.9** (Convergence to ground states (I)).

If \( K_0 \) has compact resolvent, then any quasi-classical Wigner measure \( m \in \mathcal{W}(\Psi_{0,\delta}(1), \epsilon \in (0,1)) \), corresponding to approximate ground states \( \Psi_{0,\delta}(1) \) satisfying (1.13) with \( \delta = \delta_0(1) \), is such that \( \text{tr}_{L^2(\mathbb{R}^{dN})} m(h_\omega) = 1 \) and it is concentrated on the set of ground states \( \{\psi_{qc}, z_{qc}\} \in \mathcal{D}_{qc} \) of \( \mathcal{E}_{qc}[\psi, z] \). Consequently, \( \mathcal{E}_{qc}[\psi, z] \) has at least one ground state and both (VP2) and (VP2) hold true.

**Corollary 1.10** (Convergence of ground states (I)).

If \( K_0 \) has compact resolvent and \( H_{\epsilon} \) has a ground state \( \Psi_{gs} \), then any corresponding quasi-classical Wigner measure \( m \in \mathcal{W}(\Psi_{gs}, \epsilon \in (0,1)) \) is such that \( \text{tr}_{L^2(\mathbb{R}^{dN})} m(h_\omega) = 1 \) and it is concentrated on the set of ground states \( \{\psi_{qc}, z_{qc}\} \in \mathcal{D}_{qc} \) of \( \mathcal{E}_{qc}[\psi, z] \).

**Remark 1.11** (Uniqueness and gauge invariance).

Concerning uniqueness, we point out that both the microscopic and the quasi-classical variational problems are gauge invariant, namely the multiplication by a constant phase factor of \( \Psi \) or \( \psi \) does not change the energy. Hence, even if one could prove uniqueness of the quasi-classical minimizer \( (\psi_{qc}, z_{qc}) \) up to gauge transformations, one could not conclude that the set of limit points \( \mathcal{W}(\Psi_{0,\delta}(1), \epsilon \in (0,1)) \) or \( \mathcal{W}(\Psi_{gs}, \epsilon \in (0,1)) \) are just given by a Dirac delta measure centered at \( (\psi_{qc}, z_{qc}) \). Indeed, because of gauge invariance, the quasi-classical Wigner measures would be supported over the unit one-dimensional sphere generated by the configurations \( (e^{i\vartheta}\psi_{qc}, z_{qc}) \), \( \vartheta \in \mathbb{R} \).
Remark 1.12 (Condition on $K_0$).

The assumption that $K_0$ has compact resolvent is reasonable, since that is typically the case in which one can also prove the existence of a microscopic minimizer at least for massive systems (see Remark 1.13 below), e.g., in presence of a trapping potential. However, it is also needed in a technical step in the proof to ensure that there is no loss of mass along the convergence (1.46), i.e., $\text{tr}_{L^2(\mathbb{R}^dN)} m(h_\omega) = 1$. Similar assumptions are present also in [CFO19b] (see in particular the discussion in [CFO19b, Rmks. 1.9 – 1.10 & §1.6]).

Remark 1.13 (Existence of $\Psi_{gs}$).

In all the three cases (a) – (c), if the Bose field is massive, i.e., $\exists m > 0$ such that $\omega \geq m > 0$ (which is always the case for the polaron), then it is known [DG99, Thm. 4.1] that the microscopic Hamiltonian $H_\varepsilon$ admits a ground state $\Psi_{gs} \in \mathcal{H}_\varepsilon$, if $K_0$ has compact resolvent. Hence, in the massive case, one can remove the assumption on the existence of $\Psi_{gs}$. When the field is massless, on the other hand, it is also known that microscopic ground states might not exist or belong to a non-Fock representation of the algebra of observables [Piz03]. This second case is not covered by the above Corollary 1.10, but it may be treated with our techniques. We plan to come back to such a question in a future work.

Remark 1.14 (Existence of quasi-classical minimizers).

Our analysis shows that the quasi-classical energy functionals $E_{qc}[\psi, z]$ always have at least one minimizer, provided that $K_0$ has compact resolvent, i.e., provided that the quantum subsystem is trapped. This gives an additional evidence of the fact that nonexistence or non-Fock-representability (see Remark 1.13 above) of the microscopic ground-state is one of the many complications encountered in quantizing fields.

As anticipated, if we drop the assumption on the operator $K_0$, there is still convergence, but the variational problem (1.20) has to be generalized: we thus set, for any pure state $\rho \in \mathcal{F}_{L^1} + L^2(\mathbb{R}^dN)$ and any $z \in h_\omega$, $E_{gqc}[\rho, z] := \rho[H_z].$ (1.52)

We consider the corresponding variational problem: setting (recall the definition (1.47)) $D_{gqc} := \left\{ (\rho, z) \in \mathcal{F}_{L^1} + L^2(\mathbb{R}^dN) \oplus h_\omega \mid \|\rho\|_{B'} = 1, |\rho[H_z]| < +\infty \right\},$ (1.53)

we define $E_{gqc} := \inf_{(\rho, z) \in D_{gqc}} E_{gqc}[\rho, z],$ (1.54)

and denote by $(\rho_{gqc}, z_{gqc}) \in D_{gqc}$ a minimizing sequence satisfying $E_{gqc}[\rho_{gqc}, z_{gqc}] < E_{gqc} + \delta,$

and by $(\rho_{gqc}, z_{gqc}) \in D_{gqc}$ any corresponding minimizing configuration.

Theorem 1.15 (Convergence of approximate ground states (II)).

If $K_0$ does not have compact resolvent, then, for any $\delta > 0$ and for any family of approximate ground states $\Psi_{\varepsilon, \delta}$ satisfying (1.13), $\mathcal{W}(\Psi_{\varepsilon, \delta}, \varepsilon \in (0, 1)) \neq \emptyset$. Moreover, any family of generalized quasi-classical Wigner measures $\{n_\delta\}_{\delta > 0} \in \bigcup_{\delta > 0} \mathcal{W}(\Psi_{\varepsilon, \delta}, \varepsilon \in (0, 1))$ is such that, for all $\delta > 0$, $\|n_\delta(h_\omega)\|_{B'} = 1$ and it is an approximate ground state of $E_{gqc}[\rho, z]$, i.e., $\int_{h_\omega} \text{dn}_\delta(z)[H_z] < E_{gqc} + \delta.$ (1.55)
Corollary 1.16 (Convergence to ground states (II)).
If $K_0$ does not have compact resolvent, then any generalized quasi-classical Wigner measure $\mathfrak{n} \in \mathcal{GW}(\Psi, \varepsilon \in (0, 1))$, corresponding to approximate ground states $\Psi(\varepsilon, o) \in \mathcal{GW}[\varepsilon, 0]$. Consequently, the functional $E_{gqc}[\rho, z]$ admits at least one ground state in $\mathcal{D}_{gqc}$.

Corollary 1.17 (Convergence of ground states (II)).
If $K_0$ does not have compact resolvent and $H_\varepsilon$ has a ground state $\Psi_{gs}$, then any generalized Wigner measure $\mathfrak{n} \in \mathcal{GW}(\Psi_{gs}, \varepsilon \in (0, 1))$ is such that $\|\mathfrak{n}(\hbar_\omega)\|_{\mathcal{B}'} = 1$ and it is concentrated on the set of ground states $(\rho_{gqc}, z_{gqc}) \in \mathcal{D}_{gqc}$.

Remark 1.18 (Quasi-classical energy and generalized quasi-classical energy).
As proved in §2 below (see Proposition 2.8),

$$E_{qc} = E_{gqc},$$

which is in fact crucial to prove convergence of the ground state energy for systems without trapping on the quantum particles.

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2. QUASI-CLASSICAL MINIMIZATION PROBLEMS

In this section we consider minimization problems in the quasi-classical setting: we study the functionals introduced in §1.1 and the relative minimizations, but also define and investigate more general problems.

2.1. Quasi-classical functionals, states and related minimization problems. A quasi-classical system behaves like an open system in which a classical environment (of infinite dimension) drives a quantum small system, described by an Hilbert space $L^2(\mathbb{R}^{dN})$. The classical environment is described by a space of configurations $h_\omega$, usually a complex Hilbert space identifiable with the complex phase space of the environment’s degrees of freedom. A probability distribution $\mu$ on $h_\omega$ tells how probable each environment’s configuration is, while a state-valued function $h_\omega \ni z \mapsto \gamma(z) \in L^1(\mathcal{L}^2(\mathbb{R}^{dN}))$ tells how each environment’s configuration drives the small system’s quantum state. Analogously, both the value of observables $\mathcal{F}(z)$ and the small system’s dynamics $U_t(z)$ are driven by the environment.

A quasi-classical minimization problem is the problem of finding the lowest energy and possibly the ground states of a suitable functional $\mathcal{E}[\psi, z] : L^2(\mathbb{R}^{dN}) \oplus h_\omega \to \mathbb{R}$ depending on the configuration of both the small system and the environment. The first energy functional to consider is $E_{qc}[\psi, z]$, as defined in (1.19):

$$E_{qc}[\psi, z] := \langle \psi | H_z | \psi \rangle_{L^2(\mathbb{R}^{dN})}, \quad (\psi, z) \in \mathcal{D}_{qc},$$
where $\mathcal{H}_z$ and $\mathcal{D}_{qc}$ are given in (1.1.5) and (1.2.1), respectively. We also recall that the ground state energy and minimizer of $\mathcal{E}_{qc}$ are denoted by $E_{qc}$ and $(\psi_{qc}, z_{qc})$, respectively.

Although the above is the foremost functional coming to mind in this context, another minimization problem emerges naturally in studying the quasi-classical limit. To this purpose, we recall the notion of state-valued measure [Fal18b, CFO19b], already mentioned in §1.2, a state-valued probability measure $\mathbf{m} \in \mathcal{P}(\mathfrak{h}_\omega; L^1_+(L^2(\mathbb{R}^dN)))$ is a vector Borel Radon measure on $\mathfrak{h}_\omega$, taking values in the density matrices $L^1_+(L^2(\mathbb{R}^dN))$ of the small system, such that

$$\|\mathbf{m}(\mathfrak{h}_\omega)\|_{L^1} = 1. \quad (2.1)$$

Thanks to the Radon-Nikodým property enjoyed by the separable dual space $L^1(L^2(\mathbb{R}^dN))$, it is possible to decompose $\mathbf{m}$ in a scalar Borel Radon probability measure $\mu_{\mathbf{m}} \in \mathcal{P}(\mathfrak{h}_\omega)$, such that $\mu_{\mathbf{m}}(\mathfrak{h}) = 1$, and in an a.e.-defined function (the Radon-Nikodým derivative)

$$\mathfrak{h}_\omega \ni z \mapsto \gamma_{\mathbf{m}}(z) \in L^1_+(L^2(\mathbb{R}^dN))$$

taking values in the normalized density matrices of the small system:

$$d\mathbf{m}(z) = \gamma_{\mathbf{m}}(z) d\mu_{\mathbf{m}}(z).$$

The quasi-classical energy $\mathcal{E}_{qc}$, constrained to $\|\psi\|_{L^2(\mathbb{R}^dN)} = 1$, is the expectation of the quasi-classical Hamiltonian $\mathcal{H}_z$. Therefore, its generalization to state valued measures obviously reads

$$\mathcal{E}_{svm}[\mathbf{m}] := \int_{\mathfrak{h}_\omega} d\mu_{\mathbf{m}}(z) \ tr_{L^2(\mathbb{R}^dN)} [\gamma_{\mathbf{m}}(z) \mathcal{H}_z]. \quad (2.2)$$

This leads to the following minimization problem: setting

$$\mathcal{D}_{svm} := \left\{ \mathbf{m} \in \mathcal{P}(\mathfrak{h}_\omega; L^1_+(L^2(\mathbb{R}^dN))) \mid tr_{L^2(\mathbb{R}^dN)}\mathbf{m}(\mathfrak{h}_\omega) = 1, |\mathcal{E}_{svm}[\mathbf{m}]| < +\infty \right\}, \quad (2.3)$$

we define

$$E_{svm} := \inf_{\mathbf{m} \in \mathcal{D}_{svm}} \mathcal{E}_{svm}[\mathbf{m}] > -\infty, \quad (vp1)$$

such that

$$\exists \mathbf{m}_{svm} \in \mathcal{D}_{svm} \text{ s.t. } \mathcal{E}_{svm}[\mathbf{m}_{svm}] = E_{svm}. \quad (vp2)$$

A variant of the above problem is obtained by assuming that $\gamma_{\mathbf{m}}(z) = |\psi\rangle\langle\psi|$ for some $\psi \in L^2(\mathbb{R}^dN)$ independent of $z$, in which case the functional depends only on a wave function $\psi$ and a probability measure $\mu$ over $\mathfrak{h}_\omega$. We thus set

$$\mathcal{E}_{pm}[\psi, \mu] := \int_{\mathfrak{h}_\omega} d\mu(z) \ \langle \psi | \mathcal{H}_z | \psi \rangle_{L^2(\mathbb{R}^dN)}. \quad (2.4)$$

The variational problem reads

$$E_{pm} := \inf_{(\psi, \mu) \in \mathcal{D}_{pm}} \mathcal{E}_{pm}[\psi, \mu] > -\infty, \quad (vp'1)$$

where

$$\mathcal{D}_{pm} := \left\{ (\psi, \mu) \in L^2(\mathbb{R}^dN) \oplus \mathcal{P}(\mathfrak{h}_\omega) ; \|\psi\|_2 = 1, \mu(\mathfrak{h}_\omega) = 1, |\mathcal{E}_{pm}[\psi, \mu]| < +\infty \right\}, \quad (2.5)$$

and

$$\exists (\psi_{pm}, \mu_{pm}) \in \mathcal{D}_{pm} \text{ s.t. } \mathcal{E}_{pm}[\psi_{pm}, \mu_{pm}] = E_{pm}. \quad (vp'2)$$

Note that the functional $\mathcal{E}_{qc}$ and the corresponding variational problems (VP1) and (VP2) are recovered by simply imposing in $\mathcal{E}_{pm}$ above that $\mu$ is a Dirac delta, i.e., $\exists z_0 \in \mathfrak{h}_\omega$ such that $\mu = \delta_{z_0}$. Yet another minimization problem can be formulated by substituting the
Proposition 2.1 (Quasi-classical energies).

Under the assumptions made,

\[ E_{\text{qc}} = E_{\text{svm}} = \inf_{m \in \mathcal{P}_{\text{svm}} \cap \mathcal{P}_{\text{atom}}} \mathcal{E}_{\text{svm}}[m] = E_{\text{pm}} = \inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}}} \mathcal{E}_{\text{pm}}[\psi, \mu] = E_{\text{Pekar}} = \inf_{\mu \in \mathcal{P}_{\text{atom}}} \mathcal{I}[\mu]. \] (2.7)

Proof. We use the weak density of atomic scalar measures, supported on a finite number of points, in the space of all finite measures, that holds for \( \mathcal{P}_{\text{atom}} \) separable [Par67]. Thanks to that it is possible to prove the following (see [CFT18, Lemma 3.20] for a detailed proof):

\[ E_{\text{svm}} = \inf_{m \in \mathcal{P}_{\text{svm}}} \mathcal{E}_{\text{svm}}[m] = \inf_{m \in \mathcal{P}_{\text{svm}} \cap \mathcal{P}_{\text{atom}}} \mathcal{E}_{\text{svm}}[m]; \]

\[ E_{\text{pm}} = \inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}}} \mathcal{E}_{\text{pm}}[\psi, \mu] = \inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}} \cap \mathcal{P}_{\text{atom}}} \mathcal{E}_{\text{pm}}[\psi, \mu]. \] (2.8)

Let \( \delta > 0 \) and let \( m_\delta = \sum_{k=1}^{K} \lambda_k \gamma_k \delta z_k \), with \( \gamma_k \in \mathcal{L}^1_{\text{qc}}(L^2(\mathbb{R}^d)) \), \( \lambda_k \geq 0 \) (recall that \( m_\delta \) takes values in positive operators) and \( \sum_{k=1}^{K} \lambda_k = 1 \), be an atomic state-valued measure, such that

\[ \mathcal{E}_{\text{svm}}[m_\delta] = \sum_{k=1}^{K} \lambda_k \text{tr}_{L^2(\mathbb{R}^d)}[\gamma_k \mathcal{H}_{z_k}] < \inf_{m \in \mathcal{P}_{\text{svm}} \cap \mathcal{P}_{\text{atom}}} \mathcal{E}_{\text{svm}}[m] + \delta. \]

For fixed \( k \), since \( \gamma_k \) is a normalized density matrix,

\[ \inf_{\psi \in L^2(\mathbb{R}^d)} \langle \psi | \mathcal{H}_{z_k} | \psi \rangle \leq \text{tr}_{L^2(\mathbb{R}^d)}[\gamma_k \mathcal{H}_{z_k}]. \]

Therefore,

\[ \inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}} \cap \mathcal{P}_{\text{atom}}} \mathcal{E}_{\text{pm}}[\psi, \mu] = \inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}} \cap \mathcal{P}_{\text{atom}}} \mu \int_{\mathcal{P}_{\text{atom}}} \text{d} \mu' \langle \psi | \mathcal{H}_{z_k} | \psi \rangle \leq \sum_{k=1}^{K} \lambda_k \text{tr}_{L^2(\mathbb{R}^d)}[\gamma_k \mathcal{H}_{z_k}] < \inf_{m \in \mathcal{P}_{\text{svm}} \cap \mathcal{P}_{\text{atom}}} \mathcal{E}_{\text{svm}}[m] + \delta. \] (2.9)
Since $\delta > 0$ is arbitrary, we conclude that
\[
\inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}}, \mu \in \mathcal{P}_{\text{atom}}(h_\omega)} \mathcal{E}_{\text{pm}}[\psi, \mu] \leq \inf_{m \in \mathcal{P}_{\text{svm}} \cap \mathcal{P}_{\text{atom}}(h_\omega; L^1_\omega(L^2(\mathbb{R}^d N)))} \mathcal{E}_{\text{svm}}[m]. \tag{2.10}
\]

To prove the opposite inequality, we follow a similar reasoning. Let $\delta > 0$ and $\mu_\delta = \sum_{k=1}^K \lambda_k \delta_{z_k}$ be a scalar atomic measure and $\psi_{\delta, z_k} \in L^2(\mathbb{R}^d N)$ a family of normalized wave functions, such that $\mu_\delta(h_\omega) = 1$ and
\[
\sum_{k=1}^K \lambda_k \langle \psi_{\delta, z_k} | \mathcal{H}_{z_k} | \psi_{\delta, z_k} \rangle_{L^2(\mathbb{R}^d N)} < \inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}}, \mu \in \mathcal{P}_{\text{atom}}(h_\omega)} \mathcal{E}_{\text{pm}}[\psi, \mu] + \delta.
\]

Now, $m_\delta := \sum_{k=1}^K \lambda_k |\psi_{\delta, z_k}|^2 \delta_{z_k}$ is an atomic state-valued measure belonging to $\mathcal{P}_{\text{svm}}$. Therefore,
\[
\inf_{m \in \mathcal{P}_{\text{svm}} \cap \mathcal{P}_{\text{atom}}(h_\omega; L^1_\omega(L^2(\mathbb{R}^d N)))} \mathcal{E}_{\text{svm}}[m] \leq \mathcal{E}_{\text{svm}}[m_\delta] = \sum_{k=1}^K \lambda_k \langle \psi_{\delta, z_k} | \mathcal{H}_{z_k} | \psi_{\delta, z_k} \rangle_{L^2(\mathbb{R}^d N)} < \inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}}, \mu \in \mathcal{P}_{\text{atom}}(h_\omega)} \mathcal{E}_{\text{pm}}[\psi, \mu] + \delta, \tag{2.11}
\]

which yields the desired inequality.

To complete the proof, we show that
\[
\inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}}, \mu \in \mathcal{P}_{\text{atom}}(h_\omega)} \mathcal{E}_{\text{pm}}[\psi, \mu] = \inf_{z \in h_\omega} \mathcal{T}[z] = E_{\text{Pekar}} = E_{\text{qc}}. \tag{2.12}
\]

Let us prove the first equality beforehand. Let $\mu_\delta = \sum_{k=1}^K \lambda_k \delta_{z_k}$ be the atomic minimizing family of measures defined before and $\psi_{\delta, z_k}$ the corresponding minimizing vectors. Then,
\[
\sum_{k=1}^K \lambda_k \inf_{\psi \in L^2(\mathbb{R}^d N), \|\psi\|_2 = 1} \{ \psi | \mathcal{H}_{z_k} | \psi \}_2 \leq \sum_{k=1}^K \lambda_k \langle \psi_{\delta, z_k} | \mathcal{H}_{z_k} | \psi_{\delta, z_k} \rangle_{L^2(\mathbb{R}^d N)} < \inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}}, \mu \in \mathcal{P}_{\text{atom}}(h_\omega)} \mathcal{E}_{\text{pm}}[\psi, \mu] + \delta. \tag{2.13}
\]

Since the l.h.s. is a convex combination and $\delta$ is arbitrary, we immediately deduce that
\[
\inf_{z \in h_\omega} \mathcal{T}[z] \leq \inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}}, \mu \in \mathcal{P}_{\text{atom}}(h_\omega)} \mathcal{E}_{\text{pm}}[\psi, \mu]. \tag{2.14}
\]

On the other hand, since a measure concentrated in a single point is atomic,
\[
\inf_{(\psi, \mu) \in \mathcal{P}_{\text{pm}}, \mu \in \mathcal{P}_{\text{atom}}(h_\omega)} \mathcal{E}_{\text{pm}}[\psi, \mu] \leq \inf_{z \in h_\omega} \inf_{\psi \in L^2(\mathbb{R}^d N), \|\psi\|_2 = 1} \mathcal{E}_{\text{qc}}[\psi, z] = \inf_{z \in h_\omega} \mathcal{T}[z],
\]

which implies the first identity in (2.12).

Now, let us prove the second equality above, namely
\[
\inf_{z \in h_\omega} \mathcal{T}[z] = E_{\text{Pekar}}. \tag{2.15}
\]

Let again $\delta > 0$ and let $z_\delta$ be a minimizing family of vectors for $\mathcal{T}$, i.e., such that $\mathcal{T}[z_\delta] < \inf_{z \in h_\omega} \mathcal{T}[z] + \delta$. For each $z_\delta$, let $\psi_{\delta, z_\delta}$ be a minimizing vector for $\mathcal{E}_{\text{qc}}[\cdot, z_\delta]$, i.e., such that
\[
\mathcal{E}_{\text{qc}}[\psi_{\delta, z_\delta}, z_\delta] < \mathcal{T}[z_\delta] + \delta.
\]

Now,
\[
E_{\text{Pekar}} \leq \mathcal{E}_{\text{Pekar}}[\psi_{\delta, z_\delta}] \leq \mathcal{E}_{\text{qc}}[\psi_{\delta, z_\delta}, z_\delta].
\]
so that,
\[ E_{\text{Pekar}} \leq \inf_{z \in \mathcal{H}} I[z]. \]  
(2.16)

On the other hand, let \( \psi_\delta \) be a minimizing family of states for \( E_{\text{Pekar}} \), and, once fixed \( \psi_\delta \), let \( z_{\delta,\psi_\delta} \) be a minimizing family for \( \mathcal{E}_{qc}[\psi_\delta, \cdot] \):
\[ \mathcal{E}_{qc}[\psi_\delta, z_{\delta,\psi_\delta}] < E_{\text{Pekar}} + \delta. \]  
(2.17)

As above, we then get
\[ \inf_{z \in \mathcal{H}} I[z] \leq \inf_{\psi \in L^2(\mathbb{R}^dN), \|\psi\|_2 = 1} \mathcal{E}_{qc}[\psi, z_{\delta,\psi_\delta}] \leq \mathcal{E}_{qc}[\psi_\delta, z_{\delta,\psi_\delta}] < E_{\text{Pekar}} + \delta. \]  
which yields
\[ \inf_{z \in \mathcal{H}} I[z] \leq E_{\text{Pekar}}. \]  
(2.18)

Finally, we prove that
\[ E_{\text{Pekar}} = E_{qc}. \]  
(2.19)

Now, let \( (\psi_\delta, z_{\delta,\psi_\delta}) \) be as above, i.e., such that (2.17) holds true. Hence,
\[ E_{qc} \leq \mathcal{E}_{qc}[\psi_\delta, z_{\delta,\psi_\delta}] < E_{\text{Pekar}} + \delta, \]
and thus \( E_{qc} \leq E_{\text{Pekar}} \). On the other hand, let \( (\psi_\delta, z_\delta) \) be a minimizing family of configurations for \( \mathcal{E}_{qc} \):
\[ \mathcal{E}_{qc}[\psi_\delta, z_\delta] < E_{qc} + \delta. \]
Clearly, now one has
\[ E_{\text{Pekar}} \leq \mathcal{E}_{\text{Pekar}}[\psi_\delta] \leq \mathcal{E}_{qc}[\psi_\delta, z_\delta] < E_{qc} + \delta, \]
yielding the opposite inequality, i.e., \( E_{\text{Pekar}} \leq E_{qc} \).

**Remark 2.2 (Stability).**
In the above proof we have implicitly assumed that the energies under considerations are bounded from below, but in fact it is easy to see that, if one of the functionals in unbounded from below, then all the others must be unstable as well. We do not provide any detail of such an argument, because our main result (Theorem 1.3) implies that (VP1) holds true, so that (VP’1), (vp1) and (vp’1) immediately follow.

The other important result concerns equivalences for the existence of minimizers in the variational problems above.

**Proposition 2.3 (Quasi-classical minimizers).**
*Under the assumptions made,*
\[ \text{(VP2)} \iff \text{(VP’2)} \iff \text{(vp2)} \iff \text{(vp’2)}. \]  
(2.20)

Furthermore, any minimizer \( m_{\text{svm}} \) of (vp2) is concentrated on the set of minimizers \( (\psi_{qc}, z_{qc}) \) of (VP2).

Before proving Proposition 2.3, we state a useful result about the quasi-classical functional defined in (1.19) or, more precisely, about its variant \( F_{qc} \) introduced in (1.26), which is important to explore the connection with the Pekar-like functionals (1.29) and (1.35).

**Lemma 2.4.**
For any fixed \( \psi \), the functional \( F_{qc}[\psi, \eta] \) is strictly convex in \( \eta \in \mathcal{H}. \)
Proof. In cases (a) and (b) the proof is trivial, since \( \mathcal{F}_{\text{qe}} \) contains only two terms depending on \( \eta \); one is quadratic in \( \eta \) (the free field energy) and therefore strictly convex, while the other (the interaction) is linear and thus convex.

So we have to investigate in detail only case (c), namely the Pauli-Fierz quasi-classical energy, and, specifically, only the kinetic part of the energy involving the interaction, which reads

\[
\sum_{j=1}^{N} \frac{1}{2m_j} \left( -i \nabla_j + 2 \Re \left\langle \eta \right| (\mathbf{\Omega}^{1/2} \lambda_j) (x_j) \right|_h \right)^2.
\]

Let us then set \( \eta = \beta \eta_1 + (1 - \beta) \eta_2 \) for some \( \eta_1, \eta_2 \in \mathfrak{h} \) and \( \beta \in (0, 1) \). Expanding the square and setting \( \xi_j(x) := \mathbf{\Omega}^{1/2} \lambda_j(x) \) for short, we get (for any non-zero \( \psi \))

\[
\left\langle \psi \left| \left( -i \nabla_j + 2 \Re \left\langle \eta \right| \xi_j(x_j) \right|_h \right) \right|_2 \left\langle \psi \left| \left( -\Delta_j \right) \psi \right| _{L^2(\mathbb{R}^3_N)} - 2 \left\langle \psi \right| i \beta \Re \left\langle \eta \right| \xi_j(x_j) \right|_h \cdot \nabla_j + i(1 - \beta) \Re \left\langle \eta_2 \right| \xi_j(x_j) \right|_h \cdot \nabla_j \left| \psi \right| _{L^2(\mathbb{R}^3_N)} + 4 \left\langle \psi \right| \beta \left( \Re \left\langle \eta_1 \right| \xi_j(x_j) \right|_h \right)^2 + (1 - \beta) \left( \Re \left\langle \eta_2 \right| \xi_j(x_j) \right|_h \right)^2 \left| \psi \right| _{L^2(\mathbb{R}^3_N)} \right)
\]

again by the strict convexity of the square, i.e., the bound \((\beta a + (1 - \beta)b)^2 < \beta a^2 + (1 - \beta)b^2\), valid for any \( a, b \in \mathbb{R} \) and \( \beta \in (0, 1) \). The result easily follows, since the remaining term in the functional depending on \( \eta \) is the free field energy, which is quadratic in \( \eta \) and thus strictly convex as well.

Proof of Proposition 2.3. Some implications are easy to prove. Let us first prove that \([\text{VP2}] \implies [\text{vp2}]\). Let \( (\psi_{\text{qc}}, z_{\text{qc}}) \) be a minimizer of \( \mathcal{E}_{\text{qc}} \) in \( \mathcal{D}_{\text{qc}} \). Then, evaluating the energy \( \mathcal{E}_{\text{pm}} \) on the configuration \( (\psi_{\text{pm}}, \mu_0) \), with \( \mu_0 = \delta_{z_{\text{qc}}} \), we get

\[
\mathcal{E}_{\text{pm}} [\psi_{\text{pm}}, \mu_0] = \int_{\mathfrak{h}_{\omega}} d\mu_0(z) \mathcal{E}_{\text{qc}} [\psi_{\text{qc}}, z] = \mathcal{E}_{\text{qc}} [\psi_{\text{qc}}, z_{\text{qc}}] = E_{\text{qc}}.
\]

By Proposition 2.1 \((\psi_{\text{qc}}, \mu_0)\) is thus solving \([\text{vp2}]\). Analogously, let us prove \([\text{vp2}] \implies [\text{vp2}]\): let \( (\psi_{\text{pm}}, \mu_{\text{pm}}) \) be a minimizer for \([\text{vp2}]\); then, the state-value measure \( \mu_{\text{pm}} \), with \( \mu_{\text{pm}} = \mu_{\text{pm}} \) and \( \gamma_{\text{pm}}(z) = \langle \psi_{\text{pm}} \rangle \langle \psi_{\text{pm}} \rangle \), solves \([\text{vp2}]\) by Proposition 2.1.

We prove now that \([\text{vp2}] \implies [\text{VP2}]\). Given a minimizer \( \mu_{\text{svm}} \) of \( \mathcal{E}_{\text{svm}} \), for \( \mu_{\text{svm}} \)-a.e. \( z \in \mathfrak{h}_{\omega} \) there exist \( \{\lambda_k(z)\}_{k \in \mathbb{N}} \), \( \lambda_k(z) \geq 0 \), \( \sum_{k \in \mathbb{N}} \lambda_k(z) = 1 \) and \( \{\psi_k(z)\}_{k \in \mathbb{N}} \), \( \|\psi_k(z)\|_{L^2(\mathbb{R}^3_N)} = 1 \), such that

\[
E_{\text{svm}} = \mathcal{E}_{\text{svm}}[\mu_{\text{svm}}] = \int_{\mathfrak{h}_{\omega}} d\mu_{\text{svm}}(z) \sum_{k \in \mathbb{N}} \lambda_k(z) \mathcal{E}_{\text{qc}} [\psi_k(z), z].
\]

The above is due to the fact that \( \gamma_{\text{pm}}(z) \) is a density matrix on \( L^2(\mathbb{R}^3_N) \) for \( \mu_{\text{svm}} \)-a.e. \( z \). The measure \( \mu_{\text{svm}} \in \mathcal{D}(\mathfrak{h}_{\omega}) \) is a probability measure, hence the r.h.s. of the above equation is a (double) convex combination of numerical values of the real-valued function \( \mathcal{E}_{\text{qc}} \). However, a convex combination of values of a function equals its infimum, if and only if the infimum is a minimum, and all variables appearing in the convex combination are minimizers. Therefore, \( \mathcal{E}_{\text{qc}} \) admits at least one minimizer. Actually, the measure \( \mu_{\text{svm}} \) is concentrated on the set of minimizers \( (\psi_{\text{qc}}, z_{\text{qc}}) \), in the above sense.

Finally, we consider the Pekar-like variational problem \([\text{VP2}]\) and its equivalence with \([\text{VP2}]\). Let us first prove that \([\text{VP2}] \implies [\text{VP2}]\): given a Pekar minimizer \( \psi_{\text{Pekar}} \in L^2(\mathbb{R}^d_N) \), we immediately deduce that \( \psi_{\text{Pekar}} \in H^1(\mathbb{R}^d_N) \) by boundedness from above of the energy and
regularity of the classical field $a(x)$, which is continuous and vanishing at infinity [CFO19a, Rmk. 1.5]. Furthermore, Lemma 2.4 guarantees the existence (and uniqueness) of $\eta_{\text{Pekar}}[\psi_{\text{Pekar}}] \in h$ minimizing $E_{qc}[\psi_{\text{Pekar}}, z]$ w.r.t. $z$. Therefore, the configuration $(\psi_{\text{Pekar}}, \eta_{\text{Pekar}}[\psi_{\text{Pekar}}])$ is admissible for $E_{qc}$ and we deduce from Proposition 2.1 that $E_{qc}[\psi_{\text{Pekar}}, \eta_{\text{Pekar}}[\psi_{\text{Pekar}}]] = E_{QC}$.

Conversely, given a minimizer $(\psi_{qc}, z_{qc}) \in \mathcal{D}$ of $E_{qc}$, we know that the configuration must satisfy the Euler-Lagrange equations $L(1.24)$ at least in weak sense. However, the second equation in (1.24) is easily seen to coincide with $L(1.28)$ or the first equation in $L(1.35)$, when the change of variable $\eta = \omega^{1/2}z$ has been done. Furthermore, any weak solution $\eta$ of such equations is in fact a strong solution, i.e., $\eta \in h$, under the assumptions made. Hence, by strict convexity of $F_{qc}[\psi, \eta]$ in $\eta$ proven in Lemma 2.4 and then uniqueness of $\eta_{\text{Pekar}}$, we deduce that $\eta_{\text{Pekar}}[\psi_{qc}] = \omega^{1/2}z_{qc}$ and the equivalence $\text{(VP2)} \implies \text{(VP2)}$ is readily proven via Proposition 2.1.

\textbf{Remark 2.5} (Minimizers for (vp2)). The existence of a solution for (vp2) obtained here is trivial, i.e., it involves a measure concentrated in a single point $z_{qc} \in h_\omega$ and a $\psi_{qc}$ dependent on such a point. It would be interesting, but outside the scope of this paper, to know whether there are non-trivial minimizers in which $\mu_0$ is not concentrated at a single point. This is obviously related to the question of uniqueness of the minimizing configuration $(\psi_{qc}, z_{qc})$. Note that this would not be in contradiction with Lemma 2.4 since we prove there strict convexity of $F_{qc}[\psi, \eta]$ only in $\eta$, while the full functional $E_{qc}[\psi, z]$ is in general not jointly convex in $\psi$ and $z$ nor in $|\psi|^2$ and $z$ (see also Remark 1.2).

Note that the combination of Proposition 2.1 with Proposition 2.3 provides the proof of Proposition 1.1 stated in §1.

2.2. Minimization problem for generalized state-valued measures. We discuss now the generalization of the concepts introduced above needed to deal with the minimization $L(1.32)$, that is particularly useful to treat small systems consisting of unconfined particles. Taking the double dual, it is well known that $\mathcal{L}^1(L^2(\mathbb{R}^d))$ can be continuously embedded in $\mathcal{B}(L^2(\mathbb{R}^d))'$, the dual of bounded operators, in a positivity preserving way. By an abuse of notation, we will write $\mathcal{L}^1(L^2(\mathbb{R}^d)) \subset \mathcal{B}(L^2(\mathbb{R}^d))'$. We recall that we denoted by $\mathcal{F}_{\mathcal{T}}(L^2(\mathbb{R}^d))$ the closure of $\mathcal{L}^1(L^2(\mathbb{R}^d))$ with respect to the weak-* topology $\sigma(\mathcal{B}(L^2(\mathbb{R}^d))', \mathcal{B}(L^2(\mathbb{R}^d)))$ on $\mathcal{B}(L^2(\mathbb{R}^d))'$. Also, $\mathcal{F}_{\mathcal{T}}(L^2(\mathbb{R}^d))$ and $\mathcal{F}_{\mathcal{T},1}(L^2(\mathbb{R}^d))$ stand for the subsets of positive and normalized positive elements, respectively. A generalized state-valued measure is then a measure on $h_\omega$ with values in the space of generalized states $\mathcal{F}_{\mathcal{T}}(L^2(\mathbb{R}^d))$. Properties of generalized state-valued measures are discussed in Appendix A. Since the dual space $\mathcal{B}(L^2(\mathbb{R}^d))'$ is not separable, it does not have the Radon-Nikodým property, therefore integration of functions $F : h_\omega \to \mathcal{B}(L^2(\mathbb{R}^d))$ is restricted only to ones with separable range.

Such integration can be extended to functions valued in unbounded operators in the following sense.

\textbf{Definition 2.6} (Domains of generalized Wigner measures).
Let $\mathcal{T}$ be a strictly positive unbounded operator on $L^2(\mathbb{R}^d)$. A generalized state-valued measure $\sigma$ is in the domain of $\mathcal{T}$, if and only if there exists a measure $\sigma_{\mathcal{T}} \in \mathcal{D}(h_\omega, \mathcal{F}_{\mathcal{T},1}(L^2(\mathbb{R}^d)))$, 

\textbf{Remark 2.5} (Minimizers for (vp2)).
such that for all \( B \in \mathcal{B}(L^2(\mathbb{R}^d)) \) and all Borel sets \( S \subseteq \mathfrak{h}_\omega \),

\[
\mathfrak{n}_\mathcal{T}(S) \left[ T^{-1/2} ST^{-1/2} \right] = \mathfrak{n}(S)[B] .
\]

(2.22)

Therefore, if \( \mathfrak{n} \) is in the domain of \( \mathcal{T} \), with a little abuse of notation, we may write

\[
\mathfrak{n}(S) \left[ T^{1/2} \cdot T^{-1/2} \right] = \mathfrak{n}_\mathcal{T}(S)[\cdot]
\]

(2.23)

as a state valued measure “absorbing a singularity” of order \( \mathcal{T} \). Now, let \( F(z) \) be a function with values in unbounded operators such that for all \( z \in \mathfrak{h}_\omega \):

- \( T^{-1/2} F(z) T^{-1/2} \in \mathcal{B}(L^2(\mathbb{R}^d)) \);
- the range of \( z \mapsto T^{-1/2} F(z) T^{-1/2} \) is separable;
- \( T^{-1/2} F(z) T^{-1/2} \) is \( \mathfrak{n}_\mathcal{T} \)-absolutely integrable.

Then, it follows that we can define the integral of \( F \) with respect to \( \mathfrak{n} \) as

\[
\int_{\mathfrak{h}_\omega} d\mathfrak{n}(z) [F(z)] := \int_{\mathfrak{h}_\omega} d\mathfrak{n}_\mathcal{T}(z) [T^{-1/2} F(z) T^{-1/2}] .
\]

(2.24)

A simple but useful example of such \( F(z) \) is the following: let \( S \) be a self-adjoint operator, and let \( \mathfrak{n} \) be in the domain of \( \mathcal{T} = |S| + 1 \); then the function \( F(z) = S \) satisfies all above hypotheses and thus it makes sense to write, for all Borel set \( S \subseteq \mathfrak{h}_\omega \),

\[
\int_S d\mathfrak{n}(z)[S] = \mathfrak{n}(S)[S] := \mathfrak{n}_\mathcal{T}(S)[T^{-1/2} ST^{-1/2}] \in \mathbb{R} .
\]

(2.25)

The other cases useful for our analysis are discussed in § 3.

We are now in a position to define another quasi-classical minimization problem. Recall the definition (1.33) of the domain \( \mathcal{D}_{gqc} \), the ground state energy \( E_{gqc} \) given by (1.54) and any corresponding minimizing configuration \( (\rho_{gqc}, z_{gqc}) \in \mathcal{D}_{gqc} \); then the analogues of (VP1) and (VP2) are

\[
E_{gqc} > -\infty ,
\]

(GVP1)

\[
\frac{1}{2} (\rho_{gqc}, z_{gqc}) \in \mathcal{D}_{gqc} ;
\]

(GVP2)

The functional \( \mathcal{E}_{gqc} \) can indeed be seen as the generalized quasi-classical energy: let \( H_z \) be the abstract realization of \( \mathcal{H}_z \) as an operator affiliated to the abstract \( C^* \)-algebra \( \mathcal{B}(L^2(\mathbb{R}^d)) \). Then, given a normalized pure state \( \rho \in \mathcal{L}^\times_+ (L^2(\mathbb{R}^d)) \), we define the corresponding irreducible GNS representation by \( (\mathcal{K}_\rho, \pi_\rho, \psi_\rho) \), where \( \mathcal{K}_\rho \) is a suitable Hilbert space, \( \pi_\rho : \mathcal{B}(L^2(\mathbb{R}^d)) \to \mathcal{B}(\mathcal{K}_\rho) \) is a \( C^* \)-homomorphism (that can be extended to operators affiliated to the algebra) and \( \psi_\rho \in \mathcal{K}_\rho \) is the normalized cyclic vector associated to \( \rho \). Therefore, it follows that

\[
\mathcal{E}_{gqc}[\rho, z] = \langle \psi_\rho | \pi_\rho (H_z) | \psi_\rho \rangle_{\mathcal{K}_\rho} .
\]

This expression is analogous to the one for \( \mathcal{E}_{qc} \) (see (1.19)) and it reduces exactly to the latter whenever \( \rho \) is a pure state belonging to \( \mathcal{L}^1(\mathbb{R}^d) \) (see next Remark 2.7).

The generalization of the variational problems for state-valued measures (vP1) and (vP2) is obtained as follows: setting

\[
\mathcal{D}_{gvm} := \left\{ n \in \mathcal{L}^\times_+ (L^2(\mathbb{R}^d)) \mid \|n(\mathfrak{h}_\omega)\|_{\mathcal{B}'} = 1, \int_{\mathfrak{h}_\omega} d\mathfrak{n}(z) |\mathcal{H}_z| < +\infty \right\} ,
\]

(2.26)

we consider

\[
E_{gvm} := \inf_{n \in \mathcal{D}_{gvm}} \int_{\mathfrak{h}_\omega} d\mathfrak{n}(z) |\mathcal{H}_z| > -\infty ,
\]

(gvp1)
On the other hand, let \( \rho \) is possible to approximate any measure which implies the claim.

**Remark 2.7** (State-valued and generalized state-valued measures).

We point out that, if a generalized state-valued measure \( n \in \mathcal{D}_{gsvm} \) is actually a state-valued measure, *i.e.*, such that, for all Borel sets \( S \subseteq \mathcal{h}_\omega \),

\[
n(S) \in \mathcal{L}_+^1(L^2(\mathbb{R}^dN)),
\]

then \( n \in \mathcal{D}_{svm} \) and

\[
\int_{\mathcal{h}_\omega} dn(z) [H_z] = \mathcal{E}_{svm}[n].
\]

**Proposition 2.8** (Generalized quasi-classical ground state energy).

*Under the assumptions made above,*

\[
E_{qc} = E_{gqc} = E_{gsvm}.
\]

**Proof.** Firstly, let us prove that

\[ E_{qc} = E_{gqc}. \]

Since \( \rho \) belongs to the weak-* closure of \( \mathcal{L}_+^1(L^2(\mathbb{R}^dN)) \), there exists a filter base \( \mathcal{G} \subset 2^{\mathcal{L}_+^1(L^2(\mathbb{R}^dN))} \) such that \( \mathcal{G} \rightarrow \rho \) in weak-* topology. Hence, for any fixed \( z \in \mathcal{h}_\omega \),

\[
\lim_{\mathcal{G} \rightarrow \rho} \text{tr}_{L^2(\mathbb{R}^dN)} [\mathcal{G}(H_z)] = \rho [H_z].
\]

Now, on one hand, each \( |\psi\rangle \langle \psi|, \psi \in L^2(\mathbb{R}^dN), \) is also a pure generalized state and therefore

\[
E_{gqc} \leq \inf_{(\psi, z) \in \mathcal{G}_{qc}} \mathcal{E}_{qc}[\psi, z] = E_{qc}. \tag{2.27}
\]

On the other hand, let \( (\rho_\delta, z_\delta) \in \mathcal{D}_{gqc} \) be a minimizing sequence:

\[
\mathcal{E}_{gqc}[\rho_\delta, z_\delta] = \rho_\delta [H_{z_\delta}] < E_{gqc} + \delta;
\]

for some \( \delta > 0 \), and \( \mathcal{G}_{\delta} \subset 2^{\mathcal{L}_+^1(L^2(\mathbb{R}^dN))} \) the corresponding approximating filter base for \( \rho_\delta \).

Then,

\[
E_{qc} = \inf_{(\psi, z) \in \mathcal{G}_{qc}} E_{qc}[\psi, z] = \inf_{(\gamma, z) \in \mathcal{L}_+^1(L^2(\mathbb{R}^dN)) \in \mathcal{h}_\omega} \text{tr}_{L^2(\mathbb{R}^dN)} [\gamma H_z] \leq \sup_{X \in \mathcal{G}_{\delta}} \inf_{\gamma \in X} \text{tr}_{L^2(\mathbb{R}^dN)} [\gamma H_{z_\delta}]
\]

\[
= \liminf_{\mathcal{G}_{\delta} \rightarrow \rho_\delta} \text{tr}_{L^2(\mathbb{R}^dN)} [\mathcal{G}_{\delta}(H_{z_\delta})] = \lim_{\mathcal{G}_{\delta} \rightarrow \rho_\delta} \text{tr}_{L^2(\mathbb{R}^dN)} [\mathcal{G}_{\delta}(H_{z_\delta})] = \rho_\delta [H_{z_\delta}] < E_{gqc} + \delta. \tag{2.28}
\]

Since the above chain of inequalities is valid for all \( \delta > 0 \), it follows that the opposite inequality of \( 2.28 \) holds true, *i.e.,*

\[
E_{qc} \leq E_{gqc} \tag{2.29}
\]

which implies the claim.

The proof of the identity \( E_{gsvm} = E_{qc} \) is perfectly analogous, keeping in mind that it is possible to approximate any measure \( n \in \mathcal{P}(\mathcal{h}_\omega, \mathcal{L}_+^1(L^2(\mathbb{R}^dN))) \) with a filter base \( \mathcal{F} \subset 2^{\mathcal{P}(\mathcal{h}_\omega, \mathcal{L}_+^1(L^2(\mathbb{R}^dN)))} \) w.r.t. the product of weak-* topologies

\[
\prod_{S \subset \mathcal{h}_\omega \text{ Borel}} \sigma(B(L^2(\mathbb{R}^dN))^t, B(L^2(\mathbb{R}^dN))),
\]

4The notation \( \text{tr}_{L^2(\mathbb{R}^dN)} [\mathcal{G}(H_z)] \) stands for filter base that is image of \( \mathcal{G} \) on \( \mathbb{R} \) via the map \( \gamma \rightarrow \text{tr}_{L^2(\mathbb{R}^dN)} [\gamma H_z] \): given any \( X \in \mathcal{G} \), \( \{ \text{tr}_{L^2(\mathbb{R}^dN)} [\gamma H_z], \gamma \in X \} \in \text{tr}_{L^2(\mathbb{R}^dN)} [\mathcal{G}(H_z)]. \)
that implies the convergence of integrals:

\[
\lim_{T \to n} \text{tr}_{L^2} \left[ \int_{\mathcal{F}} d\mathcal{F}(z) \mathcal{H}_z \right] = \int_{\mathcal{F}} dn(z) \left[ \mathcal{H}_z \right].
\]

Finally, also for the generalized minimization problems, it is possible to prove equivalence of existence of minimizers.

**Proposition 2.9** (Generalized quasi-classical minimizers).

*Under the assumptions made,\(^5\) \([GVP2] \iff [gvp2].\) \(\) Furthermore, any minimizer \(n_{\text{gsvm}}\) of \([gvp2]\) is concentrated on the set of minimizers \((\rho_{\text{gqc}}, z_{\text{gqc}})\) of \([GVP2].\)

**Proof.** The \((\implies)\) implication is trivial: let \((\rho_{\text{gqc}}, z_{\text{gqc}})\) be a minimizer for \([GVP2]\), then, evaluating the energy of the generalized state-valued measure \(n_0 = \delta_{z_{\text{gqc}}} \rho_{\text{gqc}},\) we get

\[
\int_{\mathcal{F}} dn_0(z) \left[ \mathcal{H}_z \right] = \rho_{\text{gqc}} \left[ \mathcal{H}_{z_{\text{gqc}}} \right] = E_{\text{gqc}}.
\]

By **Proposition 2.8** \(n_0\) is thus a minimizer for \([gvp2].\)

To prove the converse implication, note that the integral w.r.t. a generalized state-valued probability measure is a convex combination of expectations over possibly mixed generalized states. Since the mixed states are themselves convex combinations of pure states, it follows that the measure \(n_{\text{gsvm}}\) must be concentrated on the set of minimizers for \([gvp2],\) and thus the latter is not empty.

\[\]

### 3. Ground States Energies and Ground States in the Quasi-Classical Regime

In this section we study the quasi-classical limit of ground state energies and ground states of the microscopic models introduced in \([8.1].\)

The microscopic interaction is described by a fully quantum system, in which both the small system and the environment are quantum. The Hilbert space is thus (see \([1.2]\))

\[
H_\varepsilon = L^2(\mathbb{R}^dN) \otimes \mathcal{G}(\mathfrak{h}),
\]

where \(\mathcal{G}(\mathfrak{h}) = \bigoplus_{n \in \mathbb{N}} \mathfrak{h}^\otimes n\) is the symmetric Fock space over \(\mathfrak{h}\) and \(\varepsilon\) is the quasi-classical parameter whose dependence is yielded by a semiclassical choice of canonical commutation relations \([1.3],\) i.e., \([a_\varepsilon(z), a_\varepsilon^\dagger(w)] = \varepsilon \langle z|w \rangle_\mathfrak{h},\) with \(a_\varepsilon^\dagger\) the annihilation and creation operators on the Fock space. A state of the whole system is given by a density matrix

\[
\Gamma_\varepsilon \in \mathcal{L}^1_{+,1} \left( L^2(\mathbb{R}^dN) \otimes \mathcal{G}(\mathfrak{h}) \right),
\]

the positive trace-class operators with unit trace.

The dynamics of the system is described by a self-adjoint Hamiltonian operator \(H_\varepsilon\) whose general form is given in \([1.14].\) Such operator is the partial Wick quantization of the quasi-classical Schrödinger energy operator \(\mathcal{H}_z\) provided in \([1.15].\) Wick quantization consists in substituting each \(z\) appearing in \(\mathcal{H}\) with \(a_\varepsilon\) and each \(\bar{z}\) with \(a_\varepsilon^\dagger\), and of ordering all \(a_\varepsilon-s\) to the left of all \(a_\varepsilon^\dagger-s.\) Such quantization procedure is well-defined for symbols \(\mathcal{F}_z\) that are polynomial

\[\]

\(^5\)As before, the integral w.r.t. to \(d\mathcal{F}\) is just a short-hand notation to denote the integral over elements belonging to the filter \(\mathcal{F}.\)
in $z$ and $z^*$, as it is the case in concrete models we are considering (see §4 for additional details and [AN08] for the rigorous procedure). Hence, we can write,

$$H_\varepsilon = \text{Op}^\text{Wick}_\varepsilon (H_z), \quad (3.1)$$

and, more precisely, $H_z$ can be split in three terms, at least in the sense of quadratic forms, i.e.,

$$H_z = K_0 + \sum_{i=1}^N \mathcal{V}_z(x_i) + \langle z | \omega | z \rangle_h, \quad (3.2)$$

with $K_0$ self-adjoint and bounded from below, yielding

$$H_\varepsilon = K_0 \otimes 1 + 1 \otimes \text{Op}^\text{Wick}_\varepsilon (\langle z | \omega | z \rangle_h) + \sum_{i=1}^N \text{Op}^\text{Wick}_\varepsilon (\mathcal{V}_z(x_i)), \quad (3.3)$$

as a quadratic form. The first and second terms on the r.h.s are the free energies of the small system and environment, respectively, and the third term is the small system-environment interaction.

The minimization problem for the quantum system described by $H_\varepsilon$ is defined in (1.12): the microscopic ground state energy is $E_\varepsilon := \inf \sigma(H_\varepsilon)$, while $\Psi_{gs}$ stands for any corresponding minimizer. Such a minimization problem has been thoroughly studied, for the concrete models under consideration in this paper; for bibliographical references the reader shall consult §4. The results are as follows.

**Proposition 3.1** (Stability and existence of the ground state).

*Under the assumptions made, there exist finite constants $c, C > 0$ independent of $\varepsilon$, such that*

$$-c \leq E_\varepsilon \leq C. \quad (3.4)$$

*Furthermore, under suitable conditions on the operator $K_0$ (e.g., if $K_0$ has compact resolvent), then $E_\varepsilon \in \sigma_{\text{pp}}(H_\varepsilon)$ and thus $\exists \Psi_{gs}$ ground state of $H_\varepsilon$.***

The proof of the above results is model-dependent and therefore it is postponed to §4.

We now investigate the link between the microscopic ground state problem and the quasi-classical minimization problems described in §2, starting from the proof of Theorem 1.3. Recall the definition of quasi-classical and generalized quasi-classical Wigner measure defined in Definition 1.5 and Definition 1.7, respectively. Although both cases could be treated at once, we provide a separate discussion of the main results for trapped and non-trapped particle systems, whose difference is apparent in the statements of Corollary 1.10 and Corollary 1.17.

### 3.1. Trapped particle systems.

The proof of Theorem 1.3 is divided into two steps (upper and lower bounds for the microscopic energy). At the end of this section, we also complete the proof of Corollary 1.10 about the convergence of minimizers.

#### 3.1.1. Energy upper bound.

In the following, we denote by $\Psi_{\varepsilon,\delta} \in \mathcal{D}(H_\varepsilon)$, $\delta > 0$, a minimizing sequence for $H_\varepsilon$:

$$\langle \Psi_{\varepsilon,\delta} | H_\varepsilon | \Psi_{\varepsilon,\delta} \rangle_{\mathcal{H}_\varepsilon} < E_\varepsilon + \delta. \quad (3.5)$$

The first step towards the proof of the energy convergence is given by the proposition below.
Proposition 3.2 (Energy upper bound). Under the above assumptions,
\[ \limsup_{\varepsilon \to 0} E_\varepsilon \leq E_{\text{qc}}. \] (3.6)

In order to prove the upper bound we use a coherent trial state: let us denote by \( \Omega_\varepsilon \in \mathcal{G}_\varepsilon(\mathfrak{h}) \) the Fock vacuum and let
\[ \Xi_\varepsilon[\psi, z] := \psi \otimes W_\varepsilon(\hat{z}) \Omega_\varepsilon, \] (3.7)
be a coherent product state constructed over the particle state \( \psi \) and the classical configuration \( z \in \mathfrak{h} \). We shall restrict to \( \psi \in Q(K_0) \), where \( Q(K_0) \) is the form domain of \( K_0 \), and \( z \in \mathfrak{h} \) such that \( \omega^{1/2} z \in \mathfrak{h} \). As discussed in §4, this is sufficient to make \( \Xi_\varepsilon[\psi, z] \in \mathcal{D}(H_\varepsilon) \) and \( (\psi, z) \in \mathcal{D}_{\text{qc}} \). The energy of the above trial state is provided in the next lemma.

Lemma 3.3. Under the above assumptions,
\[ \langle \Xi_\varepsilon[\psi, z] | H_\varepsilon | \Xi_\varepsilon[\psi, z] \rangle_{\mathcal{H}_\varepsilon} = \mathcal{E}_{\text{qc}}[\psi, z] + o_\varepsilon(1). \] (3.8)

Proof. The proof of the result depends on the microscopic model involved. The computation of the expectation over the trial states (3.7) can be found in [CF18, Prop. 3.11 & Sect. 3.6], for the Nelson and polaron models, and in [CFO19a, Proof of Thm. 1.9], for the Pauli-Fierz model, respectively.

Proof of Proposition 3.2. By Lemma 3.3 we have that
\[ E_\varepsilon \leq \inf_{(\psi, z) \in \mathcal{D}_{\text{qc}}} \langle \Xi_\varepsilon[\psi, z] | H_\varepsilon | \Xi_\varepsilon[\psi, z] \rangle_{\mathcal{H}_\varepsilon} = \inf_{(\psi, z) \in \mathcal{D}_{\text{qc}}} \mathcal{E}_{\text{qc}}[\psi, z] + o_\varepsilon(1) = E_{\text{qc}} + o_\varepsilon(1). \] (3.9)
The result is then obtained by taking the \( \limsup_{\varepsilon \to 0} \) on both sides.

3.1.2. Energy lower bound. The symmetric result of Proposition 3.2 is stated in the following proposition.

Proposition 3.4 (Energy lower bound). Under the above assumptions,
\[ \liminf_{\varepsilon \to 0} E_\varepsilon \geq E_{\text{qc}}. \] (3.10)

Although not necessary in principle, we find convenient to present two different proofs of (3.10), one valid only when \( K_0 \) has compact resolvent, e.g., when the small system is trapped, one valid for non-trapped small systems as well. The main reason is that the former does not require the use of generalized Wigner measures, since conventional state-valued measures are sufficient, resulting in a more accessible proof.

If \( K_0 \) has compact resolvent, the set of quasi-classical Wigner measures (as in Definition 1.5) associated with minimizing sequences for \( H_\varepsilon \) is not empty. In addition, the expectation of \( \text{Op}_W(v_z) \) converges to the quasi-classical integral of \( v_z \). Let us formulate some preliminary results about the convergence of the expectation values of the operators involved. Such results rely on suitable a priori bounds on the family of states \( \Psi_\varepsilon \in \mathcal{H}_\varepsilon \), as \( \varepsilon \) varies in \((0,1)\). Lemma 3.8 below guarantees that there exists a minimizing sequence \( \Psi_{\varepsilon,\delta} \) in the sense of (3.5) satisfying such bounds.
Lemma 3.5.
If $\mathcal{K}_0$ has compact resolvent and there exist $C < +\infty$ such that, uniformly w.r.t $\varepsilon \in (0,1)$,
\[
|\langle \Psi_{\varepsilon} \mid (\mathcal{K}_0 + dG_{\varepsilon}(\omega)) + 1 \mid \Psi_{\varepsilon} \rangle_{\mathcal{H}}| \leq C,
\]  
(3.11)
then $\mathcal{W}(\Psi_{\varepsilon}, \varepsilon \in (0,1)) \neq \emptyset$. Furthermore, if $\Psi_{\varepsilon_n} \xrightarrow{qc} m$, then $\text{tr}_{L^2(\mathbb{R}^d)}[\gamma_m(z)\mathcal{K}_0]$ is $\mu_m$-a.e. finite and $\mu_m$-absolutely integrable, and
\[
\lim_{n \to +\infty} \langle \Psi_{\varepsilon_n} \mid \mathcal{K}_0 \mid \Psi_{\varepsilon_n} \rangle_{\mathcal{H}} = \int_{\mathfrak{h}_\omega} d\mu_m(z) \text{tr}_{L^2(\mathbb{R}^d)}[\gamma_m(z)\mathcal{K}_0].
\]  
(3.12)

Proof. For $\omega = 1$ this proposition is proved in [CFO19b Props. 2.3 & 2.6]. For a generic $\omega \geq 0$, the proof (in presence of semiclassical degrees of freedom only) can be found in [Fal18a Thm. 3.3]; the extension to the quasi-classical setting is straightforward, testing with compact observables of the small system, as in the aforementioned [CFO19b Props. 2.3 & 2.6]. Let us stress that the fact that all Wigner measures are probability measures, i.e., there is no loss of mass and $m(\mathfrak{h}_\omega) = 1$, is due to the fact that $\mathcal{K}_0$ has compact resolvent. Otherwise, there may be a loss of probability mass due to the interplay between the particle system and the environment (see [CFO19b Cor. 1.7 & Rmk. 1.9] for additional details). \(\dashv\)

In order to control the convergence of the free field energy, we first have to regularize it: we pick a sequence of positive self-adjoint compact operators $\{1_r\}_{r \in \mathbb{N}} \subset \mathcal{B}(\mathfrak{h})$ approximating the identity: for all $r \in \mathbb{N}$, $1_r \leq 1$, and for all $z \in \mathfrak{h}_\omega$,
\[
\lim_{r \to +\infty} \langle z \mid 1_r \mid z \rangle_{\mathfrak{h}} = \lim_{r \to +\infty} \langle z \mid 1 \mid z \rangle_{\mathfrak{h}_\omega} = \|z\|_{\mathfrak{h}_\omega}^2 = \langle z \mid \omega \mid z \rangle_{\mathfrak{h}},
\]  
(3.13)
where we have denoted $\omega_r := \omega^{\frac{1}{2}} 1_r \omega^{\frac{1}{2}}$. Recall also that $\text{Op}_{\varepsilon}^{\text{Wick}}(\langle z \mid \omega \mid z \rangle_{\mathfrak{h}}) = 1 \otimes dG_{\varepsilon}(\omega)$, where $dG_{\varepsilon}(\omega)$ stands for the second quantization of $\omega$ as above.

Lemma 3.6.
If $\mathcal{K}_0$ has compact resolvent and there exist $C < +\infty$ such that, uniformly w.r.t $\varepsilon \in (0,1)$,
\[
|\langle \Psi_{\varepsilon} \mid (\mathcal{K}_0 + dG_{\varepsilon}(\omega)) + 1 \mid \Psi_{\varepsilon} \rangle_{\mathcal{H}}| \leq C,
\]  
(3.14)
then, if $\Psi_{\varepsilon_n} \xrightarrow{qc} m \in \mathcal{W}(\Psi_{\varepsilon}, \varepsilon \in (0,1))$, it follows that
\[
\int_{\mathfrak{h}_\omega} d\mu_m(z) \langle z \mid \omega \mid z \rangle_{\mathfrak{h}} \leq C,
\]  
(3.15)
and, for all $r \in \mathbb{N},$
\[
\lim_{n \to +\infty} \langle \Psi_{\varepsilon_n} \mid 1 \otimes dG_{\varepsilon_n}(\omega_r) \mid \Psi_{\varepsilon_n} \rangle_{\mathcal{H}} = \int_{\mathfrak{h}_\omega} d\mu_m(z) \langle z \mid \omega_r \mid z \rangle_{\mathfrak{h}} = \int_{\mathfrak{h}_\omega} d\mu_m(z) \langle z \mid 1_r \mid z \rangle_{\mathfrak{h}_\omega}.
\]  
(3.16)

Proof. The proof of $\mu_m$-integrability of $\langle z \mid \omega \mid z \rangle_{\mathfrak{h}}$ (and the relative bound) is a consequence of the corresponding result for semiclassical (scalar) Wigner measures proved in [AN08, Fal18a]. Analogously, the convergence holds because $\langle z \mid 1_r \mid z \rangle_{\mathfrak{h}_\omega}$ is a compact scalar symbol (see [Fal18a] for the convergence of compact symbols in $\mathfrak{h}_\omega$, and [CFO19b Props. 2.3 & 2.6] for additional details on the generalization of results in semiclassical analysis to the quasi-classical case). \(\dashv\)
Lemma 3.7.
If $K_0$ has compact resolvent and there exists $C < +\infty$, such that, uniformly w.r.t $\varepsilon \in (0, 1)$,
\[ \left| \langle \Psi_{\varepsilon} \left| (K_0 + dG_{\varepsilon}(\omega)^2 + 1) \right| \Psi_{\varepsilon} \rangle \right| \leq C, \] (3.17)
then, if $\Psi_{\varepsilon_n} \xrightarrow{qc} m$, for any $i = 1, \ldots, N$,
\[ \lim_{n \to +\infty} \left\langle \Psi_{\varepsilon_n} \left| \Omega_{\varepsilon_n}^{Wick}(V_z(x_i)) \right| \Psi_{\varepsilon_n} \right\rangle_{\mathcal{H}_{\varepsilon_n}} = \int_{h_\omega} d\mu_m(z) \operatorname{tr}_{L^2(\mathbb{R}^{dN})} [\gamma_m(z) V_z(x_i)]. \] (3.18)

Lemma 3.8.
There exists a minimizing sequence $\{\Psi_{\varepsilon, \delta}\}_{\varepsilon, \delta \in (0, 1)}$, such that, for all fixed $\delta \in (0, 1)$, (3.5) holds true and there exists $C_\delta < +\infty$, such that
\[ \left| \langle \Psi_{\varepsilon} \left| (K_0 + dG_{\varepsilon}(\omega)^2 + 1) \right| \Psi_{\varepsilon} \rangle \right| \leq C_\delta. \] (3.19)

The proofs of Lemma 3.7 and Lemma 3.8 above, like the form of the quasi-classical potential $V_z$, depend on the model considered. We thus provide them in § 4.

We are now in a position to prove the lower bound in the trapped case.

Proof of Proposition 3.4 Let $\Psi_{\varepsilon, \delta}$ be the minimizing sequence for $H_\varepsilon$ of Lemma 3.8. Since for any $r \in \mathbb{N}$, $\omega_r \leq \omega$, it follows that $dG_{\varepsilon}(\omega_r) \leq dG_{\varepsilon}(\omega)$. Hence,
\[ \left| \left\langle \Psi_{\varepsilon, \delta} \left| \left( K_0 + dG_{\varepsilon}(\omega_r) + \Omega_{\varepsilon}^{Wick}(\sum_i V_z(x_i)) \right) \right| \Psi_{\varepsilon, \delta} \right\rangle \right|_{\mathcal{H}_{\varepsilon}} \leq \left| \left\langle \Psi_{\varepsilon, \delta} \left| H_\varepsilon \right| \Psi_{\varepsilon, \delta} \right\rangle \right|_{\mathcal{H}_{\varepsilon}} < E_\varepsilon + \delta. \] (3.20)

Now, let us recall that, by Lemmas 3.5 to 3.8

- for any $\delta > 0$, $\mathcal{W}(\Psi_{\varepsilon, \delta}, \varepsilon \in (0, 1)) \neq \emptyset$;
- the expectation value of each term in the Hamiltonian converges as $\varepsilon \to 0$ or, more precisely, there exists $m \in \mathcal{W}(\Psi_{\varepsilon, \delta}, \varepsilon \in (0, 1))$ such that
\[ \int_{h_\omega} d\mu_m(z) \operatorname{tr}_{L^2(\mathbb{R}^{dN})} [\gamma_m \left( K_0 + \frac{d}{2} + \sum_i |z| V_z(x_i) \right)] \leq \liminf_{\varepsilon \to 0} \left| \left\langle \Psi_{\varepsilon, \delta} \left| \left( K_0 + dG_{\varepsilon}(\omega_r) + \Omega_{\varepsilon}^{Wick}(\sum_i V_z(x_i)) \right) \right| \Psi_{\varepsilon, \delta} \right\rangle \right|_{\mathcal{H}_{\varepsilon}}. \] (3.21)

Hence, we deduce that
\[ \int_{h_\omega} d\mu_m(z) \operatorname{tr}_{L^2(\mathbb{R}^{dN})} [\gamma_m \left( K_0 + \frac{d}{2} + \sum_i |z| V_z(x_i) \right)] \leq \liminf_{\varepsilon \to 0} E_\varepsilon + \delta. \] (3.22)

Now, by construction, $\langle z | \omega_r | z \rangle_{h} \xrightarrow{r \to +\infty} \langle z | \omega | z \rangle_{h}$ for any $z \in h_\omega$, and, by Lemma 3.5, any $m \in \mathcal{W}(\Psi_{\varepsilon}, \varepsilon \in (0, 1))$ is concentrated on $h_\omega$. Furthermore,
\[ \int_{h_\omega} d\mu_m(z) \langle z | \omega_r | z \rangle_{h} \leq \int_{h_\omega} d\mu_m(z) \langle z | \omega | z \rangle_{h} \leq C < +\infty. \]

Hence, by dominated convergence,
\[ \lim_{r \to +\infty} \int_{h_\omega} d\mu_m(z) \langle z | \omega_r | z \rangle_{h} = \int_{h_\omega} d\mu_m(z) \langle z | \omega | z \rangle_{h}. \] (3.23)

Thus, one gets
\[ \inf_{m \in \mathcal{W}(\Psi_{\varepsilon, \delta}, \varepsilon \in (0, 1))} \int_{h_\omega} d\mu_m(z) \operatorname{tr}_{L^2(\mathbb{R}^{dN})} [\gamma_m \mathcal{H}_z] < \liminf_{\varepsilon \to 0} E_\varepsilon + \delta. \] (3.24)
which, via Proposition 2.1 implies that
\[
E_{qc} \leq \inf_{m \in \mathcal{W}} \left( \Psi_{\varepsilon, \delta, \varepsilon} \right) \int_{\mathbb{R}^2} d\mu_m(z) \, \text{tr} L^2(\mathbb{R}^2) [\gamma_m H_z] < \lim_{\varepsilon \to 0} E_\varepsilon + \delta .
\]
Since \( \delta > 0 \) is arbitrary, the claim follows.

### 3.1.3. Convergence of minimizing sequences and minimizers
Once the energy convergence is proven, we investigate the behavior of minimizing sequences and minimizers, if any. We may thus assume that the microscopic system admits a ground state \( \Psi_{gs} \).

**Proof of Theorem 1.8.** Let \( \Psi_{\varepsilon, \delta} \in \mathcal{D}(H) \) be a minimizing sequence. Then by Lemmas 3.5 to 3.8 any \( m_\delta \in \mathcal{W} \left( \Psi_{\varepsilon, \delta}, \varepsilon \in (0, 1) \right) \), corresponding to a sequence \( \{\Psi_{\varepsilon, \delta}^n\}_{n \in \mathbb{N}}, \varepsilon \to 0 \), satisfies
\[
\int_{\mathbb{R}^2} d\mu_{m_\delta}(z) \, \text{tr} L^2(\mathbb{R}^2) [\gamma_m(z) H_z] = \lim_{n \to +\infty} (\Psi_{\varepsilon, \delta}^n | H_{\varepsilon, \delta} | \Psi_{\varepsilon, \delta}^n) \leq \lim_{n \to +\infty} E_{\varepsilon, \delta} + \delta = E_{qc} + \delta ,
\]
as proven in Theorem 1.3.

**Proof of Corollary 1.9.** If \( \delta = o_\varepsilon(1) \), then considering \( m_0 \in \mathcal{W} \left( \Psi_{\varepsilon, o_\varepsilon(1)}, \varepsilon \in (0, 1) \right) \), corresponding to a sequence \( \{\Psi_{\varepsilon, o_\varepsilon(1)}^n\}_{n \in \mathbb{N}}, \varepsilon \to 0 \), it satisfies
\[
\int_{\mathbb{R}^2} d\mu_{m_0}(z) \, \text{tr} L^2(\mathbb{R}^2) [\gamma_m(z) H_z] \leq \lim_{n \to +\infty} (E_{\varepsilon, o_\varepsilon(1)} + o_\varepsilon(1)) = E_{qc} .
\]
By Proposition 2.1 it follows that \( m_0 \) is a minimizer of (VP2) and, by Proposition 2.3, it is concentrated on the set \( (\psi_{qc}, z_{qc}) \) of minimizers of (VP2).

**Proof of Corollary 1.10.** Let \( \Psi_{gs} \) be a ground state of \( H \). Then, it is also a (exact) minimizing sequence with \( \delta = 0 \), and thus as above \( m_0 \) is a minimizer of (VP2), and it is concentrated on the set \( (\psi_{qc}, z_{qc}) \) of minimizers of (VP2).

### 3.2. Non-trapped particle systems
In the non-trapped case, the strategy of proof is very similar, however it is not ensured that the set of quasi-classical Wigner measures for the minimizing sequence is not empty. It is then necessary to use generalized Wigner measures (recall Definition 1.7). We note however that the proof of the upper bound stated in Proposition 3.2 applies to the non-trapped case too and therefore we have just to provide an alternative proof of Proposition 3.4 without the assumption of compactness of the resolvent of \( K_0 \).

We first generalize the preparatory lemmas that we needed in the trapped case to the general situation. Note that for Lemma 3.8 it is not necessary that \( K_0 \) has compact resolvent and therefore we can use it directly also in the non-trapped case. We also use the same notation as in the trapped case; in particular, we make use of the same compact approximation \( \omega_r \) of \( \omega \) we introduced in 3.1.3.

**Lemma 3.9.**
If there exist \( C < +\infty \) such that, uniformly w.r.t. \( \varepsilon \in (0, 1) \),
\[
| \langle \Psi_{\varepsilon} | (K_0 + dG_{\varepsilon}(\omega) + 1) | \Psi_{\varepsilon} \rangle_{\mathcal{H}_z} | \leq C ,
\]
then \( \mathcal{W} \left( \Psi_{\varepsilon}, \varepsilon \in (0, 1) \right) \neq \emptyset \). Furthermore, if \( \Psi_{\varepsilon} \xrightarrow{qc} n \), then \( n \) is in the domain of \( K_0 + 1 \) in the sense of Definition 2.6 and
\[
\lim_{n \to +\infty} \langle \Psi_{\varepsilon} | K_0 | \Psi_{\varepsilon} \rangle_{\mathcal{H}_z} = \int_{\mathbb{R}^2} d \omega(z) [K_0] .
\]
**Lemma 3.10.**

If there exist $C < +\infty$ such that, uniformly w.r.t $\varepsilon \in (0, 1)$,

$$|\langle \Psi_\varepsilon | (\mathcal{K}_0 + dG_\varepsilon (\omega) + 1) | \Psi_\varepsilon \rangle_{\mathcal{H}_\varepsilon} | \leq C,$$

(3.27)

then if $\Psi_{\varepsilon_n} \xrightarrow{\text{gqc}} n \in \mathcal{W}(\Psi_\varepsilon, \varepsilon \in (0, 1))$, it follows that

$$\int_{\mathcal{H}_\omega} \text{d}n(z)[1] \langle z | \omega | z \rangle_{\mathcal{H}} \leq C,$$

(3.28)

and, for all $r \in \mathbb{N}$,

$$\lim_{n \to +\infty} \langle \Psi_{\varepsilon_n} | 1 \otimes dG_{\varepsilon_n} (\omega_r) | \Psi_{\varepsilon_n} \rangle_{\mathcal{H}_{\varepsilon_n}} = \int_{\mathcal{H}_\omega} \text{d}n(z)[1] \langle z | \omega_r | z \rangle_{\mathcal{H}}.$$

(3.29)

**Proof of Lemmas 3.9 and 3.10.** These lemmas extend to generalized Wigner measures Lemmas 3.5 and 3.6 respectively. Their proofs are, mutatis mutandis, completely analogous to the ones of the latter. Contrarily to Lemma 3.5, since now $\mathcal{K}_0$ has a non-compact resolvent, the set of Wigner measures of $\Psi_\varepsilon$ may be empty and there might be a loss of mass along the quasi-classical convergence. The set of generalized Wigner measures is, however, always non-empty: no mass is lost due to the fact that

$$\| \Psi_\varepsilon \|^2_{\mathcal{H}_\varepsilon} = \langle \Psi_\varepsilon | 1 \otimes W_\varepsilon (0) | \Psi_\varepsilon \rangle_{\mathcal{H}_\varepsilon} = 1,$$

and the identity operator belongs to $\mathcal{B}(L^2(\mathbb{R}^d))$ but it is not compact. More precisely, the above quantity can be immediately identified, in the limit $\varepsilon \to 0$, with the total mass of all generalized Wigner measures associated to $\Psi_\varepsilon$, as defined in Definition 1.7, whereas it is a priori only bigger or equal than the total mass of measures defined by the convergence in Definition 1.5 (if all cluster points for the aforementioned convergence have total mass strictly less than one, the set of Wigner measures associated to $\Psi_\varepsilon$, that are required by Definition 1.5 to have total mass one, is thus empty).

\[\square\]

**Lemma 3.11.**

If there exists $C < +\infty$, such that, uniformly w.r.t $\varepsilon \in (0, 1)$,

$$|\langle \Psi_\varepsilon | (\mathcal{K}_0 + dG_\varepsilon (\omega)^2 + 1) | \Psi_\varepsilon \rangle_{\mathcal{H}_\varepsilon} | \leq C,$$

(3.30)

then, if $\Psi_{\varepsilon_n} \xrightarrow{\text{gqc}} n$, for any $i = 1, \ldots, N$,

$$\lim_{n \to +\infty} \langle \Psi_{\varepsilon_n} | \text{Op}_{\varepsilon_n}^{\text{Wick}} (\mathcal{V}_\varepsilon (x_i)) | \Psi_{\varepsilon_n} \rangle_{\mathcal{H}_{\varepsilon_n}} = \int_{\mathcal{H}_\omega} \text{d}n(z) [\mathcal{V}_\varepsilon (x_i)].$$

(3.31)

As for its analogue Lemma 3.7, the proof of Lemma 3.11 is model-dependent and thus given in § 4.

The proof of the lower bound for the non-trapped case is now equivalent to the one in the trapped case, using generalized Wigner measures.

**Proof of Proposition 3.4.** Let $\Psi_{\varepsilon, \delta}$ be the minimizing sequence for $H_\varepsilon$ of Lemma 3.8 satisfying (3.20). Now, by Lemmas 3.8 to 3.11

- for any $\delta > 0$, $\mathcal{W}(\Psi_{\varepsilon, \delta}, \varepsilon \in (0, 1)) \neq \emptyset$;
as for Wigner measures, there exists $n \in \mathcal{G} W (\Psi_{\varepsilon, \delta}, \varepsilon \in (0, 1))$ such that

$$
\int_{h_\omega} \text{d}n(z) \left[ K_0 + \langle z | \omega_r | z \rangle_h + \sum_i \mathcal{V}_z(x_i) \right] \\
\leq \liminf_{\varepsilon \to 0} \left\langle \Psi_{\varepsilon, \delta} \left| \left( K_0 + \text{d}G_\varepsilon(\omega_r) + \text{Op}^{\text{Wick}} (\sum_i \mathcal{V}_z(x_i)) \right) \right| \Psi_{\varepsilon, \delta} \right\rangle_{\mathcal{H}_\varepsilon},
$$

and therefore

$$
\int_{h_\omega} \text{d}n(z) \left[ K_0 + \langle z | \omega_r | z \rangle_h + \sum_i \mathcal{V}_z(x_i) \right] < \liminf_{\varepsilon \to 0} E_\varepsilon + \delta.
$$

However, by dominated convergence, see Theorem A.18 in Appendix A,

$$
\lim_{r \to +\infty} \int_{h_\omega} \text{d}n(z)[1] \langle z | \omega_r | z \rangle_h = \int_{h_\omega} \text{d}n(z)[1] \langle z | \omega | z \rangle_h.
$$

Hence,

$$
E_{\text{gqc}} \leq \inf_{n \in \mathcal{G} W (\Psi_{\varepsilon, \delta, \varepsilon \in (0, 1)})} \int_{h_\omega} \text{d}n(z)[\mathcal{H}_z] < \liminf_{\varepsilon \to 0} E_\varepsilon + \delta,
$$

and the result follows from the arbitrariness of $\delta > 0$, via Proposition 2.8.

The proof of Theorem 1.15 is also completely analogous to the proof of Theorem 1.8 for trapped systems:

**Proof of Theorem 1.15.** If $K_0$ does not have compact resolvent, then by Lemmas 3.9 to 3.11 and Lemma 3.8, any $n_\delta \in \mathcal{G} W (\Psi_{\varepsilon, \delta, \varepsilon \in (0, 1)})$ satisfies

$$
\int_{h_\omega} \text{d}n_\delta(z)[\mathcal{H}_z] < \liminf_{\varepsilon \to 0} E_{\varepsilon_n} + \delta = E_{\text{qc}} + \delta = E_{\text{gqc}} + \delta,
$$

by Theorem 1.3 and Proposition 2.8.

**Proof of Corollary 1.16.** If $\delta = o_\varepsilon(1)$, it follows that $n_0 \in \mathcal{G} W (\Psi_{\varepsilon, o_\varepsilon(1), \varepsilon \in (0, 1)})$ satisfies

$$
\int_{h_\omega} \text{d}n_0(z)[\mathcal{H}_z] = \lim_{\varepsilon_n \to 0} E_{\varepsilon_n} = E_{\text{gqc}}.
$$

Therefore $n_0$ solves (gvp2), and thus it is concentrated on minimizers solving (GVP2).

**Proof of Corollary 1.17.** This proof is completely analogous to the one above.

4. **Concrete Models**

In this section we discuss the concrete models introduced in §1, and in particular we provide the proof of results used in §3 that require a model-dependent treatment.

4.1. **The Nelson model.** The simplest model under consideration is the so-called Nelson model [Nel64]. It consists of a small system of $N$ non-relativistic particles coupled with a scalar bosonic field, both moving in $d$ spatial dimensions.

We recall the explicit expression of the quasi-classical energy (1.15) in the Nelson model:

$$
\mathcal{H}_z = \sum_{j=1}^N \{-\Delta_j + \mathcal{V}_z(x_j)\} + \mathcal{W}(x_1, \ldots, x_N) + \langle z | \omega | z \rangle_h,
$$
irrespective of compactness of the resolvent of \( \mathcal{R} \), or if \( 0 \in \sigma(\omega) \) we get, for all \( \alpha > 0 \) and all \( \Psi \in \mathcal{D}(\mathcal{A}) \), a detailed discussion of the existence of ground states for the Nelson model.

The quasi-classical Wick quantization of \( \mathcal{H}_\varepsilon \) yields the quantum field Hamiltonian

\[
H_\varepsilon = \sum_{j=1}^{N} \left\{ -\Delta_j \otimes 1 + a_\varepsilon(\lambda(x_j)) + a_\varepsilon^\dagger(\lambda(x_j)) \right\} + W(x_1, \ldots, x_N) \otimes 1 + 1 \otimes \mathcal{G}_\varepsilon(\omega),
\]

acting on \( \mathcal{H}_\varepsilon = L^2(\mathbb{R}^dN) \otimes \mathcal{G}_\varepsilon(\mathfrak{h}) \), where we have explicitly highlighted the trivial action of some terms of \( H_\varepsilon \) on either the particle’s or the field’s degrees of freedom. Whenever \( \lambda \in L^\infty(\mathbb{R}^d; \mathfrak{h}) \), the operator \( H_\varepsilon \) is self-adjoint, with domain of essential self-adjointness \( \mathcal{D}(-\Delta + W + dG_\varepsilon(\omega)) \cap C_0^\infty(dG_\varepsilon(1)) \), where the latter is the set of vectors with finite number of field’s excitations [Fal15], but it may be unbounded from below, if \( 0 \in \sigma(\omega) \). It is however well-known that, if for a.e. \( x \in \mathbb{R}^d, \lambda(x) \in \mathcal{D}(\omega^{-1/2}) \), then \( H_\varepsilon \) is bounded from below by Kato-Rellich’s theorem. Nonetheless, it may still not have a ground state, if \( 0 \notin \sigma(\omega) \) or if \( W \) is not regular enough. We refer to the list of works [Ara01, BHL+02, Der03, Piz03, GGM04, Mol05, Hir06, GHPS11, AH12, HM19] and references therein for a detailed discussion of the existence of ground states for the Nelson model.

We simply remark here that the ground state exists, if \( 0 \notin \sigma(\omega) \) and \(-\Delta + W\) has compact resolvent (trapped particle system), or if \( 0 \in \sigma(\omega) \) and \( \lambda \) and \( W \) satisfy suitable conditions, irrespective of compactness of the resolvent of \(-\Delta + W\).

**Proof of Proposition 3.1** The upper and lower bounds in [3.3] are well known (see, e.g., [GNV06, AF14, CF18]). The lower bound is a direct consequence of Kato-Rellich’s inequality, while the upper bound is proved using coherent states for the field. We provide some details for the sake of completeness.

Setting\( \check{} \)

\[
H_{\text{free}} := \mathcal{K}_0 \otimes 1 + 1 \otimes d\mathcal{G}_\varepsilon(\omega),
\]

we get, for all \( \alpha > 0 \) and all \( \Psi \in \mathcal{D}(H_{\text{free}}) \),

\[
\left\| \sum_{j=1}^{N} \left( a_\varepsilon(\lambda(x_j)) + a_\varepsilon^\dagger(\lambda(x_j)) \right) \right\| \Psi_\varepsilon \right\|_{\mathcal{H}_\varepsilon} \leq 2N \left\| \omega^{-1/2} \lambda \right\|_{L^\infty(\mathbb{R}^d; \mathfrak{h})} \left\| d\mathcal{G}_\varepsilon(\omega)^{1/2} \Psi_\varepsilon \right\|_{\mathcal{H}_\varepsilon} \left( \sqrt{\varepsilon} \right) \left\| \lambda \right\|_{L^\infty(\mathbb{R}^d; \mathfrak{h})} \left\| \Psi_\varepsilon \right\|_{\mathcal{H}_\varepsilon} \leq \alpha \left( \left\| d\mathcal{G}_\varepsilon(\omega) \right\| \Psi_\varepsilon \right)_{\mathcal{H}_\varepsilon} + \left( \frac{N^2}{\alpha} \right) \left\| \omega^{-1/2} \lambda \right\|_{L^\infty(\mathbb{R}^d; \mathfrak{h})}^2 + \left( \sqrt{\varepsilon} \right) \left\| \lambda \right\|_{L^\infty(\mathbb{R}^d; \mathfrak{h})} \left\| \Psi_\varepsilon \right\|_{\mathcal{H}_\varepsilon}. \quad (4.2)
\]

\( ^6 \) Of course we may allow for a negative part of the potential \( W \), provided it is bounded, but we choose a positive potential for the sake of simplicity.

\( ^7 \) Even if not stated explicitly, we use the notation \( H_{\text{free}} \) also in [8.4.2 and 11.3] with the same meaning.
Therefore, choosing $\alpha = 1$, we deduce that (recall that $\varepsilon \in (0,1)$)
\[
E_\varepsilon \geq - N^2 \left\| \omega^{-1/2} \lambda \right\|_{L^\infty(\mathbb{R}^d;\mathbb{R})}^2 - ||\lambda||_{L^\infty(\mathbb{R}^d;\mathbb{R})}.
\] (4.3)

The upper bound is trivial to show by exploiting (4.2) and evaluating the energy on any state such that $\langle \Psi_\varepsilon | dG_\varepsilon(\omega) \Psi_\varepsilon \rangle_{\mathscr{H}_\varepsilon} \leq C < +\infty$, e.g., a product state $\Psi_\varepsilon = \psi \otimes \Omega$, with $\psi \in \mathcal{D}(K_0)$ and $\Omega_\varepsilon$ the field vacuum. Note that the uniform boundedness of $E_\varepsilon$ from above could as well be deduced by the boundedness of $E_0$, which in turn follows from the evaluation of $E_{qc}$ on, e.g., a configuration $(\psi,0)$, with $\psi \in \mathcal{D}(K_0)$.

We now prove Lemmas 3.7 and 3.8 for the Nelson model. We have however to state first a technical result, which generalizes the convergence of expectation values proven in [CFO19b]: indeed, in [CFO19b] Prop. 2.6 it is shown that
\[
\lim_{n \to +\infty} \left\langle \Psi_{\varepsilon n} \Big| \text{Op}^{Wick}_\varepsilon (\mathcal{V}_x) \mathcal{K} \Psi_{\varepsilon n} \right\rangle_{\mathscr{H}_{\varepsilon n}} = \int_{B_\omega} d\mu_m(z) \text{tr}_{L^2(\mathbb{R}^d)} [\gamma_m(z)\mathcal{V}_z \mathcal{K}],
\] (4.4)
but our goal is to apply the above convergence to the identity, which is not compact. We have then to approximate it with compact operators.

**Lemma 4.1.**

Let $K_0$ have compact resolvent. If there exist $C < +\infty$ and $\delta \geq 1$, such that, uniformly w.r.t. $\varepsilon \in (0,1)$,
\[
\left\| \left| \Psi_\varepsilon \right| \left( K_0 + dG_\varepsilon(\omega)\lambda + 1 \right) \Psi_\varepsilon \right\|_{\mathscr{H}_\varepsilon} \leq C,
\] (4.5)
and $\Psi_{\varepsilon n} \xrightarrow{n \to +\infty} \Psi_m$, then, for all $\mathcal{K} \in \mathscr{L}_P(L^2(\mathbb{R}^d))$ and any $j = 1, \ldots, N$,
\[
\lim_{n \to +\infty} \left| \Psi_{\varepsilon n} \Big| \text{Op}^{Wick}_{\varepsilon n} (\mathcal{V}_x) \mathcal{B} \Psi_{\varepsilon n} \right\rangle_{\mathscr{H}_{\varepsilon n}} = \int_{B_\omega} d\mu_m(z) \text{tr}_{L^2(\mathbb{R}^d)} [\gamma_m(z)\mathcal{V}_z \mathcal{B}].
\] (4.6)

**Proof.** Let us introduce compact approximate identities $\{1_m\}_{m \in \mathbb{N}} \subset \mathscr{L}_P(L^2(\mathbb{R}^d))$ as follows:
\[
1_m := 1_{[-m,m]}(K_0),
\]
where $1_{[-m,m]} : \mathbb{R} \to \{0,1\}$ is the characteristic function of the interval $[-m,m]$, so that the r.h.s. of the above expression is the usual spectral projector of $K_0$ constructed via spectral theorem. For later convenience, let us also define $\mathcal{B}_m := \mathcal{B}1_m$. Therefore, we have that
\[
\left| \Psi_{\varepsilon n} \Big| \text{Op}^{Wick}_{\varepsilon n} (\mathcal{V}_x) \mathcal{B} \Psi_{\varepsilon n} \right\rangle_{\mathscr{H}_{\varepsilon n}} = \left| \Psi_{\varepsilon n} \Big| \text{Op}^{Wick}_{\varepsilon n} (\mathcal{V}_x) \mathcal{B} \Psi_{\varepsilon n} \right\rangle_{\mathscr{H}_{\varepsilon n}} + \left| \Psi_{\varepsilon n} \Big| \text{Op}^{Wick}_{\varepsilon n} (\mathcal{V}_x) \mathcal{B} \Psi_{\varepsilon n} \right\rangle_{\mathscr{H}_{\varepsilon n}}.
\] (4.7)

The first term on the r.h.s. converges when $n \to +\infty$, for any fixed $m \in \mathbb{N}$, since $\mathcal{B}_m \in \mathscr{L}_P(L^2(\mathbb{R}^d))$ (see [Fal18a]), i.e.,
\[
\lim_{n \to +\infty} \left| \Psi_{\varepsilon n} \Big| \text{Op}^{Wick}_{\varepsilon n} (\mathcal{V}_x) \mathcal{B}_m \Psi_{\varepsilon n} \right\rangle_{\mathscr{H}_{\varepsilon n}} = \int_{B_\omega} d\mu_m(z) \text{tr}_{L^2(\mathbb{R}^d)} [\gamma_m(z)\mathcal{V}_z \mathcal{B}_m].
\]

\[\text{In [CFO19b] Prop. 2.6 the result is proved for } \omega = 1. \text{ The extension to a generic } \omega \text{ is done straightforwardly combining the proof of Prop. 2.6 with the techniques introduced in [Fal18a].}\]
By dominated convergence, we can then take the limit \( m \to +\infty \), to obtain
\[
\lim_{m \to +\infty} \frac{1}{n} \left( \langle \Psi \varepsilon, (\mathcal{B} - \mathcal{B}_m) \Psi \varepsilon \rangle \mathcal{H}_\varepsilon \right) = \int_{\mathbb{R}^d} d\mu_m(z) \text{ tr}_{L^2(\mathbb{R}^d)} [\gamma_m(z) \mathcal{V}_\varepsilon \mathcal{B}] .
\]
(4.8)

It remains to prove that
\[
\lim_{m \to +\infty} \sup_{\varepsilon \in (0,1)} \left| \langle \Psi \varepsilon, (\mathcal{B} - \mathcal{B}_m) \Psi \varepsilon \rangle \mathcal{H}_\varepsilon \right| = 0 .
\]
(4.9)

For any \( 0 < s \leq \frac{1}{2} \) and for any \( c_0 > |\inf \sigma(\mathcal{K}_0)| \),
\[
\begin{align*}
\left| \langle \Psi \varepsilon, (\mathcal{B} - \mathcal{B}_m) \Psi \varepsilon \rangle \mathcal{H}_\varepsilon \right| &\leq 2 \left\| (\mathcal{B} - \mathcal{B}_m) (\mathcal{K}_0 + c_0)^{-\frac{1}{2}} \right\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \\
&\times \left( \left\| (\mathcal{B} - \mathcal{B}_m) \mathcal{V}_\varepsilon \right\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \right)^{\frac{1}{2}} \left\| (\mathcal{B} - \mathcal{B}_m) \mathcal{V}_\varepsilon \right\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \\
&\leq C \left\| \mathcal{B} \mathcal{V}_\varepsilon \right\|_{\mathcal{B}(L^2(\mathbb{R}^d))} \left( 1 - m \right) (\mathcal{K}_0 + c_0)^{-\frac{1}{2}} \left\| \omega^\frac{1}{2} \right\|_{L^\infty(\mathbb{R}^d,b)} \langle \Psi \varepsilon, \mathcal{K}_0^2 \mathcal{V}_\varepsilon \mathcal{V}_\varepsilon \rangle \mathcal{H}_\varepsilon \\
&\leq C \sup_{|\eta| \leq (1 - m)(\mathcal{K}_0 + c_0)^{-\frac{1}{2}}} \frac{1}{(\eta + c_0)^2} \leq Cm^{-\frac{1}{2}},
\end{align*}
\]
for \( m \) large enough, e.g., larger than \( |\inf \sigma(\mathcal{K}_0)| \). Therefore, since the above quantity vanishes as \( m \to +\infty \) uniformly w.r.t. \( \varepsilon \in (0,1) \), we conclude that (4.9) holds true and the result follows.

**Proof of Lemma 3.7** The result follows by taking \( \mathcal{B} = 1 \) in Lemma 4.1. Again, this makes crucial use of the fact that \( \mathcal{K}_0 = -\Delta + \mathcal{W} \) has compact resolvent, and that \( \Psi \varepsilon \) is regular enough w.r.t. \( \mathcal{K}_0 \).

**Proof of Lemma 3.8** The proof of Lemma 3.8 stems from a known result that allows to compare the expectation of the square of the free energy \( H^2_\varepsilon \) with the expectation of the square of the full Hamiltonian \( H^2_\varepsilon \). This is a consequence of Kato-Rellich’s inequality: there exists \( C > 0 \) (independent of \( \varepsilon \)), such that
\[
\langle \Psi \varepsilon, H^2_\varepsilon \rangle_\mathcal{H}_\varepsilon \leq C \langle \Psi \varepsilon, H^2_\varepsilon + 1 \rangle_\mathcal{H}_\varepsilon .
\]
(4.10)
The idea of the proof of this standard inequality goes as follows: from the triangular inequality, we get
\[
\langle \Psi \varepsilon, H^2_\varepsilon \rangle_\mathcal{H}_\varepsilon \leq 2 \langle \Psi \varepsilon, H^2_\varepsilon \rangle_\mathcal{H}_\varepsilon + 2 \langle \Psi \varepsilon, (H_\varepsilon - H^2_\varepsilon) \rangle_\mathcal{H}_\varepsilon .
\]
Now, using inequality (4.2), we get that for any \( \alpha < 1/\sqrt{2} \),
\[
(1 - 2\alpha^2) \langle \Psi \varepsilon, H^2_\varepsilon \rangle_\mathcal{H}_\varepsilon \leq 2 \langle \Psi \varepsilon, H^2_\varepsilon \rangle_\mathcal{H}_\varepsilon + C_\alpha \langle \Psi \varepsilon \rangle_\mathcal{H}_\varepsilon ,
\]
with \( C_\alpha \) independent of \( \varepsilon \). The result then easily follows.

It remains to prove that there exists a minimizing sequence \( \{ \Psi_{\varepsilon,\delta} \}_{\varepsilon,\delta \in (0,1)} \subset \mathcal{D}(H_\varepsilon) \) for \( H_\varepsilon \), such that
\[
\langle \Psi \varepsilon, H^2_\varepsilon \rangle_\mathcal{H}_\varepsilon \leq \max \left\{ E^2_\varepsilon, (E_\varepsilon + \delta)^2 \right\} \leq C ,
\]
(4.11)
with the last inequality given by Proposition 3.1. Indeed, combining the above estimate with (4.10), we immediately deduce that (3.17) holds true. Let us denote by \( \mathcal{I}_{(a,b)}(H_\varepsilon) \) the spectral projections of \( H_\varepsilon \), and by \( \mathcal{D}_{(a,b)} := \mathcal{I}_{(a,b)}(H_\varepsilon) \mathcal{H}_\varepsilon \) the associated spectral subspaces. Let now
choose, for any $\delta > 0$,
\[ \Psi_{\varepsilon,\delta} \in \left\{ \Psi \in \mathcal{S}(E_{\varepsilon-\delta, E_{\varepsilon+\delta}}) \mid \|\Psi\|_{\mathcal{H}_z} = 1 \right\}. \]

Each spectral subspace above is not empty by definition of $E_{\varepsilon} = \inf \sigma(H_{\varepsilon})$. Therefore, on one hand,
\[ (\Psi_{\varepsilon,\delta} \mid H_{\varepsilon} \mid \Psi_{\varepsilon,\delta})_{\mathcal{H}_z} \leq E_{\varepsilon} + \delta, \]
and, on the other,
\[ \|H_{\varepsilon}\Psi_{\varepsilon,\delta}\|_{\mathcal{H}_z}^2 \leq \max \{E_{\varepsilon}^2, (E_{\varepsilon} + \delta)^2\}. \]

It remains only to prove Lemma 3.11 used in the non-trapped case.

**Proof of Lemma 3.11** To prove the result, it is sufficient to show that, if $\Psi_{\varepsilon}$ is such that
\[ \left| \left( \Psi_{\varepsilon} \mid (dG_{\varepsilon}(\omega) + 1)^{\delta} \mid \Psi_{\varepsilon} \right)_{\mathcal{H}_z} \right| \leq C, \]
for some $\delta \geq 1/2$ and some finite constant $C$, and, if $\Psi_{\varepsilon_{n}} \xrightarrow{n \to +\infty} \Psi_{\varepsilon}$, then (3.31) holds true, i.e., for all $\mathcal{B} \in \mathcal{B}(L^2(\mathbb{R}^{dN}))$,
\[ \lim_{n \to +\infty} \left\langle \Psi_{\varepsilon_{n}} \left| \text{Op}_{\varepsilon_{n}}^{\text{Wick}}(\mathcal{V}_{\varepsilon}) \right| \mathcal{B} \Psi_{\varepsilon_{n}} \right\rangle_{\mathcal{H}_z} = \int_{\mathbb{R}} \text{d}n |\mathcal{V}_{\varepsilon} \mathcal{B}|. \]

Such a result is however a special case of [CFO19b, Prop. 2.6], if in that statement Wigner measures are substituted by generalized Wigner measures, the test with compact operators of the small system is replaced with the test with bounded operators, and $dG_{\varepsilon}(1)$ is replaced by $dG_{\varepsilon}(\omega)$. The proof given there is generalized to this setting straightforwardly, recalling the properties of generalized Wigner measures outlined in Appendix A. There is only one thing that is worth to remark explicitly: the integration of operator-valued functions w.r.t. generalized Wigner measures makes sense only if $\text{Ran}(z \mapsto \mathcal{V}_{\varepsilon}) \subset \mathcal{B}(L^2(\mathbb{R}^{dN}))$ is separable in the norm topology of $\mathcal{B}(L^2(\mathbb{R}^{dN}))$. Let us check explicitly that $\text{Ran}(z \mapsto \mathcal{V}_{\varepsilon})$ is indeed separable: since $\mathfrak{h}_\omega$ is separable, let us denote by $\mathfrak{k} \subset \mathfrak{h}_\omega$ a countable dense subset and denote by
\[ \mathcal{V}_k := \left\{ \mathcal{V}_\zeta(\mathbf{x}) \in \mathcal{B}(L^2(\mathbb{R}^{dN})), \zeta \in \mathfrak{k} \right\}, \]
the image of $\mathfrak{k}$ by means of $z \mapsto \mathcal{V}_z$. Now, for any $z \in \mathfrak{h}_\omega$, $\zeta \in \mathfrak{k}$, we have that
\[ \|\mathcal{V}_z - \mathcal{V}_\zeta\|_{\mathcal{B}(L^2(\mathbb{R}^{dN}))} \leq 2 \left\| \omega^{-\frac{1}{2}} \lambda \right\|_{L^\infty(\mathbb{R}^d; \mathfrak{h})} \|z - \zeta\|_{\mathfrak{h}_\omega}, \]
which implies that $\mathcal{V}_k$ is dense in $\text{Ran}(z \mapsto \mathcal{V}_{\varepsilon})$ w.r.t. the $\mathcal{B}(L^2(\mathbb{R}^{dN}))$-norm topology.

### 4.2. The polaron model.

The polaron model, introduced in [Fro37], describes $N$ electrons (spinless for simplicity) subjected to the vibrational (phonon) field of a lattice. This model is similar to Nelson’s, however the coupling is slightly more singular. The one-excitation space is $\mathfrak{h} = L^2(\mathbb{R}^d)$, while the form factor is given by (1.9): the quasi-classical energy has the same form as in the Nelson model, as well as the effective potential $\mathcal{V}_{\varepsilon}$ (see (1.16)), although now
\[ \lambda(\mathbf{x}; \mathbf{k}) = \sqrt{\alpha} e^{-\beta k^2/2}, \quad \omega = 1, \]
where $\alpha > 0$ is a constant measuring the coupling’s strength. The assumptions on $K_0 = -\Delta + W$ are the same as in the Nelson model. Let us remark that in this case since $\omega = 1$, $\mathfrak{h}_\omega = \mathfrak{h}$.

The key difference with the aforementioned Nelson model is thus that $\exists z \in \mathfrak{h}$ such that

$$V_z(\cdot) \notin L^\infty(\mathbb{R}^d),$$

due to the fact that $\lambda \notin L^\infty(\mathbb{R}^d; \mathfrak{h})$. However, it is possible to write $V_z$ as the sum of an $L^\infty$ function and the commutator between an $L^\infty$ vector function and the momentum operator $-i\nabla_x$:

$$V_z(x) = \sqrt{\alpha} (V_{<,z}(x) + [-i\nabla_x, V_{>,z}(x)]), \quad (4.12)$$

where

$$V_{<,z}(x) = 2\text{Re} \mathcal{F}^{-1} [\lambda_{<}z](x), \quad \lambda_{<}(k) := 1_{[|k| \leq \epsilon]} |k|^{-\frac{d+1}{2}},$$

$$V_{>,z}(x) = 2\text{Re} \mathcal{F}^{-1} [\lambda_{>}z](x), \quad \lambda_{>}(k) := 1_{[|k| > \epsilon]} |k|^{-\frac{d+1}{2}}k,$$

where $k := \frac{\epsilon}{|k|}$ and $\mathcal{F}$ stands for the Fourier transform in $\mathbb{R}^d$. Note that, for any $\rho > 0$, $\lambda_{<} \in \mathfrak{h}$ and $\lambda_{>} \in \mathfrak{h} \otimes \mathbb{C}^d$. By KMLN theorem, it then follows that $H_z$ is self-adjoint and bounded from below for all $z \in \mathfrak{h}$, with $z$-independent form domain $\mathcal{D}(H_z) = \mathcal{D}(K_0)$. Let us remark that, choosing $\rho$ suitably large (independent of $z$) in the above decomposition, it is possible to make the operator $H_z$ bounded from below uniformly w.r.t. $z \in \mathfrak{h}$ (see, e.g., [LF98, Prop. 3.21]).

The quasi-classical Wick quantization of $H_z$ formally yields the same expression as in the Nelson model (with $\omega = 1$ and $\lambda$ as above). Such a formal operator gives rise to a closed and bounded from below quadratic form, via the decomposition (4.12) (this can also be proved by KLMN theorem, choosing $\rho$ sufficiently large (see, e.g., [LT97, PS14])). We still denote the corresponding self-adjoint operator by $H_z$ with a little abuse of notation. The polaron Hamiltonian $H_\varepsilon$ has a ground state, if $-\Delta + W$ has compact resolvent by an application of the HVZ theorem analogous to the one for the Nelson model (see the aforementioned result in [DG99]). It is known that ground states exist also for non-confining but suitably regular external potentials $W$.

**Proof of Proposition 3.1.** These lower and upper bounds are well-known (see, e.g., [LT97, CF18]). The lower bound is a direct consequence of KLMN theorem, while the upper bound is proved using coherent states for the field in a fashion that is completely analogous to the one discussed for the Nelson model. Thus here we focus on the lower bound.

Let us introduce the unperturbed operator $H_{\text{free}} = K_0 \otimes 1 + 1 \otimes \text{d}G_\varepsilon(1)$, as in the Nelson model. Then, for any $\Psi_\varepsilon \in \mathcal{D}(H_{\text{free}})$, for all $\rho > 0$, and for all $\beta > 0$, we can bound the interaction term in the polaron quadratic form via

$$\left| \left\langle \Psi_\varepsilon \left| \text{Op}_\varepsilon \text{Wick} (V_{<,z}(x)) - i\nabla_x \cdot \text{Op}_\varepsilon \text{Wick} (V_{>,z}(x)) + i\text{Op}_\varepsilon \text{Wick} (V_{>,z}(x)) \cdot \nabla_x \right| \Psi_\varepsilon \right\rangle \right|_{\mathcal{H}_\varepsilon}$$

$$\leq 2 \| \lambda_{<} \|_\mathfrak{h} \left| \left\langle \Psi_\varepsilon \left| H^\frac{1}{2}_{\text{free}} \right| \Psi_\varepsilon \right\rangle \right|_{\mathcal{H}_\varepsilon} + 4 \| \lambda_{>} \|_\mathfrak{h} \left| \left\langle \Psi_\varepsilon \right| H_{\text{free}} \left| \Psi_\varepsilon \right\rangle \right|_{\mathcal{H}_\varepsilon}$$

$$\leq \frac{1}{\beta} \| \lambda_{<} \|_\mathfrak{h}^2 \| \Psi_\varepsilon \|_{\mathcal{H}_\varepsilon}^2 + \left( \beta + 4 \| \lambda_{>} \|_\mathfrak{h} \right) \left| \left\langle \Psi_\varepsilon \right| H_{\text{free}} \left| \Psi_\varepsilon \right\rangle \right|_{\mathcal{H}_\varepsilon}. \quad (4.13)$$
Obviously, the norms of $\lambda_<$ and $\lambda_>$ depend on $\varrho$. However, since the norm of $\lambda_>$ diverges as $\varrho \to 0$ and vanishes as $\varrho \to +\infty$, we can always choose $\varrho = \varrho(\beta)$, such that

$$4 \| \lambda_\|_h = \beta.$$  \hspace{1cm} (4.14)

Hence, we can bound

$$| \langle \Psi_\varepsilon | H_1 | \Psi_\varepsilon \rangle_{\mathcal{H}_d} | \leq \sqrt{\alpha} N \left[ 2\beta \langle \Psi_\varepsilon | H_{\text{free}} | \Psi_\varepsilon \rangle_{\mathcal{H}_d} + \frac{1}{\beta} \| \lambda_\|_h^2 \| \Psi_\varepsilon \|_{\mathcal{H}_d}^2 \right],$$

so that, taking $\beta = (2\sqrt{\alpha} N)^{-1}$, we conclude that

$$E_\varepsilon \geq -2\alpha N^2 \| \lambda_\|_h^2,$$  \hspace{1cm} (4.15)

where the last norm is evaluated at $\varrho((2\sqrt{\alpha} N)^{-1})$.

Let us now prove Lemmas 3.7 and 3.8. The assumption in the former takes the following simplified form for the polaron model: assuming that there exists a finite constant $C$, such that

$$\left| \langle \Psi_\varepsilon | (K_0 + dG_\varepsilon(1)^2 + 1) | \Psi_\varepsilon \rangle_{\mathcal{H}_d} \right| \leq C,$$  \hspace{1cm} (4.16)

then the convergence (3.18) holds true for any limit point in $\mathcal{V} (\Psi_\varepsilon, \varepsilon \in (0, 1))$.

**Proof of Lemma 3.7.** Using again the splitting (4.12), we immediately see that the term involving the quantization of $\mathcal{V}_{<,\varepsilon}$ converges by Lemma 4.1. Let us consider then the other term. Analogously to the proof of Lemma 4.1, we define compact approximate identities $\{1_m\}_{m \in \mathbb{N}} \subset L^\infty(L^2(\mathbb{R}^d))$ as $1_m := I_{[-m,m]}(K_0)$.

We can now rewrite explicitly the term involving the quantization of $\mathcal{V}_{>,\varepsilon}$, by introducing $\xi \in L^\infty(\mathbb{R}^d)$ given by

$$\xi(x; k) := \lambda_> e^{-ik \cdot x},$$  \hspace{1cm} (4.17)

as

$$\sqrt{\alpha} \sum_{j=1}^{N} \left| \langle \Psi_{\varepsilon n} | -i \nabla_j, \text{Op}(Wick) (\mathcal{V}_>(x_j)) | \Psi_{\varepsilon n} \rangle_{\mathcal{H}_n} \right| = 2\sqrt{\alpha} \sum_{j=1}^{N} \text{Re} \left| \langle -i \nabla_j \Psi_{\varepsilon n} | [a^\dagger_{\varepsilon n}(\xi(x_j)) + a_{\varepsilon n}(\xi(x_j))] \Psi_{\varepsilon n} \rangle_{\mathcal{H}_n} \right|. \hspace{1cm} (4.18)$$

In order to prove its convergence, we estimate

$$\left| \langle -i \nabla_j \Psi_{\varepsilon n} | [a^\dagger_{\varepsilon n}(\xi(x_j)) + a_{\varepsilon n}(\xi(x_j))] \Psi_{\varepsilon n} \rangle_{\mathcal{H}_n} \right| \leq \left| \langle -i \nabla_j \Psi_{\varepsilon n} | [a^\dagger_{\varepsilon n}(\xi(x_j)) + a_{\varepsilon n}(\xi(x_j))] 1_m \Psi_{\varepsilon n} \rangle_{\mathcal{H}_n} \right| + \left| \langle -i \nabla_j \Psi_{\varepsilon n} | [a^\dagger_{\varepsilon n}(\xi(x_j)) + a_{\varepsilon n}(\xi(x_j))] (1 - 1_m) \Psi_{\varepsilon n} \rangle_{\mathcal{H}_n} \right|. \hspace{1cm} (4.19)$$

The first term on the r.h.s. converges, when $n \to +\infty$ and $m \in \mathbb{N}$ is fixed, thanks to [CFO19], Prop. 7.1; then, a dominated convergence argument allows to take the limit $m \to +\infty$, yielding the sought result. It remains therefore to prove that the second term on the
Thus, for all $\beta > 0$, the proof for the Nelson model and it does not depend on the model at hand. We omit further details.

Proof of the formula extends to the polaron model immediately (see [Oli20] for additional renormalized Nelson model “contains” all type of terms appearing in the polaron model, the Kato-Rellich inequality for the Nelson model. The fact that there exists a minimizing sequence

where we have chosen $\beta = \beta_m := \| (1 - 1_m)(K_0 + c_0)^{-\frac{1}{2}} \|_\beta = \sup_{\eta \in [(-\infty, -m) \cup (m, +\infty)] \cap \sigma(K_0)} \frac{1}{\eta + c_0} \to 0$.

Since the r.h.s. of (4.22) is independent of $\varepsilon$ and converges to zero as $m \to +\infty$, the result is proven.

Proof of [Lemma 3.8] The proof is analogous to the one for the Nelson model. The expectation of the number operator squared is bounded via the pull-through formula by means of the expectation of $H^2$. As discussed in [CFO19b], the pull-through formula was originally proved for the renormalized Nelson Hamiltonian with a bound that is $\varepsilon$-dependent in [Amm00]; the uniformity of such bound with respect to $\varepsilon \in (0, 1)$ has been proved in [AF17]. Since the renormalized Nelson model “contains” all type of terms appearing in the polaron model, the proof of the formula extends to the polaron model immediately (see [Oli20] for additional details).

The pull-through formula reads as follows: there exists a finite constant $C$ (independent of $\varepsilon$), such that

\[ \langle \Psi_\varepsilon | dG_\varepsilon(1)^2 | \Psi_\varepsilon \rangle_{\mathcal{H}_\varepsilon} \leq C \langle \Psi_\varepsilon | (H_\varepsilon + 1)^2 | \Psi_\varepsilon \rangle_{\mathcal{H}_\varepsilon}. \]  

(4.23)

The expectation of $H_{\text{free}}$ is bounded by means of the expectation of $H_\varepsilon$, using the KMLN inequality, already discussed in the proof of [Proposition 3.1] in the very same way we used Kato-Rellich inequality for the Nelson model. The fact that there exists a minimizing sequence such that the expectation of $H^2$ is bounded uniformly w.r.t. $\varepsilon \in (0, 1)$ is also discussed in the proof for the Nelson model and it does not depend on the model at hand. We omit further details for the sake of brevity.

It remains only to prove [Lemma 3.11] for non-trapping potentials.
Proof of Lemma 3.11. The proof is done using the following fact: if \( \Psi_\varepsilon \) is such that there exists \( \delta \geq 1 \) and a finite constant \( C \), such that
\[
\left| \left\langle \Psi_\varepsilon \left| \left( K_0 + dG_\varepsilon(1)^\delta + 1 \right) \right| \Psi_\varepsilon \right\rangle_{\mathcal{F}_\varepsilon} \right| \leq C ,
\]
then, if \( n \in \mathcal{W} \left( \Psi_\varepsilon, \varepsilon \in (0, 1) \right) \) and \( \Psi_\varepsilon \overset{\text{gqc}}{\xrightarrow{n \to +\infty}} n \), one has that (3.31) holds true.

Such a result is proved by a combination of [CFO19b] Props. 2.6 & 7.1, if in these propositions Wigner measures are substituted by generalized Wigner measures and the test with compact operators of the small system is substituted by the test with the identity operator. The proof given there is generalized to this setting straightforwardly, recalling the properties of generalized Wigner measures outlined in Appendix A.

As in the proof for the Nelson model, let us check explicitly that \( \text{Ran}(z \mapsto \nu_z) \) is separable in the norm operator topology.\(^9\) By using the decomposition (4.12), we see that the part containing \( \nu_{<,z} \) has separable range, since it is equivalent to the one appearing in the Nelson model. Let us focus then on the remaining one containing the expectation of the operator \([-i\nabla_j, \nu_{>,z}]\). Such an operator is not bounded. Nonetheless, it is \( n_{\Gamma} \)-integrable with \( \Gamma = K_0 + 1 \) by [Lemma 3.10], provided that
\[
\mathfrak{h} \ni z \mapsto \sum_{j=1}^N T^{-\frac{1}{2}} \left[ -i\nabla_j, \nu_{>,z}(x_j) \right] T^{-\frac{1}{2}} \in \mathcal{B}(L^2(\mathbb{R}^{dN}))
\]
has separable range. Since \( \mathfrak{h} \) is separable, let us denote by \( \mathfrak{t} \subset \mathfrak{h} \) a countable dense subset and denote by
\[
T^{-\frac{1}{2}} \tilde{\nu}_t T^{-\frac{1}{2}} := \left\{ \sum_j T^{-\frac{1}{2}} \left[ -i\nabla_j, \nu_{>,z}(x_j) \right] T^{-\frac{1}{2}} \in \mathcal{B}(L^2(\mathbb{R}^{dN})), \zeta \in \mathfrak{t} \right\}
\]
the image of \( \mathfrak{t} \) through \( T^{-\frac{1}{2}} \sum_j [-i\nabla_j, \nu_{>,z}(x_j)] T^{-\frac{1}{2}} \). Now, for any \( z \in \mathfrak{h}, \zeta \in \mathfrak{t} \) and \( j = 1, \ldots, N \), we have that (recall (4.11))
\[
\left\| T^{-\frac{1}{2}} \left[ -i\nabla_j, \nu_{>,z}(x_j) \right] T^{-\frac{1}{2}} - T^{-\frac{1}{2}} \left[ -i\nabla_j, \nu_{>,\zeta}(x_j) \right] T^{-\frac{1}{2}} \right\|_{\mathcal{B}(L^2(\mathbb{R}^{dN}))} \leq 4 \| \zeta \|_{L^\infty} \| z - \zeta \|_{\mathfrak{h}} ,
\]
which implies that \( T^{-\frac{1}{2}} \tilde{\nu}_t T^{-\frac{1}{2}} \) is dense in the image of the map \( \mathfrak{t} \) w.r.t. the norm topology in \( \mathcal{B}(L^2(\mathbb{R}^{dN})) \).

4.3. The Pauli-Fierz model. The Pauli-Fierz model describes \( N \) spinless charges (with an extended and sufficiently smooth charge distribution) interacting with the electromagnetic field in the Coulomb gauge, in three dimensions. Generalizations to other gauges, to particles with spin or to two dimensions are possible without much effort. The one-excitation Hilbert space is thus \( \mathfrak{h} = L^2(\mathbb{R}^3; \mathbb{C}^2) \). Let the charge density of each particle be given by \( \lambda_j(x) \), with \( \lambda_j \in L^\infty(\mathbb{R}^3; L^2(\mathbb{R}^3)) \), \( j = 1, \ldots, N \), such that \( -i\nabla_j \lambda_j(x; k) = k \lambda_j(x; k) \) and let the polarization vectors be denoted \( e_p \in L^\infty(\mathbb{R}^3; \mathbb{R}^3), p = 1, 2 \), such that for a.e. \( k \in \mathbb{R}^3 \), \( e_p(k) \cdot e_{p'}(k) = \delta_{pp'} \), \( k \cdot e_p(k) = 0 \) (Coulomb gauge). The quasi-classical energy functional is then given by (1.18), i.e.
\[
\mathcal{H}_z = \sum_{j=1}^N \frac{1}{2m_j} \left( -i\nabla_j + a_j + (x_j) \right)^2 + W(X) + \langle z | \omega | z \rangle \theta
\]
\(^9\)More precisely, we prove that \( \text{Ran}(z \mapsto (K_0 + 1)^{-\frac{1}{2}} \nu_z(K_0 + 1)^{-\frac{1}{2}}) \) has separable range. This is sufficient to prove that \( \nu_z \) is integrable w.r.t. \( n \), since the latter is in the domain of \( K_0 + 1 \).
\(^{10}\)W.l.o.g. we fix the charge \( e = 1 \) since it does not play any relevant role in these arguments.
where the classical field is

$$a_{z,j}(x) = 2\text{Re} \langle z | \lambda_j(x) \rangle_h = 2\text{Re} \sum_{p=1}^{2} \langle z_p | \lambda_j(x) e_p \rangle L^2(\mathbb{R}^3) \in \mathbb{C}^3$$

and, as usual, $W$ is an external positive potential acting on the particles. Note that the field free energy reads

$$\langle z | \omega | z \rangle_h = 2\sum_{p=1}^{2} \langle z_p | \omega | z_p \rangle L^2(\mathbb{R}^3).$$

The operator $H_z$ is self-adjoint for all $z \in \mathbb{C}$, with domain of self-adjointness $D(K_0)$, where we recall that $K_0 = -\Delta + W$, where in this case we adopt the notation $-\Delta = \sum_{j=1}^{N} -\Delta_j / 2m_j$.

The quasi-classical Wick quantization of $H_z$ yields the Pauli-Fierz Hamiltonian in (1.10):

$$H_\varepsilon = \sum_{j=1}^{N} \frac{1}{2m_j} \left( -i \nabla_j + A_{\varepsilon,j}(x_j) \right)^2 + \mathcal{W}(x_1, \ldots, x_N) + 1 \otimes dG_\varepsilon(\omega),$$

where

$$A_{\varepsilon,j}(x) = \langle z | \lambda_j(x) \rangle + \langle z | \lambda_j(x) \rangle = \sum_{p=1}^{2} \left( \langle z_p | \lambda_j(x) e_p \rangle + \langle z_p | \lambda_j(x) e_p \rangle \right)$$

is the quantized magnetic potential. The Pauli-Fierz Hamiltonian has a ground state for suitable choices of the potential $W$, e.g., if it is the sum of single particle and pair potentials with suitable properties (clustering, binding, etc.) (see, e.g., [AHH99, Ger00, GLL01, Hir01] and references therein). In particular, this holds true when the field is massive [Ger00], i.e., for $\omega > 0$. As for the other models, we refrain from giving a detailed description of the conditions allowing to have a ground state, since for our purposes it is sufficient that a ground state do exist in some cases.

**Proof of Proposition 3.1.** The lower bound follows from the diamagnetic inequality [Mat17]:

$$\langle \Psi_{\varepsilon} | -\Delta_j | \Psi_{\varepsilon} \rangle_{\mathcal{H}_\varepsilon} \leq \left| \langle \Psi_{\varepsilon} | (-i \nabla_j + A_{\varepsilon,j}(x_j))^2 | \Psi_{\varepsilon} \rangle_{\mathcal{H}_\varepsilon} \right|,$$

which in particular implies that $H_\varepsilon$ is positive. The upper bound is proved using coherent states for the field, analogously to the Nelson model and the polaron. $\dashv$

Let us now prove Lemmas 3.7 and 3.8 for the Pauli-Fierz model. The former takes the following form.

**Proof of Lemma 3.7.** The “potential” (1.17) is composed of two parts:

$$V_\varepsilon(x) = 2 \sum_{j=1}^{N} \frac{1}{m_j} \left[ -i \text{Re} \langle z | \lambda_j(x) \rangle_h \cdot \nabla_j + \left( \text{Re} \langle z | \lambda_j(x) \rangle_h \right)^2 \right]$$

as well as its Wick quantization. The convergence of the quantization of the second term is perfectly analogous to the one given for the Nelson model in Lemma 4.1. The proof of
convergence for the quantization of the term involving the gradient is given in the proof of Lemma 3.7 for the polaron.

Proof of Lemma 3.8. The proof follows from the following estimate, due to F. Hiroshima, and whose detailed proof will be given in [AFH20]. There exists a finite constant \( C > 0 \) such that, for all \( \Psi \in \mathcal{D}(H_{\text{free}}) \),

\[
\| H_{\text{free}} \Psi \|_{\mathcal{H}_0} \leq C \| H_\varepsilon \Psi \|_{\mathcal{H}_\varepsilon} \ .
\] (4.27)

Let us remark that the expectation of \( K_0 = -\sum_j \frac{1}{2m_j} \Delta_j + W \) could also be bounded by means of the expectation of \( H_\varepsilon \) using the diamagnetic inequality (4.26). Hence if \( \omega > 0 \), (3.19) could be proved combining the diamagnetic inequality and the pull-through formula (4.23).

Finally, the fact that there exists a minimizing sequence such that the expectation of \( H^2 \) is bounded uniformly w.r.t. \( \varepsilon \in (0,1) \) is also discussed in the proof of Lemma 3.8 for the Nelson model.

It remains only to prove Lemma 3.11 for non-trapped systems.

Proof of Lemma 3.11. The proof here is obtained combining the proofs given for the Nelson and polaron models. In fact, the quadratic terms can be treated exactly as the linear terms in the Nelson model and the gradient terms are equivalent to the ones appearing in the polaron.

Appendix A. Algebraic State-Valued Measures

The quasi-classical Wigner measures are state-valued by construction [Fal18b, CFO19b]. In other words, quasi-classical measures are countably additive (in a sense to be clarified below) measures on the measurable phase space of classical fields, taking values in quantum states, or, more generally, in the Banach cone \( \mathfrak{A}'_+ \) of positive elements in the dual of a C*-algebra \( \mathfrak{A} \). In addition, the quasi-classical symbols are measurable functions from the phase space to a W*-algebra \( \mathfrak{B} \supseteq \mathfrak{A} \) of observables (operators), where \( \mathfrak{A} \) is supposed to be an ideal of \( \mathfrak{B} \). It is therefore necessary to properly define integration of operator-valued symbols w.r.t. a state-valued measure. In this appendix we collect some technical properties of state-valued measures and integration, from a general algebraic standpoint that includes both state-valued and generalized state-valued measures, as used throughout the paper. The ideas developed here in great generality are particularly suited for what we called generalized state-valued measures, and they are mostly taken from [Bar56] and [Nee98]. In fact, if state-valued measures have been already studied in semiclassical analysis and adiabatic theories (see [Bal85, FG02, Gér91, GMS91, Ten03] and references therein contained), the reader might not be so familiar with generalized state-valued measures. Since for the latter there is no Radon-Nikodým property, their description is more abstract, and there are some limitations, especially concerning integration of operator-valued functions. This justifies the abstract approach followed in this appendix.

A.1. Algebraic State-Valued Measures. Let \( \mathfrak{A} \) be a C*-algebra and denote by \( \mathfrak{A}'_+ \) the cone of positive elements in the dual of \( \mathfrak{A} \). In addition, let \( (X, \Sigma) \) be a measurable space. There are two equivalent ways of defining an \( \mathfrak{A}'_+ \)-valued measure on \( (X, \Sigma) \).

Definition A.1 (State-valued measure [Nee98]). A family of real-valued measures \( (\mu_A)_{A \in \mathfrak{A}_+} \) defines a weak-\( \sigma \)-additive measure \( m : \Sigma \to \mathfrak{A}'_+ \)
as
\[ [m(S)] (A_1 - A_2 + iA_3 - iA_4) = \mu_{A_1}(S) - \mu_{A_2}(S) + i\mu_{A_3}(S) - i\mu_{A_4}(S), \]
for any \( S \in \Sigma \) and \( A_1, A_2, A_3, A_4 \in \mathfrak{A}_+ \), iff for any \( A, B \in \mathfrak{A}_+ \) and \( \lambda \in \mathbb{R}_+ \), \( \mu_{\lambda A+B} = \lambda \mu_A + \mu_B \).

**Definition A.2** (Algebraic state-valued measure [Bar56]).
An application \( m : \Sigma \to \mathfrak{A}'_+ \) is a measure iff \( m(\emptyset) = 0 \), and for any family \( (S_n)_{n \in \mathbb{N}} \subset \Sigma \) of mutually disjoint measurable sets,
\[ m \left( \bigcup_{n \in \mathbb{N}} S_n \right) = \sum_{n \in \mathbb{N}} m(S_n), \]
where the r.h.s. converges unconditionally in the norm of \( \mathfrak{A}' \).

It is clear that any \( m \) satisfying [Definition A.2] satisfies also [Definition A.1] since \( \sigma \)-additivity in norm implies weak-* \( \sigma \)-additivity. The converse, i.e., that a \( m \) satisfying [Definition A.1] also satisfies [Definition A.2] is nontrivial, and follows from properties of uniform boundedness in Banach spaces, as proved by [Dun38, Chapter II]. We use these two definitions interchangeably, depending on the context. Let us remark that with the definitions above, any state-valued measure is automatically finite, since \( m(X) \in \mathfrak{A}'_+ \). Actually, in the main body of the paper, we consider probability measures, i.e., \( \|m(X)\|_{\mathfrak{A}'} = 1 \).

**Remark A.3** (State-valued and generalized state-valued measures).
The state-valued measures used in the paper correspond to choosing \( \mathfrak{A} = L^\infty(\mathcal{H}) \); generalized state-valued measures are in a subset of the measures obtained by picking \( \mathfrak{A} = L^1(\mathcal{H}) \).

For algebraic state-valued (cylindrical) measures on vector spaces, Bochner’s theorem holds, and the Fourier transforms are completely positive maps that are weak-* continuous when restricted to any finite-dimensional subspace (see [Fal18b] for additional details). An algebraic state-valued measure is also monotone:

**Lemma A.4.**
For any \( S_1 \subseteq S_2 \in \Sigma \),
\[ m(S_1) \leq m(S_2), \]
\[ i.e., m(S_2) - m(S_1) \in \mathfrak{A}'_+. \]

**Proof.** The scalar measures \( \mu_A, A \in \mathfrak{A}_+ \), are monotonic. Therefore, for all \( A \in \mathfrak{A}_+ \),
\[ [m(S_2)](A) := \mu_A(S_2) \geq \mu_A(S_1) =: [m(S_1)](A). \] (A.1)
Hence, for all \( A \in \mathfrak{A}_+ \),
\[ [m(S_2) - m(S_1)](A) \geq 0. \]

We can now introduce the scalar norm measure \( m \), satisfying \( \mu_A(S) \leq \|A\|_\mathfrak{A} m(S) \), for any \( S \in \Sigma \), that proves to be a very useful tool to compare vector integrals with scalar integrals.

**Definition A.5** (Norm measure).
Let \( m \) be an algebraic state-valued measure. Then, its norm measure \( m : \Sigma \to \mathbb{R}_+ \) is defined as
\[ m(S) := \|m(S)\|_{\mathfrak{A}'} , \] (A.2)
for any measurable set $S$.

Using the cone properties of positive states in a C*-algebra, it is possible to prove that $m$ is a finite measure. Let us recall that the C*-algebra $\mathfrak{A}$ may not be unital, so from now on we assume that there exists a W*-algebra $\mathfrak{B} \supseteq \mathfrak{A}$. If $\mathfrak{A} = \mathcal{L}^\infty(\mathcal{X})$, the compact operators on a separable Hilbert space $\mathcal{X}$, and $\mathfrak{B} = \mathcal{B}(\mathcal{X})$, it is well-known that the aforementioned property is satisfied: $\mathfrak{A}$ is actually in this case a two-sided ideal of $\mathfrak{B}$. Let us denote by $e \in \mathfrak{B}$ the identity element.

**Proposition A.6** (Properties of the norm measure). Let $m$ be an algebraic state-valued measure. Then, its norm measure $m$ is a finite measure on $(X, \Sigma)$ and $m \ll m$.

**Proof.** The proof that $m(\emptyset) = 0$ and $m(X) < +\infty$ follows immediately from the definition, while $\sigma$-additivity is proved as follows: let $(S_n)_{n \in \mathbb{N}} \subset \Sigma$ be a family of mutually disjoint measurable sets, we are going to prove that, for any $N \in \mathbb{N}$,

$$m \left( \bigcup_{n=1}^{N} S_n \right) = \sum_{n=1}^{N} m(S_n) \quad \text{(A.3)}$$

Indeed, let $(e_\alpha)_{\alpha \in I} \subset \mathfrak{A}_+$ be an approximate identity of $e \in \mathfrak{B}$. It is well-known that for any $\omega \in \mathfrak{A}^{' +}$, $\|\omega\|_{\mathfrak{B}} = \lim_{\alpha \in I} \omega(w_\alpha)$. Hence, by Definition A.1 and Definition A.5

$$m \left( \bigcup_{n=1}^{N} S_n \right) = \lim_{\alpha \in I} m \left( \bigcup_{n=1}^{N} S_n \right) (e_\alpha) = \lim_{\alpha \in I} \mu_{e_\alpha} \left( \bigcup_{n=1}^{N} S_n \right) = \lim_{\alpha \in I} \sum_{n=1}^{N} \mu_{e_\alpha}(S_n) = \sum_{n=1}^{N} m(S_n) .$$

Next, we show

$$\lim_{N \to \infty} m \left( \bigcup_{n=1}^{N} S_n \right) - \sum_{n=1}^{N} m(S_n) = 0 , \quad \text{(A.4)}$$

which directly implies $\sigma$-additivity: using again the approximate identity on the left hand side, we obtain

$$\lim_{N \to \infty} \lim_{\alpha \in I} m \left( \bigcup_{n=1}^{N} S_n \right) - \sum_{n=1}^{N} \mu_{e_\alpha}(S_n) .$$

We know that every $\mu_{e_\alpha}, \alpha \in I,$ is $\sigma$-additive, and therefore that $\lim_{N \to \infty} \sum_{n=1}^{N} \mu_{e_\alpha}(S_n) = \mu_{e_\alpha} \left( \bigcup_{n=1}^{N} S_n \right)$, and $\lim_{\alpha \in I} \mu_{e_\alpha} \left( \bigcup_{n=1}^{N} S_n \right) = m \left( \bigcup_{n=1}^{N} S_n \right)$. Hence, it remains to show that the limits in $N$ and $\alpha$ can be exchanged. In order to do that, it is sufficient to show that the limit in $\alpha$ exists uniformly w.r.t. $N$:

$$\lim_{N \to \infty} \limsup_{\alpha \in I} \left| m \left( \bigcup_{n=1}^{N} S_n \right) - \sum_{n=1}^{N} \mu_{e_\alpha}(S_n) \right| = \lim_{N \to \infty} \limsup_{\alpha \in I} \left| m \left( \bigcup_{n=1}^{N} S_n \right) - \mu_{e_\alpha} \left( \bigcup_{n=1}^{N} S_n \right) \right|$$

$$= \limsup_{N \in \mathbb{N}} \left( m - \mu_{e_\alpha} \right) \left( \bigcup_{n=1}^{N} S_n \right) \leq \limsup_{\alpha \in I} \left( m - \mu_{e_\alpha} \right) (X) = 0 , \quad \text{(A.5)}$$

where we have used finite additivity of $m$ and $m - \mu_{e_\alpha}$ and the fact that for any $S \in \Sigma$, $\mu_{e_\alpha}(S) \leq m(S)$.

It remains to prove that $m$ is absolutely continuous w.r.t. $m$. For absolute continuity of a vector measure with respect to a scalar one, we adopt the definition of [DU77] Section I.2, Definition 3]. Since both $m$ and $m$ are countably additive, it is sufficient to prove that, for
any $S \in \Sigma$, $m(S) = 0$ implies $m(S) = 0$. However, since $m(S) = \|m(S)\|_{\mathcal{M}}$, and $\| \cdot \|_{\mathcal{M}}$ is a norm, then the aforementioned implication follows directly by the properties of norms.

### A.2. Integration of Scalar Functions.

The theory of integration for algebraic state-valued measures could be done in a unified way for scalar- and operator-valued functions. However, it is instructive to deal with scalar functions first. Let us recall that a function $g : X \rightarrow \mathbb{R}^+$ is simple if there exist a number $N \in \mathbb{N}$, mutually disjoint measurable sets $S_1, \ldots, S_N \in \Sigma$ and non-negative numbers $c_1, \ldots, c_N \in \mathbb{R}^+$, such that for all $x \in X$, $g(x) = \sum_{j=1}^{N} c_j \mathbb{1}_{S_j}(x)$, \hspace{1cm} (A.6)

where $\mathbb{1}_{S_j}$ is the characteristic function of the set $S_j$. Integration of simple functions w.r.t. an algebraic state-valued measure $\mu$ is straightforwardly defined as

$$\int_X \text{d}m(x) \ g(x) = \sum_{j=1}^{N} c_j m(S_j) \in \mathfrak{A}_+.$$ \hspace{1cm} (A.7)

The integral of a non-simple function can be defined again in two equivalent ways:

**Definition A.7 (Integrability I [Nee98, Lemma I.12]).**

A measurable function $f : X \rightarrow \mathbb{R}^+$ is m-integrable iff $f$ is $\mu_A$-integrable for any $A \in \mathfrak{A}_+$. Furthermore, its integral belongs to $\mathfrak{A}_+$ and is uniquely defined by the integral w.r.t. $\mu_A$, i.e.,

$$\left( \int_S \text{d}m(x) \ f(x) \right) (A_1 - A_2 + iA_3 - iA_4) = \int_S \text{d}\mu_{A_1}(x) \ f(x) - \int_S \text{d}\mu_{A_2}(x) \ f(x)$$

$$+ i \int_S \text{d}\mu_{A_3}(x) \ f(x) - i \int_S \text{d}\mu_{A_4}(x) \ f(x).$$ \hspace{1cm} (A.8)

for any $A_1, A_2, A_3, A_4 \in \mathfrak{A}_+$.

**Definition A.8 (Integrability II [Bar56, Definition 1]).**

A measurable function $f : X \rightarrow \mathbb{R}^+$ is m-integrable iff for any $S \in \Sigma$ the sequence of simple integrals

$$\left\{ \int_X \text{d}m(x) f_n(x) \mathbb{1}_S(x) \right\}_{n \in \mathbb{N}} \in \mathfrak{A}_',$$

where $(f_n)_{n \in \mathbb{N}}$ is any approximation of $f$ in terms of simple functions, is Cauchy. The integral is then defined as

$$\int_S \text{d}m(x) f(x) = \lim_{n \rightarrow \infty} \int_X \text{d}m(x) f_n(x) \mathbb{1}_S(x),$$ \hspace{1cm} (A.9)

and it is independent of the chosen approximation.

In both cases one says that a complex function $f : X \rightarrow \mathbb{C}$ is $\mu$-integrable if and only if $|f|$ is m-integrable and, in this case, its integral is given by the complex combination of the integrals of its real positive, real negative, imaginary positive and imaginary negative parts.

Since the weak-\* and strong limits coincide if they both exist, it follows that the integrals of a function that is m-integrable w.r.t. [Definition A.7] and [Definition A.8] coincide. In addition, if $f$ is m-integrable in the “strong” sense of [Definition A.8] then it is also m-integrable in the weak-\* sense of [Definition A.7]. It remains to show that if $f$ is m-integrable in the sense of [Definition A.7] then it is m-integrable in the sense of [Definition A.8] but this can be done exploiting the norm measure $m$. 

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**Integrability I** [Nee98, Lemma I.12]

**Integrability II** [Bar56, Definition 1]
Lemma A.9.
If a measurable function \( f: X \to \mathbb{R}^+ \) is \( m \)-integrable in the sense of Definition A.7, then it is \( m \)-integrable as well.

\[ \text{Proof.} \] If \( f \) is \( m \)-integrable, then for any \( S \in \Sigma \), \( \int_S d\mu(x) f(x) \) is finite and non-negative for any \( A \in \mathcal{A}_+ \). Applying \cite{Nee98} Lemma I.5, we deduce that there exists a finite constant \( C \), depending only on \( S \), \( m \), and \( f \), such that
\[
\int_S d\mu(x) f(x) \leq C \| A \|_{\mathbb{R}}.
\]
(A.10)

Now, let \( (f_n)_{n \in \mathbb{N}} \) be a simple pointwise non-decreasing approximation of \( f \) from below. Then, by monotone convergence theorem,
\[
\int_S \text{dm}(x) f(x) = \lim_{n \to \infty} \int_X \text{dm}(x) f_n(x) \mathbb{1}_S(x).
\]
Hence, by Definition A.5 and \( \mu_{e_n} \)-integrability of \( f \),
\[
\int_X \text{dm}(x) f_n(x) \mathbb{1}_S(x) = \lim_{\alpha \in \mathcal{I}} \int_X \text{dm}(e_{\alpha n}) f_n(x) \mathbb{1}_S(x) \leq \lim_{\alpha \in \mathcal{I}} \int_S \text{dm}(e_{\alpha n}) f(x) \leq C \lim_{\alpha \in \mathcal{I}} \| e_{\alpha} \|_{\mathbb{R}} \leq C,
\]
(A.11)
and taking the limit \( n \to +\infty \), we get the result. \( \dashv \)

Proposition A.10 (Equivalence of Definition A.7 and Definition A.8).
If a measurable function \( f: X \to \mathbb{R}^+ \) is \( m \)-integrable in the sense of Definition A.7, then it is \( m \)-integrable in the sense of Definition A.8. In addition, for any \( S \in \Sigma \),
\[ \left\| \int_S \text{dm}(x) f(x) \right\|_{\mathbb{R}} \leq \int_S \text{dm}(x) f(x). \]
(A.12)

\[ \text{Proof.} \] We prove that
\[ \left\{ \int_S \text{dm}(x) f_n(x) \right\}_{n \in \mathbb{N}} \in \mathcal{A}_+, \]
where \( (f_n)_{n \in \mathbb{N}} \) is a non-decreasing simple approximation of \( f \), is a Cauchy sequence. Observe that for any \( n \geq m \in \mathbb{N} \), \( f_n - f_m \) is a simple positive function, which can be written as
\[ f_n - f_m = \sum_{j=1}^{N(n,m)} c_j^{(n,m)} \mathbb{1}_{S_j^{(n,m)}}. \]
(A.13)

Hence,
\[
\left\| \int_S \text{dm}(x) (f_n(x) - f_m(x)) \right\|_{\mathbb{R}} \leq \sum_{j=1}^{N(n,m)} c_j^{(n,m)} m(S_j^{(n,m)} \cap S) \]
\[ = \int_S \text{dm}(x) (f_n - f_m)(x) \mathop{\rightarrow}_{n,m \to \infty} 0, \quad \text{A.14} \]
where in the last limit we have used the dominated convergence theorem, since \( f_n - f_m \leq 2f \), and \( f \) is \( m \)-integrable by Lemma A.9. This proves both \( m \)-integrability of \( f \) in the sense of Definition A.8 and the bound (A.12). \( \dashv \)

Therefore, the two definitions are indeed equivalent: Definition A.8 has the advantage of identifying constructively the integral as the limit of the integrals of simple approximations of
Lemma A.11.
Let \( f, g : X \to \mathbb{R} \) be two \( \mathfrak{m} \)-integrable functions. If for \( \mathfrak{m} \)-a.e. \( x \in X \) \( g(x) \leq f(x) \), then
\[
\int_X \mathfrak{m}(x) (f(x) - g(x)) \in \mathfrak{A}'_+.
\] (A.15)

Proof. The result follows from Definition A.7 and monotonicity of the usual integral \( \dashv \)

The dominanted convergence theorem holds in a general form (see Theorem A.17 and Theorem A.18 below), which in particular implies that it applies to scalar functions.

A.3. Integration of Operator-Valued Functions. The integration of operator-valued functions is defined similarly to Definition A.8. Let us discuss first the integration of simple operator-valued functions and the approximation with simple functions in this context.

An operator valued function \( g : X \to \mathfrak{B} \) is simple if there exist \( N \in \mathbb{N} \), mutually disjoint measurable sets \( S_1, \ldots, S_N \in \Sigma \), and \( c_1, \ldots, c_N \in \mathfrak{B} \) such that for all \( x \in X \),
\[
g(x) = \sum_{j=1}^{N} c_j 1_{S_j}(x) .
\] (A.16)

Let us recall that since \( \mathfrak{A} \subset \mathfrak{B} \), for any \( \omega \in \mathfrak{A}' \) and \( B \in \mathfrak{B} \), we can define \( \omega \circ B \in \mathfrak{A}' \) as
\[
(\omega \circ B)(\cdot) := \omega(\cdot B) \quad \text{or} \quad (\omega \circ B)(\cdot) := \omega(B \cdot) ,
\] (A.17)
depending on which side \( \mathfrak{A} \) is an ideal of \( \mathfrak{B} \). If it is a two-sided ideal, both definitions are equivalent. Keeping this definition in mind, we can define the integral of simple functions as
\[
\int_X \mathfrak{m}(x) g(x) = \sum_{j=1}^{N} \mathfrak{m}(S_j) \circ c_j \in \mathfrak{A}'.
\] (A.18)

Next, we recall hypotheses under which an operator-valued function admits a simple approximation.

Proposition A.12 (Simple approximation [Coh13, Proposition E.2]).
Let \( f : X \to \mathfrak{B} \) be a measurable function. If \( f(X) \) is separable, then \( f \) admits a simple approximation, i.e., there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) of simple functions such that for all \( x \in X \) and \( n \in \mathbb{N} \)
\[
\|f_n(x)\|_{\mathfrak{B}} \leq \|f(x)\|_{\mathfrak{B}} , \quad \lim_{n \to \infty} \|f(x) - f_n(x)\|_{\mathfrak{B}} = 0 .
\] (A.19)

Due to this result, in the following we only consider operator-valued functions with separable range, even if not stated explicitly.

Definition A.13 (Integrability III).
A measurable function with separable range \( f : X \to \mathfrak{B} \) is \( \mathfrak{m} \)-integrable iff, for any \( S \in \Sigma \), the sequence of simple integrals
\[
\left\{ \int_X \mathfrak{m}(x) f_n(x) 1_S(x) \right\}_{n \in \mathbb{N}} \in \mathfrak{A}' ,
\] (A.20)
where \( \{f_n\}_{n \in \mathbb{N}} \) is any approximation of \( f \) in terms simple functions, is Cauchy. The integral is then defined as

\[
\int_S \mathcal{d}m(x)f(x) = \lim_{n \to \infty} \int_S \mathcal{d}m(x)f_n(x)\mathbb{1}_S(x),
\]

and it is independent of the chosen approximation.

**Definition A.14** (Absolute integrability).

A measurable function with separable range \( f : X \to \mathcal{B} \) is \( \mathcal{m} \)-absolutely integrable iff \( \|f(\cdot)\|_{\mathcal{B}} \) is \( \mathcal{m} \)-integrable.

In fact, any \( \mathcal{m} \)-absolutely integrable function is also \( \mathcal{m} \)-integrable.

**Proposition A.15** (Integrability and absolute integrability).

Let \( f : X \to \mathcal{B} \) be a \( \mathcal{m} \)-absolutely integrable function. Then, \( f \) is also \( \mathcal{m} \)-integrable and, for all \( S \in \Sigma \),

\[
\left\| \int_S \mathcal{d}m(x)f(x) \right\|_{\mathcal{B}} \leq \int_S \mathcal{d}m(x)\|f(x)\|_{\mathcal{B}}.
\]

**Proof.** The proof is completely analogous to the proof of Proposition A.10. We omit it for the sake of brevity.

\( \square \)

**Corollary A.16** (Integrability of bounded functions).

Any function with separable range \( f : X \to \mathcal{B} \) such that \( \|f(\cdot)\|_{\mathcal{B}} \) is \( \mathcal{m} \)-a.e. uniformly bounded is \( \mathcal{m} \)-integrable.

We are now in a position to state two versions of the dominated convergence theorem for operator-valued functions. The second, that makes crucial use of absolute integrability, is the most convenient in our concrete applications. Note that both results easily applies to the special case of scalar functions discussed in the previous section.

**Theorem A.17** (Dominated convergence I [Bar56, Theorem 6]).

Let \( \{f_n\}_{n \in \mathbb{N}}, f_n : X \to \mathcal{B} \) for all \( n \in \mathbb{N} \), be a sequence of \( \mathcal{m} \)-integrable operator-valued functions strongly converging \( \mathcal{m} \)-a.e. to \( f : X \to \mathcal{B} \). If there exists a \( \mathcal{m} \)-integrable operator-valued function \( g \), such that for all \( n \in \mathbb{N} \) and \( S \in \Sigma \)

\[
\left\| \int_S \mathcal{d}m(x)f_n(x) \right\| \leq \left\| \int_S \mathcal{d}m(x)g(x) \right\|,
\]

then, \( f \) is \( \mathcal{m} \)-integrable and for any \( S \in \Sigma \)

\[
\int_S \mathcal{d}m(x)f(x) = \lim_{n \to \infty} \int_S \mathcal{d}m(x)f_n(x).
\]

**Theorem A.18** (Dominated convergence II).

Let \( \{f_n\}_{n \in \mathbb{N}}, f_n : X \to \mathcal{B} \) for all \( n \in \mathbb{N} \), be a sequence of operator-valued functions strongly converging \( \mu \)-a.e. to \( f : X \to \mathcal{B} \). If there exists a \( m \)-integrable function \( G : X \to \mathbb{R}^+ \) such that \( \mu \)-a.e.

\[
\|f_n(x)\|_{\mathcal{B}} \leq G(x),
\]

then, for any \( n \in \mathbb{N} \), \( f_n, f \) are \( \mathcal{m} \)-absolutely integrable, and

\[
\int_S \mathcal{d}m(x)f(x) = \lim_{n \to \infty} \int_S \mathcal{d}m(x)f_n(x).
\]

**Proof.** By dominated convergence theorem for scalar measures and functions, applied to \( m \) and \( \{\|f_n(\cdot)\|_{\mathcal{B}}\}_{n \in \mathbb{N}} \), respectively, we get that the \( \|f_n(\cdot)\|_{\mathcal{B}}, \|f(\cdot)\|_{\mathcal{B}} \) are both \( m \)-integrable.
and therefore, by Proposition A.15, it follows that \( f_n, f \) are also \( m \)-integrable. Now, for any \( S \in \Sigma \), again by Proposition A.15,
\[
\left\| \int_S dm(x)(f - f_n)(x) \right\|_{\mathcal{A}'} \leq \int_S dm(x) \| (f - f_n)(x) \|_{\mathcal{B}}.
\]

Hence by dominated convergence theorem for \( m \), applied to the sequence of scalar functions \( \{ \| (f - f_n)(x) \|_{\mathcal{B}} \}_{n \in \mathbb{N}} \), it follows that in the strong topology of \( \mathcal{A}' \),
\[
\int_S dm(x)f(x) = \lim_{n \to \infty} \int_S dm(x)f_n(x).
\]

### A.4. Integration of functions with values in unbounded operators.

Let us restrict the attention, for this section, to the concrete case \( \mathfrak{A} = \mathscr{B}(L^2(\mathbb{R}^d)) \). In the applications described above, it is sometimes necessary to integrate functions from some measurable space \( X \) to the unbounded operators on \( L^2(\mathbb{R}^d) \) (albeit with a rather explicit form). It is possible to define the integration of such functions with respect to suitable generalized state-valued measures, as already outlined in \[\S\ 2.2\]. Let us repeat here the argument for the sake of completeness.

Let \( T > 0 \) be an operator on \( L^2(\mathbb{R}^d) \), possibly unbounded. A generalized state-valued measure is in the domain of \( T \) iff there exists a generalized state-valued measure \( \eta_T \) such that for all \( \mathcal{B} \in \mathscr{B}(L^2(\mathbb{R}^d)) \), and for any \( S \in \Sigma \),
\[
\eta_T(S)\left[T^{-\frac{1}{2}}BT^{-\frac{1}{2}}\right] = \eta(S)[\mathcal{B}].
\]

Given a measure in the domain of \( T \), we can integrate functions singular “at most as \( T \”). Let \( \mathcal{F} \) be a function from \( X \) to the (closed and densely defined) operators on \( L^2(\mathbb{R}^d) \). Then \( \mathcal{F} \) is \( \eta \)-absolutely integrable, with \( \eta \) in the domain of \( T \), iff for \( \eta \)-a.e. \( x \in X \):

- \( T^{-\frac{1}{2}}\mathcal{F}(x)T^{-\frac{1}{2}} \in \mathscr{B}(L^2(\mathbb{R}^d)) \);
- \( T^{-\frac{1}{2}}\mathcal{F}(x)T^{-\frac{1}{2}} \) is \( \eta_T \)-absolutely integrable.

Given an absolutely integrable function, we can define the integral as follows: for any \( S \in \Sigma \),
\[
\int_S d\eta(x)[\mathcal{F}(x)] = \int_S d\eta_T(x)\left[T^{-\frac{1}{2}}\mathcal{F}(x)T^{-\frac{1}{2}}\right].
\]

### A.5. Two-Sided Integration.

If \( \mathfrak{A} \) is a two-sided ideal of \( \mathfrak{B} \), we can give a slight generalization of the operator-valued integration, to accommodate integration of one function to the left and one function to the right of the measure. We use the notations and definitions of Appendix A.3. Let \( g, h : X \to \mathfrak{B} \) be two simple functions,
\[
g(x) = \sum_{j=1}^{N} c_j \mathbb{1}_{S_j}(x), \quad h(x) = \sum_{j=1}^{M} d_j \mathbb{1}_{T_j}(x).
\]

In addition, for any \( B, C \in \mathfrak{B} \) and for any \( \omega \in \mathfrak{A}' \), let us define \( B \circ \omega \circ C \in \mathfrak{A}' \) by
\[
(B \circ \omega \circ C)(\cdot) := \omega(B \cdot C).
\]  

(Hence, it is possible to define two-sided simple integration as
\[
\int_X g(x) \ d\mu(x) \ h(x) = \sum_{j=1}^{N} \sum_{k=1}^{M} c_j \circ \mu(S_j \cap T_k) \circ d_k.
\]  

(A.28)
Moreover, if $f_1, f_2 : X \to \mathcal{B}$ have separable range, it is straightforward to extend to define the two-sided integral
\[
\int_S f_1(x) \, dm(x) f_2(x) \in \mathcal{A}^* .
\] (A.29)

If the above integral exists, we say that the pair $f_1, f_2$ is $m$-two-sided-integrable (the order is relevant). This notion also preserves positivity: for all $f$ such that $f^* \cdot f$ is $m$-two-sided-integrable, then
\[
\int_S f^*(x) \, dm(x) \, f(x) \in \mathcal{A}^*_+ .
\] (A.30)

A pair of functions with separable range $f_1, f_2 : X \to \mathcal{B}$ are $m$-two-sided-absolutely integrable iff $\|f_1(\cdot)\|_{\mathcal{B}} \|f_2(\cdot)\|_{\mathcal{B}}$ is $m$-integrable. The analogue of Proposition A.15 is the following

**Proposition A.19** (Integrability and absolute integrability).

Let $f_1, f_2 : X \to \mathcal{B}$ be $m$-two-sided-absolutely integrable. Then, $f_1, f_2$ and $f_2, f_1$ are both $m$-two-sided-integrable and, for all $S \in \Sigma$,
\[
\left\| \int_S f_1(x) \, dm(x) f_2(x) \right\|_{\mathcal{A}^*} \leq \int_S dm(x) \|f_1(x)\|_{\mathcal{B}} \|f_2(x)\|_{\mathcal{B}},
\] (A.31)

with analogous bound when $f_1$ and $f_2$ are exchanged on the left hand side.

Finally, dominated convergence applies to two-sided integration too.

**Theorem A.20** (Dominated convergence III).

Let $\{f_n\}_{n \in \mathbb{N}}, \{g_n\}_{n \in \mathbb{N}}, f_n, g_n : X \to \mathcal{B}$ for all $n \in \mathbb{N}$, be two sequences of operator-valued functions strongly converging $m$-a.e. to $f, g : X \to \mathcal{B}$, respectively. If there exists a $m$-square-integrable function $G : X \to \mathbb{R}^+$ such that $m$-a.e.
\[
\|f_n(x)\|_{\mathcal{B}} \leq G(x), \quad \|g_n(x)\|_{\mathcal{B}} \leq G(x),
\] (A.32)

then, for any $n \in \mathbb{N}, f_n, g_n$ and $f, g$ are $m$-two-sided-absolutely integrable, and
\[
\int_S f(x) \, dm(x) \, g(x) = \lim_{n \to \infty} \int_S f_n(x) \, dm(x) \, g_n(x) ;
\] (A.33)
\[
\int_S g(x) \, dm(x) \, f(x) = \lim_{n \to \infty} \int_S g_n(x) \, dm(x) \, f_n(x) .
\] (A.34)

**A.6. Radon-Nikodým Property and Push-forward.** If an operator-valued function does not have a separable range, it may fail to have an approximation with simple functions. It is possible to give an alternative definition of integration if $\mathcal{A}$ is a separable space, as it is the case for the trace class operators on a separable Hilbert space $\mathcal{L}^1(\mathcal{H})$, thanks to the following property.

**Theorem A.21** (Radon-Nikodým property [DP40, Theorem 2.1.0]).

If $\mathcal{A}$ is separable, then it has the Radon-Nikodým property: for every algebraic state-valued measure $m$, there exists a function $\varrho : X \to \mathcal{A}^*_+$, which is $m$-Bochner-integrable and such that, for all $S \in \Sigma$,
\[
m(S) = \int_S dm(x) \varrho(x) .
\] (A.35)

The function $\varrho$ is the Radon-Nikodým derivative of $m$ w.r.t. $m$, denoted by $\varrho = \frac{dm}{dm}$. Therefore, it is natural to give the following alternative definition of integrability. Recall that for any $\Gamma \in \mathcal{A}$, and $B \in \mathcal{B}$ we define $(\Gamma \circ B)(\cdot) = \Gamma(B \cdot)$, if $\mathcal{A}$ is a left ideal of $\mathcal{B}$,
and \((\Gamma \circ B)(\cdot) = \Gamma(\cdot)\), if \(\mathfrak{A}\) is a right ideal of \(\mathfrak{B}\). If \(\mathfrak{A}\) is a two-sided ideal, the notation \(\Gamma B\) denotes indifferently any of the two. In this case, for any \(B, C \in \mathfrak{B}\) we can define \((B \circ \Gamma \circ C)(\cdot) = \Gamma(B \cdot C)\).

**Definition A.22** (Integrability IV).

Suppose that \(\mathfrak{A}'\) is separable, and let \(f, g : X \to \mathfrak{B}\) be measurable functions (possibly with non-separable range) and \(\mathfrak{m}\) an algebraic state-valued measure with Radon-Nikodym derivative \(\varrho = \frac{d\mathfrak{m}}{d\mathfrak{m}}\). Then, \(f\) is \(\mathfrak{m}\)-integrable iff \(\varrho \circ f \in \mathfrak{A}'\) is \(\mathfrak{m}\)-Bochner-integrable and, for any \(S \in \Sigma\),

\[
\int_S \mathfrak{m}(x) f(x) := \int_S \mathfrak{m}(x) \varrho(x) \circ f(x) \in \mathfrak{A}'. \tag{A.36}
\]

If in addition \(\mathfrak{A}\) is a two-sided ideal of \(\mathfrak{B}\), then \(f, g\) is \(\mathfrak{m}\)-two-sided-integrable iff \(fg \in \mathfrak{A}'\) is \(\mathfrak{m}\)-Bochner-integrable, and, for any \(S \in \Sigma\),

\[
\int_S f(x) \mathfrak{m}(x) g(x) := \int_S \mathfrak{m}(x) f(x) \varrho(x) \circ g(x) \in \mathfrak{A}'. \tag{A.37}
\]

It is straightforward to see that Definition A.22 is equivalent to Definition A.13 and the analogous one for the two-sided integral for any \(f, g\) with separable range, and therefore Definition A.22 extends Definition A.13 to any separable \(\mathfrak{A}'\). In addition, since \(\mathfrak{m}\)-Bochner-integrability is equivalent to \(\mathfrak{m}\)-absolute integrability, it follows that, if \(\mathfrak{A}'\) is separable, then \(\mathfrak{m}\)-integrability is equivalent to \(\mathfrak{m}\)-absolute-integrability. Hence, all the results of Appendices A.2, A.3 and A.5 extend, if \(\mathfrak{A}'\) is separable, to functions with non-separable range.

Suppose now that \(X\) is a topological vector space and \(\Sigma\) the corresponding Borel \(\sigma\)-algebra. In this context, Bochner’s theorem holds for algebraic state-valued measures [Fal18b]: the Fourier transform, with \(\xi \in X'\),

\[\hat{\mathfrak{m}}(\xi) := \int_X \mathfrak{m}(x) e^{2\pi i \xi(x)} \in \mathfrak{A}'\] \hspace{1cm} (A.38)

identifies uniquely a measure. Therefore, the push-forward of an algebraic state-valued measure \(\mathfrak{m}\) by means of a linear continuous map \(\Phi : X \to Y\), where \(Y\) is again a topological vector space with the Borel \(\sigma\)-algebra, is conveniently defined using the Fourier transform, and this definition suffices for the purposes of this paper: more precisely, the push-forward measure \(\Phi \sharp \mathfrak{m}\) is the measure on \(Y\) whose Fourier transform is defined by, with \(\eta \in Y'\),

\[(\Phi \sharp \mathfrak{m})(\eta) := \int_X \mathfrak{m}(x) e^{2\pi i \eta(\Phi(x))} \in \mathfrak{A}'\]. \hspace{1cm} (A.39)

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