Spatial Factor Modeling: A Bayesian Matrix-Normal Approach for Misaligned Data

Lu Zhang
UCLA Department of Biostatistics
Lu.Zhang@ucla.edu

Sudipto Banerjee
UCLA Department of Biostatistics
sudipto@ucla.edu

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Abstract

Multivariate spatially-oriented data sets are prevalent in the environmental and physical sciences. Scientists seek to jointly model multiple variables, each indexed by a spatial location, to capture any underlying spatial association for each variable and associations among the different dependent variables. Multivariate latent spatial process models have proved effective in driving statistical inference and rendering better predictive inference at arbitrary locations for the spatial process. High-dimensional multivariate spatial data, which is the theme of this article, refers to data sets where the number of spatial locations and the number of spatially dependent variables is very large. The field has witnessed substantial developments in scalable models for univariate spatial processes, but such methods for multivariate spatial processes, especially when the number of outcomes are moderately large, are limited in comparison. Here, we extend scalable modeling strategies for a single process to multivariate processes. We pursue Bayesian inference which is attractive for full uncertainty quantification of the latent spatial process. Our approach exploits distribution theory for the Matrix-Normal distribution, which we use to construct scalable versions of a hierarchical linear model of coregionalization (LMC) and spatial factor models that deliver inference over a high-dimensional parameter space including the latent spatial process. We illustrate the computational and inferential benefits of our algorithms over competing methods using simulation studies and an analysis of a massive vegetation index data set.

Key words: Bayesian inference; Factor models; Linear Models of Coregionalization; Matrix-Normal distribution; Multivariate spatial processes; Scalable spatial modeling
I. INTRODUCTION

Statistical modeling for multiple spatially-oriented data are required to capture underlying spatial associations in each variable and accounting for inherent associations among the different variables. As an example, to which we return later, consider a set of spatially indexed spectral variables for vegetation activity on the land. Such variables exhibit strong spatial dependence as customarily exhibited through plots of spatial variograms and other exploratory maps. In addition, the variables are assumed to be associated with each other because of shared physical processes that manifest through the observations.

Modeling each variable separately captures the spatial distribution of that variable independent of other variables. Such analysis ignores associations among the variables and can impair prediction or interpolation (see, e.g., Chiles and Delfiner [2009], Wackernagel [2003], Gelfand and Banerjee [2010] Cressie and Wikle [2015]). Each of the aforementioned works provide ample evidence, theoretical and empirical, in favor of joint modeling of multiple spatially indexed variables. Joint modeling, or multivariate spatial analysis is especially pertinent in the presence of spatial misalignment, where not all variables have been observed over the same set of locations. For example, suppose \( Y(s_0) \) is Normalized Difference Vegetation Index (NDVI) and \( X(s_0) \) is red reflectance. If location \( s_0 \) has yielded a measurement for \( X(s_0) \) but not for \( Y(s_0) \), then optimal imputation of \( Y(s_0) \) should proceed from \( p(Y(s_0) \mid Y, X) \), where \( Y \) and \( X \) comprise all measurements on \( Y(s) \) and \( X(s) \). If the processes \( Y() \) and \( X() \) are modeled as independent, then the predictive distribution \( p(Y(s_0) \mid Y, X) = p(Y(s_0) \mid Y) \) and will not exploit the possible predictive information present in \( X(s_0) \) for \( Y(s_0) \). This specific issue has also been discussed, with examples, in Banerjee and Gelfand [2002].

Joint modeling is driven by vector-valued latent spatial stochastic processes, such as a multivariate Gaussian process. These are specified with matrix-valued cross-covariance functions (see, e.g., Genton and Kleiber [2015], Salvanà and Genton [2020], Le and Zidek [2006] and references therein) that models pairwise associations at distinct locations. Theoretical properties of cross-covariances are well established, but practical modeling implications and computational efficiency require specific considerations depending upon the application (see, e.g., Le et al. [1997], Sun et al. [1998], Le et al. [2001], Gamerman and Moreira [2004], Schmidt and Gelfand [2003], Banerjee et al. [2014]).

High-dimensional multivariate spatial models will deal with a large number of dependent variables over a massive number of locations. While analyzing massive spatial and spatio-temporal databases have received attention (see, e.g., Sun et al. [2011], Banerjee [2017], Heaton et al. [2019], Zhang et al. [2020], the bulk of methods has focused on one or very few (two or three) spatially dependent variables and often have to rely upon restrictive assumptions that preclude full inference on the latent process. With larger numbers of dependent variables, modeling the cross-covariance becomes challenging. Even for stationary cross-covariance functions, where we assume that the associations among the variables do not change over space and the spatial association for each variable depends only on the difference of two positions, matters become computationally challenging.

This manuscript builds upon the popular linear models of coregionalization (Bourgault and Marcotte [1991], Goulard and Voltz [1992], Wackernagel [2003], Gelfand et al. [2004], Chiles and Delfiner [2009], Genton and Kleiber [2015]). Our contributions include: (i) developing a hierarchical model with a Matrix-Normal distribution as a prior for an unknown linear transformation on latent spatial processes; (ii) extending classes of spatial factor models for spatially misaligned data; (iii) accounting for multiple outcomes over very large number of locations. Spatial factor models have been explored by Wang and Wall [2003], Lopes et al. [2008], Ren and Banerjee [2013] and Taylor-Rodriguez et al. [2019]. Lopes et al. [2008] provides an extensive discussion on how
hierarchical models emerged from dynamic factor models. Ren and Banerjee (2013) proposed low-rank specifications for spatially-varying factors to achieve dimension reduction, but such low-rank specifications tend to over-smooth the latent process from massive data sets containing millions of locations. More recently, Taylor-Rodriguez et al. (2019) consider Nearest-Neighbor Gaussian process (Datta et al., 2016a) for spatial factors with the usual constrained loading matrices in non-spatial factor models. These are more restrictive than needed for identifying spatially correlated factors (see, e.g. Ren and Banerjee, 2013).

We develop our modeling framework in Section II. Section III presents some theoretical results about posterior consistency for the proposed models. Simulation studies for exploring the performance of proposed models are summarized in Section IV. Section V presents an application to remote-sensed vegetation analysis on land surfaces. We conclude with some discussion in Section VI.

II. Multivariate spatial processes

Let \( z(s) = (z_1(s), \ldots, z_q(s))^T \) be a \( q \times 1 \) stochastic process, where each \( z_i(s) \) is a real-valued random variable at location \( s \in \mathcal{D} \subseteq \mathbb{R}^d \). The process is specified by its mean \( \text{E}[z_i(s)] = \mu_i(s) \) and, customarily, second-order stationary covariances \( C_{ij}(h) = \text{Cov}(z_i(s), z_j(s+h)) \) for \( i, j = 1, 2, \ldots, q \). These covariances define the matrix-valued \( q \times q \) cross-covariance function \( C(h) = \{C_{ij}(h)\} \) with \( (i,j) \)-th entry \( C_{ij}(h) \). While there is no loss of generality in assuming the process mean to be zero by absorbing the mean into a separate regression component in the model, as we will do here, modeling the cross-covariance function requires care. From its definition, \( C(h) \) need not be symmetric, but must satisfy \( C(h)^T = C(-h) \). Also, since \( \text{var}\{\sum_{i=1}^q a_i^T z(s_i)\} \geq 0 \) for any set of finite locations \( s_1, s_2, \ldots, s_n \in \mathcal{D} \) and any set of constant vectors \( a_1, a_2, \ldots, a_n \in \mathbb{R}^q \), we have \( \sum_{i,j=1}^n a_i^T C(s_i - s_j)a_i \geq 0 \). Genton and Kleiber (2015) provide a comprehensive review of cross-covariance functions.

Perhaps the most widely used approach for constructing multivariate random fields is the linear model of coregionalization (LMC). This hinges on invertible linear maps of independent spatial processes yielding valid spatial processes. If \( f(s) = (f_1(s), f_2(s), \ldots, f_K(s))^T \) is a \( K \times 1 \) vector of independent spatial processes so that \( \text{cov}(f_i(s), f_j(s')) = 0 \) for all \( i \neq j \) and any two locations \( s \) and \( s' \) (same or distinct), then LMC (Bourgault and Marcotte, 1991) specifies

\[
z(s) = \sum_{k=1}^K \lambda_k f_k(s) = \Lambda^T f(s), \tag{II.1}
\]

where \( z(s) = q \times 1, \Lambda \) is \( K \times q \), \( \lambda_k^T \) is the \( k \)-th row of \( \Lambda \) and each \( f_k(s) \) is an independent Gaussian process with correlation function \( \rho_{\psi_k} (\cdot, \cdot) \) with parameters \( \psi_k \). The cross-covariance for \( z(s) \) yields non-degenerate process-realizations whenever \( K \geq q \) and \( \Lambda \) is nonsingular. To achieve dimension reduction in the number of variables, we restrict \( K < q \) so we have non-degenerate realizations in a \( K \) dimensional sub-space.

Schmidt and Gelfand (2003) propose multivariate spatial processes through a hierarchical spatial conditional model, whereupon \( \Lambda^T \) in (II.1) is a \( q \times q \) lower triangular matrix. Other variants of LMC (e.g. Goulard and Voltz, 1992) can also be recast as (II.1) using linear algebra. The flexibility offered in modeling \( \Lambda \) is appealing and, in particular, can accrue computational benefits in high-dimensional settings. Hence, we build upon (II.1).
I. A Bayesian LMC factor model (BLMC)

Let \( y(s) = (y_1(s), \ldots, y_q(s))^\top \in \mathbb{R}^q \) denote the \( q \times 1 \) vector of dependent outcomes in location \( s \in \mathcal{D} \subset \mathbb{R}^d \), \( x(s) = (x_1(s), \ldots, x_p(s))^\top \in \mathbb{R}^p \) be the corresponding explanatory variables, and \( \beta \) be a \( p \times q \) regression coefficient matrix in the multivariate spatial model

\[
y(s) = \beta^\top x(s) + \Lambda^\top f(s) + e(s), \quad s \in \mathcal{D},
\]

where the latent process \( \Lambda^\top f(s) \) is an LMC as described above. Elements in \( f(s) \) are as described in (II.1), while the noise process \( e(s) \) iid \( N(0, \Sigma) \) with covariance matrix \( \Sigma \). We model \{\( \beta, \Lambda, \Sigma \)\} using a Matrix-Normal-Inverse-Wishart family. To be precise,

\[
\beta | \Sigma \sim MN(\mu_\beta, V_\beta, \Sigma); \quad \Lambda | \Sigma \sim MN(\mu_\Lambda, V_\Lambda, \Sigma); \quad \Sigma \sim IW(\Psi, \nu),
\]

where \( \mu_\Lambda \) a \( q \times K \) matrix and \( V_\Lambda \) a \( K \times K \) positive definite matrix. A random matrix \( Z_{n \times p} \sim MN_{n,p}(M, U, V) \) has the probability density function [Dawid 1981]

\[
p(Z | M, U, V) = \frac{\exp \left[ -\frac{1}{2} \text{tr} \{ V^{-1}(Z - M)^\top U^{-1}(Z - M) \} \right]}{(2\pi)^{np/2}|V|^{n/2}|U|^{p/2}},
\]

where \( \text{tr}(\cdot) \) is the trace function, \( M \) is the mean matrix, \( U \) is the first scale matrix with dimension \( n \times n \) and \( V \) is the second scale matrix with dimension \( p \times p \). This distribution is equivalent to \( \text{vec}(Z) \sim N_{np}(\text{vec}(M), V \otimes U) \), where \( \otimes \) is the Kronecker product and \( \text{vec}(Z) = [z_1^\top, \ldots, z_p^\top]^\top \) is the vectorized \( n \times p \) random matrix \( Z = [z_1 : \cdots : z_p] \). We refer to the model specified through (II.2)–(II.3) as the Bayesian LMC (BLMC) factor model.

Without misalignment, the observation model in (II.2) can be cast as

\[
Y_{n \times q} = X_{n \times p} \beta_{p \times q} + F_{n \times K} \Lambda_{K \times q} + e_{n \times q},
\]

where \( Y = y(S) = [y(s_1), \ldots : y(s_n)]^\top \) is the \( n \times q \) response matrix, \( X = x(S) = [x(s_1), \ldots : x(s_n)]^\top \) is the corresponding design matrix with full rank \( (n > p) \), and \( F \) is the \( n \times K \) matrix with \( j \)-th column being the \( n \times 1 \) vector comprising \( f_j(s_i) \)'s for \( i = 1, 2, \ldots, n \).

The parameters \( \Lambda \) and \( F \) are not jointly identified in factor models and some constraints are required to ensure identifiability [Lopes and West 2004; Ren and Banerjee 2013]. These constraints are not without problems. For example, a lower-trapezoidal (triangular for \( K = q \)) specification for \( \Lambda \) imposes possibly unjustifiable conditional independence on the spatial processes. Alternatively, ordering the spatial range parameters can ensure identifiability but creates difficulties in computation and interpretation. We avoid such constraints and transform \( \omega = FA \) to obtain inference for the latent process. This parametrization yields conditional conjugate distributions and, therefore, efficient posterior sampling. We elucidate below in the context of misaligned data.

II. Inference for spatially misaligned data

Let \( S = \{s_1, \ldots, s_n\} \) be the set of locations that have recorded at least one of the observed outcomes and let \( S_i \) be the subset of locations that have recorded the \( i \)-th response. Then \( \bigcup_{i=1}^q S_i = S \) and let \( n_i = |S_i| \). Let \( M_i = S \setminus S_i \) denote the set of locations where at least one response, but not the \( i \)-th response, is recorded so that \( M = \bigcup_{i=1}^q M_i \) is the set of all locations with incomplete data. We derive the conditional distribution of \( F \) and of the unobserved
responses \( \{ y_i(M_i) \}_{i=1}^q \) conditional on \( \{ \beta, \Lambda, \Sigma, \{ \psi_k \}_{k=1}^K \} \). Let \( P \) be the \( (\sum_{i=1}^q n_i) \times nq \) matrix such that \( P \text{vec}(Y) = \{ y(s_1)_{os_1}^T, y(s_2)_{os_2}^T, \ldots, y(s_n)_{os_n}^T \}^T \), where the suffix \( os_i \) indexes of the observed responses at \( s_i \in S \). Thus, \( P \) extracts the observed responses from vec(\( Y \)) in each of the locations \( \{ s_1, \ldots, s_n \} \). The joint distribution of vec(\( F \)) and \( \{ y(s_i)_{os_i} \}_{i=1}^q \), given \( \{ \beta, \Lambda, \Sigma, \{ \psi_k \}_{k=1}^K \} \), can be represented through the augmented linear system,

\[
\begin{bmatrix}
\{ (y(s_i) - x(s_i)\beta)_{os_i} \}_{i=1}^q \\
0
\end{bmatrix} = \begin{bmatrix} P(A^T \otimes I_n) & e_1 \\
I_k \otimes I_n & e_2 \end{bmatrix} \text{vec}(F) + \begin{bmatrix} \epsilon_1 \\
\epsilon_2 \end{bmatrix},
\]

where \( \epsilon_1 \sim N(0, \oplus_{i=1}^q \{ \Sigma_{os_i} \}) \), \( \epsilon_2 \sim N(0, \oplus_{k=1}^K (\rho(\psi_k(S,S))) \), \( \rho(\psi_k(S,S)) \) is the \( n \times n \) spatial correlation matrix corresponding to \( f_k = (f_k(s_1), f_k(s_2), \ldots, f_k(s_n))^T \), and \( \oplus_{i=1}^n \) represents the block diagonal operator stacking matrices along the diagonal. Letting \( D_{\Sigma_0}^{-\frac{1}{2}} = \oplus_{i=1}^q \{ \Sigma_{os_i}^{-\frac{1}{2}} \} \) and \( V_F = \oplus_{k=1}^K \{ V_k \} \), where \( \rho^{-1}(S,S) = V_k^T V_k \), we obtain

\[
\begin{bmatrix}
D_{\Sigma_0}^{-\frac{1}{2}} \{ (y(s_i) - x(s_i)\beta)_{os_i} \}_{i=1}^q \\
0
\end{bmatrix} = \begin{bmatrix} D_{\Sigma_0}^{-\frac{1}{2}}P^\Lambda \otimes I_n & V_F \\
V_F & \chi \end{bmatrix} \text{vec}(F) + \begin{bmatrix} \eta_1 \\
\eta_2 \\
\eta \end{bmatrix}.
\]

The elements of \( \eta \) are independent error terms, each with unit variance. The full conditional distribution vec(\( F \) \( | \{ y(s_i)_{os_i} \}_{i=1}^q, \beta, \Lambda, \Sigma, \{ \psi_k \}_{k=1}^K \) for the LMC model in (II.2) then follows

\[
\text{vec}(F) \mid \{ y(s_i)_{os_i} \}_{i=1}^q, \beta, \Lambda, \Sigma, \{ \psi_k \}_{k=1}^K \sim N((\tilde{X}^T \tilde{X})^{-1} \tilde{X}^T \tilde{Y}, (\tilde{X}^T \tilde{X})^{-1}).
\] (II.8)

For misaligned data, we will perform Bayesian updating of the outcomes missing at a location \( s \in M \). Let \( ms \) be the suffix that indexes outcomes that are missing at \( s \in M \). The conditional distribution of \( y(s)_{ms} \) given the parameters \( \{ F, \{ y(s_i)_{os_i} \}_{i=1}^q, \beta, \Lambda, \Sigma, \{ \psi_k \}_{k=1}^K \} \) is

\[
N(\mu_s + \Sigma_{[ms,os]} \Sigma_{[os,os]}^{-1} (y(s)_{os} - \mu_s)_{os}), \Sigma_{[ms,ms]} - \Sigma_{[ms,os]} \Sigma_{[os,os]}^{-1} \Sigma_{[os,ms]}^{-1}),
\]

where \( \mu_s = \beta^T x(s) + \Lambda^T f(s) \), \( \Sigma_{[ms,os]} \) is the sub-matrix of \( \Sigma \) extracted with row and column indices \( ms \) and \( os \), respectively. With the priors given in (II.3), we let \( L_\Lambda = L_A^T \Lambda^{-1} \) and define \( \gamma = [\beta^T, \Lambda^T]^T \). The conditional posterior distribution \( \gamma \mid \Sigma, F, Y \) can be found from

\[
\begin{bmatrix}
Y \\
L_A^{-1} \mu_B \\
L_A^{-1} \mu_A \end{bmatrix} = \begin{bmatrix} X & F & \beta \\
0 & L_B^{-1} & 0 \\
0 & L_A^{-1} & \gamma \end{bmatrix} + \begin{bmatrix} \eta_1 \\
\eta_2 \\
\eta_3 \end{bmatrix},
\]

where \( \eta \sim MN(0, (n+p+k) \times q, I_{n+p+k}, \Sigma) \). Using standard distribution theory, we can show that \( \gamma, \Sigma \mid F, Y \) follows MNIW(\( \mu^*, \Psi^*, \Psi^* \), \( \nu^* \)), where

\[
V^* = [X^T X]^\frac{1}{2}, ~ \mu^* = V^* [X^* Y^*]^\frac{1}{2}, \quad \Psi^* = \Psi + S^*, \quad \text{and} \quad \nu^* = \nu + n
\]

with \( S^* = (Y^* - X^* \mu^*)^T (Y^* - X^* \mu^*) \). In particular, if \( \Sigma = \oplus_{i=1}^q \{ \sigma_i^2 \} \) and each \( \sigma_i^2 \sim IG(a, b_i) \) for \( i = 1, \ldots, q \), then the conditional distribution of \( \sigma_i^2 \) given \( Y, F \) follows IG(\( a^*, b_i^* \)), where

\[
a^* = a + \frac{n}{2}, b_i^* = b_i + \frac{1}{2} (Y^* - X^* \mu^*)^T (Y^* - X^* \mu^*)_i, \quad i = 1, \ldots, q,
\]

(II.12)
and \((Y^* - X^* \mu^*)_i\) is the \(i\)-th column of \(Y^* - X^* \mu^*\). From (II.10), \(\gamma | \Sigma, F, Y \sim \text{MN}(\mu^*, V^*, \Sigma)\).

The parameters \(\psi_k, k = 1, 2, \ldots, K\), by themselves, are not consistently estimable under in-fill asymptotics. Therefore, irrespective of the sample size (within a fixed domain), inference on \(\psi_k\) will be sensitive to the choice of the prior. Furthermore, without placing restrictions on the loading matrix or ordering these parameters (Ren and Banerjee 2013), these parameters are identifiable primarily through the prior. We treat these as unknown and model them using priors based upon customary spatial domain considerations. The full conditional distributions for \(\{\psi_k\}_{k=1}^K\) are not available in closed form. However, since \(\{\psi_k\}_{k=1}^K\) and \(Y\) are conditionally independent given \(\{F, \gamma, \Sigma\}\), and \(f_k\) are independent for \(k = 1, 2, \ldots, K\), we obtain \(p(\psi_k | F, Y, \gamma, \Sigma, \{\psi_i\}_{i \neq k})\) up to a proportionality constant as

\[
p(Y | F, \gamma, \Sigma) \times p(\gamma, \Sigma) \times \prod_{k=1}^{K} p(f_k | \psi_k) \times p(\psi_k) \propto p(f_k | \psi_k) \times p(\psi_k) , \tag{II.13}
\]

for each \(k = 1, \ldots, K\), where \(p(\psi_k)\) is the prior for \(\psi_k\).

Turning to predictions, if \(\mathcal{U} = \{u_t, \ldots, u_{n'}\}\) is a set of new locations, then \(Y_{\mathcal{U}} = y(\mathcal{U})\) is independent of \(\{y(s_i)_{os_i}\}_{i=1}^{n}\) given \(\{\beta, \Lambda, \Sigma\}\) and \(F_{\mathcal{U}} = [f_1(\mathcal{U}) : \ldots : f_K(\mathcal{U})]^T\). Then,

\[
f_k(\mathcal{U}) | f_k, \psi_k \sim N(\rho_{\psi_k}(U, S)\rho_{\psi_k}(S, S)^{-1}f_k, \rho_{\psi_k}(S, S)^{-1}) , \tag{II.14}
\]

for each \(k = 1, 2, \ldots, K\). It follows that \(p(Y_{\mathcal{U}}, F_{\mathcal{U}} | \{y(s_i)_{os_i}\}_{i=1}^{n})\) is proportional to

\[
p(Y_{\mathcal{U}} | F_{\mathcal{U}}, \beta, \Lambda, \Sigma) \times p(F_{\mathcal{U}} | F, \psi_k) \times p(\beta, \Lambda, \Sigma, F, \psi_k) \times p(\psi_k) \times p(\psi_k) , \tag{II.15}
\]

where we have used the independence between \(F_{\mathcal{U}}\) and \(\{y(s_i)_{os_i}\}_{i=1}^{n}\) given \(F\) and \(\{\psi_k\}_{k=1}^K\). The distributions in (II.14) and (II.15) help in sampling from the posterior predictive distribution over \(\mathcal{U}\) using the posterior samples of \(\{\beta, \Lambda, \Sigma, F, \psi_k\}_{k=1}^K\). We elaborate below.

III. The block update MCMC algorithm

We formulate an efficient MCMC algorithm for obtaining full Bayesian inference as follows. From the \(l\)th iteration with \(\{\beta^{(l)}, \Lambda^{(l)}, \Sigma^{(l)}, \{\psi_k^{(l)}\}_{k=1}^K\}\), we generate \(F_{(l+1)}^{(l)}\) from (II.8). Next, we draw \(\{y(s_i)_{mi}^{(l+1)}\}_{i \in \mathcal{M}}\) on \(\mathcal{M}\) using (II.9) and then update \(\{\beta^{(l+1)}, \Lambda^{(l+1)}, \Sigma^{(l+1)}\}\) using (II.11). We complete the \((l + 1)\)th iteration by drawing \(\{\psi_k^{(l+1)}\}_{k=1}^K\) through a Metropolis random walk using (II.13). Upon convergence, these iterations will generate samples from the desired joint posterior distribution \(p(\beta, \Lambda, F, \Sigma, \{\psi_k\}_{k=1}^K | Y)\).

For inference on \(\mathcal{U}\), we sample \(F_{\mathcal{U}}\) from (II.14), given the posterior samples of \(F\) and \(\{\psi_k\}_{k=1}^K\), then generate posterior predictions of \(Y_{\mathcal{U}}\) given the posterior samples of \(\{\beta, \Lambda, \Sigma, F_{\mathcal{U}}\}\). Applying the SCAM algorithm introduced in Haario et al. (2005), one can avoid tuning parameters in Metropolis algorithms by warming up each MCMC chain of \(\{\psi_k\}_{k=1}^K\) with an adaptive proposal distribution. In our implementation, we use the proposal distribution defined by equation (2.1) in Roberts and Rosenthal (2009).

We sample \(F\) as a single block through a linear transformation of the \(n \times K\) independent parameters from the model in (II.7). Sampling \(\{\beta, \Lambda\}\) follows analogously. We significantly improve convergence by reducing the posterior dependence among the parameter in this Gibbs with Metropolis algorithm (Gelman et al. 2013). Since \(F\) is sensitive to the value of the intercept, we recommend using an intercept-centered latent process to obtain inference for the latent spatial pattern.
IV. Scalable Modeling

We use a conjugate gradient method (Nishimura and Suchard, 2018) to facilitate sampling of \( F \) when \( p_{\psi_k}(S, S) \) is sparse for \( k = 1, \ldots, K \). Here, we develop a scalable BLMC model with each element of \( f(s) \) modeled as a Nearest-Neighbor Gaussian Process (NNGP).

Let each \( f_k(s), s \in D \) be an NNGP \((0, p_{\psi_k} (\cdot, \cdot))\), which implies that \( f_k \sim N(0, \tilde{\rho}_k) \) for each \( k = 1, 2, \ldots, K \), where \( \tilde{\rho}_k = (I - A_{\rho_k})^{-1} D_{\rho_k} (I - A_{\rho_k})^{-\top} \), \( A_{\rho_k} \) is a sparse-lower triangular matrix with no more than a specified small number, \( m \), of nonzero entries in each row and \( D_{\rho_k} \) is a diagonal matrix. The diagonal entries of \( D_{\rho_k} \) and the nonzero entries of \( A_{\rho_k} \) are obtained from the conditional variance and conditional expectations for a Gaussian process with covariance function \( \rho_{\psi_k}(s, s') \). We consider a fixed order of locations in \( S \) and let \( N_m(s_i) \) be the set of at most \( m \) neighbors of \( s_i \) among locations \( s_j \in S \) such that \( j < i \). The \((i, j)\)-th entry of \( A_{\rho_k} \) is 0 whenever \( s_j \not\in N_m(s_i) \). If \( j_1 < j_2 < \cdots < j_m \) are the \( m \) column indices for the nonzero entries in the \( i \)-th row of \( A_{\rho_k} \), then the \((i, j_k)\)-th element of \( A_{\rho_k} \) is the \( k \)-th element of the \( 1 \times m \) vector \( a_i = \rho_{\psi_k}(s_i, N_m(s_i)) \rho_{\psi_k}(N_m(s_i), N_m(s_i))^{-1} \). The \((i, i)\)-th diagonal element of \( D_{\rho_k} \) is given by \( \rho_{\psi_k}(s_i, s_i) - a_i \). Repeating these calculations for each row completes the construction of \( A_{\rho_k} \) and \( D_{\rho_k} \) and yields a sparse \( \tilde{\rho}_k^{-1} \). This construction is performed in parallel and requires storage or computation of at most \( m \times m \) matrices, where \( m << n \), costing \( O(n) \) flops and storage. See Appendix S.1 for details.

Sampling \( F \) is computationally expensive, but is expedited by solving \((X^\top X)^{-1}X^\top v \) efficiently for any vector \( v \). If \( \rho_{\psi_k}(S, S) = L_k L_k^\top \) has a sparse Cholesky factor \( L_k \), then calculating \( X^\top v \) is efficient. To be precise, the Woodbury matrix identity yields

\[
(X^\top X)^{-1} = (DF_{\Sigma_k}^{-1} F^\top + \bigoplus_{k=1}^K (\rho_{k}^{-1}))^{-1} = \bigoplus_{k=1}^K \{\rho_{k}\}^{-1} F G^{-1} F^\top \bigoplus_{k=1}^K \{\rho_{k}\}, \tag{II.16}
\]

where \( F = (A \otimes I_n) P^\top \) is sparse, \( G = D_{\Sigma_k} + P \{ \sum_{k=1}^K \lambda_{ijk} \rho_{k} \}_{i,j=1}^p P^\top \) with \( \rho_k = \rho_{\psi_k}(S, S) \). If all the \( \rho_k \)'s have similar structures, then permuting the \( \{ \sum_{k=1}^K \lambda_{ijk} \rho_{k} \}_{i,j=1}^p P^\top \) in \( P \) in rows and columns often renders structures in \( \rho_k \)'s that can be exploited by BLMC for very large spatial data sets. For example, if \( \rho_k \)'s are banded matrices with bandwidth \( b \), then \( \{ \sum_{k=1}^K \lambda_{ijk} \rho_{k} \}_{i,j=1}^p P^\top \) is also banded with bandwidth \( bq \). Moreover, \( D_{\Sigma_k} \) is a banded matrix with bandwidth \( \leq q \). Hence, adding \( D_{\Sigma_k} \) hardly increases the computational burden in the Cholesky decomposition of \( G \) when \( q \) is small. Assembling all features of \( \rho_{\psi_k} \), \( F \) and \( G \), the calculation of \((X^\top X)^{-1} u \) for any \( u = X^\top v \) is scalable when multiplying \( u \) with \( II.16 \).

We conclude this section with a remark on the BLMC model with diagonal \( \Sigma \). This specification is desirable for data sets with a massive number of responses \( q \). A diagonal \( \Sigma \) avoids the quadratic growth of the number of parameters in \( \Sigma \) as \( q \) increases. We illustrate an NNGP based BLMC with diagonal \( \Sigma \) in Section II.

III. On posterior consistency: Large-sample properties of posterior estimates

We present some theoretical results for the models constructed in the previous section. Specifically, we investigate the behavior of the posterior distribution as the sample size increases and establish its convergence to an oracle distribution. Here, for establishing the results, we will assume conjugate MNIW models with no misalignment. First, we assume that \( y(s) \) itself is modeled as a spatial process without explicitly introducing a latent process. Let

\[
y(s) \sim \text{GP}(\beta^\top x(s), C(\cdot, \cdot)), \quad C(s, s') = \{\rho_{\psi}(s, s') + (\alpha^{-1} - 1)\delta_{s=s'}\} \Sigma, \tag{III.1}\n\]
where $\rho_\psi(\cdot, \cdot)$ is a spatial correlation function defined through hyperparameter $\psi$, $\delta$ denotes Dirac’s delta function, and $\kappa^{-1}\Sigma$ is the non-spatial covariance matrix of $y(s)$. The fixed scalar $\alpha$ represents the proportion of total variability allocated to the spatial process. This implies that $Y|\beta, \Sigma \sim MN_{n,p}(X\beta, \mathcal{K}, \Sigma)$, where $\mathcal{K} = \rho_\psi(S,S) + (\kappa^{-1} - 1)I_n$. We model $\{\beta, \Sigma\}$ using the conjugate MNIW prior

$$
\beta|\Sigma \sim MN_{n,p}(\mu_\beta, V_r, \Sigma), \quad \Sigma \sim IW(\Psi, v),
$$

with prefixed $\{\mu_\beta, V_r, \Psi, v\}$. Closely following the developments in Gamerman and Moreira (2004), we obtain the posterior distribution of $\{\beta, \Sigma\}$ as $\text{MNIW}(\mu^*, V^*, \Psi^*, v^*)$, where

$$
V^* = (X^T \mathcal{K}^{-1} X + V_r^{-1})^{-1}, \quad \mu^* = V^*(X^T \mathcal{K}^{-1} Y + V_r^{-1} \mu_\beta),
$$

$$
\Psi^* = \Psi + Y^T \mathcal{K}^{-1} Y + \mu_\beta^T V_r^{-1} \mu_\beta - \mu^T V^* \mu^*, \quad \text{and} \quad v^* = v + n.
$$

We refer to the above model as the “response” model.

Next, we consider the spatial regression model with the latent process,

$$
y(s) = \beta^T x(s) + \omega(s) + e(s), \quad s \in D,
$$

where $\omega(s) \sim \text{GP}(0_{q \times 1}, \rho_\psi(\cdot, \cdot) \Sigma)$ is a latent process and $e(s) \sim N(0_{q \times 1}, (\kappa^{-1} - 1)\Sigma)$ is measurement error. Define $\omega = (\omega(s_1) : \cdots : \omega(s_n))^T$. For theoretical tractability, we restrict posterior inference on $\{\beta, \omega, \Sigma\}$, assuming that the scalar $\alpha$ is fixed. Assuming that the joint distribution of $\beta$ and $\Sigma$ are given in (III.3) and that $\omega|\Sigma \sim MN_{n \times q}(0, \rho_\psi(S,S), \Sigma)$, the posterior distribution of $\gamma^T = [\beta^T, \omega^T]$ is $p(\gamma, \Sigma | Y) = \text{MNIW}(\mu_\gamma^*, V^*, \Psi^*, v^*)$, where

$$
V^* = \left[ \begin{array}{c}
\frac{\alpha}{1-\alpha} X^T X + V_r^{-1} \\
\frac{\alpha}{1-\alpha} X \\
\rho^{-1}(S,S) + \frac{\alpha}{1-\alpha} I_n
\end{array} \right]^{-1}, \quad \mu_\gamma^* = V^* \left[ \begin{array}{c}
\frac{\alpha}{1-\alpha} X^T Y + V_r^{-1} \mu_\beta \\
\frac{\alpha}{1-\alpha} Y
\end{array} \right],
$$

$$
\Psi^* = \Psi + \frac{\alpha}{1-\alpha} Y^T Y + \mu_\beta^T V_r^{-1} \mu_\beta - \mu_\gamma^T V^* \mu_\gamma^* \quad \text{and} \quad v^* = v + n,
$$

We refer to the above model as the “latent” model.

We establish the posterior consistency of $\{\beta, \Sigma\}$ for the response model (III.1) and the latent model (III.4). For distinguishing the variables based on the number of observations, we make the dependence upon $n$ explicit. Denote $X(n)_{n \times p} = [x(s_1) : \cdots : x(s_n)]^T$, $Y(n)_{n \times q} = [y(s_1) : \cdots : y(s_n)]^T$, $S(n) = [s_1, \ldots, s_n]$, $\mathcal{K}(n) = C(S(n), S(n)) + (\kappa^{-1} - 1)I_n$ and $J(n) = X(n)^T \mathcal{K}(n)^{-1} X(n)$. Proofs and technical details are available in Appendix S.2.

**Theorem III.1.** [Theorem S.1, Theorem S.2] Parameter set $\{\beta, \Sigma\}$ is posterior consistent for both conjugate response and latent models if and only if $\lim_{n \to \infty} \lambda_{\min}\{J(n)\} = \infty$, where $\lambda_{\min}\{J(n)\}$ is the smallest eigenvalue of $J(n)$.

When the explanatory variables share the same spatial correlation with the responses, the necessary and sufficient conditions for Theorem III.1 hold (see Remark S.2). When the explanatory variables are themselves regarded as independent observations, the necessary and sufficient conditions in Theorem III.1 hold (see Remark S.3).

**IV. Simulation**

We present two simulation examples. The first compares BLMC model with other multivariate Bayesian spatial models. The second assesses our BLMC model when $K$ is not excessively large.
BLMC models were implemented in Julia 1.2.0 (Bezanson et al., 2017). We modeled the univariate processes in the proposed BLMC by NNGP. We took the Bayesian LMC model proposed by Schmidt and Gelfand (2003) as a benchmark in the first simulation example. The benchmark model was implemented in R 3.4.4 through function spMisalignLM in the R package spBayes (Finley et al., 2007). The posterior inference for each model was based on MCMC chains with 5,000 iterations after a burn-in of 5,000 iterations. All models were run on a single 8 Intel Core i7-7700K CPU @ 4.20GHz processor with 32 Gbytes of random-access memory running Ubuntu 18.04.2 LTS. Convergence diagnostics and other posterior summaries were implemented within the Julia statistical environment. Model comparisons were based on parameter estimates (posterior mean and 95% credible interval), root mean squared prediction error (RMSE), mean squared error of intercept-centered latent processes (MSEL), prediction interval coverage (CVG; the percent of intervals containing the true value), interval coverage for intercept-centered latent process of observed response (CVGL), average continuous rank probability score (CRPS; see Gneiting and Raftery, 2007) for responses, and the average interval score (INT; see Gneiting and Raftery, 2007) for responses and run time. We assessed convergence of MCMC chains by visually monitoring auto-correlations and checking the accuracy of parameter estimates using effective sample size (ESS) (Gelman et al., 2013, Sec. 10.5) and Monte Carlo standard errors (MCSE) with batch size 50 (Flegal et al., 2008). To calculate the CRPS and INT, we assumed that the associated distribution was well approximated by a Gaussian distribution with mean centered at the predicted value and standard deviation equal to the predictive standard error. All NNGP models were specified with at most \( m = 10 \) nearest neighbors.

I. Simulation Example 1

We simulated the response \( y(s) \) from the LMC model in (II.2) with \( q = 2, p = 2, K = 2 \) over 1200 randomly generated locations over a unit square. The size of the data set was kept moderate to enable comparisons with the expensive full GP based LMC models for experiments conducted on the computing setup described earlier. The explanatory variable \( x(s) \) consists of an intercept and a single predictor generated from a standard normal distribution. An exponential correlation function was used to model \( \{ \rho_{\phi_k}(\cdot, \cdot) \}_{k=1}^K \), i.e., \( \rho_{\phi_k}(s, s') = \exp(-\phi_k \| s - s' \|) \), for \( s, s' \in D \), where \( \| s - s' \| \) is the Euclidean distance between \( s \) and \( s' \), and \( \phi_k = \phi_k \) is the decay for each \( k \). We randomly picked 200 locations for predicting each response to examine the predictive performance. Appendix S.3 presents the fixed parameters generating the data and the subsequent posterior estimates.

For NNGP based BLMC model, we assigned a flat prior for \( \beta \), which makes \( L^{-1} \beta \) in (II.10) a zero matrix. The prior for \( \Lambda \) followed (II.3) with \( \mu_\Lambda \) a zero matrix and \( V_\Lambda \) a diagonal matrix whose diagonal elements are 25. The prior for \( \Sigma \) was set to follow \( \text{IW}(\Psi, \nu) \) with \( \Psi = \text{diag}([1.0, 1.0]) \) and \( \nu = 3 \). For the benchmark LMC, we assigned a flat prior for \( \beta \), \( \text{IW}(\Psi, \nu) \) with \( \Psi = \text{diag}([1.0, 1.0]) \) and \( \nu = 3 \) for the cross-covariance matrix \( \Lambda^{-1} \Lambda \), and \( \text{IG}(2, 0.5) \) for each diagonal element of \( \Sigma \). We assigned \( \text{unif}(2.12, 212) \) as priors of decays for both models. This implies that the “effective spatial range”, which is the distance where spatial correlation drops below 0.05, will be bounded above by \( \sqrt{2} \) (the maximum inter-site distance within a unit square) and bounded below by 1/100th of that to ensure a wide range.

Table I presents posterior estimates of parameters and performance metrics for all candidate models. Both models provided similar posterior inferences for \( \{\beta_{21}, \beta_{21}\} \). The 95% credible intervals of \( \{\beta_{11}, \beta_{12}\} \) all include the true value used to generate the data. The NNGP based BLMC model and the benchmark LMC model cost 2.38 minutes and around 18.25 hours, respectively. Despite the shorter running time, we observed superior performance of the NNGP based BLMC
than the benchmark LMC for inferring on the latent process using CVGL, MSEL, CRPSL and INTL. Moreover, the interpolated map of the recovered intercept-centered latent processes (Figure 1) by BLMC and benchmark LMC are almost indistinguishable from each other. BLMC and benchmark LMC produce very similar RMSPEs, CRPSs and INTs. The differences in estimates between the two models is likely emerging from the different prior settings and sampling schemes. Benchmark LMC restricts the loading matrix $\Lambda$ to be upper triangular, while BLMC does not, resulting in greater flexibility in fitting latent process. On the other hand, the unidentifiable parameter setting of BLMC cause less somewhat less stable inference for the hyperparameters $\{\phi_1, \phi_2\}$. The inferences for $\{\beta_{11}, \beta_{12}\}$ are also less stable due to the sensitivity of intercept to latent process. For all other parameters including the intercept-centered latent process on 1200 locations, the median ESS is 4111.5. All MCSEs were consistently less than 0.02. These diagnostics suggest adequate convergence of the MCMC algorithm.

### Table 1: Simulation study summary table: posterior mean (2.5%, 97.5%) percentiles

|                  | true | BLMC inference | MCSE   | benchmark LMC inference | MCSE   |
|------------------|------|----------------|--------|-------------------------|--------|
| $\beta_{11}$     | 1.0  | 0.705 (0.145, 1.233) | 0.034  | 0.806 (0.502, 1.131) | 0.002  |
| $\beta_{12}$     | -1.0 | -1.24 (-1.998, -0.529) | 0.045  | -1.1 (-1.533, -0.646) | 0.001  |
| $\beta_{21}$     | -5.0 | -4.945 (-5.107, -4.778) | 0.002  | -4.949 (-5.113, -4.787) | 0.004  |
| $\beta_{22}$     | 2.0  | 1.979 (1.78, 2.166) | 0.004  | 1.974 (1.785, 2.167) | 0.002  |
| $\Sigma_{11}$    | 0.4  | 0.346 (0.283, 0.409) | 0.002  | 0.306 (0.248, 0.364) | 0.003  |
| $\Sigma_{12}$    | 0.15 | 0.133 (0.072, 0.194) | 0.003  | 0.0 –          | –      |
| $\Sigma_{22}$    | 0.3  | 0.29 (0.198, 0.386) | 0.004  | 0.233 (0.159, 0.334) | 0.005  |
| $\phi_1$         | 6.0  | 8.723 (4.292, 14.065) | 0.343  | 12.839 (8.805, 17.471) | 0.23   |
| $\phi_2$         | 18.0 | 22.63 (15.901, 29.555) | 0.416  | 18.075 (12.99, 23.741) | 0.301  |
| RMSPE$^a$        | –    | [0.728, 0.756, 0.742] |        | [0.725, 0.762, 0.744] |        |
| MSEL$^b$         | –    | [0.136, 0.168, 0.152] |        | [0.147, 0.192, 0.169] |        |
| CRPS$^a$         | –    | [-0.412, -0.423, -0.418] |       | [-0.41, -0.427, -0.418] |       |
| CRPSL$^b$        | –    | [-0.035, -0.038, -0.036] |       | [-0.216, -0.248, -0.232] |       |
| CVG$^a$          | –    | [0.915, 0.955, 0.935] |        | [0.925, 0.96, 0.9425] |        |
| CVGL$^b$         | –    | [0.946, 0.962, 0.954] |        | [0.756, 0.773, 0.765] |        |
| INT$^a$          | –    | [3.378, 3.756, 3.567] |        | [3.347, 3.823, 3.585] |        |
| INTL$^b$         | –    | [0.282, 0.329, 0.305] |        | [1.875, 2.023, 1.949] |        |
| time(s)          | 143  | [42047, 23664] |        | [42047, 23664] |        |

*a* [response 1, response 2, all responses]  
*b* intercept + latent process on 1000 observed locations for [response 1, response 2, all responses]  
*c* [time for MCMC sampling, time for recovering predictions]

II. Simulation Example 2

We generated 100 different data sets using (II.2) with $\{q = 10, p = 3, K = 50\}$ and a diagonal $\Sigma$ (i.e., independent measurement errors across outcomes). Appendix S.3 presents the parameter values used to generate the data sets. We fixed a set of 1200 irregularly situated locations inside a unit square. The explanatory variable $x(s)$ comprised an intercept and two predictors generated independently from a standard normal distribution. The same set of locations and explanatory variables were used for the 100 data sets. Each $f_k(s)$ was generated using an exponential covariance
(a) $\omega_1 + \beta_{11}$ true
(b) $\omega_1 + \beta_{11}$ BLMC
(c) $\omega_1 + \beta_{11}$ benchmark LMC
(d) $\omega_2 + \beta_{12}$ true
(e) $\omega_2 + \beta_{12}$ BLMC
(f) $\omega_2 + \beta_{12}$ benchmark LMC
(g) fitted correlation with $K = 2$
(h) fitted correlation with $K = 4$
(i) fitted correlation with $K = 6$
(j) fitted correlation with $K = 8$
(k) fitted correlation $K = 10$
(l) correlation of the raw data

Figure 1: Interpolated maps of (a) & (d) the true generated intercept-centered latent processes, the posterior means of the intercept-centered latent process $\omega$ from the (b) & (e) NNGP based BLMC model and the (c) & (f) benchmark LMC model. Heat-maps of the (l) actual finite sample correlation among latent processes and (g)-(k) posterior mean of finite sample correlation among latent processes based on the posterior samples of $\Omega$. 
Table 2: Simulation study summary table 2: posterior mean (2.5%, 97.5%) percentiles

| K = | 1             | 2             | 3             | 4             |
|-----|---------------|---------------|---------------|---------------|
| CVGL| 0.28(0.14, 0.93) | 0.39(0.17, 0.94) | 0.49(0.2, 0.95) | 0.58(0.25, 0.96) |
| CVG | 0.95(0.92, 0.98) | 0.95(0.92, 0.98) | 0.95(0.92, 0.98) | 0.95(0.92, 0.98) |
| RMSPE| 2.07(1.94, 2.18) | 1.96(1.86, 2.05) | 1.87(1.78, 1.97) | 1.78(1.7, 1.85) |
| MCSE| 0.004(0.002, 0.008) | 0.005(0.002, 0.01) | 0.005(0.003, 0.01) | 0.004(0.003, 0.01) |

| K = | 5             | 6             | 7             | 8             |
|-----|---------------|---------------|---------------|---------------|
| CVGL| 0.67(0.29, 0.96) | 0.74(0.33, 0.96) | 0.81(0.38, 0.96) | 0.86(0.44, 0.96) |
| CVG | 0.95(0.92, 0.98) | 0.95(0.92, 0.98) | 0.95(0.92, 0.98) | 0.95(0.91, 0.98) |
| RMSPE| 1.7(1.62, 1.77) | 1.63(1.55, 1.69) | 1.56(1.5, 1.63) | 1.51(1.45, 1.57) |
| MCSE| 0.005(0.003, 0.01) | 0.005(0.003, 0.013) | 0.005(0.003, 0.011) | 0.005(0.003, 0.011) |

| K = | 9             | 10            |
|-----|---------------|---------------|
| CVGL| 0.91(0.59, 0.96) | 0.95(0.92, 0.96) |
| CVG | 0.95(0.91, 0.98) | 0.95(0.91, 0.98) |
| RMSPE| 1.46(1.41, 1.51) | 1.43(1.38, 1.48) |
| MCSE| 0.005(0.003, 0.01) | 0.005(0.003, 0.01) |

function, \(\{\rho_{\psi_k}(\cdot, \cdot)\}_{k=1}^{K}\), where \(\psi_k = \phi_k\) was the decay for \(k = 1, \ldots, K\). We held out 200 locations for assessing predictive performances.

For each simulated data set, we fitted the BLMC model specifying a diagonal \(\Sigma\) with \(K\) from 1 to 10. Each \(\phi_k\) has a Gamma prior with shape and scale equaling 2 and 4.24, respectively, so that the expected effective spatial range is half of the maximum inter-site distance. We assigned flat prior for \(\beta\), a vague prior for \(\Lambda\) which follows the prior of \(\Lambda\) in the preceding example and \(IG(2, 1.0)\) priors for the diagonal elements of \(\Sigma\).

The posterior mean and the 95% credible interval of CVGL, CVG, RMSPE and diagnostics metric MCSE for regression slopes and \(\Sigma\) for 100 simulation studies are summarized by \(K\) in Table 2. Inference for CVG and MCSE were robust to the choice of \(K\). All of the 95% credible intervals for CVG and MCSE were within [0.9, 0.99] and [0.0, 0.02], respectively. As shown in Table 2, the performance metrics were quickly improved as \(K\) increased from 1 to 10. On average, RMSPE decreased by about 30.9% and CVGL increased from 28% to 95%. Given that our data comes from an LMC model with \(K = 50\), we can conclude that BLMC with diagonal \(\Sigma\) is efficient in obtaining inference for the latent processes even when \(K\) is not adequately large. We also create heat-maps of the posterior mean of our finite sample correlation matrix among the latent processes based on posterior samples of \(\Omega\), where \(\Omega = \frac{1}{n} \sum_{i=1}^{n} (\omega(s_i) - \bar{\omega})(\omega(s_i) - \bar{\omega})^\top\) with \(\bar{\omega}\) the vector of column means of \(\omega\). Figures 1g–1k depict such heat maps from one of the 100 simulated data sets. As \(K\) increases from 2 to 10, the estimated correlation matrix approaches the true correlation matrix. The plots also reveal that the performance of BLMC is sensitive to the choice of \(K\). We recommend choosing \(K\) based on scientific considerations for the problem at hand and exploratory data analyses, or checking the RMSPE value for different \(K\) and picking \(K\) by an elbow rule (Thorndike, 1953).

V. Remote-sensed Vegetation Data Analysis

We apply our proposed models to analyze Normalized Difference Vegetation Indices (NDVI) and Enhanced Vegetation Indices (EVI) measuring vegetation activity on the land surface, which can help us understand the global distribution of vegetation types as well as their biophysical and structural properties and spatial variations. Apart from vegetation indices, we consider Gross
Primary Productivity data, Global Terrestrial Evapotranspiration (ET) Product, and landcover data (see Ramon Solano et al., 2010; Mu et al., 2013; Sulla-Menashe and Friedl, 2018, for further details). The geographic coordinates of our variables were mapped on a Sinusoidal (SIN) projection grid. We focus on zone h08v05, which covers 11,119,505 to 10,007,555 meters south of the prime meridian and 3,335,852 to 4,447,802 meters north of the equator. The land is situated in the western United States. Our explanatory variables included an intercept and a binary indicator for no vegetation or urban area through the 2016 land cover data. All other variables were measured through the MODIS satellite over a 16-days period from 2016.04.06 to 2016.04.21. Some variables were rescaled and transformed in exploratory data analysis for the sake of better model fitting. The data sets were downloaded using the R package MODIS and the code for the exploratory data analysis is provided as supplementary material to this paper.

Our data comprises 1,020,000 observed locations to illustrate the proposed model. Our spatially dependent outcomes were the transformed NDVI \((\log(\text{NDVI} + 1))\) labeled as NDVI and red reflectance \((\text{red refl})\). A Bayesian multivariate regression model, defined by (II.2) excluding \(\Lambda^T f(s)\), was also fitted for comparisons. All NNGP based models used \(m = 10\) nearest neighbors. We randomly held out 10% of each response and then held all responses over the region 10,400,000 to 10,300,000 meters south of the prime meridian and 3,800,000 to 3,900,000 meters north of the equator to evaluate the models’ predictive performance over a missing region (white square) and randomly missing locations. Figure 2a illustrates the map of the transformed NDVI data.

We fit both models with 5,000 iterations after 5,000 iterations as burn-in. The priors for all parameters except decays followed those in the simulation section. We assigned Gamma\((200, 0.02)\) and Gamma\((200, 0.04)\) for \(\phi_1\) and \(\phi_2\) for BLMC based on fitted variograms to the raw data. All the code were run with single thread. No other processes were simultaneously run so as to provide an accurate measure of computing time.

Table 3 presents results on the BLMC. The regression coefficients of the index of no vegetation or urban area show relatively low biomass (low NDVI) and high red reflectance over no vegetation or urban area. Estimates of \(\Sigma\) and the finite sample process covariance matrix \(\Omega\), as defined in Section II, show a negative association between the residuals and latent processes of transformed NDVI and red reflectance, which satisfies the underlying relationship between two responses. BLMC captured a high negative correlation \((\approx -0.87)\) between the latent processes of two responses, indicating that the spatial pattern of the latent processes of NDVI and red-reflectance are almost the reverse of each other. The maps of the latent processes recovered by BLMC, presented in Figure 2, also support this relationship.

We provide RMSPE, CVG, CRPS, INT, MCSE and run time in Table 3. Apparently BLMC substantially improved predictive accuracy. BLMC’s RMSPEs were over 50% less than the Bayesian linear model. CVG is similar between two models, while INT and CRPS also favored BLMC over the Bayesian linear model. Figure 2 presents the estimated latent processes from BLMC. Notably, the BLMC smooths out the predictions in the held-out region. The model’s run time was around 38.6 hours, which is still impressive given the full model-based analysis it offers for such a massive multivariate spatial data set.

We also fitted a BLMC with diagonal \(\Sigma\) to explore the underlying latent processes of ten (transformed) responses: (i) NDVI, (ii) EVI, (iii) Gross Primary Productivity (GPP), (iv) Net Photosynthesis (PsnNet), (v) red reflectance (red refl), (vi) blue reflectance (blue refl), (vii) average daily global evapotranspiration (ET), (viii) latent heat flux (LE), (ix) potential ET (PET) and (x) potential LE (PLE). There are, in total, 12,057 locations with no responses and 656,366 observed locations with misaligned data (at least one but not all responses), which covers 65.12% of observed locations. We provide a heat-map (Figure 2) to present the status of misalignment over the study domain.
### Table 3: Vegetation data analysis summary table 1: posterior mean (2.5%, 97.5%) percentiles

| Variable                        | Bayesian linear model | BLMC | MCSE |
|---------------------------------|-----------------------|------|------|
|                                | inference             | inference | |
| intercept<sub>1</sub>           | 0.2515(0.2512, 0.2517) | 0.1433(0.1418, 0.1449) | 1.145e-4 |
| intercept<sub>2</sub>           | 0.1395(0.1394, 0.1396) | 0.1599(0.159, 0.1608) | 6.17e-5 |
| no vege or urban area<sub>1</sub> | -0.1337(-0.1346, -0.1328) | -1.385e-2 (-1.430e-2, -1.342e-2) | 1.69e-5 |
| no vege or urban area<sub>2</sub> | 6.035e-2 (5.992e-2, 6.075e-2) | 7.831e-3 (7.584e-3, 8.097e-3) | 8.24e-6 |
| Σ<sub>11</sub>                  | 1.599e-2 (1.594e-2, 1.603e-2) | 3.514e-4 (3.477e-4, 3.553e-4) | 1.93e-7 |
| Σ<sub>12</sub>                  | -6.491e-3(-6.512e-3, -6.471e-3) | -1.084e-4 (-1.100e-4, -1.067e-4) | 8.19e-8 |
| Σ<sub>22</sub>                  | 3.656e-3(3.646e-3, 3.667e-3) | 1.074e-4 (1.063e-4, 1.084e-4) | 4.79e-8 |
| Ω<sub>11</sub>                  | –                     | 1.675e-2(1.674e-2, 1.676e-2) | 4.17e-7 |
| Ω<sub>12</sub>                  | –                     | -6.873e-3(-6.879e-3, -6.867e-3) | 1.77e-7 |
| Ω<sub>22</sub>                  | –                     | 3.764e-3 (3.760e-3, 3.768e-3) | 9.06e-8 |
| φ<sub>1</sub>                   | –                     | 3.995 (3.887, 4.075) | 7.535e-3 |
| φ<sub>2</sub>                   | –                     | 12.376 (11.512, 13.320) | 7.60e-3 |

RMSPE<sup>a</sup> = [0.074, 0.0359, 0.0581]  
CRPS<sup>a</sup> = [-0.04135, -0.01988, -0.03061]  
CVG<sup>a</sup> = [0.956, 0.958, 0.957]  
INT<sup>a</sup> = [0.3468, 0.1711, 0.2589]  
Time(mins) = 10.83  

<sup>a</sup>[1st response transformed NDVI, 2nd response red reflectance, all responses]

Based on the exploratory analysis, we observed two groups of responses that have high within-group correlations but relatively low between-group correlations (see Figure 2g). Hence we picked $K = 2$. Estimates from the BLMC model are presented in Table 4. No vegetation or urban area exhibits lower vegetation indexes (lower NDVI and EVI) and lower production of chemical energy in organic compounds by living organisms (lower GPP and PsnNet). We observe a trend of higher blue reflectance, red reflectance, evapotranspiration (higher ET LE) and lower potential evapotranspiration (lower PET PLE) in urban area and area with no vegetation. We provide maps of posterior predictions for all 10 variables in Appendix S.5. The latent processes corresponding to transformed NDVI and red reflectance fitted in two analyses in Figure 2 share a similar pattern. Finally, the heat map of the posterior mean of the finite sample correlation among the latent processes (elements of $\Omega$ as defined in Section II) based on BLMC with diagonal $\Sigma$, presented in Figure 2h, reveals a high underlying correlation among NDVI, EVI, GPP, PsnNet, red and blue reflectance, and that LE and ET are slightly more correlated with NDVI and EVI than PLE and PET. The total run time for BLMC with diagonal $\Sigma$ was around 60.7 hours (3642.25 minutes).

### VI. Summary and Discussion

We have proposed scalable models for analyzing massive and possibly misaligned multivariate spatial data sets. Our framework offers flexible covariance structures and scalability by modeling the loading matrix of spatial factors using Matrix-Normal distributions and the factors themselves as NNGPs. This process-based formulation allows us to resolve spatial misalignment by fully model-based imputation. Through a set of simulation examples and an analysis of a massive misaligned data set comprising remote-sensed variables, we demonstrated the inferential and computational benefits accrued from our proposed framework.

This work can be expanded further in at least two important directions. The first is to extend the current methods to spatiotemporal data sets, where multiple variables are indexed by spatial...
coordinates, as considered here, as well as by temporal indices. Associations are likely to be exhibited across space and time as well as among the variables within a location and time-point. In addition, these variables are likely to be misaligned across time and space. Regarding the scalability of the spatiotemporal process, we can build a dynamic nearest-neighbor Gaussian process (DNNGP) (Datta et al., 2016b) to model spatiotemporal factors and one can also envisage...
A second direction will consider spatially-varying coefficient models. We model the regression coefficients $\beta$ using a spatial (or spatiotemporal) random field to capture spatial (or spatiotemporal) patterns in how some of the predictors impact the outcome. We can assign the prior of the regression coefficients $\beta$ using a multivariate Gaussian random field with a proportional cross-covariance function. Then the prior of $\beta$ over observed locations follows a Matrix-Normal distribution, which is the prior we designed for $\beta$ in all of the proposed models in this article. While the modification seems to be easy, the actual implementation requires a more detailed exploration, and we leave these topics for further explorations.

From a computational perspective, we clearly need to further explore high-performance computing and high-dimensional spatial models amenable to such platforms. The programs provided in this work are for illustration and have limited usage in Graphical Processing Units (GPU) computing and parallelized CPU computing. A parallel CPU computing algorithm for the BLMC model can simultaneously sample multiple MCMC chains, improving the performance of the actual implementations. Implementations with modeling methods such as MRA (Katzfuss, 2017) also requires dedicated programming with GPU. Other scalable modeling methods that build graphical Gaussian models on space, time and the number of variables can lead to sparse models for high-dimensional multivariate data and scale not only up to millions of locations and time points, but also to hundreds or even thousands of spatially or spatiotemporally oriented variables. The idea here will be to extend current developments in Vecchia-type models to graphs building dependence among a large number of variables so that the precision matrices across space, time and variables is sparse. Research on scalable statistical models and high-performance computing algorithms for such models will be of substantial interest to statisticians and environmental scientists.

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SUPPORTING INFORMATION

The MODIS vegetation indices data analyzed in Section V and the Julia code implementing our models are available at [https://github.com/LuZhangstat/Multi_NNGP](https://github.com/LuZhangstat/Multi_NNGP).

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We discuss the posterior predictions before going to the detailed algorithm. We use $N_m(u_i)$ to denote the $m$ neighbors of $u_i \in U$ among $S$. The posterior prediction for $f_k(U)$ given in (II.14) follows

$$f_k(U) \mid f_k, \psi_k \sim \mathcal{N}(\hat{A}f_k, D),$$

where the $(i,j)$-th entry of $\hat{A}$ is 0 when $s_j \notin N_m(u_i)$, and, similar to $A_{\psi_k}$, the $m$ nonzero entries in the $i$-th row of $\hat{A}$ corresponds to the elements of the $1 \times m$ vector $\hat{a}_l^T = \rho_{\psi_k}(u_i, N_m(u_i))\rho_{\psi_k}(N_m(u_i), N_m(u_i))^{-1}$. The $(i,i)$-th diagonal element of $\hat{D}$ equals $\rho_{\psi_k}(u_i, u_i) - \hat{a}_l^T \rho_{\psi_k}(N_m(u_i), u_i)$. And the posterior sample of $Y_U$ after giving posterior sample of $\beta, \Lambda, \Sigma$ and $F_U$ can be sampled through

$$MN(X_{ul}B + F_U\Lambda, I_n', \Sigma).$$

The following gives the detailed algorithm.

**Algorithm 1:** Obtaining posterior inference of \( \{\gamma, \Sigma, \omega\} \) and predictions on a new set $U$ for NNGP based BLMC model

1. Precalculation and preallocation for the MCMC algorithm
   
   (a) Find location sets $S, M$ and the index of the observed and missing response $\{os_i\}_{i=1}^n$ and $\{ms_i\}_{i=1}^m$.
   
   (b) Build the nearest neighbor for $S$.
   
   (c) Calculate Cholesky decompositions $V_A = L_A L_A^T$ and $V_\beta = L_\beta L_\beta^T$.
   
   (d) Preallocate MCMC samples and initialize MCMC chain with $\beta(0), \Lambda(0), \Sigma(0)$ and $\{\psi_k(0)\}_{1 \leq k \leq L}$.

2. Block update MCMC algorithm. For $l = 1 : L$

   (a) Update $F(l)$ and impute missing response $\{y(s_i)^{(l)}\}_{s_i \in M}$
   
   - Construct $\hat{X}$ and $\hat{Y}$ in (II.7) with $V_k = D_k^{1/2}(I - A_{\psi_k})$.
   
   - Use LSMR [Fong and Saunders, 2011] to generate sample of $F(l)$
   
   - Sample $u \sim \mathcal{N}(0, I_{kn})$.
   
   - Solve $\text{vec}(F(l)) - \text{vec}(F(l)) = \hat{Y} + u$ by LSMR.
   
   - Impute missing response $\{y(s_i)^{(l)}\}_{s_i \in M}$ over $M$ through (II.9)
   
   - Calculate $\mu_s = \beta(l-1)X(s) + \Lambda(l-1)F(s)$ for $s \in M$.
   
   - Sample $y(s)^{(l)}$ by (II.9) for $s \in M$.

   (b) Use MNIW to update $\{\beta(l), \Lambda(l), \Sigma(l)\}$
   
   - Construct $X^*$ and $Y^*$ in (II.10).
   
   - Generate $\Sigma(l)$
   
   - (When $\Sigma$ is a positive symmetric matrix)
• Calculate $\mu^*$, $V^{*-1}$, $\Psi^*$ and $v^*$ by \((\ref{eq:post})\) $O(n(p + K)(p + K + q))$

• Sample $\Sigma^{(i)}$ from IW($\Psi$, $\psi^*$)
  - (When $\Sigma$ is diagonal)
• Calculate $\mu^*$ by \((\ref{eq:post})\) $O(n(p + K)(p + K + q))$

• Sample elements of $\Sigma^{(i)}$ from Inverse-Gamma with parameters provided in \((\ref{eq:post})\)

• Sample $\gamma^{(i)} = [\beta^{(i)}^T, A^{(i)}^T]^T$ from MN($\mu^*$, $V^*$, $\Sigma^{(i)}$)

  i. Sample $u \sim MN(0, I_p, \psi_{2})$

  ii. Calculate Cholesky decomposition $V^{*-1} = L_V L_V^T$ and $\Sigma^{(i)} = L_{\Sigma^{(i)}} L_{\Sigma^{(i)}}^T$

  iii. Generate $\gamma^{(i)} = \mu^* + L_V^{\top} u L_{\Sigma^{(i)}}^{\top}$

 (c) Use Metropolis random walk to update $\{\Psi_i^{(k)}\}_{k=1}^K$

  i. Propose new $\{\Psi_i^{(k)}\}_{k=1}^K$ based on $\{\Psi_i^{(k-1)}\}_{k=1}^K$

  ii. Calculate the likelihood of the new proposed $\{\Psi_i^{(k)}\}_{k=1}^K$ and $\{\Psi_i^{(k-1)}\}_{k=1}^K$ given $F^{(i)}$ using \((\ref{eq:lik})\) $O(K(N^3))$

  iii. Accept the new $\{\Psi_i^{(k)}\}_{k=1}^K$ as $\{\Psi_i^{(k)}\}_{k=1}^K$ with the probability of the ratio of the likelihood of $\{\Psi_i^{(k)}\}_{k=1}^K$ and $\{\Psi_i^{(k-1)}\}_{k=1}^K$.

  Let $\{\Psi_i^{(k)}\}_{k=1}^K = \{\Psi_i^{(k-1)}\}_{k=1}^K$ when the new proposal is rejected.

3. Generate posterior samples of $\{f_i^{(l)}(u), y_i^{(l)}(u)\}$ on a new set $U$

(a) Construct $A$ and $D$ in \((\ref{eq:lin})\) $O(n'm^3K)$

(b) Generate $f_k(U)^{[l]} \sim N(Af_k, D)$ for $k = 1, \ldots, K$ $O(n'Km)$

(c) Sample $Y^{(l)} | u^{(l)}, \gamma^{(l)}, \Sigma^{(l)}, F^{(l)} \sim MN(X_0\beta + F_k\Lambda, I_q, \Sigma^{(l)})$

• Sample $u \sim MN(0, I_q, \psi_{4})$ $O(n'q)$

• Generate $Y^{(l)} = X_k\beta + F_k\Lambda + u L_{\Sigma^{(l)}}$ with $F^{(l)} = [f_1(U)^{[l]} : \cdots : f_K(U)^{[l]}]$ $O(n'(pq + Kq + q^2))$

\section*{S.2. Technical details and proofs of results in Section III}

Let us begin with a representation of posterior distributions of the latent model in Section III

Let $V_B$ be a non-singular square matrix such that $\rho^{-1}(S, S') = V_B V_B$. Treat the prior of $\gamma$ as additional observations and recast $p(Y, \gamma | \Sigma) = p(Y | \gamma, \Sigma) \times p(\gamma | \Sigma)$ into an augmented linear model

\[
\begin{bmatrix}
\sqrt{\frac{1}{1-n}} Y \\
L_{\gamma}^{-1} \mu_{\beta} \\
0
\end{bmatrix} = \begin{bmatrix}
\sqrt{\frac{1}{1-n}} X \\
L_{\gamma}^{-1} \\
0
\end{bmatrix} \begin{bmatrix}
\beta \\
\omega \\
\gamma
\end{bmatrix} + \begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix}, \quad \text{\(S.1\)}
\]

where $L_{\gamma}$ is the Cholesky decomposition of $V_{\gamma}$, and $\eta \sim MN(0, I_{2n+p} \Sigma)$. When having a flat prior for $\beta$, $L_{\gamma}^{-1}$ degenerates to a zero matrix, showing no information from $\beta$’s prior contributes to the linear system. The expression in \((\ref{eq:post})\) can be simplified as

\[
V^* = (X^* X^*)^{-1}, \quad \mu^* = (X^* X^*)^{-1} X^* Y^*, \\
\Psi^* = \Psi + (Y^* - X^* \mu^*)^T (Y^* - X^* \mu^*) \quad \text{\(S.2\)}
\]

We explore the behavior of the above posterior density as the number of observations becomes large under a true data generating distribution. Assume that the true distribution of the dependent variables is included in the parametric family $f(Y) = p(Y | \beta_0, \Sigma_0)$ for some $\Sigma_0$ and $\beta_0$. For
where the regression slopes \( \beta \) are posterior consistent for both conjugate models if and only if

\[ \text{Var}(\beta_{ij}|Y(n)) = 0 \]

for \( 1 \leq i,j \leq q \). When \( \beta \) is posterior consistent, \( \lim_{n \to \infty} \text{Var}(\beta_{ij}|Y(n)) = 0 \) a.s. is a sufficient condition for the posterior consistency of \( \beta \) through Chebyshev's inequality. In the conjugate model, \( \beta | Y(n) \sim T_{\rho,q}(\nu^*(n) - q + 1, \mu^*(n), V^*(n), \Psi^*(n)) \)

Theorem S.2. The matrix \( \Sigma \) in the conjugate multivariate models is posterior consistent if and only if \( \Psi^*(n)_{ij} \sim N(0, \Sigma_{ij}) \) a.s. with respect to the true distribution of \( Y(n) \). When \( \beta \) is posterior consistent, \( \lim_{n \to \infty} \text{Var}(\beta_{ij}|Y(n)) = 0 \) a.s. with respect to the true distribution of \( Y(n) \). Moreover, \( \lim_{n \to \infty} \text{Var}(\beta_{ij}|Y(n)) = 0 \) a.s. is a sufficient condition for the posterior consistency of \( \beta \) through Chebyshev's inequality. In the conjugate model, \( \beta | Y(n) \sim T_{\rho,q}(\nu^*(n) - q + 1, \mu^*(n), V^*(n), \Psi^*(n)) \)

Theorem S.3. The regression slopes \( \beta \) is posterior consistent for both conjugate models if and only if

\[ \lim_{n \to \infty} \lambda_{\min}(J(n)) = \infty, \text{where } \lambda_{\min}(J(n)) \text{ is the smallest eigenvalue of } J(n). \]
with parameters given in (III.3). From Theorem III.1, \( \lim_{n \to \infty} \Psi^* (n) / n = \Sigma_{ij} \) a.s. for \( 1 \leq i, j \leq q \), hence \( \lim_{n \to \infty} \text{Var} [\beta_{ij} | \mathbf{Y} (n)] = 0 \) a.s. if and only if \( \lim_{n \to \infty} \{ \mathbf{V}^s (n) \}_{ij} = 0 \) for all \( i = 1, \ldots, q \). Following Eicker (1963) (see his proof of Theorem 1), the sufficient and necessary condition is \( \lim_{n \to \infty} \lambda_{\text{min}} \{ \mathbf{V}^{s-1} (n) \} = \infty \). Since \( \lambda_{\text{min}} \{ \mathbf{J} (n) \} + \lambda_{\text{max}} \{ \mathbf{V}^{-1} \} \geq \lambda_{\text{min}} \{ \mathbf{V}^{s-1} (n) \} = \lambda_{\text{min}} \{ \mathbf{J} (n) + \mathbf{V}^{-1} \} \geq \lambda_{\text{min}} \{ \mathbf{J} (n) \} \), the condition simplifies to \( \lim_{n \to \infty} \lambda_{\text{min}} \{ \mathbf{J} (n) \} = \infty \). □

The following remarks reveal that the posterior consistency in Theorem III.1 satisfies with common conditions.

**Remark S.4.** \( \lambda_{\text{min}} \{ \mathbf{J} (n) \} \) is non-decreasing, and when \( \mathbf{J} \) is posterior consistent, \( \lim_{n \to \infty} \mathbf{J} (n) = \infty \) since \( \mathbf{J} (n)_{ij} = \lambda_{\text{min}} \{ \mathbf{J} (n) \} \)

**Proof.** Let \( \mathbf{X} (n + 1) = [\mathbf{X} (n)^\top, x_{n+1}]^\top, \mathbf{K} (n + 1) = \begin{bmatrix} \mathbf{K} (n) & \mathbf{K} (n)^{\top} \mathbf{a} \mathbf{a}^{-1} \\ \mathbf{a} \mathbf{a}^{-1} & \mathbf{a} \mathbf{a}^{-1} \end{bmatrix}. \) Then

\[
\mathbf{J} (n + 1) = \begin{bmatrix} \mathbf{X} (n)^\top, x_{n+1} \end{bmatrix} \begin{bmatrix} \mathbf{K} (n) & \mathbf{K} (n)^{\top} \mathbf{a} \mathbf{a}^{-1} \\ \mathbf{a} \mathbf{a}^{-1} & \mathbf{a} \mathbf{a}^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{X} (n) \\ x_{n+1} \end{bmatrix} = \mathbf{J} (n) + \{ \mathbf{X} (n) \top \mathbf{K} (n) \top \mathbf{K} (n, n+1) - x_{n+1} \} d \{ \mathbf{K} (n+1) \mathbf{K} (n) \mathbf{X} (n) - x_{n+1} \}
\]

(3.3)

where \( d = \{ \mathbf{a} \mathbf{a}^{-1} - \mathbf{K} (n, n+1) \mathbf{K} (n) - 1 \mathbf{K} (n, n+1) \} > 0 \) and \( \mathbf{A} (n) \) is positive semi-definite symmetric matrix. Thus, \( \lambda_{\text{min}} \{ \mathbf{J} (n + 1) \} = \lambda_{\text{min}} \{ \mathbf{J} (n) + \mathbf{A} (n) \} \geq \lambda_{\text{min}} \{ \mathbf{J} (n) \}. \) □

**Remark S.5.** When \( \mathbf{X} (n) \sim \mathbf{MN} (0, \mathbf{K} (n), \mathbf{\Sigma}^*) \) for some \( \mathbf{\Sigma}^* \), \( \mathbf{J} \) is posterior consistent.

**Proof.** Let \( \mathbf{K} (n)^{-1} \) be the square root of \( \mathbf{K} (n) \). Then, \( \mathbf{K} (n)^{-1} \mathbf{X} (n) \sim \mathbf{MN} (0, \mathbf{I}_n, \mathbf{\Sigma}^*) \) and the strong law of large numbers ensures \( \lim_{n \to \infty} \{ \frac{1}{n} \mathbf{J} (n) \}_{ij} = \Sigma_{ij}^* \) a.s. for \( 1 \leq i, j \leq p \). Hence, \( \lambda_{\text{min}} \{ \mathbf{J} (n) \} \to \infty \). □

**Remark S.6.** If \( \mathbf{X} (n) \sim \mathbf{MN} (0, \mathbf{I}_n, \mathbf{\Sigma}^*) \) for some \( \mathbf{\Sigma}^* \), then \( \mathbf{J} \) is posterior consistent.

**Proof.** For any \( n \), there exists an orthogonal matrix \( \mathbf{Q} (n) \) and a diagonal matrix \( \mathbf{D} (n) \) with diagonal entries \( d_i \) for \( i = 1, 2, \ldots, n \) such that \( \mathbf{C} (\mathbf{S} (n), \mathbf{S} (n)) = \mathbf{Q} (n)^\top \mathbf{D} (n) \mathbf{Q} (n) \). This yields

\[
\mathbf{J} (n) = \mathbf{X} (n)^\top \mathbf{Q} (n)^\top \{ \mathbf{D} (n) + (\mathbf{a}^{-1} - 1) \mathbf{I}_n \}^{-1} \mathbf{Q} (n) \mathbf{X} (n) = \mathbf{Z} (n)^\top \text{diag} \left\{ \frac{1}{d_i + (\mathbf{a}^{-1} - 1)} \right\}^n \mathbf{Z} (n) = \sum_{i=1}^n \frac{1}{d_i + (\mathbf{a}^{-1} - 1)} \mathbf{z}_i \mathbf{z}_i^\top \text{,}
\]

(4.4)

where \( \mathbf{Z} (n) = [\mathbf{z}_1 : \cdots : \mathbf{z}_n]^\top \sim \mathbf{MN} (0, \mathbf{I}_n, \mathbf{\Sigma}^*) \) and \( \sum_{i=1}^n d_i = n, d_i \geq 0, i = 1, \ldots, n \). Letting \( \mathbf{V}_i = \mathbf{z}_i \mathbf{z}_i^\top \) for \( i = 1, \ldots, n \) and applying the matrix version of Cauchy-Schwarz inequality (see, e.g., equation 4 in [Marshall and Olkin 1990]), we obtain

\[
\sum_{i=1}^n \frac{1}{d_i + (\mathbf{a}^{-1} - 1)} \sum_{i=1}^n \mathbf{V}_i \geq \left\{ \sum_{i=1}^n \sqrt{d_i + (\mathbf{a}^{-1} - 1)} \right\}^2 \mathbf{V}_i \geq \left\{ \sum_{i=1}^n \sqrt{d_i + (\mathbf{a}^{-1} - 1)} \mathbf{V}_i \right\}^2 \sqrt{n} \mathbf{u}_i \mathbf{u}_i^\top \text{,}
\]

where \( \mathbf{V}_i \mathbf{V}_i^\top = \mathbf{V}_i \), and, hence, \( \mathbf{J} (n) \geq \mathbf{a} \left\{ \sum_{i=1}^n \mathbf{V}_i \right\}^2 \mathbf{u}_i \mathbf{u}_i^\top \text{.} \) Letting \( \mathbf{V}_i = \mathbf{z}_i \mathbf{z}_i^\top = \lambda_i \mathbf{u}_i \mathbf{u}_i^\top \) where \( \lambda_i = \mathbf{z}_i^\top \mathbf{z}_i = ||\mathbf{z}_i||^2 \) and \( \mathbf{u}_i = \frac{\mathbf{z}_i}{||\mathbf{z}_i||} \), we have \( \mathbf{V}_i \mathbf{V}_i^\top = \sqrt{\lambda_i} \mathbf{u}_i \mathbf{u}_i^\top \). Changing \( n \) into \( np \) and rewriting
\[ \sum_{i=1}^{n} V_i^2 = \sum_{i=1}^{n} \sum_{k=1}^{p} V_{ik}^2 \] where \( \{ \sum_{k=1}^{p} V_{ik}^2 \} \) for each \( i \) is a full rank \( p \times p \) matrix with probability 1, we obtain
\[ J(np) \geq \frac{\alpha}{p} \left( \sum_{i=1}^{n} \sum_{k=1}^{p} \frac{\lambda_{ik} u_{ik} u_{ik}^T}{\sqrt{n}} \right)^2 \] \( \text{(S.5)} \)

We now argue that the smallest eigenvalue of the matrix on the right side goes to infinity as \( n \to \infty \), which will imply that \( \lambda_{\min}\{J(np)\} \to \infty \). Since \( \sum_{k=1}^{p} V_{ik}^2 = \sum_{k=1}^{p} \sqrt{\lambda_{ik}} u_{ik} u_{ik}^T \sim W_p(S^2, p) \), where \( W_p \) is Wishart distribution, \( \{u_{i1}, \ldots, u_{ip}\} \) make up the bases of the space \( \mathbb{R}^p \) with probability 1. For any \( u \in \mathbb{R}^p, ||u|| = 1 \), we have
\[ \sum_{k=1}^{p} \left( \sqrt{\lambda_{ik}} u_{ik} u_{ik}^T \right) u \geq \min_{k=1,\ldots,p} (\sqrt{\lambda_{ik}}) \]
Hence, \( \lambda_{\min}(\sum_{i=1}^{n} \sum_{k=1}^{p} \sqrt{\lambda_{ik}} u_{ik} u_{ik}^T) \geq \sum_{i=1}^{n} \min_{k=1,\ldots,p} (\sqrt{\lambda_{ik}}) \). Since \( \lambda_{ik} = ||z_{ik}||^2 \) where \( z_{ik} \sim N(0, \Sigma^*) \), \( \min_{k=1,\ldots,p} (\sqrt{\lambda_{ik}}) \) are independent and identically distributed with a positive mean \( \mathbb{E}(\min_{k=1,\ldots,p} (\sqrt{\lambda_{ik}})) = c^* > 0 \) and a finite variance \( \sigma^2 \). By law of large numbers, we have \( \lim_{n \to \infty} \sum_{i=1}^{n} \min_{k=1,\ldots,p} (\sqrt{\lambda_{ik}})/n = c^* \) a.s. Therefore,
\[ \lambda_{\min} \left\{ \left( \sum_{i=1}^{n} \sum_{k=1}^{p} \frac{\lambda_{ik} u_{ik} u_{ik}^T}{\sqrt{n}} \right)^2 \right\} \geq \frac{1}{n} \left\{ \sum_{i=1}^{n} \min_{k=1,\ldots,p} (\sqrt{\lambda_{ik}}) \right\}^2 \to \infty \]

By \( \text{(S.5)} \), \( \lim_{n \to \infty} \lambda_{\min}\{J(n)\} = \infty \).

\[ \text{(S.5)} \]

**S.3. Values of parameters in simulation examples**

**I. Values of parameters to generate simulations in simulation example 1**

\[ \Sigma = \begin{bmatrix} 0.4 & 0.15 \\ 0.15 & 0.3 \end{bmatrix} \quad \beta = \begin{bmatrix} 1.0 & -1.0 \\ -5.0 & 2.0 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 1.0 & 1.0 \\ 0.0 & 2.0 \end{bmatrix} \quad \phi_1 = 6.0, \phi_2 = 18.0 \]

**II. Values of parameters to generate simulations in simulation example 2**

\[ \{\Sigma_{ii}\}_{i=1}^{10} = (0.5, 1, 0.4, 2, 0.6, 2.5, 3.0, 0.45, 1.5, 0.5) \]

\[ \{\phi_k\}_{k=1}^{50} = (11.36, 13.43, 10.22, 6.87, 5.89, 10.09, 9.17, 2.75, 5.35, 3.43, 4.09, 7.81, 12.52, 9.54, 5.56, 7.7, 5.44, 7.49, 9.12, 5.2, 10.61, 5.63, 5.5, 11.65, 4.64, 13.16, 9.51, 11.77, 8.8, 13.43, 7.89, 11.62, 6.4, 12.95, 8.48, 2.5, 12.95, 13.42, 9.59, 6.31, 8.98, 4.57, 6.63, 11.25, 4.43, 4.94, 3.3, 9.66, 13.5, 8.7) \]

\[ \beta = \begin{bmatrix} 1.0 & -1.0 & 1.0 & -0.5 & 2.0 & -1.5 & 0.5 & 0.3 & -2.0 & 1.5 \\ -5.0 & 2.0 & 3.0 & -2.0 & -6.0 & 4.0 & 5.0 & -3.0 & 6.0 & -4.0 \\ 8.0 & 6.9 & -12.0 & 0.0 & -4.0 & 7.7 & -8.8 & 3.3 & 6.6 & -5.5 \end{bmatrix} \]

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\[
\{\Lambda_{ij}\}_{i,j=10,11} = \\
\begin{bmatrix}
-0.38 & -0.33 & 0.23 & -0.38 & -0.13 & 0.31 & 0.28 & 0.0 & 0.42 & 0.18 \\
-0.39 & -0.13 & -0.13 & -0.31 & -0.15 & 0.42 & 0.13 & -0.07 & 0.21 & 0.1 \\
0.01 & 0.48 & 0.32 & 0.13 & -0.46 & 0.27 & 0.09 & 0.12 & 0.25 & 0.38 \\
0.13 & 0.48 & -0.47 & -0.48 & -0.34 & -0.09 & -0.28 & -0.21 & -0.19 & 0.44 \\
-0.47 & 0.46 & 0.24 & -0.45 & 0.44 & -0.29 & -0.36 & -0.46 & 0.34 & -0.44 \\
0.21 & 0.12 & -0.46 & 0.29 & 0.36 & 0.17 & 0.03 & 0.2 & -0.12 & -0.23 \\
-0.2 & 0.48 & 0.18 & -0.1 & 0.13 & -0.13 & -0.41 & -0.04 & -0.07 & -0.22 \\
-0.19 & 0.28 & 0.47 & -0.42 & 0.17 & -0.18 & -0.03 & -0.13 & -0.04 & 0.3 \\
-0.04 & 0.27 & -0.23 & -0.07 & -0.09 & -0.39 & -0.48 & -0.27 & 0.19 & 0.21 \\
-0.03 & -0.18 & -0.08 & -0.12 & 0.35 & 0.3 & -0.33 & 0.34 & 0.38 & 0.31 \\
\end{bmatrix}
\]

\[
\{\Lambda_{ij}\}_{i,j=10,11} = \\
\begin{bmatrix}
-0.15 & -0.43 & 0.27 & 0.18 & 0.38 & -0.4 & -0.27 & -0.3 & -0.38 & -0.09 \\
0.28 & -0.24 & -0.32 & -0.08 & -0.01 & -0.31 & 0.2 & 0.31 & 0.11 & -0.38 \\
-0.38 & -0.42 & -0.16 & -0.37 & -0.22 & 0.09 & -0.08 & -0.07 & -0.33 & 0.01 \\
-0.42 & -0.22 & 0.44 & 0.09 & 0.25 & 0.12 & 0.1 & -0.33 & -0.41 & 0.42 \\
0.42 & 0.48 & -0.06 & 0.07 & 0.43 & 0.12 & -0.15 & 0.29 & 0.1 & -0.32 \\
-0.15 & -0.03 & -0.42 & 0.01 & 0.05 & 0.33 & -0.46 & 0.12 & 0.22 & -0.44 \\
0.28 & -0.08 & -0.41 & 0.13 & 0.03 & 0.22 & 0.08 & 0.32 & 0.02 & -0.41 \\
0.35 & -0.39 & 0.37 & -0.47 & -0.08 & -0.01 & 0.09 & 0.06 & -0.21 & 0.38 \\
-0.38 & 0.2 & -0.21 & 0.21 & -0.11 & 0.27 & 0.2 & 0.17 & 0.31 & -0.12 \\
0.36 & -0.09 & -0.16 & -0.06 & 0.43 & -0.04 & -0.07 & 0.4 & -0.39 & -0.06 \\
\end{bmatrix}
\]

\[
\{\Lambda_{ij}\}_{i,j=10,21} = \\
\begin{bmatrix}
0.08 & 0.15 & 0.11 & 0.37 & 0.25 & 0.28 & 0.13 & -0.18 & 0.35 & 0.17 \\
-0.32 & -0.31 & 0.24 & -0.29 & -0.38 & -0.1 & -0.19 & 0.18 & -0.37 & -0.34 \\
0.45 & 0.19 & 0.34 & -0.36 & 0.43 & 0.44 & -0.13 & -0.26 & -0.46 & -0.08 \\
0.38 & -0.48 & -0.22 & -0.14 & 0.5 & 0.08 & 0.02 & -0.07 & 0.07 & -0.3 \\
-0.49 & -0.48 & 0.34 & 0.1 & -0.01 & 0.2 & 0.33 & 0.37 & 0.1 & 0.21 \\
0.1 & 0.11 & -0.33 & -0.16 & 0.06 & 0.25 & -0.37 & -0.1 & -0.16 & -0.13 \\
0.45 & 0.02 & -0.21 & 0.16 & 0.37 & -0.2 & -0.44 & -0.37 & 0.46 & 0.25 \\
0.34 & 0.31 & 0.06 & -0.25 & 0.37 & 0.12 & 0.27 & -0.35 & 0.09 & -0.28 \\
-0.2 & -0.12 & -0.41 & 0.23 & -0.23 & -0.07 & -0.34 & 0.37 & -0.43 & 0.18 \\
0.36 & 0.14 & 0.47 & 0.3 & 0.36 & -0.09 & 0.1 & -0.01 & 0.11 & 0.43 \\
\end{bmatrix}
\]

\[
\{\Lambda_{ij}\}_{i,j=10,31} = \\
\begin{bmatrix}
0.42 & 0.17 & -0.24 & 0.05 & -0.0 & -0.41 & -0.03 & -0.0 & -0.22 & 0.2 \\
0.26 & -0.22 & 0.33 & -0.06 & -0.06 & -0.36 & -0.31 & 0.14 & -0.14 & -0.1 \\
0.09 & 0.43 & 0.04 & -0.35 & 0.42 & 0.19 & 0.33 & -0.12 & 0.4 & -0.32 \\
-0.13 & 0.36 & 0.02 & 0.02 & 0.34 & 0.06 & -0.32 & -0.47 & 0.02 & 0.34 \\
0.27 & -0.35 & -0.12 & 0.5 & 0.33 & 0.33 & -0.27 & 0.39 & 0.45 & 0.27 \\
0.38 & 0.11 & 0.05 & 0.38 & -0.34 & -0.19 & -0.12 & 0.39 & 0.2 & 0.31 \\
0.16 & 0.31 & 0.02 & -0.43 & 0.13 & 0.33 & -0.34 & -0.1 & 0.41 & -0.46 \\
0.32 & -0.2 & -0.18 & -0.05 & 0.2 & -0.17 & -0.06 & 0.49 & -0.06 & 0.3 \\
0.44 & -0.05 & 0.06 & -0.22 & -0.16 & -0.43 & 0.04 & -0.23 & -0.22 & 0.11 \\
-0.23 & -0.34 & 0.45 & -0.47 & 0.03 & -0.09 & -0.47 & 0.28 & 0.27 & -0.4 \\
\end{bmatrix}
\]
We offer a brief discussion on jointly modeling outcomes and independent univariate modeling of outcomes. We use the setting in our first simulation experiment in Section I of the main manuscript. We fit a latent NNGP model using the R package spNNGP (Finley et al., 2017) for each outcome individually for the simulated data there. The priors for regression coefficients and decay were the same as that of the BLMC model. The priors for the partial sill and nugget were IG(2, 1) and IG(2, 0.5), respectively. The maximum number of nearest neighbors was set to be \( m = 10 \). The posterior inference was based on an MCMC chain with 5,000 iterations after an initial burn-in of 5,000 iterations. Table 5 compares the posterior inference along with performance metrics of the extended data analysis with that of the BLMC model.

Jointly modeling all the outcomes, as discussed in Section I of the main manuscript, often yields better predictions compared to modeling each outcome separately. In this experiment, we observed that the BLMC model provided more precise predictions than the univariate latent NNGP model based on RMSPEs, MSELs and INTs. For example, the RMSPE for the second outcome underwent a reduction of 8.5% in the joint model as compared to the independent model. All performance metrics for measuring the posterior inferences on latent processes favor multivariate modeling than independent univariate models in this simulation study.

S.5. Maps of predictions for 10 responses of the factor BLMC model in Real Data Analysis
### Table 5: Simulation study summary table: posterior mean (2.5%, 97.5%) percentiles

|                  | BLMC true inference | BLMC MCSE | univariate latent NNGP inference | univariate latent NNGP MCSE |
|------------------|---------------------|-----------|----------------------------------|-----------------------------|
| \( \beta_{11} \) | 1.0                 | 0.705 (0.145, 1.233) | 0.034 | 0.764 (0.372, 1.199) | 0.025 |
| \( \beta_{12} \) | -1.0                | -1.24 (-1.998, -0.529) | 0.045 | -1.101 (-1.511, -0.596) | 0.027 |
| \( \beta_{21} \) | -5.0                | -4.945 (-5.107, -4.778) | 0.002 | -4.96 (-5.133, -4.795) | 0.003 |
| \( \beta_{22} \) | 2.0                 | 1.979 (1.78, 2.166) | 0.004 | 1.975 (1.777, 2.168) | 0.004 |
| \( \Sigma_{11} \) | 0.4                 | 0.346 (0.283, 0.409) | 0.002 | 0.361 (0.303, 0.421) | 0.002 |
| \( \Sigma_{12} \) | 0.15                | 0.133 (0.072, 0.194) | 0.003 | 0.00 | 0.0 |
| \( \Sigma_{22} \) | 0.3                 | 0.29 (0.198, 0.386) | 0.004 | 0.299 (0.208, 0.392) | 0.004 |
| \( \phi_1 \)    | 6.0                 | 8.723 (4.292, 14.065) | 0.343 | 9.393 (4.906, 13.976) | 0.247 |
| \( \phi_2 \)    | 18.0                | 22.63 (15.901, 29.555) | 0.416 | 14.086 (10.114, 18.366) | 0.226 |
| RMSPE\(^a\)     | -                   | [0.728, 0.756, 0.742] |          | [0.733, 0.826, 0.781] |          |
| MSEL\(^b\)      | -                   | [0.136, 0.168, 0.152] |          | [0.139, 0.172, 0.156] |          |
| CRPS\(^a\)      | -                   | [-0.412, -0.423, -0.418] |          | [-0.41, -0.427, -0.418] |          |
| CRPSL\(^b\)     | -                   | [-0.035, -0.038, -0.036] |          | [-0.21, -0.235, -0.222] |          |
| CVG\(^a\)       | -                   | [0.915, 0.955, 0.935] |          | [0.945, 0.96, 0.9525] |          |
| CVGL\(^b\)      | -                   | [0.946, 0.962, 0.954] |          | [0.787, 0.798, 0.792] |          |
| INT\(^a\)       | -                   | [3.378, 3.756, 3.567] |          | [3.396, 4.083, 3.739] |          |
| INTL\(^b\)      | -                   | [0.282, 0.329, 0.305] |          | [1.75, 1.917, 1.834] |          |
| time(s)          | -                   | 143       |          | 139 |          |

\(^a\)[response 1, response 2, all responses]  
\(^b\)intercept + latent process on 1000 observed locations for [response 1, response 2, all responses]  
\(^c\)[time for MCMC sampling, time for recovering predictions]
Figure 3: Maps (a)-(j) of predicted value on 1,020,000 observed locations for 10 variables in Section V. The deeper the color, the higher the value. Some variables are transformed for better model fitting. All values are estimated by posterior mean. Each map has its own color scale.