Instanton-torus knot duality in 5d SQED and $SU(2)$ SQCD

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Abstract

We briefly review the instanton-torus knot duality found in 5d SUSY gauge theories with one compact dimension. The fermion condensate turn out to be the generating function for the torus knot polynomials colored by the fundamental representation.
I. INTRODUCTION

The field theory derivation of the knot invariants was initiated long time ago [1] when the Jones polynomial was identified as the vev of Wilson loop along the knot in the $SU(2)$ 3d Chern-Simons (CS) topological theory. Similar consideration of $SU(N)$ CS theory yields the HOMFLY polynomials of knot colored by the arbitrary representation. The CS approach has been embedded into the topological string framework in [2, 3] where the knot is realized via the intersection of branes in the Calabi-Yau 3-manifold. The invention of the Khovanov homologies allowed to introduce the generalization of the HOMFLY knot polynomials - superpolynomials [4]. For instance, the Jones polynomial can be expressed in terms of the Khovanov homologies $H^{i,j}$ as

$$ J(q) = \sum_{i,j} (-1)^i q^j \dim H^{i,j} \quad (1) $$

The refinement of the CS partition function which allows to evaluate the torus knot superpolynomials has been suggested in [5] however the refined CS was defined via matrix model and its explicit field theory representation is still questionable. The candidate for this field theory has been discussed recently in [6].

The Wilson loop approach is not the only way to recognize the knot polynomials in the field theory. It was shown in [7] that the knot polynomials count the multiplicities of the BPS states in SUSY YM theories and can be considered as the refined BPS index depending on the set of chemical potentials which serve as the generating parameters. A bit schematically this approach can be summarized in the representation of the superpolynomial for the knot $K$ as

$$ P_K(a,q,t) = \sum_{ijk} c_{ijk} a^i q^j t^k \quad (2) $$

where the three gradings correspond to three equivariant parameters for the global symmetries. The numbers $c_{ijk}$ count the dimensions of the corresponding spaces of BPS states.

Later the subject got twisted in one more direction and it was argued in [8] that via 3d/3d duality the knot invariants can be used for the classification of 3d SUSY gauge theories. The particular knot at one side of the correspondence fixes the matter content in the particular dual theory and the knot polynomials serve as some index. More recently there was attempt [9] to relate the refined knot invariants to the instanton counting lifting the CS theory to the topologically twisted $N = 4$ $D = 4$ SYM theory and its $D = 5$ counterpart. The nice
recent reviews summarizing the physical and mathematical aspects of the knot homologies can be found in \[10\] and \[11\].

Here we shall briefly review new reincarnation of the knot polynomials found in \[12,13\] based on the observation made in \[14\]. The new place for the knot invariants is the evaluation of the fermion condensates in Omega-deformed 5D SQED and \(SU(2)\) SQCD. In an interesting way this novel approach glues together all previously considered viewpoints concerning the roles of knot polynomials and add new flavor to this issue. The motivation for this new approach goes as follows. It was argued in \[15\] that bottom row \((a=0)\) of superpolynomial for the \((n,n+1)\) torus knots is expressed in terms of the \((q,t)\) deformed Catalan numbers \(C_n(q,t)\) which are intimately related with the particular equivariant integrals over the moduli space of centered points on \(C^2\) \[16\]. On the other hand the evaluation of the HOMFLY polynomials of generic torus knots in terms of \(Hilb^n(C^2)\) has been developed in \[17–20\]. It has been extended to the evaluation of superpolynomial in \[22\].

Hence it is natural to suggest that in the physical language we are evaluating vev of some observable in the K-theory of centered abelian instantons in 5d gauge theory with some matter content. The problem concerns the identification of the particular gauge theory and the particular observable which does the job. The answers to these questions have been found in \[12,13\].

It turned out that the proper field theory is the Omega-deformed 5d SQED or \(SU(2)\) SQCD with some number of flavors on \(R^4 \times S^1\) supplemented in some cases by 5d CS term. The proper observables are the derivatives of the Nekrasov partition functions \[23\] with respect to the masses of the fundamentals. A bit loosely it could be considered as IR counterpart of the fermion condensate defined in the UV. The two integers characterizing \((n,m)\) torus knot were identified as instanton and electric quantum numbers and summation over quantum numbers of BPS particles necessary for evaluation of observable implies the summation over the torus knot polynomials with all \((n,m)\). More precisely the derivative of the Nekrasov partition function is nothing but the generating function for the torus knot polynomials. The three generating parameters in the superpolynomial \(P_{n,m}(a,q,t)\) of \((n,m)\) torus knot are identified with two equivariant parameters of the Omega-background and the mass of the matter in antifundamental representation.
II. SUPERPOLYNOMIAL OF \((n, nk + 1)\) TORUS KNOTS

Recall that the torus knots in \(S^3\) can be obtained as follows. The \(S^3\) manifold can be represented as two solid tori glued together with the help of S-matrix changing the cycles. The boundary of one torus is equipped with the curve with winding numbers \((n, m)\) over the cycles. Therefore we can present the torus knot schematically as \(<n, m|S|0, 0>\) where \(|0, 0>\) is the state of pure solid torus. The series \((n, nk + 1)\) plays the special role since the corresponding knots can be obtained from the unknot \((n, 1)\) in a simple manner. We start with \((n, 1)\) unknot and take into account that due to the Witten’s effect the instanton with topological charge \(n_I\) acquires the electric charge \(n_I k\) where \(k\) is the level of 5d CS term. For the \((n, nk + 1)\) series we are able to identify in physical terms the superpolynomials depending on three generating parameters \(P_{n,nk+1}(a, q, t)\). It was shown in [12, 13] that the \(T_{n,nk+1}\) superpolynomials are encoded in the UV properties of the condensate of the massless flavor in the 5d SQED with one compact dimension and 5d CS term at level \(k\).

To explain the physical picture it is necessary to remind the spectrum of BPS particles in the 5d SQCD. The corresponding central charge involves the quantum numbers corresponding to the instantons, W-bosons and fundamentals (for SU(2)) [25]

\[
Z = \frac{1}{g^2} n_I + n_e a + \sum_i n_f m_f
\]  
(3)

where \(g\) is coupling constant and \(a\) is vev of the real adjoint scalar. The instantons in 5d theory are particles which carry the charge corresponding to the conserved topological current

\[
J = * Tr F \wedge F
\]  
(4)

If we add to the action the CS term

\[
S_{CS} = k \int Tr A \wedge F \wedge F
\]  
(5)

it implies that the instanton particle with charge \(n_I\) carries the electric charge \(n_I k\) and the central charge can be written as

\[
Z = (n_e + kn_I)a + \frac{1}{g^2} n_I + \sum_i n_f m_f
\]  
(6)

Generically a particle carries quantum numbers \((n_I, n_e, n_f)\). The specific wall crossing phenomena in the BPS spectrum have been investigated in [26, 27]. The one loop contribution
to the 5d effective action involving all BPS states reproduces the full 4d effective actions upon reduction \[28\].

In \[12\] the new instanton-torus knot duality has been formulated for the Omega-deformed 5d SUSY QED with \(N_F = 3\) on \(S^1_\beta \times R^4_\Omega\) with the Chern-Simons term at level \(k\). It has been proved that the second derivative of the Nekrasov instanton partition function with respect to the masses of the hypermultiplets is the generating function for the superpolynomials of the torus \(T_{n,nk+1}\) knots where \(n\) is the instanton charge.

\[
\frac{e^{\beta M}}{(1 + A)\beta^2} \frac{d^2 Z_{nek}(q, t, m_f, M, m_a, Q, k)}{dM dm_f} \bigg|_{m_f \to 0, M \to \infty} = \sum_n Q^n (tq)^{n/2} P_{n,nk+1}(q, t, A) \tag{7}
\]

where \(m_a, m_f, M\) are masses of three hypermultiplets in antifundamental \((m_a)\) and fundamental representations \((m_f, M)\) and \(Q\) is the counting parameter for the instantons. The mapping between the parameters at the lhs and rhs goes as follows

\[
t = \exp(-\beta \epsilon_1) \tag{8}
\]
\[
q = \exp(-\beta \epsilon_2) \tag{9}
\]
\[
A = -\exp(\beta m_a) \tag{10}
\]
\[
Q = \exp(-\beta/g^2) \tag{11}
\]

where \(\beta\) corresponds to the radius of \(S^1\) and \(\epsilon_1, \epsilon_2\) are two equivariant parameters of the Omega-background which physically mean two independent angular velocities in \(R^4\).

The selection of two special masses provides the reduction of the generic double sum over quantum numbers to the summation over the instanton number without the summation over the electric charges. One could say that we select the particular observable in the gauge theory with the fixed value of the electric charge. Indeed it is possible to trade the regulator field with mass \(M\) to the insertion of the particular operator \(\exp(\beta \phi)\), where \(\phi\) is the scalar in adjoint, inserted at some point in \(R^4\) which is the particular example of qq-character \[24\]. This realization of the generating function for the superpolynomials explains why the answer deals with the instantons centered around the point with the operator insertion.

Hence we can claim that from the 5d theory viewpoint the refined knot invariants are sensitive to the interference of two independent rotations in \(R^4\) and the regulator UV scale
M. Since the abelian instantons are centered we could say a bit loosely that the refined knot invariants govern the renormalization effects from the point-like instantons or sheaves on Hilbert scheme of $C^2$ in more mathematical terms. The rich life at point in $\mathbb{R}^4$ can be also recognized from the CY side since in M theory the instantons actually are M2 branes extended in CY manifold. To conclude this Section we mention that it is worth to think that the information about the knot superpolynomial is encoded in the non-perturbative UV properties of the fermionic condensate of the massless flavor.

III. THE HOMFLY POLYNOMIALS FOR GENERIC $(n,m)$ TORUS KNOTS FROM $SU(2)$ SQCD

A. $N_f = 2$ theory with Lagrangian brane

In this Section we shall consider the unrefined case and corresponding HOMFLY polynomials. We switched off the second independent parameter of the background however as a bonus are able to represent the fermion condensate as the double series over two quantum numbers and therefore over all torus knots. To this aim instead of the insertion of the particular operator in $N_f = 2$ theory we can consider the $N_f = 2$ $SU(2)$ SQCD supplemented by the Lagrangian brane with zero framing with some value of Fayet-Iliopoulos(FI) parameter $z$. To get the HOMFLY polynomials we make two step procedure. First, we consider the decoupling limit $1/g^2 \to \infty$ in $SU(2)$ theory when it effectively decouples into the product of two $U(1)$ theories and pure 4d instantons decouple. However due to the additional Lagrangian brane we have the FI parameter which counts the instantons on the Lagrangian brane $[29]$. Considering the derivative of the Nekrasov partition function in this case with respect to mass and expanding it into the double series $z^m Q^n$ we obtain the HOMFLY polynomials of the generic $(n, m)$ knots as the coefficients of the expansion.

\[
\langle \tilde{\psi} \psi \rangle_{LB} = \left. \frac{\partial Z_{\text{inst}}}{\partial m_f} \right|_{m_f = 0} = \sum_{n,m} Q^n z^m P_{n,nk+m}(A, q, t) \tag{12}
\]

\[
P(A, q, t)_{n,nk+m} = \sum_{\lambda, |\lambda| = m} \frac{t^{(k+1)} \Sigma q^{(k+1)} \Sigma a (1 - t) \prod_{0}^{0,0} (1 + A q^{-a' t' - t'}) \prod_{0}^{0,0} (1 - q^{a' t'}) \prod (q^a - t^{l+1}) \prod (t^{l'} - q^{a'+1})}{\prod (q^a - t^{l+1}) \prod (t^{l'} - q^{a'+1})} \times \text{Coef}_{z^m M(z)} \]  

6
where $M(z)$ is the contribution from the Lagrangian brane with zero framing:

$$M(z) = \prod_{j=1}^{l(\lambda)} \frac{1 - zt^{j-1}q^{\lambda_j}}{1 - zt^{j-1}}$$  \hspace{1cm} (14)$$

In this case the parameter $z$ counts the 2d instantons while the parameter $Q$ equals to $\exp(\beta a)$ and counts the number of W-bosons in the decoupling perturbative limit of $SU(2)$ theory. In this approach we can say that HOMFLY polynomials provide the entropic factor in the condensate in the sector with the particular defect.

**B. SU(2) $N_f = 4$ SQCD**

The most effective way to evaluate the effective action in 5d gauge theory is the use of the refined vertex \[30\]. The counting of the torus knot invariants along this way can be found in \[31, 36\]. We can use some useful properties of the toric diagrams for $SU(2)$ SQCD to embed the abelian theory into the SU(2) with $N_f = 4$. Remind that the abelian 5d theory is engineered by $O(-1) \times O(-1) \to CP_1$ CY manifold. The 5d CS term adds the nontrivial framing while each flavor corresponds to blow-up of one point.

To get the $SU(2)$ theory with $N_f = 4$ it is necessary to perform two transformations of the toric diagrams \[13\] trading the Lagrangian brane for the additional flavor. We end up with two flavors in fundamental and two in antifundamental. The corresponding masses are $m_1 \to 0$, $m_2 = m_a$, $\exp(\beta m_3) = q\sqrt{q^t}$, $\exp(\beta m_4) = \sqrt{q^t}$. Hence two masses of fundamentals are fixed by parameters of the $\Omega$-deformation, one mass tends to zero and one mass is arbitrary. The toric diagram with the corresponding Kahler parameters is presented at the Figure.

If we expand the derivative of the partition function into the double series $e^{m\beta a}Q^n$ corresponding to the expansion in the electric and 4d instanton charges we get the HOMFLY polynomial for the generic torus knots \[13\]. No decoupling of 4d instantons occurs since $Q = \exp(\beta g^{-2})$ is finite. In terms of the toric diagram in IIB picture the instantons correspond to the horizontal D1 strings while the W-bosons to the vertical F1 strings in the box of the toric diagram. The complicated string network in IIB has to be taken into account for the generic quantum numbers. The representation of the HOMFLY polynomials in terms of $SU(2)$ $N_f = 4$ SQCD explains their relation with the q-Liouville conformal blocks via AGT relation \[34, 35\] which has been observed in \[12\].
IV. FRACTIONAL 5D CS TERM AND THE CALOGERO MODEL

Another approach involves the famous Jones-Rosso formula for the torus knots. It was shown in [13] that Jones-Rosso representation for the HOMFLY polynomial corresponds to the $N_f = 2$ SQED without the additional operator insertions but with the fractional 5d CS term. The Jones-Rosso formula reads as

$$H_{\square}^{(n,m)}(A, q) = (-1)^{n-1} q^n \sum_{|\lambda|=n} q^{\frac{m}{2}(n+1)} \frac{\prod_{l=1}^{0} (1-q^{l-a}) \prod_{l=1}^{0} (1+ AQ^{l-a})}{\prod(q^{-l-1}-q^a)(q^{-l}-q^{a+1})}$$  \hspace{1cm} (15)

and fits with the instanton partition function of 5D $N = 1$ $U(1)$ gauge theory on $\mathbb{R}^4_{\Omega} \times S^1_{\beta}$ in self-dual Omega deformation $\epsilon_1 = -\epsilon_2$ with antifundamental matter of mass $m_a$ and fundamental matter of mass $m_f$ supplemented with fractional CS term:

$$\left. \frac{\partial \tilde{Z}^{\text{inst}}_{\square}}{\partial m_f} \right|_{m_f=0} = (1+A) \beta \sum_{|\lambda|=n} q^{\frac{m}{2}(n+1)} \prod_{l=1}^{0} (1-q^{l-a}) \prod_{l=1}^{0} (1+ AQ^{l-a}) \frac{\prod_{l=1}^{0} (1-q^{l-a}) \prod_{l=1}^{0} (1+ AQ^{l-a})}{\prod(q^{-l-1}-q^a)(q^{-l}-q^{a+1})}$$  \hspace{1cm} (16)

where the parameters are identified as follows

$$q = \exp(-\beta \epsilon_2), \quad A = -\exp(\beta m_a)$$  \hspace{1cm} (17)

The generalization of this formula for the superpolynomial of generic $T_{n,m}$ knot has been found in [22].
In this representation we obtain the $n$-instanton contribution to the fermionic condensate as the HOMFLY invariant of $T_{n,nk+1}$ knot when the fractional 5d CS term is $k = m + 1/n$. Note that the denominator in the CS coupling is equal to the number of instantons. This approach does not allow to get the whole instanton sums but provides the additional framework for the evaluation of the separate terms in the instanton expansion of the fermion condensate.

Due to the non-vanishing CS term the instanton acquires the electric charge $k + 1$ therefore in this case we have the system of $n$ instanton dyons. It is known that the 5d CS term induces the connection on the instanton moduli space which results in our case to the interacting instanton dyons in $R^4$. Remarkably it turns out that the system of the interacting instanton dyons in $R^4$ is described by the integrable $n$-particle Calogero model with the attractive interaction. The coupling constant is determined by the coefficient in front of the 5d CS term. The HOMFLY polynomial of $T_{n,m}$ knots appears in the spectrum of the Calogero model with the rational $m/n$ coupling constant in an interesting way as the weighted degeneracy of the $E = 0$ states [21]. More algebraically HOMFLY polynomials can be identified with the twisted character of the finite-dimensional representation of DAHA which exists at rational coupling constant in the Calogero model [37]. Physically the situation corresponds to the "falling to the center" problem and reflects the centering of the instantons. Let us emphasize that the Calogero dynamics takes place in the $R^4$ while the torus knots are pictured inside the CY space by the membrane instantons which are actual degrees of freedom in the Calogero model.

This representation can be potentially useful for the description of the FQHE in (4+1) dimensions. The Calogero coupling constant corresponds to the filling fraction and it is well-known that the Hall conductivity $\sigma_H$ is identified with the coefficient in front of the 5d CS term [33]. It seems that the interacting instanton dyons are some analogue of the composite fermions in the (2+1) FQHE. The changing of the 5d CS level corresponds to the transition between two plateau which has on the other hand the interesting geometrical interpretation as the cut-and-join operation [32].
V. WHERE THE DUALITY COMES FROM?

Let us briefly summarize the qualitative physical picture behind the duality found. The most unusual features are the necessity to sum over all torus knots to get the vev of physical observable and the relation between the mass of the field in fundamental representation and the rank of the gauge group involved into the 3d CS derivation of the HOMFLY invariants. Actually these features are related: the instantons and instanton dyons are represented by the wrapped M2 branes while the mass of the fundamental being the Kahler modulus of $CP_1$ gets traded for the rank of the corresponding gauge group in 3d CS theory or for the number of M5 branes upon the inverse geometrical transition. The M2 branes draw the torus knots on the M5 branes representing the fundamental (or antifundamental) matter inside the CY manifold. The necessity to sum over the all knots is now clear. The similar picture can be seen also in IIB model where the relation with the toric diagrams for CY manifold makes the underlying geometry even more transparent \[13\] and the BPS states are represented by the string web.

The knots are placed in CY space and polynomials of $T_{n,m}$ knots in CY space count the multiplicity of the particles with $(n, m)$ charges in the 5d gauge theory. The most subtle point concerns the choice of the observable however upon this choice is done the knot HOMFLY polynomials can be equally evaluated in terms of the spectrum of Calogero model in physical space-time and describes the peculiarities of the "falling to the center" phenomena for the instanton dyons.

To explain the interplay between the mass of the antifundamental and the rank of 3d CS group consider first one-loop effective action in QED in constant external electric and magnetic fields. In a self-dual background field the effective action can be identified as the topological string at $T^*S^3$ or equivalently $SU(N)$ 3d CS at $S^3$ when the rank of the group appears to be the ratio of the fermion mass and the external field $N \propto \frac{m^2}{eE} [38]$. This is the toy example of the inverse geometrical transition with the mass dependent rank of the group in the 3d CS theory.

Assume now that we have the second fermion loop in the same external field probably of the different flavor. We take the derivative of the second loop with respect to the mass which yields the insertion of fermion bilinear. At the next step we assume that there is the web of interacting particles of two types between the loop with operator insertion and the first loop.
They can braid providing the torus knots $T_{n,m}$ if we have $n$ propagating particles of one type and $m$ propagating particles of another type. Since we have prepared 3d CS theory in CY space from the first loop in the external field the ends of the propagating particles picture the torus knot in $S^3$ inside CY. From the viewpoint of the second loop with the operator inserted we evaluate the contribution to the condensate from the ”tadpole” connected to the loop by some web involving particles of two types. The knot invariants count the entropy of the web with fixed two quantum numbers which are attached to the loop. Equivalently, it can be thought as the particular entropic factor in the fermion condensate. To some extend this picture can be considered as the generalization of the Schwinger-type evaluation of the BPS multiplicities in $39$.

In our case we have the loop of the antifundamental in the external graviphoton field (at least at weak field) and the loop of the fundamental in the same external field with the inserted bilinear operator. Due to the inverse geometric transition the loop of antifundamental provides the $SU(N)$ 3d CS action in CY when the rank of the group is $N \propto \frac{m_a}{\epsilon}$ similar to the QED case above. The insertion of the fermion bilinear and the loop of antifundamental are connected by the instanton-W-boson web with electric and instanton charges $(n,m)$ which pictures the torus knot at the antifundamental side. From the viewpoint of the fundamental we evaluate the condensate of the bilinear in the external field taking into account the tadpole of the antifundamental connected by the W-boson-instanton web. The configuration of the web has some peculiarities, for instance, one has to have in mind that instantons are almost sitting at the top of each other in $C^2$. The multiplicity of the web yields the entropic factor in the condensate.

Our findings suggest that the role of the polynomials of the torus knots drawn in the ”‘ momentum space”‘ could be fairly important for the description of the different vacuum condensates in the field theory. The key phenomena behind the appearance of the knots (at least torus knots) is the interesting interplay between the perturbative and non-perturbative contributions to the physical observables. This could be crucial for the resurgence phenomena in generic QFT.
VI. CONCLUSION AND ACKNOWLEDGEMENTS

We have briefly summarized the recent developments concerning the relation between the instanton counting in 5d SQED and $SU(2)$ SQCD and torus knot invariants. There are a lot of open questions, some of them are under investigation now [40].

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12
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