Access to the Kaon Radius with Kaonic Atoms

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A method to determine the kaon radius from the spectra of kaonic atoms is put forward. The few lowest-lying transitions in kaonic atoms and their sensitivity to the size of the kaon for ions in the nuclear charge range $Z = 1 – 100$ are analyzed, taking into account finite-nuclear-size, finite-kaon-size, recoil, and leading-order quantum-electrodynamic effects. Additionally, the opportunities for extracting the kaon mass and nuclear radii are demonstrated by examining the sensitivity of the transition energies in kaonic atoms.

1. Introduction

Kaons are not elementary particles, nevertheless they are of particular interest for fundamental research as the lightest meson with non-zero strangeness. Associated studies could give access to the strangeness sector of quantum chromodynamics, and lepton number violation, properties of other particles, and improve our knowledge of physical properties or constants, or improve our knowledge thereof. Kaon mass and radius values have not been updated since they were reported more than 30 years ago. For kaon mass determination, exotic-atom X-ray spectroscopy was used whereas the kaon radius was measured by the direct scattering of kaons on electrons.

Another class of exotic systems, namely muonic atoms, have been proved to be extremely sensitive to nuclear parameters, and therefore their study allowed to retrieve information about atomic nuclei (see, e.g., refs. [23, 24]). Since kaons are twice as heavy as muons, kaonic atoms feature even stronger dependence on nuclear parameters, opening an alternative path for the extraction of these parameters.

In the current manuscript, we consider the spontaneous decay spectra of kaonic atoms for nuclear charges in the range $Z = 1–100$ in order to establish how these systems can be used to determine the kaonic mass, kaonic radius, and nuclear radius. This opens access to the fundamental properties of kaons and puts forward a new path to probe essential properties of atomic nuclei.

1.1. Klein–Gordon–Fock Equation

As a spinless particle, a kaon with a mass $m_K$ is described by the stationary Klein–Gordon–Fock equation (in the natural system of units, $\hbar = c = 1$) as ref. [25]

$$[(E - V(r))^2 + \Delta - m_K^2] \varphi(r) = 0 \quad (1)$$

In the case of a spherically-symmetric potential $V(r) = V(r)$, the angular variables can be separated from the radial ones as $\varphi(r) = \frac{R(r)}{r} Y_{lm}(\theta, \varphi)$, where $l$ and $m$ are the orbital quantum number and its projection, respectively. The angular part $Y_{lm}$ of the bound kaon wave function is given by spherical harmonics. The radial part, represented as $R(r) = \phi(r) + \chi(r)$, satisfies the Schrödinger-form set of equations:

$$\left[ -\frac{D_l^2}{2m_k} + m_k + V(r) \right] \phi(r) + \frac{D_l}{2m_k} \chi(r) = E\phi(r) \quad (2a)$$

$$\frac{D_l}{2m_k} \phi(r) + \left[ \frac{D_l}{2m_k} - m_k + V(r) \right] \chi(r) = E\chi(r) \quad (2b)$$

with the differential operator $D_l(r) = d^2/dl(l + 1)/r^2$.

Assuming provisionally the nucleus to be point-like and infinitely heavy, one can describe the kaon-nucleus interaction with the Coulomb potential $V = -\alpha Z/r$, where $\alpha = e^2/(4\pi) \approx 1/137$ is the fine-structure constant, and $Z$ is the nuclear charge.
Then, Equation (2) can be solved analytically, resulting in the energies \[^{25}\]

\[ E_{nl}(aZ) = m_e \left( 1 + \frac{(aZ)^2}{n_k - 1/2 + \mu} \right)^{-1/2} \]  

(3)

where \( n \) is the principal quantum number,

\[ n_k = n - l, \]

\[ \mu = \sqrt{(l+1/2)^2 - (aZ)^2} \]

(4) and the wavefunctions are given in terms of a hypergeometric function:

\[ \psi_{nl}(r) = N_0 r^{l+1/2} e^{-r/\lambda} F_1(1-n_k, 2\mu+1; \rho) \]

(5)

where \( \rho = 2r \sqrt{1-E_{nl}^2/m_e^2} \), and \( N_0 \) is the normalization constant. Equation (3) contains a singularity in the denominator, and for \( l = 0 \) it breaks at \( aZ = 1/2 \), or at \( Z \approx 69 \). This indicates that the point-like-nucleus approximation is not valid anymore, and one has to consider finite nuclear size effects.

### 1.2. Finite-Nuclear-Size Effect

One of the simplest nuclear models is the homogeneously-charged-sphere model, with the corresponding charge density of the nucleus

\[ \rho_{nucle}(r) = \frac{3Ze}{4\pi r_0^2} \theta(r_0 - r) \]  

(6)

Here \( r_0 \) is the effective radius of the nucleus, which is related to the root-mean-square radius \( \sqrt{\langle r^2 \rangle} \) through

\[ r_0 = \sqrt{\frac{5}{3} \langle r^2 \rangle} \]  

(7)

The interaction between electron and nucleus can therefore be described by the potential

\[ V_{sphere}(r) = \begin{cases} -\frac{Ze}{r} \left( 3 - \frac{r}{r_0} \right), & \text{while } r \leq r_0, \\ -\frac{Ze}{r_0}, & \text{while } r > r_0 \end{cases} \]  

(8)

With this potential, the Equation (2) can be solved numerically or semi-analytically in analogy to refs. \([26, 27]\) for electrons and muons, described by the Dirac equation. However, even for a muon, which is more than twice as light as a kaon, and therefore located on a larger distance from the nucleus, the semi-analytical method in a first order of FNS correction turns out to be not sufficient.

### 1.3. Finite-Kaon-Size Effect

Additionally to the finite-nuclear-size effect, one should take into account the finite-kaon-size (FKS) effect. To estimate its order of magnitude, we used a comparably simple two-sphere approach to build a potential, presented in ref. [29].

We assume that the nucleus and the kaon are two spheres which interact without deformation via electromagnetic forces. Denoting the radius of the nucleus as \( R_n \), the radius of the kaon as \( R_k \), and their ratio as \( \lambda = R_k/R_n \), three different regions should be considered:

(i) \( 0 \leq r/R_n \leq 1 - \lambda \), the kaon is entirely inside the nucleus,

(ii) \( 1 - \lambda \leq r/R_n \leq 1 + \lambda \), the kaon and the nucleus partly overlap, and

(iii) \( r/R_n \geq 1 + \lambda \), the kaon is completely outside the nucleus.

The corresponding potential \( V_{2spheres}(r) \) is determined as ref. [29]:

\[ V_{2spheres}(r) = \begin{cases} -V_0 (C_0 - (r/R_n)^2), & \text{for } 0 \leq r/R_n \leq 1 - \lambda, \\ -\frac{V_0}{(r/R_n)^2} (C_1 + C_2 (r/R_n)^2), & \text{for } 1 - \lambda \leq r/R_n \leq 1 + \lambda, \\ -\frac{V_0}{(r/R_n)^2}, & \text{for } r/R_n \geq 1 + \lambda, \end{cases} \]  

(9)

Here \( V_0 = -aZ/R_n \), and the coefficients in Equation (9) are:

\[ C_0 = 3/2 - 3\lambda^2/10, \]

\[ C_1 = (1 - 9\lambda^2 + 16\lambda^4 - 9\lambda^6)/12, \]

\[ C_2 = (-2 + 10\lambda^2 + 10\lambda^4 - 2\lambda^6)/5, \]

\[ C_3 = (3 + 6\lambda^2 + 3\lambda^4)/4, \]

\[ C_4 = (-2 - 9\lambda^2 - 2\lambda^4)/3, \]

\[ C_5 = (1 + \lambda^2)/4, \]

\[ C_6 = 0, \]

\[ C_7 = -1/60 \]  

(10)

By calculating the energy of a given state \( E_{nl} \) with a homogeneously-charged sphere (8) and two-spheres (9) potentials, one can evaluate the FKS effect:

\[ \delta_{FKS} = 1 - \frac{E_{nl}[V_{2spheres}]}{E_{nl}[V_{sphere}]} \]  

(11)

### 1.4. Quantum-Electrodynamic Effects

Another important contribution to the energies of kaonic atoms originates from quantum-electrodynamics (QED) corrections. In the first order in \( a \), there are self-energy (SE) and vacuum polarization (VP) corrections. For light hydrogen-like electronic ions, the SE correction is dominant, whereas for heavy ions these two corrections are of the same order of magnitude. However, it is known (see, e.g., refs. [28, 31]) that, for muonic atoms, the hierarchy is different: due to the large muon–electron mass ratio, the VP with a virtual electron–positron pair is a few orders of magnitude larger than the VP with a virtual muon–antimuon pair and the SE correction. The same stands also for kaonic atoms.
and therefore the leading QED correction can be described by the Uehling potential[32]

\[ V_{\text{Uehl}}(r) = \frac{\alpha}{3\pi} \int_0^{\infty} dr' 4\pi r' \int_1^{\infty} dt \left( 1 + \frac{1}{2t^2} \right) \]

\[ \times \frac{\sqrt{r^2 - 1}}{t^2} \exp(-2m_e |r - r'|t) - \exp(-2m_e (r + r')t)}{4m_e rt} \]

(12)

with \( m_e \) being the mass of an electron. In our calculation, this potential has also been included in Equation (2). Therefore, the calculated energies account for the dominant QED effect to all orders.

1.5. Recoil

Due to the large mass of a kaon compared to that of a proton, the recoil effect is important for very light ions, but becomes negligible for medium and heavy ions. To evaluate it, we used a simple reduced-mass formalism[33] replacing the nuclear mass \( m_N \) with

\[ m_r = \frac{m_N m_k}{m_N + m_k} \]

(13)

Standard atomic weights[34] have been used in the current manuscript.

1.6. Sensitivities

Taking into account FNS, FKS, the leading QED and recoil effects, we calculated the kaonic atom spectrum. To characterize how values of physical observable change depending on the parameters of the theory used, one can introduce sensitivity coefficients

\[ \frac{\Delta E}{E} = K_N \frac{\Delta R_N}{R_N} + K_m \frac{\Delta m_k}{m_k} \]

(14)

Here, \( K_N \) stands for the sensitivity coefficient to the radius of the nucleus \( R_N \), and \( K_m \) for the sensitivity coefficient to the kaon mass \( m_k \). By varying different parameters of our calculations, we can estimate the corresponding sensitivity factors, similarly as it was done in earlier works for other physical constants, for example, in ref. [35]. We use only the most general sensitivity coefficients \( K_N \) and \( K_m \) in our current work, since the kaonic atoms' spectrum feature complicated non-linear dependence on \( R_N \) and \( m_k \). For example, the mass of the kaon is a scaling factor for all energies, however, it is also included in the nuclear potential via scaling of the radius and reduced mass. Analogously with the nuclear radius, for simple atomic systems, like electronic H-like ions, one can describe the FNS effect via simple term \( \Delta E_{\text{FNS}} \propto (aZ)^2 \).[36] For kaonic atoms the FNS correction has a much stronger impact, and therefore one should take into account also higher-order terms in \( aZ \) (see, e.g., refs. [37, 38]). As an outcome, the sensitivity coefficients depend significantly on the specific ion and transition, and can vary considerably.

1.7. Strong Shift

The strong interaction effects in the spectra of kaonic atoms are also important and can significantly affect the binding and transition energies. For the 2p state of kaonic \( ^4\text{He} \), it was estimated from the difference of experimentally measured and theoretically predicted values to be \( 0(8) \text{ eV} \)[9] and can hence be neglected for this system. However, the situation changes drastically with higher charge numbers, where the strong shift effects can be a few orders of magnitude larger. Namely, for kaonic lead \( ^{206}\text{Pb} \), the binding energy of the 1s state is \( E_{1s}[^{206}\text{Pb}] = -17 \text{ MeV} \). The hadronic contribution of \( \approx 10 \text{ MeV} \) decreases it to only \( E_{1s}[^{206}\text{Pb}] = -7 \text{ MeV} \).[16–18] The strong shift will decrease transition energies analogously for other kaonic atoms. The strong shift effect, together with its uncertainty, represents currently a bottleneck for the calculations of the structure of kaonic atoms, and individual treatment for different charge numbers and states under consideration would be advantageous. In the present paper, we focus only on the atomic and QED calculations, however, in the future, it will need to be taken into account.

2. Results

Using the above described method, we calculated the spectra and transition energies for \( Z = 1–100 \) nuclei. In order to work with general expressions, we assumed the nuclear radius to be \( R_N = 1.22Z^{1/3} \text{ fm} \), and for the mass of the nucleus the standard atomic weight has been used. Such simple assumptions allowed us to analyze the general trends for our observables. The transition energies, including the FNS, FKS, the leading QED and recoil effects for the circular transitions from \( 2p \rightarrow 1s \) up to \( 6h \rightarrow 5g \), are plotted as functions of nuclear charge \( Z \) in Figure 1. In Table 1, the same energies, FKS correction \( \delta_{\text{FKS}} \) and the sensitivities to the nuclear radius \( K_N \) and to the mass of a kaon \( K_m \) are listed for few kaonic atoms: helium \( ^7\text{He} \), titanium \( ^{22}\text{Ti} \), xenon \( ^{34}\text{Xe} \) and uranium \( ^{93}\text{U} \). In the last column, we give an order-of-magnitude estimate of the strong shift based on the data from ref. [18]. Our value for \( 3d \rightarrow 2p \) transition of \( 6.465 \text{ keV} \) for kaonic helium is in perfect agreement with previously reported experimental and
Due to the limitations of the used assumptions, the transition energies are given with four significant digits.

### 2.1. Determination of the Kaon’s Radius

In Figure 2 one can see the relative FKS correction (in percent) to the transition energy as a function of nuclear charge $Z$. Since

theoretical values.$^{[9]}$ Due to the limitations of the used assumptions, the transition energies are given with four significant digits.

### Table 1. Transition energies $\Delta E$, finite-kaon-size effect $\delta_{FKS}$, and sensitivities to the nuclear radius $K_R$ and mass of a kaon $K_m$ for the first few circular transitions in kaonic helium $^2$He, titanium $^{22}$Ti, xenon $^{54}$Xe, and uranium $^{92}$U. The number between square brackets indicates a power of 10. Last column gives an order-of-magnitude estimate of the strong shift effect for the corresponding transition energy.

| Ion   | Transition | $\Delta E$, [keV] | $\delta_{FKS}$, [%] | $K_R$   | $K_m$   | $\Delta E_{str}$ |
|-------|------------|-------------------|---------------------|---------|---------|-----------------|
| $^2$He | $2p \to 1s$ | 34.84             | 0.07                | $-0.01$ | 0.99    | 0.0             |
|       | $3d \to 2p$ | 6.465             | 2[-5]               | $-2[-6]$| 1.00    |                 |
|       | $4f \to 3d$ | 2.259             | 0                   | 0       | 1.00    |                 |
|       | $5g \to 4f$ | 1.045             | 0                   | 0       | 1.00    |                 |
|       | $6h \to 5g$ | 0.5714            | 0                   | 0       | 1.00    |                 |
| $^{22}$Ti | $2p \to 1s$ | 2315              | 0.80                | $-0.71$ | 0.27    | 1700            |
|       | $3d \to 2p$ | 846.8             | 0.20                | $-0.12$ | 0.87    | 200             |
|       | $4f \to 3d$ | 308.6             | 5[-3]               | $-3[-3]$| 0.99    | 4               |
|       | $5g \to 4f$ | 142.5             | 2[-5]               | $-9[-6]$| 1.00    |                 |
|       | $6h \to 5g$ | 78.06             | 3[-8]               | 0       | 1.00    |                 |
| $^{54}$Xe | $2p \to 1s$ | 4253              | 0.49                | $-1.2$  | 0.23    | 4000            |
|       | $3d \to 2p$ | 3109              | 0.57                | $-0.80$ | 0.18    | 1800            |
|       | $4f \to 3d$ | 1746              | 0.26                | $-0.25$ | 0.74    | 400             |
|       | $5g \to 4f$ | 868.8             | 0.03                | $-0.02$ | 0.98    |                 |
|       | $6h \to 5g$ | 476.3             | 6[4]                | $-4[4]$ | 1.00    |                 |
| $^{92}$U | $2p \to 1s$ | 4825              | 0.23                | $-1.4$  | 0.39    | 4500            |
|       | $3d \to 2p$ | 4340              | 0.41                | $-1.2$  | 0.21    | 3300            |
|       | $4f \to 3d$ | 3470              | 0.46                | $-0.84$ | 0.15    | 2000            |
|       | $5g \to 4f$ | 2322              | 0.26                | $-0.34$ | 0.65    | 700             |
|       | $6h \to 5g$ | 1386              | 0.05                | $-0.05$ | 0.95    | 90              |

Figure 2. Finite kaon size effect $\delta_{FKS}$ (in percent), estimated within the two-spheres model, for the transition energies of the first few circular transitions for kaonic atoms in the range $Z = 1–100$.

Figure 3. Sensitivity coefficient $K_R$ to the radius of the nucleus, for the transition energies of the first few circular transitions for kaonic atoms in the range $Z = 1–100$.

the potential in Equation (9) is a function of $R_K / R_N$, and the radius of the kaon $R_K = 0.34(5)$ fm is smaller than any nuclear radius, one could naively expect, that the FKS effect would be the largest when the kaon-nucleus size ratio is minimal, namely, for hydrogen. However, as one can see from Figure 2, this is not the case. The relative FKS effect to the $2p \to 1s$ transition grows with increasing $Z$ due to the increasingly strong attraction of the nucleus, reaching the maximum value of 0.8% at $Z \approx 22^{[39]}$ and then starts to decrease. Also, unlike the FNS effect, which is always maximized for the 1s shell, and is getting smaller and finally is simply negligible for the higher atomic shells, we can observe quite a different trend for the FKS effect. All other transitions exhibit the same qualitative behavior with respect to the FKS effect as the $2p \to 1s$ transition, however with different positions and values of their maxima. Thus, for the $3d \to 2p$ transition the FKS effect has its maximum of 0.57% at $Z \approx 54$, and for $4f \to 3d$ the maximum of 0.46% can be reached at $Z \approx 92$.

Since the strong shift decreases the transition energies, accounting for previously neglected strong contributions would lead to the further enhancement in the transition's sensitivity to the FKS effect.

The FNS effect and the uncertainty on the nuclear radius value can in principle affect a proposed kaon-radius extraction scheme from a given transition, though the wide range of the possible systems allows one to avoid such problems. For example, in uranium, the sensitivity of the $4f \to 3d$ transition to the nuclear radius is 0.8, which corresponds to a 0.5% uncertainty on the transition energy, and it is comparable to the size of the FKS effect. However, one can also consider lighter ions: for lead, the uncertainty associated with the nuclear radius is below 0.05%, and therefore significantly smaller than the expected FKS effect. This opens various possibilities for the determination of the size of the kaon based on different transitions with different nuclear charge $Z$.

### 2.2. Extraction of Nuclear Radii

As one can see from Table 1 and from Figure 3, for all ions the transitions to the low-lying states are sensitive to the nuclear radius, and therefore they can be used for its extraction in addi-
2.3. Extraction of the Mass of a Kaon

Due to the complicated dependence of the energy on the kaon mass, the sensitivity coefficient $K_m$ differs from unity, especially for the lowest-lying transitions, see Figure 4. However, even for the heaviest elements considered, it is close to unity for transitions starting with $6h \rightarrow 5g$, and therefore the analysis of kaonic atom spectra can be used for the determination of its mass. The fact that the dependence holds for any nucleus can be used to enlarge the statistics and choose the system with the most suitable parameters for an experiment. A similar procedure was already used before in ref. [21], however, with continuous progress in both experimental techniques and theoretical calculations one can aim at improving the existing accuracy.

2.4. Further Improvements

So far, we showed the principal idea of using kaonic atoms for the extraction of particle and nuclear parameters, considering the leading size and QED effects. However, for high-precision theoretical predictions to be compared with experimental data, one has to take into account effects already calculated in this manuscript and other effects, with higher accuracy, similarly to what was done, for example, for muonic atoms in ref. [24]. First of all, the strong and weak interaction contributions can be either taken from the previously reported data\cite{17,19} or calculated to the required accuracy with current state-of-the-art methods. Then, the FNS effect has been calculated using tabulated nuclear data\cite{40} for root-mean-square radius values, with a Fermi or even with a deformed Fermi nuclear potential\cite{28,41} or with more realistic predictions based on the Skyrme-type nuclear potential\cite{42} not forgetting about the nuclear deformation correction\cite{38}. There is room for an improvement in the evaluation of the FKS effect as well, from a simple two-sphere model to a more sophisticated and realistic one. The higher-order QED effects, such as self-energy, Wichmann–Kroll, Källén–Sabry, muonic and hadronic Uehling potentials\cite{16,41,43} should be also included. The electron screening effects were shown to be negligible in the spectra of muonic atoms,\cite{28,44} therefore we expect them to be even smaller for kaonic atoms. The recoil effect should be included within a rigorous relativistic approach\cite{16,45} for more precise values for light kaonic atoms. Finally, the effects of nuclear polarization have to be taken into account, as it was done, for example, in refs. [46, 47]. All above mentioned atomic structure effects have been calculated before for electronic or muonic atoms, and therefore, can be straightforwardly implemented also for the case of kaonic atoms. Based on our experience, we estimated the accuracy of QED effects to be safely below 1%, and mainly determined by the uncertainty in nuclear parameters. Therefore, they do not represent a limitation for the suggested extraction of nuclear and kaonic radii and kaonic mass from the spectra.

3. Conclusions

We considered kaonic atoms with the nuclear charge $Z = 1–100$. Taking into account finite-nuclear size, finite-kaon size, leading-order quantum-electrodynamic, and recoil effects, we calculated transition energies and sensitivity coefficients to the nuclear radius and mass of a kaon. We analyzed the finite-kaon-size effect, showing that the value of the kaon’s radius can be extracted with almost equal efficiency from a few different transitions and for a few different ions. Similarly, the decay spectra of kaonic atoms can be used for the determination of nuclear radii and the kaon mass in addition to the existing methods. Since the strong interaction effects and the corresponding uncertainties can represent the main difficulty for the successful analysis of the spectra, one should consider systems where they are well known or minimized: for example, light or well-studied nuclei, high-lying kaonic levels or some combination of these. The choice of the most suitable system for any of these purposes should be made based on many important parameters, such as the accuracy of theoretical predictions, the natural linewidths, the experimental accessibility of particular nuclei, and the possibility to carry out high-precision measurements. In conclusion, the knowledge of atomic structure of kaonic atoms would give access to fundamental nuclear and particle parameters, and therefore could motivate new experiments and high-precision calculations.

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