ADDENDUM TO
“SUPERCONNECTIONS AND PARALLEL TRANSPORT”

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Abstract. In this addendum to our article “Superconnections and Parallel Transport” we give an alternate construction to the parallel transport of a superconnection contained in Corollary 4.4 of [2], which has the advantage that is independent on the various ways a superconnection splits as a connection plus a bundle endomorphism valued form.

Consider as in Section 4 of [2] a superconnection \( \mathbb{A} \) in the sense of Quillen (see [3] and [1]) on a \( \mathbb{Z}/2 \)-graded vector bundle \( E \) over a manifold \( M \), i.e. an odd first-order differential operator

\[
\mathbb{A} : \Omega^*(M, E) \to \Omega^*(M, E)
\]

satisfying Leibniz rule

\[
\mathbb{A}(\omega \otimes s) = d\omega \otimes s \pm \omega \otimes \mathbb{A}(s),
\]

with \( \omega \in \Omega^*(M) \) differential form on \( M \) and \( s \in \Gamma(M; E) \) arbitrary section of the bundle \( E \) over \( M \). For such a superconnection we defined in [2] a notion of parallel transport along (families of) superpaths \( c : S \times \mathbb{R}^{1|1} \to M \) that is compatible under gluing of superpaths. Let us briefly recall this construction. First, let us write \( \mathbb{A} = \mathbb{A}_1 + A \), with \( \mathbb{A}_1 = \nabla \) the connection part of the superconnection \( \mathbb{A} \) and \( A \in \Omega^*(M, \text{End } E)^{odd} \) the linear part of the superconnection. For an arbitrary superpath \( c \) in \( M \) consider the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{c^*} & c^*E \\
\downarrow \pi^*E & & \downarrow \pi \\
M & \xrightarrow{c} & S \times \mathbb{R}^{1|1} \\
\downarrow \pi & & \downarrow \pi \\
\Pi TM & \xrightarrow{\tilde{c}} & \Pi TM
\end{array}
\]

with \( \tilde{c} \) a canonical lift of the path \( c \) to \( \Pi TM \), the “odd tangent bundle” of \( M \). Then parallel transport along \( c \) is defined by parallel sections \( \psi \in \Gamma(c^*E) \)

Date: January 4, 2011.
along $c$ which are solutions to the following differential equation
\[(c^*\nabla)_D\psi - (c^*A)\psi = 0.\]
Here $D = \partial_\theta + \theta \partial_t$ denotes the standard (right invariant) vector field on $\mathbb{R}^{1|1}$, see Section 2.4 of [2].

Our alternate construction goes as follows. We first write $A = A_0 + \bar{A}$, where $A_0$ denotes the zero part of the superconnection and $\bar{A}$ the remaining part. Define then a connection $\bar{\nabla}$ on the bundle $\pi^*E$ over $\Pi TM$ as follows
\[\nabla_{L_X}(\omega \otimes s) = L_X\omega \otimes s \pm \iota_X\bar{A}s,\]
\[\nabla_{\iota_X}(\omega \otimes s) = \iota_X\omega \otimes s,\]
for $\omega \in \Omega^*(M)$ and $s \in \Gamma(M; E)$. Here, for a vector field $X$ on $M$, $L_X$ and $\iota_X$ denote the Lie derivative respectively contraction in the $X$-direction acting as even respectively odd derivations on $\Omega^*(M) = C^\infty(\Pi TM)$, i.e. as vector fields on $\Pi TM$. These relations are enough to define a connection $\bar{\nabla}$ on the bundle $\pi^*E$ over $\Pi TM$ since the algebra of vector fields on $\Pi TM$ is generated over $C^\infty(\Pi TM)$ by vector fields of the type $L_X$ and $\iota_X$, for $X$ arbitrary vector field on $M$, i.e.

\[\text{Vect}(\Pi TM) = C^\infty(\Pi TM) < L_X, \iota_X \mid X \in \text{Vect}(M) > .\]

Parallel transport along a superpath $c : S \times \mathbb{R}^{1|1} \to M$ is defined by parallel sections $\psi \in \Gamma(c^*E)$ along $c$ which are solutions to the following differential equation
\[(c^*\nabla)_D\psi - (c^*A_0)\psi = 0.\]
As before, the parallel transport is well-defined (cf. Proposition 4.2 of [2]) by this “half-order” differential equation and is compatible under glueing of superpaths (i.e. it satisfies properties (i) and (ii) of Theorem 4.3 in [2]). The advantage of this construction resides in the fact that the parallel transport so defined is invariant under the various ways in which a superconnection can be written as a sum of a connection plus a linear part, as $\bar{A}$ is invariant under such splittings.

Denote by $D$ the de Rham differential on $\Pi TM$. If $\omega$ is a function on $\Pi TM$, then the 1-form $D\omega$ on $\Pi TM$ evaluated on the standard odd vector field $d$ on $\Pi TM$ gives us
\[(D\omega)(d) = d\omega,\]
the differential of $\omega$, understood as a function on $\Pi TM$. This allows us to conclude that for any $s$ a section of $E$,
\[\nabla ds = \bar{A}s.\]
Let us note that the connection $\bar{\nabla}$ is torsion free in the odd directions, i.e.
\[\left[\nabla_{\iota_X}, \nabla_{\iota_Y}\right] = \nabla_{[\iota_X, \iota_Y]} (= 0),\]
where $X$ and $Y$ are vector fields on $M$. 
The two constructions coincide when we consider connections instead of superconnections on the bundle $E$ over $M$. When the manifold $M$ reduces to a point, a graded vector bundle with superconnection reduces to a $\mathbb{Z}/2\mathbb{Z}$-vector space $V$ together with an odd endomorphism $A (= A_0)$ of $V$. The two constructions of parallel transport we considered also coincide in this situation giving rise to the supergroup homomorphism of Example 3.2.9 in [4]

$$R^{1|1} \ni (t, \theta) \mapsto e^{-tA^2 + \theta A} \in GL(V),$$

as solution to the “half-order” differential equation $D\psi = A\psi$.

We can also recover the superconnection from its associated parallel transport by first recovering the $A_0$ from looking at constant superpaths in $M$, and then recover $\hat{A}$ by looking at parallel transport along the superpath given by $R^{1|1} \times \Pi TM \to R^{0|1} \times \Pi TM \to M$, where the two maps are the obvious projection maps. The lift of such a superpath to $\Pi TM$ gives, after the obvious projection map, the flow of the vector field $d$ on $\Pi TM$. Given that $\nabla ds = \hat{A}s$, this recovers $\hat{A}$. Compare with Section 4.4 of [2].

Acknowledgements. The construction presented here is a mere continuation of an idea of Stephan Stolz who first thought to interpret a Quillen superconnection on a bundle $E$ over $M$ as a connection on the pullback bundle $\pi^*E$ over $\Pi TM$. I would like to thank Peter Teichner for suggesting to write up this Addendum.

References

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