UNIVERSAL PROPERTY OF TRIANGULATED DERIVATORS VIA KELLER’S TOWERS

MARCO PORTA

ABSTRACT. In [20] B. Keller solved the universal problem of the extension of an exact category to its (bounded) derived category by introducing the notions of tower of exact and triangulated categories and proving the universal property in this setting. In this note we show that his result extends to the corresponding universal problem for Grothendieck’s derivators.

CONTENTS

1. Introduction 1
1.1. Acknowledgements 3
2. Grothendieck’s derivators 3
2.1. Triangulated derivators 5
2.2. Morphisms of derivators 5
2.3. Exact categories and their derived categories 10
2.4. The triangulated derivator associated with an exact category 12
3. Keller’s towers 16
4. Epivalence and Recollement 17
4.1. The redundancy of the connecting morphism 22
5. The Universal Property 31
6. Derived extension of an exact category 34
6.1. Full faithfulness of the diagram functor 35
6.2. Epivalence of the diagram functor 40
6.3. Invertibility of the diagram functor 47
References 51

1. INTRODUCTION

Triangulated categories were invented in the early sixties by Grothendieck–Verdier [33] and, simultaneously and independently, by Dold–Puppe [8]. Grothendieck–Verdier sought to axiomatize the properties of derived categories of sheaves, while Dold–Puppe were motivated by examples from topology, and notably the stable homotopy category of finite CW-complexes.

In spite of the success of derived and triangulated categories in recent years, during the thirty years that followed, triangulated categories were largely considered to be too poor in structure to allow the development of a significant general theory, analogous to that of abelian categories established by Freyd, Mitchell, Gabriel, . . . . As an example, let us
consider a triangulated category $\mathcal{T}$ and let $I$ be a finite directed diagram (i.e., a small finite category with no nontrivial loops). We can form the category $\operatorname{Hom}(I, \mathcal{T})$ of contravariant functors from the diagram $I$ into $\mathcal{T}$. In general, there is no canonical triangulated structure on this category [20], even in the simple case where $I$ is the category containing only two objects, their identities, and an arrow connecting them, the main problem being that the mapping cone is a non-functorial construction. Another important consequence of this fact is the following: Let $\mathcal{E}$ be an exact category and $\mathcal{D}^b(\mathcal{E})$ its bounded derived category. Then, the universal property does not hold for the triangulated category $\mathcal{D}^b(\mathcal{E})$, i.e., the natural functor $\operatorname{Hom}_{ex}(\mathcal{D}^b(\mathcal{E}), \mathcal{T}) \to \operatorname{Hom}_{ex}(\mathcal{E}, \mathcal{T})$, relating categories of exact functors, is not quasi-invertible for all triangulated categories $\mathcal{T}$.

It follows that, in order to have a satisfactory theory, the notion of triangulated category must be modified or enhanced. A long list of attempts appeared in the literature in recent years in order to define new kinds of structures assuring the universal property to the construction $\mathcal{D}^b$, e.g., DG-categories and $A_{\infty}$-categories. Derivators were introduced in a systematic fashion by Alexander Grothendieck [15] in the nineties in a long letter addressed to Quillen and by Alex Heller in [16]. However, the idea of a ‘derivatory notion’ was already around for a few years, and possibly others also contributed. For instance, there is the paper [1] by D. W. Anderson from the late seventies and other papers of his in which the philosophy was already very ‘derivatory’.

Their advantage, compared to the other constructions, is that they have the least amount of added structure. I will not try to give a complete definition of what a derivator is in this introduction. It suffices to say that it is a 2-functor $\mathcal{D}$ from the 2-category of diagrams into the 2-category of categories $\mathcal{C}AT$ satisfying a list of axioms inspired by the properties of the formalism of (homotopy) Kan extensions in the context of Quillen’s definition of a model category [30] together with its homotopy category.

Similar constructions were independently introduced by Jens Franke and Bernhard Keller in the nineties. Keller’s construction [20], called ‘epivalent tower’, only considers categories ‘indexed’ over hyper-cubical diagrams, but nevertheless he was able to prove the universal property for the epivalent towers enhancing the derived category. Some years later, G. Maltsiniotis [26], principally inspired by works of Grothendieck [15] and Franke [10], gave the axioms for derivators that we follow in this paper. He introduced a notion of triangulated derivator, i.e., a derivator taking values in triangulated categories and satisfying two more axioms [26]. At the same time [21], Keller associates a derivator $\mathcal{D}_E$ with an exact category $\mathcal{E}$ by means of the following definition: $\mathcal{D}_E(I) := \mathcal{D}^b(\operatorname{Hom}(I, \mathcal{E}))$, and proves that this derivator is triangulated.

The principal aim of this paper is to show that, similarly to the case of ‘towers’, the triangulated derivator associated to an exact category $\mathcal{E}$ has a universal property. This is Theorem 2.16 in the text. As a corollary, in the particular case of the diagram $I = e$ (the terminal category), we obtain a result (Theorem 2.17) that we can consider as the correct universal property for the (bounded) derived category $\mathcal{D}_E(e) = \mathcal{D}^b(\mathcal{E})$ among the ‘basic’ triangulated categories, i.e., categories of the form $\mathcal{T}(e)$ for some triangulated derivator $\mathcal{T}$. This result is similar in spirit to a string of other results about universal properties of derivator-like structures obtained by other authors. A. Heller proved his main results on ‘homotopy theories’, a notion very near to Grothendieck’s notion of ‘dérivateur’, in [16] already in the ’80s. J. Franke introduced in 1996 his notion of ‘systems of triangulated diagram categories’ in [10]. Both these authors proved universal properties for the structures they have introduced, the main difference being that in the former the base is given by the homotopy category of simplicial sets of arbitrary size rather than of finite spectra as in the latter. We summarize and report these historical results in Theorem 2.20 in the
text. Finally, in 2008 D.-C. Cisinski [6] proved a universal property for the derivator $\mathbb{H}ot_I$ associated to the homotopy theory of presheaves over a diagram $I$ with values in small categories which models classical homotopy theory.

Let us briefly describe the contents of the sections of the present article. In section 2 we fix the notations and recall the definitions of derivator, pointed derivator, triangulated derivator, morphism of derivators. Then, after recalling the notion of exact category, we introduce the associated triangulated derivator defined by Keller in [21], which is the central object of interest in this paper. Section 3 is dedicated to the description of the notion of Keller’s tower and its related morphisms. We prove our main result about the universal property of this derivator in section 5. Let us remark that one could write down a proof completely expressed in the language of derivators, following the lines of Keller’s proof. However, in this paper we prefer to show how the theorem in the setting of derivators naturally follows from the case of towers.

Section 4 is devoted to a very useful property of morphisms of derivators that we call, following Keller’s [20], ‘redundancy of the connecting morphism’. This means that a morphism of derivators which preserves bicartesian squares (in the sense of derivators) automatically preserves connecting morphisms, hence preserves distinguished triangles. This fact entails that an additive morphism of exact or triangulated derivators $F$ is automatically triangulated provided that it locally preserves distinguished triangles, even when it does not preserves connecting morphisms functorially. The proof involves the formalism said ‘recollement of triangulated categories’ or ‘6-gluing functors’ for low dimensional diagrams.

The proof of the universal property for (bounded) derived categories is left to section 6. The techniques used in the proof are new with respect to Keller’s approach in [20]. Indeed, Keller could prove his result using induction over the integer dimensions of his diagrams since he was dealing with hypercubes. This is not possible if we consider all finite directed diagrams (cf. definition in Section 2) and we have to change our strategy. A key ingredient in our proof is a careful analysis of the property of epivalence (i.e., essential surjectivity and fullness) of the diagram functor $\text{dia}_I$, for any $I$, which sends an object $X$ in the category $\mathbb{T}(I)$ to its ‘diagram’ in the base $\mathbb{T}(e)$. This is done in subsections 6.1 and 6.2 under a special hypothesis which we shall commonly refer to as ‘Toda condition’ (cf. Remark 2.19 for an explanation of the name).

1.1. Acknowledgements. I am very happy to thank Bernhard Keller, for helpful discussions and inspiration about this project. I thanks the anonymous referees for their useful suggestions to improve the readability of the paper.

2. Grothendieck’s derivators

In this section we briefly recall the definition of derivator in the sense of Grothendieck, following the exposition in [20]. The reader is invited to look at the original manuscript [15] for a complete exposition. We begin by reminding some useful categorical notions and fixing the notations.

We denote by $\mathcal{C}at$ the 2-category of small categories. Among its objects there are $\emptyset$, the empty category, and $e$, the terminal category with one object $*$ and only the identity morphism. If $\mathcal{A}$ is a small category, we write $\mathcal{A}^o$ to indicate the opposite category. Given two small categories $\mathcal{A}$ and $\mathcal{B}$, then $\text{Hom}(\mathcal{A}, \mathcal{B})$ denotes the category of functors (1-morphisms in the 2-category $\mathcal{C}at$) from $\mathcal{A}$ to $\mathcal{B}$ with natural transformations as morphisms (2-morphisms in $\mathcal{C}at$). For any functor $u : \mathcal{A} \rightarrow \mathcal{B}$ and a fixed object $b$ in the category $\mathcal{B}$, the objects of the category $\mathcal{A}/b$ are the pairs $(a, f)$, where $a$ is an object of $\mathcal{A}$ and $f$ a morphism from $u(a)$ to $b$ in the category $\mathcal{B}$. A morphism $\varphi : (a, f) \rightarrow (a', f')$ in the
category \( \mathcal{A}/b \) is determined by an arrow \( g : a \to a' \) such that \( f' \circ u(g) = f \). Composition of morphisms in \( \mathcal{A}/b \) is clearly induced by composition in \( \mathcal{A} \). Dually, we have the category \( b\setminus \mathcal{A} \) defined by \( b\setminus \mathcal{A} = (\mathcal{A}^o/b)^o \). There are canonical forgetful functors from \( \mathcal{A}/b \) and \( b\setminus \mathcal{A} \) to \( \mathcal{A} \), defined by \((a,f) \mapsto a \). By considering the identity functor of \( \mathcal{B} \), we can canonically associate the categories \( \mathcal{B}/\text{sl⊗sh.ri∅htA} \) defined by \( \mathcal{B}/\text{sl⊗sh.ri∅htA} = \mathcal{A}\bigcirc/\text{sl⊗sh.left} \mathcal{B} \bigcirc/\text{sl⊗sh.right} \mathcal{A} \). There are canonical forgetful functors from \( \mathcal{A}/\text{sl⊗sh.left} \mathcal{B} \) and \( \mathcal{B}/\text{sl⊗sh.ri∅htA} \) to \( \mathcal{A} \), defined by \((a,f) \mapsto a \). By considering the identity functor of \( \mathcal{B} \), we can canonically associate the categories \( \mathcal{B}/\text{sl⊗sh.left} \mathcal{B} \) and \( \mathcal{B}/\text{sl⊗sh.ri∅htB} \) with \( \mathcal{B} \), for any arbitrarily fixed object \( b \in \mathcal{B} \). Let us call \( u/\text{sl⊗sh.left} \mathcal{B} : \mathcal{A}/\text{sl⊗sh.left} \mathcal{B} \to \mathcal{B}/\text{sl⊗sh.left} \mathcal{B} \) and \( b/\text{sl⊗sh.ri∅htB} : \mathcal{B}/\text{sl⊗sh.ri∅htA} \to \mathcal{B}/\text{sl⊗sh.ri∅htB} \) the induced functors which associate the pair \((a,f) \) with the pair \((u(a), f) \). It is easy to check that the following commutative squares are cartesian:

\[
\begin{array}{c}
\mathcal{A}/b \quad \xrightarrow{u/b} \quad \mathcal{A} \\
\mathcal{B}/b \quad \xrightarrow{b\setminus u} \quad \mathcal{B}
\end{array}
\]

In the spirit of the theory of derivators we usually call diagrams the objects of \( \mathcal{C}at \). Let us name \( \mathcal{D}ia \) the full 2-subcategory of \( \mathcal{C}at \) whose objects are finite directed diagrams, i.e., categories whose nerves have only a finite number of non-degenerate simplices. Equivalently, we can say that the objects in \( \mathcal{D}ia \) are the finite categories whose underlying quiver (vertices: objects, arrows: non identical morphisms) does not have oriented cycles (e.g., finite posets). These objects, together with the functors of categories as 1-morphisms and the natural transformations as 2-morphisms, endow \( \mathcal{D}ia \) with the structure of a 2-category.

Clearly, more general full sub-2-categories \( \mathcal{D}ia \) of the 2-category \( \mathcal{C}at \) can be used as diagrams in the definition of a derivator (see, e.g., [26] for a discussion and references therein). We simply write \( \mathcal{D}ia \) whenever it is possible to work in those wider situations.

**Definition 2.1.** A prederivator of type \( \mathcal{D}ia \) is a 2-functor

\[
\mathcal{D} : \mathcal{D}ia \to \mathcal{C}at
\]

to the 2-category of categories.

More explicitly, this means that there are a category \( \mathcal{D}(I) \) associated with any diagram \( I \), a functor \( u^* : \mathcal{D}(u) \to \mathcal{D}(I) \), associated with any functor \( u : I \to J \), and a natural transformation \( \alpha^* : \mathcal{D}(\alpha) \),

\[
\begin{array}{ccc}
\mathcal{D}(J) & \xrightarrow{u^*} & \mathcal{D}(I) \\
\downarrow{\alpha^*} & & \downarrow{\mathcal{D}(\alpha)} \\
\mathcal{D}(I) & \xrightarrow{\beta^*} & \mathcal{D}(K)
\end{array}
\]

associated with any natural transformation

\[
\begin{array}{ccc}
I & \xrightarrow{\alpha} & J \\
\downarrow{\beta} & & \downarrow{\mathcal{D}(\beta)} \\
J & \xrightarrow{\beta} & K
\end{array}
\]

These data have to verify the following coherence axioms:

- \( 1^*_I = 1_{\mathcal{D}(I)} \), for any object \( I \) in \( \mathcal{D}ia \);
- \( 1^*_u = 1_{u^*} \), for any arrow \( I \to J \) in \( \mathcal{D}ia \);
- \( (vu)^* = u^*v^* \), for any diagram \( I \to J \to K \) in \( \mathcal{D}ia \);
- \( (\beta\alpha)^* = \alpha^*\beta^* \), for any diagram \( I \to J \) in \( \mathcal{D}ia \);
• \((\beta \ast \alpha)^* = \alpha^* \ast \beta^*\), for any diagram \(\begin{array}{ccc} I & \xrightarrow{\psi} & J \\ \alpha \downarrow & & \beta \downarrow \\ u & \xleftarrow{\psi} & v \end{array}\) in \(\mathcal{D}ia\).

If \(u : I \to J\) is a morphism in \(\mathcal{D}ia\), we respectively denote \(u_*\) and \(u_!\) the right and left adjoint functor to \(u^*\), when they exist.

2.1. Triangulated derivators. We have to introduce some more notations and terminology in order to define the notion of derivator.

For any object \(x\) of a small category \(I\) lying in \(\mathcal{D}ia\), we write \(i_x : e \to I\) to indicate the functor that is uniquely determined by \(x\) and \(I\). Sometimes, we will simply write \(i_x\), or even \(x\), when the context is clear. Given an arbitrary prederivator \(\mathbb{D}\) and an object \(F\) of \(\mathbb{D}(I)\), the object \(F_x := i_x^*(F)\) is called the fiber of \(F\) at the point \(x\). If \(\varphi : F \to F'\) is a morphism in \(\mathbb{D}(I)\), the morphism \(\varphi_x := i_x^*(\varphi) : F_x \to F'_x\) is the morphism of fibers induced by \(\varphi\).

For the rest of this discussion we shall always assume that the restriction functors admit the necessary adjoints. For any 2-square in \(\mathcal{D}ia\),

\[
\begin{array}{ccc}
\mathcal{D} & = & I' \\
\xrightarrow{v} & & \xleftarrow{u} \\
\xrightarrow{w} & & \\ \\
\xleftarrow{u} & & \xrightarrow{v} \\
\xrightarrow{\alpha} & & \xrightarrow{\beta} \\
J & & I
\end{array}
\]

we denote by \(\varepsilon : u^*u_* \to 1_{\mathbb{D}(I)}\), \(\eta : 1_{\mathbb{D}(J)} \to u_*u^*\), \(\varepsilon' : u'^*u'_* \to 1_{\mathbb{D}(I')}\), \(\eta' : 1_{\mathbb{D}(J')} \to u'_*u'^*\) the adjunction morphisms. Let us define the base change morphism \(c_D : w^*u_* \to u'_*v^*\) to be the composition \((u'_*v^* \ast \varepsilon)(u'_* \ast \alpha^* \ast u_*)\) of the following morphisms

\[
\begin{array}{c}
w^*u_* \\
\xrightarrow{u'^*u_*} \\
u'^*w^*u_* = u'_*(wu)_1 \\
\xrightarrow{\alpha^*} \\
u'_*(wu)^*u_* = u'_*v^*u'_*u_* \\
\xrightarrow{u'^*v^*} \\
u'_*v^*.
\end{array}
\]

Clearly, we have a dual morphism \(c'_D : v_!u'^* \to u_!w_!\).

Given a small category \(I\) in \(\mathcal{D}ia\), we indicate as \(p_I : I \to e\) the unique functor from \(I\) to the terminal category \(e\). In order to remain compatible with the standard notations of model category theory, we can set, for any object \(F\) in \(\mathbb{D}(I)\),

\[
\text{holim}_IF := (p_I)_*(F) \quad \text{and} \quad \text{holim}^IF := (p_I)_!(F)
\]

in the category \(\mathbb{D}(e)\) and talk about homotopical projective and inductive limit of \(F\). Clearly, our notation is inspired by the theory of model categories and the corresponding homotopy derivators. Let us emphasize that in those examples the functors above correspond to the classical homotopy (co-)limits but respectively over the corresponding opposite categories.

We remark that sometimes the notations \(\Gamma_*(I,F) := (p_I)_*(F)\) and \(\Gamma!(I,F) := (p_I)_!(F)\) are also used. In this way, these objects can be thought of as the global sections of \(F\) over \(I\). The topologically inclined reader might speak of the (co-)homology of \(I\) with coefficients in \(F\). Other common notations that we also use in this paper are \(\text{holim}_IF\) and \(\text{holim}^IF\), respectively for \(\text{holim}_IF\) and \(\text{holim}^IF\).

Note that the notions of homotopy limit and fiber of an object \(F\) are not completely unrelated. Indeed, we can construct a comparison morphism in the following way. Let \(u : I \to J\) be a morphism in \(\mathcal{D}ia\), \(y\) an object of \(J\) and \(F\) an object in \(\mathbb{D}(I)\). There are the forgetful functors \(j : I/y \to I\) and \(j : y/I \to J\). We denote by \(F|_{I/y}\) (resp., \(F|_{y/I}\)) the image
\[ j^*(F) \text{ in } \mathbb{D}(I/y) \text{ (resp., in } \mathbb{D}(y/I)). \] Let us consider the 2-squares

\[
\begin{array}{ccc}
\mathcal{D}_{u/y} & \xrightarrow{j} & I \\
\downarrow \phi & & \downarrow u \\
J & \xrightarrow{\alpha} & J
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{D}_{y/u} & \xrightarrow{j} & I \\
\downarrow \alpha' & & \downarrow u \\
I & \xrightarrow{\phi} & J
\end{array}
\]

where the 2-morphisms \( \alpha \) and \( \alpha' \) are defined, for any object \( x \) of \( I \) and any morphism \( f : u(x) \to y \) or, respectively, \( g : y \to u(x) \), by the formulae \( \alpha(x,fu(x)\to y) := f \) and \( \alpha'(x,gy\to u(x)) := g \). For any \( F \) in \( \mathbb{D}(I) \), there are the associated canonical base change morphisms

\[
c_{\mathcal{D}_{u/y}} : (u_*F)_y \xrightarrow{(p_{I/y})_*} j^*(F) = \text{holim}_{I/y}(F|_{I/y}) \quad \text{and} \quad
c'_{\mathcal{D}_{y/u}} : \text{holim}_{y/I}(F|_{y/I}) = (p_{y/I})_*j^*(F) \xrightarrow{(uF)_y} .
\]

Suppose that we have fixed a prederivator \( \mathbb{D} \). Let us consider two arbitrary diagrams \( I,J \) in \( \mathcal{D}ia \). For any object \( x \) of \( I \), there is the canonical functor \( x_{I,J} : J \to I \times J \), which associates any object \( y \) in the category \( J \) with the pair \( (x,y) \). By 2-functoriality of \( \mathbb{D} \), there exist the functor \( x_{I,J} : \mathbb{D}(I \times J) \to \mathbb{D}(J) \) and the morphism of functors \( \alpha_{I,J} : x'_{I,J} \to x_{I,J} \), for every morphism \( \alpha : x \to x' \) in \( I \). In other words, we have a functor

\[
\mathbb{D}(I \times J) \times I^o \longrightarrow \mathbb{D}(J),
\]

which associates the object \( x_{I,J}^*(F) \) of \( \mathbb{D}(J) \) with a pair \( (F,x) \). By adjunction, we get the functor

\[ \text{dia}_{I,J} : \mathbb{D}(I \times J) \longrightarrow \text{Hom}(I^o, \mathbb{D}(J)), \]

where \( \text{Hom}(I^o, \mathbb{D}(J)) \) denotes the category of contravariant functors from \( I \) to \( \mathbb{D}(J) \). When \( J = e \), the functor \( \text{dia}_{I,e} \) induces a functor

\[ \text{dia}_I : \mathbb{D}(I) \longrightarrow \text{Hom}(I^o, \mathbb{D}(e)), \]

whose image \( \text{dia}_I(F) \), for any \( F \) in \( \mathbb{D}(I) \), is called underlying diagram of \( F \) or simply diagram of \( F \). It is the presheaf

\[ \text{dia}(F) := \text{dia}_I(F) : I^o \longrightarrow \mathbb{D}(e) \]

defined by the equality \( \text{dia}_I(F)(x) = F_x \), for any object \( x \) in \( I \).

**Definition 2.2.** A Grothendieck’s derivator of type \( \mathcal{D}ia \) is a prederivator \( \mathbb{D} \) of type \( \mathcal{D}ia \) which satisfies the following axioms.

**Der 1**

a) If \( I \) and \( J \) are in \( \mathcal{D}ia \), the functor

\[ \mathbb{D}(I \coprod J) \xrightarrow{(i^*,j^*)} \mathbb{D}(I) \times \mathbb{D}(J), \]

induced by the canonical functors \( i : I \to I \coprod J \) and \( j : J \to I \coprod J \), is an equivalence of categories;

b) the category \( \mathbb{D}(\emptyset) \) is equivalent to the point category \( e \).

**Der 2**

For any category \( I \) in \( \mathcal{D}ia \), the family of functors \( i^*_x : \mathbb{D}(I) \to \mathbb{D}(e) \), indexed by the objects \( x \) lying in \( I \), is conservative. Explicitly, this means that any morphism \( \varphi : F \to F' \) in \( \mathbb{D}(I) \) is an isomorphism iff the morphism of fibers \( \varphi_x = i^*_x(f) : F_x \to F'_x \) is an isomorphism in \( \mathbb{D}(e) \), for all \( x \) in \( I \).

**Der 3**

For any morphism \( u : I \to J \) in \( \mathcal{D}ia \), the induced functor \( u^* : \mathbb{D}(J) \to \mathbb{D}(I) \) admits a right adjoint \( u_* : \mathbb{D}(I) \to \mathbb{D}(J) \) and a left adjoint \( u_0 : \mathbb{D}(I) \to \mathbb{D}(J) \).
**Der 4** For any morphism $u : I \to J$ in $\mathcal{D}ia$, any object $y$ in $J$ and any $F$ in $\mathcal{D}(I)$, the associated canonical base change morphisms

$$c_{\mathcal{D}_{u/y}} : (u_! F)_y \xrightarrow{\text{can}} (p_{1/y})_\ast j^\ast (F) = \underset{\text{holim}_{I/y}}{\text{holim}} p_{1/y} F_{1/y}$$

and

$$c_{\mathcal{D}_{y/u}} : \underset{\text{holim}_{y/I}}{\text{holim}} p_{y/I} F_{y/I} = (p_{y/I})_\ast j^\ast (F) \xrightarrow{\text{can}} (u_! F)_y$$

are invertible.

We usually refer to the notion of a Grothendieck’s derivator simply as ‘derivator’.

**Definition 2.3.** A prederivator (resp., derivator) is an epivalent prederivator (resp., epivalent derivator) if it enjoys the property stated in the following axiom.

**Der 5** (Epivalence axiom). For any $J$ in $\mathcal{D}ia$, the functor

$$\text{dia}_{\Delta_1, J} : \mathcal{D}(\Delta_1 \times J) \to \text{Hom}(\Delta_1, \mathcal{D}(J)),$$

where $\Delta_1$ denotes the category $\{0 < 1\}$ of $\mathcal{D}ia$, is full and essentially surjective.

We recall that some authors (e.g., Moritz Groth in [11], Andrei Radulescu-Banu in [32] for the notion he calls ‘Heller derivator’ and Kevin Carlson in [4]) reserve the name ‘strong derivator’ for the notion of derivator which satisfies (some version of) axiom **Der 5**.

**Remark 2.4.** We agree that the status of Axiom **Der 5** is perhaps not in its final form. Indeed, we use the axiom version of Franke [10] but Heller in [17] uses a variant of the axiom for finite free categories. Moreover, one can show that a yet more general version of this axiom is satisfied by typical examples such as homotopy derivators of model categories (cf. [32] by Radulescu-Banu), and this more refined version might be useful in future applications. Recently, yet a different version of the fifth axiom was proposed by Carlson, and this version is enjoyed by homotopy derivators of quasi-categories (certain partial underlying diagram functors are assumed to be smothering functor in the language of Riehl-Verity). This version was used to offer a different perspective on the relation between derivators and homotopy theories of homotopy theories. (This version, however, is not satisfied by homotopy derivators of model categories.)

Let us remark that axioms **Der 1** and **Der 2** above assure us that, for any diagram $I$ in $\mathcal{D}ia$, the category $\mathcal{D}(I)$ has finite products and coproducts and in particular it admits initial and final objects.

A morphism $j : U \to I$ of $\mathcal{D}ia$ is an open immersion if it is injective on objects, fully faithful, and if any morphism $f : y \to j(x)$ in $I$ is in the image of $j$, i.e., it is of the form $j(g) : j(x') \to j(x)$, for some morphism $g : x' \to x$ in $U$. Dually, a morphism $i : Z \to I$ of $\mathcal{D}ia$ is a closed immersion if $i^0 : Z^0 \to I^0$ is an open immersion. Open immersions and closed immersions are stable under composition and pullback.

**Definition 2.5.** A derivator $\mathcal{D}$ is pointed if the following axiom holds.

**Der 6** For any closed immersion $i : Z \to I$ in $\mathcal{D}ia$, the induced functor $i_*$ admits a right adjoint $i^!$. Dually, for any open immersion $j : U \to I$ in $\mathcal{D}ia$, the induced functor $j_!$ admits a left adjoint $j^\ast$.

Notice that in [11] Groth proves that axiom **Der 6** is equivalently stated by requiring that the base $\mathcal{D}(e)$ admits a zero object.
Coming back to notations and terminology, we write $\square$ to indicate the category $\Delta_1 \times \Delta_1$, \(\top, \bot\) to indicate the two subcategories $(0,0)\leftarrow (0,1)$, $(0,1)$, $(1,0)\rightarrow (1,1)$ of $\square$, and $i_\top : \top \rightarrow \square$, $i_\bot : \bot \rightarrow \square$ to indicate the inclusion functors. An object $F$ of $D(\square)$ is homotopically cartesian (resp., homotopically cocartesian), or, more simply, cartesian (resp., cocartesian), if the adjunction morphism

$$\eta^F : F \rightarrow (i_\top)_*(i_\top)^* F$$

(resp., $\varepsilon^F : (i_\bot)_!(i_\bot)^* F \rightarrow F$) is an isomorphism.

**Definition 2.6.** An equivalent derivator is *triangulated* if it is pointed and if the following axiom holds.

**Der 7** Any object $F$ in $D(\square)$ is homotopically cartesian if and only if it is homotopically cocartesian (i.e., homotopically bicartesian).

We prefer to follow Maltsiniotis–Keller [26] and Cisinski–Neeman [7] and include axiom **Der 5** in the very definition of a triangulated derivator. The main argument in favour of this choice is that in this work we are above all interested in triangulated derivators and the equivalence axiom is central for the construction of the triangulated structure of the local categories $D(I)$, as in the proof of Theorem 2.8.

**Example 2.7.** The following example by Cisinski [5] is of particular interest. Let $\mathcal{M}$ be a category such that there exists a stable Quillen closed model structure on with $W$ as weak equivalences. Then, the prederivator $D(\mathcal{M}, W)$ defined, on any small category $I$, by

$$D(\mathcal{M}, W)(I) := \text{Hom}(I^\circ, \mathcal{M})[W^{-1}]$$

and, on any functor $u : I \rightarrow J$, by

$$D(\mathcal{M}, W)(J) \xrightarrow{u^* := (u^\circ)^r} D(\mathcal{M}, W)(I),$$

is a triangulated derivator. Here the overline means the induced functor between the localized categories. The fact that these derivators are triangulated is verified in all detail in [14] by Groth–Ponto–Shulman.

For example, the stable model category of spectra [3, 9, 24, 27, 30] gives rise to a triangulated derivator $\mathcal{S}p$. The following theorem was announced in [26]. A precursor version, based on Franke’s theory, can be found in [10]. The interested reader can find a proof in [11, Theorem 4.15]. The fact that the triangulations are ‘canonical’ was made more precise in the paper [12] by Groth (this includes a 2-functoriality statement as opposed to a mere functoriality, a ‘2-naturality statement’, and a uniqueness statement).

**Theorem 2.8 (Maltsiniotis).** If $D$ is a triangulated derivator, for every $I$ in $\mathfrak{Dia}$, there is a canonical structure of triangulated category on $D(I)$, such that for every morphism $u : I \rightarrow J$ in $\mathfrak{Dia}$, the functor $u^* : D(J) \rightarrow D(I)$ is canonically endowed with the structure of triangulated functor.

We refer to the articles of Maltsiniotis and Groth for all the details concerning the canonical triangulated structure of the values of a triangulated derivator. In what follows we are going to use notation an terminology of [26].
2.2. Morphisms of derivators.

**Definition 2.9.** Let $\mathcal{D}$ and $\mathcal{E}$ be two prederivators of type $\mathcal{D}ia$. A morphism $F : \mathcal{D} \to \mathcal{E}$ is given by

- a functor $F_I : \mathcal{D}(I) \to \mathcal{E}(I)$, for each $I$ in $\mathcal{D}ia$;
- an invertible transformation of functors
  $$\varphi_u : F_J u^* \to u^* F_J,$$

  for each morphism $u : I \to J$ in $\mathcal{D}ia$;

such that

- $\varphi_1 = 1_{F_I}$, for each $I$ in $\mathcal{D}ia$;
- $\varphi_u = (u^* \varphi_v)(\varphi_v u^*)$, for each pair of morphisms $K \to I \xrightarrow{\varphi} J$ in $\mathcal{D}ia$;
- $\varphi_u \alpha^* = \alpha^* \varphi_v$, for each 2-morphism $I \xrightarrow{\varphi} J$ in $\mathcal{D}ia$.

Let us remark that this is a special case of the notion of a ‘pseudo–natural’ transformation.

The composition of two morphisms of prederivators $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{E} \to \mathcal{F}$ is defined by setting $(GF)_I = G_I F_I$, for all $I$ in $\mathcal{D}ia$, and $(\gamma \varphi)_u = (\gamma_u F_I)(G_I \varphi_v)$, for all morphisms $u : I \to J$ in $\mathcal{D}ia$.

If $F$ and $G$ are two morphisms of prederivators from $\mathcal{D}$ to $\mathcal{E}$, a 2-morphism $\alpha : F \to G$ is given by a natural transformation $\alpha_I : F_I \to G_I$, for each $I$ in $\mathcal{D}ia$, such that $(u^* \alpha_I) \varphi_u = \gamma_u(\alpha_I u^*)$, for all morphisms $u : I \to J$ of $\mathcal{D}ia$ (this is a special case of the notion of a ‘modification’). The composition of two such morphisms is clear. Thus, the morphisms between any fixed pair of prederivators $\mathcal{D}$ and $\mathcal{E}$, and their 2-morphisms, are the objects and, respectively, the morphisms of a category that we denote $\mathcal{H}om(\mathcal{D}, \mathcal{E})$.

A morphism of derivators is just a morphism of the underlying prederivators and similarly for 2-morphisms. Thus, given a pair of derivators $\mathcal{D}$ and $\mathcal{E}$, we use the same notation $\mathcal{H}om(\mathcal{D}, \mathcal{E})$ to indicate the category with morphisms of derivators from $\mathcal{D}$ to $\mathcal{E}$ as objects and the related 2-morphisms as morphisms.

Let us remark that, in the presence of another morphism $v : K \to I$, for any object $X$ lying in $\mathcal{D}(J)$, the isomorphism $\varphi^X_{uv}$ is explicitly given by the composition

$$F_K v^* u^* X \xrightarrow{\varphi^X_{uv}} v^* F_I u^* X \xrightarrow{v^*(\varphi^X_\gamma)} v^* u^* F_J X.$$ 

The morphism $\varphi_u$ induces, via the adjunction morphisms $\eta_u : 1 \to u^* u_!$ and $\varepsilon_u : u u^* \to 1$, another morphism

$$\varphi^u : u_! F_I \to F_J u_!,$$

whose action on any object $X$ of $\mathcal{D}(I)$ is given by the composition

$$u_! F_I X \xrightarrow{\eta_u F_I X} u_! F_I u^* u_! X \xrightarrow{u_!(\varphi u^* X)} u_! u^* F_J u_! X \xrightarrow{\varepsilon_u F_J u_! X} F_J u_! X.$$ 

Analogously to the contravariant case, we have the relation

$$\varphi^X_{uv} \circ u_!(\varphi^X_u) = \varphi^X_{uv},$$ 

valid for all objects $X$ of $\mathcal{D}(K)$. Indeed, functoriality of the morphisms involved and the relation $\varphi^X_{uv} = v^*(\varphi^X_u) \circ \varphi^X_{uv}$ give us commutative diagrams that we can use in the following.
sequence of equalities

\[
\varphi^{uv,X} = e_{uv}^F(\varphi^{uv})_{\cdot}X \circ (uv)_!\left[\varphi^{(uv)}_{\cdot}X\right] \circ (uv)_!J(\eta_{uv}^X) \\
= e_{uv}^F(\varphi^{uv})_{\cdot}X \circ u_!\left[q^*F_{\cdot}(\varphi^{uv})_{\cdot}X\right] \circ (uv)_!\left[\varphi^{uv}_X\right] \circ o((uv)_!J(\eta_{uv}^X)) \\
= e_{uv}^F(\varphi^{uv})_{\cdot}X \circ u_!\left[\varphi^{uv}_X\right] \circ (uv)_!\left[\varphi^{uv}_X\right] \circ u_!\left(\varphi^{uv}_X\right) \circ o((uv)_!J(\eta_{uv}^X)) \\
= e_{uv}^F(\varphi^{uv})_{\cdot}X \circ u_!\left[\varphi^{uv}_X\right] \circ u_!\left(\varphi^{uv}_X\right) \circ u_!\left(\varphi^{uv}_X\right) \circ o((uv)_!J(\eta_{uv}^X)) \\
= \varphi^{uv,X} \circ u_!(\varphi^{uv}_X). \\
\]

Let us remark that one might be able to find the last formula (and quite a few others of this kind) by using the ‘calculus of mates’, i.e., a 2-categorical machinery which provides a framework for pasting diagrams arising in the presence of adjoint pairs in 2-categories. Nevertheless, since we intend this paper useful for the applications and we are trying to get a high readability in the exposition, we prefer to maintain all the higher categorical details at the very least.

2.3. Exact categories and their derived categories. In this subsection we recall the notion of exact category and the construction of the related derived category. This notion was introduced by Quillen in [31] and modified by Keller in [19].

**Definition 2.10.** An exact category is an additive category \(\mathcal{A}\) endowed with a class \(\Phi\) of short exact sequences, also said exact pairs,

\[
\begin{array}{c}
X \xrightarrow{i} Y \\
\downarrow p \\
Z
\end{array}
\]

(i.e., \(\ker(p) = i\), \(\cok(i) = p\)), which is closed under isomorphisms. Elements in this class are called conflations and the arrows of type \(\rightarrow\) (resp., \(\rightarrow\)) are called inflations (resp., deflations), such that the following axioms hold.

**Ex 0** The identity morphism \(\begin{array}{c}
0 \\
\downarrow 0
\end{array}\) is a deflation.

**Ex 1** The class of deflations is stable under composition.

**Ex 2** For any deflation \(\begin{array}{c}
Y \xrightarrow{d} Z
\end{array}\) and any morphism \(\begin{array}{c}
Z' \xrightarrow{f} Z
\end{array}\) in \(\mathcal{A}\), there exists a cartesian square (pull-back)

\[
\begin{array}{c}
Y' \\
\downarrow f' \\
Y \\
\downarrow d \\
Z
\end{array}
\xrightarrow{d'} \begin{array}{c}
\rightarrow Z'
\end{array}
\]

where \(d'\) is a deflation.

**Ex2°** For any inflation \(\begin{array}{c}
X \xrightarrow{i} Y
\end{array}\) and any morphism \(\begin{array}{c}
X \xrightarrow{f} X'
\end{array}\) in \(\mathcal{A}\), there exists a cocartesian square (push-out)

\[
\begin{array}{c}
X \\
\downarrow f \\
\rightarrow Y
\end{array}
\xrightarrow{i} \begin{array}{c}
\rightarrow Y'
\end{array}
\]

\[
\begin{array}{c}
X'
\downarrow f' \\
\rightarrow Y'
\end{array}
\]

\[
\begin{array}{c}
X
\downarrow f \\
\rightarrow Y
\end{array}
\xrightarrow{i} \begin{array}{c}
\rightarrow Y'
\end{array}
\]

\[
\begin{array}{c}
X' \\
\downarrow f' \\
\rightarrow Y'
\end{array}
\]

\[
\begin{array}{c}
X
\downarrow f \\
\rightarrow Y
\end{array}
\xrightarrow{i} \begin{array}{c}
\rightarrow Y'
\end{array}
\]

\[
\begin{array}{c}
X'
\downarrow f' \\
\rightarrow Y'
\end{array}
\]

\[
\begin{array}{c}
X
\downarrow f \\
\rightarrow Y
\end{array}
\xrightarrow{i} \begin{array}{c}
\rightarrow Y'
\end{array}
\]

\[
\begin{array}{c}
X' \\
\downarrow f' \\
\rightarrow Y'
\end{array}
\]
for any conflation in \( A \)

Let us remark that the derived category might be a large category of derived category and triangles obtained from the componentwise split short exact sequences.

in Verdier sense [33, Ch. 2.2], with suspension functor \( \Sigma \) given by

\[ (\Sigma X)^p := X^{p+1}, \quad d_{\Sigma X} := -d_X, \]

and triangles obtained from the componentwise split short exact sequences.

A complex \( N \) is strictly acyclic if there exist conflations

\[ \cdots \rightarrow M^p \xrightarrow{dp} M^{p+1} \rightarrow \cdots, \quad dp+1dp = 0, \quad p \in \mathbb{Z}, \]

over an exact category \( A \), where \( M^p = 0 \) for all \( |p| > 0 \). The category \( \mathcal{H}^b A \) is the quotient of \( \mathcal{C}^b A \) by the ideal of nullhomotopic morphisms. Note that the category \( \mathcal{H}^b A \) is triangulated in Verdier sense [33, Ch. 2.2], with suspension functor \( \Sigma \) given by

\[ (\Sigma X)^p := X^{p+1}, \quad d_{\Sigma X} := -d_X, \]

and triangles obtained from the componentwise split short exact sequences.

A complex \( N \) is strictly acyclic if there exist conflations

\[ \cdots \rightarrow M^p \xrightarrow{dp} M^{p+1} \rightarrow \cdots, \quad dp+1dp = 0, \quad p \in \mathbb{Z}, \]

such that \( dp = ip+1q^p \), for all \( p \in \mathbb{Z} \). A complex \( N \) is acyclic if it is isomorphic in \( \mathcal{H}^b A \) to a strictly acyclic complex. One can show that, if the category \( A \) is idempotent complete, then any complex is acyclic iff it is strictly acyclic (cf. [21] for details on the constructions of this paragraph). Moreover, (without the idempotent completeness condition) it is true that the images in \( \mathcal{H}^b A \) of the acyclic complexes form a thick subcategory \( N \) of \( \mathcal{H}^b A \), i.e., \( N \) is a triangulated subcategory of \( \mathcal{H}^b A \) which is stable under retracts. The (bounded) derived category of \( A \), that we denote \( \mathcal{D}^b A \), is the Verdier quotient [33, Ch. 2.2] \( \mathcal{H}^b A / N \).

Let us remark that the derived category might be a large category.

Note that there is a canonical embedding [21]

\[ \text{can}: A \rightarrow \mathcal{D}^b A, \quad X \mapsto (\ldots \rightarrow 0 \rightarrow X \rightarrow 0 \rightarrow \ldots), \quad \deg(X) = 0. \]

For any conflation in \( A \),

\[ \varepsilon: X \xrightarrow{i} Y \xrightarrow{p} Z, \]

we have a canonical distinguished triangle in \( \mathcal{D}^b A \),

\[ \text{can}(\varepsilon): \quad \text{can}(X) \xrightarrow{\text{can}(i)} \text{can}(Y) \xrightarrow{\text{can}(p)} \text{can}(Z) \xrightarrow{\partial_{\varepsilon}} \Sigma \text{can}(X). \]

In this way, we have constructed a 2-functor

\[ \mathcal{E} \mathcal{X} \Rightarrow \mathcal{D}^b A, \quad A \mapsto \mathcal{D}^b A, \]

from the 2-category of exact categories to the 2-category of triangulated categories.

It may be that some explanations are in order, here. The objects of \( \mathcal{E} \mathcal{X} \Rightarrow \mathcal{A} \) are pairs \((A, \Phi)\) formed by a (possibly large) additive category \( A \) with a class of conflations \( \Phi \) as in definition [21]. 1-morphisms \( F: (A, \Phi) \rightarrow (A', \Phi') \) are pairs formed by an additive morphism \( F: A \rightarrow A' \) such that \( F(\Phi) \subseteq \Phi' \) with an obvious morphism of conflation,
2-morphisms \( \alpha : F \to F' \) are natural transformations of additive functors which are compatible with the exact structure. Analogously, objects of the 2-category \( \mathcal{T}RIA \) are pairs \( (T, \Psi) \) formed by a (possibly large) additive category \( T \) with a class of distinguished triangles \( \Psi \), 1-morphisms \( (F, \delta) : (T, \Psi) \to (T', \Psi') \) are pairs formed by an additive morphism \( F : T \to T' \) such that \( F(\Psi) \subseteq \Psi' \) in the sense of [23] with an obvious morphism of distinguished triangles, 2-morphisms \( \alpha : (F, \delta) \to (F', \delta') \) are natural transformations of functors which are compatible with the triangulated structure (cf. [20] for complete definitions). The action of the 2-functor \( D^b \) on the functor \( F \) gives a derived functor \( R^b \) and functoriality is expressed by the commutativity of the following diagram

\[\begin{array}{ccc}
A & \xrightarrow{F} & A' \\
\downarrow{\text{can}} & & \downarrow{\text{can'}} \\
D^b A & \xrightarrow{RF} & D^b A'
\end{array}\]

As usual in the literature, we do not write all the heavy 2-categorical notation and we leave to the reader filling in the details. The canonical functors \( \text{can} \) and \( \text{can}' \) are examples of exact or \( \partial \)-functors (cf. definitions just after theorem 2.16).

The reason we consider derivators or other structures like towers is that the canonical embedding \( \text{can} \) does not have the universal extension property we are looking for.

2.4. The triangulated derivator associated with an exact category. Let \( A \) be an exact category. Here is one central notion of this paper.

**Definition 2.12.** \( D_A : \mathcal{Dia} \to \mathcal{CAT} \) is the prederivator that associates the bounded derived category \( D^b(\text{Hom}(I^0, A)) \) with any \( I \) in \( \mathcal{Dia} \).

In the Appendix [21] to the article [20], B. Keller proves the following

**Theorem 2.13.** Let us consider the restriction of the derivator in the definition 2.12 to the 2-subcategory \( \mathcal{Dia} \). Then, the prederivator \( D_A \) we obtain this way is a triangulated derivator.

Remark that triangularity is a property of a derivator as opposed to the triangulated extra structure that one can put on an additive category. In order to emphasize that all axioms of a triangulated derivator ask for properties some authors (e.g., [11]) refer to it as a stable derivator (in analogy to the theory of model categories and quasi-categories).

Let us call \( \mathcal{A} \) the full 2-subcategory of \( \mathcal{Dia} \) which contains the additive categories. We can think of the category \( \mathcal{E} \mathcal{A} \mathcal{A} \) consisting of the exact categories, i.e., additive categories endowed with some exact structure. Morphisms in this category are known as ‘exact functors’, i.e., additive functors which preserve conflations. Notice that, thanks to Proposition A.1 of [19], this amounts to ask that the additive functors preserve bicartesian squares of the structure, i.e., bicartesian squares with two parallel maps consisting of inflations whereas the other two parallel maps are deflations.

An additive derivator is a derivator \( \mathcal{A} \) whose base \( \mathcal{A}(e) \) is an additive category. In [12] it is shown that this amounts to ask that the image of the derivator \( \mathcal{A} \) lies in the full 2-subcategory \( \mathcal{A} \mathcal{D} \mathcal{A} \) of additive categories, i.e., it is componentwise given by additive categories and by additive functors among them.

Let us emphasize that in this article we consider a particular case of the yet not defined notion of an exact (pre)derivator, i.e., whenever we mention this notion in this article we always mean the represented exact prederivator associated with an exact category. We leave the discussion on the general notion of an exact derivator for future work.
Let us denote $\mathbb{A}: \mathcal{D} \to \mathcal{C} \mathcal{A} \mathcal{T}$ the prederivator that associates the category $\mathbb{A}(I) := \text{Hom}(I^\circ, \mathcal{A})$ of contravariant functors from $I$ to an exact category $\mathcal{A}$, with any diagram $I$ in $\mathcal{D}$. It is clear that, even if the image of $\mathbb{A}$ is in the 2-category $\mathcal{C} \mathcal{A} \mathcal{T}$, we may think, for any diagram $I$, of the exact category $\mathbb{A}(I)$ endowed with short exact sequences which are argumentwise conflations. Moreover, it is important to stress that the prederivator just constructed is an epivalent prederivator (2.3).

Clearly, the canonical embedding of an exact category into its bounded derived category gives rise to a canonical morphism of prederivators, which we still call $\text{can} : \mathbb{A} \to \mathbb{D}_A$, given, for any diagram $I$, by the canonical embedding $\text{can}_I : \mathbb{A}(I) \to \mathbb{D}_A(I) = \mathcal{D}^b(\mathbb{A}(I))$.

For each bicartesian square in $\mathbb{A}(\square)$ of type

\[
\begin{array}{c}
\begin{array}{c}
X_00 \to X_01 \downarrow \\
\downarrow \\
X_{10} \to X_{11}
\end{array}
\end{array}
\]

its image $\text{can}_G(X)$ is bicartesian in $\mathcal{D}^b(\mathbb{A}(\square))$, i.e., isomorphisms

\[
X_{00} \rightsquigarrow X_{10} \bigcup_{X_{11}} X_{01} \quad \text{and} \quad X_{10} \bigcup_{X_{00}} X_{01} \rightsquigarrow X_{11}
\]

hold. We say that the additive morphism $\text{can} : \mathbb{A} \to \mathbb{D}_A$ is exact. More generally we have the following

**Definition 2.14.** Let $\mathbb{E}$ and $\mathbb{F}$ be triangulated derivators.

a) An additive morphism $F : \mathbb{A} \to \mathbb{E}$ is said exact or $\partial$-morphism if $F_G(X)$ is bicartesian in $\mathbb{E}(\square)$ for each $X$ of type (1).

b) An additive morphism of derivators $F : \mathbb{E} \to \mathbb{F}$ is triangulated if $F_G(X) \in \mathbb{F}(\square)$ is bicartesian for each bicartesian $X \in \mathbb{E}(\square)$.

c) An additive 2-morphism $\alpha : F \to G$ of $\partial$-morphisms of derivators $F$ and $G$ from $\mathbb{A}$ to $\mathbb{E}$ is exact (or $\partial$) if, for each bicartesian square $X$ in $\mathbb{A}(\square)$ of type (1), the morphism

\[\alpha^X_G : F_G X \to G_G X\]

is a bicartesian morphism of bicartesian squares in $\mathbb{E}(\square)$. More explicitly this means that the canonical morphisms

\[\eta^X_G : \alpha^X_G \rightarrow i_{\downarrow \downarrow} i_{\downarrow \downarrow}^* \alpha^X_G,
\]

\[\varepsilon^X_G : i_{\uparrow \uparrow} i_{\uparrow \uparrow}^* \alpha^X_G \rightarrow \alpha^X_G,
\]

induced by the adjunction isomorphisms $\eta^X : X \to i_{\downarrow \downarrow} i_{\downarrow \downarrow}^* X$ and $\varepsilon^X : i_{\uparrow \uparrow} i_{\uparrow \uparrow}^* X \to X$, are invertible.

d) An additive 2-morphism $\alpha : F \to G$ of triangulated morphisms of derivators $F$ and $G$ from $\mathbb{E}$ to $\mathbb{F}$ is triangulated if, for each bicartesian square $X$ in $\mathbb{E}(\square)$, the morphism

\[\alpha^X_G : F_G X \to G_G X\]

is a bicartesian morphism of bicartesian squares in $\mathbb{F}(\square)$. 

Remark 2.15. Since \( \varepsilon : i_r i_r^* \to 1 \) is a natural transformation the following square

\[
\begin{array}{ccc}
i_r i_r^* F X & \xrightarrow{\varepsilon F X} & F X \\
\downarrow \alpha^X & & \downarrow \alpha^X \\
i_r i_r^* G X & \xrightarrow{\varepsilon G X} & G X
\end{array}
\]

is commutative. It follows that if the horizontal arrows are invertible then the canonical morphism \( \varepsilon^X : i_r i_r^* \alpha^X \to \alpha^X \) is invertible, too. An analogous observation holds for the natural transformation \( \eta : 1 \to i_* i_*^* \). Therefore, the conditions in items c) and d) are always true for all additive 2-morphisms \( \alpha : F \to G \), i.e., we get that all additive 2-morphisms of exact/triangulated (pre)derivators are automatically exact or triangulated.

We shall prove in proposition 4.6 that our definition 2.14 is correct, in the sense that if an additive morphism of (pre)derivators \( F \) preserves bicartesian squares then it locally preserves conflations/distinguished triangles of the canonical structure, thanks to the fact that the involved derivators enjoy the ‘epivalence axiom’ (i.e., if axiom Der 5 holds).

The composition of two morphisms is clear. Thus, the \( \partial \)-morphisms between any fixed pair of derivators \( A \) and \( E \), and their \( \partial \)-morphisms, are the objects and, respectively, the morphisms of a category that we indifferently denote \( \text{Hom}_{\text{ex}}(A, E) \) or \( \text{Hom}_{\partial}(A, E) \). Analogously, the category \( \text{Hom}_{\text{tr}}(E, F) \) is the category of triangulated morphisms of derivators from \( E \) to \( F \) and their triangulated morphisms.

The following is the main result of this article.

Theorem 2.16. Let \( E \) be a triangulated derivator of type \( \mathcal{D}_{\text{iaf}} \). Then, the canonical morphism can : \( A \to \mathcal{D}_A \) induces an equivalence of categories

\[
\text{Hom}_{\text{ex}}(\mathcal{D}_A, E) \xrightarrow{\sim} \text{Hom}_{\text{ex}}(A, E).
\]

Let \( T \) be a triangulated category and \( A \) an exact category. A \( \partial \)-functor (also called exact functor) is an additive functor \( F : A \to T \) endowed with functorial distinguished triangles

\[
F(X) \xrightarrow{\partial \varepsilon} F(Y) \xrightarrow{\partial} F(Z) \xrightarrow{\varepsilon} \Sigma F(X)
\]

for each conflation \( \varepsilon : X \to Y \to Z \).

A 2-morphism of \( \partial \)-functors \( \alpha : F \to F' \) is a natural transformation such that the square

\[
\begin{array}{ccc}
F Z & \xrightarrow{\partial \varepsilon} & \Sigma F X \\
\downarrow \alpha^Z & & \downarrow \Sigma \alpha^X \\
F' Z & \xrightarrow{\partial' \varepsilon} & \Sigma F' X
\end{array}
\]

commutes, for each conflation \( \varepsilon \). This also gives a category that we indifferently denote \( \text{Hom}_{\text{ex}}(A, T) \) or \( \text{Hom}_{\partial}(A, T) \).

Let us recall that a (triangulated) category (resp., a (triangulated) functor) is basic if it occurs as the evaluation over the terminal diagram \( e \) of a (triangulated) derivator (resp., of a morphism of (triangulated) derivators). Apparently, it is not known whether all (triangulated) categories are basic, nor if the ‘extension’ of a (triangulated) category to a (triangulated) derivator having it as a base is unique. However, we have some reasons to think of a negative answer to both of these statements.

We want to stress that ‘basic’ always implies a choice of a derivator. As a consequence, when we claim that some basic morphism extends ‘uniquely’, we always mean ‘uniquely with respect to a chosen derivator’.
The main aim of the second part of this article is to show that from Theorem 2.16 we can prove the following theorem, whose proof is completed by the end of the paper.

**Theorem 2.17.**

a) Let \( F : \mathcal{A} \to \mathcal{T} \) be a \( \partial \)-functor. Suppose that \( \mathcal{T} = \mathbb{T}(e) \) for some triangulated derivator \( \mathbb{T} \) of type \( \mathbb{D}_\text{ia} \), and that

\[
\text{Hom}_\mathcal{T}(F(X), \Sigma^n F(Y)) = 0
\]

for each \( n < 0 \), for all \( X, Y \) in \( \mathcal{A} \) (Toda condition). Then, the functor \( F \) uniquely extends (up to a unique basic natural isomorphism) to a basic triangulated functor

\[
\tilde{F} : \mathbb{D}^b \mathcal{A} \to \mathcal{T}.
\]

b) Let \( F, F' : \mathbb{D}^b \mathcal{A} \to \mathcal{T} \) be two basic triangulated functors such that the (Toda) conditions

\[
\text{Hom}_\mathcal{T}(\Sigma^n F(X), F(Y)) = 0, \quad n > 0,
\]

\[
\text{Hom}_\mathcal{T}(\Sigma^n F'(X), F'(Y)) = 0, \quad n > 0,
\]

\[
\text{Hom}_\mathcal{T}(\Sigma^n F(X), F'(Y)) = 0, \quad n > 0,
\]

hold for all \( X, Y \) in \( \mathcal{A} \). Suppose that \( \mu \) is a 2-morphism between the restrictions of \( F \) and \( F' \) to \( \mathcal{A} \). Then, the 2-morphism \( \mu \) uniquely extends to a basic 2-morphism of triangulated functors

\[
\tilde{\mu} : F \to F'.
\]

Let us notice that item b) in Theorem 2.17 differs from the analogous fact, true for the towers, which is contained in Corollary 2.7 in [20]. In fact, the same conclusion there follows from the last of our three Toda conditions alone. The analogous situation holds for the item b) in our Theorem 6.5 when compared with Keller’s Theorem 2.7 in [20].

**Remark 2.18.** The extension of \( F : \mathcal{A} \to \mathcal{T} \) to the derived category is unique once the derivator \( \mathcal{T} \) is fixed. The question whether the extension is not unique since there might be more than one triangulated derivator with the same base \( \mathcal{T} \) remains open.

**Remark 2.19.** The name Toda condition is inspired by the use made by topologists who refer to the obstructions to building a Postnikov system as Toda brackets. Let us recall that the same condition already appears in the foundational work [18] by M. Kapranov. More importantly to us for the similarity to the results of the present paper are the theorems about unicity of enhancement of some triangulated categories and their functors to DG-categories proved by V. Lunts and D. Orlov in [25] which are based on the same condition as ours. We refer to this article for a thorough explanation of some important cases in which the Toda condition holds.

Let us recall that Heller [17], Franke [10], Cisinski [9], . . . have shown universality properties of many important derivators. The following result is proved by Jens Franke in [10] in the case that \( \mathbb{D}_\text{ia} \) is the full subcategory of \( \mathcal{C}at \) consisting of finite posets and by Alex Heller in [17] when \( \mathbb{D}_\text{ia} = \mathcal{C}at \).

**Theorem 2.20.** Let \( Sp : \mathbb{D}_\text{ia} \to \mathcal{C}at \) be the triangulated derivator of spectra associated with the homotopy category of finite spectra, i.e., spectra with finitely many cells (more precisely with a localizer defining this homotopy category). Let \( D : \mathbb{D}_\text{ia} \to \mathcal{C}at \) be a triangulated derivator. Then, we have the equivalence of categories

\[
\text{Hom}(Sp, D) \xrightarrow{\sim} D(e), \quad F \mapsto (F_e)(S^0).
\]
Here, the decoration ! indicates commutativity with all homotopy Kan extensions. More precisely, we can refer to these as ‘cocontinuous morphisms’ (because this is the correct universal property if we drop the finiteness assumptions). In the stable case and given our choice of diagram categories, it follows by proposition 5.1 from the paper [29] by Ponto-Shulman that such morphisms also preserve all finite limits, and by the pointwise formula for right Kan extensions also all right Kan extensions in this diagram category (see [13]).

3. Keller’s towers

We begin by defining a 2-category that we call \( \mathcal{C}ubes \). For any positive integer \( n \), let us denote \( C_n \) the \( n \)-product \( \Delta_1 \times \Delta_1 \times \ldots \times \Delta_1 \). Clearly, \( C_0 \) is the terminal category \( e = \{0\} \) and \( C_1 \) is the diagram \( \square \). Sometimes, we will refer to these diagrams as \( n \)-cubes. These diagrams are the objects of the 2-category \( \mathcal{C}ubes \).

Let us think of the objects \( C_n \) as the partially ordered \( n \)-dimensional cubes with side of length one, so that any vertex \( x \) is identified by an \( n \)-tuple \((x_1, \ldots, x_n)\) of numbers in the set \( \{0,1\} \). The morphisms in \( \mathcal{C}ubes \) are all possible compositions of the following order preserving maps. For any \( n \in \mathbb{N} \),

\[
\begin{align*}
i_0^n & : C_n \to C_{n+1}, & \quad (x_1, \ldots, x_n) & \mapsto (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n), \\
p_0^n & : C_{n+1} \to C_n, & \quad (x_1, \ldots, x_n) & \mapsto (x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n),
\end{align*}
\]

where \( \varepsilon \in \{0,1\} \). Clearly, there are relations among these functors (see [20]). The 2-morphisms in \( \mathcal{C}ubes \) are given by the following order relation: for any pair of morphisms \( u, v \) from \( C_l \) to \( C_m \), we write \( u \Rightarrow v \) if there is an arrow \( u(x) \to v(x) \), for all \( x \in C_l \).

Notice that \( \mathcal{C}ubes \) is not a full 2-subcategory of \( \mathcal{D}iat \).

**Definition 3.1.** A **Keller’s tower of additive categories** or simply a **tower of additive categories** \( \mathbb{E} \) is a 2-functor from the opposite 2-category \( \mathcal{C}ubes^\circ \) to the 2-category of additive categories \( \mathcal{A}^\circ \).

For any morphism \( u : C_l \to C_m \) and any 2-morphism \( \alpha : u \Rightarrow v \) in \( \mathcal{C}ubes \), we denote \( u^* : \mathbb{E}(C_m) \to \mathbb{E}(C_l) \) the induced morphism and \( \alpha^* : v^* \Rightarrow u^* \) the induced 2-morphism, as in the case of derivators.

Clearly, any additive (pre-)derivator \( \mathbb{E} \) gives rise to a tower \( \mathbb{E} \) when it is restricted to \( \mathcal{C}ubes^\circ \). An important example of additive tower is the restriction of the prederivator \( \mathcal{A} \) defined in [2,4] to \( \mathcal{C}ubes^\circ \).

If the category \( \mathcal{A} \) has an exact structure, then we can speak of a **tower of exact categories** \( \mathcal{A} \) by endowing each \( \mathcal{A}(C_n) \) with the pairs of composable 2-morphisms of functors (i.e., natural transformations) whose evaluation at each \( x \in C_n \) is a conflation of \( \mathcal{A} \). Analogously, we can define the notion of **tower of triangulated categories** as a contravariant 2-functor from \( \mathcal{C}ubes \) to the 2-category of triangulated categories \( \mathcal{T} \mathcal{R} \mathcal{I} \mathcal{A} \). As an important example we can consider, for any arbitrary exact category \( \mathcal{A} \), the triangulated tower defined by \( D_\mathcal{A} : \mathcal{C}ubes^\circ \to \mathcal{C} \mathcal{A} \mathcal{T} \), i.e., the tower that associates the bounded derived category \( D^b(\mathcal{H}om(C_n, \mathcal{A})) \) with any \( C_n \) in \( \mathcal{C}ubes \). With analogy to the derivator case, we can define the **morphisms of towers** (called ‘towers of morphisms’ in [20]) and their compositions.

**Definition 3.2.** Let \( \mathbb{D} \) and \( \mathbb{E} \) be two towers. A **morphism of towers** \( F : \mathbb{D} \to \mathbb{E} \) consists of

- an additive functor \( F_n : \mathbb{D}_n \to \mathbb{E}_n \), for each \( n \in \mathbb{N} \);
- an isomorphism of functors

\[
\phi_u : F_m u^* \cong u^* F_n,
\]

for each morphism \( u : C_m \to C_n \) in \( \mathcal{C}ubes \); such that
• \( \varphi_1 = 1_{F_n} \), for each \( n \in \mathbb{N} \);
• \( \varphi_{uv} = (v^* \varphi_u)(\varphi_v u^*) \), for each pair of morphisms \( C_l \to C_m \to C_n \) in \( \mathcal{Cubes} \);
• \( \varphi_u \alpha^* = \alpha^* \varphi_v \), for each 2-morphism \( C_m \xrightarrow{[u]} C_n \) in \( \mathcal{Cubes} \).

The composition of two morphisms of towers \( F : D \to E \) and \( G : E \to F \) is defined by setting \( (GF)_n = G_n F_n \), for all \( n \in \mathbb{N} \), and \( (\gamma \varphi)_u = (G_n \varphi_u)(\gamma_u F_m) \), for all morphisms \( u : C_m \to C_n \) in \( \mathcal{Cubes} \).

If \( F \) and \( G \) are two morphisms of towers from \( D \) to \( E \), a 2-morphism \( \alpha : F \to G \) is given by a functor \( \alpha_n : F_n \to G_n \), for each \( n \in \mathbb{N} \), such that \( (u^* \alpha_n) \varphi_u = \gamma_u (\alpha_m u^*) \), for all morphisms \( u : C_m \to C_n \) of \( \mathcal{Cubes} \). The composition of two such morphisms is clear. Thus, the morphisms between any fixed pair of towers \( D \) and \( E \) and the 2-morphisms among them are the objects and, respectively, the morphisms of an additive category that we denote \( \text{Hom}_{add}(D, E) \), or simply \( \text{Hom}(D, E) \).

It is clear how to define morphisms of exact towers (resp., morphisms of triangulated towers) and 2-morphisms among them. Thus, for any pair of exact (resp., triangulated) towers \( D \) and \( E \), we get the category \( \text{Hom}_{ex}(D, E) \) (resp., \( \text{Hom}_{tr}(D, E) \)). A morphism of towers \( F : E \to F \) from a tower of exact categories \( E \) to a tower of triangulated categories \( F \) is a \( \partial \)-morphism or exact morphism of towers from \( E \) to \( F \) if \( F_n : E_n \to F_n \), \( n \in \mathbb{N} \), is a sequence of \( \partial \)-functors. We denote \( \text{Hom}_{ex}(E, F) \) the category of these functors.

**Theorem 3.3** (Keller, [20]). Let \( \mathcal{A} \) be an exact category and \( \mathcal{E} \) a tower of triangulated categories. Then, the canonical morphism \( \gamma : \mathcal{A} \to \mathcal{D}_\mathcal{A} \) induces an equivalence of categories \( \text{Hom}_{tr}(\mathcal{D}_\mathcal{A}, \mathcal{E}) \to \text{Hom}_{ex}(\mathcal{A}, \mathcal{E}) \).

Clearly, this is an important theorem. The aim of the present article is to show that it also holds in the context of derivators.

4. Epivalence and Recollement

In this section we want to show a very important property of triangulated derivators and their morphisms. We will see that the property of recollement, which is enjoyed by derivators by the very definition, is important in this setting.

**Definition 4.1.** A recollement of triangulated categories \( \mathcal{T}' \), \( \mathcal{T} \) and \( \mathcal{T}'' \) is a diagram of triangulated functors

\[
\begin{array}{ccc}
\mathcal{T}'' & \xrightarrow{i^*} & \mathcal{T} \\
\downarrow{j^*} & & \downarrow{j_l} \\
\mathcal{T} & \xrightarrow{i_l} & \mathcal{T}'
\end{array}
\]

such that

- the pairs \( j_l^* \hookrightarrow j_l, j_l \hookrightarrow j^*, i_l \hookrightarrow i^*, i_l \hookrightarrow i_+ \) are adjunctions;
- \( i_l j_l = 0 \);
- the functors \( i_+ \), \( i_l \) and \( j_l \) are fully faithful;
- for every object \( X \) lying in \( \mathcal{T} \), there are distinguished triangles

\[
\begin{align*}
&i_l i^* X \xrightarrow{i_l^*} X \xrightarrow{j_l} j_l j^* X \to \Sigma i_l i^* X, \\
&j_l j^* X \xrightarrow{j_l^*} X \xrightarrow{i_l^*} i_l i^* X \to \Sigma j_l j^* X.
\end{align*}
\]
where \((\varepsilon_i^X, \eta_i^X)\) and \((\varepsilon_j^X, \eta_j^X)\) are pairs of adjunction morphisms related to \(i\) and \(j\), respectively.

The functors \(j_i^\ast, j_j^*, i_i^!, i_i^*, i_i^\ast, i_i^\rightarrow\) are also known as the 6-gluing functors.

Let us recall that in the context of derivators open / closed inclusions of diagrams naturally give rise to recollements of triangulated categories. This allows us constructing the shift and loop autoequivalences in terms of the 6-gluing functors as in the following lemma that we state even if we are not going to use it in what follows.

**Lemma 4.2.** Let \(\mathcal{S}\) be a triangulated derivator. Fix an arbitrary diagram \(I\) in \(\mathcal{D}ia\), let \(j : I \to I \times \Delta_1\) and \(i : I \to I \times \Delta_1\) be the obvious open and closed inclusions, respectively. Then, there are canonical isomorphisms of functors

\[
j^3 i_\ast \to \Sigma_I , \quad \Omega_I = \Sigma_j^1 \to i_j^1.
\]

**Proof.** It is well known (e.g., look at [7, Section 9]) that an open inclusion \(j\) with a closed inclusion \(i\) such that the diagram spanned by the union of their images is all of \(I \times \Delta_1\) give rise to a recollement of triangulated categories

\[
\begin{array}{c}
\mathcal{S}(I) \\
\downarrow j^! \\
\mathcal{S}(I \times \Delta_1) \\
\downarrow i^! \\
\mathcal{S}(I)
\end{array}
\]

Thus, for any object \(X\) in \(\mathcal{S}(I \times \Delta_1)\) there is a distinguished triangle

\[
i_i i^\ast X \longrightarrow X \longrightarrow j^3 j^! X \longrightarrow \Sigma_{I \times \Delta_1} i_i i^\ast X.
\]

Here, by applying the functor \(\text{Hom}(-, j^3 j^! X)\) and passing to long exact sequence, we see that the connecting morphism \(j^3 j^! X \to \Sigma_{I \times \Delta_1} i_i i^\ast X\) is unique because \(\text{Hom}(\Sigma_{I \times \Delta_1} i_i i^\ast X, j^3 j^! X) = 0\) by adjunction and by recollement.

Let us apply the triangulated functor \(j^*\) to this triangle. Because of the recollement axioms we get a distinguished triangle in \(\mathcal{S}(I)\)

\[
i_i i^\ast X \longrightarrow j^* X \longrightarrow j^3 X \longrightarrow \Sigma_{I \times \Delta_1} i_i i^\ast X.
\]

In particular, when the object \(X\) is of the form \(i_i^\ast Y\), for some object \(Y\) in \(\mathcal{S}(I)\), the triangle becomes

\[
i_i^\ast i_i^\ast Y \longrightarrow j^* i_i^\ast Y \longrightarrow j^3 i_i^\ast Y \longrightarrow \Sigma_{I \times \Delta_1} i_i^\ast Y.
\]

Here \(i_i^\ast i_i^\ast Y = Y\) because the functor \(i_i^\ast\) is fully faithful and \(j^* i_i^\ast Y = 0\) by adjunction. It survives a functorial iso \(j^3 i_i^\ast Y \to \Sigma_{Y} Y\), for all \(Y\) in \(\mathcal{S}(I)\). The conclusion follows easily.

The analogous proof of the second isomorphism starts by applying the triangulated functor \(i_\ast\) to the canonical distinguished triangle

\[
\Sigma_{-1} i_i^\ast X \longrightarrow j_i^\ast X \longrightarrow \varepsilon_j^X \longrightarrow X \longrightarrow \eta_j^X \longrightarrow i_i^* X
\]

and taking \(X = j_\ast Y\).

If we are willing to study commutativity of derivator morphisms with the 6-gluing functors we need another concept.

**Definition 4.3.** Let \(\mathcal{A}\) be an exact category and let \(\mathcal{S}, \mathcal{T}\) be triangulated categories. We say that
a) an additive functor \( F : \mathcal{S} \to \mathcal{T} \) is \textit{weakly triangulated} if, for each distinguished triangle
\[
X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X
\]
of \( \mathcal{S} \), there is \textit{some} distinguished triangle
\[
F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{w'} \Sigma F(X)
\]
of \( \mathcal{T} \);
b) an additive functor \( F : \mathcal{A} \to \mathcal{S} \) is a \textit{weak \( \partial \)-functor} or \textit{weakly exact} if, for each conflation
\[
X \xrightarrow{u} Y \xrightarrow{v} Z
\]
of \( \mathcal{A} \), there is \textit{some} distinguished triangle
\[
F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{z} \Sigma F(X)
\]
of \( \mathcal{S} \).

Clearly, the notion of weakly triangulated functor is much weaker than the related notion of triangulated functor, which is described in [23] under the name of \( S \)-functor. Nevertheless, we will see that (Proposition 4.7), in order to extend an additive morphism of triangulated derivators to a triangulated morphism, it is redundant to ask that all the functors \( F_f \) are triangulated. We shall see that this extension is possible if all these functors are \textit{weakly} triangulated. Indeed, even less is required for this happens, it is enough to ask that \( F_{\Delta_1} \) and \( F_0 \) are \textit{weakly} triangulated functors.

Let \( \mathcal{S} \) and \( \mathcal{T} \) be triangulated derivators and \( \mathcal{A} \) a represented exact derivator. Let \( F : \mathcal{S} \to \mathcal{T} \) and \( F : \mathcal{A} \to \mathcal{S} \) be \textit{additive} functors of derivators. Let us fix an arbitrary diagram \( I \) in \( \mathcal{G} \). Let \( j : I \to I \times \Delta_1 \) and \( i : I \to I \times \Delta_1 \) be the obvious open and closed inclusions, respectively. We remark that in this situation the natural transformation (cf. subsection 2.2) \( \psi^j : jF_I \to F_{I \times \Delta_1} j \) is invertible. Indeed, since every point inclusion \( e \to I \times \Delta_1 \) either factors through \( i \) or \( j \), it follows that \( i^* \) and \( j^* \) detect isomorphisms and we can easily check that \( i^*(\psi^j_X) = 0 \) and \( j^*(\psi^j_X) \) are isomorphisms, for all \( X \) in \( \mathcal{S}(I \times \Delta_1) \).

Thus, there is an induced natural transformation
\[
\psi^j_X : j^jF_{I \times \Delta_1} \to F_{Ij}^j,
\]
whose action \( \psi^j_X \) on an object \( X \) is given by the composition
\[
j^jF_{I \times \Delta_1}X \xrightarrow{j^jF_{I \times \Delta_1}(\psi^j_X)} j^jF_{I \times \Delta_1}j_j^jX \xrightarrow{j^j(\psi^j_X)^{-1}} j^jF_{Ij}^jX \xrightarrow{\epsilon_{F_I}^{j,j}X} F_{Ij}^jX.
\]
Also, there is an induced natural transformation
\[
\psi_{j,a} : j^jF_{I \times \Delta_1} \to F_{I \times \Delta_1}j_j^j,
\]
whose action on every \( X \) in \( \mathcal{S}(I \times \Delta_1) \) is defined by the composition
\[
j^jF_{I \times \Delta_1}X \xrightarrow{j^j(\psi^j_X)} j^jF_{Ij}^jX \xrightarrow{\phi^{j,j}X} F_{I \times \Delta_1}j_j^jX.
\]

Lemma 4.4. In the situation just described for the additive morphism \( F : \mathcal{S} \to \mathcal{T} \) of triangulated derivators, suppose that the functor \( F_{I \times \Delta_1} \) is weakly triangulated. Then, for any \( X \) in \( \mathcal{S}(I \times \Delta_1) \), there is a unique isomorphism
\[
\psi^{j,a}_X = \phi^{j,j}X \circ \psi^j_X : j^jF_{I \times \Delta_1}X \to F_{I \times \Delta_1}j_j^jX.
\]
such that the relation $\psi_j^X \circ \eta_j = F_{i \Delta_1}(\eta_j^X)$ holds. In particular, the 2-morphism of functors

$$\psi_j^X : j F_{i \Delta_1} \to F_j j^X$$

is invertible, for all $X$ in $\mathbb{S}(I \times \Delta_1)$. More generally, $\psi_{jj} : j j^2 F_{i \Delta_1} \to F_{i \Delta_1} j j^2$ and $\psi_j : j^2 F_{i \Delta_1} \to F_j j^2$ are 2-isomorphisms of functors.

Proof. For any object $X$ in $\mathbb{S}(I \times \Delta_1)$, let us start by considering the canonical distinguished triangle that we have just seen in the proof of Lemma 4.2

$$i \iota^* X \xrightarrow{\varepsilon_i^X} X \xrightarrow{\eta_j^X} j j^3 \xrightarrow{\partial^X} \Sigma i \iota^* X,$$

whose connecting morphism $\partial^X : j j^3 X \to \Sigma i \iota^* X$ is unique because $\text{Hom}(\Sigma i \iota^* X, j j^3 X) = 0$ by adjunction and by recollement.

Now apply the weakly triangulated functor $F_{i \Delta_1}$ and get a distinguished triangle in $\mathbb{T}(I \times \Delta_1)$

$$F_{i \Delta_1} i \iota^* X \xrightarrow{F(\varepsilon)} F_{i \Delta_1} X \xrightarrow{F(\eta)} F_{i \Delta_1} j j^3 X \xrightarrow{w} \Sigma F_{i \Delta_1} i \iota^* X,$$

where we shortly write $F(\varepsilon)$ for $F_{i \Delta_1}(\varepsilon_i^X)$, $F(\eta)$ for $F_{i \Delta_1}(\eta_j^X)$. Recall that here the morphism $w : F_{i \Delta_1} j j^3 X \to \Sigma F_{i \Delta_1} i \iota^* X$ just exists, i.e., it is not unique, nor canonically constructed.

Let us consider the object $F_{i \Delta_1} X$ and write the related distinguished triangle

$$i \iota^* F_{i \Delta_1} X \xrightarrow{\varepsilon'} F_{i \Delta_1} X \xrightarrow{\eta'} j j^3 F_{i \Delta_1} X \xrightarrow{\partial'} \Sigma i \iota^* F_{i \Delta_1} X.$$

Here we identify $\varepsilon' = \varepsilon_i F_{i \Delta_1} X$, $\eta' = \eta_j F_{i \Delta_1} X$ and $\partial' = \partial F_{i \Delta_1}$ for the unique connecting morphism. Since there is a natural transformation (cf. subsection 2.2)

$$\varphi_{ii} := (\varphi_i^i) \circ i \iota(\varphi_i^{-1}) : i \iota^* F_{i \Delta_1} \to i \iota F_{i \Delta_1} \to F_{i \Delta_1} i \iota^*,$$

for every object $X$ in $\mathbb{S}(I \times \Delta_1)$ there are a morphism $\varphi_i^X = ((\varphi_i^i) \circ i \iota(\varphi_i^{-1}))^X$ and an induced morphism of distinguished triangles of $\mathbb{T}(I \times \Delta_1)$

$$F_{i \Delta_1} i \iota^* F_{i \Delta_1} X \xrightarrow{F(\varepsilon)} F_{i \Delta_1} X \xrightarrow{F(\eta)} F_{i \Delta_1} j j^3 X \xrightarrow{w} \Sigma F_{i \Delta_1} i \iota^* X.$$

Indeed, let us check commutativity of the square on the left. We have

$$F_{i \Delta_1} (\varepsilon_i^X) \circ \varphi_i^X = F_{i \Delta_1} (\varepsilon_i^X) \circ \varepsilon_i F_{i \Delta_1} i \iota^* X \circ i \iota (\varphi_i^i) \circ i \iota F_i (\eta_i^X) \circ i \iota [(\varphi_i^{-1})^X]$$

$$= \varepsilon_i F_{i \Delta_1} X \circ i \iota^* F_{i \Delta_1} (\varepsilon_i^X) \circ i \iota (\varphi_i^i) \circ i \iota F_i (\eta_i^X) \circ i \iota [(\varphi_i^{-1})^X]$$

$$= \varepsilon_i F_{i \Delta_1} X \circ i \iota (\varphi_i^X) \circ i \iota F_i (\eta_i^X) \circ i \iota [(\varphi_i^{-1})^X]$$

$$= \varepsilon_i F_{i \Delta_1} X \circ i \iota (\varphi_i^X) \circ i \iota [(\varphi_i^{-1})^X]$$

$$= \varepsilon_i F_{i \Delta_1} X.$$

Here, every canonical isomorphism comes from functoriality, as $\varphi_i$ is a 2-isomorphism of functors, and from the relation $i \iota^*(\varepsilon_i^X) \circ \eta_i^X = 1^{i \iota^* X}$.

Let us remark the useful fact that, if we call $p$ the obvious projection functor $I \times \Delta_1 \to I$, then there are the adjunction morphisms $j \to p \to i$. This fact implies that $j^* \to p^* \to i^*$ also
are adjunction morphisms. In particular, we get the equality $i_i = p^*$, which entails that the canonical morphisms $\varphi := \varphi_{ii}^X = \varphi_{\varphi_{ii}}^X : (i \circ p)^* F_{I \times \Delta_1} X \to F_{I \times \Delta_1} (i \circ p)^* X$ and $\Sigma \varphi$ are invertible.

Thus, we have shown that $\psi$ is an isomorphism. Moreover, it is unique because we can see that the group

$$
\text{Hom}(\Sigma F_{I \times \Delta_1} i_i^* X, j j^2 F_{I \times \Delta_1} X) = \text{Hom}(\Sigma i_i^* F_{I \times \Delta_1} X, j j^2 F_{I \times \Delta_1} X)
$$

is canonically trivial via the canonical isomorphism $\varphi$, adjunction and the canonical isomorphism $i_i^* j j = 0$, thanks to the recollement axioms.

Therefore, if we are able to show that the morphism $\psi_{jj}^X$ makes the central square of the diagram commutative too, we get $\psi = \psi_{jj}^X$ canonically and the statement follows. Indeed, there is a commutative diagram, by functoriality of $\eta_j$,

$$
\begin{array}{ccc}
F_{I \times \Delta_1} j j^2 X & \xrightarrow{\sim} & j j^2 F_{I \times \Delta_1} X \\
\eta_{j j^2} X & & \eta_{j} j j^2 X \\
\end{array}
$$

We already know that the three arrows labelled with the symbol $\sim$ are isomorphisms. Therefore, the arrows $F_{I \times \Delta_1} j j^2 X$ and $j j^2 F_{I \times \Delta_1} X \to j j^2 F_{I \times \Delta_1} X$ are invertible, too. In particular, we get that the composition $\varphi_{jj}^X j j^2 X \circ \eta_j j j^2 X \circ \varphi_{jj}^X_{jj} j j^2 X^{-1}$ gives us an inverse to $\eta_{j j^2} F_{I \times \Delta_1} X$. Now, since the relation $j j^2 F_{I \times \Delta_1} X \circ \eta_{j j^2} F_{I \times \Delta_1} X = j j^2 F_{I \times \Delta_1} X$ holds, it follows that $\psi_{jj}^X \circ \eta_{j j^2} F_{I \times \Delta_1} X = F_{I \times \Delta_1} (\eta_{j j} X)$ also holds.

Since we have shown that $\psi_{jj}^X$ is invertible, we get that the morphism $j (\psi_{jj}^X)$ must be invertible, too. Now apply $j^*$ to see that $\psi_{jj}^X$ is an isomorphism. The claim follows. 

We need a similar lemma for morphisms defined over exact derivators.

**Lemma 4.5.** Let $\mathcal{A}$ be the prederivator represented by an exact category $\mathcal{A}$ and let $\mathcal{S}$ be a triangulated derivator. Let $F : \mathcal{A} \to \mathcal{S}$ be an additive morphism of derivators such that the functor $F_{I \times \Delta_1}$ is weakly exact. Consider an arbitrary object $X$ in $\mathcal{A}(I \times \Delta_1)$ such that the ‘vertical parallel arrows’ of his diagram are deflations. More precisely, this means that $i_k^* X \to i_k^* j^2 X$ is a deflation, for all $k \in I$ (here, $i$ and $j$ are as in lemma [4.4] and $i_k : e \to I$ is the obvious map).

Then, there is a unique isomorphism

$$
\psi_{jj}^X := \varphi_{jj}^X \circ \psi_j^X : j j^2 F_{I \times \Delta_1} X \to F_{I \times \Delta_1} j j^2 X
$$

such that the relation $\psi_{jj}^X \circ \eta_{j j^2} X = F_{I \times \Delta_1} (\eta_{j j} X)$ holds. In particular, the natural transformation of functors

$$
\psi_{jj}^X : j j^2 F_{I \times \Delta_1} X \to F_{I \times \Delta_1} j j^2 X
$$

is invertible. More generally, when restricted on the subcategory of squares as above, $\psi_{jj} : j j^2 F_{I \times \Delta_1} \to F_{I \times \Delta_1} j j^2$ and $\psi_j : j j^2 F_{I \times \Delta_1} \to F_{I \times \Delta_1}$ are 2-isomorphisms of functors.
Proposition 4.6. Let \( A \) be the represented exact prederivator associated to an exact category \( A \). Let \( S \) and \( T \) be triangulated derivators. Let us suppose that all these derivators be of some type \( \mathcal{D} \).a)

Let \( F : A \to S \) be a \( \Theta \)-morphism according to Def. \([2, 14]\), item a). Then, for all diagram \( I \) in \( \mathcal{D} \) the local functor \( F_I : A(I) \to S(I) \) has a canonical structure of \( \Theta \)-functor with respect to the exact structure of \( A(I) \) and to the canonical triangulated structure of \( T(I) \).
b) Let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated morphism according to Def. 2.12 item b). Then, for all diagram $I$ in $\mathcal{D}$ the local functor $F_I : \mathcal{S}(I) \to \mathcal{T}(I)$ has a canonical structure of triangulated functor with respect to the canonical triangulated structures of the categories $\mathcal{S}(I)$ and $\mathcal{T}(I)$.

**Proof.** a) We begin with an exact morphism of prederivators $F : \mathcal{A} \to \mathcal{S}$ as in the hypothesis. We have to show that the functor $F_I : \mathcal{A}(I) \to \mathcal{S}(I)$ is exact, for all diagrams $I$. This means that, given a conflation $\varepsilon$ in $\mathcal{A}(I) = \mathcal{A}^I(e)$

$$X \xrightarrow{i} Y \xrightarrow{d} Z,$$

there is a natural transformation $\delta_I$ which assigns a morphism $\delta_I \varepsilon : F_I Z \to \Sigma I F_I X$ in such a way that

$$F_I X \xrightarrow{F(i)} F_I Y \xrightarrow{F(d)} F_I Z \xrightarrow{\delta_I \varepsilon} \Sigma I F_I X$$

is a distinguished triangle of $\mathcal{S}(I)$. Altogether, all these natural transformations define a canonical $\delta$-morphism $\delta$.

Associated with the conflation $\varepsilon$ there is a bicartesian square

$$
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
0 & \rightarrow & Z
\end{array}
$$

lying in $\text{Hom}(\Box^0, \mathcal{A}^I(e))$. Since we can write this category as $\text{Hom}(\Delta_1^0, \mathcal{A}^I(\Delta_1))$, thanks to the axiom $\text{Der 5}$ there exists a bicartesian object $S$ in $K^I(\Box)$ whose diagram is the square above.

Clearly, we have a canonical iso $\varepsilon^S : i_{r^{-1}} i_{e}^{-*} S \to S$. For the sake of simplicity in the notation, here we write $i_{r^{-1}}$, $i_{e^{-1}}$, etc., instead of the correct ones $i_{r^{-1}}^e = i_{i_{r^{-1}} e^{-1}}$, $i_{e^{-1}}^e = i_{i_{e^{-1}} e^{-1}}$, etc. Since the morphism $F$ is supposed to be exact, we get an object $F^I_S$ in $\mathcal{S}^I(\Box)$ having a commutative square as diagram

$$
\begin{array}{ccc}
F^I_e X & \rightarrow & F^I_e Y \\
\downarrow & & \downarrow \\
0 & \rightarrow & F^I_e Z
\end{array}
$$

such that $\varepsilon^{F^I_S} : i_{r^{-1}} i_{e}^{-*} F^I_S \to F^I_S$ is invertible. Our local claim is that there exists a distinguished triangle in $\mathcal{S}^I(e)$

$$F^I_e X \xrightarrow{\delta^I_e \varepsilon} F^I_e Y \xrightarrow{\delta^I_e \varepsilon} F^I_e Z$$

for some morphism $\delta^I_e \varepsilon : F^I_e Z \to \Sigma F^I_e X$, functorial in $\varepsilon$, i.e., that the functor $F^I_e$ is exact.

Let us fix some notation. We write $\Box$ for the diagram $\Delta_2 \times \Delta_1$ and $\Box$ for the diagram $\Box$ after we erase the object $(1,1)$. There are obvious fully faithful inclusions $i_{\Box} : \Box \to \Box$, $i_{r} : \Gamma \to \Box$ and $i_{\Gamma} : \Gamma \to \Box$, injective on objects. Moreover, we define other inclusions $l_{\Box} : \Box \to \Box$ and $r_{\Box} : \Box \to \Box$ defined by the evident overlap of the image of the square over the small squares on the left and on the right of the two-square, respectively. These functors further induce other inclusions $l_{\Gamma} : \Gamma \to \Box$ and $r_{\Gamma} : \Gamma \to \Box$, which are the compositions $l_{\Box} i_{r}$ and $r_{\Box} i_{r}$, respectively. We also consider the ‘global’ inclusion $g_{\Box} : \Box \to \Box$ mapping
the square on the exterior bord of the two-square and the inclusion $i_{\mathsf{r} \cdot \mathsf{t}} : \mathsf{r} \to \mathsf{t}$ mapping
the diagram $\mathsf{r}$ to the right corner of the diagram $\mathsf{t}$. Let us define the object $P := \iota_{\mathsf{r} \cdot \mathsf{t}} \cdot F^I \cdot S$. This is a 

\[ \varepsilon^{l_0} : i_{\mathsf{r} \cdot \mathsf{t}} \cdot i_{\mathsf{r}} \cdot i_0 \cdot P \to l_0 \cdot P, \quad \varepsilon^{r_0} : i_{\mathsf{r} \cdot \mathsf{t}} \cdot r_0 \cdot P \to r_0 \cdot P, \quad \varepsilon^{g_0} : i_{\mathsf{r} \cdot \mathsf{t}} \cdot g_0 \cdot P \to g_0 \cdot P. \]

It suffices to check the first and the second isos. We compute

\[ \varepsilon^{l_0} = i_{\mathsf{r} \cdot \mathsf{t}} \cdot i_{\mathsf{r}} \cdot i_0 \cdot P = i_{\mathsf{r} \cdot \mathsf{t}} \cdot i_{\mathsf{r}} \cdot i_0 \cdot F^I \cdot S = i_{\mathsf{r} \cdot \mathsf{t}} \cdot i_{\mathsf{r}} \cdot F^I \cdot S \]

Here we use the relation $l_0 = i_{\mathsf{r} \cdot \mathsf{t}}$ in the first and the last equalities. The crucial iso $\varepsilon^{F^I \cdot S}$ is the fourth, which is our hypothesis. As for the square on the right, let us compute

\[ \varepsilon^{r_0} = i_{\mathsf{r} \cdot \mathsf{t}} \cdot r_0 \cdot P = i_{\mathsf{r} \cdot \mathsf{t}} \cdot r_0 \cdot i_{\mathsf{r}} \cdot i_0 \cdot F^I \cdot S = i_{\mathsf{r} \cdot \mathsf{t}} \cdot r_0 \cdot i_{\mathsf{r}} \cdot F^I \cdot S \]

Here, the only non trivial iso comes from the invertible natural transformation $i_{\mathsf{r} \cdot \mathsf{t}} \cdot r_0 \cdot i_{\mathsf{r}} \cdot F^I \cdot S \sim r_0 \cdot i_{\mathsf{r}} \cdot S$. To see why this transformation is invertible, let us consider the co-cartesian square in the category $\mathcal{D}ia$

\[
\begin{array}{c}
\mathsf{r} \\
\downarrow \mathsf{r} \\
\mathsf{t} \\
\downarrow \\
\mathsf{t} \\
\end{array}
\]

By the “commentaires” after the axiom Der 7 in [26] or, in greater detail, by the dual of Prop. 6.9 in [7, Prop. 6.9] the required isomorphism follows. The local claim follows by the description of the triangulated structure over $S^I(\mathcal{I})$ as, e.g., in [26]. Indeed, the diagram of the object $P$ is

\[
\begin{array}{c}
\begin{array}{c}
F^I X \\
\downarrow \varepsilon_X \\
F^I Y \\
\downarrow \varepsilon_Y \\
F^I Z \\
\downarrow \\
P^I_{12}
\end{array}
\end{array},
\]

\[
\begin{array}{c}
\begin{array}{c}
F^I X \\
\downarrow \varepsilon_X \\
F^I Y \\
\downarrow \varepsilon_Y \\
F^I Z \\
\downarrow \\
P^I_{12}
\end{array}
\end{array},
\]
and there is an isomorphism $\theta^l : F^l_{12} \to \Sigma^l_{c} F^l_{c} X$ which induces a *standard* distinguished triangle in $\mathbb{S}^l(e)$

\[
F^l_{c} X \longrightarrow F^l_{c} Y \longrightarrow F^l_{c} Z \longrightarrow \Sigma^l_{c} F^l_{c} X ,
\]

where the morphism $\delta^l_{c} : F^l_{c} Z \to \Sigma F^l_{c} X$ is given by the composition $\theta^l \circ f_2$. Thus, this triangle provides a canonical construction of the distinguished triangle $[2]$.

It remains to show the functoriality of the construction. So, let us consider an exact 2-morphism $\mu : F \to F'$, where $F' : \mathbb{A} \to \mathbb{S}$ is another exact morphism of prederivators. We desire to show that the functor $\mu_I : F_I \to F'_I$ actually is a natural transformation of $\partial$-functors. Indeed, given a conflation $\varepsilon$ in $\mathbb{A}(I) = \mathbb{A}^l(e)$

\[
X \longrightarrow Y \longrightarrow Z ,
\]

thanks to the axiom Der 5 there is a bicartesian object $S$ in $\mathbb{A}^l(\square)$ whose diagram is as above. Again, we have a canonical iso $\varepsilon^S : i_{\square}! i_{\square}^* S \to S$. We already know that there are isomorphisms $\varepsilon F^l_{\square} : i_{\square}! i_{\square}^* F^l_{\square} S \to F^l_{\square} S$ and $\varepsilon F'^l_{\square} : i_{\square}! i_{\square}^* F'^l_{\square} S \to F'^l_{\square} S$ because $F$ and $F'$ are exact.

Since the 2-morphism of prederivators $\mu : F \to F'$ is supposed to be exact, it induces a morphism $\mu^l S : F^l_{\square} S \to F'^l_{\square} S$ in $\mathbb{S}^l(\square)$, whose diagram in $\mathbb{S}^l(e)$ is

\[
\text{dia}^l_{\square}(\mu^l S) = \mu^l_{\square} \text{dia}^l S = \begin{array}{ccc}
F^l_{e} X & \longrightarrow & F^l_{e} Y \\
\downarrow & & \downarrow \\
0 & \longrightarrow & F^l_{e} Z \\
\downarrow & & \downarrow \\
F'^l_{e} X & \longrightarrow & F'^l_{e} Y \\
0 & \longrightarrow & F'^l_{e} Z
end{array}
\]

such that the morphism $i_{\square}! i_{\square}^* \mu^l S \to \mu^l S$ is invertible (cf. item c) of Def. 2.14.

Our local claim is that there exists a morphism of distinguished triangles in $\mathbb{S}^l(e)$

\[
\begin{array}{ccc}
F^l_{c} X & \longrightarrow & F^l_{c} Y \\
\downarrow \mu^X & & \downarrow \mu^Y \\
F'^l_{c} X & \longrightarrow & F'^l_{c} Y
end{array} \quad \begin{array}{ccc}
F^l_{c} Z & \longrightarrow & \Sigma F^l_{c} X \\
\downarrow \mu^Z & & \downarrow \Sigma \mu^X \\
F'^l_{c} Z & \longrightarrow & \Sigma F'^l_{c} X
end{array}
\]

As above, let us define the polycartesian objects $P := i_{\square}! i_{\square}^* F^l_{\square} S$ and $P' := i_{\square}! i_{\square}^* F'^l_{\square} S$. We also define a morphism $\mu^P : P \to P'$ to be $i_{\square}! i_{\square}^* \mu^l S$. This morphism is *polycartesian* in the following sense: there are isomorphisms

\[
\varepsilon^P \mu^P : i_{\square}! i_{\square}^* l_{\square}^* \mu^P \to l_{\square}^* \mu^P , \quad \varepsilon^P \circ \mu^P : i_{\square}! i_{\square}^* r_{\square}^* \mu^P \to r_{\square}^* \mu^P .
\]
It suffices to check the first and the second isos. We compute

\[
i_{\Delta \times I}^* \circ i_{\Delta}^* \circ \mu_S^P = i_{\Delta \times I}^* \circ i_{\Delta}^* \circ i_{\Delta \times I}^* \circ \mu_S^S
\]

\[
\Rightarrow i_{\Delta}^* \circ i_{\Delta}^* \circ \mu_S^S
\]

\[
\Rightarrow i_{\Delta}^* \circ \mu_S^S
\]

\[
\Rightarrow i_{\Delta}^* \circ \mu_S^S
\]

\[
\Rightarrow i_{\Delta}^* \circ \mu_S^S
\]

\[
\Rightarrow i_{\Delta}^* \circ \mu_S^S
\]

Here we use the relation \( l_{\Delta} = i_{\Delta} \circ i_{\Delta} \) in the first and the last equalities. The crucial iso \( i_{\Delta \times I}^* \circ \mu_S^S \rightarrow \mu_S^S \) is the fourth, which is our hypothesis. As for the square on the right, let us compute

\[
i_{\Delta \times I}^* \circ r_{\Delta}^* \circ \mu_S^P = i_{\Delta \times I}^* \circ r_{\Delta}^* \circ i_{\Delta \times I}^* \circ \mu_S^S
\]

\[
\Rightarrow i_{\Delta \times I}^* \circ r_{\Delta}^* \circ \mu_S^S
\]

\[
\Rightarrow i_{\Delta \times I}^* \circ r_{\Delta}^* \circ \mu_S^S
\]

\[
\Rightarrow i_{\Delta \times I}^* \circ r_{\Delta}^* \circ \mu_S^S
\]

\[
\Rightarrow r_{\Delta}^* \circ \mu_S^S
\]

Here, the isos we are using are in complete analogy with those at the corresponding point in the proof of the object case. The local claim follows by the description of the triangulated structure over \( S^I(e) \) as, e.g., in [26] and an analogous construction to the one we have just done for the object case that we leave to the interested reader. Altogether, these natural transformations of functors \( \mu_I : F^I \rightarrow F'^I \) clearly define an exact 2-morphism of \( \partial \)-morphisms \( \mu : F \rightarrow F' \).

b) Since the statement and the proof are analogous to those in item a) we just sketch out the proof of this case.

The main difference is that we start with a triangulated morphism \( F : S \rightarrow T \) of triangulated derivators. Suppose we are given a distinguished triangle with respect to the canonical triangulated structure of the category \( S(I) \),

\[
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\epsilon} \Sigma_I X,
\]

This means that, up to iso, there is a polycartesian object \( P \) in \( S(\square \times I) \) whose diagram is

\[
\text{dia}_{I}(P) = \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
0 & \xrightarrow{f_2} & P^I_{12}
\end{array}
\]

with an isomorphism \( \theta^I : P^I_{12} \rightarrow \Sigma^I X \). From now on the proof is the same as in item a) by considering the polycartesian object \( F_I(P) \) instead of the previous one \( P \). The construction
provides us with an invertible natural transformation $\delta_I : F_I \Sigma_I \to \Sigma_I F_I$. Altogether, all these transformations define a canonical triangulated 2-isomorphism $\delta : F\Sigma \to \Sigma F$. □

Clearly, we may forget the property of a morphism of derivators of being triangulated. Consequently, we also forget the canonical 2-morphism $\delta$ and think of the underlying additive structure of categories and morphisms only. We indicate the related forgetful functor by means of the vertical bar ‘|’.

**Proposition 4.7.** Let $\mathcal{A}$ be the exact prederivator represented by an exact category $\mathcal{A}$. Let $\mathcal{S}$ and $\mathcal{T}$ be triangulated derivators. Let us suppose that all these (pre)derivators be of some type $\mathcal{G}_{\text{via}}$.

a) The forgetful functor

\[
\text{Hom}_{\mathcal{G}}(\mathcal{S}, \mathcal{T}) \to \text{Hom}_{\text{add}}(\mathcal{S}|, \mathcal{T}|)
\]

is an isomorphism onto the full subcategory consisting of the additive morphisms of derivators $F$ such that $F_{\Delta_1} : \mathcal{S}(\Delta_1) \to \mathcal{T}(\Delta_1)$ and $F_\Box : \mathcal{S}(\Box) \to \mathcal{T}(\Box)$ are weakly triangulated functors with respect to the canonical triangulated structures of these categories.

b) The forgetful functor

\[
\text{Hom}_{\mathcal{G}}(\mathcal{A}, \mathcal{S}) \to \text{Hom}_{\text{add}}(\mathcal{A}|, \mathcal{S}|)
\]

is an isomorphism onto the full subcategory consisting of the additive morphisms of derivators $F$ such that $F_{\Delta_1} : \mathcal{A}(\Delta_1) \to \mathcal{S}(\Delta_1)$ and $F_\Box : \mathcal{A}(\Box) \to \mathcal{S}(\Box)$ are weak $\partial$-functors with respect to the canonical additive and triangulated structures of these categories.

**Proof.** Let us note that we already know from proposition 4.6 that the forgetful functors (in the statements a) and b) actually take a triangulated (resp., exact) morphism of (pre)derivators $F$ to an additive morphism $F$ that is locally triangulated (resp., locally exact), and, a fortiori, locally weakly triangulated (resp., locally weakly exact). Moreover, since we know from remark 2.10 that our forgetful functors are fully faithful, it remains to check their essential surjectivity only.

a) Suppose we are given an arbitrary additive morphism $F : \mathcal{S} \to \mathcal{T}$ such that $F_{\Delta_1}$ and $F_\Box$ are weakly triangulated functors. We have to check that it preserves (homotopy) bicartesian objects according to the condition in the item b) of Definition 2.13. Since the derivators $\mathcal{S}$ and $\mathcal{T}$ have the property of being triangulated, thanks to the axiom Der 7 it is sufficient to check that $F$ preserves (homotopy) cocartesian objects.

Let us denote by $ab$ the objects in the diagrams $\triangleleft$, $\triangleleft$ and $\square$ whose coordinates are $(a, b)$. Accordingly, we denote $i_{ab,\triangleleft}$, $i_{ab,\triangleleft}$ and $i_{ab,\square}$ the functors defined by the object $ab$ having coordinates $(a, b)$, which send the diagram $e$ into the diagrams $\triangleleft$, $\triangleleft$ and $\square$, respectively. We simply write $i_{ab}$ when there is no possibility of confusion.

Remark that the pair of open / closed immersions $e \xleftarrow{i_{11}} \square \xrightarrow{i_{\triangleleft}} \triangleleft$ gives rise to a recollement of triangulated categories

\[
\begin{array}{ccc}
\mathcal{S}(e) & \xleftarrow{i_{11}} & \mathcal{S}(\square) \\
\downarrow & & \downarrow \\
i_{11} & & i_{\triangleleft}
\end{array}
\]

Thus, for any object $X$ in $\mathcal{S}(\Box)$, there is a distinguished triangle

\[
i_{\triangleleft}i_{\triangleleft}^*X \xrightarrow{e} X \xrightarrow{\eta} i_{11}i_{11}^?X \xrightarrow{\partial} \Sigma_{\Box}i_{\triangleleft}i_{\triangleleft}^*X.
\]
Here, we respectively write $\varepsilon, \eta, \partial$ as for $\varepsilon^X, \eta^X_i, \partial^X$. Remark that the connecting morphism $\partial$ is unique since $\text{Hom}(\Sigma_{\partial^X} i_r^* X, i_{11!} i_{11!}^* X) = 0$ by adjunction and by the canonical 2-isomorphism $i_r^* i_{11!} = 0$, because of recollement.

Let us apply the weakly triangulated functor $F_\square$ and get a distinguished triangle in $T(\square)$

$$F_\square i_r^* X \xrightarrow{F_\square \varepsilon} F_\square X \xrightarrow{F_\square \eta} F_\square i_{11!} i_{11!}^* X \xrightarrow{\varepsilon^X} \Sigma_{\partial} F_\square i_r^* X.$$ 

Now, consider the canonical distinguished triangle associated to the object $F_\square X$,

$$\xymatrix@C=1em{i_r^* F_\square X \ar[r]^\varepsilon & F_\square X \ar[r]^\eta & i_{11!} i_{11!}^* F_\square X \ar[r]^{\varepsilon^X} & \Sigma_{\partial} i_r^* F_\square X},$$

whose connecting morphism is also unique. Again, we use $\varepsilon^X, \eta^X, \partial^X$ as for $\varepsilon^X, \eta^X_i, \partial^X$.

Let us call $i_1 : e \to \Delta_1$ the open immersion which identifies $e$ with 1 and $i_{\Delta_1} : \Delta_1 \to \square$ the open immersion which identifies 0 and 1 with 10 and 11, respectively. Having fixed this notation, we can write $i_{11!} = i_{\Delta_1} i_1$. Let us fix an arbitrary object $X$ lying in $S(\Delta_1)$ or, respectively, in $S(\square)$. By applying Lemma 4.4 twice to the diagram $I = e$ with $j = i_1$ and to the diagram $I = \Delta_1$ with $j = i_{\Delta_1}$, respectively, we get two canonical isomorphisms

$$\psi^X_{i_{11!}} : i_{11!}^* F_{\Delta_1} X \sim F_{\Delta_1} i_{11!} i_{11!}^* X$$

and

$$\psi^X_{i_{\Delta_1}} : i_{\Delta_1!} i_{\Delta_1!}^* F_{\Delta_1} X \sim F_{\Delta_1} i_{\Delta_1!} i_{\Delta_1!}^* X$$

such that the two commutation relations

$$\psi^X_{i_{11!}} \circ \eta_{i_{11!}} = F_{\Delta_1}(\eta^X_{i_{\Delta_1}})$$

and

$$\psi^X_{i_{\Delta_1}} \circ \eta_{i_{\Delta_1}} = F_{\Delta_1}(\eta^X_{i_{11!}})$$

hold. Moreover, the functorial images of the two isomorphisms under $i_1^*$ and $i_{\Delta_1}^*$, respectively, furnish isomorphisms

$$\psi^X_{i_1} : i_1^* F_{\Delta_1} X \sim F_{\Delta_1} i_1^* X$$

and

$$\psi^X_{i_{\Delta_1}} : i_{\Delta_1!}^* F_{\Delta_1} X \sim F_{\Delta_1} i_{\Delta_1!}^* X.$$

We can combine all these isomorphisms via the compositions of natural transformations

$$\xymatrix{1 \ar[r]^\eta_{i_{11!}} & i_{11!} i_{11!}^* \ar[r]^{\eta_{i_{11!}} i_{11!} i_{11!}} & i_{11!} i_{11!} i_{11!}^* \ar[r]^{\eta_{i_{11!}} i_{11!} i_{11!}} & i_{11!} i_{11!} i_{11!} i_{11!}^* = i_{11!} i_{11!}^*}
$$

and

$$\xymatrix{i_{11!} i_{11!} i_{11!}^* \ar[r]^{i_{11!}^* \varepsilon i_{\Delta_1}} & i_{11!} i_{11!}^* \ar[r]^{\varepsilon i_{\Delta_1}} & 1}
$$

We get, canonically, an isomorphism $\psi^X_{i_{\Delta_1}} \circ i_{\Delta_1}^{-1}(\psi^X_{i_1})$ as follows

$$i_{11!} i_{11!}^* F_\square X = (i_{\Delta_1} i_1)^{\sim} F_\square X = i_1^{\sim} F_\square i_{\Delta_1}^{-1} F_\square X$$

$$\sim i_1^{\sim} F_\square i_{\Delta_1}^{-1} F_\square X$$

$$\sim F_{\Delta_1} i_{\Delta_1!}^* F_\square X = F_{\Delta_1} i_{\Delta_1!}^* F_\square X,$$

which is canonically isomorphic to $\psi^X_{i_{11!}}$.

Clearly, there is another functorial isomorphism $\psi^X_{i_{11!} i_{11!}} : i_{11!} i_{11!}^* F_\square X \sim F_\square i_{11!} i_{11!}^* F_\square X$, explicitly given by the composition

$$\xymatrix{i_{11!} i_{11!}^* F_\square X \ar[r]^{i_{11!} (\psi^X_{i_{11!}})} & i_{11!} i_{11!} i_{11!}^* F_\square X \ar[r]^{\psi^X_{i_{11!} i_{11!}} F_\square X} & F_\square i_{11!} i_{11!}^* F_\square X}$$
Let us remark the important fact that the relation $\psi_{i_{11}}X \circ \eta_{i_{11}}^{F_{\square}} X = F_{\square}(\eta_{i_{11}})$ holds. Indeed, we can form the following diagram:

```
\[
\begin{array}{cccc}
F_{\square}X & \xrightarrow{\eta_{i_{11}}^{F_{\square}} X} & i_{\Delta_1}! i_{\Delta_1} ? F_{\square}X & \xrightarrow{i_{\Delta_1} i_{\Delta_1} ? i_{\Delta_1} ? i_{\Delta_1} ? F_{\square}X} & i_{\Delta_1} i_{\Delta_1} ? i_{\Delta_1} ? X & \xrightarrow{i_{\Delta_1} i_{\Delta_1} ? i_{\Delta_1} ? F_{\square}X} & X
\end{array}
\]
```

Let us consider the first square on the left. By definition, we have $\varphi^{i_{\Delta_1}, i_{\Delta_1}}X \circ i_{\Delta_1}! (\psi^{X}_{i_{\Delta_1}}) = \psi^{X}_{i_{\Delta_1}},$ and we know (cf. lemma 4.4) that this morphism is an iso and makes the square commute.

The three squares in the center of the diagram also commute. Indeed, the square at the top commutes by functoriality of $\eta_{i_{11}},$ the square in the middle commutes by lemma 4.4 and the square at the bottom commutes because $\varphi^{i_{\Delta_1}}$ is a natural transformation.

The last square on the right also is commutative. This fact is easy to check by using the general relation proved in subsection 2.2 in our special case $v = i_1$ and $u = i_{\Delta_1},$ which gives the formula

$$\varphi^{i_{\Delta_1}, i_{\Delta_1}}X \circ i_{\Delta_1}! (\varphi^{i_{\Delta_1}}X) = \varphi^{i_{\Delta_1}, X},$$

for all $X$ in $\mathcal{S}(e).$ By using this formula, we get a similar relation, valid for any $X$ in $\mathcal{S}(e),$

$$\psi_{i_{11}}^{X} = \varepsilon_{i_{11}}^{F_{\square}(i_{11}) X} \circ (i_{11})^?((\varphi^{i_{11}, i_{11}}X)^{-1}) \circ (i_{11})^? F_{\square}(\eta_{i_{11}})$$

$$\begin{align*}
&= \varepsilon_{i_{11}}^{F_{\square}(i_{11}) X} \circ (i_{11})^? (\varphi^{i_{11}, i_{11}}X)^{-1} \circ (i_{11})^? F_{\square}(\eta_{i_{11}}) \\
&= (i_{11})^?((\varphi^{i_{11}, i_{11}}X)^{-1}) \circ (i_{11})^? F_{\square}(\eta_{i_{11}}) \\
&= \varepsilon_{i_{11}}^{F_{\square}(i_{11}) X} \circ (i_{11})^? (\varphi^{i_{11}, i_{11}}X)^{-1} \circ (i_{11})^? F_{\square}(\eta_{i_{11}}) \\
&= (i_{11})^?((\varphi^{i_{11}, i_{11}}X)^{-1}) \circ (i_{11})^? F_{\square}(\eta_{i_{11}}) \\
&= \varepsilon_{i_{11}}^{F_{\square}(i_{11}) X} \circ (i_{11})^? (\varphi^{i_{11}, i_{11}}X)^{-1} \circ (i_{11})^? F_{\square}(\eta_{i_{11}}) \\
&= (i_{11})^?((\varphi^{i_{11}, i_{11}}X)^{-1}) \circ (i_{11})^? F_{\square}(\eta_{i_{11}}) \\
&= \varepsilon_{i_{11}}^{F_{\square}(i_{11}) X} \circ (i_{11})^? (\varphi^{i_{11}, i_{11}}X)^{-1} \circ (i_{11})^? F_{\square}(\eta_{i_{11}}) \\
&= (i_{11})^?((\varphi^{i_{11}, i_{11}}X)^{-1}) \circ (i_{11})^? F_{\square}(\eta_{i_{11}}) \\
&= \varepsilon_{i_{11}}^{F_{\square}(i_{11}) X} \circ (i_{11})^? (\varphi^{i_{11}, i_{11}}X)^{-1} \circ (i_{11})^? F_{\square}(\eta_{i_{11}}) \\
&= (i_{11})^?((\varphi^{i_{11}, i_{11}}X)^{-1}) \circ (i_{11})^? F_{\square}(\eta_{i_{11}})
\end{align*}$$

Altogether, all these formulae show commutativity of the last square on the right.

Coming back to our fixed object $X$ of $\mathcal{S}(\square),$ let us consider the morphism $\varphi_{\square}^{X} : i_{r}! i_{r}! F_{\square}X \to F_{\square}i_{r}! i_{r}! X$ that means, in our notations, the composition $\varphi^{i_{r}, i_{r}! X} \circ i_{r}! (\varphi_{\square}^{X})^{-1}.$
We can see that the morphism $\varepsilon^{F_{\square}}$ factors through $\varphi_{i_{\square}X}$. Indeed, we can compute

$$F_{\square}(\varepsilon^{X}) \circ \varphi_{i_{\square}X} = F_{\square}(\varepsilon^{X}) \circ \varphi_{i_{\square}i_{\square}X} \circ i_{\square}[(\varphi_{X}^{X})^{-1}]$$

$$= F_{\square}(\varepsilon^{X}) \circ \varphi_{i_{\square}i_{\square}X} \circ i_{\square}[(\varphi_{X}^{X})^{-1}]$$

$$= \varphi_{X}^{X} \circ i_{\square}[(\varphi_{X}^{X})^{-1}]$$

$$= \varepsilon^{X}$$

Let us consider an extension of this factorization to a morphism of distinguished triangles of $T(\square)$

$$
\begin{array}{cccc}
\varepsilon^{F_{\square}} & F_{\square}X & \eta^{F_{\square}} & i_{\square}i_{\square}^{?}F_{\square}X & \partial^{F_{\square}} & \Sigma^{2} \circ i_{\square}i_{\square}^{?}F_{\square}X \\
\varphi_{i_{\square}X} & F_{\square}i_{\square}i_{\square}^{?}X & F_{\square}\gamma & F_{\square}X & \psi_{X} & \Sigma^{2} \circ \varphi_{i_{\square}X} \\
\end{array}
$$

The extension morphism $\psi_{X}^{X}$ is unique making commute the central square because the group

$$\text{Hom}(\Sigma \circ i_{\square}i_{\square}^{?}F_{\square}X, F_{\square}i_{\square}i_{\square}^{?}X) = \text{Hom}(\Sigma \circ i_{\square}i_{\square}^{?}F_{\square}X, i_{\square}i_{\square}^{?}F_{\square}X)$$

$$= \text{Hom}(i_{\square} \circ \Sigma \circ i_{\square}^{?}F_{\square}X, i_{\square}i_{\square}^{?}F_{\square}X)$$

$$= \text{Hom}(\Sigma \circ i_{\square}^{?}F_{\square}X, i_{\square}^{?}F_{\square}X)$$

$$= 0$$

is canonically trivial. Here, we use the canonical isomorphism $\psi_{X}^{X}$ in the first equality, the canonical invertible natural transformation $\delta_{i_{\square}} : i_{\square} \circ \Sigma \rightarrow \Sigma \circ i_{\square}$ in the second equality, adjunction in the third one and the canonical 2-isomorphism $i_{\square}^{?}i_{\square}^{?} = 0$ of the recollement axioms in the last. Hence, we must canonically identify $\psi_{X}^{X}$ with the isomorphism $\psi_{X}^{X}$. It follows that $\varphi_{i_{\square}X}$ is invertible, too. Therefore, $\varepsilon^{X}$ is invertible if and only if $\varepsilon^{F_{\square}X}$ is.

b) Since Axiom Der 7 holds for the derivator $\mathcal{S}$, it suffices to show that, given any cocartesian (hence cartesian) square $X \in \mathcal{A}(\square)$ as in Definition 2.14 part a), whose diagram is

$$
\begin{array}{ccc}
\text{dia}_{\square}X & = & \\
\downarrow & & \\
i_{00}^{X} & \Rightarrow & i_{01}^{X} \Rightarrow \\
i_{10}^{X} & \Rightarrow & i_{11}^{X} \\
\end{array}
$$

the adjunction morphism

$$\varepsilon^{F_{\square}} : i_{\square}^{?}i_{\square}^{?}F_{\square}X \rightarrow F_{\square}X$$

is invertible.

Let us start with a slightly more general object $X$ in $\mathcal{A}(\square)$, whose diagram is a commutative square with inflations as horizontal arrows and deflations as vertical arrows, which is not required to be cartesian, nor cocartesian. Let us suppose that the adjoint morphism $\varepsilon^{X} : i_{\square}i_{\square}^{?}X \rightarrow X$ is a deflation. This is not restrictive. Indeed, it occurs under the mild additional hypothesis that the exact categories we are considering have split idempotents, or, which is the case we will be dealing with, when the square object $X$ is bicartesian. In fact, the object $i_{\square}^{?}i_{\square}^{?}X$ is the homotopy colimit $\text{hocolim}_{i_{\square}^{?}}X$. 
Let us explain this, in general. Let us consider the category \( \mathbf{11}/\mathbf{r} \), defined by the inclusion \( i_r : \mathbf{r} \to \square \). There is a (noncommutative) square
\[
\begin{array}{ccc}
\mathbf{11}/\mathbf{r} & \xrightarrow{p_{11}/\mathbf{r}} & \mathbf{r} \\
p_{11}/\mathbf{r} \downarrow & \searrow & \downarrow i_r \\
\mathbf{11}/\square & \xrightarrow{p_r} & \square
\end{array}
\]
where the forgetful functor \( j \) is an isomorphism of categories. By Axiom \textbf{Der 4}, we find a functorial invertible natural transformation of functors
\[
e_{i_{11}/\mathbf{r}} : (p_r)! = (p_{11}/\mathbf{r})j^* \to i_{11,\square}i_r! .
\]
Moreover, it is not difficult to see that there is a canonical iso \( i_{10,r}Z \coprod_{i_{10,0}Z} Z \to p_r!Z \), for any object \( Z \) in \( \mathbf{A}(\mathbf{r}) \). Thus, by the uniqueness of push-outs in exact categories it follows that the (unique) universal morphism \( i_{r!}\xi_X \) is a deflation, if the exact category \( \mathbf{A}(e) = \mathbf{A} \) has splitting idempotents, or even an iso, if the the square is supposed to be cocartesian.

Let us consider the conflation induced by the deflation \( \xi_X \),
\[
\Omega i_{11}/\mathbf{r}i_{11}^* : i_{r!}\xi_i^* X \to X ,
\]
and apply the weak \( \partial \)-functor \( F_{\square} \). After shifting, we get a distinguished triangle
\[
F_{\square}i_{r!}\xi_i^* X \to F_{\square}X \to F_{\square}i_{11!/\mathbf{r}}i_{11}^* X \to \Sigma_{\square}F_{\square}i_{r!}\xi_i^* X ,
\]
The proof goes on along the same lines of the proof of item a), with the difference that this time we will apply lemma 4.3 anytime we were applying lemma 4.4. \( \square \)

5. The Universal Property

In this section we give a proof of the main theorem 2.16. Let us put together some trivial observations at first.

\textbf{Lemma 5.1.} Let \( \mathbf{A} \) be the represented exact prederivator associated to an exact category \( \mathbf{A} \). Let \( \mathbf{E} \) be a triangulated derivator. Fix an arbitrary diagram \( I \) in \( \mathbf{Dia} \). Then, the sequence of exact (resp., triangulated) categories defined by
\[
\mathbf{A}_I^n := \mathbf{A}(C_n \times I) , \quad \mathbf{E}_I^n := \mathbf{E}(C_n \times I) , \quad n \in \mathbb{N} ,
\]
with the induced functors defined by
\[
u_I^* := (u \times 1_I)^* : \mathbf{A}_I^n \to \mathbf{A}_m^n , \quad u_I^* := (u \times 1_I)^* : \mathbf{E}_I^n \to \mathbf{E}_m^n ,
\]
for any morphism \( u : C_m \to C_n \) in \( \mathbf{Cubes} \), and the natural transformations defined by
\[
\alpha_I^* := (\alpha \times 1_I)^* : v_I^* \Rightarrow u_I^* ,
\]
for any 2-morphism \( \alpha : u \Rightarrow v \) in \( \mathbf{Cubes} \), give rise to an exact (resp., triangulated) tower \( \mathbf{E}_I \) (resp., \( \mathbf{A}_I^\prime \)) such that \( \mathbf{E}_I^n(C_n) = \mathbf{E}_I^n \) (resp., \( \mathbf{A}_I^\prime = \mathbf{A}_I^\prime(C_n) \)).

\textbf{Proof.} It reduces to a straightforward checking that the axioms of an exact (resp., triangulated) tower hold for \( \mathbf{E}_I \). \( \square \)
Lemma 5.2. Let $F : D \to E$ be a triangulated morphism of triangulated derivators. Fix an arbitrary diagram $I$ in $\mathcal{D}_{iaF}$. Then, the sequence of associated triangulated functors defined by

$$ F^I_n := F_{C_n \times I} : D^I_n \to E^I_n, \quad n \in \mathbb{N}, $$

with the induced invertible natural transformations of functors defined by

$$ \varphi^I_n := \varphi_{n \times I} : F^I_{m \circ i} \xrightarrow{\sim} F^I_{n \circ i}, $$

for any morphism $u : C_m \to C_n$ in $\mathcal{Cubes}$, gives rise to a morphism of triangulated towers $F^I : D^I \to E^I$ such that $F^I_n = F_{C_n \times I}$.

Proof. Again, it is straightforward to check that the axioms of a triangulated morphism of towers hold for $F^I$. □

Similarly, an analogous version of the last lemma holds for exact morphisms of represented exact prederivators associated to additive categories. Let $I$ be any diagram in $\mathcal{D}_{iaF}$. Let us associate an additive (resp., exact) tower $\tilde{A}^I$ with any additive (resp., exact) category $\mathcal{A}$, according to the following definition

$$ \tilde{A}^I(C_n) := \text{Hom}(C_n, \text{Hom}(I^o, \mathcal{A})), \quad n \in \mathbb{N}. $$

Lemma 5.3. Let $\mathcal{A}$ be an exact category and $E$ a triangulated derivator. Suppose that $F : \mathcal{A} \to E$ is an exact morphism of prederivators. Fix an arbitrary diagram $I$ in $\mathcal{D}_{iaF}$. Then, there are an exact tower $\tilde{A}^I$, a triangulated tower $E^I$ and an induced exact morphism of towers

$$ F^I : \tilde{A}^I \to E^I $$

where $\tilde{A}^I$ equals $\tilde{A}^I$ (according to the notations of Lemma 5.1).

Proof. For each $n \in \mathbb{N}$, we have

$$ \tilde{A}^I(C_n) = \tilde{A}(C_n \times I) = \text{Hom}((C_n \times I)^o, \mathcal{A}) = \text{Hom}(C_n^o \times I^o, \mathcal{A}) = \text{Hom}(C_n^o, \text{Hom}(I^o, \mathcal{A})) = \tilde{A}^I(C_n). $$

Now use Lemma 5.1 and Lemma 5.2. □

We can now prove the main Theorem 2.16.

Proof. Let us consider an exact morphism of prederivators $F : \mathcal{A} \to E$. We functorially construct an object $\tilde{F}$ lying in $\text{Hom}_{\mathcal{D}_{A}}(\mathcal{D}_{A}, E)$ such that $\text{can}^*(\tilde{F}) = \tilde{F} \circ \text{can}$ is $F$ (up to a unique iso). By Lemma 5.3 if we fix a diagram $I$ we have the induced exact morphism of towers $F^I : \tilde{A}^I \to E^I$. This is an object in the category $\text{Hom}_{\mathcal{D}_{A}}(\tilde{A}^I, E^I)$. By Keller’s theorem 4.3, this morphism extends uniquely to a morphism of triangulated towers $\tilde{F}^I : \mathcal{D}_{A} \to E^I$.

Here $\mathcal{D}_{A}$ is a tower that associates the bounded derived category $\mathcal{D}_b(\text{Hom}(C_n, \text{Hom}(I^o, \mathcal{A})))$ with any cube $C_n$ and equals the tower $\mathcal{D}_A(C_n)$ which is defined by the relation $\mathcal{D}_A(C_n) = \mathcal{D}_b(\text{Hom}((C_n \times I)^o, \mathcal{A}))$. Thus, the morphism $\tilde{F}^I : \mathcal{D}_{A} \to E^I$ identifies with a morphism $\tilde{F}^I : \mathcal{D}_{A} \to E^I$. The base of the morphism $\tilde{F}^I$ (i.e., the evaluation at $C_0$) gives us a triangulated functor $(\tilde{F}^I)_0 : \mathcal{D}_{A}(I) \to E(I)$.

It remains to check that, by letting $I$ running in $\mathcal{D}_{iaF}$, we get a triangulated morphism of derivators $\tilde{F} : \mathcal{D}_{A} \to E$, defined by the equality $\tilde{F}_I := (\tilde{F}^I)_0$, whose image in $\text{Hom}_{\mathcal{D}_{A}}(\tilde{A}^I, E)$ is $F$. Notice that, by the construction, the composition $\tilde{F}_I \circ \text{can}_I$ is $F_I$, for all diagrams $I$ (up to a canonical iso). So, we can start with the second item in Definition 2.14. Let
$u: I \to J$ be a morphism of finite diagrams. Since $F$ is a morphism of prederivators, there is a square

$$\begin{array}{ccc}
\mathcal{A}(J) & u^* & \mathcal{A}(I) \\
F & \downarrow & F \\
E(J) & u^* & E(I)
\end{array}$$

which is commutative (up to a unique iso) by the definition, i.e., there is an invertible natural transformation of functors $\varphi_u: F_J u^* \to u^* F_J$. Now, the previous square is (canonically isomorphic to) the base of the following commutative square of towers

$$\begin{array}{ccc}
\mathcal{A}(J) & u^* \wedge & \mathcal{A}(I) \\
F & \downarrow & F \\
E(J) & u^* \wedge & E(I)
\end{array}$$

where $u^* \wedge$ is the induced exact morphism of towers $\text{Hom}((-)^*, u^*)$. By Theorem 3.3 the morphisms $F^J$ and $F^I$ uniquely extend to morphisms $\tilde{F}^J$ and $\tilde{F}^I$. Since the morphism $u^* \wedge$ is exact it extend to the derived morphism $Ru^* \wedge$. So, we get a square of triangulated towers

$$\begin{array}{ccc}
D_A & Ru^* \wedge & D_A \\
\tilde{F}^J & \downarrow & \tilde{F}^J \\
E(J) & u^* \wedge & E(I)
\end{array}$$

whose base (evaluation at $C_0$) is a square of triangulated categories

$$\begin{array}{ccc}
D_A & Ru^* \wedge & D_A \\
\tilde{F}^J & \downarrow & \tilde{F}^J \\
E(J) & u^* \wedge & E(I)
\end{array}$$

The functor $Ru^*$ exists since $u^*: \mathcal{A}(J) \to \mathcal{A}(I)$ is an exact functor of exact categories.

It remains to check that this square is commutative up to a unique iso. Let $\varphi_u: \tilde{F}^J Ru^* \to u^* \tilde{F}^J$ be the natural transformation of functors induced by $\varphi_u$ via the fully faithful functor $\text{can}_J$. We apply item b) in [21, Lemme 2], where we identify the functor $G$ with $\tilde{F}^J Ru^*$, the functor $G'$ with $u^* \tilde{F}^J$ and the class of objects $\mathcal{X}$ with the set of objects of the category $\mathcal{A}(J)$. Indeed, the image of the subcategory $\mathcal{A}(J)$ generates $D_A(J)$ as a triangulated category. Since the restriction of the natural transformation $\varphi_u$ to the evident restriction subfunctors over the subcategory $\mathcal{A}(J)$ equals the invertible natural transformation $\varphi_u$, it follows by the cited Lemma that $\varphi_u: \tilde{F}^J Ru^* \to u^* \tilde{F}^J$ is an invertible natural transformation of functors, too. It is then clear how to check the remaining axioms of a morphism of derivators.
The additive morphism $\tilde{F}$ extends to a triangulated morphism. Indeed, the functor $\tilde{F}_I$ is triangulated with respect to the canonical triangulated structures of the categories $\mathbb{D}_A(I)$ and $\mathbb{E}(I)$, for all diagrams $I$. Therefore, we can apply the item a) of Prop. 17 which (largely) suffices to tell us that the morphism $\tilde{F}$ is triangulated.

It is clear that the functor $F \mapsto \tilde{F}$ is a quasi-inverse to the functor $F \mapsto F \circ \text{can}$ since this is locally true for every diagram $I$. \hfill \square

6. Derived extension of an exact category

In this section we work with derivators of type $\mathcal{D}_{ia}$, i.e., derivators which are defined over the 2-subcategory of $\mathcal{D}_{ia}$ that consists of finite directed diagrams.

Let us denote by $\text{Mor}_I(\mathbb{T}(e))$ the category of $I^\circ$-morphisms in $\mathbb{T}(e)$, i.e., an object in this category is a family of morphisms of $\mathbb{T}(e)$

$$\{F_{\sigma(u),\tau(u)} : F_{\sigma(u)} \to F_{\tau(u)}\}_{\sigma \in \text{Mor}(I^\circ)} \in \prod_{j \in I} \prod_{i \in I(j,i)} \text{Hom}_{\mathbb{T}(e)}(F_i, F_j)$$

indexed over all arrows in $I^\circ$, with the convention that $F_{1_i} = 1_{F_i}$ for all $i \in I$. (It is possible that some morphism in the diagram is zero if in the category $I$ there are objects with no arrows in between.) A morphism in $\text{Mor}_I(\mathbb{T}(e))$ is given by a family of morphisms

$$\{g_i : F_i \to G_i\}_{i \in I} \in \prod_{i \in I} \text{Hom}_{\mathbb{T}(e)}(F_i, G_i)$$

such that the square

$$\begin{array}{ccc}
F_{\sigma(u)} & \xrightarrow{F_{\sigma(u),\tau(u)}} & F_{\tau(u)} \\
g_{\sigma(u)} \downarrow & & \downarrow g_{\tau(u)} \\
G_{\sigma(u)} & \xrightarrow{G_{\sigma(u),\tau(u)}} & G_{\tau(u)}
\end{array}$$

is commutative, for each arrow $u \in \text{Mor}(I^\circ)$. It is clear that there is a natural isomorphism of categories

$$\text{Hom}(I^\circ, \mathbb{T}(e)) \xrightarrow{\sim} \text{Mor}_{I^\circ}(\mathbb{T}(e)),$$

$$F \mapsto \{F(u) : F(\sigma(u)) \to F(\tau(u))\}_{\sigma \in \text{Mor}(I^\circ)}$$

whose inverse is given by the functor which maps $\{F_{\sigma(u),\tau(u)}\}_{\sigma \in \text{Mor}(I^\circ)}$ to the presheaf $F$ defined by $F(u) = F_{\sigma(u),\tau(u)}$, for all $u \in I^\circ$. Let us consider the functor

$$\mathbb{T}(I) \xrightarrow{\text{mor}_I} \text{Mor}_{I^\circ}(\mathbb{T}(e)),$$

$$X \mapsto \{\alpha_{\sigma(u),\tau(u)}^* X : \tau(u)^* X \to \sigma(u)^* X\}_{\sigma \in \text{Mor}(I)},$$

where, for each arrow $u \in \text{Mor}(I)$, the natural transformation $\alpha_{\sigma(u),\tau(u)}^* : \tau(u)^* \to \sigma(u)^*$ is induced by the 2-arrow

$$\alpha_{\sigma(u),\tau(u)} : \sigma(u) \Rightarrow \tau(u).$$

There is a commutative diagram

$$\begin{array}{ccc}
\text{Hom}(I^\circ, \mathbb{T}(e)) & \xrightarrow{\sim} & \text{Mor}_{I^\circ}(\mathbb{T}(e)) \\
\text{dia}_I \downarrow & & \downarrow \text{mor}_I \\
\mathbb{T}(I).
\end{array}$$

and a canonical bijection

$$\text{Hom}_{\text{Hom}(I^\circ, \mathbb{T}(e))}(\text{dia}_I X, \text{dia}_I Z) \cong \text{Hom}_{\text{Mor}_{I^\circ}(\mathbb{T}(e))}(\text{mor}_I X, \text{mor}_I Z).$$
Let us prepare the crucial ingredients for the proof of the main theorem of this section.

6.1. Full faithfulness of the diagram functor. The following proposition gives a sufficient condition which allows us to lift morphisms of presheaves uniquely, when this is possible, from $\text{Hom}(I^0, \mathbb{T}(e))$ to morphisms of $\mathbb{T}(I)$. This will be helpful in the main theorem of this section in order to tell whether the functor $\text{dia}_I$ is fully faithful when we consider its restriction to some subcategory of $\mathbb{T}(I)$ which enjoys this property (Toda condition).

**Proposition 6.1.** Let $\mathbb{T}$ be a triangulated derivator of type $\mathcal{D}ia_I$. Fix an arbitrary finite diagram $I$. Let $X$ and $Z$ be objects lying in $\mathbb{T}(I)$. Suppose that the (Toda) condition

$$\text{Hom}_{\mathbb{T}(e)}(\sum^n i^*_X, j^*_Z) = 0, \quad n > 0,$$

holds for each pair $(i, j) \in I \times I$. Then, the functor $\text{dia}_I : \mathbb{T}(I) \to \text{Hom}(I^0, \mathbb{T}(e))$ induces a bijection

$$\text{Hom}_{\mathbb{T}(I)}(X, Z) \cong \text{Hom}_{\text{Hom}(I^0, \mathbb{T}(e))}(\text{dia}_I X, \text{dia}_I Z).$$

**Proof.** Let us endow the additive category $\mathbb{T}(e)$ with an exact structure by defining conflations as the split short exact sequences. In this way, once a diagram $I$ of $\mathcal{D}ia_I$ has been fixed, we can consider $\text{Hom}(I^0, \mathbb{T}(e))$ as an exact category with the pointwise (split) exact structure.

Let $X$ be an arbitrary object in the triangulated category $\mathbb{T}(I)$. For each arrow $a : j \to i$ in the diagram $I$, let $a_i \in \text{Hom}_{\mathbb{T}(I)}(ji^*_X, ij^*_X)$ be the image of the identity $1_{ji^*_X}$ under the composition of homomorphisms

$$\text{Hom}_{\mathbb{T}(I)}(ij^*_X, ij^*_X) \to \text{Hom}_{\text{Hom}(I^0, \mathbb{T}(e))}(ij^*_X, ij^*_X) \to \text{Hom}_{\mathbb{T}(I)}(i^*_X, ij^*_X) \to \text{Hom}_{\mathbb{T}(I)}(ji^*_X, ij^*_X)$$

induced by the adjoint pairs $i_! \dashv i^*$ and $j_! \dashv j^*$ and the 2-arrow $a^{*}_j$

$$\mathbb{T}(I) \xrightarrow{a^{*}_j} \mathbb{T}(e)$$

associated with the natural transformation $\alpha_{ji}$

$$\xymatrix{ I \ar[rr]^-{j^*} & & \mathbb{T}(e) \ar[ll]^-{i^*} }$$

defined by the morphism $a$.

By using the formal properties of adjoint functors, we can find the formula for $a_i$

$$a_i = \varepsilon^Y_{iX} \circ j_!(\alpha^Y_j(i^*_X)) \circ j_!(\eta^Y_{lX}),$$

where, for each $l \in I$ and $Y \in \mathbb{T}(I)$, the morphisms $\varepsilon^Y_l$ (resp., $\eta^Y_l$) are defined by using the counit (resp., the unit) of the adjunction $l \dashv l^*$.

Similarly, let $a_j \in \text{Hom}_{\mathbb{T}(I)}(ji^*_X, jj^*_X)$ be the image of the identity $1_{ji^*_X}$ under the composition of homomorphisms

$$\text{Hom}_{\mathbb{T}(I)}(ji^*_X, jj^*_X) \to \text{Hom}_{\text{Hom}(I^0, \mathbb{T}(e))}(ji^*_X, jj^*_X) \to \text{Hom}_{\mathbb{T}(I)}(j^*_X, jj^*_X) \to \text{Hom}_{\mathbb{T}(I)}(ji^*_X, jj^*_X)$$

induced by the adjoint pairs $j_! \dashv j^*$ and $i_! \dashv i^*$ and the 2-arrow $a^{*}_j$.

$$\mathbb{T}(I) \xrightarrow{a^{*}_j} \mathbb{T}(e)$$

associated with the natural transformation $\alpha_{ji}$

$$\xymatrix{ I \ar[rr]^-{j^*} & & \mathbb{T}(e) \ar[ll]^-{i^*} }$$

defined by the morphism $a$. 
induced by the adjoint pair $j_! \dashv j^*$ and the 2-arrow defined by the morphism $a$ as above. Here, it is easy to see that the formula

$$a_j = j_![\alpha^*_{j,i}(X)]$$

for $a_j$ holds. Thus, for each morphism $a : j \to i$ in the diagram $I$, we get an arrow

$$j i^* X \xrightarrow{[a_i, -a_j]'} i i^* X \oplus j j^* X.$$

We claim that the composition of morphisms

$$j j^* X \xrightarrow{[a_i, -a_j]'} i i^* X \oplus j j^* X \xrightarrow{[\varepsilon_i, \varepsilon_j]} X$$

is zero. Indeed, by 2-functoriality of $\alpha^*_{ji}$ there is the equality

$$j^*(\varepsilon_i) \circ \alpha^*_{ji}(i i^* X) = \alpha^*_{ji}(X) \circ i^*(\varepsilon_i).$$

Compose with the morphism $\eta^*_{i,j} X$ and use $i^*(\varepsilon_i) \circ \eta^*_i X = 1^* X$ to find the relation for $\alpha^*_{ji}$

$$j^*(\varepsilon_i) \circ \alpha^*_{ji}(i i^* X) \circ \eta^*_j X = \alpha^*_{ji}(X).$$

Hence, we can compute

$$[\varepsilon_i, \varepsilon_j] \circ [a_i, -a_j]' = \varepsilon_i \circ \varepsilon_i^{i* X} \circ j_![\alpha^*_{ji}(i i^* X)] \circ j_![\eta^*_i X] - \varepsilon_j \circ j_![\alpha^*_{ji}(X)]$$

$$= \varepsilon_j \circ j_![j^*(\varepsilon_i)] \circ j_![\alpha^*_{ji}(i i^* X)] \circ j_![\eta^*_i X] - \varepsilon_j \circ j_![\alpha^*_{ji}(X)]$$

$$= 0.$$

Here the second equality comes from the functoriality of $\varepsilon_j$ and the third by the relation for $\alpha^*_{ji}$.

Recall that the set of objects and the $\text{Hom}$-sets in $I$ are finite and that the triangulated category $T(I)$ has finite coproducts. So, we obtain a morphism in $T(I)$

$$\coprod_{I,I} j i^* X \xrightarrow{\text{can}} i i^* X \oplus j j^* X,$$

whose components are $[a_i, -a_j]'$. All these morphisms are the components of an arrow

$$\coprod_{I,I} \coprod_{I,I} j i^* X \xrightarrow{u} \coprod_{I,I} i i^* X.$$
where $\alpha_{k,i}$ is the 2-arrow

$$
e \xymatrix{ & I \\
\ar@{..>}[ru]_k & }$$

defined by a morphism $k \to i$ in the diagram category $I$. Since in the diagram $I$ there are no nontrivial loops, we have that $k^*(\varepsilon)$ is a section, with retraction given by a morphism

$$k^*X \rightarrow \coprod_{i \in I} \bigcup_{l(k,i)} l^*X,$$

all of whose components are zero but one, the identity on $k^*X$. It follows that $k^*(v)$ is actually the kernel of $k^*(\varepsilon)$ and that the connecting morphism $\partial k^*X$ must be zero. Hence, we have that locally the kernel is given by the equality

$$j^*(LX) = \coprod_{i \in I \setminus \{j\}} i^*X.$$

Now it is clear that

$$QLX = \coprod_{j \in I} j^*LX = \coprod_{j \in I \setminus \{j\}} \coprod_{l(j,i)} j^iLX$$

is just the domain of the morphism $u$ which we have constructed above.

Since we have seen that the composition $\varepsilon \circ u$ vanishes, there exists an arrow $\varepsilon^1 : QLX \rightarrow LX$ such that, by composition with $v$, it gives $u$, as in the following commutative diagram

$$\begin{array}{c}
QLX \\
\overset{\varepsilon^1}{\rightarrow}
\end{array} \xymatrix{ & u \\
\ar@{..>}[ru]^v & \ar@{..>}[rd]^0 & \\
LX & X & \Sigma LX.}
\end{array}$$

By iterating this construction we get a ‘resolution’ as in the following commutative diagram in $\mathcal{T}(I)$

$$\begin{array}{c}
QL^{n-1}X \\
\overset{\varepsilon^n}{\rightarrow}
\end{array} \xymatrix{ & u^n \\
\ar@{..>}[ru]^{L^nX} & \ar@{..>}[rd]^{u^{n-1}} & \\
QL^{n-2}X & \cdots & \ar@{..>}[ru]^{QLX} & \ar@{..>}[rd]^{u^2} & \\
L^{n-1}X & \cdots & \ar@{..>}[ru]^{LX} & \ar@{..>}[rd]^{\varepsilon^1} & \\
L^nX & \cdots & \ar@{..>}[ru]^{LX} & \ar@{..>}[rd]^{v^1} & \\
QL^{n-1}X & \cdots & \ar@{..>}[ru]^{QLX} & \ar@{..>}[rd]^{u^1} & \\
QLX & \cdots & \ar@{..>}[ru]^{QLX} & \ar@{..>}[rd]^{u^0} & \\
X & X & \cdots & \ar@{..>}[ru]^{X} & \\
\Sigma LX & \Sigma LX & \cdots & \ar@{..>}[ru]^{\Sigma LX} & \\
& \Sigma LX.}
\end{array}$$

Here, the maps $\varepsilon^0, u^1, v^1$ are $\varepsilon, u$ and $v$, respectively. Moreover, compositions $u^k \circ u^{k+1}$ vanish, and all consecutive arrows $v^{l+1}, \varepsilon^{l}$ fit in a distinguished triangle, for all $l$,

$$L^{l+1}X \xrightarrow{v^{l+1}} QL^lX \xrightarrow{\varepsilon^{l}} L^lX \rightarrow \Sigma L^{l+1}X, \quad l \in \mathbb{N},$$

such that $u^l = v^l\varepsilon$, for all $l \in \mathbb{N} - \{0\}$. In this sequence of maps we can easily check that

$$m^*(L^nX) = \bigcup_{i_l \in I} \bigcup_{i_{l+1} \in I} \cdots \bigcup_{i_2 \in I} \bigcup_{i_1 \in I} \bigcup_{i_0 \in I} \bigcup_{i_{-1} \in I} \bigcup_{i_{-2} \in I} i_0^*X,$$

for all $m \in I$. This shows that our ‘resolution’ of $X$ must be finite since the diagram $I$ is supposed to be finite.

If $l = n$ or $l = n - 1$ we have something more, since in these cases distinguished triangles can be written as

$$QLW \xrightarrow{v} QW \xrightarrow{\varepsilon} W \rightarrow \Sigma QLW,$$

where $W$ can be $L^nX$ or $L^{n-1}X$. We recall that this means a distinguished triangle

$$\bigcup_{j \in I} \bigcup_{i \in I \setminus \{j\}} j^*i^*W \xrightarrow{u} \bigcup_{j \in I} \bigcup_{i \in I \setminus \{j\}} i^*W \xrightarrow{\varepsilon} W \rightarrow \Sigma \bigcup_{j \in I} \bigcup_{i \in I \setminus \{j\}} \bigcup_{l(j,i)} j^iW.$$
Let us apply the functor $\text{Hom}_T(\cdot, Z)$ to this distinguished triangle and get a long exact sequence of abelian groups

\[
\begin{array}{c}
\text{Hom}_T(\sum \coprod_{j \in I} \coprod_{i \in I-\{j\}} \coprod_{I(j,i)} j_i^*W, Z) \\
\downarrow \\
\text{Hom}_T(W, Z) \\
\downarrow \varepsilon \\
\text{Hom}_T(\coprod_{i \in I} i_i^*W, Z) \\
\downarrow u \\
\text{Hom}_T(\coprod_{j \in I} \coprod_{i \in I-\{j\}} \coprod_{I(j,i)} j_i^*W, Z).
\end{array}
\]

Here, the morphisms $\varepsilon$ and $u$ are induced by composition from $\varepsilon$ and $u$. By using the bijections induced by the adjunction pairs $j_i \dashv j_i^*$ and $i_i \dashv i_i^*$, we obtain an exact sequence of abelian groups

\[
\begin{array}{c}
\prod_{j \in I} \prod_{i \in I-\{j\}} \prod_{I(j,i)} \text{Hom}_T(e(\cdot), (\sum i_i^*W, j_i^*Z)) \\
\downarrow \\
\text{Hom}_T(W, Z) \\
\downarrow \varepsilon' \\
\prod_{i \in I} \text{Hom}_T(e(\cdot), (i_i^*W, i_i^*Z)) \\
\downarrow u' \\
\prod_{j \in I} \prod_{i \in I-\{j\}} \prod_{I(j,i)} \text{Hom}_T(e(\cdot), (i_i^*W, j_i^*Z)).
\end{array}
\]

Since this sequence is exact, it follows that the image of the homomorphism $\varepsilon'$ precisely consists of the families of morphisms $\{f_i : i_i^*W \to i_i^*Z\}_{i \in I}$ which specify a morphism in the category $\text{Mor}_{T^e}(\mathbb{T}(e))$. Indeed, if $u'\{f_i\}_{i \in I}$ is zero, we have a commutative square

\[
\begin{array}{ccc}
i_i^*W & \longrightarrow & i_i^*W \oplus j_i^*W \\
\alpha_{ij}^*W & \downarrow & \downarrow 0 \\
i_j^*Z & \longrightarrow & i_j^*Z \oplus j_j^*Z, 
\end{array}
\]

for each arrow $a \in I(j,i)$. This clearly gives the equality $f_j \circ \alpha_{ij}^*W = \alpha_{ij}^*Z \circ f_i$, which is exactly the condition for the family $\{f_i\}_{i \in I}$ to be a morphism lying in the category $\text{Mor}_{T^e}(\mathbb{T}(e))$. 
Thus, we get an exact sequence

$$\prod_{j \in I} \prod_{i \in I - \{j\}} \prod_{(j,i)} \text{Hom}_{\mathbb{T}(e)}(\Sigma i^* W, j^* Z)$$

Thus, we get an exact sequence

$$\text{Hom}_{\mathbb{T}(I)}(W, Z)$$

$$\text{Hom}_{\text{Mor}_{\mathbb{T}(e)}}(\mathbb{T}(e))(\text{mor}_I W, \text{mor}_I Z)$$

$$0.$$

Notice that, passing in long exact sequence and using the Toda condition of the hypothesis, it is not difficult to see that there are vanishing groups

$$\text{Hom}_{\mathbb{T}(e)}(\Sigma^n i^* W, j^* Z) = 0, \quad n > 0,$$

for each pair \((i, j) \in I \times I\). This condition forces the first group in the sequence to be zero and we get a bijection

$$\text{Hom}_{\mathbb{T}(I)}(W, Z) \xrightarrow{\sim} \text{Hom}_{\text{Mor}_{\mathbb{T}(e)}}(\mathbb{T}(e))(\text{mor}_I W, \text{mor}_I Z).$$

Going from \(l = n - 2\) on, the situation is new. In this case a distinguished triangle is the horizontal one in the commutative diagram

$$\begin{array}{ccc}
QL^{n-1} X & \xrightarrow{\varepsilon^{n-1}} & \Sigma QL^{n-1} X \\
\downarrow \varepsilon^{n-1} & & \downarrow \Sigma \varepsilon^{n-1} \\
L^{n-1} X & \xrightarrow{\varepsilon^{n-2}} & L^{n-2} X \\
\downarrow \varepsilon^{n-2} & & \downarrow \Sigma \varepsilon^{n-1} \\
QL^{n-2} X & \xrightarrow{\varepsilon^{n-1}} & L^{n-1} X.
\end{array}$$

After applying the functor \(\text{Hom}_{\mathbb{T}(I)}(?, Z)\) as in the initial step, we get a long exact sequence of abelian groups, that we report as the vertical sequence in the commutative diagram

$$\begin{array}{cc}
\text{Hom}_{\mathbb{T}(I)}(\Sigma L^{n-1} X, Z) & \xrightarrow{\Sigma \varepsilon^{n-1}} \text{Hom}_{\mathbb{T}(I)}(QL^{n-1} X, Z) \\
\downarrow & \downarrow \\
\text{Hom}_{\mathbb{T}(I)}(L^{n-2} X, Z) & \xrightarrow{\varepsilon^{n-2}} \text{Hom}_{\mathbb{T}(I)}(L^{n-1} X, Z) \\
\downarrow \varepsilon^{n-2} & \downarrow \Sigma \varepsilon^{n-1} \\
\text{Hom}_{\mathbb{T}(I)}(QL^{n-2} X, Z) & \xrightarrow{\varepsilon^{n-1}} \text{Hom}_{\mathbb{T}(I)}(QL^{n-1} X, Z).
\end{array}$$

By the preceding step we know that the maps \(\varepsilon^{n-1}\) and \(\Sigma \varepsilon^{n-1}\) are mono. This entails that the group \(\text{Hom}_{\mathbb{T}(I)}(\Sigma L^{n-1} X, Z)\) must be zero since the group \(\text{Hom}_{\mathbb{T}(I)}(QL^{n-1} X, Z)\)
is trivial. Indeed, using again the adjunction morphisms, we have

\[
\text{Hom}_{T(I)}(\Sigma Q L^{n-1} X, Z) = \text{Hom}_{T(I)}(\coprod_{i \in I} i^* L^{n-1} X, Z)
\]

\[
\cong \prod_{i \in I} \text{Hom}_{T(I)}(i^* L^{n-1} X, Z)
\]

\[
\cong \prod_{i \in I} \text{Hom}_{T(I)}(i^* L^{n-1} X, i^* Z)
\]

\[
= 0.
\]

The last equality comes from the condition found in the initial step. It follows that the homomorphism \( \varepsilon^{n-2} \) is a mono.

Moreover, the morphisms in \( \text{Hom}_{T(I)}(Q L^{n-2} X, Z) \) that are killed by the map \( \mu^{n-1} \) are exactly the same which are killed by \( \mu^{n-1} \). Hence, the homomorphism \( \varepsilon^{n-2} \circ \mu^{n-1} \), which is zero, must factor, up to iso, through the group \( \text{Hom}_{\text{Mor}_I(T(e))}(\text{mor}_I L^{n-2} X, \text{mor}_I Z) \).

If we put everything together we find an exact sequence

\[
0 \rightarrow \text{Hom}_{T(I)}(L^{n-2} X, Z) \rightarrow \text{Hom}_{\text{Mor}_I(T(e))}(\text{mor}_I L^{n-2} X, \text{mor}_I Z) \rightarrow 0.
\]

Now we can proceed by induction until we get an isomorphism

\[
\text{Hom}_{T(I)}(X, Z) \rightarrow \text{Hom}_{\text{Mor}_I(T(e))}(\text{mor}_I X, \text{mor}_I Z).
\]

Clearly, this isomorphism is induced by the functor \( \text{mor}_I \). Thus, we can use the canonical isomorphism between the categories \( \text{Hom}(I, T(e)) \) and \( \text{Mor}_I(T(e)) \) and the claim follows.

\[\square\]

### 6.2. Equivalence of the diagram functor.

Suppose we are given a triangulated derivator \( T \) of type \( \mathcal{D}ia \). Let us recall, from Appendix 1 in [22] to which we inspire, that, for each \( i \) in a small category \( I \) lying in \( \mathcal{D}ia \), the evaluation functor of presheaves \( F \mapsto F_i \) has a left adjoint denoted as \( ? \otimes i \),

\[
\begin{array}{ccc}
\text{Hom}(I^\circ, T(e)) & \xrightarrow{\varepsilon \otimes i} & \text{Hom}(I^\circ, (\text{mor}_I L^{n-2} X, \text{mor}_I Z)) \\
\uparrow & & \uparrow \\
\gamma \otimes i & & \gamma \otimes i \\
T(e) & & T(e)
\end{array}
\]

For each \( j \in I \), there is a canonical isomorphism

\[
(F \otimes i)_j = \coprod_{I(j,i)} F.
\]

**Lemma 6.2** (Keller and Nicolàs, [22]). Let \( I \) be any small category in \( \mathcal{D}ia \). For each \( i \in I \) the triangle

\[
\begin{array}{ccc}
T(I) & \xrightarrow{\text{dia}_I} & \text{Hom}(I^\circ, T(e)) \\
\text{id} & \xrightarrow{? \otimes i} & \text{id}
\end{array}
\]

commutes up to a canonical isomorphism.

**Remark 6.3.** Everything we have said in this section is true if we replace the triangulated derivator \( T \) with the represented exact prederivator \( A \). In particular, Lemma 6.2 holds
in this case. Moreover, suppose that $F_e : \mathcal{A}(e) \to \mathcal{T}(e)$ is an exact functor and $F_{a,i}$ is the induced functor defined on presheaves. It is not difficult to check that the diagram

\[
\begin{array}{ccc}
\text{Hom}(I^o, \mathcal{A}(e)) & \xrightarrow{\mathcal{F}_e} & \text{Hom}(I^o, \mathcal{T}(e)) \\
\mathcal{A}(e) & \xrightarrow{F_e} & \mathcal{T}(e)
\end{array}
\]

commutes in both directions.

The following proposition gives a sufficient condition which allows presheaves and their morphisms lift from $\text{Hom}(I^o, \mathcal{T}(e))$ to objects and morphisms of $\mathcal{T}(I)$. This tells us that the functor $\text{dia}_I$ is an equivalence when we consider its restriction to the preimage of some subcategory of presheaves which enjoys this property (that we call ‘Toda condition’). This result is one of the key ingredients in the proof of the main theorem of this section.

**Proposition 6.4.** Let $\mathcal{T}$ be a triangulated derivator of type $\text{Dia}_I$. Fix an arbitrary finite diagram $I$. Then,

a) given an object $F$ in $\text{Hom}(I^o, \mathcal{T}(e))$, such that the (Toda) condition

\[\text{Hom}_{\mathcal{T}(e)}(\Sigma^n F_i, F_j) = 0, \quad n > 0,\]

holds for each pair $(i, j) \in I \times I$, there exists an object $\tilde{F}$ in $\mathcal{T}(I)$ such that $\text{dia}_I(\tilde{F})$ is canonically isomorphic to $F$;

b) given a morphism $f : F \to F'$ in $\text{Hom}(I^o, \mathcal{T}(e))$, such that the (Toda) conditions

\[\text{Hom}_{\mathcal{T}(e)}(\Sigma^n F_i, F_j) = 0, \quad \text{Hom}_{\mathcal{T}(e)}(\Sigma^n F'_i, F'_j) = 0, \quad \text{Hom}_{\mathcal{T}(e)}(\Sigma^n F_i, F'_j) = 0, \quad n > 0,\]

hold for each pair $(i, j) \in I \times I$, there exists a morphism $\tilde{f} : \tilde{F} \to \tilde{F}'$ in $\mathcal{T}(I)$ such that $\text{dia}_I(\tilde{f})$ is canonically isomorphic to $f$.

**Proof.** a) Step 1. An exact category with finite global dimension. Recall that every additive category can be endowed with an exact structure by considering as conflations the split exact pairs (cf. Example 2.11). In this way, we can consider $\mathcal{T}(e)$ as an exact category and endow $\text{Hom}(I^o, \mathcal{T}(e))$ with an exact structure by taking as conflations the pointwise split exact pairs.

Let us construct a projective-like resolution of the object $F$ lying in $\text{Hom}(I^o, \mathcal{T}(e))$. As a first step we can consider the morphism

\[PF := \bigsqcup_{i \in I} F_i \otimes i \xrightarrow{p^0} F\]

defined by using the counit of the adjunctions ? $\otimes i \dashv (?)_i$. Clearly, the morphism $p^0$ is a deflation. Indeed, evaluating $PX$ over $j$, we get a retraction in $\mathcal{T}(e)$

\[(PF)_j = \bigsqcup_{i \in I} \bigsqcup_{(j,i)} F_i \longrightarrow F_j.\]

We can form the (possibly non split) conflation in $\text{Hom}(I^o, \mathcal{T}(e))$

\[KF \xrightarrow{p^0} PF \xrightarrow{p^0} F.\]

It may happen that $KF$ is already a projective-like object of type $PKF$ and consequently the global dimension of the category $\text{Hom}(I^o, \mathcal{T}(e))$ is one even if the maximal length of the diagram $I$ is infinite (cf. the case where $I$ is $\mathbb{N}$ in [22, Appendix 1]). But in general
this is not the case and we have to iterate the construction. Suppose that \( n \) is the maximal length of a chain of nonidentical arrows

\[
i_n \rightarrow \ldots \rightarrow i_1 \rightarrow i_0
\]

in \( I \). Then, it is not difficult to check by induction that the object \( K^{n+1}F \) is zero. This fact shows that \( K^nF \) is projective and that the length of a projective resolution is bounded by \( n \), as in the following chain complex

\[
0 \rightarrow PK^nF = K^nF \xrightarrow{d^{n-1}} PK^{n-1}F \xrightarrow{u_n-1} \ldots \rightarrow PKF \xrightarrow{u_1} PF \xrightarrow{p^0} F.
\]

Let us give a description of the morphisms in the above resolution. We can start with \( u^1 \). For each arrow \( a^1 : j \rightarrow i \) in \( I \), let \( a^1_j \in \text{Hom}(F_i \otimes j, F_j \otimes j) \) be the image of the identity \( 1_{F_i} \otimes j \) under the composition of the morphisms

\[
\text{Hom}_{\text{Hom}(I^o,T(e))}(F_j \otimes j, F_j \otimes j) \xrightarrow{(?)_j} \text{Hom}_{T(e)}(F_j, (F_j \otimes j)_j) \xrightarrow{(?)_j} \text{Hom}_{\text{Hom}(I^o,T(e))}(F_j \otimes j, F_i \otimes j)
\]

induced by the adjoint pair \( ? \otimes j \dashv (?)_j \) and the 2-arrow

\[
\text{Hom}(I^o,T(e)) \xrightarrow{(?)_j} T(e)
\]

associated with the natural transformation

\[
e \xrightarrow{?} I
\]

defined by the morphism \( a^1 \). Similarly, the morphism \( a^1 \) and the adjoint pair \( ? \otimes i \dashv (?)_i \) induce a morphism \( a^1_j : F_i \otimes j \rightarrow F_i \otimes i \), image of \( 1_{F_i} \otimes i \). Thus, for each morphism \( a^1 : j \rightarrow i \) in \( I \), we get an arrow

\[
F_i \otimes j \xrightarrow{[a^1_j,a^1_j^t]^t} F_i \otimes i \oplus F_j \otimes j \xrightarrow{\text{can}} \bigsqcup_{i \in I} F_i \otimes i.
\]

These are the components of the arrow

\[
(KF)_j \otimes j = \bigsqcup_{i \in I \setminus \{j\}} \bigsqcup_{I(j,i)} F_i \otimes j \xrightarrow{u^1_j} \bigsqcup_{i \in I} F_i \otimes i
\]

describing the morphism

\[
PKF = \bigsqcup_{j \in I} (KF)_j \otimes j \xrightarrow{u^1} PF.
\]

We recall that here \( I(j,i) \) denotes the discrete category of arrows \( j \rightarrow i \) in \( I \).

An explicit diagram, associated to a maximal length chain of nonidentical arrows in \( I \),

\[
i_n \rightarrow \ldots \rightarrow i_1 \rightarrow i_0,
\]
which describes a part of the whole diagram associated to \( I \), might help

\[
\begin{array}{c}
\xymatrix{
PKF & (KF)_{i_1} \ar[r] & (KF)_{i_2} \oplus (K^2F)_{i_2} \ar[r] & \ldots \ar[r] & \ldots \\
KF & (KF)_{i_1} \ar[r] & (KF)_{i_2} \ar[r] & \ldots \\
PF & F_{i_0} \ar[r] & F_{i_1} \oplus (KF)_{i_1} \ar[r] & F_{i_2} \oplus (KF)_{i_2} \ar[r] & \ldots \\
F & F_{i_0} \ar[r] & F_{i_1} \ar[r] & F_{i_2} \ar[r] & \ldots 
}
\end{array}
\]

By induction, one can get the description of all the maps in the projective resolution of \( F \).

**Step 2. Lifting a morphism of projective objects along the diagram functor.** Define two objects in \( \mathbb{T}(I) \)

\[
\widetilde{PKF} := \coprod_{j \in I} j_!(KF)_j \quad \mathbb{T}(I) \xrightarrow{\widetilde{PKF}} \mathbb{T}(e)
\]

and

\[
\widetilde{PF} := \coprod_{i \in I} i_!(F_i).
\]

Lemma \[6.2\] tells us that these objects lift the diagrams \( PKF \) and \( PF \) along the functor \( \text{dia}_I \). Let us construct the morphism \( \widetilde{a}^1 \) that lifts \( a^1 \). For each arrow \( a^1 : j \to i \) in the diagram \( I \), let \( \widetilde{a}^1_i \) be the image of the identity \( 1^{a_1} \) under the composition of morphisms

\[
\begin{align*}
\text{Hom}_{\mathbb{T}(I)}(i_!F_i, i_!F_i) & \xrightarrow{\sim} \text{Hom}_{\mathbb{T}(e)}(F_i, i^*i_!F_i) \\
& \xrightarrow{\sim} \text{Hom}_{\mathbb{T}(e)}(F_i, j^*i_!F_i) \\
& \xrightarrow{\sim} \text{Hom}_{\mathbb{T}(I)}(j_!F_i, i_!F_i)
\end{align*}
\]

induced by the adjoint pairs \( i_! \to i^* \) and \( j_! \to j^* \) and the 2-arrow

\[
\mathbb{T}(I) \xrightarrow{\widetilde{a}^1} \mathbb{T}(e)
\]

associated with the natural transformation

\[
e \xrightarrow{\widetilde{a}^1} I
\]

defined by the morphism \( a^1 \). Let us denote by \( \widetilde{a}^1_j \) the morphism \( j_!(F_{a^1}) \). Thus, for each morphism \( a^1 : j \to i \) in \( I \), we get an arrow

\[
\xymatrix{j_!F_i \\
\coprod_{i \in I} i_!F_i \ar[r]^-{i_!F_i \oplus j_!F_i} & \coprod_{i \in I} i^*_!F_i
\]

These are the components of an arrow

\[
\xymatrix{j_!(KF)_j = \coprod_{i \in I \to (j)} \coprod_{I(j,i)} j_!F_i \\
\coprod_{i \in I \to (j)} i^*_!F_i
\]
describing the morphism

$$PKF = \Pi_{j \in J_f} (KF)_j \xrightarrow{\widetilde{u}^i} PF.$$ 

It is clear by the construction and by Lemma 6.2 that this morphism lifts \( u^1 \), i.e., the morphism \( \text{dia}_f(u^1) \) is isomorphic to \( u^1 \).

Analogously, we can lift all the morphisms \( u^l : PK^l F \to PK^{l-1} F \), \( l \in \{1, \ldots n\} \), of the projective resolution of \( F \) to morphisms \( \widetilde{u}^l : PK^l F \to PK^{l-1} F \), \( l \in \{1, \ldots n\} \), in \( T(I) \).

**Step 3. Lifting a projective resolution along the diagram functor.** We construct a sequence of morphisms in \( T(I) \) which lifts the projective resolution that we have constructed in \( \text{Hom}(I^o, T(e)) \),

$$0 \to PK^n F = K^n F \xrightarrow{u^n} PK^{n-1} F \xrightarrow{u^{n-1}} \ldots \xrightarrow{u^2} PK F \xrightarrow{u^1} PF \xrightarrow{u^0} F.$$ 

Recall that this is a strictly acyclic resolution, i.e., there are conflations

$$K^{l+1} F \xrightarrow{i_l} PK^l F \xrightarrow{p^l} K^l F, \quad l \in \mathbb{N},$$

such that \( u^l = i^{l-1} p^l \), for all \( l \in \mathbb{N} \setminus \{0\} \).

We can start by lifting the morphism \( u^n = i^{n-1} \) in the conflation

$$K^n F \xrightarrow{u^n} PK^{n-1} F \xrightarrow{p^{n-1}} K^{n-1} F$$

along the diagram functor to a morphism \( \widetilde{u}^n : K^n F \to PK^{n-1} F \), following the step 2. Let us consider a distinguished triangle

$$K^n F \xrightarrow{\widetilde{u}^n} PK^{n-1} F \xrightarrow{\widetilde{p}^{n-1}} K^{n-1} F \xrightarrow{\Sigma K^n F},$$

which extends \( \widetilde{u}^n \) in \( T(I) \). The image of the composition of morphisms \( p^{n-1} \circ \widetilde{u}^n \) under the functor \( \text{dia}_f \) is the zero morphism \( \text{dia}_f(p^{n-1}) \circ \widetilde{u}^n \). Therefore, there exists a unique morphism \( \varphi^{n-1} : K^{n-1} F \to \text{dia}_f(K^{n-1} F) \) such that \( \text{dia}_f(p^{n-1}) \) equals the composition \( \varphi^{n-1} \circ p^{n-1} \).

For each \( m \in I \), after applying the (triangulated) functor \( m^* \) to the distinguished triangle above, we get a distinguished triangle in \( T(e) \)

$$\Sigma(K^n F)_m \xrightarrow{(\widetilde{u}^n)_m} (PK^{n-1} F)_m \xrightarrow{(\text{dia}_f(p^{n-1}))_m} (\text{dia}_f(K^{n-1} F))_m \xrightarrow{\Sigma(K^n F)_m}$$

which is split since \( u^n \) is an inflation. This shows that \( (\text{dia}_f(K^{n-1} F))_m \) is the cokernel of \( (u^n)_m \), for all \( m \in I \). Thus, \( (\varphi^{n-1})_m \) must be a (canonical) isomorphism, for all \( m \in I \). It follows that \( \text{dia}_f(p^{n-1}) \) is the cokernel of \( u \). Hence the canonical morphism \( \varphi^{n-1} \) is invertible.

Let us lift the morphism \( u^{n-1} \) along \( \text{dia}_f \) to a morphism \( \widetilde{u}^{n-1} \) as in step 2. It is not difficult to verify, using the hypotheses by induction over \( \mathbb{N} \), that

$$\text{Hom}_{T(e)}(\Sigma^n i^*(PK^{l+2} F), j^*(PK^F)) = 0, \quad n > 0,$$

for all \( l \in \mathbb{N} \). Since \( \text{dia}_f(u^{n-1} \circ \widetilde{u}^l) \) is zero, Proposition 6.1 with \( l = n - 2 \) applies and tells us that the composition \( \widetilde{u}^{n-1} \circ \widetilde{u}^l \) already vanishes in \( T(I) \). As a consequence, we get a morphism \( \widetilde{u}^{n-2} : K^{n-2} F \to PK^{n-2} F \) such that the morphism \( u^{n-1} : PK^{n-1} F \to PK^{n-2} F \) equals the composition \( \widetilde{u}^{n-2} \circ p^{n-1} \), whose image under \( \text{dia}_f \) is the inflation \( \widetilde{u}^{n-2} : K^{n-1} F \to PK^{n-2} F \).
It is clear that we can iterate this construction until we get a distinguished triangle

\[ \tilde{\mathcal{K}F} \xrightarrow{p_{0}} \mathcal{P}\mathcal{F} \xrightarrow{p_{0}} \mathcal{F} \xrightarrow{\Sigma\mathcal{K}F}, \]

whose image under \text{dia}_{I} gives (up to a canonical iso) a conflation

\[ \mathcal{K}F \xrightarrow{s_{0}} \mathcal{P}\mathcal{F} \xrightarrow{p_{0}=u_{0}} \mathcal{F}. \]

Thus, by lifting a projective resolution of \( F \), we have constructed an object \( \tilde{\mathcal{F}} \) in the category \( \mathcal{T}(I) \) which lifts \( F \) along the diagram functor \text{dia}_{I}.

b) **Step 1. Lifting a square of projective objects along the diagram functor.** Suppose that we are given a commutative square lying in \( \text{Hom}(I^{*}, \mathcal{T}(e)) \)

\[ \begin{array}{ccc}
PKF & \xrightarrow{u} & PF \\
\downarrow{h} & & \downarrow{g} \\
PKF' & \xrightarrow{u'} & PF'.
\end{array} \]

By looking at the structure of the objects \( PKF \) and \( PF \) (cf. with Step 1 in the proof of Claim a), we can see that the morphisms \( g \) and \( h \) are completely determined by their components, which are of the type \( g_{i} \otimes j \) and \( h_{i} \otimes j \), respectively. So, by Lemma 6.2, it is sufficient to lift these components to \( j_{i}(g_{i}) \) and \( j_{i}(h_{i}) \) and then reconstruct to obtain morphisms \( \tilde{h} \) and \( \tilde{g} \) which lift \( h \) and \( g \), respectively.

Thanks to the hypotheses, we can use the second step in the proof of Claim a) of this proposition and lift the morphisms \( u \) and \( u' \) to the category \( \mathcal{T}(I) \). We get a commutative square

\[ \begin{array}{ccc}
\mathcal{P}\mathcal{K}\mathcal{F} & \xrightarrow{\tilde{u}} & \mathcal{P}\mathcal{F} \\
\downarrow{\tilde{h}} & & \downarrow{\tilde{g}} \\
\mathcal{P}\mathcal{K}\mathcal{F}' & \xrightarrow{\tilde{u}'} & \mathcal{P}\mathcal{F}'.
\end{array} \]

in \( \mathcal{T}(I) \) whose image under the diagram functor is canonically isomorphic to the given square.

To verify commutativity, let us use the hypotheses about \( F \) and \( F' \) in the same way as above. It is easy to check that the vanishing

\[ \text{Hom}_{\mathcal{T}(e)}(\Sigma^{n-l}(PKF^{l+2}), j^{*}(PKF')) = 0, \quad n > 0, \]

holds for all \( l \in \mathbb{N} \). Since \text{dia}_{I}(\tilde{g} \circ \tilde{u} - \tilde{g} \circ \tilde{h}) \) is zero, Proposition 6.1 with \( l = n - 2 \) applies and tells us that the square commutes.

Step 2. **Lifting a morphism of projective resolutions along the diagram functor.** Let \( f : F \rightarrow F' \) be an arbitrary morphism in the category \( \text{Hom}(I^{*}, \mathcal{T}(e)) \). We know from the proof of Claim a) that we can construct projective resolutions of \( F \) and \( F' \) whose lengths are bounded by the maximal length of a chain of nonidentical arrows in \( I \),

\[ 0 \rightarrow PK^{0}F = K^{0}F \xrightarrow{u^{0}} PK^{1}F \xrightarrow{u^{1}} \ldots \xrightarrow{u^{n}} PKF \xrightarrow{u^{n}} PF \xrightarrow{u^{0}} F \]

and

\[ 0 \rightarrow PK^{0}F' = K^{0}F' \xrightarrow{u'^{0}} PK^{1}F' \xrightarrow{u'^{1}} \ldots \xrightarrow{u'^{n}} PKF' \xrightarrow{u'^{n}} PF' \xrightarrow{u'^{0}} F'. \]
Since these resolutions are made of projective objects, the morphism $f : F \to F'$ extends to a morphism of resolutions lying in $\text{Hom}(F', T(e))$

\[
\begin{array}{c}
0 \rightarrow PK^0 F = K^0 F \oplus \cdots \oplus K^{n-1} F \oplus K^n F \oplus PF \rightarrow F \\
\downarrow f^n \quad \downarrow f^{n-1} \quad \downarrow f^1 \quad \downarrow f^0 \quad \downarrow f \\
0 \rightarrow PK^0 F' = K^0 F' \oplus \cdots \oplus K^{n-1} F' \oplus K^n F' \oplus PF' \rightarrow F'.
\end{array}
\]

Let us start by lifting the square of projective objects on the left side of the commutative diagram

\[
\begin{array}{ccc}
K^n F \oplus \cdots \oplus K^{n-1} F & \xrightarrow{\tilde{u}} & PK^{n-1} F \\
\downarrow f^n & & \downarrow f^{n-1} \\
K^n F' \oplus \cdots \oplus K^{n-1} F' & \xrightarrow{\tilde{u}'} & PK^{n-1} F',
\end{array}
\]

where the morphism $g^{n-1}$ is induced by the universal property of the cokernel. It is routine to verify that the sufficient conditions which allow us using the first step of this claim hold.

We get a commutative square

\[
\begin{array}{ccc}
\tilde{K}^n F & \xrightarrow{\tilde{u}} & PK^{n-1} F \\
\downarrow \tilde{f}^n & & \downarrow \tilde{f}^{n-1} \\
\tilde{K}^n F' & \xrightarrow{\tilde{u}'} & PK^{n-1} F',
\end{array}
\]

whose image under the diagram functor is canonically isomorphic to the given one.

Let us consider an extension in $\mathbb{T}(I)$ of the latter square to a morphism of distinguished triangles

\[
\begin{array}{ccc}
\tilde{K}^n F & \xrightarrow{\tilde{u}} & PK^{n-1} F \\
\downarrow \tilde{f}^n & & \downarrow \tilde{f}^{n-1} \\
\tilde{K}^n F' & \xrightarrow{\tilde{u}'} & PK^{n-1} F',
\end{array}
\]

The images of the compositions of morphisms $p^{n-1} \circ \tilde{u}$ and $p^{n-1} \circ \tilde{u}'$ under the functor $\text{dia}_I$ are the zero morphisms $\text{dia}_I(p^n) \circ \tilde{u}$ and $\text{dia}_I(p^{n-1}) \circ \tilde{u}'$, respectively. It follows that there exist unique morphisms $\varphi^{n-1} : K^n F \to \text{dia}_I(K^{n-1} F)$ and $\varphi^{n-1} : K^n F' \to \text{dia}_I(K^{n-1} F')$ such that $\text{dia}_I(p^n) \circ \varphi^{n-1}$ and $\text{dia}_I(p^{n-1})$ respectively equal compositions $\varphi^{n-1} \circ p^n$ and $\varphi^{n-1} \circ p^{n-1}$. Moreover, there is an isomorphism $\text{dia}_I(g^n) \circ \varphi^{n-1} = \varphi^{n-1} \circ g^{n-1}$.

For each $m \in I$, we apply the (triangulated) functor $m^*$ to the morphism of distinguished triangles above and get a morphism of distinguished triangles in $\mathbb{T}(e)$

\[
\begin{array}{ccc}
(K^n F)_m & \xrightarrow{(u^n)_m} & (PK^{n-1} F)_m \\
\downarrow (f^n)_m & & \downarrow (f^{n-1})_m \\
(K^n F')_m & \xrightarrow{(u'_n)_m} & (PK^{n-1} F')_m
\end{array}
\]

\[
\begin{array}{ccc}
\text{dia}_I(K^{n-1} F)_m & \xrightarrow{(\text{dia}_I(p^n))_m} & \text{dia}_I(K^{n-1} F')_m \\
\downarrow (\text{dia}_I(g^n))_m & & \downarrow (\text{dia}_I(g^{n-1}))_m \\
\Sigma(K^n F)_m & \xrightarrow{\Sigma(f^n)_m} & \Sigma(K^n F')_m.
\end{array}
\]

Here, distinguished triangles are split since $u^n$ and $u'_n$ are inflations. This shows that the morphisms $(\text{dia}_I(K^{n-1} F)_m$ and $(\text{dia}_I(K^{n-1} F'))_m$ are the cokernels of $(u^n)_m$ and $(u'_n)_m$, for all $i \in I$. Thus, $(\varphi^{n-1})_m$ and $(\varphi^{n-1})_m$ must be (canonical) isomorphisms, for all $m \in I$.

It follows that the morphisms $\text{dia}_I(p^{n-1})$ and $\text{dia}_I(p^{n-1})$ are the cokernels of $u^n$ and $u'_n$. 


Hence, the morphisms $\varphi'^{-1}$ and $\varphi'^{-1}$ are invertible. Finally, the universal property of cokernels induces a (canonical) isomorphism from $g'^{-1}$ to $\text{dia}_I(g'^{-1})$.

At this point we can proceed by lifting the square of projective objects

$$
\begin{array}{ccc}
PK^{n-1}F & \overset{u^{-1}}{\longrightarrow} & PK^{n-2}F \\
\downarrow f^{-1} & & \downarrow f^{-2} \\
PK^{n-1}F' & \overset{u'^{-1}}{\longrightarrow} & PK^{n-2}F',
\end{array}
$$

according to the Step 1 of this Claim b), in order to get a commutative square

$$
\begin{array}{ccc}
PK^{n-1}F & \overset{u^{-1}}{\longrightarrow} & PK^{n-2}F \\
\downarrow f^{-1} & & \downarrow f^{-2} \\
PK^{n-1}F' & \overset{u'^{-1}}{\longrightarrow} & PK^{n-2}F',
\end{array}
$$

whose image under the diagram functor is isomorphic to the given one.

Sufficient conditions in order to apply Proposition 6.1 hold. Since the images $\text{dia}_I(u^{-1} \circ u^n)$ and $\text{dia}_I(u'^{-1} \circ u'^n)$ are both isomorphic to zero, it follows that the compositions $u^{-1} \circ u^n$ and $u'^{-1} \circ u'^n$ already vanish in $\mathcal{T}(I)$. Therefore, there exist morphisms $\hat{u}^{-2}$ and $\hat{u}'^{-2}$, whose images are the inflations $i^{-2}$ and $i'^{-2}$, such that $u^{-1}$ and $u'^{-1}$ are respectively isomorphic to the compositions $\hat{u}^{-2} \circ \hat{u}'^{-1}$ and $\hat{u}'^{-2} \circ \hat{u}'^{-1}$.

It is clear that we can continue by iterating this construction until the cohomological degree is 0. In the end, we will get a morphism $\tilde{f} : F \rightarrow F'$, which lifts the given morphism $f : F \rightarrow F'$ to the triangulated category $\mathcal{T}(I)$ along the diagram functor $\text{dia}_I$. 

$\square$

6.3. Invertibility of the diagram functor. Let us also recall that we say that an additive functor $F : \mathcal{E} \rightarrow \mathcal{T}$ from an exact category $\mathcal{E}$ to a triangulated category $\mathcal{T}$ is exact or, equivalently, a $\partial$-functor if to any conflation in $\mathcal{E}$,

$$
\begin{array}{ccc}
X & \overset{u}{\longrightarrow} & Y \\
\downarrow v & & \downarrow Z,
\end{array}
$$

it functorially associates a morphism $\partial(\varepsilon)$ such that the diagram

$$
\begin{array}{ccc}
F(X) & \overset{F(u)}{\longrightarrow} & F(Y) \\
\downarrow F(v) & & \downarrow F(\varepsilon) \\
F(Z) & \overset{\partial(\varepsilon)}{\longrightarrow} & \Sigma F(X)
\end{array}
$$

is a distinguished triangle of $\mathcal{T}$. It is straightforward to see that an additive 2-morphism $\mu : F \rightarrow F'$ of exact functors is an exact 2-morphism if to the conflation above it functorially associates a morphism of distinguished triangles of $\mathcal{T}$,

$$
\begin{array}{ccc}
F(X) & \overset{F(u)}{\longrightarrow} & F(Y) \\
\downarrow \mu_X & & \downarrow \mu_Y \\
F'(X) & \overset{F'(u)}{\longrightarrow} & F'(Y) \\
\downarrow \mu' (\varepsilon) & & \downarrow \mu' X \\
F'(Z) & \overset{\partial(\varepsilon)}{\longrightarrow} & \Sigma F'(X)
\end{array}
$$

In any case, let us remark that, in the presence of the Toda condition as in the items a) and b) of the following theorem, it would be equivalent if the basic exact 2-morphism $\mu$ of basic exact functors $F, F'$ that we want to extend were taken weakly exact only. Indeed, it is not difficult to see that all the weakly exact functors are exact in the presence of such a condition.

Theorem 6.5. Let $\mathcal{A}$ be an exact category and $\mathcal{T}$ a triangulated category such that there exists an isomorphism $\mathcal{T} \rightarrow \mathcal{T}(\varepsilon)$ for some triangulated derivator $\mathcal{T}$ of type $\mathcal{D} \text{ia}_I$. 

a) Suppose that $F : \mathcal{A} \to \mathcal{T}$ is an exact functor such that the (Toda) condition

$$\text{Hom}_\mathcal{T}(\Sigma^n F(X), F(Y)) = 0, \quad n > 0,$$

holds for all $X,Y$ in $\mathcal{A}$. Then, there exists an exact morphism of prederivators $\overline{F} : \mathcal{A} \to \mathcal{T}$ (of type $\mathcal{D}ia$) having base $F$ (up to iso). The morphism $\overline{F}$ is unique up to a unique natural isomorphism.

b) Suppose that $F$ and $F'$ are exact morphisms $\mathcal{A} \to \mathcal{T}$ and that the (Toda) conditions

$$\text{Hom}_\mathcal{T}(\Sigma^n F_e(X), F_e(Y)) = 0, \quad n > 0,$$

$$\text{Hom}_\mathcal{T}(\Sigma^n F'_e(X), F'_e(Y)) = 0, \quad n > 0,$$

hold for all $X,Y$ in $\mathcal{A}(e) = \mathcal{A}$. Then, the map

$$\text{Hom}^{-\mu_{F,F'}}(\mathcal{T}) \to \text{Hom}^{-\mu_{F,F'}}(\mathcal{A}(e), \mathcal{A}(e))$$

is bijective.

Proof. a) For any diagram $I$ in $\mathcal{D}ia$, let us denote by

$$E_I : \text{Hom}(I^o, \mathcal{A}(e)) \to \text{Hom}(I^o, \mathcal{T}(e))$$

the induced functor defined on presheaves. Let $\mathcal{V}(I)$ be the essential image of the functor $E_I$, which is an additive subcategory of $\text{Hom}(I^o, \mathcal{A}(e))$. Because of the hypotheses of claim a), every object lying in $\mathcal{V}(I)$ clearly satisfies the hypotheses of claim a) of Proposition [6.1]. Therefore, according to that claim they lift to $\mathcal{T}(I)$. Thus, we can form an additive category $\mathcal{U}(I)$ by taking all the preimages under $\mathcal{D}ia_I$ of all the objects lying in $\mathcal{V}(I)$ and then considering the additive full subcategory they span in $\mathcal{T}(I)$.

By construction, the restriction of the diagram functor $\mathcal{D}ia_I$ to $\mathcal{U}(I)$ is essentially surjective. Actually, it is fully faithful, too. Indeed, for every pair of objects $X$ and $Z$ in $\mathcal{U}(I)$, the hypotheses of Proposition [6.1] hold. Hence, the conclusion gives us a bijection

$$\text{Hom}_{\mathcal{T}(I)}(X,Z) \cong \text{Hom}^{-\mu_{F,F'}}(\mathcal{T}(I), \mathcal{D}ia_I X, \mathcal{D}ia_I Z),$$

induced by $\mathcal{D}ia_I$, which clearly reduces to a bijection

$$\text{Hom}_{\mathcal{U}(I)}(X,Z) \cong \text{Hom}_{\mathcal{V}(I)}(\mathcal{D}ia_I X, \mathcal{D}ia_I Z),$$

induced by its restriction to the subcategory $\mathcal{U}(I)$, for all pairs of objects $X$ and $Z$ in $\mathcal{U}(I)$.

Said otherwise, the restriction of the diagram functor to $\mathcal{U}(I)$ is an (additive) equivalence of (additive) categories. Nevertheless, we will show that its (inverse) composition with $E_I \circ \mathcal{D}ia_I$ gives rise to a $\partial$-functor from $\mathcal{A}(I)$ to $\mathcal{T}(I)$.

Consider the following commutative diagram of additive categories

$$\begin{array}{ccc}
\text{Hom}(I^o, \mathcal{A}(e)) & \cong & \mathcal{V}(I) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{A}(I) & - & \mathcal{U}(I) \\
\mathcal{D}ia_I & & \mathcal{T}(I)
\end{array}$$

that we have constructed for any diagram $I$ in $\mathcal{D}ia_I$. The dashed arrow is obtained by composition with the functor which is inverse to the vertical equivalence in the centre of the diagram. It clearly induces an additive functor $\overline{F}_I : \mathcal{A}(I) \to \mathcal{T}(I)$ for every diagram $I$ lying in $\mathcal{D}ia_I$. Here and in the sequel, the symbol $\mid$ means the image under the forgetful functor.
This construction gives rise to a diagram of additive prederivators,
\[
\begin{array}{ccc}
A & \rightarrow & V \\
\downarrow & & \downarrow \\
U & \leftarrow & T
\end{array}
\]

After inverting the equivalence and composing, we get an additive morphism of additive prederivators \( \tilde{F} : A \rightarrow T \). It is indeed easy to check that, for all the morphisms \( u : I \rightarrow J \) in \( \mathcal{D}iag \), we have an invertible natural transformation of functors \( u^*\tilde{F}_J \rightarrow \tilde{F}_Ju^* \). Moreover, it is clear that the base of \( \tilde{F} \) is naturally isomorphic to the functor \( F \) as an additive functor.

We want to show that, with respect to the exact and triangulated structures of the categories \( A(I) \) and \( T(I) \), the functor \( \tilde{F}_J \) is weakly exact according to definition 4.3. Suppose we are given a pair of morphisms in \( U(I) \)
\[
X \xrightarrow{f} Y \xrightarrow{g} Z
\]
which is the image under \( \tilde{F}_J \) of some conflation \( \varepsilon \) lying in \( A(I) \). We can extend the morphism \( f \) to a distinguished triangle
\[
X \xrightarrow{f} Y \xrightarrow{h} W \xrightarrow{i} \Sigma X
\]
in the triangulated category \( T(I) \). Since the composition of \( f \) with \( g \) vanishes there is an arrow \( \varphi : W \rightarrow Z \) which lifts \( g \). After applying to this distinguished triangle the (triangulated) functor \( i^* \), we get a distinguished triangle in \( T(\varepsilon) \)
\[
i^*X \xrightarrow{i^*f} i^*Y \xrightarrow{i^*h} i^*W \xrightarrow{\delta_i} \Sigma i^*X,
\]
for any \( i \in I \). As the morphisms \( f \) and \( g \) are in the image of a conflation of \( A(I) \) and \( F : A \rightarrow T \) is supposed to be a \( \partial \)-functor, we get that this triangle fits in a morphism of distinguished triangles
\[
i^*X \xrightarrow{i^*f} i^*Y \xrightarrow{i^*h} i^*W \xrightarrow{\delta_i} \Sigma i^*X
\]
\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]
\[
i^*X \xrightarrow{i^*f} i^*Y \xrightarrow{i^*g} i^*Z \xrightarrow{\psi_i} \Sigma i^*X,
\]
Here, the invertible arrow \( \psi_i : i^*W \rightarrow i^*Z \) which makes the diagram commute exists thanks to the axioms of triangulated categories. Since the objects \( X \) and \( Z \) belong to the subcategory \( U(I) \), their images under the (restriction of the) functor \( U(I) \) are in \( V(I) \).

Hence, we know from the Toda condition that the abelian group \( \operatorname{Hom}_{T(\varepsilon)}(\Sigma i^*X, i^*Z) \) must be zero for each \( i \in I \). This condition makes clear that the isomorphism \( \psi_i \) is uniquely determined. It follows that \( \psi_i \) must be canonically isomorphic to \( i^*\varphi \), for all \( i \in I \). Now we can use Axiom Der 2 of derivators which ensures that \( \varphi \) actually is an isomorphism. This shows that the pair of morphisms \( (f, g) \) actually extends to a distinguished triangle of \( T(I) \)
\[
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{m} \Sigma X,
\]
for some morphism \( m \).

Now, since we have checked that the hypotheses of Proposition 4.7 b) are fulfilled, we know that the morphism \( \tilde{F} : A \rightarrow T \) has the property of being an exact morphism of prederivators. We have already checked at the beginning of the proof of 4.7 b) that the base of our morphism \( \tilde{F} \) is canonically an exact functor. Therefore, it must coincide (up to a canonical iso) with \( F \) as an exact functor. The uniqueness of our construction also follows by the Toda condition.
b) We want to show that the functor
\[ \text{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{T}) \xrightarrow{\text{ev}_{e}} \text{Hom}(\mathcal{A}(e), \mathcal{T}(e)), \quad F \mapsto F_{e} \]
induces a bijection
\[ \text{Hom}_{\text{Hom}(\mathcal{A}, \mathcal{T})}(F, F') \xrightarrow{\sim} \text{Hom}_{\text{Hom}(\mathcal{A}(e), \mathcal{T}(e))}(F_{e}, F'_{e}). \]

It is known (cf. [19, 8.1]) the easy fact that, under the hypotheses of b) (Toda conditions), the set of natural transformations of \( \partial \)-functors from \( F_{e} \) to \( F'_{e} \) is in bijection with the set of natural transformations of their underlying additive functors. Moreover, after item b) of Prop. 4.7 we know that the set of 2-morphisms of exact morphisms from \( F \) to \( F' \) is in bijection with the subset of 2-morphisms of their underlying additive 2-morphisms from \( F \) to \( F' \).

Thus, to prove the claim in item b) of this theorem, we only have to show that the functor
\[ \text{Hom}_{\text{add}}(\mathcal{A}, \mathcal{T}) \xrightarrow{\text{ev}_{e}} \text{Hom}_{\text{add}}(\mathcal{A}(e), \mathcal{T}(e)), \quad F \mapsto F_{e} \]
induces a bijection
\[ \text{Hom}_{\text{Hom}_{\text{add}}(\mathcal{A}, \mathcal{T})}(F, F') \xrightarrow{\sim} \text{Hom}_{\text{Hom}_{\text{add}}(\mathcal{A}(e), \mathcal{T}(e))}(F_{e}, F'_{e}). \]

From now on we omit the symbol \( | \). Let us factor the functor \( \text{ev}_{e} \) as follows
\[ \text{Hom}_{\text{add}}(\mathcal{A}, \mathcal{T}) \xrightarrow{\text{ev}_{e}} \text{Hom}_{\text{add}}(\mathcal{A}(e), \mathcal{T}(e)) \xrightarrow{\sim} \text{Hom}_{\text{add}}(\mathcal{A}(e), \mathcal{T}(e)). \]

Here, \( \mathcal{T}(e) \) is the additive prederivator which associates the additive category \( \text{Hom}(\mathcal{I}, \mathcal{T}(e)) \) with a diagram \( I \in \mathcal{D}_{\text{iaF}} \). The additive category \( \text{Hom}_{\text{add}}(\mathcal{A}, \mathcal{T}(e)) \) contains as objects the morphisms of the underlying additive prederivators, \( i.e., \) morphisms \( F : \mathcal{A} \to \mathcal{T}(e) \) such that \( F_{i} \) are additive functors for all diagrams \( I \in \mathcal{D}_{\text{iaF}} \) with compatibility conditions. The additive functor \( \text{dia}_{\mathcal{A}} \) is induced by the morphism of the underlying additive prederivators \( \text{dia} : \mathcal{T} \to \mathcal{T}(e) \) by composition. Moreover, the additive functor of additive categories on the right of the last diagram above is given by evaluation on the terminal diagram \( e \). It is not hard to directly check that it is an equivalence of categories.

Thus, in order to prove the claim, it is enough to show that the morphism \( \text{dia} \) induces a bijection
\[ \text{Hom}_{\text{Hom}_{\text{add}}(\mathcal{A}, \mathcal{T})}(F, F') \xrightarrow{\sim} \text{Hom}_{\text{Hom}_{\text{add}}(\mathcal{A}(e), \mathcal{T}(e))}(\text{dia} \circ F, \text{dia} \circ F'). \]

By the contravariant version of Lemma A.5 in [20], we can check this isomorphism locally, \( i.e., \) we have to show that the functor \( \text{dia}_{I} : \mathcal{T}(I) \to \mathcal{T}(e)(I) \) induces a bijection
\[ \text{Hom}_{\text{Hom}_{\text{add}}(\mathcal{A}(I), \mathcal{T}(I))}(F_{I} \circ u^{*}, F'_{I} \circ u^{*}) \xrightarrow{\sim} \text{Hom}_{\text{Hom}_{\text{add}}(\mathcal{A}(e), \mathcal{T}(e)(I))}((\text{dia} \circ F)_{I} \circ u^{*}, (\text{dia} \circ F')_{I} \circ u^{*}), \]
for each morphism \( u : I \to J \) in \( \mathcal{D}_{\text{iaF}} \). This is a posteriori true if the map
\[ \text{Hom}_{\mathcal{T}(I)}(\Sigma^{n}F_{I}X, F'_{I}Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{T}(e)(I)}(\Sigma^{n}\text{dia}_{I}(F_{I}X), \text{dia}_{I}(F'_{I}Y)) \]

is an equivalence for all \( I \).
is a bijection for all $X, Y$ in $\mathcal{A}(I)$. But this is true, since, by the argument of the proof of Proposition 6.1, we know that, for all $n \in \mathbb{N}$, there is an exact sequence

$$\prod_{j \in I} \prod_{i \in I \setminus \{j\}} \prod_{(i,j) \in I^2} \text{Hom}_{\mathcal{T}}(\Sigma^{n+1}(\text{dia}_I(F_I X)), (\text{dia}_I(F'_I Y)))_j$$

$$\text{Hom}_{\mathcal{T}(I)}(\Sigma^n F_I X, F'_I Y)$$

$$\text{Hom}_{\mathcal{T}(e)(I)}(\Sigma^n \text{dia}_I(F_I X), \text{dia}_I(F'_I Y))$$

$$0,$$

where the first group vanishes by the third Toda condition in the hypotheses. The assertion follows. □

This theorem, combined with Theorem 2.16 immediately yields Theorem 2.17 as a corollary.

**References**

[1] Don W. Anderson, *Axiomatic homotopy theory*, in Algebraic Topology, Waterloo, 1978, Lecture Notes in Mathematics, no. 741, Springer-Verlag, Berlin, 1979.

[2] Marcel Bökstedt, Amnon Neeman, *Homotopy limits in triangulated categories*, Compositio Math. 86 (1993), 209–234.

[3] Aldridge Bousfield, Eric Friedlander, *Homotopy theory of $\Gamma$-spaces, spectra, and bisimplicial sets*, Springer Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, 80-130.

[4] Kevin Carlson, *An embedding of quasicategories in prederivators*, preprint, [arXiv:1612.06980v1 [math.CT]] 21 Dec 2016.

[5] Denis-Charles Cisinski, *Images directs cohomologiques dans les catégories de modèles*, Ann. math. Blaise Pascal. 10 (2003), 195–244 (in French).

[6] , *Propriétés universelles et extensions de Kan dérivées*, Theory Appl. Categ., 20 (2008), no. 17, 605–649 (in French).

[7] Denis-Charles Cisinski, Amnon Neeman, *Additivity for derivator $K$-theory*, Adv. Math., 217 (2008), 1381–1475.

[8] Albrecht Dold, Dieter Puppe, *Homologie nicht–additiver Funktoren. Anwendungen*, Ann. Inst. Fourier, tome 11 (1961), 201–312 (in German).

[9] William Dwyer, *Localizations*, in Axiomatic, enriched and motivic homotopy theory, vol. 131 of NATO Sci. Ser. II Math. Phys. Chem., Kluwer Acad. Publ., Dordrecht (2004), 3–28.

[10] Jens Franke, *Uniqueness theorems for certain triangulated categories possessing an Adams spectral sequence*, preprint, available at [http://www.math.uiuc.edu/K-theory/0139/](http://www.math.uiuc.edu/K-theory/0139/) 1996.

[11] Moritz Groth, *Derivators, pointed derivators, and stable derivators*, Algebr. Geom. Topol. 13(1) (2013), 313–374 [arXiv:1112.3840v2 [math.KT] 13 Feb 2012].

[12] , *Monoidal derivators and additive derivators*, preprint, [arXiv:1203.5071v1 [math.AT]] 22 Mar 2012.

[13] , *Revisiting the canonicity of canonical triangulations*, preprint, [arXiv:1602.04846v1 [math.AT]] 15 Feb 2016.

[14] Moritz Groth, Kate Ponto, Michael Shulman, *Mayer-Vietoris sequences in stable derivators*, Homology, Homotopy Appl., vol. 16, no. 1 (2014) 265-294.

[15] Alexander Grothendieck, *Les dérivateurs*, manuscript (~1990), ed. by M. Künzer, J. Malgoire, G. Maltsiniotis, available at [http://www.math.jussieu.fr/~maltsin/groth/Derivateurs.html](http://www.math.jussieu.fr/~maltsin/groth/Derivateurs.html) (in French).

[16] Alex Heller, *Homotopy theories*, Mem. Amer. Math. Soc., vol. 71, no. 383 (1988).

[17] , *Stable homotopy theories and stabilization*, J. Pure Appl. Alg., 115 (1997), 113–130.

[18] Mikhail M. Kapranov, *On the derived categories of coherent sheaves on some homogenous spaces*, Invent. Math. 92 (1988), 479–508.
Bernhard Keller, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), no. 4, 379–417.

Bernhard Keller, *Derived categories and universal problems*, Comm. in Alg. **19** (1991), 379–417.

Bernhard Keller, Pedro Nicolás, *Weight structures and simple DG modules for positive DG algebras*, Int. Math. Res. Not. (2013), no. 5, 1028–1078 [arXiv:1009.5994v3 [math.RT] 14 Sep 2011].

Bernhard Keller, Dieter Vossieck, *Sous les catégories dérivées*, C.R. Acad. Sci. Paris **305**, Série I, (1987), 225–228 (in French).

Manos Lydakis, *Simplicial functors and stable homotopy theory* preprint, available via Hopf archive at [http://hopf.math.purdue.edu/Lydakis/sfunctors.pdf](http://hopf.math.purdue.edu/Lydakis/sfunctors.pdf) (1998).

Valery A. Lunts, Dmitri O. Orlov, *Uniqueness of enhancement for triangulated categories*, J. Amer. Math. Soc., vol. 23, no. 3 (2010), 853–908.

Georges Maltsiniotis, *La K-théorie d’un dérivateur triangulé*, Categories in Algebra, geometry and mathematical physics, vol. 431, Contemp. Math., Amer. Math. Soc., Providence, RI, 2007, 341–368 (in French).

Michael Mandell, Peter May, Stefan Schwede, Brooke Shipley, *Model categories of diagram spectra*, Proceedings of the London Mathematical Society, **82** (2001), 441–512.

Amnon Neeman, *Triangulated Categories*, Annals of Math. Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

Kate Ponto, Michael Shulman, *The linearity of traces in monoidal categories and bicategories*, Theory Appl. Categ., **31** (2016), no. 23, 594–689.

Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, no. 43, Springer-Verlag, Berlin, 1967.

Andrei Radulescu-Banu, *Cofibrations in homotopy theory*, preprint [arXiv:math/0610009v4 [math.AT] 8 Feb 2009].

Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque, vol. 239, Société Mathématique de France, 1996 (in French).

Marco Porta
E-mail address: marcoporta1@libero.it