On the $\mathcal{D}$-module and formal-variable approaches to vertex algebras

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1 Introduction

In a program to formulate and develop two-dimensional conformal field theory in the framework of algebraic geometry, Beilinson and Drinfeld [BD] have recently given a notion of “chiral algebra” in terms of $\mathcal{D}$-modules on algebraic curves. This definition consists of a “skew-symmetry” relation and a “Jacobi identity” relation in a categorical setting, and it leads to the operator product expansion for holomorphic quantum fields in the spirit of two-dimensional conformal field theory, as expressed in [BPZ]. Because this operator product expansion, properly formulated, is known to be essentially a variant of the main axiom, the “Jacobi identity” [FLM], for vertex (operator) algebras ([Bord], [FLM]; see [FLM] for the proof), the chiral algebras of [BD] amount essentially to vertex algebras.

In this paper, we show directly that the chiral algebras of [BD] are essentially the same as vertex algebras without vacuum vector (and without grading), by establishing an equivalence between the skew-symmetry and Jacobi identity relations of [BD] and the (similarly-named, but different) skew-symmetry and Jacobi identity relations in the formal-variable approach to vertex operator algebra theory (see [FLM], [FHL]). In particular, among the equivalent formulations of the notion of vertex (operator) algebra, the $\mathcal{D}$-module notion of chiral algebra corresponds the most closely to the formal-variable notion, rather than to, say, the operator-product-expansion notion (based on the “commutativity” and “associativity” relations, as explained in [FLM], [FHL]) or to the geometric or operadic notion ([Hu1], [Hu2], [HL]).

More precisely, we prove that for any nonempty open subset $X$ of $\mathbb{C}$, the category of vertex algebras without vacuum over $X$ (see Definitions 2.2
and \([3,1]\) and the category of chiral algebras over \(X\) (see Definition \([3,1]\)) are equivalent (see Section 5). Beilinson-Drinfeld’s notion of chiral algebra is in general formulated over higher-genus curves. But since chiral algebras in this sense are essentially local objects, the equivalence proved in this paper shows that the notion in \([BD]\) is indeed essentially equivalent to the notion of vertex algebra without vacuum.

We hope that the present expository exercise helps to illuminate the relations between the theories and philosophies of \(\mathcal{D}\)-modules and of vertex operator algebras. For example, in the Jacobi identity for vertex algebras, the three formal variables are on equal footing because of an intrinsic \(S_3\)-symmetry (see \([FHL]\), Section 2.7), while in the \(\mathcal{D}\)-module approach, there are only two (complex) variables, as in the operator-product-expansion approach. (See Remark \([3,17]\)). To see the \(S_3\)-symmetry explicitly in the algebro-geometric framework, we would have to introduce an analogue of the notion of \(\mathcal{D}\)-module allowing global translations of a variable rather than just “infinitesimal translations.” The Jacobi identity would then be interpreted as an identity in terms of such “modified \(\mathcal{D}\)-modules,” so that the three variables involved would play symmetric roles. This will be discussed in future publications. The Jacobi identity for vertex operator algebras and its \(S_3\)-symmetry in fact play a central role in the theory of vertex operator algebras, in particular, in the construction of “vertex tensor categories” (see \([HL2]\), \([HL3]\), \([HL4]\)).

Even without the introduction of such global translations of variables, all of the many calculations in this paper involving binomial expansions can be greatly simplified if we systematically introduce formal (not complex) variables playing the role of “formal global translations.” For instance, the expression \((z_1 - z_2)^n(A_1 \otimes A_2), n \in \mathbb{Z},\) occurring starting in Section 4 can be viewed as the coefficient of \(x^{-n-1}\) in \(x^{-1}\delta(\frac{z_1 - z_2}{x})(A_1 \otimes A_2),\) where \(x\) is a formal variable and \(\delta(\frac{z_1 - z_2}{x})\) is defined in Section 2.

In order to make this work reasonably self-contained, we include elementary definitions and notions needed in both theories. The reader can consult \([FLM]\) and \([FHL]\), for example, for the motivation and development of the theory of vertex operator algebras, and \([HR]\) for sheaves and \([Bore]\) for algebraic \(\mathcal{D}\)-modules, whose theory was developed by Beilinson and Bernstein.

This paper is organized as follows: In Section 2, we recall some basic notations and elementary tools and give the definitions of vertex algebra and vertex algebra without vacuum. In Section 3, we recall some basic concepts
in the theory of $\mathcal{D}$-modules and give examples which we shall need later. Beilinson-Drinfeld’s notion of chiral algebra over $X$ for a nonempty open subset $X \subset \mathbb{C}$ is given in Section 4. In Section 5, we define the notion of vertex algebra without vacuum over $X$ and prove the equivalence theorem stated above.

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## 2 Vertex algebras and vertex algebras without vacuum

Following the treatment in [FLM] and [FHL], we describe the basic notations and elementary tools needed to formulate the notion of vertex algebra. We work over $\mathbb{C}$. In this paper, the symbols $x, x_0, x_1, \ldots$ are independent commuting formal variables, and all expressions involving these variables are to be understood as formal Laurent series. (Later we shall also use the symbols $z, z_1, \ldots$, which will denote complex numbers, not formal variables.) We use the “formal $\delta$-function”

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n,$$

which has the following simple and fundamental property: For any Laurent polynomial $f(x) \in \mathbb{C}[x, x^{-1}]$,

$$f(x)\delta(x) = f(1)\delta(x).$$

This property has many important variants. For example, for any

$$X(x_1, x_2) \in (\text{End } W)[[x_1, x_1^{-1}, x_2, x_2^{-1}]]$$

(where $W$ is a vector space) such that

$$\lim_{x_1 \to x_2} X(x_1, x_2) = X(x_1, x_2) \bigg|_{x_1 = x_2}$$
exists, we have
\[ X(x_1, x_2)\delta \left( \frac{x_1}{x_2} \right) = X(x_2, x_2)\delta \left( \frac{x_1}{x_2} \right). \]  
(2.1)

The existence of this “algebraic limit” means that for an arbitrary vector \( w \in W \), the coefficient of each power of \( x_2 \) in the formal expansion \( X(x_1, x_2)w \big|_{x_1=x_2} \) is a finite sum.

We use the convention that negative powers of a binomial are to be expanded in nonnegative powers of the second summand. For example,
\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) = \sum_{n \in \mathbb{Z}} \frac{(x_1 - x_2)^n}{x_0^{n+1}} = \sum_{m,n \in \mathbb{N}, n \in \mathbb{Z}} (-1)^m \frac{n}{m} x_0^{-n-1} x_1^{n-m} x_2^m. \]

We have the following elementary identities:
\[ x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right), \]  
(2.2)
\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right). \]  
(2.3)

Here and below, it is important to note that the relevant sums and products, etc., of formal series, are well defined. See [FLM] and [FHL] for extensive discussions of the calculus of formal \( \delta \)-functions.

The following version of the definition of vertex algebra, using formal variables and the Jacobi identity of [FLM] is equivalent to Borcherds’ original definition [Borc]:

**Definition 2.1** A vertex algebra is a vector space \( V \), equipped with a linear map \( V \otimes V \rightarrow V[[x, x^{-1}]] \), or equivalently,
\[ V \rightarrow (\text{End } V)[[x, x^{-1}]] \]
\[ v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \text{ (where } v_n \in \text{End } V), \]
\( Y(v, x) \) denoting the vertex operator associated with \( v \), and equipped also with a distinguished homogeneous vector \( 1 \in V_{(0)} \) (the vacuum) and a linear map \( D : V \rightarrow V \). The following conditions are assumed for \( u, v \in V \): the lower truncation condition holds:
\[ u_n v = 0 \text{ for } n \text{ sufficiently large} \]
(or equivalently, \(Y(u, x)v \in V((x))\));

\[ Y(1, x) = 1 \text{ (1 on the right being the identity operator);} \]

the creation property holds:

\[ Y(v, x)1 \in V[[x]] \text{ and } \lim_{x \to 0} Y(v, x)1 = v \]

(that is, \(Y(v, x)1\) involves only nonnegative integral powers of \(x\) and the constant term is \(v\)); the Jacobi identity (the main axiom) holds:

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2)Y(u, x_1)
\]

\[
= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)
\]

(note that when each expression in (2.8) is applied to any element of \(V\), the coefficient of each monomial in the formal variables is a finite sum; on the right-hand side, the notation \(Y(\cdot, x_2)\) is understood to be extended in the obvious way to \(V[[x_0, x_0^{-1}]]\); and

\[
\frac{d}{dx} Y(v, x) = Y(Dv, x)
\]

(the \(D\)-derivative property).

The vertex algebra just defined is denoted by \((V, Y, 1, D)\) (or simply by \(V\)). Homomorphisms of vertex algebras are defined in the obvious way.

A consequence of the definition above is the skew-symmetry [Borc]:

\[ Y(u, x)v = e^{xD}Y(v, -x)u \quad (2.4) \]

for \(u, v \in V\). The proof uses the Jacobi identity, properties of the \(\delta\)-function, the \(D\)-derivative property and the creation property (see [FHL]).

In the definition above, we have required that the vertex algebra has a vacuum, but in Section 5 we shall see that the notion of chiral algebra formulated using \(\mathcal{D}\)-modules does not give a vacuum. Thus we need the following notion:
Definition 2.2 A *vertex algebra without vacuum* is a vector space $V$, a vertex operator map $Y : V \otimes V \to V[[x, x^{-1}]]$ and an operator $D$ on $V$ satisfying all the axioms for a vertex algebra which do not involve the vacuum and in addition the skew-symmetry (2.4).

We denote the vertex algebra without vacuum by $(V, Y, D)$ or simply by $V$.

3 $\mathcal{D}$-modules

We recall the elementary notions in the theories of sheaves (see [Ha]) and $\mathcal{D}$-modules (see [Borel]). We begin with the definition of presheaf. Following [Borel], we shall work over nonsingular quasi-projective varieties over $\mathbb{C}$ of pure dimension. Below, by a variety we shall mean a variety of this type.

Definition 3.1 Let $X$ be a variety. A *presheaf $\mathcal{F}$ of abelian groups* on $X$ consists of the data

1. for every open subset $U \subset X$, an abelian group $\mathcal{F}(U)$,
2. for every inclusion $V \subset U$ of open subsets of $X$, a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,

satisfying the conditions

1. $\mathcal{F}(\emptyset) = 0$,
2. $\rho_{UU}$ is the identity map from $\mathcal{F}(U)$ to itself,
3. if $W \subset V \subset U$ are open subsets of $X$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

We call the elements of $\mathcal{F}(U)$ *sections over $U$* and the maps $\rho_{UV}$ the *restriction maps*. If $\mathcal{F}$ is a presheaf on $X$, and if $P$ is a point of $X$, we define the *stalk $\mathcal{F}_P$ of $\mathcal{F}$ at $P$* to be the direct limit of the groups $\mathcal{F}(U)$ for all open sets $U$ containing $P$ via the restriction maps $\rho$. An element of $\mathcal{F}_P$ is called a *germ of sections of $\mathcal{F}$ at the point $P$*. 
Presheaves with values in any fixed category can be defined by replacing the references to abelian groups in the definition by the analogous references for the given category. Most of the considerations below hold for presheaves of (commutative or noncommutative) rings, presheaves of vector spaces and presheaves of (commutative or noncommutative) algebras, for example; when we consider the zero object or subobjects or quotient objects, it will be understood that we are always working in a category where these exist. We sometimes write $s|_{V}$ instead of $\rho_{UV}(s)$ for $s \in \mathcal{F}(U)$. When it is necessary to distinguish the restriction maps of different sheaves, we shall write the restriction maps for a sheaf $\mathcal{F}$ as $\rho_{UV}$.

**Definition 3.2** A presheaf on a variety $X$ is a sheaf if it satisfies the following additional conditions:

1. If $U$ is an open subset of $X$, $\{V_i\}$ an open covering of $U$ and $s \in \mathcal{F}(U)$ satisfying $s|_{V_i} = 0$ for all $i$, then $s = 0$.  

2. If $U$ is an open subset of $X$, $\{V_i\}$ an open covering of $U$ and we have elements $s_i \in \mathcal{F}(V_i)$ for each $i$, with the property that for each $i, j$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each $i$.

**Example 3.3** Let $X$ be a variety. We define the sheaf $\mathcal{O}_X$ of regular functions on $X$ as follows: For any open set $U$ of $X$, let $\mathcal{O}_X(U)$ be the ring of regular functions on $U$, which we shall also write as $\mathcal{O}(U)$, and for any open subset $V \subset U$ and $f \in \mathcal{F}(U)$, let $\rho_{UV}(f)$ be the restriction of $f$ to $V$. It is clear that this gives a sheaf of rings (actually of commutative associative algebras).

**Definition 3.4** Morphisms and isomorphisms of presheaves are defined in the obvious way. Morphisms and isomorphisms of sheaves are defined to be morphisms and isomorphisms of the underlying presheaves. Presheaf kernels, presheaf cokernels and presheaf images of morphisms of presheaves are defined in the obvious ways.
Note that presheaf kernels of morphisms of sheaves are sheaves but in general
presheaf cokernels and presheaf images of morphisms of sheaves are
not sheaves.

The following result associates a sheaf naturally to a presheaf in terms of
a universal property:

**Proposition 3.5** Given a presheaf \( F \), there is a sheaf \( F^+ \) and a morphism \( \theta : F \to F^+ \)
with the property that for any sheaf \( G \) and any morphism \( \varphi : F \to G \), there is a unique morphism
\( \psi : F^+ \to G \) such that \( \varphi = \psi \circ \theta \). The pair \( (F^+, \theta) \) is unique up to unique isomorphism.

See [Ha], for example, for a proof. The sheaf \( F^+ \) is called the sheaf
associated with the presheaf \( F \). It is determined completely by the germs of
sections of \( F \). Thus we shall also call \( F^+ \) the sheaf of germs of sections of \( F \).

**Definition 3.6** A subsheaf of a sheaf \( F \) is a sheaf \( F' \) such that for every
open set \( U \subset X \), \( F'(U) \) is a subgroup (or subobject in the category we are
discussing) of \( F(U) \), and the restriction maps of \( F' \) are induced from those
of \( F \). Let \( \varphi : F \to G \) be a morphism of sheaves. The kernel of \( \varphi \) is the
presheaf kernel of \( \varphi \), which is a sheaf, and in fact a subsheaf. The image of \( \varphi \) is the sheaf associated to the presheaf image of \( \varphi \). The image can be identified with a subsheaf of \( G \) by means of the universal property
of the sheaf associated to a presheaf. Let \( F' \) be a subsheaf of a sheaf \( F \).
The quotient sheaf \( F/F' \) is the sheaf associated to the presheaf given by
\( U \mapsto F(U)/F'(U) \). The cokernel of \( \varphi \) is the sheaf associated to the presheaf
coke kernel of \( \varphi \).

**Definition 3.7** Let \( X \) and \( Y \) be varieties and \( f : X \to Y \) a morphism.
For any sheaf \( F \) on \( X \), the direct image \( f_*(F) \) is the sheaf on \( Y \) given by
\( V \mapsto (f_*(F))(V) = F(f^{-1}(V)) \) for any open set \( V \subset Y \). For any sheaf \( G \) on
\( Y \), the inverse image \( f^{-1}(G) \) is the sheaf associated to the presheaf given by
\( U \mapsto \lim_{V \supset f(U)} G(V) \), where \( U \) is any open set of \( X \) and the limit is taken
over all open sets \( V \) of \( Y \) containing \( f(U) \).

For any open subset \( V \subset Y \) and \( s \in F(f^{-1}(V)) \), we denote \( s \) by \( f_*(s) \)
when it is viewed as an element of \( (f_*(F))(V) \), and we shall call \( s \) the preimage
of \(f_*(s)\). For any open subset \(U \subset X\) and any open subset \(V \subset Y\) such that \(V \supset f(U)\), an element \(s \in \mathcal{G}(V)\) determines an element of \((f^{-1}(\mathcal{G}))(U)\). We denote this element by \(f^{-1}(s)\).

We now define the notions of left and right \(\mathcal{O}\)-module for any sheaf \(\mathcal{O}\) of rings. Modules will always be left modules unless “right module” is specified. The concepts below have obvious analogues for right modules.

**Definition 3.8** Let \(\mathcal{O}\) be a sheaf of rings on \(X\). A sheaf \(\mathcal{F}\) on \(X\) is an \(\mathcal{O}\)-module if for any open set \(U\), \(\mathcal{F}(U)\) is an \(\mathcal{O}(U)\)-module, and for any open sets \(V \subset U\) and \(f \in \mathcal{O}(U)\), \(s \in \mathcal{F}(U)\), we have \(\rho_{UV}(fs) = \rho_{UV}(f)\rho_{UV}(s)\).

Morphisms and isomorphisms of \(\mathcal{O}\)-modules are defined in the obvious ways. Note that kernels, images and cokernels of morphisms of \(\mathcal{O}\)-modules are again \(\mathcal{O}\)-modules. If \(\mathcal{F}'\) is a subsheaf of \(\mathcal{O}\)-modules of an \(\mathcal{O}\)-module \(\mathcal{F}\), then the quotient sheaf \(\mathcal{F}/\mathcal{F}'\) is also an \(\mathcal{O}\)-module.

Now we restrict our attention to the sheaf \(\mathcal{O}_X\) (recall Example 3.3), for which left and right modules are the same.

**Definition 3.9** An \(\mathcal{O}_X\)-module \(\mathcal{F}\) is said to be quasi-coherent if for any open affine subset \(U\) of \(X\) and \(f \in \mathcal{O}_X(U)\), the following conditions hold:

1. Let \(V\) be the open set \(\{x \in U \mid f(x) \neq 0\}\). For any \(s \in \mathcal{F}(V)\), there exists \(n \in \mathbb{N}\) and \(\bar{s} \in \mathcal{F}(U)\) such that \(\rho_{UV}(\bar{s}) = f^n s\).

2. For any \(s \in \mathcal{F}(U)\) such that \(\rho_{UV}(s) = 0\), there exists \(n \in \mathbb{N}\) such that \(f^n s = 0\).

**Definition 3.10** Let \(\mathcal{F}\) and \(\mathcal{G}\) be \(\mathcal{O}_X\)-modules. The tensor product \(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}\) (or simply \(\mathcal{F} \otimes \mathcal{G}\)) is the sheaf associated to the presheaf given by \(U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)\).

**Definition 3.11** Let \(X\) and \(Y\) be varieties, \(f : X \rightarrow Y\) a morphism. This induces a natural morphism \(f^\# : \mathcal{O}_Y \rightarrow f_* (\mathcal{O}_X)\) of sheaves. If \(\mathcal{F}\) is an \(\mathcal{O}_X\)-module, then \(f_*(\mathcal{F})\) is an \(f_*(\mathcal{O}_X)\)-module, and thus an \(\mathcal{O}_Y\)-module via \(f^\#\). This \(\mathcal{O}_Y\)-module \(f_*(\mathcal{F})\) is called the direct image of \(\mathcal{F}\) by \(f\). If \(\mathcal{G}\) is a \(\mathcal{O}_Y\)-module, then \(f^{-1}(\mathcal{G})\) is an \(f^{-1}(\mathcal{O}_Y)\)-module. Since \(f^\#\) induces a natural morphism from \(f^{-1}(\mathcal{O}_Y)\) to \(\mathcal{O}_X\), we can form the tensor product \(f^{-1}(\mathcal{G}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X\) and it is an \(\mathcal{O}_X\)-module. This \(\mathcal{O}_X\)-module is called the inverse image of \(\mathcal{G}\) by \(f\) and is denoted \(f^*(\mathcal{G})\).
For any open subset \( U \subset X \), any open subset \( V \subset Y \) such that \( V \supset f(U) \), and any element \( s \in \mathcal{G}(V) \), we denote \( f^{-1}(s) \otimes_{(f^{-1}(\mathcal{O}_Y))(U)} 1 \in (f^*(\mathcal{G}))(U) \) by \( f^*(s) \).

**Example 3.12** Let \( X \) be a variety. Here we define the sheaf of (noncommutative) algebras \( \mathcal{D}_X \)—the sheaf of germs of algebraic differential operators on \( X \) (see [Borel]). If \( X \) is an affine variety, we define \( \mathcal{D}_X(X) \) to be the algebra \( \mathcal{D}(X) \) of operators on \( \mathcal{O}(X) = \mathcal{O}_X(X) \) generated by the elements of \( \mathcal{O}(X) \) (acting by multiplication) and the derivations of \( \mathcal{O}(X) \). This is the ring of (algebraic) differential operators on \( X \). For any open affine subset \( U \subset X \), we have \( \mathcal{D}(U) = \mathcal{O}(U) \otimes_{\mathcal{O}(X)} \mathcal{D}(X) \), and there exists a unique quasi-coherent \( \mathcal{O}_X \)-module \( \mathcal{D}_X \) on \( X \) such that \( \mathcal{D}(U) \) is the \( \mathcal{O}(U) \)-module of sections over an open affine subset \( U \subset X \). In general, for any variety \( X \) (of the type we are considering), there exists a unique sheaf \( \mathcal{D}_X \) of algebras on \( X \) whose restriction to every open affine subset \( U \subset X \) is \( \mathcal{D}_U \). The algebra \( \mathcal{D}_X(X) \) consists of the algebraic differential operators on \( X \). The sheaf \( \mathcal{D}_X \) is quasi-coherent as an \( \mathcal{O}_X \)-module and its restriction to an open subset \( U \subset X \) is \( \mathcal{D}_U \).

We shall use the following notion of (algebraic) \( \mathcal{D} \)-module:

**Definition 3.13** Let \( X \) be a variety. A \( \mathcal{D}_X \)-module is a \( \mathcal{D}_X \)-module as defined above which is also quasi-coherent as an \( \mathcal{O}_X \)-module. Similarly for a right \( \mathcal{D}_X \)-module.

Note that a (left or right) \( \mathcal{D}_X \)-module is a \( \mathcal{D}_X \)-module in the sense of Definition 3.8 satisfying an extra condition. Morphisms and isomorphisms of \( \mathcal{D}_X \)-modules are defined in the obvious ways. It is clear that the set of all morphisms from a \( \mathcal{D}_X \)-module to another one has a natural abelian group structure. For any (left) \( \mathcal{D}_X \)-modules \( \mathcal{F}_1, \mathcal{F}_2 \) and right \( \mathcal{D}_X \)-modules \( \mathcal{G}_1, \mathcal{G}_2 \), \( \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2 \) and \( \text{Hom}_{\mathcal{O}_X}(\mathcal{G}_1, \mathcal{G}_2) \) are naturally (left) \( \mathcal{D}_X \)-modules, and \( \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{G}_1 \) and \( \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{G}_2) \) are naturally right \( \mathcal{D}_X \)-modules. In particular, we have a tensor product operation in the category of (left) \( \mathcal{D}_X \)-modules.

**Example 3.14** Let \( \omega_X \) be the sheaf of germs of differential forms of top degree on \( X \). Let \( U \) be an open affine subset of \( X \). A derivation \( \xi \in \mathcal{D}_X(U) \) acts on \( \omega_X(U) \) from the left by the Lie derivative \( L_\xi \), but this does not give a (left) \( \mathcal{D}_X \)-module structure to \( \omega_X \). Rather, the action \( -L_\xi \) of \( \xi \) on \( \omega_X(U) \) gives a right \( \mathcal{D}_X \)-module structure to \( \omega_X \). Thus for any (left) \( \mathcal{D}_X \)-module \( \mathcal{F} \)
and right $\mathcal{D}_X$-module $\mathcal{G}$, $\mathcal{F} \otimes_{\mathcal{O}_X} \omega_X$ is a right $\mathcal{D}_X$-module and $\text{Hom}_{\mathcal{O}_X}(\omega_X, \mathcal{G})$ is a (left) $\mathcal{D}_X$-module. (See [Borel].)

We have the following notions of inverse image and direct image of a $\mathcal{D}_X$-module:

**Definition 3.15** Let $X$ and $Y$ be varieties and $f : X \to Y$ a morphism. For any $\mathcal{D}_Y$-module $\mathcal{G}$, we have the inverse image $\mathcal{O}_X$-module $f^*(\mathcal{G})$, which is quasi-coherent. There is a natural $\mathcal{D}_X$-module structure on $f^*(\mathcal{G})$ (see [Borel]), which we call the inverse image $\mathcal{D}_X$-module and write as $f^!(\mathcal{G})$.

Now, to define the direct image, we begin with the $\mathcal{O}_X$-module $f_*(\mathcal{D}_Y)$, which has a natural (left) $\mathcal{D}_X$-module structure, as above. On the other hand, the right multiplication on $\mathcal{D}_Y$ carries over to a right $f^{-1}(\mathcal{D}_Y)$-module structure on $f_*(\mathcal{D}_Y)$. We denote $f_*(\mathcal{D}_Y)$ by $\mathcal{D}_X \to Y$ when it is equipped with this $\mathcal{D}_X \times f^{-1}(\mathcal{D}_Y)$-module structure. Let $\mathcal{D}_{Y \leftarrow X} = \mathcal{D}_X \to Y \otimes_{\mathcal{O}_X} \text{Hom}_{\mathcal{O}_X}(f^*(\omega_Y), \omega_X)$. Then $\mathcal{D}_{Y \leftarrow X}$ is a (left) $f^{-1}(\mathcal{D}_Y)$-module and a right $\mathcal{D}_X$-module. Let $\mathcal{F}$ be a (left) $\mathcal{D}_X$-module. Then $\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{F}$ (defined in the obvious way) is a (left) $f^{-1}(\mathcal{D}_Y)$-module. We also have a canonical morphism $\tilde{f} : \mathcal{D}_Y \to f_*(f^{-1}(\mathcal{D}_Y))$ of sheaves of rings (recall Definition 3.7). We define the direct image $\mathcal{D}_Y$-module $f_*(\mathcal{F})$ to be $f_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{F})$, viewed as a $\mathcal{D}_Y$-module via $\tilde{f}$ (cf. the definition of direct image in Definition 3.11).

**Example 3.16** Let $X$ be a nonempty open subset of $\mathbb{C}$, $Y = X \times X$, $f = \Delta : X \to Y = X \times X$ the diagonal map, defined by $\Delta(z) = (z, z)$, and $\mathcal{F}$ a $\mathcal{D}_X$-module. Note that $X$ is necessarily affine, and that for a local theory over curves, such varieties $X$ are enough. In this case, $\omega_X(X)$ and $\omega_{X \times X}(X \times X)$ are generated by $dz$ and $dz_1 \wedge dz_2$, respectively, and

$$(\text{Hom}_{\mathcal{O}_X}(\Delta^*(\omega_{X \times X}), \omega_X))(X) = \text{Hom}_{\mathcal{O}_X(X)}(\Delta^*(\omega_{X \times X}(X \times X)), \omega_X(X))$$

is generated by $\phi \in \text{Hom}_{\mathcal{O}_X(X)}((\Delta^*(\omega_{X \times X}(X \times X)), \omega_X(X))$ defined by

$$\phi(\Delta^*(dz_1 \wedge dz_2)) = dz.$$

Thus $(\Delta_*(\mathcal{F}))(X \times X)$ is generated by elements of the form

$$\Delta_*(\Delta^*(1) \otimes_{\mathcal{O}_X(X)} \phi) \otimes_{\mathcal{D}_X(X)} A$$

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for $A \in \mathcal{F}(X)$. The element of $\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$ of $\mathcal{D}_{\times X}(X \times X)$ is in fact in $(\Delta_*(\Delta^{-1}(\mathcal{D}_{X \times X}))(X \times X)$ and from the definition, we have

$$
\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right) \Delta_*(\Delta^*(\xi \otimes_{\mathcal{O}_X(X)} \phi) \otimes_{\mathcal{D}_X(X)} A)
= \Delta_*(\Delta^*(\xi \otimes_{\mathcal{O}_X(X)} \phi) \otimes_{\mathcal{D}_X(X)} A)
= \Delta_*(\Delta^*(\xi \otimes_{\mathcal{O}_X(X)} \phi) \otimes_{\mathcal{D}_X(X)} \frac{\partial}{\partial z_1} A)
$$

for any $\xi \in \mathcal{D}_{\times X}(X \times X)$ commuting with $\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$ and $A \in \mathcal{F}(X)$, so that the elements

$$
\frac{\partial^n}{\partial z_1^n} \Delta_*(\Delta^*(1) \otimes_{\mathcal{O}_X(X)} \phi) \otimes_{\mathcal{D}_X(X)} A), \ n \in \mathbb{Z},
$$

span $(\Delta_*(\mathcal{F}))(X \times X)$. Note that since $X$ is an open subset of $\mathbb{C}$, $(\Delta_*(\mathcal{D}_X))(X \times X)$ can be embedded into $\mathcal{D}_{\times X}(X \times X)$ such that $\Delta_*(\frac{\partial}{\partial z})$ is mapped to $\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}$. Thus we have a $\mathcal{D}_{\times X}(X \times X)$-module

$$
\mathcal{D}_{\times X}(X \times X) \otimes_{(\Delta_*(\mathcal{D}_X))(X \times X)} (\Delta_*(\mathcal{F}))(X \times X),
$$

and if $\{A_\alpha\}$ is a basis of $\mathcal{F}(X)$, this $\mathcal{D}_{\times X}(X \times X)$-module has as a basis $\frac{\partial^n}{\partial z_1^n} \otimes_{(\Delta_*(\mathcal{D}_X))(X \times X)} A_\alpha$ for $n \in \mathbb{N}$. It is easy to see that

$$
\frac{\partial^n}{\partial z_1^n} \Delta_*(\Delta^*(1) \otimes_{\mathcal{O}_X(X)} \phi) \otimes_{\mathcal{D}_X(X)} A) \mapsto \frac{\partial^n}{\partial z_1^n} \otimes_{(\Delta_*(\mathcal{D}_X))(X \times X)} A
$$

gives an isomorphism from $(\Delta_*(\mathcal{F}))(X \times X)$ to

$$
\mathcal{D}_{\times X}(X \times X) \otimes_{(\Delta_*(\mathcal{D}_X))(X \times X)} (\Delta_*(\mathcal{F}))(X \times X).
$$

Thus $(\Delta_*(\mathcal{F}))(X \times X)$ has as a basis the elements of the form

$$
\frac{\partial^n}{\partial z_1^n} \Delta_*(\Delta^*(1) \otimes_{\mathcal{O}_X(X)} \phi) \otimes_{\mathcal{D}_X(X)} A_\alpha)
$$

for $n \in \mathbb{N}$, and we can identify the $\mathcal{D}_{\times X}$-module $\Delta_*(\mathcal{F})$ with $\mathcal{D}_{\times X} \otimes_{\Delta_*(\mathcal{D}_X)} \Delta_*(\mathcal{F})$. Now let us take $\mathcal{F} = \mathcal{O}_X$. Then we have the element

$$
\Delta_*(\Delta^*(1) \otimes_{\mathcal{O}_X(X)} \phi) \otimes_{\mathcal{D}_X(X)} 1).
$$

(3.1)
For any $f(z_1, z_2) \in \mathcal{O}_{X \times X}(X \times X)$, we have

$$f(z_1, z_2) \Delta_*((\Delta^*(1) \otimes \mathcal{O}_X(\phi) \otimes \mathcal{D}_X(1))$$

$$= \Delta_*((f(z, z)(\Delta^*(1) \otimes \mathcal{O}_X(\phi) \otimes \mathcal{D}_X(1))))$$

$$= f(z_1, z_2) \Delta_*((\Delta^*(1) \otimes \mathcal{O}_X(\phi) \otimes \mathcal{D}_X(1)). \quad (3.2)$$

Similarly, the left-hand side of (3.2) is also equal to

$$f(z_2, z_2) \Delta_*((\Delta^*(1) \otimes \mathcal{O}_X(\phi) \otimes \mathcal{D}_X(1)).$$

So we see that the element (3.1) has the property similar to the property (2.1) of the $\delta$-function $x^{-1} \delta (x_1 x_2)$. In fact it is easy to show that the derivatives

$$\frac{\partial^m}{\partial z_1^m} \Delta_*((\Delta^*(1) \otimes \mathcal{O}_X(\phi) \otimes \mathcal{D}_X(1)),$$

$m \in \mathbb{N}$, and

$$\frac{\partial^n}{\partial z_2^n} \Delta_*((\Delta^*(1) \otimes \mathcal{O}_X(\phi) \otimes \mathcal{D}_X(1)),$$

$n \in \mathbb{N}$, also have the corresponding properties of the derivatives of $x^{-1} \delta (\frac{x_1}{x_2})$ (see [FLM], Proposition 8.2.2). Thus we can identify (3.1) with $x^{-1} \delta (\frac{x_1}{x_2})$. In particular, we obtain a $\mathcal{D}_{X \times X}$-module generated by $x^{-1} \delta (\frac{x_1}{x_2})$ which is isomorphic to $\Delta_0(\mathcal{O}_X)$.

**Remark 3.17** In the example above, we have given a $\mathcal{D}$-module-theoretic interpretation of the $\delta$-function $x^{-1} \delta (\frac{x_1}{x_2})$ and its derivatives. But in the formal variable approach to vertex algebras, it is most natural to use $x^{-1} \delta (\frac{x_1-z_0}{x_2})$ (see above), which is in fact a formal infinite linear combination of derivatives of $x^{-1} \delta (\frac{x_1}{x_2})$, by a formal Taylor’s theorem (see e.g. [FLM], Propositions 8.2.2 and 8.3.1). This $\delta$-function and the other two in the Jacobi identity for a vertex algebra apparently do not have direct interpretations in terms of $\mathcal{D}$-modules. The $S_3$-symmetry of (2.3) and of the Jacobi identity for vertex algebras (see [FLM], [FHL]) involves all the three formal variables $x_0, x_1, x_2$ on equal footing, and we note that the $\mathcal{D}$-module approach does not naturally have three variables on equal footing.
We also need the concept of exterior tensor product:

**Definition 3.18** Let \(X\) and \(Y\) be varieties and \(\mathcal{F}\) and \(\mathcal{G}\) \(\mathcal{D}_X\)- and \(\mathcal{D}_Y\)-modules, respectively. Let \((\mathcal{F} \boxtimes \mathcal{G})(U \times V) = \mathcal{F}(U) \otimes \mathcal{G}(V)\) for all open affine subsets \(U \subset X\) and \(V \subset Y\). Then there is a unique \(\mathcal{D}_{X \times Y}\)-module \(\mathcal{F} \boxtimes \mathcal{G}\) such that \((\mathcal{F} \boxtimes \mathcal{G})(U_1 \times U_2)\) is the \(\mathcal{O}_{X \times Y}(U \times V)\)-module of sections on \(U \times V\) for any open affine subsets \(U \subset X\) and \(V \subset Y\). We call this \(\mathcal{D}_{X \times Y}\)-module the **exterior tensor product of** \(\mathcal{F}\) and \(\mathcal{G}\). From the definition, we see that \(\mathcal{F} \boxtimes \mathcal{G}\) is determined by sections on open affine subsets of the form \(U \times V\).

### 4 Beilinson-Drinfeld’s chiral algebra

We now give Beilinson-Drinfeld’s definition of chiral algebra. For simplicity and for the purpose of comparing it with the definition of vertex algebra without vacuum, we only give the definition of chiral algebra over a nonempty open subset \(X \subset \mathbb{C}\). But since the definition is sheaf-theoretic and thus is local in nature, this definition carries over naturally to the general case. The axioms in the definition are natural in the language of quasi-tensor categories in Beilinson-Drinfeld’s formulation.

Let \(X \subset \mathbb{C}\) be a nonempty open subset. Let \(F^2\) be the complement of the diagonal in \(X \times X\). We denote the embedding map from \(F^2\) to \(X \times X\) by \(j\) and the diagonal map from \(X\) to \(X \times X\) by \(\Delta\). Let

\[
F^3 = \{(z_1, z_2, z_3) \in X \times X \times X \mid z_k \neq z_l \text{ for } k \neq l, k, l = 1, 2, 3\}.
\]

We denote the embedding map from \(F^3\) to \(X \times X \times X\) by \(j_3\) and the diagonal map from \(X\) to \(X \times X \times X\) by \(\Delta_3\).

Let \(\mathcal{A}\) be a \(\mathcal{D}_X\)-module. The exterior tensor product \(\mathcal{D}_{X \times X}\)-module \(\mathcal{A} \boxtimes \mathcal{A}\) is by definition determined by

\[
U_1 \times U_2 \subset X \times X
\]

\[
\mapsto \{\sum_{i=1}^n f_i(A_i \otimes B_i) \mid A_i \in \mathcal{A}(U_1), B_i \in \mathcal{A}(U_2), f_i \in \mathcal{O}(U_1 \times U_2) \text{ for } i = 1, \ldots, n\}
\]

for all open subsets \(U_1, U_2 \subset X\). The inverse image \(\mathcal{O}_{F^2}\)-module \(j^*(\mathcal{A} \boxtimes \mathcal{A})\) is the \(\mathcal{O}_{F^2}\)-module obtained by restricting the sections of \(\mathcal{A} \boxtimes \mathcal{A}\) over open
subsets of $X \times X$ to open subsets of $F^2$, and the direct image $\mathcal{O}_{X \times X}$-module
$j_* j^*(A \boxtimes A)$ is the $\mathcal{O}_{X \times X}$-module determined by

$$U_1 \times U_2 \subset X \times X$$

$$\mapsto \{ \sum_{i=1}^{n} f_i(A_i \otimes B_i) \mid A_i \in \mathcal{A}(U_1), B_i \in \mathcal{A}(U_2), f_i \in \mathcal{O}((U_1 \times U_2) \cap F^2) \text{ for } i = 1, \ldots, n \}$$

for all open subsets $U_1, U_2 \subset X$. It has a natural $\mathcal{D}_{X \times X}$-module structure. As an
$\mathcal{O}_{X \times X}(U_1 \times U_2)$-module, $j_* j^*(A \boxtimes A)(U_1 \times U_2)$ is generated by the elements of the form

$$(z_1 - z_2)^m(A_1 \otimes A_2)$$

for $m < 0$, $A_1 \in \mathcal{A}(U_1)$ and $A_2 \in \mathcal{A}(U_2)$. Similarly, if $U = U_1 \times U_2 \times U_3$
where $U_1, U_2, U_3$ are open subsets of $X$, then $((j_3)_* j_3^*(A \boxtimes A \boxtimes A))(U)$ as an
$\mathcal{O}_{X \times X \times X}(U)$-module is generated by the elements of the form

$$(z_1 - z_2)^{m_1}(z_2 - z_3)^{m_2}(z_1 - z_3)^{m_3}(A_1 \otimes A_2 \otimes A_3)$$

for $m_i < 0$, $A_i \in \mathcal{A}(U_i)$, $i = 1, 2, 3$. This has a natural $\mathcal{D}_{X \times X \times X}(U)$-module
structure, and so we have the $\mathcal{D}_{X \times X \times X}$-module $(j_3)_* j_3^*(A \boxtimes A \boxtimes A)$.

In the definition of Lie algebra, we need compositions $[\cdot, \cdot, \cdot]$ and $[\cdot, [\cdot, \cdot]]$ of
the Lie bracket $[\cdot, \cdot]$ to formulate the Jacobi identity. Here we need analogues of these compositions.

For any morphism $\mu : j_* j^*(A \boxtimes A) \rightarrow \Delta_0 A$ of $\mathcal{D}_{X \times X}$-modules, we define
a natural morphism

$$\mu(\mu(\cdot, \cdot), \cdot) : (j_3)_* j_3^*(A \boxtimes A \boxtimes A) \rightarrow (\Delta_3)_0(A)$$

of $\mathcal{D}_{X \times X \times X}$-modules as follows: Let $U$ be an open subset of $X \times X \times X$
of the form $U_1 \times U_2 \times U_3$ where $U_1, U_2$ and $U_3$ are open subsets of $X$. We identify $\Delta_0(A)$
with $\mathcal{D}_{X \times X} \otimes (\Delta_*(\mathcal{D}_X)) \Delta_*(A)$ (see Example 3.16). In particular,
$(\Delta_0(A))(U_1 \times U_2)$ is spanned by the elements of the form

$$\frac{\partial^n}{\partial z_1^n} \otimes (\Delta_*(\mathcal{D}_X))(U_1) \Delta_*(A)$$

for $n \in \mathbb{N}$ and $A \in \mathcal{A}(\Delta^{-1}(U_1 \times U_2))$. Thus we see that for any $n_1 \in \mathbb{Z}$ there
exist $p_{n_1} \in \mathbb{N}$ and

$$B_k^{n_1} \in \mathcal{A}(\Delta^{-1}(U_1 \times U_2)) = \mathcal{A}(U_1 \cap U_2), \ k = 0, \ldots, p_{n_1},$$

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such that
\[
\mu((z_1 - z_2)^{n_1}(A_1 \otimes A_2)) = \sum_{k=0}^{p_{n_1}} \frac{\partial^k}{\partial z_1^k} \otimes (\Delta_*(\mathcal{D}_X))(U_1 \times U_2) \Delta_*(B_k^{n_1}); \quad (4.1)
\]
The nonzero elements $B_k^{n_1}$ are uniquely determined. Note that when $n_1 \geq 0$, $(z_1 - z_2)^{n_1}$ is regular, so that when $n_1 > p_0$, we have
\[
\mu((z_1 - z_2)^{n_1}(A_1 \otimes A_2)) =
= (z_1 - z_2)^{n_1} \mu(A_1 \otimes A_2)
= (z_1 - z_2)^{n_1} \sum_{k=0}^{p_0} \frac{\partial^k}{\partial z_1^k} \otimes (\Delta_*(\mathcal{D}_X))(U_1 \times U_2) \Delta_*(B_k^0)
= 0.
\]
Similarly, for any $n_2 \in \mathbb{Z}$ we have, for the elements $B_k^{n_1}$,
\[
\mu((z_2 - z_3)^{n_2}(B_k^{n_1} \otimes A_3)) = \sum_{l=0}^{q_{n_1,n_2}} \frac{\partial^l}{\partial z_3^l} \otimes (\Delta_*(\mathcal{D}_X))(U_2 \times U_3) \Delta_*(C_{kl}^{n_1,n_2}) \quad (4.2)
\]
where $q_{n_1,n_2} \in \mathbb{N}$ and $C_{kl}^{n_1,n_2} \in \mathcal{A}(\Delta^{-1}((U_1 \cap U_2) \times U_3) = \mathcal{A}(U_1 \cap U_2 \cap U_3)$, $k = 0, \ldots, p_{n_1}$, $l = 0, \ldots, q_{n_1,n_2}$. Since $\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \in (\Delta_*(\mathcal{D}_X))(U_2 \times U_3)$ and its preimage in $D_X(U_2 \cap U_3)$ is $\frac{\partial}{\partial z}$, the right-hand side of (4.2) is equal to
\[
\sum_{l=0}^{q_{n_1,n_2}} \left( \left( \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \right) - \frac{\partial}{\partial z_3} \right)^l \otimes (\Delta_*(\mathcal{D}_X))(U_2 \times U_3) \Delta_*(C_{kl}^{n_1,n_2})
= \sum_{l=0}^{q_{n_1,n_2}} \sum_{j=0}^{l} (-1)^j \binom{l}{j} \frac{\partial^j}{\partial z_3^j} \left( \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \right)^{l-j} \otimes (\Delta_*(\mathcal{D}_X))(U_2 \times U_3) \Delta_*(C_{kl}^{n_1,n_2})
= \sum_{l=0}^{q_{n_1,n_2}} \sum_{j=0}^{l} (-1)^j \binom{l}{j} \frac{\partial^j}{\partial z_3^j} \otimes (\Delta_*(\mathcal{D}_X))(U_2 \times U_3) \Delta_* \left( \frac{\partial^{l-j}}{\partial z_2^{l-j}} C_{kl}^{n_1,n_2} \right). \quad (4.3)
\]
Let $C_{kl}^{n_1,n_2} = 0$ for $k > p_{n_1}$ or $l > q_{n_1,n_2}$. As in Example 3.19, $(\Delta_3)_0(\mathcal{A})$ is canonically isomorphic to $D_{X \times X \times X} \otimes (\Delta_3)_*(\mathcal{D}_X) \mathcal{A}$, and we shall identify $(\Delta_3)_0(\mathcal{A})$ with $D_{X \times X \times X} \otimes (\Delta_3)_*(\mathcal{D}_X) \mathcal{A}$. We define
\[
(\mu(\mu(\cdot,\cdot),\cdot))((z_1 - z_2)^{m_1}(z_2 - z_3)^{m_2}(z_1 - z_3)^{m_3}(A_1 \otimes A_2 \otimes A_3))
= \sum_{i \in \mathbb{N}} \sum_{k,l \in \mathbb{N}} \sum_{j=0}^{l} \binom{m_3}{i} (-1)^j \binom{l}{j} \frac{\partial^k}{\partial z_1^k} \frac{\partial^j}{\partial z_3^j} \otimes ((\Delta_3)_*(\mathcal{D}_X))(U) \Delta_3 \left( \frac{\partial^{l-j}}{\partial z_2^{l-j}} C_{kl}^{m_1+m_2+m_3-i} \right), \quad (4.4)
\]
using the expansion of
\[(z_1 - z_3)^{m_3} = ((z_2 - z_3) + (z_1 - z_2))^{m_3}\]
in nonnegative powers of \(z_1 - z_2\). Note that the right-hand side of (4.4) is
in fact a finite sum and is indeed an element of \(((\Delta_3)_*(A))(U)\). It is easy
to verify directly that (4.4) indeed gives a morphism of \(D_X \times X \times X\)-modules.
Thus we obtain the morphism we want.

The morphism given by (4.4) is natural but we need another express- 
on of the right-hand side of (4.4) in Section 5. Note that \(\partial_1 + \partial_2 + \partial_3 \in \((\Delta_3)_*(D_X))(U)\) and its preimage in \(D_X(U_1 \cap U_2 \cap U_3)\) is \(\partial_1\). So the right-
hand side of (4.4) is equal to

\[
\sum_{i \in \N} \sum_{k,l \in \N} \frac{m_3}{i} (-1)^j \binom{l}{j} \left( \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3} \right)_{-j}^{l-j} \otimes((\Delta_3)_*(D_X))(U) (\Delta_3)_*(C_{kl}^{m_1+i,m_2+m_3-i}).
\]  

(4.5)

From
\[
\mu((z_1 - z_2)^n(A_1 \otimes A_2)) = (z_1 - z_2)\mu((z_1 - z_2)^n(A_1 \otimes A_2))
\]  

(4.6)
and (4.1), we obtain

\[
\sum_{k=0}^{p_a+i+1} \frac{\partial^k}{\partial z_1^k} \otimes((\Delta_3)_*(D_X))(U_1 \times U_2) \Delta_*(B_{k+1}^{n+1})
\]

\[
= (z_1 - z_2) \sum_{k=0}^{p_a+i} \frac{\partial^k}{\partial z_1^k} \otimes((\Delta_3)_*(D_X))(U_1 \times U_2) \Delta_*(B_{k}^{n+1}).
\]  

(4.7)

Since for any \(k \in \N\),

\[
\frac{\partial^k}{\partial z_1^k} (z_1 - z_2) \otimes((\Delta_3)_*(D_X))(U_1 \times U_2) \Delta_*(B_{m_1}^{n+1}) = 0,
\]  

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the right-hand side of (4.7) is equal to
\[
- \sum_{k=0}^{p_{n_1}} k \frac{\partial^{k-1}}{\partial z_{1}^{k-1}} \otimes (\Delta_*(\mathcal{D}_X))(U_1 \times U_2) \Delta_*(B_k^{n_1}).
\] (4.8)

Note that for any \( A_i \in \mathcal{A}(U_1 \cap U_2), m_i \in \mathbb{N}, i = 1, \ldots, n, \)
\[
\frac{\partial^{m_i}}{\partial z_i^{m_i}} \otimes (\Delta_*(\mathcal{D}_X))(U_1 \times U_2) \Delta_*(A_i), \ i = 1, \ldots, n
\]
are linearly independent when \( A_i \neq 0 \) for \( i = 1, \ldots, n \) and \( m_i \neq m_j \) for \( i \neq j, \)
\( i, j = 1, \ldots, n, \) and they are equal to 0 if and only if \( A_i = 0, i = 1, \ldots, n. \)
Thus from (4.7) and (4.8), we obtain
\[
B_k^{n_1+1} = -(k + 1)B_k^{n_1}
\] (4.9)
for any \( n_1 \in \mathbb{Z}, k = 0, \ldots, p_{n_1} - 1. \) Similarly we have
\[
C_{kl}^{m_1+1,n_2+1} = -(k + 1)C_{k+l+1}^{m_1,n_2+1}
= -(l + 1)C_{k+l+1}^{m_1+1,n_2}.
\] (4.10)

Using (4.10) repeatedly, we can write the right-hand side of (4.3) as
\[
\sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \sum_{k, l \in \mathbb{N}} \left( \begin{array}{c} m_3 \\ i \\ j \\ k \\ l \\
\end{array} \right) \frac{\partial^{k+j}}{\partial z_1^{k+j}} \frac{\partial^{l-j}}{\partial z_2^{l-j}} \otimes ((\Delta_3)_*(\mathcal{D}_X))(U) (\Delta_3)_*(C_{k+j, l-j}^{m_1+1-i-j,m_2+m_3-i+j})
= \sum_{i' \in \mathbb{N}} \sum_{k', l' \in \mathbb{N}} \sum_{0 \leq i, k \leq i'} \left( \begin{array}{c} m_3 \\ i' \\ k' \\
\end{array} \right) \frac{\partial^{k'}}{\partial z_1^{k'}} \frac{\partial^{l'}}{\partial z_2^{l'}} \otimes ((\Delta_3)_*(\mathcal{D}_X))(U) (\Delta_3)_*(C_{k'+l'}^{m_1+1-i'-k',m_2+m_3-i'+k'})
= \sum_{i' \in \mathbb{N}} \sum_{k', l' \in \mathbb{N}} \left( \begin{array}{c} m_3 + k' \\ i' \\
\end{array} \right) \frac{\partial^{k'}}{\partial z_1^{k'}} \frac{\partial^{l'}}{\partial z_2^{l'}} \otimes ((\Delta_3)_*(\mathcal{D}_X))(U) (\Delta_3)_*(C_{k'+l'}^{m_1+1-i'-k',m_2+m_3-i'+k'}),
\] (4.11)
where in the last step, we have used the identity
\[
\sum_{i' \in \mathbb{N}} \sum_{k', l' \in \mathbb{N}} \left( \begin{array}{c} m_3 \\ i \\ k \\ l \\
\end{array} \right) = \left( \begin{array}{c} m_3 + k' \\ i' \\
\end{array} \right).
\]
By (4.4), (4.3) and (1.11), we obtain

\[
(\mu(\cdot, \cdot))(\cdot)\left((z_1 - z_2)^{m_1}(z_2 - z_3)^{m_2}(z_1 - z_3)^{m_3}(A_1 \otimes A_2 \otimes A_3)\right) = \sum_{i \in \mathbb{N}} \sum_{k, l \in \mathbb{N}} \left(\frac{m_3 + k}{i}\right) \frac{\partial^k}{\partial z_1^k} \frac{\partial^l}{\partial z_2^l} \otimes((\Delta_3)_{(D_X)}(U)) (\Delta_3)_* (C^{m_1+i, m_2+m_3+k-i})_k
\]

(4.12)

We also need another morphism (the other “composition”)

\[
\mu(\cdot, \mu(\cdot, \cdot)) : (j_3)_* j_3^*(A \otimes A \otimes A) \rightarrow (\Delta_3)_* A
\]

of \(D_{X \times X \times X}\)-modules defined naturally as follows: Let \(U\) be an open subset of \(X \times X \times X\) of the form \(U_1 \times U_2 \times U_3\) where \(U_1, U_2\) and \(U_3\) are open subsets of \(X\). For any \(n_2 \in \mathbb{Z}\) there exist \(s_{n_2} \in \mathbb{N}\) and

\[
D^m_{l^2} \in A(\Delta^{-1}(U_2 \times U_3)) = A(U_2 \cap U_3), \ l = 0, \ldots, s_{n_2},
\]

such that the element \(\mu((z_2 - z_3)^{n_2}(A_1 \otimes A_2)) \in (\Delta_*(A))(U_1 \times U_2)\) is equal to

\[
\sum_{l=0}^{s_{n_2}} \frac{\partial^l}{\partial z_2^l} \otimes((\Delta_*(D_X))(U_2 \times U_3)) \Delta_*(D^m_{l^2}).
\]

For any \(n_1 \in \mathbb{Z}\), we have

\[
\mu((z_1 - z_3)^{n_1}(A_1 \otimes D^{n_2}_{l^2})) = \sum_{k=0}^{r_{n_1, n_2}} \frac{\partial^k}{\partial z_1^k} \otimes((\Delta_*(D_X))(U_1 \times U_3)) \Delta_*(E^{m_1 n_2}_{k l})
\]

where \(r_{n_1, n_2} \in \mathbb{N}\) and \(E^{m_1 n_2}_{k l} \in A(\Delta^{-1}((U_1 \cap U_2) \times U_3)) = A(U_1 \cap U_2 \cap U_3),\ k = 0, \ldots, r_{n_1, n_2}, \ l = 0, \ldots, s_{n_2},\). Let \(E^{m_1 n_2}_{k l} = 0\) for \(k > r_{n_1, n_2}\) or \(l > s_{n_2}\). We define

\[
(\mu(\cdot, \mu(\cdot, \cdot)))((z_1 - z_2)^{m_1}(z_2 - z_3)^{m_2}(z_1 - z_3)^{m_3}(A_1 \otimes A_2 \otimes A_3)) = \sum_{i \in \mathbb{N}} (-1)^i \left(\begin{array}{c} m_1 \\ i \end{array}\right) \sum_{k, l \in \mathbb{N}} \frac{\partial^k}{\partial z_1^k} \frac{\partial^l}{\partial z_2^l} \otimes((\Delta_3)_*(D_X))(U) (\Delta_3)_*(E^{m_1+m_3-i, m_2+i}_{k l})
\]

(4.13)

using the expansion of

\[
(z_1 - z_2)^{m_1} = ((z_1 - z_3) - (z_2 - z_3))^{m_1}
\]

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in nonnegative powers of \( z_2 - z_3 \).

We also have an action of the symmetric group \( S_2 \) on \( j_* j^*(A \boxtimes A) \) and an action of \( S_3 \) on \( (j_3)_* j^*_3(A \boxtimes A \boxtimes A) \): In fact, the elements of \( S_2 \) act on \( X \times X \), and we have direct image sheaves of \( j_* j^*(A \boxtimes A) \) by these actions. It is easy to see that these direct image sheaves are the same as \( j_* j^*(A \boxtimes A) \), so that we obtain an action of \( S_2 \) on \( j_* j^*(A \boxtimes A) \). Explicitly, this action can be described as follows: For any element \( f(z_1, z_2)(z_1 - z_2)^n(A_1 \otimes A_2) \) of \((j_* j^*(A \boxtimes A))(U_1 \times U_2)\), we define \( \sigma(f(z_1, z_2)(z_1 - z_2)^n(A_1 \otimes A_2)) \) to be the element \( f(z_1, z_2)(z_1 - z_2)^n(A_2 \otimes A_1) \) of \((j_* j^*(A \boxtimes A))(U_2 \times U_1)\). The action of \( S_3 \) on \((j_3)_* j^*_3(A \boxtimes A \boxtimes A)\) is defined similarly.

**Definition 4.1 (Beilinson-Drinfeld [BD])** Let \( X \) be a nonempty open subset of \( \mathbb{C} \). A chiral algebra over \( X \) is a \( \mathcal{D}_X \)-module \( A \) equipped with a morphism \( \mu : j_* j^*(A \boxtimes A) \to \Delta \circ A \) of \( \mathcal{D}_{X \times X} \)-modules satisfying the following axioms:

1. (Skew-symmetry) Let \( \sigma_{12} \) be the nontrivial element of the symmetric group \( S_2 \). Then
   \[ \mu \circ \sigma_{12} = -\mu. \]

2. (Jacobi identity) Let \( \sigma_{12} \) be the element of \( S_3 \) permuting the first two letters. Then
   \[ \mu(\mu(\cdot, \cdot), \cdot) = \mu(\cdot, \mu(\cdot, \cdot)) - \mu(\cdot, \mu(\cdot, \cdot)) \circ \sigma_{12}. \]

5 Vertex algebras without vacuum and chiral algebras in the sense of Beilinson-Drinfeld

In this section, \( X \) is a nonempty open subset of \( \mathbb{C} \). We need the following notion:

**Definition 5.1** A vertex algebra without vacuum over \( X \) is a vertex algebra without vacuum \((V, Y, D)\) equipped with an \( \mathcal{O}(X) \)-module structure on \( V \) such that for any \( f, g \in \mathcal{O}(X) \) and \( u, v \in V \), \( Y(fu, x)gv = fgY(u, x)v \) and \( D(fu) = (\frac{\partial}{\partial z})fu + fDu \).
Any vertex algebra without vacuum tensored with the commutative associative algebra \( \mathcal{O}(X) \), viewed as a vertex algebra (see [Borc]), is a vertex algebra without vacuum over \( X \) (see [Borc], [FHLL]).

Let \((V,Y,D)\) be a vertex algebra without vacuum over \( X \). For any open (necessarily affine) subset \( U \subset X \), let \( A(U) = \mathcal{O}(U) \otimes \mathcal{O}(X) \). We define \( A(u,U) = 1 \otimes \mathcal{O}(X) u \in A(U) \). Note that \( A(X) = \mathcal{O}(X) \otimes \mathcal{O}(X) V = V \) and \( A(u,X) = u \). Since \( \mathcal{O}(X) \) is quasi-coherent, we see that the \( \mathcal{O}(X) \)-module defined by \( U \mapsto A(U) \) for all open subsets \( U \subset X \) is also quasi-coherent.

We define \( \partial/\partial z A(u,U) = A(Du,U) \), so that \( \partial/\partial z \) acts on \( A(U) \). Thus the \( \mathcal{O}(X) \)-module defined by \( U \mapsto A(U) \) for all open subsets \( U \subset X \) is a \( \mathcal{D}(X) \)-module.

We denote this \( \mathcal{D}(X) \)-module by \( A \).

For the vertex algebra \( V \) without vacuum, we write the Jacobi identity in the following form:

\[
(x_1 - x_2)^n Y(u, x_1) Y(v, x_2) - (-x_2 + x_1)^n Y(u, x_2) Y(v, x_1) = \text{Res}_{x_0} x_0^n x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)
\]

for \( u, v \in V, n \in \mathbb{Z} \). We rewrite the right-hand side as

\[
\text{Res}_{x_0} x_0^n e^{-x_0 \partial/\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) Y(Y(u, x_0)v, x_2) = \sum_{m \in \mathbb{N}} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial x_1^m} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) Y(u_{m+n}v, x_2).
\]

We define \( \mu : j_* j^*(\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_*(\mathcal{A}) \) as follows: For any open subset \( U \) of the form \( U_1 \times U_2 \subset X \times X \), \( j_* j^*(\mathcal{A} \boxtimes \mathcal{A})(U) \) is spanned by elements of the form

\[
(z_1 - z_2)^n (A(u, U_1) \otimes A(v, U_2)),
\]

\( u, v \in V, n \in \mathbb{Z} \) (or \( n < 0 \)). We define

\[
\mu((z_1 - z_2)^n (A(u, U_1) \otimes A(v, U_2))) = \sum_{m \in \mathbb{N}} -1^m \frac{1}{m!} \frac{\partial^m}{\partial z_1^m} \otimes (\Delta_*(\mathcal{P}(U))) \Delta_*(A(u_{m+n}v, \Delta^{-1}(U))). \tag{5.1}
\]

It is easy to verify that \( \mu \) is well-defined and is indeed a morphism of \( \mathcal{D}(X \times X) \)-modules.

We have:
Proposition 5.2 The pair $(\mathcal{A}, \mu)$ is a chiral algebra over $X$ in the sense of Definition \[\ref{chiral algebra}.\]

Proof We first verify the skew-symmetry using the skew-symmetry for the vertex algebra without vacuum $V$. By definition, for $U = U_1 \times U_2$ and $u, v \in V$,

\[
(\mu \circ \sigma_{12})((z_1 - z_2)^n(A(u, U_1) \otimes A(v, U_2)))
= \mu((z_1 - z_2)^n(A(v, U_2) \otimes A(u, U_1)))
= (-1)^n \mu((z_2 - z_1)^n(A(v, U_2) \otimes A(u, U_1)))
= (-1)^n \sum_{m \in \mathbb{N}} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial z_2^m} \otimes (\Delta_*(\mathcal{D}_X))(U) \Delta_*(A(v_{m+n}u, \Delta^{-1}(\sigma_{12}(U))))
\]

\[
= \sum_{m \in \mathbb{N}} \frac{(-1)^{m+n}}{m!} \frac{\partial^m}{\partial z_2^m} \otimes (\Delta_*(\mathcal{D}_X))(U) \Delta_*(A(v_{m+n}u, \Delta^{-1}(U)))
\]

\[
= \sum_{m \in \mathbb{N}} \sum_{k \in \mathbb{N}} \frac{(-1)^{m+n}}{m!} \left( \begin{array}{c} m \\ k \end{array} \right) \left( \begin{array}{c} m-k \\ k \end{array} \right) \Delta_*(A(v_{m+n}u, \Delta^{-1}(U)))
\]

Since $\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \in (\Delta_*(\mathcal{D}_X))(U)$ and its preimage in

$\mathcal{D}_X(\Delta^{-1}(U)) = \mathcal{D}_X(U_1 \cap U_2)$

is $\frac{\partial}{\partial z}$, the right-hand side of (5.2) is equal to

\[
\sum_{l \in \mathbb{N}} \sum_{k \in \mathbb{N}} \frac{(-1)^{k+n}}{l!k!} \left( \begin{array}{c} \frac{\partial}{\partial z_1} \\
\frac{\partial}{\partial z_2} \end{array} \right)^{l+k} \otimes (\Delta_*(\mathcal{D}_X))(U) \Delta_*(A(v_{m+n}u, \Delta^{-1}(U)))
\]

\[
= \sum_{l \in \mathbb{N}} \sum_{k \in \mathbb{N}} \frac{(-1)^{k+n}}{l!k!} \left( \begin{array}{c} \frac{\partial}{\partial z_1} \\
\frac{\partial}{\partial z_2} \end{array} \right)^{l+k} \otimes (\Delta_*(\mathcal{D}_X))(U) \Delta_*(A(D^k v_{m+n}u, \Delta^{-1}(U)))
\]

\[22\]
\[- \sum_{l \in \mathbb{N}} \frac{(-1)^l}{l!} \left( \frac{\partial}{\partial z_1} \right)^l \otimes (\Delta_+(\mathcal{D}X))(U) \Delta_+ \left( A \left( \sum_{k \in \mathbb{N}} \frac{(-1)^{k+l+n+1}}{k!} D_k v_{k+l+n}, \Delta^{-1}(U) \right) \right). \]  

(5.3)

But from the skew-symmetry (2.4), we obtain its component form

\[ \sum_{k \in \mathbb{N}} \frac{(-1)^{k+m+1}}{k!} D^k v_{k+m} u = u_m v \]

for all \( m \in \mathbb{Z} \). Thus the right-hand side of (5.3) is equal to

\[- \sum_{l \in \mathbb{N}} \frac{(-1)^l}{l!} \left( \frac{\partial}{\partial z_1} \right)^l \otimes (\Delta_+(\mathcal{D}X))(U) \Delta_+ \left( A \left( u_{l+n}, \Delta^{-1}(U) \right) \right). \]  

(5.4)

On the other hand,

\[ (-\mu)((z_1 - z_2)^n A(u, U_1) \otimes A(v, U_2)) \]

\[ = -\mu((z_1 - z_2)^n A(u, U_1) \otimes A(v, U_2)) \]

\[ = - \sum_{m \in \mathbb{N}} \frac{(-1)^m}{m!} \frac{\partial^m}{\partial z_1^m} \otimes (\Delta_+(\mathcal{D}X))(U) \Delta_+ \left( A(u_{m+n}, \Delta^{-1}(U)) \right). \]  

(5.5)

We see that the right-hand side of (5.5) is equal to (5.4). By the calculation from (5.3) to (5.4), we see that the left-hand side of (5.1) and the left-hand side of (5.4) are equal, proving the skew-symmetry.

Next we prove the Jacobi identity using the Jacobi identity for vertex algebra without vacuum. By (5.1) and (4.13), for any open subset \( U \) of the form \( U_1 \times U_2 \times U_3 \subset X \times X \times X \),

\[ (\mu(\cdot, \mu(\cdot, \cdot)))((z_1 - z_2)^{m_1} (z_2 - z_3)^{m_2} (z_1 - z_3)^{m_3} \cdot (A(u, U_1) \otimes A(v, U_2) \otimes A(w, U_3))) \]

\[ = \sum_{i \in \mathbb{N}} (-1)^i \left( \begin{array}{c} m_1 \\ i \end{array} \right) \sum_{k, l \in \mathbb{N}} \frac{(-1)^{k+l}}{k!l!} \frac{\partial^k}{\partial z_1^k} \frac{\partial^l}{\partial z_2^l} \]

\[ \otimes ((\Delta_3)_+(\mathcal{D}X))(U)(\Delta_3)_+ \left( A(u_{m_1+m_3-i-k}, v_{m_2+i+l}, \Delta^{-1}_3(U)) \right). \]
\[
= \sum_{k,l \in \mathbb{N}} \frac{(-1)^{k+l}}{k!l!} \frac{\partial^k}{\partial z_1^k} \frac{\partial^l}{\partial z_2^l} \otimes ((\Delta_3)_*(\mathcal{D}_X))(U)(\Delta_3)_* \left( A \left( \sum_{i \in \mathbb{N}} (-1)^i \binom{m_1}{i} \cdot u_{m_1+m_3-i+k} v_{m_2+i+l} w, \Delta_3^{-1}(U) \right) \right).
\]

Similarly, we have

\[
(\mu(\cdot, \mu(\cdot, \cdot)) \circ \sigma_{12})((z_1 - z_2)^{m_1}(z_2 - z_3)^{m_2}(z_1 - z_3)^{m_3} \cdot \cdot (A(u, U_1) \otimes A(v, U_2) \otimes A(w, U_3)))
\]

\[
= \sum_{k,l \in \mathbb{N}} \frac{(-1)^{k+l}}{k!l!} \frac{\partial^k}{\partial z_1^k} \frac{\partial^l}{\partial z_2^l} \otimes ((\Delta_3)_*(\mathcal{D}_X))(U)(\Delta_3)_* \left( A \left( (-1)^{m_1} \sum_{i \in \mathbb{N}} (-1)^i \binom{m_1}{i} \cdot v_{m_1+m_2-i+l} u_{m_3+i+k} w, \Delta_3^{-1}(U) \right) \right).
\]

On the other hand, by (5.11) and (4.12), we have

\[
(\mu(\mu(\cdot, \cdot), \cdot))((z_1 - z_2)^{m_1}(z_2 - z_3)^{m_2}(z_1 - z_3)^{m_3} \cdot \cdot (A(u, U_1) \otimes A(v, U_2) \otimes A(w, U_3)))
\]

\[
= \sum_{k,l \in \mathbb{N}} \frac{(-1)^{k+l}}{k!l!} \frac{\partial^k}{\partial z_1^k} \frac{\partial^l}{\partial z_2^l} \otimes ((\Delta_3)_*(\mathcal{D}_X))(U)(\Delta_3)_* \left( A \left( \sum_{i \in \mathbb{N}} \binom{m_3+k}{i} \cdot (u_{m_1+i} v)_{m_3+m_2-i+k+l} w, \Delta_3^{-1}(U) \right) \right).
\]

From the Jacobi identity for vertex algebras without vacuum, we can obtain its component form

\[
\sum_{i \in \mathbb{N}} \binom{m}{i} (u_{t+i} v)_{m+n-i} w =
\]

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for all \( l, m, n \in \mathbb{Z} \) and \( u, v, w \in V \) (see \([FLM]\)). From (5.9), we obtain

\[
\sum_{i \in \mathbb{N}} \binom{m_3 + k}{i} (u_{m_1 + i}v)_{m_3 + m_2 - i + k + l}w = \sum_{i \in \mathbb{N}} (-1)^i \binom{m_1}{i} u_{m_1 + m_3 - i + k}v_{m_2 + i + l}w
\]

\[
-(-1)^i \sum_{i \in \mathbb{N}} (-1)^i \binom{m_1}{i} v_{m_1 + m_2 - i + l}u_{m_3 + i + k}w
\] (5.10)

for any \( m_1, m_2, m_3 \in \mathbb{Z} \), \( k, l \in \mathbb{N} \) and \( u, v, w \in V \). By (5.6)–(5.10), we obtain

\[
\begin{align*}
(\mu(\mu(\cdot, \cdot), \cdot))((z_1 - z_2)^{m_1}(z_2 - z_3)^{m_2}(z_1 - z_3)^{m_3}.
& \cdot (A(u, U_1) \otimes A(v, U_2) \otimes A(w, U_3))) \\
& = (\mu(\cdot, \mu(\cdot, \cdot))((z_1 - z_2)^{m_1}(z_2 - z_3)^{m_2}(z_1 - z_3)^{m_3}.
& \cdot (A(u, U_1) \otimes A(v, U_2) \otimes A(w, U_3))) \\
& - (\mu(\cdot, \mu(\cdot, \cdot)) \circ \sigma_{12})((z_1 - z_2)^{m_1}(z_2 - z_3)^{m_2}(z_1 - z_3)^{m_3}.
& \cdot (A(u, U_1) \otimes A(v, U_2) \otimes A(w, U_3))),
\end{align*}
\]

proving the Jacobi identity for \( A \). \( \square \)

Conversely, suppose that we have a chiral algebra \((A, \mu)\) over a nonempty open subset \( X \) of \( \mathbb{C} \) in the sense of Definition 4.1, and let \( V = A(X) \) be the space of all global sections over \( X \). Then \( V \) is an \( \mathcal{O}(X) \)-module and \( \frac{\partial}{\partial z} \) acts on \( V \). For any element \( v \in V \), we define \( Dv = \frac{\partial}{\partial z}v \). For any \( f \in \mathcal{O}(X) \) and \( u \in V \), we have \( D(fu) = (\frac{\partial}{\partial z}f)u + fDu \). For \( u, v \in V \), there exist \( B^n_m(u, v) \in V, n \in \mathbb{Z}, m = 1, \ldots, p_n \), such that

\[
\mu((z_1 - z_2)^n(u \otimes v)) = \sum_{m=0}^{p_n} \frac{\partial^m}{\partial z_1^m} \otimes (\Delta_*(\mathcal{O}_X))(X \times X) \Delta_*(B^n_m(u, v)),
\] (5.11)

and the nonzero \( B^n_m(u, v) \) are uniquely determined. The equation (1.9) in this case becomes

\[
B^n_{m+1}(u, v) = -(m + 1)B^n_m(u, v)
\] (5.12)
for any \( u, v \in V, n \in \mathbb{Z}\) and \( m = 0, \ldots, p_n - 1 \).

We define \( u_n v = B_0^n(u, v) \) and

\[
Y(u, x)v = \sum_{n \in \mathbb{Z}} B_0^n(u, v)x^{-n-1}.
\]

From the definition, we have \( Y(fu, x)gv = fgY(u, x)v \) for any \( f, g \in \mathcal{O}(X) \) and \( u, v \in V \).

**Proposition 5.3** The triple \((V, Y, D)\) is a vertex algebra without vacuum over \( X \).

**Proof** When \( n > p_0 \), we have

\[
\mu((z_1 - z_2)^n(u \otimes v)) = (z_1 - z_2)^n \mu(u \otimes v) = (z_1 - z_2)^n \sum_{m=0}^{p_n} \frac{\partial^m}{\partial z_1^m} \otimes (\Delta_*(\mathcal{D}_X))(X \times X) \Delta_*(B_0^m(u, v)) = 0.
\]

Thus for \( n > p_0 \), \( u_n v = B_0^n(u, v) = 0 \).

To show the \( D \)-derivative property, we need only show its component form:

\[
-(n + 1)B_0^n(u, v) = B_0^{n+1}(Du, v) \quad (5.13)
\]

for all \( n \in \mathbb{Z} \). In fact, for any \( n \in \mathbb{Z} \),

\[
\frac{\partial}{\partial z_1} \mu((z_1 - z_2)^{n+1}(u \otimes v)) = (n + 1)\mu((z_1 - z_2)^n(u \otimes v)) + \mu((z_1 - z_2)^{n+1} \frac{\partial}{\partial z_1}(u \otimes v)) = (n + 1)\mu((z_1 - z_2)^n(u \otimes v)) + \mu((z_1 - z_2)^{n+1}((Du) \otimes v)). \quad (5.14)
\]

On the other hand,
\[
\frac{\partial}{\partial z_1} \left( \sum_{m=0}^{p_0} \frac{\partial^m}{\partial z_1^m} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(B^n_m(u,v)) \right) \\
= \sum_{m=0}^{p_0} \frac{\partial^{m+1}}{\partial z_1^{m+1}} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(B^n_m(u,v)). \tag{5.15}
\]

From (5.14), (5.15) and the expansions of \(\mu((z_1 - z_2)^n(u \otimes v))\) and \(\mu((z_1 - z_2)^{n+1}((Du) \otimes v))\), we obtain

\[
(n+1) \sum_{m=0}^{p_0} \frac{\partial^m}{\partial z_1^m} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(B^n_m(u,v)) \\
+ \sum_{m=0}^{p_0+1} \frac{\partial^m}{\partial z_1^m} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(B^{n+1}_m(u,v)) \\
= (n+1) \mu((z_1 - z_2)^n(u \otimes v)) + \mu((z_1 - z_2)^{n+1}((Du) \otimes v)) \\
= \sum_{m=0}^{p_0} \frac{\partial^{m+1}}{\partial z_1^{m+1}} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(B^n_m(u,v)). \tag{5.16}
\]

The equality (5.16) implies

\[
(n+1)B^n_0(u,v) + B^{n+1}_0(u,v) = 0
\]

which is equivalent to (5.13).

We now prove that the skew-symmetry for vertex algebras without vacuum is satisfied by \(V\), using the skew-symmetry for \((\mathcal{A}, \mu)\). By (5.11) and the skew-symmetry for \((\mathcal{A}, \mu)\), we obtain

\[
\sum_{m=0}^{p_0} \frac{\partial^m}{\partial z_1^m} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(B^n_m(u,v)) \\
= (-1)^n \sum_{m=0}^{q_0} \frac{\partial^m}{\partial z_2^m} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(B^n_m(v,u)). \tag{5.17}
\]

The right-hand side of (5.17) is equal to

\[
-(-1)^n \sum_{m=0}^{q_0} \left( -\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right)^m \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(B^n_m(v,u)) \\
= (-1)^{n+1} \sum_{m=0}^{q_0} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k}.
\]
\[
\frac{\partial^{m-k}}{\partial z_1^{m-k}} \left( \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right)^k \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(B^n_m(v, u))
\]
\[
= (-1)^{n+1} \sum_{m=0}^{q_n} \sum_{k=0}^{m} (-1)^{m-k} \left( \frac{m}{k} \frac{\partial^{m-k}}{\partial z_1^{m-k}} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(\frac{d^k}{dz^k}B^n_m(v, u)) \right)
\]
\[
= (-1)^{n+1} \sum_{m=0}^{q_n} \sum_{k=0}^{m} (-1)^{m-k} \left( \frac{m}{k} \frac{\partial^{m-k}}{\partial z_1^{m-k}} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(D^k B^n_m(v, u)) \right).
\]

Comparing the left-hand side of (5.17) and the right-hand side of (5.18), the terms involving \( \frac{\partial}{\partial z_1} \) give
\[
B^n_0(u, v) = \sum_{k=0}^{q_n} (-1)^{n+k} D^k B^n_k(v, u).
\]

Using (5.12) repeatedly, we obtain
\[
B^n_k(v, u) = \frac{(-1)^k}{k!} D^k B^n_0(v, u).
\]

Combining (5.19) and (5.20), we obtain
\[
B^n_0(u, v) = \sum_{k=0}^{q_n} \frac{(-1)^{k+n+1}}{k!} D^k B^n_0(v, u).
\]

Since when \( k > q_n \), we have
\[
\mu((z_1 - z_2)^{k+n}(v \otimes u)) =
\]
\[
= (z_1 - z_2)^k \mu((z_1 - z_2)^n(v \otimes u))
\]
\[
= (z_1 - z_2)^k \sum_{m=0}^{q_n} \frac{\partial^m}{\partial z_1^m} \otimes (\Delta_*(\mathcal{D}X))(X \times X) \Delta_*(D^k B^n_m(v, u))
\]
\[
= 0.
\]

Thus \( B^n_0^{k+n}(v, u) = 0 \) when \( k > q_n \). So (5.21) becomes
\[
B^n_0(u, v) = \sum_{k \in \mathbb{N}} \frac{(-1)^{k+n+1}}{k!} D^k B^n_0^{k+n}(v, u),
\]
or equivalently

\[ u_n v = \sum_{k \in \mathbb{N}} \frac{(-1)^{k+n+1}}{k!} D^k v_{k+n} u \]

which is the component form of the skew-symmetry.

Finally, we prove that the Jacobi identity for vertex algebra without vacuum is satisfied by \( V \), using the Jacobi identity for \((A, \mu)\). By (4.12), (4.13) and the Jacobi identity for \((A, \mu)\), we obtain the following identity:

\[
\sum_{i \in \mathbb{N}} \sum_{k,l \in \mathbb{N}} \binom{m_3 + k}{i} \frac{\partial^k}{\partial z_1^k} \frac{\partial^l}{\partial z_2^l} \otimes (\Delta_3)_* (\mathcal{D}_X)(U) \]

\[
\sum_{i \in \mathbb{N}} (-1)^i \binom{m_1}{i} \frac{\partial^k}{\partial z_1^k} \frac{\partial^l}{\partial z_2^l} \otimes (\Delta_3)_* (\mathcal{D}_X)(U) \]

\[
\sum_{i \in \mathbb{N}} (-1)^i \binom{m_1}{i} \frac{\partial^k}{\partial z_1^k} \frac{\partial^l}{\partial z_2^l} \otimes (\Delta_3)_* (\mathcal{D}_X)(U) \]

or equivalently, the component form of the Jacobi identity for vertex algebra without vacuum:

\[
\sum_{i \in \mathbb{N}} \binom{m_3}{i} (u_{m_3 + i} v)_{m_2 + m_3 - i} w = \]

\[
\sum_{i \in \mathbb{N}} (-1)^i \binom{m_1}{i} u_{m_1 + m_3 - i} v_{m_2 + i} w \]

\[
- (-1)^m \sum_{i \in \mathbb{N}} (-1)^i \binom{m_1}{i} v_{m_1 + m_2 - i} u_{m_3 + i} w. \]
Since \((V,Y,D)\) satisfies the lower-truncation condition for vertex operators, the \(D\)-derivative property, the skew-symmetry and the Jacobi identity for vertex algebras without vacuum, it is a vertex algebra without vacuum. We already know that \(V\) is an \(\mathcal{O}(X)\)-module and for any \(f,g \in \mathcal{O}(X), u,v \in V, Y(fu,x)gv = fgY(u,x)v\) and \(D(fu) = (\frac{\partial}{\partial z} f)u + fDu\). So \((V,Y,D)\) is a vertex algebra without vacuum over \(X\).  

From the two propositions above, we obtain the following main result of this paper:

**Theorem 5.4** Let \(X\) be a nonempty open subset of \(\mathbb{C}\). The category of vertex algebras without vacuum over \(X\) and the category of chiral algebras over \(X\) are equivalent.  

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