ON PREMET CONJECTURE FOR FINITE W-SUPERALGEBRAS

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Abstract. Let $\bullet^\dagger$ be the map in sense of the Losev, which sends the set of two sided ideals of a finite W-algebras to that of the universal enveloping algebra of corresponding Lie algebras. The Premet conjecture which was proved in [Lo11], says that, restricted to the set of primitive ideals with finite codimension, any fiber of the map $\bullet^\dagger$ is a single orbit under an action of a finite group. In this article we formulate and prove a similar fact in the super case.

1. Introduction

Let $g = g_0 \oplus g_1$ be a basic Lie superalgebra over an algebraically closed field $\mathbb{K}$, $U$ and $U_0$ be the enveloping algebra of $g$ and $g_0$ respectively. Denote by $(\cdot, \cdot)$ the Killing form on it. Let $e \in g_0$ and $\chi \in g_0^*$ be the corresponding element to $e$ via the the Killing form. Pick an $sl_2$-triple $\{f, h, e\} \subset g_0$ and let $\tilde{W}$ be the extended W superalgebra defined in $\mathcal{A}_\dagger$ §3 [SX] (or §6 [Lo15]). It was obtained in [SX] that there is a following relation among the three kind of W algebras. (1), we have embedding $W_0 \hookrightarrow \tilde{W}$ and the later is generated over the former by $\dim(g_1)$ odd elements. (2), we have decomposition $\tilde{W} = Cl(V_1^\circ) \otimes W$ of associative algebras, where $Cl(V_1^\circ)$ is the Clifford algebra over a vector space $V_1$ with a non-degenerate symmetric two form, see Theorem 2.3 for the details. Essentially, this makes $W_0$ to $\tilde{W}$ as $U_0$ to $U$.

For a given associative algebra $A$, denote by $id(A)$ the set of two sided ideals of $A$ and by $Prim^{fin}(A)$ the set of primitive ideals of $A$ with finite codimension in $A$. It is well known that $Prim^{fin}(A)$ is bijective with the set $Ir^{fin}(A)$ of isomorphism classes of finite dimensional irreducible $A$ modules. In [Lo10] Losev constructed a ascending map $\bullet^\dagger : id(W_0) \rightarrow id(U_0)$ and descending map $\bullet^\downarrow : id(U_0) \rightarrow id(W_0)$. Those two maps are crucial to his study for representations of $W_0$. The ascending map $\bullet^\dagger$ sends $Prim^{fin}(W_0)$ to the set $Prim_0(U_0)$ of primitive ideals of $U_0$ supported on the Zariski closure of the adjoint orbit $O = G_0 \cdot e$. Denote by $Q = Z_{G_0}\{e, h, f\}$ the stabilizer of the triple $\{e, h, f\}$ in $G_0$ under the adjoint action. Let $C_e = Q/Q^\circ$, where $Q^\circ$ is the identity component of $Q$. The Premet conjecture which was proved in [Lo11], is saying that for any $I \in Prim_0(U_0)$ the set $\{J \mid J \in Prim^{fin}(W) \quad and \quad J^\dagger = J\}$ is a single $C_e$ orbit. This gives an almost complete classification of $Ir^{fin}(W_0)$.

In this paper we generalize the above fact to the super case. The super analogue of the maps $\bullet^\dagger$ and $\bullet^\downarrow$ were established in [SX]. By abuse of notation, we also
denote it by $\mathbf{\cdot}^*$ and $\mathbf{\cdot}$ from now on. Denote by $\text{Prim}_0(\mathcal{U})$ the set of primitive ideals of $\mathcal{U}$ supported on the Zariski closure of the adjoint orbit $\mathcal{O} = G_0 \cdot e$, see §2 for the precise definition of term ‘supported’ in the super context. In §2 we will construct an action of $Q$ on $\mathcal{W}$ with a property that $Q^g$ leaves any two sided ideal of $\mathcal{W}$ stable, see Proposition 2.4. This provide us an action of $C_e$ on $\mathfrak{i}(\mathcal{W})$. The main result of the present paper is following.

**Theorem 1.1.** For any $\mathcal{J} \in \text{Prim}_0(\mathcal{U})$, the set $\{ \text{Cl}(V_f) \otimes \mathcal{J} \mid \mathcal{J} \in \text{Prim}^\text{fin}(\mathcal{W}) \text{ and } \mathcal{J}^\dagger = \mathcal{J} \}$ is a single $C_e$-orbit.

Our strategy to prove the theorem is that we apply [Theorem 4.1.1 [Lo11]] to the Harish-Chandra bimodule module $\mathcal{U}$ over $\mathcal{U}_0$ and the relation among $\mathcal{W}, \mathcal{W}_0$ and $\mathcal{W}$ obtained in Theorem 3.11 [SX]. Our approach is highly inspired by §6 [Lo15].

Since we can recover $\mathcal{J}$ from $\text{Cl}(V_f) \otimes \mathcal{J}$ by the procedure of Case 1 in the proof of Theorem 1.6 [SX]. It was proved in Theorem 4.8 [SX] that the map $\mathbf{\cdot}^*$ sends $\text{Prim}^\text{fin}(\mathcal{W})$ to $\text{Prim}_0(\mathcal{U})$. Thus Theorem 1.1 almost completely reduced the problem of classification of $\text{Prim}_0^\text{fin}(\mathcal{W}) = \text{Irr}^\text{fin}(\mathcal{W})$ to that of $\text{Prim}(\mathcal{U})$. For the recent studies on the later, see [CM] and [Mu], for example.

## 2. Proof on the main result

We first recall the Poisson geometric realization of finite $W$-(super)algebra in the sense of Losev. Denote by $A_0$ (resp. $\tilde{A}$) the Poisson (resp. super) algebra $S[g_0]$ (resp. $S[g]$) with the standard bracket $\{,\}$ given by $\{x, y\} = [x, y]$ for all $x, y \in g_0$ (resp. $g$). Let $\tilde{A}_0$ (resp. $\tilde{A}$) be the completion of $A_0$ (resp. $\tilde{A}$) with respect to the point $\chi \in g_0^*(\text{resp. } g)$. Denote by $\mathcal{U}_h^0$ (resp. $\mathcal{U}_h^\dagger$) the formal quantization of $\tilde{A}_0$ (resp. $\tilde{A}$) given by $x * y - y * x = \hbar^2 [x, y]$ for all $x, y \in g_0$. Equip all the above algebras the Kazdan $\mathbb{K}^*$ actions arise from the good $\mathbb{Z}$-grading on $g$ and $t : \hbar = \theta \hbar$ for all $t \in \mathbb{K}^*$.

Denote by $\omega$ the super even symplectic form on $[f, g]$ given by $\omega(x, y) = \chi([x, y])$. Let $V = V_0 \oplus V_1$ be the superspace $[f, g]$ if $\dim(g(-1))$ is even. If $\dim(g(-1))$ is odd, let $V \subset [f, g]$ be a super space with a standard basis $v_i$ with $\omega(v_i, v_j) = \delta_{i, -j}$ for all $i, j \in \{ \pm 1, \ldots, \pm (\dim([f, g]) - 1)/2 \}$. We chose such a $V$ in the present paper for considering the definition of $W$ superalgebra given in [ZS]. All the statements in the present paper still valid even if we just take $V = [f, g]$.

For a superspace $V$ with an even symplectic form, we denote by $A_h(V)$ the corresponding Weyl superalgebra, see Example 1.5 [SX] for the definition. Specially, if $V$ is pure odd, we denote by $\text{Cl}_h(V)$ the Weyl algebra $A_h(V)$.

It is (§2.3 [Lo11]) proved in that there is a $Q \times \mathbb{K}^*$-equivariant

$$\Phi_{0, h} : A_h^\dagger(V_0) \otimes W_0^\dagger \longrightarrow \mathcal{U}_{0, h}^\dagger$$

isomorphism of quantum algebras.

**Proposition 2.1.**

1. We have a $Q \times \mathbb{K}^*$-equivariant

$$\Phi_h : A_h^\dagger(V_0) \otimes \mathcal{W}_h^\dagger \longrightarrow \mathcal{U}_h^\dagger$$
and \(K^*\)-equivariant isomorphism

\[
\Phi_{1,h} : \text{Cl}_h(V_1) \otimes \mathcal{W}^\wedge_h \rightarrow \tilde{\mathcal{W}}_h
\]

of quantum algebras. Finally this give us a \(K^*\)-equivariant isomorphism

\[
\Phi_h : A^\wedge_h(V) \otimes \mathcal{W}^\wedge_h \rightarrow \mathcal{U}^\wedge_h
\]

of quantum algebras.

(2) There are isomorphisms

\[
(W^\wedge_h)^{K^* - \text{lf}}/(h-1) = \tilde{\mathcal{W}}; \quad \left(W^\wedge_{0,h}\right)^{K^* - \text{lf}}/(h-1) = \mathcal{W}_0 \quad \text{and} \quad \left(W^\wedge_h\right)^{K^* - \text{lf}}/(h-1) = \mathcal{W}
\]

of associative algebra. Where, for a vector space \(V\) with a \(K^*\)-action, we denote by \(V^{K^* - \text{lf}}\) the sum of all finite dimensional \(K^*\)-stable subspaces of \(V\).

(3) There is an embedding \(q := \text{Lie}(Q) \hookrightarrow \tilde{\mathcal{W}}\) of Lie algebras such that the adjoint action of \(q\) coincides with the differential of the \(Q\)-action.

Proof. (1) Suppose that \(V_0\) has a basis \(\{v_i\}_{1 \leq i \leq L}\) with \(\omega(v_i, v_j) = \delta_{i,j} - h\). The isomorphism \(\Phi_{0,h}\) gives us \(Q\)-equivariant embedding \(\Phi_h : V \hookrightarrow \mathcal{U}^\wedge_h\) with \([\Phi_h(v_i), \Phi_h(v_j)] = \delta_{i,j} - h\). Now the isomorphism \(\Phi_h\) can be constructed as in the proof of Theorem 1.6 [SX]. For the construction of isomorphism \(\Phi_{1,h}\), see Case 1 in the proof of Theorem 1.6 [SX]. The isomorphism \(\Phi_h\) can be constructed from the embedding \(\Phi_h : V \hookrightarrow \mathcal{U}^\wedge_h\) given by \(\Phi_h|_{V_0} = \tilde{\Phi}_h\) and \(\Phi_h|_{V_1} = \Phi_{1,h}\).

(2) The first isomorphism was proved in [Lo11]. The remaining statements follow by a similar argument as in the proof of Theorem 3.8 [SX].

(3) View \(\mathcal{U}\) as Harish-Chandra \(\mathcal{U}_0\) bimodule and use §2.5 [Lo11]. \(\square\)

Remark 2.2. In the proposition above we are not claiming that \(\Phi_h\) is \(Q\)-equivariant, although this is probably true.

The above decompositions give us

Theorem 2.3 (Theorem 4.1 [SX]). (1) We have an embedding \(\mathcal{W}_0 \hookrightarrow \tilde{\mathcal{W}}\) of associative algebras. The later is generated over the former by \(\dim(g_1)\) odd elements.

(2) Moreover we have isomorphism

\[
\Psi : \tilde{\mathcal{W}} \rightarrow \text{Cl}(V_1) \otimes \mathcal{W}
\]

of algebras. Here \(\text{Cl}(V_1)\) is the Clifford algebra on the vector space \(V_1 = [g, f]_1\) with symmetric two from \(\chi([\cdot, \cdot])\).

For the proof of second statement of (1), see the proof of Theorem 4.1 [SX].

2.1. The maps \(\bullet^\dagger\) and \(\bullet_\downarrow\). Now we recall the construction of maps \(\bullet^\dagger\) and \(\bullet_\downarrow\) maps between \(i\mathfrak{d}(\mathcal{W})\) and \(i\mathfrak{d}(\mathcal{U})\) in [SX].

For \(J \in i\mathfrak{d}(\mathcal{W})\), we denote by \(R_h(J) \subset \mathcal{W}_h\) the Rees algebra associated with \(J\) and \(R_h(J)^\wedge \subset \mathcal{W}_h\) by completion of \(R_h(J)\) at 0. Let \(A(J)^\wedge_h = A_h(V) \otimes R_h(J)^\wedge\) and set \(J^\dagger = (\mathcal{U}_h \cap \Phi_h(A(J)^\wedge_h))/(h - 1)\). For an ideal \(J \in i\mathfrak{d}(\mathcal{U})\), denote by \(\bar{J}_h\) the closure of
R\textsubscript{\(h\)}\((J)\) in \(U\textsuperscript{\wedge}_{h}\). Define \(I_{t}\) to be the unique (by Proposition 3.4(3) \[SX\]) ideal in \(W\) such that

\[ R\textsubscript{\(h\)}(J_{t}) = \Phi_{h}^{-1}(\tilde{J}_{h}) \cap R\textsubscript{\(h\)}(W). \]

A \(g\textsubscript{0}\) bimodule \(M\) is said to be Harish-Chandra(HC)-bimodules if \(M\) is finitely generated and the adjoint action of \(g\) on \(M\) is locally finite. For any two sided ideal \(J \subset \mathfrak{u} (\text{resp. } J \subset \tilde{W})\), we denote by \(J_{t} (\text{resp. } \tilde{J}_{t})\) image of \(J\) under the functor \(\bullet_{t} (\text{resp. } \bullet^{\dagger})\) in \(\mathfrak{g}(\mathfrak{u})\). Here we view \(J\) and \(J\) as a (HC)-bimodules over \(\mathfrak{g}_{0}\) and \(W_{0}\) respectively. The following lemma is a direct result of the above construction and Theorem 2.3.

**Lemma 2.4.** We have that \((\text{Cl}(V_{1}) \otimes J)^{\dagger} = J^{\dagger}\) and \(I_{t} = \text{Cl}(V_{1}) \otimes J_{t}\).

### 2.2. Properties of \(\bullet^{\dagger}\) and \(\bullet\) after \[SX\].

For an associative algebra \(A\), we denote by \(GK\text{dim}(A)\) the Gelfand-Kirillov dimension of \(A\) (for the definition, see \[KL\]). The *associated variety* \(V(J)\) of a two sided ideal \(J \in \mathfrak{m}(\mathfrak{u})\), is defined to be the associated variety \(V(J)\) of \(J = J \cap \mathfrak{u}\). We say that \(J\) is supported on \(V(J)\).

**Lemma 2.5.** For any two sided ideal of \(J \subset \mathfrak{u}\), we have

\[ GK\text{dim}(\mathfrak{u}/J) = GK\text{dim}(\mathfrak{u}(\mathfrak{g}_{0})/J_{0}) = \text{dim}(V(J)). \]

*Proof.* Note that we have embedding \(\mathfrak{u}(\mathfrak{g}_{0})/J_{0} \hookrightarrow \mathfrak{u}/J\). The first equality follows from the definition of Gelfand-Kirillov dimension (see P14 Definition \[KL\] and the remark following it ) and the PBW base theorems for \(\mathfrak{u}(\mathfrak{g}_{0})\) and \(\mathfrak{u}\). The second equality follows form Corollary 5.4 \[BK\].

The following Proposition and it’s proof are super version of Theorem 1.2.2 (vii) \[Lo10\] in a special case.

**Proposition 2.6.** For any \(J \in \text{Prim}_{0}(\mathfrak{g})\), the preimage of \(J\) under \(\bullet^{\dagger}\) is exactly the minimal prime ideals containing \(J_{t}\).

*Proof.* Suppose that \(\tilde{J}\) is prime ideal of \(\tilde{W}\) with \(J^{\dagger} = \tilde{J}\). Proposition 4.5 \[SX\] implies that \(J_{t} \subset \tilde{J}\). So \(\tilde{J}\) has finite codimension in \(\tilde{W}\). Hence we deduce that \(\tilde{J}\) is minimal by Corollary 3.6 \[BK\]. Now suppose that the minimal prime ideal \(J \subset W\) with \(J_{t} \subset \tilde{J}\). It is follows from Proposition 4.6 \[SX\] that \(J_{t}\) has finite codimension in \(W\). It is easy to check that \(\tilde{J} = \text{Cl}(V_{1}) \otimes J\) has finite codimension in \(\tilde{W}\). Whence \(\tilde{J}_{0} = \mathfrak{W}_{0} \cap \tilde{J}\) has finite codimension in \(\mathfrak{W}_{0}\). Since \(J^{\dagger} \cap \mathfrak{u}_{0} = (\tilde{J}_{0})^{\dagger}\), we obtain that \(J^{\dagger}\) is supported on \(G_{0} \cdot \chi\) by the proof of Theorem 1.2.2 (vii) \[Lo10\]. Thus by Lemma 2.5 and Corollary 3.6 \[BK\], we have \(J^{\dagger} = J\).

### 2.3. Proof of the main result.

**Theorem 2.7.** We have \((J^{\dagger})_{t} = J\) if and only if \(\text{Cl}(V_{1}) \otimes J\) is \(C_{e}\)-invariant.

*Proof.* Pick \(J \in \mathfrak{m}(\mathfrak{W})\) with finite codimension. Theorem 2.3(3) implies that \(J\) is stable under the adjoint action of \(\mathfrak{q}\). Hence \(J\) is stable under the \(C_{e}\) action. Thus the ‘only if’ part follows. Note that \(\mathfrak{u}\) is a HC \(\mathfrak{g}_{0}\)-bimodule. So by the pure even
result, Theorem 4.1.1 [Lo11], we have \((\mathrm{Cl}(V_1) \otimes \mathcal{J})_\mathfrak{f}) = \mathrm{Cl}(V_1) \otimes \mathcal{J}\). So the ‘if’ part follows by Lemma 2.3.

□

Now we are ready to prove the main result.

*Proof of Theorem 1.1*

By Theorem 2.7 and Proposition 2.6, the theorem follows by similar argument as in the proof of [Conjecture 1.2.1 [Lo11]. Indeed, let \(I_1, \ldots, I_l\) be the minimal prime ideal containing \(J_{\mathfrak{f}}\), for a fixed \(\mathcal{J} \in \text{Prim}_\mathfrak{f}(\mathfrak{U})\). Since \(\mathrm{Cl}(V_1) \otimes I_1\) is stable under \(Q^\circ\), \(\bigcap_{\gamma \in C_\mathfrak{f}} \gamma(\mathrm{Cl}(V_1) \otimes I_1)\) is \(Q\)-stable. Set \(J_1 = (\bigcap_{\gamma \in C_\mathfrak{f}} \gamma(\mathrm{Cl}(V_1) \otimes I_1))_\mathfrak{f}\), then by Theorem 4.1.1 [Lo11] we have \((J_1)_\mathfrak{f} = \bigcap_{\gamma \in C_\mathfrak{f}} \gamma(\mathrm{Cl}(V_1) \otimes I_1)\). This implies \(J = J_1\) and hence \(J_1 = \bigcap_{\gamma \in C_\mathfrak{f}} \gamma(\mathrm{Cl}(V_1) \otimes I_1)\). We have that \(\gamma(\mathrm{Cl}(V_1) \otimes I_1) = \mathrm{Cl}(V_1) \otimes I_{\gamma(1)}\) for some \(\gamma(1) \in \{1, \ldots, l\}\) by Proposition 3.1.10 [Di] and Case 1 in the proof of Theorem 1.6 [SX]. Thus we have \(J = \bigcap_{\gamma \in C_\mathfrak{f}} J_{\gamma(1)}\) by Proposition 3.1.10 [Di] and Lemma 2.4. Thus the proof is completed by Proposition 2.6. □

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**References**

[BL] W. Borho, H. Kraft, *Über die Gelfand-Kirillov-Dimension*, Math. Ann. 220(1976), 1-24.

[CM] K. Coulembier, I. Musson, *The primitive spectrum for \(\mathfrak{gl}(m|n)\)*, Tohoku Math. J. (2)70 (2018), no. 2, 225-266.

[Di] J. Dixmier, *Enveloping algebras*. North-Holland Mathematical Library, Vol. 14.

[KL] G. R. Krause, T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, revised edition, Graduate Studies in Mathematics, vol. 22(2000), American Mathematical Society, Providence.

[Lo08] I. Losev, *On the structure of the category O for W-algebras*, Semin. Congr. 24 (2012), 351-368.

[Lo10] I. Losev, *Quantized symplectic actions and W-algebras*, J. Amer. Math. Soc., 23 (2010), 35-59.

[Lo11] I. Losev, *I. Losev, Finite dimensional representations of W-algebras*, Duke Math. J. 159 (2011), 99C143

[Lo15] I. Losev, *Dimensions of irreducible modules over W-algebras and Goldie ranks*, Invent. Math, 200(3)(2015), 849-923.

[Mu] I. Musson *The enveloping algebra of the Lie superalgebra \(\mathfrak{osp}(1|2r)\)*, Repr. Theory 1(97), 405–423.

[SX] B. Shu, H. Xiao *Super formal Darboux-Weinstein theorems and finite W superalgebras*, J. Algebra 550(2020), 242-265.

[ZS] Y. Zeng, B. Shu, *Finite W-superalgebras for basic Lie superalgebras*, J. Algebra, 438 (2015), 188-234.
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