\textbf{\(\mathcal{M}_{15}\) IS RATIONALLY CONNECTED}

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1. Introduction.
Let \(D\) be a smooth, irreducible complex projective curve, it is certainly an unexpected property that \(D\) moves in a linear system

\[ |D| \]

on a smooth irreducible surface \(S\) which is not birational to \(D \times \mathbb{P}^1\). For a curve \(D\) of genus \(g\) with general moduli such a property is indeed equivalent to the existence of a rational curve \(R\) such that

\[ [D] \in R \subset \mathcal{M}_g, \]

where \(\mathcal{M}_g\) is the moduli space of \(D\) and \([D]\) denotes the moduli point of \(D\). Due to the fundamental theorem of Eisenbud, Harris and Mumford on the Kodaira dimension of \(\mathcal{M}_g\), there is no rational curve through a general point of \(\mathcal{M}_g\) if \(g \geq 23\). Therefore we have not to expect the above property for a given curve \(D\) of genus \(g \geq 23\).

Of course the existence of a rational curve through a general point of \(\mathcal{M}_g\) just means that \(\mathcal{M}_g\) is a uniruled variety. The uniruledness of \(\mathcal{M}_g\) is somehow expected for \(g \leq 23\), more precisely it is implied by two famous conjectures: the conjecture that every variety of negative Kodaira dimension is uniruled and the so called slope conjecture on effective divisors of \(\mathcal{M}_g\). The latter one implies that \(\mathcal{M}_g\) has negative Kodaira dimension for \(g \leq 22\), (cfr. [HM] and [FP]).

In spite of being expected, the uniruledness of \(\mathcal{M}_g\) for \(g \leq 22\) persists to be an open problem if \(16 \leq g \leq 22\). Let us briefly recall some history and some known facts about this matter. For \(g \leq 14\) one knows more: \(\mathcal{M}_g\) is not only uniruled but also unirational. The proof of the unirationality of \(\mathcal{M}_g\) goes back to Severi for \(g \leq 10\), while the cases between 11 and 14 admit more recent proofs due to Sernesi (\(g = 12\)), Chang and Ran (\(g = 11, 13\)) and Verra (\(g = 14\); ([S], [Se], [CR], [V]).

In addition Chang and Ran showed that the Kodaira dimension of \(\mathcal{M}_g\) is negative for \(g = 15, 16\) and asked about the uniruledness of these moduli spaces, ([CR1]). In this note we positively answer this question in the case of genus 15. Actually we will show a stronger result:

(1.1) \textbf{MAIN THEOREM} \(\mathcal{M}_{15}\) is rationally connected.

The proof relies on a property which can be observed for a general curve \(D\) of any genus \(g \leq 15\), namely that there exists an embedding

\[ D \subset S \subset \mathbb{P}^r \]

where \(S\) is a smooth, regular, canonical surface and \(|D|\) is at least 2-dimensional. For instance it is possible to show that a general \(D\) of genus \(g \leq 10\) admits such an embedding in a quintic surface \(S \subset \mathbb{P}^3\). Moreover it is shown in [V] that, for \(g = 11, 12, 14\), \(S\) can be chosen among the few other canonical surfaces which
are also complete intersections, (the case \( g = 13 \), though not covered, admits a completely analogous description).
Let \( D \) be a general curve of genus 15 and let \( L \) be a general line bundle on \( D \) having degree 9 and satisfying \( h^0(L) = 2 \), it will be shown in section 3 that
\[
D \subset S \subset \mathbb{P}^6
\]
where \( S \) is a complete intersection of four quadrics and \( \mathcal{O}_D(D) \cong L \). Birationally speaking we can consider the moduli space
\[
W
\]
of pairs \((D, L)\), which is open in the universal Brill-Noether locus \( W_{d,g}^{r} \) with \( r = 1, d = 9, g = 15 \). Let
\[
h : |D| \to W
\]
be the natural map sending \( A \) to the isomorphism class of \((A, \mathcal{O}_A(A))\). It turns out that \( h \) is generically finite onto its image and that the same is true for \( f \cdot h \), where
\[
f : W \to \mathcal{M}_{15}
\]
is the forgetful map, (see 4.12). This implies a quite interesting property: through a general point of \( W \) always passes a rational surface. In particular it follows that both \( W \) and \( \mathcal{M}_{15} \) are uniruled. To prove that \( \mathcal{M}_{15} \) is rationally connected we consider the divisor
\[
\Delta_0 \subset \overline{\mathcal{M}}_{15}
\]
parametrizing isomorphism classes of nodal stable curves of arithmetic genus 15. As is well known \( \Delta_0 \) is dominated by the moduli space \( \mathcal{M}_{14,2} \) of 2-pointed curves of genus 14. We show in section 3 that \( \mathcal{M}_{14,2} \) is unirational. Hence \( \Delta_0 \) is unirational and the proof of the rational connectedness of \( \mathcal{M}_{15} \) can be sketched as follows: let \( x_i = [D_i], i = 1, 2 \), be general in \( \mathcal{M}_{15} \) and let \( L_i \) be a line bundle on \( D_i \) with \( \deg L_i = 9 \) and \( h^0(L_i) = 2 \). As above we have an embedding
\[
D_i \subset S_i \subset \mathbb{P}^6
\]
where \( S_i \) is a smooth complete intersection of four quadrics and \( \mathcal{O}_{D_i}(D_i) \cong L_i \). In the 2-dimensional linear system \(|D_i|\) we can choose a Lefschetz pencil \( P_i \) containing \( D_i \) and \( D'_i \), where \( D'_i \) is a nodal curve defining a general point \( x'_i \) of \( \Delta_0 \). In particular this implies that \( x'_i \) is smooth for \( \overline{\mathcal{M}}_{15} \), \( P_i \) defines an irreducible rational curve \( R_i \) in \( \overline{\mathcal{M}}_{15} \) containing \( x_i \) and \( x'_i \). Since \( \Delta_0 \) is unirational \( x'_1 \) and \( x'_2 \) are connected by an irreducible rational curve \( R' \). Therefore \( x_1, x_2 \) are connected by a chain \( R = R_1 \cup R_2 \cup R' \) of irreducible rational curves. Moreover \( U \cap R \) is connected, where \( U \) is the regular locus of \( \overline{\mathcal{M}}_{15} \). Hence \( \mathcal{M}_{15} \) is rationally connected.
We do not know at the moment whether \( \mathcal{M}_{15} \) is unirational, nor if this is plausible. In view of affording the uniruledness of \( \mathcal{M}_g \), in the cases where this is unknown, it could be interesting to have some indications about the possible embeddings \( D \subset S \subset \mathbb{P}^r \), where \( S \) is a canonical surface and \( D \) is general of genus \( g \in [16, 22] \).
Finally it seems worth to address the following natural

(1.2) **Problem** For which values of \( g \) is \( \mathcal{M}_g \) rationally connected?
2. Curves of degree 19 and genus 15 in $\mathbb{P}^6$.
In the Hilbert scheme $\text{Hilb}_{19,15,6}$ of all curves of degree 19 and genus 15 of $\mathbb{P}^6$ we consider the open set $\mathcal{H}$ whose points are stable, non degenerate, smoothable curves and its open subset

$$\mathcal{D} = \{D \in \mathcal{H} / h^1(T_{\mathbb{P}^6} \otimes \mathcal{O}_D) = 0\}.$$  \hspace{1cm} (2.1)

Assume that

$$\mathcal{D} \neq \emptyset$$

and consider any $D \in \mathcal{D}$. As is well known the condition $h^1(T_{\mathbb{P}^6} \otimes \mathcal{O}_D) = 0$ implies that the Kodaira-Spencer map $dt_D : H^0(N_D) \rightarrow H^1(T_D)$ is surjective. Since $dt_D$ is the tangent map at $D$ of the natural map

$$t : \mathcal{D} \rightarrow \mathcal{M}_{15},$$  \hspace{1cm} (2.2)

it follows that $t$ is dominant. On the other hand, for these values of the degree and of the genus of $D$, there exists a unique irreducible component of $\mathcal{H}$ which dominates $\mathcal{M}_{15}$. Therefore $\mathcal{D}$ is an irreducible open subset of such a component.

We will show in a moment that $\mathcal{D}$ is non empty. Previously we want to explain more of the geometric situation: fix a general curve $D$ of genus 15, then its Brill-Noether locus

$$W^1_9(D) = \{L \in \text{Pic}^9(D) / h^0(L) = 2\}$$  \hspace{1cm} (2.3)

is a smooth, irreducible curve. This indeed follows from the general Brill-Noether theory because the Brill-Noether number $\rho(d,g,r)$ is one if $(d,g,r) = (9,15,1)$. It is not difficult to check that, since $D$ is general, the line bundle $\omega_D \otimes L^{-1}$ is very ample and defines an embedding

$$D \subset \mathbb{P}^6$$

as a curve of degree 19. $D$ is a general point in the irreducible component of $\mathcal{H}$ which dominates $\mathcal{M}_{15}$. A priori we could have $h^1(T_{\mathbb{P}^6} \otimes \mathcal{O}_D) = h^1(N_D) \geq 1$ for every $D$ in such a component, but this is not the case if $\mathcal{D} \neq \emptyset$. To show that $\mathcal{D}$ is non empty we consider the nodal, reducible curve

$$D_o = R \cup A$$  \hspace{1cm} (2.4)

where $R$ is a rational normal sextic curve in $\mathbb{P}^6$, $A$ is a smooth, irreducible, non degenerate curve of degree 13 and genus 8 and

$$R \cap A = \{z_1, \ldots, z_8\} =: Z$$  \hspace{1cm} (2.5)

is a set of 8 distinct points on $R$. Note that these points are in general position, indeed this happens for any set of $n+2$ distinct points lying on a smooth, irreducible rational normal curve of $\mathbb{P}^n$. Notice also that

$$\mathcal{O}_A(1) \cong \omega_A(-p)$$

where $p$ is a point. We want now to show that $D_o \in \mathcal{D}$. 

(2.6) **PROPOSITION** \( h^1(T_{P^6} \otimes \mathcal{O}_{D_a}) = 0. \)

**PROOF** Tensoring by \( T_{P^6} \) the standard exact sequence

\[
0 \to \mathcal{O}_{D_a} \to \mathcal{O}_R \oplus \mathcal{O}_A \to \mathcal{O}_Z \to 0
\]

and passing to the associated long exact sequence we obtain

\[
(2.7) \quad 0 \to H^0(T_{P^6} \otimes \mathcal{O}_{D_a}) \to H^0(T_{P^6} \otimes \mathcal{O}_A) \oplus H^0(T_{P^6} \otimes \mathcal{O}_R) \to
\]

\[
\to H^0(T_{P^6} \otimes \mathcal{O}_Z) \to H^1(T_{P^6} \otimes \mathcal{O}_{D_a}) \to H^1(T_{P^6} \otimes \mathcal{O}_A) \oplus H^1(T_{P^6} \otimes \mathcal{O}_R) \to 0.
\]

Since \( \mathcal{O}_R(1) \) is non special, the standard Euler sequence

\[
0 \to \mathcal{O}_R \to \mathcal{O}_R(1)^7 \to T_{P^6} \otimes \mathcal{O}_R \to 0
\]

implies that \( h^1(T_{P^6} \otimes \mathcal{O}_R) = 0. \) It is also standard that the restriction

\[
b : H^0(T_{P^6} \otimes \mathcal{O}_R) \to H^0(T_{P^6} \otimes \mathcal{O}_Z)
\]

is an isomorphism. Indeed the Euler sequence induces a diagram

\[
H^0(\mathcal{O}_R(1))^7 \xrightarrow{a} H^0(T_{P^6} \otimes \mathcal{O}_R) \xrightarrow{b} H^0(T_{P^6} \otimes \mathcal{O}_Z)
\]

such that \( a \) is surjective and \( b \cdot a \) is the natural evaluation map. Then \( b \cdot a \) is surjective and \( b \) is surjective, hence an isomorphism. From the previous arguments it then suffices to show that \( h^1(T_{P^6} \otimes \mathcal{O}_A)) = 0. \) Let \( A' \) be the canonical model of \( A \) and recall that \( A \) is obtained from \( A' \) by projection from \( p \). Consider the natural diagram

\[
\begin{array}{cccccc}
0 & \to & H^0(A', \omega_{A'}(-p)) \otimes \mathcal{O}_{A'} & \to & H^0(A', \omega_{A'}) \otimes \mathcal{O}_{A'} & \to & H^0(A', \omega_{A'}|_p) \otimes \mathcal{O}_{A'} & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \omega_{A'}(-p) & \to & \omega_{A'} & \to & \omega_{A'}|_p & \to & 0
\end{array}
\]

where the first two vertical arrows are given by evaluation and are both surjective. If we dualize and twist the exact sequence that one obtains from the snake lemma we obtain the sequence

\[
0 \to \mathcal{O}_{A'}(1) \to T_{P^7} \otimes \mathcal{O}_{A'}(-p) \to T_{P^6} \otimes \mathcal{O}_A \to 0.
\]

We are then reduced to show that \( h^1(T_{P^7} \otimes \mathcal{O}_{A'}(-p)) = 0, \) but this is a consequence of the fact that, since \( A' \) is canonical, \( h^1(T_{P^7} \otimes \mathcal{O}_{A'}) = 0 \) and we can factor the natural surjection \( H^0(P^7, T_{P^7}) \to T_{P^7}|_p \to 0 \) through \( H^0(T_{P^7} \otimes \mathcal{O}_{A'}) \to T_{P^7}|_p. \)

The next theorem is a well known consequence of the vanishing of \( h^1(T_{P^6} \otimes \mathcal{O}_{D_a}), \) (cfr. [HH]1.1).

(2.8) **THEOREM** \( D_a \) is smoothable.
The theorem implies that the natural morphism \( t : D \to M_{15} \) is dominant. Let \( \mathcal{W} \) be the moduli space of pairs \((D, L)\) such that \( D \) is a general curve of genus 15 and \( L \in W^1_D \), in addition we can consider the natural morphism

\[
(2.9) \quad u : D \to \mathcal{W}
\]

sending \( D \in D \) to the moduli point of \((D, L)\), where \( L = \omega_D(-1) \). It is clear from the previous results and remarks that \( u \) is also dominant. So we can summarize the main achievements of this section as follows:

\[
(2.10) \text{COROLLARY} \quad D \text{ is non empty and dominates both } \mathcal{W} \text{ and } M_{15}.
\]

3. Embedding \( D \) in a smooth \((2,2,2,2)\) complete intersection of \( \mathbb{P}^6 \).

In this section we show that a general \( D \in D \) is embedded in a smooth complete intersection of 4 quadrics and that \( h^0(\mathcal{I}_D(2)) = 4 \), where \( \mathcal{I}_D \) is the ideal sheaf of \( D \). We will use a reducible curve \( D_o = R \cup A \) of the type considered in the previous section, proving that \( D_o \) embeds in a reducible complete intersection of 4 quadrics \( S_o \) and that the pair \((D_o, S_o)\) is smoothable. We start with a smooth, non degenerate sextic Del Pezzo surface

\[
(3.1) \quad Y \subset \mathbb{P}^6.
\]

The ideal sheaf \( \mathcal{I}_Y \) of \( Y \) is generated by quadrics: this implies the next property.

\[
(3.2) \text{LEMMA} \quad \text{The intersection scheme of four general quadrics containing } Y \text{ is a reduced surface}
\]

\[
(3.3) \quad X \cup Y,
\]

where \( X \) is a smooth, irreducible component. Moreover the intersection scheme of \( X \) and \( Y \) is a smooth, irreducible curve

\[
(3.4) \quad B = X \cap Y.
\]

**PROOF** Let \( \sigma : P \to \mathbb{P}^6 \) be the blowing up of \( Y \), \( E \) the exceptional divisor of \( \sigma \), \( H \) the pull-back of a hyperplane by \( \sigma \). Since \( \mathcal{I}_Y \) is generated by quadrics the strict transform

\[
| \mathcal{I}_Y(2) |
\]

is a base point free linear system and coincides with

\[
| 2H - E |.
\]

By Bertini theorem the intersection of 4 general elements \( Q'_1, \ldots, Q'_4 \in | 2H - E | \) is a smooth surface \( X' \). Since \((2H - E)^6 = 4 \) is positive, \( X' \) is connected. Furthermore we can assume that

\[
B' = Q'_1 \cap \ldots Q'_4 \cap E
\]

is a smooth, connected curve and that \( \sigma/B' : B' \to \mathbb{P}^6 \) is an embedding. Then \( \sigma/X' : X' \to \mathbb{P}^6 \) is an embedding too and \( X = \sigma(X') \) has the following properties:
(i) $X \cup Y$ is the complete intersection of the quadrics $Q_i = \sigma(Q'_i)$, $i = 1, \ldots, 4$. (ii) $B = \sigma(B')$ is the intersection scheme of $X$ and $Y$. This completes the proof.

The complete intersection of four quadrics $X \cup Y$ is a canonical surface i.e. $\omega_{X \cup Y} \cong \mathcal{O}_{X \cup Y}(1)$. This implies that $\omega_Y(B) \cong \omega_{X \cup Y} \otimes \mathcal{O}_Y \cong \mathcal{O}_Y(1)$. Since $\omega_Y \cong \mathcal{O}_Y(-1)$ it follows that

$$B \in \mathcal{O}_Y(2) |$$

is a quadratic section of $Y$ and a canonical curve of genus 7. The surface $X$ is described in detail in [V], in particular $X$ is obtained from the blowing up

$$\sigma : X \to \mathbb{P}^2$$

of 11 points $e_1 \ldots e_{11}$ in general position. Let $E_i$ be the exceptional divisor over $e_i$, $L$ the pull-back of a line by $\sigma$, $H$ a hyperplane section of $X$ then

$$|H| = |6L - 2(E_1 + \cdots + E_5) - E_6 - \cdots - E_{11}|.$$

It is easy to see that a general curve

$$R \in |2L - E_1 - E_2 - E_{10} - E_{11}|$$

is a smooth, irreducible rational sextic curve.

(3.5) **PROPOSITION** $R$ is non degenerate.

**PROOF** $R$ is non degenerate if $|H - R| = |4L - E_1 - E_2 - 2E_3 - 2E_4 - 2E_5 - E_6 - E_7 - E_8 - E_9|$ is empty. Note that $(H - R)E_{10} = (H - R)E_{11} = 0$ and that $(H - R)^2 = -2$. Then consider the contraction $f : X \to X'$ of $E_{10}$ and $E_{11}$. Let $C' = f_*C$ with $C \in |H - R|$, then $C'' = C^2 = -2$. As is well known there is no effective $-2$ curve on the blowing up of $\mathbb{P}^2$ in $n \leq 9$ general points. Hence $C$ cannot exist and $|H - R|$ is empty.

Notice also that

(3.6) $$\omega_X(B) \cong \mathcal{O}_X(H)$$

so that $H - B \sim -3L + E_1 + \cdots + E_{11}$ is the canonical class.

(3.7) **PROPOSITION** Let $\mathcal{I}_{Y \cup R}$ be the ideal sheaf of $Y \cup R$, then

$$h^0(\mathcal{I}_{Y \cup R}(2)) = 4.$$

**PROOF** Since $X \cup Y$ is the complete intersection of four quadrics and $X \cap Y = B$, it suffices to show that $h^0(\mathcal{O}_X(2H - B - R)) = 0$. Note that $2H - B - R \sim L - E_3 - E_4 - E_5 + E_{10} + E_{11}$. Moreover $(2H - B - R)E_{10} = (2H - B - R)E_{11} = -1$. This implies that an effective $C \in |2H - B - R|$ contains $E_{10}$ and $E_{11}$. But then $C = C' + E_{10} + E_{11}$ with $C' \in |L - E_3 - E_4 - E_5|$. This is a contradiction because, since the points $e_3, e_4, e_5$ are not collinear, $|L - E_3 - E_4 - E_5|$ is empty. Hence $|2H - B - R|$ is empty and $h^0(\mathcal{O}_X(2H - B - R)) = 0$. 


PROPOSITION \( | \mathcal{O}_B(R) | \) is a base point free pencil of degree 8.

PROOF One easily computes \( RB = 8 \). Note that \( K_X - (R - B) \sim H - R \) so that, by Serre duality, \( h^i(\mathcal{O}_X(R - B)) = h^{2-i}(\mathcal{O}_X(H - R)) \). Since \( R \) is irreducible and \( R(R - B) = -8 \), we have \( h^2(\mathcal{O}_X(H - R)) = 0 \). On the other hand proposition 3.5 implies that \( h^0(\mathcal{O}_X(H - R)) = 0 \). Since \( \chi(\mathcal{O}_X(H - R)) = 0 \), it follows \( h^1(\mathcal{O}_X(H - R)) = 0 \) and hence \( h^i(\mathcal{O}_X(R - B)) = 0 \) for \( i = 0, 1, 2 \). Then the statement follows considering the long exact sequence of

\[
0 \rightarrow \mathcal{O}_X(R - B) \rightarrow \mathcal{O}_X(R) \rightarrow \mathcal{O}_B(R) \rightarrow 0
\]

Now we consider on \( Y \) the linear system

\[
| B + N |
\]

where \( N \) is one of the 6 lines contained in \( Y \). We have \( BN = 2, \quad p_a(B + N) = 8, \quad \deg B + N = 13 \). Since \( B \) is a quadratic section of \( Y \) we have also \( h^1(\mathcal{O}_Y(B)) = 0 \) and the exact sequence

\[
0 \rightarrow H^0(\mathcal{O}_Y(B)) \rightarrow H^0(\mathcal{O}_Y(B + N)) \rightarrow H^0(\mathcal{O}_N(B)) \rightarrow 0.
\]

It easily follows from the sequence that \( | B + N | \) is base point free and that its general element is smooth, irreducible. As above we consider a general \( R \) in the pencil \( | 2L - E_1 - E_2 - E_{10} - E_{11} | \). Since \( | \mathcal{O}_B(R) | \) is base point free of degree 8 and \( R \) is a rational normal sextic, we can assume that the intersection scheme

\[
Z = B \cap R
\]

is smooth and supported on 8 points in general position. Moreover we can also assume that \( B \cap N \) and \( Z \) do not intersect. Let \( \mathcal{I}_{Z/Y} \) be the ideal sheaf of \( Z \) in \( Y \), we have the following:

PROPOSITION The base locus of \( | \mathcal{I}_{Z/Y}(B + N) | \) is \( Z \).

PROOF Note that \( B + N \) is smooth along \( Z \) and that the linear system \( | \mathcal{O}_B(B + N - Z) | \) is base point free. Then the statement follows if the restriction \( \rho : | B + N | \rightarrow | \mathcal{O}_B(B + N) | \) is surjective. Since \( h^1(\mathcal{O}_Y(N)) = 0 \), the surjectivity of \( \rho \) follows from the long exact sequence of

\[
0 \rightarrow \mathcal{O}_Y(N) \rightarrow \mathcal{O}_Y(B + N) \rightarrow \mathcal{O}_B(B + N) \rightarrow 0.
\]

The proposition implies that a general

\[
A \in | \mathcal{I}_Z(B + N) |
\]

is a smooth, irreducible curve of degree 13 and genus 8. Moreover the curve

\[
D_o = A \cup R,
\]
is nodal with \( \deg D_o = 19, p_a(D_o) = 15 \), \( \text{Sing } D_o = Z \). We know from section 2 that
\[
D_o \in \mathcal{D},
\]
where \( \mathcal{D} \) is the open subset of the Hilbert scheme of \( D_o \) parametrizing nodal, non degenerate, smoothable curves \( D \) satisfying \( h^1(T_{P^6} \otimes \mathcal{O}_D) = 0 \).

(3.10) **PROPOSITION** \( h^0(\mathcal{I}_{D_o}(2)) = 4 \) and \( h^i(\mathcal{I}_{D_o}(2)) = 0, i > 0 \).

**PROOF** We have \( h^1(\mathcal{O}_{D_o}(2)) = 0 \) and \( h^0(\mathcal{O}_{D_o}(2)) = 24 \), this can be easily proved considering the standard exact sequence
\[
0 \to \mathcal{O}_{D_o}(2) \to \mathcal{O}_{A}(2) \oplus \mathcal{O}_R(2) \to \mathcal{O}_Z(2) \to 0
\]
and its associated long exact sequence. Then, by the standard exact sequence
\[
0 \to \mathcal{I}_{D_o}(2) \to \mathcal{O}_{P^6}(2) \to \mathcal{O}_{D_o}(2) \to 0,
\]
it follows that \( h^i(\mathcal{I}_{D_o}(2)) = 0 \) for \( i > 1 \) and that \( h^1(\mathcal{I}_{D_o}(2)) = 0 \) iff \( h^0(\mathcal{I}_{D_o}(2)) = 4 \). Finally we observe that the natural inclusion \( | \mathcal{I}_{Y \cup R}(2) \mid \subseteq | \mathcal{I}_{D_o}(2) \mid \) is an equality. Indeed let \( Q \) be a quadric containing \( D_o \), then \( Q \cap Y \) contains \( A \). But \( A \) is linearly equivalent to \( B + N \) where \( B \) is a quadratic section of \( Y \) and \( N \) is effective. Therefore \( Q \) contains \( Y \). Since \( R \subset D_o \), it follows that \( Q \) contains \( Y \cup R \) and hence \( h^0(\mathcal{I}_{D_o}(2)) = h^0(\mathcal{I}_{Y \cup R}(2)) \). By proposition 3.7 \( h^0(\mathcal{I}_{Y \cup R}(2)) = 4 \) and this completes the proof.

The proposition implies that a general \( D \in \mathcal{D} \) is contained in a unique complete intersection \( S \) of four quadrics. More precisely one can show the following:

(3.11) **THEOREM** For a general \( D \in \mathcal{D} \) one has \( D \subset S \), where \( S \) is a smooth complete intersection of 4 quadrics. Moreover it holds \( h^0(\mathcal{I}_D(2)) = 4 \) for the ideal sheaf \( \mathcal{I}_D \) of \( D \).

**PROOF** The curve \( D_o \) is contained in \( X \cup Y \) which is a complete intersection of 4 quadrics. Moreover, by 3.10, its ideal sheaf \( \mathcal{I}_{D_o} \) satisfies \( h^0(\mathcal{I}_{D_o}(2)) = 4 \) and \( h^i(\mathcal{I}_{D_o}(2)) = 0, i > 0 \). Then, by standard semicontinuity arguments, a general \( D \in \mathcal{D} \) satisfies the same properties. In particular the base locus \( S \) of \( | \mathcal{I}_D(2) | \) is a complete intersection of 4 quadrics. It remains to show that \( S \) is smooth. Let us consider the standard exact diagram of tangent and normal bundles,
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & T_{D_o} & \longrightarrow & T_{P^6} \otimes \mathcal{O}_{D_o} & \longrightarrow & N_{D_o} & \longrightarrow & T^1_Z & \longrightarrow & 0 \\
0 & \longrightarrow & T_{S_o} \otimes \mathcal{O}_{D_o} & \longrightarrow & T_{P^6} \otimes \mathcal{O}_{D_o} & \longrightarrow & N_{S_o} \otimes \mathcal{O}_{D_o} & \longrightarrow & T^1_{S_o} \otimes \mathcal{O}_{D_o} & \longrightarrow & 0 \\
\end{array}
\]
where \( S_o = X \cup Y \) and \( T^1_Z, T^1_{S_o} \) are the sheaves defined by the \( T^1 \)-functor of Lichtenbaum-Schlessinger. \( T^1_Z \) and \( T^1_{S_o} \) are respectively supported on the singular loci of \( D_o \) and of \( S_o \), it is easy to check that \( T^1_Z = \mathcal{O}_Z \) and that \( T^1_{S_o} \) is a line bundle on \( B \). Furthermore we have \( T^1_{S_o} \otimes \mathcal{O}_{D_o} = \mathcal{O}_{D_o \cap B} \) and the vertical arrow
\[
T^1_Z \rightarrow T^1_{S_o} \otimes \mathcal{O}_{D_o}
\]
is the natural injection $\mathcal{O}_Z \to \mathcal{O}_{D_o \cap B}$. We know that $D_o$ is smoothable, let $\delta \in H^0(N_{D_o})$ be an infinitesimal deformation which is induced by an effective smoothing of $D_o$. Then, as is well known, the image of $\delta$ in $T^1_Z$ is non zero at each $z \in Z$, (cfr. [S2]). Let

$$\delta_1 \in H^0(N_{S_o} \otimes \mathcal{O}_{D_o})$$

be the image of $\delta$. Then, by the commutativity of the diagram, the image of $\delta_1$ in $T^1_{S_o} \otimes \mathcal{O}_{D_o}$ is non zero at each $z \in Z \subset D_o \cap B$. On the other hand every $x \in D_o \cap B - Z$ is smooth for $D_o$ and it is obvious that we can choose $\delta$ such that $\delta_1(x) \neq 0$. Finally we observe that the restriction map

$$r : H^0(N_{S_o}) \to H^0(N_{S_o} \otimes \mathcal{O}_{D_o})$$

is surjective. To see this it suffices to tensor by $N_{S_o}$ the standard exact sequence

$$0 \to \mathcal{I}_{D_o/S_o} \to \mathcal{O}_{S_o} \to \mathcal{O}_{D_o} \to 0.$$  

Passing to the long exact sequence the surjectivity of $r$ follows if $h^1(\mathcal{I}_{D_o/S_o} \otimes N_{S_o}) = 0$. Now $N_{S_o} = \mathcal{O}_{S_o}(2)^4$ because $S_o$ is the complete intersection of 4 quadrics. So it suffices to show that $h^1(\mathcal{I}_{D_o/S_o}(2)) = 0$. This easily follows from the long exact sequence of

$$0 \to \mathcal{I}_{S_o}(2) \to \mathcal{I}_{D_o}(2) \to \mathcal{I}_{D_o/S_o}(2) \to 0$$

and proposition 3.10. Since $r$ is surjective, $\delta_1$ lifts to an infinitesimal deformation

$$\sigma \in H^0(N_{S_o}).$$

By the commutativity of the diagram the image of $\sigma$ in $H^0(T^1_{S_o})$ is not zero at each $x \in B \cap D_o$. Finally let

$$(S_t, D_t), \ t \in T,$$

be an effective deformation of $(S_o, D_o)$ induced by $\sigma$. Then such a deformation smooths $Z = \text{Sing} \ D_o$ so that $D_t$ is smooth. Moreover it smooths $D_o \cap B$ as a subset of $B = \text{Sing} \ S_o$. This implies that $S_t$ has at most finitely many singular points and that they are not in $D_t$ for $t \neq o$. Let $t \neq o$, it is easy to show that $(D_t, S_t)$ deforms to a pair $(D, S)$ such that $\text{Sing} \ S$ is empty: we leave this to the reader.

4 Proof of the main theorem.

In this section we prove the main theorem of this paper i.e. that $\mathcal{M}_{15}$ is rationally connected. At first we need to show that the moduli space

$$\mathcal{M}_{14,2}$$

of 2-pointed curves of genus 14 is unirational. With this purpose we consider the Hilbert scheme $\text{Hilb}_{14,8,6}$ of curves in $\mathbb{P}^6$ having degree 14 and arithmetic genus 8. It is known that a non degenerate, linearly normal, smooth, irreducible curve

$$C \subset \mathbb{P}^6$$
of degree 14 and genus 8 is projectively normal and generated by quadrics, ([V]). Let
\[ C \subset \text{Hilb}_{14,8,6} \]
be the open subset parametrizing all curves with the above properties, then (see [V]):

(4.3) **PROPOSITION** $C$ is irreducible and unirational.

It is standard to construct a projective bundle $h : P \to C$ such that the fibre of $P$ at $C$ is
\[ P_C = |I_C(2)|, \]
$I_C$ being the ideal sheaf of $C$. Since $C$ is projectively normal, $\dim P_C = 6$. We have:

(4.4) **PROPOSITION** Let $V \subset P_C$ be a general 4-dimensional linear system of a general $C \in C$, then the base locus of $V$ is
\[ C \cup D \]
where $D$ is a smooth, irreducible curve of genus 14 and degree 18. Conversely let $D \subset P^6$ be a curve of genus 14 and degree 18 with general moduli, then $D$ is in the base locus of $V$ for some pair $(C, V)$.

**PROOF** See [V].

Using the previous results we can easily construct a rational dominant map

(4.5) \[ \phi : C \times P^6 \times P^6 \to \mathcal{M}_{14,2}. \]

Indeed let $(C, x, y) \in C \times P^6 \times P^6$ be a sufficiently general element, then we can assume that $x, y$ are not in $C$ and moreover that the linear system
\[ V \]
of all quadrics containing $C \cup \{x, y\}$ is 4-dimensional. In view of the previous theorem we can also assume that the base locus of $V$ is
\[ C \cup D \]
where $D$ is a general curve of genus 14. Then we define $\phi$ by setting

(4.6) \[ \phi(C, x, y) = [D, x, y] \]

where $[D, x, y]$ denotes the moduli of the 2-pointed curve $(D, x, y)$. It is clear, by proposition 4.4, that $\phi$ is dominant. Since $C$ is unirational, it follows that

(4.7) **THEOREM** $\mathcal{M}_{14,2}$ is unirational.

Now we consider the divisor

(4.8) \[ \Delta_0 \subset \overline{\mathcal{M}_{15}} \]
parametrizing stable singular curves of arithmetic genus 15. As is well known there
exists a natural rational map of degree two

\[ \psi : \mathcal{M}_{14,2} \to \Delta_0 \]

sending \([D, x, y]\) to the moduli point of the stable curve obtained from \(D\) by glueing
\(x\) to \(y\). In particular we have:

\[ (4.10) \text{COROLLARY } \Delta_0 \text{ is unirational.} \]

Let \(D \in \mathcal{D}\) be a general smooth curve of degree 19 and genus 15, we know from
section 3 that

\[ D \subset S \]

where \(S\) is a smooth complete intersection of four quadrics. Since \(D\) is non degenerate,
the sheaf \(\mathcal{O}_D(1)\) is special and \(\omega_D(-1)\) is an element of the Brill-Noether locus
\(W_9^1(D)\). We want to point out that \(|\omega_D(-1)|\) is a base-point-free pencil and that
\(D\) is linearly normal. This follows because a general \(D \in \mathcal{D}\) has general moduli.
Then \(D\) has no \(g_3^9\) nor a \(g_k^1\) with \(k \leq 8\) and hence \(|\omega_D(-1)|\) is a base-point-free \(g_0^9\).
Moreover, by Riemann Roch, \(\dim |\mathcal{O}_D(1)| = 6\), therefore \(D\) is linearly normal.
Since \(S\) is a canonical surface we have \(\omega_D(-1) \cong \mathcal{O}_D(D)\). Then, from the standard,
extact sequence

\[ 0 \to \mathcal{O}_S \to \mathcal{O}_S(D) \to \mathcal{O}_D(D) \to 0 \]

it follows that

\[ (4.11) \quad \dim |D| = 2. \]

Notice also that \(|D|\) is base point free, because \(|\mathcal{O}_D(D)|\) is base point free. We
expect that a general pencil in \(|D|\) is a Lefschetz pencil and that a singular element
of \(|D|\) defines a general point of \(\Delta_0\). We will see that this is actually true.

\[ (4.12) \text{LEMMA } \text{Let } D \text{ be as above, then a general singular element of } |D| \text{ is an} \]

irreducible curve with exactly one ordinary node and no other singularity.

\[ \text{PROOF } \text{Let } f : S \to \mathbb{P}^2 \text{ be the covering of degree 9 defined by } |D|. \text{ Since } D \text{ is general, the linear series } |\mathcal{O}_D(D)| \text{ has simple ramification. Hence the branch curve } B \text{ of } f \text{ is reduced. This implies that a general tangent line to } B \text{ intersects } B \text{ transversally except for the tangency point. Hence a general singular element of } |D| \text{ is integral with exactly one ordinary node.} \]

Let us recall once more that a general smooth element \(D \in \mathcal{D}\) has the following
properties:

(i) \(D \subset S\), where \(S\) is a smooth complete intersection of 4 quadrics and \(h^0(I_D(2)) = 4\).
(ii) \(|\omega_D(-1)|\) is a base-point-free pencil and \(D\) is linearly normal.
(iii) The Petri map

\[ \mu_D : H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_D(1)) \to H^0(\omega_D) \]

is injective.
(iv) The family of the singular, irreducible curves

\[ \Gamma \in |D| \]

having an ordinary node as a unique singularity is non empty and hence 1-dimensional.

(i) holds by theorem 3.10 and (iv) by the previous lemma, while (ii) has been just remarked above. The injectivity of \( \mu_D \) follows from Gieseker-Petri theorem because \( D \) has general moduli, notice also that the Brill-Noether locus \( W^1_9(D) \) is a smooth, irreducible curve. The family of all curves \( \Gamma \) as in (iv) will be denoted as

\[ D_o. \]

(4.13) LEMMA For each \( \Gamma \in D_o \) (ii) holds for \( \Gamma \) and the Petri map \( \mu_\Gamma \) is injective.

PROOF Recall that \( S \) is a canonical surface, so that \( \omega_C(-1) \cong O_C(C) \) for any curve \( C \subset S \). A general \( D \in |\Gamma| \) satisfies conditions (ii) and (iii), in particular \( |O_D(D)| \) is a base-point-free pencil. Then it follows from the standard long exact sequence

\[ 0 \to H^0(O_S) \to H^0(O_S(D)) \to H^0(O_D(D)) \to 0 \]

that \( |D| \) is 2-dimensional and base-point-free. Replacing \( O_D \) by \( O_\Gamma \) in the above sequence, it follows that \( |\omega_\Gamma(-1)| \) is a base-point-free pencil. Let \( H \) be a hyperplane section of \( S \), by Serre duality \( h^1(O_S(H)) = h^1(O_S) = 0 \). Hence we have the long exact sequence

\[ 0 \to H^0(O_S(H - D)) \to H^0(O_S(H)) \to H^0(O_D(H)) \to H^1(O_S(H - D)) \to 0. \]

Since a general \( D \in |\Gamma| \) is non degenerate and linearly normal, it follows \( h^0(O_S(H - D)) = h^1(O_S(H - D)) = 0 \). Then, replacing as above \( O_D \) by \( O_\Gamma \), we deduce that \( \Gamma \) is non degenerate and linearly normal. It remains to show that the Petri map

\[ \mu_\Gamma; H^0(O_\Gamma(D)) \otimes H^0(O_\Gamma(H)) \to H^0(O_\Gamma(H + D)) \]

is injective. By the base-point-free pencil trick we have \( Ker \mu_\Gamma \cong H^0(O_\Gamma(H - D)) \), cfr. [ACGH] p.126. For the same trick we have \( dim Ker \mu_D = h^0(O_D(H - D)) \) for each \( D \in |\Gamma| \). Since \( \mu_D \) is injective for a general \( D \), the map \( D \to h^0(O_D(H - D)) \) is generically zero on \( |\Gamma| \). Since \( h^1(O_S(H - D)) = 0 \), the standard exact sequence

\[ 0 \to O_S(H - 2D) \to O_S(H - D) \to O_D(H - D) \to 0 \]

yields the long exact sequence

\[ 0 \to H^0(O_D(H - D)) \to H^1(O_S(H - 2D)) \to 0 \to \ldots \]

Then we have \( h^0(O_\Gamma(H - D)) = h^1(O_S(H - 2D)) = 0 \) and \( \mu_\Gamma \) is injective.

Let \( \overline{D} \) be the closure of \( D \) in the Hilbert scheme, then \( \overline{D} \) contains the dense open set

\[ U \]
parametrizing those integral curves $D \in \mathcal{D}$ which satisfy the previous conditions (i), (ii), (iii), (iv) and have at most one ordinary node as their only singularity. Note that

$$\mathcal{D}_0 \subset U$$

as a divisor. Indeed (i) implies that $U$ is ruled by the family of surfaces $U_D =: U \cap |D|$, where $D \in U$. Moreover $U_D \cap \mathcal{D}_0$ is pure of dimension 1. We consider the natural map

$$f : U \to \overline{\mathcal{M}}_{15}.$$ 

Since $U$ is open in $\mathcal{D}$ the map $f$ is dominant, we want to show something more:

(4.14) **PROPOSITION** The map $f/\mathcal{D}_0 : \mathcal{D}_0 \to \Delta_0$ is dominant.

**PROOF** It suffices to show that $f : U \to \overline{\mathcal{M}}_{15}$ has fibres of constant dimension. Then, since $\mathcal{D}_0$ is a divisor in $U$ and $f(\mathcal{D}_0)$ is contained in the irreducible divisor $\Delta_0$, the statement follows from a count of dimensions. For any $D \in U$ let us consider the fibre $F_D = f^{-1}(f(D))$ and the natural morphism

$$h : F_D \to W^1_9(D) \subset Pic^9(D)$$

sending $D' \in F_D$ to the point $\omega_{D'}(-1)$ of the Brill-Noether locus $W^1_9(D)$. We know that $D'$ is linearly normal, therefore $h^{-1}(h(D'))$ is just the family of all curves projectively equivalent to $D'$. In particular $\dim h^{-1}(D') = \dim PGL(7)$. On the other hand the Petri map

$$\mu_{D'} : H^0(\omega_{D'}(-1)) \otimes H^0(\mathcal{O}_{D'}(1)) \to H^0(\omega_{D'})$$

is injective i.e. its corank is one. This implies that $W^1_9(D)$ is smooth of dimension one at its point $\omega_{D'}(-1)$. Let $L$ be in a small neighborhood $N$ of $\omega_{D'}(-1)$ in $W^1_9(D)$. Then, by standard semicontinuity arguments, we can assume that $|L|$ is a base-point-free pencil and that $\omega_D \otimes L^{-1}$ defines an embedding of $D$ in $\mathbb{P}^6$ as a linearly normal curve $D''$. By the same semicontinuity arguments we can also assume that $D'' \in U$ so that $h(D'') = L$. But then $N \subset h(F_D)$ and it follows that each irreducible component of $h(F_D)$ is a curve. This implies that $\dim F_D = \dim PGL(7) + 1 = 49$, for each fibre $F_D$.

(4.15) **PROOF OF THE MAIN THEOREM** We can finally conclude this note by proving that $\mathcal{M}_{15}$ is rationally connected. Fix two general points $[D_1]$ and $[D_2]$ in $\mathcal{M}_{15}$, then fix a general $L_i$ in the Brill-Noether locus $W^1_9(D_i)$, $i = 1, 2$. Consider the embedding

$$D_i \subset \mathbb{P}^6$$

defined by the line bundle $\omega_{D_i} \otimes L_i^{-1}$. $D_i$ is a general element of $\mathcal{D}$, in particular

$$D_i \subset S_i$$

where $S_i$ is a smooth complete intersection of 4 quadrics. Take a general pencil $P_i \subset |D_i|$ containing $D_i$ and consider one element $D^0_i \in P_i$ which is a singular curve. Then $[D^0_i]$ is general in $\Delta_0$ and hence smooth for $\overline{\mathcal{M}}_{15}$. Since $\Delta_0$ is unirational,
there exists an irreducible rational curve \( R \) containing \([D_1]^0\) and \([D_2]^0\). Let \( R_i \) be the image of \( P_i \) in \( \overline{\mathcal{M}}_{15} \) and let \( U \subset \overline{\mathcal{M}}_{15} \) be the open set of regular points. Then

\[
U \cap (R_1 \cup R_2 \cup R)
\]

is a connected chain of rational curves joining \([D_1]\) to \([D_2]\). This implies the rational connectedness of \( \mathcal{M}_{15} \).

\[(4.16)\] **Remark** The unirationality of \( \mathcal{W} \) and \( \mathcal{M}_{15} \) would follow if there exists a unirational variety \( V \subset D \) which intersect a general \( |D| \) in finitely many points. It is not clear to us whether such a \( V \) exists.

5. References

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris *Geometry of Algebraic Curves I* Springer-Verlag, Berlin (1984), 1-386

[CR1] M.C. Chang, Z. Ran *Unirationality of the moduli space of curves of genus 11, 13 (and 12)*, Invent. Math. 76 (1984) 41-54,

[FP] G. Farkas, M. Popa *Effective divisors on \( M_g \) and a counterexample to the Slope Conjecture* preprint (2002)

[HH] R. Hartshorne, A. Hirschowitz *Smoothing Algebraic Space Curves*, in Algebraic Geometry, Sitges 1983 (E. Casas-Alvero, G.E. Welters, S. Xambo-Descamps eds.) L.N.M. 1124 (1985), 98-131

[HM1] J. Harris, I. Morrison *Moduli of Curves*, Springer-Verlag Berlin (1991)

[HM2] *Slopes of effective divisors on the moduli space of curves*, Invent. Math. 99 (1990), 321-335

[MM] S. Mori, S. Mukai *The uniruledness of the moduli space of curves of genus 11*, in Algebraic Geometry, Proceedings Tokio/Kyoto 1982 L.N.M 1016 (M.Raynaud, T. Shioda eds.) (1983), 334-353

[S1] E. Sernesi *L’unirazionalità della varietà dei moduli delle curve di genere 12*, Ann. Sc. Norm. Sup. Pisa 8 (1981), 405-439

[S2] E. Sernesi *On the existence of certain families of curves*, Invent. Math. 75 (1984), 25-57

[Se] F. Severi *Vorlesungen über Algebraische Geometrie*, (E. Loeffler ubersetzzung), Teubner, Leipzig (1921),

[ST] F.O. Schreyer, F. Tonoli *Needles in a haystack: special varieties via small fields*, in Mathematical computations with Macaulay 2, (D. Eisenbud, D. Grayson, M. Stillman, B. Sturmfels eds.), Springer-Verlag, Berlin (2002)

[V] A. Verra *The unirationality of the moduli space of curves of genus \( g \leq 14 \) preprint (2004)