How to Define Global Lie Group Actions on Functions

Elemér E Rosinger

Department of Mathematics and Applied Mathematics
University of Pretoria
0002 Pretoria
South Africa
eerosinger@hotmail.com

Abstract

A particularly easy, even if for long overlooked way is presented for defining globally arbitrary Lie group actions on smooth functions on Euclidean domains. This way is based on the appropriate use of the usual parametric representation of functions. As a further benefit of this way, one can define large classes of genuine Lie semigroup actions. Here "genuine" means that, unlike in the literature, such Lie semigroups need no longer be sub-semigroups of Lie groups, and instead, can contain arbitrary noninvertible smooth functions on Euclidean domains.

1. Introduction

The advantages of being able to define global actions for arbitrary Lie groups are well known for at least six decades by now, as presented systematically in the celebrated text of Chevalley, for instance. Yet, even in the case of Lie groups acting on Euclidean spaces, and not on manifolds in general, the customary approach has not been able to go beyond a mere local definition, when it comes to actions on functions by arbitrary Lie groups, see for instance Bluman & Kumei, Ibragimov, or Olver [1-3].
Rather surprisingly, this failure to define globally the action on functions of arbitrary Lie groups is due to an elementary difficulty, which can easily be overcome by a parametric definition of functions, as shown for the first time in Rosinger [6], see also Rosinger [7].

This parametric approach proves to have in fact two important advantages, namely, one of calculus, and the other of functorial nature. The calculus advantage relates to the simple and well known fact that the partial derivatives of any order of a parametrically given function can be computed from it, without first having to bring the function to the usual, nonparametric form. The functorial advantage, relating perhaps even to a simpler fact, is the one which will actually allow the most easy, direct and natural global definition of arbitrary Lie group actions on functions. In fact, as shown in Rosinger [6,7] and mentioned in the sequel, it allows as well for the equally easy global definition of a far larger class of Lie semigroup actions.

As a general remark about the parametric approach to the global definition of arbitrary Lie group actions on functions, it is rather ironic to note that, in an embryonic, partial and local manner, this approach has in fact been in use for a long time by now.

Indeed, suppose given a smooth function \( f : \Omega \rightarrow \mathbb{R} \), with \( \Omega \subseteq \mathbb{R}^n \) nonvoid, open. Further, suppose given an arbitrary Lie group \( G \) acting on \( M = \Omega \times \mathbb{R} \) according to

\[
G \times M \rightarrow M
\]

Then the usual way this Lie group action on \( M \) is extended to such functions \( f \), and thus to \( C^\infty(\Omega, \mathbb{R}) \), is as follows. We consider the graph of \( f \), that is, the set

\[
\gamma_f = \{ (x, f(x)) \mid x \in \Omega \} \subseteq M
\]

Therefore, for any \( g \in G \), we can define point-wise the action \( g\gamma_f \) and obtain again a subset of \( M \).
Unfortunately however, in general, it will not be true that

\[ g\gamma f = \gamma h \]

for a certain smooth function \( h : \Omega \rightarrow \mathbb{R} \), which function \( h \) if it existed, it would obviously correspond to the global action of \( g \) on \( f \), that is, we would have

\[ gf = h \]

And then, the usual way to define arbitrary Lie group actions on functions overcomes this difficulty at the cost of no less than a double localization, Olver [1-3], namely

- \( g \) is restricted to a neighbourhood of the identity \( e \in G \), and in addition
- \( f \) is restricted to suitable nonvoid, open subsets \( \Delta \) of \( \Omega \).

It is clear, however, that the consideration of the graph \( \gamma f \) of \( f \) amounts to replacing \( f : \Omega \rightarrow \mathbb{R} \) by the following special parametric form of it, see (3.3), (3.4) in the sequel, namely \( f_* : \Omega \rightarrow M \), where \( \Omega \ni x \mapsto f_*(x) = (x, f(x)) \in M \). Furthermore, in this case \( g\gamma f \) is nothing else but \( gf_* \), that is, the action of \( g \) on \( f_* \), which can always be defined globally, irrespective of the function \( f \), or of the Lie group action \( G \) on \( M \).

Thus it becomes clear that the only difficulty we have ever faced when trying to define globally arbitrary Lie group actions on functions is not at all related to Lie groups or functions, but solely to our rather unformulated, and yet quite implacable intent to have \( gf_* \) retranslated into a usual, nonparametric function \( h : \Omega \rightarrow \mathbb{R} \).

On the other hand, the parametric approach to Lie group actions introduced in Rosinger [6], is adopted and pursued in its full extent, that is, without any sort of localization, this being the simple and fundamental reason for the fact that arbitrary Lie group actions can be defined globally on smooth functions.
Furthermore, as shown in Rosinger [6], this possibility to define globally arbitrary Lie group actions on smooth functions can easily be extended to actions on large classes of generalized functions, and in particular, distributions, one of the effects of such an extension being the first general solution of Hilbert’s Fifth Problem, Rosinger [6].

Also as mentioned and shown briefly in the sequel, one can define globally on functions the action of far larger classes of Lie semigroups. This comes as a rather unexpected bonus, and the effect of the mentioned functorial nature of the parametric approach to Lie group actions which allows the definition of arbitrary smooth - thus typically noninvertible - actions. Such noninvertible actions can, of course, no longer belong to Lie group actions, but only to Lie semigroup actions, Rosinger [6,7].

Let us mention here in passing that the interest in such Lie semigroups of actions comes from the fact that they range over a significantly larger class of actions than those corresponding to Lie groups. Therefore, when applied to the study of solutions of PDEs - this time as semisymmetries - they can offer new additional insights. Furthermore, as pointed out by P J Olver, semigroups of actions appear quite naturally in several aspects of the classical Lie theory, see for details Rosinger [6, chap. 13], [7].

2. Difficulties with Actions on Usual Functions

**Classical Lie Group Actions.** For convenience, let us consider the familiar and important setup when Lie group actions are used in the study of PDEs. In such cases, we are given a linear or nonlinear PDEs of the general form

\[(2.1) \quad T(x, D)U(x) = 0, \quad x \in \Omega \subseteq \mathbb{R}^n\]

where \(\Omega\) is nonvoid open, \(U : \Omega \rightarrow \mathbb{R}\) is the unknown function, while \(T(x, D)\) is a \(C^\infty\)-smooth linear or nonlinear partial differential operator. The relevant Lie groups \(G\) act on the open subset \(M = \Omega \times \mathbb{R} \subseteq \mathbb{R}^{n+1}\), according to
\[(2.2) \quad G \times M \ni (g, (x, u)) \mapsto g(x, u) = (g_1(x, u), g_2(x, u)) \in M\]

where \(x \in \Omega, \ u \in \mathbb{R}\) are the independent and dependent variables, respectively, and

\[(2.3) \quad G \times M \ni (g, (x, u)) \mapsto g_1(x, u) \in \Omega \quad G \times M \ni (g, (x, u)) \mapsto g_2(x, u) \in \mathbb{R}\]

with \(g_1\) and \(g_2\) being \(C^\infty\)-smooth.

We note that, given \(g \in G\), in view of the Lie group axioms, it follows that the mapping

\[(2.4) \quad M \ni (x, u) \mapsto g(x, u) \in M\]

is a \(C^\infty\)-smooth diffeomorphism.

A first basic problem in Lie group theory, when applied to PDEs, is how to extend the action in (2.2), (2.3) of the Lie group \(G\) on the open subset \(M\), to an action of \(G\) on the \(C^\infty\)-smooth functions

\[(2.5) \quad U : \Omega \longrightarrow \mathbb{R}\]

or more generally, on \(C^\infty\)-smooth functions

\[(2.6) \quad U : \triangle \longrightarrow \mathbb{R}\]

where \(\triangle \subseteq \Omega\) is nonvoid, open. And unless one solves this problem, one simply cannot speak about the Lie group invariance of classical solutions of PDEs.

From this point of view, the Lie group actions (2.2), (2.3) are divided in two types, Olver [1,2].

The simpler ones, called projectable, or fibre preserving, satisfy the condition, see (2.3)

\[(2.7) \quad g_1(x, u) = g_1(x), \quad g \in G, \ (x, u) \in M\]
The special interest in Lie group actions (2.7) comes from the fact that they allow an easy global extension to action on $C^\infty$-smooth functions. Indeed, in this case, in view of (2.4), it follows that for $g \in G$, we obtain the $C^\infty$-smooth diffeomorphism

\[(2.8) \quad \Omega \ni x \mapsto g_1(x) \in \Omega\]

Now, given $g \in G$ and $U$ in (2.6), it is easy to define the respective global Lie group action

\[(2.9) \quad g \ U = \tilde{U} : \tilde{\Delta} = g_1(\Delta) \longrightarrow \mathbb{R}\]

by

\[(2.10) \quad \tilde{U}(g_1(x)) = g_2(x, U(x)), \quad x \in \Delta\]

Indeed, (2.4) implies that in (2.9), we have $\tilde{\Delta} \subseteq \Omega$ nonvoid, open, while (2.10) is equivalent with

\[(2.11) \quad \tilde{U}(\tilde{x}) = g_2(g_1^{-1}(\tilde{x}), U(g_1^{-1}(\tilde{x}))), \quad \tilde{x} \in \tilde{\Delta}\]

However, an arbitrary Lie group action (2.2), (2.3) need not be projectable. And in such a case the global extension of the Lie group action (2.2), (2.3) to $C^\infty$-smooth functions (2.5), or in general (2.6), will typically fail. In this way, we are obliged, Olver [1,2], to limit ourselves to local Lie group actions on functions, and thus return to the pre-Chevalley stage of Lie group theory.

Indeed, in the case of general, nonprojectable Lie group actions (2.2), (2.3), we may immediately run into the problem of possible noninvertibility. Namely, certain $C^\infty$-smooth mappings involved in the definition of the group action $g \ U = \tilde{U} : \tilde{\Delta} \longrightarrow \mathbb{R}$ may fail to have inverses, let alone, $C^\infty$-smooth ones. Let us illustrate this phenomenon in more detail. Given $g \in G$, let us write (2.3) in the form

\[(2.12) \quad \tilde{x} = g_1(x, u) \quad \tilde{u} = g_2(x, u)\]

where $(x, u), (\tilde{x}, \tilde{u}) \in M$. Given now $U : \Delta \longrightarrow \mathbb{R}$ as in (2.6), the
natural way to define the group action \( gU = \tilde{U} : \tilde{\Delta} \to \mathbb{R} \) would be by the relation, see (2.12)

\[
(2.13) \quad \tilde{U}(g_1(x, U(x))) = g_2(x, U(x)), \quad x \in \Delta
\]

which means that \( \tilde{U}(\tilde{x}) = \tilde{u} \). However, in order that (2.13) be a correct definition, we have to be able to obtain \( x \in \Delta \) as a \( C^\infty \)-smooth function of \( \tilde{x} \in \tilde{\Delta} \), by using the first equation in

\[
(2.14) \quad \begin{align*}
\tilde{x} &= g_1(x, U(x)) \\
\tilde{u} &= g_2(x, U(x))
\end{align*}
\]

and thus by replacing \( x \in \Delta \) in the second equation above, in order to obtain \( \tilde{u} \) as a function of \( \tilde{x} \), that is, the relation (2.13). Furthermore, one also has to obtain \( \tilde{\Delta} \subseteq \Omega \) as being nonvoid, open. The crucial issue here is, therefore, the \( C^\infty \)-smooth invertibility of the mapping

\[
(2.15) \quad \Delta \ni x \mapsto \tilde{g}_1(x, U(x)) \in \Omega
\]

which obviously depends on \( g \) and \( U \). And as seen in the very simple example next, this in general is not possible.

**Example 2.1.**

Let us consider the following nonprojectable case of the Lie group action (2.2), (2.3), where \( \Omega = \mathbb{R} \), \( M = \Omega \times \mathbb{R} = \mathbb{R}^2 \), \( G = (\mathbb{R}, +) \), and for \( \epsilon = g \in G = \mathbb{R} \), \( (x, u) \in M \), we have

\[
\begin{align*}
\tilde{x} &= x + \epsilon u^2 \\
\tilde{u} &= u
\end{align*}
\]

Let us take \( \Delta = \Omega = \mathbb{R} \) and the simple function \( U : \Delta \to \mathbb{R} \) defined by \( U(x) = x \), with \( x \in \Delta \). Then (2.15) becomes

\[
\mathbb{R} \ni x \mapsto x + \epsilon x^2 \in \mathbb{R}
\]

which is not invertible as a function, let alone as a \( C^\infty \)-smooth function, except for the trivial group action corresponding to \( \epsilon = 0 \), that is, to the identical group transformation.
The usual way to deal with this situation, Olver [1,2], is to consider the group action (2.2), (2.3) as well as the mapping $\alpha$ in (2.15), and therefore the function to be acted upon $U : \triangle \rightarrow \mathbb{R}$, only locally, that is, to restrict all of them to such suitable neighbourhoods of the neutral element $e \in G$, as well as of points $x \in \triangle$, on which $\alpha$ is $C^\infty$-smooth invertible.

It is useful to note however that, depending also on the function $U$ in (2.6), the mapping $\alpha$ in (2.15) can sometime happen to have a global, and not only local $C^\infty$-smooth inverse, even in the case of a nonprojectable Lie group action. For instance, this happens if in the above Example 2.1., we consider $\bar{x} = x + \epsilon u$.

Let us mention what happens when the mapping $\alpha$ in (2.15) is invertible, regardless of the Lie group action being projectable or not, and when its inverse $\alpha^{-1}$ is also a $C^\infty$-smooth mapping. Then we can indeed turn to (2.13) in order to define the group action $g U = \tilde{U}$ by

\begin{equation}
\tilde{U}(\bar{x}) = g_2(\alpha^{-1}(x), U(\alpha^{-1}(\bar{x}))), \quad \bar{x} \in \tilde{\triangle}
\end{equation}

where

\begin{equation}
\tilde{\triangle} = \alpha(\triangle) \text{ is open}
\end{equation}

Obviously, the case of projectable Lie group actions in (2.7) - (2.11) is included in (2.16), (2.17).

As mentioned in the Introduction, here, following Rosinger [6,7], we take a new route, when dealing with the difficulties in (2.12) - (2.15), which we face in the case of general, nonprojectable Lie group actions (2.2), (2.3). This new route will not require the above mentioned traditional localisation of $g \in G$, $\alpha$ or $U$. In other words, we are able to perform globally arbitrary Lie group actions on functions $U$ defined on the whole of their unrestricted, original domains, as for instance in (2.5) and (2.6). Fortunately, this construction is particularly simple and applicable without any undue restrictions.
**A Simple, Basic Observation.** To summarize. The basis upon which we can develop this global approach is the following rather simple observation:

- The usual impediment which prevents us from extending arbitrary Lie group actions (2.2), (2.3) to global actions on functions (2.5) or (2.6) is not at all related to Lie groups, but to the usual way of representing functions, by discriminating between independent and dependent variables. Once one does away with such a discrimination, by using a parametric representation of functions, the way to a natural and easy global Lie group action on functions is open.

**Parametrisation** in its essence amounts to the following *embedding* of the usual definition of a function into a larger concept. Namely, a usual function

\[(2.18) \quad A \ni x \mapsto y = f(x) \in B\]

is actually *constrained* to be a correspondence from the set \(A\) of its independent variable \(x\), to the set \(B\) of its dependent variable \(y\).

On the other hand, a parametric representation of \(f\) can be given by any *pull-back* type mapping

\[(2.19) \quad P \ni p \mapsto (x(p), y(p)) \in A \times B\]

which maps any suitably given parameter domain \(P\) into the graph of \(f\), under the following two conditions:

\[(2.20) \quad y(p) = f(x(p)), \quad p \in P\]

and

\[(2.21) \quad P \ni p \mapsto x(p) \in A \text{ is surjective}\]
With respect to $P$, this, in general, only implies that its cardinal is not smaller than that of $A$.

However, when dealing with Lie group actions, the parameter domain $P$ is required to be a suitable open subset in an Euclidean space, while the parametrisation $h$ is assumed $C^\infty$-smooth.

It follows that, in general, a parametric representation will introduce an additional variable $p$, ranging over $P$, which this time is mapped into the pair $(x(p), y(p))$ of the original independent and dependent variables, pair which is an element in the cartesian product $A \times B$.

This kind of embedding, obtained by introducing an additional variable, and thus going beyond the constraint of only dealing with the usual independent and dependent variables, proves to have an important and naturally built in advantage. Namely, it allows for the first time - and in a straightforward manner - the global definition of arbitrary Lie group actions on functions.

In the usual, that is, nonparametric approach, however, when one want to define the Lie group action on a function, and obtain again a function, one cannot in general do so, unless at the end one is able to separate the independent and dependent variables, by expressing the latter as a function of the former. And in the nonprojectable case of Lie group actions, this typically is not possible, except locally in the independent variable, and also, near to the trivial, identical Lie group transformation.

On the other hand, if one starts, and ends, with parametrically given functions, then as shown in Rosinger [6,7] and seen in the sequel, one has no difficulties at all.

### 3. Parametric Functions

**Need for a Global Approach.** It is instructive to give another simple example, which by its particularly familiar setup, can further highlight the rather basic, yet extreme difficulties one may face when
trying to define \textit{globally} the action of a \textit{nonprojectable} Lie group on a function.

\textbf{Example 3.1.}

Let us consider the Lie group action given by the usual \textit{rotation of the plane}. In terms of (2.2), (2.3), it means that $\Omega = \mathbb{R}$, $M = \Omega \times \mathbb{R} = \mathbb{R}^2$, $G = (\mathbb{R}, +)$ and for $\theta = g \in G = \mathbb{R}$, $(x, u) \in M$, we have

$$
\begin{align*}
\tilde{x} &= x \cos \theta - u \sin \theta \\
\tilde{u} &= x \sin \theta + u \cos \theta
\end{align*}
$$

therefore, here again, we are dealing with a nonprojectable Lie group action, since for a given $\theta = g$, obviously $\tilde{x}$ depends not only on $x$, but also on $u$, see (2.7).

Let $\triangle = \Omega = \mathbb{R}$, and $U : \triangle \rightarrow \mathbb{R}$ be given by the \textit{parabola} $U(x) = x^2$, with $x \in \triangle$.

Then (2.15) takes the form

$$
\mathbb{R} \ni x \xrightarrow{\alpha} x \cos \theta - x^2 \sin \theta \in \mathbb{R}
$$

which, again, is \textit{not} invertible, except for the trivial group actions, for which $\theta = k\pi$, with $k \in \mathbb{Z}$.

It follows that, except for a trivial rotation of $\theta = \pm \pi$, which in this case amounts to nothing else but a mere symmetry with respect to the origin of coordinates, the parabola

$$
\triangle = \mathbb{R} \ni x \longmapsto x^2 \in \mathbb{R}
$$

when taken as a whole, \textit{cannot} be rotated at all in the plane, without ceasing to be the \textit{graph} of a function from $\triangle = \mathbb{R}$ to $\mathbb{R}$. Yet it is clear that, as a geometric object, by arbitrarily rotating in the plane a parabola, one again gets a parabola.

Therefore, the difficulty must lie with the particular way one happens to \textit{represent} the parabola, that is, as a function from $\triangle = \mathbb{R}$ to $\mathbb{R}$.
Parametric Representations. It turns out that the alternative way to represent functions $U : \triangle \rightarrow \mathbb{R}$ in (2.5), (2.6), namely, parametrically, avoids the above difficulties related to the possible lack of $C^\infty$-smooth inverse of the mapping in (2.15), thus allows for the definition of global Lie group actions on the respective functions.

Let us recall here that parametric representation, and not only of functions, is a rather familiar method in differential geometry, among others, where it is used to define, for instance, the concept of submanifold.

Here, parametric representation is only employed for functions such as those in (2.5), (2.6).

Given therefore a $C^\infty$-smooth function

$$ U : \triangle \rightarrow \mathbb{R} $$

where $\triangle \subseteq \mathbb{R}^n$ is nonvoid, open, we denoted its graph by

$$ \gamma_U = \{ (x, U(x)) \mid x \in \triangle \} \subseteq M $$

Now, a parametric representation of $U$ is given by any $C^\infty$-smooth function

$$ V : \Lambda \rightarrow M $$

where the set $\Lambda \subseteq \mathbb{R}^n$ of parameters is nonvoid, open, and such that

$$ V(\Lambda) = \gamma_U $$

As seen in (3.22) - (3.25) and (3.31) - (3.34) in the sequel, the above condition (3.4), although seemingly quite weak, has certain useful implications.

It is important to note that in (3.3), the set $\Lambda$ of parameters is assumed to be $n$-dimensional. This however, is in line with the fact that the domain of definition $\triangle$ of $U$ in (3.1) is also $n$-dimensional. In par-
ticular, since $M$ in (3.2) is $n+1$-dimensional, it follows that condition (3.4) is quite natural. Later, when in (3.6) we define the class of arbitrary parametrically given functions, which are of interest here, we shall hold to this assumption on the dimension of the set of parameters.

**Canonical Parametrisations.** Clearly, an immediate, simple and natural parametric representation of $U$ in (3.1) is given by

$$\triangle \ni x \mapsto U_*(x) = (x, U(x)) \in M$$

and we shall call $U_* : \triangle \to M$ the *canonical* parametric representation of $U : \triangle \to \mathbb{R}$. Thus in terms of (3.3), we have $\Lambda = \triangle$ and $V = U_*$, and clearly, condition (3.4) is satisfied.

However, it is obvious that a function $U$ in (3.1) can have many other parametric representations (3.3), (3.4). Details in this respect can be found in the sequel. In particular, we shall see in (3.22) and (3.24) that in a certain sense $U_*$ is the *simplest possible* parametric representation of $U$.

**Classes of Parametrisations.** Clearly, the set of functions in (3.3), (3.4) is larger than that in (3.1). More precisely, not every function $V$ in (3.3), (3.4) is the parametric representation of a function $U$ in (3.1). For instance, a nontrivially rotated parabola can be written as a function in (3.3), (3.4), but not as a function in (3.1).

Let us, therefore, denote by

$$C_\infty^n(M)$$

the set of all $C^\infty$-smooth functions $V : \Lambda \to M$, where $\Lambda \subseteq \mathbb{R}^n$ is nonvoid, open, and call them $n$-dimensional *parametric representations* in $M$. Also, let us denote by

$$C_\infty^\text{par}(\Omega)$$

the set of $C^\infty$-smooth *partial* functions $U : \triangle \to \mathbb{R}$, see (3.1), where $\triangle \subseteq \Omega$ is nonvoid, open. Then (3.5) yields an *embedding*
while on the other hand, in view of (3.3), (3.4), we obtain a mapping

\[ C^\infty_{\text{par}}(\Omega) \ni U \mapsto P_U \subseteq C^\infty_n(M) \]

where \( P_U \) is the set of mappings \( V \) in (3.3), which satisfy (3.4). In other words, \( P_U \) is the set of all parametric representations of \( U \). And in view of (3.5), it is clear that

\[ U_* \in P_U \neq \phi, \quad U \in C^\infty_{\text{par}}(\Omega) \]

The important point is that the construction of arbitrary nonlinear Lie group actions on the set of functions \( C^\infty_n(M) \) will no longer suffer from the above difficulties related to the possible lack of a \( C^\infty \)-smooth inverse of the mappings in (2.15).

Similar to (3.7), it will be useful, for \( \ell \in \mathbb{N}, \ \ell \geq 1 \) and \( N \subseteq \mathbb{R}^\ell \) nonvoid, open, to denote by

\[ C^\infty_{\text{par}}(\Omega, N) \]

the set of all partial functions \( U : \Delta \rightarrow N \) which are \( C^\infty \)-smooth, where \( \Delta \subseteq \Omega \) is any nonvoid, open subset.

Obviously \( C^\infty_{\text{par}}(\Omega, M) \subseteq C^\infty_n(M) \).

**Comparing Parametrisations.** Here we further clarify the meaning of the parametric representation of functions, defined in (3.1) - (3.6).

Let us define a preorder \( \leq \) on \( C^\infty_n(M) \), that is, a reflexive and transitive binary relation, as follows. Given \( \Lambda \xrightarrow{V} M \) and \( \Lambda' \xrightarrow{V'} M \), with \( \Lambda, \Lambda' \subseteq \mathbb{R}^n \) nonvoid, open, then

\[ V \leq V' \]
if and only if there exists a surjective $C^\infty$-smooth mapping $\Lambda \xrightarrow{\varphi} \Lambda'$, such that the diagram is commutative

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{V} & M \\
\downarrow \varphi & & \downarrow \varphi' \\
\Lambda' & \xrightarrow{V'} & \\
\end{array}
\]

(3.13)

It is easy to see that, owing to the surjectivity of $\varphi$, we obtain

\[
V(\Lambda) = V'(\Lambda')
\]

(3.14)

A natural interpretation of this preorder $V \leq V'$ is that the parametrisation $V'$ is simpler than $V$. This is illustrated in Example 3.2.

Let $\Lambda = \Omega \subseteq \mathbb{R}^n$ be nonvoid, open, and let us take any $\varphi : \Omega \rightarrow \Omega$ which is $C^\infty$-smooth and surjective, but it is not injective. Also, let us take any $C^\infty$-smooth $U : \Omega \rightarrow \mathbb{R}$. We can now define the parametric function in $C^\infty_n(M)$, namely

\[
\Lambda = \Omega \ni x \xmapsto{V} V(x) = (\varphi(x), U(\varphi(x))) \in M
\]

(3.15)

Then clearly

\[
V \leq U
\]

(3.16)
However

\[ U_* \not\subseteq V \]  

indeed, assume \( \Omega \xrightarrow{\Psi} \Lambda \) is surjective and \( \mathcal{C}^\infty \)-smooth, and such that \( U_* = V \circ \Psi \). Then \( x = \varphi(\Psi(x)), x \in \Omega \), which means that, contrary to the assumption, \( \varphi \) is injective, since \( \Psi \) is surjective.

\[ \square \]

Recalling that, see (3.5)

\[ \Omega \ni x \mapsto (x, U(x)) \in M \]  

and comparing it with (3.15), where \( \varphi \) can be arbitrary under the mentioned assumptions, it follows that a natural meaning of (3.16), (3.17) is that the canonical parametric representation \( U_* \) of \( U \) is simpler than all the other parametric representations of \( U \), given by \( V \) in (3.15), see also (3.22) and (3.24) below.

**Basic Properties of Parametric Representations.** It is useful to note that the simple looking condition (3.4) is precisely the one which leads to the situation in Example 3.2. Indeed, let \( U : \Delta \rightarrow \mathbb{R} \) be a \( \mathcal{C}^\infty \)-smooth function, with \( \Delta \subseteq \Omega \) nonvoid, open, and let \( \Lambda \rightarrow M \) be any function in \( \mathcal{C}^\infty_n(M) \), which therefore acts according to

\[ \Lambda \ni y \mapsto V(y) = (V_1(y), V_2(y)) \in M \]  

\[ \Lambda \ni y \mapsto V_1(y) \in \Omega \]  

\[ \Lambda \ni y \mapsto V_2(y) \in \mathbb{R} \]  

Then it is easy to see that the following four conditions are equivalent:

\[ V \text{ is a parametric representation of } U \]  

\[ V(\Lambda) = \gamma_U \]  

\[ V \leq U_* \]
(3.23) $V_1$ is surjective and $V_2 = U \circ V_1$

In view of the above and (3.9), it follows that

$$\forall \ V \in \mathcal{P}_U :$$

(3.24)

$$V \leq U_*$$

In this way, in view of (3.22) and (3.24), and in the sense of Example 3.2., it is clear that for any given function $U$, its canonical parametric representation $U_*$ is simpler than any other parametric representation of that function.

Moreover, given two $C^\infty$-smooth functions $U_1, U_2 : \triangle \rightarrow \mathbb{R}$ in $C^\infty_{\text{par}}(\Omega)$, then

(3.25) $(U_1)_* \leq (U_2)_* \implies U_1 = U_2$

which shows to what a large extent the canonical parametric representation does in fact determine a function.

**Staying with Usual Functions.** We conclude that given $\Lambda \xrightarrow{V} M$ in $C^\infty_n(M)$, then a sufficient condition for the existence of a function $U : \triangle \rightarrow \mathbb{R}$ in $C^\infty_{\text{par}}(\Omega)$, such that $V$ is a parametric representation of $U$, is given by, see (3.23)

(3.26) $\Lambda \xrightarrow{V_1} \triangle$ is a $C^\infty$-smooth diffeomorphism

In this case it also follows that, see (3.19)

(3.27) $U = V_2 \circ V_1^{-1}$

as well as, see (3.22)

(3.28) $V \leq U_*$

However, when one deals with arbitrary nonlinear, and possibly non-projectable Lie group actions on functions, one can encounter the general situation of mappings $\Lambda \xrightarrow{V} M$ in $C^\infty_n(M)$ which may fail
to satisfy (3.26). Thus this condition (3.26) can be seen as the more general reformulation of the $C^\infty$-smooth invertibility problem in (2.15).

**Equivalent Parametrisations.** Let us consider two parametric functions $V : \Lambda \to M$ and $V' : \Lambda' \to M$ in $C^\infty_n(M)$, see (3.6). We say that $V$ and $V'$ are *equivalent*, and write

\[(3.29) \quad V \approx V'\]

if and only if, see (3.4)

\[(3.30) \quad V(\Lambda) = V'(\Lambda')\]

Clearly, if we have a usual function $U$ in (3.1), then in view of (3.4), $V$ will be a parametric representation of $U$, if and only if, see (3.5)

\[(3.31) \quad V \approx U_\ast\]

Also, owing to (3.12) - (3.14) and (3.30), it follows that

\[(3.32) \quad V \leq V' \implies V \approx V'\]

4. Actions on Parametric Representations

**Natural Definition.** Now with the use of parametric representations, we come to the *basic idea* in this paper, namely, we can define the *arbitrary Lie group actions on functions*

\[(4.1) \quad G \times C^\infty_n(M) \to C^\infty_n(M)\]

in the following simple and natural way. Given $g \in G$ and a function $\Lambda \overset{V}{\to} M$ from $C^\infty_n(M)$, we define

\[(4.2) \quad g \cdot V = g \circ V\]

where in the right hand term, $g$ is the mapping in (2.4). In other words, we use as definition of the Lie group action the commutative
Clearly, with the definition (4.2), (4.3), we have

\[(4.4) \quad gV \in C_\infty^\infty(M), \quad \Lambda \xrightarrow{gV} M\]

that is, \(gV\) and \(V\) have the same domain of definition \(\Lambda\), and the same range \(M\).

**Properties.** We show now that the Lie group actions (4.1) contain as a particular case the usual Lie group actions on functions, Olver [1,2], namely

\[G \times C_\infty^\infty(\Omega) \longrightarrow C_\infty^\infty(\Omega)\]

whenever the latter can be defined globally. Indeed, assume given \(g \in G\) and \(U : \triangle \longrightarrow \mathbb{R}\) in (2.26), such that the mapping \(\alpha\) in (2.15) is a \(C^\infty\)-smooth diffeomorphism. In view of (3.5), we obtain \(U_* \in C_\infty^\infty(M)\) and then (2.4) and (4.2) give

\[(4.5) \quad \triangle \xrightarrow{gU_*} M\]

where

\[(4.6) \quad (gU_*)(x) = g(U_*(x)) = g(x, U(x)) = (g_1(x, U(x)), g_2(x, U(x)))\]

with \(x \in \triangle\). On the other hand, in view of our assumption on \(\alpha\), we can apply (2.16), (2.17) and obtain

\[(4.7) \quad (gU)(\tilde{x}) = g_2(\alpha^{-1}(\tilde{x}), U(\alpha^{-1}(\tilde{x}))), \quad \tilde{x} \in \tilde{\triangle} = \alpha(\triangle)\]
therefore (3.5) gives

\[
\begin{array}{ccc}
\widetilde{\Delta} & \xrightarrow{(gU)_*} & M \\
\tilde{x} & \xrightarrow{} & (\tilde{x}, g_2(\alpha^{-1}(\tilde{x}), U(\alpha^{-1}(\tilde{x})))))
\end{array}
\]

(4.8)

Now, from (2.17), (4.5) and (4.8) it is clear that, in general

(4.9) \hspace{1cm} gU_* \neq (gU)_*

since, for instance, their domains of definition need not be the same. However, we have

(4.10) \hspace{1cm} (gU_*)(\Delta) = \gamma_{gU}

since a direct computation gives, see (3.2), (2.2)

(4.11) \hspace{1cm} \gamma_{gU} = g\gamma_U = \{ (g_1(x, U(x)), g_2(x, U(x))) \mid x \in \Delta \}

and on the other hand, see (4.6), (2.2)

(4.12) \hspace{1cm} (gU_*)(\Delta) = \{ (g_1(x, U(x)), g_2(x, U(x))) \mid x \in \Delta \}

It follows that, in view of (3.3), (3.4), the Lie group action \( gU_* \) of \( g \) on the parametric representation \( U_* \) of \( U \), is itself a parametric representation of \( gU \), which is the Lie group action of \( g \) on \( U \). In other words, in general, the diagram

\[
\begin{array}{ccc}
C^\infty_{\text{par}}(\Omega) & \ni U & \xrightarrow{g} & gU \in C^\infty_{\text{par}}(\Omega) \\
\downarrow (\ )_* & & \downarrow (\ )_* \\
C^\infty_n(M) & \ni U_* & \xrightarrow{g} & gU_* \neq (gU)_* \in C^\infty_n(M)
\end{array}
\]

(4.13)
is not commutative, see (4.9). Nevertheless, we have, see (3.9), (3.24), (4.11), (4.12)

\[(4.14) \quad g U_\ast \in \mathcal{P}_{gU}, \quad g U_\ast \leq (g U)_\ast\]

Further, we note that, regardless of (4.9) and (4.13), we obtain the following commutative diagram

\[(4.15) \quad \alpha \quad \alpha^{-1} \quad M \quad (gU)_\ast \quad gU_\ast \quad \Delta' \quad \Delta\]

which follows easily from (4.6), (4.8) and (3.5). In this way, in view of (3.12), (3.13), the above commutative diagram means that

\[(4.16) \quad g U_\ast \leq (g U)_\ast \leq g U_\ast\]

in other words, in case the usual Lie group action \(g U\) of \(g \in G\) on the function \(U\) exists globally, then its canonical parametric representation \((g U)_\ast\) is both more simple and less simple than \(g U_\ast\), which is the Lie group action on the canonical parametric representation \(U_\ast\) of \(U\), and which always exists.

**Remark 4.1.**

In view of the commutative diagram (4.15), and the double inequality in (4.16), it appears to be natural to use the global Lie group action
\( g \, U_* \), which always exists, instead of the Lie group action \( g \, U \), since as seen in section 2, the latter need not always exist.

In fact, the essential interest in using parametric representations is that we can abandon \( (g \, U)_* \) in (4.9), (4.13) - (4.16), and instead, use \( g \, U_* \), which always exist globally, and which also happens to be a parametric representation of \( g \, U \), whenever the latter exists globally in the classical sense, Olver [1,2].

\[ \square \]

Finally, related to the commutative diagram (4.15), and the double inequality (4.16), we have the following additional universality type properties. Given \( \Lambda \xrightarrow{V} M \) a function from \( C^\infty_n(M) \), such that the diagram of \( C^\infty \)-smooth mappings

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{\lambda} & M \\
\downarrow & & \downarrow \\
\Lambda & & M
\end{array}
\]

is commutative, then

\[ V \circ \lambda \circ \alpha = U_* \]

Indeed, (4.15) - (4.17) yield

\[ g \, U_* = g \, V \circ \lambda \circ \alpha \]

hence (4.18) follows from (4.2). Similarly, if the diagram of \( C^\infty \)-smooth
(4.19) \( \Lambda \) is commutative, then
\[
(4.20) \quad V = U_* \circ \alpha^{-1} \circ \lambda
\]
Also, if the diagram of \( \mathcal{C}^\infty \)-smooth mappings
\[
(4.21) \quad \lambda
\] is commutative, then
Finally, if the diagram of $C^\infty$-smooth mappings

\[
\begin{array}{c}
\Lambda \\
\downarrow \\
M \\
\downarrow \\
\triangle \\
\end{array}
\]

is commutative, then

\[
V = U_\ast \circ \lambda
\]

The commutative diagram (4.15), the double inequality (4.16) and the universal properties (4.17) - (4.24) give both the explanation and remedy for the failures in (4.9) and (4.13).

5. Comments

The *novelty* of the extension of Lie group actions to parametric functions, as defined in (4.2), (4.3), when compared with the usual one in section 2, becomes now clear. Indeed, in the usual approach, Bluman & Kumei, Ibragimov, Olver [1-3], one proceeds as follows.

First, one defines the Lie group action (2.2) on the set $M$ of independent and dependent variables, respectively, $x \in \Omega$ and $u \in \mathbb{R}$.

Then as a second step, one extends this initial Lie group action to
functions $U : \triangle \rightarrow \mathbb{R}$ in (2.6).

This extension is done by replacing $U$ with its graph $\gamma_U \subseteq M$, and then letting the Lie group act pointwise on $\gamma_U$, seen as a subset of $M$. Certainly, for every $g \in G$, we obtain in this way a well defined subset $g\gamma_U \subseteq M$. However, for nonprojectable Lie groups it need not in general happen that

\begin{equation}
\exists \quad \tilde{U} \in \mathcal{C}_{\text{par}}^\infty(\Omega) : \\
g\gamma_U = \gamma_{\tilde{U}}
\end{equation}

that is, the subset $g\gamma_U$ need not be the graph of any function $\tilde{U} : \tilde{\triangle} \rightarrow \mathbb{R}$, where $\tilde{\triangle} \subseteq \Omega$ is nonvoid, open. In this way, the usual method of extending the Lie group action (2.2) from the set $M$ to functions $U$ in $\mathcal{C}_{\text{par}}^\infty(\Omega)$, by using the graph $\gamma_U \subseteq M$ of $U$, is severely limited in its globality, in the case of nonprojectable Lie groups.

The way out of this nonglobality impasse, as presented in this paper, is based on the observation that the functions $U$ in $\mathcal{C}_{\text{par}}^\infty(\Omega)$ need not be seen as being defined in terms which are necessarily internal or confined to the set $M$ of independent and dependent variables $x \in \Omega$ and $u \in \mathbb{R}$, respectively.

Indeed, by introducing the parametric representation of functions in $\mathcal{C}_{\text{par}}^\infty(\Omega)$, as done in section 3, we can embed $\mathcal{C}_{\text{par}}^\infty(\Omega)$ into the space of parametric functions $\mathcal{C}_{\text{par}}^\infty(M)$, see (3.8). These parametric functions have arbitrary domains, which are nonvoid, open in $\mathbb{R}^n$, however, their range is always in the set $M$ of independent and dependent variables.

And as seen in (4.1) - (4.4), extending globally arbitrary Lie group actions (2.2) to functions in $\mathcal{C}_{\text{par}}^\infty(M)$ is a rather simple and straightforward procedure, being merely the composition of two mappings, each of which always exists. Furthermore, as seen in (4.5) - (4.24), this extension contains as a particular case the usual way Lie group actions (2.2) are extended to functions in $\mathcal{C}_{\text{par}}^\infty(\Omega, \mathbb{R})$, whenever these latter extensions happen to exist globally.
6. Semigroup Actions

It is obvious that (4.3) remains valid, that is, the composition $g \cdot V$ will still be a $C^\infty$-smooth function, and thus an element of $C^\infty_n(M)$, even if the mapping

\[ (6.1) \quad M \xrightarrow{g} M \]

is no longer restricted to being given by the Lie group action (2.2), through the $C^\infty$-smooth diffeomorphism (2.4), but instead, it is simply an arbitrary $C^\infty$-smooth function.

In other words, (4.3) actually defines the following extension of (4.1)

\[ (6.2) \quad C^\infty(M, M) \times C^\infty_n(M) \rightarrow C^\infty_n(M) \]

And since $C^\infty(M, M)$ is a noncommutative semigroup with identity, and not a group, when considered with the usual composition of functions, it is clear that (6.2) is a vast extension of (4.1), no matter which would be the Lie group $G$ considered in (2.2).

Here, it should be noted that there has been an interest in certain Lie semigroup type actions, Hilgert & Neeb, Weinstein. None of them, however, aims anywhere near to the generality of (6.2). Indeed, in Hilgert & Neeb, which follows the work of the school of K H Hofmann, the Lie semigroups considered must be subsemigroups of Lie groups, thus they cannot include the semigroup $C^\infty(M, M)$ which acts in (6.2). As for the concept of grupoid, presented in the survey of Weinstein, it is similarly not capable of including the mentioned semigroups which act in (6.2).

Needless to say, the extension of the symmetry concept from the framework of the Lie group actions in (4.1), to that of the vastly more general semigroup actions in (6.2), can be of a significant interest, among others, in the study of PDEs, even in the case of their classical solutions.

A start in this direction was presented in Rosinger [6, chap. 13] and Rosinger [7].
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