Quantum Expanders and Geometry of Operator Spaces II

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In this appendix to [5] we give a quick proof of an inequality that can be substituted to Hastings’s result from [2], quoted as Lemma 1.9 in [5]. Our inequality is less sharp but also appears to apply with more general (and even matricial) coefficients. It shows that up to a universal constant all moments of the norm of a linear combination of the form

$$S = \sum_j a_j U_j \otimes \bar{U}_j (1 - P)$$

are dominated by those of the corresponding Gaussian sum

$$S' = \sum_j a_j Y_j \otimes \bar{Y}_j'.$$

The advantage is that $S'$ is now simply separately a Gaussian random variable with respect to the independent Gaussian random matrices $(Y_j)$ and $(Y_j').$

We recall that we denote by $P$ the orthogonal projection onto the orthogonal of the identity. Also recall we denote by $S_2^N$ the space $M_N$ equipped with the Hilbert-Schmidt norm ($S_2^N$ can also be naturally identified with $\ell_2^N \otimes_2 \ell_2^N$). We will view elements of the form $\sum x_j \otimes y_j$ with $x_j, y_j \in M_N$ as linear operators acting on $S_2^N$ as follows

$$T(\xi) = \sum_j x_j \xi y_j^*,$$

so that

$$\| \sum x_j \otimes \bar{y}_j \| = \| T \|_{B(S_2^N)}.$$  

We denote by $(U_j)$ a sequence of i.i.d. random $N \times N$-matrices uniformly distributed over the unitary group $U(N)$. We will denote by $(Y_j)$ a sequence of i.i.d. Gaussian random $N \times N$-matrices, more precisely each $Y_j$ is distributed like the variable $Y$ that is such that $\{Y(i,j)N^{1/2}\}$ is a standard family of $N^2$ independent complex Gaussian variables with mean zero and variance 1. In other words $Y(i,j) = (2N)^{-1/2}(g_{ij} + \sqrt{-1} g'_{ij})$ where $g_{ij}, g'_{ij}$ are independent Gaussian normal $N(0,1)$ random variables.

We denote by $(Y_j')$ an independent copy of $(Y_j)$.

We will denote by $\| . \|_q$ the Schatten $q$-norm ($1 \leq q \leq \infty$), i.e. $\| x \|_q = (\text{tr}(|x|^q))^{1/q}$, with the usual convention that for $q = \infty$ this is the operator norm.
Lemma 0.1. There is an absolute constant $C$ such that for any $p \geq 1$ we have for any scalar sequence $(a_j)$ and any $1 \leq q \leq \infty$

$$
\E \left\| \sum_{1}^{n} a_j U_j \otimes \bar{U}_j (1 - P) \right\|_q^p \leq C^p \E \left\| \sum_{1}^{n} a_j Y_j \otimes \bar{Y}_j' \right\|_q^p,
$$

(in fact this holds for all $k$ and all matrices $a_j \in M_k$ with $a_j \otimes$ in place of $a_j$).

Proof. We assume that all three sequences $(U_j)$, $(Y_j)$ and $(\bar{Y}_j)$ are mutually independent. The proof is based on the well known fact that the sequence $(Y_j)$ has the same distribution as $U_j|Y_j|$, or equivalently that the two factors in the polar decomposition $Y_j = U_j|Y_j|$ of $Y_j$ are mutually independent. Let $\mathcal{E}$ denote the conditional expectation operator with respect to the $\sigma$-algebra generated by $(U_j)$. Then we have $U_j \mathcal{E}|Y_j| = \mathcal{E}(U_j|Y_j|) = \mathcal{E}(Y_j)$, and moreover

$$(U_j \otimes \bar{U}_j) \mathcal{E}(|Y_j| \otimes |\bar{Y}_j|) = \mathcal{E}(U_j|Y_j| \otimes \bar{U}_j|Y_j|) = \mathcal{E}(Y_j \otimes \bar{Y}_j).$$

Let

$$T = \mathcal{E}(|Y_j| \otimes |\bar{Y}_j|) = \mathcal{E}(|Y| \otimes |\bar{Y}|).$$

Then we have

$$\sum_{1}^{n} a_j(U_j \otimes \bar{U}_j) T (I - P) = \mathcal{E}\left(\sum_{1}^{n} a_j Y_j \otimes \bar{Y}_j \right)(I - P).$$

Note that by rotational invariance of the Gaussian measure we have $(U \otimes \bar{U})T(U^* \otimes \bar{U}^*) = T$. Indeed, since $UYU^*$ and $Y$ have the same distribution it follows that also $UYU^* \otimes \bar{U}YU^*$ and $Y \otimes \bar{Y}$ have the same distribution, and hence so do their modulus.

Viewing $T$ as a linear map on $S_2^\infty = \ell_2^N \otimes \ell_2^N$, this yields

$$\forall U \in U(N) \quad T(U \xi U^*) = UT(\xi)U^*.$$

Representation theory shows that $T$ must be simply a linear combination of $P$ and $I - P$. Indeed, the unitary representation $U \mapsto U \otimes \bar{U}$ on $U(N)$ decomposes into exactly two distinct irreducibles, by restricting either to the subspace $\mathbb{C}I$ or its orthogonal. Thus, by Schur’s Lemma we know a priori that there are two scalars $\chi_N', \chi_N$ such that $T = \chi_N' P + \chi_N (I - P)$. We may also observe $\mathbb{E}(|Y|^2) = I$ so that $T(I) = I$ and hence $\chi_N' = 1$, therefore

$$T = P + \chi_N (I - P).$$

Moreover, since $T(I) = I$ and $T$ is self-adjoint, $T$ commutes with $P$ and hence $T(I - P) = (I - P)T$, so that we have

$$(0.2) \quad \sum_{1}^{n} a_j(U_j \otimes \bar{U}_j)(1 - P) T = \mathcal{E}\sum_{1}^{n} a_j(Y_j \otimes \bar{Y}_j)(I - P).$$

We claim that $T$ is invertible and that there is an absolute constant $C$ so that

$$||T^{-1}|| = \chi_N^{-1} \leq C.$$

From this and $(0.2)$ follows immediately that for any $p \geq 1$

$$\sum_{1}^{n} a_j(U_j \otimes \bar{U}_j)(1 - P) ||_q^p \leq C^p \mathbb{E} \sum_{1}^{n} a_j(Y_j \otimes \bar{Y}_j)(1 - P) ||_q^p.$$
To check the claim it suffices to compute $\chi_N$. For $i \neq j$ we have a priori $T(e_{ij}) = e_{ij} \langle T(e_{ij}), e_{ij} \rangle$ but (since $\text{tr}(e_{ij}) = 0$) we know $T(e_{ij}) = \chi_N e_{ij}$. Therefore for any $i \neq j$ we have $\chi_N = \langle T(e_{ij}), e_{ij} \rangle$, and the latter we can compute

$$\langle T(e_{ij}), e_{ij} \rangle = \mathbb{E}\text{tr}([Y|e_{ij}|Y^* e_{ij}]) = \mathbb{E}(|Y|_{ii}|Y|_{jj}).$$

Therefore,

$$N(N-1)\chi_N = \sum_{i \neq j} \mathbb{E}(|Y|_{ii}|Y|_{jj}) = \sum_{i \neq j} \mathbb{E}(|Y|_{ii}|Y|_{jj}) - \sum_{j} \mathbb{E}(|Y|_{jj}^2) = \mathbb{E}(\text{tr}|Y|)^2 - N\mathbb{E}(|Y|_{11}^2).$$

Note that $\mathbb{E}(|Y|_{11}^2) = \mathbb{E}(|Y|_{e_1, e_1})^2 \leq \mathbb{E}(|Y|^2 e_{1, e_1}) = \mathbb{E}|Y(1)|^2 = 1$, and hence

$$N(N-1)\chi_N = \sum_{i \neq j} \mathbb{E}(|Y|_{ii}|Y|_{jj}) \geq \mathbb{E}(\text{tr}|Y|)^2 - N.$$

Now it is well known that $E|Y| = b_N I$ where $b_N$ is determined by $b_N = N^{-1} \mathbb{E}\text{tr}|Y| = N^{-1}||Y||_1$ and $\inf_N b_N > 0$ (see e.g. [3] p.80). Actually, by Wigner’s limit theorem, when $N \to \infty$, $N^{-1}||Y||_1$ tends almost surely to $\tau|c_1|$. Therefore, $N^{-2}\mathbb{E}(\text{tr}|Y|)^2$ tends to $(\tau|c_1|)^2$. We have

$$\chi_N = (N(N-1))^{-1} \sum_{i \neq j} \mathbb{E}(|Y|_{ii}|Y|_{jj}) \geq (N(N-1))^{-1} \mathbb{E}(\text{tr}|Y|)^2 - (N-1)^{-1},$$

and this implies

$$\liminf_{N \to \infty} \chi_N \geq (\tau|c_1|)^2.$$

In any case, we have

$$\inf_N \chi_N > 0,$$

proving our claim.

We will now deduce from (10) the desired estimate by a classical decoupling argument for multilinear expressions in Gaussian variables.

We first observe $\mathbb{E}((Y \otimes Y')(I - P)) = 0$. Indeed, by orthogonality, a simple calculation shows that $\mathbb{E}(Y \otimes Y') = \sum_{ij} E(Y_{ij} Y_{ij}') e_{ij} \otimes e_{ij} = \sum_{ij} N^{-1} e_{ij} \otimes e_{ij} = P$, and hence $\mathbb{E}((Y \otimes Y')(I - P)) = 0$.

We will use

$$(Y_j, Y'_j)^{\text{dist}} = ((Y_j + Y'_j)/\sqrt{2}, (Y_j - Y'_j)/\sqrt{2})$$

and if $E_Y$ denotes the conditional expectation with respect to $Y$ we have (recall $E(Y_j \otimes \bar{Y}_j)(I - P) = 0$)

$$\sum_{1}^{n} a_j Y_j \otimes \bar{Y}_j(I - P) = E_Y \left( \sum_{1}^{n} a_j Y_j \otimes \bar{Y}_j(I - P) - \sum_{1}^{n} a_j Y'_j \otimes \bar{Y}'_j(I - P) \right).$$

Therefore

$$\mathbb{E}\| \sum_{1}^{n} a_j Y_j \otimes \bar{Y}_j(1 - P) \|^q_p \leq \mathbb{E}\| \sum_{1}^{n} a_j Y_j \otimes \bar{Y}_j(1 - P) - \sum_{1}^{n} a_j Y'_j \otimes \bar{Y}'_j(I - P) \|^q_p$$

$$= \mathbb{E}\| \sum_{1}^{n} a_j (Y_j + Y'_j)/\sqrt{2} \otimes (Y_j + Y'_j)/\sqrt{2}(1 - P) - \sum_{1}^{n} a_j (Y_j - Y'_j)/\sqrt{2} \otimes (Y_j - Y'_j)/\sqrt{2}(I - P) \|^q_p.$$
E\| \sum_{j=1}^{n} a_j (Y_j \otimes \overline{Y_j} + Y'_j \otimes \overline{Y_j}) (1 - P) \|^p_q

and hence by the triangle inequality

\leq 2^p E\| \sum_{j=1}^{n} a_j (Y_j \otimes \overline{Y_j}) (1 - P) \|^p_q.

Thus we conclude a fortiori

E\| \sum_{j=1}^{n} a_j U_j \otimes \overline{U_j} (1 - P) \|^p_q \leq (2C)^p E\| \sum_{j=1}^{n} a_j (Y_j \otimes \overline{Y_j}) \|^p_q.

Theorem 0.2. Let C be as in the preceding Lemma. Let

\hat{S}^{(N)} = \sum_{j=1}^{n} a_j U_j \otimes \overline{U_j} (1 - P).

Then

(0.4) \limsup_{N \to \infty} E\| \hat{S}^{(N)} \| \leq 4C(\sum |a_j|^2)^{1/2}.

Moreover we have almost surely

(0.5) \limsup_{N \to \infty} \| \hat{S}^{(N)} \| \leq 4C(\sum |a_j|^2)^{1/2}.

Proof. A very direct argument is indicated in Remark 0.4 below, but we prefer to base the proof on [1] in the style of [5] in order to make clear that it remains valid with matrix coefficients. By [5, (3.1)] applied twice (for k = 1) (see also Remark 3.5 in [5]) one finds for any even integer p

(0.6) E\| \sum_{j=1}^{n} a_j (Y_j \otimes \overline{Y_j}) \|^p \leq (E\|Y\|^p)^2 (\sum |a_j|^2)^{p/2}

Therefore by the preceding Lemma

E\| \hat{S}^{(N)} \|^p \leq C^p (E\|Y\|^p)^2 (\sum |a_j|^2)^{p/2},

and hence a fortiori

E\| \hat{S}^{(N)} \|^p \leq N^2 C^p (E\|Y\|^p)^2 (\sum |a_j|^2)^{p/2}.

We then complete the proof, as in [5], using only the concentration of the variable \|Y\|. We have an absolute constant \beta' and \epsilon(N) > 0 tending to zero when N \to \infty, such that

(\sum |a_j|^2)^{1/2} \leq 2 + \epsilon(N) + \beta' \sqrt{p/N},

and hence

(\sum |a_j|^2)^{1/2} \leq N^2 C^p (2 + \epsilon(N) + \beta' \sqrt{p/N})^2 (\sum |a_j|^2)^{1/2}.
Fix $\varepsilon > 0$ and choose $p$ so that $N^{2/p} = \exp \varepsilon$, i.e. $p = 2\varepsilon^{-1} \log N$ (note that this is $\geq 2$ when $N$ is large enough) we obtain

$$E\|\hat{S}^{(N)}\| \leq (E\|\hat{S}^{(N)}\|^p)^{1/p} \leq 4\varepsilon C(1 + \varepsilon^{-1}\varepsilon'(N))\left(\sum |a_j|^2\right)^{1/2}$$

where $\varepsilon'(N) \to 0$ when $N \to \infty$, and (0.2) follows.

Let $R_N = 4C(1 + \varepsilon^{-1}\varepsilon'(N))\left(\sum |a_j|^2\right)^{1/2}$. By Tshebyshev’s inequality $(E\|\hat{S}^{(N)}\|^p)^{1/p} \leq e^\varepsilon R_N$ implies

$$\mathbb{P}\{\|\hat{S}^{(N)}\| > e^{2\varepsilon} R_N\} \leq \exp -\varepsilon p = N^2.$$ 

From this it is immediate that almost surely

$$\limsup_{N \to \infty} \|\hat{S}^{(N)}\| \leq e^{2\varepsilon} 4C\left(\sum |a_j|^2\right)^{1/2}$$

and hence (0.5) follows.

Remark 0.3. The same argument can be applied when $a_j \in M_k$ for any integer $k > 1$. Then we find

$$\limsup_{N \to \infty} \mathbb{E}\left\|\sum_{1}^{n} a_j \otimes U_j \otimes \bar{U}_j (1 - P)\right\| \leq 4C \max\{\|\sum a_j a_j^*\|^{1/2}, \|\sum a_j a_j^*\|^{1/2}\}.$$

Moreover we have almost surely

$$\limsup_{N \to \infty} \left\|\sum_{1}^{n} a_j \otimes U_j \otimes \bar{U}_j (1 - P)\right\| \leq 4C \max\{\|\sum a_j a_j^*\|^{1/2}, \|\sum a_j a_j^*\|^{1/2}\}.$$

Remark 0.4. In the case of scalar coefficients $a_j$ the proof extends also to double sums of the form

$$\sum_{ij} a_{ij} U_i \otimes \bar{U}_j (I - P).$$

We refer the reader to [4, Theorem 16.6] for a self-contained proof of (0.6) for such double sums.

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