FINITE TIME BLOWUP OF THE $n$-HARMONIC FLOW ON $n$-MANIFOLDS

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ABSTRACT. In this paper, we generalize the no-neck result of Qing-Tian [24] to show that there is no neck during blowing up for the $n$-harmonic flow as $t \to \infty$. As an application of the no-neck result, we settle a conjecture of Hungerbühler [17] by constructing an example to show that the $n$-harmonic map flow on an $n$-dimensional Riemannian manifold blows up in finite time for $n \geq 3$.

1. INTRODUCTION

Let $M$ be an $n$-dimensional Riemannian manifold without boundary, and let $N$ be another $m$-dimensional compact Riemannian manifold without boundary (isometrically embedded into $\mathbb{R}^L$). In local coordinates, a smooth Riemannian metric $g$ of $M$ can be represented by

$$g = g_{ij} dx_i \otimes dx_j,$$

where $(g_{ij})$ is a positive definitive symmetric $n \times n$ matrix. The volume element of $(M; g)$ is defined by

$$dv = \sqrt{|g|} dx \quad \text{with} \quad |g| = \det (g_{ij}).$$

For a map $u : M \to N \subset \mathbb{R}^L$, the $n$-energy functional of $u$ is defined by

$$E_n(u; M) = \frac{1}{n} \int_M |\nabla u|^{n} dv,$$

where $|\nabla u|$ is the gradient norm given by

$$|\nabla u(x)|^2 = \sum_{\alpha,i,j} g^{ij}(x) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_j},$$

with $(g^{ij}) = (g_{ij})^{-1}$ the inverse matrix of $(g_{ij})$. A $C^1$-map $u$ from $M$ to $N$ is said to be an $n$-harmonic map if $u$ is a critical point of the $n$-energy functional; i.e. it satisfies

$$(1.1) \quad \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left[ |\nabla u|^{n-2} g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j} u \right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u) = 0 \quad \text{in} \ M,$$

where $A$ is the second fundamental form of $N$.

In 1964, Eells and Sampson [9] investigated the existence problem of harmonic maps in a homotopic class; i.e. “Given a smooth map $u_0 : M \to N$, is there a harmonic map $u$, which is homotopic to $u_0$?” (See [8]).

For the target manifold $N$ with non-positive sectional curvature, Eells and Sampson [9] proved the first existence result of harmonic maps in a homotopic class by
introducing the “heat flow method”. The heat flow method transforms the existence problem to an evolution problem. Since then, questions on existence and regularity of harmonic maps and their flows have been attracted a great attention (See [8]). One of the key components of the heat flow method for answering the Eells-Sampson question is to prove existence of a global solution to the harmonic map flow. In 1975, Hamilton [11] proved local existence of the heat flow of harmonic map; i.e. the solutions of the heat flow of harmonic map exists locally. If the solution exists only in a finite interval \([0, T_{\text{max}}]\) with \(T_{\text{max}} < \infty\) and cannot be extended any further, then we say that the solution blows up in finite time \(T_{\text{max}}\). In the two dimensional case (i.e. \(n = 2\)), Struwe [26] proved global existence of a unique weak solution to the harmonic map flow, where the solution is smooth except for a finite set of point singularities. In 1989, Coron-Ghidaglia [4] constructed the first example to show that for \(n \geq 3\), the harmonic map heat flow from \(S^n\) into \(S^n\) blows up in finite time. However, when \(n = 2\), the Dirichlet energy \(E_2\) on the 2-dimensional manifold is conformally invariant on its critical dimension. In addition, Hélein [12] proved that any weak harmonic map from surfaces is smooth. Thus, it was widely believed during the time that the harmonic map heat flow would not blow up in finite time on the 2-dimension manifold. In 1992, Chang, Ding and Ye [1] made a breakthrough by constructing a counter-example that harmonic map heat flow on \(S^2\) can blow up in finite time.

In higher dimensions (i.e. \(n > 2\)), \(E_n\) is also conformally invariant on the \(n\)-dimensional manifold \(M\). Motivated by the Eells-Sampson question on harmonic maps, one can ask whether a given map from an \(n\)-dimensional manifold to another manifold can be deformed into an \(n\)-harmonic map. Related to this question, Hungerbühler [17] studied the \(n\)-harmonic flow in the following setting:

\[
(1.2) \quad \frac{\partial u}{\partial t} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left[ |\nabla u|^{n-2} g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j} u \right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u)
\]

and generalized the result of Struwe [26] by proving that there exists a global weak solution \(u : M \times [0, +\infty) \to N\) of (1.2) with initial value \(u_0\) such that \(u \in C^{1,\alpha}(M \times (0, +\infty) \setminus \{\Sigma_k \times T_k\}_{k=1}^L)\) for a finite number of times \(\{T_k\}_{k=1}^L\) and a finite number of singular closed sets \(\Sigma_k \subset M\) for \(k = 1, \ldots, L\) with an integer \(L\), depending only \(M\) and \(u_0\). However, it is still unknown whether the singular set \(\Sigma_k\) of the flow (1.2) at the singular \(T_k\) is finite. In order to sort out this issue, the second author [13] introduced a rectified \(n\)-harmonic map flow from an \(n\)-dimensional from \(M\) to \(N\) and proved existence of a global solution, which is regular except for a finite number of points, of the rectified \(n\)-harmonic map flow.

Based on the fundamental result of Chang-Ding-Ye [1] for \(n = 2\), it is an interesting question whether the \(n\)-harmonic flow (1.2) blows up in finite time for \(n \geq 3\). Supported by some numerical evidence, Hungerbühler ([16], [17]) conjectured the phenomenon of finite time blow-up of the \(n\)-harmonic flow for \(n \geq 3\). Later, Chen, Cheung, Choi and Law [2] followed the method of Chang-Ding-Ye to construct an example that the \(n\)-harmonic flow (1.2) blows up in finite time for \(n = 3\). However, due to the nonlinearity and degeneracy of the \(n\)-harmonic maps, they [2] also pointed out that their proofs could not be applied to the cases when \(n > 3\). Therefore, the conjecture of Hungerbühler for \(n > 3\) has remained open since then.
On the other hand, Qing-Tian [24] suggested a program to prove the finite time blow-up of the harmonic map flow for \( n = 2 \) through an application of the no-neck result for the harmonic map flow as \( t \to \infty \) and constructing a special target manifold \( N \) with a proper topology. Recently, Chen-Li [3] verified the Qing-Tian program by constructing a special target manifold \( N \) with a proper topology to show that the harmonic map flow blows up at finite time for \( n = 2 \). Later, Liu and Yin [20] successfully applied this idea to construct a proper manifold \( N \) to show that the bi-harmonic maps flow on 4-manifolds blows up at finite time.

In this paper, we apply the Qing-Tian program to confirm the conjecture of Hungerbühler on the \( n \)-harmonic map flow. Firstly, we define:

**Definition 1.1.** \( u \) is said to be a regular solution to the \( n \)-harmonic map flow (1.2) in \( M \times (0,T] \) if \( u \in C^0(M \times (0,T]; N) \) with \( T \leq \infty \) is a solution of (1.2) satisfying

\[
\int_0^T \int_M \left| \nabla (|\nabla u|^{\frac{2}{n-2}} \nabla u) \right|^2 + |\nabla u|^{2n} \ dv \ dt \leq C(T).
\]

We generalize the no-neck result of Qing-Tian [24] to the \( n \)-harmonic map flow as follow:

**Theorem 1.** Let \( u \) be a regular solution to the flow (1.2) in \( M \times [0,\infty) \) with initial value \( u_0 \in C^1(M,N) \). For a sequence \( t_i \to \infty \), there is a sub-sequence, still denoted by \( t_i \), such that as \( t_i \to \infty \), \( u(x,t_i) \) converges to an \( n \)-harmonic map \( u_\infty \) in \( C^{1,\alpha}_{\text{loc}}(M\setminus\{x^1, \cdots, x^L\}, N) \) for some positive \( \alpha < 1 \), where \( u_\infty \) can be extended to \( C^{1,\alpha}(M,N) \). Moreover, we have

(i) **(Energy identity)** There are a finite number of \( n \)-harmonic maps \( \omega_{k,l} \) (also called bubbles) on \( S^n \) for \( k = 1, \cdots, L \) and \( l = 1, \cdots, J_k \) such that

\[
\lim_{t_i \to \infty} E_n(u(\cdot,t_i); M) = E_n(u_\infty; M) + \sum_{k=1}^{L} \sum_{l=1}^{J_k} E_n(\omega_{k,l}; S^n).
\]

(ii) **(No-neck result)** There is no neck between the limiting map and bubbles; i.e. the image

\[
u_\infty(M) \cup \bigcup_{k,l} \omega_{k,l}(S^n)
\]

is a connected set.

One of the fundamental rules for bubble blowing is the bubble-neck decomposition. During the bubbling procedure, the energy identity implies that the energy is conserved. This means that the loss of energy under the limiting process can be recovered by the energy of a finite number of bubbles. Readers can refer to the pioneering work on the energy identity by Jost [18], Parker [22] regarding the harmonic maps from surfaces and by Ding-Tian [6] for the harmonic map flow. For \( n \)-harmonic maps with \( n \geq 3 \), the isolated singularities are removable due to Duzaar-Fuchs [7] and the energy identity was provided by Wang-Wei [27] for a sequence of approximate \( n \)-harmonic maps. In particular, one can use the standard blow-up argument as in Ding-Tian [6] to reduce the multiple bubble problem to the single bubble case. See more details for the bubble-neck decomposition of \( n \)-harmonic maps in [13]. These results allow us to construct the bubbling argument in the setting of the \( n \)-harmonic maps.

In order to provide an example to show that the \( n \)-harmonic flow can blow up in finite time, the key step is to generalize the no-neck result of Qing-Tian [24].
However, those no-neck results in [19] and [24] heavily rely on a key estimate in Ding-Tian’s work (Lemma 2.1, [6]) which only works for the case of harmonic maps. To settle this open problem, we generalize the Ding-Tian estimate to the context of \( n \)-harmonic maps (Lemma 3.1) and then apply it to prove the no-neck property for the \( n \)-harmonic map flow.

Secondly, we apply Theorem 1 to prove the main result of this paper:

**Theorem 2.** Let \( X \) be any closed manifold of dimension \( m > n \) with nontrivial \( \pi_n(X) \), and let \( N = X \# T^m \) be the connected sum of \( X \) with the torus \( T^m \). Then there are infinitely many initial maps \( u_0 : S^n \to N \) such that the \( n \)-harmonic map flow (1.2) with initial value \( u_0 \) blows up in finite time.

Besides the finite time blow-up result on the harmonic map flow by Chen-Li [3], another related evolution problem to the \( n \)-harmonic map flow is the bi-harmonic map flow on 4-dimensional manifolds. Liu-Yin in [21] established the no-neck result of a sequence of biharmonic maps. Later, Liu-Yin [20] generalized the no-neck result to a sequence of approximate biharmonic maps. By combining the no-neck result with a construction of a proper target manifold, they introduced a concept of width of bi-harmonic maps in the covering space to show that the bi-harmonic map flow blows up in finite time. These results provide a skeleton for the proof of Theorem 2.

This paper is organized as follows. In Section 2, we show asymptotical behavior of the solution of the \( n \)-harmonic flow as \( t \to \infty \). In Section 3, we generalize Ding-Tian’s estimate and apply it to prove the no-neck result for the \( n \)-harmonic flow. In Section 4, we construct an example to prove Theorem 2 and settle the Hungerbühler conjecture.

### 2. SOME ESTIMATES AND ASYMPTOTIC BEHAVIOR OF THE \( n \)-HARMONIC MAP FLOW

In order to study asymptotic behavior of the \( n \)-harmonic map flow, we begin with some basic estimates. We recall some results from [17] on the \( n \)-harmonic map flow.

**Lemma 2.1.** Let \( u(t) \) be a regular solution to the \( n \)-harmonic map flow (1.2) in \( M \times [0, T] \) with initial value \( u(0) = u_0 \). Then for each \( s \) with \( 0 < s \leq T \), we have

\[
\int_M \frac{1}{n} |\nabla u(s)|^n \, dv + \int_0^t \int_M \left| \frac{\partial u}{\partial t} \right|^2 \, dv \, dt \leq \int_M \frac{1}{n} |\nabla u_0|^n \, dv.
\]

**Lemma 2.2.** Let \( u \) be a regular solution to the \( n \)-harmonic map flow (1.2). Let \( \eta \) be a cut-off function in \( B_r \) such that \( \eta = 1 \) in \( B_{\frac{r}{2}} \), \( |\nabla \eta| \leq \frac{C}{r} \) and \( |\eta| \leq 1 \) in \( B_r \). Then we have

\[
(2.1) \quad \int_{B_r} |\nabla u|^{2n} \eta^n \, dv \leq C \left( \int_{B_r} |\nabla u|^n \, dv \right)^\frac{2}{n} \int_{B_r} \left( |\nabla^2 u|^2 |\nabla u|^{2n-4} \eta^n + |\nabla u|^n |
abla \eta|^n \right) \, dv.
\]
and

\[
\int_{B_r} |\nabla^2 u|^2 |\nabla u|^{2n-4} \eta^n \, dv \leq C \int_{B_r} |\nabla u|^{2n} \eta^n + |\nabla u|^n (\eta^n + |\nabla \eta|^n) \, dv.
\]

**Proof.** By using the Hölder and Sobolev inequalities, we have

\[
\int_{B_r} |\nabla u|^{2n} \eta^n \, dv = \int_{B_r} |\nabla u| (|\nabla u|^{2n-1} \eta^n) \, dv
\leq \left( \int_{B_r} |\nabla u|^n \, dv \right)^{\frac{1}{n}} \left( \int_{B_r} (|\nabla u|^{2n-1} \eta^n)^{\frac{2}{n-1}} \, dv \right)^{\frac{n-1}{n}}
\leq C \left( \int_{B_r} |\nabla u|^n \, dv \right)^{\frac{1}{n}} \int_{B_r} |\nabla (|\nabla u|^{2n-1} \eta^n)| \, dv
\leq C \left( \int_{B_r} |\nabla u|^n \, dv \right)^{\frac{1}{n}} \int_{B_r} (|\nabla^2 u| |\nabla u|^{2n-2} \eta^n + |\nabla u|^{2n-1} |\nabla \eta| \eta^{n-1}) \, dv.
\]

By Young’s inequality, we have

\[
\left( \int_{B_r} |\nabla u|^n \, dv \right)^{\frac{1}{n}} \int_{B_r} |\nabla^2 u| |\nabla u|^{2n-2} \eta^n \, dv
\leq \left( \int_{B_r} |\nabla u|^n \, dv \right)^{\frac{1}{n}} \int_{B_r} |\nabla^2 u|^2 |\nabla u|^{2n-4} \eta^n \, dv + \frac{1}{2} \int_{B_r} |\nabla u|^{2n} \eta^n \, dv.
\]

Similarly, we have

\[
\int_{B_r} |\nabla u|^{2n-1} |\nabla \eta| \eta^{n-1} \, dv = \int_{B_r} |\nabla u|^n |\nabla u|^{n-1} |\nabla \eta| \eta^{n-1} \, dv
\leq C \int_{B_r} |\nabla u|^n |\nabla \eta|^n \eta^{n-1} \, dv + C \int_{B_r} |\nabla u|^{2n} \eta^{n-1} \, dv.
\]

Combining (2.3), (2.4) with (2.5), we have

\[
\int_{B_r} |\nabla u|^{2n} \eta^n \, dv
\leq C \left( \int_{B_r} |\nabla u|^n \, dv \right)^{\frac{2}{n}} \int_{B_r} (|\nabla^2 u|^2 |\nabla u|^{2n-4} \eta^n + |\nabla u|^n |\nabla \eta|^n) \, dv.
\]

Using the Ricci identity, we have

\[
\nabla_k \nabla_l (|\nabla u|^{n-2} \nabla u) = \nabla_l \nabla_k (|\nabla u|^{n-2} \nabla u) + R_M \nabla_l (|\nabla u|^{n-2} \nabla u)
\]
with the Riemannian curvature $R_M$. Integrations by parts twice yield that

\[
\int_{B_r} \langle \nabla k(\nabla u|^{n-2}\nabla_k u), \nabla l(\nabla u|^{n-2}\nabla_l u) \rangle \eta^n \, dv \\
= -\int_{B_r} \langle \nabla l \nabla k(\nabla u|^{n-2}\nabla_k u), |\nabla u|^{n-2}\nabla_l u \rangle \eta^n \, dv \\
- \int_{B_r} \langle \nabla k(\nabla u|^{n-2}\nabla_k u), |\nabla u|^{n-2}\nabla_l u \rangle \nabla l \eta^n \, dv \\
= \int_{B_r} \langle \nabla l(\nabla u|^{n-2}\nabla_k u), \nabla k(\nabla u|^{n-2}\nabla_l u) \rangle \eta^n \, dv \\
+ \int_{B_r} \langle R_M \#(\nabla u|^{n-2}\nabla_k u), |\nabla u|^{n-2}\nabla_l u \rangle \eta^n \, dv \\
+ \int_{B_r} \langle \nabla l(\nabla u|^{n-2}\nabla_k u), |\nabla u|^{n-2}\nabla_l u \rangle \nabla k \eta^n \, dv \\
- \int_{B_r} \langle \nabla k(\nabla u|^{n-2}\nabla_k u), |\nabla u|^{n-2}\nabla_l u \rangle \nabla l \eta^n \, dv.
\]

Note that

\[(2.7) \int_{B_r} \langle \nabla l(\nabla u|^{n-2}\nabla_k u), \nabla k(\nabla u|^{n-2}\nabla_l u) \rangle \eta^n \, dv \\
= \int_{B_r} \sum |\nabla u|^{2n-4} |\nabla k u|^2 \eta^n + \langle \nabla l(\nabla u|^{n-2}\nabla_k u), \nabla k(\nabla u|^{n-2}\nabla_l u) \rangle \eta^n \, dv \\
+ 2 \int_{B_r} \langle \nabla l(\nabla u|^{n-2}\nabla_k u), |\nabla u|^{n-2}\nabla l u \rangle \eta^n \, dv \\
\geq \int_{B_r} (|\nabla u|^{2n-4} |\nabla^2 u|^2 + |\nabla k(\nabla u|^{n-2}\nabla_k u|^2) \eta^n \, dv \\
+ 2(n-2) \int_{B_r} |\nabla u|^{n-4} |\nabla^2 u|^2 \eta^n \, dv.
\]

Combining (2.6) with (2.7), this implies

\[(2.8) \int_{B_r} |\nabla^2 u|^2 |\nabla u|^{2n-4} \eta^n \, dv \\
\leq \int_{B_r} |\nabla \cdot (|\nabla u|^{n-2}\nabla u)|^2 \eta^n \, dv + C \int_{B_r} |\nabla u|^{2n-2} \eta^{n-2} (\eta^2 + |\nabla \eta|^2) \, dv \\
\leq C \int_{B_r} |\nabla u|^{2n} \eta^n + |\nabla u|^n (\eta^n + |\nabla \eta|^n) \, dv.
\]

We finish the proof by combining (2.6) with (2.8).

\[\square\]

**Lemma 2.3.** There exists a sufficiently small constant $\varepsilon_1 > 0$ such that if $u$ is a regular solution of (1.2) on $B_{2R_0}(x_0) \times [t_0 - 2R_0^2, t_0]$ satisfying

\[
\sup_{t_0 - 2R_0^2 \leq t \leq t_0} \int_{B_{2R_0}(x_0)} |\nabla u(x, t)|^n \, dv < \varepsilon_1,
\]
we have
\[ \int_{t_0-2R_0^n}^{t_0} \int_{B_{R_0}(x_0)} |\nabla^2 u|^2 |\nabla u|^{2n-4} + |\nabla u|^{2n} \, dv \, dt \]
\[ < C \sup_{t_0-2R_0^n \leq t \leq t_0} \int_{B_{2R_0}(x_0)} |\nabla u(x,t)|^n \, dv \]
for some constant \( C > 0 \).

**Proof.** Lemma 2.3 was proved by Hungerbühler by using an extension of the Ladyzhenskaya-Solomonikov-Nikolaevna inequality (see Lemma 5 of [12]). Herewith, we would like to give a slightly different approach by using Lemma 2.2.

Multiplying (1.2) by \( \phi^n \nabla \cdot (|\nabla u|^{n-2} \nabla u) \) and using Lemma 2.2 by choosing a sufficiently small \( \varepsilon_1 \) in above inequalities yields that
\[ \int_{B_{2R_0}(x_0)} |\nabla \cdot (|\nabla u|^{n-2} \nabla u)|^2 \phi^n \, dv \]
\[ \leq \frac{1}{2} \int_{B_{2R_0}(x_0)} |\nabla \cdot (|\nabla u|^{n-2} \nabla u)|^2 \phi^n \, dv + C \int_{B_{2R_0}(x_0)} \left( |\nabla u|^2 + |\nabla u|^{2n} \right) \phi^n \, dv \]
\[ \leq \frac{3}{4} \int_{B_{2R_0}(x_0)} |\nabla \cdot (|\nabla u|^{n-2} \nabla u)|^2 \phi^n \, dv + C \left( 1 + \frac{1}{R_0^n} \right) \int_{B_{2R_0}(x_0)} |\nabla u|^n \, dv \]
\[ + C \int_{B_{2R_0}(x_0)} |\nabla u|^2 \frac{\partial u}{\partial t} \, dv. \]

Together with Lemma 2.2, we obtain
\[ (2.9) \]
\[ \int_{B_{R_0}(x_0)} |\nabla u|^{2n} + |\nabla^2 u|^2 |\nabla u|^{2n-4} \, dv \]
\[ \leq C \int_{B_{2R_0}(x_0)} (R_0^n + 1)|\nabla u|^n + |\nabla u|^2 \frac{\partial u}{\partial t} \, dv \]
\[ \leq C(R_0^n + 1) E_n(u_0) + C \int_{B_{2R_0}(x_0)} |\nabla u|^2 \frac{\partial u}{\partial t} \, dv. \]

**Lemma 2.4.** Let \( u \) be a regular solution to (1.2). Then there exists a positive constant \( \varepsilon_1 \) such that if for some \( R_0 > 0 \) the inequality
\[ \sup_{t_0-2R_0^n \leq t \leq t_0} \int_{B_{2R_0}(x_0)} |\nabla u(x,t)|^n \, dv < \varepsilon_1 \]
holds, we have
\[ \sup_{[t_0-R_0^n,t_0] \times B_{R_0}(x_0)} |\nabla u|^n \, dv \leq CR_0^{-n}, \]
where \( C \) depending on \( M \) is a constant independent of \( R_0 \).

**Proof.** The proof is due to Hungerbühler in [12] for \( R_0 = 1 \). If \( R_0 \neq 1 \), one can prove it by a re-scaling argument.
Lemma 2.5. Let \( u : M \to N \) be a regular solution to the equation (1.2). Then there is a small constant \( \varepsilon_1 > 0 \) such that if the inequality

\[
\sup_{t_0 - R_0^2 < t \leq t_0} \int_{B_{2R_0}(x_0)} |\nabla u(x, t)|^n < \varepsilon_1,
\]

holds for some positive \( R_0 \), then \( \|u\|_{C^{1,\alpha}([t_0 - \frac{1}{2}R_0^2, t_0] \times B_{R_0}(x_0))} \) is bounded by a constant depending on \( E(u_0) \) and \( R_0 \).

Proof. As pointed out by Hungerbühler in [12], we can apply the result of DiBenedetto-Friedman [5] to obtain a bound of \( \|u\|_{C^{1,\alpha}([t_0 - \frac{1}{2}R_0^2, t_0] \times B_{R_0}(x_0))} \). \( \square \)

Lemma 2.6. (Local energy inequality under small condition) Let \( u \) be a regular solution of (1.2) on \( B_{2R_0}(x_0) \times [0, T] \). There exists a sufficiently small constant \( \varepsilon_1 > 0 \) such that if

\[
\sup_{t_0 - T < t \leq t_0} \int_{B_{2R_0}(x_0)} |\nabla u(x, t)|^n < \varepsilon_1,
\]

then we have for every \( x \in B_R(x_0) \), any \( R \leq R_0 \) and any two constants \( \tau \) and \( s \) in \( (t_0 - T, t_0) \)

\[
\int_{B_R(x_0)} |\nabla u|^n(\cdot, s) \, dv \leq \int_{B_R(x_0)} |\nabla u|^n(\cdot, \tau) \, dv + C \int_s^\tau \int_{B_R(x_0)} |\partial_t u|^2 \, dv \, dt + C \left( \frac{(\tau - s)}{R^n} \int_{B_R(x_0)} |\nabla u|^n \, dv \right)^{1/2} \int_s^\tau \int_{B_R(x_0)} |\partial_t u|^2 \, dv \, dt \right)^{1/2}
\]

for some constant \( C \).

Proof. Let \( \phi \) be a cut-off function with support in \( B_{2R_0}(x_0) \) such that \( \phi = 1 \) in \( B_{R_0}(x_0) \), \( |\nabla \phi| \leq CR_0^{-1} \) and \( |\phi| \leq 1 \) in \( B_{2R_0}(x_0) \). Multiplying (1.2) by \( \phi^n \partial_t u \), we have

\[
\int_{B_{2R_0}(x_0)} \frac{\partial u}{\partial t} \phi^n \, dv = \int_{B_{2R_0}(x_0)} \nabla \cdot (|\nabla u|^{n-2} \nabla u, \frac{\partial u}{\partial t} \phi^n) \, dv
\]

\[
= - \int_{B_{2R_0}(x_0)} \langle |\nabla u|^{n-2} \nabla u, \frac{\partial \nabla u}{\partial t} \phi^n + \frac{\partial u}{\partial t} \phi^{n-1} \nabla \phi \rangle \, dv
\]

\[
\geq - \frac{1}{n} \frac{d}{dt} \int_{B_{2R_0}(x_0)} |\nabla u|^n \phi^n \, dv - C \int_{B_{2R_0}(x_0)} |\nabla u|^{n-1} |\frac{\partial u}{\partial t} \phi^{n-1} | \nabla \phi | \, dv.
\]

Note

\[
\int_{B_{2R_0}(x_0)} |\nabla u|^{n-1} |\frac{\partial u}{\partial t} \phi^{n-1} | \nabla \phi | \, dv
\]

\[
\leq \left( \int_{B_{2R_0}(x_0)} |\frac{\partial u}{\partial t} \phi^n |^2 \, dv \right)^{1/2} \left( \int_{B_{2R_0}(x_0)} |\nabla u|^{2n-2} \phi^{2n-2} | \nabla \phi |^2 \, dv \right)^{1/2}
\]

since

\[
\int_{B_{2R_0}(x_0)} |\nabla u|^{2n-2} \phi^{2n-2} | \nabla \phi |^2 \, dv \leq C \int_{B_{2R_0}(x_0)} |\nabla u|^{2n} \phi^n + |\nabla u|^n | \nabla \phi |^n \, dv.
\]

Therefore, the claim is proved. \( \square \)
Proposition 2.1. Let $u$ be a regular solution to (1.2) in $M \times [0, \infty)$. For a sequence $t_i \to \infty$, there is a sub-sequence, still denoted by $t_i \to \infty$, such that $u(\cdot, t_i)$ converges to an $n$-harmonic maps $u_\infty$ locally in $C^{1, \alpha}(M \setminus \{x^1, \ldots, x^J\}; N)$ with some positive $\alpha < 1$, where $u_\infty$ can be extended regularly on $M$.

Proof. By Lemma 2.1, we know that $\int_0^\infty \int_M |\partial_t u|^2 \, dv \, dt$ is finite, so we may choose a sub-sequence $\{t_i\}$ such that as $t_i \to \infty$, $\partial_t u(\cdot, t_i) \to 0$ strongly in $L^2(M)$ and $\int_{t_i-1}^{t_i} \int_M |\partial_t u(\cdot, t)|^2 \, dv \, dt \to 0$. Moreover, there is a constant $\varepsilon_0 > 0$ such that the singular points $\{x^1, \ldots, x^J\}$ are defined by the condition

$$\limsup_{t_i \to \infty} E_n(u(t_i); B_R(x^k)) \geq \varepsilon_0$$

for any $R \in (0, 2R_0]$ with some fixed $R_0 > 0$.

By Lemma 2.6, for any $s \in [t_i - 2R_0^2, t_i]$ and for sufficiently large $i$

$$\int_{B_R(x_0)} |\nabla u|^n(\cdot, s) \, dv \leq \int_{B_{2R_0}(x_0)} |\nabla u|^n(\cdot, t_i) \, dv + C \int_{t_i-1}^{t_i} \int_{B_{2R_0}(x_0)} |\partial_t u|^2 \, dv \, dt$$

$$+ C \left( \frac{(t_i - s)}{R_0^2} \int_{B_{2R_0}(x_0)} |\nabla u|^n \, dv \int_{t_i-1}^{t_i} \int_{B_{2R_0}(x_0)} |\partial_t u|^2 \, dv \, dt \right)^{1/2}$$

where $\varepsilon_1$ is the constant defined in Lemma 2.6.

By Lemma 2.4, we have

$$\sup_{t \in [t_i - 2R_0^2, t_i], x \in B_{R_0}(x_0)} |\nabla u|^n(x, t) \, dv \leq CR_0^n.$$

Then using Lemma 2.5, there is a uniform bound of $\|u(\cdot, t_i)\|_{C^{1, \alpha}(B_{2R_0}(x_0))}$, so $u(x, t_i)$ converges to $u_\infty$ in $C^{1, \beta}(B_{2R_0})$ and hence in $C^{1, \beta}_{\text{loc}}(M \setminus \{x^1, \ldots, x^J\})$ with $\beta < \alpha$, where $u_\infty \in C^{1, \beta}_{\text{loc}}(M \setminus \{x^1, \ldots, x^J\})$ is an $n$-harmonic map. By the removable singularities of an $n$-harmonic map, $u_\infty$ can be extended to $C^{1, \alpha}(M)$.

3. NO NECK RESULT BETWEEN THE LIMITING MAP AND BUBBLES AS $t \to \infty$

In this section, we generalize the no-neck result of Qing-Tian [24] to the case of the $n$-harmonic flow. As suggested by Struwe [26] and Qing [23], the existence of solutions of the heat flow for harmonic maps can be proved by a method of “Palais-Smale sequences” with tension fields $\tau(u) \in L^2$. In the context of $n$-harmonic maps, the tension field $\tau(u)$ of $u$ is defined as follows:

$$\tau(u) := \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_1} \left[ \nabla u |^{n-2} g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j} u \right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u),$$

where $A$ is the second fundamental form of $N$.

If $\tau(u) = 0$, $u$ is an $n$-harmonic map. When $\tau(u) \in L^2(M)$ and $u$ satisfies an extra smoothness assumption in (3.1), we define $u$ to be a regular approximated $n$-harmonic map as follow (See [27] for details):
Definition 3.1. We define a map \( u \in W^{1,n}(M; N) \cap C^0(M; N) \) to be a regular approximated \( n \)-harmonic map if it satisfies the following conditions:

1. \( \nabla \left( \| \nabla u \|_2^{n-2} \nabla u \right) \in L^2(M) \);
2. There exist \( \varepsilon > 0 \), \( \alpha \in (0,1) \) and \( C > 0 \) depending only on \( M, N \) and \( \| \tau(u) \|_{L^2} \) such that for any \( B_{2r} \subset M \) and \( E_n(u; B_{2r}) \leq \varepsilon \), then
   \[
   u \in C^\alpha(B_{r}; N) \quad \text{and} \quad [u]_{C^\alpha(B_{r}(x))} \leq C.
   \]

Let \( \{u_i\} \) be a sequence of regular approximated \( n \)-harmonic maps with uniform bounds of \( E_n(u_i) \) and \( \| \tau(u_i) \|_{L^2(M)} \). Wang-Wei [27] proved that \( \{u_i\} \) converges to an \( n \)-harmonic map \( u_\infty \) strongly in \( W^{1,q}_{loc}(M \setminus \{x^1, \ldots, x^L\}) \) for any \( q < 2n \), where \( u_\infty \) can be extended to \( C^{1,\alpha}(M) \). By reducing multi bubbles into a single bubble, they proved that there are a finite number of \( n \)-harmonic maps \( \omega_{k,l} \) on \( S^n \) with \( k = 1, \ldots, L \) and \( l = 1, \ldots, J_k \) such that

\[
\lim_{i, t_i \to \infty} E_n(u_i; M) = E_n(u_\infty; M) + \sum_{k=1}^L \sum_{l=1}^{J_k} E_n(\omega_{k,l}, S^n).
\]

Then we have

Theorem 3. Let \( \{u_i\} \) be the sequence of regular approximated \( n \)-harmonic maps with uniform bounds of \( E_n(u_i) \) and \( \| \tau(u_i) \|_{L^2(M)} \), and let \( \omega_{k,l} \) be the above bubbles. Then there is no neck between the limiting map \( u_\infty \) and bubbles \( \omega_{k,l} \);

i.e. the image

\[
u_\infty(M) \cup \bigcup_{k,l} \omega_{k,l}(S^n)
\]

is a connected set.

We begin with the following \( \varepsilon \)-regularity estimate for approximated \( n \)-harmonic maps. In particular, we generalize the Ding-Tian estimate (see [6], Lemma 2.1), which is a crucial estimate to the proof of no-neck result.

Lemma 3.1. For \( n \geq 2 \), let \( u \in W^{1,n}(M, N) \cap C^0(M, N) \) be an approximated \( n \)-harmonic map. Then there exists a small constant \( \varepsilon > 0 \) such that if \( E_n(u, B_r) \leq \varepsilon \) then

\[
\|u\|_{\text{osc}(B_{2r})} \leq C \left( \int_{B_r} |\nabla u|^n \, dv \right)^{\frac{1}{n(1-\frac{2}{n})}} + C r^{-\alpha} \left( \int_{B_r} |\tau(u)|^2 \, dv \right)^{\frac{1}{1-\frac{2}{n}}}.
\]

Proof. Let \( \phi \) be a cut-off function in \( C^\infty_0(B_r) \) with \( \phi \equiv 1 \) in \( B_{\frac{3r}{4}} \) and \( |\nabla \phi| \leq C r^{-1} \) and set \( \tilde{u} = \frac{1}{|B_{\frac{2r}{3}}|} \int_{B_{\frac{2r}{3}}} u \, dv \). For a sufficient small \( a > 0 \), we apply Theorem 7.17 in [10] with \( p = \frac{2n(n-1)}{n-2-a} > n, \gamma = 1 - \frac{a}{p} \), and the Poincaré inequality to obtain
\[ \|u\|_{\text{osc}(B_r)} = \sup_{x, y \in B_r} |u(x) - u(y)| \leq 2 \sup_{x \in B_r} |(u(x) - \overline{u}) \phi(x)| \]

\[ \leq Cr^{1 - \frac{n - 2 + a}{2(n - 1)}} \left( \int_{B_{\frac{3}{4}r}} |\nabla ((u - \overline{u}) \phi)|^{\frac{2n(n - 1)}{n - 2 + a}} \, dv \right)^{\frac{n - 2 + a}{2(n - 1)}} \]

\[ \leq Cr^{\frac{n - a}{2(n - 1)}} \left( \int_{B_{\frac{3}{4}r}} |\nabla u(x)|^{\frac{2n(n - 1)}{n - 2 + a}} \, dv \right)^{\frac{n - 2 + a}{2n(n - 1)}} \]

\[ + Cr\frac{n - a}{2(n - 1)} \left( \int_{B_{\frac{3}{4}r}} |(u(x) - \overline{u}) \nabla \phi|^{\frac{2n(n - 1)}{n - 2 + a}} \, dv \right)^{\frac{n - 2 + a}{2n(n - 1)}} \]

\[ \leq Cr^{\frac{n - a}{2(n - 1)}} \left( \int_{B_{\frac{3}{4}r}} |\nabla u|^{2n(n - 1)} \, dv \right)^{\frac{n - 2 + a}{2n(n - 1)}} \]

\[ = Cr^{\frac{n - a}{2(n - 1)}} \left( \int_{B_{\frac{3}{4}r}} |\nabla u|^{n - 2 n - 2} \nabla u|^{\frac{2n}{(n - 1)}} \, dv \right)^{\frac{n - 2 + a}{2n(n - 1)}}. \]
Then substituting (3.4) into (3.3), we obtain

\[
\|u\|_{\text{osc}(B_{\frac{3}{4}r})} \leq Cr^{\frac{n-a}{2(n-1)}} \left( \int_{B_{\frac{3}{4}r}} |\nabla u|^{n-2} \nabla u|^q \, dv \right)^{\frac{1}{q}} \frac{1}{(n-1)},
\]

\[
\leq Cr^{\frac{n-a}{2(n-1)}} r^{\frac{n}{q(n-1)} - 1} r^{\frac{n}{2(n-1)}} \left( \int_{B_r} |\nabla (|\nabla u|^{n-2} \nabla u)|^2 \, dv \right) \frac{1}{2^{(n-1)}}
\]

\[
+ Cr^{\frac{n-a}{n-1}} r^{\frac{n}{q(n-1)} - 1} \left( \frac{1}{r} \int_{B_r} |\nabla u|^{n-1} \, dv \right) \frac{1}{n-1}
\]

\[
= Cr^{\frac{n-a}{2(n-1)}} \left( \int_{B_{\frac{3}{4}r}} |\nabla (|\nabla u|^{n-2} \nabla u)|^2 \, dv \right) \frac{1}{n-1} + C \left( \frac{1}{r} \int_{B_r} |\nabla u|^{n-1} \, dv \right) \frac{1}{(n-1)}
\]

\[
\leq C \left( r^n \int_{B_{\frac{3}{4}r}} |\nabla (|\nabla u|^{n-2} \nabla u)|^2 \, dv \right) \frac{1}{n-1} + C \left( \int_{B_{\frac{3}{4}r}} |\nabla u|^n \, dv \right) \frac{1}{n-1}
\]

by noting that \( \frac{n-a}{2(n-1)} + \frac{n}{q(n-1)} - 1 = 0 \) with \( q = \frac{2n}{n-2+a} \).

Multiplying (3.1) by \( \nabla \cdot (|\nabla u|^{n-2} \nabla u) \eta^n \), we have

\[
\int_{B_r} |\nabla \cdot (|\nabla u|^{n-2} \nabla u)|^2 \eta^n \, dv \leq \int_{B_r} |\nabla \cdot (|\nabla u|^{n-2} \nabla u)| (\tau(u) + C|\nabla u|^n) \eta^n \, dv.
\]

Now, using Young's inequality, we have

\[
\int_{B_r} |\nabla \cdot (|\nabla u|^{n-2} \nabla u)|^2 \eta^n \, dv \leq C \int_{B_r} (|\tau(u)|^2 + |\nabla u|^{2n}) \eta^n \, dv.
\]

Using Lemma 2.2 again, it yields that

\[
\int_{B_{\frac{3}{4}r}} |\nabla u|^{2n} + |\nabla^2 u|^2 |\nabla u|^{2n-4} \, dv \leq C \int_{B_r} (1 + r^{-n}) |\nabla u|^n + |\tau(u)|^2 \, dv.
\]

Therefore

\[
\left( r^n \int_{B_{\frac{3}{4}r}} |\nabla u|^{2n} + |\nabla^2 u|^2 |\nabla u|^{2n-4} \, dv \right) \frac{1}{2^{(n-1)}} \leq C \left( \int_{B_r} |\nabla u|^n + r^n |\tau(u)|^2 \, dv \right) \frac{1}{2^{(n-1)}}
\]

\[
\leq C \left( \int_{B_r} |\nabla u|^n \, dv \right) \frac{1}{n-1} + Cr\frac{n}{(n-1)} \left( \int_{B_r} |\tau(u)|^2 \, dv \right) \frac{1}{n-1}.
\]

We finish the proof by putting these estimates together. \( \Box \)

To analyze the behavior of approximated \( n \)-harmonic maps on the neck region, we need the following Pohozaev type inequality, which was proved in [27]:
Lemma 3.2. For $n \geq 2$, let $u \in W^{1,n}(M, N) \cap C^{1,\alpha}(M, N)$ to be a regular approximated $n$-harmonic map with tension field $\tau(u) \in L^2(M)$. Then, for any ball $B_r \subset M$, we have
\begin{equation}
\int_{\partial B_r} |\nabla u|^n \, ds \leq C(n) \left( \int_{\partial B_r} |\nabla_T u|^n \, ds + \int_{B_r} |\tau(u)||\nabla u| \, dv \right),
\end{equation}
where $\nabla_T u$ is the tangential gradient on the boundary $\partial B_r$.

Proof. For completeness, we sketch the proof here. Multiplying (3.1) by $x \cdot \nabla u$ and integrating over $B_r$, we have
\begin{align*}
\int_{B_r} \langle \tau(u), x \cdot \nabla u \rangle \, dv &= \frac{1}{n} \int_{B_r} \langle x, \nabla(|\nabla u|^n) \rangle \, dv + \int_{B_r} |\nabla u|^n \, dv - r \int_{\partial B_r} |\nabla u|^{n-2} \left| \frac{\partial u}{\partial r} \right|^2 \, ds \\
&= \frac{r}{n} \int_{\partial B_r} |\nabla u|^n \, ds - r \int_{\partial B_r} |\nabla u|^{n-2} \left| \frac{\partial u}{\partial r} \right|^2 \, ds,
\end{align*}
where we use the fact that
\begin{equation*}
\int_{B_r} \langle x, \nabla(|\nabla u|^n) \rangle \, dv = r \int_{\partial B_r} |\nabla u|^n \, ds - n \int_{B_r} |\nabla u|^n \, dv
\end{equation*}
and $|\nabla u|^2 = \left| \frac{\partial u}{\partial r} \right|^2 + |\nabla_T u|^2$.

Rearranging the inequality and by adding $(n-1) \int_{\partial B_r} |\nabla u|^{n-2} |\nabla_T u|^2 \, ds$ to the both sides we have
\begin{equation*}
(n-1) \int_{\partial B_r} |\nabla u|^n \, ds \leq n \int_{B_r} |\tau(u)||\nabla u| \, dv + n \int_{\partial B_r} |\nabla u|^{n-2} \left| (\nabla_T u) \right|^2 \, ds.
\end{equation*}
Then the claim follows from using Young’s inequality. \hfill \Box

Now we prove Theorem 3.

Proof. By using the standard bubbling arguments as in [6] and [27], one can reduce multiple bubbles to a single bubble. We assume that 0 is the single blowing up point of $\{u_i\}$ and there is only one bubble in $B_1$. Then, we follow the approach of [24] and [19] to extend the no-neck result to the case of the $n$-harmonic map flow.

Suppose $r_n R = 2^{-j_n}$ and $\delta = 2^{-j_0}$ for any $j_0 < j < j_n$. Then, we denote
\begin{equation*}
L_j = \min\{j - j_0, j_n - j\} \quad \text{and} \quad P_{j,t} = B_{2^{-j-i}} \setminus B_{2^{-j-1-i}} \quad \text{for} \quad t \in (0, L_j).
\end{equation*}

For sufficiently large $i$, we assume that
\begin{equation}
E_n(u_i, B_{2^{j-i}} \setminus B_{2^{-i}}) \leq \varepsilon^{2(n-1)}, \quad \text{for any} \ j_0 \leq j \leq j_n.
\end{equation}

Let
\begin{equation*}
h_{i,j,t}(2^{j-t-i}) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} u_i(2^{j-t-i}, \theta) \, d\theta
\end{equation*}
and
\begin{equation}
h_{i,j,t}(r) = h_{i,j,t}(2^{j-t}) + (h_{i,j,t}(2^{j-t}) - h_{i,j,t}(2^{j-t-i})) \frac{\ln(2^{-i+j} r)}{-2t \ln 2}.
\end{equation}
Note that the tangential derivative of \( h_{i,j,t}(r) \) is zero in \( n \)-dimensional spherical coordinates. Therefore, the Laplace operator can be reduced to the following form:

\[
\Delta h_{i,j,t} = \frac{d^2 h_{i,j,t}}{d^2 r} + \frac{n-1}{r} \frac{dh_{i,j,t}}{dr},
\]

which yields that

\[
\text{div}(\nabla h_{i,j,t} |^{n-2} \nabla h_{i,j,t}) = \left| \frac{dh_{i,j,t}}{dr} \right|^{n-2} \left( \frac{d^2 h_{i,j,t}}{d^2 r} + \frac{n-1}{r} \frac{dh_{i,j,t}}{dr} \right)\]

\[
+ \frac{n-2}{2} \left( \frac{dh_{i,j,t}}{dr} \right)^{n-4} \frac{dh_{i,j,t}}{dr} \left( \frac{dh_{i,j,t}}{dr} \right)^2
\]

\[
= (n-1) \left( \frac{dh_{i,j,t}}{dr} \right)^{n-2} \left( \frac{d^2 h_{i,j,t}}{d^2 r} + \frac{1}{r} \frac{dh_{i,j,t}}{dr} \right).
\]

This implies that \( h_{i,j,t}(r) \) is also a symmetric \( n \)-harmonic map for \( r \in [2^{1-t-j}, 2^{t-j}] \).

By the well-known result of the \( n \)-Laplace operator, we note that

\[
\int_{P_{j,t}} |\nabla (u_i - h_{i,j,t})|^n \, dv
\]

\[
\leq C \int_{P_{j,t}} \langle |\nabla u_i|^{n-2} \nabla u_i - |\nabla h_{i,j,t}|^{n-2} \nabla h_{i,j,t}, \nabla (u_i - h_{i,j,t}) \rangle \, dv.
\]

for some constant \( C > 0 \).

By integration by parts, we have

\[
\int_{\partial P_{j,t}} \langle |\nabla u_i|^{n-2} \nabla u_i - |\nabla h_{i,j,t}|^{n-2} \nabla h_{i,j,t}, \nabla (u_i - h_{i,j,t}) \rangle \, dv.
\]

\[
= - \int_{P_{j,t}} \langle \nabla (\nabla u_i)|^{n-2} \nabla u_i, (u_i - h_{i,j,t}) \rangle \, dv
\]

\[
+ \int_{\partial P_{j,t}} \langle |\nabla u_i|^{n-2} (u_i)_{r} - |\nabla h_{i,j,t}|^{n-2} (h_{i,j,t})_{r}, (u_i - h_{i,j,t}) \rangle \, dv
\]

\[
= \int_{P_{j,t}} \langle (A(u_i)) (\nabla u_i, \nabla u_i) |\nabla u|^2 + \tau(u_i), (u_i - h_{i,j,t}) \rangle \, dv
\]

\[
+ \int_{\partial P_{j,t}} \langle |\nabla u_i|^{n-2} (u_i)_{r} - |\nabla h_{i,j,t}|^{n-2} (h_{i,j,t})_{r}, (u_i - h_{i,j,t}) \rangle \, dv.
\]

By Lemma 3.1, we obtain

\[
\| u_i - h_{i,j,t} \|_{C^0(P_{j,t})} \leq \| u_i - h_{i,j,t}(2^{j-t}) \|_{C^0(P_{j,t})} + \| u_i - h_{i,j,t}(2^{-j-t}) \|_{C^0(P_{j,t})}
\]

\[
\leq 2 \| u_i \|_{\text{osc}(P_{j,t})}
\]

\[
\leq C \left( \int_{P_{j-1,t} \cup P_{j,t} \cup P_{j+1,t}} |\nabla u|^n \right) \frac{1}{2^{(n-1)}}
\]

\[
+ C \left( 2^{t-j+1} \right)^{\frac{n}{2(n-1)}} \left( \int_{B_{2^{1-t-j+1}}} |\tau(u)|^2 \right)^{\frac{1}{2(n-1)}}
\]

\[
\leq C \left( \varepsilon + \delta_{2(t-j+1)}^{\frac{n(n+1)}{2(n-1)}} \right) \leq C \varepsilon.
\]

(3.9)
By (3.9), we have
\[ I := \int_{P_{j,t}} |\nabla(u_i - h_{i,j,t})|^n \, dv \]
\[ \leq C \int_{P_{j,t}} \langle |\nabla u_i|^{n-2} \nabla u_i - |\nabla h_{i,j,t}|^{n-2} \nabla h_{i,j,t}, \nabla(u_i - h_{i,j,t}) \rangle \, dv \]
\[ \leq C \int_{P_{j,t}} \langle (A(u_i)(\nabla u_i, \nabla u_i) |\nabla u_i|^{n-2} + \tau(u_i), (u_i - h_{i,j,t}) \rangle \, dv \]
\[ + C \int_{\partial P_{j,t}} \langle |\nabla u_i|^{n-2} (u_i)_r, (u_i - h_{i,j,t}) \rangle \, ds \]
\[ \leq C \left( \varepsilon \int_{P_{j,t}} |\nabla u_i|^n \, dv + \varepsilon \|\tau(u_i)\|_{L^2(P_{j,t})} 2^{(t-j)\frac{n}{2}} \right) \]
\[ + C \int_{\partial P_{j,t}} |\nabla u_i|^{n-2} |(u_i)_r| |u_i - h_{i,j,t}| \, ds \]
\[ = C(\varepsilon I_1 + 2^{(t-j)\frac{n}{2}} + I_2), \]

where we set
\[ I_1 := f_j(t) = \int_{P_{j,t}} |\nabla u_i|^n \, dv \quad \text{and} \quad I_2 := \int_{\partial P_{j,t}} |\nabla u_i|^{n-2} |(u_i)_r| |u_i - h_{i,j,t}| \, ds. \]

Using the fact that \( \frac{d2^x}{dx} = \ln 2(2^x) \), this implies
\[ f'_j(t) = \ln 2 \left( 2^{t-j} \int_{\{2^{-j}\} \times S^{n-1}} |\nabla u_i|^n \, ds + 2^{-t-j} \int_{\{2^{-j}\} \times S^{n-1}} |\nabla u_i|^n \, ds \right). \]

By the Poincaré inequality and Hölder’s inequality, we have
(3.10)
\[ I_2 = \int_{\partial P_{j,t}} \langle |\nabla u_i|^{n-2} |(u_i)_r| |u_i - h_{i,j,t}| \rangle \, ds \]
\[ \leq \int_{\{2^{-j}\} \times S^{n-1}} |\nabla u_i|^{n-1} |u_i - h_{i,j,t}| \, ds + \int_{\{2^{-j}\} \times S^{n-1}} |\nabla u_i|^{n-1} |u_i - h_{i,j,t}| \, ds \]
\[ \leq \left( \int_{\{2^{-j}\} \times S^{n-1}} |\nabla u_i|^{n-1} \, ds \right)^{\frac{n-1}{n}} \left( \int_{\{2^{-j}\} \times S^{n-1}} |u_i - h_{i,j,t}|^n \, ds \right)^{\frac{1}{n}} \]
\[ + \left( \int_{\{2^{-j}\} \times S^{n-1}} |\nabla u_i|^{n-1} \, ds \right)^{\frac{n-1}{n}} \left( \int_{\{2^{-j}\} \times S^{n-1}} |u_i - h_{i,j,t}|^n \, ds \right)^{\frac{1}{n}} \]
\[ \leq \left( \int_{\{2^{-j}\} \times S^{n-1}} |\nabla u_i|^n \, ds \right)^{\frac{n-1}{n}} \left( \int_{\{2^{-j}\} \times S^{n-1}} |u_i - h_{i,j,t}|^n \, ds \right)^{\frac{1}{n}} \]
\[ + \left( \int_{\{2^{-j}\} \times S^{n-1}} |\nabla u_i|^n \, ds \right)^{\frac{n-1}{n}} \left( \int_{\{2^{-j}\} \times S^{n-1}} |u_i - h_{i,j,t}|^n \, ds \right)^{\frac{1}{n}} \]
\[ \leq C f'_j(t). \]
Note that \( h_{i,j,t} \) is the average of \( u_i \) over \( S^{n-1} \). Then we can estimate the tangential energy by

\[
|\nabla_T u_i|^n \leq \int_{P_{j,t}} \left( |(u_i - h_{i,j,t})_r|^2 + |\nabla_T u_i|^2 \right) \frac{\tau}{2} \, dv
\]

\[
= \int_{P_{j,t}} |\nabla (u_i - h_{i,j,t})|^n \, dv
\]

\[
\leq C(\varepsilon I_1 + 2^{(t-j)\frac{n}{2}} + I_2).
\]

By Lemma 3.2, given a regular approximated \( n \)-harmonic map \( u_i \), for all \( r \in [\lambda_n R, \delta] \), we have

\[
\int_{\partial B_r} |\nabla u_i|^n \, ds \leq C(n) \left( \int_{\partial B_r} |\nabla_T u_i|^n \, ds + \int_{B_r} |\tau(u_i)| |\nabla u_i| \, dv \right).
\]

Integrating (3.12) in \( r \) from \( r = 2^{t-j} \) to \( r = 2^{t-j} \), using Hölder’s inequality and (3.11), we obtain

\[
f_j(t) = \int_{P_{j,t}} |\nabla u_i|^n \, dv
\]

\[
\leq C \left( \int_{P_{j,t}} |\nabla_T u_i|^n \, dv + \int_{2^{t-j}}^{2^{t-j}} \|\tau(u_i)\|_{L^2(B_r)} \|\nabla u_i\|_{L^2(B_r)} \, dr \right)
\]

\[
\leq C(\varepsilon I_1 + 2^{(t-j)\frac{n}{2}} + I_2) + C \int_{0}^{2^{t-j}} \|\tau(u_i)\|_{L^2(B_r)} \|\nabla u_i\|_{L^2(B_r)} r^{\frac{n-2}{2}} \, dr
\]

\[
\leq C(\varepsilon I_1 + 2^{(t-j)\frac{n}{2}} + I_2) \leq C(\varepsilon f_j(t) + 2^{(t-j)\frac{n}{2}} + I_2) + C f_j(t).
\]

Let \( \lambda_n = \frac{\pi}{2(n-1)} \ln 2 \). Choosing \( \varepsilon \) sufficiently small in (3.13), we have

\[
0 \leq f_j'(t) - \frac{1}{C} f_j(t) + C e^{\lambda_n (t-j)}.
\]

Now, assuming that \( \lambda_n > \frac{1}{C} \) for a sufficiently large \( C \), it implies

\[
0 \leq \left( e^{-\frac{t}{n}} f_j(t) \right)' + C e^{\lambda_n (t-j)} e^{-\frac{t}{n}}.
\]

Integrating (3.14) in \( t \) over \([2, L_j]\), this gives

\[
f_j(2) \leq C \left( e^{-\frac{L_j}{n}} f_j(L_j) + e^{-\lambda_n j} e^{(\lambda_n - \frac{1}{n}) L_j j} \right)
\]

\[
\leq C \left( e^{-\frac{L_j}{n}} f_j(L_j) + e^{-\lambda_n j} e^{\frac{t}{n}} \right),
\]

where we note that

\[
P_j = B_{2^{t-j}} \setminus B_{2^{-j}}, \quad P_{j-1} \cup P_j \cup P_{j+1} = B_{2^{t-j}} \setminus B_{2^{-j}} \quad \text{and} \quad f_j(2) = \int_{P_{j,2}} |\nabla u_i|^n \, dv.
\]
Applying Lemma 3.1 on $P_j$, we have

\begin{align*}
\|u_i\|_{osc(P_j)} &\leq C \left( \int_{P_{j-1} \cup P_j \cup P_{j+1}} |\nabla u_i|^n \, dv \right)^{\frac{1}{2(n-1)}} \\
&\quad + (2^{-j})^{\frac{1}{2(n-1)}} C \left( \int_{B_{2^{j-1}}} |\tau(u_i)|^2 \, dv \right)^{\frac{1}{2(n-1)}} \\
&\leq C(f_j^{\frac{1}{2(n-1)}}(2) + e^{-\lambda n j}).
\end{align*}

For $j \geq L_j$, under the assumption (3.7) at the beginning of the proof, we can choose a small $\delta$ such that

\begin{align*}
C \left( \int_{B_{2^{j-1}}} |\tau(u_i)|^2 \, dv \right)^{\frac{1}{2(n-1)}} &\leq C(f_j^{\frac{1}{2(n-1)}}(2) + e^{-\lambda n j})
\end{align*}

Substituting (3.17) into (3.16) and summing over $j_0 \leq j \leq j_n$, we have

\begin{align*}
\|u_i\|_{osc(B_{2^n} \setminus B_{2^{n-1}})} &\leq \sum_{j=j_0}^{j_n} \|u_i\|_{osc(P_j)} \\
&\leq C \sum_{j=j_0}^{j_n} \left( (e^{-\frac{L_j}{2}} + e^{-\lambda n}) + e^{-\lambda n j} \right) \\
&\leq C \left( e^{-\frac{L_j}{2}} + \sum_{j=j_0}^{\infty} e^{-\lambda n j} \right) \\
&\leq C \left( \varepsilon + \delta + \varepsilon \right).
\end{align*}

Since

\begin{align*}
\|u_i\|_{osc(B_1 \setminus B_{2^n})} = \sup_{x, y \in B_1 \setminus B_{2^n}} |u_i(x) - u_i(y)|
\end{align*}

is controlled by $\delta$, this implies that

\begin{align*}
u(B_1) \cup \omega_1(\mathbb{R}^n)
\end{align*}

is a connected set. Thus, there is no neck between the limiting map and the bubbles for regular approximated $n$-harmonic maps with tension fields bounded in $L^2$.

Now we complete the proof of Theorem 1.

**Proof of Theorem 1.** We briefly describe the procedure of “bubble blowing” by following the idea from Ding-Tian [6]. First, we recall that the removable singularity theorem of $n$-harmonic maps [7]. Moreover, recall the the gap theorem: there is a constant $\varepsilon_g > 0$ such that if $u$ is an $n$-harmonic map on $S^n$ satisfying $\int_{S^n} |\nabla u|^n < \varepsilon_g$, then $u$ is a constant on $S^n$.

Let $u(x, t)$ be a regular solution of the $n$-harmonic flow in $M \times [0, \infty)$. As $t_i \to \infty$, it was showed in Proposition 2.1 that a subsequence of $u_i := u(t_i)$ converges to an $n$-harmonic map $u_\infty$ locally in $C^{1, \alpha}(M \setminus \{x^1, \cdots, x^L\})$. Furthermore, there is a
constant \( \varepsilon_0 > 0 \) such that the singular points (energy concentration points) \( \{x^k\} \) are defined by the condition
\[
\limsup_{t_i \to \infty} E_n(u(t_i); B_R(x^k)) \geq \varepsilon_0
\]
for any \( R \in (0, R_0) \), with some fixed \( R_0 > 0 \).

Let \( x^1 \) be a singular point. Then we find sequences \( x_i^1 \to x^1 \) such that
\[
|\nabla u(x_i^1)| = \max_{B_{R_0}(x^1)} |\nabla u(x, t_i)|, \quad r_i^1 = \frac{1}{|\nabla u(x_i^1)|} \to 0.
\]
In the neighborhood \( B_{R_0}(x^1) \) of the singularity \( x^1 \), we define the rescaled map
\[
(3.18) \quad u_i^1(x) = u(x_i^1 + r_i^1 x, t_i).
\]
Then the rescaled map \( u_i^1 \) satisfies
\[
(3.19) \quad (r_i^1)^n \frac{\partial u_i}{\partial t} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left[ |\nabla u_i|^{n-2} g^{ij} \sqrt{|g|} \frac{\partial u_i}{\partial x_j} \right] + |\nabla u_i|^{n-2} A(u)(\nabla u_i, u).
\]
Now, \( u_i^1 \) converge to \( u_{1,\infty} \) locally in \( \mathbb{R}^n \) as \( i \to \infty \), and \( u_{1,\infty} \) can be extended to a nontrivial \( n \)-harmonic map on \( S^n \) (see [7]). We call \( u_{1,\infty} \) to be the first bubble, which satisfies
\[
(3.20) \quad E_n(u_{1,\infty}; \mathbb{R}^n) = \lim_{R \to \infty} \lim_{i \to \infty} E_n(u_i^1; B_R(0)) = \lim_{R \to \infty} \lim_{i \to \infty} E_n(u_i; B_{R^2}(1^1)).
\]
At each singular point \( x^k \), there are finitely many blow-up points \( x_i^{k,l} \) and bubbles \( \{\omega_{k,l}\}_{l=1}^{J_k} \) on \( \mathbb{R}^n \) (see details in [13]); i.e. at each \( k \), there are sequences \( x_i^{k,l} \to p_i \), for some \( p_i \) and \( r_i^{k,l} \to 0 \) with \( \lim_{i \to \infty} \frac{r_i^{k,l}}{r_i^1} = \infty \) such that passing to a subsequence, \( u_i^{k,l}(x_i^1) := u_i(x_i^1 + r_i^{k,l} x_i) \) converges to \( \omega_{k,l} \), where \( \omega_{k,l} \) is an \( n \)-harmonic map in \( \mathbb{R}^n \).

These mean that there are finite numbers \( r_i, \) finite points \( x_i^{k,l} \), positive constants \( R_{k,l}, \delta_{k,l} \) and finitely many number of non-trivial \( n \)-harmonic maps \( \omega_{k,l} \) on \( \mathbb{R}^n \) such that
\[
(3.21) \quad \lim_{i \to \infty} E_n(u_i; M)
\]
\[
= E_n\left(u_{\infty}; M \setminus \{x_k\}_{k=1}^L\right) + \sum_{k=1}^L \sum_{l=1}^{J_k} E_n(\omega_{k,l}; \mathbb{R}^n)
\]
\[
+ \sum_{k=1}^L \sum_{l=1}^{J_k} \lim_{R_{k,l} \to \infty} \lim_{\delta_{k,l} \to 0} \lim_{i \to \infty} E_n(u_i^{k,l}; B_{R_{k,l}} \setminus B_{R_{k,l} r_i^{k,l}}(x_i^{k,l}))
\]

Moreover, at each neck region \( B_{\delta_{k,l}} \setminus B_{R_{k,l} r_i^{k,l}}(x_i^{k,l}) \) in (3.21), for all \( i \) sufficiently large, we have
\[
(3.22) \quad \int_{B_{2R} \setminus B_{R}(x_i^{k,l})} |\nabla u_{k,l,i}|^n dv \leq \varepsilon
\]
for all \( r \in \left( \frac{R_{k,l} r_i^{k,l}}{2}, 2R_{k,l} \right) \), where \( \varepsilon \) is a fixed constant to be chosen sufficiently small. In fact, (3.22) is a crucial observation by Ding and Tian [6]. This implies that the neck energy can be controlled during bubbling procedure by reducing multiple bubbles to a single bubble case, which leads to the proof of the energy identity for
We can find a sequence of $n$-harmonic maps in [6]. For the case of $n$-harmonic maps, we complete the proof of the energy identity by using a result of Wang and Wei (See Theorem B of [27]) . Now, we can choose a subsequence of time $t_i \to \infty$ such that $\lim_{i \to \infty} \| \partial_t u_i \|_{L^2(M)}$ is bounded. This completes a proof of Theorem 1 by using Theorem 3.

\[ \square \]

4. Finite-time blow-up of the $n$-harmonic map flow

As an application of the “no-neck” result, we will construct an example that the $n$-harmonic flow with initial value $u_0$ blows up in finite time. The proof here is to use similar ideas in [20]. Due to that there are several modifications for the case of $n$-harmonic maps, we give a proof for completeness here.

4.1. Width of $n$-harmonic maps in the covering space.

We follow the geometric setting as in Sections 3-4 of [20] to construct an example of finite time blowup of the $n$-harmonic flow. The idea is to construct a proper target manifold $N$ such that we can find infinitely many initial maps $u_0 : S^n \to N$ such that the $n$-harmonic flow blows up in finite time.

For $m > n$, let the target manifold $N = X \# T^m$ be the connected sum of $X$ with the torus $T^m$. Here $X$ is a closed $m$-dimensional manifold with nontrivial $\pi_n(X)$. Thus, there exists a smooth map $h : S^n \to X$ such that it is not homotopic to a constant map. Note that $N$ can be separated into $N_1$ and $N_2$ by an embedding sphere $S^{m-1} \subset N$. In particular, $N \setminus N_1$ and $N \setminus N_2$ are homeomorphic to $X$ and $T^m$ respectively. For each $l = 0, 1, 2, \ldots$, let $U_l$ denote a small neighborhood of $p_l$, which is diffeomorphic to a $m$-dimensional ball and $V \subset X$ denotes an open set which is diffeomorphic to a ball.

$\mathbb{R}^m$ is the universal cover of $T^m$ with $G = \mathbb{Z}^m$ as the covering transformations group. Now, for any point $p_0 \in \mathbb{R}^m$, its orbit under the transformation group $G$ is the set $\{ p_l \}_{l=0}^{\infty} \subset \mathbb{R}^m$. Let $U_0$ be a small ball in $\mathbb{R}^m$ and its orbit under the transformation group $G$ is a family of balls $\{ U_l \}_{l=0}^{\infty} \subset \mathbb{R}^m$. Now, we can find a cover of $N$ by modifying $\mathbb{R}^m$. For each $l = 0, \ldots, \infty$, we remove the small ball $U_l$ from $\mathbb{R}^m$ for $l = 1, 2, \ldots$ by adding a copy of $X \setminus V$, which we identify $\partial U_l$ by the boundary of $X \setminus V$. We denote by $X_l$ the copy of $X \setminus V$ through $\partial U_l$. This new complete and non-compact manifold is denoted by $\tilde{N}$ and the transformation group $G$ act naturally on $\tilde{N}$. Let $\tilde{N}$ to be a cover of $\tilde{N}$ and $\tilde{g}$ be the corresponding lift metric.

For a continuous map $u : S^n \to N$, we define its “width” of $u$ in a set $S \subset S^{n-1}$ through its lift map $\tilde{u}$ in the covering space $(\tilde{N}, \tilde{g})$ by

\[ \mathcal{W}(u; S) := \sup_{x, y \in S} d_{(\tilde{N}, \tilde{g})}(\tilde{u}(x), \tilde{u}(y)). \]

We begin with the following lemma that gives an upper bound for the width.

Lemma 4.1. (Bounded width lemma) If $u : \mathbb{R}^n \to N$ is an $n$-harmonic map with $E_n(u) < C_1$ for a constant $C_1 > 0$, there exists a constant $C_2$, depending only on $C_1$ and $N$ such that the $\mathcal{W}(u; S^n) < C_2$.

Proof. We prove this by contradiction. Suppose that the statement is not true. Then we can find a sequence of $n$-harmonic maps $\{ u_i \}_{i=1}^{\infty}$ with their energy bounded by the constant $C_1$ such that their width $\mathcal{W}(u_i; S^n)$ can not be bounded as $i \to \infty$. 
According the above bubble-neck decomposition, as \( t_i \to \infty \), it was showed in Proposition 2.1 that a subsequence of \( u_i \) converges to an \( n \)-harmonic map \( u_\infty \) locally in \( C^{1,\alpha}(M \setminus \{x^1, \ldots, x^L\}) \).

At each singular point \( x_k \), there are sequences \( x^{k,l}_i \to p^{k,l} \) for some \( p^{k,l} \) and \( r^{k,l}_i \to 0 \) with \( \lim_{i \to \infty} r^{k,l}_i = \infty \) such that passing to a subsequence, \( u^{k,l}_i(x) := u_i(x^{k,l}_i + r^{k,l}_i x) \) converges to \( \omega^{k,l} \), where \( \omega^{k,l} \) is an \( n \)-harmonic map in \( \mathbb{R}^n \). These mean that there are finite numbers \( r_{i,k} \), \( p^{k,l}_i \), positive constants \( R_{k,l} \), \( \delta_{k,l} \) and finitely many number of non-trivial \( n \)-harmonic maps \( \omega^{k,l} \) on \( \mathbb{R}^n \). Moreover, at each neck region \( B_{\delta_{k,l}} \setminus B_{R_{k,l}r_{i,k}}(x^{k,l}_i) \) in (3.21), for all \( i \) sufficiently large, we have

\[
\int_{B_{2r} \setminus B_r(x^{k,l}_i)} |\nabla u^{k,l}_i| dv \leq \varepsilon
\]

for all \( r \in \left( \frac{R_{k,l}r_{i,k}^4}{4}, 2\delta_{k,l} \right) \), where \( \varepsilon \) is a fixed constant to be chosen sufficiently small. Then

\[
\lim_{i \to \infty} W(u_i; S^n) = \lim_{i \to \infty} \sup_{x,y \in S^n} d_{(\hat{N},\hat{g})}(\hat{u}_i(x), \hat{u}_i(y)) \\
\leq \lim_{\delta \to 0} \lim_{i \to \infty} W\left(u_i; S^n \setminus \bigcup_{k=1}^L B_\delta(x_k)\right) \\
+ \lim_{\delta_{k,l} \to 0} \lim_{i \to \infty} \sum_{k=1}^L \sum_{l=1}^{J_k} W\left(u^{k,l}_i; B_{R_{k,l}}(0) \setminus \bigcup_{j=1}^{J_{k'} \in \delta_{k,l}} B_{\delta_{k,l}}(x^{k,l}_i)\right) \\
+ \sum_{k=1}^L \sum_{l=1}^{J_k} \lim_{\delta_{k,l} \to 0} \lim_{i \to \infty} W\left(u^{k,l}_i; B_{\delta_{k,l}} \setminus B_{R_{k,l}r_{i,k}}(x^{k,l}_i)\right),
\]

where we note that \( \{x^{k,l}_i\}_{l=1}^{J_k} = \bigcup_{l=1}^{J_k} \{x^{k,l}_i\}_{l=1}^{J_{k'}} \) is the set of totally blowing points and that \( L_{k,l} \) may not exist (This corresponds to the case of a single bubble).

Now we will estimate the width of the above region of bubbling, the neck domain and the base separately. Let \( \hat{u}^{k,l}_i \) denote the lift of \( u^{k,l}_i \). Since \( u^{k,l}_i \to \omega^{k,l} \) locally in \( C^{1,\alpha}(\mathbb{R}^n \setminus \{p^{k,l}_i\}_{k,l \in \mathbb{Z}}) \), the lift \( \hat{u}^{k,l}_i \) convergence to the lift \( \hat{\omega}^{k,l}_i \) in the covering space with lift metric \( \hat{g} \) as well, so

\[
\lim_{\delta_{k,l} \to 0} \lim_{i \to \infty} \sup_{x \in \mathbb{R}^n \setminus \bigcup_{l=1}^{J_{k,l}} B_{\delta_{k,l}}(x^{k,l}_i)} d_{(\hat{N},\hat{g})}(\hat{u}^{k,l}_i(x), \hat{\omega}^{k,l}_i(x)) = 0.
\]

By the triangle inequality, we have

\[
\lim_{\delta_{k,l} \to 0} \lim_{i \to \infty} \sum_{k=1}^L \sum_{l=1}^{J_k} W\left(u^{k,l}_i; \mathbb{R}^n \setminus \bigcup_{l=1}^{J_{k,l}} B_{\delta_{k,l}}(x^{k,l}_i)\right) \leq \sum_{k=1}^L \sum_{l=1}^{J_k} W(\omega^{k,l}; \mathbb{R}^n).
\]

Similarly, we have

\[
\limsup_{i \to \infty} \sup_{x,y \in \mathbb{R}^n \setminus \bigcup_{k=1}^L B_\delta(x_k)} d_{(\hat{N},\hat{g})}(\hat{u}_i(x), \hat{u}_i(y)) \leq W(u_\infty).
\]

By the no-neck result in Theorem 1, we have

\[
\lim_{\delta_{k,l} \to 0} \lim_{i \to \infty} W\left(u^{k,l}_i; B_{\delta_{k,l}} \setminus B_{R_{k,l}r_{i,k}}(x^{k,l}_i)\right) = 0
\]
These imply that
\[
\lim_{t \to \infty} W(u; S^n) \leq W(u_\infty; S^n) + \sum_{k=1}^{L} \sum_{l=1}^{J_k} W(\omega_{k,l}; \mathbb{R}^n),
\]
which is contradicted with the assumption. This proves the claim. \qed

As a consequence, we have

**Lemma 4.2.** Let \( u \) be a regular solution to (1.2) in \( M \times [0, \infty) \) with initial value \( u_0 \) satisfying \( E_n(u_0) < C_1 \) for a constant \( C_1 > 0 \). Then there is a sequence \( t_i \to \infty \) such that \( u(\cdot, t_i) \) converges to an \( n \)-harmonic map \( u_\infty \) in \( C^{1,\alpha}_{\text{loc}}(M \setminus \{x^1, \ldots, x^L\}) \). Moreover, there exists a constant \( C_3 \), depending only on \( C_1 \), such that the
\[
\limsup_{t \to \infty} W(u(\cdot, t); S^n) \leq C_3.
\]

**Proof.** By using Theorem 1, there exists a sequence \( t_i \to \infty \) such that \( u(t_i) \) converges to an \( n \)-harmonic map \( u_\infty \) in \( C^{1,\alpha}(M \setminus \{x^1, \ldots, x^L\}) \) for some positive \( \alpha < 1 \). Moreover, there are a finite number of \( n \)-harmonic maps \( \omega_{k,l} \) on \( S^n \) with \( k = 1, \ldots, L \) and \( l = 1, \ldots, J_k \). By applying Lemma 4.1, we have
\[
\limsup_{t \to \infty} W(u; S^n) \leq W(u_\infty; S^n) + \sum_{k=1}^{L} \sum_{l=1}^{J_k} W(\omega_{k,l}; S^n) \leq C_3,
\]
where \( C_3 \) depends on \( C_2 \) and total numbers of bubbles. \qed

With this bounded width lemma, we are now ready to construct the example of the \( n \)-harmonic map flow with initial map \( u_0 : S^n \to N \) which blows up in finite time. The basic idea is as follows: We construct an initial \( u_0 : S^n \to N \) which has finite energy. Then we see if a map \( u' \), which is homotopic to \( u_0 \), could have a large width.

**4.2. Proof of Theorem 2.** Since \( X \) is a closed manifold of dimension \( m > n \) with nontrivial \( \pi_n(X) \), we can find a smooth map \( h : S^n \to X \) such that

(a) \( h \) is non-subjective;

(b) \( h \) is not homotopic to any constant map;

(c) \( h(S^n) \subset X \setminus \overline{V} \);

(d) \( h \) maps the southern hemisphere of \( S^n \) to a point \( q \in X \setminus V \).

For each \( l = 0, 1, \ldots \), we denote \( h_l : S^n \to X_l \subset \tilde{N} \) as a copy of \( h \) and \( q_l \in X_l \) as a copy of \( q \), and also denote by \( S_p \) the south pole of \( S^n \).

For any large constant \( K > 0 \), there is a sufficiently large \( l \) such that
\[
d((\tilde{N}, \tilde{g}) (X_0, X_1) \geq K.
\]
Let \( q_0 \in X_0 \) and \( q_l \in X_l \) be copies of \( q \). Let \( \Psi \) and \( \Phi \) be two stereographic projections from \( S^n \) to \( \mathbb{R}^n \) given by
\[
\begin{align*}
\Phi(x^1, \ldots, x^n, x^{n+1}) &= \left( \frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}} \right), \\
\Psi(x^1, \ldots, x^n, x^{n+1}) &= \left( \frac{x^1}{1 + x^{n+1}}, \ldots, \frac{x^n}{1 + x^{n+1}} \right),
\end{align*}
\]
which map the north pole \( N_p \) and the south pole \( S_p \) of \( S^n \) to the infinity respectively.
In order to construct an initial map \( u_0 : S^n \to N \), we define a map \( \tilde{u}_0 : S^n \to \tilde{N} \) by

\[
\tilde{u}_0 = \begin{cases} 
  h_0(x), & \text{for } x \in S^n \setminus B_\sigma(S_p); \\
  \gamma \circ \varphi \left( \frac{\log \sigma - \log |x|}{-\log \sigma} \right), & \text{for } x \in B_\sigma(S_p) \setminus B_{\sigma^2}(S_p); \\
  h_1 \circ \Phi^{-1} \circ \left( \frac{\Psi(x)}{\sigma^2} \right), & \text{for } x \in B_{\sigma^2}(S_p).
\end{cases}
\]  

(4.6)

Here \( \gamma : [0, 1] \to \tilde{N} \) is the shortest geodesics connecting \( q_0 \) to \( q_1 \) in \( \tilde{N} \), and \( \varphi \) is a smooth cut-off function on \([0, 1]\) that satisfies:

1. \( \varphi' \) is non-negative and \(|\varphi| \leq 1\);
2. \( \varphi(x) = 0 \), for \( x \in \left[0, \frac{1}{8}\right] \) and \( \varphi(x) = 1 \) for \( x \in \left[\frac{7}{8}, 1\right] \);
3. \(|\varphi'| \leq C \), where \( C \) is a constant.

Under the definition of \( \tilde{u}_0 \), we can see \( \tilde{u}_0|_{\partial B_{\sigma}(S_p)} = q_0 \). Moreover, for small \( \sigma \), the metric was flattened out which gives \( \tilde{u}_0|_{\partial B_{\sigma^2}(S_p)} = q_1 \).

Given that we have \( \tilde{g} \) as the pullback metric for the covering of \((N, g)\), there exists an isometric projection map \( \pi : \tilde{N} \to N \). For sufficiently small \( \sigma \), we can find a smooth \( u_0 : S^n \to N \) defined by

\[
u_0 = \begin{cases} 
  \pi \circ h_0(x), & \text{for } x \in S^n \setminus B_\sigma(S_p); \\
  \pi \circ \gamma \circ \varphi \left( \frac{\log \sigma - \log |x|}{-\log \sigma} \right), & \text{for } x \in B_\sigma(S_p) \setminus B_{\sigma^2}(S_p); \\
  \pi \circ h_1 \circ \Phi^{-1} \circ \left( \frac{\Psi(x)}{\sigma^2} \right), & \text{for } x \in B_{\sigma^2}(S_p).
\end{cases}
\]  

(4.7)

Now we claim that there is a constant \( C_1 \) depending on \( h_0 \) such that

\[
E_n(u_0) < E_n(h_1) + E_n(h_0) + 1 = C_1.
\]  

(4.8)

Due to the fact that \( E_n(u) \) is conformally invariant, the energy \( E_n(u_0) \) over \( S^n \setminus B_\sigma(S_p) \) and \( B_{\sigma^2}(S_p) \) for small \( \sigma \) can be bounded by

\[
\int_{S^n \setminus B_\sigma(S_p)} |\nabla u_0|^n \ dv + \int_{B_{\sigma^2}(S_p)} |\nabla u_0|^n \ dv \leq E_n(h_0) + E_n(h_1) + \frac{1}{2}.
\]  

(4.9)

Now we have to check if the condition (4.8) is satisfied. We do this by estimating the energy over \( B_\sigma(S_p) \setminus B_{\sigma^2}(S_p) \), then compare it with (4.9).

Let \( L \) be the shortest distance between \( q_0 \) and \( q_1 \). Since \( \gamma \) is the shortest geodesics connecting \( q_0 \) to \( q_1 \) in \( \tilde{N} \), there is a parametrization \( \tilde{s} \) such that

\[
\int_0^1 |(\pi \circ \gamma)'| \ dv = d_{(\tilde{N}, \tilde{g})}(q_0, q_1) = L, \quad |(\pi \circ \gamma)'| = L.
\]

Therefore, we have

\[
|\partial_r u_0| \leq |(\pi \circ \gamma)'| |\varphi'| \frac{1}{r(-\log \sigma)} \leq \frac{CL}{r(-\log \sigma)}
\]

which gives us to estimate the energy of \( u_0 \) on the annulus domain; i.e.

\[
\int_{B_\sigma \setminus B_{\sigma^2}} |\nabla u_0|^n \ dv \leq C \int_{\sigma^2} |\partial_r u_0|^n r^{n-1} \ dr
\]

\[
\leq \frac{CL^n}{(-\log \sigma)^n} \int_{\sigma^2} \frac{1}{r} \ dr \leq \frac{CL^n}{(-\log \sigma)^{n-1}}.
\]
Therefore, the energy on the annulus domain \( B_\sigma \setminus B_{\sigma^2} \) can be controlled for any \( L \) with a sufficiently small \( \sigma \). Together with (4.9), we obtained an upper bound \( C_1 \) for \( E(u_0) \).

Now, for any \( u' \) (with a lift \( \tilde{u}' \)) which is homotopic to \( u_0 \), we claim that \( \tilde{u}' \) intercepts with \( X_0 \) and \( X_l \), which implies

\[
W(u'; S^n) \geq d_\beta(\tilde{u}', \beta)(X_0, X_l) \geq K > C_3.
\]

We prove this claim by contradiction. Assume that \( \tilde{u}' \) does not intercept with \( X_0 \). Set a continuous map \( \pi : \tilde{N} \to X \) so that \( \pi \) maps \( \tilde{N} \setminus X_0 \) to a single point \( p \in X \). Since \( u' \cap X_0 = \emptyset \), it follows that \( \pi \circ u' \) maps to \( p \) which is a constant map. However, consider \( \pi \circ \tilde{u}' \) is homotopic to \( \pi \circ \tilde{u}_0 \) which is homotopic to \( h_0 \) as well. This contradicts with the property (b) of the definition of \( h_0 \). This shows that \( \tilde{u}' \) must intercept with \( X_0 \). By a similar argument, \( \tilde{u}' \) must intercept with \( X_l \).

Assume that the \( n \)-harmonic map flow with initial value \( u_0 \) does not blow up in finite time. Let \( u(x, t) \) be a regular solution to the flow (1.2) in \( M \times [0, \infty) \) with initial value \( u_0 \in C^1(M, N) \). By Theorem 1, there is a sub-sequence \( t_i \) such that as \( t_i \to \infty \), \( u(x, t_i) \) converges to an \( n \)-harmonic map \( u_\infty \) in \( C^1(M \setminus \{x^1, \ldots, x^L\}) \) for some positive \( \alpha < 1 \). Since \( u_i := u(x, t_i) \) is homotopic to \( u_0 \), we have

\[
W(u_i; S^n) \geq K > C_3.
\]

On the other hand, by Lemma 4.2, \( \limsup_{i \to \infty} W(u_i; S^n) \leq C_3 \). This is a contradiction. Therefore, we have constructed an initial maps \( u_0 : S^n \to N \) such that the \( n \)-harmonic map flow with initial value \( u_0 \) must blow up in finite time. This completes a proof of Theorem 2.

**Remark.** It is an interesting question whether the the heat flow for \( H \)-systems ([14]) on \( n \)-manifolds blows up in finite time for \( n \geq 3 \).

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