THE IMPORTANCE OF THE SELBERG INTEGRAL

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Abstract. It has been remarked that a fair measure of the impact of Atle Selberg’s work is the number of mathematical terms which bear his name. One of these is the Selberg integral, an $n$-dimensional generalization of the Euler beta integral. We trace its sudden rise to prominence, initiated by a question to Selberg from Enrico Bombieri, more than thirty years after publication. In quick succession the Selberg integral was used to prove an outstanding conjecture in random matrix theory, and cases of the Macdonald conjectures. It further initiated the study of $q$-analogues, which in turn enriched the Macdonald conjectures. We review these developments and proceed to exhibit the sustained prominence of the Selberg integral, evidenced by its central role in random matrix theory, Calogero–Sutherland quantum many body systems, Knizhnik–Zamolodchikov equations, and multivariable orthogonal polynomial theory.

1. Discovery and reappearance

1941 and 1944. With the passing of Atle Selberg on August 6th 2007 at age 90, it is timely to reflect on his mathematical legacy. Indeed a number of brief articles highlighting some of his most influential mathematical discoveries were written shortly after the news of his death, see e.g., [74]. It is our aim to add to these tributes by giving a more comprehensive account of the mathematics, both pure and applied, related to what now is referred to as the Selberg integral

\begin{equation}
S_n(\alpha, \beta, \gamma) := \int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} \, dt_1 \cdots dt_n
\end{equation}

The evaluation of this integral is valid for complex parameters $\alpha, \beta, \gamma$ such that

\begin{equation}
\text{Re}(\alpha) > 0, \quad \text{Re}(\beta) > 0, \quad \text{Re}(\gamma) > -\min\{1/n, \text{Re}(\alpha)/(n-1), \text{Re}(\beta)/(n-1)\},
\end{equation}

corresponding to the domain of convergence of the integral.

The proof of (1.1) is the subject of Selberg’s 1944 paper “Bemerkninger om et multipelt integral” (Remarks on a multiple integral) [134] — the only one of Selberg’s works written in Norwegian — published in Norsk Matematisk Tidsskrift. The latter has been compared [22] to the Scandinavian equivalent of the Mathematical Gazette, with contents ranging from short research papers on subjects of

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general interest to discussions on teaching problems. Selberg himself remarks in his collected works [136] that

This paper was published with some hesitation, and in Norwegian, since I was rather doubtful that the results were new. The journal is one which is read by mathematics-teachers in the gymnasium, and the proof was written out in some detail so it should be understandable to someone who knew a little about analytic functions and analytic continuation.

Selberg’s proof of (1.1) proceeds by supposing \( \gamma \) is a positive integer, and expanding

\[
\prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} = \sum_{0 \leq k_1, \ldots, k_n \leq 2(n-1)\gamma} c_{k_1, \ldots, k_n} t_1^{k_1} \cdots t_n^{k_n}.
\]

Substituting this expansion in the definition of \( S_n(\alpha, \beta, \gamma) \) allows the resulting integrals to be evaluated by the Euler beta integral [44]

\[
B(\alpha, \beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},
\]

which itself is (1.1) with \( n = 1 \). The details, in English, of the proof from here on can be found in [54, 107], for example. Perhaps the most significant feature is the final step. It requires analytically continuing off the integers. Thus with (1.1) established for \( \gamma \) a positive integer, the remaining task is to establish its validity for all complex \( \gamma \) such that both sides are well defined.

For this purpose, after noting that both the left- and right-hand side of (1.1) are bounded analytic functions of \( \gamma \) for \( \text{Re}(\gamma) \geq 1 \) at least, Carlson’s theorem can be used [26]. The latter applies to functions \( f(z) \) analytic for \( \text{Re}(z) \geq 0 \) satisfying the bound \( |f(z)| = O(e^{\mu |z|}) \), \( \mu < \pi \). The theorem asserts that if furthermore \( f(z) = 0 \) on the nonnegative integers then, identically, \( f(z) = 0 \). Note that the function \( f(z) = \sin \pi z \) shows that the bound \( \mu < \pi \) is optimal. Selberg did not make direct use of Carlson’s theorem, but rather derived from first principles the same result in the case that \( f(z) \) is bounded in the right half-plane, which is all that is required to finalise the proof of (1.1).

Interestingly, although [134] contains the first proof of (1.1), it is not the first time it appeared in print. This occurred three years earlier (albeit with the change of variables \( t_i = s_i/(1 + s_i) \) so that \( s_i \in [0, \infty) \)) in Selberg’s 1941 paper “Über einen Satz von A. Gelfond” (On a theorem of A. Gelfond) [133]. Like [134], this earlier paper appeared in a Norwegian journal, this time Archiv für Mathematik og Naturvitenskap, known for having Sophus Lie as one of its founders. In a footnote Selberg remarks

Leider habe ich die Formel (11) [The Selberg integral] nirgends in der Litteratur finden können, ein Beweis hier zu bringen scheint aber nicht angebracht, da die Arbeit sonst zu sehr anschwellen würde; sollte sich aber herausstellen, dass die Formel neu wäre, beabsichtige ich später ein Beweis zu veröffentlichen.

[Unfortunately I have been unable to find formula (11) [The Selberg integral] in the literature. To present a proof here, however, seems inappropriate, as it would make this paper significantly longer. If
it turns out that the formula is new, I intend to publish a proof at a later date.]

Curiously, Selberg used his integral in [133] to prove a result of some similarity to Carlson’s theorem. As already noted, the latter is itself an ingredient in Selberg’s proof of (1.1). Selberg’s result relates to entire functions \( f(z) \) of exponential type \( \sigma(f) \), defined by

\[
\sigma(f) := \limsup_{r \to \infty} \frac{1}{r} \log \left( \max_{|z|=r} |f(z)| \right).
\]

A theorem of Hardy and Pólya [21,122] states that if \( \sigma(f) < \log 2 \) and \( f \) takes integer values at the nonnegative integers, then \( f(z) \) is polynomial. The transcendental function \( f(z) = 2^z \) shows that this bound is optimal. A. Gelfond [64] generalized this by proving that if \( \sigma(f) < n \log(1 + \exp(1/n - 1)) \) and \( f \), together with its first \( n-1 \) derivatives, take integer values at the nonnegative integers, then \( f(z) \) is a polynomial. However, for \( n > 1 \) this bound is not optimal. By using his integral Selberg improved Gelfond’s bound for \( n > 1 \) to \( \sigma(f) < \log m_n \), where \( m_n \) is the minimum value of \( \prod_{i=1}^{n}(1 + y_i) \) under the conditions \( y_i > 0 \), \( y_1 \cdots y_n = e^{1-n} \) and \( \prod_{1 \leq i < j \leq n}|y_i^{-1} - y_j^{-1}| = 1 \). This improves Gelfond’s result since \( \prod_{i=1}^{n}(1 + y_i) > (1 + e^{1/n-1})^n \).

The 1950s to the late 1970s — the Mehta integral. For over thirty years the Selberg integral went essentially unnoticed. It was used only once — in the special case \( \alpha = \beta = 1, \gamma = 2 \) — in a study by S. Karlin and L.S. Shapley relating to the volume of a certain moment space, published in 1953 [87].

Around ten years later there was again good reason to make use of (1.1). Building upon the earlier work of E.P. Wigner in the 1950s, F.J. Dyson wrote a series of papers on the statistical theory of energy levels of complex systems. These papers ranged from the theory’s foundations to its practical use in the analysis of experimental data. This last topic was addressed in part V of the series, in a work, written jointly with M.L. Mehta and published in 1963, which also summarizes both the status of the theory and open problems from that date.

A basic point is that random Hermitian matrices are used to model the highly excited states of complex nuclei. These matrices are taken to have real, complex or real quaternion elements, and correspond to the quantum system having time reversal symmetry, no time reversal symmetry, or time reversal symmetry with an odd number of spin 1/2 particles respectively.

In the real case all independent elements are chosen from independent standard normals \( N[0,1] \), and in the complex case the diagonal elements are chosen independently from \( N[0,1] \) while the off-diagonal elements are chosen independently from \( N[0,1/\sqrt{2}] + i N[0,1/\sqrt{2}] \). Real quaternion elements are themselves \( 2 \times 2 \) blocks of the form

\[
\begin{bmatrix}
z & w \\
-\bar{w} & \bar{z}
\end{bmatrix}.
\]

In general, the eigenvalues of matrices with real quaternion elements are doubly degenerate. On the diagonal, each independent entry is chosen from \( N[0,\sqrt{2}] \), while on the off-diagonal, each independent entry is chosen from \( N[0,1] + i N[0,1] \). These ensembles of random matrices are referred to as the Gaussian orthogonal, unitary and symplectic ensembles (abbreviated GOE, GUE and GSE) respectively.
For each of the three Gaussian ensembles the joint probability density function (PDF) for the eigenvalues can be computed explicitly as \[107\]
\[
\frac{1}{(2\pi)^{n/2}F_n(\beta/2)} \prod_{i=1}^{n} e^{-t_i^2/2} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{\beta}.
\]
Here $\beta = 1, 2, 4$ for the GOE, GUE and GSE respectively, while $F_n$ is the normalization \[107\]
\[
F_n(\gamma) := \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n} e^{-t_i^2/2} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} dt_1 \cdots dt_n,
\]
referred to as Mehta’s integral. In \[108\] Mehta and Dyson evaluated $F_n(\beta/2)$ for each of the three special values of $\beta$. Combining this with the evaluations for $n = 2$ and $n = 3$ for general $\beta$ led them to conjecture that \[108\]
\[
F_n(\gamma) = \prod_{j=1}^{n} \frac{\Gamma(1 + j\gamma)}{\Gamma(1 + \gamma)}.
\]
Assuming the validity of \[108\] for general $\gamma$, certain averages associated with \[107\] at the special random matrix couplings $\beta = 1, 2, 4$ are accessible. This becomes apparent by writing \[107\] in the form
\[
e^{-\beta U} \frac{1}{(2\pi)^{n/2}F_n(\beta/2)}, \quad U = \frac{1}{2\beta} \sum_{i=1}^{n} t_i^2 - \sum_{1 \leq i < j \leq n} \log|t_i - t_j|.
\]
The mean $\langle U \rangle$ for a given $\beta$ is now computed by taking the logarithmic derivative of the normalization $F_n(\beta/2)$. A further differentiation with respect to $\beta$ then yields the fluctuation $\langle U^2 \rangle - \langle U \rangle^2$.

The form \[108\] highlights an analogy with the equilibrium statistical mechanics of a classical gas of $n$ particles on the line, at inverse temperature $\beta$, interacting via a repulsive logarithmic Coulomb potential and confined by a harmonic well. The quantity $\exp(-\beta U)$ is referred to as the Boltzmann factor. This interpretation plays a prominent role in Dyson’s series of works. Indeed, the notation for the averages used above stems from the statistical physics literature (and corresponds to the mean energy and specific heat of the Coulomb gas) and may be substituted by the mean $\mu(U)$ and variance $\sigma^2(U)$ respectively.

It is not hard to see that the Selberg integral can be used to evaluate Mehta’s integral thus proving the conjecture \[108\]. By the change of variables $t_i \mapsto (1 - t_i/L)/2$ in \[108\]
\[
limit_{L \to \infty} 2^{L^2}(2L)^{n(n-1)\gamma} S_n(L^2/2, L^2/2, \gamma) = F_n(\gamma).
\]
Use of Stirling’s formula to compute the same limit on the right-hand side of \[108\] then gives \[108\]. However, in 1963 when Mehta and Dyson published their conjecture the Selberg integral was essentially unknown and so this method of proof was not available.

The Mehta–Dyson conjecture received more prominence with its appearance in the first edition of Mehta’s book *Random Matrices and the Statistical Theory of Energy Levels*, published in 1967 [105]. In 1974 Mehta submitted the conjecture to the problems section of *SIAM Review* [106], thus gaining exposure to an even wider mathematical audience. A proof, exactly the one mentioned in the previous
paragraphs, was finally uncovered in the late 1970s by Selberg’s IAS colleague Enrico Bombieri. The remarkable story behind this proof is best told in Bombieri’s own words [23]:

Since 1976 I had been studying elementary methods in prime number theory and in particular a several variable extension of Chebyshev’s well-known method to obtain upper and lower bounds for the number of primes up to a given bound. In the course of my researches I came across the problem of the asymptotic computation of certain multiple integrals, the simplest being

$$\int \prod_{i=1}^{n} z_i^{r-1}(1 - z_i)^p \prod_{i,j=1}^{n} (z_i - z_j)^q \, dz_1 \cdots dz_n$$

where $p, q, r$ are large positive integers and the integral is extended to the product of the unit circles $|z_i| = 1$ or to $[0, 1]^n$.

The integral is related to a partition function for the one-dimensional Coulomb gas on the unit circle $|z| = 1$ with a fixed point charge at $z = 1$, as it was explained to me by my friend and colleague Tom Spencer, so I went to Dyson and asked him whether physicists had encountered such things before; maybe he could save me some efforts.

Dyson told me that for $q = 1$ and 2 an integral of this type, over the real line with a gaussian measure, had indeed been studied and he referred me to a book by Mehta. Then I went to see Atle to ask his opinion about what I was doing in order to study the distribution of primes and whether he felt it was of any interest and whether he had any opinion on it.

He immediately recognized my integral as a complex version of the generalized beta integral he had studied before and he gave me an off-print of his paper. It was not difficult to follow his proof, given for an integral over $[0, 1]^n$, and use a classical method to write a Beta integral as a complex integral to solve my problem of computing my integral exactly. The multiple integral over $[0, 1]^n$ is of course Selberg’s integral, as in that case arithmetical applications require $r$ to be a large negative (not positive) integer. It was also quite easy to get a confluent form of the Selberg integral and compute exactly the Mehta integrals for a general value of the parameter and make physicists happy.

Since this was of interest to Dyson, I went back to Dyson and told him that using the Selberg integral one could compute the integral of interest to physicists.

More from the 1960s and 70s — constant term identities. The consideration of time reversal symmetry leading to three ensembles of Hermitian matrices applies equally well to unitary matrices [42]. A conventional time reversal symmetry requires that $U = U^T$, no time reversal symmetry imposes no constraint, whilst a time reversal symmetry for a system with an odd number of spin 1/2 particles requires $U = U^D$ (here $D$ denotes the quaternion dual; see e.g., [54, Ch. 2]). Choosing such matrices with a uniform probability then gives what are referred
to as the circular orthogonal, unitary and symplectic ensembles (COE, CUE and CSE) respectively. Their joint eigenvalue PDFs are given explicitly by

\[ \frac{1}{(2\pi)^n} C_n(\beta/2) \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^\beta, \]

where \( C_n \) is the normalization

\[ C_n(\gamma) = \frac{1}{(2\pi)^n} \int_{-\pi}^\pi \cdots \int_{-\pi}^\pi \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^{2\gamma} \, d\theta_1 \cdots d\theta_n, \]

and \( \beta = 1, 2, 4 \) for the COE, CUE and CSE respectively.

As for (1.5), the random matrix calculations give (1.10) in terms of gamma functions for the three special values of \( \beta \). Furthermore, the case \( n = 2 \) for general \( \beta \) can be related to the Euler beta integral (1.3), whilst the case \( n = 3 \) gives a sum which is a special instance of an identity of Dixon for a well-poised \( 3F_2 \) series [6],

\[ 3F_2 \left( \begin{array}{c} a, b, c \\ 1 + a - b, 1 + a - c \end{array} \right) = \frac{\Gamma(1 + \frac{a}{2})\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + \frac{a}{2} - b - c)}{\Gamma(1 + a)\Gamma(1 + \frac{a}{2} - b)\Gamma(1 + \frac{a}{2} - c)\Gamma(1 + a - b - c)}. \]

Based on all of these results, Dyson, in part I of his series of papers, made the conjecture [42]

\[ C_n(\gamma) = \frac{\Gamma(1 + n\gamma)}{\Gamma^n(1 + \gamma)}. \]

In the same paper, Dyson observed that with \( \gamma \) a nonnegative integer, say \( k \), (1.10) can be rewritten as the constant term (CT) in a Laurent expansion. This allows (1.10) to be rewritten as

\[ \text{CT} \prod_{1 \leq i < j \leq n} \left( 1 - \frac{x_i}{x_j} \right)^k \left( 1 - \frac{x_j}{x_i} \right)^k = \frac{(kn)!}{(k!)^n}. \]

This constant term identity, and thus, by Carlson’s theorem, the conjecture (1.12), was soon proved by J. Gunson and K. Wilson [159], and later in a very efficient analysis by I.J. Good [65]. Gunson’s proof is mentioned in [42], but the work is unpublished; reference often given to [66] in this context actually refers to the proof of another conjecture of Dyson. Twenty years after his proof Wilson was to receive the Nobel Prize in physics for his work on the renormalisation group approach to the study of critical phenomena; [159] is his first publication.

In their proof, Wilson and Good both took advantage of the extra degrees of freedom offered by Dyson’s more general conjecture, also contained in [42],

\[ \text{CT} \prod_{i,j=1 \atop i \neq j}^n \left( 1 - \frac{x_i}{x_j} \right)^{a_i} \left( \frac{a_1 + \cdots + a_n}{a_1! \cdots a_n!} \right). \]

The formulation of this was in turn motivated by the extra degrees of freedom permitted by Dixon’s identity (1.11), to which (1.14) reduces in the case \( n = 3 \).

In fact, as observed by R. Askey [11], the Selberg integral can be used to prove Dyson’s conjecture (1.12) directly without the need for (1.14). Askey’s observation
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is based on the easily established general identity

\[(1.15) \quad \int_0^1 \cdots \int_0^1 (t_1 \cdots t_n)^{\zeta-1} f(t_1, \ldots, t_n) \, dt_1 \cdots dt_n = \left( \frac{1}{2 \sin \pi \zeta} \right)^n \int_{-\pi}^\pi \cdots \int_{-\pi}^\pi e^{i \zeta (\theta_1 + \cdots + \theta_n)} f(-e^{i \theta_1}, \ldots, -e^{i \theta_n}) \, d\theta_1 \cdots d\theta_n,\]

valid for \( f \) a Laurent polynomial and \( \text{Re}(\zeta) \) large enough so that the left-hand side exists. Applying (1.15) to the Selberg integral with \( \beta \) a positive integer and \( \gamma \) a nonnegative integer shows that

\[(1.16) \quad S_n(\alpha, \beta, \gamma) = (-1)^n \left( \frac{\pi}{\sin \pi b} \right)^n M_n(a, b, \gamma),\]

where \( \alpha := -b - (n - 1)\gamma, \beta := a + b + 1 \) and

\[(1.17) \quad M_n(a, b, \gamma) := \frac{1}{(2\pi)^n} \int_{-\pi}^\pi \cdots \int_{-\pi}^\pi \prod_{i=1}^n e^{i \theta_i (a-b)} |1 + e^{i \theta_i}|^{-a+b} \times \prod_{1 \leq i < j \leq n} |e^{i \theta_i} - e^{i \theta_j}|^{2\gamma} \, d\theta_1 \cdots d\theta_n.\]

From (1.16), the Selberg integral, the reflection formula and finally Carlson’s theorem, it follows that

\[(1.18) \quad M_n(a, b, \gamma) = \prod_{j=0}^{n-1} \frac{\Gamma(1+a+b+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(1+a+j\gamma)\Gamma(1+b+j\gamma)\Gamma(1+\gamma)},\]

for \( a, b, \gamma \in \mathbb{C} \) such that

\[\text{Re}(a + b + 1) > 0, \quad \text{Re}(\gamma) > -\min\{1/n, \text{Re}(a+b+1)/(n-1)\}.\]

For \( a = b = 0 \) this is Dyson’s conjecture (1.12).

The change of variables \( e^{i \theta_i} = (i-t_i)/(i+t_i) \) maps the unit circle onto the real line via a stereographic projection. Applying this to the integral (1.17) leads to

\[(1.19) \quad \frac{1}{(2\pi)^n} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \prod_{i=1}^n \frac{1}{(1+it_i)^{a+n(1-it_i)}} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} \, dt_1 \cdots dt_n = 2^{-n(a+\beta-1)+n(n-1)\gamma} \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+\beta-1-(n+j-1)\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)\Gamma(1+\gamma)}.\]

When \( n = 1 \) this is the Cauchy beta integral.

In the letter to Dyson reprinted on the next page Selberg communicated the multiple Cauchy integral (1.19). Subsequently, in a letter to Askey dated 25 March 1980, he mentioned both (1.17) and (1.19), and pointed out their exact relationship. The first time (1.17) appeared in print was in W.G. Morris’ 1982 PhD thesis [113]. In his thesis Morris provided a proof of (1.17) along the lines of Selberg’s proof of (1.1), and applied it to obtain constant term identities. For these reasons (1.17) is now commonly referred to as the Morris integral.
Dear Freeman,

Thanks for your note. Actually I found the formula in 1947. I had not really planned to publish anything about it, but was later asked to contribute an article to a Norwegian journal and thought this might be suitable for that audience.

Of course, I did have a bit more along these lines, but did not wish to make the article too lengthy. Obviously the limiting cases like the analog of Euler's integral for the R-function

\[ \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n (t_i) \cdot \left( \sum_{k=1}^\infty \frac{1}{(i+1)^k} \right) \frac{d\Delta(t_1)}{dt_1} \cdots \frac{d\Delta(t_n)}{dt_n}, \]

which can be derived in a similar way as the other limiting case you mention.

Another related (though not as obviously)

formula in

\[ \int_0^\infty \cdots \int_0^\infty \left( \frac{\Delta(t)}{t_1 \cdots t_n} \right)^{2^m} \frac{d\Delta(t_1)}{dt_1} \cdots \frac{d\Delta(t_n)}{dt_n} = \]

\[ = \left( \frac{\prod_{k=1}^n (1+it_k)}{\prod_{k=1}^n (1-it_k)} \right)^{2^m} \prod_{k=1}^m \frac{\Gamma(1+y_k) \cdot \Gamma(1-x_k) \cdot \Gamma(1-\alpha_k)}{\Gamma(1+y_k-x_k) \cdot \Gamma(1-y_k-x_k) \cdot \Gamma(1-\alpha_k)} \]

This is valid for complex \( x, y, \alpha \) for which the integral converges absolutely (conditions easy to find a bit tedious to write down). From (2) one can once (using it for the case \( x = y \)) again obtain the formula you refer to in your letter as a limiting case.

I had not thought about these things now for more than thirty years when Bombieri consulted me about a problem of his which it seemed to me could be handled by using my formula from the 1944 paper. That there had been some interest in analogous integrals among physicists was completely unknown to me earlier.

Yours sincerely,

Åke
A culmination — The Macdonald Conjectures. In 1982 I.G. Macdonald [100] published his now famous ex-conjectures, generalizing the Mehta integral (1.5) to all finite reflection or Coxeter groups $G$, and the Dyson constant term identity (1.13) to all finite root systems.

Let $G$ be a finite group of isometries of $\mathbb{R}^n$, generated by reflections in $N$ hyperplanes. Normalise (up to sign) so that the equations for the hyperplanes take the form

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0 \quad \text{with} \quad a_{i1}^2 + \cdots + a_{in}^2 = 2,$$

where $i$ labels the hyperplanes, and form the polynomial

$$P(x) = \prod_{i=1}^{N} (a_{i1}x_1 + \cdots + a_{in}x_n).$$

Geometrically, $2^{-N/2}P(x)$ gives the product of the distances of the point $x \in \mathbb{R}^n$ to the $N$ hyperplanes.

By its action on $\mathbb{R}^n$ the group $G$ acts on polynomials in $x = (x_1, \ldots, x_n)$. The polynomials that are invariant under the action of $G$ are referred to as $G$-invariant polynomials. They form an $\mathbb{R}$-algebra $\mathbb{R}[f_1, \ldots, f_n]$ generated by $n$ algebraically independent polynomials $f_1, \ldots, f_n$ of degrees $d_1, \ldots, d_n$. Unlike the set of $f_i$’s, the set of $d_i$’s is uniquely determined by the underlying reflection group.

A final ingredient required in the Macdonald integral conjectures is the Gaussian measure $\varphi$ on $\mathbb{R}^n$

$$d\varphi(x) := \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}} dx_1 \cdots dx_n,$$

where $|x|^2 := \sum_{i=1}^{n} x_i^2$.

With the above notations Macdonald’s (ex)-conjecture [100, Conjecture 5.1] states that for each finite reflection group $G$

$$\int_{\mathbb{R}^n} |P(x)|^{2\gamma} d\varphi(x) = \prod_{i=1}^{n} \frac{\Gamma(1 + d_i\gamma)}{\Gamma(1 + \gamma)}.$$

For the three infinite families of crystallographic reflection groups (or reflection groups of Weyl type) $A_{n-1}$, $B_n$ and $D_n$ the Macdonald conjecture follows as a limit of the Selberg integral. For type $A_{n-1}$ this corresponds to Bombieri’s proof of the Mehta conjecture mentioned earlier. For types $B_n$ and $D_n$ this is due to A. Regev, although the actual proof appeared for the first time in the paper by Macdonald, to whom Regev communicated his results.

Around the same time as Macdonald formulated his conjectures Regev was studying the large $n$ behaviour of sums of the form $S^\beta_{\ell}(n) := \sum_\lambda (d_\lambda)^\beta$ where the sum is over partitions $\lambda$ of at most $\ell$ parts, and $d_\lambda$ is the dimension of the irreducible $\mathfrak{S}_n$ character $[\lambda]$. Combining the hook-length formula for $d_\lambda$ with a careful asymptotic analysis, Regev showed [127] (see also [29]) that the asymptotics of sums like $S^\beta_{\ell}(n)$ leads exactly to Mehta’s integral. Regev remarks [128]

From reactions to preprints and talks on [127], first from Richard Stanley (who in 1978 attended my seminar talk at UCSD) then from Freeman Dyson, I learned about the Mehta and the Macdonald conjectures. In a letter, Dyson also mentioned that the Mehta conjecture had just been verified — by applying the Selberg integral. William Beckner then showed me the details of how to deduce
the Mehta — and some other integrals — from the Selberg integral.
I worked on the other classical cases of the Macdonald conjecture
and managed to verify these shortly afterwards, in 1979.

The Coxeter group $A_{n-1}$ is the symmetry group of the $(n-1)$-simplex. It is a
group of order $n!$ generated by the $n(n-1)/2$ hyperplanes
$$x_i - x_j = 0 \quad \text{for} \quad 1 \leq i < j \leq n,$$
and is isomorphic to the symmetric group $S_n$. All of the ingredients in (1.20)
can thus easily be determined explicitly. The polynomial $P(x)$ is given by the
Vandermonde product

$$P(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j) =: \Delta(x)$$

and the $G$-invariant polynomials are given by the symmetric polynomials in $x$.
One of the classical bases for the algebra of symmetric functions is given by the
elementary symmetric functions $\{e_1, \ldots, e_n\}$ with
$$e_r(x) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}.$$ 
Accordingly the set of degrees $d_i$ is given by $\{1, 2, \ldots, n\}$, and (1.20) reduces to
Mehta’s integral (1.5).

The Coxeter groups $B_n$ and $D_n$ are the symmetry groups of the $n$-cube and
$n$-demicube, and for these groups (1.20) takes the form

$$\int_{\mathbb{R}^n} \prod_{i=1}^n (2|x_i|^2)^\gamma \prod_{1 \leq i < j \leq n} |x_i^2 - x_j^2|^{2\gamma} d\varphi(x) = \prod_{i=1}^n \frac{\Gamma(1 + 2i\gamma)}{\Gamma(1 + \gamma)},$$

and

$$\int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq n} |x_i^2 - x_j^2|^{2\gamma} d\varphi(x) = \frac{\Gamma(1 + n\gamma)}{\Gamma(1 + \gamma)} \prod_{i=1}^{n-1} \frac{\Gamma(1 + 2i\gamma)}{\Gamma(1 + \gamma)},$$

respectively. Making the changes $t_i = x_i^2/(2L)$, $\alpha = c + 1/2$ and $\beta = L + 1$ in the
Selberg integral and letting $L$ tend to infinity gives

$$\int_{\mathbb{R}^n} \prod_{i=1}^n (2|x_i|^2)^c \prod_{1 \leq i < j \leq n} |x_i^2 - x_j^2|^{2\gamma} d\varphi(x) = \prod_{j=0}^{n-1} \frac{\Gamma(1 + 2c + 2j\gamma)\Gamma(1 + (j + 1)\gamma)}{\Gamma(1 + c + j\gamma)\Gamma(1 + \gamma)},$$

where on the left use has been made of Legendre’s duplication formula. The above
integral, known as the BC$_n$ Mehta integral, leads to the $B_n$ and $D_n$ integrals by
setting $c = \gamma$ and $c = 0$ respectively.

In his original paper Macdonald established several other instances of his conjecture,
not relying on the Selberg integral. For $\gamma = 1$ Macdonald presented a uniform
proof for all crystallographic reflection groups. Another case of (1.20) — one that
may be verified by purely elementary means — is that of the dihedral group $I_2(m)$,
the symmetry group of a regular $m$-gon.

A uniform proof of Macdonald’s conjecture for all crystallographic reflection
groups — $A_{n-1}$, $B_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$ — was found in 1989 by E. Opdam
[119] using the Heckman–Opdam theory of hypergeometric shift operators [71,119].
Several years later, combined theoretical and computer efforts by Opdam [120] and
In his paper Macdonald put forward many further conjectures related to root systems. One of these [100, Conjecture 2.7] is the generalization of Dyson’s constant term identity (1.13) to arbitrary (finite) root systems. Let Φ be a root system (not necessarily reduced) with corresponding Weyl group \( W \). For \( \alpha \in \Phi \) let \( \exp(\alpha) \) be a formal exponential. Denote the degrees of the fundamental invariants of \( W \) by \( d_1, \ldots, d_l \). The \( d_i \) may, for example, be obtained from the simple formula

\[
\prod_{\alpha \in \Phi^+} \frac{1 - e^{ht(\alpha) + s(\alpha)}}{1 - e^{ht(\alpha)}} = \prod_{i=1}^{l} \frac{1 - t^{d_i}}{1 - t},
\]

where \( \Phi^+ \) is the set of positive roots of the root system, \( ht(\alpha) \) is the height of the root \( \alpha \) and \( s(\alpha) = 1 \) if \( \alpha/2 \not\in \Phi^+ \) and \( s(\alpha) = 2 \) if \( \alpha/2 \in \Phi^+ \). (The latter can only occur for nonreduced root systems.) Then Macdonald’s constant term conjecture asserts that

\[
(1.22) \quad \text{CT} \prod_{\alpha \in \Phi} (1 - e^{\alpha})^k = \prod_{i=1}^{l} \binom{d_i k}{k}.
\]

For the root system \( A_{n-1}, \Phi = \{ \epsilon_i - \epsilon_j | 1 \leq i, j \leq n, \ i \neq j \} \) with \( \epsilon_i \) the \( i \)th standard unit vector in \( \mathbb{R}^n \). The degrees in this case are given by 2, 3, \ldots, \( n \) so that, after the identification \( \exp(\epsilon_i - \epsilon_j) = x_i/x_j \), one recovers Dyson’s conjecture.

When \( k = 1 \) equation (1.22) simply follows from the classical Weyl denominator formula. Macdonald also proved the \( k = 2 \) case using algebraic techniques. Once again the Selberg integral implies the conjecture for all infinite series: \( B_n, C_n, D_n \) and \( BC_n \). Since the first three are all contained in the latter the most succinct derivation arises by slightly generalizing the problem — Macdonald does this for all root systems in [100, Conjecture 2.3] — and considering the constant term of

\[
\text{CT} \prod_{\alpha \in \Phi_{BC_n}} (1 - e^{\alpha})^{k_n}.
\]

Here

\[ \Phi_{BC_n} = \{ \pm \epsilon_i | 1 \leq i \leq n \} \cup \{ \pm 2\epsilon_i | 1 \leq i \leq n \} \cup \{ \pm \epsilon_i \pm \epsilon_j | 1 \leq i < j \leq n \} \]

is the \( BC_n \) root system and \( k_n = k_1 \) if \( \alpha \) is of type \( \pm \epsilon_i \), \( k_n = k_3 \) if \( \alpha \) is of type \( \pm 2\epsilon_i \) and \( k_n = k_2 \) otherwise. The root systems \( B_n, C_n, D_n \) are then obtained by taking \( k_3 = 0 \) or \( k_1 = 0 \) or \( k_1 = k_3 = 0 \) respectively. By the substitution \( \exp(\epsilon_i) \mapsto \exp(2i\theta_i) \) it follows that

\[
(1.23) \quad \text{CT} \prod_{\alpha \in \Phi_{BC_n}} (1 - e^{\alpha})^{k_n} = \frac{2^{n(n+b)+(n-1)c}}{\pi^n} \times \int_{0}^{\pi} \cdots \int_{0}^{\pi} \prod_{i=1}^{n} \sin^a(\theta_i) \cos^b(\theta_i) \prod_{1 \leq i < j \leq n} \sin^c(\theta_i - \theta_j) \sin^c(\theta_i + \theta_j) \, d\theta_1 \cdots d\theta_n,
\]

with \( a = 2k_1 + 2k_3 \), \( b = 2k_3 \) and \( c = 2k_2 \). Introducing new integration variables \( t_i = \sin^2(\theta_i) \) for all \( 1 \leq i \leq n \) the integral on the right transforms into the Selberg
integral, so that by (1.1) and the Legendre duplication formula
\[
\prod_{\alpha \in \Phi_{BC}} (1 - e^{\alpha})^{k_{\alpha}} = 4^{n(k_1 + 2k_3 + n(n-1)k_2)} \pi^n S_n(k_1 + k_3 + \frac{1}{2}, k_3 + \frac{1}{2}, k_2)
\]
\[
= \prod_{i=0}^{n-1} \frac{(k_2 + ik_2)!}{k_2!(k_1 + k_3 + ik_2)!(k_3 + ik_2)!(k_1 + 2k_3 + (n + i - 1)k_2)!}
\]

In a not dissimilar manner D. Zeilberger [163] showed that the \( n = 3 \) case of the Morris integral (1.17) leads to the Macdonald conjecture for the exceptional root system \( G_2 \). This result later found application in a study linking random matrix theory to number theoretical \( L \)-functions [91] (see also the section on the value distribution of \( \log \zeta(1/2 + it) \) below).

A unified proof of (1.22) for all root systems, based on hypergeometric shift operators, is again due to Opdam [119]. Pages 18–20 below contain an outline of this proof for the root system \( A_{n-1} \).

2. Underpinnings of the Selberg integral

The Dixon–Anderson integral. The Euler beta integral (1.3) has for its integrand the product of power functions \( x^{\alpha-1} y^{\beta-1} \) with \( y = 1 - x \). It is evaluated as a ratio of gamma functions, which in turn are integrals over the product of a power function and exponential function. In the theory of finite fields, the role of power and exponential functions are played by multiplicative and additive characters. These can be used to define the finite field analogue of the gamma and beta integrals, known as the Gauss and Jacobi sums respectively. Moreover, these finite field quantities satisfy an analogue of the beta integral. From Selberg’s commentary [136], we know that in the 1940s he investigated finite field analogues of (1.1), and formulated a conjecture which he could prove only for \( n = 2 \). The existence of such finite field analogues was revealed by Selberg in the letter to Askey dated 25 March 1980, referred to on page 7. Selberg also mentioned this in some colloquium lectures. A member of the audience on one of these occasions, Ron Evans, has provided us with the following recollection [51]:

Somewhere around 1980, Selberg came to UCSD for a colloquium talk. Some department members at the UCSD talk were shocked by the subject matter. They were expecting to hear about his recent work, but instead his entire talk was on the Selberg integral. I was fascinated to learn of this integral, and ended up writing several papers on \( q \)-analogues and on finite field analogues. One of these (published in 1981) formulated \( n \)-dimensional finite field analogues, which I was able to prove for \( n = 2 \). Selberg had mentioned in his talk that he had finite field analogues for \( n = 2 \), so I was reluctant to write up my proof. However, some people who knew Selberg told me that he’d never ever get around to publishing his proof, so I took the bold step of asking permission to include my proof for \( n = 2 \) with my general conjecture (with due credit, of course). He generously wrote back that he didn’t mind if I publish a proof of the “right” version of the theorem, but that he didn’t want to
be credited with my version, which was too weak! So I proved the stronger theorem for \( n = 2 \) that he supplied in his letter, and that led to stronger conjectures for general \( n \) (ultimately proved by Anderson [2]).

The finite field paper of Evans referred to above is [46], and Selberg in his commentary [136] references this as being his state of knowledge from the 1940s. In fact, the Anderson paper left open some of the conjectures from [46] and Evans himself was able to apply Anderson’s approach to provide the remaining proofs [47]. For a detailed account of the finite field Selberg integral, we refer to [6].

In 1991, motivated by the quest for a proof of the finite field conjecture, G.W. Anderson [3] published a proof of the Selberg integral based on another multiple integral, namely

\[
\int_X \prod_{1 \leq i < j \leq n} (t_i - t_j) \prod_{i=1}^{n+1} |t_i - a_j|^{s_j-1} \, dt_1 \cdots dt_n
\]

\[
= \prod_{i=1}^{n+1} \Gamma(s_i) \prod_{1 \leq i < j \leq n+1} (a_i - a_j)^{s_i + s_j - 1},
\]

where \( X \) is the domain of integration \( a_1 > t_1 > a_2 > t_2 > \cdots > t_n > a_{n+1} \), and \( \text{Re}(s_i) > 1 \) for \( 1 \leq i \leq n+1 \). Anderson’s idea was to use (2.24) to compute in two different ways the integral

\[
K(\alpha, \beta, \gamma) := \int_{X'} \prod_{i=1}^{n+1} x_i^{\alpha-1}(1-x_i)^{\beta-1} \prod_{i=1}^{n+1} |y_i - x_j|^{\gamma-1}
\]

\[
\times \prod_{1 \leq i < j \leq n} |y_i - y_j| \prod_{1 \leq i < j \leq n+1} |x_i - x_j| \, dx_1 \cdots dx_{n+1} dy_1 \cdots dy_n,
\]

where \( X' \) denotes the domain of integration

\( 1 > x_1 > y_1 > x_2 > y_2 > \cdots > y_n > x_{n+1} > 0 \).

First integrating over the \( y \)-variables gives

\[
K(\alpha, \beta, \gamma) = \frac{\Gamma^{n+1}(\gamma)}{(n+1)! \Gamma((n+1)\gamma)} S_{n+1}(\alpha, \beta, \gamma),
\]

while first integrating over the \( x \)-variables gives

\[
K(\alpha, \beta, \gamma) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{n! \Gamma(\alpha + \beta + n\gamma)} S_n(\alpha + \gamma, \beta + \gamma, \gamma).
\]

Equating the two forms reveals a first order recurrence for the Selberg integral in \( n \). Together with the initial condition \( S_0(\alpha, \beta, \gamma) = 1 \) this reclaims (1.1).

A large portion of Anderson’s paper is devoted to a derivation of (2.24). This same multiple integral, written in the form

\[
(2.25) \quad \det_{1 \leq i, j \leq n} \left( \int_{a_{i+1}}^{a_i} t^{j-1} \prod_{l=1}^{n+1} |t - a_l|^{s_l-1} \, dt \right)
\]

was evaluated at around the same time by A. Varchenko [152, 153] in his work on hyperplane arrangements. That (2.25) is equal to the integral in (2.24) is a simple consequence of the Vandermonde determinant, a fact made explicit in [129].
Remarkably, in a 1998 paper by M.C. Bergère [20] proving a conjecture from the theory of Calogero–Sutherland models (see page 18), reference is made to (2.24), citing neither Anderson nor Varchenko, but a paper of A.L. Dixon [37] published in 1905! Indeed, consulting [37] reveals both (2.24) — obtained via essentially the same analysis as that used in [3] — and its equivalent determinant form (2.25).

A study by Forrester and E.M. Rains [57] provides additional links between the Selberg and the Dixon–Anderson integrals. These apply at the level of the corresponding normalized integrands, referred to as the Selberg and Dixon–Anderson densities. The former will be denoted by $S_n(\alpha, \beta, \gamma; t)$ for $t = (t_1, \ldots, t_n)$.

The first point of note is that the computations of Dixon and Anderson can be interpreted as giving the density of zeros of the random rational function

$$R_{n+1}(x) := \sum_{i=1}^{n+1} \frac{w_i}{a_i - x},$$

where the $w_i$ are distributed according to the Dirichlet distribution — to be denoted $D_{n+1}[s_1, \ldots, s_{n+1}]$ —

$$\Gamma(s_1 + \cdots + s_{n+1}) \prod_{i=1}^{n+1} \frac{w_i^{s_i - 1}}{\Gamma(s_i)}.$$

with $w_1, \ldots, w_{n+1} > 0$ such that $w_1 + \cdots + w_{n+1} = 1$. Motivated by this interpretation, a family of random polynomials $A_j(x)$, $1 \leq j \leq n$ were defined in [57] such that the zeros of $A_j(x)$ have PDF $S_j(\alpha_j, \beta_j, \gamma; t)$ with $\alpha_j := (n-j)\gamma + \alpha$, $\beta_j := (n-j)\gamma + \beta$. Setting $A_{-1}(x) := 0$, $A_0(x) := 1$, and specifying that $(w_0^{(j)}, w_1^{(j)}, w_2^{(j)})$ be distributed according to the Dirichlet distribution $D_3[\beta_j, (j-1)\gamma, \alpha_j]$, the polynomials $A_j(x)$ are determined by the random three-term recurrence

$$A_j(x) = w_2^{(j)}(x-1)A_{j-1}(x) + w_0^{(j)}x A_{j-1}(x) + w_1^{(j)}x(x-1)A_{j-2}(x).$$

Let $\alpha \mapsto \gamma\alpha/2 + 1$, $\beta \mapsto \gamma\beta/2 + 1$, and let the integrand of the Selberg integral be written in the form $\exp(-2\gamma U)$ so that

$$U = -\frac{\alpha}{2} \sum_{i=1}^{n} \log|t_i| - \frac{\beta}{2} \sum_{i=1}^{n} \log|1-t_i| - \sum_{1 \leq i < j \leq n} \log|t_i - t_j|.$$

Then in the limit $\gamma \to \infty$ the Selberg density crystallizes to the minimum of $U$ subject to the constraint that $0 < t_i < 1$ for each $t_i$. According to a classical result of Stieltjes (see e.g., [6]) this minimum is unique and occurs at the zeros of the Jacobi polynomial $P_n(\alpha-1, \beta-1)(x)$. Indeed in the same limit the three term recurrence (2.26) is no longer random, and has solution $A_j(x) = \tilde{P}_j^{(n-\gamma+\alpha-1, n-\gamma+\beta-1)}(x)$ with $\tilde{P}^{(\alpha, \beta)}(x)$ the Jacobi polynomial normalized to be monic [57].

The change of variables and limiting procedure giving rise to (1.8) reduces the Selberg density to the PDF (1.4). The Dixon–Anderson density permits a similar limit, and applied with $n \mapsto n+1$, $a_0 = 0$ and $a_1 = 1$ this results in the PDF on
The importance of the Selberg integral \( \{t_j\}_{j=1}^{n+1} \) given by

\[
\frac{1}{\sqrt{2\pi} \Gamma(s_1) \cdots \Gamma(s_n)} \prod_{1 \leq i < j \leq n+1} (t_i - t_j) \prod_{1 \leq i < j \leq n} (a_i - a_j)^{s_i + s_j - 1} \\
\times \prod_{i=1}^{n+1} \prod_{j=1}^n |t_i - a_j|^{s_j - 1} \exp \left( -\frac{1}{2} \sum_{j=1}^{n+1} t_j^2 + \frac{1}{2} \sum_{j=1}^n a_j^2 \right)
\]

supported on \( t_1 > a_1 > t_2 > \cdots > a_n > t_{n+1} \).

The corresponding limit of \( R_{n+1}(x) \) gives the random rational function

\[
\tilde{R}_{n+1}(x) := x - \mu_0 + \sum_{i=1}^n \frac{\mu_i}{a_i - x},
\]

where \( \mu_0 \) has distribution \( N[0,1] \) and \( \mu_i \) the Gamma distribution \( \Gamma[s_i, 1] \). Indeed, the fact that the zeros of (2.28) have PDF (2.27) can readily be checked directly by adopting the strategy of Dixon and Anderson, see [49,55]. Finally, the limiting form of the three term recurrence (2.26) is

\[
C_j(x) = (x - r) C_{j-1}(x) - s^{(j-1)} C_{j-2}(x)
\]

with \( C_{-1}(x) = 0, C_0(x) = 1, r \) having distribution \( N[0,1] \) and \( s^{(j)} \) distribution \( \Gamma[j\gamma, 1] \). The random polynomial \( C_j(x) \) has as the PDF for its zeros the density (1.4) with \( n = j \) and \( \beta = 2\gamma \).

It should be remarked that since (2.27) integrates over \( t_1, \ldots, t_{n+1} \) gives unity, a limiting form of the Dixon–Anderson integral follows. Evans [49] used this, together with the strategy of Anderson, to give the first proof of the Mehta integral evaluation (1.6) which is independent of the Selberg integral. The random polynomial \( C_{n+1}(x) \) can be interpreted as the characteristic polynomial for a family of random matrices defined inductively by [57]

\[
M_{n+1} = \begin{bmatrix}
\lambda_1^{(n)} & b_1 \\
\vdots & \vdots \\
\lambda_n^{(n)} & b_n \\
b_1 & \ldots & b_n & c
\end{bmatrix}
\]

Here the \( \lambda_i^{(n)} \) are the eigenvalues of \( M_n \), \( c \) has distribution \( N[0,1] \) and \( b_j^2 \) has distribution \( \Gamma[\beta/2, 1] \). Indeed it is straightforward to show that the eigenvalues of (2.30) are given by the zeros of (2.28), with \( \mu_0 = c, \mu_i = b_i \) and \( a_i = \lambda_i^{(n)} \). In the case \( \beta = 1 \), the invariance of members of the GOE with respect to conjugation by orthogonal matrices shows that (2.30) is similar to GOE matrices, and an analogous understanding of the relationship between (2.30) in the case \( \beta = 2 \) and GUE matrices can be given. Moreover, it is generally true that a three-term recurrence

\[
p_j(x) = (x - a_j)p_{j-1}(x) - b_{j-1}p_{j-2}(x)
\]
with \( p_{-1}(x) := 0, \ p_0(x) = 1 \) is satisfied by the characteristic polynomial for the tridiagonal matrix

\[
\begin{bmatrix}
    a_n & b_{n-1} \\
    b_{n-1} & a_{n-1} & b_{n-2} \\
    \ddots & \ddots & \ddots \\
    b_2 & a_2 & b_1 \\
    b_1 & a_1 &
\end{bmatrix}
\]

Hence \( C_n(x) \) is also the characteristic polynomial for the above random tridiagonal matrix with each \( a_j \) having distribution \( N[0,1] \) and with \( b_j^2 \) distributed as in (2.30). This is a result due to I. Dumitriu and A. Edelman [39], obtained without the use of (2.27). In this regard it should be mentioned that R. Killip and I. Nenciu [92], in a study which does not make use of the Dixon–Anderson integral (2.24), give the explicit construction of a family of random orthogonal matrices with eigenvalue PDF equal to the BC \( n \) Selberg density, which itself is proportional to the integrand in (1.23). The methods of [39] and [92], which at a technical level proceed via a change of variables from a general tridiagonal matrix and unitary Hessenberg matrix to their eigenvalue/eigenvector decomposition, yield too the evaluations of the Mehta and Selberg integrals respectively.

\section{Dotsenko–Fateev integrals.} In the course of studies in conformal field theory, V.S. Dotsenko and V.A. Fateev [38] were lead to consider the multiple integral

\[
P V \int_{[0,1]^p} \int_{[1,\infty)^{n-p}} \int_{[0,1]^r} \int_{[1,\infty)^{m-r}} \prod_{i=1}^n t_i^\alpha (1-t_i)^\beta \prod_{i=1}^m \tau_i^\alpha' (\tau_i-1)^\beta' \prod_{i=1}^n \prod_{j=1}^m (\tau_j-t_i)^{-2} \\
\times \prod_{1 \leq i < j \leq n} |t_i-t_j|^{2\gamma} \prod_{1 \leq i < j \leq m} |\tau_i-\tau_j|^{2\gamma'} dt_1 \cdots dt_n \, d\tau_1 \cdots d\tau_m,
\]

where PV denotes the principal value, \( \alpha/\alpha' = \beta/\beta' = -\gamma, \gamma' = 1 \) and \( 0 \leq p \leq n \), \( 0 \leq r \leq m \). Note that the case \( p = n \) and \( m = 0 \) is, up to a shift by 1 in \( \alpha \) and \( \beta \), precisely the Selberg integral. Dotsenko and Fateev evaluated the above integral as a product of gamma and sine functions reclaiming the Selberg integral as a special case.

The method employed by Dotsenko and Fateev for evaluating their integral suggests an approach [54] to the Selberg integral by studying the simpler \( m = 0 \) case

\[
S_{n,p}(\alpha, \beta, \gamma) := \int_{[0,1]^p} \int_{[1,\infty)^{n-p}} \prod_{i=1}^n \prod_{1 \leq i < j \leq n} |1-t_i|^{\alpha-1} |1-t_j|^{\beta-1} \prod_{1 \leq i < j \leq n} |t_i-t_j|^{2\gamma} dt_1 \cdots dt_n
\]

for \( 0 \leq p \leq n \). Note that \( S_{n,n}(\alpha, \beta, \gamma) = S_n(\alpha, \beta, \gamma) \), which is the Selberg integral. Also note that the change of variables \( t_i \mapsto 1/t_i \) for all \( 1 \leq i \leq n \) implies the transformation

\[
S_{n,p}(\alpha, \beta, \gamma) = S_{n,n-p}(1-\alpha - \beta - 2(n-1)\gamma, \beta, \gamma).
\]

Singling out the integration variable \( t_p \), viewing the integrand as an analytic function and replacing the interval \([0,1]\) by a contour along a positively oriented, indented semi-circle of infinite radius (with indentations at the branch points \( t_p = \)
as a product of two Selberg integrals that up to a product of trigonometric functions the complex Selberg integral factors

\[ \vec{u} \]

where independently by K. Aomoto [8], can be written as an

\[ n \]

referred to as the complex Selberg integral. This integral, which was also studied with integration variables given by 2-dimensional vectors

\[ \text{a fact already noted and used for the same purpose in Selberg’s original proof [134].} \]

In their paper Dotsenko and Fateev considered a further generalization of (1.1), referred to as the complex Selberg integral. This integral, which was also studied independently by K. Aomoto [8], can be written as an \( n \)-dimensional real integral with integration variables given by 2-dimensional vectors

\[ A_n(\alpha, \beta, \gamma) := \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} \prod_{i=1}^{n} |\vec{r}_i|^{2(\alpha-1)} |\vec{u} - \vec{r}_i|^{2(\beta-1)} \prod_{1 \leq i < j \leq n} |\vec{r}_i - \vec{r}_j|^{4\gamma} \, d\vec{r}_1 \cdots d\vec{r}_n, \]

where \( \vec{u} \) is an arbitrary unity vector. Dotsenko and Fateev as well as Aomoto showed that up to a product of trigonometric functions the complex Selberg integral factors as a product of two Selberg integrals

\[ A_n(\alpha, \beta, \gamma) = S_n^2(\alpha, \beta, \gamma) \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\sin \pi(\alpha + j\gamma) \sin \pi(\beta + j\gamma) \sin \pi(j + 1) \gamma}{\sin \pi(\alpha + \beta + (n + j - 1) \gamma) \sin \pi \gamma}, \]

provided (1.2) is supplemented by

\[ \text{Re}(\alpha + \beta + (n - 1) \gamma) < 1 \quad \text{and} \quad \text{Re}(\alpha + \beta + 2(n - 1) \gamma) < 1. \]

K. Mimachi and M. Yoshida [111] (see also [110]) apply the theory of intersection numbers of twisted cycles to the conformal field theory study of Dotsenko and Fateev to give the evaluation of the product \( S_n(\alpha, \beta, \gamma)S_n(-\alpha, -\beta, -\gamma) \), appropriately analytically continued. This is achieved without requiring the actual evaluation of the Selberg integral itself.
Jack polynomial theory. It has been known since the early 1970s [25, 148, 149] that (1.9) with \( \beta = 2 \gamma \) — to be denoted \( \exp(-2\gamma W) \) in analogy with (1.7) — is the absolute value squared of the ground-state wave function for the Schrödinger operator

\[
H = -\sum_{i=1}^{n} \frac{\partial^2}{\partial \theta_i^2} + \frac{1}{2} \frac{\gamma (\gamma - 1)}{\sin^2 \left( \frac{1}{2}(\theta_i - \theta_j) \right)}.
\]

This operator, known as the Calogero–Sutherland Hamiltonian, describes a system of \( n \) identical quantum particles on the unit circle, with \( \theta_i \in [0, 2\pi) \) for \( 1 \leq i \leq n \) the (angular) positions of the particles. The interaction between the particles is described by a \( 1/r^2 \) two-body potential, \( 2|\sin((\theta_i - \theta_j)/2)| \) being the cord-length between particles located at \( \theta_i \) and \( \theta_j \).

B. Sutherland [148] showed that the eigenvalue \( E_0 \) corresponding to the ground-state wave function is given by \( E_0 = n(n^2 - 1)\gamma^2/12 \). Subsequently he showed [149] that the conjugated operator

\[
e^{iW}(H - E_0) e^{-iW} = \sum_{i=1}^{n} \left( x_i \frac{\partial}{\partial x_i} \right)^2 + 2\gamma \sum_{i,j=1}^{n} \frac{x_i + x_j}{x_i - x_j} \frac{\partial}{\partial x_i},
\]

where \( x_j := \exp(i\theta_j) \), admits a complete set of symmetric polynomial eigenfunctions \( P_{\lambda}^{(1/\gamma)}(x) \). These polynomials, now referred to as Jack polynomials, depend on \( x = (x_1, \ldots, x_n) \) and are indexed by partitions \( \lambda \) of at most \( n \) parts; \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \). With \( m_\lambda \) denoting the monomial symmetric function indexed by \( \lambda \) and \( \prec \) the dominance ordering on partitions, the Jack polynomials have the structure

\[
P_{\lambda}^{(1/\gamma)}(x) = m_\lambda(x) + \sum_{\mu \prec \lambda} a_{\lambda\mu} m_\mu(x)
\]

for some coefficients \( a_{\lambda\mu} = a_{\lambda\mu}(\gamma) \).

One fundamental property of the Jack polynomials is that they are orthogonal with respect to the inner product

\[
\langle f, g \rangle_\gamma := \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(e^{i\theta_1}) g(e^{-i\theta_n}) \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^{2\gamma} \, d\theta_1 \cdots d\theta_n,
\]

where \( f(e^{i\theta}) = f(e^{i\theta_1}, \ldots, e^{i\theta_n}) \). To state the orthogonality as well as the quadratic norm evaluation let

\[
[\delta]_\lambda = \prod_{i \geq 1} (b + (1 - i)\gamma)_{\lambda_i}
\]

with \( (b)_n = b(b + 1) \cdots (b + n - 1) \) a Pochhammer symbol. Also let \( c_\lambda(\gamma) \) and \( c'_\lambda(\gamma) \) be given by

\[
c_\lambda(\gamma) = \prod_{s \in \lambda} (a(s) + l(s)\gamma + \gamma),
\]

\[
c'_\lambda(\gamma) = \prod_{s \in \lambda} (a(s) + l(s)\gamma + 1),
\]

where \( a(s) \) and \( l(s) \) are the arm-length and leg-length of the square \( s \) in the diagram of the partition \( \lambda \), and \( |\lambda| \) is the total number of boxes in the diagram of \( \lambda \) [102].
Then
\[
\langle P^{(1/\gamma)}_\lambda, P^{(1/\gamma)}_\mu \rangle_\gamma = \delta_{\lambda\mu} \frac{c'_\lambda(\gamma)}{[1 + (n - 1)\gamma]^{1/\lambda}} \frac{\Gamma(1 + n\gamma)}{\Gamma^n(1 + \gamma)} P^{(1/\gamma)}_\lambda(1^n),
\]
where \(\delta_{\lambda\mu}\) is the Kronecker delta function and \((1^n)\) is shorthand for \((1, 1, \ldots, 1)\). The orthogonality relation is consistent with, but not an immediate consequence of, the operator \((2.34)\) being self-adjoint with respect to \((2.36)\). The complication is that not all eigenvalues of \((2.34)\) are distinct. This degeneracy can be removed by introducing the mutually commuting Cherednik operators \(\xi_i\) for \(1 \leq i \leq n\) \([27, 40]\)
\[
\xi_i = 1 - i + \frac{x_i}{\gamma} \frac{\partial}{\partial x_i} + \sum_{j=1}^{i-1} \frac{x_i}{x_i - x_j} (1 - s_{ij}) + \sum_{j=i+1}^{n} \frac{x_j}{x_i - x_j} (1 - s_{ij}),
\]
where \(s_{ij}\) acts by permutation \(x_i\) and \(x_j\) and 1 represents the identity operator. Any symmetric combination of the \(\xi_i\), and in particular \(\prod_{i=1}^{n} (1 - u_i \xi_i)\), has the Jack polynomials as simultaneous eigenfunctions.

The Cherednik operators can be used to construct the Jack polynomial shift operator — a special case of the shift operators studied by Heckman and Opdam, and used by the latter to prove the Macdonald integral and constant term conjectures. Properties of the Jack shift operator not only imply \((1.12)\) or, equivalently, \((1.13)\), but also the more general quadratic norm evaluation of the Jack polynomials corresponding to \((2.38)\) with \(\lambda = \mu\) \([84]\). (For \(\lambda = 0\) this yields \((1.12)\).) With \(\Delta(x)\) the Vandermonde product \((1.22)\) and \(Y_\pm := \gamma^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (\xi_i - \xi_j \pm 1)\), the Jack shift operators are defined by \(G_+ := \Delta^{-1} Y_+, G_- := Y_\Delta\). They have an adjoint type property with respect to the inner product \((2.36)\).
\[
\langle G_+ f, g \rangle_{\gamma + 1} = \langle f, G_- g \rangle_\gamma.
\]
Also, with
\[
a^{\pm}_{\lambda}(\gamma) = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j \pm 1 + (j - i + 1)\gamma)
\]
and \(\delta\) the staircase partition \((n, n-1, n-2, \ldots, 1, 0)\), the shift operators act on the Jack polynomials as
\[
G_+ P^{(1/\gamma)}_{\lambda+\delta} = a^+_\lambda(\gamma + 1) P^{(1/(\gamma+1))}_{\lambda},
\]
\[
G_- P^{(1/(\gamma+1))}_{\lambda} = a^-_\lambda(\gamma + 1) P^{(1/\gamma)}_{\lambda+\delta}.
\]
It follows from \((2.39)\) and \((2.41)\) that
\[
\langle P^{(1/(\gamma+1))}_{\lambda}, P^{(1/(\gamma+1))}_{\lambda} \rangle_{\gamma + 1} = \frac{a^+_\lambda(\gamma + 1)}{a^-_\lambda(\gamma + 1)} \langle P^{(1/\gamma)}_{\lambda+\delta}, P^{(1/\gamma)}_{\lambda+\delta} \rangle_{\gamma}
\]
and thus
\[
\langle P^{(1/(\gamma+k))}_{\lambda}, P^{(1/(\gamma+k))}_{\lambda} \rangle_{\gamma+k} = \langle P^{(1/\gamma)}_{\lambda+k\delta}, P^{(1/\gamma)}_{\lambda+k\delta} \rangle_{\gamma} \prod_{j=1}^{k-1} \frac{a^-_{\lambda+j\delta}(\gamma + k - j)}{a^+_{\lambda+j\delta}(\gamma + k - j)}
\]
Taking \(\gamma = 0\), using that \(P^{(\infty)}_{\lambda} = m_\lambda\) (the monomial symmetric function) and
\[
\langle m_\mu, m_\mu \rangle_0 = \text{CT} \left( m_\mu(x) m_\mu(x^{-1}) \right) = m_\mu(1^n)
\]
which is \( n! \) for \( \mu = \lambda + k\delta \), it follows that for nonnegative integer \( k \)

\[
\langle P^{(1/k)}_\lambda, P^{(1/k)}_\lambda \rangle_k = n! \prod_{j=0}^{k-1} \frac{a^{-j}}{a^j(k-j)}.
\]

Using the evaluation formula \cite{141}

\[
P^{(1/\gamma)}_\lambda(1^n) = \frac{[n\gamma]_\lambda}{\gamma(\gamma)}
\]

and the definitions \( \gamma \)

\[
(2.37) \quad (2.40)
\]

it is now a straightforward exercise to verify that for \( \gamma = k \)

\[
(2.43)
\]

using the evaluation formula \cite{141} with \( n \rightarrow m \) and a standard analytic argument to replace \( m \) by \( a\alpha \), leads to Z. Yan’s binomial theorem for Jack polynomials \cite{161}

\[
\sum_{\lambda} \frac{|a|^{(\alpha)}}{c^{(\alpha)}(\alpha)} P^{(\alpha)}_\lambda(x) = \prod_{i=1}^{n} \frac{1}{(1-x_i)^a},
\]

This, together with the orthogonality \( \gamma \), the property

\[
P^{(\alpha)}_{(\lambda_1+a,\ldots,\lambda_n+a)}(x) = (x_1 \cdots x_n)^a P^{(\alpha)}_\lambda(x), \quad a = 0, 1, 2, \ldots
\]

and the gamma reflection formula, implies a generalization of the Morris integral \cite{117} \cite{17}

\[
\frac{1}{(2\pi)^a} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} P^{(1/\gamma)}_\lambda(-e^{i\theta_1} \cdots e^{i\theta_n}) \prod_{i=1}^{n} e^{\frac{1}{2}i\theta_i(a-b)} |1 + e^{i\theta_i}|^{a+b} \times \prod_{1 \leq i < j \leq n} |e^{i\theta_i} - e^{i\theta_j}|^{2\gamma} d\theta_1 \cdots d\theta_n
\]

\[
= \frac{[-b]_\lambda^{(\gamma)}}{[1 + a + (n-1)\gamma]_\lambda^{(\gamma)}} P^{(1/\gamma)}_\lambda(1^n) M_n(a, b, \gamma).
\]

Applying \( \gamma \) finally results in a generalization of the Selberg integral

\[
\int_{0}^{1} \cdots \int_{0}^{1} P^{(1/\gamma)}_\lambda(t) \prod_{i=1}^{n} t_i^{a_i} (1-t_i)^{b_i-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} dt_1 \cdots dt_n
\]

\[
= \frac{|\alpha + (n-1)\gamma|_\lambda^{(\gamma)}}{|\alpha + \beta + 2(n-1)\gamma|_\lambda^{(\gamma)}} P^{(1/\gamma)}_\lambda(1^n) S_n(\alpha, \beta, \gamma).
\]

This evaluation is usually referred to as Kadell’s integral \cite{83} after its first prover, but as a conjecture is due to Macdonald \cite[Conjecture (C5)]{101}. When \( \lambda = (1^r) \), in which case the Jack polynomial is nothing but the \( r \)th elementary symmetric function, the above is known as Aomoto’s integral, who used it to give what is
arguably the first elementary proof of the Selberg integral [7]. A proof of Kadell’s integral along the lines of Anderson’s proof of the Selberg integral as described on page 13 has recently been obtained in [156] through use of the Okounkov–Olshanski integral formula for Jack polynomials [116].

\[
P^{(1/\gamma)}(x) = \prod_{i=1}^{n-1} \frac{\Gamma(\lambda_i + (n-i+1)\gamma)}{\Gamma(\lambda_i + n)\Gamma(\gamma)} \prod_{1 \leq i < j \leq n} (x_j - x_i)^{1 - 2\gamma}
\]

\[
\times \int_Y P^{(1/\gamma)}(y) \prod_{1 \leq i < j \leq n-1} (y_j - y_i) \prod_{i=1}^{n-1} \prod_{j=1}^{n} |y_i - x_j|^{-\gamma} dy_1 \cdots dy_{n-1}
\]

where \( Y \) denotes the domain \( x_1 < y_1 < x_2 < \cdots < x_{n-1} < y_{n-1} < x_n \) and \( \lambda \) is a partition of at most \( n - 1 \) parts.

An open problem, settled only in the 2-variable case [50], is to find (and prove) a finite field analogue of Kadell’s integral.

The binomial theorem for Jack polynomials (2.45) can succinctly be written in hypergeometric notation as

\[
\binom{a}{x} = \prod_{i=1}^{n} \frac{1}{(1 - x_i)^a}
\]

where \( \binom{a}{x} \) is an example of a hypergeometric function with Jack polynomial argument

\[
\binom{a_1, \ldots, a_r}{b_1, \ldots, b_s} := \sum_{\lambda} [a_1]^{(\gamma)} \cdots [a_r]^{(\gamma)} \frac{P^{(1/\gamma)}(x)}{c^{(\gamma)}}.
\]

When \( n = 1 \), so that \( x \) is a scalar, this function reduces to the classical hypergeometric function, \( \binom{a_1, \ldots, a_r}{b_1, \ldots, b_s} \). For general \( n \), hypergeometric functions of the type (2.47) have their genesis in the work of A.G. Constantine [30], C.S. Herz [72] and R.J. Muirhead [114], but were first studied in their full form presented here by Kaneko [85], A. Korányi [94] and Yan [161]. The case \( r = 2, s = 1 \) of (2.47) shares many properties in common with its \( n = 1 \) counterpart, the Gauss hypergeometric function.

One such property is that \( _2F_1^{(\gamma)}(a, b; c; x) \) solves the \( n \)-dimensional analogue of Euler’s hypergeometric equation. Specifically, Yan [161] and Kaneko [85] independently showed that \( _2F_1^{(\gamma)}(a, b; c; x) \) is the unique symmetric function, analytic at the origin, that solves the system of \( n \) partial differential equations

\[
x_i(1 - x_i) \frac{\partial^2 F}{\partial x_i^2} + \left(c - (n-1)\gamma - (a + b + 1 - (n-1)\gamma)x_i\right) \frac{\partial F}{\partial x_i} - abF
\]

\[
+ \gamma \sum_{j=1}^{n} \frac{1}{x_i - x_j} \left(x_i(1 - x_i) \frac{\partial F}{\partial x_i} - x_j(1 - x_j) \frac{\partial F}{\partial x_j}\right) = 0
\]

for \( 1 \leq i \leq n \).

In the one-variable theory the Gauss hypergeometric function admits an integral representation due to Euler [45]

\[
_2F_1\left(a, b; c; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b} (1 - zt)^{-a} dt,
\]
for $\text{Re}(c) > \text{Re}(b) > 0$ with a branch cut in the complex $z$-plane from 1 to infinity. When $z = 1$ the integral on the right is the beta integral \([1.3]\), resulting in a closed form evaluation of the $2\mathbf{F}_1$ due to Gauss. In the multivariable theory an analogous results holds, where now the key integral-evaluation is provided by the Selberg integral. Multiplying both sides of Kadell’s integral by $z^{|\lambda|}/c_\lambda(1)$, summing the left-hand side using the binomial theorem \((2.45)\), and using the definition \((2.47)\) on the right-hand side, shows that Euler’s integral extends to \([85]\)

\[
2\mathbf{F}_1\left(\begin{array}{c} a, b \\ c \end{array}; z^n \right) = \prod_{j=0}^{n-1} \frac{\Gamma(c-j\gamma)\Gamma(1+\gamma)}{\Gamma(b-j\gamma)\Gamma(c-b-j\gamma)\Gamma(1+(j+1)\gamma)} \\
\times \int_0^1 \cdots \int_0^1 \left(1-t_1\right)^{\beta-1} \left(1-zt_1\right)^{-a} \prod_{1 \leq i < j \leq n} |t_i-t_j|^2 \, dt_1 \cdots dt_n,
\]

with $\alpha = b-(n-1)\gamma$ and $\beta = c-b-(n-1)\gamma$. Evaluating the $z = 1$ instance of the integral by the Selberg integral (which, incidentally, follows by taking $z = 0$ in \((2.49)\)) implies a generalized Gauss summation \([161]\)

\[
2\mathbf{F}_1\left(\begin{array}{c} a, b \\ c \end{array}; 1^n \right) = \prod_{j=0}^{n-1} \frac{\Gamma(c-j\gamma)\Gamma(c-a-b-j\gamma)}{\Gamma(c-a-j\gamma)\Gamma(c-b-j\gamma)}.
\]

In addition to relating to the Selberg integral, the Cauchy identity also gives rise to a special limiting case of the Selberg density, referred to as the Laguerre PDF \([57,77]\). To motivate the origin of this we remark that the Jack polynomial at $\gamma = 1$ is equal to the Schur polynomial $s_\lambda$ while $c_\lambda(1)/c_\lambda'(1) = 1$. The normalised summand of \((2.44)\) then reads

\[
s_\lambda(x)s_\lambda(y) \prod_{i=1}^{n} \prod_{j=1}^{m} (1-x_i y_j)
\]

which may be recognised as the measure on partitions induced by the Robinson–Schensted–Knuth correspondence, see e.g., \([78,118]\). As such the Schur measure holds a special place in certain studies relating to the representation theory of the symmetric group \([24,127]\).

For general $\gamma$ the normalised summand of \((2.44)\) implies the more general measure on partitions

\[
\frac{c_\lambda(\gamma)}{c_\lambda'(\gamma)} \prod_{\lambda}^{(1/\gamma)}(x) F^{(1/\gamma)}(y) \prod_{i=1}^{n} \prod_{j=1}^{m} (1-x_i y_j)^\gamma,
\]

where $n \leq m$ and $\lambda$ a partition of at most $n$ parts. To obtain the Laguerre PDF \([57,77]\) one needs to specialize $x$ and $y$ in \((2.50)\) to

\[
x_i = q^{1/2}, \quad 1 \leq i \leq n \quad \text{and} \quad y_j = q^{1/2}, \quad 1 \leq j \leq m.
\]

Use of the Jack polynomial evaluation formula \((2.43)\) allows all terms in \((2.50)\) to be made explicit. The remaining step is to take the scaling limit, turning the discrete measure on partitions into a continuous one on the positive real line. This is done by setting $q = \exp(-1/L)$, introducing the scaled variables $t_j := \lambda_j/L$ and then

\[

\text{Re}(c) > \text{Re}(b) > 0 \quad \text{with a branch cut in the complex } z \text{-plane from 1 to infinity.}
\]
taking the large $L$ limit for fixed $t_j$. One finds that (2.50) multiplied by $L^n$ tends to the PDF, supported on $t_1 > t_2 > \cdots > t_n > 0$,

$$\frac{1}{W_n((m-n+1)\gamma, \gamma)} \prod_{i=1}^n e^{-t_i} t_i^{(m-n+1)\gamma-1} \prod_{1 \leq i < j \leq n} (t_i - t_j)^{2\gamma},$$

where $W_n$ is the normalization

$$W_n(\alpha, \gamma) = \frac{1}{n!} \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j\gamma) \Gamma((j+1)\gamma)}{\Gamma(\gamma)}.$$

To obtain this same result as a limit of the Selberg density, order the integration variables in the latter and write $\beta = L$. Then the change of variables $t_i \rightarrow t_i/L$ followed by the limit $L \rightarrow \infty$ results in (2.51) after identifying $\alpha$ with $(m-n+1)\gamma$.

This limiting “Laguerre” case of the Selberg integral, leading to the evaluation (2.52), is contained as equation (1) in the letter from Selberg to Dyson reprinted on page 8.

Askey and D. Richards [13] (see also [107]) have shown that after some fairly straightforward manipulations and a change of variables the Laguerre limit of the Selberg integral leads to the evaluation

$$\int_D \prod_{i=1}^n t_i^{\alpha-1} \left(1 - \sum_{i=1}^n t_i\right)^{\beta-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} dt_1 \cdots dt_n$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta + n(n-1)\gamma)} \prod_{j=0}^{n-1} \frac{\Gamma(\alpha + j\gamma) \Gamma(1 + (j+1)\gamma)}{\Gamma(1 + \gamma)},$$

where $D$ is the domain $t_i \geq 0$ for $1 \leq i \leq n$ such that $t_1 + \cdots + t_n \leq 1$, and $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\gamma) > -\min\{1/n, Re(\alpha)/(n-1)\}$.

According to Askey and Richards, the first statement of (2.53) is due to Selberg at a meeting held in Sri Lanka during December 1987. The intriguing point is that while Selberg did not give his derivation of (2.53), he is reported to have said that it was different to (1.1), and had the advantage of working in the finite field case. The derivation given in [13] does not work in the finite field case, and therefore must be different to that known to Selberg.

$q$-Integrals and constant terms. Motivated by the Selberg integral and its success in dealing with Dyson and Macdonald type constant term identities, Askey in 1980, was led to consider several $q$-analogues of the Selberg integral and to study connections to $q$-constant term identities. In fact, one learns from [12] that he had earlier spent time searching for a proof of the Mehta integral upon its appearance in the problem section of SIAM Review. To describe some of Askey’s work we require the multiple Jackson or $q$-integral

$$\int_0^1 \cdots \int_0^1 f(t) dt_1 \cdots dt_n := (1-q)^n \sum_{k_1, \ldots, k_n=0}^{\infty} f(q^{k_1} \cdots q^{k_n})$$

with $t = (t_1, \ldots, t_n)$, $q^k = (q^{k_1}, \ldots, q^{k_n})$ and $0 < q < 1$, and where it is assumed that the multiple sum on the right is absolutely convergent. Also needed is the
$q$-shifted factorial

$$(a; q)_z = \prod_{j=0}^{\infty} \frac{1 - a q^j}{1 - a q^{z+j}}$$

for $z \in \mathbb{C}$. Probably the most important of the $q$-Selberg integrals considered by Askey is

$$(2.54) \quad S_n(\alpha, \beta, \gamma; q) := \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} q^{\alpha - 1}(qt_i; q)_{\beta-1} \prod_{1 \leq i < j \leq n} t_i^2(q^{-1}t_j/t_i; q)_2 \, dt_1 \cdots dt_n.$$

It is immediate, at least formally, that

$$\lim_{q \to 1} S_n(\alpha, \beta, \gamma; q) = S_n(\alpha, \beta, \gamma).$$

For $\Re(\alpha) > 0$, $\gamma$ a nonnegative integer, say $k$, and $\beta \in \mathbb{C}$ excluding the nonpositive integers Askey conjectured (and proved for $k = 2$) that [11, Conjecture 1]

$$S_n(\alpha, \beta, k; q) = q^{\alpha k(\gamma/2) + 2k^2/3} \prod_{j=0}^{n-1} \frac{\Gamma_q(\alpha + (j - 1)k) \Gamma_q(\beta + (j - 1)k) \Gamma_q(1 + jk)}{\Gamma_q(\alpha + \beta + (n + j - 2)k) \Gamma_q(1 + k)},$$

where $\Gamma_q(x)$ is the $q$-gamma function

$$\Gamma_q(x) = \frac{(q; q)_{x-1}}{(1 - q)^{x-1}}.$$

For Askey’s other $q$-Selberg integrals and many further results relating to Jackson-integral type extensions of beta integrals, see [5, 9–11, 48, 76, 79, 86, 155].

In 1988 Askey’s conjecture was proved independently by L. Habsieger [70] and K. Kadell [80]. Both then applied the $q$-analogue of the identity (1.15) to (2.54) to obtain a $q$-generalization of the Morris integral (1.17). Expressing this integral as a constant term identity they thus proved Morris’ $q$-constant term conjecture [113]

$$\text{CT} \prod_{j=1}^{n} (x_j; q)_{\alpha}(x_j/x; q)_k \prod_{1 \leq i < j \leq n} (x_i/x_j; q)_k(qx_j/x_i; q)_k = \prod_{j=0}^{n-1} \frac{\Gamma_q(1 + a + b + jk) \Gamma_q(1 + (j + 1)k)}{\Gamma_q(1 + a + b + jk) \Gamma_q(1 + b + jk) \Gamma_q(1 + k)}.$$

When $a = b = 0$ this is precisely the $A_{n-1}$ case of the $q$-Macdonald constant term conjecture [100, Conjecture 3.1]

$$(2.55a) \quad \text{CT} \prod_{\alpha \in \Phi^+} \prod_{i=1}^{k} (1 - q^{-1}e^{-\alpha})(1 - q^{i}e^{\alpha}) = \prod_{i=1}^{l} \left[ \frac{d_i k}{\sqrt{q}} \right],$$

where

$$\left[ \frac{n}{k} \right]_q = \prod_{i=0}^{k} \frac{1 - q^{n-k+i}}{1 - q^i}$$

is a $q$-binomial coefficient, and $\Phi$ is a reduced (finite) root system. To also include the root systems of type BC one again needs the numbers $s(\alpha)$ as defined above.
equation (2.55b) [100, Conjecture 3.4]

(2.55b) \[ \text{CT} \prod_{a \in \Phi^+} \prod_{i=1}^{k} (1 - q^{i s(a) - 1} e^{-a})(1 - q^{i-1} s(a) + 1 e^a) = \prod_{i=1}^{l} \left[ \frac{d_k}{k! q^j} \right]. \]

The $A_{n-1}$ case of (2.55), was in fact proved prior to the work of Habsieger and Kadell by Zeilberger and D.M. Bressoud [162], who proved the more general $q$-Dyson conjecture formulated by G.E. Andrews [4].

R.A. Gustafson [67], at around the same time as Anderson’s work on the Dixon–Andersson integral, made use of a further $q$-generalization of (2.24) and invented the same general strategy as used in [3] to derive the BC$_n$-type constant term identity

(2.56) \[ \text{CT} \Delta(x; t, t_1, \ldots, t_4) = 2^n n! \prod_{j=1}^{n} \frac{(t; q)_{n+j-2} t_1 t_2 t_3 t_4; q)_{\infty}}{((t_j; q)_{\infty}) (q; q)_{\infty} \prod_{1 \leq r < s \leq 4} (t^{-1}_r t_s; q)_{\infty}}, \]

where

(2.57) \[ \Delta(x; t, t_1, \ldots, t_4) := \prod_{i=1}^{n} \frac{(x_i^2; q)_{\infty} (x_i^{-2}; q)_{\infty}}{t_{i} x_i; q)_{\infty} (t_{i} x_i^{-1}; q)_{\infty}} \times \prod_{1 \leq i < j \leq n} \frac{(x_i x_j^{-1}; q)_{\infty} (x_i^{-1} x_j; q)_{\infty} (x_i x_j; q)_{\infty} (x_i^{-1} x_j^{-1}; q)_{\infty}}{(t x_i x_j^{-1}; q)_{\infty} (t x_i^{-1} x_j; q)_{\infty} (t x_i x_j; q)_{\infty} (t x_i^{-1} x_j^{-1}; q)_{\infty}}. \]

This is a generalization of the so-called Macdonald–Morris constant term identity and implies the B$_n$, C$_n$, D$_n$ and BC$_n$ cases of (2.55) through specialisation [67].

Most other cases of (2.55) were proved on a case by case basis, often using methods based on $q$-integrals of Selberg type [48,61,62,69,82,145,163–165], but the three exceptional root systems E$_6$, E$_7$ and E$_8$ refused to surrender until Cherednik gave a uniform proof for all reduced root systems based on his theory of double affine Hecke algebras [28].

Returning to Askey’s $q$-Selberg integral we remark that the density function corresponding to the integrand of (2.54) can be deduced from Macdonald polynomial theory [102], following a procedure similar to that of deducing (2.51) from (2.50) [57]. Macdonald polynomials $P_{\lambda}(x; q, t)$ are generalizations of Jack polynomials — the latter being reclaimed according to $\lim_{q \to 1} P_{\lambda}(x; q, q^{\gamma}) = P_{\lambda}^{(1/\gamma)}(x)$ — exhibiting the structure (2.35) and the orthogonality

\[ \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} P_{\lambda}(e^{i \theta_1}, q, t) P_{\mu}(e^{-i \theta_1}, q, t) \prod_{1 \leq i < j \leq n} (e^{i(\theta_i - \theta_j)}, q)_{\infty} d\theta_1 \cdots d\theta_n \propto \delta_{\lambda\mu}. \]

It is the connection between affine Hecke algebras and Macdonald type orthogonal polynomials that is at the heart of Cherednik’s proof of the $q$-constant term conjectures for arbitrary root systems [28], see also [103].
Multivariable orthogonal polynomials. By the change of variables \( t = (1 - x)/2 \) and a shift in \( \alpha \) and \( \beta \) by 1 the Euler beta integral \([1.3]\) takes the form

\[
J(\alpha, \beta) := \int_{-1}^{1} (1 - x)^{\alpha}(1 + x)^{\beta} \, dx = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.
\]

The integrand on the left is the weight function of the Jacobi polynomials \( P_n^{(\alpha,\beta)}(x) \) \([6, 75]\). Up to normalization these are the unique functions, analytic around the origin, solving the second order ODE

\[
(1 - x^2)y'' + (\beta - \alpha - x(\alpha + \beta + 2))y' + n(n + \alpha + \beta + 1)y = 0.
\]

Defining the inner product

\[
\langle f, g \rangle_{\alpha, \beta} := \int_{-1}^{1} f(x)g(x)(1 - x)^{\alpha}(1 + x)^{\beta} \, dx
\]

the Jacobi polynomials satisfy the orthogonality relation

\[
\langle P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)} \rangle_{\alpha, \beta} = \delta_{m,n} \frac{(\alpha + 1)_n(\beta + 1)_n}{n!(\alpha + \beta + 1)_n} \frac{\alpha + \beta + 1}{\alpha + \beta + 2n + 1} \cdot J(\alpha, \beta).
\]

All other classical orthogonal polynomials, such as the Laguerre and Hermite polynomials (corresponding to weights \( x^n \exp(-x) \) and \( \exp(-x^2) \) respectively) follow from the Jacobi polynomials by taking appropriate limits.

Several people have studied multivariable generalizations of the Jacobi, Laguerre and Hermite polynomials \([16, 33, 35, 41, 95–97, 104, 121]\). The most general of these are the multivariable Jacobi polynomials \( P^{(\alpha,\beta,\gamma)}_{\lambda}(x) \) which arise as the eigenfunctions of the operator

\[
\sum_{i=1}^{n} \left( (1 - x_i^2) \frac{\partial^2}{\partial x_i^2} + (\beta - \alpha - x_i(\alpha + \beta + 2)) \frac{\partial}{\partial x_i} \right) + 2\gamma \sum_{i,j} \frac{1 - x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}.
\]

The \( P^{(\alpha,\beta,\gamma)}_{\lambda}(x) \) are orthogonal with respect to an inner product with weight function derived from the Selberg integral. With \((2.58)\)

\[
\langle f, g \rangle_{\alpha, \beta, \gamma} := \int_{[-1, 1]^n} f(x)g(x) \prod_{i=1}^{n} (1 - x_i)^{\alpha}(1 + x_i)^{\beta} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\gamma} \, dx_1 \cdots dx_n
\]

for \( x = (x_1, \ldots, x_n) \), the multivariable Jacobi polynomials satisfy

\[
\langle P^{(\alpha,\beta,\gamma)}_{\lambda}, P^{(\alpha,\beta,\gamma)}_{\mu} \rangle_{\alpha, \beta, \gamma} = 0 \quad \text{if} \ \lambda \neq \mu.
\]

The quadratic norm evaluation can be computed explicitly in term of Pochhammer symbols and gamma functions using the shift operators of Heckman and Opdam \([119]\). From the Selberg integral it of course immediately follows that

\[
(1, 1)_{\alpha, \beta, \gamma} = 2^{n(\alpha+\beta+1+(n-1)\gamma)} S_n(\alpha + 1, \beta + 1, \gamma).
\]

Two important limiting cases of the inner product \((2.58)\) are

\[
\langle f, g \rangle_{\gamma} := \int_{\mathbb{R}^n} f(x)g(x) \prod_{i=1}^{n} e^{-x_i^2} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\gamma} \, dx_1 \cdots dx_n
\]
and
\[ \langle f, g \rangle_{\alpha, \gamma} := \int_{[0, \infty)^n} f(x)g(x) \prod_{i=1}^{n} x_i^\alpha e^{-x_i} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\gamma} dx_1 \cdots dx_n. \]

The corresponding families of orthogonal polynomials are referred to as the multi-variable Hermite and Laguerre polynomials, respectively. In particular note that
\[ \langle 1, 1 \rangle_{\gamma} = 2^{\binom{n}{2}/2} F_n(\gamma) \]
with \( F_n \) the Mehta integral (1.5), and
\[ \langle 1, 1 \rangle_{\alpha, \gamma} = n! W_n(\alpha + 1, \gamma), \]
with \( W_n \) given by (2.52).

All of the orthogonal polynomials mentioned above admit \( q \)-analogues. In the \( q \)-theory the role of the Selberg integral is played by Askey’s \( q \)-Selberg integral (2.54) and generalisations thereof. In the case of the Jacobi polynomials these \( q \)-analogues are known as the multivariable big and little \( q \)-Jacobi polynomials, and were introduced by J.V. Stokman [146]. Stokman [147] also showed how the big and little \( q \)-Jacobi polynomials arise as special limits of the Koornwinder polynomials [93].

The latter are multivariable analogues of the Askey–Wilson orthogonal polynomials [14] and may be viewed as the generalizations of the Macdonald polynomials to the root system BC\( n \). The relevant inner product in this case is given by [34, 93]
\[ (2.59) \]
\[ \langle f, g \rangle_{t, t_1, \ldots, t_4} := \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(e^{ix})g(e^{ix}) \Delta(e^{ix}; t, t_1, \ldots, t_4) dx_1 \cdots dx_n, \]
where \( \exp(i x) = (\exp(i x_1), \ldots, \exp(i x_n)) \), \( \Delta(x; t, t_1, \ldots, t_4) \) is the weight function (2.57) of Gustafson’s constant term identity, and \( f \) and \( g \) are BC\( n \) symmetric functions. \( (f(z) \) is BC\( n \) symmetric if \( f(\exp(i x)) \) is symmetric under signed permutations of \( x = (x_1, \ldots, x_n) \).) The evaluation of \( \langle 1, 1 \rangle_{t, \ldots, t_4} \) is of course provided by the right-hand side of (2.56).

### 3. Recent and current research directions

**The case of \( \gamma \) a positive integer.** Two recent studies have identified special features of the Selberg integral for \( \gamma \) a positive integer. The first of these is due to J.-G. Luque and J.-Y. Thibon [98], and exhibits an inter-relation with a special class of hyperdeterminants. The second, due to Stanley [143], gives a probabilistic interpretation of Selberg’s integral.

For a \( k \)th order tensor \( A = [A_{i_1 i_2 \cdots i_k}] \) on an \( n \)-dimensional space (so that \( 1 \leq i_p \leq n \)) the hyperdeterminant has been defined by Cayley (see references in [98]) as
\[ \text{det}_k(A) := \sum_{\sigma_2, \ldots, \sigma_k \in \mathfrak{S}_n} \epsilon(\sigma_2) \cdots \epsilon(\sigma_k) \prod_{i=1}^{n} A_{i_1, \sigma_2(i), \ldots, \sigma_k(i)} \]
where \( \epsilon(\sigma) \) denotes the signature of the permutation \( \sigma \). For \( k \) odd this vanishes while \( k = 2 \) corresponds to the usual definition of a determinant.

For an arbitrary measure \( \mu(x) \) it is easy to see by use of the Vandermonde determinant formula that in the so-called Hankel case
\[ A_{i_1 i_2 \cdots i_k} = \mu_{i_1 + i_2 + \cdots + i_k - k}, \quad \mu_j := \int_{-\infty}^{\infty} x^j \, d\mu(x) \]
the corresponding hyperdeterminant is equal to a multiple integral,

\[
\det_{2k}(A) = \frac{1}{n!} \int \cdots \int \prod_{1 \leq i < j \leq n} (x_i - x_j)^{2k} d\mu(x_1) \cdots d\mu(x_n).
\]

For \(d\mu(x) = x^{a-1}(1-x)^{b-1}dx\) on \(x \in [0,1]\) this is precisely the Selberg integral with \(\gamma = k\) an integer.

The probabilistic interpretation of the Selberg integral given in [143] applies for \(\alpha = \beta = 1\) and \(\gamma\) a positive integer. In fact, as communicated to us by Stanley [144], this same probabilistic interpretation, extended to \(\alpha, \beta\) general nonnegative integers, is already implied by appropriately interpreting supplementary problem I.25 of [142]. Following [144], the setting is to choose labelled points independently and with uniform distribution from the interval \([0,1]\). Specifically, for each \(1 \leq p \leq n\) and \((i,j)\) such that \(1 \leq i < j \leq n\), choose \(\alpha - 1\) points labelled \(y_p\), \(\beta - 1\) points labelled \(z_p\), \(n\) points labelled \(t\) and \(2\gamma\) points labelled \(a_{ij}\). Let \(t_i\) be the \(i\)-th smallest point labelled \(t\). The probability that any one of the points labelled \(y_p\) is to the left of \(t_i\) is \(1 - t_i\); the probability that any one of the points labelled \(a_{ij}\) is between \(t_i\) and \(t_j\) for \(i < j\) is \(t_i - t_j\). It follows immediately that the Selberg integral \(S_n(\alpha, \beta, \gamma)\) is the probability, \(P_S\) say, that all the points labelled \(y_p\) are to the left of \(t_p\), and all the points labelled \(z_p\) are to the right of \(t_p\), and all the points labelled \(a_{ij}\) lie between \(t_i\) and \(t_j\). Note that this statement remains valid for \(2\gamma\) an odd integer. An equivalent formulation (the one given in [142]) is to regard the selection of the labelled points uniformly and independently from \([0,1]\) as a random re-arrangement of the symbols themselves (i.e. the \(t\)'s, \(y_p\)'s, \(z_p\)'s and \(a_{ij}\)'s), in which case \(P_S\) corresponds to the probability that the re-arrangement complies with the prescribed rule.

**Random matrix theory.** A number of interplays between random matrix theory and the Selberg integral appearing in papers published in the last few years were discussed previously under the heading of the Dixon–Anderson integral. Below two further applications of the Selberg integral to random matrix theory as they have occurred in current works will be outlined.

The first of these is a study by Forrester and Rains [58] focusing on the family of multi-dimensional integrals

\[
(3.60) \quad I_{n,p}(x) := \int_{[0,x]^p} \int_{[x,1]^{n-p}} \prod_{i=1}^{n} t_i^{a-1}(1-t_i)^{b-1} |x - t_i|^{\tau-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} dt_1 \cdots dt_n
\]

for \(0 \leq p \leq n\) and \(x \in [0,1]\). In the case \(\tau = 1\) the integral \((3.60)\) is proportional to the probability that for the Selberg density, interpreted as an eigenvalue PDF, there are \(p\) eigenvalues in the interval \([0,x]\) and \((n-p)\) eigenvalues in the interval \([x,1]\); for \(\tau = 1 + 2\gamma\) the integral \((3.60)\) relates to the derivative of this quantity. The case \(\tau = 2\gamma = 1\) of this was first studied in the mathematical statistics literature for its relevance to canonical correlation analysis [32].

Theory connecting \(I_{n,p}\) to a certain Fuchsian differential equation [32,53,109] implies that the integral is expressible as a linear combination of Frobenius solutions.
These are solutions to the differential equation of the form

\[ g_i(x) = x^{\sigma_i} \sum_{k=0}^{\infty} a_{i,k} x^k, \quad \sigma_i = i(\alpha - 1 + \tau + (i - 1)\gamma) \]

for \(0 \leq i \leq n\), and are normalized such that

\[ (3.61) \lim_{x \to 0} \frac{g_i(x)}{I_{n,p}(x)} = 1 \quad \text{for} \quad \Re(\sigma_i) > 0. \]

A basic task — essentially equivalent to finding the monodromy matrix for the basis of integral solutions of the matrix Fuchsian system, of which (3.60) forms the top row — is to give the explicit form of the coefficients \( c_{p,i} \) in the expansion

\[ (3.62) I_{n,p}(x) = \sum_{i=0}^{n} c_{p,i} g_i(x). \]

One approach to this problem is to seek a regime in parameter space such that for \( x \to 0 \) the leading behaviour of \( I_{n,p}(x) \) is proportional to \( x^{\sigma_i} \). This is achieved by changing variables \( t_j = xu_j \) for \( 1 \leq j \leq i \) where \( p + 1 \leq i \leq n \). A simple scaling of the integrand then shows that

\[ I_{n,p}(x) \sim x^{\sigma_i} S_{n-i}(\alpha + \tau + 2\gamma - 1, \beta, \gamma) S_{i,p}(\alpha, \tau, \gamma), \]

where \( S_{n-i} \) is the Selberg integral and \( S_{i,p} \) the Dotsenko–Fateev integral (2.31). Recalling the normalization (3.61) and the recurrence (2.33) allows the sought coefficients to be calculated as

\[
\begin{align*}
    c_{p,i} &= 0, & 0 \leq i \leq p - 1 \\
    c_{p,i} &= (-1)^{i-p} \prod_{j=1}^{i-p} \frac{\sin \pi(i - j + 1)\gamma \sin \pi(\alpha + (i - j)\gamma)}{\sin \pi j\gamma \sin \pi(\alpha + \tau - 1 + (2i - j - 1)\gamma)}, & p \leq i \leq n
\end{align*}
\]

thus solving the problem at hand.

The second of the applications stems from a question posed by Bálint Virág at a recent AMS–IMS–SIAM summer research conference [1]. By way of background to his question, let us recall a result of Mehta and Dyson [108] which gives the circular ensembles identity \( \text{alt}(\text{COE}_{2n}) = \text{CSE}_n \). Here \( \text{alt} \) is the operation of integrating out every second eigenvalue, and the subscripts on the names of the ensembles indicate the total number of eigenvalues. Let us more generally introduce the notation \( \text{CE}_{\beta,n} \) for the PDF (1.9). The question posed by Virág was to investigate extensions of the result of Mehta and Dyson, in which blocks of eigenvalues in \( \text{CE}_{\beta,n} \) are integrated out to obtain another circular ensemble \( \text{CE}_{\beta',n'} \). The Selberg integral is relevant for this purpose.

Let \( p(k; s; \text{CE}_{\beta,n}) \) denote the PDF for the spacing between eigenvalues which are \((k+1)\)-st neighbours in the ensemble \( \text{CE}_{\beta,N} \). Let \( \text{alt}_m(\text{CE}_{\beta,n}) \) denote the joint marginal distribution of every \( m \)-th eigenvalue in \( \text{CE}_{\beta,n} \). With this notation, if it were true that

\[ (3.63) \text{alt}_m(\text{CE}_{\beta,mn}) = \text{CE}_{\beta',n} \]

for some \( m, \beta, \beta' \), then

\[ (3.64) p(mk + m - 1; s; \text{CE}_{\beta,mn}) = p(k; s; \text{CE}_{\beta',n}). \]
Now the $k$-point correlation $\rho_k$ is obtained from (1.9) by integrating out the variables $\theta_{k+1}, \ldots, \theta_n$ and multiplying by $n!/(n-k)!$. It follows from this definition that $\rho_k$ is related to the small $s$ expansion of $p$ according to

$$
(3.65) \quad p(k; s; CE_{\beta,n}) \sim \frac{2\pi^{k}}{nk!} \int_0^s \cdots \int_0^s \rho_{k+2}(0, s, \theta_1, \ldots, \theta_k) \, d\theta_1 \cdots d\theta_k.
$$

But for $\theta_1, \ldots, \theta_n$ small the definition of $\rho_k$ also implies that

$$
(3.66) \quad \rho_k(\theta_1, \ldots, \theta_k) \sim \frac{1}{(2\pi)^{k}} \frac{n!}{(n-k)!} \frac{M_{n-k}(k\beta/2, k\beta/2, \beta/2)}{M_n(0, 0, \beta/2)} \prod_{1 \leq i < j \leq k} |\theta_i - \theta_j|^\beta,
$$

where $M_n$ refers to the Morris integral (1.17). Substituting (3.66) in (3.65) and scaling the integrand an example of the Selberg integral is obtained, giving the formula

$$
(3.67) \quad p(k; s; CE_{\beta,n}) \sim \frac{1}{(2\pi)^{k+1}} \frac{(n-1)!}{k!(n-k-2)!} s^{k+\beta(k+2)(k+1)/2} \times \frac{M_{n-k-2}((k+2)\beta/2, (k+2)\beta/2, \beta/2)}{M_n(0, 0, \beta/2)} S_k(\beta + 1, \beta + 1, \beta/2).
$$

Using the gamma function evaluations (1.11) and (1.18), together with the duplication formula for the gamma function, one can check that in the case $m = r + 1$, $\beta = 2/(r+1)$ and $\beta' = 2(r+1)$, (3.64) is compatible with (3.67). Thus, this investigation based on the Selberg integral reveals parameters for which the validity of (3.63) may be expected. One can in fact proceed further and prove, using a generalization of the Dixon–Anderson integral, that for these parameters (3.63) is indeed valid [56].

**KZ equations and the Mukhin–Varchenko conjecture.** On page 22 we have seen that hypergeometric integrals of Selberg type arise naturally as solutions of (systems) of partial differential equations. There is a well-developed theory extending much of this to the setting of partial differential equations — referred to as Knizhnik–Zamolodchikov (KZ) equations — based on Lie algebras [43,115,132,154].

Let $\mathfrak{g}$ be a simple Lie algebra of rank $n$, with simple roots, fundamental weights and Chevalley generators given by $\alpha_i, \Lambda_i$ and $e_i, f_i, h_i$ for $1 \leq i \leq n$. Let $V_\lambda$ and $V_\mu$ be highest weight representations of $\mathfrak{g}$ with highest weights $\lambda$ and $\mu$, and let $u = u(z, w)$ be a function taking values in $V_\lambda \otimes V_\mu$ solving the KZ equation

$$
\kappa \frac{\partial u}{\partial z} = \frac{\Omega}{z-w} u, \quad \kappa \frac{\partial u}{\partial w} = \frac{\Omega}{w-z} u,
$$

where $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ is the Casimir element. (For the sake of simplicity we only consider the KZ equation in two variables, $z$ and $w$; for the more general case of $p$ variables $z_1, \ldots, z_p$, see e.g., [154].) Let $\text{Sing}_{\lambda,\mu}[\nu]$ denote the space of singular vectors of weight $\nu$ in $V_\lambda \otimes V_\mu$

$$
\text{Sing}_{\lambda,\mu}[\nu] := \{ v \in V_\lambda \otimes V_\mu : h_i v = \nu(h_i)v, \ e_i v = 0, \ 1 \leq i \leq n \}.
$$

Then, according to a theorem of V.V. Schechtman and Varchenko [132], solutions $u$ with values in $\text{Sing}_{\lambda,\mu}[\lambda + \mu - \sum_{i=1}^n k_i \alpha_i]$ are expressible in terms of multiple hypergeometric integrals

$$
u(z, w) = \sum u_{IJ}(z, w) f^I v_\lambda \otimes f^J v_\mu.$$
with coordinate functions \( u_{IJ} \) given by

\[
u_{IJ}(z, w) = \int_\gamma \Phi^{1/\kappa}(z, w; t)A_{IJ}(z, w; t) \, dt_1 \cdots dt_k.
\]

Here \( k := k_1 + \cdots + k_n, t := (t_1, \ldots, t_k) \), the sum is over all ordered multisets \( I \) and \( J \) with elements taken from \( \{1, \ldots, n\} \) such that their union contains the number \( i \) exactly \( k_i \) times, \( v_\lambda \) and \( v_\mu \) are the highest weight vectors of \( V_\lambda \) and \( V_\mu \),

\[
f^I_v := (\prod_{i \in I} f_i)v \quad \text{and} \quad \gamma \quad \text{a suitable integration domain.}
\]

The functions \( \Phi \) and \( A_{IJ} \) in the integrand of \( u_{IJ} \) are explicitly known. \( A_{IJ} \) is a rational function whose general form is too involved to explicitly state here (an example will be given below), and the function \( \Phi \), known as the master function, is defined as follows. The first \( k_1 \) integration variables are attached to the simple root \( \alpha_1 \), the next \( k_2 \) integration variables are attached to the simple root \( \alpha_2 \), and so on, such that \( \alpha_{k_j} := \alpha_j \) if \( k_1 + \cdots + k_{j-1} < j \leq k_1 + \cdots + k_i \). With this understood

\[
\Phi(z, w; t) = (z - w)^{(\lambda, \mu)} \prod_{i=1}^k (t_i - z)^{-(\lambda, \alpha_i)}(t_i - w)^{-(\mu, \alpha_i)} \times \prod_{1 \leq i < j \leq k} (t_i - t_j)^{\alpha_i, \alpha_j},
\]

where \( (\cdot, \cdot) \) is the bilinear symmetric form on \( \mathfrak{h}^* \) (the dual of the Cartan subalgebra \( \mathfrak{h} \)) normalised such that \( \langle \theta, \theta \rangle = 2 \) for the maximal root \( \theta \).

The simplest possible example of a KZ solution contained in the Schechtman–Varchenko theorem corresponds to the rank 1 Lie algebra \( \mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{A}_1 \), with simple root \( \alpha_1 \) and fundamental weight \( \Lambda_1 = \alpha_1/2 \). Taking \( \lambda = m_1\Lambda_1 \) and \( \mu = m_2\Lambda_1 \) it follows that \( u(z, w) \) takes values in the space of singular vectors of weight \( (m_1 + m_2 - 2k_1)\Lambda_1 \). Since \( n = 1 \) it follows that \( I = \{1\} \) and \( J = \{k_1-1\} \) with \( 1 \leq r \leq k_1 \), so that \( u_{IJ}, A_{IJ} \) and \( f^I \) can simply be denoted by \( u_r, A_r \) and \( f^r \). (In the case of rank one there is no need for the index in \( f_1 \).) Using \( n \) instead of \( k_1 \) (so that \( n \) no longer denotes the rank of the Lie algebra) and writing \( v_1 \) and \( v_2 \) instead of \( v_\lambda = v_{m_1\Lambda_1} \) and \( v_\mu = v_{m_2\Lambda_2} \) one finds, upon the assumption that \( z < w \) are both real,

\[
u(z, w) = \sum_{r=0}^n u_r(z, w) f^r v_1 \otimes f^{n-r} v_2
\]

with

\[
u_r(z, w) = (z - w)^{m_1m_2/(2\kappa)} \int_\gamma A_r(z, w; t) \prod_{i=1}^n (t_i - z)^{-m_1/\kappa}(t_i - w)^{-m_2/\kappa} \times \prod_{1 \leq i < j \leq n} (t_i - t_j)^2/\kappa \, dt_1 \cdots dt_n.
\]

Here the domain of integration is the simplex \( \gamma = \{t \in \mathbb{R}^n \mid z \leq t_n \leq \cdots \leq t_1 \leq w\} \), and the rational function \( A_r(z, w; t) \) is given by

\[
A_r(z, w; t) = \sum_{I \subseteq \{1, \ldots, n\}, \#I = r} \frac{1}{\prod_{i \in I} t_i - z} \prod_{i \in I} \frac{1}{t_i - w}.
\]

The coordinate functions \( u_r \) are easily recognised as generalizations of the Selberg integral. In fact, for the extremal cases \( r = 0 \) and \( r = n \) they are exactly the
Making the change of variables \( t_i = (w - z)s_i + z \) for \( 1 \leq i \leq n \) this yields

\[
u_0(z, w) = \frac{(-1)^A(z - w)^B}{n!} S_n \left( 1 - \frac{m_1}{\kappa}, \frac{m_2}{\kappa}, \frac{1}{\kappa} \right),
\]

where \( A = n(n - 1 - m_1)/\kappa + n \) and \( B = (m_1m_2 - 2n(m_1 + m_2) + 2n(n - 1))/(2\kappa) \).

In 2000 E. Mukhin and Varchenko [115] formulated a surprising conjecture regarding the scaled master function

\[
\Phi(t) = \prod_{i=1}^k t_i^{-(\lambda, \alpha_{t_i})} (1 - t_i)^{-(\mu, \alpha_{t_i})} \prod_{1 \leq i < j \leq n} (t_i - t_j)^{\alpha_{t_i} \cdot \alpha_{t_j}}.
\]

They conjectured that if the space \( \text{Sing}[\lambda + \mu - \sum_{i=1}^n k_i \alpha_i] \) of singular vectors is one-dimensional, then

\begin{equation}
\int |\Phi(t)|^{1/\kappa} \, dt_1 \cdots dt_k
\end{equation}

is expressible as a product of gamma functions. Neither the exact integration domain nor the specific form for the product of gamma functions is contained in the Mukhin–Varchenko conjecture.

For \( g = sl_2 = A_1 \) the conjecture corresponds to the evaluation of the Selberg integral. For \( g = sl_{n+1} = A_n \), and \( \text{Sing}_{\lambda, \mu} [\lambda + \mu - \sum_{i=1}^n \alpha_i] \) with \( \lambda = A_1, \mu = \sum_{i=1}^n \mu_i A_i \), the conjecture simply follows by iterating the beta integral [113], see [115]. For \( g = B_n, C_n \) or \( D_n \) and

\[
\text{Sing}_{A_1, A_1} \left[ 2A_1 - r\alpha_n - s\alpha_n - \sum_{i=1}^{n-2} \alpha_i \right] \quad \text{with} \quad (r, s) = \begin{cases} (2, 2) & B_n \\ (2, 1) & C_n \\ (1, 1) & D_n \end{cases}
\]

(corresponding to the tensor product of the vector representation of \( g \)) Mimachi and T. Takamuki [112] established the Mukhin–Varchenko conjecture iterating the Selberg integral for \( n = 2 \) (\( B_n \) case) or the beta integral (\( C_n \) and \( D_n \) cases).

In 2003 V. Tarasov and Varchenko employed KZ equations and the closely related dynamical equations to settle the conjecture for \( g = sl_3 = A_2 \). In recent work by Warnaar [157, 158] an approach to the \( sl_{n+1} = A_n \) case of the Mukhin–Varchenko conjecture was developed, based on the theory of Macdonald polynomials and generalized hypergeometric series. Specifically, the integral (3.68) for \( g = A_n \) can be evaluated in closed form when \( \lambda = \lambda_n A_n \) and \( \mu = \sum_i \mu_i A_i \) (or when \( \lambda = \lambda_1 A_1 \) and \( \mu = \sum_i \mu_i A_i \)). Stripping the integral from its Lie algebra notation and using \( \alpha_i \) and \( \beta_i \) \((1 \leq i \leq n)\) for exponents in the integral (so that the \( \alpha_i \) no longer denote
the simple roots) the $A_n$ Selberg integral can be stated explicitly as

$$\int \prod_{s=1}^{n} \left[ \Delta(t^{(s)}) \right]^{2} k_s \prod_{i=1}^{k_s} \left( t_i^{(s)} \right)^{\alpha_s-1} (1 - t_i^{(s)})^{\beta_s-1} \prod_{s=1}^{n-1} \left[ \Delta(t^{(s)}, t^{(s+1)}) \right]^{-\gamma} \, dt$$

$$= \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s} \frac{\Gamma(\beta_s + \cdots + \beta_r + (i + s - r - 1)\gamma)}{\Gamma(\alpha_r + \beta_s + \cdots + \beta_r + (i + s - r + k_r - k_{r+1} - 2)\gamma)}$$

$$\times \prod_{s=1}^{n} k_s \frac{\Gamma(\alpha_s + (i - k_{s+1} - 1)\gamma)\Gamma(\gamma)}{\Gamma(\gamma)}.$$

Here $k_1, \ldots, k_{n+1}$ are nonnegative integers such that $k_{n+1} = 0$ and $k_1 \leq k_2 \leq \cdots \leq k_n$, the exponents $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma \in \mathbb{C}$ such that $\alpha_1 = \cdots = \alpha_{n-1} = 1$ and such that both sides of the identity are well-defined. Furthermore, $t^{(s)} = (t_1^{(s)}, \ldots, t_{k_s}^{(s)})$ is the set of variables attached to the $s$th simple root of $A_n$,

$$\Delta(u) = \prod_{1 \leq i < j \leq \ell_u} (u_i - u_j) \quad \text{and} \quad \Delta(u, v) = \prod_{i=1}^{\ell_u} \prod_{j=1}^{\ell_v} (u_i - v_j)$$

for sets of variables $u = (u_1, \ldots, u_{\ell_u})$ and $v = (v_1, \ldots, v_{\ell_v})$, and $dt = dt^{(1)} \cdots dt^{(n)}$ with $dt^{(s)} = dt_1^{(s)} \cdots dt_{k_s}^{(s)}$ so that the integral is $(k_1 + \cdots + k_n)$-dimensional.

Not yet specified in the $A_n$ Selberg integral is the domain of integration, which, unfortunately, is rather involved. A key ingredient is the set of maps

$$M_s : \{1, \ldots, k_s\} \rightarrow \{1, \ldots, k_{s+1}\}$$

such that

$$M_s(i) \leq M_s(i+1) \quad \text{and} \quad 1 \leq M_s(i) \leq k_{s+1} - k_s + i.$$

A standard counting argument shows that there are exactly $c_{k_{s+1}, k_s}$ admissible $M_s$, where $c_{n,k}$ is the row $(n,k)$ entry in the Catalan triangle, or, equivalently, the number of standard Young tableaux of shape $(n,k)$. Given $M_s$ fix an ordering among the $t_1^{(s)}$ and $t_1^{(s+1)}$ as

$$(3.69) \quad t_1^{(s+1)}_{M_s(i)} \leq \cdots \leq t_1^{(s+1)}_{M_s(i-1)} \quad \text{for} \ 1 \leq i \leq k_s,$$

where $t_0^{(s+1)} := \infty$. Given admissible maps $M_1, \ldots, M_{n-1}$ define $D_{M_1, \ldots, M_{n-1}}^{k_1, \ldots, k_n}$ as the set of points

$$\left( t_1^{(1)}, \ldots, t_{k_1}^{(1)}, t_1^{(2)}, \ldots, t_{k_2}^{(2)}, \ldots, t_1^{(n)}, \ldots, t_{k_n}^{(n)} \right)$$

such that $3.69$ holds for all $1 \leq s \leq n-1$ and

$$0 \leq t_1^{(s)} \leq \cdots \leq t_1^{(s)} \leq 1$$

holds for all $1 \leq s \leq n$. Then the domain of integration, written as a chain, is given by

$$\sum_{M_1, \ldots, M_{n-1}} F_{M_1, \ldots, M_{n-1}}^{k_1, \ldots, k_n} (\gamma) D_{M_1, \ldots, M_{n-1}}^{k_1, \ldots, k_n},$$

where

$$F_{M_1, \ldots, M_{n-1}}^{k_1, \ldots, k_n} (\gamma) = \prod_{s=1}^{n-1} \prod_{i=1}^{k_s} \frac{\sin(\pi (i + k_{s+1} - k_s - M_s(i) + 1)\gamma)}{\sin(\pi (i + k_{s+1} - k_s)\gamma)}.$$
In complete analogy with the ordinary Selberg integral, the evaluation of the $A_n$ Selberg integral can be generalized to include a Jack polynomial in the integrand, thus generalizing the Kadell integral (3.46), see [158].

Elliptic Selberg integrals. In the last few years there has been rapid progress in the field of elliptic generalizations of hypergeometric series, see [63,140]. Classical hypergeometric series
\[
\sum_{n=0}^{\infty} c_n
\]
are characterized by the ratio $c_{n+1}/c_n$ being a rational function of $n$. Their elliptic counterparts have the same ratio equal to an elliptic function of $n$.

As well as the classical hypergeometric series permitting elliptic generalizations, so do related integrals such as the Euler beta integral (1.3). In the elliptic theory the ordinary gamma function must be replaced by what is known as the elliptic gamma function
\[
\Gamma(z; p,q) = \prod_{i,j=0}^{\infty} \frac{1}{1 - z^{-1}p^i+1q^j+1 - p^iz q^j},
\]
declared for $|p|, |q| < 1$. This function can be traced back to E.W. Barnes in 1904 [18], but was given prominence through the recent work of S.N.M. Ruijsenaars [131]. It permits the extension of the standard gamma recurrence to
\[
\Gamma(qz; p,q) = \frac{\theta(z; p)}{\Gamma(qz; p,q)},
\]
where $\theta(z; p) = (z; p)_{\infty}(p/z; p)_{\infty}$ is a normalised theta function. Another fundamental property of the elliptic gamma function is the functional equation
\[
\Gamma(z; p,q) = 1 \Gamma(pq/z; p,q)
\]
which follows immediately from the definition (3.70).

The elliptic analogue of the beta integral (1.3) was discovered in 2000 by V.P. Spiridonov [137]
\[
\int_C \prod_{r=1}^{6} \frac{\Gamma(t_r z^{\pm 1}; p,q)}{\Gamma(z^{\pm 1}; p,q)} \frac{dz}{2\pi i z} = \frac{2}{(p;p)_{\infty}(q;q)_{\infty}} \prod_{1 \leq r < s \leq 6} \Gamma(t_r t_s; p,q),
\]
where each $|t_r| < 1$, $C$ is the positively oriented unit circle, $\prod_{r=1}^{6} t_r = pq$ and
\[
\Gamma(tz^{\pm m}; p,q) := \Gamma(tz^{-m}; p,q)\Gamma(tz^{m}; p,q).
\]
The $p \to 0$ limit is the well-known Rahman [99, 123] integral, which itself is an extension of the Askey–Wilson integral [14]. For the reduction of this last integral to the beta integral (1.3) see [63].

J.F. van Diejen and Spiridonov [36] have given an $n$-dimensional generalization of (3.72) which may be viewed as an elliptic extension of the Selberg integral. This integral, the $p \to 0$ limit of which was first obtained by Gustafson [68], takes the form
\[
\int_{C^n} \prod_{1 \leq i < j \leq n} \frac{\Gamma(tz_i^{\pm 1} z_j^{\pm 1}; p,q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p,q)} \prod_{i=1}^{n} \frac{\Gamma(tz_i^{\pm 2}; p,q)}{\Gamma(z_i^{\pm 2}; p,q)} \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_n}{2\pi i z_n} = \frac{2^n n!}{(p;p)_{\infty}(q;q)_{\infty}} \prod_{j=1}^{n} \left( \frac{\Gamma(t^j; p,q)}{\Gamma(t; p,q)} \prod_{1 \leq r < s \leq 6} \Gamma(t^{j-1} t_r t_s; p,q) \right),
\]
where \(|t|, |t_1|, \ldots, |t_6| < 1\) and \(t^{2n-2} \prod_{k=1}^6 t_j = pq\). van Diejen and Spiridonov provided a proof of (3.73) along the lines of the Anderson and Gustafson proofs of (1.1) and (2.56) respectively. This required an elliptic generalization of the Dixon–Anderson integral (2.24) which, initially, was proved making an assumption about the vanishing of certain elliptic integrals. A complete proof of the elliptic Dixon–Anderson integral was subsequently given by Rains [124] and by Spiridonov [139].

The reduction of the elliptic Selberg integral (3.73) to the ordinary Selberg integral is rather cumbersome, requiring several limits, variables changes and specializations of parameters [126]. Fairly straightforward, however, is to see that (3.73) provides an elliptic extension of Gustafson’s BC\(_n\) constant term identity (2.56). To see this one first needs to eliminate \(t_6\) using \(t^{2n-2} \prod_{k=1}^6 t_j = pq\). This gives rise to several elliptic gamma functions of the form \(\Gamma(\lambda, \tau)\) which, by (3.71), may be replaced by \(1/\Gamma(1/\lambda, \tau)\). After these elementary manipulations the \(p \to 0\) limit can be carried out, using that \(\Gamma(z; 0, q) = 1/(z; q)_\infty\). Finally taking \(t_5 = 0\) and interpreting the resulting integral as a constant term identity yields (2.56).

Analogous to (2.59), the integrand of (3.73) can be used to define an inner product. Rains [124] has specified a family of abelian functions which are biorthogonal with respect to this inner product, extending the Rahman–Spiridonov theory [123, 138] of such functions to the multivariable setting, as well as generalising the Koornwinder polynomials and Okounkov BC\(_n\) interpolation polynomials [117] to the elliptic level. (These functions were independently introduced by H. Coskun and Gustafson in [31] without the use of elliptic Selberg type integrals.) Rains also extended the integrand of (3.73) analogous to the \(2F_1\) extension (2.49) of the Selberg integral, and obtained transformation formulas for the resulting elliptic hypergeometric integrals. By considering the reduction of his theory to the Selberg level Rains obtained, for example, [125]

\[
\int_0^1 \cdots \int_0^1 P^{(1/\gamma)}(t) P^{(1/\gamma)}(t) \prod_{i=1}^n t_i^{\gamma-1}(1-t_i)^{\gamma-1} \prod_{1 \leq i < j \leq n} |t_i - t_j|^{2\gamma} dt_1 \cdots dt_n = \prod_{i,j=1}^n \frac{\Gamma(\alpha + (2n-i-j) + \lambda_i + \mu_j)}{\Gamma(\alpha + (2n-i-j+1) + \lambda_i + \mu_j)} \prod_{j=0}^{n-1} \frac{\Gamma((j+1)\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(1+\gamma)} \times P^{(1/\gamma)}(1^n) P^{(1/\gamma)}(1^n).
\]

This integral, which generalises the \(\beta = \gamma\) case of Kadell’s integral (2.46) is originally due to Kadell [81] and (for \(\gamma = 1\)) L.K. Hua [73]. Kadell’s integral (2.46) also has an elliptic analogue, which has the feature that the Dotsenko–Fateev integral (2.31) is a special case [125].

There are other integrals in the literature referred to as elliptic Selberg integrals, although they do not contain the actual Selberg integral as a limiting case. These integrals arise as solutions to the Knizhnik–Zamolodchikov–Bernard (KZB) heat equation for \((2n+1)\)-dimensional \(\mathfrak{sl}_2\) modules

\[
2\pi i \kappa \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2} + n(n+1)\rho(\lambda, \tau) u.
\]

Here \(u = u(\lambda, \tau), \rho(\lambda, \tau) = \vartheta'(\lambda, \tau)/\vartheta(\lambda, \tau)\) with differentiation with respect to \(\lambda\), and \(\vartheta(\lambda, \tau) = \theta_1(\pi \lambda, \tau)\) is a theta function [160].
To describe the relevant solutions to the KZB equations let $\Phi$ be the elliptic master function
\[
\Phi(t_1, \ldots, t_n; \tau) = \prod_{i=1}^{n} E(t_i, \tau)^{-2n} \prod_{1 \leq i < j \leq n} E(t_i - t_j, \tau)^2,
\]
where $E(t, \tau)$ is the elliptic analogue of $t E(t, \tau) = \vartheta(t, \tau) \vartheta'(0, \tau)$.

The solutions considered by G. Felder, L. Stevens and Varchenko [52] are the linear combinations
\[
u_{\kappa,m}(\lambda, \tau) = J_{\kappa,m}(\lambda, \tau) + (-1)^{n+1} J_{\kappa,m}(-\lambda, \tau),
\]
where
\[
J_{\kappa,m}(\lambda, \tau) := \int_{0 < t_0 < \cdots < t_1 < 1} \Phi^{1/\kappa}(t_1, \ldots, t_n; \tau) \theta_{\kappa,m}(\lambda + \frac{2|t|}{\kappa}, \tau) \prod_{i=1}^{n} \sigma_{\lambda}(t_i, \tau) \, dt_1 \cdots dt_n.
\]
Whenever necessary this integral is understood in the sense of analytic continuation from the region where the exponents in $\Phi^{1/\kappa}$ have positive real part [52].

$\theta_{\kappa,m}(\lambda, \tau)$ is a theta function of degree $\kappa$ and characteristic $m$
\[
\theta_{\kappa,m}(\lambda, \tau) = \sum_{j \in \mathbb{Z}^+} e^{2\pi i \kappa (\tau j + \lambda) j}
\]
and $\sigma_{\lambda}(t, \tau) = \theta(t - \lambda, \tau) / (\theta(\lambda, \tau) E(t, \tau))$.

In several instances Felder, Stevens and Varchenko found that the “elliptic Selberg integrals” $u_{\kappa,m}(\lambda, \tau)$ permit closed form evaluations in terms of theta functions and ordinary gamma functions. The simplest case of such an evaluation corresponds to
\[
u_{2n+2,n+1}(\lambda, \tau) = \left(2\pi\right)^{n/2} e^{-\pi i \frac{n(n-1)}{2(n+1)}} e^{\pi i \frac{n+1}{2}} \theta(\lambda, \tau)^{n+1}
\times S_n \left(\frac{n+2}{2(n+1)}, -\frac{n}{n+1}, \frac{1}{2(n+1)}\right) \prod_{i=1}^{n} \left(1 - e^{2\pi i \frac{n+1+i}{2(n+1)}}\right),
\]
where $S_n$ is the Selberg integral $\langle 1 \rangle$.

The value distribution of $\log \zeta(1/2 + it)$ on the critical line. The final topic to be reviewed, following J.P. Keating and N.C. Snaith [89], is a link between the Selberg integral in its trigonometric form $\langle 1 \rangle$, and one of Selberg’s theorems relating to the Riemann zeta function [135]. The latter gives the value distribution of $\log \zeta(1/2 + it)$ for large $t$, asserting that for any rectangle $B \subseteq \mathbb{C}$
\[
\lim_{T \to \infty} \frac{1}{T} \left\{ t : T \leq t \leq 2T, \frac{\log \zeta(1/2 + it)}{\sqrt{\log \log T}} \in B \right\} = \frac{1}{2} \int_B e^{-\frac{1}{2}(x^2 + y^2)} \, dx \, dy.
\]
To relate (3.74) to (1.17), first note that (1.18) can be used to evaluate
\[
\left\langle \prod_{i=1}^{n} e^{\pm i k \theta_i} |1 + e^{i \theta_i}|^i \right\rangle,
\]
where the average is with respect to the eigenvalue PDF (1.9) [15]. Three further ingredients are then required: the interpretation of (3.75) as specifying a distribution in random matrix theory, an hypothesis linking the value distribution of \( \log \zeta(1/2 + it) \) to the value distribution of \( \log \Lambda(z) \), \( \Lambda(z) := \prod_{i=1}^{n} (\exp(i\theta_i) - z) \) being the characteristic polynomial for the random matrices, and the large \( n \) form of (3.75) deduced from (1.18). Regarding the interpretation, note that

\[
(3.76) \quad \text{Re} \log \Lambda(-1) = \sum_{i=1}^{n} \log|e^{i\theta_i} + 1|, \quad \text{Im} \log \Lambda(-1) = \frac{1}{2} \sum_{i=1}^{n} \theta_i.
\]

It follows immediately that

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\langle \delta(s - \text{Re} \log \Lambda(-1))\delta(t - \text{Im} \log \Lambda(-1)) \right\rangle e^{is} e^{ikt} \, ds \, dt
\]

is equal to (3.75). In other words, the characteristic function for the joint distribution of the quantities (3.76) is equal to (3.75).

The hypothesis of Keating and Snaith [89], which extends the Montgomery–Odlyzko law linking the Riemann zeros to eigenvalues of large complex Hermitian random matrices (see e.g., [88]), asserts that the value distribution of \( \log \zeta(1/2 + it) \) for large \( t \) will coincide with the value distribution of \( \log \Lambda(z) \), \( |z| = 1 \), for \( \Lambda(z) \) the characteristic polynomial of matrices from the CUE (\( \beta = 2 \) case of (1.9)) for large \( n \). Further, the value of \( n \) in the CUE is to be related to the value of \( t \) in \( \zeta(1/2 + it) \) by \( n = \log t \), which ensures that to leading order the density of eigenvalues and zeta function zeros is equal.

Thus the task at hand is to compute the large-\( n \) limit of (3.75) with \( k \mapsto k/(\frac{1}{2} \log n)^{1/2} \), \( l \mapsto l/(\frac{1}{2} \log n)^{1/2} \), which for \( \beta = 2 \) and with the identifications of the previous paragraph corresponds to the scaling of \( \log \zeta(1/2 + it) \) by \( \left( \frac{1}{4} \log \log T \right)^{1/2} \) in (3.74). Using (1.18) this limit has been computed in [15] as being equal to \( \exp(-k^2 + l^2)/2 \). Hence the Selberg integral evaluation in its form (1.18) implies that the joint distribution of the scaled logarithm of the characteristic polynomial is equal to \( \exp(-s^2 + l^2)/2 \), giving quantitative agreement between the hypothesis of Keating and Snaith and Selberg’s theorem (3.74).

The value distribution of \( |\Lambda(z)| \) for \( |z| = 1 \) is also of relevance to zeta function theory [89]. The characteristic function of this quantity does not lead to a tractable integral. On the other hand, with \( p(s) \) a distribution supported on \( s > 0 \), knowledge of the Mellin transform (complex moments)

\[
m(x) = \int_{0}^{\infty} s^{x-1} p(s) \, ds
\]

as a function in the complex plane gives, via the inverse Mellin transform,

\[
p(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-x} m(x) \, dx.
\]

With \( p(s) \) the distribution of values of \( |\Lambda(-1)|^2 \) for the CUE, \( m(x + 1) \) is equal to (3.75) with \( k = 0, \ im = 2x \) and thus from (1.18) is given explicitly in terms of gamma functions (for a discussion of computing the corresponding inverse Mellin transform, see [130]). It should also be remarked that the value distribution of \( \Lambda(\pm 1) \) for \( \Lambda(z) \) the characteristic polynomial of a random orthogonal or unitary symplectic matrix, chosen with Haar measure, is a special case of (1.23), and thus similarly is an example of the Selberg integral. Keating and Snaith [90] make use...
of this fact to provide a quantitative link between the value distribution of families of \( L \)-functions on the critical line and random matrix theory.

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