Hagen-Hurley equations and the $W$ boson

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Abstract

We proceed with our study of the Hagen-Hurley equations describing spin one bosons. In this work, we describe a general decay of the Hagen-Hurley boson. It is important that the transformation conserves spin $s = 1$ of the decaying boson and provides information about the kinematics of the decay. We explain the high instability of the Hagen-Hurley particle and identify it as the $W$ boson.

1 Introduction

The intermediate vector bosons, $W^\pm$, as well as $Z^0$, are extremely short-lived particles with a half-life of about $3 \times 10^{-25}$ s [1]. Very recently, an analysis of high-precision measurement of the $W$ boson mass suggests that the mass is slightly higher than the Standard Model predicts [2]. This result hints at the need for reconsideration of the Standard Model.

This work aims to propose a relativistic equation that could effectively describe the $W$ boson.

We base our approach on spin one Hagen-Hurley equation [3–7]. In Section 2, for the sake of completeness, we rewrite the Hagen-Hurley equations as two coupled Dirac equations involving higher-order spinors as in [8].

Then, in Section 3 we describe decays of real $W$ bosons, such as appearing in the top quark decay or produced in collisions of protons and antiprotons. More precisely, we transform two coupled Dirac equations involving non-standard spinors, obtained in Section 2 into two Dirac equations for two fermions, generalizing our earlier work [9]. It is important that this transformation conserves spin $s = 1$ of the decaying boson and provides information about the kinematics of the decay, as described in Section 4.

In what follows, we are using definitions and conventions of Ref. [8].
2 Rearrangement of the Hagen-Hurley equations as two coupled Dirac equations

We start with the spinor formulation of the Hagen-Hurley equations, cf. [10] or Subsection 6 iii) in [11] or Eqs. (18), (19) in [8], see also Eqs. (2.1), (2.24) in [6] for the 7 × 7 form. These equations violate parity $P$, where

$$P: x^0 \rightarrow -x^0, \quad x^i \rightarrow -x^i \quad (i = 1, 2, 3).$$

Since parity is violated in weak interactions [12, 13], these equations can describe weakly interacting particles.

We write one of these equations (Eq. (19) of Ref. [8]), in the form:

$$p^A_{\cdot B} \zeta_{A\cdot D} = m \chi_{B\cdot D}$$

$$p^A_{\cdot D} \chi_{B\cdot D} = -m \zeta_{A\cdot B}$$

(1a)

$$\chi_{B\cdot D} = \chi_{D\cdot B}$$

(1b)

where Eq. (1b) is the spin-1 constraint [11]. In Eqs. (1a) we have $p^A_{\cdot B} = (\sigma^0 p^0 + \vec{p} \cdot \vec{\sigma})^A_{\cdot B}$, where $\sigma^k$ ($k = 1, 2, 3$) are the Pauli matrices, and $\sigma^0$ is the 2 × 2 unit matrix. Equations (1), which were first proposed by Dirac [10], can be written in the 7 × 7 Hagen-Hurley form:

$$\beta^\mu p_\mu \Psi = m \Psi,$$  

(2)

the $\beta^\mu$ matrices given in [6], with $\Psi = (\chi_{11}, \chi, \chi_{22}, \zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22})^T$, where $\chi_{ij} = \chi_{ij}$ and $^T$ stands for transposition of a matrix.

Equations (1a) in explicit form read:

$$- (p^1 + ip^2) \chi_{11} - (p^0 - p^3) \chi_{21} = -m \zeta_{11}$$

$$- (p^0 + p^3) \chi_{11} + (p^1 - ip^2) \chi_{21} = -m \zeta_{21}$$

$$- (p^1 - ip^2) \zeta_{11} - (p^0 - p^3) \zeta_{21} = m \chi_{11}$$

$$- (p^0 + p^3) \zeta_{11} + (p^1 + ip^2) \zeta_{21} = m \chi_{21}$$

(3a)

$$- (p^1 + ip^2) \chi_{12} - (p^0 - p^3) \chi_{22} = -m \zeta_{12}$$

$$- (p^0 + p^3) \chi_{12} + (p^1 - ip^2) \chi_{22} = -m \zeta_{22}$$

$$- (p^1 - ip^2) \zeta_{12} - (p^0 - p^3) \zeta_{22} = m \chi_{12}$$

$$- (p^0 + p^3) \zeta_{12} + (p^1 + ip^2) \zeta_{22} = m \chi_{22}$$

(3b)

These equations are coupled due to the condition $\chi_{12} = \chi_{21}$ which ensures that $s = 1$.

3 Decays of real spin-1 Hagen-Hurley bosons

We realize that solutions of two Dirac equations (3) are non-standard since they involve higher-order spinors rather than spinors $\xi_A, \eta_{\cdot B}$. There are several possibilities to reduce these higher-order spinors via the de Broglie method of fusion [14, 15].
Attempting to describe leptonic decays of the W boson, \( W^\pm \rightarrow l + \bar{\nu}_l \), we made the following substitution [9]:

\[
\begin{align*}
\chi_{\bar{B}B} (x) &= \eta_{\bar{B}} (x) \alpha_{\bar{D}} (x), \\
\zeta_{AB} (x) &= \xi_A (x) \alpha_{\bar{B}} (x),
\end{align*}
\]

where \( \alpha_{\bar{A}} (x) \) is the Weyl spinor, describing massless neutrinos, while \( \eta_{\bar{B}} (x) \), and \( \xi_A (x) \) are the Dirac spinors. Since \( \chi_{\bar{B}B} (x) \neq \chi_{\bar{D}D} (x) \) we had to assume that the spin of the decaying boson belonged to \( 0 \pm 1 \) space. We suggested that our formalism can describe a decay of a virtual \( W^\pm \) boson into a lepton and antineutrino in the mixed beta decay [9].

On the other hand, to describe the decay of a real \( W \) boson, we carry out another, conserving spin, substitution:

\[
\begin{align*}
\chi_{\bar{B}B} (x) &= \eta_{\bar{B}} (x) \alpha_{\bar{D}} (x) + \alpha_{\bar{B}} (x) \eta_{\bar{D}} (x), \\
\zeta_{AB} (x) &= \xi_A (x) \alpha_{\bar{B}} (x) + \lambda_A (x) \eta_{\bar{B}} (x).
\end{align*}
\]

Note that in this case \( \chi_{12} = \chi_{21} \) and thus \( s = 1 \). Both terms in \( \chi_{\bar{B}B} (x) \) are necessary to ensure \( s = 1 \) condition.

Substituting (5) into Eqs. (5) and rearranging terms we obtain:

\[
\begin{align*}
&- (p^1 + i p^2) \eta_1 \alpha_A - (p^0 - p^3) \eta_2 \alpha_A = -m \xi_1 \alpha_A, \\
&(p^0 + p^1) \eta_1 \alpha_A + (p^1 - i p^2) \eta_2 \alpha_A = -m \xi_2 \alpha_A, \\
&- (p^1 - i p^2) \xi_1 \alpha_A - (p^0 - p^3) \xi_2 \alpha_A = m \eta_1 \alpha_A, \\
&(p^0 + p^1) \xi_1 \alpha_A + (p^1 + i p^2) \xi_2 \alpha_A = m \eta_2 \alpha_A, \\
&- (p^1 - i p^2) \alpha_A \eta_B - (p^0 - p^3) \alpha_2 \eta_B = -m \lambda_1 \eta_B, \\
&(p^0 + p^1) \alpha_1 \eta_B + (p^1 - i p^2) \alpha_2 \eta_B = -m \lambda_2 \eta_B, \\
&- (p^1 - i p^2) \lambda_1 \eta_B - (p^0 - p^3) \lambda_2 \eta_B = m \alpha_1 \eta_B, \\
&(p^0 + p^1) \lambda_1 \eta_B + (p^1 + i p^2) \lambda_2 \eta_B = m \alpha_2 \eta_B,
\end{align*}
\]

where \( \bar{A} \), \( \bar{B} = 1, 2 \).

Assuming solutions as:

\[
\begin{align*}
\alpha_{\bar{A}} (x) &= \tilde{\alpha}_\bar{A} e^{-ikx}, \quad \lambda_A (x) = \tilde{\lambda}_A e^{-ikx}, \quad k^\mu k_\mu = m_1^2, \\
\eta_{\bar{B}} (x) &= \tilde{\eta}_{\bar{B}} e^{-iqx}, \quad \xi_A (x) = \tilde{\xi}_A e^{-iqx}, \quad q^\mu q_\mu = m_2^2,
\end{align*}
\]

we get two Dirac equations for spinors \( (\xi_A, \eta_{\bar{B}})^T \) and \( (\lambda_C, \alpha_{\bar{D}})^T \):

\[
\begin{align*}
&- (\tilde{p}^1 + i \tilde{p}^2) \eta_1 (x) - (\tilde{p}^0 - \tilde{p}^3) \eta_2 (x) = -m \xi_1 (x), \\
&(\tilde{p}^0 + \tilde{p}^1) \eta_1 (x) + (\tilde{p}^1 - i \tilde{p}^2) \eta_2 (x) = -m \xi_2 (x), \\
&- (\tilde{p}^1 - i \tilde{p}^2) \xi_1 (x) - (\tilde{p}^0 - \tilde{p}^3) \xi_2 (x) = m \eta_1 (x), \\
&(\tilde{p}^0 + \tilde{p}^1) \xi_1 (x) + (\tilde{p}^1 + i \tilde{p}^2) \xi_2 (x) = m \eta_2 (x), \\
&- (\tilde{p}^1 + i \tilde{p}^2) \alpha_1 (x) - (\tilde{p}^0 - \tilde{p}^3) \alpha_2 (x) = -m \lambda_1 (x), \\
&(\tilde{p}^0 + \tilde{p}^1) \alpha_1 (x) + (\tilde{p}^1 - i \tilde{p}^2) \alpha_2 (x) = -m \lambda_2 (x), \\
&- (\tilde{p}^1 - i \tilde{p}^2) \lambda_1 (x) - (\tilde{p}^0 - \tilde{p}^3) \lambda_2 (x) = m \alpha_1 (x), \\
&(\tilde{p}^0 + \tilde{p}^1) \lambda_1 (x) + (\tilde{p}^1 + i \tilde{p}^2) \lambda_2 (x) = m \alpha_2 (x)
\end{align*}
\]
with rescaled momentum operators:
\[ \tilde{p}^\mu = i \frac{\partial}{\partial x^\mu} + k^\mu, \quad \tilde{q}^\mu = i \frac{\partial}{\partial x^\mu} + q^\mu. \] (12)

4 Kinematics of decay of the real spin-1 Hagen-Hurley bosons

The forms (8), and (9) solve equations (10), and (11) provided that:
\[ (k^\mu + q^\mu) (k_\mu + q_\mu) = m^2. \] (13)

We thus define a new four-vector:
\[ p^\mu_{df} = k^\mu + q^\mu, \] (14)
and we have \( p^\mu p_\mu = m^2 \) where \( m \) is the decaying boson mass.

Taking into account Eqs. (8), (9), and (13) we obtain two-body decay kinematic equality:
\[
\begin{align*}
    m_1^2 + m_2^2 + 2 \left( k^0 q^0 - \vec{k} \cdot \vec{q} \right) = m^2, \\
    k^0 = +\sqrt{\vec{k}^2 + m_1^2}, \quad q^0 = +\sqrt{\vec{q}^2 + m_2^2}.
\end{align*}
\] (15)

It follows that transition from Eqs. (8), (9), via the substitution (5), can be interpreted as the decay of a real (not virtual) spin one boson with mass \( m \) into two spin one-half fermions with four-momenta \( k^\mu, q^\mu \), and masses \( m_1, m_2 \) because the inequality:
\[ m^2 - m_1^2 - m_2^2 = 2 \left( k^0 q^0 - \vec{k} \cdot \vec{q} \right) > 0, \] (16a)
is fulfilled.

Indeed, we have
\[ k^0 q^0 - \vec{k} \cdot \vec{q} = \sqrt{\vec{k}^2 + m_1^2} \sqrt{\vec{q}^2 + m_2^2} - |\vec{k}| |\vec{q}| \cos \varphi > 0, \] (16b)
where we have used the definition of the scalar product \( \vec{k} \cdot \vec{q} \) and \( \varphi \) is an angle between vectors \( \vec{k} \) and \( \vec{q} \).

Let us reconsider Eq. (15). For small \( m_1, m_2 \) we have:
\[ m^2 = 2 \frac{\vec{k}^2}{|\vec{k}|} |\vec{q}| (1 - \cos \varphi). \] (17)

Imposing the transverse momentum conservation as in (16) or, for negligible \( k_3, q_3 \), we get the expression for the classical transverse mass:
\[ m_\perp^2 = 2 \frac{\vec{k}_\perp^2}{|\vec{k}_\perp|} |\vec{q}_\perp| (1 - \cos \varphi), \] (18)
and this is exactly the equation 1 proposed in Ref. [16] to determine the mass of the \( W \) boson from the kinematic distribution of the decay leptons [2,16] (note that the neutrino from the \( W \) boson is not directly detectable and longitudinal momentum balance cannot be imposed because of the geometry of the experiment [2], therefore the Eq. (18) is needed).

In conclusion, we identify the Hagen-Hurley spin one particle as the \( W \) boson.

5 Summary

We have demonstrated that the Hagen-Hurley equations describe the decay of a spin-1 boson into two massive spin-\( \frac{1}{2} \) fermions. The Hagen-Hurley boson is unstable, as manifested by inequalities (16). Instability of the boson is enhanced by the presence in the coupled Dirac equations (3) of non-standard higher-order spinors \( \zeta_{AB}, \chi_{CD} \), rather than Dirac bispinors \((\xi_A, \eta_B)^T\). Moreover, in our formalism, the classical kinematic condition for the transverse mass (18) appears naturally.

Summing up, we think that the Hagen-Hurley equations in spinor form (11) or in alternative 7 x 7 matrix formulation [6] can describe the extremely unstable intermediate vector \( W \) boson. Moreover, the decay of \( W^+ \) into two massive spin-\( \frac{1}{2} \) fermions \((W^+ \rightarrow l^+\nu, W^+ \rightarrow \text{hadrons})\) as can be described by our formalism (with massive neutrinos), amounts to almost 100% of all decay modes of the \( W \) boson [1]. It is interesting, that our approach, based on the de Broglie method of fusion [14,15], also describes the kinematics of the decay.

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