Analytic solution for grand confluent hypergeometric function

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Abstract

In Ref.[21] I construct an approximative solution of the power series expansion in closed forms of Grand Confluent Hypergeometric (GCH) function only up to one term of $A_n$’s. And I obtain normalized constants and orthogonal relations of GCH function.

In this paper I will apply three term recurrence formula (3TRF) [22] to the power series expansion in closed forms of GCH function (for infinite series and polynomial which makes $B_n$ term terminated) including all higher terms of $A_n$’s.

In general most of well-known special function with two recursive coefficients only has one eigenvalue for the polynomial case. However this new function with three recursive coefficients has infinite eigenvalues that make $B_n$’s term terminated at specific value of index $n$ because of 3TRF [22].

This paper is 9th out of 10 in series “Special functions and three term recurrence formula (3TRF)”. See section 6 for all the papers in the series. The previous paper in series deals with generating functions of Lame polynomial in the Weierstrass form[28]. The next paper in the series describes the integral formalism and the generating function of GCH function[30].

Keywords: Biconfluent Heun Equation, Three term recurrence formula, Asymptotic expansion

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1. Introduction

Biconfluent Heun (BCH) function, a confluent form of Heun function[1, 2], is the special case of Grand Confluent Hypergeometric (GCH)[21]: this has a regular singularity at $x = 0$, and an irregular singularity at $\infty$ of rank 2. For example, BCH function is included in the radial Schrödinger equation with rotating harmonic oscillator and a class of confinement potentials: recently it’s started to appear in theoretical modern physics [3, 5, 6, 7, 8].

In [23, 24], I construct the power series expansion in closed form and its integral representation of Heun function by applying 3TRF. Heun function is applicable to diverse areas such as...
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theory of black holes, lattice systems in statistical mechanics, addition of three quantum spins, solutions of the Schrödinger equation of quantum mechanics. [7, 9, 10, 11]

In Ref.[22] I show an analytic solution of GCH function only up to one term of $A_n$’s. In this paper I construct the power series expansion of GCH equation in closed forms and asymptotic behaviors including all higher terms of $A_n$’s by applying 3TRF [22].

$$x \frac{d^2y}{dx^2} + \left( \mu x^2 + \varepsilon x + \nu \right) \frac{dy}{dx} + (\Omega x + \varepsilon \omega) y = 0 \quad (1)$$

(1) is a Grand Confluent Hypergeometric (GCH) differential equation where $\mu$, $\varepsilon$, $\nu$, $\Omega$ and $\omega$ are real or imaginary parameters.[21] It has a regular singularity at the origin and an irregular singularity at the infinity. Biconfluent Heun Equation is derived, the special case of GCH equation, by putting coefficients $\mu = 1$ and $\omega = -q/\varepsilon$.[20]

$y(x)$ has a series expansion of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+1} \quad (2)$$

where $\lambda$ is an indicial root. Plug (2) into (1).

$$c_{n+1} = A_n \cdot c_n + B_n \cdot c_{n-1} \quad ; n \geq 1 \quad (3)$$

where,

$$A_n = -\frac{\varepsilon(n + \omega + \lambda)}{(n + 1 + \lambda)(n + \nu + \lambda)} \quad (4a)$$

$$B_n = -\frac{\Omega + \mu(n - 1 + \lambda)}{(n + 1 + \lambda)(n + \nu + \lambda)} \quad (4b)$$

$$c_1 = A_0 \cdot c_0 \quad (4c)$$

We have two indicial roots which are $\lambda = 0$ and $1 - \nu$

2. Power series

2.1. Polynomial which makes $B_n$ term terminated

**Theorem 1.** In Ref.[22], the general expression of power series of $y(x)$ for polynomial which makes $B_n$ term terminated is defined by

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+1} = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots$$

$$= c_0 \left\{ \sum_{i_0=0}^{\beta_0} \left( \prod_{i_1=0}^{\beta_{i_1}-1} B_{2i_1+1} \right) x^{2i_0+1} \right\} + \sum_{k=0}^{\beta_2} \left\{ A_{2k} \prod_{i_1=0}^{\beta_{i_1}-1} B_{2i_1+1} \left( \prod_{i_2=0}^{\beta_{i_2}-1} B_{2i_2+2} \right) x^{2i_0+1} \right\} + \sum_{N=2}^{\infty} \left\{ A_{2N-1} \prod_{i_1=0}^{\beta_{i_1}-1} B_{2i_1+1} \left( \prod_{i_2=0}^{\beta_{i_2}-1} B_{2i_2+2} \right) \left( \sum_{k=0}^{\beta_N} A_{2k+1} \prod_{i_2=0}^{\beta_{i_2}-1} B_{2i_2+2} \right) \right\} \times \prod_{i_2=0}^{\beta_N} \left( \prod_{i_3=0}^{\beta_{i_3}-1} B_{2i_3+3} \right) \left( \prod_{i_4=0}^{\beta_{i_4}-1} B_{2i_4+4} \right) \prod_{i_5=0}^{\beta_{i_5}-1} B_{2i_5+5} \left( \prod_{i_6=0}^{\beta_{i_6}-1} B_{2i_6+6} \right) \right\} \quad (5)
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For a polynomial, we need a condition:

\[ B_{2\beta_i+1} = 0 \quad \text{where } i, \beta_i = 0, 1, 2, \cdots \quad (6) \]

In this paper Pochhammer symbol \((x)_n\) is used to represent the rising factorial: \((x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}\). On the above \(\beta_i\) is an eigenvalue that makes \(B_{\beta_i}\) term terminated at certain value of index \(n\). (6) makes each \(y_i(x)\) where \(i = 0, 1, 2, \cdots\) as the polynomial in (5). Substitute (4a)-(4c) into (5) by using (6). The general expression of power series of GCH equation for polynomial which makes \(B_{\beta_i}\) term terminated is given by

\[
y(x) = c_0 x^\lambda \left\{ \sum_{k_0=0}^{\beta_0} \frac{(-\beta_0)_{k_0}}{(1 + \frac{1}{2})_{k_0}(\gamma + \frac{1}{2})_{k_0}} z^{k_0} + \left\{ \sum_{k_{i-1}=0}^{\beta_{i-1}} \frac{(i_0 + \frac{1}{2} + \frac{\gamma}{2})}{(i_0 + \frac{1}{2} + \frac{\gamma}{2})(i_0 - \frac{1}{2} + \gamma + \frac{1}{2}) (1 + \frac{1}{2})_{k_{i-1}}(\gamma + \frac{1}{2})_{k_{i-1}}} \right. \right. \\
\times \sum_{k_i=0}^{\beta_i} \frac{(-\beta_i)_{k_i}(\frac{1}{2} + \frac{1}{2})_{k_i}(\gamma + \frac{1}{2})_{k_i}}{(\frac{1}{2} + \frac{1}{2})_{k_i}(1 + \frac{1}{2})_{k_i}(\gamma + \frac{1}{2})_{k_i}} z^{k_i} \\
\left. \left. + \sum_{n=2}^{\infty} \left( \sum_{k_0=0}^{\beta_0} \frac{(i_0 + \frac{1}{2} + \frac{\gamma}{2})}{(i_0 + \frac{1}{2} + \frac{\gamma}{2})(i_0 - \frac{1}{2} + \gamma + \frac{1}{2}) (1 + \frac{1}{2})_{k_0}(\gamma + \frac{1}{2})_{k_0}} \right) \right. \right. \\
\times \sum_{n=2}^{\infty} \left( \sum_{k_{i-1}=0}^{\beta_{i-1}} \frac{(-\beta_{i-1})(1 + \frac{1}{2} + \frac{\gamma}{2})_{k_{i-1}}(\gamma + \frac{1}{2})_{k_{i-1}}}{(1 + \frac{1}{2} + \frac{\gamma}{2})_{k_{i-1}}(\gamma + \frac{1}{2})_{k_{i-1}}} \right) \right. \\
\left. \left. \times \sum_{k_i=0}^{\beta_i} \frac{(-\beta_i)_{k_i}(\frac{1}{2} + \frac{1}{2})_{k_i}(\gamma + \frac{1}{2})_{k_i}}{(\frac{1}{2} + \frac{1}{2})_{k_i}(1 + \frac{1}{2})_{k_i}(\gamma + \frac{1}{2})_{k_i}} z^{k_i} \right. \right. \right. \\
\left. \left. \right. \right. \right. \right. \right. \left. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right.

where

\[
\begin{align*}
\zeta &= -\frac{1}{4} \mu x^2 \\
\bar{\delta} &= -\frac{1}{2} \bar{\varepsilon} x \\
\gamma &= \frac{1}{2} (1 + \nu) \\
\Omega &= -\mu (2 \beta_i + i + \lambda) \quad \text{as } i = 0, 1, 2, \cdots \quad \text{and } \beta_i = 0, 1, 2, \cdots \\
as \beta_i \leq \beta_j \quad \text{only if } i \leq j
\end{align*}
\]

Put \(c_0 = \frac{\Gamma(1+\nu)\beta_i}{\Gamma(1)}\) as \(\lambda=0\) in (7).

**Remark 1.** The power series expansion of GCH equation of the first kind for polynomial which
Remark 2. The power series expansion of GCH equation of the second kind for polynomial which makes $B_n$ term terminated about $x = 0$ as $\Omega = -2\mu(\beta_i + \frac{1}{2})$ where $i, \beta_i = 0, 1, 2, \cdots$ is

$$y(x) = QW_{i\mu} \left( \beta_i = \frac{-\Omega}{2\mu} - \frac{i}{2}, \omega, \gamma = \frac{1}{2}(1 + \nu); \bar{e} = \frac{-1}{2}e x; z = \frac{-1}{2}e x^2 \right)$$

$$= \frac{\Gamma(y + \beta_n)}{\Gamma(y)} \sum_{n=0}^{\beta_n} \frac{(-\beta_0)_n}{(1)_n(y)_n} z^n + \sum_{n=0}^{\beta_n} \frac{(-\mu + \frac{\gamma}{2})_{n-1}}{(1)_n(y)_n} \left( \sum_{i=0}^{\beta_n} \frac{(-\mu + \frac{\gamma}{2})_{n-1}}{(1)_n(y)_n} \right)$$

$$\times \sum_{i=0}^{\beta_n} \frac{(-\beta_1)_i \left( \frac{1}{2} \right)_i (y + \frac{1}{2})_i}{(1)_i(y + \frac{1}{2})_i} \bar{e} + \sum_{n=2}^{\beta_n} \frac{(-\mu + \frac{\gamma}{2})_{n-1}}{(1)_n(y)_n} \left( \sum_{i=0}^{\beta_n} \frac{(-\mu + \frac{\gamma}{2})_{n-1}}{(1)_n(y)_n} \right)$$

$$\times \prod_{k=1}^{n-1} \left\{ \frac{(-\beta_1)_k \left( \frac{1}{2} \right)_k (y + \frac{1}{2})_k}{(1)_k(y + \frac{1}{2})_k} \bar{e} + \sum_{i=0}^{\beta_n} \frac{(-\mu + \frac{\gamma}{2})_{n-1}}{(1)_n(y)_n} \left( \sum_{i=0}^{\beta_n} \frac{(-\mu + \frac{\gamma}{2})_{n-1}}{(1)_n(y)_n} \right) \right\} \bar{e}^k$$

put $c_0 = \left( \frac{-1}{2\mu} \right)^{1-\gamma} \frac{\Gamma(\mu + 2 - \gamma)}{\Gamma(2 - \gamma)}$ as $\lambda = 1 - \nu = 2(1 - \gamma)$ in (7) with replacing $\beta_i$ by $\psi_i$.

Remark 2. The power series expansion of GCH equation of the second kind for polynomial which makes $B_n$ term terminated about $x = 0$ as $\Omega = -2\mu(\psi_i + 1 - \gamma + \frac{1}{2})$ where $i, \psi_i = 0, 1, 2, \cdots$ is

$$y(x) = RW_{i\mu} \left( \psi_i = \frac{\Omega}{2\mu} + 1 - \frac{i}{2}, \omega, \gamma = \frac{1}{2}(1 + \nu); \bar{e} = \frac{-1}{2}e x; z = \frac{-1}{2}e x^2 \right)$$

$$= z^{1-\gamma} \frac{\Gamma(\psi_0 + 2 - \gamma)}{\Gamma(2 - \gamma)} \sum_{n=0}^{\psi_n} \frac{(-\psi_0)_n}{(1)_n(2 - \gamma)_n} z^n$$

$$+ \sum_{n=0}^{\psi_n} \frac{(i_0 + 1 - \gamma + \frac{1}{2})_{n-1}}{(1)_n(2 - \gamma)_n} \left( \sum_{i=0}^{\psi_n} \frac{(-\psi_1)_i \left( \frac{1}{2} \right)_i (y + \frac{1}{2})_i}{(1)_i(y + \frac{1}{2})_i} \right) \bar{e}$$

$$+ \sum_{n=2}^{\psi_n} \frac{(i_0 + 1 - \gamma + \frac{1}{2})_{n-1}}{(1)_n(2 - \gamma)_n} \left( \sum_{i=0}^{\psi_n} \frac{(-\psi_1)_i \left( \frac{1}{2} \right)_i (y + \frac{1}{2})_i}{(1)_i(y + \frac{1}{2})_i} \right) \bar{e}^k$$

$$\times \prod_{k=1}^{n-1} \left\{ \frac{(i_0 + 1 - \gamma + \frac{1}{2})_{n-1}}{(1)_n(2 - \gamma)_n} \left( \sum_{i=0}^{\psi_n} \frac{(-\psi_1)_i \left( \frac{1}{2} \right)_i (y + \frac{1}{2})_i}{(1)_i(y + \frac{1}{2})_i} \right) \right\} \bar{e}^k$$

$$\times \prod_{i=0}^{\psi_n} \frac{(-\psi_1)_i \left( \frac{1}{2} \right)_i (2 - \gamma + \frac{1}{2})_i}{(1)_i(2 - \gamma + \frac{1}{2})_i} z^n \bar{e}^k$$
2.2. Infinite series

**Theorem 2.** In Ref. [22], the general expression of power series of $y(x)$ for infinite series is defined by

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots$$

$$= c_0 \left( \sum_{l_0=0}^{\infty} \prod_{l_1=0}^{l_0-1} B_{2l_1+1} \right) x^{2l_0+1} + \sum_{l_0=0}^{\infty} c_{l_0} \left( \sum_{l_1=0}^{l_0-1} \prod_{l_1=0}^{l_0-1} B_{2l_1+1} \right) x^{2l_0+1}$$

$$+ \sum_{N=2}^{\infty} \left\{ \sum_{i=0}^{N-1} \left( A_{2i} \prod_{l_1=0}^{i-1} B_{2l_1+1} \right) \prod_{l_1=0}^{N-1} \sum_{l_2=2l_1+1} A_{2i} \prod_{l_1=0}^{i-1} B_{2l_1+1} \right\} x^{2i+N+1}$$

Substitute (4a)-(4c) into (9). The general expression of power series of GCH equation for infinite series is given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \cdots$$

$$= c_0 x^l \left\{ \sum_{i=0}^{\infty} \left( \frac{\gamma + 1 + \frac{1}{2} \lambda}{\gamma + \frac{1}{2}} \right) \sum_{l_1=0}^{i-1} A_{2i} \prod_{l_1=0}^{i-1} B_{2l_1+1} \right\} x^{2i}$$

$$+ \sum_{n=2}^{\infty} \left\{ \sum_{l_0=0}^{\infty} \left( \frac{\gamma + 1 + \frac{1}{2} \lambda}{\gamma + \frac{1}{2}} \right) \sum_{i=0}^{n-1} \prod_{l_1=0}^{i-1} A_{2i} \prod_{l_1=0}^{i-1} B_{2l_1+1} \right\} x^{n}$$

Put $c_0 = \frac{\Gamma(\gamma - \frac{1}{2})}{\Gamma(\gamma + \frac{1}{2})}$ as $\lambda = 0$ for the first independent solution of GCH equation and $c_0 = \left( -\frac{1}{4} \mu \right)^{1-\gamma} \frac{\Gamma(1-\frac{1}{2})}{\Gamma(1+\frac{1}{2})}$ as $\lambda = 1 - \gamma = 2(1 - \gamma)$ for the second one in (10).

**Remark 3.** The power series expansion of GCH equation of the first kind for infinite series about
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\[ x = 0 \] is

\[ y(x) = QW \left( \omega, \gamma = \frac{1}{2}(1 + \nu); \bar{\varepsilon} = -\frac{1}{\nu}x; z = -\frac{1}{2}\mu x^2 \right) \]

\[ = \frac{\Gamma(\gamma - \frac{\omega}{2})}{\Gamma(\gamma)} \left\{ \sum_{n=0}^{\infty} \frac{(\frac{\beta}{2})_n}{(1)_n(\gamma)_n} e^{\beta n} + \left\{ \sum_{n=0}^{\infty} \frac{(i_0 + \frac{\nu}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{1}{2} + \gamma)} (1)^n(\gamma)_n \right\} \right\} \]

Remark 4. The power series expansion of GCH equation of the second kind for infinite series about \( x = 0 \) is

\[ y(x) = R\left( \omega, \gamma = \frac{1}{2}(1 + \nu); \bar{\varepsilon} = -\frac{1}{\nu}x; z = -\frac{1}{2}\mu x^2 \right) \]

\[ = z^{1-\gamma} \frac{\Gamma(1 - \frac{\omega}{2})}{\Gamma(2 - \gamma)} \left\{ \sum_{n=0}^{\infty} \frac{(\frac{\beta}{2} + 1 - \gamma)_n}{(1)_n(2 - \gamma)_n} z^n + \left\{ \sum_{n=0}^{\infty} \frac{(i_0 + 1 - \gamma + \frac{\nu}{2})}{(i_0 + \frac{1}{2})(i_0 + \frac{1}{2} - \gamma)} \right\} \right\} \]

When \( \nu \) is integer, one of two solution of the GCH equation does not have any meaning, because \( RW_{\beta} \left( \omega = -\frac{\beta}{2} + \gamma - 1 - \frac{\nu}{2}, \gamma = \frac{1}{2}(1 + \nu); \bar{\varepsilon}; z \right) \) can be described as \( QW_{\beta} \left( \beta = -\frac{\beta}{2}, \gamma = \frac{1}{2}(1 + \nu); \bar{\varepsilon}; z \right) \) as long as \( |\lambda_1 - \lambda_2| = |\nu - 1| \) is integer. As we see remarks 1–4, it is required that \( \nu \neq 0, -1, -2, \cdots \) for the first kind of independent solution of GCH equation for polynomial and infinite series. By similar reason, \( \nu \neq 2, 3, 4, \cdots \) is required for the second kind of independent solution of GCH equation.
3. Asymptotic behavior of the function $y(x)$ and the boundary condition for $x$

3.1. The case of $|\mu| \ll 1$ or $|\mu| \ll |\varepsilon|$

As $n \gg 1$, (4a) and (4b) are

$$\lim_{n \gg 1} A_n = A = -\frac{\varepsilon}{n}$$ (11a)

And,

$$\lim_{n \gg 1} B_n = B = -\frac{\mu}{n}$$ (11b)

Since $|\mu| \ll 1$ or $|\mu| \ll |\varepsilon|$, (11b) is negligible. Its recurrence relation is

$$c_{n+1} \approx -\frac{\varepsilon}{n} c_n$$ (12)

Plug (12) into the power series expansion where $\sum_{n=0}^{\infty} c_n x^n$, putting $c_0 = 0$ and $c_1 = 1$ for simplicity.

$$\lim_{n \gg 1} y(x) \approx x (e^{-\varepsilon x} - 1) \quad \text{where } -\infty < x < \infty$$ (13)

3.2. The case of $|\varepsilon| \ll 1$ or $|\varepsilon| \ll |\mu|$

Let assume that $|\varepsilon| \ll 1$ or $|\varepsilon| \ll |\mu|$. Then (11a) is negligible. Its recurrence relation is

$$c_{n+1} \approx -\frac{\mu}{n} c_n$$ (14)

We can classify $c_n$ as to even and odd terms in (14).

$$c_{2n} = \frac{(-\frac{1}{2})^n}{(n - \frac{1}{2})!} \left(-\frac{1}{2}\mu\right)^n c_0$$

$$c_{2n+1} = \frac{1}{n!} \left(-\frac{1}{2}\mu\right)^{1+} c_1$$

where $n \geq 1$ (15)

$c_1 = Ac_0 = 0$ in (15). Because $A$ is negligible since $|\varepsilon| \ll 1$ or $|\varepsilon| \ll |\mu|$. Put $c_{2n}$ in (15) into the power series expansion where $\sum_{n=0}^{\infty} c_n x^n$, putting $c_0 = 1$ for simplicity.

$$\lim_{n \gg 1} y(x) = 1 + \sqrt{\frac{-\pi}{2\mu x^2}} \text{Erf} \left( \sqrt{-\frac{1}{2}\mu x^2} \right) e^{-\frac{1}{2}\mu x^2}$$ (16)

where $-\infty < x < \infty$

On the above Erf(y) is an error function which is

$$\text{Erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y dt \ e^{-t^2}$$

4. Application

I show the power series expansion in closed forms and asymptotic behaviors of the GCH function in this paper. We can apply this new special function into many physics areas. I show three examples of GCH equation as follows:
4.1. The rotating harmonic oscillator

For example, there are quantum-mechanical systems whose radial Schrödinger equation may be reduced to a Biconfluent Heun function \[12, 13\], namely the rotating harmonic oscillator and a class of confinement potentials. Its radial Schrödinger equation is given by

\[
\Psi''(r) + \left(\frac{2\lambda_m + 1}{2\omega} - \frac{(r-1)^2}{4\omega^2} - \frac{l_m(l_m + 1)}{r^2}\right)\Psi(r) = 0 \tag{17}
\]

where \(0 \leq r < \infty\), \(\lambda_m\) is the eigenvalue, \(l_m\) is the rotational quantum number and \(\omega\) is a coupling parameter. By means of the changes of variable,

\[
\Psi(r) = r^{l_m+1} \exp\left(-\frac{(r-1)^2}{2\omega}\right) U(r) \quad \text{and} \quad r = \sqrt{2}\omega x \tag{18}
\]

the above becomes the following Biconfluent Heun equation:

\[
xU''(x) + (1 + \alpha - \beta x - 2x^2)U'(x) + \left\{\left(\gamma - \alpha - 2\right)x - \frac{1}{2}\left[\delta + \beta(1 + \alpha)\right]\right\} U(x) = 0 \tag{19}
\]

where the four Heun parameters are

\[
\alpha = 2l_m + 1 \quad \beta = -\sqrt{\frac{2}{\omega}} \quad \delta = 0 \quad \gamma = 1 + 2\lambda_m \tag{20}
\]

If we compare (19) with (1), all coefficients on the above are correspondent to the following way.

\[
\begin{align*}
\mu &\leftarrow -2 \\
\varepsilon &\leftarrow -\beta \\
\nu &\leftarrow 1 + \alpha \\
\Omega &\leftarrow \gamma - \alpha - 2 \\
\omega &\leftarrow \frac{1}{2\beta}\left[\delta + \beta(1 + \alpha)\right]
\end{align*} \tag{21}
\]

Put (20) in (21).

\[
\begin{align*}
\mu &\leftarrow -2 \\
\varepsilon &\leftarrow \sqrt{\frac{2}{\omega}} \\
\nu &\leftarrow 2(l_m + 1) \\
\Omega &\leftarrow 2(l_m - l_m - 1) \\
\omega &\leftarrow l_m + 1
\end{align*} \tag{22}
\]

Let’s investigate function \(\Psi(r)\) as \(n\) and \(r\) go to infinity. I assume that \(U(x)\) is infinite series in (19). Since \(s \ll 1\) in (22), put (16) in (18) with replacing \(\mu\) by \(-2\).

\[
\lim_{n \to 1} \Psi(r) \approx r^{l_m+1} \exp\left(-\frac{(r-1)^2}{2\omega}\right) \left(1 + \sqrt{\frac{\alpha}{2\omega}} \text{Erf}\left(\frac{r}{\sqrt{2\omega}}\right)r e^{\frac{r^2}{2\omega}}\right) \tag{23}
\]
In (23) if \( r \to \infty \), then \( \lim_{n=1} \Psi(r) \to \infty \). It is unacceptable that wave function \( \Psi(r) \) is divergent as \( r \) goes to infinity in the quantum mechanical point of view. Therefore the function \( U(x) \) must to be polynomial in (19) in order to make the wave function \( \Psi(r) \) being convergent even if \( r \) goes to infinity. \( RW_{\beta}(\psi, \omega, \gamma; \hat{\varepsilon} = -\frac{\omega}{2}; z = \frac{r^2}{2\omega}) \to \infty \) as \( r \to 0 \) because of \( \gamma = \lambda_m + \frac{1}{2} \) in Remark 2. But \( QW_{\beta}(\beta, \omega, \gamma; \hat{\varepsilon} = -\frac{\omega}{2}; z = \frac{r^2}{2\omega}) \to 0 \) as \( r \to 0 \) in Remark 1. So I choose Remark 1 as eigenfunction for (18). Put (22) in (18) replacing \( x \) and \( y(x) \) by \( \frac{r^2}{2\omega} \) and \( U(r) \).

\[
U(r) = QW_{\beta}(\beta, \omega, \gamma; \hat{\varepsilon} = -\frac{\omega}{2}; z = \frac{r^2}{2\omega})
\]

\[
= \frac{\Gamma(\gamma + \beta_0)}{\Gamma(\gamma)} \left( \sum_{i=0}^{\beta_0} (-\beta_0)_{\text{in}} \tilde{\psi}_n \right) + \left( \sum_{i=0}^{\beta_0} \left( \frac{i \omega}{2} \right) (i \omega)_{\text{in}} \right) \hat{\varepsilon} + \sum_{n=2}^{\infty} \left( \sum_{i=0}^{\beta_0} \left( \frac{i \omega}{2} \right) (i \omega)_{\text{in}} \right) \hat{\varepsilon}^n \right)
\]

Put (24) in (18). The wave function for the rotating harmonic oscillator is given by

\[
\Psi(r) = N r^{\lambda_m} \exp \left( \frac{(r - 1)^2}{2\omega} \right) QW_{\beta}(\beta, \omega, \gamma; \hat{\varepsilon} = -\frac{r^2}{2\omega}; z = \frac{r^2}{2\omega})
\]

\[
\lambda_m = 2 \beta_0 + \lambda_m + 1 + i \quad \text{where} \ i, \beta_0 = 0, 1, 2, \cdots
\]

N is normalized constant. Eigenvalue \( \lambda_m \) is

4.2. Confinement potentials

Following Chaudhuri and Mukherjee, there is the radial Schrödinger equation,[12, 14, 15]:

\[
\Psi(r) \left( \frac{2m}{\hbar^2} \right) \left( E + \frac{\alpha}{r} - br - cr^2 \right) - \frac{k(k + 1)}{r^2} \Psi(r) = 0
\]

with \( E \) being the energy. By means of the consecutive changes of variable

\[
\Psi(r) = r^{\lambda_m + 1} \exp \left( -\frac{1}{2} \alpha_F r^2 - \beta_F r \right) U(r) \quad \text{and} \quad x = \sqrt{\alpha_F} r
\]
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the above becomes also the following Biconfluent Heun equation:

\[ xU''(x) + (1 + \alpha - \beta x - 2x^2)U'(x) + \left\{ (\gamma - \alpha - 2)x - \frac{1}{2}[\delta + \beta(1 + \alpha)] \right\} U(x) = 0 \quad (28) \]

where the four Heun parameters are

\[ \alpha = 2l + 1, \quad \gamma = \frac{\epsilon_F}{\alpha_F}, \]
\[ \beta = 2\frac{\beta_F}{\sqrt{\alpha_F}}, \quad \delta = -4\frac{\mu}{\hbar^2} \frac{a}{\sqrt{\alpha_F}} \quad (29) \]

where,

\[ \alpha_F = \left( \frac{2\mu c}{\hbar^2} \right)^{1/2}, \quad \beta_F = \beta \left( \frac{\mu}{\hbar^2} \right)^{1/2}, \quad \epsilon_F = \beta_F^2 + \frac{2\mu}{\hbar^2}E \quad (30) \]

Put (29) and (30) in (21).

\[ \mu \rightarrow -2 \]
\[ \epsilon \rightarrow -2\frac{\beta_F}{\sqrt{\alpha_F}} \]
\[ \nu \rightarrow 2(l + 1) \]
\[ \Omega \rightarrow \frac{\epsilon_F}{\alpha_F} - 2 \left( l + \frac{3}{2} \right) = \frac{1}{\alpha_F} \left( \beta_F^2 + \frac{2\mu}{\hbar^2}E \right) - 2 \left( l + \frac{3}{2} \right) \]
\[ \omega \rightarrow -\frac{\mu a}{\hbar^2 \beta_F} + l + 1 \quad (31) \]

Let’s investigate function \( \Psi(r) \) as \( n \) and \( r \) go to infinity. I assume that \( U(x) \) is infinite series in (28). Since \(|\epsilon| \ll 1 \) in (31), put (16) in (27) with replacing \( \mu \) and \( x \) by \(-2\) and \( \sqrt{\alpha_F}r \).

\[ \lim_{n \rightarrow 1} \Psi(r) \approx r^{3l+1} \exp \left( \frac{\alpha_F}{2} r^2 - \beta_F r \right) \left( 1 + \sqrt{\alpha_F} \text{Erf} \left( \sqrt{\alpha_F}r \right) r e^{\alpha_F r^2} \right) \quad (32) \]

In (32) if \( r \rightarrow \infty \), then \( \lim_{n \rightarrow 1} \Psi(r) \rightarrow \infty \). It is unacceptable that wave function \( \Psi(r) \) is divergent as \( r \) goes to infinity in the quantum mechanical point of view. Therefore the function \( U(x) \) must to be polynomial in (28) in order to make the wave function \( \Psi(r) \) being convergent even if \( r \) goes to infinity. \( \text{RW}_\beta \left( \beta, \omega, \gamma; \tilde{\epsilon} = -\beta_F r; z = \alpha_F r^2 \right) \rightarrow \infty \) as \( r \rightarrow 0 \) because of \( \gamma = l + \frac{3}{2} \) in Remark 2. But \( \text{QW}_\beta \left( \beta, \omega, \gamma; \tilde{\epsilon} = -\beta_F r; z = \alpha_F r^2 \right) \rightarrow 0 \) as \( r \rightarrow 0 \) in Remark 1. So I choose Remark 1.
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as eigenfunction for (27). Put (31) in Remark 1 replacing x and y(x) by $\sqrt{\alpha_F} r$ and U(r).

$$U(r) = QW_{\beta_i} \left( \beta_i = \frac{1}{4\alpha_F} \left( \beta_F^2 + \frac{2\mu}{h^2} E \right) - \frac{1}{2} \left( i + l + \frac{3}{2} \right) , \omega = -\frac{\mu a}{h^2 \beta_F} + l + 1 \right)$$

$$, \gamma = l + \frac{3}{2}; \; \xi = -\beta_F r; \; \; z = \alpha_F r^2$$

$$= \frac{\Gamma(\gamma + \beta_0)}{\Gamma(\gamma)} \left( \sum_{i=0}^{\beta_0} (-\beta_0)_i \frac{(i + \frac{\omega}{2})}{(i_0 + \frac{\omega}{2})(i_0 - \frac{1}{2} + \gamma)(1)_i(\gamma)_i} \right) \Gamma(\gamma - \beta_0)$$

$$\times \sum_{i=0}^{\beta_0} (-\beta_0)_i \frac{(i + \frac{\omega}{2})}{(i_0 + \frac{\omega}{2})(i_0 - \frac{1}{2} + \gamma)(1)_i(\gamma)_i} \times \sum_{i=0}^{\infty} \left( \frac{(-\beta_0)_i}{(i_0 + \frac{\omega}{2})(i_0 - \frac{1}{2} + \gamma)(1)_i(\gamma)_i} \right)^{i+1}$$

Put (33) in (27). The wave function for confinement potentials is given by

$$\Psi(r) = N r^{l+1} \exp \left( -\frac{1}{2} r^2 \alpha_F - \beta_F r \right) QW_{\beta_i}$$

$$, \omega = -\frac{\mu a}{h^2 \beta_F} + l + 1, \gamma = l + \frac{3}{2}; \; \xi = -\beta_F r; \; z = \alpha_F r^2$$

$$N$$ is normalized constant. Energy E is

$$E = \frac{\hbar^2}{2\mu} \left( 4\alpha_F \left( \beta_i + \frac{i + l + \frac{3}{2}}{2} \right) - \beta_F^2 \right)$$

where $i, \beta_i = 0, 1, 2, \ldots$

Again the GCH function with three recursive coefficients has infinite eigenvalues.

4.3. The spin free Hamiltonian involving only scalar potential for the $q - \bar{q}$ system

Following Gürsy and his colleagues, there is the spin free Hamiltonian involving only scalar potential for the $q - \bar{q}$ system,[16, 17, 18, 19]

$$H^2 = 4 \left( (m + \frac{1}{2} b r) + P_r^2 + \frac{l(l + 1)}{r^2} \right)$$

where $P_r = -\frac{\partial}{\partial r} - \frac{a}{r}, m= mass, b= real positive, and l= angular momentum quantum number.

When wave function $\Psi(r) = \exp \left( -\frac{l}{2} \left( r + \frac{a}{r} \right) \right) r^l y(r) y^m(\theta, \phi)$ acts on both sides of (35), it becomes

$$i \frac{\partial^2 \psi}{\partial r^2} + \left( -b r^2 + 2m r + 2(l + 1) \right) \frac{\partial \psi}{\partial r} + \left( \frac{E^2}{4} - b \left( l + \frac{3}{2} \right) \right) \frac{\partial \psi}{\partial y} = 0$$

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If we compare (36) with (1), all coefficients on the above are correspondent to the following way.

\[
\begin{align*}
\mu &\leftrightarrow -b \\
\varepsilon &\leftrightarrow -2m \\
\nu &\leftrightarrow 2(l+1) \\
\Omega &\leftrightarrow \frac{E^2}{4} - b \left( l + \frac{3}{2} \right) \\
\omega &\leftrightarrow l + 1
\end{align*}
\]  

(37)

Let’s investigate function \( \Psi(r) \) as \( n \) and \( r \) go to infinity. I assume that \( y(r) \) is infinite series in (36). Since \( \varepsilon \ll 1 \) in (37), put (16) in \( \Psi(r) = \exp \left( -\frac{\varepsilon}{4} \left( r + \frac{2m^2}{b} \right)^2 \right) r^l y(r) Y^m_\gamma(\theta, \phi) \) with replacing \( x \) and \( \mu \) by \( r \) and \(-b\).

\[
\lim_{n \to 1} \Psi(r) = r^l \exp \left( -\frac{b}{4} \left( r + \frac{2m^2}{b} \right) \right) \left\{ 1 + \sqrt{\frac{1}{2 \pi b r^2}} e^{+ib} \right\} Y^m_\gamma(\theta, \phi)
\]  

(38)

In (38) if \( r \to \infty \), then \( \lim \Psi(r) = \infty \). It is unacceptable that wave function \( \Psi(r) \) is divergent as \( r \) goes to infinity in the quantum mechanical point of view. Therefore the function \( y(r) \) must to be polynomial in (36) in order to make the wave function \( \Psi(r) \) being convergent even if \( r \) goes to infinity. \( RW_{\beta_0} \left( \psi_{l, \omega, \gamma}; \tilde{e} = mr, z = \frac{b}{2 \mu r^2} \right) \to \infty \) as \( r \to 0 \) because of \( \gamma = l + \frac{3}{2} \) in Remark 2. But \( QW_{\beta_0} \left( \beta, \omega, \gamma; \tilde{e} = mr, z = \frac{b}{2 \mu r^2} \right) \to 0 \) as \( r \to 0 \) in Remark 1. So I choose Remark 1 as eigenfunction for (36). Put (37) in Remark 1 with replacing \( x \) by \( r \).

\[
y(r) = QW_{\beta_0} \left( \beta, \frac{1}{2} \left( \frac{E^2}{4b} - \left( l + \frac{3}{2} \right) \right), \omega = l + 1; \gamma = l + \frac{3}{2}; \tilde{e} = mr; z = \frac{b}{2 \mu r^2} \right)
\]

(39)

Put (39) in \( \Psi(r) = \exp \left( -\frac{\varepsilon}{4} \left( r + \frac{2m^2}{b} \right)^2 \right) r^l y(r) Y^m_\gamma(\theta, \phi) \). The wave function for the spin free Hamiltonian involving only scalar potential for the \( q - \bar{q} \) system is given by

\[
\Psi(r) = N^l \exp \left( -\frac{b}{4} \left( r + \frac{2m^2}{b} \right)^2 \right) QW_{\beta_0} \left( \beta, \frac{1}{2} \left( \frac{E^2}{4b} - \left( l + \frac{3}{2} \right) \right), \omega = l + 1; \gamma = l + \frac{3}{2} \right)
\]

(40)
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N is normalized constant. Energy $E^2$ is

$$E^2 = 4b \left( 2\beta_i + i + l + \frac{3}{2} \right)$$

where $i, \beta_i = 0, 1, 2, \cdots$

The GCH function with three recursive coefficients has infinite eigenvalues.

5. Conclusion

Any special functions with two recursive coefficients (such as Bessel, Legendre, Kummer, Laguerre, hypergeometric, Coulomb wave function, etc) only have one eigenvalue for the polynomial case. However the GCH function with three recursive coefficients has infinite eigenvalues that make $B_n$'s term terminated as we see (25), (34) and (40).

I show the power series expansion in closed forms of the GCH function in this paper. As we see analytic power series expansion of the GCH function by applying 3TRF [22], denominators and numerators in all $B_n$ terms arise with Pochhammer symbol: the meaning of this is that the analytic solutions of GCH equation with three recursive coefficients can be described as hypergeometric function in a strict mathematical way. Since this function is described as hypergeometric function, we can transform this function to other well-known special functions having two term recurrence relation: understanding the connection between other special functions is important in the mathematical and physical points of views as we all know.

In my next paper I derive the integral representation of GCH equation including all higher terms of $A_n$s by applying 3TRF [22]. From integral forms of the GCH function, we can investigate how this function is associated with other well known special functions such as Bessel, Laguerre, Kummer, hypergeometric functions, etc. And I show the generating function for the GCH polynomial. The generating function is really useful in order to derive orthogonal relations, recursion relations and expectation values of any physical quantities as we all recognize; i.e. the normalized wave function of hydrogen-like atoms and expectation values of its physical quantities such as position and momentum.

6. Series “Special functions and three term recurrence formula (3TRF)”

This paper is 9th out of 10.

1. “Approximative solution of the spin free Hamiltonian involving only scalar potential for the $q - \bar{q}$ system” [21] - In order to solve the spin-free Hamiltonian with light quark masses we are led to develop a totally new kind of special function theory in mathematics that generalize all existing theories of confluent hypergeometric types. We call it the Grand Confluent Hypergeometric Function. Our new solution produces previously unknown extra hidden quantum numbers relevant for description of supersymmetry and for generating new mass formulas.

2. “Generalization of the three-term recurrence formula and its applications” [22] - Generalize three term recurrence formula in linear differential equation. Obtain the exact solution of the three term recurrence for polynomials and infinite series.
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3. “The analytic solution for the power series expansion of Heun function” [23] - Apply three term recurrence formula to the power series expansion in closed forms of Heun function (infinite series and polynomials) including all higher terms of $A_n$'s.

4. “Asymptotic behavior of Heun function and its integral formalism”, [24] - Apply three term recurrence formula, derive the integral formalism, and analyze the asymptotic behavior of Heun function (including all higher terms of $A_n$).

5. “The power series expansion of Mathieu function and its integral formalism”, [25] - Apply three term recurrence formula, analyze the power series expansion of Mathieu function and its integral forms.

6. “Lame equation in the algebraic form” [26] - Applying three term recurrence formula, analyze the power series expansion of Lame function in the algebraic form and its integral forms.

7. “Power series and integral forms of Lame equation in Weierstrass’s form and its asymptotic behaviors” [27] - Applying three term recurrence formula, derive the power series expansion of Lame function in Weierstrass’s form and its integral forms.

8. “The generating functions of Lame equation in Weierstrass’s form” [28] - Derive the generating functions of Lame function in Weierstrass’s form (including all higher terms of $A_n$’s). Apply integral forms of Lame functions in Weierstrass’s form.

9. “Analytic solution for grand confluent hypergeometric function” [29] - Apply three term recurrence formula, and formulate the exact analytic solution of grand confluent hypergeometric function (including all higher terms of $A_n$’s). Replacing $\mu$ and $\epsilon\omega$ by 1 and $-q$, transforms the grand confluent hypergeometric function into Biconfluent Heun function.

10. “The integral formalism and the generating function of grand confluent hypergeometric function” [30] - Apply three term recurrence formula, and construct an integral formalism and a generating function of grand confluent hypergeometric function (including all higher terms of $A_n$’s).

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