Universal description of $S$-wave meson spectra in a renormalized light-cone QCD-inspired model

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Abstract

A light-cone QCD-inspired model, with the mass squared operator consisting of a harmonic oscillator potential as confinement and a Dirac-delta interaction, is used to study the $S$-wave meson spectra. The two parameters of the harmonic potential and quark masses are fixed by masses of $\rho(770)$, $\rho(1450)$, $J/\psi$, $\psi(2S)$, $K^*(892)$ and $B^*$. We apply a renormalization method to define the model, in which the pseudo-scalar ground state mass fixes the renormalized strength of the Dirac-delta interaction. The model presents an universal and satisfactory description of both singlet and triplet states of $S$-wave mesons and the corresponding radial excitations.

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I. INTRODUCTION

In the effective light-cone QCD theory [1,2] the lowest Fock component of the hadron wave function is an eigenfunction of an effective mass squared operator with constituent quark degrees of freedom and parameterized in terms of an interaction which contains a Coulomb-like potential and a Dirac-delta term. The Fock-state components of the hadron light-front wave function can be constructed recursively from the lowest Fock-state component. The interaction in the mass operator comes from an effective one-gluon-exchange where the Dirac-delta term corresponds to the hyperfine interaction. The masses of the ground state of the pseudo-scalar mesons and in particular the pion structure [3] were described reasonably, with a small number of free parameters, which is only the canonical number plus one — the renormalized strength of the Dirac-delta interaction.
The model was extended to include the confining interaction and used to study the splitting of the excited pseudo-scalar states from the excited $^3S_1$ vector meson states as a function of the ground state pseudo-scalar mass [4]. In Ref. [4], the Coulomb-like and the confining interactions were substituted by a harmonic oscillator potential, which allowed an analytic formulation. The parameters of the confining interaction in the mass squared operator, were fitted to the $^3S_1$-meson ground state mass and to the slope of the trajectory of excited states with the radial quantum number [5]. With the renormalized strength of the Dirac-delta interaction fixed by pseudo-scalar masses, it was shown that the $\pi$-$\rho$ mass splitting, due to the attractive Dirac-delta interaction, is the source of the splitting between the masses of the excited states. A reasonable agreement with the data [6] was obtained. Besides, in light-cone framework (mass squared operator appears in the Hamiltonian), the Dirac-delta plus confining harmonic oscillator potential gave a natural explanation of the observation of the almost linear relationship between the mass squared of excited states with radial quantum number $n$ [5]. This reveals some of the physics that are brought by the work of Ref. [5] and showed the relation between the $\pi$ and $\rho$ spectrum, through the pion mass scale which defines the renormalization condition of the model.

In the present paper, this simple model, with the Dirac-delta interaction acting in the $^1S_0$ channel only and harmonic oscillator potential as confinement, is used to investigate the $S$-wave meson spectra from $\pi$-$\rho$ to $\eta_b$-$\Upsilon$ and make predictions for $\eta$-$\Theta$ (we do not study $\eta$-$\omega$ and $\eta'$-$\phi$). Instead of using flavor-dependent parameters, parameters are found for the harmonic oscillator potential which is valid for not only light mesons but also heavy mesons. In other words, the parameters are universal. It is shown that the linear relationship between the mass squared of excited states with radial quantum number is still qualitatively valid even for heavy mesons like $\Upsilon$. The simply model presents reasonable agreement with available data and/or with the meson mass spectra given by Godfrey and Isgur [7].

This paper is organized as follows. In section II, we give very briefly the extension of the light-cone QCD-inspired theory for which the mass squared operator of a constituent quark-antiquark system includes confinement. In the effective theory the bare mass operator equation for the lowest Light-Front Fock-state component of a bound system of a constituent quark and antiquark of masses $m_1$ and $m_2$, is described as [1,2]

\[ M^2 \psi(x, k_\perp) = \left[ \frac{k_\perp^2 + m_1^2}{x} + \frac{k_\perp^2 + m_2^2}{1 - x} \right] \psi(x, k_\perp) \]
\[- \int \frac{d^4x'd^4k'_\perp \theta(x')\theta(1-x')}{\sqrt{x(1-x)x'(1-x')}} \left( \frac{4m_1m_2}{3\pi^2} \frac{\alpha}{Q^2} - \lambda - W_{\text{conf}}(Q^2) \right) \psi(x', \vec{k}'_\perp) , \tag{1}\]

where \( M \) is the mass of the bound-state and \( \psi \) is the projection of the light-front wavefunction in the quark-antiquark Fock-state. The confining interaction is included in the model by \( W_{\text{conf}}(Q^2) \). The momentum transfer \( Q \) is the space-part of the four momentum transfer and the strength of the Coulomb-like potential is \( \alpha \). The singular interaction is active only in the pseudo-scalar meson channel with \( \lambda \) as the bare coupling constant.

For convenience the mass operator equation is transformed to the instant form representation [9], which in operatorial form is written as [4]

\[
(M_0^2 + V + V^2 + W_{\text{conf}}) |\varphi\rangle = M^2 |\varphi\rangle , \tag{2}\]

where the free mass operator, \( M_0 \) (= \( E_1 + E_2 \)), is the sum of the energies of quark 1 and 2 \( (E_i = \sqrt{m_i^2 + k^2}, i=1, 2 \text{ and } k \equiv |\vec{k}|) \), \( V \) is the Coulomb-like potential, \( V^2 \) is the short-range singular interaction and \( W_{\text{conf}} \) gives the quark confinement.

We simplify Eq. (2) by omitting the Coulomb term to the form [4]

\[
(M_{\text{ho}}^2 + g\delta(r)) \varphi(r) = M^2\varphi(r) , \tag{3}\]

where the bare strength of the Dirac-delta interaction is \( g \), and the mass squared operator is [9]

\[
M_{\text{ho}}^2 = (C(k)k^2 + m_s^2) + 2m_sv(r) , \tag{4}\]

in units of \( \hbar = c = 1 \), \( m_s = m_1 + m_2 \). The dimensionless factor of \( k \) is

\[
C(k) = 2 + \frac{E_1 + m_1}{E_2 + m_2} + \frac{E_2 + m_2}{E_1 + m_1} . \tag{5}\]

In the following, we approximate \( C(k) \) as \( m_s/m_r \) [9], with \( m_r = m_1m_2/(m_1 + m_2) \).

The harmonic oscillator potential is introduced as a confinement

\[
v(r) = -c_0 + \frac{1}{2}c_2r^2 , \tag{6}\]

where \( c_0 \) and \( c_2 \) are two universal parameters valid for all of the mesons. The eigenvalue Eq. (4) is given now by

\[
2m_s \left( -\frac{1}{2m_r} \nabla^2 + \frac{1}{2}c_2r^2 + \frac{1}{2}m_s - c_0 \right) \Psi_n(r) = M_n^2 \Psi_n(r) , \tag{7}\]

with \( \Psi_n(r) \) the eigenstate of the harmonic oscillator potential and the corresponding eigenvalue

\[
M_n^2 = 2m_s \left( \left( 2n + \frac{3}{2} \right) \sqrt{c_2/m_r} + \frac{1}{2}m_s - c_0 \right) = nw + m_s^2 + 3m_s\sqrt{c_2/m_r} - 2m_sc_0 , \tag{8}\]

where \( n \) (0, 1, 2, \cdots) is the radial quantum number and \( w = 4m_s\sqrt{c_2/m_r} \). Note that here \( n \) begins from 0 for convenience, while in discussions in Section III, it begins from 1 to keep
accordance with literature. In the present model, Eq. (8) gives the vector meson spectrum since Dirac-delta interaction acts only on pseudo-scalar mesons. The mass squared of the ground state \((n = 0)\) of the \(^3S_1\) meson can be written as

\[
M_{gs}^2 = m^2_s + 3m_s \sqrt{c_2/m_r - 2m_sc_0}.
\]

With the subtraction point, \(\mu\), taken to be the mass of the pseudo-scalar meson ground state, the reduced \(T\) matrix was derived in Ref. [4] as

\[
t_{R}^{-1}(M^2) = \left(\frac{2\pi}{\mu}\right)^3 \sum_n \frac{1}{M_n^2 - M_{gs}^2} \left(\frac{1}{\mu^2 - M_n^2} - \frac{1}{M^2 - M_n^2}\right), (10)
\]

The value of the \(S\)-wave eigenfunction at the origin is given by [10]

\[
\Psi_n(0) = \alpha^{\frac{3}{2}} \left[\frac{2^{2-n} n!!}{\sqrt{\pi} n!}\right]^\frac{1}{2}, (11)
\]

where \(\alpha^{-1}\) is the oscillator length. The final form the reduced \(T\) matrix is

\[
t_{R}^{-1}(M^2) = (2\pi\alpha)^3 \sum_{n=0}^{\infty} \frac{2^{2-n} (2n + 1)!!}{\sqrt{\pi} n!} \left(\frac{1}{\mu^2 - nw - M_{gs}^2} - \frac{1}{M^2 - nw - M_{gs}^2}\right). (12)
\]

The zeros of Eq. (12) gives the eigenvalues of the the mass squared operator of Eq. (3). In Ref. [4], \(\mu\) was changed continuously to study the splitting between the \(^1S_0\) and \(^3S_1\) spectrum of the \(\pi - \rho\) mesons. In the following section, we are going to discuss this splitting in more detail and apply the model to mesons with heavy quarks as well.

### III. RESULTS AND DISCUSSIONS

#### A. Parameters and nomenclature

The masses of \(\rho(770), \rho(1450), J/\psi(1S)\) and \(\psi(2S)\) [6] and Eqs. (8, 9) are used to fix \(c_0\), \(c_2\) and up, down and charm quark masses with the assumption of \(m_u = m_d\). Then masses of strange and bottom quarks are determined by the masses of \(K^*\) and \(B^*\) [6] and Eq. (9). In order to predict \(t\)-quark meson spectrum, we adopt an estimate from Godfrey and Isgur [7], \(m_t = 35\) GeV. The parameters used in the present model are listed in Table I.

The physical nomenclature of mesons are given in Table II to facilitate the following discussion. Since we assume that up and down quarks are the same, and since mixing between different flavors can not be dealt with within this simple model, we investigate only \(I = 1\) states among the diagonal meson sectors containing \(u, d\) and \(s\) quark. \(m_u = m_d\) also means that in our model the spectrum of \(\pi^0\) (\(\rho^0, K^0, K^{0*}, D^0, D^{0*}, B^0, B^{0*}, T^0, T^{0*}\)) is same as that of \(\pi^\pm\) (\(\rho^\pm, K^\pm, K^{\pm*}, D^\pm, D^{\pm*}, B^\pm, B^{\pm*}, T^\pm\) and \(T^{\pm*}\)). In the following discussions, we will omit the charge signs of mesons for simplicity.
B. S-wave meson spectra

The splitting of the light mesons, \( \pi - \rho \) and \( K-K^* \) spectra, due to the Dirac-delta interaction acting in the pseudo-scalar \( ^1S_0 \) channel, are studied in the previous work [4]. There the empirical slope \( (w) \) from Ref. [5] or Ref. [6] are used directly. In the present paper, \( \pi - \rho \) and \( K-K^* \) are reinvestigated in the same framework as other \( \bar{S} \)-state mesons. The results are presented in Fig. 1 and Fig. 2 respectively. Similar agreement with the data as that in Ref. [4] is found in Fig. 1, simply because the present model gives \( w = 1.55 \) GeV\(^2 \) which is very close to \( w = 1.39 \) GeV\(^2 \) used in Ref. [4]. For \( K^* \), the present model gives a large value of \( w, 1.92 \) GeV\(^2 \), (due to the fact that strange quark mass is larger than up-down quark mass, \( w \) for \( K^* \) must be larger than that for \( \rho \) from the present model), compared to that extracted from the data, 1.19 GeV\(^2 \). However, as pointed out in Refs. [5,6], there are ambiguities about the quantum number assignment of exited states of \( K \) and \( K^* \). If we follow the identification of the quark model [6] for \( K(1460) \) \((2^1S_0)\), \( K(1830) \) \((3^1S_0)\) and \( K^*(1410) \) \((2^3S_1)\), the spectra of \( K-K^* \) from present model are in good agreement with the data.

In Fig. 3 the results are shown for the \( \eta_c - \psi \) mass splitting as a function of the ground-state pseudo-scalar mass \( \mu \), which interpolates from the \( \eta_c \) to the \( J/\psi \) meson spectrum. Compared to the light \( \pi - \rho \) and \( K-K^* \) mesons, the splitting is smaller even for the ground state (\( \sim 100 \) MeV) and becomes weaker in the excited states, although the model attributes consistently smaller masses for the \( ^1S_0 \) states compared to the respective \( ^3S_1 \) ones. An excited state of \( \eta_c \) is observed without definite spin and parity assignment [6]. From our model, this state might be \( \eta_c(2S) \), which is consistent with the assignment of quark model [6]. There are many exited states for \( \psi \), such as \( \psi(3770), \psi(4040), \psi(4160), \) and \( \psi(4415) \), all of them are assigned to \( J^\pi = 1^- \). Considering that \( \psi(3770) \) is \( \psi(1^3D_1) \) [6,7], \( \psi(4040) \) or \( \psi(4160) \) seems to be \( \psi(3^3S_1) \) from the present model.

From Eqs. (8, 9) and quark masses listed in Table I, one obtains the spectrum of \( D^* \). The splitting of \( D-D^* \) and spectrum of \( D \) meson can be calculated from Eq. (10) with the renormalized strength of \( \delta \) potential fixed by the mass of corresponding pseudo-scalar ground state \( D \) and is shown in Fig. 4. One finds good agreement for \( D^*(1S) \) with the data. The predicted mass of \( D(2S) \) is about 10\% larger than the unconfirmed data.

Similarly, the present model gives \( D_s^*(1S) \) with good agreement with the data as is seen from Fig. 5 in which the spectra for \( D_s^*-D_s^* \) are presented.

As mentioned before, the bottom quark mass is fixed by mass of \( B^*(1S) \) and Eq. (9). There are no more data for \( b \)-quark mesons. The predicted spectra for \( B \) and \( B^* \) are presented in Fig. 6.

The spectrum of \( \Upsilon \) can be calculated from our model with parameters listed in Table I and is shown in Fig. 7. We should note that the agreement with the data is good, from \( \Upsilon(1S) \) to \( \Upsilon(4S) \), considering that no parameter is adjusted specially for \( \Upsilon \), the heaviest meson states observed up to now. The unconfirmed bottomonium state \( \eta_b(9300) \) is used to predict the spectrum of \( \eta_b \) and also presented in the same figure.

No confirmed data are available for the triplet states of \( B_s \) and \( B_c \). With the bottom quark mass \( m_b = 5068 \) MeV, the present model predicts the triplet ground state masses for \( B_s \) and \( B_c \), being 5342.1 MeV and 6.346 GeV, respectively. Although the predicted mass for \( B_s(1S) \) is comparable to the unconfirmed data 5416.6 MeV, both predictions are respectively
smaller than their corresponding singlet ground state masses, 536.9 MeV and 6.4 GeV. Therefore we can not calculate within our model the singlet spectra $B_s$ and $B_c$. However, for completeness and in order to give a reference to the reader, we list the calculated values for triplets of the bottom strange and bottom charged mesons in Table III and Table IV respectively.

The vector top meson spectra from the present model with the top quark mass $m_t = 35$ GeV [7] and $m_t = 175$ GeV [6] are given in Table V. The top quark mass $m_t = 35$ GeV is used in order to compare the present spectra with predictions of Ref. [7]. Qualitative agreement between the two models is found.

C. Further discussions

From the above results, it is clear that our model, with the mass squared operator consisting of a harmonic oscillator confining potential and a Dirac-delta interaction, could be used to describe universally and satisfactorily both singlet and triplet states of $S$-wave mesons as well as radial excitations. However, lattice QCD calculations predict that the quark-antiquark potential increases linearly with the distance between quark and antiquark $r$ when $r$ is large. One may ask, does this contradict our model? The answer is no. In the following, we justify roughly that the two interactions are consistent with each other for large $r$ disregarding the case for small $r$ which is not important in the present work.

In the front form of QCD, the mass squared $M^2 = m_s^2 + 2m_sE = m_s^2 + 2m_s(T_{FF} + V_{FF})$ ($m_s$ is the sum of the quark and antiquark mass, $T$ is the kinetic energy and $V_{FF}$ is the interaction) [9], while in the instant form of QCD, $M = m_s + E = m_s + (T_{IF} + V_{IF})$. The relation between the interactions $V_{FF}$ and $V_{IF}$ is found for large $r$ (where $T \ll V \ll V^2$) as $V_{FF} \sim V_{IF}^2$, from which the harmonic like potential for the mass squared operator in the front form can be derived from the linear confinement potential in the instant form of QCD.

Very recently the radial excitations of light mesons were studied in detail in the framework of the QCD string approach [12]. There the spin-averaged meson masses were calculated with a modified confining potential and the calculated slopes of the radial Regge trajectories are in agreement with Ref. [5]. In Ref. [12], the linear relation between mass squared $M^2_n$ and the radial quantum number $n$ comes mainly from properties of the approximated eigenvalues of the spinless Salpeter equation with the linear confining potential. Since in our light-cone QCD model, the harmonic oscillator potential is included in the mass squared operator, one arrives naturally at the same linear relation between $M^2_n$ and $n$ for vector mesons. This gives an explicit explanation of the radial Regge trajectories found for light vector mesons in Ref. [5]. This relation is still valid even for heavy vector mesons as shown in the previous subsection. In addition, an extension of the present model with orbital excitation included could also be used to describe the orbital Regge trajectory.

For pseudoscalar mesons, particularly for the light ones, the present model does not support the simple linear relation because the Dirac-delta interaction plays an important role now. However, this is not in contradiction with the data because light pseudoscalar mesons follow the radial Regge trajectories poorly. Let’s take the pion as an example, the slope $w = 1.67$ GeV$^2$ is derived from the masses of $\pi$ ($M = 0.14$ GeV) and its first radial excited state ($M = 1.3$ GeV), while a slope smaller by $\sim 10\%$, $w = 1.55$ GeV$^2$ can be calculated from its first and second excited states ($M = 1.8$ GeV).
IV. SUMMARY

We applied the renormalized light-cone QCD-inspired effective theory with confinement in the mass squared operator to study mesons with heavy quarks. For the confinement, the harmonic oscillator potential is used which allows an analytic solution of our model. The Coulomb-like potential is omitted in the present work while the Dirac-delta interaction is kept which acts on the singlet S-wave states and plays important role in the splitting of the singlet-triplet spectra. The two parameters of the harmonic potential and quark masses are fixed by masses of $\rho(770)$, $\rho(1450)$, $J/\psi$, $\psi(2S)$, $K^{*}(892)$ and $B^{*}$. A $T$ matrix renormalization method is used to renormalize the model, in which the pseudo-scalar ground state mass fixes the renormalized strength of the Dirac-delta interaction.

The model is applied to study the S-wave meson spectra from $\pi$-$\rho$ to $\eta_b$-$\Upsilon$ and is also used to predict top quark meson spectra. The linear relationship between the mass squared of excited states with radial quantum number [5]—the radial Regge trajectory [12]—is apparent from our model and is found to be qualitatively valid even for heavy mesons like $\Upsilon$.

The simple model presents satisfactory agreement with available data and/or with the meson mass spectra given by Godfrey and Isgur [7]. Therefore, the recently proposed extension of the light-cone QCD-inspired model which includes confinement while keeping simplicity and renormalizability, gives a reasonable picture of the spectrum of both light and heavy mesons. An extension of the present model with orbital excitations included could also be used to describe the orbital Regge trajectories.

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[12] A.M. Badalian, B.L.G. Bakker and Yu.A. Simonov, Phys. Rev. D 66, 034026 (2002).
TABLE I. Parameters used in the present paper. $c_0$ and $c_2$ of harmonic oscillator potential and masses of up, down and charm quarks are fixed from masses of $\rho(770)$, $\rho(1450)$, $J/\psi(1S)$ and $\psi(2S)$, with the assumption of $m_u = m_d$. Strange and bottom quark masses are determined by masses of $K^*$ and $B^*$. The top quark mass, $m_t = 35$ GeV, same as that in Godfrey et al. [7], and $m_t = 175$ GeV from [6], are used to predict spectra for $t$-quark mesons. The data for meson masses are taken from Hagiwara et al. [6].

| Parameters | $c_0$ [MeV] | $c_2$ [GeV$^{-2}$] | $m_u$ = $m_d$ [MeV] | $m_c$ [MeV] | $m_b$ [MeV] | $m_t$ [GeV] |
|------------|-------------|-----------------|-----------------|-------------|-------------|-------------|
| Values     | 807         | 7.13 x 10^{-2}  | 265             | 478         | 1749        | 5068        | 35/175     |

TABLE II. The physical nomenclature of the mesons as a reminder. Pseudo-scalar mesons are given on the left, vector mesons on the right of each sector. Diagonal sectors are marked with frames for guiding eye.

| $d$ | $\pi^0, \eta, \eta'$ | $K^0, \rho^0, \omega, \phi$ | $\pi^-$ | $K^0$ | $D^0, D^{*-}$ | $B^0, B^{*-}$ | $T^0, T^{*-}$ |
|-----|---------------------|-------------------------------|--------|------|-------------|---------------|--------------|
| $u$ | $\pi^+$             | $\pi^0, \eta, \eta'$        | $\rho^+$ | $K^+$ | $D^0, D^{*-}$ | $B^0, B^{*-}$ | $T^0, T^{*-}$ |
| $s$ | $K^0$               | $K^0, \eta, \eta'$          | $K^0$   | $K^*$ | $D^0, D^{*-}$ | $B^0, B^{*-}$ | $T^0, T^{*-}$ |
| $c$ | $D^0, D^{*-}$       | $D^0, D^{*-}$               | $D^0$   | $D^0$ | $D^0, D^{*-}$ | $B^0, B^{*-}$ | $T^0, T^{*-}$ |
| $b$ | $B^0$               | $B^0, B^{*-}$               | $B^0$   | $B^0$ | $B^0, B^{*-}$ | $T^0, T^{*-}$ | $T^0, T^{*-}$ |
| $t$ | $T^0$               | $T^0, T^{*-}$               | $T^0$   | $T^0$ | $T^0, T^{*-}$ | $T^0, T^{*-}$ | $T^0, T^{*-}$ |

TABLE III. The present spectra of vector bottom strange mesons $B_s^*(nS)$ compared with available data from Hagiwara et al. [6] and predictions of Godfrey et al. [7]. Note that the data for $B_s^*(1S)$ is not confirmed [7]. Masses are in MeV.

| $B_s^*$ | Ours | Data [6] | Godfrey et al. [7] |
|---------|------|----------|-------------------|
| $1^3S_1$ | 5342.2 | 5416.6 | 5450 |
| $2^3S_1$ | 6123.8 | — | 6010 |
| $3^3S_1$ | 6816.4 | — | — |
| $4^3S_1$ | 7444.7 | — | — |
| $5^3S_1$ | 8024.2 | — | — |
| $6^3S_1$ | 8564.6 | — | — |
TABLE IV. The present spectra of vector bottom charmed mesons $B_c^*(nS)$ compared with predictions of Godfrey et al. [7] (no data available). Masses are in MeV.

| $B_c^*$ | Ours  | Godfrey et al. [7] |
|---------|-------|--------------------|
| $1^3S_1$ | 6345.8 | 6340 |
| $2^3S_1$ | 6830.4 | 6890 |
| $3^3S_1$ | 7282.9 | — |
| $4^3S_1$ | 7709.0 | — |
| $5^3S_1$ | 8112.6 | — |
| $6^3S_1$ | 8497.1 | — |

TABLE V. The present spectra of vector $t$-quark mesons compared with predictions (in parenthesis) of Godfrey et al. [7]. Results with $m_t = 35$ GeV [7] and $m_t = 175$ GeV [6] are given in the first and the third row of each sector. The masses are in GeV. The spectra of $T^{*0}$ and $T^{*±}$ are the same since $m_u = m_d$ is assumed.

| Mesons | $M(1^3S_1)$ | $M(2^3S_1)$ | $M(3^3S_1)$ | $M(4^3S_1)$ | $M(5^3S_1)$ | $M(6^3S_1)$ |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|
| $T^*$  | 35.24       | 36.27       | 37.27       | 38.24       | 39.19       | 40.12       |
|        | (35.39)     |             |             |             |             |             |
|        | 175.24      | 176.27      | 177.30      | 178.33      | 179.34      | 180.36      |
| $T_s^*$| 35.25       | 36.03       | 36.79       | 37.53       | 38.26       | 38.97       |
|        | (35.46)     |             |             |             |             |             |
|        | 175.25      | 176.02      | 176.79      | 177.56      | 178.32      | 179.08      |
| $T_c^*$| 36.25       | 36.67       | 37.08       | 37.49       | 37.89       | 38.29       |
|        | (36.30)     |             |             |             |             |             |
|        | 176.25      | 176.65      | 177.06      | 177.46      | 177.87      | 178.27      |
| $T_b^*$| 39.45       | 39.71       | 39.96       | 40.21       | 40.47       | 40.72       |
|        | (39.31)     |             |             |             |             |             |
|        | 179.44      | 179.68      | 179.92      | 180.16      | 180.40      | 180.64      |
| $\theta$ | 69.29      | 69.42       | 69.54       | 69.67       | 69.80       | 69.93       |
|        | (68.70)     | (69.39)     | (69.71)     | (69.93)     | (70.10)     | (70.40)     |
|        | 349.24      | 349.29      | 349.35      | 349.41      | 349.46      | 349.52      |
FIG. 1. Mass of the excited $q\bar{q}$ states ($M^*$) as a function of the mass ($\mu$) of the pseudoscalar meson ground state for $I = 1$. The data is taken from Hagiwara et al. [6] and is shown by solid circles on the left and right of the figure for $^1S_0$ and $^3S_1$ mesons, respectively. $\rho(1700)$ (labelled with empty diamond) might be $D$-wave dominant [5,11]. Calculated $^1S_0$ and $^3S_1$ spectra are given in parentheses within the dashed lines.
FIG. 2. Mass of the excited $q\bar{q}$ states ($M^*$) as a function of the mass ($\mu$) of the pseudoscalar meson ground state for $I = 1/2$ of the strange mesons. The data is taken from Hagiwara et al. [6] and is shown by solid circles on the left and right of the figure for $^1S_0$ and $^3S_1$ mesons, respectively. Errors are not available for $K(1460)$ and $K(1830)$. There is ambiguity about the radial quantum number of $K^*(1410)$ and $K^*(1680)$ [6] (labelled with empty diamond). $K(3100)$ is not confirmed and represented by empty square. Calculated $^1S_0$ and $^3S_1$ spectra are given in parentheses within the dashed lines.
FIG. 3. Mass of the excited $q\bar{q}$ states ($M^*$) as a function of the mass ($\mu$) of the pseudoscalar meson ground state for $I = 0$ of the charmed mesons. The data is taken from Hagiwara et al. [6] and is shown by solid circles on the left and right of the figure for $^1S_0$ and $^3S_1$ mesons, respectively. No radial quantum numbers assigned to $\psi(3770), \psi(4040), \psi(4160)$ and $\psi(4415)$ (labelled with empty diamond). $\eta_c(3594)$ is not confirmed and represented by empty square. Calculated $^1S_0$ and $^3S_1$ spectra are given in parentheses within the dashed lines.
FIG. 4. Mass of the excited $q\bar{q}$ states ($M^*$) as a function of the mass ($\mu$) of the pseudoscalar meson ground state for $I = 1/2$ of the charmed mesons. The data is taken from Hagiwara et al. [6] and is shown by solid circles on the left and right of the figure for $^1S_0$ and $^3S_1$ mesons, respectively. $D^*(2637)$ is not confirmed and represented by empty square. Calculated $^1S_0$ and $^3S_1$ spectra are given in parentheses within the dashed lines.
FIG. 5. Mass of the excited $q\bar{q}$ states ($M^*$) as a function of the mass ($\mu$) of the pseudoscalar meson ground state for $I = 0$ of the charmed strange mesons. The data is taken from Hagiwara et al. [6] and is shown by solid circles on the left and right of the figure for $^1S_0$ and $^3S_1$ mesons, respectively. Calculated $^1S_0$ and $^3S_1$ spectra are given in parentheses within the dashed lines.
FIG. 6. Mass of the excited $qar{q}$ states ($M^*$) as a function of the mass ($\mu$) of the pseudoscalar meson ground state for $I = 1/2$ of the bottom mesons. The data is taken from Hagiwara et al. [6] and is shown by solid circles on the left and right of the figure for $^1S_0$ and $^3S_1$ mesons, respectively. Calculated $^1S_0$ and $^3S_1$ spectra are given in parentheses within the dashed lines.
FIG. 7. Mass of the excited $q\overline{q}$ states ($M^*$) as a function of the mass ($\mu$) of the pseudoscalar meson ground state for $I = 0$ of the bottom mesons. The data is taken from Hagiwara et al. [6] and is shown by solid circles on the left and right of the figure for $^1S_0$ and $^3S_1$ mesons, respectively. No radial quantum numbers assigned to $\Upsilon(10865)$, and $\Upsilon(11020)$ (labelled with empty diamond). $\eta_b(9300)$ is not confirmed and represented by empty square. Calculated $^1S_0$ and $^3S_1$ spectra are given in parentheses within the dashed lines.