New Solution based on Hodge Decomposition for Abstract Games

Yihao Luo\textsuperscript{1}, Jinhui Pang\textsuperscript{2}, Weibin Han\textsuperscript{3*} and Huafei Sun\textsuperscript{1}

\textsuperscript{1}School of Mathematics and Statistics, Beijing Institute of Technology, Street, City, 100190, State, Country.
\textsuperscript{2}School of Computer Science and Technology, Beijing Institute of Technology, Street, City, 10587, State, Country.
\textsuperscript{3*}School of Economics and Management, South China Normal University, Street, City, 610101, State, Country.

*Corresponding author(s). E-mail(s): weibinhan@m.scnu.edu.cn; pangjinhui@bit.edu.com; huafeisun@bit.edu.cn;
Contributing authors: knowthingless@bit.edu.cn;

Abstract

This paper proposes Hodge Potential Choice (HPC), a new solution for abstract games with irreflexive dominance relations. This solution is formulated by involving geometric tools like differential forms and Hodge decomposition onto abstract games. We provide a workable algorithm for the proposed solution with a new data structure of abstract games. From the view of gaming, HPC overcomes several weaknesses of conventional solutions. HPC coincides with Copeland Choice in complete cases and can be extended to solve games with marginal strengths. It will be proven that the Hodge potential choice possesses three prevalent axiomatic properties: neutrality, strong monotonicity, dominance cycle s reversing independence, and sensitivity to mutual dominance. To compare the HPC with Copeland Choice in large samples of games, we design digital experiments with randomly generated abstract games with different sizes and completeness. The experimental results present the advantage of HPC in the statistical sense.

Keywords: Abstract game, Copeland Choice, Hodge Potential Choice, Hodge decomposition
1 Introduction

Decision science is one of the essential fields for people to understand human society and analyze various social behaviors using natural science and engineering techniques. It is a highly interdisciplinary field involving social science, economics, game theory, psychology, etc. With data science booming, increasingly advanced and deep mathematical tools should be introduced into decision science to solve more extensive and complex problems. Hence, we will utilize geometric methods to solve decision problems in this paper with such a background.

The main object studied in this field is the abstract game, which provides canonical models for many social problems. An abstract decision problem consists of two essential elements: a set of viable alternatives $X$ and an irreflexive dominance relation $R$ over $X$ that reflects assessments or preferences concerning alternatives. The explanation for alternatives depends on the problem formulations and certain contents considered. It may be a set of competing projects or a collection of allocations of goods among individuals, etc. A dominance relation may be interpreted in several ways by considering all the characteristics of the two alternatives. It may be seen as a social preference of some collectivity or be generated by some coalition consisting of organized members where some members can enforce their preference as one alternative over another when only two alternatives are considered. We will leave aside how a dominance relation is formed and assume it is an exogenous variable. Many games in practice can be fundamentally modeled as such a pair. Noted examples are tournaments\(^1\) and majority voting problems in de MelloLuiz Flvio Autran Monteiro Gomes et al (2005) and Laslier (1997), coalition formation problems in Deemen (2013) and Dutta and Jackson (2013), page rank problems in Brandt and Fischer (2007), multi-criteria games in Arrow and Raynaud (1986) and Figueira et al (2005) and exchange market problems with indivisible goods in Subiza and Peris (2014) and the finite coalitional games with non-transferable utility \(^2\) in Brandt and Harrenstein (2010).

Since the dominance relation is associated with evaluating alternatives, it is logical to postulate a link between the information in this relation and the solution of abstract games. To be more to the point, given a dominance relation, what in dominance relation makes an alternative so special that it might be chosen. If there exists an alternative that dominates any other alternatives, referred to as the best alternatives, then such an alternative should be chosen, certainly. Many examples of applications demonstrate that the nonexistence of the best alternative is quite common. For instances, we see majority voting activity in Laslier (1997), the coalition formation problems in Deemen (2013), finite coalitional games in Brandt and Harrenstein (2010), the aggregation of multiple criteria decision-making problems in Figueira et al (2005), etc.

\(^1\)It has been shown in McGarvey (1953) that any irreflexive and asymmetric relation can be obtained as a relation induced by pairwise majority rule.

\(^2\)Brandt and Harrenstein (2010) has shown that any irreflexive relation can be obtained as a relation induced by a finite coalitional game with non-transferable utility.
it is necessary to formulate broader notions generalizing the best alternative. As the extensions of the best alternatives, solutions of abstract games were involved. A solution for abstract games is a certain function that assigns a subset of alternatives to any abstract game. A series of widely used solutions were proposed in the literature based on the additional information contained in the dominance relation. See Laslier (1997); Brandt (2011); Brandt et al (2014, 2016a) for excellent reviews of the state of knowledge about these dominance-based solutions. So far, the existing research restrictively assumed that the dominance relation is either complete or asymmetric. Whenever the dominance relation is relaxed to incomplete or not necessarily asymmetric situations, the existing solutions usually cannot maintain their original natures and characteristics. Moreover, they are either poorly discriminative or even lose effects in the conditions with large sizes or lower completeness of dominance relation.

Among traditional solutions, one frequently used approach is to assign alternatives with scores that may stand for their superiority in the perspective of the dominant position. Copeland Choice in Copeland (1951) is a well-noted example. Recall that the Copeland score of an alternative \(x\) in a given game is gained by the difference from the number of alternatives dominated by \(x\) minus the number of alternatives dominating \(x\). The Copeland winner set contains all alternatives with the greatest Copeland score. Copeland (1951) originally proposed Copeland Choice for pairwise majority comparison relation (See Fishburn (1977) for more details). Rubinstein (1980) characterized this solution in cases of asymmetric and complete dominance relations, and later Henriët (1985) proposed a complete characterization of Copeland Choice valid for cases of complete but not necessary asymmetric dominance relations. Although it has many advantages against other solutions Han et al (2016), Copeland Choice cannot discriminate the mutual dominance relation against empty-rounds when applied to irreflexive and incomplete games. Due to this, it may sometimes determine the winner sets in an imprecise or inconvincible way (See Example Exp:weakness).

To overcome the mentioned disadvantage of the conventional solutions, we propose a new solution for abstract games, called Hodge Potential Choice, by using the Hodge decomposition in Schwarz (1995). Hodge Potential Choice follows an idea about decomposing an abstract game into two subgames and ignoring the regular subgame representing the counterbalance among alternatives. If the remained part only depends on a series of scores on alternatives, the alternatives with the maximal score will be selected into the winner set. Later, it will be shown that Hodge Potential Choice degenerates to Copeland Choice for complete games. Moreover, the newly proposed solution possesses three classical axiomatic properties of Copeland Choice, while it avoids the main weakness of conventional solutions. To explicit the advantages of Hodge Potential Choice in discriminating alternatives, we conduct digital experiments from the perspectives of computations and statistics and make a comparison with Copeland Choice in these aspects.
The rest of this paper is organized as follows. Section 2 presents the basic concepts and notions of abstract games and prior knowledge of Hodge decomposition on graphs. Section 3 is dedicated to Hodge Potential Choice. First, we show the existence of Hodge decomposition of abstract games, which derives a feasible algorithm for the newly proposed solution. Moreover, we prove that Hodge Potential Choice possesses three axiomatic properties: neutrality, strong monotonicity, and cycles reversing independence, while it keeps sensitive to mutual dominance. Section 4 contains digital experiments with large samples of abstract games generated randomly, which shows the computational and statistical characteristics of Hodge Potential Choice compared to Copeland Choice. Section 5 is a short conclusion.

2 Preparation

This section is dedicated to necessary notations, concepts, techniques and data structures concerning Hodge Potential Choice of abstract games.

2.1 Foundations of abstract games

The concepts and notations in this subsection are mainly from Deemen (2013); Han et al (2016); Bouyssou and Vincke (2010).

In previous paper, an abstract games\(^3\) is presented by a pair \((X, R)\) where \(X\) is a nonempty finite set of alternatives, and the dominance relation \(R\) is a set of irreflexive binary relationships among \(X\) (i.e., \(R = \{x_1Ry_1, \ldots, x_mRy_m \mid x_i, y_i \in X, x_i \neq y_i, \forall 0 \leq i \leq m\}\)). For any \(xRy \in R\), we call it \(x\) dominates \(y\).

Let \(\Omega(X)\) be the collection of these abstract games with the same alternatives set \(X\). For any game \((X, R) \in \Omega(X)\) and any \(B \subseteq X\), if \(\sigma(B)\) is a permutation of \(B\), the permuted dominance relation \(R_{\sigma} \in \Omega(X)\) is defined as that for any \(x, y \in X\), \(xR_{\sigma}y\) if and only if \(\sigma^{-1}(x)R\sigma^{-1}(y)\).

A finite sequence \([x_1Rx_2, \ldots, x_{m-1}Rx_m, x_mRx_1] \subseteq R\) is called a (dominance) cycle \(C\) in \((X, R)\). The reversed cycle of \(C\) is defined as \(C^T = [x_mRx_{m-1}, \ldots, x_2Rx_1, x_1Rx_m]\). Specifically, a mutual dominance between \(x\) and \(y\) is a two-elements cycle \(xIy := [xRy, yRx] \subseteq R\). To clarification, an empty-round \(xEy\) existing between \(x, y \in X\) means neither \(xRy\) nor \(yRx\) in \(R\). We denote \(xEy = yEx, xEy \cap R = \emptyset\) and \(xEy \cup R = R\).

For any alternative \(x \in X\) in \((X, R)\), the Copeland score \(cs\) in Copeland (1951) is defined by

\[
    cs(x) := |\{y \in X | xRy\}| - |\{y \in X | yRx\}|, \tag{1}
\]

which presents the purely dominating number of \(x\). An abstract game \((X, R)\) is said to be

- asymmetric: \(\forall x, y \in X\), if \(xRy\), then not \(yRx\) (i.e., without any mutual dominance);

\(^3\)Abstract games are also known as abstract decision problems or decision systems in Richardson (1953) and Shenoy (1980).
A new solution based on Hodge decomposition

- **complete**: \( \forall x, y \in X, \text{if } x \neq y, \text{then either } xRy \text{ or } yRx \) (i.e., without empty-round);
- **a tournament**: if it is asymmetric and complete;
- **strongly connected**: \( \forall x, y \in X, \text{if } \exists \text{ a sequence } [x_1, \ldots, x_m] \subset X, \text{ s.t. } xRx_1, x_1Rx_2, \ldots, x_mRy; \)
- **(weakly) connected**: \( \forall x, y \in X, \text{if } \exists \text{ a sequence } [x_1, \ldots, x_m] \subset X, \text{ s.t. } R \cap \{xRx_1, x_1Rx\} \neq \emptyset, R \cap \{x_1Rx_2, x_2Rx_1\} \neq \emptyset, \ldots, R \cap \{x_mRy, yRx_m\} \neq \emptyset; \)
- **regular**: \( \forall x, y \in X, \text{cs}(x) = \text{cs}(y). \)
- **irregular**: it is not regular.

Abstract games without weak connectedness can be separated into several irrelevant subgames, so a global winner set becomes nonsense. Hence, this paper focuses exclusively on connected abstract games for simplicity.

One of the main tasks of studies about abstract game is to decide a universal and reasonable winner set choice rule Arrow and Raynaud (1986) for all abstract games. A choice rule is called a solution terminologically.

A solution for abstract games is a mapping \( S : \Omega(X) \rightarrow 2^X \), for any \( X \), i.e., the image set \( S(X, R) \subseteq X \) for any \( (X, R) \), where we call \( S(X, R) \) as the winner set of the game \( (X, R) \) under solution \( S \). Solution \( S' \) is called refinement of \( S \), if \( S'(X, R) \subseteq S(X, R) \) for any \( (X, R) \).

A reasonable solution will have some good axiomatic properties. Here we list several prevalent axioms mainly defined in Henriet (1985).

A solution \( S \) is said to satisfy

- **neutrality**: for any abstract game \( (X, R) \) and any permutation \( \sigma(X) \), if \( S(X, R_\sigma) = \sigma(S(X, R)) \);
- **strong monotonicity**: for any \( (X, R) \), if \( x \in S(X, R) \), \( yRx \in R \) and \( R' = R - \{yRx\} + \{xRy\} \), then \( S(X, R') = \{x\} \);
- **cycle reversing independence**: for any \( (X, R) \) with a cycle \( C \), if \( R' \) equals to \( R \) except \( C \) reversed, i.e., \( R' = R' - C + C^T \), then \( S(X, R') = S(X, R) \);
- **cycle adding and removing independence**: for any \( (X, R) \) and \( (X, R') \) with a cycle \( C \), if \( R = R' - C \), then \( S(X, R') = S(X, R) \);
- **mutual dominance independence**: for any \( (X, R) \) with a mutual dominance \( xIy \), if \( xIy \) is replaced by empty-round \( xEy \) in \( (X, R') \), i.e., \( R' = R - xEy \), then \( S(X, R') = S(X, R) \);
- **mutual dominance sensitivity**: if it does not satisfy mutual dominance independence.

Remark 1 A solution \( S \) satisfying cycle independence will be independent on mutual dominance naturally.

Intuitively, the neutrality presents the fairness of a solution, while monotonicity shows its accuracy, and cycle (reversing)independence means the reducibility. It is remarkable that the reducibility may decrease the accuracy or the sensitivity for a solution of abstract games, which will be demonstrated in section 3. Therefore, the cycle revering independence, rather than the cycle
A new solution based on Hodge decomposition

independence, keeps the balance between the reducibility and the accuracy. mutual dominance sensitivity shows that the solution can distinguish mutual dominances against empty-rounds.

**Copeland Choice** $Cp$, mapping abstract games onto their Copeland winner sets, is one of the most famous and widely applied solutions of abstract games. For any abstract game $(X, R)$, the **Copeland winner set** $Cp(X, R)$ is defined as the maximal set of Copeland scores:

$$Cp(X, R) = \{x \in X \mid cs(x) \geq cs(y), \forall y \in X\}.$$  

Copeland Choice was originally proposed by Copeland (1951) for pairwise majority comparison relation (See Fishburn (1977) for more details). Our definition is a natural extension of Copeland Choice for general abstract games. Copeland Choice is widely applied in fields of social choice and sport analysis, with its aliases as pure net winnings or goal difference, see Ladouceur et al (1988). The solution satisfies neutrality, strong monotonicity, cycle adding and removing independence. Hence, it is not mutual dominance sensitive. See the proof of axiomatic properties of Copeland Choice in Henriet (1985) for details.

Traditionally, any $(X, R) \in \Omega(X)$ can be represented by a digraph where $X$ is a vertex set and $R$ is the set of directed edges. To clarify this point and above concepts, see the following example:

**Example 1** The abstract game $(X, R)$ is represented by the digraph in Figure 1, where the alternatives set $X = \{x_1, \cdots, x_5\}$ and the dominance relation

$$R = \{x_1Rx_2, x_1Rx_3, x_1Ix_5, x_2Ix_5, x_3Rx_2, x_3Rx_4, x_5Rx_3, x_5Rx_4\}.$$  

By definitions, the $Cp(X, R) = \{x_1, x_5\}$ with the highest Copeland score $cs(x_1) = cs(x_5) = 2$. Obviously, the Copeland winner set would still keep invariant if mutual dominances $\{x_1Ix_5, x_2Ix_5\}$ were replaced by empty-rounds $\{x_1Ex_5, x_2Ex_5\}$ or the cycle $[x_2Rx_5, x_5Rx_3, x_3Rx_2]$ were reversed or removed.

From Example 1, we can see that Copeland Choice can not distinguish mutual dominances against empty-rounds, while it admits cycle adding and removing as well. Intuitively speaking, these phenomenons shows Copeland Choice can not extract information precisely and completely enough, especially for the counterbalanced information of abstract games from cycles and mutual dominances.

To make up the above-mentioned weakness of Copeland Choice and persist its main advantage, we would propose a new solution for abstract games. In order to formulate this new solution and pave its background naturally, we need to adopt a new data structure for abstract games from topological viewpoint and introduce some necessary mathematical tools. For these points, we present the following subsections.
A new solution based on Hodge decomposition

Fig. 1: A example of an abstract game presented by a digraph, where the bidirectional arrows represent mutual dominances, the three green arrows provides an example of the cycle. $x_1$ and $x_5$, with the highest Copeland score, compose the Copeland winner set, which are marked by red nodes.

2.2 Games representation: differential forms on graphs

The concepts and notations in this subsection are mainly from Diestel (2000) for graph theory, Hatcher (2002) for topology and Lee (1997) for geometry.

Conventionally, abstract games are always represented by digraphs. Hence, we introduce concepts about graphs firstly.

A digraph $G$ is a pair $G = (V_G, E_G)$, where we call $V_G$ as vertexes and $E_G \subset V_G \times V_G$ as directed edges. If any $(x, y), (y, x) \in E_G$ were identified to each other (i.e., $(x, y) \sim (y, x)$ and $E_G \subset V_G \times V_G/\sim$), then $G$ degenerate to an undirected graph. Both digraphs and undirected graphs are called as graph. The adjacent matrix is one of classical data structure for graphs, see Tarjan (1984) for details. We will not discriminate a graph against its adjacent matrix in follows.

Any abstract game $(X, R)$ has its digraph representation $G$, where $V_G \cong X$ and $(x, y) \in E_G$ iff $xRy \in R$.

For the digraph showed in Example 1, the adjacent matrix is

$$G = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}, \tag{2}$$

where every $G_{ij} = 1$ represents $x_iRy_j \in R$. The adjacent matrix $G$ contains all information of the abstract game $(X, R)$, such as the Copeland score presented by a vector $cs = (G - G^T) \cdot [1, \cdots, 1]^T = [2, -2, 0, -2, 2]^T$.

From the game viewpoint, we tend to introduce another data structure for abstract games which further involves differential forms to understand games.
A new solution based on Hodge decomposition

We firstly define an undirected graph $W$ called the base space for any abstract game $(X, R)$, where $W_{ij} = W_{ji} = 1$ if $x_i R x_j \in R$. In another words, the base space $W$ equals to the digraph representation $G$ forgetting its directions on edges.

**Remark 2** In many specific situations, such as sports, the base space is abstracted from the “game arrangement”, where any $W_{ij} = 1$ shows a round existing between the alternative $x_i$ and $x_j$.

The base space does not contain all information from the abstract game $(X, R)$, since $W_{ij} = 1$ never ensures whether $x_i R x_j \in R$ exists. We should endows additional structure on base spaces to provide residential information about game. Therefore, we will introduce differential forms on graphs and define the local dominance difference of abstract games.

In follows, we provide the definitions of edge chains and (discrete) differential 1-forms on graphs at first. Then provide the game-relevant definition, called local dominance difference. Then we illustrate the new representation by examples following Example 1.

An edge chain $\gamma = \sum k_{\alpha} \cdot E_{\alpha} \in C_1(G, \mathbb{R})$ on any graph $G = (V_G, E_G)$ is a formal combination of some directed edges paired with real coefficients $k_{\alpha} \in \mathbb{R}$, where $k_{\alpha}(v_i, v_j) \sim -k_{\alpha}(v_j, v_i)$ are identified naturally. In mathematical words, edge chains constitute a finitely generated Abelian group:

$$C_1(G, \mathbb{R}) := \bigoplus_{e_j \in E_G} \mathbb{R} \sim,$$

where the identity means $k(v_i, v_j) = -k(v_j, v_i)$ for any $k \in \mathbb{R}$ and $(v_i, v_j) \in E_G$.

**Remark 3** This identity can be immigrated onto the base spaces of abstract games, meaning that a round $x_i$ against $x_j$ can be identified as a negative round $x_j$ against $x_i$. In sport games, this identification symbolizes the “Home alternative” opposing against the “visiting alternative”.

An 1-form $\phi \in F_1(G, \mathbb{R})$ is a linear function from edge chains to $\mathbb{R}$, i.e., $\phi(\gamma_1 + k \cdot \gamma_2) = \phi(\gamma_1) + k \cdot \phi(\gamma_2)$ for any $\gamma_1, \gamma_2 \in C_1(G, \mathbb{R})$. By linearity, any 1-form is determined by its values on every edge $e_j = 1 \cdot e_j \in C_1(G, \mathbb{R})$ and $\phi(v_i, v_j) = -\phi(v_j, v_i)$ holds naturally. An 1-form can be represented by a skew-symmetric matrix $\phi$ with the same size as the adjacent matrix representation of the base space, where $\phi_{ij} = -\phi_{ji} = \phi(v_i, v_j)$. In terminology of categories and algebra, $F_1(G, \mathbb{R}) = Hom(C_1(G, \mathbb{R}))$, called the dual of $C_1(G, \mathbb{R})$, is still a finitely generated Abelian group.
By the 1-forms on base space of abstract games, we can provide residential information of dominance relations over base space, for which we define the local dominance difference.

The **local dominance difference** $R$ of an abstract game $(X, R)$ is an 1-form on the base space $W$, where

1. $R(x_i, x_j) = -1$ and $R(x_j, x_i) = 1$, if $x_iRx_j \in R$ but $x_iRx_j \notin R$,
2. $R(x_i, x_j) = 1$ and $R(x_j, x_i) = -1$, if $x_jRx_i \in R$ but $x_iRx_j \notin R$,
3. $R(x_i, x_j) = R(x_j, x_i) = 0$, if $x_jIx_i \subset R$,
4. $R(x_i, x_j)$ for any $x_jEx_i$ is illegal, since $(x_i, x_j) \notin E_W$ and $W_{ij} = 0$.

The pair of the base space and local dominance difference can be written as two matrices $(W, R)$, where $W$ is symmetric and $R$ is skew-symmetric while $R_{ij} \neq 0$ only if $W_{ij} = 1$. We say $(W, R)$ contains all information of an abstract game $(X, R)$ according following theorem.

**Theorem 1** Any two matrices pair $(W, R)$ with the size $n \times n$ determines a unique abstract game $(X, R)$, if $(W, R)$ satisfies

1. $W = W^T$, $W_{ii} = 0$ and $W_{ij} = 0$ or 1 for $\forall i \neq j$,
2. $R = -R^T$, $R_{ij} = 0, -1$ or 1,
3. $R_{ij} \neq 1 \implies W_{ij} = 1$.

**Proof** Constructively, initialize the alternatives $X = x_1 \cdots x_n$ and the dominance relations $R = \emptyset$. For any $W_{ij} = 1, j > i$, if $R_{ij} \leq 0$, then add $x_iRx_j$ into the dominance relations $R$, i.e., $R = R \cup x_iRx_j$; and if $R_{ij} \geq 0$, then add $x_jRx_i$ into the dominance relations $R$, i.e., $R = R \cup x_jRx_i$. It is obvious that the final generated abstract game $(X, R)$ has the base space $W$ and the local dominance difference $R$.

By the way, for an abstract game $(X, R)$, its digraph representation $G$ and the pair of the base space with its local dominance difference $(W, R)$ determine each other. If $G$ is known, $W = \|(G^T + G)\|$ ($\| \cdot \|$ turns each non-zero component to 1) and $R = G^T - G$.

**Remark 4** Usually, after the “game arrangements” is known from the base space $W$, the local dominance difference give us the information of the “game results” by providing the scores difference in every arranged round.

**Example 2** Reconsidering the game $(X, R)$ from Example 1, the base space $W$ represented by following symmetric matrix can be regarded as the “game arrangements”, and the local dominance difference $R$ is an 1-form on $W$ showing “game results” (see

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4We use the same notation for both the local dominance difference and the dominance relation without ambiguity, due to them contain same information.
A new solution based on Hodge decomposition

Figure 2). Hence, we have

\[ W = \begin{bmatrix}
0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}, \quad R = \begin{bmatrix}
0 & -1 & -1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & -1 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & -1 & -1 & 0
\end{bmatrix}, \]

where \( W_{ij} = W_{ji} = 1 \) means a round existing between alternatives \( x_i \) and \( x_j \); \( R_{ij} = -R_{ji} = -1 \) means \( x_iRy \) (-1 presenting that \( x_j \) has one minus point higher than \( x_i \) in the round between them).

\[ \begin{array}{cccccc}
& x_5 & -1 & x_4 & \quad & \\
\text{-1} & & -1 & & \text{-1} & \\
x_3 & & 0 & & & \\
\text{-1} & & \text{-1} & & \text{-1} & \\
& x_2 & \quad & x_1 & & \\
\end{array} \]

\textbf{Fig. 2:} Abstract game \((X, R)\) can be informed by the local dominance difference \( R \) which is an 1-form on the base space \( W \). Every \(-1\) on the edge from \( x_i \) to \( x_j \) means \( R_{ij} = -1 \) equal to \( x_iRx_j \in R \), while every 0 means a mutual dominance \( x_iIx_j \) existing between \( x_i \) and \( x_j \).

To consider the density of the “game arrangement”, we can define the \textbf{completeness} \( \eta \) of an abstract game \((X, R)\) as

\[ \eta(X, R) = \frac{\text{tr}(W^TW)}{n(n-1)}, \quad (3) \]

where \( W \) is the base space and \( n \) is the cardinalities of alternatives \( X \). Any complete abstract game \((X, T)\) has the highest completeness \( \eta(X, T) = 1 \).

\textit{Example 3} The game in Example 1 has the completeness as \( \eta(X, R) = \frac{4}{5} \).

In the following passage, we will frequently use the matrices pair \((W, R)\) as the data structure of an abstract game \((X, R)\), which is called the base and form representation. We will not distinguish the dominance relation set \( R \) and the 1-form \( R \) as the local dominance difference.
A new solution based on Hodge decomposition

Besides edge chains and 1-form, concepts of vertex chains and 0-form is worth to introduce. With similar definitions, a vertex chain \( \tau = \sum k_{\alpha} \cdot V_{\alpha} \in C_0(G, \mathbb{R}) \) is a formal combination of vertexes in the graph \( G \) with coefficient of \( \mathbb{R} \), where

\[
C_0(G, \mathbb{R}) := \bigoplus_{v_i \in V_G} \mathbb{R}.
\]

A 0-form \( \psi \in F_0(G, \mathbb{R}) \) is a linear function on \( C_0(G, \mathbb{R}) \) which has the representation as an array (or a vector) with the length of \(|V_G| \). When the graph \( G \) is specialized to be the base space \( W \) of an abstract game, a 0-form \( \psi \) sometimes represents a sequence of “scores” over alternatives in the game. For example, the Copeland scores of any abstract game \((W, R)\) can be regarded as a 0-form on \( W \).

We omit the real coefficient \( \mathbb{R} \) and adopt denotations like \( C_0(G), C_1(G), F_0(G) \) and \( F_1(G) \) in follows.

More importantly, there exist deep connections among \( C_0(G), C_1(G), F_0(G) \) and \( F_1(G) \). In algebraic topology, they construct chain and cochain complexes. The chain complex on the graph \((V_G, E_G)\) includes \( C_0(G), C_1(G) \) and a homomorphism \( \partial : C_1(G) \to C_0(G) \) called as the boundary operator, where \( \partial(v_i, v_j) = v_j - v_i \) for any \((v_i, v_j) \in E_G\). Further, the cochain complex is constructed by \( F_0(G), F_1(G) \) and a homomorphism \( d : F_0(G) \to F_1(G) \) called the differential operator s.t., for any \( \psi \in F_0(G) \) and \( \gamma \in C_1(G) \),

\[
d\psi(\gamma) = \psi(\partial\gamma).
\]

Example 4 Following Example 1 and 2, the Copeland scores of the game \((X, R)\) is a 0-form \( cs = [2, -2, 0, -2, 2]^T \in F_0(W) \) on the base space \( W \). Then we have the differential of \( cs \):

\[
d(cs) = \begin{bmatrix}
0 & -4 & -2 & -4 & 0 \\
4 & 0 & 2 & 0 & 4 \\
2 & -2 & 0 & -2 & 2 \\
4 & 0 & 2 & 0 & 4 \\
0 & -4 & -2 & -4 & 0
\end{bmatrix} \in F_1(W).
\]

We will rank alternatives and establish the new reasonable solution in Section 3. More technically, we tend to extract the most informative 0-form (scores) from the local dominance difference 1-form \( R \). This is why we introduce differential forms and adopt the new data structure of games.

2.3 Laplacian operators on abstract games

For approaching the main goal to establish the new solution for abstract games, we need to introduce some useful concepts and geometric tools including the divergence operator, inner products of forms and Laplacian operators. In this subsection, we will provide definitions and several lemmas to present their characteristics, which will be frequently used later. The concepts and notations in this subsection mainly come from Lee (1997); do Carmo (1993); Lee (2003); Taubes (2011).
Without ambiguity, we will not discriminate the abstract game \((X, R)\) with its graph and form representation \((W, R)\). We will constantly adopt \(W\) to denote the base space and its adjacent matrix, and use \(n\) to represent the cardinalities of alternatives \(X\).

In subsection 2.2, for any abstract game \((X, R)\), we have introduced the differential operator \(d\) from \(F_0(W)\) to \(F_1(W)\) on the base space \(W\). Besides, there naturally exists another homomorphism \(\delta : F_1(W) \rightarrow F_0(W)\) called the divergence operator, which is defined by: For any \(\phi \in F_1(W)\) and any \(x_i \in X\),

\[
(\delta \phi)(x_i) := \sum_{k=0}^{n} \phi(x_k, x_i)W_{ik},
\]

where \(\phi(x_i, x_j) := 0\), for any empty-round \(x_i E x_j\) (i.e., \(W_{ij} = 0\)).

From above definitions\(^5\), it is easily check that \(\delta \phi = -\phi \cdot [1, \cdots, 1]^T\).

Specifically, we have Copeland scores \(cs = \delta R\).

In addition, we have Euclidean inner products for \(F_0(W)\) and \(F_1(W)\):

For any \(\psi_1, \psi_2 \in F_0(W)\),

\[
\langle \psi_1, \psi_2 \rangle := \int_X \psi_1 \psi_2 = \sum_{i=1}^{n} \psi_1(x_i)\psi_2(x_i),
\]

For any \(\phi_1, \phi_2 \in F_1(W)\),

\[
\langle \phi_1, \phi_2 \rangle := \int_{E_W} \phi_1 \phi_2 = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \phi_1(x_i, x_j)\phi_2(x_i, x_j)W_{ij}.
\]

Both of the two products satisfy axioms of inner product, including bilinear, symmetric, non-degenerated. See Hirsch and Smale (1974). It is obviously that they have explicit expression as

\[
\langle \psi_1, \psi_2 \rangle = \psi_1^T \cdot \psi_2,
\]

\[
\langle \phi_1, \phi_2 \rangle = \text{tr}(\phi_1^T \cdot \phi_2).
\]

It would be said that the divergence \(\delta\) is the dual operator of the differential \(d\) due to the following lemma.

**Lemma 1** For any game \((X, R)\), \(\phi \in F_1(W)\) and \(\psi \in F_0(W)\), \(\langle \phi, d\psi \rangle = \langle \delta \phi, \psi \rangle\).

---

\(^5\)In Riemannian geometry, \(d\) and \(\delta\) are defined for any \(k\)-differential forms \(\{F_k(M) \mid k = 0, 1, \cdots, \dim M\}\) on a manifold \(M\). The definitions in this paper can be regarded as some simplifications in discrete version for graphs. The original definition of \(\delta\) operator on a manifold depends on metric while \(d\) is more natural and independent on metric choices. A metric \(g\) determines a series of isomorphism \(*\) between \(F_k\) and \(F_{\dim M - k}\), called the Hodge star operator. Then the divergence defined as \(\delta := (-1)^{k\dim + n} \star d\star\). Definition (4) is exactly derived from defaulted Euclidean metric. For simplicity, we ignore complicated mathematical discussions and present new definitions directly, see Lee (2003) for details.
Proof By definitions, we check that
\[
\langle \phi, d\psi \rangle = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \phi(x_i, x_j) d\psi(x_i, x_j) W_{ij}
\]
\[
= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \phi(x_i, x_j) (\psi(x_j) - \psi(x_i)) W_{ij} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi(x_i, x_j) (\psi(x_j) - \psi(x_i)) W_{ij}
\]
\[
= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi(x_i, x_j) W_{ij} - \sum_{j=1}^{n} \psi(x_j) \sum_{i=1}^{n} \phi(v_i, v_j) W_{ij} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi(v_i, v_j) W_{ij}
\]
\[
= \sum_{i=1}^{n} \psi(x_i) \sum_{j=1}^{n} \phi(x_j, x_i) W_{ij} + \sum_{j=1}^{n} \psi(x_j) \sum_{i=1}^{n} \phi(v_i, v_j) W_{ij}
\]
where we repeatedly utilize the skew-symmetric of 1-form \(\phi\).

With the differential \(d\) and the divergence \(\delta\), we consider the following two sequences which will induce Laplacian operators on abstract games:
\[
F_0(W) \xrightarrow{d} F_1(W) \xrightarrow{\delta} F_0(W);
\]
\[
F_1(W) \xrightarrow{\delta} F_1(W) \xrightarrow{d} F_0(W),
\]
where \(d \circ \delta\) and \(\delta \circ d\) are endomorphic on \(F_0(W)\) and \(F_1(W)\) respectively. They are called Laplacian operators according to classical Laplacian operator \(\Delta := \text{div} \circ \nabla\) (\(\text{div}\) and \(\nabla\) are the divergence and the gradient in analysis). More exactly, for any abstract game \((X, R)\), we define that Laplacian operators by \(\Delta^0 := \delta \circ d : F_0(W) \to F_0(W)\) and \(\Delta^1 := d \circ \delta : F_1(W) \to F_1(W)\). When no ambiguity exists, we omit the upper indexes and write all them as \(\Delta\). Any form in ker \(\Delta\) is said to be a harmonic form.

Consider an abstract game \((W, R)\) where the local dominance difference \(R\) is a harmonic 1-form (i.e., \(\Delta R = 0\)). Then we have the Copeland scores \(cs = \delta R \in \text{ker } d\), which means for any \(x_i, x_j \in X\) s.t. \(cs(x_i) = cs(x_j)\) for any \(W_{ij} = 1\). In addition, if \((W, R)\) is weakly connected as well, the the game \((W, R)\) is regular. These derive that all alternatives are contained in the Copeland winner set \(Cp(X, R) = X\). From the perspective of Copeland Choice, there is neither “winner” nor “loser” in the game and all alternative equivalent to each other, which explains the meaning of “harmonic”.

Now we prove that Laplacian \(\Delta^0\) is exactly expressed by the Graph-Laplacian matrix Anderson and Morley (1985) \(L\) of the base space \(W\). The Graph-Laplacian matrix is defined by \(L := D - W\), where \(D\) is a diagonal matrix called degree matrix in Diestel (2000), s.t. \(D_{ii} = \sum_{j=1}^{n} W_{ij}\). By the definitions, we have the following lemma:

Lemma 2 For any game \((X, R), \psi \in F_0(W), \Delta \phi = L \cdot \phi.\)
A new solution based on Hodge decomposition

Proof For any $x_i$ in $X$, we check that

$$
\Delta \psi(x_i) := \delta d \psi(x_i) = \sum_{k=1}^{n} (d \psi(x_k, x_i) W_{ik} = \sum_{k=1}^{n} (\psi(x_i) - \psi(x_k)) W_{ik} = D_{ii} \phi(x_i) - \sum_{k=1}^{n} (W_{ik} \phi(x_k)) = \sum_{k=1}^{n} L_{ik} \phi(x_k) = (L \cdot \phi)_i.
$$

From the arbitrary of $x_i$, we complete the proof of Lemma 2. □

Combining the definition of Laplacian operators with Lemma 1, we gain the following lemma directly.

Lemma 3 For any game $(X, R)$, $\ker L = \ker \Delta^0 = \ker d$, then $\text{rank}\, L = n - m$, where $m$ is the number of connected components\(^6\) of the base space $W$. Specifically, for connected games, $\text{rank}\, L = n - 1$ and $\ker L = \{k[1, \cdots, 1]^T | k \in \mathbb{R}\}$.

Proof Firstly we suppose $\psi \in \ker \Delta^0 \subset F_0(W)$ and tend to prove $\psi \in \ker d$. For $\Delta \psi = 0$, $\langle \Delta^0 \psi, \psi \rangle = 0$. By Lemma 1, $\langle \delta d \psi, \psi \rangle = \langle d \psi, d \psi \rangle = 0$, then $\psi \in \ker d$. Thus, $\ker d = \ker \Delta^0 = \ker L$. On the other hand, $d \psi = 0$ derives $\psi(x_i) = \psi(x_j)$ for any $W_{ij} = 1$, which is called as locally constant (constant on each connected component). Therefore, we prove that $\dim(\ker d) = \dim(\ker L) = k$, then $\text{rank}\, L = n - m$. Specifically, if $m = 1$, i.e., $(X, R)$ is weakly connected, we have $\text{rank}\, L = n - 1$. According to $L = D - W$, it is obviously that $L \cdot [1, \cdots, 1]^T = 0$. Due to $\dim(\ker L) = 0$, $\ker L = \{k[1, \cdots, 1]^T | k \in \mathbb{R}\}$. □

Similarly, for $\phi \in \ker \Delta^1$, according to Lemma 1 working on $\langle d \delta \phi, \phi \rangle = 0$, we have $\langle \delta \phi, \delta \phi \rangle = 0$ and the lemma bellow.

Lemma 4 For any abstract game $(X, R)$, $\ker \Delta^1 = \ker \delta$.

These are all preparations we need to involve the Hodge decomposition for abstract games and establish the new solution in next section.

3 HPC: the new solution

In this section, we propose a new reasonable solution, called Hodge Potential Choice, by using the Hodge decomposition on abstract games.

---

\(^6\)A connected component of an undirected graph is a maximal connected subgraph. Two distinct connected components never intersect each other. An undirected graph can be decomposed into the disjoint union of all its connected components. See Diestel (2000).
3.1 Hodge decomposition for abstract games

Following above discussions, we tend to design a reasonable solution by ranking alternatives $X$. We will use the tools introduced in last section to find a 0-form expressing a convincing “scores” for alternatives $X$ which contains main information from the dominance relation $R$.

The most ideal cases is that we find a 0-form $P \in F^0(W)$ s.t. $R = dP$, which means the dominance difference $R$ is the difference of the scores $P$. Then we can set the solution as searching for the maximal set of $P$. Although this is not established for general cases (i.e., $R \notin \text{Im}(d)$), we can actually find a 0-form $P$ for each game $(X,R)$ such that $dP$ approaching to $R$ except for a harmonic 1-form $H$. Here we will use a famous geometric technique called the Hodge decomposition Schwarz (1995).

To show the processing of the decomposition, we directly prove the uniqueness and existence of the Hodge decomposition for abstract games constructively.

**Lemma 5** For any $\phi \in F^1(W)$ on a connected game $(X,R)$, there exist an unique $Q \in \text{Im}(d)$ and an unique harmonic form $H \in \ker \Delta^1$ s.t.

$$\phi = Q + H = dP + H,$$

for some $P \in F^0(W)$. We call $P$ as the potential 0-form of $\phi$.

**Proof** Firstly, we suppose the existence and tend to prove the uniqueness. If there exist two decomposition

$$\phi = dP + H = d\tilde{P} + \tilde{H},$$

According to Lemma 2 and Lemma 3, if we use divergence $\delta$ to operate on both sides of the equation

$$d(P - \tilde{P}) = \tilde{H} - H,$$

we get that

$$L \cdot (P - P') = 0.$$

By Lemma 3, $(P - \tilde{P}) \in \ker(d)$, we have $dP = d\tilde{P}$ and $H = \tilde{H}$. It proves the uniqueness.

Then we prove the existence constructively by giving the algorithm to searching for the potential 0-form $P$. If $\phi \in \ker \Delta^1$, it is a decomposition of itself, where $dP = 0$ and $P$ can be any constant 0-form due to the connectedness. For any $\phi \notin \ker \Delta^1$, by definition (4) and the skew-symmetric of $\phi$, we have

$$\langle [1, \cdots, 1]^T, \delta \phi \rangle = [1, \cdots, 1] \cdot \phi \cdot [1, \cdots, 1]^T = 0.$$

Therefore, the divergence $\delta \phi$ is orthogonal to all constant 0-forms which compose $\ker L$ by Lemma 3. This fact derives that there exists a solution $P$ for the nonhomogeneous linear equation

$$L \cdot P = \delta \phi,$$

due to $\text{rank}L = \text{rank}[L, \delta \phi]$, see Hirsch and Smale (1974) for details. For any solution $P$ satisfying (10), the divergence $\delta (\phi - dP) = 0$, which equivalents to that $\phi - dP$ is harmonic by Lemma 4. The proof is completed, and it is obviously that the Hodge decomposition for a 1-form attributes to solving the Laplacian equation. □
A new solution based on Hodge decomposition

Intuitively speaking, the Hodge decomposition\(^7\) separates a 1-form into two parts. One of them is determined by potential scores and the another part is harmonic. If we ignore the harmonic part, the potential scores contain main information from original 1-form.

Particularly, we can decompose the local dominance difference $R$ by Lemma 5 and gain the Hodge potential scores $P$ for any connected abstract game $(X, R)$. Hodge potential scores are defined as any 0-form $P \in F_0(W)$, s.t. $R - dP$ is harmonic. It is noticeable that Hodge potential scores are not unique for a fixed connected game. The difference between any two Hodge potential scores is a constant 0-form. Thus, the difference term $dP$ of $R$ is still unique.

Example 5 Following Example 2, the Graph-Laplacian matrix $L$ on the game $(X, R)$ can be gained by definition

$$L = \begin{bmatrix}
3 & -1 & -1 & 0 & -1 \\
-1 & 3 & -1 & 0 & -1 \\
-1 & -1 & 4 & -1 & -1 \\
0 & 0 & -1 & 2 & -1 \\
-1 & -1 & -1 & 1 & 4
\end{bmatrix}.$$  

On the other hand, the divergence or the Copeland scores $cs = \delta R = [2, -2, 0, -2, 2]^T$. Then the Laplacian equation $L \cdot P = cs$ has solutions that $P = \begin{bmatrix}
7/10 & 3/5 & 0 & 0 & 2/5 \\
3/10 & 0 & 3/5 & 0 & 7/5 \\
-3/10 & 0 & 1/5 & 0 & 4/5 \\
0 & 0 & -1/5 & 0 & 3/5 \\
3/10 & -7/5 & -3/5 & 0 & 5/5
\end{bmatrix}$, $H := R - dP = \begin{bmatrix}
0 & 0 & -3/10 & 0 & 3/10 \\
0 & 0 & 3/10 & 0 & -7/10 \\
7/10 & -7/10 & 0 & -1/5 & 2/5 \\
0 & 0 & 1/5 & 0 & -3/5 \\
3/10 & 7/5 & 3/5 & 0 & -1/5
\end{bmatrix}$,

where it is easily to check that $H$ is harmonic (i.e., $H \cdot [1, \cdots, 1]^T = 0$). The decomposition $R = dP + H$ can be regarded as separating $R$ into two subgames presented in Figure 3.

From the viewpoint of gaming, the harmonic part $H$ is not important for ranking alternatives, because alternatives counterbalance each other in the subgame $(X, H)$. Therefore, we can define a notion to describe the tenseness of a game by the proportion of the harmonic remainder in the original game. The tenseness $ts$ of a connected abstract game $(X, R)$ is defined as

$$ts(X, R) := \frac{\langle H, H \rangle}{\langle R, R \rangle},$$

where $R = H + dP$. The tenseness $ts(X, R) \in [0, 1]$ for any game. Intuitively speaking, A game with higher tenseness means strength differences among

---

\(^7\)The general Hodge decomposition works for any $k$-differential form $F$ on an Riemannian manifold, with the form as $F = dP + \delta Q + H$ where $P$ is a $(k-1)$-form and $Q$ is a $(k+1)$-form while $H$ is a harmonic $k$-form satisfying the Beltrami-Laplacian equation. See Lee (2003); Schwarz (1995) for details. The second term $\delta Q$ vanishing in this paper attributes to no 2-dimensional cell existing on graphs.
A new solution based on Hodge decomposition

![Hodge decomposition diagram](image)

Fig. 3: Hodge decomposition $R = dP + H$: The red numbers on nodes in (a) show one of the Hodge potential scores $P$. The 1-form presented on edges in (a) is the unique potential part $dP$ of $R$. In (b), $H$ is the harmonic part of $R$ whose divergence $\delta H$ equals to 0 constantly.

alternatives are relatively smaller and the competition is tenser. Particularly, a regular game has its tenseness as 1, which means it is a balanced game without any meaningful winner.

Example 6 The game in Example 1 has the tenseness as $ts(X, R) = \frac{4}{15}$.

With the idea to ignore harmonic part symbolizing counterbalances but consider ranking and determine winners from the Hodge potential scores, we design a new solution for abstract games in the next subsection.

3.2 Hodge Potential Choice and its characteristics

From above discussions, the new solution called Hodge Potential Choice (HPC) $\mathcal{H}_p : \Omega(X) \to 2^X$ for any alternatives set $X$, can be naturally introduced by ranking alternatives according to any Hodge potential scores. Similar with the definition of Copeland Choice, the Hodge potential winner set $\mathcal{H}_p(X, R)$ is defined as the maximal set of any Hodge potential scores $P$ for any connected game $(X, R)$:

$$\mathcal{H}_p(X, R) = \{x \in X \mid P(x) \geq P(y), \forall y \in X, R - dP \in \ker \Delta\}.$$

Despite more than one Hodge potential scores exist for any irregular connected game, the maximal set is unified, so the winner set is well-defined, due to the constant differences among Hodge potential scores. For any regular game $(X, H)$, $P \equiv 0$ is the unique Hodge potential scores and $\mathcal{H}_p(X, H) = X$. 
Lemma 5 ensures the existence and uniqueness of Hodge potential winner set for any connected game. We directly provide the algorithm to search for Hodge potential winner set, which shows the feasibility of Hodge potential choice. This algorithm basically comes from the constructive processing in the proof of Lemma 5.

**Algorithm 1** Hodge potential choice (HPC)

**Input:** The form representation \((W, R)\) of a connected game \((X, R)\), where \(W\) is a symmetric matrix and \(R\) is a skew-symmetric matrix with same size.

**Output:** Hodge potential winner set \(Hp(X, R)\)

1. compute Copeland winner \(cs = -R \cdot [1, \cdots, 1]^T\);
2. if \(cs\) is constant then **return** \(X\)
3. else get the degree matrix \(D = \text{diag}(W \cdot [1, \cdots, 1]^T)\) and the Graph-Laplacian matrix \(L = D - W\);
4. end if
5. solve the linear equation \(L \cdot P = cs\) to obtain a particular solution \(P\);
6. get the maximal index set \(J = \text{index}(P_{\max})\) and **return** \(\{x_i \in X \mid i \in J\}\).

The main complexity of Algorithm 1 for HPC focus on the third step: solving the Laplacian equation (10), while the first two steps of HPC contain only additions without multiply and matrix operating. Hence, HPC has its complexity approximating to the counterpart of computational methods to solve linear equations such as the Jacobi method and the Gauss-Seidel method Neumaier (2001), which means HPC has the complexity about \(O(n^2)\). The complexity can be reduced when the dominance relation \(R\) is sparse enough, i.e., with a lower completeness. There exists a large space to optimize the algorithm, although it is not the key point on in this paper.

**Example 7** Continuing with Example 5, the Copeland winner set \(Cp(X, R) = \{x_1, x_5\}\) and the Hodge potential winner set \(Hp(X, R) = \{x_1\}\). This provides an example for that HPC is more refined than Copeland Choice.

Next, we show the main characteristics possessed by HPC, including its degeneration into Copeland Choice, neutrality, strong monotonicity, cycle reversing independence and mutual dominance sensitivity.

**Theorem 2** For any complete abstract game \((X, T)\) with \(n\) alternatives, HPC coincides with Copeland Choice, i.e., if the completeness \(\eta(X, T) = 1\),

\[Hp(X, T) = Cp(X, T).\]

**Proof** By \(\eta(X, T) = 1\), we have \(\langle W, W \rangle = n(n - 1)\), \(W_{ij} = 1\) if \(i \neq j\), and \(W_{ii} = 0\). Then we get the Laplacian matrix \(L\) satisfying \(L_{ij} = -1\) if \(i \neq j\), and \(L_{ii} = n - 1\).
Subsequently, $L$ has only two eigenvalues 0 and $n$, with the multiplications\(^8\) of 1 and $n - 1$ respectively. For any vector $v \in \mathbb{R}^n$, $L \cdot v = v$ iff $[1, \cdots, 1] \cdot v = 0$, due to $L \cdot [1, \cdots, 1]^T = 0$ and eigenvectors corresponding to different eigenvalues orthogonal to each other. On the other hand, the Copeland scores $cs = \delta T = T \cdot [1, \cdots, 1]^T$, and then

$$[1, \cdots, 1] \cdot cs = [1, \cdots, 1] \cdot T \cdot [1, \cdots, 1]^T = 0,$$

according to the skew-symmetric $T = -T^T$. This derives

$$L \cdot cs = n \cdot cs,$$

so $P_{cs} := \frac{1}{n} cs$ satisfying the Laplacian equation (10) is a Hodge potential scores for $(X, T)$. Thus, there exist a Hodge potential scores proportional to the Copeland scores, which proves $\mathcal{H}p(X, T) = \mathcal{C}p(X, T) = \arg_X \max(cs)$.

This property provides the reason why we say Hodge Potential Choice is one of variations of Copeland Choice.

**Proposition 1** Hodge Potential Choice is neutral for any connected game $(X, R)$, i.e., for any permutation $\sigma(X)$,

$$\sigma(\mathcal{H}p(X, R)) = \mathcal{H}p(X, R_\sigma).$$

The proof of the neutrality attributes to that any permutation $\sigma(X)$ can be regarded as an elementary transform. By finite group theory (see Scott (2012)), every permutation of $X$ can be decomposed into products of commutations such as $(x_i x_j)$. Without loss of generality, we check that $\sigma_0(\mathcal{H}p(X, R)) = \mathcal{H}p(X, R_{\sigma_0})$ for any $\sigma_0 = (x_i x_j)$.

**Proof** We know $\sigma_0^{-T} = \sigma_0^{-1} = \sigma_0$ by $(x_i x_j) = (x_j x_i)$ commuting. It is obvious that $\delta R_{\sigma_0} = \sigma_0^{-T} \cdot \delta R$, $W_{\sigma_0} = \sigma_0^{-T} W_{\sigma_0}$ and $L_{\sigma_0} = \sigma_0^{-T} L_{\sigma_0}$ according to definitions. Then if $P$ is a Hodge potential scores for $(X, R)$, $\sigma_0^{-T} P$ is a Hodge potential scores for $(X, R_{\sigma_0})$ due to

$$L_{\sigma_0} \cdot \sigma_0^{-T} P = \sigma_0^{-T} L_{\sigma_0} \sigma_0^{-1} P = \sigma_0^{-T} L \cdot P = \sigma_0^{-T} \delta R = \delta R_{\sigma_0},$$

which proves the neutrality. \(\square\)

**Proposition 2** Hodge Potential Choice is strongly monotonic. Namely, for any $x_i \in \mathcal{H}p(X, R)$ and $\exists x_j \in X$ s.t $x_j Rx_i \in R$, we have $\mathcal{H}p(X, R') = \{x_i\}$ where $R' = R - \{x_j Rx_i\} + \{x_i Rx_j\}$.

**Proof** When we regard the dominance $B_{ji} = \{x_j Rx_i\}$ as a 1-form, it has the matrix expression as

$$B_{ji}(x_k, x_l) = \begin{cases} 
1, & k = i, l = j; \\
-1, & k = j, l = i; \\
0, & \text{else}.
\end{cases}$$

\(^8\)Multiplication of an eigenvalue means the dimension of the characteristic subspace corresponding to the eigenvalue. See Greub (2012).
Therefore, we have
\[ R' = \begin{cases} 
R - 2B_{ji}, & x_iRx_j \notin R; \\
R - B_{ji}, & x_iRx_j \in R,
\end{cases} \]
where \( W' = W \) and \( L' = L \) establish naturally. Subsequently, the Copeland scores
\[ c's' = \delta R' = \delta R - (2)\delta B_{ji}. \]
Suppose there exists a Hodge potential scores \( P' \) for \((X, R')\) and denote \( \tilde{P} := P - P' \). Then we have the difference of two Laplacian equations as
\[ L \cdot \tilde{P} = (2)\delta B_{ji}, \]
where this vector equation equivalents to following components equations:
\[ \sum_{l=1}^{n} L_{kl} \tilde{P}_l = \begin{cases} 
-(2)1, & k = i, \\
(2)1, & k = j, \\
0, & \text{else},
\end{cases} \quad (11) \]
According to the structure of \( L \), when components in any row are regarded as weights for the summations, the first case in (11) shows that \( \tilde{P}_i \) is smaller than the average of \( \tilde{P}_l \) for all \( W_{il} = 1 \). Similarly, \( \tilde{P}_j \) is larger than the average of \( \tilde{P}_l \) for all \( W_{jl} = 1 \), while else \( \tilde{P}_k \) equals to average of \( \tilde{P}_l \) for all \( W_{kl} = 1 \). It is easily derived from the connectedness of \((X, R)\) that \( \tilde{P}_i \) is the unique minimal and \( \tilde{P}_j \) is the unique maximal. On the other side, from \( x_i \in H_p(X, R) \), we have known \( P_i = P_{\text{max}} \) and \( \tilde{P}_i \) is the unique minimal. Therefore, \( P'_i = P_i - \tilde{P}_i \) is the unique maximal for the Hodge potential scores \( P' \) of \((X, R')\). In consequence, \( H_p(X, R') = \{x_i\} \) exclusively. \( \square \)

More intuitively, the strong monotonicity means that a shared winner in a game will occupy the solo winner position if who can turn the tide in one more round. This property points out that Hodge Potential Choice is a discriminative solution, which only arises shared winners in exceptional cases.

**Proposition 3** Hodge Potential Choice is independent to dominance cycle reversing. Namely, for any connected abstract game \((X, R)\) with a cycle \( C \) and \( R' = R-C+C^T \), we have
\[ H_p(X, R) = H_p(X, R'). \]

**Proof** By definitions, the cycle \( C \) can be regarded as 1-form on the base space \( W \). We know that for any \( x_iRx_j \in C \), there exist \( x_k, x_l \in X \) s.t. \( x_kRx_i, x_jRx_l \in C \). Therefore, 1 and \(-1\) appear in pairs in every row in skew-symmetric matrix \( C \), which means \( \delta C = 0 \) and \( C \) is harmonic by Lemma 4. We assume that \( R = dP + H \) is the Hodge decomposition for \((X, R)\). From \( C^T \) with its form representation as \(-C\), we have the Hodge decomposition \( R' = dP+(H-2C) \) for \((X, R')\) and the decomposition is proven to be unique by Lemma 5. Thus, \( P \) becomes one of the Hodge potential scores of \( R' \), which maintains the Hodge potential winner set. \( \square \)

Cycle reversing independence shows that Hodge Potential Choice admits dominance cycle reversing, which means HPC will not concentrate on details about the local dominance deference from a cycle. However, the base space
might change if cycles were removed or added directly. The base space, symbolizing the underlying game organization, influences almost everything about an abstract game, especially the Graph-Laplacian matrix, which determines the Hodge point scores. Thus, HPC is sensitive to changing of the base space, which will be illustrated in the following two properties. The proof of the two properties will be finished by arising examples.

**Proposition 4** Hodge Potential Choice is dependent on cycles removing or adding, i.e., there exists connected abstract game \((X, R)\) with a cycle \(C\) such that if \(R' = R - C\) then

\[ \mathcal{H}_P(X, R') \neq \mathcal{H}_P(X, R). \]

**Example 8** Recalling the game \((X, R)\) in Example 1 and 2, if we remove the cycle \(C_0 = [x_2Rx_5, x_5Rx_3, x_3Rx_2]\) and add \(C_1 = [x_2Rx_1, x_1Rx_4, x_4Rx_2]\), we get the new game \((X, R')\) where \(R' = R - C_0 + C_1\) presented in Figure 4.

![Fig. 4:](image)

Fig. 4: When the cycle \(C_0 = [x_2Rx_5, x_5Rx_3, x_3Rx_2]\) is replaced by \(C_1 = [x_2Rx_1, x_1Rx_4, x_4Rx_2]\) in the game \((X, R)\) in Figure 1, the new game \((X, R')\) has the different Hodge potential scores with the maximal as 0.5 at \(x_5\), which turns \(\mathcal{H}_P(X, R) = \{x_1\}\) into \(\mathcal{H}_P(X, R') = \{x_5\}\).

\(R'\) has its form representation as

\[
W' = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0
\end{bmatrix}, \quad R' = \begin{bmatrix}
0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & -1 & 1 \\
1 & -1 & 1 & 0 & 1 \\
0 & -1 & 0 & -1 & 0
\end{bmatrix}.
\]

Taking \((W', R')\) as the input of Algorithm 1, we gain the Hodge potential scores \(P' = [0.4, -0.5, 0, -0.4, 0.5]^T\) which derives \(\mathcal{H}_P(X, R') = \{x_5\}\). This provides an example for that the Copeland scores and Copeland winner set hold but Hodge potential winner set changes when dominance cycles are removed or added.
As the special case of the dominance cycles, the captivity to distinguish mutual dominances against empty rounds is one of the fundamental difference from HPC to traditional solutions such as Copeland Choice. Here we have

**Proposition 5** Hodge Potential Choice is sensitive to mutual dominances, i.e., for any mutual dominance $x_iIx_j \subset R$ in a connected game $(X, R)$, $R' = R - x_iIx_j$, it is generally incorrect that $Hp(X, R) = Hp(X, R')$.

**Example 9** Similarly with Example 8, if mutual dominances $x_1Ix_5$ and $x_2Ix_5$ in the game $(X, R)$ from Example 1 is removed, the new game $(X, R')$ presented in Figure 5 has its Hodge potential scores $P' = [1, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, 1]^T$ and the Hodge potential winner set $Hp(X, R) = \{x_1, x_5\}$.

![Fig. 5](image)

**Fig. 5**: If mutual dominances $x_1Ix_5$ and $x_2Ix_5$ are removed from the game $(X, R)$ in Figure 1, the Hodge potential winner set $Hp(X, R') = \{x_1, x_5\}$ equals to the Copeland winner set $Cp(X, R) = Cp(X, R') = \{x_1, x_5\}$. The red numbers denote the Hodge potential scores $P'$ with the maximal 1 at $x_1$ and $x_5$.

Based on the above examples, we may conclude that HPC admits changes when strength counterbalances among alternatives persist. The cycles containing balanced information, specifically mutual dominances, have essential differences from empty rounds. On the contrary, HPC coincides with Copeland Choice when an abstract game is complemented into a tournament by adding mutual dominances. This processing brings some nonexistent balanced information into the original game. “It is ridiculous to assume that a little girl might break even the world boxing champion just for no round organized between them.” The main problem of conventional solutions like Copeland Choice is that they ignore the difference between cycles and empty rounds. But Hodge Potential Choice solves it. Logically, counterbalanced information adding or vanishing will not be present in the “game arrangement” $W$ rather than the “game results” $R$. When the base space changes, its Graph-Laplacian matrix...
A new solution based on Hodge decomposition

3.3 Extended HPC for games with marginal strengths

In technical words, Hodge Potential Choice based on the form representation involve the differential tools and language to analyze abstract games, rather than consider the combinatorial properties of games merely. This idea can naturally extended to solving games with marginal strengths.

A **game with marginal strengths** is an abstract game \((X, R)\) endowed with a non-negative **marginal strength** \(M_{ij} \geq 0 \in \mathbb{R}\) for every dominance \(x_iRx_j \in R\), where \(M_{ij} = 0\) iff \(x_iRx_j \subset R\). A game with marginal strengths is denoted as \((X, R, M)\) and every pair \((x_iRx_j, M_{ij})\) in \((X, R, M)\) is read as the alternative \(x_i\) dominates the alternative \(x_j\) by \(M_{ij}\).

A game with marginal strengths can induce the abstract game naturally by forgetting the marginal strengths \(M\), and the non-negative regulation of \(M\) guarantees that the local dominance difference \(R\) can be recovered by normalizing \(-M + MT\). Therefore, a game with marginal strengths has its form representation \((W, \overline{M})\), where \(W\) is the base space presented by a symmetric adjacent matrix and \(\overline{M} := -M + MT\) is an 1-form presented by the skew-symmetric matrix whose normalization of non-zero elements is \(R\).

By Hodge decomposition, we can analyze the 1-form \(\overline{M}\) to gain the Hodge potential scores and winner set for the game \((W, \overline{M})\). Utilizing HPC on games with marginal strengths are called **Extended Hodge Potential Choice** (EHPC).

The algorithm of for EHPC games with marginal strengths has no fundamental difference from Algorithm 1. We directly provide an example to explain concepts and show the process of EHPC.

**Example 10** We have a game with marginal strengths \((X, R, M)\) presented in Figure 6 generated by endowing the marginal strengths \(M\) onto the abstract game \((X, R)\) from Example 1, where the marginal strengths \(M = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix}\), \(\overline{M} := -M + MT = \begin{bmatrix} 0 & -2 & -1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 \\ 1 & -3 & 0 & -5 & 2 \\ 0 & 0 & 5 & 0 & 2 \\ 0 & 0 & -2 & -2 & 0 \end{bmatrix}\).

The base space of \((X, R, M)\) is inherited from \(W\) in Example 2 and the game with marginal strengths \((X, R, M)\) has its form expression \((W, \overline{M})\).

The 1-form \(\overline{M}\) on \(W\) can be decomposed by Lemma 5 into the potential term \(dP\) and the harmonic term \(H\) (See Figure 7), which means we have

\[-M = dP + H = \begin{bmatrix} 0 & -2 & -\frac{2}{5} & 0 & -\frac{2}{5} \\ 2 & 0 & \frac{8}{5} & 0 & \frac{7}{5} \\ \frac{2}{5} & -\frac{8}{5} & 0 & -\frac{18}{5} & -\frac{8}{5} \\ 0 & 0 & \frac{18}{5} & 0 & \frac{14}{5} \\ \frac{3}{5} & -\frac{7}{5} & \frac{1}{5} & -\frac{17}{5} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\frac{2}{5} & 0 & \frac{3}{5} \\ 0 & 0 & \frac{7}{5} & 0 & -\frac{7}{5} \\ 3 & -\frac{7}{5} & 0 & -\frac{7}{5} & \frac{11}{5} \\ 0 & 0 & \frac{7}{5} & 0 & -\frac{7}{5} \\ -\frac{3}{5} & \frac{7}{5} & -\frac{11}{5} & \frac{7}{5} & 0 \end{bmatrix},\]

\[-L\] will change correspondingly, which naturally impacts HPC results despite the local dominance difference \(R\) invariant.
A new solution based on Hodge decomposition

where the Hodge potential scores $P = [0.6, -1.4, 0.2, 3.4, 0]^T$ determines the winner set $\mathcal{H}p(X, M) = \{x_1\}$. Besides, we have the tenseness of $(X, R, M)$

$$ts(X, M) := \langle H, H \rangle_{\langle M, M \rangle} \approx 0.2851,$$

which shows there exist wide strengths gaps among alternatives and the competition is relatively relaxed.

In Example 10, the alternative $x_3$ has highest pure marginal strength as 5, and it dominates $x_3$ and $x_2$ by significant strength gaps. However, $x_3$ is dominated by both $x_1$ and $x_5$. Obverse that EHPC determines the winner set excluding $x_3$. The result of EHPC unveils the mask of ”pseudo powerhouse” over $x_3$ in some extent.
Despite EHPC attributing to solve Laplacian equation (10) as same as HPC, there remains a large research space to explore the characteristics and applications of EHPC in sense of game theory.

4 Digital Experiments

In this section, we attempt to present some experimental results to show the computational advantages of Hodge Potential Choice. Due to the proposition 2, we take Copeland Choice as the reference for comparison. We will concern about the statistical performances of HPC to deal with games in distinct sizes and completeness.

In our experiments, we randomly generate matrix pairs \((W, R)\) to represent connected irregular abstract games with size \(n\) and completeness \(m\) as two variable parameters. With algorithm 1, we record cardinalities \(N(n, m)\) of Hodge potential winner set and statistic the frequencies of three difference cases:

1. the frequency \(F_t(n, m)\) of cases that \(C_p(W, R) \subset \mathcal{H}p(W, R)\) which means HPC is rougher than Copeland Choice;
2. the frequency \(F_e(n, m)\) of cases that \(C_p(W, R) = \mathcal{H}p(W, R)\) meaning equivalence between HPC and Copeland Choice;
3. the frequency \(F_r(n, m)\) of cases that \(C_p(W, R) \supset \mathcal{H}p(W, R)\) which means HPC is the refinement of Copeland Choice.

If the experiment shows that \(F_r(n, m) \gg F_t(n, u) \approx 0\) is always held for various \((n, m)\), we may claim that HPC is more precise than Copeland Choice from the statistical and computational views in Brandt et al (2016b).

Figure 8 shows the conditional averages of \(F_e, F_t, F_r\) along game sizes \(n \in [10, 20, \cdots, 190, 200]\) and the completeness \(m \in [0.1, 0.15, \cdots, 0.9, 0.95]\).
A new solution based on Hodge decomposition

changing. The experimental results show the closed relationship between the two solutions even for games far from complete. It is obviously that $F_t(u)$ increases along games tending to complete. This phenomenon conforms to proposition 2.

By the way, we record the cardinalities of winner sets in Figure 9, which supports the above argument that HPC is more refined than Copeland Choice and HPC rarely determine shared winners regardless of how Copeland winner set varies. Additionally, the almost constant average alternatives near 1 of HPC shows it probably decides a solo winner usually, which present the accuracy of HPC.

![Graphs showing average cardinalities of winner sets varying](image)

(a) along sizes of game changing  
(b) along completeness changing

**Fig. 9:** Average cardinalities of winner sets varying: the blue for Hodge Potential Choice and the red for Copeland Choice.

To sum up, the experimental results illustrate that Hodge Potential Choice have good statistical properties including rarely admitting shared winners and possibly more refined than Copeland Choice.

## 5 Conclusion

In this paper, we proposed a new solution, Hodge Potential Choice (HPC), for abstract games. We prove it is a universal and well-defined solution for connected abstract games according to the existence and uniqueness of the Hodge decomposition of 1-forms on graphs. According to our discussions, HPC possesses its natural mathematical structure combined with the good properties from viewpoints of gaming. It has been shown that Hodge Potential Choice coincides with Copeland Choice in complete cases and can be extended to games with marginal strengths. HPC satisfies three main axiomatic properties: neutrality, strong monotonicity, and cycle revering independence. Most importantly, HPC is sensitive to dominance cycles removing or adding, and it can discriminate mutual dominances against empty-rounds. To display the performances of HPC in large samples, we provide simulating examples and design
the digital experiments. The results show its advantages over conventional methods from the viewpoint of statistics.

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References

Anderson WN, Morley TD (1985) Eigenvalues of the laplacian of a graph. Linear & Multilinear Algebra 18(2):141–145

Arrow K, Raynaud H (1986) Social Choice and Multicriterion Decision-making. MIT Press, Cambridge

Bouyssou D, Vincke P (2010) Binary relations and preference modeling. Decision-making Process: Concepts and Methods pp 49–84

Brandt F (2011) Minimal stable sets in tournaments. Journal of Economic Theory 146(4):1481–1499

Brandt F, Fischer F (2007) Pagerank as a weak tournament solution pp 300–305

Brandt F, Harrenstein P (2010) Characterization of dominance relations in finite coalitional games. Theory and Decision 69(2):233–256

Brandt F, Brill M, Fischer F, et al (2014) Minimal retentive sets in tournaments. Social Choice and Welfare 42(3):551–574

Brandt F, Conitzer V, Endriss U, et al (2016a) Handbook of Computational Social Choice. Cambridge University Press, Cambridge

Brandt F, Conitzer V, Endriss U, et al (2016b) Handbook of Computational Social Choice. Cambridge University Press, Cambridge

do Carmo MP (1993) Riemannian Geometry. Cambridge University Press, Cambridge

Copeland AH (1951) A reasonable social welfare function. In: University of Michigan Seminar on Applications of Mathematics to the social sciences

Deemen AV (2013) Coalition Formation and Social Choice, vol 19. Springer Science & Business Media
A new solution based on Hodge decomposition

Diestel R (2000) Graph Theory. Spring-Verlag, New York

Dutta B, Jackson MO (2013) Networks and Groups: Models of Strategic Formation. Springer Science & Business Media

Figueira J, Greco S, Ehrgott M (2005) Multiple Criteria Decision Analysis: State of the Art Surveys, vol 78. Springer Science & Business Media

Fishburn PC (1977) Condorcet social choice functions. SIAM Journal on applied Mathematics 33(3):469–489

Greub WH (2012) Linear algebra, vol 23. Springer Science & Business Media

Han W, Deemen AV, Samsura DAA (2016) A note on extended stable sets. Social Choice and Welfare 47(2):265–275

Hatcher A (2002) Algebraic Topology. Cambridge University Press

Henriet D (1985) Copeland choice function an axiomatic characterization. Social Choice and Welfare (2):49–63

Hirsch MW, Smale S (1974) Differential Equations, Dynamical Systems, and Linear Algebra. New York

Ladouceur R, Gaboury A, Dumont M, et al (1988) Gambling: Relationship between the frequency of wins and irrational thinking. The Journal of Psychology 122(4):409–414

Laslier JF (1997) Tournament Solutions and Majority Voting. 7, Springer Verlag, Berlin

Lee JM (1997) Riemannian Manifolds. Springer, New York

Lee JM (2003) Introduction to Smooth Manifolds. Springer, Berlin, Germany

Mcgarvey D (1953) A theorem on the construction of voting paradoxes. Econometrica 21(4):608–610

de MelloLuiz Flyio Antran Monteiro Gomes JCCBS, Gomes, Mello EGG, et al (2005) Use of ordinal multi-criteria methods in the analysis of the formula 1 world championship. Cadernos Ebape BR 3(2):01–08

Neumaier A (2001) Introduction to Numerical Analysis. Cambridge University Press, Cambridge

Richardson M (1953) Solutions of irreflexive relations. Annals of Mathematics 58(3):275–290
Rubinstein (1980) Ranking the participants in a tournament. Siam Journal on Applied Mathematics 38(1):108–111

Schwarz G (1995) Hodge Decomposition: a Method For Solving Boundary Value Problems. Springer Berlin, Heidelberg

Scott WR (2012) Group theory. Courier Corporation

Shenoy PP (1980) A dynamic solution concept for abstract games. Journal of Optimization Theory and Applications 32(2):151–169

Subiza B, Peris JE (2014) A solution for general exchange markets with indivisible goods when indifferences are allowed. Mathematical Economics Letters 2(3-4):77–81

Tarjan RE (1984) Data structures and network algorithms. Society for Industrial and Applied Mathematics, Philadelphia

Taubes CH (2011) Differential Geometry: Bundles, Connections, Metrics and Curvature. Oxford University Press, Oxford