Homologically arc-homogeneous ENRs

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We prove that an arc-homogeneous Euclidean neighborhood retract is a homology manifold.

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1 Introduction

The so-called Modified Bing–Borsuk Conjecture, which grew out of a question in [1],
asserts that a homogeneous Euclidean neighborhood retract is a homology manifold.
At this mini-workshop on exotic homology manifolds, Frank Quinn asked whether a
space that satisfies a similar property, which he calls homological arc-homogeneity, is a
homology manifold. The purpose of this note is to show that the answer to this question
is yes.

2 Statement and proof of the main result

Theorem 2.1 Suppose that $X$ is an $n$–dimensional homologically arc-homogeneous
ENR. Then $X$ is a homology $n$–manifold.

Definitions A homology $n$–manifold is a space $X$ having the property that, for each
$x \in X$,

$$H_k(X, X - x; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

A Euclidean neighborhood retract (ENR) is a space homeomorphic to a closed subset
of Euclidean space that is a retract of some neighborhood of itself. A space $X$ is
homologically arc-homogeneous provided that for every path $\alpha : [0, 1] \to X$, the
inclusion induced map

$$H_k(X \times [0, 1], X \times [0, 1] - \Gamma(\alpha))$$

is an isomorphism, where $\Gamma(\alpha)$ denotes the graph of $\alpha$. The local homology sheaf $\mathcal{H}_k$
in dimension $k$ on a space $X$ is the sheaf with stalks $H_k(X, X - x), x \in X$. 

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By a result of Bredon [2, Theorem 15.2], if an $n$–dimensional space $X$ is cohomologically locally connected (over $\mathbb{Z}$), has finitely generated local homology groups $H_k(X, X - x)$ for each $k$, and if each $\mathcal{H}_k$ is locally constant, then $X$ is a homology manifold. We shall show that an $n$–dimensional, homologically arc-connected ENR satisfies the hypotheses of Bredon’s theorem.

Assume from now on that $X$ represents an $n$–dimensional, homologically arc-homogeneous ENR. Unless otherwise specified, all homology groups are assumed to have integer coefficients. The following lemma is a straightforward application of the definition and the Mayer–Vietoris theorem.

**Lemma 2.2** Given a path $\alpha: [0, 1] \to X$ and $t \in [0, 1]$, the inclusion induced map

$$H_s(X \times t, X \times t - (\alpha(t), t)) \to H_s(X \times I, X \times I - \Gamma(\alpha))$$

is an isomorphism.

Given points $x, y \in X$, an arc $\alpha: I \to X$ from $x$ to $y$, and an integer $k \geq 0$, let $\alpha_*: H_k(X, X - x) \to H_k(X, X - y)$ be defined by the composition

$$H_k(X, X - x) \xrightarrow{\times 0} H_s(X \times I, X \times I - \Gamma(\alpha)) \xrightarrow{\times 1} H_k(X, X - y).$$

Clearly $(\alpha^{-1})_* = \alpha_*^{-1}$ and $(\alpha\beta)_* = \beta_*\alpha_*$, whenever $\alpha/\beta$ is defined.

**Lemma 2.3** Given $x \in X$ and $\eta \in H_k(X, X - x)$, there is a neighborhood $U$ of $x$ in $X$ such that if $\alpha$ and $\beta$ are paths in $U$ from $x$ to $y$, then $\alpha_*(\eta) = \beta_*(\eta) \in H_k(X, X - y)$.

**Proof** We will prove the equivalent statement: for each $x \in X$ and $\eta \in H_k(X, X - x)$ there is a neighborhood $U$ of $x$ with $\alpha_*(\eta) = \eta$ for any loop $\alpha$ in $U$ based at $x$.

Suppose $x \in X$ and $\eta \in H_k(X, X - x)$. Since $H_k(X, X - x)$ is the direct limit of the groups $H_k(X, X - W)$, where $W$ ranges over the (open) neighborhoods of $x$ in $X$, there is a neighborhood $U$ of $x$ and an $\eta_U \in H_k(X, X - U)$ that goes to $\eta$ under the inclusion $H_k(X, X - U) \to H_k(X, X - x)$.

Suppose $\alpha$ is a loop in $U$ based at $x$. Let $\eta_\alpha \in H_k(X \times I, X \times I - \Gamma(\alpha))$ correspond to $\eta$ under the isomorphism $H_k(X, X - x) \xrightarrow{\times 0} H_k(X \times I, X \times I - \Gamma(\alpha))$ guaranteed by homological arc-homogeneity.

Let

$$\eta_{U \times t} = \eta_U \times 0 \in H_k(X \times I, X \times I - U \times I).$$
Then the image of $\eta_{0,t}$ in $H_k(X \times I, X \times I - \Gamma(\alpha))$ is $\eta_{\alpha}$, as can be seen by chasing the following diagram around the lower square.

\[
\begin{array}{ccc}
H_k(X, X - U) & \xrightarrow{\times 1} & H_k(X, X - x) \\
\cong & \downarrow & \cong \\
H_k(X \times I, X \times I - U \times I) & \xrightarrow{\times 0} & H_k(X \times I, X \times I - \Gamma(\alpha)) \\
\cong & \uparrow & \cong \\
H_k(X, X - U) & \xrightarrow{\times 1} & H_k(X, X - x)
\end{array}
\]

But from the upper square we see that $\eta_{\alpha}$ must also come from $\eta$ after including into $X \times 1$. That is, $\alpha_s(\eta) = \eta$. \hfill \Box

**Corollary 2.4** Suppose the neighborhood $U$ above is path connected and $F$ is the cyclic subgroup of $H_k(X, X - U)$ generated by $\eta_i$. Then, for every $y \in U$, the inclusion $H_k(X, X - U) \to H_k(X, X - y)$ takes $F$ one-to-one onto the subgroup $F_y$ generated by $\alpha_s(\eta)$, where $\alpha$ is any path in $U$ from $x$ to $y$.

**Lemma 2.5** Suppose $x, y \in X$ and $\alpha$ and $\beta$ are path-homotopic paths in $X$ from $x$ to $y$. Then $\alpha_s = \beta_s : H_k(X, X - x) \to H_k(X, X - y)$.

**Proof** By a standard compactness argument it suffices to show that, for a given path $\alpha$ from $x$ to $y$ and element $\eta \in H_k(X, X - x)$, there is an $\varepsilon > 0$ such that $\alpha_s(\eta) = \beta_s(\eta)$ for any path $\beta$ from $x$ to $y$ $\varepsilon$–homotopic (rel $\{x, y\}$) to $\alpha$.

Given a path $\alpha$ from $x$ to $y$, $\eta \in H_k(X, X - x)$, and $t \in I$, let $U_t$ be a path-connected neighborhood of $\alpha(t)$ associated with $(\alpha_s)_s(\eta) \in H_k(X, X - \alpha(t))$ given by Lemma 2.3, where $\alpha_t$ is the path $\alpha|[0, t]$. There is a subdivision

\[
\{0 = t_0 < t_1 < \cdots < t_m = 1\}
\]

of $I$ such that $\alpha([t_{i-1}, t_i]) \subseteq U_t$ for each $i = 1, \ldots, m$, where $U_t = U_t$ for some $t$. There is an $\varepsilon > 0$ so that if $H : I \times I \to X$ is an $\varepsilon$–path-homotopy from $\alpha$ to a path $\beta$, then $H([t_{i-1}, t_i] \times I) \subseteq U_t$.

For each $i = 1, \ldots, m$, let $\alpha_i = \alpha|[t_{i-1}, t_i]$ and $\beta_i = \beta|[t_{i-1}, t_i]$, and for $i = 0, \ldots, m$, let $\gamma_i = H[t_i \times I]$ and $\eta_i = (\alpha_i)_s(\eta)$. By Corollary 2.4,

\[
(\alpha_s)_{\eta_{i-1}} = (\gamma_{i-1}\beta\gamma_i^{-1})_s(\eta_{i-1}) = \eta_i
\]

where $\eta_0 = \eta$. Since $\gamma_0$ and $\gamma_m$ are the constant paths, it follows easily that

\[
\alpha_s(\eta) = (\alpha_m)_s \cdots (\alpha_1)_s(\eta) = (\beta_m)_s \cdots (\beta_1)_s(\eta) = \beta_s(\eta).
\]
Proof of Theorem 2.1  As indicated at the beginning of this note, we need only show that the hypotheses of [2, Theorem 15.2] are satisfied.

Since $X$ is an ENR, it is locally contractible, and hence cohomologically locally connected over $\mathbb{Z}$.

Given $x \in X$, let $W$ be a path-connected neighborhood of $x$ such that $W$ is contractible in $X$. If $\alpha$ and $\beta$ are two paths in $W$ from $x$ to a point $y \in W$, then $\alpha$ and $\beta$ are path-homotopic in $X$. Hence, by Lemma 2.5, $\alpha^* : H_k(X, X - x) \to H_k(X, X - y)$ is a well-defined isomorphism that is independent of $x$ for every $k \geq 0$. Hence, $H_k|W$ is the constant sheaf, and so $\mathcal{H}_k$ is locally constant.

Finally, we need to show that the local homology groups of $X$ are finitely generated. This can be seen by working with a mapping cylinder neighborhood of $X$. Assume $X$ is nicely embedded in $\mathbb{R}^{n+m}$, for some $m \geq 3$, so that $X$ has a mapping cylinder neighborhood $N = C_\phi$ of a map $\phi : \partial N \to X$, with mapping cylinder projection $\pi : N \to X$ (see [3]). Given a subset $A \subseteq X$, let $A^* = \pi^{-1}(A)$ and $\check{A} = \phi^{-1}(A)$.

Lemma 2.6  If $A$ is a closed subset of $X$, then $H_k(X, X - A) \cong \check{H}_{n+m-k}(A^*, \check{A})$.

Proof  Suppose $A$ is closed in $X$. Since $\pi : N \to X$ is a proper homotopy equivalence, $H_k(X, X - A) \cong H_k(N, N - A^*)$.

Since $\partial N$ is collared in $N$, $H_k(N, N - A^*) \cong H_k(\text{int } N, \text{int } N - A^*)$,

and by Alexander duality,

$H_k(\text{int } N, \text{int } N - A^*) \cong \check{H}_{n+m-k}(A^* - \check{A})$

$\cong \check{H}_{n+m-k}(A^*, \check{A})$

(since $\check{A}$ is also collared in $A^*$). 

Since $X$ is $n$–dimensional, we get the following corollary.

Corollary 2.7  If $A$ is a closed subset of $X$, then $\check{H}_q(A^*, \check{A}) = 0$, if $q < m$ or $q > n + m$.

Thus, the local homology sheaf $\mathcal{H}_k$ of $X$ is isomorphic to the Leray sheaf $\check{H}_{n+m-k}$ of the map $\pi : N \to X$ whose stalks are $\check{H}_{n+m-k}(x^*, x)$. For each $k \geq 0$, this sheaf is also locally constant, so there is a path-connected neighborhood $U$ of $x$ such that $\mathcal{H}_k|U$ is constant for all $q \geq 0$. Given such a $U$, there is a path-connected neighborhood $V$
of $x$ lying in $U$ such that the inclusion of $V$ into $U$ is null-homotopic. Thus, for any coefficient group $G$, the inclusion $H^p(U, G) \to H^p(V, G)$ is zero if $p \neq 0$ and is an isomorphism for $p = 0$.

The Leray spectral sequences of $\pi|\pi^{-1}(U)$ and $\pi|\pi^{-1}(V)$ have $E_2$ terms

$$E_2^{p,q}(U) \cong H^p(U; \mathcal{H}^q), \quad E_2^{p,q}(V) \cong H^p(V; \mathcal{H}^q)$$

and converge to

$$E_\infty^{p,q}(U) \subset H^{p+q}(U^*, \hat{U}; \mathbb{Z}), \quad E_\infty^{p,q}(V) \subset H^{p+q}(V^*, \hat{V}; \mathbb{Z}),$$

respectively (see [2, Theorem 6.1]). Since the sheaf $\mathcal{H}^q$ is constant on $U$ and $V$, $H^p(U; \mathcal{H}^q)$ and $H^p(V; \mathcal{H}^q)$ represent ordinary cohomology groups with coefficients in $G_q \cong \check{H}^q(x^*, \check{x})$.

By naturality, we have the commutative diagram

$$
\begin{array}{ccc}
E_2^{0,q}(U) & \longrightarrow & E_2^{2,q-1}(U) \\
\downarrow \cong & & \downarrow 0 \\
E_2^{0,q}(V) & \longrightarrow & E_2^{2,q-1}(V)
\end{array}
$$

which implies that the differential $d_2: E_2^{0,q}(V) \to E_2^{2,q-1}(V)$ is the zero map. Hence,

$$E_3^{0,q}(V) = \ker(E_2^{0,q}(V) \to E_2^{2,q-1}(V)) / \text{im}(E_2^{0,q}(V) \to E_2^{2,q-1}(V)) = E_2^{0,q}(V),$$

and, similarly, $E_3^{0,q}(V) = E_4^{0,q}(V) = \cdots = E_\infty^{0,q}(V)$. Thus,

$$H^q(V^*, \hat{V}; \mathbb{Z}) \supseteq E_\infty^{0,q}(V) \cong E_2^{0,q}(V) \cong H^q(V; \mathcal{H}^q) \cong H^q(V; G_q) \cong G_q.$$

Applying the same argument to the inclusion $(x^*, \check{x}) \subset (V^*, \hat{V})$ yields the commutative diagram

$$
\begin{array}{ccc}
E_2^{0,q}(V) & \longrightarrow & E_2^{2,q-1}(V) \\
\downarrow \cong & & \downarrow 0 \\
E_2^{0,q}(x) & \longrightarrow & E_2^{2,q-1}(x)
\end{array}
$$

which, in turn, gives

$$G_q \cong H^0(V; G_q) \xrightarrow{\cong} H^q(V^*, \hat{V}; \mathbb{Z})$$

$$\cong$$

$$G_q \cong H^0(x; G_q) \xrightarrow{\cong} H^q(x^*, \check{x}; \mathbb{Z}) \cong G_q$$
from which it follows that the inclusion \( H^q(V^*, V; \mathbb{Z}) \to H^q(x^*, \dot{x}; \mathbb{Z}) \cong G_q \) is an isomorphism of \( G_q \). Since \((x^*, \dot{x})\) is a compact pair in the manifold pair \((V^*, \dot{V})\), it has a compact manifold pair neighborhood \((W, \partial W)\). Since the inclusion \( H^q(V^*, V) \to H^q(x^*, \dot{x}) \) factors through \( H^q(W, \partial W) \), its image is finitely generated for each \( q \). Hence, \( H_k(X, X - x) \cong \tilde{H}^{n+m-k}(x^*, \dot{x}) \) is finitely generated for each \( k \).

The following theorem, which may be of independent interest, emerges from the proof of Theorem 2.1.

**Theorem 2.8** Suppose \( X \) is an \( n \)-dimensional ENR whose local homology sheaf \( \mathcal{H}_k \) is locally constant for each \( k \geq 0 \). Then \( X \) is a homology \( n \)-manifold.

**References**

1. R H Bing, K Borsuk, *Some remarks concerning topologically homogeneous spaces*, Ann. of Math. (2) 81 (1965) 100–111 MR0172255
2. G E Bredon, *Sheaf theory*, McGraw-Hill Book Co., New York (1967) MR0221500
3. R T Miller, *Mapping cylinder neighborhoods of some ANR’s*, Ann. of Math. (2) 103 (1976) 417–427 MR0402757

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