Stability of the cohomology rings of Hilbert schemes of points on surfaces

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Abstract

We establish some remarkable properties of the cohomology rings of the Hilbert scheme $X^{[n]}$ of $n$ points on a projective surface $X$, from which one sees to what extent these cohomology rings are (in)dependent of $X$ and $n$.

1 Introduction

Lehn [Leh] and more recently the authors [LQW1] have developed (vertex) algebraic calculus to study the cup products in the Hilbert schemes $X^{[n]}$ of $n$ points on a projective surface $X$. This approach was built on the earlier beautiful formula of Göttsche [Got] on the Betti numbers of $X^{[n]}$ and an important construction of Heisenberg algebra of Nakajima [Na1, Na2] and Grojnowski [Gro]. In [LQW1], we obtained a set of ring generators for the rational cohomology ring $H^*(X^{[n]})$, which has not been accessible in general by classical algebro-geometric methods (see however [Mar]). Using this set of generators, an algorithm, first pointed out by Lehn [Leh] in a restricted case, can be given to compute the cup product of any two cohomology classes in $H^*(X^{[n]})$ for an arbitrary projective surface $X$. This algebraic approach has been surprisingly effective in establishing new purely geometric results, as indicated in the further developments of Lehn-Sorger and the authors [LS1, LQW2, LS2] on the cohomology rings $H^*(X^{[n]})$. We refer to [Wa] for a detailed overview and further references on closely related topics.

In this paper, we establish some remarkable properties of the cohomology rings of the Hilbert scheme $X^{[n]}$ of $n$-points on a projective surface $X$, from which one sees to what extent these cohomology rings are (in)dependent of $X$ and $n$. As a consequence, we are able to introduce a ring $\mathcal{H}_X$ which encodes all the cohomology ring structures of $H^*(X^{[n]})$ for all $n$, and further determine its structure. To achieve

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these, we will extensively use and sharpen the techniques developed in the earlier works of Lehn and of the authors. Needless to say, the Heisenberg operators of Nakajima and Grojnowski are part of our basic vocabulary used in this paper.

We first obtain a quantitative description of the cup product of the ring generators given in $[\text{LQW1}]$, which indicate to what extent the cup product of cohomology classes of $X^{[n]}$ is (in)dependent of the canonical class $K_X$ and Euler class $e_X$ of $X$ (see Theorem $4.1$). As a corollary (which has been implicit in the earlier work $[\text{Leh}, \text{LQW1}]$), we see clearly that if there exists a ring isomorphism from $H^*(X)$ to $H^*(Y)$ for two projective surfaces $X$ and $Y$ which sends the canonical class $K_X$ to $K_Y$, then the cohomology rings of the Hilbert schemes $X^{[n]}$ and $Y^{[n]}$ are isomorphic for any $n$. In addition, we obtain the general structure of intersection numbers on $X^{[n]}$ in terms of intersection numbers on $X$. This general structure bears some similarities with the general structure of the Donaldson invariants from Donaldson theory (compare with $[\text{EGL}]$).

Using Theorem $4.1$, we work out the cup products of two cohomology classes which are monomials of Heisenberg generators, and observe that the cup products are independent of $n$ in an appropriate sense (see Theorem $5.1$). Roughly speaking, Theorem $5.1$ says that the cup product of certain cohomology classes in $H^*(X^{[n]})$ with $n$ being large can be read off from the cup product of cohomology classes in $H^*(X^{[m]})$ with $m$ being small. In other words, the cup product on $X^{[n]}$ partially determines the cup product on $X^{[n]}$ when $n > m$. This stability result enables us to construct a super-commutative associative ring $\mathcal{H}_X$, called the Hilbert ring associated to $X$, which captures all the information about the cohomology ring of the Hilbert scheme $X^{[n]}$ for each $n$. We further prove that $\mathcal{H}_X$ is isomorphic to a super-symmetric algebra with a simple set of generators which essentially comes from the set of ring generators for the cohomology rings $H^*(X^{[n]})$ found in $[\text{LQW2}]$.

In a sequel, we shall develop a counterpart of our results in terms of the orbifold cohomology rings $[\text{CR}]$ of the symmetric products, and clarify the connections with our present work. In another direction, it is natural to expect that results similar to those in the present paper hold as well for the quantum cohomology rings of the Hilbert schemes of points on projective surfaces.

The layout of the paper is as follows. In Sect. $3$, we collect some known results and definitions. In Sect. $4$, we establish a series of technical lemmas related to Heisenberg generators and pushforwards. In Sect. $5$, we work out the cup product of certain Chern characters in the cohomology ring of the Hilbert scheme, and derive some consequences. In Sect. $6$, we establish the stability of the cohomology ring of $X^{[n]}$. In Sect. $7$, we introduce the Hilbert ring and determine its structure.

**Conventions:** All cohomology groups are in $\mathbb{Q}$-coefficients. For a continuous map $p : Y_1 \to Y_2$ between two smooth compact manifolds and for $\alpha_1 \in H^*(Y_1)$, we define $p_*(\alpha_1)$ to be $\text{PD}^{-1} p_*(\text{PD}(\alpha_1))$ where $\text{PD}$ stands for the Poincaré duality.

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2 Generalities

Let $X$ be a smooth projective surface over $\mathbb{C}$, and $X^{[n]}$ be the Hilbert scheme of $n$-points in $X$. An element in the Hilbert scheme $X^{[n]}$ is represented by a length-$n$ 0-dimensional closed subscheme of $X$. It is well-known that $X^{[n]}$ is smooth. Let $\mathcal{Z}_n = \{ (\xi, x) \subset X^{[n]} \times X \mid x \in \text{Supp}(\xi) \}$, and $X^n$ be the $n$-th Cartesian product.

Definition 2.1

(i) Let $\mathbb{H} = \bigoplus_{n,i \geq 0} \mathbb{H}^{n,i}$ denote the double graded vector space with $\mathbb{H}^{n,i} \overset{\text{def}}{=} H^i(X^{[n]})$, and $\mathbb{H}_n \overset{\text{def}}{=} H^*(X^{[n]}) \overset{\text{def}}{=} \bigoplus_{i=0}^n H^i(X^{[n]})$. The element $1$ in $H^0(X^{[0]}) = \mathbb{Q}$ is called the vacuum vector and denoted by $|0\rangle$;

(ii) $f \in \text{End}(\mathbb{H})$ is homogeneous of bidegree $(\ell, m)$ if $f(\mathbb{H}^{n,i}) \subset \mathbb{H}^{n+\ell,i+m}$;

(iii) For $f$ and $g \in \text{End}(\mathbb{H})$ of bidegrees $(\ell, m)$ and $(\ell_1, m_1)$ respectively, define the Lie superalgebra bracket $[f, g]$ by putting $[f, g] = f \cdot g - (-1)^{mm_1} g \cdot f$.

A non-degenerate super-symmetric bilinear form $(,)$ on $\mathbb{H}$ is induced from the standard one on $\mathbb{H}_n = H^*(X^{[n]})$ defined by $(\alpha, \beta) = \int_{X^{[n]}} \alpha \beta$ for $\alpha, \beta \in H^*(X^{[n]})$. For $f \in \text{End}(\mathbb{H})$ of bidegree $(\ell, m)$, we can define its adjoint $f^\dagger \in \text{End}(\mathbb{H})$ by $(f(\alpha), \beta) = (-1)^{m-|\alpha|} \cdot (\alpha, f(\beta))$ where $|\alpha| = s$ if $\alpha \in H^s(X^{[n]})$. Note that the bidegree of $f^\dagger$ is $(-\ell, m - 4\ell)$. Also, for $g \in \text{End}(\mathbb{H})$ of bidegree $(\ell_1, m_1)$, we have

\begin{equation}
(f \cdot g)^\dagger = (-1)^{mm_1} \cdot g^\dagger \cdot f^\dagger \quad \text{and} \quad [f, g]^\dagger = -[f^\dagger, g^\dagger].
\end{equation}

We recall that the Heisenberg operators $a_n(\alpha) \in \text{End}(\mathbb{H})$ with $n \in \mathbb{Z}$ and $\alpha \in H^*(X)$ were defined in [Na1, Gro, Na2]. These operators satisfy the property $a_n(\alpha) = (-1)^s \cdot a_{-n}(\alpha)^\dagger$. In the next two sets of definitions, we collect various operators from [Leh, LQW1, LQW2]. We have adopted here the usual convention in the theory of vertex algebras on the signs of indices. For example, our indices for the Heisenberg and Virasoro generators coincide with those used in the paper [LS2], but differ exactly by a sign from the notations adopted in [Leh, LQW1].

Definition 2.2

(i) The normally ordered product : $a_{m_1} a_{m_2} :$ is defined by

\[ a_{m_1} a_{m_2} : = \left\{ \begin{array}{ll}
a_{m_1} a_{m_2}, & m_1 \leq m_2 \\
a_{m_2} a_{m_1}, & m_1 \geq m_2.
\end{array} \right. \]

For $n \in \mathbb{Z}$, define $\mathcal{L}_n : H^*(X) \rightarrow \text{End}(\mathbb{H})$ by $\mathcal{L}_n = -\frac{1}{2} \cdot \sum_{m \in \mathbb{Z}} : a_m a_{n-m} : \tau_{2s}$. Here for $k \geq 1$, $\tau_{k*} : H^*(X) \rightarrow H^*(X^k)$ is the map induced by the diagonal embedding $\tau_k : X \rightarrow X^k$, and $a_{m_1} \cdots a_{m_k}(\tau_{k*}(\alpha)) = \sum_j a_{m_1}(\alpha_{j,1}) \cdots a_{m_k}(\alpha_{j,k})$ when $\tau_{k*}(\alpha) = \sum_j \alpha_{j,1} \otimes \cdots \otimes \alpha_{j,k}$ via the Künneth decomposition of $H^*(X^k)$;
(ii) Define the linear operator \( \delta \in \text{End}(\mathbb{H}) \) by \( \delta = \bigoplus_n c_1(p_1, \mathcal{O}_X) \), where \( p_1 \) is the projection of \( X^{[n]} \times X \) to \( X^{[n]} \), and the first Chern class \( c_1(p_1, \mathcal{O}_X) \) of \( p_1 \mathcal{O}_X \) acts on \( \mathbb{H}_n = H^*(X^{[n]}) \) by the cup product.

(iii) For a linear operator \( f \in \text{End}(\mathbb{H}) \), define its derivative \( f' \) by \( f' = [\delta, f] \). The higher derivative \( f^{(k)} \) of \( f \) is defined inductively by \( f^{(k)} = [\delta, f^{(k-1)}] \).

**Definition 2.3**

(i) Fix \( i, n \geq 0 \) and \( \alpha \in H^*(X) \). Let \( G_i(\alpha, n) \) denote the \( H^{[\alpha]+2i}(X^{[n]}) \)-component of \( p_{1*}(\text{ch}(\mathcal{O}_X(p_1) \mathcal{O}_X(p_2)) \in \mathbb{H}_n \), where \( p_1 \) and \( p_2 \) are the two projections of \( X^{[n]} \times X \). Let \( B_i(\alpha, n) = 0 \) when \( i \geq n \), and \( B_i(\alpha, n) = 1/(n-i+1)! \cdot a_{-i}(1 \mathcal{X})^{n-i} a_{-(i+1)}(\alpha) \) when \( i < n \).

(ii) For \( i \geq 0 \) and \( \alpha \in H^*(X) \), the Chern character operator \( \varphi_i(\alpha) \in \text{End}(\mathbb{H}) \) is defined to be the operator which acts on the component \( \mathbb{H}_n \) by the cup product by \( G_i(\alpha, n) \). The operator \( \varphi_i(\alpha) \in \text{End}(\mathbb{H}) \) is defined to be the operator which acts on the component \( \mathbb{H}_n \) by the cup product by \( B_i(\alpha, n) \).

**Theorem 2.1** Let \( K_X \) be the canonical divisor of the smooth projective surface \( X \). Let \( k \geq 0, n, m \in \mathbb{Z} \) and \( \alpha, \beta \in H^*(X) \). Then,

(i) \([a_n(\alpha), a_m(\beta)] = -n \cdot \delta_{n+m} \cdot \int_X (\alpha \beta) \cdot \text{Id}_{\mathbb{H}} \) where \( \text{Id}_{\mathbb{H}} \) stands for the identity map of \( \mathbb{H} \), and \( \delta_{n+m} \) is 1 when \( n + m = 0 \) and 0 when \( n + m \neq 0 \);

(ii) \([\varphi_n(\alpha), a_m(\beta)] = -m \cdot a_{n+m}(\alpha \beta) \);

(iii) \( a'_n(\alpha) = n \cdot \varphi_n(\alpha) - n(n-1)/2 \cdot a_n(K_X \alpha) \);

(iv) \([\varphi_k(\alpha), a_{-1}(\beta)] = 1/k! \cdot a_{-1}^{(k)}(\alpha \beta) \);

(v) \[\ldots [\varphi_k(\alpha), a_{n_1}(\alpha_1)], \ldots, a_{n_{k+1}}(\alpha_{k+1})] = - \prod_{\ell=1}^{k+1} n_\ell \cdot a_{n_1 + \ldots + n_{k+1}}(\alpha \alpha_1 \ldots \alpha_{k+1}) \]

for all \( n_1, \ldots, n_{k+1} \in \mathbb{Z} \) with \( \sum_{\ell=1}^{k+1} n_\ell \neq 0 \) and all \( \alpha_1, \ldots, \alpha_{k+1} \in H^*(X) \).

Theorem 2.1 (i) was proved in [Na2]. The next two formulas in Theorem 2.1 were obtained in [Leh]. Theorem 2.1 (iv) and (v) were from [LQW1]. Also, as observed in [Na1, Gre], \( \mathbb{H} \) is an irreducible representation of the Heisenberg algebra generated by the \( a_i(\alpha) \)'s with the vacuum vector \([0] \in H^0(X^{[0]})\) being the highest weight vector. Our next Theorem was proved in [LQW1, LQW2].

**Theorem 2.2** For \( n \geq 1 \), the cohomology ring \( \mathbb{H}_n = H^*(X^{[n]}) \) is generated by the cohomology classes \( G_i(\alpha, n) \) (respectively, the cohomology classes \( B_i(\alpha, n) \)) where \( 0 \leq i < n \) and \( \alpha \) runs over a fixed linear basis of \( H^*(X) \).
3 Pushforwards and multi-commutators

In this section, we establish several technical lemmas concerning the pushforward maps $\tau_{k*}$ and multiple commutators. These lemmas will be used throughout the paper. We shall also introduce the concept of a universal linear combination.

Our first lemma about the pushforward maps $\tau_{k*}$ is elementary but plays an essential role in the entire paper. We remark that in this lemma and hereafter, $\tau_{k*}(\alpha)$ is understood to be $\int_X \alpha$ when $k = 0$ and $\alpha \in H^*(X)$.

**Lemma 3.1** Let $k, u \geq 1$ and $\alpha, \beta \in H^*(X)$. Assume that $\tau_{k*}(\alpha) = \sum_i \alpha_{i,1} \otimes \ldots \otimes \alpha_{i,k}$ under the Künneth decomposition of $H^*(X^k)$. Then for $0 \leq j \leq k$, we have

\[
\tau_{k*}(\alpha \beta) = \sum_i (-1)^{\beta_i} \sum_{\ell=j+1}^k |\alpha_{i,\ell}| \cdot \left( \otimes_{s=1}^{j-1} \alpha_{i,s} \right) \otimes (\alpha_{i,j} \beta) \otimes \left( \otimes_{t=j+1}^k \alpha_{i,t} \right)
\]

We have

\[
\tau_{(k-1)*}(\alpha \beta) = \sum_i (-1)^{\beta_i} \sum_{\ell=j+1}^k |\alpha_{i,\ell}| \int_X \alpha_{i,j} \beta \cdot \otimes_{1 \leq s \leq k, s \neq j} \alpha_{i,s}
\]

\[
\tau_{(k+u-1)*}(\alpha) = \sum_i \left( \otimes_{s=1}^{j-1} \alpha_{i,s} \right) \otimes (\tau_{u*} \alpha_{i,j}) \otimes \left( \otimes_{t=j+1}^k \alpha_{i,t} \right).
\]

**Proof.** The basic idea is to use the projection formula. We have

\[
\sum_i (-1)^{\beta_i} \sum_{\ell=j+1}^k |\alpha_{i,\ell}| \cdot \left( \otimes_{s=1}^{j-1} \alpha_{i,s} \right) \otimes (\alpha_{i,j} \beta) \otimes \left( \otimes_{t=j+1}^k \alpha_{i,t} \right) = \left( \sum_i \alpha_{i,1} \otimes \ldots \otimes \alpha_{i,k} \right) \cdot p_j^*(\beta) = \tau_{k*}(\alpha) \cdot p_j^*(\beta) = \tau_{k*}(\alpha \beta)
\]

where $p_j$ is the projection of $X^k$ to the $j$th factor. This proves the first formula. The proofs of the second formula and the third formula are similar. \(\square\)

**Lemma 3.2** Let $k, s \geq 1$, $n_1, \ldots, n_k, m_1, \ldots, m_s \in \mathbb{Z}$, and $\alpha, \beta \in H^*(X)$. Then,

(i) $[a_{n_1} \cdots a_{n_k}(\tau_{k*} \alpha), a_{m_1} \cdots a_{m_s}(\tau_{s*} \beta)]$ is equal to

\[
- \sum_{t=1}^k \sum_{j=1}^s n_t a_{n_t+m_j} \cdot \left( \prod_{l=1}^{j-1} a_{m_l} \prod_{1 \leq u \leq k, u \neq t} a_{n_u} \prod_{t=j+1}^s a_{m_l} \right) (\tau_{(k+s-2)*}(\alpha \beta));
\]

(ii) the derivative $(a_{n_1} \cdots a_{n_k}(\tau_{k*} \alpha))'$ is equal to

\[
- \sum_{j=1}^k n_j \cdot \sum_{m_1+m_2=n_j} a_{n_1} \cdots a_{n_{j-1}} : a_{m_1} a_{m_2} : a_{n_{j+1}} \cdots a_{n_k}(\tau_{(k+1)*}(\alpha))
\]

\[
- \sum_{j=1}^k \frac{n_j(n_j-1)}{2} a_{n_1} \cdots a_{n_k}(\tau_{k*}(K_X \alpha)).
\]
Definition 3.1 Let $X$ be a projective surface, $s \geq 1$, and $\alpha_1, \ldots, \alpha_s \in H^*(X)$. Let $k_1, \ldots, k_s \geq 0$, and $n_{ij} \in \mathbb{Z}$ with $1 \leq i \leq s$ and $1 \leq j \leq k_i$. Then, a universal linear combination of $a_{n_{i1}} \cdots a_{n_{ik_i}}(\tau_{k_i*}(\alpha_i))$, $1 \leq i \leq s$ is a linear combination of the form $\sum_{i=1}^s f_i(k_i, n_{i1}, \ldots, n_{ik_i})a_{n_{i1}} \cdots a_{n_{ik_i}}(\tau_{k_i*}(\alpha_i))$ where the coefficients $f_i(k_i, n_{i1}, \ldots, n_{ik_i})$ are independent of $X, \alpha_1, \ldots, \alpha_s$. A universal linear combination of $a_{n_{i1}} \cdots a_{n_{ik_i}}(\tau_{k_i*}(\alpha_i))|0)$, $1 \leq i \leq s$ is defined in a similar way.

Lemma 3.3 Let $k, s \geq 0$, $n, m_1, \ldots, m_s \in \mathbb{Z}$, and $\alpha, \beta_1, \ldots, \beta_s \in H^*(X)$. Then,

(i) $a^{(k)}_n(\alpha)$ is a universal linear combination of $a_{n_1} \cdots a_{n_{k-r+1}}(\tau_{k-r+1*}(K_X^r\alpha))$ where $0 \leq r \leq 2$ and $n_1 + \ldots + n_{k-r+1} = n$;

(ii) $[\cdots [a^{(k)}(\alpha), a_{m_1}(\beta_1)], \cdots, a_{m_s}(\beta_s)]$ is a universal linear combination of $a_{n_1} \cdots a_{n_{k-s-r+1}}(\tau_{k-s-r+1*}(K_X^{r-s}\alpha_1 \cdots \alpha_s))$ where $0 \leq r \leq 2$ and $n_1 + \ldots + n_{k-s-r+1} = n + m_1 + \ldots + m_s$.

Proof. Since $K_X^3 = 0$, (i) follows from repeatedly applying Lemma 3.2 (ii). Now (ii) follows from (i) and repeatedly applying Lemma 3.2 (i).

Lemma 3.4 Let $e_X$ denote the Euler class of $X$. Fix $k \geq 2$, $n_1, \ldots, n_k \in \mathbb{Z}$, and $\alpha \in H^*(X)$. Let $j$ satisfy $1 \leq j < k$. Then, $a_{n_1} \cdots a_{n_k}(\tau_{k*}\alpha)$ is equal to

\[\left(\prod_{1 \leq s < j} a_{n_s} \cdot a_{n_{j+1}}a_{n_j} \cdot \prod_{j+1 \leq s \leq k} a_{n_s}\right)(\tau_{k*}\alpha) - n_j \delta_{n_j+n_{j+1}} \prod_{1 \leq s \leq k, s \neq j, j+1} a_{n_s}(\tau_{k-2*}(e_X\alpha)).\]

Proof. Note that $\sum_\ell \beta_{t,1}^\ell \beta_{t,2} = e_X\beta$ if $\tau_{2*}(\beta) = \sum_\ell \beta_{t,1}^\ell \otimes \beta_{t,2}$. Now our result follows from Theorem 2.1 (i) and the second and third formulas in Lemma 3.1.

Lemma 3.5 Let $n \geq 1$, $\alpha \in H^*(X)$, and $\mathfrak{f} \in \text{End}(\mathfrak{h})$ with $\mathfrak{f}' = 0$. Then,

$[\mathfrak{f}, a_{-(n+1)}(\alpha)] = -\frac{1}{n} \cdot \{[\mathfrak{f}, a_{-(1)}(1_X)]', a_{-n}(\alpha)] + [a'_{-1}(1_X), [\mathfrak{f}, a_{-n}(\alpha)]]\}.$

Proof. Appeared implicitly in [Leh], and follows from Theorem 2.1 (ii), (iii).
4 Products of Chern characters

In this section, we prove that the products of Chern characters $G_k(\alpha, n)$ can be written as some universal finite linear combination of monomials of Heisenberg generators (see Theorem 4.1 below). As an application, we obtain the general written as some universal finite linear combination of monomials of Heisenberg

Theorem 4.1 will also be used substantially in later sections.

The following lemma is a variation of the Lemma 5.26 in [LQW1].

Lemma 4.1 Fix $k \geq 0$ and $b \geq 1$. Let $g \in \mathrm{End}(\mathbb{H})$ be of bidegree $(s, s)$ satisfying

\begin{equation}
[[\cdots [g, a_{n_1}(\alpha_1)], \cdots], a_{n_{k+1}}(\alpha_{k+1})] = 0
\end{equation}

for any $n_1, \ldots, n_{k+1} < 0$ and $\alpha_1, \ldots, \alpha_{k+1} \in H^*(X)$. Let $A = a_{m_1}(\beta_1) \cdots a_{m_b}(\beta_b)[0]$ where $m_1, \ldots, m_b < 0$ and $\beta_1, \ldots, \beta_b \in H^*(X)$. Then, $g(A)$ is equal to

\[
\sum_{i=0}^{k} \sum_{\sigma_i} (-1)^{s} \sum_{i \in \sigma_i^0} \sum_{j=1}^{i} \sum_{l \geq \sigma_i(j)} |\beta_{\sigma_i(j)}||\beta_{l}| \cdot \prod_{\ell \in \sigma_i^0} a_m(\beta_{\ell})[[\cdots [g, a_{m}(\beta_{\sigma_i(1)}), \cdots], a_{m}(\beta_{\sigma_i(i)})]|0\}
\]

where for each fixed $i$, $\sigma_i$ runs over all the maps $\{1, \ldots, i\} \to \{1, \ldots, b\}$ satisfying $\sigma_i(1) < \cdots < \sigma_i(i)$, and $\sigma_i^0 = \{\ell \mid 1 \leq \ell \leq b, \ell \neq \sigma_i(1), \ldots, \sigma_i(i)\}$. \hfill \square

Lemma 4.2 Let $s \geq 1$, and $\alpha, \beta \in H^*(X)$. Then, $[\mathfrak{g}_k(\alpha), a_{n_1} \cdots a_{n_s}(\tau_{s, \beta})]$ is a universal linear combination of expressions $a_{m_1} \cdots a_{m_{k+s-r}}(\tau_{k+s-r,s}(K^{s}_X \alpha \beta))$ where $0 \leq r \leq 2$ and $m_1 + \cdots + m_{k+s-r} = n_1 + \cdots + n_s$.

Proof. First of all, let $s = 1$. Note that $a_0(\beta) = 0$. Since $\mathfrak{g}_k(\alpha)^\dagger = \mathfrak{g}_k(\alpha)$ and $a_{n_1}(\beta)^\dagger = (-1)^{n_1} a_{-n_1}(\beta)$, we see from (4) that $[\mathfrak{g}_k(\alpha), a_{n_1}(\beta)]^\dagger = -[\mathfrak{g}_k(\alpha)^\dagger, a_{n_1}(\beta)^\dagger] = (-1)^{1+n_1} [\mathfrak{g}_k(\alpha), a_{-n_1}(\beta)]$. Since $a_{m_1} \cdots a_{m_{k+s-r}}(\tau_{k+s-r,s}(K^{s}_X \alpha \beta))^\dagger$ is equal to

\[
(-1)^{m_1+\cdots+m_{k+s-r}} \cdot a_{-m_{k+s-r}} \cdots a_{-m_1}(\tau_{k+s-r,s}(K^{s}_X \alpha \beta)),
\]

we need only to prove the statement for $[\mathfrak{g}_k(\alpha), a_{n_1}(\beta)]$ with $n_1 \leq -1$. When $n_1 = -1$, $[\mathfrak{g}_k(\alpha), a_{n_1}(\beta)] = 1/k! \cdot a^{(k)}_{-1}(\alpha \beta)$. So the statement for $s = 1$ and $n_1 = -1$ follows from Lemma 3.3 (i). When $n_1 \leq -2$, we see from Lemma 3.5 that

\[
[\mathfrak{g}_k(\alpha), a_{n_1}(\beta)]
\]

\[
= \frac{1}{n_1+1} \cdot \left\{ [[\mathfrak{g}_k(\alpha), a_{-1}(1_X)], a_{n_1+1}(\beta)] + [a_{-1}(1_X), [\mathfrak{g}_k(\alpha), a_{n_1+1}(\beta)]] \right\}
\]

\[
= \frac{1}{n_1+1} \cdot \left\{ [[\mathfrak{g}_k(\alpha), a_{-1}(1_X)], a_{n_1+1}(\beta)]
\]

\[
+ [a_{-1}(1_X), [\mathfrak{g}_k(\alpha), a_{n_1+1}(\beta)] - [a_{-1}(1_X), [\mathfrak{g}_k(\alpha), a_{n_1+1}(\beta)]] \right\}.
\]
So the statement for \( s = 1 \) and \( n_1 \leq -2 \) follows from induction and Lemma 3.2.

Next, let \( s \geq 2 \). Let \( \tau_{s*}(\beta) = \sum_i \beta_{i,1} \otimes \cdots \otimes \beta_{i,s} \in H^*(X^s) \). Then, we have \([\mathcal{F}_k(\alpha), a_{n_1} \cdots a_{n_s}(\tau_{s*})] = \sum_i [\mathcal{F}_k(\alpha), a_{n_1}(\beta_{i,1}) \cdots a_{n_s}(\beta_{i,s})] \). By symmetry, it suffices to show that \( \sum_i [\mathcal{F}_k(\alpha), a_{n_1}(\beta_{i,1})]a_{n_2}(\beta_{i,2}) \cdots a_{n_s}(\beta_{i,s}) \) is a universal linear combination of the forms \( a_{m_1} \cdots a_{m_{k+s-r}}(\tau_{(k+s-r)})(K_X^r \alpha \beta) \) where \( 0 \leq r \leq 2 \) and \( m_1 + \cdots + m_{k+s-r} = n_1 + \cdots + n_s \). To prove this, we apply what we have already proved in the preceding paragraph to \([\mathcal{F}_k(\alpha), a_{n_1}(\beta_{i,1})] \). So \([\mathcal{F}_k(\alpha), a_{n_1}(\beta_{i,1})] \) equals

\[
\sum_{0 \leq r \leq 2} f_r(k, n_1, m_1, \ldots, m_{k-r+1}) a_{m_1} \cdots a_{m_{k-r+1}}(\tau_{(k-r+1)})(K_X^r \alpha \beta_{i,1})
\]

where \( f_r(k, n_1, m_1, \ldots, m_{k-r+1}) \) stands for universal rational numbers independent of \( X, \alpha, \) and \( \beta_{i,1} \). In particular, these universal numbers are independent of \( i \). So

\[
\sum_i [\mathcal{F}_k(\alpha), a_{n_1}(\beta_{i,1})]a_{n_2}(\beta_{i,2}) \cdots a_{n_s}(\beta_{i,s}) = \sum_{0 \leq r \leq 2} f_r(k, n_1, m_1, \ldots, m_{k-r+1}).
\]

By the first formula in Lemma 3.1, \( \sum_i (K_X^r \alpha \beta_{i,1}) \otimes \beta_{i,2} \otimes \cdots \otimes \beta_{i,s} = \tau_{s*}(K_X^r \alpha \beta) \). So \( \tau_{(k+r-1)}(K_X^r \alpha \beta_{i,1}) \otimes \beta_{i,2} \otimes \cdots \otimes \beta_{i,s} = \tau_{(k+s-r)}(K_X^r \alpha \beta) \). It follows that \( \sum_i [\mathcal{F}_k(\alpha), a_{n_1}(\beta_{i,1})]a_{n_2}(\beta_{i,2}) \cdots a_{n_s}(\beta_{i,s}) \) equals

\[
\sum_{0 \leq r \leq 2} f_r(k, n_1, m_1, \ldots, m_{k-r+1}) \left( \prod_{\ell=1}^{k-r+1} a_{m_{\ell+r-1}} \prod_{t=2}^{s} a_{n_t} \right)(\tau_{(k+s-r)})(K_X^r \alpha \beta))
\]

where \( f_r(k, n_1, m_1, \ldots, m_{k-r+1}) \) are independent of \( X, \alpha, \) and \( \beta_{i,1} \).

\[\square\]

**Definition 4.1**

(i) Let \( s \geq 1 \), and \( \alpha_1, \ldots, \alpha_s \in H^*(X) \) be homogeneous. For a partition \( \pi = \{\pi_1, \ldots, \pi_j\} \) of the set \( \{1, \ldots, s\} \), we fix the orders of the elements listed in each subset \( \pi_i \) \((1 \leq i \leq j) \) once and for all, and define \( \ell(\pi) = j \), \( \alpha_{\pi_i} = \prod_{m \in \pi_i} \alpha_m \), and \( \operatorname{sign}(\alpha, \pi) \) by \( \prod_{i=1}^{\ell(\pi)} \alpha_{\pi_i} = \operatorname{sign}(\alpha, \pi) \cdot \prod_{i=1}^{s} \alpha_i \).

(ii) We denote \( 1_{-n} = 1/n! \cdot a_{-1}(1_X)^n \) when \( n \geq 0 \), and \( 1_{-n} = 0 \) when \( n < 0 \).

The geometric meaning of \( 1_{-n} \) is that \( 1_{-n}(0) = 1_{X[n]} \) (the fundamental class of \( X^{[n]} \)). Also, the choice of the orders for the elements listed in each \( \pi_i, 1 \leq i \leq \ell(\pi) \) will affect \( \operatorname{sign}(\alpha, \pi) \), but will not affect the expression \( \prod_{i=1}^{s} G_{k_i}(\alpha_i, n) \) stands for \( 1_{X[n]} = 1_{-n}(0) \) by convention.
Theorem 4.1 Let \( n \geq 1, s \geq 0, k_1, \ldots, k_s \geq 0 \), and \( \alpha_1, \ldots, \alpha_s \in H^s(X) \) be homogeneous. Then, \( \prod_{i=1}^{s} G_k(\alpha_i, n) \) is a finite linear combination of expressions:

\[
\text{sign}(\alpha, \pi) \cdot \mathbf{1}_{-(n-\bar{n})} \left( \prod_{i=1}^{\ell(\pi)} \left( \prod_{j=1}^{m_i-r_i} a_{-n_{i,j}} \right) \left( \tau_{(m_i-r_i)}(\epsilon_i \alpha_{\pi_i}) \right) \right) |0\)
\]

whose coefficients are independent of \( X, \alpha_1, \ldots, \alpha_s, \) and the integer \( n \). Here \( \pi \) runs over all partitions of \( \{1, \ldots, s\} \), \( \epsilon_i \in \{1_X, K_X, K^2_X, e_X\} \), \( r_i = |\epsilon_i|/2 \leq m_i \leq 2 + \sum_{j \in \pi_i} k_j \), \( 0 < n_{i,1} \leq \ldots \leq n_{i,m_i-r_i} \), \( \sum_{j \in \pi_i} n_{i,j} \leq \sum_{j \in \pi_i} (k_j + 1) \) for each \( i \), and

\[
\sum_{i=1}^{\ell(\pi)} (m_i - 2 + \sum_{j=1}^{m_i-r_i} n_{i,j}) = \sum_{i=1}^{s} k_i.
\]

Proof. Use induction on \( s \). When \( s = 0 \), the statement is trivial by our convention. Next, let \( s \geq 1 \). By induction, \( \prod_{i=2}^{s} G_k(\alpha_i, n) \) is a linear combination of expressions:

\[
\text{sign}(\alpha, \sigma) \cdot \mathbf{1}_{-(n-\bar{n})} \left( \prod_{i=1}^{\ell(\sigma)} \left( \prod_{j=1}^{m_i-r_i} a_{-n_{i,j}} \right) \left( \tau_{(m_i-r_i)}(\epsilon_i \alpha_{\sigma_i}) \right) \right) |0\)
\]

where \( \sigma \) runs over all partitions of \( \{2, \ldots, s\} \), \( \epsilon_i \in \{1_X, K_X, K^2_X, e_X\} \), \( r_i = |\epsilon_i|/2 \leq m_i \leq 2 + \sum_{j \in \pi_i} k_j \), \( 0 < n_{i,1} \leq \ldots \leq n_{i,m_i-r_i} \), \( \sum_{j \in \pi_i} n_{i,j} \leq \sum_{j \in \pi_i} (k_j + 1) \), and \( \bar{n} = \sum_{i=1}^{\ell(\sigma) m_i-r_i} \sum_{j=1}^{m_i-r_i} n_{i,j} \). Moreover, the coefficients in the linear combination are independent of \( X, \alpha_2, \ldots, \alpha_s \) and \( n \). Now apply \( \mathfrak{S}_{k_1}(\alpha_1) \) to \( \prod_{i=2}^{s} G_k(\alpha_i, n) \), and move \( \mathfrak{S}_{k_1}(\alpha_1) \) to the right by using Lemma 4.1. Note that \( |\tau_{(m_i-r_i)}(\epsilon_i \alpha_{\pi_i})| \equiv |\alpha_{\pi_i}| \pmod{2} \). By Theorem 2.1 (v), \( \left[ [\mathfrak{S}_{k_1}(\alpha_1), a_{\ell_1}((\beta_1)), \ldots, a_{\ell_{k_1+2}}((\beta_{k_1+2}))] = 0 \right. \) when \( \ell_1, \ldots, \ell_{k_1+2} < 0 \). Since \( \mathfrak{S}_{k_1}(\alpha_1)|0\) = 0 and \( \prod_{i=1}^{s} G_k(\alpha_i, n) = \mathfrak{S}_{k_1}(\alpha_1) \left( \prod_{i=2}^{s} G_k(\alpha_i, n) \right) \), we see from Lemma 4.1 that \( \prod_{i=1}^{s} G_k(\alpha_j, n) \) is a universal linear combination of expressions:

\[
\frac{\text{sign}(\alpha, \sigma)}{(n-\bar{n})!} \left( \begin{array}{c} n - \bar{n} \\ t \end{array} \right) (-1)^{|\alpha_1| \sum_{1 \leq i \leq \ell(\sigma), i \notin U} |\alpha_{\sigma_i}| + \sum_{v=1}^{w} \sum_{w>v \in U} |\alpha_{\sigma_w}| |\alpha_{\sigma_v}|}
\]
where \(0 \leq t \leq (k_1 + 1), u \geq 0, (t + u) \geq 1\), \(U = \{i_1, \ldots, i_u\} \subset \{1, \ldots, \ell(\sigma)\}\) with \(i_1 < \ldots < i_u\). Let \(\pi\) be the partition of \(\{1, \ldots, s\}\) consisting of all the \(\sigma_i\) with \(1 \leq i \leq \ell(\sigma)\) and \(i \notin U\), and \(\{1\} \prod_{i \in U} \left( \prod_{\sigma_i \in U} \right)\). Then, \(\alpha_{\pi(\sigma)} = \alpha_{\sigma_1} \cdots \alpha_{\sigma_u}\) and

\[
\text{sign}(\alpha, \pi) = \text{sign}(\alpha, \sigma) \cdot (-1)^{|\alpha_2| \sum_{1 \leq i < \ell(\sigma), \sigma_i \in U} |\alpha_{\sigma_i}| + \sum_{u=1}^{n} \sum_{w=g \in U} |\alpha_{\sigma_w}| |\alpha_{\sigma_u}|}.
\]

In view of (5) and (6), \(\prod_{i=1}^{s} G_{k_i}(\alpha, n)\) is a linear combination of

\[
\text{sign}(\alpha, \pi) \cdot 1_{-(n-\tilde{n}-t)} \left( \prod_{1 \leq i < \ell(\pi)} \left( \prod_{j=1}^{m_i-r_i} a_{-n_{i,j}} \right) (\tau(m_{i-r_i})* (\epsilon_{i\alpha_{\pi_i}})) \right) \\
\cdot \left[ \cdots [G_{k_1}(\alpha_1), a_{-1}(1X)], \cdots, a_{-1}(1X) \right] \left( \prod_{j=1}^{m_{i_1}-r_{i_1}} a_{-n_{i_1,j}} \right) (\tau(m_{i_1-r_{i_1}})* (\epsilon_{i_1\alpha_{\pi_{i_1}}}))], \\
\cdots, \left( \prod_{j=1}^{m_{i_u}-r_{i_u}} a_{-n_{i_u,j}} \right) (\tau(m_{i_u-r_{i_u}})* (\epsilon_{i_u\alpha_{\pi_{i_u}}})) \right] |0\).
\]

with all the coefficients being independent of \(X, \alpha_1, \ldots, \alpha_s\) and \(n\). Also notice that in the expression (7), the only factor depending on \(n\) is \(1_{-(n-\tilde{n}-t)}\).

Let \(t = 0\). Then, \(u \geq 1\). By Lemma 4.2, Lemma 3.2 (i) and Lemma 3.4, each expression (7) is a universal linear combination of expressions of the form

\[
\text{sign}(\alpha, \pi) \cdot 1_{-(n-\tilde{n})} \left( \prod_{1 \leq i < \ell(\pi)} \left( \prod_{j=1}^{m_i-r_i} a_{-n_{i,j}} \right) (\tau(m_{i-r_i})* (\epsilon_{i\alpha_{\pi_i}})) \right) \\
\cdot a_{-n_1} \cdots a_{-n_{m-r}} (\tau(m-r)* (\epsilon_{\alpha_{\pi_{1}}\cdots\alpha_{\pi_{m}}}) |0\}
\]

\[
= \text{sign}(\alpha, \pi) \cdot 1_{-(n-\tilde{n})} \left( \prod_{1 \leq i < \ell(\pi)} \left( \prod_{j=1}^{m_i-r_i} a_{-n_{i,j}} \right) (\tau(m_{i-r_i})* (\epsilon_{i\alpha_{\pi_i}})) \right) \\
\cdot a_{-n_1} \cdots a_{-n_{m-r}} (\tau(m-r)* (\epsilon_{\alpha_{\pi_{1}}\cdots\alpha_{\pi_{m}}}) |0\}
\]
which is of the form (3). Here \( \epsilon = \bar{\epsilon}e_{i_1} \cdots e_{i_u} \) with \( \bar{\epsilon} \in \{ 1_X, K_X, K_X^2, e_X \} \), \( r = |\epsilon|/2 + \sum_{j=1}^{u} r_{ij} \), \( m = k + \sum_{j=1}^{u} m_{ij} - 2(u - 1) \), \( 0 < n_1 \leq \ldots \leq n_{m-r} \), and

\[
n_1 + \ldots + n_{m-r} = \sum_{i \in U} \sum_{j=1}^{m_i-r_i} n_{i,j} \leq \sum_{i \in U} \sum_{j \in \sigma_i} (k_j + 1) < (k_1 + 1) + \sum_{i \in U} \sum_{j \in \sigma_i} (k_j + 1) = \sum_{j \in \pi_i(\alpha)} (k_j + 1).
\]

Note that either \( \epsilon = 0 \) or \( \epsilon \in \{ 1_X, K_X, K_X^2, e_X \} \). When \( \epsilon \in \{ 1_X, K_X, K_X^2, e_X \} \), we have \( r = |\epsilon|/2 \leq m \). Since \( m_{iu} \leq 2 + \sum_{j \in \sigma_{iu}} k_j \) for \( 1 \leq u \leq u \), we obtain

\[
m = k_1 + m_{i_1} + \ldots + m_{i_u} - 2(u - 1) \leq 2 + \sum_{j \in \pi_i(\alpha)} k_j.
\]

Next, assume that \( t \geq 1 \). Then by Theorem 2.1 (iv), we have

\[
[\ldots [\mathfrak{G}_{k_1}(\alpha_1), a_{-1}(1_X)], \ldots, a_{-1}(1_X)] = \frac{1}{k_1!} [\ldots [a_{(k_1)}(\alpha_1), a_{-1}(1_X)], \ldots, a_{-1}(1_X)]
\]

which by Lemma 3.3 (ii), is a universal linear combination of expressions of the form

\[
a_{n_1} \cdots a_{n_{k_1-t+2-t}}(\tau_{(k_1-t+2-t)}(K_X^r \alpha_1)) \quad \text{where} \quad 0 \leq r \leq 2 \quad \text{and} \quad n_1 + \ldots + n_{k_1-t+2-t} = -t.
\]

So by Lemma 3.2 (i) and Lemma 3.4, (7) is a universal linear combination of

\[
\operatorname{sign}(\alpha, \pi) \cdot \mathbf{1}_{-(n-\bar{n})} \left( \prod_{1 \leq i < \ell_1(\pi)} \left( \prod_{j=1}^{m_i-r_i} a_{-n_{i,j}} \right) (\tau_{(m_i-r_i)}(\epsilon_i \alpha_{\pi_i})) \right) \cdot \\
\cdot a_{-n_1} \cdots a_{-n_{m-r}}(\tau_{(m-r)}(\epsilon \alpha_{\sigma_1} \cdots \alpha_{\sigma_{iu}})[0])
\]

which again is of the form (3). Here \( m = (k_1 - t + 2) + m_{i_1} + \ldots + m_{i_u} - 2u < 2 + \sum_{j \in \pi_i(\alpha)} k_j \), \( \epsilon \in \{ 1_X, K_X, K_X^2, e_X \} \), \( r = |\epsilon|/2 \leq m \), \( 0 < n_1 \leq \ldots \leq n_{m-r} \), and \( n_1 + \ldots + n_{m-r} = t + \sum_{i \in U} \sum_{j=1}^{m_i-r_i} n_{i,j} \leq \sum_{j \in \pi_i(\alpha)} (k_j + 1) \) since \( t \leq (k_1 + 1) \).

Finally, the cohomology degree of \( \prod_{i=1}^{s} G_{k_i}(\alpha_i, n) \) is equal to \( \sum_{i=1}^{s} (2k_i + |\alpha_i|) \). Comparing this with the cohomology degree of (3), we obtain (4). \( \square \)

**Corollary 4.1** Let \( X \) and \( Y \) be two complex projective surfaces. Assume that there exists a ring isomorphism \( \Phi : H^*(X) \rightarrow H^*(Y) \) with \( \Phi(K_X) = K_Y \). Then for every \( n \geq 1 \), the two cohomology rings \( H^*(X[n]) \) and \( H^*(Y[n]) \) are isomorphic.
Proof. Note that \( \Phi(e_X) = e_Y \). Since the Chern characters \( G_k(\alpha, n) \) generate the cohomology ring \( H^*(X^{[n]}) \), our result follows from Theorem 4.1. \( \square \)

Next, we apply Theorem 4.1 to study intersection numbers in the Hilbert scheme \( X^{[n]} \). For this purpose, we establish the notation \( \langle w \rangle = \int_Y w \) where \( w \in H^*(Y) \) and \( Y \) stands for a smooth projective variety.

Corollary 4.2 Let \( n, s \geq 1, k_1, \ldots, k_s \geq 0, \) and let \( \alpha_1, \ldots, \alpha_s \in H^*(X) \) be homogeneous cohomology classes. Assume \( \sum_{i=1}^s (2k_i + |\alpha_i|) = 4n \). Then, \( \left\langle \prod_{i=1}^s G_k(\alpha_i, n) \right\rangle \) is a finite linear combination of \( \text{sign}(\alpha, \pi) \cdot \prod_{i=1}^{\ell(\pi)} \langle \epsilon_i \alpha_{\pi_i} \rangle \) where \( \pi \) runs over all partitions of \( \{1, \ldots, s\} \), \( \epsilon_i \in \{1_X, K_X, K^2_X, e_X\} \). Moreover, all the coefficients in this linear combination are independent of \( X, \alpha_1, \ldots, \alpha_s \) and \( n \).

Proof. Note that the positive generator of \( H^{4n}(X^{[n]}) \cong \mathbb{Q} \) is \( a_{-1}([x])^{n}|0) \) where \( [x] \in H^4(X) \) stands for the cohomology class corresponding to a point \( x \in X \). So by Theorem 4.1, an expression (3) nontrivially contributing to \( \left\langle \prod_{i=1}^s G_k(\alpha_i, n) \right\rangle \) must satisfy: i) \( n_{i,j} = 1 \) for all \( 1 \leq i \leq \ell(\pi) \) and \( 1 \leq j \leq m_i - r_i \); ii) \( \epsilon_i \alpha_{\pi_i} = \langle \epsilon_i \alpha_{\pi_i} \rangle \cdot [x] \) for all \( 1 \leq i \leq \ell(\pi) \); and iii) \( n - \sum_{i=1}^{\ell(\pi)} (m_i - r_i) = 0 \). Since \( \tau_{k_*}([x]) = [x] \otimes \cdots \otimes [x] \) for all \( k \geq 0 \), our conclusion follows immediately from (3). \( \square \)

5 The stability

In this section, we establish a remarkable stability for the cohomology rings of the Hilbert schemes of \( n \)-points on projective surfaces as \( n \) varies.

We need two lemmas which sharpen the Lemma 3.20 and Lemma 3.5 in [LQW2]. In the first lemma, we determine the leading monomial of Heisenberg generators in \( \prod_{i=1}^s G_k(\alpha_i, n) \). In the second lemma, we express \( 1_{-(n - \sum_{i=1}^s n_i)} \left( \prod_{i=1}^s a_{-n_i}(\alpha_i) \right) |0) \) as a universal finite linear combination of cup products of the form \( \prod_{j=1}^t \prod_{i=1}^s G_{m_i}(\beta_j, n) \).

Lemma 5.1 Let notations be the same as in Theorem 4.4.

(i) An expression of the form (3) satisfying \( \sum_{i=1}^{\ell(\pi)} \sum_{j=1}^{m_i - r_i} n_{i,j} = \sum_{i=1}^s (k_i + 1) \) is equal to

\[
1_{-(n - n_0)} \prod_{i=1}^s a_{-(k_i + 1)}(\alpha_i) \cdot |0), \quad \text{where} \quad n_0 \overset{\text{def}}{=} \sum_{i=1}^s (k_i + 1).
\]
(ii) The coefficient of \( 1_{-(n-n_0)} \prod_{i=1}^{s} a_{-(k_i+1)}(\alpha_i) \cdot |0\rangle \) in \( \prod_{i=1}^{s} G_{k_i}(\alpha_i, n) \) is \( \prod_{i=1}^{s} \frac{(-1)^{k_i}}{(k_i + 1)!} \).

Proof. (i) We may let \( s \geq 1 \). Since \( \sum_{i=1}^{\ell(\pi)} \sum_{j=1}^{m_i-r_i} n_{i,j} = \sum_{i=1}^{s} (k_i + 1) \), we see from Theorem 4.1 that \( \sum_{j=1}^{m_i-r_i} n_{i,j} = \sum_{j \in \pi_i} (k_j + 1) \) for every \( i \). By (4), we obtain

\[ \sum_{i=1}^{\ell(\pi)} (m_i - 2 + |\pi_i|) = 0 \]

where \( |\pi_i| \) stands for the number of elements in the subset \( \pi_i \). Note that for every \( i \) with \( 1 \leq i \leq \ell(\pi) \), we have \( m_i \geq 1 \) since \( \sum_{j=1}^{m_i-r_i} n_{i,j} = \sum_{j \in \pi_i} (k_j + 1) \geq 1 \). So for every \( i \), \( m_i \geq 1 \), and \( r_i = 0 \) if \( m_i = 1 \). By (8), \( m_i = |\pi_i| = 1 \) for \( 1 \leq i \leq \ell(\pi) \). Thus for \( 1 \leq i \leq \ell(\pi) \), we have \( r_i = 0 \), \( t_i = 1 \), and \( n_{i,1} = (k_j + 1) \) if \( \pi_i = \{j\} \).

Now let \( \pi_i = \{t_i\} \) for \( 1 \leq i \leq \ell(\pi) = s \). Then the expression (8) is

\[ \text{sign}(\alpha, \pi) \cdot 1_{-(n-n_0)} \left( \prod_{i=1}^{s} a_{-(k_i+1)}(\alpha_i) \right) |0\rangle = 1_{-(n-n_0)} \left( \prod_{i=1}^{s} a_{-(k_i+1)}(\alpha_i) \right) |0\rangle. \]

(ii) The idea is to use induction on \( s \) and track the proof of Theorem 4.1 more carefully. When \( s = 0 \), the statement is trivial. Next, let \( s \geq 1 \) and \( \tilde{n}_0 = \sum_{j=2}^{s} (k_j + 1) \). Assume that the coefficient of \( 1_{-(n-n_0)} \left( \prod_{j=2}^{s} a_{-(n_i)}(\alpha_i) \right) |0\rangle \) in the cup product \( \prod_{i=2}^{s} G_{k_i}(\alpha_i, n) \) is equal to \( \prod_{i=2}^{s} \frac{(-1)^{k_i}}{(k_i + 1)!} \). Tracking the proof of Theorem 4.1 and applying Theorem 2.1 (v), we conclude that the coefficient of \( 1_{-(n-n_0)} \left( \prod_{i=1}^{s} a_{-(k_i+1)}(\alpha_i) \right) |0\rangle \) in \( \prod_{i=1}^{s} G_{k_i}(\alpha_i, n) \) is equal to \( \prod_{i=1}^{s} \frac{(-1)^{k_i}}{(k_i + 1)!} \).

\[ \prod_{j=1}^{t} G_{m_{j}}(\beta_{j}, n) \]

whose coefficients are independent of \( X, \alpha_1, \ldots, \alpha_s \) and \( n \). Here \( \sum_{j=1}^{t} (m_{j} + 1) \leq n_0 \), and \( \beta_1, \ldots, \beta_t \) depend only on \( e_X, K_X, \alpha_1, \ldots, \alpha_s \) and \( \tau_{i*} \) with \( 1 \leq i \leq n_0 \).
Proof. We use induction on \( n_0 \). When \( n_0 = 1 \), \( s = n_1 = 1 \). By the Lemma 3.20 (i) in \([LQW2]\), \( \mathbf{1}_{-(n-1)a^{-1}}(\alpha) |0\rangle = G_0(\alpha, n) \). So the lemma holds for \( n_0 = 1 \).

Next, let \( n_0 > 1 \). Let \( k_i = n_i - 1 \). Then, \( k_i \geq 0 \) for every \( i \). By Theorem 1.1, \( \prod_{i=1}^{s} G_{k_i}(\alpha_i, n) \) is a finite linear combination of expressions of the form (3) such that the coefficients in this linear combination are independent of \( X, \alpha_1, \ldots, \alpha_s \) and \( n \).

Note that \( \sum_{i=1}^{t} \sum_{j=1}^{\ell(\pi)} n_{i,j} \leq \sum_{i=1}^{t} \sum_{j=1}^{\ell(\pi)} (k_{j} + 1) = \sum_{i=1}^{s} (k_{i} + 1) = n_0 \). By induction, those expressions (3) with \( \sum_{i=1}^{t} \sum_{j=1}^{\ell(\pi)} n_{i,j} < n_0 \) are linear combinations of the form (9) where \( \sum_{j=1}^{t} (m_{j} + 1) \leq (n_0 - 1) \), and \( \beta_1, \ldots, \beta_t \) depend only on \( e_X, K_X, \alpha_1, \ldots, \alpha_s, \tau_{ir} \) with \( 1 \leq i \leq (n_0 - 1) \). Moreover, the coefficients in these linear combinations are independent of \( X, \alpha_1, \ldots, \alpha_s \) and \( n \). Now our lemma follows from Lemma 5.1. \( \square \)

Remark 5.1 By Lemma 5.2, an expression (9) in \( \mathbf{1}_{-(n-n_0)} \left( \prod_{i=1}^{s} a_{-n_i}(\alpha_i) \right) |0\rangle \) satisfies \( \sum_{j=1}^{t} (m_{j} + 1) \leq n_0 \). In fact, we see from the proof of Lemma 5.2 that an expression (9) satisfies the upper bound \( \sum_{j=1}^{t} (m_{j} + 1) = n_0 \) if and only if it is equal to \( \prod_{i=1}^{s} G_{n_i-1}(\alpha_i, n) \) whose coefficient is \( \prod_{i=1}^{s} ((-1)^{n_i-1} n_i!) \) in view of Lemma 5.1 (ii).

Next, we prove a lemma which says that the Chern character \( G_k(\alpha, n) \) can be expressed as a universal finite linear combination of cup products \( \prod_{j=1}^{t} B_{m_j}(\beta_j, n) \) (see Definition 2.3 (i)). In other words, our next lemma essentially reverses the process in Lemma 5.2. This lemma will be used later in the proof of Theorem 6.2.

Lemma 5.3 The Chern character \( G_k(\alpha, n) \) is a finite linear combination of products \( \prod_{j=1}^{t} B_{m_j}(\beta_j, n) \) whose coefficients are independent of \( X, \alpha \) and \( n \). Here \( \sum_{j=1}^{t} m_j \leq k \), and \( \beta_1, \ldots, \beta_t \) depend only on \( e_X, K_X \) and \( \alpha \). In addition, \( \sum_{j=1}^{t} m_j = k \) if and only if the product \( \prod_{j=1}^{t} B_{m_j}(\beta_j, n) \) equals \( B_k(\alpha, n) \) whose coefficient is \( (-1)^k/(k+1)! \).

Proof. Use induction on \( k \). When \( k = 0 \), we have \( G_0(\alpha, n) = B_0(\alpha, n) \) by the Lemma 3.20 (i) in \([LQW2]\). Next, we assume that the lemma is true for \( 0, \ldots, k-1 \) for some fixed \( k \geq 1 \). We shall prove that the lemma holds for \( k \) as well. We apply Lemma 5.2 and Remark 5.1 to \( B_k(\alpha, n) = \mathbf{1}_{-(n-k-1)a^{-1}(k+1)}(\alpha)|0\rangle \). We see that
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\[G_k(\alpha, n) - \frac{(-1)^k}{(k+1)!} \cdot B_k(\alpha, n)\] is a finite linear combination of \(\prod_{j=1}^{u} G_{n_j}(\gamma_j, n)\) where \(\sum_{j=1}^{u} (n_j + 1) < (k+1)\), and \(\gamma_1, \ldots, \gamma_u\) depend only on \(e_X, K_X\) and \(\alpha\). Moreover, the coefficients in this linear combination are independent of \(X, \alpha\) and \(n\). Note that \(n_j < k\) for all \(1 \leq j \leq u\). So by induction hypothesis, the lemma holds for \(k\). \(\square\)

\textbf{Remark 5.2} Lemma 5.2 and Lemma 5.3 provide a new proof to Theorem 2.2 which was originally proved in [LQW1, LQW2].

Our stability result below indicates that the cup product on the Hilbert scheme \(X^{[n]}\) are independent of \(n\) in an appropriate sense. Furthermore, we find an explicit form of the leading term in the cup product. This result enables us to construct a ring and determine its structure in the next section.

\textbf{Theorem 5.1} Let \(s \geq 1\) and \(k_i \geq 1\) for \(1 \leq i \leq s\). Fix \(n_{i,j} \geq 1\) and \(\alpha_{i,j} \in H^*(X)\) for \(1 \leq j \leq k_i\), and fix \(n\) with \(n \geq \sum_{i,j} n_{i,j}\) for all \(1 \leq i \leq s\). Then the cup product

\[1 \rightarrow (n - \sum_{i,j=1}^{k_i} n_{i,j}) \left( \prod_{j=1}^{k_i} a_{-n_{i,j}}(\alpha_{i,j}) \right) |0\]

in \(H^*(X^{[n]})\) is equal to a finite linear combination of monomials of the form

\[1 \rightarrow (n - \sum_{p=1}^{N} m_p) \left( \prod_{p=1}^{N} a_{-m_p}(\gamma_p) \right) |0\]

whose coefficients are independent of \(X, \alpha_{i,j}\) and \(n\). Here \(\sum_{p=1}^{N} m_p \leq \sum_{i=1}^{s} \sum_{j=1}^{k_i} n_{i,j}\), and \(\gamma_1, \ldots, \gamma_N\) depend only on \(e_X, K_X, \alpha_{i,j}, 1 \leq i \leq s, 1 \leq j \leq k_i\). In addition, the expression \([1]\) satisfies the upper bound \(\sum_{p=1}^{N} m_p = \sum_{i=1}^{s} \sum_{j=1}^{k_i} n_{i,j}\) if and only if it is equal to \(1 \rightarrow (n - \sum_{i=1}^{s} \sum_{j=1}^{k_i} n_{i,j}) \left( \prod_{i=1}^{s} \prod_{j=1}^{k_i} a_{-n_{i,j}}(\alpha_{i,j}) \right) |0\) whose coefficient is 1.

\textbf{Proof.} Put \(N_i = \sum_{j=1}^{k_i} n_{i,j}\) for \(1 \leq i \leq s\). For each \(i\), we see from Lemma 5.2 that \(1 \rightarrow (n - \sum_{j=1}^{k_i} n_{i,j}) \left( \prod_{j=1}^{k_i} a_{-n_{i,j}}(\alpha_{i,j}) \right) |0\) is a finite linear combination of products

\[\prod_{j=1}^{t_i} G_{m_{i,j}}(\beta_{i,j}, n)\] where \(\sum_{j=1}^{t_i} (m_{i,j} + 1) \leq N_i\), and \(\beta_{i,1}, \ldots, \beta_{i,t_i}\) depend only on \(e_X, K_X, \alpha_{i,1}, \ldots, \alpha_{i,k_i}\) and \(\tau_{js}\) with \(1 \leq j \leq N_i\). Moreover, the coefficients in the linear combinations are independent of \(X, \alpha_{i,1}, \ldots, \alpha_{i,k_i}\) and \(n\). By Remark 5.1,
the product satisfies the upper bound \( \prod_{j=1}^{k_i} G_{n_{i,j}-1}(\alpha_{i,j}, n) \). Furthermore, the coefficient of \( \prod_{j=1}^{k_i} G_{n_{i,j}-1}(\alpha_{i,j}, n) \) in
\[
1 - (n - \sum_{j=1}^{k_i} n_{i,j}) \left( \prod_{j=1}^{k_i} a_{-n_{i,j}}(\alpha_{i,j}) \right) |0\) is equal to \( \prod_{j=1}^{k_i} ((-1)^{n_{i,j}-1} n_{i,j}) \).

So (10) is a universal finite linear combination of products \( \prod_{i=1}^{s} \prod_{j=1}^{t_i} G_{m_{i,j}}(\beta_{i,j}, n) \)
where \( \sum_{i=1}^{s} \sum_{j=1}^{t_i} (m_{i,j} + 1) \leq \sum_{i=1}^{s} \sum_{j=1}^{t_i} n_{i,j} \). Also, \( \sum_{i=1}^{s} \sum_{j=1}^{t_i} (m_{i,j} + 1) = \sum_{i=1}^{s} \sum_{j=1}^{t_i} n_{i,j} \) if and only if the product \( \prod_{i=1}^{s} \prod_{j=1}^{t_i} G_{m_{i,j}}(\beta_{i,j}, n) \) is equal to \( \prod_{i=1}^{s} \prod_{j=1}^{t_i} G_{n_{i,j}-1}(\alpha_{i,j}, n) \). The
coefficient of \( \prod_{i=1}^{s} \prod_{j=1}^{t_i} G_{n_{i,j}-1}(\alpha_{i,j}, n) \) in (10) is equal to \( \prod_{i=1}^{s} \prod_{j=1}^{t_i} ((-1)^{n_{i,j}-1} n_{i,j}) \).

It follows from Theorem 4.1 that (10) is a universal linear combination of expressions (11). The statement for the expression (11) reaching the upper bound \( \sum_{p=1}^{N} m_p = \sum_{i=1}^{s} \sum_{j=1}^{k_i} n_{i,j} \) follows from Lemma 5.1.

6 The Hilbert ring

Using the stability result proved in the previous section, we shall introduce and
determine the Hilbert ring \( S_X \) associated to a projective surface \( X \).

Given a finite set \( S \) which is a disjoint union of subsets \( S_0 \) and \( S_1 \), we denote by \( P(S) \) the set of partition-valued functions \( \rho = (\rho(c))_{c \in S} \) on \( S \) such that for
every \( c \in S_1 \), the partition \( \rho(c) \) is required to be strict in the sense that \( \rho(c) = (1^m_1(2^m_2(c) \ldots) \) with \( m_r(c) = 0 \) or 1 for all \( r \geq 1 \).

Now let us take a linear basis \( S = S_0 \cup S_1 \) of \( H^*(X) \) such that \( 1_X \in S_0 \), \( S_0 \subset H^{even}(X) \) and \( S_1 \subset H^{odd}(X) \). If we write \( \rho = (\rho(c))_{c \in S} \) and \( \rho(c) = (r^m_r(c))_{r \geq 1} = (1^m_1(2^m_2(c) \ldots) \) then we put \( \ell(\rho) = \sum_{c \in S} m_r(c) \) and
\[
\|\rho\| = \sum_{c \in S} |\rho(c)| = \sum_{c \in S, r \geq 1} r \cdot m_r(c), \quad P_n(S) = \{ \rho \in P(S) \mid \|\rho\| = n \}.
\]

Given \( \rho = (\rho(c))_{c \in S} = (r^m_r(c))_{c \in S, r \geq 1} \in P(S) \) and \( n \geq 0 \), we define
\[
\begin{align*}
a_{-\rho(c)}(c) &= \prod_{r \geq 1} a_{-r}(c)^{m_r(c)} = a_{-1}(c)^{m_1(c)} a_{-2}(c)^{m_2(c)} \ldots \\
a_{\rho}(n) &= 1_{-n - \|\rho\|} \prod_{c \in S} a_{-\rho(c)}(c) \cdot |0| \in H^{*}(X^{[n]})
\end{align*}
\]
where we fix the order of the elements \( c \in S_1 \) appearing in \( \prod_{c \in S} \) once and for all.

Note from Definition 4.1 (ii) that \( a_\rho(n) = 0 \) for \( 0 \leq n < \|\rho\| \).

As \( \rho \) runs over all partition-valued functions on \( S \) with \( \|\rho\| \leq n \), the corresponding \( a_\rho(n) \) linearly span \( H^*(X^{[n]}) \) as a corollary to the theorem of Nakajima and Grojnowski [Na2]. By Theorem 5.1 (for \( s = 2 \)), we have the cup product

\[
(12) \quad a_\rho(n) \cdot a_\sigma(n) = \sum_\nu d^\nu_{\rho\sigma} a_\nu(n)
\]

in \( H^*(X^{[n]}) \), where \( \|\nu\| \leq \|\rho\| + \|\sigma\| \) and the structure coefficients \( d^\nu_{\rho\sigma} \) are independent of \( n \). Even though the cohomology classes \( a_\nu(n) \) with \( \|\nu\| \leq n \) in \( H^*(X^{[n]}) \) are not linearly independent in general, we have the following.

**Lemma 6.1** The structure constants \( d^\nu_{\rho\sigma} \) in the formula (12) are uniquely determined by the requirement that they are independent of \( n \).

**Proof.** Assume that there exist a finite subset \( I \subset \mathcal{P}(S) \) and some constants \( c^\nu \in \mathbb{Q} \) independent of \( n \) such that for all \( n \geq 0 \), we have

\[
(13) \quad \sum_{\nu \in I} c^\nu a_\nu(n) = 0.
\]

As an immediate consequence of the theorem of Nakajima and Grojnowski [Na2], the Heisenberg monomials \( \prod_{c \in S} a_{\rho(c)}(c) \cdot [0] \), where \( \rho = (\rho(c))_c \in \mathcal{P}_n(S) \), are linearly independent in the cohomology ring \( H^*(X^{[n]}) \). Therefore, by the definition of \( a_\nu(n) \), we may assume in (13) that any two distinct \( \nu \) and \( \tilde{\nu} \) in \( I \) satisfy \( \nu(c) = \nu(c) \) for \( c \neq 1_X \), \( \nu(1_X) = (1^{m_1}2^{m_2} \cdots) \) and \( \tilde{\nu}(1_X) = (1^{m_1+\|\tilde{\nu}\|}2^{\|\tilde{\nu}\|} \cdots) \) (here we assume for definiteness that \( n \geq \|\tilde{\nu}\| > \|\nu\| \)). In this case, we have \( a_\nu(n) = (n - \|\tilde{\nu}\|)!/(n - \|\nu\|)! \cdot a_\nu(n) \). Letting \( n \to \infty \), we see from (13) that \( c^\nu = 0 \) for the \( \tilde{\nu} \in I \) with the largest size \( \|\tilde{\nu}\| \). So all the constants \( c^\nu \) are zero. \( \square \)

Now we are ready to introduce the Hilbert ring.

**Definition 6.1** The Hilbert ring associated to a projective surface \( X \), denoted by \( \mathfrak{H}_X \), is defined to be the ring with a linear basis formed by the symbols \( a_\rho, \rho \in \mathcal{P}(S) \) and with the multiplication defined by \( a_\rho \cdot a_\sigma = \sum_\nu d^\nu_{\rho\sigma} a_\nu \) where the structure constants \( d^\nu_{\rho\sigma} \) are from the relations (12).

Note that the Hilbert ring does not depend on the choice of a linear basis \( S \) of \( H^*(X) \) containing \( 1_X \) since the operator \( a_\alpha(\alpha) \) depends on the cohomology class \( \alpha \in H^*(X) \) linearly. It follows from the super-commutativity and associativity of the cohomology ring \( H^*(X^{[n]}) \) that the Hilbert ring \( \mathfrak{H}_X \) itself is also super-commutative and associative. The ring \( \mathfrak{H}_X \) captures all the information of the cohomology rings of \( X^{[n]} \) for all \( n \), as we easily recover the relations (12) from the ring \( \mathfrak{H}_X \). We summarize these observations into the following.
Theorem 6.1 (Stability) For a given projective surface $X$, the cohomology rings $H^*(X^{[n]})$, $n \geq 1$ give rise to a Hilbert ring $\mathcal{H}_X$ which completely encodes the cohomology ring structure of $H^*(X^{[n]})$ for each $n$.

We further have the following result on the structure of the Hilbert ring $\mathcal{H}_X$.

For convenience, in the case when $\ell(\rho) = 1$, that is, when the partition $\rho(c)$ is a one-part partition $(r)$ for some element $c \in S$ and is empty for all the other elements in $S$, we will simply write $a_{\rho} = a_{r,c}$ and $a_{\rho}(n) = a_{r,c}(n)$.

Theorem 6.2 The Hilbert ring $\mathcal{H}_X$ is isomorphic to the tensor product $P \otimes E$, where $P$ is the polynomial algebra generated by $a_{r,c}$, $c \in S_0$, $r \geq 1$ and $E$ is the exterior algebra generated by $a_{r,c}$, $c \in S_1$, $r \geq 1$.

Proof. Note that $a_{r,c}(n) = 1_{-(n-r)}a_{r,c}(0) = B_{r-1}(c,n)$. By Lemmas 5.2 and 5.3, the ring $\mathcal{H}_X$ is generated by the elements $a_{r,c}$, where $c \in S = S_0 \cup S_1$ and $r \geq 1$.

By the super-commutativity of $\mathcal{H}_X$, we have $a_{r,c}^2 = 0$ for $c \in S_1$ and $r \geq 1$. It remains to show that as $\rho = (r^{m_{\nu}(c)})_{\nu \in S, r \geq 1}$ runs over $\mathcal{P}(S)$, the monomials

$$\prod_{c \in S, r \geq 1} a_{r,c}^{m_{\nu}(c)}$$

are linearly independent in $\mathcal{H}_X$. Assume $\sum_i d_i \prod_{c \in S, r \geq 1} a_{r,c}^{m_i(n)} = 0$ where $d_i \in \mathbb{Q}$ and $\rho_i = (r^{m_i(c)})_{c \in S, r \geq 1}$ runs over a finite set $I$ of distinct elements in $\mathcal{P}(S)$.

By the definition of the structure constants in $\mathcal{H}_X$ and $1_{-(n-r)}a_{r,c}(0) = a_{r,c}(n)$,

$$\sum_{i \in I} d_i \cdot \prod_{c \in S, r \geq 1} (1_{-(n-r)}a_{r,c}(0))^{m_i(n)} = \sum_{i \in I} d_i \cdot \prod_{c \in S, r \geq 1} a_{r,c}(n)^{m_i(n)} = 0.$$

Take an integer $n$ large enough such that $n \geq n_i \overset{\text{def}}{=} \sum_{r,c} r m_i^* (r)$ for all $i \in I$. By Theorem 6.1, Eq. (14) can be rewritten as

$$\sum_{i \in I} d_i \left(1_{-(n-n_i)} \prod_{c \in S, r \geq 1} (a_{r,c}(0))^{m_i^*(n)} \cdot |0| + w_i \right) = 0.$$

where each $w_i$ is a finite linear combination of $1_{-(n-\sum_{p=1}^N m_p)} \prod_{p=1}^N a_{r,c}(\gamma_p) \cdot |0|$ with $\sum_{p=1}^N m_p < n_i$ and $\gamma_p \in S$ for every $p$. Recall that $1_{-k} = 1/k! \cdot a_{-1}(1_X)^k$, $k \geq 0$. If we multiply (15) by $n!$, and locate in the resulting summation those terms whose coefficients contain the largest power of $n$, then we see that

$$\sum_{i \in I} d_i \cdot 1_{-(n-n_i)} \prod_{c \in S, r \geq 1} (a_{r,c}(0))^{m_i(n)} \cdot |0| = 0$$

where $i$ satisfies $n_i = \max\{n_j | j \in I\}$. Since all the integers $n_i$ in (16) are equal, the Heisenberg monomials in (16) are linearly independent as a corollary to the theorem of Nakajima and Grojnowski [Na2]. Thus all the coefficients $d_i$ in (16) are zero. By repeating the above argument, we obtain that $d_i = 0$ for all $i \in I$. $\square$
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