Factorization in hard exclusive processes.
Computation of one-loop corrections to the diphoton photoproduction on proton.

Master’s thesis
in the field of Physics

The thesis was written under the supervision of
Dr hab. Jakub Wagner
National Centre for Nuclear Research
and Prof. Piotr H. Chankowski
University of Warsaw, Faculty of Physics
Institute of Theoretical Physics
Chair of Theory of Hadrons and Leptons

Warsaw, July 2021
Summary

Generalized Parton Distributions (GPDs) carry information on the internal structure of hadrons such as the angular momentum of quarks and gluons, or their spacelike distribution. They can be experimentally studied in exclusive experiments with hadrons, i.e. processes in which all initial and final states are measured. The tool that creates the necessary bridge between the theoretical predictions and the experiments is the collinear factorization. It allows disentangling perturbatively computable parts of the amplitude, which describes interactions of quark and gluons with the external particles, from the non-perturbative quantities, which are identified as GPDs.

In this work, I extend the theoretical analysis of the process photoproduction of photon pairs on a proton to the next-to-leading order in perturbative Quantum Chromodynamics within the framework of collinear factorization. I give the proof that all collinear divergences which arise in one-loop computations cancel at the level of the amplitude. This result enlarges the family of reactions, which can be studied using the collinear factorization by processes of the type $2 \rightarrow 3$, which have not been previously studied within this theoretical framework beyond the leading QCD order. Furthermore, I compute the full form of the amplitude of the discussed process at the one-loop order. That improves the accuracy of theoretical predictions for this experiment, which may be used for planning future experiments in JLAB or EIC.

Keywords

Quantum Chromodynamics, Generalised Parton Distributions, Factorization, Renormalization

The title of the thesis in Polish

Faktoryzacja w twardych procesach ekskluzywnych. Poprawki jednopętlowe do fotoprodukcji par fotonów na protonie.
# Contents

1 Introduction

2 The QCD factorization in hard exclusive processes
   2.1 Leading order factorization and generalized parton distributions
   2.2 Higher order QCD corrections
      2.2.1 Gauge link
      2.2.2 UV and soft collinear divergences
   2.3 Renormalization of non-local product of operators
   2.4 From evolution of operators in the position space to evolution of GPDs

3 Photoproduction of photon pairs
   3.1 Kinematics, amplitude, and the differential cross section
   3.2 The LO amplitude
   3.3 Self-energy corrections
      3.3.1 Integral
      3.3.2 Traces
   3.4 Vertex corrections - 3.L and 3.R
   3.5 Vertex corrections - 3.M
   3.6 Box diagrams - 4.L and 4.R
      3.6.1 Localizing divergences inside the integral
      3.6.2 Trace - terms leading to divergences
      3.6.3 The finite part of the amplitude
   3.7 The 5-point diagram
      3.7.1 The divergent part
      3.7.2 The finite part
   3.8 Factorization theorem at the 1-loop order

4 Conclusions

A Derivation of the finite part of 4- and 5- point loop diagrams.

B The full form of the amplitude
Chapter 1

Introduction

As far as we know, hadrons are the smallest composite objects in the Universe. The first indication that nucleons, unlike electrons, are not pointlike particles came from the measurement of the magnetic moment of the proton performed by Stern and collaborators \[1\] already in 1933. More than twenty years later, analysis of elastic scattering of several hundreds MeV electrons off nuclei allowed R. Hofstadter to determine the electric charge distributions in protons and neutrons \[2\] (the charge r.m.s. radius of the proton is now estimated to be approximately \(0.87 \times 10^{-15}\) m \[3\]). Finally, at the end of the sixtieth of the XX century, the experiments, in some respect analogous to the historic one of Rutherford which revealed the existence of atomic nuclei, with deeply inelastic scattering (the so-called DIS) of leptons on nucleons performed at Stanford Linear Accelerator Center (SLAC) \[4\] inspired Feynman to formulate the parton model \[5–8\] in which hadrons probed by highly virtual photons can be treated as composed of almost noninteracting pointlike constituents. These, called by Feynman partons, were subsequently identified with quarks, first proposed by Zweig and Gell-Mann \[10,11\], and spin 1 bosons (gluons) which are now known to be the elementary constituents of hadrons.

The proper theory of interactions of quarks and gluons is a non-abelian quantum gauge field theory known as Quantum Chromodynamics (QCD). The effective coupling constant of this theory becomes large at long distances (or, equivalently, at small transfers of momenta) \[12\]. Therefore, applications of the perturbative expansion to QCD are limited to only special classes of high energy processes which are characterized by the existence of at least one “hard scale” \(Q\), e.g. of a large (compared to a typical hadronic mass scale set by \(\Lambda_{\text{QCD}}\) - the intrinsic energy scale of QCD) four-momentum transfer \(-q^2 = Q^2 (\sqrt{q^2} \gg \Lambda_{\text{QCD}})\). This makes the study of hadrons, their reactions and structure, a great challenge: rather advanced quantum field theory tools are required in order to extract the relevant information from QCD. Already to obtain precise theory predictions for scaling violation in DIS experiments (i.e. to go beyond the simple Feynman parton picture) one needs to employ the so-called operator product expansion. It allows in this case to rigorously split the computed cross section of the inclusive process into the part (Wilson coefficients) which is dominated by the aforementioned hard scale \(Q\) and can, owing to this, be computed perturbatively, and the other part which can only be phenomenologically parametrized in terms of the parton distribution functions (PDFs) \(f_p(x)\) introduced originally at the intuitive level by Feynman in his (so-called naive) parton model and which are given the interpretation \[6–9\] of the probability densities that inside the hadron a parton \(p\) of a given type carries a fraction \(x\) of the total hadron four-momentum. Measurements which confirmed the scaling violation, i.e. the predicted by QCD
changes of the distributions $f_p(x)$ with the changes of the hard scale $Q$, were important tests of QCD which together with the observation by the DORIS and Petra experiments in Hamburg of the three-jet events in $e^- e^+$ collisions essentially established it as a correct theory of strong interactions.

Amplitudes of more complicated processes, in particular of exclusive processes in which characteristics of not only the initial but also of the final state hadron(s) are specified (and measured experimentally), usually cannot be analysed as rigorously as DIS. Nevertheless, one commonly believes that amplitudes of the studied process which are characterized by a hard scale can still be split into a perturbatively computable (within QCD) hard part (dominated by the scale $Q$) and the rest which is nonperturbative and must be treated phenomenologically, i.e. parametrized in agreement with the known symmetries of the underlying theory (QCD), determined experimentally in one group of experiments and then used to predict results of another group of experiments. For example, analysis of the data accumulated over years in DIS-type experiments carried out with the help of the HERA accelerator (experiments ZEUS and H1) allowed to obtain parametrizations of PDFs of the parton model and are now routinely being used to predict rates of processes studied experimentally in the proton-proton inelastic collisions in LHC.

The principle on which the possibility of splitting amplitudes of exclusive hadronic processes into perturbative and nonperturbative parts relies is the so-called factorization which operates at the level of amplitudes. It is not as rigorous as the OPE, the use of which is limited to only a rather narrow class of directly experimentally measurable processes, but can, at least in the case of some special processes characterized by a hard scale $Q$, be proven with a fair degree of rigour (enough to satisfy most physicists but few pure mathematicians - as my and prof. Chankowski’s hero, A.B. Pippard, would say). Demonstrating factorization requires, however, a careful analysis of Feynman diagrams and their singularities in various kinematical regimes, both ultraviolet (UV), infrared and/or collinear, in order to be able to isolate them and properly absorb into the nonperturbative part of the amplitude. As far as exclusive processes with one and the same single hadron (usually nucleon) in the initial and final state are concerned, the nonperturbative parts of their amplitudes are parametrized in terms of the Generalized Parton Distributions (GPDs), which are also known as Off-Forward Parton Distributions. GPDs which, similarly to PDFs, describe properties of a given hadron and are, therefore, process-independent, significantly generalize PDFs (which can be obtained from the GPDs by taking their forward limit) and as such contain much more detailed information on the hadron structure, e.g. on angular momenta of partons of a given kind or on their spacelike distribution inside the hadron [14–17], than do the PDFs. Still, the strategy of their use is similar to that of the latter: they should be extracted from the data on one class of processes and applied to predict rates of other classes of processes.

Extraction of GPDs from experimental data is complicated, because cross sections of processes sensitive to parton distributions are rather small and because what actually can be extracted are only convolutions of GPDs with some other, process-dependent, distributions [7, 17, 18]. Hence, to reliably determine generalized distributions, processes of different kinds must be considered in conjunction. Moreover, since at energy scales of order few GeV ($|q|^2 \sim 10 \text{ GeV}^2$) relevant for these processes [17, 20], the strong coupling constant $\alpha_s$ is of order $\sim 0.1$, higher order corrections must necessarily be taken into account. One-loop results concerning electroproduction of a real photon, of photoproduction of lepton pair or electroproduction of a meson are already available - they have been given in the papers [22], [23] and [24], respectively. Recently, the process of photoproduction of two photons on nucleon has been considered, but only at the leading order [20, 21].
The main goal of this work is to extend the analysis of factorization of the amplitude of diphoton photoproduction on proton, i.e. of the process $\gamma p \rightarrow \gamma'\gamma''p'$, given in \cite{20,21}, to the first nontrivial order. The work consists of two parts. In Chapter 2 I describe in some details the collinear factorization. In particular I explain how factorization should be handled in higher orders, when the perturbatively computed part of the amplitude develops in addition to UV also infrared divergences resulting from diagrams containing loops with propagators of partons treated as massless quanta. It will be seen that if factorization is to hold also in higher orders, the infrared (collinear) divergences must cancel against the ones present in parton distributions defined as (Fourier transforms - see below - of) matrix elements between hadron states of appropriately defined renormalized nonlocal composite operators. Parton distributions defined in this way are frequently called, somewhat misleadingly, renormalized distributions\footnote{Parton distributions defined in terms of unrenormalized operators are accordingly and also typically misleadingly called bare parton distributions. To be sure: whether defined in the leading order or in higher orders they are extracted from the same measured quantities. It it thus only their theoretical definition which is given either in terms of bare or in terms of renormalized operators; the GPDs are always the same quantities.} If such a cancellation does indeed hold, the amplitude can still be split (i.e. factorized) into the perturbative (hard) part computable with the help of renormalized perturbative expansion and the nonperturbative one parametrized now in terms of the renormalized GPDs. Since this cancellation is not obvious (and there are known situations, in which it does not obtain \cite{25}), it must be checked for each process independently. Proofs that factorization holds in the leading order in $|Q|^{-1}$ and up to an arbitrary order in perturbative QCD can be carried out only for a limited family of processes. Since renormalization of GPDs (i.e. the construction of renormalized operators in terms of which GPDs must be defined in higher orders) is more complicated than, for example, renormalization of conventional $S$-matrix elements or Green’s functions of elementary or local composite operators (here the renormalized operators must be defined in a gauge invariant way, i.e. must be supplemented by the Wilson gauge link operator), I provide in Section 2.2 a detailed demonstration of the necessary renormalization procedure at the one-loop level, and derive the resulting evolution equations (analogous to the ones governing the scale dependence of the ordinary PDFs) satisfied by the resulting generalized parton densities.

The second part of the work is devoted to the analysis in the next-to-leading order of photoproduction of photon pairs with large invariant mass (which sets the hard scale $Q$) on proton. This process is especially interesting from the point of view of the theoretical analysis, because it is the simplest one of all $2 \rightarrow 3$ processes that can be studied within the framework of collinear factorization discussed in the first part of the work. The corresponding amplitude is sensitive only to combinations of partons distributions, which are odd with respect to the charge conjugation. That provides the additional source of information on GPDs. The leading-order analysis \cite{20} shows, that this process can in principle be studied experimentally at the JLab facilities which provide intense beams of quasi-real photons and in the planned Electron-Ion Collider (EIC). Moreover, some experimental data pertaining to this process probably already exists, as exclusive experiments with protons and highly energetic quasi-real photons have already been performed to measure another process, the Timelike Compton Scattering. In Chapter 3 I present a detailed computation of one loop corrections to the amplitude of the $\gamma p \rightarrow \gamma'\gamma''p'$ process. I give a proof of the validity of the collinear factorization in it at NLO, compute the finite part of the amplitude and the differential cross section of the process. I also introduce some techniques which may, I believe, prove useful also in theoretical studies of other processes.
Before proceeding with the main part of the work, it is convenient to define the notation used throughout it. I will introduce here also the definitions of the Generalized Parton Distributions since, in my opinion, it is better to do it before explaining how they arise in actual computations of amplitudes of specific physical processes.

In defining GPDs it is customary to introduce the so-called Sudakov decomposition of four-vectors \( v^\mu \) defined in the following way:

\[
v^\mu = v^+ n_+^\mu + v^- n_-^\mu + v_\perp^\mu ,
\]

where

\[
n_\pm^\mu \equiv \frac{1}{\sqrt{2}}[1,0,0,\pm 1]^\mu ,
\]

and \( v_\perp^\mu \) denotes the spatial part of the vector projected onto the \( xy \) plane. In the analogous way one introduces the lightcone basis of the gamma matrices:

\[
\gamma^\pm \equiv \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^3) .
\]

Consider now two states \( |N(p_1, \sigma_1)\rangle \) and \( |N(p_2, \sigma_2)\rangle \) of the same hadron \( N \) with the four-momenta \( p_1^\mu = (E_1, \mathbf{p}_1) \) and \( p_2^\mu = (E_2, \mathbf{p}_2) \) \( (E_i = \sqrt{m_N^2 + \mathbf{p}_i^2}, \ i = 1,2, \text{where } m_N \text{ is the mass of the hadron}), \) and \( \sigma_i \) its spin projection. The ”+” components \( p_i^+ = (E_i + \not{p}_i)/\sqrt{2}, \ i = 1,2 \) of the four-momenta \( p_1 \) and \( p_2 \) will be parametrized with the help of two parameters: \( \xi (-1 < \xi < 1) \) and \( p^+ \geq 0 \) defined by

\[
p_1^+ = (1 + \xi) p^+, \quad p_2^+ = (1 - \xi) p^+ .
\]

It follows that \( p^+ = (p_1^+ + p_2^+)/2 \geq 0 \) and \( \xi = (p_1^+ - p_2^+)/p_1^+ + p_2^+ \). We assume that both hadrons move nearly with the speed of light in the direction of the \( z \)-axis, so that \( p^+ \gg m_N \).

Generalized Parton Distribution can be defined by considering in the limit \( p^+ \to \infty \) of the following Fourier transforms of matrix elements of non-local composite operators:

\[
\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle N(p_2, \sigma_2) | \bar{\psi}_q(z/2) [z^-/2, z/2] \gamma^+ \psi_q(z/2) | N(p_1, \sigma_1) \rangle \bigg|_{z^+, z_\perp = 0} , \tag{1.5}
\]

\[
\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle N(p_2, \sigma_2) | \bar{\psi}_q(z/2) [z^-/2, z/2] \gamma^+ \gamma_5 \psi_q(z/2) | N(p_1, \sigma_1) \rangle \bigg|_{z^+, z_\perp = 0} , \tag{1.6}
\]

\[
\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle N(p_2, \sigma_2) | G^{\mu+}(z/2) [z^-/2, z/2] G_\mu^+(z/2) | N(p_1, \sigma_1) \rangle \bigg|_{z^+, z_\perp = 0} , \tag{1.7}
\]

\[
\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle N(p_2, \sigma_2) | G^{\mu+}(z/2) [z^-/2, z/2] \tilde{G}_\mu^+(z/2) | N(p_1, \sigma_1) \rangle \bigg|_{z^+, z_\perp = 0} . \tag{1.8}
\]

\( \psi_q \) denotes here the bare (i.e. the Heisenberg picture operator the scale of which is fixed by the canonical kinetic term in the QCD lagrangian density) quark field operator of flavor \( q \), \( G^{\mu\nu} \)

is the bare field strength tensor of the gluon field and \( G^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\kappa\lambda} G_{\kappa\lambda} \) the tensor dual to it; \( \epsilon^{\mu\nu\kappa\lambda} \) is the completely antisymmetric tensor (I use \( \epsilon^{0123} = +1 \)). The matrix \( \gamma_5 \) is defined as \( \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \). Strictly speaking, in the leading order the operator denoted in (1.5)-(1.8) \( \langle -z/2, z/2 \rangle \), called the gauge link is replaced by the unit operator. It must be included in higher
orders (together with appropriate operator counterterms which complete the definitions of
the renormalized nonlocal operators) and is defined in the following way:

\[ [z_1, z_2] \equiv P \exp \left( ig \int_{z_2}^{z_1} d\xi \mu A^\mu(\xi) \right). \]  

(1.9)

The symbol \( P \) denotes here the path ordering and

\[ A^\mu \equiv \sum_a t^a A^{\mu,a}, \]  

(1.10)

where \( t^a \) are the generators (taken in the fundamental representation) of the \( SU(3) \) gauge group; \( a = 1, \ldots, 8 \) is the adjoint color index. Presence of this operator ensures gauge invariance of GPDs. How it arises in actual calculations is explained in Section 2.2.1. In the literature, see e.g. [29,30] products of two operators separated by a light-like distance and the gauge link, as in the matrix elements (1.5-1.8), are often called string operators.

In most applications the hadron \( N \) is a spin-1/2 particle (proton or neutron). In such cases the most general forms of the Fourier transforms (1.5)- (1.8) can be, taking into account Lorentz invariance, parity, time reversal and charge conjugation symmetries of the underlying theory, parametrized in terms of the functions \( H^{q,g}, \tilde{H}^{q,g}, E^{q,g} \) and \( \tilde{E}^{q,g} \) (their definitions given here agree with those used in [27]) called generalized parton distributions (\( \Delta^\kappa \equiv (p_2 - p_1)^\kappa \)):

\[
(1.5) = \frac{1}{p^+} \left[ H^q(x, \xi, t) \bar{u}(p_2, \sigma_2) \gamma^+ u(p_1, \sigma_1) + E^q(x, \xi, t) \bar{u}(p_2, \sigma_2) \frac{i\sigma^+ \Delta^\kappa}{2m} u(p_1, \sigma_1) \right]; \\
(1.6) = \frac{1}{p^+} \left[ \tilde{H}^q(x, \xi, t) \bar{u}(p_2, \sigma_2) \gamma^+ \gamma_5 u(p_1, \sigma_1) + \tilde{E}^q(x, \xi, t) \bar{u}(p_2, \sigma_2) \frac{\gamma_5 \Delta^+}{2m} u(p_1, \sigma_1) \right]; \\
(1.7) = \frac{1}{2} \left[ H^q(x, \xi, t) \bar{u}(p_2, \sigma_2) \gamma^+ u(p_1, \sigma_1) + E^q(x, \xi, t) \bar{u}(p_2, \sigma_2) \frac{i\sigma^+ \Delta^\kappa}{2m} u(p_1, \sigma_1) \right]; \\
(1.8) = \frac{i}{2} \left[ \tilde{H}^q(x, \xi, t) \bar{u}(p_2, \sigma_2) \gamma^+ \gamma_5 u(p_1, \sigma_1) + \tilde{E}^q(x, \xi, t) \bar{u}(p_2, \sigma_2) \frac{\gamma_5 \Delta^+}{2m} u(p_1, \sigma_1) \right].
\] 

(1.11-1.14)

Owing to the Lorentz invariance, the distributions \( H, \tilde{H}, E, \tilde{E} \) depend on only 3 variables: \( x, \xi \) and \( t = \Delta^2 \); moreover, as can be shown [28], \( -1 < x < 1 \).

As already said, beyond the leading order GPDs (called then renormalized Generalized Parton Distributions) must be defined in the analogous way but as matrix elements of renormalized string operators. The form of these renormalized operators is established in Section 2.3. Because of cancellation of divergences, which is an essential element of factorization, all observables must be then expressed in terms of these renormalized GPDs. How the matrix elements (1.5)- (1.8), and therefore the generalized parton distributions, appear in actual computation of amplitudes of physical processes, and how the generalized parton distributions vary with changes of the hard scale \( Q \) (i.e. what renormalization group they satisfy) will be shown in Chapter 2.
Chapter 2

The QCD factorization in hard exclusive processes

2.1 Leading order factorization and generalized parton distributions

One of the simplest processes allowing to extract GPDs from the experimental data is the so-called Deeply Virtual Compton Scattering (DVCS) \[17\] in which a nucleon $N$ of mass $m_N$ absorbs a highly virtual photon carrying the four-momentum $q (-q^2 \equiv Q^2 \gg m_N^2)$ and subsequently emits a real photon: $\gamma^*(q)N(p) \rightarrow \gamma(q')N(p')$. This process can be experimentally studied as a sub-process shown in Fig. 2.1 of the photon electroproduction that is, in the reaction $e^-N \rightarrow e^-\gamma N$.

\[
\begin{array}{c}
\begin{array}{c}
\text{Figure 2.1: DVCS as a subprocess of the photon electroproduction on nucleon.} \\
\text{$k$ and $k'$ are the} \\
\text{four-momenta of the initial and final electron. The hard scale is in} \\
\text{this case set by the virtuality of} \\
\text{the exchanged photon $-q^2 = -(k-k')^2$. The gray blob is the part of the} \\
\text{amplitude which cannot be} \\
\text{computed perturbatively. Another subprocess contributing to the same} \\
\text{photon electroproduction on nucleon is the Bethe-Heitler subprocess} \\
\text{in which nucleon absorbs only the virtual photon and} \\
\text{the final state photon is emitted from one of the electron lines and,} \\
\text{therefore, the “hadronic part”} \\
\text{of the amplitude is parametrized in term of well-known elastic nucleon form-factors \[2\].}
\end{array}
\end{array}
\]

Another process sensitive to the generalized parton distributions \[18\] is the Timelike Compton Scattering (TCS), $\gamma(q)N(p) \rightarrow \gamma^*(q')N(p')$, which can be regarded as a “reversed” DVCS: the nucleon absorbs a real photon and emits a virtual photon, which subsequently produces a lepton pair. Both DVCS and TCS are the limiting cases of a more general process called Double Deeply
Virtual Compton Scattering (DDVCS), which can be represented as $\gamma^*(q)N(p) \to \gamma^*(q')N(p')$, in which at least one photon is far off-shell, so that $|q^2| + |q'^2| \gg m_N^2$. An analysis of DDVCS at the next-to-leading order, together with the proof of factorization of its amplitude at the 1-loop level, has been presented in [31].

Before considering the process $\gamma(q)N(p) \to \gamma(q')\gamma(q'')N(p')$ which is the main goal of this work, let us discuss factorization of the photon electroproduction amplitude starting from the basic principles of quantum field theory. In view of the fact that hadrons are not “quanta” of elementary QCD field operators, the $S$-matrix element

$$
\langle \{N(p', \sigma')e^- (k')\gamma(q')\} \rangle_{\text{out}} \left| \langle N(p, \sigma)e^- (k)\rangle_{\text{in}} \right|,
$$

(2.1)

(the subscripts “in” and “out” denote the in and out asymptotic eigenstates of the QCD+QED Hamiltonian) corresponding to the experimentally realizable process $N(p)e^- (k) \to N(p')e^- (k')\gamma(q')$ must be extracted from the (connected part of the) vacuum Green’s function

$$
\langle \Omega_{\text{out}} \left| T \left[ O(y')O(y)\psi(e)(w')\bar{\psi}(e)(w)A^\alpha(x') \right] \right| \Omega_{\text{in}} \rangle,
$$

where $O(y')$ is an appropriately constructed composite Heisenberg picture operator ($\psi(e)$ and $A^\alpha$ are the electron and photon Heisenberg picture elementary field operators) using the Lehmann-Symanzik-Zimmermann reduction formula [33]. If $N$ stands for proton, as $O$ one can take e.g. the composite operator transforming as the spin 1/2 representation of the Lorentz group constructed out of three quark (Heisenberg picture) field operators $\varepsilon_{ijk}(C^{-1})^{\beta\gamma}\psi(i\alpha)\bar{\psi}(j\beta)\bar{\psi}(k\gamma)$, where $C^{-1}$ is the charge conjugation matrix (which makes a Lorentz singlet out of two quark fields) and $\varepsilon_{ijk}$ makes the operator a colour singlet. Because the quantum numbers match, the operator $O(y')$ certainly has a nonzero matrix element between the vacuum state $|\Omega\rangle$ and the (in or out) state $|N(p, \sigma)\rangle$ (an eigenstate of the complete Hamiltonian of QCD+QED) which by symmetry principles must take the form

$$
\langle \Omega_{\text{out}} \left| O^{H}(y') \right| (N(p', \sigma'))_{\text{out}} \right| (N(p, \sigma)e^- (k))_{\text{in}} \rangle = Z_O^{1/2} e^{-i\vec{p} \cdot \vec{y}'} u_\alpha(p', \sigma'),
$$

(2.2)

in which $u_\alpha(p', \sigma')$ is the ordinary spinor and $Z_O$ is some non-zero constant (which is the property of the operator $O$). However since the electromagnetic interaction is weak, it is sufficient to consider the amplitude of the process $N(p)e^- (k) \to N(p')e^- (k')\gamma(q')$ in the lowest possible order in the coupling $e$. This allows to write it as a sum of two terms: one the main part of which is the amplitude of the DVCS subprocess and another one corresponding to the Bethe-Heitler subprocess. The latter is well-known (it can be written in terms of the elastic nucleon form-factors) and will not be discussed here. Thus in this approximation\footnote{As usually we consider only connected parts of $S$-matrix elements and of the corresponding Green’s functions; this will not be indicated explicitly.}

$$
\langle \{N(p', \sigma')e^- (k')\gamma(q')\} \rangle_{\text{out}} \left| (N(p, \sigma)e^- (k))_{\text{in}} \right| = (2\pi)^4 \delta^4(q' + k' + p' - k - p)
\times \left\{ \epsilon_\nu^\alpha(q') (-ie) W^{\mu\nu}(p', p'; q', q) (-ie) - \frac{i}{q^2} i e [\bar{u}(\vec{k'}, s') \gamma_\mu u(\vec{k}, s)] - i A_{\text{BeHe}} \right\},
$$

where $A_{\text{BeHe}}$ stands for the invariant amplitude of the Bethe-Heitler subprocess, $\epsilon_\nu^\alpha(q')$ is the polarization vector of the outgoing photon, $-i/q^2$ is the virtual photon propagator (marked $q$ in Fig. 2.1) and the tensor $W^{\mu\nu}(p', p'; q', q)$ is defined by the double Fourier transform

$$
(2\pi)^4 \delta^{(4)}(q' + p' - q - p) W^{\mu\nu}(p', p'; q', q) = \int d^4x'd^4x e^{iq' \cdot x'} e^{-iq \cdot x} W^{\mu\nu}(p', p'; x', x),
$$

(2.3)
of the matrix element

\[ W^{\mu\nu}(p', p; x', x) = \langle (N(p', \sigma'))_{\text{out}} | T J_H^\mu(x) J_H^\nu(x') | (N(p, \sigma))_{\text{in}} \rangle , \]  

(2.4)

of the chronological product of two (Heisenberg picture) operators \( J_H^\mu \) of the hadronic electromagnetic current

\[ J_H^\mu(x) = \sum_f e_f \bar{\psi}_f(x) \gamma^\mu \psi_f(x) , \]

(2.5)

where the sum is over all light quark flavours \( f \) and \( e_f \) are the quark electric charges (in units of \( e > 0 \)). The matrix element (2.4) can be extracted using the LSZ prescription [33] from the appropriate vacuum Green’s function similarly as the \( S \)-matrix element (2.1), e.g. as

\[ W^{\mu\nu}(p', p; x', x) = \left( i Z_O^{-1/2} \right)^2 \int d^4y'd^4y \, e^{ip'y'} e^{-ip'y} \bar{u}(p', \sigma') \]

\[ \left( i \hat{\phi}_y - m_N \right) \langle \Omega_{\text{out}} | T J_H^\mu(x) J_H^\nu(x') O_H(y') \bar{O}_H(y) | \Omega_{\text{in}} \rangle \left( i \hat{\phi}_y + m_N \right) u(p, \sigma) . \]

The application of the Gell-Man - Low formula\(^2\) (see e.g. [33]) allows in principle to compute vacuum matrix elements like the one in (2.6) perturbatively expressing them through the chronological products of the interaction picture operators according to the standard rule

\[ \langle \Omega_{\text{out}} | T J_H^\mu(x) J_H^\nu(x') O_H(y') \bar{O}_H(y) | \Omega_{\text{in}} \rangle \]

\[ = \langle \Omega_0 | T J_I^\mu(x) J_I^\nu(x') O_I(y') \bar{O}_I(y) \exp \left( -i \int d^4w \mathcal{H}^\text{int}_I(w) \right) | \Omega_0 \rangle , \]  

(2.7)

in which \( \mathcal{H}^\text{int}_I(w) \) is the interaction hamiltonian density of QCD expressed in terms of the interaction picture quark and gluon field operators and the operators \( O_I \) and \( J_I^\mu \) in this picture are obtained by simply replacing in their Heisenberg counterparts the elementary Heisenberg picture operators by the interaction picture ones. The Green’s function can be therefore computed perturbatively by expanding the exponens and using the Wick theorem. Because of the large value of the QCD coupling \( g_s \), it is clear, however, that the expansion obtained in this way cannot be truncated to its few first terms only and for this reason the entire prescription for computing \( W^{\mu\nu}(p', p; q', q) \) remains rather formal. To make some progress, one has therefore to reorganize the formal perturbative expansion obtained from (2.7) in order to pick up this part of it which through which the large momentum transfer \( q \) flows and which, owing to this, can be reliably computed evaluating only a few subdiagrams (as a small value of the QCD coupling can be used for this purpose), and the rest which can be identified as giving the expansion of matrix elements analogous to the left hand side of (2.7), but with the electromagnetic currents replaced by some other operators. In other words one wants to represent the right hand side of the formula (2.3) defining the tensor \( W^{\mu\nu}(p', p; q', q) \) by the sum of convolutions of “hard” and “nonperturbative” parts of the schematic form (possible contractions of spinor and/or Lorentz indices other than \( \mu\nu \)

\(^{2}\)It is interesting to note that the Gell-Mann - Low formula can be directly applied here owing to nonzero quark masses which are sources of explicit chiral symmetry breaking which removes degeneracy of a continuum of QCD vacua which, if quark masses were vanishing, would exist making impossible adiabatic reaching the full QCD vacuum from the free one.
between the amplitudes $T_W^{\mu\nu}$ and $W$ are not displayed

$$(2\pi)^4\delta^{(4)}(q' + p' - q - p) W^{\mu\nu}(p', p; q', q)$$

$$= \sum_{W} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} (2\pi)^4\delta^{(4)}(q' - q - k_n - \ldots - k_1) T_W^{\mu\nu}(q', q, k_n, \ldots, k_1)$$

$$\times (2\pi)^4\delta^{(4)}(p' + k_1 + \ldots + k_n - p) W(k_n, \ldots, k_1; p', p),$$

(2.8)

where the expressions in the second line are Fourier transforms of amplitudes defined similarly as (2.4) but with the product of two electromagnetic currents replaced by various products of different numbers of other operators. One such amplitude (in fact the most important one in the leading order) is

$$W_{(f)\beta\alpha}(p', p; z) = \langle(N(p', \sigma'))_{out}|T\bar{\psi}^H_{(f)\beta}(-z/2) \psi^H_{(f)\alpha}(z/2)|(N(p, \sigma))_{in}\rangle.$$  (2.9)

It too, similarly as the amplitude (2.6), can formally be represented by a perturbative expansion by applying the Gell-Mann - Low theorem to the appropriate vacuum matrix element. The contribution of (2.9) to $W^{\mu\nu}(p', p; q', q)$ can be written in the form

$$W^{\mu\nu}(p', p; q', q) = \int \frac{d^4k}{(2\pi)^4} T_{\beta\alpha}^{\mu\nu}(q', q, k) \int d^4z e^{ikz/2+i(k+q-q')\cdot z/2} W_{(f)\beta\alpha}(p', p; z),$$  (2.10)

To see this, let’s write explicitly the corresponding term in the sum (2.8) in the form (see Fig. 2.2):

![Figure 2.2: Graphical representation of the contribution (2.10) to the amplitude $W^{\mu\nu}(p', p; q', q)$: a quark of flavour $f$ and four-momentum $k_1$ emerges from the nonperturbative part, interacts absorbing a large momentum transfer $q$ (the “hard” part of the amplitude) and reimmerses in the nonperturbative part with four-momentum $k_2 = k_1 + q - q'$.

$$\int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (2\pi)^4\delta^{(4)}(q' - q - k_2 - k_1) T_{\beta\alpha}^{\mu\nu}(q', q, k)$$

$$\int d^4z_2 d^4z_1 e^{ik_2z_2} e^{ik_1z_1} \langle(N(p', \sigma'))_{out}|T\bar{\psi}^H_{(f)\beta}(z_2) \psi^H_{(f)\alpha}(z_1)|(N(p, \sigma))_{in}\rangle.$$  

Making the change of variables: $z_2 = \frac{1}{2}(w - z)$, $z_1 = \frac{1}{2}(w + z)$ (the Jacobian is $2^{-4}$) and using the Poincaré covariance of the matrix element

$$\langle(N(p', \sigma'))_{out}|T\bar{\psi}^H_{(f)\beta}(z_2) \psi^H_{(f)\alpha}(z_1)|(N(p, \sigma))_{in}\rangle$$

$$= e^{i(p' - p) \cdot w/2} \langle(N(p', \sigma'))_{out}|T\bar{\psi}^H_{(f)\beta}(-z/2) \psi^H_{(f)\alpha}(z/2)|(N(p, \sigma))_{in}\rangle,$$

(2.10)
enables one to take explicitly the integral over $w$ which gives $(2\pi)^4 \delta^{(4)}(q' + p' - q - p)$ (times $2^4$ which cancels the Jacobian) and comparing the result with the definition (2.3) one arrives at the contribution (2.10) to the tensor $W^\mu\nu(p', p; q', q)$.

The contribution of the amplitude (2.9) to the tensor $W^\mu\nu(p', p; q', q)$, taking into account only the zeroth order (in $g_s$) contributions to $T^\mu\nu_{\beta\alpha}(q', q, k)$, is shown in Figs. 2.3a and 2.3b. Fig. 2.3c shows one of possible contributions to $W^\mu\nu(p', p; q', q)$ of the amplitude analogous to (2.9) but with four quark fields under the chronological product (and with the lowest order contribution to the associated amplitude $T^\mu\nu_{\alpha_0\beta_0}(k_1, k_2, q, q')$). This contribution, which can be interpreted as absorption and emission by nucleon of two different quarks, is subleading in the limit of high $|q^2|$ [8], and will therefore be neglected in our analysis.

![Diagram](image)

Figure 2.3: Contributions to the DVCS amplitude (to the tensor $W^\mu\nu(p', p; q', q)$) of the nonperturbative amplitude (2.9) combined with the lowest order contributions (diagrams a and b) to the associated perturbative amplitude $T^\mu\nu_{\beta\alpha}(q', q, k)$. Diagram c is the possible contribution of the nonperturbative amplitude with four quark fields. Such contributions can be shown to be of a lower order in the hard scale [8], and are, hence, neglected.

The perturbative amplitudes $T^\mu\nu_{\beta\alpha}$ can be computed using the standard Feynman rules and are simply equivalent to connected Green’s functions with amputated external lines. The lowest order contribution to $T^\mu\nu_{(f)\beta\alpha}(q', q, k)$ therefore is (cf. Figs 2.3a and 2.3b):

$$T^\mu\nu_{(f)\beta\alpha}(q', q, k) = e_f^2 \left[ \gamma^\nu \frac{i(k + q + m_f)}{(k + q)^2 - m_f^2 + i0} + \gamma^\mu \frac{i(k - q' + m_f)}{(k - q')^2 - m_f^2 + i0} \right]_{\beta\alpha}.$$  (2.11)

It can be viewed as the amplitude of the elastic scattering of the (highly) virtual photon of four-momentum $q$ on an off-shell quark of flavour $f$ and four-momentum $k$.

In the limit $-q^2 \to \infty$ (physically realized when $-q^2 \gg m_N$) it is convenient to go to the infinite momentum frame, in which the ”$+$” components of the four-momenta $p$, $p'$ and $q$, as well as the ”$-$” component of $q$ are much larger than $m_N$. Recall, that in this frame the momenta of hadrons are parametrized in the following way:

$$p^\mu \approx (1 + \xi) p^+ n_+^\mu, \quad p'^\mu \approx (1 - \xi) p'^+ n_+^\mu.$$  (2.12)

At this point one invokes the basic assumption of the parton model, namely that in the hard amplitude, that is in the lowest order in the formula (2.11), the four-momentum of the incoming quark can be taken in the form $k^\mu = k^+ n_+^\mu$ (see: [7], page 112, Fig. 5.2). Heuristically one can argue that this quark, being a constituent of a hadron with large ”$+$” component of its (on-shell)
four-momentum $p^\mu$, should also have a large "+" component of the four-momentum $k^\mu$, while all other components of $k^\mu$ should be much smaller compared to the components of the $q$ and $q'$ four-momenta (note, that $q - q' \approx 2\xi p^+ n_+$). It should, however, be clearly said, that this is a model assumption which cannot be justified from the first principles. Using this approximation, the integration over $dk^- d^2 k_\perp$ in the formula (2.10) can be explicitly performed giving the Dirac delta which allows to perform the integration over $dz^+ d^2 z_\perp$. Parametrizing the four-momentum of the quark 'entering' the hard part using the variable $x$ as $k^+ = (x + \xi) p^+$, one can represent the contribution (2.10) to $W^{\mu\nu}(p', p, q', q)$ in the form

$$W^{\mu\nu}(p', p, q', q) = p^+ \int dx \mathcal{T}^\mu_\alpha_\beta (q', q, k = (x + \xi) p^+ n_+) \int \frac{dz^-}{2\pi} e^{ixp^+ z^-} W_{\beta\alpha}(p', p, z = z^- n_-).$$

(2.13)

In order to perform the contraction of the spinor indices in (2.13) one can write the product $\bar{\psi}_\beta \psi_\alpha$ of field operators in (2.9) using the Fierz decomposition (see for example [7], p. 33):

$$\bar{\psi}_\beta \psi_\alpha = \frac{1}{4} \left( 1 \right)_{\alpha\beta} \bar{\psi} \psi + \frac{1}{4} \left( \gamma_\mu \right)_{\alpha\beta} \bar{\psi} \gamma^\mu \psi + \frac{1}{4} \left( \gamma_5 \gamma_\mu \right)_{\alpha\beta} \bar{\psi} \gamma^\mu \gamma_5 \psi + \frac{1}{4} \left( \gamma_5 \right)_{\alpha\beta} \bar{\psi} \gamma_5 \psi + \frac{8}{16} (\sigma^{\mu\nu})_{\alpha\beta} \bar{\psi} \sigma^{\mu\nu} \psi,$$

(2.14)

where 1 denotes the unit matrix and $\sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$. If the quark mass $m_f$ in the numerator of (2.11) is neglected, only two structures, the one with $\gamma^\lambda$ and that with $\gamma^\lambda \gamma_5$, survive after taking the trace. Moreover, Lorentz invariance implies that the expressions

$$\begin{aligned}
\int \frac{dz^-}{2\pi} e^{ixp^+ z^-} \langle N(p', \sigma')_{\text{out}} | T \bar{\psi}_f^H (-z^- n_- / 2) \gamma_5 \psi_f^H(z^- n_- / 2) | N(p, \sigma)_{\text{in}} \rangle, \\
\int \frac{dz^-}{2\pi} e^{ixp^+ z^-} \langle N(p', \sigma')_{\text{out}} | T \bar{\psi}_f^H (-z^- n_- / 2) \gamma^\lambda \gamma_5 \psi_f^H(z^- n_- / 2) | N(p, \sigma)_{\text{in}} \rangle,
\end{aligned}$$

(2.15)

can depend on the index $\lambda$ only through $p^\lambda$ and $p'^\lambda$. Therefore, in the Sudakov frame in which the "transverse" ("\perp") and "−" components of $p^\lambda$ and $p'^\lambda$ can be neglected compared to the "+" one one can approximate (2.13) by

$$W^{\mu\nu}(p', p, q', q) \approx \sum_f \left[ \frac{p^+_4}{4} \int d^4x \text{tr} \left[ \mathcal{T}^\mu_\alpha_\beta (q', q, (x + \xi) p^+ n_+) \gamma^- \right] ight.$$

$$\times \int \frac{dz^-}{2\pi} e^{ixp^+ z^-} \langle N(p', \sigma')_{\text{out}} | T \bar{\psi}_f^H (-z^- n_- / 2) \gamma^+ \psi_f^H(z^- n_- / 2) | N(p, \sigma)_{\text{in}} \rangle$$

$$+ \frac{p^+_4}{4} \int d^4x \text{tr} \left[ \mathcal{T}^\mu_\alpha_\beta (q', q, (x + \xi) p^+ n_+) \gamma^- \gamma_5 \right]$$

$$\times \int \frac{dz^-}{2\pi} e^{ixp^+ z^-} \langle N(p', \sigma')_{\text{out}} | T \bar{\psi}_f^H (-z^- n_- / 2) \gamma^+ \gamma_5 \psi_f^H(z^- n_- / 2) | N(p, \sigma)_{\text{in}} \rangle \right].$$

(2.16)

The 1/4 from the Fierz decomposition is conventionally absorbed in the hard part. As shown e.g. in [28], in the kinematics of the Sudakov frame in the matrix elements (2.15) the symbols $T$ of the chronological products can be removed.
Figure 2.4: The lowest order (in $g_s$) diagram which introduces the dependence of $W^\mu\nu(p', p, q', q)$ on the gluon generalized distributions.

Higher order (in $g_s$) corrections to the hard amplitude (2.11) will be discussed in section 2.2. Since gluons can interact with photons through quark loops, in the sum (2.8) one has to take into account also other terms. In particular in order $g^2_s$ there will appear terms corresponding to the emergence from the nonperturbative part of two gluons. The order $g^2_s$ contribution to the corresponding hard amplitude $T^\rho\kappa_{\mu\nu}(q', q, k)$ is shown in Fig. 2.4. The additional Lorentz structure of this amplitude (the indices $\rho\kappa$) can be decomposed into $g_{\rho\kappa}$ and $\epsilon^\rho_{\kappa\sigma\lambda}n^\sigma_+n^\lambda_-$ and the analysis similar to the one discussed above shows that the corresponding contribution to the $W^\mu\nu(p', p, q', q)$ amplitude takes the form analogous to (2.16) but with the nonperturbative part represented by the matrix elements

$$\int \frac{dz}{-2\pi} e^{ixp^+z^-} \langle N(p', \sigma)_{\text{out}} | G^{+\lambda}(z^- n^\lambda_-/2) | N(p, \sigma)_{\text{in}} \rangle, \quad (2.17)$$

$$\int \frac{dz}{-2\pi} e^{ixp^+z^-} \langle N(p', \sigma')_{\text{out}} | G^{+\lambda}(z^- n^\lambda_-/2) \tilde{G}^{+\lambda}(z^- n^\lambda_-/2) | N(p, \sigma)_{\text{in}} \rangle, \quad (2.18)$$

the gluon field strength tensor $G^{\lambda\rho}$ and and its dual $\tilde{G}^{\lambda\rho}$ are used instead of the gauge potentials – for reference see Sec. 3.2.6 in [35].

Summarizing, the factorization, sketched above on the example of the DVCS subprocess, should in general allow to represent the amplitude $A(W^\mu\nu(p', p, q', q)$ in the case of DVCS) of an exclusive process in the schematic form (depicted in Fig. 2.5)

$$A = \sum_i \int_{-1}^{1} dx \ GPD^i(x, \xi, t) T_i(x, \xi, \ldots), \quad (2.19)$$

where the summation over $i$ runs over several kinds of quark and gluon generalized distributions $GPD^i(x, \xi, t)$ ($H^{q,g}, \tilde{H}^{q,g}, E^{q,g}$ defined by the decompositions (1.11)-(1.14) in the case of DVCS), $t = (p' - p)^2$, and $T_i(x, \xi, \ldots)$ are the corresponding hard parts computable perturbatively (the ellipses stand for all kinematics other kinematical characteristics on which the hard amplitude may depend).
\[ \sum_i \int_{-1}^{1} p^+ dx \begin{array}{c} x \xi \end{array} (x + \xi) p^+ (x - \xi) p^+ \]

\[ \text{GPD}^i(x, \xi, t) \]

Figure 2.5: Graphical representation of the factorization of the DVCS amplitude. \( \int_{-1}^{1} p^+ dx \) corresponds to integration over all possible values of the ”+” components of momentum of the parton entering the hard part of the amplitude (the upper green blob).

### 2.2 Higher order QCD corrections

In this section we explain, how perturbative corrections in QCD affect the theoretical predictions of amplitudes of exclusive processes. It turns out, that whether the collinear factorization can be applied at higher orders of perturbative calculation is a non-trivial question, since validity of the factorization requires that infrared and collinear divergences from bare parton distributions cancel with those present in the hard amplitude. For example, see [22,26,31].

Figure 2.6: Examples of diagrams in the hard-part of the amplitude at the \( \alpha_S \) order in QCD. There are two kinds of divergences. The first one – ultraviolet divergence – is present in diagrams (a) and (b). By power counting one sees that the diagram (c) is UV safe. However, there are also so-called collinear divergences present in diagrams (b) and (c), which result from the integration region, where the momentum of the gluon is nearly collinear with the one of the on-shell quark. These divergences are discussed in more detail in Subsec. 2.2.2.

In this work we use the \( \overline{MS} \) renormalization scheme together with the dimensional regularization, where the number of dimensions is \( d = 4 - \varepsilon \). In general, one should use the renormalized perturbation theory, so that the hard amplitude is UV-finite, however, at the one-loop level the renormalization constants cancel, so that in calculations presented here, for our convenience, we can use the bare fields and unrenormalized hamiltonian (so that there is no need to include the diagrams with counter-terms). For reference, see the beginning of Part III in [22]. At higher orders, that argument does not necessarily work, so that one has to use the renormalized theory. Computation of regularized loop corrections to the hard part is a well-known procedure, which, at this point, is not discussed. In Chapter 3 we present such calculations in the process of photoproduction of photon pairs. The NLO analysis of DDVCS can be found in [31].
Apart from hard amplitudes, also bare parton distributions develop divergences at higher orders in strong coupling $\alpha_s$. To control them, one needs to renormalize string operators used in Eqs. (1.5)-(1.8) by finding such combinations of these operators, that their matrix elements are finite. Subsequently, one defines renormalized GPDs denoted by $\text{GPD}_R^i$ using Green’s functions of the renormalized string operators. The relation between the bare parton distributions used in the formula (2.21) and the renormalized ones reads:

$$\text{GPD}^i(x, \xi, t) = \text{GPD}_R^i(x, \xi, t; \mu_F) + \frac{2}{\varepsilon} \left( \frac{\mu_F^2 e^\gamma}{4\pi \mu_R^2} \right)^{-\varepsilon/2} \int dx' \ K^{ij}(x, x') \text{GPD}_R^j(x', \xi, t; \mu_F) + \mathcal{O}(\alpha_s^2),$$

(2.20)

where $\gamma$ is the Euler constant, $\mu_R$ and $\mu_F$ denote the renormalization scale and factorization scale, respectively. Meaning of these scales is elaborated in Section 2.3. The matrix $K^{ij}$ is called the evolution kernel, and there is summation over index $j$. In Section 2.3 it is explained, how it is derived by considering Green’s functions of string operators defining GPDs.

Recall the factorization formula (2.19):

$$\mathcal{A} = \sum_i \int_{-1}^1 \text{GPD}^i(x, \xi, t) \mathcal{T}_i(x, \xi, ... ) dx. \quad (2.21)$$

The divergent hard part can be organized in the following way (for simplicity we omit arguments other than $x$):

$$\mathcal{T}^i(x) = C_0^i(x) + \left[ -\frac{2}{\varepsilon} \left( \frac{|Q^2| e^\gamma}{4\pi \mu_R^2} \right)^{-\varepsilon/2} C_{\text{coll.}}^i(x) + C_1^i(x) \right], \quad (2.22)$$

where $C_1^i$ and $C_{\text{coll.}}^i$ are of order $\alpha_s$. Let us write the divergent part of the amplitude at the order $\alpha_s$:

$$\mathcal{A}_{\text{div}} = -\frac{2}{\varepsilon} \sum_j \int_{-1}^1 dx \ \text{GPD}_R^j(x) C_{\text{coll.}}^j(x) +$$

$$+ \frac{2}{\varepsilon} \sum_{i,j} \int_{-1}^1 dx \ dx' \ K^{ij}(x, x') \text{GPD}_R^j(x') C_0^i(x'). \quad (2.23)$$

The cancellation of divergences occurs if

$$C_{\text{coll.}}^j(x) = \sum_i \int_{-1}^1 K^{ij}(y, x) C_0^i(y) dy. \quad (2.24)$$

If that is the case, the the amplitude can be expressed using the finite quantities:

$$\sum_i \int_{-1}^1 \text{GPD}_R^i(x, \xi, t; \mu_F) \left[ C_0^i(x, \xi, ...) + C_1^i(x, \xi, ...) + \log \left( \frac{|Q^2|}{\mu_F^2} \right) C_{\text{coll.}}^i(x, \xi, ...) \right]. \quad (2.25)$$

Dependence on the scale $\mu_R^2$ vanished due to the simple relation

$$\log \left( \frac{|Q^2|}{\mu_R^2} \right) - \log \left( \frac{\mu_F^2}{\mu_R^2} \right) = \log \left( \frac{|Q^2|}{\mu_F^2} \right).$$

Generalization to higher QCD orders is straightforward, and can be described in the following points:
1. Using the renormalized fields and couplings, compute all diagrams in the hard part of the amplitude within a given regularization scheme. Renormalization makes them only UV-finite, but the still possess infrared/collinear divergences.

2. Using the same regularization, express the bare parton densities in terms of renormalized ones. In particular, the divergent part of it must be expressed as a product of an infinite constant and a term linear in renormalized GPDs.

3. Verify cancellation of infrared/collinear divergences and write the amplitude in terms of finite quantities.

### 2.2.1 Gauge link

It turns out, that at higher orders in perturbative expansion the leading (in the hard scale) contributions stem not only from terms, in which there are just two field operators between the hadronic states. Equally important are graphs with an arbitrary number of gluons leaving the hard diagram and entering the nonperturbative part, presented graphically in Fig. (2.7). In this part we argue, that these diagrams result in presence of the gauge link in operators defining GPDs.

![Diagram](image.png)

Figure 2.7: At higher orders one needs to include contributions, in which apart from two “main” partons (denoted by straight lines connecting the hard and nonperturbative parts) there is an arbitrary number of gluons leaving the hard part and entering the nonperturbative matrix element.

To simplify these contributions, let us work on the matrix element from Eq. (2.7) in the coordinate space. Let \( \mathcal{T}^{(i)}(z_1, z_2) \) denote a hard diagram with one parton \( i \) incoming at the position \( z_1 \) and one (of the same kind) outgoing at \( z_2 \) (we do not explicitly write free spinor or vector indices that get contracted with their counterparts in the nonperturbative part). By \( \mathcal{T}_{(i)\mu_1...\mu_n}^{(i)}(z_1, z_2; y_1, \ldots, y_n) \) we denote the sum of all diagrams with \( n \) outgoing gluons obtained by attaching gluon fields \( A_{\mu_i}(y_i) \) at positions \( y_1, \ldots, y_n \) on propagators lines inside of the diagram \( \mathcal{T}^{(i)}(z_1, z_2) \). To make this statement more clear, let us present an example: let \( \mathcal{T}(z_1, z_2) \) correspond to a diagram presented in Fig. 2.8.

![Diagram](image.png)

Figure 2.8: The exemplary diagram \( \mathcal{T}(z_1, z_2) \).
Then, $\mathcal{T}_1^{\mu}(z_1, z_2; y_1)$ is a sum of diagrams shown in Fig. 2.9:

\[ \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
+ \text{other attachments}
\end{array} \]

Figure 2.9: The sum $\mathcal{T}_1^{\mu}(z_1, z_2; y)$ defined by $\mathcal{T}(z_1, z_2)$ in Fig. 2.8.

For a given kind of parton $i$, summation over all possible hard part diagrams involving this parton and having an arbitrary number of outgoing gluon lines is equivalent to summation over all possible diagrams $\mathcal{T}^{(i)}$ and the corresponding $\mathcal{T}^{(i)}$. Inside of the time ordered product of operators in the matrix element (2.7) one finds the expression

\[ \sum_{\mathcal{T}^{(i)}} \left( \mathcal{T}^{(i)}(z_1, z_2) + \sum_{n=1}^{\infty} \left( \prod_{j=1}^{n} \int d^4 y_j A_{\mu_j}(y_j) \right) \mathcal{T}^{(i)}_{\mu_1 \ldots \mu_n}(z_1, z_2; y_1, \ldots, y_n) \right). \tag{2.26} \]

Note, that in the analysis presented in Section 2.1 only the first term from (2.26) was present. The reasoning shown in the work of Efremov and Radyushkin [32] allows to prove, that for any diagram $\mathcal{T}^{(i)}$, the sum in (2.26) can be written as

\[ \mathcal{T}^{(i)}(z_1, z_2) P \exp \left( ig \int_{z_1}^{z_2} dz A_{\mu}(z) \right) + \text{remainder}, \tag{2.27} \]

where the remainder consists of operators which matrix elements yield sub-leading terms.

To show it, one has to analyze how attachments of gluon vertices modify propagators of the particles. Consider the sum of quark propagators with additional attachments of gluons, as it is shown in Fig. 2.2.1:

\[ \begin{array}{c}
\text{Propagator} \\
+ \text{Attachments} \\
+ \text{Propagator} \\
+ \ldots
\end{array} \]

Sum of such diagrams can be written as

\[ S(x_1 - x_2) = S(x_1 - x_2) + g \int d^4 y_1 S(x_1 - y_1) \mathbb{A}(y_1) S(y_1 - x_2) + \ldots \tag{2.28} \]

Observe that for a massless quark the free propagator $S$ is the solution of the equation

\[ i\partial S(x_1 - x_2) = -\delta^4(x_1 - x_2), \tag{2.29} \]

while $S$ is the propagator of the quark in the background gauge field, and

\[ i\mathbb{D} S(x_1 - x_2) = -\delta^4(x_1 - x_2), \tag{2.30} \]

where $D_\mu = \partial_\mu + igA_\mu$ is the covariant derivative. It can be shown [32] that

\[ S(x_1 - x_2) = P \exp \left( ig \int_{x_2}^{x_1} dx A_\mu(x) \right) \left( S(x_1 - x_2) + R(x_1, x_2) \right), \tag{2.31} \]
where \( P \exp \left( ig \int_{x_1}^{x_2} dx_\mu A_\mu(x) \right) \) is the path-ordered Wilson line (also known as the gauge link). The “remainder” \( R(x_1, x_2) \) depends on the gauge field by the field strength tensor only, which gives a sub-leading terms when inserted between the hadron states. The analogous reasoning applied to gluon and ghost propagators yields the same result. Hence, summation over all possible attachments of external gluon lines is (to the leading order in the hard scale) equivalent to substituting of propagators inside of the diagram \( T^{(i)}(z_1, z_2) \) with ones multiplied by the gauge link. That, on the other hand can be shown to give the overall factor

\[
P \exp \left( ig \int_{z_1}^{z_2} dz_\mu A_\mu(z) \right)
\]

standing by the amplitude \( T^{(i)}(z_1, z_2) \). It is crucial that it does not depend on the parton type \( i \) or the exact form of the diagram \( T^{(i)} \), and hence can included inside the nonperturbative part without any further complication. Therefore, after using the collinear approximation and applying the LSZ reduction formula to the vacuum Green’s function of the product of parton fields, hadron operators \( O \) and the gauge link one finally obtains the full definition of Generalised Parton Distributions shown in Formulas (1.5)-(1.8).

### 2.2.2 UV and soft collinear divergences

Here we briefly describe, how divergences arise in all the considered amplitudes and how to classify them. It allows to clarify, why collinear divergences are present in our calculations, and say in which diagram they occur.

A general form of a momentum integral encountered in a diagram with \( L \) loops, which contain \( N \) propagators (all of massless particles), can be written in the following form:

\[
G\left(\{p_i\}\right) = \prod_{i=1}^{L} \int \frac{d^d k_i}{(2\pi)^d} \frac{P(\{k_i\})}{\prod_{n=1}^{N} \left( Q_n^2(\{k_i\}) + i0 \right)},
\]

where \( \{p_i\} \) denotes the set of external momenta entering the loops, \( Q_n^2(\{k_i\}) \) is the momentum squared in \( n \)-th propagator, which depends on momenta over which we integrate, and \( P(\{k_i\}) \) denotes momenta-dependent polynomial. There are two classes of regions, which may lead to divergences. The first one, corresponding to the limit \( k^2 \to \infty \) is the well-known ultraviolet divergence, which is fixed by the hamiltonian renormalization. The other source of divergence are poles of the integrand corresponding to \( Q_n^2 = 0 \) (particle on the mass shell). They can be studied using the method developed by Landau [34], which we describe now. Using the Feynman parametrization, one can write formula (2.32) as

\[
G\left(\{p_i\}\right) = \prod_{j=1}^{N} \int_{0}^{1} d\alpha_j \delta \left( 1 - \sum_{i=1}^{N} \alpha_i \right) \prod_{l=1}^{L} \frac{d^d k_l}{(2\pi)^d} \frac{P(\{k_i\}, \{p_i\})}{\left[ \sum_{n=1}^{N} \alpha_n Q_n^2(\{k_i\}) + i0 \right]^N}.
\]

Let us denote the denominator as \( D(\{k_i\}) + i0 \). The integrand is an analytical function of \( \{k_i\} \) and \( \{\alpha_i\} \), so that one can modify the contour of the integration in a way that it avoids singular points. For example, assume that, for some fixed momenta, \( D \) becomes 0 and there is some \( \alpha_j \).
such that at this problematic point \( Q_j^2 \neq 0 \). Then, if \( \alpha_j \neq 0 \) (it is not at the endpoint), one can shift the contour to avoid the singularity, see Fig. \ref{fig:contour_change}

\[ \text{Re} \alpha_j \]

\( \times \)

\( \text{Re} k_i^\mu \)

\( \times \)

\( \times \)

\( \text{Re} k_i^\mu \)

\( \times \)

Figure 2.10: An example of how one can change the contour avoiding integration in a neighborhood of the pole, which is marked by the cross slightly above the real axis.

Modification of the integration contour avoiding the pole is impossible, if that occurs at one of the endpoints (0 or 1) or the term \( Q_j^2 \), which multiplies \( \alpha_j \) is zero. In the case \( \alpha_j = 1 \), all other Feynman parameters are equal to 0. It means that, if \( D = 0 \), then \( Q_j^2 = 0 \). Hence, modification of \( \alpha_j \) does not allow to avoid the singular point if

\[ \alpha_j = 0 \quad \text{or} \quad Q_j^2 = 0 \quad \text{for all } j. \] (2.34)

If that is the case, one can try to avoid the singularity by modifying the contour of integration over some component of the loop momentum \( k_i^\mu \). Since the domain of the integration is the whole real axis, then there are no problems with the endpoints. However, there still can be a situation, when it is impossible to avoid the singularity – because \( D \) is a quadratic function of momenta \( \{k_i\} \), then for each considered component of momentum there are 2 corresponding roots of \( D \). If their real values are equal (and their corresponding imaginary parts are separated only by an infinitesimally small value \( i0 \)), then the line of integration lays between these two poles and one cannot change the contour, see Fig. \ref{fig:pinch}

\[ \text{Re} k_i^\mu \]

\( \times \)

\( \times \)

\( \times \)

\( \text{Re} k_i^\mu \)

Figure 2.11: The illustration of a pinch singularity: in the case presented on the left hand side, one can modify the contour avoiding poles, while in the case on the right hand side it is impossible, because the initial contour is 'pinched' between two poles.

The pinch singularity corresponds to the situation, when all components of considered loop momenta \( k_i^\mu \) are double roots of \( D \), which translates into the condition

\[ \frac{\partial}{\partial k_i^\mu} D = 0. \] (2.35)
If we assume that all loop momenta come with the positive sign into propagator momenta $Q$, we obtain that Eq. (2.35) is equivalent to

$$
\sum_{n=0}^{N} \alpha_n Q_n^\mu = 0. \quad (2.36)
$$

Equations (2.34) and (2.36) are known as Landau equations. We use them to analyze, what kind of divergences are present in the 1-loop hard diagrams in DVCS.

Let us start from the simplest case, namely the self-energy correction, where the momentum of the quark $p$ is off-shell. In dimension $d \geq 4$ this diagram naturally has the UV divergence, but we want to focus only on the on-shell singularities. The corresponding denominator can be written using the Feynman parameters as

$$
D = \alpha_1 k^2 + \alpha_2 (p + k)^2. \quad (2.37)
$$

Using the Landau equations we find, that there are two singular points, which cannot be avoided:

$$
\begin{align*}
\alpha_1 &= 0, \quad p^\mu + k^\mu = 0, \\
\alpha_2 &= 0, \quad k^\mu = 0.
\end{align*} \quad (2.38)
$$

However, note that this is a singularity at a single point, therefore, for a number of dimensions $d$ greater than 2 it is integrable and there are no resulting infrared divergences. Hence, the considered self-energy diagram has only UV divergence for $d \geq 4$, is finite for $d = 3$ and develops the infrared divergence for $d = 2$.

Now let us consider a vertex diagram with the incoming quark line on-shell. Assume that $p = p^+ n_+^\mu$.

Landau equation again tell us that there are pinch singularities when $k$, $p + k$ or $p + q + k$ are zero, but in dimension 4 it does not lead to IR divergences. However, there is another solution which corresponds to the situation, when the gluon’s momentum is collinear with that of the incoming quark: $k^\mu = k^+ n_+^\mu$. In such situation the pinch singularity occurs at

$$
\alpha_1 k^+ + \alpha_2 (p^+ + k^+) = 0, \quad \alpha_3 = 0, \quad \iff \quad \alpha_2 = -\frac{k^+}{p^+}. \quad (2.39)
$$
To obtain the second equation we used the fact that \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \). That leads to so-called collinear divergences, which correspond to the situation when two or more momenta of massless particles become collinear and on-shell. Note, that if the momentum of the incoming quark was off-shell, then the vertex diagram would have only the UV divergence. Collinear divergence is present also in the case of the box diagram (i.e. the graph (c) in Fig. 2.6). As it was shown in the previous part, the key point of factorization theorem is proving that collinear divergences from the hard part can be absorbed in the divergences present in the non-renormalized parton distributions.

### 2.3 Renormalization of non-local product of operators

As it was discussed at the beginning of Section 2.2, to gain control over divergences in parton distributions, one need to find renormalized string operators used in the definition of GPDs. In this Section we describe the procedure of construction of these operators. For further convenience, let use denote string operators according to the following formula:

\[
O^{(f)}(z_1, z_2) := \bar{\psi}^{(f)}(z_1) \gamma^+ [z_1, z_2] \psi^{(f)}(z_2),
\]

\[
\tilde{O}^{(f)}(z_1, z_2) := \bar{\psi}^{(f)}(z_1) \gamma^+ \gamma_5 [z_1, z_2] \psi^{(f)}(z_2),
\]

\[
O^{gg}(z_1, z_2) := G_\mu^+(z_1) [z_1, z_2] G_\mu^+(z_2),
\]

\[
\tilde{O}^{gg}(z_1, z_2) := \bar{G}_\mu^+(z_1) [z_1, z_2] \tilde{G}_\mu^+(z_2).
\]

Our goal is to express them in terms of renormalized operators minus counter-terms. To do so, we follow the reasoning shown in Appendix G in [35]. It allows to write bare GPDs in terms of renormalized parton distributions plus divergent terms.

Let us start with description of methods used in renormalization of local operators. For example will serve the operator \( \phi^3(0) \) in the scalar theory \( \phi^4 \) in dimension 4. We demand that all vacuum Green’s function of this operator be finite to a given order in perturbation theory. At the first order in perturbation theory, the following two Green’s functions become divergent:

\[
G^1(p) := \int d^4x e^{-ipx} \langle \Omega_{\text{out}} | T \phi^3_H(0) \phi_H(x) | \Omega_{\text{in}} \rangle,
\]

\[
G^3(p_1, p_2, p_3) := \int d^4x_1 d^4x_2 d^4x_3 e^{-ip_1 x_1 - ip_2 x_2 - ip_3 x_3} \langle \Omega_{\text{out}} | T \phi^3_H(0) \phi_H(x_1) \phi_H(x_2) \phi_H(x_3) | \Omega_{\text{in}} \rangle,
\]

Using the Gell-Mann and Low theorem [33] one can perform computation using standard Feynman rules. The corresponding diagrams leading to divergences are shown in Fig. 2.12.
Figure 2.12: Diagrams contributing to Green’s function. The gray dots denote insertion of the operator $\phi^3$. The first diagram from the left represents $G^1(p)$ and two other: $G^3(p_1,p_2,p_3)$—note that for each of them there are 3 possible permutations of insertions of momenta. The divergence from the self-energy correction in the third diagram is cancelled by the field-strength renormalization constant. Divergences present in the rest of diagrams are not cured by the lagrangian renormalization, so that additional counter-terms are required.

The renormalized operator $\phi^3(0)$ at the arbitrary order needs to be corrected by counter-terms proportional to renormalized operators $\phi_R^3$ and $\phi_R^3$:

$$[\phi^3(0)]_R = \phi^3_R(0) + c_1\phi_R(0) + c_3\phi^3_R(0), \quad (2.46)$$

where $c_{1/3}$ are the renormalization constants. Equivalently, one can write it in terms of the bare operators $\phi$, using $\phi_R = Z_{\phi}^{-1/2}\phi$, where $Z_{\phi}$ is the renormalization constant.

In the more general case of two operators separated by a lightlike vector $z_1 - z_2$, one has to use a more complicated form of counter-terms, which become an product of operators integrated over positions:

$$\text{counterterm} \sim \int_0^1 d\alpha_1 d\alpha_2 K(\alpha_1, \alpha_2)\mathcal{O}(\alpha_1 z_1 + (1 - \alpha_1)z_2, \alpha_2 z_2 + (1 - \alpha_2)z_1). \quad (2.47)$$

Distributions $K(\alpha_1, \alpha_2)$ are called the evolution kernels.

It turns out, that the Green’s functions of fields of different kinds do not vanish, hence, for example, the renormalized quark operator consists of gluon fields. Because of that, it is convenient to consider combinations of operators defined in Eqs (2.40)-(2.43) of definite charge-parity. The non-singlet operators, which are $\mathcal{C}$-odd and do not mix under renormalization, are the following:

$$\mathcal{O}^{(f)}_{NS}(z_1, z_2) = \mathcal{O}^{(f)}(z_1, z_2) + \mathcal{O}^{(f)}(z_2, z_1), \quad (2.48)$$

$$\tilde{\mathcal{O}}^{(f)}_{NS}(z_1, z_2) = \tilde{\mathcal{O}}^{(f)}(z_1, z_2) - \tilde{\mathcal{O}}^{(f)}(z_2, z_1). \quad (2.49)$$

$\mathcal{C}$-even (singlet) combinations are:

$$\mathcal{O}^{(f)}_{S}(z_1, z_2) = \mathcal{O}^{(f)}(z_1, z_2) - \mathcal{O}^{(f)}(z_2, z_1), \quad (2.50)$$

$$\tilde{\mathcal{O}}^{g g}_{S}(z_1, z_2) = \tilde{\mathcal{O}}^{g g}(z_1, z_2), \quad (2.51)$$

$$\tilde{\mathcal{O}}^{(f)}_{S}(z_1, z_2) = \tilde{\mathcal{O}}^{(f)}(z_1, z_2) + \tilde{\mathcal{O}}^{(f)}(z_2, z_1), \quad (2.52)$$

$$\tilde{\mathcal{O}}^{g g}_{S}(z_1, z_2) = \tilde{\mathcal{O}}^{g g}(z_1, z_2). \quad (2.53)$$

(it is worth to stress, that gluon operators are always charge-even).

Similarly as in the simpler case of local operator, one needs to find counter-terms necessary to make the vacuum Green’s functions of the time-ordered product of bare fields and operators
\[ \tilde{\psi}(z_1)[z_1, z_2]\psi(z_2), G^{+\mu}(z_1)[z_1, z_2]G^+_\mu(z_2) \] finite. In the case of the quark string operator one considers two kinds of such Green’s functions:

\[ G^{f'f}(p_1, p_2, z_1, z_2) := \int d^4x_1 \int d^4x_2 e^{-ip_1x_1 - ip_2x_2} \langle \Omega_{\text{out}} | T\mathcal{O}^{(f)}(z_1, z_2)\tilde{\psi}(f')(x_1)\psi(f')(x_2) \rangle |\Omega \rangle, \]

\[ G^{fg}(p_1, p_2, z_1, z_2) := \int d^4x_1 \int d^4x_2 e^{-ip_1x_1 - ip_2x_2} \langle \Omega_{\text{out}} | T\mathcal{O}^{(f)}(z_1, z_2)G^{+\mu}(z_1)G^+_\mu(x_2) |\Omega \rangle, \]

(2.54)
in the limit when \( p_{1,2} \) have only the ”+” component of momentum (that is due to the fact, that we assume partons taking part in the hard process to be massless and nearly collinear with the hadron). The function \( G^{fg} \) corresponds to the mixing of quark \( f \) and gluon operators and is present only in the case of the singlet combination. At the \( \alpha_s \) order, Green’s functions with quark operators do not vanish only as \( f = f' \), however, at higher orders there are also contributions responsible for mixing of quark flavors in the singlet sector, as it is presented in Fig. 2.13.

Figure 2.13: Diagrams appearing in computation of Green’s functions of string operators at higher orders, which contribute to the mixing of the operators. The diagram in the middle corresponds to mixing of different quark flavors, while the other two result in mixing of quarks and gluons.

For simplicity, we consider amputated Green’s functions, which are ordinary Green’s functions, but with removed external propagators and spinors (or polarization vectors). For example, in the leading order in the strong coupling constant, the only non-vanishing Green’s function can be represented by the following diagram:

\[ G^{ff}_{\text{amp.}}(p_1, p_2, z_1, z_2) \bigg|_{\alpha_s^0} = \gamma^+ e^{-iz_1^+ p_1^+ - iz_2^+ p_2^+}. \]

(2.55)

Crosses labeled \( z_1 \) and \( z_2 \) denote insertions of operators \( \tilde{\psi}(z_1) \) and \( \psi(z_2) \). It is easy to find, that

\[ \gamma^+ e^{-iz_1^+ p_1^+ - iz_2^+ p_2^+} = \gamma^+ e^{-iz_1^+ p_1^+ - iz_2^+ p_2^+}. \]

(2.56)

At the 1-loop order, when computing \( G^{ff}_{\text{amp.}}(p_1, p_2, z_1, z_2) \), one has to consider diagrams presented in Fig. 2.14. Let us present a detailed computation of the diagram (a). It allows to show methods used in computation of considered Green’s functions and explain, how one obtains the necessary counter-terms. In the Sudakov basis, which is the most convenient for analysis of operators on the lightcone, usually one has to treat ”±” and ”⊥” components independently. It turns out, that the simplest way to achieve it, is to use the Schwinger parametrization of propagator:

\[ \frac{i}{k^2 + i\delta} = \int_0^\infty e^{iy(k^2 + i\delta)} dy. \]

(2.57)
Figure 2.14: Diagrams present at the 1-loop computation of $G^{gf}_{amp.}(p_1, p_2, z_1, z_2)$. Diagrams (a) and (b) contain insertion of a gluon field coming from expanding the gauge link up to the order $g$ (it is denoted by the cross labelled by $\xi$). Diagram (c) corresponds to gluon exchange between two on-shell quarks. Diagrams (d) and (e) are the standard self-energy corrections. As it was said at the beginning of Section 2.2, at the one-loop level one can use the unrenormalized perturbation theory, so that here we do not include diagrams with the self-energy counter-terms.

When more propagators are present, it is useful to perform the following substitution in the integration:

$$
\prod_{j=1}^{N} \int_{0}^{\infty} dy_j \to \int_{0}^{\infty} d\rho \rho^{N-1} \prod_{j=1}^{N} \int_{0}^{1} d\alpha_j \delta \left( \sum_{j=1}^{N} \alpha_j - 1 \right). \quad (2.58)
$$

Because of divergences, we use the dimensional regularization. For gluon field we use the Feynman gauge. Averaging over colors we get that the contribution to the Green’s function from the considered diagram, which we denote by $M^{ff}_{a}$, is

$$
M^{ff}_{a} = g^2 C_F \int_{z_1}^{z_2} d\xi \mu_R^{4-d} \int \frac{d^d k}{(2\pi)^d} \gamma^+ e^{-iz_1^+(p_1^+ - k^+)} e^{-iz_2^+ p_2^+} e^{-i\xi^+ k^+} \frac{p_1^+ - k^+}{(p_1^+ - k^+)^2 + i0} \gamma^+ \frac{1}{k^2 + i0}, \quad (2.59)
$$

where $C_F = \frac{4}{3}$ is the color factor and $\mu_R$ is the renormalization scale added to keep the proper dimension of the amplitude. Phases are the result of insertion of fields with given momenta at points $z_1, z_2$ and $\xi$ – the parameter $\xi$ is the position, at which one inserts the gluon field from the gauge link. Note that, since $(\gamma^+)^2 = 0$, we can write

$$
\gamma^+ (\not{p}_1 - \not{k}) \gamma^+ = 2 (p_1^+ - k^+) \gamma^+, \quad (2.60)
$$

so that the gamma matrix structure of (2.59) is the same as that of (2.55). Performing the integral over $d\xi^-$ we get

$$
(2.59) = 2ig^2 C_F \gamma^+ \mu_R^{4-d} \int \frac{d^d k}{(2\pi)^d k^2 (p_1^+ - k^+)^2} e^{-iz_1^+(p_1^+ - k^+)} e^{-iz_2^+ p_2^+} \frac{1}{k^+} \left( e^{-iz_1^+ k^+} - e^{-iz_2^+ k^+} \right). \quad (2.61)
$$
Let us denote $\Delta z = z^+ - z^{-}$ and use the following identity:

$$
\frac{1}{k^+} \left( 1 - e^{-i\Delta z k^+} \right) = i\Delta z \int_0^1 d\tau e^{-i\Delta z k^+\tau}.
$$

(2.62)

The resulting integrals can be done using the previously discussed Schwinger parametrization:

$$
\int \frac{d^d k}{(2\pi)^d} \int_0^1 d\tau e^{-i\Delta z k^+\tau} \frac{p^+_1 - k^+}{k^2(p_1 - k)^2} =
$$

$$
- \int \frac{d^d k}{(2\pi)^d} \int_0^1 d\tau \int_0^1 d\alpha ((1 - \alpha)p^+_1 - k^+) e^{i\rho k^2} e^{-i\tau\Delta z (k^+ + \alpha p^+_1)\tau}. \tag{2.63}
$$

Going to the second line we used the substitution $k \to k - \alpha p_1$. Integration over $dk^-$ produces a delta distribution:

$$
\int \frac{dk^-}{2\pi} e^{2i\rho k^+} = \frac{1}{2} \rho^{-1} \delta(k^+), \tag{2.64}
$$

which allows to easily perform the integral over $dk^+$. Keep in mind that $k^2 = 2k^+k^- - k_1^2$. Integration over perpendicular momenta is just a standard Gaussian integral:

$$
\int \frac{d^{d-2} k_\perp}{(2\pi)^{d-2}} e^{-i\rho k_\perp} = \frac{1}{(4\pi)^{d/2-1}} (-i\rho)^{-d/2+1}. \tag{2.65}
$$

The resulting integral over $d\rho$ is ill-defined:

$$
\frac{1}{(4\pi)^{d/2-1}} \int_0^\infty d\rho (-i\rho)^{-d/2+1}. \tag{2.66}
$$

To solve this problem, let us multiply the integrand by $e^{-i\rho\mu^2_F}$, where $\mu_F$ is called the factorization scale. It is equivalent with the assumption that all partons in the diagram have masses $\mu_F$ [35]. This way, instead of the integral in formula (2.66), we have

$$
\frac{1}{(4\pi)^{d/2-1}} \int_0^\infty d\rho (-i\rho)^{-d/2+1} e^{-i\rho\mu^2_F} = \frac{i}{4\pi} \left( \frac{\mu_F^2}{4\pi} \right)^{d/2-2} \Gamma \left( 2 - \frac{d}{2} \right), \tag{2.67}
$$

where $\Gamma(z)$ is the gamma function. It is clear, that in the limit of $d \to 4$, this expression becomes divergent. Now we perform the integration over the parameter $d\tau$:

$$
p^+_1 (1 - \alpha) \int_0^1 d\tau e^{-i\alpha p^+_1 \Delta z \tau} = \frac{i}{\Delta z} \frac{1 - \alpha}{\alpha} (e^{-i\alpha p^+_1 \Delta z} - 1). \tag{2.68}
$$

All these integrals yield the following result:

$$
(2.63) = - \frac{i}{(4\pi)^2} \left( \frac{\mu_F^2}{4\pi} \right)^{d/2-2} \Gamma \left( 2 - \frac{d}{2} \right) \frac{i}{\Delta z} \frac{1 - x}{x} (e^{-ixp^+_1 \Delta z} - 1). \tag{2.69}
$$

Hence, after multiplying by the remaining constants, we obtain

$$
(2.59) = \frac{1}{8\pi^2} C_F g^2 \left( \frac{\mu_F^2}{4\pi \mu_R^2} \right)^{d/2-2} \Gamma \left( 2 - \frac{d}{2} \right) \gamma^+ \int_0^1 d\alpha \frac{1 - \alpha}{\alpha} (e^{-i(z^- - z^{-})\alpha p^+_1} - 1) e^{-i\alpha p^+_1 + iz^- p^+_1}. \tag{2.70}
$$
It can be conveniently written using so-called 'plus prescription':

\[
\int_0^1 \frac{d\alpha}{\alpha} \left[ \frac{1 - \alpha}{\alpha} \right]_+ f(\alpha) := \int_0^1 \frac{d\alpha}{\alpha} \frac{1 - \alpha}{\alpha} (f(\alpha) - f(0)).
\]  

(2.71)

Moreover, using the Laurent expansion of the gamma function \((d = 4 - \varepsilon)\),

\[
\Gamma \left( 2 - \frac{d}{2} \right) = \Gamma \left( \frac{\varepsilon}{2} \right) = \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon),
\]

(2.72)

we can write

\[
\left( \frac{\mu_F^2}{4\pi\mu_R^2} \right)^{d/2-2} \Gamma \left( 2 - \frac{d}{2} \right) = \frac{2}{\varepsilon} \left( \frac{\varepsilon\mu_F^2}{4\pi\mu_R^2} \right)^{-\varepsilon/2}.
\]

(2.73)

Finally, using \(\alpha_S = g^2/4\pi\), we can re-express the result in a compact form

\[
(2.70) = \frac{2\alpha_S}{\varepsilon} C_F \left( \frac{\varepsilon\mu_F^2}{4\pi\mu_R^2} \right)^{-\varepsilon/2} \int_0^1 d\alpha \left[ \frac{1 - \alpha}{\alpha} \right] + G_{\text{amp.}}(p_1, p_2, (1 - \alpha)\bar{z}_1^- + \alpha \bar{z}_2^-; z_2^\prime).
\]

(2.74)

At the first sight, this expression looks complicated and finding the counter-term may appear a cumbersome task. However, let us notice, that Eq. (2.74) can be written in terms of the LO amputated Green’s function, see Eq. (2.56):

\[
(2.74) = \frac{2\alpha_S}{\varepsilon} C_F \left( \frac{\varepsilon\mu_F^2}{4\pi\mu_R^2} \right)^{-\varepsilon/2} \int_0^1 d\alpha \left[ \frac{1 - \alpha}{\alpha} \right] + G^{\text{ff}}_{\text{amp.}}(p_1, p_2, (1 - \alpha)\bar{z}_1^- + \alpha \bar{z}_2^-, z_2^\prime)\,\bar{G}_{\text{amp.}}(p_1, p_2, (1 - \alpha)\bar{z}_1^- + \alpha \bar{z}_2^-, z_2^\prime).
\]

(2.75)

Using this result, one can easily notice, that the counter-term necessary to cancel divergence from the diagram \((a)\) at the order \(\alpha_S^2\) is

\[
- \frac{2}{\varepsilon} \left( \frac{\varepsilon\mu_F^2}{4\pi\mu_R^2} \right)^{-\varepsilon/2} \int_0^1 d\alpha \frac{\alpha_S}{2\pi} C_F \left[ \frac{1 - \alpha}{\alpha} \right] + \mathcal{O}^{\text{ff}}((1 - \alpha)\bar{z}_1^- + \alpha \bar{z}_2^-, z_2).
\]

(2.76)

After considering all diagrams presented in Fig. 2.14 one obtains the renormalized non-singlet operator:

\[
O_{NS,R}^{(f)}(z_1, z_2; \mu_F) = - \frac{2}{\varepsilon} \left( \frac{\varepsilon\mu_F^2}{4\pi\mu_R^2} \right)^{-\varepsilon/2} \int_0^1 d\alpha_1 \, d\alpha_2 \, K^{\text{ff}}_{\text{NS}}(\alpha_1, \alpha_2) \, O_{\text{NS}}^{(f)}(\bar{\alpha}_1 z_1 + \alpha_1 z_2, \bar{\alpha}_2 z_2 + \alpha_2 \bar{z}_1),
\]

(2.77)

where we denote \(\bar{\alpha}_i = 1 - \alpha_i\) and the evolution kernel is given by

\[
K^{\text{ff}}_{\text{NS}}(\alpha_1, \alpha_2) = \frac{\alpha_S}{2\pi} C_F \left( 1 + \delta(\alpha_1) \left[ \frac{\bar{\alpha}_2}{\alpha_2} \right]_+ + \delta(\alpha_2) \left[ \frac{\bar{\alpha}_1}{\alpha_1} \right]_+ + \frac{3}{2} \delta(\alpha_1)\delta(\alpha_2) \right).
\]

(2.78)

Other evolution kernels in the position space can be found in [27,35]. The relation between renormalized and bare operators can be inverted:

\[
O_{NS}^{(f)}(z_1, z_2) = O_{NS,R}^{(f)}(z_1, z_2; \mu_F) + \frac{2}{\varepsilon} \left( \frac{\varepsilon\mu_F^2}{4\pi\mu_R^2} \right)^{-\varepsilon/2} \int_0^1 d\alpha_1 \, d\alpha_2 \, K^{\text{ff}}_{\text{NS}}(\alpha_1, \alpha_2) \, O_{NS,R}^{(f)}(\bar{\alpha}_1 z_1 + \alpha_1 z_2, \bar{\alpha}_2 z_2 + \alpha_2 \bar{z}_1; \mu_F),
\]

(2.79)

where we used the fact that, at the order \(\alpha_S\), in the counter-term we can use either renormalized operator, or the bare one (the difference is of order \(\alpha_S^2\)). This result allows to write bare GPDs in terms of renormalized parton distributions and their convolutions with the corresponding evolution kernels.
2.4 From evolution of operators in the position space to evolution of GPDs

In this section we finally obtain the relation between bare and renormalized parton distributions and formulate their evolution equations. We define renormalized GPDs in the same way as bare one, but using renormalized string operator. For example, the renormalized non-singlet vector quark distribution is

\[ F_R^f(x, \xi, t; \mu_F) := \int \frac{dz}{2\pi} e^{izp^z} \langle N(p_2, \sigma_2)_{\text{out}} | O^{(f), H}_{NS,R}(-z^-/2, z^-/2; \mu_F) | N(p_1, \sigma_1)_{\text{in}} \rangle. \]  

(2.80)

From the Formula (2.79) we obtain the difference between bare and renormalized GPD:

\[
F_{NS,R}^f(x, \xi, t; \mu_F) - F_{NS}^f(x, \xi, t) = \\
-\frac{2}{\varepsilon} (\frac{e^\varepsilon \mu_R^2}{4\pi})^{-\varepsilon/2} \int \frac{dz}{2\pi} e^{izp^z} \int_0^1 d\alpha_1 d\alpha_2 K_N^{ff}(\alpha_1, \alpha_2) \times \\
\langle N(p_2, \sigma_2)_{\text{out}} | O^{(f), H}_{NS,R}(-z^-(1/2 - \alpha_1), z^-(1/2 - \alpha_2); \mu_F) | N(p_1, \sigma_1)_{\text{in}} \rangle. 
\]  

(2.81)

To write it as a parton distribution convoluted with some kernel, let us first use the Poincaré covariance of the matrix element:

\[
\langle N(p_2, \sigma_2)_{\text{out}} | O^{(f), H}_{NS,R}(-z^-(1/2 - \alpha_1), z^-(1/2 - \alpha_2)) | N(p_1, \sigma_1)_{\text{in}} \rangle = \\
e^{-zp^z \xi(\alpha_1 - \alpha_2)} \langle N(p_2, \sigma_2)_{\text{out}} | O^{(f), H}_{NS,R}(-z^-(1 - \alpha_1 - \alpha_2)/2, z^-(1 - \alpha_1 - \alpha_2)/2; \mu_F) | N(p_1, \sigma_1)_{\text{in}} \rangle. 
\]  

(2.82)

Using the substitution \( z' = z(1 - \alpha_1 - \alpha_2) \) we rewrite the integrals from the Eq. (2.81) in the following form:

\[
\int \frac{dz'}{2\pi} e^{izp^z} \frac{z'(\alpha_1 - \alpha_2)}{1 - \alpha_1 - \alpha_2} \int_0^1 d\alpha_1 d\alpha_2 K_N^{ff}(\alpha_1, \alpha_2) \langle N(p_2, \sigma_2)_{\text{out}} | O^{(f), H}_{NS,R}(-z'^-/2, z'^-/2; \mu_F) | N(p_1, \sigma_1)_{\text{in}} \rangle. 
\]  

(2.83)

It is easy to verify that it can be written as

\[
\int_{-1}^1 dx' \int_0^1 d\alpha_1 d\alpha_2 \delta \left( x - x' (1 - \alpha_1 - \alpha_2) - \xi(\alpha_1 - \alpha_2) \right) K_N^{ff}(\alpha_1, \alpha_2) F_{NS,R}^f(x', \xi, t) = \\
\int_{-1}^1 dx' \frac{1}{|\xi|} V_N^f \left( \frac{x}{\xi}, \frac{x'}{\xi} \right) F_{NS,R}^f(x', \xi, t; \mu_F), 
\]  

(2.84)

where the evolution kernel for the non-singlet GPD is defined as

\[
\frac{1}{|\xi|} V_N^f \left( \frac{x}{\xi}, \frac{x'}{\xi} \right) = \int_0^1 d\alpha_1 d\alpha_2 \delta \left( x - x' (1 - \alpha_1 - \alpha_2) - \xi(\alpha_1 - \alpha_2) \right) K_N^{ff}(\alpha_1, \alpha_2). 
\]  

(2.85)

The resulting formula takes the same form as the one in Eq. (2.20):

\[
F_{NS}^f(x, \xi, t) = F_{NS,R}^f(x, \xi, t; \mu_F) + \frac{2}{\varepsilon} (\frac{e^\varepsilon \mu_F^2}{4\pi \mu_R^2})^{-\varepsilon/2} \int_{-1}^1 dx' \frac{1}{|\xi|} V_N^f \left( \frac{x}{\xi}, \frac{x'}{\xi} \right) F_{NS,R}^f(x', \xi, t; \mu_F). 
\]  

(2.86)
Demanding that the bare operators are independent of the factorization scale, one obtains the evolution equation:

\[ \mu_F^2 \frac{d}{d\mu_F^2} F_{NS,R}^f(x, \xi; \mu_F) = \int_{-1}^{1} dx' \frac{1}{|\xi'|} V_{NS}\left(\frac{x}{\xi}; \frac{x'}{\xi}\right) F_{NS,R}^f(x', \xi, t; \mu_F). \]  

(2.87)

In the case of singlet GPDs all these relations take the same form, but with the evolution kernels in the form of a matrix with indices corresponding to quark and gluon distributions. All kernels can be found in [27,35].
Chapter 3

Photoproduction of photon pairs

After introducing the collinear factorization, generalized parton distributions and methods used in the computation of higher-order QCD corrections, we present a detailed analysis of photoproduction of photon pair with large invariant mass on a nucleon, i.e. in the process

$$\gamma(q, \epsilon) N(p_1, \sigma_1) \rightarrow \gamma'(q_1, \epsilon_1(q_1)) \gamma''(q_2, \epsilon(q_2)) N(p_2, \sigma_2).$$

$$\epsilon$$ denotes polarization vectors of photons. It is the simplest reaction, which allows to study the factorization in a 2 $\rightarrow$ 3 process – and hence a good starting point in studies of this general class of processes. Because of the charge conjugation symmetry, it is sensitive only to non-singlet combinations of GPDs (in contrary to e.g. DVCS, TCS or DVMP), so it might be useful in extraction of GPDs from experimental data. In Section 3.1 we describe kinematics of this process. In Section 3.2 we show the leading-order results [20, 21]. The remaining part of the work are my original results. Throughout Sections 3.3 - 3.7 we present a detailed computation of one loop QCD corrections to the hard-part. The full result can be found in Appendix B. The proof of validity of the collinear factorization at NLO is shown in Section 3.8.

3.1 Kinematics, amplitude, and the differential cross section

We parametrize particles’ momenta and polarizations using the following light-cone four-vectors:

$$p^\mu = \frac{\sqrt{s}}{2} (1, 0, 0, 1), \quad n^\mu = \frac{\sqrt{s}}{2} (1, 0, 0, -1),$$

(3.1)

the parameter $s$ will be described in more detail later. We use the Sudakov decomposition:

$$p_1^\mu = (1 + \xi) p^\mu + \frac{m_N^2}{s(1 + \xi)} n^\mu, \quad p_2^\mu = (1 - \xi) p^\mu + \frac{m_N^2 + \Delta_\perp^2}{s(1 - \xi)} n^\mu + \Delta_\perp^\mu, \quad q^\mu = n^\mu,$$

(3.2)

$$q_1^\mu = \alpha n^\mu + \frac{(\vec{p}_t - \frac{1}{2} \vec{\Delta}_t)^2}{\alpha s} p^\mu + p_\perp^\mu - \frac{1}{2} \Delta_\perp^\mu,$$

$$q_2^\mu = \bar{\alpha} n^\mu + \frac{(\vec{p}_t + \frac{1}{2} \vec{\Delta}_t)^2}{\bar{\alpha} s} p^\mu - p_\perp^\mu - \frac{1}{2} \Delta_\perp^\mu.$$
\[ \Delta^\mu = p_2^\mu - p_1^\mu, \quad m_N \] is the mass of the nucleon, and the parameter \( \xi \) is called skewedness. We denote the Euclidean product as \( \vec{p}_2^2 = -p_1^2 \geq 0 \). We use the gauge \( A^- = 0 \) for the electromagnetic field. It implies that for each photon’s polarization vector

\[ \epsilon \cdot p = 0. \]

Hence, the polarization vectors are the following:

\[ e^\mu(q_1) = e^\mu_1(q_1) + \frac{2 \vec{\epsilon}_i(q_1) (\vec{p}_t - \frac{1}{2} \vec{\Delta}_t)}{\alpha s} p^\mu, \]
\[ e^\mu(q_2) = e^\mu_1(q_2) - \frac{2 \vec{\epsilon}_i(q_2) (\vec{p}_t + \frac{1}{2} \vec{\Delta}_t)}{\bar{\alpha} s} p^\mu, \]
\[ e^\mu(q) = e^\mu_1(q). \]  

The process is described in terms of 4 Mandelstam variables:

\[ S_{\gamma N} = (p_1 + q)^2, \quad t = (p_1 - p_2)^2, \quad M^2_{\gamma\gamma} = (q_1 + q_2)^2, \quad u' = (q_2 - q)^2. \]  

For further convenience we relate them to other invariants:

\[ S_{\gamma N} = (1 + \xi) s + m^2_N \quad \Rightarrow \quad s = \frac{S_{\gamma N} - m^2_N}{1 + \xi}, \]
\[ t' = (q_1 - q)^2 = t - M^2_{\gamma\gamma} - u'. \]  

Like in the work \[20\], we consider the situation when \( M^2_{\gamma\gamma} \) is large and \( t \) is small (with respect to the mass squared of the nucleon). It implies that transverse momenta of outgoing photons are large and approximately opposite. Let us discuss the case \( \Delta^\mu_\perp = 0 \). It is justified, since in the considered kinematics one has \( |p_2^2| \gg |\Delta^2_\perp| \). The simplified kinematical relations used in the calculation of the hard part are:

\[ \xi = \frac{\tau}{2 - \tau}, \quad \tau = \frac{M^2_{\gamma\gamma} - t}{S_{\gamma N} - m^2_N}, \]
\[ -t' = \frac{\vec{p}_t^2}{\alpha}, \quad -u' = \frac{\vec{p}_t^2}{\bar{\alpha}}, \quad \alpha + \bar{\alpha} = 1. \]  

The second line of Eq. (3.6) yields

\[ \bar{p}_t^2 = -\frac{u't'}{w' + t'}, \quad \alpha = \frac{u'}{w' + t'}, \quad \bar{\alpha} = \frac{t'}{w' + t'}. \]  

Because of the charge conjugation symmetry, the amplitude of the photoproduction of two real photons on a proton is sensitive only to non-singlet (odd with respect to the charge conjugation) parton distributions. To avoid proliferation of indices, from now on by

\[ H^q(x, \xi, t; \mu_F) \quad \text{and} \quad E^q(x, \xi, t; \mu_F) \]

we denote the \textit{renormalized, non-singlet} vector GPDs corresponding to quark of flavour \( q \), see Eq. (1.11). We neglect the axial GPDs, since the LO analysis \[20\] has shown, that their contribution
in this process is negligible. The scattering amplitude of the full process \( \mathcal{M} \) can be written in the following way:

\[
\mathcal{M} = \frac{1}{2s} \sum_{q} \left[ \mathcal{H}^{q} \bar{u}(\mathbf{p}_{2}, \sigma_{2}) \gamma^{+} u(\mathbf{p}_{1}, \sigma_{1}) + \mathcal{E}^{q} \bar{u}(\mathbf{p}_{2}, \sigma_{2}) \frac{i \sigma^{\mu} \Delta_{\mu}}{2m_{N}} u(\mathbf{p}_{1}, \sigma_{1}) \right],
\]

(3.8)

where \( u(\mathbf{p}, \sigma) \) denotes the bispinor of the nucleon, and

\[
\mathcal{H}^{q} = \int_{-1}^{1} dx H^{q}(x, \xi, t; \mu_{F}) \mathcal{T}_{R}^{q}(x, \ldots; \mu_{F}),
\]

\[
\mathcal{E}^{q} = \int_{-1}^{1} dx E^{q}(x, \xi, t; \mu_{F}) \mathcal{T}_{R}^{q}(x, \ldots; \mu_{F}),
\]

(3.9)

where \( \mathcal{T}_{R}^{q}(x, \ldots; \mu_{F}) \) denotes the “renormalized” amplitude of the hard sub-process:

\[
\mathcal{T}_{R}^{q}(x, \xi, \ldots; \mu_{F}) \equiv \mathcal{C}_{0}^{q}(x, \xi, \ldots) + \mathcal{C}_{1}^{q}(x, \xi, \ldots) + \log \left( \frac{s}{\mu_{F}^{2}} \right) \mathcal{C}_{\text{coll.}}^{q}(x, \xi, \ldots),
\]

(3.10)

as it is described by the Formula (2.25). Ellipsis denotes all parameters other than \( x \), on which depends the hard amplitude. Here, the hard scale is provided by \( s \) instead of \( |Q|^{2} \) used in Chapter 2.

The amplitude squared, summed (averaged) over the final (initial) nucleon spins \( \sigma_{1}, \sigma_{2} \) reads:

\[
\frac{1}{2} \sum_{\sigma_{1}, \sigma_{2}} |\mathcal{T}|^{2} = \frac{1}{4} \left[ (1 - \xi^{2}) \sum_{q} |\mathcal{H}^{q}|^{2} - 2 \xi^{2} \Re \sum_{q} \mathcal{H}^{q} \sum_{q'} \mathcal{E}^{q'} \ast + \frac{\xi^{4}}{1 - \xi^{2}} \sum_{q} |\mathcal{E}^{q}|^{2} \right].
\]

(3.11)

The differential cross section, averaged over the initial photon polarization and summed over the final polarizations, has the following form [20]:

\[
\frac{d\sigma}{dM_{\gamma\gamma}^{2} dt d(-u')} = \frac{1}{2 \cdot 32(2\pi)^{3} S_{\gamma N}^{2} M_{\gamma}^{2}} \sum_{\text{pol.}} \sum_{\sigma_{1}, \sigma_{2}} \frac{1}{4} |\mathcal{T}|^{2} =
\]

\[
= \frac{1}{(16\pi)^{3} S_{\gamma N}^{2} M_{\gamma}^{2}} \sum_{\text{pol.}} \left[ (1 - \xi^{2}) \sum_{q} |\mathcal{H}^{q}|^{2} - 2 \xi^{2} \Re \sum_{q} \mathcal{H}^{q} \sum_{q'} \mathcal{E}^{q'} \ast + \frac{\xi^{4}}{1 - \xi^{2}} \sum_{q} |\mathcal{E}^{q}|^{2} \right].
\]

(3.12)

In Section 3.2 we recall results of [20] and [21] concerning the LO amplitude, in particular the term \( \mathcal{C}_{0}^{q}(x, \xi, \ldots) \). In the subsequent part we present the computation of the NLO amplitude of the hard part, i.e. \( \mathcal{C}_{1}^{q}(x, \xi, \ldots) \) and \( \mathcal{C}_{\text{coll.}}^{q}(x, \xi, \ldots) \).

To reduce the number of considered NLO diagrams (originally 48), we consider a process with 3 incoming photons with momenta \( (k_{1}, k_{2}, k_{3}) \) and polarisation vectors \( (\epsilon_{1}, \epsilon_{2}, \epsilon_{3}) \), and then we sum the amplitudes over all combinations

\[
((k_{1}, \epsilon_{1}), (k_{2}, \epsilon_{2}), (k_{3}, \epsilon_{3}))
\]

corresponding to all permutations of

\[
((-q_{1}, \epsilon^{*}(q_{1})), (-q_{2}, \epsilon^{*}(q_{2})), (q, \epsilon(q))).
\]

31
We introduce the following parametrization of these invariants:

\[ 2pk_i := \beta_is, \quad 2k_1k_2 := \kappa_3s, \quad 2k_2k_3 := \kappa_1s, \quad 2k_1k_3 := \kappa_2s. \]  

(3.13)

It is easy to check that

\[ \kappa_i = -2\xi\beta_i. \]

Scalar products of the corresponding momenta are:

\[ 2pq_1 = \alpha s, \quad 2pq_2 = \bar{\alpha} s, \quad 2pq = s; \]
\[ 2q_1q_2 = M_{\gamma\gamma}^2, \quad -2qq_2 = u', \quad -2qq_1 = t'. \]  

(3.14)

Hence, parameters \( \beta_i \) and \( \kappa_i \) are the following permutations:

\[ \{\beta_i\} = \text{permutations of } \{-\alpha, -\bar{\alpha}, 1\}; \]
\[ \{\kappa_i\} = \text{permutations of } \left\{ \frac{M_{\gamma\gamma}^2}{s}, \frac{u'}{s}, \frac{t'}{s} \right\}. \]  

(3.15)

We often use the following abbreviations: \( \bar{p} := (x + \xi)p, \quad \underline{p} := (x - \xi)p. \)
3.2 The LO amplitude

At the leading order there are 6 tree diagrams, each corresponding to a different permutation of photons attached to the fermion line, as it is shown in Fig. 3.1.

![Diagram](image)

Figure 3.1: The general form of the LO diagram. As it was described in Sec. 3.1, to simplify the computation we first consider a diagram with 3 incoming photons, and at the end of the computations we use appropriate substitution and sum over permutations.

The hard-part amplitude corresponding to a given permutation reads

$$Tr[i \mathcal{M}_{1,2,3}^0 \bar{\psi}] = ie_q^3 \frac{1}{2(x + \xi)p + i0} - 2(x - \xi)pk_3 + i0 \frac{1}{\tilde{\beta}_1 \beta_3 x + \xi + i0_1 x - \xi - i0_3} Tr\left(\epsilon_3(\bar{\psi} - \tilde{k}_3)\epsilon_2(\bar{\psi} + k_1)\epsilon_1\bar{\psi}\right).$$  \hspace{1cm} (3.16)

Let us denote $i0_i = i0 \times sgn(\beta_i)$. Using definitions from Sec. 3.1 and $\bar{\psi}\psi = 0$, we rewrite (3.16) in a compact form:

$$Tr[i \mathcal{M}_{1,2,3}^0 \bar{\psi}] = ie_q^3 s^{-2} \frac{1}{\tilde{\beta}_1 \beta_3} x + \xi + i0_1 x - \xi - i0_3 Tr\left(\epsilon_3 k_3\epsilon_2 k_1\epsilon_1\bar{\psi}\right).$$  \hspace{1cm} (3.17)

The "−" sign standing by $\tilde{k}_3$ in the trace got absorbed by the "−" in the propagator $\frac{1}{-(x - \xi) + i0_3}$. It turns out that the trace structure present here often appears in loop calculations, and because of that we write

$$4A_{1,2,3} := -Tr\left(\epsilon_3 k_3\epsilon_2 k_1\epsilon_1\bar{\psi}\right).$$  \hspace{1cm} (3.18)

Let us observe that the interchange of indices $1 \leftrightarrow 3$ in Eq. 3.17 does not change that expression if $sgn(\beta_1) = sgn(\beta_3)$, and results in complex conjugation of the term

$$\frac{1}{x + \xi + i0_1} x - \xi - i0_3$$

if $sgn(\beta_1) = -sgn(\beta_3)$. The full amplitude of the hard sub-process is obtained by summing over all permutations of photons $k_i$, so that the only terms proportional to the imaginary part of propagators which survive, are those from diagrams in which the incoming photon is the middle one (so that $k_2 = q$). It turns out, that terms proportional to the real part of propagators cancel as well, since in the considered kinematics

$$\sum_{\text{perm.}} \frac{1}{\tilde{\beta}_1 \beta_3} Tr\left(\epsilon_3 k_3\epsilon_2 k_1\epsilon_1\bar{\psi}\right) = \frac{1}{\tilde{\beta}_1 \beta_2 \beta_3} \sum_{\text{perm.}} \beta_2 Tr\left(\epsilon_3 k_3\epsilon_2 k_1\epsilon_1\bar{\psi}\right) = 0.$$  \hspace{1cm} (3.19)
That can be checked by a straightforward computation, for reference see [21] (note that the original formula in [20] contained a mistake). Hence, after summing over all the possible photons’ permutations, one arrives at the following hard part:

\[
C_q^0(x, \xi, \ldots) = \sum_{\text{perm.}} \text{Tr} [\mathcal{M}_{1,2,3}^0 \rho] = -\frac{e_q^3}{s^2 \alpha \bar{\alpha}} \left( \frac{1}{x + \xi - i0} \frac{1}{x - \xi + i0} - \text{c.c.} \right) \text{Tr} \left( \epsilon^*_2(q_2) \epsilon^*_1(q) \epsilon^*_1(q_1) \rho \right)
\]

\[
= -\frac{2\pi e_q^3 m}{s \alpha \bar{\alpha}} \frac{1}{\xi} \left( \delta(x - \xi) + \delta(x + \xi) \right) \times
\]

\[
\left[ (\alpha - \bar{\alpha}) (\epsilon^*_t(q_1) \epsilon^*_t(q_2)) (\bar{\epsilon}_t \epsilon^*_t(q)) - (\bar{\epsilon}_t \epsilon^*_t(q_1)) (\bar{\epsilon}_t \epsilon^*_t(q_2)) + (\bar{\epsilon}_t \epsilon^*_t(q_2)) (\bar{\epsilon}_t \epsilon^*_t(q_1)) \right].
\]

(3.20)
3.3 Self-energy corrections

Before presenting the computation of NLO diagrams, let us recapitulate our conventions. We use the dimensional regularization with the number of dimensions \( d = 4 - \varepsilon \). It results in the modification of the strong coupling \( g \rightarrow g\mu^{2-d/2} \), which ensures the correct dimensionality of the amplitude. For gamma matrices in \( d \) dimensions one has

\[
\gamma^\mu \gamma^\nu \gamma_\mu = (2 - d) \gamma^\nu = (-2 + \varepsilon) \gamma^\nu, \\
\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\delta \gamma_\mu = -2\delta^\gamma \gamma^\alpha + \varepsilon \gamma^\alpha \gamma^\beta \gamma^\delta.
\]  

(3.21)

We use the Feynman formula for combining denominators

\[
\frac{1}{a_1 \ldots a_n} = \Gamma(n) \int_0^1 du_1 \cdots \int_0^1 du_n \delta(1 - u_1 - \cdots - u_n) \frac{1}{(u_1 a_1 + \cdots + u_n a_n)^n}.
\]  

(3.22)

Finally, we perform momentum integration according to the following formulas:

\[
\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i0)^n} = \frac{i(-1)^n (\Delta - i0)^{d/2-n} \Gamma(n - d/2)}{\Gamma(n)},
\]

\[
\int \frac{d^d k}{(2\pi)^d} k^{\mu} k^{\nu} (k^2 + \Delta + i0)^n = -\frac{i}{2} \delta^{\mu\nu} (-1)^n \frac{\Gamma(n - 1 - d/2)}{\Gamma(n)} (-\Delta - i0)^{d/2+1-n}.
\]  

(3.23)

Let us emphasise that, since the hard amplitude is integrated over the parameter \( x \) and may contain singularities, it is important to keep the \( i0 \) terms through all the calculation.

We use the following naming convention: a diagram with a loop at left/right/middle involving \( n \) propagators will be denoted by superscript \( n.L/n.R/n.M \). Let us recall one more time, that we often use the short-hand notation \( \bar{p} := (x + \xi)p, \bar{q} := (x - \xi)p \).

After this discussion, we can proceed with the computation of loop corrections. The simplest ones, self-energy diagrams, are shown in Fig. 3.2.

![Diagram 3.2: Diagrams 2.L and 2.R.](image)

Amplitude corresponding to 2.L reads

\[
\text{Tr} \left[ iM_{1,2,3}^{2.L} \bar{p} \right] = -C_F (ie_q)^3 (ig)^2 \left( \frac{i}{(x + \xi)p + k_1 + i0} \right)^2 \left( \frac{i}{(x - \xi)p - k_3 + i0} \right) \times
\]

\[
\text{Tr} \left\{ \gamma \bar{\gamma}_3 [(x - \xi)p - k_3] \gamma_3 [(x + \xi)p + k_1] \right\} \times
\]

\[
\mu^{4-d} \int \frac{i^2 d^d k}{(2\pi)^d} \frac{\gamma^\mu [(x + \xi)p + k_1 + k] \gamma_\mu}{(x + \xi)p + k_1 + k + i0) \cdot (x - \xi)p + k_3 + i0)} \cdot \left[ (x + \xi)p + k_1 \right] \gamma_1 \bar{p} \right\}.
\]  

(3.24)
3.3.1 Integral

Using the discussed integration formula [(3.23)], we first employ the Feynman parametrization and shift momentum to complete the square in denominator:

\[ I = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu (\not{P} + \not{k}) \gamma_\mu}{(k^2 + i\epsilon)(P^2 + k^2 + i\epsilon)} = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{(2 - d)(1 - x) \not{P}}{(k^2 + x(1 - x)P^2 + i\epsilon)^2}, \tag{3.25} \]

and then integrate over the loop momentum \( k \):

\[ I = \mu^{4-d} (2 - d) \int_0^1 dx \frac{i \not{P}(1 - x)}{(4\pi)^{d/2}} \frac{-x(1 - x)P^2 - i\epsilon}{x^{\frac{d}{2} - 1} \Gamma(2 - d/2)}. \tag{3.26} \]

In the next step we use the fact that \( x(1 - x) \geq 0 \), so that it can be taken outside the parenthesis without changing the sign of \( i\epsilon \)

\[ I = (\frac{2}{\epsilon} + \varepsilon) \int_0^1 dx \frac{i \not{P}(1 - x)}{(4\pi)^{d/2}} \frac{-x(1 - x)P^2 - i\epsilon}{x^{-\frac{d}{2} - 1/2} \Gamma(\varepsilon/2)} \tag{3.27} \]

and finally we perform the integral over \( dx \) obtaining the result expressed in terms of Euler gamma functions:

\[ I = (\frac{2}{\epsilon} + \varepsilon) \int_0^1 dx \frac{i \not{P}(1 - x)}{(4\pi)^{d/2}} \frac{-x(1 - x)P^2 - i\epsilon}{x^{-\frac{d}{2} - 1/2} \Gamma(\varepsilon/2)} \frac{\Gamma(1 - \varepsilon/2)\Gamma(2 - \varepsilon/2)}{\Gamma(3 - \varepsilon)}. \tag{3.28} \]

Taking the limit \( \varepsilon \to 0 \) we get

\[ (\frac{2}{\epsilon} + \varepsilon) \frac{\Gamma(1 - \varepsilon/2)\Gamma(2 - \varepsilon/2)}{\Gamma(3 - \varepsilon)} = -\frac{2}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon). \tag{3.29} \]

Finally, we have

\[ I = \frac{i \not{P}}{(4\pi)^{d/2}} \left( -\frac{2}{\varepsilon} + \log(-P^2/4\pi \mu^2 - i\epsilon) - 1 + \gamma \right). \tag{3.30} \]

3.3.2 Traces

In the case of the self-energy contributions we have the following traces:

\[ Tr \left\{ \gamma_3 \left[(x - \xi) \not{\psi} - \not{k}_3 \right] \gamma_2 \left[(x + \xi) \not{\psi} + \not{k}_1 \right] \gamma_1 \not{\psi} \right\}, \]

\[ Tr \left\{ \gamma_3 \left[(x - \xi) \not{\psi} - \not{k}_3 \right] \gamma_2 \left[(x + \xi) \not{\psi} + \not{k}_1 \right] \gamma_1 \not{\psi} \right\}. \tag{3.31} \]

They are simplified using \( \not{P} \not{P} = P^2 \), and the problem reduces to computation of

\[ Tr \left\{ \gamma_3 \left[(x - \xi) \not{\psi} - \not{k}_3 \right] \gamma_2 \left[(x + \xi) \not{\psi} + \not{k}_1 \right] \gamma_1 \not{\psi} \right\}. \tag{3.32} \]

Using \( \not{\psi} \not{\psi} = 0 \) we obtain

\[ -Tr \left\{ \gamma_3 \gamma_2 \gamma_1 \gamma_1 \not{\psi} \right\} = 4A_{1,2,3}. \tag{3.33} \]
Putting together the results of the integration and the trace yields

\[
Tr \left[ iM_{1,2,3}^{2L} \right] = -C_F e_q^3 g^2 \frac{i}{(4\pi)^2} \frac{1}{(x + \xi) p k_1 + i0} \frac{1}{(x - \xi) p k_3 - i0} A_{1,2,3} \times \\
\left( -\frac{2}{\varepsilon} + \log(-(x + \xi)(p k_1)/2\pi \mu^2 - i0) - 1 + \gamma \right).
\]

It can be further simplified using \( p k_1/2\pi = s\beta_1/4\pi \) and

\[
-\frac{2}{\varepsilon} + \log(-(x + \xi)s\beta_1/4\pi \mu^2 - i0) + \gamma = \left( \frac{se^\gamma}{4\pi \mu^2} \right)^{-\frac{2}{\varepsilon}} \left( -\frac{2}{\varepsilon} + \log(-(x + \xi)\beta_1 - i0) \right) + \mathcal{O}(\varepsilon), \tag{3.34}
\]

so that

\[
Tr \left[ iM_{1,2,3}^{2L} \right] = -C_F e_q^3 g^2 \frac{i}{(4\pi)^2} \frac{1}{x + \xi + i0} \frac{1}{x - \xi - i0} s^2 \beta_1 \beta_3 \times \\
\left( \frac{se^\gamma}{4\pi \mu^2} \right)^{-\frac{2}{\varepsilon}} \left( -\frac{2}{\varepsilon} + \log(-(x + \xi)\beta_1 - i0) - 1 \right) \tag{3.35}
\]

Using this results one can write the second amplitude in a straightforward way:

\[
Tr \left[ iM_{1,2,3}^{2R} \right] = -C_F (ie_q)^3 (ig)^2 \frac{i}{(x + \xi)p + k_1} \frac{1}{((x - \xi)p - k_3 + i0)^2} \times \\
Tr \{ f_3 [(x - \xi)p - k_3] \mu^{4-d} \int d^d k \frac{\gamma^\mu [(x - \xi)p - k_3 + k]}{(2\pi)^d (k^2 + i0)((x - \xi)p - k_3 + k)^2 + i0} \times \\
\left[ (x - \xi)p - k_3 \right] f_2 [(x + \xi)p + k_1] f_1 p \} = \\
-C_F e_q^3 g^2 \frac{i}{(4\pi)^2} \frac{1}{x + \xi + i0} \frac{1}{x - \xi - i0} s^2 \beta_1 \beta_3 \times \\
A_{1,2,3} \left( -\frac{2}{\varepsilon} + \log((x - \xi)(p k_3)/2\pi \mu^2 - i0) - 1 + \gamma \right) \\
\times \\
\left( \frac{se^\gamma}{4\pi \mu^2} \right)^{-\frac{2}{\varepsilon}} \left( -\frac{2}{\varepsilon} + \log((x - \xi)\beta_3 - i0) - 4 \right). \tag{3.36}
\]
3.4 Vertex corrections - 3.L and 3.R

The vertex corrections with the gluon line attached to the on-shell quark develop both the UV and collinear divergence, as it was explained in Subsec. 2.2.2. The considered diagrams are shown in Fig. 3.3.

![Figure 3.3: Diagrams 3.L and 3.R.](image)

The full form of the amplitude corresponding to 3.L is

\[
Tr [i M_{123}^{3.L} \not{p}] = -C_F (i e_q)^2 \frac{i}{(x + \xi \not{p} + k_1 + i0)^2 + i0 (x - \xi \not{p} - k_2 + i0)^2 + i0} \frac{i}{(x - \xi \not{p} - k_3 + i0)} \frac{\gamma_{\mu} (x + \xi \not{p} + k_1 + k_2 + k_3) \gamma_{\nu}}{(x - \xi \not{p} - k_3 + i0)} \frac{\gamma_{\nu}}{(x - \xi \not{p} - k_3 + i0)} \left\{ \epsilon_3 [x - \xi \not{p} - k_3] \epsilon_2 \right\}.
\]

(3.37)

First, let us first simplify the traces present in Eq. (3.37). Using the formula

\[
\gamma_{\mu} A B C \gamma_{\nu} = -2 A B C + \varepsilon A B C,
\]

for arbitrary numbers \(a, b, c, d\) we obtain

\[
Tr \left\{ \epsilon_3 [(x - \xi \not{p} - k_3)] \epsilon_2 [(x + \xi \not{p} + k_1)] [a \not{p} + b k_1] \epsilon_1 [c \not{p} + d k_1] \not{p} \right\} = -2ad(pk_1) Tr \left\{ \epsilon_3 \epsilon_2 \epsilon_1 \not{p} \right\} = -8ad(pk_1) A_{1,2,3}.
\]

(3.38)

We proceed with the computation by applying the Feynman parametrization to the integral:

\[
\mu^{1-d} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_{\mu} [(x + \xi \not{p} + k_1 + k_2 + k_3) \gamma_{\nu}}{(x - \xi \not{p} - k_3 + i0)} \frac{\gamma_{\nu}}{(x - \xi \not{p} - k_3 + i0)} \left\{ \epsilon_3 \epsilon_2 \epsilon_1 \not{p} \right\} = 2 \mu^{1-d} \int_0^1 dy \int_0^{1-y} dy' \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_{\mu} [(x + \xi \not{p} + k_1 + k_2 + k_3) \gamma_{\nu}}{(x - \xi \not{p} - k_3 + i0)} \frac{\gamma_{\nu}}{(x - \xi \not{p} - k_3 + i0)} \left\{ \epsilon_3 \epsilon_2 \epsilon_1 \not{p} \right\}.
\]

(3.39)

Substituting \(k' = k + (y + z)(x + \xi \not{p} + y k_1)\) and using the trace formula (3.38) we reduce the problem of computation of the integral inside the trace to the following one:

\[
I = 2 \mu^{1-d} \int_0^1 dy \int_0^{1-y} dy' \int \frac{d^d k}{(2\pi)^d} \frac{2(1 - y - z)(x + \xi)(1 + (\varepsilon/2 - 1)y) \not{p} \not{k}_1 \not{k}_1 + (-2 + \varepsilon) k' \varepsilon \gamma_{\mu} \gamma_{\nu}}{(k^2 + 2(x + \xi)(p k_1)(1 - y - z)y + i0)^3}.
\]

(3.40)
It breaks into 2 term which are proportional to different gamma structures:

\[ I \equiv (x + \xi) \phi \cdot k_1 I_1 + I_2 \phi. \tag{3.41} \]

After performing the momentum integration, \( I_1 \) is found to be

\[ I_1 = \frac{2i\mu^{4-d}}{(4\pi)^{d/2}} \Gamma(3 - \frac{d}{2}) \int_0^1 dy \int_0^{1-y} dz \frac{(1-y-z)(1+\varepsilon/2-1)y}{(-2(x+\xi)(pk_1)(1-y-z)y-i0)^{3-d/2}}. \tag{3.42} \]

The integration over Feynman parameters \( y \) and \( z \) can be simplified by the substitution:

\[ y = \bar{y}(1-\bar{y}), \quad z = \bar{z} \bar{y}, \quad \int_0^1 dy \int_0^{1-y} dz \to \int_0^1 d\bar{y} \int_0^1 \bar{z} d\bar{y}. \tag{3.43} \]

\( I_1 \) has now the following form (for simplicity we omit the lines over the reparametrized variables \( \bar{y}, \bar{z} \)):

\[ I_1 = \Gamma(3 - \frac{d}{2}) \frac{2i\mu^{4-d}}{(4\pi)^{d/2}} \int_0^1 dy \int_0^1 z dz \frac{(1-z)(1+\varepsilon/2-1)z(1-y)}{(-2(x+\xi)(pk_1)(1-z)(1-y)z-i0)^{3-d/2}}. \]

The integration over \( dx \) and \( dz \) can be now factorized into a product of 2 integrals:

\[ I_1 = \frac{2i\mu^{4-d}}{(4\pi)^{d/2}} ( -2(x+\xi)(pk_1) - i0 )^{d/2-3} \Gamma(3 - \frac{d}{2}) \left( \int_0^1 (1-y)^{d/2-3} dy \int_0^1 z^{d/2-2}(1-z)^{d/2-2} + (\varepsilon/2 - 1) \int_0^1 (1-y)^{d/2-2} \int_0^1 z^{d/2-1}(1-z)^{d/2-2} dz \right). \]

These integral are easily performed and one obtains

\[ I_1 = \frac{2i\mu^{4-d}}{(4\pi)^{d/2}} ( -2(x+\xi)(pk_1) - i0 )^{d/2-3} \times \left( \frac{\Gamma(3 - \frac{d}{2}) \Gamma(d/2-1)\Gamma(d/2-1)}{d/2 - 2} \frac{\Gamma(3 - \frac{d}{2}) \Gamma(d/2)\Gamma(d/2-1)}{\Gamma(d-2)} + (\varepsilon/2 - 1) \frac{\Gamma(3 - \frac{d}{2}) \Gamma(d/2)\Gamma(d/2-1)}{d/2 - 1} \frac{\Gamma(d)\Gamma(d/2-1)}{\Gamma(d-1)} \right). \tag{3.44} \]

For \( d = 4 - \varepsilon \), in the limit \( \varepsilon \to 0 \) one has

\[ \frac{\Gamma(3 - \frac{d}{2}) \Gamma(d/2-1)\Gamma(d/2-1)}{d/2 - 2} \frac{\Gamma(3 - \frac{d}{2}) \Gamma(d/2)\Gamma(d/2-1)}{\Gamma(d-2)} \approx -\frac{2}{\varepsilon} - 2 + \gamma, \]

\[ (\varepsilon/2 - 1) \frac{\Gamma(3 - \frac{d}{2}) \Gamma(d/2)\Gamma(d/2-1)}{d/2 - 1} \frac{\Gamma(d)\Gamma(d/2-1)}{\Gamma(d-1)} \approx -\frac{1}{2}. \tag{3.45} \]

In this way, we obtain the final form of \( I_1 \):

\[ I_1 = \frac{2i}{(4\pi)^2} ( -2(x+\xi)(pk_1) - i0 )^{-1} \left( 1 - \frac{\varepsilon}{2} \log \left( - (x+\xi)(pk_1)/2\pi\mu^2 - i0 \right) \right) \left( -\frac{2}{\varepsilon} + \gamma - \frac{5}{2} \right) \]

\[ = \frac{2i}{(4\pi)^2} ( -2(x+\xi)(pk_1) - i0 )^{-1} \left( -\frac{2}{\varepsilon} + \log \left( - (x+\xi)(pk_1)/2\pi\mu^2 - i0 \right) + \gamma - \frac{5}{2} \right). \tag{3.46} \]
Let us note, that the loop momentum integral inside of $I_1$ was UV safe, from the argument of power counting, and hence the divergence present therein is the result of the region of integration, where $k$ and $(x + \xi)p + k$ become collinear and lightlike (i.e. the collinear divergence). In such case, the divergence manifested in the integration over the Feynman parameters – one can easily check, that if one assumed a non-zero mass of partons, the relevant integral would be finite.

The second integral, after using $\gamma^\mu \phi_1 \gamma_\mu = (-2 + \varepsilon)\phi_1$ and integrating over momenta becomes

$$I_2 \phi_1 = \frac{i\mu^{4-d}}{2(4\pi)^{d/2}} \Gamma(2 - d/2)(2 - \varepsilon)^2 \phi_1 \int_0^1 dy \int_0^{1-y} dz \frac{1}{(2(x + \xi)(pk_1)(1 - y - z)y - i0)^{2-d/2}}.$$ 

We see that this integral is not UV-safe and there is a divergent term $\Gamma(2 - d/2)$ resulting from the integration over the loop momentum. Now we use the same substitution as in Eq. (3.43):

$$I_2 \phi_1 = \frac{i\mu^{4-d}}{2(4\pi)^{d/2}} \Gamma(2 - d/2)(2 - \varepsilon)^2 \phi_1 \int_0^1 dy \int_0^{1-y} dz (-2(x + \xi)(pk_1)(1 - z)(1 - y)z - i0)^{d/2-2},$$

because $(1 - z)(1 - y)z \geq 0$, we can bring that term outside the parenthesis without changing the sign of $i0$, and as a result we get

$$I_2 \phi_1 = \frac{i\mu^{4-d}}{2(4\pi)^{d/2}} \Gamma(2 - d/2)(2 - \varepsilon)^2 \phi_1 (-2(x + \xi)(pk_1) - i0)^{d/2-2} \frac{\Gamma(d/2)\Gamma(d/2 - 1)}{\Gamma(d - 1)} \frac{1}{d/2 - 1}.$$ 

We expand with respect to $\varepsilon$:

$$\Gamma(2 - d/2)(2 - \varepsilon)^2 \frac{\Gamma(d/2)\Gamma(d/2 - 1)}{\Gamma(d - 1)} \frac{1}{d/2 - 1} \approx 2\left(\frac{2}{\varepsilon} + 1 - \gamma\right).$$

In this case, the integrals over Feynman parameters are finite, since the $k^2$ from the denominator cancels with the $k^2$ from the trace, and the integrand does not have a collinear singularity. The final expression for the second integral is then

$$I_2 \phi_1 = \frac{i\mu^{4-d}}{2(4\pi)^{d/2}} \phi_1 (-2(x + \xi)(pk_1) - i0)^{d/2-2} \left(\frac{2}{\varepsilon} + 1 - \gamma\right) - \log\left(-\frac{x + \xi}{2\pi}\right).$$

Let us recall that

$$I = (x + \xi)\phi_1 k_1 I_1 + I_2 \phi_2.$$

Using the previous results concerning traces we obtain

$$Tr\left[iM_{1,2,3}^{3,L} \not{\!p}\right] = -CFe g^2 \int \frac{1}{p k_1 + i0} \frac{1}{p k_3 - i0} A_{1,2,3}\left(-\frac{2}{\varepsilon} + \log\left(-\frac{\beta_1 - i0}{2\pi}\right) + \gamma - 4\right)$$

\[= Tr\left[iM_{1,2,3}^{0} \not{\!p}\right] \cdot \frac{g^2 CF}{(4\pi)^2} \left(-\frac{2}{\varepsilon} + \log\left(-\frac{x + \xi}{2\pi}\right) + \gamma - 4\right) \]  

(3.48)
The computation of the diagram $3.R$ is done in the same way, and it yields

$$Tr\left[i\mathcal{M}^{3,R}_{1,2,3}\right] = -C_F e_q^3 g^2 \frac{i}{(4\pi)^2} \left( \frac{1}{p k_1 + i 0} \right) \left( \frac{1}{p k_3 - i 0} \right) A_{1,2,3} \left( -\frac{2}{\varepsilon} + \log \left( (x - \xi)(p k_3)/2\pi\mu^2 - i 0 \right) + \gamma - 4 \right)$$

$$= Tr\left[i\mathcal{M}^0_{1,2,3}\right] \cdot \frac{g^2 C_F}{(4\pi)^2} \left( \frac{\text{se}^7}{4\pi\mu^2} \right)^{-\frac{1}{2}} \left( -\frac{2}{\varepsilon} + \log \left( (x - \xi)\beta_3 - i 0 \right) - 4 \right).$$

(3.49)

### 3.5 Vertex corrections - 3.M

The considered diagram is shown in fig. 3.4

![Diagram 3.M](image)

Figure 3.4: Diagram 3.M.

The fact that the gluon line in the diagram 3.M is connected to quarks that are off-shell implies that in this term there is no collinear divergence here. The integration over the Feynman parameters becomes more complicated, but it turns out that an appropriate substitution allows to simplify it significantly.

The considered amplitude reads

$$Tr\left[i\mathcal{M}^0_{3,3,3}\right] = -C_F (ie_q)^3 (ig)^2 \frac{i}{((x + \xi)p + k_1)^2 + i 0} \frac{i}{((x - \xi)p - k_3)^2 + i 0} Tr\left\{ f_3 [(x - \xi)\not{p} - k_3] \times \right.$$ 

$$\left( \frac{\gamma_{\mu}}{\sqrt{\varepsilon}} \right) [(x + \xi)\not{p} + k_1 + k_2 + k]\not{f}_2 [(x + \xi)\not{p} + k_1 + k]\not{f}_1 [(x + \xi)\not{p} + k_1 + k_2 + k]\not{f}_3 [(x + \xi)\not{p} - k_3].$$

(3.50)

First, we simplify the trace:

$$Tr\left\{ f_3 (-k_3) [a(x + \xi)\not{p} + a k_1 + b k_2] \not{f}_2 [a(x + \xi)\not{p} + a k_1 + c k_2] k_1 \not{f}_1 \right\} =$$

$$= 2 p k_1 Tr\left\{ f_3 k_3 [a(x + \xi)\not{p} + a k_1 + b k_2] \not{f}_2 [a(x + \xi)\not{p} + c k_2] \not{f}_1 \right\} +$$

$$- Tr\left\{ f_3 k_3 [a(x + \xi)\not{p} + a k_1 + b k_2] \not{f}_2 c k_2 \not{p} k_1 \not{f}_1 \right\}.$$
Using identities $(x + \xi)p + k_1 = (x - \xi)p - k_2 - k_3$ and $k_i k_i = 0$ we obtain

\[ (3.51) = 2pk_1 Tr \{ \epsilon_3 k_3 [a(x - \xi)p + (b - a) k_2] \epsilon_2 [a(x + \xi)p + c k_2] \epsilon_1 \} - Tr \{ \epsilon_3 k_3 a(x - \xi)p \epsilon_2 c k_2 \epsilon_1 \} \]

\[ = 2a(x - \xi)c(pk_1) Tr \{ \epsilon_3 k_3 \epsilon_2 k_2 \epsilon_1 \} + 2a(x + \xi)(b - a)(pk_1) Tr \{ \epsilon_3 k_3 \epsilon_2 k_1 \epsilon_1 \} + \]

\[ - 2ac(x - \xi)(pk_2) Tr \{ \epsilon_3 k_3 \epsilon_2 k_1 \epsilon_1 \} \equiv a(\mathcal{B}(b - a) + Cc). \]  

(3.52)

$\mathcal{B}$ and $C$ are auxiliary parameters, which do not appear in the final expression for the amplitude. Let us further simplify $C$. First, apply $pk_2 = -(p(k_1 + k_3)$ to the second trace inside the parenthesis:

\[ C = 2(x - \xi) \left( (pk_1) Tr \{ \epsilon_3 k_3 \epsilon_2 k_2 \epsilon_1 \} - (pk_2) Tr \{ \epsilon_3 k_3 \epsilon_2 k_1 \epsilon_1 \} \right) = \]

\[ = 2(x - \xi) \left( (pk_1) Tr \{ \epsilon_3 k_3 \epsilon_2 (k_1 + k_2) \epsilon_1 \} + (pk_3) Tr \{ \epsilon_3 k_3 \epsilon_2 k_1 \epsilon_1 \} \right). \]

Then, from the relation $\epsilon_2 (k_1 + k_2) = -\epsilon_2 k_3$ follows that

\[ C = 2(x - \xi) \left( -8(pk_1)(pk_3)(k_3 \epsilon_1)(\epsilon_2 \epsilon_3) + (pk_3) Tr \{ \epsilon_3 k_3 \epsilon_2 k_1 \epsilon_1 \} \right) = \]

\[ = 2(x - \xi)pk_3 Tr \{ \epsilon_3 \epsilon_2 k_1 \epsilon_1 \}. \]  

(3.53)

To perform the integration, let us denote $P = (x + \xi)p + k_1, Q = P + k_2$. Obviously $(P - Q)^2 = 0$, so that $P^2 + Q^2 = 2PQ$. We take the integral (3.50), employ the Feynman parametrization and shift the loop momentum to complete the square. The part of the resulting integral without the $k^2$ term in numerator, after computing the trace according to the previous results, reads

\[ I_1 = -2i \mu^{d - 4} \int \frac{dk}{(2\pi)^d} \int_0^1 dy \int_0^{1-y} dz \frac{-2(1 - y - z)((C - \mathcal{B}) + (1 - \varepsilon/2)By - (1 - \varepsilon/2)Cz)}{(k^2 + (1 - y - z)(yP^2 + zQ^2) + i0)^3}. \]  

(3.54)

Performing the momenta integration and making the substitution (3.43) one obtains

\[ I_1 = \frac{2i \mu^{d - 4}}{(4\pi)^{d/2}} \int_0^1 dy \int_0^1 dz \left( - (1 - z)z((1 - y)P^2 + yQ^2) - i0 \right)^{d/2 - 3} \Gamma(3 - d/2) \times \]

\[ (1 - z)\left( (C - \mathcal{B}) + z(1 - \frac{\varepsilon}{2})\mathcal{B}(1 - y) - y\mathcal{C} \right). \]  

(3.55)
The integrals over $dz$ are finite:

$$I_1 = \frac{2\mu^{4-d}}{(4\pi)^{d/2}} \Gamma(3-d/2) \left( \frac{\Gamma(d/2-1)\Gamma(d/2-1)}{\Gamma(d-2)} \right) \int_0^1 dy \left( y(P^2 - Q^2) - (P^2 + i0) \right)^{d/2-3} (C - B) +$$

$$+ \frac{\Gamma(d/2)\Gamma(d/2-1)}{\Gamma(d-1)} (1 - \frac{\varepsilon}{2}) \left[ B \int_0^1 dy \left( y(P^2 - Q^2) - (P^2 + i0) \right)^{d/2-3} +$$

$$- (C + B) \int_0^1 dy \left( y(P^2 - Q^2) - (P^2 + i0) \right)^{d/2-3} \right].$$  \hspace{1cm} (3.56)

Now let us calculate integrals over $dy$. The first one (second line of Eq. (3.56)) is

$$\frac{\mu^{4-d}}{(4\pi)^{d/2}} \int_0^1 dy \left( y(P^2 - Q^2) - (P^2 + i0) \right)^{d/2-3} =$$

$$= \frac{\mu^{4-d}}{(4\pi)^{d/2}} \frac{1}{d/2 - 2} \frac{1}{P^2 - Q^2} \left( (-Q^2 - i0)^{d/2-2} - (P^2 - i0)^{d/2-2} \right).$$

In the limit $d \to 4$ this expression is finite and equal to

$$\frac{1}{(4\pi)^2} \frac{1}{P^2 - Q^2} \left( \log(-Q^2/4\pi\mu^2 - i0) - \log(-P^2/4\pi\mu^2 - i0) \right)^{d/2-2}.$$  \hspace{1cm} (3.57)

The second integral (in the last line of Eq. (3.56)) yields

$$\frac{\mu^{4-d}}{(4\pi)^{d/2}} \int_0^1 y dy \left( y(P^2 - Q^2) - (P^2 + i0) \right)^{d/2-3} =$$

$$= \frac{\mu^{4-d}}{(4\pi)^{d/2}} \frac{1}{d/2 - 2} \frac{1}{P^2 - Q^2} \left( (-Q^2 - i0)^{d/2-2} - \frac{1}{d/2 - 1} \frac{1}{P^2 - Q^2} \left( (-Q^2 - i0)^{d/2-1} - (-P^2 - i0)^{d/2-1} \right) \right)$$

$$= \frac{1}{(4\pi)^2} \frac{1}{(P^2 - Q^2)^2} \left( P^2 \log \frac{Q^2 + i0}{P^2 + i0} + P^2 - Q^2 \right).$$  \hspace{1cm} (3.58)

Notice that both values are finite. Finally, we get:

$$I_1 = \frac{4}{(4\pi)^2} \left( 2(C - \frac{1}{2}\varepsilon) \frac{\log(-Q^2/4\pi\mu^2 - i0) - \log(-P^2/4\pi\mu^2 - i0)}{P^2 - Q^2} +$$

$$- (B + C) \frac{1}{(P^2 - Q^2)^2} \left( P^2 \log \frac{Q^2 + i0}{P^2 + i0} + P^2 - Q^2 \right) \right).$$  \hspace{1cm} (3.59)

Now let us compute the second part of the loop momentum integral with the term $k^2$ in the
After taking the traces one obtains

\[ I_2 = 2\mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \int_0^1 dy \int_0^{1-y} dz \frac{\gamma_{\mu \nu \rho \sigma}}{(k^2 + (1 - y - z)(y P^2 + z Q^2) + i0)^3} \]

\[ = \frac{i\mu^{4-d}}{(4\pi)^{d/2}} \frac{\Gamma(2 - d/2)}{2} (-2 + \varepsilon)^2 \int_0^1 dy \int_0^1 dz \tau_2 (1 - z) \frac{(-Q^2 - i0)^{d/2-1} - (-P^2 - i0)^{d/2-1}}{P^2 - Q^2} \]

\[ = \frac{i}{(4\pi)^2} \left( \frac{2}{\varepsilon} + 1 - \gamma + \frac{Q^2 \log(Q^2/4\pi^2) - i0 - P^2 \log(-P^2/4\pi^2 - i0)}{P^2 - Q^2} \right). \]

After taking the traces one obtains

\[ Tr [iM_{1,2,3}^3, \gamma^\nu] = -C_F e_q g^2 \frac{i}{(4\pi)^2} \frac{1}{\not{p} k_1 + i0} \frac{1}{\not{p} k_3 - i0} \times \]

\[ \left( -\frac{P^2}{(P^2 - Q^2)^2} Tr \left\{ \frac{1}{\not{p} k_3 \not{p} k_1 \not{p} k_2 \not{p} k_1} \right\} \left( (Q^2 - 2P^2) \log \frac{Q^2 + i0}{P^2 + i0} + Q^2 - P^2 \right) + \right. \]

\[ \left. -\frac{Q^2}{(P^2 - Q^2)^2} Tr \left\{ \frac{1}{\not{p} k_3 \not{p} k_1 \not{p} k_2 \not{p} k_1} \right\} \left( (P^2 - 2Q^2) \log \frac{Q^2 + i0}{P^2 + i0} + Q^2 - P^2 \right) + \right. \]

\[ + A_{1,2,3} \left( \frac{2}{\varepsilon} + 1 - \gamma + \frac{Q^2 \log(Q^2/4\pi^2) - i0 - P^2 \log(-P^2/4\pi^2 - i0)}{P^2 - Q^2} \right) \right). \]

To obtain a more compact and readable expression, let us define

\[ D_1 = P^2/s = (x + \xi)\beta_1, \quad D_3 = Q^2/s = -(x - \xi)\beta_3, \]

\[ B_{1,2,3} = Tr \left\{ \not{p} k_3 \not{p} k_1 \not{p} k_2 \not{p} k_1 \right\}, \]

\[ C_{1,2,3} = Tr \left\{ \not{p} k_3 \not{p} k_1 \not{p} k_2 \not{p} k_1 \right\}. \]

Finally, we get

\[ Tr [iM_{1,2,3}^3, \gamma^\nu] = Tr [iM_{1,2,3}^0, \gamma^\nu] \cdot \frac{g^2 C_F}{(4\pi)^2} \left\{ \left( \frac{se^\gamma}{4\pi^2} \right)^{-\frac{\varepsilon}{2}} \left( \frac{2}{\varepsilon} + 1 - \frac{D_1 \log(-D_1 - i0) - D_3 \log(-D_3 - i0)}{D_1 - D_3} \right) \right. \]

\[ + \left( B_{1,2,3} \right) \frac{D_1}{D_1 - D_3} \left( 1 + \frac{-2D_1 + D_3}{D_1 - D_3} \log \frac{D_1 + i0}{D_3 + i0} \right) \]

\[ + \left( C_{1,2,3} \right) \frac{D_3}{D_1 - D_3} \left( 1 + \frac{D_1 - 2D_3}{D_1 - D_3} \log \frac{D_1 + i0}{D_3 + i0} \right) \right\}. \]

(3.63)
3.6 Box diagrams - 4.L and 4.R

4-point diagrams, also called box diagrams, contain collinear divergence only. Because of more complicated structure of the corresponding traces and integrals, as compared to 2- and 3-point diagrams, we first focus only on the divergent part, and show how it can be handled analytically. Full derivation of the finite part of the amplitude is presented in Appendix A. The box diagrams are presented in Fig. 3.5.

![Box diagrams - 4.L and 4.R.](image)

Figure 3.5: Diagrams 4.L and 4.R.

Amplitude corresponding to 4.L takes the following form:

\[
Tr[iM_{1,2,3}^{4L, p}] = -C_F(i\epsilon_q)^3(i\gamma^5)^2 \frac{i}{-2\not{p}\not{k}_3 + i0} \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} Tr \left\{ \eta_{\mu \nu} \not{p} \not{k}_3 (\not{p} - \not{k}_3) \gamma^\mu \times 
\right. \\
\left. \frac{i^4 (\not{p} + \not{k}_1 + \not{k}_2 + k) \not{\epsilon}_2 (\not{p} + \not{k}_1 + \not{k}_2) \not{\epsilon}_1 (\not{p} + \not{k}_2)}{(k^2 + i0)(\not{p} + \not{k})^2 + i0)} \right\}
\]

(3.64)

3.6.1 Localizing divergences inside the integral

Let use denote by \(Tr(k)\) the trace present in Formula (3.64). Using the Feynman parametrization we write the considered integral in the following form:

\[
\int \frac{d^d k}{(2\pi)^d} \frac{Tr(k)}{(k^2 + i0)(\not{p} + \not{k})^2 + i0)(\not{p} + \not{k} + \not{k}_1)^2 + i0)} \left( k^2 + 2k\left( (r + y + z)\not{p} + (y + z)k_1 + zk_2 \right) + 2(z + y)\not{p}k_1 + 2z(\not{p} + \not{k}_1)k_2 + i0 \right)^{-4}
\]

(3.65)

The expression in the last line can be written as

\[
\left( k + (r + y + z)\not{p} + (y + z)k_1 + zk_2 \right)^2 + 2(1 - r - y - z)(y + z)\not{p}k_1 + 2(1 - r - y - z)z\not{p}k_2 - 2(1 - y - z)z\not{p}k_2.
\]

We use the following substitution for Feynman parameters:

\[
r = \bar{y}(1 - \bar{z}), \quad y = (1 - \bar{r})(1 - \bar{y}), \quad z = \bar{r}(1 - \bar{y}).
\]

(3.66)
Note that
\[ y + z = 1 - \bar{y}, \quad r + y + z = 1 - \bar{z}\bar{y}. \]

After reparametrization, the considered integral takes the following form (I omit the overline in \( r, y, z \)):
\[
6 \int \frac{d^d k'}{(2\pi)^d} \int_0^1 y(1-y)dr dy dz \text{Tr}(k') \left( k'^2 + 2y(1-y) \left( z\bar{p}k_1 + zr\bar{p}k_2 + rk_1k_2 \right) + i0 \right)^{-4}, \tag{3.67}
\]
where
\[ k' = k + (1-yz)\bar{p} + (1-y)k_1 + r(1-y)k_2. \]

Analogously to the case of diagrams 3.L and 3.R, the collinear divergence manifests after performing the integration over Feynman parameters – let us localize where it does occur. The general form of the momentum integral reads
\[
6 \int \frac{d^d k'}{(2\pi)^d} k'^{2n} \left( k'^2 + 2y(1-y) \left( z\bar{p}k_1 + zr\bar{p}k_2 + rk_1k_2 \right) + i0 \right)^{-4} =
\]
\[
\frac{(-1)^n i}{(4\pi)^{d/2}} \left( -2y(1-y) \left( z\bar{p}k_1 + zr\bar{p}k_2 + rk_1k_2 \right) - i0 \right)^{d/2-4+n} \frac{\Gamma(4-n-d/2)\Gamma(n+d/2)}{\Gamma(d/2)}, \tag{3.68}
\]
where \( n \) can be equal to 0 or 1. To see it, note that from the trace one can get terms which are at most of power 3 in \( k' \). However, since we have shifted the momentum variable to complete the square in denominator, terms with \( k'\alpha \) and \( k'\alpha k'\beta k'\gamma \) vanish. The term with the even power of \( k' \) in the numerator, i.e. \( k'\alpha k'\beta \), is proportional to the metric tensor \( g^{\alpha\beta} \) times the momentum integral with \( k'^2 \) in numerator, see Eq. (3.23). Reparametrization of the Feynman parameters (3.66) allows to factor-out the integral over \( dy \), which becomes
\[
\int_0^1 dy y^{d/2-3+n+n_1}(1-y)^{d/2-3+n+n_2} = \frac{\Gamma(d/2-2+n+n_1)\Gamma(d/2-2+n+n_2)}{\Gamma(d-4+2n+n_1+n_2)}. \tag{3.69}
\]
Indices \( n_{1,2} \) are different for various terms in the trace, and will be discussed later. We see that the divergence in (3.69) occurs if both conditions: \( n = 0 \) and \( (n_1 \text{ or } n_2 = 0) \) hold.

Integrals over \( dr \) and \( dz \) are more complicated and take the following general form:
\[
\int_0^1 dr \int_0^1 dz r^a z^b \left( -2 \left( z\bar{p}k_1 + zr\bar{p}k_2 + rk_1k_2 \right) - i0 \right)^{d/2-4+n}. \tag{3.70}
\]
\( a, b, n \) are non-negative integers. It can be checked that (3.70) is divergent only if \( a = b = n = 0 \). Further discussion of divergences requires analysis of the trace.

### 3.6.2 Trace - terms leading to divergences

From the previous part, we see that divergences come from terms in the trace without the loop momentum components \( k'^\mu \) in numerator, so that to find the divergent part of the amplitude we need to consider only the trace with all terms \( \bar{k}' \) removed:
\[
\text{Tr} \left\{ \gamma_{\mu} \gamma_{3}(-\bar{k}_3)\gamma' \left( yz\bar{p}+y\bar{k}_1+(1-r(1-y))\bar{k}_2 \right) \gamma_{2} \left( yz\bar{p}+y\bar{k}_1-(1-y)\bar{k}_2 \right) \gamma_{1} (yz\bar{p}-(1-y)\bar{k}_1-r(1-y)\bar{k}_2) \right\}. \tag{3.71}
\]
After contraction of matrices $\gamma^\mu$ in dimension $d = 4 - \varepsilon$ we get two terms:

$$
2Tr\left\{ k_3\gamma_3\bar{p}(y\gamma_1 + (1 - r(1 - y))\gamma_2)\gamma_2(y\gamma_1 + y\gamma_1 - r(1 - y)\gamma_2)\gamma_1(y\gamma_1 - (1 - y)\gamma_1 - r(1 - y)\gamma_2)\right\} + 
- \varepsilon Tr\left\{ p\gamma_3\gamma_3(y\gamma_1 + y\gamma_1 + (1 - r(1 - y))\gamma_2)\gamma_2(y\gamma_1 + y\gamma_1 - r(1 - y)\gamma_2)\gamma_1(- (1 - y)\gamma_1 - r(1 - y)\gamma_2)\right\}.
$$

(3.72)

After expanding the traces we conclude that there are only 2 combinations which do not vanish in the considered kinematics (due to $k_i k_i = \gamma^0 = 0$) and lead to divergent integral over $dy$:

$$
2y^2 z Tr\left\{ k_3\gamma_3\bar{p}\gamma_2\gamma_2\gamma_1\gamma_1\right\} + 2y^3 z Tr\left\{ k_3\gamma_3\bar{p}\gamma_2\gamma_2\gamma_1\gamma_1\bar{p}\right\}.
$$

(3.73)

They are present only in the first trace, while all terms inside the trace proportional to $\varepsilon$ yield finite integrals, so that the second line of [3.72] can be neglected. There are no terms corresponding to $n = a = b = 0$ in Eq. [3.70], so that integrals over $dz$ and $dr$ are always convergent.

The divergent part of the amplitude $4L$ is hence proportional to

$$
\frac{2i}{(4\pi)^{d/2}} \Gamma(4 - d/2) \int_0^1 dy y^{d/2 - 1}(1 - y)^{d/2 - 3} \times
$$

$$
\int_0^1 dr \int_0^1 dz \left(- 2(z\bar{p}k_1 + zr\bar{p}k_2 + rk_1 k_2) - i0\right)^{d/2 - 4} Tr\left\{ k_3\gamma_3\bar{p}\gamma_2\gamma_2\gamma_1\gamma_1\bar{p}\right\} +
$$

$$
\frac{2i}{(4\pi)^{d/2}} \Gamma(4 - d/2) \int_0^1 dy y^{d/2}(1 - y)^{d/2 - 3} \times
$$

$$
\int_0^1 dr \int_0^1 dz \left(- 2(z\bar{p}k_1 + zr\bar{p}k_2 + rk_1 k_2) - i0\right)^{d/2 - 4} Tr\left\{ k_3\gamma_3\bar{p}\gamma_2\gamma_2\gamma_1\gamma_1\bar{p}\right\}.
$$

(3.74)

For now, we focus on the divergent part of this expression only, so that we neglect terms in the expansion around $\varepsilon = 0$ which are not divergent. They contribute to the finite part, which is computed later. We obtain

$$
(3.74)_{div.} = -\frac{2}{\varepsilon} i\frac{2}{8\pi^2} Tr\left\{ k_3\gamma_3\bar{p}(\gamma_1 + \gamma_2)\gamma_2\gamma_1\bar{p}\right\} \int_0^1 dr dz \left(- 2(z\bar{p}k_1 + zr\bar{p}k_2 + rk_1 k_2) - i0\right)^{-2}.
$$

(3.75)

Let us observe that we can equivalently write the expression in the integrand as:

$$
-2(z(x + \xi)pk_1 + zr(x + \xi)pk_2 + rk_1 k_2) - i0 \rightarrow
$$

$$
-2(z(x + \xi + i0/pk_i)pk_1 + zr(x + \xi + i0/pk_2)pk_2 + rk_1 k_2)
$$

(3.76)

Denote

$$(x + \xi + i0/pk_i)pk_i = \bar{p}k'_i.$$

The integral yields

$$
\int_0^1 dr dz \left(- 2(z\bar{p}k'_1 + zr\bar{p}k'_2 + rk_1 k_2)\right)^{-2} =
$$

$$
= \frac{1}{4(\bar{p}k'_1 + \bar{p}k'_2)\bar{p}k'_1} \log \left(\frac{\bar{p}k'_1 + \bar{p}k'_2 + k_1 k_2}{k_1 k_2}\right).
$$

(3.77)
Let us simplify Formula (3.77). Using \( \bar{p}k_1' + \bar{p}k_2' = -\bar{p}k_3 + i0 \) one obtains

\[
\bar{p}k_1' + \bar{p}k_2' + \bar{k}_1k_2 = (x + \xi)p(k_1 + k_2) + k_1k_2 + i0 = \\
\frac{1}{2}\left((x + \xi)p + k_1 + k_2\right)^2 + i0 = \frac{1}{2}\left((x - \xi)p - k_3\right)^2 + i0 = -(x - \xi)pk_3 + i0.
\] (3.78)

Moreover,

\[(x + \xi)p + k_1 + k_2 = (x - \xi)p - k_3 \implies 2\xi p + k_3 = -(k_1 + k_2) \implies 2\xi pk_3 = k_1k_2,
\]
so that

\[
\frac{\bar{p}k_1' + \bar{p}k_2' + \bar{k}_1k_2}{k_1k_2} = -\frac{(x - \xi)pk_3 + i0}{2\xi pk_3} = \frac{1}{2\xi}(\xi - x + i0 \cdot \text{sgn}(pk_3)).
\] (3.79)

Hence, the divergent part of the integral (3.65) reads

\[-\frac{2}{\varepsilon} \frac{i}{32\pi^2} \text{Tr}\left\{\kappa_3\kappa_2\bar{p}(\kappa_1 + \kappa_2)\ell_1\ell_2\right\} \frac{-1}{(p + i0)(\bar{p}k_1 + i0)} \log\left(\frac{1}{2\xi}(\xi - x + i0 \cdot \text{sgn}(pk_3))\right). \quad (3.80)
\]

Using \((x + \xi)p + k_1 + k_2 = (x - \xi)p - k_3\) and anti-commuting \( pk_3 \) we can write it in a simpler way

\[-\frac{2}{\varepsilon} \frac{i}{16\pi^2} \text{Tr}\left\{\kappa_3\kappa_2\ell_1\ell_2\right\} \frac{1}{\bar{p}k_1 + i0} \log\left(\frac{1}{2\xi}(\xi - x + i0 \cdot \text{sgn}(pk_3))\right). \quad (3.81)
\]

We recognize that the resulting trace is \( A_{1,2,3} \). Finally, the divergent part of the amplitude \( 4.L \)

\[
\text{Tr}[i\mathcal{M}^{4.L}_{1,2,3} \bar{p}]_{\text{div.}} = iC_F\epsilon_q^3g^2 \frac{1}{(4\pi)^2} A_{1,2,3} \frac{1}{\bar{p}k_1 + i0} \frac{1}{2\xi pk_3 - i0} \frac{4}{\epsilon} \log\left(\frac{1}{2\xi}(\xi - x + i0 \cdot \text{sgn}(pk_3))\right)
\]

\[-\frac{4}{\varepsilon} \frac{g^2C_F}{(4\pi)^2} \text{Tr}[i\mathcal{M}^0_{1,2,3} \bar{p}] \log\left(\frac{1}{2\xi}(\xi - x + i0 \cdot \text{sgn}(pk_3))\right). \quad (3.82)
\]

Amplitude corresponding to the diagram \( 4.R \) reads

\[
\text{Tr}[i\mathcal{M}^{4.R}_{1,2,3} \bar{p}] = -C_F(i\epsilon_q)^3(i\gamma^5)^2 \frac{i}{2\bar{p}k_1 + i0} \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \text{Tr}\left\{\gamma_{\mu}(\bar{p} + \kappa_1)\ell_1\bar{p}\gamma^\mu \times \right\}
\]

\[
\frac{i^4(p + k)\kappa_3(p - \kappa_3 + k)\ell_2(p - \kappa_3 - \kappa_2 + k)}{(k^2 + i0)((p + k)^2 + i0)((p + k - k_3)^2 + i0)((p + k - k_3 - k_2)^2 + i0)}
\]

(3.83)

Let us notice, that this expression can be obtained by taking the following substitutions:

\[(\xi \to -\xi, \ k_1 \to -k_3, \ k_2 \to -k_2)\]

in the Formula (3.64). It allows to easily obtain the divergent part of the second box diagram:

\[
\text{Tr}[i\mathcal{M}^{4.R}_{1,2,3} \bar{p}]_{\text{div.}} = iC_F\epsilon_q^3g^2 \frac{1}{(4\pi)^2} A_{1,2,3} \frac{1}{\bar{p}k_1 + i0} \frac{1}{2\xi pk_3 - i0} \frac{4}{\epsilon} \log\left(\frac{1}{2\xi}(\xi + x + i0 \cdot \text{sgn}(pk_1))\right)
\]

\[-\frac{4}{\varepsilon} \frac{g^2C_F}{(4\pi)^2} \text{Tr}[i\mathcal{M}^0_{1,2,3} \bar{p}] \log\left(\frac{1}{2\xi}(\xi + x + i0 \cdot \text{sgn}(pk_1))\right).
\] (3.84)
3.6.3 The finite part of the amplitude

Contributions to the finite part can be divided into 2 classes: the first one will be terms in traces that yield finite integrals over Feynman parameters. From the discussion concerning the loop integrals in Subsec. 3.6.1 one concludes, that they are proportional to the following functions:

\[
\int_0^1 dr dz r^a z^b \left( -2(z \bar{\rho} k_1 + zr \bar{\rho} k_2 + rk_1 k_2) - i0 \right)^{-n} =
\]

\[
= (-1)^n s^{-n} \int_0^1 dr dzr^a z^b \left( (z(x + \xi) \beta_1 + zr(x + \xi) \beta_2 + r \kappa_3) + i0 \right)^{-n} =
\]

\[
=: (-1)^n s^{-n} \mathcal{F}(n, a, b; x, \xi, \{\beta_i\}).
\]

In the definition we used the fact that \( \kappa_3 = -2 \xi \beta_3 \). All the functions \( \mathcal{F} \) present in the considered amplitude are collected in Appendix B.

The second class of finite terms are those from the finite part of divergent integrals. In particular, since the integral

\[
\int_0^1 dr \int_0^1 dz z \left( -2(z \bar{\rho} k_1 + zr \bar{\rho} k_2 + rk_1 k_2) - i0 \right)^{d/2-4}
\]

is multiplied by a divergent factor \( 1/\varepsilon \), to finite part will contain a contribution resulting from expansion of the integrand in \( \varepsilon \) according to

\[
X^{-2-\varepsilon/2} = X^{-2} - \frac{\varepsilon}{2} X^{-2} \log X + \mathcal{O}(\varepsilon^2).
\]

To account for this term, let us define the function

\[
\mathcal{G}(x, \xi, \{\beta_i\}) :=
\]

\[
\int_0^1 dr \int_0^1 dz z \left( (z(x + \xi) \beta_1 + zr(x + \xi) \beta_2 + r \kappa_3 + i0)^{-2} \ln \left( - z(x + \xi) \beta_1 - zr(x + \xi) \beta_2 + r \kappa_3 - i0 \right). \right.
\]

(3.87)

For the exact form of (3.87), see Appendix B. Using Formulas (3.85) and (3.87) we can write the
finite part of the amplitude $4L$:

$$Tr[\mathcal{M}_{1,2,3}^{4L} \phi]_{\text{fin.}} = -C_F e_4^3 g^2 \frac{i}{(4\pi)^2} \frac{1}{2p_k^3 - i0} s^{-2} \left\{ \right.$$

$$F(2, 1, 0; x, \xi, \{\beta_i\}) \left( Tr(k_{3f3}p_k_{1f2}k_{2f1}k_{1}) - Tr(k_{3f3}p_k_{1f2}k_{1f2}k_{2}) - 2Tr(k_{3f3}p_k_{1f2}k_{2f1}k_{1}) \right) +$$

$$F(2, 0, 1; x, \xi, \{\beta_i\}) \left( -Tr(k_{3f3}p_k_{1f2}k_{2f1}k_{1}) - 2Tr(k_{3f3}p_k_{2f2}k_{1f1}k_{1}) + 
- 4Tr(k_{3f3}p_k_{2f2}k_{1f1}k_{1}^2) - 5Tr(k_{3f3}p_k_{2f2}k_{1f1}p) \right)$$

$$F(2, 1, 1; x, \xi, \{\beta_i\}) \left( -Tr(k_{3f3}p_k_{1f2}k_{2f1}k_{1}) - Tr(k_{3f3}p_k_{1f2}k_{2f2}k_{2f1}^2) \right) + 
+ Tr(k_{3f3}p_k_{2f2}k_{2f1}k_{1}) - 2Tr(k_{3f3}p_k_{2f2}k_{2f1}k_{2}) - Tr(k_{3f3}p_k_{2f2}k_{1f1}k_{2}) +$$

$$F(2, 2, 0; x, \xi, \{\beta_i\}) \left( Tr(k_{3f3}p_k_{1f2}k_{2f2}k_{2}) + Tr(k_{3f3}p_k_{2f2}k_{1f1}k_{2}) \right) +$$

$$F(2, 2, 1; x, \xi, \{\beta_i\}) \left( Tr(k_{3f3}p_k_{2f2}k_{1f1}k_{2}) \right) +$$

$$sF(1, 1, 0; x, \xi \{\beta_i\}) \left( Tr(k_{3f3}p_k_{2f1}k_{2}) + Tr(p_k_{3f3}k_{1f2}k_{1}f1) + Tr(k_{3f3}p_k_{2f2}k_{1f1}) \right) +$$

$$sF(1, 0, 0; x, \xi \{\beta_i\}) \left( Tr(p_k_{3f3}k_{2f2}k_{1f1}) - Tr(p_k_{3f3}k_{1f2}k_{1}f1) - Tr(k_{3f3}p_k_{2f2}k_{1f1}) - 2Tr(k_{3f3}p_k_{2f2}k_{1f1}) \right) +$$

$$- 2G(x, \xi \{\beta_i\}) Tr(k_{3f3}p_k_{2f2}k_{1f1}p) \right\}. \tag{3.88}$$

The expression for the finite part of $Tr[\mathcal{M}_{1,2,3}^{4L} \phi]$ is obtained by changing

$$\xi \rightarrow -\xi, \quad k_1 \rightarrow -k_3, \quad k_2 \rightarrow -k_2, \quad k_3 \rightarrow -k_1,$$

according to the previous observation. The full derivation of (3.88) is presented in the Appendix A. Traces present in the Formula (3.88) can be further simplified using Mathematica – the resulting form of the amplitude is shown in Appendix B.
3.7 The 5-point diagram

The last integral, which corresponds to the diagram shown in Fig. 3.6, in which the loop contains 5 propagators, might (a priori) develop divergences $1/\varepsilon^2$, since 3 momenta can be simultaneously collinear and lightlike ($\bar{p} + k$, $k$, and $p - k$). Fortunately, it turns out that the trace contains terms which cancel those in the denominator.

\[ \text{Figure 3.6: The 5-point diagram.} \]

The amplitude reads

\[
Tr[iM_{1,2,3}^5\slashed{p}] = C_F e_0^3 g^2 \mu^{4-d} \int \frac{d^4 k}{(2\pi)^d} \frac{Tr\left\{\gamma_\mu \gamma_\mu (\bar{p} + \bar{k}) f_3 (\bar{p} + \bar{k}_1 + \bar{k}_2 + \bar{k}) f_2 (\bar{p} + \bar{k}_1 + \bar{k}) f_1 (\bar{p} + \bar{k})\right\}}{(k^2 + i0)(D_+ + i0)(D_- + i0)(D_1 + i0)(D_{12} + i0)},
\]

where

\[
D_\pm = ((x \pm \xi)p + k)^2, \quad D_1 = ((x + \xi)p + k_1 + k)^2, \quad D_{12} = ((x + \xi)p + k_1 + k_2 + k)^2.
\]

We use the following gamma-identities inside the trace:

\[
\gamma_\mu \gamma_\mu = (-2 + \varepsilon)\slashed{p}, \quad \slashed{k}\slashed{p}\slashed{k} = 2(pk)k - k^2\slashed{p}.
\]

Let us observe, that

\[
2(pk) = \frac{1}{2\xi}(D_+ - D_-), \quad k^2 = \frac{1}{2\xi}((x + \xi)D_- - (x - \xi)D_+).
\]

It follows that

\[
2(pk)k - k^2\slashed{p} = \frac{D_+}{2\xi}(\slashed{k} + \slashed{p}) - \frac{D_-}{2\xi}(\slashed{k} + \slashed{p}). \tag{3.90}
\]

The amplitude breaks into two 4-point integrals:

\[
Tr[iM_{1,2,3}^5\slashed{p}] = Tr[iM_{1,2,3}^{5,L}\slashed{p}] + Tr[iM_{1,2,3}^{5,R}\slashed{p}], \tag{3.91}
\]
where

\[
\mathcal{M}^{5,L}_{1,2,3}(p) := -\frac{2 + \varepsilon}{2 \xi} C_F e^3 g^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr} \left\{ (\bar{p} + \bar{k}) \bar{f}_3 (\bar{p} + \bar{k}_1 + \bar{k}_2 + \bar{k}) \bar{f}_2 (\bar{p} + \bar{k}_1 + \bar{k}) \bar{f}_1 \right\}}{(k^2 + i0)(D_+ + i0)(D_1 + i0)(D_2 + i0)},
\]

(3.92)

\[
\mathcal{M}^{5,R}_{1,2,3}(p) := -\frac{2 + \varepsilon}{2 \xi} C_F e^3 g^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr} \left\{ (\bar{p} + \bar{k}) \bar{f}_3 (\bar{p} + \bar{k}_1 + \bar{k}_2 + \bar{k}) \bar{f}_2 (\bar{p} + \bar{k}_1 + \bar{k}) \bar{f}_1 \right\}}{(k^2 + i0)(D_- + i0)(D_1 + i0)(D_2 + i0)},
\]

(3.93)

The names 5.L and 5.R come from the structure of denominators in integrands, which are the same as in amplitudes 4.L and 4.R.

### 3.7.1 The divergent part

Let us start from the term 5.L: performing steps analogous to those in the case of 4.L we obtain

\[
\int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr} \left\{ (\bar{p} + \bar{k}) \bar{f}_3 (\bar{p} + \bar{k}_1 + \bar{k}_2 + \bar{k}) \bar{f}_2 (\bar{p} + \bar{k}_1 + \bar{k}) \bar{f}_1 \right\}}{(k^2 + i0)(D_+ + i0)(D_1 + i0)(D_2 + i0)} =
\]

\[
6 \int \frac{d^d k}{(2\pi)^d} \int_0^1 y(1-y) dr dy dz \text{Tr} \left\{ (\bar{p} + \bar{k}) \bar{f}_3 (\bar{p} + \bar{k}_1 + \bar{k}_2 + \bar{k}) \bar{f}_2 (\bar{p} + \bar{k}_1 + \bar{k}) \bar{f}_1 \right\} \frac{1}{(k^2 + 2y(1-y) \left( z\bar{p}k_1 + zr\bar{p}k_2 + rk_1 k_2 \right) + i0)^4},
\]

(3.94)

where

\[
k' = k + (1-y)\bar{p} + (1-y)k_1 + r(1-y)k_2.
\]

(3.95)

Following the same reasoning as in the case of 4.L, we analyze the following trace

\[
\text{Tr} \left\{ (yz\bar{p} - (1-y)\bar{k}_1 - r(1-y)\bar{k}_2) \bar{f}_3 (y\bar{z}\bar{p} + y\bar{k}_1 + (1-r(1-y))\bar{k}_2) \bar{f}_2 (yz\bar{p} + y\bar{k}_1 - r(1-y)\bar{k}_2) \bar{f}_1 \right\}.
\]

(3.96)

Divergent integrals over the parameter \(y\) come from sum of the following terms:

\[
y^3 z \text{Tr} \left\{ \bar{p} \bar{f}_3 \bar{k}_1 \bar{f}_2 \bar{k}_1 \bar{f}_1 \right\} + y^2 z \text{Tr} \left\{ \bar{p} \bar{f}_3 \bar{k}_2 \bar{f}_2 \bar{k}_1 \bar{f}_1 \right\}.
\]

(3.97)

As in the Section 3.6, all resulting integrals over \(dz\) and \(dr\) are convergent. The divergent part of 3.94 can be written as

\[
-\frac{i}{(4\pi)^2 \varepsilon} \int_0^1 dz \left( -2(\bar{z}p \bar{k}_1 + zr\bar{p}k_2 + rk_1 k_2) - i0 \right)^{-2}.
\]

(3.98)

It is the same integral as the one in Eq. (3.75). Using results of the previous section, we obtain the divergent part of Eq. (3.94) is

\[
\frac{i}{(4\pi)^2 \varepsilon} \mathcal{A}_{1,2,3} \left( \frac{x - \xi}{(pk_3 - i0)(\bar{p}k_1 + i0)} \right) \log \left( \frac{1}{2\xi} \left( \xi - x + i0 \cdot \text{sgn}(pk_3) \right) \right)
\]

(3.99)
After multiplying it by $-\frac{2+\epsilon}{2\epsilon} C_F \epsilon_q^3 g^2$ we get the divergent part of $Tr[i\mathcal{M}^{5, L}_{1,2,3} \not{p}]$

$$
Tr[i\mathcal{M}^{5, L}_{1,2,3} \not{p}]_{\text{div.}} = \frac{i}{(4\pi)^2} C_F \epsilon_q^3 g^2 A_{1,2,3} \frac{1}{(p k_3 - i 0)(\bar{p} k_1 + i 0)} \frac{2x - \xi}{\xi} \log \left( \frac{1}{2\xi} (\xi - x + i 0 \cdot \text{sgn}(p k_3)) \right) \log \left( \frac{1}{2\xi} (\xi + x + i 0 \cdot \text{sgn}(p k_1)) \right)
$$

(3.100)

The divergent part of $Tr[i\mathcal{M}^{5, R}_{1,2,3} \not{p}]$ by analogous steps is found to be equal to

$$
Tr[i\mathcal{M}^{5, R}_{1,2,3} \not{p}]_{\text{div.}} = -\frac{i}{(4\pi)^2} C_F \epsilon_q^3 g^2 A_{1,2,3} \frac{1}{(p k_3 - i 0)(\bar{p} k_1 + i 0)} \frac{2x + \xi}{\xi} \log \left( \frac{1}{2\xi} (\xi + x + i 0 \cdot \text{sgn}(p k_1)) \right)
$$

(3.101)

Hence the divergent part of amplitude $Tr[i\mathcal{M}^{5}_{1,2,3} \not{p}]$:

$$
Tr[i\mathcal{M}^{5}_{1,2,3} \not{p}]_{\text{div.}} = \frac{i}{(4\pi)^2} C_F \epsilon_q^3 g^2 A_{1,2,3} \frac{1}{(p k_3 - i 0)(\bar{p} k_1 + i 0)} \times
\frac{2}{\xi} \left( \frac{x - \xi}{\xi} \log \left( \frac{1}{2\xi} (\xi - x + i 0 \cdot \text{sgn}(p k_3)) \right) - \frac{x + \xi}{\xi} \log \left( \frac{1}{2\xi} (\xi + x + i 0 \cdot \text{sgn}(p k_1)) \right) \right)
$$

(3.102)

$$
= -\frac{g^2 C_F}{(4\pi)^2} Tr[i\mathcal{M}^0_{1,2,3} \not{p}] \times
\frac{2}{\xi} \left( \frac{x - \xi}{\xi} \log \left( \frac{1}{2\xi} (\xi - x + i 0 \cdot \text{sgn}(p k_3)) \right) - \frac{x + \xi}{\xi} \log \left( \frac{1}{2\xi} (\xi + x + i 0 \cdot \text{sgn}(p k_1)) \right) \right)
$$

It concludes the computation of the divergent part of the amplitude at the one loop order. The divergence is proportional to the same trace structure, that is present in the LO amplitude – as it will be discussed in Section 3.8, this feature will be crucial in the proof of the QCD collinear factorization for the considered process at NLO.
3.7.2 The finite part

Before proceeding with the proof of factorization, let us write the expression for the remaining finite part of $\text{Tr}[i\mathcal{M}_{1,2,3}^R \bar{\psi}]$, which is

$$
\text{Tr}[i\mathcal{M}_{1,2,3}^R \bar{\psi}]_{\text{fin.}} = -\frac{i}{(4\pi)^2} C_F e_q^3 g_2 \frac{1}{\xi s^2} \left\{ \right.
\frac{1}{2} \mathcal{F}(2, 0, 1; x, \xi, \{\beta_i\}) \left( \text{Tr}(\bar{k}_1 \bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) + \text{Tr}(\bar{k}_1 \bar{\varphi}_3 \bar{\varphi}_1 \bar{\varphi}_2 \bar{k}_1) + 2\text{Tr}(\bar{k}_1 \bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) + 
+ 5\text{Tr}(\bar{\varphi}_3 \bar{k}_1 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) + 4\text{Tr}(\bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) \right) + 
\frac{1}{2} \mathcal{F}(2, 1, 0; x, \xi, \{\beta_i\}) \left( -\text{Tr}(\bar{k}_1 \bar{\varphi}_3 \bar{k}_1 \bar{\varphi}_2 \bar{k}_1) + \text{Tr}(\bar{k}_2 \bar{\varphi}_3 \bar{\varphi}_1 \bar{\varphi}_2 \bar{k}_1) + 2\text{Tr}(\bar{k}_2 \bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) \right) + 
\frac{1}{2} \mathcal{F}(2, 1, 1; x, \xi, \{\beta_i\}) \left( \text{Tr}(\bar{\varphi}_3 \bar{\varphi}_1 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) + \text{Tr}(\bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\varphi}_1) - \text{Tr}(\bar{k}_1 \bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) \right) + 
- \text{Tr}(\bar{k}_1 \bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\varphi}_1) + \text{Tr}(\bar{k}_2 \bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) + +\text{Tr}(\bar{k}_2 \bar{\varphi}_3 \bar{\varphi}_1 \bar{\varphi}_2 \bar{k}_1) + 2\text{Tr}(\bar{k}_2 \bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) \right) + 
\frac{1}{2} \mathcal{F}(2, 2, 0; x, \xi, \{\beta_i\}) \left( \text{Tr}(\bar{k}_2 \bar{\varphi}_3 \bar{\varphi}_1 \bar{\varphi}_2 \bar{k}_1) - \text{Tr}(\bar{k}_2 \bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) \right) + 
- \frac{1}{2} \mathcal{F}(2, 2, 1; x, \xi, \{\beta_i\}) \left( \text{Tr}(\bar{k}_2 \bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) + \text{Tr}(\bar{k}_2 \bar{\varphi}_3 \bar{\varphi}_1 \bar{\varphi}_2 \bar{k}_1) \right) + 
4s \mathcal{F}(1, 0, 0; x, \xi, \{\beta_i\}) \left( 3(k_1 \epsilon_2)(\epsilon_2 \epsilon_3) - 3(k_1 \epsilon_3)(\epsilon_1 \epsilon_2) - 2(k_2 \epsilon_1)(\epsilon_2 \epsilon_3) \right) + 
- 4s \mathcal{F}(1, 1, 0; x, \xi, \{\beta_i\}) \left( (k_2 \epsilon_1)(\epsilon_2 \epsilon_3) + (k_2 \epsilon_3)(\epsilon_1 \epsilon_2) \right) + 
\mathcal{G}(x, \xi, \{\beta_i\}) \text{Tr}(\bar{\varphi}_3 \bar{\varphi}_2 \bar{k}_1 \bar{\ell}_1) + 
\mathcal{A} \frac{s^2(x - \xi)}{(pk_3 - i0)(\bar{p}k_1 + i0)} \log \left( \frac{1}{2\xi^2} (\xi - x + i0 \cdot \text{sgn}(pk_3)) \right) \left\} \right.
\end{align}

Derivation of this formula is presented in Appendix A. The expression for $\text{Tr}[i\mathcal{M}_{1,2,3}^R \bar{\psi}]$ is obtained by changing $\xi \rightarrow -\xi$, $k_1 \rightarrow -k_3$, $k_2 \rightarrow -k_2$, $k_3 \rightarrow -k_1$. Simplification of traces leads to the formula (B.10) presented in Appendix B.
3.8 Factorization theorem at the 1-loop order

The divergent part of the amplitude corresponding to a given permutation of photons entering the hard part is equal to

\[
Tr\left[i\mathcal{M}_{1,2,3}\right]_{\text{div.}} = \frac{2}{\varepsilon} \cdot \frac{i\alpha_s}{4\pi} C_F \varepsilon_q^3 \mathcal{A}_{1,2,3} \frac{1}{\beta_1 \beta_3} \frac{1}{x - \xi - i0 \cdot sgn(\beta_3)} \frac{1}{x + \xi + i0 \cdot sgn(\beta_1)} \times \\
\left(3 + \frac{x + \xi}{\xi} \log \left(\frac{1}{2\xi}(\xi - x + i0 \cdot sgn(\beta_3))\right) - \frac{x - \xi}{\xi} \log \left(\frac{1}{2\xi}(\xi + x + i0 \cdot sgn(\beta_1))\right)\right). 
\]

(3.104)

It can also be written as

\[
Tr\left[i\mathcal{M}_{1,2,3}\right]_{\text{div.}} = -\frac{2}{\varepsilon} \cdot \frac{\alpha_s}{4\pi} C_F \cdot Tr\left[i\mathcal{M}_{1,2,3}^0\right] \times \\
\left(3 + \frac{x + \xi}{\xi} \log \left(\frac{1}{2\xi}(\xi - x + i0 \cdot sgn(\beta_3))\right) - \frac{x - \xi}{\xi} \log \left(\frac{1}{2\xi}(\xi + x + i0 \cdot sgn(\beta_1))\right)\right). 
\]

(3.105)

The term \(\mathcal{C}_{\text{coll.}}^q\) is defined by

\[
\sum_{\text{permutations}} Tr\left[\mathcal{M}_{1,2,3}\right]_{\text{div.}} = -\frac{2}{\varepsilon} \mathcal{C}_{\text{coll.}}^q. 
\]

(3.106)

Let us observe, that Formula (3.104) bears an important similarity to that of Eq. (3.17) – it is proportional to the same trace, and the only result of interchange of photons \(k_1 \leftrightarrow 3\) is the change of poles’ positions if \(sgn(\beta_1) \neq sgn(\beta_3)\). It follows, that

\[
\mathcal{C}_{\text{coll.}}^q = 4 \cdot \frac{\varepsilon_q^3}{\alpha_s 4\pi^2} C_F \left[ (\alpha - \bar{\alpha})(\bar{e}^\tau_{i}(\mathbf{q}_1)\bar{e}^\tau_{i}(\mathbf{q}_2))(\bar{p}_i \bar{e}_i(\mathbf{q})) + \\
- (\bar{p}_i \bar{e}_i(\mathbf{q}_1))(\bar{e}_i(\mathbf{q})\bar{e}^\tau_{i}(\mathbf{q}_2)) + (\bar{p}_i \bar{e}_i(\mathbf{q}_2))(\bar{e}_i(\mathbf{q})\bar{e}^\tau_{i}(\mathbf{q}_1)) \right] \times \\
\text{Im} \left[ \frac{1}{x - \xi + i0} \frac{1}{x + \xi - i0} \left(3 + \frac{x + \xi}{\xi} \log \left(\frac{1}{2\xi}(\xi - x - i0)\right) - \frac{x - \xi}{\xi} \log \left(\frac{1}{2\xi}(\xi + x - i0)\right)\right) \right]. 
\]

(3.107)

The exact form of the imaginary part in the last line of the Formula (3.107) is rather lengthy, and it is more convenient to write it as an imaginary part of a simpler expression in order to verify the factorization at the NLO order. Let us recall the form of \(\mathcal{C}_{0}^q\):

\[
\mathcal{C}_{0}^q(x, \xi, \ldots) = 4 \cdot \frac{\varepsilon_q^3}{\alpha_s 4\pi^2} \text{Im} \left( \frac{1}{x + \xi} \frac{1}{x - \xi + i0} \right) \times \\
\left[ (\alpha - \bar{\alpha})(\bar{e}^\tau_{i}(\mathbf{q}_1)\bar{e}^\tau_{i}(\mathbf{q}_2))(\bar{p}_i \bar{e}_i(\mathbf{q})) - (\bar{p}_i \bar{e}_i(\mathbf{q}_1))(\bar{e}_i(\mathbf{q})\bar{e}^\tau_{i}(\mathbf{q}_2)) + (\bar{p}_i \bar{e}_i(\mathbf{q}_2))(\bar{e}_i(\mathbf{q})\bar{e}^\tau_{i}(\mathbf{q}_1)) \right]. 
\]

(3.108)

The non-singlet quark kernel reads (see Eqs. (99-101) in [27]):

\[
K^{qq}(x, x') = \frac{1}{|\xi|} \frac{\alpha_s}{4\pi} C_F \left[ \rho \left(\frac{x}{\xi}, \frac{x'}{\xi}\right) \left\{ \frac{x + \xi}{x'} \left(1 + \frac{2\xi}{x' - x}\right) \right\} + \rho \left(-\frac{x}{\xi}, -\frac{x'}{\xi}\right) \left\{ -\frac{x + \xi}{x' + \xi} \left(1 + \frac{2\xi}{x - x'}\right) \right\} \right], 
\]

(3.109)

\[
K^{qg}(x, x') = \frac{\alpha_s}{4\pi} C_F \left[ \rho \left(\frac{x}{\xi}, \frac{x'}{\xi}\right) \left\{ \frac{x + \xi}{x'} \left(1 + \frac{2\xi}{x' - x}\right) \right\} + \rho \left(-\frac{x}{\xi}, -\frac{x'}{\xi}\right) \left\{ -\frac{x + \xi}{x' + \xi} \left(1 + \frac{2\xi}{x - x'}\right) \right\} \right]. 
\]

(3.110)
where
\[ \rho(x, x') = \theta\left(\frac{x'}{x} - 1\right) \theta\left(\frac{x'}{x} + 1\right) - \theta\left(\frac{x'}{x} - 1\right) \theta\left(1 - \frac{x'}{x}\right), \] (3.110)
where \( \theta \) is the step function, and
\[ \int dx \left[ f(x, x') \right] g(x) = \int dx f(x, x')(g(x) - g(x')). \] (3.111)

In the considered kinematics one has \( \xi \geq 0 \). To prove the factorization, according to the Formula 2.24, and using Eqs. 3.108 and 3.107 we need to verify if
\[ \frac{\alpha_s}{4\pi} C_F \Im \left[ \frac{1}{y + \xi - i\delta y - \xi - i\delta} \right] \]
\[ = \frac{\alpha_s}{4\pi} C_F \Im \left[ \frac{1}{x - \xi + i\delta x + \xi - i\delta} \left(3 + \frac{x + \xi}{\xi} \log \left(\frac{1}{2\xi} (\xi - x - i\delta)\right) - \frac{x - \xi}{\xi} \log \left(\frac{1}{2\xi} (\xi + x - i\delta)\right)\right) \right]. \] (3.112)

Since the evolution kernel is real-valued, we can first compute the integral and then take the imaginary part. Moreover, to avoid complications associated with singularities in the integrand, we will substitute \( i\delta \to i\delta \), for some positive constant \( \delta \), and then check, if the resulting integral has the appropriate limit \( \delta \to 0 \). Let us denote
\[ g_\delta(x) = \frac{1}{x + \xi - i\delta x - \xi + i\delta}. \] (3.113)

The resulting integrals are (we omit \( \frac{\alpha_s}{4\pi} C_F \)):
\[ \int_{-\xi}^{x} dy \frac{\xi + y}{\xi + x} \left(1 + \frac{2\xi}{x - y}\right) \left(g_\delta(y) - g_\delta(x)\right) = \]
\[ = \frac{1}{x - \xi + i\delta} \log \left(\frac{x - \xi + i\delta}{2\xi}\right) + \frac{1}{x + \xi + x - \xi - i\delta} \left(2\xi^2 - 1\right) + O(\delta), \] (3.114)
\[ \int_{x}^{\xi} dy \frac{\xi - y}{\xi - x} \left(1 + \frac{2\xi}{y - x}\right) \left(g_\delta(y) - g_\delta(x)\right) = \]
\[ = -\frac{1}{x - \xi + i\delta} \log \left(\frac{x + \xi - i\delta}{2\xi}\right) + \frac{1}{x - \xi + x - \xi - i\delta} \left(-2\xi^2 + 1\right) + O(\delta). \] (3.115)

Adding these terms we obtain
\[ \left(\frac{1}{4\pi} C_F\right)^{-1} \int_{-1}^{1} dy K^{qq}(y, x) g_\delta(y) = \]
\[ \frac{1}{x - \xi + i\delta} \log \left(\frac{x - \xi + i\delta}{-2\xi}\right) - \frac{1}{x + \xi - i\delta} \log \left(\frac{x + \xi - i\delta}{2\xi}\right) + \frac{3\xi}{(x + \xi - i\delta)(x - \xi + i\delta)} + O(\delta). \] (3.116)

Hence, in the limit \( \delta \to 0 \) we indeed obtain the relation (3.112), so that the factorization indeed holds at the one-loop level. What is interesting, in the considered limit both imaginary and real parts of the resulting expression are equal to those in \( C_{coll}^q \).
Chapter 4

Conclusions

In Chapter 1 we briefly reviewed the history of studies on the nuclei structure, the relevant theoretical concepts, and, most importantly, definitions of the Generalized Parton Distributions.

In Chapter 2 we presented, how one relates the definition of GPDs to experimentally measured differential cross sections, and what approximations one need to take, to do so. We discussed, how the perturbative corrections affect the aforementioned definitions, in particular, how the gauge link (also known as the Wilson line) appears in calculations after performing a resummation of a relevant class of diagrams. We described the problem with the collinear divergences, which are present because of the assumption that all partons forming hadrons are massless, and how those divergences vanish at the level of the amplitude after taking into account corrections to parton distributions. We showed, how one extracts the divergent part of GPD by using so-called renormalized string operators, presented an example of calculation of the relevant counter-terms, and explained how one obtains the renormalization group equations of the renormalized generalized parton distributions. It allows to write the final form of the amplitude of a given exclusive process in terms of finite quantities, given that the collinear divergences in fact do cancel.

Results obtained in Chapter 3 show that the collinear factorization holds in the case of the photoproduction of photon pairs $\gamma p \rightarrow \gamma'\gamma'' p$ at the one-loop QCD order. That process is recently of particular interest, since, due to the charge conjugation symmetry, it sensitive to $C$-odd combinations of Generalized Parton Distributions, and hence provides a complementary (with respect to e.g. DVCS or TCS) source of data that can be used in extraction of GPDs from experiments, which will be further supported by the improved accuracy of the theoretical predictions, due to knowledge of the full amplitude at the order $\alpha_s$ in strong coupling constant. Moreover, this is the simplest exclusive process, in which the hard sub-process on the parton is a $2 \rightarrow 3$ reaction – hopefully these results will be helpful in studies of this new class of processes. This work is also a ground for the implementation of observables related to $\gamma p \rightarrow \gamma'\gamma'' p$ in the PARTONS software framework [36].

Acknowledgements

This work was supported by the budget for science in 2020-2021, as a research project under the ”Diamond Grant” program.
Appendix A

Derivation of the finite part of 4- and 5-point loop diagrams.

In the Appendix I often omit Feynman slashed inside of traces, if that does not lead to ambiguity.

The full form of the amplitude $4L$:

\[
\text{Tr}\left[i\mathcal{M}_{1,2,3}^{4L} \not{p}\right] = -C_F(i\epsilon_q)^3(i\epsilon_g)^2 \frac{i}{-2p k_3 + i0} \mu^{d-4} 6 \int \frac{d^d k'}{(2\pi)^d} \times \\
\int_0^1 y(1-y)dy dy dz \text{Tr}(k^2 + 2y(1-y)(z\not{p}k_1 + zr\not{p}k_2 + rk_1 k_2) + i0)^{-4} \\
2\text{Tr}\left\{k_3\epsilon_3 p(yk_1 + (1-r(1-y))k_2 + k')\epsilon_2(yz\not{p} + yk_1 - r(1-y)k_2 + k')\times \\
\epsilon_1(yz\not{p} - (1-y)k_1 - r(1-y)k_2 + k')\right\}. \tag{A.1}
\]

Let me define the following functions:

\[
\int_0^1 dr dz r^a z^b \left(- 2(z\not{p}k_1 + zr\not{p}k_2 + rk_1 k_2) - i0\right)^{-n} = \\
(-1)^n s^{-n} \int_0^1 dr dz r^a z^b \left((z(x + \xi)\beta_1 + zr(x + \xi)\beta_2 + r\kappa_3) + i0\right)^{-n} = \tag{A.2}
\]

\[
= (-1)^n s^{-n} F(n, a, b; x, \xi, x, \xi, \beta_i).
\]

$n$ depends on the power of $k'^2$ in the nominator of the loop momentum integral. We have $n = 0$ and $n = 1$ terms; all of $n = 1$ terms are finite; there are two different $n = 0$-terms with a collinearly divergent part. Note that they have also finite parts which we neglected in the previous part – they are analyzed separately in here. Let me start from the finite $n = 0$-terms – to do so, let us take a look at the following trace:

\[
\text{Tr}\left\{k_3\epsilon_3 p(yk_1 + (1-r(1-y))k_2)\epsilon_2(yz\not{p} + yk_1 - r(1-y)k_2)\epsilon_1(yz\not{p} - (1-y)k_1 - r(1-y)k_2)\right\}. \tag{A.3}
\]

We have the general expression

\[
\int_0^1 dr dz dy y^{-1}(1-y)^{-1} s^{-2} (z(x + \xi)\beta_1 + zr(x + \xi)\beta_2 + r\kappa_3) + i0)^{-2} Tr(\ldots). \tag{A.4}
\]
Figure A.1: Graphical representation of all nonvanishing $n = 0$ terms. * and ** denote terms with divergences - those are treated separately.

All the resulting traces and integrands are

\[
\begin{align*}
-y^2(1-y)zTr(k_3 \epsilon_3 p_1 \epsilon_2 \bar{p} \epsilon_1 k_1) &\rightarrow -\frac{1}{2} Tr(k_3 \epsilon_3 p_1 \epsilon_2 \bar{p} \epsilon_1 k_1) F(2,0,1;x,\xi,\{\beta_i\}) \\
y^2(1-y)zTr(k_3 \epsilon_3 p_1 \epsilon_2 \bar{p} \epsilon_1 k_2) &\rightarrow -\frac{1}{2} Tr(k_3 \epsilon_3 p_1 \epsilon_2 \bar{p} \epsilon_1 k_2) F(2,1,1;x,\xi,\{\beta_i\}) \\
y^2(1-y)zTr(k_3 \epsilon_3 p_1 \epsilon_2 k_2 \epsilon_1 \bar{p}) &\rightarrow -\frac{1}{2} Tr(k_3 \epsilon_3 p_1 \epsilon_2 k_2 \epsilon_1 \bar{p}) F(2,1,1;x,\xi,\{\beta_i\}) \\
y(1-y)^2rTr(k_3 \epsilon_3 p_1 \epsilon_2 k_2 \epsilon_1 k_1) &\rightarrow \frac{1}{2} Tr(k_3 \epsilon_3 p_1 \epsilon_2 k_2 \epsilon_1 k_1) F(2,1,0;x,\xi,\{\beta_i\}) \\
y(1-y)^2r^2Tr(k_3 \epsilon_3 p_1 \epsilon_2 k_2 \epsilon_1 k_2) &\rightarrow \frac{1}{2} Tr(k_3 \epsilon_3 p_1 \epsilon_2 k_2 \epsilon_1 k_2) F(2,2,0;x,\xi,\{\beta_i\}) \\
y^2(1-y)rTr(k_3 \epsilon_3 p_1 \epsilon_2 k_1 \epsilon_2 k_2) &\rightarrow -\frac{1}{2} Tr(k_3 \epsilon_3 p_1 \epsilon_2 k_1 \epsilon_2 k_2) F(2,1,0;x,\xi,\{\beta_i\}) \\
y(1-y)^2zTr(k_3 \epsilon_3 p_2 \epsilon_2 \bar{p} \epsilon_1 k_1) &\rightarrow -Tr(k_3 \epsilon_3 p_2 \epsilon_2 \bar{p} \epsilon_1 k_1) F(2,0,1;x,\xi,\{\beta_i\}) \\
y(1-y)^2zTr(k_3 \epsilon_3 p_2 \epsilon_2 \bar{p} \epsilon_1 k_2) &\rightarrow -Tr(k_3 \epsilon_3 p_2 \epsilon_2 \bar{p} \epsilon_1 k_2) F(2,1,1;x,\xi,\{\beta_i\}) \\
y(1-y)^2z^2Tr(k_3 \epsilon_3 p_2 \epsilon_2 \bar{p} \epsilon_1 k_2) &\rightarrow -Tr(k_3 \epsilon_3 p_2 \epsilon_2 \bar{p} \epsilon_1 k_2) F(2,2,1;x,\xi,\{\beta_i\}) \\
y^2(1-y)zTr(k_3 \epsilon_3 p_2 \epsilon_2 k_1 \epsilon_1 \bar{p}) &\rightarrow -\frac{1}{2} Tr(k_3 \epsilon_3 p_2 \epsilon_2 k_1 \epsilon_1 \bar{p}) F(2,1,1;x,\xi,\{\beta_i\}) \\
y(1-y)rTr(k_3 \epsilon_3 p_2 \epsilon_2 k_1 \epsilon_1 k_2) &\rightarrow -Tr(k_3 \epsilon_3 p_2 \epsilon_2 k_1 \epsilon_1 k_2) F(2,1,0;x,\xi,\{\beta_i\}) \\
y(1-y)^2r^2Tr(k_3 \epsilon_3 p_2 \epsilon_2 k_1 \epsilon_1 k_2) &\rightarrow \frac{1}{2} Tr(k_3 \epsilon_3 p_2 \epsilon_2 k_1 \epsilon_1 k_2) F(2,2,0;x,\xi,\{\beta_i\}).
\end{align*}
\]
After performing the integration, we obtain the contribution from the finite integrals with $n = 0$: 

$$Tr[i M_{1;2,3}^4 \psi] = -C F e_q g^2 \frac{i}{(4\pi)^2} \frac{1}{2p^{k_3} - i0} s^{-2}$$

$$F(2, 1, 0; x, \xi, \{\beta_i\}) \left( Tr(k_{3\mu}p_{k_1}^\mu \epsilon_2 k_{1\epsilon_1}k_1) - Tr(k_{3\mu}p_{k_1}^\mu \epsilon_2 k_{1\epsilon_1}k_2) - 2Tr(k_{3\mu}p_{k_2}^\mu \epsilon_2 k_{1\epsilon_1}k_2) \right) +$$

$$+ F(2, 0, 1; x, \xi, \{\beta_i\}) \left( -Tr(k_{3\mu}p_{k_1}^\mu \epsilon_2 \bar{\epsilon}_1 k_1) - 2Tr(k_{3\mu}p_{k_2}^\mu \epsilon_2 \bar{\epsilon}_1 k_1) \right) +$$

$$+ F(2, 1, 1; x, \xi, \{\beta_i\}) \left( -Tr(k_{3\mu}p_{k_1}^\mu \epsilon_2 \bar{\epsilon}_1 k_2) - Tr(k_{3\mu}p_{k_1}^\mu \epsilon_2 \bar{\epsilon}_1 k_2) - 2Tr(k_{3\mu}p_{k_2}^\mu \epsilon_2 \bar{\epsilon}_1 k_2) \right) +$$

$$+ F(2, 2, 0; x, \xi, \{\beta_i\}) \left( Tr(k_{3\mu}p_{k_1}^\mu \epsilon_2 k_{1\epsilon_1}k_2) + Tr(k_{3\mu}p_{k_2}^\mu \epsilon_2 k_{1\epsilon_1}k_2) \right) +$$

$$+ F(2, 2, 1; x, \xi, \{\beta_i\}) Tr(k_{3\mu}p_{k_2}^\mu \epsilon_2 \bar{\epsilon}_1 k_2) \right).$$

(A.6)

Now let us look on the terms with $k^2$ in the nominator (the $n = 1$ terms):

$$Tr(k_{3\mu}p_{k_1}^\mu \epsilon_2 k_{1\epsilon_1}k_1(yz\bar{p} - (1 - y)k_1 - r(1 - y)k_2)) +$$

$$+ Tr(k_{3\mu}p_{k_1}^\mu \epsilon_2 yz\bar{p} + k_1 - r(1 - y)k_2) \epsilon_1 k') +$$

$$+ Tr(k_{3\mu}p(yk_1 + (1 - r(1 - y)) k_2) \epsilon_2 k'_1 k').$$

(A.7)

Use the following integration formula:

$$6 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k'^\nu}{k^2 + \Delta + i0} = -i g^{\mu\nu} \frac{1}{2(4\pi)^2} \left( -\Delta - i0 \right)^{-1}. \tag{A.8}$$

The previous traces become

$$2Tr(k_{3\epsilon_3}p_{\epsilon_2} \epsilon_1 ((1 - y)k_1 + r(1 - y)k_2)) +$$

$$- 2Tr(p_{\epsilon_3}k_{3} \epsilon_2 (yk_1 - r(1 - y)k_2) \epsilon_1) +$$

$$- 2Tr(k_{3\epsilon_3}p(yk_1 + (1 - r(1 - y) k_2) \epsilon_2 \epsilon_1). \tag{A.9}$$

We have the following integral

$$\int_0^1 d\rho d\gamma d\phi s^{-1}(z(x + \xi)\beta_1 + zr(x + \xi)\beta_2 + \rho r_3 + i0)^{-1} 2Tr(\ldots). \tag{A.10}$$

Note that there was a minus sign in the expression in the biggest parenthesis – later on I multiply the result by an additional $-1$. Recalling the previous definitions of $F$ we obtain

$$F(1, 0, 0; x, \xi, \{\beta_i\}) Tr(k_{3\epsilon_3}p_{\epsilon_2} \epsilon_1 k_1) + F(1, 1, 0; x, \xi, \{\beta_i\}) Tr(k_{3\epsilon_3}p_{\epsilon_2} \epsilon_1 k_2) +$$

$$- F(1, 0, 0; x, \xi, \{\beta_i\}) Tr(p_{\epsilon_3}k_{3\epsilon_2} \epsilon_1 k_1) + F(1, 1, 0; x, \xi, \{\beta_i\}) Tr(p_{\epsilon_3}k_{3\epsilon_2} \epsilon_2 k_1) +$$

$$- F(1, 0, 0; x, \xi, \{\beta_i\}) Tr(k_{3\epsilon_3}p_{\epsilon_2} \epsilon_1 k_1) - 2F(1, 0, 0; x, \xi, \{\beta_i\}) Tr(k_{3\epsilon_3}p_{\epsilon_2} \epsilon_2 k_1) +$$

$$+ F(1, 1, 0; x, \xi, \{\beta_i\}) Tr(k_{3\epsilon_3}p_{\epsilon_2} \epsilon_1 k_1). \tag{A.11}$$
Now, we multiply it by \(-C_F e_q^3 g^2 \frac{1}{2p_{k3} - i0}(-1)^{\frac{-i}{4\pi^2}}s^{-1}\) and obtain the \(k'^2\)-part of the amplitude:

\[
Tr[i\mathcal{M}_{1,2,3}^{L,\rho}] = -C_F e_q^3 g^2 \frac{i}{(4\pi)^2} \frac{1}{2p_{k3} - i0} s^{-1} \left( \mathcal{F}(1, 1, 0; x, \xi, \{\beta_i\}) \left( Tr(k_3e_3pe_2e_1k_2) + Tr(p\epsilon k_3e_2k_2\epsilon_1) + Tr(k_3e_3pk_2e_2\epsilon_1) \right) + \mathcal{F}(1, 0, 0; x, \xi, \{\beta_i\}) \left( Tr(k_3e_3pe_2e_1k_1) - Tr(p\epsilon k_3e_2k_1\epsilon_1) - Tr(k_3e_3pk_1\epsilon_1\epsilon_1) - 2Tr(k_3e_3pk_2e_2\epsilon_1) \right) \right).
\]

Finally, we extract the finite part of divergent terms. Recall that they were \(-C_F e_q^3 g^2 (2p_{k3} - i0)^{-1}\) times

\[
\frac{2i}{(4\pi)^{d/2}} \Gamma(4 - d/2) Tr \left\{ k_3e_3pk_2e_2k_1\epsilon_1\bar{p} \right\} \times
\int_0^1 dy \frac{y^{d/2-1}(1 - y)^{d/2-3}}{2^2} \int_0^1 dr dz \left( -2(z\bar{p}k_1 + zr\bar{p}k_2 + rk_1k_2) - i0 \right)^{d/2-4},
\]

We can rewrite these terms as

\[
\frac{2i}{(4\pi)^{2}} \left( \frac{se^\gamma}{4\pi\mu^2} \right)^{-\frac{d}{2}} (1 + \gamma\epsilon/2) \Gamma(2 + \epsilon/2) \frac{\Gamma(2 - \epsilon/2)\Gamma(-\epsilon/2)}{\Gamma(2 - \epsilon)} Tr \left\{ k_3e_3pk_2e_2k_1\epsilon_1\bar{p} \right\} s^{-2} \times
\int_0^1 dr dz \left( (z(x + \xi)\beta_1 + zr(x + \xi)\beta_2 + r\kappa_3 + i0) \right)^{-2} \left( 1 - \frac{\epsilon}{2} \ln(-z(x + \xi)\beta_1 - zr(x + \xi)\beta_2 - r\kappa_3 - i0) \right),
\]

\[
\frac{2i}{(4\pi)^{2}} \left( \frac{se^\gamma}{4\pi\mu^2} \right)^{-\frac{d}{2}} (1 + \gamma\epsilon/2) \Gamma(2 + \epsilon/2) \frac{\Gamma(3 - \epsilon/2)\Gamma(-\epsilon/2)}{\Gamma(3 - \epsilon)} Tr \left\{ k_3e_3pk_1\epsilon_2k_1\epsilon_1\bar{p} \right\} s^{-2} \times
\int_0^1 dr dz \left( (z(x + \xi)\beta_1 + zr(x + \xi)\beta_2 + r\kappa_3 + i0) \right)^{-2} \left( 1 - \frac{\epsilon}{2} \ln(-z(x + \xi)\beta_1 - zr(x + \xi)\beta_2 - r\kappa_3 - i0) \right).
\]

Expanding with respect to \(\epsilon\) we obtain

\[
(1 + \gamma\epsilon/2) \Gamma(2 + \epsilon/2) \frac{\Gamma(2 - \epsilon/2)\Gamma(-\epsilon/2)}{\Gamma(2 - \epsilon)} = -\frac{2}{\epsilon} - 2,
\]

\[
(1 + \gamma\epsilon/2) \Gamma(2 + \epsilon/2) \frac{\Gamma(3 - \epsilon/2)\Gamma(-\epsilon/2)}{\Gamma(3 - \epsilon)} = -\frac{2}{\epsilon} - \frac{5}{2}.
\]

Defining the following function:

\[
\mathcal{G}(x, \xi, \{\beta_i\}) = \int_0^1 dr dz \left( (z(x + \xi)\beta_1 + zr(x + \xi)\beta_2 + r\kappa_3 + i0) \right)^{-2} \ln(-z(x + \xi)\beta_1 - zr(x + \xi)\beta_2 - r\kappa_3 - i0),
\]

(A.16)
we obtain

\[
Tr [i\mathcal{M}_{1;2,3}^{4L} \, \hat{p}] \supset - \frac{i}{(4\pi)^2} s^{-2} \left( \mathcal{F}(2,0,1,x,\xi,\{\beta_i\}) \left( 4Tr(k_3\epsilon_3p_2\epsilon_2k_1\epsilon_1\bar{p}) + 5Tr(k_3\epsilon_3p_1\epsilon_2k_1\epsilon_1\bar{p}) \right) + \\
- 2G(x,\xi,\{\beta_i\})Tr(k_3\epsilon_3p_3\epsilon_2k_1\epsilon_1\bar{p}) \right) .
\]

(A.17)

In the last line I used the trick with interchanging \( \hat{p}(k_1 + k_2) \to -\hat{p}k_3 \) inside the trace. Hence, we get the finite part:

\[
Tr [i\mathcal{M}_{1;2,3}^{4L} \, \hat{p}]_{\text{fin.}} = -C_F e_3^2 g^2 \frac{i}{(4\pi)^2} \frac{1}{2\sqrt{p}k_3 - i0} s^{-2} \left( \mathcal{F}(2,0,1,x,\xi,\{\beta_i\}) \left( Tr(k_3\epsilon_3p_1\epsilon_2k_2\epsilon_1k_1) - Tr(k_3\epsilon_3p_1\epsilon_2k_2\epsilon_1k_2) - 2Tr(k_3\epsilon_3p_2\epsilon_2k_1\epsilon_1k_2) \right) + \\
+ \mathcal{F}(2,1,1;x,\xi,\{\beta_i\}) \left( -Tr(k_3\epsilon_3p_1\epsilon_2\bar{p}_1k_2) - Tr(k_3\epsilon_3p_1\epsilon_2\bar{p}_1k_2) + \\
+ 4Tr(k_3\epsilon_3p_2\epsilon_2k_1\epsilon_1\bar{p}) - 5Tr(k_3\epsilon_3p_1\epsilon_2k_1\epsilon_1\bar{p}) \right) + \\
+ \mathcal{F}(2,2,0;x,\xi,\{\beta_i\}) \left( Tr(k_3\epsilon_3p_1\epsilon_2k_2\epsilon_1k_2) + Tr(k_3\epsilon_3p_2\epsilon_2k_1\epsilon_1k_2) \right) + \\
+ \mathcal{F}(2,2,1;x,\xi,\{\beta_i\}) Tr(k_3\epsilon_3p_2\epsilon_2\bar{p}_1k_2) + \\
+ s\mathcal{F}(1,1,0;x,\xi,\{\beta_i\}) \left( Tr(k_3\epsilon_3p_2\epsilon_1k_2) + Tr(p_3k_3\epsilon_3k_2\epsilon_1) + Tr(k_3\epsilon_3p_2\epsilon_2\epsilon_1) \right) + \\
+ s\mathcal{F}(1,0,0;x,\xi,\{\beta_i\}) \left( Tr(k_3\epsilon_3p_2\epsilon_1k_1) - Tr(p_3k_3\epsilon_3k_1\epsilon_1) - Tr(k_3\epsilon_3p_2\epsilon_2\epsilon_1) - 2Tr(k_3\epsilon_3p_2\epsilon_2\epsilon_1) \right) + \\
- 2G(x,\xi,\{\beta_i\})Tr(k_3\epsilon_3p_3\epsilon_2k_1\epsilon_1\bar{p}) \right) .
\]

(A.18)

The expression for \( Tr [i\mathcal{M}_{1;2,3}^{4R} \, \hat{p}] \) is obtained by changing \( \xi \to -\xi, \quad k_1 \to -k_3, \quad k_2 \to -k_2, \quad k_3 \to -k_1 \).

Now we find the finite part of the 5.L part of the amplitude \( Tr [i\mathcal{M}_{1;2,3}^{5L} \, \hat{p}] \). Let us remind that we consider the following:

\[
Tr [i\mathcal{M}_{1;2,3}^{5L} \, \hat{p}] = \\
- \frac{2 + e}{2\xi} C_F e_3^2 g^2 \mu^{4-d} 6 \int \frac{d^d k'}{(2\pi)^d} \int_0^1 y(1-y) dr dy dz (k'^2 + 2y(1-y) \left( z\bar{p}_1 + zr\bar{p}_2 + rk_1k_2 \right) + i0)^{-4} \times \\
Tr \left\{ (yz\bar{p} - (1-y)k_1 - r(1-y)k_2 + k')\epsilon_3 \times \\
(yz\bar{p} + yk_1 + (1 - r(1-y))k_2 + k')\epsilon_2 (yz\bar{p} + yk_1 - r(1-y)k_2 + k')\epsilon_1 \right\} .
\]

(A.19)
As in the analysis in the 4-point loop case, we start from $n = 0$ terms, which are presented in the following graph:

![Graphical representation of all nonvanishing $n = 0$ terms. * and ** denote terms with divergences - those are treated separately.](image)

We expand the trace inside of the integral

$$6 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dr dy dz (1-y)(y(1-y)\Delta+i0)^{-4}Tr(...)= \frac{i}{(4\pi)^2} \int_0^1 dr dy dz^{-1}(1-y)^{-1}(\Delta+i0)^{-2}Tr(...).$$
After computing the integrals it yields
\[
- Tr(\bar{p} e_3 k_1 e_2 k_1) F(1, 1; x, \xi, \{\beta_i\}),
- Tr(\bar{p} e_3 k_2 e_2 k_1) F(1, 1; x, \xi, \{\beta_i\}),
- Tr(k_1 e_3 \bar{p} e_2 k_1) F(0, 1; x, \xi, \{\beta_i\}),
Tr(k_1 e_3 \bar{p} e_2 k_2) F(1, 1; x, \xi, \{\beta_i\}),
- Tr(k_1 e_3 k_1 e_2 \bar{p} e_1) F(0, 1; x, \xi, \{\beta_i\}),
- Tr(k_1 e_3 k_1 e_2 k_2) F(1, 0; x, \xi, \{\beta_i\}),
- 2Tr(k_1 e_3 k_2 e_2 \bar{p} e_1) F(0, 1; x, \xi, \{\beta_i\}),
Tr(k_1 e_3 k_2 e_2 \bar{p} e_1) F(1, 1; x, \xi, \{\beta_i\}),
- Tr(k_2 e_3 \bar{p} e_2 k_1) F(1, 1; x, \xi, \{\beta_i\}),
Tr(k_2 e_3 \bar{p} e_2 k_2) F(2, 1; x, \xi, \{\beta_i\}),
- Tr(k_2 e_3 k_1 e_2 \bar{p} e_1) F(1, 1; x, \xi, \{\beta_i\}),
- Tr(k_2 e_3 k_1 e_2 k_2) F(2, 0; x, \xi, \{\beta_i\}),
- 2Tr(k_2 e_3 k_2 e_2 \bar{p} e_1) F(1, 1; x, \xi, \{\beta_i\}),
Tr(k_2 e_3 k_2 e_2 \bar{p} e_1) F(2, 1; x, \xi, \{\beta_i\}),
- 2Tr(k_2 e_3 k_2 e_2 k_1) F(1, 0; x, \xi, \{\beta_i\}),
Tr(k_2 e_3 k_2 e_2 k_1) F(2, 0; x, \xi, \{\beta_i\}).
\]  

Now, let us recall the divergent terms:
\[
\int_0^1 dr dz \left(z(x + \xi) \beta_1 + zr(x + \xi) \beta_2 + r \kappa_3 + i0\right)^{-2} \left(1 - \frac{\varepsilon}{2} \ln \left(-z(x + \xi) \beta_1 - zr(x + \xi) \beta_2 - r \kappa_3 - i0\right)\right),
\]
\[
\int_0^1 dr dz \left(z(x + \xi) \beta_1 + zr(x + \xi) \beta_2 + r \kappa_3 + i0\right)^{-2} \left(1 - \frac{\varepsilon}{2} \ln \left(-z(x + \xi) \beta_1 - zr(x + \xi) \beta_2 - r \kappa_3 - i0\right)\right).
\]  

Expansion with respect to \(\varepsilon\) yields the following:
\[
(1 + \gamma \varepsilon/2) \Gamma(2 + \varepsilon/2) \Gamma(3 - \varepsilon/2) \Gamma(-\varepsilon/2) \Gamma(3 - \varepsilon) = -\frac{2}{\varepsilon} - 2,
\]
\[
(1 + \gamma \varepsilon/2) \Gamma(2 + \varepsilon/2) \Gamma(3 - \varepsilon/2) \Gamma(-\varepsilon/2) \Gamma(3 - \varepsilon) = -\frac{2}{\varepsilon} - \frac{5}{2}.
\]
In this way, we obtain
\[
Tr [i \mathcal{M}^{5L}_{1,2,3} p] \supset -\frac{i}{(4\pi)^2} \frac{5}{2} Tr (\bar{\psi} \gamma_5 k_1 \gamma_5 k_1) s^{-2} \mathcal{F}(2, 0, 1; x, \xi, \{\beta_i\}) + \frac{i}{(4\pi)^2} Tr (\bar{\psi} \gamma_5 k_2 \gamma_5 k_1) s^{-2} \mathcal{F}(2, 0, 1; x, \xi, \{\beta_i\}) (A.23) \\
+ \frac{i}{(4\pi)^2} Tr (\bar{\psi} \gamma_5 k_2 \gamma_5 k_1) s^{-2} \mathcal{G}(x, \xi, \{\beta_i\}).
\]

Mind that there was an overall factor \(\frac{-1}{2 \xi} \) multiplying everything in (A.19). To take that \(\varepsilon\) into account, we will just take the divergent part \((3.100)\) of the expression and multiply it by \(-\frac{1}{2}\), which results in an additional term
\[
Tr [i \mathcal{M}^{5L}_{1,2,3} p] \supset -\frac{i}{(4\pi)^2} C_F \epsilon^3 g^2 A_{1,2,3} \frac{1}{(pk_3 - i0)(pk_1 + i0)} x - \xi \log \left(\frac{1}{2\xi} (\xi - x + i0 \cdot sgn(pk_3))\right). 
\]

(A.24)

Now we focus on the \(n = 1\) terms. Non-vanishing traces (see \((3.100)\)) are
\[
y Tr (k' \gamma_5 k_2 \gamma_5 k_1), \quad -r(1 - y) Tr (k' \gamma_5 k_1 \gamma_5 k_2), \\
y Tr (\gamma_5 k_1 \gamma_5 k_2), \quad (1 - r(1 - y) Tr (\gamma_5 k_2 \gamma_5 k_1 k'), \\
- (1 - y) Tr (\gamma_5 k_1 \gamma_5 k_1 k), \quad -r(1 - y) Tr (\gamma_5 k_2 \gamma_5 k_2 \gamma_5 k_1).
\]

(A.25)

We use again the well-known formula
\[
6 \int \frac{d^4 k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 + \Delta + i0)^4} = -\frac{ig^{\mu\nu}}{2(4\pi)^2} (\Delta - i0)^{-1}, 
\]
and \(\gamma_\mu A_\mu = -2A\), so that we obtain
\[
Tr [i \mathcal{M}^{5L}_{1,2,3} p] \supset -\frac{i}{(4\pi)^2} s^{-1} \left( \mathcal{F}(1, 0, 0; x, \xi, \{\beta_i\}) \left( Tr (\gamma_5 k_2 \gamma_5 k_1) + Tr (\gamma_5 k_1 \gamma_5 k_2) - Tr (\gamma_5 k_1 \gamma_5 k_1) + 2Tr (\gamma_5 k_2 \gamma_5 k_1) \right) + \mathcal{F}(1, 1, 0; x, \xi, \{\beta_i\}) \left( Tr (\gamma_5 k_2 \gamma_5 k_1) + Tr (\gamma_5 k_2 \gamma_5 k_1) + Tr (\gamma_5 k_2 \gamma_5 k_1) \right) \right).
\]

(A.27)

After the trace algebra:
\[
(A.27) = -\frac{i}{(4\pi)^2} s^{-4} \left( \mathcal{F}(1, 0, 0; x, \xi, \{\beta_i\}) \left( 3(k_2 \gamma_5)(\gamma_5 k_2) - 3(k_1 \gamma_5)(\gamma_5 k_1) - 2(k_2 \gamma_5)(\gamma_5 k_2) \right) + \mathcal{F}(1, 1, 0; x, \xi, \{\beta_i\}) \left( (k_2 \gamma_5)(\gamma_5 k_2) + (k_2 \gamma_5)(\gamma_5 k_2) \right) \right).
\]

(A.28)
Appendix B

The full form of the amplitude

In this part we summarize all results concerning the amplitude at the higher order.

Let us first make some simplifications with polarization vectors. In the calculation of the hard part it is assumed that \( p_1 - p_2 \approx 2\xi p \), so that \( 2\xi p = q_1 + q_2 - q \). Hence, for each polarization vector \( \epsilon_i \):

\[
\epsilon_i(k_1 + k_2 + k_3) = 0,
\]

moreover \( k_i\epsilon_i = 0 \) (there is no summation over index \( i \)).

In the considered frame, only the transverse parts of polarization vectors contribute to their products with momenta. We have:

\[
q_1\epsilon(q) = -q_2\epsilon(q) = -\vec{p}_t\vec{e}_t(q), \\
q\epsilon(q_1) = q_2\epsilon(q_1) = \frac{1}{\alpha}\vec{p}_t\vec{e}_t(q_1), \\
q\epsilon(q_2) = q_1\epsilon(q_2) = -\frac{1}{\bar{\alpha}}\vec{p}_t\vec{e}_t(q_2). 
\]

\( \vec{v}\vec{u} \) denotes the Euclidean product. Choosing the x-axis as the direction of \( p_\perp \):

\[
\vec{p}_t = |p_\perp|\vec{e}_x,
\]

and introduce the basis in which the perpendicular parts of polarization vectors are either in x or y direction:

\[
\epsilon^\mu(q_1) = \epsilon^\mu_i + \frac{2|p_\perp|\delta_{ix}}{\alpha s}p^\mu, \\
\epsilon^\mu(q_2) = \epsilon^\mu_i - \frac{2|p_\perp|\delta_{ix}}{\bar{\alpha} s}p^\mu, \\
\epsilon^\mu(q) = \epsilon^\mu_i, 
\]

\( i = x, y \), and \( \delta \) denotes the Kronecker delta. Let me denote by \( \sigma_i \) the polarization of the photon \( i \).

In this real-valued basis \( \epsilon = \epsilon^* \). We obtain:

\[
q_1\epsilon(q) = -|p_\perp|\delta_{\sigma_1,x}, \\
q_2\epsilon(q) = |p_\perp|\delta_{\sigma_2,x}, \\
q\epsilon(q_1) = \frac{1}{\alpha}|p_\perp|\delta_{\sigma_1,x}, \\
q\epsilon(q_2) = -\frac{1}{\bar{\alpha}}|p_\perp|\delta_{\sigma_2,x}. 
\]

Product of two polarization vectors are \( \epsilon_i\epsilon_j = -\delta_{\sigma_i,\sigma_j} \).
At NLO, the summation over photons’ permutations does not yield such simplifications as it did at the leading order, so that it will be performed numerically. To make the formulae more readable and easier in numerical implementation, we will denote products of polarization vectors and considered momenta as

\[ \epsilon_1 k_2 = -\epsilon_1 k_3 = e k_1, \quad \epsilon_2 k_1 = -\epsilon_2 k_3 = e k_2, \quad \epsilon_3 k_1 = -\epsilon_3 k_2 = e k_3. \]  

(B.6)

Product of two polarization vectors \( i \) and \( j \) will be denoted as \( eij \).

We have shown, that the sum over permutations of \( Tr[i(M_{1,2,3}^{2L} + M_{1,2,3}^{2R} + M_{1,2,3}^{3L} + M_{1,2,3}^{3R})\tilde{\rho}] \) is proportional to the LO amplitude, so that one can easily sum that combination over all photons’ permutations to obtain

\[
\sum_{\text{perm.}} Tr[i(M_{1,2,3}^{2L} + M_{1,2,3}^{2R} + M_{1,2,3}^{3L} + M_{1,2,3}^{3R})\tilde{\rho}] = 
\]

\[
iC_F e_q^3 \frac{\alpha_s}{4\pi} s^{-2} \frac{1}{\alpha\alpha} \left[(\alpha - \bar{\alpha})(e^+_{\perp}(q_1)e^+_{\perp}(q_2)) (\bar{p}_\perp e^+_{\perp}(q_1))(\bar{q}(q)e^+_{\perp}(q_2)) - (\bar{p}_\perp e^+_{\perp}(q_2))(\bar{q}(q)e^+_{\perp}(q_1))\right] \times 
\]

\[
-\frac{i\pi}{\xi} \left(-10 + \log(\alpha\bar{\alpha})\right) \left(\delta(x - \xi) + \delta(x + \xi)\right) + i \text{Im} \frac{1}{x + \xi - i0} \frac{1}{x + \xi + i0} \log\left(-\frac{x + \xi - i0}{x + \xi + i0}\right) .
\]

(B.7)

\( Tr[iM_{1,2,3}^{3M}\tilde{\rho}] \) does no longer admit the previously used symmetry in interchange of photons 1 and 3, and it cannot be summed in such a simply manner. For a given permutation of photons one has

\[
Tr[iM_{1,2,3}^{3M}\tilde{\rho}]_{\text{fin.}} = -iC_F e_q^3 \frac{\alpha_s}{4\pi} s^{-2} \frac{1}{\beta_1 \beta_2} \frac{1}{x + \xi + i0} \frac{1}{x - \xi,i0} \times 
\]

\[
2s \left(\frac{(x + \xi)\beta_1}{(x + \xi)\beta_1 + (x - \xi)\beta_3}\right)^2 \left(\beta_1 e 13 \cdot ek2 - \beta_1 e 23 \cdot ek1 + \beta_3 e 12 \cdot ek3 + \beta_3 e 13 \cdot ek2\right) \times 
\]

\[
\left((2x + \xi)\beta_1 + (x - \xi)\beta_3\right) \log\left(-\frac{(x - \xi)\beta_3 + i0}{(x + \xi)\beta_1 + i0}\right) + (x + \xi)\beta_1 + (x - \xi)\beta_3 + 
\]

\[
-2s \left(\frac{(x - \xi)\beta_3}{(x + \xi)\beta_1 + (x - \xi)\beta_3}\right)^2 \left(\beta_1 e 13 \cdot ek2 + \beta_1 e 23 \cdot ek1 - \beta_3 e 12 \cdot ek3 + \beta_3 e 13 \cdot ek2\right) \times 
\]

\[
\left((x + \xi)\beta_1 + 2(x - \xi)\beta_3\right) \log\left(-\frac{(x + \xi)\beta_1 + i0}{(x - \xi)\beta_3 + i0}\right) + (x + \xi)\beta_1 + (x - \xi)\beta_3 + 
\]

\[
+ A_{1,2,3} \left(1 - \frac{(x - \xi)\beta_3 \log\left((-\xi) \beta_3 - i0\right) + (x + \xi)\beta_1 \log\left((-\xi) \beta_1 - i0\right)}{(x + \xi)\beta_1 + (x - \xi)\beta_3}\right) \right) .
\]

(B.8)

Graphs containing 4- and 5-point loops result in far more complicated integrals over Feynman parameters than the previously presented ones, so that these amplitudes will be written using
functions $\mathcal{F}$ and $\mathcal{G}$ defined in Section 3.6. Their exact form is presented at the end of this Appendix.

\[
Tr[i\mathcal{M}^{4L}_{1,2,3} \varphi]_{\text{fin.}} = -2i \frac{C_F e^3 g^2}{(4\pi)^2} \frac{1}{2p_{k_3} - i0} \left\{ \right.
\]

\[
- \mathcal{F}(2, 1, 0; x, \xi, \{\beta_i\})^{-1} \left[ \beta_3 (\kappa_3 s(e_{13} \cdot e_{k2} - 3e_{12} \cdot e_{k3}) + 4e_{k1} \cdot e_{k2} \cdot e_{k3}) + \right.
\]

\[
- s(e_{23} \cdot e_{k1} - e_{13} \cdot e_{k2})(\beta_1(\kappa_3 + 2\kappa_1) - 2\beta_2(\kappa_3 + \kappa_2)) +
\]

\[
+ \mathcal{F}(2, 0, 1; x, \xi, \{\beta_i\})(x + \xi) \left[ \beta_1^2 (e_{13} \cdot e_{k2} - e_{23} \cdot e_{k1}) + \beta_1 (2\beta_2 e_{13} \cdot e_{k2} - 2\beta_2 e_{23} \cdot e_{k1} + 
\]

\[
+ 3\beta_3 e_{12} \cdot e_{k3} - 9\beta_3 e_{13} \cdot e_{k2} + 4\beta_3 e_{23} \cdot e_{k1}) + 2\beta_2 \beta_3 (e_{12} \cdot e_{k3} - e_{13} \cdot e_{k2}) \right]
\]

\[
- \mathcal{F}(2, 1, 1; x, \xi, \{\beta_i\})(x + \xi) \left[ \beta_1 \beta_3 (e_{23} \cdot e_{k1} - e_{12} \cdot e_{k3}) + 2\beta_2^2 (e_{23} \cdot e_{k1} - e_{13} \cdot e_{k2}) + 
\]

\[
+ \beta_2 \beta_3 (-3e_{12} \cdot e_{k3} + 3e_{13} \cdot e_{k2} + 2e_{23} \cdot e_{k1}) \right] +
\]

\[
+ \mathcal{F}(2, 2, 0; x, \xi, \{\beta_i\})(x + \xi) \left[ \beta_2 \kappa_3 s(e_{23} \cdot e_{k1} - e_{13} \cdot e_{k2}) + 
\]

\[
+ \beta_3 (-e_{12} \cdot e_{k3} \cdot \kappa_3 s + e_{23} \cdot e_{k1} \cdot \kappa_3 s + 4e_{k1} \cdot e_{k2} \cdot e_{k3}) \right] +
\]

\[
+ \mathcal{F}(2, 2, 1; x, \xi, \{\beta_i\})(x + \xi) \beta_2 \left[ -\beta_2 e_{13} \cdot e_{k2} + \beta_2 e_{23} \cdot e_{k1} - \beta_3 e_{12} \cdot e_{k3} + \beta_3 e_{23} \cdot e_{k1} \right] \]

\[
+ \mathcal{F}(1, 0, 0; x, \xi, \{\beta_i\}) \left[ -\beta_1 e_{13} \cdot e_{k2} + \beta_1 e_{23} \cdot e_{k1} - 2\beta_2 e_{13} \cdot e_{k2} + 
\]

\[
+ 2\beta_2 e_{23} \cdot e_{k1} - 3\beta_3 e_{12} \cdot e_{k3} + 3\beta_3 e_{13} \cdot e_{k2} - 2\beta_3 e_{23} \cdot e_{k1} \right] +
\]

\[
+ \mathcal{F}(1, 1, 0; x, \xi, \{\beta_i\}) \left[ 3\beta_2 e_{13} \cdot e_{k2} - 3\beta_2 e_{23} \cdot e_{k1} + \beta_3 e_{12} \cdot e_{k3} - \beta_3 e_{23} \cdot e_{k1} \right] +
\]

\[
- 2\mathcal{G}(x, \xi, \{\beta_i\})(x + \xi) \beta_3 \left[ -\beta_1 e_{13} \cdot e_{k2} + \beta_1 e_{23} \cdot e_{k1} - \beta_3 e_{12} \cdot e_{k3} + \beta_3 e_{13} \cdot e_{k2} \right] \right\}.
\]
\[ Tr[iM^{5L}_1] = \frac{i}{4\pi^2} Cubes^2 \left\{ \int_0^1 dz \int_0^1 dr \left( az + br + czr \right)^{-2} \right\} \]

The \( R \) counter-part is obtained by substituting \( k_1 \rightarrow -k_3, k_2 \rightarrow -k_2, k_3 \rightarrow -k_1, \xi \rightarrow -\xi \).

Integrals used in the definition of functions \( F \) and \( G \) can be computed analytically and written in terms of logarithms and dilogarithms \( Li_2 \).

\[
\int_0^1 \frac{dz}{az + br + czr} = \frac{\log(a + b + c) - \log(a)}{b(b + c)}.
\] (B.11)

\[
\int_0^1 dr \int_0^1 dz \left( az + br + czr \right)^{-2} = \frac{1}{6e^2(a + c)(b + c)} \left[ -6(a + c)(b + c) \left( -\text{Li}_2 \left( -\frac{c(a + b + c)}{ab} \right) + \text{Li}_2 \left( -\frac{c}{a} \right) + \text{Li}_2 \left( \frac{b}{b + c} \right) \right) + (a + c)(b + c) \left( -3 \log^2 \left( \frac{b + c}{b} \right) + 6 \left( \log \left( \frac{c}{b} \right) - 1 \right) \log \left( \frac{b + c}{b} \right) + 6 \log(b + c) + \pi^2 \right) + 6(b + c) \log(b) \left( -(a + c) \log \left( \frac{a + c}{a} \right) - a \right) + 6(a + c) \log(a)(b + c)(\log(b) - \log(b + c)) + c) + 6 \left( (a + c)(b + c) \log \left( \frac{(a + c)(b + c)}{ab} \right) - c(a + b + 2c) \right) \log(a + b + c) \right].
\] (B.12)
\[
\begin{align*}
\int_0^1 dr \int_0^1 dz \ r^2 \left(az + br + czr\right)^{-2} &= \\
&= \frac{-ac^2 \log(a + b + c) + (b + c) \left(c^2 - a(b + c) \left( - \log(b + c) + \log \left( \frac{b+c}{b} \right) \right) + \log(b) \right) + ac^2 \log(a)}{bc^2(b + c)^2} \\
&= (B.13)
\end{align*}
\]

\[
\begin{align*}
\int_0^1 dr \int_0^1 dz \ r^2 \left(az + br + czr\right)^{-2} &= \\
&= \frac{2a}{c^3} \left( -\text{Li}_2 \left( -\frac{c(a + b + c)}{ab} \right) + \text{Li}_2 \left( -\frac{c}{a} \right) + \text{Li}_2 \left( \frac{b}{b+c} \right) \right) + \\
&\quad + \frac{1}{c^3(a + c)(b + c)^2} \left[ a(a + c)(b + c)^2 \log^2 \left( \frac{b + c}{b} \right) - 3a(a + c)(b + c)^2 \log(b + c) + \\
&\quad - a(a + c)(b + c)^2 \left( 2 \log \left( \frac{c}{b} \right) - 3 \right) \log \left( \frac{b + c}{b} \right) + \\
&\quad + \left( c \left( \frac{a^2(2b + 3c)}{c} + a \left( \frac{2b^2 + 6bc + 5c^2}{c} \right) + c(b + c)^2 \right) - 2a(a + c)(b + c)^2 \log \left( \frac{(a + c)(b + c)}{ab} \right) \right) \log(a + b + c) \right] \\
&\quad + \frac{1}{3c^3(a + c)(b + c)^2} \left[ 3(b + c)^2 \log(b) \left( 3a^2 + ac + 2a(a + c) \log \left( \frac{a + c}{a} \right) - c^2 \right) + \\
&\quad - (a + c)(b + c) \left( \pi^2 a(b + c) + 3c^2 \right) + 3a(a + c) \log(a) \left( -c(2b + 3c) - 2(b + c)^2 \left( \log(b) - \log(b + c) \right) \right) \right] \right] \\
&= (B.14)
\end{align*}
\]

\[
\begin{align*}
\int_0^1 dr \int_0^1 dz \ (az + br + czr)^{-1} &= \\
&= -\frac{1}{c} \left[ \log \left( \frac{b}{a} \right) \log \left( \frac{a + c}{a} \right) - \log \left( \frac{(a + c)(b + c)}{ab} \right) \log \left( \frac{a + b + c}{a} \right) + \\
&\quad \text{Li}_2 \left( -\frac{c}{a} \right) + \text{Li}_2 \left( -\frac{c}{b} \right) - \text{Li}_2 \left( -\frac{c(a + b + c)}{ab} \right) \right] \right]. \\
&= (B.15)
\end{align*}
\]

\[
\begin{align*}
\int_0^1 dr \int_0^1 dz \ r (az + br + czr)^{-1} &= \\
&= \frac{1}{c^2(b + c)} \left[ a(b + c) \left( -\text{Li}_2 \left( -\frac{c(a + b + c)}{ab} \right) + \text{Li}_2 \left( -\frac{c}{a} \right) + \text{Li}_2 \left( -\frac{c}{b} \right) \right) + \\
&\quad a \log \left( \frac{a}{b} \right) \left( (b + c) \log \left( \frac{b + c}{b} \right) - c \right) + \left( c(a + b + c) - a(b + c) \log \left( \frac{(a + c)(b + c)}{ab} \right) \right) \log \left( \frac{a + b + c}{b} \right) \right] \right] \right]. \\
&= (B.16)
\end{align*}
\]
\[ \int_0^1 dr \int_0^1 dz \left( az + br + czr \right)^{-1} \log \left( -az - br - czr \right) = \]
\[ \frac{1}{6ac(a + c)} \left[ 6(a + c) \left( -\text{Li}_2 \left( -\frac{c(a + b + c)}{ab} \right) \right) + \text{Li}_2 \left( -\frac{c}{a} \right) + \text{Li}_2 \left( \frac{b}{b + c} \right) \right] + \]
\[ 3c \log^2(-a - b - c) + 6(a + c) \log(-b) \log \left( \frac{a + c}{a} \right) + 6a \log(-a) \log \left( \frac{b + c}{b} \right) + \]
\[ 3 \log \left( \frac{b + c}{b} \right) \left( (a + c) \left( \log \left( \frac{b + c}{b} \right) - 2 \log \left( \frac{c}{b} \right) \right) + 2c \log(-a) \right) + \]
\[ -6(a + c) \log(-a - b - c) \log \left( \frac{(a + c)(b + c)}{ab} \right) + 6c \log(a + b + c) + \]
\[ -\pi^2(a + c) - 3c \log^2(-b) - 6c \log(b) \right]. \]
Bibliography

[1] O. Stern, R. Frisch, Über die magnetische Ablenkung von Wasserstoffmolekül en und das magnetische Moment des Protons, Zeitschrift für Physik 85 (1933),

[2] see e.g. E.E. Chambers and R. Hofstadter Structure of the proton, Phys. Rev. 103, 1454 (1956),

[3] H.W. Hammer, U.G. Meißner, The proton radius: From a puzzle to precision, Science Bulletin 65 (2020) pp. 257-258,

[4] M. Breidenbach et al. Observed Behavior of Highly Inelastic Electron-Proton Scattering, Phys. Rev. Lett. 23, 935 (1969),

[5] R.P. Feynman Photon-Hadron Interactions, W.A. Benjamin Inc., Reading Massachussets, (1972),

[6] G. Altarelli, G. Parisi, Asymptotic Freedom in Parton Language, Nuclear Physics B1 26, 298-318 (1977),

[7] S. Wallon, Hard exclusive processes in perturbative QCD: from medium to asymptotical energies, Doctoral Schools ED 107 (Physique de la Region Parisienne) and ED 517 (Particules, Noyaux et Cosmos) (2017),

[8] T. Muta, Foundations of Quantum Chromodynamics 3rd edition, World Scientific Lectures in Physics - Vol. 78 (2018), ISBN-13 978-981-279-353-9,

[9] J.C. Collins, D. E. Soper, G. Sterman, Factorization of Hard Processes in QCD, Adv. Ser. Direct. High Energy Phys.5:1-91 (1988),

[10] M.Gell-Mann, A schematic model of baryons and mesons, Reson 24, 923–925 (2019).

[11] G. Zweig, An SU(3) Model for Strong Interaction Symmetry and its Breaking, report number: CERN-TH-401 (1964).

[12] H.D. Politzer, Reliable Perturbative Results for Strong Interactions, Phys. Rev. Lett. 30 (1973), D. Gross and F. Wilczek, Ultraviolet Behavior of Non-Abelian Gauge Theories, Phys. Rev. Lett. 30, (1973),

[13] N. Chouika, Generalized Parton Distributions and their covariant extension : towards nucleon tomography, Université Paris-Saclay (2018). <NNT : 2018SACLS259>.
[14] M.V. Polyakov, P. Schweizer, *Forces inside hadrons: pressure, surface tension, mechanical radius, and all that*, Int.J.Mod.Phys. A33 (2018) 1830025, https://arxiv.org/abs/1805.06596.

[15] J. A. Osborne, *A study of nucleon spin structure from quantum chromodynamics*, Dissertation submitted to the Faculty of the Graduate School of The University of Maryland, 2000, https://arxiv.org/abs/hep-ph/0109255

[16] X. Ji, *Gauge-Invariant Decomposition of Nucleon Spin*, Phys.Rev.Lett. 78 (1997) 610-613.

[17] X. Ji, *Deeply Virtual Compton Scattering*, Phys.Rev.D55:7114-7125 (1997).

[18] E. R. Berger, M. Diehl, B. Pire, *Timelike Compton scattering: exclusive photoproduction of lepton pairs*, Eur.Phys.J.C23:675-689, 2002, https://arxiv.org/abs/hep-ph/0110062v1.

[19] S.V. Goloskokov, P. Kroll, *Vector Meson Electroproduction at small Bjorken-x and Generalized Parton Distributions*, Eur.Phys.J.C42:281-301 (2005).

[20] A. Pedrak, B. Pire, L. Szymanowski, and J. Wagner, *Hard photoproduction of a diphoton with a large invariant mass*, Phys.Rev. D96 (2017) no. 7.

[21] A. Pedrak, B. Pire, L. Szymanowski, and J. Wagner, *Erratum: Hard photoproduction of a diphoton with a large invariant mass*, Phys.Rev. D100 (2019) no.3, 039901.

[22] X. Ji, J. Osborne, *One-Loop Corrections and All Order Factorization In Deeply Virtual Compton Scattering*, Phys.Rev.D 58 (1998) 094018.

[23] O. Grocholski, H. Moutarde, B. Pire, P. Sznajder, J. Wagner, *Data-driven study of timelike Compton scattering*, European Physical Journal C 80(2) (2020).

[24] J. C. Collins, L. Frankfurt, M. Strikman, *Factorization for hard exclusive electroproduction of mesons in QCD*, Phys.Rev. D56 (1997) 2982-3006.

[25] L. Mankiewicz, G. Piller, *Comments on exclusive electroproduction of transversely polarized vector mesons*, Phys.Rev. D61 (2000) 074013.

[26] J.C. Collins, A. Freund, *Proof of factorization for deeply virtual Compton scattering in QCD*, Phys. Rev. D 59, 074009 (1999).

[27] M. Diehl, *Generalized Parton Distributions*, Phys.Rept.388:41-277, 2003, https://arxiv.org/abs/hep-ph/0307382v2.

[28] M. Diehl, T. Gousset, *Time ordering in off-diagonal parton distributions*, Phys.Lett. B428 (1998) 359-370.

[29] N. Kivel, L. Mankiewicz, *Conformal string operators and evolution of skewed parton distributions*, Nucl.Phys. B557 (1999) 271-295.

[30] I.I. Balitsky, V.M. Braun, *Evolution equations for QCD string operators*, Nuclear Physics B311 (1988/89) 541-584.
[31] B. Pire, L. Szymanowski, and J. Wagner, *NLO corrections to timelike, spacelike and double deeply virtual Compton scattering*, Phys.Rev.D83:034009 (2011).

[32] A. V. Efremov, A. V. Radyushkin, *Hard Processes, Patton Model and QCD.*, Riv. Nuovo Cim. 3, 1 (1980).

[33] P. Chankowski, *Green’s functions and the LSZ prescription*, lecture notes https://www.fuw.edu.pl/chank/qft15.pdf, accessed: 04.07.2021.

[34] L.D. Landau, *On analytic properties of vertex parts in quantum field theory*, Nucl. Phys. 13 (1959) 181,

[35] A.V. Belitsky, A.V. Radyushkin, *Unraveling hadron structure with generalized parton distributions*, Physics Reports 418 (2005) 1–387.

[36] PARTONS webpage: http://partons.cea.fr/partons/doc/html/index.html, accessed: 05.07.2021.