GLOBAL STABILITY OF STEADY TRANSONIC EULER SHOCKS IN QUASI-ONE-DIMENSIONAL NOZZLES

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Abstract. We prove global in time dynamical stability of steady transonic shock solutions in divergent quasi-one-dimensional nozzles. We assume neither the smallness of the relative slope of the nozzle nor the weakness of the shock. Key ingredients of the proof are an exponentially decaying energy estimate for a linearized problem together with methods from [12].

1. Introduction and Main Results

Compressible isentropic Euler flows in quasi-one-dimensional nozzles are governed by

\[
\begin{align*}
\rho_t + (\rho u)_x &= -\frac{a'(x)}{a(x)} \rho u, \\
(\rho u)_t + (\rho u^2 + p(\rho))_x &= -\frac{a'(x)}{a(x)} \rho u^2,
\end{align*}
\]

where \(\rho, u\) and \(p\) denote the density, velocity and pressure, and, \(a(x)\) is the cross-sectional area of the nozzle. The typical examples are 1D, 2D rotationally symmetric, and 3D spherical symmetric compressible Euler equations, where \(a(x)\) are 1, \(x\) and \(x^2\), respectively.

We assume that \(p\) satisfies

\[\forall \rho > 0, \quad p(\rho) > 0, \quad p'(\rho) > 0, \quad \text{and}, \quad p''(\rho) \geq 0.\]

Examples are \(p(\rho) := A\rho^\gamma\) with \(\gamma \geq 1\), in particular the polytropic (\(\gamma > 1\)) and isothermal (\(\gamma = 1\)) gases. The sound speed is \(c(\rho) := \sqrt{p'(\rho)}\). A state \(\rho, u\) at which the flow speed is larger than the sound speed, \(|u| > \sqrt{p'(\rho)}\), is called supersonic. The opposite case, \(|u| < \sqrt{p'(\rho)}\), is subsonic.

For steady states, (1) becomes the system of ordinary differential equations

\[
\begin{align*}
(\rho u)_x &= -\frac{a'(x)}{a(x)} \rho u, \\
(\rho u^2 + p(\rho))_x &= -\frac{a'(x)}{a(x)} \rho u^2.
\end{align*}
\]

studied intensively in [1, 9]. Some properties needed later are the following.
The first equation in (2), \((aρu)_x = 0\), holds if and only if
\[\exists M ∈ \mathbb{R}, \quad aρu = M.\] (3)

After a possible reflection we may assume without loss of generality that \(M > 0\). The flow is from left to right. Using (3) to eliminate \(u\) from the second equation in (2) yields a scalar equation for \(ρ\),
\[\left(\frac{M^2}{aρ}\right)_x + ap(ρ)_x = 0.\] (4)

**Definition 1.** A steady transonic shock solution is a piecewise smooth solution of (4) with two smooth solutions separated by a shock connecting a supersonic state on the left to a subsonic state on the right.

The steady transonic shock solutions of (2) on \([l, L]\) are the piecewise smooth solutions
\[(\bar{ρ}, \bar{u}) = \begin{cases} (ar{ρ}_-, \bar{M}/(a\bar{ρ}_-)), & \text{if } l < x < x_0, \\ (ar{ρ}_+, \bar{M}/(a\bar{ρ}_+)), & \text{if } x_0 < x < L, \end{cases}\] (5)
satisfying the Rankine-Hugoniot condition
\[\left(p(\bar{ρ}_-) + \frac{M}{(a\bar{ρ}_-)^2}\right)(x_0-) = \left(p(\bar{ρ}_+) + \frac{M}{(a\bar{ρ}_+)^2}\right)(x_0+),\] (6)
with \(\bar{ρ}_-\) is supersonic \((p'(\bar{ρ}_-) < \frac{M^2}{(a\bar{ρ}_-)^2})\) and \(\bar{ρ}_+\) is subsonic \((p'(\bar{ρ}_+) < \frac{M^2}{(a\bar{ρ}_+)^2})\). This implies the entropy condition
\[\bar{ρ}_-(x_0) < \bar{ρ}_+(x_0).\] (7)

Steady transonic shock solutions were constructed in \([1, 9]\) satisfying the boundary conditions
\[(ρ, u)(l) = (ρ_l, u_l) = (ρ_l, \frac{\bar{M}}{a(l)ρ_l}), \quad \text{and,} \quad ρ(L) = ρ_r,\] (8)
with \((ρ_l, u_l)\) and \((ρ_r, u_r)\) supersonic and subsonic respectively. There have been many studies of the stability of steady transonic shocks for (1) (see [10]). In [10] a wave front tracking variant of Glimm’s scheme was used to prove that when \(|a'(x)/a(x)|\) is small, a weak transonic shock is dynamically stable if \(a'(x_0) > 0\) and dynamically unstable if \(a'(x_0) < 0\). Studies on the solutions of general hyperbolic conservation laws with source terms include [9] [11] [8] [3] [4] and references therein. For piecewise smooth initial data, in the case that \(a(x) = x\) or \(a(x) = x^2\), Xin and Yin [18] proved dynamical stability of weak shocks in nozzles with large \(l\) (thus \(|a'(x)/a(x)|\) small) in the class of piecewise smooth transonic shock solutions. The main result of this paper proves global stability in time without these restrictions.
Let \((\bar{\rho}, \bar{u})\) be a steady transonic shock solution of the form (5) satisfying the boundary conditions (8). Suppose that the solution stays away from vacuum,
\[
\inf_{x \in [l, L]} \bar{\rho}(x) > 0.
\] (9)
Solving an initial value problem for (2) we can extend \((\bar{\rho} - \bar{u}, \bar{u} - \bar{u})\) to be a smooth supersonic solution to (2) on \([l, x_0 + \delta]\) for some \(\delta > 0\), that coincides with \((\bar{\rho} - \bar{u}, \bar{u} - \bar{u})\) on \([l, x_0]\). The notation \((\bar{\rho} - \bar{u}, \bar{u} - \bar{u})\) is used for the extended solution as well. Similarly, denote by \((\bar{\rho} + \bar{u}, \bar{u} + \bar{u})\) a subsonic solution of (2) on \([x_0 - \delta, L]\) for some \(\delta > 0\), that coincides with \((\bar{\rho} + \bar{u}, \bar{u} + \bar{u})\) on \([x_0, L]\).

Consider the mixed initial boundary value problem defined by (1) with initial data,
\[
(\rho(u))(0, x) = (\rho_0(u_0))(x),
\] (10)
and boundary conditions
\[
(\rho(u))(t, l) = (\rho_0, \frac{M}{a(\rho_l)}), \quad \rho(t, L) = \rho_r,
\] (11)
with \(\rho_l\) and \(\rho_r\) from (8).

Assume that the initial data satisfy
\[
(\rho_0(u_0))(x) = \begin{cases} 
(\rho_{0-}, u_{0-}))(x), & \text{if } l < x < \bar{x}_0, \\
(\rho_{0+}, u_{0+}))(x), & \text{if } \bar{x}_0 < x < L,
\end{cases}
\] (12)
These data are small perturbations of \((\bar{\rho}, \bar{u})\) in the sense that
\[
|x_0 - \bar{x}_0| + \|(\rho_{0+}, u_{0+}) - (\bar{\rho}_+, \bar{u}_+))\|_{H^{k+2}([\bar{x}_0, L])} \\
+ \|(\rho_{0-}, u_{0-}) - (\bar{\rho}_-, \bar{u}_-)\|_{H^{k+2}([l, \bar{x}_0])} < \varepsilon,
\] (13)
for some small \(\varepsilon > 0\), and some integer \(k \geq 15\), where \(\bar{x}_0 = \min\{x_0, \bar{x}_0\}\) and \(\bar{x}_0 = \max\{x_0, \bar{x}_0\}\). Moreover, \((\rho_0, u_0)\) is assumed to satisfy the Rankine-Hogoniot condition. That is, at the shock location \(x = \bar{x}_0\),
\[
((p(\rho_{0+}) + \rho_{0+}u_{0+}^2 - (p(\rho_{0-}) + \rho_{0-}u_{0-}^2) (\rho_{0+} - \rho_{0-})) = (\rho_{0+}u_{0+} - \rho_{0-}u_{0-})^2.
\] (14)

Our dynamical stability theorem is the following.

**Theorem 2.** Let \((\bar{\rho}, \bar{u})\) be a steady transonic shock solution to system (2) satisfying (5), (7), (8), and (9). The key hypothesis is that the nozzle is widening at the shock location,
\[
a'(x_0) > 0.
\] (15)
There exists an \( \varepsilon_0 > 0 \) so that for any \( 0 < \varepsilon \leq \varepsilon_0 \), if the initial data \( (\rho_0, u_0) \) satisfy (13), (14) and the compatibility conditions at \( x = \bar{x}_0 \) and \( x = L \) or order \((k+2)\), then the initial boundary value problem (1), (10) and (11) has a unique piecewise smooth solution \( (\rho, u)(x,t) \) for \( (x,t) \in [l,L] \times [0,\infty) \) containing a single transonic shock \( x = s(t) \) with \( s(0) = \bar{x}_0 \) and \( l < s(t) < L \) satisfying the pair of Rankin-Hougoniot conditions for \( t \geq 0 \),

\[
(p(\rho) + pu^2)(t, s(t+)) - (p(\rho) + pu^2)(t, s(t-)) = (\rho u(t, s(t+)) - \rho u(t, s(t-))) \dot{s}(t),
\]

\[
\rho u(t, s(t+)) - \rho u(t, s(t-)) = (\rho(t, s(t+)) - \rho(t, s(t-))) \dot{s}(t),
\]

and the Lax geometric shock conditions,

\[
(u-\sqrt{p'(\rho)})(t, s(t-)) > \dot{s}(t) > (u-\sqrt{p'(\rho)})(t, s(t+)), \quad (u+\sqrt{p'(\rho)})(t, s(t+)) > \dot{s}(t).
\]

Denote

\[
(\rho, u) = \begin{cases} 
(\rho_-, u_-), & \text{if } l < x < s(t), \\
(\rho_+, u_+), & \text{if } s(t) < x < L.
\end{cases}
\]

There is a \( T_0 > 0 \) so that for \( t > T_0 \),

\[
(\rho_-, u_+)(t, x) = (\bar{\rho}_-, \bar{u}_-)(x), \quad \text{for } l \leq x < s(t).
\]

The solution approaches the steady transonic flow at an exponential rate, that is, there exist positive constants \( C > 0 \) and \( \lambda > 0 \) independent of the solution so that for \( t > 0 \)

\[
\|(\rho_+, u_+)(\cdot, t) - (\bar{\rho}_+, \bar{u}_+)(\cdot)\|_{W^{k-7,\infty}(s(t),L)} + \sum_{m=0}^{k-6} |\partial_t^m(s(t) - x_0)| \leq C \varepsilon e^{-\lambda t},
\]

where \( (\bar{\rho}_\pm, \bar{u}_\pm) \) is the unperturbed solution.

**Remark 1.** The results in Theorem 2 are also true if we impose small perturbations in the boundary conditions (8).

**Remark 2.** The regularity assumption in (13) is not optimal. Adapting the paradifferential methods from [14], one can decrease the regularity required in (13).

**Remark 3.** Compatibility conditions for the initial boundary value problems for hyperbolic equations are discussed in detail in [16, 13, 14].

**Remark 4.** We require neither the smallness of \(|a'/a|\), largeness of \( l \), nor the weakness of the shock strength as in [18]. Similar results for steady transonic shock solutions for the Euler-Poisson equations were proved in [12]. The strategy of the proof of Theorem 1 is inspired by [12].
Remark 5. If the condition (13) is violated, i.e. \( \alpha'(x_0) < 0 \), then the transonic shock solution is unstable, see [10][18].

The paper is organized as follows. Section 2 transforms the problem to free boundary problem for a second order scalar hyperbolic partial differential equation. The weighted energy equality for the associated linearized problem is derived in §3. This yields Theorem 1 with the aid of ideas from [12].

2. Transformation of the Problem

Let \((\bar{\rho}, \bar{u})\) be a steady transonic shock solution of the form (2) satisfying (5), (6), (8), and (9). If the initial data \((\rho_0, u_0)\) satisfies (13) and the compatibility condition of order \(k+2\). It follows from the argument in [7] that there exists a piecewise smooth solution of the equations on \([0, \bar{T}]\) for some \(\bar{T} > 0\), which can be written as

\[
(\rho, u) = \begin{cases} 
(\rho_-, u_-), & \text{if } l < x < s(t), \\
(\rho_+, u_+), & \text{if } s(t) < x < L.
\end{cases}
\] (16)

Since the flow is supersonic on the left it follows that there is a \(T_0 > 0\) so that, when \(t > T_0\), \((\rho_-, u_-)\) depends only on the boundary conditions at \(x = l\). Moreover, when \(\varepsilon\) is small, by the standard lifespan argument, we have \(T_0 < \bar{T}\) (cf. [7]). Therefore,

\[
(\rho_-, u_-) = (\bar{\rho}_-, \bar{u}_-) \text{ for } t > T_0.
\] (17)

It suffices to study the solution for \(T_0 > 0\) so without loss of generality we suppose that \(T_0 = 0\). We want to extend the local-in-time solution to all \(t > 0\). In view of (17), it suffices to obtain uniform estimates in the region \(x > s(t), t > 0\). We formulate an initial boundary value problem in this region. First, the Rankine-Hugoniot conditions for (16) read

\[
[\rho u] = [\rho]s'(t), \quad [\rho u^2 + p] = [\rho u]s'(t),
\] (18)

using standard notation for the jump \([f] := f(s(t+), t) - f(s(t-), t)\), so

\[
[p + \rho u^2] [\rho] = [\rho u]^2.
\]
Denote \( M := a \rho u \). Then
\[
\left( p(\rho_+)(t, s(t)) + \frac{M_+^2(t, s(t))}{a^2(s(t))\rho_+(s(t), t)} - p(\bar{\rho}_+)(s(t)) - \frac{M_+^2}{a^2(\bar{\rho}_+)}(s(t)) \right) + p(\bar{\rho}_+)(s(t)) + \frac{M_+^2}{a^2\bar{\rho}_+}(s(t)) - p(\bar{\rho}_-)(s(t)) - \frac{M_+^2}{a^2\bar{\rho}_-}(s(t)) \right) \left( \rho_+(t, s(t)) - \rho_-(s(t)) \right)
= \left( \frac{M_+(t, s(t)) - M_+(s(t))}{a(s(t))} \right)^2.
\]

The implicit function theorem and the momentum equation in (2) imply,
\[
(M_+ - 2M_+)(t, s(t)) = \mathcal{A}_1((\rho_+ - \bar{\rho}_+)(t, s(t)), s(t) - x_0)
\]
where \( \mathcal{A}_1 \) regarded as a function of two variables satisfies \( \mathcal{A}_1(0, 0) = 0 \) and
\[
\left. \frac{\partial \mathcal{A}_1}{\partial (\rho_+ - \bar{\rho}_+)} \right|_{(0,0)} = -\frac{a(p'(\bar{\rho}_+) - \bar{u}_+^2)}{2\bar{u}_+}(x_0), \quad \left. \frac{\partial \mathcal{A}_1}{\partial (s - x)} \right|_{(0,0)} = -\frac{a'\bar{u}_-(\rho_+ - \bar{\rho}_-)}{2}(x_0).
\]

Note that \( \bar{M}_+ = \bar{M}_- = \bar{M} \). And so substituting (19) into the first equation in [18] yields,
\[
\psi'(t) = \mathcal{A}_2(\rho_+ - \bar{\rho}_+, s(t) - x_0)
\]
where \( \mathcal{A}_2 \) satisfies \( \mathcal{A}_2(0, 0) = 0 \) and
\[
\left. \frac{\partial \mathcal{A}_2}{\partial (\rho_+ - \bar{\rho}_+)} \right|_{(0,0)} = -\frac{p'(\bar{\rho}_+) - \bar{u}_+^2}{2\bar{u}_+(\rho_+ - \bar{\rho}_-)}(x_0), \quad \left. \frac{\partial \mathcal{A}_2}{\partial (s - x)} \right|_{(0,0)} = -\frac{a'\bar{u}_-(x_0)}{2a}(x_0).
\]

Define
\[
\psi_-(t, x) := -\bar{M}t + \int_t^x a(y)\rho_-(y)dy
\]
and
\[
\tilde{\psi}_+(t, x) := \psi_-(t, x_0) + \int_{x_0}^x a(y)\bar{\rho}_+(y)dy, \quad \psi_+(t, x) = \psi_-(t, s(t)) + \int_{s(t)}^x a(y)\rho_+(y, t)dy.
\]

Set
\[
\Psi := \psi_+(x, t) - \tilde{\psi}_+(t, x).
\]

Then
\[
\Psi_t = \bar{M} - M_+, \quad \Psi_x = a\rho_+ - a\bar{\rho}_+.
\]

The second equation in (1) is equivalent to
\[
-(\psi_+ + \Psi)_{tt} + \partial_x \left( \frac{\partial \psi_+ + \partial_t \Psi}{\partial_x \psi_+ + \partial_x \Psi} \right)^2 + a(x)\partial_x p \left( \frac{\partial \psi_+ + \partial_t \Psi}{a(x)} \right) = 0.
\]
Write this equation as follows. Setting $\xi = (\xi_0, \xi_1) = (t, x)$,
\[ \sum_{ij} a_{ij}(x, \Psi_t, \Psi_x) \partial_{ij} \Psi + \sum_i b_i(x, \Psi_t, \Psi_x) \partial_i \Psi = 0, \tag{21} \]
where $a_{ij}, b_i$ and $c$ are smooth with respect to each variable, and satisfy (using the Einstein summation convention),
\[ \mathcal{L}_0 \Psi = \bar{a}_{ij}(x, 0, 0) \partial_{ij} \Psi + \bar{b}_i(x, 0, 0) \partial_i \Psi + c\bar{y}(x, 0, 0) \Psi = \partial_t \Psi + 2\bar{u} \partial_x \Psi + (\bar{u}^2 - p'(\bar{\rho}^+)) \partial_{xx} \Psi + \partial_x \bar{\psi}_+ + \partial_x \psi_1 + B \partial_x \Psi \]
with
\[ B := \frac{\alpha'}{\alpha} p'(\bar{\rho}^+) - \partial_x p'(\bar{\rho}^+) + \partial_x \bar{u}_+^2. \]

In terms of $\Psi$, the equations (19) and (20) can be written as
\[ \Psi_t(t, s(t)) = -\mathcal{A}_1(\frac{\Psi_x(t, s(t))}{a(s(t))}, s(t) - x_0), \tag{22} \]
and
\[ s' = \mathcal{A}_2(\frac{\Psi_x(t, s(t))}{a(s(t))}, s(t) - x_0), \tag{23} \]
respectively. A direct computation yields
\[ \Psi(t, s(t)) = \psi_+(t, s(t)) - \bar{\psi}_+(t, s(t)) = \psi_-(t, s(t)) - \bar{\psi}_-(t, s(t)) \]
\[ = \psi_-(t, x_0) + \psi_-(t, s(t)) - \psi_-(t, x_0) - \bar{\psi}_+(t, x_0) - \bar{\psi}_+(t, s(t)) + \bar{\psi}_+(t, x_0) \]
\[ = \partial_x \psi_-(t, x_0) \cdot (s(t) - x_0) - \partial_x \bar{\psi}_+(t, x_0) \cdot (s(t) - x_0) + R_1, \]
where $R_1$ is quadratic in $s(t) - x_0$. This implies
\[ s(t) - x_0 = \mathcal{A}_3(\Psi(s(t), t)), \tag{24} \]
where $\mathcal{A}_3$ satisfies $\mathcal{A}_3(0) = 0$ and
\[ \left. \frac{\partial \mathcal{A}_3}{\partial \Psi} \right|_{\Psi=0} = \frac{1}{a(\bar{\rho}^- - \bar{\rho}^+)}(x_0). \]
It follows from (22), (23) and (24) that
\[ \partial_t \Psi = \mathcal{A}_4(\Psi_x, \Psi), \quad \text{at} \quad x = s(t), \tag{25} \]
where
\[ \mathcal{A}_4(0, 0) = 0, \quad \frac{\partial \mathcal{A}_4}{\partial \Psi_x} = \frac{c^2(\bar{\rho}^+) - \bar{u}_+^2}{2\bar{u}_+}(x_0), \quad \frac{\partial \mathcal{A}_4}{\partial \Psi} = -\frac{a'u_-}{2a}(x_0). \]

On the right boundary, $x = L$, $\Psi$ satisfies
\[ \partial_x \Psi = 0, \quad \text{at} \quad x = L. \tag{26} \]
It suffices to derive uniform estimates for $\Psi$ and $s$ that satisfy (21), (24), (25) and (26).

To this end, first transform this free boundary value problem into a fixed boundary value problem. Set

$$\tilde{t} := t, \quad \tilde{x} := (L - x_0) \frac{x - s(t)}{L - s(t)} + x_0, \quad \sigma(\tilde{t}) := s(t) - x_0,$$

and

$$q_1(\tilde{x}, \sigma) := \frac{L - \tilde{x}}{L - x_0 - \sigma(t)}, \quad q_2(\sigma) = \frac{L - x_0}{L - x_0 - \sigma(t)}. \quad (27)$$

Then

$$\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial \tilde{t}^2} - \sigma'(\tilde{t})q_1 \frac{\partial}{\partial \tilde{x}}, \quad \frac{\partial}{\partial \tilde{x}} = q_2 \frac{\partial}{\partial \tilde{x}},$$

$$\frac{\partial^2}{\partial \tilde{x}^2} = \frac{\partial^2}{\partial \tilde{t}^2} + (q_1 \sigma'(\tilde{t}))^2 \frac{\partial^2}{\partial \tilde{x}^2} - 2q_1 \sigma'(\tilde{t}) \frac{\partial^2}{\partial \tilde{x} \partial \tilde{t}} - q_1 \left( \sigma''(\tilde{t}) + \frac{(\sigma'(\tilde{t}))^2}{L - x_0 - \sigma(t)} \right) \frac{\partial}{\partial \tilde{x}},$$

$$\frac{\partial^2}{\partial x \partial t} = q_2 \left( \frac{\partial^2}{\partial \tilde{x} \partial \tilde{t}} + \frac{\sigma'(\tilde{t})}{L - x_0 - \sigma(t)} \frac{\partial}{\partial \tilde{x}} - q_1 \sigma'(\tilde{t}) \frac{\partial^2}{\partial \tilde{x}^2} \right), \quad \frac{\partial^2}{\partial x^2} = q_2^2 \frac{\partial^2}{\partial \tilde{x}^2}.$$ 

So (21) becomes

$$\partial_{\tilde{t}} \Psi + q_2 \partial_{\tilde{x}} \left( \frac{(-\tilde{M} + \partial_{\tilde{t}} \Psi - \sigma'(\tilde{t})q_2 \partial_{\tilde{x}} \Psi)^2}{a \tilde{\rho}_+ + q_2 \partial_{\tilde{x}} \Psi} \right) + a(x)q_2 \partial_{\tilde{x}} p \left( \frac{a \tilde{\rho}_+ + q_2 \Psi_{\tilde{x}}}{a(x)} \right)$$

$$- 2\sigma'(t)q_1 \partial_{\tilde{t}} \Psi + (q_1 \sigma'(t))^2 \partial_{\tilde{x}} \Psi - 2 \frac{(\sigma'(\tilde{t}))^2}{L - x_0 - \sigma(t)} q_1 \partial_{\tilde{x}} \Psi = \sigma''(\tilde{t})q_1 \partial_{\tilde{x}} \Psi.$$ 

Equation (24) takes the form

$$\sigma = \mathcal{A}_3(\Psi(t, \tilde{x} = x_0)),$$

and the equation for the shock front, (23), becomes

$$\frac{d\sigma}{dt} = \mathcal{A}_2 \left( \frac{q_2(\sigma) \Psi_{\tilde{x}}}{\sqrt{a}}, \sigma(\tilde{t}) \right).$$

Using (24) to represent the quadratic terms for $\sigma$ in terms of $\Psi$, we have, at $\tilde{x} = x_0$,

$$\frac{d\sigma}{dt} + \frac{a' \tilde{u}_-}{2a}(x_0)\sigma = \mathcal{C}_2(\Psi_{\tilde{x}}, \Psi), \quad (28)$$

where $\mathcal{C}_2$ satisfies

$$\left| \mathcal{C}_2(\Psi_{\tilde{x}}, \Psi) + \frac{c^2(\bar{\rho}_+) - \tilde{u}_+^2}{2(\bar{\rho}_+ - \bar{\rho}_-)\tilde{u}_+ a}(x_0)\Psi_{\tilde{x}} \right| \leq C(\Psi_{\tilde{x}}^2 + \Psi^2).$$

It follows from (24) and (28) that one can represent $\sigma$ and $\sigma'$ in terms of $\Psi$ and its derivatives at $\tilde{x} = x_0$. Thus, after manipulating (25) with (24) and (28), one finds

$$\Psi_{\tilde{t}} = \mathcal{C}_1(\Psi_{\tilde{x}}, \Psi), \quad \text{at} \quad \tilde{x} = x_0.$$
By the implicit function theorem this is equivalent to
\[ \Psi_{\tilde{x}} = \mathcal{G}_3(\Psi_{\tilde{t}}, \Psi), \quad \text{at} \quad \tilde{x} = x_0, \]
where \( \mathcal{G}_3 \) satisfies
\[ \left| \mathcal{G}_3(\Psi_{\tilde{t}}, \Psi) - \frac{2\bar{u}_+}{c^2(\bar{\rho}_+) - \bar{u}_+^2}(x_0)\Psi_{\tilde{t}} - \frac{a'\bar{u}_-\bar{u}_+}{(c^2(\bar{\rho}_+) - \bar{u}_+^2)a}(x_0)\Psi \right| \leq C(\Psi_{\tilde{t}}^2 + \Psi^2). \]

In the following, drop the \( \tilde{\cdot} \) in \( \tilde{x} \) and \( \tilde{\cdot} \) for ease of reading.

In summary, the problem has been transformed to the following compact form
\[
\begin{cases}
\mathcal{L}(x, \Psi, \sigma) \Psi = \sigma''(t)q_1\partial_x \Psi, \quad (t, x) \in [0, \infty) \times [x_0, L], \\
\partial_x \Psi = d_1(\Psi_{\tilde{t}}, \Psi)\Psi_{\tilde{t}} + e_1(\Psi_{\tilde{t}}, \Psi)\Psi, \quad \text{at} \quad x = x_0, \\
\partial_x \Psi = 0, \quad \text{at} \quad x = L, \\
\sigma(t) = \mathcal{A}_3(\Psi(t, x_0)), \quad \sigma(0) = \sigma_0,
\end{cases}
\] (29)

where,
\[
\mathcal{L}(x, \Psi, \sigma) \Phi := \sum_{i,j=0}^1 a_{ij}(x, \nabla \Psi, \sigma, \sigma') \partial_{ij} \Phi + \sum_{i=0}^1 b_i(x, \nabla \Psi, \sigma, \sigma') \partial_i \Phi,
\]
with
\[
d_1(\Psi_{\tilde{t}}, \Psi) := \int_0^1 \frac{\partial \mathcal{G}_3}{\partial \Psi_{\tilde{t}}}(\theta \Psi_{\tilde{t}}, \theta \Psi)d\theta, \quad e_1(\Psi_{\tilde{t}}, \Psi) := \int_0^1 \frac{\partial \mathcal{G}_3}{\partial \Psi}(\theta \Psi_{\tilde{t}}, \theta \Psi)d\theta.
\]

Furthermore, one has
\[
\mathcal{L}(x, 0, 0) \Phi = \mathcal{L}_0 \Phi
\]
and
\[
\begin{align*}
a_{00}(x, \nabla \Psi, \sigma, \sigma') &= 1, \quad a_{01}(x, 0, 0, 0) = a_{10}(x, 0, 0, 0) = \bar{u}_+, \\
a_{11}(x, 0, 0, 0) &= -(p'(\bar{\rho}_+) - \bar{u}_+^2), \\
b_0(x, 0, 0, 0) &= 2\partial_x \bar{u}_+, \quad b_1(x, 0, 0, 0) = B, \\
d_1(0, 0) &= \frac{2\bar{u}_+}{c^2(\bar{\rho}_+) - \bar{u}_+^2}(x_0), \quad e_1(0, 0) = \frac{a'(x_0)\bar{u}_+\bar{u}_-}{(c^2(\bar{\rho}_+) - \bar{u}_+^2)a}(x_0).
\end{align*}
\]

3. PROOF OF THE MAIN THEOREM

The key first step in the proof of Theorem 2 is the exponential decay of solutions of a linearized problem. It yields in non trivial manner \textit{a priori} estimates for the nonlinear problem which lead to the global stability of the steady shock.
3.1. **Linear Estimate.** The first step is the following energy identity for the linearization at the steady shock.

**Lemma 3.** Assume that \( a \) satisfies (15). Let \( \Psi \) be a smooth solution of the linearized problem

\[
\begin{aligned}
\mathcal{L}_0 \Psi &= 0, \quad x_0 < x < L, \quad t > 0, \\
\partial_x \Psi &= \frac{2\bar{u}_+}{c^2(\bar{\rho}_+) - \bar{u}_+^2} (x_0) \partial_t \Psi + \frac{a'(x_0)\bar{u}_-}{c^2(\bar{\rho}_+) - \bar{u}_+^2} a(x_0) \Psi, \quad \text{at } x = x_0, \\
\partial_x \Psi &= 0, \quad \text{at } x = L, \\
\Psi(0, x) &= h_1(x), \quad \Psi_t(0, x) = h_2(x), \quad x_0 < x < L.
\end{aligned}
\]  

(30)

Then the following dissipation identity holds,

\[
E(\Psi, t) = E(\Psi, 0) - D(\Psi, t),
\]

(31)

where \( E \) and \( D \) are defined as,

\[
E(\Psi, t) := \left( \frac{a' \bar{u}_+^2 \bar{u}_- \Psi^2}{a} \right)(t, x_0) + \int_{x_0}^{L} \bar{u}_+ \left\{ (\partial_t \Psi)^2 + (\bar{p}'(\bar{\rho}_+) - \bar{u}_+^2)(\partial_x \Psi)^2 \right\} (t, x) \, dx, \\
D(\Psi, t) := 2 \int_{0}^{t} \bar{u}_+^2(x_0)(\partial_t \Psi)^2(\tau, x_0) + \bar{u}_+^2(L)(\partial_t \Psi)^2(\tau, L) \, d\tau.
\]

**Proof:** Multiplying the first equation in (30) by \( \bar{u}_+(x) \partial_t \Psi \) and integrating by parts yields

\[
\int_{0}^{t} \int_{x_0}^{L} \mathcal{L}_0 \Psi \cdot \bar{u}_+ \partial_t \Psi \, dx \, d\tau = \int_{x_0}^{L} \frac{\bar{u}_+ (\partial_t \Psi)^2 + \bar{u}_+ (c^2(\bar{\rho}_+) - \bar{u}_+^2)(\partial_x \Psi)^2}{2} (t, x) \, dx
\]

\[
+ \int_{0}^{t} \int_{x_0}^{L} [B\bar{u}_+ + \partial_x(\bar{u}_+(c^2(\bar{\rho}_+) - \bar{u}_+^2))](\partial_t \Psi) \partial_x \Psi \, dx \, d\tau
\]

\[
+ \int_{0}^{t} \left( \bar{u}_+^2(\partial_t \Psi)^2 + \bar{u}_+ (c^2(\bar{\rho}_+) - \bar{u}_+^2)(\partial_x \Psi)^2 \right) \bigg|_{x=x_0}^{x=L} d\tau
\]

\[
- \int_{x_0}^{L} \frac{\bar{u}_+(\partial_t \Psi)^2 + \bar{u}_+(c^2(\bar{\rho}_+) - \bar{u}_+^2)(\partial_x \Psi)^2}{2} (0, x) \, dx.
\]

(32)
Note that
\[ B\bar{u}_+ + \partial_x (\bar{u}_+ (x) (c^2 (\bar{\rho}_+) - \bar{u}_+^2)) = B\bar{u}_+ + \bar{u}_+ (x) \partial_x (c^2 - \bar{u}_+^2) + \partial_x \bar{u}_+ (x) (c^2 - \bar{u}_+^2) \]
\[ = \bar{u}_+ \left[ \frac{d'}{a} p'(\bar{\rho}_+) - p''(\bar{\rho}_+) \partial_x \bar{\rho}_+ + \partial_x \left( \frac{M^2}{(\partial_x \bar{\psi}_+)^2} \right) \right] + \bar{u}_+ \partial_x \left( \frac{M^2}{(\partial_x \bar{\psi}_+)^2} \right) + \partial_x \bar{u}_+ (p'(\rho) - \bar{u}_+^2) \]
\[ = \bar{u}_+ \frac{d'}{a} p'(\bar{\rho}_+) + \frac{d}{dx} \left( \frac{M}{a \bar{\rho}_+} \right) (p'(\bar{\rho}_+) - \bar{u}_+^2). \tag{33} \]

The momentum equation can be written as
\[ \partial_x \left( \frac{M^2}{a(x) \bar{\rho}_+} \right) + a(x) p'(\bar{\rho}_+) \bar{\rho}_+ = 0, \quad \text{so,} \quad \left( p'(\bar{\rho}_+) - \frac{M^2}{(a \bar{\rho}_+)^2} \right) \bar{\rho}_+ = \frac{M^2}{a^3 \bar{\rho}_+} a'. \]

Therefore,
\[ \bar{u}_+ \frac{d'}{a} p'(\bar{\rho}_+) + \frac{d}{dx} \left( \frac{M}{a \bar{\rho}_+} \right) (p'(\bar{\rho}_+) - \bar{u}_+^2) = 0. \tag{34} \]

The boundary terms in (32) are
\[ \bar{u}_+^2 (x) (\partial_t \Psi)^2 \bigg|_{x=L}^{x=x_0} + \bar{u}_+ (\bar{u}_+^2 - c^2) \partial_x \Psi \partial_t \Psi \bigg|_{x=L}^{x=x_0} \]
\[ = \bar{u}_+^2 (\partial_t \Psi)^2 (\tau, L) + \bar{u}_+^2 (x_0) (\partial_t \Psi)^2 (\tau, x_0) + \frac{d'}{a} \frac{\bar{u}_+^2 \bar{u}_-}{2} \partial_t \Psi^2 (\tau, x_0) \tag{35} \]
\[ = \bar{u}_+^2 (\partial_t \Psi)^2 (\tau, L) + \bar{u}_+^2 (x_0) (\partial_t \Psi)^2 (\tau, x_0) + \partial_t \left( \left( \frac{a' \bar{u}_+^2 \bar{u}_-}{2a} \right) \Psi^2 \right)(\tau, x_0). \]

The lemma follows from (32), (33), (34), and (35). \(\square\)

The estimate in Lemma 3 and the method of [12] imply that solutions of the linear problem (30) decay exponentially to zero. This in turn implies the exponential decay of solutions of the nonlinear problem. We describe the form that these arguments take in the present context.

**Lemma 4.** If \( a' \) satisfies (15) then there exist constants \( \lambda_0 > 0 \) and \( C > 0 \) so that solutions \( \Psi \) of the problem (30) satisfy,
\[ E(\Psi, t) \leq C e^{-\lambda_0 t} E(\Psi, 0) \]
Proof: Step 1. Rauch-Taylor estimates. Thanks to the boundary condition at \( x = x_0 \), the estimate (31) implies
\[
E(\Psi, t) + C_1 \int_0^t (\Psi_x^2 + \Psi_t^2)(s, x_0) \, ds \leq E(\Psi, 0) + C_2 \int_0^t \Psi^2(s, x_0) \, ds.
\]
The argument in [17] and the details in [12, Appendix], one can show that there exist \( T > 0 \) and \( \delta \in (0, T/4) \) such that
\[
\int_0^T (\Psi_t^2 + \Psi_x^2)(t, x_0) \, dt \geq \delta E(\Psi, T) - C_3 \int_0^T \Psi^2(t, x_0) \, dt.
\] (36)
Combining (31) and (36) yields,
\[
(1 + C_4)E(\Psi, T) \leq E(\Psi, 0) + C_5 \int_0^T \Psi^2(t, x_0) \, dt,
\] (37)
for some positive constants \( C_4 \) and \( C_5 \), independent of \( t \).

Step 2. Spectrum of the evolution operator. Define a new norm \( \| \cdot \|_X \) for the function \( h = (h_1, h_2) \in H^1 \times L^2([x_0, L]) \),
\[
\| h \|_X^2 = \frac{a^2}{a} (|h_1|^2(x_0) + \int_{x_0}^L \{ |h_2|^2 + (p'(\rho_+) - \bar{u}_+^2)|h_1'|^2 \} (x) \, dx).
\]
The associated complex Hilbert space will be denoted by \((X, \| \cdot \|_X)\). Define the solution operator \( S_t : X \mapsto X \) as
\[
S_t(h) = (\Psi(t, \cdot), \Psi_t(t, \cdot))
\]
where \( \Psi \) is the solution of the problem (30) with the initial data \( h = (h_1, h_2) \). Applying the Lemma on page 81 in [15], there are at most a finite set of generalized eigenvalues for the operator \( S_T \) in the annulus \( \{ \frac{1}{1+\varepsilon} < |z| \leq 1 \} \subset \mathbb{C} \), each with finite multiplicity.

Step 3. Refined estimate for the spectrum of \( S_T \). We show that the spectrum does not touch the unit circle. Otherwise there would exist \( \omega \in \mathbb{R} \) and \( V \in X \) such that
\[
(S_T - e^{i\omega}I)V = 0.
\]
Note that the identity (31) still holds in the complex setting if we replace the square terms in \( E \) and \( D \) by the square of modulus. Thus
\[
E(\Psi, 0) - E(\Psi, nT) = nD(\Psi, T).
\]
Since \( E(\Psi, nT) \) and \( E(\Psi, 0) \) are both positive and finite, it follows that
\[
D(\Psi, T) = 0.
\] (38)
Therefore $E(\Psi, t) = E(\Psi, 0)$, for all $t$. Let $\mathcal{V} := \ker(S_T - e^{i\omega}I)$. Note that the coefficients in the problem (30) do not depend on $t$, so

$$(S_T - e^{i\omega}I)S_t = S_t(S_T - e^{i\omega}I)$$

In particular, $S(t)\mathcal{V} \subset \mathcal{V}$ for any $t$, so $\mathcal{V}$ is invariant with respect to $S_t$. Therefore $S_t|_\mathcal{V}$ is a semigroup on a finite dimensional subspace. This yields that $S(t)|_\mathcal{V} = e^{tA}$ for some $A \in \text{Hom} \mathcal{V}$. The definition of $\mathcal{V}$ implies that $e^TA = e^{i\omega}I$ so the spectrum of $A$ is purely imaginary. Choose an eigenvector $w$ of $A$, $Aw = \lambda w$, $\lambda \in i\beta \subset i\mathbb{R}$ Then $S_t w = e^{\lambda t} w$. From $S_t w = e^{i\beta t} w$ and (38) it follows that $w(x_0) = 0$. The boundary condition at $x_0$ then yields $w'(x_0) = 0$. The uniqueness of solutions of the linear homogeneous second order ordinary differential equation satisfied by eigenfunctions implies that $w = 0$. This contradicts the assumption that $w$ is an eigenvector hence not equal to zero.

The contradiction shows that there exists $0 < \beta_0 < 1$ so that the spectrum, $\sigma(S_T)$, of $S_T$ satisfies $\sigma(S_T) \subset \{|z| \leq \beta_0\}$. The formula for the spectral radius implies that $\|S(nT)\| = \|S(T)^n\|$ decays exponentially as $n \to \infty$. This is equivalent to the assertion Lemma 4. $\square$

**Corollary 5.** Assume that $a'$ satisfies (17) and $0 \leq k \in \mathbb{N}$. Define

$$E_k(\Psi, t) := \sum_{m=0}^{k} E(\partial_t^m \Psi, t).$$

Then, with $\lambda_0$ from Lemma 4, solutions $\Psi$ of the linearized problem (30) satisfy

$$E_k(\Psi, t) \leq C e^{-\lambda_0 t} E_k(\Psi, 0),$$

and,

$$\int_0^\infty e^{\lambda_0 t/4} \sum_{l=0}^{k} \left( |\partial_t^l \Psi|^2(t, x_0) + |\partial_t^l \Psi|^2(t, L) \right) dt \leq C E_k(\Psi, 0).$$

**Proof:** Since the coefficients of the equation and the boundary conditions are independent of $t$, applying this estimate to the solution $\partial_t^m \Psi$ yields

$$E(\partial_t^m \Psi, t) \leq C e^{-\lambda_0 t} E(\partial_t^m \Psi, 0).$$

Summing on $m$ yield the first estimate of the Corollary. The second follows from (31), Sobolev imbedding, and the first. $\square$
3.2. Uniform A Priori Estimates. The existence of local-in-time solutions for the problem (29) is proved as in [7]. In order to get global existence for the nonlinear problem (29), it suffices to prove global a priori estimates for (29) supplemented with initial data

\[ \Psi(0, x) = h_1(x), \quad \Psi_t(0, x) = h_2(x). \]  

(39)

With \( C, \lambda_0 \) from Lemma 4, choose \( T > 0 \) so that

\[ \alpha_0 := C e^{-\lambda_0 T} < 1. \]  

(40)

For \( t \geq T \) and integer \( k \geq 15 \), introduce

\[ \|\| (\Psi, \sigma) \|\| = \tilde{(Y, \sigma)} + \hat{(Y, \sigma)} \]

where

\[ \tilde{(\Psi, \sigma)} := \sup_{\tau \in [0, t]} \sum_{0 \leq m \leq k-6} \left( \sum_{0 \leq l \leq m} e^{\lambda \tau} \| \partial_l \partial_x^{m-l} \Psi(\tau, \cdot) \|_{L^\infty([x_0, L])} + e^{\lambda \tau} \| \frac{d \sigma}{dt} \|_{L^2([x_0, L])} \right) \]

and

\[ \hat{(\Psi, \sigma)} = \sup_{0 \leq \tau \leq t} \left( \sum_{0 \leq l \leq m, 0 \leq m \leq k} \| \partial_l \partial_x^{m-l} \Psi(\tau, \cdot) \|_{L^2([x_0, L])} + \sum_{0 \leq l \leq k} \| \partial_l \partial_x^{k+1-l} \Psi(\tau, \cdot) \|_{L^2([x_0, L])} \right) \]

\[ + \sup_{0 \leq \tau \leq t} \| \partial \partial_x^{k+1} \Psi(\tau, \cdot) - \frac{d^{k+1} \sigma}{dt^{k+1}} \|_{L^2([x_0, L])} \]

\[ + \sum_{0 \leq l \leq m, 0 \leq m \leq k+1} (\| \partial_l \partial_x^{m-l} \Psi(\cdot, x_0) \|_{L^2[0, t]} + \| \partial_l \partial_x^{m-l} \Psi(\cdot, L) \|_{L^2[0, t]} \right) + \sum_{0 \leq m \leq k+1} \left( \left. \| \frac{d \sigma}{dt} \right|_{L^2[0, t]} \right)^2, \]

with a positive constant \( \lambda \) to be defined in (52) below.

In this subsection, when there are no specific indications to the contrary, we assume \( a_{ij}, b_i, \) and \( g \) are functions of \((x, \Psi, \nabla \Psi, \sigma, \dot{\sigma})\), \( d_1 \) and \( e_1 \) are functions of \((\Psi, \Psi_t)\). Define

\[ \mathcal{E}(\Phi, t) := \frac{e_1 \Phi^2}{d_1 \beta_+} (t, x_0) + \int_{x_0}^L \bar{u}_+ \{(\partial_t \Phi)^2 - a_{11}(\partial_x \Phi)^2\} \]  

(41)

Furthermore, for any \( l \in \mathbb{N} \) and given \( \Psi \) and \( \sigma \) such that \( \|\| (\Psi, \sigma) \|\| < \infty \), define

\[ \mathcal{E}_l(\Phi, t) := \sum_{m=0}^l \mathcal{E}(\partial_l^m \Phi, t), \quad \text{and} \quad \mathcal{E}_l(\Phi, t) := \mathcal{E}_{l-1}(\Phi, t) + \mathcal{E}_0(\partial_t^l \Phi - q_1(x, \sigma) \Psi_x \frac{d \sigma}{dt}, t). \]

(42)
It is easy to see that if \(||(Ψ, σ)||\leq ε\) for sufficiently small \(ε > 0\), then
\[
\mathcal{E}(Φ, t)(t) \geq C \int_{x_0}^L (\partial_t Φ)^2 + (\partial_x Φ)^2 + Φ^2)(t, x) \, dx
\]
for some constant \(C > 0\) independent of \(t\).

**Proposition 6.** Assume that \(a'\) satisfies (15). There exists an \(ε_0 > 0\) so that for any \(0 < ε < ε_0\), if \((Ψ, σ)\) is a smooth solution of the problem (29) and (39) satisfying
\[
|σ_0| + \|h_1\|_{H^{k+2}} + \|h_2\|_{H^{k+1}} \leq ε^2 \leq ε_0^2, \quad \text{and}, \quad ||(Ψ, σ)|| \leq ε,
\]
then
\[
||(Ψ, σ)|| \leq ε/2.
\]

**Proof:** The proof has four steps.

Step 1. Lower order energy estimate. Define
\[
\mathcal{D}(Φ, t) := \int_0^t -\bar{u}_+(a_{11}d_1 + a_{01})(\partial_t Φ)^2(τ, x_0)d\tau + \int_0^t \bar{u}_+ a_{01}(\partial_t Φ)^2(τ, L)d\tau,
\]
and \(\mathcal{D}_m(Φ, t) = \sum_{l=0}^m \mathcal{D}(\partial_l^t Φ, t)\). Taking the \(m\)-th \((0 \leq m \leq k - 1)\) order derivative of the equation (29) with respect to \(t\), then multiplying the both sides by \(\bar{u}_+ \partial_l^m Ψ\) and integrating on \(Ω = [0, t] \times [x_0, L]\), noticing that \(a_{00} = 1\) yields
\[
\mathcal{E}_m(Ψ, t) + \mathcal{D}_m(Ψ, t) \leq \mathcal{E}_m(Ψ, 0)
\]
\[
+ C ||(Ψ, σ)|| \left[ \int_0^t e^{-\frac{2ετ}{ε_0}} \left[ \mathcal{E}_m(Ψ, τ) + \sum_{l=0}^{m+2} \left| \frac{d^l σ}{dt^l} \right|^2 \right] dτ + \mathcal{D}_m(Ψ, t) \right],
\]
for \(m = 0, 1, \cdots, k - 1\).

Step 2. The highest order energy estimates. Take the \(k\)-th order derivative for the equation (29) with respect to \(t\) to find,
\[
\mathcal{L}(x, Ψ, σ)∂_t^k Ψ = \mathcal{F}_k(x, Ψ, σ) + \frac{d^{k+2} σ}{dt^{k+2}} q_1(x, σ) Ψ_x + \sum_{0 \leq l \leq k-1} C_k^l \frac{d^{k+2} σ}{dt^{k+2}} \partial_t^{k-l} (q_1(x, σ) Ψ_x),
\]
where
\[
\mathcal{F}_k(x, Ψ, σ) := \sum_{1 \leq l \leq k} C_k^l \left\{ - \frac{1}{i,j=0} \partial_t^i a_{ij} \partial_t^{k-l} Y - \frac{1}{i=0} \partial_t^i b_i \partial_t^{k-l} Y \right\}.
\]
In order to handle the term \(\frac{d^{k+2} σ}{dt^{k+2}}\), study the equation for \(\Psi = ∂_t^k Ψ - q_1(x, σ) Ψ_x \frac{d^k σ}{dt^k}\),
\[
\mathcal{L}(x, Ψ, σ) \Psi = \mathcal{F}_k(x, Ψ, σ) + \mathcal{F}(x, Ψ, σ),
\]
(44)
where
\[
\mathcal{F}(x, \Psi, \sigma) = \sum_{0 \leq l \leq k-1} C^l_k \frac{d^{l+2}\sigma}{dt^{l+2}} \partial^{k-l}_t (q_1(x, \sigma)) \Psi_x - \frac{d^{k+1}\sigma}{dt^{k+1}} \partial_t (q_1(x, \sigma)) \Psi_x
\]
\[
- \frac{d^k\sigma}{dt^k} \partial^2_t (q_1(x, \sigma)) \Psi_x - 2a_{01} \partial_t \partial_x x + a_{11} \partial^2_x x + \sum_{i=0}^1 b_i \partial_i \left( \frac{d^k\sigma}{dt^k} q_1(x, \sigma) \Psi_x \right).
\]

Multiplying both sides of (44) by \( \bar{u} \) and integrating on \( \Omega \) yields
\[
\mathcal{E}(\Psi, t) + \mathcal{D}(\Psi, t) \leq \mathcal{E}(\Psi, 0) + C \left\| (\Psi, \sigma) \right\| \left[ \int_0^t e^{-\lambda \tau} \left[ \hat{E}_k(\Psi, \tau) + \sum_{l=0}^{k+1} \left| \frac{d^l\sigma}{dt^l} \right|^2 \right] d\tau + \hat{\mathcal{D}}(\Psi, t) \right].
\]
(45)

where \( \hat{\mathcal{D}}_k \) is defined as follows
\[
\hat{\mathcal{D}}_k(\Phi, t) = \mathcal{D}_{k-1}(\Phi, t) + \mathcal{D}_0(\partial_t^k \Phi - q_1(x, \sigma) \Psi_x \frac{d^k\sigma}{dt^k}, t).
\]

Adding the estimates (43) and (45) yields
\[
\hat{E}_k(\Psi, t) + \hat{\mathcal{D}}_k(\Psi, t) \leq \hat{E}_k(\Psi, 0) + C \left\| (\Psi, \sigma) \right\| \left[ \int_0^t e^{-\lambda \tau} \left[ \hat{E}_k(\Psi, \tau) + \sum_{l=0}^{k+1} \left| \frac{d^l\sigma}{dt^l} \right|^2 \right] d\tau + \hat{\mathcal{D}}_k(\Psi, t) \right].
\]
(46)

Step 3. Boundedness of the energy. Differentiating the equation for the shock front
\[
\sigma(t) = \mathcal{A}_0(\Psi(t, x_0))
\]
(47)

with respect to \( t \), yields
\[
\sum_{l=0}^{k+1} \left| \frac{d^l\sigma}{dt^l} \right|^2 (\tau) \leq C \left( |\sigma(\tau)|^2 + \sum_{l=0}^{k+1} |\partial^l_t \psi(\tau, x_0)|^2 \right).
\]

Using (47) again yields
\[
\sum_{l=0}^{k+1} \left| \frac{d^l\sigma}{dt^l} \right|^2 \leq C \left( \sum_{l=0}^k |\partial^l_t \psi(\tau, x_0)|^2 + |\partial_t \psi|^2 \right).
\]

Thus
\[
\int_0^t e^{-\frac{\lambda \tau}{64}} \sum_{l=0}^{k+1} \left| \frac{d^l\sigma}{dt^l} \right|^2 d\tau \leq C \left( \int_0^t e^{-\frac{\lambda \tau}{64}} \hat{E}_0(\Psi, \tau; \Psi, \sigma) d\tau + \hat{\mathcal{D}}(\Psi, t; \Psi, \sigma) \right).
\]

Therefore, the energy estimate (46) is equivalent to
\[
\hat{E}_k(\Psi, t) + \hat{\mathcal{D}}_k(\Psi, t) \leq \hat{E}_k(\Psi, 0) + C \left\| (\Psi, \sigma) \right\| \left( \hat{\mathcal{D}}_k(\Psi, \tau) + \int_0^t e^{-\frac{\lambda \tau}{64}} \hat{E}_k(\Psi, \tau) d\tau \right).
If \( \| (\Psi, \sigma) \| \leq \varepsilon \), then

\[
\hat{E}_k(\Psi, t) + \hat{D}_k(\Psi, t) \leq 2 \hat{E}_k(\Psi, 0) \leq C \varepsilon^4.
\]

This yields

\[
\| (\Psi, \sigma) \| \leq C \left( \sup_{0 \leq \tau \leq t} \hat{E}_k^{1/2}(\Psi, \tau) + \hat{D}_k^{1/2}(\Psi, t) \right) \leq C \varepsilon^2 \leq \frac{\varepsilon}{4}. \tag{48}
\]

Step 4. Decay of the lower energy and the shock position. The basic idea to get the decay is to control the deviation of the solution \( \Psi \) of the nonlinear problem (29) from that of the linearized problem (30) (denoted by \( \bar{\Psi} \)). The contraction of the energy for \( \bar{\Psi} \) will also yields the contraction of the energy for \( \Psi \). This gives the decay of the lower energy of \( \Psi \). The decay of the shock position is a consequence of the governing equation for the shock front and the decay of the lower energy.

At time \( \tau = t_0 \), we can choose \( \bar{h}_1 \in H^k \) and \( \bar{h}_2 \in H^{k-1} \) so that there exists a solution \( \bar{\Psi} \in C^{k-1-i}(t_0, \infty); H^i([x_0, L]) \) of the linear problem (30) satisfying \( \bar{\Psi}(t_0, \cdot) = \bar{h}_1 \) and \( \bar{\Psi}_t(t_0, \cdot) = \bar{h}_2 \), and \( \bar{\Psi} \) satisfies

\[
\sum_{l=0}^{k-1} \sum_{i=0}^l \| \partial_i^l \partial_x^{l-i} \bar{\Psi}(t_0, \cdot) \|_{L^2[x_0, L]} \leq C \| (\Psi, \sigma) \| \tag{49}
\]

for some uniform constant \( C \), and

\[
\hat{E}_{k-4}(\Psi - \bar{\Psi}, t_0) \leq C \| (\Psi, \sigma) \| \hat{E}_{k-4}(\Psi, t_0). \tag{50}
\]

As that in Steps 1 and 2, energy estimates for the equation for \( \Psi - \bar{\Psi} \) gives

\[
\hat{E}_{k-4}(\Psi - \bar{\Psi}, t_0 + T) + \hat{D}_{k-4}(\Psi - \bar{\Psi}, t_0 + T) - \hat{D}_{k-4}(\Psi - \bar{\Psi}, t_0)
\]

\[
\leq \hat{E}_{k-4}(\Psi - \bar{\Psi}, t_0) + C \| (\Psi, \sigma) \| \left[ \int_{t_0}^{t_0+T} \left( \sum_{l=0}^{k-3} \frac{d^l \sigma}{dt^l} \right)^2 + \hat{E}_{k-4}(\Psi - \bar{\Psi}, \tau) \ d\tau \right.
\]

\[+ \left. \int_{t_0}^{t_0+T} \hat{E}_{k-4}^{1/2}(\Psi, \tau) \hat{E}_{k-4}^{1/2}(\Psi - \bar{\Psi}, \tau) d\tau + (\hat{D}_{k-4}(\Psi, t_0 + T) - \hat{D}_{k-4}(\Psi, t_0)) \right]
\]

\[+ (\hat{D}_{k-4}(\Psi - \bar{\Psi}, t_0 + T) - \hat{D}_{k-4}(\Psi - \bar{\Psi}, t_0)) \right].
\]

Using the contraction of energy for \( \bar{\Psi} \) and noting that the deviation of \( \Psi \) and \( \bar{\Psi} \) at \( t_0 \) is of higher order (cf. (51)), one has

\[
\frac{34 + 30\alpha_0}{64} \hat{E}_{k-4}(\Psi, t_0 + T) \leq \frac{2 + 30\alpha_0}{32} \hat{E}_{k-4}(\Psi, t_0), \tag{51}
\]

\[
\frac{34 + 30\alpha_0}{64} \hat{E}_{k-4}(\Psi(t_0 + T) - \bar{\Psi}, t_0) \leq \frac{2 + 30\alpha_0}{32} \hat{E}_{k-4}(\Psi(t_0) - \bar{\Psi}, t_0),
\]

\[
\frac{34 + 30\alpha_0}{64} \hat{E}_{k-4}(\Psi, t_0 + T) \leq \frac{2 + 30\alpha_0}{32} \hat{E}_{k-4}(\Psi, t_0).
\]
if $\epsilon$ is sufficiently small. As in Corollary 5 it follows from (51) that

$$\hat{E}_{k-4}(\Psi_t, t) + \sigma^2(t) \leq C (\hat{E}_{k-4}(\Psi, 0) + \sigma^2(0)) e^{-2M},$$

where with $\alpha_0$ from (40),

$$\lambda := -\frac{\ln(1 + \alpha)/2}{2T}, \quad \text{and} \quad \alpha := \frac{2 + 30\alpha_0}{17 + 15\alpha_0}. \quad (52)$$

Thus

$$\sum_{l=0}^{k-6} \left[ \frac{d^l \sigma}{dt^l} + \|\Psi(t, \cdot)\|_{L^\infty[x_0, L]} \right] \leq C\epsilon^2 e^{-\lambda t}. \quad (53)$$

Combining (48) and (53), one has (42). This finishes the proof of the Proposition 6.

Once one has the Proposition 6, Theorem 2 follows from the standard continuation argument and local existence result.

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