A CHARACTERIZATION OF ROUGH FRACTIONAL TYPE INTEGRAL OPERATORS AND CAMPANATO ESTIMATES FOR THEIR COMMUTATORS ON THE VARIABLE EXPONENT VANISHING GENERALIZED MORREY SPACES

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Abstract. In this paper, applying some properties of variable exponent analysis, we first dwell on Adams and Spanne type estimates for a class of fractional type integral operators of variable orders, respectively and then, obtain variable exponent generalized Campanato estimates for the corresponding commutators on the vanishing generalized Morrey spaces $V^p_{11}(\cdot,w(\cdot))(E)$ with variable exponent $p(\cdot)$ and bounded set $E$. In fact, the results in this paper are generalizations of some known results on an operator basis.

1. Introduction

In this paper we mainly focus on some operators and commutators on the variable exponent generalized Morrey type space. Precisely, our aim is to characterize the boundedness for the maximal operator, fractional integral operator and fractional maximal operator with rough kernel as well as the corresponding commutators on the variable exponent vanishing generalized Morrey spaces.

Now, we list some background material needed for later sections. We assume that our readers are familiar with the foundation of real analysis. Since it is impossible to squeeze everything into just a few pages, sometimes we will refer the interested readers to some papers and references.

Notation 1. · Let $x = (x_1, x_2, \ldots, x_n)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ . . . etc. be points of the real $n$-dimensional space $\mathbb{R}^n$. Let $x, \xi = \sum_{i=1}^{n} x_i \xi_i$ stand for the usual dot product in $\mathbb{R}^n$ and $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ for the Euclidean norm of $x$.

· By $x'$, we always mean the unit vector corresponding to $x$, i.e. $x' = \frac{x}{|x|}$ for any $x \neq 0$.

· $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ represents the unit sphere in Euclidean $n$-dimensional space $\mathbb{R}^n$ $(n \geq 2)$ and $d\sigma'$ is its surface measure.

· Denote by $|E|$ the Lebesgue measure and by $\chi_E$ the characteristic function for a measurable set $E \subset \mathbb{R}^n$.

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Given a function $f$, we denote the mean value of $f$ on $E$ by

$$f_E := \frac{1}{|E|} \int_E f(x) \, dx.$$  

$B(x,r) = \{ y \in \mathbb{R}^n : |x-y| < r \}$ denotes $x$-centred Euclidean ball with radius $r$, $B^C(x,r)$ denotes its complement and $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$, $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)| = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ and $B(x,r) = B(x,r) \cap E$, where $E \subset \mathbb{R}^n$ is an open set. Finally, we use the notation

$$f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{\tilde{B}(x,r)} f(y) \, dy.$$  

$C$ stands for a positive constant that can change its value in each statement without explicit mention.

The exponents $p'(\cdot)$ and $s'(\cdot)$ always denote the conjugate index of any exponent $1 < p(x) < \infty$ and $1 < s(x) < \infty$, that is, $p'(x) := 1 - \frac{1}{p(x)}$ and $s'(x) := 1 - \frac{1}{s(x)}$.

In the sequel, for any exponent $1 < p(x) < \infty$ and bounded sets $E \subset \mathbb{R}^n$, if we use

$$|p(x) - p(y)| \leq \frac{-C}{\log(|x-y|)} \quad |x-y| \leq \frac{1}{2}, \quad x,y \in E,$$

where $C = C(p) > 0$ does not depend on $x, y$, then we call that $p(\cdot)$ satisfies local log-Hölder continuity condition or Dini-Lipschitz condition. The important role of local log-Hölder continuity of $p(\cdot)$ is well known in variable analysis. On the other hand, the condition

$$|p(x) - p(y)| \leq \frac{C}{\log(e+|x|)} \quad |y| \geq |x|, \quad x,y \in E,$$

introduced by Cruz-Uribe et al. in [30] is known as the log-Hölder decay condition used for unbounded sets $E$. It is equivalent to the condition that there exists a number $p_\infty \in [1,\infty)$ such that

$$\left| \frac{1}{p_\infty} - \frac{1}{p(x)} \right| \leq \frac{C_\infty}{\log(e+|x|)} \quad \text{for all } x \in E,$$

where $p_\infty = \lim_{|x| \to \infty} p(x)$.

If $p(\cdot)$ satisfies both (1.1) and (1.2), then we say that it is log-Hölder continuous.

Here and henceforth, $F \approx G$ means $F \gtrless G \gtrless F$; while $F \gtrless G$ means $F \geq CG$ for a constant $C > 0$.

Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous function of degree $0$ on $\mathbb{R}^n$ and satisfy the integral zero property over the unit sphere $S^{n-1}$. Moreover, note
that \( \| \Omega \|_{L_s(S^{n-1})} := \left( \int_{S^{n-1}} |\Omega(z')|^s \, d\sigma(z') \right)^{\frac{1}{s}} \) and

\[
\|\Omega(z-y)\|_{L_s(B(x,r))} = \left( \int_{B(x,r)} (\Omega((z-y))^s \, dz \right)^{\frac{1}{s}}
\]

\[
\lesssim \left( \int_{B(x,r)} \Omega(\sigma)^s \int_0^r \rho^{n-1} \, d\rho \, d\sigma \right)^{\frac{1}{s}}
\]

(1.3)

for \( z \in B(x,r) \).

Suppose that \( 0 < \alpha(x) < n \), \( x \in E \subset \mathbb{R}^n \). Then, the rough Riesz type potential operator with variable order \( I_{\Omega,\alpha(\cdot)} \) and the corresponding rough fractional maximal operator with variable order \( M_{\Omega,\alpha(\cdot)} \) are defined, respectively, by

\[
I_{\Omega,\alpha(\cdot)} f(x) = \int_E \frac{\Omega(x-y)}{|x-y|^{n-\alpha(x)}} f(y) \, dy
\]

and

\[
M_{\Omega,\alpha(\cdot)} f(x) = \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\Omega(y)| \, |f(y)| \, dy,
\]

where \( E \subset \mathbb{R}^n \) is an open set. On the other hand, if \( \alpha(\cdot) = 0 \), then the rough Calderón-Zygmund type singular integral operator \( T_\Omega \) in the sense of principal value Cauchy integral is defined by

\[
T_\Omega f(x) = \text{p.v.} \int_E \frac{\Omega(x-y)}{|x-y|^n} f(y) \, dy,
\]

and especially in the limiting case \( \alpha(\cdot) = 0 \), the rough fractional maximal operator with variable order \( M_{\Omega,\alpha} \) reduces to the rough Hardy-Littlewood maximal operator \( M_{\Omega} \) and \( M_\Omega \) is also defined by

\[
M_{\Omega} f(x) = \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |\Omega(y)| \, |f(x-y)| \, dy,
\]

where \( E \subset \mathbb{R}^n \) is an open set. In fact, we can easily see that when \( \Omega \equiv 1; M_{1,\alpha(\cdot)} \equiv M_{\alpha(\cdot)} \) and \( I_{1,\alpha(\cdot)} \equiv I_{\alpha(\cdot)} \) are the fractional maximal operator with variable order and the Riesz type potential operator with variable order, and similarly \( M \) and \( T \) are the Hardy-Littlewood maximal operator and the standard Calderón-Zygmund type singular integral operator, respectively.

Let \( b \) be a locally integral function on \( E \). Define the rough commutators \([b,T_\Omega]\), \([b,M_\Omega]\) generated by the function \( b \) and the operators \( T_\Omega, M_\Omega \) with rough kernel \( \Omega \) via

\[
[b,T_\Omega] f(x) = b(x) T_\Omega f(x) - T_\Omega (b f)(x)
\]

\[
= \text{p.v.} \int_E \frac{\Omega(x-y)}{|x-y|^n} (b(x) - b(y)) f(y) \, dy
\]
and

\[ [b, M_\Omega] f(x) = b(x) M_\Omega f(x) - M_\Omega (bf)(x) \]

\[ = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |\Omega(x - y)||b(x) - |b(y)|| |f(y)|| dy, \]

similarly, define the rough commutators \([b, I_{\Omega, \alpha(\cdot)}]\), \([b, M_{\Omega, \alpha(\cdot)}]\) generated by the function \(b\) and the fractional integral operator \(I_{\Omega, \alpha(\cdot)}\), the fractional maximal operator \(M_{\Omega, \alpha(\cdot)}\) with rough kernel \(\Omega\) and variable order \(\alpha(\cdot)\) \((0 \leq \alpha(\cdot) < n)\) as follows.

\[ [b, I_{\Omega, \alpha(\cdot)}] f(x) = b(x) I_{\Omega, \alpha(\cdot)} f(x) - I_{\Omega, \alpha(\cdot)} (bf)(x) \]

\[ = \int_{E} \frac{\Omega(x - y)}{|x - y|^{n - \alpha(x)}} (b(x) - |b(y)|| f(y) dy \]

and

\[ [b, M_{\Omega, \alpha(\cdot)}] f(x) = b(x) M_{\Omega, \alpha(\cdot)} f(x) - M_{\Omega, \alpha(\cdot)} (bf)(x) \]

\[ = \sup_{r > 0} |B(x, r)|^{-\frac{\alpha(x)}{n}} \int_{B(x, r)} |\Omega(x - y)||b(x) - |b(y)|| |f(y)|| dy. \]

Morrey spaces can complement the boundedness properties of operators that Lebesgue spaces can not handle. Morrey spaces which we have been handling are called classical Morrey spaces (see [20]). In this sense, the classical Morrey spaces (see [20]) ever were applied to study the local regularity behavior of solutions to second order elliptic partial differential equations (see [14] and [29]). For the boundedness of various classical operators in Morrey or Morrey type spaces, refer to for maximal, potential, singular integral and others, [1, 2, 3, 4, 7, 9, 19, 28] and references therein. In [31] the vanishing Morrey space was introduced by Vitanza to character the regularity results for elliptic partial differential equations. Moreover, Ragusa ([22] and Samko et al [20, 27] and references therein) ever systematically obtain the boundedness of various classical operators in such these spaces. Recently, while we try out to resolve somewhat modern problems emerging inherently such that nonlinear elasticity theory, fluid mechanics etc., it has become that classical function spaces are not anymore suitable spaces. It thus became essential to introduce and analysis the diverse function spaces from diverse viewpoints. One of such spaces is the variable exponent Lebesgue space \(L^{p(\cdot)}\). This space is a generalization of the classical \(L^p(\mathbb{R}^n)\) space, in which the constant exponent \(p\) is replaced by an exponent function \(p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)\), it consists of all functions \(f\) such that \(\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx\). This theory got a boost in 1931 when Orlicz published his seminal paper [24]. The next major step in the investigation of variable exponent spaces was the comprehensive paper by Kováčik and Rákosník in the early 90’s [15]. Since then, the theory of variable exponent spaces was applied to many fields, refer to [8, 33] for the image processing, [9] for thermorheological fluids, [23] for electrorheological fluids and [14] for the differential equations with nonstandard growth. For the nonweighted and weighted variable exponent settings, refer to [10, 11, 12, 13]. On the other hand, Kováčik and Rákosník [15] established many of the basic properties of Lebesgue and Sobolev spaces. Moreover, since these authors clarified fundamental properties
of the variable exponent Lebesgue and Sobolev spaces, there are many spaces studied, such as variable exponent Morrey, generalized Morrey, vanishing generalized Morrey, Herz-Morrey spaces, etc. see [5, 15, 17, 25, 32]. In the last decade, when the parameters that define the operator have changed from point to point, there has been a strong interest in fractional type operators and the “variable setting” function spaces. The field called variable exponent analysis has become a fairly branched area with many interesting results obtained in the last decade such as harmonic analysis, approximation theory, operator theory, pseudo-differential operators, etc. But, the results in this paper lie in these spaces known as variable exponent Morrey type spaces on the rough fractional type operators, etc. But, the results in this paper lie in these spaces known as variable exponent Morrey type spaces on the rough fractional type operators with variable order of harmonic analysis, which has been extensively developed during the last ten years and continues to attract attention of researchers from various fields of mathematics. Many of problems about such spaces have been solved both in the classical setting and in the Euclidean setting, including fractional upper and lower dimensions. For example, in 2008 variable exponent Morrey spaces $L^{p(\cdot),\lambda}(\cdot)$ were introduced to study the boundedness of $M$ and $I_{\alpha(\cdot)}$ in the Euclidean setting by Almeida et al. [5]. In 2010, variable exponent generalized Morrey spaces $L^{p(\cdot),w(\cdot)}(E)$ were introduced to consider the boundedness of $M$, $I_{\alpha(\cdot)}$, $T$ for bounded sets $E \subset \mathbb{R}^n$ on $L^{p(\cdot),w(\cdot)}(E)$ in [15]. In 2016, variable exponent vanishing generalized Morrey spaces $VL^{p(\cdot),w(\cdot)}(E)$ were introduced to characterize the boundedness of $M$, $I_{\alpha(\cdot)}$, $T$ for bounded or unbounded sets $E$ on $VL^{p(\cdot),w(\cdot)}(E)$ in [25].

After the boundedness of $M$, $I_{\alpha(\cdot)}$, $T$ for bounded sets $E \subset \mathbb{R}^n$ both on $L^{p(\cdot),w(\cdot)}(E)$ and $VL^{p(\cdot),w(\cdot)}(E)$ have been established in [15, 25], a natural question is: Can these results be generalized? In other words, what properties do the more general operators $M_{\Omega,\alpha(\cdot)}$ and $I_{\Omega,\alpha(\cdot)}$ have for bounded sets $E \subset \mathbb{R}^n$ both on $L^{p(\cdot),w(\cdot)}(E)$ and $VL^{p(\cdot),w(\cdot)}(E)$? We give answers to these questions in this paper. In view of the definitions of $M_{\Omega}$, $I_{\Omega,\alpha(\cdot)}$ and $T_{\Omega}$ above, we see that these operators are generalizations of the operators $M$, $I_{\alpha(\cdot)}$, $T$. On the other hand, recently, Rafeiro and Samko [21] proved that the boundedness of $I_{\Omega,\alpha(\cdot)}$, $M_{\Omega,\alpha(\cdot)}$ and $M_{\Omega}$ for bounded sets $E \subset \mathbb{R}^n$ both on $L^{p(\cdot)}$ and $L^{p(\cdot),\lambda(\cdot)}$, respectively.

2. Preliminaries and Main results

In this section, we recall the definitions and some properties of basic spaces that we need and also give the main results.

2.1. Preliminaries on variable exponent Lebesgue spaces $L^{p(\cdot)}$.

We first define variable exponent Lebesgue space.

**Definition 1.** Given an open set $E \subset \mathbb{R}^n$ and a measurable function $p(\cdot) : E \to [1, \infty)$. We assume that $1 \leq p_{-}(E) \leq p_{+}(E) < \infty$, where $p_{-}(E) = \essinf_{x \in E} p(x)$ and $p_{+}(E) = \esssup_{x \in E} p(x)$. The variable exponent Lebesgue space $L^{p(\cdot)}(E)$ is the collection of all measurable functions $f$ such that, for some $\lambda > 0$, $\rho(f/\lambda) < \infty$, where the modular is defined by

$$\rho(f) = \rho_{p(\cdot)}(f) = \int_{E} |f(x)|^{p(x)} dx.$$
Then, the spaces $L^{p(·)}(E)$ and $L^{p(·)}_{loc}(E)$ are defined by
\[
L^{p(·)}(E) = \left\{ f \text{ is measurable} : \rho_{p(·)}(f/\lambda) < \infty \text{ for some } \lambda > 0 \right\}
\]
and
\[
L^{p(·)}_{loc}(E) = \left\{ f \text{ is measurable} : f \in L^{p(·)}(K) \text{ for all compact } K \subset E \right\},
\]
with the Luxemburg norm
\[
\|f\|_{L^{p(·)}(E)} = \inf \left\{ \lambda > 0 : \rho_{p(·)}(f/\lambda) = \int_E (|f(x)|/\lambda)^{p(x)} \, dx \leq 1 \right\} \quad f \in L^{p(·)}(E).
\]

Since $p_-(E) \geq 1$, $\|\cdot\|_{L^{p(·)}(E)}$ is a norm and $\left( L^{p(·)}(E), \|\cdot\|_{L^{p(·)}(E)} \right)$ is a Banach space. However, if $p_-(E) < 1$, then $\|\cdot\|_{L^{p(·)}(E)}$ is a quasinorm and $\left( L^{p(·)}(E), \|\cdot\|_{L^{p(·)}(E)} \right)$ is a quasi Banach space. The variable exponent norm has the following property
\[
\|f^\lambda\|_{L^{p(·)}(E)} = \|f\|_{L^{p(·)}(E)},
\]
for $\lambda \geq \frac{1}{p_-}$. Moreover, these spaces are referred to as variable $L^p$ spaces, since they generalize the standard $L^p$ spaces: if $p(x) = p$ is constant, then $L^{p(·)}(E)$ is isometrically isomorphic to $L^p(E)$. As a result, using notations above ($p_-(E)$ and $p_+(E)$), we define a class of variable exponent as follows:
\[
\Phi(E) = \{ p(·) : E \to [1, \infty), \quad p_-(E) \geq 1, \quad p_+(E) < \infty \}.
\]

Now, we define two the sets of exponents $p(x)$ with $1 \leq p_-(E) \leq p_+(E) < \infty$. These will be denoted by as follows:
\[
\mathcal{P}^{\log}(E) = \left\{ p(·) : p_-(E) \geq 1, p_+(E) < \infty \text{ and } p(·) \text{ satisfy both the conditions } \text{[11]} \text{ and } \text{[12]} \right\}
\]
and
\[
\mathcal{B}(E) = \left\{ p(·) : p(·) \in \mathcal{P}^{\log}(E), \quad M \text{ is bounded on } L^{p(·)}(E) \right\},
\]
where $M$ is the Hardy-Littlewood maximal operator. We recall that the generalized Hölder inequality on Lebesgue spaces with variable exponent
\[
\left| \int_E f(x)g(x) \, dx \right| \leq \int_E |f(x)g(x)| \, dx \leq C_p \|f\|_{L^{p(·)}(E)} \|g\|_{L^{p'(·)}(E)} \quad C_p = 1 + \frac{1}{p_-} - \frac{1}{p_+},
\]
is known to hold for $p(·) \in \Phi(E)$, $f \in L^{p(·)}(E)$ and $g \in L^{p'(·)}(E)$, see Theorem 2.1 in [13]. Now, we recall some recent results for the rough Riesz type potential operator with variable order $I_{\Omega,\alpha(·)}$ and the corresponding rough fractional maximal operator with variable order $M_{\Omega,\alpha(·)}$ on variable exponent Lebesgue space $L^{p(·)}(E)$. The order $\alpha(x)$ of the potential is not assumed to be continuous. We assume that it is a measurable function on $E$ satisfying the following assumptions
\[
\begin{align*}
\alpha_0 &= \text{essinf}_{x \in E} \alpha(x) > 0 \\
\text{esssup}_{x \in E} \alpha(x) p(x) &< n
\end{align*}
\]
(2.2)
First, the norm in the space $L^p(E)$ seems to be complicated in a sense, to be calculated or estimated. So the following basic estimation of the boundedness of an operator $B$:

$$\|Bf\|_{L^p(E)} \lesssim \|f\|_{L^p(E)}$$

is not easy. However, in the case of linear operators, the above inequality between the norm and the modular and the homogeneity property

$$\|B\|_{X \rightarrow X} = \sup_{f \in X} \|Bf\|_X / \|f\|_X = \sup_{\|f\|_X = 1} \|Bf\|_X$$

allow us to replace checking of (2.3) by a work with a modular:

$$\int_E |Bf(x)|^{p(x)} \, dx, \text{ for all } f \text{ with } \|f\|_{L^p(E)} \leq 1,$$

which is certainly easier. In that respect, the boundedness of the rough Riesz-type potential operator from the space $L^p(E)$ into $L^q(E)$ was an open problem for a long time. It was solved in the case of bounded domains. First, in [21], in the case of bounded domains $E$, there has the following conditional result.

**Theorem 1.** Let $E$ be a bounded open set, $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$, $p(x) \in \mathcal{P}^{\log}(E)$, $\alpha(x)$ satisfy the assumptions (2.2) and $(p')_+ \leq s$. Define $q(x)$ by (2.4). Then, the rough Riesz-type potential operator $I_{\Omega,\alpha(\cdot)}$ is $(L^{p(\cdot)}(E) \rightarrow L^{q(\cdot)}(E))$-bounded, that is, the Sobolev type theorem

$$\|I_{\Omega,\alpha(\cdot)}f\|_{L^{q(\cdot)}(E)} \lesssim \|f\|_{L^{p(\cdot)}(E)}$$

is valid.

**Corollary 1.** Let $E$ be a bounded open set, $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous function of degree 0 on $\mathbb{R}^n$, $\frac{\alpha}{p} \in \mathcal{B}(E)$ and $(p')_+ \leq s$. Under the conditions of Theorem 1 (taking $\alpha(\cdot) = 0$ there), the operator $T_{\Omega}$ is $(L^{p(\cdot)}(E) \rightarrow L^{p(\cdot)}(E))$-bounded, that is,

$$\|T_{\Omega}f\|_{L^{p(\cdot)}(E)} \lesssim \|f\|_{L^{p(\cdot)}(E)}$$

is valid.

On the other hand, the pointwise inequalities on variable exponent Lebesgue spaces are very useful. Indeed, we have

$$|f(x)| \leq |h(x)| \implies \|f\|_{L^{p(\cdot)}(E)} \lesssim \|h\|_{L^{p(\cdot)}(E)}.$$

Thus, if one operator is pointwise dominated by another one:

$$|Bf(x)| \leq |Df(x)|,$$

and we know that the operator $D$ is bounded, then the boundedness of the operator $B$ immediately follows. For example, by Theorem 1 we get the following:
Corollary 2. Let $E$ be a bounded open set, $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous function of degree 0 on $\mathbb{R}^n$, $\frac{n}{p} \in B(E)$ and $(p')_+ \leq s$. Under the conditions of Theorem 2 (taking $\alpha(\cdot) = 0$ there), the operator $M_\Omega$ is $(L^p(E) \to L^{p'}(E))$-bounded, that is,
\[
\|M_\Omega f\|_{L^{p'}(E)} \lesssim \|f\|_{L^p(E)}
\]
is valid.

Remark 1. The conclusion of (2.7) is a direct consequence of the following Lemma 1 and (2.5). In order to do this, we need to define an operator by
\[
\|M_\Omega f\|_{L^{p'}(E)} \lesssim \|f\|_{L^p(E)}
\]
is valid.

We are now in a place of proving (2.7) in Theorem 2.

Remark 2. The conclusion of (2.7) is a direct consequence of the following Lemma 1 and (2.5). In order to do this, we need to define an operator by
\[
\|M_\Omega f\|_{L^{p'}(E)} \lesssim \|f\|_{L^p(E)}
\]
is valid.

\[
\frac{n}{p} \in B(E) \quad \text{and} \quad (p')_+ \leq s.
\]

Under the conditions of Theorem 1, the operator $M_\Omega$ is $(L^p(E) \to L^{p'}(E))$-bounded, that is,
\[
\|M_\Omega f\|_{L^{p'}(E)} \lesssim \|f\|_{L^p(E)}
\]
is valid.

Remark 1. The conclusion of (2.7) is a direct consequence of the following Lemma 1 and (2.5). In order to do this, we need to define an operator by
\[
\|M_\Omega f\|_{L^{p'}(E)} \lesssim \|f\|_{L^p(E)}
\]
is valid.
The above theorems (Theorem 11 and Theorem 12) allows to use the known results for the boundedness of the operators $M_{\Omega, \alpha(\cdot)}$ and $I_{\Omega, \alpha(\cdot)}$ transfer to the various function spaces. The following fact is known, see Lemma 3.1. in [25].

**Lemma 2.** Let $E$ be a bounded open set, $p(x) \in P^{\log}(E)$ and $\alpha(x)$ satisfy assumptions (2.2). Then,

$$
\| x - \cdot \|^{\alpha(x) - \alpha} \chi_{B(x,r)} \rceil_{L^{p(\cdot)}(E)} \lesssim r^{\alpha(x) - \frac{n}{p(\cdot)}}.
$$

We will also make use of the estimate provided by the following fact (see [25]),

$$
(2.10) \quad \| \chi_{B(x,r)} \rceil_{L^{p(\cdot)}(E)} \lesssim r^\psi_p(x,r), \quad x \in E, \; p(x) \in P^{\log}(E),
$$

where

$$
\psi_p(x,r) = \begin{cases} \frac{p}{\alpha(x)} & r \leq 1 \\ \frac{n}{\alpha(x)} & r > 1. 
\end{cases}
$$


**2.2. Preliminaries on variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}$.**

We define variable exponent Morrey space as follows.

**Definition 2.** Let $E$ be a bounded open set and $\lambda(x)$ be a measurable function on $E$ with values in $[0,n]$. Then, the variable exponent Morrey space $L^{p(\cdot),\lambda(\cdot)}(E)$ is defined by

$$
L^{p(\cdot),\lambda(\cdot)} \equiv L^{p(\cdot),\lambda(\cdot)}(E) = \left\{ f \in L^{p(\cdot)}(E) : \sup_{x \in E, \, r > 0} r^{-\psi_p(x,r)} \| \chi_{B(x,r)} \rceil_{L^{p(\cdot)}(E)} < \infty \right\}.
$$

Note that $L^{p(\cdot),0}(E) = L^{p(\cdot)}(E)$ and $L^{p(\cdot),n}(E) = L^{\infty}(E)$. If $\lambda(x) > n$, then $L^{p(\cdot),\lambda(\cdot)}(E) = \{0\}$.

**Lemma 3.** Let $E$ be a bounded open set, $\Omega \in L_s(S^{n-1})$ with $1 < s < \infty$, $p(x), q(x) \in P^{\log}(E)$, $\alpha(x)$ satisfy the following assumptions

$$
(2.11) \quad \alpha_0 = \text{essinf}_{x \in E} \alpha(x) > 0 \quad \text{esssup}_{x \in E} [\lambda(x) + \alpha(x) p(x)] < n
$$

and $(p')_+ \leq s$. Define $q(x)$ by $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n - \lambda(x)}$. Then, the rough Riesz-type potential operator $I_{\Omega, \alpha(\cdot)}$ is $(L^{p(\cdot),\lambda(\cdot)}(E) \rightarrow L^{q(\cdot),\lambda(\cdot)}(E))$-bounded. Moreover,

$$
\| I_{\Omega, \alpha(\cdot)} f \|_{L^{q(\cdot),\lambda(\cdot)}(E)} \leq \| f \|_{L^{p(\cdot),\lambda(\cdot)}(E)}.
$$

**Proof.** By the embedding property in Lemma 7 in [5], we only need to prove that the operator $I_{\Omega, \alpha(\cdot)}$ is bounded in $L^{p(\cdot),\lambda(\cdot)}(E)$.

**Hedberg’s trick:**

$$
I_{\Omega, \alpha(\cdot)} f(x) = \int_{B(x,2r)} \frac{\Omega(x - y)}{|x - y|^{n - \alpha(x)}} f(y) dy + \int_{B^c(x,2r)} \frac{\Omega(x - y)}{|x - y|^{n - \alpha(x)}} f(y) dy
$$

$$
= \mathcal{F}(x,r) + \mathcal{G}(x,r).
$$
We may assume that \( \|f\|_{L^{p(\cdot),\lambda(\cdot)}(E)} \leq 1 \). For \( \mathcal{F}(x, r) \), we first have to prove the following:

\[
\mathcal{F}(x, r) := \left| \int_{|x-y|<r} \frac{\Omega(x-y)}{|x-y|^{n-\alpha(x)}} f(y) dy \right| \leq \frac{2^n r^{\alpha(x)}}{2^\alpha(x) - 1} M_\Omega f(x).
\]

Indeed, for \( f(x) \geq 0 \) we have

\[
\mathcal{F}(x, r) = \sum_{j=0}^{\infty} \int_{2^{-j}r \leq |x-y|<2^{-j}r} \frac{\Omega(x-y)}{|x-y|^{n-\alpha(x)}} f(y) dy
\]

\[
\quad \leq \sum_{j=0}^{\infty} \frac{1}{(2^{-j}r)^{\alpha(x)}} \int_{|x-y|<2^{-j}r} \Omega(x-y) f(y) dy
\]

\[
\quad \leq 2^{n-\alpha(x)} M_\Omega f(x) \sum_{j=0}^{\infty} \left( 2^{-j} \right)^{\alpha(x)}.
\]

Hence by \( |B(x, 2^{-j}r)| \lesssim (2^{-j}r)^{n} \), we obtain

\[
\mathcal{F}(x, r) \lesssim 2^{n-\alpha(x)} r^{\alpha(x)} M_\Omega f(x) \sum_{j=0}^{\infty} \left( 2^{-j} \right)^{\alpha(x)},
\]

which gives the estimate (2.13). Then by (2.13):

\[
|\mathcal{F}(x, r)| \lesssim r^{\alpha(x)} M_\Omega f(x).
\]

For \( \mathcal{G}(x, r) \), from Lemma 2 and the procedure of Theorem 3 in [5], we may show that

\[
|\mathcal{G}(x, r)| \lesssim r^{\alpha(x)} \frac{\alpha(x)}{p(\cdot) - \alpha(x)}.
\]

Then, from (2.14) we get

\[
I_{\Omega,\alpha(\cdot)} f(x) \lesssim \left[ r^{\alpha(x)} M_\Omega f(x) + r^{\alpha(x)} \frac{\alpha(x)}{p(\cdot) - \alpha(x)} \right].
\]

As usual in Hedberg approach, we choose

\[
r = [M_\Omega f(x)]^{\frac{\alpha(x)}{p(\cdot) - \alpha(x)}}.
\]

Substituting this into the (2.14), we get

\[
|I_{\Omega,\alpha(\cdot)} f(x)| \lesssim (M_\Omega f(x))^{\frac{\alpha(x)}{p(\cdot) - \alpha(x)}},
\]

here we need the (2.9). Therefore, by Theorem 5.1 in [24] we know that

\[
\int_{B(x,r)} |I_{\Omega,\alpha(\cdot)} f(y)|^{q(y)} dy \lesssim \int_{B(x,r)} |M_\Omega f(y)|^{p(y)} dy \lesssim r^{\lambda(x)},
\]

which completes the proof of Lemma 3.

\[\square\]

**Theorem 3.** Let \( E \) be a bounded open set, \( \Omega \in L_s(S^{n-1}) \) with \( 1 < s < \infty \), \( p(\cdot), q(\cdot) \in P_{\log}(E) \), \( \alpha(\cdot) \) satisfy (2.11) and \( (p')_+ \leq s \). Define \( q(\cdot), \mu(\cdot) \) by

\[
\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha(\cdot)}{n}, \quad \frac{\alpha(\cdot)}{p(\cdot)} = \frac{\alpha(\cdot)}{q(\cdot)} - \alpha(\cdot),
\]

respectively. Then, the rough Riesz-type potential operator \( I_{\Omega,\alpha(\cdot)} \) is \( (L^{p(\cdot),\lambda(\cdot)}(E) \rightarrow L^{q(\cdot),\mu(\cdot)}(E)) \)-bounded. Moreover,

\[
\|I_{\Omega,\alpha(\cdot)} f\|_{L^{p(\cdot),\lambda(\cdot)}(E)} \lesssim \|f\|_{L^{p(\cdot),\lambda(\cdot)}(E)}.
\]
where
\[ 1 \leq q(\cdot) \leq \frac{p(\cdot)(n - \lambda(\cdot))}{n - \lambda(\cdot) - \alpha(\cdot)p(\cdot)}. \]

**Proof.** Since
\[ \frac{p(\cdot)(n - \lambda(\cdot))}{n - \lambda(\cdot) - \alpha(\cdot)p(\cdot)} < \frac{np(\cdot)}{n - \alpha(\cdot)p(\cdot)}, \]
from Lemma 3 and Theorem 1 we obtain
\[ \Pi \subset \text{cond}_{\alpha(\cdot)} \subset \Pi. \]

Under the conditions of Theorem 3, \( \Box \)
Clearly, Theorem 3 holds.

**Proof.** Since

\[ (2.16) \]

we can immediately obtain (2.16), which completes the proof.

\[ \tilde{\text{it is easy to see that the conclusion (2.15) also holds for}} \]
\[ \text{sequence of (2.9) and (2.15). Indeed, from the process proving (2.15) in Theorem 3,} \]
\[ \text{similar to the proof of Theorem 2, the conclusion (2.16) is a direct consequence of (2.9) and (2.15). Indeed, from the process proving (2.15) in Theorem 3, it is easy to see that the conclusion (2.15) also holds for} \]
\[ \text{combining this with (2.9), we can immediately obtain (2.16), which completes the proof.} \]

\[ \Box \]

**Theorem 4.** Under the conditions of Theorem 3,
\[ (2.16) \]

Prove. Similar to the proof of Theorem 4, the conclusion (2.16) is a direct consequence of (2.9) and (2.15). Indeed, from the process proving (2.15) in Theorem 3, it is easy to see that the conclusion (2.15) also holds for \( T_{\alpha(\cdot)} \). Combining this with (2.9), we can immediately obtain (2.16), which completes the proof. \( \Box \)

\[ \Box \]

2.3. Preliminaries on variable exponent vanishing generalized Morrey spaces.

In this section we first consider the generalized Morrey spaces \( L^{p(\cdot),w(\cdot)}(E) \) with variable exponent \( p(x) \) and a general function \( w(x, r) : \Pi \times (0, \text{diam}(E)) \to \mathbb{R}_+ \), \( \Pi \subset E \subset \mathbb{R}^n \), defining the Morrey type norm on sets \( E \subset \mathbb{R}^n \) which may be both bounded and unbounded; see the definition of the spaces \( L^{p(\cdot),w(\cdot)}(E) \) in (2.18)
below.

Everywhere in the sequel the functions \( w(x, r), w_1(x, r), w_2(x, r) \) used in the body of this paper, are non-negative measurable functions on \( E \times (0, \infty) \), where \( E \subset \mathbb{R}^n \) is an open set. We recall the definition of variable exponent generalized Morrey space in the following.

**Definition 3.** Let \( 1 \leq p(x) \leq p_+ < \infty, \Pi \subset E \subset \mathbb{R}^n, x \in \Pi, w(x, r) : \Pi \times (0, \text{diam}(E)) \to \mathbb{R}_+ \), where
\[ (2.17) \]
\[ \inf_{x \in \Pi} w(x, r) > 0 \quad r > 0. \]

Then, the variable exponent generalized Morrey space \( L^{p(\cdot),w(\cdot)}(E) \) is defined by
\[ (2.18) \]
\[ L^{p(\cdot),w(\cdot)}(E) = \left\{ f \in L^{p(\cdot)}_{\text{loc}}(E) : \inf_{x \in E, r > 0} w(x, r) \cdot \frac{\|f\|_{L^{p(\cdot)}(B(x, r))}}{\|f\|_{L^{p(\cdot)}(B(x, r))}} < \infty \right\}. \]
and one can also see that for bounded exponents $p$ there holds the following equivalence:

$$f \in L^{p(\cdot),w(\cdot)}_\Pi \text{ if and only if } \sup_{x \in \Pi, r > 0} \int_{B(x,r)} \frac{|f(y)|}{w(x,r)}^p \, dy < \infty.$$ 

On the other hand, the above definition recover the definition of $L^{p(\cdot),\lambda(\cdot)}(E)$ if we choose $w(x,r) = r^{\frac{\lambda(x)}{p(x)}}$ and $\Pi = E$, that is

$$L^{p(\cdot),\lambda(\cdot)}(E) = L^{p(\cdot),w(\cdot)}_\Pi (E) \bigg|_{w(x,r) = r^{\frac{\lambda(x)}{p(x)}}}.$$ 

Also, when $\Pi = \{x_0\}$ and $\Pi = E$, $L^{p(\cdot),w(\cdot)}_\Pi$ turns into the local generalized Morrey space $L^{p(\cdot),w(\cdot)}(E)$ and the global generalized Morrey space $L^{p(\cdot),w(\cdot)}_E(E)$, respectively. Moreover, we point out that $w(x,r)$ is a measurable non-negative function and no monotonicity type condition is imposed on these spaces. Note that by the above definition of the norm in $L^{p(\cdot)}(E)$ (see 2.11), we can also write that

$$\|f\|_{L^{p(\cdot),w(\cdot)}_\Pi} = \sup_{x \in \Pi, r > 0} \inf_{\lambda = \lambda(x,r)} \left\{ \lambda = \lambda(x,r) : \int_{B(x,r)} \frac{|f(y)|}{\lambda w(x,r)}^p \, dy \leq 1 \right\}.$$ 

Then, recall that the concept of the variable exponent vanishing generalized Morrey space $V L^{p(\cdot),w(\cdot)}_{\Pi}(E)$ has been introduced in [25] in the following form:

**Definition 4.** Let $1 \leq p(x) \leq p_+ < \infty$, $\Pi \subset E \subset \mathbb{R}^n$, $x \in \Pi$, $w(x,r) : \Pi \times (0, \text{diam}(E)) \to \mathbb{R}_+$. Then, the variable exponent vanishing generalized Morrey space $V L^{p(\cdot),w(\cdot)}_{\Pi}(E)$ is defined by

$$\left\{ f \in L^{p(\cdot),w(\cdot)}_{\Pi}(E) : \limsup_{r \to 0} M_{p(\cdot),w(\cdot)}(f; x, r) = 0 \right\},$$

where

$$M_{p(\cdot),w(\cdot)}(f; x, r) := \sup_{y \in B(x,r)} \frac{r^{-\frac{p(x)}{w(x,r)}} \|f\|_{L^{p(\cdot)}(B(x,r))}}{w(x,r)^{\frac{p(x)}{w(x,r)}}}.$$ 

Naturally, it is suitable to impose on $w(x,t)$ with the following conditions:

$$\limsup_{t \to 0} \frac{t^{-\psi_p(x,t)}}{w(x,t)^{\frac{p(x)}{w(x,t)}}} = 0 \quad (2.19)$$

and

$$\inf_{t > 1} \sup_{x \in \Pi} w(x,t) > 0. \quad (2.20)$$

From (2.19) and (2.20), we easily know that the bounded functions with compact support belong to $V L^{p(\cdot),w(\cdot)}_{\Pi}(E)$, which make the spaces $V L^{p(\cdot),w(\cdot)}_{\Pi}(E)$ non-trivial.

The spaces $V L^{p(\cdot),w(\cdot)}_{\Pi}(E)$ are Banach spaces with respect to the norm

$$\|f\|_{V L^{p(\cdot),w(\cdot)}_{\Pi}(E)} = \sup_{x \in \Pi, r > 0} M_{p(\cdot),w(\cdot)}(f; x, r).$$

The spaces $V L^{p(\cdot),w(\cdot)}_{\Pi}(E)$ are also closed subspaces of the Banach spaces $L^{p(\cdot),w(\cdot)}_{\Pi}(E)$, which may be shown by standard means.
Furthermore, we have the following embeddings:

$$ V_{\Pi}^{p(\cdot),w(\cdot)} \subset L_{\Pi}^{p(\cdot),w(\cdot)}, \quad \|f\|_{V_{\Pi}^{p(\cdot),w(\cdot)}} \leq \|f\|_{L_{\Pi}^{p(\cdot),w(\cdot)}}. $$

In 2016, for bounded or unbounded sets $E$, Long and Han [25] considered the Spanne type boundedness of operators $M_{\alpha(\cdot)}$ and $I_{\alpha(\cdot)}$ on $V_{\Pi}^{p(\cdot),w(\cdot)}(E)$.

Now, in this section we extend Theorem 4.3. in [25] to rough kernel versions. In other words, the Theorem 4.3. in [25] allows to use the known results for the boundedness of the operators $I_{\alpha(\cdot)}$ and $M_{\alpha(\cdot)}$ in generalized variable exponent Morrey spaces to transfer them to the operators $I_{\Omega,\alpha(\cdot)}$ and $M_{\Omega,\alpha(\cdot)}$. We give two versions of such an extension, the one being a generalization of Spanne’s result for rough potential operators with variable order, the other extending the corresponding Adams’ result, respectively.

In this extension, we will give some answers to the above explanations as follows:

**Theorem 5. (Spanne type result with variable $\alpha(x)$) (our main result)** Let $E$ be a bounded open set, $\Omega \in L_\infty(S^{n-1})$, $1 < s \leq \infty$, $\Omega(\mu x) = \Omega(x)$ for any $\mu > 0$, $x \in \mathbb{R}^n \setminus \{0\}$, $p(x) \in p_{\log}(E)$, $\alpha(x)$ satisfy the assumption (2.2). Define $q(x)$ by (2.4). Suppose that $q(\cdot)$ and $\alpha(\cdot)$ satisfy (2.11). For $\frac{1}{s-1} < p^- \leq p(\cdot) < \frac{n}{\alpha(\cdot)}$, the following pointwise estimate

$$\begin{equation}
(2.21) \quad \|I_{\Omega,\alpha(\cdot)} f\|_{L^p(\cdot)(\overline{B(x,r)})} \lesssim r^{-n/s} \int_r^{diam(E)} \|f\|_{L^p(\cdot)(\overline{B(x,t)})} \frac{dt}{t^{n/s + 1}}
\end{equation}$$

holds for any ball $\overline{B(x,r)}$ and for all $f \in L^{p(\cdot)}_{loc}(E)$.

If the functions $w_1(x,r)$ and $w_2(x,r)$ satisfy (2.17) as well as the following Zygmund condition

$$\begin{equation}
(2.22) \quad \int_r^{diam(E)} \frac{w_1^{p(\cdot)}(x,t)}{t^{1-\alpha(x)}} dt \lesssim w_2^{p(\cdot)}(x,r), \quad r \in (0, diam(E)]
\end{equation}$$

and additionally these functions satisfy the conditions (2.17), (2.20),

$$\begin{equation}
(2.23) \quad \delta \geq \int_{\delta}^{diam(E)} \sup_{x \in \Pi} \frac{w_1^{p(\cdot)}(x,t)}{t^{1-\alpha(x)}} dt < \infty, \quad \delta > 0
\end{equation}$$

then the operators $I_{\Omega,\alpha(\cdot)}$ and $M_{\Omega,\alpha(\cdot)}$ are $(V_{\Pi}^{p(\cdot),w_1(\cdot)}(E) \to V_{\Pi}^{p(\cdot),w_2(\cdot)}(E))$-bounded. Moreover,

$$\begin{equation}
(2.24) \quad \|I_{\Omega,\alpha(\cdot)} f\|_{V_{\Pi}^{p(\cdot),w_1(\cdot)}(E)} \lesssim \|f\|_{V_{\Pi}^{p(\cdot),w_1(\cdot)}(E)},
\end{equation} \quad \|M_{\Omega,\alpha(\cdot)} f\|_{V_{\Pi}^{p(\cdot),w_2(\cdot)}(E)} \lesssim \|f\|_{V_{\Pi}^{p(\cdot),w_1(\cdot)}(E)}.
$$

**Proof.** Since inequality (2.21) is the key of the proof of (2.24), we first prove (2.21).

For any $x \in E$, we write as

$$\begin{equation}
(2.25) \quad f(y) = f_1(y) + f_2(y),
\end{equation}
$$

where $f_1(y) = f(y) \chi_{\overline{B(x,2r)}(y)}$, $r > 0$ such that

$$\begin{equation}
(2.26) \quad I_{\Omega,\alpha(\cdot)} f_1(y) = I_{\Omega,\alpha(\cdot)} f_1(y) + I_{\Omega,\alpha(\cdot)} f_2(y).
\end{equation}$$
By using triangle inequality, we get
\[
\|I_{\Omega, \alpha} f_1\|_{L^q(E)} \leq \|I_{\Omega, \alpha} f_1\|_{L^q(B(x,r))} + \|I_{\Omega, \alpha} f_2\|_{L^q(B(x,r))}. 
\]

Now, let us estimate \(\|I_{\Omega, \alpha} f_1\|_{L^q(B(x,r))}\) and \(\|I_{\Omega, \alpha} f_2\|_{L^q(B(x,r))}\), respectively.

By Hardy-Littlewood-Sobolev type inequality and Theorem 1, we obtain that
\[
\|I_{\Omega, \alpha} f_1\|_{L^q(B(x,r))} \leq \|f\|_{L^p(E)} = \|f\|_{L^p(B(x,2r))} 
\]
\[\approx r^{-\frac{n}{p'}} \|f\|_{L^p(B(x,2r))} \int_{2r}^{\text{diam}(E)} \frac{dt}{t^{\frac{n}{p'}} + 1}. \]
\[
\leq r^{-\frac{n}{p'}} \int_{2r}^{\text{diam}(E)} \|f\|_{L^p(B(x,t))} \frac{dt}{t^{\frac{n}{p'}} + 1}, \]

where in the last inequality, we have used the following fact:
\[
\|f\|_{L^p(B(x,2r))} \leq \|f\|_{L^p(B(x,t))}, \text{ for } t > 2r. 
\]

Now, let us estimate the second part. For the estimate used in \(\|I_{\Omega, \alpha} f_2\|_{L^q(B(x,r))}\),
we first have to prove the below inequality:

\[
(2.26) \quad \|I_{\Omega, \alpha} f_2\|_{L^q(B(x,r))} \leq \|\Omega\|_{L^n(S^{n-1})} \int_{2r}^{\text{diam}(E)} \|f\|_{L^p(B(x,t))} \frac{dt}{t^{\frac{n}{p'}} + 1}. 
\]

Indeed, if \(|x - z| \leq r\) and \(|z - y| \geq r\), then \(|x - y| \leq |x - z| + |y - z| \leq 2|y - z|\).

By generalized Minkowski’s inequality we get
\[
\|I_{\Omega, \alpha} f_2\|_{L^q(B(x,r))} = \left\| \int_{E \setminus B(x,2r)} \frac{\Omega(z - y)}{|z - y|^{n-\alpha(x)}} f(y) dy \right\|_{L^q(B(x,r))} 
\]
\[\lesssim \int_{E \setminus B(x,2r)} \frac{|\Omega(z - y)||f(y)|}{|x - y|^{n-\alpha(x)}} dy \|\chi_B(x,r)\|_{L^q(E)}. \]

Put \(\gamma > \frac{1}{q} - \frac{n}{p'}\). Provided that \(1 < s' < p^- \leq p^+ < \infty\), \(\sup_{x \in E} (\alpha(x) + \gamma - n) < \infty\)
and \(\inf_{x \in E} \left( n + (\alpha(x) + \gamma - n) \left(\frac{p}{s} - 1\right) \right) < \infty\), by generalized Hölder’s inequality for
we get (2.21) and thus (2.24) holds. On the other hand, since

\[ \| f \|_{L^{p(\cdot)}(E)} \leq \| f \|_{L^{p(\cdot)}(\Omega, \alpha)} \leq \| f \|_{L^{p(\cdot)}(\Omega, \alpha)} \quad \text{for} \quad \frac{\alpha(x)}{p(x)} + \frac{1}{q(x)} = 1. \]

Thus, by (2.10) we get

\[ \| f \|_{L^{p(\cdot)}(\hat{B}(x, t))} \lesssim \left( \frac{\text{diam}(E)}{r} \right)^{\frac{1}{q(x)}} \int_{\Omega} \| f \|_{L^{p(\cdot)}(\hat{B}(x, t))} \frac{dt}{t^{\frac{1}{q(x)} + 1}}. \]

Combining all the estimates for \( \| I_{\Omega, \alpha(\cdot)} f_1 \|_{L^{p(\cdot)}(\hat{B}(x, r))} \) and \( \| I_{\Omega, \alpha(\cdot)} f_2 \|_{L^{p(\cdot)}(\hat{B}(x, r))} \), we get (2.21).

At last, by Definition 1 (2.21) and (2.22) we get

\[ \| I_{\Omega, \alpha(\cdot)} f \|_{V_{2, 2}^{p(\cdot)}(E)} = \sup_{x \in \Omega, r > 0} \frac{r^{-\frac{\alpha(x)}{p(x)}} \| I_{\Omega, \alpha(\cdot)} f \|_{L^{p(\cdot)}(\hat{B}(x, r))}}{w_2(x, r)^{\frac{1}{q(x)}}} \]

\[ \lesssim \sup_{x \in \Omega, r > 0} \frac{1}{w_2(x, r)^{\frac{1}{q(x)}}} \int_{\hat{B}(x, t)} \| f \|_{L^{p(\cdot)}(\hat{B}(x, t))} \frac{dt}{t^{\frac{1}{q(x)} + 1}} \]

\[ \lesssim \| f \|_{V_{1}^{p(\cdot), w_1(\cdot)}(E)} \sup_{x \in \Omega, r > 0} \frac{1}{w_2(x, r)^{\frac{1}{q(x)}}} \int_{\hat{B}(x, t)} u_1^{\frac{\alpha(x)}{p(x)}}(x, t) dt \]

and

\[ \limsup_{r \to 0} \frac{r^{-\frac{\alpha(x)}{p(x)}} \| I_{\Omega, \alpha(\cdot)} f \|_{L^{p(\cdot)}(\hat{B}(x, r))}}{w_2(x, t)^{\frac{1}{q(x)}}} \lesssim \limsup_{r \to 0} \frac{r^{-\frac{\alpha(x)}{p(x)}} \| f \|_{L^{p(\cdot)}(\hat{B}(x, r))}}{w_1(x, t)^{\frac{1}{q(x)}}} = 0. \]

Thus, (2.24) holds. On the other hand, since \( M_{\Omega, \alpha(\cdot)}(f) \lesssim I_{\Omega, \alpha(\cdot)}(\| f \|) \) (see Lemma 1) we can also use the same method for \( M_{\Omega, \alpha(\cdot)} \), so we omit the details. As a result, we complete the proof of Theorem 4. \( \square \)
Definition 5. (Rough \((p, q)\)-admissible \(T_{\Omega, \alpha(\cdot)}\)-potential type operator with variable order) Let \(1 \leq p^- (E) \leq p (\cdot) \leq p^+ (E) < \infty\). A rough sublinear operator with variable order \(T_{\Omega, \alpha(\cdot)}\), i.e., \(\|T_{\Omega, \alpha(\cdot)} (f + g)\| \leq \|T_{\Omega, \alpha(\cdot)} (f)\| + \|T_{\Omega, \alpha(\cdot)} (g)\|\) and for all \(\lambda \in \mathbb{C}\) \(\|T_{\Omega, \alpha(\cdot)} (\lambda f)\| = |\lambda| \|T_{\Omega, \alpha(\cdot)} (f)\|\), will be called rough \((p, q)\)-admissible \(T_{\Omega, \alpha(\cdot)}\)-potential type operator with variable order if
\[
\cdot \quad T_{\Omega, \alpha(\cdot)} \text{ fulfills the following size condition:}
\]
\[
(2.28)
\]
\[
\chi_{B(z, r)} (x) |T_{\Omega, \alpha(\cdot)} \left( f \chi_{E \setminus B(z, 2r)} \right) (x) | \leq C \chi_{B(z, r)} (x) \int_{E \setminus B(z, 2r)} \frac{|\Omega(x - y)|}{|x - y|^{n - \alpha(\cdot)}} |f(y)| \, dy,
\]
\[
\cdot \quad T_{\Omega, \alpha(\cdot)} \text{ is } (L^p(\cdot) (E) \rightarrow L^q(\cdot) (E))\text{-bounded.}
\]

Remark 3. Note that rough \((p, q)\)-admissible potential type operators were introduced to study their boundedness on Morrey spaces with variable exponents in [17]. The operators \(M_{\Omega, \alpha(\cdot)}\) and \(T_{\Omega, \alpha(\cdot)}\) are also rough \((p, q)\)-admissible potential type operators. Moreover, these operators satisfy (2.28).

Corollary 3. Obviously, under the conditions of Theorem 2, if the rough \((p, q)\)-admissible \(T_{\Omega, \alpha(\cdot)}\)-potential type operator is \((L^p(\cdot) (E) \rightarrow L^q(\cdot) (E))\)-bounded and satisfies (2.28), the result in Theorem 2 still holds.

For \(\alpha (x) = 0\) in Theorem 5, we get the following new result:

Corollary 4. Let \(E, \Omega, p (x)\) be the same as in Theorem 5. Then, for \(\frac{p^-}{p^+} < p^- \leq p (\cdot) \leq p^+ < \infty\), the following pointwise estimate
\[
\|T_{\Omega} f\|_{L^p(\cdot) (\bar{B}(x, r))} \lesssim r^\frac{\text{diam}(E)}{\text{diam}(\bar{B}(x, r))} \int_{r}^{\text{diam}(E)} t^{-\frac{p^-}{p^+} - 1} \|f\|_{L^p(\cdot) (\bar{B}(x, t))} \, dt
\]
holds for any ball \(\bar{B}(x, r)\) and for all \(f \in L^p(\cdot)_{\text{loc}} (E)\).

If the function \(w (x, r)\) satisfies (2.17) as well as the following Zygmund condition
\[
\int_{r}^{\text{diam}(E)} \frac{w^{\frac{1}{p^-}} (x, t)}{t} \, dt \lesssim w^{\frac{1}{p^-}} (x, r), \quad r \in (0, \text{diam} (E)]
\]
and additionally this function satisfies the conditions (2.19)-(2.21),
\[
c_0 := \int_{\delta} \sup_{x \in E} \frac{w^{\frac{1}{p^-}} (x, t)}{t} \, dt < \infty, \quad \delta > 0
\]
then the operators \(T_{\Omega}\) and \(M_{\Omega}\) are bounded on \(VL^p(\cdot),w^{\cdot}(\cdot) (E)\). Moreover,
\[
\|T_{\Omega} f\|_{VL^p(\cdot),w^{\cdot}(\cdot) (E)} \lesssim \|f\|_{VL^p(\cdot),w^{\cdot}(\cdot) (E)},
\]
\[
(2.29)
\]
\[
\|M_{\Omega} f\|_{VL^p(\cdot),w^{\cdot}(\cdot) (E)} \lesssim \|f\|_{VL^p(\cdot),w^{\cdot}(\cdot) (E)}.
\]

Theorem 6. (Adams type result with variable \(\alpha (x)\)) (our main result) Let \(E, \Omega, p (x), q (x), \alpha (x)\) be the same as in Theorem 5. Then, for \(\frac{p^-}{p^+} < p^- \leq p (\cdot) < p^+ < \infty\), the following
\[
\|T_{\Omega} f\|_{VL^p(\cdot),w^{\cdot}(\cdot) (E)} \lesssim \|f\|_{VL^p(\cdot),w^{\cdot}(\cdot) (E)},
\]
\[
(2.30)
\]
\[
\|M_{\Omega} f\|_{VL^p(\cdot),w^{\cdot}(\cdot) (E)} \lesssim \|f\|_{VL^p(\cdot),w^{\cdot}(\cdot) (E)}.
\]
\[
\frac{n}{\alpha(x)} \text{, the following pointwise estimate}
\]
\[
(2.30) \quad |I_{\Omega, \alpha(\cdot)} f(x)| \lesssim \frac{\alpha(x)}{\alpha(x)} M_\Omega f(x) + \int_r^{diam(E)} t^{\alpha(x) - \frac{\alpha(x)}{p(x)} - 1} \|f\|_{L_p(B(x, t))} \, dt
\]
holds for any ball \(B(x, r)\) and for all \(f \in L^{p(\cdot)}(E)\).

The function \(w(x, t)\) satisfies (2.17), (2.19)–(2.20) as well as the following conditions:
\[
(2.31) \quad \int_r^{diam(E)} \frac{w^{\frac{1}{p(x)}}(x, t)}{t} \, dt \lesssim w^{\frac{1}{p(x)}}(x, r),
\]
where \(p(x) < q(x)\). Then the operators \(I_{\Omega, \alpha(\cdot)}\) and \(M_{\Omega, \alpha(\cdot)}\) are \((VL^{p(\cdot)}_{\Omega}, w^{\frac{1}{p(x)}}(E) \rightarrow VL^{q(\cdot)}_{\Omega}, w^{\frac{1}{q(x)}}(E))\)-bounded. Moreover,
\[
\|I_{\Omega, \alpha(\cdot)} f\|_{VL^{p(\cdot)}_{\Omega}, w^{\frac{1}{p(x)}}(E)} \lesssim \|f\|_{VL^{p(\cdot)}_{\Omega}, w^{\frac{1}{p(x)}}(E)},
\]
\[
\|M_{\Omega, \alpha(\cdot)} f\|_{VL^{q(\cdot)}_{\Omega}, w^{\frac{1}{q(x)}}(E)} \lesssim \|f\|_{VL^{q(\cdot)}_{\Omega}, w^{\frac{1}{q(x)}}(E)}.
\]

**Proof.** As in the proof of Theorem 5 we represent the function \(f\) in the form (2.25) and have
\[
I_{\Omega, \alpha(\cdot)} f(x) = I_{\Omega, \alpha(\cdot)} f_1(x) + I_{\Omega, \alpha(\cdot)} f_2(x).
\]
For \(I_{\Omega, \alpha(\cdot)} f_1(x)\), similar to the proof of (2.14), we obtain the following pointwise estimate:
\[
(2.32) \quad |I_{\Omega, \alpha(\cdot)} f_1(x)| \lesssim t^{\alpha(x)} M_\Omega f(x).
\]
For \(I_{\Omega, \alpha(\cdot)} f_2(x)\), similar to the proof of (2.27), applying Fubini’s theorem, Hölder’s inequality and (1.3), we get
\[
(2.33) \quad |I_{\Omega, \alpha(\cdot)} f_2(x)| \lesssim \int_r^{diam(E)} t^{\alpha(x) - \frac{\alpha(x)}{p(x)} - 1} \|f\|_{L_p(B(x, t))} \, dt
\]
and by (2.32) and (2.33) complete the proof of (2.30).

Since \(M_{\Omega, \alpha(\cdot)} (f) \lesssim I_{\Omega, \alpha(\cdot)} (|f|)\) (see Lemma 1), it suffices to treat only the case of the operator \(I_{\Omega, \alpha(\cdot)}\). In this sense, by (2.30) and (2.31), we obtain
\[
|I_{\Omega, \alpha(\cdot)} f(x)| \lesssim r^{\alpha(x)} M_\Omega f(x) + r^{-\frac{\alpha(x)}{q(x)-p(x)}} \|f\|_{VL^{q(\cdot)}_{\Omega}, w^{(\cdot)}(E)}.
\]
Then, choosing \(r = \left(\frac{\|f\|_{VL^{q(\cdot)}_{\Omega}, w^{(\cdot)}(E)}}{M_\Omega f(x)}\right)^{\frac{q(x)-p(x)}{\alpha(x)p(x)}}\) for every \(x \in E\) supposing that \(f\) is not equal 0, thus we have
\[
(2.34) \quad |I_{\Omega, \alpha(\cdot)} f(x)| \lesssim (M_\Omega f(x))^{\frac{q(x)}{q(x)-p(x)}} \|f\|_{VL^{q(\cdot)}_{\Omega}, w^{(\cdot)}(E)}.
\]
Finally, by Definition \[ \text{(2.34)} \] and \[ \text{(2.29)} \] we get

\[
\|I_{\Omega,(\cdot)}f\|_{VL_{\Pi}^{p,(\cdot),w^{\frac{1}{q(\cdot)}}}(E)} = \sup_{x \in \Pi, r > 0} \frac{r^{-\frac{n}{p(x)}} \|I_{\Omega,(\cdot)}f\|_{L^{p(\cdot)}(B(x,r))}}{w(x,r)^{\frac{1}{q(\cdot)}}} 
\]

\[
\lesssim \|f\|_{VL_{\Pi}^{p,(\cdot),w^{\frac{1}{q(\cdot)}}}(E)} \sup_{x \in \Pi, r > 0} \frac{r^{-\frac{n}{p(x)}} \|M_{\Omega}f\|_{L^{p(\cdot)}(B(x,r))}}{w(x,r)^{\frac{1}{q(\cdot)}}} 
\]

\[
\lesssim \|f\|_{VL_{\Pi}^{p,(\cdot),w^{\frac{1}{q(\cdot)}}}(E)} \left( \sup_{x \in \Pi, r > 0} \frac{r^{-\frac{n}{p(x)}} \|M_{\Omega}f\|_{L^{p(\cdot)}(B(x,r))}}{w(x,r)^{\frac{1}{q(\cdot)}}} \right) 
\]

\[
\lesssim \|f\|_{VL_{\Pi}^{p,(\cdot),w^{\frac{1}{q(\cdot)}}}(E)} \|M_{\Omega}f\|_{VL_{\Pi}^{p,(\cdot),w^{\frac{1}{q(\cdot)}}}(E)} 
\]

if \( p(x) < q(x) \) and

\[
\limsup_{r \to 0} \frac{r^{-\frac{n}{p(x)}} \|I_{\Omega,(\cdot)}f\|_{L^{p(\cdot)}(B(x,r))}}{w_2(x,t)^{\frac{1}{q(\cdot)}}} \lesssim \limsup_{r \to 0} \frac{r^{-\frac{n}{p(x)}} \|f\|_{L^{p(\cdot)}(B(x,r))}}{w_1(x,t)^{\frac{1}{q(\cdot)}}} = 0, 
\]

which completes the proof of Theorem \( \text{[2]} \) \( \square \)

**Corollary 5.** Obviously, under the conditions of Theorem \( \text{[2]} \) if the rough \((p,q)\)-admissible \(T_{\Omega,(\cdot)}\)-potential type operator is \((L^{p(\cdot)}(E) \to L^{q(\cdot)}(E))\)-bounded and satisfies \( \text{(2.29)} \), the result in Theorem \( \text{[7]} \) still holds.

**Remark 4.** Let \( E \) be a bounded open set and \( \lambda(x) \) be a measurable function on \( E \) with values in \( [0,n] \). Then, the variable exponent vanishing Morrey space \( VL_{\Pi}^{p(\cdot),\lambda(\cdot)}(E) \) is defined by

\[
VL_{\Pi}^{p(\cdot),\lambda(\cdot)}(E) = \left\{ f \in L^{p(\cdot),\lambda(\cdot)}(E) : \|f\|_{VL_{\Pi}^{p(\cdot),\lambda(\cdot)}(E)} = \lim_{r \to 0} \sup_{0 < t < r} t^{-\frac{\lambda(t)}{p(t)}} \|f\chi_{B(x,t)}\|_{L^{p(\cdot)}(E)} = 0 \right\}. 
\]

**Corollary 6.** Let \( E, \Omega, p(x), \alpha(x) \) be the same as in Theorem \( \text{[4]} \). Define \( q(x) \) by \( \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n-\lambda(x)} \). Let also the following conditions hold:

\[
\lambda(x) \geq 0, \quad \text{esssup}_{x \in \mathbb{E}} [\lambda(x) + \alpha(x)p(x)] < n. 
\]

Then for \( (p_-)' \leq s \), the operators \( I_{\Omega,(\cdot)} \) and \( M_{\Omega,(\cdot)} \) are \( (VL_{\Pi}^{p(\cdot),\lambda(\cdot)}(E) \to VL_{\Pi}^{p(\cdot),\lambda(\cdot)}(E))\)-bounded. Moreover,

\[
\|I_{\Omega,(\cdot)}f\|_{VL_{\Pi}^{p(\cdot),\lambda(\cdot)}(E)} \lesssim \|f\|_{VL_{\Pi}^{p(\cdot),\lambda(\cdot)}(E)}, \quad \|M_{\Omega,(\cdot)}f\|_{VL_{\Pi}^{p(\cdot),\lambda(\cdot)}(E)} \lesssim \|f\|_{VL_{\Pi}^{p(\cdot),\lambda(\cdot)}(E)}. 
\]

In the case of \( \lambda(x) \equiv 0 \), for the spaces \( L^{p(\cdot)}(E) \), from Corollary \( \text{[4]} \) we get the following:
Theorem 7. Let $E$, $\Omega$, $p(x)$, $q(x)$, $\alpha(x)$ be the same as in Theorem 3. Then, the operators $I_{\Omega,\alpha(x)}$ and $M_{\Omega,\alpha(x)}$ are $(L^p(E) \to L^q(E))$-bounded. Moreover,

\[
\|I_{\Omega,\alpha(x)}f\|_{L^q(E)} \lesssim \|f\|_{L^p(E)},
\]

\[
\|M_{\Omega,\alpha(x)}f\|_{L^q(E)} \lesssim \|f\|_{L^p(E)}.
\]

2.4. Preliminaries on variable exponent generalized Campanato spaces $C^{q(\cdot),\gamma(\cdot)}_E$.

In this section, we first introduce the variable exponent generalized Campanato spaces and then obtain the boundedness of the commutators of the operators $I_{\Omega,\alpha(x)}$, $M_{\Omega,\alpha(x)}$, $T_{\Omega}$ and $M_{\Omega}$ on the spaces $VL^{p(\cdot),u(\cdot)}(E)$.

Definition 6. Let $1 \leq q(\cdot) \leq q^+ < \infty$ and $0 \leq \gamma(\cdot) < \frac{1}{n}$. Define the generalized Campanato space $C^{q(\cdot),\gamma(\cdot)}_E$ with variable exponents $q(\cdot)$, $\gamma(\cdot)$ as follows:

\[
C^{q(\cdot),\gamma(\cdot)}_E(E) = \left\{ f \in L^q_{\text{loc}}(\tilde{B}(x,r)) : \|f\|_{C^{q(\cdot),\gamma(\cdot)}_E(E)} < \infty \right\},
\]

where

\[
\|f\|_{C^{q(\cdot),\gamma(\cdot)}_E(E)} = \sup_{x \in \Omega, r > 0} \|B(x,r)^{-\frac{1}{q(x)} - \gamma(x)} \|f - f_{B(x,r)}\|_{L^q(B(x,r))}.
\]

such that

\[
\|f - f_{B(x,r)}\|_{L^q(B(x,r))} \lesssim r^{\frac{q^+ + n\gamma(x)}{q(x)}} \|f\|_{C^{q(\cdot),\gamma(\cdot)}_E(E)}.
\]

When $\Omega = \{x_0\}$ and $E = \Omega$, $C^{q(\cdot),\gamma(\cdot)}_E(E)$ turns into the local generalized Campanato space $C^{q(\cdot),\gamma(\cdot)}_E(\{x_0\})$ and the global generalized Campanato space $C^{q(\cdot),\gamma(\cdot)}_E(E)$, respectively. If $q(\cdot)$, $\gamma(\cdot)$ are constant functions and $\Omega = E$, then the variable exponent generalized Campanato space $C^{q(\cdot),\gamma(\cdot)}_E(E)$ is exactly the usual Campanato space $C^{q,\gamma}(E)$. If $\gamma(\cdot) \equiv 0$ and $q(\cdot) \equiv q$, the generalized Campanato space $C^{q(\cdot),\gamma(\cdot)}_E(E)$ is just the central BMO $(E)$ (the local version of BMO $(E)$).

Theorem 7. Let $E$, $\Omega$, $p(x)$, $q(x)$, $\alpha(x)$ be the same as in Theorem 3. Let also $p_1 = \frac{1}{p} - \frac{1}{q} \frac{\alpha(x)}{n} + \frac{1}{p_2}$, $q_1 = \frac{1}{p} - \frac{\alpha(x)}{n}$ and $b \in C^{q(\cdot),\gamma(\cdot)}_E(E)$. Suppose that $p_1(\cdot)$, $p_2(\cdot)$, $q(\cdot)$, $q_1(\cdot)$ and $\alpha(\cdot)$ satisfy (2.17). Then, for $\frac{n}{n-1} < p^- \leq p(\cdot) < \frac{n}{\alpha(\cdot)}$, the following pointwise estimate

\[
\| [b, I_{\Omega,\alpha(x)}] f \|_{L^q(B(x,r))} \lesssim \| b \|_{C^{q(\cdot),\gamma(\cdot)}_E(E)} \frac{\text{diam}(E)}{2r} \int_0^{\text{diam}(E)} \left( 1 + \ln \frac{t}{r} \right)^{q(x)-\frac{1}{q_1(x)} - 1} \| f \|_{L^{p_1(\cdot)}(B(x,t))} dt
\]

holds for any ball $B(x,r)$ and for all $f \in L^{p_1(\cdot)}_{\text{loc}}(E)$.

If the functions $w_1(x,r)$ and $w_2(x,r)$ satisfy (2.17) as well as the following Zygmund condition

\[
\int_0^{\text{diam}(E)} \left( 1 + \ln \frac{t}{r} \right)^{\frac{1}{1-\alpha(x)+\gamma(x)}} dt \lesssim w_2^{\frac{1}{\alpha(x)+\gamma(x)}}(x,r), \quad r \in (0, \text{diam}(E))
\]
and additionally these functions satisfy the conditions \([2.17], [2.20]\),

\[
d_{\delta} := \int_{\delta}^{\text{diam}(E)} \sup_{x \in \Omega} \left( 1 + \ln \frac{t}{r} \right) \frac{w_{1,\alpha}(x, t)}{t^{1-(\alpha(x) + \eta(x))}} dt < \infty, \quad \delta > 0,
\]

then the operators \([b, I_{\Omega,\alpha(\cdot)}]\) and \([b, M_{\Omega,\alpha(\cdot)}]\) are \(\left( VL^{p_{1}(\cdot),w_{1}(\cdot)}(E) \rightarrow VL^{q_{1}(\cdot),w_{2}(\cdot)}(E) \right)\) bounded. Moreover,

\[
\| [b, I_{\Omega,\alpha(\cdot)}] f \|_{VL^{q_{1}(\cdot),w_{2}(\cdot)}(E)} \lesssim \| b \|_{C^{p_{2}(\cdot),\gamma(\cdot)}} \| f \|_{VL^{p_{1}(\cdot),w_{1}(\cdot)}(E)},
\]

\[
\| [b, M_{\Omega,\alpha(\cdot)}] f \|_{VL^{q_{1}(\cdot),w_{2}(\cdot)}(E)} \lesssim \| b \|_{C^{p_{2}(\cdot),\gamma(\cdot)}} \| f \|_{VL^{p_{1}(\cdot),w_{1}(\cdot)}(E)}.
\]

**Proof.** Since \([b, M_{\Omega,\alpha(\cdot)}](f) \lesssim [b, I_{\Omega,\alpha(\cdot)}](f)\), it suffices to treat only the case of the operator \([b, I_{\Omega,\alpha(\cdot)}]\). As in the proof of Theorem \([\ref{thm}4]\) we represent the function \(f\) in the form \([2.25]\) and have

\[
[b, I_{\Omega,\alpha(\cdot)}] f(x) = (b(x) - b_{B(x,r)}) I_{\Omega,\alpha(\cdot)} f_{1}(x) - I_{\Omega,\alpha(\cdot)} (\{(b(\cdot) - b_{B(x,r)}) f_{1}\})(x) + (b(x) - b_{B(x,r)}) I_{\Omega,\alpha(\cdot)} f_{2}(x) - I_{\Omega,\alpha(\cdot)} (\{(b(\cdot) - b_{B(x,r)}) f_{2}\})(x)
\]

\[
= F_{1} + F_{2} + F_{3} + F_{4}.
\]

Hence we get

\[
\| [b, I_{\Omega,\alpha(\cdot)}] f \|_{L^{p}(B(x,r))} \leq \| F_{1} \|_{L^{p}(\hat{B}(x,r))} + \| F_{2} \|_{L^{p}(\hat{B}(x,r))} + \| F_{3} \|_{L^{p}(\hat{B}(x,r))} + \| F_{4} \|_{L^{p}(\hat{B}(x,r))}.
\]

First, we use the Hölder’s inequality such that \(\frac{1}{q_{1}(\cdot)} = \frac{1}{p_{1}(\cdot)} + \frac{1}{q_{2}(\cdot)}\), the boundedness of \(I_{\Omega,\alpha(\cdot)}\) from \(L^{p}(\cdot)\) into \(L^{q}(\cdot)\) (see Theorem \([\ref{thm}4]\)) and \([2.25]\) to estimate \(\| F_{1} \|_{L^{p_{1}}(\hat{B}(x,r))}\), and we obtain

\[
\| F_{1} \|_{L^{p_{1}}(\hat{B}(x,r))} = \| (b(\cdot) - b_{B}) I_{\Omega,\alpha(\cdot)} f_{1}(\cdot) \|_{L^{p_{1}}(\hat{B}(x,r))} \lesssim \| b(\cdot) - b_{B} \|_{L^{p_{2}}(\hat{B}(x,r))} \| I_{\Omega,\alpha(\cdot)} f_{1}(\cdot) \|_{L^{q_{1}}(\hat{B}(x,r))} \lesssim r^{-\frac{\alpha}{p_{2}}(\cdot) + \eta(x)} \| b \|_{C^{p_{2}(\cdot),\gamma(\cdot)}} \| f_{1} \|_{L^{p_{1}}(\hat{B}(x,r))}
\]

\[
= \| b \|_{C^{p_{2}(\cdot),\gamma(\cdot)}} r^{-\frac{\alpha}{p_{2}}(\cdot) + \eta(x)} \| f \|_{L^{p_{1}}(\hat{B}(x,r))} \int_{2r}^{\text{diam}(E)} t^{-1 - \frac{\alpha}{p_{2}(\cdot)} - \eta(x)} dt \lesssim \| b \|_{C^{p_{2}(\cdot),\gamma(\cdot)}} r^{-\frac{\alpha}{p_{2}}(\cdot) + \eta(x)} \| f \|_{L^{p_{1}}(\hat{B}(x,r))} \int_{2r}^{\text{diam}(E)} t^{-\frac{\alpha}{p_{2}(\cdot)} + \eta(x) - \frac{\alpha}{q_{2}(\cdot)} - 1} dt.
\]

Second, for \(\| F_{2} \|_{L^{p_{1}}(\hat{B}(x,r))}\), applying the boundedness of \(I_{\Omega,\alpha(\cdot)}\) from \(L^{p}(\cdot)\) into \(L^{q}(\cdot)\) (see Theorem \([\ref{thm}4]\)), generalized Hölder’s inequality such that \(\frac{1}{p_{1}(\cdot)} = \frac{1}{p_{1}(\cdot)} + \frac{1}{p_{2}(\cdot)}\),
\[
\frac{1}{q(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{q_1(\cdot)} \quad \text{and} \quad (2.35), \text{ we know that}
\]
\[
\|F_2\|_{L^q(\cdot)}(B(x,r)) = \left\| I_{\Omega, \alpha(\cdot)} \left( b(\cdot) - b_{B(x,r)} \right) f_1 \right\|_{L^q(\cdot)}(B(x,r)) \\
\lesssim \left\| (b(\cdot) - b_{B(x,r)}) f_1 \right\|_{L^{p_2(\cdot)}(B(x,r))} \\
\lesssim \left\| (b(\cdot) - b_{B(x,r)}) \right\|_{L^{p_2(\cdot)}(B(x,r))} \left\| f_1 \right\|_{L^{p_1(\cdot)}(B(x,r))} \\
\lesssim \|f\|_{C^{p_2(\cdot), \gamma(\cdot)} \cap L^{p_2(\cdot), n\gamma(\cdot)}} \|f\|_{L^{p_1(\cdot)}(B(x,2r))} \int_{2r}^{\text{diam}(E)} t^{-1} - \frac{\text{diam}(E)}{q_1(\cdot)} dt.
\]

Third, for \( \|F_3\|_{L^q(\cdot)}(B(x,r)) \), similar to the proof of (2.27), when \( \frac{1}{q_1(\cdot)} \leq p_1(\cdot) \), by Fubini’s theorem, generalized Hölder’s inequality and (1.3), we have
\[
\left| I_{\Omega, \alpha(\cdot)} f_2(x) \right| \lesssim \int_{E \cap B(x,2r)} \frac{\left| \Omega(z - y) \right|}{|x - y|^{\alpha(x)}} dy \\
\int_{\text{diam}(E)} \|f\|_{L^{p_1(\cdot)}(B(x,t))} t^{-1} - \frac{\text{diam}(E)}{q_1(\cdot)} dt.
\]
(2.38)

Thus, by generalized Hölder’s inequality such that \( \frac{1}{q(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{q_1(\cdot)} \), (2.35) and (2.38), we obtain
\[
\|F_3\|_{L^q(\cdot)}(B(x,r)) = \left\| I_{\Omega, \alpha(\cdot)} \left( (b(\cdot) - b_{B(x,r)}) f_2(\cdot) \right) \right\|_{L^q(\cdot)}(B(x,r)) \\
\lesssim \left\| (b(\cdot) - b_{B(x,r)}) \right\|_{L^{p_2(\cdot)}(B(x,r))} \|I_{\Omega, \alpha(\cdot)} f_2(\cdot)\|_{L^{p_1(\cdot)}(B(x,r))} \\
\lesssim \left\| (b(\cdot) - b_{B(x,r)}) \right\|_{L^{p_2(\cdot)}(B(x,r))} \|I_{\Omega, \alpha(\cdot)} f_2(\cdot)\|_{L^{p_1(\cdot)}(B(x,r))} \\
\lesssim \|f\|_{C^{p_2(\cdot), \gamma(\cdot)} \cap L^{p_2(\cdot), n\gamma(\cdot)}} \|f\|_{L^{p_1(\cdot)}(B(x,t))} t^{-1} - \frac{\text{diam}(E)}{q_1(\cdot)} dt.
\]

Finally, we consider the term \( \|F_4\|_{L^q(\cdot)}(B(x,r)) \), \( \|I_{\Omega, \alpha(\cdot)} \left( (b(\cdot) - b_{B(x,r)}) f_2(\cdot) \right) \|_{L^q(\cdot)}(B(x,r)) \).

For \( z \in B(x,r) \), when \( \frac{1}{q(\cdot)} \leq p(\cdot) \), by the Fubini’s theorem, applying the generalized Hölder’s inequality and from (1.3) and (2.35) we have
\[
\left| I_{\Omega, \alpha(\cdot)} \left( (b(\cdot) - b_{B(x,r)}) f_2(\cdot) \right) \right| \lesssim \int_{2r}^{\text{diam}(E)} \left| b(y) - b_{B(x,r)} \right| \left| \Omega(z - y) \right| \frac{|f(y)|}{|x - y|^{\alpha(x)}} dy \\
\approx \int_{2r}^{\text{diam}(E)} \int_{2r < |x - y| < t} \left| b(y) - b_{B(x,r)} \right| \left| \Omega(z - y) \right| |f(y)| dy \frac{dt}{|x - y|^{\alpha(x)+1}}.
\]
Then, by (2.39) we have

\[ \int_{2r}^{diam(E)} \int_{B(x,t)} \left| b(y) - b_{B(x,t)} \right| \left| \Omega(z - y) \right| |f(y)| \frac{dy}{r(t)} \frac{dt}{r(t)} \]

Combining all the estimates of \( \|b\|_{C^2_{\Pi}(\cdot \cdot \cdot)} \), we get (2.36). At last, by Definition 4, (2.36) and (2.37) we get (2.39). Then, by (2.39) we have

\[ \|F_4\|_{L^q(\cdot \cdot \cdot)}(B(x,r), t) \leq \|I_{\Omega, \alpha}(\cdot \cdot \cdot) f_2(\cdot \cdot \cdot) \|_{L^q(\cdot \cdot \cdot)}(B(x,r)) \]

Combining all the estimates of \( \|F_1\|_{L^q(\cdot \cdot \cdot)}(B(x,r)), \|F_2\|_{L^q(\cdot \cdot \cdot)}(B(x,r)), \|F_3\|_{L^q(\cdot \cdot \cdot)}(B(x,r)) \), we get (2.36). At last, by Definition 4 (2.36) and (2.37) we get

\[ \left\| \left[ b, I_{\Omega, \alpha}(\cdot \cdot \cdot) f \right] \right\|_{V_{L^q(\cdot \cdot \cdot), w_2(\cdot \cdot \cdot)}^\gamma(\cdot \cdot \cdot)} = \sup_{x \in \Pi, r > 0} \frac{r^{-\frac{n\gamma}{q} - 1}}{w_2(x, r)^{\frac{\gamma}{q}}} \int_{2r}^{diam(E)} \left(1 + \ln \frac{t}{r}\right) t^{n\gamma(x) - \frac{n}{q_1} - 1} dt \]

\[ \times \left\| [b, I_{\Omega, \alpha}(\cdot \cdot \cdot) f] \right\|_{L^q(\cdot \cdot \cdot)}(B(x,r), t) \]

\[ \leq \left\| [b]_{C^2_{\Pi}(\cdot \cdot \cdot)} \left\| f \right\|_{V_{L^q(\cdot \cdot \cdot), w_2(\cdot \cdot \cdot)}^\gamma(\cdot \cdot \cdot)} \sup_{x \in \Pi, r > 0} \frac{1}{w_2(x, r)^{\frac{\gamma}{q}}} \int_{2r}^{diam(E)} \left(1 + \ln \frac{t}{r}\right) t^{n\gamma(x) - \frac{n}{q_1} - 1} dt \]

\[ \times t^{n\gamma(x) - \frac{n}{q_1} - 1} \frac{1}{w_2(x, r)^{\frac{\gamma}{q}}} \left\| f \right\|_{V_{L^q(\cdot \cdot \cdot), w_2(\cdot \cdot \cdot)}^\gamma(\cdot \cdot \cdot)} \]

\[ \times \frac{1}{w_2(x, r)^{\frac{\gamma}{q}}} \int_{2r}^{diam(E)} \left(1 + \ln \frac{t}{r}\right) t^{n\gamma(x) - \frac{n}{q_1} - 1} dt \]
and

$$\limsup_{r \to 0} \frac{r^{-\frac{\alpha(x)}{p}}} {w_2(x,t)^{\frac{1}{p}}} \| [b, I_{\Omega,\alpha(x)}] f \|_{L^p(\bar{B}(x,r))} \lesssim \limsup_{r \to 0} \frac{r^{-\frac{\alpha(x)}{p}}} {w_1(x,t)^{\frac{1}{p}}} \| f \|_{L^p(\bar{B}(x,r))} = 0,$$

which completes the proof of Theorem 7.

**Corollary 8.** Let $E, \Omega, p(x), q(x), \alpha(x)$ be the same as in Theorem 7. Suppose that $q(\cdot)$ and $\alpha(\cdot)$ satisfy (1.7). Then, for $\frac{s-1}{s+1} < p^- \leq p(\cdot) < \frac{s}{\alpha(\cdot)}$ and $b \in BMO(E)$, the following pointwise estimate

$$\| [b, I_{\Omega,\alpha(x)}] f \|_{L^p(\bar{B}(x,r))} \lesssim \| b \|_{BMO} \frac{diam(E)}{r^\frac{\alpha(x)}{p}} \int_{2r}^r \left(1 + \ln \frac{t}{r}\right) \frac{w_1(x,t)^{\frac{1}{p}}}{t^{1-\frac{\alpha(x)}{p}}} \| f \|_{L^p(\bar{B}(x,t))} dt$$

holds for any ball $\bar{B}(x,r)$ and all $f \in L^p_{loc}(E)$.

If the functions $w_1(x,r)$ and $w_2(x,r)$ satisfy (2.17) as well as the following Zygmund condition

$$\int_{r}^{diam(E)} \left(1 + \ln \frac{t}{r}\right) \frac{w_1^{\frac{1}{p}}(x,t)}{t^{1-\frac{\alpha(x)}{p}}} dt \lesssim w_2^{\frac{1}{p}}(x,r), \quad r \in (0, diam(E))$$

and additionally these functions satisfy the conditions (2.19), (2.20),

$$d_\delta := \int_\delta^{diam(E)} \sup_{x \in \Omega} \left(1 + \ln \frac{t}{r}\right) \frac{w_1^{\frac{1}{p}}(x,t)}{t^{1-\frac{\alpha(x)}{p}}} dt < \infty, \quad \delta > 0,$$

then the operators $[b, I_{\Omega,\alpha(x)}]$ and $[b, M_{\Omega,\alpha(x)}]$ are $\left(VL^p_{\Omega}(w_1(\cdot)(E) \to VL^q_{\Omega}(w_2(\cdot)(E))\right)$ bounded. Moreover,

$$\| [b, I_{\Omega,\alpha(x)}] f \|_{VL^p_{\Omega}(w_2(\cdot)(E))} \lesssim \| b \|_{BMO} \| f \|_{VL^p_{\Omega}(w_1(\cdot)(E))},$$

$$\| [b, M_{\Omega,\alpha(x)}] f \|_{VL^p_{\Omega}(w_2(\cdot)(E))} \lesssim \| b \|_{BMO} \| f \|_{VL^p_{\Omega}(w_1(\cdot)(E))}.$$

For $\alpha(x) = 0$ in Theorem 7 we get the following new result:

**Corollary 9.** Let $E, \Omega, p(x)$ be the same as in Theorem 7. Let also $\frac{n}{p_1(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$ and $b \in C^{p_2(\cdot),\gamma(\cdot)}(\Omega)$. Suppose that $p_1(\cdot)$ and $p_2(\cdot)$ satisfy (1.7). Then, for $\frac{s-1}{s+1} < p^- \leq p(\cdot) \leq p^+ < \infty$, the following pointwise estimate

$$\| [b, T_{\Omega}] f \|_{L^p(\bar{B}(x,r))} \lesssim \| b \|_{C^{p_2(\cdot),\gamma(\cdot)}} \frac{diam(E)}{r^\frac{n}{p_1(\cdot)}} \int_{2r}^r \left(1 + \ln \frac{t}{r}\right) \frac{w_1^{\frac{1}{p}}(x,t)^{\frac{n}{p_1(\cdot)}}}{t^{1-n\gamma(x)}} \| f \|_{L^p(\bar{B}(x,t))} dt$$

holds for any ball $\bar{B}(x,r)$ and all $f \in L^p_{loc}(E)$.

If the function $w(x,r)$ satisfies (2.17) as well as the following Zygmund condition

$$\int_{r}^{diam(E)} \left(1 + \ln \frac{t}{r}\right) \frac{w_1^{\frac{1}{p}}(x,t)^{\frac{n}{p_1(\cdot)}}}{t^{1-n\gamma(x)}} dt \lesssim w_1^{\frac{1}{p}}(x,r), \quad r \in (0, diam(E))$$
and additionally this function satisfies the conditions \([2.19] - [2.22]\),
\[
\begin{align*}
\delta (E) & := \int_{\delta} \sup_{t \in \Omega} \left( 1 + \ln \frac{t}{r} \right) \frac{w^{\varphi_{E}}(x, t)}{t^{1-n\gamma(x)}} dt < \infty, \quad \delta > 0,
\end{align*}
\]
then the operators \([b,T_{\Omega}]\) and \([b,M_{\Omega}]\) are \((V L_{H}^{p(\cdot),w(\cdot)}(E) \to V L_{H}^{p(\cdot),w(\cdot)}(E))\)-bounded. Moreover,
\[
\begin{align*}
\| [b,T_{\Omega}] f \|_{V L_{H}^{p(\cdot),w(\cdot)}(E)} & \lesssim \| b \|_{C_{H}^{p(\cdot),w(\cdot)}(\cdot)} \| f \|_{V L_{H}^{p(\cdot),w(\cdot)}(E)}, \\
\| [b,M_{\Omega}] f \|_{V L_{H}^{p(\cdot),w(\cdot)}(E)} & \lesssim \| b \|_{C_{H}^{p(\cdot),w(\cdot)}(\cdot)} \| f \|_{V L_{H}^{p(\cdot),w(\cdot)}(E)}.
\end{align*}
\]
From Corollary 10 we get the following:

**Corollary 10.** Let \(E, \Omega, p(x)\) be the same as in Theorem 4. Then, for \(\frac{\gamma}{n} < p_{-} \leq p_{+} < \infty\) and \(b \in BMO(\Omega)\), the following pointwise estimate
\[
\| [b,T_{\Omega}] f \|_{L^{p(\cdot)}(B(x,r))} \lesssim \| b \|_{BMO} \| f \|_{L^{p(\cdot)}(B(x,r))} 
\]
holds for any ball \(B(x,r)\) and for all \(f \in L^{p(\cdot)}(E)\).

If the function \(w(x,r)\) satisfies \([2.17]\) as well as the following Zygmund condition
\[
\begin{align*}
\int_{r}^{diam(E)} \left( 1 + \ln \frac{t}{r} \right) \frac{w^{\varphi_{E}}(x, t)}{t} dt & \lesssim w^{\varphi_{E}}(x, r), \quad r \in (0, diam(E))
\end{align*}
\]
and additionally this function satisfies the conditions \([2.19] - [2.22]\),
\[
\begin{align*}
\delta(E) & := \int_{\delta} \sup_{t \in \Omega} \left( 1 + \ln \frac{t}{r} \right) \frac{w^{\varphi_{E}}(x, t)}{t^{1-n\gamma(x)}} dt < \infty, \quad \delta > 0,
\end{align*}
\]
then the operators \([b,T_{\Omega}]\) and \([b,M_{\Omega}]\) are bounded on \(V L_{H}^{p(\cdot),w(\cdot)}(E)\). Moreover,
\[
\begin{align*}
\| [b,T_{\Omega}] f \|_{V L_{H}^{p(\cdot),w(\cdot)}(E)} & \lesssim \| b \|_{BMO} \| f \|_{V L_{H}^{p(\cdot),w(\cdot)}(E)}, \\
\| [b,M_{\Omega}] f \|_{V L_{H}^{p(\cdot),w(\cdot)}(E)} & \lesssim \| b \|_{BMO} \| f \|_{V L_{H}^{p(\cdot),w(\cdot)}(E)}.
\end{align*}
\]

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