Spatially modulated two- and three-component Rabi-coupled Gross–Pitaevskii systems

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Abstract
Vector rogue wave (RW) formation and its dynamics in Rabi-coupled two- and three-species Bose–Einstein condensates with spatially varying dispersion and nonlinearity are studied. For this purpose, we obtain the RW solution of the two- and three-component inhomogeneous Gross–Pitaevskii (GP) systems with Rabi coupling by introducing suitable rotational and similarity transformations. Then, we investigate the effect of inhomogeneity (spatially varying dispersion, trapping potential and nonlinearity) on vector RWs for two different forms of potential strengths, namely periodic (optical lattice) with specific reference to hyperbolic-type potentials and parabolic cylinder potentials. First, we show an interesting oscillating boomeronic behaviour of dark–bright solitons due to Rabi coupling in two-component condensate with constant nonlinearities. Then in the presence of inhomogeneity but in the absence of Rabi coupling we demonstrate the creation of new daughter RWs co-existing with a dark (bright) soliton part in the first (second) component of the two-component GP system. Further, the hyperbolic modulation (sech type) of the parameter along with the Rabi effect leads to the formation of dromion (two-dimensional localized structure) trains even in the (1 + 1) dimensional two-component GP system, which is a striking feature of Rabi coupling with spatial modulation. Next, our study on three-component condensate reveals the fact that the three RWs can be converted into broad-based zero-background RW appearing on top of a bright soliton by introducing spatial modulation only. Further, by including Rabi coupling we observe beating behaviour of solitons with internal oscillations mostly at the wings. Also, we show that by employing parabolic cylinder modulation with model parameter $n$, one can produce $(n + 1)$ RWs. Our explores that the spatially varying environment

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leads to the possibility of realization of a two-dimensional nonlinear structure in (1 + 1) dimensional setting.

Keywords: Rabi coupled Bose–Einstein condensates, rogue waves, spatially modulated nonlinearity, similarity transformation

(Some figures may appear in colour only in the online journal)

1. Introduction

The past two decades have witnessed a growing interest in rogue wave (RW) research [1–4]. These RWs are indeed giant nonlinear waves that appear from nowhere and disappear without a trace [5]. It is generally recognized that modulation instability is one of the prime mechanisms leading to the formation of RW excitation [1–6]. The versatile nonlinear Schrödinger (NLS) equation is a prototype nonlinear equation that features such RWs. Study of RWs has attracted significant attention not only in the more standard ocean-surface-dynamical problem [1, 2], but also in other physical contexts. Indeed, there exists a vast amount of theoretical and experimental works on RWs in various fields ranging from optics [4, 6–9] and Bose–Einstein condensates (BECs) [1, 10] to plasmas [11] and atmospheric dynamics [12] (see also the recent short review [13]). Recently, higher-order RWs [14] were excited successfully in a water wave tank. This suggests that a higher-order analytic RW solution is of physical relevance and can be realized experimentally [14, 15].

These comprehensive theoretical and experimental studies on single-component RWs stimulated increasing interest in investigating multi-component RWs. Rogue waves of the two-component Manakov system [16] have been studied extensively in the focusing regime [17, 18] and in the defocusing regime [19–21]. The higher-order RW solutions of the Manakov system have also been constructed by employing the Darboux transformation (DT) method [18]. Furthermore, RWs for the three-component NLS (3-CNLS) system in the focusing regime feature an interesting four-petal-pattern [22, 23]. Very recently the single, doublet, triplet and quadruple RWs of a fairly general M-component NLS system has been obtained by using the Kadomtsev–Petviashvili hierarchy reduction method [24]. Such wave profiles are otherwise not possible in single- and two-component NLS systems. These studies indicate that there exists very abundant pattern dynamics for RWs in the multi-component nonlinear systems, which is quite distinct from those of single-component (scalar) nonlinear dynamical systems.

Bose–Einstein condensates act as a fertile arena for the theoretical and experimental realization of multi-component nonlinear systems. These multi-component BECs consist of atoms in different internal states or even of different species. There are several experiments related to such multi-component condensates [25–27]. An interesting experiment [28] on $^{87}$Rb shows that the irradiation of the condensate with an electric field induces a linear coupling (proportional to the Rabi frequency) in the overlap region, which causes Rabi oscillations between the two components. This Rabi coupling transfers an arbitrary fraction of atoms from one component to another component, e.g. from the $|1, 1\rangle$ state to the $|2, 1\rangle$ state. In [29], it was shown that in analogy with systems arising in the field of nonlinear fibre optics (such as a twisted fibre with two linear polarizations), exact Rabi oscillations between two condensates can be analytically found when inter-species coupling are equal to unity. Further, [30, 31] generalized this concept to multi-component BECs with time-varying Rabi coupling and showed that the
Rabi switch is very robust with high efficiency when transferring a condensate wave function between the components.

The techniques for managing nonlinearity (spatial or temporal dependence of nonlinearity) to produce nonlinear waves in BECs have also gained considerable attention [32]. Specifically, the Gross–Pitaevskii (GP) equation describing the dynamics of BEC in the mean field approximation has been examined in the presence of temporally [33] or spatially [28, 34, 35] varying nonlinearity coefficients. Such spatial or temporal variations of nonlinearities can be realized experimentally very well in the atomic condensate by employing the Feshbach resonance mechanism, where the $s$-wave scattering length is tuned in space or time with the aid of external magnetic [29] or optical [30] fields (for a review see [36]). In particular, much attention has been paid to spatially modulated nonlinearities in BECs which lead to collisionally inhomogeneous condensates [37–41]. Recently, the scattering of such matter wave solitons at an interface has been studied in [42]. Then the dark solitons in such a spatially varying environment with linear and nonlinear potential steps and a double potential barrier have been studied in [43, 44] respectively. Very recently, the scattering of such matter wave solitons at an interface has been studied in [42]. Then the dark solitons in such a spatially varying environment with linear and nonlinear potential steps and a double potential barrier have been studied in [43, 44] respectively. Very recently, single and multi-peak two-dimensional (2D) solitons and vortex solitons have been studied in parity–time ($PT$) symmetric atomic condensates and condensates with spatially modulated nonlinearities [45].

On the other hand, the experimental demonstration of controlling the dispersion of a matter wave packet by engineering the effective mass [46] and the observation of variation of the sign of the effective mass of BEC from positive to negative when the acceleration is no longer proportional to the force [41, 47] suggest the possibility of varying mass also along the spatial direction in the description of BEC. This varying mass will ultimately lead to the coefficient of the kinetic energy term (spatial dispersion coefficient) in the GP system responsible for the dispersion of the matter wave to be a function of spatial co-ordinates. From another experimental view point, spatial variation of mass can also be achieved by imposing a nonuniform optical lattice created by laser beams over a condensate in a cigar-shaped trapping potential [48]. In addition to atomic condensates, the NLS-type equations with trapping potential, varying dispersion and nonlinearity (referred as variable coefficient GP system) have been studied extensively in the context of nonlinear optics, especially in dispersion managed systems [49], and the notion of nonlinearity management was also theoretically developed [50] and was well established experimentally later on [51].

These fascinating concepts of varying dispersion and nonlinearity management can be profitably employed to stabilize RWs by making their corresponding coefficients as functions of the propagation distance or transverse coordinate, or both. One of the mathematical approaches used to deal with such problems is to employ a similarity transformation which is indeed a nonlinear transformation that transforms the original variable coefficient nonlinear evolution equation into an integrable system that admits soliton, breather, rogue wave solutions, etc [52–59]. This method works very well in higher dimensions too [60, 61]. In this way, the exact RW solutions have been obtained in several (1 + 1) dimensional scalar and vector GP systems by choosing appropriate forms of trapping potential and nonlinearity coefficients [19, 35, 59, 62–64]. In particular, dynamics of bright–bright, bright–dark solitons and vector boomeronic solitons in non-autonomous Rabi-coupled GP system are discussed in [59, 63] in a detailed manner. Later, the tuning mechanism of dispersion along with nonlinearity is employed to study RWs in the Manakov system with spatially varying coefficients for a parabolic cylinder potential [35]. Subsequently, the binary condensates in a three-dimensional setting with spatially varying nonlinearity is considered in [65] and vector solitary waves are obtained along with a stability analysis. As a next step, it is quite natural to pose the question of how the spatial variation of the nonlinearity and dispersion influences the RW phenomenon in Rabi-coupled condensates. This study will also be relevant to pico-second...
pulse propagation in tapered and twisted birefringent fibres as the governing equation for both these settings are identical.

Motivated by the above reasons, in this work we study the multi-component BECs, particularly two- and three-component BECs, with a spatially dependent dispersion coefficient, scattering length and trapping potential by including Rabi coupling.

The rest of this paper is organized as follows. In section 2, we introduce two successive transformations and reduce the two-component inhomogeneous (spatially modulated) GP system (1) to the canonical Manakov system, and discuss the general RW solutions by considering two different types of trapping potentials, namely optical lattice and parabolic cylinder potentials. Then, in section 3, we consider the three-component inhomogeneous GP equation with Rabi coupling and explore interesting dynamical features of RWs. Finally, we conclude our study in section 4.

2. Two-component GP equation with Rabi coupling

2.1. Description of the system

To start with, we consider a spatially modulated two-component GP (2-CGP) equation with Rabi coupling following [31, 34, 35]

\[
\begin{align*}
\frac{\partial \psi_j}{\partial t} &+ \gamma(x) \frac{\partial^2 \psi_j}{\partial x^2} + 2 \sum_{l=1}^{2} g_{jl}(x) |\psi_l|^2 \psi_j + V(x,t) \psi_j - \sum_{l=1,l\neq j}^{2} \sigma \psi_l = 0, \quad j = 1, 2, \\
\end{align*}
\]

(1)

where \( \psi_j \) is the macroscopic wave function of the \( j \)th component in the context of BEC and it represents the complex electric field envelope of the \( j \)th component in an optical setting. In equation (1), \( t \) is the time coordinate and \( x \) is the spatial co-ordinate in the BEC setting, while in optics respectively they are the longitudinal and transverse coordinates. The dispersion parameter \( \gamma(x) = \frac{\hbar}{2m(x)} \), where \( m \) is the effective mass, \( \sigma \) is the Rabi coupling coefficient responsible for the interaction of two or more different hyperfine state atoms, and \( g_{jl}(x) \) are the nonlinearity coefficients related to the \( s \)-wave scattering lengths \( a_{jl}(x) \) through \( g_{jl}(x) = \frac{4\pi \hbar^2 a_{jl}(x)}{m} \), where \( a_B \) is the Bohr radius. Based on experimental results pertaining to two-component \(^{87}\text{Rb} \) BECs [66–68], we can assume that the scattering length ratios are almost equal to one, and also they can be tuned through Feshbach resonance, as it was demonstrated in experimental works [69, 70]. Hence, without loss of generality, we consider equal interaction strengths, i.e. \( g_{jl}(x) = g(x) \) in equation (1). Here \( V(x,t) \) represents external spatially varying trapping potential.

2.2. Transformation to the Manakov system

Here our aim is to transform equation (1) to an integrable nonlinear evolution equation with constant coefficients, namely the canonical Manakov system. For this purpose, we first perform the rotational transformation

\[
\begin{align*}
\begin{pmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{pmatrix} &= \begin{pmatrix} \cos(\sigma t) & -i \sin(\sigma t) \\ -i \sin(\sigma t) & \cos(\sigma t) \end{pmatrix} \begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{pmatrix},
\end{align*}
\]

(2)

in equation (1). This results in the following inhomogeneous 2-CGP system.
\[ i \frac{\partial \phi_j}{\partial t} + \gamma(x) \frac{\partial^2 \phi_j}{\partial x^2} + 2g(x) \sum_{l=1}^{2} |\phi_l|^2 \phi_j + V(x, t) \phi_j = 0, \quad j = 1, 2. \] (3)

The above equation (3) can then be transformed into the celebrated Manakov system with the canonical form [16]

\[ i \frac{\partial q_j}{\partial T} + \frac{\partial^2 q_j}{\partial Y^2} + 2 \sum_{l=1}^{2} |q_l|^2 q_j = 0, \quad j = 1, 2, \] (4)

by employing the similarity transformation

\[ \phi_j(x, t) = N(x) q_j(T, Y), \quad j = 1, 2, \] (5)

where \( Y = Y(x) \) and \( T = t \) are self-similar variables.

The real amplitude \( N(x) \) in (5) satisfies the following set of coupled ordinary differential equations

\[ \gamma(x) \frac{d^2 N(x)}{dx^2} + V(x, t) N(x) = 0, \] (6a)
\[ 2 \frac{dN(x)}{dx} \frac{dY(x)}{dx} + N(x) \frac{d^2 Y(x)}{dx^2} = 0, \] (6b)

where the similarity variable \( Y(x) \) is defined in terms of amplitude as

\[ Y(x) = \int \frac{dx}{[N(x)]^2}. \] (6c)

The dispersion parameter \( \gamma(x) \) and nonlinearity function \( g(x) \) are related to \( N(x) \) respectively as

\[ \gamma(x) = [N(x)]^4, \quad g(x) = [N(x)]^{-2}. \] (6d)

Here we confine ourselves to only spatially dependent potentials, i.e. \( V(x, t) \equiv V(x) \), in view of experiments on BEC [46]. The conditions on \( g(x) \), \( \gamma(x) \) and \( Y(x) \) given by equation (6) can be viewed as the integrability conditions for the inhomogeneous 2-CGP system (3) and hence that of (1). The above equations clearly show that all the variable coefficients can be expressed in terms of \( N(x) \). Thus for a given form of trapping potential, one can determine the form of \( N(x) \) from (6a) which in turn results in the required dispersion parameter function and necessary nonlinearity management for obtaining exact nonlinear waves in equation (1).

In this work, we consider two types of trapping potentials namely optical lattice potential [71–73] and parabolic cylinder potential [74] for illustrative purposes.

(i) Periodic potentials (optical lattice potential)

First we are interested in the lattice potential having the general form

\[ V(x) = \gamma(x)(\lambda - \ell(\ell + 1))\text{msn}^2(x, m), \] (7)

which is similar to the Lamé potential with \( \lambda \) being the eigenvalue of the Lamé equation [75, 76], \( \ell \) is the order of the Lamé polynomial and \( \text{sn}(x, m) \) denotes the Jacobian ellipticity sine function with elliptic modulus \( m, 0 \leq m \leq 1 \). In this work, we consider first-order Lamé polynomials \( \text{sn}(x, m), \text{cn}(x, m) \) and \( \text{dn}(x, m) \) as eigenfunctions with eigenvalues \( \lambda = m + 1, 1 \) and \( m \), respectively. The potential is sinusoidal for the case \( m = 0 \) and thus describes an optical lattice potential. For \( m = 1 \), it results in the hyper-
bolic potential and the potential becomes doubly periodic if $m$ lies between 0 to 1. Such potentials expressed in terms $\text{sn}^2(x, m)$ function can be well approximated by a periodic lattice potential created from four interacting laser beams [48]. Experimentally, one can clearly observe that the matter wave dynamics can be changed and also be enhanced considerably when an optical lattice is imposed on the condensate [77].

The profiles of optical lattice potentials for $\ell = 1$ and the resulting $\gamma(x)$ are shown respectively in the left and right panels of figure 1.

(ii) Next, we consider the parabolic cylinder potential having a fairly general form:

$$V(x) = \gamma(x)(ax^2 + b),$$

where $a$ and $b$ are real constants and are chosen as $\frac{1}{2}$ and $n + \frac{1}{2}$ respectively, $n$ is the quantum model parameter which is a non-negative integer. Figures 2(a) and (b) represent the form of the parabolic cylinder potential $V(x)$ and the corresponding dispersion parameter function $\gamma(x)$ for $n = 0, 1, 2$, respectively. The functional forms of $N(x)$ and $\gamma(x)$ corresponding to the above two potentials are listed in table 1.

2.3. General rogue wave solutions

The general RW (semi-rational) solution of the canonical Manakov system (4) has been obtained by the DT method in [17] in a semi-rational form as

$$\begin{pmatrix}
q_1(Y, T) \\
q_2(Y, T)
\end{pmatrix} = \begin{bmatrix}
L & a_1 \\
B & a_2
\end{bmatrix} \begin{pmatrix}
M & a_2 \\
B & -a_1
\end{pmatrix} e^{2i\omega T},$$

where $L = \frac{1}{2} - 8\omega^2 T^2 - 2a^2 Y^2 + 8i\omega T + [f]_2 e^{2\omega Y}$, $M = 4f(aY - 2i\omega T - \frac{1}{2})e^{iY + i\omega T}$, $B = \frac{1}{2} + 8\omega^2 T^2 + 2a^2 Y^2 + [f]_2 e^{2\omega Y}$, $\omega = a^2$, $a = \sqrt{a_1^2 + a_2^2}$. Here $a_1$ and $a_2$ are real arbitrary parameters, $f$ is an arbitrary complex constant. This solution features standard RW for the choice $a_1 = a_2 \neq 0, f = 0$ and interesting boomeronic solitons (comprising of both bright and dark parts) accompanied by RWs for the parametric choice $a_1 = a_2 = f \neq 0$.

2.3.1. Constant coefficients 2-CGP system with Rabi coupling. It is of relevance to briefly discuss the important results of the Rabi-coupled two-component GP system with constant coefficients before addressing the variable coefficient system. For this purpose we write down the corresponding solution for this case with $\sigma \neq 0$, $g$ and $\gamma$ being constants in the absence of trapping potential.
\[ \psi_1(x,t) = \cos(\sigma t) q_1(x,t) - i \sin(\sigma t) q_2(x,t), \quad (10a) \]
\[ \psi_2(x,t) = -i \sin(\sigma t) q_1(x,t) + \cos(\sigma t) q_2(x,t). \quad (10b) \]

Figure 3 shows a special dynamical feature of the Manakov system in the presence of Rabi coupling alone. The top panels of figure 3 show that RW co-exists with an oscillating boomeronic soliton even for the choice \( a_1 = f \neq 0, a_2 = 0 \) in both the components for which the boomeronic soliton does not exist in the standard Manakov system. A careful look into the plot shows that at the centre \( (x,t) = (0,0) \) there exists RW and a dark soliton. Then during the first cycle \( (t = 0–15) \) of oscillation, that central dark soliton transforms into a boomeronic soliton \( (t = 2–5) \), which later becomes a completely bright soliton \( (t = 7–9) \) and finally attains
the original dark soliton form with an intermediate appearance of a boomeronic soliton in the range \((t = 11–14)\). This first cycle of oscillation is illustrated in the middle panels of figure \(3\). This process repeats periodically. On the other hand, the second component \(\psi_2\) displays a reverse dynamical behaviour as shown in the bottom panels of figure \(3\). Here the central bright soliton transforms into a boomeronic soliton, followed by an intermediate dark soliton form and finally retains the original bright form at the end of first cycle.

2.3.2. Rabi-coupled spatially modulated 2-CGP system. Next, we turn our attention to the variable coefficient 2-CGP system (1) for different trapping potentials. Now using the similarity transformation equation (5) and the rotational transformation equation (2), we construct the RW solution of the two-component inhomogeneous GP equation with Rabi coupling (1) as

\[
\psi_1(x, t) = N(x) \left( \frac{L}{B} (a_1 \cos(\sigma t) - ia_2 \sin(\sigma t)) + \frac{M}{B} (ia_1 \sin(\sigma t) + a_2 \cos(\sigma t)) \right) e^{2i\omega t},
\]

\(11a\)

\[
\psi_2(x, t) = N(x) \left( \frac{L}{B} (-ia_1 \sin(\sigma t) + a_2 \cos(\sigma t)) - \frac{M}{B} (a_1 \cos(\sigma t) + ia_2 \sin(\sigma t)) \right) e^{2i\omega t}.
\]

\(11b\)

Here the amplitude function \(N(x)\) depends on the various choices of external trapping potential.

2.3.3. Optical lattice potential.

(i) First, we consider the optical lattice potential discussed in the preceding section with the following form

\[
V(x) = ((m + 1) - 2m \text{sn}^2(x, m))\text{sn}^4(x, m).
\]

\(12\)

Then from equation (6a), the amplitude parameter is found to be \(N(x) = \text{sn}(x, m)\) and the integrability condition (6a) results in \(Y(x) = \frac{(x - E(\varphi, m)\text{sn}(x, m)) - \text{dn}(x, m)\text{cn}(x, m)}{\text{sn}(x, m)}\), where \(E\) is the elliptic integral of the second kind which in turn gives the spatially varying nonlinearity coefficient function as \(g(x) = \frac{1}{\text{sn}^2(x, m)}\) and the dispersion parameter function \(\gamma(x) = \text{sn}^4(x, m)\).

The nature of the vector RWs differs for different values of the modulus parameter \(m\). Let us consider the modulus parameter \(m\) to be unity. Ultimately \(V(x)\) becomes

Figure 2. Profiles of trapping potential \(V(x)\) and the dispersion parameter function \(\gamma(x)\) for different values of model parameter \(n\).
The standard Manakov soliton resulting for \( \sigma = V(x) = 0 \) and \( \gamma = g = 1 \) is shown in figures 4(a) and (b) for comparison purposes. In the absence of Rabi coupling and trapping potential the resulting solution supports RW with dark soliton part and RW with bright soliton part in the first and second components respectively (see figures 4(a) and (b)). Note that due to the inclusion of trapping potential, a new daughter RW with dark part and RW with bright part appears in first and second components respectively, thereby shifting the original parent wave (RW combined with soliton) to the right with a suppression in RW amplitude and an enhancement in soliton part amplitude (see at region \( x = 3 \)). This is shown in figures 4(c) and (d).

Next we include Rabi coupling, say \( \sigma = 0.045 \). Here the oscillating boomeronic behaviour discussed in figure 3, is observed predominantly only in the shifted parent RW–soliton pair located at \( x = 3 \) (see figure 4). The newly created daughter RW along with the dark soliton in an oscillating background remain almost unaffected. These are shown in figures 4(e) and (f).

Further analysis of solution (11) for trapping potentials resulting for other values of \( m \) (\( 0 < m < 1 \)) as given by (12) is straightforward. This choice will result in a structure resembling a two-dimensional optical lattice with modulations around the central region by RW and solitons which features the oscillating boomeronic soliton behaviour in that region. This is shown in figure 5.

(ii) Next, we consider a different form for the optical lattice potential.
Figure 4. Exact RW solutions in the system (1) for constant coefficients and in the absence of Rabi coupling (top panels), in the presence of variable coefficients without Rabi coupling (middle panels), and in the presence of both variable coefficients and Rabi coupling (bottom panels). The potential form is given by equation (13). The parameter values are chosen as $a_1 = 2.5$, $a_2 = 0$, $f = 0.8$, and $\sigma = 0.045$.

Figure 5. Exact RW solutions of the system (1) in the presence of both variable coefficients and Rabi coupling for the potential given by equation (12). The parameter values $a_1 = 2.5$, $a_2 = 0$, $f = 0.8$, $m = 0.9$, and $\sigma = 0.045$. 

$\psi_1 |^2$
\[ V(x) = (1 - 2m cn^2(x, m))cn^4(x, m), \]

which is depicted in figure 1(e). The amplitude parameter is calculated from equation (6a) as \( N(x) = cn(x, m) \) and the integrability condition yields \( Y(x) = x + \frac{1}{(m-1)} \left( E(\varphi, m) - \frac{sn(x, m)dn(x, m)}{cn(x, m)} \right) \). This determines the spatially varying nonlinearity coefficient function and the corresponding dispersion parameter function as \( g(x) = \frac{1}{cn^2(x, m)} \) and \( \gamma(x) = cn^4(x, m) \), respectively.

For the modulus parameter \( m = 1 \), the potential (14) takes the form

\[ V(x) = (1 - 2sech^2(x))sech^4(x). \]

To construct the solution of the spatially modulated two-component GP equation (1) with and without Rabi coupling, we start with the semi-rational solution (11) of the Manakov system where the first (second) component supports a dark (bright) soliton and a RW. In the absence of Rabi coupling \( (\sigma = 0) \), but in the presence of spatial modulation and external trapping potential, in the first component \( (\psi_1) \) the background of RW disappears and a zero-background (ZB) RW superimposes the bright soliton at \( ((x, t) = (0, 0)) \) and in the \( \psi_2 \) component the ZB RW appears with the bright soliton. Thus due to the trapping potential the original dark soliton disappears (see figure 6(a)) in the first component and the bright soliton gets shifted from its position in the second component (see figure 6(b)). By including Rabi coupling \( (\text{say } \sigma = 0.2) \), we observe the creation of very low intensity dromion (localized structure in \( x-t \) plane) trains which co-exist with RW at centre in the \( \psi_1 \)-component soliton as shown in figure 6(c). However, in the \( \psi_2 \) component we get dromion trains with significant intensity accompanied by an oscillating bright soliton. Clearly from figure 6(d) we note that when the dromion attains maximum intensity, the oscillating bright soliton reaches minimum intensity. Thus Rabi coupling induces creation of dromion trains even in this \( (1 + 1) \) dimensional spatially modulated 2-CGP system.

### 2.3.4. Parabolic cylinder potential

Next, we focus our attention on another form of spatial modulation of the trapping potential strength namely parabolic cylinder potential (8)

\[ V(x) = k^2D_n^2(x) \left( n + \frac{1}{2} - \frac{1}{4}k^2 \right). \]

Interestingly, for this choice the integrability condition \( Y(x) = \int [kD_n(x)]^{-2}dx \) turns out to be the standard Weber equation having the following solution [78] as \( N(x) = kcn(x, n), n = 0, 1, 2 \). This in turn gives the spatially varying nonlinearity coefficient function as \( g(x) = (1/k^2)D_n^2(x) \) and also the dispersion parameter function as \( \gamma(x) = k^2D_n^2(x) \). Here \( D_n(x) \) is the parabolic cylinder function of order \( n \) (control parameter), \( c_1 \) is an arbitrary integration constant, and \( k = \sqrt{2\pi n} \) is the normalization constant. Without loss of generality, one can choose the constant \( c_1 \) to be unity. Clearly, as \( |x| \rightarrow \infty \) for any non-negative integer \( n \), \( D_n(x) \) is stable.

Here, we construct different types of RW profiles depending on the parabolic cylinder potential by varying the quantum model parameter \( n \) and also the Rabi term \( \sigma \). We would like to recall that the parabolic cylinder function has \( (n+1) \) modes as shown in figure 2. Such a trapping potential with a barrier form can be realized experimentally [74, 79]. In [35], the RW solutions of the two-component Manakov system with variable coefficients, and an external potential albeit without Rabi coupling, were obtained. In particular, the previous study [35] considered the parametric choice \( a_1 = f \neq 0, n = 2 \) and \( a_2 = 0 \) without \( \sigma \), and showed that the first component has a bell-shaped peak on top of a long crest extended along the \( t \) direction.
Figure 6. Exact rogue wave solutions in the system (1) for $\sigma = 0$ (top panels) and for $\sigma = 0.2$ (bottom panels). The potential form is given by equation (15). The parameters values are chosen as $a_1 = 1, a_2 = 0, f = 0.3$.

Figure 7. Evolution of two-component condensates in spatially modulated 2-CGP system with Rabi coupling for the potential form given by equation (16). The parameters are chosen as $a_1 = 2.8, a_2 = 0, f = 0.5, n = 2$, and $\sigma = 0.2$. The left panels show surface plots, while right panels show the density plots.
and the second component featured a peak with a dip. Here, for comparison purposes we consider the same parametric choices of $a_1, a_2, f, n$ as that of [35] but in the presence of Rabi coupling. Interestingly, our study shows that the Rabi coupling induces oscillating boomeronic soliton behaviour. Figure 7 shows that for $n = 2$, we get three as the number of oscillating boomeronic soliton structures co-existing with bell-shaped peak at the centre ($(x, t) = (0, 0)$) in the background of oscillating PCM. By increasing $n$ we identify that for a given $n$ value there can arise $(n + 1)$ oscillating boomeronic solitons. This is the consequence of the specific form of the potentials, and the production of oscillating boomeronic behaviour is due to the Rabi coupling caused by the applied electro-magnetic field.

3. Spatially varying three-component Rabi-coupled GP systems

The three-component generalization of the two-coupled GP system (3) with Rabi coupling describing the dynamics of the three-species condensates in the presence of an external electro-magnetic field can be expressed as

$$\begin{align*}
\imath \frac{\partial \psi_j}{\partial t} + \gamma(x) \frac{\partial^2 \psi_j}{\partial x^2} + 2 \sum_{l=1}^{3} g_{jl}(x) |\psi_l|^2 \psi_j + V(x, t) \psi_j - \sum_{l=1, l \neq j}^{3} \sigma \psi_l &= 0, \quad j = 1, 2, 3, \\
(17)
\end{align*}$$

where $g_{jl}(x) = g(x), j, l = 1, 2, 3$, are spatially varying interacting strengths, $\sigma$ is the Rabi coupling term and $V(x, t)$ is the trapping potential. By employing the following unitary transformation in the above equation

$$\begin{pmatrix}
\psi_1(x, t) \\
\psi_2(x, t) \\
\psi_3(x, t)
\end{pmatrix} = \frac{1}{3} B \begin{pmatrix}
\phi_1(x, t) \\
\phi_2(x, t) \\
\phi_3(x, t)
\end{pmatrix},
\quad (18)
$$

where

$$B = \begin{pmatrix}
(2e^{i\sigma t} + e^{-2i\sigma t}) & (e^{-i\sigma t} - e^{i\sigma t}) & (e^{-i\sigma t} - e^{i\sigma t}) \\
(e^{-i\sigma t} - e^{i\sigma t}) & (2e^{i\sigma t} + e^{-2i\sigma t}) & (e^{-i\sigma t} - e^{i\sigma t}) \\
(e^{-i\sigma t} - e^{i\sigma t}) & (e^{-i\sigma t} - e^{i\sigma t}) & (2e^{i\sigma t} + e^{-2i\sigma t})
\end{pmatrix},
\quad (19)
$$

the Rabi coefficient can be absorbed and the resulting inhomogeneous three-component NLS system can be cast as

$$\begin{align*}
\imath \frac{\partial \phi_j}{\partial t} + \gamma(x) \frac{\partial^2 \phi_j}{\partial x^2} + 2g(x) \sum_{l=1}^{3} |\phi_l|^2 \phi_j + V(x, t) \phi_j &= 0, \quad j = 1, 2, 3.
\end{align*}
(20)
$$

After performing the similarity transformation given by equation (5) with $j = 1, 2, 3$, to equation (20), we obtain the following set of the integrable 3-CNLS system

$$\begin{align*}
\imath \frac{\partial q_j}{\partial T} + \frac{\partial^2 q_j}{\partial Y^2} + 2 \sum_{l=1}^{3} |q_l|^2 q_j &= 0, \quad j = 1, 2, 3,
\end{align*}
\quad (21)
$$

where $Y$ is given by equation (60a) and $T = t$. The soliton solutions of the above system have been obtained in [80] and fascinating energy-sharing collisions have been explored. Then in [22], the following explicit three-component RW solutions of the above equation (21) were obtained by the DT.
\begin{align}
q_1(Y,T) &= 1 - \frac{H_1(Y,T)}{G_1(Y,T)} \exp \left( \frac{9T}{2} - i \frac{Y}{\sqrt{2}} \right), \\
q_2(Y,T) &= 1 - \frac{H_2(Y,T)}{G_2(Y,T)} \frac{\exp(5iT)}{\sqrt{2}}, \\
q_3(Y,T) &= 1 - \frac{H_3(Y,T)}{G_3(Y,T)} \exp \left( \frac{9T}{2} + i \frac{Y}{\sqrt{2}} \right),\end{align}

where \( H_j(Y,T) \) and \( G_j(Y,T), \ j = 1, 2, 3, \) are the sixth-order polynomial functions given in equation (22) of appendix of [22]. It is a straightforward task to extend our analysis to solitons and other nonlinear wave solutions with this procedure. This solution has four hidden parameters \( A_1, A_2, A_3 \) and \( A_4 \), which are also given in appendix of [22], and due to their cumbersome form we refrain from presenting their explicit forms here again. The different profiles of these resulting nonlinear waves with choice three RWs are localized along the time axis as shown in figures 8(a) of RW solution of the three-component GP equation with variable coefficients. This reads as

\begin{bmatrix}
\psi_1(x,t) \\
\psi_2(x,t) \\
\psi_3(x,t)
\end{bmatrix} = \frac{N(x)}{3} B \begin{bmatrix}
q_1(Y,T) \\
q_2(Y,T) \\
q_3(Y,T)
\end{bmatrix}.

Here too we consider the same two different forms of the trapping potentials discussed in the previous section.

### 3.1. Sech-type hyperbolic potential (optical lattice potential with \( m = 1 \))

First, we consider the trapping potential \( V(x) = (1 - 2\text{sech}^2(x))\text{sech}^4(x) \) resulting for \( m = 1 \) in equation (14) in the absence of Rabi coupling. For constructing \( \psi_j \), we assume specific patterns of RWs in \( q_j \)s resulting for \( A_1 = A_3 = 0 \) and \( A_2 = A_4 \neq 0 \). For this choice three RWs are localized along the time axis as shown in figures 8(a)–(c) [22]. The resulting nonlinear waves with \( \sigma = 0 \) and in the presence of potential and spatial modulation are depicted in figures 8(d)–(f). In the \( \psi_1 \) component now we have only one zero-background RW that gets superposed on the bright soliton that broadens the base of the RW, and the other two RWs diminish in their amplitudes appearing as tiny peaks at the wings. The three RWs superpose on a bright soliton around the centre \((0,0)\) which broadens the base of the RW and makes the background existing earlier in the absence of potential vanish. With the introduction of Rabi coupling (non-zero value for \( \sigma \)) the amplitude of the central zero-background RW is suppressed considerably, and the soliton exhibits internal oscillations predominantly at the wings. For example in the \( \psi_1 \) component one can observe the periodic recurrence of six solitary waves with oscillating wings. The behaviour of \( \psi_3 \) is similar to that of \( \psi_1 \). These are shown in figures 8(g) and (i) respectively. But in the \( \psi_2 \) component, the internal oscillations of the soliton are vibrant resulting in multi-peak solitons as shown in figure 8(h). To facilitate understanding we separately present the corresponding two-dimensional plots of the three components in figure 9.
3.2. Parabolic cylinder potential in three-coupled GP system

Next, we consider the parabolic cylinder potential of the form $V(x) = k^4(n + \frac{1}{2} - \frac{1}{4}x^2)D_n(x)$. 

3.2.1. Case (i) $A_3 = A_4 = 0$. In [22], it has been shown that for this choice there arises a single RW in the canonical 3-CNLS system (21) without Rabi coupling and trapping potential. In the presence of trapping and inhomogeneities, for the choice $n = 1$, a single RW at $t = 0$ and its replica appear at different spatial positions (see at $x = (-2, 2)$) as shown in figures 10(a)–(c). For the same choice, but with $n = 2$, as shown in figures 10(d)–(f) the smaller parabolic cylinder mode (PCM) appears at the centre, while the waves with maximum value
appear at either side of this central PCM. The RWs appear on top of the three PCMs, and as a result of which we get a shifted three-RW structure.

3.2.2. Case (ii) $A_1 \leq 0$ and $A_2 = A_3 \neq 0, A_4 = 0$. We consider the parametric choice $A_1 \leq 0$ or $A_1 \geq 0$, $A_2 = A_3 \neq 0, A_4 = 0$ for which two four-petal rogue waves exist in the 3-CNLS system with constant coefficients [22]. Indeed, here for $A_1 \leq 0$ the two RWs emerge at same time. Now for the choice of $n = 0$, we obtain a zero-background rogue wave appearing on top of the zeroth-order PCM at $(0, 0)$. The zero-background RW appearing in the $-ve$ x-axis at $t = 0$ has very low intensity at the tail of the PCM. These are shown in the top panels.
of figure 11. The corresponding two-dimensional plots are also given in the bottom panels to enhance the understanding of this structure.

3.2.3. Case (iii) $A_4 \neq 0$. Next, we consider the RW for the choice $A_4 \neq 0$. This choice leads to three rogue waves in a constant coefficient 3-CNLS system (see figure 12). For the choice $n = 0$ and $A_2 A_4 < 0$, the centre of the three RWs displayed in the top panel of figure 7 almost coalesces and appears on the top of the PCM as a single shifted RW. As before, here too for $n = 3$ we find an increase in the number of RWs as we move along the spatial direction. Thus from these three cases we note that for arbitrary $n$, one can have $(n+1)$ RWs appearing on top of the corresponding PCM. Finally we consider system (17) in the presence of Rabi coupling for any choice of $n$ and $A$ values. Here the internal oscillation with beating behaviour discussed in figures 7(d)–(f), is noted predominantly in PCMs as shown in figure 13.

Figure 12. The top panels show the surface plot of the exact rogue wave solutions in the system (17) in the presence of a parabolic cylinder potential $n = 0$ and in the absence of Rabi coupling ($\sigma = 0$). The corresponding two-dimensional plots are shown in the bottom panels for $t = 0$. The $A$ parameter values are $A_1 = 0, A_2 = 120, A_3 = 0$ and $A_4 = -1$.

Figure 13. Exact rogue wave solutions in the system (17) in the presence of parabolic cylinder potential $n = 0$ and in the presence of Rabi coupling $\sigma = 0.2$, and the $A$ parameter values are $A_1 = 0, A_2 = 120, A_3 = 0$ and $A_4 = -1$. 
4. Conclusion

In the present work, we have investigated the dynamics of RWs in two- and three-component GP equations with Rabi coupling and spatially varying parameter functions. The exact solutions have been obtained by using a rotation transformation followed by a similarity transformation. We have examined the behaviour of RWs, for two types of spatially varying trapping potentials which are the lattice potential including the hyperbolic type as special case and the parabolic cylinder potential. To gain insight into the problem first we consider a constant coefficient two-component GP system with Rabi coupling and identify the RW co-existing with oscillating boomeronic solitons. Then we include a spatial variation of the parameter functions which reveals that new daughter RW with a dark (bright) soliton part is produced. In particular, for $\gamma(x) = \text{sech}^4(x)$ we demonstrate the possibility of producing oscillating dromions, two-dimensional nonlinear structure in the 2-CGP system. Subsequently, we studied the interesting behaviour of RWs for a spatially varying three-component GP system without Rabi coupling and we have observed different behaviours of RWs. The three RWs are converted into broad-based zero-background single RW superposed on a bright soliton for a sech-type hyperbolic potential. For the parabolic cylinder potential, we observed the appearance of a single RW replica in the spatial direction. Then the introduction of Rabi coupling makes the internal oscillations and beating behaviour in the solitons. Thus our study shows the possibility of the formation of 2D structures by a spatially varying environment with Rabi coupling even in $(1 + 1)$ dimensional nonlinear systems. This study can be extended to investigate any kind of nonlinear wave, say solitons or elliptic waves, supported by the standard integrable CNLS systems and explore their behaviour in the presence of Rabi coupling and spatially varying parameters with different choices of potentials. Additionally, it is of interest to generalize this study to the N-component case with arbitrary $N$ for which rogue waves have been reported recently [24]. This requires the identification of a more general unitary transformation. It will be intriguing to study various modifications of the system (1) by including spin-orbit coupling, four-wave mixing effects (so-called pair-transition BECs) and also $\mathcal{PT}$ symmetric potentials. One can envisage the application of the present study in dispersion management fibres, dispersion control of matter-wave packets and matter wave switches with Rabi coupling. Work is in progress along these directions.

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