Twisted Reed-Solomon Codes With One-dimensional Hull

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Abstract—The hull of a linear code is defined to be the intersection of the code and its dual. When the size of the hull is small, it has been proved that some algorithms for checking permutation equivalence of two linear codes and computing the automorphism group of a linear code are very effective in general. Maximum distance separable (MDS) codes are codes meeting the Singleton bound. Twisted Reed-Solomon codes is a generalization of Reed-Solomon codes, which is also a nice construction for MDS codes. In this short letter, we obtain some twisted Reed-Solomon MDS codes with one-dimensional hull. Moreover, these codes are not monomially equivalent to Reed-Solomon codes.

Index Terms—twisted Reed-Solomon codes, one-dimensional hull, monomially equivalent.

I. INTRODUCTION

Given a linear code \( C \) of length \( n \) over the finite field \( \mathbb{F}_q \), the dual code of \( C \) is defined by

\[
C^\perp = \{ x \in \mathbb{F}_q^n \mid xy^T = 0 \text{ for all } y \in C \}
\]

where \( xy^T \) denotes the standard inner product of two vectors \( x \) and \( y \). The hull of the linear code \( C \) is defined to be

\[
\text{Hull}(C) := C \cap C^\perp.
\]

It is clear that \( \text{Hull}(C) \) is also a linear code over \( \mathbb{F}_q \). The hull was originally introduced in 1990 by Assmus, Jr. and Key \[11\] to classify finite projective planes. It had been shown that the hull plays an important role in determining the complexity of algorithms for checking permutation equivalence of two linear codes and computing the automorphism group of a linear code (see \[10, 11, 22, 24\]), which are very effective in general when the dimension of the hull is small.

It is worth mentioning that the special case of the hulls of linear codes is of much interest. Namely the codes with trivial intersection with its dual, which is also named linear complementary dual (LCD) codes. Massey \[18\] first introduced this class of codes and proved that there exist asymptotically good LCD codes. A practical application of binary LCD codes against side-channel attacks (SCAs) and fault injection attacks (FIAs) was investigated by Carlet et al. \[3\] and Carlet and Guilley \[4\]. Since then, the study of LCD codes is thus becoming a hot research topic in coding theory (\[6, 9, 12\]).

II. PRELIMINARIES

Let \( \mathbb{F}_q \) be the finite field of order \( q \), where \( q \) is a prime power. An \([n, k]_q\) linear code \( C \) over \( \mathbb{F}_q \) is a \( k \)-dimensional subspace of \( \mathbb{F}_q^n \). The minimum distance \( d \) of a linear code \( C \) is bounded by the so-called Singleton bound: \( d \leq n - k + 1 \). If \( d = n - k + 1 \), then the code \( C \) is called a maximum distance separable (MDS) code.

The following lemma on the hull of linear codes, which is very important for obtaining our main results.

Lemma 2.1: \[14\] Proposition 1] Let \( C \) be an \([n, k]_q\) linear code over \( \mathbb{F}_q \) with generator matrix \( G \). Then the code \( C \) has one-dimensional hull if and only if the rank of the matrix \( GG^T \) is \( k - 1 \), where \( G^T \) denotes the transpose of \( G \).

Recall that a monomial matrix is a square matrix which has exactly one nonzero entry in each row and each column.

[13, 15, 16, 26, 29]. Some nice progress on linear codes with small hulls has been made, for examples (\[15, 14\]).

A maximum distance separable (MDS) code has the greatest error correcting capability when its length and dimension is fixed. MDS codes are extensively used in communications (for example, Reed-Solomon codes are all MDS codes), and they have good applications in minimum storage codes and quantum codes. There are many known constructions for MDS codes; for instance, Generalized Reed-Solomon (GRS) codes \[19\], based on the equivalent problem of finding \( n \)-arcs in projective geometry \[17\], circulant matrices \[20\], Hankel matrices \[21\], or extending GRS codes.

Recently the authors in \[8, 16\] investigated the hull of MDS codes via generalized Reed-Solomon codes over finite fields. Beelen et al. \[2\] first gave the definition of twisted Reed-Solomon codes, which is a generalization of the Reed-Solomon codes, and they proved under some conditions twisted Reed-Solomon codes could be not monomially equivalent to the Reed-Solomon codes. However, the hull of twisted Reed-Solomon codes have not been studied in that paper. Recently, Wu, Hyun and Lee \[25\] constructed some LCD twisted Reed-Solomon codes.

In this letter, as a follow-up work we will focus on the hull of twisted Reed-Solomon codes. In particular, we will consider to construct some twisted Reed-Solomon MDS codes with one-dimensional hull, which are not monomially equivalent to Reed-Solomon codes. The rest of this letter is organized as follows. In Section II, we introduce basic concepts on the hull of linear codes and twisted Reed-Solomon codes. In Sections III, we present our main results and give some examples. We conclude the letter in Section IV.
Definition 2.2: Let $C_1$ and $C_2$ be two linear codes of the same length over $\mathbb{F}_q$, and let $G_1$ be a generator matrix of $C_1$. Then $C_1$ and $C_2$ are monomially equivalent if there is a monomial matrix $M$ such that $G_1 M$ is a generator matrix of $C_2$.

Next we will recall some constructions of MDS codes. We begin with the well-known generalized Reed-Solomon codes.

Definition 2.3: Let $\alpha_1, \ldots, \alpha_n$ be distinct elements in $\mathbb{F}_q \cup \{\infty\}$ and $v_1, \ldots, v_n$ be nonzero elements in $\mathbb{F}_q$. For $1 \leq k \leq n$, the corresponding generalized Reed-Solomon (GRS) code over $\mathbb{F}_q$ is defined by

$$G_{\text{RS}}(\alpha, v) := \left\{ (v_1 f(\alpha_1), \ldots, v_n f(\alpha_n)) : f(x) \in \mathbb{F}_q[x], \deg(f(x)) < k \right\},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{F}_q \cup \{\infty\})^n$ and $v = (v_1, v_2, \ldots, v_n)$, and the quantity $f(\infty)$ is defined as the coefficient of $x^{k-1}$ in the polynomial $f$.

If $v_i = 1$ for every $i = 1, \ldots, n$, then $G_{\text{RS}}(\alpha, v)$ is called a Reed-Solomon code. In fact, $G_{\text{RS}}(\alpha, v)$ has a generator matrix as follows:

$$G_{\alpha, v} = \begin{pmatrix}
\begin{array}{cccc}
v_1 & v_2 & \ldots & v_n \\
v_1 \alpha_1 & v_2 \alpha_2 & \ldots & v_n \alpha_n \\
v_1 \alpha_1^2 & v_2 \alpha_2^2 & \ldots & v_n \alpha_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
v_1 \alpha_1^{k-1} & v_2 \alpha_2^{k-1} & \ldots & v_n \alpha_n^{k-1}
\end{array}
\end{pmatrix} \begin{pmatrix}
1 & 1 & \ldots & 1 \\
0 & 1 & \ldots & 0 \\
0 & 0 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0
\end{pmatrix}$$

It is well-known that a generalized Reed-Solomon code $G_{\alpha, v}$ is an $[n, k, n-k+1]$ MDS code and it is monomially equivalent to a Reed-Solomon code.

In 2007, Beelen et al. [2] presented a generalization of Reed-Solomon codes, so-called twisted Reed-Solomon codes.

Definition 2.4: Let $\eta$ be a nonzero element in the finite field $\mathbb{F}_q$. Let $k, l$ and $h$ be nonnegative integers such that $0 \leq h < k \leq q$, $k < n$, and $0 < t \leq n - k$. Let $\alpha_1, \ldots, \alpha_n$ be distinct elements in $\mathbb{F}_q \cup \{\infty\}$, and we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then the corresponding twisted Reed-Solomon code over $\mathbb{F}_q$ of length $n$ and dimension $k$ is given by $C_{\alpha}(\alpha, t, h, \eta) = \{(f(\alpha_1), \ldots, f(\alpha_n)) : f(x) = \sum_{i=0}^{l-1} a_i x^i + \eta q^{-2^{t(1+h)}}x^{k-t} \in \mathbb{F}_q[x]\}$.

In fact, $G = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\alpha_1^{-1} & \alpha_2^{-1} & \ldots & \alpha_n^{-1} \\
\alpha_1^h & \alpha_2^h & \ldots & \alpha_n^h \\
\alpha_1^h + \eta q^{-2^h} & \alpha_2^h + \eta q^{-2^h} & \ldots & \alpha_n^h + \eta q^{-2^h} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{h-k} & \alpha_2^{h-k} & \ldots & \alpha_n^{h-k} \\
\alpha_1^{h-k} + \eta q^{-2^{t+k-h}} & \alpha_2^{h-k} + \eta q^{-2^{t+k-h}} & \ldots & \alpha_n^{h-k} + \eta q^{-2^{t+k-h}}
\end{pmatrix}$ (1)

is the generator matrix of the twisted Reed-Solomon code $C_{\alpha}(\alpha, t, h, \eta)$.

Note that in general, the twisted Reed-Solomon codes are not MDS. Beelen et al. got some results on the twisted Reed-Solomon codes as follows:

Lemma 2.5: [2] Theorem 17] Let $F_s \subset \mathbb{F}_q$ be a proper subfield and $\alpha_1, \ldots, \alpha_n \in F_s$. If $\eta \in \mathbb{F}_q \setminus F_s$, then the twisted Reed-Solomon code $C_{\alpha}(\alpha, t, h, \eta)$ is MDS.

Lemma 2.6: [2] Theorem 18] Let $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$ and $2 \leq k < n - 2$. Furthermore, let $H \subseteq \mathbb{F}_q$ satisfy that the twisted Reed-Solomon code $C_{\alpha}(\alpha, t, h, \eta)$ is MDS for every $\eta \in H$. Then there are at most 6 choices of $\eta \in H$ such that $C_{\alpha}(\alpha, t, h, \eta)$ is monomially equivalent to a Reed-Solomon code.

Lemma 2.7: [2] Corollary 20] Let $F_s \subset \mathbb{F}_q$ with $|F_s \setminus F_s| > 6$. Let $2 < k < n - 2$ and $n \leq s$. Then there exists $\eta \in \mathbb{F}_q \setminus F_s$ such that $C_{\alpha}(\alpha, t, h, \eta)$ is MDS but not monomially equivalent to a Reed-Solomon code.

Throughout the paper, if a code is not monomially equivalent to a Reed-Solomon code, then we call it a code of non-Reed-Solomon type or a non-Reed-Solomon code.

III. Twisted Reed-Solomon codes with one-dimensional hull

Let $\gamma$ be a primitive element of $\mathbb{F}_q$ and $k \mid (q - 1)$. Then $\gamma^{\frac{1}{k^i}}$ generates a subgroup of $\mathbb{F}_q$ of order $k$. Let $\alpha_i = \gamma^{\frac{1}{k^i}}$, for $1 \leq i \leq k$. One can easily check that

$$\theta_f = \alpha_1^f + \cdots + \alpha_k^f = \begin{cases} k & \text{if } f \equiv 0 \pmod{k}, \\ 0 & \text{otherwise}. \end{cases} (2)$$

Lemma 3.1: Let $q$ be a power of two. If $k$ is a positive integer with $k \mid (q - 1)$, then there exists a $[2k, k]_q$ twisted Reed-Solomon code $C_{\alpha}(\alpha, t, h, \eta)$ over $\mathbb{F}_q$ with one-dimensional hull for $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_n)$, where $\gamma$ is a primitive element of $\mathbb{F}_q$ and $\alpha_i = \gamma^{\frac{1}{k^i}}$, for $1 \leq i \leq k$.

Proof By Definition 2.4, to make sure that $C_{\alpha}(\alpha, t, h, \eta)$ is a twisted Reed-Solomon code, we need $k \neq q - 1$. From (1), we recall that $G$ is a generator matrix of the twisted Reed-Solomon code $C_{\alpha}(\alpha, t, h, \eta)$ over $\mathbb{F}_q$.

Let $A_\beta = \begin{pmatrix} 1 & \beta \alpha_1 & \beta \alpha_2 & \ldots & \beta \alpha_{k-1} & \beta \alpha_k \\ 0 & \beta \alpha_1^2 & \beta \alpha_2^2 & \ldots & \beta \alpha_{k-1}^2 & \beta \alpha_k^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \beta h & \beta \alpha_1^{h-k} & \beta \alpha_2^{h-k} & \ldots & \beta \alpha_k^{h-k} \\ \end{pmatrix}$

By (2), we have

$$A_\beta A_\beta^T = \begin{pmatrix} k & 0 & 0 & \ldots & 0 & 0 \\ 0 & k & 0 & \ldots & 0 & 0 \\ 0 & 0 & k & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & k & 0 \\ 0 & 0 & 0 & \ldots & 0 & k \\ \end{pmatrix}.$$
Since every $\theta_t$ for $l \leq t \leq l+k-1$ is zero except exactly one $\theta_{t'}$, we can rewrite

$$C_\beta C_\beta^T = \begin{pmatrix} k & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & \beta^k k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \beta^k k & \ldots & \ldots & 0 \\ \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & *_\beta & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & *_\beta & \Delta_\beta & 0 \\ \end{pmatrix},$$

where $*_\beta$ and $\Delta_\beta$ are all elements in $F_q$, the $*_\beta$ and $\Delta_\beta$ are respectively entries located in the $(i+1, h+1)th$, $(h+1, i+1)th$ and $(h+1, h+1)th$ positions, and the other elements are all zero.

Let $G = [C_1 : C_\gamma]$ and $h > 0$. Then

$$GG^T = C_1 C_1^T + C_\gamma C_\gamma^T$$

$$= \begin{pmatrix} 2k & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & (1+\gamma^k)k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (1+\gamma^k)k & \ldots & 0 & 0 \\ \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & *_1 + *_\gamma & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & *_1 + *_\gamma & \Delta_1 + \Delta_\gamma & 0 \\ \end{pmatrix}.$$

It is easy to find an elementary matrix $P$ such that

$$P G G^T P^T = \begin{pmatrix} 2k & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & (1+\gamma^k)k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (1+\gamma^k)k & \ldots & 0 & 0 \\ \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & \Delta_1 + \Delta_\gamma & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 0 \\ \end{pmatrix}.$$

By the given conditions, we have $k \mid (q-1)$, $k < (q-1)$, $q$ is even, and $\gamma$ is a primitive element of $F_q$. Hence, $\gamma^k + 1 \neq 0$ and $2k = 0$. Then we have rank$(GG^T) = k - 1$. The result follows from Lemma 2.1.

**Lemma 3.2:** Let $F_q$ be a finite field of odd order $q$ and $k$ be a positive integer with $k \mid (q-1)$ and $2 < k < (q-1)/2$. If $h > 1$, then there exists a $[2k, k-1]_q$ twisted Reed-Solomon code $C_{k-1}(\alpha, t, h, \eta)$ over $F_q$ with one-dimensional hull for $\alpha = (\alpha_1, \ldots, \alpha_k, \gamma \alpha_1, \ldots, \gamma \alpha_k)$, where $\gamma$ is a primitive element of $F_q$ and $C_1 = \gamma^{k \frac{q-1}{2}}$ for $1 \leq i \leq k$.

**Proof:** Let

$$D_\beta = \begin{pmatrix} 1 & \beta \alpha_1 & \beta \alpha_2 & \ldots & 1 & \beta \alpha_k \\ 1 & \beta \alpha_1 & \beta \alpha_2 & \ldots & 1 & \beta \alpha_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (\beta \alpha_1)^{k-2} & (\beta \alpha_2)^{k-2} & \ldots & (\beta \alpha_{k-1})^{k-2} & (\beta \alpha_k)^{k-2} \\ \end{pmatrix}.$$ (4)

Then

$$D_\beta D_\beta^T = \begin{pmatrix} k & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \beta^k k & \ldots & \ldots & 0 & 0 \\ \end{pmatrix}.$$

Let $H_\beta = D_\beta + E_\beta$, where

$$E_\beta = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ \end{pmatrix} \leftarrow (h+1)th.$$

Let $G = [H_1 : H_2]$. By the proof of Lemma 3.1,

$$GG^T = H_1 H_1^T + H_2 H_2^T$$

$$= \begin{pmatrix} 2k & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & (1+\gamma^k)k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (1+\gamma^k)k & \ldots & 0 & 0 \\ \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & *_1 + *_\gamma & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & *_1 + *_\gamma & \Delta_1 + \Delta_\gamma & 0 \\ \end{pmatrix}.$$

Note that $q$ is odd and $k < \frac{q-1}{2}$. We have $2k \neq 0$ and $1+\gamma^k \neq 0$. By the same process of the proof of Lemma 3.1, the rank of $GG^T$ is $k - 1$ and result follows from Lemma 2.1.

**Remark 3.3:** By the process of Lemma 3.2, we can also construct some twisted Reed-Solomon codes with small hulls.

An effective method for construction of twisted Reed-Solomon codes with MDS property is to use the lifting of the finite field (refer to [2]). Hence, we obtain the following theorem by Lemmas 2.5, 2.7, 3.1 and 3.2.

**Theorem 3.4:** Let $q$ be a power of a prime and $F_s \subset F_q$ with $|F_q/F_s| > 6$. Suppose that $k$ is a positive integer with $k \mid (q-1)$.

1. If $q$ is even and $2 < k < (s-1)$, then there exists a $[2k, k]_q$ MDS non-Reed-Solomon code with one-dimensional hull.

2. If $q$ is odd and $2 < k < (s-1)/2$, then there exists a $[2k, k-1]_q$ MDS non-Reed-Solomon code with one-dimensional hull.
In the following, we will present some examples to show our main results.

**Example 3.5:** Let $q = 2^4 = 16$, $k = 5$, and $\gamma$ be a primitive element of $\mathbb{F}_q$. Consider a twisted Reed-Solomon code $C_5(\alpha, 1, 3, \eta)$, when $\alpha = (1, \gamma, \gamma^6, \gamma^9, \gamma^{12}, \gamma^3, \gamma^6, \gamma^9, \gamma^{12})$ and $\eta = \gamma^4 \in \mathbb{F}_{16}$. By Lemma 3.1, $C_5(\alpha, 1, 3, \eta)$ has one-dimensional hull for all $i$. By Magma, it follows that the codes $C_5(\alpha, 1, 3, \eta)$ are MDS non-Reed-Solomon code parameters $[10, 5, 5]_{16}$.

**Example 3.6:** Let $q = 3^4 = 81$, $k = 5$, and $\gamma$ be a primitive element of $\mathbb{F}_q$. Consider a twisted Reed-Solomon code $C_4(\alpha, 2, 2, \eta)$, when $\alpha = (1, \gamma, \gamma^9, \gamma^{12}, \gamma^3, \gamma^6, \gamma^9, \gamma^{12})$ and $\eta = \gamma^4 \in \mathbb{F}_{81}$ and $w = \gamma^{16}$. By Lemma 3.2, $C_4(\alpha, 2, 2, \gamma)$ has one-dimensional hull for all $i$. By Magma, it follows that the codes $C_4(\alpha, 2, 2, \gamma)$ are MDS non-Reed-Solomon code parameters $[10, 4, 4]_{81}$.

**IV. CONCLUDING REMARKS**

For a given linear code, in general case it is hard to show that if the code is monomially equivalent to a Reed-Solomon code with the same parameters. In this paper, we applied twisted Reed-Solomon codes to construct some MDS codes, which have one-dimensional hull and are not monomially equivalent to Reed-Solomon codes. We also presented some examples by using Magma.

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**REFERENCES**

[1] E. F. Assmus, Jr., and J. D. Key, “Affine and projective planes,” Discrete Math., vol. 83, pp. 161-187, Aug. 1990.

[2] P. Beelen, S. Puchinger, and J. R. nén Nielsen, “Twisted Reed-Solomon codes,” in IEEE Int. Symp. on Information Theory (ISIT), pp. 336-340, Jun. 2017.

[3] J. Bringer, C. Carlet, H. Chabanne, S. Guilley, H. Maghrebi, Orthogonal direct sum masking: A smartcard friendly computation paradigm in a code, with built-in protection against side-channel and fault attacks, in Information Security Theory and Practice. Securing the Internet of Things (Lecture Notes in Computer Science), vol. 8501. Heraklion, Greece: Springer, 2014, pp. 40-56.

[4] C. Carlet, S. Guilley, Complementary dual codes for counter-measures to side-channel attacks, in Coding Theory and Applications (CIM Series in Mathematical Sciences), vol. 3, E. R. Pinto, Ed. Cham, Switzerland: Springer-Verlag, 2014, pp. 97-105.

[5] C. Carlet, C. Li, S. Mesnager, “Linear codes with small hulls in semi-primitive case,” Des. Codes Cryptogr., vol. 87 no. 12, pp. 3063-3075, 2019.

[6] C. Carlet, S. Mesnager, C. Tang, Y. Qi, R. Pellikaan, Linear codes over $\mathbb{F}_q$ are equivalent to LCD codes for $q > 3$, IEEE Trans. Inf. Theory, vol. 64, no. 4, pp. 3010-3017, Apr. 2018.

[7] B. Chen and H. Liu, New constructions of MDS codes with complementary duals, IEEE Trans. Inf. Theory, vol. 64, no. 8, pp. 5776-5782, Aug. 2018.

[8] W. Fang, F.-W. Fu, L. Li, and S. Zhu, “Euclidean and Hermitian hulls of MDS codes and their applications to EAQECCs,” IEEE Trans. Inf. Theory, vol. 66, no. 6, pp. 3527-3537, Jun. 2020.

[9] L. Jin, Construction of MDS codes with complementary duals, IEEE Trans. Inf. Theory, vol. 63, no. 5, pp. 2843-2847, May 2017.

[10] J. S. Leonard, “Permutation group algorithms based on partitions,” in Coding Theory and Algorithms,” J. Symbolic Comput., vol. 12, pp. 533-583, Oct. 1991.

[11] C. Li, Hermitian LCD codes from cyclic codes, Designs, Codes Cryptography, vol. 86, no. 10, pp. 2261-2278, Oct. 2018.

[12] C. Li, C. Ding, and S. Li, LCD cyclic codes over finite fields, IEEE Trans. Inf. Theory, vol. 63, no. 7, pp. 4344-4356, Jul. 2017.

[13] C. Li, P. Zeng, “Constructions of linear codes with one-dimensional hull,” IEEE Trans. Inf. Theory, vol. 65, no. 3, pp. 1668-1676, Mar. 2019.

[14] S. Li, C. Li, C. Ding, and H. Liu, “Two families of LCD BCH codes,” IEEE Trans. Inf. Theory, vol. 63, no. 9, pp. 5699-5717, Sep. 2017.

[15] G. Luo, X. Cao, and X. Chen, “MDS codes with hulls of arbitrary dimensions and their quantum error correction,” IEEE Trans. Inf. Theory, vol. 65, no. 5, pp. 2944-2952, May 2019.

[16] F. J. MacWilliams and N. A. Sloane, The Theory of Error-Correcting Codes, Amsterdam, The Netherlands: North Holland, 1977.

[17] J. L. Massey, Linear codes with complementary duals, Discrete Math., 106-107, pp. 337-342, Sep. 1992.

[18] S. Reed and G. Solomon, “Polynomial codes over certain finite fields,” J. Soc. Ind. Appl. Math., vol. 8, no. 2, pp. 300-304, Jun. 1960.

[19] R. M. Roth and A. Lempel, “A construction of non-Reed-Solomon type MDS codes,” IEEE Trans. Inf. Theory, vol. 35, no. 3, pp. 653-657, May 1989.

[20] R. M. Roth and G. Seroussi, “On generator matrices of MDS codes (Corresp.),” IEEE Trans. Inf. Theory, vol. 31, no. 6, pp. 826-830, Nov. 1985.

[21] N. Sendrier, “On the dimension of the hull,” SIAM J. Discrete Math., 10, no. 2, pp. 282-293, 1997.

[22] N. Sendrier, “Finding the permutation between equivalent linear codes: The support splitting algorithm,” IEEE Trans. Inf. Theory, vol. 46, no. 4, pp. 1193-1205, Jul. 2000.

[23] N. Sendrier and G. Skeresys, “On the computation of the automorphism group of a linear code,” in Proc. IEEE Int. Symp. Inf. Theory, Washington, DC, USA, p. 13, Jun. 2001.

[24] Y. Wu, J. Y. Hyun, Y. Lee, “New LCD MDS codes of non-Reed-Solomon type,” submitted for publication.

[25] Y. Wu, Q. Yue, “Factoring linear cyclic polynomials and enumerations of LCD and self-dual constacyclic codes,” IEEE Trans. Inf. Theory, vol. 65, no. 3, pp. 1740-1751, Mar. 2019.

[26] Y. Wu, Q. Yue, S. Fan, “Further factorization of $x^n - 1$ over a finite field,” Finite Fields Appl., vol. 54, pp. 197-215, Nov. 2018.

[27] Y. Wu, Q. Yue, S. Fan, “Self-reciprocal and self-conjugate-reciprocal irreducible factors of $x^n - \lambda$ and their applications,” Finite Fields Appl., vol. 63, pp. 101648, Mar. 2020.

[28] Y. Wu, Q. Yue, X. Zhu, S. Yang, “Weight enumerators of reducible cyclic codes and their dual codes,” Discrete Math., vol. 342, no. 3, pp. 671-682, Mar. 2019.