We review the occurrence of negative energies in Pais-Uhlenbeck oscillator. We point out that in the absence of interactions negative energies are not problematic, neither in the classical nor in the quantized theory. However, in the presence of interactions that couple positive and negative energy degrees of freedom the system is unstable, unless the potential is bounded from below and above. We review some approaches in the literature that attempt to avoid the problem of negative energies in the Pais-Uhlenbeck oscillator.

Keywords: Pais-Uhlenbeck oscillator; negative energies; ghosts; indefinite Hamiltonian; higher derivative theories; vacuum decay

1 Introduction

Higher derivative theories could be one of possible roads to quantum gravity. But the wrong sign in the Ostrogradsky Hamiltonian of such theories has prevented them to be considered as viable physical theories. A toy model for higher-derivative theories is the Pais-Uhlenbeck (PU) oscillator. It has been a common believe that, because of the presence of negative energy states, the PU oscillator is unstable. In the absence of interaction, negative energies are not problematic. But a completely isolated physical system is unrealistic, therefore even a small interaction causes the transitions between positive and negative energies. If the interaction potential is unbounded, like in harmonic oscillator, then it causes a runaway behaviour of the system. But an unbounded potential, either from below or form above, is physically unrealistic. We know that the potential of a harmonic oscillator is an idealization; there is no such a potential in the nature. Therefore, in the case of the PU oscillator, or any other system that has negative energy states, a realistic potential is bounded from above and from below. In such a case, the system is stable in the sense that the runaway behavior with increasing positive energies (and decreasing negative energies) cannot superced the height of the potential.

In this paper I review how various authors have attacked the problem of negative energies in PU oscillators. We focus to those works in which the original Pais-Uhlenbeck theory is assumed, and is not generalized to its non-Hermitian, but PT
symmetric version [6]–[9]. In most approaches with the original PU Lagrangian [10]–[18] (see also [19]) the Hamiltonian of the non interacting PU oscillator is transformed into a form in which only positive energies occur. Those authors say nothing about how such procedures would work in the case of an interacting or self-interacting PU oscillator. In Ref. [5] it was shown that though a free PU oscillator Lagrangian can be transformed into a form which leads to positive energies only, this cannot be done for a self-interacting PU oscillator with a quartic interaction term. In the presence of an interaction term, a system with indefinite Hamiltonian cannot behave as a system with positive definite Hamiltonian. The approaches of Refs. [10]–[16] and also of [20] thus reconfirm the known fact [3, 4, 5] that a free system with indefinite Hamiltonian behaves just like a system with positive definite Hamiltonian. Therefore, in an attempt to solve the negative energy problem, one must consider the PU oscillator with an interaction term right from the beginning. The investigations by Pagani et al. [21] showed that such a system can be stable for certain interaction potentials and choices of parameters. Smilga [22]–[24] has found that the classical interacting PU oscillator is stable for certain range of initial data. Such a system would still not be stable after quantization because of the tunnel effect. But in Ref. [25] an unconditionally stable interacting system was found. Further, in Ref. [5] an example the PU oscillator with an interaction term bounded from above and below, given by the forth power of sine, was considered and shown that such a system is stable for all initial conditions. The Pais-Uhlenbeck oscillator as a fourth order system in derivatives with respect to time can be written as a system of two oscillators with equal masses. If the masses of the the oscillators in such a system are different, then the system is stable even in the presence of a coupling term [26, 5].

Usually, when speaking about higher derivative systems, it is stated that upon quantization such theories contain ghosts, and therefore violate unitarity. However, as shown in Refs. [27, 28, 29, 30, 31], whether in a quantized theory we have negative probabilities (ghosts) and positive energies, or vice versa, positive probabilities and negative energies, depends on choice of vacuum. Woodard [4] was very explicit in saying that a correct quantization should always have positive probabilities, while the energies can be negative, otherwise the correspondence principle is not satisfied.

Another important issue had been raised by Nesterenko [30] who considered the effect of damping on the PU oscillator. He found that in the presence of damping the PU oscillator is unstable. This was further explored in Ref. [31] where it was found that the instability does not occur if besides a damping term with the first order derivative we have also a term with the third order derivative. In the presence of both terms the system becomes stable.

Finally, it is pointed out how the issue of instantaneous vacuum decay in higher derivative field theories can be clarified.
2 Pais-Uhlenbeck oscillator as a toy model for higher derivative theories

Whereas the theory based on the Einstein-Hilbert action is not renormalizable, the \( R + R^2 \) gravity is renormalizable. In spite of this nice property, such a theory, because of the presence of ghosts, has been generally considered as problematic and hence dismissed by majority of researchers. The rôle of ghosts can be studied on a simpler example, namely, the fourth-order scalar field theory with the action \[ I = \int d^4x \left[ \frac{1}{2} \phi(\Box + m_1^2)(\Box + m_2^2)\phi - \lambda \phi^4 \right]. \] (1)

Defining new variables \[ \psi_1 = \frac{(\Box + m_1^2)\phi}{[2(m_2^2 - m_1^2)]^{1/2}}, \quad \psi_2 = \frac{(\Box + m_2^2)\phi}{[2(m_2^2 - m_1^2)]^{1/2}}, \] (2)

where \( m_2 > m_1 \), the action (1) becomes

\[ I = \int d^4x \left[ \frac{1}{2} \psi_1(\Box + m_1^2)\psi_1 - \frac{1}{2} \psi_2(\Box + m_2^2)\psi_2 - \frac{4\lambda}{(m_2^2 - m_1^2)^2}(\psi_1 - \psi_2)^4 \right]. \] (3)

This gives the following coupled equations of motion

\[ (\Box + m_1^2)\psi_1 - \frac{16\lambda}{(m_2^2 - m_1^2)^2}(\psi_1 - \psi_2)^3 = 0 \] (4)
\[ (\Box + m_2^2)\psi_2 - \frac{16\lambda}{(m_2^2 - m_1^2)^2}(\psi_1 - \psi_2)^3 = 0 \] (5)

In the absence of interactions, there would be no problem. We would just have two independent, uncoupled equations. The problem arises if \( \lambda \neq 0 \). Then the energy flows between the two fields and supposedly causes a runaway behaviour of the system.

In order to facilitate the study, instead of the field action (1), one suppresses the spatial dependence of the fields and considers the action

\[ I = \int dt \left[ \frac{1}{2} \phi \left( \frac{d^2}{dt^2} + m_1^2 \right) \left( \frac{d^2}{dt^2} + m_2^2 \right) \phi - \lambda \phi \right] \]
\[ = \int dt \left[ \frac{1}{2} \dddot{x}^2 - \frac{1}{2}(m_1^2 + m_2^2)\dot{x}^2 + \frac{1}{2}m_1^2m_2^2\dot{\phi}^2 - \lambda \phi^4 \right] + \text{boundary term.} \] (6)

This is the action for a higher derivative harmonic oscillator, in the literature known as Pais-Uhlenbeck oscillator \[ \text{[2].} \] Notation is usually adapted to such a simplified system and the following Lagragian is considered:

\[ L = \frac{1}{2} \left[ \dddot{x}^2 - (\omega_1^2 + \omega_2^2)\dot{x}^2 + \omega_1^2\omega_2^2x \right] - \frac{\Lambda x^4}{4} \] (7)
According to the Ostrogradski second order formalism we have

\[ p = \frac{\partial L}{\partial \dot{x}} = (\omega_1^2 + \omega_2^2) \dot{x}, \quad P = \frac{\partial L}{\partial \ddot{x}} = \ddot{x} \] (8)

\[ H = p\dot{x} + P\ddot{x} - L = p_\varphi q + \frac{1}{2} [p_\varphi^2 + (\omega_1^2 + \omega_2^2)q^2 - \omega_1^2 \omega_2^2 x] + \Lambda x^4. \] (9)

Because the momentum \( p_x \) occurs linearly in (9) the Hamiltonian is not positive definite. It can have positive or negative values. This is a manifestation of the fact that the PU oscillator possesses the so called Ostrogradski instability. Much effort has been devoted in attempts to circumvent this problem.

3 Non interacting Pais-Uhlenbeck oscillator

If the coupling constant \( \lambda \) is zero, then the equation of motion is

\[ x^{(4)} + (\omega_1^2 + \omega_2^2) \ddot{x} + \omega_1^2 \omega_2^2 x = 0, \] (10)

which can be written as

\[ \left( \frac{d^2}{dt^2} + \omega_1^2 \right) \left( \frac{d^2}{dt^2} + \omega_2^2 \right) x = 0, \] (11)

In Refs. [12, 10, 16] it has been shown that Eq. (10) can be derived from a positive definite Hamiltonian. An equivalent procedure was considered by Stephen [33], who distinguishes between Hamiltonian and the “pseudo-mechanical energy”. The procedure by Stephen and Mostafazadeh is employed and generalized in Ref. [5] by using a different notation as follows.

The fourth order equation (10) can be written as a system of two second order equations

\[ \ddot{x} + \mu_1 x - \rho_1 y = 0, \] (12)

\[ \ddot{y} + \mu_2 y - \rho_2 x = 0, \] (13)

provided that the real constants \( \mu_1, \mu_2, \rho_1, \rho_2 \) satisfy the relations

\[ \mu_1 + \mu_2 = \omega_1^2 + \omega_2^2 \] (14)

\[ \mu_1 \mu_2 - \rho_1 \rho_2 = \omega_1^2 \omega_2^2 \] (15)

The solution is

\[ \omega_{1,2}^2 = \frac{1}{2} (\mu_1 + \mu_2) \pm \frac{1}{2} \sqrt{(\mu_1 + \mu_2)^2 - 4(\mu_1 \mu_2 - \rho_1 \rho_2)}. \] (16)
Eqs. (12), (13) can be derived from two different types of Lagrangians, one associated a positive definite Hamiltonian, and the other one with an indefinite Hamiltonian.

Case I. One possible Lagrangian is

\[
L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\mu_1 x^2 + \mu_2 y^2 - 2\rho_1 xy),
\]

which gives the equations of motion (12), (13) if \( \rho_2 = \rho_1 \). Then Eq. (16) reads

\[
\omega_{1,2}^2 = \frac{1}{2}(\mu_1 + \mu_2) \pm \frac{1}{2}\sqrt{(\mu_1 - \mu_2)^2 + 4\rho_1^2}.
\]

The latter relation admits real frequencies \( \omega_1, \omega_2 \) and thus oscillatory motion.

The second term in (17) can be diagonalized if we perform a rotation in the \((x, y)\)-space,

\[
x' = x \cos \alpha + y \sin \alpha \\
y' = -x \sin \alpha + y \cos \alpha,
\]

such that

\[
\mu_1 x^2 + \mu_2 y^2 - 2\rho_1 xy = ax'^2 + by'^2.
\]

This gives the system of equations

\[
a \cos^2 \alpha + b \sin^2 \alpha = \mu_1 \\
a \sin^2 \alpha + b \cos^2 \alpha = \mu_2 \\
(a - b) \cos \alpha \sin \alpha = \rho_1,
\]

whose solution is

\[
a = \frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{2}\sqrt{(\mu_1 - \mu_2)^2 + 4\rho_1^2} = \omega_1^2,
\]

\[
b = \frac{1}{2}(\mu_1 + \mu_2) - \frac{1}{2}\sqrt{(\mu_1 - \mu_2)^2 + 4\rho_1^2} = \omega_2^2,
\]

\[
\cos 2\alpha = \frac{\mu_1 - \mu_2}{\sqrt{(\mu_1 - \mu_2)^2 + 4\rho_1^2}}.
\]

In the new coordinates the Lagrangian is thus

\[
L = \frac{1}{2}(\dot{x}'^2 + \dot{y}'^2) - \frac{1}{2}(\omega_1^2 x'^2 + \omega_2^2 y'^2),
\]

the corresponding Hamiltonian being

\[
H = \frac{1}{2}(\dot{x}'^2 + \dot{y}'^2) + \frac{1}{2}(\omega_1^2 x'^2 + \omega_2^2 y'^2).
\]
The energy of such a system is always positive. Let us mention that the system described by the Lagrangian (25) is equivalent to the PU oscillator if $\omega_1^2 \neq \omega_2^2$. The case $\omega_1^2 = \omega_2^2$ is degenerate, because then $\rho_1 = 0$, $\rho_2 = 0$, and the system of equations (12), (13) does not give the equation of motion of the PU oscillator. Such a degenerated case has been considered in Ref. [34].

Case II. Another possible Lagrangian is

$$L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - \frac{1}{2}(\mu_1 x^2 - \mu_2 y^2 - 2\rho_1 xy),$$

which gives the equations of motion (12), (13) if $\rho_2 = -\rho_2$. Then Eq. (16) becomes

$$\omega_{1,2}^2 = \frac{1}{2}(\mu_1 + \mu_2) \mp \frac{1}{2}\sqrt{(\mu_1 - \mu_2)^2 - 4\rho_1^2}.$$ (28)

The frequencies $\omega_1$, $\omega_2$ are real if $(\mu_1 - \mu_2)^2 > 4\rho_1^2$, and $\mu_1 + \mu_2 > \sqrt{(\mu_1 - \mu_2)^2 - 4\rho_1^2}$.

Now the second term in the Lagrangian (27) can be diagonalized if we perform the hyperbolic rotation in the $(x,y)$-space,

$$x' = x \cosh \alpha + y \sinh \alpha$$

$$y' = x \sinh \alpha + y \cosh \alpha,$$ (29)

such that

$$\mu_1 x^2 - \mu_2 y^2 - 2\rho_1 xy = ax'^2 - by'^2,$$ (30)

which gives the system of equations

$$\omega_1^2 \cosh^2 \alpha - \omega_2^2 \sinh^2 \alpha = \mu_1$$

$$-\omega_1^2 \sinh^2 \alpha + \omega_2^2 \cosh^2 \alpha = \mu_2$$

$$(\omega_1^2 - \omega_2^2) \cosh \alpha \sinh \alpha = -\rho_1.$$ (33)

The solution of the latter system is $a = \omega_1^2$, $b = \omega_2^2$, where $\omega_1^2$, $\omega_2^2$ are given by Eq. (28).

The Lagrangian and the Hamiltonian in the new coordinates are now

$$L = \frac{1}{2}(\dot{x}'^2 - \dot{y}'^2) - \frac{1}{2}(\omega_1^2 x'^2 - \omega_2^2 y'^2),$$ (34)

$$H = \frac{1}{2}(\dot{x}'^2 - \dot{y}'^2) + \frac{1}{2}(\omega_1^2 x'^2 - \omega_2^2 y'^2).$$ (35)

The latter system can have either positive or negative energy, depending on which degrees of freedom, $x'$ or $y'$ are more excited.

By employing a procedure analogous to the one reviewed above, many authors [10]–[16] have come to the conclusion that the Pais-Uhlenbeck oscillator satisfying the equations of motion (10) can be described in terms of a positive definite
Hamiltonian. Since quantization of such a system is straightforward, those authors concluded that this resolves “the ghost problem” of the PU oscillator.

But in the literature it is widely recognized [4] that the quantization of a non interacting PU oscillator or any system with indefinite energy is not problematic, if performed correctly so that the correspondence principle is satisfied. This means that the classical as well as the quantized system has both, positive and negative energy states, whereas the norm of the quantum states is always positive. According to those authors, the problem occurs in the presence of an interaction that couples the positive and negative energy degrees of freedom, because then the system becomes unstable. But some authors have observed that interacting systems with positive and negative energies can be stable as well.

4 Illustrative example: The system of two equal frequency oscillators

4.1 Uncoupled oscillators

Let us consider a toy model, described by the following Lagrangian and the Hamiltonian

\[ L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - \frac{1}{2}\omega^2(x^2 - y^2). \] (36)

\[ H = p_x\dot{x} + p_y\dot{y} - L = \frac{1}{2}(p_x^2 - p_y^2) + \frac{\omega^2}{2}(x^2 - y^2) \] (37)

The Hamilton equations of motion are

\[ \dot{x} = \{x, H\} = \frac{\partial H}{\partial p_x} = p_x, \quad \dot{y} = \{y, H\} = \frac{\partial H}{\partial p_y} = -p_y \] (38)

\[ \dot{p}_x = \{p_x, H\} = -\frac{\partial H}{\partial x} = -\omega^2 x, \quad \dot{p}_y = \{p_y, H\} = -\frac{\partial H}{\partial y} = \omega^2 y \] (39)

where the Poisson brackets are defined as usual,

\[ \{x, p_x\} = 1, \quad \{y, p_y\} = 1 \] (40)

In the quantized theory we have commutators

\[ [x, p_x] = i, \quad [y, p_y] = i \] (41)

\[ ^2 \text{Though this system, because of the degeneracy } \omega_1^2 = \omega_2^2 = \omega^2 \text{ is not equivalent to the PU oscillator, it serves well for illustration of the main points. It can be straightforwardly generalized to the case of unequal frequencies } \omega_1^2 \neq \omega_2^2 \text{ of the two oscillators.} \]
Introducing
\[ c_x = \frac{1}{\sqrt{2}}(\sqrt{\omega}x + \frac{i}{\sqrt{\omega}}p_x), \quad c_x^\dagger = \frac{1}{\sqrt{2}}(\sqrt{\omega}x - \frac{i}{\sqrt{\omega}}p_x) \]  
\[ c_y = \frac{1}{\sqrt{2}}(\sqrt{\omega}y + \frac{i}{\sqrt{\omega}}p_y), \quad c_y^\dagger = \frac{1}{\sqrt{2}}(\sqrt{\omega}y - \frac{i}{\sqrt{\omega}}p_y) \]  
we have
\[ [c_x, c_x^\dagger] = 1, \quad [c_y, c_y^\dagger] = 1, \quad [c_x, c_y] = 0, \quad [c_x^\dagger, c_y^\dagger] = 0. \]  
The Hamiltonian (37) can be written as
\[ H = \omega (c_x^\dagger c_x + c_x c_x^\dagger - c_y^\dagger c_y - c_y c_y^\dagger). \]  
Defining the vacuum according to
\[ c_x|0\rangle = 0, \quad c_y|0\rangle = 0, \]  
the Hamiltonian becomes
\[ H = \omega (c_x^\dagger c_x - c_y^\dagger c_y). \]  
All states have positive norms, e.g.,
\[ \langle 0|cc|0\rangle = \langle 0|[c, c^\dagger]|0\rangle = \langle 0|0\rangle = 1, \]  
regardless of whether the subscript of \( c, c^\dagger \) is \( x \) or \( y \), because both commutators in Eq. (44) are equal to 1.

In the coordinate representation the momenta are
\[ p_x = -i \frac{\partial}{\partial x}, \quad p_y = -i \frac{\partial}{\partial y}, \]  
whereas the vacuum state and its defining equations become
\[ \langle x, y|0\rangle = \psi_0(x, y) \]
\[ \frac{1}{2} \left( \sqrt{\omega}x + \frac{1}{\sqrt{\omega}} \frac{\partial}{\partial x} \right) \psi_0(x, y) = 0 \]  
\[ \frac{1}{2} \left( \sqrt{\omega}y + \frac{1}{\sqrt{\omega}} \frac{\partial}{\partial y} \right) \psi_0(x, y) = 0. \]  
The solution is
\[ \psi_0 = \frac{2\pi}{\omega} e^{-\frac{1}{2}\omega(x^2 + y^2)}, \]
and it has positive norm
\[ \int \psi_0^2 \, dx \, dy = 1. \] (55)

The vacuum \( \psi_0(x, y) \) and the states \( \psi(x, y) \) excited by successive action of the operators \( c_x^\dagger, c_y^\dagger \) on \( \psi_0(x, y) \) are not invariant under the hyperbolic rotations in the \( (x, y) \)-space, under which the Lagrangian (36) and the Hamiltonian (37) are invariant. However, even if solutions are not invariant, the theory is covariant, because the set of possible solutions in a new reference frame corresponds to the set of possible solutions in the old reference frame. Thus, though in a new frame \( S' \) the vacuum (54) becomes
\[ \psi_0(x', y') = 2\pi e^{-\frac{1}{2}\omega((x'^2 + y'^2)}} \] (56)
there also exists the solution
\[ \psi'_0(x', y') = 2\pi e^{-\frac{1}{2}\omega(x'^2 + y'^2)}} \] (57)
which has the same form as \( \psi_0(x, y) \) of Eq. (54). The same is true for all excited states.

Such a model can be straightforwardly generalized to higher dimensional spaces with signature \((r, s)\). The Lagrangian and the Hamiltonian are
\[ L = \frac{1}{2} \dot{x}^a \dot{x}_a - \frac{1}{2} \omega^2 x^a x_a, \] (58)
\[ H = \frac{1}{2} p^a p_a + \frac{1}{2} \omega^2 x^a x_a, \] (59)
where indices are lowered and raised with the metric \( \eta_{ab} \) and its inverse \( \eta^{ab} \). The momentum is \( p_a = \partial L / \partial \dot{x}^a = \dot{x}_a = \eta_{ab} \dot{x}^b \). Upon quantization we have
\[ [x^a, p_b] = i\delta_a^b \] (60)

The creation/annihilation operators defined according to
\[ c^a = \frac{1}{\sqrt{2}} \left( \sqrt{\omega} x^a + \frac{i}{\sqrt{\omega}} p_a \right) \] (61)
\[ c^a \dagger = \frac{1}{\sqrt{2}} \left( \sqrt{\omega} x^a - \frac{i}{\sqrt{\omega}} p_a \right) \] (62)
are a generalization of the operators \( c_x, c_x^\dagger, c_y, c_y^\dagger \) given in Eqs. (42), (43). They satisfy the commutation relations
\[ [c^a, c^b \dagger] = \delta^{ab}, \quad [c^a, c^b] = [c^a \dagger, c^b \dagger] = 0. \] (63)
The Hamiltonian is

\[ H = \frac{1}{2} \omega (c_a^\dagger c^a + c^a c_a^\dagger) = \omega \left( c_a^\dagger c^a + \frac{r}{2} - \frac{s}{2} \right). \]  

(64)

If vacuum is defined as

\[ c^a |0\rangle = 0, \]  

(65)

then the vacuum expectation value of \( H \) is

\[ \langle H \rangle = \frac{\omega}{2} (r - s), \]  

(66)

which vanishes if \( r = s \), that is when the signature is neutral.

In the literature an alternative definition of annihilation/creation operators is usually employed, namely

\[ a^a = \frac{1}{2} \left( \sqrt{\omega} x^a + \frac{i}{\sqrt{\omega}} p^a \right) \]  

(67)

\[ a^{a\dagger} = \frac{1}{2} \left( \sqrt{\omega} x^a - \frac{i}{\sqrt{\omega}} p^a \right), \]  

(68)

that satisfy

\[ [a^a, a_b^{\dagger}] = \delta^a_b, \quad [a^a, a_b] = \eta^{ab}. \]  

(69)

There are two possible definitions of vacuum:

**Definition I.** This is the usual definition,

\[ a^a |0\rangle = 0. \]  

(70)

The Hamiltonian

\[ H = \frac{1}{2} \omega (a^{a\dagger} a_a + a_a a^{a\dagger}) = \omega \left( a^{a\dagger} a_a + \frac{r}{2} + \frac{s}{2} \right), \]  

(71)

acting on the states created by \( a^{a\dagger} \) has always positive eigenvalues. But the states corresponding to negative signature have negative norms, and are therefore called ghosts.

**Definition II.** This is the Cangemi-Jackiw-Zwiebach definition [28], discussed in Refs. [3, 4]. We split the operators into the positive and negative signature parts,

\[ a^a = (a^{\bar{a}}, a_a), \quad \bar{a} = 1, 2, ..., r ; \]  

\[ a = r + 1, r + 2, ..., r + s. \]  

(72)

Then the Hamiltonian operators (71) which can be written as

\[ H = \omega \left( a^{\bar{a}\dagger} a_{\bar{a}} + a_2 a^{2\dagger} + \frac{r}{2} - \frac{s}{2} \right), \]  

(73)
has positive and negative eigenvalues. There are no negative norm states. If the signature is neutral, \( r = s \), then the vacuum energy vanishes.

The quantization of the system (58) according to Definition II has the correct classical limit, and satisfies the correspondence principle. Woodard [4] argues that this is the correct quantization, whereas the quantization according to Definition I is incorrect.

Unfortunately, many authors who have been aware of the Definition II, considered such a system as problematic anyway, because in physically realistic situations there are interactions between positive and negative energy degrees of freedom. According to the prevailing opinion, such a system is necessarily unstable in the presence of interactions. But it has turned out that this is not true. Behaviour of the system (36) in the presence of various interactions has been studied in Ref. [26, 35], where it was found that for suitable interactions the system is stable.

Figure 1: The solution of the system described by eqs. (75), (76), for the initial conditions \( \dot{x}(0) = 1 \), \( \dot{y}(0) = -1.2 \), \( x(0) = 0 \), \( y(0) = 0.5 \).

### 4.2 Inclusion of interactions

Let us now include into the Lagrangian (36) and additional term \( V_1(x, y) \) that couples the positive energy degree of freedom \( x \) and the negative energy degree of freedom \( y \), and also include masses \( m_1, m_2 \):

\[
L = \frac{1}{2}(m_1 \dot{x}^2 - m_2 \dot{y}^2) - V, \quad V = \frac{\omega}{2}(x^2 - y^2) + V_1. \tag{74}
\]
The equations of motion are then
\begin{align*}
m_1 \ddot{x} + \omega^2 x + \frac{\partial V_1}{\partial x} &= 0, \quad (75) \\
m_2 \ddot{y} + \omega^2 y - \frac{\partial V_1}{\partial y} &= 0. \quad (76)
\end{align*}

As an example let us first take
\[ V_1 = \frac{\lambda}{4} (x^2 - y^2)^2. \quad (77) \]
and consider two cases:

a) \( m_1 = m_2 \)

In Fig. 1 are shown the results [26] of numerical solutions for \( m_1 = m_2 \). We see that the trajectory in the \((x, y)\)-space and the kinetic energy run into infinity. Such a system is thus unstable.

b) \( m_1 \neq m_2 \)

If masses are different, then something peculiar happens. As shown in Fig. 2, the oscillations of the trajectory in the \((x, y)\)-space first increase along one arm [26]. After

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3 Calculations were executed with Mathematica by using NDSolve.
some time, the second arm forms. The trajectory remains confined within these two arms. The envelop of the kinetic energy oscillations does not run into infinity, but forms the peaks. If the masses differ very slightly, then the peaks are nearly separated (Fig. 2). We see that the system is stable. We checked this numerically for various initial conditions and positive coupling constants $\lambda$.

![Figure 3](image1.png)

Figure 3: An example of the potential that is bounded from below and from above (see Eq. [78]).

So far we have considered the unbounded potential that was of the form [74], [77]. But a realistic potential is bounded from below and from above. In nature there are

![Figure 4](image2.png)

Figure 4: The trajectory in the ($x, y$)-space of the solution of the system described by Eqs. [79], [80], for the initial conditions $\dot{x}(0) = 0.8$, $\dot{y}(0) = 0.1$, $x(0) = 2.50$, $y(0) = -0.50$.

no potentials that go into infinity. As an example, the potential of the form (Fig. 3)

$$V = \frac{\lambda}{4}(\sin^2 x - \sin^2 y + \lambda_1 \sin x \sin y)$$

(78)

was studied in Ref. [35]. The equations of motion are then

$$\ddot{x} + \frac{\lambda}{2}(2 \sin x \cos x + \lambda_1 \cos x \sin y) = 0,$$

(79)

$$\ddot{y} + \frac{\lambda}{2}(2 \sin y \cos y - \lambda_1 \sin x \cos y) = 0,$$

(80)
The results of calculations show that the system, as expected, is stable. If the initial speed is not too high, the trajectory remains confined within a finite region of the \((x, y)\)-space (Fig. 4). In the case of an increased initial speed, the position is not confined, the trajectory can run into infinity (Fig. 5), whereas the velocity remains finite. The system behaves as a quasi free particle subjected to perturbative forces arising from the potential \((78)\). Such a system is also stable in the sense that its speed and energy do not increase into infinity.

\[
\lambda = 2, \quad \lambda_1 = 0.1
\]

Figure 5: The solution of the system described by Eqs. \((79),(80)\), for the initial conditions \(\dot{x}(0) = 1, \dot{y}(0) = 0.9, x(0) = 2.50, y(0) = -0.505\). Left: the trajectory in the \((x, y)\)-space. Right: the velocity \(\dot{x}\) as function of time \(t\).

These results show that the systems with a bounded potential do not exhibit any unusual behaviour even if certain degrees of freedom have negative energies. There are no runaway solutions and energies do not increase (or decrease) beyond the boundaries determined by the potential. Upon correct quantization, such systems are expected to be stable as well because of the correspondence principle.

5 Pais-Uhlenbeck oscillator in the presence of quartic self-interaction

After having discussed in Sec. 4.2 a toy model interacting system with the positive energy degree of freedom \(x\) and a negative energy degree of freedom \(y\), let us now return to a similar system, which should be equivalent to an interacting PU oscillator. We will again employ the notation \(x', y'\). As discussed in Sec. 3, many authors have observed that the non-interacting PU oscillator can be described either by a Lagrangian which leads to a positive definite Hamiltonian, or it can be described by a Lagrangian which gives indefinite Hamiltonian. In Ref. [5] it was shown that in the presence of an interaction, the system with positive definite Hamiltonian is not equivalent to a PU oscillator.

i) If we start from the Lagrangian

\[
L = \frac{1}{2}(\dot{x}'^2 + \dot{y}'^2) - \frac{1}{2}(\omega_1^2 x'^2 + \omega_2^2 y'^2) - \frac{\lambda}{4} (x' + y')^4, \quad (81)
\]
whose equations of motion are
\[ \ddot{x}' + \omega_1^2 x' + \lambda(x' + y')^3 = 0, \tag{82} \]
\[ \ddot{y}' + \omega_2^2 y' + \lambda(x' + y')^3 = 0, \tag{83} \]
and introduce the new variables
\[ u = \frac{x' + y'}{\sqrt{2}}, \quad v = \frac{x' - y'}{\sqrt{2}}, \tag{84} \]
we obtain
\[ L = \frac{1}{2}(\dot{u}^2 + \dot{v}^2) - \frac{1}{4}[(\omega_1^2 + \omega_2^2)(u^2 + v^2) + 2(\omega_1^2 - \omega_2^2)uv] - \lambda u^4 \tag{85} \]
Introducing
\[ \mu_1 = \mu_2 = \frac{1}{2}(\omega_1^2 + \omega_2^2), \quad -\rho_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2), \tag{86} \]
the equation of motion derived from (85) are
\[ \ddot{u} + \mu_1 u - \rho_1 v + 4\lambda u^3 = 0 \tag{87} \]
\[ \ddot{v} + \mu_2 v - \rho_1 u = 0, \tag{88} \]
which, after elimination of \( u \), becomes
\[ v^{(4)} + (\mu_1 + \mu_2)\ddot{v} + (\mu_1 \mu_2 - \rho_1^2)v + 4\lambda \rho_1 (\ddot{v} + \mu_2 v)^3 = 0, \tag{89} \]
The interaction term is nonlinear in the second order derivative \( \ddot{v} \).
In the system (87),(88) we may as well eliminate \( v \). Then we obtain the equation
\[ u^{(4)} + (\mu_1 + \mu_2)\ddot{u} + (\mu_1 \mu_2 - \rho_1^2)u + 4\mu_2 \lambda u^3 + 4\lambda \frac{d^2}{dt^2} (u^3) = 0, \tag{90} \]
which also contains a nonlinear term.
Neither Eq. (89) nor (90) does contain a simple interaction term that could be derived from a potential. This reveals that the system described by the Lagrangian (81) is not equivalent to a self-interacting PU oscillator.
(ii) The situation is different if we start from the Lagrangian
\[ L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - \frac{1}{2}(\omega_1^2 x^2 - \omega_2^2 y^2) - \frac{\lambda}{4}(x' + y')^4, \tag{91} \]
which in terms of the new variables (84) reads
\[ L = \dot{u}\dot{v} - \frac{1}{4}[(\omega_1^2 - \omega_2^2)(u^2 + v^2) + 2(\omega_1^2 + \omega_2^2)uv] - \lambda u^4. \tag{92} \]
The equations of motion are now

\[ \ddot{u} + \mu_1 u - \rho_1 v = 0 \quad (93) \]

\[ \ddot{v} + \mu_2 v - \rho_1 u + 4\lambda u^3 = 0, \quad (94) \]

where \( \mu_1, \mu_2 \) and \( \rho_1 \) are given in Eq. (86). Eliminating \( v \), we have

\[ u^{(4)} + (\mu_1 + \mu_2)\ddot{u} + (\mu_1\mu_2 - \rho_1^2)u + 4\rho_1\lambda u^3 = 0. \quad (95) \]

This can be rewritten in terms of \( \omega_1^2, \omega_2^2 \) by using Eq. (86):

\[ u^{(4)} + (\omega_1^2 + \omega_2^2)\ddot{u} + \omega_1\omega_2^2 u - \Lambda u^3 = 0, \quad (96) \]

where \( \Lambda = 2(\omega_1^2 - \omega_2^2)\lambda \). The corresponding Lagrangian is

\[ L = \frac{1}{2} \left[ \dot{u}^2 - (\omega_1^2 + \omega_2^2)u^2 + \omega_1\omega_2^2 u^2 \right] + \frac{1}{4}\Lambda u^4. \quad (97) \]

This is the Lagrangian for a PU oscillator with a quartic self-interaction term. The variable \( u \) here is in fact the same as the variable \( x \) used in Eqs. (7)–(9) in Sec. 3.

Figure 6: Solutions of the equations of motion derived from the Lagrangian (91) for different values of the coupling constant \( \lambda \) and different initial conditions. Left and middle: the trajectories in the \( (x', y') \) space. Right: The kinetic energy \( \dot{x}'^2/2 \) as function of time. The oscillations within the envelope are so fine that they fill the diagram.

By applying the Ostrogradski formalism to (97) we obtain the Hamiltonian which is not positive definite. Inclusion of an interaction term into the free PU
Lagrangian reveals that such a second order system has to be processed à la Ostrogradski. Alternative ways discussed in the literature [10]–[16], where the PU oscillator is described in terms of positive energies only, are applicable to the free oscillator, but not to a (self) interacting one.

It used to be taken for granted that a system with a Lagrangian of the sort (91), or, equivalently (96) is unstable. But Smilga [22] has found that this is not necessarily so. For a certain range of initial velocity and coupling constant the system (91) can be stable. This has been studied in Refs. [5], where it was found by numerical calculations that the system was stable if the initial velocity $\dot{x}'(0)$, $\dot{y}'(0)$ and the coupling constant $\lambda$ were below certain critical values (Fig. 6). Above the critical

values the system exhibited runaway behaviour (Fig. 7). Very close to the critical value of $\lambda$ the solution oscillated seemingly stably within a confined region of the $(x', y')$-space, but after a long time it escaped into infinity (Fig. 8). It is reasonable to expect that a similar behaviour could occur in certain solid state system that can eventually be described by a higher order derivative field theory à la Eq. (1). This could have far reaching consequences and applications (see also Ref. [26]).
Figure 8: At certain values of $\lambda$ and the initial conditions, the system behaves stably for a long time, before it finally escapes to infinity. The total energy $E_{\text{tot}}$ remains constant within the numerical error.

6 The Cases of Stable Interacting Pais-Uhlenbeck Oscillator

(i) A Bounded Interaction Term

Let us now investigate how the system behaves if the interaction potential is bounded from below and form above. As an example the interaction term $\frac{1}{4}\sin^4(x' + y')$ was considered in Ref. [5]. Instead of the Lagrangian (91) we then have

$$L = \frac{1}{2}(\dot{x}'^2 - \dot{y}'^2) - \frac{1}{2}(\omega_1^2 x'^2 - \omega_2^2 y'^2) - \frac{\lambda}{4}\sin^4(x' + y'),$$

which gives the following equations of motion:

$$\ddot{x}' + \omega_1^2 x' + \lambda\sin^3(x' + y')\cos(x' + y') = 0,$$

$$\ddot{y}' + \omega_2^2 y' - \lambda\sin^3(x' + y')\cos(x' + y') = 0.$$  

By solving the above equation numerically, it was found [3] that such a system is stable for all initial velocities and for all positive values of the coupling constant $\lambda$ (see Fig. 9).

(ii) Unequal masses

The case when masses are different was also considered in Ref. [5]. Instead of the Lagrangian (91) we then have

$$L = \frac{1}{2}(m_1 \dot{x}'^2 - m_2 \dot{y}'^2) - \frac{1}{2}(\omega_1^2 x'^2 - \omega_2^2 y'^2) - \frac{\lambda}{4}(x' + y')^4$$

Introducing

$$m = \frac{1}{2}(m_1 - m_2), \quad M = \frac{1}{2}(m_1 + m_2),$$

$$\lambda=0.02299$$

$$x'(0)=0, \quad y'(0)=1$$

$$x'(0)=0.999851, \quad y'(0)=0$$

$$E_{\text{tot}}$$
and the variables $u,v$, defined in Eq. (84), the Lagrangian (81) becomes

$$L = \frac{1}{2} [m(\dot{u}^2 + \dot{v}^2) + 2M \ddot{u} \dot{v} + \rho_1 (u^2 + v^2) - 2\mu_1 uv] - \lambda u^4,$$

where $\mu_1$ and $\rho_1$ are defined in Eq. (86). This gives the system of two coupled second order equation of motion

$$m\ddot{u} + M\ddot{v} - \rho_1 u + \mu_1 v + 4\lambda u^3 = 0$$

$$m\ddot{v} + M\ddot{u} - \rho_1 v + \mu_1 u = 0.$$

which is equivalent to the fourth order equation

$$u^{(4)}(M^2 - m^2) + 2\ddot{u}(\mu_1 M + \rho_1 m) + uM(\mu_2^2 - \rho_1^2) + 4M\rho_1 \lambda u^3 - 4Mm\lambda \frac{d^2}{dt^2} (u^3) = 0.$$

By putting $M = 1, m = 0$, we verify that the latter equation becomes the equation (95) of the PU oscillator with the quartic self-Interaction term. But if $M \neq 0, m \neq 0$, then Eq. (106), because of the last term, looks like a generalization of the equation of motion (99) that was derived from the Lagrangian (85), equivalent to the Lagrangian (81) with the signature (+++) in the space of the variables $x',y'$. This suggests that the
system with different masses is stable. Stability of such a system was confirmed [5] by numerical solutions to the equation of motion

\[ m_1 \ddot{x} + \omega_1^2 x + \lambda (x' + y')^3 = 0, \]  
\[ m_2 \ddot{y} + \omega_2^2 y' - \lambda (x' + y')^3 = 0, \]
derived from the Lagrangian (101).

The results of many calculations, reported in Ref. [5] showed stability for all values of \( \lambda > 0 \) and initial velocities. Some examples are shown in Fig. 10. If masses are equal, \( m_1 = m_2 = 1 \), then the system of equations (107), (108) with \( \omega_1 = 1, \omega_2 = \sqrt{2} \) exhibits runaway solution when \( \lambda > 0.03 \), whilst it is stable for lower values of \( \lambda \). But if masses are slightly different, then the system is stable.

Similar behaviour had been previously shown [26] for a similar system, but with the coupling term \( \frac{1}{4} (x^2 - y^2)^2 \) (see also Sec. 4 of this review).

Let us now show also analytically that the system of equations (107), (108) is stable for \( m_1 < m_2 \) and \( \lambda \gg 0 \). Assuming very high \( \lambda \), such that the contribution of the terms \( \omega_1^2 x' \) and \( \omega_2^2 y' \) are negligible, the system (107), (108) becomes

\[ \ddot{x} + \frac{1}{m_1} \lambda (x' + y')^3 = 0, \]
\[ \ddot{y} - \frac{1}{m_2} \lambda (x' + y')^3 = 0. \]

In terms of the new variables \( \xi = x' + y' \) and \( \eta = x' - y' \), the latter system reads

\[ \ddot{\xi} + \left( \frac{1}{m_1} - \frac{1}{m_2} \right) \lambda \xi^3 = 0, \]
\[ \ddot{\eta} + \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \lambda \xi^3 = 0. \]

In the first equation only the variable \( \xi \) occurs, and the quartic potential is bounded from below if \( m_1 < m_2 \) and \( \lambda > 0 \). The solution is stable. Consequently, also \( \eta(t) \) in the second equation is stable. This is true regardless of how high is \( \lambda \).

If we do not neglect the terms \( \omega_1^2 x' \) and \( \omega_2^2 y' \), and rewrite the equations (107) and (108) in terms of the variables \( \xi, \eta \), we obtain

\[ \ddot{\xi} + \frac{1}{2} \left( \frac{\omega_1^2}{m_1} + \frac{\omega_2^2}{m_2} \right) \xi + \frac{1}{2} \left( \frac{\omega_1^2}{m_1} - \frac{\omega_2^2}{m_2} \right) \eta + \left( \frac{1}{m_1} - \frac{1}{m_2} \right) \lambda \xi^3 = 0, \]
\[ \ddot{\eta} + \frac{1}{2} \left( \frac{\omega_1^2}{m_1} - \frac{\omega_2^2}{m_2} \right) \xi + \frac{1}{2} \left( \frac{\omega_1^2}{m_1} + \frac{\omega_2^2}{m_2} \right) \eta + \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \lambda \xi^3 = 0. \]
If
\[
\frac{\omega_1^2}{m_1} = \frac{\omega_2^2}{m_2},
\] 
(115)
then the variable \( \eta \) disappears from Eq. (113), and what remains is the differential equation for \( \xi \) only, which has stable oscillatory solution if \( \lambda > 0 \). Once we have a solution for \( \xi(t) \), we can plug it into Eq. (114), which then becomes a harmonic oscillator equation for the variable \( \eta \) with a time dependent force term. The system (113), (114), or equivalently (107), (108) is thus stable despite the fact that it is derived from the Lagrangian (101) in which the degrees of freedom \( x \) and \( y \) have positive and negative energy, respectively.

Numerical solutions shown in Fig. 10 reveal that even if we relax the relation (115), the system is stable. Moreover, in Sec. 4.2 it is shown that the system in which the interacting potential is \( \frac{\lambda}{2}(x^2 - y^2)^2 \) is also stable, provided that the masses are different. Ilhan and Kovner [34] studied a system of the same form as in Eq. (74), but with the interaction potential \( V_1 = \lambda_1 x^4 - \lambda_2 y^4 + \mu x^2 y^2 \), and they found stability in many numerical runs.

\[
\begin{align*}
&\lambda = 5, \ m_1 = 0.7, \ m_2 = 1.3 \\
x'(0) = 0.3, \ y'(0) = 1 \\
\dot{x}'(0) = 4, \ \dot{y}'(0) = -0.5 \\
\omega_1 = 1, \ \omega_2 = \sqrt{1.5} \\
&\lambda = 500, \ m_1 = 0.7, \ m_2 = 1.3 \\
x'(0) = 0.3, \ y'(0) = 1 \\
\dot{x}'(0) = 4, \ \dot{y}'(0) = -0.5 \\
\omega_1 = 1, \ \omega_2 = \sqrt{1.5} \\
&\lambda = 5, \ m_1 = 0.7, \ m_2 = 1.3 \\
x'(0) = 0.3, \ y'(0) = 1 \\
\dot{x}'(0) = 40, \ \dot{y}'(0) = 55 \\
\omega_1 = 1, \ \omega_2 = \sqrt{1.5} \\
&\lambda = 5, \ m_1 = 0.7, \ m_2 = 1.3 \\
x'(0) = 0.3, \ y'(0) = 1 \\
\dot{x}'(0) = 40, \ \dot{y}'(0) = -0.5 \\
\omega_1 = \omega_2 = 0
\end{align*}
\]

Figure 10: Solutions of Eqs. (107), (108) for different values of the coupling constant \( \lambda \) and different initial conditions. We show here the kinetic energy \( \dot{x}^2/2 \) as function of time.

Though the system (74) is equivalent to a self interacting PU oscillator only if \( V_1 \) is a function of \( x' + y' \) or \( x' - y' \), and if \( m_1 = m_2 \), the studies revealing stability of the systems such as (74) or (101) are important, because they show that the presence of
negative energies does not automatically destroy physical viability of such systems.

Returning now to the system of equations (111), (112) and taking $m_1 = m_2$, we find

$$\ddot{\xi} = 0, \quad \ddot{\eta} + \frac{2}{m_1} \lambda \xi^3 = 0, \quad (116)$$

with the general solution

$$\xi = \xi_0 + c_1 t, \quad \eta = -\frac{2}{m_1} \frac{\lambda}{20} (\xi_0 + c_1 t)^5 + c_2 t \quad (117)$$

which is a runaway trajectory. In the presence of the terms $\omega_1^2 x$ and $\omega_2^2 y$, the above runaway behavior is modulated by oscillations.

But if instead of $\frac{1}{4}(x^2 + y^4)$ we take the bounded interaction potential $\frac{1}{4}\sin^4(x+y)$, then instead of Eqs. (116) we have

$$\ddot{\xi} = 0, \quad \ddot{\eta} + \frac{2}{m_1} \lambda \sin^3 \xi \cos \xi = 0, \quad (118)$$

whose general solution is

$$\xi = \xi_0 + c_1 t, \quad \dot{\eta} = -\frac{2}{m_1} \frac{1}{4c_1} \lambda \sin^4 (\xi_0 + c_1 t), \quad \eta = -\frac{2}{m_1} \frac{1}{128c_1^3} \lambda \left[12(\xi_0 + c_1 t) - 8 \sin (2(\xi_0 + c_1 t) + \sin (4(\xi_0 + c_1 t)) \right] \quad (119)$$

This is a free particle trajectory modulated by finite oscillations. The kinetic energy $\frac{m_1}{2} \dot{x}^2 - \frac{m_1}{2} \dot{y}^2 = \frac{m_1}{2} \dot{\xi} \dot{\eta}$ remains finite.

## 7 Pais-Uhlenbeck oscillator in the presence of damping

### 7.1 Inclusion of a damping term

An important factor that must be taken into account is the influence of the environment, which manifests itself as dissipative forces, acting on a system. In the case of an oscillator, the environment acts as a damping force. Nesterenko [30] studied the behaviour of the Pais-Uhlenbeck oscillator in the presence of a damping term that is linear in velocity. But Stephen showed [33] that not only the linear term $\dot{x}$, but also the term $\ddot{x}$ has to be present in the PU oscillator. Indeed, if we generalize the equation (11) so to included damping as well, we arrive at [31]

$$\left(\frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_1^2\right) \left(\frac{d^2}{dt^2} + 2\alpha \frac{d}{dt} + \omega_2^2\right) x = 0, \quad (120)$$
which gives the following fourth order differential equation with two damping terms

\[ x^{(4)} + 2(\alpha + \beta)\ddot{x} + (\omega_1^2 + \omega_2^2 + 4\alpha\beta)\dot{x} + 2(\omega_1^2\beta + \omega_2^2\alpha)\dot{x} + \omega_1^2\omega_2^2x = 0, \tag{121} \]

The general solution of the latter equation is

\[ x = e^{-\alpha t} \left( C_1e^{\sqrt{\alpha^2 - \omega_1^2} t} + C_2e^{-\sqrt{\alpha^2 - \omega_1^2} t} \right) + e^{-\beta t} \left( C_3e^{\sqrt{\beta^2 - \omega_2^2} t} + C_4e^{-\sqrt{\beta^2 - \omega_2^2} t} \right). \tag{122} \]

This is oscillatory function if \( \alpha^2 < \omega_1^2, \beta^2 < \omega_2^2 \). If in addition \( \alpha \) and \( \beta \) are positive, then \( x(t) \) has an exponentially decaying envelop. Thus, under the above conditions, the solution (122) is stable.

In particular, if \( \alpha = -\beta \), then the solution (122) has an exponentially decreasing and an exponentially increasing part:

\[ x = e^{\beta t} \left( C_1e^{\sqrt{\beta^2 - \omega_1^2} t} + C_2e^{-\sqrt{\beta^2 - \omega_1^2} t} \right) + e^{-\beta t} \left( C_3e^{\sqrt{\beta^2 - \omega_2^2} t} + C_4e^{-\sqrt{\beta^2 - \omega_2^2} t} \right). \tag{123} \]

Such a solution is thus unstable for every real \( \beta \). If we plug \( \alpha = -\beta \) into Eq. (121) we obtain

\[ x^{(4)} + (\omega_1^2 + \omega_2^2 - 4\beta^2)\ddot{x} + 2\beta(\omega_1^2 - \omega_2^2)\dot{x} + \omega_1^2\omega_2^2x = 0, \tag{124} \]

which has no \( \ddot{x} \) term. This can be rewritten into the form considered by Nesterenko [30]:

\[ x^{(4)} + (\Omega_1^2 + \Omega_2^2)\ddot{x} + 2\gamma\dot{x} + \Omega_1^2\Omega_2^2x = 0, \tag{125} \]

where \( \gamma = \beta(\omega_1^2 - \omega_2^2) \), and

\[ \Omega_1^2 + \Omega_2^2 = \omega_1^2 + \omega_2^2 - 4\beta^2, \tag{126} \]

\[ \Omega_1^2\Omega_2^2 = \omega_1^2\omega_2^2. \tag{127} \]

The latter system of equations has the solution

\[ \Omega_{1,2}^2 = \frac{1}{2} \left[ \omega_1^2 + \omega_2^2 - 4\beta^2 \pm \sqrt{(\omega_1^2 + \omega_2^2 - 4\beta^2)^2 - 4\omega_1^2\omega_2^2} \right]. \tag{128} \]

Nesterenko studied an approximate solution to Eq. (125) for weak dumping, by using the perturbation theory. His results is in agreement with the exact solution (122) considered in Ref. [31], which in turn is a special case (for \( \alpha = -\beta \)) of the solution (122) of the oscillator with the \( \dot{x} \) and \( \ddot{x} \) damping terms.

### 7.2 Presence of an arbitrary external force

Nesterenko [30] considered also the Pais-Uhlenbeck oscillator which experiences an arbitrary external force \( f(t) \). He expressed the solution \( x(t) \) in terms of the propagator \( G(t - t') \), or equivalently, in terms of its Fourier transform \( \tilde{G}(\omega) \). He found that \( \tilde{G}(\omega) \)
contains a positive and a negative term, and thus the displacement \( x(t) \) consists of two contributions. Nesterenko concluded that one of these contributions to \( x(t) \) was unphysical.

In Ref. [31] and arbitrary time dependent force \( f(t) \) was added to the equation of motion (120) for the PU oscillator with damping:

\[
\left( \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_2^2 \right) \left( \frac{d^2}{dt^2} + 2\alpha \frac{d}{dt} + \omega_1^2 \right) x = f(t). \tag{129}
\]

Two cases, (i) without damping, \( \alpha = \beta = 0 \), and (ii) with damping, \( \alpha \neq 0 \), \( \beta \neq 0 \), were considered in some more details [31].

The general solution of Eq. (129) is

\[
x(t) = x_0(t) + \int_{-\infty}^{\infty} G(t-t')f(t')dt',
\]

where \( x_0(t) \) is a general solution of the homogeneous equation (120), i.e., Eq. (122), and where

\[
G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \tilde{G}(\omega)d\omega,
\]

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \tilde{f}(\omega)d\omega,
\]

In Case (i), the Fourier transformed Green function, derived from Eq. (129) for \( \alpha = \beta = 0 \), is

\[
\tilde{G}(\omega) = \frac{1}{\sqrt{2\pi(\omega_1^2 - \omega_2^2)}} \left( \frac{1}{\omega^2 - \omega_1^2} - \frac{1}{\omega^2 - \omega_2^2} \right), \tag{133}
\]

which for \( t > 0 \) gives

\[
G(t) = \frac{1}{2(\omega_1^2 - \omega_2^2)} \left( \frac{\sin \omega_2 t}{\omega_2} - \frac{\sin \omega_1 t}{\omega_1} \right). \tag{134}
\]

As an example, the following force was considered [31]:

\[
f(t) = a \cos \omega_1 t + b \cos \omega_2 t, \tag{135}
\]

which according to Eq. (132) gives

\[
\tilde{f}(\omega) = \frac{a\sqrt{2\pi}}{2} \left[ \delta(\omega - \omega_1) + \delta(\omega + \omega_1) \right] + \frac{b\sqrt{2\pi}}{2} \left[ \delta(\omega - \omega_2) + \delta(\omega + \omega_2) \right]. \tag{136}
\]

Using (134) and (135), the solution (130) reads explicitly

\[
x(t) = x_0(t) - \frac{1}{2\omega_1\omega_2(\omega_1^2 - \omega_2^2)} \left[ (\omega_1^2 - \omega_2^2)(a\omega_2 t \sin \omega_1 t - b\omega_1 t \sin \omega_2 t) \right].
\]
+ 2(a − b)\omega_1\omega_2(\cos \omega_1 t − \cos \omega_2 t)], \tag{137}

where \(x_0(t)\) is given by Eq. (122) in which we set \(\alpha = 0, \beta = 0\). This solution was also obtained directly from Eq. (129) by using the command DSolve in Mathematica.

In Case (ii) we have

\[
\tilde{G}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{(\omega^2 - \omega_2^2 - 2i\beta)(\omega^2 - \omega_1^2 - 2i\alpha)}
\]

Then the relation (130) is

\[
x(t) = x_0(t) + \int \tilde{G}(\omega)\tilde{f}(\omega) d\omega
\]

\[
x(t) = x_0(t) + \frac{a}{4i\alpha\omega_1} \left( -\frac{e^{i\omega_1 t}}{-\omega_1^2 + \omega_2^2 + 2i\beta\omega_1} - \frac{e^{-i\omega_1 t}}{-\omega_1^2 + \omega_2^2 + 2i\beta\omega_1} \right)
\]

\[
+ \frac{b}{4i\beta\omega_2} \left( -\frac{e^{i\omega_2 t}}{-\omega_2^2 + \omega_1^2 + 2i\alpha\omega_2} - \frac{e^{-i\omega_2 t}}{-\omega_2^2 + \omega_1^2 + 2i\alpha\omega_2} \right)
\]

\[
x(t) = x_0(t) - \frac{2\beta\omega_1 \cos \omega_1 t - a(\omega_1^2 - \omega_2^2) \sin \omega_1 t}{2\alpha\omega_1 [4\beta^2\omega_1^2 + (\omega_1^2 - \omega_2^2)^2]}
\]

\[
+ \frac{2\alpha\omega_2 \cos \omega_2 t + b(\omega_2^2 - \omega_1^2) \sin \omega_2 t}{2\beta\omega_2 [4\alpha^2\omega_2^2 + (\omega_1^2 - \omega_2^2)^2]}, \tag{139}
\]

where \(x_0(t)\) is given by Eq. (122) for \(\alpha > 0, \beta > 0\). The same solution (139) can be obtained by applying the Mathematica command DSolve to the 4th order differential equation (129) with \(f(t)\) given in Eq. (135), which confirmed the correctness of the above result.

We see from Eq. (137) that the amplitude in the solution \(x(t)\) for \(\alpha = \beta = 0\) increases linearly with time \(t\), and thus such a system is unstable. This is not the case in the presence of damping, i.e., when \(\alpha > 0, \beta > 0\). Then the solution is (139) which exhibits decent oscillatory behaviour. Here the behaviour of the displacement \(x(t)\) is very similar as in the case of a harmonic oscillator in the presence of a force that oscillates with the resonant frequency. In the absence of damping, the amplitude of \(x(t)\) grows linearly with time, whereas in the presence of damping it remains confined.

In Eq. (135) we considered the time dependent force \(f(t)\) whose spectral density is sharply localized around \(\omega_1^2\) and \(\omega_2^2\) according to Eq. (136). In Ref. [31] also a more general case

\[
\tilde{f}(\omega) = \frac{a\sqrt{2\pi}}{2} \sqrt{\frac{c}{\pi}} \left( e^{-c(\omega - \omega_1)^2} + e^{-c(\omega + \omega_1)^2} \right) + \frac{b\sqrt{2\pi}}{2} \sqrt{\frac{c}{\pi}} \left( e^{-c(\omega - \omega_2)^2} + e^{-c(\omega + \omega_2)^2} \right), \tag{140}
\]

was considered. Then the time dependent force has exponentially decreasing envelop:

\[
f(t) = e^{-\frac{t^2}{2}} (a \cos \omega_1 t + b \cos \omega_2 t). \tag{141}
\]
In such a case the system is stable even in the absence of damping, i.e., when $\alpha = \beta = 0$. In Ref. [31], numerical solutions of Eq. (129) were found by using the Mathematic command NDSolve, and they were stable for all finite positive values of the width parameter $c$ and initial velocities. Some examples are shown in Fig. 11.

Figure 11: Solution of the undamped Pais-Uhlenbeck oscillator ($\alpha = \beta = 0$) in the presence of an external force with the spectral density localized around $\omega^2_1 = 1$ and $\omega^2_2 = 3$ according to (140) for two different values of the width parameter $c$. We took the constants $a = b = 1$, and the initial conditions $x(0) = 0.5$, $\dot{x}(0) = 0.4$, $\ddot{x}(0) = -1.2$, $\dddot{x}(0) = 1$.

Another possibility is to consider the external force of the form

$$f(t) = a \cos \omega_1^t t + b \cos \omega_2^t t,$$

where the frequencies $\omega_1^t$, $\omega_2^t$ are different from the resonant frequencies $\omega_1$, $\omega_2$. Then in the case $\alpha = \beta = 0$, the exact solution is

$$x(t) = x_0(t) - \frac{\cos \omega_1^t t}{(\omega_1^t - \omega_1^2)(\omega_1^2 - \omega_2^2)} - \frac{\cos \omega_2^t t}{(\omega_2^t - \omega_2^2)(\omega_1^2 - \omega_2^2)}$$

In the case $\alpha \neq 0$, $\beta \neq 0$, we obtain the following exact solution:

$$x(t) = x_0(t) - \left[ (\omega_1^2 - \omega_2^2 - (4\alpha \beta + \omega_1^2 + \omega_2^2)\omega_1^2 + \omega_1^4) \left( (\omega_1^2 - \omega_2^2)^2 + 4\alpha^2 \omega_2^2 \right) \right. \left. \times \left( (\omega_2^2 - \omega_2^2)^2 + 4\beta^2 \omega_2^2 \right) \cos \omega_1^t t \right. \left. - (\omega_1^2 \omega_2^2 - (4\alpha \beta + \omega_1^2 + \omega_2^2)\omega_2^2 + \omega_1^4) \left( (\omega_1^2 - \omega_2^2)^2 + 4\alpha^2 \omega_1^2 \right) \times \left( (\omega_2^2 - \omega_2^2)^2 + 4\beta^2 \omega_2^2 \right) \cos \omega_2^t t \right. \left. - 2\omega_1 \left( \beta \omega_1^2 + \alpha \omega_2^2 - (\alpha + \beta) \omega_1^2 \right) \left( (\omega_1^2 - \omega_2^2)^2 + 4\beta^2 \omega_2^2 \right) \times \left( (\omega_2^2 - \omega_2^2)^2 + 4\beta^2 \omega_2^2 \right) \sin \omega_1^t t \right. \left. - 2\omega_2 \left( \beta \omega_1^2 + \alpha \omega_2^2 - (\alpha + \beta) \omega_2^2 \right) \left( (\omega_1^2 - \omega_2^2)^2 + 4\alpha^2 \omega_2^2 \right) \times \left( (\omega_2^2 - \omega_2^2)^2 + 4\beta^2 \omega_1^2 \right) \sin \omega_2^t t \right] \times \left[ (\omega_1^4 - 2\omega_1^2 \omega_2^2 + \omega_2^4 + 2\omega_1^2 \omega_1^2 \omega_2^2) \left( \omega_1^2 - 2\omega_2 \omega_1^2 \omega_2^2 + \omega_1^2 \omega_2^2 (4\beta^2 + \omega_2^2) \right) \times \left( \omega_1^4 - 2\omega_1^2 \omega_2^2 + \omega_2^4 + 2\omega_1^2 \omega_1^2 \omega_2^2) \left( \omega_1^2 - 2\omega_2 \omega_1^2 \omega_2^2 + \omega_1^2 \omega_2^2 (4\beta^2 + \omega_2^2) \right) \right]^{-1}$$

(144)
In both cases we used Mathematica’s DSolve, and verified that the solution so obtained indeed satisfied the 4th order differential equation (129) that we started from.

Nesterenko [30] remarked that due to the expression (133) for the propagator the forces with spectral densities localized around $\omega_1^2$ and $\omega_2^2$ give rise to displacement $x$ of opposite signs. According to Nesterenko this implies that one of these displacements is unphysical. But our calculations reveal that nothing unphysical happens with the classical displacement $x(t)$. The displacement rises into infinity in the physically unrealistic case of the $\delta$-function localization of $\tilde f(\omega)$, i.e., when the width parameter $c$ in Eq. (140) and (141) is infinite, i.e., when $f(t)$ is a sum (up to a phase) of pure $\cos \omega_1 t$ and $\cos \omega_2 t$. In practice this is impossible, because a realistic time dependent force cannot have exactly the resonant frequencies $\omega_1$ and $\omega_2$, the frequencies are at least slightly different, or the spectral density is spread around them. Moreover, there is always some damping. In either case, the solution is stable. The contributions of both terms in the Green’s function $\tilde G(\omega)$ together give rise to a physical displacement.

8 On the stability of higher derivative field theories

A higher derivative field theory of the form (1) is an infinite set of Pais-Uhlenbeck oscillators. If instead of the interaction potential $\lambda \phi^4$ we take a potential $V_1(\phi)$ that is bounded from below and form above, then just as in the case of the PU oscillator also such a field theoretic system is stable. However, now that we have infinite dimensional system, the number of available final states, and hence the phase space, can be infinite. In the studies of physical processes, one usually sums over final states. With both positive and negative energies, the phase space can be infinite. The formulas for transition and decay rates thus besides the transition amplitude squared contain the integration over the phase space. A common conclusion is that because in the case of higher derivative theory the phase space is infinite, vacuum would instantaneously decay into positive and negative energy particles, which means that such theories are not physically viable.

Such a conclusion has been questioned in Ref. [26]. When studying a physical system, for instance an excited nucleus, we usually do not measure the momenta of the decay particles, therefore in the transition probability formula

$$\text{Transition probability} = \int |S_{fi}|^2 d^3 p_1 d^3 p_2 ... \quad (145)$$

we integrate over them. If we measure the final state particle’s momenta, the integration goes over a narrower portion of the phase space, which reduces the transition probability.
As an example let us consider the scalar fields described by the action
\[ I = \frac{1}{2} \int d^4x [g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^b \gamma_{ab} + V(\varphi)], \]
(146)
If the metric \( \gamma_{ab} \) in the space of the fields \( \varphi^a \) is indefinite with signature \((r, s)\), then the latter action is a generalization of the action (3) that comes from the higher derivative action (1). Let the potential be of the form
\[ V(\varphi) = m^2 \varphi^a \varphi^b \gamma_{ab} + \varphi^a \varphi^b \varphi^c \varphi^d \lambda_{abcd}, \]
(147)
or similar.

After performing the standard quantization procedure, the system is described by a state vector \( |\Psi\rangle \) expanded in terms of the Fock space basis vectors, \( |P\rangle \equiv |p_1p_2...p_n\rangle \) that are eigenvectors of the free field Hamiltonian \( H_0 \) (without the interaction quartic term \( \varphi^a \varphi^b \varphi^c \varphi^d \lambda_{abcd} \)):
\[ |\Psi\rangle = \sum |P\rangle \langle P|\Psi\rangle, \]
(148)
The system evolves according to the Schrödinger equation
\[ i\frac{\partial|\Psi\rangle}{\partial t} = H|\Psi\rangle, \]
(149)
with the formal solution
\[ |\Psi(t)\rangle = e^{-iH(t-t_0)}|\Psi(t_0)\rangle. \]
(150)
Because the signature of the field space metric \( \gamma_{ab} \) is \((r, s)\), a Fock space state vector contains particles with positive and negative energies. A consequence is that if the initial state is the vacuum \( |\psi(t_0)\rangle = |0\rangle \), it can evolve into a superposition of many particle states, because the transitions
\[ \langle P|e^{-iH(t-t_0)}|0\rangle = \langle P|\Psi(t)\rangle \]
(151)
can satisfy the energy and momentum conservation.

The vacuum thus decays into a superposition of many particle states:
\[ |\Psi(t)\rangle = \sum_{n=0}^\infty |p_1p_2...p_n\rangle \langle p_1p_2...p_n|\Psi(t)\rangle, \]
(152)
where \( \langle p_1p_2...p_n|\Psi(t)\rangle \) is the probability amplitude of observing at time \( t \) the many particle state \( |p_1p_2...p_n\rangle \). The probabilities that the vacuum decays into any of the states \( |p_1\rangle, |p_1p_2\rangle, |p_1p_2...p_n\rangle, ... \), are not very different from each other, and, after a proper normalization, they sum to \( 1 - |\langle 0|\Psi\rangle|^2 \). We thus have
\[ |\langle 0|\Psi\rangle|^2 + \sum_{p_1} |\langle p_1|\Psi\rangle|^2 + \sum_{p_1,p_2} |\langle p_1,p_2|\Psi\rangle|^2 + \sum_{p_1,p_2,...,p_n} |\langle p_1,p_2,...,p_n|\Psi\rangle|^2 + ... = 1. \]
(153)
This is so, because at any time $t$ the system must be in one of the states $|0\rangle$, $|p_1\rangle$, $|p_1, p_2\rangle$, ..., $|p_1, p_2, ..., p_n\rangle$. Therefore the total probability of finding the system in any of those states is 1. However, the probability that vacuum decays into 2, 4, 6, 8, or any finite number of particles is infinitely small in comparison with the probability that it decays into infinite numbers of particles, because such configurations occupy the vast majority of the phase space. Therefore, for an outside observer, who does not measure momenta of the particles, the vacuum $|0\rangle$ instantly decays into infinitely many particles. The usual reasoning in the literature then goes along the lines that because of such vacuum instability the theories involving ultrahyperbolic space, and higher derivative theories in particular, are not physically viable.

However, every reasoning is based on certain assumptions, often implicit or tacit. The reasoning against higher derivative field theories based on the instantaneous vacuum decay due to the infinite phase space tacitly assumes the existence of an observer, outside the system described by the higher derivative field theory, who does not measure the number of particles and their momenta. But in reality, such an outside observer cannot exist. Within our universe there can be no observer to whom the higher derivative theory does not apply, if such a theory is the theory describing our universe. Every conceivable observer is thus a part of our universe. If so, he is then coupled to the particles in the universe, and thus he at least implicitly measures their number and momenta. For such an observer, there is no instantaneous vacuum decay for the reason discussed in Ref. [26], quoted below:

Let us consider a generalization of the field action (146). We can rewrite it in a more compact notation:

$$I = \frac{1}{2} \partial_\mu \phi^a(x) \partial_\nu \phi^b(x') \gamma_{\mu\nu}^{ab}(x) - U[\phi]$$

(154)

Here $\phi^a(x) \equiv \phi^a(x)$, where $(x)$ is the continuous index, denoting components of an infinite dimensional vector. In addition, for every $(x)$, the components are also denoted by a discrete index $a$. Altogether, vector components are denoted by the double index $a(x)$. Alternative notation, often used in the literature, is $\phi^a(x) \equiv \varphi^{ax}$ or $\phi^a(x) \equiv \varphi^{(ax)}$.

The action (154) may be obtained from a higher dimensional action

$$I_\phi = \frac{1}{2} \partial_\mu \phi^A(x) \partial_\nu \phi^B(x') G_{AB}^{\mu\nu}(x(x'))$$

(155)

where $A = (a, \bar{A})$, and $\phi^A(x) = (\phi^a(x), \phi^{\bar{A}}(x))$. The higher dimensional metric is a functional of $\phi^A(x)$. Performing the Kaluza-Klein split,

$$G_{\mu\nu}^{AB} = \begin{pmatrix}
\gamma_{\mu\nu}^{ab} + A_a^A A_b^B \bar{G}_{AB}^{\mu\nu}, & A_a^B \bar{G}_{AB}^{\mu\nu} \\
A_b^B \bar{G}_{AB}^{\mu\nu}, & \bar{G}_{AB}^{\mu\nu}
\end{pmatrix},$$

(156)
where for simplicity we have omitted the index \((x)\), we find,

\[
I_\phi = \frac{1}{2} \partial_\mu \phi^a(x) \partial_\nu \phi^b(x') \gamma^{\mu\nu} a(x)b(x') + \frac{1}{2} \partial_\mu \phi_{\bar{A}}(x) \partial_\nu \phi_B(x) \bar{G}^{\mu\nu \bar{A}}(x)B(x). \tag{157}
\]

Identifying \(\phi^a(x) \equiv \varphi^a(x)\), and denoting \(\frac{1}{2} \partial_\mu \phi_{\bar{A}}(x) \partial_\nu \phi_B(x) \bar{G}^{\mu\nu \bar{A}}(x)B(x) = -\mathcal{U}[\varphi]\), we obtain the action (154).

Thus, the field action (146) is embedded in a higher dimensional action with a metric \(G^{\mu\nu \bar{A}}(x)B(x)\) in field space. A question arises as to which field space metric to choose. The lesson from general relativity tells us that the metric itself is dynamical. Let us therefore assume that this is so in the case of field theory \([36]\) as well. Then (155) must be completed by a kinetic term, \(I_G\), for the field space metric \([36]\). The total action is then

\[
I[\phi, G] = I_\phi + I_G. \tag{158}
\]

According to such dynamical principle, not only the field \(\phi\), but also the metric \(G^{\mu\nu \bar{A}}(x)B(x)\) changes with the evolution of the system. This implies that also the potential \(\mathcal{U}[\varphi]\) of eq. (154) changes with evolution, and so does the potential \(V(\varphi)\), occurring in eq. (146).

Let us assume that the action (158) describes the whole universe.\footnote{Of course, such a model universe is not realistic, because our universe contains fermions and accompanying gauge fields as well.} Then \(|\psi(t)\rangle\) of eq. (152) contains everything in such universe, including observers. There is no external observer, \(O_{\text{ext}}\), according to whom the coefficients \(\langle p_1...p_n|\psi(t)\rangle\) in eq. (152) could be related to the probability densities \(|\langle p_1...p_n|\psi(t)\rangle|^2\) of finding the system in an \(n\)-particle states with momenta \(p_1, ..., p_n\), \(n = 0, 1, 2, 3, ..., \infty\). There are only inside observers, \(O\), incorporated within appropriate multiparticle states \(|p_1...p_n\rangle\), \(n = 0, 1, 2, 3, ..., \infty\), of the “universal” state \(|\psi(t)\rangle\). According to the Everett interpretation of quantum mechanics, all states, \(|0\rangle, |p_1p_2\rangle, ..., |p_1...p_n\rangle\), ..., \(n = 0, 1, 2, ..., \infty\), in the superposition (152) actually occur, each in a different world. The Everett interpretation is now getting increasing support among cosmologists (see, e.g., Ref. \([37, 38]\)). For such an inside observer, \(O\), there is no instantaneous vacuum decay into infinitely many particles. For \(O\), at a given time \(t\), there exists a configuration \(|P\rangle\) of \(n\)-particles (that includes \(O\) himself), and a certain field potential \(V(\varphi)\) (coming from \(G^{\mu\nu \bar{A}}(x)B(x)\)). At some later time, \(t + \Delta t\), there exists a slightly different configuration \(|P'\rangle\) and potential \(V'(\varphi)\), etc. Because \(O\) nearly continuously measures the state \(|\psi(t)\rangle\) of his universe, the evolution of the system is being “altered”, due to the notorious “watchdog effect”
or “Zeno effect”\cite{39} of quantum mechanics (besides being altered by the evolution of the potential $V(\varphi)$). The peculiar behavior of a quantum system between two measurements has also been investigated in Refs.\cite{40,41}.

In Discussion of the same reference \cite{26} it is then written:

After having investigated how the theory works on the examples of the classical and quantum pseudo Euclidean oscillator, we considered quantum field theories. If the metric of field space is neutral, then there occur positive and negative energy states. An interaction causes transitions between those states. A vacuum therefore decays into a superposition of states with positive and negative energies. Because of the vast phase space of infinitely many particles, such a vacuum decay is instantaneous—for an external observer. But we, as a part of our universe, are not external observers. We are internal observers entangled with the “wave function” of the rest of our universe. According to the Everett interpretation of quantum mechanics, we find ourselves in one of the branches of the universal wave function. In the scenario with decaying vacuum, our branch can consist of a finite number of particles. Once being in such a branch, it is improbable that at the next moment we will find ourselves in a branch with infinitely many particles. For us, because of the “watchdog effect” of quantum mechanics, the evolution of the universal wave function relative to us is frozen to the extent that instantaneous vacuum decay is not possible.

Another important point is that the field potential in the action \cite{146} need not be fixed during the evolution of the universe, but can change. Also, a realistic potential is not unbounded, it should be bounded from below and from above, which then presumably gives stability as we have shown that it is the case in the classical theory (see Sec. 4.2). Moreover, as pointed by Carroll et al.\cite{42}, decay rates of the processes with negative energy particles (“phantoms” in their terminology) would not diverge if there were a cutoff to the phase space integrals. Maximum momenta imply minimum length, and in Ref.\cite{26} it was pointed out that such a finite system (though with many degrees of freedom) oscillates between the “vacuum” state and the very many (but not infinitely many) particle state.

A realistic description of our universe must include fermions as well. A well known fact is that fermions are just particular Clifford numbers (algebraic spinors or Clifford aggregates)\cite{13,56}. An element of the Clifford algebra generated by the basis vectors of 4-dimensional spacetime can be represented by a 4 matrix, and spinors are just elements of one column (an “ideal” of a Clifford algebra). Because
there are four columns, there are four kinds of spinors. In Ref. [26] we find the following discussion (after a lengthy formalism supporting it):

Description of our universe requires fermions and accompanying gauge fields, including gravitation. According to the Clifford algebra generalized Dirac equation—Dirac-Kähler equation [57]–[59]—there are four sorts of the 4-component spinors, with energy signs a shown in eq. (111) [of Ref. [26]]. The vacuum of such field has vanishing energy and evolves into a superposition of positive and negative energy fermions, so that the total energy is conserved. A possible scenario is that the branch of the superposition in which we find ourselves, has the sea of negative energy states of the first and the second, and the sea positive energy states of the third and forth minimal left ideal of Cl(1,3). According to ref. [?, 44], the former states are associated with the familiar, weakly interacting particles, whereas the latter states are associated with mirror particles, coupled to mirror gauge fields, and thus invisible to us. According to the field theory based on the Dirac-Kähler equation, the unstable vacuum could be an explanation for Big Bang.

Although the concept of the Dirac vacuum as a sea of negative energy particles is nowadays considered as obsolete, it finds its natural place within the Clifford algebra description of spinors and spinor fields [43]–[46]. If a higher derivative field theory describes our universe, then as a possible scenario it predicts that an initial “bare” vacuum state evolved into the present state, which involves the sea of negative energy fermions of the 1st and 2nd ideal (column) of Cl(1,3), and the sea of positive energy fermions of the 3rd and 4th ideal, so that their total energies sum to zero. The initially “bare” vacuum thus decayed into infinitely many particles not completely, but only partially, because half of the available fermion states remained finite, namely the positive energy states of the 1st and 2nd (that form our visible universe, including us), and the negative energy states of the 3rd and 4th ideal (that according to Ref. [43, 44, 46]) form mirror particles. Our sector of particles has remained finite, because we live in it and we observe (or measure) it. In other words, we are composed of those particles and we observe the same kind of particles in our environment and measure (at least implicitly) their momenta. But in this scenario we have not measured the momenta of those other kinds of fermions which then formed the sea. So people when talking about an (instantaneous) vacuum decay were only “half right”. Vacuum indeed (presumably instantaneously) decayed into infinitely many particles, but only into half of the available four kinds of (fermionic) particles.
9 Conclusion

We have reviewed various approaches in the literature to the problem of negative energies that occur in higher derivative theories, a toy model for them being the Pais-Uhlenbeck oscillator. We have pointed out to the well-known fact that negative energies themselves are not problematic, if there are no interactions between positive and negative energy degrees of freedom. The problems are expected to arise in the presence of interactions. Therefore, the approaches in which the authors show how the description of the PU oscillator in terms of an indefinite Hamiltonian can be replaced with a positive definite Hamiltonian, do not solve the problem. Namely, such a reformulation of a free PU oscillator does not work for an interacting PU oscillator. For instance, if a self-interacting PU oscillator is attempted to be reformulated in terms of a positive definite Hamiltonian, the Lagrangian acquires additional non-linear terms that are not present in the original Lagrangian. Therefore, an interacting PU oscillator as a higher derivative system has to be described in terms of the Ostrogradski or an equivalent formalism that contains negative energies. It was shown that if the potential is unbounded either from below or from above, then such systems are not necessarily unstable; they can be either stable or unstable, depending on the values of the coupling constant and the initial velocity. However, when quantized, the system becomes unstable because of the tunneling effect, even if it was stable in the classical case.

But unbounded potentials are not physically realistic. For instance, a harmonic oscillator is an idealization, not realized in nature. An actual oscillator always has a potential that is bounded. In the case of systems whose energies are positive, the potential is required to be bounded from below, but usually it is taken for granted that it need not be bounded from above, despite that a physically realistic system is bounded form above as well. When considering higher derivative theories that contain negative energies, if we assume the presence of an unbounded potential, then the systems described by such theories are not stable. As we mentioned, they can be stable within certain ranges of initial conditions and coupling constant, but they cannot be absolutely stable, i.e., stable for all values of those parameters. In this review we pointed out that a physically realistic potential has to be bounded not only from bellow, but also from above. Then even the systems whose energies can be positive or negative, are absolutely stable. When such systems are quantized, they remain stable. An alternative approach to the interacting higher derivative systems, including the PU oscillator has, been considered in Refs. [60]–[63].

The issue regarding the infinite decay rate of vacuum due to the infinite phase space in higher derivative fields theories has also been clarified. The integration over phase space has to be performed when momenta of decaying products are not measured. If they are measured, then the integration does not run over the entire
infinite phase space, but only over a small finite part of it, and therefore the decay
rates are not infinite. Assume that our universe is governed by a higher derivative field
theory. Then we observers living in such a universe are part of one of the infinitely
many components of such a decayed vacuum, and we in fact implicitly measured
momenta of the particles in our universe, therefore for us there was no instantaneous
vacuum decay. Moreover, if there is a momentum cutoff due to the minimal length,
the phase space is not infinite anyway. We conclude that higher derivative theories are
physically viable with many important physical implications for further development
of quantum gravity and its unification with the rest of physics.

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