Generalized ILW hierarchy: Solutions and limit to extended lattice GD hierarchy

Kanehisa Takasaki
Department of Mathematics, Kindai University
3-4-1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan

Abstract

The intermediate long wave (ILW) hierarchy and its generalization, labelled by a positive integer $N$, can be formulated as reductions of the lattice KP hierarchy. The integrability of the lattice KP hierarchy is inherited by these reduced systems. In particular, all solutions can be captured by a factorization problem of difference operators. A special solution among them is obtained from Okounkov and Pandharipande’s dressing operators for the equivariant Gromov-Witten theory of $\mathbb{CP}^1$. This indicates a hidden link with the equivariant Toda hierarchy. The generalized ILW hierarchy is also related to the lattice Gelfand-Dickey (GD) hierarchy and its extension by logarithmic flows. The logarithmic flows can be derived from the generalized ILW hierarchy by a scaling limit as a parameter of the system tends to 0. This explains an origin of the logarithmic flows. A similar scaling limit of the equivariant Toda hierarchy yields the extended 1D/bigraded Toda hierarchy.

2010 Mathematics Subject Classification: 14N35, 37K10
Key words: ILW hierarchy, lattice KP hierarchy, lattice GD hierarchy, equivariant Toda hierarchy, logarithmic flow, Gromov-Witten theory

*E-mail: takasaki@math.kindai.ac.jp
1 Introduction

The intermediate long wave (ILW) equation is an integro-differential equation that describes internal long waves in a stratified fluid of finite depth. The KdV and Benjamin-Ono equations may be thought of as its limit as the depth tends to 0 and $\infty$ respectively. Soon after the equation was proposed \[1, 2\], many features of integrability, such as soliton solutions, Bäcklund transformations, inverse scattering transformations, conservation laws and Hamiltonian structures, were discovered \[3, 4, 5, 6\]. Since then, further new aspects have been reported on this equation, its hierarchy of higher flows (the ILW hierarchy) and related integrable systems \[7, 8, 9, 10\].

Recently, Buryak and Rossi presented a new Lax representation of the ILW hierarchy \[11\]. Their Lax representation is formulated in terms of difference operators just like integrable hierarchies of the Toda type \[12\]. They compared it to the equivariant bigraded Toda hierarchy \[13\] and argued that the ILW hierarchy is a degenerate case thereof. The bigraded equivariant Toda hierarchy, as well as the ordinary equivariant 1D Toda hierarchy \[14\], is a reduction of the 2D Toda hierarchy. An approach to the ILW hierarchy from the 2D Toda hierarchy was indeed carried out by Liu et al. \[15\]. The original form of Buryak and Rossi’s Lax representation, however, rather resembles the lattice KP hierarchy (aka the discrete KP hierarchy, the modified KP hierarchy, etc. \[16\]).

We consider the ILW hierarchy and its generalization, labelled by a positive integer $N$, as reductions of the lattice KP hierarchy. These reduced systems have a parameter $\nu$. If $\nu = 0$, the same reduction procedure yields the lattice Gelfand-Dickey (GD) hierarchy. In the previous work \[17\], we constructed an extension of the lattice GD hierarchy. A main subject of this paper is to elucidate the relationship among these systems.

The extended lattice GD hierarchy is obtained by adding, by hand, a set of logarithmic flows. We are interested in the origin of these flows. A similar extension by logarithmic flows is known for the 1D Toda hierarchy \[18, 19\] and its bigraded generalization \[20, 21\]. We shall show that the $N$-th generalized ILW hierarchy turns into the $N$-th extended lattice GD hierarchy in a limit as the parameter $\nu$ tends to 0. This is achieved by a kind of scaling limit of the time variables of flows. In this limit, part of the flows of the generalized ILW hierarchy are transmuted into the logarithmic flows. Actually, we can show that the extended 1D/bigraded Toda hierarchy, too, can be derived from the equivariant 1D/bigraded Toda hierarchy by a similar scaling limit. The parameter $\nu$ therein plays the role of equivariant parameter. Thus the extended Toda hierarchy turns out to be a non-equivariant limit of the equivariant Toda hierarchy. To the best of our knowledge, this fact has not
been explained in such a direct way.

Another subject of this paper is the description of solutions of the generalized ILW hierarchy. We shall consider two types of solutions. The first one are soliton solutions. Soliton solutions of the lattice KP hierarchy are well known. We show a condition under which they become solutions of the generalized ILW hierarchy. This condition is described by a set of equations for the parameters of the soliton solutions. We can use these equations to illustrate the aforementioned scaling limit to the extended lattice GD hierarchy. Solutions of the second type are characterized by a factorization problem of difference operators. The factorization problem itself can generate all solutions of the generalized ILW hierarchy. We find a special solution of this type in the context of our previous work [22] on the equivariant Toda hierarchy. We constructed therein a pair of difference operators that play the role of Okounkov and Pandharipande’s “dressing operators” [23, 24] in their fermionic description of the equivariant Gromov-Witten theory of \( \mathbb{C}P^1 \).

One of the two operators can be used for a special setup of the factorization problem, which generates a solution of the generalized ILW hierarchy. We show a generalization of this special solution as well.

This paper is organized as follows. Section 2 is a review of the lattice KP and generalized ILW hierarchies. Section 3 introduces the factorization problem in a general form. In Section 4, special solutions of the two types are presented. In Section 5, the scaling limit to the extended lattice GD hierarchy is formulated and illustrated for soliton solutions. An appendix is added for a supplementary explanation on the bigraded equivariant Toda hierarchy and its scaling limit to the extended bigraded Toda hierarchy.

2 Lattice KP and generalized ILW hierarchies

The 2D Toda hierarchy has two Lax operators \( L, \bar{L} \) and two sets of time variables \( \mathbf{t} = (t_k)_{k=-1}^\infty, \bar{\mathbf{t}} = (\bar{t}_k)_{k=-1}^\infty \) (see Appendix). The lattice KP hierarchy can be obtained from the 2D Toda hierarchy by discarding the second Lax operator \( \bar{L} \) and the second set of time variables \( \bar{\mathbf{t}} \). We here consider an \( \hbar \)-dependent formulation of these integrable hierarchies [25]. Let \( s \) be the spatial coordinate therein and \( \Lambda \) denote the shift operator

\[
\Lambda = e^{\hbar \partial_s}, \quad \partial_s = \partial / \partial s,
\]

that acts on a function \( f(s) \) of \( s \) as \( \Lambda^n f(s) = f(s + n\hbar) \).
2.1 Lattice KP hierarchy

The \( \hbar \)-dependent lattice KP hierarchy consists of the Lax equations

\[
\hbar \frac{\partial L}{\partial t_k} = [B_k, L], \quad k = 1, 2, \ldots,
\]

(1)

for the difference Lax operator

\[
L = \Lambda + \sum_{n=1}^{\infty} u_n \Lambda^{1-n}
\]

(which is an analogue of the pseudo-differential Lax operator of the KP hierarchy). The coefficients \( u_n = u_n(\hbar, s, t) \) are dynamical variables that depend on the space-time coordinates \( s \) and \( t = (t_k)_{k=1}^{\infty} \) and the parameter \( \hbar \). \( B_k \)'s are defined by the Lax operator as

\[
B_k = (L^k)_{\geq 0},
\]

where \( (\, )_{\geq 0} \) is the projection

\[
\left( \sum_{n=-\infty}^{\infty} a_n \Lambda^n \right)_{\geq 0} = \sum_{n \geq 0} a_n \Lambda^n
\]

to the linear combination of non-negative powers of \( \Lambda \). Let \( (\, )_{< 0} \) denote the complementary projection

\[
\left( \sum_{n=-\infty}^{\infty} a_n \Lambda^n \right)_{< 0} = \sum_{n < 0} a_n \Lambda^n.
\]

The wave function \( \Psi \) of the auxiliary linear equations

\[
\hbar \frac{\partial \Psi}{\partial t_k} = B_k \Psi, \quad L \Psi = z \Psi
\]

(2)

is related to the tau function \( \tau = \tau(\hbar, s, t) \) as

\[
\Psi = \frac{\tau(\hbar, s, t - \hbar[z^{-1}])}{\tau(\hbar, s, t)} z^{s/h} e^{\xi(t, z)/\hbar},
\]

(3)

where

\[
[z] = (z^k / k)_{k=1}^{\infty}, \quad \xi(t, z) = \sum_{k=1}^{\infty} t_k z^k.
\]
The amplitude part of $\Psi$ can be expanded into negative powers of $z$ as

$$\frac{\tau(h, s, t - \frac{h}{z^2})}{\tau(h, s, t)} = 1 + \sum_{n=1}^{\infty} w_n z^{-n}$$

and used to construct the dressing operator

$$W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}.$$  

The wave function, in turn, can be written as

$$\Psi = W \frac{z^s}{\hbar} e^{\xi(t, z)/\hbar}.$$  

$L$ and $B_k$’s are expressed in the dressed form as

$$L = W \Lambda W^{-1}, \quad B_k = (W \Lambda^k W^{-1})_{\geq 0}.$$  

The Lax equations are thus converted to the Sato equations

$$\hbar \frac{\partial W}{\partial t_k} = (W \Lambda^k W^{-1})_{\geq 0} W - W \Lambda^k = -(W \Lambda^k W^{-1})_{< 0} W$$

for the dressing operator $W$.

### 2.2 Generalized ILW hierarchy

Let $N$ be a positive integer and allow the dynamical variables to depend on the new parameter $\nu$ as $u_n = u_n(h, \nu, s, t)$, etc. Reduction to the $N$-th generalized ILW hierarchy with parameter $\nu$ is achieved by the reduction condition

$$(L^N - \nu \log L)_{< 0} = 0.$$  

If $\nu = 0$, this condition becomes the $N$-reduction condition $(L^N)_{< 0} = 0$ to the lattice GD hierarchy. The notation used here in the $\nu \neq 0$ case will need explanation [11, 17]. First, the logarithm $\log L$ of $L$ is defined as

$$\log L = W \log AW^{-1} = Wh\partial_s W^{-1}.$$  

Second, the projections $(\quad)_{\geq 0}$ and $(\quad)_{< 0}$ of difference operators are extended to operators of the form $A\partial_s + B$, $A$ and $B$ being genuine difference operators, as

$$(A\partial_s + B)_{\geq 0} = (A)_{\geq 0}\partial_s + (B)_{\geq 0},$$

$$(A\partial_s + B)_{< 0} = (A)_{< 0}\partial_s + (B)_{< 0}.$$
This extended projection should be used carefully. Namely, the shift operators $\Lambda^n = e^{n\hbar \partial_s}$ should not be expanded into powers of $\partial_s$; such careless expansion leads to ambiguity in the definition of the projection.

Under the reduction condition (5), the lattice KP hierarchy reduces to a system of evolution equations for a finite number of dynamical variables. Note that $L^N - \nu \log L$ can be expressed as

$$L^N - \nu \log L = B_N - \nu \log \Lambda + \text{negative powers of } \Lambda.$$  

The reduction condition implies that the “negative powers of $\Lambda$” on the right hand side disappear. We thus obtain the reduced Lax operator

$$\mathcal{L} = L^N - \nu \log L = B_N - \nu \log \Lambda$$  

that satisfies the reduced Lax equations

$$\hbar \frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}], \quad k = 1, 2, \ldots.$$  

(7)

$B_N$ is a difference operator of the form

$$Q = \Lambda^N + b_1\Lambda^{N-1} + \cdots + b_N,$$

and $B_k$’s can be thereby redefined as

$$B_k = (Q^{k/N})_{\geq 0}.$$  

Fractional powers and logarithm of $Q$ [18, 20, 15] can be used to reconstruct $L$ and $\log L$ as well. Thus the Lax equations (7) become a system of evolution equations for $b_n$’s.

The reduction condition (5) can be translated to the language of the dressing operator and the tau function as follows.

**Proposition 1.** The reduction condition (5) is equivalent to the equation

$$\frac{\partial W}{\partial t_N} - \nu \frac{\partial W}{\partial s} = 0$$  

(8)

for the dressing operator.

**Proof.** The condition (5) implies that

$$W(\Lambda^N - \nu \log \Lambda)W^{-1} = B_N - \nu \log \Lambda,$$  

(9)

hence

$$W(\Lambda^N - \nu \log \Lambda) = (B_N - \nu \log \Lambda)W.$$  

Since

$$B_NW = \hbar \frac{\partial W}{\partial t_N} + W\Lambda^N, \quad \log \Lambda W = \hbar \frac{\partial W}{\partial s} + W \log \Lambda,$$

the last relation boils down to (8). 

☐
Corollary 1. If the tau function of the lattice KP hierarchy satisfies the equation
\[ \frac{\partial \tau}{\partial t_N} - \nu \frac{\partial \tau}{\partial s} = 0, \quad (10) \]
the associated Lax and dressing operators satisfy the reduction conditions (5) and (8) to the generalized ILW hierarchy.

The reduction conditions (8) and (10) mean that the tau function \( \tau = \tau(\hbar, \nu, s, t) \) and all dynamical variables of the lattice KP hierarchy depend on \( t_N \) and \( s \) through the linear combination \( t_N + s/\nu \). In particular, the tau function can be expressed as
\[ \tau = f(\hbar, \nu, t_1, \ldots, t_{N-1}, t_N + s/\nu, t_{N+1}, \ldots) \quad (11) \]
with some function \( f(\hbar, \nu, t) \) of \( \hbar, \nu \) and \( t \).

Remark 1. The Lax equations (7) define isospectral deformations of the spectral problem
\[ \mathfrak{L} \Psi = \lambda \Psi. \quad (12) \]
The spectral variable \( \lambda \) is related to the spectral parameter \( z \) as
\[ \lambda = z^N - \nu \log z. \quad (13) \]
We shall encounter avatars of this relation in the construction of soliton solutions.

2.3 Relation to ILW equation

The ILW equation [1, 2] is related to the \( N = 1 \) case. The reduced Lax operator (6) in this case takes the form
\[ \mathfrak{L} = \Lambda + u - \nu \log \Lambda, \quad u = u_1, \]
proposed by Buryak and Rossi [11]. The Lax equations (7) for \( k = 1, 2 \) yield the equations
\[ \hbar \frac{\partial u}{\partial t_1} = \nu \hbar \frac{\partial u}{\partial s}, \quad (14) \]
\[ \hbar \frac{\partial u}{\partial t_2} = \nu \hbar \frac{\partial b_{22}}{\partial s} \quad (15) \]
for \( u \), where \( b_{22} \) denotes the last term of
\[ B_2 = \Lambda^2 + b_{12} \Lambda + b_{22}. \]
Actually, \( b_{22} \) can be expressed in terms of \( u \) as follows.
Proposition 2.

\[ b_{22} = u^2 + \nu(1 + \Lambda)(1 - \Lambda)^{-1}h \frac{\partial u}{\partial s}. \]  

(16)

Proof. Since \( L^2 \) can be expanded into powers of \( \Lambda \) as

\[ L^2 = \Lambda^2 + (u_1(s) + u_1(s + h))\Lambda + (u_1(s)^2 + u_2(s) + u_2(s + h)) + \cdots, \]

\( b_{22} \) can be written as

\[ b_{22} = u^2 + (1 + \Lambda)u_2. \]

To find an expression of \( u_2 \), let us recall the relation (6) between \( L \) and \( \mathcal{L} \):

\[ L - \nu \log L = \mathcal{L} = \Lambda + u - \nu \log \Lambda. \]

Since

\[ \log L = W \Lambda W^{-1} = \log \Lambda - \hbar \frac{\partial W}{\partial s} W^{-1} = \log \Lambda - \hbar \frac{\partial w_1}{\partial s} \Lambda^{-1} + \cdots, \]

the left hand side of the last relation is an operator of the form

\[ L - \nu \log L = -\nu \log \Lambda + \Lambda + u_1 + \left( w_2 - \nu \hbar \frac{\partial w_1}{\partial s} \right) \Lambda^{-1} + \cdots. \]

There is, however, no \( \Lambda^{-1} \)-term in \( \mathcal{L} = \Lambda + u - \nu \log \Lambda \). Consequently, we find that

\[ u_2 = \nu \hbar \frac{\partial w_1}{\partial s}, \]

so that

\[ b_{22} = u^2 + \nu(1 + \Lambda)\hbar \frac{\partial w_1}{\partial s}. \]

To find an expression of \( w_1 \), let us notice that the \( \Lambda^0 \)-part of the intertwining relation \( W \Lambda = LW \) reads

\[ w_1(s) = u_1(s) + w_1(s + h). \]

This implies that

\[ (1 - \Lambda)w_1 = u_1. \]

Solving this relation for \( w_1 \) as

\[ w_1 = (1 - \Lambda)^{-1}u \]

and plugging it into the foregoing expression of \( b_{22} \), we obtain (16). \( \square \)
By (16), the equation (15) of the $t_2$-flow becomes a closed equation for $u$:

$$
\hbar \frac{\partial u}{\partial t_2} = \nu \hbar \frac{\partial}{\partial s} \left( u^2 + \nu(1 + \Lambda)(1 - \Lambda)^{-1}\hbar \frac{\partial u}{\partial s} \right).
$$

(17)

If $\hbar$ takes an imaginary value $\hbar = 2i\delta$, $\delta > 0$, the non-local operator $(1 + \Lambda)(1 - \Lambda)^{-1}$ can be identified with $(i$ times) the integral operator [5, 6]

$$
T[u](x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{2\delta} \coth \left( \frac{y - x}{2\delta} \right) u(y)dy
$$

in the usual formulation

$$
\frac{\partial u}{\partial t} + \frac{1}{\delta} \frac{\partial u}{\partial x} + 2u \frac{\partial u}{\partial x} + T \left[ \frac{\partial^2 u}{\partial x^2} \right] = 0
$$

of the ILW equation. Taking a linear combination with the equation (14) of the $t_1$-flow amounts to adding the advection term $\partial u/\partial x$. Thus the $N = 1$ case of the generalized ILW hierarchy turns out to contain the ILW equation in the lowest part.

## 3 Factorization problem

### 3.1 Factorization problem for lattice KP hierarchy

We now turn to the issue of solving the generalized ILW hierarchy by a factorization problem of difference operators.

To this end, let us start from the factorization problem

$$
U = W^{-1}\bar{W}.
$$

(18)

for the lattice KP hierarchy. The first factor on the right hand side is the inverse of a difference operator $W$ of the type considered in the previous section. The second factor $\bar{W}$ is a difference operator

$$
\bar{W} = \sum_{n=0}^{\infty} \bar{w}_n \Lambda^n, \quad \bar{w}_0 \neq 0,
$$

of the opposite type. The solution-generating operator $U$ is assumed to satisfy the simple evolution equations

$$
\hbar \frac{\partial U}{\partial t_k} = \Lambda^k U, \quad k = 1, 2, \ldots
$$

(19)
and the problem is to factorize it as shown in (18).

One can confirm, as illustrated in our previous work [17], that the first factor $W$ of the solution $(W, \bar{W})$ of this factorization problem does satisfy the Sato equations (4). Moreover, as explained in the remarks below, all solutions of the lattice KP hierarchy can be thus captured.

To fulfill the reduction condition (5) to obtain a solution of the generalized ILW hierarchy, one needs an extra condition on $\bar{U}$. We shall present a necessary and sufficient condition in the next subsection.

**Remark 2.** The second factor $\bar{W}$, which is absent in the formulation of the lattice KP hierarchy, emerges from the Sato equations (4) directly as follows. One can rewrite these equations as

$$h \frac{\partial}{\partial t_k} (We^{\xi(t, \Lambda)}/\hbar) = B_k We^{\xi(t, \Lambda)}/\hbar.$$

Let $\bar{W}$ be the difference operator

$$\bar{W} = We^{\xi(t, \Lambda)/h}W^{-1}_{\text{in}},$$

where $W_{\text{in}}$ denotes the initial value

$$W_{\text{in}} = W|_{t=0}$$

of $W$ at $t = 0$. $\bar{W}$ satisfies the evolution equations

$$h \frac{\partial \bar{W}}{\partial t_k} = B_k \bar{W}$$

and the initial condition

$$\bar{W}|_{t=0} = 1.$$

This implies that all Taylor coefficients of $\bar{W}$ at $t = 0$ contain no negative powers of $\Lambda$. Thus $\bar{W}$ turns out to be a difference operator of the aforementioned form. One can rewrite (20) as

$$e^{\xi(t, \Lambda)/h}W^{-1}_{\text{in}} = W^{-1}\bar{W}.$$ 

Since the operator on the left hand side satisfies the differential equations (19), this is exactly a factorization relation of the form shown in (18).

**Remark 3.** (22) is a formulation of the factorization problem that solves the Sato equations (4) under the initial condition $W|_{t=0} = W_{\text{in}}$. Moreover, $W$ does not change even if this special choice of $U$ is modified to

$$U = e^{\xi(t, \Lambda)/h}W^{-1}_{\text{in}}C,$$

(23)
where $C$ is an arbitrary difference operator of the form

$$C = \sum_{n=0}^{\infty} c_n \Lambda^n, \quad c_0 \neq 0, \quad \frac{\partial C}{\partial t_k} = 0, \quad k = 1, 2, \ldots.$$ 

In the context of the initial value problem, (23) is the most general choice of $U$.

### 3.2 Reduction condition on $U$-operator

We now allow $U$ to depend on $\nu$ as well, and seek for a condition on $U$ under which the solution of the factorization problem (18) yields a solution of the generalized ILW hierarchy. Such a condition can be formulated as follows.

**Proposition 3.** Let $L$ be the Lax operator $L = W \Lambda W^{-1}$ of the lattice KP hierarchy obtained from the factorization problem (18). $L$ satisfies the reduction condition (5) to the generalized ILW hierarchy if and only if there is a difference operator

$$\varphi = \sum_{n=0}^{\infty} \varphi_n \Lambda^n, \quad \frac{\partial \varphi}{\partial t_k} = 0, \quad k = 1, 2, \ldots, \quad (24)$$

such that $U$ satisfies the intertwining relation

$$(\Lambda^N - \nu \log \Lambda) U = U(\varphi - \nu \log \Lambda). \quad (25)$$

**Proof.** Suppose that $L$ satisfies the reduction condition (5). Let us recall the algebraic relation (9) mentioned in the proof of (8). Substituting $W = \bar{W} U^{-1}$ and $W^{-1} = U \bar{W}^{-1}$ therein yields the equation

$$U^{-1}(\Lambda^N - \nu \log \Lambda) U = \bar{W}^{-1}(B_N - \nu \log \Lambda) \bar{W}.$$ 

Since

$$\bar{W}^{-1} \log \Lambda \bar{W} = \bar{W}^{-1} \hbar \frac{\partial \bar{E}}{\partial s} + \log \Lambda,$$

we can rewrite the right hand side of the last equation as

$$\bar{W}^{-1}(B_N - \nu \log \Lambda) \bar{W} = \bar{W}^{-1} B_N \bar{W} - \bar{W}^{-1} \nu \hbar \frac{\partial \bar{W}}{\partial s} - \nu \log \Lambda.$$ 

Thus (25) holds by letting

$$\varphi = W^{-1} B_N W - W^{-1} \nu \hbar \frac{\partial W}{\partial s}. \quad (26)$$
Obviously, \( \varphi \) is a difference operator without negative powers of \( \Lambda \). Moreover, differentiating both sides of (25) with respect to \( t_k \) and using (19), we obtain the equation

\[
(L^N - \nu \log \Lambda) \Lambda^k U = \Lambda^k U(\varphi - \nu \log \Lambda) + Uh \frac{\partial \varphi}{\partial t_k}.
\]

Since the left hand side and the first term on the right hand side cancel out by (25) itself, we can conclude that \( \partial \varphi / \partial t_k = 0 \).

Conversely, suppose that there is a difference operator \( \varphi \) as shown in (24) such that (25) holds. (25) turns into the equation

\[
W(L^N - \nu \log \Lambda) W^{-1} = \bar{W}(\varphi - \nu \log \Lambda) \bar{W}^{-1}
\]

for \( W \) and \( \bar{W} \). We can further rewrite this equation as

\[
L^N - \nu \log L = \bar{W}\varphi\bar{W}^{-1} + \nu \hbar \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1} - \nu \log \Lambda.
\]

Since the first and second terms on the right hand side contain no negative powers of \( \Lambda \), (5) holds.

Remark 4. If \( \nu = 0 \), (25) reduces to the reduction condition to the lattice GD hierarchy in our previous work [17].

Remark 5. (25) is equivalent to the intertwining relation

\[
(L^N - \nu \log \Lambda) U_{in} = U_{in}(\varphi - \nu \log \Lambda)
\]

for the initial value \( U_{in} = U|_{t=0} \).

Remark 6. By the Sato equations (21) for \( \bar{W} \), one can rewrite (26) as

\[
\varphi = \bar{W}^{-1} \hbar \left( \frac{\partial \bar{W}}{\partial t_N} - \nu \frac{\partial \bar{W}}{\partial s} \right).
\]

Thus the dependence of \( \bar{W} \) on \( t_N \) and \( s \) is not as simple as that of \( W \), cf. (8).

Remark 7. The coefficients \( \varphi_n \) of \( \varphi \) can depend \( s \). If they are constants, (27) becomes the algebraic relation

\[
L^N - \nu \log L = \sum_{n=0}^{\infty} \varphi_n \tilde{L}^n - \nu \log \tilde{L}
\]

between \( L \) and

\[
\tilde{L} = \bar{W} \Lambda \bar{W}^{-1}.
\]
\( \bar{L} \) amounts to the second Lax operator of the 2D Toda hierarchy,

\[
\bar{L}^{-1} = \sum_{n=0}^{\infty} \bar{u}_n \Lambda^{n-1},
\]

though it plays no dynamical role in the present setup where the second time variables \( \bar{t} = (\bar{t}_k)_{k=1}^\infty \) of the 2D Toda hierarchy are turned off. The algebraic relation (29) in the \( \phi = 0 \) (and \( N = 1 \)) case coincides with the reduction condition proposed by Liu et al. [15] to characterize the ILW hierarchy in the 2D Toda hierarchy. Actually, as far as \( \phi_n \)'s are constants, \( \phi \) can be eliminated by a reparametrization transformation

\[
\bar{L} \to c_0 \bar{L} + c_1 \bar{L}^2 + \cdots, \quad c_0 \neq 0,
\]

of \( \bar{L} \) with constant coefficients \( c_n, n = 0, 1, \ldots \). The setup of Liu et al. can be thus restored when \( \phi \) is a difference operator with constant coefficients.

### 4 Special solutions

#### 4.1 Soliton solutions

Soliton solutions provide a readily accessible playground in the theory of integrable hierarchies [26]. It will be instructive to construct soliton solutions for the generalized ILW hierarchy as well. To this end, we start from soliton solutions of the lattice KP hierarchy, and examine how they turn into solutions of the generalized ILW hierarchy.

The soliton solutions of the lattice KP hierarchy can be obtained from those of the 2D Toda hierarchy [27] by letting \( \bar{t} = 0 \). The tau function of the \( M \)-soliton solution thus takes the following form:

\[
\tau = 1 + \sum_{i=1}^{M} a_i e_i + \sum_{1 \leq i < j \leq M} a_i a_j c_{ij} e_i e_j + \sum_{1 \leq i < j < k \leq M} a_i a_j a_k c_{ijk} e_1 e_2 e_3 + \cdots + a_1 \cdots a_M e_1 \cdots e_M. \quad (30)
\]

\( a_i \)'s are amplitude parameters and \( e_i \)'s are the following exponential factors:

\[
e_i = (p_i/q_i)^{s_i/h} \exp \left( \hbar^{-1}(\xi(t, p_i) - \xi(t, q_i)) \right). \]

\( p_i \)'s and \( q_i \)'s are another set of parameters with mutually distinct values, and \( c_{ij} \cdots k \)'s are thereby defined as

\[
c_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}, \quad c_{i_1 \cdots i_m} = \prod_{1 \leq a < b \leq m} c_{i_a i_b}.
\]
It is easy to find a set of conditions on the parameters under which this tau function satisfies the reduction condition (10).

**Proposition 4.** If the parameters \( p_i, q_i \) satisfy the equations
\[
p_i^N - \nu \log p_i = q_i^N - \nu \log q_i, \quad i = 1, \ldots, M,
\]
the tau function (30) satisfies the reduction condition (11) to the generalized ILW hierarchy.

**Proof.** (31) implies that
\[
p_i^N - q_i^N = \nu \log(p_i/q_i),
\]
so that \( e_i \)'s can be reassembled as
\[
e_i = (p_i/q_i)^{s/h} \exp \left( h^{-1}(p_i^N - q_i^N)t_N \right) e_i'
= \exp \left( h^{-1}(p_i^N - q_i^N)(t_N + s/\nu) \right) e_i',
\]
where \( e_i' \) is a factor that does not depend on \( t_N \) and \( s \). Thus the tau function (30) depends on \( t_N \) and \( s \) through the linear combination \( t_N + s/\nu \).

Note that avatars of the right hand side of (13) show up in (31). By letting \( \nu = 0 \), (31) turns into the \( N \)-reduction condition
\[
p_i^N = q_i^N
\]
for soliton solutions of the lattice GD hierarchy.

### 4.2 Dressing operators of Okounkov-Pandharipande

A special solution of quite different nature is hidden in the work of Okounkov and Pandharipande [23] on the equivariant Gromov-Witten theory of \( \mathbb{CP}^1 \). They use what they call “dressing operators” to convert a fermionic formula of the generating function of equivariant Gromov-Witten invariants to the standard fermionic formalism of 2D Toda tau functions [25, 12]. In our previous work [22], those operators (denoted by \( V \) and \( \bar{V} \) therein) are reformulated as difference operators in the spatial variable \( s \), and used to explain how the equivariant Toda hierarchy emerges in the equivariant Gromov-Witten theory of \( \mathbb{CP}^1 \).

\( V \) is a difference operator of the form
\[
V = 1 + \sum_{n=1}^{\infty} v_n A^{-n}
\]
and constructed to satisfy the intertwining relation

\[(\Lambda^N + H - \nu \log \Lambda) V = V (\Lambda^N - \nu \log \Lambda), \tag{33}\]

where \(H\) is the multiplication operator

\[H = s/\hbar + 1/2\]

that corresponds to the energy operator in the fermionic formalism. More precisely, the work of Okounkov and Pandharipande [23] amounts to the case of \(N = 1\); the case of \(N > 1\) is an orbifold generalization [24].

Let us briefly recall the construction of \(V\) [22]. \(V\) is constructed order-by-order in the \(\nu\)-expansion

\[V = \sum_{k=0}^{\infty} \nu^k V_k.\]

By this expansion, the intertwining relation (33) can be decomposed into an infinite number of equations of the following form:

\[(\Lambda^N + H) V_0 = V_0 \Lambda^N \tag{34}\]

\[(\Lambda^N + H) V_k - \log \Lambda V_{k-1} = V_k \Lambda^N - V_{k-1} \log \Lambda, \quad k = 1, 2, \ldots. \tag{35}\]

We can multiply both sides of (35) by \(V_0^{-1}\) and use the first equation (34) to rewrite (35) as

\[[\Lambda^N, V_0^{-1} V_k] = V_0^{-1} [\log \Lambda, V_{k-1}], \tag{36}\]

which is simpler than (35). Starting with (34), as explained below, we can solve these equations step-by-step and find that the \(k\)-th operator \(V_k\) can be expressed as

\[V_k = \sum_{n=kN}^{\infty} v_{kn} \Lambda^{-n}\]

with the coefficients \(v_{kn}\) being polynomials in \(s\). Consequently, \(V\) becomes a difference operator of the form

\[V = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{n/N} v_{kn} \Lambda^{-n},\]

---

\(^1\)The definition of \(H\) is slightly different from our previous work [22] in two aspects. First, \(s\) is rescaled by \(h\) so as to preserve the commutation relation \([\log \Lambda, s] = 1\). Second, the constant term is modified from \(-1/2\) to \(1/2\) to be consistent with the relation between the wave function and the tau function in this paper.
where \([n/N]\) denotes the integral part of \(n/N\). Thus the coefficients of \(V\) itself turn out to be polynomials in \(s\).

The operator equations (34) and (36) consist of an infinite number of difference equations for the coefficients \(v_{kn}\). To find a solution of this system of equations, we have to solve a difference equation of the form

\[ v(s + N\hbar) - v(s) = f(s), \]

(37)

for \(v(s)\) repeatedly, where \(f(s)\) is a given polynomial. As far as the right hand side is a polynomial in \(s\), we can use the difference identities

\[ B_k(x + 1) - B_k(x) = kx^{k-1} \]

of the Bernoulli polynomials \(B_k(x)\), defined by the generating function

\[ \frac{te^t}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}, \]

to find a polynomial solution \(v(s)\) of (37).

The intertwining relation readily implies that the operator

\[ U = e^{\xi(t, \Lambda)/\hbar}V^{-1} \]

(38)

satisfies the reduction condition (25) with \(\varphi = \Lambda^N + H\).

The factorization problem (18) thereby generates a solution of the generalized ILW hierarchy.

**Remark 8.** A more interesting choice of the \(U\)-operator will be

\[ U = e^{\xi(t, \Lambda)/\hbar}V^{-1}e^{\Lambda N/N}, \]

(39)

which is the left half of the \(U\)-operator (76) for the equivariant Gromov-Witten theory of \(\mathbb{C}P^1\). This operator satisfies the reduction condition (25) with \(\varphi = H\).

Actually, (38) and (39) yield the same factor \(W\) in the factorization problem (18). The right action \(U \rightarrow UC\) of the \(U\)-operator by a difference operator of the form

\[ C = \sum_{n=0}^{\infty} c_n \Lambda^n, \quad c_0 \neq 0, \quad \frac{\partial C}{\partial t_k} = 0 \quad , k = 1, 2, \ldots \]

leaves \(W\) invariant and changes \(\bar{W}\) as \(\bar{W} \rightarrow \bar{W}C\).

\(^2\)Our previous work [22] uses a different set of polynomials.

\(^3\)Solving the factorization problem explicitly is a difficult task. The solution is encoded in the fermionic formula of tau functions [25, 12].
4.3 Generalization

The foregoing construction of $V$ (and the special solution of the generalized ILW hierarchy) can be extended to the case where $\varphi$ takes a form more general than $\varphi = \Lambda^N + H$. Let us assume that $\varphi$ is a difference operator of the form

$$\varphi = \Lambda^N + \sum_{n=1}^{N} \varphi_n \Lambda^{N-n}$$  \hspace{1cm} (40)

and that $\varphi_n$’s do not depend on $\nu$. The second assumption can be relaxed, but we impose it to simplify the subsequent computations. The problem is to construct a difference operator

$$V = 1 + \sum_{n=1}^{\infty} v_n \Lambda^{-n}$$

that satisfies the intertwining relation

$$(\varphi - \nu \log \Lambda)V = V(\Lambda^N - \nu \log \Lambda).$$  \hspace{1cm} (41)

Plugging the $\nu$-expansion

$$V = \sum_{k=0}^{\infty} \nu^k V_k,$$

into (41) yields the following generalization of (34) and (35):

$$\varphi V_0 = V_0 \Lambda^N$$

$$\varphi V_k - \log \Lambda V_{k-1} = V_k \Lambda^N - V_{k-1} \log \Lambda, \quad k = 1, 2, \ldots.$$  \hspace{1cm} (42) \hspace{1cm} (43)

The second equations can be converted to the equations

$$[\Lambda^N, V_0^{-1}V_k] = V_0^{-1} [\log \Lambda, V_{k-1}]$$

of the same form as (36). If $\varphi_n$’s are polynomials in $s$, we can solve these equations in much the same way as the foregoing case. Thus we arrive at the following result.

**Proposition 5.** If $\varphi$ takes the form as shown in (40) and $\varphi_n$’s are polynomials in $s$, there is a difference operator $V$ of the form

$$V = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{[n/N]} v_{kn} \Lambda^{-n}$$
such that the intertwining relation (41) holds and $v_{km}$’s are polynomials in $s$. The operator

$$U = e^{\xi(t,\Lambda)/\hbar}V^{-1}$$

(44)

satisfies the reduction condition (25) to the generalized ILW hierarchy.

Remark 9. We can also construct $U_{in} = V^{-1}$ directly by the $\nu$-expansion

$$U_{in} = \sum_{k=0}^{\infty} \nu^k U_{in}^k.$$ 

$U_{in}^k$’s are required to satisfy the equations

$$\Lambda^N U_{in}^0 = U_{in}^0 \varphi,$$

$$\Lambda^N U_{in}^k - \log \Lambda U_{in}^{k-1} = U_{in}^k \varphi - U_{in}^{k-1} \log \Lambda, \quad k = 1, 2, \ldots.$$ 

Solving these equations for $U_{in}^k$’s is mostly parallel to the construction of $V_k$’s.

Remark 10. It is an interesting issue to construct a solution of the bigraded equivariant Toda hierarchy from the foregoing generalization of $V$ and its partner $\bar{V}$. $\bar{V}$ is required to satisfy the intertwining relation

$$(\Lambda^{-\bar{\bar{N}}} - \nu \log \Lambda)\bar{V} = \bar{V}(\bar{\varphi} - \nu \log \Lambda),$$

(45)

where $\bar{\bar{N}}$ is another positive integer, and $\bar{\varphi}$ is a difference operator of the form

$$\bar{\varphi} = \Lambda^{-\bar{\bar{N}}} + \sum_{n=1}^{\bar{\bar{N}}} \bar{\varphi}_n \Lambda^{n-\bar{\bar{N}}}$$

with polynomial coefficients $\bar{\varphi}_n$. If $\varphi$ and $\bar{\varphi}$ are intertwined by another difference operator $S$ as

$$(\varphi - \nu \log \Lambda)S = S(\bar{\varphi} - \nu \log \Lambda),$$

(46)

one can build the $U$-operator

$$U = e^{\xi(t,\Lambda)/\hbar}V^{-1}S\bar{V}^{-1}e^{-\xi(t,\Lambda^{-1})/\hbar}$$

(47)

that satisfies the reduction condition (70) from the 2D Toda hierarchy to the equivariant bigraded Toda hierarchy. The factorization problem for the 2D Toda hierarchy thereby generates a solution of the equivariant bigraded Toda hierarchy of type $(N,\bar{\bar{N}})$. In order to accommodate the equivariant Gromov-Witten theory of $\mathbb{C}P^1$, however, one has to modify the intertwining relation (46) by a constant term [13, 22] (see Appendix, Remark 11).
5 Limit to extended lattice GD hierarchy

Let us turn to the issue of the limit to the lattice GD hierarchy. To avoid a trivial situation, we here consider the case where $N > 1$.

In the naive limit as $\nu \to 0$, the reduction condition (5) becomes the $N$-reduction condition

\[ (L^N)_< = 0 \] (48)


to the lattice GD hierarchy. Since this condition implies that $B_{kN} = L^{kN}$, $k = 1, 2, \ldots$, time evolutions with respect to $t_{kN}$’s are trivialized, namely,

\[ \frac{\partial L}{\partial t_{kN}} = 0, \quad \frac{\partial W}{\partial t_{kN}} = 0. \]

More precisely, the naive limit means that all dynamical variables, i.e., the coefficients $u_n = u_n(\hbar, \nu, s, t)$ and $w_n = w_n(\hbar, \nu, s, t)$ of the Lax and the dressing operators have a smooth limit

\[ u_n = u_n^{(0)} + O(\nu), \quad w_n = w_n^{(0)} + O(\nu) \] (49)

as $\nu \to 0$, where $u_n^{(0)} = u^{(0)}(\hbar, s, t)$ and $w_n^{(0)} = w^{(0)}(\hbar, s, t)$ are functions independent of $\nu$. Although the same notations are used, $L$ and $W$ in the equations shown above denote the operators with $u_n$ and $w_n$ being replaced by $u_n^{(0)}$ and $w_n^{(0)}$.

We argue in the following that a more careful prescription of the $\nu \to 0$ limit leads to emergence of an extended set of flows in place of the trivialized $t_{kN}$-flows. The outcome is the extended lattice GD hierarchy introduced in our previous work [17].

5.1 Extended lattice GD hierarchy

Before formulating the scaling limit, let us recall the construction of the extended lattice GD hierarchy.

Let $\mathcal{L}$ denote the reduced Lax operator

\[ \mathcal{L} = L^N = B_N = \Lambda^N + b_1 \Lambda^{N-1} + \cdots + b_N \]

under the reduction condition (48). $\mathcal{L}$ satisfies the reduced Lax equations

\[ \hbar \frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}]. \]

The lattice KP hierarchy is thus reduced to evolution equations for $b_n$’s.
The extended lattice GD hierarchy is obtained by adding an infinite number of time variables \( x = (x_k)_{k=1}^{\infty} \) and Lax equations

\[
h \frac{\partial \mathcal{L}}{\partial x_k} = [C_k, \mathcal{L}]
\]

in place of the trivialized \( t_k \)-flows. The generators \( C_k \) of these flows are given by

\[
C_k = (L^{kN} \log L)_{\geq 0}.
\]

Since

\[
L^{kN} \log L = W A^{kN} \log A W^{-1} = \mathcal{L}^k \log A - \mathcal{L}^k h \frac{\partial W}{\partial s} W^{-1},
\]

we can rewrite \( C_k \) into the expression

\[
C_k = \mathcal{L}^k \log A - \left( \mathcal{L}^k h \frac{\partial W}{\partial s} W^{-1} \right)_{\geq 0}
\]

presented in our previous work [17]. Because of the presence of the logarithmic terms, these flows are called logarithmic flows. Since \( C_0 = \log A \), the \( x_0 \)-flow can be identified with the translation in the \( s \)-direction.

The Sato equations, too, are extended to the logarithmic flows as

\[
h \frac{\partial W}{\partial x_k} = C_k W - W A^{kN} \log A. \tag{53}
\]

These equations imply that \( \mathcal{L} = W A^N W^{-1} \) satisfies the Lax equations [50].

### 5.2 Scaling limit of Lax and Sato equations

We consider the following scaling transformations of time variables from \( \{t_k\}_{k=1}^{\infty} \) to \( \{T_k\}_{k \neq 0 \mod N} \) and \( \{X_k\}_{k=1}^{\infty} \):

\[
t_k = \begin{cases} T_k & \text{if } k \neq 0 \mod N, \\ X_{k/N}/\nu & \text{if } k \equiv 0 \mod N. \end{cases}
\]

The derivatives are thus transformed as

\[
\frac{\partial}{\partial t_k} = \frac{\partial}{\partial T_k} \quad \text{for } k \neq 0 \mod N, \tag{55}
\]

\[
\frac{\partial}{\partial t_{kN}} = \nu \frac{\partial}{\partial X_k} \quad \text{for } k = 1, 2, \ldots . \tag{56}
\]
We assume that $u_n$ and $w_n$, viewed as functions of the rescaled variables $T_k, X_k$, have a smooth limit
\begin{align}
&u_n = u_n^{(0)}(h, s, T_1, \ldots, T_{N-1}, X_1, T_{N+1}, \ldots, T_{2N-1}, X_2, \ldots) + O(\nu), \\
&w_n = w_n^{(0)}(h, s, T_1, \ldots, T_{N-1}, X_1, T_{N+1}, \ldots, T_{2N-1}, X_2, \ldots) + O(\nu)
\end{align}
(57)
as $\nu \to 0$. This condition is stronger than the condition (49) for the naive limit.

Let $L(0)$ and $W(0)$ denote the difference operators with $u_n$ and $w_n$ being replaced by $u_n^{(0)}$ and $w_n^{(0)}$:
\begin{align}
L(0) &= \Lambda + \sum_{n=1}^{\infty} u_n^{(0)} \Lambda^{1-n}, \\
W(0) &= 1 + \sum_{n=1}^{\infty} w_n^{(0)} \Lambda^{-n}.
\end{align}

$L(0)$ and $W(0)$ are connected by the dressing relation
\begin{equation}
L(0) = W(0)\Lambda W(0)^{-1}.
\end{equation}
The reduction condition (5) for $L$ turns into the condition
\begin{equation}(L(0)^N)_{<0} = 0
\end{equation}
for $L(0)$. This is again a reduction condition to the lattice GD hierarchy. Moreover, we can derive a set of Lax and Sato equations for $L(0)$ and $W(0)$ as follows.

**Proposition 6.** $W(0)$ satisfies the Sato equations
\begin{align}
\hbar \frac{\partial W(0)}{\partial T_k} &= B_k(0)W(0) - W(0)\Lambda^k \quad \text{for } k \not\equiv 0 \mod N, \\
\hbar \frac{\partial W(0)}{\partial X_k} &= kC_k(0)W(0) - kW(0)\Lambda^{kN} \log \Lambda \quad \text{for } k = 1, 2, \ldots,
\end{align}
(59)
where
\begin{align}
B_k(0) &= (L(0)^k)_{\geq 0}, \\
C_k(0) &= (L(0)^{(k-1)N} \log L(0))_{\geq 0}.
\end{align}

**Proof.** In view of (55), the Sato equation (59) for $W(0)$ is an immediate consequence of the Sato equations (4) for $W$ with respect to $t_k, k \not\equiv 0 \mod N$. In order to derive (59), we consider the operator
\begin{equation}
\mathcal{L}_k = (L^N - \nu \log L)^k = L^{kN} - k\nu L^{(k-1)N} \log L + O(\nu^2).
\end{equation}
The reduction condition (5) implies that
\begin{equation}
\mathcal{L}_k = ((L^N - \nu \log L)^k)_{\geq 0} = B_{kN} - k\nu C_k + O(\nu^2),
\end{equation}
(60)
where

\[ C_k = (L^{(k-1)N} \log L)_{\geq 0}. \]

Let us examine the operator identity

\[ \mathcal{L}_k W = W(\Lambda^N - \nu \log \Lambda)^k. \]

(61)

We can use the Sato equation (4) and the scaling relation (56) to rewrite the left hand side of this identity as

\[
\begin{align*}
\mathcal{L}_k W &= B_k W - k\nu C_k W + O(\nu^2) \\
&= h \frac{\partial W}{\partial \lambda_N} + W \Lambda^{kN} - k\nu C_k W + O(\nu^2) \\
&= \hbar \nu \frac{\partial W}{\partial X_k} + W \Lambda^{kN} - k\nu C_k W + O(\nu^2).
\end{align*}
\]

The right hand side of (61) can be expanded as

\[
W(\Lambda^N - \nu \log \Lambda)^k = W \Lambda^{kN} - k\nu W \Lambda^{(k-1)N} \log \Lambda + O(\nu^2).
\]

We can thus extract the coefficients of \(\nu\) in the \(\nu\)-expansion of (61) to obtain (60).

**Corollary 2.** \(L(0)\) satisfies the Lax equations

\[
\begin{align*}
\hbar \frac{\partial L(0)}{\partial T_k} &= [B_k(0), L(0)] \quad \text{for } k \not\equiv 0 \mod N, \\
\hbar \frac{\partial L(0)}{\partial X_k} &= [kC_k(0), L(0)] \quad \text{for } k = 1, 2, \ldots.
\end{align*}
\]

(62)

(63)

We thus obtain two sets of flows defined by the Lax equations (62), (63) and the Sato equations (59), (60) alongside the reduction condition (58). These equations are essentially the same as those of our previous work [17], cf. (50) – (53), except that the time variables of the logarithmic flows are rescaled and renumbered as

\[ x_{k-1} = kX_k, \quad k = 1, 2, \ldots. \]

**5.3 Scaling limit of soliton solutions**

We can apply the foregoing prescription of scaling limit to the soliton solution (30) as well. Suppose that the parameters \(p_i, q_i\) of the soliton solution (30) depend on \(\nu\), behave as

\[ p_i = p_i(0) + O(\nu), \quad q_i = q_i(0) + O(\nu) \]

(30)
as $\nu \to 0$, and satisfy the reduction condition $[31]$. $p_i(0)$ and $q_i(0)$ thereby satisfies the reduction condition

$$p_i(0)^N = q_i(0)^N$$

to the lattice GD hierarchy.

In this setup, we do the scaling transformation $[54]$ of time variable $s$.

The exponential factors $e_i$ in the tau function turn out to have the following limit as $\nu \to 0$.

**Proposition 7.**

$$\lim_{\nu \to 0} e_i = \left( \frac{p_i(0)}{q_i(0)} \right)^{s/h} \exp \left( \hbar^{-1} \sum_k' \left( p_i(0)^k - q_i(0)^k \right) T_k \right)$$

$$\times \exp \left( \hbar^{-1} \sum_{k=1}^{\infty} k \left( p_i(0)^{(k-1)N} \log p_i(0) - q_i(0)^{(k-1)N} \log q_i(0) \right) X_k \right),$$

where $\sum_k'$ denotes the sum over $k = 1, 2, \ldots, k \neq 0 \mod N$.

**Proof.** Let us consider the obvious consequence

$$(p_i^N - \nu \log p_i)^k = (q_i^N - \nu \log q_i)^k$$

of the reduction condition $[31]$. Both sides can be expanded as

$$(p_i^N - \nu \log p_i)^k = p_i^{kN} - k\nu p_i^{(k-1)N} \log p_i + O(\nu^2),$$

$$(q_i^N - \nu \log q_i)^k = q_i^{kN} - k\nu q_i^{(k-1)N} \log q_i + O(\nu^2),$$

hence

$$p_i^{kN} - q_i^{kN} = \nu k(p_i^{(k-1)N} \log p_i - q_i^{(k-1)N} \log q_i) + O(\nu^2)$$

$$= \nu k(p_i(0)^{(k-1)N} \log p_i(0) - q_i(0)^{(k-1)N} \log q_i(0)) + O(\nu^2).$$

Consequently, the part of $\xi(t, p_i) - \xi(t, q_i)$ containing $t_{kN}$'s becomes

$$\sum_{k=1}^{\infty} (p_i^{kN} - q_i^{kN})t_{kN} = \sum_{k=1}^{\infty} k(p_i(0)^{(k-1)N} \log p_i(0) - q_i(0)^{(k-1)N} \log q_i(0))X_k + O(\nu).$$

The other part of $\xi(t, p_i) - \xi(t, q_i)$ and $(p_i/q_i)^{s/h}$ have the simpler asymptotic form

$$(p_i/q_i)^{s/h} = (p_i(0)/q_i(0))^{s/h} + O(\nu),$$

$$\sum_k' (p_i^k - q_i^k)t_k = \sum_k' (p_i(0)^k - q_i(0)^k)T_k + O(\nu).$$

\[\square\]
Corollary 3. In the limit as $\nu \to 0$, the tau function (30) converges to

$$
\mathcal{T} = 1 + \sum_{i=1}^{M} a_i E_i + \sum_{1 \leq i < j \leq M} a_i a_j C_{ij} E_i E_j + \sum_{1 \leq i < j < k \leq M} a_i a_j a_k C_{ijk} E_i E_j E_k + \cdots + a_1 \cdots a_M C_{1 \cdots M} E_1 \cdots E_M,
$$

(65)

where

$$
C_{ij} = \frac{(p_i(0) - p_j(0))(q_i(0) - q_j(0))}{(p_i(0) - q_j(0))(q_i(0) - p_j(0))},
$$

and $E_i$ denotes the right hand side of (64).

Thus, as anticipated, we obtain the tau function (65) of the $N$-soliton solution of the extended lattice GD hierarchy.

6 Conclusion

We have formulated the ILW hierarchy and its generalization as reductions of the lattice KP hierarchy. The integrability of the lattice KP hierarchy is inherited by these reduced systems. In particular, all solutions can be described by the factorization problem for the lattice KP hierarchy. The $U$-operator therein is required to satisfy a reduction condition. The situation is thus parallel to the equivariant 1D/bigraded Toda hierarchy.

We have uncovered some features hidden in these somewhat exotic integrable hierarchies.

First, we can thereby explain an origin of the logarithmic flows of the lattice GD hierarchy [17] and the 1D/bigraded Toda hierarchy [18, 20]. These integrable hierarchies are a naive limit of the generalized ILW hierarchy and the equivariant Toda hierarchy as the parameter $\nu$ tends to 0. In this limit, part of the flows are trivialized, and the logarithmic flows are added by hand in place of these trivialized flows. We have found that the logarithmic flows can be derived by a more refined formulation of the limit.

Second, a special solutions related to the equivariant Toda hierarchy can be obtained from the factorization problem. The $U$-operator therein is built from one of the dressing operators of Okounkov and Pandharipande [23, 24, 22]. The dressing operators were originally introduced for the construction of a solution of the equivariant Toda hierarchy. This indicates a somewhat puzzling, but quite intriguing relation between the generalized ILW hierarchy and the equivariant Toda hierarchy. Moreover, we have shown a generalization of the dressing operators, which can lead to new special solutions of these integrable hierarchies.
Acknowledgements
This work is partly supported by the JSPS Kakenhi Grant JP18K03350 and JP21K03261.

A Equivariant Toda hierarchy

A.1 Reduction from 2D Toda hierarchy

The 2D Toda hierarchy consists of the Lax equations
\[ \hbar \frac{\partial L}{\partial t_k} = [B_k, L], \quad \hbar \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L], \]
\[ \hbar \frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \quad \hbar \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}], \]
for two Lax operators \( L \) and \( \bar{L} \) of the form mentioned in the main text. \( B_k \) and \( \bar{B}_k \) are defined as
\[ B_k = (L^k)_{\geq 0}, \quad \bar{B}_k = (\bar{L}^{-k})_{< 0}. \]

The Lax operators can be expressed in the dressed form
\[ L = W\Lambda W^{-1}, \quad \bar{L} = \bar{W}\Lambda \bar{W}^{-1}. \]

These Lax equations are thereby converted to the Sato equations
\[ \frac{\partial W}{\partial t_k} = B_k W - W \Lambda^k, \quad \frac{\partial W}{\partial \bar{t}_k} = \bar{B}_k \bar{W}, \]
\[ \frac{\partial \bar{W}}{\partial t_k} = B_k \bar{W}, \quad \frac{\partial \bar{W}}{\partial \bar{t}_k} = \bar{B}_k \bar{W} - W \Lambda^{-k}. \]

The dressing operators are also characterized by the factorization problem
\[ U = W^{-1}\bar{W}, \]
where \( U \) satisfies the evolution equations
\[ \hbar \frac{\partial U}{\partial t_k} = \Lambda^k U, \quad \hbar \frac{\partial U}{\partial \bar{t}_k} = -U \Lambda^{-k}. \]

Given the initial values \( W_{in} = W|_{t=\bar{t}=0} \) and \( \bar{W}_{in} = \bar{W}|_{t=\bar{t}=0} \), the initial value problem of the Sato equations can be solved by the factorization problem with
\[ U = e^{\xi(t,\Lambda)/\hbar} W_{in}^{-1} \bar{W}_{in} e^{-\xi(t,\Lambda^{-1})/\hbar}. \]
The equivariant bigraded Toda hierarchy of type \((N, \bar{N})\) \[13\] can be derived by the reduction condition
\[
L^N - \nu \log L = \bar{L}^{-\bar{N}} - \nu \log \bar{L}.
\] (67)

The original equivariant Toda hierarchy amounts to the \(N = \bar{N} = 1\) case \[14\]. Since
\[
L^N - \nu \log L = B_N - \nu \log \Lambda + \text{negative powers of } \Lambda,
\]
and
\[
\bar{L}^{-\bar{N}} - \nu \log \bar{L} = \bar{B}_N - \nu \log \Lambda + \text{non-negative powers of } \Lambda,
\]
the reduction condition yields the reduced Lax operator
\[
\mathcal{L} = B_N + \bar{B}_N - \nu \log \Lambda
\] (68)
that satisfies the reduced Lax equations
\[
\hbar \frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}], \quad \hbar \frac{\partial \mathcal{L}}{\partial \bar{t}_k} = [\bar{B}_k, \mathcal{L}].
\] (69)

In the language of the factorization problem, the reduction condition reads
\[
(\Lambda^N - \nu \log \Lambda) U = U(\Lambda^{-\bar{N}} - \nu \log \Lambda - \nu \log Q).
\] (70)

The reduction condition (67) can be translated to the dressing operators and the tau function as well. In particular, the reduction condition for the tau function reads
\[
\frac{\partial \tau}{\partial t_N} + \frac{\partial \tau}{\partial \bar{t}_N} - \nu \frac{\partial \tau}{\partial s} = 0,
\] (71)
hence the tau function depends on \(t_N, \bar{t}_N\) and \(s\) through the linear combinations \(t_N + s/\nu\) and \(\bar{t}_N + s/\nu\):
\[
\tau = f(\hbar, \nu, t_1, \ldots, t_{N-1}, t_N + s/\nu, t_{N+1}, \ldots, \bar{t}_1, \ldots, \bar{t}_{\bar{N}-1}, \bar{t}_\bar{N} + s/\nu, \bar{t}_{\bar{N}+1}, \ldots).
\] (72)

**Remark 11.** In order to deal with the equivariant Gromov-Witten theory of \(\mathbb{C}P^1\), one has to modify (67) and (70) as
\[
L^N - \nu \log L = \tilde{L}^{-\tilde{N}} - \nu \log \tilde{L} - \nu \log Q,
\] (73)
\[
(\Lambda^N - \nu \log \Lambda) U = U(\Lambda^{-\tilde{N}} - \nu \log \Lambda - \nu \log Q),
\] (74)
where \(Q\) is a constant \[13\ \[17\]. The reduced form (72) of the tau function is accordingly modified as
\[
\tau = Q^{(s/\hbar)^2/2} f(\hbar, \nu, t_1, \ldots, t_{N-1}, t_N + s/\nu, t_{N+1}, \ldots, \bar{t}_1, \ldots, \bar{t}_{\bar{N}-1}, \bar{t}_\bar{N} + s/\nu, \bar{t}_{\bar{N}+1}, \ldots).
\] (75)
The solution of the equivariant Gromov-Witten theory is obtained from the U-operator
\[ U = e^{\xi(t, \Lambda)/\hbar} V^{-1} e^{\Lambda^N/N} Q^H e^{\Lambda^{-\bar{N}}/\bar{N}} \bar{V}^{-1} e^{-\xi(t, \Lambda^{-1})/\hbar}, \tag{76} \]
where \( \bar{V} \) is a partner of \( V \) as explained in Section 4, and satisfies the intertwining relation
\[ (\Lambda^{-\bar{N}} - \nu \log \Lambda) \bar{V} = \bar{V} (\Lambda^{-\bar{N}} + H - \nu \log \Lambda). \tag{77} \]
The operator
\[ S = e^{\Lambda^N/N} Q^H e^{\Lambda^{-\bar{N}}/\bar{N}} \tag{78} \]
intertwines \( \Lambda^N + H - \nu \log \Lambda \) on the left hand side of (73) and \( \Lambda^{-\bar{N}} + H - \nu \log \Lambda \) on the right side of (77) up to a constant term as
\[ (\Lambda^N + H - \nu \log \Lambda) S = S (\Lambda^{-\bar{N}} + H - \nu \log \Lambda - \nu \log Q). \tag{79} \]

### A.2 Limit to extended bigraded Toda hierarchy

The naive limit of (67) as \( \nu \to 0 \) is the reduction condition
\[ L^N = \bar{L}^{-\bar{N}} \]
to the bigraded Toda hierarchy of type \( (N, \bar{N}) \), which becomes the 1D Toda hierarchy when \( N = \bar{N} = 1 \). In this reduced hierarchy, the sum of the \( t_kN \)-flow and the \( \bar{t}_k\bar{N} \)-flow is trivialized for \( k = 1, 2, \ldots \), namely,
\[ \frac{\partial L}{\partial t_{kN}} + \frac{\partial L}{\partial \bar{t}_{k\bar{N}}} = 0, \quad \frac{\partial \bar{L}}{\partial t_{kN}} + \frac{\partial \bar{L}}{\partial \bar{t}_{k\bar{N}}} = 0 \tag{80} \]
and
\[ \frac{\partial W}{\partial t_{kN}} + \frac{\partial W}{\partial \bar{t}_{k\bar{N}}} = 0, \quad \frac{\partial \bar{W}}{\partial t_{kN}} + \frac{\partial \bar{W}}{\partial \bar{t}_{k\bar{N}}} = 0. \tag{81} \]
The extended bigraded Toda hierarchy is constructed by adding logarithmic flows in place of these trivialized flows \cite{20, 21}.

A more careful prescription of the \( \nu \to 0 \) limit enables us to derive the logarithmic flows. This prescription is based on the scaling transformations
\[ t_k = \begin{cases} T_k & \text{if } k \not\equiv 0 \mod N, \\ T_k + X_{k/N}/\nu & \text{if } k \equiv 0 \mod N, \end{cases} \tag{82} \]
and
\[ \bar{t}_k = \begin{cases} \bar{T}_k & \text{if } k \not\equiv 0 \mod \bar{N}, \\ X_{k/\bar{N}}/\nu & \text{if } k \equiv 0 \mod \bar{N}. \end{cases} \tag{83} \]
of the time variables from \( \{ t_k \}_{k=1}^{\infty} \) and \( \{ \tilde{t}_k \}_{k=1}^{\infty} \) to \( \{ T_k \}_{k=1}^{\infty}, \{ \tilde{T}_k \}_{k \neq 0 \bmod \bar{N}} \) and \( \{ X_k \}_{k=1}^{\infty} \). The derivatives are accordingly transformed as

\[
\frac{\partial}{\partial t_k} = \frac{\partial}{\partial T_k} \quad \text{for } k = 1, 2, \ldots, \quad (84)
\]

\[
\frac{\partial}{\partial \tilde{t}_k} = \frac{\partial}{\partial \tilde{T}_k} \quad \text{for } k \not\equiv 0 \bmod \bar{N}, \quad (85)
\]

\[
\frac{\partial}{\partial t_{k \bar{N}}} = \nu \frac{\partial}{\partial X_k} - \frac{\partial}{\partial T_{k \bar{N}}} \quad \text{for } k = 1, 2, \ldots. \quad (86)
\]

The dynamical variables \( u_n, \) etc., viewed to be functions of the new variables \( T_k, \tilde{T}_k \) and \( X_k \), are assumed to have a smooth limit

\[
u \rightarrow 0.
\]

Let \( L(0), \tilde{L}(0), W(0) \) and \( \check{W}(0) \) denote the difference operators with the coefficients being replaced by \( u_n^{(0)}, \) etc. \( L(0) \) and \( \tilde{L}(0) \) are expressed in the dressed form

\[
L(0) = W(0)\Lambda W(0)^{-1}, \quad \tilde{L}(0) = \check{W}\Lambda\check{W}^{-1}
\]

and satisfies the reduction condition

\[
L(0)^N = \tilde{L}(0)^{-\bar{N}} \quad (87)
\]

to the bigraded Toda hierarchy of type \((N, \bar{N})\). These operators satisfy a set of Lax and Sato equations of the following form.

**Proposition 8.** \( W(0) \) and \( \check{W}(0) \) satisfy the Sato equations

\[
\hbar \frac{\partial W(0)}{\partial T_k} = B_k(0)W(0) - W(0)\Lambda^k \quad \text{for } k = 1, 2, \ldots, \quad (88)
\]

\[
\hbar \frac{\partial \check{W}(0)}{\partial \tilde{T}_k} = \check{B}_k(0)W(0) \quad \text{for } k \not\equiv 0 \bmod \bar{N}, \quad (89)
\]

\[
\hbar \frac{\partial W(0)}{\partial X_k} = kC_k(0)W(0) - kW\Lambda^{(k-1)N} \log \Lambda \quad \text{for } k = 1, 2, \ldots, \quad (90)
\]

\[4\]The same transformations are used in the work of Okounkov and Pandharipande [23] to consider some particular elements in the equivariant cohomology of \( \mathbb{C}P^1 \).
and

\[ \hbar \frac{\partial \bar{W}(0)}{\partial T_k} = B_k(0) \bar{W}(0) \text{ for } k = 1, 2, \ldots, \tag{91} \]

\[ \hbar \frac{\partial \bar{W}(0)}{\partial T_k} = \bar{B}_k(0) \bar{W}(0) - \bar{W}(0) \Lambda^{-k} \text{ for } k \neq 0 \mod \bar{N}, \tag{92} \]

\[ \hbar \frac{\partial \bar{W}(0)}{\partial X_k} = kC_k(0) \bar{W}(0) - k \bar{W} \Lambda^{-(k-1)\bar{N}} \log \Lambda \text{ for } k = 1, 2, \ldots, \tag{93} \]

where

\[ B_k(0) = \left( L(0)^k \right)_{\geq 0}, \quad \bar{B}_k(0) = \left( \bar{L}(0)^{\bar{k}} \right)_{< 0}, \]

\[ C_k(0) = \left( L(0)^{(k-1)N} \log L(0) \right)_{\geq 0} + \left( \bar{L}(0)^{-(k-1)N} \log \bar{L}(0) \right)_{< 0}. \]

**Proof.** (88), (89), (91) and (92) are a direct consequence of the Sato equations for \( W \) and \( \bar{W} \). In order to derive (90) and (93), we consider the operator

\[ L_k = (L^N - \nu \log L)^k = (\bar{L}^{-\bar{N}} - \nu \log \bar{L})^k. \]

This operator, just like \( L \), can be expressed as

\[ L_k = \left( (L^N - \nu \log L)^k \right)_{\geq 0} + \left( (\bar{L}^{-\bar{N}} - \nu \log \bar{L})^k \right)_{< 0}, \]

where

\[ C_k = \left( L^{(k-1)N} \log L \right)_{\geq 0} + \left( \bar{L}^{-(k-1)N} \log \bar{L} \right)_{< 0}. \]

This operator and the dressing operators satisfy the operator identities

\[ L_k W = W(\Lambda^N - \nu \log \Lambda)^k, \]

\[ L_k \bar{W} = \bar{W}(\Lambda^{-\bar{N}} - \nu \log \Lambda). \]

By the Sato equations, the left hand side of the first identity can be expressed as

\[ L_k W = B_k N W + \bar{B}_k \bar{N} W - k \nu C_k W + O(\nu^2) \]

\[ = \hbar \frac{\partial W}{\partial t_k N} + \hbar \frac{\partial W}{\partial \bar{t}_k \bar{N}} + W \Lambda^k N - k \nu C_k W + O(\nu^2) \]

\[ = \hbar \nu \frac{\partial W}{\partial X_k} + W \Lambda^k N - k \nu C_k W + O(\nu^2), \]

whereas the right hand side can be expanded as

\[ W(\Lambda^N - \nu \log \Lambda)^k = W \Lambda^k N - k \nu W \Lambda^{(k-1)N} \log \Lambda + O(\nu^2). \]

Extracting the coefficients of \( \nu \) in these expressions, we obtain (90). In the much same way, we can derive (93). \( \square \)
Corollary 4. \(L(0)\) and \(\bar{L}(0)\) satisfy the Lax equations

\[
\hbar \frac{\partial L(0)}{\partial T_k} = [B_k(0), L(0)] \quad \text{for } k = 1, 2, \ldots, \tag{94}
\]

\[
\hbar \frac{\partial L(0)}{\partial \bar{T}_k} = [\bar{B}_k(0), L(0)] \quad \text{for } k \not\equiv 0 \mod \bar{N}, \tag{95}
\]

\[
\hbar \frac{\partial L(0)}{\partial X_k} = [kC_k(0), L(0)] \quad \text{for } k = 1, 2, \ldots \tag{96}
\]

and

\[
\hbar \frac{\partial \bar{L}(0)}{\partial T_k} = [B_k(0), \bar{L}(0)] \quad \text{for } k = 1, 2, \ldots, \tag{97}
\]

\[
\hbar \frac{\partial \bar{L}(0)}{\partial \bar{T}_k} = [\bar{B}_k(0), \bar{L}(0)] \quad \text{for } k \not\equiv 0 \mod \bar{N}, \tag{98}
\]

\[
\hbar \frac{\partial \bar{L}(0)}{\partial X_k} = [kC_k(0), \bar{L}(0)] \quad \text{for } k = 1, 2, \ldots. \tag{99}
\]

Thus we can derive all Lax and Sato equations of the extended bigraded Toda hierarchy [20] from the equivariant bigraded Toda hierarchy in the scaling limit as \(\nu \to 0\).  

Remark 12. The soliton solutions of the 2D Toda hierarchy are given by tau functions of the same form as shown in (30) with the exponential factors

\[e_i = (p_i/q_i)^{s_i/\hbar} \exp \left(\hbar^{-1}(\xi(t, p_i) - \xi(t, q_i) + \xi(\bar{t}, p_i^{-1}) - \xi(\bar{t}, q_i^{-1}))\right).\]

This tau function takes the reduced form (72) if the parameters \(p_i, q_i\) satisfy the equations

\[p_i^N + p_i^{-N} - \nu \log p_i = q_i^N + q_i^{-N} - \nu \log q_i, \quad i = 1, \ldots, M. \tag{100}\]

Thus we obtain soliton solutions of the equivariant bigraded Toda hierarchy. The prescription of the scaling limit will be applicable to these soliton solutions as well (though in a technically more complicated way).

References

[1] R. I. Joseph, Solitary waves in a finite depth fluid, J. Phys. A: Math. Gen. 10 (1977), 225–227.

[2] T. Kubota, D. R. S. Ko and L. D. Dobbs, Weakly-nonlinear, long internal gravity waves in stratified fluids of finite depth, J. Hydronautics 12 (1978), 157–165.
[3] J. Satsuma, M. J. Ablowitz and Y. Kodama, On an internal wave equation describing a stratified fluid with finite depth, Phys. Lett. A73 (1979), 283–286.

[4] H. H. Chen and Y. C. Lee, Internal-wave solitons of fluids with finite depth, Phys. Rev. Lett. 43 (1979), 264–266.

[5] Y. Kodama, J. Satsuma and M. J. Ablowitz, Nonlinear intermediate long-wave equation: analysis and method of solution, Phys. Rev. Lett. 46 (1981), 677–690.

[6] D. R. Lebedev and A. O. Radul, Generalized internal long waves equations: Construction, Hamiltonian structure, and conservation laws, Comm. Math. Phys. 91 (1983), 543–555.

[7] A. Degasperis, D. Lebedev, M. Olshanetsky, S. Pakuliak, A. Perelomov and P. Santini, Nonlocal integrable partners to generalized MKdV and two dimensional Toda lattice equation in the formalism of a dressing method with quantized spectral parameter, Comm. Math. Phys. 141 (1991), 133–151.

[8] A. Degasperis, D. Lebedev, M. Olshanetsky, S. Pakuliak, A. Perelomov and P. Santini, Generalized intermediate long-wave hierarchy in zero-curvature representation with noncommutative spectral parameter, J. Math. Phys. 33 (1992), 3783–3793.

[9] Y. Tutiya and J. Satsuma, On the ILW hierarchy, Phys. Lett. A313 (2003) 45–54.

[10] J. Shiraishi and Y. Tutiya, Periodic ILW equation with discrete Laplacian, J. Phys. A: Math. Theor. 42 (2009), 404018 (15 pages).

[11] A. Buryak and P. Rossi, Simple Lax description of the ILW hierarchy, SIGMA 14 (2018), 120 (7 pages).

[12] K. Takasaki, Toda hierarchies and their applications, J. Phys. A: Math. Theor. 51 (2018), 203001 (35 pages).

[13] T. Milanov and H. H. Tseng, Equivariant orbifold structures on the projective line and integrable hierarchies, Adv. Math. 226 (2011), 641–672.

[14] E. Getzler, The equivariant Toda lattice, Publ. RIMS, Kyoto University, 40 (2004), 507–534.
[15] S.-Q. Liu, Z. Wang and Y. Zhang, Reduction of the 2D Toda hierarchy and linear Hodge integrals, SIGMA 18 (2022), 037 (18 pages).

[16] L. A. Dickey, Soliton equations and Hamilton systems, 2nd edition, World Scientific, Singapore, 2003.

[17] K. Takasaki, Extended lattice Gelfand-Dickey hierarchy, J. Phys. A: Math. Theor. 55 (2022), 305203 (14 pages).

[18] G. Carlet, B. Dubrovin and Y. Zhang, The extended Toda hierarchy, Moscow Math. J. 4 (2004), 313-332 and 534.

[19] K. Takasaki, Two extensions of 1D Toda hierarchy, J. Phys. A: Math. Theor. 43 (2010), 434032.

[20] G. Carlet, The extended bigraded Toda hierarchy, J. Phys. A: Math. Gen. 39 (2006), 9411–9435.

[21] C.-Z. Li, J.-S. He, K. Wu and Y. Cheng, Tau function and Hirota bilinear equations for the extended bigraded Toda hierarchy, J. Math. Phys. 51 (2010), 043514.

[22] K. Takasaki, Dressing operators in equivariant Gromov-Witten theory of $\mathbb{C}P^1$, J. Phys. A: Math. Theor. 54 (2021), 35LT02 (9 pages).

[23] A. Okounkov and R. Pandharipande, The equivariant Gromov–Witten theory of $\mathbb{P}^1$, Ann. Math. 163 (2006), 561–605.

[24] P. D. Johnson, Equivariant Gromov-Witten theory of one dimensional stacks, Comm. Math. Phys. 327 (2014), 333–386.

[25] K. Takasaki and T. Takebe, Integrable hierarchies and dispersionless limit, Rev. Math. Phys. 7 (1995), 743–808.

[26] T. Miwa, M. Jimbo and E. Date, Solitons: Differential equations, symmetries, and infinite-dimensional algebras, Cambridge University Press, 2000.

[27] K. Ueno and K. Takasaki, Toda lattice hierarchy, K. Okamoto (ed.), Group Representations and Systems of Differential Equations, Advanced Studies in Pure Math. 4, Kinokuniya, Tokyo, 1984, pp. 1–95.