Gravitational forces between weights in Force Standard Machines – Simplified analytical and numerical approaches

To cite this article: D Röske 2018 J. Phys.: Conf. Ser. 1065 042012

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Gravitational forces between weights in Force Standard Machines – Simplified analytical and numerical approaches

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Abstract. The gravitational field at the installation site of deadweight force standard machines has been investigated at PTB in the past. For this purpose, the local gravitational acceleration was measured prior to and after the installation of the machines. The effect of the masses of the machine frame and the load masses was simulated for the 200 kN force standard machine (FSM). In this paper, an attempt is made to calculate the gravitational forces between the weights of PTB’s 2 MN force standard machine using simplified analytical and numerical approaches and models. The aim in doing so is to determine the order of magnitude of these forces and to compare them with the force variations that arise due to tidal changes of the local gravitational acceleration.

1. Introduction

In force standard machines that have the lowest possible uncertainty of the force realized, the weight of mass bodies (“weights”) in the local gravitational field of the Earth is used to physically represent the force. The gravitational acceleration \( g_{\text{loc}} \) in the place where the mass bodies are to be located can be measured very precisely with a relative uncertainty of less than \( 10^{-8} \) [1]. This uncertainty is usually increased to a value of about \( 1.2 \times 10^{-7} \) because the gravitational acceleration shows temporal variations due to the rotation of the Earth around its axis and the changing position of other bodies – mainly that of the Moon – in relation to the place under consideration on the Earth’s surface. This tidal effect is well understood and measurable [2] but not significant for force standard machines, whose best relative standard uncertainty is currently \( 5 \times 10^{-6} \).

The gravitational field and its change have been investigated at PTB in the past ([1], [2]) by Leibniz University Hanover, Institute of Geodesy (Institut für Erdmessung); here, the gravitational acceleration was measured at several points prior to and after the installation of force standard machines. One important result was that the vertical gradient (i.e., the dependency of the value of \( g_{\text{loc}} \) on the vertical coordinate \( z \)) must be considered if the uncertainty in \( g_{\text{loc}} \) is to be limited only by the tidal variations. For PTB’s new 200 kN force standard machine [3], the forces acting between the weights of the machine and the machine frame were modeled and found to be negligible.

In this paper, an attempt is made to calculate the gravitational forces between the weights of PTB’s 2 MN force standard machine [4]. Due to the large masses involved in the realization of the force, it is expected that the effect will be greater than in the 200 kN machine. The calculations were performed using a simplified numerical approach with R, a free statistical software program [5]. The aim was to determine the order of magnitude of the forces and compare them with the force variations that arise due to tidal changes of the local gravitational acceleration.
2. Analytical considerations

The force between two point particles with the masses $m_1$ and $m_2$, respectively, can be calculated according to Newton’s Law of Gravitation (1)

$$\vec{F} = -G \frac{m_1 \cdot m_2}{r^2} \hat{e}_r,$$

where $G$ is the gravitational constant, $r$ is the distance between the two points and $\hat{e}_r$ is the unit vector of the vector connecting the two points.

![Figure 1. Relative position and dimensions of two disc-shaped weights.](image1)

![Figure 2. Two points (one in each of the two discs), their coordinates and the distance vector.](image2)

For two disc-shaped weights placed in the machine close to each other, as shown in figure 1, the force can be found by replacing (in (1)) their masses with the mass distributions over the volume given by the local quantity density $\rho(\vec{r})$, and integrating over the two volumes according to (2)

$$\vec{F} = -G \int_{V_2} \int_{V_1} \frac{\rho_1(\vec{r}_1) \cdot \rho_2(\vec{r}_2)}{r^2} \hat{e}_r \, dV_1 \, dV_2.$$

(2)

Here, $\rho_1(\vec{r})$ and $\rho_2(\vec{r})$ are the local densities of the material of the respective weights. Figure 2 shows the coordinates of one point in each disc and the distance vector $\vec{r} = \vec{P}_1 - \vec{P}_2$ between these points.

For homogeneous materials, these densities do not depend on the coordinates. Under this presumption, and considering the shape and the axially aligned position of the weights, (2) can be rewritten in cylindrical coordinates as

$$\vec{F} = -G \rho_1 \rho_2 \int_{h_1+z_0}^{R_2} \int_{h_1}^{R_1} \int_0^{2\pi} \int_0^{2\pi} \frac{\vec{r}}{r^3} \, dz_1 \, d\phi_1 \, r_1 \, dr_1 \, dz_2 \, d\phi_2 \, r_2 \, dr_2.$$

(3)

For reasons of symmetry, the force $\vec{F}$ can only have a $z$-component. Therefore, only the $z$-components of the distance vector $\vec{r}$ need to be considered. For the same reasons of symmetry, in order to reduce the calculation time in the numerical integration, only one sector (for example, the quarter of the disc located in the first quadrant of the coordinate system) of the upper disc can be taken when the result is multiplied by the corresponding factor (4, in the case of a quarter; the second integral will then be calculated from 0 to $\pi/2$). Then, (3) can be transformed into (4)
where the distance \( r \) is given by (5)
\[
r = \sqrt{(z_2 - z_1)^2 + r_1^2 + r_2^2 - 2 \cdot r_1 \cdot r_2 \cdot \cos(\varphi_1 - \varphi_2)} \] .

If the dimensions of the discs and the densities are given, the resulting force will depend only on the distance \( z_0 \) between the weights. Equation (4) can be simplified analytically by integrating over the heights of the discs. The weights of the 2 MN force standard machine are ring-shaped (i.e., discs with a cylindrical hole in the center). Therefore, integration over the radius \( r \) starts from the inner radius \( l_1 \) and \( l_2 \), respectively. The result is given by (6)
\[
F(z_0) = -4G \rho_1 \rho_2 \int_{l_2}^{l_1} \int_{l_1}^{l_2} r_1 \cdot r_2 \cdot f(\varphi_1, r_1, \varphi_2, r_2) \, d\varphi_1 \, d\varphi_2 \, dr_1 \, dr_2
\]
with
\[
f(\varphi_1, r_1, \varphi_2, r_2) = \ln \left( \frac{z_0 + h_2 + [(z_0 + h_2)^2 + d(\varphi_1, r_1, \varphi_2, r_2)]^{1/2}}{z_0 + h_1 + [(z_0 + h_1)^2 + d(\varphi_1, r_1, \varphi_2, r_2)]^{1/2}} \right) \]
\[
d(\varphi_1, r_1, \varphi_2, r_2) = r_1^2 + r_2^2 - 2 \cdot r_1 \cdot r_2 \cdot \cos(\varphi_1 - \varphi_2)
\]

Below, the minus sign in (6) will be omitted. The direction of the force should be clear from the arrangement.

If weight 2 is considered a point particle that has the mass of this weight, the general formula (2) changes for homogeneous material into (9)
\[
\vec{F} = -G m_2 \rho_1 \int_{V_1} \frac{\vec{r}}{r^3} \, dV_1
\]

Considering a ring as in (6) and presuming that the point particle is located on the \( z \) axis and has the coordinate \( z_0 \) allows (9) to be transformed into (10)
\[
F(z_0) = -G m_2 \rho_1 \int_{l_2}^{l_1} \int_{l_1}^{l_2} \frac{(z_0 - z_1) \, r_1}{[(z_0 - z_1)^2 + r_1^2]^{3/2}} \, dz_1 \, d\varphi_1 \, dr_1
\]

The integral in (10) can be solved analytically. The result is (11)
\[
F(z_0) = 2\pi G m_2 \rho_1 \left( \sqrt{(z_0 - h_1)^2 + R_1^2} - \sqrt{(z_0 - h_1)^2 + l_1^2} - \sqrt{z_0^2 + R_1^2} + \sqrt{z_0^2 + l_1^2} \right)
\]

The exact result (11) was used to evaluate and improve the accuracy of the numerical integration using (10). The optimized procedure was the basis for the subsequent integral calculations using (6) to (8).
3. Numerical integration
The following parameters were taken for the largest weights of the 2 MN force standard machine:

- outer radius $R = 1400$ mm
- inner radius: $I = 276.7$ mm
- height $h = 220$ mm.

No boreholes, bevels, recesses or taring cavities were considered; the inner radius $I$ was adjusted to a value that gave the correct weight value for the known material density and for the given parameters of the mass body including $g_{\text{loc}}$. Integration was carried out using a Monte Carlo (MC) method in R.

In the first step, the integral calculations of the double integral over $r_1$ and $z_1$ in (10) –integration over $\phi_1$ is trivial – were carried out for a point particle and a ring (figure 3, left). In each run, 100 000 points $P_1$ from the ring were randomly found by the program, which then calculated the corresponding functional value of the integrand. In the result, the mean overall attraction force was calculated and recorded. The run was repeated 100 times with the aim of obtaining an additional statistical parameter. This procedure was repeated for positions of $P_2$ between 0.22 m and 2.2 m. The standard uncertainty (spread) of the calculation results was below $7.6 \times 10^{-6}$ N (relative 0.4 %), the maximum span (maximum – minimum for each distance) was below 0.000 04 N (0.04 mN, relative 1.8 %). The good agreement between the analytical and numerical results (see figure 3, right) was a successful proof and validation of the method.

In the second step, the fourfold integral in (6) was calculated for two rings (as shown in figure 1 and figure 2) using the same Monte Carlo method. In each run, 100 000 pairs of points $P_1$ and $P_2$ from the two rings were randomly found by the program, which then calculated the corresponding functional value of the integrand. In the result, the mean overall attraction force was calculated and recorded. The run was repeated 100 times with the aim of obtaining an additional statistical parameter. This procedure was repeated for distances between 0.22 m and 2.2 m. The standard uncertainty (spread) of the calculation results was below $8.3 \times 10^{-6}$ N (relative 2.2 %); the maximum span (maximum – minimum for each distance) was below 0.000 49 N (0.49 mN, relative 12.7 %).

4. Results
The attraction forces calculated are shown in figure 4 (left, orange dots). To be able to calculate the force for any position of the second ring, or even for a stack of rings at given positions within a range, a fitting function was found for these values. This function is given analytically in (12) and shown in figure 4 (left, blue line)
\[ F(z) = 0.004485 \text{ N} \cdot e^{-\frac{0.8407 z}{m}}. \]  

(12)

Figure 4. Left: Gravitational force between two axially aligned rings as function of their vertical distance (distance = position - 0.22 m). Right: Force of a stack of 9 rings acting on one ring below the stack.

The right part of figure 4 shows which force is acting on weight 1 when all other weights are put together into one stack without gaps and when the distance to weight 1 is 60 mm. This force amounts to 0.017 N, which is about $1.7 \times 10^{-7}$ relative to the weight value and slightly more than the tidal effect of $1.2 \times 10^{-7}$ mentioned in the introduction. The effect can be neglected in uncertainty considerations.

5. Conclusions
The investigation presented here shows that, under certain conditions, the gravitational forces between the weights of deadweight force standard machines can reach an order of magnitude comparable to that of the tidal variation of the local gravitational acceleration. Due to the small values, neglecting them in uncertainty considerations is justified.

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