Quantum Sphere $S^4$ as a Non-Levi Conjugacy Class

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Abstract. We construct a $U_h(\mathfrak{sp}(4))$-equivariant quantization of the four-dimensional complex sphere $S^4$ regarded as a conjugacy class, $Sp(4)/Sp(2) \times Sp(2)$, of a simple complex group with non-Levi isotropy subgroup, through an operator realization of the quantum polynomial algebra $C_h[S^4]$ on a highest weight module of $U_h(\mathfrak{sp}(4))$.

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1. Introduction

There are two types of closed conjugacy classes in a simple complex algebraic group $G$. One type consists of classes that are isomorphic to orbits in the adjoint representation on the Lie algebra $\mathfrak{g}$. They are homogeneous spaces of $G$ whose stabilizer of the initial point is a Levi subgroup in $G$. Our concern is equivariant quantization of classes of the second type, i.e. whose isotropy subgroup is not Levi. Regarding the classical infinite series, such classes are present only in the orthogonal and symplectic groups.

The group $G$ supports a (Drinfeld–Sklyanin) Poisson bivector field $\pi_0 \in \Lambda^2(G)$ associated with a solution of the classical Yang–Baxter equation. This structure makes $G$ a Poisson group, whose multiplication $G \times G \to G$ is a Poisson map (here, $G \times G$ is equipped with the Poisson structure of Cartesian product). The Drinfeld–Sklyanin bracket gives rise to the quantum group $U_h(\mathfrak{g})$, which is a deformation, along the parameter $h$, of the universal enveloping algebra $U(\mathfrak{g})$ in the class of Hopf algebras [4].

There is a Poisson structure $\pi_1 \in \Lambda^2(G)$ compatible with the conjugacy action of the Poisson group on itself [15]. It means that action map from the Cartesian product of $(G, \pi_0)$ and $(G, \pi_1)$ to $(G, \pi_1)$ is Poisson. Then $G$ is said to be a Poisson space over the Poisson group $G$, under the conjugacy action.

The Poisson bivector field $\pi_1$ restricts to every closed conjugacy class making it a Poisson $G$-variety [1]. In this sense, the group $G$ is analogous to $\mathfrak{g} \simeq \mathfrak{g}^*$ equipped with the canonical $G$-invariant bracket.
A quantization of conjugacy classes with Levi isotropy subgroups has been constructed in various settings, namely, as a star product and in terms of generators and relations [6,12]. Both approaches rely upon the representation theory of the quantum group $U_h(\mathfrak{g})$ and make use of the following facts: (a) the universal enveloping algebra $U(I)$ of the isotropy subgroup is quantized to a Hopf subalgebra $U_h(I) \subset U_h(\mathfrak{g})$, (b) there is a triangular factorization of $U(\mathfrak{g})$ relative to $U(I)$, which amounts to a factorization of quantum groups and facilitates the parabolic induction. In particular, quantum conjugacy classes of the Levi type have been realized by operators on scalar parabolic Verma modules in [12].

The above-mentioned conditions are violated for non-Levi conjugacy classes, which makes the conventional methods of quantization inapplicable in this case. In this paper, we show how to overcome these obstructions for the simplest non-Levi conjugacy class $Sp(4)/Sp(2) \times Sp(2)$. This is the class of invertible symplectic $4 \times 4$-matrices with eigenvalues $\pm 1$, each of multiplicity 2. As an affine variety, it is isomorphic to the four-dimensional complex sphere $S^4$. Although the quantization of $S^4$ can be obtained by other methods, e.g. as in [7], we are interested in $S^4$ as an illustration of our approach to the general non-Levi classes [13].

The idea is to find a suitable highest weight $U_h(\mathfrak{g})$-module where the quantum sphere could be represented by linear operators. This module is constructed as follows. Let $\mathfrak{l} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)$ denote the Lie algebra of the stabilizer and let $\mathfrak{l} \subset \mathfrak{t}$ denote the Levi subalgebra $\mathfrak{l} = \mathfrak{gl}(1) \oplus \mathfrak{sp}(2)$. We consider an auxiliary parabolic Verma module $\hat{M}_\lambda$ of highest weight $\lambda$ relative to $U_h(\mathfrak{l}) \subset U_h(\mathfrak{g})$. For a special value of $\lambda$, the module $\hat{M}_\lambda$ has a singular vector of weight $\lambda - \delta$, where $\delta$ is the root of the first copy of $\mathfrak{sp}(2) \subset \mathfrak{t}$, which is not in $\mathfrak{l}$. Note that $\delta$ is not a simple root of $\mathfrak{g}$. The singular vector generates a submodule $\hat{M}_{\lambda - \delta} \subset \hat{M}_\lambda$. The quotient $M_\lambda = \hat{M}_\lambda/\hat{M}_{\lambda - \delta}$ is irreducible and it is the module that supports the quantization. Namely, a deformation of the polynomial algebra $\mathbb{C}[S^4]$ is realized by a $U_h(\mathfrak{g})$-invariant subalgebra in $\text{End}(M_\lambda)$. This also allows us to describe the quantized polynomial algebra $\mathbb{C}_h[S^4]$ in terms of generators and relations.

Irreducibility of $M_\lambda$ implies non-degeneracy of the “Shapovalov form” on it. In the simple case of $Sp(4)/Sp(2) \times Sp(2)$ this form can be calculated explicitly. This provides a bi-differential operator relating the multiplication in $\mathbb{C}_h[S^4]$ to the multiplication in the dual Hopf algebra $U_h^*(\mathfrak{g})$, as explained in [10].

We start from description of the classical conjugacy class $Sp(4)/Sp(2) \times Sp(2)$ and the Poisson structure on it. Next, we collect the necessary facts about the quantum group $U_h(\mathfrak{g})$. Further, we describe the quantization of the polynomial algebra $\mathbb{C}_h[G]$ and its properties. After that we construct the module $M_\lambda$ and analyze the submodule structure of the tensor product $\mathbb{C}_h^{\otimes M_\lambda}$. This allows us to realize $\mathbb{C}_h[S^4]$ by operators on $M_\lambda$ and describe it in generators and relations. In conclusion, we calculate the invariant pairing between $M_\lambda$ and its dual and discuss the star product on $\mathbb{C}_h[S^4]$. 
2. The Classical Conjugacy Class $Sp(4)/Sp(2) \times Sp(2)$

Let $Sp(4)$ denote the complex algebraic group of matrices preserving the anti-symmetric skew-diagonal bilinear form $C_{ij} = \epsilon_i \delta_{ij}'$, where $i' = 5 - i$, $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 1, -1, -1)$, and $\delta_{ij}$ is the Kronecker symbol. We are interested in the conjugacy class of symplectic matrices with eigenvalues $\pm 1$ each of multiplicity 2. It is an $Sp(4)$-orbit with respect to the conjugation action on itself. The initial point $A_o$ of the class and its isotropy subgroup can be taken as

$$A_o = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Sp(2) \times Sp(2) = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & * & 0 & * \\ * & 0 & 0 & * \end{pmatrix} \subseteq Sp(4).$$

This conjugacy class is a subvariety in $Sp(4)$ defined by the system of equations

$$ACA^t - C = 0, \quad \text{Tr}(A) = 0, \quad A^2 - 1 = 0,$$  \hspace{1cm} (2.1)

where 1 in the third equality is the matrix unit. This is a system of polynomial equations on the matrix coefficients $A_{ij}$, which can be written in an alternative way:

$$A^t + CAC = 0, \quad \text{Tr}(A) = 0, \quad A^2 - 1 = 0.$$  

The first two equations are linear and allow for the following non-zero entries:

$$A = \begin{pmatrix} a & b & y & 0 \\ c & -a & 0 & -y \\ z & 0 & -a & b \\ 0 & -z & c & a \end{pmatrix}.$$  

The quadratic equation is then equivalent to

$$a^2 + bc + yz - 1 = 0.$$  \hspace{1cm} (2.2)

Thus, the conjugacy class of $A_o$ is isomorphic to the complex sphere $S^4$. The ideal generated by the entries of the matrix equations (2.1) along with the zero trace condition is, in fact, generated by a single irreducible polynomial of $a, b, c, y, z$ and is the defining ideal of the class.

Consider the $r$-matrix

$$r = \sum_{i=1}^{4} (e_{ii} \otimes e_{ii} - e_{ii} \otimes e_{i'i'}) + 2 \sum_{i,j=1, i>j}^{4} (e_{ij} \otimes e_{ji} - e_i e_j e_{ij} \otimes e_{i'j'}) \in \mathfrak{sp}(4) \otimes \mathfrak{sp}(4)$$

solving the classical Yang–Baxter equation [4]. It induces a Drinfeld–Sklyanin bivector field $\pi_0$ on $Sp(4)$ making it a Poisson group [4]. We are concerned with the following Poisson structure, $\pi_1$, on $Sp(4)/Sp(2) \times Sp(2) \simeq S^4$:

$$\{A_1, A_2\} = \frac{1}{2} (A_2 r_{21} A_1 - A_1 r A_2 + A_2 A_1 r - r_{21} A_1 A_2).$$  \hspace{1cm} (2.3)
This equation is understood in $\text{End}(\mathbb{C}^4) \otimes \text{End}(\mathbb{C}^4) \otimes \mathbb{C}[S^4]$ and is a shorthand matrix form of the system of $n^2 \times n^2$ identities defining the Poisson brackets $\{A_{ij}, A_{kl}\}$ of the coordinate functions. The subscripts indicate the copy of $\text{End}(\mathbb{C}^4)$ in the tensor square, as usual in the quantum group literature. Explicitly, the brackets of the generators $a, b, c, y, z \in \mathbb{C}[S^4]$ read

$$
\begin{align*}
\{a, b\} &= ab, \quad \{a, c\} = -ac, \quad \{a, y\} = ay, \quad \{a, z\} = -az, \\
\{b, y\} &= by, \quad \{b, z\} = -bz, \quad \{c, y\} = cy, \quad \{c, z\} = -cz, \\
\{y, z\} &= 2a^2 + 2bc, \quad \{b, c\} = 2a^2.
\end{align*}
$$

This Poisson structure restricts from $Sp(4)$ and makes $S^4$ a Poisson manifold under the conjugacy action of the Poisson group $Sp(4)$ [15]. In can be shown that such a Poisson structure on $S^4$ is unique.

### 3. Quantum Group $U_h(\mathfrak{sp}(4))$

Throughout the paper, $\mathfrak{g}$ stands for the Lie algebra $\mathfrak{sp}(4)$. We are looking for quantization of the polynomial algebra $\mathbb{C}[S^4]$ along the Poisson bracket (2.3) that is invariant under an action of the quantized universal enveloping algebra $U_h(\mathfrak{g})$. In this section, we recall the definition of $U_h(\mathfrak{g})$, following [4].

The root system of $\mathfrak{g}$ is generated by the simple positive roots $\alpha, \beta$, which are defined in the orthogonal basis $\varepsilon_1, \varepsilon_2$ as

$$
\alpha = \varepsilon_1 - \varepsilon_2, \quad \beta = 2\varepsilon_2.
$$

The other positive roots are $\gamma = \alpha + \beta$ and $\delta = 2\alpha + \beta$. The root vectors and the Cartan elements are represented by the matrices

$$
\begin{align*}
e_\alpha &= e_{12} - e_{34}, & e_\beta &= e_{23}, & e_\gamma &= e_{13} + e_{24}, & e_\delta &= e_{14}, \\
f_\alpha &= e_{21} - e_{43}, & f_\beta &= e_{32}, & f_\gamma &= e_{31} + e_{42}, & f_\delta &= e_{41}, \\
h_\alpha &= e_{11} - e_{22} + e_{33} - e_{44}, & h_\beta &= 2e_{22} - 2e_{33},
\end{align*}
$$

(3.4)

where $\{e_{ij}\}_{i,j=1}^4$ is the standard matrix basis.

The quantized universal enveloping algebra (quantum group) $U_h(\mathfrak{g})$ is a $\mathbb{C}[[h]]$-algebra generated by the elements $e_\alpha, e_\beta, f_\alpha, f_\beta, h_\alpha, h_\beta$ subject to the commutator relations

$$
\begin{align*}
[h_\alpha, e_\alpha] &= 2e_\alpha, & [h_\alpha, f_\alpha] &= -2f_\alpha, & [h_\beta, e_\beta] &= 4e_\beta, & [h_\beta, f_\beta] &= -4f_\beta, \\
[h_\alpha, e_\beta] &= -2e_\beta, & [h_\alpha, f_\beta] &= 2f_\beta, & [h_\beta, e_\alpha] &= -2e_\alpha, & [h_\beta, f_\alpha] &= 2f_\alpha, \\
[e_\alpha, f_\alpha] &= \frac{q^{h_\alpha} - q^{-h_\alpha}}{q - q^{-1}}, & [e_\alpha, f_\beta] &= 0 = [e_\beta, f_\alpha], & [e_\beta, f_\beta] &= \frac{q^{h_\beta} - q^{-h_\beta}}{q^2 - q^{-2}},
\end{align*}
$$

plus the Serre relations.
\[ e_\alpha^3 e_\beta - (q^2 + 1 + q^{-2}) e_\alpha^2 e_\beta e_\alpha + (q^2 + 1 + q^{-2}) e_\alpha e_\beta e_\alpha^2 - e_\beta e_\alpha^3 = 0, \]
\[ e_\beta^3 e_\alpha - (q^2 + q^{-2}) e_\beta e_\alpha e_\beta + e_\alpha e_\beta^2 = 0, \]
and the similar relations for \( f_\alpha, f_\beta \). Here and further on \( q = e^h \).

The comultiplication \( \Delta \) and antipode \( \gamma \) are defined on the generators by
\[
\Delta(h_\mu) = h_\mu \otimes 1 + 1 \otimes h_\mu, \quad \gamma(h_\mu) = -h_\mu, \\
\Delta(e_\mu) = e_\mu \otimes 1 + q^{\mu h} \otimes e_\mu, \quad \gamma(e_\mu) = -q^{-\mu h} e_\mu, \\
\Delta(f_\mu) = f_\mu \otimes q^{-\mu h} + 1 \otimes f_\mu, \quad \gamma(f_\mu) = -f_\mu q^{\mu h},
\]
for \( \mu = \alpha, \beta \). The counit homomorphism \( \varepsilon : U_h(\mathfrak{g}) \to \mathbb{C}[[h]] \) is nil on the generators.

Remark 3.1. The quantum group \( U_h(\mathfrak{g}) \) is regarded as a \( \mathbb{C}[[h]] \)-algebra, bearing in mind its application to deformation quantization. Accordingly, all its modules are understood as free \( \mathbb{C}[[h]] \)-modules. However, we will suppress the reference to \( \mathbb{C}[[h]] \) in order to simplify the formulas. For instance, the vector representation of \( U_h(\mathfrak{g}) \) will be denoted simply as \( \mathbb{C}^4 \). The tensor products and linear maps are also understood over \( \mathbb{C}[[h]] \).

Let us introduce higher root vectors \( e_\gamma, f_\gamma, e_\delta, f_\delta \in U_h(\mathfrak{g}) \) (the coincidence in the notation for the weight and the antipode should not cause a confusion) by
\[
f_\gamma = f_\beta f_\alpha - q^{-2} f_\alpha f_\beta, \quad f_\delta = f_\gamma f_\alpha - q^2 f_\alpha f_\gamma, \]
\[
e_\gamma = e_\alpha e_\beta - q^2 e_\beta e_\alpha, \quad e_\delta = e_\alpha e_\gamma - q^{-2} e_\gamma e_\alpha.
\]
Our definition of \( e_\delta, f_\delta \) is different from the usual definition \( e_\delta = [e_\alpha, e_\gamma], f_\delta = [f_\gamma, f_\alpha] \), corresponding to \( (\alpha, \gamma) = 0 \) [3]. The reason for that will be clear later on. The elements \( h_\alpha, h_\beta \) span the Cartan Lie algebra \( \mathfrak{h} \) and generate the Hopf subalgebra \( U_h(\mathfrak{h}) \subset U_h(\mathfrak{g}) \). The vectors \( e_\alpha, e_\beta \) along with \( \mathfrak{h} \) generate the positive Borel subalgebra \( U_h(\mathfrak{b}^+) \) in \( U_h(\mathfrak{g}) \). Similarly, \( f_\alpha, f_\beta \), and \( \mathfrak{h} \) generate the negative Borel subalgebra \( U_h(\mathfrak{b}^-) \). They are Hopf subalgebras of \( U_h(\mathfrak{g}) \).

**Lemma 3.2.** The root vectors satisfy the relations
\[
e_\gamma e_\beta - q^{-2} e_\beta e_\gamma = 0, \quad [e_\alpha, e_\delta] = 0, \quad [e_\beta, e_\delta] = 0, \quad [e_\gamma, e_\delta] = 0, \\
f_\beta f_\gamma - q^2 f_\gamma f_\beta = 0, \quad [f_\alpha, f_\delta] = 0, \quad [f_\beta, f_\delta] = 0, \quad [f_\gamma, f_\delta] = 0.
\]

**Proof.** The first two equalities in both lines are simply a rephrase of the Serre relations in the new terms. The last equalities follow from the second and third. Let us check the third equality, say, in the first line:
\[
e_\beta e_\delta = e_\beta (e_\alpha e_\gamma - q^{-2} e_\gamma e_\alpha) = e_\beta e_\alpha e_\gamma - e_\gamma e_\beta e_\alpha \\
= q^{-2} e_\alpha e_\beta e_\gamma - q^{-2} e_\gamma e_\beta e_\alpha + q^{-2} e_\gamma^2 = q^{-2} e_\alpha e_\beta e_\gamma - q^{-2} e_\gamma e_\alpha e_\beta \\
= e_\alpha e_\gamma e_\beta - q^{-2} e_\gamma e_\alpha e_\beta = e_\delta e_\beta,
\]
as required. \( \square \)
Denote by $U_h(n_0^\pm)$ the subalgebras generated by, respectively, positive and negative Chevalley generators. The Borel subalgebras $U_h(b^\pm)$ are freely generated by $U_h(n_0^\pm)$ over $U_h(h)$ with respect to the right or left multiplication. These equalities facilitate the following

**COROLLARY 3.3.** The positive (respectively, negative) root vectors generate a Poincarè–Birkhoff–Witt basis in $U_h(n_0^+)$ (respectively, $U_h(n_0^-)$).

**Proof.** The presence of PBW basis in the quantum group is a well-known fact. However, we use a non-standard definition of the root vectors $e_\delta, f_\delta$, therefore the lemma is substantial. To prove it, say, for $U_h(n_0^-)$ one should check that the system of monomials $f_a^4 f_c^4 f_d^2 f_b = f_a^4 f_c^4 f_d^2 f_b$, where $a, b, c,$ and $d$ are non-negative integers, is linearly independent and complete in $U_h(n^-)$. The ordered sequence of the elements $f_\alpha, f_\beta'=[f_\gamma, f_\alpha], f_\gamma, f_\beta$ does generate a PBW basis [3]. Using this fact along with the relations of Lemma 3.2, one can easily check the statement via the substitution $f_\delta' = f_\delta + (q^2-1)f_\alpha f_\gamma$. 

\[\square\]

4. The Algebra of Quantized Polynomials on $Sp(4)$

We adopt the convention throughout the paper that $G$ stands for the complex algebraic group $Sp(4)$. The conjugacy class of our interest is a closed affine variety in $G$, and its polynomial ring is a quotient of the polynomial ring $\mathbb{C}[G]$ by a certain ideal. Our goal is to obtain an analogous description of the quantum conjugacy class. To that end, we need to describe the quantum analog of the algebra $\mathbb{C}[G]$ first.

Recall from [2,9] that the image of the universal R-matrix of the quantum group $U_h(g)$ in the natural vector representation is equal, up to a scalar factor, to

$$R = \sum_{i,j=1}^4 q^{\delta_{ij} - \delta_{ij'}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i,j=1}^4 (e_{ij} \otimes e_{ji} - q^{\rho_i - \rho_j} e_i e_j e_{ij} \otimes e_{i'j'}),$$

where $(\rho_1, \rho_2, \rho_3, \rho_4) = (2,1,-1,-2)$.

Denote by $S$ the $U_h(g)$-invariant operator $PR \in \text{End}(\mathbb{C}^4) \otimes \text{End}(\mathbb{C}^4)$, where $P$ is the ordinary flip of $\mathbb{C}^4 \otimes \mathbb{C}^4$. This operator has three invariant projectors to its eigenspaces, among which there is a one-dimensional projector $\sim \sum_{i,j=1}^4 q^{\rho_i - \rho_j} e_i e_j e_{ij} \otimes e_{i'j'}$ to the trivial $U_h(g)$-submodule, call it $\kappa$.

Denote by $\mathbb{C}_h[G]$ the associative algebra generated by the entries of the matrix $K = ||k_{ij}|_{i,j=1}^4 \in \text{End}(\mathbb{C}^4) \otimes \mathbb{C}_h[G]$ modulo the relations

$$S_{12} K_2 S_{12} K_2 = K_2 S_{12} K_2 S_{12}, \quad K_2 S_{12} K_2 K = -q^{-5} \kappa = \kappa K_2 S_{12} K_2. \quad (4.5)$$

These relations are understood in $\text{End}(\mathbb{C}^4) \otimes \text{End}(\mathbb{C}^4) \otimes \mathbb{C}_h[G]$, and the indices distinguish the two copies of $\text{End}(\mathbb{C}^4)$, as usual.
The algebra $\mathbb{C}[G]$ is an equivariant quantization of $\mathbb{C}[G]$ [7,16], which is different from the RTT-quantization and is not a Hopf algebra. It carries a $U_h(g)$-action, which is a deformation of the conjugation $U(g)$-action on $\mathbb{C}[G]$. The algebra $\mathbb{C}[G]$ admits a $U_h(g)$-equivariant algebra monomorphism to $U_h(g)$, where the latter is regarded as the adjoint module. The monomorphism is implemented by the assignment

$$K \mapsto (\phi \otimes \text{id})(R_{21}R) = Q \in \text{End}(\mathbb{C}^4) \otimes U_h(g),$$

where $\phi: U_h(g) \rightarrow \text{End}(\mathbb{C}^4)$ is the natural vector representation and $R$ is the universal $R$-matrix of $U_h(g)$. The matrix $Q$ is important for our presentation, and the reader is referred to [12] for a detailed explanation of its role in the quantization and for its basic characteristics.

5. The Generalized Verma Module $M_\lambda$

Denote by $l$ the Levi subalgebra in $g=\mathfrak{sp}(4)$ spanned by $e_\beta, f_\beta, h_\beta, h_\alpha$. It is a Lie subalgebra of maximal rank, and its semisimple part is isomorphic to $\mathfrak{sl}(2) \cong \mathfrak{sp}(2)$. The universal enveloping algebra $U(l)$ is quantized as a Hopf subalgebra $U_h(l) \subset U_h(g)$. Denote by $n^+$ and $n^-$ the nilpotent subalgebras in $g$ spanned, respectively, by $\{e_\alpha, e_\gamma, e_\delta\}$ and $\{f_\alpha, f_\gamma, f_\delta\}$. The sum $l + n^\pm$ is a parabolic subalgebra $p^\pm \subset g$ whose universal enveloping algebra is quantized to a Hopf subalgebra in $U_h(p^\pm) \subset U_h(g)$.

Let $U_h(n^\pm)$ be the subalgebras in $U_h(g)$ generated by the quantum root vectors $\{e_\alpha, e_\gamma, e_\delta\}$ and $\{f_\alpha, f_\gamma, f_\delta\}$, respectively. The quantum group $U_h(g)$ is a free $U_h(n^-) - U_h(n^+)$-bimodule generated by $U_h(l)$:

$$U_h(p^-) = U_h(n^-)U_h(l), \quad U_h(g) = U_h(n^-)U_h(l)U_h(n^+), \quad U_h(p^+) = U_h(l)U_h(n^+).$$

(5.6)

The factorizations of $U_h(p^\pm)$ have the structure of smash product.

Fix a weight $\lambda \in \mathfrak{h}^*$ orthogonal to $\beta$. It can be regarded as a one-dimensional representation of $U_h(l)$,

$$\lambda: e_\beta, f_\beta, h_\beta \mapsto 0, \quad \lambda: h_\alpha \mapsto (\alpha, \lambda),$$

which can be extended to a representation of $U_h(p^+)$ by the assignment $\lambda: e_\alpha \mapsto 0$. Let $\mathbb{C}_\lambda$ denote the one-dimensional $\mathbb{C}[[\hbar]]$-module supporting this representation.

Consider the scalar parabolic Verma module $\hat{M}_\lambda$ induced from $\mathbb{C}_\lambda$,

$$\hat{M}_\lambda = U_h(g) \otimes_{U_h(p^+)} \mathbb{C}_\lambda.$$

As a module over $U_h(n^-)$, it is freely generated by its highest weight vector $v_\lambda$. As a module over the Cartan subalgebra, it isomorphic to $U_h(n^-) \otimes \mathbb{C}_\lambda$, where $U_h(n^-)$ is the natural adjoint module over $U_h(l)$. 

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The \( U_\hbar(\mathfrak{g}) \)-module \( \hat{M}_\lambda \) is irreducible except for special values of \( \lambda \), when \( \hat{M}_\lambda \) contains singular vectors. Recall that a weight vector is called singular if it is annihilated by the positive Chevalley generators. Such vectors generate submodules in \( \hat{M}_\lambda \), where they carry the highest weight. We are looking for such \( \lambda \) that \( \hat{M}_\lambda \) admits a singular vector of weight \( \lambda - \delta \). Quotienting out the corresponding submodule yields a module that supports the quantization of \( \mathbb{C}[S^4] \).

**Proposition 5.1.** The module \( \hat{M}_\lambda \) admits a singular vector of weight \( \lambda - \delta \) if and only if \( q^{2(\alpha, \lambda)} = -q^{-2} \). Then \( f_\delta v_\lambda \) is the singular vector.

**Proof.** Up to a scalar multiplier, the general expression for the vector of weight \( \lambda - \delta \) in \( \hat{M}_\lambda \) is

\[
(f_\alpha^2 f_\beta - (a + b) f_\alpha f_\beta f_\alpha + ab f_\beta f_\alpha^2)v_\lambda = (- (a + b) f_\alpha f_\beta f_\alpha + ab f_\beta f_\alpha^2)v_\lambda,
\]

where \( a \) and \( b \) are some scalars. For this vector being singular, we have a system of two equations on \( a \) and \( b \) resulted from the action of \( e_\beta \) and \( e_\alpha \):

\[
\begin{align*}
(-(a + b)f_\alpha [e_\beta, f_\alpha] f_\alpha + ab[f_\beta, f_\alpha^2])v_\lambda &= 0, \\
(-(a + b)[e_\alpha, f_\alpha] f_\beta f_\alpha + abf_\beta [e_\alpha, f_\alpha] f_\alpha + abf_\beta f_\alpha [e_\alpha, f_\alpha])v_\lambda &= 0.
\end{align*}
\]

The non-zero solution of this system is unique (up to the permutation \( a \leftrightarrow b \)) and equal to

\[
q^{2(\alpha, \lambda)} = -q^{-2}, \quad a = q^2, \quad b = q^{-2},
\]

as required. Finally, notice that \( f_\delta = f_\alpha^2 f_\beta - (q^2 + q^{-2}) f_\alpha f_\beta f_\alpha + f_\beta f_\alpha^2 \). This completes the proof.

Denote by \( M_\lambda \) the quotient of \( \hat{M}_\lambda \) by the submodule \( \hat{M}_{\lambda - \delta} = U_\hbar(\mathfrak{g}) f_\delta v_\lambda \). By Corollary 3.3, the vectors \( f_\alpha^k f_\gamma^l f_\beta^m v_\lambda \) for all non-negative integer \( k, l, m \) form a basis in \( \hat{M}_\lambda \). Therefore, \( M_\lambda \) is spanned by \( f_\alpha^k f_\gamma^l v_\lambda, k, l \geq 0 \).

**Proposition 5.2.** The module \( M_\lambda \) is irreducible.

**Proof.** The irreducibility follows from the non-degeneracy of the invariant bilinear pairing of \( M_\lambda \) with its dual, see Section 8. Alternatively, one can directly check that \( M_\lambda \) has no singular vector. Omitting the details, the action of the positive Chevalley generators on \( M_\lambda \) is given by

\[
e_\alpha f_\alpha^k f_\gamma^m v_\lambda = q^{(\alpha, \gamma) + 1} q^{2k - q^{-2k}} (q - q^{-1})^2 f_\gamma^{k-1} f_\alpha^m v_\lambda,
\]

\[
e_\beta f_\alpha^k f_\gamma^m v_\lambda = q^{2m - q^{-2m}} q^2 f_\alpha f_\gamma^{k+1} f_\alpha^{m-1} v_\lambda
\]
Here, we assume that $k > 0$ in the first line and $m > 0$ in the second; otherwise, the right hand side is nil. This immediately implies the absence of singular vectors in $M_\lambda$.

\section{The $U_\hbar(g)$-Module $\mathbb{C}^4 \otimes M_\lambda$}

The tautological assignment (3.4) defines the four-dimensional irreducible representation of $U(g)$. The similar assignment on the quantum Chevalley generators and Cartan elements defines a representation of $U_\hbar(g)$. Our next object of interest is the $U_\hbar(g)$-module $\mathbb{C}^4 \otimes M_\lambda$. In particular, we shall study a decomposition of $\mathbb{C}^4 \otimes M_\lambda$ into a direct sum of irreducible submodules.

Choose the standard basis $\{w_i\}_{i=1}^4 \subset \mathbb{C}^4$ of columns with the only non-zero entry in the $i$th place from the top. Their weights are $\varepsilon_1, \varepsilon_2, -\varepsilon_2, -\varepsilon_1$, respectively. As a $U_\hbar(l)$-module, $\mathbb{C}^4$ splits into the sum of two one-dimensional blocks of weights $\pm \varepsilon_1$ and one two-dimensional block of the highest weight $\varepsilon_2$. The parabolic Verma module contains three blocks of highest weights $\varepsilon_1 + \lambda, \varepsilon_2 + \lambda, -\varepsilon_1 + \lambda$, which we denote by $\hat{V}_{\varepsilon_1 + \lambda}, \hat{V}_{\varepsilon_2 + \lambda}$, and $\hat{V}_{-\varepsilon_1 + \lambda}$. For generic $\lambda$ these submodules are irreducible, and

$$\mathbb{C}^4 \otimes \hat{M}_\lambda = \hat{V}_{\varepsilon_1 + \lambda} \oplus \hat{V}_{\varepsilon_2 + \lambda} \oplus \hat{V}_{-\varepsilon_1 + \lambda}. \quad (6.7)$$

All these blocks are parabolic Verma modules corresponding to the $U_\hbar(l)$-submodules of $\mathbb{C}^4$.

Clearly, $\lambda + \varepsilon_1$ is the highest weight of $\mathbb{C}^4 \otimes \hat{M}_\lambda$ and $w_1 \otimes v_\lambda$ is the highest weight vector. The other singular vectors in $\mathbb{C}^4 \otimes \hat{M}_\lambda$ are given next.

\begin{lemma}

The vectors

\begin{align*}
 u_{\varepsilon_1} &= w_1 \otimes v_\lambda, \\
 u_{\varepsilon_2} &= w_1 \otimes f_\alpha v_\lambda - q^{(\alpha,\lambda)} - q^{-\alpha,\lambda} w_2 \otimes v_\lambda, \\
 u_{-\varepsilon_1} &= w_1 \otimes f_\delta v_\lambda + \left( q^{(\lambda,\alpha)+1} + q^{-\lambda,\alpha-1} \right) \\
 &\quad \times \left( q w_2 \otimes f_\alpha v_\lambda - q^3 w_3 \otimes f_\alpha v_\lambda - q^4 q^{(\lambda,\alpha)} - q^{-\lambda,\alpha} \right) w_4 \otimes v_\lambda,
\end{align*}

are singular and generate the submodules $\hat{V}_{\varepsilon_1 + \lambda}, \hat{V}_{\varepsilon_2 + \lambda}, \hat{V}_{-\varepsilon_1 + \lambda}$, respectively.

\end{lemma}

\begin{proof}

One should check that $u_{\varepsilon_1}, u_{\varepsilon_2}, u_{-\varepsilon_1}$ are annihilated by $e_\alpha$ and $e_\beta$. That is obvious for $u_{\varepsilon_1}$ and relatively easy for $u_{\varepsilon_2}$. The case of $u_{-\varepsilon_1}$ requires a bulky but straightforward calculation, which is omitted here.

We denote by $V_{\varepsilon_1 + \lambda}, V_{\varepsilon_2 + \lambda}, V_{-\varepsilon_1 + \lambda}$ the images of $\hat{V}_{\varepsilon_1 + \lambda}, \hat{V}_{\varepsilon_2 + \lambda}, \hat{V}_{-\varepsilon_1 + \lambda}$ under the projection $\mathbb{C}^4 \otimes \hat{M}_\lambda \rightarrow \mathbb{C}^4 \otimes M_\lambda$, assuming $q^{2(\alpha,\lambda)} = -q^{-2}$. An important fact is that,
for \( q^{2(\alpha, \lambda)} = -q^{-2} \), the singular vector \( u_{-\xi_1} \) turns into \( w_1 \otimes f_\beta v_\lambda \) and thus disappears from \( \mathbb{C}^4 \otimes M_\lambda \). Hence, the submodule \( \tilde{V}_{-\xi_1+\lambda} \) is killed by the projection \( \mathbb{C}^4 \otimes M_\lambda \rightarrow \mathbb{C}^4 \otimes M_\lambda \), so \( V_{-\xi_1+\lambda} = \{0\} \).

**Proposition 6.2.** The module \( \mathbb{C}^4 \otimes M_\lambda \) is a direct sum of the submodules \( V_{\xi_1+\lambda} \) and \( V_{\xi_2+\lambda} \).

**Proof.** The modules \( V_{\xi_1+\lambda} \) and \( V_{\xi_2+\lambda} \) have zero intersection, as they carry different eigenvalues of the invariant matrix \( \Omega \), see below. We must show that the sum \( V = V_{\xi_1+\lambda} \oplus V_{\xi_2+\lambda} \) exhausts all of \( \mathbb{C}^4 \otimes M_\lambda \). To that end, it is sufficient to show that \( \mathbb{C}^4 \otimes v_\lambda \) lies in \( V \). Then for all \( u \in U_h(g) \) and all \( w \in \mathbb{C}^4 \)

\[
 w \otimes u v_\lambda = \Delta(u^{(2)})(\gamma^{-1}(u^{(1)})w \otimes v) \in V,
\]

as required.

In what follows \( \equiv \) will mean equality modulo \( V \). Obviously, \( w_1 \otimes v_\lambda \equiv 0 \). Applying \( f_\alpha \) to \( w_1 \otimes v_\lambda \) gives \( w_1 \otimes f_\alpha v_\lambda + q^{-(\alpha, \lambda)} w_2 \otimes v_\lambda \equiv 0 \). Comparing this with \( u_{\xi_2+\lambda} \in V \) we conclude that \( w_2 \otimes v_\lambda \equiv 0 \). Applying \( f_\beta \) to \( w_2 \otimes v_\lambda \) gives \( w_3 \otimes v_\lambda \equiv 0 \).

Thus, we are left to check that \( w_4 \otimes v \in V \). We have

\[
0 \equiv f_\alpha (w_1 \otimes v_\lambda) \equiv w_1 \otimes f_\alpha v_\lambda, \quad 0 \equiv f_\beta (w_2 \otimes v_\lambda) = w_2 \otimes f_\beta v_\lambda,
\]

\[
0 \equiv f_\alpha f_\beta (w_1 \otimes v_\lambda) \equiv f_\alpha (w_1 \otimes f_\beta v_\lambda) = w_1 \otimes f_\alpha f_\beta v_\lambda + q^{-(\alpha, \lambda)} w_2 \otimes f_\beta v_\lambda = w_1 \otimes f_\beta f_\alpha v_\lambda + q^{-(\alpha, \lambda)} w_2 \otimes f_\beta f_\alpha v_\lambda.
\]

(6.8)

Further,

\[
0 \equiv f_\beta f_\alpha (w_1 \otimes v_\lambda) \equiv f_\beta (w_1 \otimes f_\alpha v_\lambda) = w_1 \otimes f_\beta f_\alpha v_\lambda,
\]

(6.9)

Combining (6.8) and (6.9), we calculate \( f_\delta (w_1 \otimes v_\lambda) \in V \):

\[
0 \equiv (-q^2 + q^{-2}) f_\alpha f_\beta f_\alpha + f_\beta f_\alpha^2)(w_1 \otimes v_\lambda) = w_1 \otimes f_\delta v_\lambda + (q^2 + q^{-2}) q^{-(\alpha, \lambda)} w_2 \otimes f_\beta f_\alpha v_\lambda.
\]

The first equality takes place because \( f_\beta (w_1 \otimes v_\lambda) = 0 \). Therefore, \( w_2 \otimes f_\beta f_\alpha v_\lambda \equiv 0 \), and

\[
0 \equiv f_\beta (w_2 \otimes f_\alpha v_\lambda) = w_2 \otimes f_\beta f_\alpha v_\lambda + q^2 w_3 \otimes f_\alpha v_\lambda \equiv q^2 w_3 \otimes f_\alpha v_\lambda.
\]

Finally,

\[
0 \equiv f_\alpha (w_3 \otimes v_\lambda) = w_3 \otimes f_\alpha v_\lambda - q^{-(\alpha, \lambda)} w_4 \otimes v_\lambda \equiv -q^{-(\alpha, \lambda)} w_4 \otimes v_\lambda,
\]

as required. \( \square \)
Now consider the action of the matrix $Q$ on $\mathbb{C}^4 \otimes \hat{M}_\lambda$. It satisfies a cubic polynomial equation, and its eigenvalues in $\mathbb{C}^4 \otimes \hat{M}_\lambda$ can be found in [12]:

$$
q^{2(\lambda,\epsilon_1)} = -q^{-2}, \\
q^{2(\lambda+\rho,\epsilon_2) - 2(\rho,\epsilon_1)} = q^{2(\rho,\epsilon_2) - 2(\rho_1)} = q^{-2}, \\
q^{2(\lambda+\rho,-\epsilon_1) - 2(\rho,\epsilon_1)} = q^{-2(\lambda,\epsilon_1) - 4(\rho,\epsilon_1)} = -q^{-6}.
$$

The operator $Q$ is semisimple on $\mathbb{C}^4 \otimes \hat{M}_\lambda$ for generic $\lambda$. Due to Proposition 6.2, it is semisimple on $\mathbb{C}^4 \otimes M_\lambda$ and has the eigenvalues $\pm q^{-2}$.

### 7. Quantization of $S^4$

Let $\phi$ denotes the representation homomorphism $U_\hbar(\mathfrak{g}) \to \text{End}(\mathbb{C}^4)$. The $q$-trace of $Q$ is a weighted trace $\text{Tr}_q(Q) = \text{Tr}(DQ)$, where $D$ is the diagonal matrix diag$(q^4, q^2, q^{-2}, q^{-4})$. It belongs to the center of $U_\hbar(\mathfrak{g})$ and hence the center of $\mathbb{C}_h[G] \subset U_\hbar(\mathfrak{g})$.

A module of highest weight $\lambda$ defines a central character $\chi_\lambda$ of the algebra $\mathbb{C}_h[G]$, which returns zero on $\text{Tr}_q(Q)$:

$$
\chi_\lambda(\text{Tr}_q(Q)) = \text{Tr}(\phi(q^{h_\lambda + h_\rho})) = q^{2(\lambda,\rho,\epsilon_1)} + q^{2(\lambda+\rho,\epsilon_2)} + q^{2(\lambda+\rho,-\epsilon_2)} + q^{2(\lambda+\rho,-\epsilon_1)} \\
= q^{2(\lambda,\epsilon_1) + 4} + q^2 + q^{-2} + q^{-2(\lambda,\epsilon_1) - 4} = -q^2 + q^2 + q^{-2} - q^{-2} = 0,
$$

cf. [12]. Thus, the $q$-trace of the matrix $Q$ vanishes in $M_\lambda$. Also, the entries of the matrix $Q^2 - q^{-4}$ are annihilated in $\text{End}(M_\lambda)$, as discussed in the previous section.

**PROPOSITION 7.1.** The image of $\mathbb{C}_h[G]$ in $\text{End}(M_\lambda)$ is a quantization of $\mathbb{C}_h[S^4]$. It is isomorphic to the quotient of the subalgebra in $U_\hbar(\mathfrak{g})$ generated by the entries of $Q$, modulo the relations

$$
Q^2 = q^{-4}, \quad \text{Tr}_q(Q) = 0. \tag{7.10}
$$

**Proof.** The center of $\mathbb{C}_h[G]$ is formed by $U_\hbar(\mathfrak{g})$-invariants, which are also central in $U_\hbar(\mathfrak{g})$. Therefore, $\ker \chi_\lambda$ lies in the kernel of the representation $\mathbb{C}_h[G] \to \text{End}(M_\lambda)$. The quotient of $\mathbb{C}_h[G]$ by the ideal generated by $\ker \chi_\lambda$ is free over $\mathbb{C}[[h]]$ and is a direct sum of isotypical $U_\hbar(\mathfrak{g})$-components of finite multiplicities [11]. Therefore, the image of $\mathbb{C}_h[G]$ in $\text{End}(M_\lambda)$ is a direct sum of isotypical $U_\hbar(\mathfrak{g})$-components which are free and finite over $\mathbb{C}[[h]]$.

The ideal in $\mathbb{C}_h[G]$ generated by (7.1) lies in the kernel of the homomorphism $\phi: \mathbb{C}_h[G] \to \text{End}(M_\lambda)$ and turns into the defining ideal of $S^4$ modulo $\hbar$. Therefore, this ideal coincides with $\ker \phi$, and the quotient of $\mathbb{C}_h[G]$ by this ideal is a quantization of $\mathbb{C}[S^4]$, see [12] for details.

We will give a more explicit description of $\mathbb{C}_h[S^4]$. The matrix $Q$ is the image of the matrix $K$ from Section 4 under the embedding $\mathbb{C}_h[G] \to U_\hbar(\mathfrak{g})$. The algebra
\( \mathbb{C}_h[\mathbb{S}^4] \) is generated by elements \( a, b, c, y, z \) arranged in the matrix

\[
\begin{pmatrix}
  a & b & y & 0 \\
  c & -q^2a & 0 & -y \\
  z & 0 & -q^2a & q^2b \\
  0 & -z & q^2c & q^4a \\
\end{pmatrix}.
\]

This matrix is obtained from \( K \) by imposing the linear relations on its entries derived from (4.5) by the substitution \( K^2 = q^{-4}, \text{Tr}_q(K) = 0 \). The generators of \( \mathbb{C}_h[\mathbb{S}^4] \) obey the relations

\[
ab = q^2ba, \quad ac = q^{-2}ca, \quad ay = q^2ya, \quad az = q^{-2}za,
\]

\[
by = q^2yb, \quad bz = q^{-2}zb, \quad cy = q^2yc, \quad cz = q^{-2}zc,
\]

\[
[b, c] = (q^4 - 1)a^2, \quad [y, z] = (q^4 - 1)a^2 + (q^4 - 1)bc,
\]

plus

\[
a^2 + bc + yz = q^{-4},
\]

which is a deformation of (2.2).

Remark that \( \mathbb{C}_h[\mathbb{S}^4] \) has a 1-dimensional representation \( a, b, c \mapsto 0, y, z \mapsto q^{-2} \). Therefore, it can be realized as a subalgebra in the Hopf algebra dual to \( U_\hbar(\mathfrak{g}) \), as explained in [5].

8. On Invariant Star Product on \( \mathbb{S}^4 \)

It follows from [10] that the star product on the conjugacy class \( Sp(4)/Sp(2) \times Sp(2) \) can be calculated by means of the invariant pairing between the modules \( M^-_\lambda \) and \( M^+_\lambda \), where \( M^+_\lambda = M_\lambda \) and \( M^-_\lambda \) is its restricted dual. The module \( M^-_\lambda \) is the quotient of the lower parabolic Verma module \( \hat{M}^-_\lambda = U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{p}^-) \mathbb{C}_{-\lambda} \) by the submodule \( U_\hbar(\mathfrak{g})e_\delta v_{-\lambda} \). Explicitly, the pairing is given by the assignment

\[
xv_{-\lambda} \otimes yv_\lambda \mapsto \langle xv_{-\lambda}, yv_\lambda \rangle = \lambda([y(x)y]).
\]

Here, \( x \mapsto [x]_1 \) is the projection \( U_\hbar(\mathfrak{g}) \to U_\hbar(\mathfrak{l}) \) along \( U^+_\hbar(\mathfrak{n}^-)U_\hbar(\mathfrak{g}) + U_\hbar(\mathfrak{g})U^-_\hbar(\mathfrak{n}^+), \)

where the prime designates the kernel of the counit. This projection is facilitated by the triangular factorization (5.6) and it is a homomorphism of \( U_\hbar(\mathfrak{l}) \)-bimodules.

The modules \( M^\pm_{\pm\lambda} \) are irreducible if and only if the pairing is non-degenerate [8].

Our next goal is to calculate it explicitly. Put

\[
x_1 = e_\alpha, \quad x_2 = e_\gamma, \quad \bar{x}_1 = e_\alpha, \quad \bar{x}_2 = q^4e_\beta e_\alpha - q^2e_\alpha e_\beta, \quad y_1 = f_\alpha, \quad y_2 = f_\gamma.
\]

The twiddled root vectors are related to non-twiddled via the antipode:

\[
\gamma(\bar{e}_\gamma) = q^4 q^{-h_\beta} e_\beta q^{-h_\alpha} e_\alpha - q^2 q^{-h_\alpha} e_\alpha q^{-h_\beta} e_\beta = -q^{-h_\gamma} e_\gamma,
\]

with a similar relation for the root \( \alpha \).
The following system of monomials constitutes bases in $M^+_{-\lambda}$ and $M^-_{\lambda}$:

$$(y^k_1 y^m_2 v_\lambda)_{k,m=0}^\infty \subset M^+_{-\lambda}, \quad (\tilde{x}^k_1 \tilde{x}^m_2 v_\lambda)_{k,m=0}^\infty \subset M^-_{-\lambda}.$$ 

Further, we need the identities

$$[e_\alpha, f_\gamma] = -(q + q^{-1})q^{-2} f_\beta q^{-h_\alpha}, \quad [e_\beta, f_\gamma] = f_\alpha q^h_\beta, \quad (8.11)$$

$$[e_\gamma, f_\alpha] = -(q + q^{-1})e_\beta q^{h_\alpha}, \quad [e_\gamma, f_\beta] = q^2 e_\alpha q^{-h_\beta}, \quad (8.12)$$

which can be derived directly from the defining relations of $U_\hbar(g)$ and the definition of $e_\gamma$ and $f_\gamma$. Also, one can check that

$$[e_\gamma, f_\gamma] = q h_\gamma - q^{-h_\gamma}. \quad (8.13)$$

Therefore, for $\nu = \alpha, \gamma$ and any positive integer $k$ we have

$$[e_\nu, f_k \nu] = q^{h_\nu + 1} \frac{1 - q^{-2k}}{(q - q^{-1})^2} + q^{-h_\nu - 1} \frac{1 - q^{2k - 1}}{(q - q^{-1})^2}. \quad (8.14)$$

**LEMMA 8.1.** The matrix coefficient $\langle \tilde{x}^i_1 \tilde{x}^j_2 v_{-\lambda}, y^k_1 y^m_2 v_\lambda \rangle$ is zero unless $i = k, j = m$.

**Proof.** It follows that $[e_\alpha, f_k^k]$ belongs to the left ideal $U_\hbar(g) f_\beta$, hence $x_1 y^k_1 v_\lambda = 0$. We have

$$x_1 y^m_2 v_\lambda = e_\alpha f_k^k f_m^m v_\lambda = [e_\alpha, f_k^k] f_m^m v_\lambda + f_k^k [e_\alpha, f_m^m] v_\lambda \sim f_k^{k-1} f_m^m v_\lambda = y^k_1 y^m_2 v_\lambda.$$ 

Using this, we find

$$\langle \tilde{x}^i_1 \tilde{x}^j_2 v_{-\lambda}, y^k_1 y^m_2 v_\lambda \rangle = \langle v_{-\lambda}, x^j_1 x^i_2 y^k_1 y^m_2 v_\lambda \rangle \sim \langle v_{-\lambda}, x^j_1 x^i_2 y^m_2 v_\lambda \rangle = 0,$$

assuming $i > k$. If $i < k$, then $y_1$ is pulled to the left in a similar way, so the matrix coefficient is zero if $i \neq k$. If $i = k$ but $j \neq m$, then the total weight of the product $x^j_1 x^i_2 y^k_1 y^m_2$ is not zero and the matrix coefficient is nil. □

Define $[2l]_q!! = \prod_{i=1}^l [2l]_q$ for all positive integer $l$ and put $[0]_q!! = 1$.

**PROPOSITION 8.2.** The matrix coefficients of the invariant pairing are given by the formula

$$\langle \tilde{x}^k_1 \tilde{x}^m_2 v_{-\lambda}, y^k_1 y^m_2 v_\lambda \rangle = q^{m(m-2) + k(k-2)} \frac{1}{(q - q^{-1})^{k+m}} \left[2k]_q!![2m]_q!! \right.$$

for all $k, m = 0, 1, \ldots$.
Proof. The matrix coefficient \( \langle \tilde{x}^k \tilde{x}^m v_{-\lambda}, \tilde{y}^k \tilde{y}^m v_{\lambda} \rangle \) is equal to
\[
\langle v_{-\lambda}, (-q^{-h_{\gamma}} e_{\gamma})^m (-q^{-h_{\alpha}} e_{\alpha}) f_{\alpha} f_{\gamma}^m v_{\lambda} \rangle = c \langle v_{-\lambda}, e_{\gamma}^{m} e_{\alpha}^k f_{\alpha} f_{\gamma}^m v_{\lambda} \rangle,
\]
where \( c = (-1)^{k+m} q^{(k+m)(\alpha, \lambda)+m(m-1)+k(k-1)} \). Further, \( \langle v_{-\lambda}, e_{\gamma}^{m} e_{\alpha}^k f_{\alpha} f_{\gamma}^m v_{\lambda} \rangle \) is found to be
\[
\langle v_{-\lambda}, e_{\gamma}^{m} e_{\alpha}^k f_{\alpha}^k f_{\gamma}^m v_{\lambda} \rangle = \left( q^{h_{\alpha}+1} \frac{1-q^{-2k}}{(q-q^{-1})^2} + q^{-h_{\alpha}-1} \frac{1-q^{-2k}}{(q-q^{-1})^2} \right) f_{\gamma}^m v_{\lambda} + \ldots
\]
The omitted term is zero, as it involves \( e_{\alpha} f_{\gamma}^m v_{\lambda} = 0 \). We continue in this way and get
\[
[k]_q! \prod_{i=1}^k (\alpha, \lambda) + 1 - i] q [m]_q! \prod_{i=1}^m (\alpha, \lambda) + 1 - i] q = [2k]_q! [2m]_q! \left( \frac{q^{(\alpha, \lambda)} q}{q-q^{-1}} \right)^{k+m},
\]
since \([\alpha, \lambda] + 1 - i] q = \frac{q^{(\alpha, \lambda)+1-i-q^{-1}(\alpha, \lambda)-1+i}}{q-q^{-1}} = q^{(\alpha, \lambda)+1-i+q^i(q-q^{-1})]} q^{-i} = \frac{q^{(\alpha, \lambda)+1} [2i]_q}{(q-q^{-1})]} q^{-1} \]
Combining this result with the multiplier \( c \) calculated earlier and taking into account that \( q^2 (\alpha, \lambda) = -q^{-2} \) we prove the statement. \( \square \)

Let \( C_h[G_{DS}] \) denote the affine coordinate ring on the quantum group \( Sp_q(4) \), i.e. the quantization of the algebra of polynomial functions on \( G \) along the Drinfeld–Sklyanin bracket. It is the Hopf dual to the quantized universal enveloping algebra \( U(h) \), and the reader should not confuse it with \( C_h[G] \) defined in Section 4. It is known that the multiplication in \( C_h[G_{DS}] \), call it \( \cdot_h \), is a star product. Denote by \( C_h[G_{DS}]^l \) the subalgebra of \( U_h(l) \)-invariants in \( C_h[G_{DS}] \) under the left co-regular action. It is a natural right \( U_h(g) \)-module algebra under the right co-regular action and is also a quantization a closed conjugacy class (the coset space by the Levi subgroup).

The Shapovalov form on \( \hat{M}_\lambda \) is invertible for generic \( \lambda \), and its inverse provides an associative multiplication on \( C_h[G_{DS}]^l \). For the special value of \( \lambda \) as in Proposition 5.1, it has a pole, while its regular part still defines an associative multiplication on a certain subspace of \( C_h[G_{DS}]^l \), as argued in [10]. Here we describe this subspace. Denote by \( \mathfrak{t} \) the Lie algebra of the classical stabilizer \( Sp(2) \times Sp(2) \). For any \( U_h(g) \)-module \( V \), put \( V^\perp = V^\perp \cap \ker e_{\delta} \cap \ker f_{\delta}, \) where the quantum root vectors \( e_{\delta}, f_{\delta} \) are defined in Sect. 3. Note that \( U(\mathfrak{t}) \) is not quantized as a Hopf subalgebra in \( U_h(g) \), and we do not know any natural candidate for \( U(\mathfrak{t}) \) in \( U_h(g) \). Thus, \( V^\perp \) is just a definition, at this stage. Nevertheless, it can be proved that the \( \mathbb{C}[[h]] \)-module \( C_h[G_{DS}]^l \) is a (flat) deformation of \( \mathbb{C}[S^4] \simeq \mathbb{C}[G]^l \). The product
\[
f \otimes g \mapsto f \ast_h g = \sum_{k,m=0}^{\infty} q^{-m(m-2)-k(k-2)} \frac{(q-q^{-1})^{k+m}}{[2k]_q!! [2m]_q!!} (y^k y^m f) \cdot_h (x^k x^m g)
\]
belongs to \( C_h[G_{DS}]^l \) for all \( f, g \in C_h[G_{DS}]^l \). This multiplication makes \( C_h[G_{DS}]^l \) an associative algebra.
Remark that the product (8.15) is not perfectly explicit because the exact expression of $\mathcal{O}^2$ through the classical multiplication in $\mathbb{C}[G]$ is unknown. Also, the new multiplication should be isomorphic to $\cdot \hbar$, because $S^4$ has a unique structure of Poisson manifold over the Poisson group $G$. Therefore, neither (8.15) nor (7.1) is particularly new with regard to an abstract quantization. For instance, one can apply the method of characters (which is doable in the special case under consideration) and realize the quantized polynomial algebra on $S^4$ both as a quotient of $\mathbb{C}_\hbar[G]$ and as a subalgebra in $\mathbb{C}_\hbar[G_{DS}]$ [5]. Alternatively, one can quantize $S^4$ through the quantum plane, along the lines of [7]. The novelty of the present work is a realization of $\mathbb{C}[S^4]$ by operators on a highest weight module. This approach admits far reaching generalization that unites the non-Levi conjugacy classes and the classes with Levi isotropy subgroups in a common quantization context [12–14].

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