TRAPEZOIDAL NUMBERS, DIVISOR FUNCTIONS, AND A PARTITION THEOREM OF SYLVESTER

MEIVYN B. NATHANSON

To Krishnaswami Alladi on his 60th birthday

Abstract. A partition of a positive integer \( n \) is a representation of \( n \) as a sum of a finite number of positive integers (called parts). A trapezoidal number is a positive integer that has a partition whose parts are a decreasing sequence of consecutive integers, or, more generally, whose parts form a finite arithmetic progression. This paper reviews the relation between trapezoidal numbers, partitions, and the set of divisors of a positive integer. There is also a complete proof of a theorem of Sylvester that produces a stratification of the partitions of an integer into odd parts and partitions into disjoint trapezoids.

1. Partition theorems of Euler and Sylvester

Let \( \mathbb{N}, \mathbb{N}_0, \) and \( \mathbb{Z} \) denote, respectively, the sets of positive integers, nonnegative integers, and integers. A partition of a positive integer \( n \) is a representation of \( n \) as a sum of a finite number of positive integers (called parts), written in decreasing order. The usual left-justified Ferrers diagram of the partition

\[ n = a_1 + a_2 + \cdots + a_k \]

with

\[ a_1 \geq a_2 \geq \cdots \geq a_k \geq 1 \]

consists of \( k \) rows of dots, with \( a_i \) dots on row \( i \). For example, the Ferrers diagram of the partition

\[ 57 = 11 + 11 + 11 + 9 + 5 + 5 + 5 \]

is

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & & & & & & \\
\cdot & \cdot & & & & & & \\
\cdot & & & & & & & \\
\end{array}
\]

Perhaps the best known result about partitions is the following theorem of Euler.

**Theorem 1** (Euler). The number of partitions of \( n \) into odd parts equals the number of partitions of \( n \) into distinct parts.

\[ 2010 \text{ Mathematics Subject Classification.} \quad 05A17, \text{ 11P81, 11A05, 11B75.} \]

\[ \text{Key words and phrases.} \quad \text{Partitions, Sylvester, trapezoidal numbers, divisor functions.} \]
Proof. Let \( p_{\text{odd}}(n) \) denote the number of partitions of \( n \) into odd parts, and let \( p_{\text{dis}}(n) \) denote the number of partitions into distinct parts. A deceptively simple proof uses formal power series:

\[
\sum_{n=0}^{\infty} p_{\text{odd}}(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}} = \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{(1 - q^{2n-1})(1 - q^{2n})} = \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{1 - q^n} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} = \sum_{n=0}^{\infty} p_{\text{dis}}(n)q^n.
\]

This argument is valid only after one understands infinite products, inversion, and composition of formal power series. \( \square \)

Every positive integer \( n \) has a unique \( g \)-adic representation in the form \( n = \sum_{i=0}^{\infty} \varepsilon_i g^i \), where \( \varepsilon_i \in \{0,1,\ldots,g-1\} \) for \( i \in \mathbb{N}_0 \) and \( \varepsilon_i = 0 \) for all sufficiently large \( i \). Glaisher \([11]\) generalized Euler’s theorem by using the uniqueness of the \( g \)-adic representation. Theorem 1 is the special case \( g = 2 \).

**Theorem 2 (Glaisher).** Let \( g \geq 2 \). The number of partitions of \( n \) into parts not divisible by \( g \) equals the number of partitions of \( n \) such that every part occurs less than \( g \) times.

Proof. Every positive integer \( a \) can be written uniquely in the form \( a = g^vs \), where \( s \) is not divisible by \( g \). Sylvester calls \( s \) the *nucleus* of \( a \). A partition of \( n \) in which every part occurs at most \( g-1 \) times can be written uniquely in the form

\[
(1)
n = \varepsilon_1 a_1 + \cdots + \varepsilon_k a_k
\]

where the parts \( a_1, \ldots, a_k \) are pairwise distinct and \( \varepsilon_i \in \{1,\ldots,g-1\} \) for \( i = 1,\ldots,k \). Let

\[
a_i = g^{v_i} s_i = \underbrace{s_i + \cdots + s_i}_{g^{v_i} \text{ summands}}
\]

where \( s_i \) is the nucleus of \( a_i \). The nuclei \( s_1, \ldots, s_k \) are not necessarily distinct. Let \( S = \{s_1,\ldots,s_k\} \). For each \( s \in S \), let

\[
\delta(s) = \sum_{i \in \{1,\ldots,k\}, s_i = s} \varepsilon_i g^{v_i}.
\]

Then

\[
n = \varepsilon_1 a_1 + \cdots + \varepsilon_k a_k
= \varepsilon_1 g^{v_1} s_1 + \cdots + \varepsilon_k g^{v_k} s_k
= \underbrace{s_1 + \cdots + s_1}_{\varepsilon_1 g^{v_1} \text{ summands}} + \cdots + \underbrace{s_k + \cdots + s_k}_{\varepsilon_k g^{v_k} \text{ summands}}
= \sum_{s \in S} \left( \sum_{i \in \{1,\ldots,k\}, s_i = s} \varepsilon_i g^{v_i} \right) s
= \sum_{s \in S} \delta(s)s.
\]
Thus, from the partition (1) of $n$ into parts occurring less than $g$ times we have constructed a partition of $n$ as a sum of integers not divisible by $g$.

Conversely, let $n = \sum_{s \in S} \delta(s)s$ be a partition of $n$ with parts in a set $S$ of integers not divisible by $g$, and where each $s \in S$ has multiplicity $\delta(s)$. Consider the $g$-adic representation 

$$\delta(s) = \sum_{i \in I_s} \varepsilon_i g^i$$

where $\varepsilon_i \in \{1, \ldots, g-1\}$. If $(i_1, s_1) \neq (i_2, s_2)$, then $g^{i_1} s_1 \neq g^{i_2} s_2$ and so 

$$n = \sum_{s \in S} \delta(s)s = \sum_{s \in S} \sum_{i \in I_s} \varepsilon_i g^i s$$

is a partition of $n$ into distinct parts $g^i s$ with multiplicities at most $g - 1$. These two partition transformations are inverse maps, and establish a one-to-one correspondence between partitions into parts not divisible by $g$ and parts occurring with multiplicities less than $g$. □

Sylvester [21, sections 45–46] discovered and proved a different, very beautiful, and insufficiently known generalization of Euler’s theorem. We prove this theorem in Section 4.

2. TRAPEZOIDAL NUMBERS

For integers $k \in \mathbb{N}$, $t \in \mathbb{N}_0$, and $a \in \mathbb{Z}$, the finite arithmetic progression with length $k$, difference $t$, and first term $a$ is the set

$$\{a, a + t, a + 2t, \ldots, a + (k-1)t\}.$$  

The sum of this arithmetic progression is

$$s_{k,t}(a) = \sum_{i=0}^{k-1} (a + it) = ka + \frac{k(k-1)t}{2}.$$  

The integer $a$ is the smallest element of the set (2) because $t \geq 0$.

Let $t \in \mathbb{N}_0$. A positive integer $n$ is a $k$-trapezoid with difference $t$ if it is the sum of a finite arithmetic progression of integers of length $k$ and difference $t$, that is, if it can be represented in the form (3) for integers $k \in \mathbb{N}$, $t \in \mathbb{N}_0$, and $a \in \mathbb{Z}$. A trapezoid with difference $t$ is a $k$-trapezoid with difference $t$ for some $k \in \mathbb{N}$. A $k$-trapezoid is a $k$-trapezoid with difference 1. For example, every odd integer is a 2-trapezoid, because $2n - 1 = (n - 1) + n$. A trapezoid is an integer that is a $k$-trapezoid for some $k$, that is, an integer that can be represented as the sum of a strictly decreasing sequence of consecutive integers.

A $k$-trapezoid with difference $t$ is positive if $a \geq 1$ and nonpositive if $a \leq 0$. If $a$ is positive, then the Ferrers diagram of this partition of $n$ has a trapezoidal shape. For example, $32 = 11 + 9 + 7 + 5$ is a positive 4-trapezoid with difference 2. Its Ferrers diagram is

```
• • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • • •
Every positive integer \( n \) has a trivial positive trapezoidal representation with length 1 and difference 1, namely, \( n = n \). Sylvester [22] and Mason [16] proved that a positive integer \( n \) is a \( k \)-trapezoid for some \( k \geq 2 \) if and only if \( n \) is not a power of 2, and that the number of positive trapezoidal representations of \( n \) is exactly the number of odd positive divisors of \( n \). Bush [9] extended this result to trapezoidal representations with difference \( t \). We prove their theorems below.

In Section 4 we show how a special case of a partition theorem of Sylvester establishes another bijection between the number of trapezoidal representations of \( n \) and the number of positive odd divisors of \( n \).

For every positive integer \( n \), let \( \Phi_t(n) \) denote the number of representations of \( n \) as a trapezoid with difference \( t \), and let \( \Phi^+_t(n) \) denote the number of representations of \( n \) as a positive trapezoid with difference \( t \). Thus,

\[
\Phi_t(n) = \left| \{(k, a) \in \mathbb{N} \times \mathbb{Z} : s_{k,t}(a) = n\} \right|
\]

\[
\Phi^+_t(n) = \left| \{(k, a) \in \mathbb{N} \times \mathbb{N} : s_{k,t}(a) = n\} \right|
\]

For \( t = 1 \), these functions count partitions into consecutive integers.

Let \( d(n) \) denote the number of positive divisors of \( n \), and let \( d_1(n) \) denote the number of odd positive divisors of \( n \). Let \( d(n, \theta) \) denote the number of positive divisors \( d \) of \( n \) such that \( d < \theta \). If \( n/2 < k \leq n \), then

\[
d(n, k) = d(n, n) = d(n) - 1.
\]

Let \( [x] \) denote the integer part of the real number \( x \).

**Lemma 1.** Let \( t \) and \( n \) be positive integers. For every positive integer \( k \), there is at most one representation of \( n \) as a sum of a \( k \)-term arithmetic progression of integers with difference \( t \).

**Proof.** This is true because the function \( s_{k,t}(a) \) defined by (3) is a strictly increasing function of \( a \). \( \square \)

**Theorem 3.** Let \( t \) be an even positive integer. For every positive integer \( n \),

(4) \[ \Phi_t(n) = d(n) \]

and

(5) \[ \Phi^+_t(n) = d(n, \theta) \]

where

\[
\theta = \frac{1}{2} + \sqrt{\frac{2n}{t} + \frac{1}{4}}.
\]

**Proof.** For every positive divisor \( k \) of \( n \),

\[
a_{k,t}(n) = \frac{n}{k} - \frac{(k - 1)t}{2}
\]

is an integer and

\[
s_{k,t}(a_{k,t}(n)) = \sum_{i=0}^{k-1} \left( \frac{n}{k} - \frac{(k - 1)t}{2} + it \right) = n.
\]

Moreover, if \( k \) and \( d \) are distinct positive divisors of \( n \), then \( a_{k,t}(n) \neq a_{d,t}(n) \). Thus,

\[
d(n) \leq \Phi_t(n).
\]
Conversely, if \( n \) is the sum of a \( k \)-term arithmetic progression with even difference \( t \) and first term \( a \), then

\[
 n = s_{k,t}(a) = k \left( a + \frac{(k-1)t}{2} \right)
\]

and so \( k \) is a positive divisor of \( n \) and \( a = a_{k,t}(n) \). Thus, \( \Phi_t(n) \leq d(n) \), and so there is a one-to-one correspondence between the positive divisors of \( n \) and representations of \( n \) as a sum of a finite arithmetic progression with difference \( t \). This proves (4).

Let \( n = \sum_{i=0}^{k-1} (a + it) \). The first term \( a = a_{k,t}(n) \) is positive if and only if

\[
 \frac{n}{k} > \frac{(k-1)t}{2}
\]

or, equivalently,

\[
 k < \frac{1}{2} + \sqrt{\frac{2n}{t} + \frac{1}{4}}
\]

This proves (5). \( \Box \)

Lemma 2. Let \( t \) be an odd positive integer. Let \( n \) be a positive integer, and let \( s_{k,t}(a) = n \) for some integer \( a \) and some positive integer \( k \). If \( k \) is odd, then \( k \) is an odd positive divisor of \( n \). If \( k \) is even, then \( 2n/k \) is an odd positive divisor of \( n \).

Proof. If \( k \) is odd, then \((k - 1)/2\) is an integer and the identity

\[
 n = s_{k,t}(a) = ka + \frac{k(k-1)t}{2} = k \left( a + \frac{(k-1)t}{2} \right)
\]

implies that \( k \) is a positive divisor of \( n \).

If \( k \) is even, then \( d = 2a + (k - 1)t \) is odd and the identity

\[
 n = \frac{k}{2} (2a + (k - 1)t)
\]

implies that \( 2n/k = 2a + (k - 1)t \) is an odd positive divisor of \( n \). This completes the proof. \( \Box \)

Theorem 4. Let \( t \) be an odd positive integer. For every odd positive divisor \( k \) of \( n \), there is exactly one representation of \( n \) as a sum of a \( k \)-term arithmetic progression of integers with difference \( t \), and there is exactly one representation of \( n \) as a sum of a \((2n/k)\)-term arithmetic progression of integers with difference \( t \).

The number of representations of \( n \) as a \( t \)-trapezoid is

\[
 \Phi_t(n) = 2d_1(n).
\]

Proof. Let \( k \) be a odd positive divisor of \( n \), and let \( n = kq \). If \( k = 2e + 1 \), then

\[
 n = \sum_{i=-e}^{e} (q + it)
\]

is a representation of \( n \) as a sum of an arithmetic progression with difference \( t \), length \( k \), and first term

\[
 a_{k,t}(n) = q - et = \frac{n}{k} - \frac{(k-1)t}{2}.
\]
Let
\begin{equation}
(8) \quad b_{k,t}(n) = \frac{n}{2q} - \frac{(2q-1)t}{2} = \frac{k + t}{2} - \frac{nt}{k}.
\end{equation}

Then \( b_{k,t}(n) \) is an integer, and
\begin{equation}
(9) \quad n = \sum_{i=0}^{2q-1} (b_{k,t}(n) + it)
\end{equation}
is a representation of \( n \) as a sum of an arithmetic progression with difference \( t \), length \( 2q = 2n/k \), and first term \( b_{k,t}(n) \). Applying Lemma 2 we see that there is a one-to-one correspondence between the odd positive divisors of \( n \) and the representations of \( n \) as a sum of an arithmetic progression with difference \( t \) and odd length, and there is also a one-to-one correspondence between the odd positive divisors of \( n \) and the representations of \( n \) as a sum of an arithmetic progression with difference \( t \) and even length. This completes the proof.

For example, the only odd positive divisor of 1 is 1, and so \( \Phi_t(1) = 2d_1(1) = 2 \). The two representations of 1 as a sum of a finite arithmetic progression with odd difference \( t \) are \( 1 = 1 \) and
\[
1 = \left(\frac{1-t}{2}\right) + \left(\frac{1+t}{2}\right).
\]

The only odd positive divisor of 2 is 1, and so \( \Phi_t(2) = 2d_1(1) = 2 \). The two representations of 2 as a sum of a finite arithmetic progression with odd difference \( t \) are \( 1 = 1 \) and
\[
2 = \left(\frac{1-3t}{2}\right) + \left(\frac{1-t}{2}\right) + \left(\frac{1+t}{2}\right) + \left(\frac{1+3t}{2}\right).
\]

The trapezoidal representations with odd difference \( t \) of an odd prime \( p \) are
\[
p = \frac{p - t}{2} + \frac{p + t}{2} = \sum_{i=0}^{p-1} \left( 1 + \frac{(2i - p + 1)t}{2} \right) = \sum_{i=0}^{2p-1} \frac{1 + (2i - 2p + 1)t}{2}.
\]

Thus, the four trapezoidal representations with difference 3 of the prime 5 are
\[
5 = 1 + 4 = (-5) + (-2) + 1 + 4 + 7 = (-13) + (-10) + (-7) + (-4) + (-1) + 2 + 5 + 8 + 11 + 14.
\]

**Theorem 5.** For every positive integer \( n \),
\[
\Phi_t^+(n) = d_1(n).
\]

In particular, \( \Phi_t^+(n) = 1 \) if and only if \( n \) is a power of 2.

Equivalently, the positive integer \( n \) is a sum of \( k \geq 2 \) consecutive positive integers if and only if \( n \) is not a power of 2.
Proof. Let \( k \) be an odd positive divisor of \( n \). The identities
\[
a_{k,1}(n) = \frac{n}{k} - \frac{(k-1)}{2} \quad \text{and} \quad b_{k,1}(n) = \frac{k+1}{2} - \frac{n}{k}
\]
imply that
\[
a_{k,1}(n) + b_{k,1}(n) = 1
\]
and so exactly one of the integers \( a_{k,1}(n) \) and \( b_{k,1}(n) \) is positive. Thus, for each odd positive divisor \( k \) of \( n \) there is exactly one sequence of consecutive positive integers that sums to \( n \). This proves that \( \Phi^+_1(n) = d_1(n) \). \( \square \)

**Theorem 6.** For every odd positive integer \( t \), let
\[
\theta_t(n) = \sqrt{\frac{2nt}{t} + \frac{1}{4} + \frac{1}{2}} \quad \text{and} \quad \psi_t(n) = \sqrt{2nt + \left(\frac{t-2}{2}\right)^2 - \left(\frac{t-2}{2}\right)}.
\]
The number of representations of \( n \) as a positive trapezoid with difference \( t \) is
\[
(10) \quad \Phi^+_t(n) = d_1(n) + d_1(n, \theta_t(n)) - d_1(n, \psi_t(n)).
\]

**Proof.** Let \( k \) be an odd divisor of \( n \). All of the summands in the length \( k \) representation (6) are positive if and only if \( a_{k,t}(n) > 0 \), or, equivalently, \( k < \theta_t(n) \). The number of such divisors is \( d_1(n, \theta_t(n)) \).

All of the summands in the length \( 2n/k \) representation (9) are positive if and only if \( b_{k,t}(n) > 0 \) or, equivalently, \( k \geq \psi_t(n) \). The number of such divisors is \( d_1(n) - d_1(n, \psi_t(n)) \). This completes the proof. \( \square \)

Note that if \( t = 1 \), then \( \theta_1(n) = \psi_1(n) \), and so, for every odd divisor \( k \) of \( n \), exactly one of the inequalities \( k < \theta_1(n) \) and \( k \geq \psi_1(n) \) will hold. This gives another proof that \( \Phi^+_1(n) = d_1(n) \).

In a *Comptes Rendus* note in 1883, Sylvester [22] proved that “...le nombre de suites de nombres consécutifs dont la somme est \( N \) est égal au nombre de diviseurs impairs de \( N \).” This result (Theorem 5) has been rediscovered many times. A special case is in *Number Theory for Beginners* [25] by André Weil: Problem III.4 is to prove that an “integer \( > 1 \) which is not a power of 2 can be written as the sum of 2 or more consecutive integers.”

MacMahon [15, vol. 2, p. 28] used generating functions to prove Theorem 5. Here is a nice generalization. Let \( \Phi^+_1,0(n) \) (resp. \( \Phi^+_1,1(n) \)) denote the number of representations of \( n \) as the sum of an even (resp. odd) number of consecutive positive integers. Thus,
\[
\Phi^+_1(n) = \Phi^+_1,0(n) + \Phi^+_1,1(n).
\]
Andrews, Jiménez-Urroz, and Ono [7] proved analytically that
\[
\Phi^+_1,0(n) - \Phi^+_1,1(n) = d(n, \sqrt{2n}) - d(n, \sqrt{(n/2)}).
\]
Chapman [10] gave a combinatorial proof of this result.

3. **Hook numbers and the Durfee square**

Before describing Sylvester’s algorithm, we recall some properties of the Durfee square of a partition of a positive integer \( n \). Let
\[
n = r_1 + \cdots + r_k
\]
be a partition of $n$ into $k$ positive and decreasing parts. We have $r_1 \geq 1$. Let $s$ be the greatest integer such that $r_s \geq s$. The square array of $s^2$ dots in the upper left corner of the Ferrers graph is called the Durfee square of the partition, and the positive integer $s$ is the side of the Durfee square. If $s + 1 \leq i \leq k$, then $r_1 \leq r_{s+1} \leq s$ and all of the dots on the $i$th row of the Ferrers graph lie on the first $s$ columns of the graph. It follows that every dot in the Ferrers graph lies on one of the first $s$ rows or on one of the first $s$ columns of the graph. Therefore, the row numbers $r_1, \ldots, r_s$ and the column numbers $c_1, \ldots, c_s$ determine the partition. We extend this observation as follows.

**Lemma 3.** Let $s_1$ and $s_2$ be positive integers, and let $(r_i)_{i=1}^{s_1}$ and $(c_j)_{j=1}^{s_2}$ be sequences of integers such that

$$r_1 \geq r_2 \geq \cdots \geq r_{s_1} \geq s_2$$

and

$$c_1 \geq c_2 \geq \cdots \geq c_{s_2} \geq s_1.$$

The positive integer

$$n = \sum_{i=1}^{s_1} r_i + \sum_{j=1}^{s_2} c_j - s_1 s_2$$

has a unique partition with parts $r_1, \ldots, r_{s_1}, r_{s_1}+1, \ldots, r_{c_1}$, where, for $i = s_1 + 1, \ldots, c_1$,

$$r_i = \max(j : c_j \geq i).$$

If $s_1 = s_2 = s$, then the Durfee square of this partition has side $s$, and the row numbers $r_1, \ldots, r_s$ and column numbers $c_1, \ldots, c_s$ determine the partition.

**Proof.** Note that

$$n = \sum_{i=1}^{s_1} r_i + \sum_{j=1}^{s_2} (c_j - s_1) \geq \sum_{i=1}^{s_1} r_i \geq s_1 s_2.$$

Construct the Ferrers diagram with $r_i$ dots on row $i$ for $i = 1, \ldots, s_1$, and with $c_j$ dots on column $j$ for $j = 1, \ldots, s_2$. The Ferrers diagram has $c_1$ rows, and so the partition of $n$ has $c_1$ parts. For $i = s_1 + 1, \ldots, c_1$, there is a dot on the $j$th column of row $i$ if and only if $j \leq s_2$ and $c_j \geq i$. Therefore, $r_i = \max(j : c_j \geq i) \leq s_2$.

If $s_1 = s_2 = s$, then $r_{s_1+1} = r_{s_1}+1 \leq s_2 \leq r_{s_1} = r_s$, and so this partition has a Durfee square with side $s$. This completes the proof.

The upper left corner of a Ferrers diagram of a partition contains a unique minimal square array of dots (the Durfee square) whose rows and columns determine the partition. The upper left corner of a Ferrers diagram also contains minimal rectangular arrays of dots whose rows and columns determine the partition. The Ferrers diagram contains a “Durfee rectangle” with sides $(s_1, s_2)$ if

$$r_{s_1+1} \leq s_2 \leq r_{s_1} \quad \text{and} \quad c_{s_2+1} \leq s_1 \leq c_{s_2}.$$

These Durfee rectangles are not unique. For example, the partition

$$23 = 5 + 5 + 4 + 3 + 3 + 2 + 1$$

has Durfee square of side 3, and Durfee rectangles of sides $(s_1, s_2) = (2, 4)$ and $(s_1, s_2) = (5, 2)$.

For $1 \leq i \leq k$ and $1 \leq j \leq r_i$, let $R_{i,j}$ be the set of dots on the $i$th row that are on and to the right of the $j$th dot, and let $C_{i,j}$ be the set of dots on the $j$th column.
that are on and below the $i$th dot. The $(i,j)$th hook number is the cardinality of the set $H_{i,j} = R_{i,j} \cup C_{i,j}$. The number of dots on row $i$ is $r_i = |R_{i,1}|$. Denote the number of dots on column $j$ by $c_j = |C_{1,j}|$. We obtain
\[|H_{i,j}| = r_i + c_j - i - j + 1.\]

For $i = 1, \ldots, s$, we define the diagonal hook number
\[h_i = |H_{i,i}| = r_i + c_i - 2i + 1.\]
The set of diagonal hooks $\{H_{i,i} : i = 1, \ldots, s\}$ partitions the dots in the Ferrers diagram and produces the hook partition of $n$:
\[n = h_1 + h_2 + \cdots + h_s.\]

Lemma 4. Let $n = r_1 + \cdots + r_k$ be a partition of $n$, let $s$ be the side of the Durfee square of the Ferrers diagram of this partition, and let $h = h_1 + \cdots + h_s$ be the associated hook partition of $n$. For $i = 1, \ldots, s - 1$,
\[h_i - h_{i+1} \geq 2\]
and
\[h_i - h_{i+1} = 2\]
if and only if $r_i = r_{i+1}$ and $c_i = c_{i+1}$.

Proof. For $i = 1, \ldots, s - 1$ we have
\[h_i - h_{i+1} = (r_i + c_i - 2i + 1) - (r_{i+1} + c_{i+1} - 2i - 1)\]
\[= (r_i - r_{i+1}) + (c_i - c_{i+1}) + 2\]
\[\geq 2.\]
Moreover, $h_i - h_{i+1} = 2$ if and only if $r_i = r_{i+1}$ and $c_i = c_{i+1}$. \qed

For example, the partition into odd parts
\[57 = 11 + 11 + 11 + 9 + 5 + 5 + 5\]
has the left-justified Ferrers graph

```
  •
  • • • • • • • • • •
  • • • • • • • • • •
  • • • • • • • • • •
  • • • • • • • • • •
  • • • • • • • • • •
  • • • • • • • • • •
  • • • • • • • • • •
  • • • • • • • • • •
  • • • • • • • • • •
```

We have $5 = r_5 = r_6 < 6$ and so the Durfee square has side 5 contains $5^2 = 25$ dots. The hook partition is
\[57 = 17 + 15 + 13 + 9 + 3.\]

Note that the hook partition of a partition does not determine the partition. For example, the partitions $5 + 2$ and $4 + 2 + 1$ both have Durfee squares of side 2 and hook partitions $6 + 1$. 
Theorem 7. The number of partitions of \( n \) into exactly \( k \) parts differing by at least 2 is the number of partitions of \( n - k^2 \) into at most \( k \) parts.

Proof. The first construction converts a partition of \( n - k^2 \) into at most \( k \) parts into a partition of \( n \) into exactly \( k \) parts differing by at least 2. Let \( n > k^2 \), and let \( n - k^2 = \sum_{i=1}^{k} b_i \) be a partition with \( 1 \leq r \leq k \) and \( b_1 \geq \cdots \geq b_r \). For \( r + 1 \leq i \leq k \) we define \( b_i = 0 \), and for \( i = 1, \ldots, k \) we define \( a_i = b_i + 2(k - i) + 1 \).

It follows that
\[
a_i - a_{i+1} = (b_i + 2(k - i) + 1) - (b_{i+1} + 2(k - i - 1) + 1) = b_i - b_{i+1} + 2 \geq 0
\]
for \( i = 1, \ldots, k - 1 \). The identity
\[
k^2 = \sum_{i=1}^{k} (2i - 1) = \sum_{i=1}^{k} (2(k - i) + 1)
\]
implies that
\[
n = (n - k^2) + k^2 = \sum_{i=1}^{k} (b_i + 2(k - i) + 1) = \sum_{i=1}^{k} a_i.
\]
This is a partition of \( n \) into exactly \( k \) parts differing by at least 2.

The second construction converts a partition of \( n \) into exactly \( k \) parts differing by at least 2 into a partition of \( n - k^2 \) into at most \( k \) parts. Let \( n = \sum_{i=1}^{k} a_i \) be a partition of \( n \) into exactly \( k \) parts differing by at least 2. We have \( a_k \geq 1 \). If \( 1 \leq i \leq k - 1 \) and \( a_{i+1} \geq 2(k - (i + 1)) + 1 \), then
\[
a_i \geq a_{i+1} + 2 \geq (2(k - (i + 1)) + 1) + 2 = 2(k - i) + 1.
\]
It follows by downward induction that \( a_i \geq 2(k - i) + 1 \) and so
\[
b_i = a_i - (2(k - i) + 1) \geq 0
\]
for \( i = 1, \ldots, k \). We have
\[
\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} (2(k - i) + 1) = n - k^2.
\]
This is a partition of \( n - k^2 \) into at most \( k \) parts.

It is straightforward to check that the first and second constructions are inverses of each other. This completes the proof. \( \square \)

Consider a partition of \( n \) whose Ferrers diagram has Durfee square of side \( s \). Let \( r_1, \ldots, r_s \) be the number of dots on the first \( s \) rows of the Ferrers diagram, and let \( c_1, \ldots, c_s \) be the number of dots on the first \( s \) columns. The Frobenius symbol of the partition is the \( 2 \times s \) matrix
\[
\begin{pmatrix}
  r_1 - 1 & r_2 - 2 & \cdots & r_s - s \\
  c_1 - 1 & c_2 - 2 & \cdots & c_s - s
\end{pmatrix}.
\]
Note the rows are strictly decreasing sequences of nonnegative integers, and that
\[
n = s + \sum_{i=1}^{s} (r_i - 1) + \sum_{i=1}^{s} (c_i - 1).
\]
The Frobenius symbol is related to the construction in Lemma 3. See Andrews [3, 4].

4. Sylvester's algorithm

Sylvester discovered a graphical algorithm, sometimes called the fish-hook method, that transforms a partition of \( n \) with odd parts into a partition of \( n \) with distinct parts, and showed that this transformation is a bijection between the set of partitions into odd parts and the set of partitions into distinct parts. Moreover, he proved that this transformation has the extraordinary property that if the original partition of \( n \) into odd parts contains exactly \( \ell \) different odd integers, then the new partition of \( n \) into distinct parts contains exactly \( \ell \) maximal subsequences of consecutive integers.

Here is the algorithm. Let

\[ n = a_1 + \cdots + a_k \]

be a partition of \( n \) into odd parts, with

\[ a_1 \geq \cdots \geq a_k \geq 1 \]

and

\[ a_i = 2r_i - 1 \]

for \( i = 1, \ldots, k \). Then

\[ r_1 \geq \cdots \geq r_k \geq 1. \]

Because the summands \( a_i \) are odd, we can draw a center-justified Ferrers diagram, and divide it into two sub-diagrams. The major right half consists of the vertical central line and the dots to its right. The minor left half consists of the dots that are strictly to the left of the central line. We compute the hook numbers of the major half, and denote them in decreasing order by \( h_1 > h_3 > h_5 > \cdots \). We compute the hook numbers of the minor half, and denote them in decreasing order by \( h_2 > h_4 > h_6 > \cdots \). We shall prove that \( h_1 > h_2 > h_3 > h_4 > h_5 > h_6 \cdots \), and so the hook numbers create a partition of \( n \) into distinct parts.

Before proving this statement, we consider an example:

\[ 57 = 11 + 11 + 11 + 9 + 5 + 5 + 5 \]

is a partition into odd parts. The center-justified Ferrers diagram is
The major half is the Ferrers diagram of the partition $32 = 6 + 6 + 5 + 3 + 3 + 3$:

\[ \begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array} \]

The remainder of the original Ferrers diagram is the minor half, associated with the partition $25 = 5 + 5 + 5 + 4 + 2 + 2 + 2$:

\[ \begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array} \]

which we rearrange as the Ferrers diagram of the partition $5 + 5 + 5 + 4 + 2 + 2 + 2$:

\[ \begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array} \]

Note that deleting the first column of the major half produces the minor half.

The Durfee square of the major half consists of $4^2 = 16$ vertices. Every dot in this diagram lies on one of the first four rows or on one of the first four columns. We partition the vertices of the major half into the four hooks of the Durfee square and obtain the hook partition

\[ 32 = 12 + 10 + 8 + 2. \]

The minor left half is the major half with the left column removed, and the Durfee square of the minor half also consists of 16 vertices. Separating the minor
half into hooks, we obtain

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

with hook partition

\[25 = 11 + 9 + 4 + 1.\]

Notice that not only are the parts in the hook partitions strictly decreasing, but they are also interlaced in magnitude. Their union gives a partition of 57 into distinct parts:

\[57 = 12 + 11 + 10 + 9 + 8 + 4 + 2 + 1.\]

Thus, the original partition with odd parts has been transformed into a partition with distinct parts. We also observe that the original partition of 57 used only the three odd integers 11, 9, and 5, and that the new partition of 57 into distinct parts consists of three maximal decreasing sequences of consecutive integers: (12, 11, 10, 9, 8), (4), and (2, 1).

MacMahon \[15, \text{vol. 2, pp. 13–14}\] contains a description of Sylvestre’s fish-hook method. Andrews \[3, \text{Section 4}\] uses the Frobenius symbol of a partition to explain the fish-hook method.

5. \textbf{Sylvester’s proof of Euler’s theorem}

\textbf{Theorem 8.} Let \(n\) be a positive integer, let \(U(n)\) be the set of all partitions of \(n\) into odd parts, and \(V(n)\) be the set of all partitions of \(n\) into distinct parts. The function \(f : U(n) \to V(n)\) defined by Sylvester’s algorithm is a bijection.

\textit{Proof.} Consider a partition of \(n\) into \(k\) odd parts of the form \((12) - (15)\). Let \(s\) be the side of the Durfee square of the major half. Every dot in the major half lies on one of the first \(s\) rows or on one of the first \(s\) columns. For \(i = 1, \ldots, s\), the number \(r_i\) of dots on the \(i\)th row of the major half satisfies

\[r_1 \geq \cdots \geq r_s \geq s \geq r_{s+1}.\]

For \(i = 1, \ldots, r_1\), let \(c_i\) be the number of dots in the \(i\)th column of the major half. Note that \(r_{s+1} \leq s\) implies that \(c_{s+1} \leq s\), and so

\[k = c_1 \geq \cdots \geq c_s \geq s \geq c_{s+1}.\]

For \(i = 1, \ldots, s\), we have the hook numbers

\[h_i = r_i + c_i - 2i + 1.\]  

By Lemma it is shown that these numbers satisfy

\[h_i - h_{i+1} \geq 2 \text{ for } i = 1, \ldots, s - 1.\]

The minor half of the original Ferrers diagram is exactly the major half with the first column removed. Therefore, every dot in the minor half lies on one of the first \(s\) rows of the minor half or on one of the first \(s - 1\) columns of the graph. For \(i = 1, \ldots, r_2\), let \(r_i' = r_i - 1\) denote the number of dots on the \(i\)th row of the minor half. For \(i = 1, \ldots, r_1'\), let \(c_i'\) denote the number of dots on the \(i\)th column of the minor half.
Let $s'$ be the side of the Durfee square of the minor half. Because
\[ r'_{s-1} \geq r'_s = r_s - 1 \geq s - 1 \geq r_{s+1} - 1 = r'_{s+1} \]
it follows that $s' = s - 1$ or $s' = s$. Moreover, $s' = s$ if and only if $r_s \geq s + 1$ and $c_{s+1} = s$. Similarly, $s' = s - 1$ if and only if $r_s = s$ and $c_{s+1} = s - 1$.

For $i = 1, \ldots, s'$, there are the hook numbers
\[(17) \quad h'_i = r'_i + c'_i - 2i + 1 = r_i + c_{i+1} - 2i. \]
By Lemma 4, we have $h'_i - h'_i+1 \geq 2$ for $i = 1, \ldots, s' - 1$.

If $s' = s$, then $r_s \geq s + 1$ and $c_{s+1} = s$, and so
\[ h'_s = r_s + c_{s+1} - 2s = r_s - s. \]
If $s' = s - 1$, then $r_s = s$. We define $h'_s = 0$, and again have
\[ h'_s = r_s - s \]
and
\[ h'_{s-1} - h'_s = h'_{s-1} = r_{s-1} + c_s - 2s + 2 \]
\[ \geq (r_s - s) + (c_s - s) + 2 \]
\[ \geq 2. \]

We shall prove that
\[ h_1 > h'_1 > h_2 > h'_2 > \cdots > h_s > h'_s \geq 0. \]
For $i = 1, \ldots, s - 1$, we have
\[ h_i - h'_i = (r_i + c_i - 2i + 1) - (r'_i + c'_i - 2i + 1) \]
\[ = (r_i - r'_i) + (c_i - c'_i) \]
\[ = 1 + c_i - c_{i+1} \]
\[ \geq 1. \]
Also,
\[ h_s - h'_s = (r_s + c_s - 2s + 1) - (r_s - s) \]
\[ = c_s - s + 1 \geq 1. \]
For $i = 1, \ldots, s - 1$, we have
\[ h'_i - h_{i+1} = (r'_i + c'_i - 2i + 1) - (r_{i+1} + c_{i+1} - 2i - 1) \]
\[ = (r_i - 1 + c_{i+1} - 2i + 1) - (r_{i+1} + c_{i+1} - 2i - 1) \]
\[ = r_i - r_{i+1} + 1 \]
\[ \geq 1. \]
Therefore,
\[(18) \quad n = h_1 + h'_1 + h_2 + h'_2 + \cdots + h_{s-1} + h'_{s-1} + h_s + h'_s \]
is a partition into $2s$ or $2s - 1$ distinct positive parts, and we have transformed a partition with only odd parts to a partition into distinct parts. We shall prove that this transformation is one-to-one and onto.

Consider a partition of $n$ into $2s$ distinct nonnegative parts:
\[ n = h_1 + h'_1 + h_2 + h'_2 + \cdots + h_s + h'_s \]
where
\[ h_1 > h'_1 > h_2 > h'_2 > \cdots > h_{s-1} > h'_{s-1} > h_s > h'_s \geq 0. \]
If the number of positive parts is even, then \( h'_s \geq 1. \) If the number of positive parts is odd, then \( h'_s = 0. \)

If this partition is constructed by Sylvester’s algorithm from a partition of \( n \) into odd parts, then there are positive integers \( r_1, r_2, \ldots, r_s \) and \( c_1, c_2, \ldots, c_s \) such that
\[ h_1 = r_1 + c_1 - 1 \]
\[ h'_1 = r_1 + c_2 - 2 \]
\[ \vdots \]
\[ h_i = r_i + c_i - (2i - 1) \]
\[ h'_i = r_i + c_{i+1} - 2i \]
\[ \vdots \]
\[ h_s = r_s + c_s - (2s - 1) \]
\[ h'_s = r_s - s. \]

Conversely, given the \( 2s \) parts \( h_1, h'_1, \ldots, h'_s, \) we can solve these \( 2s \) equations recursively, and obtain unique integers \( r_1, \ldots, r_s, c_1, \ldots, c_s. \) For \( i = 1, \ldots, s, \) the inequality \( h_i > h'_i \) implies that
\[ r_i + c_i - (2i - 1) > r_i + c_{i+1} - 2i \]
and so
\[ c_i \geq c_{i+1} \]
For \( i = 1, \ldots, s - 1, \) the inequality \( h'_i > h_{i+1} \) implies that
\[ r_i + c_{i+1} - 2i > r_{i+1} + c_{i+1} - (2i + 1) \]
and so
\[ r_i \geq r_{i+1}. \]
Because
\[ r_s = h'_s + s \geq s \]
and
\[ c_s = h_s - r_s + 2s - 1 \]
\[ = h_s - (h'_s + s) + 2s - 1 \]
\[ = h_s - h'_s + s - 1 \]
\[ \geq s \]
it follows that \( r_1 \geq \cdots \geq r_s \geq s \) and \( c_1 \geq \cdots \geq c_s \geq s \) are decreasing sequences of positive integers.

Thus, every partition into odd parts determines a unique partition into distinct parts, and every partition into distinct parts can be obtained uniquely from a partition into odd parts. \( \square \)

For example, consider the partition
\[ 50 = 22 + 17 + 8 + 3. \]
We have $d = 2$ and
\begin{align*}
22 &= r_1 + c_1 - 1 \\
17 &= r_1 + c_2 - 2 \\
8 &= r_2 + c_2 - 3 \\
3 &= r_2 - 2.
\end{align*}

Solving these equations, we obtain
\begin{align*}
r_2 &= 5 \\
c_2 &= 6 \\
r_1 &= 13 \\
c_1 &= 10.
\end{align*}

Thus, the major half has 10 rows, of lengths
\begin{align*}
r_1 &= 13 \\
r_2 &= 5 \\
r_i &= 2 \quad \text{for } i = 3, \ldots, 6 \\
r_i &= 1 \quad \text{for } i = 7, \ldots, 10.
\end{align*}

Defining $a_i = 2r_i - 1$ for $i = 1, \ldots, 10$, we obtain the following partition of 50 into odd parts:

$$50 = 25 + 9 + 3 + 3 + 3 + 3 + 1 + 1 + 1 + 1.$$ 

Note that the partition into distinct parts consists of four maximal sequences of consecutive integers, and that the corresponding partition into odd parts contains four distinct odd numbers.

Here is another example:

$$31 = 9 + 8 + 7 + 4 + 3.$$ 

We have $d = 3$ and
\begin{align*}
9 &= r_1 + c_1 - 1 \\
8 &= r_1 + c_2 - 2 \\
7 &= r_2 + c_2 - 3 \\
4 &= r_2 + c_3 - 4 \\
3 &= r_3 + c_3 - 5 \\
0 &= r_3 - 3.
\end{align*}

Solving these equations, we obtain
\begin{align*}
r_3 &= 3 \\
c_3 &= 5 \\
r_2 &= 3 \\
c_2 &= 7 \\
r_1 &= 3 \\
c_1 &= 7.
\end{align*}
Thus, the major half has 7 rows, of lengths
\[ r_i = 3 \quad \text{for} \quad i = 1, 2, 3, 4 \]
\[ r_i = 2 \quad \text{for} \quad i = 5, 6, 7. \]

Defining \( a_i = 2r_i - 1 \) for \( i = 1, \ldots, 7 \), we obtain the following partition of 31 into odd parts:
\[ 31 = 5 + 5 + 5 + 5 + 5 + 3 + 3. \]

Note that the partition into distinct parts consists of two maximal sequences of consecutive integers, and that the corresponding partition into odd parts contains two distinct odd numbers.

Another example: The partition into distinct parts
\[ 30 = 10 + 8 + 7 + 4 + 1 \]
is mapped to the following partition into odd parts:
\[ 30 = 9 + 9 + 5 + 3 + 3 + 1. \]

6. Sylvester’s stratification of Euler’s theorem

For every positive integer \( n \), let \( p_{\text{odd}}(n) = |U(n)| \), where \( U(n) \) is the set of partitions of \( n \) into not necessarily distinct odd parts. Let \( p_{\text{dis}}(n) = |V(n)| \), where \( V(n) \) is the set of partitions of \( n \) into distinct parts. Euler proved (Theorem 1) that these two sets have the same cardinality, that is, \( p_{\text{odd}}(n) = p_{\text{dis}}(n) \). In the proof of Theorem 8 we proved that the function \( f : U(n) \to V(n) \) defined by Sylvester’s algorithm is a bijection.

For positive integers \( n \) and \( \ell \), let \( U_{\ell}(n) \) denote the set of partitions of \( n \) into not necessarily distinct odd parts with exactly \( \ell \) distinct odd parts, and let \( U_{\ell}(n) = |U_{\ell}(n)| \). We have
\[ p_{\text{odd}}(n) = \sum_{\ell=1}^{\infty} U_{\ell}(n). \]

Similarly, if \( V_{\ell}(n) \) denotes the set of partitions of \( n \) into distinct parts and \( V_{\ell}(n) = |V_{\ell}(n)| \), then
\[ p_{\text{dis}}(n) = \sum_{\ell=1}^{\infty} V_{\ell}(n). \]

Sylvester’s “stratification” of Euler’s theorem is that \( U_{\ell}(n) = V_{\ell}(n) \) for all positive integers \( n \) and \( \ell \).

For example, the set \( U_3(57) \) contains the partition
\[ 11 + 11 + 11 + 9 + 5 + 5 + 5 \]
which is a partition of 57 into odd parts whose three distinct parts are 11, 9, and 5. Similarly, the set \( V_3(57) \) contains the partition
\[ 12 + 11 + 10 + 9 + 8 + 4 + 2 + 1 \]
which is a partition of 57 with three maximal subsequences of consecutive integers: \((12, 11, 10, 9, 8), (4), \) and \((2, 1)\).

There are three partitions of 5 into odd parts: \( 5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1 \).
The partitions with one distinct part are 5 and \( 1 + 1 + 1 + 1 + 1 \), and so \( U_1(5) = 2 \).
The partition with two distinct parts is \( 3 + 1 + 1 \), and so \( U_2(5) = 1 \).
There are three partitions of 5 into distinct parts: $5 = 4 + 1 = 3 + 2$. The partitions with one maximal subsequence of consecutive integers are 5 and $3 + 2$, and so $V_1(5) = 2$. The partition with two maximal subsequences of consecutive integers is $4 + 1$, and so $V_2(5) = 1$.

The proof of Sylvester’s theorem uses the following combinatorial observation.

**Lemma 5.** Let $U$ and $V$ be sets, and let $\{U_i : i = 1, 2, 3, \ldots\}$ and $\{V_i : i = 1, 2, 3, \ldots\}$ be partitions of $U$ and $V$, respectively. Let $f : U \rightarrow V$ be a bijection. For every positive integer $\ell$, let $f_\ell : U_\ell \rightarrow V_\ell$ be the restriction of $f$ to $U_\ell$. If $f_\ell(U_\ell) \subseteq V_\ell$ for all $\ell \in \mathbb{N}$, then $f_\ell : U_\ell \rightarrow V_\ell$ is a bijection for all $\ell \in \mathbb{N}$.

**Proof.** Because $f$ is a bijection, it follows that $f$ is one-to-one, and so $f_\ell$ is one-to-one for all $\ell \in \mathbb{N}$. Let $v \in V_\ell \subseteq V$. Because $f$ is onto, there exists $u \in U$ such that $f(u) = v$. Because $U = \bigcup_{i=1}^{\infty} U_i$ is a partition of $U$, there is a unique integer $j$ such that $u \in U_j$. Therefore, $v = f(u) = f_j(u) \in V_j$ and so $v \in V_\ell \cap V_j$. Because $V = \bigcup_{i=1}^{\infty} V_i$ is a partition of $V$, it follows that $\ell = j$ and $u \in U_\ell$. Therefore, $f_\ell : U_\ell \rightarrow V_\ell$ is one-to-one and onto. This completes the proof. \hfill $\Box$

**Theorem 9.** Let

$$ n = a_1 + \cdots + a_k $$

be a partition of $n$ into $k$ not necessarily distinct odd parts, and let $\ell$ be the number of distinct odd parts in this partition. The major-minor hook partition consists of exactly $\ell$ pairwise disjoint maximal sequences of consecutive integers.

**Proof.** Let

$$ a_1 \geq a_2 \geq \cdots \geq a_k \geq 1 $$

and, for $i = 1, \ldots, k$, let

$$ a_i = 2r_i - 1. $$

We have

$$ r_1 \geq r_2 \geq \cdots \geq r_k \geq 1. $$

Let $\ell$ be the number of distinct odd parts in the partition \ref{n}. The proof is by induction on $\ell$.

If $\ell = 1$, then $a_i = a_1 = 2r_1 - 1$ for $i = 1, \ldots, k$, and $n = ka_1$. The Ferrers diagram for the partition is a rectangular array consisting of $k$ rows of $a_1$ dots. The major half of the diagram is a rectangular array consisting of $k$ rows of $r_1$ dots, and the minor half is a rectangular array consisting of $k$ rows of $r_1 - 1$ dots. The Durfee square of the major half has side $s = \min(k, r_1)$ and the Durfee square of the minor half has side $s' = \min(k, r_1 - 1)$. Let $n = h_1 + h'_1 + h_2 + h'_2 + \cdots$ be the major-minor hook partition. By Lemma \ref{inclusion} we have

$$ h_i - h_{i+1} = 2 $$

for $i = 1, \ldots, s - 1$, and

$$ h'_i - h'_{i+1} = 2 $$

for $i = 1, \ldots, s' - 1$. Because

$$ h_1 - h'_1 = (r_1 + c_1 - 1) - (r_1 + c_1 - 2) = 1 $$

it follows that the parts in the major-minor hook partition of $n$ form a strictly decreasing sequence of consecutive integers.
For example, if $n = 21 = 7 + 7 + 7$, then $k = 3$, $r_1 = 4$, and the major-minor hook partition is $21 = 6 + 5 + 4 + 3 + 2 + 1$. If $n = 21 = 3 + 3 + 3 + 3 + 3 + 3 + 3$, then $k = 7$, $r_1 = 2$, and the major-minor hook partition is $21 = 8 + 7 + 6$.

Let $\ell \geq 2$, and assume that the Theorem is true for partitions into at most $\ell - 1$ distinct odd parts. The smallest part in the partition is $a_k = 2r_k - 1$. We also know that $a_k < a_1$ because $\ell \geq 2$. If $j$ is the greatest integer such that $a_k < a_j$, then

$$1 \leq r_k < r_j$$

and

$$a_i = a_k \quad \text{for } i = j + 1, \ldots, k$$

and

$$(20) \quad m = n - (k - j)a_k = a_1 + \cdots + a_j$$

is a partition of $m$ into odd parts with exactly $\ell - 1$ distinct parts. By the induction hypothesis, the Theorem is true for this partition of $m$.

There are three cases.

Case 1:

Because $j < r_j$, both the major and the minor halves of the partition of $m$ have Durfee squares with side $j$. Let

$$(21) \quad m = g_1 + g_1' + \cdots + g_j + g_j'$$

be the major-minor hook partition for $m$, where

$$(22) \quad g_1 > g_1' > g_2 > \cdots > g_j > g_j'.$$

For $i = 1, \ldots, j$ we have

$$(23) \quad g_i = r_i + j - 2i + 1$$

$$(24) \quad g_i' = r_i + j - 2i. $$

The partition $(21)$ is a partition of $m$ into odd parts with exactly $\ell - 1$ distinct parts. By the induction hypothesis, the major-minor hook partition $(21)$ consists of exactly $\ell - 1$ pairwise disjoint maximal sequences of consecutive integers.

Because

$$j + 1 \leq r_k = r_{j+1}$$

the Durfee square for the major half of the partition of $n$ has side $s = \min(k, r_k) \geq j + 1$, and the Durfee square for the minor half of the partition of $n$ has side $s' = \min(k, r_k - 1) \geq j$. Let

$$(25) \quad n = h_1 + h_1' + \cdots + h_j + h_j' + h_{j+1} + \cdots$$

be the major-minor hook partition for $n$, where

$$h_1 > h_1' > h_2 > \cdots > h_j > h_j' > h_{j+1} > \cdots .$$

For $i = 1, \ldots, j$ we have

$$(26) \quad h_i = r_i + k - 2i + 1 = g_i + (k - j)$$

$$(27) \quad h_i' = r_i + k - 2i = g_i' + (k - j).$$

It follows that the number of pairwise disjoint maximal sequences of consecutive integers in the sequence $(g_1, \ldots, g_1', \ldots, g_j, g_j')$ of parts in the major-minor hook partition for $m$ is equal to the number of pairwise disjoint maximal sequences of
consecutive integers in the sequence $(h_1, \ldots, h'_1, \ldots, h_j, h'_j)$. For $i = j + 1, \ldots, s$ we have

$$h_i = r_k + k - 2i + 1$$

and for $i = j + 1, \ldots, s'$ we have

$$h'_i = r_k + k - 2i.$$

We observe that, for $i > j$,

$$h_i - h'_i = h'_i - h_{i+1} = 1$$

and so

(28) \quad (h_{j+1}, h'_{j+1}, h_{j+2}, \ldots)

is a sequence of consecutive integers. Moreover,

$$h'_j - h_{j+1} = (r_j + k - 2j) - (r_k + k - 2j - 1) = r_j - r_k + 1 \geq 2$$

and so (28) is a maximal sequence of consecutive integers in the major-minor hook partition of $n$. It follows that the number of pairwise disjoint maximal sequences of consecutive integers in the major-minor hook partition of $n$ is exactly one more than the number of pairwise disjoint maximal sequences of consecutive integers in the major-minor hook partition of $m$. By the induction hypothesis, the latter partition consists of $\ell - 1$ maximal disjoint sequences, and so the partition (28) consists of $\ell$ maximal disjoint sequences.

For example, if

$$n = 49 = 13 + 13 + 9 + 7 + 7$$

then $k = 5$, $\ell = 3$, and

$$j = 3 < r_k = 4 < r_j = 5.$$  

We have $k - j = 2$ and

$$m = 35 = 13 + 13 + 9.$$ 

The major-minor hook partition for $m$ is

$$m = 35 = 9 + 8 + 7 + 6 + 3 + 2$$

and contains two maximal sequences of consecutive integers: $(9, 7, 6)$ and $(3, 2)$. The major-minor hook partition for $n$ is

$$n = 49 = 11 + 10 + 9 + 8 + 5 + 4 + 2$$

and contains three maximal sequences of consecutive integers: $(11, 10, 9, 8)$, $(5, 4)$, and (2). Note that $(11, 10, 9, 8) = (9, 7, 6) + (2, 2, 2, 2)$ and $(5, 4) = (3, 2) + (2, 2)$.

Case 2:

$$r_k \leq j < r_j$$

Because $j < r_j$, the major-minor hook partition of $m$ satisfies the relations (21), (22), (23), and (24).

The major and minor halves of the center-justified Ferrers diagram for the partition (19) of $n$ also have Durfee squares with side $j$. The associated major-minor hook partition for $n$, denoted

$$n = h_1 + h'_1 + \cdots + h'_{r_k-1} + h_{r_k} + h'_{r_k} + h_{r_k+1} + \cdots + h'_j$$
is a partition into strictly decreasing parts, where
\[
h_i = \begin{cases} r_i + k - 2i + 1 & \text{for } i = 1, \ldots, r_k \\ r_i + j - 2i + 1 & \text{for } i = r_k + 1, \ldots, j. \end{cases}
\]
\[
h_i' = \begin{cases} r_i + k - 2i & \text{for } i = 1, \ldots, r_k - 1 \\ r_i + j - 2i & \text{for } i = r_k, \ldots, j. \end{cases}
\]
Applying (23) and (24), we obtain, for \(i = 1, \ldots, r_k - 1,\)
\[
h_i - g_i = h_i' - g_i' = h_{r_k} - g_{r_k} = k - j
\]
and, for \(i = r_k + 1, \ldots, j,\)
\[
h_i = g_i = r_i + j - 2I + 1
\]
and
\[
h_i' = g_i' = r_i + j - 2I.
\]
The critical observations are that
\[
h_{r_k} = g_{r_k} = r_{r_k} + j - 2r_k
\]
\[
g_{r_k} - g_{r_k} = (r_{r_k} + j - 2r_k + 1) - (r_{r_k} + j - 2r_k) = 1
\]
and
\[
h_{r_k} - h_{r_k}' = (r_{r_k} + k - 2r_k + 1) - (r_{r_k} + k - 2r_k) = k - j + 1 \geq 2.
\]
These imply that the number of pairwise disjoint maximal sequences of consecutive integers in the major-minor hook partition for \(n\) is exactly one more than the number in the major-minor hook partition for \(m.\) By the induction hypothesis, the hook partition for \(m\) contains exactly \(\ell - 1\) such sequences, and so the hook partition for \(n\) contains exactly \(\ell\) pairwise disjoint maximal sequences of consecutive integers.

For example, if
\[
n = 57 = 11 + 11 + 11 + 9 + 5 + 5 + 5
\]
then \(k = 7, \ell = 3\) and
\[
r_7 = 3 < j = 4 < r_j = 5.
\]
We have \(k - j = 3\) and
\[
m = 42 = 11 + 11 + 11 + 9.
\]
The major-minor hook partition for \(m\) is
\[
m = 42 = 9 + 8 + 7 + 6 + 5 + 4 + 2 + 1
\]
and contains two maximal sequences of consecutive integers: \((9, 8, 7, 6, 5, 4)\) and \((2, 1).\) The major-minor hook partition for \(n\) is
\[
n = 57 = 12 + 11 + 10 + 9 + 8 + 4 + 2 + 1
\]
and contains three maximal sequences of consecutive integers: \((12, 11, 10, 9, 8), (4),\) and \((2, 1).\) Note that \(r_k = r_7 = 3, h_3 = 8, g_3 = 5,\) and \(h_3' = g_3' = 4.\)

Case 3:

Because \(r_k = r_{j+1} < r_j = j,\) it follows that the sides of the Durfee squares of the major halves of the partitions of both \(m\) and \(n\) are \(s = j.\) Because \(r_j' = r_j - 1 = j - 1\)
and \( r'_{j-1} = r_{j-1} - 1 \geq r_j - 1 = j - 1 \), it follows that the sides of the Durfee squares of the minor halves of the partitions of both \( m \) and \( n \) are \( s' = j - 1 \). The hook numbers of the major halves are

\[
\begin{align*}
g_i &= r_i + j - 2i + 1 \quad \text{for } i = 1, \ldots, j \\
h_i &= \begin{cases} 
    r_i + k - 2i + 1 & \text{if } 1 \leq i \leq r_k \\
    r_i + j - 2i + 1 & \text{if } r_k + 1 \leq i \leq j
\end{cases}
\end{align*}
\]

and so

\[
h_i - g_i = \begin{cases} 
    k - j & \text{if } 1 \leq i \leq r_k \\
    0 & \text{if } r_k + 1 \leq i \leq j
\end{cases}
\]

The hook numbers of the minor halves are

\[
\begin{align*}
g'_i &= r_i + j - 2i \quad \text{for } i = 1, \ldots, j - 1 \\
h'_i &= \begin{cases} 
    r_i + k - 2i & \text{if } 1 \leq i \leq r_k - 1 \\
    r_i + j - 2i & \text{if } r_k \leq i \leq j
\end{cases}
\end{align*}
\]

and so

\[
h'_i - g'_i = \begin{cases} 
    k - j & \text{if } 1 \leq i \leq r_k - 1 \\
    0 & \text{if } r_k \leq i \leq j
\end{cases}
\]

Because

\[
g_{r_k} - g'_{r_k} = (r_{r_k} + j - 2r_k + 1) - (r_{r_k} + j - 2r_k) = 1
\]

and

\[
h_{r_k} - h'_{r_k} = (r_{r_k} + k - 2r_k + 1) - (r_{r_k} + j - 2r_k) = k - j + 1 \geq 2
\]

it follows that the major-minor hook partition for \( n \) contains exactly more maximal sequence of consecutive integers than the hook partition for \( m \).

For example, if

\[
n = 13 = 9 + 3 + 1
\]

then \( k = \ell = 3 \) and

\[
r_3 = 1 < r_j = 2 = j.
\]

We have \( k - j = 1 \) and

\[
m = 12 = 9 + 3.
\]

The major-minor hook partition for \( m \) is

\[
m = 12 = 6 + 5 + 1
\]

and contains two maximal sequences of consecutive integers: \((6, 5)\) and \((1)\). The major-minor hook partition for \( n \) is

\[
n = 13 = 7 + 5 + 1
\]

and contains three maximal sequences of consecutive integers: \((7), (5), \) and \((1)\).

Case 4:

\[
\boxed{r_k < r_j < j}
\]

Let \( s \) be the side of the Durfee square of the major half of partition of \( n \). The inequality \( r_j < j \) implies that \( s \leq j - 1 \), and so

\[
r_k < r_j \leq r_{s+1} \leq s < j.
\]


The side of the Durfee square of the major half of the partition of \( m \) is also \( s \). Let \( c_i \) be the number of dots in the \( i \)th column of the Ferrers diagram of the major half of the partition of \( n \). We have

\[
g_i = \begin{cases} 
    r_i + j - 2i + 1 & \text{if } 1 \leq i \leq r_j \\
    r_i + c_i - 2i + 1 & \text{if } r_j + 1 \leq i \leq s 
\end{cases}
\]

and

\[
h_i = \begin{cases} 
    r_i + k - 2i + 1 & \text{if } 1 \leq i \leq r_k \\
    r_i + j - 2i + 1 & \text{if } r_k + 1 \leq i \leq r_j \\
    r_i + c_i - 2i + 1 & \text{if } r_j + 1 \leq i \leq s 
\end{cases}
\]

Thus,

\[
h_i - g_i = \begin{cases} 
    k - j & \text{if } 1 \leq i \leq r_k \\
    0 & \text{if } r_k + 1 \leq i \leq s 
\end{cases}
\]

Let \( c'_i = c_{i+1} \) be the number of dots in the \( i \)th column of the minor half of the partition of \( n \). Let \( s' \) denote the side of the Durfee square of the minor half of the partition of \( n \). If \( r_s \geq s + 1 \), then

\[
r'_s = r_s - 1 \geq s \geq r_{s+1} > r'_{s+1}
\]

and so

\[
s' = s.
\]

If \( r_s = s \), then

\[
r'_{s-1} = r_{s-1} - 1 \geq r_s - 1 = s - 1 = r'_s
\]

and so

\[
s' = s - 1.
\]

In both cases we have \( r_j - 1 \leq s - 1 \leq s' \) and

\[
g'_i = \begin{cases} 
    r_i + j - 2i & \text{if } 1 \leq i \leq r_j - 1 \\
    r_i + c'_i - 2i & \text{if } r_j \leq i \leq s'
\end{cases}
\]

and

\[
h'_i = \begin{cases} 
    r_i + k - 2i & \text{if } 1 \leq i \leq r_k - 1 \\
    r_i + j - 2i & \text{if } r_k \leq i \leq r_j - 1 \\
    r_i + c'_i - 2i & \text{if } r_j + 1 \leq i \leq s'.
\end{cases}
\]

Thus,

\[
h'_i - g'_i = \begin{cases} 
    k - j & \text{if } 1 \leq i \leq r_k - 1 \\
    0 & \text{if } r_k \leq i \leq s'.
\end{cases}
\]

Because

\[
g_{r_k} - g'_{r_k} = (r_{r_k} + j - 2r_k + 1) - (r_{r_k} + j - 2r_k) = 1
\]

and

\[
h_{r_k} - h'_{r_k} = (r_{r_k} + k - 2r_k + 1) - (r_{r_k} + j - 2r_k) = k - j + 1 \geq 2
\]

it follows that the major-minor hook partition for \( n \) contains exactly one more sequence of consecutive integers than the hook partition for \( m \).

For example, if

\[
n = 50 = 11 + 11 + 9 + 7 + 7 + 5
\]

then \( k = 6, \ell = 4, \) and

\[
r_6 = 3 < r_5 = 4 < j = 5.
\]
We have \( k - j = 1 \) and 
\[
m = 45 = 11 + 11 + 9 + 7 + 7.
\]
The major-minor hook partition for \( m \) is 
\[
m = 45 = 10 + 9 + 8 + 7 + 5 + 4 + 2
\]
and contains three pairwise disjoint maximal sequences of consecutive integers: \((10, 9, 8, 7), (5, 4), \) and \((2)\). The major-minor hook partition for \( n \) is 
\[
n = 50 = 11 + 10 + 9 + 8 + 6 + 4 + 2
\]
and contains four disjoint maximal sequences: \((11, 10, 9, 8), (6), (4), \) and \((2)\).
This completes the proof. □

**Theorem 10** (Sylvester). For all positive integers \( n \) and \( \ell \), 
\[
U_\ell(n) = V_\ell(n).
\]

**Proof.** By Theorem 10, Sylvester’s one-to-one and onto function \( f : U(n) \to V(n) \) maps \( U_\ell(n) \) into \( V_\ell(n) \). We simply apply Lemma 5 to complete the proof. □

There are several recent proofs of Theorem 10, for example, Andrews [1], Andrews and Eriksson [6], and Hirschhorn [13]. V. Ramamani and K. Venkatachalipattu [20] obtained a combinatorial proof. Their method is discussed in Andrews [2, pp. 448–449] and [5, pp. 24–25].

For other recent work on trapezoidal numbers, see Apostol [8], Guy [12], Leveque [14], Moser [17], Pong [18, 19], and Tsai and Zaharescu [23, 24].

7. A PROBLEM

An odd integer is an integer of the form \( r + (r - 1) \). Thus, a partition into odd parts is a partition into parts, each of which is a sum of two consecutive integers. A different generalization of Euler’s theorem about partitions into odd parts would be a theorem about partitions into parts, each of which is a sum of \( e \) consecutive integers, or, equivalently, a sum of \( e \)-trapezoids. Thus, we consider positive parts of the form
\[
a_i = \sum_{j=0}^{e-1} (r_i - j)
\]
with 
\[
r_i \geq e
\]
and partitions of the form
\[
n = \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} \left( \sum_{j=0}^{e-1} (r_i - j) \right).
\]
Interchanging summations, we obtain a partition of \( n \) into \( e \) parts, each of which inherits a well-defined partition:
\[
n = \sum_{j=0}^{e-1} n_j
\]
where

\[ n_j = \sum_{i=1}^{k} (r_i - j). \]

Partitions into 2-trapezoids (that is, partitions into odd numbers) are equinumerous with partitions into distinct parts. What kind of partition are in one-to-one correspondence with partitions into \( e \)-trapezoids for \( e \geq 3 \)?

Acknowledgement. I thank the referee for providing many references to the current literature on Sylvester’s theorem.

References

[1] G. E. Andrews, On generalizations of Euler’s partition theorem, Michigan Math. J. 13 (1966), 491–498.
[2] G. E. Andrews, Applications of basic hypergeometric functions, SIAM Rev. 16 (1974), 441–484.
[3] G. E. Andrews, Use and extension of Frobenius’ representation of partitions, in: Enumeration and design, Academic Press, Toronto, 1984, pages 51–65.
[4] G. E. Andrews, Generalized Frobenius partitions, Mem. Amer. Math. Soc. 49 (1984), number 301.
[5] G. E. Andrews, The Theory of Partitions, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998 (Reprint of the 1976 original).
[6] G. E. Andrews and K. Eriksson, Integer Partitions, Cambridge University Press, Cambridge, 2004.
[7] G. E. Andrews, J. Jiménez-Urroz, and K. Ono, \( q \)-series identities and values of certain \( L \)-functions, Duke Math. J. 108 (2001), 395–419.
[8] T. M. Apostol, Sums of consecutive integers, Math. Gazette 87 (2003), 98–101.
[9] L. E. Bush, On the expression of an integer as the sum of an arithmetic series, Amer. Math. Monthly 37 (1930), 353–357.
[10] R. Chapman, Combinatorial proofs of \( q \)-series identities, J. Combin. Theory Ser. A 99 (2002), 1–16.
[11] J. W. L. Glaisher, A theorem in partitions, Messenger Math. 12 (1883), 158–170.
[12] R. Guy, Sums of consecutive integers, Fibonacci Quart. 20 (1982), no. 1, 36–38.
[13] M. D. Hirschhorn, Sylvester’s partition theorem, and a related result, Michigan Math. J. 21 (1974), 133–136.
[14] W. J. LeVeque, On representations as a sum of consecutive integers, Canadian J. Math. 2 (1950), 399–405.
[15] P. A. MacMahon, Combinatory Analysis, Cambridge University Press, Cambridge, 1916; reprinted by Chelsea Publishing Co., New York, 1960.
[16] T. E. Mason, On the representation of an integer as the sum of consecutive integers, Amer. Math. Monthly 19 (1912), 46–50.
[17] L. Moser, Notes on number theory. III. On the sum of consecutive primes, Canad. Math. Bull. 6 (1963), 159–161.
[18] W. Y. Pong, Sums of consecutive integers, College Math. J. 38 (2007), no. 2, 119–123.
[19] W. Y. Pong, Length spectra of natural numbers, Int. J. Number Theory 5 (2009), no. 6, 1089–1102.
[20] V. Ramamani, V. and K. Venkatachaliengar, On a partition theorem of Sylvester, Michigan Math. J. 19 (1972), 137–140.
[21] J. J. Sylvester, A constructive theory of partitions, arranged in three acts, an interact, and an exodion, Amer. J. Math. 5 (1882), 251–330.
[22] J. J. Sylvester, Sur un théorème de partitions, Comptes Rendus Acad. Sci. Paris 96 (1883), 674–675.
[23] M.-T. Tsai and A. Zaharescu, On the sum of consecutive integers in sequences, Int. J. Number Theory 8 (2012), no. 3, 643–652.
[24] M.-T. Tsai and A. Zaharescu, On the sum of consecutive integers in sequences II, Int. J. Number Theory 8 (2012), no. 5, 1281–1299.
[25] A. Weil, Number Theory for Beginners, Springer-Verlag, New York, 1979.
DEPARTMENT OF MATHEMATICS, LEHMAN COLLEGE (CUNY), BRONX, NY 10468

E-mail address: melvyn.nathanson@lehman.cuny.edu