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Special cases in order statistics for the alternative parametrization of the Generalized Power Function Distribution

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ABSTRACT
This paper investigates the order statistics based on the moment-generating function for the Generalized Power Function Distribution for the two different forms of this distribution. In this paper, we continue our investigation on the distribution of order statistics assuming that the original sample \( Y = (Y_1, Y_2, \ldots, Y_n) \) of size \( n \) is taken from a population that follows the Generalized Power Function Distribution. Two different forms of the Generalized Power Function Distribution with its special cases are presented and some order statistics related to these different forms are discussed. The main technique is the consideration of the moment-generating function.

1. Introduction
The Generalized Power Function Distribution (GPFD) is one of the useful lifetime distribution models which offers a good fit to various sets of failure data. This paper investigates the moment generating function and moments for the special cases of the order statistics for the GPFD by the classical method of moment generating functions \[1\].

The GPFD \[2\] is defined as
\[
f(y) = \frac{p}{\sigma^p} \left( \frac{y - \theta}{\sigma} + a \right)^{p-1}, \quad \theta - \sigma a \leq y \leq \theta + (b - a) \sigma, 0 \text{ otherwise,}
\]
where \( p \geq 1 \), and \( a \) and \( b \) are defined as:
\[
b = (p + 1) \sqrt{\frac{p + 2}{p}}, \quad a = \sqrt{p(p + 2)}.
\]

It is well known that the GPFD is a special case of Beta \((p, q)\) when \( q = 1 \) distribution \[2\].

The following is the alternative parametrization of the GPFD \[3\]:
\[
f(y) = \frac{p}{(b - a)^p} (y - a)^{p-1}, \quad p > 1, \quad a \leq y \leq b.
\]

In the following, we will always assume that parameter \( p > 1 \), and we will not specify this anymore.

Figure 1 shows the probability density function for the GPFD for the parametrization (3) for different values for the shape \( p \), scale \( b \) and location \( a \) parameters.

2. Parametrization (3)
Let \( Y = (Y_1, Y_2, \ldots, Y_n) \) be a random sample of size \( n \) from a population which follows the GPFD with the probability density function (3). Al Mutairi \[4\] derived the probability density function of the \( r \)-th order statistic \( Y_{(r)} \), \( 1 \leq r \leq n \), \( Y_1 \leq Y_2 \leq \cdots \leq Y_{(1)} \leq \cdots \leq Y_{(n)} \), obtained from this random sample in the following form:
\[
g_{Y_{(r)}}(y) = rp \left( \frac{n}{r} \right) \sum_{k=0}^{n-r} \binom{n-r}{k} \left( \frac{1}{b-a} \right)^{n-kp} (y - a)^{n-kp-1},
\]
\[
a \leq y \leq b, \quad 0 \text{ otherwise.}
\]

In addition, Al Mutairi \[4\] derived the moment generating function for the order statistic \( Y_{(r)} \), \( 1 \leq r \leq n \) as follows:
\[
M_{Y_{(r)}}(u) = \sum_{k=0}^{n-r} C_k e^{au} \left\{ \frac{(b - a)^{n-kp}}{n!} \left( \frac{-1}{(b-a)^p} \right)^{n-kp} \right\},
\]
where
\[
C_k = \frac{(-1)^k}{n-k} \binom{n}{r} \binom{n-r}{k}.
\]
Hence the m-th moment of the order statistic \( Y_{(r)} \) is given as:

\[
E(Y_{(r)}^m) = \sum_{k=0}^{n-r} C_k (n-k) \beta \sum_{v=0}^{(n-k)p-1} \left( \begin{array}{c} (n-k)p - 1 \\ v \end{array} \right) \left( \frac{(-a)^v}{(n-k)p-v+m} \right) \left( \frac{b^{n-kp-v+m} - a^{n-kp-v+m}}{(n-k)p-v+m} \right). 
\]

(5)

In particular, the first moment of the order statistics \( Y_{(r)} \) is obtained by setting \( m = 1 \) in Equation (5) and this leads to

\[
E(Y_{(r)}) = \sum_{k=0}^{n-r} C_k (n-k) \beta \sum_{v=0}^{(n-k)p-1} \left( \begin{array}{c} (n-k)p - 1 \\ v \end{array} \right) \left( \frac{(-a)^v}{(n-k)p-v+1} \right). 
\]

The second non-central moment \( \mu_{Y_{(r)}}^{(2)} \) is obtained by setting \( m = 2 \) in Equation (5). The same as in Al Mutairi [4], we obtain

\[
E(Y_{(r)}^2) = \sum_{k=0}^{n-r} C_k (n-k) \beta \sum_{v=0}^{(n-k)p-1} \left( \begin{array}{c} (n-k)p - 1 \\ v \end{array} \right) \left( \frac{(-a)^v}{(n-k)p-v+2} \right). 
\]

2.1. Special case

Let \( a = 0 \) and \( b = 1 \) in Equation (3) to get one of the most important special cases which is known as a standard Power Function distribution that has many applications. For example, see ref. [3] where data representing the failure time in minutes of an electrical insulation device are considered and have been modelled as a standard Power Function distribution.

To derive some properties for this special case, we consider the moment generating function of the order statistic \( Y_{(r)} \) (Equation (4)) as follows:

\[
M_{Y_{(r)}}(u) = \sum_{k=0}^{n-r} C_k \sum_{v=0}^{\infty} \frac{(u)^v (n-k)p}{v! ((n-k)p + v)} 
= \sum_{k=0}^{n-r} C_k \left[ 1 + \frac{(n-k)p u}{((n-k)p + 1)!} \right] 
+ \frac{(n-k)p u^2}{((n-k)p + 2)!} + \ldots. 
\]

(6)

Then, by differentiating Equation (6) and setting \( u = 0 \), the first moment (mean) \( \mu_{Y_{(r)}} \) of the order statistic \( Y_{(r)} \) for the GPFD with \( a = 0 \) and \( b = 1 \) is obtained as

\[
\frac{d}{du} M_{Y_{(r)}}(u) \bigg|_{u=0} = \sum_{k=0}^{n-r} C_k \frac{(n-k)p}{(n-k)p + 1}. 
\]

In the same way, the second non-central moment \( \mu_{Y_{(r)}}^{(2)} \) for the order statistic for the Standard Power Function Distribution is computed as

\[
\mu_{Y_{(r)}}^{(2)} = \sum_{k=0}^{n-r} C_k \frac{(n-k)p}{(n-k)p + 2}. 
\]

The variance \( \sigma_{Y_{(r)}}^2 \) of the order statistic \( Y_{(r)} \) for the GPFD with \( a = 0 \) and \( b = 1 \) can be obtained as follows:

\[
\sigma_{Y_{(r)}}^2 = \mu_{Y_{(r)}}^{(2)} - (\mu_{Y_{(r)}})^2 
= \sum_{k=0}^{n-r} C_k \frac{(n-k)p}{(n-k)p + 2} - \sum_{k=0}^{n-r} C_k \left[ \frac{(n-k)p}{(n-k)p + 1} \right]^2 
= \sum_{k=0}^{n-r} C_k \left[ \frac{(n-k)p(n-k)p + 1)^2}{((n-k)p + 2)((n-k)p)^2} \right]. 
\]
3. Alternative parametrization (1)

A special case of the alternative parametrization of the GPFD can be obtained by setting \( \theta = 0 \) and \( \sigma = 1 \) in Equation (1). Then we obtain

\[
f(y) = \frac{p}{bp} (y + a)^{p-1}, \quad -a \leq y \leq b - a, \tag{7}
\]

where \( p \) is the shape parameter, \( b \) is the scale parameter and \( a \) is the location parameter \([3]\). Moreover, a special case of Equation (7) can be obtained by setting \( a = 0 \) and \( b = 1 \) which leads to the same formula of Standard GPFD that is discussed in Section 2.1.

The probability density function of the \( r \)-th order statistic \( Y_{(r)} \) is given \([4]\) as follows:

\[
g_{Y_{(r)}}(y) = rp \left( \sum_{t=0}^{n-r} \binom{n-r}{t} \left( \frac{-1}{r} \right)^t \frac{(y + a)^{n-1}}{b^{n-1}} \right)
\]

\[
- a \leq y \leq b - a. \tag{8}
\]

See refs. \([5]\) and \([6]\).

With preliminaries accounted for, we can now formulate and prove the following theorem.

**Theorem 1:** If \( Y_{(r)} \) is the \( r \)-th order statistic for a sample from the population with GPFD of the form (7), then:

(i) The moment generating function \( M_{Y_{(r)}}(u) \) of the order statistic \( Y_{(r)} \) is given as:

\[
M_{Y_{(r)}}(u) = \sum_{t=0}^{n-r} C_t e^{-tu} \left\{ \sum_{v=0}^{\infty} \frac{[bu]^v}{v!} \frac{(n-t)p}{(n-t)p + v} \right\}. \tag{9}
\]

(ii) The first moment (mean) \( \mu_{Y_{(r)}} \), second non-central moment \( \mu^2_{Y_{(r)}} \) and the variance \( \sigma^2_{Y_{(r)}} \) of the order statistic \( Y_{(r)} \) are given in \([7]\) as:

\[
\mu_{Y_{(r)}} = \sum_{t=0}^{n-r} C_t \left( \frac{(n-t)p}{(n-t)p + 1} - a \right) \tag{10}
\]

\[
\mu^2_{Y_{(r)}} = \sum_{t=0}^{n-r} C_t \left( a^2 - \frac{2a(n-t)p}{(n-t)p + 1} + \frac{(n-t)b^2p}{(n-t)p + 1} + \frac{2n-r}{n-t} \right) \tag{11}
\]

\[
\sigma^2_{Y_{(r)}} = \sum_{k=0}^{n-r} C_t \left[ (n-t)^2 p^2 \left( (b-a)^2 - \sum_{t=0}^{n-r} C_t (b-a)^2 \right) + (n-t)^2 p^2 \left( (b^2 - 4ab + 3a^2) + (b-a)^2 \right) \right] \tag{12}
\]

(iii) The \( m \)-th moment \( E(Y_{(r)}^m) \) of the order statistic \( Y_{(r)} \) is

\[
E(Y_{(r)}^m) = \sum_{v=0}^{n-r} C_t (n-t)p \left( \frac{r-1}{r} \right)^{r-1} \left[ \frac{(n-t)p}{(n-t)p + v} \right] \times \frac{a^m}{(n-t)p + v + m} \times \left[ (b-a)^{n-t-p-v+m} - (a)^{n-t-p-v+m} \right]. \tag{13}
\]

**Proof:** (i) The moment generating function of the order statistic \( Y_{(r)} \) can be obtained by the considering the probability density function

\[
g_{Y_{(r)}}(y) = \frac{n!}{(r-1)!(n-r)!} \left[ \frac{(y - a)p}{(b-a)^2} \right]^{r-1} \left[ 1 \frac{(y - a)^{n-r}}{(b-a)^2} \right] \times \frac{p}{(b-a)^2} (y - a)^{n-r-1} \quad -a \leq y \leq b - a. \tag{14}
\]

This leads to

\[
M_{Y_{(r)}}(u) = E(e^{uy}) = \int_{-a}^{b-a} e^{uy} \sum_{t=0}^{n-r} C_t f(y_t) dy_t,
\]

where \( f(y_t) \) is given by

\[
f(y_t) = \frac{(n-k)p(y_t - a)^{n-k-1}}{(b-a)^{n-k-1}}. \tag{15}
\]

Then using the following substitution:

\[
y_t = bx - a \quad \text{hence} \quad dy_t = bdx.
\]

The limits of integration become as follows:

\[
y_t = -a \Rightarrow x = 0 y_t = b - a \Rightarrow x = 1.
\]
Thus, the previous equation becomes

\[ M_{Y_{(r)}}(u) = \sum_{t=0}^{n-r} C_t \int_0^{1} e^{bu x} \frac{(n-t)p}{b^{(n-t)p-1}} \ dx \]

\[ = \sum_{t=0}^{n-r} C_t (n-t)pe^{-au} \int_0^{1} e^{bu x} x^{n-t-1} \ dx \]

\[ = \sum_{t=0}^{n-r} C_t (n-t)pe^{-au} \frac{1}{(n-t)p+1} + \frac{bu}{(n-t)p+2} + \ldots \]

\[ = \sum_{t=0}^{n-r} C_t e^{-au} \left\{ 1 + \frac{(n-t)bpu}{(n-t)p+1} + \frac{b^2 u^2}{(n-t)p+2} + \ldots \right\} \]

\[ = \sum_{t=0}^{n-r} C_t e^{-au} \left\{ \sum_{v=0}^{n} \left( \frac{bu}{v!} \right)^v \frac{(n-t)p}{(n-t)p+v} \right\}. \quad (16) \]

This formula proves Equation (9) in Theorem 1, which provides the moment generating function

\[ M_{Y_{(r)}}(u) \]

(iii) The first moment (mean) \( \mu_{Y_{(r)}} \) for the order statistic \( Y_{(r)} \) is calculated by differentiating Equation (9) with respect to \( u \) and setting \( u = 0 \):

\[ \sigma_{Y_{(r)}}^2 = \mu_{Y_{(r)}}^2 - (\mu_{Y_{(r)}})^2 \]

\[ = \sum_{k=0}^{n-r} \frac{(n-t)^3 p^3 \left[ (b-a)^2 - \sum_{t=0}^{n-r} C_t (b-a)^2 \right] + (n-t)^2 p^2 \left[ (b^2 - 4ab + 3a^2) + (b-a)^2 + 2 \sum_{t=0}^{n-r} C_t a(b-a) - 2 \sum_{t=0}^{n-r} C_t (b-a)^2 \right]}{[n-t](p+2)[n-t](p+1)^2} + (n-t)p[2a + (b^2 - 4ab + 3a^2) - \sum_{t=0}^{n-r} C_t a^2 + 4 \sum_{t=0}^{n-r} C_t a(b-a)] + 2a^2 - 2 \sum_{t=0}^{n-r} C_t a^2 \]

\[ = \sum_{t=0}^{n-r} C_t a^2 \left[ 2 - 2a(n-t)b - \frac{n-t}{n-t}(b^2) \right]. \]

(iii) Finally, the \( m \)-th moment \( E(Y_{(r)}^m) \) for the order statistic is

\[ E(Y_{(r)}^m) = \int_{-a}^{b-a} \frac{y^m}{n-r} \sum_{t=0}^{n-r} C_t f_{Y_{(r)}}(y) \ dy. \]
Using the binomial expansion, we obtain the following:

\[ E(Y_m^m) = \frac{\sum_{t=0}^{n-t} C_t(n-t)p}{b^{n-t}p} \int_{-a}^{b} y^n \sum_{y=0}^{(n-t)p-1} \binom{(n-t)p-1}{v} \times a^n y^{(n-t)p-v-1} dy. \]

This result proves Equation (13) in Theorem 1, which represents the m-th moment \( E(Y_m^m) \) of the order statistic \( Y_m \).

Now we consider first two moments for (7). We can obtain the first moment (mean) \( \mu_Y \) of the order statistic \( Y \) by setting \( m = 1 \) in Equation (13) as follows:

\[ E(Y) = \frac{\sum_{t=0}^{n-t} C_t(n-t)p}{b^{n-t}p} \int_{-a}^{b} y^n \sum_{y=0}^{(n-t)p-1} \binom{(n-t)p-1}{v} \times \frac{a^n}{(n-t)p - v + 1} \times \left[ (b - a)^{(n-t)p-v+1} - (a)^{(n-t)p-v+1} \right]. \]

Similarly, by setting \( m = 2 \) in Equation (13), we get:

\[ E(Y^2) = \frac{\sum_{t=0}^{n-t} C_t(n-t)p}{b^{n-t}p} \int_{-a}^{b} y^n \sum_{y=0}^{(n-t)p-1} \binom{(n-t)p-1}{v} \times \frac{a^n}{(n-t)p - v + 2} \times \left[ (b - a)^{(n-t)p-v+2} - (a)^{(n-t)p-v+2} \right]. \quad (17) \]

This gives the second non-central moment \( \mu_{Y^2} \) for GPFD.

4. Conclusion

In this paper, two different forms of the GPFD with its special cases are presented and some order statistics related to these different forms are discussed. The main technique is the consideration of the moment generating function.

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