FRACTIONAL YAMABE SOLITONS AND FRACTIONAL NIRENBERG PROBLEM

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Abstract. In this paper, we first study the fractional Yamabe solitons, which are the self-similar solutions to fractional Yamabe flow. We prove some rigidity results and Liouville type results for such solitons. We then consider the fractional Nirenberg problem: the problem of prescribing fractional order curvature on the sphere. More precisely, we prove that there exists a conformal metric on the unit sphere such that its fractional order curvature is \( f \), when \( f \) possesses certain reflection or rotation symmetry.

1. Introduction. In the last decade several attempts have been developed to understand fractional order curvatures which interpolate between well-known curvatures associated to differential quantities such as scalar curvature and \( Q \)-curvature. This has been investigated for instance in [1, 3, 20]. This led naturally to a fractional version of the Yamabe problem, which has been investigated in [19]. A natural extension of the program is to consider the flow case. The present paper belongs to the last of this program.

Suppose that \( X \) is an \((n+1)\)-dimensional smooth manifold with smooth boundary \( M \), where \( n \geq 3 \). A function \( \rho \) is a defining function on the boundary \( M \) in \( X \) if

\[
\rho > 0 \text{ in } X, \quad \rho = 0 \text{ on } M, \quad d\rho \neq 0 \text{ on } M.
\]

We say that a Riemannian metric \( h^+ \) on \( X \) is conformally compact if, for some defining function \( \rho \), the metric \( \overline{h} = \rho^2 h^+ \) extends smoothly to \( X \). This induces a conformal class of metrics \( g = \overline{h}|_{TM} \) on \( M \) as defining function vary. The manifold \((M, [g])\) equipped with the conformal class \([g]\) is called the conformal infinity of \((X, h^+)\).

A metric \( h^+ \) is called asymptotically hyperbolic if it is conformally compact and its sectional curvature approaches \(-1\) at infinity, which is equivalent to \(|d\rho|_{\overline{h}} = 1\).
on $M$. If $\text{Ric}(h^+) = -nh^+$, then we call $(X, h^+)$ a conformally compact Einstein manifold. In this setting, given a representative $g$ of the conformal infinity, there exists a unique defining function $\rho$ such that, in a tubular neighborhood near $M$, such that the metric $h^+$ has the normal form

$$h^+ = \frac{d\rho^2 + g_{\rho}}{\rho^2}$$

where $g_{\rho}$ is a one-parametric family of metrics on $M$ satisfying $g_0 = g$.

The conformal fractional Laplacian $P^g_\sigma$ is constructed as the Dirichlet-to-Neumann operator for the scattering problem for $(X, h^+)$. In particular, it follows from [22, 38] that given $f \in C^\infty(M)$, for all but a discrete set of values $s \in \mathbb{C}$, the generalized eigenvalue problem

$$-\Delta_{h^+} u - s(n - s)u = 0 \text{ in } X$$

has a solution of the form

$$u = F \rho^{n-s} + G \rho^s, \ F, G \in C^\infty(\overline{X}), \ F|_{\rho=0} = f.$$  

The scattering operator on $M$ is defined as

$$S_g(s)f = G|_M,$$

and it is a meromorphic family of pseudo-differential operators in the whole complex plane.

The conformal fractional Laplacian on $(M, g)$ is defined as

$$P^g_\sigma = 2^{2\sigma} \frac{\Gamma(\sigma)}{\Gamma(-\sigma)} S_g \left( \frac{n}{2} + \sigma \right).$$

These operators satisfy the conformal transformation relation

$$P^g_{\sigma-\frac{4}{n-2}} g(\phi) = \rho^{\frac{n+2\sigma}{n-2}} P^g_\sigma(\rho \phi), \ \forall \ \rho, \phi \in C^\infty(S^n) \text{ with } \rho > 0.$$  

(1.1)

As proven in [18, 22], when $h^+$ is Poincaré–Einstein, $P^g_\sigma$ is the conformal Laplacian, $P^g_2$ is the Paneitz operator, and $P^g_k$, where $k$ are positive integers, are the GJMS operator discovered in [21]. One can then define the fractional order curvature of $g$ by

$$R^g_\sigma = P^g_\sigma(1).$$  

(1.2)

Hence, for the case when $\sigma = 1$, the fractional order curvature is the scalar curvature.

The normalized fractional Yamabe flow on $M$ is defined as the evolution of the metric $g = g(t)$:

$$\frac{\partial}{\partial t} g = -(R^g_\sigma - \overline{R^g_\sigma}) g, \ g|_{t=0} = g_0,$$  

(1.3)

where $\overline{R^g_\sigma}$ is the average of the fractional order curvature $R^g_\sigma$ given by

$$\overline{R^g_\sigma} = \frac{\int_M R^g_\sigma dV_g}{\int_M dV_g}.$$  

This is used to study the fractional Yamabe problem, and has been studied in [2, 16, 23, 30]. Note that the normalized fractional Yamabe flow is equivalent to the (unnormalized) fractional Yamabe flow

$$\frac{\partial}{\partial \ell} g = -R^g_\sigma g, \ g|_{\ell=0} = g_0$$  

(1.4)

by rescaling.
The fractional Yamabe soliton is a self-similar solution of the fractional Yamabe flow: \( g = g(t) \) is called fractional Yamabe soliton if there exists a one-parameter family of diffeomorphism \( \{ \psi_t \} \) on \( X \) and a one-parameter family of positive real numbers \( \alpha(t) \) such that

\[
g(t) = \alpha(t)\psi_t^*(g_0) \tag{1.5}
\]

is a solution of the fractional Yamabe flow (1.4), with \( \alpha(0) = 1 \) and \( \psi_0 = \text{id}_X \). In particular, \( g|_{t=0} = g_0 \) since \( \alpha(0) = 1 \) and \( \psi_0 = \text{id}_X \). It was proved in [23] that any compact fractional Yamabe soliton must have constant fractional order curvature.

We can also look at the fractional Yamabe soliton from the equation point of view. Differentiating (1.5) gives

\[
-\mathcal{R}_g = \frac{\partial}{\partial t} g = \dot{\alpha}(t)\psi_t^*(g_0) + \alpha(t)\mathcal{L}_W g_0.
\]

Evaluating it at \( t = 0 \) gives

\[
-\mathcal{R}^0_g = \dot{\alpha}(0)g_0 + \mathcal{L}_W g_0 \tag{1.6}
\]

where \( \mathcal{L}_W \) is the Lie derivative with respect to \( W \), and \( W \) is the vector field generated by \( \{ \psi_t \} \).

Yamabe soliton, which is a self-similar solution of the Yamabe flow, has been studied extensively by many authors. See [15, 27, 37, 36] and the references therein. However, there are not many results related to the fractional Yamabe soliton. Therefore, we study in this paper the fractional Yamabe soliton. After proving some basic properties of the fractional Yamabe soliton, we prove some rigidity results and Liouville type results in section 2. More precisely, we have the following:

**Theorem 1.1.** If \((M, g)\) is a compact, gradient fractional Yamabe flow soliton with constant scalar curvature such that its first nonzero eigenvalue \( \lambda_1 \) satisfies \( \lambda_1 > R_g/(n-1) \), then \((M, g)\) has constant fractional order curvature, and hence is trivial.

We also have the following theorem for non-compact case, where the corresponding theorem for Yamabe soliton was established in [36, Theorem 3]:

**Theorem 1.2.** Let \((M, g, f)\) be a complete, non-compact, gradient fractional Yamabe soliton such that \( \mathcal{R}^0_g - \lambda \in L^1(M, g) \), \( \int_M \text{Ric}(\nabla f, \nabla f) dV_g \leq 0 \) and the potential function \( f \) satisfies the following growth condition in \( M \):

\[
|\nabla f(x)|_g \leq C \tag{1.7}
\]

near infinity, where \( C \) is a uniform constant. Then the fractional order curvature of \( g \) is constant and is given by \( \mathcal{R}^0_g = \lambda \).

The following analogous theorem was established in [36, Theorem 5] for Yamabe soliton:

**Theorem 1.3.** If \((M, g, f)\) is a fractional Yamabe soliton (2.1) on a complete noncompact Riemannian manifold with nonpositive Ricci curvature such that

\[
\int_{M \setminus B_{r_0}(x_0)} d(x, x_0)^{-2} |W|^2_g dV_g < \infty \tag{1.8}
\]

for some \( B_{r_0}(x_0) \) in \( M \), then the fractional order curvature is constant and is given by \( \mathcal{R}^0_g = \lambda \).
In section 2, we also consider the fractional Yamabe soliton which is conformally flat and rotationally symmetric.

In another direction, we study the fractional Nirenberg problem. Recall that the classical Nirenberg’s problem is to find a conformal metric of the $n$-dimensional unit sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$ such that its scalar curvature is a given function $f$. This has been studied extensively for the last few decades. See [4, 5, 6, 7, 41, 43] and the references therein. In particular, Chang-Yang obtained in [5] a perturbation theorem which asserts that there exists a conformal metric whose scalar curvature is equal to $f$, provided that the degree condition holds for $f$ which is a positive Morse function and is sufficiently closed to $n(n-1)$ in $C^0$ norm.

A geometric flow has been introduced to study the Nirenberg’s problem by Struwe in [42] for $n = 2$, and has been generalized to $n \geq 3$ by Chen-Xu in [10]. More precisely, when $n \geq 3$, the scalar curvature flow is defined as $g = g(t)$ for which

$$\frac{\partial}{\partial t} g = (\alpha f - R_g)g,$$

where $\alpha$ is a constant chosen to preserve the volume along the flow. Using the scalar curvature flow, Chen-Xu [10] was able to prove Chang-Yang’s result with the quantitative bound on $\| f - n(n-1) \|_{C^0(\mathbb{S}^n)}$. More precisely, they proved in [10] the following:

**Theorem 1.4** (Chen-Xu [10]). Let $f$ be a positive smooth Morse function in $\mathbb{S}^n$ with only non-degenerate critical points. Suppose that

1. $\max_{\mathbb{S}^n} f / \min_{\mathbb{S}^n} f < 2^{\frac{n}{n-2}}$. \hfill (1.9)
2. For any integer $0 \leq i \leq n$, denote

$$\gamma_i = \#\{ \theta \in \mathbb{S}^n; \nabla_{g_{\mathbb{S}^n}} f(\theta) = 0, \Delta_{g_{\mathbb{S}^n}} f(\theta) < 0, \text{ind}(f, \theta) = n - i \},$$

where $\text{ind}(f, \theta)$ stands for the Morse index of $f$ at the critical point $\theta$. There are no nonnegative constants $k_1, k_2, \ldots, k_n$ satisfying

$$\gamma_0 = 1 + k_0, \quad \gamma_i = k_{i-1} + k_i \quad \text{for } 1 \leq i \leq n, \quad k_n = 0.$$

Then there exists a conformal metric $g \in [g_{\mathbb{S}^n}]$ such that its scalar curvature is equal to $f$.

Hereafter, $[g_{\mathbb{S}^n}]$ is the conformal class of the standard metric $g_{\mathbb{S}^n}$. Again using the scalar curvature flow, Leung-Zhou [31] proved an existence result for prescribing scalar curvature with symmetry. To describe their result, we have the following:

**Assumption 1.1.** $f$ is symmetric under a mirror reflection upon a hyperplane $\mathcal{H} \subset \mathbb{R}^{n+1}$ passing through the origin.

Under Assumption 1.1, without loss of generality, we may assume that $\mathcal{H}$ is the hyperplane perpendicular to the $x_1$-axis. Then the symmetry can be expressed as

$$f(\gamma(x)) = f(x)$$

where $\gamma : S^n \to S^n$ is given by

$$\gamma(x_1, x_2, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1}) \quad \text{for } x = (x_1, x_2, \ldots, x_{n+1}) \in S^n.$$

Then

$$\mathcal{F} = \mathcal{H} \cap S^n = \{ (0, x_2, \ldots, x_{n+1}) \in S^n \}$$

is the fixed point set.
Assumption 1.2. $f$ is invariant under a rotation $\gamma_\theta$ of angle $\theta = \pi/k$ with the rotation axis being a straight line in $\mathbb{R}^{n+1}$ passing through the origin. Here $k > 1$ is an integer.

Under Assumption 1.2, without loss of generality, we may assume that the rotation axis is the $x_{n+1}$-axis. In this case, $\mathcal{F} = \{N, S\}$ is the fixed point set, where $N = (0, \ldots, 0, 1)$ is the north pole and $S = (0, \ldots, 0, -1)$ is the south pole.

With these assumptions, we can state the result of Leung-Zhou in [31].

**Theorem 1.5 (Leung-Zhou [31]).** Suppose that $f$ is a positive smooth function on $S^n$ satisfying Assumption 1.1 or 1.2. Assume that $x \in \mathcal{F}$ with $f(x) = \max_{\mathcal{F}} f \Rightarrow \Delta g_{S^n} f(x) > 0$ where $\Delta g_{S^n}$ is the Laplacian of the standard metric of $S^n$, and

$$\max_{S^n} f < 2\pi^{2-\sigma}/(\min_{S^n} f).$$

Then $f$ can be realized as the scalar curvature of some metric conformal to the standard metric of $S^n$.

Note that existence results for prescribing scalar curvature with symmetry were obtained earlier by Moser [39] and by Escobar-Schoen [17].

As a generalization of the Nirenberg’s problem, the fractional Nirenberg’s problem is to find a conformal metric of the $n$-dimensional unit sphere $(S^n, g_{S^n})$ such that its fractional order curvature is a given function $f$. This has been studied in [11, 12, 13, 14, 28, 29, 33, 34, 35]. For simplicity, we write $P_\sigma = P_{g_{S^n}}$ and $R_\sigma = R_{g_{S^n}}$.

A geometric flow has been defined to study the fractional Nirenberg’s problem. Given a positive smooth function $f$ on $S^n$ and $g_0 \in [g_{S^n}]$, we define the flow $g(t)$:

$$\frac{\partial}{\partial t} g = (\alpha f - R_\sigma^g) g, \ g|_{t=0} = g_0,$$ (1.10)

where $\alpha = \alpha(t)$ is given by

$$\alpha = \frac{\int_{S^n} R_\sigma^g dV_g}{\int_{S^n} f dV_g}.$$ (1.11)

We remark that when $f$ is a constant function, (1.10) reduces to the fractional Yamabe flow (1.3) defined in $S^n$. Using the flow (1.10), it was proved in [9] the following theorem, which can be viewed as a generalization of Theorem 1.4:

**Theorem 1.6.** Suppose $\sigma \in (1/2, 1)$. With the same assumptions as in Theorem 2.2, except the condition (1.9) is replaced by

$$\max_{S^n} f/\min_{S^n} f < 2\pi^{2-\sigma}/\pi.$$ (1.12)

Then $f$ can be realized as the fractional order curvature $R_\sigma^g$ of some conformal metric $g \in [g_{S^n}]$.

Inspired by the result of Leung-Zhou stated above, we will prove the following in section 3:

**Theorem 1.7.** Suppose that $\sigma \in (1/2, 1)$ and $f$ is a positive smooth function on $S^n$ satisfying Assumption 1.1 or 1.2. Assume that $x_m \in \mathcal{F}$ with $f(x_m) = \max_{\mathcal{F}} f \Rightarrow \Delta g_{S^n} f(x_m) > 0$,
and
\[
\max_{\mathbb{S}^n} f < 2^{\frac{n-2}{2}} \left( \max_{\mathbb{S}^n} f \right).
\] (1.13)

Then \( f \) can be realized as the fractional order curvature \( R^g_{\sigma} \) of some conformal metric \( g \in [g_{\mathbb{S}^n}] \).

We remark that same technique has been applied to obtain existence results of prescribing curvature in [24, 25, 26].

2. Fractional Yamabe solitons. Recall that the fractional Yamabe soliton is given by
\[
(R^g_{\sigma} - \lambda)g = \mathcal{L}_W g, \tag{2.1}
\]
where \( R^g_{\sigma} \) is the fractional order curvature of \( (M, g) \), \( \lambda \) is a constant and \( \mathcal{L}_W \) is the Lie derivative with respect to the vector field \( W \). We emphasize that we do not assume that \( (M, g) \) is compact, i.e. \( (M, g) \) can be a complete, noncompact manifold. Note that the fractional order curvature may not be defined for a general complete, noncompact manifold. Therefore, we are going to assume in this section that there is a conformal compactification of the complete manifold \( (M, g) \), for which the fractional order curvature can still be defined.

If \( W \) is a gradient vector field, then (2.1) can be written as
\[
(R^g_{\sigma} - \lambda)g = \nabla^2 f \tag{2.2}
\]
for some smooth function \( f \). Here \( \nabla^2 f \) is the Hessian of \( f \) with respect to the metric \( g \). The triple \( (M, g, f) \) satisfying (2.2) is called gradient fractional Yamabe soliton.

We have the following:

**Proposition 1.** Suppose that \( (M, g, f) \) is a gradient fractional Yamabe soliton. There hold
\[
\nabla_g G = 2(R^g_{\sigma} - \lambda)\nabla_g f, \tag{2.3}
\]
where \( G = |\nabla_g f|^2 \),
\[
\Delta_g f = n(R^g_{\sigma} - \lambda), \tag{2.4}
\]
and
\[
(n - 1)\nabla_g R^g_{\sigma} = -\text{Ric}(\nabla_g f, \cdot). \tag{2.5}
\]

**Proof.** Fix a point \( p \in M \) and choose normal coordinates around \( p \) so that the metric matrix of \( g \) is diagonal at \( p \). Then
\[
\nabla_i G = 2\nabla_i \nabla_j f \nabla_j f = 2(R^g_{\sigma} - \lambda)g_{ij}\nabla_j f
\]
by (2.2). This proves (2.3). Taking the trace of (2.2) with respect to \( g \), we get (2.4). Moreover, continuing to compute in normal coordinates around \( p \in M \), if we apply \( \nabla_k \) to (2.2), we obtain
\[
\nabla_k \nabla_i \nabla_j f = \nabla_k R^g_{\sigma} g_{ij},
\]
which implies that
\[
\nabla_i \nabla_k \nabla_j f + R^i_{kij} \nabla^j f = \nabla_k R^g_{\sigma} g_{ij}.
\]
Taking the trace in \( k \) and \( j \), we obtain
\[
\nabla_i \Delta_g f + R^i_{dl} \nabla^l f = \nabla_i R^g_{\sigma}.
\]
Substituting (2.3) into this equation, we obtain
\[
(n - 1)\nabla_i R^g_{\sigma} = -R^i_{dl} \nabla^l f \tag{2.6}
\]
which proves (2.5).
We say that the gradient fractional Yamabe soliton is trivial if $f$ is a constant function. Note that it follows from the definition in (2.2) that a trivial, gradient fractional Yamabe soliton must have constant fractional order curvature, i.e. $R^g_\sigma \equiv \lambda$. The converse is true when $M$ is compact, i.e. if the fractional order curvature $R^g_\sigma$ is constant, then it must be trivial. To see this, note that if $M$ is compact and $R^g_\sigma$ is constant, then we can integrate (2.4) to deduce that $R^g_\sigma \equiv \lambda$. Combining this with (2.4), we can deduce that $f$ is a harmonic function in $M$. Since $M$ is compact, $f$ must be constant.

Taking the covariant derivative $\nabla_i$ of (2.6) and summing it up over $i$, we obtain

$$(n-1)\Delta_g R^g_\sigma = -R_{il,i}\nabla_l f - R_{il}\nabla_i f.$$  

(2.7)

Combining (2.2), (2.7) and the contracted Bianchi identity:

$$R_{il,i} = \frac{1}{2} \nabla_l R_g,$$

we get

$$(n-1)\Delta_g R^g_\sigma + \frac{1}{2} \langle \nabla_g R_g, \nabla_g f \rangle + R_g (R^g_\sigma - \lambda) = 0,$$

(2.8)

where $R_g$ is the scalar curvature of $g$.

Proof of Theorem 1.1. Since the scalar curvature $R_g$ is constant by assumption, it follows from (2.8) that

$$(n-1)\Delta_g (R^g_\sigma - \lambda) + R_g (R^g_\sigma - \lambda) = 0.$$  

(2.9)

Since $M$ is compact, we can multiply (2.9) with $R^g_\sigma - \lambda$ and integrate it over $M$ to get

$$0 = (n-1) \int_M (R^g_\sigma - \lambda) \Delta_g (R^g_\sigma - \lambda) dV_g + R_g \int_M (R^g_\sigma - \lambda)^2 dV_g$$

$$= -(n-1) \int_M |\nabla_g (R^g_\sigma - \lambda)|^2 dV_g + R_g \int_M (R^g_\sigma - \lambda)^2 dV_g.$$  

(2.10)

Therefore, if $R^g_\sigma$ is not constant, then $R^g_\sigma - \lambda$ is not a constant function. By (2.4), we have

$$\int_M (R^g_\sigma - \lambda) dV_g = 0.$$

Hence, the first eigenvalue $\lambda_1$ of $(M,g)$ satisfies

$$\lambda_1 \int_M (R^g_\sigma - \lambda)^2 dV_g \leq \int_M |\nabla_g (R^g_\sigma - \lambda)|^2 dV_g.$$  

(2.11)

It follows from (2.10) and (2.11) that $\lambda_1 \leq R_g / (n-1)$. This proves the assertion.

As a corollary, we have the following result, which partially recovers the main result in [23].

Corollary 1. Any compact gradient fractional Yamabe soliton with constant negative scalar curvature must have constant fractional order curvature.

Proof of Theorem 1.2. For any $r > 0$, we consider the geodesic ball $B_r(x_0)$ in $M$. By integration by parts, we have

$$\int_{B_r(x_0)} |\Delta_g f|^2 dV_g = \int_{\partial B_r(x_0)} \Delta_g f \frac{\partial f}{\partial \nu_g} d\sigma_g - \int_{B_r(x_0)} \langle \nabla (\Delta_g f), \nabla f \rangle_g dV_g.$$  

(2.12)
where $\nu_g$ is the outward unit normal on $\partial B_r(x_0)$. Recall the Bochner formula:

$$\frac{1}{2} \Delta_g |\nabla f|^2 = |\nabla^2 f|^2_g + \text{Ric}(\nabla f, \nabla f) + \langle \nabla(\Delta_g f), \nabla f \rangle_g.$$  \hspace{1cm} (2.13)

Substituting (2.4) and (2.13) into (2.12), we obtain

$$\int_{B_r(x_0)} |\Delta_g f|^2 dV_g = n \int_{\partial B_r(x_0)} (R^g_{\sigma} - \lambda) \frac{\partial f}{\partial \nu_g} d\sigma_g$$

$$+ \int_{B_r(x_0)} \left( |\nabla^2 f|^2_g + \text{Ric}(\nabla f, \nabla f) - \frac{1}{2} \Delta_g |\nabla f|^2_g \right) dV_g.$$ \hspace{1cm} (2.14)

Note that

$$\int_{B_r(x_0)} \Delta_g |\nabla f|^2_g dV_g$$

$$= \int_{\partial B_r(x_0)} \frac{\partial}{\partial \nu_g} |\nabla f|^2 d\sigma_g = 2 \int_{\partial B_r(x_0)} \left( \frac{\partial}{\partial \nu_g} (\nabla f), \nabla f \right)_g d\sigma_g$$ \hspace{1cm} (2.15)

$$= 2 \int_{\partial B_r(x_0)} \nabla^2 f (\nu_g, \nabla f) d\sigma_g = 2 \int_{\partial B_r(x_0)} (R^g_{\sigma} - \lambda) \frac{\partial f}{\partial \nu_g} d\sigma_g$$

by (2.2). Substituting (2.15) into (2.14), we get

$$\int_{B_r(x_0)} |\Delta_g f|^2 dV_g$$

$$= (n - 1) \int_{\partial B_r(x_0)} (R^g_{\sigma} - \lambda) \frac{\partial f}{\partial \nu_g} d\sigma_g + \int_{B_r(x_0)} (|\nabla^2 f|^2_g + \text{Ric}(\nabla f, \nabla f)) dV_g.$$ \hspace{1cm} (2.16)

Combining (2.2), (2.4) and (2.16), we obtain

$$n \int_{B_r(x_0)} (R^g_{\sigma} - \lambda)^2 dV_g$$

$$= \int_{\partial B_r(x_0)} (R^g_{\sigma} - \lambda) \frac{\partial f}{\partial \nu_g} d\sigma_g + \frac{1}{n - 1} \int_{B_r(x_0)} \text{Ric}(\nabla f, \nabla f) dV_g.$$ \hspace{1cm} (2.17)

By the assumption (1.7), we have

$$\int_{\partial B_r(x_0)} (R^g_{\sigma} - \lambda) \frac{\partial f}{\partial \nu_g} d\sigma_g \leq C \int_{\partial B_r(x_0)} |R^g_{\sigma} - \lambda| d\sigma_g.$$ \hspace{1cm} (2.18)

By the assumption that $\int_M |R^g_{\sigma} - \lambda| dV_g < \infty$ and Fubini’s Theorem, we can choose a sequence $r_i$ such that $r_i \to \infty$ as $i \to \infty$ with the following property

$$\int_{\partial B_{r_i}(x_0)} |R^g_{\sigma} - \lambda| d\sigma_g \to 0 \text{ as } i \to \infty.$$ \hspace{1cm} (2.19)

Combining (2.17)-(2.19) and the assumption that $\int_M \text{Ric}(\nabla f, \nabla f) dV_g \leq 0$, we have $\int_M (R^g_{\sigma} - \lambda)^2 dV_g \leq 0$, which implies that $R^g_{\sigma} = \lambda$. This proves the assertion. \hspace{1cm} \square

**Proof of Theorem 1.3.** By taking the trace of (2.1), we get

$$\text{div} W = n(R^g_{\sigma} - \lambda).$$ \hspace{1cm} (2.20)

Recall the following Bochner formula:

$$\text{div} (\mathcal{L}_W g)(W) = \frac{1}{2} \Delta_g |W|^2_g - |\nabla W|^2_g + \text{Ric}(W, W) + \langle W, \nabla \text{div}(W) \rangle.$$
Combining this with (2.20), we get
\[ |\nabla W|^2 = \frac{1}{2} \Delta_g |W|^2 + \text{Ric}(W, W) + (n-2)\langle W, \nabla R_g^\sigma \rangle. \tag{2.21} \]

Choose a cut-off function \( \phi = \phi_r \) on the ball \( B_{2r}(x_0) \), where \( r > 0 \) such that
\[
\phi = 1 \text{ in } B_r(x_0), \quad \phi = 0 \text{ in } M \setminus B_{2r}(x_0), \quad |\nabla \phi|^2 \leq \frac{C}{r^2}, \quad \text{and } \Delta \phi \leq \frac{C}{r^2}. \tag{2.22} \]

Since
\[
\text{div}((R_g^\sigma - \lambda)W\phi^2) = (R_g^\sigma - \lambda)\phi^2 \text{div}(W) + \phi^2\langle W, \nabla R_g^\sigma \rangle + 2\phi(R_g^\sigma - \lambda)\langle W, \nabla \phi \rangle,
\]
we have
\[
\int_M \phi^2\langle W, \nabla R_g^\sigma \rangle dV_g = -2\int_M \phi(R_g^\sigma - \lambda)\langle W, \nabla \phi \rangle dV_g. \tag{2.23}
\]

Substituting (2.20) into (2.23), we have
\[
\int_M \phi^2\langle W, \nabla R_g^\sigma \rangle dV_g = -n\int_M (R_g^\sigma - \lambda)\phi^2 dV_g - 2\int_M \phi(R_g^\sigma - \lambda)\langle W, \nabla \phi \rangle dV_g. \tag{2.24}
\]

Multiplying (2.21) by \( \phi^2 \) and integrating it over \( M \), we obtain
\[
\int_M |\nabla W|^2 \phi^2 dV_g = \frac{1}{2} \int_M \phi^2 \Delta_g |W|^2 dV_g + \int_M \text{Ric}(W, W)\phi^2 dV_g + (n-2)\int_M \phi^2\langle W, \nabla R_g^\sigma \rangle dV_g. \tag{2.25}
\]

Combining (2.24) and (2.25), we get
\[
\int_M |\nabla W|^2 \phi^2 dV_g + \frac{3}{2}n(n-2)\int_M (R_g^\sigma - \lambda)^2 \phi^2 dV_g
\]
\[
\leq \frac{1}{2} \int_M |W|^2 \Delta_g \phi^2 dV_g + \int_M \text{Ric}(W, W)\phi^2 dV_g - 2(n-2)\int_M \phi(R_g^\sigma - \lambda)\langle W, \nabla \phi \rangle dV_g
\]
\[
\leq \frac{1}{2} \int_M |W|^2 \Delta_g \phi^2 dV_g + \int_M \text{Ric}(W, W)\phi^2 dV_g + (n-2)\int_M (R_g^\sigma - \lambda)^2 \phi^2 dV_g
\]
\[
+ (n-2)\int_M D^2 \langle W, \nabla \phi \rangle + (n-2)\int_M (R_g^\sigma - \lambda)^2 \phi^2 dV_g + C(n)\int_M |W|^2 ||\nabla \phi||^2 dV_g
\]
where we have used integration by parts and Cauchy-Schwarz’s inequality. Here \( C(n) \) is an uniform constant depending only on \( n \). This implies that
\[
\int_M |\nabla W|^2 \phi^2 dV_g + \frac{3}{2}n(n-2)\int_M (R_g^\sigma - \lambda)^2 \phi^2 dV_g
\]
\[
\leq \frac{1}{2} \int_M |W|^2 \Delta_g \phi^2 dV_g + \int_M \text{Ric}(W, W)\phi^2 dV_g + C(n)\int_M |W|^2 ||\nabla \phi||^2 dV_g. \tag{2.26}
\]

Since the Ricci curvature is nonpositive, it follows from (1.8) and (2.22) that the right hand side of (2.26) tends to zero as \( r \to \infty \). This implies that left hand side
of (2.26) tends to zero as \( r \to \infty \). Since \( \phi = 1 \) in \( B_r(x_0) \) by (2.22), we can conclude that \( R_g^\sigma = \lambda \), as required.

2.1. PDE formulation of the fractional Yamabe soliton. In this subsection, we consider the fractional Yamabe soliton which is conformally flat and rotationally symmetric.

Recall that a Riemannian manifold \((M,g_{ij})\) is called a gradient Yamabe soliton if there exists a smooth scalar function \( f : M \to \mathbb{R} \) and a constant \( \lambda \in \mathbb{R} \) such that
\[
(R - \lambda) g_{ij} = \nabla_i \nabla_j f,
\]
where \( R \) is the scalar curvature of \( g_{ij} \). In [15], Daskalopoulos and Sesum proved the following: (see Theorem 1.3 in [15])

**Theorem 2.1.** Any locally conformally flat complete Yamabe gradient solitons with positive sectional curvature must be rotational symmetric.

They also provided the PDE formulation of Yamabe solitons (see Proposition 1.4 in [15]).

**Theorem 2.2.** Let \( dx^2 \) be the standard metric on \( \mathbb{R}^n \) and \( g_{ij} = u^{\frac{4}{n+2}} dx^2 \) be a conformally flat rotationally symmetric Yamabe gradient soliton. Then, \( u \) is a smooth solution to the elliptic equation
\[
\frac{n-1}{m} \Delta u^m + \beta x \cdot \nabla u + \gamma u = 0 \quad \text{on } \mathbb{R}^n \tag{2.28}
\]
where \( \beta \geq 0 \) and
\[
\gamma = \frac{2\beta + \lambda}{1 - m}
\]
and
\[
m = \frac{n - 2}{n + 2}.
\]

Theorem 2.2 reduces the classification of Yamabe solitons to studying the solutions of (2.28). By classifying the global smooth solutions of the elliptic equation (2.28), Daskalopoulos and Sesum were able to classify the locally conformally flat complete Yamabe gradient solitons with positive sectional curvature. See Proposition 1.5 and Theorem 1.7 in [15].

We would like to play the same game for the fractional Yamabe soliton. While we are not able to prove Theorem 2.1 in our case, we would like to provide the PDE formulation of the fractional Yamabe soliton which is conformally flat rotationally symmetric.

To this end, assume that \( g \) is conformal to \( \mathbb{R}^n \) and is rotationally symmetric. Then we can write
\[
g = u(r) \frac{4}{n+2} (dr^2 + r^2 g_{S^{n-1}}) = u(r) \frac{4}{n+2} g_{\mathbb{R}^n}.
\]
We can also write
\[
g = w(s)(ds^2 + g_{S^{n-1}})
\]
where \( ds^2 + g_{S^{n-1}} \) is the cylindrical metric. Then we have the relation:
\[
w(s) = u(r) \frac{4}{n+2} r^2 \quad \text{where } r = e^s. \tag{2.29}
\]
Assume the potential function \( f \) depends only on \( s \), i.e. \( f = f(s) \). Then we have
\[
(\nabla^2 f)_{ss} = f_{ss} - \Gamma^s_{ss} f_s \quad \text{and} \quad (\nabla^2 f)_{ii} = -\Gamma^s_{ii} f_s \quad \text{for } i \geq 2.
\]
Since
\[ \Gamma_{ss}^s = \frac{w_s}{2w} \]
and
\[ \Gamma_{ii}^s = -\frac{w_s}{2w} \text{ for } i \geq 2, \]
we have
\[ (\nabla_g^2 f)_{ss} = f_{ss} - \frac{w_s f_s}{2w} \]
and
\[ (\nabla_g^2 f)_{ii} = \frac{w_s f_s}{2w} \text{ for } i \geq 2. \tag{2.30} \]
Recall the equation of the fractional Yamabe soliton:
\[ -R_g^s g = \dot{\alpha}(0) g + 2\nabla_g^2 f. \tag{2.31} \]
Combining (2.30) and (2.31), we obtain
\[ -R_g^s g = \dot{\alpha}(0) w + 2 \left( f_{ss} - \frac{w_s f_s}{2w} \right) \] \tag{2.32}
and
\[ -R_g^s g = \dot{\alpha}(0) w + \frac{w_s f_s}{w}. \tag{2.33} \]
Subtracting (2.32) from (5), we obtain
\[ f_{ss} - \frac{w_s f_s}{w} = 0. \]
This implies that
\[ \frac{f_s}{w} \equiv C_0 \]
for some constant $C_0$. Combining this with (2.33), we obtain
\[ -R_g^s g = \dot{\alpha}(0) w + C_0 w_s. \tag{2.34} \]
By (2.29), we have
\[ w_s = \frac{\partial r}{\partial s} \left( \frac{4}{n-2\sigma} u^{\frac{4}{n-2\sigma} - 1} u_r r^2 + 2 u^{\frac{4}{n-2\sigma}} r \right) = e^s \left( \frac{4}{n-2\sigma} u^{\frac{4}{n-2\sigma} - 1} u_r r^2 + 2 u^{\frac{4}{n-2\sigma}} \right) \]
\[ = \frac{4}{n-2\sigma} u^{\frac{4}{n-2\sigma} - 1} u_r r^3 + 2 u^{\frac{4}{n-2\sigma}} r^2, \tag{2.35} \]
which implies that
\[ \frac{w_s}{w} = \frac{4}{n-2\sigma} u^{\frac{4}{n-2\sigma} - 1} u_r + 2. \tag{2.36} \]
Substituting (2.36) into (2.34), we get
\[ -R_g^s = \dot{\alpha}(0) + \frac{4C_0}{n-2\sigma} u^{-1} u_r + 2C_0. \tag{2.37} \]
Since $g = u^{\frac{4}{n-2\sigma}} g_{\mathbb{R}^n}$, then the fractional order curvature is given by
\[ R_g^s = P_g^s(1) = u^{-\frac{n+2\sigma}{n-2\sigma}} (-\Delta_{g_{\mathbb{R}^n}})^{\sigma} u. \]
Combining this with (2.37), we obtain
\[ 0 = (-\Delta_{g_{\mathbb{R}^n}})^{\sigma} u + (\dot{\alpha}(0) + 2C_0) u^{\frac{n+2\sigma}{n-2\sigma}} + \frac{4C_0}{n-2\sigma} u^{\frac{4}{n-2\sigma}} u_r. \tag{2.38} \]
This is the fractional PDE formulation of the fractional Yamabe soliton in $\mathbb{R}^n$. We remark that we have difficulty classifying solutions $u$ of (2.38) even $u$ is assumed to...
be radically symmetric, because we do not know if some of the results in the case of Laplacian, such as Lemmas 4.1 and 4.2 in [15], are still true in the case of fractional Laplacian.

3. Fractional Nirenberg problem.

3.1. Some facts. In this subsection, we collect some basic facts and prove some preliminary results which will be needed to prove Theorem 1.7.

When $\sigma \in (0, 1)$, Pavlov and Samko [40] proved that

$$P_\sigma(v)(x) = c_{n, -\sigma} \int_{\mathbb{S}^n} \frac{v(x) - v(y)}{|x - y|^{n + 2\sigma}} dV_{g_{\mathbb{S}^n}}(y) + R_\sigma v(x) \quad \text{for } v \in C^2(\mathbb{S}^n),$$

(3.1)

where $|x - y| = \sqrt{\sum_{i=1}^{n+1} (x_i - y_i)^2}$, $c_{n, -\sigma} = \frac{2^{2\sigma} \pi^{n+1} \Gamma(n+\sigma) \Gamma(n+2\sigma)}{\pi^{n+1} \Gamma(n+\sigma) \Gamma(n+2\sigma)}$, and

$$R_\sigma = P_\sigma(1) = \frac{\Gamma\left(\frac{n}{2} + \sigma\right)}{\Gamma\left(\frac{n}{2} - \sigma\right)}$$

is the fractional order curvature of the standard metric $g_{\mathbb{S}^n}$. Furthermore, $P_\sigma$ is the pull back of the fractional Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$ via the stereographic projection through $$(P_\sigma(\phi) \circ \Psi) = (\det d\Psi)^{-\frac{n+2\sigma}{n}} (-\Delta)^\sigma ((\det d\Psi)^{-\frac{n+2\sigma}{n}} \phi \circ \Psi) \quad \text{for } \phi \in C^2(\mathbb{S}^n),$$

(3.2)

where $\Psi : \mathbb{R}^n \to \mathbb{S}^n$ defined as

$$\Psi(z) = \left(\frac{2z_1}{1 + |z|^2}, \ldots, \frac{2z_n}{1 + |z|^2}, 1 - |z|^2 \right), \quad z = (z_1, \ldots, z_n) \in \mathbb{R}^n$$

(3.3)

is the inverse of the stereographic projection from the south pole and $\det d\Psi$ is the determinant of the Jacobian of $\Psi$.

If we write $g = u^{n+2\sigma} g_{\mathbb{S}^n}$ where $0 < u = u(t) \in C^\infty(\mathbb{S}^n)$, it follows from (1.1) and (1.2) that the flow (1.10) is equivalent to

$$\frac{4}{n+2\sigma} \frac{\partial}{\partial t} (u^{\frac{n+2\sigma}{n}}) = -P_\sigma(u) + \alpha f u^{\frac{n+2\sigma}{n}}, \quad u|_{t=0} = u_0$$

(3.4)

where $g_0 = u_0^{\frac{1}{n+2\sigma}} g_{\mathbb{S}^n}$, and (1.11) can be written as

$$\alpha = \frac{\int_{\mathbb{S}^n} u P_\sigma(u) dV_{g_{\mathbb{S}^n}}}{\int_{\mathbb{S}^n} f u^{\frac{n+2\sigma}{n}} dV_{g_{\mathbb{S}^n}}}. \quad (3.5)$$

Define the functional

$$E_f[u] = \frac{\int_{\mathbb{S}^n} u P_\sigma(u) dV_{g_{\mathbb{S}^n}}}{(\int_{\mathbb{S}^n} f u^{\frac{n+2\sigma}{n}} dV_{g_{\mathbb{S}^n}})^{\frac{n+2\sigma}{n}}}.$$

(3.6)

It was proved in [9] that the flow (3.4) is the negative gradient flow of this functional. More precisely, it follows from Proposition 3.1(3) in [9] that

$$\frac{d}{dt} E_f[u] = -\frac{n-2\sigma}{2} \frac{\int_{\mathbb{S}^n} (\alpha f - R^g_\sigma)^2 dV_g}{(\int_{\mathbb{S}^n} f dV_g)^{\frac{n+2\sigma}{n}}} \leq 0$$

(3.7)

along the flow (3.4). Hence, it follows from (3.7) that

$$E_f[u] \leq E_f[u_0]$$

(3.8)

along the flow (3.4).
3.2. The flow and its properties. First, we have the following:

Lemma 3.1. Given an isometry $\gamma : (S^n, g_{S^n}) \rightarrow (S^n, g_{S^n})$, we assume that

$$f(\gamma(x)) = f(x) \quad \text{and} \quad u_0(\gamma(x)) = u_0(x) \quad \text{for all } x \in S^n. \quad (3.9)$$

Suppose $\sigma \in (0, 1)$. Let $u(x, t)$ be the solution of (1.10)-(3.4) with initial data $u_0$. Then

$$u(\gamma(x), t) = u(x, t) \quad \text{for all } (x, t) \in S^n \times [0, \infty). \quad (3.10)$$

Proof. For any $v \in C^2(S^n)$, it follows from (3.1) that

$$P_\sigma(v)(\gamma(x)) = c_{n,-\sigma} \int_{S^n} \frac{v(\gamma(x)) - v(y)}{|\gamma(x) - y|^{n+2\sigma}} dV_{g_{S^n}}(y) + R_\sigma v(\gamma(x))$$

$$= c_{n,-\sigma} \int_{S^n} \frac{v(\gamma(x)) - v(\gamma(y))}{|\gamma(x) - \gamma(y)|^{n+2\sigma}} dV_{g_{S^n}}(y) + R_\sigma v(\gamma(x))$$

$$= c_{n,-\sigma} \int_{S^n} \frac{(v \circ \gamma)(x) - (v \circ \gamma)(y)}{|x - y|^{n+2\sigma}} dV_{g_{S^n}}(y) + R_\sigma (v \circ \gamma)(x)$$

$$= P_\sigma(v \circ \gamma)(x)$$

where we have used the change of variables $y \rightarrow \gamma(y)$ in the second equality, and we have used the fact $|\gamma(x) - \gamma(y)| = |x - y|$ for the third equality. Thus, it follows from (3.5), (3.10) and (3.11) that $u(\gamma(x), t)$ is also a solution to (3.4) with initial data $u_0$. By [9, Theorem 1.1], we knew that (3.4) has a unique smooth positive solution for all $t \geq 0$ when $\sigma \in (0, 1)$. Hence, we can conclude that $u(\gamma(x), t) = u(x, t)$ for all $(x, t) \in S^n \times [0, \infty)$, as required. \qed

The following was proved in [9, Lemma 5.2].

Lemma 3.2. Let $u$ be a positive smooth solution of (3.4) with initial data $u_0$. For any $t_k \rightarrow \infty$ as $k \rightarrow \infty$, let $u_k := u(t_k)$. Then, after passing to a subsequence, there exist a non-negative integer $L$, a convergent sequence $\{x_{k,\nu}\} \subset S^n$ and a non-negative smooth function $\nu_\infty$, a sequence of real numbers $\{\lambda_{k,\nu}\}$ with $\lambda_{k,\nu} \rightarrow \infty$ and $\alpha(t_k) \rightarrow \alpha_\infty$ as $k \rightarrow \infty$ for any fixed $\nu = 1, 2, \ldots, L$ such that

$$u_k = \sum_{\nu=1}^L u_{x_{k,\nu}, \lambda_{k,\nu}} + u_\infty + o(1) \quad \text{in } H^\sigma(S^n),$$

where

$$u_{x_{k,\nu}, \lambda_{k,\nu}}(x) = b_k u_{x_{k,\nu}, \lambda_{k,\nu}}(x) = b_k \left( \frac{2\lambda_{k,\nu}}{2 + (\lambda_{k,\nu}^2 - 1)(1 - \langle x, x_{k,\nu} \rangle)} \right)^{\frac{n-2\sigma}{2}}$$

with $b_k = \left( \frac{R_{x_{k,\nu}}}{\alpha_\infty f(-x_{k,\nu})} \right)^{\frac{n-2\sigma}{\sigma}}$ satisfies

$$P_\sigma(u_{x_{k,\nu}, \lambda_{k,\nu}}) = \alpha_\infty f(-x_{k,\nu}) u_{x_{k,\nu}, \lambda_{k,\nu}}^{\frac{n+2\sigma}{2}} \quad \text{on } S^n, \quad (3.12)$$

and $u_\infty$ satisfies

$$P_\sigma u_\infty = \alpha_\infty f u_\infty^{\frac{n+2\sigma}{2}} \quad \text{on } S^n. \quad (3.13)$$

From Lemma 3.2, we have the following:

Lemma 3.3. For a point $x_0 \in S^n$ with

$$\max_{S^n} f < 2^{\frac{2\sigma}{n-2\sigma}} f(x_0), \quad (3.14)$$

...
suppose the initial data \( u_0 \) satisfies

\[
E[u_0] \leq R^n_\sigma \omega^n_n (1 + \epsilon)^{n-2\sigma} f(x_0)^{-\frac{n-2\sigma}{n}},
\]

(3.15)

where \( \omega_n \) is the volume of the \( n \)-dimensional standard unit sphere \( S^n \), i.e.

\[
\omega_n = \text{Vol}(S^n, g_{S^n}).
\]

Then we have \( L \leq 1 \).

Proof. Suppose \( L > 0 \), otherwise it is trivial. It was proved in [9, Lemma 4.3] that

\[
\int_{S^n} |\alpha f - R^n_\sigma|^p dV_g \to 0 \quad \text{as} \quad t \to \infty,
\]

(3.16)

for all \( p \geq 2 \). By (1.10), we have

\[
\frac{\partial}{\partial t} dV_g = \frac{n}{2} (\alpha f - R^n_\sigma) dV_g,
\]

which together with (1.11) implies that the volume of \( M \) with respect to \( g \) is fixed along the flow (1.10), i.e.

\[
\int_{S^n} dV_g = \int_{S^n} u_0^{\frac{2n}{n-2\sigma}} dV_{g_{S^n}}
\]

(3.17)

for all \( t \geq 0 \). Hence, it follows from Hölder’s inequality and (3.16) that

\[
\int_{S^n} |\alpha f - R^n_\sigma|^p dV_g \to 0 \quad \text{as} \quad t \to \infty,
\]

(3.18)

for all \( p \geq 1 \). On the other hand, for \( k \) sufficiently large, we have (see (5.8) in [9])

\[
\left| \int_{S^n} \varphi u^{\frac{2n}{n-2\sigma}} x_{k,\nu} \lambda_{k,\nu} dV_{g_{S^n}} - \varphi(-x_{k,\nu}) \omega_n \right| = o(1)
\]

for any \( \varphi \in C^0(S^n) \), which implies that

\[
\left| \int_{S^n} \varphi^u x_{k,\nu} \lambda_{k,\nu} dV_{g_{S^n}} - \varphi(-x_{k,\nu}) b_k^{\frac{2n}{n-2\sigma}} \omega_n \right| = o(1)
\]

(3.19)

for any \( \varphi \in C^0(S^n) \). If we write \( g(t_k) = u_k^{\frac{1}{n-2\sigma}} g_{S^n} \), we can compute

\[
\int_{S^n} |R^n_\sigma|^{\frac{n}{n-2\sigma}} dV_{g(t_k)}
\]

\[
= \int_{S^n} |\alpha(t_k)f|^{\frac{n}{n-2\sigma}} dV_{g(t_k)} + o(1)
\]

\[
= \sum_{\nu=1}^L \int_{S^n} |\alpha(t_k)f|^\frac{n}{2} u_{x_{k,\nu}}^{\frac{2n}{n-2\sigma}} dV_{g_{S^n}} + \int_{S^n} |\alpha(t_k)f|^\frac{n}{n-2\sigma} u_{\infty}^{\frac{2n}{n-2\sigma}} dV_{g_{S^n}} + o(1)
\]

(3.20)

\[
\geq \sum_{\nu=1}^L |\alpha(t_k)f(-x_{k,\nu})|^{\frac{n}{n-2\sigma}} b_k^{\frac{2n}{n-2\sigma}} \omega_n + o(1) = LR^n_\sigma \omega_n + o(1),
\]
where the first equality follows from (3.18), the second equality follows from Lemma 3.2, and the last inequality follows from (3.19). On the other hand, we have

\[
\alpha(t_k) \left( \int_{\mathbb{S}^n} |f|^\frac{2n}{n-2\sigma} dV_{g(t_k)} \right)^\frac{2\sigma}{n} = E_f[u_k] \left( \int_{\mathbb{S}^n} f dV_{g(t_k)} \right)^{-\frac{2n}{n-2\sigma}} \left( \int_{\mathbb{S}^n} |f|^\frac{2n}{n-2\sigma} dV_{g(t_k)} \right)^\frac{2\sigma}{n} \leq E_f[u_0] \left( \int_{\mathbb{S}^n} f dV_{g(t_k)} \right)^{-\frac{2n}{n-2\sigma}} \left( \max_{\mathbb{S}^n} f \right)^{\frac{n-2\sigma}{n}} \left( \int_{\mathbb{S}^n} f dV_{g(t_k)} \right)^\frac{2\sigma}{n} \leq R_0 \omega_n\left(1 + \epsilon \right)^{\frac{n-2\sigma}{2\sigma}} \left( \max_{\mathbb{S}^n} f \right)^{\frac{n-2\sigma}{n}} + o(1),
\]

where the first equality follows from (3.5) and (3.6), the second last inequality follows from (3.8), and the last inequality follows from (3.15).

Combining (3.18), (3.20) and (3.21), we obtain

\[
LR_0 \omega_n \leq R_0 \omega_n\left(1 + \epsilon \right)^{\frac{n-2\sigma}{2\sigma}} \left( \max_{\mathbb{S}^n} f \right)^{\frac{n-2\sigma}{n}} + o(1).
\]

This together with (3.14) implies that \( L \leq 2 \), when \( \epsilon > 0 \) is small enough. \( \square \)

The following was proved in [9, Lemma 5.4].

**Lemma 3.4.** With the same assumptions as in Lemma 3.3, if \( L = 0 \), then as \( k \to \infty \), up to a subsequence, \( u_k \to u_{\infty} \) in \( H^{2\sigma,p}(\mathbb{S}^n) \), \( R_0^{g(t_k)} \to f \) in \( L^p(\mathbb{S}^n) \) for any \( p \geq 1 \) and \( u_{\infty} > 0 \) is a smooth solution of (3.13).

Therefore, if \( L = 0 \), then Theorem 1.7 follows from Lemma 3.4. Hence, from now on, we assume that \( L = 1 \).

For \( t \geq 0 \), we define

\[
\Theta(t) = \int_{\mathbb{S}^n} x dV_{g(t)}, \quad \text{where } x \in \mathbb{S}^n.
\]

Let \( Q = \lim_{t \to \infty} \Theta(t) \) be the unique limit of the shadow flow \( \Theta(t) \) associated with \( u(t) \), where \( u(t) \) is the solution of the flow (3.4).

**Lemma 3.5.** Under the assumptions of Lemma 3.2 and Lemma 3.3, if \( L = 1 \), then there is a point \( Q \in \mathbb{S}^n \) such that the following statements hold:

(i) As \( t \to \infty \), the metric \( g(t) \) concentrate at a critical point \( Q \) of \( f \) where \( \Delta_{g(t)} f(Q) \leq 0 \) in the sense that, for any \( r_0 > 0 \), \( \max_{B_{r_0}(Q)} u_k \) cannot be uniformly bounded from above for all \( k \gg 1 \).

(ii) There holds

\[
\lim_{t \to \infty} E_f[u(t)] = R_0 \omega_n^{\frac{2\sigma}{n}} f(Q)^{-\frac{n-2\sigma}{n}}.
\]

**Proof.** It follows from Proposition 6.10(ii) in [9] that, as \( t \to \infty \), the metric \( g(t) \) concentrate at a critical point \( Q \) of \( f \) where \( \Delta_{g(t)} f(Q) \leq 0 \) in the sense that (see the proof of Lemma 6.6 in [9])

\[
\text{Vol}(B_{r_0}(Q), g(t)) \to \omega_n \quad \text{as } t \to \infty.
\]

for any \( r_0 > 0 \). Now (i) follows from (3.22). On the other hand, (ii) follows from Lemma 6.11 in [9]. \( \square \)
3.3. Proof of Theorem 1.7. We also have the following:

**Lemma 3.6.** For any point $x_0 \in S^n$ and any $\epsilon > 0$, there exists $0 < u_0 \in C^\infty(S^n)$ such that

$$E_f[u_0] \leq R_\sigma \omega_n^\frac{2\sigma}{n}(1 + \epsilon) \frac{n-2\sigma}{n} f(x_0)^{-\frac{n-2\sigma}{n}}.$$

Moreover, we can choose $u_0$ to be invariant under the reflection upon a hyperplane passing through $x_0$ and the origin $0 \in \mathbb{R}^{n+1}$, and invariant under rotations with axis passing through $x_0$ and 0.

**Proof.** As the situation is unchanged after a rotation of $S^n$, we may assume that $x_0 = N = (0, \cdots, 0, 1)$ is the north pole. It follows from [29, Theorem 1.5] (see also [8] and [32]) that for any $\lambda > 0$,

$$v(z) := (N_\sigma R_\sigma 2^{2\sigma})^{\frac{n-2\sigma}{n}} \left(\frac{\lambda}{\lambda^2 + |z|^2}\right)^{\frac{n-2\sigma}{2}}$$

is a solution of

$$N_\sigma (-\Delta)^\sigma v = v^{n+2\sigma}$$

in $\mathbb{R}^n$, (3.24)

where $N_\sigma = 2^{1-2\sigma} \Gamma(1-\sigma)/\Gamma(\sigma)$. It is well known that (see (3.9) in [31] for example)

$$\int_{\mathbb{R}^n} v^{\frac{n}{n-2\sigma}} dz = (N_\sigma R_\sigma)^{\frac{n}{2\sigma}} 2^n \int_{\mathbb{R}^n} \left(\frac{\lambda}{\lambda^2 + |z|^2}\right)^n dz = (N_\sigma R_\sigma)^{\frac{n}{2\sigma}} \omega_n,$$

(3.25)

By (3.24) and (3.25), we have

$$\int_{\mathbb{R}^n} v(-\Delta)^\sigma v dz = \frac{1}{N_\sigma} \int_{\mathbb{R}^n} v^{\frac{n}{n-2\sigma}} dz = N_\sigma^{\frac{n-2\sigma}{n-2\sigma}} R_\sigma^\frac{n}{n} \omega_n.$$  

(3.26)

Now we define $u_0 : S^n \to \mathbb{R}$ by

$$v(z) = (\det d\Psi)^{\frac{n-2\sigma}{n}} u_0 \circ \Psi(z)$$

(3.27)

where $\Psi$ is defined as in (3.3). Then we compute

$$\int_{S^n} u_0 P_\sigma u_0 dV_{S^n} = \int_{\mathbb{R}^n} (\det d\Psi)(u_0 \circ \Psi)(P_\sigma(u_0)) \circ \Psi dz$$

$$= \int_{\mathbb{R}^n} (\det d\Psi)(\det d\Psi)^{-\frac{n-2\sigma}{n}} v (\det d\Psi)^{-\frac{n+2\sigma}{n}} (-\Delta)^\sigma v dz$$

(3.28)

$$= \int_{\mathbb{R}^n} v(-\Delta)^\sigma v dz = N_\sigma^{\frac{n-2\sigma}{2\sigma}} R_\sigma^\frac{n}{2\sigma} \omega_n,$$

where the second equality follows from (3.2) and (3.27), and the last equality follows from (3.26). Let $B_\delta(0)$ be the ball in $\mathbb{R}^n$ centered at the origin 0 with radius $\delta > 0$. For any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(\Psi(z)) - f(\Psi(0))| \leq \epsilon \quad \text{for all } z \in B_\delta(0).$$

(3.29)

On the other hand, it follows from (3.23) that

$$|v(z)| \leq C \left(\frac{\lambda}{\delta^2}\right)^{\frac{n-2\sigma}{2}} \quad \text{for all } z \in \mathbb{R}^n \setminus B_\delta(0),$$

(3.30)
for some constant $C$ independent of $z$. We estimate
\[
\int_{\mathbb{R}^n} f|u_0|^{\frac{2n}{n-s}} dV_{g_{\mathbb{R}^n}} = \int_{\mathbb{R}^n} f \circ \Psi(z)|u_0 \circ \Psi(z)|^{\frac{2n}{n-s}} (\det d\Psi) dz
\]
\[
= \int_{\mathbb{R}^n} f \circ \Psi(z)|v(z)|^{\frac{2n}{n-s}} dz
\]
\[
= \int_{\mathbb{R}^n} |v(z)|^{\frac{2n}{n-s}} \left( f(\Psi(z)) - f(\Psi(0)) \right) dz + f(\Psi(0)) \int_{\mathbb{R}^n} |v(z)|^{\frac{2n}{n-s}} dz
\]
\[
= \int_{B_\delta(0)} |v(z)|^{\frac{2n}{n-s}} \left( f(\Psi(z)) - f(\Psi(0)) \right) dz
\]
\[
+ \int_{\mathbb{R}^n \setminus B_\delta(0)} |v(z)|^{\frac{2n}{n-s}} \left( f(\Psi(z)) - f(\Psi(0)) \right) dz + f(N) \int_{\mathbb{R}^n} |v(z)|^{\frac{2n}{n-s}} dz
\]
\[
= O(\epsilon) + O \left( \left( \frac{\lambda}{\delta} \right)^{\frac{n-2}{n-s}} \right) + f(N)(N \sigma R \sigma)^{\frac{2n}{n-s}} \omega_n,
\]
where the second equality follows from (3.27), and the last equality follows from (3.26), (3.29) and (3.30). By first choosing $\delta > 0$ to be small enough so that $\epsilon > 0$ is small and then choosing a small $\lambda$ so that $\lambda/\delta^2$ to be small, we obtain from (3.28) and (3.31) that
\[
E_f[u_0] \leq \frac{R \sigma \omega_n^{\frac{2n}{n-s}}}{f(x_0)^{\frac{n-2}{n-s}}} + \epsilon.
\]

Note that it follows from (3.3), (3.23) and (3.27) that
\[
v(z) = (N \sigma R \sigma 2^{2\sigma})^{\frac{n-2}{4\sigma}} \left( \frac{\lambda}{\lambda^2 + |z|^2} \right)^{\frac{n-2}{2}} \left( 1 + \frac{|z|^2}{2} \right)^{\frac{n-2}{2}}.
\]
From (3.3), we have
\[
|z|^2 = \frac{1 - x_{n+1}}{1 + x_{n+1}}, \text{ where } x = (x_1, ..., x_n, x_{n+1}) \in S^n.
\]
Hence, $u_0$ defined in (3.27) depends on $x_{n+1}$ only. One can then verify the claimed symmetries directly. This proves the assertion.

Now we are ready to prove Theorem 1.7.

**Proof of Theorem 1.7.** As the situation is unchanged after a rotation, we can assume that
\[
S \in \mathcal{F} \text{ and } f(S) = \max_{\mathcal{F}} f
\]
where $S = (0, ..., 0, -1) \in S^n$ is the south pole. We claim that there is $\delta > 0$ such that
\[
f(S) \geq f(x_c) + \delta \text{ for any point } x_c \in \mathcal{F} \text{ with } \Delta g_{\mathbb{R}^n} f(x_c) \leq 0. \tag{3.33}
\]
If not, then there exists a sequence of points $x_{c_i} \in \mathcal{F}$ such that
\[
\lim_{i \to \infty} f(x_{c_i}) = f(S) = \max_{\mathcal{F}} f \text{ and } \Delta g_{\mathbb{R}^n} f(x_{c_i}) \leq 0. \tag{3.34}
\]
By passing to subsequence, we assume that $x_{c_i} \to x_m \in \mathcal{F}$ as $i \to \infty$. Hence, it follows from (3.34) that
\[
f(x_m) = \lim_{i \to \infty} f(x_{c_i}) = \max_{\mathcal{F}} f \text{ and } \Delta g_{\mathbb{R}^n} f(x_m) \leq 0, \tag{3.35}
\]
which contradicts (1.12). This proves (3.33).

It follows from (3.33) that
\[ f(S)^{-\frac{n-2}{n}} + \epsilon \leq f(x_c)^{-\frac{n-2}{n}} \text{ for } x_c \in F \text{ with } \Delta_{g^n} f(x_m) \leq 0, \]
where \( \epsilon > 0 \) is a small positive number. Let \( u_0 \) be the positive smooth function constructed in Lemma 3.6. We claim that, with this choice of initial data, we have \( L = 0 \) in Lemma 3.2. Suppose not, then it follows from Lemma 3.3 that \( L = 1 \). Let \( Q \) be the blow up point in Lemma 3.5.

We are going to show that \( Q \in F \). Suppose \( Q \not\in F \). There exists an isometry described in Assumption 1.1 or 1.2 such that
\[ \gamma(Q) \neq Q, \]
which implies that
\[ \max_{B_r(\gamma(Q))} u_k \leq C \text{ for all } k \gg 1 \]
whenever \( r \) is small enough, since \( \{u_k\} \) is uniformly bounded on any compact subsets of \( S^n \setminus \{Q\} \) by Lemma 3.2. But Lemma 3.1 implies that
\[ u_k(\gamma(x)) = \gamma(x), t_k = u(x,t_k) = u_k(x) \text{ for all } x \in S^n. \]
This together with (3.37) implies that \( \max_{B_r(Q)} u_k \) is uniformly bounded, which contradicts Lemma 3.5(i). This proves that \( Q \in F \).

Hence, by Lemma 3.5(i), \( Q \in F \) and \( \Delta_{g^n} f(Q) \leq 0 \). This together with (3.36) implies that
\[ f(S)^{-\frac{n-2}{n}} + \epsilon \leq f(Q)^{-\frac{n-2}{n}}. \]
Combining (3.38), Lemma 3.6 and Lemma 3.5(ii), we obtain
\[ E_f[u_0] \leq R^n_\omega \omega_n^{\frac{2\nu}{n}} (1 + \epsilon') \frac{n-2\nu}{n-2} f(S)^{-\frac{n-2}{n}} < R^n_\omega \omega_n^{\frac{2\nu}{n}} f(Q)^{-\frac{n-2\nu}{n}} = \lim_{t \to \infty} E_f[u(t)], \]
where the second inequality holds by choosing \( \epsilon' \) small enough, which contradicts (3.8).

Therefore, \( L = 0 \). Now Theorem 1.7 follows from Lemma 3.4. \( \square \)

REFERENCES

[1] J. Case and S. Y. A. Chang, On fractional GJMS operators, Commun. Pure Appl. Math., 69 (2016), 1017–1061.
[2] H. Chan, Y. Sire and L. Sun, Convergence of the fractional Yamabe flow for a class of initial data, preprint, arXiv:1809.05753v1.
[3] S. Y. A. Chang and M. González, Fractional Laplacian in conformal geometry, Adv. Math., 226 (2011), 1410–1432.
[4] S. Y. A. Chang, M. J. Gursky and P. C. Yang, The scalar curvature equation on 2-and 3-spheres, Calc. Var. Partial Differ. Equ., 1 (1993), 205–229.
[5] S. Y. A. Chang and P. C. Yang, A perturbation result in prescribing scalar curvature on \( S^n \), Duke Math. J., 64 (1991), 27–69.
[6] S. Y. A. Chang and P. C. Yang, Conformal deformation of metrics on \( S^2 \), J. Differ. Geom., 27 (1988), 259–296.
[7] S. Y. A. Chang and P. C. Yang, Prescribing Gaussian curvature on \( S^2 \), Acta Math. 159 (1987), 215–259.
[8] W. Chen, C. Li and B. Ou, Classifications of solutions for an integral equation, Commun. Pure Appl. Math., 59 (2006), 330–343.
[9] X. Chen, P. T. Ho and J. Xiong, A fractional conformal curvature flow on the unit sphere, preprint, arXiv:1906.08434.
[10] X. Chen and X. Xu, The scalar curvature flow on \( S^n \)—perturbation theory revisited, Invent. Math., 187 (2012), 395–506.
[11] Y. H. Chen, C. Liu and Y. Zheng, Existence results for the fractional Nirenberg problem, *J. Funct. Anal.*, **270** (2016), 4043–4086.

[12] Y. H. Chen and Y. Zheng, Peak solutions for the fractional Nirenberg problem, *Nonlinear Anal.*, **122** (2015), 100–124.

[13] H. Chtioui and W. Abdelhedi, On a fractional Nirenberg type problem on the n-dimensional sphere, *Complex Var. Elliptic Equ.*, **62** (2017), 1015–1036.

[14] H. Chtioui and W. Abdelhedi, On a fractional Nirenberg problem on n-dimensional spheres: existence and multiplicity results, *Bull. Sci. Math.*, **140** (2016), 617–628.

[15] P. Daskalopoulos and N. Sesum, The classification of locally conformally flat Yamabe solitons, *Adv. Math.*, **240** (2013), 346–369.

[16] P. Daskalopoulos, Y. Sire and J. L. Vázquez, Weak and smooth solutions for a fractional Yamabe flow: the case of general compact and locally conformally flat manifolds, *Commun. Partial Differ. Equ.*, **42** (2017), 1481–1496.

[17] J. F. Escobar and R. Schoen, Conformal metrics with prescribed scalar curvature, *Invent. Math.*, **86** (1986), 243–254.

[18] C. Fefferman and C. R. Graham, The Ambient Metric, Princeton Univ. Press, Princeton, NJ, 2012.

[19] M. González and J. Qing, Fractional conformal Laplacians and fractional Yamabe problems, *Anal. PDE*, **6** (2013), 1535–1576.

[20] M. González, R. Mazzeo and Y. Sire, Singular solutions of fractional order conformal Laplacians, *J. Geom. Anal.*, **22** (2012), 845–863.

[21] C. R. Graham, R. Jenne, L. Mason and G. Sparling, Conformally invariant powers of the Laplacian. I. Existence, *J. Lond. Math. Soc. (2)*, **46** (1992), 557–565.

[22] C. R. Graham and M. Zworski, Scattering matrix in conformal geometry, *Invent. Math.*, **152** (2003), 89–118.

[23] P. T. Ho, Soliton to the fractional Yamabe flow, *Nonlinear Anal.*, **139** (2016), 211–217.

[24] P. T. Ho, Prescribed mean curvature equation on the unit ball in the presence of reflection or rotation symmetry, *Proc. Roy. Soc. Edinburgh Sect. A*, **149** (2019), 781–789.

[25] P. T. Ho, Prescribed Webster scalar curvature on $S^{n+1}$ in the presence of reflection or rotation symmetry, *Bull. Sci. Math.*, **140** (2016), 506–518.

[26] P. T. Ho, Prescribing $Q$-curvature on $S^n$ in the presence of symmetry, *Commun. Pure Appl. Anal.*, **19** (2020), 715–722.

[27] S. Y. Hsu, Some properties of the Yamabe soliton and the related nonlinear elliptic equation, *Calc. Var. Partial Differ. Equ.*, **49** (2014), 307–321.

[28] T. Jin, Y. Y. Li and J. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions, *J. Eur. Math. Soc. (JEMS)*, **16** (2014) 1111–1171.

[29] T. Jin, Y. Y. Li and J. Xiong, On a fractional Nirenberg problem, Part II: existence of solutions, *Int. Math. Res. Not. IMRN*, (2015) 1555–1589.

[30] T. Jin and J. Xiong, A fractional Yamabe flow and some applications, *J. Reine Angew. Math.*, **696** (2014), 187–223.

[31] M. C. Leung and F. Zhou, Prescribed scalar curvature equation on $S^n$ in the presence of reflection or rotation symmetry, *Proc. Amer. Math. Soc.*, **142** (2014), 1607–1619.

[32] Y. Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, *J. Eur. Math. Soc. (JEMS)*, **6** (2004), 153–180.

[33] C. Liu and Q. Ren, Infinitely many non-radial solutions for fractional Nirenberg problem, *Calc. Var. Partial Differ. Equ.*, **56** (2017), 40 pp.

[34] C. Liu and Q. Ren, Multi-bump solutions for fractional Nirenberg problem, *Nonlinear Anal.*, **171** (2018), 177–207.

[35] Z. Liu, Concentration of solutions for the fractional Nirenberg problem, *Commun. Pure Appl. Anal.*, **15** (2016), 563–576.

[36] L. Ma and V. Miquel, Remarks on scalar curvature of Yamabe solitons, *Ann. Global Anal. Geom.*, **42** (2012), 195–205.

[37] S. Maeta, Three-dimensional complete gradient Yamabe solitons with divergence-free Cotton tensor, *Ann. Global Anal. Geom.*, **58** (2020), 227–237.

[38] R. Mazzeo and R. B. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, *J. Funct. Anal.*, **75** (1987), 260–310.

[39] J. Moser, On a nonlinear problem in differential geometry, Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), Academic Press, New York, (1973), 273–280.
[40] P. Pavlov and S. Samko, A description of spaces $L^p_0(S_{n-1})$ in terms of spherical hypersingular integrals (Russian), Dokl. Akad. Nauk SSSR, 276 (1984), 546–550; translation in Soviet Math. Dokl., 29 (1984), 549–553.

[41] R. Schoen and D. Zhang, Prescribed scalar curvature on the $n$-sphere, Calc. Var. Partial Differ. Equ. 4 (1996), 1–25.

[42] M. Struwe, A flow approach to Nirenberg’s problem, Duke Math. J., 128 (2005), 19–64.

[43] J. C. Wei and X. Xu, On conformal deformations of metrics on $S^n$, J. Funct. Anal., 157 (1998), 292–325.

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