On the Generalized Miura Transformation

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Abstract

By generalizing the Miura transformation for $W_N$ to other classical $W$ algebras obtained by hamiltonian reduction, we find realisations of these algebras in terms of relatively simple non-abelian current algebras, e.g. $\widehat{sl}(2) \times \widehat{u}(1)^N$, generalizing the free field realisation of $W_N$. As an example, we present the $\widehat{sl}(2) \times \widehat{u}(1)$ realisation of $W_2^2$, which we also quantize. By a specific example, we also show how the realisation of $W_N$ with the currents of $W_{N-1}$ and a free boson can be generalized to certain classes of “extended” $W_N$ algebras.
1 Introduction

Recently, a large class of $W$ algebras has been shown to arise from the hamiltonian reduction of current algebras \[1, 2, 3, 4, 5\]. Specifically, different classical $W$ algebras can be associated to all inequivalent $\text{sl}(2)$ subalgebras of any Lie algebra \[5\]. These algebras have been shown to be the symmetry algebras of non-abelian Toda field theories or, equivalently, constrained WZW models \[3\]. Also, it has been shown that to classical $W$ algebras and quantum $W$ algebras existing for generic central charge, a Lie algebra can be associated with a specific $\text{sl}(2)$ embedding \[5, 6\].

To study these $W$ algebras, free field realisations can be a very powerful tool. In fact, the well-known $W_N$ algebras were originally defined in a realisation with $N - 1$ free bosons, using the quantum Miura transformation \[8\]. The purpose of this paper is to generalize this at the classical level to other $W$ algebras. As it turns out, this in general doesn’t lead directly to a free field realisation, but rather a realisation in terms of a relatively simple non-abelian current algebra. For example, the algebra $W_2^{3/2}$, first introduced by Polyakov \[9\] and constructed explicitly by Bershadsky \[10\], with conformal spin content $(1, \frac{3}{2}, \frac{3}{2}, 2)$, has a realisation with $\text{sl}(2) \times \text{u}(1)$.

A beautiful consequence of the Miura transformation was found in \[11\], where it was proven that $W_N$ can be realised from the currents of $W_{N-1}$ plus an extra free boson. We generalize this kind of reduction to other classes of $W$ algebras obtained by hamiltonian reduction.

This paper is organised as follows. In section 2, we briefly review, in an appropriate formalism, how classical $W$ algebras appear via hamiltonian reduction. Next, in section 3, we introduce the Miura transformation using two examples, the $W_N$ series and a specific class of algebras containing, amongst others, $W_2^{3/2}$. Using this same class, we describe in section 4 how the reduction $W_N \Rightarrow W_{N-1}$ generalizes. We also give an explicit example. In section 5 we say a few words on the possibility of a generalized quantum Miura transformation, and, finally, we conclude with a number of remarks.

2 Hamiltonian reduction

The treatment in this section is necessarily sketchy. Emphasis will be laid on the aspects needed further along. A full treatment of classical hamiltonian reduction in the present context was given in \[1\].

With $T_a$ being the generators of a Lie algebra $\mathcal{G}$, consider a current

$$J(x) = J^a(x)T_a,$$  \hspace{1cm} (1)

with

$$J_a(x) = <T_a, J(x)>,$$

$$g_{ab} = <T_a, T_b>,$$

$$g_{ab}g^{bc} = \delta_a^c,$$

$$J^a(x) = g^{ab}J_b(x),$$ \hspace{1cm} (2)

\(^1\)By classical algebras we will always mean Poisson Bracket algebras.
where $<\cdot,\cdot>$ is the invariant bilinear form. The current components $J_a(x)$ form a closed algebra under Poisson Brackets (PB)

$$\{<J(x), T_a>, <J(y), T_b>\} =<[T_a, T_b], J(x)>, \delta(x-y) + k<T_a, T_b>\partial_x\delta(x-y).$$

(3)

Now suppose we choose a subalgebra $S=sl(2)$ of $\mathcal{G}$, generated by $\{M_0, M_\pm\}$. We can then split $\mathcal{G}$ into irreducible representations of $S$, thus obtaining a *graded* basis of $\mathcal{G}$ w.r.t. $M_0$:

$$\mathcal{G} = \sum \mathcal{G}_{hi},$$

$$\mathcal{G}_{hi} = \{T | [M_0, T] = h_i T\}. \quad (4)$$

In an obvious notation, we have

$$\mathcal{G} = \mathcal{G}_- + \mathcal{G}_0 + \mathcal{G}_+. \quad (5)$$

In a first stage, we impose a number of first class constraints, the first of which is

$$<J(x), M_-> = k \quad (6)$$

If $M_0$ defines an integer grading ($h_i \in \mathbb{Z}$), the constraints

$$\phi_\gamma(x) \equiv <J(x), \gamma> - k <M_+, \gamma> = 0, \quad \forall \gamma \in \mathcal{G}_- \quad (7)$$

are all first class. However, if $Ad(M_0)$ also has half-integer eigenvalues, not all constraints with $\gamma \in \mathcal{G}_{-\frac{1}{2}}$ are first class, since there will always be $\gamma, \gamma' \in \mathcal{G}_{-\frac{1}{2}}$ such that

$$\{\phi_\gamma(x), \phi_{\gamma'}(y)\} =<J(x), M_-> \delta(x-y) + \lambda \partial_x \delta(x-y), \quad (8)$$

which is not weakly zero. This is no obstacle for the construction of $\mathcal{W}$ algebras [5], but for the Miura transformation we will make use of the gauge transformations generated by first class constraints. The problem of finding a complete set of first class constraints was solved in [4] by introducing the notion of *symplectic halving*. Using the fact that

$$\omega(\cdot,\cdot) =<[\cdot,\cdot], M_+> \quad (9)$$

is a symplectic form on $\mathcal{G}_{-\frac{1}{2}}$, we can apply the Darboux theorem to split $\mathcal{G}_{-\frac{1}{2}}$ into two subspaces of equal dimension,

$$\mathcal{G}_{-\frac{1}{2}} = \mathcal{P}_{-\frac{1}{2}} + \mathcal{Q}_{-\frac{1}{2}}, \quad (10)$$

such that $\omega$ vanishes on $\mathcal{P}_{-\frac{1}{2}}$ and $\mathcal{Q}_{-\frac{1}{2}}$ separately. This halving is by no means unique, but it is sufficient to choose one particular halving. Instead of the constraints (7), we only set

$$\phi_\gamma = 0, \quad \forall \gamma \in \Gamma \equiv \mathcal{G}_{\leq -1} + \mathcal{P}_{-\frac{1}{2}}. \quad (11)$$

The set of constraints (11) is now completely first class.
Next, we demand these constraints to be \textit{conformally invariant}. For this, $\langle J(x), M_\perp \rangle$ will have to have conformal dimension 0 (since we want to constrain this component to a non-zero constant). This is not the case when using the Sugawara tensor

$$L_{\text{Sug}}(x) = \frac{1}{2k} g^{ab} J_a(x) J_b(x),$$

so we introduce the \textit{improved Virasoro tensor}

$$L_{\text{imp}}(x) = L_{\text{Sug}}(x) + \langle M_0, J'(x) \rangle.$$

After imposing the constraints, $J(x)$ is of the form

$$J(x) = kM_+ + j(x),$$

$$j(x) \in \Gamma^\perp \equiv G_- + G_0 + Q_{\frac{1}{2}},$$

where

$$Q_{\frac{1}{2}} = [M_+, P_{-\frac{1}{2}}].$$

As always, first class constraints generate gauge transformations.

$$J(x) \Rightarrow e^{\gamma(x)} J(x) e^{-\gamma(x)} + k\gamma'(x), \quad \forall \gamma(x) \in \Gamma.$$

Using this gauge freedom we can fix $J(x)$ to a number of different gauge choices. \mathcal{W} algebras appear when choosing a so called \textit{Drinfeld-Sokolov} (DS) gauge [1, 3, 4]:

$$J_{\text{DS}}(x) = kM_+ + j_{\text{DS}}(x),$$

$$j_{\text{DS}}(x) \in g_{\text{DS}},$$

where $g_{\text{DS}}$ is a graded complement of $[M_+, \Gamma]$ in $\Gamma^\perp$:

$$\Gamma^\perp = g_{\text{DS}} + [M_+, \Gamma], \quad g_{\text{DS}} \cap [M_+, \Gamma] = \{0\}.$$  \hfill (18)

In a graded basis $\gamma^i$ of $g_{\text{DS}},$

$$j_{\text{DS}} = u_i(x)\gamma^i,$$

$$[M_0, \gamma^i] = -h_i\gamma^i,$$

the components $u_i(x)$ form the algebra \mathcal{W}^S_{\text{DS}} under Dirac brackets. An equivalent point of view is to consider the $u_i(x)$ as a particular set of \textit{gauge invariant polynomial} in the original current components and their derivatives. These polynomials form the same \mathcal{W} algebra under ordinary PBs [3, 4]. Note that different choices of DS gauge will give other gauge fixed polynomials, but this only corresponds to a change of basis in the \mathcal{W} algebra.

In the above graded basis, $u_i(x)$ will have scaling dimension $(h_i + 1)$, but it will not in general be primary. This will only be the case in one specific DS gauge, called \textit{highest weight} gauge (HW):

$$j_{\text{HW}}(x) = W_i(x)T^i, \quad T^i \in \ker(Ad(M_\perp)).$$

\footnote{The fact that they are polynomial is one of the main properties of DS gauges [3, 4].}
or

\[
< j_{HW}(x), T_i > = W_i(x), \quad T_i \in ker(Ad(M_+)),
\]

\[
= 0, \quad T_i \notin ker(Ad(M_+)),
\]

(21)

where the \(T_i\) is the highest weight of a spin \(h_i\) representation of \(S\). \(W_i(x)\) is now a primary of dimension \((h_i + 1)\) w.r.t. \(L_{imp}\), except for the coefficient of \(M_−\), which is basically the improved e.m. tensor \(L_{imp}\). To get \(L_{imp}\) completely, one must still add the e.m tensor of the dimension 1 primaries (which correspond to \(h_i = 0\), as explained in [3, 4, 5].

3 sl(2) subalgebras

Lie algebras generally admit a number of inequivalent \(sl(2)\) subalgebras. A particular one is the principal \(sl(2)\) subalgebra, which can be defined as \(\{M_±, M_0\}\) with \(M_+\) given by

\[
M_+ = \sum_{\alpha \in \Phi_+} E_\alpha,
\]

(22)

where \(\Phi_+\) is the set of positive simple roots. This particular embedding has the additional property that the fundamental representation of \(G\) is also an irreducible representation of \(S\). In the case \(sl(2) \subset_p sl(N)[3]\), \(W^S_G\) is the well-known \(W_N\) algebra. Up to a few exceptions (in \(D_n\) and \(E_{6,7,8}\)) all \(sl(2)\) subalgebras of a semisimple Lie algebra \(G\) can be found as the principal \(sl(2)\) of a regular subalgebra \(H\) of \(G\) [12]. In all these cases, the algebra \(W^S_H\) associated to the reduction scheme \(sl(2) \subset_p H\) will be a subalgebra of the final \(W^S_G\) [4]. For example, in a reduction scheme \(S = sl(2) \subset_p sl(n) \subset G\), \(W^S_G\) will have a \(W_n\) subalgebra. See [4] for a detailed discussion of some reduction schemes, and [3, 4] for an exhaustive list of \(sl(2)\) (and \(sl(2) \times u(1)\)) embeddings and their resulting \(W\) algebras.

Throughout this paper we will use an example based on the reduction scheme \(S = sl(2) \subset_p sl(N - 1) \subset sl(N)\). In the fundamental representation of \(sl(N)\) and with the matrices \(E_{i,i+1}\) as positive simple roots, the constrained current [4] has the form

\[
J(x) = \begin{pmatrix}
* & 1 & 0 & 0 & \cdots & 0 & 0 \\
* & * & 1 & 0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
* & \cdots & * & 1 & \vdots \\
* & \cdots & * & 0 & \cdots & * \\
* & \cdots & * & 0 & \cdots & * \\
\end{pmatrix}.
\]

(23)

The topleft \((N - 1) \times (N - 1)\) part forms the \(sl(N - 1)\) algebra, whose principal \(sl(2)\) subalgebra we choose as \(S\), while the last row and column form two \((N - 1)\) dimensional representations of \(sl(N - 1)\), and thus also of \(S\). Also, the diagonal matrix \((\frac{1}{N-1}, \cdots, \frac{1}{N-1}, -1)\)

\(\subset_p\) denotes a principal embedding.
forms a 1 dimensional representation. In the constrained current \(23\), there are \(\left\lfloor \frac{N-1}{2} \right\rfloor\) zeros in the last column and \(\left\lfloor \frac{N}{2} - 1 \right\rfloor\) in the last row.

The resulting \(\mathcal{W}\) algebra has conformal spin content \(\{1, 2, \ldots, (N-1), 2 \times \frac{N}{2}\}\), the currents of spin 2 to \((N-1)\) forming the subalgebra \(\mathcal{W}_{N-1}\). Note that for \(N\) odd, there will be two bosonic currents with half integer conformal spin. The simplest algebra in this class, at \(N = 3\), is the algebra \(\mathcal{W}_2^3\) introduced in \([9, 10]\).

4 The Miura transformation

There is another useful gauge choice which is not a DS gauge:

\[
J_D(x) = kM_0 + j_D(x),
\]

where

\[
j_D(x) \in g_D \equiv G_0 + Q_{\frac{1}{2}} + Q_{-\frac{1}{2}}.
\]

In the case of \(\mathcal{W}_N\) there are no half-integer grades, and \(G_0\) is exactly the Cartan subalgebra formed by the diagonal matrices (with trace zero). Hence the name diagonal gauge \((D)\). We will continue to use this name, calling \(g_D\) the diagonal subspace, but keeping in mind that \(j_D(x)\) isn’t purely diagonal in the general case. The Miura transformation now appears when comparing the \(D\) and DS gauge \([4]\).

4.1 Miura transformation for \(sl(2) \subset_p sl(N)\)

For the present purpose, the most convenient DS gauge is not the highest weight gauge, but what is sometimes called the Wronskian gauge \([1, 2, 3]\):

\[
J_{DS}(x) = \begin{pmatrix}
0 & k & 0 & 0 & \cdots & 0 \\
0 & 0 & k & 0 & 0 & \cdots \noalign{\vdots}
0 & 0 & 0 & \cdots & \cdots \noalign{\vdots}
\vdots & \vdots & \cdots & \cdots & k & 0 \\
0 & 0 & 0 & 0 & k & u_N & u_{N-1} & u_{N-2} & \cdots & u_2 & 0
\end{pmatrix}.
\]

The diagonal gauge is given by

\[
J_D(x) = \begin{pmatrix}
\phi_1(x) & k & 0 & \cdots \\
0 & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & k \\
0 & \cdots & 0 & \phi_N(x)
\end{pmatrix}, \quad \sum_{i=1}^N \phi_i(x) = 0.
\]

The fields \(\phi_i\) form a closed subalgebra of \(\hat{G}\), so their Dirac brackets are identical to their PBs:

\[
\{\phi_i(x), \phi_j(y)\} = k \left( \delta_{ij} - \frac{1}{N} \right) \partial_x \delta(x-y).
\]
There is a unique gauge transformation of the form (16) converting $J_D$ into the form $J_{DS}$ [3, 4]. This would then give us the currents $u_i$ in a polynomial realisation with the boson fields $\phi_i$. In [4] a general algorithm is described to find this gauge transformation. However, in this case one usually uses a more subtle way to proceed [2], which does not involve finding the generator $\gamma$ explicitly. Consider the set of differential equations

\[(k\partial_x - J_{GF}(x))v(x) = 0,\]  

where $v(x)$ is the column matrix

\[v(x) = \begin{pmatrix} v_1(x) \\ \vdots \\ v_N(x) \end{pmatrix}.
\]

If we change to another gauge using a gauge transformation (16), then $v(x)$ changes according to

\[v(x) \rightarrow e^{\gamma(x)}v(x),\]  

which leaves $v_1(x)$ unchanged, since $\Gamma$ contains the strictly lower triangular matrices. If we express the set of differential equations (28) as a differential operator $L(J_{GF})$ working on $v_1(x)$, then this differential operator is invariant under gauge transformations:

\[L(J_D) = L(J_{DS}),\]  

\[\prod_{i=1}^{N} \left( \partial_x - \frac{1}{k}\phi_i(x) \right) = \partial_x^N + \frac{1}{k} \sum_{i=0}^{N} u_i(x)(\partial_x)^{N-i},\]  

where $u_0 \equiv -1$ and $u_1 \equiv 0$. The left hand side of (30) is to be read in the order

\[\left( \partial_x - \frac{1}{k}\phi_N(x) \right) \cdots \left( \partial_x - \frac{1}{k}\phi_1(x) \right).\]

Equation (30) is exactly the Miura transformation for $W_N$.

4.2 Miura transformation for $sl(2) \subset_p sl(N - 1) \subset sl(N)$

To keep a margin of clarity in the formulas, we set $k = 1$ [4]. The two different gauge choices are now

\[J_{DS} = \begin{pmatrix} \frac{H}{N-1} & 1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \frac{H}{N-1} & 1 & 0 \\ u_{N-1} & u_{N-2} & \cdots & u_2 & \frac{H}{N-1} & W_- \\ W_+ & 0 & \cdots & \cdots & 0 & -H \end{pmatrix} ,
\]

\(^4\)To reinsert $k$, one simply puts a factor $1/k$ in front of every field in the final formula.
and

\[
J_D = \begin{pmatrix}
\phi_1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\
0 & \phi_2 & 1 & 0 & \vdots & \vdots & 0 \\
0 & 0 & \ddots & \ddots & \vdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \phi_{N-2} & 1 & \vdots & 0 \\
0 & \cdots & 0 & \phi_{N-1} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad \sum_{i=1}^N \phi_i = 0 \tag{33}
\]

where \(J_-\) is on the \(\left[\frac{N+1}{2}\right]^{th}\) row, and \(J_+\) on the \(\left[\frac{N+1}{2}\right]^{th}\) column.

The components in the diagonal gauge again form a subalgebra, but this time it is no longer abelian. It is easy to see that \(\{\phi_i, J_x\}\) form the affine algebra \(\widehat{sl}(2) \times \widehat{u(1)}^{N-2}\), where the \(\widehat{sl}(2)\) part is formed by \(J_{\pm}\) and \(J_0 \equiv \frac{1}{2} \left(\phi_{[N/2]} - \phi_N\right)\).

We again consider the set of differential equations (28) in these two gauges. It can easily be seen that in this case, too, the transformation (29) leaves \(v_1(x)\) unchanged. So in principal we could again consider the two differential operators on \(v_1(x)\). However, in the D-gauge this is rather cumbersome, and it turns out that the Miura transformation in this example is most easily expressed using Pseudo-Differential Operators\(^5\):

\[
L(J_D) = N^{-1} \prod_{i=1}^{N-1} (\partial_x - \phi_i) - N^{-1} \prod_{i=\left[\frac{N+1}{2}\right]}^{N-1} (\partial_x - \phi_i)J_- (\partial_x - \phi_N)^{-1} J_+ \prod_{i=1}^{\left[\frac{N+1}{2}\right]} (k\partial_x - \phi_i),
\]

\[
L(J_{DS}) = -\sum_{i=0}^{N-1} u_i \left(\partial_x - \frac{1}{N-1} H\right)^{N-i-1} W_-(\partial_x + H)^{-1} W_+ \tag{34}
\]

where again \(u_0 \equiv -1\) and \(u_1 \equiv 0\).

The basic rules to work with these PDO’s are

\[
(\partial - A)^{-1} = \partial^{-1} \circ \left[1 + \prod_{i=1}^\infty \left(\frac{A}{i!}\right)^{-1}\right],
\]

and the generalized Leibniz rule

\[
\partial^n \circ A = A \partial^n + \sum_{i=1}^\infty \frac{q(q-1)\cdots(q-i+1)}{i!} A^{(i)} \partial^{n-i}. \tag{36}
\]

For \(N = 3\), the result is an \(\widehat{sl}(2) \times \widehat{u(1)}\) realisation of the algebra \(W_3^2\):

\[
H = \phi_1 + \phi_2, \\
u_2 = J_- J_+ + \frac{1}{4}(\phi_1 - \phi_2)^2 + \frac{1}{2}(\phi_1 - \phi_2)', \\
W_+ = -J_+ + (-2\phi_1 - \phi_2)J_+, \\
W_- = J_- \tag{37}
\]

\(^5\)This was also done for some classes of superconformal algebras in [18].
To make the underlying $\mathfrak{sl}(2) \times \mathfrak{u}(1)$ structure more explicit, we identify $J_0 = \frac{1}{2} (\phi_2 - \phi_3)$ and $U = \phi_1$. Also, we reinsert $k$.

\[
H = J_0 + \frac{1}{2} U, \quad
T = u_2 + \frac{3}{4k} H^2 = \frac{1}{k} \left( J_- J_+ + (J_0)^2 + \frac{3}{4} U^2 \right) - \frac{1}{2} J_0' + \frac{3}{4} U',
\]

\[
W_+ = -J_+ - \frac{1}{k} (J_0 + \frac{3}{2} U) J_+,
\]

\[
W_- = J_-.
\]

(38)

4.3 The diagonal subspace

As was already mentioned above, the diagonal subspace $g_D$ for both examples is in fact a subalgebra of $G$, so we don’t have to bother about Dirac Brackets. However, this need not always be the case. If $M_0$ defines an integral grading, there is no problem. In that case, $g_D = G_0$ is just the centralizer of $M_0$.

If $M_0$ defines a half-integral grading, however, things are not that clear. In [4] so-called $H$-compatible halvings are discussed. An $\mathfrak{sl}(2)$ embedding and a particular symplectic halving are said to be $H$-compatible if a new grading operator $H$ can be found such that

1. $Ad(H)$ has only integer eigenvalues,
2. $H - M_0$ commutes with $S$,
3. $\dim \ker (Ad(H)) = \dim \ker (Ad(M_\pm))$,
4. $\mathcal{P}_{-\frac{1}{2}} + G_{\leq -1} = G_{\leq -1}^H$ and 
   $\mathcal{P}_{\frac{1}{2}} + G_{\geq -1} = G_{\geq 1}^H$.

Clearly, if an embedding and a symplectic halving are $H$-compatible, $g_D$ is a subalgebra, since it is then the centraliser of $H$. In [4] all $H$-compatible gradings are listed. In particular, for any $\mathfrak{sl}(2)$ embedding of $G = \mathfrak{sl}(N)$, one can always find a $H$-compatible halving. In other cases, Dirac brackets must be computed.

5 The reduction $N \Rightarrow N - 1$

In [11] it was shown using the Miura transformation how the free field realisation of $\mathcal{W}_N$ can be rewritten as a realisation with one free boson and a $\mathcal{W}_{N-1}$ current algebra. In this section, we will generalize their argument for the class of $\mathcal{W}$ algebras introduced in the previous sections.

First, we will consider the case when $N$ is even (still using $k = 1$). Consider $\mathcal{L}(J_D)$ of (34). We can then define a new set of bosonic fields by

\[
\tilde{\phi}_i = \phi_i + \frac{1}{N - 1} \phi_{N-1} , \quad i = 1, \ldots, N - 2
\]
\[ \tilde{\phi}_{N-1} = \phi_N + \frac{1}{N-1} \phi_{N-1}, \]
\[ \Phi = \frac{1}{N-1} \phi_{N-1}. \quad (39) \]

The PBs of these new fields are
\[
\begin{align*}
\{ \tilde{\phi}_i(x), \tilde{\phi}_j(y) \} &= \left( \delta_{ij} - \frac{1}{N-1} \right) \partial_x \delta(x-y), \\
\{ \tilde{\phi}_i(x), \Phi(y) \} &= 0,
\end{align*}
\]
and we also have
\[ \sum_{i=1}^{N-1} \tilde{\phi}_i = 0. \quad (41) \]

Writing \( \mathcal{L} (J_D) \) in these new variables, we get
\[
\mathcal{L} (J_D) = (\partial_x - (N-1) \Phi) \left[ \prod_{i=1}^{N-2} (\partial_x - \tilde{\phi}_i + \Phi) - \prod_{i=\frac{N}{2}+1}^{N-2} (\partial_x - \tilde{\phi}_i + \Phi) J^- (\partial_x - \tilde{\phi}_{N-1} + \Phi)^{-1} J^+ \prod_{i=1}^{\frac{N-1}{2}} (\partial_x - \tilde{\phi}_i + \Phi) \right]. \quad (42)
\]

Using the identity
\[
\prod_{i=a}^{b} (\partial_x - \tilde{\phi}_i + \Phi) = e^{-\Phi} \prod_{i=a}^{b} (\partial_x - \tilde{\phi}_i) e^{\Phi}, \quad (43)
\]
this reduces to
\[
\begin{align*}
\mathcal{L} (J_D) &= (\partial_x - (N-1) \Phi) e^{-\Phi} \left[ \prod_{i=1}^{N-2} (\partial_x - \tilde{\phi}_i) - \prod_{i=(N-1)/2}^{(N-1)/2} (\partial_x - \tilde{\phi}_i) J^- (\partial_x - \tilde{\phi}_{N-1})^{-1} J^+ \prod_{i=1}^{(N-1)/2} (\partial_x - \tilde{\phi}_i) \right] e^{\Phi} \\
&= (\partial_x - (N-1) \Phi) e^{-\Phi} \left[ \mathcal{L} \left( J_{DS}^{N-1} \right) \right] e^{\Phi}. \quad (44)
\end{align*}
\]

The part in square brackets is now exactly the PDO \( \mathcal{B} \) for extended \( \mathcal{W}_{N-2} \), and can be written in any gauge. If we put \( \mathcal{L} \left( J_{DS}^{N-1} \right) \), we thus obtain a realisation of extended \( \mathcal{W}_{N-1} \) with the currents of the extended \( \mathcal{W}_{N-2} \) and the free boson \( \Phi \).

If \( N \) is odd, the method is essentially the same, except that we must now extract \( \phi_1 \) as the free boson field instead of \( \phi_{N-1} \).

So we see that in the same way that \( Vir \) is the “base” for all \( \mathcal{W}_N \) algebras (in the sense that all \( \mathcal{W}_N \) algebras can be written as a Virasoro algebra and \( N-2 \) commuting free bosons), \( \mathcal{W}_{N-2}^3 \) is the base for the new class of extended \( \mathcal{W}_N \) algebras introduced in the previous sections. We might ask ourselves at this point whether it is possible to extract the \( \mathfrak{sl}(2) \) subalgebra too, but looking at \( \mathcal{B} \), it does not seem possible to realise \( \mathcal{W}_{N-2}^3 \) with a Virasoro tensor and a \( \mathfrak{sl}(2) \) current algebra.

9
An example: \( N = 4 \)

As an example of this reduction, we write the algebra resulting from the scheme \( sl(2) \subset_p \)

\( sl(3) \subset sl(4) \) (which is a particular extension of \( \mathcal{W}_3 \)) in terms of \( \mathcal{W}_3^2 \) and a free boson \( \Phi \) with normalisation

\[ \{ \Phi(x), \Phi(y) \} = \frac{1}{12} \partial_x \delta(x - y). \] (45)

The currents of \( \mathcal{W}_3^2 \) are written in capital letters.

\[
\begin{align*}
    h &= H + \Phi, \\
    w_+ &= W_+, \\
    w_- &= W_- - W_-(H + 4\Phi), \\
    t &= T + \frac{1}{2}H' + 6\Phi^2 - 4\Phi', \\
    w_3 &= \frac{T'}{2} + W_+W_- + \frac{1}{3}(H - 8\Phi)T - \\
    &- \frac{1}{12}(H - 8\Phi)^{''} - \frac{5}{6}HH' + \frac{2}{3}(H\Phi)' + \frac{16}{3}\Phi' - \\
    &- \frac{7}{27}H^3 + \frac{20}{9}H^2\Phi - \frac{16}{9}H\Phi^2 + \frac{128}{27}\Phi^3.
\end{align*}
\] (46)

Note that \( w_\pm \) are dimension 2 primaries w.r.t. \( t \), whereas \( W_\pm \) are dimension 3/2 primaries w.r.t. \( T \).

Other classes of algebras

It is now logical to ask whether there might exist other classes of \( \mathcal{W} \) algebras connected in the same way. Without giving a proof, we can expect such behaviour to present itself for classes of algebras whose diagonal subalgebra is \( g \times (u(1))^n \), with \( g \) a semisimple Lie algebra. It is easy to find many different examples. We present two, both based on \( sl(N) \) algebras. In the first column of the table, we give the regular subalgebra of \( sl(N) \) of which \( S \) is the principal \( sl(2) \) subalgebra.

| Regular subalgebra | Diagonal subspace | Spin content |
|--------------------|--------------------|--------------|
| \( sl(N - 2) \)    | \( sl(3) \times (u(1))^{N-3} \) | \( 2, 3, \ldots, (N - 2), 4 \times 1, 4 \times \left( \frac{N-1}{2} \right) \) |
| \( sl(N - 2) \times sl(2) \) | \( (sl(2))^2 \times (u(1))^{N-3} \) | \( 1, 2 \times 2, 3, \ldots, (N - 2), 2 \times \frac{N}{2}, 2 \times \left( \frac{N-2}{2} \right) \) |

*table 1. Some examples of other possible classes of extended \( \mathcal{W} \) algebras.*

It appears that any subalgebra of \( sl(N) \) of maximal rank \( (N - 1) \) can be used as the diagonal subspace for a particular \( \mathcal{W} \) algebra. In particular, if \( g_D \) is the Cartan subalgebra, we get \( \mathcal{W}_N \), and if \( g_D \) is \( sl(N) \) itself, we get the current algebra \( sl(N) \) (whose universal covering algebra contains all others).
6  An example of quantization

In the $W_N$ case, the Miura transformation can be quantized directly by interpreting all products of fields as normal ordered products. In the general case, this is no longer possible. For instance, it is well-known that the quantum version of the Sugawara tensor
\[ L^\text{Sug}_C = \frac{1}{2k} g^{ab} J_a J_b \]
is
\[ L^\text{Sug}_Q = \frac{1}{2k + Q} g^{ab} : J_a J_b : , \]
where
\[ Qg^{ab} = \sum f^{ac}_{\cdot \cdot} f^{bd}_{\cdot \cdot} . \]
So it is to be expected that the realisations obtained in the previous sections, where the Miura form is no longer abelian, also don’t quantize trivially. As an example of how the coefficients can change dramatically, we present here a quantum construction of $W^2_3$. We choose the following normalisation for the $sl(2) \times u(1)$ current algebra:
\begin{align*}
J_+(z)J_-(w) &= \frac{k}{(z-w)^2} + \frac{2J_0(w)}{(z-w)} + \cdots , \\
J_0(z)J_\pm(w) &= \pm \frac{J_\pm(w)}{(z-w)} + \cdots , \\
J_0(z)J_0(w) &= \frac{k/2}{(z-w)^2} + \cdots , \\
U(z)U(w) &= \frac{2(k+2)/3}{(z-w)^2} + \cdots .
\end{align*}

We now want a construction of $W^2_3$ with exactly the same terms as the classical construction (38), but permitting the coefficients to change. It turns out that there is exactly one\footnote{Up to some equivalences.} solution for which the algebra closes:
\begin{align*}
H &= J_0 + \frac{1}{2} U , \\
T &= \frac{1}{k+2} \left( [J_+J_-]_S + [J_0J_0]_S + \frac{3}{4} [UU]_S \right) + \frac{1}{2} J'_0 - \frac{3k}{4(k+2)} U' , \\
W_+ &= -\frac{2k+1}{2} J'_+ + [J_0J_+]_S + \frac{3}{2} [UJ_+]_S , \\
W_- &= J_-, \tag{48}
\end{align*}
where \([ \ ]_S\) denotes symmetric normal ordering. The formula for $H$ appears identical to the classical case (38), but this is a consequence of the chosen normalisation of $U$ in (47).
Up to a redefinition $k \to (k + 1)$, the currents (48) close under the same OPEs as the currents realised in [10]. In fact, if we realise the $sl(2)$ algebra using one free boson $\phi$ and a $(\beta, \gamma)$ system [15],

$$
\begin{align*}
J_+ &= -\beta \gamma^2 + i \sqrt{2k + 4} \gamma \partial \phi + k \partial \gamma, \\
J_- &= \beta, \\
J_0 &= \beta \gamma - i \frac{1}{2} \sqrt{2k + 4} \partial \phi,
\end{align*}
$$

we get exactly the same construction as [10], based on two free bosons and a $(\beta, \gamma)$ system.

7 Conclusions and outlook

We have shown how the Miura transformation arises for classical $W$ algebras constructed via hamiltonian reduction. This has permitted us to realise these algebras with a relatively simple current algebra which we called the diagonal subspace $g_D$. In the simplest case of $sl(N)$, the Dirac brackets of $g_D$ are identical to the Poisson brackets, which makes it easy to identify the algebraic structure of $g_D$.

Using this realisation, we have shown for a particular example how $W$ algebras with a diagonal subalgebra of the form $\hat{g} \times \hat{u}(1)^n$ can all be reduced to a certain “base” algebra, realised by $\hat{g} \times \hat{u}(1)$, and a number of free bosons. We expect this to be a general principle.

For the simple case $W_3^2$ we have also given the quantum version of the Miura realisation. However, in general the quantum Miura transformation will be very hard to compute systematically without applying the BRST methods of [2], if indeed quantization is always possible without having to introduce new terms.

In e.g. [16], the reduction $W_N \Rightarrow W_{N-1}$ was also generalized in other ways. For the classes of (quantum) algebras called $W_N$ and $WD_N$ realisations were found not only with one smaller $W$ algebra and some free bosons, but also with a number of commuting smaller $W$ algebras. It is natural to expect realisations of this kind to appear in the general $W_S^G$ case as well.

Another possibility for further investigation is the supersymmetric version of the arguments presented in this paper. Hamiltonian reduction of superalgebras, based on $osp(1|2)$ subalgebras of Lie superalgebras, was discussed in [14, 17]. In [18], the Miura transformation for certain classes of superconformal algebras was computed.

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