LIMIT CYCLES AND THE DISTRIBUTION OF ZEROS OF FAMILIES OF ANALYTIC FUNCTIONS

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Abstract

We estimate the expected number of limit cycles situated in a neighbourhood of the origin of a planar polynomial vector field. Our main tool is a distributional inequality for the number of zeros of some families of univariate holomorphic functions depending analytically on a parameter. We obtain this inequality by methods of Pluripotential Theory. This inequality also implies versions of a strong law of large numbers and the central limit theorem for a probabilistic scheme associated with the distribution of zeros.

1. Introduction.

1.1. The second part of Hilbert’s sixteenth problem asks whether the number of isolated closed trajectories (limit cycles) of a planar polynomial vector field is always bounded in terms of its degree. This is the most prominent finiteness problem, related to a fairly general class of algebraic differential equations and is one of the few Hilbert’s problems, which remain unsolved. Recent results of Y. Il’yashenko [I] and of Écalle, Martinet, Moussu and Ramis [EMMR] give global finiteness of the number of limit cycles for each individual vector field (but leave open the question of the existence of a bound depending on the degree only). In our paper we consider the local version of the problem in which one asks for explicit bounds (in terms of degree only) on the number of limit cycles situated in a small neighbourhood of a singular point of the vector field. Except for the result of Bautin [B], the answer to this problem is not known. According to Smale (see [S]) the global estimate should be polynomial in the degree $d$ of the components of the vector field. Our estimate below (local case) is “in the mean” but gives a substantially better estimate of the

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order \((\log d)^2\). Moreover, Example 1.2 shows that locally the maximal number of limit cycles can be essentially bigger than \((\log d)^2\). We now formulate our result precisely.

Consider a system of ODE’s in \(\mathbb{R}^2\)

\[
\begin{align*}
\dot{x} &= -y + F(x, y) \\
\dot{y} &= x + G(x, y)
\end{align*}
\]

(1.1)

where \(F\) and \(G\) are polynomials of degree \(d\) whose Taylor expansions at 0 begin with terms of degree \(\geq 1\). The origin \((0, 0)\) is a singular point of (1.1). Assume that

\[
F = \sum_{k=1}^{d} F_k, \quad G = \sum_{k=1}^{d} G_k
\]

where

\[
F_k(x, y) = \sum_{i=0}^{k} a_{ki} x^i y^{k-i}, \quad G_k(x, y) = \sum_{i=0}^{k} b_{ki} x^i y^{k-i}
\]

with real \(a_{ki}, b_{ki}\). Let

\[
|F_k| := \left( \sum_{i=0}^{k} |a_{ki}|^2 \right)^{1/2}, \quad |G_k| := \left( \sum_{i=0}^{k} |b_{ki}|^2 \right)^{1/2}
\]

Assume also that

\[
\sum_{k=1}^{d} (a^{k-1}|F_k|)^2 + \sum_{k=1}^{d} (a^{k-1}|G_k|)^2 \leq N^2 .
\]

(1.2)

Condition (1.2) determines an ellipsoid \(E(a, N) \subset \mathbb{R}^s\), \(s := d(d + 3)\), in the space of parameters. In what follows we identify pairs \(F, G\) with points \(v \in \mathbb{R}^s\). Further, for any \(F, G\) corresponding to \(v \in \mathbb{R}^s\) denote by \(C(v, K)\) the number of limit cycles of (1.1) in the disk \(D_K := \{ (x, y) \in \mathbb{R}^2; |x|^2 + |y|^2 \leq K^2 \}\). We will write \(C(v, K) = +\infty\) if any trajectory in \(D_K\) is closed (the case of the center). Let \(|\cdot|\) denote the Lebesgue measure on \(\mathbb{R}^s\).

**Theorem A** Let \(N \leq \frac{1}{16 \pi \log d}\) be a positive number. There are absolute positive constants \(c_1, c_2\) such that for any \(T \geq 0\)

\[
|\{ v \in E(a, N); C(v, a/2) \geq T \}| \leq c_1 se^{-c_2 T/\log d} \cdot |E(a, N)|
\]

**Corollary 1.1** Under assumptions of Theorem A the expected number of limit cycles of a random vector field (1.1) in the disk \(D_{a/2}\) is bounded from above by \(c(\log d)^2\) with an absolute constant \(c > 0\).

The corollary follows from the distributional inequality of Theorem A which we apply to estimate the integral \(\left[ \int_{E(a, N)} C(v, a/2) dv \right] / |E(a, N)|\) by \(c(\log d)^2\).
Example 1.2  *(The maximal number of limit cycles in $D_{a/2}$ can be essentially bigger than $(\log d)^2$.)* Consider the system

$$\begin{align*}
\dot{x} &= -y + x f(x^2 + y^2) \\
\dot{y} &= x + y f(x^2 + y^2)
\end{align*}$$

where $f$ is a real polynomial whose degree $l$ is the integer part of $(d - 1)/2$. Assume also that $f$ has $l$ different positive roots $x_1, ..., x_l$ in open interval $(0, 1/2)$. Then the above system has $l$ limit cycles \( s_i := \{ (x, y) \in \mathbb{R}^2; \ |x|^2 + |y|^2 = x_i \} \) for $i = 1, ..., l$ in $D_{1/2}$ (see, e.g. [Le], Ch.X, Sec.5). Clearly we can choose coefficients of $f$ so small that the vector of coefficients of polynomials determining the above vector field belongs to $E(1, N)$.

1.2. The proof of Theorem A is based on a distributional inequality for the number of zeros of some families of univariate holomorphic functions depending analytically on parameter. Let us note that in recent years the problem of estimating the number of zeros for families of analytic functions depending analytically on a parameter has been extensively studied in connection with different aspects of modern analysis; e.g. 2nd part of Hilbert’s 16th problem, dynamical and control systems etc (see [BY], [FY], [IY], [G], [NY], [Y]). However, in many cases the existing estimates are essentially differ from desired. In view of the inequality of Theorem B below it seems likely that “good” estimates in similar problems can be obtained in the mean. To formulate the result, let \( f := \{ f_v ; v \in B_c(0, r) \} \), $r > 1$, be a family of holomorphic in the open unit disk $\mathbb{D}_1$ functions depending holomorphically on parameter $v$ varying in the open Euclidean ball $B_c(0, r) \subset \mathbb{C}^N$. Assume that for some $\mathbb{D}_s := \{ z \in \mathbb{C} ; |z| < s \}$ with $s < 1$

$$\sup_{v \in B_c(0, r)} \sup_{z \in \mathbb{D}_1} |f_v(z)| \leq M < \infty \quad \text{and} \quad \sup_{v \in B_c(0, 1)} \sup_{z \in \mathbb{D}_s} |f_v(z)| \geq 1. \quad (1.3)$$

Denote the set of these functions by $\mathcal{H}(M, r, s)$.

Set, further, for $f \in \mathcal{H}(M, r, s)$

$$N_f(v) := \# \{ z \in \mathbb{D}_s ; \ f_v(z) = 0 \}; \quad (1.4)$$

in addition, $N_f(v) = +\infty$, if $f_v \equiv 0$ identically on $\mathbb{D}_1$. Let $|\cdot|$ denote the Lebesgue measure on $\mathbb{C}^N$.

**Theorem B**  For every $T \geq 0$ inequality

$$|\{ v \in B_c(0, 1) ; \ N_f(v) \geq T \}| \leq N_c_1 e^{-c_2 T/\log M} \cdot |B_c(0, 1)|,$$

holds with constants $c_1, c_2$ depending only on $s, r$.

**Remark 1.3** According to the doubling inequality from [FN] and inequality \((2.4)\) below, the function $N_f$ is uniformly bounded on the set $B_c(0, 1) \setminus V_f$, where $V_f := \{ v \in B_c(0, 1) ; f_v \equiv 0 \}$. As we will show (see Proposition \((2.7)\) below) the set $V_f$ has measure 0. The following example shows, nevertheless, that $N_f$ can assign arbitrary big values on $B_c(0, 1) \setminus V_f$. 
Example 1.4 Let $O^N(D_1; B_c(0, 1))$ denote the set of holomorphic mappings $f : D_1 \to B_c(0, 1) \subset \mathbb{C}^N$. For any $f \in O^N(D_1; B_c(0, 1))$ consider the number of intersections of $f(D)$, $s < 1$, with hyperplane $\{(z_1, ..., z_N) \in \mathbb{C}^N; l(a, z) := a_1z_1 + ... + a_nz_N + a_{N+1} = 0\}$, where $a = (a_1, ..., a_{N+1}) \in \mathbb{C}^{N+1}$. It coincides with the number of zeros in $D$, the function $l(a, f(z))$. For any $f \in O^N(D_1; B_c(0, 1))$ the function $l(a, f(z))$ satisfies (1.3) with $M = 3, r = 2$ if $a \in B_c(0, 2)$. Let $B_k(z)$ be a Blashke product with $k$ zeros in $D$. Then the mapping $f(z) := (B_k(z), 0, ..., 0)$ belongs to $O^N(D_1; B_c(0, 1))$ and $l(1, 0, ..., 0, f(z))$ has $k$ zeros in $D$.

We prove Theorem B in Section 2 and then in Section 3 we prove Theorem A.

1.3. In 1943 M.Kac [Ka] proved that the expected number of real zeros of a random polynomial of degree $n$ whose coefficients are independent standard normals asymptotically equals $\frac{2}{3} \log n$ as $n \to \infty$. This theorem was a starting point for many results on zeros of random polynomials and other functions linearly depending on random variables (e.g., see [EK]). Nonlinear problems of this kind appear in many important fields of pure and applied mathematics. Here we mention only a class of problems related to distribution of eigenvalues of random matrices (among other applications see [Gi], [M], [Mu] and [EK] for the corresponding results and applications to quantum physics and multivariate statistics).

Based on Theorem B we study similar problems for the family of analytic functions $\{f_v\}$ of Section 1.2.

Let $f^{(k)} = \{f_v^{(k)} : v \in B_c(0, r)\}$, $k = 1, 2, ..., be a sequence of functions from $\mathcal{H}(M, r, s)$. Consider a sequence $\{N_k\}_{k=1}^\infty$ of random variables defined on the probability space $B_c(0, 1)$ as follows. For a nonnegative integer $l$ probability

$$P(N_k = l) := \frac{1}{|B_c(0, 1)|}\{|v \in B_c(0, 1) ; \mathcal{N}_{f^{(k)}}(v) = l\}.$$  (1.5)

Denote, as usual, the expectation of $N_k$ by $E(N_k)$ and the variance of $N_k$ by $D(N_k)$.

**Theorem 1.5** There are positive constants $c = c(r, s), \bar{c} = \bar{c}(r, s)$ such that

$$\sup_k E(N_k) \leq c \log M \log (N + 1), \quad \sup_k D(N_k) \leq \bar{c}(\log M \log (N + 1))^2.$$

Let us consider $\Omega = \prod_{k=1}^\infty \mathbb{Z}_+$ and the product probability $\mathbb{P}$ on $\Omega$ associated with distribution (1.3). We study the following probabilistic scheme related to distribution $\mathbb{P}$. For every $k = 1, 2, ...$ we decompose $B_c(0, 1)$ into (finite) number of domains where $\mathcal{N}_{f^{(k)}}$ is constant (removing a set of measure zero where $\mathcal{N}_{f^{(k)}} = +\infty$, see Remark 1.3). Then for each $k = 1, 2, ...$ we choose at random and independently one of these domains. Theorem 1.5 guarantees for this scheme fulfillment of

**Corollary 1.6** The following inequality

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^n N_k \leq c \log M \log (N + 1)$$

holds with $\mathbb{P}$ probability one (see also Example 4.4).
Assume in addition that the family \( \{ f^{(k)} \} \subset \mathcal{H}(M, r, s) \) satisfies the following conditions:

there are a constant \( \delta > 0 \) and an open disk \( \mathbb{D}_{s'} \) with \( s' < s \) such that for every \( k \)

\[
(a) \quad \min_{z \in \mathbb{D}_s} |f^{(k)}(z)| > \delta
\]

for some \( v = v(k) \in \overline{B}_c(0, 1) \);

\[
(b) \quad \max_{z \in \mathbb{D}_{s'}} |f^{(k)}(z)| > \delta \quad \text{and} \quad f^{(k)}(z) = 0
\]

for some \( v' = v'(k) \in \overline{B}_c(0, 1) \) and \( z = z(k) \in \mathbb{D}_{s'} \).

Under these assumptions and in the above notations the following result holds.

**Theorem 1.7** The sequence \( \{ N_k \}_{k=1}^{\infty} \) of independent random variables satisfies the normal distribution law, that is,

\[
P\left\{ \frac{1}{B_n} \sum_{k=1}^{n} (N_k - E(N_k)) < x \right\} \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz, \quad (n \to \infty)
\]

uniformly in \( x \). Here \( B_n^2 := \sum_{k=1}^{n} D(N_k) \) is the sum of the variances of \( N_k \).

**Example 1.8** Let \( \{ f_k(z) := (cz, f_{1,k}(z), ..., f_{N-1,k}(z)) \}_{k=1}^{\infty}, \ c \neq 0 \), be a sequence of holomorphic mappings from \( \mathcal{O}^N(\mathbb{D}_1; B_c(0,1)) \), i.e. \( \sup_{z \in \mathbb{D}_1} (|cz|^2 + \sum_{i=1}^{N-1} |f_{i,k}(z)|^2) < 1 \) for each \( k \geq 1 \) (see Example 1.7). We set \( f^{(k)}(z) := l(v, f_k(z)) \). Then, clearly, the sequence \( \{ f^{(k)} \}_{k \geq 1} \) satisfies the conditions of Theorem 1.7.

Consider another theorem of this type generalizing a classical result on behavior of zeros of random polynomials.

Let \( P_{k,v}(z) := \sum_{i=0}^{k} a_{ik}(v)z^i \) be a polynomial in \( z \in \mathbb{C} \) and \( v \in \mathbb{C}^{n(k)} \) of degree \( d(k) \) in \( v \) satisfying

\[
(a) \quad d(k) \log n(k) = o(k) \quad \text{as} \quad k \to \infty;
\]

\[
(b) \quad \sup_{||v|| \leq 1} |a_{ik}(v)| \leq 1, \quad \text{if} \quad 1 \leq i \leq k - 1;
\]

\[
(c) \quad \sup_{||v|| \leq 1} |a_{ik}(v)| = 1, \quad \text{if} \quad i = 0, k.
\]

Let \( \epsilon > 0 \) be arbitrary small. Denote by \( A_\epsilon \) the annulus \( \{ z \in \mathbb{C} \; ; \; 1 - \epsilon < |z| < 1 + \epsilon \} \).

Consider the counting function \( \widehat{N}_k(v) \) defined by

\[
\widehat{N}_k(v) := \# \{ z \in A_\epsilon ; P_{k,v}(z) = 0 \}.
\]

Further, determine as before a random variable \( \widehat{N}_k \) defined on the probability space \( B^{n(k)}_c := \{ v \in \mathbb{C}^{n(k)} ; ||v|| < 1 \} \) by

\[
P(\widehat{N}_k = l) := \frac{1}{|B^{n(k)}_c|} \text{mes}_{2n(k)}(\{v \in B^{n(k)}_c ; \widehat{N}_k(v) = l\}),
\]

where \( l = 0, 1, ..., 5 \).
Theorem 1.9  Expectation $E(\tilde{N}_k)$ of the numbers of zeros of $P_{k,v}$ in $A_\epsilon$ is bounded from below by $k(1 - \alpha_1)$ as $k \to \infty$.

Remark 1.10  Thus, as in the linear case (i.e., for $a_{ik}(v) = v_{i+1}$, $0 \leq i \leq n(k) - 1 := k$) the zeros of polynomials $P_{k,v}$ concentrate on the unit circle, as their degrees grow. It is possible to prove that if $\{a_{ik}\}_{i=0}^n$ are homogeneous and mutually independent on $B_c^{n(k)}$ ($k = 1, 2, \ldots$) then roots of random polynomials $P_{k,v}$ are asymptotically uniformly distributed on the unit circle as $k \to \infty$.

2. Proof of Theorem B.

Let us verify, first, that the set

$$\omega_T := \{ v \in B_c(0,1) : \mathcal{N}_f(v) \geq T \}$$

is measurable.

Proposition 2.1  For every $f \in H(M, r, s)$ there is a closed subset $V = V_f \subset B_c(0, r)$ of measure 0 such that $\mathcal{N}_f$ is upper semicontinuous on $B_c(0, r) \setminus V_f$.

Proof. Let

$$V_f := \{ v \in B_c(0, r) ; \ f_v \equiv 0 \text{ on } D_1 \}.$$ 

Prove that $V_f$ is a proper complex analytic subset of $B_c(0, r)$. To this end we set

$$\tilde{V}_f := \bigcap_{k \geq 0} \{(v, z) \subset B_c(0, r) \times D_1 ; \ \frac{\partial^k}{\partial z^k} f_v(z) = 0 \}.$$ 

According to the definition $\tilde{V}_f$ is a complex analytic subset of $B_c(0,1) \times D_1$. If $\Pi : B_c(0, r) \times D_1 \to B_c(0, r)$ is the natural projection, then $\Pi(\tilde{V}_f)$, clearly, coincides with $V_f$. Hence $\Pi^{-1}(v) \subset \tilde{V}_f$ for every $v \in V_f$. Therefore $V_f$ is biholomorphically isomorphic to the set $\tilde{V}_f \cap (B_c(0, r) \times \{0\})$ which is analytic as intersection of analytic sets. It remains to check that $V_f$ is a proper subset of $B_c(0, r)$. If, to the contrary, $\tilde{V}_f$ coincides with $B_c(0, r) \times D_1$, then $\{f_v\}_v = \{0\}$ which contradicts to the second inequality of (1.3).

Prove now that $\mathcal{N}_f$ is upper semicontinuous on $B_c(0, r) \setminus V_f$. This means that

$$\limsup_{k \to \infty} \mathcal{N}_f(v_k) \leq \mathcal{N}_f(v) \quad (2.1)$$

for every $\{v_k\} \subset B_c(0, r) \setminus V_f$ converging to $v \in B_c(0, r) \setminus V_f$. Denote by $\gamma_v$ the boundary of the disk $\{z ; |z| \leq r\}$, $r > s$, containing as the same number of zeros of $f_v$ as $\overline{D}_s$. Then we can represent $\mathcal{N}_f$ by

$$\frac{1}{2\pi i} \int_{\gamma_v} \frac{\partial}{\partial z} f_v(z) \, dz. \quad (2.2)$$
By continuity of \( f_w(z) \) there is an open connected neighbourhood \( U_v \) of \( v \) such that the function \( f_w \) has no zeroes on \( \gamma_v \) for each \( w \in U_v \). Therefore the right hand side of (2.2) is well defined for such \( f_w \) and is a continuous function in \( w \) on \( U_v \). Moreover, this function assigns only integer values and \( U_v \) is connected. Hence,

\[
\frac{1}{2i\pi} \int_{\gamma_v} \frac{\partial}{\partial z} f_w(z) dz = N_f(v)
\]

(2.3)

for all \( w \in U_v \). Let now \( \{v_k\} \) be a sequence from (2.1). Because of the precompactness of the family \( \{f_v\} \) in the topology of uniform convergence on compact subsets of \( \mathbb{D} \), see (1.3), we can assume that \( \{f_{v_k}\} \) converges to \( f \) in this topology. In particular, \( \{f_{v_k}|_{\gamma_v}\} \) converges uniformly to \( f|_{\gamma_v} \). Let \( k_0 \) be such that \( v_k \) belongs to \( U_v \) for any \( k \geq k_0 \). Then the number of zeros \( N_k \) of \( f_{v_k} \), \( k \geq k_0 \), inside \( \gamma_v \) coincides with \( N_f(v) \). Moreover, \( \mathbb{D}_s \) is also containing in the disk bounded by \( \gamma_v \); therefore

\[ N_f(v_k) \leq N_k = N_f(v). \]

The proposition is proved. \( \Box \)

**Remark 2.2** Let

\[ \widetilde{N}_k(v) := \{ z \in \mathbb{D}_s : f_v(z) = 0 \} \]

denote the number of zeros in the open disk \( \mathbb{D}_s \). Then by similar arguments one can establish that \( \widetilde{N}_k \) is lower semicontinuous on the set \( B_c(0, r) \setminus V_f \).

Verify now that \( \omega_T \) is a compact subset of \( \overline{B}_c(0, 1) \). In fact, let \( \{v_k\} \subset \omega_T \) be a sequence converging to \( v \in \overline{B}_c(0, 1) \). If \( v \in B_c(0, r) \setminus V_f \) then, according to upper semicontinuity of \( N_f \), it also belongs to \( \omega_T \). Otherwise, \( v \in V_f \) and therefore \( N_f(v) = +\infty \) implying \( v \in \omega_T \).

The next part of the proof involves arguments of Pluripotential Theory. We, first, recall the following classical estimate for the number of zeros of a holomorphic function \( h \) in the closed disk \( \overline{D}_s \) (a consequence of Jensen’s inequality).

Let \( M_1 := \sup_{\mathbb{D}_s} \log |h| \) and \( M_2 := \sup_{\mathbb{D}_s} \log |h| \). Then

\[ \# \{ z \in \overline{D}_s : h(z) = 0 \} \leq c(M_1 - M_2) \]

(2.4)

with \( c = c(s) \).

Let now the family \( f = \{f_v\} \) belongs to \( H(M, r, s) \). For a fixed \( v \in B_c(0, r) \) we set

\[ M_{f,1}(v) := \frac{\sup_{\mathbb{D}_s} \log |f_v| - \log M}{\log M} \quad \text{and} \quad M_{f,2}(v) := \frac{\sup_{\mathbb{D}_s} \log |f_v| - \log M}{\log M}. \]

Since \( f_v(z) \) is continuous on \( \mathbb{D}_1 \times B_c(0, r) \), the functions \( e^{M_{f,1}} \) and \( e^{M_{f,2}} \) are continuous on \( B_c(0, r) \). In particular, \( M_{f,1} \) and \( M_{f,2} \) are upper semicontinuous. Moreover, each of these functions is, by definition, supremum of a family of plurisubharmonic on \( B_c(0, r) \) functions. Therefore they are also plurisubharmonic. Further, inequality (2.4) gets

\[ N_f(v) \leq c \log M(M_{f,1}(v) - M_{f,2}(v)). \]

(2.5)
Note that from (1.3) it follows that \( \sup_{B_c(0, r)} M_{f, 1} \leq 0 \). Moreover, inequalities (1.3) imply that there is a point \( x_f \) in the open ball \( B_c(0, 1) \) such that

\[
\sup_{B_c(0, 1)} M_{f, 2} > M_{f, 2}(x_f) \geq -2.
\]

From Lemma 3 of [BG] it follows that there exists a ray \( l_f \) with the origin at \( x_f \) such that

\[
\frac{\text{mes}_1(B_c(0, 1) \cap l_f)}{\text{mes}_1(\omega_T \cap l_f)} \leq \frac{2N|B_c(0, 1)|}{|\omega_T|}. \tag{2.6}
\]

Consider the one-dimensional affine complex line \( l_f' \) containing \( l_f \). Let \( D_1, D_2 \) be the intersections of \( l_f' \) with \( B_c(0, r) \) and \( B_c(0, 1) \), respectively. If \( z_f \) denotes the point of \( l_f' \) such that \( d_f := \text{dist}(0, l_f') = |z_f| \), then \( D_1, D_2 \) are open disks in \( l_f' \) centered at \( z_f \) with radii

\[
r_1 := \sqrt{r^2 - d_f^2}, \quad r_2 := \sqrt{1 - d_f^2},
\]

respectively. Since \( r > 1 \), the ratio \( r_1/r_2 \geq r \). Consider now the disks

\[
\bar{D}_2 := \frac{1}{r r_2} (D_2 - z_f) \quad \text{and} \quad \bar{D}_1 := \frac{1}{r r_2} (D_1 - z_f)
\]

We can thought of them as open disks in \( \mathbb{C} \). Then, clearly,

\[
\mathbb{D}_{1/r} = \bar{D}_2 \subset \mathbb{D}_1 \subset \bar{D}_1.
\]

Further, consider the function \( M' \) defined on \( \mathbb{D}_1 \) by \( M'(z) := M_{f, 2}(r r_2 z + z_f) \). Since \( M_{f, 2} \) is non-positive plurisubharmonic, the function \( M' \) has the same property and satisfies:

\[
\sup_{\mathbb{D}_{1/r}} M' \geq -2.
\]

Set \( K := \omega_T \cap l_f \) and \( K' := \frac{1}{r r_2}(K - z_f) \) and \( I_r = \frac{1}{r r_2}((D_2 \cap l_f) - z_f) \). Then according to Theorem 1.2 of [Br] (see also section 2.3 there) there is a constant \( c(r) > 0 \) such that

\[
-2 \leq \sup_{\mathbb{D}_{1/r}} M' \leq c(r) \log \frac{4|I_r|}{|K'|} + \sup_K M'. \tag{2.7}
\]

Noting that

\[
\sup_{K'} M' = \sup_K M_{f, 2} \quad \text{and} \quad \frac{|I_r|}{|K'|} = \frac{\text{mes}_1(B_c(0, 1) \cap l_f)}{\text{mes}_1(\omega_T \cap l_f)}
\]

and taking into account (2.6) we obtain from (2.7)

\[
\sup_K M_{f, 2} \geq -2 - c(r) \log \frac{8N|B_c(0, 1)|}{|\omega_T|}. \tag{2.8}
\]

Further, assume without loss of generality that \( \sup_K M_{f, 2} = M_{f, 2}(\bar{v}) \) for some \( \bar{v} \in \omega_T \cap l_f \). Then from (2.5) and (2.8) it follows that

\[
T \leq \mathcal{N}_f(\bar{v}) \leq c \log M(M_{f, 1}(\bar{v}) - M_{f, 2}(\bar{v})) \leq -(c \log M)M_{f, 2}(\bar{v}) \\
\leq c \log M \left( 2 + c(r) \log \frac{8N|B_c(0, 1)|}{|\omega_T|} \right).
\]

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The previous inequality implies
\[ |\omega_T| \leq Nc_1 e^{-c_2 T/\log M} |B_c(0, 1)| \]
with \( c_1, c_2 \) depending only on \( s, r \).

The proof of the theorem is complete. \( \Box \)

Let \( f \in H(M, r, s) \) be such that
\[ \sup_{v \in B_c(0, 1)} \sup_{z \in D_s} |f_v(z)| \geq 1 \] \hspace{1cm} (2.9)
where \( B(0, 1) \subset B_c(0, 1) \subset \mathbb{C}^N \) is the real Euclidean ball. Then one can prove

**Theorem 2.3** For every \( T \geq 0 \)
\[ |\{ v \in B(0, 1) \ ; \ N_f(v) \geq T \}| \leq N\tilde{c}_1 e^{-\tilde{c}_2 T/\log M} \cdot |B(0, 1)|, \]
where \( \tilde{c}_1, \tilde{c}_2 \) are positive constants depending on \( s \) and \( r \) only.

Here \( |\cdot| \) is the Lebesgue measure on \( \mathbb{R}^N \).

**Proof.** The proof repeats word-for-word our proof of Theorem B. We have to use only that the above point \( x_f \) can be taken from \( B(0, 1) \) and also instead of (2.6) the real version of Brudnyi-Ganzburg lemma (see [BG])
\[ \frac{mes_1(B(0, 1) \cap l_f)}{mes_1(\omega_T \cap l_f)} \leq \frac{N|B(0, 1)|}{|\omega_T|} \] \hspace{1cm} \( \Box \)

### 3. Proof of Theorem A.

By the change of variables \( x \mapsto x/a, y \mapsto y/a \) we reduce (1.1) to the equivalent system with polynomial terms
\[ F = \sum_{k=1}^d F_k, \quad G = \sum_{k=1}^d G_k \]
satisfying
\[ \sum_{k=1}^d |F_k|^2 + \sum_{k=1}^d |G_k|^2 \leq N^2. \]

Hence it suffices to prove the theorem for \( a = 1 \). Note that in this case the ellipsoid \( E(a, N) \) coincides with the Euclidean ball \( B(0, N) \subset \mathbb{R}^s \) with center 0 and radius \( N \). So we must estimate the number of limit cycles \( C(v, 1/2) \) in the disk \( D_{1/2} = \{(x, y) \in \mathbb{R}^2 ; \ x^2 + y^2 \leq 1/4\} \).

Writing system (1.1) in polar coordinates \( x = r \cos \theta, y = r \sin \theta \) we get
\[ \dot{x} = \dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta} = -r \sin \theta + F(r \cos \theta, r \sin \theta), \]
\[ \dot{y} = \dot{r} \sin \theta + r \cos \theta \cdot \dot{\theta} = r \cos \theta + G(r \cos \theta, r \sin \theta). \]

Multiplying the first equation by \( \cos \theta \), the second by \( \sin \theta \) and adding we get
\[ \dot{r} = F(r \cos \theta, r \sin \theta) \cos \theta + G(r \cos \theta, r \sin \theta) \sin \theta = \sum_{k=1}^d r^k (F_k(r \cos \theta, \sin \theta) \cos \theta + G_k(r \cos \theta, \sin \theta) \sin \theta) = \sum_{k=1}^d r^k f_k(\theta). \]
Similarly,
\[
\dot{\theta} = 1 + \sum_{k=1}^{d} r^{k-1} (-F_k(\cos \theta, \sin \theta) \sin \theta + G_k(\cos \theta, \sin \theta) \cos \theta) = 1 + \sum_{k=1}^{d} r^{k-1} g_k(\theta) .
\]

Finally, we get
\[
\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{r P(r, \theta)}{1 + Q(r, \theta)} = \frac{r P(v, r, \theta)}{1 + Q(v, r, \theta)},
\]

where
\[
P(r, \theta) := \sum_{k=1}^{d} r^{k-1} f_k(\theta), \quad Q(r, \theta) := \sum_{k=1}^{d} r^{k-1} g_k(\theta).
\]

The functions \(P\) and \(Q\) is clear to depend on \(v = (F, G) \in \mathbb{R}^s\) linearly. So we will write \(f_k(\theta) := f_k(v, \theta), g_k(\theta) := g_k(v, \theta), k = 1, \ldots, d,\) and \(P(r, \theta) := P(v, r, \theta), Q(r, \theta) := Q(v, r, \theta), v \in \mathbb{R}^s.\)

**Lemma 3.1** Let \(v \in B(0, N), N \leq \frac{1}{192\pi d^2}.\) Then the trajectory of system \((1.1)\) in the disk \(D_1\) is closed if and only if the corresponding solution of \((3.1)\) satisfies \(r(2\pi) = r(0)\) (and hence is periodic).

**Proof.** It suffices to check that denominator \(1 + Q(v, r, \theta)\) differs from 0 if \(v \in B(0, N), 0 \leq r \leq 1, \theta \in \mathbb{R}.\) By definition,
\[
G_k(\cos t, \sin t) = \sum_{i=0}^{k} b_{ki} (\cos t)^i (\sin t)^{k-i}
\]

to see that \(G_k(\cos \theta, \sin \theta)\)
\[
|G_k(\cos \theta, \sin \theta)| \leq \sqrt{\sum_{i=0}^{k} \frac{|b_{ki}|^2}{C_k}}, \quad \sqrt{\sum_{i=0}^{k} C_k^i (\cos \theta)^{2i} (\sin \theta)^{2(k-i)}},
\]

and therefore
\[
\max_{\theta} |G_k(\cos \theta, \sin \theta)| \leq |v|, \quad i = 0, 1, \ldots, k.
\]

Similarly we get
\[
\max_{\theta} |F_k(\cos \theta, \sin \theta)| \leq |v|, \quad i = 0, 1, \ldots, k.
\]

From (3.2) we now have for \(v \in B(0, N)\)
\[
\max_{0 \leq r \leq 1, 0 \leq \theta \leq 2\pi} |Q(v, r, \theta)| \leq 2d|v| \leq 2dN < 1,
\]

since \(N \leq \frac{1}{192\pi d^2}.\) \(\square\)

Assuming that \(v\) belongs to the complex Euclidean ball \(B_c(0, 2N) \subset \mathbb{C}^s\) we extend \(f_k\) and \(g_k\) to holomorphic in \(v\) functions on \(B_c(0, 2N).\) This also gives an
extension of $P$ and $Q$ in (3.1) to holomorphic in $v$ functions on $B_c(0, 2N)$. Consider
the complexification of equation (3.1):

$$\frac{dr}{dt} = \frac{rP(v, r, t)}{1 + Q(v, r, t)}$$  (3.4)

where $v \in B_c(0, 2N)$, $t \in \mathbb{R}$. We are looking for a complex valued solution $r$
of (3.4) defined in interval $I = [0, 2\pi] \subset \mathbb{R}$ and satisfying the initial condition
$r(0) = w \in \mathbb{D}_{3/4}$. Solve (3.4) by the method of successive approximations. Namely,
set $r_0(v, w, t) = w$ and let

$$r_{n+1}(v, w, t) := w + \int_0^t r_n(v, w, \theta) P(v, r_n(v, w, \theta), \theta) \frac{1}{1 + Q(v, r_n(v, w, \theta), \theta)} d\theta \quad (v \in B_c(0, 2N), w \in \mathbb{D}_{3/4}, t \in I).$$

For a function $f$ defined in $B_c(0, 2N) \times \mathbb{D}_h \times I$, $h > 0$, we set

$$||f||_h := \sup_{v \in B_c(0, 2N), z \in \mathbb{D}_h, \theta \in I} |f(v, z, \theta)| .$$

**Proposition 3.2** The sequence $\{r_n\}_{n \geq 0}$ converges uniformly on $B_c(0, 2N) \times \mathbb{D}_{3/4} \times I$
to a function $r$ which is analytic on this set. Function $r$ uniquely solves (3.4) with
the initial condition $r(v, w, 0) = w$ and satisfies

(a) $||r||_{3/4} \leq 1$;  (b) $r(\cdot, \cdot, t)$ is holomorphic in $B_c(0, 2N) \times \mathbb{D}_{3/4}$ for any $t \in I$.

**Proof.** We begin with

**Lemma 3.3**

$$||P||_1 \leq 4Nd \leq 1/(48\pi d), \quad ||Q||_1 \leq 4Nd < 1/2 .$$

**Proof.** Note that inequalities (3.2), (3.3) are also valid for $v \in C^s$. Applying these
inequalities with $v \in B_c(0, 2N)$ and using definitions of $f_k$ and $g_k$ we then have

$$\sup_{t \in \mathbb{R}} |f_k(v, t)| \leq 4N, \quad \sup_{t \in \mathbb{R}} |g_k(v, t)| \leq 4N .$$

Hence

$$||P||_1 \leq 4N \sum_{i=1}^d 1^i = 4Nd \leq 1/(48\pi d),$$

$$||Q||_1 \leq 4N \sum_{i=1}^d 1^i = 4Nd < 1/2 .$$

by the choice of $N$.  \(\Box\)

Check now that

$$||r_n||_{3/4} < 1 \quad \text{for any } n .$$  (3.5)

Indeed, from inequalities of Lemma 3.3 it follows that

$$8\pi ||P||_1 + ||Q||_1 < 1 .$$
Therefore for \( v \in B_c(0, 2N), w \in \mathbb{D}_{3/4}, t \in I \)
\[
|r_{n+1}(v, w, t)| = |w + \int_0^t \frac{r_n(v, w, \theta)P(v, r_n(v, w, \theta), \theta)}{1 + Q(v, r_n(v, w, \theta), \theta)}d\theta| \leq \frac{3}{4} + \frac{2\pi||P||_1}{1 - ||Q||_1} < 1,
\]
provided \( ||r_n||_{3/4} < 1 \). Since \( |r_0| = |w| \leq 3/4 \) the inequality (3.5) is proved by induction.

Using (3.5) we now prove that
\[
||r_{n+1} - r_n||_{3/4} < \frac{1}{2}||r_n - r_{n-1}||_{3/4}.
\]
Applying the mean-valued inequality we obtain
\[
||r_{n+1} - r_n||_{3/4} \leq 2\pi||r_n - r_{n-1}||_{3/4} \left| \frac{\partial (rP)}{\partial r}(1 + Q) - \frac{\partial Q}{\partial r}(rP) \right|_{1}.
\] (3.6)
Since \( |r| \leq 1 \), we can use the classical Bernstein inequality for holomorphic polynomials (in \( r \))
\[
\left| \frac{\partial (rP(v, r, \theta))}{\partial r} \right| \leq d \||P||_1, \quad \left| \frac{\partial (Q(v, r, \theta))}{\partial r} \right| \leq (d - 1) \||Q||_1
\]
for \((v, r, \theta) \in B_c(0, 2N) \times \mathbb{D}_1 \times I\) to estimate the right-hand side of (3.6). From here and Lemma 3.3 we have
\[
||r_{n+1} - r_n||_{3/4} \leq ||r_n - r_{n-1}||_{3/4} \frac{4\pi d||P||_1(1 + ||Q||_1)}{(1 - ||Q||_1)^2} < \frac{1}{2}||r_n - r_{n-1}||_{3/4}.
\]
Hence the sequence \( \{r_n\}_{n \geq 0} \) converges uniformly on \( B_c(0, 2N) \times \mathbb{D}_{3/4} \times I \) to a complex valued analytic function \( r \) such that \( ||r||_{3/4} \leq 1 \). This function uniquely solves equation (3.4) with \( r(v, w, 0) = w \). Since each \( r_n(\cdot, \cdot, t) \) is holomorphic on \( B_c(0, 2N) \times \mathbb{D}_{3/4} \) for any \( t \in I \), the function \( r(\cdot, \cdot, t) \) is holomorphic on \( B_c(0, 2N) \times \mathbb{D}_{3/4} \) for any \( t \in I \), as well.

The proposition is proved. \( \square \)

**Remark 3.4** It is worth noting that restriction of \( r \) to \( B(0, 2N) \times [0, 3/4] \times I \) is the real valued solution of (3.4).

We proceed to the proof of Theorem A for \( a = 1 \).

Let \( r = r(v, w, t), v \in B_c(0, 2N), w \in \mathbb{D}_{3/4}, t \in I \) be the solution of (3.4). Let us rewrite this equation as
\[
p(v, w) := r(v, w, 2\pi) - w = \int_0^{2\pi} \frac{r(v, w, \theta)P(v, r(v, w, \theta), \theta)}{1 + Q(v, r(v, w, \theta), \theta)}d\theta \quad (3.7)
\]
To estimate the number of limit cycles \( C(v, 1/2) \) of (1.1) we must, according to Lemma 3.1 and Remark 3.4, estimate the number of zeros of \( p(v, \cdot) \) in \( \mathbb{D}_{1/2} \). From (3.7) and Lemma 3.3 one has
\[
||p||_{3/4} = \sup_{B_c(0, 2N) \times \mathbb{D}_{3/4}} |p| \leq 16\pi d N.
\]
Consider now the system
\[
\begin{align*}
\dot{x} &= -y + (N/2)x \\
\dot{y} &= x + (N/2)y
\end{align*}
\]
and denote by \(v_0 \in B(0, N) \subset \mathbb{R}^s\) the vector of coefficients of polynomials which determines this system. Reducing this system to the equation
\[
\frac{dr}{d\theta} = \frac{N}{2} r
\]
we easily see that its solution is
\[
r(v_0, w, \theta) = e^{N\theta/2}w.
\]
Then
\[
|p(v_0, w)| = |1 - e^{\pi N}||w| \geq \pi N|w|.
\]
In particular, for \(w = 1/2\) we have
\[
|p(v_0, 1/2)| \geq \frac{\pi N}{2}.
\]
Thus
\[
\frac{||p||_{3/4}}{|p(v_0, 1/2)|} \leq 32d. \tag{3.8}
\]
Further, we set
\[
f_v(z) := \frac{p(Nv, 3z/4)}{p(v_0, 1/2)} \quad (|z| < 1, v \in B_c(0, 2)).
\]
Now inequality (3.8) shows that
\[
\sup_{v \in B_c(0, 2)} \sup_{z \in \mathbb{D}_1} |f_v(z)| \leq 32d.
\]
Moreover,
\[
\sup_{v \in B_c(0, 1)} \sup_{z \in \mathbb{D}_{2/3}} |f_v(z)| \geq |f_{v_0/N}(2/3)| = 1.
\]
Hence \(f\) belongs to \(\mathcal{H}(32d, 2, 2/3)\) and satisfies (2.9). So we can apply Theorem 2.3 to estimate \(\mathcal{N}_f\) in \(B(0, 1)\) which, in turn, coincides with the number of zeros \(N_p(v)\) of \(p(v, \cdot), v \in B(0, N)\), in the disk \(\mathbb{D}_{1/2}^1\):
\[
|\{v \in B(0, N) ; C(v, 1/2) \geq T\}| \leq |\{v \in B(0, N) ; N_p(v) \geq T\}| = N^s|\{v \in B(0, 1) ; \mathcal{N}_f(v) \geq T\}| \leq d(d + 3)\tilde{c}_1 e^{-\tilde{c}_2 T / \log(32d)}|B(0, N)|,
\]
where \(\tilde{c}_1, \tilde{c}_2\) are absolute positive constants.

The proof is complete. \(\Box\)

**Proof of Corollary 1.1.** The proof is similar to the proof of Theorem 1.5 below and we do not reproduce it. \(\Box\)
4. Proofs.

In this section we prove results of Section 1.3. Many of our implications are well known in Probability Theory but the whole point is that they require in their assumptions the inequality of Theorem B.

**Proof of Theorem 1.5.** We first estimate the expectation of the random variable \( N_k, k \geq 1 \). By definition

\[
E(N_k) = \frac{1}{|B_c(0,1)|} \sum_{l=0}^{\infty} l \cdot |\{v \in B_c(0,1) : N_f^{(k)}(v) = l\}| = \frac{1}{|B_c(0,1)|} \int_{B_c(0,1)} N_f^{(k)}(v) dv.
\]

The latter integral equals

\[
\frac{1}{|B_c(0,1)|} \int_0^{|B_c(0,1)|} N_f^{*(k)}(t) dt,
\]

where \( g^* \) denotes the nonincreasing rearrangement of a function \( g \). Since \( g^* \) is, by definition, the right inverse to the distribution function \( T \mapsto |\{v \in B_c(0,1) : |g(v)| \geq T\}| \), we easily deduce from Theorem B that

\[
N_f^{*(k)}(t) \leq \frac{\log M}{c_2} \log \left( \frac{c_1 N |B_c(0,1)|}{t} \right).
\]

Putting together this inequality and the previous identity we prove that

\[
E(N_k) \leq c \log M \log(N + 1),
\]

where \( c \) depends only on \( r, s \).

Using similar arguments one can show that

\[
E(N_k^2) \leq c'(\log M \log(N + 1))^2 \quad (4.1)
\]

with \( c' \) depending only on \( r, s \). Then

\[
D(N_k) \leq 2E(N_k^2) + 2(E(N_k))^2 \leq \tilde{c}(\log M \log(N + 1))^2, \quad (k \geq 1)
\]

where \( \tilde{c} \) depends only on \( r, s \). \( \square \)

**Proof of Corollary 1.6.** According to Kolomogorov's theorem (see, e.g., [Gn], Ch. VI, Sec. 34) the condition

\[
\sum_{i=1}^{\infty} \frac{D(N_i)}{i^2} < \infty
\]

guarantees fulfilment of the strong law of large numbers for the sequence \( \{N_k\} \). This condition is a direct consequence of Theorem 1.5. Furthermore, Theorem 1.5 implies

\[
\frac{1}{n} \sum_{k=1}^{n} E(N_k) \leq c \log M \log(N + 1).
\]

Then the required statement follows from the strong law of large numbers and the above inequality. \( \square \)
Example 4.1 Let \( f : \mathbb{D}_1 \times B_c(0, r) \rightarrow B_c(0, K) \) be a holomorphic mapping, where \( B_c(0, r) \) and \( B_c(0, K) \) are complex balls in \( \mathbb{C}^N \). Let \( Df_z(v) \) denote the Jacobi matrix of \( f(z, \cdot) : B_c(0, r) \rightarrow B_c(0, K), z \in \mathbb{D}_1, \) at \( v \in B_c(0, r) \). Assume that

\[
K_1 := \sup_{z \in \mathbb{D}_1} \sup_{v \in B_c(0, r)} ||Df_z(v)||_2 < \infty .
\]

Here \( ||A||_2 \) stands for the norm of a linear mapping \( A : l_2(\mathbb{C}^N) \rightarrow l_2(\mathbb{C}^N) \).

Let \( f := (f_1, ..., f_N) \). Consider the system of ordinary differential equations

\[
\frac{dx_i}{dz} = f_i(z, x_1, ..., x_N), \quad i = 1, ..., N
\]

with an initial condition

\[
(x_1(0), ..., x_N(0)) = v \in B_c(0, 3r/4) .
\]

In the special case that the system of equations above expresses the property that the \( N^{th} \) derivative of a function is equal to zero, the solutions are polynomials of degree \( \leq N \). Therefore the inequalities below apply to describe distribution of zeros of random polynomials.

For \( |z| \) being sufficiently small one can solve the above system by the method of successive approximations. Namely, we choose \( x_0(z, v) = v \) and then

\[
x_{n+1}(z, v) := v + \int_0^z f(w, x_n(w, v))dw .
\]

Here the integral is taken over the segment connecting 0 and \( z \). Further, if we take \( R := R(r, K, K_1) < \min\{r/4K, 1/K_1, 1\} \) then \( \{x_n\}_{n \geq 0} \) converges uniformly on \( \mathbb{D}_R \times B_c(0, 3r/4) \) to a holomorphic mapping \( x : \mathbb{D}_R \times B_c(0, 3r/4) \rightarrow B_c(0, r) \) which solves the above initial value problem (and this solution is unique).

Let \( x_1 \) be the first coordinate of \( x \). Denote by \( N_{x_1}(v) \) the number of zeros of \( x_{1v} := x_1(\cdot, v), v \in B_c(0, r/2) \), in a closed disk \( \mathbb{D}_t, t < R \).

By definition we have

\[
M := \sup_{v \in B_c(0, 3r/4)} \sup_{z \in \mathbb{D}_R} |x_{1v}(z)| \leq r \quad \text{and} \quad M_1 := \sup_{v \in B_c(0, r/2)} |x_{1v}(0)| = r/2
\]

such that \( M/M_1 \leq 2 \). Therefore according to Theorem B, for every \( T \geq 0 \)

\[
|\{v \in B_c(0, r/2) ; N_{x_1}(v) \geq T\}| \leq Nc_1e^{-c_2T}|B_c(0, r/2)|,
\]

where \( c_1, c_2 \) depend only on \( t/R, r \).

Consider now a particular case of the scheme described in Corollary 1.6. Let \( f_1^{(k)}(z) = x_{1v}(z), \ k = 1, 2, ..., \) We define distributions of \( N_k \) as follows: for a nonnegative \( l \) probability

\[
P(N_k = l) := \frac{1}{|B_c(0, r/2)|}|\{v \in B_c(0, r/2) ; N_{x_1}(v) = l\}| .
\]
Then Corollary 1.6 implies that the inequality
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} N_k \leq c \log(N + 1) \]
holds with probability one. Here \( c \) depends on \( t/R \) and \( r \) only.

The constant on the right estimates the expected number of zeros in \( \mathbb{B}_t \) of a random function \( x_{1v}, \ v \in B_c(0, r/2) \). In the special case that our system of ODE’s depends linearly on the parameter \( v \), e.g. in the polynomial case mentioned above, one can obtain sharper estimates by making use of explicit formulae (cf. [EK]).

**Proof of Theorem 1.7.** A simple calculation based on the inequality of Theorem B shows that the sequence \( \{ E(|N_k - E(N_k)|^2) \} \) of absolute moments of third order is bounded. We will show now that there is an \( \epsilon > 0 \) such that \( D(N_k) \geq \epsilon \) for every \( k \).

This and the above boundedness of third moments imply Lindeberg’s condition for the sequence \( \{ N_k - E(N_k) \} \) (see, e.g., [Gu], Ch. VIII, Sec. 42). Therefore the central limit theorem will be valid in this case.

Assume, to the contrary, that there exists a subsequence \( \{ N_{k_i} \} \geq 1 \) such that \( \lim_{i \to \infty} D(N_{k_i}) = 0 \). Since \( H(M, r, s) \) is a compact in the topology of uniform convergence on compact subsets of \( B_c(0, r) \times \mathbb{D}_1 \), we can assume, without loss of generality, that \( f^{(k_i)} \) converges in this topology to a function \( f \in H(M, r, s) \). Similarly to the definition of \( N_{f^{(k)}} \), introduce the function \( N_f \) counting the number of zeros of \( f_v \) (\( v \in B_c(0, 1) \)) in \( \mathbb{D}_s \). We will show that under the above assumption the following result holds.

**Lemma 4.2** \( N_f \) equals a constant almost everywhere on \( B_c(0, 1) \).

Based on this lemma let us, first, complete the proof of the theorem.

According to Proposition 2.1 \( N_f \) is a nonnegative upper semicontinuous on \( B_c(0, r) \) function assigning only integer values. This and assumption (a) of the theorem imply that the set \( \{ v \in \overline{B_c}(0, 1) ; N_f(v) = 0 \} \) is nonempty relatively open in \( \overline{B_c}(0, 1) \). Thus \( N_f = 0 \) almost everywhere on \( B_c(0, 1) \). Further, for some \( s'' \) satisfying \( s' < s'' < s \) consider the function \( \tilde{N}_f \) which counts the number of zeros of \( f_v \) in the open disk \( \mathbb{D}_{s''} \). Then \( \tilde{N}_f \) is bounded from above and lower semicontinuous on \( \overline{B_c}(0, 1) \setminus V_f \) (see Remarks 1.3 and 2.2); here \( V_f := \{ v \in B_c(0, r) ; f_v = 0 \} \) is a proper analytic subset of \( B_c(0, r) \). Clearly, \( 0 \leq \tilde{N}_f \leq N_f \) which implies \( \tilde{N}_f = 0 \) almost everywhere on \( B_c(0, 1) \). Observe now that condition (b) of the theorem implies existence of \( \tilde{v} \in \overline{B_c}(0, 1) \) such that the holomorphic function \( f_{\tilde{v}} \neq 0 \) on \( \mathbb{D}_{s''} \) and has at least one zero there. In particular, \( \tilde{v} \notin V_f \) and \( \tilde{N}_f(\tilde{v}) \geq 1 \). The latter shows that \( \tilde{N}_f \) attains its maximum (which is \( > 0 \)) on a nonempty relatively open subset of \( \overline{B_c}(0, 1) \setminus V_f \). It contradicts to the assumption \( \tilde{N}_f = 0 \) almost everywhere on \( B_c(0, 1) \). This proves that \( \inf_k D(N_k) = \epsilon > 0 \).

The theorem is proved.

**Proof of Lemma 4.2.** Let us consider the integral
\[ \frac{1}{2\pi i} \int_{\Gamma} \frac{f'_v(z)}{f_v(z)} \, dz; \] (4.2)
where $\Gamma := \partial \mathbb{D}_s$. Note that the expression above makes sense only for $v \in B_c(0, 1)$ satisfying $\min_{z \in \Gamma} |f_v(z)| \neq 0$. Denote the set of these points by $R$. If $v \in R$ then (4.2) coincides with $N_f(v)$. Consider now the set $V \subset B_c(0, 1) \times \Gamma$ defined by

$$\{(v, z) \in B_c(0, 1) \times \Gamma : f_v(z) = 0\}.$$ 

Clearly, $V$ is a real analytic subset of $B_c(0, 1) \times \Gamma$. Moreover, real dimension of $V$ is at most $2N - 1$ (recall that $\dim_{\mathbb{R}} B_c(0, 1) = N$). In fact, consider the natural projection $\pi_2 : V \rightarrow \Gamma$. By the definition of $V$, for any $\gamma \in \Gamma$ the set $\pi_2^{-1}(\gamma)$ is a complex analytic subset of $B_c(0, 1)$. From assumption (a) of the theorem it follows that $\pi_2^{-1}(\gamma)$ does not coincide with $B_c(0, 1)$. In particular, $\dim_{\mathbb{R}} \pi_2^{-1}(\gamma) = 2N - 2$. Applying Sard’s theorem to $\pi_2$ we then conclude that $\dim_{\mathbb{R}} V \leq \dim_{\mathbb{R}} \Gamma + 2N - 2 = 2N - 1$. Further, the natural projection $\pi_1 : V \rightarrow B_c(0, 1)$ maps $V$ onto a subanalytic subset $\pi_1(V)$ of $B_c(0, 1)$ and therefore $\dim_{\mathbb{R}} \pi_1(V) \leq \dim_{\mathbb{R}} V \leq 2N - 1$. In particular, we obtain that $R(= B_c(0, 1) \setminus \pi_1(V))$ is everywhere dense in $B_c(0, 1)$. In addition, the sequence $\{f_v^{(k)}\}_{i \geq 1}$ converges to $f_v$ uniformly on $\Gamma$ for every $v \in R$. Hence, there is a number $i_0 = i_0(v)$ such that for any $i \geq i_0$ the function $f_v^{(k)}$ has no zeros on $\Gamma$. From here and (4.2) it follows that $N_{f(k)}$ converges to $N_f$ almost everywhere on $B_c(0, 1)$. Moreover, from Theorem 1.3 it follows that the sequence $\{E(N_{k_i})\}$ of nonnegative numbers is uniformly bounded from above. Thus, without loss of generality, we can assume that $\lim_{i \to \infty} E(N_{k_i}) = c$ for some nonnegative $c$. This implies that the sequence of functions $\{(N_{f(k)} - E(N_{k_i}))(1)\}_{i \geq 1}$ converges to $(N_f - c)^2$ almost everywhere on $B_c(0, 1)$. From here and Fatou’s lemma one obtains

$$\int_{B_c(0, 1)} (N_f - c)^2 dx = \int_{B_c(0, 1)} \lim_{i \to \infty} (N_{f(k)} - E(N_{k_i}))(1) dx \leq |B_c(0, 1)| \liminf_{i \to \infty} D(N_{k_i}) = 0.$$ 

This implies $N_f = c$ almost everywhere on $B_c(0, 1)$. \hfill \Box

**Proof of Theorem 1.9.** Consider, first, $\mathbb{D}_{1-\epsilon} = A_c \cap \mathbb{D}_1$ and estimate expectation $E_{1,k}$ of the number of zeros of $P_{k,v}$ in $\mathbb{D}_{1-\epsilon}$. To this end we apply Bernstein’s doubling inequality for polynomials along with the assumptions of the theorem to obtain

$$\sup_{\|v\| \leq 2} |a_{ik}(v)| \leq 2^{d(k)} \sup_{\|v\| \leq 1} |a_{ik}(v)| \leq 2^{d(k)}, \quad (1 \leq 0 \leq k).$$ 

Then

$$\sup_{z \in \mathbb{D}_{1-\epsilon}} |P_{k,v}(z)| \leq \max_{\|v\| \leq 2} \sup_{\|v\| \leq 2} |a_{ik}(v)| \sum_{s=0}^{k} (1 - \epsilon/2)^s < \frac{2}{\epsilon} 2^{d(k)}.$$ 

Similarly, according to assumption (c) of the theorem

$$\sup_{z \in \mathbb{D}_{1-\epsilon}} |P_{k,v}(z)| \geq \sup_{\|v\| \leq 1} |P_{k,v}(0)| = \sup_{\|v\| \leq 1} |a_{kk}(v)| = 1.$$ 

These inequalities show that, up to some dilation of the space of parameters and $C$ with coefficients depending only on $\epsilon$, the family $\{P_{k,v}\}_v$ belongs to $\mathcal{H}(M, r, s)$ with $M = \frac{2^{d(k)+1}}{\epsilon}$ and with corresponding $r, s$ depending on $\epsilon$. Applying now Theorem 1.3 we estimate the expectation $E_{1,k}$ of the number of zeros of $P_{k,v}$ in $\mathbb{D}_{1-\epsilon}$ by
c(\epsilon)d(k) \log(n(k) + 1).

To estimate expectation $E_{2,k}$ of the number of zeros of $P_{k,v}$ outside of $\mathbb{D}_{1+ \epsilon}$ consider the family of polynomials $P'_{k,v}(z) := z^k P_{k,v}(1/z)$. This, clearly, reduces the problem to that of estimating the expectation of the number of zeros of $P'_{k,v}$ in $\mathbb{D}_{1- \epsilon}$. As above we get in this case the inequality $E_{2,k} \leq c(\epsilon)d(k) \log(n(k) + 1)$.

Finally, the required expectation $E(\tilde{N}_k) = k - E_{1,k} - E_{2,k} \geq k(1 - 2c(\epsilon)\frac{d(k) \log(n(k) + 1)}{k})$.

It remains to note that from here and assumption (a) of the theorem it follows $E(\tilde{N}_k) = k(1 - o(1))$ as $k \to \infty$.

The theorem is proved. □

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