On Orbit Equivalence of Quasiconformal Anosov Flows

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Abstract – We classify up to $C^\infty$ orbit equivalence the volume-preserving quasiconformal Anosov flows whose strong stable and strong unstable distributions are at least three-dimensional. If one of the strong distributions is two-dimensional, then we get a partial classification. Using these classification results, we obtain the following rigidity result:

Let $\Phi$ be the orbit foliation of the geodesic flow of a closed hyperbolic manifold of dimension at least three. Let $\Psi$ be another $C^\infty$ one-dimensional foliation. If $\Phi$ is $C^1$ conjugate to $\Psi$, then $\Phi$ is $C^\infty$ conjugate to $\Psi$.

Résumé – Nous classifions, à équivalence orbitale $C^\infty$ près, les flots d’Anosov quasiconformes topologiquement transitifs dont les distributions stable forte et instable forte sont de dimension au moins 3. Si ces deux distributions sont de dimension au moins 2, alors nous obtenons une classification partielle. Nous déduisons de ces résultats de classification le résultat de rigidité suivant :

Soit $\Phi$ le feuilletage orbitale du flot géodésique d’une variété hyperbolique fermée de dimension au moins 3. Soit $\Psi$ un autre feuilletage $C^\infty$ de dimension 1. Si $\Phi$ et $\Psi$ sont $C^1$ conjugué, alors $\Phi$ et $\Psi$ sont $C^\infty$ conjugué.

1. Introduction

1.1. Motivation

Let $M$ be a $C^\infty$-closed manifold. A $C^\infty$-flow $\phi_t$ generated by the non-singular vector field $X$ is said to be an Anosov flow if there exists a $\phi_t$-invariant splitting of the tangent bundle

$$TM = \mathbb{R}X \oplus E^+ \oplus E^-,$$
a Riemannian metric on $M$ and two positive numbers $a$ and $b$ such that for any $u^\pm \in E^\pm$ and for any $t \geq 0$,
\[
    \| D\phi_{\pm t}(u^\pm) \| \leq ae^{-bt} \| u^\pm \|,
\]
where $E^-$ and $E^+$ are said to be the strong stable and strong unstable distributions of the flow. For any $x \in M$ the leaves containing $x$ of $E^+$ and $E^-$ are denoted respectively by $W^+_x$ and $W^-_x$.

Define two functions on $M \times \mathbb{R}$ as following:
\[
    K^+(x, t) = \max\{ \| D\phi_t(u) \| : u \in E^+_x, \| u \| = 1 \}
\]
and
\[
    K^-(x, t) = \max\{ \| D\phi_t(u) \| : u \in E^-_x, \| u \| = 1 \}
\]

If $K^-(K^+)$ is bounded, then the Anosov flow $\phi_t$ is said to be quasiconformal on the stable (unstable) distribution. If $K^+$ and $K^-$ are both bounded, then $\phi_t$ is said to be quasiconformal. If it is the case, then the superior bound of $K^+$ and $K^-$ is said to be the distortion of $\phi_t$. The corresponding notions for Anosov diffeomorphisms are defined similarly (see [Sa]).

Recall that two $C^\infty$ Anosov flows $\phi_t : M \to M$ and $\psi_t : N \to N$ are said to be $C^k$ flow equivalent ($k \geq 0$) if there exists a $C^k$ diffeomorphism $h : M \to N$ such that $\phi_t = h^{-1} \circ \psi_t \circ h$ for all $t \in \mathbb{R}$. They are said to be $C^k$ orbit equivalent ($k \geq 0$) if there exists a $C^k$ diffeomorphism $h : M \to N$ sending the orbits of $\phi_t$ onto the orbits of $\psi_t$ such that the orientations of the orbits are preserved. Similarly, two $C^\infty$ foliations $\Phi$ and $\Psi$ are said to be $C^k$ conjugate if there exists a $C^k$ diffeomorphism $h : M \to N$ sending the leaves of $\Phi$ onto those of $\Psi$. By convention, a $C^0$ diffeomorphism means a homeomorphism. We can prove easily the following

**Lemma 1.1.** Let $\phi_t$ and $\psi_t$ be two $C^\infty$ Anosov flows. If they are $C^1$ orbit equivalent and $\psi_t$ is quasiconformal, then $\phi_t$ is also quasiconformal.

**Proof.** Denote by $\phi : M \to N$ the $C^1$ orbit conjugacy between $\phi_t$ and $\psi_t$ and by $\mathcal{F}^\pm$ and $\mathcal{F}^{\pm,0}$ the Anosov foliations of $\psi_t$. Then we have the following

**Sublemma.** Under the notations above, $\phi(\mathcal{F}^{\pm,0}) = \mathcal{F}^{\pm,0}$.

**Proof.** Define $\hat{\phi}_t = \phi \circ \phi_t \circ \phi^{-1}$. Then $\hat{\phi}_t$ is a $C^1$ flow on $N$ with the same orbits as $\psi_t$. So there exists a $C^1$ map $\alpha : \mathbb{R} \times N \to \mathbb{R}$ such that
\[ \hat{\phi}_t(x) = \psi_{\alpha(t,x)}(x). \] Define \( \hat{E}^- = (D\phi)(E^-) \). Then it is the \( C^0 \) tangent bundle of the \( C^1 \) foliation \( \hat{F}^- = \phi(F^-) \).

Let us prove first that \( \hat{E}^- \subseteq \hat{E}^-_0 \). Fix a \( C^0 \) Riemannian metric \( g \) on \( N \) such that \( \hat{E}^+ \) and \( \hat{E}^- \) and \( \hat{X} \) are orthogonal to each other. Since \( \phi \) is \( C^1 \), then it is bi-Lipschitz. We deduce that for all \( x \in N \) and \( \hat{u} \in \hat{E}^-_x \), \( \| D\hat{\phi}_t(\hat{u}) \| \to 0 \) if \( t \to +\infty \).

If \( \hat{u} = \hat{u}^+ + a\hat{X} + \hat{u}^- \) and \( \hat{u}^- \neq 0 \), then by a simple calculation we get for a certain function \( b_x : \mathbb{R} \to \mathbb{R} \),

\[
D\hat{\phi}_t(\hat{u}) = D(\psi_{\alpha(t,x)})(\hat{u}^+) + b_x(t) \cdot \hat{X}_{\hat{\phi}_t(x)} + D(\psi_{\alpha(t,x)})(\hat{u}^-).
\]

So we get \( \| D\hat{\phi}_t(\hat{u}) \| \geq\| D(\psi_{\alpha(t,x)})(\hat{u}^+) \| \to +\infty \) if \( t \to +\infty \), which is a contradiction. We deduce that \( \hat{E}^- \subseteq \hat{E}^-_0 \). So \( \phi(F^-_0) \subseteq \hat{F}^-_0 \). Then it is easy to see that \( \phi(F^-_0) = \hat{F}^-_0 \), i.e. \( \phi \) sends \( C^1 \) diffeomorphically each leaf of \( F^-_0 \) onto a leaf of \( \hat{F}^-_0 \). Similarly, we have \( \phi(F^+_0) = \hat{F}^+_0 \). □

In particular we deduce from the sublemma above that \( \hat{E}^+ \oplus \mathbb{R}\hat{X} = \hat{E}^+ \oplus \mathbb{R}\hat{X} \). By projecting parallel to the direction of \( \hat{X} \), we get a \( C^0 \) section \( P \) of \( \text{End}(\hat{E}^+, \hat{E}^+) \) and two positive constants \( A_1 \) and \( A_2 \) such that

\[
A_1 \| u \| \leq\| P(u) \| \leq A_2 \| u \|, \forall u \in \hat{E}^+.
\]

Then it is easy to verify that for all \( x \in N \), \( P \circ D_x\hat{\phi}_t = D_x\psi_{\alpha(t,x)} \circ P \).

If \( \psi_t \) is quasiconformal with distortion \( K \), then we have the following estimation

\[
\hat{K}^+(x,t) = \sup\left\{ \frac{\| D\hat{\phi}_t(\hat{u}^+) \|}{\| \hat{u}^+ \|= 1, \hat{u}^+ \in \hat{E}^+_x} \right\}
\]

\[
\geq A_2 \frac{\sup\left\{ \| P(D\hat{\phi}_t(\hat{u}^+)) \|}{\| \hat{u}^+ \|= 1, \hat{u}^+ \in \hat{E}^+_x} \right\}
\]

\[
\leq (A_2 A_1)^2 \cdot \frac{\inf\left\{ \| D_x\psi_{\alpha(t,x)}(\frac{P(\hat{u}^+) \|}{\| \hat{u}^+ \|= 1, \hat{u}^+ \in \hat{E}^+_x} \right\}}{\| \hat{u}^+ \|= 1, \hat{u}^+ \in \hat{E}^+_x}
\]

\[
\leq (A_2 A_1)^2 \cdot K.
\]

Similarly \( \hat{K}^- \) is also bounded. So \( \hat{\phi}_t \) is quasiconformal. Since \( \phi \) is a \( C^1 \) diffeomorphism, then it is bi-Lipschitz. We deduce that \( \phi_t \) is also quasiconformal. □
It is easy to see that the geodesic flow of a closed hyperbolic manifold is quasiconformal (even conformal). Then by the previous lemma, each $C^\infty$ time change of such a flow is also quasiconformal. It is easily seen that an Anosov diffeomorphism is quasiconformal iff its suspension is quasiconformal. So if $\phi$ denotes a semisimple hyperbolic automorphism of a torus with two eigenvalues, then its suspension is a quasiconformal Anosov flow.

It seems to be a common phenomena in mathematics that things can only be effectively studied and understood when placed in a suitable and flexible environment. Conformal structures (Anosov flows) are pretty rigid while quasiconformal structures (Anosov flows) seem to be much more flexible. We wish to better understand the classical conformal Anosov flows, notably the geodesic flows of closed hyperbolic manifolds, by using quasiconformal techniques, which is our motivation to study general quasiconformal Anosov systems.

1.2. Main theorems

In our previous paper [Fa1], we have studied the rigidity of volume-preserving Anosov flows with smooth $E^+ \oplus E^-$. In particular we have obtained the following

**Theorem 1.1.** ([Fa1], Corollary 1) Let $\phi$ be a $C^\infty$ volume-preserving quasiconformal Anosov diffeomorphism on a closed manifold $\Sigma$. If the dimensions of $E^+$ and $E^-$ are at least two, then up to finite covers $\phi$ is $C^\infty$ conjugate to a hyperbolic automorphism of a torus.

In [K-Sa], B. Kalinin and V. Sadovskaya have classified the topologically transitive quasiconformal Anosov diffeomorphisms whose strong stable and unstable distributions are of dimension at least 3. Their argument, though quite elegant, meets an essential difficulty in the case that one of the strong stable and unstable distributions is two dimensional.

Based on the classification result in [K-Sa], we classify in this paper completely the quasiconformal Anosov flows whose strong stable and unstable distributions have relatively high dimensions. More precisely, we prove

**Theorem 1.2.** Let $\phi_t$ be a $C^\infty$ topologically transitive quasiconformal Anosov flow such that $E^+$ and $E^-$ are at least three dimensional. Then up to finite covers, $\phi_t$ is $C^\infty$ orbit equivalent either to the geodesic flow of a hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.
Under the conditions above, if $E^+ \oplus E^-$ is in addition $C^1$, then up to a constant change of time scale and finite covers, $\phi_t$ is $C^\infty$ flow equivalent either to a canonical time change of the geodesic flow of a hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.

Recall that if $\alpha$ is a closed $C^\infty$ 1-form on $M$ such that $1 + \alpha(X) > 0$, then the flow of $\frac{X}{1+\alpha(X)}$ is said to be a canonical time change of $\phi_t$, where $X$ denotes the generator of $\phi_t$.

If one of the strong distributions is two-dimensional, then by using our Theorem 1.1, we get the following partial result.

**Theorem 1.3.** Let $\phi_t$ be a $C^\infty$ volume-preserving quasiconformal Anosov flow such that $E^+$ is of dimension two and $E^-$ is of dimension at least two. If $\phi_t$ has the sphere-extension property, then up to finite covers, $\phi_t$ is $C^\infty$ orbit equivalent either to the geodesic flow of a three-dimensional hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.

Under the conditions above, if $E^+ \oplus E^-$ is in addition $C^1$, then up to a constant change of time scale and finite covers, $\phi_t$ is $C^\infty$ flow equivalent either to a canonical time change of the geodesic flow of a three-dimensional hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.

The sphere-extension property will be defined in Subsection 2.3. Let us just mention that this property is invariant under $C^1$ orbit equivalence.

Now we can get some concrete applications of our results above. Recall at first that flow conjugacies between Anosov flows have been and being extensively studied. The philosophical conclusion is that they exist rarely, even $C^0$ ones. Let us just mention two of the most beautiful supporting results (see also [L1] and [L2]) :

**Theorem 1.4.** (U. Hamenstädt, [Ham2]) Let $M$ be a closed negatively curved manifold. If the geodesic flow of $M$ is $C^0$ flow equivalent to that of a locally symmetric space of rank one $N$, then $M$ is isometric to $N$.

**Theorem 1.5.** (R. de la Llave and R. Moriyon, [LM]) Let $\phi_t$ and $\psi_t$ be two $C^\infty$ three-dimensional volume-preserving Anosov flows. If they are $C^1$ flow equivalent, then they are $C^\infty$ flow equivalent.

However there exist plenty of $C^0$ orbit conjugacies between Anosov flows. For example, if two $C^\infty$ Anosov flows are sufficiently $C^1$-near, then they are
Hölder-continuous orbit equivalent by the celebrated structural stability (see [An]). A natural question to ask is whether $C^1$ orbit conjugacies between Anosov flows are rare. We can deduce from Theorems 1.1 and 1.2 the following result showing that $C^1$ orbit conjugacies are surely rare in some cases, while Hölder-continuous orbit conjugacies are abundant.

**Theorem 1.6.** Let $\phi_t$ be a $C^\infty$ Anosov flow and $\psi_t$ be the geodesic flow of a closed hyperbolic manifold of dimension at least three. If $\phi_t$ and $\psi_t$ are $C^1$ orbit equivalent, then they are $C^\infty$ orbit equivalent.

In order to state the next result, let us recall firstly some notions.

**Definition 1.1.** Let $(X,d_X)$ and $(Y,d_Y)$ be two metric spaces. Then they are said to be quasi-isometric if there a map $f : X \to Y$ and two positive numbers $C$ and $D$ such that the following two conditions are satisfied:

1. \[ \frac{1}{C} d_X(x,y) - D \leq d_Y(f(x),f(y)) \leq C d_X(x,y) + D, \quad \forall x,y \in X. \]
2. For any $y \in Y$, there exists $x \in X$ such that $d_Y(y,f(x)) \leq D$.

Roughly speaking, two metric spaces are quasi-isometric if and only if they are bi-Lipschitz equivalent in the large scale. Recall that for any $n \geq 2$, a $n$-dimensional Riemannian manifold $M$ is said to be hyperbolic if it has constant sectional curvature $-1$. We denote by $\mathbb{H}^n$ the unique simply connected hyperbolic manifold. By combining some classical results with our previous theorem, we get the following

**Corollary 1.1.** Let $M$ be a $n$-dimensional closed Riemannian manifold of negative curvature such that $n \geq 3$. Then we have the following relations between dynamics and geometry:

1. The geodesic flow of $M$ is Hölder-continuously orbit equivalent to that of a hyperbolic manifold if and only if the universal covering space $\tilde{M}$ with its lifted metric is quasi-isometric to $\mathbb{H}^n$.
2. The geodesic flow of $M$ is $C^1$ orbit equivalent to that of a hyperbolic manifold if and only if $M$ has constant negative curvature.

Let $g$ be the Riemannian metric of a closed hyperbolic manifold of dimension at least three. Let $g'$ be a perturbed Riemannian metric of $g$ with non-constant negative curvature. Thus by the proof of the previous corollary, the orbit foliation of the geodesic flow of $g$ is $C^0$ conjugate to that
of $g'$. However, by the previous corollary, these two $C^\infty$ one-dimensional foliations are not $C^1$ conjugate.

By combining a result of R. Mané with our Theorem 1.6, we get finally the following corollary, which is the key objective of this article.

**Corollary 1.2.** Let $\Phi$ be the orbit foliation of the geodesic flow of a closed hyperbolic manifold of dimension at least three. Let $\Psi$ be another $C^\infty$ one-dimensional foliation. If $\Phi$ and $\Psi$ are $C^1$ conjugate, then they are $C^\infty$ conjugate.

1.3. The organization of the paper

In Section two we recall some fundamental facts concerning quasiconformal Anosov flows and some properties of transverse $(G,T)$-structures of foliations. Then in Section three we prove Theorems 1.2 and 1.3. Finally in Section four we apply these two results to geodesic flows to deduce Theorem 1.6 and Corollaries 1.1 and 1.2.

2. Preliminaries

2.1. Linearizations and smooth conformal structures

Let us recall firstly the following two results established in [Sa]:

**Theorem 2.1.** ([Sa], Theorem 1.3) Let $f$ be a topologically transitive $C^\infty$ Anosov diffeomorphism ($\phi_t$ be a topologically mixing $C^\infty$ Anosov flow) on a closed manifold $M$ which is quasiconformal on the unstable distribution. Then it is conformal with respect to a Riemannian metric on this distribution which is continuous on $M$ and $C^\infty$ along the leaves of the unstable foliation.

**Theorem 2.2.** ([Sa], Theorem 1.4) Let $f$ ($\phi_t$) be a $C^\infty$ Anosov diffeomorphism (flow) on a closed manifold $M$ with $\dim E^+ \geq 2$. Suppose that it is conformal with respect to a Riemannian metric on the unstable distribution which is continuous on $M$ and $C^\infty$ along the leaves of the unstable foliation. Then the (weak) stable holonomy maps are conformal and the (weak) stable distribution is $C^\infty$.

Let us recall briefly the steps to prove these two theorems in the case of flow. Denote by $\phi_t$ a topologically mixing quasiconformal Anosov flow. Then by some classical arguments (see [Su] and [Tu]), V. Sadovskaya found two measurable $\phi_t$-invariant conformal structures $\tilde{\tau}^+$ and $\tilde{\tau}^-$ along respectively the leaves of $F^+$ and $F^-$. Then as is usual for Anosov flows, these two con-
formal structures were perturbed to continuous \( \phi_t \)-invariant ones denoted by \( \tau^+ \) and \( \tau^- \). Using the linearizations, she proved that along each leaf of \( \mathcal{F}^+ \), \( \tau^+ \) is isometric to a vector space with its canonical conformal structure, which has permitted her to blow up the smoothness of weak holonomy maps. Then by using a result of J. L. Journé, she proved the smoothness of the weak stable and unstable distributions.

In [Fa1], we have proved the following lemma based on [Pl1]. For the sake of completeness, let us recall the arguments.

**Lemma 2.1.** Let \( \phi_t \) be a \( C^\infty \) topologically transitive Anosov flow. Then we have the following alternative:

1. \( \phi_t \) is topologically mixing,
2. \( \phi_t \) admits a \( C^\infty \) closed global section with constant return time.

**Proof.** If Case (2) is true, then up to a constant change of time scale, \( \phi_t \) is \( C^\infty \) flow equivalent to the suspension of a \( C^\infty \) Anosov diffeomorphism. Thus it is not topologically mixing. So the alternative is exclusive.

If there exists \( x \in M \) such that \( W^+_x \) is not dense in \( M \), then by Theorem 1.8 of [Pl1] \( E^+ \oplus E^- \) is the tangent bundle of a \( C^1 \) foliation \( \mathcal{F} \). In addition the leaves of \( \mathcal{F} \) are all compact. So \( \phi_t \) admits a \( C^1 \) closed global section with constant return time. Then to realize Case (2) we need only prove that \( E^+ \oplus E^- \) is \( C^\infty \).

Denote by \( \lambda \) the canonical 1-form of \( \phi_t \). Then \( \lambda \) is, a priori, a continuous 1-form on \( M \). For each point \( y \in M \) we take a small neighborhood \( F_y \) of \( y \) in the leaf containing \( y \) of \( \mathcal{F} \). Then we can construct a local \( C^1 \) chart \( \theta_y : (-\epsilon, \epsilon) \times F_y \to M \) such that \( \theta_y(t, z) = \phi_t(z) \). In this chart we have \( \lambda = dt \). We deduce that \( \int_{\gamma} \lambda = 0 \) for each piecewise \( C^1 \) closed curve \( \gamma \) contained in the image of \( \theta_y \). So \( \lambda \) is locally closed (see Section two of [Pl1] for the definition). Then by Proposition 2.1 of [Pl1], \( \lambda \) is seen to be closed in a weak sense, i.e. for every \( C^1 \) immersed two-disk \( \sigma \) such that \( \partial \sigma \) is piecewise \( C^1 \),

\[
\int_{\partial \sigma} \lambda = 0.
\]

So by integrating along the closed curves, \( \lambda \) gives an element in \( \text{Hom}(\pi_1(M), \mathbb{R}) \), i.e. the space of group homomorphisms of \( \pi_1(M) \) into \( \mathbb{R} \), where \( \pi_1(M) \) denotes the fundamental group of \( M \). However we have naturally

\[
\text{Hom}(\pi_1(M), \mathbb{R}) \cong (H_1(M, \mathbb{R}))^* \cong H^1(M, \mathbb{R}),
\]

where \( H^1(M, \mathbb{R}) \) denotes the first de Rham cohomology group of \( M \). So
there exists a $C^\infty$ 1-form $\beta$ such that for each $C^\infty$ closed curve $\gamma$,

$$\int_\gamma \lambda = \int_\gamma \beta.$$

So by integrating $(\lambda - \beta)$ along curves, we get on $M$ a well-defined continuous function $f$. Then for any $y \in M$ and any $t \in \mathbb{R}$ we have

$$(f \circ \phi_t)(y) - f(y) = \int_0^t (1 - \beta(X))(\phi_s(y))ds.$$

Since the right-hand side of this identity is a $C^\infty \mathbb{R}$-cocycle, then by [LMM] $f$ is smooth. However by the definition of $f$ we have

$$\lambda - \beta = df.$$

So $\lambda$ is also smooth. We deduce that $E^+ \oplus E^- (= \text{Ker}\lambda)$ is $C^\infty$. So Case (2) is realized if $W^+_x$ is not dense for a certain point $x \in M$.

Now suppose that for all $x \in M$, $W^+_{x}$ is dense in $M$. Fix a Riemannian metric on $M$. For any $x \in M$ and any $r > 0$ we denote by $M_{x,r}$ and $W^+_{x,r}$ the balls of center $x$ and radius $r$ in $M$ and $W^+_x$. Take arbitrarily two open subsets $U$ and $V$ in $M$ and a small ball $M_{y,\varepsilon}$ in $V$. Since $M$ is closed and each strong unstable leaf is supposed to be dense in $M$, then we can find $R < +\infty$ such that

$$W^+_{x,R} \cap M_{y,\varepsilon} \neq \emptyset, \forall x \in M.$$

Take a small disk $W^+_{x,\delta}$ in $U$. Then by the Anosov property there exists $T > 0$ such that

$$\phi_t(W^+_{x,\delta}) \supset W^+_{\phi_t(x),R}, \forall t \geq T.$$

So $\phi_t U \cap V \neq \emptyset, \forall t \geq T$, i.e. $\phi_t$ is topologically mixing. □

If $\phi_t$ is a topologically transitive quasiconformal Anosov flow such that the dimensions of $E^+$ and $E^-$ are at least two, then by the previous lemma and Theorem 2.1, it preserves two continuous conformal strutures $\tau^+$ and $\tau^-$ which are $C^\infty$ along the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$. Then by Theorem 2.2, these two conformal structures are invariant under the weak holonomy maps. In addition we can deduce from the previous lemma and Theorem 2.2 the following

**Lemma 2.2.** Let $\phi_t$ be a $C^\infty$ topologically transitive quasiconformal Anosov flow such that the dimensions of $E^+$ and $E^-$ are at least two. Then $E^{+,0}$ and $E^{-,0}$ are both $C^\infty$. 

Based on [Sa], we have found in [Fa1] for each quasiconformal Anosov flow \( \phi_t \) an unstable linearization \( \{ h^+_x \}_x \in M \) such that for any \( x \in M \), \( h^+_x : W^+_x \to E^+_x \) is a \( C^\infty \) diffeomorphism and the following conditions are satisfied:

1. \( h^+_x (\phi_s (x)) = D_x \phi_s \circ h^+_x, \forall s \in \mathbb{R} \),
2. \( h^+_x (x) = 0 \) and \( (Dh^+_x)_x \) is the identity map,
3. \( h^+_x \) depends continuously on \( x \) in the \( C^\infty \) topology.

Recall that the family of diffeomorphisms satisfying these three conditions is unique. Similarly we have the stable linearization \( \{ h^-_x \}_x \in M \) of \( \phi_t \).

Suppose that \( \phi_t \) satisfies the conditions of Lemma 2. For any \( x \in M \) we can extend the conformal structure \( \tau^+_x \) at 0 to all other points of \( E^+_x \) via linear translations. If the resulting translation-invariant conformal structure on \( E^+_x \) is denoted by \( \sigma^+_x \), then \( (E^+_x, \sigma^+_x) \) is isometric to the canonical conformal structure of \( \mathbb{R}^n \) if \( E^+ \) is \( n \)-dimensional. Similarly we have \( (E^-_x, \sigma^-_x) \) for any \( x \in M \).

Then by Lemma 3.1 of [Sa] we get for any \( x \in M \), \( (h^+_x)_* \tau^+ |_{W^+_x} = \sigma^+_x \) and \( (h^-_x)_* \tau^- |_{W^-_x} = \sigma^-_x \).

### 2.2. Transverse (G, T)-structures

In this subsection, we consider the transverse \((G, T)\)-structures of foliations. Let \( \mathcal{F} \) be a \( C^\infty \) foliation on a connected manifold \( M \). Denote by \( Q_\mathcal{F} \) the leaf space of \( \mathcal{F} \) and by \( \mathcal{F}(A) \) the saturation of \( \mathcal{F} \) on \( A \) for any \( A \subseteq M \). We assume that the holonomy maps of \( \mathcal{F} \) are defined on connected transverse sections.

Let \( G \) be a real Lie group acting effectively and transitively on a connected manifold \( T \). If \( \Sigma \) is a \( C^\infty \) transverse section of \( \mathcal{F} \) and \( \phi \) is a \( C^\infty \) diffeomorphism of \( \Sigma \) onto its open image in \( T \), then \( (\Sigma, \phi) \) is said to be a transverse \( T \)-chart. Two transverse \( T \)-charts \( (\Sigma_1, \phi_1) \) and \( (\Sigma_2, \phi_2) \) are said to be compatible if for each holonomy map \( h \) of a germ of \( \Sigma_1 \) to a germ of \( \Sigma_2 \), the map \( \phi_1 \circ h \circ \phi_2^{-1} \) is locally the restriction of elements of \( G \).

A family of transverse sections is said to be covering if each leaf of \( \mathcal{F} \) intersects at least one of the sections in this family. By definition, a transverse \((G, T)\)-structure on \( \mathcal{F} \) is a maximal family of compatible transverse \( T \)-charts of which the underlying family of transverse sections is covering.

In order to define a transverse \((G, T)\)-structure, we need just separate out a family of covering compatible \( T \)-charts. Then by considering all the \( T \)-charts compatible with this family, we get automatically a transverse \((G, T)\)-structure.
Denote by $\tilde{F}$ the lifted foliation on the universal covering space $\tilde{M}$ of $M$ and denote by $\pi$ the projection of $\tilde{M}$ onto $M$. For each transverse $(G,T)$-structure on $F$, we get naturally a lifted transverse $(G,T)$-structure on $\tilde{F}$ by considering the composition of $\pi$ with the $T$-charts of the given transverse $(G,T)$-structure on $F$. Then by [Go] there exists a $C^\infty$ submersion $D : \tilde{M} \to T$ and a group homomorphism $H : \pi_1(M) \to G$ satisfying the following two conditions:

1. $D(\gamma x) = H(\gamma)D(x)$, $\forall x \in \tilde{M}, \forall \gamma \in \pi_1(M)$.
2. The lifted foliation $\tilde{F}$ is defined by $D$.

This submersion $D$ is said to be the developing map of the transverse $(G,T)$-structure of $F$ and $H$ is said to be the holonomy representation of $D$. The transverse $(G,T)$-structure of $F$ is said to be complete if $D$ is a $C^\infty$ fibre bundle over $D(\tilde{M})$.

If $D'$ denotes another developing map with holonomy representation $H'$, then by [Go] there exists a unique element $g \in G$ such that $D' = g \circ D$ and $H' = g \cdot H \cdot g^{-1}$.

Since $D$ is obtained by analytic continuation along curves (see [Go]), then for each transverse section $\Sigma$ of $\tilde{F}$ such that $D|_\Sigma$ is a $C^\infty$ diffeomorphism onto its image, $D|_\Sigma$ is a transverse $T$-chart of the lifted transverse $(G,T)$-structure of $\tilde{F}$.

Since $\tilde{F}$ is defined by the submersion $D$, then $D$ sends each leaf of $\tilde{F}$ to a point of $T$. Thus for any $x \in \tilde{M}$ there exists a small $C^\infty$ transverse section $\Sigma$ containing $x$ of $\tilde{F}$ such that each leaf of $\tilde{F}$ intersects $\Sigma$ at most once. Then it is easily seen that each leaf of $\tilde{F}$ is closed in $\tilde{M}$.

Denote by $Q_{\tilde{F}}$ the leaf space of $\tilde{F}$. Then we have the quotient map $D : Q_{\tilde{F}} \to D(\tilde{M})$. Since each leaf of $\tilde{F}$ is closed, then $D$ is bijective iff the $D$-inverse image of each point of $T$ is connected. If this is the case, then by considering the projections of small transverse $T$-charts, $Q_{\tilde{F}}$ becomes naturally a $C^\infty$ (seperable) manifold such that $D$ is a $C^\infty$ diffeomorphism of $Q_{\tilde{F}}$ onto $D(\tilde{M})$. In addition, the fundamental group $\pi_1(M)$ of $M$ acts naturally on $Q_{\tilde{F}}$.

By [Hae] we have the following

**Proposition 2.1.** Let $(M_1,F_1)$ and $(M_2,F_2)$ be two $C^\infty$ foliations with complete transverse $(G,T)$-structures. Suppose that their developing maps have both connected fibres and the holonomy covers of their leaves are all contractible. If the $\pi_1(M_1)$-action on $Q_{\tilde{F}_1}$ is $C^\infty$ conjugate to the $\pi_1(M_2)$-action on $Q_{\tilde{F}_2}$, then there exists a $C^\infty$ map $f : M_1 \to M_2$ such that the following conditions are satisfied:
(1) $f$ is a surjective homotopy equivalence.
(2) $f$ sends each leaf of $\mathcal{F}_1$ onto a leaf of $\mathcal{F}_2$ and $f$ sends different leaves to different leaves.
(3) $f$ is transversally a local $C^\infty$ diffeomorphism conjugating the two transverse $(G,T)$-structures.

The lemma below is self-evident and will be used several times in the following.

**Lemma 2.3.** Let $(M,\mathcal{F})$ be a $C^\infty$ foliation with a transverse $(G,T)$-structure. If $T_1$ is an open subset of $T$ and $G_1$ is a closed Lie subgroup of $G$ acting transitively on $T_1$ such that $\mathcal{D}(M) \subseteq T_1$ and $\mathcal{H}(\pi_1(M)) \subseteq G_1$, then $\mathcal{F}$ admits a transverse $(G_1,T_1)$-structure with the same developing map $\mathcal{D}$ and the same holonomy representation $\mathcal{H}$, which is compatible with the initial transverse $(G,T)$-structure.

By adapting the arguments in [Gh2] we can prove the following

**Lemma 2.4.** Let $(M,\mathcal{F})$ be a $C^\infty$ foliation with a transverse $(G,T)$-structure. If the closed leaves of $\mathcal{F}$ are dense in $M$ and the $\mathcal{D}$-inverse image of each point of $T$ is connected, then $\mathcal{H}(\pi_1(M))$ is a discrete subgroup of $G$.

**Proof.** Denote by $\pi_1$ the projection of $\tilde{M}$ onto $Q_{\tilde{\mathcal{F}}}$ and by $\pi_2$ the projection of $Q_{\tilde{\mathcal{F}}}$ onto $Q_{\mathcal{F}}$ the leaf space of $\mathcal{F}$. Take a closed leaf $F_x$ of $\mathcal{F}$ and $\tilde{x} \in \tilde{M}$ such that $\pi(\tilde{x}) = x$. Since $F_x$ is closed, then we can find a fine transverse section $\Sigma$ passing through $\tilde{x}$ such that $\pi$ sends $\Sigma$ diffeomorphically onto its image and $F_x \cap \pi(\Sigma) = \{x\}$. So for each $y \in \Sigma$ and $y \neq \tilde{x}$, $\pi(y)$ is not in $F_x$. Thus $\pi_2^{-1}(F_x)$ is discrete in $Q_{\tilde{\mathcal{F}}}$.

Since $F_x$ is closed and $\pi^{-1}(F_x) = \pi_1^{-1}(\pi_2^{-1}(F_x))$, then $\pi_2^{-1}(F_x)$ is closed in $Q_{\tilde{\mathcal{F}}}$. So the $\pi_1(M)$-orbit of $F_x$, i.e. $\pi_2^{-1}(F_x)$ is closed and discrete in $Q_{\tilde{\mathcal{F}}}$.

Since the closed leaves of $\mathcal{F}$ are dense in $M$ i.e. the union of all the closed leaves is dense, then $\pi_2$-inverse images of these closed leaves form a dense subset $P$ of $Q_{\tilde{\mathcal{F}}}$ such that the $\pi_1(M)$-orbit of each point of $P$ is closed and discrete.

Suppose on the contrary that $\mathcal{H}(\pi_1(M))$ is not discrete in $G$. Since the $\mathcal{D}$-inverse image of each point of $T$ is connected, then $\mathcal{D}$ induces a $C^\infty$ diffeomorphism $\bar{\mathcal{D}} : Q_{\tilde{\mathcal{F}}} \to \mathcal{D}(\tilde{M})$. So $\mathcal{D}(P)$ is dense in $\mathcal{D}(\tilde{M})$ and the $\mathcal{H}(\pi_1(M))$-orbit of each point of $\bar{\mathcal{D}}(P)$ is discrete and closed in $\mathcal{D}(\tilde{M})$.

Take a non-trivial one-parameter subgroup $g_t$ of the closure of $\mathcal{H}(\pi_1(M))$
in $G$. For each $t \in \mathbb{R}$, $g_t$ preserves the closed complement of $D(\tilde{M})$. So we have

$$g_t(D(\tilde{M})) = D(\tilde{M}).$$

Thus $g_t$ fixes each point in $\tilde{D}(P)$. We deduce that $g_t$ is a trivial one-parameter subgroup, which is a contradiction. □

2.3. Sphere-extension property

Let $M$ be a $C^\infty$ manifold. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous foliations with $C^1$ leaves on $M$ such that

$$T\mathcal{F}_1 \oplus T\mathcal{F}_2 = TM.$$

If $\mathcal{F}_1$ is a foliation by planes, i.e. each leaf of $\mathcal{F}_1$ is $C^1$ diffeomorphic to a certain $\mathbb{R}^n$, then $(\mathcal{F}_1, \mathcal{F}_2)$ is said to be a plane foliation couple. The local leaves of $\mathcal{F}_1$ are natural transverse sections of $\mathcal{F}_2$ and we consider only the holonomy maps of $\mathcal{F}_2$ with respect to these special transverse sections.

For each leaf $F_{1,x}$ of $\mathcal{F}_1$ we denote by $S_{1,x}$ its one-point compactification which is homeomorphic to a standard sphere. The point at infinity of $S_{1,x}$ is denoted by $\infty$.

**Definition 2.1.** Under the notations above, a plane foliation couple $(\mathcal{F}_1, \mathcal{F}_2)$ is said to have the sphere-extension property if for each holonomy map $\theta$ of $\mathcal{F}_2$ sending $x$ to $y$ there exists a homeomorphism $\Theta : S_{1,x} \to S_{1,y}$ which coincides locally with $\mathcal{F}_2$-holonomy maps on $S_{1,x} \setminus \{\infty, \Theta^{-1}(\infty)\}$ and extends the germ of $\theta$ at $x$.

If $\tilde{\phi}_t$ is a lifted flow of a $C^\infty$ Anosov flow $\phi_t$, then $\tilde{\phi}_t$ is said to have the sphere-extension property if $(\tilde{\mathcal{F}}^+, \tilde{\mathcal{F}}^{-,0})$ and $(\tilde{\mathcal{F}}^-, \tilde{\mathcal{F}}^{+,0})$ have both the sphere-extension property.

Recall that $\tilde{\mathcal{F}}^+$ and $\tilde{\mathcal{F}}^-$ are both foliations by planes. The corresponding notion for Anosov diffeomorphisms is defined similarly.

Denote by $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ the lifted couple on $\tilde{M}$ of $(\mathcal{F}_1, \mathcal{F}_2)$. Then it is easily seen that if $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)$ has the sphere-extension property, then $(\mathcal{F}_1, \mathcal{F}_2)$ has also this property. So by considering the lifted flows and drawing pictures, we can easily see that the geodesic flows of closed negatively curved manifolds have the sphere-extension property.

It is easily verified that hyperbolic infra-nilautomorphisms have the sphere-extension property. However by [Man] each Anosov diffeomorphism defined on an infra-nilmanifold is topologically conjugate to a hyperbolic infra-nilautomorphism. We deduce that the suspensions of Anosov diffeomorphisms on infra-nilmanifolds have the sphere-extension property.
Lemma 2.5. Let \( \phi_t \) and \( \psi_t \) be two C\(^1\) orbit equivalent C\(^\infty\) Anosov flows. Suppose that the strong stable and strong unstable distributions of \( \psi_t \) are of dimension at least two. If \( \psi_t \) has the sphere-extension property, then \( \phi_t \) has also this property.

Proof. Denote by \( \phi \) the C\(^1\) orbit equivalence. Define \( \hat{\phi}_t = \phi \circ \phi_t \circ \phi^{-1} \) and denote by \( \hat{E}^\pm \) the strong distributions of \( \psi_t \). Define \( \hat{F}^\pm = \phi(F^\pm) \) and \( \hat{F}^{\pm,0} = \phi(F^{\pm,0}) \). It is clear that \( \phi_t \) has the sphere-extension property iff \( \hat{\phi}_t \) has this property in the natural sense.

Since \( \phi_t \) and \( \psi_t \) are C\(^1\) orbit equivalent, then \( \hat{F}^{\pm,0} = \hat{F}^{\pm,0} \). Take a leaf of \( \hat{F}^+ \) and take a non-periodic point \( x \) in this leaf. We can identify \( \hat{W}_x^+ \) and \( \hat{W}_x^- \) naturally as following.

For all \( y \in \hat{W}_x^+ \), there exists \( t \in \mathbb{R} \) such that \( \psi_t(y) \in \hat{W}_x^+ \). If \( \hat{W}_x^{\pm,0} \) contains no periodic orbit, this number \( t \) is unique for each \( y \) in \( \hat{W}_x^+ \). If \( \hat{W}_x^{\pm,0} \) contains a periodic orbit, then it contains exactly one periodic orbit. Denote by \( T \) its minimal positive period with respect to \( \psi_t \). So if \( \psi_t(y) \in W_x^+ \), then for all \( k \in \mathbb{Z} \), \( \psi_{t+kT}(y) \in \hat{W}_x^+ \).

Conversely if \( \psi_{t_1}(y) \in W_x^+ \) and \( \psi_{t_2}(y) \in \hat{W}_x^+ \), then \( \psi_{t_2-t_1}\hat{W}_x^+ = \hat{W}_x^+ \). Thus \( t_2 - t_1 \in T \cdot \mathbb{Z} \). So by associating \( t + T \cdot \mathbb{Z} \) to \( y \), we get a well-defined C\(^\infty\) map from \( \hat{W}_x^+ \) to \( \mathbb{R}/T\mathbb{Z} \).

Thus by taking a lift if necessary, there exists a unique C\(^1\) map \( \theta_x : \hat{W}^+ \rightarrow \mathbb{R} \) such that

\[
\theta_x(x) = 0, \quad \psi_{\theta_x(y)}(y) \in \hat{W}_x^+, \quad \forall \ y \in \hat{W}_x^+.
\]

Define a C\(^1\) map \( \hat{\eta}_x : \hat{W}^+ \rightarrow \hat{W}_x^+ \) such that

\[
\hat{\eta}_x(y) = \psi_{\theta_x(y)}(y).
\]

Then \( \hat{\eta}_x \) is easily seen to be a local C\(^1\) diffeomorphism such that \( \hat{\eta}(x) = x \).

Similar we get \( \eta : \hat{W}^+_x \rightarrow \hat{W}^+_x \) such that \( \eta(x) = x \). If \( \hat{W}_x^{\pm,0} \) contains no \( \psi_t \)-periodic orbit, then \( \eta \) and \( \hat{\eta} \) are both C\(^1\) diffeomorphisms.

Suppose that \( \hat{W}_x^{\pm,0} \) contains a unique \( \psi_t \)-periodic orbit of period \( T \). Denote by \( z \) the unique intersection point of this periodic orbit with \( \hat{W}_x^+ \). For each \( k \in \mathbb{Z} \), we define

\[
\Lambda_k = \{ y \in \hat{W}_x^+ \backslash \{z\} \mid \hat{\eta} \circ \eta(y) = \psi_{kT}(y) \}.
\]

Then \( \hat{W}_x^+ \backslash \{z\} \) is the disjoint union of \( \{\Lambda_k\}_{k \in \mathbb{Z}} \). Each \( \Lambda_k \) is closed in \( \hat{W}_x^+ \backslash \{z\} \) and \( x \in \Lambda_0 \). Take \( y \in \Lambda_0 \) and since \( \psi_{-T} \) is a contracting diffeomorphism of \( \hat{W}_x^+ \), then a small ball containing \( y \) intersects with at most finitely many \( \Lambda_I \) non-trivially. We deduce that \( \Lambda_0 \) is open.
Since $E^+$ is at least two-dimensional, then $W^+_x \setminus \{z\}$ is connected. We deduce that $A_0 = W^+_x \setminus \{z\}$, i.e. $\hat{\eta} \circ \hat{\eta} = Id$. Similarly we have $\hat{\eta} \circ \hat{\eta} = Id$. We identify $W^{-}_x$ and $W^{\pm}_x$ under these two sliding $C^1$ diffeomorphisms $\eta$ and $\hat{\eta}$. We can identify $W^{-}_x$ and $W^\pm_x$ similarly.

Since these identifications conjugate the holonomy maps and $\psi_t$ has the sphere extension property, then $\phi_t$ has also this property. We deduce that $\phi_t$ has the sphere-extension property. □

3. Proofs of Theorems 1.2 and 1.3

3.1. Construction of a transverse geometric structure

Denote by $\phi_t$ a $C^\infty$ topologically transitive quasiconformal Anosov flow such that $E^+$ and $E^-$ are of dimensions at least three. Then by Theorem 2.1 and Lemma 2.2, $E^{+0}$ and $E^{-0}$ are both $C^\infty$ and there exist $\tau^+$ and $\tau^-$ two continuous $\phi_t$-invariant conformal structures on $E^+$ and $E^-$ which are $C^\infty$ along the leaves of $F^+$ and $F^-$. Denote by $\Phi$ the orbit foliation of $\phi_t$. For each transverse section $\Sigma$ of $\Phi$ we get two $C^\infty$ foliations $F^+_\Sigma$ and $F^-_\Sigma$ on $\Sigma$ by intersecting $F^{+0}$ with $\Sigma$. Denote their tangent distributions by $E^+_\Sigma$ and $E^-_\Sigma$ respectively.

We can identify $E^+_\Sigma$ and $E^-_\Sigma$ by projecting $E^\pm_\Sigma$ onto $E^\pm_\Sigma$ parallel to the direction of the flow. Under this identification, we get two conformal structures $\tau^+_\Sigma$ and $\tau^-_\Sigma$ on $E^+_\Sigma$ and $E^-_\Sigma$. Since $\tau^-_\Sigma$ is easily seen to be invariant under the $\Phi$-holonomy maps and the $F^+_\Sigma$-holonomy maps, then $\tau^-_\Sigma$ is $C^\infty$ on $\Sigma$. Similarly we can see that $\tau^+_\Sigma$ is also $C^\infty$ on $\Sigma$. So we get on each transverse section $\Sigma$ a $C^\infty$ geometric structure $(F^\pm_\Sigma, \tau^\pm_\Sigma)$ which is invariant under the $\Phi$-holonomy maps.

Denote by $c_n$ the canonical conformal structure on the $n$-dimensional sphere $S^n$ and by $M_n$ the isometry group of $c_n$. Then $M_n$ acts transitively on $S^n$ and is called the Möbius group. Suppose that $E^+$ is of dimension $n$ and $E^-$ is of dimension $m$. Then we can construct as following a transverse $(M_n \times M_m, S^n \times S^m)$-structure on $\Phi$.

For any $x \in M$ we denote by $S^+_x$ and $S^-_x$ the one-point compactifications of $E^+_x$ and $E^-_x$. Then they admit naturally $C^\infty$ conformal structures extending $\sigma^+_x$ and $\sigma^-_x$. Since

$$(h^+_x)_*(\tau^+) = \sigma^+_x \quad \text{and} \quad (h^-_x)_*(\tau^-) = \sigma^-_x,$$

then $S^+_x$ and $S^-_x$, i.e. the one-point compactifications of $W^+_x$ and $W^-_x$ admit also natural conformal structures isometric to those of $S^+_x$ and $S^-_x$ under the natural extensions of $h^+_x$ and $h^-_x$, which are denoted by $\tilde{h}^+_x$ and $\tilde{h}^-_x$.

By fixing two conformal frames of $E^+_x$ and $E^-_x$ we get two $C^\infty$ conformal isometries $\phi^+_x : S^+_x \to S^n$ and $\phi^-_x : S^-_x \to S^m$. 

Take a $C^\infty$ small transverse section $\Sigma_x$ containing $x$ and pieces of $W^+_x$ and $W^-_x$. Thus for $\delta \ll 1$ we get the local diffeomorphism

$$\theta_x : W^+_{x,\delta} \times W^-_{x,\delta} \to \Sigma_x$$

$$(y, z) \to W^-_{\Sigma_x,y, 2\delta} \cap W^+_{\Sigma_x,z, 2\delta}.$$ 

Then we define $\phi_x : \Sigma_x \to S^n \times S^m$ such that $\phi_x = (\phi_x^+ \times \phi_x^-) \circ (\bar{h}_x^+ \times \bar{h}_x^-) \circ \theta_x^{-1}$.

Since $\tau_\Sigma^+$ and $\tau_\Sigma^-$ are invariant under respectively the $F^+_{\Sigma}$-holonomy maps and the $F^-_{\Sigma}$-holonomy maps, then by its definition, $\phi_x$ is easily seen to be a local isometry of $(F^\pm_{\Sigma_x}, \tau^\pm_{\Sigma_x})$ to $(\{S^n \times \ast\}, \{\ast \times S^m\}, c_n \times c_m)$.

Let $h$ be any $\Phi$-holonomy map from a germ of $\Sigma_x$ to a germ of $\Sigma_y$. Then it is easy to see that $\theta_x^{-1} \circ h \circ \theta_x$ is given by weak holonomy maps. We deduce that $\phi_y \circ h \circ \phi_x^{-1} = \phi \times \psi$, where $\phi$ and $\psi$ are respectively local conformal isometries of $S^n$ and $S^m$. Since $n, m \geq 3$, then by the following classical theorem of Liouville, $\phi$ and $\psi$ can be both extended to global conformal isometries of $S^n$ and $S^m$. So $\{(\Sigma_x, \phi_x)\}_{x \in M}$ gives a transverse $(M_n \times M_m, S^n \times S^m)$-structure of $\Phi$.

**Theorem 3.1.** (Liouville) For $n \geq 3$, each local conformal isometry of $S^n$ defined on a connected open subset can be extended uniquely to a global conformal isometry.

### 3.2. Completeness

Fix a developing map $D$ of the transverse $(M_n \times M_m, S^n \times S^m)$-structure of $\Phi$ defined in the previous subsection. Denote by $H$ the associated holonomy representation.

**Lemma 3.1.** Under the notations above, each leaf of $\bar{F}^+$ intersects each leaf of $\bar{F}^-\cdot 0$ at most once.

**Proof.** By the definition of $D$, for any $x \in \bar{M}$, $D(\bar{W}^+_x)$ must be contained in a certain subset $S^n \times b$. Then for any $y \in \bar{W}^+_x$, there exists $\gamma_1 \times \gamma_2 \in M_n \times M_m$ such that

$$(\gamma_1 \times \gamma_2) \circ D = \bar{\phi}_y,$$

where $\bar{\phi}_y$ is defined similarly as above. Denote by $D_1$ the composition $pr_1 \circ D$. Then there exists an open neighborhood $V_y$ of $y$ in $\bar{W}^+_x$ such that

$$\gamma_1 \circ D_1 \mid V_y = \bar{\phi}_y^+ \circ \bar{h}_y^+.$$
Since $\tilde{h}_x^+ \circ h_y^{-1}$ sends $\tilde{\sigma}_y^+$ to $\tilde{\sigma}_x^+$ and the dimension of $E^+$ is at least two, then $\tilde{h}_x^+ \circ h_y^{-1}$ is an affine map. Thus there exists $\gamma \in M_n$ such that
\[ \gamma \circ D_1 |_{\tilde{W}_x^+} = \tilde{\phi}_x^+ \circ \tilde{h}_x^+. \]

So $D$ sends $\tilde{W}_x^+$ diffeomorphically onto a set of the form $(S^n \setminus a) \times b$.

For any $y \in S^n \setminus a$ such that $D(z) = y$ and $z \in \tilde{W}_x^+$, $D$ sends $\tilde{W}_x^-$ diffeomorphically onto a set of the form $y \times (S^n \setminus \omega(y))$. So we get a well-defined map $\omega : S^n \setminus a \to S^n$.

Now suppose that $\tilde{W}_x^+$ intersects $\tilde{W}_x^{-,0}$ at a point $x'$ other than $x$. Then there exist $y, y' \in S^n$ such that $y \neq y'$ and
\[ D(W_x^-) \subseteq y \times S^n, \quad D(W_x'^-) \subseteq y' \times S^n. \]

Denote by $x''$ the intersection of the $\tilde{\phi}_{\epsilon}$-orbit of $x'$ with $\tilde{W}_x^-$. Then we have $D(x'') \neq D(x')$. However by the definition of $D$, $D(x'') = D(x')$, which is a contradiction. We deduce that each leaf of $\tilde{F}^+$ intersects each leaf of $\tilde{F}^{-,0}$ at most once. □

The following lemma is a direct consequence of the previous lemma, which is firstly observed by T. Barbot in [Ba].

**Lemma 3.2.** Under the notations above, the lifted orbit space $Q_{\tilde{\Phi}}$ is Hausdorff.

**Proof.** Suppose on the contrary that there exist two different orbits $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ such that each $\Phi$-saturated open neighborhood of $\tilde{\Phi}_1$ intersects that of $\tilde{\Phi}_2$. We want to see that these two orbits are contained in the same leaf of $\tilde{F}^{-,0}$.

Suppose that it is not the case. Denote by $F_1$ and $F_2$ the leaves of $\tilde{F}^{-,0}$ containing respectively these two orbits. Then by assumption the $\tilde{F}^+$-saturated sets of $F_1$ and $F_2$ intersect non-trivially. We deduce that there exists a leaf $\tilde{W}_x^+$ intersecting $F_1$ and $F_2$. Denote by $V_1$ and $V_2$ two disjoint open subsets of $\tilde{W}_x^+$ containing respectively the intersection of $\tilde{W}_x^+$ with $F_1$ and that of $\tilde{W}_x^+$ with $F_2$. Then by assumption the $\tilde{F}^{-,0}$-saturated set of $V_1$ intersects that of $V_2$ non-trivially, which contradicts Lemma 3.1.

Thus $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are contained in the same leaf of $\tilde{F}^{-,0}$. Similar we can prove that they are contained in the same leaf of $\tilde{F}^{+,0}$. Then by Lemma 3.1, we have $\tilde{\Phi}_1 = \tilde{\Phi}_2$, which is a contradiction. □
For each $x \in \tilde{M}$ we construct an open subset $U_x$ of $\tilde{M}$ such that $U_x$ is the union of the leaves of $\tilde{F}^{-b}$ intersecting $\tilde{W}^+_b$. Then we can find a sequence $\{x_i\}_{i=1}^\infty \subseteq \tilde{M}$ satisfying the following conditions:

1. $\bigcup_{i \geq 1} U_{x_i} = \tilde{M}$.
2. For each $k \geq 1$, $\Omega_k = \bigcup_{i=1}^k U_{x_i}$ is connected.

In the following we denote $U_{x_i}$ by $U_i$. Largely inspired by the arguments in [Gh1], we prove the following lemma.

**Lemma 3.3.** For each $k \geq 1$, $\mathcal{D}_{|\Omega_k} : \Omega \to \mathcal{D}(\Omega_k)$ is a $C^\infty$ fiber bundle with fiber $\mathbb{R}$ over $\mathcal{D}(\Omega_k)$. In addition $\mathcal{D}(\Omega_k)$ is either the complement in $S^n \times S^m$ of the graph of a continuous map from $S^n$ to $S^m$ or the complement of the union of $\{\ast\} \times S^m$ and of the graph of a continuous map from $(S^n \setminus \{\ast\})$ to $S^m$.

**Proof.** We prove this lemma by induction. For $k = 1$ we have $\Omega_1 = U_1$. In the proof of Lemma 3.1 we have seen that $\mathcal{D}$ sends $\tilde{W}^+_x$ diffeomorphically onto a set of the form $(S^n \setminus a) \times b$. For any $y \in S^n \setminus a$ such that $\mathcal{D}(z) = y$ and $z \in \tilde{W}^+_x$, $\mathcal{D}$ sends $\tilde{W}^-_x$ diffeomorphically onto a set of the form $y \times (S^m \setminus \omega(y))$. So we get a well-defined map $\omega : S^n \setminus a \to S^m$.

Denote by $Gr(\omega)$ the graph of $\omega$. Then the complement of $Gr(\omega)$ in $(S^n \setminus a) \times S^m$ is the open set $\mathcal{D}(U_1)$. So $\omega$ is continuous. By the definition of $U_1$ the inverse images of $\mathcal{D}_{|U_1}$ are all connected. Then by the existence of fine transverse sections $\mathcal{D}_{|U_1}$ is seen to be a fiber bundle of fiber $\mathbb{R}$. So the lemma is true for $k = 1$.

Suppose that the lemma is true for $\Omega_k$. Then $\mathcal{D}_{|\Omega_k}$ is a fiber bundle with fiber $\mathbb{R}$ and $\mathcal{D}(\Omega_k)$ is the complement in $S^n \times S^m$ of the graph of a $C^0$ map $u_k : S^n \to S^m$ or of the union of a vertical $a_k \times S^m$ and the graph of a $C^0$ map $u_k : S^n \setminus a_k \to S^m$.

In addition by the argument above, we know that $\mathcal{D}(U_{k+1})$ is the complement of the union of $b_{k+1} \times S^m$ and of the graph of a $C^0$ map $v_{k+1} : S^n \setminus b_{k+1} \to S^m$ and $\mathcal{D}_{|U_{k+1}}$ is a fiber bundle with fiber $\mathbb{R}$.

Since $\mathcal{D}(U_{k+1}) \cap \mathcal{D}(\Omega_k)$ is the complement in $S^n \times S^m$ of a finite union of topological submanifolds of codimension at least two, then $\mathcal{D}(U_{k+1}) \cap \mathcal{D}(\Omega_k)$ is connected and open.

Firstly we want to see that $\mathcal{D}_{|\Omega_{k+1}}$ is a fiber bundle with fiber $\mathbb{R}$. Take $x \in \Omega_k$ and $y \in U_{k+1}$ such that $\mathcal{D}(x) = \mathcal{D}(y)$. Since $\Omega_{k+1}$ is connected, then $\Omega_k \cap U_{k+1} \neq \emptyset$. So we can take $z \in \Omega_k \cap U_{k+1}$ and a $C^0$ curve $\gamma$ in $\mathcal{D}(U_{k+1}) \cap \mathcal{D}(\Omega_k)$ connecting $D(z)$ and $D(x)$.
Since $\mathcal{D} \mid_{\Omega_k}$ and $\mathcal{D} \mid_{U_{k+1}}$ are fiber bundles with fiber $\mathbb{R}$, then we can lift $\gamma$ to two $C^0$ curves $\gamma_1 \subseteq \Omega_k$ and $\gamma_2 \subseteq U_{k+1}$ such that $\gamma_1(0) = \gamma_2(0) = z$. Thus $\gamma_1(1)$ and $x$ are contained in the same $\tilde{\phi}_t$-orbit and so are $\gamma_2(1)$ and $y$.

Denote by $\Lambda$ the subset of $t \in [0, 1]$ such that $\gamma_1(t)$ and $\gamma_2(t)$ are in the same orbit of $\tilde{\phi}_t$. By the section property, $\Lambda$ is easily seen to be open in $[0, 1]$. Suppose that $\{t_n\}_{n=1}^\infty \subseteq \Lambda$ and $t_n \to t$. If $\gamma_1(t)$ and $\gamma_2(t)$ are not in the same $\tilde{\phi}_t$-orbit, then by Lemma 3.2 there exist disjoint $\tilde{\phi}_t$-saturated open neighborhoods of $\gamma_1(t)$ and $\gamma_2(t)$. Thus for $n \gg 1$, $\gamma_1(t_n)$ and $\gamma_2(t_n)$ are not in the same orbit of $\tilde{\phi}_t$, which is a contradiction. We deduce that $\Lambda$ is closed. Thus $\Lambda = [0, 1]$. So $x$ and $y$ are contained in the same $\tilde{\phi}_t$-orbit. We deduce that $\mathcal{D} \mid_{\Omega_{k+1}}$ is a fiber bundle with fiber $\mathbb{R}$.

Now we want to see the form of $\mathcal{D}(\Omega_{k+1})$. Suppose at first that $\mathcal{D}(\Omega_k) = (Gr(u_k))^c$. Take $p \in S^n \setminus b_{k+1}$. If $u_k(p) \neq v_{k+1}(p)$, then $\mathcal{D}(\Omega_{k+1})$ contains the vertical $p \times S^m$. Since $p \neq b_{k+1}$, then there exists $x \in U_{k+1}$ such that $\mathcal{D}(\tilde{W}^{-}_x) = p \times (S^m \setminus v_{k+1}(p))$. So there exists $y \in \Omega_k$ such that $p \times v_{k+1}(p) \in \mathcal{D}(\tilde{W}^{-}_y)$. In particular, $\mathcal{D}(\tilde{W}^{-}_x) \cap \mathcal{D}(\tilde{W}^{-}_y) \neq \emptyset$. So there exists $t \in \mathbb{R}$ such that $\tilde{\phi}_t(\tilde{W}^{-}_x) = \tilde{W}^{-}_y$. We deduce that $\mathcal{D}(\tilde{W}^{-}_x) = p \times S^m$, which is absurd. So in this case, $\mathcal{D}(\Omega_{k+1}) = (Gr(u_k))^c$.

Suppose that $\mathcal{D}(\Omega_k) = (Gr(u_k) \cup (a_k \times S^m))^c$. For each $p \in S^n \setminus \{a_k, b_{k+1}\}$ we get as above that $u_k(p) = v_{k+1}(p)$.

If $a_k \neq b_{k+1}$, then $u_k$ and $v_{k+1}$ can be extended to the same continuous map $\tilde{u}_k$ on $S^n$. In this case $\mathcal{D}(\Omega_{k+1}) = (Gr(\tilde{u}_k))^c$.

If $a_k = b_{k+1}$, then we certainly have $\mathcal{D}(\Omega_{k+1}) = (Gr(u_k) \cup (a_k \times S^m))^c$.

□

We deduce from the previous lemma that $\mathcal{D} : \tilde{M} \to \tilde{\mathcal{M}}$ is a $C^\infty$ fiber bundle with fiber $\mathbb{R}$. So the transverse $(M_n \times M_m, S^n \times S^m)$-structure of $\Phi$ is complete. In addition by the proof of the previous lemma we see that if $b_k$ is equal to $b_1$ for each $k \geq 1$ then $\mathcal{D}(\tilde{M}) = (Gr(u_1) \cup (a_1 \times S^m))^c$. If there exists $k > 1$ such that $b_k \neq b_1$ then $\mathcal{D}(\tilde{M}) = (Gr(u_1))^c$.

By exchanging the roles of $E^+$ and $E^-$ in the previous lemma, we get the following two cases:

1. $\mathcal{D}(\tilde{M}) = (S^n \setminus a) \times (S^m \setminus b)$.
2. $\mathcal{D}(\tilde{M}) = (Gr(f))^c$ where $f$ is a homeomorphism of $S^n$ onto $S^m$. In particular $n = m$ in this case.

Let us consider firstly Case (1). By changing the developing map we can suppose that $a = b = \infty$. Denote by $CO_n$ the isometry group of the canonical conformal structure of $\mathbb{R}^n$. Then by Lemma 2.3 we get a compatible transverse $(CO_n \times CO_m, \mathbb{R}^n \times \mathbb{R}^m)$-structure of $\Phi$. In particular, the weak
stable and weak unstable foliations admit transverse affine structures. So by [Pl2] the flow \( \phi_t \) admits a \( C^\infty \) global section \( \Sigma \). Since the Poincaré map \( \phi \) of \( \Sigma \) is also topologically transitive and quasiconformal, then by [K-Sa], \( \phi_t \) is \( C^\infty \) conjugate to a finite factor of a hyperbolic automorphism of a torus. We deduce that up to finite covers, \( \phi_t \) is \( C^\infty \) orbit equivalent to the suspension of a hyperbolic automorphism of a torus.

Now we consider Case (2). Denote by \( \Gamma \) the fundamental group of \( M \). Then by Lemma 2.4 the group \( \mathcal{H}(\Gamma) \) is discrete in \( M_n \times M_n \). Define \( \mathcal{H}_1 = pr_1 \circ \mathcal{H} \) and \( \mathcal{H}_2 = pr_2 \circ \mathcal{H} \). Then we have

\[
  \pi \circ \mathcal{H}_1(\gamma) \circ f^{-1} = \mathcal{H}_2(\gamma), \quad \forall \gamma \in \Gamma.
\]

We deduce that \( \mathcal{H}_1(\Gamma) \) and \( \mathcal{H}_2(\Gamma) \) are both discrete in \( M_n \). Denote \( \mathcal{H}_1(\Gamma) \) by \( \Gamma_1 \) and \( \mathcal{H}_2(\Gamma) \) by \( \Gamma_2 \). Since \( \phi_t \) is topologically transitive, then \( \Phi \) admits at least a simply connected leaf. We deduce that \( \mathcal{H} \) is injective. So \( \Gamma \) and \( \Gamma_1 \) and \( \Gamma_2 \) are all isomorphic.

We can prove that \( \Gamma_1 \) is uniform in \( M_n \) as following. Suppose on the contrary that \( \Gamma_1 \) is not uniform. Then \( \Gamma_1 \) admits a finite index torsion free subgroup \( \Gamma'_1 \) such that \( cd(\Gamma'_1) \leq n \), where \( cd(\Gamma'_1) \) denotes the cohomological dimension of \( \Gamma'_1 \). So by passing to a finite index subgroup if necessary, we can suppose that \( cd(\Gamma) \leq n \).

Denote by \( BG \) the classifying space of \( \Gamma \) and by \( ET \) the universal covering space of \( BG \). Then we have

\[
  BG \cong \Gamma_1 \backslash \mathbb{H}^{n+1}, \quad ET \cong \mathbb{H}^{n+1},
\]

where \( \mathbb{H}^{n+1} \) denotes the simply connected hyperbolic space of dimension \( n + 1 \). Denote by \( ET \times \Gamma \tilde{M} \) the quotient manifold of \( ET \times \tilde{M} \) under the diagonal action of \( \Gamma \). Then we have the following fibre bundle with fiber \( \tilde{M} \)

\[
  \pi_1 : ET \times \Gamma \tilde{M} \to BG,
\]

\[
  \Gamma((a,x)) \to \Gamma(a).
\]

By using the cohomology Leray-Serre spectral sequence to this fibre bundle (see [Mc]), we get that

\[
  E_2^{p,q} = H^p(\Gamma, H^q(\tilde{M}))
\]

converges to \( H^{p+q}(ET \times \Gamma \tilde{M}) \). Since \( \tilde{M} \) is a fibre bundle with fiber \( \mathbb{R} \) and base \( (S^n \times S^n) \backslash (Gr(f)) \), then \( \tilde{M} \) is homotopically equivalent to the sphere.
$S^n$. Since we have in addition $cd(\Gamma) \leq n$, then we deduce from the spectral sequence above that $H^{2n+1}(E \times \tilde{M})$ is trivial.

However by projecting onto the second factor $E \times \tilde{M}$ is easily seen to be also a fibre bundle over $M$ and with contractible fiber $E \Gamma$. So $E \times \tilde{M}$ is homotopically equivalent to $M$. We deduce that $H^{2n+1}(M)$ is trivial, which is absurd. So $\Gamma_1$ is uniform in $M_n$. Similarly $\Gamma_2$ is also uniform in $M_n$.

Since $f$ conjugates $\Gamma_1$ to $\Gamma_2$, then by Mostow's rigidity theorem (see [Mo]) $f$ is contained in $M_n$. So by replacing $D$ by $(Id \times f^{-1}) \circ D$, we can suppose that $f = Id$ and

$$D(\tilde{M}) = (S^n \times S^n) \setminus \Delta,$$

where $\Delta$ denotes the diagonal of $S^n \times S^n$. In addition, we have $\mathcal{H}_1 = \mathcal{H}_2$. So by Lemma 6.3.1, $\Phi$ admits a compatible transverse $(M_n, (S^n \times S^n) \setminus \Delta)$-structure with respect to the diagonal action of $M_n$ on $(S^n \times S^n) \setminus \Delta$.

Lift $\phi_t$ to a finite cover to eliminate the torsion of $\Gamma$ and define $V = \mathcal{H}(\Gamma) \setminus \mathbb{H}^{n+1}$. Then $V$ is a closed hyperbolic manifold. In addition, the $\Gamma$-action on $Q_\Phi$ is $C^\infty$ conjugate to the $\mathcal{H}(\Gamma)$-action on the leaf space of the lifted geodesic flow of $V$ under $D$ and $\mathcal{H}$. Since the holonomy of each periodic orbit of $\phi_t$ is non-trivial, then the holonomy covering of each leaf of $\Phi$ is contractible. Denote by $\psi_t$ the geodesic flow of $V$. So by Proposition 2.1 there exists a $C^\infty$ homotopy equivalence $h$ conjugating the leaf space of $\phi_t$ with that of $\psi_t$. However $h$ is not in general a $C^\infty$ diffeomorphism. In order to get a $C^\infty$ orbit conjugacy between $\phi_t$ and $\psi_t$, we use a classical diffusion argument discovered by É. Ghys. Let us recall briefly this argument (see [Gh2] and [Ba] for details):

There exists a $C^\infty$ function $u : \mathbb{R} \times M \to \mathbb{R}$ such that

$$h(\phi_t(x)) = \psi_{u(t,x)}(h(x)), \quad \forall \ t \in \mathbb{R}, \ \forall \ x \in M.$$ 

Define for $T \gg 1$, $u_T(x) = \frac{1}{T} \int_0^T u(s,x)ds$ and $h_T : M \to T^1V$ such that $h_T(x) = \psi_{u_T(x)}(h(x))$. If $T \gg 1$, then we can see that $h_T$ satisfies the same conditions as $h$ and is a $C^\infty$ diffeomorphism.

So up to finite covers, $\phi_t$ is $C^\infty$ orbit equivalent to the geodesic flow of a closed hyperbolic manifold, which finishes the proof of the first part of Theorem 1.2.

### 3.3. Smoothness blowing up

In this subsection we prove the second part of Theorem 1.2. Suppose that $\phi_t$ satisfies the conditions of Theorem 1.2 such that $E^+ \oplus E^-$ is in
addition $C^1$. Then because of the first part of Theorem 1.2, $\phi_t$ is seen to be volume-preserving. So in order to prove the second part of Theorem 1.2, we need only prove the $C^\infty$ smoothness of $E^+ \oplus E^-$ and then use the following classification result established in [Fa1]:

**Theorem 3.2.** ([Fa1], Theorem 1) Let $\phi_t$ be a $C^\infty$ volume-preserving uniformly quasiconformal Anosov flow on a closed manifold $M$. If $E^+ \oplus E^-$ is $C^\infty$ and the dimensions of $E^+$ and $E^-$ are at least 2, then up to a constant change of time scale and finite covers, $\phi_t$ is $C^\infty$ flow equivalent either to the suspension of a hyperbolic automorphism of a torus, or to a canonical perturbation of the geodesic flow of a hyperbolic manifold.

**Lemma 3.4.** Under the notations above, $E^+ \oplus E^-$ is $C^\infty$.

**Proof.** Suppose at first that $\phi_t$ is $C^\infty$ orbit equivalent to the geodesic flow $\psi_t$ of a hyperbolic manifold (up to finite covers). Denote by $\lambda$ the canonical 1-form of $\phi_t$ and by $X$ the generator of $\psi_t$. Up to $C^\infty$ flow conjugacy we suppose that $\phi_t$ is generated by $fX$.

Since $E^+ \oplus E^-$ is supposed to be $C^1$, then $\lambda$ is $C^1$ and $\lambda(X) = \frac{1}{f}$ is $C^\infty$. It is easily seen that $d\lambda$ is $\phi_t$-invariant. Then by the Anosov property of $\phi_t$, we get $i_{fX}d\lambda = 0$. Thus $i_Xd\lambda = 0$. We deduce that $d\lambda$ is $\psi_t$-invariant. Denote by $\lambda'$ the canonical 1-form of $\psi_t$. Then by [Ham1] there exists $a \in \mathbb{R}$ such that $d\lambda = a \cdot d\lambda'$.

Define $\beta = \lambda - a \cdot \lambda'$. Then $\beta$ is a $C^1$ 1-form such that $d\beta = 0$. In addition $\beta(X) = \lambda(X) - a$ is $C^\infty$.

**Sublemma.** Let $\phi_t$ be a $C^\infty$ volume-preserving Anosov flow on $M$. If $\alpha$ is a $C^1$ 1-form on $M$ such that $d\alpha = 0$ and $\alpha(X)$ is $C^\infty$, then $\alpha$ is $C^\infty$.

**Proof.** Since $d\alpha = 0$ and the Stokes formula is valid for $C^1$ forms (even for Lipschitz forms), then there exists a $C^\infty$ 1-form $\beta$ giving the same element of $(H^1_t(M, \mathbb{R}))^*$ as that of $\alpha$. So by integrating $(\alpha - \beta)$ along curves, we get a well-defined $C^2$ function $f$ on $M$. Thus for any $x \in M$ and any $t \in \mathbb{R}$ we have

$$f(\phi_t(x)) - f(x) = \int_0^t (\alpha(X) - \beta(X)) \circ \phi_s(x) ds.$$  

Since $\alpha(X)$ is supposed to be $C^\infty$, then by [LMM], $f$ is seen to be $C^\infty$. However by the definition of $f$, we have $\alpha - \beta = df$. Thus $\alpha$ is $C^\infty$. $\square$
We deduce from this sublemma that $\beta$ is $C^\infty$. Thus $\lambda$ is $C^\infty$. So $E^+ \oplus E^-$ is also $C^\infty$.

If $\phi_t$ is $C^\infty$ orbit equivalent to the suspension $\psi_t$ of a hyperbolic automorphism of a torus (up to finite covers), then by similar arguments as above, we can see that $d\lambda$ is $\psi_t$-invariant.

Take a leaf $\Sigma$ of the foliation of the sum of the strong stable and the strong unstable distributions of $\psi_t$ and denote by $\psi$ its Poincaré map. Then $\lambda|_\Sigma$ is $C^1$ and $d(\lambda|_\Sigma)$ is $\psi$-invariant. Thus by the same arguments as in [Fa2] we get $d(\lambda|_\Sigma) = 0$. We deduce that $d\lambda = 0$. Since in addition $\lambda(X) = \frac{1}{4}$ is $C^\infty$, then by the previous sublemma $\lambda$ is $C^\infty$. Thus $E^+ \oplus E^-$ is also $C^\infty$. □

**Proof of Theorem 1.3.** Suppose that $\phi_t$ satisfies the conditions of Theorem 1.3. Similar to the previous section, we can construct a $C^\infty$ geometric structure $(\mathcal{F}^\pm, \tau^\pm)$ on each transverse section $\Sigma$ of $\Phi$. Similarly we can construct a family of transverse charts $\{\Sigma_x, \phi_x\}_{x \in \mathcal{M}}$. Then because of the sphere-extension property, the chart changes of these charts are easily seen to be given by the restrictions of the elements of $M_2 \times M_m$ with respect to the natural action of $M_2 \times M_m$ on $S^2 \times S^m$. So in this way we get a transverse $(M_2 \times M_m, S^2 \times S^m)$-structure on $\Phi$. Then as in the previous subsection the proof splits into Case (1) and Case (2). Each of them is understood in the same manner as in the previous subsection.

4. Applications to the geodesic flows of hyperbolic manifolds

Now let us begin to prove Theorem 1.6.

**Lemma 4.1.** Let $\phi_t$ and $\psi_t$ be two $C^\infty$ Anosov flows which are $C^1$ orbit equivalent. If $\psi_t$ is volume-preserving, then so is $\phi_t$.

**Proof.** By conjugating $\phi_t$ by the $C^1$ orbit conjugacy, we can suppose that $\phi_t$ is a $C^1$ flow and a time change of $\psi_t$. Denote by $\nu$ the $\psi_t$-invariant volume form and by $X$ the generator of $\psi_t$. Then by taking $i_X \nu$ we get a family of $\Psi$-holonomy invariant volume forms on the transverse sections of $\Psi$. This family of transversal volume forms is also $\Phi$-holonomy invariant. Denote by $dt_\phi$ the normalized foliated measure along the leaves of $\Phi$ such that $dt_\phi(Y) \equiv 1$, where $Y$ denotes the generator of $\phi_t$. In each flow box of $\phi_t$ we take the product measure $\nu_\Sigma \otimes dt_\phi$. Then it is easily seen that in the intersection of two flow boxes the two measures coincide. Then we can extend this family of local measures to a measure $\mu$ on $M$ which is in the Lebesgue class and easily seen to be $\phi_t$-invariant. □
Proof of Theorem 1.6. Since $\psi_t$ is conformal, then by Lemma 1.1, $\phi_t$ is quasiconformal. In addition by lemmas 2.5 and 4.1, $\phi_t$ verifies the conditions of Theorem 1.2 or Theorem 1.3. So up to finite covers, $\phi_t$ is $C^\infty$ orbit equivalent either to a suspension or to the geodesic flow of a hyperbolic manifold $\hat{\psi}_t$. Since $\psi_t$ is contact, then it admits no $C^1$ global section. So up to finite covers, $\phi_t$ is $C^\infty$ orbit equivalent to $\hat{\psi}_t$.

However in the proofs of Theorems 1.2 and 1.3, we passed to a finite cover only in order to eliminate the torsion in the fundamental group of $M$. But in the current case, the fundamental group has no torsion by the classical Cartan theorem. So $\phi_t$ is $C^\infty$ orbit equivalent to $\hat{\psi}_t$. Then by Mostow’s rigidity theorem (see [Mo] and [M]), $\hat{\psi}_t$ is $C^\infty$ flow equivalent to $\psi_t$. We deduce that $\phi_t$ is $C^\infty$ orbit equivalent to $\psi_t$. □

Proof of Corollary 1.1. Let us prove firstly (1). Suppose that the geodesic flow of $M$ is Hölder-continuously orbit equivalent to that of a hyperbolic manifold $N$. Since $n \geq 3$, then the fundamental group of $M$ is isomorphic to that of $N$. Since $\pi_1(M)$ with its word metric is quasi-isometric to $\tilde{M}$ and $\pi_1(N)$ is quasi-isometric to $\mathbb{H}^n$, then we deduce that $\tilde{M}$ is quasi-isometric to $\mathbb{H}^n$.

Conversely, if $\tilde{M}$ is quasi-isometric to $\mathbb{H}^n$, then $\pi_1(M)$ is also quasi-isometric to $\mathbb{H}^n$. Thus by [Su] and [Tu], there exists a uniform lattice $\Gamma$ in the isometric group of $\mathbb{H}^n$ and a surjective group homomorphism $\rho : \pi_1(M) \to \Gamma$ such that the kernel of $\rho$ is finite. However by a classical result of É. Cartan, $\pi_1(M)$ is without torsion. We deduce that $\pi_1(M)$ is isomorphic to $\Gamma$. In particular, $\Gamma$ is also without torsion. So $N = \Gamma \backslash \mathbb{H}^n$ is a closed hyperbolic manifold.

Denote respectively by $\phi_t$ and $\psi_t$ the geodesic flows of $M$ and $N$. Since $\pi_1(M) \cong \pi_1(N)$, then by [Gr], $\phi_t$ is $C^0$ orbit equivalent to $\psi_t$. Since each continuous orbit conjugacy between Anosov flows can be $C^0$ approximated by Hölder-continuous orbit conjugacies (see [HK]), then (1) is true.

Now let us prove (2). We need only prove the necessity. Suppose that the geodesic flow $\phi_t$ of $M$ is $C^1$ orbit equivalent to the geodesic flow $\psi_t$ of a closed hyperbolic manifold $N$. Since $\psi_t$ is conformal, then by Lemma 1.1, $\phi_t$ is quasiconformal. Thus by Corollary 1.1, it is $C^\infty$ orbit equivalent to the geodesic flow of $N$. Since $C^\infty$ orbit conjugacy preserves weak stable and weak unstable distributions, then $\phi_t$ is Anosov-smooth. So by [BFL], it is $C^\infty$ flow equivalent to the geodesic flow of a Riemannian manifold of constant negative curvature. Then by [BCG], $M$ has constant negative...
Before the proof of Corollary 1.2, let us recall firstly some notions. A $C^\infty$ flow $\phi_t$ defined on a closed $n$-dimensional manifold is said to be quasi-Anosov if there exists a continuous $(n-1)$-dimensional distribution $\nu$ transversal to the flow, such that for any non-zero vector $v$ in $\nu$, the set $\{\|D\phi_t(v)\|, \; t \in \mathbb{R}\}$ is unbounded with respect to a certain (then all) Riemannian metric. In [Ma], R. Manè proved the following important result.

**Theorem 4.1.** (R. Manène) If $\phi_t$ is quasi-Anosov and volume-preserving, then $\phi_t$ is Anosov.

**Proof of Corollary 1.2.** Denote $h : M \to N$ a $C^1$ conjugacy sending the leaves of $\Phi$ onto those of $\Psi$. Since $\Phi$ is the orbit foliation of the geodesic flow of a hyperbolic manifold, then $\Psi$ is orientable. Thus we can find a $C^\infty$ non-where vanishing vector field $Y$ tangent to $\Psi$, whose flow is $C^1$ orbit equivalent to $\phi_t$ under $h$. We denote by $\psi_t$ the flow of $Y$.

There exists a $C^1$ map $\alpha : M \times \mathbb{R} \to \mathbb{R}$ such that $\psi_t \circ h = h \circ \phi_{\alpha(t, \cdot)}$.

Since $h$ is $C^1$, then there exists $A > 0$ such that for any $u \in TM$, we have

$$\frac{1}{A} \| u \| \leq \| Dh(u) \| \leq A \| u \|.$$ 

Denote by $E^+$ and $E^-$ the strong unstable and stable distributions of $\phi_t$. Then for any $x \in M$ and any $(u^+ + u^-) \in (E^+ \oplus E^-)_x$,

$$\| D\psi_t(Dh(u^+ + u^-)) \| = \| Dh(D(\phi_{\alpha(t, \cdot)})(u^+ + u^-)) \|$$

$$\geq \frac{1}{A} \| D(\phi_{\alpha(t, x)})(u^+) + D(\phi_{\alpha(t, x)})(u^-) + b(t) \cdot X(\phi_{\alpha(t, x)}(x)) \|,$$

where $X$ denotes the generator of $\phi$. Since $\phi_t$ is Anosov, then it is easy to see that for any $v \in Dh(E^+ \oplus E^-)$, $\{\| D\psi_t(v) \|, \; t \in \mathbb{R}\}$ is unbounded. Thus $\psi_t$ is quasi-Anosov. In addition, we know by Lemma 4.1 that $\psi_t$ is volume-preserving. Thus we deduce from Theorem 4.1 that $\psi_t$ is a $C^\infty$ Anosov flow, which is $C^1$ orbit equivalent to $\phi_t$. Then we deduce from Theorem 1.6 that $\psi_t$ is $C^\infty$ orbit equivalent to $\phi_t$. Thus their orbit foliations $\Psi$ and $\Phi$ are $C^\infty$ conjugate.

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