The Non-negative Inverse Eigenvalue Problem for Tridiagonal Matrix, Circulant Matrix and Symmetric Matrix

Pengyu Zhu¹*, Yingjing Yong², Zhifeng Guo³ and Baicheng Ren⁴

¹School of Mathematics, Shandong University, Jinan, Shandong 250100, China
²School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, China
³Catholic Memorial High school, West Roxbury, MA, 02132, the United States
⁴Brisbane Boy’s college, Kensington Terrace, Toowong, Brisbane QLD 4066, Australia

*Corresponding author, Email: 201700090089@mail.sdu.edu.cn

Abstract. The Non-negative Inverse Eigenvalue Problem (NIEP) is a sub-problem extracted from inverse eigenvalue problem with a long history from 1930s determining the sufficient and necessary conditions in order that, 
\[ \sigma = \{\lambda_1, \ldots, \lambda_n\} \] to be the spectrum of an entry wise non-negative \( n \times n \) matrix. There had been many excellent researchers and scholars who contributed to discover many practical theories. The following of this paper would be then separated into two parts to further analyze the NIEP. In the first section of the paper, some important pre-existing conclusions would be demonstrated and the groups’ understanding of these indispensable theories would be expressed. In the second section, three special and wildly used matrix, including tridiagonal matrix, circulant matrix, and symmetric matrix would be considered. The solution of the NIEP of these matrices done by the group would be expressed.

1. Introduction

1.1. Application and history
The Non-negative Inverse Eigenvalue Problem (NIEP)[1,2] proposes the question that whether a spectrum composed with n complex numbers (repeating allowed) can be realized by an n-by-n matrix, where all of whose entries are non-negative real numbers.

As for its application, in cybernetics, vibration theory and structural design, it is often required to construct matrices according to given eigenvalues. That’s why it’s still an attractive problem, though it’s probably the hardest problem in the matrix analysis.

The NIEP has a long history since its proposal by Kolmogorov in the 1930s [3]. It is a sub-problem extracted from inverse eigenvalue problem of matrix, but compared with many other inverse eigenvalue problems, tools for NIEP are more limited. Nonetheless, these difficulties have been attacked by many wonderful researchers and there have been prior surveys of work on the sufficient and necessary conditions for NIEP.

1.2. Four necessary conditions
Loewy-London, and Johnson show us the initial four necessary conditions for the NIEP, including reality, trace, Perron root [4], and JLL condition [5,6], which plays an important role in the next chapter.
Study of necessary conditions of non-negative inverse eigenvalue problem, this so called inverse problem, is the study of conditions which eigenvalues of a non-negative matrix should satisfy, which is:

Assume $A$ is a $N \times N$ non-negative square matrix, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ (repeatable) is the eigenvalue of $A$, then according to fundamental theorem of algebra, $\lambda_1, \lambda_2, \ldots, \lambda_n$ is $n$ zeros $\in C$ of a $n^{th}$ polynomial $\text{Det}(A - \lambda I)$. Since complex zeros appear in pairs and have equal number of multiplicity, we can conclude the first condition of eigenvalue of a non-negative matrix:

$$\text{(reality)} \quad \sigma = \bar{\sigma} \quad \text{i.e.} \quad \{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{\overline{\lambda_1}, \overline{\lambda_2}, \ldots, \overline{\lambda_n}\} \quad (1)$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_n$ is $n$ zeros $\in C$ of a $n^{th}$ polynomial $\text{Det}(A - \lambda I)$, therefore:

$$\text{Det}(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

$$= (-1)^n \lambda^n + (-1)^{n-1} \sum_{i=1}^{n} \lambda_i \times \lambda^{n-1} + \cdots \quad (2)$$

Similarly, if we expand polynomial $\text{Det}(A - \lambda I)$ directly,

$$\text{Det}(A - \lambda I) = \begin{vmatrix}
    a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \\
\end{vmatrix}$$

$$= (-1)^n \lambda^n + (-1)^{n-1} \sum_{i=1}^{n} a_{ii} \times \lambda^{n-1} + \cdots \quad (3)$$

Compare the coefficient of correspondence, we have:

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i \quad (4)$$

$$\text{i.e.} \quad \text{Tr}(\sigma) = \text{Tr}(A) = \sum_{i=1}^{n} a_{ii} \geq 0 \quad (5)$$

Then, we talk about $A^k (k \in N)$. Recall that over $C$ we can always change bases so that we achieve diagonal representation. This is to say that for any $N \times N$ matrix $A$, we can find a transition matrix $X$ to get $B = X^{-1}AX$, which is a diagonal matrix and the new bases $(e'_1, e'_2, \ldots, e'_n) = (e_1, e_2, \ldots, e_n)X$.

Then, we have $A = XBX^{-1} \Rightarrow A^k = XB^kX^{-1} \Rightarrow B^k = \begin{pmatrix}
\lambda^k_1 \\
\lambda^k_2 \\
\vdots \\
\lambda^k_n \\
\end{pmatrix} = X^{-1}AX \quad \text{i.e.} \quad \lambda^k_1, \lambda^k_2, \ldots, \lambda^k_n$ is the spectrum of $A^k$, which is non-negative too.

$$(k\text{-th moment}) \quad s_k(\sigma) = \lambda^k_1 + \lambda^k_2 + \cdots + \lambda^k_n \geq 0, k \in N^+ \quad (6)$$

According to Perron-Frobenius theorem [4], the spectral radius of a non-negative matrix must, itself, be an eigenvalue. Therefore, there must be a non-negative one, at least as big in absolute value as any others. Without loss of generality, we let $\lambda^k_1$ be the non-negative one, then we have:

$$\text{(Perron)} \quad \lambda_1 \geq |\lambda_i|, i = 2, 3, \ldots, n \quad (7)$$

The last one is the famous JLL condition [5,6]:

$$\text{(JLL)} \quad (s_k(\sigma))^m \leq n^{m-1}s_{km}(\sigma), k, m = 1, 2, \ldots \quad (8)$$

Overall, we get the four necessary conditions:

Lemma (1.2.1)
The necessary conditions for whether a propose spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ can be realized by a non-negative matrix is that:

- $\sigma = \sigma^\top$
- $s_k(\sigma) = \lambda_k^1 + \lambda_k^2 + \cdots + \lambda_k^k \geq 0, k \in N^*$
- $\lambda_i \geq |\lambda_i^k|, i = 2, 3, \ldots, n$
- $(s_i(\sigma))^n \leq n^{n-1}s_{km}(\sigma), k, m = 1, 2, \ldots$

### 1.3. Low dimensional results

When $n \leq 5$, things may become easier and the results can be inspiring. So why not focus on the low dimensional situation first.

We find something interesting, when $n=2$.

**Theorem (1.3.1)**

If $\{\lambda_1, \lambda_2\}$ can be realized by a non-negative matrix, then $\lambda_1, \lambda_2$ must be real numbers.

**Proof:**

We assume that $\lambda_1, \lambda_2$ is a complex conjugate. i.e. $\lambda_1 = x + iy, \lambda_2 = x - iy (x, y \in R, y \neq 0)$

If $\{\lambda_1, \lambda_2\}$ can be realized by non-negative matrix $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, then $\lambda_1, \lambda_2$ are the zeros of characteristic polynomial

$$Det(\lambda I - A) = \begin{vmatrix} \lambda - a_1 & -a_2 \\ -a_3 & \lambda - a_4 \end{vmatrix} = \lambda^2 - (a_1 + a_4)\lambda + (a_1a_4 - a_2a_3)$$

(9)

According to Vieta’s formulas, we have:

$$\begin{cases} \lambda_1 + \lambda_2 = a_1 + a_4 = 2x \geq 0 \\ \lambda_1\lambda_2 = a_1a_4 - a_2a_3 = x^2 + y^2 \geq 0 \end{cases}$$

(10)

According to am-gm inequality, we have that

$$a_1 + a_4 = 2x \Rightarrow a_1a_4 \leq x^2$$

(11)

Meanwhile,

$$a_1a_4 - a_2a_3 = x^2 + y^2 \Rightarrow a_1a_4 = x^2 + y^2 + a_2a_3 \geq x^2$$

(12)

The only way to make both inequality (1.3.1) and (1.3.2) possible is

$$y^2 + a_2a_3 = 0$$

(13)

However, obviously, $y \neq 0 \Rightarrow y^2 > 0 \Rightarrow y^2 + a_2a_3 > 0$.

It doesn’t make sense. Therefore both $\lambda_1$ and $\lambda_2$ are real numbers.

**Theorem (1.3.2)**

For the real numbers $\lambda_1, \lambda_2$ (without loss of generality, we assume that $\lambda_1 \geq \lambda_2$), the sufficient and necessary conditions for whether $\sigma = \{\lambda_1, \lambda_2\}$ can be achieved is $\lambda_1 \geq |\lambda_2|$

**Proof:**

Sufficiency
• When \( \lambda_2 \geq 0 \), then \( A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \) is the non-negative matrix that we are looking for.

• When \( \lambda_2 < 0 \), we assume that \( A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \).

\[
a_1 = a_4 = \frac{\lambda_1 + \lambda_2}{2} \geq 0, a_2 = 1, a_3 = \frac{(\lambda_1 + \lambda_2)^2}{4} - \lambda_1 \lambda_2
\]

We let \( \frac{\lambda_1 + \lambda_2}{2} - \lambda_1 \lambda_2 \), then \( A = \begin{pmatrix} \frac{\lambda_1 + \lambda_2}{2} & 1 \\ \frac{(\lambda_1 + \lambda_2)^2}{4} - \lambda_1 \lambda_2 & \frac{\lambda_1 + \lambda_2}{2} \end{pmatrix} \) is the non-negative matrix we are looking for Necessity.

We assume that \( \lambda_1 < |\lambda_2| \), then we have \( \lambda_1 \geq 0 > \lambda_2 > -\lambda_1 \Rightarrow \lambda_1 + \lambda_2 = a_1 + a_4 < 0 \), but \( a_1, a_4 \geq 0 \). We launch the contradiction.

In fact, when \( n=2, \lambda_1 \geq |\lambda_2| \) is equivalent to \( \lambda_1 + \lambda_2 \geq 0 \). Both of them are necessary conditions as we mentioned in 1.2, but now, when \( n=2 \), they become necessary sufficient and necessary condition.

The above discussion is just an example, actually, researchers have drawn more advanced conclusions.

**Theorem (1.3.3)**

• When \( n \) is an odd number and \( Tr(\sigma) = 0 \), then JLL condition can be written as (LM) \( (s_2(\sigma))^3 \leq (n-1)s_4(\sigma) \) [7].

• When \( n \leq 3 \) or \( n = 4 \) and \( Tr(\sigma) = 0 \), then the four necessary conditions become sufficient and necessary conditions [6].

2. The inverse eigenvalue problem of three kinds of specific matrix

2.1. The non-negative tridiagonal matrix

Before we start, let’s define the tridiagonal matrix first.

2.1.1. Definition. The matrix \( T_n \) shaped like following, it is tridiagonal matrix.

\[
T_n = \begin{pmatrix}
  x_1 & y_1 & & \\
  z_1 & x_2 & y_2 & \\
  & z_2 & x_3 & y_3 & \cdots \\
  & & \ddots & \ddots & \ddots \\
  & & & z_{n-2} & x_{n-1} & y_{n-1} \\
  & & & & z_{n-1} & x_n
  \end{pmatrix}  \quad (14)
\]

Especially, if \( x_i \geq 0 \) \( (i = 1, 2, \cdots, n) \), \( y_i \geq 0 \), \( z_i \geq 0 \) \( (i = 1, 2, \cdots, n-1) \), then \( T_n \) is a non-negative tridiagonal matrix.

If \( x_i \geq 0 \) \( (i = 1, 2, \cdots, n) \), \( y_i = z_i \geq 0 \) \( (i = 1, 2, \cdots, n-1) \), then \( T_n \) is a symmetric non-negative tridiagonal matrix.
According to the definition, second order non-negative tridiagonal matrix is just a non-negative matrix without any other special properties. Therefore, we will talk about the NIEP from third order non-negative tridiagonal matrix.

We assume that

\[
\begin{pmatrix}
  a_1 & b_1 & 0 \\
  c_1 & a_2 & b_2 \\
  0 & c_2 & a_3
\end{pmatrix}
\]

is a non-negative tridiagonal matrix, (i.e. \(a_1, a_2, a_3, b_1, b_2, c_1, c_2 \geq 0\)), and \(\sigma = \{\lambda_1, \lambda_2, \lambda_3\}\) is the eigenvalue of A. Consider that characteristic polynomial of A:

\[
\text{Det}(\lambda I - A) = \begin{vmatrix}
  \lambda - a_1 & -b_1 & 0 \\
  -c_1 & \lambda - a_2 & -b_2 \\
  0 & -c_2 & \lambda - a_3
\end{vmatrix} = (\lambda - a_1)(\lambda - a_2)(\lambda - a_3) - b_2c_2(\lambda - a_1) - b_1c_1(\lambda - a_3)
\]

\[= \lambda^3 - (a_1 + a_2 + a_3)\lambda^2 + (a_1a_2 + a_1a_3 + a_2a_3 - b_1c_1 - b_2c_2) + (a_1b_2c_2 + a_2b_1c_1 - a_1a_2a_3)
\]

According to Vieta’s formulas,

\[
\begin{align*}
\lambda_1 + \lambda_2 + \lambda_3 &= a_1 + a_2 + a_3 \\
\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 &= a_1a_2 + a_1a_3 + a_2a_3 - b_1c_1 - b_2c_2 \\
\lambda_1\lambda_2\lambda_3 &= -a_1b_2c_2 - a_2b_1c_1 + a_1a_2a_3
\end{align*}
\]

For simplicity, let:

\[
\begin{align*}
d_1 &= \lambda_1 + \lambda_2 + \lambda_3 \\
d_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\
d_3 &= \lambda_1\lambda_2\lambda_3 \\
e_1 &= b_1c_1 \\
e_2 &= b_2c_2
\end{align*}
\]

Then, we have

\[
\begin{align*}
d_1 &= a_1 + a_2 + a_3 \\
d_2 &= a_1a_2 + a_1a_3 + a_2a_3 - e_1 - e_2 \\
d_3 &= -a_1e_2 - a_2e_1 + a_1a_2a_3
\end{align*}
\]

Firstly, we assume that \(\sigma = \{\lambda_1, \lambda_2, \lambda_3\}\) satisfy the four necessary conditions mentioned before. Then we can discuss with different conditions:

When \(\lambda_1, \lambda_2, \lambda_3\) are real numbers (assume \(\lambda_1 \geq \lambda_2 \geq \lambda_3\))

1) \(\lambda_2 < 0\)

Then we get:
According to (3), let’s talk about the value of $a_1, a_2, a_3$. When $a_1 = 0$, $d_3 = -a_3 e_1 \leq 0$, but we already knew that $d_3 > 0$. It doesn’t make sense. So that $a_1 > 0$. Similarly, we know that $a_2 > 0, a_3 > 0$. Then, we want to work out $e_1, e_2$, by simultaneous equations (2), (3).

i) $a_1 = a_3$

According to equation (2), we have $e_1 + e_2 = -d_2 + 2 a_1 a_2 + a_1^2 (4)$. According to equation (3), we have $e_1 + e_2 = \frac{a_1^3 a_2 - a_1 d_2 + d_3}{a_1} (5)$.

Let (4)-(5), we have $\frac{a_1^3 + a_1^2 a_2 - a_1 d_2 + d_3}{a_1} = 0$

However, we already know that

$$a_1^3 > 0, a_1^2 a_2 > 0, -a_1 d_2 > 0, d_3 > 0 \Rightarrow \frac{a_1^3 + a_1^2 a_2 - a_1 d_2 + d_3}{a_1} > 0$$

(20)

It’s impossible.

$a_1 \neq a_3$

Due to simultaneous equations (2), (3), we have that

$$\begin{cases} e_1 = \frac{a_1 d_2 - d_3 - a_1^2 (a_2 + a_3)}{a_1 - a_1} \\ e_2 = \frac{a_2^2 (a_1 + a_2) + d_3 - a_2 d_2}{a_1 - a_1} \end{cases}$$

(21)

Focusing on the numerator, we find that in equation(6)

$$a_1 d_2 < 0, -d_3 < 0, -a_1^2 (a_2 + a_3) < 0 \Rightarrow a_1 d_2 - d_3 - a_1^2 (a_2 + a_3) < 0$$

(22)

In equation (7) $a_1^2 (a_1 + a_2) > 0, d_2 > 0, -a_1 d_2 > 0 \Rightarrow a_1^2 (a_1 + a_2) + d_3 - a_3 d_2 > 0$. When $a_3 > a_1$, then $e_1 < 0$, but $e_2 > 0$. When $a_1 > a_3$, then $e_1 > 0$, but $e_2 < 0$. However, we need both $e_1 \geq 0$ and $e_2 \geq 0$. As mentioned above, when $\lambda_2 < 0$, $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ can’t be achieved by non-negative tridiagonal matrix.

2) $\lambda_2 \geq 0, \lambda_3 < 0$

Then, we have $\lambda_1 > |\lambda_3| > 0$, $d_1 \geq 0, d_3 < 0$, sign of $d_2$ cannot be determined. Following the same way as situation 1), we try to work out $e_1, e_2$, immediately.

i) $a_1 \neq a_3$

Then, we have
\[
\begin{cases}
    e_1 = \frac{a_3 d_2 - d_1 - a_1^2 (a_2 + a_3)}{a_3 - a_1} \quad (6) \\
    e_2 = \frac{a_1^2 (a_1 + a_2) + d_3 - a_2 d_2}{a_3 - a_1} \quad (7)
\end{cases}
\]

(23)

However, since the sign of \(d_2\) is unknown, things are different from situation 1), it is hard for us to determine the sign of numerator and we can’t launch contradiction anymore. So we try to find a solution of equations (6), (7). Having failed over and over again, we finally find a proper solution.

We let \(e_2 = 0\), i.e. \(d_3 + a_2^2 (a_1 + a_2) = a_3 d_2\)

\[
\lambda_1 \lambda_2 \lambda_3 + a_3^2 (a_1 + a_2) = a_3 (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) \quad (8)
\]

(24)

According to equations\((1),(8)\), we find a solution for them

\[
\begin{cases}
    a_1 + a_2 = \lambda_1 + \lambda_3 \\
    a_3 = \lambda_2
\end{cases}
\]

(25)

Here, we let \(a_1 = a_2 = \frac{\lambda_1 + \lambda_3}{2}\), \(a_3 = \lambda_2\), then we have \(e_2 = 0\). Meanwhile,

\[
e_1 = \frac{a_3 d_2 - d_1 - a_1^2 (a_2 + a_3)}{a_3 - a_1}
\]

\[
= \frac{2\lambda_1^2 \lambda_2 + 2\lambda_2 \lambda_3^2 + 2\lambda_3 \lambda_1^2 + \lambda_1 \lambda_2 - \lambda_1 - \lambda_2 - 4\lambda_1 \lambda_2 \lambda_3}{4(2\lambda_2 - \lambda_1 - \lambda_3)}
\]

\[
= \frac{(2\lambda_2 - \lambda_1 - \lambda_3)(\lambda_1 - \lambda_3)^2}{4(2\lambda_2 - \lambda_1 - \lambda_3)}
\]

\[
= \frac{(\lambda_1 - \lambda_3)^2}{4}
\]

(26)

Here, we let \(b_1 = c_1 = \frac{\lambda_1 - \lambda_3}{2} > 0\), \(A = \begin{pmatrix}
\frac{\lambda_1 + \lambda_3}{2} & \frac{\lambda_1 - \lambda_3}{2} & 0 \\
\frac{\lambda_1 - \lambda_3}{2} & \frac{\lambda_1 + \lambda_3}{2} & 0 \\
0 & 0 & \lambda_2
\end{pmatrix}\) is such non-negative tridiagonal matrix whose eigenvalue is \(\sigma = \{\lambda_1, \lambda_2, \lambda_3\}\). It can be inferred that the actual values of \(b_1, c_1, b_2, c_2\) are not critical. In fact, it is the values of \(e_1 = b_1 c_1\) and \(e_2 = b_2 c_2\) that matters. When the values of \(e_1\) and \(e_2\) are determined, any values of \(b_1, c_1, b_2, c_2\) satisfy equations \(b_1 c_1 = e_1\) and \(b_2 c_2 = e_2\) are appropriate.

Then we have:

**Deduction (2.1.2).** Whether a proposed spectrum \(\sigma = \{\lambda_1, \lambda_2, \lambda_3\}\) can be realized by a non-negative tridiagonal matrix is equivalent to whether they can be realized by a symmetric non-negative tridiagonal matrix. More specifically, \(\sigma = \{\lambda_1, \lambda_2, \lambda_3\}\) can be realized by a non-negative tridiagonal matrix \(\Leftrightarrow\) they can be realized by a symmetric non-negative tridiagonal matrix \(\sigma = \{\lambda_1, \lambda_2, \lambda_3\}\) cannot be realized by a
non-negative tridiagonal matrix $\iff$ they cannot be realized by a symmetric non-negative tridiagonal matrix.

3) $\lambda_3 \geq 0$

It’s a piece of cake. Obviously, $\lambda_1$, $\lambda_2$, $\lambda_3$ are all equal to or greater than zero. So,

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

is such non-negative tridiagonal matrix whose eigenvalue is $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$.

We have discussed all situations, then we can conclude that Theorem (2.1.3).

For a proposed spectrum $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ (assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3$), the sufficient and necessary condition under which they can be realized by a (symmetric) non-negative tridiagonal matrix is

(a) $\lambda_i \geq |\lambda_i|$, ($i = 2,3$)

(b) $\lambda_1 + \lambda_2 + \lambda_3 \geq 0$

(c) $\lambda_2 \geq 0$

Obviously, it is not complete to discuss only the real numbers. What if the spectrum is composed with a real number and a complex conjugate?

We assume that $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ is composed with a real number $\lambda_1$, and a complex conjugate $\lambda_2 = x + iy$, $\lambda_3 = x - iy$ ($y > 0$). Then,

- $\lambda_1 \geq \lambda_1 = \sqrt{x^2 + y^2} > 0$
- $d_1 = \lambda_1 + \lambda_2 + \lambda_3 \geq 0$
- $d_2 = \lambda_1 \lambda_2 \lambda_3 = \lambda_1 (x^2 + y^2) > 0$
- $d_3 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = 2 \lambda_1 x + (x^2 + y^2)$

According to the JLL condition, we have

$$S_1 \leq 3S_2$$

$$\Rightarrow d_2 \leq \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

(27)

Unfortunately, we cannot determine the sign of $d_2$. Firstly, we pay attention to $a_1, a_2, a_3$. If $a_1 = 0$, due to (3), we have $d_3 = -a_1a_2 \leq 0$, but $d_3 > 0$. It’s impossible. So, we have $a_1 > 0$. In a similar way, we can conclude that $a_2 > 0, a_3 > 0$. So, we know that $d_1 = a_1 + a_2 + a_3 > 0$. If $d_2 \leq 0$, like situation 1), we know that it cannot be achieved by a non-negative tridiagonal matrix.

However, if $d_2 > 0$, we can neither find a proper solution nor launch contradiction. So, the conclusion is:

Theorem (2.1.4)

For a given spectrum composed of a real number and a complex conjugate $\sigma = \{\lambda_1, \lambda_2, \lambda_3\} (\lambda_1 \in R, \lambda_2 = x + iy, \lambda_3 = x - iy, y > 0)$, the necessary condition for it to be achieved by a (symmetric) non-negative tridiagonal matrix is

- $\lambda_1 \geq \sqrt{x^2 + y^2}$
- $\lambda_1 + \lambda_2 + \lambda_3 > 0$
- $2 \lambda_1 x + x^2 + y^2 > 0$

At this stage, we want to conclude about the methods in the above discussion. The method of using vieta’s formula to represent the elements in the non-negative tridiagonal matrix and then determine the sign of the elements to find contradiction when researching the necessary conditions of the NIEP is quiet
convenient and effective. However, when researching the sufficient conditions for the NIEP, it requires an appropriate solution which needs large amount of calculation and aimless attempts.

2.2. Circulant matrix

Before we start, we have to define the circulant matrix.

Definition (2.2.1) The matrix $C$ shaped like $C$.

$$C = \begin{pmatrix}
\varepsilon_0 & \varepsilon_1 & \cdots & \varepsilon_{n-1} \\
\varepsilon_{n-1} & \varepsilon_0 & \cdots & \varepsilon_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_0
\end{pmatrix}_{n \times n}$$ (28)

It is circulant matrix.

**Step one:**

We refer to the simplest form of this kind of matrices, which we name it $K_1$.

$$K_1 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}_{n \times n} = \begin{pmatrix}
0 & I_{n-1} \\
1 & 0^n
\end{pmatrix}_{n \times n}$$ (29)

With $K$, we can construct any circulant matrix with addition and multiplication. For example,

$$C = \begin{pmatrix}
\varepsilon_0 & \varepsilon_1 & \cdots & \varepsilon_{n-1} \\
\varepsilon_{n-1} & \varepsilon_0 & \cdots & \varepsilon_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_0
\end{pmatrix}_{n \times n}$$

$C = \varepsilon_0 \cdot \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}_{n \times n} + \varepsilon_1 \cdot \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{pmatrix}_{n \times n} + \cdots + \varepsilon_{n-1} \cdot \begin{pmatrix}
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}_{n \times n}$, where

$$K_1^0 = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}_{n \times n}$$, $K_1 = \begin{pmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{pmatrix}_{n \times n}$, $K_1^2 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \cdots & 0
\end{pmatrix}_{n \times n}$, $\ldots$

$K_1^{-n} = \begin{pmatrix}
0 & 0 & \cdots & 1 \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}_{n \times n}$

All the matrices in this form are circulant matrix, and for convenience, we name $K_1^2$ as $K_2$, $K_1^3$ as $K_3$, $\ldots$, $K_1^{-n}$ as $K_{-n}$, $K_1^0$ as $K_0$. It’s easy to get the equation below:
\[ C = \varepsilon_0 \cdot K_0 + \varepsilon_1 \cdot K_1 + \cdots + \varepsilon_{n-1} \cdot K_{n-1} = \sum_{j=0}^{n-1} \varepsilon_j \cdot K_j. \] So, let’s abbreviate this equation as

\[ C = p_\varepsilon(K). \]

**Step two:**

For any unit cycle matrix \( K \), which is also the simplest form of circulant matrices, we can calculate its eigenvalue through this equation

\[ \text{Det}(\lambda I - C) = \lambda^n - 1, \]

where \( \lambda \) refers to the eigenvalue of matrix \( K \). It’s easy to get that

\[ \lambda_1 = 1, \quad \lambda_2 = \omega, \quad \lambda_3 = \omega^2, \ldots, \quad \lambda_n = \omega^{n-1}, \]

where \( \omega = e^{-\frac{2\pi i}{n}} \), by calculating this equation

\[ |\lambda I - K| = \lambda^n - 1. \] Thus we have a new matrix.

\[
F = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2}
\end{pmatrix}
\]

(30)

It is the transformation matrix of \( K \). In other words, \( F^{-1}KF = \begin{pmatrix} \lambda_1 & \ddots & \\
\vdots & \ddots & \\
\lambda_n & & \end{pmatrix} \)

\[ K = F \begin{pmatrix} \lambda_1 & \ddots & \\
\vdots & \ddots & \\
\lambda_n & & \end{pmatrix} F^{-1}. \]

**Theorem (2.2.2)**

The sufficient and necessary condition, under which \( C(\varepsilon) \) is n-dimensional circulant matrix, is that

(a) \( C = C(\varepsilon) = FP_{\varepsilon}(\Omega)F^{-1} \)

(b) \( \Omega = \text{diag}\{1, \omega, \omega^2, \cdots, \omega^{n-1}\} \)

**Step three:**

We can just solve the inverse eigenvalue problem of circulant matrix by discussing the first line or row of it, thus we can get every element of circulant matrix. Specifically, we assume that the matrix we need is \( C(\varepsilon) \), then all we need is \( e_1C(\varepsilon) \) or \( e_1^TC(\varepsilon) \), where \( e_1 = (1,0,0,\cdots,0) \). Now, we already know \( \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n) \) and looking for \( C(\varepsilon) \) which eigenvalue of it is \( \lambda_1, \cdots, \lambda_n \). We have this equation based on properties of matrix eigenvalue

\[ C(\varepsilon) = FA F^{-1}, \]

then,

\[ e_1 C(\varepsilon) = e_1 FAF^{-1} = (1, \cdots, 1)_x \Lambda F^{-1} = (\lambda_1, \cdots, \lambda_n)F^{-1}. \]

Since we have gotten the elements of the first line of \( C(\varepsilon) \), we can structure a circulant matrix based on it.
2.3. Symmetric matrix

We already know \( \Lambda = \begin{pmatrix} \lambda_1 & \cdots & \cdots \lambda_n \end{pmatrix} \) and look for a symmetric matrix \( A \) whose eigenvalue is \( \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \). The transformation matrix of \( A \) is \( P \). Based on properties of eigenvalue, we get an equation \( P^{-1}AP = \begin{pmatrix} \lambda_1 & \cdots & \cdots \lambda_n \end{pmatrix} \Rightarrow A = P \begin{pmatrix} \lambda_1 & \cdots & \cdots \lambda_n \end{pmatrix} P^{-1}. \)

Here, we introduce a property of symmetric matrices. Noticing that \( \begin{pmatrix} \lambda_1 & \cdots & \cdots \lambda_n \end{pmatrix} \) is not only a diagonal matrix but also a symmetric matrix as well, it applies to this property.

**Lemma (2.3.1)**

If \( X \) is a symmetric matrix, then for any matrix \( B \), \( BXB^T \) is also a symmetric matrix. In this case, for any matrix \( B \), \( B^T \Lambda B \) is a symmetric matrix. Based on these, in order to find a symmetric matrix \( A \) through eigenvalue, \( P^T \) must be equal to \( P^{-1} \), so that \( P^T \Lambda P = P \Lambda P^T = A \).

**Definition (2.3.2)**

If a matrix \( P \) satisfies \( P^T = P^{-1} \), we call it orthogonal matrix. So the method is, we have already know \( \Lambda = diag(\lambda_1, \cdots, \lambda_n) \), create a random orthogonal matrix \( P \), and calculate \( P^T \Lambda P \), then we get a symmetric matrix \( A \) whose eigenvalue is \( \{\lambda_1, \cdots, \lambda_n\} \).

3. Conclusion

Although the non-negative inverse eigenvalue problem (NIEP) is a long-standing problem and corresponding researching tools are limited, in our research, you will find that when the matrix is special matrix, such as tridiagonal matrix, circulant matrix and symmetric matrix, and the dimension is limited, this long-standing puzzle is entirely solvable.

3.1. The non-negative inverse eigenvalue problem for tridiagonal matrix

In this paper, 3-dimension non-negative tridiagonal matrix is considered. According to the Vieta’s formulas, we build a bridge between the eigenvalue and the elements of the matrix. While finding a proper solution, we separate the eigenvalue into different groups, based on the different construction of spectrum. On the one hand, in some situations, we find the corresponding matrix, which means this kind of eigenvalue can be achieved by non-negative tridiagonal matrix, on the other hand, sometimes we get a paradox, which means this kind of eigenvalue isn’t realizable. However, unfortunately, when the eigenvalue is composed by a real number and a pair of conjugate complex numbers, we can neither find a suitable solution nor derive a contradiction.

3.2. The inverse eigenvalue problem for cyclic matrix

For circular matrix, in this paper, we start with the most basic unit cyclic matrix. We solve the inverse eigenvalue problem of unit cyclic matrix at first, then simplifying the general cyclic matrix inverse eigenvalue problem base on that general cyclic matrix can be written as a combination of unit matrix. In this way we find the transformation matrix \( F \) and get a solution for general cyclic matrix inverse eigenvalue problem.
When it comes to the inverse eigenvalue problem of cyclic matrix, the uniqueness of the obtained cyclic matrix remains to be studied. For example, under what circumstances the obtained cyclic matrix \( C \) is unique?

3.3. The inverse eigenvalue problem for symmetric matrix

In this paper, by using the special characters of symmetric matrices, the conditions that transformation matrices should satisfy are obtained. Thus a general solution to the inverse eigenvalue problem of symmetric matrices is derived.

3.4. Expectation and application

Furthermore, based on our research, it’s possible to establish an R package or set up a function in MATLAB. When entering a set of eigenvalues and the desired matrix category (like tridiagonal matrix, circulant matrix, and symmetric), the corresponding matrix can be output. However, when the input eigenvalues do not meet the corresponding necessary conditions, it shows not found. In short, our theory can be realized by computer language, which is convenient for the following researchers to verify and calculate.

References

[1] Egleston P D, Lenker T D, Narayan S K (2004) The nonnegative inverse eigenvalue problem. Linear Algebra and its Application, PP 475-490.
[2] Minc H (1988) Nonnegative Matrices. John Wiley and Sons, New York.
[3] Kolmogorov A N (1937) Markov chains with a countable number of possible states. Bull. Moskow Gosu-darstvennogo Univ. Mat. Meh., PP 1-16.
[4] Horn R A and Johnson C R (1990) Matrix Analysis. Cambridge University Press, Cambridge.
[5] Johnson C R (1981) Row stochastic matrices similar to doubly stochastic matrices. Linear Multilinear Algebra, PP 113-130.
[6] Loewy R and London D (1978) A note on the inverse problem for nonnegative matrices. Linear and Multilinear Algebra PP 83-90.
[7] Laffey T J and Meehan E (1998) A refinement of an inequality of Johnson, Loewy and London on nonnegative matrices and some applications. Electron. J. Linear Algebra, PP 119-128.