Chiral Magnetic Effect and Anomalous Hall Effect in Antiferromagnetic Insulators with Spin-Orbit Coupling

Akihiko Sekine* and Kentaro Nomura
Institute for Materials Research, Tohoku University, Sendai 980-8577, Japan
(Dated: January 29, 2016)

We search for dynamical magnetoelectric phenomena in three-dimensional correlated systems with spin-orbit coupling. We focus on the antiferromagnetic insulator phases where the dynamical axion field is realized by the fluctuation of the antiferromagnetic order parameter. It is shown that the dynamical chiral magnetic effect, an alternating current generation by magnetic fields, emerges due to the time dependences of the order parameter such as antiferromagnetic resonance. It is also shown that the anomalous Hall effect arises due to the spatial variations of the order parameter such as antiferromagnetic domain walls. Our study indicates that spin excitations in antiferromagnetic insulators with spin-orbit coupling can result in nontrivial charge responses. Moreover, observing the chiral magnetic effect and anomalous Hall effect in our system is equivalent to detecting the dynamical axion field in condensed matter.

PACS numbers: 75.47.-m, 73.43.-f, 75.70.Tj, 71.27.+a

Introduction.— Antiferromagnets (AFMs) have attracted much attention from the viewpoints of both purely scientific and applied research. Theoretically, it is known that antiferromagnetic (AF) phases are often favored in systems with strong on-site repulsive interactions. In the vicinity of AF phases, the emergence of exotic phases and phenomena such as high-temperature superconductivity and spin liquids is well acknowledged [1–3]. On the other hand, AFMs have recently been studied intensively in the field of spintronics [4–7], as a possible new class of materials alternative to ferromagnets. These studies suggest that the staggered magnetization can play essential roles, although AFMs had not been considered suitable for practical use due to the lack of net magnetization unlike ferromagnets.

Recent intensive and extensive studies have revealed the importance of spin-orbit coupling (SOC) in condensed matter. Especially, the discovery that strong SOC is essential to realize topologically nontrivial phases opened a new direction in modern physics [8–10]. Since topological invariants are determined from electronic band structures, studies of topologically correlated systems, for example, the axionic polariton, a total polarization of electronic waves, has been studied in frustrated or noncollinear AFMs [11–13]. The CME was originally proposed in gapless Dirac fermion systems [17], and its possibility has been discussed in Weyl semimetals [18–23]. Recent intensive and extensive studies have revealed the occurrence of the anomalous Hall effect (AHE) in the AFI phase. It is known that the AHE occurs usually in ferromagnetic metals [24], while the AHE arising from nontrivial spin textures has been studied in frustrated or noncollinear AFMs [25, 26]. We propose that spatial variations of the staggered magnetization lead to the AHE.

Realization of the Dynamical Axion Field and its Consequences.— Let us consider 3D electron systems having both on-site interactions and SOC, such as 5d transition metal oxides [12, 13, 27]. We focus on systems that become magnetically ordered Mott insulators when on-site interactions are strong, while they are topological band insulators when on-site interactions are weak. Once a magnetic order is formed, the mean-field approximation of the interaction term can capture the essential physics of the system. In this work, we particularly consider AFIs whose mean-field lattice Hamiltonian is given by $H(k) = \epsilon_0(k)1 + \sum_{\mu=1}^{5} R_{\mu}(k)\sigma_\mu$, Here $k = (k_1, k_2, k_3)$ is a wave vector in the Brillouin zone, $\epsilon_0$ is the 4 × 4 identity matrix, and the 4 × 4 matrices $\sigma_\mu$ satisfy the Clifford algebra $[\sigma_\mu, \sigma_\nu] = 2\delta_{\mu\nu}$ with $\sigma_5 = \sigma_1\sigma_2\sigma_3\sigma_4$. The Hamiltonian of this form can be realized, for example, in the AFI phases of Bi$_2$Se$_3$ family doped with magnetic impurities such as Fe [16] and transition metal oxides with corundum structure such as $\alpha$-Fe$_2$O$_3$ [28]. In this case, we can derive 3D massive Dirac Hamiltonians of the form

$$\mathcal{H}_{\text{eff}}(q) = q_1\sigma_1 + q_2\sigma_2 + q_5\sigma_3 + M_0\sigma_4 + M_5\sigma_5$$

arXiv:1508.04590v2 [cond-mat.str-el] 28 Jan 2016
tries induced by mean-field AF order parameter. We require that the system is a topological insulator when $M_0 > 0$ and $M_{1f} = 0$.

In what follows, we consider consequences arising from the existence of the $M_{1f}\alpha_5$ mass term. The effective action of the system in the presence of an external electromagnetic potential $A_\mu$ is written as

$$S_{ef} = \int dt d^3r \left[ i\bar{\psi}_f (r,t) \left( D_{\mu} \psi_f - M_f e^{i\theta(t)} \right) \right] \psi_f (r,t), \tag{2}$$

where $t$ is real time, $\psi_f (r,t)$ is a four-component spinor, $\bar{\psi}_f = \psi_f^\dagger y_0$, $D_{\mu} = \partial_{\mu} + i e A_\mu$, $M_f = \sqrt{(M_0)^2 + (M_{1f}^2)^2}$, $\cos \theta_f = M_0/M_f$, $\sin \theta_f = -M_{1f}/M_f$, and we have used the fact that $\alpha_4 = \gamma^0$, $\alpha_5 = -i\gamma^0\gamma^3$ and $\alpha_j = \gamma^0\gamma^j$ ($j = 1, 2, 3$). By applying the Fujikawa’s method [29] to the action (2), the theta term is obtained as [30]

$$S_\theta = \int dt d^3r \frac{e^2}{2\hbar} \theta \mathbf{E} \cdot \mathbf{B}, \tag{3}$$

where $\theta = \frac{\pi}{2} [1 + \text{sgn}(M_0)] - \sum_i \tan^{-1}(M_{1f}/M_0)_i$, and $\mathbf{E}$ ($\mathbf{B}$) is an external electric (magnetic) field. From this action, we obtain the magnetoelectric responses expressed by $P = \theta \epsilon^2/(2\hbar)\mathbf{B}$ and $M = \theta \epsilon^2/(2\hbar)\mathbf{E}$, with $\mathbf{P}$ the electric polarization and $\mathbf{M}$ the magnetization. In 3D time-reversal invariant topological (normal) insulators, $\theta = \pi$ ($\theta = 0$) [36]. However, the value of $\theta$ can be arbitrary when time-reversal and parity symmetries of the system are broken [37–39]. Furthermore, when the value of $\theta$ depends on space and time, it can be said that the dynamical axion field is realized in condensed matter [16]. Some consequences of the realization have been studied so far [16, 14, 40].

Notice that, when the dynamical axion field is realized, the theta term can be rewritten in the Chern-Simons form as

$$S_\theta = -\int dt d^3r \frac{e^2}{4\pi\hbar} \epsilon^{\mu\nu\rho\lambda} [\partial_\mu \theta(r,t) A_\nu \partial_\rho A_\lambda]. \tag{4}$$

Then the induced four-current density $j^\mu$ can be obtained from the variation of the above action with respect to the four-potential $A_\mu$: $j^\mu = \frac{\delta S}{\delta A_\mu} = -\frac{e^2}{2\pi\hbar} [\partial_\mu \theta(r,t)] \epsilon^{\mu\nu\rho\lambda} \partial_\lambda A_\nu$. The induced current density is given by [41, 42]

$$j(r,t) = \frac{e^2}{2\pi\hbar} \left[ \partial \theta(r,t) \mathbf{B} + \nabla \theta(r,t) \times \mathbf{E} \right], \tag{5}$$

where $\partial \theta = \partial \theta(r,t)/\partial t$. The magnetic-field-induced term is the CME [17]. The electric-field-induced term is the AHE, since it is perpendicular to the electric field. The induced current of the form (5) has been also studied in Weyl semimetals [18–21], where the chemical potential difference between the band touching points and the separation of the points in momentum space are required for the CME and AHE, respectively. However, the existence of the CME in Weyl semimetals is still being discussed theoretically [18–23]. Note that the situation we consider in this paper is completely different, since the system is gapped, i.e., the above conditions required in the case of Weyl semimetals are not needed.

**Theoretical Model.**— To study the induced current (5) more concretely, let us consider a 3D lattice model with SOC and electron correlations. The model we adopt is the Fu-Kane-Mele-Hubbard model on a diamond lattice at half-filling, whose Hamiltonian is given by [43–45]

$$H = \sum_{i,j} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + \frac{4A}{a^2} \sum_{\langle i,j \rangle} c_{i\sigma}^\dagger \mathbf{\sigma} \cdot (\mathbf{d}_{ij} \times \mathbf{d}_{ij}) c_{j\sigma}\notag + U \sum_{n} n \tilde{n},$$

where $c_{i\sigma}^\dagger$ is an electron creation operator at a site $i$ with spin $\sigma = \uparrow, \downarrow$, $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$, and $a$ is the lattice constant of the fcc lattice. The first through third terms represent the nearest-neighbor hopping, the next-nearest-neighbor SOC, and the on-site electron-electron interaction, respectively. $\mathbf{d}_{ij}$ and $\mathbf{d}_{ij}$ are the two vectors which connect two sites $i$ and $j$ on the same sublattice. Namely they are given by two of the four nearest-neighbor bond vectors. $\mathbf{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices for the spin degrees of freedom. We introduce a lattice distortion such that $t_{ij} = t + \delta t_1$ for the [111] direction and $t_{ij} = t$ for the other three directions, which induces a bandgap of $2|M_0| (M_0 \equiv \delta t_1)$ in the noninteracting spectrum.

We perform the mean-field approximation to the interaction term as $H_U = U \sum_{n} n \tilde{n} \equiv U \sum_{n} \left( n \tilde{n} + (n_i) \Delta n_i - (n_i) (\Delta n_i) - \langle c_{i\uparrow}^\dagger c_{i\downarrow}\rangle \langle c_{i\downarrow}^\dagger c_{i\uparrow}\rangle - \langle c_{i\uparrow}^\dagger c_{i\uparrow}\rangle \langle c_{i\downarrow}^\dagger c_{i\downarrow}\rangle \right)$. SOC breaks spin $\text{SU}(2)$ symmetry and the orientations of the spins are coupled to the lattice structure. Hence we should parametrize the AF ordering between the two sublattices $A$ and $B$ in terms of the spherical coordinate ($\theta, \varphi$):

$$\langle S_{\theta\varphi} \rangle = \langle \mathbb{S}_{\theta\varphi} \rangle \equiv n_1 e_x + n_2 e_y + n_3 e_z \equiv (n),$$

where $\langle S_{\theta\varphi} \rangle = \frac{1}{2} \langle \psi_{\theta\varphi}^\dagger \sigma_{\alpha\beta} \psi_{\theta\varphi} \rangle (\mu = A, B)$ with $i'$ denoting the $i'$-th unit cell. In the following we consider the ground state given by $(n_0, \theta_0, \varphi_0)$. The low-energy effective Hamiltonian of the AFI phase is written in the form (1): $H_{\text{eff}} = \sum_{q} \sum_{j=1,2,3} \psi_{j}^\dagger H_j(q) \psi_{jq}$, where $\psi_{jq}$ is a four-component spinor [45]. Therefore the value of $\theta$ in the AFI phase of the Fu-Kane-Mele-Hubbard model is given by [45]

$$\theta = \pi \left[ 1 + \text{sgn}(M_0) \right] - \sum_{j=1,2,3} \tan^{-1} \left( U n_j / M_0 \right). \tag{8}$$

From this equation, we see that the dynamical axion field is realized by the fluctuation of the AF order parameter $n_j$, i.e., by the spin excitations [16, 28].

**Dynamical Chiral Magnetic Effect.**— First we focus on the magnetic-field-induced term in Eq. (5), i.e., the CME in the AFI phase:

$$j_{\text{CME}}(r,t) = -\frac{e^2}{2\pi\hbar} \sum_{M_0} \frac{UM_0}{M_0^2 + (U n_j)^2} \hat{n}_j(r,t) \mathbf{B}. \tag{9}$$
Here we have neglected the di

\[ \tan \theta_A = \frac{\delta m_{B\perp}}{\delta m_{A\perp}} \quad \text{and} \quad \tan \theta_B = \frac{\delta m_{A\perp}}{\delta m_{B\perp}} \ \text{at} \ n_f \approx 0.1 \ [50, \ 51, \ 52], \]  

where $D_1 = \sqrt{p^2 + q^2}$ and $\alpha = q/p$ with $p = (\cos \phi_0 + \sin \phi_0) \cos \theta_0 - \sin \theta_0$ and $q = \sin \phi_0 - \cos \phi_0$. Equation (12) means that an alternating current is induced by the AF resonance. Schematic figure of a possible experimental setup to observe the CME in our system is shown in Fig. 1(b). \( \delta m_\perp \) is a function of the external microwave frequency $\omega$ with Lorentzian structure, i.e., \( \delta m_\perp (\omega) \approx a/(\omega^2 + \omega_0^2) \) with $a$ being a constant. Therefore two peaks will appear in the intensity $|\langle \langle j_\text{CME} \rangle \rangle|$.

Here we let us estimate the maximum value of the CME (12): 

\[ |\langle j_\text{CME,max} \rangle| = \frac{e^2}{2\pi h} B_0 \sum_{\omega \approx \omega_0} \omega_1 \delta m_{\perp} \sin \omega_1 t + \alpha. \]  

From the relation such that $n = n_1 e_x + n_2 e_y + n_3 e_z$, we have $n_1 = n_3 \sin \theta_0 \cos \phi_0 + n_2 \cos \theta_0 \sin \phi_0$, $n_2 = n_3 \sin \theta_0 \sin \phi_0 + n_1 \cos \theta_0 \sin \phi_0$, and $n_3 = n_0 \cos \theta_0 - n_2 \cos \theta_0 \sin \phi_0$. Substituting these quantities into Eq. (9), we obtain the analytical expression for $j_\text{CME}(t)$. Especially, in the vicinity of the phase boundary where $Un_f/M_0 \approx 1$ [49], Eq. (9) is simplified as 

\[ j_\text{CME}(t) = \frac{e^2}{2\pi h} B_0 \sum_{\omega \approx \omega_0} \omega_1 \delta m_{\perp} \sin \omega_1 t + \alpha. \]  

where $D_1 = \sqrt{p^2 + q^2}$ and $\alpha = q/p$ with $p = (\cos \phi_0 + \sin \phi_0) \cos \theta_0 - \sin \theta_0$ and $q = \sin \phi_0 - \cos \phi_0$. Equation (12) means that an alternating current is induced by the AF resonance. Schematic figure of a possible experimental setup to observe the CME in our system is shown in Fig. 1(b). \( \delta m_\perp \) is a function of the external microwave frequency $\omega$ with Lorentzian structure, i.e., \( \delta m_\perp (\omega) \approx a/(\omega^2 + \omega_0^2) \) with $a$ being a constant. Therefore two peaks will appear in the intensity $|\langle \langle j_\text{CME} \rangle \rangle|$.

Here we let us estimate the maximum value of the CME (12): 

\[ |\langle j_\text{CME,max} \rangle| = \frac{e^2}{2\pi h} B_0 \sum_{\omega \approx \omega_0} \omega_1 \delta m_{\perp} \sin \omega_1 t + \alpha. \]  

where $D_1 = \sqrt{p^2 + q^2}$ and $\alpha = q/p$ with $p = (\cos \phi_0 + \sin \phi_0) \cos \theta_0 - \sin \theta_0$ and $q = \sin \phi_0 - \cos \phi_0$. Equation (12) means that an alternating current is induced by the AF resonance. Schematic figure of a possible experimental setup to observe the CME in our system is shown in Fig. 1(b). \( \delta m_\perp \) is a function of the external microwave frequency $\omega$ with Lorentzian structure, i.e., \( \delta m_\perp (\omega) \approx a/(\omega^2 + \omega_0^2) \) with $a$ being a constant. Therefore two peaks will appear in the intensity $|\langle \langle j_\text{CME} \rangle \rangle|$.

Here we let us estimate the maximum value of the CME (12): 

\[ |\langle j_\text{CME,max} \rangle| = \frac{e^2}{2\pi h} B_0 \sum_{\omega \approx \omega_0} \omega_1 \delta m_{\perp} \sin \omega_1 t + \alpha. \]  

where $D_1 = \sqrt{p^2 + q^2}$ and $\alpha = q/p$ with $p = (\cos \phi_0 + \sin \phi_0) \cos \theta_0 - \sin \theta_0$ and $q = \sin \phi_0 - \cos \phi_0$. Equation (12) means that an alternating current is induced by the AF resonance. Schematic figure of a possible experimental setup to observe the CME in our system is shown in Fig. 1(b). \( \delta m_\perp \) is a function of the external microwave frequency $\omega$ with Lorentzian structure, i.e., \( \delta m_\perp (\omega) \approx a/(\omega^2 + \omega_0^2) \) with $a$ being a constant. Therefore two peaks will appear in the intensity $|\langle \langle j_\text{CME} \rangle \rangle|$.

Here we let us estimate the maximum value of the CME (12): 

\[ |\langle j_\text{CME,max} \rangle| = \frac{e^2}{2\pi h} B_0 \sum_{\omega \approx \omega_0} \omega_1 \delta m_{\perp} \sin \omega_1 t + \alpha. \]  

where $D_1 = \sqrt{p^2 + q^2}$ and $\alpha = q/p$ with $p = (\cos \phi_0 + \sin \phi_0) \cos \theta_0 - \sin \theta_0$ and $q = \sin \phi_0 - \cos \phi_0$. Equation (12) means that an alternating current is induced by the AF resonance. Schematic figure of a possible experimental setup to observe the CME in our system is shown in Fig. 1(b). \( \delta m_\perp \) is a function of the external microwave frequency $\omega$ with Lorentzian structure, i.e., \( \delta m_\perp (\omega) \approx a/(\omega^2 + \omega_0^2) \) with $a$ being a constant. Therefore two peaks will appear in the intensity $|\langle \langle j_\text{CME} \rangle \rangle|$.

Here we let us estimate the maximum value of the CME (12): 

\[ |\langle j_\text{CME,max} \rangle| = \frac{e^2}{2\pi h} B_0 \sum_{\omega \approx \omega_0} \omega_1 \delta m_{\perp} \sin \omega_1 t + \alpha. \]  

where $D_1 = \sqrt{p^2 + q^2}$ and $\alpha = q/p$ with $p = (\cos \phi_0 + \sin \phi_0) \cos \theta_0 - \sin \theta_0$ and $q = \sin \phi_0 - \cos \phi_0$. Equation (12) means that an alternating current is induced by the AF resonance. Schematic figure of a possible experimental setup to observe the CME in our system is shown in Fig. 1(b). \( \delta m_\perp \) is a function of the external microwave frequency $\omega$ with Lorentzian structure, i.e., \( \delta m_\perp (\omega) \approx a/(\omega^2 + \omega_0^2) \) with $a$ being a constant. Therefore two peaks will appear in the intensity $|\langle \langle j_\text{CME} \rangle \rangle|$.

Here we let us estimate the maximum value of the CME (12): 

\[ |\langle j_\text{CME,max} \rangle| = \frac{e^2}{2\pi h} B_0 \sum_{\omega \approx \omega_0} \omega_1 \delta m_{\perp} \sin \omega_1 t + \alpha. \]  

where $D_1 = \sqrt{p^2 + q^2}$ and $\alpha = q/p$ with $p = (\cos \phi_0 + \sin \phi_0) \cos \theta_0 - \sin \theta_0$ and $q = \sin \phi_0 - \cos \phi_0$. Equation (12) means that an alternating current is induced by the AF resonance. Schematic figure of a possible experimental setup to observe the CME in our system is shown in Fig. 1(b). \( \delta m_\perp \) is a function of the external microwave frequency $\omega$ with Lorentzian structure, i.e., \( \delta m_\perp (\omega) \approx a/(\omega^2 + \omega_0^2) \) with $a$ being a constant. Therefore two peaks will appear in the intensity $|\langle \langle j_\text{CME} \rangle \rangle|$.
There is a relative angle $\delta$ between the AF order and the electric field $E$ perpendicular to the AF order as given by spherical coordinates. A static electric field is applied perpendicular to the AF order. There is a relative angle $\delta$ in the magnetization directions of ferromagnet 1 (FM1) and ferromagnet 2 (FM2).

At the two edges has a relative angle $\delta$. Namely, we have $\theta(Z=0)=\theta_0$ and $\theta(Z=L)=\theta_0+\delta$ in the original spherical coordinate. A static electric field $E$ is applied perpendicular to the AF order. For simplicity, we assume that the system lies near the phase boundary where $U n_f/M_0 \ll 1$ [49]. Noting that only the $X$ component $J_{\text{AHE}}^X$ survives, we see that Eq. (13) is simplified to be

$$J_{\text{AHE}}^X(Z) = \frac{e^2}{2\pi \hbar} U D_2 \frac{E_Y}{M_0} \sum_f \partial n_f(Z)/\partial Z.$$  

(14)

The total current in the $X$ direction is given by

$$J_{\text{AHE}}^X = \int_0^L dZ J_{\text{AHE}}^X(Z) = \frac{e^2}{2\pi \hbar} U D_2 \frac{E_Y}{M_0},$$

where $D_2 = \sum_f \int_{-\pi/\theta_0}^{\pi/\theta_0} d\phi n_f = \sum_f [n_f(\theta_0+\delta) - n_f(\theta_0)] = n_0 [\sqrt{2} \sin(\frac{\phi_0}{2}) - \sin(\theta_0+\delta) - \sin(\theta_0) \cos(\frac{\phi_0}{2}) - \cos(\theta_0+\delta)]$.

The Hall conductivity is estimated as $\sigma_{XY} = \frac{e^2}{2\pi \hbar} U D_2 \sim 1 \times 10^{-2} e^2/h$, since $U n_f/M_0 \sim 0.1$ [50] and $|D_2|/n_0 \sim 1$.

Schematic figure of a possible experimental setup to observe the AHE in our system is shown in Fig. 2(b). Two ferromagnets with a relative angle $\delta$ in the magnetization directions are attached to the AFI [55]. In experiments, the $\delta$ dependence of the Hall conductivity will be a direct evidence for the observation of the axion field. Note that, in contrast to preceding works on the AHE in AFMs [25, 26], the AHE studied here does not occur in uniform ground states. Namely, spatial variations of the AF order parameter $n_f$ need to be realized by external forces.

**Discussions and Summary.**—Let us discuss briefly the realization of our predictions in realistic correlated systems with SOC. It has been suggested that the dynamical axion field can be realized by spin excitations in the AFI phases of Bi$_2$Se$_3$ family doped with magnetic impurities [16] and transition metal oxides with corundum structure such as $\alpha$-Fe$_2$O$_3$ [28]. In the same manner as above, we can derive similar expressions for the CME and AHE in these systems. What about the possibility in other systems? First of all, time-reversal and inversion symmetries of the system must be broken to induce the deviation of $\theta$ from 0 or $\pi$. Theoretically, the value of $\theta$ can be calculated numerically in any insulating systems [37–39].

The point is that the emergence of the CME and AHE depends on whether $\theta$ is a function of physical quantities such as AF order parameter, as in our case. If $\theta$ is a function of a physical quantity, then the fluctuation of the physical quantity realizes the dynamical axion field. It should be noted that, even if the value of $\theta$ is zero in ground states, the realization of dynamical axion fields is possible.

In summary, we have studied theoretically 3D AFIs with SOC, focusing on a role of the staggered magnetization. We have revealed that, in the presence of SOC, spin excitations in AFIs can result in nontrivial charge responses. It is shown that the dynamical CME, an alternating current generation by magnetic fields, emerges due to the time dependences of the AF order parameter. It is also shown that the AHE arises due to the spatial variations of the order parameter. These two phenomena are the consequences of the realization of the dynamical axion field in the AFI phase. The magnetic-field-induced and electric-field-induced currents in this study are understood as a polarization current in the bulk and a magnetization current in the bulk, respectively, which can flow in insulators. Observing these phenomena is equivalent to detecting the dynamical axion field in condensed matter. In other words, we propose a new way to detect the dynamical axion field.

The authors thank T. Chiba, Y. Araki, O. A. Tretiakov, S. Takahashi, and J. Barker for valuable discussions. A.S. is supported by a JSPS Research Fellowship. This work was supported in part by Grant-in-Aid for Scientific Research (No. 26107505 and No. 26400308) from MEXT, Japan.
1. Alternative Derivation of Eq. (3)

In this section, we derive the effective action consisting of the Néel field $\mathbf{n}$ and an external electromagnetic potential $A_\mu$. For this purpose, it is convenient to adopt a perturbative method rather than the Fujikawa’s method. We start with the total action of an antiferromagnetic (AF) insulator described by Eq. (1) in the presence of an external electromagnetic potential $A_\mu$:

$$S_{\text{eff}}[\psi, \bar{\psi}, \mathbf{n}, A_\mu] = \int dt d^3r \sum_f \bar{\psi}_f(r, t) \left[ i\gamma^\mu D_\mu - M_0 + i\gamma^5 M_5 \right] \psi_f(r, t).$$

(S1)
By integrating out the fermionic field, we obtain the effective action for $n$ and $A_{\mu}$ as

$$Z[n, A_{\mu}] = \int [\psi, \bar{\psi}] e^{iS_{\text{eff}}} = \exp \left\{ \sum_f \text{Tr} \ln \left[ G_{0f}^{\dagger}(1 + G_{0f} V_f) \right] \right\} = \exp \left\{ \sum_f \text{Tr} \left( \ln G_{0f}^{\dagger} \right) + \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left( G_{0f} V_f \right)^n \right\}$$

$$= e^{iW_{\text{eff}}[n, A_{\mu}]},$$

where $G_{0f} = (i \gamma^a \partial_{\mu} - M_0)^{-1}$ is the Green’s function of the noninteracting part, $V_f = -e \gamma^a A_{\mu} + i \gamma^5 M_5 f$ is the perturbation term, and we have used $i \gamma^a \partial_{\mu} M_0 + i \gamma^5 M_5 f = (G_{0f}^{-1} + V_f)$. Next we use the following identities for the traces of gamma matrices:

$$\text{tr}(\gamma^0) = \text{tr}(\gamma^5) = 0, \quad \text{tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu \nu}, \quad \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = -4 i e^{\mu \nu \rho \sigma}.$$ 

The relevant terms up to the leading order read

$$i W_{\text{eff}}[n, A_{\mu}] = \frac{1}{2} \sum_f \text{Tr} \left( G_{0f} i \gamma^5 M_5 f \right)^2 + \sum_f \text{Tr} \left[ (-G_{0f} e \gamma^\mu A_{\mu}) \gamma^5 M_5 f \right].$$

The first and second terms correspond to a bubble-type diagram and a triangle-type diagram, respectively.

For concreteness, we consider the case of $M_{5,1} = U n_1$, $M_{5,2} = U n_2$, and $M_{5,3} = U n_3$ ($U$ is the on-site interaction strength), which is applied to the Fu-Kane-Mele-Hubbard model. Here the Néel field is given by $n = n_1 e_x + n_2 e_y + n_3 e_z$. The first term in Eq. (S4) is given explicitly by

$$\text{Tr} \left( G_{0f} i \gamma^5 M_5 f \right)^2 = U^2 \int \frac{d^4 q}{(2 \pi)^4} \int \frac{d^4 k}{(2 \pi)^4} \text{tr} \left[ i \gamma^\mu k_{\mu} + M_0 \right] i \gamma^5 n_f i \gamma^\nu (k + q) + M_0 [i \gamma^5 n_f (-q)]$$

$$= 4 U^2 \int \frac{d^4 q}{(2 \pi)^4} \int \frac{d^4 k}{(2 \pi)^4} \frac{\text{tr} \left[ k_{\mu} (k + q)^{\mu} + M_0^2 n_f n_f (-q) \right]}{(k^2 - M_0^2)(k^2 + q^2 - M_0^2)}$$

$$= \int \frac{d^4 q}{(2 \pi)^4} I(q) n_f n_f (-q),$$

where $k^2 = g^{\mu \nu} k_{\mu} k_{\nu} = k_{\alpha} k_{\alpha} = k_0^2 - k^2$. We have used $G_{0f} (k) = i \gamma^\mu k_{\mu} + M_0) / (k^2 - M_0^2)$, $\{\gamma^\mu, \gamma^5\} = 0$, and $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu \nu}$. Furthermore, we introduce the Feynman parameter to combine the denominator as

$$\frac{1}{(k^2 - M_0^2)(k + q)^2 - M_0^2} = \int_0^1 \frac{1}{[k^2 + a(1 - a) q^2 - M_0^2]},$$

where $l = k + a q$. On the other hand, in terms of $l$, we obtain $k_{\mu} (k + q)^{\mu} = l^2 - a(1 - a) q^2 + (1 - 2a) q^2$. Then the integral $I(q)$ can be represented as

$$I(q)/i = 4 U^2 \int \frac{d^4 l_E}{(2 \pi)^4} \int_0^1 \frac{d^4 q}{(2 \pi)^4} \frac{\text{tr} \left[ \left( l_E^2 + a(1 - a) q^2 - M_0^2 \right) \right]}{[l_E^2 + M_0^2 - a(1 - a) q^2]^2} = A + B q^2 + O(q^4),$$

where we have Wick-rotated as $l_E^0 = i l^0$. The term that contains $l_E q^2$ vanishes, because it is an odd function of $l_E$. Finally, by substituting Eq. (S7) into Eq. (S5), we arrive at the action of the form

$$\sum_f \text{Tr} \left( G_{0f} i \gamma^5 M_5 f \right)^2 = \frac{i}{g} \int dt d^3 r \left[ \left( \partial_{\nu} n \right) \cdot \left( \partial^\nu n \right) + m^2 n^2 \right]$$

where $1/g = B$ and $m^2 = A/B$. This action is nothing but the action of the Néel field (i.e., the nonlinear sigma model) [31]. In the present low-energy effective model [Eq. (S1)], the information on the anisotropy of the Néel field is not included. On the other hand, many (actual) AF insulators have the easy-axis anisotropy. Hence the term $m^2 n^2$ will be replaced by a term like $m^2 (n \cdot e_A)^2$ with $e_A$ denoting the easy axis.

The second term in Eq. (S4) is the so-called triangle anomaly, which gives the theta term. The final result is [32, 33]

$$\sum_f \text{Tr} \left[ (-G_{0f} e \gamma^\mu A_{\mu}) \gamma^5 M_5 f \right] = i \int dt d^3 r \frac{e^2}{4 \pi h} \left[ \frac{\pi}{2} [1 + \text{sgn}(M_0)] - \sum_f \frac{U n_f (r, t)}{M_0} \right] e^{i \eta_{\mu \nu} \partial_{\mu} A_{\nu} \partial_{\rho} A_{\rho}}$$

$$= i \int dt d^3 r \frac{e^2}{2 \pi h} \theta(r, t) E \cdot B.$$
where \( \theta(r, t) = \frac{x}{2} [1 + \text{sgn}(M_0)] - \sum_{f} U n_f(r, t)/M_0 \). Note that this value of \( \theta \) is consistent with that obtained by the Fujiawa’s method \( \theta_f = \frac{x}{2} [1 + \text{sgn}(M_0)] - \sum_{f} \tan^{-1}(U n_f/M_0) \), since \( \tan^{-1}(x) = x - x^3/3 + \cdots \).

### 2. From the Fu-Kane-Mele-Hubbard Model to the Heisenberg Model

In this section, we consider the validity of the dynamics of the sublattice magnetizations described by Eq. (10) in the Fu-Kane-Mele-Hubbard model. A mean-field study has shown that the AF insulator phase of the Fu-Kane-Mele-Hubbard model develops when \( U/t = 4 \) (with \( \lambda/t \sim 0.3 \)) [34]. Hence we may consider a strong coupling picture. When \( U \gg t \) and \( U \gg \lambda \), we can derive the Heisenberg model by treating the nearest-neighbor (NN) hopping term \( H \), when

\[
\text{Mele-Hubbard model. A mean-field study has shown that the AF insulator phase of the Fu-Kane-Mele-Hubbard model develops}
\]

\[
\text{ground state direction. For example, in the case of the AF insulator phase of Bi}
\]

\[
\text{coupling anisotropy [35]. Therefore, the e}
\]

\[
\text{of spins are coupled to the lattice structure. More generally, as well known, the presence of spin-orbit coupling leads to magnetic}
\]

\[
\text{in the presence of spin-orbit coupling [34]. Furthermore, spin-orbit coupling breaks spin SU(2) symmetry, i.e., the orientations}
\]

\[
\text{with the NN exchange interaction which favors ferromagnetic alignments between the same sublattices. Such a competition is}
\]

\[
\text{exchange interaction can favor AF alignments between the same sublattices. Namely, the NNN exchange interaction competes}
\]

\[
\text{competition is consistent with a numerical result from the weak coupling that shows the increase of the critical strength}
\]

\[
\text{in the presence of spin-orbit coupling [34]. Furthermore, spin-orbit coupling breaks spin SU(2) symmetry, i.e., the orientations}
\]

\[
\text{of spins are coupled to the lattice structure. More generally, as well known, the presence of spin-orbit coupling leads to magnetic}
\]

\[
\text{with the easy-axis anisotropy such that}
\]

\[
\text{easy axis is the}
\]

\[
\text{direction (perpendicular to the quintuple layers) [16]. Then, in the presence of an external magnetic field}
\]

\[
\text{B, the spin Hamiltonian of the system could be written effectively as}
\]

\[
\text{where}
\]

\[
\text{includes the effect of spin-orbit coupling and we have assumed isotropic exchange interaction for simplicity. It should}
\]

\[
\text{be noted that the continuum action of this Hamiltonian is given by Eq. (S8) (this is the famous Haldane’s mapping) [31].}
\]

The spin dynamics is determined from the equation of motion:

\[
S_i = i \left[ H_{\text{eff}}, S_i \right] = S_i \times f_i,
\]

where \( f_i \) is the effective magnetic field given by

\[
f_i = -\partial H_{\text{eff}}/\partial S_i = -\bar{J} \sum_{(i,j)} S_j + 2K(S_i \cdot e_m) e_{m_0} + g\mu_B B.
\]

In order to consider the AF resonance state, where all the spins are precessing around the easy axis with the same frequency, we can replace the spins \( S_i \) by the mean-field values \( m_A \) and \( m_B \) with \( A, B \) denoting two sublattices of a diamond lattice. Finally,
we arrive at Eq. (10):

\[
\begin{align*}
\dot{m}_A &= m_A \times \{-\omega J m_B + [g\mu_B B + \omega_A (m_A \cdot e_{m})] e_{m} \}, \\
\dot{m}_B &= m_B \times \{-\omega J m_A + [g\mu_B B + \omega_A (m_B \cdot e_{m})] e_{m} \},
\end{align*}
\]

(S16)

where \( \omega_J = J_{2NN} \) (\( J_{2NN} \) is the number of NN bonds), \( \omega_A = 2K \), and a static magnetic field is applied along \( e_{m} \) as \( B = B e_{m} \). Note that the easy-axis anisotropy term in Eq. (S13) is not essential to cause the dynamics. What is essential is an external magnetic field, as is understood from the resonance frequency \( \omega = \omega_0 = g\mu_B B \pm \sqrt{(2\omega_J + \omega_A)\omega_A} \).