Explicit 3-colorings for exponential graphs

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Abstract

For a graph $H$ and integer $k \geq 1$, two functions $f, g$ from $V(H)$ into $\{1, \ldots, k\}$ are adjacent if for all edges $uv$ of $H$, $f(u) \neq g(v)$. The graph of all such functions is the exponential graph $K^H_k$. El-Zahar and Sauer proved that if $\chi(H) \geq 4$, then $K^H_3$ is 3-chromatic [ES85]. Tardif showed that, implicit in their proof, is an algorithm for 3-coloring $K^H_3$ whose time complexity is polynomial in the size of $K^H_3$ [Tar06]. Tardif then asked if there is an “explicit” algorithm for finding such a coloring: Essentially, given a function $f$ belonging to a 3-chromatic component of $K^H_3$, can we assign a color to this vertex in time polynomial in the size of $H$? The main result of this paper is to present such an algorithm, answering Tardif’s question affirmatively. Our algorithm yields an alternative proof of the theorem of El-Zahar and Sauer that the categorical product of two 4-chromatic graphs is 4-chromatic.

1 Introduction

For a graph $G$, we use $V(G)$ and $E(G)$ to denote its vertex and edge sets, respectively, and we use $\chi(G)$ to denote its chromatic number. A homomorphism from a graph $G$ to a graph $G'$ is a function $\phi$ from $V(G)$ to $V(G')$ such that for every edge $uv$ in $G$, $\phi(u)\phi(v)$ is an edge in $G'$. We denote by $G \to G'$ the existence of a homomorphism from $G$ to $G'$. Note that a graph $G$ admits a proper $k$-coloring if and only if $G \to K_k$.

The categorical product of two graphs $G \times H$ has vertex set $V(G) \times V(H)$ and edge set $\{(u, v), (\bar{u}, \bar{v})\}$ for $u\bar{u}$ and $v\bar{v}$ belonging to $E(G)$ and $E(H)$, respectively. Observe that $G \times H$ admits a homomorphism to both $G$ and $H$. Since a proper colouring corresponds to a homomorphism to a complete graph, and since $G \times H \to G$ and $G \times H \to H$, it is therefore immediate that $\chi(G \times H) \leq \min(\chi(G), \chi(H))$. The following conjecture is due to Hedetniemi [Hed66] and was also posed as a question by Greenwell and Lovász [GL74].

Conjecture 1.1. $\chi(G \times H) = \min(\chi(G), \chi(H))$.

For two graphs $H$ and $K$, there exists an exponential graph $K^H$ with the following property: $G \times H \to K$ only if $G \to K^H$. The vertices of $K^H$ are functions from $V(H)$ into $V(K)$, and two

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functions \( f, g \) are adjacent if for every edge \( uv \) of \( H \), \( f(u)g(v) \) is an edge of \( K \). With this definition of an exponential graph, observe that Hedetniemi’s conjecture can be rewritten: If \( \chi(H) > k \) and \( G \to K^H_k \), then \( G \to K_k \). But now we see that it is sufficient to replace \( G \) with \( K^H_k \), and Conjecture 1.1 can be restated as follows.

**Conjecture 1.2.** If \( \chi(H) > k \), then \( \chi(K^H_k) \leq k \).

The connection between exponential graphs and Hedetniemi’s conjecture was observed by El-Zahar and Sauer who proved Conjecture 1.2 when \( k = 3 \) [ES85]. (Exponential graphs have also been studied in other contexts [Lov67].) Specifically, El-Zahar and Sauer proved that if \( \chi(H) > 3 \), then \( K^H_3 \) is 3-colorable. Their proof is based on a global parity argument concerning so-called fixed points of odd cycles and does not immediately yield a 3-coloring of \( K^H_3 \). Attributing the question of efficiently finding a 3-coloring to Edmonds, Tardif presented an algorithm, implicit in the work of El-Zahar and Sauer, for 3-coloring \( K^H_3 \) [Tar06]. Basically, Tardif noted that if we consider an (arbitrary) edge \( ab \) in any odd cycle of \( H \), then functions \( f \) in which \( f(a) = f(b) \) form a hitting set of the odd cycles in \( K^H_3 \). A 3-coloring can easily be found based on a bipartition of the remaining vertices in \( K^H_3 \). The time complexity of this algorithm depends on the time to find a bipartition, which is polynomial in the size of \( K^H_3 \), but can be exponential in \( |V(H)| \). A thorough description of this algorithm is provided in Section 1.2.

Tardif then posed the problem of finding an “explicit” 3-coloring of \( K^H_3 \). Essentially, given a function \( f \) belonging to a 3-chromatic component of \( K^H_3 \), can we assign a color to this vertex in time polynomial in the size of \( H \)? (His precise question is a bit more involved and presented in detail in Section 1.2.) In this paper, we present an algorithm whose time complexity is linear in \( |V(H)| \) for finding such an explicit 3-coloring.

### 1.1 From coloring \( K_3^{C_{2n+1}} \) to coloring \( K^H_3 \)

Let \( n \) be a positive integer, and let \( C_{2n+1} \) denote the odd cycle on the vertices \( u_1, u_2, \ldots, u_{2n+1} \). The vertex set \( V(K_3^{C_{2n+1}}) \) consists of all \( 3^{2n+1} \) functions from \( V(C_{2n+1}) \) into \( \{1, 2, 3\} \), except for \( u_i \in C_{2n+1} \) a fixed point in \( f \in V(K_3^{C_{2n+1}}) \) if \( f(u_{i-1}) \neq f(u_{i+1}) \), where indices are computed modulo \( 2n+1 \). Let \( K_{Even}^{C_{2n+1}} \) denote the subgraph of \( K_3^{C_{2n+1}} \) induced on the following vertex set.

\[
V(K_{Even}^{C_{2n+1}}) = \left\{ f \in V(K_3^{C_{2n+1}}) : f \text{ has an even number of fixed points} \right\}.
\]

The problem of finding a 3-coloring of \( K_3^H \) when \( \chi(H) > 3 \) can be reduced to the problem of finding a 3-coloring for \( K_{Even}^{C_{2n+1}} \) [ES85, Tar06]. Any non-isolated function \( f \) from \( V(H) \) into \( \{1, 2, 3\} \) contains an odd cycle with an even number of fixed points and such an odd cycle can be found in time polynomial in the size of \( H \) (see Proposition 4.1 in [ES85] or Claim 2 from [Zhu98]). For the connected component of \( K_3^H \) containing this function \( f \), let us fix this odd cycle in \( H \) to be \( C_{2n+1} \).

\(^1\)Observe that the proof of this claim becomes easier when \( \chi(H) \geq 5 \). Consider any function \( f \) from \( V(H) \) into \( \{1, 2, 3\} \) and partition \( V(H) \) into the set of vertices colored by 1 and 2 and the set of vertices colored by 3. Either the first set or the second set contains an odd cycle (which then has an even number of fixed points because it has at most two colors) or each set is bipartite, and we can color with four colors, which is a contradiction.
Now we find a 3-coloring for $K_C^{2n+1}$. Each function $g$ in the same connected component of $K_H^3$ as $f$ must also have an even number of fixed points on $C_{2n+1}$ (see Lemma 3.3 in [ES85] or Claim 1 from [Zhu98]). So for each function $g \in V(K_H^3)$ in the same connected component as $f$, the restriction of $g$ onto $C_{2n+1}$ belongs to $V(K_C^{2n+1})$. Therefore, the vertex in $V(K_H^3)$ corresponding to the function $g$ can be assigned the same color that the restriction of $g$ onto $C_{2n+1}$ receives in the 3-coloring of $K_C^{2n+1}$.

### 1.2 Algorithm for 3-coloring $K_E^{2n+1}$

Now we are ready to present the algorithm for 3-coloring $K_E^{2n+1}$ from [Tar06]. Let $ab$ denote a fixed (but arbitrarily chosen) edge in $C_{2n+1}$.

#### COLOR-GRAPH($K_E^{2n+1}$)

1. For all $f \in V(K_E^{2n+1})$:
   
   (i) If $f(a) = f(b)$, assign the vertex $f$ in $V(K_E^{2n+1})$ the color $f(a)$.

2. Remove all the colored vertices from $V(K_E^{2n+1})$.

3. Find a bipartition $(A, B)$ of the subgraph induced on the remaining vertices.

4. For each vertex $f$ in $A$, assign the vertex color $f(a)$.

5. For each vertex $f$ in $B$, assign the vertex color $f(b)$.

The correctness of this algorithm follows from the fact that the copies of $C_{2n+1}$ in which $f(a) = f(b)$ form a hitting set for the odd cycles in $K_E^{2n+1}$, which follows from the main result of El-Zahar and Sauer (e.g., see Lemma 3.1 and Proposition 3.4 in [ES85]).

### 1.3 Tardif’s open problem

Tardif asked if there is an algorithm, whose running time is polynomial in $n$, to assign a color to $f \in V(K_E^{2n+1})$ so that a 3-coloring is maintained for any subset of colored vertices of $K_E^{2n+1}$. (See Problem 6 in [Tar06] and also [Tar11].) He defines the vertex set

$$V(B_{2n+1}) = \left\{ f \in V(K_E^{2n+1}) : f(a) \neq f(b) \right\},$$

for a fixed (but arbitrarily chosen) edge $ab \in C_{2n+1}$. $B_{2n+1}$ is a bipartite subgraph of $K_E^{2n+1}$ induced on $V(B_{2n+1})$. If we can decide in $O(n)$ time to which side of the bipartition $f \in V(B_{2n+1})$ belongs, then we can resolve Tardif’s question affirmatively. We present an algorithm for this task in the next section. Our approach is inspired by ideas from reconfiguration of 3-recolorings [CvdHJ07].
We note that Tardif showed that the main result from [ES85] implies that $B_{2n+1}$ is bipartite. Conversely, our algorithm gives another proof that $B_{2n+1}$ is bipartite, and consequently, we give an alternative proof of the main result of [ES85]. Note, however, that our proof is not completely independent as it uses Lemma 3.3 and Proposition 4.1 from [ES85].

2 Properties of adjacent functions

In this section, we state and prove two properties of functions that are adjacent in $K_{3}^{C_{2n+1}}$. These properties are key to the design and analysis of our algorithm, which we present in Section 3. For an ordered pair of vertices $uv$ (i.e., an arc $uv$) with colors $c(u)$ and $c(v)$, respectively, we say the value of $uv$ is $\Delta(c(u)c(v))$, where

$$\Delta(12) = \Delta(23) = \Delta(31) = +1 \quad \text{and} \quad \Delta(21) = \Delta(32) = \Delta(13) = -1.$$  

Monochromatic pairs have value 0.

Let $f$ denote a function from $V(C_{2n+1})$ into $\{1, 2, 3\}$. For a vertex $u \in C_{2n+1}$, its color in $f$ is denoted by $f(u)$. We fix the orientation for the chords of length two in the cycle $C_{2n+1}$ so that they form the directed cycle $\{u_1, u_3, \ldots, u_{2n+1}, u_2, u_4, \ldots, u_{2n}, u_1\}$, which we refer to as $\vec{C}_{2n+1}$. Then we have the following definitions. Recall that $ab$ is a fixed (but arbitrarily chosen) edge in $C_{2n+1}$.

**Definition 2.1.** The label of $f$, denoted by $\ell_f$, is the total value of the arcs in $\vec{C}_{2n+1}$ based on $f$. Formally,

$$\ell_f = \sum_{uv \in \vec{C}_{2n+1}} \Delta(f(u)f(v)).$$

**Definition 2.2.** The little path of $f$, denoted by $\vec{p}_{ab}$, is the directed path from $a$ to $b$ in $\vec{C}_{2n+1}$ containing $n$ arcs (e.g., see Figure 1). The value of $\vec{p}_{ab}$, denoted by $p_f$, is computed as follows.

$$p_f = \sum_{uv \in \vec{p}_{ab}} \Delta(f(u)f(v)).$$

Figure 1: The dotted edges denote $C_9$. The directed edges denote $\vec{C}_9$. The little path $\vec{p}_{ab}$ is shown in red.
Observation 2.3. \( \ell_f \equiv 0 \pmod{3} \).

Observation 2.4. If \( f(a) \neq f(b) \), then \( p_f \not\equiv 0 \pmod{3} \).

Now let \( f \) and \( g \) be two functions from \( V(C_{2n+1}) \) into \( \{1, 2, 3\} \) such that \( f \) and \( g \) are adjacent. Recall that \( \{u_1, u_2, \ldots, u_{2n+1}\} \) denotes the vertex set \( V(C_{2n+1}) \). Let \( f_i \) denote the copy of \( u_i \) in \( f \) and let \( g_i \) denote the copy of \( u_i \) in \( g \). Define the directed cycle \( D \) as follows.

\[
D = \{f_1, g_2, f_3, \ldots, g_{2n}, f_{2n+1}, g_1, f_2, \ldots, f_{2n}, g_{2n+1}, f_1\}.
\]

We have

\[
\Delta(D) = \sum_{i=1}^{2n+1} \Delta(f(u_i)g(u_{i+1})) + \sum_{i=1}^{2n+1} \Delta(g(u_i)f(u_{i+1})),
\]

where subscripts are computed modulo \( 2n + 1 \). We can relate the value of the arcs in \( C_{2n+1} \) to the value of the arcs in \( D \) using the following claim.

Claim 2.5. Let \( f \) and \( g \) be two functions from \( V(C_{2n+1}) \) into \( \{1, 2, 3\} \) such that \( f \) and \( g \) are adjacent. Then

\[
\Delta(f(u_i)f(u_{i+2})) = -\frac{\Delta(f(u_i)g(u_{i+1})) + \Delta(g(u_{i+1})f(u_{i+2}))}{2}.
\]

Proof. Since \( f \) and \( g \) are adjacent, we have \( f(u_i) \neq g(u_{i+1}) \) and \( f(u_i) \neq g(u_{i-1}) \) for \( i \in \{1, \ldots, k\} \). If \( f(u_i) = f(u_{i+2}) \), then \( \Delta(f(u_i)g(u_{i+1})) + \Delta(g(u_{i+1})f(u_{i+2})) = 0 \). Furthermore, we have the following observations.

1. \( \Delta(f(u_i)f(u_{i+2})) = +1 \Rightarrow \Delta(f(u_i)g(u_{i+1})) + \Delta(g(u_{i+1})f(u_{i+2})) = -2 \), and
2. \( \Delta(f(u_i)f(u_{i+2})) = -1 \Rightarrow \Delta(f(u_i)g(u_{i+1})) + \Delta(g(u_{i+1})f(u_{i+2})) = +2 \).

Next, we present two key properties of adjacent functions via the following lemmas.

Lemma 2.6. Let \( f \) and \( g \) be two functions from \( V(C_{2n+1}) \) into \( \{1, 2, 3\} \) such that \( f \) and \( g \) are adjacent. Then \( \ell_f = \ell_g \).

Proof. Recall that \( \ell_f \) is defined as follows.

\[
\ell_f = \sum_{i=1}^{2n+1} \Delta(f(u_i)f(u_{i+2})).
\]

By Claim 2.5, we observe that \( \ell_f = -\Delta(D)/2 \). The same argument shows that \( \ell_g = -\Delta(D)/2 \).

Lemma 2.7. Let \( f \) and \( g \) be two functions from \( V(C_{2n+1}) \) into \( \{1, 2, 3\} \) such that \( f \) and \( g \) are adjacent and let \( \ell = \ell_f = \ell_g \). Then \( \ell - 1 \leq p_f + p_g \leq \ell + 1 \).

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Proof. Without loss of generality, let \( a = u_1 \) and \( b = u_{2n+1} \). Then
\[
p_f = \Delta(f(u_1)f(u_3)) + \Delta(f(u_3)f(u_5)) + \cdots + \Delta(f(u_{2n-1})f(u_{2n+1})),
\]
and
\[
p_g = \Delta(g(u_1)g(u_3)) + \Delta(g(u_3)g(u_5)) + \cdots + \Delta(g(u_{2n-1})g(u_{2n+1})).
\]
Applying Claim 2.5, we have
\[
p_f + p_g = -\Delta(D) - \Delta(f(b)g(a)) - \Delta(g(b)f(a)) + \ell + \frac{\Delta(f(b)g(a)) + \Delta(g(b)f(a))}{2}.
\]
Since
\[-2 \leq \Delta(f(b)g(a)) + \Delta(g(b)f(a)) \leq 2,
\]
the lemma follows. \( \square \)

3 Coloring a vertex of \( K_{2n+1}^{C} \) in time \( O(n) \)

Now we present an algorithm to assign a color to a function \( f \) when \( f \in V(K_{2n+1}^{C}) \) (i.e., \( f \in V(K_{3}^{C_{2n+1}}) \) and \( \ell_f \equiv 0 \mod 2 \)). Our algorithm will produce a 3-coloring for \( K_{2n+1}^{C} \) and the time required to assign a color to \( f \) is \( O(n) \). We recall that edge \( ab \in C_{2n+1} \) is a fixed edge and is considered as input to the algorithm.

**COLOR-VERTEX** \((f \in K_{2n+1}^{C})\)

1. If \( f(a) = f(b) \), assign \( f \) color \( f(a) \).
2. Otherwise, compute value \( p_f \).
   (a) If \( p_f < \ell_f/2 \), assign color \( f(a) \).
   (b) If \( p_f > \ell_f/2 \), assign color \( f(b) \).

The correctness of the algorithm will be shown via the following theorem, whose proof is based on the observation that if \( f \) and \( g \) are adjacent in \( B_{2n+1} \), then by Lemma 2.7, \( p_f \) and \( p_g \) are anticorrelated. For example, if \( \ell_f = 0 \), then either \( p_f \) is positive and \( p_g \) is negative or vice versa. Theorem 3.1 implies that using **COLOR-VERTEX** to color the vertices of \( K_{2n+1}^{C} \) results in a proper 3-coloring of \( K_{2n+1}^{C} \).

**Theorem 3.1.** Let \( f \) and \( g \) be two functions from \( V(K_{2n+1}^{C}) \) such that \( f \) and \( g \) are adjacent. Then **COLOR-VERTEX** assigns \( f \) and \( g \) different colors.
First, we show via Claim 3.2 that Step 2. of Color-Vertex is well-defined.

**Claim 3.2.** Let $f$ be a function in $V(B_{2n+1})$. Then either $p_f < \ell_f/2$ or $p_f > \ell_f/2$.

**Proof.** By Observation 2.3, $\ell_f$ is a multiple of 3. Since $f \in V(B_{2n+1})$, it follows that $\ell_f$ is a multiple of 2. Thus, $\ell_f/2$ is an integer and is a multiple of 3.

By Observation 2.4 and the fact that $f \in V(B_{2n+1})$, $p_f$ is not a multiple of 3. Therefore, $p_f \neq \ell_f/2$. Thus, we have

\[
p_f \leq \ell_f/2 \implies p_f \leq \ell_f/2 - 1, \quad \text{and} \quad
p_f \geq \ell_f/2 \implies p_f \geq \ell_f/2 + 1.
\]

Next, we show that two adjacent functions in $B_{2n+1}$ will not be assigned to the same side of the bipartition, and thus will not be assigned the same color.

**Claim 3.3.** Let $f$ and $g$ be two adjacent functions in $V(B_{2n+1})$ and let $\ell = \ell_f = \ell_g$. Then $p_f$ and $p_g$ cannot both be greater (or both smaller) than $\ell/2$.

**Proof.** By Lemma 2.7, we have

\[
\ell - 1 \leq p_f + p_g \leq \ell + 1.
\]

Let us consider two cases. The first case is when $p_f \leq \ell/2 - 1$. Then we have

\[
\ell - 1 - p_g \leq p_f \leq \ell/2 - 1 \implies p_g \geq \ell/2.
\]

By Claim 3.2, we conclude that $p_g > \ell/2$. The second case is when $p_f \geq \ell/2 + 1$. Then we have

\[
\ell/2 + 1 \leq p_f \leq \ell + 1 - p_g \implies p_g \leq \ell/2.
\]

By Claim 3.2, we conclude that $p_g < \ell/2$.

\[\diamondsuit\]

### 4 Explicit homomorphisms for odd cycles

For a graph $H$ and a cycle $C_k$ for odd integer $k \geq 5$, one can define (as in the introduction) the exponential graph $C_k^H$. The vertices of $C_k^H$ are functions from $V(H)$ into $V(C_k)$ and two such functions $f$ and $g$ are adjacent if for all edges $uv$ of $H$, $f(u)$ and $g(v)$ are adjacent in $C_k$. Häggkvist, Hell, Miller and Neumann Lara proved that if there is no homomorphism from $H$ to $C_k$, then $C_k^H$ has a homomorphism to $C_k$ [HHML88]. Their proof can be viewed as a generalization of the work of El-Zahar and Sauer, who proved the same statement when $k = 3$. In fact, as in the case in the latter proof of El-Zahar and Sauer, it is also implicit in the proof of Häggkvist et al. that if we consider an (arbitrary) edge $ab$ in any odd cycle of $H$, then functions $f$ in which $f(a) = f(b)$ form a
hitting set of the odd cycles in $C^H_k$. It is therefore not surprising that we can extend our framework for obtaining explicit homomorphisms to odd cycles.

As in the case of $C_3$, we can find a homomorphism from $C^H_k$ to $C_k$ by considering an arbitrary odd cycle in $H$. Applying Lemma 7 from [HHML88], we see that if there is no homomorphism from $H$ to $C_k$, then $H$ contains an odd cycle with an even number of fixed points.\footnote{In [HHML88], a fixed point is called a 2-point.} We refer to this odd cycle as $C_{2n+1}$. It remains to generalize the two key properties of adjacent functions (i.e., Lemmas 2.6 and 2.7). For an ordered pair of vertices $uv$ (i.e., an arc $uv$) with values $c(u)$ and $c(v)$ (from $V(C_k)$), respectively, we say the value of $uv$ is $\Delta(c(u)c(v))$, where

$$\Delta(ij) = \begin{cases} +1, & \text{if } j-i \equiv 2 \text{ mod } k, \\ -1, & \text{if } j-i \equiv 1 \text{ mod } k, \\ -\frac{1}{2}, & \text{if } j-i \equiv 1 \text{ mod } k, \\ 0, & \text{if } j-i \equiv 0 \text{ mod } k. \end{cases}$$ (3)

For example, we have

$$\Delta(13) = \Delta(24) = \Delta(k2) = +1 \text{ and } \Delta(31) = \Delta(42) = \Delta(2k) = -1.$$ 

Let $f$ and $g$ be two adjacent functions from $V(C_{2n+1})$ to $V(C_k)$. We can apply the rules from (3) to compute the values $\ell_f, \ell_g$ and $p_f, p_g$ via Definitions 2.1 and 2.2. Note that if $|f(u) - f(v)| \not\in \{0, 2\}$ mod $k$ for some arc $uv \in C_{2n+1}$, then $f$ is an isolated function in $C^2_{2n+1}$. Moreover, note that if $|f(u) - g(v)| \not\equiv 1 \text{ mod } k$ for some arc $uv \in C_{2n+1}$, then $f$ and $g$ are not adjacent.

The next lemmas are the generalizations of Lemma 2.6 and Lemma 2.7 for homomorphisms to an odd cycle $C_k$.

**Lemma 4.1.** Let $f$ and $g$ be two functions from $V(C_{2n+1})$ into $V(C_k)$ such that $f$ and $g$ are adjacent. Then $\ell_f = \ell_g$.  

**Proof.** Recall the definition of the directed cycle $D$ and $\Delta(D)$ from (1) and (2). It is straightforward to prove that $\ell_f = \ell_g = \Delta(D)$. \hfill $\square$

**Lemma 4.2.** Let $f$ and $g$ be two functions from $V(C_{2n+1})$ into $V(C_k)$ such that $f$ and $g$ are adjacent and let $\ell = \ell_f = \ell_g$. Then $\ell - 1 \leq p_f + p_g \leq \ell + 1$.

**Proof.** Let $a = u_1$ and $b = u_{2n+1}$. We observe that

$$p_f + p_g = \Delta(D) - \Delta(f(b)g(a)) - \Delta(g(b)f(a))$$

$$= \ell - \Delta(f(b)g(a)) - \Delta(g(b)f(a)).$$

Since

$$-1 \leq \Delta(f(b)g(a)) + \Delta(g(b)f(a)) \leq 1,$$

the lemma follows. \hfill $\square$
It is straightforward to extend Claims 3.2 and 3.3 to this generalized setting and we obtain the following theorem.

**Theorem 4.3.** Let $f$ and $g$ be two functions from $V(C_{2n+1})$ into $V(C_k)$ such that $f$ and $g$ are adjacent. Then Color-Vertex assigns $f$ and $g$ to adjacent vertices in $V(C_k)$.

5 Discussion: Explicit versus efficient colorings

For a graph $H$ such that $\chi(H) > 3$, the question of finding an explicit 3-coloring of $K_3^H$ is closely related to—but not exactly the same as—the question of finding an efficient 3-coloring of $K_3^H$. A connected component of $K_3^H$ is either (i) isolated (i.e., a single vertex), (ii) bipartite, or (iii) 3-chromatic. For a function $f$ from $V(H)$ into $\{1, 2, 3\}$, it can be efficiently determined (in time polynomial in the size of $H$) whether or not $f$ is isolated. For any given connected component of $K_3^H$ that is bipartite or 3-chromatic, there exists an odd cycle that can be found efficiently (as discussed in Section 1.1) and this odd cycle can be used to obtain an explicit and efficient 3-coloring for this component. In other words, for a given connected component, after a polynomial amount of preprocessing time (i.e., time to find an odd cycle and to fix an orientation of its chord cycle and an edge $ab$ to use as input for the VERTEX-COLOR routine), we give an explicit reason (i.e., certificate) for assigning a particular color to a function $f$ in the given connected component. In particular, the value $\vec{p}_{ab}$ of the little path from $a$ to $b$ is such a short certificate. In terms of efficiency, for any subset $S$ of functions in $V(K_3^H)$ belonging to a fixed connected component, the total time required to color the subgraph induced on $S$ is $O(|S| \cdot |H|)$.

Moreover, for a function $f$ belonging to any 3-chromatic component of $K_3^H$, we can actually use an arbitrary fixed cycle $C_{2n+1}$ from $H$ for the COLOR-VERTEX routine. However, the fact that we can use the same cycle in $H$ for any such $f$ follows from the main result of El-Zahar and Sauer; Proposition 3.4 in [ES85] states that for such an $f$, all odd cycles in $H$ have an even number of fixed points. Note that the results we have presented here do not imply a proof of this proposition. Thus, while such a function $f$ can in fact be assigned a color efficiently (i.e., in time $O(|H|)$), this time complexity is not implied solely by the results we have presented here.

For $f$ belonging to an arbitrary bipartite component of $K_3^H$, using the approach presented in this paper, we can assign a color to $f$ in time $O(|B| \cdot |H|)$, where $B$ is the set of functions for which we searched for a new odd cycle containing an even number of fixed points. Note that $|B|$ is upper bounded by the number of functions also belonging to bipartite components previously colored by the algorithm. In other words, the algorithm is input-sensitive: before invoking the COLOR-VERTEX routine on $f$, we need to check all odd cycles used so far (in the order used) until we find one with an even number of fixed points with respect to $f$. We leave it as an open problem to find an algorithm that assigns a color in time $O(|H|)$ to a vertex $f$ from a bipartite component of $K_3^H$, so that the resulting coloring is a proper 3-coloring or possibly even a 2-coloring. Finally, we note that we do not know how to efficiently determine if a function $f$ belongs to a bipartite component (i.e., whether or not it contains at least one odd cycle with an odd number of fixed points).
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