On the universal principally polarized abelian variety of dimension 4 *

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1 Introduction

Let $A_g$ be the coarse moduli space for principally polarized abelian varieties of dimension $g$ over an algebraically closed field $k$, $\text{char} \ k \neq 2, 3$. In this paper we deal with the universal family

$$f : \mathcal{X}_g \rightarrow A_g,$$

$T_g \subset \mathcal{X}_g$ defined by the following property: for $u \in U$ let $A = f^*u$ and let $\Theta = A : T_g$. Then $\Theta$ is a principal polarization on $A$ and $u$ is the moduli point of the p.p.a.v. $(A, \Theta)$. We will say that $T_g$ is the universal theta divisor on $A_g$.

In some sense the study of $\mathcal{X}_g$ parallels that of the universal Jacobian $Y_g$ over the moduli space $M_g$ of curves of genus $g$. For low $g$ some natural questions are still open: for instance one would like to know the Kodaira dimension of $\mathcal{X}_g$ and $Y_g$ for any $g$. Let us discuss a related problem:

For which values of $g$ is $\mathcal{X}_g$ uniruled, rationally connected, unirational or even rational?

Since $f$ is dominant none of these properties appears if the Kodaira dimension of $A_g$ is $\geq 0$, in particular if $A_g$ is of general type. This is actually the case for $g \geq 7$ due to the results of Freitag, Mumford and Tai ([F], [M1], [T]). So we are confined to $g \leq 6$.

If $g \leq 3$ then $\mathcal{X}_g$ is birational to $Y_g$. The rationality of $\mathcal{X}_1$ is well known, while the results about the rationality of moduli of pointed curves, [CF], should also imply the rationality of $\mathcal{X}_2$ and $\mathcal{X}_3$.

The genus 4, 5 and 6 cases are open. Notice also that $A_4$ and $A_5$ are known to be unirational, (see the survey [G] and [C], [D], [L], [ILS], [V]), while the Kodaira dimension of $A_6$ is unsettled. In this note we contribute to the genus 4 case of the previous problem proving the following:

Theorem (1) The universal principally polarized abelian variety over $A_4$ is unirational.
(2) The universal theta divisor over $A_4$ is unirational.

The proof relies on the theory of Prym varieties, on the beautiful geometry related to étale double covers of curves of low genus and on K3 surfaces endowed with a Nikulin involution. We conclude this introduction with a brief summary of it.

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Due to the Prym construction the universal family $\mathcal{X}_4$ is dominated by the moduli space of pairs

$$(\pi, d)$$

where $\pi : \tilde{C} \to C$ is a non split étale double cover of a smooth, irreducible curve of genus 5 and $d$ is an isolated, effective divisor on $\tilde{C}$ such that $\pi_* d \in |\omega_C|$. We prefer the moduli of triples

$$(\pi, d, L)$$

where $L = \mathcal{O}_{\tilde{C}}(l) \in \text{Pic}^8(\tilde{C})$ satisfies $h^0(L) = 3$ and $\pi_* l \in |\omega_C|$. By the Brill-Noether theory for special curves $\tilde{C}$ as above a 1-dimensional family of line bundles $L$ does exist on $\tilde{C}$. Therefore this new moduli space dominates $\mathcal{X}_4$ via the forgetful map $(\pi, d, L) \to (\pi, d)$.

The additional line bundle $L$ is geometrically useful: indeed let $i : \tilde{C} 	o \tilde{C}$ be the fixed point free involution induced by $\pi$, it turns out that $L \otimes i^* L \cong \omega_{\tilde{C}}$ and that the Petri map

$$
\mu : H^0(L) \otimes H^0(i^* L) \to H^0(\omega_{\tilde{C}})
$$

has corank one for a general $\tilde{C}$ as above. This is an important point: with a little extra work one can deduce from this property that the linear system $|\text{Im} \mu|$ defines an embedding

$$d \subset \tilde{C} \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8,$$

where the linear span of $\tilde{C}$ is a hyperplane in $\mathbb{P}^8$. Let $\iota$ be the projective involution exchanging the factors of $\mathbb{P}^2 \times \mathbb{P}^2$, in addition we can assume that $i = \iota/\tilde{C}$. In sections 3, 4 and 5 we use this embedding to construct a rational parameter space for the family of the 0-cycles $d$ as above. Essentially we construct a rational family of triples $(a, \tilde{S}, d)$ with the following properties:

1. $o = (y, B)$ where $B \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a smooth conic and $y = (a, b_1, b_2)$ is a triple such that: $(b_1, b_2) \in B \times B$, a is a set of 6 independent points in $\mathbb{P}^2 \times \mathbb{P}^2$, $q(a)$ is a set of 6 coplanar points where $q : \mathbb{P}^8 \to \mathbb{P}^5$ is the projection from the 2-dimensional projectivized eigenspace of $\iota$.

2. $\tilde{S}$ is a smooth K3 surface which is a complete intersection of a quadratic and a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$. Moreover $\iota$ restricts to a Nikulin involution on $\tilde{S}$ and $a \cup B \subset \tilde{S}$.

3. $a \cup b_1 \cup b_2$ is contained in a unique, smooth hyperplane section $\tilde{F}$ of $\tilde{S}$. Moreover $\iota$ restricts to a fixed-point-free involution on $\tilde{F}$ and finally

$$d \in |\mathcal{O}_F(a + b_1 + \iota(b_2))|.$$

We will say that $(a, \tilde{S}, d)$ is a marking 0-cycle of type 1. It turns out that the latter linear system is a pencil of divisors of degree 8 and that the family $\mathbb{D}_1$ of marking 0-cycle of type 1 is rational.

Given an element $(a, \tilde{S}, d) \in \mathbb{D}_1$ we have from it the pencil $\Lambda$ of all curves

$$\tilde{C} \in |\mathcal{O}_S(\tilde{F} + B + \iota^* B)|$$

passing through $d$ and having $\iota^*$-invariant equation. As we will see a general $\tilde{C} \in \Lambda$ is a smooth, irreducible curve of genus 9. In addition $\tilde{C}$ is endowed with the fixed-point-free involution $\iota/\tilde{C}$ and it is marked by the divisor $d$. Let $C = \tilde{C}/\iota >$ and let $\pi : \tilde{C} \to C$
be the quotient map, it turns out that \( \pi_*(d) \in \omega_C \) and that \( h^0(\mathcal{O}_C(d)) = 1 \). Hence \( \mathcal{O}_C(d) \) defines a point of the Prym variety \( P(\pi) \) of \( \pi \). Since \( P(\pi) \) is a 4-dimensional p.p.a.v. the pair \((\pi, d)\) defines a point of the universal family \( \mathcal{X}_4 \). Finally we consider the family \( \tilde{\mathcal{C}}_1 \) of all 4-tuples \((o, \tilde{S}, d, \tilde{C})\) and the map

\[ \phi_1 : \tilde{\mathcal{C}}_1 \to \mathcal{X}_4, \]

sending \((o, \tilde{S}, d, \tilde{C})\) to the moduli point of the pair \((P(\pi), d)\). \( \tilde{\mathcal{C}}_1 \) is a \( \mathbb{P}^1 \)-bundle over \( D_1 \), hence it is rational. We show in section 6 that \( \phi_1 \) is dominant, so that \( \mathcal{X}_4 \) is unirational.

The proof of the unirationality of the universal theta divisor over \( \mathcal{A}_4 \) is identical: in this case we use the family of marking 0-cycles of type 2. These are triples \((o, \tilde{S}, d)\) defined by the same conditions (1), (2), (3): the only difference is that \( d \in |\mathcal{O}_{\tilde{C}}(a + b_1 + b_2)| \). We omit further details.

**Acknowledgements** Prym varieties are nowadays a subject having a long recent history, starting perhaps in the early seventies of the last century. One of the main actors of this history is certainly Roy Smith, whose 65-th birthday is celebrated in this volume. So it is a pleasure and a honour, for the author of this paper, to contribute and to wish all the best to Roy.

**Remark 1.1.** As the referee pointed out, the results of this paper imply that the Prym moduli space \( \mathcal{A}_5 \) is unirational. Indeed \( \mathcal{A}_5 \) is the moduli space of étale double covers \( \pi : \tilde{C} \to C \) as above and it turns out that the map \( \rho : \tilde{\mathcal{C}}_1 \to \mathcal{A}_5 \), sending \((o, \tilde{S}, d, \tilde{C})\) to the moduli point of \( \pi \), is dominant.

About this one has however to mention that an independent and simple proof of the unirationality of \( \mathcal{A}_5 \) was recently worked out by Marco Lo Giudice in his Tesi di Dottorato, ([L]). An amplified and revised version of such a proof is contained in the preprint *The moduli space of étale double covers of genus 5 curves is unirational*, by E. Izadi, M. Lo Giudice and G. Sankaran, ([ILS]).

At the end of this introduction we wish to thank the referee: for his previous remark on \( \mathcal{A}_5 \), for some useful observations and finally for his careful and patient reading of the paper.

## 2 Preliminaries on Pryms and notations

Usually we will denote as

\[ \pi : \tilde{C} \to C \]

a non split étale double cover of a smooth, irreducible curve \( C \) of genus \( g \) and by

\[ i : \tilde{C} \to \tilde{C} \]

the fixed-point-free involution exchanging the sheets of \( \pi \). To give \( \pi \) is equivalent to giving a non trivial order two element \( \eta \in \text{Pic}^0(C) \). The Prym variety associated to \( \pi \) will be denoted by

\[ P(\pi), \]

as is well known \( P(\pi) \) is a principally polarized abelian variety of dimension \( g - 1 \). Let us recall a useful construction of \( P(\pi) \) described by Mumford in [M2]: consider the Norm map

\[ Nm : \text{Pic}^{2g-2}(\tilde{C}) \to \text{Pic}^{2g-2}(C), \]
sending $\mathcal{O}_{\tilde{C}}(\sum x_i)$ to $\mathcal{O}_C(\sum \pi(x_i))$. Each fibre of $Nm$ is the disjoint union of 2 copies of $P(\pi)$. For the point $\omega_C \in Pic^{2g-2}(C)$ it turns out that $Nm^{-1}(\omega_C) = P^+ \cup P^-$. where 

$$P^+ = \{ M \in Pic^{2g-2}(\tilde{C}) \mid Nm(M) = \omega_C, \ h^0(M) \text{ is even} \}$$

and

$$P^- = \{ M \in Pic^{2g-2}(\tilde{C}) \mid Nm(M) = \omega_C, \ h^0(M) \text{ is odd} \}.$$ 

For $M$ general in $P^-$ one has $h^0(M) = 1$ and one has $h^0(M) = 0$ for $M$ general in $P^+$. Let

$$\Xi = \{ M \in P^+ / h^0(M) \geq 2 \},$$

then $\Xi$ is a principal polarization on $P^+$: by definition $P(\pi)$ is the pair $(P^+, \Xi)$. We will use the following well known property, sometimes called Parity Lemma, (cfr. [B] prop. 3.4 and [M3]).

**Lemma 2.1.** Let $x \in \tilde{C}$ then $M \in P^-$ if and only if $M(x - i(x)) \in P^+$. 

**Proof.** Let $n \in Div^{2g-2}(\tilde{C})$ be an effective divisor such that $N := O_{\tilde{C}}(n) \in P^-$. If $N$ is general in $P^-$ then $x, i(x)$ are not in $Supp \ n$ and $h^0(N(x)) = 1$. Therefore $h^0(N(x - i(x))) = 0$ and the translation $M \rightarrow M(x - i(x))$ induces a birational isomorphism between $P^-$ and $P^+$. This implies the lemma.

The Brill-Noether theory is known for $\tilde{C}$, though $\tilde{C}$ is not a general curve ([W]). Let 

$$\mu : H^0(M) \otimes H^0(i^*M) \rightarrow H^0(\omega_{\tilde{C}})$$

be the Petri map, for a given $M \in P^+ \cup P^-$. Then $i^*$ acts as an involution on the above vector spaces and $\mu$ preserves eigenspaces. By definition the Prym-Petri map

$$\mu_- : [H^0(M) \otimes H^0(i^*M)]^- \rightarrow H^0(\omega_{\tilde{C}})^-$$

is the induced map between the corresponding $-1$ eigenspaces. As is well known $\mu$ is injective for a general $\tilde{C}$ as above and every $M$. Moreover consider the Prym-Brill-Noether scheme

$$W^r(\pi) := \{ N \in P^+ (P^-) \mid h^0(N) \geq r + 1, \ h^0(N) = r + 1 \mod 2 \}.$$ 

For any $\tilde{C}$ as above $Coker \mu$ is the tangent space to $W_r(\pi)$ at $M$. For general $\tilde{C}$ and $M$ the rank of $\mu$ is $(r+1)$ so that $dim Coker \mu = g - 1 - (r+1)$. 

Finally we recall that the Prym moduli space $R_g$ is by definition the moduli space of the étale double covering $\pi$. $R_g$ is irreducible. Let $p_g : R_g \rightarrow A_{g-1}$ be the Prym map sending the moduli point of $\pi$ to the moduli point of $P(\pi)$. Then $p_g$ is dominant for $g \leq 6$. In particular its degree is 27 for $g = 6$, (see [DS]).

**Some frequent notations:**
3 Conics and some useful 0-cycles in $\mathbb{P}^2 \times \mathbb{P}^2$

First, we fix some further notations and conventions: we begin by fixing an auxiliary projective plane $\Pi$. Let $V := H^0(\mathcal{O}_\Pi(1))$ and $\mathbb{P}^2 := \mathbb{P} V$ we will consider the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 := \mathbb{P}(V \otimes V)$.

The natural projection of $\mathbb{P}^2 \times \mathbb{P}^2$ onto the $i$-th factor will be denoted as $\pi_i : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$, $i = 1, 2$.

From the decomposition $V \otimes V = \wedge^2 V \oplus Sym^2 V$ we obtain the subspaces

$P^- := P \wedge^2 V = \Pi$ and $P^+ := P Sym^2 V = |\mathcal{O}_\Pi(2)|$

which are the projectivized eigenspaces of the involution $v_1 \otimes v_2 \to v_2 \otimes v_1$. We denote by $\iota : \mathbb{P}^8 \to \mathbb{P}^8$ the induced projective involution, while the natural linear projections onto $P^-$ and $P^+$ will be $p : \mathbb{P}^8 \to P^-$ and $q : \mathbb{P}^8 \to P^+$.

It is clear that $\mathbb{P}^2 \times \mathbb{P}^2$ is the space of ordered pairs $(l, l')$ of lines of $\Pi$ and that $i(l, l') = (l', l)$. It turns out that: (1) $p(l, l')$ is the point $x = l \cap l'$ of $\Pi$, (2) $q(l, l')$ is the singular conic $l + l'$ of $\Pi$. Notice that $\Delta := \mathbb{P}^2 \times \mathbb{P}^2 \cap P^+$

is the diagonal of $\mathbb{P}^2 \times \mathbb{P}^2$. $\Delta$ is embedded as a Veronese surface in the 5-dimensional space $P^+$. The variety of the bisecant lines to $\Delta$ is a well known cubic hypersurface, we denote it by $\Sigma$.

Finally we mention another well known fact i.e. that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbb{P}^- & \xrightarrow{p} & \mathbb{P}^8 & \xrightarrow{q} & \mathbb{P}^+ \\
\cup & & \cup & & \cup \\
\mathbb{P}^- & \xrightarrow{p/\mathbb{P}^2 \times \mathbb{P}^2} & \mathbb{P}^2 \times \mathbb{P}^2 & \xrightarrow{q/\mathbb{P}^2 \times \mathbb{P}^2} & \Sigma \\
\end{array}
$$

In particular $q/\mathbb{P}^2 \times \mathbb{P}^2$ is a finite 2:1 cover of $\Sigma$ branched along $\Delta$ and $\Delta = Sing \Sigma$.

Now we begin our constructions by defining a suitable family of 0-dimensional subschemes of length 6 in $\mathbb{P}^2 \times \mathbb{P}^2$. In the Grassmannian of 4-spaces of $\mathbb{P}^8$ let $\sigma$ be the Schubert cycle parametrizing all 4-spaces $\Lambda$ such that $dim \Lambda \cap P^- \geq 1$. Then $\sigma$ is rational and contains the open set

$\Lambda := \{ \Lambda \in \sigma / \Lambda$ is transversal to $\mathbb{P}^2 \times \mathbb{P}^2, \Lambda \cap P^- \text{ is a line, } \Lambda \cap i(\Lambda) \cap \mathbb{P}^2 \times \mathbb{P}^2 = \emptyset \}.$
For each $\Lambda \in A$ the scheme $Z = \Lambda \cdot \mathbb{P}^2 \times \mathbb{P}^2$ has length 6 and spans $\Lambda$. Moreover $\Lambda = \langle Z - z \rangle$ for each $z \in Z$: this just follows because $\mathbb{P}^2 \times \mathbb{P}^2$ is a 4-fold of minimal degree 6 in $\mathbb{P}^8$. The definition of $A$ also implies that
\[
q/\Lambda : Z \to q(Z)
\]
is bijective: if not we would have $q(x) = q(y)$ for two distinct points $x, y \in Z$ and hence $y = \iota(x)$. This implies $x, y \in \Lambda \cap \iota(\Lambda) \cap \mathbb{P}^2 \times \mathbb{P}^2$, which is excluded. From now on we identify $A$ to the family of schemes $Z = \Lambda \cdot \mathbb{P}^2 \times \mathbb{P}^2$, where $\Lambda \in A$. Over $A$ we consider the universal family:

**Definition 3.1.** $\mathcal{A}_0 := \{(Z, z) \in A \times \mathbb{P}^2 \times \mathbb{P}^2 / z \in Z\}$.

**Proposition 3.1.** $\mathcal{A}_0$ is rational.

**Proof.** Consider the map $\phi : \mathcal{A}_0 \to \mathbb{P}^*-\times \mathbb{P}^2 \times \mathbb{P}^2$ defined as follows. Let $(Z, z) \in \mathcal{A}_0$, then $z$ is a point of the scheme $Z \subset U$ and $l_Z := \langle Z \rangle \cap \mathbb{P}^-$ is a line: by definition $\phi(Z, z) := (l_Z, z)$. Note that the fibre of $\phi$ at $(l_Z, z)$ is open in the Grassmann variety $G(1, 5)$ parametrizing all the 4-spaces containing the plane $\langle l_Z, z \rangle$. It is standard to deduce from this that $\mathcal{A}_0$ is open in a $G(1, 5)$-bundle over $\mathbb{P}^* \times \mathbb{P}^2 \times \mathbb{P}^2$: we omit the details. Then $\mathcal{A}_0$ is rational.

Let $(Z, z) \in \mathcal{A}_0$: in $Z$ we can replace $z$ by $\iota(z)$, obtaining a second scheme we denote as $Z'$.

**Definition 3.2.** $\mathcal{A} := \{(Z', \iota(z)), (Z, z) \in \mathcal{A}_0\}$.

It is clear from the definition that $\mathcal{A}$ is birigular to $\mathcal{A}_0$, so that $\mathcal{A}$ is rational too.

**Proposition 3.2.** Let $(Z, z) \in \mathcal{A}_0$. Then the following properties hold:

1. $\langle Z' \rangle$ is a 5-dimensional space containing $\mathbb{P}^-$,
2. $\langle Z \rangle$ is a 4-dimensional space and $\langle Z \rangle \cap \mathbb{P}^-$ is a line,
3. $\langle q_s Z \rangle = \langle q_s \rangle$, moreover this is a plane in $\mathbb{P}^+$.

**Proof.** (1) Recall that $\langle Z \rangle$ is a 4-space intersecting $\mathbb{P}^-$ along a line and not containing $\iota(z)$. Since $\langle Z \rangle = \langle Z - z \rangle$ it follows that $\langle Z' \rangle$ is the 5-space spanned by $Z$ and $\iota(z)$. In particular $\langle Z' \rangle$ contains the line $l = \langle z, \iota(z) \rangle$ and $l$ intersects $\mathbb{P}^-$ in a point not in $\langle Z \rangle \cap \mathbb{P}^-$. Hence it follows $\mathbb{P}^- \subset \langle Z' \rangle$. (2) and (3) are obvious consequences of the definitions and of (1).

For simplicity we will use the notation $a$ both for an element of $\mathcal{A} \cup \mathcal{A}_0$ and for its corresponding scheme $Z$ or $Z'$, omitting to indicate the distinguished point of such a scheme unless it is necessary.

**Definition 3.3.** Let $a \in \mathcal{A}$, then $\mathbb{P}^2_a := \langle a \rangle$ and $\mathbb{P}^2_a^+ := \mathbb{P}^5_a \cap \mathbb{P}^+$.

**Remark 3.1.** $\mathbb{P}^2_a^+$ is the plane spanned by $q_s a$. By proposition 3.2 (1) $\mathbb{P}^5_a$ contains $\mathbb{P}^-$. Then $\mathbb{P}^5_a$ is spanned by $\mathbb{P}^2_a^+$ and $\mathbb{P}^-$ and hence it is $\iota$-invariant: the set of fixed points of $\iota/\mathbb{P}^5_a$ is $\mathbb{P}^- \cup \mathbb{P}^2_a^+$.
Let $P$ be a general 5-space through $P^5$: it is clear that $P$ is transversal to $P^2 \times P^2$ and that $P = P_a$ for some $a \in \mathbb{A}$. This implies that $P_a^5$ is transversal to $P^2 \times P^2$ for a general $a \in \mathbb{A}$.

**Assumption 3.1.** We always assume that $a$ is general in the above sense.

Now we consider in $\mathbb{A} \times P^2 \times P^2$ the open set $U$ of pairs $(a, b_1)$ such that the linear span $< a \cup b_1 >$ is 6-dimensional and transversal to $P^2 \times P^2$. This implies that for each $u = (a, b_1) \in U$

$$Y_u := < a \cup b_1 > \cap P^2 \times P^2$$

is a smooth sextic Del Pezzo surface. Then we consider over $U$ the universal family

$$\mathcal{Y} = \{ (u, b_2) \in U \times P^2 \times P^2 / b_2 \in Y_u \}.$$

**Proposition 3.3.** $\mathcal{Y}$ is rational.

**Proof.** Let $p : \mathcal{Y} \to U$ be the natural projection and let $\alpha : U \to \mathcal{Y}$ be the section which is so defined: if $u = (a, b_1)$, $\alpha(u)$ is the distinguished point of $a$. For each $u = (a, b_1)$ consider the map

$$\phi_u : Y_u \to P^2$$

defined by the linear system of hyperplane sections of $Y_u$ which are singular at $b_2$ and contain the point $\alpha(u)$. It is easy to see that this map is birational and to deduce that there exists a birational map $\phi : \mathcal{Y} \to U \times P^2$ such that $\phi/Y_u = \phi_u$. Since $\mathbb{A}$ is rational, this implies that $\mathcal{Y}$ is rational.

With some abuse we will still denote by $\mathcal{Y}$ its open set defined by the following condition (*):

(*) let $(u, b_2) \in \mathcal{Y}$ with $u = (a, b_1)$ then:

1. $b_1, b_2$ are distinct points not in $P^5_a \cup \Delta$ and such that $b_2 \neq \iota(b_1)$,
2. their projections $\pi_i(b_1)$ and $\pi_i(b_2)$ are distinct points in $P^2$, for $i = 1, 2$.
3. Any 7 points of $a \cup \{ b_1, \iota(b_1), b_2, \iota(b_2) \}$, no two exchanged by $\iota$, are linearly independent.

**Remark 3.2.** For completeness we show that (*) is satisfied on a non empty open set of $\mathcal{Y}$: Let $u = (a, b_1) \in \mathcal{Y}$, it is clear that $(u, b_2)$ satisfies (1) and (2) if $b_2$ is general in $\mathcal{Y}_u$. It follows from (1) that any $x \in \beta := \{ b_1, b_2, \iota(b_1), \iota(b_2) \}$ is not in $P^5_a$, so that the seven points of $a \cup x$ are linearly independent. Let $(x, y)$ be one of the following pairs: $(b_1, b_2)$, $(b_1, \iota(b_2))$, $(\iota(b_1), b_2)$, $(\iota(b_1), \iota(b_2))$. Since $(b_1, b_2)$ is general in $\mathcal{Y}_u \times \mathcal{Y}_u$, the same is true for the other pairs. Hence it suffices to prove (3) for a set of seven points $(a - a_i) \cup x \cup y$, where $a_i$ is a point of $a$ for some $i = 1 \ldots 6$, $x = b_1$ and $y = b_2$. Let $H_i$ be the 4-space in $P^5_a$ spanned by $a - a_i$, then (3) is satisfied if $H_i \cap < xy > = \emptyset$. Now $< xy >$ is a general bisecant line to $Y_u$ because $(b_1, b_2)$ is general. So it is obvious that such a bisecant does not intersect $H_i$.

Equivalently $\mathcal{Y}$ is the family of triples $(a, b_1, b_2) \in \mathbb{A} \times (P^2 \times P^2)^2$ such that:

- the linear span $< a \cup b_1 \cup b_2 >$ is 6-dimensional and transversal to $P^2 \times P^2$,
Proposition 3.4. Let \( y = (a, b_1, b_2) \in \mathbb{Y} \), then \( \mathbb{P}^6_y := \langle a \cup b_1 \cup b_2 \rangle \) and \( \mathbb{P}^{3+}_y := \mathbb{P}^6_y \cap \mathbb{P}^+ \).

To continue with our elementary constructions we need now to consider the Hilbert scheme of conics in \( \mathbb{P}^2 \times \mathbb{P}^2 \). This is split in 3 irreducible components, according to the bidegree of a conic with respect to first Chern classes of \( \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 0) \) and \( \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, 1) \). We will be interested only in the component of bidegree \((1, 1)\), that is, in conics \( \mathbb{B} \) such that \( \pi_1(\mathbb{B}) \) and \( \pi_2(\mathbb{B}) \) are lines.

Definition 3.5. \( \mathbb{B} \) is the Hilbert scheme of smooth conics of bidegree \((1, 1)\) in \( \mathbb{P}^2 \times \mathbb{P}^2 \).

Proposition 3.4. \( \mathbb{B} \) is rational.

Proof. If \( B \in \mathbb{B} \) then \( B \subset L_1 \times L_2 \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \), where \( L_i \) is the line \( \pi_i(B) \) and \( i = 1, 2 \). Note that \( L_1 \times L_2 \) is embedded in \( \mathbb{P}^2 \times \mathbb{P}^2 \) as a smooth quadric, so that \( B \in (\mathcal{O}_{L_1 \times L_2}(1)) = \mathbb{P}^3 \). On the other hand \( (L_1, L_2) \) is a point of \( \mathbb{P}^{2*} \times \mathbb{P}^{2*} \). It is easy to conclude that \( \mathbb{B} \) is biregular to an open set of the \( \mathbb{P}^3 \)-bundle on \( \mathbb{P}^{2*} \times \mathbb{P}^{2*} \), whose fibre at \((L_1, L_2)\) is \( |\mathcal{O}_{L_1 \times L_2}(1)| \). \( \square \)

Lemma 3.5. Let \( b_1, b_2 \) be distinct points in \( \mathbb{P}^2 \times \mathbb{P}^2 \) such that their projections \( \pi_i(b_1) \) and \( \pi_i(b_2) \) are distinct points of \( \mathbb{P}^2 \), for \( i = 1, 2 \). Then the family of conics

\[ \{ B \in \mathbb{B} / b_1, b_2 \in B \} \]

is a pencil of plane sections of a smooth quadric surface in \( \mathbb{P}^2 \times \mathbb{P}^2 \).

Proof. Let \( B \in \mathbb{B} \) be a conic containing \( b_1, b_2 \). Then \( L_i := \pi_i(B) \) is the line joining \( \pi_i(b_1) \) to \( \pi_i(b_2) \), for \( i = 1, 2 \). Moreover it is obvious that \( B \subset L_1 \times L_2 \) and that \( L_1 \times L_2 \) is embedded in \( \mathbb{P}^2 \times \mathbb{P}^2 \) as a smooth quadric surface. This implies the statement. \( \square \)

As a further step we introduce a suitable family of pairs \( (y, B) \in \mathbb{Y} \times \mathbb{B} \).

Definition 3.6. \( \mathcal{O} \) is the family of pairs \((y, B) \in \mathbb{Y} \times \mathbb{B} \) such that:

\( y = (a, b_1, b_2) \) and \( \{ b_1, b_2 \} \subset B \),

\( \text{the linear span of } \langle \mathbb{P}^6_y \cup B \rangle \text{ is a hyperplane in } \mathbb{P}^8 \text{ transversal to } \mathbb{P}^2 \times \mathbb{P}^2. \)

Proposition 3.6. \( \mathcal{O} \) is rational.

Proof. Let \( p : \mathcal{O} \rightarrow \mathbb{Y} \) be the projection map, \( \forall y = (a, b_1, b_2) \) we have that \( p^*(y) \) is open in \( \mathbb{P}^4 \). Indeed let \( \pi_i(B) = L_i \), \( (i = 1, 2) \), then \( p^*(y) \) is the family of pairs \((a, B) \) such that \( B \) is a smooth plane section of \( L_1 \times L_2 \) passing through \( b_1, b_2 \). It is standard to conclude that then \( p : \mathcal{O} \rightarrow \mathbb{Y} \) realizes \( \mathcal{O} \) as an open subset of a \( \mathbb{P}^1 \)-bundle over \( \mathbb{Y} \). Hence \( \mathcal{O} \) is rational. \( \square \)

Definition 3.7. Let \( o = (y, B) \in \mathcal{O} \), we denote the hyperplane of \( \mathbb{P}^8 \) spanned by \( \mathbb{P}^6_y \cup B \) as \( \mathbb{P}^7_o \), moreover we denote the 4-dimensional intersection of \( \mathbb{P}^7_o \) with \( \mathbb{P}^+ \) as \( \mathbb{P}^{4+}_o \).
Remark 3.3. Let us point out that, since $P_7^−$ contains $P^−$, its equation is $ι^∗$-invariant.

The fixed set of $ι/ P_7^−$ is $P^− \cup P_4^+$. Let us consider in addition the threefolds $\tilde{T}_o := P_7^− \cap P^2 \times P^2$ and $T_o := P_4^+ \cap \Sigma$.

$\tilde{T}_o$ is a smooth 3-fold whose hyperplane sections are sextic Del Pezzo surfaces. Moreover $ι$ acts on $\tilde{T}_o$ and the set of fixed points of $ι/\tilde{T}_o$ is the smooth rational normal quartic curve $R_o := \Delta \cap \tilde{T}_o \subset P_4^+$.

It is clear that $T_o$ is the quotient of $\tilde{T}_o$ by $ι/\tilde{T}_o$, in particular we have the commutative diagram

Proposition 3.7. The cubic 3-fold $T_o$ is the variety of bisecant lines to $R_o \subset P_4^+$.

Proof. Since $\Delta = Sing \Sigma$ it follows that $P_4^+ \cap \Sigma$ is the secant variety of $\Delta \cap P_4^+$. □

4 K3 surfaces and marking 0-cycles in $P^2 \times P^2$

Let $o \in \mathbb{O}$ then a smooth quadratic section of $\tilde{T}_o$ is a K3 surface. The K3 surfaces we want form a special family of such quadratic sections. To construct it let us define the universal divisor $\tilde{T} := \{(o, x) \in \mathbb{O} \times P^2 \times P^2 / x \in \tilde{T}_o\}$ and the universal cycle $Z := \{(o, x) \in \tilde{T} / x \in o \cup B, \text{ where } o = (y, B) \text{ with } y = (a, b_1, b_2)\}$ over the parameter space $\mathbb{O}$. Let $\alpha : \tilde{T} \to \mathbb{O}$ and $\beta : \tilde{T} \to P^2 \times P^2$ be the natural projections. Then, at the point $o = (a, B)$, the fibre of $\alpha$ is $\tilde{T}_o$, and the fibre of $\alpha/Z$ is $Z_o := a \cup B$. We consider the ideal sheaf $I_{Z/T}$ of $Z$ and then the sheaf $V := \alpha_*(I_{Z/T} \otimes \beta^* \mathcal{O}_{P^2 \times P^2}(2))$.

At the point $o \in \mathbb{O}$ the fibre of $V$ is the vector space $H^0(I_{Z_o/T_o}(2))$. Since $o$ is general the dimension of such a vector space is constant. Hence $V$ is a vector bundle. The involution $ι^*$ acts on each fibre of $V$. So we have in $V$ a natural subbundle whose fibre at $o$ is $H^0(I_{Z_o/T_o}(2))^+$, i.e. the $+1$ eigenspace of $ι^*_o$. The projectivization of such a subbundle will be denoted as $\tilde{S}_o$.

If $o \in \mathbb{O}$ then $\tilde{S}_o = |I_{Z_o/T_o}(2)|^+$. We describe some properties of such a linear system.
Proposition 4.1. If \( o \in \mathcal{O} \) and \( \tilde{S} \in \tilde{\mathcal{S}}_o \) are general then \( \tilde{S} \) is a smooth K3 surface. Furthermore:

1. \( \tilde{S} \) is the section of \( \tilde{T}_o \) by \( q^*Q \), for some \( Q \in |\mathcal{O}_{\mathbb{P}^4}(2)| \).
2. \( \iota/\tilde{S} \) is an involution with exactly eight fixed points, which are the points of \( \tilde{S} \cap \Delta \).
3. let \( S \) be the quotient of \( \tilde{S} \) by \( \iota/\tilde{S} \) then \( S = q(\tilde{S}) \), moreover \( S = \mathbb{P}^4_{y} \cap \Sigma \cap Q \).
4. let \( y = (a, b_1, b_2) \) and let \( \mathcal{I}_{a/\tilde{S}} \) be the ideal sheaf of \( a \) in \( \tilde{S} \) then the linear system

\[
| \mathcal{I}_{a/\tilde{S}}(1) |^+ 
\]

is a pencil and its general element is a smooth, irreducible canonical curve \( \tilde{F} \) of genus 7.

Proof. (1) Let \( \tilde{T} \) be a general section of \( \mathbb{P}^2 \times \mathbb{P}^2 \) by a hyperplane through \( \mathbb{P}^- \), then \( \tilde{T} \) is smooth and contains a smooth conic \( B \in \mathbb{B} \). Let \( \mathcal{J} \) be the ideal sheaf of \( \tilde{B} \) in \( \tilde{T} \), since \( \tilde{T} \) is generated by quadrics it follows that the base locus of \( | \mathcal{J}(2) | \) is exactly \( B \) and that the base locus of \( | \mathcal{J}(2) |^+ \) is \( B \cup \iota(B) \). This easily implies the following property: \( \forall \ x \in B \cup \iota(B) \) the set of the elements of \( | \mathcal{J}(2) |^+ \) which are singular at \( x \) has codimension 2. Then the locus of the elements which are singular at some \( x \in B \) has codimension 1. Hence a general \( \tilde{S} \in | \mathcal{J}(2) |^+ \) is smooth along \( B \cup \iota(B) \) and then it is smooth by Bertini’s theorem. Such a \( \tilde{S} \) is a K3 surface and a quadratic section of \( \tilde{T} \) defined by a \( \iota^* \)-invariant equation.

Hence \( \tilde{S} = \tilde{T_o} \cap q^*Q \), for some \( Q \in |\mathcal{O}_{\mathbb{P}^4}(2)| \).

(2) It follows from (1) that \( \iota/\tilde{S} \) is an involution on \( \tilde{S} \) whose set of fixed points is \( \Delta \cap \tilde{T} \cap \tilde{Q} \).

(3) Finally the surface \( S := q(\tilde{S}) \) is the quotient of \( \tilde{S} \) by \( \iota/\tilde{S} \) and it is a sextic K3 surface. More precisely \( S = \mathbb{P}^4 \cap \Sigma \cap Q \), where \( \mathbb{P}^4 \) is the 4-dimensional linear space spanned by \( q(\tilde{S}) \) in \( \mathbb{P}^4 \).

(4) Note that, for a general \( \tilde{T} \) as above, we certainly have \( \tilde{T} = \tilde{T_o} \) for some \( o = (y, B) \), where \( B \) is as above, \( y = (a, b_1, b_2) \) and \( a \in \mathbb{A}_1 \). This implies the statement. \( \square \)

The proposition describes a well known situation: by definition a Nikulin involution \( \iota \) on a K3 surface \( \tilde{X} \) is an involution with exactly 8 fixed points. In particular \( X := \tilde{X}/<\iota> \) is a K3 surface and the image of the set of fixed points by the quotient map is an even set of 8 nodes on \( X \).

Remark 4.1. Let \( V \) be the family of all K3 surfaces \( \tilde{X} \subset \mathbb{P}^2 \times \mathbb{P}^2 \) which are complete intersections of a hyperplane and a quadratic section and such that \( \iota(\tilde{X}) = \tilde{X} \). \( GL(3) \) admits a diagonal action on \( \mathbb{P}^2 \times \mathbb{P}^2 \) and hence on \( V \): it turns out that its GIT-quotient is the moduli space of K3 surfaces endowed with a Nikulin involution and with the genus two polarization \( (\pi_1/\tilde{X})^*\mathcal{O}_{\mathbb{P}^2}(1) \).

Fix \( o = (y, B) \in \mathcal{O} \) and \( \tilde{S} \in \tilde{\mathcal{S}}_o \) then \( \mathbb{P}^4_y \) is a hyperplane in the ambient space \( \mathbb{P}^7_o \) of \( \tilde{S} \). Let

\[
\tilde{F} := \mathbb{P}^6_y \cap \tilde{S} \]

then the equation of \( \tilde{F} \) is \( \iota^* \)-invariant and \( \iota/\tilde{F} \) is a fixed-point-free involution on \( \tilde{F} \). The curve

\[
F := q(\tilde{F}) \subset \Sigma \cap \mathbb{P}^3_y
\]

is the quotient of \( \tilde{F} \) by \( \iota/\tilde{F} \) and the quotient map is the étale double covering

\[
q/\tilde{F} : \tilde{F} \rightarrow F.
\]
Proposition 4.2. For general \( o = (y, B) \in \mathcal{O} \) and \( \tilde{S} \in \tilde{S}_o \) we have:

- \( \tilde{F} \) is a smooth canonical curve of genus 7 in \( P^6 \).
- \( \tilde{F} \) is endowed with the fixed point free involution \( \iota / \tilde{F} \).
- \( F \) is a smooth, canonical curve of genus 4 in \( P^3 + \).

Proof. Let \( \tilde{F} \) be a general section of \( \tilde{S} \) by a hyperplane containing \( P^- \). Then \( \tilde{F} \) is a smooth canonical curve of genus 7, \( \iota / \tilde{F} \) is a fixed-point-free involution and \( q / \tilde{F} : \tilde{F} \to F = q(\tilde{F}) \) is its quotient map. As in section 2 we consider the Norm map \( Nm : Pic^6(\tilde{F}) \to Pic^6(F) \) and the two connected components \( P^+ \) and \( P^- \) of \( Nm^{-1}(\omega_F) \). For a general \( N \in P^- \) we have \( N = O_{\tilde{F}}(a') \), where \( a' \) is a smooth divisor, \( q : a' \to q(a') \) is bijective and \( h^0(N) = 1 \). The latter equality implies that the 6 points of \( a' \) are linearly independent. Let \( z \in a' \) and let \( a'' = a' - z + \iota(z) \), then \( h^0(O_{\tilde{F}}(a'')) = 2 \). Indeed it follows from prop. 2.1 that \( N(i(z) - z)) \in P^+ \), that is, \( h^0(N(i(z) - z)) \) is even. On the other hand \( a'' \) is effective and \( h^0(O_{\tilde{F}}(a'')) \leq 2 \), because \( a'' - \iota(z) \) consists of 5 linearly independent points. Hence \( h^0(O_{\tilde{F}}(a'')) = 2 \).

Applying geometric Riemann-Roch to the canonical curve \( \tilde{F} \), the linear span \( \Lambda = \langle a'' \rangle \) is a 4-space, moreover \( \Lambda \) intersects \( P^- \) along a line. It is easy to conclude that the pair \( (a'', z) \), is a point of \( \mathcal{K} \). Finally let \( y' = (a', b_1, b_2) \) where \( b_1 + b_2 = B : \tilde{F} \), then \( \iota' = (y', B) \) is a point of \( \mathcal{O} \) and \( \tilde{S} \in \tilde{S}_o \). The statement clearly holds true for \( \iota' \). Hence it holds on an open dense subset of \( \tilde{S} \).

Keeping the above notation we have the commutative diagram

\[
\begin{array}{cccc}
P^- & \xrightarrow{p} & P^7_o & \xrightarrow{q} & P^4_o^+ \\
\cup & & \cup & & \cup \\
P^- & \xrightarrow{p / \tilde{S}} & \tilde{S} & \xrightarrow{q / \tilde{S}} & S \\
\cup & & \cup & & \cup \\
P^- & \xrightarrow{p / F} & \tilde{F} & \xrightarrow{q / \tilde{F}} & F \\
\end{array}
\]

We are now in position to define the main family of 0-cycles of this paper: in particular these are subschemes \( \mathcal{D} \subset P^2 \times P^2 \) of length 8 such that \( q_d \) is the base locus of a net of quadrics of \( P^3 \). So, for a general \( d, q, \mathcal{D} \) is a hyperplane section of a smooth canonical curve in \( P^3 \). Consider a general \( o = (y, B) \in \mathcal{O} \) and a general \( \tilde{S} \in \tilde{S}_o \), then consider the curves \( F = P^6_o \cap \tilde{S} \) and \( q(\tilde{F}) \). Let \( y = (a, b_1, b_2) \) then one has on \( \tilde{F} \) the smooth divisors of degree 8

\[
m_1 := a + b_1 + \iota(b_2) \quad \text{and} \quad m_2 = a + b_1 + b_2.
\]

Definition 4.1. A marking 0-cycle of type \( i = 1 \) or 2 is a triple \( (o, S, d) \) such that

\[
d \in | O_{\tilde{F}_i}(m_i) |.
\]

The family of all marking 0-cycles of type \( i \) will be denoted as \( \mathcal{D}_i \).
Proposition 4.3. \(| \mathcal{O}_{\tilde{C}}(m_i) \) is a base-point-free pencil for \( i = 1, 2 \).

Proof. By the definition of \( \mathcal{Y} \), \( m_i \) is not in a hyperplane. Since \( \tilde{F} \) is embedded in \( \mathbb{P}^6_y \) as a canonical curve, geometric Riemann-Roch implies \( h^1(\mathcal{O}_{\tilde{C}}(m_i)) = 0 \). Hence \( h^0(\mathcal{O}_{\tilde{C}}(m_i)) = 2 \) and \( | m_i | \) is a pencil. This is base-point-free if the points of any divisor of degree 7 contained in \( m_i \) are linearly independent. Since \( y = (a, b_1, b_2) \in \mathcal{Y} \), the latter condition is satisfied by the definition of \( \mathcal{Y} \).

Consider the projection map \( \mathbb{D}_i \to \tilde{S} \) and observe that its fibre at \((o, \tilde{S})\) is \( | \mathcal{O}_{\tilde{C}}(m_i) | = \mathbb{P}^1 \). This, with some more standard work we omit for brevity, implies that

Proposition 4.4. \( \mathbb{D}_1 \) and \( \mathbb{D}_2 \) are \( \mathbb{P}^1 \)-bundles over \( \tilde{S} \), hence they are rational.

5 The family of marked curves \( \tilde{C} \)

Since we have all the ingredients, it is time to cook them up to construct a family of pairs

\[ (\pi, d) \]

where \( \pi : \tilde{C} \to C \) is an étale double covering and \( \tilde{C} \) is a curve of genus 9, marked by a divisor \( d \) satisfying the following condition: \( \pi_* d \in | \omega_C \) and \( h^0(\mathcal{O}_{\tilde{C}}(d)) = 1 \) or \( h^0(\mathcal{O}_{\tilde{C}}(d)) = 2 \).

Let \( P(\pi) \) be the Prym of \( \pi \): in the former case \( \mathcal{O}_{\tilde{C}}(d) \) is a point of the model \( P^- \) of \( P(\pi) \), in the latter \( \mathcal{O}_{\tilde{C}}(d) \) is a point of the theta divisor \( \Xi \) of \( P(\pi) \), embedded in its model \( P^+ \) (see section 2).

Once more we will consider a general point \( o = (y, B) \in \mathcal{Q} \) and a general surface \( \tilde{S} \) in the linear system \( \tilde{S}_o \). As usual let \( F = \mathbb{P}^6_y \cap \tilde{S} \) then we have on \( \tilde{S} \) the linear system

\[ | \tilde{F} + B + i^* B | . \]

Proposition 5.1. \( | \tilde{F} + B + i^* B | \) is a 9-dimensional and base-point-free linear system of smooth irreducible curves of genus 9.

Proof. Let \( H := \tilde{F} + B + i^* B \). Since \( F \) is base-point-free, \( | H | \) is base-point-free on the open set \( S - (B \cup i^* B) \). Note that \( \mathcal{O}_{\tilde{B} \cup i^* B}(H) \) is the trivial sheaf \( \mathcal{O}_{\tilde{B} \cup i^* B} \). Then \( | H | \) is base-point-free if the restriction map \( H^0(\mathcal{O}_S(H)) \to H^0(\mathcal{O}_{\tilde{B} \cup i^* B}(H)) \) is surjective. This follows from the vanishing of \( H^1(\mathcal{O}_S(\tilde{F})) \) and the long exact sequence associated to the standard exact sequence

\[ 0 \to \mathcal{O}_S(\tilde{F}) \to \mathcal{O}_S(H) \to \mathcal{O}_{\tilde{B} \cup i^* B}(H) \to 0 . \]

Since \( H^2 > 0 \) and \( | H | \) is base-point-free, a general \( \tilde{C} \in | H | \) is a smooth, irreducible curve of genus 9, moreover \( h^1(\mathcal{O}_S(H)) = 0 \), (see [SD]). In particular it follows \( \dim | H | = 9 \).

Observe that \( | F + B + i^* B | \) is invariant under the action of the involution \( \iota \). In this linear system we have therefore the projectivized eigenspace \( | F + B + i^* B | \) of \( \iota \).

Proposition 5.2. A general \( \tilde{C} \in | F + B + i^* B | \) is a smooth, irreducible curve of genus 9, endowed with a fixed-point-free involution \( \iota : \tilde{C} \to \tilde{C} \).
5 THE FAMILY OF MARKED CURVES \( \tilde{C} \)

Proof. With the usual notations let \( F = q(\tilde{F}) \) and \( S = q(\tilde{S}) \), then \( F \) is the section of \( S \subset \mathbb{P}^4_y \) by the hyperplane \( \mathbb{P}^4_y \). We know that \( S = \mathbb{P}^4_y \cap \Sigma \cap Q \) where \( Q \subset \mathbb{P}^4_y \) is a quadric hypersurface. In addition we also know that \( S \) is a K3 surface, singular exactly at the 8 nodes of \( \Delta \cap \Sigma \). Since \( \tilde{F} + B + \iota^* B \) is the projectivized +1 eigenspace of \( \iota/\tilde{S} \), it follows that \( \tilde{F} + B + \iota^* B \) spans a hyperplane in the 7-space \( F + B \), where \( B \) is the conic \( q_* B \). Since each element of \( \tilde{F} + B + \iota^* B \) is connected, the proposition follows if we show that a general \( C \subset F + B \) is smooth, irreducible. To prove such a property just apply the same proof used in the previous proposition.

Proposition 5.3. Let \( \tilde{C} \) be a smooth element of \( \tilde{F} + B + \iota^* B \) and let \( I_{\tilde{C}/T} \) be the ideal sheaf of \( \tilde{C} \) in the threefold \( \tilde{T} = \tilde{S} \cap \mathbb{P}^2 \times \mathbb{P}^2 \), then:

1. \( I_{\tilde{C}/T}(2) \) is acyclic and \( h^0(I_{\tilde{C}/T}(2)) = 3 \),
2. \( h^0(I_{\tilde{C}/T}(2))^- = 0 \).

Proof. (1) Consider the standard exact sequence of ideal sheaves

\[
0 \to \mathcal{I}_{\tilde{C}/\tilde{S}}(2) \to \mathcal{I}_{\tilde{C}/T}(2) \to \mathcal{I}_{S/T}(2) \to 0.
\]

We have \( \mathcal{I}_{\tilde{C}/\tilde{S}}(2) = \mathcal{O}_E(E) \) where \( E := F - B - \iota^* B \). Note that \( h^0(\mathcal{O}_E(E)) = 2 \). This follows because we assume \( B \) is general in its family of conics so that \( B \cap \iota^* B = \emptyset \). Then \( B \cap \iota^* B > 0 \) is a 5-space in \( \mathbb{P}^7 = \tilde{S} \) and this implies \( h^0(\mathcal{O}_E(E)) = 2 \). Since \( E^2 = 0 \), Riemann-Roch implies that \( \mathcal{O}_E(E) \) is acyclic. On the other hand \( \mathcal{I}_{S/T}(2) \) is just \( \mathcal{O}_T \), which is acyclic. Then \( \mathcal{I}_{\tilde{C}/T} \) is acyclic too and the associated long exact sequence yields \( h^0(I_{\tilde{C}}(2)) = 3 \).

(2) Note that \( C = q(\tilde{C}) \) is a canonical curve of genus 5 in the space \( \mathbb{P}^4_y \) previously considered. In particular observe that \( T = q(\tilde{T}) \) is a cubic hypersurface in \( \mathbb{P}^4_y \) and \( h^0(I_{\tilde{C}/T}(2)) = 3 \), where \( I_{\tilde{C}/T} \) is the ideal sheaf of \( C \) in \( T \). Then (1) implies that \( H^0(I_{\tilde{C}/T}(2)) = q^* H^0(I_{\tilde{C}/T}(2)) \). In particular each section \( s \in H^0(I_{\tilde{C}/T}(2)) \) is \( \iota^* \)-invariant, that is \( h^0(I_{\tilde{C}/T}(2))^- = 0 \).}

Lemma 5.4 (Parity lemma). Let \( d \subset \tilde{S} \) be a smooth scheme of length 8 such that:

1. \( \tilde{F} = \tilde{S} \cap \tilde{S} \) is a smooth hyperplane section transversal to \( B + \iota^* B \).
2. \( d \sim a + b + b'' \) where \( \{b', b''\} \subset B \cap \iota^* B \), \( a \) is effective and \( a \cap (B \cup \iota^* B) = 0 \).
3. \( a + b + b'' \) is a base-point-free pencil and \( d \) is general in it.
4. A general element of \( a + b' + b'' \) is included in a smooth \( \tilde{C} \subset \tilde{F} + B + \iota^* B \).

Then, for the ideal sheaf of \( d \) in \( \tilde{S} \), we have:

\[ h^0(I_{d/\tilde{S}}(\tilde{C})) = 2 \] if \( \#((b' + b'') \cap B) \) is odd,

\[ h^0(I_{d/\tilde{S}}(\tilde{C})) = 3 \] if \( \#((b' + b'') \cap B) \) is even.

Proof. If \( \#((b' + b'') \cap B) \) is odd we consider the standard exact sequence of sheaves

\[
0 \to I_{d/\tilde{S}}(\tilde{F}) \to I_{d/\tilde{S}}(\tilde{C}) \to J_d \to 0,
\]

where the map \( I_{d/\tilde{S}}(\tilde{F}) \to I_{d/\tilde{S}}(\tilde{C}) \) is the natural inclusion of ideal sheaves and \( J_d \) its cokernel. By (1) \( d \) spans a hyperplane in the 7-space \( \tilde{S} \), hence \( h^0(I_{d/\tilde{S}}(\tilde{F})) = 1 \). Since \( \tilde{C}B = \tilde{C}\iota^* B = 0 \) we have
To complete the proof we prove our previous claim that Proposition 5.5.

This consists of curves

By semicontinuity, the equality follows for a general $d$ if it holds for $a + b' + b''$. To prove that $h^0(I_{a+b'+b''}(\tilde{C})) = 2$ we remark that $B +\ i^* B$ is a fixed component of $| \ I_{a+b'+b''}(\tilde{C}) |$. This is clear because $\tilde{C}$ intersects both $B$ and $i^* B$, while we have $\tilde{C} B = \tilde{C} i^* B = 0$. Therefore it follows $h^0(I_{a+b'+b''}(\tilde{C})) = h^0(I_{a+i^* B}(\tilde{F}))$. Now $h^0(\mathcal{F}_{i}(a)) = 1$ because the pencil $| a + b' + b'' |$ is base-point-free, hence $h^0(I_{a+i^* B}(\tilde{F})) = 1$ that is $h^0(I_{a+i^* B}(\tilde{F})) = 2$.

If $|(b' + b'') \cap B|$ is even we can assume $b' + b'' \subset B$. We consider the standard exact sequence

and claim that $h^0(I_{d+i^* B}(\tilde{F} + B)) = 2$. This implies that $i^* B$ is a fixed component of the pencil $i^* B + \ | I_{d+i^* B}(\tilde{F} + B) |$ which is contained in $| I_{d+i^* B}(\tilde{C}) |$. Then assumption (4) implies

By semicontinuity the equality follows for a general $d$ if it holds for $d = a + b' + b''$. Since $b' + b''$ is in $B$ it follows, arguing as in the previous part of the proof, that $B$ is a fixed component of $| I_{d+i^* B}(\tilde{C}) |$ and that $h^0(I_{d+i^* B}(\tilde{C})) = h^0(I_{a+i^* B}(\tilde{F} + i^* B))$. Under assumption (2) we have also $a \cap i^* B = 0$. Therefore we have the following standard exact sequence of ideal sheaves

As above $h^0(\mathcal{O}_{\tilde{F}}(a)) = 1$ so that $h^0(I_{a+i^* B}(\tilde{F})) = 2$ and $h^1(I_{a+i^* B}(\tilde{F})) = 0$. Then it follows that $h^0(I_{a+i^* B}(\tilde{F} + i^* B)) = 3$.

To complete the proof we prove our previous claim that $h^0(I_{d+i^* B}(\tilde{F} + B)) = 2$: since $\tilde{F}$ is very ample, the map defined by $\tilde{F} + B | 0$ is a birational morphism $\sigma : \tilde{F} \to \tilde{F}_0$ onto its image $\tilde{F}_0$ and $a = \sigma(B)$ is a node, (cfr. [SD], see also [S] lemma 2.4). Since $\sigma(b') = \sigma(b'') = a, \mathcal{O}_{\tilde{F}}(a + b' + b'')$ descends to a line bundle $E$ on $\tilde{F}_0$ such that $h^0(E) = h^0(\mathcal{O}_{\tilde{F}_0}(a + b' + b'')) = 2$. By Riemann-Roch on $\tilde{F}_0$, it follows $h^0(\omega_{\tilde{F}_0} \otimes E^{-1}) = 1$. This easily implies that $h^0(I_{a+b'+b''}(\tilde{F} + B)) = 2$.

Keeping the previous notations let us consider now the linear system defined as follows:

This consists of curves $\tilde{C}$ containing $d$ and having a $i^*$-invariant equation. We want to apply the parity lemma to marking 0-cycles:

**Proposition 5.5.** Let $(o, \tilde{S}, d)$ be a general marking 0-cycle of type $i = 1$ or 2, then we have:

1. $| I_{d+i^* B}(\tilde{C}) |^{+}$ is a pencil.
2. A general $\tilde{C} \in | I_{d+i^* B}(\tilde{C}) |^{+}$ is a smooth, irreducible curve.
3. The dimension of $| I_{d+i^* B}(\tilde{C}) |$ is $i$. 


Proof. Let $\overline{B} = q(B)$ and $F = q(\overline{F})$ then we have
\[ |(I_d/S(\overline{F} + B + i^*B)) |^* \subset |\overline{F} + B + i^*B| |^* : = (q/S)^* | F + B | . \]

Since $(o, \tilde{S}, d)$ is a marking 0-cycle we have: $d \sim a + b' + b''$, where $a \in \mathbb{A}$ and $q_*, a \in |\omega_F |$. Moreover $b' \neq i(b'')$ so that $q_*(b' + b'') = F \cdot \overline{B}$. This implies that
\[ q_* d \in |\mathcal{O}_F(F + \overline{B}) | . \]

Consider the standard exact sequence of ideal sheaves
\[ 0 \rightarrow I_{F/S}(F + \overline{B}) \rightarrow I_{q_*.d/S}(F + \overline{B}) \rightarrow I_{q_.d/F}(F + \overline{B}) \rightarrow 0. \]

By construction $B \cap \text{Sing} S = \emptyset$, so that $I_{F/S}(F + \overline{B})$ is the line bundle $\mathcal{O}_S(\overline{B})$. Since $\overline{B}^2 = -2$ and $B$ is effective, Riemann-Roch and Serre duality imply $h^1(\mathcal{O}_S(\overline{B})) = 0$. On the other hand $q_*. d \in |\omega_F |$ implies that $I_{q_.d/F}(F + \overline{B}) \cong \mathcal{O}_F$. Passing to the associated long exact sequence it follows $h^0(I_{q_.d/S}(F + \overline{B})) = 2$. Hence $P := |I_{q_.d/S}(F + \overline{B}) |$ is a pencil as well as
\[ q^* P = |I_{d/S}(\tilde{C}) |^* . \]

Let us show that a general $C \in P$ is smooth: since $F + \overline{B} \in P$ it follows that $P$ has no fixed components. Indeed $F$ is not a fixed component because $B$ is isolated. If $B$ is a fixed component then the moving part of $P$ is an irreducible pencil of hyperplane sections of $S$, with base locus a scheme of length 6. This implies that two points of $q_* d$ are in $\overline{B}$, which is not true for a general $d$. Since $C^2 = 8$ we conclude that $q_* d$ is the base locus of $P$. Since $q_* d$ is smooth for a general $d$, a general $C \in P$ is smooth. This implies (1) and (2). (3) follows from parity lemma.

For a general $\tilde{C} \in |I_{d/S}(\tilde{F} + B + i^*B) |$ as above $\iota/\tilde{C}$ is a fixed-point-free involution. So $\tilde{C}$ is a smooth curve of genus 9 endowed with $\iota/\tilde{C}$ and marked by $d$. Let $C = \tilde{C}/ < \iota >$ and let
\[ \pi : \tilde{C} \rightarrow C \]
be the quotient map. Then $\pi = q/\tilde{C}$ and $C = q(\tilde{C})$ is a canonical curve of genus 5 in the 4-space $\mathbb{P}^4_u$, (here $o = (y, B)$ and $(o, \tilde{S}, d)$ is the marking 0-cycle considered in the previous statement).

**Proposition 5.6.** Let $d$ and $\pi : \tilde{C} \rightarrow C$ be as above, then $\pi_* d \in |\omega_C |$ and $h^0(\mathcal{O}_C(d)) = i$, where $i = 1, 2$ is the type of the marking 0-cycle $(o, \tilde{S}, d)$.

**Proof.** To see that $\pi_* d \in |\omega_C |$ just note that $\pi_* d$ is a hyperplane section of the canonical curve $C$. Indeed we have $\pi_* d = q_* d = q(\overline{F}) \cdot C$ and $q(\overline{F})$ is a hyperplane section of $S = q(\tilde{S})$. To see that $h^0(\mathcal{O}_C(d)) = i$ observe that, by (3) of the previous proposition, $h^0(I_{d/S}(\overline{F} + B + i^*B)) = i + 1$. But this is exactly equivalent to $h^0(\mathcal{O}_C(d)) = i$. \[ \square \]

**Remark 5.1.** Let $P(\pi)$ be the Prym variety of $\pi : \tilde{C} \rightarrow C$ and let $Nm : P^{i\mathbb{g}}(\tilde{C}) \rightarrow \text{Pic}(C)$ be the Norm map defined in section 2. Keeping the notations fixed there, the proposition implies that $\mathcal{O}_C(d) \in Nm^{-1}(\omega_C) = P^+ \cup P^-$. Since $h^0(\mathcal{O}_C(d)) = i$ with $i = 1$ or 2 we have in addition that
\[ \mathcal{O}_C(d) \in P^- \cup \Xi. \]
$P^-$ and $P^+$ are copies of $P(\pi)$ and $\Xi \subset P^+$ is a copy of the theta divisor of $P(\pi)$. So the proposition says that $\mathcal{O}_{\tilde{C}}(d)$ defines a point of $P(\pi)$ if $i = 1$ and of its theta divisor if $i = 2$.

**Definition 5.1.** Let $i = 1$ or 2 then $\tilde{C}$ is the étale double covering, where $(o, \tilde{S}, d, \tilde{C})$ is a marking 0-cycle of type $i$ and

$$\tilde{C} \in |\mathcal{I}_{d/\tilde{S}}(\tilde{F} + B_{a,o} + \iota^*B_{a,o})|^+.$$  

**Proposition 5.7.** $\tilde{C}_1$ and $\tilde{C}_2$ are rational varieties.

**Proof.** Let $i = 1$ or 2. It is clear that the natural projection map $\tilde{C}_i \rightarrow \Xi$ is a $\mathbb{P}^1$-bundle structure over $\Xi$, so we omit further details. Since $\Xi$ is rational, it follows that $\tilde{C}_i$ is rational. □

### 6 The unirationality results

Finally we use the constructions given in the previous sections to prove our main theorem which says that both the universal p.p.a.v. and the universal theta divisor over $\mathcal{A}_4$ are unirational varieties. Let

$$\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2,$$

moreover let

$$\tilde{C} \subset \tilde{C} \times \mathbb{P}^2 \times \mathbb{P}^2$$

be respectively the universal curve over $\tilde{C}$ and its image via the product map

$$\iota d \times q : \tilde{C} \times \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \tilde{C} \times \mathbb{P}^2.$$  

Restricting this map to $\tilde{C}$ we obtain a finite double covering of $\tilde{C}$-schemes

$$q : \tilde{C} \rightarrow \mathcal{C}.$$  

At $x = (o, \tilde{S}, d, \tilde{C}) \in \tilde{C}$ the fibre map $q_x : \tilde{C}_x \rightarrow \mathcal{C}_x$ is the étale double covering $\pi : \tilde{C} \rightarrow C$, where $C = q(\tilde{C})$ and $\pi = q/\tilde{C}$. Let us define the following line bundles associated to $x$:

$$M_x := \mathcal{O}_{\tilde{C}}(d) \text{ and } L_x := \mathcal{O}_{\tilde{C}} \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(1, 0).$$

$M_x$ has been studied in proposition 5.6. Let $i$ be the involution defined by $\pi$, then $i$ is exactly $\iota/\tilde{C}$. This implies that $i^*L_x = \mathcal{O}_{\tilde{C}} \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0, 1)$ and that $L_x \otimes i^*L_x \cong \mathcal{O}_{\tilde{C}}(1)$. Since $\mathcal{O}_{\tilde{C}}(1) \cong \omega_{\tilde{C}}$ it follows that

$$L_x \in Nm^{-1}(\omega_{\tilde{C}}) = P^+ \cup P^-.$$  

Here $Nm : Pic^8(\tilde{C}) \rightarrow Pic^8(C)$ is the Norm map defined in section 2. Recall also that, by definition, an element $M$ of $Nm^{-1}(\omega_{\tilde{C}})$ is in $P^+$, (in $P^-$), iff $h^0(M)$ is even, (odd). We want to study the Prym-Petri map of $L_x$, that is the multiplication map

$$\mu_{L_x} : [H^0(L_x) \otimes H^0(i^*L_x)]^- \rightarrow H^0(\omega_{\tilde{C}})^-.$$  

**Proposition 6.1.** $L_x$ is an element of $P^-$, $h^0(L_x) = 3$ and the Prym-Petri map of $L_x$ is injective.
Proof. First let us show that \( h^0(L_x) = 3 \). We put \( L := L_x \), then we recall that \( \tilde{C} \subset F + B + \epsilon^* B \) and that \( \tilde{F} \sim F_1 + F_2 \) where \( \tilde{F}_1 \in \mathcal{O}_S(1,0) \) and \( \tilde{F}_2 \in \mathcal{O}_S(0,1) \). Hence one has the standard exact sequence

\[
0 \to \mathcal{O}_S(-\tilde{F}_2 - B - \epsilon^* B) \to \mathcal{O}_S(\tilde{F}_1) \to L \to 0.
\]

Since \( h^0(\mathcal{O}_S(\tilde{F}_1)) = 3 \), it follows \( h^0(L) = 3 \) if \( h^1(\mathcal{O}_S(-\tilde{F}_2 - B - \epsilon^* B)) = 0 \). To prove this observe that \((F_2 + B + \epsilon^* B)B = (F_2 + B + \epsilon^* B)^\epsilon B = -1 \). Then the standard exact sequence

\[
0 \to \mathcal{O}_S(F_2) \to \mathcal{O}_S(F_2 + B + \epsilon^* B) \to \mathcal{O}_{B + \epsilon^* B}(F_2 + B + \epsilon^* B) \to 0
\]

implies \( h^0(\mathcal{O}_S(F_2 + B + \epsilon^* B)) = h^0(\mathcal{O}_S(F_2)) = 3 \). Applying Riemann-Roch and Serre duality we conclude that \( h^1(\mathcal{O}_S(F_2 + B + \epsilon^* B)) = h^1(\mathcal{O}_S(-F_2 - B - \epsilon^* B)) = 0 \).

We are left to show that the Prym-Petri map of \( L \) is injective. Consider the Petri map

\[
\mu : H^0(\tilde{L}) \otimes H^0(\epsilon^* \tilde{L}) \to H^0(\omega_{\tilde{C}})
\]

then \( \text{Im}(\mu) \) defines the embedding \( \tilde{C} \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \). This implies that \( \text{Ker} \, \mu \cong H^0(\mathcal{I}_{\tilde{C}}(1)) \), where \( \mathcal{I}_{\tilde{C}} \) is the ideal sheaf of \( \tilde{C} \) in \( \mathbb{P}^8 \). Now \( \text{Ker} \, \mu \) is 1-dimensional and it is generated by a \(+1\) eigenvector of \( i^* \). Indeed we know that \( \tilde{C} \subset \tilde{S} \subset \mathbb{P}^7 \), the equation of \( \mathbb{P}_7^0 \) being \( \epsilon^* \)-invariant. Hence there exists a non-zero \( h \in H^0(L) \otimes H^0(\epsilon^* L) \) such that \( i^* h = h \) and \( \mu(h) = 0 \). Since \( h \) is a \(+1\) eigenvector of \( i^* \), the injectivity of the Prym-Petri map of \( L \) follows if \( \dim \text{Ker} \, \mu = 1 \), that is \( h^0(\mathcal{I}_{\tilde{C}}(1)) = 1 \). To prove this consider the standard exact sequence of ideal sheaves on \( \mathbb{P}^8 \)

\[
0 \to \mathcal{I}_{\tilde{S}}(1) \to \mathcal{I}_{\tilde{C}}(1) \to \mathcal{I}_{\tilde{C}/\tilde{S}}(1) \to 0
\]

From \( h^0(\mathcal{I}_{\tilde{C}/\tilde{S}}(1)) = h^0(\mathcal{O}_S(-B - \epsilon^* B)) = 0 \) and \( h^0(\mathcal{I}_{\tilde{S}}(1)) = 1 \), it follows \( h^0(\mathcal{I}_{\tilde{C}}(1)) = 1 \).

\( L_x \) and \( M_x \) admit natural extensions to \( \tilde{C} \), indeed we have on \( \tilde{C} \) the following line bundles:

\[
\mathcal{L} := \mathcal{O}_{\tilde{C}} \otimes \beta^* \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0,1) \text{ and } \mathcal{M} := \mathcal{O}_{\tilde{C}}(D),
\]

where \( D = \{(o, \tilde{S}, d, \tilde{C}, z) \in \tilde{C} \times \mathbb{P}^2 \times \mathbb{P}^2 / z \in d \} \) is the universal marking 0-cycle and \( \beta \) is the second projection of \( \tilde{C} \times (\mathbb{P}^2 \times \mathbb{P}^2) \). For any \( x \in \tilde{C} \) the fibre \( \tilde{C}_x \) is \( \tilde{C} \), moreover the restrictions of \( \mathcal{L} \) and \( \mathcal{M} \) to \( \tilde{C}_x \) are respectively \( L_x \) and \( M_x \). Let us also define the following moduli spaces:

**Definition 6.1.** Let \( i \in \{1, 2\} \) then \( W_i \) is the moduli space of triples \((\pi, M, L)\) such that:

1. \( \pi : \tilde{C} \to C \) is a general étale double cover of a smooth, irreducible curve of genus 5,
2. \( M \) is a point in \( U_i := \{M \in P^- \cup P^+ / h^0(\mathcal{O}_C(d)) = i \} \).
3. \( L \) is general in \( W^2(\pi) := \{M \in P^- / h^0(M) \geq 3\} \) i.e. its Prym-Petri map is injective.

**Remark 6.1.** \( W^2(\pi) \) is a Prym-Brill-Noether locus: it is a scheme defined as in [W]. Counting dimensions it follows that the Prym-Petri map of \( L \) cannot be injective if \( h^0(L) > 3 \). Therefore we are implicitly assuming \( h^0(L) = 3 \) in the above definition.
Remark 6.2. $U_1$ is open in the model $P^-$ of the Prym of $\pi$, more precisely $U_1 = P^- - W^2(\pi)$. A point $M \in U_2$ is just a point $M \in P^+$ such that $h^0(M) = 2$. Therefore $U_2$ is open in the model $\Xi \subset P^+$ of the theta divisor of the Prym of $\pi$. Notice also that the restriction to $P^-$ of the natural theta line bundle of $Pic^8(C)$ is twice the principal polarization of the Prym, ([M2]).

Definition 6.2. $U_i$ is the moduli space of $(\pi, M)$ where $\pi$ and $M \in U_i$ are as above, $(i = 1, 2)$.

For $i = 1, 2$ let us consider the forgetful map

$$p_i : U_i \rightarrow A_5$$

onto the Prym moduli space $A_5$. At the moduli point of $\pi$ the fibre $U_1$ of $p_1$ is a copy of the Prym of $\pi$ and the fibre $U_2$ of $p_2$ is a copy of its theta divisor.

Proposition 6.2. (1) $U_1$ dominates the universal p.p.a.v. over $A_4$.
(2) $U_2$ dominates the universal theta divisor over $A_4$.

Proof. Recall that the Prym map $p_5 : A_5 \rightarrow A_4$ is dominant. Hence the family $p_i : U_i \rightarrow A_5$ dominates the universal family over $A_4$ via the natural map. \hfill \Box

Now let us consider the forgetful map

$$f_i : W_i \rightarrow U_i.$$  

The fibre of $f_i$ at the moduli point of $(\pi, M)$ is open in the Prym-Brill-Noether scheme $W^2(\pi)$. Therefore the general Prym-Brill-Noether theory can be applied to such a fibre and to $W_i$. In the next proposition we summarize what this theory implies in our particular situation:

Proposition 6.3. (0) Each irreducible component of $W_i$ has dimension $\geq \dim U_i + 1$.
(1) $W^2(\pi)$ is non empty and any of its irreducible components has dimension $\geq 1$.
(2) The Prym-Petri map $\mu_i^L$ has corank $\geq 1$ for each $\pi$ and $L \in W^2(\pi)$.
(3) $\mu_i^L$ is injective i.e. its corank is 1. This is the dimension of the tangent space at $L$ to the fibre $f_i^*(y) \subseteq W^2(\pi)$. (1) implies that $f_i^*(y)$ is a smooth curve at $L$ and (0) implies that $\dim T_{W_i, y} = \dim U_i + 1$ and $W_i$ is smooth at $y$. It follows that any irreducible component $Y$ of $W_i$ is smooth and dominates $U_i$ via $f/Y$. By (4) a general fibre of $f$ is irreducible, hence $Y = W_i$.\hfill \Box

Proposition 6.4. $f_i$ is dominant and $W_i$ is irreducible, $(i = 1, 2)$.  

Proof. Assume $(\pi, M)$ defines a general point $u$ of $U_i$. Then (1) implies that $W^2(\pi)$ is non empty. Since $\pi$ is general, (3) implies that $f_i^*(u) = W^2(\pi)$. Hence $f_i$ is dominant. Let $Y$ be any irreducible component of $W_i$ and let $y \in Y$ be the moduli point of $(\pi, M, L)$. Since $y \in W_i$, $\mu_i^L$ is injective i.e. its corank is 1. This is the dimension of the tangent space at $L$ to the fibre $f_i^*(y) \subseteq W^2(\pi)$. (1) implies that $f_i^*(y)$ is a smooth curve at $L$ and (0) implies that $\dim T_{W_i, y} = \dim U_i + 1$ and $W_i$ is smooth at $y$. It follows that any irreducible component $Y$ of $W_i$ is smooth and dominates $U_i$ via $f/Y$. By (4) a general fibre of $f$ is irreducible, hence $Y = W_i$.\hfill \Box
For $i = 1, 2$ let $x \in \tilde{C}_i$, then the fibre map $q_x : \tilde{C}_x \to C_x$ is an étale double covering. Moreover let $M_x = O_{\tilde{C}_x} \otimes \mathcal{M}$ and $L_x = O_{\tilde{C}_x} \otimes \mathcal{M}$, by propositions 6.1 and 5.4, the triple $(q_x, M_x, L_x)$ defines a point of $W_i$. Then for $i = 1, 2$ the triple $(q : \tilde{C} \to C, \mathcal{M}, L)$ defines a natural morphism

$$\phi_i : \tilde{C}_i \to W_i$$

sending $x \in \tilde{C}_i$ to the moduli point of the triple $(q_x, M_x, L_x)$. Now let us fix a general point $z \in W_i$, our goal will be to show that $z \in \phi_i(\tilde{C}_i)$. We assume that $z$ is the moduli point of the triple

$$(\pi : \tilde{C} \to C, M, L).$$

Let $\mu : H^0(L) \otimes H^0(i^*L) \to H^0(\omega_{\tilde{C}})$ be the Petri map of $L$, then $\text{Im } \mu$ defines a map

$$f_\mu : \tilde{C} \to P^2 \times P^2 \subset P^8.$$

Up to a projective isomorphism it is not restrictive to assume $i \cdot f_\mu = f_\mu \cdot i$.

**Lemma 6.5.** For a general $z \in W_i$, the map $f_\mu$ is an embedding, $(i = 1, 2)$.

**Proof.** The condition that $f_\mu$ is an embedding is open. Since it holds on $\phi_i(\tilde{C}_i)$ it is non empty. From now on we will assume that $f_\mu$ is an embedding and put $\tilde{C} = f_\mu(\tilde{C})$ so that

$$\tilde{C} \subset P^2 \times P^2 \subset P^8.$$

**Lemma 6.6.** For a general $z \in W_i$ and $i = 1, 2$ we have:

1. there exists a unique hyperplane containing $\tilde{C}$ and its equation is $i^*$-invariant.
2. Such a hyperplane is transversal to $P^2 \times P^2$ and to its diagonal $\Delta$.

**Proof.** (1) Ker $\mu$ is naturally isomorphic to the space of linear forms vanishing on $\tilde{C}$, let us show that $\text{dim Ker } \mu \geq 1$ for each $z \in U$. This follows from Brill-Noether theory, indeed note that

$$L \in W^2(\pi) \subseteq W := \{N \in \text{Pic}^3(\tilde{C}) / h^0(N) \geq 3\}.$$

$W$ is a Brill-Noether locus and the tangent space at $L$ to $W$ is Coker $\mu$. On the other hand we know that $\text{dim } W^2(\pi) \geq 1$, hence $\text{dim Coker } \mu \geq 1$ for each $z \in U$. Since $\mu$ is a linear map between vector spaces of the same dimension, $\text{dim Ker } \mu \geq 1$. We know that $\text{dim Ker } \mu = 1$ if $z \in \phi_i(\tilde{C})$. By semicontinuity the same is true for a general $z$. Finally $i^*$ acts as an involution on Ker $\mu$. Since the Prym-Petri map is injective no $-1$ vector is in Ker $\mu$ hence $h = i^*h, \forall h \in \text{Ker } \mu$.

(2) The transversality condition is open and non empty because each $z \in \phi_i(\tilde{C}_i)$ satisfies it. 

Cutting $P^2 \times P^2$ with the hyperplane $< \tilde{C} >$ we obtain a smooth threefold

$$\tilde{T} = < \tilde{C} > \cap P^2 \times P^2.$$
Lemma 6.7. For $i = 1, 2$ and a general $z \in \mathcal{W}_i$ one has:

1. $h^0(I_{\tilde{C}/\tilde{T}}(2)) = 3$ and the equation of each $\tilde{S} \in |I_{\tilde{C}/\tilde{T}}(2)|$ is $\tau^*$-invariant.

2. A general $\tilde{S} \in |I_{\tilde{C}/\tilde{T}}(2)|$ is smooth and transversal to the diagonal $\Delta$.

3. A general $\tilde{S}$ contains a smooth conic $B$ such that $B \cap \tau^* B = \emptyset$, moreover

$$|\tilde{C} - \tilde{F}| = |\tilde{B} + \tau^* B|,$$

where $\tilde{F}$ is a hyperplane section of $\tilde{S}$.

Proof. (1) By 5.3 $I_{\tilde{C}/\tilde{T}}$ is acyclic and $h^0(I_{\tilde{C}/\tilde{T}}(2)) = 3$ if $z \in \phi_1(\tilde{C}_i)$. By semicontinuity the same is true for a general $z$. $\tau^*$ acts as an involution on $H^0(I_{\tilde{C}/\tilde{T}}(2))$: by 5.3 its $-1$ eigenspace is zero if $z \in \phi(\tilde{C})$. So this is generically true on $U$ and hence $\tau^*$ is the identity for a general $z$.

(2) If $z \in \phi_1(\tilde{C}_i)$ we know that $|I_{\tilde{C}/\tilde{T}}(2)|$ has minimal dimension two and that its general element $\tilde{S}$ is smooth and transversal to $\Delta$. Then the same is true for a general $z \in U$.

(3) Let $\tilde{S} \in |I(2)|$ be general and let $\tilde{F}$ be a hyperplane section of $\tilde{S}$: at first we want to show that $|\tilde{C} - \tilde{F}| \neq \emptyset$. For this observe that $\mathcal{O}_\tilde{C}(\tilde{F}) \equiv \omega_{\tilde{C}}$ and consider the standard exact sequence

$$0 \to \mathcal{O}_S(\tilde{F} - \tilde{C}) \to \mathcal{O}_S(\tilde{F}) \to \mathcal{O}_C(\tilde{F}) \to 0.$$ 

Since $h^0(\mathcal{O}_S(\tilde{F} - \tilde{C})) = h^1(\mathcal{O}_S(\tilde{F})) = 0$ the associated long exact sequence is

$$0 \to H^0(\mathcal{O}_S(\tilde{F})) \to H^0(\omega_{\tilde{C}}) \to H^1(\mathcal{O}_S(\tilde{F} - \tilde{C})) \to 0,$$

hence $h^1(\mathcal{O}_S(\tilde{F} - \tilde{C})) = 1$. From Riemann-Roch and Serre duality it follows $h^2(\mathcal{O}_S(\tilde{F} - \tilde{C})) = h^0(\mathcal{O}_S(\tilde{F} - \tilde{C})) = 1$ and $|\tilde{C} - \tilde{F}| \neq \emptyset$. If $z \in \phi(\tilde{C})$ then $|\tilde{C} - \tilde{F}|$ consists of one element $B + \tau^* B$, where $B$ is a smooth conic and $B \cap \tau^* B = \emptyset$. So the same is true for a general $z$. \qed

Since $z$ is the moduli point of the triple $(\pi, \tilde{C} \to C, M, L)$ we now study $M$. At first let us recall that $h^0(M) = i$, since $z \in \mathcal{W}_i$. Moreover $M = \mathcal{O}_C(d)$ where $d$ is an effective divisor of degree 8 such that $\pi_* d \in |\omega_C|$, of course we have

$$d \subset \tilde{C} \subset \tilde{S} \subset \mathbf{P}^2 \times \mathbf{P}^2.$$

Lemma 6.8. Let $z \in \mathcal{W}_i$ be general, let $i = 1, 2$ and let $I_{d/\tilde{S}}$ be the ideal sheaf of $d$ in $\tilde{S}$, then:

1. $d$ is contained in a unique hyperplane section $\tilde{F}$ of $\tilde{S}$.

2. $\tilde{F}$ is smooth, transversal to $B + \tau^* B$ and its equation is $\tau^*$-invariant.

3. $|d|$ is a base-point-free pencil on $\tilde{F}$.

4. $d \sim a + b' + b''$ where $b', b'' \in \tilde{F} \cap (B \cup \tau^* B)$ and $b'' \neq \tau(b')$.

5. $z \in \mathcal{W}_1$ if $\#(\{b' + b''\} \cap B)$ is odd and $z \in \mathcal{W}_2$ if $\#(\{b' + b''\} \cap B)$ is even.

6. The linear system $P_z := |I_{d/\tilde{S}}(\tilde{C})|$ is a pencil of smooth, irreducible curves.

If $o = (y, B)$, where $y = (a, b', b'')$, it follows that $(o, \tilde{S}, d)$ is a marking 0-cycle of type $i$. 

\[ \text{\textcopyright 2023} \]
Proof. (1) Note that, by 6.7 (1) the space \( \langle \tilde{S} \rangle \) is \( \iota^* \)-invariant, in particular the projectivized eigenspaces of \( \iota/ \langle \tilde{S} \rangle \) are \( \mathbb{P}^- \) and \( \mathbb{P}^{4+} := \langle \tilde{S} \rangle \cap \mathbb{P}^+ \). Notice also that \( C \) is not in a hyperplane of \( \langle \tilde{S} \rangle \) for degree reasons. On the other hand we know that \( q/C = \pi \) and \( \pi_d \in |\omega_C| \). Hence it follows that \( q(C) \) is \( C \) canonically embedded in \( \mathbb{P}^{4+} \) and that \( \langle q_* d \rangle \) is a hyperplane in \( \mathbb{P}^{4+} \). This implies that \( d \) is always contained in the hyperplane

\[
(q/ \langle \tilde{S} \rangle)^* \langle q_* d \rangle,
\]

whose equation is \( \iota^* \)-invariant. This implies that \( h^0(\mathcal{I}_{d/\tilde{S}}(1))^+ \geq 1 \) for each \( z \in \mathcal{W}_i \). By Proposition 4.3 the equality holds if \( z \in \phi_i(\tilde{C}_i) \). Hence, by semicontinuity, it holds generically on \( \mathcal{W}_i \).

(2) We have shown in (1) that \( < d > \) is a hyperplane whose equation is \( \iota^* \)-invariant. Actually \( < d > \) is the hyperplane of \( \langle \tilde{S} \rangle \) spanned by \( \mathbb{P}^- \) and \( \langle q_* d \rangle \). Since \( z \) is general we can assume that \( d \) is general in the irreducible family \( D_i \) of the effective divisors \( d' \) on \( \tilde{C} \) such that \( \pi_{*d'} \in |\omega_C| \) and \( h^0(O_{\tilde{C}}(d')) = 1 \). It is well known that the image of \( D_i \) via the push-down map \( \pi_* \) is open in \( |\omega_C| \), (cfr. [B1]). Therefore, for a general \( d'' \in \mathcal{P}' \), \( < d'' > \) is just a general hyperplane through \( \mathbb{P}^- \). Hence it is transversal to \( \tilde{S} \) and \( B + \iota^* B \) and the same holds for \( < d > \).

(3) Obviously no hyperplane of \( < d > \) contains \( d \). Since \( \tilde{F} \) is canonically embedded in \( < d > \) it follows that \( d \) is a non special divisor and that \( h^0(O_{\tilde{F}}(d)) = 2 \). To see that \( |d| \) is base-point-free observe that this is an open condition, which is satisfied on \( \phi_i(\tilde{C}_i) \) by Proposition 4.3.

(4) As usual we put \( C = q(\tilde{C}), F = q(\tilde{F}), \overline{B} = q(B + \iota^* B) \) and \( S = q(\tilde{S}) \). Then we consider the double covering \( q/S : \tilde{S} \to S \) and observe that \( C \sim F + \overline{B} \). On the other hand \( \pi_* d = q_* d \) is a hyperplane section of \( C \), so we can conclude that \( \pi_* d = F \cdot C \). Since \( C \sim F + \overline{B} \), it follows that

\[
\pi_* d \in |\omega_F(\overline{B})| = |\omega_F(\pi(b_1) + \pi(b_2))|.
\]

Since \( F \) is non hyperelliptic the divisor \( \pi(b_1) + \pi(b_2) \) is isolated on \( F \), it is also smooth because \( \tilde{F} \) is transversal to \( B + \iota^* B \) so that \( F \) is transversal to \( \overline{B} \). We consider the curve

\[
\Gamma = \{ \pi_* m, m \in |O_{\tilde{F}}(d)| \} \subseteq |\omega_F(\overline{B})|.
\]

As is well known the divisors of \( |\omega_F(\pi(b_1) + \pi(b_2))| \) passing through \( \pi(b_1) + \pi(b_2) \) form a hyperplane \( H \). Hence \( H \cap \Gamma \) is non empty and this implies that \( d \sim \pi_* (a + b' + b'') \) for some \( a \in \text{Div} \tilde{F} \) such that \( q_* a \in |\omega_F| \) and for some \( b', b'' \in B \cup \iota^* B \) such that \( b' \neq \iota(b') \). Since \( |O_{\tilde{F}}(d)| \) is base-point-free \( h^0(O_{\tilde{F}}(a)) = 1 \), hence the linear span of \( a \) is a 5-space \( \mathbb{P}^5_a \). Notice also that \( \mathbb{P}^5 \subset \mathbb{P}^5_a \) because \( q_* a \) is a plane. Hence it follows that \( q_* a \in \mathbb{A} \) for a general \( z \).

Note that, by the previous part of the proof, assumptions (1), (2), (3) of parity lemma are satisfied, while (4) is obvious because \( d \) is contained in the smooth curve \( \tilde{C} \). Then properties (5) and (6) follow from parity lemma 5.4 and proposition 5.5 (1).

Putting together the previous lemmas we can finally deduce that:

**Lemma 6.9.** The map \( \phi_i : \tilde{C}_i \to \mathcal{W}_i \) is dominant for \( i = 1, 2 \).

**Proof.** Let \( z \) be general in \( \mathcal{W}_i \), then \( z \) is the moduli point of \( (\pi : \tilde{C} \to C, M, L) \), where \( M = O_{\tilde{C}}(d) \) and \( d \) is a smooth, effective divisor of degree 8 as above. Due to the
previous lemmas we have the embeddings

\[ d \subset \tilde{C} \subset \tilde{S} \subset \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8, \]

where \( L \cong O_{\tilde{C}}(1, 0) \) and \( i = \iota/\tilde{C} \). Moreover we have the smooth conic \( B \subset \tilde{S} \) such that \( \tilde{C} - \tilde{F} \sim B + \iota^*B \) and the smooth hyperplane section \( \tilde{F} = \langle d \rangle \cap \tilde{S} \). We have also shown in the above Lemma 6.8 that, putting \( y = (a, b', b'') \) and \( o = (y, B) \), it follows that \((o, \tilde{S}, d)\) is a marking 0-cycle of type \( i \), hence \( x = (o, \tilde{S}, d, \tilde{C}) \in \tilde{C}_i \). On the other hand \( \phi_i(x) \) is the moduli point of the triple \((q_x, M_x, L_x)\), where \( q_x : \tilde{C} \to \tilde{C}/\langle \iota \rangle \) is the quotient map, \( M_x = O_{\tilde{C}}(d) \) and \( L_x = O_{\tilde{C}}(1, 0) \). Therefore \( \phi_i(x) = z \) and \( \phi_i \) is dominant.

**Theorem 6.10.** The universal principally polarized abelian variety over \( A_4 \) and the universal theta divisor over \( A_4 \) are unirational varieties.

**Proof.** We have seen that \( W_1 \) dominates the universal p.p.a.v. over \( A_4 \) and that \( W_2 \) dominates the universal theta divisor over \( A_4 \). On the other hand \( \phi_i : \tilde{C}_i \to W_i \) is dominant and \( \tilde{C}_i \) is rational. Hence the universal p.p.a.v. and the universal theta divisor over \( A_4 \) are unirational varieties.

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