A Remark on Integrable Poisson Algebras and Two Dimensional Manifolds

Sergio Albeverio* and Shao-Ming Fei

Institute of Mathematics, Ruhr-University Bochum, D-44780 Bochum, Germany
*SFB 237 (Essen-Bochum-Düsseldorf); BiBoS (Bielefeld-Bochum);
CERFIM Locarno (Switzerland)

Abstract

The relations between integrable Poisson algebras with three generators and two-dimensional manifolds are investigated. Poisson algebraic maps are also discussed.

*Alexander von Humboldt-Stiftung fellow.
On leave from Institute of Physics, Chinese Academy of Sciences, Beijing
Poisson algebras have been discussed widely in Hamiltonian mechanics and in the quantization of classical systems such as canonical quantization and Moyal product quantization, see e.g. [1, 2, 3, 4]. In this paper we investigate the relations between manifolds and Poisson algebras. We find that there exist general relations between integrable Poisson algebras with three generators and two-dimensional smooth manifolds, which gives manifest geometric meanings to the Poisson algebras and from which the Poisson algebraic maps can easily be discussed.

A symplectic manifold \((M, \omega)\) is an even dimensional manifold \(M\) equipped with a symplectic two form \(\omega\), see e.g. [5, 6, 7, 8]. Let \(d\) denote the exterior derivative on \(M\). By definition a symplectic form \(\omega\) on \(M\) is closed, \(d\omega = 0\), and non-degenerate, \(X \omega = 0 \Rightarrow X = 0\), where \(X\) is a (smooth) vector on \(M\) and \(\omega\) denotes the left inner product defined by \((X \omega)(Y) = \omega(X, Y)\) for any two vectors \(X\) and \(Y\) on \(M\). The non-degeneracy means that for every tangent space \(T_xM\), \(x \in M\), \(X \in T_xM\), if \(\omega_x(X, Y) = 0\) for all \(Y \in T_xM\), then \(X = 0\).

Infinitesimal symplectic diffeomorphisms are given by vectors. A vector \(X\) on \(M\) corresponds to an infinitesimal canonical transformation if and only if the Lie derivative of \(\omega\) with respect to \(X\) vanishes, \(\mathcal{L}_X \omega = X \omega + d(X \omega) = 0\). (1)

A vector \(X\) satisfying (1) is said to be a Hamiltonian vector field.

Since \(\omega\) is closed, it follows from (1) that a vector \(X\) is a Hamiltonian vector field if and only if \(X \omega\) is closed. Since \(\omega\) is non-degenerate, this gives rise to an isomorphism between vector fields \(X\) and one forms on \(M\) given by \(X \rightarrow X \omega\). Let \(\mathcal{F}(M)\) denote the real-valued smooth functions on \(M\). For an \(f \in \mathcal{F}(M)\), there exists a Hamiltonian vector field \(X_f\) (unique up to a sign on the right hand of the following equation) satisfying \(X_f \omega = -df\). (2)

We call \(X_f\) the Hamiltonian vector field associated with \(f\).

Let \(f, g \in \mathcal{F}(M)\), \(X_f\) and \(X_g\) be the Hamiltonian vector fields associated with \(f\) and \(g\) respectively. The Lie bracket \([X_f, X_g]\) is the Hamiltonian vector field of \(\omega(X_f, X_g)\), in
the sense that
\[
[X_f, X_g] \omega = \mathcal{L}_{X_f} (X_g \omega) - X_g (\mathcal{L}_{X_f} \omega) \\
= X_f d(X_g \omega) + d(X_f) (X_g \omega)) - X_g d(X_f \omega)
\]
\[
= -d(\omega(X_f, X_g)),
\]
where the Cartan’s formula for the Lie derivative \( \mathcal{L}_X = i_X \circ d + d \circ i_X \) of a vector \( X \) has been used. The function \(-\omega(X_f, X_g)\) is called the Poisson bracket of \( f \) and \( g \) and denoted by \([f, g]_{P.B.}\),
\[
[f, g]_{P.B.} = -\omega(X_f, X_g) = -X_f g.
\]
Since \( \omega \) is closed, the so defined Poisson bracket satisfies the Jacobi identity
\[
[f, [g, h]_{P.B.}]_{P.B.} + [g, [h, f]_{P.B.}]_{P.B.} + [h, [f, g]_{P.B.}]_{P.B.} = 0.
\]
Therefore under the Poisson bracket operation the space \( C^\infty(M) \) of all smooth functions on \((M, \omega)\) is a Lie algebra, called the Poisson algebra of \((M, \omega)\).

In general one calls Poisson algebra any associative, commutative algebra \( A \) over \( \mathbb{R} \) with unit, equipped with a bilinear map \([,]_{P.B.}\), called Poisson bracket satisfying

1) **Antisymmetry** \([f, g]_{P.B.} = -[g, f]_{P.B.}\),

2) **Derivation property** \([fg, h]_{P.B.} = f[g, h]_{P.B.} + g[f, h]_{P.B.}\),

3) **Jacobi identity** \([f, [g, h]_{P.B.}]_{P.B.} + [g, [h, f]_{P.B.}]_{P.B.} + [h, [f, g]_{P.B.}]_{P.B.} = 0\), for any \( f, g, h \in A \).

Now let \( A \) be a Poisson algebra with three generators \((x_1, x_2, x_3) = x\) and a Poisson bracket of the form
\[
[x_i, x_j]_{P.B.} = \sum_{k=1}^{3} \epsilon_{ijk} f_k,
\]
where \( \epsilon_{ijk} \) is the completely antisymmetric tensor and \( f_i, i = 1, 2, 3 \), are smooth real-valued functions of \( x \), restricted to satisfy the Jacobi identity:
\[
[x_1, [x_2, x_3]_{P.B.}]_{P.B.} + [x_2, [x_3, x_1]_{P.B.}]_{P.B.} + [x_3, [x_1, x_2]_{P.B.}]_{P.B.}
\]
\[
= \frac{\partial f_1}{\partial x_2} f_3 - \frac{\partial f_1}{\partial x_3} f_2 + \frac{\partial f_2}{\partial x_3} f_1 - \frac{\partial f_2}{\partial x_1} f_3 + \frac{\partial f_3}{\partial x_1} f_2 - \frac{\partial f_3}{\partial x_2} f_1 = 0.
\]

**[Definition 1]**. The Poisson algebra \([\mathfrak{g}]\) is said to be integrable if \( f_i \) satisfies
\[
\frac{\partial f_i}{\partial x_j} = \frac{\partial f_i}{\partial x_i}, \quad i, j = 1, 2, 3.
\]

3
Obviously the integrability condition (3) is a sufficient condition for the Poisson algebra (1) to satisfy the Jacobi identity.

Let $\mathcal{F}$ be the space of smooth real-valued functions of $x$, $x \in \mathbb{R}^3$. We consider the realization of the Poisson algebra $\mathcal{A}$ in $\mathbb{R}^3$ and will not distinguish between the symbols $x_i$ of the coordinates of $\mathbb{R}^3$ and the generators of $\mathcal{A}$. In the following $M$ will always denote a smooth two dimensional manifold smoothly embedded in $\mathbb{R}^3$.

[Theorem 1]. For a given integrable Poisson algebra $\mathcal{A}$ there exists a two dimensional symplectic manifold $M$ described by an equation of the form $F(x) = c$, $x \in \mathbb{R}^3$, with $F \in \mathcal{F}$ and $c$ an arbitrary real number, such that the Poisson algebra generated by the coordinates of $M$ coincides with the algebra $\mathcal{A}$.

[Proof]. A general integrable Poisson algebra is of the form (1),

$$[x_i, x_j]_{P.B.} = \sum_{k=1}^{3} \epsilon_{ijk}f_k,$$

where $f_i$, $i = 1, 2, 3$, satisfy the integrability condition (3). What we have to show is that this Poisson algebra can be described by the symplectic geometry on a suitable two dimensional symplectic manifold $(M, \omega)$, in the sense that the above Poisson bracket can be described by the formula (3), i.e., the Poisson bracket $[x_i, x_j]_{P.B.}$ is given by the Hamiltonian vector field $X_{x_i}$ associated with $x_i$ such that

$$[x_i, x_j]_{P.B.} = -X_{x_i}x_j = \sum_{k=1}^{3} \epsilon_{ijk}f_k,$$

(6)

with $x_i$ the coordinates of the two dimensional manifold $M$ in $\mathbb{R}^3$.

Let $X'_{x_i} \in \mathbb{R}^3$ be given by

$$X'_{x_i} \equiv \sum_{j,k=1}^{3} \epsilon_{ijk}f_j \frac{\partial}{\partial x_k}.$$

(7)

Then $X'_{x_i}$ satisfies

$$[x_i, x_j]_{P.B.} = -X'_{x_i}x_j = \sum_{k=1}^{3} \epsilon_{ijk}f_k.$$

A general two form on $\mathbb{R}^3$ has the form

$$\omega' = -\frac{1}{2} \sum_{i,j,k=1}^{3} \epsilon_{ijk}h_i dx_j \wedge dx_k,$$

(8)
where \( h_i \in \mathcal{F}, \ i = 1, 2, 3. \)

We have to prove that \( x \) can be restricted to a suitable 2-dimensional manifold \( M \subset \mathbb{R}^3 \) in such a way that \( X'_{x_i} \) coincides with the Hamiltonian vector field \( X_{x_i} \) and \( \omega' \) is the corresponding symplectic form \( \omega \) on \( M \).

A two form on a two dimensional manifold is always closed. What we should then check is that there exists \( M \subset \mathbb{R}^3 \) such that for \( x \) restricted to \( M \) the formula (2) holds for \( f = x_i \), i.e.,

\[
X'_{x_i} \omega' = -dx_i \quad x_i \in M, \quad i = 1, 2, 3. \tag{9}
\]

Substituting formulae (8) and (7) into (9) we get

\[
X'_{x_i} \omega' = -\frac{1}{2} \sum_{l,m,n=1}^{3} \epsilon_{lmn} h_l dx_m \wedge dx_n
\]

That is,

\[
(1 - f_2 h_2 - f_3 h_3) dx_1 + f_2 h_1 dx_2 + f_3 h_1 dx_3 = 0,
\]

\[
(1 - f_3 h_3 - f_1 h_1) dx_2 + f_3 h_2 dx_3 + f_1 h_2 dx_1 = 0,
\]

\[
(1 - f_1 h_1 - f_2 h_2) dx_3 + f_1 h_3 dx_1 + f_2 h_3 dx_2 = 0. \tag{10}
\]

Let us now look at the coefficient determinant \( D \) of the \( dx_i \) in the system (10). By a suitable choice of \( h_1, h_2, h_3 \) we can obtain that \( D \) is zero. This is in fact equivalent with the equation

\[
f_1 h_1 + f_2 h_2 + f_3 h_3 = 1 \tag{11}
\]

being satisfied. The fact that \( D = 0 \) implies that there exists indeed an \( M \) as above.

Substituting condition (11) into (10) we get

\[
f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = 0. \tag{12}
\]

From the assumption (4) we know that the differential equation (12) is exactly solvable, in the sense that there exists a smooth (potential) function \( F \in \mathcal{F} \) and a constant \( c \) such that

\[
F(x) = c \tag{13}
\]
and $\partial F/\partial x_i = f_i$. The above manifold $M$ is then described by (13).

Therefore for any given integrable Poisson algebra $\mathcal{A}$ there always exists a two dimensional manifold of the form (13) on which $X'_x$, in (6) is a Hamiltonian vector field and the Poisson bracket of the algebra $\mathcal{A}$ is given by $X'_x$ according to the formula (3),

$$[x_i, x_j]_{PB} = -X'_{x_i} x_j = \sum_{k=1}^{3} \epsilon_{ijk} f_k.$$

The two dimensional manifold defined by (13) is unique (once $c$ is given). Hence an integrable Poisson algebra is uniquely given by the two dimensional manifold $M$ described by $F(x) = c$.

Before going over to investigate the Poisson algebraic structures on general two dimensional manifolds, we would like to make some remarks on special symplectic properties of two dimensional manifolds.

[Proposition 1]. Let $M$ be a two dimensional manifold embedded in $\mathbb{R}^3$. If $\omega$ is a symplectic form on $M$, then for $\alpha(x) \neq 0$, $\forall x \in \mathbb{R}^3$, $\alpha^{-1} \omega$ is also a symplectic form on $M$.

[Proof]. As $M$ is two dimensional, the two form $\alpha^{-1} \omega$ is closed, i.e., $d(\alpha^{-1} \omega) = 0$, and $\alpha^{-1} \omega$ is nondegenerate, as $\omega$ is nondegenerate. Hence $\alpha^{-1} \omega$ is also a symplectic form on $M$.

[Proposition 2]. For $f, g, h \in \mathcal{F}(\mathbb{R}^3)$, if $[f, g]_{PB} = h$ on the symplectic manifold $(M, \omega)$, then $[f, g]_{PB} = \alpha h$ on the symplectic manifold $(M, \alpha^{-1} \omega)$, $\alpha(x) \neq 0$, $\forall x \in \mathbb{R}^3$.

[Proof]. From formulae (2) and (3), on the symplectic manifold $(M, \omega)$ the symplectic vector field $X_f$ satisfies $X_f \omega = -df$ and $[f, g]_{PB} = -X_f g = h$. If $\omega$ is changed to be $\alpha^{-1} \omega$, then $X_f$ becomes $\alpha X_f$ such that $\alpha X_f \alpha^{-1} \omega = -d f$. Therefore on $(M, \alpha^{-1} \omega)$, $[f, g]_{PB} = -\alpha X_f g = \alpha h$.

From the properties in Propositions 1 and 2 we define an equivalent class of Poisson algebras on two dimensional manifolds.

[Definition 2]. On a two dimensional manifold embedded in $\mathbb{R}^3$, a Poisson algebra $A$ is equivalent to a Poisson algebra $B$ if the Poisson bracket on $A$ is the same as the one on $B$, multiplied by some common non zero factor $\alpha(x)$, $\forall x \in \mathbb{R}^3$.

[Theorem 2]. For a given smooth two dimensional manifold $M$ embedded in $\mathbb{R}^3$ of the form $F(x) = 0$, $F \in \mathcal{F}$, $x \in \mathbb{R}^3$, $x$ generates a Poisson algebra with the following Poisson
[Proof]. We assume the symplectic form $\omega$ and the Hamiltonian vector field $X_x$ on $M$ are of the following general form:

\[
\omega = -\frac{1}{2} \sum_{i,j,k=1}^{3} \epsilon_{ijk} h'_i dx_j \wedge dx_k,
\]

\[
X_{x_i} = \sum_{j,k=1}^{3} \epsilon_{ijk} f'_j \frac{\partial}{\partial x_k}, \quad i = 1, 2, 3,
\]

$h'_i, f'_i \in F, i = 1, 2, 3$. From $X_{x_1} \omega = -dx_i$ we have

\[
(1 - f'_2 h'_2 - f'_3 h'_3)dx_1 + f'_2 h'_1 dx_2 + f'_3 h'_1 dx_3 = 0,
\]

\[
(1 - f'_3 h'_3 - f'_1 h'_1)dx_2 + f'_3 h'_2 dx_3 + f'_1 h'_2 dx_1 = 0,
\]

\[
(1 - f'_1 h'_1 - f'_2 h'_2)dx_3 + f'_1 h'_3 dx_1 + f'_2 h'_3 dx_2 = 0,
\]

where $dx$ are not independent since $F(x) = 0$ implies

\[
\sum_{i=1}^{3} \frac{\partial F(x)}{\partial x_i} dx_i = 0.
\]

Therefore the coefficient determinant of the system (16) is zero, which gives

\[
\sum_{i=1}^{3} f'_i h'_i = 1.
\]

Hence the system of equations (16) becomes

\[
\sum_{i=1}^{3} f'_i dx_i = 0.
\]

Equations (17) and (18) give rise to

\[
f'_i(x) = \alpha(x) \frac{\partial F(x)}{\partial x_i}, \quad i = 1, 2, 3,
\]

where $\alpha(x) \neq 0, \forall x \in \mathbb{R}^3$.

From (19) the Hamiltonian vector field (15) takes the form

\[
X_{x_i} = \alpha(x) \sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial F(x)}{\partial x_j} \frac{\partial}{\partial x_k}.
\]

This is unique in the sense of the equivalence in definition 2.
Using formula (3) we have
\[
[x_i, x_j]_{P.B.} = \alpha(x) \sum_{k=1}^{3} \epsilon_{ijk} \frac{\partial F(x)}{\partial x_k}.
\]  
(21)

This is just the formula (14) under the equivalence definition 2 of Poisson algebras on two dimensional embedded manifolds.

It should be noted that if \( F(x) = 0 \) defines a smooth two dimensional manifold \( M \) in \( \mathbb{R}^3 \), then \( \alpha(x)F(x) = 0, \alpha(x) \neq 0, \forall x \in \mathbb{R}^3 \), also defines the same manifold \( M \). By formula (14) we see that \( F(x) = 0 \) and \( \alpha(x)F(x) = 0 \) give rise to the same Poisson algebra under the algebraic equivalence given by definition 2.

As \( F \in \mathcal{F} \), we have that
\[
\frac{\partial}{\partial x_j} \left( \frac{\partial F(x)}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left( \frac{\partial F(x)}{\partial x_j} \right), \quad i, j = 1, 2, 3.
\]

Therefore the Poisson algebra given by (14) is by definition integrable and it is uniquely given by the manifold \( M \).

[Definition 3]. \( C \in \mathcal{F} \) is said to be a center of the Poisson algebra (14) if it satisfies 
\[
[x_i, C]_{P.B.} = 0, \quad i = 1, 2, 3.
\]

[Corollary 1]. The center elements of the Poisson algebra on the two dimensional manifold \( F(x) = 0 \) are \((1, F(x))\).

[Proof]. From formulae (3) and (20) we have
\[
[x_i, F(x)]_{P.B.} = -X_{x_i}F(x) = \sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial F(x)}{\partial x_j} \frac{\partial F(x)}{\partial x_k} = 0, \quad i = 1, 2, 3,
\]

from which the proof follows.

[Corollary 2]. For \( f, g \in \mathcal{F} \), if \( x \) satisfies the Poisson algebraic relations (14), then
\[
[f, g]_{P.B.}(x) = - \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial f}{\partial x_i} \frac{\partial F(x)}{\partial x_j} \frac{\partial g}{\partial x_k}.
\]  
(22)

[Proof]. The Hamiltonian vector field associated with \( x_i \) giving rise to the algebra (21) is given by formula (20) and satisfies
\[
X_{x_i} \omega = -dx_i.
\]  
(23)
On the other hand the Hamiltonian vector field $X_f$ associated with $f$ satisfies by definition

$$X_f | \omega = -df. \quad (24)$$

From (23) and (20) the right hand side of equation (24) reads

$$-df = -\sum_{i=1}^{3} \frac{\partial f}{\partial x_i} dx_i = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} (X_{x_i} | \omega)$$

$$= \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \left( \sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial F(x)}{\partial x_j} \frac{\partial}{\partial x_k} \right) \omega.$$

Substituting it into (24) we have, as $\omega$ is non degenerate,

$$X_f = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial f}{\partial x_i} \frac{\partial F}{\partial x_j} \frac{\partial}{\partial x_k}.$$

Therefore by definition

$$[f, g]_{P.B.}(x) = -X_f g = -\sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial f}{\partial x_i} \frac{\partial F}{\partial x_j} \frac{\partial g}{\partial x_k}. \quad (25)$$

Theorem 1 and Theorem 2 establish relations between two dimensional smooth manifolds and Poisson algebras with three generators. In what follows we study some properties related to smooth Poisson algebraic maps.

Let $A$ resp. $B$ be two integrable Poisson algebras with related two dimensional manifolds $M_A$ resp. $M_B$ defined by $F_A(x) = 0$ resp. $F_B(y) = 0$ in $\mathbb{R}^3$, where $x = (x_1, x_2, x_3)$ resp. $y = (y_1, y_2, y_3)$ are the generators of the algebra $A$ resp. $B$.

[Theorem 3]. If the smooth algebraic map $\tilde{y}(x) = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)(x)$ gives rise to the algebra $B$, then $\tilde{y}$ satisfies $F_B(\tilde{y}) = 0$.

[Proof]. From Theorem 2 the Poisson algebra $A$ is given by

$$[x_i, x_j]_{P.B.} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial F_A(x)}{\partial x_k}.$$

Using formula (22) of Corollary 2 we have

$$[\tilde{y}_i, \tilde{y}_j]_{P.B.}(x) = -\sum_{l,m,n=1}^{3} \epsilon_{l,m,n} \frac{\partial \tilde{y}_i}{\partial x_l} \frac{\partial F_A}{\partial x_m} \frac{\partial \tilde{y}_j}{\partial x_n}. \quad (25)$$
Since $F_A(x) = 0$, we have that the $x_i$, $i = 1, 2, 3$, are not independent. Without losing generality we take $x_1$ and $x_2$ to be independent. By using the relation

$$
\sum_{i=1}^{3} \frac{\partial F_A(x)}{\partial x_i} dx_i = 0,
$$
equation (23) becomes

$$
[\tilde{y}_i, \tilde{y}_j]_{P.B.}(x) = \frac{\partial F_A(x)}{\partial x_3} \left( \frac{\partial \tilde{y}_i}{\partial x_1} \frac{\partial \tilde{y}_j}{\partial x_2} - \frac{\partial \tilde{y}_i}{\partial x_2} \frac{\partial \tilde{y}_j}{\partial x_1} \right).
$$

(26)

From Theorem 2 the Poisson algebra $B$ is given by

$$
[y_i, y_j]_{P.B.} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial F_B(y)}{\partial y_k}.
$$

Hence if the smooth map $\tilde{y}(x) = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)(x)$ gives rise to the algebra $B$, then

$$
[\tilde{y}_i, \tilde{y}_j]_{P.B.}(x) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial F_B(\tilde{y})}{\partial \tilde{y}_k}.
$$

(27)

From (26) and (27) we obtain

$$
\frac{\partial F_A(x)}{\partial x_3} \left( \frac{\partial \tilde{y}_i}{\partial x_1} \frac{\partial \tilde{y}_j}{\partial x_2} - \frac{\partial \tilde{y}_i}{\partial x_2} \frac{\partial \tilde{y}_j}{\partial x_1} \right) = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\partial F_B(\tilde{y})}{\partial \tilde{y}_k}.
$$

(28)

(28) has three different equations for $i = 1$, $j = 2$ resp. $i = 2$, $j = 3$ resp. $i = 3$, $j = 1$.

Multiplying these equations by $\frac{\partial \tilde{y}_l}{\partial x_1}$ resp. $\frac{\partial \tilde{y}_l}{\partial x_1}$ resp. $\frac{\partial \tilde{y}_l}{\partial x_1}$, $l = 1, 2$, and summing these equations together we get

$$
\frac{\partial F_B(\tilde{y})}{\partial x_1} = 0, \quad l = 1, 2.
$$

Therefore $F_B(\tilde{y})$ is independent of $x_1$, $l = 1, 2$, and $F_B(\tilde{y}) = \text{constant}$. This constant can be taken to be zero, since addition of a constant does not change the Poisson algebraic structure of the manifold. ■

[Theorem 4]. If the map $\tilde{y}(x) = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)(x)$ satisfies $F_B(\tilde{y}) = 0$, where $x$ satisfies $F_A(x) = 0$, then $\tilde{y}$ generates the Poisson algebra $B$.

[Proof]. From Theorem 2 we know that there is a unique Poisson algebra $B$ associated with the manifold $M_B$ (up to the algebraic equivalence given in definition 2). Hence if $\tilde{y}$ satisfies $F_B(\tilde{y}) = 0$, then $\tilde{y}$ generates the algebra $B$. ■

Summarizing, we have discussed the relations between integrable Poisson algebraic structures and two-dimensional manifolds and have proved that there is a unique relation
between integrable Poisson algebras and two-dimensional smooth manifolds. We have also shown that the sufficient and necessary condition for a smooth Poisson algebraic map $\tilde{y}(x)$ to act from an integrable Poisson algebra $A$ into an integrable Poisson algebra $B$ is that both $F_A(x) = 0$ and $F_B(\tilde{y}) = 0$ are satisfied. These conclusions can be extended to the infinite dimensional case, see [9].

ACKNOWLEDGEMENTS: We would like to thank the A.v. Humboldt Foundation for the financial support given to the second named author.

References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd ed. Addision-Wesley, Benjamin/Cummings, Reagings, Mass.

[2] J.E. Moyal, *Quantum Mechanics as a Statistical Theory*, Proc. Cambridge Phil. Soc. 45 (1949) 99.

[3] M.V. Karasev and V.P. Maslov, *Nonlinear Poisson Brackets, Geometry and Quantization*, Translations of Mathematical Monographs, Vol. 119, America Mathematical Society, 1993.

[4] K.H. Bhaskara and K. Viswanath, *Poisson Algebras and Poisson Manifolds*, Pitman Research Notes in Mathematics Series 174, 1988.

[5] J. Sniatycki, *Geometric Quantization and Quantum Mechanics*, Springer Verlag, 1980.

[6] N. Woodhouse, *Geometric Quantization*, Oxford: Clarendon Press, 1980.

[7] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser Verlag 1994.

[8] B. Aebischer, M. Borer, M. Kälín, Ch. Leuenberger and H.M. Reimann, *Symplectic Geometry*, Progress in Mathematics, Vol. 124, Birkhäuser 1994.

[9] S. Albeverio and S.M. Fei, *Current Algebraic Structures over Manifolds: Poisson Algebras, q-Deformations and Quantization*, SFB-preprint, 1995.