Identification of Sparse Continuous-Time Linear Systems with Low Sampling Rate: Exploring Matrix Logarithms

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Abstract—A continuous time linear dynamic in state space form has an \( A \)-matrix that reveals the connection between the states. Many dynamical systems allow a a sparse structure for this matrix. This means that the right-hand sides of the dynamical equations depend only on a subset of the states. If the system is unknown, one can try to identify it from data samples. This paper addresses identification of such sparse continuous time dynamical systems. We consider linear noise-driven dynamical systems evolving in continuous time. The assumption is that the sampling period is not small enough to apply methods where the continuous time system is identified directly from data. Instead a discrete time system is inferred in an intermediate step. Due to aliasing, different branches of the matrix logarithm might provide different degree of sparsity. We shed light on this issue, which has largely been overlooked in the community before and provide theoretical results for when a unique solution exists up to a finite equivalence class. We also provide a mixed integer linear programming formulation corresponding to a simplified version of our problem.

I. INTRODUCTION

A continuous time linear dynamic in state space form has an \( A \)-matrix that reveals the connection between the states. Many dynamical systems allow a sparse structure for this matrix. The property of sparsity is present in large, often complex, systems, in nature, and biological applications. An example of the latter are the interactions of species such as genes and proteins in human cells; such interactions are modeled/approximated by stochastic/ordinary differential equations by [1]–[3]. Such models or networks in biology help to understand, for instance, metabolic pathways, interactions between DNAs/proteins, and furthermore contribute to pathology of, disease detection or even clinical treatment to complicate diseases such as Parkinson’s disease or Alzheimer’s disease. In order to perform such an analysis, identification of sparse systems or at least sparse topology turns to be more critical as more techniques have been available to acquire time-series data.

How to estimate continuous time system from measurement is an important part of system identification, see e.g. [4]–[6]. The typical approaches include direct methods that directly infer a continuous time system from data, and indirect methods, where first a discrete time model is estimated, which is then converted to continuous time (e.g. using “c2d” in MATLAB System Identification Toolbox [7]).

One practical rule to choose sampling frequencies is taking ten times the bandwidth of the underlying linear systems [8]. However, in biological systems, most time-series data are sampled considerably slower than this empirical frequency, e.g. “high time-resolution” time series in [9], and identification of continuous-time systems becomes particularly challenging. Choosing too low sampling frequencies may trigger the problem of “system aliasing”, that is, multiple continuous-time systems produce the exactly same output samples. This could be roughly understood in scalar cases as the alias effect from \( e^{i2\pi n} = 1 \) (any integer \( n \)). It deserves more considerations since it cannot be solved by requesting for increasing sampling rates, considering strict limits in biological experiments.

In this paper, we first clarified the ambiguity effect when choosing low sampling frequencies and then presented a definition of “system aliasing”. A “Nyquist-Shannon-like sampling theorem” was presented to determine the minimal sampling frequency that avoids the effect of “system aliasing”. What’s more, we recognize the observation that allowing “aliased” representations may lead to \( A \)-matrices that are sparser. One could resort to this feature to estimate the ground truth, when it is obvious to see that certain dynamics are not captured in samples. We performed a theoretical study to gain more understanding of “system aliasing”, and, in the end, a mixed-integer problem is formulated to accelerate the search of sparse systems.

II. LINEAR TIME-ININVARIANT SYSTEMS

This paper addresses the problem of estimating a sparse matrix \( A \) in a linear noise-driven dynamical system on state-space form. Consider the following dynamic system

\[
\dot{x}(t) = Ax(t) + d\omega(t)
\]

where \( x(t) \in \mathbb{R}^n \), \( A \in \mathbb{R}^{n \times n} \) is stable, and \( \omega(t) \) is a Wiener process with incremental covariance \( \Sigma dt \).

The discrete-time system that relates the values of the state variable \( x \) in (1) at the sampling times [8, p. 82-85] [4, chap. 2], in the weak sense [11], is given by

\[
x(t + 1) = A_d x(t) + v(t),
\]

where \( A_d(A) = \exp(hA) \), the white noise \( v(t) \) has mean zero and covariance matrix

\[
\Sigma_d(A, \Sigma) = \int_0^h \exp(A\tau) \Sigma \exp(A^T \tau) \, d\tau,
\]

and \( h \in \mathbb{R}^+ \) is the sampling period.
Let
\[ X_1 \triangleq [x(h), x(2h), \ldots, x(Nh)], \]
\[ X_2 \triangleq [x(0), x(h), \ldots, x((N - 1)h)], \]
where \( X_1, X_2 \in \mathbb{R}^{n \times N} \). Either the ML (maximum likelihood) or PEM (Predictive Expectation Maximization) estimate of \( A_d \) in (2) gives
\[ \hat{A}_d = X_1 X_2^T (X_2 X_2^T)^{-1}, \]  
noticing that the identification problem is a multivariable case (\cite[p. 206]{I}) and \( \Sigma \) is fully unknown. Due to the invariance of ML estimates under function transformation, it also means that the ML estimate of \( A \) in (1) is any \( \hat{A} \) that satisfies \( \exp(h\hat{A}) = A_d \).

### III. Ambiguities in the Matrix Logarithm

Supposing that \( \hat{A}_d \) has been estimated from data, identification of the continuous-time system is to solve \( \hat{A} \) from
\[ \exp(h\hat{A}) = \hat{A}_d, \]  
which is indicated by (2). It seems easy to solve (5) by just taking the logarithm. However, it is well known that the logarithm is ambiguous, so (5) has several (in fact infinitely many) solutions. As shown in Figure 1, we have two matrices \( A_1 \) and \( A_2 \) that both solve (5). \( A_1 \) is the standard choice,

\[
A_1 = \begin{bmatrix}
-3.7147 & -1.1778 & 0.1625 & 0.2600 \\
-0.9883 & -5.4242 & 0.5625 & 0.9199 \\
1.7815 & -0.2501 & 4.7786 & -1.1221 \\
0.8726 & 0.0143 & -0.6277 & -3.9102
\end{bmatrix}
\]

\[
A_2 = \begin{bmatrix}
-9.1872 & -1.7824 & -2.9096 & 3.1019 \\
-0.0000 & -5.8354 & -4.0022 & -4.0112 \\
10.2474 & 13.2820 & 0.0000 & -7.4828 \\
-13.7279 & 8.6646 & -2.7852 & -3.0612
\end{bmatrix}
\]

Fig. 1: For \( h = 1 \), \( A_d = \exp(hA_1) = \exp(hA_2) \).

the one MATLAB returns from \( \logm(Ad) \) (the principal logarithm, see Section IV). Nevertheless \( A_2 \) shows an additional structure: \( x_1 \) does not directly affect “the growth” of \( x_2 \) and \( x_3 \) does not directly affect the growth of itself. This may be useful structural information. Such sparsity patterns could help us to choose the right one among aliased systems when adopting a fairly low sampling frequency, by looking for sparse solutions of (5). Consequently we now turn to an analysis and description of possible solutions to the logarithm issue, which present an important concept, “system aliasing”, in the cases of low sampling rates.

**Definition 1.**
\[
\delta(A, M, h, \mathcal{F}) = \left\{ A^* \in \mathbb{R}^{n \times n} : M \in \mathbb{R}^{n^2 \times n^2}, h \in \mathbb{R}, A^* = \arg \min_{A \in \mathcal{F}} \|M \text{vec}(\exp(hA)) - M \text{vec}(\exp(h\hat{A}))\|_2 \right\},
\]
where \( \mathcal{F} \subseteq \mathbb{R}^{n \times n} \) contains \( A \).

With the general notation given in Definition 1, we present a definition of system aliasing only in terms of the \( A \) matrix in state-space representations and the sampling period \( h \), which does not depend on specific identification methods or data.

**Definition 2 (System aliasing).** Given \( A \in \mathcal{F} \) and \( h \in \mathbb{R}^+ \), if there exists \( \hat{A} \neq A \in \delta(A, I, h, \mathcal{F}) \) and \( \hat{A} \) is called system alias of \( A \) with respect to \( \mathcal{F} \). By default, choose \( \mathcal{F}(A) := \{ \hat{A} \in \mathbb{R}^{n \times n} : \max \{ \text{Im}(\text{eig}(\hat{A})) \} \leq \max \{ \text{Im}(\text{eig}(A)) \} \} \).

We are particularly interested in \( \delta(A, I, h, \mathcal{F}) = \{ A \} \), i.e. there is no problem of system aliasing. Note that the concept of system aliasing does not depend on specific data. It only depends on system dynamics (e.g. the \( A \)-matrix in (1)) and sampling frequencies. If the \( M \) matrix is specifically constructed by data instead of \( I \), \( \delta(A, M, h, \mathcal{F}) = \{ A \} \), where \( A \) denotes the ground truth, tells that the underlying system is identifiable from the given data (see [12, Sec. III-B]). Obviously if we have system aliasing for the system with a specific sampling frequency, without extra prior information on \( A \) (see Section VI), the system is not identifiable.

### IV. Characterization of matrix logarithms

This section briefly reviews the useful results on matrix logarithms and gives notations used throughout the text. The existence of a principal logarithm is guaranteed by the following result.

**Theorem 1** (principal logarithm [13, Thm. 1.31]). Let \( P \in \mathbb{C}^{n \times n} \) have no eigenvalues on \( \mathbb{R}^- \). There is a unique logarithm \( A \) of all of whose eigenvalues lie in the strip \( \{ z : -\pi < \text{Im}(z) < \pi \} \). We refer to \( A \) as the principal logarithm of \( P \) of write \( A = \log(P) \). If \( P \) is real then its principal logarithm is real.

Let \( G(h) = \{ z \in \mathbb{C} : -\pi/h < \text{Im}(z) < 2\pi/h \}, \) and \( \mathcal{A}(h) \) denote the set of real matrices in \( \mathbb{R}^{n \times n} \) whose eigenvalues all lie in the strip \( G(h) \). For convenience, we use \( \log(\cdot) \) for general primary matrix logarithms and \( \log(\cdot) \) for principal logarithms. Following the notations, \( \log(P) \) (\( P \in \mathbb{R}^{n \times n} \)) is the one among \( \log(\cdot) \)'s that lies in \( \mathcal{A}(1) \).

When \( \hat{A}_d \) has no real negative eigenvalues, one solution to (5) naturally follows by taking principal logarithms, \( \hat{A} = \log(\hat{A}_d)/h \), which is in \( \mathcal{A}(h) \). This approach has some drawbacks. The main problem is that, considering the non-uniqueness of the matrix logarithm, for large sampling periods \( h \), it is less likely that the result obtained by the principal logarithm happens to be the ground truth we desire to get. For example, if the estimated matrix \( \hat{A} = \log(\hat{A}_d)/h \) has eigenvalues in the strip with width equal to \( 2\pi \), we can only get the correct value by taking principal logarithms when the \( h \) is chosen to be smaller than \( 1/2 \). For a larger \( h \), the width of the corresponding strip \( G(h) \) turns to be smaller. This is illustrated in Figure 2.

**Theorem 2** (Gantmacher [13, Thm. 1.27]). Let \( P \in \mathbb{C}^{n \times n} \)
be nonsingular with the Jordan canonical form

\[ Z^{-1}PZ = J = \text{diag}(J_1, J_2, \ldots, J_p) \]

\[ J_k = J_k(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix} \in \mathbb{C}^{m_k \times m_k}. \]

Then all solutions to \( e^A = P \) are given by

\[ A = ZU \text{diag}(L_1^{j_1}, L_2^{j_2}, \ldots, L_p^{j_p})U^{-1}Z^{-1}, \]

where

\[ L_k^{j_k} = \log(J_k(\lambda_k)) + 2j_k\pi iI_{m_k}; \]

\[ \log(J_k(\lambda_k)) \text{ denotes} \]

\[ f(J_k) := \begin{bmatrix} f(\lambda_k) & f'(\lambda_k) & \cdots & f^{(m_k-1)(\lambda_k)}(\lambda_k) \\ f(\lambda_k) & \ddots & \ddots & \vdots \\ & \ddots & \ddots & f'(\lambda_k) \\ & & f(\lambda_k) & f(\lambda_k) \end{bmatrix} \]

with \( f \) the principal branch of the logarithm, defined by \( \text{Im}(\log(z)) \in (-\pi, \pi] \); \( j_k \) is an arbitrary integer; and \( U \) is an arbitrary nonsingular matrix that commutes with \( J \).

**Theorem 3** (classification of logarithms [13, Thm. 1.28]). Let the nonsingular matrix \( P \in \mathbb{C}^{n \times n} \) have the Jordan canonical form (6) with \( p \) Jordan blocks, and let \( s \leq p \) be the number of distinct eigenvalues of \( A \). Then \( e^A = P \) has a countable infinity of solutions that are primary functions of \( P \), given by

\[ A_j = Z \text{diag}(L_1^{j_1}, L_2^{j_2}, \ldots, L_p^{j_p})Z^{-1}, \]

where \( L_k^{j_k} \) is defined in (8), corresponding to all possible choices of the integers \( j_1, \ldots, j_p \), subject to the constraint that \( j_i = j_k \) whenever \( \lambda_i = \lambda_k \).

If \( s < p \) then \( e^A = P \) has nonprimary solutions. They form parametrized families

\[ A_j(U) = ZU \text{diag}(L_1^{j_1}, L_2^{j_2}, \ldots, L_p^{j_p})U^{-1}Z^{-1}, \]

where \( j_k \) is an arbitrary integer, \( U \) is an arbitrary nonsingular matrix that commutes with \( J \), and for each \( j \) there exist \( i \) and \( k \), depending on \( j \), such that \( \lambda_i = \lambda_k \) while \( j_i \neq j_k \).

Throughout the text, by default, we always deal with primary matrix functions, e.g., \( \exp(\cdot), \log(\cdot) \). Primary matrix functions refer to the ones defined via Jordan Canonical Form (e.g., other equivalent ways like Polynomial Interpolation, Cauchy Integral Theorem) [13, chap. 1]. The primary notion of matrix functions is of particular interest and most useful in applications [13], [14]. By default this article deals with primary matrix functions.

**V. NO SYSTEM ALIASING: THE MINIMAL SAMPLING FREQUENCY**

Provided with the definition of system aliasing, a question comes first: for what \((A, h)\), it holds that \( \mathcal{E}(A, I, h, \mathcal{I}(A)) = \{A\} \).

To make principal matrix logarithm \( \log(\cdot) \) well-defined, assume that \( \exp(hA) \) has no negative real eigenvalues. By Theorem 1 and 2, it always holds that \( \log(\exp(hA))/h \in \mathcal{E}(A, I, h, \mathcal{I}(A)) \). To avoid system aliasing, we have to force \( \log(\exp(hA))/h = A \) to be satisfied. It is equivalent to \( \text{eig}(A) \in \mathcal{G}(h) \).

Given no other information on the system, consider the identification problem of \( A \) using full-state measurement. The only way to find the unique estimation is to decrease the sampling period \( h \) until the ground truth falls into the strip of \( \mathcal{G}(h) \), and then use the principal logarithm, as illustrated in Figure 3. Otherwise, we would be bothered by system aliases of \( A \) and unable to make a decision, unless we know extra prior information on \( A \). For full-state measurement, identifiability is guaranteed by selecting appropriate \( h \) such that there is no system aliases.

**Theorem 4** (Nyquist-Shannon-like sampling theorem). To uniquely obtain \( A \) from \( A_d \) by taking the principal matrix logarithm, where \( A_d \) is identified from sampled data, the sampling frequency \( \omega \) (rad/s) must satisfy

\[ \omega \geq 2 \max \{ |\text{Im}(\lambda_i(A))|, \ i = 1, \ldots, n \}. \]

Equivalently, the sampling period \( h \) should satisfy

\[ h \leq \min \{ \pi/|\text{Im}(\lambda_i(A))|, \ i = 1, \ldots, n \}. \]

**Proof.** The proof immediately follows from the above explanation on principal matrix logarithms.

The meaning of Theorem 4 in continuous-time system identification can be understood by comparing with the Nyquist-Shannon sampling theorem in signal processing. The Nyquist-Shannon sampling theorem gives conditions on sampling frequencies, by looking at spectral information of signals, under which the continuous signal can uniquely
reconstructed from its discrete-time signal. Similarly, Theorem 4 addresses that the continuous-time LTI system can be uniquely reconstructed from the discrete-time system under a condition that is built based on the spectrum of the $A$ matrix.

VI. SYSTEM ALIASING AND BOUNDED CONSTRAINTS

In the previous section we hinted that the conditions on no system aliasing follows as a consequence of bounded eigenvalues. In this section we follow this path and explicitly formulate an optimization problem to deal with identification in the presence of system aliases.

Consider the case of full-state measurement of (1), and the sampling period $h$ is NOT chosen small enough such that $\delta(A, I, h, \mathcal{F}(\kappa)) = \{A\}$. Then finding out $A$ among its aliases need extra information, for instance, properties of $A$ known a priori. Here assume that the ground truth $A$ is the sparsest solution in $\delta(A, I, h, \mathcal{F}(\kappa))$ and $\kappa \in \mathbb{R}$ as an upper bound can be roughly estimated, where $\mathcal{F}(\kappa)$ will be defined after giving Definition 3. It implies that $A$ is chosen by solving the following optimization problem

$$\min_{\hat{A} \in \delta(A, I, h, \mathcal{F}(\kappa))} \| \hat{A} \|_0.$$  \hspace{1cm} (11)

We need to calculate $\delta(A, I, h, \mathcal{F}(\kappa))$ from data. Given the full-state measurement $(X_1, X_2)$, let $\hat{A}_d$ be an estimation of the $A$-matrix in the corresponding discrete-time state space representation. In the deterministic case or the stochastic case with infinite samples, $\hat{A}_d = X_1 X_2^T (X_2 X_2^T)^{-1}$ and $\hat{A}_d = \exp(hA)$. Hence,

$$\delta(A, I, h, \mathcal{F}(\kappa)) = \left\{ \hat{A} \in \mathcal{F}(\kappa) : \exp(h\hat{A}) = \hat{A}_d \right\}$$  \hspace{1cm} (12)

and $\delta(A, I, h, \mathcal{F}(\kappa)) \subseteq \mathcal{S}$, where

$$\mathcal{S} := \left\{ \hat{A} \in \mathbb{R}^{n \times n} : \exp(h\hat{A}) = \hat{A}_d \right\}.$$  \hspace{1cm} (13)

To formulate $\mathcal{F}(\kappa)$, we need to introduce a special norm of $A$, which is equivalent to the Frobenius norm up to a change of coordinates.

**Definition 3** ($Z$-weighted norm). Let $h_Z(A) = Z^{-1}AZ$, where $Z$ is the matrix defined in Theorem 3. Then the norm is defined as $\|h_Z(\cdot)\|_F = \| \cdot \|_F \circ h_Z$.

Since we assume that $\hat{A}_d$ is fixed, i.e., the data $X$ is not used in the optimization problems defined here, the matrix $Z$ is constant. One can observe that

$$\|h_Z(\hat{A})\|_F = \|\text{vec}(\hat{A})^T (Z^T \otimes Z^{-1})^T (Z^T \otimes Z^{-1}) \text{vec}(\hat{A})\|_F$$

is a proper $(Z^T \otimes Z^{-1})^T (Z^T \otimes Z^{-1})$-weighted vector norm in terms of vec$(\hat{A})$. Using $\|h_Z(\cdot)\|_F$ is on the one hand simplifying the analysis we conduct throughout this section, and on the other explicitly penalizes the imaginary part of the eigenvalues without “distorting” them through the transformation by $Z$.

Now we define $\mathcal{F}(\kappa)$ using the norm $\|h_Z(\cdot)\|_F$. The basic idea is that one should exclude such $A$'s whose imaginary parts of eigenvalues are too large, which implies their system response will show wild fluctuation. To make our assumption and the problem (11) practically meaningful, instead of $\mathbb{R}^{n \times n}$, we restrict $\mathcal{F}$ to be a norm bounded subset

$$\mathcal{F}(\kappa) = \left\{ \hat{A} \in \mathbb{R}^{n \times n} : \|h_Z(\hat{A})\|_F \leq \kappa \right\}.$$  \hspace{1cm} (14)

or

$$\mathcal{F}(\kappa) = \left\{ \hat{A} \in \mathbb{R}^{n \times n} : \kappa_l \leq \|h_Z(\hat{A})\|_F \leq \kappa_u \right\}.$$  \hspace{1cm} (15)

In the following we will show that the feasible set of (11) has only finite elements, which implies it can be solved at least by brutal force methods. Recall that the set $\mathcal{S}$ is countable according to Theorem 3.

Let $M := \text{diag}(m_1, m_2, \ldots, m_p)$, $\alpha := [j_1, j_2, \ldots, j_p]$ and $\beta := [\beta_1, \beta_2, \ldots, \beta_p]$, where $\log(\lambda_k) \triangleq \alpha_k + i\pi\beta_k$, $k = 1, \ldots, p$, and $j_k, \lambda_k$ are defined in Theorem 2. Then a function $d$ can be defined as

$$\mathcal{F}(\alpha, \beta) := \delta^T M \delta + (2\beta + \beta) \delta^T M \delta,$$  \hspace{1cm} (16)

where $\alpha, \beta \in \mathbb{Z}^p$. Moreover, it satisfies $\mathcal{F}(\alpha, \beta) = \mathcal{F}(0, \beta, \alpha)$, which follows by noticing

$$\mathcal{F}(\alpha, \beta) = (\beta + \beta)^T M \delta + (\beta + \beta) - (\beta + \beta)^T M (\alpha + \beta),$$  \hspace{1cm} (17)

1For stochastic cases, $\hat{A}_d$ is consistently estimated by Prediction Error Minimization or Maximum Likelihood methods [8], and $\lim_{N \to \infty} \mathbb{E}\|\hat{A}_d(N)\| = \| \exp(hA) \|$. If only finite samples are available, we cannot obtain the exactly equivalent $\delta(\hat{A}, I, h, \mathcal{F}(\kappa))$ from data.
Moreover, let $A_0$ denote a special matrix logarithm for which all $j_k$ $(k = 1, \ldots, p)$ in (8) are equal to 0.

**Definition 4** (equivalence relations). An equivalence relation “~” is defined on $S$ as a binary relation: for any $A_1, A_2 \in S$, $j^{(1)}$ and $j^{(2)}$ are defined for $A_1, A_2$, respectively, we say $A_1 \sim A_2$ if $J(j^{(1)}, j^{(2)} - j^{(1)}) = 0$.

**Lemma 5.** Let $S$ be the set defined in (13) and parametrized by $\delta$ in Theorem 3. For any $A_1, A_2 \in S$, $\|h_Z(A_1)\|_F = \|h_Z(A_2)\|_F$ if and only if $A_1 \sim A_2$.

**Proof.** Let $A_i := Z \text{diag}(L_1^{(j)}(i), \ldots, L_p^{(j)}(i))Z^{-1}$, where $i = 1, 2$, $L_k^{(j)} := \log J_k(\lambda_k) + 2j_k^i \pi I_m$, and all other notations are given in (9). By using (19) for $A_1, A_2$, we obtain

$$
\|h_Z(A_1)\|_F = \|h_Z(A_2)\|_F \iff \|h_Z(A_1)\|_F^2 - \|h_Z(A_0)\|_F^2 = \|h_Z(A_2)\|_F^2 - \|h_Z(A_0)\|_F^2 \iff J(j^{(1)}, j^{(2)} - j^{(1)}) = 0,
$$

which implies that $A_1 \sim A_2$ by definition. The first equality in (19) is due to the linear transformation $h_Z(A)$.

**Remark 1.** It is not necessary that $A_0$ is the principal matrix logarithm (consider the case when the principal logarithm does not exist), nor does it have to be the logarithm with the smallest (weighted) Frobenius norm.

**Lemma 6.** Given any $\bar{A} \in S$, there exist finite $A_i \in S$ that satisfies $A_i \sim \bar{A}$.

**Proof.** Let $j$ denote $j_1, \ldots, j_p$ of $\bar{A}$ in (8), and $j^{(i)}$ denotes $j_1^{(i)}, \ldots, j_p^{(i)}$ of $A_i \in S$. $\delta \triangleq j^{(i)} - j$, therefore $\delta \in \mathbb{Z}^p$, where $\geq$ denote the element-wise larger-or-equal relation. By Definition 4, it is equivalent to show that $J(j, \delta) = 0$ has finite solutions, given $j$. We require $\delta$ to satisfy the following condition:

$$
|\delta_i + j_i + \beta_i/2| \leq \sqrt{\frac{(j + \beta/2)^T M(j + \beta/2)}{m_i}}
$$

(20)

for all $i = 1, \ldots, p$. Otherwise, supposing that there exists $i \in \{1, \ldots, p\}$ such that $\delta_i$ does not satisfy (20), we will have

$$
J(j, \delta) = m_i(\delta_i + j_i + \beta_i/2)^2 + \sum_{k \neq i} m_k(\delta_k + j_k\beta_k/2)^2 - \sum_{k} m_k(\delta_k + j_k\beta_k/2)^2 > \sum_{k \neq i} m_k(\delta_k + j_k\beta_k/2)^2 \geq 0.
$$

Let $S' := \{ A_i \in S : j_k^{(i)} = \delta_k + j_k, \delta_k satisfies (20) \}$. We have $\{ A_i \in S : A_i \sim \bar{A} \} \subseteq S'$ and $S'$ is a finite set.

**Lemma 7.** There exists finite $A_i \in S$ such that $\|h_Z(A_i)\|_F \leq \kappa$.

**Proof.** Let $\kappa_0 \triangleq \|h_Z(A_0)\|_F$. Then we need to show there exists a finite number of $A_i \in S$ such that $\|h_Z(A_i)\|_F^2 - \|h_Z(A_0)\|_F^2 \leq \kappa^2 - \kappa_0^2$, which is equivalent to show that there exists a finite number of solutions $\delta \in \mathbb{Z}$ to $J(0, \delta) \leq (\kappa^2 - \kappa_0^2)/4\pi$. $\delta$ must satisfy the following condition:

$$
|\delta_i + \beta_i/2| \leq \sqrt{\frac{(\beta/2)^T M(\beta/2) + (\kappa^2 - \kappa_0^2)}{m_i}}
$$

(21)

for all $i = 1, \ldots, p$. Otherwise, by supposing that there exists $i \in 1, \ldots, p$ such that $\delta_i$ does not satisfy (21) leads to $J(0, \delta) = m_i(\delta_i + \beta_i/2)^2 + \sum_{k \neq i} m_k(\delta_k + \beta_k/2)^2 - (\beta/2)^T M(\beta/2) > \sum_{k \neq i} m_k(\delta_k + \beta_k/2)^2 + (\kappa^2 - \kappa_0^2) \geq \kappa^2 - \kappa_0^2$.

Note that the set of all $\delta \in \mathbb{Z}$ that satisfies (21) is finite, which finalizes the proof.

**Remark 2.** We could have a more precise bound of $\delta_i$. Let $\mu(i) := \min\{ J(0, \delta_i) : \delta_i \in [\beta/2 - \delta_k/2], \beta/2 + \delta_k/2), k \neq i; (\delta_j) = 0 \}$. We have the solution to $\mu(i)$, $i = 1, \ldots, p$:

$$
\mu(i) = \min\{ J(0, \delta_i) : \delta_i \in [\beta/2 - \delta_k/2], \beta/2 + \delta_k/2), k \neq i; (\delta_j) = 0 \}.
$$

(23)

**Theorem 8** (lower boundness of logarithms). Let $S$ be the set defined in (13). Given any $A \in S$, there exists $M(A) > 0$, such that for any $A \in \{ A \in S : A \sim \bar{A} \}$, it holds that

$$
\|h_Z(A)\|_F - \|h_Z(A)\|_F \geq M.
$$

**Proof.** Let $j$ denote $j_1, \ldots, j_p$ of $\bar{A}$ in (8). $N_{eqv}$ be the number of A’s that satisfy $A \sim \bar{A}$. Note that $\|h_Z(A)\|_F^2 - \|h_Z(A)\|_F^2 = (\|h_Z(A)\|_F^2 - \|h_Z(A_0)\|_F^2) - (\|h_Z(A)\|_F^2 - \|h_Z(A_0)\|_F^2) = J(j, \delta), \delta \in \mathbb{Z}$, which implies it is equivalent to show that $J(j, \delta), \delta \in \mathbb{Z}$ has a non-zero lower bound if not considering the $\delta$’s that result in $J(j, \delta) = 0$. We will prove it by contradiction. Assume this is not true, i.e. $\forall \epsilon > 0$ there exists $\delta$ such that $0 < J(j, \delta) < \epsilon$. It implies that, arbitrarily given $\epsilon > 0$, there exists an infinite number of solutions $\delta$ such that $J(j, \delta) < \epsilon$, which is impossible since $J(0, j + \delta) < J(0, j) + \epsilon$ (using the fact that $J(j, \delta) = J(0, j + \delta) - J(0, j)$) has a finite number of solutions provided by Lemma 7.

**Proposition 9** (uniform lower boundness of logarithms). Let $S$ be the set defined in (13). Given that all the eigenvalues of $A_d$ are real, for any $A_1 \sim A_2 \in S$, we always have

$$
\|h_Z(A_1)\|_F - \|h_Z(A_2)\|_F \geq 4\pi^2 \min_k \{m_k\}
$$

**Proof.** Since all the eigenvalues of $A_d$ are real, we have $\beta_k \equiv 0 \forall k$. Let $A_i := Z \text{diag}(L_1^{(j)}(i), \ldots, L_p^{(j)}(i))Z^{-1}, i = 1, 2$.

$$
\|h_Z(A_1)\|_F - \|h_Z(A_2)\|_F = \sum_{k=1}^p 4\pi^2 m_k \left( j_k^{(1)} - j_k^{(2)} \right) \left( j_k^{(1)} + j_k^{(2)} \right).
$$
\[ \|h_Z(A_i)\|_F^2 - \|h_Z(A_0)\|_F^2 = \text{tr} \left( \text{diag}^*(L_1^{(i)}, \cdots, L_p^{(i)}) \text{diag}(L_1^{(i)}, \cdots, L_p^{(i)}) \right) \]
\[ - \text{tr} \left( \text{diag}^*(L_1^{(0)}, \cdots, L_p^{(0)}) \text{diag}(L_1^{(0)}, \cdots, L_p^{(0)}) \right) \]
\[ = \sum_{k=1}^p \text{tr} \left( L_k^{(i)*} L_k^{(i)} - L_k^{(0)*} L_k^{(0)} \right) \]
\[ = \sum_{k=1}^p \text{tr} \left( 2j_k^{(i)} \pi i (\log(J_k)^* - \log(J_k)) + 4\pi^2 j_k^{(i)} I_{m_k} \right) \]
\[ = \sum_{k=1}^p 4\pi j_k^{(i)} m_k(\beta_k + j_k^{(i)}) = 4\pi \mathcal{J}(0, j), \quad j = [j_1^{(i)}, \ldots, j_p^{(i)}]. \]

Noting that \(m_k, j_k^{(1)}, j_k^{(2)}\) are non-negative integers, \(\|h_Z(A_i)\|_F - \|h_Z(A_0)\|_F \geq 4\pi^2 \min_k \{m_k\}\). The equality is obtained when \(j_k^{(1)} + j_k^{(2)} = 1\) where \(\xi = \arg\min_k \{m_k\}\), and all the other \(j_k^{(i)}\) equal 0.

**Remark 3.** Note that the lower bound in Theorem 8 depends on the specific choice of \(\tilde{A}\), i.e., \(M = M(\tilde{A})\), while \(M\) in Proposition 9 does not depend on \(A_1, A_2\).

**Proposition 10.** Let \(S\) be the set defined in (13). For any \(\tilde{A} \in S\), there exist \(\kappa_1, \kappa_u \in \mathbb{R}\) such that (11) with (15) has a unique optimal point in the sense of the equivalence relation in Definition 4.

**Proof.** It immediately follows by choosing

\[ \kappa_1 > \max\{0, \|h_Z(\tilde{A})\|_F - M(\tilde{A})\}, \]
\[ \kappa_u < \|h_Z(\tilde{A})\|_F + M(\tilde{A}), \]

where \(M(\tilde{A})\) is the lower bound on the gap between \(\tilde{A}\) and any \(A \sim \tilde{A} \in S\), defined in Theorem 8.

**VII. MIXED-INTEGER LINEAR PROGRAM**

Consider the special case that \(A_d\) is diagonalizable. We could effectively find the sparsest one among the set of system aliases by formulating a mixed-integer linear program.

**Lemma 11.** Suppose \(A_d\) is diagonalizable, and let \(A_0 = \log(A_d)/h\). Every real primary solution to \(\exp(hA_i) = A_d\) (i.e., \(A_i = \log(A_d)/h\)) is real and \(\log(\cdot)\) is primary matrix logarithm) admits that

\[ A_i = Z \text{diag}(C_1^{(i)}, \ldots, C_p^{(i)}) Z^{-1}, \]

where

\[ C_k^{(i)} = \begin{bmatrix} \text{Re}(\sigma_k) & \text{Im}(\sigma_k) + 2j_k^{(i)} \pi \\ -\text{Im}(\sigma_k) - 2j_k^{(i)} \pi & \text{Re}(\sigma_k) \end{bmatrix}, \]

\(\sigma_k := \log(\lambda_k)/h\), \(\lambda_k\) is the \(k\)-th eigenvalue of \(A_d\), and \(Z \in \mathbb{R}^{n \times n}\) is given by \(A_0 = Z \text{diag}(C_1^{(0)}, \ldots, C_p^{(0)}) Z^{-1}\), \(k = 1, \ldots, p\).

Based on the observation of Lemma 11, we know that there are \(p+1\) matrices \(P_k\)'s such that

\[ A_i = Z(P_0 + j_1^{(i)}P_1 + \cdots + j_p^{(i)}P_p)Z^{-1}, \]

where \(P_0 = \text{diag}(C_1^{(0)}, \ldots, C_p^{(0)})\),

\[ P_k = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 2\pi & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -2\pi & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \]

and \(j_k^{(i)} \in \mathbb{Z}\). After vectorization we get that

\[ \text{vec}(A_i) = V P_0 + V p_1 j_1^{(i)} + \cdots + V p_p j_p^{(i)}, \]

where \(V = Z^{-T} \otimes Z\) and \(p_k = \text{vec}(P_k)\). Let \(W = [V p_1, V p_2, \ldots, V p_p]\) and \(b = \text{vec}(P_0)\), and the matrix \(W\) and the vector \(b\) are independent of the \(j_k\)'s.

Using the form (25), we obtain the following problem

\[
\begin{aligned}
\text{minimize} & \quad \|y\|_0, \\
\text{subject to} & \quad Wx = y - b,
\end{aligned}
\]

and its convex relaxation

\[
\begin{aligned}
\text{minimize} & \quad \|y\|_1, \\
\text{subject to} & \quad Wx = y - b,
\end{aligned}
\]

where \(y\) is the vectorization of matrix \(A_i\) and the elements of \(x\) correspond to the integer multiples of the \(\pm 2\pi\) (the \(j_k^{(i)}\)'s for \(A_i\)). By solving these problems, we will obtain a sparse matrix in the 0-norm and 1-norm sense, respectively. If one want to explicitly exclude the principal branch (\(j = 0\)), the constraint \(\sum_{k=1}^p (x_k) \geq \epsilon\) could be included, where \(\epsilon\) can be any real number \(0 < \epsilon \leq 1\). The latter problem (27) is a Mixed-Integer Linear Program (MILP); in essence, it can be reformulated as a standard MILP by adding additional variables.

\[
\begin{aligned}
\text{minimize} & \quad \sum_{i=1}^{n^2} z_i, \\
\text{subject to} & \quad Wx = y - b, \\
& \quad z_i \geq y_i, \\
& \quad z_i \geq -y_i, \quad \forall i = 1, \ldots, n^2.
\end{aligned}
\]

This problem can solved (sub-optimally) using any standard MILP-solvers such as Gurobi [15].
VIII. Conclusions

This paper addresses the identification of continuous time noise-driven dynamical systems with sparse topologies. The key assumption is that the sampling time is long. Under this assumption, a realization/identification problem comes to surface, which has largely been overlooked in the community. In this problem one needs to search over a, sometimes infinite, collection of matrix logarithms to find the sparsest one. We provide theoretical results for when a unique solution exists and when a solution exists up to a finite equivalence class. We also provide a mixed integer linear programming formulation, which can be solved (sub optimally) by standard solvers.

Appendix

A. Proof of Lemma 11

Proof. Firstly it is easy to see that all $A_i$’s in the form of (24) are the solutions to $\exp(hA_i) = A_d$ by noticing $\exp(C_j^{(k)}) = \exp(C_0^{(k)})$. Then we need to show that every primary solution $A_i$ has the form of (24). Theorem 2 tells that $A_i = Z\operatorname{diag}(L_1^{(i)}, \ldots, L_n^{(i)})Z^{-1}, L_k^{(i)} = \log(\lambda^i_k) + 2\gamma^i_k \pi i$ and $A_0 = Z\operatorname{diag}(L_1^{0}, \ldots, L_n^{0})Z^{-1}, L_k^{0} = \log(\lambda_k).$ Since $A_d$ is a real matrix, we can choose $Z \in \mathbb{R}^{n \times n}$. Noticing that $A_i$’s are real matrices, it follows that each $A_i$ has the form of (24) (refer to [16, p. 153]).

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