OCTONIONS AND BINOCULAR MOBILEVISION

In memory of beautiful days of my being at Tartu.

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Abstract. This paper is devoted to an interaction of two objects: the first one is octonions, the classical structure of pure mathematics, the second one is Mobilevision, the recently developed technique of computer graphics. Namely, it is shown that the binocular Mobilevision may be elaborated by use of the octonionic colour space — the seven dimensional extension of the classical one, which includes a strange overcolour besides two triples of ordinary ones (blue, green and red for left and right eyes).

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1. INTERPRETATIONAL GEOMETRY, ANOMALOUS VIRTUAL REALITIES, QUANTUM PROJECTIVE FIELD THEORY AND MOBILEVISION

1.1. Interpretational geometry.

A geometry being described below is related to a certain class of interactive information systems. Namely, let us call interactive information system computer graphical if the information stream from the computer is counted as a stream of geometrical graphical data on a screen of the display; an interactive computer graphical information system is called psychological information one if the information from the observer to computer transmits its unconsciously (such systems are widely applied, e.g., in medical diagnostics or for the psychophysiological self-regulation, computer hypnosis and suggestion, so below we shall be interested presumably in them). We shall denote the concepts of the interpretation in the computer-geometric representation of mathematical data in interactive information systems.

Mathematical data in interactive information systems exist in the form of an interrelation of an interior geometric image (figure) in the subjective space of observer and an exterior computer graphical representation (cf., e.g. [1]). The exterior computer graphical representation includes the visible elements (draws of figure) as well as of the invisible ones (e.g., analytic expressions and algorithms of the construction of such draws, cf. [2]). Process of the corresponding of a geometric image (figure) in the interior space of observer to a computer graphical representation (visible and invisible elements) will be called translation. For example, a circle as a figure is a result of the translation of its draw on a screen of the video computer (the visible object), constructed by an analytic formula (the invisible object) accordingly to the used algorithm (also the invisible object, cf. [3, 4]). It should be mentioned that the visible object may be non-identical to the figure, e.g., if a 3-dimensional body is defined by a geometric, in three projections, cross-sections or cuts, or in the window technique, which allows to scale up a concrete detail of a draw (which is a rather pityy operation for the visualization of fractals [5, 6], etc.; in this case partial visible elements may be regarded as modules, which translation is realized separately. Continuing to use the terminology of computer science we shall call the translation by interpretation if the translation of partial modules is realized depending on the result of the translation of preceding ones and by composition otherwise. An example of the interpretation may be produced by the drawing of a fractal whose structure is defined by an observer on each step of the scaling up in the window technique; the translation of visible elements in an intentional anomalous virtual reality (see below) is also an interpretation.

Definition 1. A figure, which is obtained as a result of the interpretation, will be called interpretational figure.

It should be mentioned that an interpretational figure may have no any habitual formal definition; namely, only if the process of interpretation has an equivalent compilation process then the definition of figure is reduced to sum of definitions of its draws; nevertheless, in interactive information system not each interpretation process has an equivalent compilation one. The draw of interpretational figure may be characterized only as "visual perception technology" of figure but not as an "image", such draws will be called symbolic.

The computer-geometric description of mathematical data in interactive information systems is deeply related to the concept of anomalous virtual reality. It should be mentioned that there exist not less than two approaches to foundations of geometry: in the first one the basic geometric objects are figures defined by their draws, geometry describes relations between them, in the second one the basic geometric concept is a space (a medium, a field), geometry describes various properties of a space and its states, which are called the draws of figures. For the purposes of the describing of geometry of interactive information system it is convenient to follow the second approach; the role of the medium is played by an anomalous virtual reality, the draws of figures are its certain states.

1.2. Anomalous virtual realities.

Definition 2 [9].

An anomalous virtual reality (AVR) in a narrow sense is a certain system of rules of non-standard descriptive geometry adopted to a realization on video computer (or multisensor system of "virtual reality"). An anomalous virtual reality in a wide sense contains also an image in the
cyberspace made accordingly to such system of rules. We shall use this term in a narrow sense below.

B. Naturalization is the corresponding of an anomalous virtual reality to an abstract geometry or a physical model. We shall say that the anomalous virtual reality naturalizes the model and such model transcodizes the naturalizing anomalous virtual reality.

C. Visualization is the corresponding of certain images or visual dynamics in the anomalous virtual reality to objects of the abstract geometry or processes in the physical model.

D. An anomalous virtual reality, whose image depends on an observer, is called intentional anomalous virtual reality (I AVR). Generalized perspective laws in I AVR contain the equations of dynamics of observed images besides standard (geometric) perspective laws. A process of observation in I AVR contains a physical process of observation and a virtual process of intention, which directs an evolution of images accordingly to dynamical laws of perspective.

In the intentional anomalous virtual reality objects of observation present themselves being connected with observer, who acting on them in some way, determine their observed states, so an object is thought as a potentiality of a state from the desired spectrum, but its realization depends also on observer. The symbolic drawings of interpretational figures are presented by states of a certain intentional anomalous virtual reality.

1.3. Colours in anomalous virtual realities.

It should be mentioned that the deep difference of descriptive geometry of computer graphical information system from the classical one is the presence of colours as important bearers of visual information. The reduction to shape graphics, which is adopted in standard descriptive geometry, is very inconvenient, since the use of colours is very familiar in the scientific visualization [10;13]. The approach to the computer graphical interactive information system based on the concept of anomalous virtual reality allows to consider an investigation of structure of a colour space as a rather patent problem of descriptive geometry, because such space may be much larger than the usual one and its structure may be rather complicated. Also it should be mentioned that the use of other colour spaces allows to transmit diverse information in different forms, so an investigation of the information transmission via anomalous virtual realities, which character deeply depends on a structure of colour space, becomes also an important mathematical problem (cf.[1]).

Definition 3.

A set of continuously distributed visual characteristics of image in an anomalous virtual reality is called anomalous colour space. Elements of an anomalous colour space, which have non-colour nature, are called overcolours, and quantities, which transcode them in the abstract model, are called "latent lights". Colour perspective system is a set of generalized perspective laws in anomalous colour space.

B. The transmission of information via anomalous virtual reality by "latent lights" is called AVR (photodyssey).

1.4. Quantum projective field theory.

M obilisation is an intentional anomalons virtual reality naturalizing the quantum projective field theory ([14,15]). The process of naturalization is described in [10,15]. Its key points will be presented below, here our attention is concentrated on the basic concepts of the quantum projective field theory, which naturalization mobilisation is.

Definition 4A [14,15]. QFT (operator algebra (operator algebra of the quantum field theory, vertex operator algebra, vertex algebra) is the pair (H; tij(x)): H is a linear space, tij(x) is H valued tensor such that tijm(x)en(x) = tije(x)yn(x) (duality).

Let us introduce the operators 1 = (e1e2) = tij(x)en(x), then the following relations will hold: 1 = (e11)e2(e1) = tij(x)e1(e2) (operator product expansion) and 1 = 1 (duality).

Definition 4B [14,15]. QFT (operator algebra (H; tij(u); u 2 C) is called QPFT (operator algebra (operator algebra of the quantum projective field theory) if (1) H is the sum of Verma modules V over sl2(C) with the highest vectors v and the highest weights h, (2) c(x) is a primary algebra of spin h, i.e., [Lk; l](v) = (u)k(θu + (k + 1)h)l(v), where Lk are the sl2(C) generators (L1; L2) = (ij)Lj, l = 1; 0, 1, (3) the descendant generation rule holds: 1 = 1 (f) = 1 (L1 f). QFT (operator algebra (H; tij(u); u 2 C) is called derived QPFT (operator...
tor algebra i conditions (1) and (2) as well as derived rule of descendants generation \((L_1; l_i(f)) = \lambda_i (L_1; f)\) hold.

As it was shown in the paper [14] the categories of QPFT (operator algebras) and derived QPFT (operator algebras) are equivalent. The explicit construction of equivalence was described. Therefore, QPFT (operator algebras) and derived QPFT (operator algebras) may be considered as different recordings of the same object, and one may use the most convenient one in each concrete case.

It should be mentioned also that in arbitrary QFT (operator algebra) one can define an operation depending on the parameter: \(m_u (; m_s (; j )) = \frac{m}{m_u} m_u (; j )); \). The operators are the operators of the left multiplication in the obtained algebra.

Definition 4C [15]. QPFT (operator algebra) defined QPFT (operator algebra) \((H; t^{\mathcal{H}}_{i,j}(u))\) is called projective \(G\) (hyper multiplet, i.e. the group \(G\) acts in \(H\) by automorphism, otherwise, the space \(H\) possesses a structure of the representation of the group \(G\), the representation operators commute with the action of \(\text{sl}(2;C)\) and \(L_i (\text{g}; f) = (\text{g}; l_i(f)\text{g}^{-1})\).

The linear spaces of the highest vectors of the \(\text{xed weight form subrepresentations of} G\), which are called multiplets of projective \(G\)-hyper multiplet.

1.5. Moible vision.

As it was mentioned above Moible vision is a certain anomalous virtual reality, which naturalizes the quantum projective Elki theory. Possibly, Moible vision is not its unique naturalization. Here we describe the key morphisms of the process of naturalization of the quantum projective Elki theory which is realized in Moible vision.

Unless the abstract model (quantum projective Elki theory) has a quantum character the images in its naturalization, the intentional anomalous virtual reality of Moible vision, are classical. The transition from the quantum Elki model to classical one is done by standard rules [16], namely, the classical Elki with Taylor coe cients \(j_{\text{a}} \frac{\partial^{2}}{\partial x^{2}}\) is corresponded to the element \(a_{k} L_{k}^{j} v\) of the QPFT (operator algebra).

Under the naturalization three classical Elkis are identified with Elkis of three basic colours (red, green and blue (see [17]), other Elkis with Elkis of overcolours; there are pictured only the colour characteristics for the \(\text{xed m om emt of tis m on the screen of the videocomputer as well as the perception of the overcolours by an observer is determined by the intentional character of the anomalous virtual reality of Moible vision. Namely, during the process of the evolution of the im age, produced by the observation, the vacillations of the colour Elkis take place in accordance with the dynamical perspective laws of the intentional anomalous virtual reality of Moible vision (Euler formulae [9] or Euler (Ampére) equations [15]). These vacillations depend on the character of an observation (i.e. the eye movement or another dynamical parameter); the vacillating in age depends on the distribution of the overcolours, that allows to interpret the overcolours as certain vacillations of the ordinary colours. So the overcolours in the intentional anomalous virtual reality of Moible vision are vacillations of the \(\text{xed type and structure of ordinary colours w ith the de ned dependence on the parameters of the observation process. The transcending "latent lights" are the quantized Elkis of the basic m odel of the quantum projective Elki theory.}

The presence of the SU (3)(symmetry of classical colour space (see 3.1) allows to suppose that the QPFT (operator algebra) of the initial model is the projective SU (3)(hyper multiplet).

Now we are interested in the investigation of a behaviour of overcolours under scaling transformations, the extraction of natural classes of overcolours with respect to these transformation and describe some properties of these classes.

Proposition 1. If the initial quantized Elkis of "latent light" in the abstract model of the quantum projective Elki theory has the spin \(h\), then the lightening of the corresponding overcolour in the intentional anomalous virtual reality of Moible vision increase in \(s^{j}\) times under the scaling up in \(s^{j}\) times.

This statement follows from the description of the naturalization process given above.

Therefore, there are the currents, quantized Elkis of spin 1, corresponded to the ordinary colours. Let us mention that for the Elkis of "latent lights" of negative spin (ghosts) the anomalous increasing of lightening holds under the scaling down i.e. the moving of object away from
observer. So the virtual diffusion of ghosts in cyberspace holds with an intensity according to a principle of a snow (slip). There are the overcolours, which lightening disease under the scaling down (i.e. the moving of object away from observer) quicker than for ordinary colours, corresponded to "latent lights" of spin greater than 1.

2. QUANTUM CONFORMAL AND $q_R$ {CONFORMAL FIELD THEORIES, AN INFINITE DIMENSIONAL QUANTUM GROUP AND QUANTUM FIELD ANALOGS OF EULER ARNOLD EQUATIONS

2.1. Quantum conformal field theory.

Definition 7A [18].

A. The highest vector $T$ of the weight 2 in the QPFT (operator algebra will be called the conformal stress(energy tensor if $T(u) = L_k(u) = k^2$, where the operators $L_k$ from the Virasoro algebra: $[L_i, L_j] = (i,j)L_{i+j} + \frac{i+1}{12}cI$. 

B. The set of the highest vectors $J$ of the weight 1 in the QPFT (operator algebra will be called the set of the a ne currents if $J(u) = J_k(u) = k^1$, where the operators $J_k$ form the a ne Lie algebra: $[J_i, J_j] = c(J_{i+j} + k(i+j)I$.

In view of the results of [19,20] the quantum projective eld theories with conformal stress(energy tensor are just conformal eld theories in sense of [21,22].

If there is defined a set of the a ne currents in the QPFT (operator algebra then one can construct the conformal stress(energy tensor by use of Sugawara construction [23] or more generally by use of the Virasoro master equations [24,26].

Below we shall be interested in the special deformations of the quantum conformal eld theories in class of the quantum projective ones (cf.27,29), which will be called quantum $q_R$ (conformal eld theories.

2.2. Lobachevskii algebra, the quantization of the Lobachevskii plane.

In the Poincaré realization of the Lobachevskii plane (the realization in the unit disk) the Lobachevskii metric may be written as follows:

$$w = q_{R}^{-1}dzz = (1 + j)z^2$$

We shall construct the $C$ (algebra, which may be considered as a quantization of such metric, namely, let us consider two variables $t$ and $\tau$, which obey the following commutation relations: $[\tau, t] = 0$, $[\tau, \tau] = q_{R}(\tau_{+}t - \tau_{-}t)$, $0 \leq q_{R} < 1$. One may realize such variables by bounded operators in the Verma module over $sl(2;C)$ of the weight $h = \frac{q_{R}^{-1}+1}{2}$. If such Verma module is realized in polynomials of one complex variable $z$ and the action of $sl(2;C)$ has the form $L_1 = z$, $L_0 = z\theta_z + h$, $L_{\pm} = \pm(\theta_z)^2 + 2h\theta_z$, then the variables $t$ and $\tau$ are represented by tensor operators $\theta_z$ and $z(\theta_z + 2h)$. These operators are bounded if the Verma module is unitarizable ($h > \frac{1}{2}$), $q_{R} > 0$ and therefore one can construct a Banach algebra generated by them and obeying the prescribed commutation relations. The structure of $C$ (algebra is rather obvious: an involution is defined on generators in a natural way, because the corresponding tensor operators are conjugate to each other.
2.3. Quantum $q_R$ (conformal eld theory).

**Definition 7B [18].**

A. The highest vector $T$ of the weight 2 in the QPFT (operator algebra) will be called the $q_R$ (conformal stress-energy tensor) if $T(u) = \lambda(T) = \sum L_k (u)^k$, where the operators $L_k$ form the $q_R$ (Virasoro algebra):

$$[L_i; L_j] = (i; j)L_{i+j}; \quad (i; j) \geq 1; \quad (i; j) \\ [L_2; L_2] = (L_0 + 1) \cdot (L_0) + 1) \cdot \left(\frac{t(t+1)(t+3h)}{(t+2h)(t+2h+1)}\right);$$

B. The set of the highest vectors $J$ of the weight 1 in the QPFT (operator algebra) will be called the set of the $q_R$ (a ne currents) if $J(u) = \lambda(J) = J_k(u)^k$, where the operators $J_k$ form the $q_R$ (a ne Lie algebra):

$$J_k = J_T^k f_k(t); \quad [J; J] = c; \quad f(t) = f(t+1)T; \quad [T; J] = [f(t); J] = 0; \quad f_k(t) = \cdots (t; k); \quad k; \quad 0; \quad (t+2h) \cdots (t+2h+k+1);$$

$$h = (q^1_R + 1)^2;$$

It should be mentioned that $q_R$ (a ne currents and $q_R$ (conformal stress-energy tensor) are just the $\mathfrak{sl}(2;\mathbb{C})$ (primary elds in the Verma module $V_h$ ($h = \frac{q^1_R + 1}{2}$) over $\mathfrak{sl}(2;\mathbb{C})$) of spin 1 and 2, respectively. If such a module is realized as before then

$$J_k = \Theta^k_z; \quad J_k = z^k = (t+2h) \cdots (t+2h+k);$$

$$L_2 = (t+3h)\Theta^2_z; \quad L_1 = (t+2h)\Theta_z; \quad L_0 = +h; \quad L_1 = z; \quad L_2 = z^2 \left(\frac{t+2h(t+2h+1)}{t+2h}(t+2h+1)}\right) = z\Theta_z;$$

the $q_R$ (a ne algebra may be realized in terms of Lobachevskii $C$ (algebra):

$$J_k = J^T^k; \quad k; \quad 0; \quad [J; J] = c^i J;$$

The QPFT (operator algebras generated by $q_R$ (conformal currents are called canonical projective $G$ (hypermultiplets [15]).

2.4. An in nite dimensional quantum group.

It should be mentioned that the primary elds $V_k(u)$ of nonnegative integer spins $k$ in the Verma module $V_h$ [30,31], which form a closed QPFT (operator algebra (as well as in the case of extended conformal eld theories [32,35]), the components of the $\mathfrak{sl}(2;\mathbb{C})$ (primary elds form in some sense an analogue of conformal $\mathbb{W}$ (algebra [36,39]), are not local. Nevertheless, their commutation relations may be described as follows

$$V(u)V(v) = S(u,v)V(v)V(u);$$

That means that these primary elds form a Zamolodchikov algebra [40,42, S (matrix $S(u;\Theta)$) of this Zamolodchikov algebra describes an in nite dimensional quantum group, which is a certain deformation of $GL(1)$, in a standard way [43,44].

2.5. Quantum ( eld Euler A mold top and V irasoro master equation.

Let $H$ be an arbitrary direct sum of Verma modules over $\mathfrak{sl}(2;\mathbb{C})$ and $P$ be a trivial bundle over $C$ with bers isomorphic to $H$. It should be mentioned that $P$ is naturally trivialized and possesses a structure of $\mathfrak{sl}(2;\mathbb{C})$ (homogeneous bundle. A $\mathfrak{sl}(2;\mathbb{C})$ (invariant Finsler connection $A(u;\Theta_t)$ in $P$ is called an angular eld [9]. Angular eld $A(u;\Theta_t)$ may be expanded by $(\Theta_t)^k$, the coe cients of such expansion are just $\mathfrak{sl}(2;\mathbb{C})$ (primary elds [9]. The equation

$$\Theta_t = A(u;\Theta_t) t;$$
where $t$ belongs to $H$ and $u = u(t)$ is the function of scanning, is a quantum (eik analog of the Euler formulas [9]). Such analog describes an evolution of mobilevision in age under the observation. One may also consider an a new version of the Euler formulas, which may be written as follows

$$\theta(t) = A(\theta_i u)(t) = 0;$$

Regarding canonical projective $G$, we may construct a quantum eik analog of the Euler (amold equation [15])

$$\theta_t A = fA;$$

where an angular eik $A(u;\theta_i u)$ is considered as an element of the canonical projective $G$ (hyper multiplet being expanded by $sl(2;C)$ primary eiks of this hyper multiplet, $H$ is the quadratic element of $S(g), f; g$ are canonical Poisson brackets in $S(g)$. It is quite natural to dem and $H$ be a solution of the Virasoro master equation. If we consider a projective $G$ (hyper multiplet, which is a semi/direct product of the canonical one and a trivial one (i.e. with $|\lambda_i(f)| = 0$), then it will be possible to combine Euler (amold equations with Euler formulas to receive the complete dynamical perspective laws of the M obilevision. Such construction will be used in the next paragraph for the description of the octonionic colour space of binocular M obilevision.

3. OCTONIONIC COLOUR SPACE AND BINOCULAR MOBILEVISION

3.1. Quaternionic description of ordinary colour space.

It should be mentioned that ordinary colour space may be described by use of imaginary quaternions in the following way: let us consider an arbitrary imaginary complex quaternion $x = ri + bj + gk$, $ij,jk$ are imaginary roots and $r; g; b$ are complex numbers. One may correspond to such quaternion an element of the colour space, which in RGB coordinates [17] has components $R = x + jy$, $G = yj + B = yj$. The lightening $L$ has the quadratic form in the quaternionic space, namely, $L = \frac{1}{2}(x^2 + yj^2 + yj)$. The group SU(3) is a group of its invariance.

3.2. Octonionic colour space and binocular M obilevision.

Let us construct an octonionic colour space to describe the binocular M obilevision. This space will be a semi/direct product of a canonical projective $G_2$ (hyper multiplet on the trivial one, which is a direct sum of seven copies of the suitable $Vem$ a module over $sl(2;C)$. The group $G_2$ acts in this seven dimensional space as it acts on imaginary octonions [45] (see also [64,47]). There is uniquely determined up to a multiple and a module the Poisson center an SU(3) (invariant quadratic element in $S^2(g_2)$). Moreover, it obeys the Virasoro master equation (as a solution of $G_2 = SU(3)$ (coset model)). So we can construct the Euler (amold equations in the canonical projective $G_2$ (hyper multiplet). To receive the binocular version of the same Euler formulas one should use the decomposition of $S^2(g_2)$ on the SU(3) (chiral components (right and left); the angular eiks from the chiral components will depend on chiral parameters $u_j, \theta_i u$, and $u_i, \theta_i u_i$, attributed to the left and right eyes, respectively. Six copies of $Vem$ a modules over $sl(2;C)$, mentioned above, form a pair of projective SU(3) (hyper multiplets, which correspond to ordinary colours for left and right eyes; one copy form a also projective SU(3) (hyper multiplet, its overcolour will be called a strange overcolour. So the constructed seven dimensional octonionic colour space includes a pair of ordinary three dimensional colour spaces (for left and right eyes, respectively) and one strange overcolour.

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