Adinkras and SUSY Holography:
Some Explicit Examples

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ABSTRACT
We discuss the mechanism by which adinkras holographically store the required information for the \(Spin(1, 3)\) Clifford Algebra fiber bundle in the cases of three 4D, \(\mathcal{N} = 1\) representations: the chiral, vector and tensor supermultiplets.

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1 Introduction

Some time ago, [1] an effort was made to extract the relationships between some 4D, \( \mathcal{N} = 1 \) \((N = 4)\) supermultiplets and one of the GR\((d, \mathcal{N})\) Algebras\(^4\) (or ‘Garden Algebras’) that arise in an approach to understanding the representations of one dimensional supersymmetrical quantum mechanical systems. These algebras [2], generated by L- and R-matrices, are extended real versions of the algebras that arise from the van der Waerden formalism. It has been proposed that these can be taken as the basic building blocks of a rigorous theory of off-shell representations for space-time SUSY in much the same way as quark triplets and anti-triplets are the basic building blocks for hadrons. For 4D, \( \mathcal{N} = 1 \) supermultiplets, the appropriate Garden Algebra has \( d = 4 \) and \( N = 4 \). The proposal that the same sets of mathematical objects provide the foundation for both supersymmetrical quantum mechanical systems and higher dimensional supersymmetrical quantum field theories implies the possible existence of a type of holography, given the name ‘SUSY Holography’ in the work of [3].

An unexpected and empowering development in this approach was the realization that Garden Algebras can be obtained from graph theory [4]. The required graphs have been given the names of ‘adinkras’ and these are adept at capturing an attribute (though present in all such descriptions) of these systems that is less conveniently described in other approaches. This attribute has been given the name ‘height’ and corresponds to the engineering dimension of the fields that occur in a supersymmetrical representation. Other contributions to the development of this approach that have been facilitated by adinkras are:

(a.) the discovery that adinkras (and therefore off-shell supersymmetry representation theory) describe spaces of ‘marked cubical topology’ [5],
(b.) the irreducible representations of adinkras are determined by self-dual block linear error-correcting codes [6], and
(c.) that there exist a relation to mathematical structures called posets [7].

The L-matrices have a well defined role for one dimensional quantum mechanical systems. But how can these structures be used to re-construct a Dirac-operator for a higher dimensional field theories with supersymmetry? If the ‘SUSY Holography’ conjecture in the work of [3] is correct, there must be a way to systematically achieve this goal. In fact, Ref. [8] contains explicit demonstrations along these lines for some specific examples and gives a general requirement of necessary conditions to achieve SUSY holography between 1D, \( \mathcal{N} = 4 \) models and 4D, \( \mathcal{N} = 1 \) systems. At least in the methods of implementation shown in Ref. [8] the heights of nodes plays a role. A rather different approach [9] has been achieved for relating 1D \( N \)-extend SUSY models and 2D, \((p,q)\)-supersymmetric systems. In this approach it has been shown that there exist obstructions, in the form of minimal length closed co-cycles, whose absence permit the ‘liftability’ of the data described by the L-matrices to successfully describe 2D models.

In the case of 4D, \( \mathcal{N} = 1 \) SUSY it turns out that there is sufficient information in the work of [1] to see how, by beginning from a four dimensional field theory with simple supersymmetry, one can

\(^4\)The label “GR\((d, \mathcal{N})\)” specifies the algebra generated by \( N \) L- and \( N \) R-matrices of size \( d \times d \), akin to van der Waerden’s \( \sigma \)- and \( \bar{\sigma} \)-matrices; see equations (4.1)–(4.2), below. \( \mathcal{N} \) counts supersymmetry in terms of the smallest spinors in the given (space)time, while \( \mathcal{N} \) counts the individual, real supercharge components. Finally, (space)time dimension is denoted by a capital D, such as in “1D” vs. “4D.”
uncover a relation to the L-matrices of a 1D, \( \mathcal{N} = 4 \) formalism. It is a more intricate problem to see how one can begin solely within a one dimensional ‘Plato’s Cave’ \([10] \) and then in a systematic manner ‘discover’ that all the required data is already present within those confines. This will be treated in a separate publication. So the primary purpose of this work is to give an explicit discussion based on the results in \([1] \) showing how the data associated with adinkras is embedded with the structure of some dimensional reduced \( d = 4 \), \( N = 4 \) Garden Algebras, \( \mathcal{G} \mathcal{R}(4, 4) \).

2 Spinors Outside of Plato’s Cave

For the four dimensional physicist, the beginning of describing spinors can start with the introduction of a set of Dirac gamma matrices. Let us be even more restrictive in our starting point by requiring that all four of the Dirac gamma matrices are real. It is simple to see that such a set is provided by \( \gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3) \) where

\[
\begin{align*}
(\gamma^0)^a_b &= i(\sigma^3 \otimes \sigma^2)^a_b, & (\gamma^1)^a_b &= (I_2 \otimes \sigma^1)^a_b, \\
(\gamma^2)^a_b &= (\sigma^2 \otimes \sigma^2)^a_b, & (\gamma^3)^a_b &= (I_2 \otimes \sigma^3)^a_b,
\end{align*}
\]

written in terms of the outer product of the usual \( 2 \times 2 \) Pauli matrices \( (\sigma^1, \sigma^2, \sigma^3) \) and the \( 2 \times 2 \) identity matrix \( I_2 \). These clearly satisfy the usual Dirac condition

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu \nu} I_4
\]

where the \( 4 \times 4 \) identity matrix \( I_4 \) can also be written in terms of the outer product,

\[
I_4 = I_2 \otimes I_2.
\]

Given a set of Dirac gamma matrices \( \gamma^\mu \), we can multiple higher powers of them and in particular define the usual product of four distinct ones to define \( \gamma^5 \) via the usual definition

\[
\gamma^\mu = i \gamma^0 \gamma^1 \gamma^2 \gamma^3
\]

which has the outer product representation

\[
(\gamma^5)^a_b = -(\sigma^1 \otimes \sigma^2)^a_b.
\]

The three generators of spatial rotation acting on the spinor are provided by \( \Sigma^{12}, \Sigma^{23}, \Sigma^{31} \) where where

\[
\Sigma^{12} = -i \frac{1}{2} \gamma^1 \gamma^2, \quad \Sigma^{23} = -i \frac{1}{2} \gamma^2 \gamma^3, \quad \Sigma^{31} = -i \frac{1}{2} \gamma^3 \gamma^1.
\]

At this point, a different set of \( 4 \times 4 \) matrices can be introduced via the definitions

\[
\begin{align*}
\alpha^1 &= \sigma^2 \otimes \sigma^1, & \beta^1 &= \sigma^1 \otimes \sigma^2, \\
\alpha^2 &= I \otimes \sigma^2, & \beta^2 &= \sigma^2 \otimes I, \\
\alpha^3 &= \sigma^2 \otimes \sigma^3, & \beta^3 &= \sigma^3 \otimes \sigma^2,
\end{align*}
\]

and it is a simple matter to show that the algebra for multiplying each of these is isomorphic to that of the usual \( 2 \times 2 \) Pauli matrices. Moreover, any element from the ‘\( \alpha \)-set’ commutes with any
element from the ‘β-set.’ This means that each of the two distinct sets can act as generators of an $SU(2)$ algebra. The Dirac gamma matrices can be expressed using ordinary matrix multiplication of the ‘α-set’ and ‘β-set’ as

$$\gamma^0 = i \beta^3, \quad \gamma^1 = \alpha^1 \beta^2, \quad \gamma^2 = \alpha^2 \beta^2, \quad \gamma^3 = \alpha^3 \beta^2. \quad (2.8)$$

which imply

$$\Sigma^{12} = \frac{1}{2} \alpha^3, \quad \Sigma^{23} = \frac{1}{2} \alpha^1, \quad \Sigma^{31} = \frac{1}{2} \alpha^2. \quad (2.9)$$

and these equations make manifest that the $\Sigma$-matrices are the generators of $SU(2)$. But what has happened to the other $SU(2)$?

The definition of the $\gamma^5$ matrix implies

$$\gamma^5 = - \beta^1. \quad (2.10)$$

and this together with the definition of $\gamma^0$ implies

$$-i \frac{1}{2} \gamma^0 = \frac{1}{2} \beta^3, \quad \frac{1}{2} \gamma^5 = \frac{1}{2} \beta^1, \quad \frac{1}{2} \gamma^0 \gamma^5 = \frac{1}{2} \beta^2. \quad (2.11)$$

It is not commonly noted that given a set of four dimensional gamma matrices, it is possible to use them to define two commuting $SU(2)$’s and this is independent of the representation chosen for the gamma matrices. The name “extended R-symmetry” seems a reasonable moniker for the $SU(2)_\beta$, given that it is a group-theoretic “extension” (in its precise technical sense) of the $U(1)$ symmetry generated by $\gamma^5$, which is identifiable with the “R-symmetry” or the “axial symmetry.”

In the work of [1], this second less well-recognized $SU(2)$ symmetry plays an interesting role in the analysis of four dimensional supersymmetrical field theories when reduced to 1D. When the theory is off-shell, both of these $SU(2)$ symmetries are realized on the anti-commutator algebra of the supercharges. When the theory is on-shell, only the $SU(2)$ symmetry associated with angular-momentum (rotations) is realized on the anti-commutator algebra of the supercharges.

3 Some 4D, $\mathcal{N} = 1$ Supermultiplets Outside of Plato’s Cave

For the four dimensional physicist knowledgeable about the simplest 4D, $\mathcal{N} = 1$ representations, the chiral, vector, and tensor supermultiplets can be respectively described by the sets of equations in (3.1), (3.2) and (3.3). For the superspace derivative $D_a$ (equivalent to the supercharge) acting on the fields of the chiral supermultiplet $(A, B, \psi_a, F, G)$ we have,

$$D_a A = \psi_a,$$

$$D_a B = i (\gamma^5)^b_a \psi_b,$$

$$D_a \psi_b = i (\gamma^\mu)^a_{ab} \partial_\mu A - (\gamma^5 \gamma^\mu)^a_{ab} \partial_\mu B - i C_{ab} F + (\gamma^5)^b_{ab} G,$$

$$D_a F = (\gamma^\mu)^a_{ab} \partial_\mu \psi_b,$$

$$D_a G = i (\gamma^5 \gamma^\mu)^a_{ab} \partial_\mu \psi_b.$$
For the superspace derivative $D_a$ acting on the fields of the vector supermultiplet $(A_\mu, \lambda_a, d)$ we have,

$$
D_a A_\mu = (\gamma_\mu)_a^b \lambda_b , \\
D_a \lambda_b = -i \frac{1}{4} (\gamma_\mu, \gamma_\nu)^a_{ab} (\partial_\nu A_\mu - \partial_\mu A_\nu) + (\gamma^5)_a^b d , \\
D_a d = i (\gamma_5 \gamma_\mu)_a^b \partial_\mu \lambda_b .
$$

(3.2)

Finally, for the superspace derivative $D_a$ acting on the fields of the tensor supermultiplet $(\varphi, B_{\mu\nu}, \chi_a)$ we have,

$$
D_a \varphi = \chi_a , \\
D_a B_{\mu\nu} = -\frac{1}{4} (\gamma_\mu, \gamma_\nu)_a^b \chi_b , \\
D_a \chi_b = i (\gamma_\mu)_a^b \partial_\mu \varphi - (\gamma^5 \gamma_\mu)_a^b \epsilon_{\mu}^{\rho\sigma\tau} \partial_\rho B_{\sigma\tau} .
$$

(3.3)

In the work of [1], there was introduced the notion that a structure similar to the Wigner ‘little group’ exists for all higher dimensional SUSY models. The process begins by restricting coordinate dependence of all field to only depend on the temporal direction and was given the name of ‘reducing on a 0-brane.’ For gauge fields, only the Coulomb gauge-fixed components are retained. Finally, one performs the redefinitions

$$
F \rightarrow \partial_\tau F , \quad G \rightarrow \partial_\tau G , \quad d \rightarrow \partial_\tau d .
$$

(3.4)

This step ensures that the set of all bosons remaining have the same engineering dimensions. All the fermions already possess the same engineering dimensions though of course this is different from that of the bosons. By this means, one obtains a truncation of the original four dimensional theories supermultiplets that have now been mapped into a set of one dimensional supermultiplets. The redefinitions above also have the property that it permits the re-defined $F$ and $G$, together with $B$, to form a triplet under the extended R-symmetry. Thus there occurs an enhanced symmetry within the confines of the cave.

4 Some 1D, $\mathcal{N} = 4$ Supermultiplets Inside of Plato’s Cave

Using the allusion to Plato’s Cave, here everything depends only on the temporal coordinate. All bosonic and fermionic functions only depend on a single time-like coordinate. This is the realm of supersymmetric quantum mechanics.

The ‘L-matrices’ of the Garden Algebra approach are real and correspond to the linking numbers of a topological space and can be directly read off (via a set of ‘Feynman-like rules’) from the corresponding adinkra. The parameter $N$ describes the number of equivalence classes of such linking numbers and the parameter $d$ specifies the size of the $d \times d$ L-matrices. By definition, the ‘L-matrices’ of the Garden Algebra approach satisfy the conditions,

$$
(L_i)_i^j (R_j)_j^k + (L_i)_i^j (R_j)_j^k = 2 \delta_{ij} \delta^k_l , \\
(R_j)_i^j (L_i)_j^k + (R_i)_i^j (L_j)_j^k = 2 \delta_{ij} \delta_l^k .
$$

(4.1)

where

$$
(R_i)_i^k \delta_{lk} = (L_i)_i^l \delta_{jk} .
$$

(4.2)
For a fixed size $d$, there are $2^{d-1}d!$ distinct matrices that can be used. Given a set of such L-matrices, one can introduce $d$ bosons $\Phi_i$ ($i = 1, \ldots, d$) and $d$ fermions $\Psi_k$ ($k = 1, \ldots, d$) along with $N$ superderivatives $D_I$ ($I = 1, \ldots, N$) that satisfy the equations

$$D_I \Phi_i = i \left( L_I \right)_{ik} \Psi_k, \quad D_I \Psi_k = \left( R_I \right)_{ki} \frac{d}{d\tau} \Phi_i.$$  \hspace{1cm} (4.3)

The definitions in (4.1), (4.2), and (4.3) will ensure that these $D = 1$ bosons and fermions form a representation of $N$-extended SUSY. All the bosons in this supermultiplet share have same engineering dimensions and all the fermions in this supermultiplet share have same engineering dimensions, but the latter dimensions are distinct from the former. We call such 1D representations ‘valises.’ Let us restrict ourselves to the case where $d = 4$ and $N = 4$. For the theoretical physicist inside Plato’s cave, these three sets of equations may be taken as a starting point.

Among the 1,536 quartets of L- and R-matrices that satisfy (4.1) and (4.2) it would seem likely that some of these must coincide with the mathematical systems obtained in the previous chapter. A simple question to ask is, “What set of L-matrices coincide so that the physicist in Plato’s cave is unknowingly describing the four dimensional multiplets when they are reduced on a 0-brane?” A priori, it is not at all obvious. However, the work in [1] has answered this question. We know the reduced sets correspond to the enunciations given in (4.4), (4.5), and (4.6). These can also be displayed as the Adinkras in Figs. 1, 2, and 3.

\[
\begin{align*}
(L_1)_{ik} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \\
(L_2)_{ik} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\
(L_3)_{ik} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \\
(L_4)_{ik} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \hspace{1cm} (4.4)
\]

\[
\begin{align*}
(L_1)_{ik} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \\
(L_2)_{ik} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\
(L_3)_{ik} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \\
(L_4)_{ik} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \hspace{1cm} (4.5)
\]

\[
\begin{align*}
(L_1)_{ik} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\
(L_2)_{ik} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \hspace{1cm} (4.6)
\]
\[
(L_3)_{ik} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}, \quad
(L_4)_{ik} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad (4.6)
\]

Figure 1: CM Adinkra.

Figure 2: VM Adinkra.

Figure 3: TM Adinkra.
The next intriguing question to ask is, “What remains of the (1, 3) Clifford Algebra fiber bundle that allowed the spinors in four dimensions to be defined?” Stated in a different manner we can ask, “Given that the physicist inside Plato’s Cave only has the data represented by the results in (4.4), (4.5), and (4.6), how is it possible to define the gamma matrices in (2.1) knowing about these three one dimensional $N$-extended supermultiplets?” A prescriptive solution as a positive response to these questions is the essence of the “SUSY Holography Conjecture” that was made in the work of [3].

5 How $d = 4$, $N = 4$ Adinkras Re-construct the Dirac Operator for 4D, $\mathcal{N} = 1$ Supermultiplets Outside of Plato’s Cave

The conditions in (4.1) and (4.2) define the Garden Algebra matrices. The sets of matrices for the (CM), (VM), and (TM) sets all can be shown to do satisfy these defining relations. However, we can perform a different calculation replacing the relative plus signs in the left-hand side of (4.1) by minus signs. Here something interesting occurs. For each of the representations $(R)$ (given in (CM), (VM), and (TM)) one finds there exists four sets of coefficients $\kappa(\mathcal{R})$, $\tilde{\kappa}(\mathcal{R})$, $\ell(\mathcal{R})$, $\tilde{\ell}(\mathcal{R})$ whose explicit values depend on the representation used to perform the calculation. The coeffcients appear in the equations,

\[
\begin{align*}
(L_i^{(R)})^j & (R_j^{(R)})^k = i 2 \left[ \kappa_{ij}^{(R)1} (\alpha_1)_i^k + \kappa_{ij}^{(R)2} (\alpha_2)_i^k + \kappa_{ij}^{(R)3} (\alpha_3)_i^k \right] + i 2 \left[ \tilde{\kappa}_{ij}^{(R)1} (\beta_1)_i^k + \tilde{\kappa}_{ij}^{(R)2} (\beta_2)_i^k + \tilde{\kappa}_{ij}^{(R)3} (\beta_3)_i^k \right], \\
(R_i^{(R)})^j (L_j^{(R)})^k - (R_i^{(R)})^j (L_j^{(R)})^k &= i 2 \left[ \ell_{ij}^{(R)1} (\alpha_1)_i^k + \ell_{ij}^{(R)2} (\alpha_2)_i^k + \ell_{ij}^{(R)3} (\alpha_3)_i^k \right] + i 2 \left[ \tilde{\ell}_{ij}^{(R)1} (\beta_1)_i^k + \tilde{\ell}_{ij}^{(R)2} (\beta_2)_i^k + \tilde{\ell}_{ij}^{(R)3} (\beta_3)_i^k \right].
\end{align*}
\]

The six matrices that appear on the right hand of these equations are exactly the same as the matrices denoted by the same symbols in equations in (2.7).

Moreover, the matrices on the left hand side of (5.2) from the view inside the cave, act to map the space of the spinors back onto itself. We are thus motivated by (2.9) and (2.11) to define relations of quantities inside the cave (the L-matrices) to quantities outside the cave (the $\gamma$-matrices) via the “Adinkra/$\gamma$-matrix holography equation,”

\[
(R_i^{(R)})^j (L_j^{(R)})^k - (R_i^{(R)})^j (L_j^{(R)})^k = 2 \left[ \ell_{ij}^{(R)1} (\gamma^2 \gamma^3)_i^k + \ell_{ij}^{(R)2} (\gamma^3 \gamma^1)_i^k + \ell_{ij}^{(R)3} (\gamma^1 \gamma^2)_i^k \right] - 2 \left[ i \tilde{\ell}_{ij}^{(R)1} (\gamma^5)_i^k - i \tilde{\ell}_{ij}^{(R)2} (\gamma^0 \gamma^5)_i^k - \tilde{\ell}_{ij}^{(R)3} (\gamma^0)_i^k \right].
\]

It should be noted that this is an over-constrained system of equations. There are a priori 96 independent conditions calculated on the LHS of this. But the RHS asserts that these conditions are satisfied by only 36 parameters (i. e. the $\ell$ and $\tilde{\ell}$ parameters).

Once this definition is accepted, it opens up a way for the physicist within the cave to construct a set of gamma matrices for the spinor bundle of the (1, 3) Clifford Algebra fiber bundle of a four dimensional spacetime. Let us point out that one would not wish to use the equation in (5.1) in this manner because there is no relevance to defining a set of $\gamma$-matrices that act on the space of
the bosonic fields. Furthermore the second line on the RHS of (5.3) refers to the generators of the SU(2)-extended R-symmetry which is broken in the four dimensional theory (see the remarks near equation (3.4)).

By construction, the \( \ell, \tilde{\ell}, \kappa, \text{and } \tilde{\kappa} \) coefficients satisfy \( \ell_{IJ} = -\ell_{JI}, \tilde{\ell}_{IJ} = -\tilde{\ell}_{JI}, \kappa_{IJ} = -\kappa_{JI}, \text{and } \tilde{\kappa}_{IJ} = -\tilde{\kappa}_{JI} \). For the representations \( CM, VM, \text{and } TM \), all \( \tilde{\kappa} \) coefficients vanish and Eqs. (5.4) and (5.5) show all non-vanishing independent \( \ell \), \( \tilde{\ell} \), \( \kappa \) and \( \kappa \) coefficients.

\[
\begin{align*}
\ell^{(CM)}_{12} & = 1 & \ell^{(CM)}_{13} & = 1 & \ell^{(CM)}_{14} & = 1 & \ell^{(CM)}_{23} & = 1 & \ell^{(CM)}_{24} & = 1 & \ell^{(CM)}_{34} & = 1 \\
\ell^{(VM)}_{12} & = -1 & \ell^{(VM)}_{13} & = 1 & \ell^{(VM)}_{14} & = 1 & \ell^{(VM)}_{23} & = 1 & \ell^{(VM)}_{24} & = 1 & \ell^{(VM)}_{34} & = 1 \\
\ell^{(TM)}_{12} & = 1 & \ell^{(TM)}_{13} & = 1 & \ell^{(TM)}_{14} & = 1 & \ell^{(TM)}_{23} & = 1 & \ell^{(TM)}_{24} & = 1 & \ell^{(TM)}_{34} & = 1
\end{align*}
\]  

(5.4)

\[
\begin{align*}
\kappa^{(CM)}_{12} & = -1 & \kappa^{(CM)}_{13} & = 1 & \kappa^{(CM)}_{14} & = 1 & \kappa^{(CM)}_{23} & = -1 & \kappa^{(CM)}_{24} & = 1 & \kappa^{(CM)}_{34} & = 1 \\
\kappa^{(VM)}_{12} & = -1 & \kappa^{(VM)}_{13} & = 1 & \kappa^{(VM)}_{14} & = 1 & \kappa^{(VM)}_{23} & = 1 & \kappa^{(VM)}_{24} & = -1 & \kappa^{(VM)}_{34} & = 1 \\
\kappa^{(TM)}_{12} & = 1 & \kappa^{(TM)}_{13} & = 1 & \kappa^{(TM)}_{14} & = 1 & \kappa^{(TM)}_{23} & = 1 & \kappa^{(TM)}_{24} & = -1 & \kappa^{(TM)}_{34} & = 1
\end{align*}
\]  

(5.5)

We see that these coefficients hold a rude surprise! When the \( CM \) representation is used, \( \ell, \tilde{\ell}, \kappa, \text{and } \tilde{\kappa} \) coefficients satisfy the basis dependent relations

\[
\begin{align*}
\ell^{(R)}_{IJ} & = \frac{1}{2} \epsilon_{IJKL} \ell^{(R)}_{KL}, & \ell^{(R)}_{IJ} & = -\frac{1}{2} \epsilon_{IJKL} \ell^{(R)}_{KL}
\end{align*}
\]  

(5.6)

where as the \( \kappa \) coefficients satisfy

\[
\begin{align*}
\kappa^{(CM)}_{IJ} & = -\frac{1}{2} \epsilon_{IJKL} \kappa^{(CM)}_{KL}, & \kappa^{(VM)}_{IJ} & = \frac{1}{2} \epsilon_{IJKL} \kappa^{(VM)}_{KL}, & \kappa^{(TM)}_{IJ} & = \frac{1}{2} \epsilon_{IJKL} \kappa^{(TM)}_{KL}
\end{align*}
\]  

(5.7)

All of the \( \kappa, \tilde{\kappa}, \ell, \text{and } \tilde{\ell} \) coefficients may be considered as the components of vectors in a seventy-two dimensional vector space. Knowing this allows the physicist within the cave to recognize that the three supermultiplets defined by Eqs. (4.3), (4.4), (4.5), and (4.6) are distinct representations. In the space of the \( \kappa, \tilde{\kappa}, \ell, \text{and } \tilde{\ell} \) parameters it is useful to introduce the notion of inner product between the parameters of representations. If \( \kappa^{(R)}, \tilde{\kappa}^{(R)}, \ell^{(R)}, \text{and } \tilde{\ell}^{(R)} \) are the parameters associated with the \( R \) and \( R' \) representations, we can define

\[
\begin{align*}
[\kappa^{(R)} \cdot \kappa^{(R')}] & = \frac{1}{2} \sum_{I,J,\hat{a}} \left[ \kappa^{(R)}_{IJ} \kappa^{(R')}_{IJ} + \tilde{\kappa}^{(R)}_{IJ} \tilde{\kappa}^{(R')}_{IJ} \right], \\
[\ell^{(R)} \cdot \ell^{(R')}] & = \frac{1}{2} \sum_{I,J,\hat{a}} \left[ \ell^{(R)}_{IJ} \ell^{(R')}_{IJ} + \tilde{\ell}^{(R)}_{IJ} \tilde{\ell}^{(R')}_{IJ} \right].
\end{align*}
\]  

(5.8)

The ‘angles’ between the corresponding 72-parameter family of vectors defined in (5.1)–(5.3) are defined from an inner product in the usual way:

\[
\begin{align*}
\angle[(\kappa^{(R)}), (\kappa^{(R')})_{\kappa}] & = \cos^{-1} \left( \frac{[\kappa^{(R)} \cdot \kappa^{(R')}_{\kappa}]}{|\kappa^{(R)}|_{\kappa} |\kappa^{(R')}_{\kappa}|} \right), \\
\angle[(\ell^{(R)}), (\ell^{(R')})_{\ell}] & = \cos^{-1} \left( \frac{[\ell^{(R)} \cdot \ell^{(R')}_{\ell}]}{|\ell^{(R)}|_{\ell} |\ell^{(R')}_{\ell}|} \right).
\end{align*}
\]  

(5.9)
Figure 4: A monoclinic lattice is built of two vectors (TM, VM) each perpendicular to a third (CM), but not perpendicular to each other.

We find for the length and angles between these 'vectors' the results:

\[ |(CM)|_\ell^2 = |(CM)|_\kappa^2 = |(VM)|_\ell^2 = |(VM)|_\kappa^2 = |(TM)|_\ell^2 = |(TM)|_\kappa^2 = 6, \]

\[ \angle[(CM),(VM)]_\ell = \angle[(CM),(TM)]_\ell = \frac{\pi}{2}, \quad \angle[(VM),(TM)]_\ell = \cos^{-1}\left[-\frac{1}{3}\right], \]

\[ \angle[(CM),(VM)]_\kappa = \angle[(CM),(TM)]_\kappa = \frac{\pi}{2}, \quad \angle[(VM),(TM)]_\kappa = \cos^{-1}\left[\frac{1}{3}\right]. \]  

(5.10)

which implies these three representations describe a space of monoclinic symmetry among the Bravais lattices as depicted in Fig. 4.

Now that the physicist inside the cave can construct a set of \( \gamma \)-matrices that are identical to those which provided the starting point of the physicist in the (1, 3)-dimensional spacetime, it is possible to write the massless Dirac equation for the spinors in the cave in the form

\[ \delta^{ik} \partial_r \Psi_k = 0 \rightarrow (C\gamma^\mu)^{ik} \partial_\mu \Psi_k = 0, \]  

(5.11)

where the \( \gamma \)-matrices are constructed solely from the data in (4.4), (4.5), (4.6), and (5.3). The function \( \Psi_k \) is now permitted to depend on all the four coordinates. In the final expression the \( C \)-matrix is the spinor metric (see appendix A in [1]). In other words, the three valise adinkras in Figure 1, Figure 2, and Figure 3 contain sufficient information to reconstruct the (1, 3) spinor bundle.

Once the full set of four Dirac matrices has been identified as in (5.11), the entire Dirac algebra, \( \{1, \gamma^\mu, \gamma^{[\mu\nu]}, \gamma^{[\mu\nu\rho]}, \gamma^5\} \) is obtained “for free.” This contains the six matrices \( \gamma^{[\mu\nu]} \) that generate the
full Lorentz group Spin(1, 3) in the standard way [11,12]. That is, the identification (5.11) allows the physicist from Plato’s Cave to reconstruct the Lorentz symmetry of the objects casting the shadows that she has been examining.

6 Conclusions

In this work, we have relied the results in [1] to derive the (1, 3)-dimensional Dirac operator by using data that are available to a physicist who is investigating one dimensional representations of 1D, \( N = 4 \) SUSY. Using data which follow from adinkras, we have given a demonstration that if one possesses a description of the one dimensional multiplet that corresponds to the (1, 3)-dimensional chiral, vector and tensor supermultiplets, this is sufficient to reconstruct the Dirac equations for the higher dimensional spinors. There are a number of steps that must be demonstrated to show how to use the adinkra-based data to reconstruct the appropriate (1, 3) Lorentz covariant equations for the bosons. This will be undertaken in a later work.

When contrasting this work with discussion given in [8], we see that there is a large amount of data for 1D valise supermultiplets (described by equation (4.3)) about the spinor representations that are related by dimensional extension. These works do not take advantage of this fact and it seems likely that any algorithm for constructing higher dimensional and more complicated SUSY representations might benefit from a thorough study of the structure described in this work prior to applying node lifting. Related to such efforts, it seems reasonable to bring to bare the “No Two-Color Ambidextrous Bow-Tie” Theorem discovered in [9]. Since all four dimensional SUSY theories must also possess consistent two dimensional SUSY truncations, combining the present results with the “No Two-Color Ambidextrous Bow-Tie” Theorem will likely enhance the efficacy of search algorithms with the goal of the discovery of new off-shell SUSY representations.

Let us note that although the formula of (5.3) was derived in the context of some 4D, \( \mathcal{N} = 1 \) supermultiplets, it can also be viewed as a definition of how to define adinkras outside the context of supersymmetrical theories. In particular, whenever there are spin-1/2 fermions present in some construction, this equation defines adinkras (via the L-matrices) that are associated with the \( \gamma \)-matrices that act on the spinors. In other words, given a set of \( \gamma \)-matrices one can use (5.3) to derive an associated set of adinkras and L-matrices without the presence of supersymmetry. But of course such adinkras cannot be related to any bosons in such a theory as this seems only to occur in the presence of supersymmetry.

There are indications that there is (much) more to this story, and we will continue our investigations.

“Suppose further,” Socrates says, “that the man was compelled to look at the fire: wouldn’t he be struck blind and try to turn his gaze back toward the shadows, as toward what he can see clearly and hold to be real?” — Plato
Acknowledgments: SJG’s and KS’s research was supported in part by the endowment of the John S. Toll Professorship, the University of Maryland Center for String & Particle Theory, National Science Foundation Grant PHY-09-68854. TH is grateful to the Physics Department of the Faculty of Natural Sciences of the University of Novi Sad, Serbia, for recurring hospitality and resources. Some Adinkras were drawn with the help of the Adinkramat © 2008 by G. Landweber.

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