Dirac-Bergmann Constraints in Relativistic Physics: Non-Inertial Frames, Point Particles, Fields and Gravity

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Abstract

There is a review of the physical theories needing Dirac-Bergmann theory of constraints at the Hamiltonian level due to the existence of gauge symmetries. It contains:

i) the treatment of systems of point particles in special relativity both in inertial and non-inertial frames with a Wigner-covariant way of eliminating relative times in relativistic bound states;

ii) the description of the electro-magnetic field in relativistic atomic physics and of Yang-Mills fields in absence of Gribov ambiguity in particle physics;

iii) the identification of the inertial gauge variables and of the physical variables in canonical ADM tetrad gravity in presence of the electro-magnetic field and of charged scalar point particles in asymptotically Minkowskian space-times without super-translations by means of a Shanmugadhasan canonical transformation to a York canonical basis adapted to ten of the 14 first-class constraints and the definition of the Hamiltonian Post-Minkowskian weak field limit.

Review paper for a chapter of a future book.
I. INTRODUCTION

Most of the relevant interactions in physics are described by singular Lagrangians implying the presence of Dirac-Bergmann constraints [1–3] at the Hamiltonian level, whose treatment was given in a previous chapter. This happens for electro-magnetism, for the standard model of particle physics ($SU(3) \times SU(2) \times U(1)$ Yang-Mills fields) and its extensions, for Einstein theory of gravity and for all its generally covariant variants. Also the description of relativistic classical and quantum point particles, needed for bound states in the particle approximation of quantum field theory (QFT), requires Hamiltonian constraints for the elimination of relative times (no time-like excitation is seen in spectroscopy). In all these theories the main problem at the classical level is the identification of the gauge-invariant physical degrees of freedom, the so called Dirac observables (DO). Instead the main open problem at the quantum level is whether one has to quantize only the DO’s or also the gauge variables shifting the search of the physical observables after quantization like in the BRST approach.

I will present a review of the main properties of relativistic constrained systems (from particles to gauge theories) at the classical level first in inertial and non-inertial frames in special relativity (SR) and then in the dynamical space-times of Einstein general relativity (GR), based on my personal viewpoint, with some comments on the weak points of the existing quantization approaches.

Besides Dirac’s book [1] and Ref.[4] I recommend the books in Refs.[5, 6] for an extended treatment of many aspects of the theory also at the quantum level (included the BRST approach). Other books on the subject are in Refs. [7–10]. Instead there is no good treatment of constrained systems in mathematical physics and differential geometry: there are only partial treatments for finite-dimensional systems like presymplectic geometry [11, 12] (see Refs.[13, 14] and their bibliography for recent contributions) without any extension to infinite-dimensional systems like field theory [15].

In Section II I show the importance of constraint theory in special relativity for the description of isolated systems in non-inertial and inertial frames in Minkowski space-time by means of parametrized Minkowski theories and for the development of classical and quantum relativistic mechanics of point particles.

In Section III I give the treatment of the electro-magnetic field needed to formulate relativistic atomic physics. Then, after reviewing the problems of gauge symmetries some results on Yang-Mills fields in absence of Gribov ambiguity are described.

In Section IV there is the treatment of Einstein gravity with constraint theory in a family of space-times reducing to Minkowski space-time when the Newton constant is switched off: in these asymptotically Minkowskian space-time there is the asymptotic ADM Poincare’ group needed for the inclusion of the standard particle model in the matter. I describe the Hamiltonian constraints of ADM tetrad gravity (needed for the inclusion of fermions) after a 3+1 splitting of the space-time like in SR and the Shanmugadhasan canonical transformation to the York canonical basis adapted to 10 of the 14 first-class constraints. I make some comments on the gauge variables (like the York time), on the search of DO’s and on the canonical quantization of gravity. Then I give an idea of relativistic atomic physics plus gravity, of its Hamiltonian Post-Minkowskian (HPM) and Post-Newtonian (PN) approximations and of the possible role of the York time gauge variable in reducing at least part of dark matter to a choice of conventions in relativistic metrology.
II. NON-INERTIAL FRAMES IN SPECIAL RELATIVITY AND PARAMETRIZED MINKOWSKI THEORIES

In this Section after the definition of non-inertial frames in Minkowski space-time (Subsection A), of the inertial rest frame and of the existing notions of relativistic center of mass of isolated systems (Subsection B) there is the definition of parametrized Minkowski theories for isolated systems (Subsection C) and of relativistic quantum mechanics for point particles (Subsection D).

A. Non-Inertial Frames in Minkowski Spacetime

Assume that the world-line $x^\mu(\tau)$ of an arbitrary time-like observer carrying a standard atomic clock is given either in Minkowski space-time or in the quoted class of Einstein space-times: $\tau$ is an arbitrary monotonically increasing function of the proper time of this clock. Then one gives an admissible 3+1 splitting of the asymptotically flat space-time, namely a nice foliation with space-like instantaneous 3-spaces $\Sigma_\tau$. It is the mathematical idealization of a protocol for clock synchronization: all the clocks in the points of $\Sigma_\tau$ sign the same time of the atomic clock of the observer (see Ref.[16] for a review on relativistic metrology). The observer and the foliation define a global non-inertial reference frame after a choice of 4-coordinates. On each 3-space $\Sigma_\tau$ one chooses curvilinear 3-coordinates $\sigma^r$ having the observer as origin. The quantities $\sigma^A = (\tau; \sigma^r)$ are the either Lorentz- or world-scalar and observer-dependent radar 4-coordinates, first introduced by Bondi [17, 18].

If $x^\mu \mapsto \sigma^A(x)$ is the coordinate transformation from world 4-coordinates $x^\mu$ having the observer as origin to radar 4-coordinates [19, 20], its inverse $\sigma^A \mapsto x^\mu(\tau, \sigma^r)$ defines the embedding functions $z^\mu(\tau, \sigma^r)$ describing the 3-spaces $\Sigma_\tau$ as embedded 3-manifolds into the asymptotically flat space-time. Instead $z^\mu_r(\tau, \sigma^u)$ is a time-like 4-vector skew with respect to the 3-spaces leaves of the foliation. In SR one has $z^\mu_r(\tau, \sigma^r) = [N l^\mu + N^r z^\mu_r](\tau, \sigma^r)$ with $N(\tau, \sigma^r) = \epsilon [z^\mu_r l^\mu(\tau, \sigma^r) = 1 + n(\tau, \sigma^r) > 0$ and $N_r(\tau, \sigma^r) = -\epsilon [z^\mu_r \eta_{\mu\nu} z^\mu_r](\tau, \sigma^r)$ being the lapse and shift functions respectively of the global non-inertial frame of Minkowski space-time so defined.

The induced 4-metric $^4g_{AB}(\tau, \sigma^r) = z^\mu_A(\tau, \sigma^r) z^\mu_B(\tau, \sigma^r) ^4\eta_{\mu\nu}$ has signature $\epsilon (+ - - -)$ like the flat Minkowski metric $^4\eta_{\mu\nu}$ with $\epsilon = \pm$ (the particle physics, $\epsilon = +$, and general relativity, $\epsilon = -$, conventions). From now on we shall denote the curvilinear 3-coordinates $\sigma^r$ with the notation $\vec{\sigma}$ for the sake of simplicity. Usually the convention of sum over repeated indices is used, except when there are too many summations. The symbol $\approx$ means Dirac weak equality, while the symbol $\cong$ means evaluated by using the equations of motion.
B. The Wigner-Covariant Rest-Frame Instant Form of Dynamics and the Relativistic Center of Mass for Isolated Systems

In SR we can restrict ourselves to inertial frames and define the *inertial rest-frame instant form of dynamics for isolated systems* by choosing the 3+1 splitting corresponding to the intrinsic inertial rest frame of the isolated system centered on an inertial observer: the instantaneous Euclidean 3-spaces, named Wigner 3-space due to the fact that the 3-vectors inside them are Wigner spin-1 3-vectors [19], are orthogonal to the conserved 4-momentum $P^\mu$ of the configuration. In Ref.[19] there is also the extension to admissible *non-inertial rest frames*, where $P^\mu$ is the asymptotic space-like hyper-planes to which the instantaneous non-Euclidean 3-spaces tend at spatial infinity.

The simplest form of the embedding of the Wigner 3-spaces in Minkowski space-time described in the inertial frame of an arbitrary inertial observer is

$$z^\mu(\tau, \vec{\sigma}) = Y^\mu(\tau) + \epsilon^\mu_\nu(\vec{h}) \sigma^\nu = Y^\mu(0) + \Lambda^\mu_\nu(\vec{h}) \sigma^\nu,$$

where $Y^\mu(\tau) = Y^\mu(0) + h^\mu \tau = z^\mu_0(\tau, 0)$ is the world-line of the external Fokker-Pryce 4-center of inertia. The Lorentz matrix $\Lambda^\mu_\nu(\vec{h})$ is obtained from the standard Wigner boost $\Lambda^\mu_\nu(P^\mu/Mc)$, sending the time-like 4-vector $P^\mu/Mc$ into (1; 0), by transforming the index $\nu$ into an index adapted to radar 4-coordinates ($\Lambda^\mu_\nu \mapsto \Lambda^\mu_\nu$). One has $\epsilon^\mu_\nu(\vec{h}) = \frac{P^\mu}{Mc} = w^\nu(P) = h^\mu = \sqrt{1 + \vec{h}^2}; \vec{h} = \Lambda^\mu_\tau(\vec{h}), \epsilon^\mu_\nu(\vec{h}) = \left(\delta^{\mu}_{\tau} + \frac{h^{\mu}h^\nu}{1 + \sqrt{1 + h^2}}\right) = \Lambda^\mu_\nu(\vec{h})$ with $4 \eta_{\mu\nu} \epsilon^\mu_A(\vec{h}) \epsilon^\nu_B(\vec{h}) = 4 \eta_{AB}$ and $\epsilon P^2 = M^2 c^2$.

The form of this embedding is a consequence of the clarification of the notion of relativistic center of mass of an isolated system after a century of research [21, 22]. It turns out that there are only three notions of collective variables, which can be built using only the Poincaré generators (they are *non-local* quantities expressed as integrals over the whole 3-space $\Sigma_\tau$) of the isolated system: the *canonical non-covariant Newton-Wigner 4-center of mass* (or center of spin) $\tilde{R}^\mu(\tau)$, the *non-canonical covariant Fokker-Pryce 4-center of inertia* $Y^\mu(\tau)$ and the *non-canonical non-covariant Moller 4-center of energy* $R^\mu(\tau)$. All of them tend to the Newtonian center of mass in the non-relativistic limit. These three variables can be expressed as known functions: a) of the Lorentz-scalar rest time $\tau = c T_s = h \cdot \vec{x} = h \cdot \vec{Y} = h \cdot \vec{R}$; b) of canonically conjugate Jacobi data (frozen Cauchy data) $\vec{h} = \vec{F}/Mc$ and $\vec{z} = Mc \vec{x}_{NW}(0)$ ($\{z^i, h^i\} = \delta^{ij}$) $^{1}$; c) of the invariant mass $Mc = \sqrt{\epsilon P^2}$ of the isolated system; d) of the rest spin $\vec{S}$ of the isolated system.

In each Lorentz frame one has different pseudo-world-lines describing $R^\mu$ and $\vec{S}^\mu$: the canonical 4-center of mass $\vec{x}^\mu$ lies *in between* $Y^\mu$ and $R^\mu$ in every frame. This leads to the existence of the *Moller non-covariance world-tube* [23], around the world-line $Y^\mu$ of the covariant non-canonical Fokker-Pryce 4-center of inertia $Y^\mu$. The *invariant radius* of the tube is $\rho = \sqrt{-W^2/p^2} = |\vec{S}|/\sqrt{P^2}$ where $(W^2 = -P^2 \vec{S}^2$ is the Pauli-Lubanski invariant

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$^{1}$ The 3-vector $\vec{x}_{NW}(\tau)$ is the standard Newton-Wigner non-covariant 3-position, classical counterpart of the corresponding position operator; the use of $\vec{z}$ avoids taking into account the mass spectrum of the isolated system in the description of the center of mass.
when $P^2 > 0$). This classical intrinsic radius delimits the non-covariance effects (the pseudo-world-lines) of the canonical 4-center of mass $\vec{x}_i^\mu$. They are not detectable because the Møller radius is of the order of the Compton wave-length: an attempt to test its interior would mean to enter in the quantum regime of pair production.

Every isolated system can be visualized [19] as a decoupled non-covariant collective (non-local) pseudo-particle described by the frozen Jacobi data $\vec{z}, \vec{h}$ carrying a pole-dipole structure, namely the invariant mass $M_c$ and the rest spin $\vec{S}$ of the system, and with an associated external realization of the Poincaré group (the last term in the Lorentz boosts induces the Wigner rotation of the 3-vectors inside the Wigner 3-spaces): $P^\mu = M_c h^\mu = M_c \left( \sqrt{1 + h^2}; h \right), J^{ij} = z_i^j h^j_z - z_j^i h^i_z + \epsilon^{ijk} S^k, K^i = J^{oi} = -\sqrt{1 + h^2} z_i + \frac{(\vec{S} \times \vec{h})^i}{1 + \sqrt{1 + h^2}}$, satisfying the Poincaré algebra. The universal breaking of Lorentz covariance connected to these decoupled non-local collective variables is irrelevant because all the dynamics of the isolated system lives inside the Wigner 3-spaces and is Wigner-covariant. Inside the Wigner 3-spaces the system is described by an internal 3-center of mass with a conjugate 3-momentum and by relative variables and there is an unfaithful internal realization of the Poincaré group [19] (whose generators are determined by using the energy-momentum tensor $T^{\mu\nu}$ of the isolated system): $M_c = \int d^3 \sigma T^{\tau\tau}(\tau, \vec{\sigma}), \vec{S}^\tau = \frac{1}{2} \delta^{\tau\tau} \epsilon_{svu} \int d^3 \sigma \sigma^{\tau\nu} T^{\tau\nu}(\tau, \vec{\sigma}), \vec{P}^\tau = \int d^3 \sigma \sigma^{\tau\nu} T^{\tau\nu}(\tau, \vec{\sigma}) \approx 0, \vec{K}^\tau = -\int d^3 \sigma \sigma^{\tau\nu} T^{\tau\nu}(\tau, \vec{\sigma}) \approx 0$. The internal 3-momentum, conjugate to the internal 3-center of mass, vanishes due the rest-frame condition $^3$. To avoid a double counting of the center of mass, i.e. an external one and an internal one, the internal (interaction-dependent) Lorentz boosts must also vanish. The only non-zero internal generators are the invariant mass $M_c$ and the rest spin $\vec{S}$ and the dynamics is re-expressed only in terms of internal Wigner-covariant relative variables. In the Wigner 3-spaces the effective Hamiltonian is the invariant mass of the isolated system: $H = M_c$.

As shown in Refs.[19, 24] for $N$ free positive energy spinless particles their world-lines are parametrized in terms of Wigner 3-vectors $\vec{v}_i(\tau), i = 1, \ldots, N$, in the following way $x_i^\mu(\tau) = z^\mu(\tau, \vec{v}_i(\tau))$: one eliminates the possibility to have time-like excitations in the spectrum of relativistic bound states, because inside each 3-space only space-like correlations among the particles are possible. At the Hamiltonian level the basic canonical variables describing the particle are $\vec{v}_i(\tau)$ and their canonically conjugate momenta $\vec{\kappa}_i(\tau): \{\vec{v}_i^\mu(\tau), \kappa_j^\nu(\tau)\} = \delta_{ij} \delta^{\tau\nu}$. The standard momenta of the positive-energy scalar particles are $p_i^\mu(\tau) = \Lambda^{\mu\nu}(\vec{v}_i, \vec{v}_i) \kappa_i^\nu(\tau), \kappa_i^\nu(\tau) = (E_i(\tau); \kappa_{i\nu}(\tau))$. For free particles we have $E_i(\tau) = \sqrt{m_i^2 c^2 + \vec{v}_i^2(\tau)}, \epsilon p_i^\mu = m_i c^2$ and $M_c = \sum_i E_i$. The internal generators have the following expression in the rest frame $M_c = \frac{1}{c} \mathcal{E}(\text{int}) = \sum_{i=1}^N \sqrt{m_i^2 c^2 + \vec{v}_i^2}, \vec{P} = \mathcal{P}_2 = \sum_{i=1}^N \vec{v}_i^2, \vec{K} = -\sum_{i=1}^N \vec{v}_i \sqrt{m_i^2 c^2 + \vec{v}_i^2} \approx 0$.

\footnote{The use of $z$ avoids taking into account the mass spectrum of the isolated system at the quantum kinematical level as already said. Moreover, since the center of mass is decoupled, its non-covariance is irrelevant: like for the wave function of the universe, who will observe it?}

\footnote{Due to the rest-frame condition $\vec{P} \approx 0$, we have $\vec{q}_+ \approx R_+ \approx \vec{y}_+$, where $\vec{q}_+$ is the internal canonical 3-center of mass (the internal Newton-Wigner position), $\vec{y}_+$ is the internal Fokker-Pryce 3-center of inertia and $\vec{R}_+$ is the internal Møller 3-center of energy. As a consequence there is a unique internal 3-center of mass, which is eliminated by the vanishing of the internal Lorentz boosts $\mathcal{K}^\tau \approx 0$.}
In the interacting case it is \( E_i \neq \sqrt{m_i^2 c^2 + \vec{r}_i^2(\tau)} \) and \( \epsilon p_i^2 \neq m_i^2 c^2 \). Instead \( E_i(\tau) \) must be deduced from the form of the invariant mass \( Mc \), which is the Hamiltonian for the \( \tau \)-evolution in the Wigner 3-spaces. This description is not in contrast with scattering theory, where \( \epsilon p_i^2 = m_i^2 c^2 \) holds asymptotically for the in and out free particles (no interpolating description due to Haag no-go theorem for the interaction picture: see Ref.[24] for the way out, at least at the classical level, from this theorem in the 3+1 point of view), if the action-at-a-distance potentials (and also interactions with electro-magnetic fields) go to zero for large separations of the particles inside Wigner 3-spaces (the cluster separability in action-at-a-distance theories). See Refs.[25–28] for the description of relativistic bound states. In relativistic kinetic theory and in relativistic statistical mechanics \( E_i = \sqrt{m_i^2 c^2 + \vec{r}_i^2(\tau)} \) holds for a gas of non-interacting particles, otherwise it has to be replaced with an expression dictated by the type of the existing interactions (see Ref.[29] on the relativistic microcanonical ensemble).

In the two-body case, by introducing the notation \( \vec{n}_+ , \vec{n}_+ = \vec{P} \), with a canonical transformation we get the following internal collective and relative variables \( \vec{n}_+ = \frac{m_+}{m} \vec{n}_1 + \frac{m_-}{m} \vec{n}_2 \), \( \vec{p}_+ = \vec{n}_1 - \vec{n}_2 \), \( \vec{r}_+ = \vec{r}_1 + \vec{r}_2 \approx 0 \), \( \vec{f}_+ = \frac{m_+}{m} \vec{r}_1 - \frac{m_-}{m} \vec{r}_2 \). The collective variable \( \vec{n}_+(\tau) \) has to be determined in terms of \( \vec{p}(\tau) \) by means of the gauge fixings \( \vec{K} \approx 0 \). For two free particles we get \( \vec{n}_+(\tau) \approx \vec{n}(\tau) = \frac{m}{\sqrt{m^2 c^2 + \vec{p}^2(\tau)}} \left( \frac{m^2 c^2 + \vec{p}^2(\tau)}{\sqrt{m_1^2 c^2 + \vec{p}^2(\tau)} + \sqrt{m_2^2 c^2 + \vec{p}^2(\tau)}} \right) \vec{p}(\tau) \) (\( \vec{n}_+(\tau) \approx 0 \) for \( m_1 = m_2 \)).

In the interacting case the rest-frame conditions \( \vec{K}_+ \approx 0 \) and the conditions eliminating the internal 3-center of mass \( \vec{K} \approx 0 \) will determine \( \vec{n}_+ \) in terms of the relative variables \( \vec{p}_+ \), \( \vec{f}_+ \) in an interaction-dependent way. Then the relative variables satisfy Hamilton equations with the invariant mass \( M(\vec{p}_+, \vec{f}_+) \) as Hamiltonian and the particle world-lines \( x_i^\mu(\tau) \) can be rebuilt [28] (they are 4-vectors but not canonical variables like in most of the approaches, with this non-commutative structure induced by the Lorentz signature of the space-time).

### C. Parametrized Minkowski Theories for Isolated Systems

In the global non-inertial frames of Minkowski space-time it is possible to describe isolated systems (particles, strings, fields, fluids) admitting a Lagrangian formulation by means of parametrized Minkowski theories [19, 30] (see Refs.[31] for reviews). The existence of a Lagrangian, which can be coupled to an external gravitational field, makes possible the determination of the matter energy-momentum tensor \( T^{\mu \nu} \) and of the ten conserved Poincaré generators \( P^\mu \) and \( J^{\mu \nu} \) (assumed finite) of every configuration of the isolated system.

First of all one must replace the matter variables of the isolated system with new ones knowing the clock synchronization convention defining the 3-spaces \( \Sigma_\tau \). As said a positive (or negative-) energy relativistic particle with world-line \( x^\mu(\tau) = z^\mu(\tau, \vec{n}(\tau)) \) is described by the 3-coordinates \( \eta^\mu(\tau) \) defined by the intersection of its world-line with \( \Sigma_\tau \). A Klein-Gordon field \( \phi(x) \) will be replaced with \( \phi(\tau, \vec{\sigma}) = \tilde{\phi}(z(\tau, \vec{\sigma})) \); the same for every other field. Then one replaces the external gravitational 4-metric in the coupled Lagrangian with the 4-metric \( g_{AB}(\tau, \vec{\sigma}) = z_A^\mu(\tau, \vec{\sigma}) z_B^{\nu\prime}(\tau, \vec{\sigma}) 4\eta_{\mu\nu} \), which is a functional of the embedding 4.

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4 As an example one may consider N free scalar particles with masses \( m_i \), sign of the energy \( \eta_i = \pm \) and world-lines \( x_i^\mu(\tau) = z^\mu(\tau, \vec{n}_i(\tau)) \), \( i = 1, ..., N \). In parametrized Minkowski theories they are described by
Parametrized Minkowski theories are defined by the resulting Lagrangian depending on the given matter and on the embedding \( z^\mu(\tau, \vec{\sigma}) \). The resulting action is invariant under the *frame-preserving diffeomorphisms* \( \tau \mapsto \tau'(\tau, \sigma^u) \), \( \sigma^r \mapsto \sigma^r'(\sigma^u) \) firstly introduced in Ref.[32]. As a consequence, there are four first-class constraints with exactly vanishing Poisson brackets (an Abelianized analogue of the super-Hamiltonian and super-momentum constraints of canonical gravity) determining the momenta \( \rho_\mu(\tau, \vec{\sigma}) \) conjugated to the embeddings in terms of the matter energy-momentum tensor: 

\[
\rho_\mu(\tau, \vec{\sigma}) - \sqrt{\gamma(\tau, \vec{\sigma})} \left[ l_\mu T_{\perp\perp} - z_{\tau\mu} h^{rs} T_{\perp s} \right](\tau, \vec{\sigma}) \approx 0,
\]

where \( h^{rs} \) is the inverse of the induced 3-metric on the 3-spaces, \( T_{\perp\perp} = l_\mu l_\nu T^{\mu\nu} \) and \( T_{\perp r} = l_\mu z_{\tau\nu} T^{\mu\nu} \). The ten external Poincaré generators are 

\[
P^\mu = \int d^3\sigma \rho^\mu(\tau, \vec{\sigma}), \quad J^{\mu\nu} = \int d^3\sigma \left( z^\mu \rho^\nu - z^\nu \rho^\mu \right)(\tau, \vec{\sigma}).
\]

This implies that the embeddings \( z^\mu(\tau, \sigma^u) \) are gauge variables, so that all the admissible non-inertial or inertial frames are gauge equivalent, namely physics does not depend on the clock synchronization convention and on the choice of the 3-coordinates \( \sigma^r \): only the appearances of phenomena change by changing the notion of instantaneous 3-space \(^5\). To describe the physics in a given admissible non-inertial frame described by an embedding \( z^\mu_F(\tau, \sigma^u) \) one must add the gauge-fixings \( z^\mu_F(\tau, \sigma^u) - z^\mu(\tau, \sigma^u) \approx 0 \).

The same description can be given for spinning particles \(^6\), massless particles \(^36\) and 2-level atoms \(^37\). The open and closed Nambu string have been studied \(^38, 39\) in the stratum \( \varepsilon p^2 > 0 \): Abelian Lorentz scalar constraints and gauge variables have been found and globally decoupled, and a redundant set of DO’s has been found. Then the rest-frame formulation and a basis of DO’s has been found for the open Nambu string \(^40\).

### D. Quantum Relativistic Point Particles for Bound States

A new formulation of *relativistic quantum mechanics* (RQM) in the Wigner 3-spaces of the inertial rest frame is developed in Ref.[41] in absence of the electro-magnetic field. It includes all the known results about relativistic bound states (absence of relative times) and avoids the causality problems of the Hegerfeldt theorem \(^42\) (the instantaneous spreading of wave packets). See the bibliography of Ref. [41] for all the previous non satisfactory attempts to define a consistent RQM.

In non-relativistic quantum mechanics (NRQM) the Hilbert space of a quantum two-body system can be described in the three following *unitarily equivalent* ways \(^41\): A) as the tensor product \( H = H_1 \otimes H_2 \), where \( H_i \) are the Hilbert spaces of the two particles (separability of the two subsystems as the zeroth postulate of NRQM); B) as the tensor product \( H = H_{com} \otimes H_{rel} \), where \( H_{com} \) is the Hilbert space of the decoupled free Newton center of mass and \( H_{rel} \) the Hilbert space of the relative motion (in the interacting case only this presentation implies the separation of variables in the Schrödinger equation); C) as the following action depending on the configurational variables \( \eta_i^\mu(\tau) \) and \( z^\mu(\tau, \vec{\sigma}) \): 

\[
S = \int d\tau d^3\sigma \mathcal{L}(\tau, \vec{\sigma}) = \int d\tau d^3\sigma \left( -\sum_{i=1}^N \delta^3(\sigma^u - \eta_i^\mu(\tau)) m_i c \eta_i \sqrt{c^4 g_{\tau\tau}(\tau, \sigma^u) + 2^4 g_{\tau r}(\tau, \sigma^u) \eta_i^r(\tau) + 4^4 g_{rs}(\tau, \sigma^u) \eta_i^r(\tau) \eta_i^s(\tau)} \right).
\]

\(^5\) In Refs.[33, 34] there is the definition of *parametrized Galilei theories*, non relativistic limit of the parametrized Minkowski theories. Also the inertial and non-inertial frames in Galilei space-time are gauge equivalent in this formulation.

\(^6\) The pseudo-classical description of the spin is made with Grassmann variables, whose quantization leads to Clifford algebras including Pauli and Dirac matrices.
tensor product $H = H_{\text{HJcom}} \otimes H_{\text{rel}}$, where $H_{\text{HJcom}}$ is the Hilbert space of the frozen Jacobi data of the Newton center of mass (use is made of the Hamilton-Jacobi transformation). Each of these three presentations gives rise to a different notion of entanglement due to the different notion of separable subsystems.

As shown in Ref.[41], at the relativistic level the elimination of the relative times of the particles and the treatment of the relativistic collective variables allows only the presentation C), i.e. $H = H_{\text{HJcom}} \otimes H_{\text{rel}}$ with $H_{\text{HJcom}}$ being the Hilbert space associated to the quantization of the canonically conjugate frozen Jacobi data $\vec{z}$ and $\vec{h}$ and $H_{\text{rel}}$ is the Hilbert space of the Wigner-covariant relative 3-coordinates and 3-momenta. As a consequence, at the relativistic level the zeroth postulate of non-relativistic quantum mechanics does not hold: the Hilbert space of composite systems is not the tensor product of the Hilbert spaces of the sub-systems. Contrary to the standard notion of separability (separate objects have their independent real states) one gets a kinematical spatial non-separability induced by the need of clock synchronization for eliminating the relative times in relativistic bound states and to be able to formulate a well-posed relativistic Cauchy problem. Moreover one has the non-locality of the non-covariant external center of mass which implies its non-measurability with local instruments. While its conjugate momentum operator must be well defined and self-adjoint, because its eigenvalues describe the possible values for the total momentum of the isolated system (the momentum basis is therefore a preferred basis in the Hilbert space), it is not clear whether it is meaningful to define center-of-mass wave packets. These non-locality and kinematical spatial non-separability are due to the Lorentz signature of Minkowski space-time.

Let us remark that instead of starting from the physical Hilbert space containing the frozen Jacobi data, one could first define an un-physical Hilbert space containing the Jacobi data and the 3-position and 3-momenta of the particles (in it we have the same kind of separability as in the presentation A) of NRQM) and then define the physical Hilbert space by imposing the rest-frame conditions at the quantum level with the Gupta-Bleuler method. However there is the risk to get an inequivalent quantum theory due to the complex form of the internal boosts.

The quantization defined in Ref.[41] leads to a first formulation of a theory for relativistic entanglement, which is deeply different from the non-relativistic entanglement due to the kinematical non-locality and spatial non-separability. To have control on the Poincaré group one needs an isolated systems containing all the relevant entities (for instance both Alice and Bob) of the experiment under investigation and also the environment when needed. One has to learn to reason in terms of relative variables adapted to the experiment like molecular physicists do when they look to the best system of Jacobi coordinates adapted to the main chemical bonds in the given molecule. This theory has still to be developed together with its extension to non-inertial rest frames. See Refs.[43, 44] for more details on the notion of relativistic entanglement and on the problem of localization of particles.

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7 If one considers the tensor product $H_1 \otimes H_2$ of two massive Klein-Gordon particles most of the states will have one particle allowed to be the absolute future of the other due to the lack of restrictions on the relative times. Only in S-matrix theory is this irrelevant since one takes the limit for infinite future and past times. Therefore this is a unitarily inequivalent quantization.
III. CLASSICAL GAUGE FIELDS IN THE REST-FRAME INSTANT FORM

By means of parametrized Minkowski theories it has been possible to reformulate relativistic fluids [45–47] (starting from the viewpoint of Ref.[48]) and classical fields in non-inertial frames and in the rest-frame instant form with the only condition that the 10 conserved generators of the Poincaré algebra are finite. In Ref. [49] this is done for the Klein-Gordon field and for it a canonical basis containing the relativistic collective center-of-mass and relative variables was found. The same was done for the Dirac equation in Refs.[50, 51] after the elimination of the existing second class constraints implied by the first order Dirac Lagrangian.

Inspired by Dirac [52], who showed that the DO’s of the electromagnetic field are the transverse vector potential $\vec{A}_\perp$ and the transverse electric field $\vec{E}_\perp$ of the radiation gauge, we describe Maxwell theory in the rest frame in Subsection A. This formulation has been used in the semi-classical approximation to get a consistent description of $N$ charged particles plus the electromagnetic field [19, 24], if we use Grassmann-valued electric charges to regularize the Coulomb self-energies. In Ref.[25–27] the electromagnetic degrees of freedom are expressed in terms of the particle variables by means the Lienard-Wiechert solution and this allows to find the relativistic Darwin potential (or the Salpeter potential for spinning particles) starting from classical electrodynamics and not as a reduction from QFT. In these papers it is shown that when a fermion field is interacting with the electromagnetic field, the fermionic DO is a fermion field dressed with a Coulomb cloud.

Then in Subsection B we will look at the Yang-Mills field whose rest-frame formulation was done in Ref.[53] in the framework of the quark model (with scalar quarks). This paper puts in Wigner-covariant form the results of previous papers on the use of constraint theory for the description of Yang-Mills theory with fermions [54], of Higgs models [55] and of the $SU(3) \times SU(2) \times U(1)$ model [56].

A. The Electro-Magnetic Field and its Dirac Observables

As shown in Refs.[19, 24] in parametrized Minkowski theories the configuration variable for the electro-magnetic field is the Lorentz-scalar potential $A_A(\tau, \vec{\sigma}) = z^\mu_A(\tau, \vec{\sigma}) \tilde{A}_\mu(\tau, \vec{\sigma})$, whose associated field strength is $F_{AB}(\tau, \vec{\sigma}) = \partial_A A_B(\tau, \vec{\sigma}) - \partial_B A_A(\tau, \vec{\sigma}) = z^\mu_A(\tau, \vec{\sigma}) z^\nu_B(\tau, \vec{\sigma}) \tilde{F}_{\mu\nu}(\tau, \vec{\sigma}))$. After the restriction to the inertial rest frame the conjugate electro-magnetic momentum variables are a scalar $\pi^r(\tau, \vec{\sigma})$ and a Wigner 3-vector $\pi^r(\tau, \vec{\sigma}) = E^r(\tau, \vec{\sigma})$. The primary first-class constraint is $\pi^r(\tau, \vec{\sigma}) \approx 0$, while the secondary first-class constraint is the Gauss law $\Gamma(\tau, \vec{\sigma}) = \partial \cdot \pi(\tau, \vec{\sigma}) \approx 0$. They are the generators of the Hamiltonian electro-magnetic gauge transformations. $E^r(\tau, \vec{\sigma})$ and $B^r(\tau, \vec{\sigma}) = (\partial \times \vec{A}_\perp)^r(\tau, \vec{\sigma})$ are the components of the electric and magnetic fields.

The gauge degrees of freedom $(A_r, \eta)$ have been separated from the transverse DO’s of Ref.[52] $(A_\perp, \pi^r_\perp = E^r_\perp) (\partial \cdot \vec{A}_\perp = \partial \cdot \vec{\pi}_\perp = 0)$ by means of a Shanmughadhasan canonical transformation [19, 24, 57] adapted to the two scalar first class constraints $(\Delta = -\partial^2_\sigma, \Box = \partial^2_r + \triangle)$.
\[ A_{\tau}(\tau, \vec{\sigma}) = \partial_{\tau} \eta(\tau, \vec{\sigma}) + A'_{\perp}(\tau, \vec{\sigma}), \quad \pi_{\tau}(\tau, \vec{\sigma}) = \pi'_{\perp}(\tau, \vec{\sigma}) + \frac{1}{\Delta_{\sigma}} \frac{\partial}{\partial \sigma^r} \Gamma(\tau, \vec{\sigma}), \]

\[ \eta(\tau, \vec{\sigma}) = -\frac{1}{\Delta_{\sigma}} \frac{\partial}{\partial \sigma^r} \cdot \vec{A}(\tau, \vec{\sigma}), \]

\[ A'_{\perp}(\tau, \vec{\sigma}) = (\delta^{rs} + \frac{\partial_{\sigma}^{s} \partial_{\sigma}^{r}}{\Delta_{\sigma}}) A_{s}(\tau, \vec{\sigma}), \quad \pi'_{\perp}(\tau, \vec{\sigma}) = (\delta^{rs} + \frac{\partial_{\sigma}^{s} \partial_{\sigma}^{r}}{\Delta_{\sigma}}) \pi_{s}(\tau, \vec{\sigma}), \]

\[ \{A_{\tau}(\tau, \vec{\sigma}), \pi_{\tau}(\tau, \vec{\sigma}')\} = -\{\eta(\tau, \vec{\sigma}), \Gamma(\tau, \vec{\sigma}')\} = \delta^3(\vec{\sigma} - \vec{\sigma}'), \]

\[ \{A'_{\tau}(\tau, \vec{\sigma}), \pi'_{\perp}(\tau, \vec{\sigma}')\} = -(\delta^{rs} + \frac{\partial_{\sigma}^{s} \partial_{\sigma}^{r}}{\Delta_{\sigma}}) \delta^3(\vec{\sigma} - \vec{\sigma}'). \] (3.1)

The Dirac Hamiltonian is \((\lambda_{\tau}(\tau, \vec{\sigma})\) is the arbitrary Dirac multiplier associated to the primary constraint \(\pi_{\tau}(\tau, \vec{\sigma}) \approx 0)\)

\[ H_{D} = H_{c} + \int d^{3}\sigma \left[ \lambda_{\tau} \pi_{\tau} - A_{\tau} \Gamma(\tau, \vec{\sigma}) \right], \quad H_{c} = \frac{1}{2} \int d^{3}\sigma \left[ \vec{\pi}_{\perp}^{2} + \vec{B}^{2} \right](\tau, \vec{\sigma}), \]

\[ \Downarrow \quad \text{kinematical Hamilton equations} \]

\[ \partial_{\tau} A_{\tau}(\tau, \vec{\sigma}) \overset{\circ}{=} \lambda_{\tau}(\tau, \vec{\sigma}), \quad \partial_{\tau} \eta(\tau, \vec{\sigma}) \overset{\circ}{=} A_{\tau}(\tau, \vec{\sigma}), \quad \partial_{\tau} A'_{\perp}(\tau, \vec{\sigma}) \overset{\circ}{=} -\pi'_{\perp}(\tau, \vec{\sigma}), \]

\[ \text{dynamical Hamilton equations} \]

\[ \partial_{\tau} \pi'_{\perp}(\tau, \vec{\sigma}) \overset{\circ}{=} \Delta A'_{\perp}(\tau, \vec{\sigma}), \quad \Rightarrow \quad \square A_{\perp}(\tau, \vec{\sigma}) \overset{\circ}{=} 0. \] (3.2)

To fix the gauge we must only add a gauge fixing \(\varphi_{\eta}(\tau, \vec{\sigma}) \approx 0\) to the Gauss law, which determines \(\eta\). Its time constancy, i.e. \(\partial_{\tau} \varphi_{\eta}(\tau, \vec{\sigma}) + \{\varphi_{\eta}(\tau, \vec{\sigma}), H_{D}\} = \varphi_{A_{\tau}}(\tau, \vec{\sigma}) \approx 0\), will generate the gauge fixing \(\varphi_{A_{\tau}}(\tau, \vec{\sigma}) \approx 0\) for \(A_{\tau}\) (as shown in Ref.[58] this is the correct procedure for fixing a gauge). Finally the time constancy \(\partial_{\tau} \varphi_{A_{\tau}}(\tau, \vec{\sigma}) + \{\varphi_{A_{\tau}}(\tau, \vec{\sigma}), H_{D}\} \approx 0\) will determine the Dirac multiplier \(\lambda_{\tau}(\tau, \vec{\sigma})\). By adding these two gauge fixing constraints to the first class constraints \(\pi_{\tau}(\tau, \vec{\sigma}) \approx 0, \Gamma(\tau, \vec{\sigma}) \approx 0, \) one gets two pairs of second class constraints allowing the elimination of the gauge degrees of freedom so that only the DO’s survive.

As shown in Refs.[19, 24] one can add point particles to define relativistic atomic physics and to study relativistic bound states. Moreover as shown in Ref.[59] the standard Lagrangian density for the electromagnetic field coupled to Dirac fields \(\mathcal{L}(x) = \)

\[
\begin{array}{|c|c|c|c|}
\hline
A_{\alpha} & \eta & A_{\perp r} & A_{\perp t} \\
\hline
\pi^{\alpha} & 0 & \Gamma \approx 0 & \pi_{\perp} \\
\hline
\end{array}
\]
BRS method can take them into account (they are zero measure effects). formulation and it is not clear how the ordinary path integral approach and the associated
stratified manifold with singularities
orbits, is in general a
potential
of the gauge foliation
of the allowed strata, in particular of the stratum
W
by the Pauli-Lubanski Casimir
class constraints, given by the Gauss laws and by the vanishing of the time components of
zero modes
their Hamiltonian description. It turns out that the space of solutions has a
nomenon was discovered in Refs.[60, 61] by studying the space of solutions of Yang-Mills and
ǫp
Let us consider the stratum
ǫp
in each point of this line there is a cone of solutions with one less symmetry.
structure of singularities: if we have a line of solutions with a certain number of symmetries,
three possible sources of singularities of the gauge foliation of Yang-Mills theory in the
stratum \( \epsilon p^2 > 0 \) may be: i) different classes of gauge potentials identified by different values
of the field invariants; ii) the orbit structure of the rest frame (or Thomas) spin \( \vec{S} \), identified by
the Pauli-Lubanski Casimir \( W^2 = -\epsilon p^2 \vec{S}^2 \) of the Poincare’ group. The final outcome
of this structure of singularities is that the reduced phase-space, i.e. the space of the gauge
orbits, is in general a stratified manifold with singularities [60, 61]. In the stratum \( \epsilon p^2 > 0 \)
the Yang-Mills theory these singularities survive the Wick rotation to the Euclidean
formulation and it is not clear how the ordinary path integral approach and the associated
BRS method can take them into account (they are zero measure effects).

In the Yang-Mills case there is also the problem of the gauge symmetries of a gauge
potential \( A_\mu(x) = A_\mu^a(x)T_a \), which are connected with the generators of its stability group,
\[ -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) + \bar{\psi}(x) \gamma^\mu (i \partial_\mu + e A_\mu(x)) \psi(x) - m \bar{\psi}(x) \psi(x) - \partial_\mu (\frac{i}{2} \bar{\psi}(x) \gamma^\mu \psi(x)) \]
has a generalized weak quasi-invariance: \( \delta L = (\eta - \partial_\mu e) \Gamma^\mu 0 \) under the generalized Hamiltonian
gauge transformations \( \delta A_\mu = \eta \) and \( \delta A_k = \partial_k e \) generated by the electro-magnetic first
class constraints. As a consequence the Noether identities implied by the second Noether
theorem reproduce Dirac’s algorithm and one can show that the improper strong conserved
electric charge \( Q(S) \) (the flux through the surface at infinity of the electric field) and the
improper weak conserved Noether electric charge \( Q(W) \) (the volume integral over the charged
fermion density) coincide due to the first class secondary Gauss law constraint:
\[ Q(S) = \int_{\Omega} d^{m-1} \Sigma_k E^k(x^o, \vec{x}) = \int_{\Omega} d^m x \left[ e [\psi^\dagger \psi](x^o, \vec{x}) + \Gamma(x^o, \vec{x}) \right]^-Q(W) = \int_{\Omega} d^m x e [\psi^\dagger \psi](x^o, \vec{x}). \]

B. Yang-Mills Fields

Notwithstanding Eqs.(3.1) for the electro-magnetic case, in general gauge field theories
the situation is more complicated, because some of the constraints are non-linear partial
differential equations (PDE) and the theorems ensuring the existence of the Shanmugadhasan
canonical transformation have not been extended to the infinite-dimensional case so that
one must use heuristic extrapolations of them. In Yang-Mills theory with gauge potentials
\( A_\mu(x) = A^a_\mu(x) T_a \)\(^8\) some of the constraints are elliptic PDE’s and they can have zero modes.
Let us consider the stratum \( \epsilon p^2 > 0 \) of free Yang-Mills theory as a prototype and its first
class constraints, given by the Gauss laws and by the vanishing of the time components of
the canonical momenta. The problem of the zero modes will appear as a singularity structure
of the gauge foliation of the allowed strata, in particular of the stratum \( \epsilon p^2 > 0 \). This phenomenon was discovered in Refs.[60, 61] by studying the space of solutions of Yang-Mills and
Einstein equations, which can be mapped onto the constraint manifold of these theories in
their Hamiltonian description. It turns out that the space of solutions has a cone over cone
structure of singularities: if we have a line of solutions with a certain number of symmetries,
in each point of this line there is a cone of solutions with one less symmetry.

\(^8\) \( T_a \) are the generators of the Lie algebra \( g \) of the Lie group \( G \) with \( T^a_a = -T_a, Tr(T_a T_b) = -\delta_{ab}, [T_a, T_b] = c_{abc} T_c \). The field strengths are \( F_{\mu\nu} = F^a_{\mu\nu} T_a = \partial_\mu A_\nu - \partial_\nu A_\mu [A_\mu, A_\nu] \) and the Lagrangian is
\( \mathcal{L} = \frac{1}{4} Tr(F_{\mu\nu} F_{\nu\lambda}) \). The canonical momenta are \( \pi^\mu = F_{\nu\mu} \). The EL equations are
\( L^a = L^a = D_\nu F^{\nu\mu} = \partial_\mu F^{\nu\mu} + [A_\mu, F^{\nu\mu}] \approx 0 \). The primary first-class constraints are \( \pi^a_\mu(x^o, \vec{x}) \approx 0 \), while the secondary ones are
the Gauss laws \( \Gamma_a(x^o, \vec{x}) = L o a \approx 0 \). The infinitesimal gauge transformations under which \( \delta \mathcal{L} \approx 0 \) are
\( \delta A_\mu(x) = \partial_\mu \epsilon_a(x) + c_{abc} A_\mu \epsilon_b(x) \).
i.e. with the subgroup of those special gauge transformations which leave invariant that
gauge potential. This is the Gribov ambiguity for gauge potentials [62] (see Ref.[63] for
a recent contribution to its mathematical aspects). There is also a more general Gribov
ambiguity for field strengths, the gauge copies problem due to those gauge transformations
leaving invariant the field strengths. For all these problems see Ref. [54] and its bibliography.

Since the Gauss laws \( \Gamma_a(x^o, \vec{x}) = D_{ab}^{(4)}(x^o, \vec{x}) \cdot \pi_b(x^o, \vec{x}) = \partial_o \pi_a(x^o, \vec{x}) + c_{abc} A_{br}(x^o, \vec{x}) \pi_c(x^o, \vec{x}) \approx 0 \) \( \{ \Gamma_a(x^o, \vec{x}), \Gamma_b(x^o, \vec{y}) \} = c_{abc} \Gamma_c(x^o, \vec{x}) \delta^3(\vec{x} - \vec{y}) \) are the
generators of the gauge transformations (and depend on the chosen gauge potential through
the covariant derivative), this means that for a gauge potential with non trivial stability
group those combinations of the Gauss laws corresponding to the generators of the stability
group cannot be any more first class constraints, since they do not generate effective gauge
transformations but special symmetry transformations. This problematic has still to be clar-
ified, but it seems that in this case these components of the Gauss laws become third class
constraints, which are not generators of true gauge transformations. This new kind of con-
straints was introduced in Refs.[57, 59] in the finite dimensional case as a result of the study
of some examples, in which the Jacobi equations (the linearization of the Euler-Lagrange
equations) are singular, i.e. some of their solutions are not infinitesimal deviations between
two neighboring extremals of the Euler-Lagrange equations. This interpretation seems to be
confirmed by the fact that the singularity structure discovered in Ref.[60, 61] follows from
the existence of singularities of the linearized Yang-Mills and Einstein equations. Due to
the Gribov ambiguity, to fix univocally a gauge one has to give topological numbers identifying
a stratum besides the ordinary gauge fixing constraints.

The search of a global canonical basis of DO’s for each stratum of the space of the gauge
orbits can give a definition of the measure of the phase space path integral, but at the price
of a non polynomial Hamiltonian. Therefore, if it is not possible to eliminate the Gribov
ambiguity (assuming that it is only a mathematical obstruction without any hidden physics
like in the approach to QCD confinement reviewed in Ref.[64]), the existence of global DO’s
for Yang-Mills theory is very problematic.

In Ref.[54] there is the study of Yang-Mills theory with Grassmann-valued fermion fields
in the case of a trivial principal bundle over a fixed-\( x^o \) \( R^3 \) slice of Minkowski space-time with
suitable Hamiltonian-oriented boundary conditions; this excludes monopole solutions and,
since \( R^3 \) is not compactified, one has only winding number and no instanton number. After
a discussion of the Hamiltonian formulation of Yang-Mills theory (with a suitable coordina-
tization of the Lie group and of the principal bundle), of its group of gauge transformations
and of the Gribov ambiguity, the theory has been studied in suitable weighted Sobolev spaces
where the Gribov ambiguity is absent [65, 66] and the global color charges are well defined.
In this case it is possible to find the non-Abelian analogue of the Abelian Shnudugad-
hasan canonical transformation (3.1) in an inertial frame of Minkowski space-time (these
results have been rewritten in the rest-frame instant form like in the electro-magnetic case
in Ref.[53]), namely gauge variables \( A^o_a(x^o, \vec{x}) \) and \( \eta_a(x^o, \vec{x}) \) canonically conjugated to the
first-class constraints \( \pi_{oa}(x^o, \vec{x}) \approx 0 \) and to Abelianized Gauss law constraints \( \Gamma_a(x^o, \vec{x}) \approx 0 \n(\{ \Gamma_a(x^o, \vec{x}), \Gamma_b(x^o, \vec{y}) \} = 0) \). Moreover there are the global DO’s, i.e. transverse quantities
\( A_{a\perp}(\vec{x}, x^o) \), \( E_{a\perp}(\vec{x}, x^o) \) (and fermion fields dressed with Yang-Mills (gluonic) clouds). All
these quantities are extremely complex, non-local and with a poor global control due to
the need of the Green function of the covariant divergence (it requires the use of path-dependent
non-integrable phases; see Ref.[53]). The nonlocal and non-polynomial (due to the presence
of classical Wilson lines along flat geodesics) physical Hamiltonian has been obtained: it is nonlocal but without any kind of singularities and it has the correct Abelian limit if the structure constants are turned off.

Again the Noether identities implied by the second Noether theorem [59] reproduce the Dirac algorithm for the identification of the constraints. In the suitable weighted Sobolev spaces eliminating the Gribov ambiguity the strong conserved charge is \( V^{\mu a} = \partial_\nu F^{\nu \mu a} = \partial_\nu U^{[\mu \nu]a} \), with the super-potential \( U^{[\mu \nu]a} = -F^{\mu \nu a} \) and the improper strong and weak conservation laws are \( \partial_\mu V^{\mu a} \equiv 0 \) and \( \partial_\mu G^{\mu 1 a} \equiv 0 \) respectively. The improper strong and weak conserved non-Abelian color charges (in absence of fermions; they would add a term like the one in the Abelian electric charge) are

\[
Q^{(S)}_a = \int_{\partial \Omega} d^{n-1} \Sigma_k F^{k a}_a \Sigma_k = Q^{(W)}_a = -C_{abc} \int_{\Omega} d^4 x F^{ab}_a F^{c k}_b A^{k c}.
\]

The following models have been studied with the described technology:

A) SU(3) Yang-Mills theory with scalar particles with Grassmann-valued color charges [53] for the regularization of self-energies. It is possible to show that in this relativistic scalar quark model the Dirac Hamiltonian expressed as a function of DO’s has the property of asymptotic freedom.

B) The Abelian and non-Abelian SU(2) Higgs models with fermion fields [55], where the symplectic decoupling is a refinement of the concept of unitary gauge. There is an ambiguity in the solutions of the Gauss law constraints, which reflects the existence of disjoint sectors of solutions of the Euler-Lagrange equations of Higgs models. The physical Hamiltonian and Lagrangian of the Higgs phase have been found; the self-energy turns out to be local and contains a local four-fermion interaction.

C) The standard SU(3)xSU(2)xU(1) model of elementary particles [56] with Grassmann-valued fermion fields. The final reduced Hamiltonian contains nonlocal self-energies for the electromagnetic and color interactions, but ”local ones” for the weak interactions implying the non-perturbative emergence of 4-fermions interactions.

IV. EINSTEIN GENERAL RELATIVITY, TETRAD GRAVITY AND THEIR CANONICAL ADM FORMULATION

In this Section we introduce the Hamiltonian description of Einstein’s GR. Instead of studying ADM metric gravity [67] we will look at tetrad gravity [68], because it is needed for the descriptions of the fermions of the particle standard model.

The use of Hamiltonian methods restricts the class of Einstein space-times to the globally hyperbolic ones, in which there is a global notion of a mathematical time parameter and of instantaneous 3-spaces to be used as Cauchy surfaces for the dynamics. The space-times must also be topologically trivial. Moreover the space-times must be asymptotically Minkowskian space-times without super-translations so to have the asymptotic ADM Poincare’ group [67, 69] (needed for particle physics, where elementary particles are always defined as irreducible representations of the Poincare’ group and all the properties are connected with the representations of this group in the inertial frames of Minkowski space-time), absent in loop quantum gravity where the 3-spaces are compact manifolds without boundary [9, 10].

In Subsection A we will describe the global non-inertial frames of the space-times. In Subsection B there is the parametrization of ADM tetrad gravity and in Subsection C its Hamiltonian formulation. In Subsection D there is a partial Shanmugadhasan canonical transformation adapted to 10 of the 14 first class constraints with a clarification on which are
the inertial gauge variables and the tidal physical degrees of freedom of GR. In Subsection E there are comments on the DO’s of GR and on the attempts of its quantization. In Subsection G there is the description of the 3-orthogonal Schwinger time gauges, replacing the harmonic ones in this framework to describe post-Minkowskian gravity in presence of point particles and electro-magnetic fields and its possible relevance for the problem of dark matter.

A. 3+1 Splittings of Asymptotically Minkowskian Space-Times

In asymptotically flat space-times we have the asymptotic symmetries of the SPI group [70] (direction-dependent asymptotic Killing symmetries). If we restrict this class of space-times to those not containing super-translations [69], the SPI group reduces to the asymptotic ADM Poincaré group ⁹: these space-times are asymptotically Minkowskian ¹⁰ and in the limit of vanishing Newton constant \( G = 0 \) the ADM Poincaré group becomes the special relativistic Poincaré group of the matter present in the space-time. In this restricted class the canonical Hamiltonian is the ADM energy [75], so that there is no frozen picture (in the reduced phase space there is a non-zero reduced Hamiltonian). In absence of matter a sub-class of these space-times is the (singularity-free) family of Christodoulou-Klainermann solutions of Einstein equations [76] (they are near to Minkowski space-time in a norm sense and contain gravitational waves).

In the considered class of Einstein space-times the ten strong asymptotic ADM Poincaré generators \( P^A_{\text{ADM}}, J^{AB}_{\text{ADM}} \) (they are fluxes through a 2-surface at spatial infinity) are well defined functionals of the 4-metric fixed by the boundary conditions at spatial infinity and of matter (when present). These ten strong generators can be expressed [69, 75] in terms of the weak asymptotic ADM Poincaré generators \( \hat{P}^A_{\text{ADM}}, \hat{J}^{AB}_{\text{ADM}} \) (integrals on the 3-space of suitable densities) plus first class constraints. The absence of super-translations implies that the ADM 4-momentum is asymptotically orthogonal to the instantaneous 3-spaces (they tend to a Euclidean 3-space at spatial infinity), namely there is an asymptotic rest frame condition \( \hat{P}^\nu_{\text{ADM}} \approx 0 \). As a consequence each 3-space of the global non-inertial frame is a non-inertial rest frame of the 3-universe. At spatial infinity there are asymptotic inertial observers carrying a flat tetrad whose spatial axes are identified by the fixed stars of star catalogues.

Instead in spatially compact space-times without boundary the canonical Hamiltonian is zero and the Dirac Hamiltonian is a linear combination of first class constraints. This fact gives rise to a frozen picture without a global evolution (the Dirac Hamiltonian generates only Hamiltonian gauge transformations; in the abstract reduced phase space, quotient with respect to such gauge transformations, the reduced Hamiltonian is zero). This class of space-times fits well with Machian ideas (no boundary conditions), with interpretations in which there is no physical time (see for instance Ref.[77]) and is used in loop quantum gravity [9, 10].

If a space-time without non-asymptotic Killing symmetries and with the fields belonging to suitable weighted Sobolev spaces is globally hyperbolic, topologically trivial, asymptoti-

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⁹ For recent reviews on this group see Refs.[71–73].

¹⁰ This class of space-times admits ortho-normal tetrads and a spinor structure [74].
cally Minkowskian and without super-translations then there is a well established Hamiltonian description of both metric and tetrad gravity [75, 78–83] (see Refs.[31, 84] for reviews). This is due to the fact that in these space-times one can make a consistent 3+1 splitting with instantaneous non-Euclidean 3-spaces (i.e. a clock synchronization convention [16]) centered on a time-like observer used as origin of (world scalar) radar 4-coordinates [17, 18]: in this way the notion of non-inertial frames defined in Minkowski space-time described in the previous Section can be extended to this class of curved space-times, where the equivalence principle forbids the existence of global inertial frames.

In GR the dynamical fields are the components \(4g_{\mu\nu}(x)\) of the 4-metric and not the embeddings \(x^\mu = z^\mu(\tau, \vec{\sigma})\) defining the admissible 3+1 splittings of space-time. Now the gradients \(z^\nu_A(\tau, \vec{\sigma})\) of the embeddings give the transition coefficients from radar to world 4-coordinates, so that the components \(4g_{AB}(\tau, \vec{\sigma}) = z^\mu_A(\tau, \vec{\sigma}) z^\nu_B(\tau, \vec{\sigma}) 4g_{\mu\nu}(z(\tau, \vec{\sigma}))\) of the 4-metric will be the dynamical fields in the ADM action \([67]\). Let us remark that not as a 3-sub-manifold of the space-time) 4-coordinates, are 4-scalars of the space-time \([83]\).

This is due to the fact that in these space-times one can make a consistent 3+1 splitting with a positive-definite Euclidean 3-metric.

Flat indices \((\alpha), \alpha = o, a,\) are raised and lowered by the flat Minkowski metric \(4\eta^{(\alpha)(\beta)} = \epsilon (+ - - -)\). We define \(4\eta^{(\alpha)(\beta)} = -\epsilon \delta^{(\alpha)(\beta)}\) with a positive-definite Euclidean 3-metric.

### B. ADM Tetrad Gravity

In tetrad gravity the 4-metric is decomposed in terms of cotetrads, \(4g_{AB} = E^{(a)}_A 4\eta^{(a)(\beta)} E^{(\beta)}_B\) and the ADM action, now a functional of the 16 fields \(E^{(a)}_A(\tau, \vec{\sigma})\), is taken as the action for ADM tetrad gravity. The diffeomorphism group (the gauge group of GR) is enlarged with the O(3,1) gauge group of the Newman-Penrose approach [85] (the extra gauge freedom acting on the tetrads in the tangent space of each point of space-time and reducing from 16 to 10 the number of independent fields in the restriction to metric gravity). This leads to an interpretation of gravity based on a congruence of time-like observers endowed with ortho-normal tetrads: in each point of space-time the time-like axis is the unit 4-velocity of the observer, while the spatial axes are a (gauge) convention for observer’s gyroscopes. This framework was developed in Refs.[75, 78].

In this framework the configuration variables are cotetrads, which are connected to cotetrads adapted to the 3+1 splitting of space-time (so that the adapted time-like tetrad is the unit normal \(l^\alpha(\tau, \vec{\sigma}) = (z^\nu_A l^A)(\tau, \vec{\sigma})\) to the 3-space \(\Sigma_\tau\) by standard Wigner boosts for time-like vectors \(^{11}\) of parameters \(\varphi^{(a)}(\tau, \vec{\sigma})\):

\[
4E^{(a)}_A = L^{(\alpha)}(\beta)(\varphi^{(a)}) 4\varphi^{(\beta)}_A.
\]

The adapted tetrads

\(^{11}\) In each tangent plane to a point of \(\Sigma_\tau\) the point-dependent standard Wigner boost for time-like Poincare’ orbits \(L^{(\alpha)}(\beta)(V(z(\sigma))); \vec{V} = \delta^{(\alpha)}_{(\beta)} + 2e V^{(a)}(z(\sigma)) \vec{V} = -e \frac{(V^{(a)}(z(\sigma)) + \frac{\delta^{(a)}_{(\beta)}}{1 + V^{(a)}(z(\sigma))}) d\sigma}{1 + V^{(a)}(z(\sigma))} d\sigma \equiv L^{(\alpha)}(\beta)(\varphi^{(a)})\) sends the unit future-pointing time-like vector \(\vec{V}^{(a)} = (1; 0)\) into the unit time-like vector \(V^{(a)} = 4E^{(a)}_A l^A = \sqrt{1 + \sum_{(a)} \varphi^{(a)}_A \varphi^{(a)} = -e \varphi^{(a)}},\) where \(l^A\) is the unit future-pointing normal to \(\Sigma_\tau\). We
and cotetrad}s have the expression $4 \hat{E}^A_{(o)} = \frac{1}{1+n}(1; - \sum_a n(a) \epsilon e^r_{(a)}) = l^A$, $4 \hat{E}^A_{(o)} = (0; \epsilon e^r_{(a)})$, $4 \hat{E}^{(o)}_A = (1+n)(1; \bar{0}) = \epsilon l_A$, $4 \hat{E}^{(a)}_A = (n(a); \epsilon e^r_{(a)r})$, where $3 \epsilon e^r_{(a)}$ and $\epsilon e^r_{(a)r}$ are triads and cotriads on $\Sigma_r$ (since we use the positive-definite 3-metric $\delta^{(a)(b)}$, we shall use only lower flat spatial indices: therefore for the cotriads we use the notation $\hat{E}^{(a)}_A = \delta^{(a)(b)} \epsilon e^r_{(a)r}$ where $\delta^{(a)(b)} = \epsilon e^r_{(a)r} e^s_{(b)s}$. The lapse and shift functions are $N(\tau, \sigma) = 1 + n(\tau, \sigma)$ and $N^r(\tau, \sigma) = n^r(\tau, \sigma)$ with $n(\tau, \sigma)$ and $n^r(\tau, \sigma)$ vanishing at spatial infinity due to the absence of super-translations [78];

$n(a) = n_r \epsilon e^r_{(a)} = n^r \epsilon e^r_{(a)r}$ are adapted shift functions. The adapted tetrads $4 \hat{E}^A_{(o)}$ are defined modulo SO(3) rotations $4 \hat{E}^A_{(o)} = \sum_b R_{(a)(b)}(\alpha(\epsilon)) 4 \hat{E}^A_{(b)}$, $\epsilon e^r_{(a)r} = \sum_b R_{(a)(b)}(\alpha(\epsilon)) \epsilon e^r_{(b)}$, where $\alpha(\epsilon)(\tau, \sigma)$ are three point-dependent Euler angles. After having chosen an arbitrary point-dependent origin $\alpha(\epsilon)(\tau, \sigma) = 0$, we arrive at the following adapted tetrads and cotetrad}s

$[\bar{n}(a) = \sum_b R_{(b)(a)}(\alpha(\epsilon))\; ; \; \sum_a n(a) \epsilon e^r_{(a)} = \sum_a \bar{n}(a) \epsilon e^r_{(a)}]$

$$4 \hat{E}^{(o)}_A = (1+n)(1; \bar{0}) = \epsilon l_A, \quad 4 \hat{E}^{(a)}_A = \epsilon l_A, \quad 4 \hat{E}^{(o)}_{(o)} = (\bar{n}(a); \epsilon e^r_{(a)r}),$$

(4.1)

which we shall use as a reference standard. We have $4g_{AB} = 4 \hat{E}^{(a)}_A 4 \eta^{(a)(b)} 4 \hat{E}^{(b)}_B$. The expressions for the general tetrad and for the 4-metric (the 3-metric $4g_{rs}$ has signature $(+++)$, so that we may put all the flat 3-indices down; we have $\eta^{rs} 4 g_{us} = \delta^r_s$) are

$$4 E^A_{(c)} = 4 \hat{E}^{(o)}_A \epsilon L^{(c)}(\phi(a)) = 4 \hat{E}^{(c)}_A \epsilon L^{(c)}(\phi(c)) + \sum_{ab} 4 \hat{E}^{(o)}_{(b)} \epsilon R^T_{(b)(a)}(\alpha(\epsilon)) \epsilon L^{(c)}(\phi(c)),$$

$$4 g_{rr} = \epsilon [(1+n)^2 - 3 \epsilon e^s_{(r)s} n_r n_s] = \epsilon [(1+n)^2 - \sum_a \bar{n}^2_{(a)}], \quad 4 g_{rr} = -\epsilon n_r = -\epsilon \sum_a \bar{n}_{(a)} \epsilon e^r_{(a)r},$$

$$4 g_{rs} = -\epsilon \sum_a 3 \epsilon e^r_{(a)r} e^s_{(a)s} = -\epsilon \sum_a \epsilon e^r_{(a)r} e^s_{(a)s},$$

$$\sqrt{-g} = \sqrt{4g} = \sqrt[4]{{\epsilon e^s_{(r)s}}} = \sqrt{(1+n)} = 3\epsilon (1+n), \quad 3 g = \gamma = (3\epsilon)^2, \quad 3 e = det 3\epsilon e^r_{(a)r}.$$  

(4.2)

Each 3+1 splitting of an either Minkowski or asymptotically Minkowskian space-time, i.e. each global non-inertial frame, has two associated congruences of time-like observers:

have $L^{-1}(\phi(a)) = 4 \eta L^T(\phi(a)) 4 \eta = L(-\phi(a))$. As a consequence, the flat indices $(a)$ of the adapted tetrads and cotetrad}s and of the triads and cotriads on $\Sigma_r$ transform as Wigner spin-1 indices under point-dependent SO(3) Wigner rotations $R_{(a)(b)}(V(z(\sigma))\; ; \; L(z(\sigma)))$ associated with Lorentz transformations $\Lambda^{(a)(b)}(z)$ in the tangent plane to the space-time in the given point of $\Sigma_r$. Instead the index $(o)$ of the adapted tetrads and cotetrad}s is a local Lorentz scalar index.
i) The congruence of the Eulerian observers with the unit normal \( l^\mu (\tau, \vec{\sigma}) = \left( z^\mu_A l^A \right)(\tau, \vec{\sigma}) \) to the 3-spaces as unit 4-velocity; the world-lines of these observers are the integral curves of the unit normal and in general are not geodesics; in adapted radar 4-coordinates the Eulerian observers carry the tetrads defined in Eqs.(4.1); ii) The skew congruence with unit 4-velocity \( v^\mu (\tau, \vec{\sigma}) = \left( z^\mu_A v^A \right)(\tau, \vec{\sigma}) \) (in general it is not surface-forming, i.e. it has a non-vanishing vorticity); the observers of the skew congruence have the world-lines (integral curves of the 4-velocity) defined by \( \sigma^r = \text{const.} \) for every \( \tau \), because the unit 4-velocity tangent to the flux lines \( x^\mu_{\vec{\phi}_a}(\tau) = z^\mu(\tau, \vec{\sigma}_o) \) is \( v^\mu_{\vec{\phi}_a}(\tau) = z^\mu(\tau, \vec{\sigma}_o)/\sqrt{\epsilon^A g_{\tau\tau}(\tau, \vec{\sigma}_o)} \); see Ref.[84] for their adapted tetrads and for their relevance in the description of physics in GR.

C. Hamiltonian ADM Tetrads Gravity

The 16 configurational variables in the ADM action are \( \varphi(a), 1 + n, n_i(a), 3 \epsilon_r(a) \). Their conjugate momenta are \( \pi_\varphi, \pi_n, \pi_n(a), 3 \pi^r(a) \). There are ten primary constraints (the vanishing of the 7 momenta of boosts, lapse and shift variables plus three constraints inducing the rotation on the flat indices \( a \) of the cotriads) and four secondary ones (the super-Hamiltonian and super-momentum constraints): all of them are first class in the phase space spanned by 16+16 fields. This implies that there are 14 gauge variables describing inertial effects and 2 canonical pairs of physical degrees of freedom describing the tidal effects of the gravitational field (namely gravitational waves in the weak field limit). In this canonical basis only the momenta \( 3 \pi_r(a) \) conjugated to the cotriads are not vanishing. The basis of canonical variables for this formulation of tetrad gravity, naturally adapted to 7 of the 14 first-class constraints, is

\[
\begin{array}{cccc}
\varphi(a) & n & \bar{n}(a) & 3 \epsilon_\tau(a) \\
\pi_\varphi(a) & \approx 0 & \pi_n & \approx 0 & \pi_{n(a)} & \approx 0 & 3 \pi^r(a)
\end{array}
\]  \hspace{1cm} (4.3)

From Eqs.(5.5) of Ref.[78] we assume the following (direction-independent, so to kill super-translations) boundary conditions at spatial infinity \( r = \sqrt{\sum_{r} \left( \sigma^r \right)^2} \); \( \epsilon > 0 \); \( M = \text{const.} \): \( n(\tau, \sigma^r) \to r \to \infty \) \( O(r^{-(2+\epsilon)}) \), \( \pi_n(\tau, \sigma^r) \to r \to \infty \) \( O(r^{-3}) \), \( n_i(a)(\tau, \sigma^r) \to r \to \infty \) \( O(r^{-\epsilon}) \), \( \pi_{n(a)}(\tau, \sigma^r) \to r \to \infty \) \( O(r^{-3}) \), \( \varphi(a)(\tau, \sigma^r) \to r \to \infty \) \( O(r^{-(1+\epsilon)}) \), \( \pi_{\varphi(a)}(\tau, \sigma^r) \to r \to \infty \) \( O(r^{-2}) \), \( 3 \epsilon_\tau(a)(\tau, \sigma^r) \to r \to \infty \) \( \left( 1 + M \frac{\epsilon}{r^2} \right) \delta_{ar} + O(r^{-3/2}) \), \( 3 \pi_r(a)(\tau, \sigma^r) \to r \to \infty \) \( O(r^{-5/2}) \).

D. The York Canonical Basis: Inertial Gauge Variables and Tidal Physical Degrees of Freedom

In Ref.[81] a partial Shankmugadhasan canonical transformation to a canonical basis adapted to ten of the first class constraints was found. It implements the York map of Ref.[86] (in the cases in which the 3-metric \( \delta_{\tau\tau} \) has three distinct eigenvalues) and diagonalizes the York-Lichnerowicz approach. Its final form is \( (\alpha(a)(\tau, \sigma^r) \) are the Euler angles of the previous Subsection; \( V_{aa} \) is defined \( V_{uv} \);

\[
\begin{array}{ccccccc}
\varphi(a) & \alpha(a) & n & \bar{n}(a) & \theta^r & \phi & R_0 \\
\pi_\varphi(a) & \approx 0 & \pi_{\alpha(a)} & \approx 0 & \pi_n & \approx 0 & \pi_{n(a)} & \approx 0 & \pi_\theta & \pi_\phi = \frac{c^4}{12\pi G} 3K & \Pi_0
\end{array}
\]
simplicity we shall use $V$ as a 3-sub-manifold of space-time. Its conjugate variable, to be determined by the super- used as a set of 4-coordinates \[87\]. The York time describes the effect of gauge transforma- momenta: this is a reflex of the Lorentz signature of space-time, because $\propto$ is more convenient to choose the three gauge parameters as first kind coordinates $\tau$ on $\Sigma$.

The set of numerical parameters $\gamma_{\alpha\alpha}$ satisfies \[75\] $\sum_\alpha \gamma_{\alpha u} = 0$, $\sum_u \gamma_{\alpha u} \gamma_{bu} = \delta_{\alpha\beta}$, $\sum_\alpha \gamma_{\alpha u} \gamma_{\alpha v} = \delta_{uv} - \frac{1}{3}$. Each solution of these equations defines a different York canonical basis. This canonical basis has been found due to the fact that the 3-metric $g_{rs}$ is a real symmetric $3 \times 3$ matrix, which may be diagonalized with an orthogonal matrix $V(\theta^r)$, $V^{-1} = V^T$, $\det V = 1$, depending on three parameters $\theta^r$. If we choose these three gauge parameters to be Euler angles $\theta^i(\tau, \vec{\sigma})$, we get a description of the 3-coordinate systems on $\Sigma$, from a local point of view, because they give the orientation of the tangents to the three 3-coordinate lines through each point. However, for the calculations (see Refs.\[82\]) it is more convenient to choose the three gauge parameters as first kind coordinates $\theta^i(\tau, \vec{\sigma})$ ($-\infty < \theta^i < +\infty$) on the O(3) group manifold. From now on for the sake of notational simplicity we shall use $V(\theta^i)$.

This canonical transformation realizes a York map because the gauge variable $\pi_\phi$ is proportional to York internal extrinsic time $3K$. It is the only gauge variable among the momenta: this is a reflex of the Lorentz signature of space-time, because $\pi_\phi$ and $\theta^a$ can be used as a set of 4-coordinates \[87\]. The York time describes the effect of gauge transformations producing a deformation of the shape of the 3-space along the 4-normal to the 3-space as a 3-sub-manifold of space-time. Its conjugate variable, to be determined by the super-Hamiltonian constraint, is $\phi = \bar{e}^3 = \sqrt{\det g_{rs}}$, which is proportional to Misner’s internal intrinsic time; moreover $\phi$ is the 3-volume density on $\Sigma$: $V_R = f_R d^3\sigma \phi$, $R \subset \Sigma$. Since we have $g_{rs} = \bar{\phi}^{3/3} \bar{g}_{rs}$ with $\det \bar{g}_{rs} = 1$, $\phi$ is also called the conformal factor of the 3-metric.

The two pairs of canonical variables $R_{\bar{a}}, \Pi_{\bar{a}}, \bar{a} = 1, 2$, describe the generalized tidal effects, namely the independent physical degrees of freedom of the gravitational field. They are 3-scalars on $\Sigma$, and the configuration tidal variables $R_{\bar{a}}$ depend only on the eigenvalues of the 3-metric. They are DO’s only with respect to the gauge transformations generated by 10 of the 14 first class constraints.

Since the variables $\bar{\phi}$ and $\pi_i^{(\theta)}$ are determined by the super-Hamiltonian (i.e. the Lichnerowicz equation) and super-momentum constraints respectively (they are coupled elliptic PDE’s in the 3-space $\Sigma$), the arbitrary gauge variables are $\alpha_{(a)}, \varphi_{(a)}, \theta^i, \pi_\phi$, $n$ and $\bar{n}_{(a)}$. As shown in Refs.\[75, 78, 82\], they describe the following generalized inertial effects: a) $\alpha_{(a)}(\tau, \vec{\sigma})$ and $\varphi_{(a)}(\tau, \vec{\sigma})$ are the 6 configuration variables parametrizing the O(3,1) gauge freedom in the choice of the tetrads in the tangent plane to each point of $\Sigma$, and describe the arbitrariness in the choice of a tetrad to be associated to a time-like observer, whose world-line goes through the point $(\tau, \vec{\sigma})$; they fix the unit 4-velocity of the observer and the conventions for the orientation of three gyroscopes and their transport along the world-line.

\[4.4\]
of the observer; the Schwinger time gauges are defined by the gauge fixings \( \alpha_{(a)}(\tau, \vec{\sigma}) \approx 0 \), \( \varphi_{(a)}(\tau, \vec{\sigma}) \approx 0 \).

b) \( \theta^a(\tau, \vec{\sigma}) \) describe the arbitrariness in the choice of the 3-coordinates in the instantaneous 3-spaces \( \Sigma_r \) of the chosen non-inertial frame centered on an arbitrary time-like observer; their choice will induce a pattern of relativistic inertial forces for the gravitational field, whose potentials are the functions \( V_{ra}(\theta^a) \) present in the weak ADM energy \( \hat{E}_{ADM} \).

c) \( \bar{n}_{(a)}(\tau, \vec{\sigma}) \), the shift functions, describe which points on different instantaneous 3-spaces have the same numerical value of the 3-coordinates; they are the inertial potentials describing the effects of the non-vanishing off-diagonal components \( 4g_{\tau r}(\tau, \vec{\sigma}) \) of the 4-metric, namely they are the gravito-magnetic potentials\(^{12}\) responsible of effects like the dragging of inertial frames (Lense-Thirring effect) in the post-Newtonian approximation; the shift functions are determined by the \( \tau \)-preservation of the gauge fixings determining the gauge variables \( \theta^a(\tau, \vec{\sigma}) \) \[58\].

d) \( \pi^r(\tau, \vec{\sigma}) \), i.e. the York time \( 3K(\tau, \vec{\sigma}) \), describes the non-dynamical arbitrariness in the choice of the convention for the synchronization of distant clocks which remains in the transition from SR to GR; since the York time is present in the Dirac Hamiltonian, it is a new inertial potential connected to the problem of the relativistic freedom in the choice of the shape of the instantaneous 3-space, which has no Newtonian analogue (in Galilei space-time time is absolute and there is an absolute notion of Euclidean 3-space); its effects are completely unexplored.

e) \( 1 + n(\tau, \vec{\sigma}) \), the lapse function appearing in the Dirac Hamiltonian, describes the arbitrariness in the choice of the unit of proper time in each point of the simultaneity surfaces \( \Sigma_r \), namely how these surfaces are packed in the 3+1 splitting; the lapse function is determined by the \( \tau \)-preservation of the gauge fixing for the gauge variable \( 3K(\tau, \vec{\sigma}) \) \[58\].

See the first paper in Refs.[82] for the expression of the super-momentum constraints \( \mathcal{H}_{(a)}(\tau, \vec{\sigma}) \approx 0 \) \[Eqs.(3.41)-(3.42)\] and of the super-Hamiltonian constraint \( \mathcal{H}(\tau, \vec{\sigma}) \approx 0 \) (the Lichnerowicz equation) \[Eqs.(3.44)-(3.45)\]. The weak ADM energy is given in Eqs. (3.43)-(3.45) of that paper, while the other weak Poincaré generators are given in Eqs.(3.47). The expression of the weak ADM energy in terms of the expansion \( \theta = -\epsilon^3 K = -\epsilon^3 12\pi G c^3 \tau_{\phi} \) and shear \( \sigma_{(a)(b)} = \sigma_{(b)(a)} = (3K_{rs} - \frac{1}{3} \frac{g_{rs}}{\epsilon^3} 3K)^3 \epsilon^r_{(a)} \epsilon^s_{(b)} \) with \( \sigma_{(a)(a)} = 0 \) (so that \( 3K_{rs} = -\frac{1}{3} \frac{g_{rs}}{\epsilon^3} \) \( \theta + \sigma_{(a)(b)} \) \( 3\epsilon^r_{(a)} \) \( 3\epsilon^s_{(b)} \) \( 3\epsilon^s_{(b)} \)) of the Eulerian observers (see the first paper in Ref.[82]) is

\[
\hat{E}_{ADM} = c \int d^3 \sigma \left[ \hat{\mathcal{M}} - \frac{c^3}{16\pi G} S + \frac{4\pi G}{c^3} \tilde{\phi} - 1 \sum_b \Pi_b^2 + \right. \\
\left. + \tilde{\phi} \left( \frac{c^3}{16\pi G} \sum_{ab, a \neq b} \sigma_{(a)(b)}^2 - \frac{6\pi G}{c^3} \pi^2_0 \right) \right](\tau, \vec{\sigma}),
\]

(4.5)

where \( \hat{\mathcal{M}} = \tilde{\phi} (1 + n)^2 T_{\tau \tau} \) is the energy-mass density of the matter (with energy-momentum tensor \( T^{AB} \)) and \( S(\tilde{\phi}, \theta^i, R_{ab}) \) is an inertial potential depending on the choice of the 3-

\(^{12}\) In the post-Newtonian approximation in harmonic gauges they are the counterpart of the electro-magnetic vector potentials describing magnetic fields \[88\]: A) \( N = 1 + n, n_{\Phi_G} = -\frac{4\pi G}{c^3} \Phi_G \) with \( \Phi_G \) the gravito-electric potential; B) \( n_{\Phi_G} = \frac{4\pi G}{c^3} A_G \), with \( A_G \), the gravito-magnetic potential; C) \( E_G = \partial_r \Phi_G - \partial_r (\frac{1}{2} A_G r) \) (the gravito-electric field) and \( B_G = r_{\tau \tau} \partial_r A_G - c \Omega_G r \) (the gravito-magnetic field). Let us remark that in arbitrary gauges the analogy with electro-magnetism breaks down.
coordinates in the 3-space (it is the $\Gamma - \Gamma$ term in the scalar 3-curvature of the 3-space). In $\hat{E}_{ADM}$ there is a negative kinetic term proportional to $(3K)^2$, vanishing only in the gauges $3K(\tau, \vec{\sigma}) = 0$, which comes from the term bilinear in momenta present both in the super-Hamiltonian and in the weak ADM energy: it was known that this quadratic form was not definite positive but only in the York canonical basis this can be made explicit.

Finally the Dirac Hamiltonian is
\[
H_D = \frac{1}{c} \hat{E}_{ADM} + \int d^3\sigma \left[ n \mathcal{H} - \bar{n}(a) \mathcal{H}(a) \right](\tau, \vec{\sigma}) + \lambda_r(\tau) \hat{P}_{ADM} + \int d^3\sigma \left[ \lambda_n \pi_n + \lambda_{\bar{n}}(a) \pi_{\bar{n}}(a) + \lambda_{\bar{\varphi}}(a) \pi_{\bar{\varphi}}(a) + \lambda_{\varphi}(a) \pi_{\varphi}(a) \right](\tau, \vec{\sigma}),
\]
where the $\lambda_{\cdots}(\tau, \vec{\sigma})$'s are Dirac multipliers. In particular the Dirac multiplier $\lambda_r(\tau)$ implements the rest frame condition $\hat{P}_{ADM} \approx 0$ required by the absence of super-translations.

Once a gauge is completely fixed by giving the six gauge-fixings for the $O(3,1)$ variables $\varphi^{(a)}$, $\alpha^{(a)}$ (choice of the tetrads and of their transport) and four gauge-fixings for $\theta^i$ (choice of the 3-coordinates on the 3-space) and $3K$ (determination of the shape of the 3-space as a 3-sub-manifold of space-time by means of a clock synchronization convention), the Hamilton equations generated by the Dirac Hamiltonian (replacing the standard 12 ADM equations and the matter equations $4\nabla_A T^{AB} = 0$) become a deterministic set of coupled PDE's for the lapse and shift functions (primary inertial gauge variables $^{13}$), the tidal variables and the matter. Given a solution $\pi_i^{(\theta)}$ and $\bar{\varphi}$ of the super-momentum $^{14}$ and super-Hamiltonian constraints and the Cauchy data for the tidal variables on an initial 3-space, we can find a solution for the tidal variables $R_{\bar{a}}$, $\Pi_{\bar{a}}$, of their hyperbolic evolution Hamilton equations and therefore a solution of Einstein’s equations in radar 4-coordinates adapted to a time-like observer in the chosen gauge.

In Refs.[83] there is the Hamiltonian expression of radar tensors which coincide with the Riemann and Weyl tensors on the solutions of Einstein equations. Moreover, by using the time-like normal to the 3-spaces and the space-like direction identified by the shift function (for solutions of Einstein equations in which it is not identically zero) it is possible to build null tetrads and to give the Hamiltonian formulation of the Newman-Penrose formalism [85]. Therefore we get the Hamiltonian expression of Ricci and Weyl scalars and of the eigenvalues of the Weyl tensor. It is shown that the Bergmann observables [89, 90], built with these eigenvalues cannot be DO’s [87].

E. The Search of Dirac Observables and Comments on the Canonical Quantization of Gravity

As already said the tidal variables $R_{\bar{a}}$, $\Pi_{\bar{a}}$, are DO’s $^{\text{only}}$ with respect to the gauge transformations generated by 10 of the 14 first class constraints. Let us remark that, if we

---

$^{13}$ As said in Ref.[58] their gauge fixings are induced by putting the gauge fixings for the secondary gauge variables $\theta^i$ and $3K$ into their Hamilton equations. Among the Hamilton equations there are the contracted Bianchi identities, namely the evolution equations for the solutions $\bar{\varphi}$ and $\pi_i^{(\theta)}$ of the constraints (they say that given a solution of the constraints on a Cauchy surface, it remains a solution also at later times). Instead in numerical gravity one gives independent gauge fixings for both the primary and secondary gauge variables in such a way to minimize the computer time.

$^{14}$ See Refs.[80, 81] for the generalized Gribov ambiguity in metric and tetrad gravity arising in the solution of the super-momentum constraints after having done the York map.
fix completely the gauge and we go to Dirac brackets, then the only surviving dynamical variables \( R_a \) and \( \Pi_a \) become two pairs of non canonical DO’s for that gauge: the two pairs of canonical DO’s have to be found as a Darboux basis of the copy of the reduced phase space identified by the gauge and they will be (in general non-local) functionals of the \( R_a, \Pi_a \) variables. Till now there exist only statements about the existence of global DO’s for gravity [91–93], but no determination of them.

As shown in the second paper of Ref. [83], the DO’s of GR could be found with a Shanmugadhasan canonical transformation adapted to all the constraints if a global solution \( \phi \approx f, \pi_i^{(\theta)} \approx f_i \) of the super-Hamiltonian and super-momentum constraints would be known. In the final canonical basis there would be the replacement of the variables \( \phi \) and \( \pi_i^{(\theta)} \) with the Abelian constraints \( \hat{\phi} = \phi - f \approx 0 \) and \( \hat{\pi}_i^{(\theta)} = \pi_i^{(\theta)} - f_i \approx 0 \). The global DO’s would be the tidal variables \( \hat{R}_a, \hat{\Pi}_a \), of this canonical basis, whose inertial gauge variables would be \( \hat{\phi} \) and \( \hat{\theta}^i \) and the old lapse and shift functions.

Let us add a comment on the open problem of the quantization of gravity. In most of the traditional approaches one quantizes all the components of the 4-metric, namely both inertial and tidal variables, and one is facing every type of problems from the interpretation of the wave function of the universe in the Wheeler-DeWitt approach to the disappearing of space-time in the loop quantum gravity approach.

The finding of global DO’s (or at least of approximate ones after a HPM linearization like in the second paper of Ref.[83]) would allow to quantize only these tidal physical variables leaving the inertial gauge variables as c-numbers (the multi-temporal quantization proposed in Ref. [94] and studied in Refs. [33, 34]). The space-time manifold, its 3-1 splittings and the inertial gauge variables (lapse, shift, 3-coordinates, final York time) would remain classical notions determined by relativistic metrology: only the eigenvalues of the 3-metric of the non-Euclidean 3-spaces would be quantized with an induced quantization of 3-lengths, 3-areas and 3-volumes to be compared with the results of loop quantum gravity (the study of Ashtekar variables [9, 95] in asymptotically Minkowskian space-times in the York canonical basis is still to be done).

F. The Einstein-Electromagnetism-Particle System and the Problem of Dark Matter

In the three papers of Ref.[82] there is the study of charged positive-energy point particles plus the electro-magnetic field in tetrad gravity. In the second paper there is the definition of the HPM approximation based on the fact that in the allowed 3+1 splittings of the asymptotically Minkowskian space-times the 3-spaces are asymptotically Euclidean and there is an asymptotic Minkowski 4-metric. Then one can do the Post-Newtonian (PN) approximation of the HPM one. The natural family of gauges to be used with the York canonical basis are the non-harmonic 3-orthogonal Schwinger time gauges (in them the 3-metric of the 3-spaces is diagonal due to the gauge fixings \( \theta^i(\tau, \vec{\sigma}) \approx 0 \)). By using Grassmann regularization of both the electro-magnetic and gravitational self-energies, we are able to recover the HPM description of gravitational waves (GW) (see Ref.[96]) from binaries like in Damour-Deruelle approach [97]. Also relativistic fluids have studied in this framework [47]. In Subsection IIIB of the second paper in Refs.[82] it is shown that this HPM linearization can be interpreted.
as the first term of a Hamiltonian PM expansion in powers of the Newton constant $G$ in the family of 3-orthogonal gauges. This expansion has still to be studied.

In the case of positive-energy relativistic particles in absence of the electro-magnetic field one can show that the HPM effective force acting on them contains: a) the contribution of the lapse function $\tilde{n}_{(1)}$, which generalizes the Newton force; b) the contribution of the shift functions $\tilde{n}_{(1)(\xi)}$, which gives the gravito-magnetic effects; c) the retarded contribution of GW’s; d) the contribution of the volume element $\phi_{(1)} (\tilde{\phi} = 1 + 6 \phi_{(1)} + O(\zeta^2))$, always summed to the GW’s, giving forces of Newton type; e) the contribution of the inertial gauge variable (the non-local York time) $3\tilde{K}_{(1)} = \frac{1}{\sqrt{3}} \tilde{K}_{(1)}$. Then one does the PN expansion with the following final form of PN equations of motion for the particle $i$ with mass $m_i$

$$\frac{d}{dt} \left[ m_i \left( 1 + \frac{1}{c} \frac{d}{dt} 3\tilde{K}_{(1)}(t, \tilde{\eta}_i(t)) \right) \right] = -G \frac{\partial}{\partial \tilde{\eta}_i^\nu} \sum_{j \neq i} \eta_j \frac{m_i m_j}{|\tilde{\eta}_i(t) - \tilde{\eta}_j(t)|} + O(\zeta^2).$$

(4.6)

We see that the term in the non-local York time can be interpreted as the introduction of an effective (time-, velocity- and position-dependent) inertial mass term for the kinetic energy of each particle: $m_i \mapsto m_i \left( 1 + \frac{1}{c} \frac{d}{dt} 3\tilde{K}_{(1)}(t, \tilde{\eta}_i(t)) \right) = m_i + \Delta m_i$ in each instantaneous 3-space. Instead in the Newton potential there are the gravitational masses of the particles, equal to the inertial ones in the 4-dimensional space-time due to the equivalence principle. Therefore the effect is due to a modification of the effective inertial mass in each non-Euclidean 3-space depending on its shape as a 3-sub-manifold of space-time: it is the equality of the inertial and gravitational masses of Newtonian gravity to be violated! In Galilei space-time the Euclidean 3-space is an absolute time-independent notion like Newtonian time: the non-relativistic non-inertial frames live in this absolute 3-space differently from what happens in SR and GR, where they are (in general non-Euclidean) 3-sub-manifolds of the space-time [34].

Now the rotation curves of spiral galaxies (see Refs.[99, 100] for reviews) imply that the relative 3-velocity of particles goes to constant for large $r$ (instead of vanishing like in Kepler theory). This result can be simulated by fitting $\Delta m(r)$ (i.e. the non-local York time in Eq.(4.6)) to the experimental data: as a consequence $\Delta m(r)$ is interpreted as a dark matter halo around the galaxy. With our approach this dark matter would be a relativistic inertial effect consequence of the a non-trivial shape of the non-Euclidean 3-space as a 3-sub-manifold of space-time. A similar interpretation can be given for the other two main signatures of the existence of dark matter in the observed masses of galaxies and clusters of galaxies, namely the virial theorem [101–103] and weak gravitational lensing [102, 104]. Therefore at least part of dark matter is a relativistic inertial effect connected with the inertial gauge variable York time to be solved with relativistic metrology [16] finding a convenient metrology convention for the fixation of York time. GAIA [105] will give the lacking information for extending the IAU relativistic conventions [106] inside the solar system to the Milky Way (where till now we have only non-relativistic conventions like in the extragalactic cosmological regions).

V. CONCLUSIONS AND OPEN PROBLEMS

As I have shown all the relevant systems in SR and GR require Dirac theory of constraints for their Hamiltonian formulation. When the constraints can be solved, Shanmugadhasan
canonical transformations adapted to all the existing first- and second-class constraints allow to find the DO’s, i.e. the global physical degrees of freedom of the given system.

One big open problem is whether one must quantize only the DO’s (when known) or also the gauge variables (un-physical degrees of freedom in gauge theories and inertial variables connected with relativistic metrology in GR) with a determination of the DO’s only at the quantum level like in the BRST approach. When both the possibilities are open, one faces the risk to obtain unitarily inequivalent descriptions of the given system.

In the inertial frames of SR the quantization of DO’s can be faced with the standard methods. The main open problem is the quantization of the scalar Klein-Gordon field in non-inertial frames, due to the Torre and Varadarajan [107] no-go theorem, according to which in general the evolution from an initial space-like hyper-surface to a final one is not unitary in the Tomonaga-Schwinger formulation of QFT. From the 3+1 point of view there is evolution only among the leaves of an admissible foliation and the possible way out from the theorem lies in the determination of all the admissible 3+1 splittings of Minkowski space-time satisfying the following requirements: i) existence of an instantaneous Fock space on each simultaneity surface $\Sigma_\tau$ (i.e. the $\Sigma_\tau$’s must admit a generalized Fourier transform); ii) unitary equivalence of the Fock spaces on $\Sigma_{r_1}$ and $\Sigma_{r_2}$ belonging to the same foliation (the associated Bogoljubov transformation must be Hilbert-Schmidt), so that the non-inertial Hamiltonian is a Hermitean operator; iii) unitary gauge equivalence of the 3+1 splittings with the Hilbert-Schmidt property. The overcoming of the no-go theorem would help also in QFT in curved space-times and in condensed matter (here the non-unitarity implies non-Hermitean Hamiltonians and negative energies).

In GR the described approach to tetrad gravity has to be completed by facing the following topics before trying to quantize the linearized HPM theory:

1) Find the second order of the HPM expansion to see whether in PM space-times there is the emergence of hereditary terms (see Refs.[96, 108]) like the ones present in harmonic gauges. Like in standard approaches (see the review in Appendix A of the second paper in Refs.[82]) regularization problems may arise at the higher orders.

2) Study the PM equations of motion of the transverse electro-magnetic field trying to find Lienard-Wiechert-type solutions (see Subsection VB of the second paper in Refs.[82]). Study astrophysical problems where the electro-magnetic field is relevant.

3) The next big challenge after dark matter is dark energy in cosmology [109, 110]. Even if in cosmology we cannot use canonical gravity, in the first paper of Ref.[82] it is shown that the usual non-Hamiltonian 12 ADM equations can be put in a form allowing to use the interpretations based on the York canonical basis by means of the expansion and the shear of the Eulerian observers. Let us remark that in the Friedmann-Robertson-Walker (FRW) cosmological solution the Killing symmetries connected with homogeneity and isotropy imply ($\tau$ is the cosmic time, $a(\tau)$ the scale factor) $3\dot{K}(\tau) = -\frac{a(\tau)}{a(\tau)} = -\dot{H}$, namely the York time is no more a gauge variable but coincides with the Hubble constant. However at the first order in cosmological perturbations we have $3\dot{K} = -\dot{H} + 3\dot{K}_1(1)$ with $3\dot{K}_1(1)$ being again an inertial gauge variable. Instead in inhomogeneous space-times without Killing symmetries like the Szekeres ones [?] the York time remains an inertial gauge variable. Therefore the York time has a central position also in the main quantities on which relies the interpretation of dark energy in the standard $\Lambda$CDM cosmological model (Hubble constant, the old Hubble redshift-distance relation replaced in FRW cosmology with the velocity distance relation or
Hubble law). As a consequence it looks reasonable to investigate on a possible gauge origin also of dark energy.

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