Representation of total angular momentum states of beams through a four-parameter notation

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Abstract

It has been confirmed beams carrying total angular momentums (TAMs) that consist of spin angular momentums (SAMs) and orbital angular momentums (OAMs) are widely used in classical and quantum optics. Here we propose and demonstrate a new kind of representation consisting of four real numbers to describe the TAM states of arbitrary beams. It is shown that any homogeneous polarization, scalar vortices and complex vectorial vortex field, all of which result from the TAMs of photons, can be well represented conveniently using our proposed four-parameter representation. Furthermore, the proposed representation can also reveal the internal change of TAMs as the conversion between SAMs and OAMs. The salient properties of the proposed representation is to give a universal form of TAMs associated with complicated polarizations and more exotic vectorial vortex beams, which offer an important basis for the future applications.

1. Introduction

Similar with macroscopic objects, microscopic particles like photons, electrons and so on can also carry angular momentums (AMs) [1, 2]. The AM of a photon has two forms: the spin angular momentum (SAM) with two eigenvalues \(|\sigma = -1\rangle\) and \(|\sigma = 1\rangle\) corresponding to the left and right circular polarizations [3], and orbital angular momentum (OAM) with infinite eigenvalues \(l\) corresponding to the diverse helical phases \(\exp(i\varphi)\) of a beam, where \(\varphi\) is the azimuthal angle and \(l\) is also known as the topological charge or OAM quantum number [4]. Under paraxial approximation, the total angular momentum (TAM) of a beam is the sum of spin and orbital contributions as \(J = (\sigma + l)\hbar\) per photon, with \(\hbar\) the reduced Planck constant. The TAM states of beams have lots of presentation forms, which can be divided into two cases as the separable and non-separable states [5]. The separable state refers to a beam with simply separable spin and orbital states, that is, the separable state can be expressed as a single direct product \(|\sigma|l\rangle\) of spin state and orbital state. Separable states have a uniform polarization distribution across their wavefront, for instance, the scalar optical vortices [4], such that the spin and orbital states can be determined separately. The non-separable states may have spatially-varying polarizations, as found in vectorial vortex beams and so on. TAM states of beams have already found lots of applications, for instance, classical and quantum optical data transmission [6–10], optical trapping and manipulation [11–13], and rotation detection [14–16].

Completely characterizing TAM states is an important basis of applications for these states. Both the separable and non-separable states can be reflected by polarization distributions, thus describing beams’
polarizations can reveal some information of TAMs indirectly. To represent the complicated polarization state of a beam, lots of models have been demonstrated in the past 100 years [17–26], where the most well-known and widely employed is a theoretical framework as Poincaré sphere (PS) [17, 21, 23]. Originally, PS is proposed to describe the homogeneous polarization states as linear, elliptical and circular polarizations, each of which can find a corresponding specific point on the PS surface [17]. Later, replacing the polarization with OAM, the created OAM PS represents the OAM states of vortex modes and interprets variant transverse modes on the surface by linear combination of the two poles which are LG modes with the opposite OAM states [18–20]. Afterward, with the development of laser beam modulation, inhomogeneous polarizations as polarization vortices and vectorial vortex beams are explored, and the fundamental PS is insufficient. Therefore, the high-order Poincaré sphere (HPS) [21] and hybrid-order Poincaré sphere (HyPS) [23] are proposed orderly to extend fundamental PS (the original PS representing homogeneous polarization states) to a more general form. Almost all the complicated polarization states can be described by a certain position on the surface of such PSs. For HyPS, states on the two poles (north and south) have opposite SAM ($\sigma_N = 1$, $\sigma_S = -1$) and different OAM absolute value ($|l_N| \neq |l_S|$). If $l_S = -l_N$, the HyPS is degraded into HPS, and if $l_N = l_S$, the HyPS is degraded into fundamental PS. States on fundamental PS, HPS, and HyPS are referred to uniformly as Poincaré beams (PBs), describing all polarization structured fields [27]. In other words, both the separable and non-separable TAM states are kinds of PBs. Recently, with new ray-optical beams like Gaussian mode family rays are studied [28, 29], a generalized SU(2) PS is proposed to represent multi-dimensional ray-wave structured beams [30, 31]. The points on its surface represent the modes of SU(2) PBs described by SU(2) symmetry groups, encompassing the evolution of OAM and SU(2)-symmetric ray trajectories. For SU(2) PS, states on the two poles (north and south) are the orthogonal eigenstates and have opposite OAMs (right- and left-handed). The point on the surface of northern or southern hemisphere with angular coordinates ($\theta$, $\rho$) is the SU(2) coherent state, which is a well-defined SU(2) superposed state by a set of Hermit–Laguerre Gaussian eigenstates. Such SU(2) PS can graphically represent complex ray-wave structured beams carrying OAM of not only two dimensional but also three dimensional space, with more degree of freedoms [30].

The frameworks for fundamental, high-order and hybrid-order PSs are derived from Stoke parameters $[S_0, S_1, S_2, S_3]^T$ with the corresponding orders respectively [17, 21, 23]. Usually such four elements ($S_0$, $S_1$, $S_2$ and $S_3$) are employed to describe the polarization parameters of PBs. However it still has limitations. Firstly a group of Stokes parameters focus only on the polarization states on a certain PS, for instance, $[S_0, S_1, S_2, S_3]^T = [1, 1, 0, 0]^T$ denotes horizontally linear polarization on fundamental PS ($l_S = l_N = 0$), but it can also denote radial polarization on the first order HPS ($l_S = l_N = 1$). One cannot distinguish which PS the Stokes parameters referred to without special illustration. Furthermore, Stokes parameters are proposed in the perspective of polarizations, and are hard to characterize the TAM states which the polarization result from. Thus the OAM features, polarization topological charges, and other important features of beams are ignored and cannot be read directly from Stokes parameters. These limitations restrict the applications of TAM to some extent, and motivate us to develop a more universal but simple representation to describe not only the complicated polarizations, but also the TAM states of arbitrary beams.

In this paper, we propose a new representation consisting of four real numbers to describe TAM states of arbitrary beams. Such a representation is derived from PS, and gives a universal form of arbitrary TAM states on arbitrary PSs, thus breaks through the boundedness of Stokes parameters mentioned above. Additionally, the proposed representation also reveals the internal change of TAM states or TAM delivery from photon to matter through a transforming matrix.

2. Four-parameter notation

2.1. Stokes parameter and Poincare sphere

Previous studies [17, 32–34] have shown a PB $|\psi\rangle$ can be treated as the superposition of two scalar separable TAM components with opposite SAMs and various OAMs:

$$
|\psi\rangle = \psi^\rho_R |R \rangle |m\rangle + \psi^\rho_L |L \rangle |l\rangle
$$

(1)

where $\psi^\rho_R$ and $\psi^\rho_L$ are the complex coefficients containing the amplitude and initial phase of the scalar electric field. $|m\rangle = \exp(i m \varphi)$, $|l\rangle = \exp(i l \varphi)$. $|R\rangle |m\rangle$, $|L\rangle |l\rangle$ comprise orthonormal circular polarization basis with opposite SAMs that represent the right and left circularly polarized scalar beams carrying various OAMs characterized by topological charges $m$, $l$ and azimuthal angle $\varphi$:

$$
|R\rangle |m\rangle = \exp( i m \varphi) [1, i]^T / \sqrt{2}
$$

(2)

$$
|L\rangle |l\rangle = \exp( i l \varphi) [1, -i]^T / \sqrt{2}.
$$

(3)
Equations (1)–(3) lead to the definition of Stokes parameters \([S_0, S_1, S_2, S_3]^T\) in the circular basis [17, 21, 23]:

\[
\begin{align*}
S_0 &= |\psi^m_R|^2 + |\psi^l_L|^2 \\
S_1 &= 2 |\psi^m_R| |\psi^l_L| \cos \phi \\
S_2 &= 2 |\psi^m_R| |\psi^l_L| \sin \phi \\
S_3 &= |\psi^m_R|^2 - |\psi^l_L|^2
\end{align*}
\]  

(4)

with \(\phi\) the phase difference of \(\psi^m_R\) and \(\psi^l_L\):

\[
\phi = \arg(\psi^m_R) - \arg(\psi^l_L).
\]  

(5)

Note that in equation (4) \(|\psi^m_R|^2\) and \(|\psi^l_L|^2\) denote the intensity of two scalar components respectively and thus \(S_0\) is the total intensity of PBs. \(S_1\) and \(S_2\) imply the relative initial phase difference between the OAM-carried orthonormal circular polarization basis given in equations (2) and (3). \(S_3\) gives the information of the overall degree of ellipticity of PBs. The PS is defined as \(S_0, S_1, S_2\) and \(S_3\) being the Cartesian coordinates, with \(S_0\) the radius from the origin, as sketched in figure 1, where take the case \(m = 0\) and \(l = 2\) as example. One can clearly see that states on the two poles of a PS correspond to the two decomposed scalar components respectively, and must have opposite SAMs. The OAMs on the two poles are forced only by the parameters \(m\) and \(l\). Hence in such framework, not SAMs but OAMs of the two poles are the main features and define a certain PS.

2.2. From Poincare sphere to four-parameter notation

When parameters \(m\) and \(l\) are fixed, a PS is confirmed, the complex state is only determined by the position of the point on the surface of the PS. Such position can be represented by the longitude and latitude coordinates \((\theta, \rho)\) of the sphere, which actually are the spherical angles and read from geometry as:

\[
\theta = \arctan \left( \frac{S_2}{S_1} \right) = \phi
\]  

(6)

\[
\rho = \arcsin \left( \frac{S_3}{S_0} \right).
\]  

(7)

We have already known from previous works [17, 21, 23] that each TAM or PB state corresponds to a sole point on the surface of a certain PS. Such one-to-one mapping provides a new idea for us to employ a group of parameters \(|m|l|\theta|\rho\) to describe a PB. The first two parameters \(m\) and \(l\) \((m, l \in \mathbb{Z})\) are the topological charges of the scalar circular polarization components on the two poles of PS, to represent the OAMs and thus define a certain PS. The last two parameters \(\theta\) \((\theta \in [0, 2\pi])\) and \(\rho\) \((\rho \in [-\pi/2, \pi/2])\) are the longitude and latitude coordinates of the points on the PS surface. According to the one-to-one mapping, PBs are represented completely. Compared with Stokes parameters, although such representation also needs four parameters, the PS where the PB located is well expressed, and the TAM states are also indicated. The biggest advantage is that each \(|m|l|\theta|\rho\) corresponds to one specific PB, and the phenomena that one group of Stokes parameters corresponds to many various PBs located on different PSs are nonexistent. If PBs...
locate on the two poles, the parameter $\theta$ is meaningless, since from equation (6) we know $\theta$ is the initial phase difference between $|\psi^m_R\rangle$ and $|\psi^l_L\rangle$, and represents the polarization direction when $\phi = 0$. Furthermore, PBs on the two poles are the circular polarization basis of a PS, and can be written directly as $|R\rangle|m\rangle$ and $|L\rangle|l\rangle$.

### 3. Representing TAM states

#### 3.1. Representation of single-mode TAM states

One of the utilities of $|m|\theta|l\rho\rangle$ is that the OAM properties is indicated, thus the TAM states of PBs are naturally embodied. For instance, the scalar optical vortices $|n\rangle = \exp(i n \phi)$ [4], a kind of beams carrying OAM, can be regarded as the combination of two components with opposite SAMs but identical OAMs. Therefore, according to equations (1)–(7), such vortices locate on the PS with $m = l = n$. The position on the PS is determined by the polarization of $|n\rangle$, for instance, if horizontally linear polarization, the representation of $|n\rangle$ is $\{n|n|0|0\}$; else if right circular polarization, the representation is $\{n|n|\theta|\pi/2\} = |R\rangle|n\rangle$.

Additionally, the polarization order and Pancharatnam–Berry phase (PBP) are also illustrated by $|m|\theta|l\rho\rangle$.

The polarization order, also known as polarization topological charge, is a real number showing the value and direction of cycled change of polarizations along azimuth [21], and is derived from $|m|\theta|l\rho\rangle$ as $p = (l - m)/2$. The PBP is a kind of geometry phase and was firstly proposed by Pancharatnam as a PB taking a cyclic path $C$ on the corresponding PS acquires a phase directly proportional to area $\Omega$ subtended by the circuit [35]. Such phase has already been intensively studied and expressed as [21–23]:

$$\gamma(C) = -\frac{m - (l + 2\sigma)}{4} \Omega. \quad (8)$$

The $|m|\theta|l\rho\rangle$ of a PB gives directly the value $m$ and $l$, thus from equation (8) the PBP can be well represented. Figure 2 gives some of the typical TAM states and their corresponding $|m|\theta|l\rho\rangle$. Obviously arbitrary PBs result from the TAM of photons can be conveniently described.

#### 3.2. Representation of multi-TAM superposed states

The proposed representation can also describe the multi-TAM superposed states understood from cascaded-PS [5]. Such states are the superposition of multiple PBs, where a typical example is the Hermite–Gauss (HG) mode, the traveling wave field of confocal cavity with square mirror. Actually, any HG mode can be decomposed into various Laguerre–Gauss modes, for the associations between Hermite and Laguerre polynomials [4, 36]. We know that LG mode is one of the most common OAM states, with two eigenvalues as radial index $p$ and topological charge $l$ to determine the concentric ring intensity and the helical wavefront respectively. That means scalar HG mode is the superposition of various OAM states. When taking into account the SAM-related polarizations, an HG mode can be regarded as superposed TAM states. As discussed previously, each PB (or TAM) component can be described through a group of four parameters $|m|\theta|l\rho\rangle$. Therefore it is easy to understand that the multi-TAM superposed states like HG...
Figure 3. Some of the typical multi-TAM superposed states and their corresponding $|m|,|l|,\theta,|\rho\rangle$. (a) HG11 mode with homogeneous elliptical polarization. (b) HG02 mode with homogeneous linear polarization. (c) One of superposed PBs with homogeneous polarization. (d) One of superposed PBs with inhomogeneous polarization. From top to bottom are $|m|,|l|,\theta,|\rho\rangle$, intensity and polarization distributions, respectively. In the polarization map, the atrovirens and dark blue bights denote left and right circular/elliptical polarizations.

modes can be written as the accumulation of multiple four-parameters with various weights as:

$$\sum_{k=0}^{N} a_k |m_k|,|l_k|,\theta_k,|\rho_k\rangle,$$

with $N$ the number of PB components and $a_k$ the complex weight demonstrating the specific amplitude and initial phase of the $k$th TAM components. Figure 3 presents some of the representative multi-TAM superposed states and their corresponding $|m|,|l|,\theta,|\rho\rangle$. Note that for the case of HG02 mode, its decomposition comprises the multi-ring OAM states term ($p \neq 0$). Such multi-ring structure results from the non-zeroth order associated Laguerre polynomials $L_p$ embedded in the complex amplitude of LG modes. This polynomial has no effects on the OAM state, and only determines the concentric ring intensity. Thus there is a coefficient $L_p$ for the decomposed multi-ring OAM states term.

### 3.3. The conversion of TAM states

The other utility of $|m|,|l|,\theta,|\rho\rangle$ is that, revealing the internal change of TAM states. The conversion between the SAM and OAM, or the transformation of TAMs from photons to matters, usually carried out through anisotropic geometric phase element as $q$-plates \([37]\), which can be well described through a $4 \times 4$ transforming matrix as:

$$S(q,\alpha_0) = \begin{bmatrix}
0 & |2q\rangle & 0 & 0 \\
|2q\rangle & 0 & 0 & 0 \\
0 & 0 & \exp(4i\alpha_0) & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}, \tag{9}
$$

where $q$ and $\alpha_0$ are two constants that denote the $q$ value and the initial main axis angle and thus define a $q$-plate, $|2q\rangle = \exp(2iq\varphi)$. The derivation details of the above transforming matrix are present in appendix A. Specially if $q = 0$ in equation (9), $S(0,\alpha_0)$ corresponds to an SAM switching element, namely a half wave plate (HWP) with $\alpha_0$ arranged fast axis. Note that in $|m|,|l|,\theta,|\rho\rangle$, the first two parameters $m$ and $l$ denote the OAM states of the two poles of a PS, and the third parameter $\theta$ is the initial phase difference of the two SAM basis. Hence in the transformation calculation the original form and the corresponding operation rules of such three parameters must be followed, where the proposed representations should be written as a $4 \times 1$ matrix as $E = |m\rangle,|l\rangle,\exp(i\theta),|\rho\rangle^T$. By now, for arbitrary PB $|m_1|,|l_1|,\theta_1,|\rho_1\rangle$, its matrix form is $E_{in} = |m_1\rangle,|l_1\rangle,\exp(i\theta_1),|\rho_1\rangle^T$. After passing through a $q$-plate or HWP, the output PB reads $E_{out} = S(q,\alpha_0) \cdot E_{in}$. Additionally, it is also easy to prove that, if a PB propagates through $N$ various $q$-plates $S_1, S_2, \ldots, S_N$ in sequence, the output field $E_{out}$ can be conveniently calculated from the total transforming
matrix as the reverse multiplication of $S_1, S_2, \ldots, S_N$:

$$E_{\text{out}} = S_1 \cdot S_2 \cdot \ldots \cdot S_N \cdot E_{\text{in}}.$$  \hfill (10)

Equation (10) illustrates the practicability of the proposed representations, and offers an easy tool for studying TAM evolution and derivation in optical field transformation.

4. Diagnosing the four-parameter notation

4.1. Experimental setup and principle

An important issue to be addressed is how to acquire $\{m|l|\theta|\rho\}$ of a PB practically, otherwise making no practical sense. Here we propose a scheme to measure $\{m|l|\theta|\rho\}$ of a PB on one single PS, in other words, the single TAM state. As displayed in figure 4, the setup consists of two parts, the PB generator and detector. The generator is employed to produce various PBs carrying various TAMs, whose principles are discussed in appendix B. The detector is to measure the four-parameter representation $\{m|l|\theta|\rho\}$, and is the key of this sub-section.

In the detector, a PB is distributed into two parts with equal intensity, one (I) is to measure parameters $m$, $l$, and $\rho$, and the other (II) is to measure parameter $\theta$. In part (I), a 45° arranged quarter wave plate (QWP) is placed, and the output field $|\psi_{\text{out}}\rangle$ is derived through Jones matrix as:

$$|\psi_{\text{out}}\rangle = \frac{1}{2} (1 + i) \left[ \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right] |\psi'\rangle,$$  \hfill (11)

where $|\psi'\rangle$ denotes the PB reflected by the beam splitter (BS) with SAM and OAM being opposite:

$$|\psi'\rangle = \psi_R^m |L\rangle |-m\rangle + \psi_L^l |R\rangle |-l\rangle.$$  \hfill (12)

Equation (11) with equation (12) lead to:

$$|\psi_{\text{out}}\rangle = \left[ \begin{array}{c} \exp \left( i \frac{\pi}{4} \right) \psi_L^l |-l\rangle \\ \exp \left( -i \frac{\pi}{4} \right) \psi_R^m |-m\rangle \end{array} \right].$$  \hfill (13)

with $|m\rangle$ and $|l\rangle$ the $m$th and $l$th helical phases. Equation (13) indicates $|\psi_{\text{out}}\rangle$ consists of two scalar vortices with orthogonal linear polarizations, and their separation is available through a polarizer (P1). When P1 is horizontally placed, we obtain scalar vortices $\exp \left( i \pi / 4 \right) \psi_L^l |-l\rangle$; and when P1 is vertically placed, we obtain another scalar vortices $\exp \left( -i \pi / 4 \right) \psi_R^m |-m\rangle$. By detecting the OAM state of the two scalar vortices, the value of $m$ and $l$ will be acquired. In the field of OAM detection, lots of approaches have been developed [38–42]. Here the scheme of tilted lens (TL) (a tilted placed lens L1), which focus the incident scalar vortices into OAM-related HG-like patterns in the focal plane, is utilized [39]. Additionally,
since \( |\exp(i\pi/4)\psi_L^l| = |\psi_R^m|, |\exp(-i\pi/4)\psi_R^m| = |\psi_R^m| \), and considering equation (7), by measuring the power of the two scalar vortices we can acquire parameter \( \rho \). Such power is obtained separately by analyzing the sum of every pixel’s gray value of the spot received by the CCD camera (CCD1), which is proportional to the real intensity if the received power is lower than the camera’s threshold \( [40] \).

In part (I) of the detector, a polarizer (P2), a lens (L2) and another CCD camera (CCD2) are placed to measure parameter \( \theta \). Note that lenses L1 and L2 are identical. Firstly, P2 is horizontally placed, and the intensity distribution captured by CCD2 is:

\[
I_0 = \left| \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \psi \right|^2 = \left| \begin{bmatrix} \psi_R^m |m\rangle + \psi_L^l |l\rangle \\ 0 \end{bmatrix} \right|^2
\]

\[
|\psi_R^m|^2 + |\psi_L^l|^2 + 2 |\psi_R^m||\psi_L^l| \cos \chi,
\]

where

\[
\chi = \arg(\psi_R^m) - \arg(\psi_L^l) + (m - l)\varphi = \theta + (m - l)\varphi.
\]

Next, P2 is \( 45^\circ \) arranged, and the amplitude distribution on CCD2 is:

\[
\sqrt{I_{45}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} |\psi| = \frac{\sqrt{2}}{\pi} \exp \left( \frac{i\pi}{4} \right) \begin{bmatrix} \psi_R^m |m\rangle + \exp(-i\pi/2) \psi_L^l |l\rangle \\ \psi_R^m |m\rangle + \exp(-i\pi/2) \psi_L^l |l\rangle \end{bmatrix}
\]

thus the intensity is:

\[
I_{45} = |\psi_R^m|^2 + |\psi_L^l|^2 - 2 |\psi_R^m||\psi_L^l| \sin \chi.
\]

From equations (14) and (17):

\[
\chi = -\arctan \frac{I_{45} - |\psi_R^m|^2 - |\psi_L^l|^2}{I_0 - |\psi_R^m|^2 - |\psi_L^l|^2}.
\]

Equation (15) with equation (18):

\[
\theta = -\arctan \frac{I_{45} - |\psi_R^m|^2 - |\psi_L^l|^2}{I_0 - |\psi_R^m|^2 - |\psi_L^l|^2} - (m - l)\varphi.
\]

In equation (19), \( I_0 \) and \( I_{45} \) are captured by CCD2 separately by rotating P2. |\psi_R^m|^2 and |\psi_L^l|^2 are captured by CCD1 providing that the lens L1 is untilted. Parameters \( m \) and \( l \) have already been acquired by part (I). Finally, parameter \( \theta \) is measured.

One thing should be noticed that, the practically measured \( \theta \) here is a pixel matrix to show the phase distribution, which is different from the defined \( \theta \) as a real value to imply the initial phase difference of \( \psi_R^m \) and \( \psi_L^l \). In fact, such pixel matrix gives the initial phase difference in various positions, and all of the phase difference must be identical without perturbation in theory. Hence any element contained in the measured matrix can be regarded as the value of \( \theta \). However in practice, there must be some errors that lead to the slight fluctuation of these element values, thus here the mean value of all the elements in the measured matrix is selected as the \( \theta \) value.

4.2. Experimental results

As the demonstration of \(|m||\theta|\rho\) measurement, three various PBs (TAM states) (azimuthal polarizations \( \{-1\} |\pi|0\rangle \), one of vectorial vortex beams \( \{0\} |2|0\rangle \) and right elliptically polarized scalar vortices \( \{3\} |3|\pi|\pi/6\rangle \) are generated and detected orderly as examples through the scheme in figure 4. One thing should be emphasized when measuring the longitude parameter \( \theta \), only the overlap region of the two scalar fields is accurate, due to the properties of interference-based phase diagnostic [43], which is the nature of \( \theta \) characterization here. Hence a filter pixel matrix function \( F(r, \varphi) \) obtained by AND operation of binary intensity distributions of two measured scalar components must be introduced as:

\[
\theta_a = \theta \circ F
\]

with \( \theta_a \) the finally accurate longitude matrix and \( \theta \) the measured longitude matrix from equation (19). The symbol ‘\( \circ \)’ denotes Hadamard product of two matrices. Figure 5 gives the experimentally generated various PBs with/without a polarizer (P2 in figure 4) captured by CCD2, and the corresponding results of \(|m||\theta|\rho\)
measurement separately. Such results are illustrated as the CCD1-captured intensity distribution of two scalar components and their intensity profiles after passing through a TL (L1 in figure 4). Firstly, the parameters $m$ and $l$ are acquired by the spot arrangement behind the tilted lens [39]. Then $\theta_a$ is computed from equations (19) and (20), as displayed in figure 5, where the atrovirens region denotes filter out part. The longitude $\theta$ is obtained as the mean value of effective elements of matrix $\theta_a$. Finally the latitude $\rho$ is computed directly from equation (7). The $\{m|l|\theta|\rho\}$ measurement results of the three PBs are $\{-1|1|3.017|0.031\}$, $\{0|2|0.268|0.129\}$ and $\{3|3|3.394|\ -0.529\}$ respectively. Obviously the characterization of parameters $m$ and $l$ are precise, while slightly measuring errors are presented for $\theta$ and $\rho$. Such errors are inevitable and mainly result from two aspects, one is manual adjustment error of the rotation of HWP1, QWP1 and $q$-plate (QP), and the other is the thermal noise of CCD.
To further evaluate the reliability of the proposed $\theta$ and $\rho$ measurement, an additional test is done, where the longitude $\theta$ and latitude $\rho$ are measured for various states on the same PS. In this test, the QP ($q = 0$ here) is arranged with a horizontal main axis. When measuring $\theta$, both the HWP and QWP1 are horizontally arranged. By now the generated PB is moving along the equator for one circle, if rotating QP from 0 to $\pi/2$. The QP is rotated in steps of 5 degrees and thus 19 different longitudes are obtained, as shown in figure 6(a). When measuring $\rho$, both the HWP and QP are horizontally arranged. By rotating QWP2 and making the angle between its fast axis and horizontal plane from $-\pi/4$ to $\pi/4$, the generated PB can move along the prime meridian from the north pole to the south pole. Here the QWP2 is also rotated in steps of 5 degrees and 19 different latitudes are acquired, as shown in figure 6(b). The biggest absolute error is measured as 0.3902 for $\theta$ and 0.1627 for $\rho$. Clearly from figure 6 the results of both the longitude $\theta$ and latitude $\rho$ confirm well the theoretical simulated curves.

5. Conclusions

In summary, we have demonstrated a new four-parameter notation $\{m|l|\theta|\rho\}$ to describe arbitrary PBs and represent TAM states. Such representation is derived from the framework PS, where the first two parameter $m$ and $l$ define a certain PS, and the other two parameters $\theta$ and $\rho$ give an exact position on the PS, implying any PBs resulting from TAMs can be well represented. The polarization order and PBP are also illustrated from $\{m|l|\theta|\rho\}$. As for the multi-TAM superposed states like HG modes and so on, they can be written as the accumulation of multiple four-parameters. Furthermore, a transforming matrix is introduced, leading to the well presentation of internal TAM transformation or from photons to matters. An approach of how to measure $\{m|l|\theta|\rho\}$ for a single TAM mode practically is also presented. The utility of these properties is demonstrated in their abilities to give a universal form of arbitrary TAM states including scalar vortices, complicated polarizations and more exotic vectorial vortex beams, and provide a tool to reveal their mutual transformation. These results provide a new choice, and have important implications for the future applications ranging from classical to quantum communities.

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Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

Appendix A. Derivation of transforming matrix of $q$-plate

As a spin–orbital angular momentum (SAM–OAM) conversion element, the basis transformation of a QP reads [37]:

\begin{align}
|\text{R}\rangle & \to \mid -2q\rangle|\text{L}\rangle \\
|\text{L}\rangle & \to \mid 2q\rangle|\text{R}\rangle
\end{align}

namely, after passing through a QP the turning direction of circular polarization will be opposite and meanwhile introduce OAM with topological charge $-2q$ or $2q$. Equations (21) and (22) imply that the two SAM bases are exchanged with each other. Hence the latitude coordinate on the PS turn to be opposite.
Considering the orientation of the main axis \( \alpha_0 \), an additional geometry phase will also be introduced, reflected on the moving of the point on the surface of a PS along a fixed latitude line. It can be proved the introduced additional phase is \( 2\alpha_0 \) \( [37] \), thus the change of longitude coordinate is \( 4\alpha_0 \). In other words, the QP parameter \( \alpha_0 \) only determines the longitude \( \theta \), and do nothing with latitude \( \rho \). By now one can conclude that, when a Poincare beam (PB) \( \{m\} | \theta, \rho \rangle \) propagates through a QP with the order \( q \) and main axis angle \( \alpha_0 \), \( m \rightarrow l + 2q, l \rightarrow m - 2q, \theta \rightarrow \theta + 4\alpha_0, \) and \( \rho \rightarrow -\rho \). The matrix transformation \( E_{\text{out}} = S(q, \alpha_0) \cdot E_{\text{in}} \) reads:

\[
\begin{bmatrix}
  l + 2q \\
  m - 2q \\
  \theta + 4\alpha_0 \\
  -\rho
\end{bmatrix} = S(q, \alpha_0) \cdot 
\begin{bmatrix}
  m \\
  l \\
  \theta \\
  \rho
\end{bmatrix}.
\]  

(23)

Obviously \( S(q, \alpha_0) \) is a \( 4 \times 4 \) square matrix. Considering that the first two parameters of \([m, l, \theta, \rho]\) are the OAM states of the two SAM bases, and are independent of the last two parameters as the longitude and latitude coordinates on the sphere surface. Thus such \( 4 \times 4 \) square matrix can be simplified to:

\[
S(q, \alpha_0) = 
\begin{bmatrix}
  S_{11} & S_{12} & 0 & 0 \\
  S_{21} & S_{22} & 0 & 0 \\
  0 & 0 & S_{33} & S_{34} \\
  0 & 0 & S_{43} & S_{44}
\end{bmatrix}.
\]  

(24)

Substitute equation (24) into equation (23), one can obtain:

\[
\begin{align*}
S_{11} \cdot |m\rangle + S_{12} \cdot |l\rangle &= |l + 2q\rangle \\
S_{21} \cdot |m\rangle + S_{22} \cdot |l\rangle &= |m - 2q\rangle \\
S_{33} \cdot \exp(i\theta) + S_{34} \cdot \rho &= \exp[i(\theta + 4\alpha_0)] \\
S_{43} \cdot \exp(i\theta) + S_{44} \cdot \rho &= -\rho
\end{align*}
\]  

(25)

Note that in the calculation each element in the proposed representation should be written in their corresponding proper forms, rather than the simplified format, as discussed in the main body. After solving equation (25), one can acquire: \( S_{11} = S_{22} = S_{34} = S_{43} = 0, S_{12} = |2q\rangle, S_{21} = |-2q\rangle, S_{33} = \exp(4i\alpha_0), \) and \( S_{44} = -1 \). Substitute these solutions into equation (24), the transforming matrix of QP with the order \( q \) and main axis angle \( \alpha_0 \) is obtained as:

\[
S(q, \alpha_0) = 
\begin{bmatrix}
  0 & |2q\rangle & 0 & 0 \\
  | -2q\rangle & 0 & 0 & 0 \\
  0 & 0 & \exp(4i\alpha_0) & 0 \\
  0 & 0 & 0 & -1
\end{bmatrix}.
\]  

(26)

When \( q = 0 \), QP introduces no additional OAMs for the SAM bases, and thus degrades into HWP:

\[
S_{\text{HWP}}(\alpha_0) = 
\begin{bmatrix}
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & \exp(4i\alpha_0) & 0 \\
  0 & 0 & 0 & -1
\end{bmatrix}.
\]  

(27)

where \( \alpha_0 \) corresponds to the angle of fast axis.

**Appendix B. Principles of PB generation**

In the PB generator of figure 4, the approach of cascading a spiral phase plate (SPP), an HWP (HWP1), a QWP (QWP1) and a QP is employed \( [44] \). The SPP is a kind of phase-only diffractive elements that can bring additional OAMs but will not change the polarization for the incident beams \( [45] \). When horizontal-linearly polarized Gaussian beams are incident, such transformation reads \( \{0\} |0\rangle \rightarrow |n\rangle |n\rangle, |L\rangle |0\rangle \rightarrow |L\rangle |n\rangle, \) with \( n \) the order of an SPP. The HWP along with QWP here are used to modulate the overall degree of ellipticity (intensity proportion between \( |R\rangle |n\rangle \) and \( |L\rangle |n\rangle \)) and thus to determine the latitude \( \rho \) with longitude unchanged. Hence after passing through a QP, the beam \( |n\rangle |n\rangle |0\rangle \rightarrow |n - 2q| n + 2q|2\alpha_0\rangle |\rho\rangle \), where the bases \( |R\rangle |n\rangle \rightarrow \exp(\alpha_0) |L\rangle |n + 2q\rangle \), \( |L\rangle |n\rangle \rightarrow \exp(-\alpha_0) |R\rangle |n - 2q\rangle \), with \( q \) the order and \( \alpha_0 \) the arranged angle of a QP separately. By now a PB \( |n - 2q| n + 2q|2\alpha_0\rangle |\rho\rangle \) is generated.
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