Stability Criteria for Breached Pair Superfluidity

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We present simple, concrete, two-fermion models that exhibit thermodynamically stable isotropic translationally-invariant gapless superfluid states (breached pair superfluidity). The mass ratio between the components and the momentum structure of the interaction are crucial for determining the stability of such states: Idealized, momentum-independent (“contact”) interactions are insufficient.

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INTRODUCTION

Recently there has been interest in superfluid fermion systems where there exist superfluid states that retain gapless fermionic excitations [1, 2, 3, 4]. These states embody “phase separation in momentum space”: some degrees of freedom pair, forming a superfluid, while others remain unpaired, maintaining the properties of a Fermi surface. They are likely to become experimentally accessible in the near future [5].

The important new result is that such states are stable, but to see this, one must perform a careful analysis of the momentum dependence of the interaction. Appropriate systems (such as two-band models [6, 7]) have been studied, but these states were missed because the possibility of momentum dependence was ignored. We also address the questions of instability toward phase separation [8] and local currents [9] raised in response to the proposal [1].

A breached pair superfluid state (BP) is characterized by the coexistence of a superfluid and a normal component in a translationally invariant and isotropic state. These components are accommodated in different regions of momentum space with the normal component residing in the “breaches”, bounded by gapless Fermi surfaces.

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We shall consider systems of two species that, in the absence of interactions, would have two distinct Fermi surfaces. Simple heuristic considerations suggest the possibility that pairing takes place about the Fermi surfaces, but that there is no pairing in a region between the surfaces: this led to the term “breached pair” [2].

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A state of this type was also considered by Sarma [10]. He considered the case of a superconductor in an external magnetic field, and found that, although there is a self-consistent mean-field solution with gapless modes, it is unfavored energetically to the fully gapped BCS solution. Similar results were considered in the context of color superconductivity [11], again concluding that these states are not stable at fixed chemical potential.

Since the fully gapped BCS solution enforces equal numbers of each species, one forbids the formation of a fully gapped BCS state, but admits “breached pair” states in which the excess in one species can be accommodated by the breach. In the QCD context, a similar argument has been made by imposing charge neutrality [8, 12]. Recently, however, Bedaque, Caldas and Rupak [8, 12] pointed out that a spatially mixed phase may be energetically preferable: this rules out the first possibility [2] but may not affect the QCD case due to the long-range gauge interactions.

Here we clarify, broaden, and correct this discussion. We conclude that:

• For extensive systems, one cannot stabilize a state by imposing different global constraints (such as fixed particle number): The composition of the state can be completely determined from an analysis of the grand canonical ensemble. The specific examples considered in [2] are accordingly unstable.

• With the proper momentum structure, however, one may realize breached pair superfluidity in states that are thermodynamically stable for fixed chemical potentials. We exhibit these below.

Our considerations do not apply directly to non-extensive systems. Charge conservation or color neutrality constraints enforced by long-range gauge forces might stabilize BP phases. (Of course, the possibility of a competing mixed phase must still be considered quantitatively.)

THERMODYNAMIC STABILITY

In the context of two component fermionic systems as considered in [2, 8, 12], three competing homogeneous phases have been considered: a normal state of free fermions (N), a fully gapped superfluid phase (BCS), and a gapless BP phase. The BCS phase has complete pairing between the two species, and thus enforces equal densities. The other phases admit differing densities.

Upon solving the self-consistency conditions (gap equations), one commonly finds that over a range of chemical potentials there are three distinct solutions. To determine which is stable in this grand canonical ensem-
ble, one must minimize the grand thermodynamic potential (equivalently, maximize the pressure) of the system. Typically, two of the three solutions are minima on either side of the third BP state which is a local maximum: Fig. [2] shows a typical potential. The gapless states found in [2, 10] correspond to local maxima, thus the competing state with larger gap parameter \( \Delta \) has higher pressure and renders the BP state unstable in this ensemble.

If the stable solution is fully gapped, then it has equal densities. By fixing the particle densities to be unequal, one may forbid this BCS state. Furthermore, upon comparing the Helmholtz free energies \( H \) —which must be minimized in this ensemble—one may find that the “unstable” BP state is favored over the normal state \( N \).

This apparent contradiction in the stability analysis based on different ensembles can be resolved by considering a mixed phase \( R \) which has an even lower Helmholtz free energy \( H \). That such a resolution is always possible, however, may not be apparent; indeed, it is generally hard to determine the mixed phase explicitly. By using general properties [13] of the grand thermodynamic potential \( \Omega \), however, one can argue that such a solution is always possible, as follows. By definition,

\[
-PV = \Omega(\vec{\mu}) = \min(H - \vec{\mu} \cdot \vec{N}),
\]

(1)

minimized over all competing phases. Thus, \( \Omega \) is a concave function of the chemical potentials \( \vec{\mu} = (\mu_a, \mu_b) \).

We consider here fixed \( T = 0 \), but concavity in \( T \) also follows from maximizing entropy.) Furthermore, there is a one-to-one correspondence between tangents \( \Omega \) and states of fixed particle number:

\[
\vec{N} = -\frac{\partial \Omega}{\partial \vec{\mu}}.
\]

(2)

When \( \Omega \) is not differentiable, there is a cone of possible tangent hyperplanes which contact \( \Omega \) and which bound \( \Omega \) from above (see Fig. 1). This cone of tangents describes various possible mixed phases composed of the pure phases (where \( \Omega \) is differentiable) that intersect at the singularity. To find the state that minimizes \( H \) for some fixed constraint \( \vec{N} = \vec{N}_0 \) one simply forms the hyperplane with gradient \( \vec{N}_0 \) and drops this until it contacts the surface \( \Omega \). The first point of contact will define either a pure or mixed state which satisfies the appropriate constraints. Since this state also lies on \( \Omega \), it minimizes \( \Omega \) for the fixed chemical potentials defined by the contact point. No matter what constraints we apply, there is always a stable state in the grand canonical ensemble.

This argument is valid only for extensive thermodynamic systems. Long-range interactions can render the energy of some pure phases non-extensive (due, for example, to the rapidly diverging Coulomb energy per unit volume \( V \) as \( V \to \infty \)). In such cases, a mixed phase would contain bubbles of limited size. The surface energy of these phase boundaries becomes a volume effect and must therefore be taken into account, even in the thermodynamic limit (see for example [14]). This complicates the relation between \( \vec{N} \) and \( \vec{\mu} \). In this rest of this letter, we shall consider only finite-range interactions.

FIG. 1: The cone of tangent (hyper)planes to a thermodynamic potential density \( -P = \Omega(\vec{\mu})/V \). Immediately to the left of \( \mu_0 \) is a pure phase with density \( n_L \) while immediately to the right is another pure phase with density \( n_R \). The densities are the negative slopes of the tangents at \( \mu_0 \) according to [2]. At \( \mu = \mu_0 \) there is a continuum of mixed phases: These consist of a volume fraction \( x \) at density \( n_L \) and the remaining fraction \( 1 - x \) at density \( n_R \). The average density over all space, \( n = xn_L + (1 - x)n_R \), lies within \( n \in (n_L, n_R) \).

STABLE BREACHED PAIR SUPERFLOUIDS

We now demonstrate, by example, how to realize pure BP superfluid states in extensive systems. We shall consider the mean-field analysis of two models, each with two species of fermions \( a \) and \( b \) of differing masses \( m_a < m_b \):

\[
\mathcal{H} = \int \frac{d^3 \vec{p}}{(2\pi)^3} \left( \frac{p^2}{2m_a} \hat{a}^\dagger \hat{a} + \frac{p^2}{2m_b} \hat{b}^\dagger \hat{b} \right) + \mathcal{H}_I. \quad (3)
\]

The models have different interactions \( \mathcal{H}_I \): [4] and [9]. We shall consider these systems in the grand canonical ensemble at zero temperature by minimizing the thermodynamic potential density \( \Omega(\mu_a, \mu_b)/V \). It will be natural, however, to use the parameters \( p_I^2 = \sqrt{2m_i \mu_i} \) in place of the chemical potentials \( \mu_i \).

The first model posits a spherically symmetric static two-body potential interaction \( V(r) \) between the two species \( a \) and \( b \):

\[
\mathcal{H}_I = \int d^3 \vec{x} \int d^3 \vec{y} \ V(|\vec{x} - \vec{y}|) \ \hat{a}_x^\dagger \hat{b}_y \hat{b}_y^\dagger \hat{a}_x. \quad (4)
\]

Defining \( m_{\pm} = 2m_am_b/(m_{\pm} + m_a) \), \( m_{\pm} = (m_a \pm m_b)/2 \), and

\[
c_{p, \pm} = \frac{1}{2} \left[ \frac{p^2}{2m_a} - \mu_a \right] \pm \frac{1}{2} \left[ \frac{p^2}{2m_b} - \mu_b \right] = \frac{p^2}{2m_{\pm}} - \mu_{\pm}, \quad (5)
\]

and considering only homogeneous (translationally invariant) and isotropic phases, we find that extrema of [10]...
satisfy the gap equation

\[ \Delta_p = -\int_R \frac{d^3\vec{q}}{(2\pi)^3} \tilde{V}(|\vec{p} - \vec{q}|) \frac{\Delta_q}{2\sqrt{(\epsilon_q^+)^2 + \Delta_q^2}}, \tag{6} \]

where \( \tilde{V}(p) \) is the Fourier transform of \( V(r) \). The integral runs over the region \( R \) outside any “breach”. \( R \) contains momenta where the two quasiparticle dispersions \( E_p^\pm \)

\[ E_p^\pm = \epsilon_p \pm \sqrt{(\epsilon_p^+)^2 + \Delta_p^2}, \tag{7} \]

have opposite sign. (See \( \text{Fig. 2} \) for further details about the generic breach structure.) Note from \( \text{Fig. 3} \) that \( \Delta_p \) is generally largest about \( p_0 \), where \( \epsilon_{p_0}^+ = 0 \).

Equation \( \text{Fig. 3} \) can be solved numerically to find extremal points of the thermodynamic potential. Over this set of self-consistent solutions, one can minimize \( \Omega \) to determine the phase structure.

We have done this for a variety of interactions, and find similar qualitative structure: a central strip of fully gapped BCS-like phase about \( p_a^F = p_b^F \), with normal unpaired phases outside (see Fig. \( \text{2} \)). Depending on the model parameters, these phases may be separated by a region of BP superfluid phase. To verify that these indeed contain gapless modes we plot in Fig. \( \text{3} \) a sample set of occupation numbers, quasiparticle dispersions, and the gap parameter \( \Delta_p \),

\[ \Delta_p = \int \frac{d^3\vec{q}}{(2\pi)^3} V(|\vec{p} - \vec{q}|)\langle \hat{b}_\vec{p}\hat{a}_{\vec{q}} \rangle. \tag{8} \]

The presence of gapless fermion modes depends crucially on two factors: 1) the momentum structure of \( \Delta_p \) and, 2) the mass ratio. First, \( \Delta_p \) must be large in some regions and small in others. If \( \Delta_p \) is large enough at a Fermi surface, it will induce pairing at that surface and support a superfluid. If it is also small enough at the other Fermi surface, it will not appreciably affect the normal free-fermion behaviour. The problem with previous analyses is that they assume pointlike interactions, implying a constant \( \Delta_p = \Delta \). Physical interactions, however, tend to exhibit more complicated behaviour and suitable \( \Delta_p \) are quite generic: \( \Delta_p \) tends to peak about the Fermi surface of the lighter species and fall off to at least one side. The model shown in Fig. \( \text{2} \) and Fig. \( \text{3} \), for example, has a Gaussian interaction. Longer-range forces (such as a screened Coulomb interaction) tend to plateau to the left of \( p_0 \) but still fall to the right of \( p_0 \).

Second, as was emphasized in \( \text{Fig. 1} \), the one may reduce the cost associated with shifting the Fermi sea \( p_b^F \) by increasing the mass \( m_b \). Thus, by choosing a large enough mass ratio, one may always move the Fermi surface for the heavy species to region where \( \Delta_p p_b^F \) is small enough to...
Since the variational states of model 4 are parameterized by a variable function $\Delta_p$, the set of states over which the minimization 4 must consider is enormous, and we cannot be certain to have found the global minimum. We have searched for stable fixed-points of the function $\eta$ in Fig. 2 and Fig. 3 have a mass ratio $m_b/m_a = 50$.

The second model allows us to be rigorous. We fall back to the type of factorized, cutoff interaction often considered in BCS models:

$$\mathcal{H}_I = g \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{q}}{(2\pi)^3} f(p)f(q) \hat{a}_\vec{p}^\dagger \hat{b}^\dagger_{-\vec{q}} \hat{b}_{-\vec{q}} \hat{a}_{\vec{q}}.$$  

In this model, the combination $\Delta f(p)$ plays the same role as $\Delta_p$, in first model, with the cutoff function $f(p)$ providing the required momentum dependence. The trial states are now parameterized by a single number $\Delta = \langle \Delta \rangle$ where $\Delta = g \int d^3\vec{p}/(2\pi)^3 f(p)\hat{a}_0\hat{b}_{-\vec{p}}$. One can now find the global minimum by plotting (see Fig. 4)

$$\frac{\Omega(\Delta)}{V} = \min_{\langle \theta | \hat{H} - \mu_a \hat{n}_a - \mu_b \hat{n}_b | \theta \rangle} \langle \theta | \hat{H} - \mu_a \hat{n}_a - \mu_b \hat{n}_b | \theta \rangle,$$  

where we minimization over all BCS style ansatz $|\theta\rangle$ with given expectation $\Delta$. This minimization is equivalent to comparing all solutions of the mean-field gap equation

$$\Delta = -\frac{g}{2} \int_{R} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\Delta f(q)}{\sqrt{\epsilon_1^2(q) + \Delta^2}}.$$  

We conclude that, within the mean-field approximation of homogeneous phases at zero temperature, this model has the phase diagram shown in Fig. 4. We plot the properties of a sample BCS state in Fig. 5 to illustrate that there are indeed gapless modes.

To model $\Delta_p$ more accurately one might use a function $f(p)$ where the location of the cutoff stays near $p_0$. This introduces an inconsistency in the thermodynamics because $f(p)$ is really a property of the Hamiltonian, while $p_0$ depends on the chemical potentials $\mu_i$, thus $N \neq -\partial \Omega/\partial \mu$. For small coupling and high densities, these spurious dependencies become small and the resulting phase diagram is qualitatively like Fig. 2.

Finally, we address the issue of the instability discussed in 4 where they claim that the superfluid density is negative due to a large negative contribution from the diverging density of states at $E = 0$ near the transition to the BP state. If one simply computes $d^2\Omega/d\Delta^2$, one finds exactly the same negative contribution indicating that the BP solution under consideration is an unstable maximum rather than a stable minimum. The solutions we present here are all global minima, and hence stable.

This raises an interesting point: if the BP/BCS transition were second order, then the density of states would formally diverge. Indeed, one finds $d^2\Omega(\Delta)/d\Delta^2 = -\infty$ at certain chemical potentials. Near the BP/BCS tran-

FIG. 4: $T = 0$ phase diagram for model 4 with a hard cutoff $f(p) \sim \theta(p - 10\sigma)$ that has been smoothed over the range from $p \in (9.7, 10.3)$. All momenta are expressed in units of $\sigma$ where $10\sigma$ is the cutoff scale, and all energies are expressed in units of $\sigma^2/(2m_+).$ The mass ratio is $m_b/m_a = 4$ and the coupling $g$ has been chosen so that $2m_+\Delta/p_0^2 = 0.2$ at $p_0^2 = p_0^2 = p_0 = 10\sigma$ to ensure weak-coupling. (This ratio is less that 1 throughout this diagram.) All phase transitions are first order as discussed in the text. The mixed phases of Fig. 4 would be found on the solid lines. The sample state in Fig. 3 at $(p_0^f, p_a^f) = (11.5, 9.2)$ is marked “X”.

FIG. 5: Quasi-particle dispersions $E_p^\pm$ (top), and occupation numbers $n_a$ and $n_b$ (middle), in a sample BP state. This state has gap parameter $\Delta \approx 11$ which is the global minimum of the grand thermodynamic potential density $\Omega(\Delta)/V$ (bottom) as defined in (10). The maximum at $\Delta = 5.6$ corresponds to an unstable BP state. These figures correspond to the point $(p_0^f, p_a^f) = (11.5, 9.2)$ in Fig. 4.
sitions, \( \Omega(\Delta) \) develops a cusp separating two competing local minima: one is BCS and the other is BP. Thus, in the \( T = 0 \) mean-field approximation, the transition must be first order. At finite temperature, the cusp is smoothed and we suspect that the transition line ends at a critical point. In this way, the \( T = 0 \) BP transition avoids instability. In non-extensive systems such as QCD where gapless states may be stabilized by neutrality constraints, similar instabilities have been noted \cite{15}. The resolution may be the formation of a non-homogeneous phase. This possibility requires further analysis.

Realizations of a stable BP phase require either non-extensivity, or a finite-range momentum dependent interaction with a large mass ratio. The former may occur in high-density QCD \cite{3, 4} where gauge interactions may stabilize the state. The latter may occur in a quantum gas of cold neutral atoms operating near Feshbach resonance with effective masses tuned by a laser lattice \cite{5}, in a system of trapped ions with dipolar interactions \cite{16}, or in superconductors with overlapping bands \cite{6, 7}.

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