Subsets of colossally abundant numbers

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Abstract

Let $G(n) = \sigma(n)/(n \log \log n)$. Robin made hypothesis that $G(n) < e^\gamma$ for all integer $n > 5040$. This article divides all colossally abundant numbers into three disjoint subsets CA1, CA2 and CA3, and shows that Robin hypothesis is true if and only if all CA2 numbers $> 5040$ satisfy Robin inequality.

Introduction

Define $\rho(n) = \sigma(n)/n$, where $\sigma(n) = \sum_{d|n} d$ is the sum of divisor function. Define

$$G(n) := \frac{\rho(n)}{\log \log n}. \quad (1)$$

Then Robin hypothesis is: all integers $n > 5040$ satisfy Robin inequality

$$G(n) < e^\gamma, \quad (RI)$$

where $\gamma$ is the Euler constant. Let

$$F(x, k) := \frac{\log(1 + 1/(x + x^2 + \cdots + x^k))}{\log x}. \quad (2)$$
Define a set
\[ E = \{ F(p, k) \mid \text{prime } p, \text{ integer } k \geq 1 \}. \]

Elements \( \epsilon_i \in E \) are indexed in decreasing order. Elements in \( E \) are called critical parameters. For a given critical parameter \( \epsilon_i \), we can construct a colossally abundant (abbreviate CA) number as follows: Define \( x_k \) as the solution of
\[ F(x_k, k) = \epsilon, \quad 1 \leq k \leq K, \]
where \( K \) is the largest integer such that \( x_K \leq 2 \). For each prime define
\[ a_p = \begin{cases} 
  k, & \text{if } x_{k+1} < p \leq x_k, \\
  0, & \text{if } p > x_1.
\end{cases} \]
and define
\[ n_i := \prod_p p^{a_p}. \]

It can be proved that \( n_i \) is a CA number, and \( n_i \) will be called the CA number constructed from \( \epsilon_i \). cf. [Broughan 2017] Section 6.3. For any integer \( n \geq 2 \), we will write \( P(n) \) for the largest prime factor of \( n \).

We divide CA into 3 disjoint subsets. Let \( n_i \) be the CA number constructed from \( \epsilon_i \), and \( p \) be the prime succeeding \( P(n_i) \).

\( n_i \) is called a CA1 number if \( \log n_i < P(n_i) \). Theorem 1 shows \( G(n_i) < G(n_{i-1}) \), \( \forall n_i \in CA1, i \geq 3 \).

\( n_i \) is called a CA2 number if \( P(n_i) < \log n_i < p \).

\( n_i \) is called a CA3 number if \( p < \log n_i \). Let \( n_j \) be the CA number constructed from \( F(p, 1) \). Theorem 2 shows that \( G(n_i) < G(n_j) \).

Corollary 4 shows that Robin hypothesis is true if and only if all CA2 numbers \( > 5040 \) satisfy (RI).

**Table 1. CA1 and CA2 numbers in the first 26 CA numbers**

| index i | \( \log n_i \) | \( P(n_i) \) | is CA1? | is CA2? | \( G(n_i) \) |
|---------|----------------|--------|--------|--------|--------|
| 1       | 0.6931         | 2      | Y      | N      | -4.0926|
| 2       | 1.7918         | 3      | Y      | N      | 3.4294 |
|   | X |   | Y | N |   |
|---|---|---|---|---|---|
| 3 | 2.4849 | 3 | Y | N | 2.5634 |
| 4 | 4.0943 | 5 | Y | N | 1.9864 |
| 5 | 4.7875 | 5 | Y | N | 1.9157 |
| 6 | 5.8861 | 5 | N | Y | 1.8335 |
| 7 | 7.8320 | 7 | N | Y | 1.8046 |
| 8 | 8.5252 | 7 | N | Y | 1.7910 |
| 9 | 10.9231 | 11 | Y | N | 1.7512 |
| 10 | 13.4880 | 13 | N | Y | 1.7331 |
| 11 | 14.1812 | 13 | N | Y | 1.7277 |
| 12 | 15.2798 | 13 | N | Y | 1.7235 |
| 13 | 16.8892 | 13 | N | Y | 1.7179 |
| 14 | 19.7224 | 17 | N | N | 1.7243 |
| 15 | 22.6669 | 19 | N | Y | 1.7342 |
| 16 | 25.8023 | 23 | N | Y | 1.7374 |
| 17 | 26.4955 | 23 | N | Y | 1.7371 |
| 18 | 29.8628 | 29 | N | Y | 1.7337 |
| 19 | 33.2968 | 31 | N | Y | 1.7340 |
| 20 | 35.2427 | 31 | N | Y | 1.7369 |
| 21 | 36.3413 | 31 | N | Y | 1.7364 |
| 22 | 39.9522 | 37 | N | Y | 1.7375 |
| 23 | 43.6658 | 41 | N | N | 1.7380 |
| 24 | 47.4270 | 43 | N | N | 1.7403 |
| 25 | 48.1203 | 47 | N | N | 1.7406 |
| 26 | 51.9703 | 47 | N | Y | 1.7430 |

So, the smallest CA1 number is \( n_1 = 2 \); the smallest CA2 number is \( n_6 = 360 \); the smallest CA3 number is \( n_{14} = 367\ 567\ 200 \).

We next calculate the bounds of increment for \( n_i \in CA3 \). Let \( p > 10^8 \) be the prime succeeding to \( P(n_i) \). Assume \( \epsilon_{i+1} = F(q,k) \) for some prime \( q \) and integer \( k \). Then
Theorem 3 shows a lower bound
\[ \frac{G(n_{i+1})}{G(n_i)} > \left(1 - \frac{\log q}{3p^2(\log p)^2}\right)^{-1}. \]

Theorem 4 shows an upper bound
\[ \frac{G(n_{i+1})}{G(n_i)} < \exp\left(\frac{0.126 \log q}{p(\log p)^3}\right). \]

I checked the first 5 763 320 CA numbers (i.e. with the largest prime factor up to $10^8$). They contain 120 529 CA1 numbers, 5 565 CA2 numbers and 5 637 226 CA3 numbers.

Main Content

Lemma 1. Let \( \epsilon \in E \) be a critical parameter and \( k \geq 1 \) be an integer. Let \( x_1 \) and \( x_k \) be defined by (4). Then
\[
(x_k + \cdots + x_k^k) \log x_k \geq x_1 \log x_1 + \left(1 - \frac{1}{2x_1}\right)\left(\frac{\log x_1}{2} - \frac{\log x_k}{2}\right). \tag{L1.1}
\]
\[
(x_k + \cdots + x_k^k) \log x_k < x_1 \log x_1 + \frac{\log x_1}{2} - \frac{\log x_k}{2} + \frac{\log x_1}{4x_1}. \tag{L1.2}
\]
(L1.1) and (L1.2) mean simple version:
\[
(x_k + \cdots + x_k^k) \log x_k \geq x_1 \log x_1. \tag{L1.1'}
\]
\[
(x_k + \cdots + x_k^k) \log x_k < x_1 \log x_1 + \frac{\log x_1}{2}. \tag{L1.2'}
\]

Proof. By definition of \( x_1 \) and \( x_k \), we have
\[
\frac{\log \left(1 + \frac{1}{x_k + \cdots + x_k^k}\right)}{\log x_k} = \epsilon = \frac{\log \left(1 + \frac{1}{x_1}\right)}{\log x_1}. \tag{L1.3}
\]
\[
x_k' = 1 + \frac{1}{x_k + \cdots + x_k^k}, \quad x_1' = 1 + \frac{1}{x_1}. \tag{L1.4}
\]
Hence
\[
\frac{x_k + \cdots + x_k^k}{x_1} = \frac{x_k'}{x_k'} - 1 = \frac{e^{\epsilon \log x_1} - 1}{e^{\epsilon \log x_k} - 1}.
\]
\[ \begin{align*}
\frac{\epsilon \log x_1 + (\epsilon \log x_1)^2}{\epsilon \log x_k + (\epsilon \log x_k)^2} + \cdots \\
= \log x_1 \left( \frac{1 + \frac{\epsilon \log x_1}{2!} + \cdots}{1 + \frac{\epsilon \log x_k}{2!} + \cdots} \right). 
\end{align*} \tag{L1.5} \]

Compare
\[ \frac{1 + \frac{\epsilon \log x_1}{2!} + \cdots}{1 + \frac{\epsilon \log x_k}{2!} + \cdots} \text{ and } 1 + \epsilon \left( \frac{\log x_1}{2} - \frac{\log x_k}{2} + c \right), \tag{L1.6} \]

where \( c \) is a to-be-determined real parameter.

\[ H := \left( 1 + \frac{\epsilon \log x_1 + \cdots}{2!} \right) \\
- \left( 1 + \frac{\epsilon \log x_k + \cdots}{2!} \right) \left( 1 + \epsilon \left( \frac{\log x_1}{2} - \frac{\log x_k}{2} + c \right) \right) \\
= \sum_{j=1}^{\infty} \frac{(\epsilon \log x_1)^{j-1}}{j!} - \sum_{j=1}^{\infty} \frac{(\epsilon \log x_k)^{j-1}}{j!} \\
- \epsilon \left( \frac{\log x_1}{2} - \frac{\log x_k}{2} + c \right) \sum_{j=1}^{\infty} \frac{(\epsilon \log x_k)^{j-1}}{j!} \\
= \sum_{j=2}^{\infty} \frac{(\epsilon \log x_1)^{j-1}}{j!} - \sum_{j=2}^{\infty} \frac{(\epsilon \log x_k)^{j-1}}{j!} \\
- \left( \frac{\log x_1}{2} - \frac{\log x_k}{2} + c \right) \sum_{j=1}^{\infty} \frac{\epsilon^{j-1}(\log x_k)^{j-2}}{(j-1)!} \\
= -\epsilon c + \sum_{j=3}^{\infty} \frac{(\epsilon \log x_1)^{j-1}}{j!} \\
- \sum_{j=3}^{\infty} \frac{\epsilon^{j-1}(\log x_k)^{j-2}}{j!} \left( \log x_k + \frac{j \log x_1}{2} - \frac{j \log x_k}{2} + jc \right). \tag{L1.7} \]

To prove (L1.1), set \( c = 0 \). The lower bound of \( H \) is

\[ H = \sum_{j=3}^{\infty} \frac{\epsilon^{j-1}}{j!} \left( (\log x_1)^{j-1} - (\log x_k)^{j-2} \left( \log x_k + \frac{j \log x_1}{2} - \frac{j \log x_k}{2} \right) \right) \]

\[ = \sum_{j=3}^{\infty} \frac{\epsilon^{j-1}}{j!} \left( (\log x_1)^{j-1} - (\log x_k)^{j-1} - \frac{j(\log x_k)^{j-2}}{2}(\log x_1 - \log x_k) \right) \]
\[
\sum_{j=3}^{\infty} \frac{e^{j-1}(\log x_1 - \log x_k)}{j!} \left( \sum_{m=0}^{j-2} (\log x_1)^m (\log x_k)^{j-2-m} - \frac{j(\log x_k)^{j-2}}{2} \right) > 0. \tag{L1.8}
\]

Combine (L1.5), (L1.7) and (L1.8), we have
\[
x_k + \cdots + x_k \frac{x_k}{x_1} > x_1 \log x_1 \left( 1 + \epsilon \left( \frac{\log x_1}{2} - \frac{\log x_k}{2} \right) \right). 
\]

Since
\[
\epsilon = \frac{\log \left( 1 + \frac{1}{x_1} \right)}{\log x_1} > \frac{1}{x_1 \log x_1} - \frac{1}{2x_1^2 \log x_1},
\]
we get
\[
(x_k + \cdots + x_k) \log x_k > x_1 \log x_1 \left( 1 + \epsilon \left( \frac{\log x_1}{2} - \frac{\log x_k}{2} \right) \right) > x_1 \log x_1 + \left( 1 - \frac{1}{2x_1} \left( \frac{\log x_1}{2} - \frac{\log x_k}{2} \right) \right). \tag{L1.9}
\]
That is, (L1.1) holds.

To prove (L1.2), we have from (L1.7)
\[
H < -\epsilon c + \sum_{j=3}^{\infty} \frac{(\epsilon \log x_1)^{j-1}}{j!}. \tag{L1.10}
\]

The summation in (L1.10) can be simplified as
\[
\sum_{j=3}^{\infty} \frac{(\epsilon \log x_1)^{j-1}}{j!} = \sum_{j=2}^{\infty} \frac{(\epsilon \log x_1)^j}{(j+1)!} < \frac{(\epsilon \log x_1)^2}{6} \sum_{j=0}^{\infty} \frac{(\epsilon \log x_1)^j}{j!} = \frac{(\epsilon \log x_1)^2}{6} e^{\epsilon \log x_1} = \frac{(\epsilon \log x_1)^2}{6} x_1 = \frac{(\epsilon \log x_1)^2}{6} \left( 1 + \frac{1}{x_1} \right). \tag{L1.11}
\]
By (L1.3), \( \epsilon < 1/(x_1 \log x_1) \), and we have
\[
H < -\epsilon c + \frac{(\epsilon \log x_1)^2}{6} \left( 1 + \frac{1}{x_1} \right)
\]
\[ \frac{\epsilon}{2} \left( \log x_1 + \frac{\log x_1}{3x_1} \right) \leq 0, \quad \text{for } c = \frac{\log x_1}{4x_1}, x_1 \geq 2. \quad \text{(L1.12)} \]

Combine (L1.5), (L1.6) and (L1.12), we get

\[ (x_k + \cdots + x_k) \log x_k < x_1 \log x_1 \left( 1 + c \left( \frac{\log x_1}{2} - \frac{\log x_k}{2} + \frac{\log x_1}{4x_1} \right) \right) < x_1 \log x_1 + \frac{\log x_1}{2} - \frac{\log x_k}{2} + \frac{\log x_1}{4x_1}. \quad \text{(L1.13)} \]

**Theorem 1.** Let \( i \geq 3 \) be an integer and \( n_i \) be a CA1 number, \( p = P(n_i) \). Then

\[ G(n_i) < G(n_{i-1}) \left( 1 - \left( \frac{\log q}{p \log p} \right)^2 \right), \quad \text{if } n_i/n_{i-1} = q. \quad \text{(1.1)} \]

\[ G(n_i) < G(n_{i-1}) \left( 1 - \left( \frac{\log q}{p \log p} \right)^2 \right) \left( 1 - \left( \frac{\log r}{p \log p} \right)^2 \right), \quad \text{if } n_i/n_{i-1} = qr. \quad \text{(1.2)} \]

**Proof.** \( n_i \in \text{CA1} \) means \( \log n_i < p \).

1) \( n_i/n_{i-1} = q \). Assume \( \epsilon_i = F(q, k) \) for some prime \( q \) and integer \( k \geq 1 \).

\[ \frac{G(n_i)}{G(n_{i-1})} = \frac{\rho(n_i) \log \log n_i}{\rho(n_{i-1}) \log \log n_{i-1}} < \left( \frac{\log q}{\log n_i \log \log n_i} \right) \left( 1 + \frac{1}{q + \cdots + q^k} \right) < \left( \frac{1 - \log q}{p \log p} \right) \left( 1 + \frac{1}{q + \cdots + q^k} \right). \quad \text{(1.3)} \]

By Lemma 1 (L1.1'), we have

\[ (q + \cdots + q^k) \log q \geq x_1 \log x_1 \geq p \log p. \quad \text{(1.4)} \]

Hence

\[ \frac{G(n_i)}{G(n_{i-1})} < \left( 1 - \frac{\log q}{p \log p} \right) \left( 1 + \frac{\log q}{p \log p} \right) = 1 - \frac{(\log q)^2}{(p \log p)^2}. \quad \text{(1.5)} \]
2) \( n_i/n_{i-1} = qr \). Assume \( \epsilon_i = F(q,k) = F(r,j) \) for some prime \( q, r \) and integer \( k \geq 1, j \geq 1 \). Then we have

\[
\frac{G(n_i)}{G(n_{i-1})} = \frac{\rho(n_i) \log \log n_{i-1}}{\rho(n_{i-1}) \log \log n_i} = \frac{\log \log n_i + \log \left(1 - \frac{\log q + \log r}{\log n_i}\right)}{\log \log n_i} \left(1 + \frac{1}{q + \cdots + q^k}\right) \left(1 + \frac{1}{r + \cdots + r^j}\right)
\]

\[
< \left(1 - \frac{\log q + \log r}{\log n_i \log \log n_i}\right) \left(1 + \frac{1}{q + \cdots + q^k}\right) \left(1 + \frac{1}{r + \cdots + r^j}\right)
\]

\[
< \left(1 - \frac{\log q}{p \log p}\right) \left(1 - \frac{\log r}{p \log p}\right) \left(1 + \frac{1}{q + \cdots + q^k}\right) \left(1 + \frac{1}{r + \cdots + r^j}\right)
\]

(1.6)

By Lemma 1 (L1.1'), we have

\[
(q + \cdots + q^k) \log q \geq p \log p, \quad (r + \cdots + r^j) \log r \geq p \log p.
\]

(1.7)

Hence we get

\[
\frac{G(n_i)}{G(n_{i-1})} < \left(1 - \frac{\log q}{p \log p}\right) \left(1 + \frac{\log q}{p \log p}\right) \left(1 - \frac{\log r}{p \log p}\right) \left(1 + \frac{\log r}{p \log p}\right)
\]

\[
= \left(1 - \frac{(\log q)^2}{(p \log p)^2}\right) \left(1 - \frac{(\log r)^2}{(p \log p)^2}\right)
\]

(1.8)

Corollary 1. Let \( n_i > n_8 = 5040 \) be a CA1 number. Let \( n_j \) be the largest non-CA1 number below \( n_i \). Then \( G(n_i) < G(n_j) \).

Proof. The condition \( n_i > n_8 = 5040 \) guarantees the existence of \( n_j \). By Theorem 1, we have

\[
G(n_i) < G(n_{i-1}) < \cdots < G(n_{j+1}) < G(n_j).
\]

(7)

Corollary 2. Robin hypothesis is true if and only if all non-CA1 numbers > 5040 satisfy (RI).
Proof. If one non-CA1 number $> 5040$ fails (RI), then Robin hypothesis fails by definition. Conversely, if Robin hypothesis fails, then (RI) fails for a CA number $n_i > 5040$, [NY 2014] Proposition 20. If $n_i \notin CA1$, then we are done. If $n_i \in CA1$, then by Corollary 1, there exists $n_j \notin CA1$, such that $G(n_i) < G(n_j)$. That is, (RI) fails for $n_j$. □

Lemma 2. Let $\epsilon \in E$ be a critical epsilon value. $x_k$ are solutions of

$$F(x_k, k) = \epsilon. \quad k \geq 1.$$  \hspace{1cm} (L2.1)

Then $g(t) = g_\epsilon(t) := t^\epsilon/ \log \log t$ has a unique minimum, say $t_0$, and $t_0$ satisfies

$$x_1 + \frac{1}{2} - \frac{1}{2 \log x_1} < \log t_0 < x_1 + \frac{1}{2} - \frac{1}{12x_1} + \frac{1}{24x_1^2}, \quad \forall x_1 \geq 2. \quad \text{(L2.2)}$$

Proof. Take derivative,

$$g'(t) = \frac{\epsilon t^{\epsilon-1} \log \log t - t^{\epsilon-1} \frac{1}{t \log t}}{(\log \log t)^2} = \frac{t^{\epsilon-1}}{\log t (\log \log t)^2} (\epsilon \log t \log \log t - 1). \quad \text{(L2.3)}$$

Define

$$f(t) := \epsilon \log t \log \log t - 1. \quad \text{(L2.4)}$$

It is obvious that $f(t)$ monotonically increases for $t \in (e, \infty)$, negative near $e$ and positive when $t$ sufficiently large. So $f(t)$ has a unique zero $t_0$. $g(t)$ attains minimum at $t_0$. Note $x_1$ is the solution of $F(x_1, 1) = \log(1 + 1/x_1)/\log x_1 = \epsilon$. Write $t = x_1 + 1/2 + d$, where $d = -1/(2 \log x_1)$. We have

$$f\left(e^{x_1+\frac{1}{2}+d}\right) = \frac{\log(1 + 1/x_1)}{\log x_1} \left(x_1 + \frac{1}{2} + d\right) \log \left(x_1 + \frac{1}{2} + d\right) - 1$$

$$= \frac{\log(1 + \frac{x_1}{x_1})}{\log x_1} \left(x_1 + \frac{1}{2} + d\right) \left(\log x_1 + \log \left(1 + \frac{1}{2} + d\right)\right) - 1$$

$$< \left(\frac{1}{x_1} - \frac{1}{2x_1^2} + \frac{1}{3x_1^3}\right) \left(x_1 + \frac{1}{2} + d\right)$$
\[
\times \left( 1 + \frac{1}{2x_1 \log x_1} - \frac{1}{2x_1 (\log x_1)^2} \right) - 1
\]
\[
= \left( 1 - \frac{1}{2x_1 \log x_1} + \frac{1}{12x_1^2} + \frac{1}{4x_1^2 \log x_1} + \frac{1}{6x_1^3} - \frac{1}{6x_1^3 \log x_1} \right)
\times \left( 1 + \frac{1}{2x_1 \log x_1} - \frac{1}{2x_1 (\log x_1)^2} \right) - 1
\]
\[
< 0, \quad \forall x_1 \geq 2.
\] (L2.5)

So we get the left inequality of (L2.2). For the right inequality, we have
\[
f \left( e^{1/\log(1+1/x_1)} \right) = \frac{\log(1 + 1/x_1)}{\log x_1} \frac{1}{\log(1 + 1/x_1)} \log \left( \frac{1}{\log(1 + 1/x_1)} \right) - 1
\]
\[
= \frac{1}{\log x_1} \log \left( x_1 + \frac{1}{2} - \frac{1}{12x_1} + \frac{1}{24x_1^2} - \cdots \right) - 1 > 0,
\] (L2.6)

here the expansion of \((\log(1+1/x_1))^{-1} = x + \frac{1}{2} - \frac{1}{12x_1} + \frac{1}{24x_1^2} - \cdots\) is calculated term wise from the formula \(\log (1 + \frac{1}{x}) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \cdots\). So we have
\[
\log t_0 < \frac{1}{\log(1 + \frac{1}{x_1})} = x_1 + \frac{1}{2} - \frac{1}{12x_1} + \frac{1}{24x_1^2} - \cdots
\]
\[
< x_1 + \frac{1}{2} - \frac{1}{12x_1} + \frac{1}{24x_1^2}.
\] (L2.7)

\[
\square
\]

**Lemma 3.** Let \(\epsilon \in E\) be a critical epsilon value. Let \(u\) and \(u_1 < u_2\) be positive reals. Then \(h(u) := e^{u}/\log u\) has a unique minimum at \(u_0 = u_0(\epsilon)\) implicitly defined by \(\epsilon = 1/(u_0 \log u_0)\). Assume \(u_0 > 40\). Write
\[
h_0 := h(u_0) = \frac{e^{1/\log u_0}}{\log u_0}.
\] (L3.1)

1) For \(u_0 - \frac{1}{2} < u_1 < u_0\),
\[
\frac{h(u_2)}{h_0} < 1 + 0.2532 \frac{u_0 - u_1}{u_0^2 \log u_0} + 0.5162 \frac{(u_0 - u_1)^2}{u_0^2 (\log u_0)^2}.
\] (L3.2)

2) For \(u_0 < u_2 < u_0 \log u_0\),
\[
\frac{h(u_2)}{h_0} > 1 + \frac{(u_2 - u_0)^2}{2u_0^2 \log u_0} - \frac{(u_2 - u_0)^2}{2u_0^2 (\log u_0)^2}.
\] (L3.3)
3) For \( u_0 < u_1 < u_2, \) \( u_2 - u_1 < \log u_0, \)

\[
\frac{h(u_2)}{h(u_1)} > 1 + 0.3337 \frac{(u_2 - u_1)^2}{u_0^2(\log u_0)^2}.
\]

(L3.4)

Proof. We have

\[
h(u) = \frac{e^{\epsilon u}}{\log u} = \frac{e^{\epsilon u_0}e^{(u-u_0)}}{\log u_0 + \log(u/u_0)} = \frac{e^{\epsilon u_0}}{\log u} \left( 1 + \frac{\log(u/u_0)}{\log u_0} \right)^{-1}
\]

\[
= h_0 \left( \sum_{i=0}^{\infty} \frac{(\epsilon(u-u_0))^i}{i!} \right) \left( \sum_{j=0}^{\infty} \frac{(-\log(u/u_0))^j}{\log u_0} \right).
\]

(L3.5)

1) When \( u_0 - \frac{1}{2} < u_1 < u_0, \) we have \( \log(u_0/\log u_0) < 0. \) Hence

\[
-\log(u_1/u_0) = -\log(1 - (u_0 - u_1)/u_0)
\]

\[
< \frac{u_0 - u_1}{u_0 \log u_0} \sum_{k=1}^{\infty} k \left( \frac{u_0 - u_1}{u_0} \right)^{k-1}
\]

\[
< \frac{u_0 - u_1}{u_0 \log u_0} \left( 1 + \frac{1}{4u_0} \sum_{k=2}^{\infty} \left( \frac{1}{2u_0} \right)^{k-2} \right)
\]

\[
< \frac{u_0 - u_1}{u_0 \log u_0} \left( 1 + \frac{0.2532}{u_0} \right).
\]

(L3.6)

\[
\frac{h(u_1)}{h_0} = \left( \sum_{i=0}^{\infty} \frac{(-\epsilon(u_0-u_1))^i}{i!} \right) \left( \sum_{j=0}^{\infty} \frac{u_0 - u_1}{u_0 \log u_0} \left( 1 + \frac{0.2532}{u_0} \right)^j \right)
\]

\[
< \left( 1 - \frac{u_0 - u_1}{u_0 \log u_0} \right) + \frac{(u_0 - u_1)^2}{2u_0^2(\log u_0)^2}
\]

\[
\times \left( 1 + \frac{u_0 - u_1}{u_0 \log u_0} + 0.2532 \frac{u_0 - u_1}{u_0 \log u_0} + 1.0162 \frac{(u_0 - u_1)^2}{u_0^2(\log u_0)^2} \right)
\]

\[
< 1 + 0.2532 \frac{u_0 - u_1}{u_0^2 \log u_0} + 0.5162 \frac{(u_0 - u_1)^2}{u_0^2(\log u_0)^2}.
\]

(L3.7)

2) When \( u_0 < u_2 < u_0 \log u_0, \) we have \( \log(u_2/u_0) > 0. \)

\[
-\log(u_2/u_0) = -\log(1 + (u_2-u_0)/u_0) > -\frac{u_2 - u_0}{u_0 \log u_0} + \frac{(u_0 - u_1)^2}{2u_0^2 \log u_0}
\]

(L3.8)
Since Xiaolong Wu and we have
\[ h(u_2) > \left( \sum_{i=0}^{\infty} \frac{(u_2 - u_0)^i}{i!} \right) \left( \sum_{j=0}^{\infty} \frac{(-u_2 - u_0 + (u_2 - u_0)^2)^i}{u_0 \log u_0 + 2u_0^2 \log u_0} \right) \]
\[ > \left( \sum_{i=0}^{\infty} \frac{(u_2 - u_0)^i}{i!u_0^i(\log u_0)^i} \right) \left( 1 - \frac{u_2 - u_0}{u_0 \log u_0} + \frac{(u_2 - u_0)^2}{2u_0^2(\log u_0)^2} \right) \]
\[ > \left( 1 + \frac{u_2 - u_0}{u_0 \log u_0} + \frac{(u_2 - u_0)^2}{2u_0^2(\log u_0)^2} \right) \left( 1 - \frac{u_2 - u_0}{u_0 \log u_0} + \frac{(u_2 - u_0)^2}{2u_0^2(\log u_0)^2} \right) \]
\[ > 1 + \frac{(u_2 - u_0)^2}{2u_0^2(\log u_0)^2} - \frac{(u_2 - u_0)^2}{2u_0^2(\log u_0)^2}. \quad \text{(L3.9)} \]

3) Write \( u_2 = u_1 + a \) for some real \( a < \log u_0 \).

\[ \frac{h(u_1)}{h(u_1)} = e^{c(u_2-u_1)} \frac{\log u_1}{\log u_2} = e^{a/(u_0 \log u_0)} \frac{\log u_0}{\log(u_0 + a)}. \quad \text{(L3.10)} \]

Since
\[ \frac{\log u_0}{\log(u_0 + a)} = \frac{\log u_0}{\log u_0 + \log(1 + a/u_0)} = \frac{1}{1 + \log(1 + a/u_0)/\log u_0} \]
\[ = \sum_{i=0}^{\infty} \left( \frac{-\log(1 + a/u_0)}{\log u_0} \right)^i \]
\[ > 1 - \frac{\log(1 + a/u_0)}{\log u_0} = 1 - \frac{1}{\log u_0} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \left( \frac{a}{u_0} \right)^i \]
\[ > 1 - \frac{a}{u_0 \log u_0} + \frac{a^2}{2u_0^2 \log u_0} - \frac{a^3}{3u_0^3 \log u_0}. \quad \text{(L3.13)} \]

and
\[ e^{a/(u_0 \log u_0)} = \sum_{i=0}^{\infty} \frac{1}{i!} \left( \frac{a}{u_0 \log u_0} \right)^i > 1 + \frac{a}{u_0 \log u_0} + \frac{a^2}{2u_0^2 \log u_0}, \quad \text{(L3.14)} \]

we have
\[ \frac{h(u_2)}{h(u_1)} > \left( 1 + \frac{a}{u_0 \log u_0} + \frac{a^2}{2u_0^2 \log u_0} \right) \]
\[ \times \left( 1 - \frac{a}{u_0 \log u_0} + \frac{a^2}{2u_0^2 \log u_0} - \frac{a^3}{3u_0^3 \log u_0} \right) \]
\[ > 1 + \frac{a^2}{u_0^2 \log u_0} \left( \frac{1}{2} - \frac{1}{2 \log u_0} - \frac{a}{3u_0} \right) > 1 + \frac{0.3337a^2}{u_0^2 \log u_0}. \quad \text{(L3.15)} \]
Lemma 4. Assume \( g(t) = t^{\epsilon} / \log \log t \) takes minimum at \( t_0 = t_0(\epsilon) \). Assume \( \log t_0 > 40 \). Let \( N \) and \( N_1 \) be positive integers.

1) If \( \log t_0 - \frac{1}{2} < \log N < \log t_0 \) and \( \log t_0 + 2 < \log N_1 \), then

\[
g(N_1) > g(N) \left( 1 + \frac{2.754}{(\log t_0)^2 \log \log t_0} \right). \tag{L4.1}
\]

2) If \( \log t_0 < \log N < \log N_1 \) and \( \log N_1 - \log N < \log \log t_0 \), then

\[
g(N_1) > g(N) \left( 1 + \frac{0.3337(\log N_1 - \log N)^2}{(\log t_0)^2 \log \log t_0} \right). \tag{L4.2}
\]

Proof. Write \( u = \log t, u_0 = \log t_0, h(u) = g(t), h_0 = \log t_0 \). By Lemma 3 (L3.1) and (L3.2), we have

\[
\frac{g(N_1) - g(N)}{h_0} > \frac{(\log N_1 - u_0)^2}{u_0^2 \log u_0} - \frac{(\log N_1 - u_0)^2}{2u_0^2 (\log u_0)^2} - 0.2532 \frac{u_0 - \log N}{u_0^2 \log u_0} - 0.5162 \frac{(u_0 - \log N)^2}{u_0^2 (\log u_0)^2}
\]
\[
> \frac{4 - 0.1266}{u_0^2 \log u_0} - \frac{2 + 0.1291}{u_0^2 (\log u_0)^2} > 3.2962. \tag{L4.3}
\]

By Lemma 3 (L3.1)

\[
g(N) h_0 < 1 + \frac{0.2532 \times 0.5}{40^2 \log 40} + \frac{0.5162 \times 0.5^2}{40^2 (\log 40)^2} = 1.00002738, \tag{L4.4}
\]

we have

\[
g(N_1) > g(N) \left( 1 + \frac{3.2962 h_0}{g(N) u_0^2 \log u_0} \right) > g(N) \left( 1 + \frac{3.2961}{u_0^2 \log u_0} \right). \tag{L4.5}
\]

2) follows from Lemma 3 (L3.3).

Theorem 2. Let \( n_i \) be \( CA3 \). Let \( p \) be the prime succeeding \( P(n_i) \), \( n_j \) be the \( CA \) number constructed from \( \epsilon_j = F(p, 1) \). then

\[
G(n_j) > G(n_i) \left( 1 + \frac{3.2961}{(\log t_0)^2 \log \log t_0} \right), \tag{2.1}
\]

where \( t_0 \) is defined as in Lemma 4.
Proof. \( n_i \in CA3 \) means \( p < \log n_i \). By definition of CA numbers, we have

\[
\frac{n}{n_i} = \frac{\rho(n_i)}{\rho(n_j)} \leq \frac{g(n_j)}{g(n_i)}.
\]

where \( g(t) = t^{\epsilon} / \log \log t \). By Lemma 2, \( g(t) \) attains minimum at \( t_0 \), and

\[
p + \frac{1}{2} - \frac{1}{2 \log p} < \log t_0 < p + \frac{1}{2} - \frac{1}{12p} + \frac{1}{24p^2}.
\]

The smallest CA3 number \( n_{14} \) can be directly checked. So, we may start from the next CA3 number \( n_{23} \). That is, we may assume \( n_i \geq n_{23} \) with \( p \geq 43 \) and \( \log p \geq 3.76 \).

Case 1) \( \log n_i < \log t_0 \). In this case, we have \( \log t_0 < p + \frac{1}{2} \) by (2.3). Hence

\[
\log n_j - \log t_0 > \log n_j - \log n_i - \frac{1}{2} > \log p - \frac{1}{2} > 2.
\]

Hence the conditions of Lemma 4 (L4.1) are satisfied and (2.1) holds.

Case 2) \( \log n_i \geq \log t_0 \). In this case, \( \log n_j - \log n_i < \log t_0 \). So by Lemma 4 (L4.2), we have

\[
G(n_j) > G(n_i) \left( 1 + \frac{0.3337(\log n_j - \log n_i)^2}{(\log t_0)^2 \log \log t_0} \right).
\]

Since \( 0.3337(\log n_j - \log n_i)^2 \geq 0.3337 \times (\log 43)^2 > 3.2961 \), (2.1) holds.

Corollary 3. Let \( n_i \) be a CA3 number. Then there exists \( n_j \in CA2 \) such that \( n_i < n_j \). If \( n_j \) is the smallest CA2 number above \( n_i \), then \( G(n_i) < G(n_j) \).

Proof. There are infinite CA1 numbers \( n \), i.e. \( \log n < P(n) \), [CNS 2012] Theorem 7. Let \( n_k \) be the smallest such number above \( n_i \).

We claim that \( n_{k-1} \) is CA2. \( n_{k-1} \) is not CA1 by minimality of \( n_k \). If \( n_{k-1} \) were CA3, there would exist a prime \( p \) such that \( P(n_{k-1}) < p < \log n_{k-1} \).

Then we would have

\[
\log n_k = \log n_{k-1} + \log(n_k/n_{k-1}) > \log n_{k-1} > p \geq P(n_k).
\]
This contradicts to \( n_k \in CA_1 \). So \( n_{k-1} \in CA_2 \) and we proved the existence of \( n_j \).

Write \( p_e = P(n_i), p_s = P(n_j) \). Let \( n_{i_m} \) be the CA number generated from parameter \( F(p_m, 1), r < m \leq s \). Since \( n_k \) is the smallest CA1 number above \( n_i \), and \( n_j < n_k \) is the smallest CA2 number above \( n_i \), all \( n_{i_m} < n_j \) are CA3.

By Theorem 2, we have

\[
G(n_i) < G(n_{i+1}) < \cdots < G(n_s) = G(n_j). \tag{C3.2}
\]

\[ \square \]

**Corollary 4.** Robin hypothesis is true if and only if all CA2 numbers > 5040 satisfy (RI).

Proof. If one CA2 number > 5040 fails (RI), then Robin hypothesis fails by definition. Conversely, if (RI) fails, then by Corollary 2, (RI) fails for a non-CA1 number \( n_i > 5040 \). If \( n_i \notin CA_2 \), then we are done. If \( n_i \notin CA_2 \), then by Corollary 3, there exists \( n_j \in CA_2 \), such that \( G(n_i) < G(n_j) \). That is, (RI) fails for \( n_j \). \[ \square \]

Under assumption of Theorem 2, is \( G(n_i) < G(n_{i+1})? \) Let \( \epsilon_{i+1} = F(q, k) \) If \( q \geq 3 \), Theorem 3 proves \( G(n_i) < G(n_{i+1}) \). The case \( q = 2 \) is open. Theorem 3 also shows a lower bound for \( G(n_{i+1})/G(n_i) \).

**Theorem 3.** Let \( n_i \) be CA3. Let \( p \) be the prime succeeding \( P(n_i) \).

1) If \( \epsilon_{i+1} = F(q, k), q \geq 3, \) then \( n_{i+1}/n_i = q, \) and \( G(n_i) < G(n_{i+1}) \).

2) If \( \epsilon_{i+1} = F(q, k), q \geq 23, \) then \( n_{i+1}/n_i = q, \) and

\[
G(n_i) < G(n_{i+1}) \left( 1 - \frac{(\log q)^2}{3p^2 \log p} \right). \tag{3.1}
\]

3) If \( \epsilon_{i+1} = F(q, k) = F(r, j), q \geq 23, r \geq 23, \) then \( n_{i+1}/n_i = qr, \) and

\[
G(n_i) < G(n_{i+1}) \left( 1 - \frac{(\log q)^2}{3p^2 \log p} \right) \left( 1 - \frac{(\log r)^2}{3p^2 \log p} \right). \tag{3.2}
\]
Proof. I numerically checked for all CA3 numbers \(n_i\) with \(i < 10,000\). 1) – 3) all hold. So we may assume \(i \geq 10,000\), and hence \(p > 103,049\). Since \(n_i \in \text{CA}_3\), we have \(p < \log n_i\). 1) and 2). Since \(\epsilon_{i+1} = F(q, k)\), we have \(n_{i+1} = n_iq\). Compare \(G(n_i)\) and \(G(n_{i+1})\), we have

\[
\frac{G(n_i)}{G(n_{i+1})} = \frac{\rho(n_i) \log \log n_{i+1}}{\rho(n_{i+1}) \log \log n_i} = \frac{\log(\log n_i + \log q) \left(\frac{q + \ldots + q^k}{1 + q + \ldots + q^k}\right)}{\log \log n_i} = \frac{\log \log n_i + \log \left(1 + \frac{\log q}{\log n_i}\right)}{\log \log n_i} \left(1 - \frac{1}{1 + q + \cdots + q^k}\right). \tag{3.3}
\]

Since \(\log n_i > p > x_1\), where \(x_1\) is defined by (4), Lemma 1 (L1.2’) means

\[
\frac{G(n_i)}{G(n_{i+1})} < \left(1 + \frac{\log \left(1 + \frac{\log q}{p}\right)}{\log p}\right) \left(1 - \frac{1}{1 + \frac{\log q}{\log p} \left(p \log p + \frac{\log p}{2}\right)}\right) < \left(1 + \frac{(\log q) \left(1 - \frac{\log q}{2p} + \frac{(\log q)^2}{3p^2}\right)}{p \log p}\right) \left(1 - \frac{1}{1 + \frac{p \log p}{\log q} + \frac{\log p}{2 \log q}}\right) = 1 + \frac{\left(p + \frac{1}{2}\right) \left(1 - \frac{\log q}{2p} + \frac{(\log q)^2}{3p^2}\right) - p}{p \left(1 + \frac{p \log p}{\log q} + \frac{\log p}{2 \log q}\right)} \tag{3.4}
\]

\[
\left(p + \frac{1}{2}\right) \left(1 - \frac{\log q}{2p} + \frac{(\log q)^2}{3p^2}\right) - p = \frac{1}{2} - \frac{\log q}{2} + \frac{(\log q)^2}{3p^2} - \frac{\log q}{2p} \left(1 - \frac{\log q}{3p}\right) < \frac{1}{2} - \frac{\log q}{2} + \frac{(\log q)^2}{3p}. \tag{3.5}
\]

When \(q \geq 3\), the expression in (3.5) is negative, so (3.4) means

\[
G(n_i) <
\]
\[ G(n_{i+1}). \text{ That is, 1) is true. Now for 2) we have} \]
\[ \frac{G(n_i)}{G(n_{i+1})} < 1 + \frac{\frac{1}{2} - \frac{\log q}{2} + \frac{(\log q)^2}{dp}}{p \left( 1 + \frac{\log p}{p \log q} + \frac{\log q}{2 \log q} \right)} \]
\[ = 1 - \frac{\log q}{6p^2} \left( -3p + 3p \log q - 2(\log q)^2 \right). \tag{3.6} \]

It is easy to verify that
\[ \frac{-3p + 3p \log q - 2(\log q)^2}{\log q + p \log p + \log p/2} > \frac{2 \log q}{\log p}, \forall q \geq 23. \tag{3.7} \]

Combine (3.6) and (3.7), we get
\[ \frac{G(n_i)}{G(n_{i+1})} < 1 - \frac{\log q}{6p^2} \left( 2 \log q \log p \right) = 1 - \frac{(\log q)^2}{3p^2 \log p}, \forall q \geq 23. \tag{3.8} \]

3) Assume \( \epsilon_{i+1} = F(q, k) = F(r, j), q \geq 23, r \geq 23. \) Then \( n_{i+1} = n_i q r. \)

Compare \( G(n_i) \) and \( G(n_{i+1}) \), we have
\[ \frac{G(n_i)}{G(n_{i+1})} = \frac{\rho(n_i) \log \log n_{i+1}}{\rho(n_{i+1}) \log \log n_i} \]
\[ = \frac{\log \log n_i + \log \left( 1 + \frac{\log q \log r}{\log n_i} \right)}{\log \log n_{i+1}} \]
\[ \times \left( 1 - \frac{1}{1 + q + \ldots + q^k} \right) \left( 1 - \frac{1}{1 + r + \ldots + r^j} \right) \]
\[ < \left( 1 + \frac{\log \left( 1 + \frac{\log q}{p} \right)}{\log p} \right) \left( 1 + \frac{\log \left( 1 + \frac{\log r}{p} \right)}{\log p} \right) \]
\[ \times \left( 1 - \frac{1}{1 + q + \ldots + q^k} \right) \left( 1 - \frac{1}{1 + r + \ldots + r^j} \right). \tag{3.9} \]

By Lemma 1 (L1.2'), we have
\[ q + \ldots + q^k < \frac{1}{\log q} \left( p \log p + \frac{\log p}{2} \right), r + \ldots + r^j < \frac{1}{\log r} \left( p \log p + \frac{\log p}{2} \right). \tag{3.10} \]

Then we can proceed with \( q \) and \( r \) separately as in 2) to prove (3.2).
We will prove Lemmas 5-7, then use them to prove an upper bound for \( G(n_{i+1})/G(n_i) \) in Theorem 4.

**Lemma 5.** Define

\[
f(x) := \frac{1}{\sqrt{2x}} \sum_{k=3}^{K(x)} (kx)^{1/k}, \quad x > 2.667,
\]  

(L5.1)

where \( K(x) \) is implicitly defined as the largest integer \( K \) satisfying

\[
\frac{2^K}{K} \leq x.
\]  

(L5.2)

Then

1) \( f(x) \) is a piece-wise differentiable function with discontinuous points at \( x = \frac{2^K}{K} \) for each integer \( K \geq 3 \).

2) \( f(x) \) decreases at differentiable points.

3) \( f(x) \) has local maximums at discontinuous points \( x = \frac{2^K}{K} \). \( f \left( \frac{2^K}{K} \right) > f \left( \frac{2^{K+1}}{K+1} \right) \), for \( K \geq 7 \).

4) In particular,

\[
f(x) < 0.10924, \quad \forall \ x \geq \frac{2^{31}}{31} = 6.93 \times 10^7.
\]  

(L5.3)

**Proof.** 1) and 2) are simple.

\[
f'(x) = \frac{1}{\sqrt{2x}} \left( \sum_{k=3}^{K(x)} k^\frac{1}{k} x^\frac{1}{k} \right)'
\]

\[
= \frac{1}{\sqrt{2}} \sum_{k=3}^{K(x)} \left( \frac{1}{k} - \frac{1}{2} \right) k^\frac{1}{k} x^\frac{1}{k} - \frac{1}{2} - 1, \quad \forall \ x > 2.667, \ x \neq \frac{2^K}{K}.
\]  

(L5.4)

So \( f(x) \) decreases at all differentiable points.

3) Because \( f(x) \) adds an extra summand 2 at point \( x = \frac{2^K}{K} \), it is discontinuous there. To show \( f(x) \) decreases from one discontinuous point to next, let \( x_s = 2^{K(x_s)}/K(x_s) \), i.e. \( (K(x_s)x_s)^{1/K(x_s)} = 2 \). Then the next discontinuous point is \( x_t := 2^{K(x_t)}/K(x_t) \) where \( K(x_t) = K(x_s) + 1 \). So we have

\[
x_s = \frac{2K(x_t)}{K(x_s)} = \frac{2^{K(x_t)+1}}{2K(x_s)} = \frac{2^{K(x_t)}}{2(K(x_t) - 1)} = \frac{K(x_t)}{2(K(x_t) - 1)} x_t.
\]  

(L5.5)
Now we want to show $f(x_s) > f(x_t)$.

\[
f(x_s) - f(x_t) = \frac{1}{\sqrt{2x_s}} \sum_{k=3}^{K(x_s)} (k x_s)^k - \frac{1}{\sqrt{2x_t}} \sum_{k=3}^{K(x_t)} (k x_t)^k
\]

\[
= \frac{1}{\sqrt{2}} \sum_{k=3}^{K(x_s)} k^\frac{1}{k} \left( x_s^{-\frac{k}{k+1}} - x_t^{-\frac{k}{k+1}} \right) - \frac{1}{\sqrt{2x_t}} (K(x_t)x_t)^{\frac{1}{K(x_t)}}
\]

\[
\geq \frac{1}{\sqrt{2}} 3^{\frac{1}{3}} (x_s^{-\frac{1}{3}} - x_t^{-\frac{1}{3}}) - \frac{2}{\sqrt{2x_t}}
\]

\[
= 1.44 \frac{1}{\sqrt{2}} x_t^{-\frac{1}{3}} \left( \left( \frac{2(K(x_t) - 1)}{K(x_t)} \right)^{\frac{1}{3}} - 1 \right) - \frac{2}{\sqrt{2x_t}} \quad \text{(L5.6)}
\]

When $K(x_t) \geq 8$,

\[
\left( \frac{2(K(x_t) - 1)}{K(x_t)} \right)^{\frac{1}{3}} \geq \left( \frac{14}{8} \right)^{\frac{1}{3}} = 1.0977. \quad \text{(L5.7)}
\]

\[
f(x_s) - f(x_t) \geq \frac{0.14077}{\sqrt{2}} x_t^{-\frac{1}{3}} - \frac{2}{\sqrt{2x_t}}
\]

\[
= \frac{2}{\sqrt{2x_t}} \left( 0.07038 \times x_t^{\frac{1}{3}} - 1 \right) > 0, \forall x_t \geq 2868. \quad \text{(L5.8)}
\]

For $x_t < 2868$ and $K \geq 7$, $f(2^K/K) > f(2^{K+1}/(K + 1))$ can be directly calculated:

| K  | $x = 2^K/K$ | $f(x)$ |
|----|-------------|--------|
| 3  | 2.67        | 0.87   |
| 4  | 4.00        | 1.52   |
| 5  | 6.40        | 1.94   |
| 6  | 10.67       | 2.15   |
| 7  | 18.29       | 2.21   |
| 8  | 32.00       | 2.16   |
| 9  | 56.89       | 2.03   |
| 10 | 102.40      | 1.86   |
4) Direct calculation shows $f\left(\frac{2\pi}{3\sqrt{3}}\right) = 0.10923475$. \hfill \Box

**Lemma 6.** Let $\theta(x)$ and $\psi(x)$ be Chebyshev functions. Define

$$
\psi_0(x) := \sum_{k=1}^{K} \theta((kx)^{1/k}), \tag{L6.1}
$$

where $K$ is the largest integer $k$ such that $(kx)^{1/k} \geq 2$. Then

$$
\psi_0(x) < x \left(1 + \frac{0.06323}{(\log x)^2}\right), \forall x > 10^8. \tag{L6.2}
$$

**Proof.** By [PT 2018] Theorem 1,

$$
\theta(x) < x, \quad \forall 0 < x < 1.39 \times 10^{17}. \tag{L6.3}
$$

Setting $k = 2, \eta_2 = 0.01$ in Theorem 4.2 of [Dusart 2018], we have,

$$
|\theta(x) - x| < \frac{0.01x}{(\log x)^2}, \quad \forall x \geq 7713133853. \tag{L6.4}
$$

Combine (L6.3) and (L6.4), we get

$$
\theta(x) - x < \frac{0.01x}{\log(1.39 \times 10^{17})^2} = 6.418 \times 10^{-6} x, \quad \forall x > 0. \tag{L6.5}
$$

By (L6.5) and Lemma 5, we have

$$
\psi_0(x) = \sum_{k=1}^{K} \theta((kx)^{1/k})
$$
\[
< \theta(x) + (1 + 6.418 \times 10^{-6}) ((2x)^{1/2} + \cdots + (Kx)^{1/K}) \\
< \theta(x) + (1 + 6.418 \times 10^{-6})(2x)^{1/2}(1 + 0.109235) \\
< x + \frac{0.01x}{(\log x)^2} + 1.5687x^{1/2} \\
= x + \frac{x}{(\log x)^2} \left(0.01 + \frac{1.5687(\log x)^2}{x^{1/2}}\right) \\
< x + \frac{0.06323x}{(\log x)^2}, \quad \forall x > 10^8. \quad \text{(L6.6)}
\]

**Lemma 7.** Let \( n \) be a CA number and \( p = P(n) \). Then

\[
\log n < p \left(1 + \frac{0.06323}{(\log p)^2}\right), \quad \forall p > 10^8. \quad \text{(L7.1)}
\]

**Proof.** Let \( x_k \) be defined by (4). By method of Theorem 4 of [Wu 2019], we have

\[
\log N = \theta(p) + \theta(x_2) + \cdots + \theta(x_K) \\
< \theta(p) + \theta((2p)^{1/2}) + \cdots + \theta((Kp)^{1/K}) = \psi_0(p), \quad \text{(L7.2)}
\]

where \( K \) is the largest integer \( k \) such that \((kx)^{1/k} \geq 2\) and \( \psi_0 \) is defined as in Lemma 6. By Lemma 6, we have

\[
\log N < p \left(1 + \frac{0.06323}{(\log p)^2}\right), \quad \forall p > 10^8. \quad \text{(L7.3)}
\]

**Theorem 4.** Let \( n_i \) be CA3. Let \( p \) be the prime succeeding \( P(n_i) \) and \( p > 10^8 \). Then \( p < \log n_i \). Assume \( n_{i+1} = n_i q \) and \( \epsilon_{i+1} = F(q,k) \) for some prime \( q \) and integer \( k \). Then

\[
G(n_{i+1}) < G(n_i) \exp \left(\frac{0.12646 \log q}{p(\log p)^3}\right). \quad \text{(4.1)}
\]
Proof. Write $\epsilon := \epsilon_{i+1}$. Define $g(t) := t' / \log \log t$ with minimum at $t_0$. Then we have

$$\frac{G(n_{i+1})}{G(n_i)} = \frac{g(n_{i+1})}{g(n_i)} = \frac{\log n_i}{\log \log n_{i+1}}$$

$$= \exp(\epsilon \log q + \log \log n_i - \log \log n_{i+1})$$

$$< \exp \left( \frac{\log q}{\log t_0 \log \log t_0} - \frac{\log n_{i+1} - \log n_i}{\log \log n_{i+1}} \right)$$

$$= \exp \left( \frac{\log q}{\log t_0 \log \log t_0} - \frac{\log n_{i+1} - \log n_i}{\log \log n_{i+1}} \right). \quad (4.2)$$

By Lemma 7,

$$\log n_{i+1} < \psi_0(p) < cp, \quad (4.3)$$

$c := 1 + \frac{0.06323}{(\log p)^2}$, for $p > 10^8$. Hence

$$\frac{c^2 - 1}{c^2} = \frac{1 + \frac{0.12646}{(\log p)^2} + \frac{0.004}{(\log p)^4} - 1}{1 + \frac{0.12646}{(\log p)^2} + \frac{0.004}{(\log p)^4}}$$

$$= \frac{0.12646}{(\log p)^2} \times \frac{1 + \frac{0.031615}{(\log p)^4}}{1 + \frac{0.12646}{(\log p)^2} + \frac{0.004}{(\log p)^4}} < \frac{0.12646}{(\log p)^2}. \quad (4.4)$$

By Lemma 2, $p < \log t_0$. So (4.2) means

$$\frac{G(n_{i+1})}{G(n_i)} < \exp \left( \frac{\log q}{p \log \log p} - \frac{\log q}{c^2 p \log p} \right)$$

$$= \exp \left( \frac{(c^2 - 1) \log q}{c^2 p \log p} \right) < \exp \left( \frac{0.12646 \log q}{p(\log p)^3} \right), \quad \forall p > 10^8. \quad (4.5)$$

\[ \square \]

References

[Briggs 2006] K. Briggs. Abundant numbers and the Riemann hypothesis. Experiment. Math., 15(2):251–256, 2006.

[Broughan 2017] K. Broughan, Equivalents of the Riemann Hypothesis Vol 1. Cambridge Univ. Press. (2017)
[CLMS 2007] Y.-J. Choie, N. Lichiardopol, P. Moree, and P. Solé. *On Robin’s criterion for the Riemann hypothesis*. J. Théor. Nombres Bordeaux, 19(2):357–372, 2007.

[CNS 2012] G. Caveney, J.-L. Nicolas, and J. Sondow, *On SA, CA, and GA numbers*, Ramanujan J. 29 (2012), 359–384.

[Dusart 1998] P. Dusart. *Sharper bounds for ψ, θ, π, pk*, Rapport de recherche n 1998–06, Laboratoire d’Arithmétique de Calcul formel et d’Optimisation

[Dusart 2018] P. Dusart. *Explicit estimates of some functions over primes*. Ramanujan J., 45(1):227–251, 2018.

[EN 1975] P. Erdős and J.-L Nicolas, *Repartition des nombres superabundants*. Bulletin de la S. M., tome 103 (1975), p. 65-90

[Morrill;Platt 2018] T. Morrill, D. Platt. *Robin’s inequality for 25-free integers and obstacles to analytic improvement* https://arxiv.org/abs/1809.10813

[NY 2014] S. Nazardonyavi and S. Yakubovich. *Extremely Abundant Numbers and the Riemann Hypothesis*. Journal of Integer Sequences, Vol. 17 (2014), Article 14.2.8

[PT 2016] D.J. Platt and Tim Trudgian. *On the first sign change of θ(x)−x*. Math. Comp. 85 (2016), 1539-1547

[Robin 1984] G. Robin. *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann*. Journal de mathématiques pures et appliquées. (9), 63(2):187–213, 1984.

[Wu 2019] X. Wu. *Properties of counterexample of Robin hypothesis*. https://arxiv.org/abs/1901.09832