Ultra-spinning exotic compact objects supporting static massless scalar field configurations

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Horizonless spacetimes describing highly compact exotic objects with reflecting (instead of absorbing) surfaces have recently attracted much attention from physicists and mathematicians as possible quantum-gravity alternatives to canonical classical black-hole spacetimes. Interestingly, it has recently been proved that spinning compact objects with angular momenta in the sub-critical regime $\tilde{a} \equiv J/M^2 \leq 1$ are characterized by an infinite countable set of surface radii, $\{r_c(\tilde{a}; n)\}_{n=1}^{\infty}$, that can support asymptotically flat static configurations made of massless scalar fields. In the present paper we study analytically the physical properties of ultra-spinning exotic compact objects with dimensionless angular momenta in the complementary regime $\tilde{a} > 1$. It is proved that ultra-spinning reflecting compact objects with dimensionless angular momenta in the super-critical regime $\sqrt{1 - |m/(l + 2)|^2} \leq |\tilde{a}|^{-1} < 1$ are characterized by a finite discrete family of surface radii, $\{r_c(\tilde{a}; n)\}_{n=1}^{N}$, distributed symmetrically around $r = M$, that can support spatially regular static configurations of massless scalar fields (here the integers $\{l, m\}$ are the harmonic indices of the supported static scalar field modes). Interestingly, the largest supporting surface radius $r_c^{\text{max}}(\tilde{a}) \equiv \max_{n} \{r_c(\tilde{a}; n)\}$ marks the onset of superradiant instabilities in the composed ultra-spinning-exotic-compact-object-massless-scalar-field system.

I. INTRODUCTION

Curved black-hole spacetimes with absorbing event horizons are one of the most exciting predictions of the classical Einstein field equations. The physical and mathematical properties of classical black-hole spacetimes have been extensively explored during the last five decades [1-2], and it is widely believed that the recent detection of gravitational waves [3-4] provides compelling evidence for the existence of spinning astrophysical black holes of the Kerr family. Intriguingly, however, the physical properties of highly compact horizonless objects have recently been explored by many physicists (see [5-22] and references therein) in an attempt to determine whether these exotic curved spacetimes can serve as valid alternatives, possibly within the framework of a unified quantum theory of gravity, to canonical black-hole spacetimes.

In a very interesting work, Maggio, Pani, and Ferrari [17] have recently explored the complex resonance spectrum of massless scalar fields linearly coupled to horizonless spinning exotic compact objects. The numerical results presented in [17] have explicitly demonstrated the important physical fact that, for given values $\{l, m\}$ of the scalar field harmonic indices, there is a critical compactness parameter characterizing the central reflecting objects, above which the massless scalar fields grow exponentially in time. This characteristic behavior of the fields in the horizonless spinning curved spacetimes indicates that the corresponding exotic objects may become unstable when coupled to bosonic (integer-spin) fields [23]. In particular, this superradiant instability [24-28] is attributed to the fact that the characteristic absorbing boundary conditions of classical black-hole spacetimes have been replaced in [17] by reflecting boundary conditions at the compact surfaces of the horizonless exotic objects.

The physical properties of marginally-stable spinning exotic compact objects were studied analytically in [19]. In particular, it was explicitly proved in [19] that reflecting compact objects with sub-critical angular momenta in the regime $0 < \tilde{a} \equiv J/M^2 \leq 1$ [29-30] are characterized by an infinite countable set of surface radii, $\{r_c(\tilde{a}; n)\}_{n=1}^{\infty}$, which can support spatially regular static (marginally-stable) configurations made of massless scalar fields. The ability of spinning compact objects to support static scalar field configurations is physically interesting from the point of view of the no-hair theorems discussed in [31-32]. In particular, it was proved in [31-32] that spherically-symmetric (non-spinning) horizonless reflecting objects, like black holes with absorbing horizons [34-35], cannot support spatially regular nonlinear massless scalar field configurations [37-39].

Interestingly, the parameter space of the composed spinning-exotic-compact-object-massless-scalar-field system is divided by the outermost supporting radius, $r_c^{\text{max}}(\tilde{a}) \equiv \max_{n} \{r_c(\tilde{a}; n)\}$, to stable and unstable configurations. In particular, horizonless reflecting objects whose surface radii lie in the regime $r_c > r_c^{\text{max}}(\tilde{a})$ are stable to scalar perturbation modes [17-19], whereas the ergoregion of compact enough spinning objects in the physical regime $r_c < r_c^{\text{max}}(\tilde{a})$ can trigger superradiant instabilities in the surrounding bosonic clouds [17-19].

The main goal of the present paper is to explore the physical properties of exotic ultra-spinning ($\tilde{a} > 1$) horizonless compact objects [40-42]. Interestingly, we shall explicitly prove below that spinning compact objects in the super-
critical \( \bar{a} > 1 \) regime are characterized by a finite discrete family of surface radii, \( \{ r_c(\bar{a}; n) \}_{n=1}^{N'} \), that can support the static (marginally-stable) scalar field configurations. This unique property of the ultra-spinning \( (\bar{a} > 1) \) reflecting compact objects should be contrasted with the previously proved fact \( [10] \) that sub-critical \( (\bar{a} < 1) \) spinning objects are characterized by an infinite countable family of surface radii, \( \{ r_c(\bar{a}; n) \}_{n=1}^{\infty} \), that can support spatially regular static scalar field configurations.

Using analytical techniques, we shall determine in this paper the characteristic critical (largest) surface radius, \( r_c^{\max}(\bar{a}) \equiv \max_n \{ r_c(\bar{a}; n) \} \), of the ultra-spinning reflecting objects that, for given value of the super-critical rotation parameter \( \bar{a} \), marks the boundary between stable and superradiantly unstable spinning configurations. In particular, below we shall derive a remarkably compact analytical formula for the discrete (and finite) family of supporting surface radii which characterizes exotic near-critical spinning horizonless compact objects in the physically interesting regime \( 0 < \bar{a} - 1 \ll 1 \).

II. DESCRIPTION OF THE SYSTEM

We consider a spatially regular configuration made of a massless scalar field \( \Psi \) which is linearly coupled to an ultra-spinning reflecting compact object of radius \( r_c \), mass \( M \), and dimensionless angular momentum in the super-critical regime

\[
\bar{a} \equiv \frac{J}{M^2} > 1.
\]

Following the interesting physical model of the exotic compact objects discussed by Maggio, Pani, and Ferrari \( [17] \) (see also \( [18, 20] \)), we shall assume that the external spacetime geometry of the spinning compact object is described by the Kerr line element \( [1, 2, 29, 45–50] \)

\[
ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \rho^2 dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left[ a dt - (r^2 + a^2) d\phi \right]^2 \quad \text{for} \quad r > r_c ,
\]

where the metric functions are given by \( \Delta \equiv r^2 - 2Mr + a^2 \) and \( \rho^2 \equiv r^2 + a^2 \cos^2 \theta \) with \( a \equiv M\bar{a} \).

The spatial and temporal behavior of the massless scalar field configurations in the curved spacetime \( (2) \) of the spinning reflecting object is governed by the compact Klein-Gordon wave equation \( [51, 52] \)

\[
\nabla^\nu \nabla_\nu \Psi = 0 .
\]

Using the spatial-temporal expression \( [51, 53] \)

\[
\Psi(t, r, \theta, \phi) = \sum_{l,m} e^{im\phi} S_{lm}(\theta; a \omega) R_{lm}(r; M, a, \omega) e^{-i\omega t}
\]

for the linearized massless scalar field, one finds the ordinary differential equation \( [51, 52] \)

\[
\Delta \frac{d}{dr} \left( \frac{dR_{lm}}{dr} \right) + \left\{ [\omega (r^2 + a^2) - ma]^2 + \Delta (2ma - K_{lm}) \right\} R_{lm} = 0
\]

for the radial part \( R_{lm}(r; M, a, \omega) \) of the massless scalar eigenfunction. The frequency-dependent eigenvalues \( K_{lm}(a \omega) \) of the familiar spheroidal harmonic functions \( S_{lm}(\theta; a \omega) \) \( [51, 52, 54, 58] \) are given by the small frequency \( a \omega \ll 1 \) expression

\[
K_{lm} - a^2 \omega^2 = l(l + 1) + \sum_{k=1}^{\infty} c_k(a \omega)^{2k} ,
\]

where the explicit functional expression of the coefficients \( \{ c_k = c_k(l, m) \} \) is given in \( [56] \).

Following the interesting physical models discussed in \( [17, 20] \) for horizonless curved spacetimes, we shall assume that the scalar fields vanish on the compact reflecting surfaces of the central exotic compact objects \( [59] \):

\[
R(r = r_c) = 0 .
\]

In addition, we consider asymptotically flat linearized scalar field configurations which are characterized by asymptotically decaying radial eigenfunctions:

\[
R(r \to \infty) \to 0 .
\]
III. THE RESONANCE CONDITION OF THE COMPOSED ULTRA-SPINNING-EXOTIC-COMPACT-OBJECT-MASSLESS-SCALAR-FIELD CONFIGURATIONS

In the present section we shall derive, for a given set of the dimensionless physical parameters \( \{r_c/M, \bar{a}, l, m\} \), the characteristic resonance condition for the existence of ultra-spinning reflecting exotic horizonless objects that support spatially regular static (marginally-stable) linearized scalar field configurations.

Substituting into the radial equation (5) the characteristic relation \( \omega = 0 \) (9) for the static scalar field configurations, one obtains the ordinary differential equation [19, 60]

\[
x(1 - x) \frac{d^2 F}{dx^2} + \{(1 - \gamma) - [1 + 2(l + 1) - \gamma]x\} \frac{dF}{dx} - [(l + 1)^2 - \gamma(l + 1)]F = 0 ,
\]
(10)

where

\[
R(x) = x^{-\gamma/2}(1 - x)^{l+1} F(x) ,
\]
(11)

\[
x = \frac{r - M(1 + i\sqrt{\bar{a}^2 - 1})}{r - M(1 - i\sqrt{\bar{a}^2 - 1})} ,
\]
(12)

and

\[
\gamma = \frac{m}{\sqrt{1 - \bar{a}^2}} .
\]
(13)

The physically acceptable solution of the characteristic radial scalar equation (10) which respects the asymptotic boundary condition (8) is given by [19, 56, 61, 62]

\[
R(x) = A \cdot x^{-\gamma/2}(1 - x)^{l+1} F_1(l + 1 - \gamma, l + 1; 2l + 2; 1 - x) ,
\]
(14)

where \( A \) is a normalization constant and \( F_1(a, b; c; z) \) is the hypergeometric function.

Substituting the radial solution (14) into the characteristic inner boundary condition (7) at the surface of the compact reflecting object, one obtains the remarkably compact resonance condition

\[
2F_1(l + 1 - \gamma, l + 1; 2l + 2; 1 - x_c) = 0
\]
(15)

for the composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations.

As we shall show below, the resonance equation (15) determines the discrete set of surface radii \( \{r_c = r_c(\bar{a}, l, m; n)\} \) which characterize the unique family of ultra-spinning exotic compact objects that can support the static spatially regular massless scalar field configurations.

IV. GENERIC PROPERTIES OF THE COMPOSED ULTRA-SPINNING-EXOTIC-COMPACT-OBJECT-MASSLESS-SCALAR-FIELD CONFIGURATIONS

In the present section we shall discuss two important features of the discrete resonance spectrum \( \{r_c(\bar{a}, l, m; n)\} \) of surface radii that characterize the composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations: (1) the distribution of the supporting radii, and (2) the (finite) number of supporting radii.

A. The resonance spectrum of surface radii is distributed symmetrically around \( r = M \)

Interestingly, we shall now prove that the discrete set of supporting radii \( \{r_c(\bar{a}, l, m; n)\} \), which stems from the characteristic resonance equation (15), is distributed symmetrically around \( r = M \). To this end, it is convenient to define the dimensionless symmetrical radial coordinate

\[
z \equiv \frac{r - M}{M} ,
\]
(16)
in terms of which the resonance equation 15 can be written in the form 16:

$$2F_1\left(l + 1 - \frac{m}{\sqrt{1 - \bar{a}^{-2}}}, l + 1; 2l + 2; \frac{2i\sqrt{\bar{a}^2 - 1}}{z_c + i\sqrt{\bar{a}^2 - 1}}\right) = 0.$$  (17)

Using the characteristic identity (see Eq. 15.3.15 of [56])

$$2F_1(a, b; 2b; z) = (1 - z)^{-a/2}2F_1\left(\frac{1}{2}a, \frac{1}{2}b - \frac{1}{2}a; \frac{1}{2}b + \frac{1}{2}; \frac{z^2}{4z - 4}\right)$$  (18)

of the hypergeometric function, one can express the resonance condition (17) in the symmetrical form

$$2F_1\left[\frac{1}{2}\left(l + 1 - \frac{m}{\sqrt{1 - \bar{a}^{-2}}}\right), \frac{1}{2}\left(l + 1 + \frac{m}{\sqrt{1 - \bar{a}^{-2}}}\right); l + 3 - \frac{2}{\bar{a}^2 - 1 + z_c^2}\right] = 0.$$  (19)

The resonance equation (19) is obviously invariant under the reflection symmetry $z_c \to -z_c$. We have therefore proved that if the dimensionless surface radius $z_c$ is a solution of the characteristic resonance equation (19), then $-z_c$ is also a valid resonance.

In addition, it is interesting to stress the fact that, for the static ($\omega = 0$) scalar field modes, the radial scalar equation 5 is invariant under the reflection symmetries $a \to -a$ and $m \to -m$ 64. One therefore deduces that if the dimensionless surface radius $z_c$ characterizes an ultra-spinning exotic compact object with $ma > 0$ that can support a spatially regular static (marginally-stable) scalar field configuration with harmonic indices $\{l, m\}$, then the same supporting radius also characterizes an ultra-spinning exotic compact object with $ma < 0$ that can support the same static scalar field configuration.

Taking cognizance of the three reflection symmetries, $z_c \to -z_c$, $a \to -a$, and $m \to -m$, which characterize the composed ultra-spinning-exotic-compact-object-massless-scalar-field system, we shall henceforth assume, without loss of generality, that

$$a > 0 \quad ; \quad m > 0 \quad ; \quad z_c \geq 0.$$  (20)

B. The number of discrete supporting radii is finite

As emphasized above, it has recently been proved 19 that exotic compact objects in the sub-critical regime $\bar{a} < 1$ are characterized by an infinite set of surface radii, $\{r_c(\bar{a}; r)\}_{n=\infty}$, that can support static (marginally-stable) massless scalar field configurations.

On the other hand, we shall now show that super-critical ($\bar{a} > 1$) compact reflecting objects are characterized by a finite set of surface radii that can support the static massless scalar field configurations. In particular, one finds that, for positive integer values of the dimensionless physical parameter $N(\bar{a}, l, m) \equiv \gamma - (l + 1)$, the resonance equation 15, which determines the characteristic spectrum of supporting radii of the ultra-spinning exotic compact objects, is a polynomial equation of degree $N$. Thus, in this case there is a finite number $N$ of complex solutions $\{x_c(\bar{a}, l, m; n)\}_{n=1}^{\infty}$ to the resonance condition 15 which in turn, using the relation 12, yield a finite discrete spectrum $\{r_c(\bar{a}, l, m; n)\}_{n=1}^{N}$ of supporting surface radii.

In addition, solving numerically the resonance equation 15 we find that, for positive non-integer values of the physical parameter $N$, the number of discrete surface radii that can support the static (marginally-stable) scalar field configurations is given by (see Tables I and II below) $|N|$ for even values of $|N|$ and by $|N| + 1$ for odd values of $|N|$ 65.

To summarize, the (finite) number $N_c(\bar{a}, l, m)$ of discrete supporting radii that characterize the composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations is given by the simple relations

$$N_c = \begin{cases} 
\gamma - (l + 1) & \text{if } \gamma - (l + 1) \text{ is a positive integer;} \\
\lfloor \gamma - (l + 1) \rfloor & \text{if } \lfloor \gamma - (l + 1) \rfloor \text{ is a positive even integer;} \\
\lfloor \gamma - (l + 1) \rfloor + 1 & \text{if } \lfloor \gamma - (l + 1) \rfloor \text{ is a positive odd integer.} 
\end{cases}$$  (21)

[It is important to emphasize that cases 2 and 3 in (21) refer to non-integer values of the dimensionless composed parameter $\gamma - (l + 1)$].
V. THE REGIME OF EXISTENCE OF THE COMPOSED ULTRA-SPINNING-EXOTIC-COMPACT-OBJECT-MASSLESS-SCALAR-FIELD CONFIGURATIONS

In the present section we shall derive an upper bound on the characteristic surface radii \( \{ r_{\nu}(\vec{a}, l, m; n) \}_{\nu=1}^{N} \), which characterize the ultra-spinning exotic compact objects that can support the static (marginally-stable) configurations of the massless scalar fields.

Substituting the scalar function 
\[
\Phi(r) \equiv \Delta^{1/2} \cdot R(r)
\]
into the characteristic radial equation (15), one obtains the ordinary differential equation
\[
\Delta^{2} \frac{d^{2}\Phi}{dr^{2}} + \left[ (ma)^{2} - l(l+1) \cdot \Delta - (a^{2} - M^{2}) \right] \Phi = 0
\]
for the static \( (\omega = 0) \) scalar configurations.

Using the characteristic boundary conditions (17) and (18) of the spatially regular linearized scalar field configurations, which are supported in the asymptotically flat curved spacetime (12) of the exotic ultra-spinning reflecting compact object, one deduces that the radial scalar eigenfunction \( \Phi(r) \) must have (at least) one extremum point, \( r = r_{\text{peak}} \), in the interval
\[
(24)
\]
In particular, the simple functional relations
\[
\{ \Phi \neq 0 \ ; \ \frac{d\Phi}{dr} = 0 \ ; \ \Phi \cdot \frac{d^{2}\Phi}{dr^{2}} < 0 \} \quad \text{for} \quad r = r_{\text{peak}}
\]
characterize the spatial behavior of the radial scalar eigenfunction at this extremum point.

Taking cognizance of Eqs. (23) and (25), one finds the simple relation
\[
(ma)^{2} - l(l+1) \cdot \Delta(r_{\text{peak}}) - (a^{2} - M^{2}) > 0
\]
(26)
The characteristic inequality (26) implies that \( r_{\text{peak}} \) is bounded by the relations
\[
r_{-} < r_{\text{peak}} < r_{+},
\]
where
\[
r_{\pm} = M \pm \sqrt{M^{2} - \frac{a^{2}[1 + l(l+1) - m^{2}] - M^{2}}{l(l+1)}}.
\]
(28)
Using Eqs. (16), (24), (27), and (28), one deduces that the composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations are characterized by the simple dimensionless upper bound
\[
|z_{c}| < \sqrt{1 - \frac{a^{2}[1 + l(l+1) - m^{2}] - 1}{l(l+1)}}.
\]
(29)
In particular, from the requirement \( a^{2}[1 + l(l+1) - m^{2}] - 1 \leq l(l+1) \) [see the r.h.s of (29)], one finds that the static (marginally-stable) massless scalar field configurations in the curved spacetimes of the ultra-spinning \( (\bar{a} > 1) \) exotic compact objects are characterized by the compact inequalities
\[
\sqrt{1 - \frac{m^{2}}{1 + l(l+1)}} < |\bar{a}|^{-1} < 1.
\]
(30)
Interestingly, a stronger upper bound on the dimensionless angular momentum parameter \( \bar{a} \), which characterizes the unique family of ultra-spinning exotic compact objects that can support the spatially regular static (marginally-stable) massless scalar field configurations, can be obtained from the observations that [see Eq. (15)]
\[
_{2}F_{1}[l + 1 - \gamma, l + 1; 2l + 2; 1 - x(r)] \neq 0 \quad \text{for} \quad \{ r \in \mathbb{R} \ \text{and} \ \ -1 < l + 1 - \gamma < 2l + 3 \}
\]
(31)
\[ _2 F_1 (-1, l + 1; 2l + 2; 2) = _2 F_1 (2l + 3, l + 1; 2l + 2; 2) = 0. \]  

(32)

From Eqs. (13), (31), and (32), one deduces that the composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations exist in the dimensionless physical regime

\[
\sqrt{1 - \left(\frac{m}{l + 2}\right)^2} \leq |\bar{a}|^{-1} < 1,
\]

(33)

where the equality sign in (33) corresponds to exotic ultra-spinning objects with \( r_c = M \) [or, equivalently, \( 1 - x_c = 2 \) and \( z_c = 0 \), see Eqs. (12) and (10)].

VI. THE RESONANCE SPECTRUM OF THE COMPOSED ULTRA-SPINNING-EXOTIC-COMPACT-OBJECT-MASSLESS-SCALAR-FIELD CONFIGURATIONS

As mentioned above, the infinite countable spectrum of supporting surface radii \( \{r_c(\bar{a}, l, m; n)\}_{n=1}^{\infty} \) which characterizes the composed spinning-exotic-compact-object-massless-scalar-field configurations in the sub-critical regime \( \bar{a} < 1 \) has been determined in [19]. In the present section we shall explicitly show that ultra-spinning exotic compact objects in the complementary regime \( \bar{a} > 1 \) of super-critical angular momenta are characterized by a finite [see Eq. (21)] discrete set \( \{r_c(\bar{a}, l, m; n)\}_{n=1}^{N} \) of surface radii that can support the asymptotically flat static scalar field configurations.

The compact resonance equation (15) can be solved numerically, for a given set \( \{\bar{a}, l, m\} \) of the dimensionless physical parameters that characterize the composed compact-object-scalar-field system, to yield the discrete resonant spectrum \( \{r_c(\bar{a}, l, m; n)\}_{n=1}^{N} \) of supporting radii. In Table I we present, for various super-critical values of the dimensionless angular momentum parameter \( \bar{a} \), the smallest and largest dimensionless surface radii \( \{z_{c, \text{min}}(\bar{a}, l, m), z_{c, \text{max}}(\bar{a}, l, m)\} \) of the ultra-spinning exotic compact objects that can support the static spatially regular configurations of the massless scalar fields [67]. We also present in Table I the (finite) number \( N_c(\bar{a}, l, m) \) [see Eq. (21)] of these unique supporting surface radii [68].

The data presented in Table I demonstrate the fact that, for given integer values \( l, m \) of the angular harmonic indices of the static (marginally-stable) massless scalar fields, the dimensionless supporting radius \( z_{c, \text{max}}(\bar{a}) \) of the ultra-spinning exotic compact objects is a monotonically decreasing function of the dimensionless physical parameter \( \bar{a} \). As a consistency check, it is worth noting that the numerically computed values \( z_{c, \text{max}}(\bar{a}) \) of the characteristic surface radii of the ultra-spinning reflecting compact objects, as displayed in Table I, conform to the analytically derived upper bound [29].

We would like to emphasize again that, for a given set of the physical parameters \( \{\bar{a}, l, m\} \), the critical supporting radius \( r_{c, \text{max}}(\bar{a}) \) marks the boundary between stable and superradiantly unstable composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations. In particular, the numerical results presented in the interesting work of Maggio, Pani, and Ferrari [17] indicate that ultra-spinning reflecting compact objects which are characterized by the inequality \( r_c < r_{c, \text{max}}(\bar{a}) \) are superradiantly unstable to massless scalar perturbation modes, whereas ultra-spinning exotic compact objects which are characterized by the relation \( r_c > r_{c, \text{max}}(\bar{a}) \) are stable.

In Table I we present, for various equatorial \( l = m \) modes of the supported static scalar fields, the smallest and largest dimensionless surface radii \( \{z_{c, \text{min}}(\bar{a}, l, m), z_{c, \text{max}}(\bar{a}, l, m)\} \) of the supporting marginally-stable ultra-spinning exotic compact objects [67]. Also displayed is the (finite) number [see Eq. (21)] of these unique supporting surface radii [68]. The data presented in Table I reveal the fact that, for a given value of the dimensionless physical parameter \( \bar{a} \), the critical (largest) supporting radius \( z_{c, \text{max}}(l) \) of the reflecting exotic compact objects is a monotonically increasing function of the harmonic index \( l \) which characterizes the static massless scalar field mode. It is worth noting that the numerically computed surface radii \( z_{c, \text{max}}(l) \) of the ultra-spinning marginally-stable exotic compact objects, as presented in Table I, conform to the analytically derived upper bound [29].

VII. THE RESONANCE SPECTRUM OF NEAR-CRITICAL ULTRA-SPINNING EXOTIC COMPACT OBJECTS

A. An analytical treatment

Interestingly, as we shall explicitly show in the present section, the compact resonance equation (15), which determines the discrete family \( \{x_c(\bar{a}, l, l; n)\} \) of dimensionless surface radii that characterize the marginally-stable...
The case [67]. Also presented is the finite number [see Eq. (21)] of these unique supporting surface radii. The data presented is for the case \( l = m = 1 \). The critical supporting radii \( \{ z_{c,max}(\bar{a}) \} \), which characterize the marginally-stable ultra-spinning reflecting compact objects, are found to be a monotonically increasing function of the dimensionless angular momentum parameter \( \bar{a} \). As a consistency check we note that the supporting radii of the ultra-spinning exotic compact objects conform to the analytically derived upper bound [20].

| \( \sqrt{1 - \bar{a}^{-2}} \) | \( \bar{a} \) | # of resonances | \( z_{c,min}(\bar{a}) \) | \( z_{c,max}(\bar{a}) \) |
|---|---|---|---|---|
| 1/3 | 1.0607 | 1 | 0 | 0 |
| 0.3 | 1.0483 | 2 | 0.05488 | 0.05488 |
| 0.25 | 1.0328 | 2 | 0.11547 | 0.11547 |
| 0.2 | 1.0206 | 3 | 0 | 0.15811 |
| 0.15 | 1.0114 | 4 | 0.06455 | 0.18788 |
| 0.1 | 1.0050 | 8 | 0.01608 | 0.20760 |

TABLE I: Marginally-stable ultra-spinning (\( \bar{a} > 1 \)) reflecting compact objects. We present, for various super-critical values of the dimensionless physical parameter \( \bar{a} \), the smallest and largest dimensionless radii, \( \{ z_{c,min}(\bar{a}, l, m), z_{c,max}(\bar{a}, l, m) \} \) [see Eq. (19)], of the ultra-spinning exotic compact objects that can support the static (marginally-stable) massless scalar field configurations. Also presented is the finite number [see Eq. (21)] of these unique supporting surface radii. The data presented is for the case \( l = m = 1 \). The critical supporting radii \( \{ z_{c,max}(\bar{a}) \} \), which characterize the marginally-stable ultra-spinning reflecting compact objects, are found to be a monotonically increasing function of the dimensionless angular momentum parameter \( \bar{a} \). As a consistency check we note that the supporting radii of the ultra-spinning exotic compact objects conform to the analytically derived upper bound [20].

| \( l \) | # of resonances | \( z_{c,min}(l) \) | \( z_{c,max}(l) \) |
|---|---|---|---|
| 1 | 2 | 0.11547 | 0.11547 |
| 2 | 5 | 0 | 0.28705 |
| 3 | 8 | 0.03553 | 0.38489 |
| 4 | 11 | 0 | 0.45263 |
| 5 | 14 | 0.02113 | 0.50342 |
| 6 | 17 | 0 | 0.54338 |

TABLE II: Marginally-stable ultra-spinning (\( \bar{a} > 1 \)) reflecting compact objects. We present, for various equatorial (\( l = m \)) modes of the supported scalar fields, the smallest and largest dimensionless surface radii \( \{ z_{c,min}(\bar{a}, l, m), z_{c,max}(\bar{a}, l, m) \} \) [see Eq. (19)] of the ultra-spinning exotic compact objects that can support the spatially regular static (marginally-stable) massless scalar field configurations. We also present the finite number of these unique supporting radii. The data presented is for the case \( \sqrt{1 - \bar{a}^{-2}} = 1/4 \). The critical surface radii \( \{ z_{c,max}(l) \} \), which characterize the marginally-stable ultra-spinning exotic compact objects, are found to be a monotonically increasing function of the dimensionless harmonic index \( l \) of the supported static scalar field configurations.

 ultra-spinning exotic compact objects, is amenable to an analytical treatment in the physically interesting regime

\[
0 < \bar{a} - 1 \ll 1
\]  

(34)

of near-critical horizonless spinning configurations.

In particular, in the near-critical regime

\[
\frac{m}{\sqrt{1 - \bar{a}^{-2}}} \gg l
\]  

(35)

one may use the large-\( |b| \) asymptotic expansion [69]

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(c - a)} (-b z)^{-a} [1 + O(|b z|^{-1})] + \frac{\Gamma(c)}{\Gamma(a)} (b z)^{a-c} (1 - z)^{c-a-b} [1 + O(|b z|^{-1})]
\]  

(36)

of the hypergeometric function in order to express the resonance condition [15] in the remarkably compact form [70]

\[
x^{m/\sqrt{1 - \bar{a}^{-2}}} = (-1)^{-l} \quad \text{for} \quad \sqrt{1 - \bar{a}^{-2}} \ll \frac{m}{l} .
\]  

(37)

From the asymptotic relation [67] one finds the set of complex solutions [71]

\[
x_c(n) = e^{-\pi(l+2n)\sqrt{1 - \bar{a}^{-2}}/m} ; \quad n \in \mathbb{Z}
\]  

(38)

which, taking cognizance of Eqs. [12] and [10], yields the discrete real resonance spectrum [63, 70, 72]

\[
z_c(n) = \sqrt{\bar{a}^2 - 1} \cdot \cot \left[ \frac{\pi(l + 2n)\sqrt{1 - \bar{a}^{-2}}}{2m} \right] \quad \text{for} \quad \{ \sqrt{1 - \bar{a}^{-2}} \ll \frac{m}{l} \quad \text{and} \quad \pi(l + 2n) \gg 1 \}
\]  

(39)
for the dimensionless surface radii which characterize the near-critical \((\bar{a} \gtrsim 1)\) exotic compact objects that can support the static massless scalar field configurations. Interestingly, the analytically derived resonance formula (39) can be further simplified in the \(\pi(l + 2n)\sqrt{1 - \bar{a}^{-2}}/2m \ll 1\) regime, in which case one finds the remarkably compact expression \([72, 73]\)

\[
z_c(n) = \frac{2m\bar{a}}{\pi(l + 2n)} \quad \text{for} \quad 1 \ll \pi(l + 2n) \ll \frac{2m}{\sqrt{1 - \bar{a}^{-2}}} \tag{40}
\]

for the characteristic radii of the ultra-spinning exotic compact objects that can support the static (marginally-stable) configurations of the massless scalar fields.

**B. Numerical confirmation**

It is of physical interest to verify the accuracy of the approximated (analytically derived) resonance spectrum (39) for the surface radii of the near-critical \((0 < \bar{a} - 1 \ll 1)\) ultra-spinning exotic compact objects that can support the spatially regular static (marginally-stable) configurations of the massless scalar fields. In Table III we present the dimensionless discrete surface radii \(z^{\text{analytical}}_c(n)\) of the supporting near-critical ultra-spinning exotic reflecting objects as obtained from the analytically derived resonance spectrum (39). We also present in Table III the corresponding surface radii \(z^{\text{numerical}}_c(n)\) of the ultra-spinning exotic compact objects as computed numerically from the exact characteristic resonance equation (15).

The data presented in Table III nicely demonstrate the important fact that there is a good agreement between the approximated surface radii \(\{z^{\text{analytical}}_c(n)\}\) of the ultra-spinning exotic compact objects that can support the static massless scalar field configurations [as calculated from the compact analytically derived resonance formula (39)] and the corresponding exact surface radii \(\{z^{\text{analytical}}_c(n)\}\) of the reflecting compact objects [as determined numerically directly from the resonance equation (15)].

| Formula       | \(z_c(n = 1)\) | \(z_c(n = 2)\) | \(z_c(n = 3)\) | \(z_c(n = 4)\) | \(z_c(n = 5)\) |
|---------------|----------------|----------------|----------------|----------------|----------------|
| Analytical [Eq. (39)] | 0.21206       | 0.12707        | 0.09058        | 0.07027        | 0.05730        |
| Numerical [Eq. (15)]     | 0.22240       | 0.12919        | 0.09135        | 0.07062        | 0.05818        |

**TABLE III**: Near-critical ultra-spinning \((\bar{a} \gtrsim 1)\) exotic compact objects. Displayed are the analytically calculated discrete surface radii \(\{z^{\text{analytical}}_c(n)\}\) which characterize the ultra-spinning exotic compact objects that can support the static (marginally-stable) spatially regular configurations of the massless scalar fields. Also displayed are the corresponding supporting radii \(\{z^{\text{numerical}}_c(n)\}\) of the near-critical exotic compact objects as obtained numerically directly from the characteristic resonance equation (15). The data presented is for static massless scalar field configurations with \(l = m = 1\) linearly coupled to near-critical ultra-spinning exotic compact objects with \(\sqrt{1 - \bar{a}^{-2}} = 10^{-2}\). The displayed data reveal a remarkably good agreement between the exact characteristic surface radii \(\{z^{\text{analytical}}_c(n)\}\) of the ultra-spinning exotic compact objects [as determined numerically from the resonance condition (15)] and the corresponding approximated radii \(\{z^{\text{analytical}}_c(n)\}\) of the near-critical ultra-spinning compact objects [as calculated analytically from the compact resonance formula (39)].

**VIII. THE RESONANCE SPECTRUM OF ULTRA-SPINNING EXOTIC COMPACT OBJECTS WITH**

\(r_e = M\)

Interestingly, as we shall now prove, the characteristic resonance equation (15) [or, equivalently, the symmetrical form (19) of the resonance condition] can also be solved *analytically* for the dimensionless angular momentum parameter \(\bar{a}\) in the physically interesting case of horizonless ultra-spinning exotic compact objects of mass \(M\) whose compact reflecting surfaces coincide with the corresponding horizon radius \(r_e = M\) of extremal Kerr black holes with the same mass parameter.

Substituting \(r_e = M\) [which corresponds to \(z_c = 0\), see Eq. (16)] into the resonance condition (19), and using the characteristic identity (see Eq. 15.1.20 of [56])

\[
2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)\Gamma(c - a)\Gamma(c - b)}{\Gamma(c - a - b)} \tag{41}
\]

of the hypergeometric function, one finds the compact resonance equation

\[
\frac{\Gamma\left(\frac{l}{2} + 1 + \frac{m}{2\sqrt{1 - \bar{a}^{-2}}}ight)}{\Gamma\left(\frac{l}{2} + 1 - \frac{m}{2\sqrt{1 - \bar{a}^{-2}}}ight)} = 0 \quad \text{for} \quad r_e = M \tag{42}
\]
Using the well known pole structure of the Gamma functions [namely, $1/\Gamma(-n) = 0$ for $n = 0, 1, 2, \ldots$ [74]], one obtains from (12) the remarkably simple discrete resonance spectrum

$$\sqrt{1 - a^2} = \frac{m}{l + 2 + 2n} ; \quad n = 0, 1, 2, \ldots$$

for the ultra-spinning ($a > 1$) exotic compact objects whose reflecting surfaces coincide with the corresponding horizon radius $r_c = M$ of extremal ($a = 1$) Kerr black holes [73]. Interestingly, one finds that the ultra-spinning exotic compact objects described by the resonance formula (13) with $n = 0$ saturate the previously derived bound (33).

IX. SUMMARY AND DISCUSSION

The physical and mathematical properties of horizonless highly compact exotic reflecting objects have recently been studied by some physicists (see [5–22] and references therein). The main motivation behind these diverse studies has been to examine the intriguing possibility that these exotic horizonless objects may serve as quantum-gravity alternatives to classical black-hole spacetimes.

Interestingly, Maggio, Pani, and Ferrari [17] have recently provided compelling evidence that sub-critical ($a \equiv J/M^2 < 1$) horizonless spinning spacetimes, in which the characteristic absorbing boundary conditions of classical black-hole spacetimes have been replaced by reflective boundary conditions at the surfaces of the exotic compact objects, may become superradiantly unstable [24–28] when linearly coupled to massless scalar (bosonic) field modes [23]. In particular, it has been explicitly proved in [19] that, in the sub-critical regime $a < 1$ of the spinning reflecting objects and for each harmonic indices $(l, m)$ of the massless scalar field, there exists an infinite countable set of surface radii, $\{r_c(a, l, m; n)\}_{n=1}^{\infty}$, which can support spatially regular static (marginally-stable) massless scalar field configurations.

In the present paper we have explored the physical and mathematical properties of marginally-stable composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations which are characterized by super-critical ($a > 1$) dimensionless rotation parameters. The following are the main results derived in this paper and their physical implications:

1. It has been explicitly proved that, for given dimensionless physical parameters $\{a, l, m\}$, the unique discrete family $\{r_c(a, l, m; n)\}$ of surface radii that characterize the ultra-spinning ($a > 1$) exotic compact objects that can support the static (marginally-stable) massless scalar field configurations is determined by the resonance condition [see Eqs. (1), (13), and (19)]

$$2F_1\left[\frac{1}{2}\left(l + 1 - \frac{ma}{\sqrt{a^2 - M^2}}\right), \frac{1}{2}\left(l + 1 + \frac{ma}{\sqrt{a^2 - M^2}}\right); l + \frac{3}{2}, \frac{a^2 - M^2}{a^2 - M^2 + (r_c - M)^2}\right] = 0 .$$

2. We have shown that the composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations, as determined by the resonance condition (44), are restricted to the physical regime [see Eq. (33)]

$$\sqrt{1 - \left(\frac{M}{a}\right)^2} \leq \frac{M}{a} < 1$$

of the dimensionless super-critical rotation parameter $a/M$. In addition, it has been proved that, for a given set $\{a, l, m\}$ of the dimensionless physical parameters that characterize the composed compact-object-scalar-field system, the simple relation [see Eqs. (16) and (29)]

$$\left|\frac{r_c - M}{M}\right| < \sqrt{1 - \frac{a^2[1 + l(l + 1) - m^2] - 1}{l(l + 1)}}$$

provides an upper bound on the surface radii of the supporting ultra-spinning exotic compact objects.

3. It has been pointed out that the analytically derived resonance condition in its symmetrical form (44) reveals the fact that, for ultra-spinning exotic compact objects, the discrete resonant spectrum of supporting surface radii is invariant under the reflection symmetries

$$r_c - M \rightarrow -(r_c - M) ; \quad a \rightarrow -a ; \quad m \rightarrow -m .$$

The symmetry transformations (47) imply, in particular, that if $z_c$ [see Eq. (10)] is a dimensionless supporting radius of a composed exotic-object-scalar-field system with dimensionless physical parameters $\{a, l, m\}$, then: (1) $-z_c$ is also
a valid supporting radius of the same composed physical system, and (2) \( z_c \) is also a valid supporting radius of a composed exotic-object-scalar-field system with dimensionless parameters \( \{ \pm \bar{a}, l, \pm m \} \).

(4) It has been shown that, for ultra-spinning exotic compact objects in the dimensionless physical regime \( (15) \), the finite number \( N_e(\bar{a}, l, m) \) of surface radii that can support the spatially regular static (marginally-stable) scalar field configurations is given by [see Eqs. (13) and (21)] 13, 63, 72, 76

\[
N_e = \begin{cases} 
N & \text{if } N \text{ is a positive integer} \\
[N] & \text{if } [N] \text{ is a positive even integer} \\
[N] + 1 & \text{if } [N] \text{ is a positive odd integer}, 
\end{cases} 
\tag{48}
\]

where

\[
N(\bar{a}, l, m) = \frac{ma}{\sqrt{\bar{a}^2 - M^2}} - (l + 1).
\tag{49}
\]

Interestingly, the fact that ultra-spinning \( (\bar{a} > 1) \) exotic compact objects are characterized by a finite discrete family \( \{ r_c(\bar{a}, l, m; n) \}_{n=1}^{\infty} \) of surface radii that can support the static massless scalar field configurations should be contrasted with the complementary case of sub-critical spinning objects in the \( \bar{a} < 1 \) regime which, as previously proved in [17, 19], are characterized by an infinite countable family \( \{ r_c(\bar{a}, l, m; n) \}_{n=1}^{\infty} \) of surface radii that can support the spatially regular static scalar fields [72].

(5) The ability of spinning objects to support spatially regular static scalar field configurations is physically intriguing from the point of view of the no-hair theorems that have recently been discussed in [31, 33] for horizonless regular spacetimes. In particular, it has been proved in [31, 32] that spherically-symmetric \( (\text{non-spinning}) \) horizonless reflecting stars cannot support nonlinear configurations made of massless scalar fields.

(6) It is important to stress the fact that, as shown in [17, 19], the outermost (largest) surface radius \( r_c^{\text{max}}(\bar{a}) \equiv \max_n \{ r_c(\bar{a}; n) \} \) that can support the static scalar field configurations is of central physical importance since it marks the boundary between stable \( [r_c > r_c^{\text{max}}(\bar{a})] \) and unstable \( [r_c < r_c^{\text{max}}(\bar{a})] \) composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations.

(7) Solving numerically the analytically derived resonance equation (15), we have demonstrated that the characteristic supporting radius \( r_c^{\text{max}}(\bar{a}, l, m) \) is a monotonically decreasing function of the dimensionless rotation parameter \( \bar{a} \) of the ultra-spinning exotic compact objects (see Table I). Likewise, it has been demonstrated that the critical (outermost) supporting surface radius \( r_c^{\text{max}}(\bar{a}, l, m) \) is a monotonically increasing function of the harmonic parameter \( l \) which characterizes the massless scalar field modes (see Table II).

(8) We have explicitly shown that the characteristic resonance equation (15) for the discrete family of supporting surface radii is amenable to an analytical treatment in the physically interesting regime \( 0 < \bar{a} - 1 < 1 \) of near-critical horizonless spinning objects. In particular, the remarkably compact resonance formula [see Eqs. (16) and (40)]

\[
r_c(n) = M + \frac{2ma}{\pi(l + 2n)} \quad ; \quad n \in \mathbb{Z}.
\tag{50}
\]

has been derived analytically for near-critical \( (\bar{a} \sim 1) \) composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations in the \( 1 \ll \pi(l + 2n) \ll 2m/\sqrt{1 - \bar{a}^{-2}} \) regime.

(9) We have verified that the predictions of the analytically derived resonance formula (50), which determines the unique family of surface radii of the near-critical ultra-spinning \( (0 < \bar{a} - 1 < 1) \) compact reflecting objects that can support the static (marginally-stable) massless scalar field configurations, agree remarkably well (see Table III) with the corresponding exact values of the supporting surface radii as determined numerically from the characteristic resonance condition (15).

(10) Finally, it has been proved that the resonance equation (15) can be solved analytically in the physically interesting case of ultra-spinning \( (\bar{a} > 1) \) exotic compact objects whose reflecting surfaces coincide with the corresponding horizon radius \( r_c = M \) of extremal \( (\bar{a} = 1) \) Kerr black holes with the same mass parameter. In particular, we have derived the remarkably compact discrete resonance spectrum [see Eq. (13)]

\[
\frac{a}{M} = \frac{1}{\sqrt{1 - \frac{m}{(l + 2n)^2}}} \quad ; \quad n = 0, 1, 2, ...
\tag{51}
\]

for the ultra-spinning compact configurations with \( r_c = M \) [78]. Interestingly, one finds from the analytically derived resonance formula (51) that, in the \( l + 2n \gg m \) regime, the dimensionless angular momenta \( \{ \bar{a} \}_{n=0}^{\infty} \) of the exotic ultra-spinning reflecting objects with physical parameters \( \{ M, r_c = M \} \) can be made arbitrarily close [79] to the
corresponding dimensionless angular momentum $\bar{a}_{EK} = 1$ of an absorbing extremal Kerr black hole with the same mass and radius.

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Using Eq. 15.1.1 of [54], one finds the asymptotic functional behavior

\[ P. M. Morse and H. Feshbach, \]  

For brevity, we shall henceforth omit the dimensionless harmonic indices

Here we have used the relation

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Here \( (\bar{a} > 1) \) hairy configurations can be formed dynamically from the complete classical gravitational collapse of spatially regular matter fields. To the best of our knowledge, the analogous dynamical formation of horizonless ultra-spinning exotic compact objects has not been demonstrated numerically thus far. (It is worth noting, however, that spinning weakly self-gravitating mundane objects are usually characterized by the relation \( \bar{a} > 1 \). In particular, planet Earth itself is characterized by the dimensionless relation \( \bar{a} \gg 1 \).) As previously suggested in [5–22], it may be the case that some quantum mechanism could halt the gravitational collapse of the ultra-spinning matter configurations, thus yielding horizonless quantum exotic compact objects instead of the ultra-spinning classical black holes discussed in \[ 42, 43. \]

As nicely emphasized in [17, 20] (see also [48, 50]), the Birkhoff theorem does not apply in non-spherically symmetric spacetimes, and thus the exterior metric of the spinning compact object is not unique. Following [17, 19, 20], we shall assume that the external spacetime of the horizonless exotic object is characterized by the familiar metric components (2)

assume that the external spacetime of the horizonless exotic object is characterized by the familiar metric components (2)

Here \( (\bar{a} > 1) \) hairy black-hole configurations are known to exist as stationary solutions of the nonlinearly coupled Einstein-scalar [38, 39] and Einstein-Proca [41] field equations. Moreover, using fully nonlinear numerical (dynamical) simulations [42, 43], it has recently been demonstrated explicitly that these ultra-spinning \( (a > 1) \) hairy configurations can be formed dynamically from the complete classical gravitational collapse of spatially regular matter fields. To the best of our knowledge, the analogous dynamical formation of horizonless ultra-spinning exotic compact objects has not been demonstrated numerically thus far. (It is worth noting, however, that spinning weakly self-gravitating mundane objects are usually characterized by the relation \( a > 1 \). In particular, planet Earth itself is characterized by the dimensionless relation \( a \gg 1 \).) As previously suggested in [5–22], it may be the case that some quantum mechanism could halt the gravitational collapse of the ultra-spinning matter configurations, thus yielding horizonless quantum exotic compact objects instead of the ultra-spinning classical black holes discussed in [42, 43].

It is worth emphasizing again that, as discussed above (see section IVA), the resonance equation (19) is invariant under

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For brevity, we shall henceforth omit the dimensionless harmonic indices \( \{l, m\} \). Here we have used the simple relation \( K_{lm}(\omega = 0) = l(l+1) \) for the static (marginally-stable) scalar field modes [see Eq. 6].

P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, New York, 1953).

Using Eq. 15.1.1 of [54], one finds the asymptotic functional behavior \( R(r \to \infty) \sim r^{-l(l+1)} \to 0 \) in the \( r \to \infty \) \((x \to 1)\) limit.

Here we have used the relation \( x = (z - i\sqrt{\alpha^2 - 1})/(z + i\sqrt{\alpha^2 - 1}) \) [see Eqs. 12 and 16]. It is worth pointing out that these \( \alpha \to -\alpha \) and \( m \to -m \) reflection symmetries also manifest themselves in the analytically derived resonance equation [19] [Note that \( \mathcal{F}_1(a; b; c; z) \equiv \mathcal{F}_1(b; a; c; z) \)].

Here \( [x] \) denotes the floor function (the largest integer less than or equal to \( x \)).

We are not aware of any general proof in the mathematical or physical literature for this interesting property of the hypergeometric functions. However, we have numerically verified the validity of the relation [11] for several values of the arguments of the hypergeometric function \( \mathcal{F}_1[l + 1 - \gamma, l + 1; 2l + 2; 1 - x(r)] \).

It is worth emphasizing again that, as discussed above (see section IVA), the resonance equation [10] is invariant under
the reflection symmetry \( z_c \rightarrow -z_c \). That is, if \( z_c \) is a solution of the characteristic resonance equation (19), then \(-z_c\) is also a valid resonance. Thus, without loss of generality, we consider non-negative surface radii with \( z_c \geq 0 \). In particular, the physical parameter \( z_{\text{min}}(\bar{a}, l, m) \) refers to the smallest non-negative dimensionless surface radius that can support the static scalar field configurations.

[68] It is worth pointing out that the data presented in Tables I and II reveal the interesting fact that composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations with an odd number of resonances are characterized by the simple relation \( z_{\text{min}} = 0 \). This interesting physical property will be discussed in section VIII below. In particular, from the resonance formula (13) one deduces that a new resonant mode (which is characterized by the simple dimensionless relation \( z_{\text{c}} = 0 \)) is added to the discrete set of resonances each time the non-negative integer parameter \( \left[ \gamma - (l + 2) \right] / 2 \) increases by one. When the value of \( \left[ \gamma - (l + 2) \right] / 2 \) is further increased, this new supporting radius splits into two distinct supporting radii which are distributed symmetrically around \( r = M \) (see the discussion in section IVA). Thus, composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations with the property \( z_{\text{min}} = 0 \) are characterized by an odd number of resonances (that is, for these systems, there is one special resonant mode with \( z_c = 0 \) and an additional even number of resonant supporting radii which are distributed symmetrically around \( z = 0 \)).

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[71] Here we have used the relation \( 1 = e^{2\pi i n} \), where the integer \( n \) is the resonance parameter of the near-critical composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations.

[72] It is important to stress the fact that the asymptotic expansion (35) of the hypergeometric function \( _2F_1(a, b; c; y) \) also requires the strong inequality \(|by| \gg 1\) (32). If \( \pi(l + 2n)\sqrt{1 - \bar{a}^{-2}}/m \ll 1 \) then we find from Eqs. (13), (15), and (38) the simple relation \(|by| = O[\pi(l + 2n)]\) (note that \( y \equiv 1 - x_c \) in our case), which implies that the asymptotic approximation (70) is valid in the regime \( \pi(l + 2n) \gg 1 \).

[73] Here we have used the small-\( x \) relation \( \cot(x) = 1/x + O(x) \).

[74] See Eq. 6.1.7 of [25].

[75] An independent way to deduce the (finite) number \( N_c(\bar{a}, l, m) \) of discrete supporting radii which characterize the composed ultra-spinning-exotic-compact-object-massless-scalar-field configurations is to use the analytically derived resonance spectrum (13) for the \( r_c \equiv M \) case. In particular, from Eq. (13) one finds that a new resonant supporting radius (which is characterized by the simple dimensionless relation \( z_c = 0 \)), see Eq. (19) is added to the discrete family of supporting surface radii each time the composed non-negative integer parameter \( \left[ \gamma - (l + 2) \right] / 2 \) increases by one. When we further increase the value of \( \left[ \gamma - (l + 2) \right] / 2 \), this new resonant supporting radius splits into two distinct supporting radii which, as explicitly proved in section IVA, are distributed symmetrically around \( z = 0 \). Thus, the number \( N_c(\bar{a}, l, m) \) of discrete supporting radii can be expressed in the simple relation \( z_c \equiv 0 \). This is valid in the regime \( \pi(l + 2n) \gg 1 \).

[76] It is worth emphasizing again that cases 2 and 3 in (15) refer to non-integer values of the dimensionless parameter \( N \).

[77] The fact that spinning exotic compact objects in the sub-critical \( \bar{a} < 1 \) regime are characterized by an infinite set of surface radii, \( \{r_c(\bar{a}, l, m; n)\}_{n=1}^{\infty} \), that can support the spatially regular static scalar fields can be attributed to the existence of an infinitely blue-shifted surface (a classical horizon which, in our model (15), 24), is covered by the external reflecting surface of the corresponding ultra-compact object) at \( r = r_+ \equiv M[1+(1-\bar{a}^2)^{1/2}] \). In this case, the supported fields are blue-shifted in the near-horizon region in the sense that their radial profile oscillates infinitely many times in the \( r/r_+ \rightarrow 1^+ \) limit (with decreasing wavelengths as the \( r/r_+ \rightarrow 1^+ \) limit is approached). Thus, in the \( \bar{a} < 1 \) regime, the supported fields are characterized by an infinite number of nodes that can approach arbitrarily close to the classical horizon at \( r = r_+ \). On the other hand, the curved spacetimes of ultra-spinning objects in the complementary regime \( \bar{a} > 1 \) do not contain an infinitely blue-shifted surface (a covered horizon), and thus the external supported fields are no longer characterized by an infinite set of blue-shifted (arbitrarily dense) resonant nodes.

[78] It is worth pointing out that the ultra-spinning exotic configurations described by the resonance formula (51) with \( n = 0 \) saturate the bound (45).

[79] That is, \( \bar{a} \rightarrow 1^+ \) in the asymptotic \( (l + 2n)/m \rightarrow \infty \) limit.