Field theory and $\lambda$-deformations: Expanding around the identity

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Abstract

We explore the structure of the $\lambda$-deformed $\sigma$-model action by setting up a perturbative expansion around the free field point corresponding to the identity group element. We include all field interaction terms up to sixth order. We compute the two- and three-point functions of current and primary field operators, their anomalous dimensions as well as the $\beta$-function for the $\lambda$-parameter. Our results are in complete agreement with those obtained previously using gravitational and/or CFT perturbative methods in conjunction with the non-perturbative symmetry, as well as with those obtained using methods exploiting the geometry defined in the space of couplings. The advantage of this approach is that all deformation effects are already encoded in the couplings of the interaction vertices and in the $\lambda$-dressed operators.
1 Introduction

Integrability plays a central rôle in modern theoretical physics. It is present in QCD at the high energy regime \([1]\), it emerges in several realizations of the gauge/gravity correspondence \([2,3]\) and it is also employed to model condensed matter systems by the use of integrable spin chains. Practically, it provides not only the information that a theory is solvable for any value of the coupling constant, but in addition tools for solving it are provided.

A whole class of integrable two-dimensional field theories having an explicit action representation, were systematically constructed \([4-10]\) and studied \([11-17]\). These
theories go under the name of $\lambda$-deformations. The most straightforward way to construct them is by a specific gauging procedure initiated in [4]. They are parametrized by one or several matrices, for small entries of which the models are nothing but a WZW model [18] (or several ones) perturbed by current bilinears. In that respect, these models represent the effective actions of WZW current algebra conformal field theories (CFTs) perturbed by current bilinears. As such they take into account all perturbative effects in the perturbation/deformation parameters. The simplest of these models is the single $\lambda$-deformed model [4] (for the SU(2) group case this model has appeared before in [19]). In [13, 12], two- and three-point correlation functions of single currents and primary field operators were calculated as exact functions of the deformation parameter $\lambda$. This was achieved by using low order perturbation theory around the conformal point assisted by a certain non-perturbative symmetry in the space of couplings which these models generically exhibit. More recently, the exact in $\lambda$ anomalous dimensions of certain composite current operators were calculated in [20]. To establish that, a novel method combining geometrical data in the space of couplings and the all-loop effective action was invented. This method allows in principle the calculation of the anomalous dimensions of composite operators built from an arbitrary number of currents, although the huge operator mixing problem makes the computations for general operators cumbersome.

The aim of the present work is to present yet another method to compute all the aforementioned observables, as exact functions of $\lambda$, which in certain aspects is simpler and advantageous. The main idea is that instead of using perturbation theory around the conformal point, to use the free field theory point as a reference. It turns out that this corresponds to setting up a large $k$ perturbative expansion of the effective action around the free field point associated with the identity group element. We include all field interaction terms up to sixth order. The resulting action will be a two-dimensional quantum field theory with a canonical kinetic term, but in principle one may keep an infinite number of interaction terms involving successively more and more fields. One of the virtues of our method is that it incorporates automatically all the dependence in $\lambda$. In fact all the vertices are invariant under the aforementioned non-perturbative symmetry of the model. In contradistinction with the analogous per-

\footnote{A similar expansion was recently considered in [21] aiming at studying the properties of massless tree level S-matrices for two-dimensional $\sigma$-models.}
turbative calculations around the conformal point, this non-perturbative symmetry does not have to be imposed in order to obtain the exact in $\lambda$ form of the correlators but is already built in the formalism. As a result, all the results obtained by using this action will inherit this symmetry and will be exact in $\lambda$. A second advantage of our approach is that it can be considered as the basis for systematically performing the perturbative expansion in powers of $1/k$.

The plan of the paper is as follows: In section 2, we will perform the perturbative expansion of the action around the free field point. In section 3, we first find the free field expansion of the fundamental currents of the theory. Subsequently, we use these and the above action in order to calculate three-point functions involving purely chiral or anti-chiral currents and well as mixed ones, as exact functions of the deformation parameter $\lambda$. In the same section, we will also calculate the anomalous dimensions of these fundamental currents, as well as those of primary fields. In section 4, we calculate the, exact in $\lambda$, $\beta$-function of the model by considering the renormalization of the cubic vertex. For consistency, we present the non-trivial check that the same expression for $\beta$-function can be obtained from the renormalization of the quartic coupling. All the results are in complete agreement with the expressions obtained by the other methods briefly mentioned above. Finally, in section 5 we will present our conclusions and future directions of this work.

2 Expansion around the free point

Our starting point will be the $\lambda$-deformed action for an element $g$ of a semi-simple group $G$ given by [4]

$$S_{k,\lambda}(g) = S_k(g) + \frac{k}{\pi} \int d^2\sigma \left( \lambda^{-1} - D^T \right)^{-1} J_-, \quad (2.1)$$

where the WZW action is

$$S_k(g) = -\frac{k}{2\pi} \int d^2\sigma \operatorname{Tr}(g^{-1} \partial_+ g g^{-1} \partial_- g) + \frac{k}{12\pi} \int (g^{-1}dg)^3 \quad (2.2)$$

and

$$J_+ = -i \partial_+ g g^{-1}, \quad J_- = -i g^{-1} \partial_- g, \quad D_{ab} = \operatorname{Tr}(t^a g t^b g^{-1}). \quad (2.3)$$
The representation matrices are denoted by $t^a$ and normalized to unity. In addition, they obey the relation $[t_a, t_b] = i f_{abc} t_c$ for some set of real structure constants. The above action can be rewritten as

$$S_{k,\lambda}(g) = \frac{k}{2\pi} \int d^2\sigma \left[ \frac{D + \lambda \mathbb{1}}{1 - \lambda D^2} J_- + \frac{k}{12\pi} \int (g^{-1} d g)^3 \right]. \quad (2.4)$$

In what follows, it will be particularly useful to parametrize the group element $g \in G$ in terms of normal coordinates as

$$g = e^{i t^a x^a}, \quad (2.5)$$

leading to the following expressions for the matrix in the adjoint representation and for the currents

$$D = e^f = \mathbb{1} + f + \frac{f^2}{2} + \cdots,$$

$$J_- = \frac{1 - e^{-f}}{f} \partial_- x = \left(1 - \frac{f}{2} + \frac{f^2}{6} + \cdots\right) \partial_- x, \quad (2.6)$$

$$J_+ = \frac{e^f - 1}{f} \partial_+ x = \left(1 + \frac{f}{2} + \frac{f^2}{6} + \cdots\right) \partial_+ x.$$ 

We have used the definition $f_{ab} = f_{abc} x^c$ and we have expanded the above quantities for small $x$’s around the identity.

The next step is to organize the expansion around the identity as a large $k$ expansion. We let for the normal coordinates the rescaling

$$x^a = \sqrt{\frac{1 - \lambda}{1 + \lambda}} \frac{\phi^a}{\sqrt{k}}, \quad (2.7)$$

which as we will see, allows the new fields $\phi^a$’s to have a canonically normalized kinetic term in the action.

It turns out that we will need the action (2.4) up to $O(1/k^2)$ in the large $k$ expansion. Using the above coordinates, the first term of the action (2.4) gives to the specified
order in the large-\(k\) expansion the action

\[ S_{k,\lambda}^{(2)} = \frac{1}{2\pi} \int d^2 \sigma \left( \partial_+ \phi^a \partial_+ \phi^a + \frac{S_3^{(2)}}{\sqrt{k}} f_{ab} \partial_+ \phi^a \partial_+ \phi^b \right. \]
\[ + \frac{S_4^{(2)}}{k} f_{ab} \partial_+ \phi^a \partial_+ \phi^b \left. + \frac{S_5^{(2)}}{k^{3/2}} f_{ab} \partial_+ \phi^a \partial_+ \phi^b \right) + \cdots, \tag{2.8} \]

where

\[ S_3^{(2)} = -\frac{2\lambda}{(1-\lambda)^{1/2}(1+\lambda)^{3/2}}, \quad S_4^{(2)} = \frac{1 + 10\lambda + \lambda^2}{12(1 - \lambda^2)}, \]
\[ S_5^{(2)} = -\frac{\lambda}{2(1-\lambda)^{3/2}(1+\lambda)^{1/2}}, \quad S_6^{(2)} = \frac{1 + 56\lambda + 246\lambda^2 + 56\lambda^3 + \lambda^4}{360(1 - \lambda^2)^2}. \tag{2.9} \]

Also we have redefined \(f_{ab}\) in terms of the field \(\phi\)

\[ f_{ab} = f_{abc} \phi^c, \tag{2.10} \]

a definition which will be used in the rest of the paper. In this expansion, we have kept the free part properly normalized as well as interactions up to the sixth order.

Now we turn to the expansion of the topological term, that is the second term in (2.4). The expansion of this term can be obtained by first considering the expansion of corresponding field strength

\[ H_0 = -\frac{k}{6} f_{abc} L^a \wedge L^b \wedge L^c, \quad L^a = -i \text{Tr}(t^a g^{-1} dg). \tag{2.11} \]

We need to expand \(H_0\) to \(O(1/k^2)\) and then read off the corresponding two-form antisymmetric tensor \(B_0\) and its contribution to the action. After certain algebraic manipulations using mainly the Jacobi identity for the structure constants, we find that

\[ H_0 = -\frac{1}{6\sqrt{k}} \left( \frac{1 - \lambda}{1 + \lambda} \right)^{3/2} f_{abc} d\phi^a \wedge d\phi^b \wedge d\phi^c \]
\[ + \frac{1}{24k^{3/2}} \left( \frac{1 - \lambda}{1 + \lambda} \right)^{5/2} f_{abc} f_{ad} f_{be} d\phi^d \wedge d\phi^e \wedge d\phi^c + \cdots. \tag{2.12} \]

Note that there are no \(O(1/k)\) and \(O(1/k^2)\) terms. Using the above expression we may read off the antisymmetric tensor field from the relation \(H_0 = dB_0\). We emphasize that this is a local expression. Then, we find out that the topological term contributes to
the action as

\[ S_{k,\lambda}^{(3)} = \frac{1}{2\pi} \int d^2 \sigma \left( \frac{g_3^{(3)}}{\sqrt{k}} f_{ab} \partial_+ \phi^a \partial_- \phi^b + \frac{g_5^{(3)}}{k^{3/2}} f_{ab} \partial_+ \phi^a \partial_- \phi^b \right) + \cdots , \]  

(2.13)

where

\[ g_3^{(3)} = -\frac{1}{3} \left( \frac{1 - \lambda}{1 + \lambda} \right)^{3/2}, \quad g_5^{(3)} = -\frac{1}{60} \left( \frac{1 - \lambda}{1 + \lambda} \right)^{5/2}. \]  

(2.14)

As a remark we mention that in the expansion of the quadratic action all powers of the matrix \( f \) appear, whereas for the cubic part of the action only the odd ones do so.

Putting everything altogether (2.4) assumes the expansion

\[ S_{k,\lambda} = \frac{1}{2\pi} \int d^2 \sigma \left( \partial_+ \phi^a \partial_- \phi^a + \frac{g_3}{\sqrt{k}} f_{ab} \partial_+ \phi^a \partial_- \phi^b \\
+ \frac{g_4}{k} f_{ab} \partial_+ \phi^a \partial_- \phi^b + \frac{g_5}{k^{3/2}} f_{ab} \partial_+ \phi^a \partial_- \phi^b + \frac{g_6}{k^{2}} f_{ab} \partial_+ \phi^a \partial_- \phi^b \right) + \cdots , \]  

(2.15)

where the couplings are \( g_i = g_i^{(2)} + g_i^{(3)}, i = 3, \ldots, 6 \). They assume the form

\[ g_3(\lambda) = -\frac{1}{3} \frac{1 + 4\lambda + \lambda^2}{(1 - \lambda)^{1/2}(1 + \lambda)^{3/2}}, \quad g_4(\lambda) = \frac{1}{12} \frac{1 + 10\lambda + \lambda^2}{(1 - \lambda^2)}, \]  

\[ g_5(\lambda) = -\frac{1}{60} \frac{1 + 26\lambda + 66\lambda^2 + 26\lambda^3 + \lambda^4}{(1 - \lambda)^{3/2}(1 + \lambda)^{5/2}}, \]  

(2.16)

\[ g_6(\lambda) = \frac{1}{360} \frac{1 + 56\lambda + 246\lambda^2 + 56\lambda^3 + \lambda^4}{(1 - \lambda^2)^2}. \]

Note the symmetry of (2.15) under

\[ \lambda \to \frac{1}{\lambda}, \quad k \to -k, \quad \phi^a \to -\phi^a. \]  

(2.17)

This originates form the same symmetry of (2.15) discovered in the present context in [16] and earlier using path integral arguments in [22], where the group element inversion corresponds to flipping the sign of \( \phi^i \). Note also that, since there are square roots involved in the definition of the coupling \( g_3 \), we should refine the way the aforementioned symmetry acts. In particular, we have that \( k \to e^{i\pi} k \) and \( 1 - \lambda \to e^{i\pi} (\lambda^{-1} - 1) \) implying, for instance, that \( \sqrt{k(1 - \lambda)} \to -\sqrt{k(1 - \lambda)} \lambda^{-1/2} \).

The reader might wonder for the reason we have kept in our expansion terms as high as of those of order six in the fields. As we shall see, in determining the three- and
two-point functions for currents to $O(1/\sqrt{k})$ and $O(1/k)$, respectively, only $g_3$ is necessary. However, for determining the $\beta$-function from the renormalization of the cubic interaction to $O(1/k)$ all three $g_3, g_4$ and $g_5$ should be kept. Finally, to verify that the same $\beta$-function follows from the renormalization of the quartic interaction the coupling $g_6$ is also necessary.

2.1 Computational QFT conventions

We would like to set up a perturbative expansion around the free theory. Passing to the Euclidean regime we have the following basic propagators

$$\langle \phi^a(z, \bar{z}) \phi^b(w, \bar{w}) \rangle = -\delta^{ab} \ln |z - w|^2 ,$$  \hspace{1cm} (2.18)

which is consistent with our normalizations. We will use the notation

$$J^a(z) = \partial \phi^a(z, \bar{z}) , \quad \bar{J}^a(z) = \bar{\partial} \phi^a(z, \bar{z}) .$$  \hspace{1cm} (2.19)

Then, we have that

$$\langle J^a(z) \phi^b(w, \bar{w}) \rangle = \frac{-\delta^{ab}}{z-w} , \quad \langle J^a(z) J^b(w) \rangle = \frac{-\delta^{ab}}{(z-w)^2} ,$$

$$\langle \phi^a(z, \bar{z}) f^{bce}(w, \bar{w}) \rangle = -f_{abc} \ln |z - w|^2 ,$$

$$\langle f^{ab}(z, \bar{z}) f^{cde}(w, \bar{w}) \rangle = -f_{abc} f_{cde} \ln |z - w|^2 ,$$

$$\langle J^a(z) f^{bce}(w, \bar{w}) \rangle = -\frac{f_{abc}}{z - w} .$$  \hspace{1cm} (2.20)

Finally, note the propagator

$$\langle J^a(z) \bar{J}^b(w) \rangle = C \delta^{ab} \delta^{(2)}(z - w) ,$$  \hspace{1cm} (2.21)

which couples the holomorphic and anti-holomorphic sectors. The constant $C$ is

$$C = \pi ,$$  \hspace{1cm} (2.22)

Note that we differ by a factor of 1/2 from other conventions, e.g. eq. (2.5.8) of [23]. This apparent disagreement comes from the fact that in our conventions $\delta^{(2)}(z) = \delta(x)\delta(y) := \delta^{(2)}(x)$, while in [23] $\delta^{(2)}(z) = \frac{1}{2} \delta^{(2)}(x)$ (see eq. (2.1.8) in that reference and the line below it). Accordingly, in our conventions $z = x + iy$. In addition, the measure of integration in the Euclidean regime is $d^2z = dx dy$. 

\[ \text{Note:} \text{Page 7} \]
which is proven by taking the derivative of the first propagator in the first line in (2.20) with respect to $\bar{w}$ and subsequently using that $\partial_z \frac{1}{z} = \partial_{\bar{z}} \frac{1}{z} = \pi \delta^{(2)}(z)$.  

3 Correlation functions and anomalous dimensions

We are interested in the correlation function of the currents $J_\pm$. However, these after the deformation no longer have the form (2.3), but they are instead $\lambda$-dressed. The dressed currents has been identified in [13] with the on-shell values of the gauge fields arising from integrating them out in the process of constructing the $\lambda$-deformed action (2.1). These have the following expressions

$$A_+ = i(\lambda^{-1}\mathbb{1} - D)^{-1}J_+ , \quad A_- = -i(\lambda^{-1}\mathbb{1} - D^T)^{-1}J_- ,$$

(3.1)

They are not invariant under the transformation (2.17), but they can be made so by multiplying with an appropriate $\lambda$-dependent factor. Obviously this will not affect their anomalous dimensions. One should expand them as we did for the action. For a well defined large $k$-expansion we should first multiply them with an appropriate constant. This is chosen so that such the leading order term is $\pm \partial_\pm \phi$. Indeed, multiplying $A_\pm$ by $i\sqrt{k}(1 - \lambda^{-1})$ and denoting the result by $\mathcal{J}_\pm$ we have that

$$\mathcal{J}_+ = \left(1 + \frac{h_1}{\sqrt{k}} f + \frac{h_2}{k} f^2 + \cdots \right) \partial_+ \phi ,$$

$$\mathcal{J}_- = -\left(1 - \frac{h_1}{\sqrt{k}} f + \frac{h_2}{k} f^2 + \cdots \right) \partial_- \phi ,$$

(3.2)

where

$$h_1(\lambda) = \frac{1}{2} \sqrt{\frac{1 + \lambda}{1 - \lambda}} , \quad h_2(\lambda) = \frac{1 + 4\lambda + \lambda^2}{6(1 - \lambda^2)} .$$

(3.3)

We will see that for computing their two- and three-point correlation functions and their anomalous dimensions up to $O(1/k)$ only the coefficient $h_1$ will play a role. Note also that the normalization of the fields is such that at the conformal point for $\lambda = 0$, the coefficient of the non-Abelian term in their operator product expansions is proportional to $f^{abc} / \sqrt{k}$. The precise coefficient is $-3$ and not 1 as is probably expected. The reason for the discrepancy is that in the $\lambda = 0$ limit one still has to take into account

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3 These are also consistent with the relation $(\partial_x^2 + \partial_y^2) \ln \rho = 2\pi \delta^{(2)}(x)$, proven using Stoke’s theorem in two-dimensions, as well as with the previous footnote.
the interaction terms in (2.15). Hence, (3.2) should not be thought of as a bosonization formula.

In the Euclidean regime we will use $\mathcal{J}$ and $\tilde{\mathcal{J}}$ in place of $\mathcal{J}_+ \text{ and } \mathcal{J}_-$, respectively.

### 3.1 The three-point function for currents

The easiest correlators to compute involving only currents are three-point ones. The leading result is of $\mathcal{O}(1/\sqrt{k})$. For the purely chiral correlator and employing a self-explanatory notation for the $\lambda$-dependent correlators, we have that

$$
\langle \mathcal{J}^a(x_1, \bar{x}_1) \mathcal{J}^b(x_2, \bar{x}_2) \mathcal{J}^c(x_3, \bar{x}_3) \rangle_\lambda = \langle \mathcal{J}^a(x_1, \bar{x}_1) \mathcal{J}^b(x_2, \bar{x}_2) \mathcal{J}^c(x_3, \bar{x}_3) e^{-S_{\text{int}}} \rangle,
$$

(3.4)

where the interaction terms in the Euclidean regime can be read from (2.15) to be

$$
S_{\text{int}} = \frac{g_3}{2\pi \sqrt{k}} \int d^2z f^{a_1} f^{b_1} \bar{f}^{b_1} + \frac{g_4}{2\pi k} \int d^2z f^{a_2} f^{b_2} \bar{f}^{b_2} + \frac{g_5}{2\pi k^{3/2}} \int d^2z f^{a_3} f^{b_3} \bar{f}^{b_3}.
$$

(3.5)

Then, expanding the exponential and keeping terms up to $\mathcal{O}(1/\sqrt{k})$ we have that

$$
\langle \mathcal{J}^a(x_1, \bar{x}_1) \mathcal{J}^b(x_2, \bar{x}_2) \mathcal{J}^c(x_3, \bar{x}_3) \rangle_\lambda
= \frac{h_1}{\sqrt{k}} \langle f^{a_1} f^{b_1} \bar{f}^{b_1} \rangle + [\text{cyclic in } (x_1, a), (x_2, b), (x_3, c)]
- \frac{g_3}{2\pi \sqrt{k}} \int d^2z \langle f^{a_1} f^{b} \bar{f}^{b_1} \rangle
= \frac{1}{\sqrt{k}} \left( h_1 + \frac{C}{2\pi g_3} \right) \langle f^{a_1} f^{b} \bar{f}^{b} \rangle + [\text{cyclic in } (x_1, a), (x_2, b), (x_3, c)].
$$

(3.6)

Explicitly, the necessary four-point function is given by

$$
\langle f^{a_1} f^{b_1} \rangle = \frac{f^{abc}}{x_{12} x_{13}} \left( \frac{1}{x_{13}} - \frac{1}{x_{12}} \right),
$$

(3.7)

finding finally that

$$
\langle \mathcal{J}^a(x_1, \bar{x}_1) \mathcal{J}^b(x_2, \bar{x}_2) \mathcal{J}^c(x_3, \bar{x}_3) \rangle_\lambda
= \frac{1}{\sqrt{k}} \left( 3h_1 + \frac{3C}{2\pi g_3} \right) \frac{f^{abc}}{x_{12} x_{13} x_{23}}.
$$

(3.8)
Using for $C$ the value in (2.22) and also (2.16) and (3.3) we obtain that

\[
\langle \mathcal{J}^a(x_1, \bar{x}_1) \mathcal{J}^b(x_2, \bar{x}_2) \mathcal{J}^c(x_3, \bar{x}_3) \rangle = \frac{1}{\sqrt{k}} \frac{1 + \lambda + \lambda^2}{(1 - \lambda)^{1/2} (1 + \lambda)^{3/2}} \frac{f_{abc}}{x_{12} x_{13} x_{23}}. \tag{3.9}
\]

This result agrees with the one computed before using conformal perturbation theory and the non-perturbative symmetry in the coupling space (2.17) (see eq. (3.33) of [13]). Similarly, for the mixed chirality correlator and keeping only those terms that potentially contribute to the correlator up to $O(1/\sqrt{k})$, we have that

\[
\langle \mathcal{J}^a(x_1, \bar{x}_1) \mathcal{J}^b(x_2, \bar{x}_2) \bar{\mathcal{J}}^c(x_3, \bar{x}_3) \rangle = \langle \mathcal{J}^a(x_1, \bar{x}_1) \mathcal{J}^b(x_2, \bar{x}_2) \bar{\mathcal{J}}^c(x_3, \bar{x}_3) e^{-\lambda^2} \rangle
\]

\[
= -\frac{h_1}{\sqrt{k}} \left( f_{a'b'}(x_1, \bar{x}_1) f_{b'}(x_2) \bar{f}_{c'}(x_3) \right) + \left[ (x_1, a) \leftrightarrow (x_2, b) \right]
\]

\[
+ \frac{g_3}{2 \pi \sqrt{k}} \int d^2 z \left( f_{a'}(x_1) f_{b'}(x_2) \bar{f}_{c'}(x_3) f_{a'}(z) \bar{f}_{b'}(z) \bar{f}_{c'}(z) \right)
\]

\[
= -\frac{1}{\sqrt{k}} \left( h_1 + \frac{C}{2 \pi g_3} \right) \left( f_{a'a'}(x_1, \bar{x}_1) f_{b'}(x_2) \bar{f}_{c'}(x_3) \right) + \left[ (x_1, a) \leftrightarrow (x_2, b) \right]
\]

\[
+ \frac{g_3}{2 \pi \sqrt{k}} \int d^2 z \frac{1}{(z - \bar{x}_3)^2} \left( f_{a'a'}(z, \bar{z}) f_{b'}(x_1) f_{b'}(x_2) \right). \tag{3.10}
\]

In the fourth line of (3.10) we have the correlator

\[
\langle f_{a'a'}(x_1, \bar{x}_1) f_{b'}(x_2) \bar{f}_{c'}(x_3) \rangle = -\frac{f_{abc}}{x_{12} x_{13} x_{23}} C \frac{f_{abc}}{x_{12}} \delta(2) (x_{12}). \tag{3.11}
\]

The last term will be ignored since it is a contact term of external points. To evaluate the integral in the last line of (3.10) we have used (3.11) and the complex conjugate of the last integral in (B.2). Putting everything together we find that

\[
\langle \mathcal{J}^a(x_1, \bar{x}_1) \mathcal{J}^b(x_2, \bar{x}_2) \mathcal{J}^c(x_3, \bar{x}_3) \rangle = -\frac{1}{\sqrt{k}} \left( h_1 + \frac{2 \pi + C}{2 \pi g_3} \right) \frac{f_{abc} \bar{x}_{12}}{x_{12} \bar{x}_{13} \bar{x}_{23}}. \tag{3.12}
\]

Substituting \pi for C and the expressions for the couplings $h_1$ and $g_3$ one gets the final result

\[
\langle \mathcal{J}^a(x_1, \bar{x}_1) \mathcal{J}^b(x_2, \bar{x}_2) \mathcal{J}^c(x_3, \bar{x}_3) \rangle = \frac{1}{\sqrt{k}} \frac{\lambda}{(1 - \lambda)^{1/2} (1 + \lambda)^{3/2}} \frac{f_{abc} \bar{x}_{12}}{x_{12}^2 \bar{x}_{13} \bar{x}_{23}}. \tag{3.13}
\]

Again, the result agrees with the one computed in [13] using conformal perturbation theory and the non-perturbative symmetry in the coupling space (2.17) (see eq. (3.33)
of [13]). Needless to say, it was an essential part of the computation that the $\lambda$-dressed currents were used. Had we used the bare currents the correct result would not have been obtained.

### 3.2 The single current anomalous dimension

In order to obtain the anomalous dimension of the currents we evaluate the two-point function

$$\langle J^a(x_1, \bar{x}_1) J^b(x_2, \bar{x}_2) \rangle_\lambda = \langle J^a(x_1, \bar{x}_1) J^b(x_2, \bar{x}_2) e^{-S_{int}} \rangle. \quad (3.14)$$

Since we are after the leading term of the anomalous dimension which is of order $O(1/k)$ we will expand the exponential keeping terms up to the same order. It can be easily seen that $O(1/\sqrt{k})$ contribution vanishes as it contain only terms with an odd number of fields. Then the result for the two-point correlator up to $O(1/k)$ reads

$$\langle J^a(x_1, \bar{x}_1) J^b(x_2, \bar{x}_2) \rangle_\lambda = -\frac{\delta^{ab}}{x_{12}^2} - \frac{1}{k} \left( h_1^a I_{1}^{ab} - \frac{1}{2 \pi} \frac{\delta^{ab}}{2} I_{2}^{ab} - h_1 g_3^a I_{3}^{ab} \right), \quad (3.15)$$

where we have defined

$$I_{1}^{ab} = \langle J^c(x_1) f^{ca}(x_1, \bar{x}_1) f^{bd}(x_2, \bar{x}_2) J^d(x_2) \rangle,$$

$$I_{2}^{ab} = \int d^2 z \langle J^c(x_1) f^{ca}(x_1, \bar{x}_1) J^b(x_2) J^{a_1}(z) J^{a_2}(\bar{z}) \rangle + (x_1, a) \leftrightarrow (x_2, b), \quad (3.16)$$

$$I_{3}^{ab} = \int d^2 z_1 d^2 z_2 \langle J^a(x_1) J^b(x_2) J^{a_1}(z_1) J^{a_2}(\bar{z}_1) \rangle \times J^{b_1}(z_2) J^{b_2}(\bar{z}_2) \rangle.$$

By introducing a short distance cut-off $\epsilon$ one can easily compute the first integral to be

$$I_{1}^{ab} = -\frac{c_G \delta^{ab}}{x_{12}^2} + \frac{c_G \delta^{ab}}{x_{12}^2} \ln \frac{\epsilon^2}{|x_{12}|^2}. \quad (3.17)$$

The integral in the second line of (3.16) is evaluated to give

$$I_{3}^{ab} = C \int d^2 z \delta(z - x) \langle f^{ca}(x_1, \bar{x}_1) J^b(x_2) J^{a_1}(z) J^{a_1 c}(z, \bar{z}) \rangle + C \langle J^c(x_1) J^{a_1}(x_2) J^{a_2}(x_2, \bar{x}_2) \rangle \quad (3.18)$$

$$- \int d^2 z \frac{\delta^{a_1 a_2}}{z - \bar{x}_1} \langle J^c(x_1) J^b(x_2) J^{a_1}(z) J^{a_1 a_2}(z, \bar{z}) \rangle + [(x_1, a) \leftrightarrow (x_2, b)].$$

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Using the first and second integral in (B.2) one finds that
\[ I_{3}^{ab} = 2(C - \pi) \frac{c G \delta_{ab}}{x_{12}^2} - 2(2\pi + C) \frac{c G \delta_{ab}}{x_{12}^2} \ln \left| \frac{\epsilon^2}{x_{12}} \right|^2. \] (3.19)

Using \( C = \pi \) we obtain the result
\[ I_{3}^{ab} = -6\pi \frac{c G \delta_{ab}}{x_{12}^2} \ln \left| \frac{\epsilon^2}{x_{12}} \right|^2. \] (3.20)

Finally, we should consider the integral \( I_{2}^{ab} \). This requires a more involved computation the details of which are presented in the appendix. The end result, reads
\[ I_{2}^{ab} = -18\pi^2 \frac{c G \delta_{ab}}{x_{12}^2} \ln \left| \frac{\epsilon^2}{x_{12}} \right|^2. \] (3.21)

Consequently, (3.15) gives
\[ \langle \mathcal{J}^{a}(x_1, \bar{x}_1) \mathcal{J}^{b}(x_2, \bar{x}_2) \rangle_{\lambda} = -\frac{\delta_{ab}}{x_{12}^2} \left( 1 - \frac{c G h_1^2}{k} + \frac{c G}{k} \left( h_1 + \frac{3}{2} g_3 \right)^2 \ln \left| \frac{\epsilon^2}{x_{12}} \right|^2 \right), \] (3.22)

Inserting in the last equation the expressions for \( g_3 \) and \( h_1 \) that can be found in (2.16) and (3.3) respectively, one can find that up to \( \mathcal{O}(1/k) \) the current anomalous dimension is given by
\[ \gamma = \frac{c G}{k} \frac{\lambda^2}{(1 - \lambda)(1 + \lambda)^3}. \] (3.23)

This is the same result found using either CFT perturbation theory in combination with the symmetry (2.17) in coupling space [12] or the effective action and the geometry in coupling space in [20]. Obviously, we would have reached the same conclusion for the anomalous dimension if we had considered the two-point correlator for \( \tilde{J} \), instead of that for \( J \).

One may wonder how one could have constructed (3.2) without resorting to previous work, in particular (3.1), but using the action (2.15) as the only input. The way to proceed in this direction is firstly to realize that beyond the free field limit an operator mixing occurs giving rise to a non-diagonal anomalous dimension matrix. Diagonalizing it will determine the operators with well defined anomalous dimension. The first step in this procedure is to determine the set of operators which in principle may mix. This consists by the operators having the same classical anomalous dimension in
the free theory limit. For the case at hand, the set of operators with classical dimension one is
\[ \mathcal{O}_{\pm,n}^a = (f^n)^{ab} \partial_{\pm} \phi^b, \quad n = 0, 1, 2, \ldots. \] (3.24)

In the free theory limit these have vanishing two-point functions among themselves for different \( n \)'s, so that they constitute a diagonal basis. Subsequently, one computes the same two point functions to a given order in the \( 1/k \)-expansion which generically, will no longer be diagonal. They can be subsequently diagonalized to the given order in the \( 1/k \)-expansion and their anomalous dimensions can be read off. Obviously there will be no mixing between opposite chirality operators since the theory is Lorentz invariant. The operators 3.2 with \( h_1 \) and \( h_2 \) given by (3.3) provide by construction the first operators in this hierarchy having a well defined anomalous dimension. In order to compute the corrected version of the partner operator that appears in the mixing, i.e. \( f \partial_{\pm} \phi, f^2 \partial_{\pm} \phi \) etc, we need to evaluate more expressions like the ones in (3.15) which will not do in this paper.

### 3.3 Anomalous dimensions of primary fields

Consider the field \( D^{ab} \) as defined in (2.3). Then to \( \mathcal{O}(1/k) \) we do not need any insertions coming from the exponential of the action. As a result one obtains
\[ \langle D^{ac}(x_1, \bar{x}_1) D^{bc}(x_2, \bar{x}_2) \rangle = \delta_{ab} + \frac{11 - \lambda}{k} \langle f^{ac}(x_1, \bar{x}_1) f^{bc}(x_2, \bar{x}_2) \rangle \]
\[ = \delta_{ab} \left( 1 + \frac{c_G}{k} \frac{1 - \lambda}{1 + \lambda} \ln \frac{\epsilon^2}{|x_{12}|^2} \right), \] (3.25)

from which the anomalous dimension of \( D^{ab} \) can be read
\[ \gamma_D = \frac{c_G}{k} \frac{1 - \lambda}{1 + \lambda}. \] (3.26)

In fact we may also compute the anomalous dimension of the group element \( g(x, \bar{x}) \) which is a primary field in some irreducible representation \( R \) with unitary matrices \( t_a \).
As before using (2.5) and the rescaling (2.7) we compute that

\[
\langle g^{ik}(x_1, x_1)g^{-1}_{kj}(x_2, x_2) \rangle = \delta_{ij} \left( 1 + \frac{c_R (1 - \lambda)}{k (1 + \lambda)} \ln \frac{e^2}{|x_{12}|^2} \right),
\]

(3.27)

where \((t_at_a)_{ij} = c_R \delta_{ij}\), with \(c_R\) being the quadratic Casimir in the representation \(R\). From the last equation we deduce that

\[
\gamma_g = \frac{c_R (1 - \lambda)}{k (1 + \lambda)},
\]

(3.28)

in agreement with previous work (see eq. (3.26) of [13] after setting \(c_{R'} = c_R\) since the representations matrices are the same for the left and the right transformations). Note that specializing to the adjoint representation we obtain the anomalous dimension (3.26) for the composite field of \(g\), that is the dimension of \(D_{ab}\) which is not something that one should have necessarily expected. It is not clear that this equality relation will persist beyond the leading order in the \(1/k\)-expansion. Note also that, unlike the current operators, the primary field operators we have considered here are not \(\lambda\)-dressed, i.e. they retain their bare CFT expressions.

4 The \(\beta\)-function

In this section, we will derive the expression for the running of the parameter \(\lambda\) under the renormalization group flow, by using the background field method. In the context of \(\lambda\)-deformations this method was first used in [26], albeit in a first order formulation of the equations of motion. In order to obtain the \(\beta\)-function for the coupling \(\lambda\) it is enough to look at the renormalization of the cubic coupling in (2.15). We will see that unlike the case of the correlation functions of the previous section, in this case all three couplings up to fifth order in the action (2.15) will be involved, as well. We will also show that that the renormalization of the quartic coupling in (2.15) will give rise to the same \(\beta\)-function, as it should be, in a non-trivial manner. In that case the sixth order coupling is necessary as well.

To proceed we need the equations of motion for the fields \(\phi^a\). Varying the action
up to terms of $O(1/k^2)$ we obtain that

$$
\partial_+\partial_-\phi^a - \frac{3g_3}{2\sqrt{k}}f_{abc}\partial_+\phi^b\partial_-\phi^c - \frac{g_4}{k}(f_{abcd}f_{dc} + f_{acdf}f_{db})\partial_+\phi^b\partial_-\phi^c \\
+ \frac{1}{2k^{3/2}}\left(3g_3g_4f_{abc}^2f_{dbc} - g_5(\partial_a f_{bc}^2 + \text{cyclic in } a, b, c)\right)\partial_+\phi^b\partial_-\phi^c \\
+ \frac{1}{k^2}\left(g_4^2f_{abc}(f_{ebd}f_{dc} + f_{ecd}f_{db}) - g_6\frac{1}{2}(\partial_a f_{bc}^4 - \partial_b f_{ac}^4 - \partial_c f_{ab}^4)\right)\partial_+\phi^b\partial_-\phi^c = 0. 
$$

(4.1)

Next we compute the fluctuation $\delta \phi^a$ of this equation, around a classical solution which we will still denote by $\phi^a$. We will cast them in the form

$$
\hat{D}^{ab}\delta \phi^b = 0, 
$$

(4.2)

where the operator $\hat{D}$ is second order in the worldsheet derivatives. We will present its explicit expression after the Euclidean analytic continuation and in momentum space. In the conventions of [8], we replace $(\partial_+, \partial_-)$ by $\frac{1}{2}(\tilde{p}, p) \equiv (p_+, p_-)$. Then after dividing by $p_+p_-$ the operator $\hat{D}$ takes the form

$$
\hat{D}^{ab} = \hat{\delta}^{ab} + \hat{F}^{ab}, \\
\hat{F}^{ab} = \frac{g_3}{\sqrt{k}}F_1^{ab} + \frac{g_4}{k}F_2^{ab} + \frac{g_4}{k}F_2^{ab} + \frac{1}{k^{3/2}}F_3^{ab} + \frac{1}{k^{3/2}}F_3^{ab} + \frac{1}{k^2}F_4^{ab},
$$

(4.3)

where we have used the following definitions

$$
F_1^{ab} = \frac{3}{2}f_{abc}\left(\frac{\partial_+\phi^c}{p_+} - \frac{\partial_-\phi^c}{p_-}\right), \\
F_2^{ab} = \frac{B^{abcd}}{p_+p_-}\partial_+\phi^c\partial_-\phi^d, \\
F_2^{ab} = B^{abc}\left(\frac{\partial_+\phi^c}{p_+} + \frac{\partial_-\phi^c}{p_-}\right)\phi^d, \\
F_3^{ab} = \frac{C^{abcd}}{p_+p_-}\partial_+\phi^c\partial_-\phi^d, \\
F_3^{ab} = -C^{abc}\left(\frac{\partial_+\phi^c}{p_+} - \frac{\partial_-\phi^c}{p_-}\right), \\
F_4^{ab} = \frac{D^{abcd}}{p_+p_-}\partial_+\phi^c\partial_-\phi^d.
$$

(4.4)

Note the Lorentz non-invariant terms which we have indicated by a prime. Nevertheless, as we will see they will eventually combine into a Lorentz invariant result.

---

4 A direct variation of the action (2.15) gives an equation of the form

$$(\delta_{ab} + \frac{g_4}{k}f_{ab}^2 + \frac{g_6}{k^2}f_{ab}^4)\partial_+\partial_-\phi^b + \cdots = 0.$$ 

The following expression is obtained after multiplying with the inverse of the matrix prefactor and keeping terms up to the specified order.
For that matter we did not keep the term, which would have been denoted as $F'_4$, arising from varying the derivatives $\partial_+ \phi^b \partial_- \phi^c$ in the last line in (4.1). Such a term should have to combine with $F'_1$ in order to give a Lorentz invariant result. But then apparently the result would be of order $O(1/k^{5/2})$ and as such would contribute to the fifth coupling interaction term. Moreover, we have found it convenient to define the following tensors
\begin{align*}
B^{abcd} & = f_{acm}f_{mbd} + f_{adm}f_{mbc} = B^{abdc}, \\
C^{abcd} & = \frac{3}{2}g_3g_4\partial_b f_{2am}f_{mcd} - \frac{1}{2}g_5(\partial_a \partial_b f_{3cd} + \text{cyclic in } a, c, d) = -C^{abdc}, \\
C^{abc} & = \frac{3}{2}g_3g_4f_{2ad}f_{dbc} - \frac{g_5}{2}(\partial_a f_{3bc} + \text{cyclic in } a, b, c) = -C^{acb}, \\
D^{abcd} & = \partial_b \left( \frac{g_4^2}{2}f_{ae}f_{2ge}f_{gd} + f_{edg}f_{gc} \right) - \frac{g_6}{2}\partial_b (\partial_a f_{4cd} - \partial_c f_{4ad} - \partial_d f_{4ac}) = D^{abdc}.
\end{align*}

Integrating out the fluctuations, gives the effective Lagrangian of our model
\begin{equation}
- \mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)}_{k, \lambda} + \int \frac{d^2 p}{(2\pi)^2} \ln(\det \hat{D})^{-1/2},
\end{equation}
where $\mathcal{L}^{(0)}_{k, \lambda}$ is the Lagrangian of the action (2.15). This integral is logarithmically divergent with respect to the UV mass scale $\mu$. The divergence is isolated by performing the large momentum expansion of the integrand and keeping terms proportional to $\frac{1}{|p|^2}$, where $|p|^2 = p\bar{p}$. We use the identity
\begin{equation}
\ln(\det \hat{D}) = \text{Tr} \hat{F} - \frac{1}{2} \text{Tr} \hat{F}^2 + \cdots
\end{equation}
The traces in (4.7) can be easily calculated to give
\begin{align*}
\text{Tr} \hat{F} & = \frac{g_4}{k} \text{Tr} F_2 + \frac{1}{k^{3/2}} \text{Tr} F_3 + \frac{1}{k^2} \text{Tr} F_4 \\
\text{Tr} \hat{F}^2 & = \frac{g_4^2}{k} \text{Tr} F'_2 + \frac{g_3 g_4}{k^{3/2}} \text{Tr} (F'_1 F'_2) + \frac{g_4^2}{k^2} \text{Tr} F_2^2 + 2\frac{g_3 g_4}{k^2} \text{Tr} (F'_1 F'_3)
\end{align*}
where in the first line we have not included terms having a single power of $1/p_+$ or $1/p_-$, since these terms will give zero when the angular part of the integration in (4.6) is performed.

In the following we examine the renormalization of the cubic and the quartic interac-
tion vertices and show that they both give rise to the same $\beta$-function for $\lambda$, thanks to the specific dependence of the couplings $g_i$, $i = 3, \ldots, 6$ on the single parameter $\lambda$. We will see that in the process wave-function renormalization, as well as field redefinitions will be needed. This is expected from the work of [16]. In that work the gravitational approach in computing the $\beta$-function using the full action (2.4) was employed and coordinate reparamaterizations were indeed needed. These in a field theoretical language would correspond to wave function and field redefinitions.

4.1 Renormalization of the cubic vertex

For the cubic vertex we only need to consider terms up to order $O(1/k^{3/2})$. We start by explicitly computing the relevant traces in (4.5)

\[
\begin{align*}
\text{Tr}F_2 &= \frac{2c_G}{p_+ p_-} \partial_+ \phi^a \partial_- \phi^a, \\
\text{Tr}F_3 &= \frac{c_G}{2p_+ p_-} (3g_3 g_4 + 5g_5) f_{ab} \partial_+ \phi^a \partial_- \phi^b, \\
\text{Tr}F_1' &= \frac{9c_G}{2p_+ p_-} \partial_+ \phi^a \partial_- \phi^a, \\
\text{Tr}(F_1' F_2') &= \frac{9c_G}{2p_+ p_-} f_{ab} \partial_+ \phi^a \partial_- \phi^b.
\end{align*}
\]

The next step is to use polar coordinates, i.e. $p = re^{i\phi}$, $\bar{p} = re^{-i\phi}$, in which the integration measure reads $d^2p = r dr d\phi$ and subsequently evaluate the effective action. Since we need terms up to $O(1/k^{3/2})$ we truncate the action keeping only up to the cubic interaction term. Equation (4.6) then straightforwardly gives (for clarity, we return back to the Lorentzian regime)

\[
S_{\text{eff}} = \frac{1}{2\pi} \int d^2\sigma \left[ \left[ 1 - \frac{c_G}{k} \left( 2g_4 - \frac{9}{4} g_3^2 \right) \ln \mu^2 \right] \partial_+ \phi^a \partial_- \phi^a \\
+ \frac{1}{\sqrt{k}} \left[ g_3 \frac{c_G}{k} \left( \frac{5}{2}g_5 - 3g_3 g_4 \right) \ln \mu^2 \right] f_{ab} \partial_+ \phi^a \partial_- \phi^b \right] + \cdots.
\]

The wavefunction renormalization

\[
\phi^a = Z^{1/2} \hat{\phi}^a, \quad Z = 1 + \frac{c_G}{k} \left( 2g_4 - \frac{9}{4} g_3^2 \right) \ln \mu^2.
\]
puts the kinetic term into a canonical form and (4.10) becomes

\[ S_{\text{eff}} = \frac{1}{2\pi} \int d^2 \sigma \left[ \partial_+ \hat{\phi}^a \partial_+ \hat{\phi}^a \right. \\
+ \frac{1}{\sqrt{k}} \left[ g_3 - \frac{c_G}{k} \left( \frac{5}{2} g_5 - 6 g_3 g_4 + \frac{27}{8} g_3^3 \right) \ln \mu^2 \right] \hat{f}_{ab} \partial_+ \hat{\phi}^a \partial_+ \hat{\phi}^b \left. \right] + \cdots \]  

(4.12)

We demand that the action (4.10) is \( \mu \)-independent, i.e. \( \partial_{\ln \mu^2} L_{\text{eff}} = 0 \). For \( k \gg 1 \) this derivative acts only on the coupling constant \( g_3(\lambda) \). Then, we obtain that

\[ \beta^{\lambda} g_3' = \frac{c_G}{8k} (20 g_5 - 48 g_3 g_4 + 27 g_3^3) \quad \Rightarrow \quad \beta^{\lambda} \equiv \frac{\mu}{2} \frac{d\lambda}{d\mu}, \]

(4.13)

from which using the explicit expressions (2.16) the \( \beta \)-function for the coupling \( \lambda \)

\[ \beta^{\lambda} = -\frac{c_G}{2k} \frac{\lambda^2}{(1 + \lambda)^2}, \]

(4.14)

follows. This is precisely the expression firstly computed in [24] and [25] with CFT methods and in [16] with gravitational methods.

### 4.2 Renormalization of the quartic vertex

In this case we should consider the terms of order \( O(1/k^2) \) as well. Performing the corresponding traces in (4.8) is much more difficult. Nevertheless, we have performed this task for the case where the group \( G \) is \( SU(2) \). Then, in our normalizations

\[ f_{abc} = \sqrt{2} \epsilon_{abc}, \quad c_G = 4. \]

(4.15)

Then

\[ f_{ab}^2 = 2(\phi_a \phi_b - \delta_{ab} \phi^2), \quad f_{ab}^3 = -2\sqrt{2} \epsilon_{abc} \phi_c \phi^2, \]

(4.16)

where \( \phi^2 = \phi^a \phi^a \). In addition, we compute the \( C^{abc} \) and \( B^{abcd} \) tensors which take the form

\[ C^{abc} = 3\sqrt{2}(g_5 - 8 g_3 g_4) \epsilon_{abc} \phi^2 \]
\[ + \sqrt{2}(3g_3 g_4 + 2g_5) \epsilon_{abcd} \phi^d - 2\sqrt{2} g_5 (\epsilon_{abd} \phi^c - \epsilon_{acd} \phi^b) \phi^d, \]

(4.17)

\[ B^{abcd} = 4 \delta_{ab} \delta_{cd} - 2 \delta_{ad} \delta_{bc} - 2 \delta_{ac} \delta_{bd}. \]
Subsequently, we calculate the additional traces in (4.8) to be
\[
\begin{align*}
\text{Tr} F_4 &= 12(2g_4^2 + g_6) \phi_a \phi_b \phi_c \phi_d \partial^a \phi^b \partial^c \phi^d - 4(2g_4^2 + 11g_6) \phi^2 \partial^a \phi^b \partial^c \phi^d, \\
\text{Tr} F_2^2 &= 48 \phi_a \phi_b \phi_c \phi_d \partial^a \phi^b - 32 \phi^2 \partial^a \phi^b \partial^c \phi^d, \\
\text{Tr}(F_1' F_3') &= 18g_3g_4 \phi_a \phi_b \phi_c \phi_d \partial^a \phi^b + 2(9g_3g_4 - 30g_5) \phi^2 \partial^a \phi^b \partial^c \phi^d.
\end{align*}
\] (4.18)

As a result, the quartic interaction term of the action becomes
\[
\begin{align*}
S^{(4)}_{\text{eff}} &= \frac{1}{2\pi} \int d^2 \sigma \frac{2}{k} \left[ \left( g_4 - \frac{1}{k} \left( 6g_6 - 9g_3^2g_4 \right) \ln \mu^2 \right) \phi^a \phi^b \phi^c \phi^d \partial^a \phi^b \partial^c \phi^d \\
&\quad - \left[ g_4 - \frac{1}{k} \left( 22g_6 + 9g_3^2g_4 - 4g_4^2 - 30g_3g_5 \right) \ln \mu^2 \right] \phi^2 \partial^a \phi^b \partial^c \phi^d \right].
\end{align*}
\] (4.19)

However, one should not forget to take into account the wave-function renormalization (4.11) (with \(c_G = 4\)). Doing so, the quartic interaction term of the action becomes
\[
\begin{align*}
S^{(4)}_{\text{eff}} &= \frac{1}{2\pi} \int d^2 \sigma \frac{2}{k} \left[ \left( g_4 - \frac{1}{k} \left( 6g_6 + 9g_3^2g_4 - 16g_4^2 \right) \ln \mu^2 \right) \phi^a \phi^b \phi^c \phi^d \partial^a \phi^b \partial^c \phi^d \\
&\quad - \left[ g_4 - \frac{1}{k} \left( 22g_6 + 27g_3^2g_4 - 20g_4^2 - 30g_3g_5 \right) \ln \mu^2 \right] \phi^2 \partial^a \phi^b \partial^c \phi^d \right].
\end{align*}
\] (4.20)

Clearly, demanding that this action (4.10) is \(\mu\)-independent would lead to two different expressions for the \(\beta\)-function for \(\lambda\) and in fact none of them coincides with (4.14). In order to make the two compatible one needs to perform a field redefinition. Realizing that \(f_{ab} \phi_b = 0\), an appropriate ansatz can only be of the form
\[
\phi^a = \tilde{Z}^{1/2} \phi^a, \quad \tilde{Z}^{1/2} = 1 + \frac{\hat{C}}{k^2} \phi^2 \ln \mu^2,
\] (4.21)

where \(\hat{C}\) is a coupling dependent constant. Applying this field redefinition to (4.20) we get
\[
\begin{align*}
S^{(4)}_{\text{eff}} &= \frac{1}{2\pi} \int d^2 \sigma \frac{2}{k} \left[ \left( g_4 - \frac{1}{k} \left( 6g_6 + 9g_3^2g_4 - 16g_4^2 - 2\hat{C} \right) \ln \mu^2 \right) \phi^a \phi^b \phi^c \phi^d \partial^a \phi^b \partial^c \phi^d \\
&\quad - \left[ g_4 - \frac{1}{k} \left( 22g_6 + 27g_3^2g_4 - 20g_4^2 - 30g_3g_5 + \hat{C} \right) \ln \mu^2 \right] \phi^2 \partial^a \phi^b \partial^c \phi^d \right],
\end{align*}
\] (4.22)

where the terms involving \(\hat{C}\) arise from the contribution of the quadratic term in (4.12) due to the field redefinition in (4.21) above. Demanding now that \(\partial_{\ln \mu^2} L_{\text{eff}} = 0\) leads
to two differential equations for $g_4$ whose compatibility requires that

$$\hat{C} = \frac{4}{3}g_4^2 + 10g_3g_5 - 6g_3^2g_4 - \frac{16}{3}g_6. \quad (4.23)$$

Then we obtain that

$$\beta^\lambda g_4' = \frac{1}{3k}(63g_3^2g_4 - 56g_4^2 - 60g_3g_5 + 50g_6). \quad (4.24)$$

Substituting in the last equation the values for the couplings one gets the same $\beta$-function as (4.14) (with $c_G = 4$). For general groups the factor $1/3$ on the right hand side of (4.24) is expected to be replaced by $c_G/12$, the reason being that for large $k$ the perturbative expansion is in terms of the ratio $c_G/k$.

It should be clear that we may keep the analysis general order by order in perturbation theory and derive a system of first order differential equations for the $g_i$'s, $i = 3, 4, \ldots$. Demanding that all couplings depend only on a single parameter $\lambda$ with the above $\beta$-function, this procedure generates all couplings to arbitrarily high order once the first three ones are known. For instance, (4.24) with the $\beta$-function (4.14) generates $g_6(\lambda)$ and so on and so forth. The summed up action should of course be the $\lambda$-deformed action (2.1). Note that, the RG structure of the theory is such that the flow is between the UV CFT point and a gapped IR theory when the coupling becomes strong. In our free field approach the strong coupling regime can be and indeed is achieved since the coupling constants $g_i$ are not independent, but all have a very specific dependence on a single coupling $\lambda$. Moreover, they all have singularities at $\lambda = \pm 1$.

5 Discussion and future directions

In the present work we have established a systematic method for performing various computations in a wide class of integrable $\sigma$-models that go under the name of $\lambda$-deformed models. These include computations of two- and three-point functions of current and primary filed operators, as well as the $\beta$-functions as exact functions of the deformation parameter $\lambda$. As an explicit example, we have chosen to perform our calculations in the simplest case of the single $\lambda$-deformed model [4]. All of our results are in complete agreement with the expressions obtained earlier in [13, 12] and are obtained after considerably easier computational efforts.
The main idea of our method is that instead of using perturbation theory around the conformal point it is more effective to use it around the free point. Practically, we have set up a large $k$ perturbative expansion of the effective action of the model around the free field point corresponding to the identity group element $g = \mathbb{I}$. The resulting action defines a two-dimensional quantum field theory with a canonical kinetic term and infinite number of interaction terms involving successively more and more fields. For our purposes, we have kept all field interaction terms up to sixth order. An additional ingredient in our approach was the free field expansion of the $\lambda$-dressed operators which we have also established. One of the virtues of our method is that it incorporates automatically all the dependence in $\lambda$, that is all-loop effects, since these are already built in at the action level and as such they inherit the non-perturbative symmetry in coupling space (2.17). A second advantage of our approach is that it can be considered as the basis for systematically performing the perturbative expansion of the $\beta$-function in powers of $1/k$. The reason is that, one may focus at the renormalization of a single vertex, i.e. the cubic being the simplest, instead of seeking the renormalization of the entire $\sigma$-model effective action. A similar comment concerns the higher, in $1/k$, corrections of the anomalous dimensions.

There are several directions towards this line of research can be extended. First recall that the original $\lambda$-deformed model [4] was formulated for general deformation matrix $\lambda_{ab}$, but correlation functions and operator anomalous dimensions are not known beyond the case of diagonal matrix we have studied also here. The present method is particularly suitable to handle the arbitrary matrix case as well. In addition, it would be interesting to reproduce for the case of left/right asymmetric models [6], and by using our method, the exact results for the correlation functions of currents and/or primaries, as well as that for the $\beta$-function that were obtained earlier in [14] by the use of conformal perturbation theory. Secondly, it would be important to apply our method to study the quantum properties of the wide class of coupled integrable models constructed recently in [8] and [9]. We note that in these models although the running of the couplings is known we lack exact results involving correlation functions of currents and primary fields. Finally, as mentioned above, our set up is ideal for calculating the subleading $1/k^2$ corrections to the $\beta$-functions, anomalous dimensions and correlators both for the single and the multiply $\lambda$-deformed models. Such a calculation will provide a non-trivial check of the non-perturbative symmetry of the
model beyond the leading order in the $1/k$-expansion. It will also clarify certain issues concerning the $1/k$ corrections to the effective action. Another direction would be to find other non-trivial classical solutions of the $\lambda$-deformed theories which would be the analogue of the uniton solutions found in the case of the PCM. It would be certainly interesting to apply our method and expand around these solutions instead of expanding around the identity.

In addition to $\lambda$-deformation corresponding to integrable deformations of WZW currents algebra CFTs, there is the similar construction in which the deformed CFT is based on a symmetric coset or the $AdS_5 \times S^5$ Superstring. It will be of importance to extend our present consideration in these cases as well.

Next, we comment on the relation of the free field expansion to the fact that the action from which it originates is integrable. To motivate the argument note that if all the coupling coefficients of the terms $f_{ab} \partial_+ \phi^a \partial_- \phi^b$ in the free field expansion were set to unity then this would correspond to the free field expansion of the non-Abelian T-dual of PCM for the group $G$, which is an integrable $\sigma$-model (see, for instance, appendix D of [4]). Placing arbitrary coefficients in front of the interaction terms and demanding integrability would place stringent conditions on them in the form of recursive relations. One solution to these would be the sequence of coefficients, all depending on a single coupling $\lambda$, arising from the free field expansion of the $\lambda$-deformed action, the first few terms being given by and . Another integrable case corresponds to the pseudo-chiral $\sigma$-model in which all couplings vanish except for $g_3$. Note that, as a mathematical truncation, this is consistent with the renormalization group flow equations as well. This approach could be useful in trying to find other solutions to the aforementioned recursive relations that could depend on more parameters. The corresponding integrable $\sigma$-models could in fact go beyond the $\lambda$-deformed models in the sense that it is far from obvious that they can be obtained by a gauging procedure.

Finally, we believe that, as technique, the free field expansion, with appropriate modifications will be useful in determining anomalous dimensions and quantum properties in general of operators in the so-called $\eta$-deformations using their also their relation to $\lambda$-deformations. Similar comment applies for the integrable coupled $\sigma$-models of.
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A Anomalous dimensions: Computational details

In this appendix we present the details of the computation of the most laborious integrals that are required for the evaluation of the anomalous dimensions and of the three-point correlation functions of the currents of the currents.

A.1 Current anomalous dimension

In this appendix we give the details of the computation of the integral $I_2^{ab}$ defined in (3.16) since they are rather involved compared to $I_1^{ab}$ and $I_3^{ab}$. Picking up and performing the first Wick contraction with $I_2^{b2}(z_2)$, this splits as

$$I_2^{ab} = I_{2,1}^{ab} + I_{2,2}^{ab} + I_{2,3}^{ab},$$

(A.1)

where the various terms are given by

$$I_{2,1}^{ab} = C \int d^2z_1 \langle J^b(x_2)J^{a_1}(z_1)f^{a_1a_2}(z_1, z_1)J^{a_2}(z_1)J^{b_1}(x_1)f^{b_1a}(x_1, x_1) \rangle + [(x_1, a) \leftrightarrow (x_2, b)],$$

(A.2)

by that

$$I_{2,2}^{ab} = f^{a_1a_2b_2} \int \frac{d^2z_1 d^2z_2}{z_{12}} \langle J^a(x_1)J^b(x_2)J^{a_1}(z_1)J^{a_2}(z_1)J^{b_1}(z_2)f^{b_1b_2}(z_2, z_2) \rangle$$

(A.3)

and by that

$$I_{2,3}^{ab} = - \int \frac{d^2z_1 d^2z_2}{z_{12}^2} \langle J^a(x_1)J^b(x_2)J^{a_1}(z_1)f^{a_1a_2}(z_1, z_1)J^{b_1}(z_2)f^{b_1a_2}(z_2, z_2) \rangle.$$
We observe that $I_{2,1}^{a,b}$ equals the integral $I_{3}^{a,b}$ in (3.20) times the factor $C = \pi$. Therefore, using (3.20) we immediately obtain that

$$I_{2,1}^{a,b} = -6\pi^2 \frac{c_G \delta^{a,b}}{x_{12}^2} \ln \frac{\epsilon^2}{|x_{12}|^2}. \quad (A.5)$$

Turning to the second integral and using for the Wick contraction $\bar{J}^a(z_1)$, we obtain

$$I_{2,2}^{a,b} = C f_{a_1 b_2} \int \frac{d^2 z_2}{x_1 - z_2} \langle J^b(x_2) J^{a_1}(x_1) J^{b_1}(z_2) J^{b_1}(z_2) \rangle + [(x_1, a) \leftrightarrow (x_2, b)]$$

$$- f_{a_1 a_2 b_2} f_{a_2 b_1 b_2} \int \frac{d^2 z_1 d^2 z_2}{z_{12}^2} \langle J^a(x_1) J^b(x_2) J^{a_1}(z_1) J^{b_1}(z_2) \rangle$$

$$= C f_{a_1 b_2} f_{b_2 a_1} \int \frac{d^2 z_2}{(z_2 - x_1)(z_2 - x_2)} \left( \frac{1}{z_2 - x_2} - \frac{1}{z_2 - x_1} \right)$$

$$+ [(x_1, a) \leftrightarrow (x_2, b)]$$

$$+ 2c_G \delta^{a,b} \int \frac{d^2 z_1 d^2 z_2}{z_{12}^2 (z_1 - x_1)^2 (z_2 - x_2)^2}, \quad (A.6)$$

where we have included only integrals that potentially contribute and we have also used (3.7). As a result, one obtains that

$$I_{2,2}^{a,b} = c_G C \delta^{a,b} \int d^2 z_2 \left( \frac{1}{(z_2 - x_2)^2 (z_2 - x_1)(z_2 - x_1)} - \frac{1}{(z_2 - x_1)^2 (z_2 - x_2)(z_2 - x_1)} \right)$$

$$+ [(x_1, a) \leftrightarrow (x_2, b)]$$

$$+ 2\pi^2 c_G \delta^{a,b} \int d^2 z_2 \frac{\delta^{(2)}(z_2 - x_1)}{(z_2 - x_2)^2}, \quad (A.7)$$

where we have used the last integral in (B.1). Using also the second integral in (B.2) and setting $C = \pi$ we obtain

$$I_{2,2}^{a,b} = -4\pi^2 \frac{c_G \delta^{a,b}}{x_{12}^2} \ln \frac{\epsilon^2}{|x_{12}|^2}. \quad (A.8)$$
Finally, the last integral to be computed is

\[ I_{2,3}^{ab} = 2 \int \frac{d^2z_1 d^2z_2}{z_{12}^2(z_1 - x_1)^2} \left( \int (x_2) f^{a \bar{a}_2}(z_1, z_1) f^{b \bar{a}_2}(z_2, z_2) \right) + 2 f^{a \bar{a}_1 a_2} \int \frac{d^2z_1 d^2z_2}{z_{12}^2(z_1 - z_1)} \left( \int (x_2) f^{b \bar{a}_1}(z_1) f^{b \bar{a}_2}(z_2, z_2) \right). \]  

(A.9)

The first line of (A.9) gives

\[
2c_G \delta^{ab} \int \frac{d^2z_1 d^2z_2}{z_{12}^2(z_1 - x_1)^2} \left( \frac{|z_{12}|^2}{(z_2 - x_2)^2} - \frac{1}{z_{12}(z_2 - x_2)} \right) = 2 \pi c_G \delta^{ab} \left( \int \frac{d^2z_2}{(z_2 - x_2)^2(z_2 - x_1)(z_2 - x_1)} + \int \frac{d^2z_1}{(z_1 - x_1)^2(z_1 - x_2)(z_1 - x_2)} \right) = -(2 \pi^2 + 2 \pi^2) c_G \delta^{ab} \ln \frac{e^2}{|x_{12}|^2},
\]

(A.10)

where we have used (B.5) and the second and third integrals of (B.2). We now move to the second line in (A.9) which gives

\[
-2c_G \delta^{ab} \int \frac{d^2z_1 d^2z_2}{z_{12}^2 z_{21}} \frac{1}{(z_1 - x_1)(z_2 - x_2)} \left( \frac{1}{z_2 - x_2} - \frac{1}{z_{21}} \right). \tag{A.11}
\]

Note that the first integral in (A.11) is identical to the second integral in the first line in (A.11) (after exchanging the indices 1 and 2 in both the z’s and the x’s) giving a contribution of \(-2 \pi^2 c_G \delta^{ab} \ln \frac{e^2}{|x_{12}|^2} \). The second integral in (A.11) is

\[
2c_G \delta^{ab} \int \frac{d^2z_2}{z_2 - x_2} \int \frac{d^2z_1}{z_{12}^2 z_{12}^2(z_1 - x_1)^2} \frac{1}{z_2 - x_2} \int d^2z_1 \left[ \frac{\partial z_1}{z_{12}^2 z_{12}^2(z_1 - x_1)} - \frac{1}{z_{12}^2 z_{12}^2(z_1 - x_1)} \right]
\]

\[
+ \pi \delta^{(2)}(z_1 - x_1) \int \frac{d^2z_2}{z_2 - x_2} \int \frac{d^2z_2}{z_{12}^2 z_{12}^2(z_1 - x_1)} \frac{dz_1}{z_{12}^2 z_{12}^2(z_1 - x_1)} + \frac{\pi}{(x_1 - x_2)(x_1 - x_2)^2} \frac{dz_1}{z_{12}^2 z_{12}^2(z_1 - x_1)}
\]

\[
= -2 \pi^2 c_G \delta^{ab} \ln \frac{e^2}{|x_{12}|^2} \tag{A.12}
\]

This result arises from the second integration over \(z_2\) which appears in the second line of (A.12). To evaluate this second integral we have employed the first relation
in (B.2). The line integral of (A.12) is infinite and can be absorbed by an appropriate counterterm (see below). Therefore we finally obtain

\[ I_{2,3}^{ab} = -8\pi^2 c_G \delta^{ab} \frac{e^2}{x_{12}^2} \ln \frac{e^2}{|x_{12}|^2}. \]  \hspace{1cm} (A.13)

Hence, adding up (A.5), (A.8) and (A.13) gives (3.21).

B Some useful integrals

In our computation we encounter the following divergent integrals which we have regulated by introducing a small distance cut-off \( \epsilon \)

\[
\int \frac{d^2z}{(x_1 - z)(\bar{z} - x_1)} = \pi \ln e^2, \\
\int \frac{d^2z}{(x_1 - z)(\bar{z} - \bar{x}_2)} = \pi \ln |x_{12}|^2, \\
\int \frac{d^2z}{(x_1 - z)^2(\bar{z} - \bar{x}_2)} = -\frac{\pi}{x_{12}}, \\
\int \frac{d^2z}{(x_1 - z)(\bar{z} - \bar{x}_2)^2} = -\frac{\pi}{\bar{x}_{12}}, \\
\int \frac{d^2z}{(x_1 - z)^2(\bar{z} - \bar{x}_2)^2} = \pi^2 \delta^{(2)}(x_{12}),
\]

as well as,

\[
\int \frac{d^2z}{(z - x_1)^2(z - x_2)(\bar{z} - x_1)} = \frac{\pi}{x_{12}^2} \ln \frac{e^2}{|x_{12}|^2}, \\
\int \frac{d^2z}{(z - x_1)^2(z - x_2)(\bar{z} - \bar{x}_2)} = -\frac{\pi}{x_{12}^2} \ln \frac{e^2}{|x_{12}|^2} - \frac{\pi}{x_{12}^2}, \\
\int \frac{d^2z}{(z - x_1)^2(\bar{z} - \bar{x}_1)(\bar{z} - \bar{x}_2)} = \frac{\pi}{|x_{12}|^2}, \\
\int \frac{d^2z}{(z - x_1)(z - x_2)^2(\bar{z} - x_3)^2} = -\frac{\pi}{x_{12}^2} \frac{\bar{x}_{12}}{x_{13}x_{23}} - \pi^2 \delta^{(2)}(x_{23}) \frac{x_{12}}{x_{12}}.
\]

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Finally, we will also need the integral

$$
\int d^2z \frac{\ln |z - x_1|^2}{(z - x_2)^2(z - x_1)^2} = \frac{\pi}{|x_{12}|^2}.
$$

(B.3)

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