GROWTH OF BALLS OF HOLOMORPHIC SECTIONS ON PROJECTIVE TORIC VARIETIES

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Abstract

Let $\mathcal{O}(D)$ be an equivariant line bundle which is big and nef on a complex projective nonsingular toric variety $X$. Given a continuous toric metric $\| \cdot \|$ on $\mathcal{O}(D)$, we define the energy at equilibrium of $(X, \varphi_D)$ where $\varphi_D$ is the weight of the metrized toric divisor $D = (D, \| \cdot \|)$. We show that this energy describes the asymptotic behaviour as $k \to \infty$ of the volume of the $L^2$-norm unit ball induced by $(X, k\varphi_D)$ on the space of global holomorphic sections $H^0(X, \mathcal{O}(kD))$.

Key Words: Toric varieties, Equilibrium weight, Energy functional, Bernstein-Markov property, Monge-Ampère operator.

MSC: 14M25, 52A41, 32W20

1 Introduction

Let $Q$ be a free $\mathbb{Z}$-module of rank $n$ and $P$ its dual. We consider a fan $\Sigma$ on $Q_\mathbb{R} = Q \otimes \mathbb{R}$ and we denote by $X = X_\Sigma$ the associated toric variety over $\mathbb{C}$, see for instance [S]. In the sequel, we assume that $X$ is nonsingular and projective (this is equivalent to the fact that $\Sigma$ is nonsingular and the support of $\Sigma$ is $Q_\mathbb{R}$, see [S theorems 1.10, 1.11]). We set $T_Q := \text{Hom}(P, \mathbb{C}^*) \simeq (\mathbb{C}^*)^n$ and we denote by $S_Q \simeq (S_1)^n$ its compact torus. We have an open dense immersion $T_Q \hookrightarrow X$ with an action of $T_Q$ on $X$ which extends the action of $T_Q$ on its self by translations.

The toric varieties have a rich geometry that can be related to the geometry of polytopes. Many results in algebraic geometry and complex differential geometry can be tested on them, for instance the Riemann-Roch formula.

On toric varieties, some properties of line bundles can be interpreted in terms of convex geometry. Let $D$ be an equivariant Cartier divisor on $X$ also called a toric divisor, that is a Cartier divisor which is invariant under the action of the torus $T_Q$. Let $s_D$ be the rational section of $\mathcal{O}(D)$ associated to $D$. We know that $D$ defines a $\Sigma$-linear support function $\Psi_D$ on $\Sigma$ (see [S Definition p. 66]) and $D$ is uniquely defined by this function (see [S Proposition 2.1 (v)]). Moreover the function $\Psi_D$ defines a convex polytope:

$$\Delta_D := \{ x \in P_\mathbb{R} \mid < x, u > \geq \Psi_D(u), \forall u \in Q_\mathbb{R} \},$$

where $P_\mathbb{R} := P \otimes \mathbb{R}$. $\Delta_D$ and $\psi_D$ encode many geometric informations about $D$, for instance

1. $H^0(X, \mathcal{O}(D)) = \oplus_{e \in \Delta_D \cap P} \mathbb{C} \chi^e$, where $\chi^e$ denotes the character associated to $e$, see [S Lemma 2.3].

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Moreover we have that by definition of the canonical metric one can check easily that one can show that 
\[ \| \cdot \| \]

is spanned by its global sections (equivalently \( \| \cdot \| \)) is concave (see \([8, \text{Theorem 2.7}]\)). When \( O \) is ample then the notions of admissible metrics and semipositive metrics are equivalent, see \([7, \text{Theorem 4.6.1}]\).

We say that a function \( \Psi \) on \( X \) is nonnegative. An admissible metric is by definition a uniform limit of a sequence of smooth hermitian semipositive metrics. When \( O \) is ample then the notions of admissible metrics and semipositive metrics are equivalent, see \([7, \text{Theorem 4.6.1}]\).

The function \( \Psi \) defines a continuous hermitian metric on \( O \) called the canonical metric of \( O \). It is given locally as follows: The norm of a local holomorphic section \( s \) at a point \( x \) is the following real
\[ \| s(x) \|_{\infty, D} = \frac{s}{\lambda(x)} \]

(see \([7, \text{Proposition 3.3.1}]\)). When \( O \) is spanned by its global sections (equivalently \( \Psi \) is concave) one can show that \( \| \cdot \|_{D, \infty} = \phi_D^\# \| \cdot \|_{\infty} \) where \( \phi_D : X \to \mathbb{P}^\#(\Delta_D \cap P)^{-1} \) is the equivariant morphism defined in terms of \( \Delta_D \cap P \) and \( \| \cdot \|_{\infty} \) is the canonical metric of \( O(1) \) (see \([7, \text{§3.3.3}]\) for a detailed construction). Moreover we have that \( \| \cdot \|_{\infty, D} \) is a uniform limit of a sequence of smooth semipositive metrics and \(-\log \|s_D\|^2_{\infty, D} \) is a plurisubharmonic weight on \( O \) \([7, \text{Propositions 3.3.11, 3.3.12}]\).

Let \( \| \cdot \|_{\overline{\Omega}_D} \) be a \( Q \)-invariant hermitian metric on \( O \) such that \( \| \cdot \|_{\overline{\Omega}_D}/\| \cdot \|_{\infty, D} \) is bounded on \( X \). We let \( \overline{\Omega} := (D, \| \cdot \|_{\overline{\Omega}_D}) \) the obtained hermitian line bundle and we called it a toric metrized divisor. We set \( g_{\overline{\Omega}} : Q \to \mathbb{R} \) the function defined as follows:
\[ g_{\overline{\Omega}}(u) := \log \|s_D(\exp(-u))\|_{\overline{\Omega}} \quad \forall u \in Q \).

By definition of the canonical metric one can check easily that
\[ g_{\overline{\Omega}}(u) = \inf \{ <v, u> | v \in \Delta_D \} \quad \forall u \in Q \tag{1} \]

(\( <v, u> \) denotes the pairing defined by \( Q \) and \( P \)). We denote by \( \hat{g}_{\overline{\Omega}} : P \to [-\infty, +\infty] \) the Legendre-Fenchel transform of \( g_{\overline{\Omega}} \), i.e. the function defined for any \( x \in P \) as follows
\[ \hat{g}_{\overline{\Omega}}(x) := \inf_{x \in Q} (x, u) = -\hat{g}_{\overline{\Omega}}(u) \]

We have \( \hat{g}_{\overline{\Omega}} \) vanishes on \( \Delta_D \) and equal to \(-\infty\) otherwise (One can show that this follows from the following assertion: \( \hat{g}_{\overline{\Omega}}(x) = t\hat{g}_{\overline{\Omega}}(x) \) for any \( x \in \Delta_D \) and \( t > 0 \), which is an easy consequence of \( \hat{1} \)) Combining this with Proposition \([2, \text{Proposition 2.3}]\) we can show that \( \hat{g}_{\overline{\Omega}}(x) \) is finite if and only if \( x \in \Delta_D \) and \( \hat{g}_{\overline{\Omega}} \) is concave on \( \Delta_D \)
Let $\| \cdot \|_{\mathcal{D}_0}$ and $\| \cdot \|_{\mathcal{D}_1}$ be two smooth hermitian metrics on $D$, $\phi_{\mathcal{D}_0}$ and $\phi_{\mathcal{D}_1}$ the associated weights. We define the Monge-Ampère functional $E$ by the formula

$$E(\mathcal{D}_1) - E(\mathcal{D}_0) = \frac{1}{n+1} \sum_{j=0}^{n} \int_X - \log \frac{\| \cdot \|_{\mathcal{D}_1}}{\| \cdot \|_{\mathcal{D}_0}} c_1(\mathcal{D}_0)^\wedge j \wedge c_1(\mathcal{D}_1)^\wedge {n-j}.$$ 

By the theory of Bedford-Taylor [1], this definition extends to admissible metrics, and hence to integrable ones by polarisation. By definition an integrable metric can be written, in additive notation, as a difference of two admissible metrics.

Following [2], when $\mathcal{O}(D)$ is big we set

$$E_{\text{eq}}(\mathcal{D}_1) - E_{\text{eq}}(\mathcal{D}_0) := \frac{1}{\text{Vol}(D)} (E((\mathcal{D}_1)_X) - E((\mathcal{D}_0)_X)).$$

where $(\mathcal{D}_i)_X$ is the metrized toric divisor $D$ endowed with the weight $P_X \phi_{\mathcal{D}_i}$, the equilibrium weight of $\phi_{\mathcal{D}_i}$ for $i = 0, 1$. In [2] §1.3, $E_{\text{eq}}(\mathcal{D})$ is called the energy at equilibrium of $(X, \phi_{\mathcal{D}})$ ($\phi_{\mathcal{D}}$ is the weight of $\mathcal{D}$).

Our first result is Theorem 3.4 which gives an integral representation of the variation of the energy functional $E$ in terms of some combinatorial objects defined on the polytope associated to $D$. This theorem can be seen as a toric version of [2, Theorem B].

Let $\mu$ be a probability measure with non-pluripolar support on $X$. We endow the space of global sections $H^0(X, \mathcal{O}(D))$ with the $L^2$-norm

$$\| s \|^2_{L^2(\mu, \mathcal{D})} := \int_X \| s \|^2_{\mathcal{D}} \mu.$$

Also we consider the sup norm defined as follows

$$\| s \|_{\text{sup,} \mathcal{D}} := \sup_{x \in X} \| s \|_{\mathcal{D}(x)}.$$ 

for any $s \in H^0(X, \mathcal{O}(D))$. Let $k \in \mathbb{N}^*$. We consider the following functional

$$L_k(\mu, k\mathcal{D}) := \frac{1}{2kN_k} \log \text{vol}_k B^2(\mu, k\mathcal{D}),$$

where $\text{vol}_k B^2(\mu, k\mathcal{D})$ is by definition the volume of the unit ball $B^2(\mu, k\mathcal{D})$ in $H^0(X, \mathcal{O}(kD))$ with respect to the $L^2$-norm, and $N_k := \dim H^0(X, \mathcal{O}(kD))$.

The Bergman distortion function $\rho(\mu, \mathcal{D})$ is by definition the function given at a point $x \in X$ by

$$\rho(\mu, \mathcal{D})(x) = \sup_{s \in H^0(X, \mathcal{O}(D)) \setminus \{0\}} \frac{\| s(x) \|^2_{\mathcal{D}}}{\| s \|^2_{L^2(\mu, \mathcal{D})}}.$$ 

If $\{s_1, \ldots, s_N\}$ is a $L^2(\mu, \mathcal{D})$-orthonormal basis of $H^0(X, \mathcal{O}(D))$, then it is well known that

$$\rho(\mu, \mathcal{D})(x) = \sum_{j=1}^{N} \| s_j(x) \|^2_{\mathcal{D}} \quad \forall x \in X.$$
Definition 1.1. We say that $\mu$ has the Bernstein-Markov property with respect to $\| \cdot \|_{D}$ if
$$\sup_X \rho(\mu, kD) = O(e^{k\varepsilon}).$$

If $\mu$ is a smooth positive volume form and $\| \cdot \|_{D}$ is a continuous metric on $O(D)$ then $\mu$ has the Bernstein-Markov property with respect to $\| \cdot \|_{D}$ (see [2, Lemma 3.2]).

Our main result is the following theorem:

Theorem 1.2. [Main theorem] Let $X$ be a complex projective nonsingular toric variety and $D$ a toric divisor on $X$ such that $O(D)$ is big and nef. Let $D_i := (D, \| \cdot \|_{D_i})$ be a continuous toric metrized divisor on $X$ for $i = 0, 1$. Let $\mu_j$ be a probability measure which is $S^Q$-invariant on $X$ and with the Bernstein-Markov property with respect to $\| \cdot \|_{D_j}, j = 0, 1$. Then as $k \to \infty$ we have
$$L_k(\mu_1, D_1) - L_k(\mu_0, D_0) \to E_{eq}(D_1) - E_{eq}(D_0).$$

This theorem describes the asymptotics of $L_k(\mu, D)$, the functional volume of the balls of the holomorphic sections of a continuous toric divisor $D$ when $k$ tends to $\infty$. In particular we recover partially a result of Berman and Boucksom [2, Theorem A]. Comparing to [2] our approach is completely different. In fact, our strategy is based mainly on the combinatorial structure of the toric variety, which makes the proof much easier. A crucial ingredient in the proof of Theorem 1.2 is Theorem 3.4.

2 The Monge-Ampère operator and the equilibrium weight

We keep the same notations as in the introduction. Let $X$ be a complex projective nonsingular toric variety and $L$ a holomorphic line bundle over $X$. Let $\phi$ be a weight of a continuous hermitian metric $e^{-\phi}$ on $L$. When $\phi$ is smooth we define the Monge-Ampère operator as
$$\text{MA}(\phi) := (ddc^{n})\phi.$$

The equilibrium weight of $\phi$ is defined as:
$$P_X \phi := \sup \{ \psi \mid \psi \text{ psh weight on } L, \psi \leq \phi \text{ on } X \}.$$

It is known that the equilibrium weight is upper semicontinuous psh weight with minimal singularities.

Proposition 2.1. Let $\phi_1$ and $\phi_0$ be two continuous weights on $L$ a big line bundle on $X$. We have
$$|P_X \phi_1 - P_X \phi_0| \leq \sup_{x \in X} |\phi_1 - \phi_0|.$$

Proof. This an easy consequence of the definition of the equilibrium weight.

Assume that $L$ is ample and let $\omega$ be a positive $(1, 1)$-form in $c_1(L)$. We set
$$H_\omega := \{ u \in C^\infty(X) \mid dd^c u + \omega > 0 \}.$$

Clearly, $H_\omega$ is convex subset which is identified with the set of smooth positive hermitian metrics (resp. weights) on $L$. We set $P_\omega(u) := \sup \{ v \in H_\omega \mid v \leq u \}$. Following [2], [3], this operator extends to $C^0(X)$ with image in $C^0(X) \cap H_\omega$. In other words, if $\phi$ is continuous weight on $L$ an ample line bundle then $P_X \phi$ is also continuous.
**Remark 2.2.** Let $D$ be a toric nef divisor on $X$. Let $\| \cdot \|_{\mathcal{T}}$ (resp. $\phi_D$) a continuous hermitian metric (resp. continuous weight) on $D$. Then $\| \cdot \|_{\mathcal{T}}$ is bounded on $X$, where $D_X$ is hermitian line bundle $\mathcal{O}(D)$ endowed with the metric defined by $P_X \phi_D$. Indeed, since $D$ is nef then $\| \cdot \|_{\mathcal{T}}$ is a semipositive metric. Then we can find a constant $C$ such that $\phi_{\mathcal{T}} - C \leq P_X \phi \leq \phi_{\mathcal{T}}$.

**Proposition 2.3.** Let $g$ be a real function on $Q_\mathbb{R}$. Then $g$ defines a hermitian (continuous) metric $\| \cdot \|_g$ on $\mathcal{O}(D)$ if and only if $g - \Psi_D$ extends to a bounded (continuous) function on $X$. Moreover, we have

$$\sup_{x \in \Delta_D} |g - \tilde{g}| \leq \sup_{u \in Q_\mathbb{R}} |g - g'|,$$

for any $g$ and $g'$ two functions on $Q_\mathbb{R}$ defining hermitian metrics on $\mathcal{O}(D)$.

**Proof.** The proof is an easy consequence of the definitions.

**Lemma 2.4.** Let $\mathcal{D} = (D, \| \cdot \|_{\mathcal{T}})$ be a continuous metrized divisor such that $\mathcal{O}(D)$ is big and nef on $X$. We set $\phi_D := -\log \|s_D\|_{\mathcal{T}}$ and we denote by $\mathcal{D}_X$ the metrized toric divisor $D$ endowed with $P_X \phi_D$. Then $P_X \phi_D$ is a $\mathcal{O}_Q$-invariant weight on $D$ and the following equality holds on $\Delta_D$

$$\beta_{\mathcal{T}} = \beta_{\mathcal{T}_X}.$$

**Proof.** By definition $\phi_D$ is $\mathcal{O}_Q$-invariant weight. Let $\psi$ be a psh weight on $\mathcal{O}(D)$ such that $\psi \leq \phi$. For any $t \in \mathcal{O}_Q$, we set $\psi_t := \psi(t \cdot \cdot \cdot )$. Then $\psi_t$ is a psh weight verifying $\psi_t(z) \leq \phi_D(t \cdot z) = \phi_D(z)$ for any $z \in X$. That is $\psi_t \leq \phi_D$. We conclude that $(P_X \phi_D)_t \leq P_X \phi_D$ for any $t \in \mathcal{O}_Q$. It follows that $P_X \phi_D$ is a $\mathcal{O}_Q$-invariant weight on $D$. By (2.2), $\beta_{\mathcal{T}_X}$ is well defined. Moreover, we have $\|s\|_{\sup \mathcal{T}_X} = \|s\|_{\sup \mathcal{T}_X}$ for any $k \in \mathbb{N}$ and $s \in H^0(X, \mathcal{O}(kD))$ (see for instance [2] Proposition 2.8]. Let $k \in \mathbb{N}^*$ and $e \in k\Delta_D \cap P$. We have $\|X^e\|_{\sup \mathcal{T}_X} = \sup_{x \in X} \|X^e(x)\|_{\mathcal{T}_X} = \sup_{u \in Q_\mathbb{R}} \|X^e(\exp(-u))\|_{\mathcal{T}_X} = \exp(-k \inf_{u \in Q_\mathbb{R}} (\frac{1}{k} \cdot u - \beta_{\mathcal{T}_X}(u)))$. We deduce that

$$\beta_{\mathcal{T}_X}(x) = \beta_{\mathcal{T}_X}(x) \quad \forall x \in \Delta_D \cap Q^d.$$  

Using the fact that a concave and finite function on a $\Delta_D$ is necessarily continuous on its interior, see [9] Theorem 10.1, then we get $\beta_{\mathcal{T}} = \beta_{\mathcal{T}_X}$ on $\text{Int}(\Delta_D)$. But, since $\Delta_D$ is the convex closure of $\Delta_D \cap P$ then

$$\beta_{\mathcal{T}}(x) = \beta_{\mathcal{T}_X}(x) \quad \forall x \in \Delta_D.$$

**3 The energy functional in the toric setting**

The goal of this section is to give a formula for the variation of the energy functional $E$ in terms of the Legendre-Fenchel transform. First, this formula is proved in the ample case, see Theorem 3.3] then we deduce the general case of big and nef divisors in Corollary 3.4.

**Proposition 3.1.** Let $D$ be a toric divisor on $X$. Assume that there exists $\| \cdot \|_1$ an admissible and $\mathcal{O}_Q$-invariant metric on $\mathcal{O}(D)$. Then there exists a sequence of smooth, semipositive and $\mathcal{O}_Q$-invariant hermitian metrics converging uniformly to $\| \cdot \|_1$.

**Proof.** First let recall that given a smooth hermitian metric $\| \cdot \|_1$ one can average it in order to get a $\mathcal{O}_Q$-invariant smooth metric. This is done as follows, we define the metric $\| \cdot \|_{\mathcal{T}_Q}$ given on $\mathcal{T}_Q$ by $\log \|s(x)\|_{\mathcal{T}_Q} = \int_{\mathcal{T}_Q} \log \|s(t \cdot x)\| \mu_{\text{Haar}}$. This metric extends to $X$ since $\log(\|s(x)\|_{\mathcal{T}_Q}/\|s(x)\|_1) = \int_{\mathcal{T}_Q} \log(\|s(t \cdot x)\|/\|s(x)\|_1) \mu_{\text{Haar}}$ where $\| \cdot \|_1$ is a smooth and $\mathcal{O}_Q$-invariant metric, extends to a smooth
function to \( X \). Clearly the metric \( \| \cdot \|_{S_Q} \) is \( S_Q \)-invariant and smooth. Moreover \( c_1(O(D), \| \cdot \|_{S_Q}) = \int_{S_Q} t^*c_1(O(D), \| \cdot \|_{S_Q}) \) where \( t^* \) is the pull-back defined by the multiplication by \( t \). It follows that if \( c_1(O(D), \| \cdot \|) \geq 0 \) then \( c_1(O(D), \| \cdot \|_{S_Q}) \geq 0 \).

Let \( \| \cdot \| \) be an admissible hermitian metric. By definition there exists \((\| \cdot \|)_n \in \mathbb{N} \) a sequence of smooth, semipositive and \( S_Q \)-invariant hermitian metrics converging uniformly to \( \| \cdot \| \). By averaging this sequence as before we get a sequence of smooth, semipositive and \( S_Q \)-invariant hermitian metrics which converges uniformly to \( \| \cdot \| \).

\[ \square \]

Let \( \overline{D} \) be a smooth positive toric divisor on \( X \). We set \( \Psi_D = - \log \| s_D \|_{\overline{D}}^2 \). The exponential map gives the following change of variables, \( z = \exp(-u + i\theta) \in T_Q \) for any \( z \in T_Q \) where \( u, \theta \in Q_R \). Then \[ \frac{\partial^2 \Psi_D}{\partial z_k \partial z_l} = \frac{1}{s_k^2 s_l} \frac{\partial^2 \Psi_D}{\partial u_k \partial u_l} \] for \( k, l = 1, \ldots, n \). Since \( \overline{D} \) is positive and smooth then \( \overline{g} \) is a strictly concave smooth function on \( Q_R \). Hence, for any \( x \in \text{Int}(\Delta_D) \) there exists a unique \( G_{\overline{g}}(x) \in Q_R \) such that \( \hat{g}(x) = x \cdot G_{\overline{g}}(x) - G_{\overline{g}}(G_{\overline{g}}(x)) \), and we can show that \( G_{\overline{g}} \) is smooth on \( \Delta_D \) and \( \frac{\partial^2 \Psi_D}{\partial u_k \partial u_l} \circ G_{\overline{g}} = \text{Id}_{\Delta_D} \) (this follows from [3, Theorem 26.5]). In other words, \( x := \frac{\partial \Psi_D}{\partial u} \) defines a \( C^\infty \)-diffeomorphism between \( \text{Int}(\Delta_D) \) and \( Q_R \), and we have

\[ c_1(\overline{D})^{\wedge n} = \frac{n!}{(2\pi)^n} \det \left( \frac{\partial^2 \Psi_D}{\partial u_k \partial u_l} \right)_{1 \leq k, l \leq n} \ dx_1 \wedge \cdots \wedge dx_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_n = \frac{n!}{(2\pi)^n} \ dx_1 \wedge \cdots \wedge dx_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_n. \]

Let \( \phi \) be a continuous function on \( X \) which is invariant under the action of \( S_Q \). We denote by \( \phi_D \) the function on \( \Delta_D \) given by \( \phi_D(x) = \phi(\exp(-G_{\overline{g}}(x))) \). One can show that

\[ \int_X \phi c_1(\overline{D})^{\wedge n} = n! \int_{\Delta_D} \phi_D \ dx, \]

where \( dx = dx_1 \wedge \cdots \wedge dx_n \) denotes the standard Lebesgue measure on \( Q_R \). In particular, one have the following identity \( \deg_{S_D}(X) = n! \text{vol}(\Delta_D) \), which extends easily to nef divisors.

**Lemma 3.2.** Let \( f \) be a smooth function on \( X \) and \( S_Q \)-invariant. We have

\[ \int_X df \wedge d^c f \wedge c_1(\overline{D})^{n-1} = \int_{\Delta_D} < \frac{dG_{\overline{g}}(x)}{dx}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} > \ dx, \]

where \( f(x) := f(\exp(-G_{\overline{g}}(x))) \) on \( \Delta_D \).

**Proof.** Notice that \( \frac{dG_{\overline{g}}(x)}{dx} = \text{Hess}(\hat{g}) \) and using the change of coordinates \( x := \frac{du}{dt} \) one can deduce the lemma. \( \square \)

Let \( \| \cdot \|_{\overline{D}_0} \) and \( \| \cdot \|_{\overline{D}_0} \) be two smooth, \( S_Q \)-invariant and positive hermitian metrics on \( O(D) \) and we let \( \overline{D}_i := (D, \| \cdot \|_{\overline{D}_i}) \) for \( i = 0, 1 \). For any \( t \in [0, 1] \) we set \( g_t = tg_1 + (1-t)g_0 \). Then we have the following result

**Proposition 3.3.** The following function defined on \([0, 1]\)

\[ t \mapsto \int_{\Delta_D} \hat{g}_tdx, \]

is differentiable on \([0, 1]\). Moreover, we have

\[ \frac{d}{dt} \left( \int_{\Delta_D} \hat{g}_tdx \right)_{t=0^+} = - \int_{\Delta_D} (g_1 - g_0)(G_0(x))dx. \]
Proof. We denote by $\mathcal{D}_t$ the positive and smooth toric metrized divisor $D$ endowed with $g_t$ and we set $G_t := G_{\mathcal{D}_t}$ for any $t \in [0, 1]$. We have
\[
\dot{g}_t(x) = \langle x, G_t(x) \rangle - g_t(G_t(x)) - t(g_1 - g_0)(G_t(x)) \quad \forall x \in \Delta_D, \forall t \in [0, 1].
\]

We set $F_t(x) := \langle x, G_t(x) \rangle - g_0(G_t(x))$ for any $x \in \Delta_D$ and $t \in [0, 1]$. We have $\frac{dF_t(x)}{dt}(x) = \langle x, \frac{dG_t(x)}{dt} \rangle - \frac{dg_0}{du}(x) \cdot \frac{dG_t(x)}{dt} = \langle x - G_0^{-1}(G_t(x)), \frac{dG_t(x)}{dt} \rangle$. That is
\[
\frac{dF_t}{dt}(x) = \langle x - G_0^{-1}(G_t(x)), \frac{dG_t(x)}{dt} \rangle \quad \forall x \in \Delta_D, \forall t \in [0, 1].
\] (2)

Let $u \in Q_R$ such that $x = \frac{du}{dt}(u)$. Then $x - G_0^{-1}(G_t(x)) = \frac{du}{dt}(u) - \frac{dg_0}{du}(u) = t\left(\frac{du}{dt}(u) - \frac{dg_0}{du}(u)\right)$. Recall that $G_t(\frac{du}{dt}(u)) = v$ for any $v \in Q_R$. This gives $\frac{dG_t}{du}(\frac{du}{dt}) = -\frac{dG_t}{dv}(\frac{du}{dt}) \cdot \frac{dg_0}{dv}(u)$. Then (2) becomes
\[
\frac{dF_t}{dt}(x) = -t \frac{dG_t}{dx}(\frac{du}{dv}(x)) \cdot \frac{dg_0}{dv}(u) = \frac{dG_t}{dv}(G_t(x)) - \frac{dg_0}{dv}(G_t(x)) > .
\]

Which is equivalent to
\[
\frac{dF_t}{dt}(x) = -t \frac{dG_t}{dx}(\frac{du}{dv}(x)) \cdot \left(\frac{dg_0}{dv}(G_t(x)) - \frac{dg_0}{dv}(G_t(x))\right) > .
\]

The function $f := \log \left\| \frac{x}{1} \right\|_{\mathcal{T}_0}$ is smooth and $S_Q$-invariant on $X$. Then by (3.2) we have
\[
\frac{dG_t}{dx}(G_t(x)) \cdot \left(\frac{dg_0}{dv}(G_t(x)) + \frac{dg_0}{dv}(G_t(x))\right) > dx \wedge d\theta = df \wedge d^c f \wedge c_1(\mathcal{D}_t)^n - 1
\]

which is absolutely integrable. Therefore,
\[
\frac{d}{dt} \int_{\Delta_D} F_t dx = \int_{\Delta_D} \frac{dF_t}{dt} dx = -t \int_X d(\log \left\| \frac{x}{1} \right\|_0) \wedge d^c(\log \left\| \frac{x}{1} \right\|_0) \wedge c_1(\mathcal{D}_t)^n - 1 \quad \forall t \in [0, 1].
\]

With similar arguments we can establish that $\int_{\Delta_D} (g_1 - g_0)(G_t(x)) dx$ is also differentiable on $[0, 1]$. We conclude that
\[
t \mapsto \int_{\Delta_D} \dot{g}_t dx,
\]
is differentiable, and we have
\[
\frac{d}{dt} \left(\int_{\Delta_D} \dot{g}_t dx\right)_{t=0^+} = -\int_{\Delta_D} (g_1 - g_0)(G_0(x)) dx.
\]

Theorem 3.4. Let $\mathcal{O}(D)$ be an ample line bundle on $X$. Let $\left\| \cdot \right\|_{\mathcal{T}_1}$ and $\left\| \cdot \right\|_{\mathcal{T}_0}$ be two smooth, $S_Q$-invariant and positive hermitian metrics on $\mathcal{O}(D)$. We have,
\[
\mathcal{E}(\mathcal{D}_1) - \mathcal{E}(\mathcal{D}_0) = -\int_{\Delta_D} (\dot{\mathcal{E}}_{\mathcal{T}_1}(x) - \dot{\mathcal{E}}_{\mathcal{T}_0}(x)) dx.
\]

Moreover, this equality extends to admissible metrics.
Proof. Let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be two smooth, positive and \( S_D \)-invariant hermitian metrics on \( O(D) \). Let \( s \in [0,1] \) and let \( \| \cdot \|_s \) be the metric defined by \( g_s := (1-s)g_{\| \cdot \|_1} + sg_{\| \cdot \|_2} \). This metric is smooth, positive and \( S_D \)-invariant. We denote by \( \overline{D}_s \) the metrized toric divisor \( D \) endowed with the metric \( \| \cdot \|_s \). Applying Proposition 3.3 one get for any \( s \in [0,1] \)

\[
\frac{d}{dt} \left( \int_{\Delta_D} \gamma \right)_{|t=s} = - \int_{\Delta_D} (g_1 - g_0) (G_s(x)) dx.
\]

From the definition of the Monge-Ampère functional, one get easily

\[
\frac{d}{dt} (E(\overline{D}_s) - E(\overline{D}_0))_{|t=s} = \int_X (\log \| \cdot \|_0 - \| \cdot \|_s) c_1 (L, \| \cdot \|_s)^n.
\]

Now, by using the change of variables \( u = G_s(x) \). We get \( \int_X (\log \| \cdot \|_0 - \| \cdot \|_s) c_1 (L, \| \cdot \|_s) = - \int_{\Delta_D} (g_1 - g_0) (G_s(x)) dx \). Thus,

\[
\frac{d}{dt} (E(\overline{D}_s) - E(\overline{D}_0))_{|t=s} = \frac{d}{dt} \left( \int_{\Delta_D} \gamma \right)_{|t=s}, \quad \forall s \in [0,1].
\]

Therefore

\[
E(\overline{D}_1) - E(\overline{D}_0) = - \int_{\Delta_D} (\gamma_{\overline{D}_1} - \gamma_{\overline{D}_0}) dx.
\]

(3)

Suppose now that \( \| \cdot \|_0 \) and \( \| \cdot \|_1 \) are admissible. By definition, there exists \((\| \cdot \|_i)_n \) a sequence of smooth and semipositive metrics on \( L \) converging uniformly to \( \| \cdot \|_i \), \( i = 1, 2 \). By Proposition 3.1 we can assume that the metrics of the sequences are \( S_Q \)-invariant. Moreover, we can also suppose that \( \| \cdot \|_i \) is positive for all \( n \in \mathbb{N} \) and \( i = 0, 1 \). Indeed, let \( \| \cdot \|' \) be a smooth, positive and \( S_D \)-invariant metric on \( O(D) \) (for example one can take the pull-back of the Fubini-Study by an equivariant morphism defined by \( D \)) then \( \| \cdot \|_i \) is positive, smooth and \( S_Q \)-invariant. We have (3) holds for \( g_i = g_{a, n} \) for any \( n \in \mathbb{N} \) and \( i = 0, 1 \). By the theory of Beford-Taylor the LHS converges to \( E(\overline{D}_1) - E(\overline{D}_0) \). By (2.3) the RHS converges to \( \int_{\Delta_D} (\gamma_{\overline{D}_1} - \gamma_{\overline{D}_0}) dx \).

Proposition 3.5. Let \( \overline{D} \) be a toric metrized divisor and \( \overline{A} \) a toric metrized divisor such that \( A \) is effective. We have

\[
\lim_{l \in \mathbb{N}, l \to \infty} \frac{1}{l} \int_{\Delta_D + \Delta_A} (\gamma_{\overline{D}} + \gamma_{\overline{A}}) dx = \int_{\Delta_D} \gamma_{\overline{D}} dx.
\]

(dx denotes the standard Lebesgue measure on \( Q_2 \)).

Proof. We set \( \gamma_{\overline{D} + \overline{A}} := \gamma_{\overline{D}} + \gamma_{\overline{A}} \) and \( \gamma_{\overline{D} + \frac{1}{l} \overline{A}} := \gamma_{\overline{D}} + \frac{1}{l} \gamma_{\overline{A}} \) for any \( l \in \mathbb{N}^* \).

The assumption that \( \overline{D} \) and \( \overline{A} \) are toric metrized divisors implies that \( \sup_{v \in Q_2} |\gamma_{\overline{D}}(v) - \gamma_{\overline{D} + \frac{1}{l} \overline{A}}(v)| \) are finite. There exists a constant \( C \) such that \( |\gamma_{\overline{D} + \frac{1}{l} \overline{A}}| \leq C \) on \( \Delta_D + \frac{1}{l} \Delta_A \) for any \( l \in \mathbb{N} \). This follows from the following inequality

\[
|\gamma_{\overline{D} + \frac{1}{l} \overline{A}}| \leq \sup_{v \in Q_2} |\gamma_{\overline{D}}(v) - \gamma_{\overline{D} + \frac{1}{l} \overline{A}}(v)| + \frac{1}{l} \sup_{v \in Q_2} |\gamma_{\overline{A}}(v) - \gamma_{\overline{D} + \frac{1}{l} \overline{A}}(v)|,
\]

on \( \Delta_D + \frac{1}{l} \Delta_A \) which is a consequence of Proposition 2.3 combined with the fact that \( \gamma_{\overline{D} + \frac{1}{l} \overline{A}} = 0 \) on \( \Delta_D + \frac{1}{l} \Delta_A \). Notice that \( 0 \in \Delta_A \) because \( A \) is effective (this follows from Proposition 2.1 (v)). Then \( \gamma_{\overline{A}}(0) \) is finite and it follows that \( \gamma_{\overline{A}} \) is bounded from above. If we multiply the metric of \( \overline{A} \) by a positive constant, then it is possible to assume that \( \gamma_{\overline{A}} \leq 0 \). Observe that the assertion of the proposition remains true.
By an obvious change of variables, we have

\[ \frac{1}{ln+1} \int_{\Delta_D + \Delta_A} \tilde{g}_D \tilde{\eta} dx = \int_{\Delta_D + \Delta_A} \tilde{g}_D \tilde{\eta} dx = \int_{\Delta_D} \tilde{g}_D \tilde{\eta} dx + \int_{(\Delta_D + \Delta_A) \setminus \Delta_D} \tilde{g}_D \tilde{\eta} dx. \]

Fix \( x \in \Delta_D \). Since \( 0 \in \Delta_A \) and \( \tilde{\eta} \leq 0 \), then \( \tilde{g}_D \tilde{\eta} \) \( x \geq \tilde{g}_D \tilde{\eta} (x) \geq \tilde{g}_D (x) \) for any \( l \leq l' \) in \( \mathbb{N}^* \).

Let \( u \in Q_\mathbb{R} \) such that \( \tilde{g}_D (x) = x \cdot u \). Then

\[ \tilde{g}_D \tilde{\eta} (x) \leq \tilde{g}_D (x) - \frac{1}{l} \tilde{g}_D (u) \quad \forall l \in \mathbb{N}^*. \]

Therefore \( \tilde{g}_D \tilde{\eta} \) is a decreasing function converging pointwise to \( \tilde{g}_D \) on \( \Delta_D \). Since \( 0 \in \Delta_A \) and \( |\tilde{g}_D \tilde{\eta}| \leq C \) on \( \Delta_D \) we conclude (by using the Fatou-Lebesgue theorem) that

\[ \lim_{l \to \infty} \int_{\Delta_D} \tilde{g}_D \tilde{\eta} = \int_{\Delta_D} \tilde{g}_D. \]

On other hand, we have

\[ \int_{(\Delta_D + \Delta_A) \setminus \Delta_D} \tilde{g}_D \tilde{\eta} \leq C \text{vol}((\Delta_D + \frac{1}{l} \Delta_A) \setminus \Delta_D). \]

Therefore

\[ \lim_{l \to \infty} \int_{\Delta_D + \Delta_A} \tilde{g}_D \tilde{\eta} = \int_{\Delta_D} \tilde{g}_D. \]

**Corollary 3.6.** Let \( \mathcal{O}(D) \) be a big and nef line bundle on \( X \). Let \( \| \cdot \|_2^\mathbb{R} \) and \( \| \cdot \|_0^\mathbb{R} \) be two admissible and \( \mathbb{S}_Q \)-invariant hermitian metrics on \( \mathcal{O}(D) \) and set \( \tilde{D}_i = (D, \| \cdot \|_i^\mathbb{R}) \) for \( i = 0, 1 \). We have,

\[ \mathcal{E}(\tilde{D}_1) - \mathcal{E}(\tilde{D}_0) = -\int_{\Delta_D} (\tilde{g}_D (x) - \tilde{g}_{D_0}(x)) dx. \]

**Proof.** Let \( \tilde{A} \) be a positive and smooth toric metrized divisor. We have \( \tilde{A} + \tilde{D}_i \) is a positive continous toric metrized divisor, for \( i = 0, 1 \) and any \( l \in \mathbb{N} \). Then by Theorem [3.3]

\[ \mathcal{E}(\tilde{A} + l\tilde{D}_1) - \mathcal{E}(\tilde{A} + l\tilde{D}_0) = -\int_{\Delta_A + lD} (\tilde{g}_{\tilde{A} + l\tilde{D}_1} (x) - \tilde{g}_{\tilde{A} + l\tilde{D}_0}(x)) dx \quad \forall l \in \mathbb{N}. \]

We have \( \mathcal{E}(\tilde{A} + \tilde{D}_1) - \mathcal{E}(\tilde{A} + \tilde{D}_0) = ln^+ (\mathcal{E}(\tilde{D}_1) - \mathcal{E}(\tilde{D}_0)) + O(l^n), \forall l \in \mathbb{N} \). Since \( A \) is effective and by Proposition [3.5] we conclude that

\[ \mathcal{E}(\tilde{D}_1) - \mathcal{E}(\tilde{D}_0) = -\int_{\Delta_D} (\tilde{g}_D (x) - \tilde{g}_{D_0}(x)) dx. \]

**Lemma 3.7.** Let \( \Theta \) be a convex compact subset in \( \mathbb{R}^n \) such that \( \text{vol}(\Theta) > 0 \) (\text{vol} denotes the volume induced by the standard Lebesgue measure \( dx \) of \( \mathbb{R}^n \)) and let \( \varphi \) be a bounded concave function on \( \Theta \). We have,

\[ \lim_{l \to \infty} \frac{1}{|P(l)|} \sum_{e_j \in \{e_j \}} \varphi(e_j) = \int_{\Theta} \varphi. \]
Proof. Let $\varepsilon > 0$. There exists $\Theta'$ a convex compact subset in $\text{Int}(\Theta)$ such that $\text{vol}(\Theta \setminus \Theta') < \varepsilon$. By [9, Theorem 10.1], the concave function $\varphi$ is continuous on $\Theta'$. Then $\lim_{l \to \infty} \frac{1}{l^n} \sum_{e \in \Theta\cap P} \varphi\left(\frac{e}{l}\right) - \int_{\Theta} \varphi \leq \varepsilon$ for $l \gg 1$. We have

$$\left| \frac{1}{l^n} \sum_{e \in \Theta \cap P} \varphi\left(\frac{e}{l}\right) - \int_{\Theta} \varphi \right| \leq \left| \frac{1}{l^n} \sum_{e \in \Theta \cap P} \varphi\left(\frac{e}{l}\right) - \int_{\Theta'} \varphi \right| + \left| \int_{\Theta} \varphi - \int_{\Theta'} \varphi \right| + \left| \frac{1}{l^n} \sum_{e \notin \Theta \cap P} \varphi\left(\frac{e}{l}\right) \right|$$

$$\leq \left| \frac{1}{l^n} \sum_{e \in \Theta \cap P} \varphi\left(\frac{e}{l}\right) - \int_{\Theta'} \varphi \right| + \sup_{x \in \Theta} |\varphi(\Theta \setminus \Theta')| + \sup_{x \in \Theta} \left| \left( \frac{1}{l^n} \#(\Theta \cap P) - \frac{1}{l^n} \#(\Theta' \cap P) \right) \right|,$$

and since $\lim_{l \to \infty} \frac{1}{l^n} \#(\Theta \cap P) = \text{vol}(\Theta)$ and $\lim_{l \to \infty} \frac{1}{l^n} \#(\Theta' \cap P) = \text{vol}(\Theta')$ we conclude that

$$\left| \frac{1}{l^n} \sum_{e \in \Theta \cap P} \varphi\left(\frac{e}{l}\right) - \int_{\Theta} \varphi \right| = O(\varepsilon) \quad \forall l \gg 1.$$  

\[ \square \]

**Lemma 3.8.** Let $\Theta$ be a convex compact subset in $\mathbb{R}^n$ such that $\text{vol}(\Theta) > 0$. For any $l \in \mathbb{N}^*$, let $A_l = (a_{e,e})_{e,e \in \Theta \cap P}$ be a positive symmetric matrix indexed by $l\Theta \cap P$, and let $K_l$ be a subset of $\mathbb{R}^{l\Theta \cap P} \simeq \mathbb{R}^{|(\Theta \cap P)|}$ given by

$$K_l = \{(x_{e}) \in \mathbb{R}^{l\Theta \cap P} \mid \sum_{e,e \in \Theta \cap P} a_{e,e}x_{e}x_{e} \leq 1\}.$$

We assume there is an integrable function $\varphi : \Theta \to \mathbb{R}$ such that for any $\varepsilon > 0$, there exists a constant $D$ verifying

$$|\varphi\left(\frac{e}{l}\right)| - l\varphi\left(\frac{e}{l}\right)| \leq D + \varepsilon l,$$

for any $l \gg 1$ and $e \in l\Theta \cap P$. Then, we have

$$\lim_{l \to \infty} \frac{1}{l^n} \sum_{e \in \Theta \cap P} \log\left(\frac{1}{a_{e,e}}\right) = \int_{\Theta} \varphi(x)dx.$$

**Proof.** Let $\varepsilon > 0$. By assumption, there exists constant $D$ such that

$$\varphi\left(\frac{e}{l}\right) - \frac{1}{l}D - \varepsilon \leq \frac{1}{l} \log\left(\frac{1}{a_{e,e}}\right) \leq \varphi\left(\frac{e}{l}\right) + \frac{1}{l}D + \varepsilon \quad \forall l \gg 1 \quad \forall e \in l\Theta \cap P.$$

Then

$$\frac{1}{l^n} \sum_{e \in \Theta \cap P} \varphi\left(\frac{e}{l}\right) - \frac{m_l}{l^{n+1}}D - \frac{m_l}{l^n} \varepsilon \leq \frac{1}{l^n} \sum_{e \in \Theta \cap P} \log\left(\frac{1}{a_{e,e}}\right) \leq \frac{1}{l^n} \sum_{e \in \Theta \cap P} \varphi\left(\frac{e}{l}\right) + \frac{m_l}{l^{n+1}}D + \frac{m_l}{l^n} \varepsilon \quad \forall l \gg 1.$$

where $m_l = \#(l\Theta \cap P)$. Note that

$$\lim_{l \to \infty} \frac{1}{l^n} \sum_{e \in \Theta \cap P} \varphi\left(\frac{e}{l}\right) = \lim_{l \to \infty} \sum_{x \in \Theta \cap (1/l)P} \varphi(x) = \int_{\Theta} \varphi(x)dx. \quad (4)$$

Then, we can find $l_0 \gg 1$ such that

$$\left| \frac{1}{l^{n+1}} \sum_{e \in \Theta \cap P} \log\left(\frac{1}{a_{e,e}}\right) - \int_{\Theta} \varphi(x)dx \right| \leq \varepsilon + \frac{m_l}{l^{n+1}}D + \frac{m_l}{l^n} \varepsilon \quad \forall l \geq l_0.$$
Since \( m_l = O(l^n) \). We can deduce that,
\[
\frac{1}{l^n + 1} \sum_{e \in \Theta \cap \Theta} \log \left( \frac{1}{a_{e,e}} \right) - \int_{\Theta} \varphi(x) dx = O(\varepsilon) \quad \forall l \geq l_0.
\]

Then
\[
\lim_{l \to \infty} \frac{1}{l^n + 1} \sum_{e \in \Theta \cap \Theta} \log \left( \frac{1}{a_{e,e}} \right) = \int_{\Theta} \varphi(x) dx.
\]

**4 The Proof of the Main Theorem**

Let \( X \) be a complex projective nonsingular toric variety and \( D \) a toric divisor on \( X \) such that \( \mathcal{O}(D) \) is big and nef. Let \( \| \cdot \|_{\overline{\mathcal{P}_e}} \) and \( \| \cdot \|_{\overline{\mathcal{P}_{j}}} \) be two continuous toric metrics on \( D \) and \( \overline{\mathcal{D}_i} := (D, \| \cdot \|_{\overline{\mathcal{P}_e}}) \) and \( \varphi_{\overline{\mathcal{D}_i}} \) the associated weight for \( i = 0, 1 \). Let \( \mu_j \) be a probability measure \( \mathbb{S}_{\mathcal{Q}} \)-invariant on \( X \) with the Bernstein-Markov property with respect to \( \| \cdot \|_{\overline{\mathcal{P}_j}}, j = 0, 1 \). We denote by \( \overline{\mathcal{D}_i, X} \) the metrized toric divisor endowed with the weight \( P_{X, \varphi_{\overline{\mathcal{D}_i}}} \) for \( i = 0, 1 \).

**Proposition 4.1.** We have,
\[
\lim_{k \to \infty} \mathcal{L}_k(\mu_1, \overline{\mathcal{D}_1}) - \mathcal{L}_k(\mu_0, \overline{\mathcal{D}_0}) = -\frac{1}{\text{vol}(D)} \int_{\Delta_D} (\varphi_{\overline{\mathcal{D}_1}} - \varphi_{\overline{\mathcal{D}_0}}) dx. \tag{5}
\]

**Proof.** For any \( e \in \Delta_D \cap P \), we set \( s_e := \| x^e \|_{L^2(\mu_1, \overline{\mathcal{D}_1})}^{-1} \chi^e \) (\( \chi^e \) is the global section of \( \mathcal{O}(D) \) defined by \( e \)). Recall that \( H^0(X, \mathcal{O}(D)) = \bigoplus_{e \in \Delta_D \cap P} \mathbb{C} \chi^e \). Since \( \mu_0 \) and \( \mu_1 \) are \( \mathbb{S}_{\mathcal{Q}} \)-invariant, then \( \{ s_e \mid e \in \Delta_D \cap P \} \) is an \( L^2(\mu_1, \overline{\mathcal{D}_1}) \)-orthonormal basis of \( H^0(X, L) \) and we have
\[
\frac{\text{vol}_{L^2(\mu_1, \overline{\mathcal{D}_1})}(B(\mu_1, \overline{\mathcal{D}_1}))}{\text{vol}_{L^2(\mu_0, \overline{\mathcal{D}_0})}(B(\mu_0, \overline{\mathcal{D}_0}))} = \det((s_e, s_{e'})_{L^2(\mu_0, \overline{\mathcal{D}_0}))_{e, e' \in \Delta_D \cap P} = \prod_{e \in \Delta_D \cap P} (s_e, s_e)_{L^2(\mu_0, \overline{\mathcal{D}_0})}. \]

Then for any \( k \in \mathbb{N}^* \),
\[
\mathcal{L}_k(\mu_1, \overline{\mathcal{D}_1}) - \mathcal{L}_k(\mu_0, \overline{\mathcal{D}_0}) = \frac{1}{kN_k} \log \prod_{e \in k\Delta_D \cap P} (s_e, s_e)_{L^2(\mu_0, \overline{\mathcal{D}_0})},
\]
\[(N_k := \dim H^0(X, \mathcal{O}(kD))). \]

Since \( (\mu_0, \overline{\mathcal{D}_0}) \) and \( (\mu_1, \overline{\mathcal{D}_1}) \) satisfy the Bernstein-Markov property, then for any \( \varepsilon > 0 \) there exists a constant \( D \) such that
\[
\left| \log(s_e, s_e)_{L^2(\mu_0, \overline{\mathcal{D}_0})} - k(\varphi_{\overline{\mathcal{D}_1}} - \varphi_{\overline{\mathcal{D}_0}})(\frac{e}{k}) \right| \leq D + k\varepsilon, \quad \forall e \in k\Delta_D \cap P, \forall k \gg 1.
\]

(Notice that we use the fact \( \| s_e \|_{\sup, k\mathcal{P}_e} = e(\frac{k}{k}) \), see the proof of lemma 2.3). Now let \( \Theta := \Delta_D \) and \( \phi := \varphi_{\overline{\mathcal{D}_1}} - \varphi_{\overline{\mathcal{D}_0}} \). They satisfy the assumptions of lemma 3.8 (More precisely, \( \phi \) satisfies 3.1 which is a consequence of lemma 3.7). This concludes the proof of the proposition. \( \square \)

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4.1 The Case of Ample Divisor

Suppose that $D$ is ample. We know that $\| \cdot \|_{\mathcal{L}_A}$ is a continuous and semipositive metric on $D$ for $i = 0, 1$. By [11, Theorem 4.6.1], this metric is admissible. Using Corollary 3.6 we deduce that

$$\mathcal{E}(\mathcal{D}_{1X}) - \mathcal{E}(\mathcal{D}_{0X}) = - \int_{\vartriangle_D} (\tilde{g}_{\mathcal{D}_{1X}}(x) - \tilde{g}_{\mathcal{D}_{0X}}(x))dx.$$ 

Thus,

$$\mathcal{E}(\mathcal{D}_{1X}) - \mathcal{E}(\mathcal{D}_{0X}) = - \int_{\vartriangle_D} (\tilde{g}_{\mathcal{D}_{1}}(x) - \tilde{g}_{\mathcal{D}_{0}}(x))dx,$$

by lemma 2.4. Now by Proposition 4.1 we get Lemma 4.2.

(First notice that the limit of the sequence of continuous weight

$$\phi$$

the continuous weight

$$\phi$$

satisfying

$$\phi$$

is also a psh weight on $D$.

By [7, Theorem 4.6.1], this metric is admissible. Using Corollary 3.6, we deduce that

$$\mathcal{E}(\mathcal{D}_{1X}) - \mathcal{E}(\mathcal{D}_{0X}) = - \int_{\vartriangle_D} (\tilde{g}_{\mathcal{D}_{1X}}(x) - \tilde{g}_{\mathcal{D}_{0X}}(x))dx.$$ 

Thus, we proved Theorem 1.2 for ample divisors.

4.2 The Case of Big and Nef Divisor

Let $\mathcal{D}$ a metrized toric divisor such that $D$ is big and nef. Let $\mathcal{A}$ be a positive metrized toric divisor on $X$. Let $l > 0$ and we set $\phi_l := P_X(\phi_{\mathcal{A}} + \frac{1}{l}\phi_A) - \frac{1}{l}\phi_A$ where $\phi_{\mathcal{A}}$ (resp. $\phi_A$) denotes the associated weight to $\mathcal{A}$ (resp. $A$). We have $\phi_l$ defines a continuous metric on $D$. Indeed, this is follows from the fact that $lD + A$ is ample for any $l \in \mathbb{N}$ (because $D$ is nef and $A$ is ample), which implies that $P_X(l\phi_{\mathcal{A}} + \phi_A)$ is a continuous weight on $lD + A$. We denote by $\mathcal{D}_l$ the continuous metrized toric divisor $D$ endowed with the continuous weight $\phi_l$.

Lemma 4.2. $(\phi_l)_{l \in \mathbb{N}^*}$ is a decreasing sequence of continuous weights on $\mathcal{O}(D)$ converging pointwise to $P_X \phi$.

Proof. First notice that the limit of the sequence $(\phi_l)_{l \in \mathbb{N}^*}$, if it exists, doesn’t depend on the choice of the metric on $A$. Indeed, let $\phi_{1,A}$ and $\phi_{0,A}$ be two weights on $A$ defining continuous metrics and we set $\phi_{l,i} := P_X(\phi_{\mathcal{A}} + l\phi_{A}) - \frac{1}{l}\phi_{A}$ for $i = 0, 1$. Then by Proposition 2.1 we have

$$|\phi_{l,i} - \phi_{0,A}| \leq \frac{2}{\vartriangle} \sup_{x \in X} |\phi_{1,A} - \phi_{0,A}|, \quad \forall l \in \mathbb{N}^*.$$

Suppose that $\phi_A$ is psh. Let $\psi$ be a psh weight on $D$ with $\psi \leq \phi_{\mathcal{A}}$ then $\psi + \frac{1}{l}\phi_A$ is also a psh weight satisfying $\psi + \frac{1}{l}\phi_A \leq P_X(\phi_{\mathcal{A}} + \frac{1}{l}\phi_A)$. Therefore, $\psi \leq P_X(\phi_{\mathcal{A}} + \frac{1}{l}\phi_A) - \frac{1}{l}\phi_A$ for any $\psi$ a psh weight on $D$ such that $\psi \leq \phi_{\mathcal{A}}$. Thus

$$P_X \phi_{\mathcal{A}} \leq \phi_l \leq \phi_{\mathcal{A}} \quad \forall l \in \mathbb{N}^*.$$

Let $l \geq k$, then clearly $P_X(\phi_{\mathcal{A}} + \frac{1}{l}\phi_A) + (\frac{1}{l} - \frac{1}{k})\phi_A \leq P_X(\phi_{\mathcal{A}} + \frac{1}{k}\phi_A)$. So

$$\phi_l \leq \phi_k \quad \forall l \geq k.$$

If we set $\Psi := \lim_{l \to \infty} \phi_l$, then

$$P_X \phi \leq \Psi \leq \phi_{l+1} \leq \phi_l \leq \phi_{\mathcal{A}} \quad \forall l \in \mathbb{N}^*.$$ 

(6)

Let $k \in \mathbb{N}^*$, we have $\phi_l + \frac{1}{k}\phi_A$ is psh for any $l \geq k$. Then $\Psi + \frac{1}{k}\phi_A$ is also a psh function for any $k \in \mathbb{N}^*$ (see for instance [11, Theorem 5.4]). It follows that $\Psi$ is psh weight on $D$. We conclude that

$$P_X \phi_{\mathcal{A}} = \Psi.$$
Recall that $\phi_l$ is continuous for any $l \in \mathbb{N}^*$. We have $(\phi_l + \frac{1}{k} \phi_A)_{l \geq k}$ is a decreasing sequence of continuous psh functions converging pointwise to $P_X \phi^E_T + \frac{1}{k} \phi_A$ with minimal singularities. Then as $l$ tends to $\infty$
\[ \mathcal{E}(\mathcal{D}_l + \frac{1}{k} \mathcal{A}) \to \mathcal{E}(\mathcal{D}_X + \frac{1}{k} \mathcal{A}), \tag{7} \]
for any $k \in \mathbb{N}^*$. The proof of the last assertion is similar to [24 Proposition 4.3] which is a consequence of the continuity of the Monge-Ampère operator, see [4].

Let $\phi_{0,D}$ be a continuous and $\mathcal{S}_Q$-invariant psh weight on $D$ and we set $\mathcal{D}$ the metrized toric $(D, \phi_{0,D})$. We set $g_{k,l} := \mathcal{D}_{k,l}$ for any $k, l \in \mathbb{N}^*$. Then by [24] and (6) we get
\[ g_{k,l} \leq \mathcal{D}_{k,l+1} \leq g_{k,l} \leq \mathcal{D}_{k,l+1}. \tag{8} \]

By Theorem 3.3 we have
\[ \mathcal{E}(\mathcal{D}_l + \frac{1}{k} \mathcal{A}) - \mathcal{E}(\mathcal{D}' + \frac{1}{k} \mathcal{A}) = - \int_{\Delta_D + \frac{1}{k} \mathcal{A}} (\mathcal{D}_l(x) - \mathcal{D}_{k,l}(x)) dx. \]
Now using (5), get for any $l, k \in \mathbb{N}^*$,
\[ - \int_{\Delta_D + \frac{1}{k} \mathcal{A}} (\mathcal{D}_l - \mathcal{D}_0) dx \leq \mathcal{E}(\mathcal{D}_l + \frac{1}{k} \mathcal{A}) - \mathcal{E}(\mathcal{D}' + \frac{1}{k} \mathcal{A}) \leq - \int_{\Delta_D + \frac{1}{k} \mathcal{A}} (\mathcal{D}_{k,l} - \mathcal{D}_0) dx. \]
As $l$ tends to $\infty$, we have by (7)
\[ - \int_{\Delta_D} (\mathcal{D}_l - \mathcal{D}_0) dx \leq \lim_{k} \mathcal{E}(\mathcal{D}_X + \frac{1}{k} \mathcal{A}) - \mathcal{E}(\mathcal{D}' + \frac{1}{k} \mathcal{A}) \]
\[ \leq \lim_{k} \sup_{k} (\mathcal{D}_X + \frac{1}{k} \mathcal{A}) - \mathcal{E}(\mathcal{D}' + \frac{1}{k} \mathcal{A}) \leq - \int_{\Delta_D} (\mathcal{D}_X - \mathcal{D}_0) dx. \]
Using lemma 2.4 we deduce that
\[ \lim_{k} \mathcal{E}(\mathcal{D}_X + \frac{1}{k} \mathcal{A}) - \mathcal{E}(\mathcal{D} + \frac{1}{k} \mathcal{A}) = - \int_{\Delta_D} (\mathcal{D} - \mathcal{D}_0) dx. \tag{9} \]
Since $P_X \phi^E_T - \phi_{\infty,D}$ is bounded then, by lemma 1.3 below, there exists a constant $C$ such that
\[ |\mathcal{E}(\mathcal{D}_X + \frac{1}{k} \mathcal{A}) - \mathcal{E}(\mathcal{D} + \frac{1}{k} \mathcal{A})| \leq \frac{C}{k} \quad \forall k \in \mathbb{N}^*. \]

Then
\[ \liminf_{l \to \infty} \mathcal{E}(\mathcal{D}_X + \frac{1}{k} \mathcal{A}) - \mathcal{E}(\mathcal{D} + \frac{1}{k} \mathcal{A}) = \mathcal{E}(\mathcal{D}_X) - \mathcal{E}(\mathcal{D}). \]

Combined with (9), we obtain
\[ - \int_{\Delta_D} (\mathcal{D}_X(x) - \mathcal{D}_0(x)) dx = \mathcal{E}(\mathcal{D}_X) - \mathcal{E}(\mathcal{D}_0). \]

We conclude that,
\[ \lim_{k \to \infty} \mathcal{L}_k(\mu_1, \mathcal{D}_1) - \mathcal{L}_k(\mu_0, \mathcal{D}_0) = \mathcal{E}_q(\mathcal{D}_1) - \mathcal{E}_q(\mathcal{D}_0). \]

This ends the proof of Theorem 1.2.
Lemma 4.3. Let $D$ be a big and nef divisor and $A$ an ample divisor on $X$. Let $\psi_D$ and $\phi_{0,D}$ be two psh weight with minimal singularities on $D$. Let $\phi_A$ and $\phi_{0,A}$ be two positive continuous weight on $A$. We assume that $\phi_{0,D}, \phi_A$ and $\phi_{0,A}$ are continuous and $\psi_D - \phi_{\infty,D}$ is bounded. Then there exists a real polynomial $P$ of degree $\leq n$ depending only on $\phi_A, \psi_D, \phi_{0,D}$ such that

$$ |\mathcal{E}(kD\psi + A) - \mathcal{E}(kD' + A) - k^{n+1}(\mathcal{E}(D\psi) - \mathcal{E}(D'))| \leq P(k) \quad \forall k \in \mathbb{N}, $$

where $D, D'$ and $A$ are the metrized toric divisors endowed with $\psi_D, \phi_{0,D}$ and $\phi_A$ respectively.

Proof. We have

$$ \mathcal{E}(kD\psi + A) - \mathcal{E}(kD' + A) = \sum_{i=0}^{n} \int_X (k(\psi_D - \phi_{0,D})) dd^c(k(\psi_D + \phi_A))^i dd^c(k\phi_{0,D} + \phi_A)^{n-i} $$

$$ = k^{n+1}(\mathcal{E}(D\psi) - \mathcal{E}(D')) + \int_X (\psi_D - \phi_{0,D}) T, $$

where $T$ is a linear sum of terms of the form $dd^c(\phi_A)^{\alpha} dd^c(\phi_{0,D})^{\beta} dd^c(\psi_D)_{\gamma} k^\varepsilon$ with $\alpha, \beta, \gamma, \varepsilon \in \mathbb{N}$ and $\alpha + \beta + \gamma = n$ and $\varepsilon \leq n$.

We set $f_D := \psi_D - \phi_{0,D}$. This function is bounded since we assume that $\psi_D - \phi_{\infty,D}$ is bounded and $\phi_{0,D}$ is continuous. Then,

$$ |\int_X f_D dd^c(\phi_A)^{\alpha} dd^c(\phi_{0,D})^{\beta} dd^c(\psi_D)_{\gamma}| \leq \sup_X |f_D| \int_X c_1(A)^{\alpha} c_1(D)^{\beta+\gamma}. $$

The lemma follows from the last inequality.

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