AN ARAKELOV THEORETIC PROOF OF THE EQUALITY OF CONDUCTOR AND DISCRIMINANT

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1. Introduction

Let $K$ be a number field, $\mathcal{O}_K$ be the ring of integers of $K$, and $S$ be $\text{Spec}(\mathcal{O}_K)$. Let $f : X \to S$ be an arithmetic surface. By this we mean a regular scheme, proper and flat over $S$, of relative dimension one. We also assume that the generic fiber of $X$ has genus $\geq 1$, and that $X/S$ has geometrically connected fibers.

Let $\omega_X$ be the dualizing sheaf of $X/S$. The Mumford isomorphism ([Mumf], Theorem 5.10)

$$\det Rf_*(\omega_X^\otimes 2) \otimes K \to (\det Rf_*(\omega_X))^{\otimes 13} \otimes K,$$

which is unique up to sign, gives a rational section $\Delta$ of

$$(\det Rf_*(\omega_X))^{\otimes 13} \otimes (\det Rf_*(\omega_X^\otimes 2))^{\otimes -1}.$$

The discriminant $\Delta(X)$ of $X/S$ is defined as the divisor of this rational section ([Saito]). If $p$ is a closed point of $S$, we denote the coefficient of $p$ in $\Delta(X)$ by $\delta_p$.

On the other hand $X/S$ has an Artin conductor $\text{Art}(X)$ (cf. [Bloch]), which is similarly a divisor on $S$. We denote the coefficient of $p$ in $\text{Art}(X)$ by $\text{Art}_p$. Let $S'$ be the strict henselization of complete local ring at $p$, with field of fractions $K'$. Let $s$ be its special point, $\eta$ be its generic point, and $\eta$ be a geometric generic point corresponding to an algebraic closure $\bar{K}'$ of $K'$. Let $\ell$ be a prime different from the residue characteristic at $p$.

Then

$$\text{Art}_p(X) = \sum_{i \geq 0} (-1)^i \dim_{\ell} \mathcal{H}^i_{\text{ét}}(X_{\eta}, \mathbb{Q}_\ell) - \sum_{i \geq 0} (-1)^i \dim_{\ell} \mathcal{H}^i_{\text{ét}}(X_s, \mathbb{Q}_\ell) + \sum_{i \geq 0} (-1)^i \text{Sw}_{\bar{K}/K'}(\mathcal{H}^i_{\text{ét}}(X_{\eta}, \mathbb{Q}_\ell)),$$

where $\text{Sw}_{\bar{K}/K'}$ denotes the Swan conductor of the Galois representation of $\bar{K}'/K'$. Both of these divisors are supported on the primes of bad reduction of $X$. We give another proof of Saito's theorem ([Saito], Theorem 1) in the number field case.

Theorem 1. For any closed point $p \in S$, we have $\delta_p = -\text{Art}_p$.

Fix a Kähler metric on $X$, this gives metrics on $\Omega^1_{X_\nu}$'s, for each $\nu \in S(\mathbb{C})$. For a hermitian coherent sheaf $\mathcal{E}$, we endow $\det Rf_*(\mathcal{E})$ with its Quillen metric. The proof of the theorem has the following corollaries.

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Proposition 1. We have

\[ \deg \det R_f \omega_X = \frac{1}{12} \left[ \deg f_* (\hat{c}_1(\omega_X)^2) + \log \text{Norm}(\text{Art}(X)) \right] \\
[K : \mathbb{Q}] (g - 1)[2\zeta'(1) + \zeta(-1)], \]

with \( \zeta \) the Riemann zeta function.

Proposition 1 is an arithmetic analogue of Noether’s formula in which \( \det R_f \omega_X \) is endowed with the Quillen metric. Faltings [Falt] and Moret-Bailly [M-B] proved a similar formula for the Faltings metrics.

Proposition 2. We have

\[ \frac{1}{[K : \mathbb{Q}]} \sum_{\nu \in S(\mathbb{C})} \log \| \Delta_{\nu} \| = 12(1 - g)[2\zeta'(1) + \zeta(-1)]. \]

In particular, the norm of the Mumford isomorphism does not depend on the metric.

2. Proof

First we prove Proposition 1. By duality ([Deligne], Lemme 1.3), \( \deg \det R_f \omega_X = \deg \det R_f \mathcal{O}_X \). By the arithmetic Riemann-Roch theorem of Gillet and Soulé ([G-S], Theorem 7), we get

\[ \deg \det R_f \mathcal{O}_X = \deg f_* (\hat{Td}(\Omega^1_X)^{(2)}) - \frac{1}{2} \sum_{\nu \in S(\mathbb{C})} \int_{X_{\nu}} Td(T_{X_{\nu}}) R(T_{X_{\nu}}) \]

Here \( Td \) and \( R \) are the Todd and Gillet-Soulé genera respectively, and the upper script (2) denotes the degree 2 component. Applying the definitions of these characteristic classes we obtain

\[ \deg \det R_f \mathcal{O}_X = \frac{1}{12} \deg f_* (\hat{c}_1(\Omega^1_X)^2 + \hat{c}_2(\Omega^1_X)) + [K : \mathbb{Q}] (g - 1)[2\zeta'(1) + \zeta(-1)] \]

Let \( Z \) denote the union of singular fibers of \( f \), and let \( c^Z_2,X(\Omega^1_X) \) be the localized Chern class of \( \Omega^1_X \) with support in \( Z \) (cf. [Bloch], [Fulton]). Chinburg, Pappas, and Taylor ([CPT], Proposition 3.1) prove the formula

\[ \deg f_* (\hat{c}_2(\Omega^1_X)) = \log \text{Norm}(c^Z_2,X(\Omega^1_X)). \]

Combining this with the fundamental formula of Bloch ([Bloch], Theorem 1)

\[ -\text{Art}_p(X) = \deg_p c^Z_2,X(\Omega^1_X), \]

we obtain the desired formula. Note that, since \( \det \Omega^1_X = \omega_X, \hat{c}_1(\Omega^1_X) = \hat{c}_1(\omega_X). \) □

Taking degrees in the Mumford isomorphism gives

\[ 13 \deg \det R_f \omega_X = \deg \det R_f (\omega_X^{\otimes 2}) + \log \text{Norm}(\Delta(X)) - \sum_{\nu \in S(\mathbb{C})} \log \| \Delta_{\nu} \|. \]

The arithmetic Riemann-Roch theorem gives

\[ \deg \det R_f (\omega_X^{\otimes 2}) = \deg \det R_f \omega_X + \deg f_* (\hat{c}_1(\omega_X)^2). \]
Therefore we get

\[(1) \quad \deg \det R_f \omega_X = \frac{1}{12} [\deg f_*(\mathcal{C}_1(\omega_X)^2) + \log \text{Norm}(\Delta(X)) - \sum_{\nu \in S(\mathbb{C})} \log \|\Delta_\nu\|].\]

Subtracting (1) from the expression in the statement of Proposition 1, we obtain

\[(2) \quad \log \left(\frac{\text{Norm}(\Delta(X/S))}{\text{Norm}(-\text{Art}(X/S))}\right) = \sum_{\nu \in S(\mathbb{C})} \log \|\Delta_\nu\| + 12[K : \mathbb{Q}](g - 1)[2\zeta'(-1) + \zeta(-1)].\]

Now \(X_K\) has semistable reduction after a finite base change \(K' / K\). For semistable \(X'/S'\), both \(-\text{Art}_{p'}(X')\) ([Bloch]), and \(\delta_{p'}(X')\) ([Falt], Theorem 6) are equal to the number of singular points in the geometric fiber over \(p'\). Therefore \(-\text{Art}_{p'} = \delta_{p'},\) and hence

\[(3) \quad \text{Norm}(\Delta(X'/S')) = \text{Norm}(-\text{Art}(X'/S')).\]

Applying this to a semistable model \(X'\) of \(X \otimes_K K'\), and noting that the base change multiplies the right hand side of (2) by \([K' : K]\), we see that the right hand side of (2) is equal to zero, and hence that the equality

\[(4) \quad \text{Norm}(\Delta(X/S)) = \text{Norm}(-\text{Art}(X/S))\]

holds for \(X\).

To prove the equality \(\delta_p = -\text{Art}_p\) for an arbitrary closed point \(p \in S\), we will use the following lemma.

**Lemma 1.** Fix distinct closed points \(\beta_1, .., \beta_s \in S\). And fix finite extensions \(L_i\) of the completions \(K_i\) of \(K\) at \(\beta_i\)'s, for each \(1 \leq i \leq s\), such that \([L_i : K_i] = n\) for some \(n\). Then there exists an extension \(L/K\) such that, for each \(1 \leq i \leq s\), there is only one prime \(\gamma_i\) of \(L\) lying over \(\beta_i\), and the completion of \(L\) at \(\gamma_i\) is isomorphic (over \(K_i\)) to \(L_i\).

**Proof.** The proof is an easy application of Krasner’s lemma, and the approximation lemma. Details are omitted. \(\square\)

Take \(p = \beta_1\), a prime of bad reduction. Denote the remaining primes of bad reduction by \(\beta_i, 2 \leq i \leq s\). Choose extensions \(L_i\) of the local fields \(K_i\) for all \(1 \leq i \leq s\), such that \(L_1\) is unramified over \(K_1\), \(X\) has semistable reduction over \(L_i\), for \(2 \leq i \leq s\), and \([L_i : K_i] = n\), for some \(n\). Applying the lemma to this data we obtain an extension \(L\) of \(K\). Let \(T = \text{Spec}(O_L)\). The curve \(X \otimes_K L\) has a proper, regular model \(Y\) over \(T\) such that

(i) \(Y \otimes_T T_{\gamma_i} \simeq X \otimes_S T_{\gamma_i}\), and
(ii) \(Y\) is semistable at \(\gamma_i\), for \(2 \leq i \leq s\).

Applying (4) to \(Y\) gives the equality

\[\sum_{1 \leq i \leq s} \delta_{\gamma_i} \log \text{Norm}(\gamma_i) = \sum_{1 \leq i \leq s} -\text{Art}_{\gamma_i} \log \text{Norm}(\gamma_i).\]
On the other hand because of semistability, we have $\delta_{\gamma_i} = -\text{Art}_{\gamma_i}$, for $2 \leq i \leq s$. Hence we get $\delta_{\gamma_1} = -\text{Art}_{\gamma_1}$. Since $T/S$ is étale at $\gamma_1$, (i) implies

$$\delta_{p} = \delta_{\gamma_1} = -\text{Art}_{\gamma_1} = -\text{Art}_{p}.$$ 

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