Mitigating the sign problem for non-relativistic fermions on the lattice

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Abstract

We study the fermion sign problem in a theory of non-relativistic fermions with a spin-independent repulsive interaction. We work in polar co-ordinates in momentum space, which makes it straightforward to keep only the low-energy degrees of freedom close to the Fermi surface. This is sufficient for the purpose of calculating many physically important low-energy observables. We find indications that the sign problem in this effective theory will be weaker than in the full theory, so low-energy properties of the theory could be calculated by modifying the action to make it positive semi-definite and including reweighting factors in the observables. We discuss suitable modifications of the action, and describe a possible lattice realization of the polar momentum space formulation of the theory.
I. INTRODUCTION

One of the few known first-principles techniques for calculating the properties of systems of strongly-interacting fermions is numerical evaluation of the functional integral. The standard approach, which involves Monte Carlo importance sampling of field configurations on a space-time lattice, has been very successful in zero-density and high-temperature systems, but at non-zero fermion density and low temperature it becomes impractical because the fermionic part of the integrand is not positive and therefore cannot be treated as a weight for the importance sampling (see, for example, [1, 2, 3, 4]). This has become known as the fermion sign problem.

In this article we investigate the feasibility of numerically calculating low-energy properties of a degenerate Fermi system by focusing on fermionic degrees of freedom close to the (perturbative) Fermi surface [5, 6]. This approach, known as Landau Fermi liquid effective theory, allows one to study low-energy properties of the degenerate Fermi system, such as heat capacity, density response, and transport coefficients, which are dominated by the low-energy degrees of freedom (see, e.g., [7]). We expect this approach to be valid as long as the coupling is not so strong as to cause major disruption of the Fermi surface, introducing sensitivity to degrees of freedom far from the perturbative Fermi surface. As long as this is the case, the fermion modes far from the Fermi surface are represented by irrelevant operators (see, e.g., [8]). Some observables that are sensitive to the degrees of freedom far from the Fermi surface, cannot be studied in this approach; these include bulk properties like fermion density and energy density.

We will work in spherical co-ordinates in momentum space. This makes it straightforward for us to write down the leading order terms of the Landau Fermi liquid effective theory, which includes only those degrees of freedom that have momenta close to the perturbative Fermi surface, and to distinguish between interactions that change radial momentum $p$ (distance from the Fermi surface, which is defined by the Fermi momentum $p_F$) and those that change the angular component. The price we pay is that the interaction term in the effective lagrangian is then non-local. It has been argued by Hong and Hsu [3, 4] that the corresponding effective theory for QCD has a positive semidefinite fermion determinant to leading order in $1/p_F$, if gluon interactions that change a fermion’s momentum in the angular direction are suppressed. The theory we study here is not QCD: it consists of non-relativistic fermions interacting via an auxiliary scalar field. However, it offers a convenient toy model for using the Landau Fermi liquid effective theory to study the sign problem in dense matter, and it also has physical relevance in its own right. We argue that for this theory, without any suppression of scattering between angular patches, the effective theory will have a more moderate sign problem than the full theory has. Then relatively minor changes to the action (discussed in Sec. III) will yield a positive-definite action that could be used to generate an ensemble, and observables in the original theory could be calculated by reweighting. We hope that future work will investigate this possibility.

Momentum lattice approaches have previously been applied to gauge theories [9, 10, 11, 12], but we will restrict our treatment to non-relativistic fermions that feel a spin-independent repulsive interaction, mediated by an auxiliary bosonic field $\mathcal{A}$. In Sec. II we write down the fermion matrices in momentum space, in polar co-ordinates, for such a theory, and show how the fermion sign problem manifests itself. In Sec. III we discuss ways to construct a modified theory with a positive semidefinite fermion matrix. In Sec. IV we outline a lattice realization of the polar momentum coordinate formulation of the theory. A
more detailed discussion of the lattice formulation is given in Appendix A.

II. THE NON-RELATIVISTIC FERMION ACTION

A. Fermions in a background scalar field

The Euclidean time action for a non-relativistic spin-$\frac{1}{2}$ fermion $\psi$ of mass $m$ in $d$ spatial dimensions, at temperature $T = 1/\beta$, interacting with a background configuration $A(x)$ (where $x = (t, \mathbf{x})$) of an auxiliary real bosonic field, is

$$S_{\text{ferm}} = \int_0^{\beta} dt \int d^d \mathbf{x} \psi^\dagger \left( \partial_\tau - \frac{\nabla^2}{2m} - \mu + iA(x) \right) \psi$$

The field $\psi$ has spin-up and spin-down components, but the action is spin-independent so we suppress the spin indices. The bosonic field $A$ has its own action $S_{\text{aux}}(A)$ which depends on details of the interaction between the fermions. Its specific form will not concern us here. If Eq. (1) were obtained by a Hubbard-Stratonovich transformation from a purely fermionic action with a 4-fermion repulsive interaction, then the specific form of the 4-fermion interaction would determine $S_{\text{aux}}(A)$. In the simplest case the $A$ field could be a discrete spin variable with $S_{\text{aux}}(A) = 0$.

We now rewrite this action in terms of a Nambu-Gor’kov-like field $\Psi$, defined by $\Psi^\dagger(x) = \psi^\dagger(x)$, $\Psi_1(x) = \psi_1(x)$. The fermion matrix is still diagonal in spin space but no longer proportional to the identity matrix, so we display it explicitly:

$$S_{\text{ferm}} = \int_0^{\beta} dt \int d^d \mathbf{x} \Psi^\dagger \left[ \begin{array}{cc} \partial_\tau - \frac{\nabla^2}{2m} - \mu + iA(x) & 0 \\ 0 & \partial_\tau + \frac{\nabla^2}{2m} + \mu - iA(x) \end{array} \right] \Psi$$

The partition function of the theory is then

$$Z = \int D\Psi \det M_{\text{ferm}}(A) \exp(-S_{\text{aux}}(A)).$$

where $M_{\text{ferm}}$ is the fermion matrix, given in square brackets in (2).

Because we assumed the repulsive interaction to be spin-independent, $M_{\text{ferm}}$ can be split into two independent diagonal blocks, $M_\uparrow$ and $M_\downarrow$, for spin up and for spin down respectively, whose determinants are, for general $A(x)$ and $\mu$ (including $\mu = 0$) complex. Therefore the partition function (3) has a sign problem, because the determinant of the fermion operator is the square of a complex number, which is in general not a positive number, and therefore cannot be used as a Monte-Carlo importance weight. However, when discretized on a spatial lattice of spacing $a$ there is a special value, $\mu_{\text{pos}} = d/(ma^2)$, at which this model has no sign problem because each spin component has a real fermion determinant. At $\mu = \mu_{\text{pos}}$ the diagonal part of the lattice $\nabla^2$ operator is cancelled, leaving only the hopping terms. Then, as in the half-filled Hubbard model [14], there is a particle-hole symmetry, and an associated transformation on the fermions that relates $M_\uparrow$ to its complex conjugate, and similarly for $M_\downarrow$. 

3
B. Fermions in polar momentum space

We now Fourier transform in space (not in time),

\[\Psi(x) = \int_p e^{i p \cdot x} \Psi(t, p), \quad A(x) = \int_p e^{i p \cdot x} A(t, p), \quad \int_p \equiv \int \frac{d^d p}{(2\pi)^d},\]

where the momentum-space field has components \(\Psi(t, p) = (\psi^\uparrow(t, p), \psi^\downarrow(t, -p))\), obtaining

\[S_{\text{ferm}} = \int_{\hat{v}pq} \psi^\dagger(t, \hat{v}, p) \left( \left[ \partial_t + \frac{p^2}{2m} - \mu \right] \delta^d(p - q) + iA(t, p - q) \right) \psi(t, q) + \psi(t, p) \left( \left[ \partial_t - \frac{p^2}{2m} + \mu \right] \delta^d(p - q) - iA(t, -p + q) \right) \psi^\dagger(t, q)\]

Now let us go to spherical coordinates in the spatial momentum. We write each momentum in terms of a direction \(\hat{v}\) and a distance \(p\) from the perturbative Fermi surface, defined by

\[p_F^2 = \frac{2m\mu}{\epsilon},\]

\[p = (p_F + p)\hat{v}, \quad \int_p = \int_{\hat{v}} \int_{-\Lambda}^{\Lambda} (p_F + p)^{d-1} dp,\]

where the factor of \(1/(2\pi)^d\) is included in the measure of the angular integral. One could think of the angular variable \(\hat{v}\) as indexing “patches” on the Fermi surface. We have imposed an ultraviolet cutoff \(\Lambda \ll p_F\) on \(p\), so only modes close to the Fermi surface are included. To obtain physical predictions one must integrate out modes with \(|p| > \Lambda\) in the standard way (see, e.g., Ref. [8]). Many of the low-energy observables of interest to us, such as transport properties, vanish in the \(T \to 0\) limit and are therefore insensitive to the U.V. cutoff (recall that there are no temperature-dependent divergences in a quantum field theory) as long as \(\Lambda\) is much greater than their characteristic momentum scale, typically of order the thermal momentum \(mT/p_F\).

Our momentum-space approach will therefore be valid as long as fermionic degrees of freedom near the Fermi surface play the dominant role in low energy physics. At sufficiently strong coupling it may become invalid, for example if there were Bose-Einstein condensation of spatially bound two-fermion states. Our approach is therefore most relevant to the region of parameter space where the coupling is too strong to allow perturbative approaches, but not so strong as to bring in degrees of freedom far from the Fermi surface. An example would be the onset of superfluidity in a moderately-strongly-coupled Nambu–Jona-Lasinio model [15]. It is important to note that the momentum-space approach can be used to map its own region of validity. As long as there is a range of values of the cutoff \(\Lambda\), obeying \(\Lambda \ll p_F\), where calculated values of low-energy observables such as transport properties are independent of the cutoff, we can conclude that fermion modes near the Fermi surface are dominating the physics. If we find that we have to push the cutoff up to values of order \(p_F\), and there is still \(\Lambda\)-dependence, then our assumption has broken down.

The fermion action (5) becomes

\[S_{\text{ferm}} = \int_{t,\hat{u},\hat{v}} \int dp dq (p_F + p)^{d-1} (p_F + q)^{d-1} \psi^\dagger(t, \hat{u}, p) \begin{bmatrix} M^\uparrow & 0 \\ 0 & M^\downarrow \end{bmatrix} \psi(t, \hat{v}, q)\]
where \( \Psi(t, \hat{v}, q) = (\psi(t, \hat{v}, q), \psi^\dagger(t, \hat{v}, q)) \), and

\[
M^{\uparrow}_{t, p, \hat{u}, \hat{v}, \hat{v}} = \left[ \partial_t + \frac{p_F}{m} p + \frac{p^2}{2m} \right] \delta_{\hat{u}, \hat{v}} \delta_{p, q} + i A(t, p_F(\hat{u} - \hat{v}) + p\hat{u} - q\hat{v})
\]

\[
M^{\downarrow}_{t, p, \hat{u}, \hat{v}, \hat{v}} = \left[ \partial_t - \frac{p_F}{m} p - \frac{p^2}{2m} \right] \delta_{\hat{u}, \hat{v}} \delta_{p, q} - i A(t, -p_F(\hat{u} - \hat{v}) - p\hat{u} + q\hat{v}) .
\]

(8)

where \( \delta_{\hat{u}, \hat{v}} \delta_{p, q} \) is a simplified notation for \( \delta^d(p - q) \) written in spherical coordinates. \( M^{\uparrow} \) describes particles with spin up, and \( M^{\downarrow} \) describes holes with spin down. We have explicitly shown the indices to make it clear that each is a matrix in momentum space. The indices are \( \hat{u} \) and \( \hat{v} \), which specify directions in momentum space, and \( p \) and \( q \), which specify the distance from the perturbative Fermi surface.

C. Fermion determinant for low-energy modes

We now focus on the degrees of the freedom near the Fermi surface, which we expect to be most important for the low-energy properties of the system. To do that, let us represent (8) and the integration measure (6) as an expansion in \( p/p_F \), where \( p \) is the distance in momentum space from the perturbative Fermi surface. This will eventually become an expansion in \( mT/p_F^2 \) (equivalently \( T/\mu \)) for observables that are not sensitive to the cutoff, or in \( \Lambda/p_F \) where \( \Lambda \) is the ultraviolet cutoff on \( p \). In the rest of this paper we will focus on the leading order contribution. The measure then becomes

\[
\int \frac{d^{d-1}p}{p_F} \int \int_{\Lambda} dp , \quad (9)
\]

and the fermion matrices are

\[
M^{\uparrow}_1 = \left[ \partial_t + \frac{p_F}{m(\Lambda)} p \right] \delta_{\hat{u}, \hat{v}} \delta_{p, q} + i A(t, p_F(\hat{u} - \hat{v}) + p\hat{u} - q\hat{v}) ,
\]

\[
M^{\downarrow}_1 = \left[ \partial_t - \frac{p_F}{m(\Lambda)} p \right] \delta_{\hat{u}, \hat{v}} \delta_{p, q} - i A(t, -p_F(\hat{u} - \hat{v}) - p\hat{u} + q\hat{v}) .
\]

(10)

Eqs. (9), (10) are obtained by integrating out degrees of freedom with \( |p| > \Lambda \). This means that irrelevant couplings (higher-dimension operators, such as higher derivative terms) are induced, and relevant couplings such as the fermion mass (and also the parameters of the bosonic field action \( S_{aux} \)) must be modified to follow a line of constant physics. We have dropped the irrelevant terms since they are suppressed by powers of \( \Lambda/\Lambda_0 \) where \( \Lambda_0 \) is the cutoff in the original theory, and in (10) we have written the mass as \( m(\Lambda) \) as a reminder that \( m \) flows with the cutoff. In the remainder of this paper we will not show the cutoff-dependence of the relevant parameters explicitly.

As noted in Sec. II B, the momentum space approach may become invalid if the coupling becomes so strong that modes far from the Fermi surface start to play an important role. This will be easily noticed, since it leads to \( \Lambda \)-dependence of the low-energy observables.

We expect the effective theory (10) to have a weaker sign problem than the full theory (2) because it is closer to having a particle-hole symmetry: we have discarded the degrees
of freedom far from the Fermi surface, which, in the full theory have very different phase space and dispersion relations above and below the Fermi surface. As we will see below, in the effective theory there is a variable transformation that almost guarantees the positivity of the fermion determinant: it is only violated by the $p\hat{u} - q\hat{v}$ piece in the interaction term. In Sec. III we will discuss how this piece might affect the fermion determinant; it is formally subleading in $p/p_F$, but for now we keep it in the action because we do not know a priori the typical scale of variation of the relevant $A$-modes, so it may not be rigorously negligible.

We expect low-energy degrees of freedom to be well described by the leading-order fermion matrices (10). This theory still has a sign problem: $\det M_{\text{ferm}} = \det M^{\uparrow}_1 \det M^{\downarrow}_1$ is in general complex. To see why, consider the complex conjugate of $\det(M^{\downarrow}_1)$:

$$\det(M^{\downarrow}_1)^* = \det(M^{\uparrow}_1)$$

$$= \det \left( (\partial_t + \frac{p_F}{m} p)\delta_{\hat{u},\hat{v}}\delta_{p,q} - iA^*(t, p_F(\hat{u} - \hat{v}) + p\hat{u} - q\hat{v}) \right)$$

$$= \det \left( (\partial_t + \frac{p_F}{m} p)\delta_{\hat{u},\hat{v}}\delta_{p,q} - iA(t, -p_F(\hat{u} - \hat{v}) - p\hat{u} + q\hat{v}) \right)$$

$$= \det \left( (\partial_t - \frac{p_F}{m} p')\delta_{\hat{u},\hat{v}}\delta_{p',q'} - iA(t, -p_F(\hat{u} - \hat{v}) + p'\hat{u} - q'\hat{v}) \right).$$

where we have used the fact that the auxiliary field $A$ is real, so $A^*(t, p) = A(t, -p)$, and that the relabelling $p' = -p$, $q' = -q$ does not change the determinant. Compare this with the determinant for the spin-down particles

$$\det(M^{\downarrow}_1) = \det \left( (\partial_t - \frac{p_F}{m} p)\delta_{\hat{u},\hat{v}}\delta_{p,q} - iA(t, -p_F(\hat{u} - \hat{v}) - p\hat{u} + q\hat{v}) \right).$$

(12)

For a general $A$-field configuration, $\det(M^{\downarrow}_1)^*$ differs from $\det(M^{\downarrow}_1)$, so the total weight $\det(M^{\downarrow}_1)\det(M^{\uparrow}_1)$ will not in general be positive semidefinite. However we can see that $\det(M^{\downarrow}_1)^*$ is some sense “almost” equal to $\det(M^{\downarrow}_1)$. Only the last term in the argument of the $A$ field differs between (12) and the last line of (11). If the offending terms could be ignored or suitably modified (for example if the $p/p_F$ expansion converged well enough, or if scattering between patches proved to be negligible—see below) then the fermion matrix would be positive-semidefinite.

### III. MODIFIED EFFECTIVE ACTION WITHOUT A SIGN PROBLEM

To test the idea that the effective theory (10) has a tractably weak sign problem, one must modify it to obtain an exactly positive-semidefinite weighting, and calculate observables with a reweighting factor that gives results appropriate to the action (10). If the sign problem is sufficiently weak, the reweighting factor will not be too noisy, and useful results can be obtained. One approach would be to generate ensembles weighted by $|\det M_1|$ or $|\text{Re} \det M_1|$, and then reweight back to the proper $\det M_1$ weighting. This has not yet been attempted, perhaps because in the position-space formulation $\text{[3,4]}$ the calculation of $\det M_1$ requires the construction of a non-local operator which is used in the coupling of the fermion modes to the auxiliary field. We suggest that formulating the theory in momentum space, as we have done in this paper, may make it easier to perform numerical investigations of the high density effective theory. For example, it is easy to do small-scale studies using a small number of angular patches on the Fermi surface.

In the remainder of this section we will describe an alternative to using $|\det M_1|$ or $|\text{Re} \det M_1|$, namely modifying the interaction term in (10) so as to make the fermion determinant positive-semidefinite.
A. Action with no scattering between patches

The simplest and most radical approach would be to make the different directions on the Fermi surface independent of each other, by coupling fermion modes only to auxiliary field modes that scatter them within the same angular patch on the Fermi surface.

To see that such a theory is positive semidefinite, consider a theory in one spatial dimension \( d = 1 \) where the angular variable \( \dot{a} \) can take on only two values, \( \dot{a} \) and \( -\dot{a} \), and we discard high-momentum \( A \)-field modes that could scatter fermions from \( \dot{a} \) to \( -\dot{a} \). This situation was studied in Sec. II of Ref. \[6\] (see also \[16\]). The fermion action for a single spin state is

\[
S_{\text{ferm}}^{(1)} = \int dpdq \psi^\dagger(\dot{a},p) \left( \delta(p-q) \partial_t + i A((p-q)\dot{a}) + \frac{p_F}{m} p \delta(p-q) \right) \psi(\dot{a},q) \\
+ \int dpdq \psi^\dagger(-\dot{a},p) \left( \delta(p-q) \partial_t + i A((p-q)(-\dot{a})) + \frac{p_F}{m} p \delta(p-q) \right) \psi(-\dot{a},q).
\]

In each line, the first two terms are anti-Hermitian and the last one is Hermitian. Now, in the \( -\dot{a} \) patch (second line of \[13\]) we change variables to \( p' = -p \) and \( q' = -q \) and then rename these back to \( p \) and \( q \), giving

\[
S_{\text{ferm}}^{(1)} = \int dpdq \psi^\dagger(\dot{a},p) \left( \delta(p-q) \partial_t + i A((p-q)\dot{a}) + p \delta(p-q) \right) \psi(\dot{a},q) \\
+ \int dpdq \psi^\dagger(-\dot{a},-p) \left( \delta(p-q) \partial_t + i A((p-q)\dot{a}) - p \delta(p-q) \right) \psi(-\dot{a},-q).
\]

Note that the fermion modes living at \( -\dot{a} \) are different from the modes at \( \dot{a} \), but they couple to the same auxiliary field modes. Equation \[14\] leads to a fermion matrix of the form

\[
M^{(1)} = \begin{pmatrix} M_A + M_H & 0 \\ 0 & M_A - M_H \end{pmatrix}
\]

where \( M_H \) is a Hermitian operator (\( p \) in this case) and \( M_A \) is an anti-Hermitian operator (\( \partial_t + i A \) in this case). We assume that both are of even dimension. It then follows that \( M^{(1)} \) has positive-semidefinite determinant, since the eigenvalues of \( M_A - M_H \) are (up to a sign) the complex conjugates of the eigenvalues of \( M_A + M_H \). The spin-down matrix \( M^{(1)} \) is independently positive-semidefinite, via a similar argument. This agrees with the result obtained in Ref. \[6\].

It was important in obtaining this result that the off-diagonal blocks in \[15\] were zero: in a \( d = 1 \) theory this restriction corresponds to putting a cutoff \( \Lambda_A \) on the auxiliary field modes, where \( \Lambda \ll \Lambda_A \ll p_F \), which imposes the condition that the \( A \)-field can only scatter fermions \emph{within} angular patches. In higher dimensions, however, there is no separation of scales between momenta that scatter between patches and momenta that scatter within patches: an \( A \)-field of arbitrarily small momentum can scatter a fermion from one patch to a neighboring one, and the fermion matrices are no longer positive semidefinite (see Sec. \[A3\]).

The argument made in Ref. \[6\] is that, in QCD, even in more than one dimension, scattering between neighboring patches is subleading in \( \mu R \) (where \( R \) is the in-medium screening distance for gluons) so scattering between patches can be ignored, and high-density QCD can be treated as a collection of \( d = 1 \) theories. However, in a first principles non-perturbative approach one cannot rely on such essentially perturbative results. The screening distance is \emph{a priori} unknown and should be determined in a simulation. Therefore, in this article we will not assume that scattering between patches can be neglected.
**B. Action that is symmetrized in the residual momentum**

We propose an alternative way of making the fermion determinant positive semidefinite: modify the coupling to the background field by symmetrizing it in the residual momentum \( p \). This means using a new fermion determinant \( M_{1s} \) which is constructed by replacing the background field \( A \) in (10) with

\[
A_s(p - q) = \frac{1}{2} \left( A(p_F(\hat{u} - \hat{v}) + p\hat{u} - q\hat{v}) + A(p_F(\hat{u} - \hat{v}) - p\hat{u} + q\hat{v}) \right).
\]

The fermion determinant with this symmetrized coupling to the auxiliary field is guaranteed to be non-negative for any \( A(p) \). This follows from the argument of section II C: by coupling to \( A_s \) we make the interaction term symmetric in the residual momenta \( p \) and \( q \), so now \( \det M_{1s} = (\det M_{1s})^* \), and the symmetrized full determinant \( \det M_{\text{ferm},s} = \det M_{1s}^* \det M_{1s} \) is real and positive for any \( A \)-field configuration.

The modified ensemble generated using \( A_s \) has larger violation of momentum conservation (see Appendix A) which will be corrected by reweighting back to the original fermion matrix (see Sec. V). Since the modification is (formally) subleading in \( p/p_F \) and hence \( \Lambda/p_F \), the reweighting factor may be sufficiently well-behaved that the modified theory is still suitable for calculations of low-energy observables in the original theory. That is one of the main conjectures of this paper, and we hope it can be tested by explicit Monte-Carlo calculations.

A more radical modification of the fermion action would be to retain only the leading piece in the expansion, \( A(p_F(\hat{u} - \hat{v})) \), throwing out all dependence on the residual momenta. This would correspond to having all fermion modes in a given patch \( \hat{u} \) couple to all modes in another patch \( \hat{v} \) via the same auxiliary field mode. This might serve as a good starting point in numerical investigations because of its relative simplicity.

It needs to be emphasized that, unfortunately, the proposed approximation may not be systematically improved. The approximation hinges on the observation that the theory with couplings symmetrized in \( p \) (which corresponds to discarding terms in its Taylor expansion that are odd powers of \( p \)) is positive semidefinite. Inclusion of those odd powers and other higher-order terms in \( p/p_F \) will inevitably re-introduce the complexity existing in the unmodified theory (10), even more so in (7),(8). However, our conjecture is that simulations based on the modified theory described above will capture low-energy properties of the full theory with a controllable error of order of \( T/\mu \).

**IV. LATTICE REALIZATION OF THE THEORY**

To numerically evaluate expectation values of observables by Monte-Carlo methods, it is necessary to formulate the theory on a lattice. We will discretize the Euclidean time direction in the standard way, with a lattice of \( N_t \) points covering the range \([0, 1/T]\). The spatial directions will be discretized as a lattice in momentum space. As described above we use polar co-ordinates: we discretize the angular direction \( \hat{u} \) and the radial direction \( p \), and keep only modes near the Fermi surface by imposing an ultraviolet (UV) cutoff \( \Lambda \) on the residual momentum \( p \). We choose a single infrared (IR) cutoff \( \delta p \), which is the momentum lattice spacing in the radial direction, and we choose this to also be the momentum lattice spacing in the angular direction on the inside edge of the momentum shell (at \( p = -\Lambda \)). An example of such a momentum lattice for fermions in two dimension is shown in Fig. 1.
FIG. 1: Discretization of the space of fermionic momentum modes near a two-dimensional Fermi surface. This example has momentum lattice spacing (infrared cutoff) $\delta p = p_F/8$, and ultraviolet cutoff $\Lambda = p_F/4$.

In order to simulate properties of infinite volume continuum theory the lattice parameters need to satisfy

$$\Lambda \gg p_{\text{typ}} \gg \delta p,$$

where $p_{\text{typ}}$ is the characteristic momentum scale of the physics of interest, for example the thermal momentum $mT/p_F$, or a momentum carried by a correlation function.

The modes of the auxiliary field $A$ are most conveniently chosen to lie on a cubic lattice of lattice spacing $\delta p$ in all spatial directions. In order to couple them to fermion modes that live on a polar momentum lattice, while maintaining the correct large-volume ($\delta p \to 0$) limit, we must allow a small violation of conservation of momentum, which goes to zero as $\delta p \to 0$. This arises because, in coupling the auxiliary field to the fermions, we have to treat momenta that differ from each other by an amount less than half the momentum resolution $\delta p$ as the same momentum. Then a pair of Fermion states labeled by $\hat{u}$, $\hat{p}$ and $\hat{v}$, $\hat{q}$ corresponding to a momentum difference $\Delta p = p_F(\hat{u} - \hat{v}) + \hat{p} \hat{u} - \hat{q} \hat{v}$, will be coupled to all cubic lattice modes $A(k)$ such that $|\Delta p - k| < \delta p/2$.

Momentum violation at the scale of an infrared cutoff is not unusual: it arises when one performs lattice calculations in a box with hard-wall or reflecting boundary conditions, in which case momentum conservation is violated at distance scales on the order of the size of the box, but this artifact disappears in the large-volume limit. For a more detailed
function is to define a mapping between the auxiliary fields \( \psi \) and positive determinants, but one can modify the coupling to the background field so that \( \det(M) \approx 0 \), with a width \( \approx \delta p \). The role of the \( W \) function is to define a mapping between the \( A \)-field momenta \( \vec{k} \), which lie on a cubic lattice of spacing \( \delta p \), and the polar lattice momentum differences \( \Delta \vec{p} = p_F(\hat{u} - \hat{v}) + \pi \hat{u} - q \hat{v} \), such that \( |\vec{k} - \Delta \vec{p}| < \delta p/2 \).

As in the momentum continuum (Sec. IIII), the lattice fermion operators \( (18) \) have non-positive determinants, but one can modify the coupling to the background field so that a given pair of fermion fields \( \psi_i(\hat{u}, p)\psi_i(\hat{v}, q) \) couples to the symmetrized combination of auxiliary fields \( \frac{1}{2}\{A(p_F(\hat{u} - \hat{v}) + \pi \hat{u} - q \hat{v}) + A(p_F(\hat{u} - \hat{v}) - \pi \hat{u} + q \hat{v})\} \).

The result is a pattern of coupling between the \( A \)-field and the fermions that has the fermion matrices \( M_{1s} \), where

\[
M_{1s}^{\text{latt}} = (\partial_t + \frac{p_F}{m} p)\delta_{\tilde{\sigma}, \sigma} \delta_{p, q} + \frac{i}{2} \sum_{\vec{k}} A(t, \vec{k}) \left( \begin{array}{c} W(p_F(\hat{u} - \hat{v}) + \pi \hat{u} - q \hat{v} - \vec{k}) \\ + W(p_F(\hat{u} - \hat{v}) - \pi \hat{u} + q \hat{v} - \vec{k}) \end{array} \right)
\]

\[+ \sum_{\vec{k}} A(t, \vec{k}) \left( \begin{array}{c} W(-p_F(\hat{u} - \hat{v}) - \pi \hat{u} + q \hat{v} - \vec{k}) \\ + W(-p_F(\hat{u} - \hat{v}) + \pi \hat{u} - q \hat{v} - \vec{k}) \end{array} \right) \]

It is then straightforward to verify, following the same procedure as in Eqns. (11) and (12), that \( \det(M_{1s})^* = \det(M_{1s}^{\text{latt}}) \), so the fermion matrices \( (19) \) have no sign problem, and their determinants can be used as Monte-Carlo weights to generate an ensemble of configurations. As noted in Appendix A in the modified theory an auxiliary field mode with momentum \( k \) couples not only to pairs of fermion modes whose momentum difference is within \( \delta p \) of \( k \), but also to pairs of fermion modes with momentum differences that deviate from \( k \) by as much as \( \Lambda \). This additional momentum non-conservation should be corrected by the reweighting back to the original fermion matrix (see Sec. V), and it remains to be seen whether it will lead to a significant sign problem in computing the reweighting factor.

V. CONCLUSIONS

We have formulated a theory of non-relativistic fermions with a spin-independent repulsive interaction in polar coordinates in momentum space. This makes it very straightforward for us to write down the high-density effective theory for the fermionic modes with the lowest free energy (those near the perturbative Fermi surface) which are expected to be relevant for low-energy properties of the system. We have shown that in general this effective theory still has a sign problem. We argued that various modifications of the action would yield
a positive semidefinite weighting factor $W_{\text{pos}}$ which is suitable for Monte-Carlo sampling. These include completely decoupling different angular patches to give a set of decoupled 1+1-dimensional theories (Sec. III A and [6]); symmetrizing the interaction in the residual momentum $p$, i.e. the momentum distance from the Fermi surface (Sec. III B); and taking the modulus of the fermion determinant. The last two modifications are (formally) subleading in $p/p_F$.

To obtain physical results one would have to re-weight back to the original theory, by calculating the reweighting factor $R = \langle \det M_1/W_{\text{pos}} \rangle_{\text{pos}}$, where $M_1$ is the fermion determinant for the unmodified effective theory of the low-energy degrees of freedom [10], and $\langle \cdots \rangle_{\text{pos}}$ is an average in the ensemble weighted by $W_{\text{pos}}$. If $R$ turns out to have sufficiently small fluctuations on lattices whose volume is big enough to be physically relevant, then the reweighting procedure can be used to calculate observables in the original theory. A priori, we cannot say which of the modifications proposed here will yield a positive ensemble with the closest overlap with the physical ensemble (i.e. the smallest fluctuations in $R$), or whether any of them will be close enough to be useful. This is a typical issue with reweighting schemes. One may get some information by comparing simulations that use different modifications. Our approach reduces the original theory to a Yukawa-type theory of the degrees of freedom near the Fermi surface, and it is encouraging that such theories have been found to have a fairly gentle sign problem, so that simulating using the modulus of the fermion determinant did not introduce unphysical phases [15]. Rough preliminary investigations of a two-dimensional model (see Appendix B) seem to indicate that in the momentum space formulation the sign problem is indeed very moderate. The momentum space approach may become invalid if the coupling becomes so strong that modes far from the Fermi surface start to play an important role. As we noted in Sec. III B it will be straightforward to check whether this is a problem, since such a breakdown will be signaled by $\Lambda$-dependence of the low-energy physical observables.

Working in momentum space gives the theory a perfect kinetic term. We keep only the leading order in $p/p_F$ but there are no discretization errors coming from approximating derivatives by finite spatial differences. The polar momentum formulation also makes it straightforward to develop simplified models for testing the sign problem in the effective theory, such as ones with an auxiliary field that takes on discrete values (see Appendix B) or with a small number of patches on the Fermi surface (see Sec. II A). However, it imposes some costs. Firstly, as discussed in Appendix A we have to introduce an infrared cutoff $\delta p$, and in the interaction between the auxiliary field and the fermions we treat fermion modes whose momenta that are within $\frac{1}{2}\delta p$ of each other as if they had the same momentum. This means the interactions introduce momentum violation of order $\delta p$. Secondly, our formulation is more expensive for lattice computations because the interaction term, which was local in position space, becomes non-local in momentum space. This means that $M_1$ and any modified version of it are non-sparse matrices. (Note, however, that the position-space formulation [3, 6] also requires the calculation of a non-local operator). Computation of the determinant of a generic non-sparse $N \times N$ matrix takes of order $N^3$ operations [17], whereas the algorithms for dynamical fermions with local actions typically take of order $N$ operations at fixed UV cutoff [18]. This extra cost will easily be worthwhile, however, if the resultant ensemble has a less severe sign problem. The number of configurations required for reweighting from a positive action to the original action rises extremely quickly, as $\exp(N\Delta f)$ (where $\Delta f$ is the free energy density difference between the actions [3]), so if $W_{\text{pos}}$ has a smaller value of $\Delta f$ than existing approaches then this could easily compensate for the extra
costs arising from the non-sparse fermion matrix. We hope that lattice calculations using
the high-density effective theory will test these ideas in the future.

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APPENDIX A: DISCRETIZATION IN POLAR MOMENTUM SPACE

In this appendix we discuss some of the technical issues that arise when we discretize the fermion degrees of freedom in polar coordinates in momentum space. The main question is how the fermions should be coupled to the auxiliary field, which is also discretized in momentum space.

1. Entire Fermi surface

FIG. 2: Momentum-conserving coupling of auxiliary field $A$ to modes near the Fermi surface, in two spatial dimensions. Panel (a): continuous momentum space, where an $A$-mode with given momentum will couple to any pair of fermions with the appropriate momentum difference (thick lines with embedded arrow heads, red online; just a few examples are shown). Panel (b): discretized momentum space for fermions using polar co-ordinates. If we require exact momentum conservation, then a given (non-radial) $A$-mode can couple to only one pair of fermion modes and their antipodal images.

If we discretize the whole Fermi surface then we cannot maintain exact momentum conservation in the interactions that are mediated by the auxiliary $A$ field. We illustrate the issue for the two-dimensional case in Fig. 2. In the infinite volume limit (continuum in momentum space) the $A$-field mode with a given momentum $\vec{k}$ can couple to any two fermion modes whose momenta differ by $\vec{k}$: the Lagrangian will contain many terms of the form $\psi^\dagger(\vec{p})A(\vec{k})\psi(\vec{p} - \vec{k})$ for different values of $\vec{p}$. These pairs of modes densely cover the available momentum space, as shown in Fig. 2a, by the many ways one can draw lines of momentum $\vec{k}$ (thick lines with embedded arrow heads) connecting pairs of fermion modes near the Fermi surface. However, once one discretizes the radial and angular components of the momenta, the momentum difference between any two fermion modes becomes unique (up to inversion through the origin, and apart from purely radial momentum differences). So in the
FIG. 3: Large-volume ($\delta p \to 0$) limit showing which pairs of fermion modes couple to an $A$-mode with momentum $(k_x, k_y) = (3, 2)$, if we tolerate momentum non-conservation $\epsilon_A = \frac{1}{2} \delta p$. We work in units where $p_F = 10$ and $\Lambda = 4$. The four graphs correspond to $\delta p = 2.67$, $\delta p = 1.6$, $\delta p = 1.14$, and $\delta p = 0.889$.

discretized theory (Fig. 2b) if we require exact conservation of momentum then a typical $A$-mode either cannot couple to any fermion modes, or can only couple to one pair of fermion modes, and their antipodal images. This remains true no matter how small $\delta p$ becomes. We conclude that an exactly-momentum-conserving discretization will not give the correct theory in the $\delta p \to 0$ limit. For example, if one made the coupling constant $g$ small, most fermion-fermion scattering amplitudes would be proportional to $g$ in the full theory (from single $A$ exchange) but would have no $O(g)$ contribution in the discretized theory, where single $A$ exchange would be forbidden by momentum conservation.

We suggest that this problem can be solved by allowing a small tolerance $\epsilon_A$ for momentum mismatch: we will allow a given $A$ mode of momentum $\vec{k}$ to couple to any pair of fermionic modes whose momenta differ by an amount $\Delta \vec{p}$ that is within $\epsilon_A$ of $\vec{k}$, i.e. $|\Delta \vec{p} - \vec{k}| < \epsilon_A$. For simplicity, we will choose the $A$-modes to live on a cubic lattice with the same momentum lattice spacing $\delta p$ as the fermionic lattice, and we will set the tolerance $\epsilon_A = \frac{1}{2} \delta p$, so that typically a pair of fermionic modes will couple to at most one $A$-mode. The $A$-mode momentum lattice will have a UV cutoff of $2(p_F + \Lambda)$, so that fermion modes can be scattered between distant parts of the Fermi surface. We expect that in the limit $\delta p \to 0$, infrared artifacts (arising from the momentum lattice spacing and the tolerance for momentum non-conservation) will disappear. In Fig. 3 we show how this scheme solves
FIG. 4: Figure showing which pairs of fermion modes couple to a the A-field mode with momentum $(k_x, k_y) = (3, 2)$, for $p_F = 10$, UV cutoff $\Lambda = 4$, IR cutoff $\delta p = 2$. Panel (a): unmodified theory (see (18)); momentum violation is of order $\delta p$. Panel (b): theory with interaction symmetrized in $p$ (see (16), (19)) to ensure a positive semidefinite fermion determinant; momentum violation is of order $\Lambda$.

the problem illustrated in Fig. 2. In Fig. 3, as $\delta p \rightarrow 0$, a given A-mode, in this case the one with momentum $(k_x, k_y) = (3, 2)$, couples to more and more pairs of fermion modes, approximating the dense set of Fig. 2a.

In this approach, then, the discretized coupling of the A field to the spin-up fermions is given by

$$M_{\text{int}}^{\uparrow,\hat{u},\hat{v}} = \sum_{\vec{k}} iA(t, \vec{k})W(p_F(\hat{u} - \hat{v}) + \hat{p} \hat{u} - \hat{q} \hat{v} - \vec{k})$$  \hspace{1cm} (A1)

where $W$ imposes momentum conservation with a tolerance of $\frac{1}{2}\delta p$,

$$W(\vec{s}) = \begin{cases} 1, & \text{for } |\vec{s}| < \frac{1}{2}\delta p \\ 0, & \text{for } |\vec{s}| > \frac{1}{2}\delta p \end{cases}$$  \hspace{1cm} (A2)

In section IV we discussed how the coupling of the fermions to the auxiliary field could be modified (symmetrized in the residual momentum $p$) to yield a positive semidefinite fermion matrix. In Fig. 4 we illustrate this modification for one of the auxiliary field modes. We see that in the modified theory (Fig. 4b) the A-mode with momentum $(k_x, k_y) = (3, 2)$ couples not only to pairs of fermion modes whose momentum difference is within about $\delta p$ of $k$, but also to pairs of fermion modes whose momentum difference deviates from $k$ by up to $\pm\Lambda$. This additional momentum non-conservation should be corrected by the reweighting back to the original fermion matrix, and it remains to be seen whether it will lead to a significant sign problem in computing the reweighting factor.
FIG. 5: Momentum-space discretization of modes in two antipodal small patches on the Fermi surface. Panel (a) shows the modes in polar co-ordinates; panel (b) shows the same set of modes realized as a flattened Fermi surface, in which $\hat{u}$ is a periodic Cartesian coordinate orthogonal to $p$. The double-headed arrows (red online) show some of the couplings of the $A$-modes with momentum $(k_x, k_y) = (3, 2)\delta p$.

2. Small antipodal patches

A less complicated coupling of the $A$-modes to the fermions is possible if we restrict ourselves to a pair of small antipodal patches on the Fermi surface. In Fig. 5(a) we show how this would work in two spatial dimensions, where $p$ gives the radial momentum relative to the perturbative Fermi surface, and $\hat{u}$ is the polar angle of the momentum.

The approximation of Eq. (10) neglects the difference between the amount of phase space above and below the Fermi surface, and can therefore be envisaged as a flattening of the Fermi surface to a line, as shown in Fig. 5(b), where $\hat{u}$ becomes a Cartesian momentum co-ordinate orthogonal to the radial momentum $p$. In each patch we treat $u$ as a periodic variable. For simplicity we assume that the lattice spacing has the same value $\delta p$ in both angular and radial directions, but one could use an anisotropic lattice if necessary. The generalization to three dimensions is straightforward: simply add another angular variable.

In Eq. (10), each mode of the $A$-field couples equally to all pairs of fermion modes whose momenta differ by exactly the momentum of the $A$ field. We implement this in the lattice theory after the flattening of the Fermi surface, so the $A$ field lives on a similar momentum lattice, with the same lattice spacings, generated by taking all possible momentum differences on the lattice of Fig. 5(b). Following Eq. (10), we allow each mode of the $A$-field to couple to all pairs of fermion modes with the right momentum difference. In Fig. 5(b) we show some of the pairs that one specific $A$-mode would couple to (double-headed arrows). When one takes into account the periodic boundary condition in the $u$ direction, these all have the same momentum difference.

The small patch scheme is a radical truncation of the full theory, because it only keeps a small part of the Fermi surface. One could regard it as a crude way of implementing a forward-scattering-dominated interaction. The ground state of a single-patch theory is more likely to be a particle-hole condensate \cite{19,20} than a condensate of Cooper pairs. But it
could also be used as a tractable toy model for testing the ideas that we have outlined in this article. It is simpler than the “whole Fermi surface” scheme of the previous section because there is a smaller number of degrees of freedom, and momentum is exactly conserved.

3. Positivity tests

![Diagram of Fermi surface discretization]

FIG. 6: Simplified discretization including only fermion modes with momenta \( \pm (p_F \pm p)\hat{a} \) and \( \pm (p_F \pm p)\hat{b} \). These couple to six auxiliary field modes with momenta \( k_1 \ldots k_6 \). We have shown \( k_3 \) and \( k_4 \) in the antipodal patches to minimize clutter.

We can test the positivity of the low energy effective theory in a very simple example, such as that depicted in Fig. 6. We focus on one spin component of the field, and reduce the fermionic field to two patches, \( \hat{a} \) and \( \hat{b} \), and their antipodal partners. In each patch we have two residual momenta \( \pm p \), spaced equally above and below the Fermi surface. Even if we decouple high-momentum \( A \)-field modes, so that there is no coupling between the \( (\hat{a}, \hat{b}) \) patches and the \( (-\hat{a}, -\hat{b}) \) patches, the fermion determinant is still not positive definite. To see this, note that the fermion matrix is block diagonal

\[
M^{(1)} = \begin{pmatrix}
M_+ & 0 \\
0 & M_-
\end{pmatrix}
\]

where \( M_+ \) is the fermion matrix for the \( (\hat{a}, \hat{b}) \) patches and \( M_- = (M_+)^T \) (see below) is the fermion matrix for the \( (-\hat{a}, -\hat{b}) \) patches. The full fermion determinant is therefore

\[
\det(M) = \det(M^{(1)}) \det(M^{(1)}) = \det(M^{(1)})^2, \quad \det(M^{(1)}) = \det(M_+)^2.
\]

In the basis \( \{\psi(\hat{a}, +p), \psi(\hat{a}, -p), \psi(b, +p), \psi(b, -p)\} \),

\[
M_+ = P + iA = \begin{pmatrix}
p & iA(k_1) & iA(k_2) & iA(k_3) \\
iA(-k_1) & -p & iA(k_4) & iA(k_5) \\
iA(-k_2) & iA(-k_3) & p & iA(k_6) \\
iA(-k_3) & iA(-k_5) & iA(-k_6) & -p
\end{pmatrix}
\]

where, as shown in Fig. 6, \( k_1 = -2p\hat{a} \), \( k_2 = (p_F + p)(\hat{b} - \hat{a}) \), etc. All the \( k_i \) are much less than \( p_F \). The residual momentum term, \( P \), is a diagonal Hermitian matrix with entries \( \pm p \), and \( A \) is an off-diagonal Hermitian matrix because \( A(-k) = A(k)^* \). We can see that \( M_- = P + iA^* = M_+^T \) because \( A \) field momenta are reversed when we perform the transformation to the antipodal momenta. It is easy to verify that for a generic \( A \)-field configuration,
det$(M_+)$ is a generic complex number, so the full fermion determinant $[A4]$ of the effective theory is not real. If we turn off the coupling between neighboring patches by setting $A(k_2) = A(k_3) = A(k_4) = A(k_5) = 0$ then the determinant is always real, in agreement with the argument of Sec. III A.

**APPENDIX B: TESTING USING A TWO-DIMENSIONAL EXAMPLE**

In this appendix we describe a simple theory that can be used to test the ideas set forth in this paper. We start with fermions in two spatial dimensions, feeling a point-like repulsion

$$H_{int} = g \psi_\uparrow^\dagger(\mathbf{x})\psi_\uparrow(\mathbf{x})\psi_\downarrow^\dagger(\mathbf{x})\psi_\downarrow(\mathbf{x}),$$

where $g > 0$. We employ Hirsch’s variant of the Hubbard-Stratonovich transformation, where the auxiliary field $\sigma$ is discrete, taking on values $\pm 1$. From the start we assume the Euclidean time direction is periodic with period $T$, and discretized with time spacing $\tau$. Then we find $[13]$

$$\exp[-\tau g n_\uparrow(\vec{x}) n_\downarrow(\vec{x})] = \frac{1}{2} \sum_{\sigma(\vec{x}) = \pm 1} \exp[2b \sigma(\vec{x})(n_\uparrow(\vec{x}) - n_\downarrow(\vec{x}))-\frac{\tau g}{2}(n_\uparrow(\vec{x}) + n_\downarrow(\vec{x}))]$$

(B2)

where $n_\alpha(\vec{x}) = \psi_\alpha^\dagger(\vec{x})\psi_\alpha(\vec{x})$, and $\alpha = \{\uparrow, \downarrow\}$, and

$$\tilde{b} = \frac{b}{\tau}, \quad \tanh^2(b) = \tanh \left(\frac{\tau g}{4}\right).$$

(B3)

The fermion matrices in the effective theory in momentum space are then given by

$$M_{F\uparrow} = \left[\partial_t + \frac{p_F}{m} p\right] \delta_{p,q}\delta_{\hat{u},\hat{v}} + 2\tilde{b}\sigma(t, p_F(\hat{u} - \hat{v}) + p\hat{\mu} - q\hat{v}),$$

$$M_{F\downarrow} = \left[\partial_t - \frac{p_F}{m} p\right] \delta_{p,q}\delta_{\hat{u},\hat{v}} + 2\tilde{b}\sigma(t, -p_F(\hat{u} - \hat{v}) - p\hat{\mu} + q\hat{v}),$$

(B4)

where $p_F = \sqrt{2m\hat{\mu}}$ and $\hat{\mu} = \mu - g/2$ as follows from (B2). Note that we use a Nambu-Gor’kov-like basis where the $\sigma(t, \mathbf{p} - \mathbf{q})$ term is positive in both the spin-up and spin-down fermion matrices. The auxiliary field does not have any action of its own ($S_{aux} = 0$).

We now discretize the system for lattice calculations. The radial residual momentum has an infra-red cutoff $\delta p$ and ultra-violet cutoff $\Lambda$, and the angular variable is discretized in $N_v$ steps (see Sec. [IV]) so the momentum lattice action is

$$S_{lat} = \frac{p_F L}{N_v} \sum_{N_t = N_\mu} \sum_{u,v = 0} \psi_\uparrow^\dagger \left[\delta_{u,v}\delta_{k,n} \left(\nabla_t + \frac{p_F p_n}{m}\right)ight.$$

$$\left. + 2\tilde{b}\frac{p_F L}{N_v} \sigma(p_F(\hat{u} - \hat{v}) + p\hat{\mu} - p_n\hat{\nu})\right]\psi_{v,p},$$

(B5)

where $N_p = \Lambda/\delta p$, $p_n = n\delta p$, $\hat{v}$ is a unit vector at angle $2\pi v/N_v$ to the $x$-axis, we have defined the IR cutoff length scale $L = 2\pi/\delta p$, and $\nabla_t$ is a discretized version of the time derivative. The fermion matrix dimensionality is $N_t \times N_v \times (2N_p + 1)$.

The auxiliary fields, $\sigma(t, \mathbf{p})$, are defined on a square lattice in Cartesian momentum space, with UV cutoffs $\pm \Lambda_\sigma$ on the $x$ and $y$ components of momentum, where $\Lambda_\sigma = p_F + \Lambda$, and a
momentum lattice spacing (IR cutoff) \( \delta p_\sigma \), and we define \( L_\sigma = 2\pi / \delta p_\sigma \). In practice we work with a position space square lattice on which \( \sigma(t, x) = \pm 1 \) and we Fourier transform it to obtain \( \sigma(t, p) \). We set \( \delta p_\sigma = \delta p \). The number of sites along each side of the \( \sigma \)-lattice is an odd integer, \( 2\Lambda_\sigma / \delta p_\sigma + 1 \).

We have performed very preliminary calculations for \( p_F = 2.58 \), \( m = 1.0 \), \( \bar{\mu} = 3.33 \), \( T = 1.0 \), with coupling \( g = 1.0 \). For the fermions we used a momentum space lattice with \( \delta p = 0.194 \), \( \Lambda = 0.775 \), \( N_v = 64 \). This corresponds to \( N_p = 4 \), \( \Lambda / p_F = 0.3 \), \( T / \bar{\mu} = 0.3 \). In the Euclidean time direction we set \( \tau = 0.111 \) which corresponds to \( N_t = 9 \). The dimension of the fermion matrix is 5184. For the auxiliary field we use \( \Lambda_\sigma = p_F + \Lambda = 3.35 \delta p_\sigma = \delta p = 0.194 \), so the \( \sigma \) field lives on a \( 35 \times 35 \) cubic lattice.

To get an idea of the severity of the sign problem in this formulation of the theory, we performed a Monte-Carlo calculation of the reweighting factor \( R \) (see Sec. [V]). We formulated the theory in Mathematica using its \texttt{Det} function to numerically evaluate the determinant of the \( 5184 \times 5184 \) fermion matrix. We used the Metropolis algorithm, with an accept/reject step for each update of the auxiliary field \( \sigma \) using the non-negative weighting factor \( W_{pos} = |\det M_F^{\uparrow} M_F^{\downarrow}| \). We note that even with this numerically unsophisticated approach, we are able to study a lattice that covers an entire 2D Fermi surface, with reasonable resolution (\( \Lambda \) four times bigger than \( \delta p \) (17)).

The observable that we measured is the real part of the reweighting factor, \( \text{Re} R = \cos(\arg \det M_F^{\uparrow} M_F^{\downarrow}(\sigma)) \). In our Monte-Carlo runs we started with all the \( \sigma(t, x) \) being +1, and found that when we had done enough updates (about \( 10^4 \)) the \( \sigma \) were equally distributed between +1 and −1, and the average value of \( \text{Re} R(\sigma) - 1 \) was of order \( 10^{-8} \), with occasional fluctuations no larger than about \( 10^{-2} \). These initial results have not been checked for proper thermalization, autocorrelation, and finite volume effects, but they are an encouraging indication that the theory of the degrees of freedom near the Fermi surface has a moderate sign problem, with a reweighting factor that is close to 1, rather than fluctuating around zero.

[1] I. M. Barbour, S. E. Morrison, E. G. Klepfish, J. B. Kogut, and M.-P. Lombardo, Results on finite density QCD, Nucl. Phys. Proc. Suppl. 60A (1998) 220–234, [hep-lat/9705042].
[2] M. G. Alford, A. Kapustin, and F. Wilczek, Imaginary chemical potential and finite fermion density on the lattice, Phys. Rev. D59 (1999) 054502, [hep-lat/9807039].
[3] J. Cox, C. Gattringer, K. Holland, B. Scarlet, and U. J. Wiese, Meron-cluster solution of fermion and other sign problems, Nucl. Phys. Proc. Suppl. 83 (2000) 777–791, [hep-lat/9909119].
[4] S. Hands, Simulating dense matter, Prog. Theor. Phys. Suppl. 168 (2007) 253–260, [hep-lat/0703017].
[5] D. K. Hong and S. D. H. Hsu, Positivity of high density effective theory, Phys. Rev. D66 (2002) 071501, [hep-ph/0202236].
[6] D. K. Hong and S. D. H. Hsu, Positivity and dense matter, Phys. Rev. D68 (2003) 034011, [hep-ph/0304156].
[7] J. Negele and H. Orland, Quantum Many-Particle Systems. Addison-Wesley, Reading, MA, 1st ed., 1988.
[8] J. Polchinski, Effective field theory and the Fermi surface, hep-th/9210046.
[9] D. Berube, H. Kroger, R. Lafrance, and L. Marleau, Yang-Mills Theory on a Momentup
Lattice: Gauge Invariance, Chiral Invariance and no Fermion Doubling, Phys. Rev. D43 (1991) 1385–1392.

[10] D. Berube, H. Kroger, R. Lafrance, and L. Marleau, QED vacuum polarization on a momentum lattice, Phys. Rev. D46 (1992) 5540–5549.

[11] H. Kroger, S. Lantagne, and K. J. M. Moriarty, Momentum lattice simulation on a small lattice using stochastic quantization, J. Comput. Phys. 122 (1995) 335, [hep-lat/9310012].

[12] J. B. Kogut and J. F. Lagae, QED on a momentum lattice, Nucl. Phys. Proc. Suppl. 34 (1994) 552–554, [hep-lat/9312050].

[13] J. E. Hirsch, Discrete Hubbard-Stratonovich transformation for fermion lattice models, Phys. Rev. B28 (1983) 4059–4061.

[14] M. Creutz, Global Monte Carlo algorithms for many-fermion systems, Phys. Rev. D38 (1988) 1228–1238.

[15] S. Hands and D. N. Walters, Numerical portrait of a relativistic BCS gapped superfluid, Phys. Rev. D69 (2004) 076011, [hep-lat/0401018].

[16] A. Dougall, High density effective theory on the lattice, PoS LAT2007 (2007) 180, [arXiv:0710.0140].

[17] D. S. Watkins, Fundamentals of Matrix Computations. John Wiley and Sons, 2002.

[18] K. Jansen, Lattice QCD: a critical status report, arXiv:0810.5634.

[19] T. Schafer, Hard loops, soft loops, and high density effective field theory, Nucl. Phys. A728 (2003) 251–271, [hep-ph/0307074].

[20] D. V. Deryagin, D. Y. Grigoriev, and V. A. Rubakov, Standing wave ground state in high density, zero temperature QCD at large N(c), Int. J. Mod. Phys. A7 (1992) 659–681.