Stationary Inviscid Limit to Shear Flows

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November 15, 2017

Abstract

In this note we establish a density result for certain stationary shear flows, $\mu(y)$, that vanish at the boundaries of a horizontal channel. We construct stationary solutions to 2D Navier-Stokes that are $\varepsilon$-close in $L^\infty$ to the given shear flow. Our construction is based on a coercivity estimate for the Rayleigh operator, $R[v]$, which is based on a decomposition made possible by the vanishing of $\mu$ at the boundaries.

1 Introduction

We are considering 2D, stationary flows on the strip:

$$\Omega = (0, L) \times (0, 2).$$

We consider an Euler shear flow:

$$u^0 = (\mu(y), 0).$$

Let $u^\varepsilon$ solve the Navier-Stokes equations:

$$\begin{aligned}
    u^\varepsilon \cdot \nabla u^\varepsilon + \nabla P^\varepsilon &= \varepsilon \Delta u^\varepsilon \\
    \nabla \cdot u^\varepsilon &= 0 \\
    u^\varepsilon|_{y=0} &= 0, \\n    u^\varepsilon|_{y=2} &= u_b
\end{aligned}$$

Here $u_b \geq 0$ denotes the velocity of the boundary at $\{y = 2\}$. Our main result, Theorem 1, treats the non-moving case of $u_b = 0$. We are interested in the asymptotic behavior of $u^\varepsilon$ as $\varepsilon \to 0$. In the presence of boundaries, the vanishing viscosity asymptotics are a major open problem in fluids made challenging due to the mismatch between the no-slip condition $u^\varepsilon|_{\partial \Omega} = 0$ and the no penetration condition typically satisfied by Euler flows: $u^0 \cdot n = 0$. This mismatch is typically rectified by the presence of Prandtl’s boundary layer (see [GN17], [Iy17a], [Iy16], [Iy17b] for relevant results in the 2D stationary setting). In this article, we will consider Euler flows that themselves satisfy no-slip:

$$\mu(0) = 0,$$
for which there is no leading order boundary layer. Denote now the asymptotic expansion:

\[ u^\varepsilon := \left( u^\varepsilon, v^\varepsilon \right) = \left( \mu + \varepsilon u_1^\varepsilon + \varepsilon^2 u_2^\varepsilon + \varepsilon^2 v_1^\varepsilon + \varepsilon^2 v_2^\varepsilon + \varepsilon^2 u_p^\varepsilon + \varepsilon^2 v_p^\varepsilon + \varepsilon^2 \right) . \] (5)

We denote:

\[ u_s := \mu + \varepsilon u_1^\varepsilon + \varepsilon^2 u_2^\varepsilon + \varepsilon^2 u_p^\varepsilon , \] (6)

\[ v_s := \varepsilon v_1^\varepsilon + \varepsilon^2 v_2^\varepsilon + \varepsilon^2 v_p^\varepsilon . \] (7)

We impose the boundary conditions:

\[ [u, v]|_{x=0} = [u, v]|_{y=0} = [u, v]|_{y=2} = 0 , \] (8)

\[ \partial_y u + \partial_x v = 0 , \quad P = 2\varepsilon \partial_x u. \] (9)

The system satisfied by \([u, v]\) is:

\[ \begin{cases}
- \varepsilon \Delta u + S_u + \partial_x P = f := \mathcal{N}_1(u,v) + \mathcal{F}_u \\
- \varepsilon \Delta v + S_v + \partial_y P = g := \mathcal{N}_2(u,v) + \mathcal{F}_v
\end{cases} \] in \(\Omega.\) (10)

We have defined:

\[ S_u := u_s u_x + u_s x u + v_s u_y , \quad S_v := u_s v_x + v_s v_y + v v_s y, \] (11)

\[ \mathcal{N}_1 := -\varepsilon^2 \gamma \left( u \partial_x u + v \partial_y u \right) , \quad \mathcal{N}_2 := -\varepsilon^2 \gamma \left( u \partial_x v + v \partial_y v \right) , \] (12)

and \(\mathcal{F}_u, \mathcal{F}_v\) are defined in [119]. Let us now define several norms in which we will control the solution:

\[ ||u, v||_E := ||\sqrt{\varepsilon} \nabla u||_{L^2} + ||\sqrt{\varepsilon} \nabla v||_{L^2} , \] (13)

\[ ||u, v||_P := ||\sqrt{u_s} \nabla v||_{L^2} , \] (14)

\[ ||u, v||_X := ||u, v||_E + \varepsilon \gamma ||\sqrt{\varepsilon} \{u, v\}||_\infty \] (15)

We introduce here the notation:

\[ \tilde{y} = y \cdot (2 - y) . \] (16)

The main theorems we prove are the following:

**Theorem 1** Let \(u_0 \geq 0\) in [5]. Let \(\mu(y) \in C^\infty([0, 2])\) be a given function, satisfying the conditions:

\[ \mu(0) = 0, \mu(2) = u_0 , \] (17)

\[ \partial_j^\mu(0) = \partial_j^\mu(2) = 0 \quad \text{for} \quad 2 \leq j \leq N_0 , \] (18)

\[ \partial_y^\mu(0) > 0, |\partial_y^\mu(2)| > 0 . \] (19)
where \( N_0 < \infty \) and large but unspecified. Let also standard compatibility conditions at the corners of \( \Omega \) be prescribed for the layers in \( u_s \). Then there exists a unique solution, \( u^\varepsilon \) satisfying the Navier-Stokes equations, (3), such that:

\[
||u^\varepsilon - \mu||_\infty + ||v^\varepsilon||_\infty \leq c_0(\mu)\varepsilon. \tag{20}
\]

The constant \( c_0(\mu) \) satisfies:

\[
c_0(\mu) \lesssim \frac{||\mu'''||}{\mu} \quad |W^{100,\infty}|. \tag{21}
\]

Our ultimate interest is motivated by Yudovich’s ninth problem, \( \textit{[Y03]} \). Classical experiments starting with Reynolds have shown that unsteady flows in a 2D channel that start near Couette or Poiseulle flow do not converge to these flows. This indicates the existence of infinitely many stationary solutions to Navier-Stokes “near” Couette or Poiseulle. Establishing the existence of these solutions is an open problem. Our second result, Corollary 2, produces stationary solutions sufficiently close to Couette, assuming \( x \in [0, L], L << 1 \), and a moving boundary at \( y = 2 \).

**Corollary 2** Let any \( \alpha > 0 \) be prescribed, which could depend on \( \varepsilon \). Let \( \tilde{\mu} \) be prescribed to satisfy the vanishing conditions: \( \partial_k y \tilde{\mu} \big|_{y=0} = \partial_k y \tilde{\mu} \big|_{y=2} = 0 \) for \( 0 \leq k \leq N_0 \). There exists a unique solution, \( u^\varepsilon \) to (3) with \( u_b = 2 \) such that:

\[
||u^\varepsilon - (y + \alpha \tilde{\mu}(y))||_\infty + ||v^\varepsilon||_\infty \lesssim \alpha \varepsilon. \tag{22}
\]

**Proof.** One can obtain this by applying Theorem 1 with \( \mu(y) = y + \alpha \tilde{\mu}(y) \), where \( \tilde{\mu} \) vanishes at high order near \( y = 0, 2 \). In this case, the constant \( c_0(\mu) \lesssim \alpha \). \( \blacksquare \)

**Remark 3** The requirement of \( u_b = 2 \) is so that the no-slip condition is satisfied by the Couette flow. We do not use this motion of the boundary anywhere in the proof.

The present article is structured as follows: the construction of the approximate layers, \( u_s, v_s \), in the expansion (5) is performed in the Appendix. The main analysis in Sections 2, 3 is centered around the system (10).

**Acknowledgements:** The authors thank Yan Guo for many useful discussions regarding this problem.

## 2 Linear Estimates

We will analyze the system (10). The reader is urged to consult Lemma 13 for relevant properties of the linearizations, \( u_s \), and the forcing terms, \( f, g \).

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1 We have selected not to optimize \( N_0 \). The optimal \( N_0 \) is likely between 4 and 10.

2 We omit stating the precise form of these compatibility conditions here. They can be found in [OS, Y03].

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2.1 Energy Estimate

Proposition 4 For any $\theta > 0$, solutions $[u, v]$ to (10) satisfy:

$$
\|u, v\|_2^2 + \|\sqrt{a_x}\{u, v\}\|_{L^2(x=L)}^2 \lesssim C(\theta)\varepsilon^{-\theta}\|u, v\|_2^2 + R_1,
$$

where:

$$
R_1 := \int f \cdot u + \int \varepsilon g \cdot v.
$$

Proof.

Apply $[u, v]$ to (10). The coercive quantities are:

$$
\int -\varepsilon \Delta u \times u - \int \varepsilon \Delta v \times v + \int \nabla P \cdot u
$$

$$
= \int \varepsilon \left[ -\partial_y u - 2\partial_{xx} u - \partial_{xy} u \right] \times u + \int \partial_x P u
$$

$$
+ \int \varepsilon \left[ -2\partial_{yy} v - \partial_x \{\partial_y u + \partial_x v\} \right] \times v + \int \partial_y P v
$$

$$
= \int \varepsilon \left[ |\partial_y u|^2 + |\partial_x v|^2 + 4|\partial_y v|^2 + 2\partial_x v \partial_y u \right]
$$

$$
\gtrsim \int \varepsilon \left[ |\nabla u|^2 + |\nabla v|^2 \right].
$$

(25)

Above, we have used the stress-free boundary condition in (9). We now have the convection terms:

$$
\int [u_s u_x + u_{sx} u + v_s u_y] \cdot u = \int u_{sx} u^2 + \frac{1}{2} \int_{x=L} u_s u_x^2,
$$

(26)

$$
\int [u_s v_x + v_s v_y + v_{sy} v] \cdot v = \int v_{sy} v^2 + \frac{1}{2} \int_{x=L} u_s v_x^2.
$$

(27)

We estimate the two bulk terms above using (123) - (125):

$$
|\int u_{sx} u^2| + |\int v_{sy} v^2| \leq |\int \sqrt{\varepsilon y}[u^2 + v^2]| \leq \sqrt{\varepsilon O(L)} ||\sqrt{y} u||_2^2.
$$

(28)

We now move to:

$$
|\int v_{sx} uv| \leq \varepsilon ||\sqrt{y} u_x||_2 ||\sqrt{y} v_x||_2,
$$

(29)

again by using (125). For the $u_{sy} v$ convection term, we first handle the leading order contribution from $\mu$, and we must take care to avoid the critical Hardy inequality:

$$
|\int \mu' uv| = |\int \mu' uv \left[ \chi(y \leq \frac{1}{10}) + \chi(\frac{1}{10} \leq y \leq \frac{19}{10}) + \chi(y \geq \frac{19}{10}) \right],
$$

4
For the interior contributions:

\[ \int \mu' u v \chi(\frac{1}{10} \leq y \leq \frac{19}{10}) \leq O(L) \| \sqrt{u_x} \partial_y u \|_2 \| \sqrt{u_x} \partial_y v \|_2 \]  

(30)

The \( y \leq \frac{1}{10} \) contribution is exactly analogous to the \( y \geq \frac{1}{10} \), and so we treat the former. Let \( \tilde{\chi} \) denote a fattened relative to \( \chi(y \leq 1) \). Fix an \( \omega > 0 \) small.

\[ \int \mu' u v \chi(y \leq \frac{1}{10}) \leq \| \mu' \|_\infty \| y^{-(\frac{1}{4} - \frac{1}{2})} v \tilde{\chi} \|_2 \| y^{\frac{1}{2} - \frac{1}{2}} \tilde{u} \tilde{\chi} \|_2 \]  

(31)

We estimate each \( L^2 \) term above individually.

\[ \int y^{-1 + \omega} v^2 \tilde{\chi} = \int \frac{\partial_y}{\omega} \{ y^\omega \} v^2 \tilde{\chi} = - \int \frac{y^\omega}{2} v \partial_y v \tilde{\chi} - \int \frac{y^\omega}{\omega} v^2 \tilde{\chi}' \leq \frac{1}{\omega} \| y^{-(\frac{1}{2} - \frac{1}{2})} v \tilde{\chi} \|_2 \| \sqrt{u_x} \partial_y v \|_2 + \frac{1}{\omega} O(L) \| \sqrt{u_x} \partial_y v \|_2^2 \]  

(32)

Next from (31):

\[ \| y^{\frac{1}{2} - \frac{1}{2}} u \tilde{\chi} \|_2 \lesssim \| y^{\frac{1}{2}} u \tilde{\chi} \|_2^{1 - \theta(\omega)} \| u \tilde{\chi} \|_2^{\theta(\omega)} \lesssim \varepsilon^{-\theta(\omega)} \| \sqrt{u_x} \partial_y u \|_2^{1 - \theta(\omega)} \left[ \| \sqrt{\varepsilon} \partial_y u \tilde{\chi} \|_2 + \| \sqrt{u_x} \partial_y u \tilde{\chi} \|_2 \right]^{\theta(\omega)} \]  

(33)

Inserting (32) and (33) into (31), one obtains for small \( \kappa > 0 \)

\[ \| \| u, v \|_2 \]  

(34)

For the higher-order contributions, we use the estimate (124), and subsequently split:

\[ \int [u_{xy} - \mu'] u v \leq \int \sqrt{\varepsilon} |u| |v| [\tilde{\chi}^{-} + \tilde{\chi}^{+} + \tilde{\chi}^{\#}] = (35.1) + (35.2) + (35.3). \]  

(35)

Here, \( \tilde{\chi}^{-}(y) = \tilde{\chi}^{-}(2 - y) \) and \( \tilde{\chi}^{\#} = 1 - \tilde{\chi}^{+} - \tilde{\chi}^{-} \), where:

\[ \tilde{\chi}^{\#}(y) = \begin{cases} 1 & \text{on } y \leq \delta \\ 0 & \text{on } y \geq 2\delta \end{cases} \]  

(36)

Terms (35.1) and (35.3) are identical. We estimate:

\[ \| (35.1) \| \lesssim \sqrt{\delta} O(L) \| \sqrt{\varepsilon} \} y \sqrt{\chi^{-}} \|_2 \| \sqrt{u_x} u_x \|_2, \]  

(37)

\[ \| (35.2) \| \lesssim \sqrt{\varepsilon} \delta \| \sqrt{u_x} v \|_2^2. \]  

(38)
We now estimate:

\[
\left\| \sqrt{\epsilon} \sqrt{\chi \delta} \right\|_2^2 = \int \epsilon \frac{\partial_v}{-1} \{y^{-1}\} u^2 \chi \delta \\
= \int \epsilon y^{-1} u \partial_y u \chi \delta + \int \epsilon y^{-1} u^2 \chi \delta' \\
\leq \left\| \sqrt{\epsilon} \sqrt{\chi \delta} \right\|_2 \left\| \sqrt{\epsilon} \partial_y u \right\|_2 + \frac{\epsilon}{\delta} \left\| \sqrt{u_s} \partial_x u \right\|_2^2.
\]  

(39)

We may thus take \( \delta = \epsilon \frac{1}{\epsilon} \) and insert (39) into (37) to conclude.

\[\Box\]

2.2 Positivity Estimate

**Proposition 5** Solutions \([u, v]\) to (10) satisfy, for any \(\kappa > 0\):

\[
||u, v||^2 \leq ||\sqrt{\epsilon} \partial_y v||^2_{L^2(x=0)} + ||\sqrt{\epsilon} \partial_x u||^2_{L^2(x=L)} \lesssim \epsilon^{1-\kappa} ||u, v||^2 + R_2,
\]

where:

\[
R_2 := \int f \cdot -\partial_y v + \int g \cdot \partial_x v.
\]

(40)

**Proof.**

We will apply the multiplier \(M := (-\partial_y v, \partial_x v)\) to the system (10). This gives:

\[
\int \left( -u_s \partial_y v + v \partial_y u_s \right) \cdot -\partial_y v + \int u_s \partial_x v \cdot \partial_x v \\
= \int u_s [\partial_y v]^2 + [\partial_x v]^2 + \int \frac{u_{syy}}{2} v^2 \\
\geq \int u_s |v|^2 - ||u_{syy}||_{\infty} \int v^2 \\
\gtrsim \int u_s |\nabla v|^2.
\]

(42)

We have used the splitting:

\[
\int v^2 = \int v^2 [\chi \delta + \chi \delta'] \\
\leq \frac{L^2}{\delta} \int |\partial_x v|^2 + \int \partial_y \{y\} v^2 \chi \delta \\
\leq \frac{L^2}{\delta} \int u_s |\partial_x v|^2 + \int y2v \partial_y v \chi \delta + \int yv^2 \chi \delta' \\
\lesssim \frac{L^2}{\delta} \int u_s |\partial_x v|^2 + \sqrt{\delta} ||u_s \partial_y v||_2 \times \sqrt{\text{LHS}} \\
= \frac{\delta}{L} \int u_s |\partial_x v|^2 + \sqrt{L} ||u_s \partial_y v||_2 \times \sqrt{\text{LHS}}.
\]

(43)
For the vorticity terms, repeated integration by parts gives:

\[
\int -\varepsilon \Delta u \cdot \partial_y v - \int \varepsilon \Delta v \cdot \partial_x u + \int \nabla P \cdot M = \frac{\varepsilon}{2} \int_{x=0} |\partial_x v|^2 + \frac{\varepsilon}{2} \int_{x=L} \left( |\partial_y u|^2 - |\partial_x u|^2 + |\partial_y v|^2 \right) + \int_{x=L} \partial_x u \\
= \frac{\varepsilon}{2} \int_{x=0} |\partial_x v|^2 + \int_{x=L} 2\varepsilon |\partial_x u|^2,
\]

where we have used the Stress-Free boundary condition from (10). We now come to the remaining linearized terms from (10):

\[
| \int \left( u \partial_x u + v \partial_y u \right) \cdot -\partial_y v | + | \int \left( u \partial_x v + v \partial_y v + v \partial_y v \right) \cdot \partial_x v | \\
\lesssim \varepsilon^n \times \text{LHS of (10)} + \varepsilon^{1-n} \|u, v\|_E^2,
\]

where we have used the Poincare inequality and the estimates in (123) - (125). Finally, the right-hand side of (40) follows from the definition of \( R_2 \).

**Lemma 6** For any \( \theta > 0 \),

\[
\varepsilon^\theta \|\sqrt{\varepsilon u}, \sqrt{\varepsilon v}\|_\infty \leq C_\theta \left[ \|u, v\|_E + \|f, g\|_2 \right].
\]

**Proof.** We omit the proof, this is found in [GN17] using interpolation arguments and estimates for the Stokes operator on domains with corners.

As a direct corollary to (23), (40), and taking \( \theta = \frac{4}{}\gamma \) in (45):

**Corollary 7**

\[
\|u, v\|_X^2 \lesssim R_1 + R_2 + \varepsilon^{\frac{2}{}\gamma} \|f, g\|_2^2.
\]

### 3 Evaluation of Right-Hand Sides

We first provide the nonlinear estimates:

**Lemma 8** With \( N_1, N_2 \) defined as in (12):

\[
| \int \varepsilon^{\frac{2}{}\gamma} \cdot [u + \partial_x u] | + | \int \varepsilon^{\frac{2}{}\gamma} \cdot [v + \partial_x v] | \\
+ \varepsilon^{\frac{2}{}\gamma} \|N_1, N_2\|_2 \leq \varepsilon^{\frac{2}{}\gamma} \left[ \|u, v\|_X^3 + \|u, v\|_X^2 \right].
\]

**Proof.**
We compute directly:

\[ \int \varepsilon^{2+\gamma} |u \partial_x u + v \partial_y v| \cdot |u + \partial_x u| \leq \varepsilon \pi \| \sqrt{\varepsilon^{2+\gamma}} \{ u, v \} \|_\infty \| \nabla \sqrt{\varepsilon^{2+\gamma}} \{ u, v \} \|_2 \lesssim \varepsilon \pi \| u, v \|_X^3. \quad (48) \]

Similarly:

\[ \int \varepsilon^{2+\gamma} |u \partial_x v + v \partial_y v| \cdot |v + \partial_x v| \leq \varepsilon \pi \| \sqrt{\varepsilon^{2+\gamma}} \{ u, v \} \|_\infty \| \nabla \sqrt{\varepsilon^{2+\gamma}} \{ u, v \} \|_2 \lesssim \varepsilon \pi \| u, v \|_X^3. \quad (49) \]

Finally:

\[ \| N_1, N_2 \|_2 \leq \varepsilon^{2+\gamma} \left[ \| u \{ \partial_x u, \partial_x v \} \|_2 + \| v \{ \partial_y u, \partial_y v \} \|_2 \right]. \quad (50) \]

**Proof.** First recall the decomposition of \( F_u, F_v \) given in (120). We first estimate:

\[ \int \varepsilon^{2+\gamma} T_1 \cdot u = \int \varepsilon^{2+\gamma} T_1 \cdot u [\chi_\delta + \chi_\delta^c]. \quad (52) \]

For the nonlocal part, we use estimate (120):

\[ | \int \varepsilon^{2+\gamma} T_1 u \partial_x \| \leq \delta^{-\frac{1}{2}} \varepsilon^{2+\gamma} \| T_1 \|_2 \| \sqrt{u} \partial_x u \|_2. \quad (53) \]

For the local component, we integrate by parts in \( y \):

\[ | \int \varepsilon^{2+\gamma} T_1 u \chi_\delta(y) | = \varepsilon^{2+\gamma} | \int y \partial_y T_1 \chi_\delta + y u \partial_y T_1 \chi_\delta + y u T_1 \chi_\delta^c T_1 | \leq \delta \varepsilon^{2+\gamma} \| \sqrt{\varepsilon} \partial_y u \|_2 \| T_1 \|_2 + \varepsilon^{2+\gamma} \sqrt{\delta} \| \partial_y T_1 \|_2 \| \sqrt{u} \partial_x u \|_2 \]

\[ + \delta^{-\frac{1}{2}} \varepsilon^{2+\gamma} \| T_1 \|_2 \| \sqrt{u} \partial_x u \|_2. \quad (54) \]

The same estimates can be used for \( \varepsilon^{2+\gamma} T_2 \cdot v \). We now come to the higher order terms, in which the non-local contributions are estimated via:

\[ | \int \varepsilon^{2+\gamma} T_1 \cdot \partial_x u \chi_\delta^c + \int \varepsilon^{2+\gamma} T_2 \cdot \partial_x v \chi_\delta^c | \leq \delta^{-\frac{1}{2}} \varepsilon^{2+\gamma} \| T_1 \|_2 \| \sqrt{u} \nabla v \|_2. \quad (55) \]
We now focus on the $T_1$ localized contributions individually. First:

\[ \epsilon^{\frac{1}{2} - \gamma} \left| \int \mu' v_p^2 \partial_y v \chi_\delta \right| \leq \epsilon^{-\gamma} \left\| \mu' \frac{v_p^2}{y} \right\|_\infty \sqrt{\delta} \left\| \sqrt{u_x} \partial_y v \right\|_2, \tag{56} \]

\[ \epsilon^{\frac{1}{2} - \gamma} \left| \int \partial_Y v_p^2 u_c^1 \partial_y v \chi_\delta \right| \leq \epsilon^{\frac{1}{2} - \gamma} \left\| \partial_Y u_p^2 \right\|_\infty \left\| \frac{v}_1^1 \right\|_2 \left\| \sqrt{u_x} \partial_y v \right\|_2, \tag{57} \]

\[ \epsilon^{\frac{1}{2} - \gamma} \int [\mu' v_p^2 + 3 \chi' \partial_Y u_p^2] \partial_y v \leq \epsilon^{\frac{1}{2} - \gamma} \left\| \mu' v_p^2 + 3 \chi' \partial_Y u_p^2 \right\|_\infty \left\| \sqrt{u_x} \partial_y v \right\|_2. \tag{58} \]

The remaining terms in $T_1$ are handled by integrating by parts in $y$ and proceeding as in (59):

\[ \epsilon^{\frac{1}{2} - \gamma} \int \left[ u_c^1 \partial_x u_c^1 + v_c^1 \partial_y u_c^1 - \Delta u_c^1 \right] \cdot \partial_y v \chi_\delta \]

\[ = - \int \epsilon^{\frac{1}{2} - \gamma} \partial_y \left[ u_c^1 \partial_x u_c^1 + v_c^1 \partial_y u_c^1 - \Delta u_c^1 \right] v \chi_\delta \]

\[ - \int \epsilon^{\frac{1}{2} - \gamma} \left[ u_c^1 \partial_x u_c^1 + v_c^1 \partial_y u_c^1 - \Delta u_c^1 \right] \cdot v \chi_\delta \]

\[ = (59) (1) + (59) (2). \tag{59} \]

First:

\[ (59) (1) = \int \epsilon^{\frac{1}{2} - \gamma} m_1 v \leq \epsilon^{\frac{1}{2} - \gamma} \left\| m_1 \right\|_2 \left\| v \right\|_2 \leq \epsilon^{\frac{1}{2} - \gamma} \left\| m_1 \right\|_2 \left\| \sqrt{u_x} \nabla v \right\|_2. \tag{60} \]

Second:

\[ (59) (2) \leq \epsilon^{\frac{1}{2} - \gamma} \left\| u_c^1 \partial_x u_c^1 + v_c^1 \partial_y u_c^1 - \Delta u_c^1 \right\|_2 \left\| \sqrt{u_x} \partial_y v \right\|_2 \tag{61} \]

We now consider the localized contributions from $T_2$, for which we apply estimate (126):

\[ \epsilon^{\frac{1}{2} - \gamma} \left| \int T_2 \cdot \partial_x v \right| \leq \left\| \frac{T_2}{y} \right\|_2 \left\| \sqrt{u_x} \partial_x v \right\|_2. \tag{62} \]

We now make the selection of $\delta = \epsilon^{10 \gamma}$, and $\gamma << 1$ sufficiently small, which closes all of the above estimates. Finally, the $O(\epsilon^{\frac{1}{2}})$ are handled easily via:

\[ \epsilon^{-\frac{1}{2} - \gamma} \int [\mathcal{F}_u - T_1] \cdot \left[ u + \partial_x u \right] + \int [\mathcal{F}_v - T_2] \cdot \left[ v + \partial_x v \right] \]

\[ \lesssim \int \epsilon^{1 - \gamma} \cdot \left[ u + \partial_x u + v + \partial_x v \right] \]

\[ \lesssim \epsilon^{\frac{1}{2} - \gamma} \left\| \sqrt{\nabla v} \right\|_2. \tag{63} \]

We now obtain our complete nonlinear estimate:
Corollary 10 Solutions \([u, v]\) to the system \((10)\) satisfy:

\[
\|u, v\|_X^2 \lesssim C(u_s, v_s) c_0 \left( \frac{\mu'''}{\mu} \right) + \varepsilon^{\frac{2}{3}} \|u, v\|_X^3.
\] (64)

From here, the main result, Theorem 1 follows from a straightforward application of the contraction mapping theorem.
A Construction of Layers

We start with the asymptotic expansions:

\[ u^\varepsilon := \mu + \varepsilon u_1^e + \varepsilon^3 u_2^e + \varepsilon^2 u_p^1 + \varepsilon^{\frac{3}{2}} u_p^2 + \varepsilon^{\frac{5}{2}} u, \]  

\[ v^\varepsilon := \varepsilon v_1^e + \varepsilon^3 v_1^p + \varepsilon^2 v_2^e + \varepsilon^{3/2} v_2^p + \varepsilon^{5/2} v, \]  

\[ P^\varepsilon := \varepsilon P_1^e + \varepsilon^{3/2} P_2^e + \varepsilon [P_1^p + \varepsilon P_1^{1,a}] + \varepsilon^{3/2} P_2^p + \varepsilon^{5/2} P. \]  

A.1 Formal Asymptotic Expansion

Here the Eulerian profiles are functions of \((x, y)\), whereas the boundary layer profiles are functions of \((x, Y)\), where:

\[ Y = \begin{cases} 
Y_+ := \frac{2 - y}{\sqrt{\varepsilon}} & \text{if } 1 \leq y \leq 2, \\
Y_- := \frac{y}{\sqrt{\varepsilon}} & \text{if } 0 \leq y \leq 1.
\end{cases} \]  

Due to this, we break up the boundary layer profiles into two components, one supported near \(y = 0\) and one supported near \(y = 2\):

\[ u^i_p(x, Y) = u^i_p^-(x, Y_-) + u^i_p^+(x, Y_+). \]  

As a notational convention, we use:

\[ \partial_Y u^i_p := \partial_{Y_-} u^i_p^-- \partial_{Y_+} u^i_p^+. \]  

The purpose of such a convention is to obtain the chain rule:

\[ \partial_y u^i_p = \frac{1}{\sqrt{\varepsilon}} \partial_Y u^i_p. \]  

Let us set the following notations:

\[ u^\varepsilon_E := \mu + \varepsilon u_1^1 + \varepsilon^2 u_2^2, \quad v^\varepsilon_E := \varepsilon v_1^1 + \varepsilon^2 v_2^2, \]  

\[ u^{(2)}_s := \mu + \varepsilon u_1^1 + \varepsilon^2 u_2^2 + \varepsilon^3 u_p^1 + \varepsilon^{2.5} u_p^2, \]  

\[ v^{(2)}_s := \varepsilon v_1^1 + \varepsilon^3 v_1^1 + \varepsilon^{3/2} v_2^2 + \varepsilon^{3/2} v_2^2, \]  

\[ P^{(2)}_s := P^\varepsilon := \varepsilon P_1^1 + \varepsilon^{3/2} P_2^2 + \varepsilon [P_1^p + \varepsilon P_1^{1,a}] + \varepsilon^{3/2} P_2^p. \]  

Using the expansions (65) - (67), we will first expand out the purely Euler terms:

\[ u^\varepsilon_E \partial_x u^\varepsilon_E = \left[ \mu + \varepsilon u_1^1 + \varepsilon^2 u_2^2 \right] \cdot \left[ \varepsilon u_1^{1} + \varepsilon^2 u_2^{2} \right] \]  

\[ = \varepsilon \mu u_1^{1} + \varepsilon^2 u_1^{1} u_1^{1} + \varepsilon^2 u_2^{2} u_1^{1} + \varepsilon^3 \mu u_2^{2} + \varepsilon^2 u_2^{2} u_1^{1} + \varepsilon^3 u_2^{2} u_2^{2}. \]
\[
\dot{v}_E \partial_y u_E^* = \left[ \varepsilon v_1^1 + \varepsilon^2 v_1^2 \right] \cdot \left[ \mu' + \varepsilon u_1^1 + \varepsilon^2 u_{xy}^1 \right] \\
= \varepsilon \mu' v_1^1 + \varepsilon^2 v_1^1 u_1^1 + \varepsilon^2 v_1^1 u_{xy}^1 + \varepsilon^2 \mu' v_1^2 + \varepsilon^3 v_1^2 u_{xy}^1 + \varepsilon^3 v_2^2 u_{xy}^2 \tag{77}
\]

\[
u_E^* \partial_x u_E^* = \left[ \mu + \varepsilon u_1^1 + \varepsilon^2 u_{xy}^1 \right] \cdot \left[ v_1^1 + \varepsilon^2 v_{xy}^1 \right] \\
= \varepsilon \mu v_1^1 + \mu \varepsilon^2 v_{xy}^1 + \varepsilon^2 u_1^1 v_1^1 + \varepsilon^2 \mu v_1^2 + \varepsilon^3 u_{xy}^1 v_1^1 + \varepsilon^3 \mu v_2^2 + \varepsilon^3 u_{xy}^2 v_1^2, \tag{78}
\]

\[
\dot{v}_E \partial_y v_E^* = \left[ \varepsilon v_1^1 + \varepsilon^2 v_{xy}^1 \right] \cdot \left[ \mu v_1^1 + \varepsilon^2 v_{xy}^1 \right] \\
= \varepsilon^2 v_1^1 v_1^1 + \varepsilon^2 v_1^1 u_1^2 + \varepsilon^2 v_{xy}^1 u_{xy}^1 + \varepsilon^2 \mu v_1^2 + \varepsilon^3 v_{xy}^1 u_{xy}^2 + \varepsilon^3 v_{xy}^2 u_{xy}^2 \tag{79}
\]

\begin{align*}
\partial_x P_E &= \varepsilon P_{1x}^1 + \varepsilon^2 P_{1x}^2, \tag{80} \\
\partial_y P_E &= \varepsilon P_{1y}^1 + \varepsilon^2 P_{1y}^2, \tag{81} \\
\varepsilon \Delta u_E^* &= \varepsilon \mu''(y) + \varepsilon^2 \Delta u_1^1 + \varepsilon^2 \Delta u_1^2, \tag{82} \\
\varepsilon \Delta v_E^* &= \varepsilon^2 \Delta v_1^1 + \varepsilon^2 \Delta v_1^2. \tag{83}
\end{align*}

We now expand:

\begin{align*}
u_s^{(2)} \partial_x u_s^{(2)} &= u_E^* \partial_x u_E^* + \varepsilon^2 v_{xy}^1 u_1^1 + \varepsilon^2 v_1^1 u_1^2 + \varepsilon^2 v_{xy}^1 u_{xy}^1 \\
&+ \varepsilon^2 u_1^1 u_{xy}^1 + \varepsilon^2 u_1^1 u_{xy}^2 + \varepsilon^2 u_{xy}^1 u_{xy}^2 + \varepsilon^2 u_{xy}^1 u_{xy}^2 \\
&+ \varepsilon^2 u_{1}^1 u_{1}^2 + \varepsilon^2 u_{1}^2 u_{1}^2 + \varepsilon^2 u_{xy}^1 u_{xy}^2 \tag{84}
\end{align*}

\begin{align*}
u_s^{(2)} \partial_y u_s^{(2)} &= v_E^* \partial_y u_E^* + \varepsilon^2 \mu' v_1^1 + \varepsilon^2 v_{xy}^1 u_1^1 + \varepsilon^2 v_1^1 u_{xy}^1 + \varepsilon^2 v_{xy}^1 u_{xy}^1 \\
&+ \varepsilon^2 u_1^1 u_{xy}^1 + \varepsilon^2 u_1^1 u_{xy}^2 + \varepsilon^2 \mu v_1^2 + \varepsilon^2 u_{xy}^1 u_{xy}^2 + \varepsilon^2 u_{xy}^1 u_{xy}^2 \\
&+ \varepsilon^2 u_{xy}^1 u_{xy}^2 + \varepsilon^2 u_{xy}^2 u_{xy}^2 + \varepsilon^2 v_{xy}^1 u_{xy}^2 + \varepsilon^2 v_{xy}^2 u_{xy}^2 \tag{85}
\end{align*}

\begin{align*}
u_s^{(2)} \partial_x v_s^{(2)} &= u_E^* \partial_x v_E^* + \varepsilon^2 v_{xy}^1 v_1^1 + \varepsilon^2 u_1^1 v_{xy}^1 + \varepsilon^2 u_1^1 v_{xy}^2 + \varepsilon^2 v_{xy}^1 v_{xy}^1 \\
&+ \varepsilon^2 u_{xy}^1 v_{xy}^1 + \varepsilon^2 u_{xy}^1 v_{xy}^2 + \varepsilon^2 u_{xy}^1 v_{xy}^2 \\
&+ \varepsilon^2 u_{xy}^1 v_{xy}^2 + \varepsilon^2 u_{xy}^2 v_{xy}^2 + \varepsilon^2 u_{xy}^2 v_{xy}^2 + \varepsilon^2 u_{xy}^2 v_{xy}^2 \tag{86}
\end{align*}

\begin{align*}
u_s^{(2)} \partial_y v_s^{(2)} &= v_E^* \partial_y v_E^* + \varepsilon^2 v_{xy}^1 v_1^1 + \varepsilon^2 v_1^1 v_{xy}^1 + \varepsilon^2 v_{xy}^1 v_{xy}^1 + \varepsilon^2 v_{xy}^1 v_{xy}^2 \tag{87}
\end{align*}
The system satisfied by the second Euler layer is obtained by collecting the order terms from (76) - (83), and is shown here:

\[
\begin{align*}
\varepsilon^2 v_c^1 v_p^1 + \varepsilon^3 v_c^1 v_p^2 + \varepsilon^2 v_p^1 v^2 + \varepsilon^2 v_c^2 v_p^2 \\
+ \varepsilon^2 v_c^1 v_p^2 + \varepsilon^3 v_p^1 v^2 + \varepsilon^3 v_c^2 v_p^2.
\end{align*}
\]

(87)

Finally, we have the linear terms:

\[
\begin{align*}
\partial_x P_x &= \partial_x \varepsilon P_E^c + \varepsilon^2 P^2 + \varepsilon^2 P^1, \\
\partial_y P_y &= \partial_y \varepsilon P_E^c + \varepsilon^2 P^2 + \varepsilon^2 P^1, \\
\varepsilon \Delta u^c &= \varepsilon \Delta u_E^c + \varepsilon^2 u_1^p + \varepsilon^2 u_2^p + \varepsilon^2 u_1^p Y + \varepsilon^2 u_2^p Y, \\
\varepsilon \Delta v^c &= \varepsilon \Delta v_E^c + \varepsilon^2 v_1^p + \varepsilon^2 v_1^2 + \varepsilon^2 v_2^p Y.
\end{align*}
\]

(88) - (91)

A.2 Euler Equations

The equations satisfied by the Euler layers are obtained by collecting the \(O(\varepsilon)\) order terms from (76) - (83), and is now shown:

\[
\begin{align*}
\mu \partial_x u_1^c + \mu' v_1^1 + \partial_x P_1^c &= \mu''(y) \\
\mu \partial_y v_1^c + \partial_y P_1^c &= 0, \\
\partial_x u_1^c + \partial_y v_1^c &= 0, \\
v_1^c |_{x=0} = v_1^c |_{y=0} = v_1^c |_{y=2} = v_1^c |_{x=L} = 0. \\
\end{align*}
\]

(92)

By going to the vorticity formulation, we arrive at the following problem:

\[
-\mu \Delta u_1^c + \mu'' v_1^c = \mu'''(y), \quad v_1^c |_{\partial \Omega} = 0, \quad u_1^c := \int_0^x v_1^c y.
\]

(93)

We will make the assumptions that:

\[
\frac{\mu''}{\mu}, \frac{\mu'''}{\mu} \text{ vanish at high order at } y = 0, 2.
\]

(94)

According to (94), we divide (93) by \(\mu\) to obtain:

\[
-\Delta u_1^c + \frac{\mu''}{\mu} v_1^c = \frac{\mu'''}{\mu}, \quad v_1^c |_{\partial \Omega} = 0.
\]

(95)

By evaluating (95) at \(y = 0, 2\) and recalling (91), it is clear that \(\partial_y v_1^c |_{y=0,2} = 0\). The system satisfied by the second Euler layer is obtained by collecting the \(O(\varepsilon^2)\) terms from (76) - (83), and is shown here:

\[
\begin{align*}
\mu \partial_x u_1^c + \mu' v_1^c + \partial_x P_1^c &= 0 \\
\mu \partial_y v_1^c + \partial_y P_1^c &= 0, \\
\partial_x u_1^c + \partial_y v_1^c &= 0, \\
v_1^c |_{x=0} = v_1^c |_{x=L} = v_1^c |_{y=2} = 0, \quad v_1^c |_{y=0} = -v_1^c |_{y=0}.
\end{align*}
\]

(96)
Going to vorticity produces the system:

$$-\mu \Delta v_e^2 + \mu'' v_e^2 = 0, \quad v_e^2|_{y=0.2} = -v_p^1|_{y=0.2}. \quad (97)$$

We will assume high-order compatibility conditions on the data $v_e^2|_{x=0, \; \Omega}$
with $v_e^2|_{y=0.2}$ at the four corners of the domain, $\Omega$. The first of these conditions
at the corner $x = 0, \; y = 0$ is as follows:

$$\partial_{yy}v_e^1|x=0(0) = \partial_{yy}v_e^1|y=0(0) = -\frac{\mu''}{\mu}v_p^1|y=0. \quad (98)$$

The remaining compatibility conditions may be derived in the same manner.
These will contribute higher order terms, which are the $O(\varepsilon^2)$ terms from $\mathcal{O}(\varepsilon^2)$ - $O(\varepsilon^2)$:

$$C_{1,u} := \varepsilon^2[u_e^1, \partial_x u_e^1] + \varepsilon^2[u_e^1, \partial_y u_e^1] + \varepsilon^2[u_e^1, \partial_{yy} u_e^1] + \varepsilon^2[u_e^1, \partial_{xx} u_e^1] + \varepsilon^2[u_e^1, \partial_{xx} u_e^1]$$

$$C_{1,v} := \varepsilon^2[u_e^1, \partial_x v_e^1] + \varepsilon^2[u_e^1, \partial_y v_e^1] + \varepsilon^2[u_e^1, \partial_{yy} v_e^1] + \varepsilon^2[u_e^1, \partial_{xx} v_e^1] + \varepsilon^2[u_e^1, \partial_{xx} v_e^1]$$

$$C_{1,w} := \varepsilon^2[u_e^1, \partial_z w_e^1] + \varepsilon^2[u_e^1, \partial_y w_e^1] + \varepsilon^2[u_e^1, \partial_{yy} w_e^1] + \varepsilon^2[u_e^1, \partial_{xx} w_e^1] + \varepsilon^2[u_e^1, \partial_{xx} w_e^1]$$

The following follow from standard elliptic theory:

**Lemma 11** Assuming $\mathcal{O}(\varepsilon^2)$ and compatibility conditions for both $v_e^1, v_e^2$ for arbitrary order as in $\mathcal{O}(\varepsilon^2)$, there exist unique solutions, $v_e^1, v_e^2$ to $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(\varepsilon^2)$ that are regular:

$$|\partial_x^k \partial_y^m \{u_e^i, v_e^i\}| \lesssim c_0(\frac{\mu''}{\mu}) \times C_{1,k} \text{ for } i = 1, 2. \quad (101)$$

### A.3 Boundary Layer Equations

Collecting the $\mathcal{O}(\varepsilon)$ terms from $\mathcal{O}(\varepsilon)$ - $\mathcal{O}(\varepsilon)$:

$$\mu \partial_x u_p^{1,0,-} - \partial_{Y_-} u_p^{1,0,-} = 0, \quad \partial_{Y_-} P_p^{1,0,-} = 0,$$

$$u_p^{1,0,-}|_{x=0} = -u_p^{1,0,-}|_{Y_+=0} = -u_p^{1,0,-}|_{Y_- \to \infty} = 0 \bigg\{ \begin{array}{l}
(102)
\end{array} \bigg\}$$

Here we must assume the compatibility condition:

$$u_p^{1,0,-}(0, Y_-)|_{Y_- = 0} = -u_p^{1,0,-}|_{y=0}, \quad \partial_{Y_-}^2 u_p^{1,0,-}(0, Y)|_{Y_- = 0} = 0. \quad (103)$$

We will also assume higher order compatibility conditions that can be obtained by differentiating the above system and reading the resulting equalities.
Note that we construct $u_p^{1,0,-}, v_p^{1,0,-}$ on $(0, L) \times (0, \infty)$. We now cut-off these layers and make a $O(\sqrt{\varepsilon})$-order error:

$$u_p^{1,-} = \sqrt{\varepsilon} \frac{Y}{100} u_p^{1,0,-} - \sqrt{\varepsilon} \frac{Y}{100} \int_0^Y v^{1,0,-}, \quad v_p^{1,-} = \sqrt{\varepsilon} \frac{Y}{100} v_p^{1,0,-}$$

(104)

$v_p^{1,0,+}, v_p^{1,0,-}, u_p^{1,+}, v_p^{1,+}$ are defined analogously, and we omit these details. We then define:

$$u_p^{1,0} := u_p^{1,0,-} + u_p^{1,0,+}, \quad v_p^{1} := v_p^{1,0,-} + v_p^{1,0,+},$$

(105)

$$u_p^{1} := u_p^{1,-} + u_p^{1,+}, \quad v_p^{1} := v_p^{1,-} + v_p^{1,+}.$$  

(106)

Note that due to the cut-off in (104), $u_p^{1}, v_p^{1}$ is smooth. The contributions to the next layer are:

$$C_{2,u} := \varepsilon \partial_x u_p^{1,0} + \varepsilon^2 u_p^1 \partial_x u_p^{1} + \varepsilon^2 u_p^1 \partial_x v_p^{1} + \varepsilon^2 u_p^1 \partial_x v_p^{1}$$

$$+ \varepsilon^2 \partial_x u_p^{1} + \varepsilon^2 \partial_x v_p^{1} + \varepsilon^2 \partial_x v_p^{1} + \varepsilon^2 \partial_x v_p^{1}$$

$$- \varepsilon^2 \partial_x u_p^{1} + \left[ \varepsilon^2 u_p^1 \partial_x u_p^{1} + \varepsilon^2 u_p^1 \partial_x v_p^{1} + \varepsilon^2 u_p^1 \partial_x v_p^{1} + \varepsilon^2 u_p^1 \partial_x v_p^{1} \right] + C_{cut}^1.$$  

(107)

Here $C_{cut}^1$ is the error introduced by the cut-off functions in (104):

$$C_{cut}^1 := \sqrt{\varepsilon} \mu \chi^{1,0} v_p^{0} + 3 \sqrt{\varepsilon} \chi' \partial_Y u_p^{2,0}$$

$$+ 3 \varepsilon \chi^{2,0} - \varepsilon^2 \chi'' \int_0^\infty u_p^{2,0},$$

(108)

Define the auxiliary pressure via:

$$P_p^{1,a} := - \int_Y \left[ \mu \partial_x v_p^{1} + \varepsilon^2 u_p^1 \partial_x v_p^{1} + \varepsilon^2 u_p^1 \partial_x v_p^{1} + \varepsilon^2 u_p^1 \partial_x v_p^{1} + \varepsilon^2 u_p^1 \partial_x v_p^{1}$$

$$+ \varepsilon^2 \partial_x u_p^{1} + \varepsilon^2 \partial_x v_p^{1} + \varepsilon^2 \partial_x v_p^{1} + \varepsilon^2 \partial_x v_p^{1} + \varepsilon^2 \partial_x v_p^{1} \right].$$

(109)

With such a choice,

$$C_{2,v} := 0.$$  

(110)

Collecting the $O(\varepsilon^3)$ terms from (101) - (103), the system satisfied by the second boundary layers is:

$$\begin{align*}
\mu \partial_x u_p^{2,0} - \partial_Y v_p^{2,0} &= f_2 := \varepsilon^2 \chi^1 C_{2,u}, \quad \partial_Y P_p^{2} = 0, \quad \partial_x v_p^{2,0} + \partial_Y v_p^{2,0} = 0, \\
u_p^{2,0}\big|_{x=0} &= u_p^{2,0,-}\big|_{y=0} = -u_p^{2,0,}\big|_{y=0}, \quad u_p^{2,0,}\big|_{y=2} = -u_p^{2,0,}\big|_{y=2}, \\
u_p^{2,0,}\big|_{y=0} &= 0, \quad u_p^{2,0,}\big|_{y=0} = 0 \\
u_p^{2,0,}\big|_{y=0} = 0, \quad v_p^{2,0,}\big|_{y=0} = -\int_0^Y \partial_x u_p^{2,0,-}, \quad v_p^{2,0,}\big|_{y=0} = -\int_0^Y \partial_x u_p^{2,0,+}. \\
\end{align*}$$

(111)
Note that in the same manner as in \( u^1_p, v^1_p \), we have two boundary layer variables, \( Y_- \), \( +_+ \). We compactify the notation in \( \text{[111]} \) to simultaneously address both. Define the cut-off layer via:

\[
u^2_p := \chi(\frac{\sqrt{\varepsilon}Y}{100})u^{2,0}_p - \sqrt{\varepsilon} \mu(\frac{\sqrt{\varepsilon}Y}{100}) \int_0^x v^2_p, \quad v^2_p := \chi(\frac{\sqrt{\varepsilon}Y}{100})v^{2,0}_p. \quad (112)
\]

**Lemma 12** Assume high order compatibility conditions in the sense of \( (103) \) for both \( u^1_p, v^1_p \). There exist unique solutions to \( (102) \) and \( (111) \) that are regular and satisfy the following estimates:

\[
|Y^m \partial_x^k \partial_y \{u^1_p, v^1_p\}| \leq c_0(\frac{\mu'''}{\mu}) \times C_{m,k,l} \text{ for any } k, l, m \geq 0, \quad (113)
\]

\[
|Y^m \partial_x^k \partial_y^l u^2_p| \leq c_0(\frac{\mu'''}{\mu}) \times C_{m,k,l} \text{ for any } k, l, m \geq 0, \quad (114)
\]

\[
|\partial_x^k v^2_p| \leq c_0(\frac{\mu'''}{\mu}) \times C_k \text{ for any } k \geq 0. \quad (115)
\]

**Proof.** These follow from standard heat equation estimates. ■

The following are the errors contributed to the next layer:

\[
C_{3,u} := \varepsilon^2 [u^1_p + u^1_x + \varepsilon \partial_x u^1_p + \varepsilon \partial_x^2 u^1_p] e^{\partial_x u^1_p} + \varepsilon^2 \partial_x^2 u^1_p + \varepsilon^2 \partial_x u^2, \quad (116)
\]

\[
C_{3,v} := \varepsilon^2 [v^1_p + \varepsilon \partial_x v^1_p + \varepsilon \partial_x^2 v^1_p] e^{\partial_x v^1_p} + \varepsilon^2 \partial_x^2 v^1_p + \varepsilon^2 \partial_x v^2, \quad (117)
\]

Here \( C_{cut} \) is the error contributed by cutting off the layers:

\[
C_{cut} := (1 - \chi) f_2 + \sqrt{\varepsilon \mu} Y^{2,0}_p + 3 \sqrt{\varepsilon \mu} \partial_y u^{2,0}_p + \varepsilon\frac{\mu'''}{\mu} \int_{-\infty}^{\infty} u^{2,0}_p. \quad (118)
\]

The total contributions to the remainder forcing is:

\[
F_u := C_{1,u} + C_{3,u}, \quad F_v := C_{1,v} + C_{3,v}. \quad (119)
\]

We can break up the forcing contribution into:

\[
F_u = T_{u,\varepsilon^2} + O(\varepsilon^2), \quad F_v = T_{u,\varepsilon^2} + O(\varepsilon^2), \quad (120)
\]
where the terms at $O(\varepsilon^2)$ are the following:

$$T_{u,\varepsilon^2} := \varepsilon^2 \left[ u^1_\varepsilon \partial_x u^1_\varepsilon + v^1_\varepsilon \partial_y u^1_\varepsilon - \Delta u^1_\varepsilon + \partial_y u^2_\varepsilon v^1_\varepsilon + \mu' v^2_\varepsilon + \mu \chi v^0_{\varepsilon} + 3 \chi' \partial_y u^0_{\varepsilon} \right] \tag{121}$$

$$T_{v,\varepsilon^2} := \varepsilon^2 \left[ u^1_\varepsilon \partial_x v^1_\varepsilon + v^1_\varepsilon \partial_y v^1_\varepsilon - \Delta v^1_\varepsilon + \mu \partial_x v^2_\varepsilon \right] \tag{122}$$

Summarizing the above constructions:

**Lemma 13** The following estimates are satisfied by $u_s, v_s$:

$$|\partial_x u_s| + |\partial_y v_s| + |u_s - \mu| \leq \min\{O(\sqrt{\varepsilon}), O(\varepsilon)\}, \tag{123}$$

$$|\partial_y u_s - \mu'| \lesssim \sqrt{\varepsilon}, \tag{124}$$

$$|\partial^l_x v_s| \lesssim \varepsilon \tilde{y} \text{ for } l \geq 0. \tag{125}$$

The following are satisfied by $T_{u,\varepsilon^2}, T_{v,\varepsilon^2}$:

$$||T_1, T_2||_2 \lesssim c_0 \left( \frac{\mu'''}{\mu} \right), \tag{126}$$

$$||\partial_y T_1, \partial_y T_2||_2 \lesssim c_0 \left( \frac{\mu'''}{\mu} \right) \times \varepsilon^{-\frac{1}{4}}. \tag{127}$$

**Proof.** Only $\frac{T_2}{\tilde{y}}$ is non-trivial, and it follows by examining that all terms in $T_2$ satisfy $T_2|_{y=0} = 0 = T_2|_{y=2}$. Note that we have used (123) and (124) to conclude that $\Delta v^1_{\varepsilon}|_{y=0,2} = 0$.\[\square\]
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