The Soliton-Ricci Flow over Compact Manifolds

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Abstract

We introduce a flow of Riemannian metrics over compact manifolds with formal limit at infinite time a shrinking Ricci soliton. We call this flow the Soliton-Ricci flow. It correspond to a Perelman’s modified backward Ricci type flow with some special restriction conditions. The restriction conditions are motivated by convexity results for Perelman’s $W$-functional over convex subsets inside adequate subspaces of Riemannian metrics. We show indeed that the Soliton-Ricci flow is generated by the gradient flow of the restriction of Perelman’s $W$-functional over such subspaces. Assuming long time existence of the Soliton-Ricci flow we show exponentially fast convergence to a shrinking Ricci soliton provided that the Bakry-Emery-Ricci tensor is uniformly strictly positive with respect to the evolving metric.

1 Introduction

The notion of Ricci soliton (in short RS) has been introduced by D.H. Friedon in [Fri]. It is a natural generalization of the notion of Einstein metric. The terminology is justified by the fact that the pull back of the RS metric via the flow of automorphisms generated by its vector field provides a Ricci flow.

In this paper we introduce the Soliton-Ricci flow, (in short SRF) which is a flow of Riemannian metrics with formal limit at infinite time a shrinking Ricci soliton.

Our interest in the SRF is motivated from the fact that it generates solutions of the Soliton-Kähler-Ricci flow (in short SKRF) for Kähler initial data (see [Pal2]).

A remarkable formula due to Perelman [Per] shows that the modified (and normalized) Ricci flow is the gradient flow of Perelman’s $W$ functional with respect to a fixed choice of the volume form $\Omega$. In this paper we will denote by $W_\Omega$ the corresponding Perelman’s functional. However Perelman’s work does not shows a priori any convexity concerning the functional $W_\Omega$.

The main attempt of this work is to fit the SRF into a gradient system picture. We mean by this the picture corresponding to the gradient flow of a convex functional.
The SRF correspond to a Perelman’s modified backward Ricci type flow with 3-symmetric covariant derivative of the $\Omega$-Bakry-Emery-Ricci (in short $\Omega$-BER) tensor along the flow. The notion of SRF (or more precisely of $\Omega$-SRF) is inspired from the recent work [Pal1] in which we show convexity of Perelman’s $W_\Omega$ functional along geodesics with 3-symmetric covariant derivative variation over points with non-negative $\Omega$-BER tensor.

This type of variations plays an important role also because they allow to generate the $\Omega$-SKRF via the $\Omega$-SRF thanks to an ODE flow of complex structures of Lax type (see [Pal2]).

The surprising fact is that the $\Omega$-SRF is a forward and strictly parabolic heat type flow with respect to such variations. However we can not expect to solve this flow equation for arbitrary initial data. It is well known that backward heat type equations, such as the backward Ricci flow (roughly speaking), can not be solved for arbitrary initial data.

We will call scattering data some special initial data which imply the formal existence of the $\Omega$-SRF as a formal gradient flow of the restriction of Perelman’s $W$-functional over adequate subspaces of Riemannian metrics.

To be more precise, we are looking for sub-varieties $\Sigma$ in the space of Riemannian metrics such that at each point $g \in \Sigma$ the tangent space of $\Sigma$ in $g$ is contained in the space of variations with 3-symmetric covariant derivative and such that the gradient of the functional $W_\Omega$ is tangent to $\Sigma$ at each point $g \in \Sigma$.

So at first place we want that the set of initial data allow a 3-symmetric covariant derivative of the variation of the metric along the $\Omega$-SRF. The precise definition of the set of scattering data and of the sub-varieties $\Sigma$ will be given in the next section.

2 Statement of the main result

Let $\Omega > 0$ be a smooth volume form over an oriented Riemannian manifold $(X, g)$ of dimension $n$. We remind that the $\Omega$-Bakry-Emery-Ricci tensor of $g$ is defined by the formula

$$\text{Ric}_g(\Omega) := \text{Ric}(g) + \nabla_g d \log \frac{dV_g}{\Omega}.$$ A Riemannian metric $g$ is called a $\Omega$-Shrinking Ricci soliton (in short $\Omega$-ShRS) if $g = \text{Ric}_g(\Omega)$. We observe that the set of variations with 3-symmetric covariant derivative coincides with the vector space

$$\mathbb{F}_g := \left\{ v \in C^\infty(X, S^2_{\text{anti}} T^*_X) \mid \nabla_{\tau_X, g} v^*_g = 0 \right\},$$

where $\nabla_{\tau_X, g}$ denotes the covariant exterior derivative acting on $T_X$-valued differential forms and $v^*_g := g^{-1} v$. We define also the set of pre-scattering data $S_\Omega$ as the subset in the space of smooth Riemannian metrics $\mathcal{M}$ over $X$ given by

$$S_\Omega := \left\{ g \in \mathcal{M} \mid \nabla_{\tau_X, g} \text{Ric}_g^*(\Omega) = 0 \right\}.$$
Definition 1. (The $\Omega$-Soliton-Ricci flow). Let $\Omega > 0$ be a smooth volume form over an oriented Riemannian manifold $X$. A $\Omega$-Soliton-Ricci flow (in short $\Omega$-SRF) is a flow of Riemannian metrics $(g_t)_{t \geq 0} \subset \mathcal{S}_\alpha$ solution of the evolution equation $\dot{g}_t = \operatorname{Ric}_{g_t}(\Omega) - g_t$.

We equip the set $\mathcal{M}$ with the scalar product

$$G_g(u, v) = \int_X \langle u, v \rangle_g \Omega,$$

for all $g \in \mathcal{M}$ and all $u, v \in \mathcal{H} := L^2(X, S^2_{\text{sym}} T^*_X)$. We denote by $d_G$ the induced distance function. Let $P^*_g$ be the formal adjoint of an operator $P$ with respect to a metric $g$. We observe that the operator

$$P^*_g \Omega := e^f P^*_g \left( e^{-f} \cdot \right),$$

with $f := \log \frac{dV}{dV_g}$, is the formal adjoint of $P$ with respect to the scalar product (2.1). We define also the $\Omega$-Laplacian operator

$$\Delta^\alpha_g := \nabla^*_g \nabla_g = \Delta_g + \nabla_g f \cdot \nabla_g .$$

We remind (see [Pall]) that the first variation of the $\Omega$-Bakry-Emery-Ricci tensor is given by the formula

$$2 \frac{d}{dt} \operatorname{Ric}_{g_t}(\Omega) = - \nabla^*_g \mathcal{D}_{g_t} \dot{g}_t ,$$

(2.2)

where $\mathcal{D}_g := \hat{\nabla}_g - 2 \nabla_g$, with $\hat{\nabla}_g$ being the symmetrization of $\nabla_g$ acting on symmetric 2-tensors. Explicitly

$$\hat{\nabla}_g \alpha (\xi_0, ..., \xi_p) := \sum_{j=0}^p \nabla_g \alpha (\xi_j, \xi_0, ..., \hat{\xi}_j, ..., \xi_p) ,$$

for all $p$-tensors $\alpha$. We observe that formula (2.2) implies directly the variation formula

$$2 \frac{d}{dt} \operatorname{Ric}_{g_t}(\Omega) = - \Delta^\alpha_{g_t} \dot{g}_t ,$$

(2.3)

along any smooth family $(g_t)_{t \in (0, \varepsilon)} \subset \mathcal{M}$ such that $\dot{g}_t \in \mathcal{F}_{g_t}$ for all $t \in (0, \varepsilon)$. We deduce that the $\Omega$-SRF is a forward and strictly parabolic heat type flow of Riemannian metrics. In the appendix we give a direct proof of the variation formula (2.3) which shows that the Laplacian term on the right hand side is produced from the variation of the Hessian of $f_t := \log \frac{dV_{g_t}}{dV}$. Moreover the formula (2.3) implies directly the variation formula

$$2 \frac{d}{dt} \operatorname{Ric}^*_{g_t}(\Omega) = - \Delta^\alpha_{g_t} \dot{g}_t^* - 2 \dot{g}_t^* \operatorname{Ric}^*_{g_t}(\Omega) .$$

(2.4)

It is quite crucial and natural from the technical point of view to introduce a "center of polarization" $K$ of the space $\mathcal{M}$. We consider indeed a section
$K \in C^\infty(X, \text{End}(T_X))$ with $n$-distinct real eigenvalues almost everywhere over $X$ and we define the vector space

$$\mathbb{F}^K_g := \left\{ v \in \mathbb{F}_g \mid [\nabla^p_g T, v^*_g] = 0, \ T = R_g, K, \ \forall p \in \mathbb{Z}_{\geq 0} \right\}.$$  

(From the technical point of view is more natural to introduce this space in a different way that we will explain in the next sections.) For any $g_0 \in \mathcal{M}$ we define the sub-variety

$$\Sigma_K(g_0) := \mathbb{F}^K_{g_0} \cap \mathcal{M}.$$  

It is a totally geodesic and flat sub-variety of the non-positively curved Riemannian manifold $(\mathcal{M}, G)$ which satisfies the fundamental property

$$T_{\Sigma_K(g_0), g} = \mathbb{F}^K_g, \ \forall g \in \Sigma_K(g_0),$$  

(see lemma [I] in section [I] below). We define the set of scattering data with center $K$ as the set of metrics

$$\mathcal{S}^K: = \left\{ g \in \mathcal{M} \mid \text{Ric}_g(\Omega) \in \mathbb{F}^{K}_g \right\},$$  

and the subset $\mathcal{S}^K_{\Omega, +}: = \left\{ g \in \mathcal{S}^K \mid \text{Ric}_g(\Omega) > 0 \right\}$. We observe that $\mathcal{S}^K_{\Omega, +} \neq \emptyset$ if the manifold $X$ admit a $\Omega$-ShRS. Moreover if $g \in \mathcal{S}^K_{\Omega, +}$ and if $\dim_\mathcal{M} \mathbb{F}^K_g = 1$ then $g$ solves the $\Omega$-ShRS equation up to a constant factor $\lambda > 0$, i.e $\lambda g$ is a $\Omega$-ShRS. With this notations we can state our main result.

**Theorem 1. (Main result).** Let $X$ be a $n$-dimensional compact and orientable manifold oriented by a smooth volume form $\Omega > 0$ and let $K \in C^\infty(X, \text{End}(T_X))$ with almost everywhere n-distinct real eigenvalues over $X$. If $\mathcal{S}^K_{\Omega, +} \neq \emptyset$ then hold the following statements.

(A) For any data $g_0 \in \mathcal{S}^K_{\Omega, +}$ and any metric $g \in \Sigma_K(g_0)$ hold the identities

$$\nabla G\mathcal{W}_{\Omega}(g) = g - \text{Ric}_g(\Omega) \in T_{\Sigma_K(g_0), g},$$

$$\nabla^{\Sigma_K(g_0)} G D\mathcal{W}_{\Omega}(g)(v, v) = \int_X \left( \langle v \text{Ric}_g^*(\Omega), v \rangle_g + \frac{1}{2} \|\nabla_g v\|_g^2 \right) \Omega,$$

for all $v \in T_{\Sigma_K(g_0), g}$. The functional $\mathcal{W}_{\Omega}$ is $G$-convex over the $G$-convex set

$$\Sigma_K(g_0) := \left\{ g \in \Sigma_K(g_0) \mid \text{Ric}_g(\Omega) \geq \text{Ric}_{g_0}(\Omega) \right\},$$

inside the totally geodesic and flat sub-variety $\Sigma_K(g_0)$ of the non-positively curved Riemannian manifold $(\mathcal{M}, G)$.

(B) For all $g_0 \in \mathcal{S}^K_{\Omega, +}$ with $\text{Ric}_{g_0}(\Omega) \geq \varepsilon g_0$, $\varepsilon \in \mathbb{R}_{>0}$ the functional $\mathcal{W}_{\Omega}$ is $G$-convex over the $G$-convex sets

$$\Sigma^\delta_K(g_0) := \left\{ g \in \Sigma_K(g_0) \mid \text{Ric}_g(\Omega) \geq \delta g \right\}, \ \forall \delta \in [0, \varepsilon),$$

$$\Sigma^{\varepsilon}(K, g_0) := \left\{ g \in \Sigma_K(g_0) \mid 2 \text{Ric}_g(\Omega) + g_0 \Delta_{g_0}^\alpha \log(g_0^{-1} g) \geq 0 \right\}.$$
In this case let $\Sigma^+_K(g_0)$ be the closure of $\Sigma^+_K(g_0)$ with respect to the metric $d_G$. Then there exists a natural integral extension $\mathcal{W}_\Omega : \Sigma^+_K(g_0) \to \mathbb{R}$ of the functional $\mathcal{W}_\Omega$ which is $d_G$-lower semi-continuous, uniformly bounded from below and $d_G$-convex over the $d_G$-closed and $d_G$-convex set $\Sigma^+_K(g_0)$ inside the non-positively curved length space $(\mathcal{M}^{\text{loc}}, d_G)$.

(C) The formal gradient flow of the functional $\mathcal{W}_\Omega : \Sigma^+_K(g_0) \to \mathbb{R}$ with initial data $g_0 \in \mathcal{S}^K_{\text{loc}}$ represents a smooth solution of the $\Omega$-SRF equation.

Assume all time existence of the $\Omega$-SRF $(g_t)_{t \geq 0} \subset \Sigma^+_K(g_0)$ and the existence of $\delta \in \mathbb{R}_{\geq 0}$ such that $\text{Ric}_{g_t}(\Omega) \geq \delta g_t$ for all times $t \geq 0$. Then the $\Omega$-SRF $(g_t)_{t \geq 0}$ converges exponentially fast with all its space derivatives to a $\Omega$-shrinking Ricci soliton $g_{RS} \in \Sigma^+_K(g_0)$ as $t \to +\infty$.

We wish to point out that the $G$-convexity of the previous sets is part of the statement. Moreover it is possible to define a $d_G$-lower semi-continuous and $d_G$-convex extension of the functional $\mathcal{W}_\Omega$ over the closure of $\Sigma^+_K(g_0)$ with respect to the metric $d_G$. However this statement is not needed for our purposes.

In order to show the convexity statements we need to perform a key change of variables which shows in particular that the SRF equation over $\Sigma^+_K(g_0)$ corresponds to an endomorphism-valued porous medium type equation. The assumption on the uniform positive lower bound of the $\Omega$-Bakry-Emery-Ricci tensor in the statement (C) allows to obtain the exponential decay of $C^1(X)$-norms via the maximum principle. This assumption seem to be reasonable in view of the $G$-convexity of the sets $\Sigma^+_K(g_0)$.

The presence of some curvature therm in the evolution equation of higher order space derivatives turns off the power of the maximum principle. In order to show the exponentially fast convergence of higher order space derivatives we use an interpolation method introduced by Hamilton in his proof of the exponential convergence of the Ricci flow in [Ham]. The difference with the technique in [Ham] is a more involved interpolation process due to the presence of some extra curvature therm which seem to be alien to Hamilton’s argument. We are able to perform our interpolation process by using some intrinsic properties of the $\Omega$-SRF.

### 3 Conservative differential symmetries

In this section we show that some relevant differential symmetries are preserved along the geodesics induced by the scalar product (2.1).

#### 3.1 First order conservative differential symmetries

We introduce first the cone $\mathbb{F}^\infty_g$ inside the vector space $\mathbb{F}_g$ given by:

$$\mathbb{F}^\infty_g := \left\{ v \in C^\infty(X, S^2 T^*_X) \mid \nabla_{\tau_X \cdot g} (v^*_g)^p = 0, \forall p \in \mathbb{Z}_{>0} \right\}$$

$$= \left\{ v \in C^\infty(X, S^2 T^*_X) \mid \nabla_{\tau_X \cdot e^{tv_g}} e^{tv_g} = 0, \forall t \in \mathbb{R} \right\}.$$
We need also a few algebraic definitions. Let $V$ be a real vector space. We consider the contraction operator
\[ \neg : \text{End}(V) \times \Lambda^2 V^* \to \Lambda^2 V^* , \]
defined by the formula
\[ H \neg (\alpha \wedge \beta) := (\alpha \cdot H) \wedge \beta + \alpha \wedge (\beta \cdot H) , \]
for any $H \in \text{End}(V)$, $u, v \in V$ and $\alpha, \beta \in V^*$. We can also define the contraction operator by the equivalent formula
\[ (H \neg \varphi)(u, v) := \varphi(Hu, v) + \varphi(u, Hv) , \]
for any $\varphi \in \Lambda^2 V^*$. Moreover for any element $A \in (V^*)^\otimes 2 \otimes V$ we define the following elementary operations over the vector space $(V^*)^\otimes 2 \otimes V$;
\[ (AH)(u, v) := A(u, Hv) , \]
\[ (HA)(u, v) := HA(u, v) , \]
\[ (H \cdot A)(u, v) := A(Hu, v) , \]
\[ \text{Alt} A(u, v) := A(u, v) - A(v, u) . \]
Assume now that $V$ is equipped with a metric $g$. Then we can define the $g$-transposed $A^T_g \in (V^*)^\otimes 2 \otimes V$ as follows. For any $v \in V$
\[ v \neg A^T_g := (v \neg A)^T_g . \]
We remind (see \cite{Pal1}) that the geodesics in the space of Riemannian metrics with respect to the scalar product (2.1) are given by the solutions of the equation
\[ \dot{g}_t := g_0^{-1} \dot{g}_t = g_0^{-1} \dot{g}_0 . \]
Thus the geodesic curves write explicitly as
\[ g_t = g_0 e^{tg_0^{-1} \dot{g}_0} . \] (3.1)
With this notations we can show now the following fact.

**Lemma 1.** Let $(g_t)_{t \in \mathbb{R}}$ be a geodesic such that $\dot{g}_0 \in \mathbb{F}_g^\infty$. Then $\dot{g}_t \in \mathbb{F}_g^\infty$ for all $t \in \mathbb{R}$.

**Proof.** Let $H \in C^\infty(X, \text{End}(TX))$ and let $(g_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ be an arbitrary smooth family. We expand first the time derivative
\[ \hat{\nabla}_{\dot{g}_t} H(\xi, \eta) = \hat{\nabla}_{g_t} H(\xi, \eta) - \hat{\nabla}_{g_t} H(\eta, \xi) \]
\[ = \hat{\nabla}_{g_t}(\xi, H\eta) - \hat{\nabla}_{g_t}(\eta, H\xi) \]
\[ - H \left[ \hat{\nabla}_{g_t}(\xi, \eta) - \hat{\nabla}_{g_t}(\eta, \xi) \right] \]
\[ = \hat{\nabla}_{g_t}(H\eta, \xi) - \hat{\nabla}_{g_t}(H\xi, \eta) , \]
since \( \hat{\nabla}_g \in C^\infty(X, S^2 T^*_X \otimes T_X) \) thanks to the variation identity (see \[Bes\])

\[
2 g_t \left( \hat{\nabla}_g (\xi, \eta, \mu) \right) = \nabla_g \dot{g}_t (\xi, \eta, \mu) + \nabla_g \dot{g}_t (\eta, \xi, \mu) - \nabla_g \dot{g}_t (\mu, \xi, \eta) \quad (3.2)
\]

We observe now that the variation formula (3.2) rewrites as

\[
2 \hat{\nabla}_g (\xi, \eta) = \nabla_g \dot{g}_t^* (\xi, \eta) + \nabla_g \dot{g}_t^* (\eta, \xi) - (\nabla_g \dot{g}_t^* \eta)^T g_t \xi .
\]

Thus

\[
2 \hat{\nabla}_{\tau_X, s_t} H (\xi, \eta) = \nabla_{\tau_X, s_t} \dot{g}_t^* (H \eta, \xi) + \nabla_{\tau_X, s_t} \dot{g}_t^* (\xi, H \eta) - (\nabla_{\tau_X, s_t} \dot{g}_t^* \eta)^T g_t \xi .
\]

Applying the identity

\[
(\nabla_{\tau_X, s_t} \dot{g}_t^* \eta)^T g_t \xi = - \left( \xi - \nabla_{\tau_X, s_t} \dot{g}_t^* \right)^T g_t \eta + \xi - \nabla_{\tau_X, s_t} \dot{g}_t^* ,
\]

we obtain the equalities

\[
2 \hat{\nabla}_{\tau_X, s_t} H (\xi, \eta) = \nabla_{\tau_X, s_t} \dot{g}_t^* (H \eta, \xi) + \nabla_{\tau_X, s_t} \dot{g}_t^* (\xi, H \eta) - \left( \nabla_{\tau_X, s_t} \dot{g}_t^* \eta \right)^T g_t H \eta \\
- \nabla_{\tau_X, s_t} \dot{g}_t^* (H \xi, \eta) - \nabla_{\tau_X, s_t} \dot{g}_t^* (\eta, H \xi) + \left( \nabla_{\tau_X, s_t} \dot{g}_t^* \eta \right)^T g_t H \xi .
\]

We infer the variation formula

\[
2 \hat{\nabla}_{\tau_X, s_t} H = - H - \nabla_{\tau_X, s_t} \dot{g}_t^* + \nabla_{\tau_X, s_t} (\dot{g}_t^* H) - \dot{g}_t^* \nabla_{\tau_X, s_t} H \\
+ \Alt \left( \nabla_{\tau_X, s_t} \dot{g}_t^* (H \eta) \right)^T g_t \eta ,
\]

Thus along any geodesic hold the upper triangular type infinite dimensional ODE system

\[
2 \frac{d}{dt} \left[ \nabla_{\tau_X, s_t} (\dot{g}_t^*)^p \right] = - (\dot{g}_t^*)^p - \nabla_{\tau_X, s_t} \dot{g}_t^* \\
+ \nabla_{\tau_X, s_t} (\dot{g}_t^*)^p + \dot{g}_t^* \nabla_{\tau_X, s_t} (\dot{g}_t^*)^p + \Alt \left( \nabla_{\tau_X, s_t} \dot{g}_t^* (\dot{g}_t^*)^p \right) ,
\]

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for all \( p \in \mathbb{Z}_{>0} \). We remind now that \( \dot{g}_t^* \equiv \dot{g}_0^* \) and we observe the formula
\[
\frac{d}{dt} \left( \nabla_{\tau X \cdot g_t} \dot{g}_t^* \right)_{g_t}^T = \left( \left( \nabla_{\tau X \cdot g_t} \dot{g}_t^* \right)_{g_t}^T , \dot{g}_t^* \right) + \frac{d}{dt} \left( \nabla_{\tau X \cdot g_t} \dot{g}_t^* \right)_{g_t}^T .
\]
This combined once again with the identity \( \dot{g}_t^* \equiv \dot{g}_0^* \) and with the previous variation formula implies that for all \( k, p \in \mathbb{Z}_{\geq 0} \) hold the identity
\[
\frac{d^k}{dt^k} \bigg|_{t=0} \left[ \nabla_{\tau X \cdot g_t} (\dot{g}_t^*)^p \right] = 0 .
\]
Indeed this follows from an increasing induction in \( k \). The conclusion follows from the fact that the curves
\[
t \mapsto \left( \nabla_{\tau X \cdot g_t} (\dot{g}_t^*)^p \right),
\]
are real analytic over the real line.

Let now \( A \in (V^*)^p \otimes V, B \in (V^*)^q \otimes V \) and let \( k = 1, \ldots, q \). We define the generalized product operation
\[
(A B)(u_1, \ldots, u_{p-1}, v_1, \ldots, v_q) := A(u_1, \ldots, u_{p-1}, B(v_1, \ldots, v_q)) .
\]
With this notations we define the vector space
\[
\mathbb{E}_g := \left\{ v \in C^\infty (X, S^2_{\mathbb{R}} T_X^*) \mid [R_g, v^*_g] = 0, [R_g, \nabla_{g, \xi} v^*_g] = 0, \forall \xi \in T_X \right\} ,
\]
and we show the following crucial fact.

**Lemma 2.** Let \( (g_t)_{t \in \mathbb{R}} \subset M \) be a geodesic such that \( \dot{g}_0 \in \mathbb{E}_{g_0}^\infty \cap \mathbb{E}_{g_0}^0 \). Then \( \dot{g}_t \in \mathbb{E}_{g_t}^\infty \cap \mathbb{E}_{g_t}^0 \) for all \( t \in \mathbb{R} \).

**Proof.** We observe first that the variation identity (3.2) combined with the fact that \( \dot{g}_t \in \mathbb{E}_{g_t}^\infty \) implies the variation identity
\[
2 \nabla_{g_t} \dot{g}_t = \nabla_{g_t} \dot{g}_t^* .
\]
Thus the variation formula (see [Bes])
\[
\mathcal{R}_{g_t}(\xi, \eta) \mu = \nabla_{g_t} \nabla_{g_t}(\xi, \eta, \mu) - \nabla_{g_t} \nabla_{g_t}(\eta, \xi, \mu) ,
\]
rewrites as
\[
2 \mathcal{R}_{g_t}(\xi, \eta) = \nabla_{g_t, \xi} \nabla_{g_t, \eta} \dot{g}_t^* - \nabla_{g_t, \eta} \nabla_{g_t, \xi} \dot{g}_t^* - \nabla_{g_t, [\xi, \eta]} \dot{g}_t^* = \left[ R_{g_t}(\xi, \eta), \dot{g}_t^* \right] .
\]
We remind in fact the general identity
\[
\nabla_{g_t, \xi} \nabla_{g_t, \eta} H - \nabla_{g_t, \eta} \nabla_{g_t, \xi} H = \left[ R_g(\xi, \eta), H \right] + \nabla_{g, [\xi, \eta]} H ,
\]
(3.6)
for any $H \in C^\infty(X, \text{End}(T_X))$. We deduce the variation identity

$$2 \mathcal{R}_{g_t} = [\mathcal{R}_{g_t}, \dot{g}_t^*],$$

(for any smooth curve $(g_t)$, such that $\dot{g}_t \in \mathcal{F}_{g_t}$), and the variation formula

$$2 \frac{d}{dt} [\mathcal{R}_{g_t}, \dot{g}_t^*] = \left[ [\mathcal{R}_{g_t}, \dot{g}_t^*], \dot{g}_t^* \right].$$

Thus the identity $[\mathcal{R}_{g_t}, \dot{g}_t^*] = 0$ hold for all times by Cauchy uniqueness. We infer in particular $\mathcal{R}_{g_t} = \mathcal{R}_{g_0}$ for all $t \in \mathbb{R}$ thanks to the identity (3.7). Using the variation formula

$$2 \dot{\nabla}_{g_t} H = 2 \dot{\nabla}_{g_t} \cdot H - 2 H \dot{\nabla}_{g_t} = \left[ \nabla_{g_t}, \dot{g}_t^* H \right],$$

we deduce

$$2 \frac{d}{dt} [\mathcal{R}_{g_t}, \nabla_{g_t, \xi} \dot{g}_t^*] = \left[ \mathcal{R}_{g_t}, \left[ \nabla_{g_t, \xi} \dot{g}_t^*, \dot{g}_t^* \right] \right]$$

$$= \left[ \mathcal{R}_{g_t}, \left[ \nabla_{g_t, \xi} \dot{g}_t^*, \mathcal{R}_{g_t}, \dot{g}_t^* \right] \right] - \left[ \dot{g}_t^*, \mathcal{R}_{g_t}, \nabla_{g_t, \xi} \dot{g}_t^* \right]$$

$$= \left[ \left[ \mathcal{R}_{g_t}, \nabla_{g_t, \xi} \dot{g}_t^* \right], \dot{g}_t^* \right],$$

by the Jacobi identity and by the previous result. We infer the conclusion by Cauchy uniqueness.

3.2 Conservation of the pre-scattering condition

This sub-section is the heart of the paper. We will show the conservation of the pre-scattering condition along curves with variations in $\mathcal{F}_g \cap \mathcal{E}_g$. We need to introduce first a few other product notations. Let $(e_k)_k$ be a $g$-orthonormal basis. For any elements $A \in (T_X^*)^2 \otimes T_X$ and $B \in A^2 T_X^* \otimes \text{End}(T_X)$ we define the generalized products

$$(B \ast A)(u, v) := B(u, e_k)A(e_k, v),$$

$$(B \odot A)(u, v) := [B(u, e_k), e_k \cdot A] v,$$

$$(A \ast B)(u, v) := A(e_k, B(u, v)e_k).$$

We observe that the algebraic Bianchi identity implies

$$\text{Alt}(\mathcal{R}_g \odot A) = \text{Alt}(\mathcal{R}_g \ast A) - A \ast \mathcal{R}_g.$$  \hspace{1cm} (3.9)

Let also $H \in C^\infty(X, \text{End}(T_X))$. Then hold the identity

$$\nabla_{\mathcal{T}, g} H \ast \mathcal{R}_g = 2 \nabla_g H \ast \mathcal{R}_g.$$  \hspace{1cm} (3.10)
We observe in fact the equalities
\[ \nabla_{\tau X,g}H \ast R_g = \nabla_g H \ast R_g - \nabla_g (R_g e_k, e_k) = 2 \nabla_g H \ast R_g . \]
This follows writing with respect to the $g$-orthonormal basis $(e_k)$ the identity
\[ R_g(\xi, \eta) = - (R_g(\xi, \eta))^T_g , \]
which is a consequence of the alternating property of the $(4,0)$-Riemann curvature operator.

For any $A \in C^\infty(X, (T_X^*)^p \otimes T_X^*)$ we define the divergence type operations
\[ \text{div}_g A(u_1, \ldots, u_p) := \text{Tr}_g [\nabla_g A(\cdot, u_1, \ldots, u_p)] , \]
\[ \text{div}^g_\tau A(u_1, \ldots, u_p) := \text{div}_g A(u_1, \ldots, u_p) - A(u_1, \ldots, u_p, \nabla_g f) . \]
We remind that the once contracted differential Bianchi identity writes often as
\[ \text{div}_g R_g = - \nabla_{\tau X,g} \nabla^2 g = R_g \ast \nabla g f \]
implies
\[ \text{div}^g_\tau R_g = - \nabla_{\tau X,g} \text{Ric}^*_g(\Omega) . \tag{3.11} \]

With the previous notations hold the following lemma.

**Lemma 3.** Let $(g_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ be a smooth family such that $\dot{g}_t \in \mathcal{F}_{g_t}$ for all $t \in \mathbb{R}$. Then hold the variation formula
\[ 2 \frac{d}{dt} \left[ \nabla_{\tau X,g_t} \text{Ric}^*_g(\Omega) \right] = \text{div}^g_\tau \left[ R_{g_t}, \dot{g}_t^* \right] + \text{Alt} (R_{g_t} \otimes \nabla g_t \dot{g}_t^*) - 2 \dot{g}_t^* \nabla_{\tau X,g_t} \text{Ric}^*_g(\Omega) . \]

**Proof.** We will show the above variation formula by means of the identity \[3.11\]. Consider any $B \in C^\infty(X, \Lambda^2 T_X^* \otimes \text{End}(T_X))$. Time deriving the definition of the covariant derivative $\nabla_{g_t}B$ we deduce the formula
\[ \dot{\nabla}_{g_t}B(\xi, u, v) w = \dot{\nabla}_{g_t}(\xi, (B(u, v) w) - B \left( \dot{\nabla}_{g_t}(\xi, u), v \right) w - B(u, \dot{\nabla}_{g_t}(\xi, v)) w - B(u, v) \dot{\nabla}_{g_t}(\xi, w) . \]
We infer the expression
\[ 2 \dot{\nabla}_{g_t}B(\xi, u, v) w = \nabla_{g_t, \xi} \dot{g}_t^* B(u, v) w - \dot{B}(\nabla_{g_t, \xi} \dot{g}_t^* u, v) w - B(u, \nabla_{g_t, \xi} \dot{g}_t^* v) w - B(u, v) \nabla_{g_t, \xi} \dot{g}_t^* w , \tag{3.12} \]
thanks to the formula \[3.4\]. We fix now an arbitrary space-time point $(x_0, t_0)$ and we pick a local tangent frame $(e_k)_k$ in a neighborhood of $x_0$ which is $g_{t_0}(x_0)$-orthonormal at the point $x_0$ and satisfies $\nabla_{g_t, e_j}(x_0) = 0$ at the time $t_0$ for all $j$. Then time deriving the term
\[ (\text{div}^g_\tau B)(\xi, \eta) = \nabla_{g_t, e_k} B(\xi, \eta) g_t^{-1} e_k^* \ast B(\xi, \eta) \nabla_{g_t} f_t , \]
and using the expression (3.12) we obtain the identity

$$2 \frac{d}{dt} (\text{div}^\alpha g_t B)(\xi, \eta) = \nabla_{g_t,e_k} \hat{g}_t^* B(\xi, \eta) e_k - B(\nabla_{g_t,e_k} \hat{g}_t^* \xi, \eta) e_k$$

$$- B(\xi, \nabla_{g_t,e_k} \hat{g}_t^* \eta) e_k - B(\xi, \eta) \nabla_{g_t,e_k} \hat{g}_t^* e_k$$

$$- 2 \nabla_{g_t,e_k} B(\xi, \eta) \hat{g}_t^* e_k - 2 B(\xi, \eta) \frac{d}{dt} \nabla_{g_t} f_t,$$  (3.13)

at the space-time \((x_0, t_0)\). Moreover hold the elementary formula

$$2 \frac{d}{dt} \nabla_{g_t} f_t = \nabla_{g_t} \text{Tr}_{g_t} \hat{g}_t - 2 \hat{g}_t^* \nabla_{g_t} f_t.$$  

We observe also that at the space time point \((x_0, t_0)\) hold the trivial equalities

$$\nabla_{g_t} \text{Tr}_{g_t} \hat{g}_t = e_k \cdot (\text{Tr}_{g_t} \hat{g}_t^*) e_k$$

$$= e_k \cdot g_t (\hat{g}_t^* e_j, e_j) e_k$$

$$= g_t (\nabla_{g_t,e_k} \hat{g}_t^* e_j, e_j) e_k$$

$$= \nabla_{g_t} \hat{g}_t (e_k, e_j, e_j) e_k$$

$$= \nabla_{g_t} \hat{g}_t (e_j, e_j, e_k) e_k$$

$$= g_t \left( \nabla_{g_t,e_j} \hat{g}_t^* e_j, e_k \right) e_k$$

$$= - \nabla_{g_t} \hat{g}_t^* ,$$

thanks to the assumption \(\hat{g}_t \in \mathbb{F}_{g_t}\). We deduce the identity

$$2 \frac{d}{dt} \nabla_{g_t} f_t = - \nabla_{g_t} \hat{g}_t^* - 2 \hat{g}_t^* \nabla_{g_t} f_t.$$  (3.14)

Thus the identity (3.13) rewrites as follows;

$$2 \frac{d}{dt} (\text{div}^\alpha g_t B)(\xi, \eta) = (\nabla_{g_t} \hat{g}_t^* \ast B)(\xi, \eta)$$

$$- B(\nabla_{g_t,e_k} \hat{g}_t^* \xi, \eta) e_k - B(\xi, \nabla_{g_t,e_k} \hat{g}_t^* \eta) e_k$$

$$- 2 \nabla_{g_t,e_k} B(\xi, \eta) \hat{g}_t^* e_k + 2 B(\xi, \eta) \nabla_{g_t} \hat{g}_t^* .$$

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The assumption $\dot{g}_t \in F_{g_t}$ implies that the endomorphism $\nabla_{g_t} \dot{g}_t^* \xi$ is $g_t$-symmetric. Thus we can choose a $g_{t_0}(x_0)$-orthonormal basis $(e_k) \subset T_{X,x_0}$ which diagonalize it at the space-time point $(x_0,t_0)$. It is easy to see that with respect to this basis hold the identity

$$B(\nabla_{g_t} e_k \dot{g}_t^* \xi , \eta) e_k = - (B * \nabla_{g_t} \dot{g}_t^*) (\eta, \xi) ,$$

by the alternating property of $B$. But the term on the left hand side is independent of the choice of the $g_{t_0}(x_0)$-orthonormal basis. In a similar way choosing a $g_{t_0}(x_0)$-orthonormal basis $(e_k) \subset T_{X,x_0}$ which diagonalizes $\nabla_{g_t} \dot{g}_t^* \eta$ at the space-time point $(x_0,t_0)$ we obtain the identity

$$B(\xi, \nabla_{g_t} e_k \dot{g}_t^* \eta) e_k = (B * \nabla_{g_t} \dot{g}_t^*) (\xi, \eta) .$$

We infer the equality

$$2 \left( \frac{d}{dt} \mathcal{D}_{g_t}^{\alpha} \right) B = \nabla_{g_t} \dot{g}_t^* * B - \text{Alt} (B * \nabla_{g_t} \dot{g}_t^*)$$

$$- 2 \nabla_{g_t} e_k B \dot{g}_t^* e_k + 2 B \nabla_{g_t}^{\alpha} \dot{g}_t^* ,$$

with respect to any $g_{t_0}(x_0)$-orthonormal basis $(e_k)$ at the arbitrary space-time point $(x_0,t_0)$. This combined with (3.7) and (3.9) implies the equalities

$$2 \frac{d}{dt} (\mathcal{D}_{g_t}^{\alpha} R_{g_t}) = 2 \left( \frac{d}{dt} \mathcal{D}_{g_t}^{\alpha} \right) R_{g_t} + \mathcal{D}_{g_t}^{\alpha} [R_{g_t}, \dot{g}_t^*]$$

$$= - \text{Alt} (R_{g_t} \otimes \nabla_{g_t} \dot{g}_t^*) - 2 \nabla_{g_t} e_k R_{g_t} \dot{g}_t^* e_k$$

$$+ 2 R_{g_t} \nabla_{g_t}^{\alpha} \dot{g}_t^* + \mathcal{D}_{g_t}^{\alpha} [R_{g_t}, \dot{g}_t^*] .$$

We observe now that for any smooth curve $(g_t)_{t \in \mathbb{R}} \subset M$ hold the identity

$$\mathcal{D}_{g_t}^{\alpha} [R_{g_t}, \dot{g}_t^*] = \nabla_{g_t} e_k R_{g_t} \dot{g}_t^* e_k - R_{g_t} \nabla_{g_t}^{\alpha} \dot{g}_t^*$$

$$- \nabla_{g_t} \dot{g}_t^* * R_{g_t} - \dot{g}_t^* \mathcal{D}_{g_t}^{\alpha} R_{g_t} .$$

(3.15)

But our assumption $\dot{g}_t \in F_{g_t}$ implies $\nabla_{g_t} \dot{g}_t^* \equiv 0$ thanks to the identity (3.10). Thus we obtain the formula

$$2 \frac{d}{dt} (\mathcal{D}_{g_t}^{\alpha} R_{g_t}) = - \text{Alt} (R_{g_t} \otimes \nabla_{g_t} \dot{g}_t^*) - \mathcal{D}_{g_t}^{\alpha} [R_{g_t}, \dot{g}_t^*]$$

$$- 2 \dot{g}_t^* \mathcal{D}_{g_t}^{\alpha} R_{g_t} ,$$

which implies the required conclusion thanks to the identity (3.11).
Corollary 1. *(Conservation of the pre-scattering condition)*

Let \((g_t)_{t \in \mathbb{R}} \subset \mathcal{M}\) be a smooth family such that \(\dot{g}_t \in \mathbb{E}_{g_t} \cap \mathbb{F}_{g_t}\) for all \(t \in \mathbb{R}\). If \(g_0 \in \mathcal{S}_0\) then \(g_t \in \mathcal{S}_0\) for all \(t \in \mathbb{R}\).

*Proof.* We observe that the assumption \(\dot{g}_t \in \mathbb{E}_{g_t}\) implies in particular the identity \(\mathcal{R}_{g_t} \otimes \nabla_{g_t} \dot{g}_t \equiv 0\). By lemma 3 we infer the variation formula

\[
\frac{d}{dt} \left[ \nabla_{\dot{X} \cdot g_t} \mathcal{R}^*_{g_t}(\Omega) \right] = - \dot{g}_t^* \nabla_{\dot{X} \cdot g_t} \mathcal{R}^*_{g_t}(\Omega),
\]

and thus the conclusion by Cauchy uniqueness. \(\square\)

The total variation of the pre-scattering operator is given in lemma 22 in the appendix. It provides in particular an alternative proof of the conservation of the pre-scattering condition.

### 3.3 Higher order conservative differential symmetries

In this sub-section we will show that some higher order differential symmetries are conserved along the geodesics. This type of higher order differential symmetries is needed in order to stabilize the scattering conditions with respect to the variations produced by the SRF. We observe first that given any diagonal \(n \times n\)-matrix \(\Lambda\) it hold the identity

\[
[\Lambda, M] = ((\lambda_i - \lambda_j) M_{i,j}),
\]

for any other \(n \times n\)-matrix \(M\). Thus if the values \(\lambda_j\) are all distinct then \([\Lambda, M] = 0\) if and only if \(M\) is also a diagonal matrix.

In this sub-section and in the sections 4, 5 that will follow we will always denote by \(K \in \Gamma(X, \text{End}(T_X))\) an element with point wise \(n\)-distinct real eigenvalues, where \(n = \dim_n X\).

The previous remark shows that if \((e_k) \subset T_{X,p}\) is a basis diagonalizing \(K(p)\) then it diagonalizes any element \(M \in \text{End}(T_{X,p})\) such that \([K(p), M] = 0\).

We deduce that if also \(N \in \text{End}(T_{X,p})\) satisfies \([K(p), N] = 0\) then \([M, N] = 0\).

We define now the vector space inside \(\mathbb{F}_g^\infty\)

\[
\mathbb{F}_g(K) := \left\{ v \in \mathbb{F}_g \mid [K, v_g^*] = 0, \left[ K, \nabla_g v_g^* \right] = 0 \right\} \subset \mathbb{F}_g^\infty.
\]

We observe in fact the definition implies \(\nabla_g v_g^*, v_g^* = 0\), and thus the last inclusion. With this notations hold the following corollary analogue to lemma 1.

**Corollary 2.** Let \((g_t)_{t \in \mathbb{R}}\) be a geodesic such that \(\dot{g}_0 \in \mathbb{E}_{g_0}(K)\). Then \(\dot{g}_t \in \mathbb{F}_{g_t}(K)\) for all \(t \in \mathbb{R}\).

*Proof.* By lemma 1 we just need to show the identity \([K, \nabla_g \dot{g}_t] \equiv 0\). In fact using the variation formula we obtain

\[
2 \frac{d}{dt} [K, \nabla_g \dot{g}_t] = \left[ K, [\nabla_g \dot{g}_t^*, \dot{g}_t^*] \right] = - \left[ \dot{g}_t^*, [K, \nabla_g \dot{g}_t] \right],
\]

since \([K, \dot{g}_t^*] \equiv 0\). Then the conclusion follows by Cauchy uniqueness. \(\square\)
We define now the sub-vector space \( \mathbb{F}_g^K \subset \mathcal{F}_g(K) \),

\[
\mathbb{F}_g^K := \left\{ v \in \mathbb{F}_g \mid T, \nabla_g^{p, \xi} v_g^* = 0, T = K, R_g, \forall \xi \in T_X^{\otimes p}, \forall p \in \mathbb{Z}_{\geq 0} \right\},
\]

and we show the following elementary lemmas

**Lemma 4.** If \( u, v \in \mathbb{F}_g^K \) then \( u v_g^* \), \( u e v_g^* \in \mathbb{F}_g^K \).

**Proof.** By assumption follows that \( u_g^* \) commutes with \( v_g^* \). This shows that \( u v_g^* \) is a symmetric form. Again by assumption we infer \( [\nabla_g u_g^*, v_g^*] = [\nabla_g v_g^*, u_g^*] = 0 \), and thus \( u_g^* v_g^* \in \mathbb{F}_g \). We observe now that for any \( A, B, C \in \text{End}(V) \) such that \( [C, A] = 0 \) hold the identity

\[
[C, A B] = A [C, B].
\]

Thus if also \( [C, B] = 0 \) then hold the identity

\[
[C, A B] = 0.
\]

Applying (3.17) with \( C = T \) and with \( A = \nabla_{g, \eta}^{r} u_g^*, \eta \in T_X^{\otimes r}, B = \nabla_{g, \mu}^{p-r} v_g^*, \mu \in T_X^{\otimes p-r} \) we infer the identity

\[
[T, \nabla_{g, \xi}^{p} (u_g^* v_g^*)] = 0,
\]

thus the conclusion \( u v_g^* \in \mathbb{F}_g^K \). The fact \( u e v_g^* \in \mathbb{F}_g^K \) follow directly from the previous one.

For all \( \xi \equiv (\xi_1, \ldots, \xi_p) \in C^\infty(X, T_X)^{\otimes p} \) we denote

\[
\nabla_{g, \xi}^{(p)} v_g^* := \nabla_{g, \xi_1} \cdots \nabla_{g, \xi_p} v_g^*,
\]

and we observe that a simple induction based on the formula

\[
\nabla_{g, \xi}^{p} v_g^* = \nabla_{g, \xi_1} \cdots \nabla_{g, \xi_p} v_g^* - \sum_{r=1}^{p-1} \sum_{I \in J_{p-r}} \varepsilon_I (\nabla_{g, \xi_I} \xi_I) - \nabla_{g, \xi_r} v_g^*,
\]

with \( J_{p-r} := \{ I \subset \{1, \ldots, p-1\} : |I| = p-r \} \), \( \varepsilon_I = 0, 1 \), \( \xi_I \equiv (j_1, \ldots, j_r) := \{1, \ldots, p\} \setminus I \) and with

\[
\nabla_{g, \xi_I} \xi_I := \nabla_{g, \xi_{i_1}} \cdots \nabla_{g, \xi_{i_{p-r}}} (\xi_{j_1} \otimes \cdots \otimes \xi_{j_r}).
\]

\( I \equiv (i_1, \ldots, i_{p-r}) \), shows the identity

\[
\mathbb{F}_g^K = \left\{ v \in \mathbb{F}_g \mid T, \nabla_{g, \xi}^{p} v_g^* = 0, T = K, R_g, \forall \xi \in C^\infty(X, T_X)^{\otimes p}, \forall p \in \mathbb{Z}_{\geq 0} \right\}.
\]

**Lemma 5.** Let \((g_t)_{t \in \mathbb{R}}\) be a geodesic such that \( \dot{g}_0 \in \mathbb{F}_g^K \). Then \( \dot{g}_t \in \mathbb{F}_g^K \) for all \( t \in \mathbb{R} \).
Proof. We observe that \( \dot{g}_t \in F_{g_t}(K) \cap E_{g_t} \) for all \( t \in \mathbb{R} \), thanks to corollary \( \ref{corollary} \) and lemma \( \ref{lemma} \). This implies in particular that \( \mathcal{R}_{g_t} = \mathcal{R}_{g_0} \) for all \( t \in \mathbb{R} \), by the variation formula (3.8). Then the conclusion will follow from the property
\[
\nabla_{g_0, \xi}^{(p)} \dot{g}_0^* = \nabla_{g_1, \xi}^{(p)} \dot{g}_1^*, \quad \forall \xi \in C^\infty(X, T_X)^{\otimes p}, \quad \forall t \in \mathbb{R}, \tag{3.18}
\]
This certainly hold true for \( p = 0 \) by the geodesic equation \( \dot{g}_0^* \equiv \dot{g}_1^* \). We assume now the statement (3.18) true for \( p - 1 \) and we show it for \( p > 0 \). We observe first that thanks to the variation formula (3.8) hold the identity
\[
\nabla_{g_t} H \equiv 0, \tag{3.19}
\]
along any smooth curve \( (g_t)_t \subset \mathcal{M} \) such that \( \dot{g}_t \in F_g \) and \( [\nabla_{g_t} \dot{g}_t, H] = 0 \). In our situation the identity \( [K, \nabla_{g_t} \dot{g}_t^*] \equiv 0 \) combined with the assumption on the initial data
\[
[K, \nabla_{g_0, \xi}^{(p-1)} \dot{g}_0^*] = 0,
\]
for all \( \xi \in C^\infty(X, T_X)^{\otimes (p-1)} \), implies thanks to (3.19) the equalities
\[
\nabla_{g_0} \nabla_{g_0, \xi}^{(p-1)} \dot{g}_0^* = \nabla_{g_1} \nabla_{g_0, \xi}^{(p-1)} \dot{g}_0^* \equiv \nabla_{g_1} \nabla_{g_0, \xi}^{(p-1)} \dot{g}_1^*,
\]
by the inductive hypothesis. We infer the conclusion of the induction. \( \square \)

4 The set of scattering data

We define the set of scattering data with center \( K \) as the set of metrics
\[
S^K_0 := \{ g \in \mathcal{M} \mid \text{Ric}_g(\Omega) \in F_g^K \}
\]
\[
= \{ g \in S_0 \mid [T, \nabla_{g, \xi}^p \text{Ric}_g^*(\Omega)] = 0, \quad T = K, \quad \mathcal{R}_g, \quad \forall \xi \in T_X^{\otimes p}, \quad \forall p \in \mathbb{Z}_{\geq 0} \},
\]
We observe that \( S^K_0 \neq \emptyset \) if the manifold \( X \) admit a \( \Omega \)-ShRS. We introduce now a few new product notations. For any \( A \in (V)^{\otimes p} \otimes V, \quad B \in (V)^{\otimes q} \otimes V \) and for any \( k = 1, \ldots, q \) we define the products \( A \bullet_k B \) as
\[
(A \bullet_k B)(u, v) := B(v_1, \ldots, v_{k-1}, A(u, v_k), v_{k+1}, \ldots, v_q),
\]
for all \( u \equiv (u_1, \ldots, u_{p-1}) \) and \( v \equiv (v_1, \ldots, v_q) \). We note \( \bullet := \bullet_1 \) for simplicity. For any \( \sigma \in S_{p+k-2} \) we define \( A \bullet_k^\sigma B \) as
\[
(A \bullet_k^\sigma B)(u, v) := (A \bullet_k B)(\xi_\sigma, v_k, \ldots, v_q),
\]
where \( \xi \equiv (\xi_1, \ldots, \xi_{p+k-2}) := (u_1, \ldots, u_{p-1}, v_1, \ldots, v_{k-1}) \). We should notice that \( A \bullet_k^\sigma B \equiv A \bullet_k B \) if \( p + k - 2 \leq 1 \). We define
\[
A \ast B := \sum_{k=1}^{q-1} A \bullet_k B.
\]
For $p > 1$ and $k = 1, \ldots, p - 1$ we define the trace operation

$$(\text{Tr}_{g,k} A)(u_1, \ldots, u_{p-2}) := \text{Tr}_g \left[ A(u_1, \ldots, u_{k-1}, \ldots, u_{p-1}) \right].$$

For any $v \in T_X$ and $k = 1, \ldots, p$ we define the contraction operation

$$(v \kappa_k A)(u_1, \ldots, u_{p-1}) := A(u_1, \ldots, u_{k-1}, v, u_{k}, \ldots, u_{p-1}).$$

For any $B \in (V^*)^q \otimes V$ and $k = 1, \ldots, q - 1$ we define the generalized type products

$$(A \ast_k B)(u_1, \ldots, u_p, v_1, \ldots, v_{q-1}) := B(v_1, \ldots, v_{k-1}, A(u_1, \ldots, u_{p_0}), v_k, \ldots, v_{q-1}),$$

and $\xi \equiv (\xi_1, \ldots, \xi_{p+k-1}) := (u_1, \ldots, u_p, v_1, \ldots, v_{k-1})$. We observe now that if $A \in C^\infty(X, (T_X^*)^p \otimes T_X)$ and $(g_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ is a smooth family such that $[\nabla_{g_t} \xi \xi, A] = 0$ for all $\xi \in T_X$ then

$$2 \tilde{\nabla}_{g_t} A \equiv - \nabla_{g_t} \hat{\xi_t} \hat{\nabla}_{g_t} A,$$

thanks to the variation formula (3.4). We infer by this and by the variation formula (3.19) that if $H \in C^\infty(X, \text{End}(T_X))$ satisfies $[\nabla_{g_t} \xi \xi, \nabla_{g_t} H] = 0$, for all $\xi \in T_X$ and $p = 0, 1$ then

$$2 \tilde{\nabla}^2_{g_t} H \equiv - \nabla_{g_t} \hat{\xi_t} \hat{\nabla}_{g_t} H.$$

A simple induction shows that if $[\nabla_{g_t} \xi \xi, \nabla^r_{g_t} H] = 0$, for all $\xi \in T_X$ and $r = 0, \ldots, p - 1$ then

$$2 \tilde{\nabla}^p_{g_t} H \equiv - \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C^{p,r}_{k,\sigma} \nabla^p_{g_t} \hat{\xi_t} \hat{\nabla}_{g_t} \nabla^r_{g_t} H,$$  \hspace{1cm} (4.1)

with $C^{p,r}_{k,\sigma} = 0, 1$. We show now the following fundamental result.

**Corollary 3.** Let $(g_t)_{t \in \mathbb{R}} \subset \mathcal{M}$ be a smooth family such that $\hat{g_t} \in \mathbb{F}^K$ for all $t \in \mathbb{R}$. If $g_0 \in S^K_{t_0}$ then $g_t \in S^K_{t_t}$ for all $t \in \mathbb{R}$.

**Proof.** Thanks to corollary we just need to show the condition on the brackets. We will proceed by induction on the order of covariant differentiation $p$. For notations simplicity we set $\rho_t := \text{Ric}_{g_t}(\Omega)$. We observe now that the assumption
implies $R_{g_t} = R_{g_0}$ for all $t \in \mathbb{R}$, thanks to the variation formula (3.7). This combined with the variation formula (2.4) gives

$$2 \frac{d}{dt} [T, \rho^*_t] = - [T, \Delta^g_{g_t} \dot{g}_t^* + 2 \dot{g}_t^* \rho_t^*]$$

$$= - 2 [T, \dot{g}_t^* \rho_t^*]$$

thanks to the assumption $[T, \nabla^p g_t \dot{g}_t] \equiv 0$ and (3.16). By Cauchy uniqueness we infer $[T, \rho^*_t] \equiv 0$. We assume now as inductive hypothesis $[T, \nabla^r g_t \dot{g}_t \rho^*_t] \equiv 0$, for all $r = 0, \ldots, p - 1, p > 1$ and $\eta \in T^p X$. We deduce thanks to this for $T = K$ and thanks to the assumption, the identity

$$[\nabla g_t \dot{g}_t^*, \nabla^p g_t \dot{g}_t^*] = 0.$$  

This combined with the variation formula (4.1) with $H = \rho^*_t$, combined with the inductive hypothesis and with (2.4) provides the identity

$$2 \frac{d}{dt} [T, \nabla^p g_t \dot{g}_t^*] = - 2 [T, \nabla^p \xi \dot{g}_t^*]$$

We can express the $p$-derivative of the $\Omega$-Laplacian as

$$\nabla^p g_t \Delta^g_{g_t} \dot{g}_t^* = - \operatorname{Tr}_{g_t, p+1} \nabla^{p+2} g_t^* + \nabla g_t f_t - \nabla^{p+1} g_t^*$$

$$+ \sum_{r=1}^{p} \sum_{|I|=r} C^p_{\sigma} \nabla^p g_t \dot{g}_t^*$$

with $C^p_{\sigma} = 0, 1$. Moreover the assumption combined with the expression of the $p$-derivative of the $\Omega$-Laplacian and with the identity

$$[T, \xi - (\nabla^p g_t \dot{g}_t^* \xi)] \equiv 0,$$

implies the variation formula

$$2 \frac{d}{dt} [T, \nabla^p g_t \dot{g}_t^*] = - 2 [T, \nabla^p g_t \dot{g}_t^*]$$

$$= - 2 \sum_{r=0}^{p} \sum_{|I|=r} [T, \nabla^{p-r}_{g_0, \xi_{I_1}} \dot{g}_t^* \nabla^{r}_{g_0, \xi_{I_1}} \rho_t^*].$$

Using the assumption $[T, \nabla^{p-r}_{g_t, g_0} \dot{g}_t^*] \equiv 0, \mu \in T^{p-r}_X$ and the inductive hypothesis we can apply the identity (3.17) to the products of type $\nabla^{p-r}_{g_0, \xi_{I_1}} \dot{g}_t^* \nabla^r_{g_0, \xi_{I_1}} \rho_t^*$ in order to obtain the identity

$$2 \frac{d}{dt} [T, \nabla^p g_t \dot{g}_t^*] = - 2 \dot{g}_t^* [T, \nabla^p g_t \dot{g}_t^*].$$

Then the conclusion follows by Cauchy uniqueness.\qed
5 Integrability of the distribution $\mathbb{F}^K$

We start first with a basic calculus fact.

**Lemma 6.** Let $B > 0$ be a $g$-symmetric endomorphism smooth section of $TX$ such that $[B, \nabla_{g,\xi} B] = 0$. Then hold the identity

$$\nabla_{g,\xi} \log B = B^{-1} \nabla_{g,\xi} B.$$  

**Proof.** We set $A := \log B$ and we observe that by definition $[B, A] = 0$, i.e $[e^A, A] = 0$. The assumption $[e^A, \nabla_{g,\xi} e^A] = 0$ is equivalent to the condition $[A, \nabla_{g,\xi} e^A] = 0$ since the endomorphisms $e^A, \nabla_{g,\xi} e^A$ can be diagonalized simultaneously. Thus deriving the identity $[e^A, A] = 0$ we infer $[\nabla_{g,\xi} e^A, e^A] = 0$, which is equivalent to $[\nabla_{g,\xi} A, A] = 0$. But this last implies

$$\nabla_{g,\xi} e^A = \nabla_{g,\xi} A e^A = e^A \nabla_{g,\xi} A.$$  

We infer $\nabla_{g,\xi} A = e^{-A} \nabla_{g,\xi} e^A$, i.e the required conclusion. \qed

We show now the following key lemma.

**Lemma 7.** For any $g_0 \in \mathcal{M}$ hold the identities

$$\Sigma_K(g_0) := \mathbb{F}^K_{g_0} \cap \mathcal{M} = \exp_{G, g_0} (\mathbb{F}^K_{g_0}),$$  

$$T_{\Sigma_K(g_0), g} = \mathbb{F}^K_g, \quad \forall g \in \Sigma_K(g_0).$$  

Moreover $\Sigma_K(g_0)$ is a totally geodesic and flat sub-variety inside the non-positively curved Riemannian manifold $(\mathcal{M}, G)$.

**Proof.** **Step I (A)***

We observe first the inclusion $\Sigma_K(g_0) \supseteq \exp_{G, g_0} (\mathbb{F}^K_{g_0})$. In fact let $(g_t)_{t \in \mathbb{R}}$ be a geodesic such that $\dot{g}_0 \in \mathbb{F}^K_{g_0}$. Then using the expression (3.1) of the geodesics we obtain

$$\nabla_{\mathbb{C}X \cdot g_0} (g_0^{-1} \cdot g_t) = \nabla_{\mathbb{C}X \cdot g_0} e^{t \cdot \dot{g}_0} = 0, \quad \forall t \in \mathbb{R},$$  

since the last equality is equivalent to the condition $\dot{g}_0 \in \mathbb{F}^K_{g_0}$. Moreover the condition

$$\left[ T, \nabla^p_{g_0,\xi} e^{t \cdot \dot{g}_0} \right] = 0, \quad \forall t \in \mathbb{R}, \quad \forall p \in \mathbb{Z}_{\geq 0},$$  

is equivalent to the identity

$$\left[ T, \nabla^p_{g_0,\xi} (\dot{g}_0^*)^q \right] = 0, \quad \text{for all } p, q \in \mathbb{Z}_{\geq 0}$$  

and this last follows from a repetitive use of the identity (3.17). We conclude the inclusion $\Sigma_K(g_0) \supseteq \exp_{G, g_0} (\mathbb{F}^K_{g_0})$. 

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Step I (B1)

We observe now that for any smooth curve \((u_t)_{t \in (-\varepsilon, \varepsilon)} \in \mathbb{R}^K\) the identity 
\([K, g_0^{-1} u_t] = 0\) implies \([K, g_0^{-1} u_t] = 0\) and thus \([g_0^{-1} u_t, g_0^{-1} u_t] = 0\). We infer that the differential

\[ D_u \exp_{G,g_0} : \mathbb{F}^K_{g_0} \rightarrow \mathbb{F}^K_{g_0}, \]

of the exponential map \(\exp_{G,g_0} : \mathbb{F}^K_{g_0} \rightarrow \Sigma_K(g_0)\) at a point \(u \in \mathbb{F}^K_{g_0}\) is given by the formula

\[ D_u \exp_{G,g_0}(v) = v e^{g_0^{-1} u}, \]

for all \(v \in \mathbb{F}^K_{g_0}\). We deduce by lemma 4 that this differential map is an isomorphism. This combined with the fact that the exponential map is injective implies that \(\exp_{G,g_0}(\mathbb{F}^K_{g_0})\) is an open subset of \(\Sigma_K(g_0)\). But \(\exp_{G,g_0}(\mathbb{F}^K_{g_0})\) is also a closed set of \(M\) and thus a closed set of \(\Sigma_K(g_0)\). The fact that this last is connected implies the required equality (5.1).

Step I (B2)

We give now an explicit proof of the inclusion \(\Sigma_K(g_0) \subseteq \exp_{G,g_0}(\mathbb{F}^K_{g_0})\). (This is useful also for other considerations.) Indeed we show that if \(g \in \Sigma_K(g_0)\) then \(g_0 \log(g_0^{-1} g) \in \mathbb{F}^K_{g_0}\). We observe first that the assumptions \([K, g_0^{-1} g] = 0\), and \([K, \nabla_{g_0}(g_0^{-1} g)] = 0\), imply

\[ [g_0^{-1} g, \nabla_{g_0} \xi(g_0^{-1} g)] = 0, \]

which allows to apply lemma 6 in order to obtain the formula

\[ \nabla_{g_0} \log(g_0^{-1} g) = g^{-1} g_0 \nabla_{g_0} \xi(g_0^{-1} g) . \] (5.3)

Thus the assumption \(\nabla_{x,g_0} (g_0^{-1} g) = 0\) implies the identity

\[ \nabla_{x,g_0} \log(g_0^{-1} g) = 0. \]

Let \(\varepsilon > 0\) be sufficiently small such that \(\varepsilon g < 2g_0\). Then the expansion

\[ \log(\varepsilon g_0^{-1} g) = \sum_{p=1}^{+\infty} \frac{(-1)^{p+1}}{p} (\varepsilon g_0^{-1} g - I)^p, \]

implies \([T, \log(\varepsilon g_0^{-1} g)] = 0\), and thus \([T, \log(g_0^{-1} g)] = 0\). Moreover the identity (3.10) implies

\[ g_0^{-1} g [T, g^{-1} g_0] = [T, I] = 0, \]

and thus \([T, g^{-1} g_0] = 0\) which combined with the formula

\[ \nabla_{g_0, \xi}(g^{-1} g_0) = -g^{-1} g_0 \nabla_{g_0, \xi}(g_0^{-1} g) g^{-1} g_0, \]

and with the identity (3.17) implies the equality \([T, \nabla_{g_0, \xi}(g_0^{-1} g)] = 0\). We infer \([T, \nabla_{g_0, \xi}(g_0^{-1} g)] = 0\) by a simple induction based on a repetitive use of the identity (3.17). We conclude

\[ [T, \nabla_{g_0, \xi} \log(g_0^{-1} g)] = 0, \]
by deriving the identity \(5.3\) and using \(3.17\).

**Step II**

We show now the identity \(5.2\), i.e the identity \(F^K_{g_0} = F^K_g\). We can consider, thanks to the equality \(5.1\), a geodesic \((g_t)_{t \in \mathbb{R}} \subset \Sigma_K(g_0)\) joining \(g = g_1\) with \(g_0\). We observe also that lemma \(5\) combined with the variation formula \(3.7\) implies the identity \(\mathcal{R}_{g_t} = \mathcal{R}_{g_0} = \mathcal{R}_g\). Moreover \([K, \nabla_{g_t} \dot{g}_t] = 0\), thanks to lemma \(6\). Then the variation formula \(3.19\) implies

\[
\nabla g H = \nabla_{g_0} H, \quad \forall H \in C^\infty(X, \text{End}(T_X)): [K, H] = 0.
\]

\(5.4\)

On the other hand using \(3.16\) we obtain the equalities

\[
[T, g_0^{-1}v] = [T, (g_0^{-1}g)(g^{-1}v)] = (g_0^{-1}g) [T, g^{-1}v].
\]

Thus \([T, g_0^{-1}v] = 0\) iff \([T, g^{-1}v] = 0\). This last for \(T = K\) implies

\[
\nabla g (g^{-1}v) = \nabla_{g_0} (g^{-1}v),
\]

\(5.5\)

thanks to \(5.4\). We consider the identities

\[
\nabla_{T_{X,g_0}} (g_0^{-1}v) = \nabla_{T_{X,g_0}} [(g_0^{-1}g)(g^{-1}v)]
\]

\[
= g^{-1}v \nabla_{T_{X,g_0}} (g_0^{-1}g) + g_0^{-1}g \nabla_{T_{X,g_0}} (g^{-1}v)
\]

\[
= g_0^{-1}g \nabla_{T_{X,g_0}} (g^{-1}v),
\]

since \(g \in \Sigma_K(g_0)\) and thanks to \(5.5\). We deduce \(v \in F_{g_0}\) iff \(v \in F_g\) provided that \([K, g^{-1}v] = 0\). We show now by induction on \(p \geq 0\) the properties

\[
[T, \nabla^{(r)}_{g_0, \xi} (g_0^{-1}v)] = 0 \iff [T, \nabla^{(r)}_{g, \xi} (g^{-1}v)] = 0, \quad \forall \xi \in C^\infty(X, T_X)^{\delta^r},
\]

\(5.6\)

and

\[
\nabla^{(r+1)}_{g, \xi} (g^{-1}v) = \nabla^{(r+1)}_{g_0, \xi} (g^{-1}v), \quad \forall \xi \in C^\infty(X, T_X)^{\delta^{(r+1)}},
\]

\(5.7\)

for all \(r = 0, \ldots, p\). This properties hold true for \(p = 0\) as we observed previously. The assumption \(g \in \Sigma_K(g_0)\) combined with \(3.10\) implies the identity

\[
[T, \nabla^{(p)}_{g_0, \xi} (g_0^{-1}v)] = \sum_{r=0}^{p} \sum_{|I|=r} \nabla^{(p-r)}_{g_0, \xi_I} (g_0^{-1}g) [T, \nabla^{(r)}_{g_0, \xi_I} (g^{-1}v)].
\]

We assume that the step \(p - 1\) of the induction hold true. We infer the equality

\[
[T, \nabla^{(p)}_{g_0, \xi} (g_0^{-1}v)] = g_0^{-1}g [T, \nabla^{(p)}_{g, \xi} (g^{-1}v)],
\]

which implies \(5.6\) for \(r = p\). Moreover the identity

\[
[K, \nabla^{(p)}_{g, \xi} (g^{-1}v)] = 0,
\]

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implies the equalities
\[
\nabla_g \nabla^{(p)}_{g, \xi} (g^{-1} v) = \nabla_{g_0} \nabla^{(p)}_{g_0, \xi} (g^{-1} v) = \nabla_{g_0} \nabla^{(p)}_{g_0, \xi} (g^{-1} v),
\]
thanks to (5.4) and to the inductive assumption. We obtain (5.7) for \( r = p \) and thus the conclusion of the induction.

**Step III**

We show now the last statement of the lemma. We observe indeed that the identities \( \sum K (g_0) = M \cap F^K_g = \exp_{g_0} \left( F^K_g \right) \), hold thanks to the equalities (5.2) and (5.1). But this implies that the second fundamental form of \( \sum K (g_0) \) inside \((M, G)\) vanishes identically. Thus using Gauss equation, the identity (5.2) and the expression of the curvature tensor
\[
R_M (u, v) w = - \frac{1}{4} g \left[ [u^*_g, v^*_g], w^*_g \right],
\]
we infer the equalities of the curvature forms \( R_{\sum K (g_0)} (g) = R_M (g) \mid F^K_g \equiv 0 \) for all \( g \in \sum K (g_0) \). This concludes the proof of lemma 7.

### 6 Reinterpretation of the space \( F^K_g \)

In this section we conciliate the definition of the vector space \( F^K_g \) given in the section 2 with the definition so far used.

We consider indeed \( K \in C^\infty (X, \text{End}(T_X)) \) with \( n \)-distinct real eigenvalues almost everywhere over \( X \). If \( A, B \in C^\infty (X, \text{End}(T_X)) \) commute with \( K \) then \( [A, B] = 0 \) over \( X \). We observe that this is all we need in order to make work the previous arguments. Thus all the previous results hold true if we use such \( K \). In this case hold an equivalent definition of the vector space \( F^K_g \). We show in fact the following lemma.

**Lemma 8.** Let \( K \in C^\infty (X, \text{End}(T_X)) \) with \( n \)-distinct real eigenvalues almost everywhere over \( X \). Then hold the identity
\[
F^K_g = \left\{ v \in F_g \mid \left[ \nabla^p_g T, v^*_g \right] = 0, T = R_g, K, \forall p \in \mathbb{Z}_{\geq 0} \right\}. \tag{6.1}
\]

**Proof.** It is sufficient to show by induction on \( p \geq 1 \) that
\[
\forall r = 0, \ldots, p; \left[ \nabla^p_g T, v^*_g \right] = 0 \iff \left[ T, \nabla^{r-s}_{g, \xi} v^*_g \right] = 0, \forall \xi \in T_X^{\otimes r}. \tag{6.2}
\]
We assume true this statement for \( p - 1 \) and we show it for \( p \). The inductive hypothesis implies
\[
\forall r = 0, \ldots, p - 1; \left[ \nabla^{r-s}_g T, \nabla^{s}_{g, \xi} v^*_g \right] = 0, \forall \xi \in T_X^{\otimes s}, \forall s = 0, \ldots, r. \tag{6.3}
\]
We show the statement (6.3) by a finite increasing induction on \( s \). The statement (6.3) hold true obviously for \( s = 0, 1 \). We assume (6.3) true for \( s \) and we show it for \( s + 1 \). Indeed by the inductive assumption on \( s \) hold the identity
\[
\left[ \nabla^{r-1-s}_g T, \nabla^{s}_{g, \xi} v^*_g \right] = 0,
\]

for all \( \xi \in C^\infty(X, T^*_X) \). We take in particular \( \xi \) such that \( \nabla_g \xi(x) = 0 \), at some arbitrary point \( x \in X \). Thus hold the equalities
\[
0 = \nabla_{g, \eta} \left[ \nabla_g^{r-1-s} T, \nabla^s g, v_g^* \right] \\
= \left[ \eta - \nabla_g^{r-s} T, \nabla^s g, v_g^* \right] + \left[ \nabla_g^{r-1-s} T, \nabla^s+1 g, \xi \right] v_g^*,
\]
thanks to the inductive assumption on \( s \). This completes the proof of (6.3). The conclusion of the induction on \( p \) for (6.2) will follow from the statement; for all \( s = 0, \ldots, p - 1 \) hold the equivalence
\[
\left[ \nabla_g^{p-1-s} T, \nabla^s g, v_g^* \right] = 0, \quad \forall \xi \in T^*_X \\
\iff \left[ \nabla_g^{p-s-1} T, \nabla^s+1 g, v_g^* \right] = 0, \quad \forall \eta \in T^*_X.
\]
By (6.3) for \( r = p - 1 \) and \( s = 0, \ldots, p - 1 \) hold the identity
\[
\left[ \nabla_g^{p-1-s} T, \nabla^s g, v_g^* \right] = 0,
\]
for all \( \xi \in C^\infty(X, T^*_X) \). We take as before \( \xi \) such that \( \nabla_g \xi(x) = 0 \), at some arbitrary point \( x \in X \). Thus
\[
0 = \nabla_{g, \eta} \left[ \nabla_g^{p-1-s} T, \nabla^s g, v_g^* \right] \\
= \left[ \eta - \nabla_g^{p-s} T, \nabla^s g, v_g^* \right] + \left[ \nabla_g^{p-1-s} T, \nabla^s+1 g, \xi \right] v_g^*,
\]
which shows (6.4) and thus the conclusion of the induction on \( p \) for (6.2). \( \square \)

7 Representation of the \( \Omega \)-SRF as the gradient flow of the functional \( \mathcal{W}_\Omega \) over \( \Sigma_K(g_0) \)

The following proposition enlightens the properties of the sub-variety \( \Sigma_K(g_0) \).

Proposition 1. For any \( g_0 \in S^K \) and any \( g \in \Sigma_K(g_0) \) hold the identities
\[
\nabla_G \mathcal{W}_\Omega (g) = g - \text{Ric}_g(\Omega) \in T_{\Sigma_K(g_0)},
\]
\[
\nabla^\Sigma_K(g_0)\mathcal{W}_\Omega (g)(v, v) = \int_X \left[ \langle v \text{Ric}^*_g(\Omega), v \rangle_g + \frac{1}{2} \nabla_g v^2 \right] \Omega,
\]
for all \( v \in T_{\Sigma_K(g_0)}. \) Moreover for all \( g_0 \in S^K \) the \( \Omega \)-SRF \( (g_t)_{t \in [0, T]} \subset \Sigma_K(g_0) \) with initial data \( g_0 \) represents the formal gradient flow of the functional \( \mathcal{W}_\Omega \) over the totally geodesic and flat sub-variety \( \Sigma_K(g_0) \) inside the non-positively curved Riemannian manifold \((\mathcal{M}, G)\).
Proof. By the identity (5.1) in lemma 7 there exist a geodesic \((g_t)_{t \in \mathbb{R}} \subset \Sigma_K(g_0)\), \(g_0 \in \mathbb{F}^K_{g_0}\), joining \(g = g_1\) with \(g_0\). Then lemma 5 combined with corollary 8 and with the identity (5.2) in lemma 7 implies \(\text{Ric}_g(\Omega) \in T_{\Sigma_K(g_0), g}\).

Moreover \(g \in T_{\Sigma_K(g_0), g}\) thanks to the identity (5.2). We conclude the fundamental property \(\nabla_G \Omega \in T_{\Sigma_K(g_0), g}\) for all \(g \in \Sigma_K(g_0)\).

The second variation formula in the statement follows directly from corollary 1 in [Pal1] and from the fact that \(\Sigma_K(g_0)\) is a totally geodesic sub-variety inside the Riemannian manifold \((\mathcal{M}, G)\). We observe however that it follows also from the tangency property \(\nabla_G \Omega \in T_{\Sigma_K(g_0), g}\) for any \(f \in C^2(\mathcal{M}, \mathbb{R})\) hold the identity

\[
\nabla_G^\alpha f (\xi, \eta) = \nabla_G^\alpha f (\xi, \eta) - G (\nabla_G^\alpha f, \Pi_G(\xi, \eta)) \quad \forall \xi, \eta \in T_{\Sigma},
\]

where \(\Pi_G \in C^\infty (\Sigma, S^2 T^*_{\Sigma} \otimes N_{\Sigma/\mathcal{M}, G})\) denotes the second fundamental form of \(\Sigma\) inside \((\mathcal{M}, G)\). \(\square\)

The result so far obtained does not allow to see yet the \(\Omega\)-SRF as the gradient flow of a convex functional inside a flat metric space. In order to see the required convexity picture we need to make a key change of variables that we explain in the next sections.

8 Explicit representations of the \(\Omega\)-SRF equation

In this section we show the following fundamental expression of the \(\Omega\)-BER-tensor over the variety \(\Sigma_K(g_0)\).

Lemma 9. Let \(g_0 \in \mathcal{M}\). Then for any metric \(g \in \Sigma_K(g_0)\) hold the expression

\[
\text{Ric}_g(\Omega) = \text{Ric}_{g_0}(\Omega) - \frac{1}{2} \Delta^{\alpha}_{g_0} [g_0 \log(g_0^{-1} g)]
\]

\[
- \frac{1}{4} g_0 \text{ Tr}_{g_0} \left[ \nabla_{g_0, \bullet} \log(g_0^{-1} g) \nabla_{g_0, \bullet} \log(g_0^{-1} g) \right].
\]

Proof. The fact that \(\mathcal{R}_g = \mathcal{R}_{g_0}\) for all \(g \in \Sigma_K(g_0)\) implies also \(\text{Ric}_g = \text{Ric}_{g_0}\). Moreover we observe the elementary identities

\[
\log \frac{dV_g}{\Omega} = \log \frac{dV_g}{dV_{g_0}} + \log \frac{dV_{g_0}}{\Omega},
\]

\[
\frac{dV_g}{dV_{g_0}} = \left[ \frac{\det g}{\det g_0} \right]^{1/2} = \left[ \det(g_0^{-1} g) \right]^{1/2},
\]

\[
\log \frac{dV_g}{\Omega} = \frac{1}{2} \log \det(g_0^{-1} g) + \log \frac{dV_{g_0}}{\Omega}
\]

\[
= \frac{1}{2} \text{ Tr}_{g_0} \log(g_0^{-1} g) + \log \frac{dV_{g_0}}{\Omega}.
\]
We set \( f_0 := \log \frac{dV_{g_0}}{dt} \) and \( \nabla_g = \nabla_{g_0} + \Gamma^T_g \). We infer the equality
\[
\text{Ric}_g(\Omega) = \text{Ric}_{g_0}(\Omega) + \Gamma^T_g d f_0 \\
+ \frac{1}{2} \nabla_{g_0} d \text{Tr}_\mathbb{R} \log(g_0^{-1} g) + \frac{1}{2} \Gamma^T_g d \text{Tr}_\mathbb{R} \log(g_0^{-1} g).
\]

We observe now that \( \Gamma^T_{g,g_0} = - (\Gamma^T_g)^*, \) with
\[
2 \Gamma^T_{g,g_0} \eta = g^{-1} \left[ \nabla_{g_0} g(\xi, \eta, \cdot) + \nabla_{g_0} g(\eta, \xi, \cdot) - \nabla_{g_0} g(\cdot, \xi, \eta) \right]
= (g^{-1} \nabla_{g_0} g) \eta,
\]

since \( \nabla_{T_X,g_0}(g_0^{-1} g) = 0 \). Using lemma [4] we deduce the identity
\[
2 \Gamma^T_{g,g_0} \eta = (g_0^{-1} g)^{-1} \nabla_{g_0,g_0} (g_0^{-1} g)
= \nabla_{g_0,g_0} \log(g_0^{-1} g),
\]

since \([g_0^{-1} g, \nabla_{g_0,g_0}(g_0^{-1} g)] = 0\). Indeed we observe that this last equality follows from the fact that \([K, g_0^{-1} g] = 0\) and \([K, \nabla_{g_0}(g_0^{-1} g)] = 0\). Thus for any function \( u \) hold the identity
\[
2 \left( \Gamma^T_{g,g_0} d u \right) \eta = - g_0 \left( \nabla_{g_0} u, \nabla_{g_0,g_0} \log(g_0^{-1} g) \eta \right).
\]

Using the fact that the endomorphism \( \nabla_{g_0,g_0} \log(g_0^{-1} g) \) is \( g_0 \)-symmetric (since \( \log(g_0^{-1} g) \) is also \( g_0 \)-symmetric) we deduce
\[
2 \left( \Gamma^T_{g,g_0} d u \right) \eta = - g_0 \left( \nabla_{g_0} u, \nabla_{g_0,g_0} \log(g_0^{-1} g) \nabla_{g_0} u, \eta \right).
\tag{8.1}
\]

We observe that \( g \in \Sigma_K(g_0) \) if and only if \( g_0 \log(g_0^{-1} g) \in \mathbb{P}_{g_0}^K \) by the identity [5.1]. In particular
\[
\nabla_{T_X,g_0} \log(g_0^{-1} g) = 0.
\tag{8.2}
\]

We deduce the equalities
\[
2 \left( \Gamma^T_{g,g_0} d f_0 \right) \eta = - g_0 \left( \nabla_{g_0}, \nabla_{g_0} f_0 \log(g_0^{-1} g) \xi, \eta \right)
= - \nabla_{g_0,g_0} f_0 (g_0 \log(g_0^{-1} g)) \xi, \eta.
\]

Let \((e_k)_k\) be a local frame of \( T_X \) in a neighborhood of an arbitrary point \( x \in X \) which is \( g \)-orthonormal at \( x \) and such that \( \nabla_{g_0}e_k(x) = 0 \). Deriving the identity
\[
\text{Tr}_\mathbb{R} \log(g_0^{-1} g) = g_0 \left( \log(g_0^{-1} g) e_k, g_0^{-1} e^*_k \right),
\]

we have
\[
\sum \log(g_0^{-1} g) e_k(x) = g_0 \left( \log(g_0^{-1} g) e_k, g_0^{-1} e^*_k \right).
\]
we infer at the point $x$

$$\xi \cdot \text{Tr}_\nabla \log(g_0^{-1}g) = g_0 \left( \nabla_{g_0, \xi} \log(g_0^{-1}g) e_k, e_k \right)$$

$$= g_0 \left( \nabla_{g_0, e_k} \log(g_0^{-1}g) \xi, e_k \right)$$

$$= g_0 \left( \xi, \nabla_{g_0, e_k} \log(g_0^{-1}g) e_k \right),$$

thanks to the identity (8.2) and thanks to the fact that the endomorphism $\nabla_{g_0, \xi} \log(g_0^{-1}g)$ is $g_0$-symmetric. We infer the formula

$$d \text{Tr}_\nabla \log(g_0^{-1}g) = -g_0 \nabla_{g_0}^* \log(g_0^{-1}g).$$

Thus using (8.2) we infer the equalities

$$\nabla_{g_0} d \text{Tr}_\nabla \log(g_0^{-1}g) = -g_0 \nabla_{g_0} \nabla_{g_0}^* \log(g_0^{-1}g)$$

$$= -g_0 \Delta_{g_0, g_0} \log(g_0^{-1}g)$$

$$= -g_0 \Delta_{g_0} \log(g_0^{-1}g)$$

$$= -\Delta_{g_0} \left[ g_0 \log(g_0^{-1}g) \right],$$

by the Weitzenböck formula in lemma 20 in the appendix and by the identity $[\mathcal{K}, \nabla_{g_0, \log(g_0^{-1}g)]} = 0$. (We remind that $g_0 \log(g_0^{-1}g) \in \mathcal{P}_{g_0}$.) We apply now the identity (8.1) to the function $u := \text{Tr}_\nabla \log(g_0^{-1}g)$. We infer by the formula (8.3) the equality

$$\nabla_{g_0} u = \nabla_{g_0, e_k} \log(g_0^{-1}g) e_k.$$ 

Thus we obtain the equality

$$2 \left( \Gamma^T_{g_0, \xi, g_0} d \text{Tr}_\nabla \log(g_0^{-1}g) \right) \eta = -g_0 \left( \nabla_{g_0, \xi} \log(g_0^{-1}g) \nabla_{g_0, e_k} \log(g_0^{-1}g) e_k, \eta \right).$$

Using the identity $[\mathcal{K}, \nabla_{g_0, \log(g_0^{-1}g)]} = 0$, we deduce the expression at the point $x$

$$2 \left( \Gamma^T_{g_0, \xi, g_0} d \text{Tr}_\nabla \log(g_0^{-1}g) \right) \eta = -g_0 \left( \nabla_{g_0, e_k} \log(g_0^{-1}g) \nabla_{g_0, \xi} \log(g_0^{-1}g) e_k, \eta \right)$$

$$= -g_0 \left( \nabla_{g_0, e_k} \log(g_0^{-1}g) \nabla_{g_0, e_k} \log(g_0^{-1}g) \xi, \eta \right),$$

thanks to (8.2). Combining the expressions obtained so far we infer the required formula.

\[\square\]
We remind that by proposition 1 follows that if \( g_0 \in S^K \) then
\[
\text{Ric}_g(\Omega) \in \mathbb{F}_g^K, \quad \forall g \in \Sigma_K(g_0). \tag{8.3}
\]
We give an other proof (not necessarily shorter but more explicit) of this fundamental fact based on the expression of \( \text{Ric}_g(\Omega) \) in lemma 9.

**Proof.** We remind that \( g \in \Sigma_K(g_0) \) if and only if \( g_0 \log(g_0^{-1}g) \in \mathbb{F}^K_{g_0} \) by the identity (5.1). We set for notation simplicity
\[
U := \log(g_0^{-1}g) \in g_0^{-1}\mathbb{F}^K_{g_0},
\]
and we show first the equality
\[
\nabla_{\tau_X,g_0} g_0^{-1} \text{Ric}_g(\Omega) = 0. \tag{8.4}
\]
Consider now \((x_1, \ldots, x_n)\) be \( g_0 \)-geodesic coordinates centered at an arbitrary point \( p \in X \) and set \( e_k := \frac{\partial}{\partial x_k} \). The local tangent frame \((e_k)\) is \( g_0(p) \)-orthonormal at the point \( p \) and satisfies \( \nabla_{g_0} g_j(p) = 0 \) for all \( j \).

We take now two vector fields \( \xi \) and \( \eta \) with constant coefficients with respect to the \( g_0 \)-geodesic coordinates \((x_1, \ldots, x_n)\). Therefore \( \nabla_{g_0,\xi}(p) = \nabla_{g_0,\eta}(p) = 0 \).

Commuting derivatives by means of (3.6) we infer the identities at the point \( p \)
\[
\text{Tr}_{g_0}(\nabla_{g_0,\xi}(U\nabla_{g_0,\eta})) = -\frac{1}{2} \text{Tr}_{g_0}(\Delta_{g_0}U).
\]

We expand now at the point \( p \) the therm
\[
\nabla_{g_0,\xi}(\text{Tr}_{g_0}(U\nabla_{g_0,\eta})) = -\text{Tr}_{g_0}(\nabla_{g_0,\xi}(U\nabla_{g_0,\eta})).
\]
Thus we expand first the therm

\[ - \nabla_{g_0, \xi} \nabla_{g_0, e_k} \nabla_{g_0, e_k} U \eta + \nabla_{g_0} U (\nabla_{g_0, \xi} \nabla_{g_0, e_k} e_k, \eta) \]

\[ + \nabla_{g_0, \eta} \nabla_{g_0, e_k} \nabla_{g_0, e_k} U \xi - \nabla_{g_0} U (\nabla_{g_0, \eta} \nabla_{g_0, e_k} e_k, \xi) \]

\[ = - \nabla_{g_0, e_k} \nabla_{g_0, \eta} \nabla_{g_0, e_k} U \eta + \nabla_{g_0} U (\nabla_{g_0, \xi} \nabla_{g_0, e_k} e_k, \eta) \]

\[ + \nabla_{g_0, e_k} \nabla_{g_0, \eta} \nabla_{g_0, e_k} U \xi - \nabla_{g_0} U (\nabla_{g_0, \xi} \nabla_{g_0, e_k} e_k, \xi) , \]

since

\[ \nabla_{g_0, e_k} \nabla_{g_0} U (e_k, \eta) = \nabla_{g_0, e_k} [\nabla_{g_0} U (e_k, \xi)] \]

\[ - \nabla_{g_0} U (\nabla_{g_0, e_k} e_k, \eta) - \nabla_{g_0} U (e_k, \nabla_{g_0, e_k} \eta) \]

\[ = \nabla_{g_0, e_k} \nabla_{g_0, e_k} U \eta - \nabla_{g_0} U (\nabla_{g_0, e_k} e_k, \eta) , \]

and \([\xi, e_k] = [\eta, e_k] \equiv 0\) in a neighborhood of \(p\) and since \([R_{g_0}, \nabla_{g_0, e_k} U] \equiv 0\). (We use here the identity (3.6).) Moreover using the fact that \([R_{g_0}, U] = 0\) we infer the identity at the point \(p\)

\[ \nabla_{T_X, \cdot \eta} \Delta_{g_0} U (\xi, \eta) = - \nabla_{g_0, e_k} \nabla_{g_0, e_k} \nabla_{g_0, e_k} U \eta + \nabla_{g_0} U (\nabla_{g_0, \xi} \nabla_{g_0, e_k} e_k, \eta) \]

\[ + \nabla_{g_0, e_k} \nabla_{g_0, e_k} \nabla_{g_0, \eta} U \xi - \nabla_{g_0} U (\nabla_{g_0, \eta} \nabla_{g_0, e_k} e_k, \xi) . \]

We expand now at the point \(p\) the therm

\[ 0 = \Delta_{g_0} \nabla_{T_X, \cdot \eta} U (\xi, \eta) = - \nabla_{g_0, e_k} [\nabla_{g_0, e_k} \nabla_{T_X, \cdot \eta} U (\xi, \eta)] . \]

Thus we expand first the therm

\[ \nabla_{g_0, e_k} \nabla_{T_X, \cdot \eta} U (\xi, \eta) = \nabla_{g_0, e_k} [\nabla_{T_X, \cdot \eta} U (\xi, \eta)] \]

\[ - \nabla_{T_X, \cdot \eta} U (\nabla_{g_0, e_k} \xi, \eta) - \nabla_{T_X, \cdot \eta} U (\xi, \nabla_{g_0, e_k} \eta) \]

\[ = \nabla_{g_0, e_k} [\nabla_{g_0, \xi} U \eta - \nabla_{g_0, \eta} U \xi] \]

\[ - \nabla_{g_0} U (\nabla_{g_0, e_k} \xi, \eta) + \nabla_{g_0} U (\eta, \nabla_{g_0, e_k} \xi) \]

\[ - \nabla_{g_0} U (\xi, \nabla_{g_0, e_k} \eta) + \nabla_{g_0} U (\nabla_{g_0, e_k} \eta, \xi) \]

\[ = \nabla_{g_0, e_k} \nabla_{g_0, \xi} U \eta - \nabla_{g_0, e_k} \nabla_{g_0, \eta} U \xi \]

\[ - \nabla_{g_0} U (\nabla_{g_0, e_k} \xi, \eta) + \nabla_{g_0} U (\nabla_{g_0, e_k} \eta, \xi) . \]
in a neighborhood of \( p \). We deduce the identity at the point \( p \)
\[
0 = \Delta_{g_0} \nabla_{T_X \cdot g_0} U(\xi, \eta) = - \nabla_{g_0, e_k} \nabla_{g_0, e_k} \nabla_{g_0, \xi} U \eta \\
+ \nabla_{g_0, e_k} \nabla_{g_0, \eta} U \xi \\
+ \nabla_{g_0} U(\nabla_{g_0, e_k} \nabla_{g_0, \xi} e_k, \eta) \\
- \nabla_{g_0} U(\nabla_{g_0, e_k} \nabla_{g_0, \eta} e_k, \xi),
\]
since \([\xi, e_k] = [\eta, e_k] \equiv 0\). Thus we obtain the identity
\[
\nabla_{T_X \cdot g_0} \Delta_{g_0} U(\xi, \eta) = \nabla_{g_0} U(\nabla_{g_0}(\xi) e_k, \eta) - \nabla_{g_0} U(\nabla_{g_0}(\eta) e_k, \xi) \\
= \nabla_{g_0, \eta} U \nabla_{g_0} \nabla_{g_0} \xi - \nabla_{g_0, \xi} U \nabla_{g_0} \eta,
\]
since \(g_0 U \in \mathbb{F}_{g_0}\). We expand now at the point \( p \) the term
\[
\nabla_{T_X \cdot g_0} \Delta_{g_0} U(\xi, \eta) = \nabla_{g_0} \nabla_{g_0} f_0 \nabla_{g_0} \eta \nabla_{g_0, \xi} - \nabla_{g_0} \nabla_{g_0} f_0 \nabla_{g_0, \eta} \nabla_{g_0, \xi},
\]
since \([\nabla_{g_0}, U] = 0 \) and \([\xi, \eta] \equiv 0\). We deduce the identity
\[
\nabla_{T_X \cdot g_0} \Delta_{g_0} U = - \text{Alt} \left[ \nabla_{g_0} U \nabla_{g_0} \xi \right].
\]
But \([\nabla_{g_0} U, \nabla_{g_0} \xi] \equiv 0\) if \( g_0 \in S^K \). We infer
\[
\nabla_{T_X \cdot g_0} \Delta_{g_0} U = - \nabla_{g_0} \xi \nabla_{g_0, \eta} \nabla_{g_0, \xi} U = 0.
\]
Moreover
\[
[T, \nabla_{g_0, \xi} \Delta_{g_0} U] = 0,
\]
thanks to the identity (1.2) applied to \( U \). We observe now the decomposition
\[
\nabla_{g_0, \xi}^p \nabla_{g_0} U \nabla_{g_0, \eta} U = \sum_{r=0}^p \sum_{|I|=r} \nabla_{g_0, e_k, \xi I}^r U \nabla_{g_0, e_k, \xi I}^{r+1} U.
\]
Then the conclusion follows from the identity (3.17) with $C = T$, $A = \nabla^{r+1} f, \xi U$ and $B = \nabla^{r+1} f, g_0^{e_k, \xi I} U$.

In the following lemma we introduce a fundamental change of variable $s$.

**Lemma 10.** Let $g_0 \in M$ and set $H \equiv H_g := (g^{-1}g_0)^{1/2}$ for any metric $g \in \Sigma_K(g_0)$. Then hold the expression

$$
\text{Ric}^*_g(\Omega) = H \Delta^\Omega g^0 H + H^2 \text{Ric}^*_{g_0}(\Omega).
$$

**Proof.** By lemma 9 we obtain the formula

$$
\text{Ric}^*_g(\Omega) = g^{-1} g_0 \text{Ric}^*_{g_0}(\Omega) - \frac{1}{2} g^{-1} g_0 \Delta^\Omega g_0 \log(g^{-1} g_0) - \frac{1}{4} g^{-1} g_0 \text{Tr}_{g_0} \left[ \nabla_{g_0} \cdot \log(g^{-1} g_0) \nabla_{g_0} \cdot \log(g^{-1} g_0) \right].
$$

We set

$$
A := \log H = - \frac{1}{2} \log(g^{-1} g_0) \in g^{-1}_0 \mathbb{F}_K.
$$

and we observe that $H = e^A \in g^{-1}_0 \Sigma_K(g_0)$ thanks to lemma 4. With this notations the previous formula rewrites as

$$
\text{Ric}^*_g(\Omega) = e^{2A} \left[ \Delta^\Omega A - \text{Tr}_{g_0} \left( \nabla_{g_0} \cdot A \nabla_{g_0} \cdot A \right) + \text{Ric}^*_{g_0}(\Omega) \right].
$$

(8.5)

We expand now, at an arbitrary center of geodesic coordinates, the term

$$
\Delta^\Omega e^A = - \nabla_{g_0, e_k} \nabla_{g_0, e_k} e^A + \nabla_{g_0} f_0 - \nabla_{g_0} e^A
$$

$$
= - \nabla_{g_0, e_k} (e^A \nabla_{g_0, e_k} A) + e^A (\nabla_{g_0} f_0 - \nabla_{g_0} A)
$$

$$
= e^A \Delta^\Omega A - e^A \text{Tr}_{g_0} \left( \nabla_{g_0} \cdot A \nabla_{g_0} \cdot A \right).
$$

We infer the expression

$$
\text{Ric}^*_g(\Omega) = e^A \Delta^\Omega e^A + e^{2A} \text{Ric}^*_{g_0}(\Omega),
$$

(8.6)

i.e the required conclusion.

We deduce the following corollary.

**Corollary 4.** The $\Omega$-SRF $(g_t)_t \subset \Sigma_K(g_0)$ is equivalent to the porous medium type equation

$$
2 \dot{H}_t = - H_t^2 \Delta^\Omega H_t - H_t^3 \text{Ric}^*_{g_0}(\Omega) + H_t,
$$

with initial data $H_0 = \mathbb{I}$, via the identification $H_t = (g_t^{-1} g_0)^{1/2} \in g_0^{-1} \Sigma_K(g_0)$. 29
Proof. Let \( U_t := \log(g_0^{-1} g_t) \in g_0^{-1} \mathcal{T}^K_{g_0} \) and observe that the \( \Omega \)-SRF equation \((g_t)_t \subset \Sigma_K(g_0)\) is equivalent to the evolution equation
\[
\dot{U}_t = \dot{g}^*_t = \text{Ric}^*_{g_t}(\Omega) - \mathbb{I}.
\]
Then the conclusion follows combining the identity \( 2 \dot{H}_t = -H_t \dot{U}_t \) with the previous lemma.

We observe that the change of variables
\[
g \in \Sigma_K(g_0) \mapsto H = (g^{-1} g_0)^{1/2} \in g_0^{-1} \Sigma_K(g_0) ,
\]
is the one which linearizes as much as possible the expression of the SRF equation. Indeed we can rewrite it as the porous medium equation in corollary \([4]\). However we will see in the next section that the change of variables \( g \mapsto A = \log H \) would fit us in a gradient flow picture of a convex functional over convex sets in a Hilbert space. So from now on we will consider the change of variables
\[
g \in \Sigma_K(g_0) \mapsto A = \log H = -\frac{1}{2} \log(g_0^{-1} g) \in \mathcal{T}^K_{g_0} := g_0^{-1} \mathcal{T}^K_{g_0}.
\] (8.7)
We define the \( \Omega \)-divergence operator of a tensor \( \alpha \) as
\[
\text{div}_g^\alpha := e^f \text{div}_g \left( e^{-f} \alpha \right) = \text{div}_g \alpha - \nabla_g f \alpha ,
\]
with \( f := \log \frac{dV_g}{dV_{g_0}} \). We observe that with this notation formula (8.6) rewrites as
\[
\text{Ric}^*_{g_A}(\Omega) = -e^A \text{div}^\alpha_{g_0} \left( e^A \nabla_{g_0} A \right) + e^{2A} \text{Ric}^*_{g_0}(\Omega) ,
\] (8.8)
with \( g_A := g_0 e^{-2A} \), for all \( A \in \mathcal{T}_{g_0}^K \). With this notations hold the analogue of corollary \([4]\).

**Corollary 5.** The \( \Omega \)-SRF \((g_t)_t \subset \Sigma_K(g_0)\) is equivalent to the solution \((A_t)_{t \geq 0} \subset \mathcal{T}_{g_0}^K\) of the forward evolution equation
\[
2 \dot{A}_t = e^{A_t} \text{div}^\alpha_{g_{0}} \left( e^{A_t} \nabla_{g_{0}} A_t \right) - e^{2A_t} \text{Ric}^*_{g_0}(\Omega) + \mathbb{I} ,
\] (8.9)
with initial data \( A_0 = 0 \), via the identification \((8.7)\).

**Proof.** We observe first the identity \( \dot{g}_t = -2 g_t \dot{A}_t \). Then the conclusion follows combining the identity \( -2 \dot{A}_t = \dot{g}^*_t = \text{Ric}^*_{g_t}(\Omega) - \mathbb{I} \), with formula \((8.8)\).

We observe that the assumption \( g_0 \in \mathcal{S}^K_1 \) implies
\[
\text{Ric}^*_{g_A}(\Omega) = e^{2A} g_0^{-1} \text{Ric}_{g_A}(\Omega) \in \mathcal{T}^K_{g_0} , \forall A \in \mathcal{T}^K_{g_0},
\]
thanks to the fundamental identity \((8.3)\) combined with lemma \([4]\). We deduce
\[
e^A \text{div}^\alpha_{g_0} \left( e^A \nabla_{g_0} A \right) - e^{2A} \text{Ric}^*_{g_0}(\Omega) \in \mathcal{T}^K_{g_0} , \forall A \in \mathcal{T}^K_{g_0},
\] (8.10)
thanks to the expression \((8.8)\).
9 Convexity of $W_\Omega$ over convex subsets inside $(\Sigma_K(g_0), G)$

We define the functional $W_\Omega$ over $\mathcal{T}_K$ by the formula $W_\Omega(A) := W_\Omega(g_A)$, via the identification \(8.7\). We remind now the identity

$$W_\Omega(g) = \int_X \left[ \text{Tr}_g (\text{Ric}_g(\Omega) - g) + 2 \log \frac{dV_g}{\Omega} \right] \Omega$$

Plunging (8.6) and integrating by parts we infer the expression

$$W_\Omega(A) = \int_X \left[ \left[ \left| \nabla_{g_0} e^A \right|^2_{g_0} + \text{Tr}_\mathcal{R} (e^{2A} \text{Ric}_{g_0}(\Omega) - 2A) \right] \Omega \right.$$

$$+ \left. \int_X \left[ 2 \log \frac{dV_{g_0}}{\Omega} - n \right] \Omega \right]$$

We define now the vector space $\mathcal{T}_K^{g_0}$ as the $L^2$-closure of $\mathcal{T}_K^{g_0}$ and we equip it with the constant $L^2$-product $4 \int_X \langle \cdot, \cdot \rangle_{g_0} \Omega$. From now on all $L^2$-products are defined by this formula.

Lemma 11. Let $g_0 \in \mathcal{S}_K$. Then the forward equation equation (8.9) with initial data $A_0 = 0$ is equivalent to a smooth solution of the gradient flow equation $A_t = -\nabla_{L^2}W_\Omega(A_t)$.

Proof. We compute first the $L^2$-gradient of the functional $W_\Omega$. For this purpose we consider a line $t \mapsto A_t := A + t V$ with $A, V \in \mathcal{T}_K^{g_0}$ arbitrary. Then hold the identity

$$\frac{d}{dt} W_\Omega(A_t) = 2 \int_X \left[ \nabla_{g_0} e^A, \nabla_{g_0} (e^A V) \right]_{g_0} \Omega$$

$$+ 2 \int_X \text{Tr}_\mathcal{R} \left[ (e^{2A} \text{Ric}_{g_0}(\Omega) - \mathbb{I}) V \right] \Omega.$$ (9.1)

Integrating by parts we obtain the first variation formula

$$\frac{d}{dt} W_\Omega(A_t) \big|_{t=0} = -2 \int_X \langle e^A \text{div}_{g_0} (e^A \nabla_{g_0} A) - e^{2A} \text{Ric}_{g_0}(\Omega) + \mathbb{I}, V \rangle_{g_0} \Omega.$$ (9.2)

The assumption $g_0 \in \mathcal{S}_K$ implies the expression of the gradient

$$2 \nabla_{L^2}W_\Omega(A) = -e^A \text{div}_{g_0} (e^A \nabla_{g_0} A) + e^{2A} \text{Ric}_{g_0}(\Omega) - \mathbb{I} \in \mathcal{T}_K^{g_0},$$

thanks to the identity \(8.10\). We infer the required conclusion. \qed
We show now the following convexity results.

**Lemma 12.** For any $g_0 \in \mathcal{M}$ and for all $A, V \in \mathbb{T}_{g_0}^K$ hold the second variation formula

\[
\nabla_{L^2} D W_\Omega(A)(V, V) = 2 \int_X \left[ - e^A \text{div}_{g_0} (e^A \nabla_{g_0} A) + 2 e^{2A} \text{Ric}_{g_0}^*(\Omega) \right] V, V \right]_{g_0} \Omega + 2 \int_X \left| \nabla_{g_0} (e^A V) \right|_{g_0}^2 \Omega.
\]

Moreover if $\text{Ric}_{g_0}(\Omega) > 0$ then the functional $W_\Omega$ is convex over the convex set

\[
\mathbb{T}_{g_0}^{K,+} := \left\{ A \in \mathbb{T}_{g_0}^K \mid \int_X |U\nabla_{g_0} A|_{g_0}^2 \Omega \leq \int_X \text{Tr}_\mathfrak{m} \left[ U^2 \text{Ric}_{g_0}^*(\Omega) \right] \Omega, \forall U \in \mathbb{T}_{g_0}^K \right\}.
\]

**Proof.** We observe first that the convexity of the set $\mathbb{T}_{g_0}^{K,+}$ follows directly by the convexity of the $L^2$-norm squared. We compute now the second variation of the functional $W_\Omega$ along any line $t \mapsto A_t := A + t V$ with $A, V \in \mathbb{T}_{g_0}^K$. Differentiating the formula (9.1) we infer the expansion

\[
\nabla_{L^2} D W_\Omega(A)(V, V) = \frac{d^2}{dt^2} \Big|_{t=0} W_\Omega(A_t)
\]

\[
= 2 \int_X \left[ \left| \nabla_{g_0} (e^A V) \right|_{g_0}^2 + \left\langle \nabla_{g_0} e^A, \nabla_{g_0} (e^A V^2) \right\rangle_{g_0} \right] \Omega + 4 \int_X \text{Tr}_\mathfrak{m} \left[ e^{2A} V^2 \text{Ric}_{g_0}^*(\Omega) \right] \Omega.
\]

The required second variation formula follows by an integration by parts. We expand now the term

\[
\left\langle \nabla_{g_0} e^A, \nabla_{g_0} (e^A V^2) \right\rangle_{g_0} = \left\langle \nabla_{g_0} e^A, \nabla_{g_0} (e^A V) + e^A V \nabla_{g_0} V \right\rangle_{g_0}
\]

\[
= \left\langle V \nabla_{g_0} e^A, \nabla_{g_0} (e^A V) + e^A \nabla_{g_0} V \right\rangle_{g_0}
\]

\[
= 2 \left\langle V \nabla_{g_0} e^A, \nabla_{g_0} (e^A V) \right\rangle_{g_0} - \left| V \nabla_{g_0} e^A \right|_{g_0}^2.
\]

Using the Cauchy-Schwarz and Jensen’s inequalities we obtain

\[
2 \left| \left\langle V \nabla_{g_0} e^A, \nabla_{g_0} (e^A V) \right\rangle_{g_0} \right| \leq 2 \left| \left\langle V \nabla_{g_0} e^A \right\rangle_{g_0} \left| \nabla_{g_0} (e^A V) \right|_{g_0} \right|
\]

\[
\leq \left| V \nabla_{g_0} e^A \right|_{g_0}^2 + \left| \nabla_{g_0} (e^A V) \right|_{g_0}^2.
\]
and thus the inequality
\[
\langle \nabla g_0 e^A, \nabla g_0 (e^A V^2) \rangle_{g_0} \geq - 2 |\nabla g_0 e^A|_{g_0}^2 - |\nabla g_0 (e^A V)|_{g_0}^2.
\]
We infer the estimate
\[
\nabla_{L^2} D W_\Omega (A)(V, V) \geq 4 \int_X \left\{ \text{Tr}_R \left[ (Ve^A)^2 \text{Ric}_{g_0}(\Omega) \right] - |Ve^A \nabla g_0 A^2|_{g_0}^2 \right\} \Omega,
\]
which implies the required convexity statement over the convex set $T_{g_0}^K$. Indeed $Ve^A \in T_{g_0}^K$ thanks to lemma 4.

**Corollary 6.** Let $g_0 \in \mathcal{M}$. The functional $W_\Omega$ is $G$-convex over the $G$-convex set
\[
\Sigma_{g_0} (g_0) := \left\{ g \in \Sigma_K (g_0) \mid \text{Ric}_g (\Omega) \geq - \text{Ric}_{g_0} (\Omega) \right\},
\]
inside the totally geodesic and flat sub-variety $\Sigma_K (g_0)$ of the non-positively curved Riemannian manifold $(\mathcal{M}, G)$. Moreover if $\text{Ric}_{g_0} (\Omega) \geq \varepsilon g_0$, for some $\varepsilon \in \mathbb{R}_{>0}$ then the functional $W_\Omega$ is $G$-convex over the $G$-convex and non-empty sets
\[
\Sigma_\delta^K (g_0) := \left\{ g \in \Sigma_K (g_0) \mid \text{Ric}_g (\Omega) \geq \delta g_0 \right\}, \quad \forall \delta \in [0, \varepsilon),
\]
\[
\Sigma_+^K (g_0) := \left\{ g \in \Sigma_K (g_0) \mid 2 \text{Ric}_g (\Omega) + g_0 \Delta_{g_0} \log (g_0^{-1} g) \geq 0 \right\}.
\]

**Proof.** **STEP I (G-convexity of the sets $\Sigma_{g_0} (g_0)$).** We observe first that the change of variables (8.7) send geodesics in to lines. Indeed the image of any geodesic $t \mapsto g_t = g e^{tv^*}$ via this map is the line $t \mapsto A_t := A - tv^*/2 \in T_{g_0}^K$. We infer that the $G$-convexity of the set $\Sigma_K^I (g_0)$ is equivalent to the (linear) convexity of the set
\[
T_{g_0}^K := \left\{ A \in T_{g_0}^K \mid \text{Ric}_{g_0} (\Omega) \geq - \text{Ric}_{g_0} (\Omega) \right\},
\]
with $g_A := g_0 e^{-2A}$. In the same way the $G$-convexity of the sets $\Sigma^K_\delta (g_0)$ and $\Sigma^K_+ (g_0)$ is equivalent respectively to the convexity of the sets
\[
T_{g_0}^{K, \delta} := \left\{ A \in T_{g_0}^K \mid \text{Ric}_{g_0} (\Omega) \geq \delta g_A \right\},
\]
\[
T_{g_0}^{K, +} := \left\{ A \in T_{g_0}^K \mid \text{Ric}_{g_0} (\Omega) \geq g_0 \Delta_{g_0} A \right\}.
\]
Given any metric $g \in \mathcal{M}$ and any sections $A, B \in C^\infty (X, \text{End}_g (T_X))$ we define the bilinear product operation
\[
\{A, B\}_g := g (\nabla g, \bullet A \nabla g, \bullet B),
\]

and we observe the inequality \( \{ A, A \}_g \geq 0 \). This implies the convex inequality

\[
\{ A_t, A_t \}_g \leq (1 - t) \{ A_0, A_0 \}_g + t \{ A_1, A_1 \}_g ,
\]

\[
A_t := (1 - t) A_0 + t A_1 , \quad t \in [0, 1].
\]

for any \( A_0, A_1 \in C^\infty(X, \operatorname{End}_g(T_X)) \). Indeed we observe the expansion

\[
\{ A_t, A_t \}_g = \{ A_t, A_0 \}_g + \{ A_t, t(A_1 - A_0) \}_g
\]

\[
= (1 - t) \{ A_0, A_0 \}_g + t \{ A_1, A_0 \}_g + (1 - t)(A_1 - A_0), t(A_1 - A_0) \}
\]

\[
= (1 - t) \{ A_0, A_0 \}_g + t \{ A_1, A_1 \}_g - t(1 - t) \{ A_1 - A_0, A_1 - A_0 \}_g .
\]

Using this notation in formula (8.5) we infer the expression

\[
\text{Ric}_{g_A}(\Omega) = g_0 \Delta^0 g_0 A - \{ A, A \}_g + \text{Ric}_{g_0}(\Omega) .
\]

(9.2)

We deduce the identities

\[
T^K_{g_0} = \left\{ A \in T^K_{g_0} \mid g_0 \Delta^0 g_0 A \geq \{ A, A \}_g - 2 \text{Ric}_{g_0}(\Omega) \right\},
\]

\[
T^K\delta = \left\{ A \in T^K_{g_0} \mid g_0 \Delta^0 g_0 A \geq \{ A, A \}_g - \text{Ric}_{g_0}(\Omega) + \delta g_0 e^{-2A} \right\},
\]

\[
T^{K,\delta^+} = \left\{ A \in T^K_{g_0} \mid \{ A, A \}_g \leq \text{Ric}_{g_0}(\Omega) \right\}.
\]

Let now \( A_0, A_1 \in T^K_{g_0} \). The fact that \([A_0, A_1] = 0\) implies the existence of a \( g_0 \)-orthonormal basis which diagonalizes simultaneously \( A_0 \) and \( A_1 \). Then the convexity of the exponential function implies the convex inequality

\[
g_0 e^{-2A_t} \leq (1 - t) g_0 e^{-2A_0} + t g_0 e^{-2A_1},
\]

for all \( t \in [0, 1] \). Using the previous convex inequalities we obtain

\[
g_0 \Delta^0 g_0 A_t = (1 - t) g_0 \Delta^0 g_0 A_0 + t g_0 \Delta^0 g_0 A_1
\]

\[
\geq (1 - t) \{ A_0, A_0 \}_g + t \{ A_1, A_1 \}_g - \text{Ric}_{g_0}(\Omega)
\]

\[
+ (1 - t) \delta g_0 e^{-2A_0} + t \delta g_0 e^{-2A_1}
\]

\[
\geq \{ A_t, A_t \}_g - \text{Ric}_{g_0}(\Omega) + \delta g_0 e^{-2A_t},
\]

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for all $t \in [0,1]$. We infer the convexity of the set $\mathbb{T}_{g_0}^{K,\delta}$. The proof of the convexity of the sets $\mathbb{T}_{g_0}^{K,-}$ and $\mathbb{T}_{g_0}^{K,++}$ is quite similar.

**STEP II (G-convexity of the functional $W_\Omega$).** Using again the fact that the change of variables (8.7) send geodesics in to lines we infer that the G-convexity of the functional $W_\Omega$ over the G-convex sets $\Sigma_0(g_0)$, $\Sigma_0^+(g_0)$, $\Sigma_0^-(g_0)$ is equivalent to the convexity of the functional $W_\Omega$ over the convex sets $\mathbb{T}_{g_0}^{K,-}$, $\mathbb{T}_{g_0}^{K,\delta}$, $\mathbb{T}_{g_0}^{K,++}$. Let now $g \in \Sigma_0(g_0)$ and observe that

$$0 \leq \text{Ric}_g(\Omega) + \text{Ric}_{g_0}(\Omega)$$

thanks to the identity (8.8). Then the second variation formula in lemma 12 implies the convexity of the functional $W_\Omega$ over the convex set $\mathbb{T}_{g_0}^{K,++}$. The convexity of the functional $W_\Omega$ over $\mathbb{T}_{g_0}^{K,\delta}$ is obvious at this point. We observe now the inclusion $\mathbb{T}_{g_0}^{K,++} \subset \mathbb{T}_{g_0}^{K,\delta}$. Indeed for all $A \in \mathbb{T}_{g_0}^{K,++}$ and for all $U \in \mathbb{T}_{g_0}^{K}$ hold the trivial identities

$$|U \nabla_{g_0} A|_{g_0}^2 = \sum_k |U \nabla_{g_0, e_k} A|_{g_0}^2$$

$$= \sum_k \left\langle \nabla_{g_0, e_k} A \nabla_{g_0, e_k} A, U \right\rangle_{g_0}$$

$$= \left\langle \text{Tr}_{g_0} \left( \nabla_{g_0, \bullet} A \nabla_{g_0, \bullet} A \right), U \right\rangle_{g_0}$$

$$= \text{Tr}_\mathbb{R} \left[ \text{Tr}_{g_0} \left( \nabla_{g_0, \bullet} A \nabla_{g_0, \bullet} A \right) U^2 \right]$$

$$\leq \text{Tr}_\mathbb{R} \left[ U^2 \text{Ric}_{g_0}^*(\Omega) \right],$$

where $(e_k)_k \subset T_{X,x}$ is a $g_0$-orthonormal basis at an arbitrary point $x \in X$. Then lemma 12 implies the convexity of the functional $W_\Omega$ over the convex set $\mathbb{T}_{g_0}^{K,++}$. □

### 10 The extension of the functional $W_\Omega$ to $\mathbb{T}_{g_0}^{K}$

We denote by $\text{End}_{g_0}(T_X)$ the space of $g_0$-symmetric endomorphisms. We define the natural integral extension

$$W_\Omega : \mathbb{T}_{g_0}^{K} \rightarrow (-\infty, +\infty),$$

of the functional $W_\Omega$ by the integral expression in the beginning of section 9 if $e^A \in \mathbb{T}_{g_0}^{K} \cap H^1(X, \text{End}_{g_0}(T_X))$ and $W_\Omega(A) = +\infty$ otherwise. We show now the following elementary fact.
Lemma 13. Let $g_0 \in \mathcal{M}$ such that $\text{Ric}_{g_0}(\Omega) \geq \varepsilon g_0$, for some $\varepsilon \in \mathbb{R}_{>0}$. Then the natural integral extension $W_\Omega : \mathbb{P}_{g_0}^K \rightarrow (-\infty, +\infty]$ of the functional $W_\Omega$ is lower semi-continuous and bounded from below. Indeed for any $A \in \mathbb{P}_{g_0}$ hold the uniform estimate

$$W_\Omega(A) \geq 2 \int_X \log \frac{dV_{g_0}}{\Omega} \Omega.$$ 

Proof. We observe that a function $f$ over a metric space is l.s.c. iff $f(x) \leq \liminf_{k \to +\infty} f(x_k)$ for any convergent sequence $x_k \to x$ such that $\sup_k f(x_k) < +\infty$. So let $(A_k)_k \subset \mathbb{P}_{g_0}^K$ and $A \in \mathbb{P}_{g_0}^K$ such that $A_k \to A$ in $L^2(X)$ with $\sup_k W_\Omega(A_k) < +\infty$. This combined with the assumption $\text{Ric}_{g_0}(\Omega) \geq \varepsilon g_0$, implies the estimates

$$\int_X \left[ |\nabla_{g_0} e^{A_k}|_{g_0}^2 + \varepsilon |e^{A_k}|_{g_0}^2 \right] \Omega$$

$$\leq \int_X \left[ |\nabla_{g_0} e^{A_k}|_{g_0}^2 + \langle \text{Ric}_{g_0}(\Omega) e^{A_k}, e^{A_k} \rangle_{g_0} \right] \Omega$$

$$\leq C,$$ 

(10.1)

for some uniform constant $C$. The first uniform estimate combined with the Rellich-Kondrachov compactness result, $H^1(X) \subset L^2(X)$, implies that for every subsequence of $(e^{A_k})_k$ there exists a sub-subsequence convergent to $e^A$ in $L^2(X)$. We observe indeed that the assumption on the $L^2$-convergence $A_k \to A$ implies that for every subsequence of $(e^{A_k})_k$ there exists a sub-subsequence convergent to $e^A$ a.e. over $X$. We infer that $e^{A_k} \to e^A$ in $L^2(X)$. Then the uniform estimates (10.1) imply that $\nabla_{g_0} e^{A_k} \to \nabla_{g_0} e^A$ weakly in $L^2(X)$. We deduce

$$W_\Omega(A) \leq \liminf_{k \to +\infty} W_\Omega(A_k),$$

thanks to the weak lower semi-continuity of the $L^2$-norm. We show now the lower bound in the statement. For this purpose we observe the estimates

$$W_\Omega(A) \geq \int_X \left[ \text{Tr}_n (\varepsilon e^{2A} - 2A) + 2 \log \frac{dV_{g_0}}{\Omega} - n \right] \Omega$$

$$\geq \int_X \left[ n(1 + \log \varepsilon) + 2 \log \frac{dV_{g_0}}{\Omega} - n \right] \Omega.$$ 

The last estimate follows from the fact that the convex function $x \mapsto \varepsilon e^{2x} - 2x$ admits a global minimum over $\mathbb{R}$ at the point $-(\log \varepsilon)/2$ in which takes the value $1 + \log \varepsilon$. We infer the required lower bound. \(\square\)
From now on we will assume that the polarization endomorphism $K$ is smooth. We define the vector spaces

$$W^{1,\infty}(\mathbb{T}_{g_0}) := \left\{ A \in W^{1,\infty}(X, \text{End}_{g_0}(T_X)) \mid \nabla_{T_X,g_0} A = 0 \right\},$$

$$W^{1,\infty}(\mathbb{T}^K_{g_0}) := \left\{ A \in W^{1,\infty}(\mathbb{T}_{g_0}) \mid [\nabla^p_{g_0} T, A] = 0, \quad T = R_{g_0}, K, \forall p \in \mathbb{Z}_{\geq 0} \right\},$$

We denote by $\mathbb{T}^{K,++}_{g_0}$ the $L^2$-closure of the set convex set $\mathbb{T}^{K,++}_{g_0}$. The following quite elementary lemma will be useful for convexity purposes.

**Lemma 14.** Let $g_0 \in \mathcal{M}$ such that $\text{Ric}_{g_0}(\Omega) > 0$. Then the $L^2$-closed and convex set $\mathbb{T}^{K,++}_{g_0}$ satisfies the inclusion

$$\mathbb{T}^{K,++}_{g_0} \subset W^{1,\infty}(\mathbb{T}^K_{g_0}),$$

where $W^{1,\infty}(\mathbb{T}^K_{g_0})$ is the set of points $A \in W^{1,\infty}(\mathbb{T}_{g_0})$ such that

$$\int_X |U \nabla_{g_0} A|^2_{g_0} \Omega \leq \int_X \text{Tr}_g [U^2 \text{Ric}^*_{g_0}(\Omega)] \Omega, \quad \forall U \in \mathbb{T}^K_{g_0}.$$  \hspace{1cm} (10.2)

**Proof.** We remind that for all $A \in \mathbb{T}^{K,++}_{g_0}$ hold the inequality

$$|\nabla_{g_0} A|^2_{g_0} \leq \text{Tr}_{g_0} \text{Ric}_{g_0}(\Omega),$$

which implies the uniform estimate

$$\left[ \int_X |\nabla_{g_0} A|^{2p}_{g_0} \right]^{\frac{1}{p}} \leq \left[ \int_X \text{Tr}_{g_0} \text{Ric}_{g_0}(\Omega) \Omega \right]^{\frac{1}{p}}, \hspace{1cm} (10.2)$$

for all $p \in \mathbb{N}_{\geq 1}$. Let now $A \in \mathbb{T}^{K,++}_{g_0}$ arbitrary and let $(A_k)_k \subset \mathbb{T}^{K,++}_{g_0}$ be a sequence $L^2$-convergent to $A$. Applying the uniform estimate \((10.2)\) to $A_k$ we infer that \((10.2)\) hold also for $A$, by the weak $L^{2p}$-compactness and the weak lower semi-continuity of the $L^{2p}$-norm. Furthermore taking the limit as $p \to +\infty$ in \((10.2)\) we infer the uniform estimate

$$\|\nabla_{g_0} A\|_{L^{\infty}(X,g_0)} \leq \sup_X \left[ \text{Tr}_{g_0} \text{Ric}_{g_0}(\Omega) \right]^{\frac{1}{p}},$$

for all $A \in \mathbb{T}^{K,++}_{g_0}$. It is clear at this point that $A \in L^\infty(X, \text{End}_{g_0}(T_X))$. Indeed consider an arbitrary coordinate ball $B \subset X$ with center a point $x \equiv 0$ such that $|A|_{g_0}(0) < +\infty$. Then for all $v \in B$ hold the inequalities

$$|A|_{g_0}(v) \leq |A|_{g_0}(0) + \int_0^1 |\langle \nabla_{g_0} A(tv), v \rangle_{g_0}| dt$$

$$\leq |A|_{g_0}(0) + \int_0^1 |\nabla_{g_0,v} A|_{g_0}(tv) dt,$$
which show that $A$ is bounded. In the last inequality we used the estimate
\[
| \langle \nabla g_0 | A \rangle_{g_0}, \xi \rangle_{g_0} | \leq | \nabla g_0, \xi | A \rangle_{g_0},
\]
for all $\xi \in T_{X,x}$. This last follows combining the elementary identities
\[
\xi | A |^2 = 2 | \langle \nabla g_0 | A \rangle_{g_0}, \xi \rangle_{g_0} |
\]
with the Cauchy-Schwartz inequality. It is also clear by the definition of the convex set $T^{K,++}_{g_0}$ that $A \in W^{1,\infty}(T^{K}_{g_0})$. Moreover the inclusion $T^{K,++}_{g_0} \subset T^{K,++}_{g_0}$ implies that for all $A \in T^{K,++}_{g_0}$ hold the inequality
\[
\int_X |U \nabla g_0 | A |^2 | \Omega \leq \int_X \text{Tr}_{\mathbb{R}} \left[ U^2 \text{Ric}_{g_0}^* \Omega \right] \Omega, \quad \forall U \in T^{K}_{g_0}.
\]
Indeed this follows by the weak $L^2$-compactness and the weak lower semicontinuity of the $L^2$-norm. By $L^2$-density we infer the inclusion in the statement of lemma 14.

We can show that the same result hold also for the closure of the set $T^{K,++}_{g_0}$. However the proof of this case is slightly more complicated and we omit it since we will not use it.

**Lemma 15.** Consider any $g_0 \in \mathcal{M}$ such that $\text{Ric}_{g_0}(\Omega) > 0$. Then the natural integral extension $W_\Omega : T^{K,++}_{g_0} \rightarrow \mathbb{R}$ of the functional $W_\Omega$ is lower semicontinuous, uniformly bounded from below and convex over the $L^2$-closed and convex set $T^{K,++}_{g_0}$.

**Proof.** Thanks to lemma 13 we just need to show the convexity of the natural integral extension $W_\Omega : T^{K,++}_{g_0} \rightarrow \mathbb{R}$. We consider for this purpose an arbitrary segment $t \in [0,1] \mapsto A_t := A + t V \in T^{K,++}_{g_0} \subset W^{1,\infty}(T^{K}_{g_0})$. In particular the fact that $A_t \in W^{1,\infty}(T^{K}_{g_0})$ combined with the expression
\[
W_\Omega(A_t) = \int_X \left[ |e^{tV} e^A \nabla g_0 A_t |^2_{g_0} + \text{Tr}_{\mathbb{R}} \left( e^{2tV} e^{2A} \text{Ric}_{g_0}^* \Omega \right) - 2 A_t \right] \Omega,
\]
implies that the function $t \in [0,1] \mapsto W_\Omega(A_t) \in \mathbb{R}$ is of class $C^\infty$ over the time interval $[0,1]$. Moreover $e^{A_t} \in W^{1,\infty}(T^{K}_{g_0})$ for all $t \in [0,1]$ thanks to the argument in the proof of lemma 14. This is all we need in order to apply to $A_t \in W^{1,\infty}(T^{K}_{g_0})$ the first order computations in the proof of lemma 12 which

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provide the second variation formula

\[ \frac{d^2}{dt^2} W_\Omega(A_t) \]

\[ = \ 4 \int_X \left\{ \Tr R \left[ (Ve^{A_t})^2 \operatorname{Ric}^*_g (\Omega) \right] - \left| Ve^{A_t} \nabla g_0 A_t \right|^2 \right\} \Omega \]

\[ + \ 2 \int_X \left[ \left| \nabla g_0 (V e^{A_t}) \right|^2 + \left| V \nabla g_0 e^{A_t} \right|^2 + 2 \left\langle V \nabla g_0 e^{A_t}, \nabla g_0 (V e^{A_t}) \right\rangle \right] \Omega \]

\[ \geq \ 4 \int_X \left\{ \Tr R \left[ (Ve^{A_t})^2 \operatorname{Ric}^*_g (\Omega) \right] - \left| Ve^{A_t} \nabla g_0 A_t \right|^2 \right\} \Omega , \]

thanks to the Cauchy-Schwarz and Jensen’s inequalities. We show now that

\[ U e^{A_t} \in \mathbb{F}^K_{g_0} , \quad (10.3) \]

for all \( U \in \mathbb{F}^K_{g_0} \) and all \( A \in \mathbb{F}^{K,+}_{g_0} \). Indeed let \((A_j)_j \subset \mathbb{F}^{K,+}_{g_0}\) and \((U_j)_j \subset \mathbb{F}^K_{g_0}\) be two sequences convergent respectively to \( A \) and \( U \) in the \( L^2 \)-topology. From a computation in the proof of lemma 14 we know that the uniform estimate

\[ \left| \nabla g_0 A_j \right|^2_{g_0} \leq \Tr R \operatorname{Ric}^*_g (\Omega) , \]

combined with the convergence a.e implies that the sequence \((A_j)_j\) is bounded in norm \( L^\infty \). We infer the \( L^2 \)-convergence \( \mathbb{T}^K_{g_0} \ni U_k e^{A_k} \longrightarrow U e^A \in \mathbb{F}^K_{g_0} \) thanks to the dominated convergence theorem. We observe now that the property \((10.3)\) applied to \( V e^{A_t} \) combined with the fact that \( A_t \in W^{1,\infty}(\mathbb{T}^K_{g_0}) \) for all \( t \in [0,1] \) provides the inequality

\[ \frac{d^2}{dt^2} W_\Omega(A_t) \geq 0 , \]

over the time interval \([0,1]\), which shows the required convexity statement. \( \square \)

The convexity statement over the \( d_G \)-convex set \( \Sigma^+_K(g_0) \) in the main theorem \( 1 \) follows directly from lemma 15 due to the fact that the change of variables

\[ g \in \Sigma^+_K(g_0) \quad \longrightarrow \quad A = -\frac{1}{2} \log(g_0^{-1} g) \in \mathbb{T}^{K,+}_{g_0} , \]

represents a \((d_G, L^2)\)-isometry map (where \( L^2 \) denotes the constant \( L^2 \)-product

\[ 4 \int_X \langle \cdot , \cdot \rangle_{g_0} \Omega \)]

which in particular send all \( d_G \)-geodesics segments

\[ t \in [0,1] \quad \longrightarrow \quad g_t = g e^{tv^*_g} \in \Sigma^+_K(g_0) , \]

in to linear segments \( t \in [0,1] \longrightarrow A_t := A - t v^*_g / 2 \in \mathbb{T}^{K,+}_{g_0} \).
11 On the exponentially fast convergence of the Soliton-Ricci-flow

Lemma 16. Let $g_0 \in S^K_{\mathcal{V}^+}$ and let $(g_t)_{t \geq 0} \subset \Sigma_K(g_0)$ be a solution of the $\Omega$-SRF with initial data $g_0$. If there exist $\delta \in \mathbb{R}_{>0}$ such that $\text{Ric}_{g_t}(\Omega) \geq \delta g_t$ for all times $t \geq 0$, then the $\Omega$-SRF converges exponentially fast with all its space derivatives to a $\Omega$-ShRS $g_{\text{RS}} \in \Sigma_K(g_0)$ as $t \to +\infty$.

Proof. Time deriving the $\Omega$-SRF equation by means of (2.3) we infer the evolution formula

$$2 \ddot{g}_t = - \Delta_{g_t} \dot{g}_t - 2 \dot{g}_t,$$

and thus the evolution equation

$$2 \frac{d}{dt} \dot{g}_t^* = - \Delta_{g_t}^* \dot{g}_t^* - 2 \dot{g}_t^* - 2 (\dot{g}_t^*)^2. \quad (11.1)$$

Using this we can compute the evolution of $|\dot{g}_t|^2_{g_t} = |\dot{g}_t^*|^2_{g_t} = \text{Tr}_{g_t}(\dot{g}_t^*)^2$. Indeed we define the heat operator

$$\Box_{g_t}^\alpha := \Delta_{g_t}^\alpha + 2 \frac{d}{dt},$$

and we observe the elementary identity

$$\Delta_{g_t}^\alpha |\dot{g}_t|^2_{g_t} = 2 \langle \Delta_{g_t}^\alpha \dot{g}_t^*, \dot{g}_t^* \rangle_{g_t} - 2 |\nabla_{g_t} \dot{g}_t^*|^2_{g_t}.$$

We infer the evolution formula

$$\Box_{g_t}^\alpha |\dot{g}_t|^2_{g_t} = - 2 |\nabla_{g_t} \dot{g}_t^*|^2_{g_t} - 4 |\dot{g}_t|^2_{g_t} - 4 \text{Tr}_{g_t}(\dot{g}_t^*)^3$$

$$\leq - \delta |\dot{g}_t|^2_{g_t},$$

thanks to the $\Omega$-SRF equation and thanks to the assumption $\text{Ric}_{g_t}(\Omega) \geq \delta g_t$. Applying the scalar maximum principle we infer the exponential estimate

$$|\dot{g}_t|_{g_t} \leq \sup_X |\dot{g}_0|_{g_0} e^{-\delta t/2}, \quad (11.2)$$

for all $t \geq 0$. In its turn this implies the convergence of the integral

$$\int_0^{+\infty} |\dot{g}_t|_{g_t} dt \leq C,$$

and thus the uniform estimate

$$e^{-C} g_0 \leq g_t \leq e^C g_0, \quad (11.3)$$

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for all \( t \geq 0 \), (see [Ch-Kn]). Thus the convergence of the integral

\[
\int_0^{+\infty} |\dot{g}_t|_{g_0} dt < +\infty,
\]

implies the existence of the metric

\[
g_\infty := g_0 + \int_0^{+\infty} \dot{g}_t \, dt,
\]

thanks to Bochner’s theorem. (The positivity of the metric \( g_\infty \) follows from the estimate \( e^{-C} g_0 \leq g_t \).) Moreover the estimate

\[
|g_\infty - g_t|_{g_0} \leq \int_t^{+\infty} |\dot{g}_s|_{g_0} ds \leq C' e^{-t},
\]

implies the exponential convergence of the \( \Omega \)-SRF to \( g_\infty \) in the uniform topology. We show now the \( C^1(X) \)-convergence. Indeed the fact that \( (g_t)_{t \geq 0} \subset \Sigma_K(g_0) \) implies \( \dot{g}_t \in \mathcal{F}_{g_0}^K \) for all times \( t \geq 0 \) thanks to the identity (5.2).

Thus hold the identities \([K, \dot{g}_t^*] \equiv 0\) and \([K, \nabla g_\tau \dot{g}_t^*] \equiv 0\), which imply in their turn \([\nabla g_\tau \dot{g}_t^*, \dot{g}_t^*] \equiv 0\). By the variation formula (3.19) we deduce the identity

\[
\nabla g_\tau \dot{g}_t^* \equiv \nabla g_0 \dot{g}_t^*,
\]

for all \( \tau, t \geq 0 \). This combined with (11.1) provides the equalities

\[
\frac{d}{dt} (\nabla g_\tau \dot{g}_t^*) = 2 \nabla g_\tau \frac{d}{dt} \dot{g}_t^* = -\nabla g_\tau \Delta^{\alpha}_{g_\tau} \dot{g}_t^* - 2 \nabla g_\tau \dot{g}_t^* - 2 \nabla g_\tau (\dot{g}_t^*)^2
\]

\[
= -\Delta^{\alpha}_{g_\tau} \nabla g_\tau \dot{g}_t^* - \text{Ric}_g^*(\Omega) \bullet \nabla g_\tau \dot{g}_t^*
\]

\[
- 2 \nabla g_\tau \dot{g}_t^* - 4 \dot{g}_t^* \nabla g_\tau \dot{g}_t^*.
\]

We justify the last equality. We pick geodesic coordinates centered at an arbitrary space time point \((x_0, t_0)\), let \((e_k)_k\) be the coordinate local tangent frame and let \(\xi, \eta\) be local vector fields with constant coefficients defined in a neighborhood of \(x_0\). We expand at the space time point \((x_0, t_0)\) the term

\[
\nabla_{g_\tau, \xi} \Delta^{\alpha}_{g_\tau} \dot{g}_t^* = - \nabla_{g_\tau, \xi} \nabla_{g_\tau, e_k} \nabla_{g_\tau, e_k} \dot{g}_t^* + \nabla_{g_\tau, \xi} \nabla_{g_\tau, e_k} e_k - \nabla_{g_\tau, \xi} \dot{g}_t^*
\]

\[
+ \nabla_{g_\tau, \xi} \nabla_{g_\tau, \nabla g_\tau f, \xi} \dot{g}_t^* + \nabla_{g_\tau, \xi} \nabla_{g_\tau, f, \xi} \dot{g}_t^* - \nabla_{g_\tau, \xi} \dot{g}_t^*.
\]
thanks to the identity \(3.6\) and \([R_{\gamma}, \nabla_{\gamma, ek} \dot{g}_t^*] = 0, [R_{\gamma}, \dot{g}_t^*] = 0, [e_k, \xi] \equiv 0.\) Moreover at the space time point \((x_0, t_0)\) hold the identities

\[
\nabla_{\gamma, ek} \nabla_{\gamma}^2 \dot{g}_t^*(e_k, \xi, \eta) = \nabla_{\gamma, ek} \left[ \nabla_{\gamma, ek} \nabla_{\gamma} \dot{g}_t^*(\xi, \eta) \right] = \nabla_{\gamma, ek} \nabla_{\gamma, ek} (\nabla_{\gamma, e k} \dot{g}_t^* \eta) - \nabla_{\gamma, ek} \left[ \nabla_{\gamma, ek} \nabla_{\gamma, ek} \dot{g}_t^* \eta \right] + \nabla_{\gamma, ek} \dot{g}_t^* (\xi, \nabla_{\gamma, ek} \eta) = \nabla_{\gamma, ek} \left[ \nabla_{\gamma, ek} \nabla_{\gamma, ek} \dot{g}_t^* \eta \right] - \nabla_{\gamma, ek} \dot{g}_t^* (\nabla_{\gamma, ek} \eta, \eta) + \nabla_{\gamma, ek} \dot{g}_t^* (\nabla_{\gamma, ek} \eta, \eta),
\]

which combined with the previous expression implies the formula

\[
\nabla_{\gamma} \Delta_{\gamma t}^\alpha \dot{g}_t^* = \Delta_{\gamma t}^\alpha \nabla_{\gamma} \dot{g}_t^* + Ric^\alpha_{\gamma}(\Omega) \bullet \nabla_{\gamma} \dot{g}_t^*.
\]

Thus

\[
2 \frac{d}{dt} (\nabla_{\gamma} \dot{g}_t^*) = - \Delta_{\gamma t}^\alpha \nabla_{\gamma} \dot{g}_t^* - \dot{g}_t^* \bullet \nabla_{\gamma} \dot{g}_t^* - 3 \nabla_{\gamma} \dot{g}_t^* - 4 \dot{g}_t^* \nabla_{\gamma} \dot{g}_t^*,
\]

by the \(\Omega\)-SRF equation. We use this to compute the evolution of the norm squared

\[
|\nabla_{\gamma} \dot{g}_t^*|_{\gamma t}^2 = Tr \left( \nabla_{\gamma, ek} \dot{g}_t^* \nabla_{\gamma, e k}^{-1} \dot{g}_t^* \right)
\]

Indeed at time \(t_0\) hold the identities

\[
\frac{d}{dt} |\nabla_{\gamma} \dot{g}_t^*|_{\gamma t}^2 = Tr \left[ 2 e_k - \frac{d}{dt} (\nabla_{\gamma} \dot{g}_t^*) \nabla_{\gamma, ek} \dot{g}_t^* - \nabla_{\gamma, ek} \dot{g}_t^* \nabla_{\gamma, e k} \dot{g}_t^* \right] = - \nabla_{\gamma} \Delta_{\gamma t}^\alpha \nabla_{\gamma} \dot{g}_t^* + 2 \nabla_{\gamma, e k} \dot{g}_t^* \nabla_{\gamma, e k} \dot{g}_t^* + 3 \nabla_{\gamma} \dot{g}_t^* + 4 \dot{g}_t^* \nabla_{\gamma} \dot{g}_t^* + 2 \left( \Delta_{\gamma t}^\alpha \nabla_{\gamma} \dot{g}_t^* \nabla_{\gamma} \dot{g}_t^* \right)_{\gamma t} - 3 |\nabla_{\gamma} \dot{g}_t^*|_{\gamma t}^2 - 2 \left( \dot{g}_t^* \bullet \nabla_{\gamma} \dot{g}_t^* + 2 \dot{g}_t^* \nabla_{\gamma} \dot{g}_t^* \nabla_{\gamma} \dot{g}_t^* \right)_{\gamma t}.
\]

This combined with the elementary identity

\[
\Delta_{\gamma t}^\alpha |\nabla_{\gamma} \dot{g}_t^*|_{\gamma t}^2 = 2 \left( \Delta_{\gamma t}^\alpha \nabla_{\gamma} \dot{g}_t^* \nabla_{\gamma} \dot{g}_t^* \right)_{\gamma t} - 2 |\nabla_{\gamma} \dot{g}_t^*|_{\gamma t}^2,
\]

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implies the evolution formula
\[ \Box_{g_t} \left| \nabla g_t \dot{g}_t^* \right|_{g_t} = -2 \left| \nabla^2 g_t \dot{g}_t^* \right|_{g_t} - 6 \left| \nabla g_t \dot{g}_t^* \right|_{g_t} \]
\[ - 4 \left\langle \dot{g}_t^* \cdot \nabla g_t \dot{g}_t^* + 2 \dot{g}_t^* \nabla g_t \dot{g}_t^*, \nabla g_t \dot{g}_t^* \right\rangle_{g_t} \]
\[ \leq \left[ (4 + 8 \sqrt{n}) \left| \dot{g}_t \right|_{g_t} - 6 \right] \left| \nabla g_t \dot{g}_t^* \right|^2_{g_t} \]
\[ \leq \left( C e^{-t} - 6 \right) \left| \nabla g_t \dot{g}_t^* \right|^2_{g_t}, \]
thanks to the uniform exponential estimate (11.2). An application of the scalar maximum principle implies the estimate
\[ \left| \nabla g_t \dot{g}_t^* \right|_{g_t} \leq C e^{-t}, \quad (11.5) \]
for all \( t \geq 0 \), where \( C_1 > 0 \) is a constant uniform in time. By abuse of notation we allow denote by \( C \) or \( C_1 \) such type of constants. Moreover the identity \( (11.4) \) for \( \tau = t \) combined with (11.3) provides the exponential estimate
\[ \left| \nabla g_0 \dot{g}_t \right|_{g_0} \leq C_1 e^{-t}. \]
We observe now the trivial decomposition
\[ \nabla g_0 (g_0^{-1} \dot{g}_t) = \nabla g_0 (g_0^{-1} \dot{g}_t) \dot{g}_t^* + g_0^{-1} \dot{g}_t \nabla g_0 \dot{g}_t^*. \]
Thus if we set \( N_t := \left| \nabla g_0 (g_0^{-1} \dot{g}_t) \right|_{g_0} \), we infer the first order differential inequality
\[ \dot{N}_t \leq \sqrt{n} N_t \left| \dot{g}_t \right|_{g_0} + \sqrt{n} \left| g_0^{-1} \dot{g}_t \right|_{g_0} \left| \nabla g_0 \dot{g}_t^* \right|_{g_0} \]
\[ \leq C N_t e^{-t} + C e^{-2t}, \]
and thus
\[ N_t \leq e^{C \int_0^t e^{-s} \, ds} \left[ N_0 + C \int_0^t e^{-2s} \, ds \right] \leq e^{C \left( N_0 + C \right)}, \]
by Gronwall’s inequality. We deduce in conclusion the exponential estimate \( N_t \leq C_1 e^{-t} \), i.e,
\[ \left| \nabla g_0 \dot{g}_t \right|_{g_0} \leq C_1 e^{-t}, \]
for all \( t \geq 0 \). We infer the convergence of the integral
\[ \int_0^{+\infty} \left| \nabla g_0 \dot{g}_t \right|_{g_0} \, dt < + \infty, \]
and thus the existence of the tensor
\[ A_1 := \int_0^{+\infty} \nabla g_0 \dot{g}_t \, dt, \]
thanks to Bochner’s theorem. Moreover hold the exponential estimate
\[ |A_1 - \nabla_{g_0} g_t|_{g_0} \leq \int_t^{+\infty} |\nabla_{g_0} \dot{g}_s|_{g_0} \, ds \leq C' e^{-t}. \]

A basic calculus fact implies that \( A_1 = \nabla_{g_0} g_{\infty} \). In order to obtain the convergence of the higher order derivatives we need to combine the uniform \( C^1 \)-estimate obtained so far with an interpolation method based in Hamilton’s work (see [Ham]). The details will be explained in the next more technical sections. In conclusion taking the limit as \( t \to +\infty \) in the \( \Omega \)-SRF equation we deduce that \( g_{\infty} = g_{RS} \) is a \( \Omega \)-ShRS. \( \square \)

12 The commutator \([\nabla^p, \Delta^\Omega]\) along the \( \Omega \)-Soliton-Ricci flow

We introduce first a few product notations. Let \( g \) be a metric over a vector space \( V \). For any \( A \in (V^*)^\otimes p \otimes V \), \( B \in (V^*)^\otimes q \otimes V \) and for all integers \( k, l \) such that \( 1 \leq l \leq k \leq q - 2 \) we define the product \( A \circ^q_{k,l} B \) as
\[
(A \circ^q_{k,l} B)(u,v) := \text{Tr}_g \left[ B(v_1, \ldots, v_{l-1}, v_l, \ldots, v_k, A(u, \cdot, v_k), v_{k+1}, \ldots, v_{q-1}) \right],
\]
for all \( u \equiv (u_1, \ldots, u_{p-2}) \) and \( v \equiv (v_1, \ldots, v_{q-1}) \). Moreover for any \( \sigma \in S_{p+l-3} \) we define the product \( A \circ^{q,\sigma}_{k,l} B \) as
\[
(A \circ^{q,\sigma}_{k,l} B)(u,v) := (A \circ^q_{k,l} B)(\xi, v_1, \ldots, v_{q-1}),
\]
where \( \xi \equiv (\xi_1, \ldots, \xi_{p+l-3}) := (u_1, \ldots, u_{p-2}, v_1, \ldots, v_{q-1}) \). We notice also that \( A \circ^{p,\sigma}_{k,l} B \equiv A \circ^{q,\sigma}_{k,l} B \) if \( p + l - 3 \leq 1 \). Finally we define
\[
A \overset{2}{\sim} B := \sum_{k=1}^{q-2} A \circ^q_{k,k+1} B.
\]

Let now \((X, g)\) be a Riemannian manifold and let \( A \in C^\infty(X, (T^*_X)^\otimes p \otimes T_X)\). We remind the classic formula
\[
[\nabla_g \xi, \nabla_g \eta] A = [\mathcal{R}_g(\xi, \eta), A] - \mathcal{R}_g(\xi, \eta) \overset{\sim}{\circ} A + \nabla_g [\xi, \eta] A, \quad (12.1)
\]
for all \( \xi, \eta \in C^\infty(X, T_X) \). We show now the following lemma.

**Lemma 17.** Let \((X, g)\) be an oriented Riemannian manifold, let \( \Omega > 0 \) be a smooth volume form over \( X \) and let \( A \in C^\infty(X, (T^*_X)^\otimes p \otimes T_X) \) such that \([\mathcal{R}_g, \xi \overset{\sim}{\circ} \nabla_g A] = 0 \) for all \( r = 0, 1 \) and \( \xi \in T^\otimes_{X} \). Then hold the formula
\[
[\nabla_g, \Delta^\Omega_g] A = \text{Ric}^g(\Omega) \bullet \nabla_g A + 2 \mathcal{R}_g \overset{\sim}{\circ} \nabla_g A + \nabla^\otimes_{g} \mathcal{R}_g \overset{\sim}{\circ} A.
\]
Proof. We pick geodesic coordinates centered at an arbitrary point $x_0$. Let $(e_k)_k$ be the coordinate local tangent frame and let $\xi, \eta \equiv (\eta_1, \ldots, \eta_p)$ be local vector fields with constant coefficients defined in a neighborhood of the point $x_0$. We expand at the point $x_0$ the term

$$\nabla_{g, \xi} \Delta^0_g A = - \nabla_{g, \xi} \nabla_{g, e_k} \nabla_{g, e_k} A$$

$$+ \nabla_{g, \xi} \nabla_{g, e_k} e_k - \nabla_g A + \nabla_{g, \xi} \nabla_g f A$$

$$= - \nabla_{g, e_k} \nabla_{g, \xi} \nabla_{g, e_k} A + R_g (\xi, e_k) \nabla_{g, e_k} A$$

$$+ \nabla_{g, \xi} \nabla_{g, e_k} e_k - \nabla_g A + \nabla_{g, \xi} \nabla_g f \nabla_{g, \xi} A$$

$$- R_g (\xi, \nabla_g f) \nabla_{g, \xi} A + \nabla_{g, \xi} \nabla_{g, \xi} A$$

$$= - \nabla_{g, e_k} \nabla_{g, e_k} \nabla_{g, \xi} A$$

$$+ 2 R_g (\xi, e_k) \nabla_{g, e_k} A + \nabla_{g, e_k} R_g (\xi, e_k) \nabla_{g, e_k} A$$

$$+ \nabla_{g, \xi} \nabla_{g, e_k} e_k - \nabla_g A + (\nabla_g f - \nabla^2_g A) (\xi, \cdot)$$

$$+ R_g (\nabla_g f, \xi) \nabla_{g, \xi} A + \nabla_{g, \xi} \nabla_g f - \nabla_g A,$$

thanks to the identity (12.1), to the assumptions on $A$ and to the identity $[e_k, \xi] \equiv 0$. Moreover at the point $x_0$ hold the identity

$$\nabla_{g, e_k} \nabla^2_g A (e_k, \xi, \eta) = \nabla_{g, e_k} \left[ \nabla_{g, e_k} \nabla_g A (\xi, \eta) \right]$$

$$= \nabla_{g, e_k} \left\{ \nabla_{g, e_k} \left[ \nabla_{g, \xi} A (\eta) \right] - \nabla_g A (\nabla_{g, e_k} \xi, \eta) \right\}$$

$$- \sum_{j=1}^p \nabla_{g, e_k} \left[ \nabla_{g, \xi} \left[ \nabla_{g, \xi} A (\eta) \right] - \nabla_g A (\nabla_{g, e_k} \xi, \eta) \right]$$

$$= \nabla_{g, e_k} \left[ \nabla_{g, e_k} \nabla_{g, \xi} A (\eta) \right] - \nabla_g A (\nabla_{g, e_k} \nabla_{g, \xi} e_k, \eta)$$

$$= \nabla_{g, e_k} \nabla_{g, e_k} \nabla_{g, \xi} A (\eta) - \nabla_g A (\nabla_{g, e_k} \nabla_{g, \xi} e_k, \eta),$$

which combined with the previous expression implies the required formula \(\square\)

Applying this lemma first to $A = \dot{g}_t^*\dot{t}$ and then to $A = \nabla_{g, \dot{g}_t^*\dot{t}}$ along the $\Omega$-SRF
we infer the formula
\[
\left[ \nabla^2_{g_t}, \Delta^u_{g_t} \right] \dot{g}_t^* = 2 \nabla^2_{g_t} \dot{g}_t^* + \dot{g}_t^* \nabla^2_{g_t} \dot{g}_t^* + \nabla_{g_t} \dot{g}_t^* \cdot \nabla_{g_t} \dot{g}_t^* \\
+ 2 \mathcal{R}_{g_t} \dot{g}_t^* - \dot{g}_t^* + \nabla^*_{g_t} \mathcal{R}_{g_t} \dot{g}_t^*.
\]
(We observe that the presence of the curvature factor turns off the power of the maximum principle in the exponential convergence of higher order space derivatives along the \(\Omega\)-SRF.) A simple induction shows the general formula
\[
\left[ \nabla^p_{g_t}, \Delta^u_{g_t} \right] \dot{g}_t^* = p \nabla^p_{g_t} \dot{g}_t^* + \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C_{k,\sigma}^{p-r} \nabla^p_{g_t} \dot{g}_t^* \cdot k \nabla^p_{g_t} \dot{g}_t^*
\]
\[
+ \sum_{r=1}^{p} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} K_{k,\sigma}^{p-r} \nabla^p_{g_t} \dot{g}_t^* \cdot k \nabla^p_{g_t} \dot{g}_t^*
\]
\[
+ 2 \sum_{r=2}^{p} \sum_{k=2}^{r-1} \sum_{l=1}^{r-k} \sum_{\sigma \in S_{p-r+l-1}} Q_{k,l,\sigma}^{p-r} \nabla^p_{g_t} \mathcal{R}_{g_t} \circ_{k,l} \mathcal{R}_{g_t} \nabla^r_{g_t} \dot{g}_t^* ,
\]
where \(C_{k,\sigma}^{p-r}, K_{k,\sigma}^{p-r}, Q_{k,l,\sigma}^{p-r} \in \{0, 1\}\).

13 Exponentially fast convergence of higher order space derivatives along the \(\Omega\)-Soliton-Ricci flow

We use here an interpolation method introduced by Hamilton in his proof of the exponential convergence of the Ricci flow in \[\text{Ham}\]. The difference with the technique in \[\text{Ham}\] is a more involved interpolation process due to the presence of some extra curvature terms which seem to be alien to Hamilton’s argument. We are able to perform our interpolation process by using some intrinsic properties of the \(\Omega\)-SRF.

13.1 Estimate of the heat of the derivatives norm

The fact that \((g_t)_{t \geq 0} \subset \Sigma_K(g_0)\) implies \(\dot{g}_t \in \mathbb{F}_K^K\) for all times \(t \geq 0\) thanks to the identity (5.2). Thus hold the identities \([K, \nabla_{g_t} \dot{g}_t^*] \equiv 0\) and \([K, \nabla^p_{g_t} \dot{g}_t^*] \equiv 0\), which in their turn imply
\[
\left[ \nabla_{g_t}, \xi \right] \nabla^p_{g_t} \dot{g}_t^* \nabla^p_{g_t} \dot{g}_t^* = 0 ,
\]
for all \(p \in \mathbb{Z}_{\geq 0}, \xi \in T_X\). We deduce by using the identity (4.1)
\[
2 \nabla^p_{g_t} \dot{g}_t^* = - \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C_{k,\sigma}^{p-r} \nabla^p_{g_t} \dot{g}_t^* \cdot k \nabla^p_{g_t} \dot{g}_t^* , \tag{13.1}
\]
for all $t \geq 0$ and $p \in \mathbb{N}_{>0}$. This combined with \((11.1)\) provides the identities

$$
2 \frac{d}{dt} (\nabla_{g_t}^p \dot{g}_t^*) = - \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C_{k,\sigma}^{p,r} \nabla_{g_t}^{p-r} \eta_{\sigma}^* \nabla_{g_t}^r \dot{g}_t^* + 2 \nabla_{g_t}^p \frac{d}{dt} g_t^*
$$

$$
= - \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C_{k,\sigma}^{p,r} \nabla_{g_t}^{p-r} \eta_{\sigma}^* \nabla_{g_t}^r \dot{g}_t^*
$$

$$
- \nabla_{g_t}^p \Delta_{g_t}^\alpha \dot{g}_t^* - 2 \nabla_{g_t}^p \dot{g}_t^* - 2 \nabla_{g_t}^p (\dot{g}_t^*)^2 .
$$

We consider now a $g_{t_0}$-orthonormal basis $(e_k)_k \subset TX_{x_{t_0}}$ and we define the multivectors $e_{K} := (e_{k_1}, \ldots, e_{k_p})$, $g_{t_0}^{-1} e_{K} := (g_{t_0}^{-1} e_{k_1}, \ldots, g_{t_0}^{-1} e_{k_p})$. Then at the space time point $(x_0, t_0)$ hold the identities

$$
\frac{d}{dt} \left| \nabla_{g_t}^p \dot{g}_t^* \right|_{g_t}^2
= \frac{d}{dt} \left( \nabla_{g_t}^p \dot{g}_t^* \nabla_{g_t}^p \dot{g}_t^* \right)_{g_t}
= \nabla_{g_t} \left( \frac{d}{dt} (\nabla_{g_t}^p \dot{g}_t^*) \nabla_{g_t}^p \dot{g}_t^* - \sum_{j=1}^{p} \dot{g}_t^* \nabla_{g_t} \left( \nabla_{g_t}^p \dot{g}_t^* \right)_{g_t} \right)
= \left( \nabla_{g_t}^p \dot{g}_t^* - \sum_{j=1}^{p} \dot{g}_t^* \nabla_{g_t} \left( \nabla_{g_t}^p \dot{g}_t^* \right)_{g_t} \right)_{g_t}
\leq - \sum_{r=1}^{p-1} \sum_{k=1}^{r} \sum_{\sigma \in S_{p-r+k-1}} C_{k,\sigma}^{p,r} \left( \nabla_{g_t}^{p-r} \eta_{\sigma}^* \nabla_{g_t}^r \dot{g}_t^* , \nabla_{g_t}^r \dot{g}_t^* \right)_{g_t}
- \left( \Delta_{g_t}^\alpha \nabla_{g_t} \dot{g}_t^* , \nabla_{g_t} \dot{g}_t^* \right)_{g_t}
+ \sum_{r=1}^{p-1} \left| \nabla_{g_t}^{p-r} \dot{g}_t^* \right|_{g_t} \left| \nabla_{g_t}^{p-r} \dot{g}_t^* \right|_{g_t}
+ C \left| \nabla_{g_t}^p \dot{g}_t^* \right|_{g_t}^2
+ \sum_{r=1}^{p-1} \left| \nabla_{g_t}^{p-r} \nabla_{g_t}^r \dot{g}_t^* \right|_{g_t} \left| \nabla_{g_t}^r \dot{g}_t^* \right|_{g_t}
+ \sum_{r=1}^{p} \left| \nabla_{g_t}^{p-r} \nabla_{g_t}^r \dot{g}_t^* \right|_{g_t} \left| \nabla_{g_t}^r \dot{g}_t^* \right|_{g_t}
+ \sum_{r=2}^{p} \left| \nabla_{g_t}^{p-r} \nabla_{g_t}^r \dot{g}_t^* \right|_{g_t} \left| \nabla_{g_t}^r \dot{g}_t^* \right|_{g_t},
$$

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where $C > 0$ will always denote a time independent constant. We observe now that $\mathcal{R}_{g_t} \equiv \mathcal{R}_{g_0}$ since $(g_t)_{t \geq 0} \subset \Sigma_{\mathcal{K}}(g_0)$. Using the standard identity

$$\Delta_{g_t}|\nabla_{g_t}\dot{g}_t|^2_{g_t} = 2\left\langle \Delta_{g_t} \nabla_{g_t}\dot{g}_t, \nabla_{g_t}\dot{g}_t \right\rangle_{g_t} - 2|\nabla_{g_t}^{p+1}\dot{g}_t|_{g_t}^2,$$

we infer the estimate of the heat of the norm squared

$$\square_{g_t}|\nabla_{g_t}\dot{g}_t|^2_{g_t} \leq -2|\nabla_{g_t}^{p+1}\dot{g}_t|_{g_t}^2 + C|\nabla_{g_t}\dot{g}_t|^2_{g_t} + C \sum_{r=1}^{p-1} |\nabla_{g_t}^{p-r}\dot{g}_t|^2_{g_t} |\nabla_{g_t}^p\dot{g}_t|_{g_t} + C \sum_{r=2}^{p-1} \left[ |\nabla_{g_t}^{p-r-1}\nabla_{g_t}^p\mathcal{R}_{g_t}|_{g_t} + |\nabla_{g_t}^{p-r}\mathcal{R}_{g_t}|_{g_t} \right] |\nabla_{g_t}^r\dot{g}_t|^2_{g_t} |\nabla_{g_t}^p\dot{g}_t|_{g_t} + C e^{-t}|\nabla_{g_t}^{p-2}\nabla_{g_t}^p\mathcal{R}_{g_t}|_{g_t} |\nabla_{g_t}^p\dot{g}_t|_{g_t},$$

thanks to the exponential estimate (11.5). Integrating with respect to the volume form $\Omega$ we obtain the inequality

$$2 \frac{d}{dt} \int_X |\nabla_{g_t}^p\dot{g}_t|^2_{g_t} \Omega \leq -2 \int_X |\nabla_{g_t}^{p+1}\dot{g}_t|^2_{g_t} \Omega + C \int_X |\nabla_{g_t}\dot{g}_t|^2_{g_t} \Omega + C \sum_{r=1}^{p-1} \int_X |\nabla_{g_t}^{p-r}\dot{g}_t|^2_{g_t} |\nabla_{g_t}^p\dot{g}_t|_{g_t} \Omega + C \sum_{r=2}^{p-1} \left[ \int_X |\nabla_{g_t}^{p-r-1}\nabla_{g_t}^p\mathcal{R}_{g_t}|_{g_t} + |\nabla_{g_t}^{p-r}\mathcal{R}_{g_t}|_{g_t} \right] |\nabla_{g_t}^r\dot{g}_t|^2_{g_t} |\nabla_{g_t}^p\dot{g}_t|_{g_t} \Omega + C e^{-t} \left[ \int_X |\nabla_{g_t}^{p-2}\nabla_{g_t}^p\mathcal{R}_{g_t}|_{g_t}^2 \Omega \right]^{1/2} \left[ \int_X |\nabla_{g_t}\dot{g}_t|^2_{g_t} \Omega \right]^{1/2},$$

thanks to Hölder’s inequality. In order to estimate the sums we need a Hamilton’s result in [Ham] which restates in our setting as follows.

### 13.2 Hamilton’s interpolation inequalities

**Lemma 18.** Let $p, q, r \in \mathbb{R}$ with $r \geq 1$ such that $1/p + 1/q = 1/r$. There exists a time independent constant $C > 0$ such that along the $\Omega$-SRF hold the
inequality
\[
\left( \int_X |\nabla g_t A|_{g_t}^{2r} \Omega \right)^{1/r} \leq C \left( \int_X |\nabla^2 g_t A|_{g_t}^p \Omega \right)^{1/p} \left( \int_X |A|_{g_t}^q \Omega \right)^{1/q},
\]
for all tensors \( A \).

**Proof.** The argument here is the same as in Hamilton [Ham]. We include it for readers convenience. We set \( u := |\nabla g_t A|_{g_t}^{2(r-1)} \) and we observe the identities
\[
\int_X |\nabla g_t A|_{g_t}^{2r} dV_{g_t} = \int_X \langle A, \nabla g_t (u \nabla g_t A) \rangle_{g_t} dV_{g_t}
\]
\[
= \int_X \langle A, \Delta g_t A \rangle_{g_t} |\nabla g_t A|_{g_t}^{2(r-1)} dV_{g_t}
\]
\[- \int_X \langle A, \nabla g_t u - \nabla g_t A \rangle_{g_t} dV_{g_t},
\]
and the inequalities
\[
| \langle A, \Delta g_t A \rangle_{g_t} | \leq \sqrt{n} |A|_{g_t} |\nabla^2 g_t A|_{g_t},
\]
\[
| \langle A, \nabla g_t u - \nabla g_t A \rangle_{g_t} | \leq \sqrt{n} |A|_{g_t} |\nabla g_t A|_{g_t} |\nabla g_t u|_{g_t}.
\]
Expanding the vector \( \nabla g_t u \) we infer the identities
\[
\nabla g_t u = (r - 1) |\nabla g_t A|_{g_t}^{2(r-2)} \nabla g_t |\nabla g_t A|_{g_t}^2
\]
\[
= 2 (r - 1) |\nabla g_t A|_{g_t}^{2(r-2)} \left\langle \nabla g_t e_k, \nabla g_t A, \nabla g_t A \right\rangle_{g_t} e_k ,
\]
where \((e_k)\) is a \(g_t\)-orthonormal basis. Thus hold the inequality
\[
|\nabla g_t u|_{g_t} \leq 2 (r - 1) \sqrt{n} |\nabla g_t A|_{g_t}^{2r-3} |\nabla^2 g_t A|_{g_t}.
\]
Combining the previous inequalities we infer
\[
\int_X |\nabla g_t A|_{g_t}^{2r} dV_{g_t} \leq C_{r,n} \int_X |A|_{g_t} |\nabla^2 g_t A|_{g_t} |\nabla g_t A|_{g_t}^{2r-2} dV_{g_t},
\]
with \(C_{r,n} := 2 (r - 1) n + \sqrt{n}\). This combined with (11.3) implies the inequality
\[
\int_X |\nabla g_t A|_{g_t}^{2r} \Omega \leq C \int_X |A|_{g_t} |\nabla^2 g_t A|_{g_t} |\nabla g_t A|_{g_t}^{2r-2} \Omega.
\]
We can estimate the last integral using Hölder’s inequality with
\[
\frac{1}{p} + \frac{1}{q} + \frac{r-1}{r} = 1,
\]
in order to obtain
\[
\int_X |\nabla g_t A|^{2r} \Omega \leq C \left[ \int_X |\nabla^2 g_t A|_{g_t}^p \Omega \right]^{1/p} \left[ \int_X |A|_{g_t}^q \Omega \right]^{1/q} \left[ \int_X |\nabla g_t A|^{2r} \Omega \right]^{1-1/r},
\]
and hence the required conclusion.

As a corollary of this inequality Hamilton obtains in \cite{Ham} the following two estimates. For all \(r = 1, \ldots, p-1\), hold
\[
\int_X |\nabla g_t A|^{2p/r} \Omega \leq C \left[ \max_X |A|_{g_t} \right]^{2(p/r-1)} \int_X |\nabla^p g_t A|_{g_t}^2 \Omega, \quad (13.2)
\]
\[
\int_X |\nabla g_t A|^{p} \Omega \leq C \left[ \int_X |\nabla^p g_t A|_{g_t}^2 \Omega \right]^{r/p} \left[ \int_X |A|_{g_t}^2 \Omega \right]^{1-r/p}. \quad (13.3)
\]

13.3 Interpolation of the \(H^p\)-norms

We estimate now the integral
\[
I_1 := \sum_{r=1}^{p-1} \int_X |\nabla^{p-r} \dot{g}_t |_{g_t}|\nabla^r \dot{g}_t |_{g_t}|\nabla^p \dot{g}_t |_{g_t} \Omega.
\]

Indeed using Hölder’s inequality we obtain
\[
\int_X |\nabla^{p-r} \dot{g}_t |_{g_t}|\nabla^r \dot{g}_t |_{g_t}|\nabla^p \dot{g}_t |_{g_t} \Omega \\
\leq \left[ \int_X |\nabla^{p-r} \dot{g}_t |_{g_t}^{2p/(p-r)} \Omega \right]^{(p-r)/2p} \times \\
\times \left[ \int_X |\nabla^r \dot{g}_t |_{g_t}^{2p/r} \Omega \right]^{r/2p} \left[ \int_X |\nabla^p \dot{g}_t |_{g_t}^2 \Omega \right]^{1/2}.
\]

This combined with Hamilton’s inequality (13.2) implies the estimate
\[
I_1 \leq C \int_X |\nabla^p \dot{g}_t |_{g_t}^2 \Omega.
\]

In a similar way we can estimate the integral
\[
I_2 := \sum_{r=2}^{p-1} \int_X |\nabla^{p-r-1} \nabla^* R_{g_t} |_{g_t}|\nabla^r \dot{g}_t |_{g_t}|\nabla^p \dot{g}_t |_{g_t} \Omega.
\]
Indeed as before we obtain the inequality
\[
\int_X |\nabla^{p-r-1} \nabla^* R_{g_t} |_{g_t} |\nabla^{r} \dot{g}_t^* |_{g_t} |\nabla^*_g \dot{g}_t^* |_{g_t} \Omega \\
\leq \left[ \int_X |\nabla^{p-r-1} \nabla^* R_{g_t} |_{g_t} |^{2p/(p-r)} \Omega \right]^{(p-r)/2p} \times \\
\left[ \int_X |\nabla^r \dot{g}_t^* |_{g_t} |^{2p/r} \Omega \right]^{r/2p} \times \left[ \int_X |\nabla^*_g \dot{g}_t^* |_{g_t} |^{2} \Omega \right]^{1/2}.
\]
Moreover the trivial inequality
\[
\frac{p}{p-r} \leq \frac{p-3}{p-3-r},
\]
implies the estimates
\[
\int_X |\nabla^{p-r-1} \nabla^* R_{g_t} |_{g_t} |^{2p/(p-r)} \Omega \\
\leq \int_X |\nabla^{p-r-1} \nabla^* R_{g_t} |_{g_t} |^{2(p-3)/(p-3-r)} \Omega \\
\leq \int_X \Omega + \int_X |\nabla^{p-3} \nabla^* R_{g_t} |_{g_t} |^{2} \Omega ,
\]
by (13.2) since \(|\nabla^* R_{g_t} |_{g_t} \leq C thanks to the uniform C1-bound on g_t. Using again (13.2) we infer the estimate
\[
I_2 \leq C \sum_{r=2}^{p-1} \left[ 1 + \int_X |\nabla^{p-3} \nabla^* R_{g_t} |_{g_t} |^{2} \Omega \right]^{(p-r)/2p} \times \left[ \int_X |\nabla^*_g \dot{g}_t^* |_{g_t} |^{2} \Omega \right]^{1/2+r/2p}.
\]
We estimate finally the integral
\[
I_3 := \sum_{r=2}^{p-1} \int_X |\nabla^{p-r} R_{g_t} |_{g_t} |\nabla^{r} \dot{g}_t^* |_{g_t} |\nabla^*_g \dot{g}_t^* |_{g_t} \Omega .
\]
As before using the trivial inequality
\[
\frac{p}{p-r} \leq \frac{p-2}{p-2-r},
\]
we obtain the estimates
\[
\int_X |\nabla^{p-r} R_{g_t} |_{g_t} |^{2p/(p-r)} \Omega \leq \int_X \Omega + \int_X |\nabla^{p-r} R_{g_t} |_{g_t} |^{2(p-2)/(p-2-r)} \Omega \\
\leq \int_X \Omega + \int_X |\nabla^{p-2} R_{g_t} |_{g_t} ^{2} \Omega ,
\]
by (13.2). Thus

\[
I_3 \leq C \sum_{r=2}^{p-1} \left[ 1 + \int_X |\nabla g_t^{r-2} R_{g_t}|^2 \right]^{(p-r)/2p} \left[ \int_X |\nabla g_t^{r} g_t^r|^2 \right]^{1/2 + r/2p}.
\]

In conclusion for all integers \( p > 1 \) hold the estimate

\[
\frac{2}{d} \frac{d}{dt} \int_X |\nabla g_t^{r} g_t^r|^2 \leq -2 \int_X |\nabla g_t^{r+1} g_t^r|^2 + C \int_X |\nabla g_t^{r} g_t^r|^2 + C \sum_{r=2}^{p-1} \left[ 1 + \int_X |\nabla g_t^{r-3} \nabla g_t^r R_{g_t}|^2 \right]^{(p-r)/2p} \left[ \int_X |\nabla g_t^{r} g_t^r|^2 \right]^{1/2 + r/2p} + C e^{-\frac{1}{t}} \int_X |\nabla g_t^{r-2} \nabla g_t^r R_{g_t}|^2 \left[ \int_X |\nabla g_t^{r} g_t^r|^2 \right]^{1/2}.
\]

We set \( \Gamma_t := \nabla g_t - \nabla g_0 \), and we observe the inequality

\[
|\nabla g_t^r R_{g_t}| \leq C + C \sum_{h=1}^{p-1} \sum_{q=1}^{h} \sum_{r_1 + \ldots + r_{h-q+1} = q} \prod_{j=1}^{h-q+1} |\nabla g_t^{r_j} g_t^{r_j}|,
\]

with \( r_j = 0, \ldots, q \). Thus by using Jensen’s inequality we obtain

\[
|\nabla g_t^r R_{g_t}|^2 \leq C + C \sum_{h=1}^{p-1} \sum_{q=1}^{h} \sum_{r_1 + \ldots + r_{h-q+1} = q} \prod_{j=1}^{h-q+1} |\nabla g_t^{r_j} g_t^{r_j}|^2,
\]

with \( r_j = 0, \ldots, q \). Moreover using Hölder’s inequality, the \( C^1 \)-uniform estimate on \( g_t \) and the inequality (13.2) we infer the estimates

\[
\int_X \prod_{j=1}^{h-q+1} |\nabla g_t^{r_j} g_t^{r_j}|^2 \Omega \leq \prod_{j=1}^{h-q+1} \left[ \int_X |\nabla g_t^{r_j} g_t^{r_j}|^2 \Omega \right]^{r_j/q} \leq C \int_X |\nabla g_t^r R_{g_t}|^2 \Omega.
\]
In its turn for all \( q > 1 \) hold the inequalities
\[
\int_X |\nabla^{q-1} \Gamma_t|_{g_0}^2 \Omega \\
\leq \sum_{r,h=1}^q \int_X |\nabla^{q-r} g_t|_{g_0} |\nabla^r g_t|_{g_0} |\nabla^{q-h} g_t|_{g_0} |\nabla^h g_t|_{g_0} \Omega \\
\leq \sum_{r,h=1}^q \left[ \int_X |\nabla^{q-r} g_t|_{g_0}^{2q/(q-r)} \Omega \right]^{(q-r)/2p} \left[ \int_X |\nabla^r g_t|_{g_0}^{2q/r} \Omega \right]^{r/2q} \times \\
\left[ \int_X |\nabla^{q-h} g_t|_{g_0}^{2q/(q-h)} \Omega \right]^{(q-h)/2q} \left[ \int_X |\nabla^h g_t|_{g_0}^{2q/h} \Omega \right]^{h/2q} \\
\leq C \int_X |\nabla^q g_t|_{g_0}^2 \Omega.
\]

We deduce the estimate
\[
\int_X |\nabla^p R_t|_{g_0}^2 \Omega \leq C + C \sum_{r=1}^p \int_X |\nabla^r g_t|_{g_0}^2 \Omega.
\]

In a similar way we obtain the estimate
\[
\int_X |\nabla^{p-1} \nabla^p R_t|_{g_0}^2 \Omega \leq C + C \sum_{r=1}^p \int_X |\nabla^r g_t|_{g_0}^2 \Omega.
\]

### 13.4 Exponential decay of the \( H^p \)-norms

We assume by induction the uniform exponential estimates
\[
\int_X |\nabla^r g_t|_{g_0}^2 \Omega \leq C_r e^{-\theta_r t},
\]
for all \( r = 0, \ldots, p-1 \), with \( C_r, \theta_r > 0 \). We deduce from the previous subsection that for all \( q = p, p+1 \) hold the estimate
\[
2 \frac{d}{dt} \int_X |\nabla^q g_t|_{g_0}^2 \Omega \leq -2 \int_X |\nabla^{q+1} g_t|_{g_0}^2 \Omega + C \int_X |\nabla^q g_t|^2 \Omega \\
+ C \sum_{r=2}^{q-1} \left[ \int_X |\nabla^r g_t|^2 \Omega \right]^{1/2+2/r+2q} \\
+ C e^{-t} \left[ 1 + \int_X |\nabla^{q-1} g_t|_{g_0}^2 \Omega \right]^{1/2} \left[ \int_X |\nabla^q g_t|_{g_0}^2 \Omega \right]^{1/2}.
\]
Thus for $q = p$ hold
\[
2 \frac{d}{dt} \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \leq -2 \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 + C \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \\
+ C \sum_{r=0}^{p-1} \left( \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \right)^{1/2 + r/2p}.
\]

Moreover Hamilton’s inequality (13.3) implies
\[
\int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \leq C \left( \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \right)^{\alpha} \leq \left( \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \right)^{1/2 + r/2p}. \tag{13.4}
\]

Now for any $\varepsilon > 0$ and all $x, y > 0$ hold the inequality
\[
x^p y \leq C \varepsilon x^p + C \varepsilon^{-p} y^{p+1},
\]
and applying this above gives
\[
\int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \leq C \varepsilon \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 + C \varepsilon^{-p} \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2,
\]
and also
\[
\left( \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \right)^{\alpha} \leq C \varepsilon \left( \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \right)^{\alpha} + C \varepsilon^{-p} \left( \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \right)^{\alpha}, \tag{13.5}
\]
with $\alpha := 1/2 + r/2p$. If we choose $\varepsilon$ sufficiently small we deduce
\[
2 \frac{d}{dt} \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \leq -\int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 + C e^{-\delta t} \\
+ C \sum_{r=0}^{p-1} \left( \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \right)^{1/2 + r/2p}.
\]

thanks to (11.2). In order to estimate the last integral term we distinguish two cases. If for some $t > 0$
\[
\int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \leq 1,
\]
then
\[
C \sum_{r=0}^{p-1} \left( \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \right)^{1/2 + r/2p} \leq C' e^{-\delta t/2(p+1)}.
\]
thanks to (13.3) and (11.2). Thus at this time hold the estimate
\[
2 \frac{d}{dt} \int_X |\nabla g_t \cdot \dot{g}_t|_{\Omega}^2 \leq C e^{-\delta t/2(p+1)}. \tag{13.6}
\]

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In the case \( t > 0 \) satisfies

\[
\int_X |\nabla_g^{p+1} \dot{h}_t|_{\Omega}^2 > 1 ,
\]

then

\[
C \sum_{r=0}^{p-1} \left( \int_X |\nabla_g^p \dot{h}_t|_{\Omega}^2 \right)^{1/2+r/2p} \leq C' \varepsilon \int_X |\nabla_g^{p+1} \dot{h}_t|_{\Omega}^2 \Omega + C' \varepsilon^{-p} e^{-\delta t/2} ,
\]

thanks to (13.5) and (11.2). Thus choosing \( \varepsilon > 0 \) sufficiently small we infer at this time the estimate

\[
2 \frac{d}{dt} \int_X |\nabla_g^{p} \dot{h}_t|_{\Omega}^2 \Omega \leq C e^{-\delta t/2} .
\]

We deduce that the estimate (13.6) hold for all times \( t > 0 \). This implies the uniform estimate

\[
\int_X |\nabla_g^p \dot{h}_t|_{\Omega}^2 \Omega = \int_X |\nabla_g^p \dot{h}_t|_{\Omega}^2 \Omega \leq C_p .
\]  

(13.7)

We observe now the inequality

\[
|\nabla_{g_0}^p \dot{h}_t|_{g_0} \leq |\nabla_g^p \dot{h}_t|_{g_0}
\]

\[
+ C \sum_{r=0}^{p-1} \sum_{h=0}^{h} \sum_{q=0}^{h-q+1} \prod_{j=1}^{h-q+1} |\nabla_{g_0}^{r_j} \Gamma_{g_0} |_{g_0} |\nabla_g^{p-1-h} \dot{h}_t|_{g_0} ,
\]

with \( r_j = 0, \ldots, q \). Thus using Jensen’s inequality we obtain

\[
|\nabla_{g_0}^p \dot{h}_t|_{g_0} \leq |\nabla_g^p \dot{h}_t|_{g_0}^2
\]

\[
+ C \sum_{r=0}^{p-1} \sum_{h=0}^{h} \sum_{q=0}^{h-q+1} \prod_{j=1}^{h-q+1} |\nabla_{g_0}^{r_j} \Gamma_{g_0} |_{g_0}^2 |\nabla_g^{p-1-h} \dot{h}_t|_{g_0}^2 .
\]

Let \( k := p-1-h \) and \( m := k+q \leq p-1 \). Then Hölder’s inequality combined with the \( L^2 \) estimate of \( |\nabla_{g_0}^{p-1} \Gamma_{g_0} |_{g_0} \) given in the previous subsection and combined with Hamilton’s interpolation inequality (13.2) provides the estimate

\[
\int_X \prod_{j=1}^{h-q+1} |\nabla_{g_0}^{r_j} \Gamma_{g_0} |_{g_0}^2 |\nabla_g^{p-1-h} \dot{h}_t|_{g_0}^2 \Omega
\]

\[
\leq \prod_{j=1}^{h-q+1} \left[ \int_X |\nabla_{g_0}^{r_j} \Gamma_{g_0} |_{g_0}^{2m/r_j} |\nabla_g^{p-1-h} \dot{h}_t|_{g_0}^{2m/k} \Omega \right]^{r_j/m} \left[ \int_X |\nabla_{g_0}^{k} \dot{h}_t|_{g_0}^{2m/k} \Omega \right]^{k/m}
\]

\[
\leq C \left[ \int_X |\nabla_{g_0}^{m} \dot{h}_t|_{g_0}^2 \Omega \right]^{q/m} \left[ \int_X |\nabla_{g_0}^{m} \dot{h}_t|_{g_0}^2 \Omega \right]^{k/m}
\]

\[
\leq C \left[ \int_X |\nabla_{g_0}^{m+1} \dot{h}_t|_{g_0}^2 \Omega \right]^{q/m} e^{-\theta_{k,m} t} ,
\]

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\( \theta_{k,m} > 0 \), thanks to the inductive assumption. (When \( r_j = 0 \) or \( k = 0 \) in the previous estimate it means just that we are taking the corresponding \( L^\infty \)-norms, which are bounded by the uniform \( C^1 \)-estimate.) Thus for some \( \tau_p, \rho_p > 0 \), hold the estimate
\[
\int_X |\nabla^p g_t^0|^2_{g_0} \Omega \leq C e^{-\tau_p t} \int_X |\nabla^p g_t|_{g_0}^2 \Omega + C \left( 1 + e^{-\rho_p t} \right),
\]
thanks to the uniform estimate \[13.7\]. We set
\[
N_{p,t} := \int_X |\nabla^p g_t|^2_{g_0} \Omega.
\]
We infer the first order differential inequality
\[
\dot{N}_{p,t} \leq C N_{p,t} e^{-\tau_p t} + C \left( 1 + e^{-\rho_p t} \right),
\]
We deduce by Gronwall’s inequality
\[
N_{p,t} \leq e^{C \int_0^t e^{-\tau_p s} ds} \left[ N_{p,0} + C \int_0^t (1 + e^{-\rho_p s}) ds \right] \leq C' (1 + t).
\]
The fact that the function \( t^{1/2} e^{-t} \) is bounded for \( t \geq 0 \) implies that for \( q = p + 1 \) hold the estimate
\[
2 \frac{d}{dt} \int_X |\nabla^{q+1} g_t|^2_{g_t} \Omega \leq -2 \int_X |\nabla^q g_t^1 g_t^*|^2_{g_t} \Omega + C \int_X |\nabla^q g_t^1 g_t^*|^2_{g_t} \Omega + C \sum_{r=0}^{q-1} \left[ \int_X |\nabla^r g_t^1 g_t^*|^2_{g_t} \Omega \right]^{1/2 + r/2q},
\]
We deduce from the same argument showing \[13.7\], the uniform estimate
\[
\int_X |\nabla^{p+1} g_t^1 g_t^*|^2_{g_t} \Omega \leq C_{p+1},
\]
and thus
\[
\int_X |\nabla^p g_t^1 g_t^*|^2_{g_t} \Omega = \int_X |\nabla^p g_t^1 g_t^*|^2_{g_t} \Omega \leq C_p e^{-\delta t/(p+1)},
\]
thanks to \[13.4\] and \[11.2\]. Applying the previous argument to this improved estimate we infer
\[
\dot{N}_{p,t} \leq C N_{p,t} e^{-\tau_p t} + C e^{-\rho_p t},
\]
and thus \( N_{p,t} \leq C \) thanks to Gronwall’s inequality. We deduce the conclusion of the induction.
\[
\dot{N}_{p,t} = \int_X |\nabla^p g_t|_{g_0}^2 \Omega \leq C_p e^{-\theta_p t}.
\]
Thus using the Sobolev estimate we infer for all times \( t > 0 \) the inequality

\[
|\nabla^p g_t|_{g_0} \leq C_p e^{-\varepsilon_p t},
\]

which implies the convergence of the integral

\[
\int_0^{+\infty} |\nabla^p g_t|_{g_0} dt < +\infty,
\]

and thus the existence of the tensor

\[
A_p := \int_0^{+\infty} \nabla^p g_t dt,
\]

thanks to Bochner’s theorem. Moreover hold the exponential estimate

\[
|A_p - \nabla^p g_t|_{g_0} \leq \int_t^{+\infty} |\nabla^p g_s|_{g_0} ds \leq C_p e^{-\varepsilon_p t}.
\]

A basic calculus fact combined with an induction on \( p \) implies that

\[
A_p = \nabla^p g_\infty.
\]

This concludes the proof of the exponential convergence of the \( \Omega \)-SRF.

14 Appendix

14.1 Weitzenböck type formulas

Lemma 19. For any \( g \in \mathcal{M} \) and \( u \in C^\infty(X, S^2T_X^*) \) hold the formula

\[
- \left( \nabla^\ast_{g} D_g u \right)_{g} = \nabla^\ast_{g} \nabla_{T_X} u_{g} + \left( \nabla^\ast_{g} \nabla_{T_X} u_{g} \right)^T_{g} - \Delta^\ast_{g} u_{g}.
\]

Proof. We need to show first that for any \( h \in C^\infty \left( X, (T_X^*)^\otimes 3 \right) \) hold the identity

\[
\left( \nabla^\ast_{g} h \right)_{g} = \nabla^\ast_{g} \left( \bullet - h \right)_{g} \ast,
\]

where the section \( \left( \bullet - h \right)_{g} \ast \in C^\infty \left( X, (T_X^*)^\otimes 2 \otimes T_X \right) \) is given by the formula

\[
\left( \bullet - h \right)_{g} \ast \left( \xi, \eta \right) := \left( \xi - h \right)_{g} \ast \eta.
\]

In fact let \( (e_\ell) \) be a smooth local tangent frame and let \( \xi, \eta \) be arbitrary smooth vector fields defined in a neighborhood of an arbitrary point \( x_0 \) such that \( \nabla_g e_\ell(x_0) = \nabla_g \xi(x_0) = \nabla_g \eta(x_0) = 0 \). Then at the point \( x_0 \) hold the equalities

\[
\nabla^\ast_{g} h(\xi, \eta) = - \nabla_g h(e_\ell, e_\ell, \xi, \eta)
\]

\[
= - e_\ell \cdot h(e_\ell, \xi, \eta)
\]

\[
= - e_\ell \cdot g \left( (e_\ell - h)_{g} \ast \xi, \eta \right)
\]

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\[ = - g \left( \nabla g \cdot \eta \left[ (e_i - h)^* \xi \right], \eta \right) \]
\[ = - g \left( \nabla g \cdot (\bullet - h)^* (e_i, \xi), \eta \right) \]
\[ = g \left( \nabla^* (\bullet - h)^* \xi, \eta \right). \]

Applying the identity (14.1) to \( h = D_g u \) and using the expression
\[ \nabla^* (\bullet - \nabla g \cdot \eta \left[ (e_i - h)^* \xi \right], \eta) = - g \left( \nabla^* (\bullet - h)^* \xi, \eta \right), \]
we deduce the formula
\[ \left( \nabla^*_g D_g u \right)^* = \nabla^*_g (\bullet - D_g u)^* + \left( \nabla g \cdot f - D_g u \right)^*. \]

In order to explicit this expression, for any \( \mu \in C^\infty(X, T_X) \), we expand the term
\[ D_g u (\mu, \xi, \eta) = \nabla_g u (\xi, \mu, \eta) + \nabla_g u (\eta, \mu, \xi) - \nabla_g u (\mu, \xi, \eta) \]
\[ = g \left( \nabla g u^*_g (\xi, \mu), \eta \right) + g \left( \nabla g u^*_g (\eta, \mu), \xi \right) \]
\[ - g \left( \nabla g u^*_g (\mu, \xi), \eta \right) \]
\[ = g \left( \nabla g u^*_g (\xi, \mu), \eta \right) + g \left( \nabla g u^*_g (\eta, \mu), \xi \right) \]
\[ + g \left( \nabla g u^*_g (\mu, \eta), \xi \right) \]
\[ = - g \left( \nabla g u^*_g (\mu, \xi), \eta \right) - g \left( \nabla g u^*_g (\mu, \eta), \xi \right) \]
\[ + g \left( \nabla g u^*_g (\mu, \eta), \xi \right) \]
\[ = - g \left( \nabla g u^*_g (\mu, \xi), \eta \right) - g \left( \mu - \nabla g u^*_g \right)^T \xi, \eta \]
\[ + g \left( \nabla g u^*_g (\mu, \eta), \xi \right) \]
\[ = - g \left( \nabla g u^*_g (\mu, \xi), \eta \right) - g \left( \nabla g u^*_g \right)^T \xi, \eta \]
\[ + g \left( \nabla g u^*_g (\mu, \xi), \eta \right). \]

We deduce the identity
\[ (\bullet - D_g u)^*_g = - \nabla g u^*_g - \left( \nabla g u^*_g \right)^T + \nabla g u^*_g. \]

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and thus the expression
\[
- \left( \nabla_g^* \mathcal{D}_g u \right)_g^* = \nabla_g^* \nabla_{TX,g} u_g^* + \nabla_g^* \left( \nabla_{TX,g} u_g^* \right)_g^T + \nabla_g \left( \nabla_{TX,g} u_g^* \right)_g^T
\]

We observe now that at the point \( x_0 \) hold the following equalities
\[
\nabla_g^* \left( \bullet \nabla_{TX,g} u_g^* \right)_g^T \xi = - \nabla_g, e_i \left( \nabla_{TX,g} u_g^* \right)_g^T (e_i, \xi)
\]

since at this point hold the identities
\[
\nabla_g, e_i \left( e_i \nabla_{TX,g} u_g^* \right)_g^T \eta = \nabla_g, e_i \left[ \nabla_{TX,g} u_g^* (e_i, \eta) \right] = \nabla_g, e_i \nabla_{TX,g} u_g^* (e_i, \eta).
\]

We infer the required formula. \( \square \)

We define the Hodge Laplacian (resp. the \( \Omega \)-Hodge Laplacian) operators acting on \( T_X \)-valued \( q \)-forms by the formulas
\[
\Delta_{TX,g} := \nabla_{TX,g}^* \nabla_g + \nabla_g \nabla_{TX,g}^*,
\]
\[
\Delta_{TX,g}^\Omega := \nabla_{TX,g}^* \nabla_g^\Omega + \nabla_g \nabla_{TX,g}^\Omega.
\]

**Lemma 20.** Let \((X, g)\) be a Riemannian manifold and let \( H \in C^\infty(X, \text{End}(T_X))\). Then hold the identity
\[
\Delta_{TX,g} H = \Delta_g H - \mathcal{R}_g \star H + H \text{Ric}_g^*,
\]
where \((\mathcal{R}_g \star H) \xi := \text{Tr}_g [(\xi \nabla_{\mathcal{R}_g}) H] \) for all \( \xi \in T_X \).

**Proof.** Let \((e_k)_k\) and \( \xi \) as in the proof of lemma 19. Then at the point \( x_0 \) hold
the equalities
\[ \Delta_{TX,g}^\alpha H \xi = \nabla g,\xi \nabla^* g H - \nabla g,e_k \left( \nabla_{TX,g}^\alpha H \right) (e_k, \xi) \]
\[ = - \nabla g,\xi \nabla g,e_k H e_k - \nabla g,e_k (\nabla g,\xi H e_k) \]
\[ = \Delta_g H \xi + \nabla g,e_k \nabla g,\xi H e_k - \nabla g,\xi \nabla g,e_k H e_k \]
\[ = \Delta_g H \xi + \mathcal{R}_g (e_k, \xi) H e_k - H \mathcal{R}_g (e_k, \xi) e_k , \]
which shows the required formula. \( \square \)

**Lemma 21.** Let \((X,g)\) be a orientable Riemannian manifold, let \(\Omega > 0\) be a smooth volume form and let \(H \in C^\infty(X, \text{End}(T_X))\). Then
\[ \Delta_{TX,g}^\alpha H = \Delta_g^\alpha H - \mathcal{R}_g * H + H \text{Ric}_g^* (\Omega) . \]

**Proof.** We expand the Laplacian
\[ \Delta_{TX,g}^\alpha H = \nabla_{TX,g} \nabla_{g}^* H + \nabla_{TX,g} (H \nabla_g f) \]
\[ + \nabla_g^* \nabla_{TX,g} H + \nabla g f - \nabla_{TX,g} H \]
\[ = \Delta_{TX,g} H + \nabla_g H \nabla_g f + H \nabla^2 f \]
\[ + \nabla g f - \nabla g H - \nabla g H \nabla g f \]
\[ = \Delta_g^\alpha H - \mathcal{R}_g * H + H \text{Ric}_g^* (\Omega) , \]
thanks to lemma \[ \square \]

### 14.2 The first variation of the pre-scattering operator

**Lemma 22.** For any smooth family of metrics \((g_t)_{t \in \mathbb{R}} \subset \mathcal{M}\) hold the total variation formula
\[ 2 \frac{d}{dt} \left[ \nabla_{TX,g_t} \text{Ric}_{g_t}^* (\Omega) \right] = \nabla_{TX,g_t} \left( \nabla_{g_t}^* \nabla_{TX,g_t} \hat{g}_t^* \right)^T_{g_t} + d \text{div}_{g_t} \left[ \mathcal{R}_{g_t}, \hat{g}_t^* \right] \]
\[ + \text{Alt} \left[ \mathcal{R}_{g_t} \odot \left( \nabla g_t - \nabla_{TX,g_t} \right) \hat{g}_t^* \right] \]
\[ + \text{Alt} \left[ \left( \nabla_{TX,g_t} \hat{g}_t^* \right)^T_{g_t} \text{Ric}_{g_t}^* (\Omega) \right] \]
\[ - \text{Ric}_{g_t}^* (\Omega) \nabla_{TX,g_t} \hat{g}_t^* - 2 \hat{g}_t^* \nabla_{TX,g_t} \text{Ric}_{g_t}^* (\Omega) . \]
Proof. Lemma 19 combined with the variation formulas (2.2) and (3.3) implies the equality
\[ 2 \frac{d}{dt} \left[ \nabla_{T_X, g_t} \text{Ric}^*_t(\Omega) \right] = \nabla_{T_X, g_t} \left[ \nabla_{g_t} \nabla_{T_X, g_t} \dot{g}_t^* + \left( \nabla_{g_t} \nabla_{T_X, g_t} \dot{g}_t^* \right)^T_{g_t} \right] \]
\[ - \nabla_{T_X, g_t} \left[ \Delta g_t \dot{g}_t^* + \dot{g}_t^* \text{Ric}^*_t(\Omega) \right] \]
\[ - \dot{g}_t^* \nabla_{T_X, g_t} \text{Ric}^*_t(\Omega) - \text{Ric}^*_t(\Omega) - \nabla_{T_X, g_t} \dot{g}_t^* \]
\[ + \ \text{Alt} \left[ \left( \nabla_{T_X, g_t} \dot{g}_t^* \right)^T_{g_t} \text{Ric}^*_t(\Omega) \right]. \]

Moreover using lemma 21 we deduce the identities
\[ \nabla_{T_X, g_t} \nabla_{g_t} \nabla_{T_X, g_t} \dot{g}_t^* = \nabla_{T_X, g_t} \Delta g_t \dot{g}_t^* - \nabla_{T_X, g_t} \nabla_{g_t} \dot{g}_t^* \]
\[ = \nabla_{T_X, g_t} \left[ \Delta g_t \dot{g}_t^* - \text{Ric}^*_t(\Omega) \right] \]
\[ - \text{Ric}^*_t \nabla_{g_t} \dot{g}_t^* . \]

We pick now an arbitrary space time point \((x_0, t_0)\) and let \((e_k)_k\) and \(\xi, \eta\) as in the proof of lemma 19 with respect to \(g_t\). We expand at the space time point \((x_0, t_0)\) the term
\[ \left[ \nabla_{T_X, g_t} (\text{Ric}^*_t(\Omega)) \right] (\xi, \eta) = \nabla_{g_t, \xi} (\text{Ric}^*_t(\Omega)) \eta - \nabla_{g_t, \eta} (\text{Ric}^*_t(\Omega)) \xi \]
\[ = \nabla_{g_t, \xi} \text{Ric}^*_t(\Omega)(\xi, e_k) \eta_{\xi} e_k + \text{Ric}^*_t(\Omega)(\xi, e_k) \nabla_{g_t, \xi} \dot{g}_t^* e_k \]
\[ - \nabla_{g_t, \eta} \text{Ric}^*_t(\Omega)(\xi, e_k) \eta_{\xi} e_k - \text{Ric}^*_t(\Omega)(\xi, e_k) \nabla_{g_t, \eta} \dot{g}_t^* e_k . \]

Thus using the differential Bianchi identity we infer the equalities
\[ \nabla_{T_X, g_t} (\text{Ric}^*_t(\Omega)) = - \nabla_{g_t, e_k} \text{Ric}^*_t(\Omega) e_k + \text{Alt} \left[ \text{Ric}^*_t(\Omega) \right] \]
\[ = - \nabla_{g_t, e_k} \text{Ric}^*_t(\Omega) e_k + \text{Alt} \left[ \text{Ric}^*_t(\Omega) \right] \]
\[ + \nabla_{g_t} \dot{g}_t^* \text{Ric}^*_t(\Omega) \].
by the identities (3.9) and (3.10). We infer the expression
\[
\nabla_{TX, g_t} \nabla_{g_t} \nabla_{TX, g_t} \dot{g}_t^* = \nabla_{TX, g_t} \left[ \Delta_{g_t}^\Omega \dot{g}_t^* + \dot{g}_t^* \text{Ric}_{g_t}^*(\Omega) \right] \\
+ \text{Alt} \left[ \mathcal{R}_{g_t} \otimes \left( \nabla_{g_t} - \nabla_{TX, g_t} \right) \dot{g}_t^* \right] \\
+ \nabla_{g_t, e_k} \mathcal{R}_{g_t} \dot{g}_t^* e_k - \nabla_{g_t} \dot{g}_t^* \ast \mathcal{R}_{g_t} - \mathcal{R}_{g_t} \nabla_{g_t}^* \dot{g}_t^*.
\]

This combined with the identity (3.15) and with the Bianchi type identity (3.11) implies the expression
\[
\nabla_{TX, g_t} \nabla_{g_t} \nabla_{TX, g_t} \dot{g}_t^* = \nabla_{TX, g_t} \left[ \Delta_{g_t}^\Omega \dot{g}_t^* + \dot{g}_t^* \text{Ric}_{g_t}^*(\Omega) \right] \\
+ \text{Alt} \left[ \mathcal{R}_{g_t} \otimes \left( \nabla_{g_t} - \nabla_{TX, g_t} \right) \dot{g}_t^* \right] \\
+ \text{div}_g^* \left[ \mathcal{R}_{g_t} \dot{g}_t^* \right] - \dot{g}_t^* \nabla_{TX, g_t} \text{Ric}_{g_t}^*(\Omega).
\]

This combined with the previous variation formula for \(\nabla_{TX, g_t} \text{Ric}_{g_t}^*(\Omega)\) implies the required conclusion.

**14.3 An direct proof of the variation formula (2.3)**

We observe that (2.3) it is equivalent to the variation formula (2.4) that we show now. Let \((e_k)_k\) be a local tangent frame. Using the identity (3.7) we compute the variation
\[
2 \frac{d}{dt} \text{Ric}_{g_t}^* \xi = 2 \frac{d}{dt} \mathcal{R}_{g_t}(\xi, e_k) g_t^{-1} e_k \\
= \left[ \mathcal{R}_{g_t}(\xi, e_k), \dot{g}_t^* \right] e_k - 2 \mathcal{R}_{g_t}(\xi, e_k) \dot{g}_t^* e_k.
\]

We deduce the variation formula
\[
2 \frac{d}{dt} \text{Ric}_{g_t}^* = - \dot{g}_t^* \text{Ric}_{g_t}^* - \mathcal{R}_{g_t}^* \ast \dot{g}_t^*.
\] (14.2)

On the other hand using the identity (3.14) we can compute the variation of the Hessian
\[
2 \frac{d}{dt} \nabla_{g_t}^2 \mathcal{f}_t \xi = 2 \nabla_{g_t, \xi} \nabla g_t \mathcal{f}_t - \nabla_{g_t, \xi} \left( \nabla_{g_t}^* \dot{g}_t^* + 2 \dot{g}_t^* \nabla_{g_t} \mathcal{f}_t \right) \\
= \nabla_{g_t, \xi} \dot{g}_t^* \nabla g_t \mathcal{f}_t - \nabla_{g_t, \xi} \nabla_{g_t} \dot{g}_t^* \\
- 2 \nabla_{g_t, \xi} \dot{g}_t^* \nabla g_t \mathcal{f}_t - 2 \dot{g}_t^* \nabla_{g_t}^2 \mathcal{f}_t \xi.
\]
\[-\nabla_{g_t, \xi} \dot{g}_t \nabla_{g_t, f_t} - \Delta_{g_t} \dot{g}_t^* \xi\]

\[\quad + (R_{g_t} \ast \dot{g}_t^*) \xi - \dot{g}_t^* \text{Ric}_{g_t}^* \xi - 2 \dot{g}_t^* \nabla^2_{g_t, f_t} \xi,\]

thanks to lemma [20]. We infer the variation identity

\[2 \frac{d}{dt} \nabla^2_{g_t, f_t} = -\Delta_{g_t} \dot{g}_t^* + (R_{g_t} \ast \dot{g}_t^*) - \dot{g}_t^* \text{Ric}_{g_t}^* - 2 \dot{g}_t^* \nabla^2_{g_t, f_t}.\]

This combined with the variation formula (14.2) implies the formula (2.4) and thus the variation identity (2.3).

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