Varieties of algebras without the amalgamation property

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Abstract. Let $\alpha$ be an ordinal and $\kappa$ be a cardinal, both infinite, such that $\kappa \leq |\alpha|$. For $\tau \in {}^\alpha \alpha$, let $\text{sup}(\tau) = \{ i \in \alpha : \tau(i) \neq i \}$. Let $G_\kappa = \{ \tau \in {}^\alpha \alpha : |\text{sup}(\tau)| < \kappa \}$. We consider variants of polyadic equality algebras by taking cylindrifications on $\Gamma \subseteq \alpha$, $|\Gamma| < \kappa$ and substitutions restricted to $G_\kappa$. Such algebras are also enriched with generalized diagonal elements. We show that for any variety $V$ containing the class of representable algebras and satisfying a finite schema of equations, $V$ fails to have the amalgamation property. In particular, many varieties of Halmos’ quasi-polyadic equality algebras and Lucas’ extended cylindric algebras (including that of the representable algebras) fail to have the amalgamation property.

The most generic examples of algebras of first order logic are Tarski’s cylindric algebras and Halmos’ polyadic algebras. Both algebras are well known and widely used. Polyadic algebras were introduced by Halmos [12] to provide an algebraic reflection of the study of first order logic without equality. Later the algebras were enriched by diagonal elements to permit the discussion of equality. That the notion is indeed an adequate reflection of first order logic was demonstrated by Halmos’ representation theorem for locally finite polyadic algebras (with and without equality). Tarski proved an analogous result for locally finite cylindric algebras. Daigneault and Monk proved a strong extension of Halmos’ theorem, namely, every polyadic algebra of infinite dimension (without equality) is representable [9]. However, not every cylindric algebra is representable. In fact, the class of infinite dimensional representable algebras is not axiomatizable by any finite schema, a classical result of Monk. This is a point (among others) where the two theories deviate. Monk’s result was considerably strengthened by Andréka by showing that there is an inevitable degree of complexity in any axiomatization of the class of

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representable cylindric algebras. In particular, any universal axiomatization of the class of representable quasipolyadic algebras must contain infinitely many variables. The representation theorem of Diagneault and Monk - a typical Stone-like representation theorem - shows that the notion of polyadic algebra is indeed an adequate reflection of Keisler’s predicate logic ($KL$). $KL$ is a proper extension of first order logic without equality, obtained when the bound on the number of variables in formulas is relaxed; and accordingly allowing the following as extra operations on formulas: Quantification on infinitely many variables and simultaneous substitution of (infinitely many) variables for variables. Adding equality to $KL$, proved problematic as illustrated algebraically by Johnson [10]. In op.cit, Johnson showed that the class of representable polyadic algebras with equality is not closed under ultraproducts, hence this class is not elementary, i.e. cannot be axiomatized by any set of first order sentences. However one can still hope for a nice axiomatization of the variety generated by the class of polyadic equality algebras. A subtle recent (negative) result in this direction is Németi - Sági’s [18]. In sharp contrast to $KL$, the validities of $KL$ with equality cannot be recaptured by any set of schemas analogous to Halmos' schemas, let alone a finite one. In particular, the variety generated by the class of representable polyadic algebras with equality cannot be axiomatized by a finite schema of equations. The latter answers a question originally raised by Craig [7].

It is interesting (and indeed natural) to ask for algebraic versions of model theoretic results, other than completeness. Examples include interpolation theorems and omitting types theorems. Unlike the cylindric case, omitting types for polyadic algebras prove problematic. This is the case because polyadic algebras of infinite dimension have uncountably many operations, and omitting types arguments- Baire Category arguments at heart - are very much tied to countability. On the other hand, Daigneault succeeded in stating and proving versions of Beth’s and Craig’s theorems. This was done by proving the algebraic analogue of Robinson’s joint consistency theorem: Locally finite polyadic algebras (with and without equality) have the amalgamation property. Later Johnson removed the condition of local finiteness, proving that polyadic algebras without equality have the strong amalgamation property [11]. With this stronger result, Robinson’s, Beth’s and Craig’s theorems hold for $KL$.

Yet another point where the two theories deviate, Pigozzi [19], proves that the class of representable cylindric algebras fails to have the amalgamation property. This shows that certain infinitary algebraizable extensions of first order logic, the so-called typless logics (or finitary logics with infinitary relations) fail to have the interpolation property. Further negative results concerning various amalgamation properties for cylindric-like algebras of relations can be found in [15], [16], [17].

Motivated by the quest for algebraisations that posses the positive prop-
erties of both polyadic algebras and cylindric algebras, in this paper we show, using basically Pigozzi’s techniques appropriately modified, that the interpolation property fails for many variants of KL with equality, contrasting the equality free case [2]. In such variants, formulas of infinite length are allowed, but quantification and substitutions are only allowed for \(< \kappa\) many variables where \(\kappa\) is a fixed beforehand infinite cardinal. Also (generalized) equality is available. Such logics are (natural) extensions of the typeless logics corresponding to cylindric algebras.

Our proof is algebraic addressing the amalgamation property for certain variants of the class of polyadic equality algebras, that are also proper expansions of cylindric algebras. From our proof it can be easily distilled that many varieties of algebraic logics existing in the literature fail to have the amalgamation property. Examples include Halmos’ quasi-polyadic equality algebras and Lucas’ extended cylindric algebras. These results are new.

1 Results and proofs

Let \(\alpha\) be an ordinal and \(\kappa\) be a cardinal, both infinite, such that \(\kappa \leq |\alpha|\). For \(\tau \in \alpha\), let \(\text{sup}(\tau) = \{i \in \alpha : \tau(i) \neq i\}\). Let \(G_\kappa = \{\tau \in \alpha : |\text{sup}(\tau)| < \kappa\}\). Clearly \(G_\kappa\) is a semigroup under the operation of composition; in fact it is a monoid. We write \(\Gamma \subseteq \kappa\) if \(\Gamma \subseteq \alpha\) and \(|\Gamma| < \kappa\). Let \(N = \{\mathcal{E} \subseteq \alpha \times \alpha : \mathcal{E}\) is an equivalence relation on \(\alpha\) and \(|\{(i < \alpha : i/\mathcal{E} \neq \{i\})\}| < \kappa\}\).

**Definition 1.1.** By a \(\kappa\) generalized polyadic equality algebra dimension \(\alpha\), or a \(\text{PEA}_{\kappa,\alpha}\) for short, we understand an algebra of the form

\[
\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\Gamma, s_\tau, d_E \rangle_{\Gamma \subseteq \kappa, \alpha, \tau \in G_\kappa, \mathcal{E} \in N}
\]

where \(c_\Gamma\) (\(\Gamma \subseteq \kappa\)) and \(s_\tau\) (\(\tau \in G_\kappa\)) are unary operations on \(A\), \(d_E \in A\) (\(E \in N\)), such that postulates below hold for \(x, y \in A\), \(\tau, \sigma \in G\), \(\Gamma, \Delta \subseteq \kappa\), \(\alpha\), \(E \in N\), and all \(i, j \in \alpha\).

1. \(\langle A, +, \cdot, -, 0, 1 \rangle\) is a boolean algebra
2. \(c_\Gamma 0 = 0\)
3. \(x \leq c_\Gamma x\)
4. \(c_\Gamma (x \cdot c_\Gamma y) = c_\Gamma x \cdot c_\Gamma y\)
5. \(c_\Gamma c_\Delta x = c_\Gamma \cup \Delta x\)
6. \(s_\tau\) is a boolean endomorphism
7. \(s_{Id} x = x\)
8. \( s_{\sigma \tau} = s_{\sigma} \circ s_{\tau} \)

9. if \( \sigma \upharpoonright (\alpha \sim \Gamma) = \tau \upharpoonright (\alpha \sim \Gamma) \), then \( s_{\sigma} c_{(\Gamma)} x = s_{\tau} c_{(\Gamma)} x \)

10. if \( \tau^{-1} \Gamma = \Delta \) and \( \tau \upharpoonright \Delta \) is one to one, then \( c_{(\Gamma)} s_{\tau} x = s_{\tau} c_{(\Delta)} x \)

11. \( d_{I} = 1 \) where \( I = Id \upharpoonright \alpha \times \alpha \)

12. \( c_{(\Gamma)} d_{E} = d_{F} \) where \( F = E \cap 2(\alpha \sim \Gamma) \cup Id \upharpoonright \alpha \times \alpha \)

13. \( s_{\tau} d_{E} = d_{F} \) where \( F = \{ (\tau(i), \tau(j)) : (i, j) \in E \} \cup Id \upharpoonright \alpha \times \alpha \)

14. \( x \cdot d_{ij} \leq s_{[i,j]} x \)

In the above definition, and elsewhere throughout the paper, \( d_{ij} \) denotes the element \( d_{E} \) where \( E \) is the equivalence relation relating \( i \) to \( j \), and everything else only to itself. For a class \( K \) of algebras, \( SK \) stands for the class of all subalgebras of algebras in \( K \), \( PK \) is the class of products of algebras in \( K \) and \( HK \) is the class of all homomorphic images of algebras in \( K \). The class of representable algebras is defined via set - theoretic operations on sets of \( \alpha \)-ary sequences. Let \( U \) be a set. For \( \Gamma \subseteq \alpha \), \( \tau \in \alpha \), \( i, j \in \alpha \) and \( E \in N \), we set

\[
\begin{align*}
c_{(\Gamma)} X &= \{ s \in ^{\alpha} U : \exists t \in X \ t(j) = s(j) \ \forall j \notin \Gamma \}. \\
s_{\tau} X &= \{ s \in ^{\alpha} U : s \circ \tau \in X \}. \\
d_{ij} &= \{ s \in ^{\alpha} U : s_i = s_j \}. \\
d_{E} &= \{ s \in ^{\alpha} U : s_i = s_j \ \forall (i,j) \in E \}.
\end{align*}
\]

Note that \( d_{E} = \bigcap_{(i,j) \in E} d_{ij} \). For a set \( X \), let \( \mathfrak{B}(X) \) be the boolean set algebra \( (\wp(X), \cap, \cup, \sim) \). The class of representable \( G_{\kappa} \) polyadic equality algebras, or \( RPEA_{\kappa, \alpha} \) is defined by

\[
SP\{ \langle \mathfrak{B}(^{\alpha} U), c_{(\Gamma)}, s_{\tau}, d_{E} \rangle : E \in N, \Gamma \subseteq \alpha, \tau \in G_{\kappa}, \ U \ a \ set \}.
\]

We make the following observations:

- \( RPEA_{\kappa, \alpha} \subseteq PEA_{\kappa, \alpha} \), and the inclusion is proper \([4]\).  

- If \( A \in PEA_{\kappa, \alpha} \) then \( A \) has a cylindric reduct and indeed this reduct is a cylindric algebra of dimension \( \alpha \). In fact, \( A \) has a quasipolyadic equality reduct obtained by restricting the operations to finite quantifiers (cylindrifications) , finite substitutions and ordinary diagonal elements, i.e. the \( d_{ij} \)'s.

- if \( G_{\kappa} \) contains one infinitary substitution then \( RPEA_{\kappa, \alpha} \) is not closed under ultraproducts \([20]\), hence is not closed under \( H \), lest it be a variety.
For what follows, we need:

**Definition 1.2.** Let \( K \subseteq V \) be classes of algebras. \( K \) is said to have the amalgamation property, or \( AP \) for short, with respect to \( V \), if for all \( A_0, A_1 \) and \( A_2 \in K \), and all monomorphisms \( i_1 \) and \( i_2 \) of \( A_0 \) into \( A_1, A_2 \), respectively, there exists \( A \in V \), a monomorphism \( m_1 \) from \( A_1 \) into \( A \) and a monomorphism \( m_2 \) from \( A_2 \) into \( A \) such that \( m_1 \circ i_1 = m_2 \circ i_2 \).

We will show that for any variety \( K \), \( RPEA_{\kappa,\alpha} \subseteq K \subseteq PEA_{\kappa,\alpha} \), \( K \) fails to have the amalgamation property with respect to \( PEA_{\kappa,\alpha} \). For motivations of studying such algebras, and similar reducts of polyadic equality algebras, initiated by Craig [7], see [1], [2], [3], [20], [21]. Amalgamation in varieties can be pinned down to congruences on free algebras. Congruences correspond to ideals. This prompts:

**Definition 1.3.** Let \( \mathfrak{A} \in PEA_{\kappa,\alpha} \). A subset \( I \) of \( \mathfrak{A} \) in an ideal if the following conditions are satisfied:

1. \( 0 \in I \),
2. If \( x, y \in I \), then \( x + y \in I \),
3. If \( x \in I \) and \( y \leq x \) then \( y \in I \),
4. For all \( \Gamma \subseteq \kappa \) and \( \tau \in G_\kappa \) if \( x \in I \) then \( c_{(\Gamma)}x \) and \( s_{\tau}x \in I \).

It can be checked that ideals function properly, that is ideals correspond to congruences the usual way. For \( X \subseteq \mathfrak{A} \), the ideal generated by \( X \), \( \mathfrak{I}g^\mathfrak{A}X \) is the smallest ideal containing \( X \), i.e the intersection of all ideals containing \( X \). We let \( \mathfrak{I}g^\mathfrak{A}X \) and sometimes \( \mathfrak{A}^{(X)} \) denote the subalgebra of \( \mathfrak{A} \) generated by \( X \).

**Lemma 1.4.** Let \( \mathfrak{A} \in PEA_{\kappa,\alpha} \) and \( X \subseteq A \). Then \( \mathfrak{I}g^\mathfrak{A}X = \{ y \in A : y \leq c_{(\Gamma)}(x_0 + \ldots x_{k-1}) \} : \) for some \( k \in \omega, x \in kX \) and \( \Gamma \subseteq \kappa \).

**Proof.** Let \( H \) denote the set of elements on the right hand side. It is easy to check \( H \subseteq \mathfrak{I}g^\mathfrak{A}X \). Conversely, assume that \( y \in H, \Gamma \subseteq \kappa \). It is clear that \( c_{(\Gamma)}y \in H \). \( H \) is closed under substitutions, since for any \( \tau \in G_\kappa \), any \( x \in A \) there exists \( \Gamma \subseteq \kappa \) such that \( s_{\tau}x \leq c_{(\Gamma)}x \). Indeed \( sup(\tau) \) is such a \( \Gamma \). Now let \( z, y \in H \). Assume that \( z \leq c_{(\Gamma)}(x_0 + \ldots x_{k-1}) \) and \( y \leq c_{(\Delta)}(y_0 + \ldots y_{l-1}) \), then

\[
z + y \leq c_{(\Gamma \cup \Delta)}(x_0 + \ldots x_{k-1} + y_0 \ldots + y_{l-1}).
\]

The Lemma is proved.

Fixing \( \alpha \) and \( \kappa \) throughout, in what follows we denote \( (R)PEA_{\kappa,\alpha} \) simply by \( (R)PEA \). The following about ideals will be frequently used.
• If $\mathfrak{A} \subseteq \mathfrak{B}$ are PEA’s and $I$ is an ideal of $\mathfrak{A}$, then $\mathfrak{I}_g^{\mathfrak{B}}(I) = \{b \in B : \exists a \in I(b \leq a)\}$.

• If $I$ and $J$ are ideals of a PEA then the ideal generated by $I \cup J$ is $\{x : x \leq i + j$ for $i \in I, j \in J\}$.

For a class $\mathbf{K}$ and a set $X$, $\mathfrak{F}_X \mathbf{K}$ denotes the $\mathbf{K}$ algebra freely generated by $X$, or the $\mathbf{K}$ free algebra on $|X|$ generators. As a wide spread custom, we identify $X$ with $|X|$. We understand the notion of free algebras in the sense of [13] Definition 0.4.19. In particular, free $\mathbf{K}$ algebras may not be in $\mathbf{K}$. However, they are always in $HSP(\mathbf{K})$, the variety generated by $\mathbf{K}$. We now formulate and prove our main result:

**Theorem 1.5.** Let $\mathbf{K}$ be a variety such that $RPEA \subseteq \mathbf{K} \subseteq PEA$. Then $\mathbf{K}$ does not have AP with respect to PEA.

**Proof.** The proof is an adaptation of Pigozzi’s techniques for showing failure of the amalgamation property for cylindric algebras [19]. Seeking a contradiction assume that $\mathbf{K}$ has AP with respect to PEA. Let $\mathfrak{A} = \mathfrak{F}_4 PEA$. Let $r$, $s$ and $t$ be defined as follows:

$$r = c_0(x \cdot c_1 y) \cdot c_0(x \cdot -c_1 y),$$

$$s = c_0 c_1 (c_1 z \cdot s^0_1 c_1 z \cdot -d_{01}) + c_0(x \cdot -c_1 z),$$

$$t = c_0 c_1 (c_1 w \cdot s^0_1 c_1 w \cdot -d_{01}) + c_0(x \cdot -c_1 w),$$

where $x, y, z$, and $w$ are the first four free generators of $\mathfrak{A}$. Then $r \leq s \cdot t$. This inequality is proved by Pigozzi, whose proof we include. Indeed put

$$a = x \cdot c_1 y \cdot -c_0(x \cdot -c_1 z),$$

$$b = x \cdot -c_1 y \cdot -c_0(x \cdot -c_1 z).$$

Then we have

$$c_1 a \cdot c_1 b \leq c_1(x \cdot c_1 y) \cdot c_1(x \cdot -c_1 y) \quad \text{by [13] 1.2.7}$$

$$= c_1 x \cdot c_1 y \cdot c_1 x \cdot -c_1 y \quad \text{by [13] 1.2.11}$$

and so

$$c_1 a \cdot c_1 b = 0. \quad (1)$$
From the inclusion $x \cdot -c_1 z \leq c_0(x \cdot -c_1 z)$ we get

$$x \cdot -c_0(x \cdot -c_1 z) \leq c_1 z.$$  

Thus $a, b \leq c_1 z$ and hence, by [13] 1.2.9,

$$c_1 a, c_1 b \leq c_1 z. \tag{2}$$

We now compute:

\[
c_0 a \cdot c_0 b \leq c_0 c_1 a \cdot c_0 c_1 b \quad \text{by [13] 1.2.7}
= c_0 c_1 a \cdot c_1 s_0^1 c_1 b \quad \text{by [13] 1.5.8 (i), [13] 1.5.9 (i)}
= c_1(c_0 c_1 a \cdot s_0^1 c_1 b)
= c_0 c_1(c_1 a \cdot s_0^1 c_1 b)
= c_0 c_1[c_1 a \cdot s_0^1 c_1 b \cdot (-d_01 + d_01)]
= c_0 c_1[(c_1 a \cdot s_0^1 c_1 b \cdot -d_01) + (c_1 a \cdot s_0^1 c_1 b \cdot d_01)]
= c_0 c_1[(c_1 a \cdot s_0^1 c_1 b \cdot -d_01) + (c_1 a \cdot c_1 b \cdot d_01)] \quad \text{by [13] 1.5.5}
= c_0 c_1(c_1 a \cdot s_0^1 c_1 b \cdot -d_01) \quad \text{by (1)}
\leq c_0 c_1(c_1 z \cdot s_0^1 c_1 z \cdot -d_01) \quad \text{by (2), [13] 1.2.7}
\]

We have proved that

$$c_0[x \cdot c_1 y \cdot -c_0(x \cdot -c_1 z)] \cdot c_0[x \cdot -c_1 y \cdot -c_0(x \cdot -c_1 z)] \leq c_0 c_1(c_1 z \cdot s_0^1 c_1 z \cdot -d_01).$$

In view of [13] 1.2.11 this gives

$$c_0(x \cdot c_1 y) \cdot c_0(x \cdot -c_1 y) \cdot -c_0(x \cdot -c_1 z) \leq c_0 c_1(c_1 z \cdot s_0^1 c_1 z \cdot -d_01).$$

The conclusion now follows. Let $X_1 = \{x, y\}$ and $X_2 = \{x, z, w\}$. Then

$$\mathfrak{A}^{(X_1 \cap X_2)} = \mathfrak{G}^{\mathfrak{A}}\{x\}. \tag{3}$$

We have

$$r \in A^{(X_1)} \text{ and } s, t \in A^{(X_2)}. \tag{4}$$

Let $R$ be an ideal of $\mathfrak{A}$ such that

$$\mathfrak{A}/R \cong \mathfrak{F}_{t_1} K_\alpha. \tag{5}$$

Since $r \leq s \cdot t$ we have

$$r \in \mathfrak{F}_{t_1} \{s \cdot t\} \cap A^{(X_1)}. \tag{6}$$
Let
\[ M = \mathcal{I}_g^{\mathfrak{A}(X_2)}[\{s \cdot t\} \cup (R \cap A^{(X_2)})]; \quad (7) \]
\[ N = \mathcal{I}_g^{\mathfrak{A}(X_1)}[(M \cap A^{(X_1 \cap X_2)}) \cup (R \cap A^{(X_1)})]. \quad (8) \]
Then we have
\[ R \cap A^{(X_2)} \subseteq M \quad \text{and} \quad R \cap A^{(X_1)} \subseteq N. \quad (9) \]
From the first of these inclusions we get
\[ M \cap A^{(X_1 \cap X_2)} \supseteq (R \cap A^{(X_2)}) \cap A^{(X_1 \cap X_2)} = (R \cap A^{(X_1)}) \cap A^{(X_1 \cap X_2)}. \]
By (8) we have
\[ N \cap A^{(X_1 \cap X_2)} = M \cap A^{(X_1 \cap X_2)}. \]
For \( R \) an ideal of \( \mathfrak{A} \) and \( X \subseteq A \), by \((\mathfrak{A}/R)^{(X)}\) we understand the subalgebra of \( \mathfrak{A}/R \) generated by \( \{x/R : x \in X\} \). Define
\[ \theta : \mathfrak{A}^{(X_1 \cap X_2)} \to \mathfrak{A}^{(X_1)}/N \]
by
\[ a \mapsto a/N. \]
Then \( ker \theta = N \cap A^{(X_1 \cap X_2)} \) and \( Im \theta = (\mathfrak{A}^{(X_1)}/N)^{(X_1 \cap X_2)} \). It follows that
\[ \bar{\theta} : \mathfrak{A}^{(X_1 \cap X_2)}/N \cap A^{(X_1 \cap X_2)} \to (\mathfrak{A}^{(X_1)}/N)^{(X_1 \cap X_2)} \]
defined by
\[ a/N \cap A^{X_1 \cap X_2} \mapsto a/N \]
is a well defined isomorphism. Similarly
\[ \bar{\psi} : \mathfrak{A}^{(X_1 \cap X_2)}/M \cap A^{(X_1 \cap X_2)} \to (\mathfrak{A}^{(X_2)}/M)^{(X_1 \cap X_2)} \]
defined by
\[ a/M \cap A^{X_1 \cap X_2} \mapsto a/M \]
is also a well defined isomorphism. But
\[ N \cap A^{(X_1 \cap X_2)} = M \cap A^{(X_1 \cap X_2)}, \]
Hence
\[ \phi : (\mathfrak{A}^{(X_1)}/N)^{(X_1 \cap X_2)} \to (\mathfrak{A}^{(X_2)}/M)^{(X_1 \cap X_2)} \]
defined by
\[ a/N \mapsto a/M \]
is a well defined isomorphism. Now \((\mathfrak{A}^{(X_1)}/N)^{(X_1\cap X_2)}\) embeds into \(\mathfrak{A}^{(X_1)}/N\) via the inclusion map; it also embeds in \(\mathfrak{A}^{(X_2)}/M\) via \(i \circ \phi\) where \(i\) is also the inclusion map. For brevity let \(\mathfrak{A}_0 = (\mathfrak{A}^{(X_1)}/N)^{(X_1\cap X_2)}, \mathfrak{A}_1 = \mathfrak{A}^{(X_1)}/N\) and \(\mathfrak{A}_2 = \mathfrak{A}^{(X_2)}/M\) and \(j = i \circ \phi\). Then \(\mathfrak{A}_0\) embeds in \(\mathfrak{A}_1\) and \(\mathfrak{A}_2\) via \(i\) and \(j\) respectively. Now observe that \(\mathfrak{A}_1, \mathfrak{A}_2\) and \(\mathfrak{A}_0\) are in \(K\). So by assumption, there exists an amalgam, i.e there exists \(\mathfrak{B} \in PEA\) and monomorphisms \(f\) and \(g\) from \(\mathfrak{A}_1\) and \(\mathfrak{A}_2\) respectively to \(\mathfrak{B}\) such that \(f \circ i = g \circ j\). Let

\[
\bar{f} : \mathfrak{A}^{(X_1)} \to \mathfrak{B}
\]

be defined by

\[
a \mapsto f(a/N)
\]

and

\[
\bar{g} : \mathfrak{A}^{(X_2)} \to \mathfrak{B}
\]

be defined by

\[
a \mapsto g(a/M).
\]

Let \(\mathfrak{B}'\) be the algebra generated by \(Imf \cup Img\). Then \(\bar{f} \cup \bar{g} \mid X_1 \cup X_2 \to \mathfrak{B}'\) is a function since \(\bar{f}\) and \(\bar{g}\) coincide on \(X_1 \cap X_2\). By freeness of \(\mathfrak{A}\), there exists \(h : \mathfrak{A} \to \mathfrak{B}'\) such that \(h \mid_{X_1 \cup X_2} = \bar{f} \cup \bar{g}\). Let \(P = kerh\). Then it is not hard to check that

\[
P \cap A^{(X_1)} = N, \tag{10}
\]

and

\[
P \cap A^{(X_2)} = M. \tag{11}
\]

In view of (4), (7), (11) we have \(s \cdot t \in P\) and hence by (6) \(r \in P\). Consequently from (4) and (11) we get \(r \in N\). From (8) there exist elements

\[
u \in M \cap A^{(X_1 \cap X_2)} \tag{12}
\]

and \(b \in R\) such that

\[
r \leq u + b. \tag{13}
\]

Since \(u \in M\) by (7) there is a \(\Gamma \subseteq \kappa\) \(\alpha\) and \(c \in R\) such that

\[
u \leq c_{(\Gamma)}(s \cdot t) + c.
\]

Let \(\{x', y', z', w'\}\) be the first four generators of \(\mathfrak{D} = \mathfrak{Fr}_4 K\). Let \(h\) be the homomorphism from \(\mathfrak{A}\) to \(\mathfrak{D}\) be such that \(h(i) = i'\) for \(i \in \{x, y, w, z\}\). Notice that \(kerh = R\). Then \(h(b) = h(c) = 0\). It follows that

\[
h(r) \leq h(u) \leq c_{(\Gamma)}(h(s), h(t)).
\]
Let \( r' = h(r), u' = h(u), s' = h(s) \) and \( t' = h(t) \). Let
\[
\mathcal{B} = (\wp(\alpha), \cup, \cap, \sim, \emptyset, \alpha, c(\tau), s_{\tau}, d_{\varepsilon})_{\tau \subseteq_\alpha, \tau \in G_\kappa, \varepsilon \in \mathbb{N}}
\]
that is \( \mathcal{B} \) is the full set algebra in the space \( \alpha \). Let \( E \) be the set of all equivalence relations on \( \alpha \), and for each \( R \in E \) set
\[
X_R = \{ \varphi : \varphi \in \alpha \text{ and for all } \xi, \eta < \alpha, \varphi_\xi = \varphi_\eta \text{ iff } \xi R \eta \}.
\]
More succintly
\[
X_R = \{ \varphi \in \alpha : \text{ker}\varphi = R \}.
\]
Let
\[
C = \{ \bigcup_{R \in L} X_R : L \subseteq E \}.
\]
\( C \) is clearly closed under the formation of arbitrary unions, and since
\[
\sim \bigcup_{R \in L} X_L = \bigcup_{R \in E \sim L} X_R
\]
for every \( L \subseteq E \), we see that \( C \) is closed under the formation of complements with respect to \( \alpha \). Thus \( C \) is a Boolean subuniverse (indeed, a complete Boolean subuniverse) of \( \mathcal{B} \); moreover, it is obvious that
\[
X_R \text{ is an atom of } (C, \cup, \cap, \sim, 0, \alpha) \text{ for each } R \in E. \tag{14}
\]
For all \( \varepsilon \in N \) we have \( d_{\varepsilon} = \bigcup \{ X_R : \varepsilon \subseteq R \in E \} \) and hence \( d_{\varepsilon} \in C \). Also,
\[
c(\tau)X_R = \bigcup \{ X_S : S \in E, 2(\alpha \sim \Gamma) \cap S = 2(\alpha \sim \Gamma) \cap R \}
\]
for any \( \Gamma \subseteq_\alpha \alpha \) and \( R \in E \). Thus, because \( c(\tau) \) is completely additive, \( C \) is closed under the operation \( c(\tau) \) for every \( \Gamma \subseteq_\alpha \alpha \). It is easy to show that \( C \) is closed under substitutions. For any \( \tau \in G_\kappa \),
\[
s_{\tau}X_R = \bigcup \{ X_S : S \in E, \forall i, j < \alpha (i R j \iff \tau(i) S \tau(j)) \}.
\]
The set on the right may of course be empty. Since \( s_{\tau} \) is also completely additive, therefore, we have shown that
\[
C \text{ is a subuniverse of } \mathcal{B}. \tag{15}
\]
We now show that there is a subset \( Y \) of \( \alpha \) such that
\[
X_{Id} \cap f(r') \neq 0 \text{ for every } f \in Hom(\mathcal{D}, \mathcal{B}) \text{ such that } f(x') = X_{Id} \text{ and } f(y') = Y, \tag{16}
\]
and also that for every $\Gamma \subseteq \kappa$, there are subsets $Z, W$ of $\alpha$ such that

$$X_{Id} \sim c_{(\Gamma)} g(s' \cdot t') \neq 0 \quad \text{for every} \quad g \in \text{Hom}(\mathcal{D}, \mathcal{B})$$

such that $g(x') = X_{Id}, g(z') = Z$ and $g(w') = W$. \hfill (17)

Here $\text{Hom}(\mathcal{D}, \mathcal{B})$ stands for the set of all homomorphisms from $\mathcal{D}$ to $\mathcal{B}$. Let $\sigma \in \alpha$ be such that $\sigma_0 = 0$, and $\sigma_\kappa = \kappa + 1$ for every non-zero $\kappa < \omega$ and $\sigma j = j$ otherwise. Let $\tau = \sigma \upharpoonright (\alpha \sim \{0\}) \cup \{(0, 1)\}$. Then $\sigma, \tau \in X_{Id}$. Take $Y = \{\sigma\}$.

Then

$$\sigma \in X_{Id} \cap c_1 Y \quad \text{and} \quad \tau \in X_{Id} \sim c_1 Y$$

and hence

$$\sigma \in c_0(X_{Id} \cap c_1 Y) \cap c_0(X_{Id} \sim c_1 Y). \hfill (18)$$

Therefore, we have $\sigma \in f(r')$ for every $f \in \text{Hom}(\mathcal{D}, \mathcal{B})$ such that $f(x') = X_{Id}$ and $f(y') = Y$, and that (16) holds. We now want to show that for any given $\Gamma \subseteq \kappa$, there exist sets $Z, W \subseteq \alpha$ such that (17) holds; it is clear that no generality is lost if we assume that $0, 1 \in \Gamma$, so we make this assumption. Take

$$Z = \{\varphi : \varphi \in X_{Id}, \varphi_0 < \varphi_1 \} \cap c_{(\Gamma)}\{Id\}$$

and

$$W = \{\varphi : \varphi \in X_{Id}, \varphi_0 > \varphi_1 \} \cap c_{(\Gamma)}\{Id\}.$$ 

We show that

$$Id \in X_{Id} \sim c_{(\Gamma)} g(s' \cdot t') \hfill (19)$$

for any $g \in \text{Hom}(\mathcal{D}, \mathcal{B})$ such that $g(x') = X_{Id}, g(z') = Z$, and $g(w') = W$; to do this we simply compute the value of $c_{(\Gamma)} g(s' \cdot t')$. This part of the proof is taken verbatim from Pigozzi [19]. For the purpose of this computation we make use of the following property of ordinals: if $\Delta$ is any non-empty set of ordinals, then $\bigcap \Delta$ is the smallest ordinal in $\Delta$, and if, in addition, $\Delta$ is finite, then $\bigcup \Delta$ is the largest element ordinal in $\Delta$. Also, in this computation we shall assume that $\varphi$ always represents an arbitrary sequence in $\alpha$. Then, setting

$$\Delta \varphi = \Gamma \sim \varphi[\Gamma \sim \{0, 1\}]$$

for every $\varphi$, we successively compute:

$$c_1 Z = \{\varphi : |\Delta \varphi| = 2, \varphi_0 = \bigcap \Delta \varphi \} \cap c_{(\Gamma)}\{Id\},$$
\[(X_{Id} \sim c_1 Z) \cap c_{(\Gamma)} \{Id\} = \{\varphi : |\Delta \varphi| = 2, \varphi_0 = \bigcup \Delta \varphi, \varphi_1 = \bigcap \Delta \varphi \} \cap c_{(\Gamma)} \{Id\} \]

and, finally,

\[c_0(X_{Id} \sim c_1 Z) \cap c_{(\Gamma)} \{Id\} = \{\varphi : |\Delta \varphi| = 2, \varphi_0 = \bigcup \Delta \varphi, \varphi_1 = \bigcap \Delta \varphi \} \cap c_{(\Gamma)} \{Id\}. \tag{20}\]

Similarly, we obtain

\[c_0(X_{Id} \sim c_1 W) \cap c_{(\Gamma)} \{Id\} = \{\varphi : |\Delta \varphi| = 2, \varphi_1 = \bigcup \Delta \varphi \} \cap c_{(\Gamma)} \{Id\}. \]

The last two formulas together give

\[c_0(X_{Id} \sim c_1 Z) \cap c_0(X_{Id} \sim c_1 W) \cap c_{(\Gamma)} \{Id\} = 0. \tag{21}\]

Continuing the computation we successively obtain:

\[c_1 Z \cap d_{01} = \{\varphi : |\Delta \varphi| = 2, \varphi_0 = \varphi_1 = \bigcap \Delta \varphi \} \cap c_{(\Gamma)} \{Id\}, \]

\[s^0_1 c_1 Z = \{\varphi : |\Delta \varphi| = 2, \varphi_1 = \bigcap \Delta \varphi \} \cap c_{(\Gamma)} \{Id\}, \]

\[c_1 Z \cap s^0_1 c_1 Z = \{\varphi : |\Delta \varphi| = 2, \varphi_0 = \varphi_1 = \bigcap \Delta \varphi \} \cap c_{(\Gamma)} \{Id\}; \]

hence we finally get

\[c_0 c_1(c_1 Z \cap s^0_1 c_1 Z) \cap d_{01} = c_0 c_1 0 = 0, \tag{22}\]

and similarly we get

\[c_0 c_1(c_1 W \cap s^0_1 c_1 W) \cap d_{01} = 0. \tag{23}\]

Now take \(g\) to be any homomorphism from \(\mathcal{D}\) into \(\mathcal{B}\) such that \(g(x') = X_{Id}, g(z') = Z\) and \(g(w') = W\). Let \(a = g(s' \cdot t')\). Then from the above

\[a \cap c_{(\Gamma)} \{Id\} = \emptyset. \]

Then applying \(c_{(\Gamma)}\) to both sides of this equation we get

\[c_{(\Gamma)} a \cap c_{(\Gamma)} \{Id\} = \emptyset. \]

Thus (19) holds. Now there exists \(\Gamma \subseteq \kappa \ \alpha\) and an interpolant \(u' \in \mathcal{D}(x')\), that is

\[r' \leq u' \leq c_{(\Gamma)}(s' \cdot t'). \]
There also exist \( Y, Z, W \subseteq ^{\alpha} \alpha \) such that (16) and (17) hold. Take any \( k \in Hom(D, B) \) such that \( k(x') = X_{Id}, k(y') = Y, k(z') = Z, \) and \( k(w') = W. \) This is possible by the freeness of \( D. \) Then using the fact that \( X_{Id} \cap k(r') \) is non-empty by (16) we get

\[
X_{Id} \cap k(u') = k(x' \cdot u') \supseteq k(x' \cdot r') \neq 0.
\]

And using the fact that \( X_{Id} \sim c_{(\Gamma)} k(s' \cdot t') \) is non-empty by (17) we get

\[
X_{Id} \sim k(u') = k(x' \cdot -u') \supseteq k(x' \cdot -c_{(\Gamma)}(s' \cdot t')) \neq 0.
\]

However, in view of (14), it is impossible for \( X_{Id} \) to intersect both \( k(u') \) and its complement since \( k(u') \in C \) and \( X_{Id} \) is an atom; to see that \( k(u') \) is indeed contained in \( C \) recall that \( u' \in D^{(x')} \), and then observe that because of (15) and the fact that \( X_{Id} \in C \) we must have \( k[D^{(x')}]) \subseteq C. \) This contradiction shows that \( K \) does not have the amalgamation property with respect to \( PEA. \) By this the proof is complete.

Other algebraic logics to which our proof applies are Halmos’ quasi-polyadic equality algebras and Lucas’ \( \kappa \) extended cylindric algebras [14] p.267. In particular, many varieties of those fail to have the amalgamation property. We recall that Halmos quasi-polyadic algebras are of the form

\[
\mathfrak{A} = \langle A, +, \cdot, - , 0, 1, c_{(\Gamma)}, s_{r}, d_{ij} \rangle_{i,j \in \alpha, \Gamma \subseteq \omega, r \in G_{\omega}}
\]

while Lucas, \( \kappa \) extended cylindric algebras are of the form

\[
\mathfrak{A} = \langle A, +, \cdot, - , 0, 1, c_{(\Gamma)}, d_{e} \rangle_{\Gamma \subseteq \kappa, e \in N}.
\]

Both classes of (abstract) algebras are defined by a finite schema analogous to Halmos’ schemas restricted to the appropriate similarity type, cf. Def 1.1. The representable algebras are defined as subdirect product of set algebras. In those two cases the class of representable algebras, as opposed to the class of abstract algebras, is not finite schema axiomatizable. The methods of Andreka in [5] can be used to prove this (the proof though is not trivial). But in those two cases the class of representable algebras forms a variety and using our proof it can be easily shown that any variety containing the representable algebras such that its cylindric reduct satisfies the cylindric axioms fails to have the amalgamation property. In particular, in both of these cases, both the variety of abstract algebras as well as that of the representable algebras fail to have the amalgamation property.

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