A generalization of the Collatz problem and conjecture

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Summary

We introduce an infinite set of integer mappings that generalize the well-known Collatz-Ulam mapping and we conjecture that an infinite subset of these mappings feature the remarkable property of the Collatz conjecture, namely that they converge to unity irrespective of which positive integer is chosen initially.
1 Introduction

The Collatz problem is so well known that we refer for formulation, references and bibliography to the web [1],[2],[3].

Let us quote from ref. [1]:

"The Collatz conjecture is an unsolved conjecture in mathematics. It is named after Lothar Collatz, who first proposed it in 1937. The conjecture is also known as the $3n + 1$ conjecture, as the Ulam conjecture (after Stanislaw Ulam), or as the Syracuse problem; the sequence of numbers involved is referred to as the hailstone sequence or hailstone numbers, or as wondrous numbers per G"odel, Escher, Bach [4].

We take any number $n$. If $n$ is even, we halve it ($n/2$), else we do "triple plus one" and get $3n + 1$. The conjecture is that for all numbers this process converges to 1. Hence it has been called 'Half Or Triple Plus One', sometimes called HOTPO.

Paul Erd"os said about the Collatz conjecture: 'Mathematics is not yet ready for such problems.' He offered $500 for its solution. 

In this paper we present a generalization of the Collatz map, and prove the corresponding conjecture for a set of initial values; this proof has no relevance for the original Collatz conjecture, although one might hope that it provide some hint for solving that problem.

1.1 The Collatz problem

Consider the following operation on an arbitrary positive integer:

* If the number is even, divide it by two.
* If the number is odd, triple it and add one.

For example, if this operation is performed on 3, the result is 10; if it is performed on 28, the result is 14.

In modular arithmetic notation, define the function $f(n)$ as follows ($n$ is a positive integer):

$$f(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \equiv 0 \pmod{2}, \\
3n + 1 & \text{if } n \equiv 1 \pmod{2}.
\end{cases}$$ (1)

Now, form a sequence $S_k$ by performing this operation repeatedly, beginning with any positive integer $n$, and taking the result at each step as the input for the next.

In notation ($n$ is a positive integer):

$$S_0 = n; \quad S_k = f(S_{k-1}), \quad k = 1, 2, \ldots.$$ (2)

The Collatz conjecture can then be formulated as follows:

**Conjecture 1** This process will eventually reach the number 1, irrespective of which positive integer $n$ is chosen initially.
Note that, once the value $n = 1$ is reached, the sequence (2) reduces to the limit cycle yielding sequentially the values 1, 4, 2, 1.

This Conjecture is simple to state but so difficult to prove that the issue of its validity is still open.

Many generalizations of the Collatz function (1) have been introduced: but, to the best of our knowledge, the generalizations that we propose in this paper are new.

In the following Section we introduce an infinite set of generalized Collatz maps (involving two additional integer parameters $b, m$); in our opinion they qualify as ”correct” generalizations of the original Collatz map, inasmuch as, for most of these maps, the characteristic property of the original Collatz Conjecture seems to hold. We can also prove, albeit only for a limited set of values of the starting integer $n$, that these generalized maps do indeed possess the Collatz property (“convergence to unity”),

1.2 Generalizations

Define the function $f(n, b, m)$ as follows ($n, m = 1, 2, ..., b = 2, 3, ...$):

\[
f(n, b, m) = \begin{cases} 
\frac{n}{b} & \text{if } n \equiv 0 \pmod{b} \\
 (b^m + 1)n + b^m - (n \mod b^m) & \text{if } n \not\equiv 0 \pmod{b}
\end{cases}
\]

(3)

(4)

Now, form a sequence $S_k$ by performing this operation repeatedly, beginning with any positive integer $n$, and taking the result at each step as the input for the next step.

In notation ($n$ is a positive integer):

\[S_0 = n; \quad S_k = f(S_{k-1}, b, m), \quad k = 1, 2, ..., \]

(5)

Remark Clearly for $b = 2, m = 1$ one gets the original Collatz map.

We can easily prove the limited result given by the following proposition:

Proposition If $S_0 < b^m$ then the sequence (5) shall eventually reach the number 1 (for any $b > 1, m > 0$)

Proof

Let us write the integer $S_0 < b^m$ in base $b$:

\[S_0 = a_0 + a_1b + a_2b^2 + ... + a_{m-1}b^{m-1}\]

where of course $a_i < b, i = 0, 1, ..., m - 1$ and let us assume without loss of generality that $a_0 > 0$ (otherwise we apply the rule (3)) and for convenience let us put

\[a_0 = b - j, \quad 0 < j < b.\]

(7)
Then the rule (4) yields

\[ S_1 = (b^m + 1)S_0 + b^m - (S_0 \mod b^m) \]  
\[ = b^m (S_0 + 1) + S_0 - (S_0 \mod b^m). \]  

Now, noting that (see (6))

\[ S_0 \mod b^m = S_0 \]  

we have

\[ S_1 = b^m \left[(a_0 + 1) + a_1 b + a_2 b^2 + \ldots + a_{m-1} b^{m-1}\right]. \]  

Let us assume, just for the sake of clearness in the exposition, that \( j \neq 1 \) so that \( (a_0 + 1) < b \) (see (7)). Then we have to apply \( m \) times the rule (3), reaching

\[ S_{m+1} = (a_0 + 1) + a_1 b + a_2 b^2 + \ldots + a_{m-1} b^{m-1}. \]  

Repeating the above procedure, one gets (see (7))

\[ S_{j+m+j} = (a_0 + j) + a_1 b + a_2 b^2 + \ldots + a_{m-1} b^{m-1} \]  
\[ = (b - j + j) + a_1 b + a_2 b^2 + \ldots + a_{m-1} b^{m-1} \]  
\[ = (a_1 + 1) b + a_2 b^2 + \ldots + a_{m-1} b^{m-1}. \]  

Then the rule (3) yields

\[ S_{j+m+j+1} = (a_1 + 1) b + a_2 b^2 + \ldots + a_{m-1} b^{m-2}. \]  

It is now clear that eventually the sequence reach the number 1. \( (Q.E.D) \)

**Remark** The above proof makes also clear how to compute the stopping time (namely the number of steps necessary to reach the number 1) for the sequences starting from \( S_0 < b^m \).

**Remark** It is also interesting to see the limit cycles, namely the cyclic sequences obtained starting from \( S_0 = 1 \) for different values of \( b, m \). They are:

\[ 1, (2b^m, 2b^{m-1}, \ldots, 2b), 2, (3b^m, 3b^{m-1}, \ldots, 3b), 3, \ldots, b - 1, \ldots, b, 1. \]  

Just as an example we report the limit cycle for \( b = 5, m = 3 \):

\[ 1, 250, 50, 10, 2, 375, 75, 15, 3, 500, 100, 20, 4, 625, 125, 25, 5, 1. \]  

The above proposition covers a limited set of starting points for the sequence (a very small set indeed for small values of \( b, m \) but a rapidly increasing one for increasing values of \( b, m \)). Of course this set can be trivially extended to an infinite set considering all the values \( S_0 = s b^N, s < b^m, N = 0, 1, 2, \ldots \). For all these initial values the sequence (5) obviously converges to the number 1 and thus the limit cycle (17) due to the above Proposition and to the rule (3). Let
us call 'trivial' all these values of $S_0$ (and "non-trivial" the values outside this set).

An obvious question arises: since our mapping is a natural generalization of the Collatz one and considering the result of the above Proposition, may we hope that a generalized Collatz conjecture hold true, namely that for any $b, m, S_0$ the limit cycle (17) is eventually reached?

Having failed (up to now) to extend the result of the above Proposition, we had to resort to a massive computer investigation which of course proves nothing... moreover it is very time-expensive even for relatively small values of $b, m$ if one tries to test a substantial number (say $10^6$-$10^9$) of non-trivial values of $S_0$ (consider for instance that with $b = 10, m = 9$ for the first non-trivial value of $S_0$, namely $S_0 = 10^9 + 1$, the stopping time is 5000000829 ! ). Thus we had to stop an initial systematic search for increasing values of $b, m, S_0$ : however eventually we tested the convergence for a huge number of random chosen values of $b, m, S_0$.

Of course hope is hope, not reality: realistically one should expect to find easily a large number of counter-examples, namely a large number of limit cycles different from the Collatz type one given in (17) (divergences also could possibly arise, but we never found one).

Indeed for $m = 1$ it is easy to find counter-examples even for small values of $b, S_0$ :

- $b = 3, S_0 = 5$ reaches the limit cycle

\[7, 30, 10, 42, 14, 57, 19, 78, 26, 105, 35, 141, 47, 189, 63, 21, 7\]

- $b = 4, S_0 = 11$ reaches the limit cycle

\[23, 116, 29, 148, 37, 188, 47, 236, 59, 296, 74, 372, 93, 468, 117, 588, 147, 736, 184, 45, 232, 58, 292, 73, 368, 92, 23\]

- $b = 6, S_0 = 7$ reaches the limit cycle

\[23, 162, 27, 192, 32, 228, 38, 270, 45, 318, 53, 372, 62, 438, 73, 516, 85, 606, 101, 708, 118, 828, 138, 23\]

- $b = 9, S_0 = 31$ reaches the limit cycle

\[35, 351, 39, 396, 44, 441, 49, 495, 55, 558, 62, 621, 69, 693, 77, 774, 86, 864, 96, 963, 107, 1071, 119, 1197, 133, 1332, 148, 1485, 165, 1656, 184, 1845, 205, 2052, 228, 2286, 254, 2547, 283, 2835, 315, 35.\]

Many other counter-examples can be found for $m = 1$; note however that we found none not only for $b = 2$ (this was expected since that is the original
Collatz map) but also for other small values of $b$ as $b = 5, b = 7, b = 8$ (and this was quite unexpected).

Let us go to $m = 2$. The first counter-example we found is the following one:

- $b = 2, S_0 = 23$ reaches the limit cycle

$$37, 188, 94, 47, 236, 118, 59, 296, 148, 74, 37$$ (23)

Again it is easy to find other counter-examples for $b = 2, m = 2$.

To our surprise, for $m = 2, b > 2$ and for $m > 2, b > 2$ we have found no other counter-example...

Since our computer investigation, even if massive, has not been systematic (nor obviously exhaustive), we do not dare to propose a new general conjecture of the Collatz type (however we dare to offer €1 for each of the first 100 counter-examples in the cases $m = 1, b = 5, 7, 8; m = 2, b > 2; m > 2, b > 2$).

Nevertheless, motivated by some considerations about the behavior of some CA (Cellular Automata) which mimic our generalized Collatz map (for CA related to the original Collatz map see [5]), we dare to propose the following conjecture (exactly analogous to the original Collatz conjecture):

**Conjecture 2** If $m = b - 1$ the sequence (5) shall eventually reach the number 1, irrespective of which positive integer is chosen initially.

**Remark** Note that for $b = 2$ this is just the original Collatz conjecture.

We have tested this conjecture for a large number of values of $b$ and $S_0$.
We offer €1 for a counter-example, €100 for a proof.

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**References**

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