Geometry of the inversion in a finite field and partitions of PG($2^k - 1, q$) in normal rational curves

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Abstract

Let $L = \mathbb{F}_{q^n}$ be a finite field and let $F = \mathbb{F}_q$ be a subfield of $L$. Consider $L$ as a vector space over $F$ and the associated projective space that is isomorphic to PG($n - 1, q$). The properties of the projective mapping induced by $x \mapsto x^{-1}$ have been studied in [6, 7, 9, 10, 11], where it is proved that the image of any line is a normal rational curve in some subspace. In this note a more detailed geometric description is achieved. Consequences are found related to mixed partitions of the projective spaces; in particular, it is proved that for any positive integer $k$, if $q \geq 2^k - 1$, then there are partitions of PG($2^k - 1, q$) in normal rational curves of degree $2^k - 1$. For smaller $q$ the same construction gives partitions in $(q + 1)$-tuples of independent points.

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1 Introduction

Let $q$ be a power of a prime and $n$ an integer greater than one. Let $L = \mathbb{F}_{q^n}$ be a finite field and let $F = \mathbb{F}_q$ be a subfield of $L$. The projective space PG($n - 1, q$) is isomorphic to the projective space associated to the $F$-vector
space $L$. So, the points of $\text{PG}(n-1, q)$ can be represented in the form $Fx$, $x \in L^*$. Define the mapping \( j : \text{PG}(n-1, q) \rightarrow \text{PG}(n-1, q) \) by
\[
j(Fx) = Fx^{-1}. \tag{1}
\]

In section 2 it will be shown that the map (1) appears spontaneously in the investigations on embeddings of minimum dimension of the product spaces [14]. It has been already studied in [7, 6], where it is proved that if $q$ is large enough, then the image of any line is a normal rational curve of degree $m-1$, with $m$ dividing $n$. Furthermore, the lines whose image is a line (so $m = 2$ for even $n$) form a spread. Some of the results in [7, 6] can be also derived from the description of affine chains given in [9, 10]; see also [1, 11]. The properties of $j$ which are relevant to this paper are described in section 3.

Section 4 is devoted to the investigation on partitions of a finite projective space. In [3] mixed partitions of $\text{PG}(3, q)$ consisting of two lines and $q^2 - 1$ normal rational curves are found. By the above results, the images under $j$ of the line spreads of $\text{PG}(3, q)$ are mixed partitions containing different numbers of lines. There are also partitions made only by normal rational curves. A construction is described which can be extended to a partition in normal rational curves of any $\text{PG}(2^k - 1, q)$ (theorem [18]).

## 2 Projections in minimum embeddings

The study of the geometry of the inversion in a finite field is motivated by previous work [14]; it arises from one of the projections in minimum embeddings of product spaces, as will be explained in this section. We will assume that the reader is acquainted with the basic results and notation from [14], but skipping this section should not affect the understanding of the following sections. Consider the hypersurface $Q_{n-1, q}$ of $\text{PG}(2n-1, q)$ defined in [14, (4,7,8)], generalizing the semifield embeddings in $\text{PG}(3, q)$ and $\text{PG}(5, q)$ [15]. Assume that $h$ and $h'$ are integers such that $0 < h' < h \leq n-1$ and $\gcd(h - h', n) = 1 = \gcd(h', n)$, and take $k \in L$ with $N(k) = 1$. For any point $P \in S_{h,k}$ [14, (11)] precisely one subspace $S_{P,h,k} \in S_{h'}$ exists satisfying $P \in S_{P,h,k}$. The correspondence $P \mapsto S_{P,h,k}$ is a bijection [14, prop. 7]. The map $\psi_{h,h',k} : S_{h,k} \rightarrow S_{0,1}$ defined by $\psi_{h,h',k}(P) = S_{P,h,k} \cap S_{0,1}$ is a bijection and will be called the projection of indexes $h, h'$ and $k$.

In the following proposition, $\theta_s = (q^{s+1} - 1)/(q-1)$ for any integer $s \geq -1$.

**Proposition 1.** Assume that $h$ and $h'$ are integers such that $0 < h' < h \leq n-1$ and $\gcd(h - h', n) = 1 = \gcd(h', n)$, and that $k \in L$, $N(k) = 1$. Then for any
\( x \in L^* \), it holds \( \psi_{h,h',k}(F(x, kx^q)) = F(t, t) \), where
\[
t = \ell x^{q^{h' - 1} - q^{h - h'}} e, \quad \ell^{q-1} = k^{-e}, \quad e\theta_{h'-1} \equiv 1 \mod \theta_{n-1}.
\] (2)

Proof. The definition of \( \psi \) implies
\[
\ell^{q^{h' - 1}} = k^{-1} x^{q^{h'} - q^h}.
\] (3)

Since \( t \) is defined up to a non-zero factor in \( F \), it is allowed to take \((q - 1)\)-th roots in (3), whence
\[
\ell^{\theta_{h'-1}} = \ell' x^{q^{h'} - q^h},
\] (4)
where \( \ell' \) is such that \( \ell q^{-1} = k^{-1} \). The assumption \( \gcd(h', n) = 1 \) implies \( \gcd(\theta_{h'-1}, \theta_{n-1}) = 1 \), so integers \( e \) and \( \epsilon \) exist such that \( e\theta_{h'-1} = \epsilon\theta_{n-1} + 1 \). By taking the \( e(q - 1)\)-th power of (4) and setting \( \ell q^{-1} = k^{-e} \) one obtains
\[
\ell q^{-1} = \ell q^{-1} x^{q^{h'}(1 - q^{h-h'})} e.
\]
Taking once again \((q - 1)\)-th roots gives (2).

The easiest nontrivial case is \( h' = 1, h = 2 \), leading to the composition of projectivities and \( j \):

Proposition 2. Two collineations \( S_{2,k} \xrightarrow{\alpha} \PG(n - 1, q) \xrightarrow{\beta} S_{0,1} \) exist such that \( \psi_{2,1,k} = \alpha j \beta \).

Proof. Just take \( \alpha : F(x, kx^q) \mapsto Fx \) and \( \beta : Fx \mapsto F\ell(x^q, x^q) \).

3 Geometric properties

The mapping \( j \) is well-defined and \( j^2 \) is the identity map in \( \PG(n - 1, q) \).

The one-dimensional \( F \)-subspace generated by \( u \in F \) is \( Fu \), but in cases like \( F(u + v) \) the notation \( \langle u + v \rangle_F \) will be preferred in order to avoid confusion with a simple algebraic extension of \( F \).

In [8, p. 13] the \( n \)-uple embedding of an \( r \)-dimensional projective space \( \mathbb{P}^r \) over an algebraically closed field \( K \) is defined to be the map \( \rho_d : \mathbb{P}^r \to \mathbb{P}^N \) defined by
\[
\rho_d(K(x_0, \ldots, x_r)) = K(M_0(x_0, \ldots, x_r), \ldots, M_N(x_0, \ldots, x_r)),
\]
where \( N = \binom{r+d}{d} - 1 \) and \( M_0, M_1, \ldots, M_N \) are all homogeneous monomials of degree \( d \) in \( x_0, x_1, \ldots, x_r \).

Here normal rational curves (NRCs for short) are defined in partial accord to [12, Chapter 21]. It should be noted that they are sets of points. In the following, \( F \) is the finite field \( \mathbb{F}_q \). A rational curve \( \mathcal{C}_n \) of order \( n \) in \( \text{PG}(r, q) \) is the set of points

\[ \{ F(g_0(t_0, t_1), \ldots, g_r(t_0, t_1)) \mid (t_0, t_1) \in F^2 \} \]

where each \( g_i \) is a binary form of degree \( n \) and a highest common factor of \( g_0, g_1, \ldots, g_r \) is 1. Also \( \mathcal{C}_n \) is normal if it is not the projection of a rational curve \( \mathcal{C}_n' \) in \( \text{PG}(r + 1, q) \), where \( \mathcal{C}_n' \) is not contained in any \( r \)-subspace of \( \text{PG}(r + 1, q) \).

Some issues arise from the fact that the order of a NRC is not unique, as is stated in the next propositions.

**Proposition 3.** The image of the \( r \)-uple embedding of \( \text{PG}(1, q) \):

\[ \mathcal{C} = \{ F(t_0^r, t_0^{-1}t_1, \ldots, t_1^r) \mid (t_0, t_1) \in F^2 \setminus \{(0,0)\} \} \]  (5)

is a NRC of order \( r \) in \( \text{PG}(r, q) \).

**Proposition 4.** If \( q < r \), then the curve \( \mathcal{C} \) in (5) is a NRC of order \( q \) in a \( q \)-subspace of \( \text{PG}(r, q) \).

In the following, for any NRC the smallest order will be chosen. Such order equals the dimension of its span [12, Theorem 21.1.1].

**Theorem 5.** Let \( \ell \) be the line through two distinct points \( Fa, Fb \) in \( \text{PG}(n - 1, q) \). Then \( j(\ell) \) is the image of an \((m - 1)\)-uple embedding with \( m \) equal to the extension field degree of \( F(ab^{-1}) \), \( m = [F(ab^{-1}) : F] \).

**Proof.** The assertion can be deduced from [11], as follows. The projective space \( \text{PG}(n - 1, q) \) can be considered as the hyperplane at infinity, say \( H_\infty \), of the affine space \( L \) with line set \( \{Fc + d \mid c, d \in L, c \neq 0\} \). The point at infinity of the line \( Fc + d \) is \( Fc \). The lines joining the origin and the points of \( \ell' = Fa + b \) meet \( H_\infty \) exactly in the points of \( \ell \setminus \{Fa\} \). By [11] prop. 3.6.3, theorem 3.6.5, \( j(\ell') \cup \{0\} \) is a NRC without points at infinity, say \( \mathcal{V} \). Furthermore, \( j(Fa) \) is the tangent line in 0 to \( \mathcal{V} \). The secant lines to \( \mathcal{V} \) through 0 and the tangent line \( j(Fa) \) meet \( H_\infty \) in the image of an \((m - 1)\)-uple embedding.

**Proposition 6.** If \( a, b \in L^* \) and \([F(ab^{-1}) : F] = 2\), then the restriction of \( j \) to the line containing \( Fa \) and \( Fb \) is a projectivity.
Proof. Let $\omega = ab^{-1}$ with $\omega^2 = \alpha \omega + \beta$ and $\alpha, \beta \in F$. The assertion follows from $\langle (t + u\omega)^{-1}\rangle_F = \langle t + \alpha u - u\omega\rangle_F$ for any $t, u \in F$. \qed

Now assume that $n$ is even. A unique subfield $C \cong \mathbb{F}_{q^2}$ exists in $L$. Every $x \in L^*$ is associated with the line in $\text{PG}(n-1,q)$ arising from the two-dimensional vector subspace $Cx$. The symbol $Cx$ will denote the line itself. The Desarguesian line spread of $\text{PG}(n-1,q)$ is $D = \{Cx \mid x \in L^*\}$.

Proposition 7. The restriction of $j$ to any line of $D$ is a projectivity with a line of $D$.

Proof. A line $\ell$ of $D$ contains two points of type $Fa$ and $Fb$ with $ab^{-1} \in C \setminus F$. So by prop. $6$, $j$ maps projectively $\ell$ into the line containing the points $Fa^{-1}$ and $Fb^{-1}$. The condition $ab^{-1} \in C$ implies that $j(\ell) \in D$. \qed

The following props. $8$ and $10$ as well as theorem $9$ can be found in [6]:

Proposition 8. Let $n$ be even. If $n \equiv 0 \mod 4$ and $q$ is odd, then the lines of $D$ which are fixed by $j$ are precisely two; otherwise there is a unique line which is fixed by $j$.

Proof. A line $Cx$ is fixed by $j$ if, and only if, $Cx = Cx^{-1}$, that is $x^2 \in C$. An $x \in L \setminus C$ such that $x^2 \in C$ exists if, and only if, $q$ is odd and $n \equiv 0 \mod 4$. Indeed, the divisibility of $n$ by 4 comes from the fact that $[C : F] = [C(x) : C] = 2$, so that $[C(x) : F] = 4$. Moreover, $q$ being odd is clear. If an $x \in L \setminus C$ such that $x^2 \in C$ exists, there are at least two lines that are fixed by $j$, precisely $C1$ and $Cx$. Otherwise only the line $C1$ is fixed by $j$.

Now assume that $q$ is odd and $n \equiv 0 \mod 4$. Let $Cy$ and $Cz$ be any two lines fixed by $j$ and distinct from $C1$. It holds $y, z \in L \setminus C$ and $y^2, z^2 \in C$. Then $y^2, z^2$ are non-squares in $C$, and $y^2z^{-2}$ is a square in $C$. This implies $yz^{-1} \in C$ and finally $Cy = Cz$. So in this case there are at most two lines which are fixed by $j$. \qed

It should be noted that the image of the line through two distinct points $Fa$ and $Fb$ is a line if, and only if, $F(ab^{-1}) = C$, as a consequence of the uniqueness of a subfield of order $q^2$ in $L$. This implies:

Theorem 9. The lines of $\text{PG}(n-1,q)$ whose image under $j$ are lines are precisely those in $D$.

The previous proposition can be partly generalized.

Proposition 10. Assume $n$ is any integer and let $m > 1$ be a divisor of $n$ and such that $m-1 \geq q$. Then the $(m-1)$-subspaces of $\text{PG}(n-1,q)$ whose images under $j$ are $(m-1)$-subspaces form a Desarguesian spread of $\text{PG}(n-1,q)$. 

\begin{align*}
5
\end{align*}
4 Partitions in normal rational curves

4.1 Three-dimensional case

The term mixed partition in the literature refers to distinct geometric concepts. Here a mixed partition of type \( r \) in \( \text{PG}(3,q) \), \( 0 \leq r \leq q^2 \), is a partition of \( \text{PG}(3,q) \) in \( r \) lines and \( q^2 + 1 - r \) NRCs or order three. In \([3]\) mixed partitions of type 2 are constructed and investigated. The following is immediate by means of theorem \([9]\):

Proposition 11. If a line spread \( \mathcal{F} \) in \( \text{PG}(3,q) \) exists having exactly \( r \) lines in common with a Desarguesian spread \( \mathcal{D} \), then there is a mixed partition of type \( r \).

By prop. \([1]\) in order to construct a partition of \( \text{PG}(3,q) \) in NRCs it is enough to find a line spread sharing no line with a given Desarguesian spread.

Proposition 12. The projective space \( \text{PG}(3,q) \) contains mixed partitions of type 0, for any \( q \).

Proof. Under the Klein correspondence, the spread \( \mathcal{D} \) is associated with the intersection of \( Q^+(5,q) \) with a solid \( S \), and \( S \cap Q^+(5,q) \) is an elliptic quadric \( E \). Take any line \( \lambda \) in \( S \) such that \( \lambda \cap E = \emptyset \). The solid \( \lambda^\perp \) intersects \( E \) in two points, corresponding to two lines \( \ell_1, \ell_2 \in \mathcal{D} \). The quadric \( \lambda^\perp \cap Q^+(5,q) \) is associated with a Desarguesian spread \( \mathcal{F}_1 \), and \( \mathcal{D} \cap \mathcal{F}_1 = \{\ell_1, \ell_2\} \). By replacing in \( \mathcal{F}_1 \) a regulus containing \( \ell_1 \) and \( \ell_2 \) with the opposite one, a new spread \( \mathcal{F} \) is obtained having no common line with \( \mathcal{D} \). So, the assumptions of prop. \([1]\) with \( r = 0 \) are satisfied. \( \square \)

4.2 Generalization

Now a generalization of prop. \([12]\) to any odd-dimensional finite projective space is obtained. In this subsection, \( n > 1 \) is an integer, and

\[
L = \mathbb{F}_{q^{2n}}; \quad M = \mathbb{F}_{q^n}; \quad C = \mathbb{F}_{q^2}; \quad F = \mathbb{F}_q; \quad F \subset M \subset L; \quad F \subset C \subset L.
\]

Any of the fields above is considered as a vector space over \( F \). Let \( i \in L \setminus M \), so any \( z \in L \) can be described uniquely in the form \( z = a + ib \) with \( a, b \in M \).

The points of \( \text{PG}(2n-1,q) \) are \( Fz \) for \( z \in L^* \).

For every \( a \in M \) define \( S_a = \{Fc(a+i) \mid c \in M^*\} \), furthermore let \( S_\infty = \{Fc \mid c \in M^*\} \), \( M^+ = M \cup \{\infty\} \), \( S = \{S_a \mid a \in M^+\} \).
Proposition 13.  (i) For any $a \in M^+$, $S_a$ is an $(n-1)$-subspace of $\text{PG}(2n-1,q)$.

(ii) If $a, a' \in M^+$, $a \neq a'$, then $S_a \cap S_{a'} = \emptyset$.

(iii) The collection $\mathcal{S}$ is the standard Desarguesian spread of $(n-1)$-subspaces of $\text{PG}(2n-1,q)$.

Proof. (i) The union of the one-dimensional subspaces belonging to $S_a$ is an $n$-subspace of $L$.

(ii) If $a \neq \infty$, then $\langle a + i \rangle_M \cap M = \{0\}$ implies $S_a \cap S_\infty = \emptyset$. Now assume $a \neq \infty \neq a'$. If $Fc(a+i) = Fc'(a'+i)$ for $c, c' \in M^*$, then $c(a+i) = \rho c'(a'+i)$ for some $\rho \in F$, and this implies $a = a'$.

(iii) First, the union of the elements of any $S_a$, $a \in M^+$, is of type $M_z$, so it is an element of the standard Desarguesian spread of $(n-1)$-subspaces of $\text{PG}(2n-1,q)$. Conversely, given an $Mu$ with $u = s + it \neq 0$, $s, t \in M$, it holds $Mu = \langle st^{-1} + i \rangle_M$ and the associated projective subspace is $S_{st^{-1}}$, possibly $S_\infty$.

The reader will immediately verify the next result:

Proposition 14.  (i) The map $\hat{\varphi} : L \to L$ defined by $\hat{\varphi}(a+ib) = a^q + ib$ for $a, b \in M$ is an automorphism of the $F$-vector space $L$.

(ii) Let $\varphi : \text{PG}(2n-1,q) \to \text{PG}(2n-1,q)$ be the projectivity associated with $\hat{\varphi}$. Then

$$S^\varphi_a = \{ (c^qa^q + ci) \} \cap M = \{ c \in M^* \} \text{ for } a \in M, \quad S^\varphi_\infty = S_\infty, \quad S^\varphi_0 = S_0.$$ 

Denote by $S^\varphi = \{ S^\varphi_a \mid a \in M^+ \}$ the image of the spread $\mathcal{S}$ under $\varphi$.

Proposition 15. For any $a, b \in M^*$, the intersection $S_a \cap S^\varphi_b$ contains at most one point.

Proof. The common points to both subspaces are of type $\langle c(a+i) \rangle_F = \langle d^q b^q + di \rangle_F$ for some $c, d \in M^*$. A $\rho \in F^*$ exists such that $ca = \rho d^q b^q$ and $c = \rho d$. This implies $d^{q-1} = ab^{-q}$. This equation has either zero or exactly $q-1$ roots in the form $\sigma d$, $\sigma \in F^*$.

A scattered subset with respect to a spread is a set of points intersecting any spread element in at most one point.

Theorem 16. If $n$ is even, then a line spread $\mathcal{F}$ of $\text{PG}(2n-1,q)$ exists such that any line of $\mathcal{F}$ is scattered with respect to the Desarguesian spread $\mathcal{S}$. 

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Proof. Let \( R \) be a regulus of \((n - 1)\)-subspaces such that \( S_0, S_\infty \in R \), and \( R \subset S^\circ \); such a regulus exists because \( S^\circ \) is a Desarguesian spread. Let \( M^- = \{ a \in M \mid S_a^\circ \notin R \} \). For any \( a \in M^- \) choose a line spread \( F_a \) of the \((n - 1)\)-subspace \( S_a^\circ \). Let \( F' \) be the set of transversal lines of the regulus \( R \). Define
\[
F = F' \cup \bigcup_{a \in M^-} F_a.
\]
By construction, \( F \) is a line spread of \( \text{PG}(2n - 1, q) \). Let \( \ell \) be any line of \( F \). If \( \ell \in F_a \), \( a \in M^- \), then \( \ell \) is contained in a scattered subspace with respect to \( S \) (prop. 15), so it is also scattered. If \( \ell \in F' \), then it intersects at least two elements \( S_0 \) and \( S_\infty \) of \( S \) and this implies that \( \ell \) is scattered.

In the previous theorem, \( n \) has to be even here in order to guarantee that each \((n - 1)\)-subspace admits a line spread. Now we further assume \( n = 2^{k-1} \), \( k > 1 \) an integer. So,
\[
L = \mathbb{F}_{q^{2^k}} \supset M = \mathbb{F}_{q^{2^{k-1}}} \supset C = \mathbb{F}_{q^2} \supset F = \mathbb{F}_q.
\]

**Theorem 17.** Let \( \text{PG}(2^k - 1, q) = \{ Fz \mid z \in L^* \} \). Then for any line \( \ell \) of \( \text{PG}(2^k - 1, q) \), \( j(\ell) \) is the image of a \((2^k - 1)\)-uple embedding if, and only if, \( \ell \) is scattered with respect to the Desarguesian spread \( S \).

Proof. Given a line \( \ell \) through two distinct points \( Fz_1, Fz_2 \), then \( \ell \) is scattered if, and only if, \( z_1 z_2^{-1} \notin M \). On the other hand, if \( z_1 z_2^{-1} \in M \), then, by theorem 5, \( j(\ell) \) is a NRC of degree at most \( 2^{k-1} - 1 \), whereas if \( z_1 z_2^{-1} \notin M \), since every proper subfield of \( L \) is contained in \( M \), then the extension \( F(z_1 z_2^{-1}) \) is equal to \( L \), and this implies once again by theorem 5 that \( j(\ell) \) is the image of a \((2^k - 1)\)-uple embedding.

**Theorem 18.** For any \( k > 1 \) and \( q \geq 2^k - 1 \), the projective space \( \text{PG}(2^k - 1, q) \) has a partition in normal rational curves of degree \( 2^k - 1 \). For \( q < 2^k - 1 \) the projective space has a partition in \((q + 1)\)-tuples of independent points.

Proof. This is a straightforward consequence of theorems 17 and 18.

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