COMPUTING VANISHING IDEALS FOR TORIC CODES

MESUT ŞAHİN

Abstract. Motivated by applications to the theory of error-correcting codes, we give an algorithmic method for computing a generating set for the ideal generated by \( \beta \)-graded polynomials vanishing on a subset of a simplicial complete toric variety \( X \) over a finite field \( \mathbb{F}_q \), parameterized by rational functions, where \( \beta \) is a \( d \times r \) matrix whose columns generate a subsemigroup \( \mathbb{N}^\beta \) of \( \mathbb{N}^d \).

We also give a method for computing the vanishing ideal of the set of \( \mathbb{F}_q \)-rational points of \( X \). When \( \beta = [w_1 \cdots w_r] \) is a row matrix corresponding to a numerical semigroup \( \mathbb{N}^\beta = \langle w_1, \ldots, w_r \rangle \), \( X \) is a weighted projective space and generators of its vanishing ideal is given using generators of defining (toric) ideals of numerical semigroup rings corresponding to semigroups generated by subsets of \( \{w_1, \ldots, w_r\} \).

1. Introduction

Let \( \beta = [\beta_1 \cdots \beta_r] \) be a \( d \times r \) matrix of rank \( d \) with non-negative integer entries and \( n = r - d > 0 \). The polynomial ring \( S = \mathbb{F}[x_1, \ldots, x_r] \) over a field \( \mathbb{F} \) is made into a \( \mathbb{Z}^d \)-graded ring by letting \( \deg_\beta(x_j) := \beta_j \in \mathbb{N}^d \), for \( j \in [r] := \{1, \ldots, r\} \). Thus, \( S = \bigoplus_{\alpha \in \mathbb{Z}^d} S_\alpha \), where \( S_\alpha \) is the finite-dimensional vector space spanned by the monomials \( x^\alpha := x_1^{a_1} \cdots x_r^{a_r} \) having degree \( \alpha = a_1 \beta_1 + \cdots + a_r \beta_r \) in the affine semigroup \( \mathbb{N}^\beta \) by [25, Theorem 8.6]. This leads to the following short exact sequence

\[
0 \longrightarrow \mathbb{Z}^n \xrightarrow{\phi} \mathbb{Z}^r \xrightarrow{\beta} \mathbb{Z}^d \longrightarrow 0 ,
\]

where \( \phi \) denotes a matrix such that \( \text{Im}(\phi) = \text{Ker}(\beta) \). Applying \( \text{Hom}(-, \mathbb{K}^*) \) for a field \( \mathbb{K} \), we get the dual short exact sequence

\[
1 \longrightarrow (\mathbb{K}^*)^d \xrightarrow{i} (\mathbb{K}^*)^r \xrightarrow{\pi}(\mathbb{K}^*)^n \longrightarrow 1 ,
\]

where \( \pi : (t_1, \ldots, t_r) \mapsto (t^{u_1}, \ldots, t^{u_n}) \), with \( u_1, \ldots, u_n \) being the columns of \( \phi \). Denote by \( G = \text{Ker}(\pi) \cong (\mathbb{K}^*)^d \). Then, \( G \) is an algebraic subgroup of \( (\mathbb{K}^*)^r \) acting on the affine space \( \mathbb{A}^r \) over \( \mathbb{K} \) by coordinate-wise multiplication. We denote by \( \mathbb{A}^r_G \) the set \( \mathbb{K}^r/G \) of \( G \)-orbits. More generally, \( Y_G \) denotes the set \( Y/G \) of \( G \)-orbits of elements in \( Y \subseteq \mathbb{A}^r \). In general, \( \mathbb{A}^r_G \) is not necessarily a variety, but Geometric Invariant Theory (GIT, for short) says removing some bad orbits we can get nice quotient spaces which are varieties. Toric varieties are such important nice quotient spaces lying at the crossroad of combinatorics, commutative algebra and algebraic geometry with numerous applications to areas such as biology, chemistry, coding theory, physics and statistics.

2020 Mathematics Subject Classification. Primary 14M25; 14G05; Secondary 94B27; 11T71.

The author is supported by TÜBİTAK Project No:119F177.
The algebraic setup above arise often within toric geometry which we briefly explain now. When $X$ is an $n$-dimensional simplicial complete toric variety over a field, the first map in equation (1.1) is just multiplication by the matrix $\phi$ whose rows are the primitive generators $v_1, \ldots, v_r \in \mathbb{Z}^n$ of the rays in the corresponding fan. Under suitable conditions, satisfied by smooth varieties for instance, the variety $X$ can be represented as a GIT quotient, i.e. $X \cong (\mathbb{K}^r \setminus V(B))/G$, where $B$ is a monomial ideal of $S$ determined by the cones in the fan, see Section 4 for details.

In applications to coding theory, we work with a finite field $\mathbb{F} = \mathbb{F}_q$ together with an algebraic closure $\mathbb{K} = \overline{\mathbb{F}}_q$ and identify $\mathbb{F}_q$-rational points $\mathbb{A}^r_\mathbb{F}_q$ with $\mathbb{F}_q^r/G$, where $G = \{ t \in (\mathbb{F}_q)^* : t^{u_{i1}} = \cdots = t^{u_{rn}} = 1 \}$ is the algebraic group determined in equation (1.2) for $\mathbb{K}$. Therefore, $\mathbb{F}_q$-rational points $X(\mathbb{F}_q)$ of $X$ is identified with the set of orbits $(\mathbb{F}_q^r \setminus V(B))/G = \mathbb{A}^r_\mathbb{F}_q \setminus V_G(B)$.

Toric codes, considered for the first time by Hansen [13], can be obtained by evaluating all homogeneous polynomials in the space $S$ determined in equation (1.2) for $\mathbb{K}$. Since the kernel of the map $ev_\alpha$ is the row space of the matrix, for sufficiently large $q$. Some record breaking examples are found replacing the vector space by its subspaces, see [5] and references therein. The latter corresponds to deleting rows from a generating matrix of the toric code, which is investigated by Little [23] using the theory of finite geometries. See also Hirschfeld [12] for another example relating finite geometry and vanishing ideals.

One can also add/delete columns to/from a generating matrix in order to get a better code, which correspond to considering a proper subset/superset of $\mathbb{F}_q$-rational points of the dense torus $T_X \subset X$. They are studied intensively from different points of view, see [19, 21, 33, 39, 34, 40, 18, 6, 22, 38, 7, 36]. A row of a generator matrix of the code is obtained by evaluating a monomial in a basis of $\mathbb{F}_q$-rational points so that the code is the row space of the matrix, for sufficiently large $q$. Some record breaking examples are found replacing the vector space $S_\alpha$ by its subspaces, see [5] and references therein. The latter corresponds to deleting rows from a generating matrix of the toric code, which is investigated by Little [23] using the theory of finite geometries. See also Hirschfeld [12] for another example relating finite geometry and vanishing ideals.

In this regard, Nardi offered to extend the length of a toric code by evaluating $\mathbb{F}_q$-rational points $X(\mathbb{F}_q)$ in [26] and [27]. There is yet another extension of classical toric codes, which we introduce now. As in the toric case, we evaluate polynomial functions from $S_\alpha := \mathbb{F}_q[x_1, \ldots, x_r]_\alpha$ at the $\mathbb{F}_q$-rational points $[P_1], \ldots, [P_N]$ of a subset $Y_G \subseteq \mathbb{A}^r_\mathbb{F}_q$ for $\mathbb{F}_q$-linear map

$$ev_{Y_G} : S_\alpha \to \mathbb{F}_q^N, \quad F \mapsto (F(P_1), \ldots, F(P_N)).$$

The image $ev_{Y_G}(S_\alpha) \subseteq \mathbb{F}_q^N$ denoted by $C_{\alpha,Y_G}$ is called an evaluation code on orbits. The main three parameters $[N, K, \delta]$ of these codes are the length $N$ of $C_{\alpha,Y_G}$ which is the size $|Y_G|$, the dimension $K = \dim_{\mathbb{F}_q}(C_{\alpha,Y_G})$ of the image as a subspace of $\mathbb{F}_q^N$, and the minimum distance $\delta$ which is the smallest weight among all code words $c \in C_{\alpha,Y_G} \setminus \{0\}$, where the weight of $c$ is the number of non-zero components. Since the kernel of the map $ev_{Y_G}$ is nothing but $I_{\alpha}(Y_G) := I(Y_G) \cap S_\alpha$, the code $C_{\alpha,Y_G}$ is isomorphic to $S_\alpha/I_{\alpha}(Y_G) = (S/I(Y_G))_\alpha$. Hence, computing a minimal generating set for the vanishing ideal $I(Y_G)$ is of central importance. When $X \subset Y_G \subseteq \mathbb{A}^r_\mathbb{F}_q$, the new codes are lengthier and one has the chance to choose the subset $Y_G$ so that the other parameters improves as well, see Example 5.5. As pointed out in [27], as the length increases one can build secret sharing schemes based on these codes with more participants, see [14].

In the present paper, we give an algorithmic method for computing the vanishing ideal $I(Y_G)$ of a subset $Y_G$ obtained as an image of a map from $\mathbb{F}_q^r$ given by rational functions, see Theorem 2.2. This enables us to see that $I(\mathbb{A}^r_\mathbb{F}_q)$ has a minimal generating set consisting of binomials and to compute them algorithmically, in
which is an extension of the vanishing ideal of the particular. We also give another more conceptual method to list binomial generators for $I(\mathcal{A}_G^\infty(\mathbb{F}_q))$ using the cell decomposition of the affine space $\mathcal{A}_G$, see Theorem 3.7. The vanishing ideal of the $\mathbb{F}_q$-rational points of the toric variety $X$ can be obtained as a colon ideal of $I(\mathcal{A}_G^\infty(\mathbb{F}_q))$ with respect to the monomial ideal $B$, see Theorem 4.1. As applications, we give three binomials generating $I(\mathcal{A}_G^\infty(\mathbb{F}_q))$ and thereby obtain a binomial and a polynomial with 4 terms generating $I(X(\mathbb{F}_q))$ minimally, where $X = H_t$ with $t > 1$ is the Hirzebruch surface, see Theorem 5.1 and Theorem 5.3. It is known that $I(Y_G)$ is a binomial ideal when $Y_G$ is a submonoid of $\mathcal{A}_G^\infty$, [35, Proposition 2.6] whereas $I(X(\mathbb{F}_q))$ can still be binomial even if $X$ is not a monoid, see Theorem 5.7. The last theorem generalizes to some weighted projective spaces the fact that the ideal $I(\mathbb{P}^n(\mathbb{F}_q))$ has binomial generators given explicitly by Mercier and Rolland [24]. It is worth pointing out that these binomials form a Groebner basis as shown by Beelen, Datta and Ghorpade [4] which is used to obtain a footprint bound for the minimum distance of the corresponding code. Binomial ideals appear as vanishing ideals in many works, see e.g. [42, 29, 28, 30, 2] and prove useful in studying basic parameters of the related codes.

2. IDEALS OF RATIONAL PARAMETERIZATIONS

In this section we give an algorithmic method so as to find out a generating set for the vanishing ideal $I(Y_G)$ of a subset $Y_G$ obtained as an image of a map from $\mathbb{F}^s \setminus V(g_1 \cdot \cdot \cdot g_r)$ given by rational functions $f_i/g_i$, where $f_i, g_i \in \mathbb{F}[y_1, \ldots, y_s]$ are polynomials, for all $i \in [s]$, and $V(J) = \{ \alpha \in \mathbb{F}^s : f(\alpha) = 0, \text{for all } f \in J \}$ for $J \subseteq \mathbb{F}[y_1, \ldots, y_s]$. We use $I(Y)$ to denote the set of all polynomials vanishing on the subset $Y$, which is an ideal called the vanishing ideal of $Y$. This differs from the $\beta$-graded vanishing ideal $I(Y_G)$ of $Y_G := Y/G$ that is generated by homogeneous (or $\beta$-graded) polynomials vanishing on $Y$. Notice that if $F \in S_\alpha$ then we have

$$F(g \cdot P) = g^\alpha F(P) = 0 \text{ if and only if } F(P) = 0, \text{ for any } g \in G.$$

**Remark 2.1.** Recall that $G$ is defined over any field $\mathbb{K}$ and in applications to coding theory, we take $F = \mathbb{F}_q$ and $\mathbb{K} = \mathbb{F}_q$. But the vanishing of a polynomial at a point $P$ is independent of the group $G$ by Equation 2.1. Therefore, the homogeneous vanishing ideal $I(Y_G) \subseteq \mathbb{F}[x_1, \ldots, x_r]$ would be the same even if $\mathbb{K} = \mathbb{F}_q$.

We are ready to prove our first main result. The statement and the proof follows the spirit of many papers including [32, 10, 41, 3].

**Theorem 2.2.** Let $R = \mathbb{F}[x_1, \ldots, x_r, y_1, \ldots, z_1, \ldots, z_d, w]$ be a polynomial ring which is an extension of $S$. Let

$$Y(f, g) = \left\{ \left( \frac{f_1(t)}{g_1(t)}, \ldots, \frac{f_r(t)}{g_r(t)} \right) : t \in \mathbb{F}^s \setminus V(g_1 \cdot \cdot \cdot g_r) \right\} \subseteq \mathbb{F}^r.$$

The vanishing ideal of $Y_G$ is $I(Y_G(f, g)) = J \cap S$ where

$$J = \{(x_2g_2 - f_2(z))_{j=1}^r \cup \{g_i = 0 \}_{i=1}^s, w_1 \cdots g_r - 1, \text{ if } F = \mathbb{F}_q \}
$$

$$J = \{(x_2g_2 - f_2(z))_{j=1}^r, w_1 \cdots g_r - 1, \text{ if } F \text{ is infinite.} \}
$$

Similarly, for the subset

$$Y^*(f, g) = \left\{ \left( \frac{f_1(t)}{g_1(t)}, \ldots, \frac{f_r(t)}{g_r(t)} \right) : t \in (\mathbb{F}^*)^s \setminus V(g_1 \cdot \cdot \cdot g_r) \right\} \subseteq \mathbb{F}^r \text{ and}$$
the vanishing ideal of \( Y^*_G \) is \( I(Y^*_G(f,g)) = J^* \cap S \) where
\[
J^* = \{ x_jg_j - f_j z_j^\beta_j \}_{j=1}^r \cup \{ y_i^{q-1} - 1 \}_{i=1}^s, w y_i \cdot g_i - 1 \}, \text{ if } \mathbb{F} = \mathbb{F}_q
\]
\[
J^* = \{ x_jg_j - f_j z_j^\beta_j \}_{j=1}^r, w y_i \cdot g_i - 1 \}, \text{ if } \mathbb{F} \text{ is infinite.}
\]

Proof. First, we show the inclusion \( I(Y_G(f,g)) \subset J \cap S \). Since \( I(Y_G(f,g)) \) is a homogeneous ideal, it is generated by homogeneous polynomials. Pick any generator
\[
F = \sum_{i=1}^k c_i x_i^{m_i} \text{ of degree } \alpha = \text{deg}(x_i^{m_i}) = \sum_{j=1}^r m_i \beta_j.
\]
We substitute the following
\[
x_j = \left( x_j - \frac{f_j(y_1, \ldots, y_s)z_j^\beta_j}{g_j(y_1, \ldots, y_s)} \right) + \frac{f_j(y_1, \ldots, y_s)z_j^\beta_j}{g_j(y_1, \ldots, y_s)}
\]
in \( F \) and use binomial theorem in order to expand powers of variables \( x_j \) to get:
\[
F(x_1, \ldots, x_r) = \sum_{j=1}^r F_j \left( \frac{f_j z_j^{\beta_j}}{g_j}, \ldots, \frac{g_j z_j^{\beta_j}}{g_j} \right) (x_j g_j - f_j z_j^\beta_j) + z^\alpha F \left( \frac{f_1}{g_1}, \ldots, \frac{f_r}{g_r} \right),
\]
for some polynomials \( F_1, \ldots, F_r \in S \). We multiply \( F \) by a sufficiently large power \( m \) of \( h := g_1 \cdot \ldots \cdot g_r \) to clear all the denominators and to obtain
\[
F h^m = h \sum_{j=1}^r G_j (x_j g_j - f_j z_j^\beta_j) + z^\alpha G,
\]
where \( G \in \mathbb{F}_q[y_1, \ldots, y_s] \) and \( G_j \in R \). We claim that \( G(t) = 0 \) for all \( t = (t_1, \ldots, t_s) \in (\mathbb{F}_q)^s \).

If \( t \in V(g_1 \cdot \ldots \cdot g_r) \) then \( h(t) = 0 \) and so equation (2.2) with \( z_1 = \cdots = z_d = 1 \) gives
\[
0 = F \cdot 0 = G(t_1, \ldots, t_s).
\]
If \( t \notin V(g_1 \cdot \ldots \cdot g_r) \) then using \( F \in I(Y_G(f,g)) \) and substituting \( y_i = t_i, x_j = \frac{f_j(t)}{g_j(t)} \), and \( z_k = 1 \) for all \( i, j, k \), in equation (2.2) gives
\[
0 = 0 h^m = h \sum_{j=1}^r G_j 0 + G(t_1, \ldots, t_s).
\]
Hence, \( G \) vanishes on all of \( \mathbb{F}_q^s \) as claimed and so \( G \in I(\mathbb{F}_q^s) \). Thus, \( G = \sum_{i=1}^s H_i(y_i^q - y_i) \) for some polynomials \( H_i \in \mathbb{F}_q[y_1, \ldots, y_s] \) by [17, p.136-137]. Therefore, we have the following equality for some \( G_j', H_i' \in R \) by equation (2.2)
\[
F h^m = \sum_{j=1}^r G_j' (x_j g_j - f_j z_j^\beta_j) + \sum_{i=1}^s H_i'(y_i^q - y_i).
\]
Multiplying this equation (2.3) by \( w^m \) yields
\[
F(hw)^m = \sum_{j=1}^r w^m G_j' (x_j g_j - f_j z_j^\beta_j) + w^m \sum_{i=1}^s H_i'(y_i^q - y_i).
\]
As \( F(hw)^m = F([hw - 1 + 1]^m = F[H(w - 1) + 1] = FH(hw - 1) + F \) for some \( H \in R \) by binomial theorem, it follows that
\[
F = \sum_{j=1}^r w^m G_j' (x_j g_j - f_j z_j^\beta_j) + w^m \sum_{i=1}^s H_i'(y_i^q - y_i) - FH(hw - 1)
\]
which gives rise to the inclusion $I(Y_G(f,g)) \subset J \cap S$.

So as to obtain the opposite inclusion, it suffices to reveal that $J \cap S$ is an $\mathbb{N}^{\beta}$-graded ideal whose elements vanish on $Y_G(f,g)$. To complete this, we take $F \in J \cap S$ and write

$$
(2.4) \quad F = \sum_{j=1}^{r} G_j (x_j g_j - f_j z^{\beta_j}) + \sum_{i=1}^{s} H_i (y_i q - y_i) + H(g_1 \cdots g_r w - 1),
$$

for some polynomials $G_1, \ldots, G_r, H_1, \ldots, H_s, H$ in $R$.

By inserting $x_j = f_j(t)/g_j(t)$ and $y_i = t_i \in \mathbb{F}_q$ and $z_k = 1$ in equation (2.4), for all $i, j, k$ and $w = g_1(t) \cdots g_r(t)$, we see $F$ vanishes on $Y_G(f,g)$.

Let us write $F = F_{a_1} + \cdots + F_{a_t}$ as a sum of its graded components. By inserting $x_j = x_j t^{\beta_j}$ and $z_i = z_i t_i$ in equation (2.4), for all $i \in [d]$ and $j \in [r]$, and regarding $t_1, \ldots, t_d$ as new variables we get

$$
(2.5) \quad \sum_{j=1}^{r} G_j (x_j t^{\beta_j} g_j - f_j z^{\beta_j} t^{\beta_j}) + \sum_{i=1}^{s} H_i (y_i q - y_i) + H(h w - 1),
$$

for some polynomials $G_1, \ldots, G_r, H_1, \ldots, H_s, H$ in $R[t_1, \ldots, t_d]$ and $h = g_1 \cdots g_r$.

The ideal $J$ in $R[t_1, \ldots, t_d]$ generated by the following polynomials

$$
\{ x_j t^{\beta_j} g_j - f_j z^{\beta_j} t^{\beta_j}, y_i q - y_i, g_1 \cdots g_r w - 1 : i \in [s], j \in [r] \}
$$

is graded with respect to the grading where $deg (t_k) = e_k \in \mathbb{Z}^d$, for all $k \in [d]$, and degree of the rest of the variables are zero. Therefore, $t^{a_1} F_{a_1}, \ldots, t^{a_t} F_{a_t}$ belong to $J$. Setting $t_i = 1$ in equation (2.5) for all $i \in [d]$ yields $F_{a_1}, \ldots, F_{a_t} \in J \cap S$. Consequently, $J \cap S \subset I(Y_G(f,g))$. The case where $F$ is infinite follows from the fact that $I(\mathbb{F}_q^s) = 0$ in which case $H_i = H_i' = 0$ above. The last part follows similarly since $I(\mathbb{F}_q^s)^{\ast} = \langle y_i^{q-1} - 1 : i \in [s] \rangle$.

**Remark 2.3.** When $g_j = 1$ for all $j \in [r]$, we can get rid of the polynomial $g_1 \cdots g_r w - 1$ in the generating set of the ideals $J$ and $J^{\ast}$.

**Remark 2.4.** Taking $f_j = y_j^{q_1}$ and $g_j = y_j^{q_r}$ for all $j \in [r]$, it is possible to obtain the set $Y_Q^\ast(f,g) = Y_Q$ for a matrix $Q = [q_1 \cdots q_r]$. The vanishing ideal of $Y_Q$ is proven in [32, 35] to be some special binomial ideal known as a lattice ideal, i.e., it is of the form $I_L := \langle x^m^+ - x^m^- : m = m^+ + m^- \in L \rangle$ for a lattice (finitely generated abelian group) $L$, where $m^+$ and $m^-$ record positive and negative components of $m$.

Binomial ideals play a central role at the crossroad of combinatorics, commutative algebra, convex and algebraic geometry, see the recent book [16] by Herzog, Hibi and Ohsugi for a thorough introduction to their theory and applications. It is an emerging hot topic relating as diverse areas as commutative algebra, graph theory, coding theory and statistics. They have many interesting properties discovered starting from the seminal work [11] by Eisenbud and Sturmfels, and their decompositions are studied further by other authors, see e.g. [31] and [37]. There is a Macaulay 2 package [20] for their binomial primary decomposition as well.

**Corollary 2.5.** The ideal $I(\mathcal{A}_G^\ast(\mathbb{F}_q))$ is binomial.
Proof: Taking $s = r$, $f_i = y_i$, and $g_i = 1$ for all $i \in [r]$, we obtain $Y_G(f, g) = A_r^e(F_q)$. Then, by Theorem 2.2 and Remark 2.3, we have $I(Y_G(f, g)) = J \cap S$ where
\[ J = \langle \{x_j - y_jz^3\}_{j=1}^r \cup \{y^q_i - y_i\}_{i=1}^r \rangle \text{ if } F = F_q. \]

As $J$ is binomial in this case so is its elimination ideal $I(Y_G(f, g)) = J \cap S$. \square

Indeed, $A_r^e(F_q)$ is a monoid under coordinatewise multiplication with identity element $(1, \ldots, 1)$. The vanishing ideal $I(A_r^e(F_q))$ being binomial follows also from the following result:

**Proposition 2.6.** [35, Proposition 2.6] If $Y_G$ is a submonoid of $A_r^e(F_q)$, then $I(Y_G)$ is binomial.

---

**Algorithm 1** Computing a generating set for $I(Y_G(f, g))$.

**Input** The matrix $\beta \in M_{d \times r}(\mathbb{N})$, $f_1, \ldots, f_r, g_1, \ldots, g_r$ and a prime power $q$.

**Output** A generating set of $I(Y_G(f, g)) \subseteq F_q[x_1, \ldots, x_r]$.

1. Compute a Groebner basis $\mathcal{G}$ for the ideal $J$ with respect to a lex ordering making $y_1, \ldots, y_s, z_1, \ldots, z_d, w$ bigger than $x_1, \ldots, x_r$.
2. Return $\mathcal{G} \cap S$.

The following example illustrates how to implement the algorithm in Macaulay2 in order to compute a generating set for the vanishing ideal in question and thereby to compute basic parameters of an evaluation code.

**Example 2.7.** Let $\beta = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and $q = 3$ so that $F = F_3$. We compute a generating set for the ideal $I(Y_G(f, g)) \subseteq F_3[x_1, \ldots, x_4]$, where
\[
Y(f, g) = \left\{ \left( \frac{1+y_1}{y_2}, \frac{1+y_3}{y_4} \right) : y_1, y_2, y_3, y_4 \in F \text{ and } y_2y_4 \neq 0 \right\}
\]
and the group acting on the affine space is $G = \text{Ker}(\pi) = \{(t_1, t_2, t_1^2t_2) \mid t_1, t_2 \in F \} \cong (F^*)^2$.

The following commands computes this vanishing ideal:

```macaulay2
i1 : q=3; F=GF(q,Variable => a); beta=matrix {{1,0,1,2},{0,1,0,1}};
i2 : r=numColumns beta; d=numRows beta;
i3 : R=F[x_1..x_r,y_1..y_r,z_1..z_d,w];
i4 : f1=1+y_1,f2=1,f3=y_3,f4=1+y_3,g1=y_2,g2=1,g3=1,g4=y_4;
i5 : J=ideal(x_1*g1-f1*(z_1),x_2*g2-f2*(z_2),x_3*g3-f3*(z_1),
   x_4*g4-f4*(z_1)-2*(z_2),y_1^q-y_1,y_2^q-y_2,y_3^q-y_3,y_4^q-y_4,
   w*(g1*g2*g3*g4)-1);
i6 : IYG=eliminate (J,for i from r to r+2*d+2 list R_i)
```

The final output consists of a non-binomial generator
\[
x_1^7x_2^2 - x_1x_2^2x_3^6 - x_1^3x_3^2 + x_1x_3^2x_4^2,
\]

**together with the following three binomial generators:**
\[
x_1^3x_3 - x_1x_3^3, \quad x_2^3x_4 - x_3x_4^3, \quad x_5^3x_4^2 - x_1x_4^3.
\]

When $\alpha = (4, 2)$, a basis for the vector space $(S/I_{Y_G})_\alpha$ is found by
is generated by binomials of the form $x_\beta$ where $\beta$ Thus, Proposition 3.4. With the notations above and $|P| := G \cdot P$, we have the following

3. Cellular Binomial Ideals for Orbits

In this section, we see that the vanishing ideals of points and of orbits are special binomial ideals. Throughout the section, we assume that both fields $F = K = F_q$ in the virtue of Remark 2.1. Let us start by explaining what we mean from special in this regard:

Definition 3.1. [37, Definition 2.2] An ideal $J \subseteq F[x_1, \ldots, x_r]$ is cellular if every variable $x_\ell$ is either a nonzerodivisor or nilpotent modulo $J$. If $J$ is a cellular binomial ideal, and $\emptyset \neq \varepsilon \subseteq \{r\}$ indexes the variables that are nonzerodivisor modulo $J$, then $J$ is called $\varepsilon$-cellular.

Definition 3.2. Let $S = F[x_1, \ldots, x_r]$ be a polynomial ring and $\emptyset \neq \varepsilon \subseteq \{r\}$. $S[\varepsilon]$ denotes the ring $F[x_i : i \in \varepsilon]$ and we define $m(\varepsilon) := \langle x_i : i \notin \varepsilon \rangle \subseteq S$.

Definition 3.3. The support $\varepsilon_P$ of a point $P \in \mathbb{A}^r$, is the set of indices $i \in \{r\}$ for which the $i$-th component $p_i$ of $P$ is not zero. So, $\mathbb{A}^r$ is the disjoint union of its subsets $\mathbb{A}^r(\varepsilon)$ consisting of the points supported at $\varepsilon \subseteq \{r\}$. Notice that $\mathbb{A}^r(\emptyset) = \{(0, \ldots, 0)\}$ and $\mathbb{A}^r(\{r\}) = (K^*)^r$.

We consider the projection $\pi_\varepsilon : \mathbb{A}^r \to \mathbb{A}^{\varepsilon}$ where $\pi_\varepsilon(x_1, \ldots, x_r) = (x_i, \ldots, x_k)$ for any subset $\varepsilon = \{i_1, \ldots, i_k\} \subseteq \{r\}$. By abusing the notation, we use the same notation for the homomorphism $\pi_\varepsilon : \mathbb{Z}^r \to \mathbb{Z}^{\varepsilon}$.

We distinguish $L_\beta(\varepsilon) = \{(m_1, \ldots, m_r) \in L_\beta : m_i = 0, \forall i \notin \varepsilon\}$ with its image $\pi_\varepsilon(L_\beta(\varepsilon))$ under $\pi_\varepsilon : \mathbb{Z}^r \to \mathbb{Z}^{\varepsilon}$. Note that

$$(m_1, \ldots, m_r) \in L_\beta(\varepsilon) \iff m_1 \beta_1 + \cdots + m_r \beta_r = 0.$$ 

Thus,

$$m \in L_\beta(\varepsilon) \iff \sum_{i \in \varepsilon} m_i \beta_i = 0 \iff \pi_\varepsilon(m) \in L_\beta(\varepsilon) := \text{Ker } (\beta(\varepsilon)),$$

where $\beta(\varepsilon)$ is the matrix with columns $\beta_j$ for $j \in \varepsilon$. Thus, $\pi_\varepsilon(L_\beta(\varepsilon)) = L_\beta(\varepsilon)$.

Recall that $\chi_p : L_\beta(\varepsilon) \to K^*$ is defined by $\chi_p(m) = x^m(P)$, and the ideal $I_{\chi_p,L_\beta(\varepsilon)}$ is generated by binomials of the form $x^{m^+} - x^m(P)x^{m^-}$ for $m \in L_\beta(\varepsilon)$.

Our first $\varepsilon$-cellular binomial ideals appears here:

Proposition 3.4. With the notations above and $|P| := G \cdot P$, we have the following
By Corollary 4.14, we have 

Proof. 

\[(1)\] Clearly, \(x_i\) vanishes at \(1\varepsilon\) when \(i \notin \varepsilon\). So, \(m(\varepsilon) = (x_i : i \notin \varepsilon) \subseteq I([1\varepsilon])\). Obviously, the homogeneous binomial \(x^m - x^m \in I_{L(\varepsilon)} \subseteq S[\varepsilon]\), vanishes at \(1\varepsilon\), as \(1 - 1 = 0\). Therefore, \(I_{L(\varepsilon)} \subseteq I([1\varepsilon])\) proving the first containment.

It is clear that \([1\varepsilon] = Y_G(f, g)\), where \(f_i = 0\) for \(i \notin \varepsilon\), \(f_i = 1\) for \(i \in \varepsilon\) and \(g_i = 1\) always. Hence, \(I([1\varepsilon])\) is a binomial ideal by Theorem 2.2. Now, let \(F \in I([1\varepsilon])\) be a homogeneous binomial with monomials not contained in \(m(\varepsilon)\). So, \(F = c_1x^{a_1} + c_2x^{a_2} \in I([1\varepsilon]) \cap S[\varepsilon]\). Thus, \(c_1 + c_2 = F(1\varepsilon) = 0\) implying \(F = c_1(x^{a_1} - x^{a_2})\). As \(F\) is a homogeneous polynomial supported at \(\varepsilon\), we have \(a_1 - a_2 \in L(\varepsilon)\). Thus, \(F \in I_{L(\varepsilon)}\).

\[(2)\] \(m(\varepsilon) = (x_i : i \notin \varepsilon) \subseteq I([P])\) follows from the assumption that \(\varepsilon\) is the support of \(P \in \mathbb{K}^r\). Let \(F \in I([P]) \setminus m(\varepsilon)\). We proceed as in the proof of [35, Theorem 5.1]. Then \(F \in S[\varepsilon]\) and \(F(P) = 0 \iff F'(1\varepsilon) = 0\), for \(F'(x_{i_1}, \ldots, x_{i_k}) = F(p_{i_1}x_{i_1}, \ldots, p_{i_k}x_{i_k})\) when \(\varepsilon = \{i_1, \ldots, i_k\}\). Since, the polynomial \(F' \in I_{L(\varepsilon)}\) is an algebraic combination of binomials \(x^m - x^m\) for the elements \(m \in L(\varepsilon)\), it follows that \(F \in I_{L(\varepsilon)}\), as \(F(x^m - x^m(P)x^m)(P) = 0 \iff (x^m - x^m)(1\varepsilon) = 0\).

These complete the proof. \(\square\)

Let \(T = \{(t_1, \ldots, t_r) \in \mathbb{K}^r : t_1 \cdots t_r \neq 0\}\) be the torus \((\mathbb{K}^*)^r\) of \(\mathbb{K}^r\) and let \(T_G\) denote the quotient group \(T/G\). Then \(T_G\) acts on \(\mathbb{K}_G^r\) via coordinate wise multiplication:

\[T_G \times \mathbb{K}_G^r \to \mathbb{K}_G^r, (|t|, [P]) \to [tP].\]

It is easy to see that \(\mathbb{K}_G^r(\varepsilon) = T_G \cdot [1\varepsilon] \cong (\mathbb{K}^*)^{|\varepsilon|}\), since for every \(P \in \mathbb{K}^r(\varepsilon)\), there is a unique \(t \in T\) with \(P = t \cdot 1\varepsilon\), where \(t_j = p_j\) when \(j \in \varepsilon\) and \(t_j = 1\) when \(j \notin \varepsilon\).

Next, we show that the vanishing ideals of orbits (of cells) are \(\varepsilon\)-cellular binomial.

**Theorem 3.5.** With the notations above and \(\mathbb{K} = \mathbb{F}_q\) we get the following result,

\[I(\mathbb{K}_G^r(\varepsilon)) = I(T_G \cdot [1\varepsilon]) = m(\varepsilon) + S \cdot I_{(q-1)L(\varepsilon)}(\varepsilon).\]

**Proof.** By taking \(f_i = y_i\), if \(i \in \varepsilon\) and \(f_i = 0\), if \(i \notin \varepsilon\), and \(g_i = 1\), \(\forall i \in [r]\) we see that \(T_G \cdot [1\varepsilon] = Y^*(f, g)\) and thus has a binomial vanishing ideal by Theorem 2.2. A polynomial \(F \in I(T_G \cdot [1\varepsilon])\) with monomials not contained in \(m(\varepsilon)\) lie in \(S[\varepsilon]\) so that \(F \in S[\varepsilon] \cap I([T_G \cdot 1\varepsilon]) = I(G_\varepsilon \cdot T_\varepsilon)\), where \(G_\varepsilon = \pi_\varepsilon(G)\) and \(T_\varepsilon = \pi_\varepsilon(T) = (\mathbb{K}^*)^{|\varepsilon|}\). By [35, Corollary 4.14], we have \(I(T_G) = I_{(q-1)L(\varepsilon)}(1\varepsilon)\), which corresponds to the case where \(\varepsilon = [r]\). We can prove similarly that \(I(G_\varepsilon \cdot T_\varepsilon) = I_{(q-1)L(\varepsilon)}(1\varepsilon)\), for the other \(\emptyset \neq \varepsilon \subset [r]\).

Therefore, \(F \in S \cdot I_{(q-1)L(\varepsilon)}(\varepsilon)\). \(\square\)

**Corollary 3.6.** \(I(\mathbb{K}_G^r) = \bigcap_{\varepsilon \subseteq [r]} I(T_G \cdot [1\varepsilon]).\)

**Proof.** Follows from \(\mathbb{K}_G^r = \bigcup_{\varepsilon \subseteq [r]} \mathbb{K}_G^r(\varepsilon) = \bigcup_{\varepsilon \subseteq [r]} T_G \cdot [1\varepsilon].\) \(\square\)
Theorem 3.7. Let \( x^\varepsilon := \prod_{i \in \varepsilon} x_i = x_{i_1} \cdots x_{i_k} \) for \( \varepsilon = \{i_1, \ldots, i_k\} \). Then,
\[
I(\mathcal{A}_G^r) = \sum_{\emptyset \neq \varepsilon \subseteq [r]} x^\varepsilon \cdot I(q-1)L_{\beta(\varepsilon)}.
\]

Proof. Firstly, we show that \( I(\mathcal{A}_G^r) \subseteq \sum_{\varepsilon \subseteq [r]} x^\varepsilon \cdot I(q-1)L_{\beta(\varepsilon)} \). We know that \( I(\mathcal{A}_G^r) \) is pure binomial. So, its generators are of the form \( x^n(x^{m^+} - x^{m^-}) \in I(\mathcal{A}_G^r) \). Then we claim that \( \text{supp}(x^{m^+}) \cup \text{supp}(x^{m^-}) \subseteq \varepsilon \) for \( \varepsilon = \text{supp}(x^n) \subseteq [r] \). If not, say there exists \( i \in \text{supp}(x^{m^+}) \setminus \varepsilon \), then consider the point \( P \) whose \( i \)-th coordinate is 0 and others are 1. Then \( x^n(x^{m^+} - x^{m^-})(P) = -1 \neq 0 \). The other option leads to a contradiction, similarly. Since \( \mathcal{A}_G^r = \bigcup_{\varepsilon \subseteq [r]} \mathcal{A}_G^r(\varepsilon) = \bigcup_{\varepsilon \subseteq [r]} T_G \cdot [1]\varepsilon \), it follows that \( I(\mathcal{A}_G^r) \subseteq I(T_G \cdot [1]\varepsilon) \). So, \( x^{m^+} - x^{m^-} \in I(T_G \cdot [1]\varepsilon) \), since \( x^n \neq 0 \) on \( T_G \cdot [1]\varepsilon \). Thus, \( x^{m^+} - x^{m^-} \in S[\varepsilon] \cap I(T_G \cdot [1]\varepsilon) = I(q-1)L_{\beta(\varepsilon)} \).

For the other direction, take \( F \in I(q-1)L_{\beta(\varepsilon)} = S[\varepsilon] \cap I(T_G \cdot [1]\varepsilon) \). Then the polynomial \( x^\varepsilon F = x_{i_1} \cdots x_{i_k} F \) vanishes on \( \mathcal{A}_G^r \) as we explain next. Every \( [P] \in \mathcal{A}_G^r \) lies in an orbit \( T_G \cdot [1]p \) for some \( \varepsilon_p \subseteq [r] \). If \( i \in \varepsilon \setminus \varepsilon_p \neq \emptyset \) then \( p_i = 0 \) so that \( x^\varepsilon (P) = 0 \). Otherwise, \( \varepsilon \subseteq \varepsilon_p \) and \( x^\varepsilon (P) \neq 0 \). Introduce a new point \( P' \in \mathcal{A}^r(\varepsilon) \) whose \( i \)-th coordinate coincides with that of \( P \), i.e. \( p_i = p'_i \) for all \( i \in \varepsilon \). If \( F \in S[\varepsilon] \), we have \( F(P) = F(p_1, \ldots, p_k) = F(P') = 0 \) for \( [P'] \in T_G \cdot [1]\varepsilon \). Therefore, we have \( x^\varepsilon F(P) = 0 \) in any case, completing the proof. \( \square \)

4. Vanishing Ideal of a Toric Variety

Let \( X = X_\Sigma \) be a simplicial complete toric variety over an algebraically closed field \( \mathbb{K} = \mathbb{F}_q \). Then, by a celebrated result due to Cox (see [9]), the \( \mathbb{K} \)-rational points \( X(\mathbb{K}) \) of the toric variety \( X \), is isomorphic to the geometric invariant theory quotient \( (\mathbb{K}^r \setminus V(B))/G \), for the monomial ideal
\[
B = \langle x^\sigma = \prod_{\rho \in \mathbb{K}} x_i : \sigma \in \Sigma \rangle \subset S = \mathbb{F}_q[x_1, \ldots, x_r] and
\]
\[
G = V(I_{L_\beta}) \cap (\mathbb{K}^*)^r = \{ P \in (\mathbb{K}^*)^r : (x^{m^+} - x^{m^-})(P) = 0 \text{ for all } m \in L_\beta \}
\]
\[
= \{ P \in (\mathbb{K}^*)^r : x^{m}(P) = 1 \text{ for all } m \in L_\beta \}.
\]
Therefore, \( \mathbb{K} \)-rational points of \( X \) are in bijection with the orbits \( [P] := G \cdot P \), for \( P \in \mathbb{K}^r \setminus V(B) \). Hence, we may regard them as elements of the set \( \mathcal{A}_G^r \setminus V_G(B) \). It follows that the \( \mathbb{F}_q \)-rational points of \( X \) are in bijection with the orbits \( [P] := G \cdot P \), for \( P \in \mathbb{F}_q^r \setminus V(B) \).

Theorem 4.1. If \( Y \subseteq \mathbb{K}^r \), then the vanishing ideal in \( S \) of the subset \( [Y \setminus V(B)] \) of \( \mathcal{A}_G^r \setminus V_G(B) \) is given by \( I([Y \setminus V(B)]) = I(Y) : B \).

Proof. As \( V(B) \) is \( G \)-invariant we first notice that
\[
[Y \setminus V(B)] = [Y] \setminus [V(B)] = Y_G \setminus V_G(B).
\]
First we prove the inclusion \( I([Y \setminus V(B)]) \subseteq I(Y) : B \). Let \( F \in I([Y \setminus V(B)]) \) be a homogeneous polynomial. Then \( F \) vanishes on \( Y \setminus V(B) \). Since \( F' \) vanishes on \( V(B) \), for all \( F' \in B, FF' \) vanishes on \( Y \). For \( F' = \bigoplus_{\alpha \in \mathbb{N}^d} F'_\alpha \in B \), we
have \( F'_\alpha \in B \), \( \forall \alpha \in \mathbb{N}\beta \) as \( B \) is a homogeneous ideal. So, \( FF'_\alpha \) is a homogeneous polynomial vanishing on \( Y_G \), i.e. \( FF'_\alpha \in I(Y_G) \) is a homogeneous generator, and hence \( FF' \in I(Y_G) \). Thus, \( F \in I(Y_G) : B \).

Now we show the other containment. As \( I(Y_G) : B \) is homogeneous, we start by taking a homogeneous generator \( F \) of \( I(Y_G) : B \). Then \( FF' \in I(Y_G), \forall F' \in B \). Let us take \( P \in Y \setminus V(B) \). Since \( P \notin V(B) \), there is a polynomial \( F'' \in B \) such that \( F'(P) \neq 0 \). As \( P \in Y \), we have \( F(P)F'(P) = 0 \), so \( F(P) = 0 \). Therefore, \( F \in I([Y \setminus V(B)]) \).

**Corollary 4.2.** \( I(X(\mathbb{F}_q)) = I(\mathbb{A}_q^r(\mathbb{F}_q)) : B \)

**Proof.** Follows from Theorem 4.1 by taking \( Y = \mathbb{A}_q^r(\mathbb{F}_q) \).

The following result combined with our Theorem 2.2 leads to an algorithmic method to compute a generating set for the vanishing ideal of a subset \( X(f, g) \) in \( X(\mathbb{F}_q) \) parameterized by rational functions, generalizing the nice main result of the paper [41]. When \( X \) is a projective space, the irrelevant ideal is nothing but \( B = \langle x_1, \ldots, x_r \rangle \), whose zero set is \( V_G(B) = \{0\} \) and hence we have

\[
I(X(f, g)) = I(Y_G(f, g)) : B = I(Y_G(f, g)),
\]
as in the proof of Proposition 5.6. Therefore, the main contribution we made in passing from a projective space to a more general toric variety is to observe that we need to compute the homogeneous ideal \( I(Y_G(f, g)) \) first and find its colon ideal \( I(Y_G(f, g)) : B \) in the second step.

**Corollary 4.3.** Let \( X(f, g) \) denote the subset of \( X(\mathbb{F}_q) \) which is parametrized by the rational functions \( f_i/y_i \), for polynomials \( f_i, g_i \in \mathbb{F}_q[y_1, \ldots, y_s] \) for \( i \in [r] \), i.e., \( X(f, g) = Y_G(f, g) \setminus V_G(B) \). Then, we have \( I(X(f, g)) = I(Y_G(f, g)) : B \)

**Proof.** Follows from Theorem 4.1 by taking \( Y = Y_G(f, g) \).

5. Applications

In this section, we compute vanishing ideals of \( \mathbb{F}_q \)-rational points of some famous examples of toric varieties applying the theory developed in previous sections.

5.1. Hirzebruch Surfaces. Let \( X = \mathcal{H}_\ell \) be the Hirzebruch surface whose primitive ray generators are as follows \( v_1 = (1, 0), v_2 = (0, 1), v_3 = (-1, \ell) \), and \( v_4 = (0, -1) \), for any positive integer \( \ell \). The exact sequence becomes

\[
\mathcal{P} : 0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\phi} \mathbb{Z}^4 \xrightarrow{\beta} \text{Cl}(\mathcal{H}_\ell) \longrightarrow 0,
\]

for \( \phi = [u_1 \ u_2] \) with \( u_1 = (1, 0, -1, 0), u_2 = (0, 1, \ell, -1) \) and \( \beta = \begin{bmatrix} 1 & 0 & 1 & \ell \\ 0 & 1 & 0 & 1 \end{bmatrix} \)

with \( L_\beta = [u_1 \ u_2] \). The dual sequence over \( \mathbb{K} = \mathbb{F}_q \) is

\[
\mathcal{P}^* : 1 \longrightarrow G \xrightarrow{i} (\mathbb{K}^*)^4 \xrightarrow{\pi} (\mathbb{K}^*)^2 \longrightarrow 1
\]

where \( \pi : t \mapsto (t_1 t_3^{-1}, t_2 t_4^{1-1}) \).

Then the class group is \( \text{Cl}(\mathcal{H}_\ell) \cong \mathbb{Z}^2 \) and the group acting on the affine space is

\[
G = \text{Ker}(\pi) = \{(t_1, t_2, t_1, t_2) \mid t_1, t_2 \in \mathbb{K}^* \} \cong (\mathbb{K}^*)^2.
\]
Lemma 5.2. As some binomials divide other, the proof follows. □

Then, a Groebner basis of the ideal $I$ is given by:

$$f_1 = x_3^{q-1} - x_2^{q-1}, \quad f_2 = x_4^{q-1} - x_2^{q-1} x_1^{(q-1)^\ell}, \quad f_3 = x_4^{q-1} - x_3^{(q-1)^\ell} x_2^{q-1}.$$

Proof. Recall that $\varepsilon \subseteq [4]$ gives the matrix $\beta(\varepsilon)$ with columns $\beta_j$ for $j \in \varepsilon$. For instance, if $\varepsilon = \{1, 2, 4\}$, then $\beta(\varepsilon) = \begin{bmatrix} 1 & 0 & \ell \\ 0 & 1 & 1 \end{bmatrix}$ whose kernel is as follows:

$L_{\beta(\varepsilon)} = \{(a_1, a_2, a_4) \in \mathbb{Z}^3 : a_1 + \ell a_4 = a_2 + a_4 = 0\} = \{(-\ell a_4, -a_4, a_4) : a_4 \in \mathbb{Z}\}$.

Thus, the corresponding toric ideal is $I_{L_{\beta(\varepsilon)}} = \langle x_4 - x_2 x_4^\ell \rangle$. Similarly, for $\varepsilon = \{2, 3, 4\}$, we have $I_{L_{\beta(\varepsilon)}} = \langle x_4 - x_3^2 x_2 \rangle$.

Theorem 5.1. $I(\mathbb{A}_N^2(F_q))$ is generated by $x^4 I_{\langle (q-1)^\ell \rangle} L_{\beta(\varepsilon)}$, so it is generated by the following binomials:

- $x_1 x_2 x_4 f_2$ for $\varepsilon = \{1, 2, 4\}$,
- $x_2 x_3 x_4 f_3$ for $\varepsilon = \{2, 3, 4\}$,
- $x_1 x_2 x_3 f_1$ for $\varepsilon = \{1, 2, 3\}$,
- $x_1 x_3 x_4 f_1$ for $\varepsilon = \{1, 3\}$,
- $x_1 x_2 x_3 x_4 f_1$, $x_1 x_2 x_3 x_4 f_2$ (or $x_1 x_2 x_3 x_4 f_3$) for $\varepsilon = \{1, 2, 3, 4\}$.

As some binomials divide other, the proof follows. □

We will use the following algorithm to compute generators of the intersections of ideals, that is given right after Theorem 11 of Chapter 4, Section 3 in [8]:

**Lemma 5.2.** Let $I = \langle f_1, \ldots, f_k \rangle$ and $J = \langle g_1, \ldots, g_l \rangle$ be ideals in $S = \mathbb{F}[x_1, \ldots, x_r]$. Then, a Groebner basis of the ideal $I \cap J$ consists of the polynomials from $S$ in a Groebner basis of the ideal $\langle w f_1, \ldots, w f_k, (1 - w) g_1, \ldots, (1 - w) g_l \rangle \subseteq S[w]$ with respect to a lexicographic term order making $w$ the biggest variable.

**Theorem 5.3.** Let us fix the following notation:

- $F_1 = x_3 x_1 f_1 = x_3 x_1 (x_3^{q-1} - x_2^{q-1})$,
- $F_2 = x_4 x_2 x_1 f_2 = x_4 x_2 x_1 (x_4^{q-1} - x_2^{q-1} x_1^{(q-1)^\ell})$,
- $F_3 = x_4 x_3 x_2 f_3 = x_4 x_3 x_2 (x_4^{q-1} - x_3^{(q-1)^\ell} x_2^{q-1})$,
- $F_4 = x_4^{q-1} x_2 - x_4 x_3^{(q-1)^\ell} x_2^{q-1} + x_4 x_3^{q-1} x_2^{q-1} x_1^{(q-1)^\ell} - x_4 x_2 x_1^{(q-1)^\ell} x_1^{q-1} - x_4^{2q-1} x_2^{q-1},$
- $F'_4 = x_4^{2q-1} x_2 - x_4 x_3^{(q-1)^\ell} x_2^{q-1} + x_4 x_3^{q-1} x_2^{q-1} x_1^{(q-1)^\ell} - x_4 x_2 x_1^{(q-1)^\ell} x_1^{q-1} - x_4^{2q-1} x_2^{q-1}. $
Then, a set of minimal generators for the vanishing ideals are given by:
\[
I(\mathcal{H}_t(F_q)) = \langle F_1, F_2, F_3, F_4 \rangle = \langle F_1, F_4 \rangle, \quad \text{if } t > 1, \text{ and}
\]
\[
I(\mathcal{H}_t(F_q)) = \langle F_1, F_2, F_3, F'_4 \rangle.
\]

Proof. Recall from Theorem 5.1 that \( \mathcal{J} := I(\mathcal{A}_t(F_q)) \) is generated by \( F_1, F_2, F_3 \).

By Corollary 4.2, \( I(\mathcal{H}_t(F_q)) = \mathcal{J} : B \), where \( B = (x_1 x_2, x_1 x_4, x_3 x_2, x_3 x_4) \) and so Proposition 10 of Chapter 4, Section 4 in [8] implies that
\[
I(\mathcal{H}_t(F_q)) = (\mathcal{J} : x_1 x_2) \cap (\mathcal{J} : x_1 x_4) \cap (\mathcal{J} : x_3 x_2) \cap (\mathcal{J} : x_3 x_4).
\]

In the first step, we compute these ideals using the fact that when \( \{h_1, \ldots, h_k\} \) is a basis for \( \mathcal{J} \cap \langle g \rangle \) then \( \{h_1/g, \ldots, h_k/g\} \) is a basis for \( \mathcal{J} : g \), see Theorem 11 of Chapter 4, Section 4 in [8].

In order to compute a basis for \( \mathcal{J} \cap \langle x_1 x_2 \rangle \), we use Lemma 5.2 and compute the Groebner basis of the ideal generated by \( wF_1, wF_2, wF_3, (1 - w)x_1 x_2 \) in the ring \( S[w] = F_q[x_1, x_2, x_3, x_4, w] \) with respect to the lexicographic term order with \( w > x_4 > x_3 > x_2 > x_1 \). It is a routine check that the polynomials

\[
x_2 F_1, F_2, (1 - w)x_1 x_2, wF_1, wF_3
\]

of \( S[w] \) form such a Groebner basis and thus \( x_2 F_1 \) and \( F_2 \in S \) generates the ideal \( \mathcal{J} \cap \langle x_1 x_2 \rangle \). Dividing these generators by \( x_1 x_2 \), we get \( \mathcal{J} : \langle x_1 x_2 \rangle = \langle x_3 f_1, x_4 f_2 \rangle \).

Similarly, the polynomials \( x_4 F_1, F_2, (1 - w)x_1 x_4, wF_1, wF_3 \) form the Groebner basis of the ideal generated by \( wF_1, wF_2, wF_3, (1 - w)x_1 x_4 \) in \( S[w] \) with respect to the same term order and thus \( x_4 F_1 \cap x_1 x_4 = x_3 f_1 \) and \( F_2/x_1 x_4 = x_2 f_2 \in S \) generates the ideal \( \mathcal{J} : \langle x_1 x_4 \rangle \).

Once again, the polynomials \( x_2 F_1, F_3, (1 - w)x_3 x_2, wF_1, wF_2 \) form the Groebner basis of the ideal generated by \( wF_1, wF_2, wF_3, (1 - w)x_3 x_2 \) in \( S[w] \) and thus \( x_2 F_1/x_3 x_2 = x_1 f_1 \) and \( F_3/x_3 x_2 = x_3 f_3 \in S \) generates the ideal \( \mathcal{J} : \langle x_3 x_2 \rangle \).

Finally, the polynomials \( x_4 F_1, F_3, (1 - w)x_3 x_4, wF_1, wF_2 \) form the Groebner basis of the ideal generated by \( wF_1, wF_2, wF_3, (1 - w)x_3 x_4 \) in \( S[w] \) and \( \mathcal{J} : \langle x_3 x_4 \rangle \) is generated by \( x_4 F_1/x_3 x_4 = x_1 f_1 \) and \( F_3/x_3 x_4 = x_2 f_3 \in S \).

Now, we compute \( (\mathcal{J} : \langle x_1 x_2 \rangle) \cap (\mathcal{J} : \langle x_1 x_4 \rangle) \) using Lemma 5.2. The Groebner basis of \( \{wx_3 f_1, wx_4 f_2, (1 - w)x_3 f_1, (1 - w)x_2 f_2\} \) with respect to the lexicographic term order with \( w > x_4 > x_3 > x_2 > x_1 \) is computed to be the following set \( \{x_3 f_1, x_4 x_2 f_2, x_4 (1 - w)x_3 f_1, x_1 (1 - w)x_2 f_2\} \) and thus \( (\mathcal{J} : \langle x_1 x_2 \rangle) \cap (\mathcal{J} : \langle x_1 x_4 \rangle) \) is generated by \( x_3 f_1 \) and \( x_2 x_4 f_2 \). Until now, \( \ell \) is any positive number. The rest depends on whether \( \ell > 1 \) or \( \ell = 1 \).

Case \( \ell > 1 \):

As before, the Groebner basis of \( \{wx_3 f_1, wx_2 x_4 f_2, (1 - w)x_1 f_1, (1 - w)x_4 f_3\} \) is computed to be the following set \( \{F_1, F_4, F_3, (w - 1)x_1 f_1, w x_3 f_1, F_6\} \), where
\[
F_4 = x_4 x_2 - x_4 x_2^{(q-1)\ell} x_2 + x_4 x_2^{q-1} x_2^{(q-1)\ell} - x_4 x_2^{q-1} x_2^{(q-1)\ell},
F_5 = x_4 x_3^{q-1} - x_4 x_3^{q-1} x_2^{(q-1)\ell} + x_4 x_3^{q-1} x_2^{(q-1)\ell} + x_4 x_3^{q-1} x_2^{(q-1)\ell},
F_6 = (w - 1)x_4 x_4^{q-1} - x_4 x_4^{(q-1)\ell} + x_4 x_4^{(q-1)\ell} - x_4^{(q-1)\ell} - x_4^{(q-1)\ell}.
\]

Hence, \( \langle x_3 f_1, x_2 x_4 f_2 \rangle \cap (\mathcal{J} : \langle x_1 x_4 \rangle) \) is generated by \( F_1, F_4, F_3 \), that is, we obtain
\[
(\mathcal{J} : \langle x_1 x_2 \rangle) \cap (\mathcal{J} : \langle x_1 x_4 \rangle) \cap (\mathcal{J} : \langle x_3 x_2 \rangle) = \langle F_1, F_4, F_3 \rangle.
\]

Finally, the Groebner basis of the set \( \{wF_1, wF_4, wF_5, (w - 1)x_1 f_1, (1 - w)x_2 f_3\} \) is found to be \( \{F_1, F_4, F_5, (w - 1)x_1 f_1, (w - 1)x_2 f_3\} \). Thus, \( \langle F_1, F_4, F_5 \rangle \cap (\mathcal{J} : \langle x_3 x_4 \rangle) \) is generated by \( F_1 \) and \( F_4 \), completing the proof for \( \ell > 1 \).
The difference between $\text{A}_i$:
In this case, the Groebner basis of $\{wx_3f_1, wx_2x_4f_2, (1-w)x_1f_1, (1-w)x_4f_3\}$ is computed to be the following set

$$\{F_1, F_2, F_3, F_4, (w-1)x_1f_1, wx_3f_1, (w-1)x_4f_3, (w-1)x_2x_4f_3\},$$

where

$$F'_0 = x_2^q x_3^q - x_2^q x_3 x_1^q - x_4 x_3^{(q-1)(\ell+1)+1} x_2^q + x_4 x_3 x_1^{q-1} (q-1)(\ell+1).$$

Hence, $\langle x_3f_1, x_2x_4f_2 \rangle \cap \langle J : \langle x_3x_2 \rangle \rangle$ is generated by $F_1, F_2, F_3, F'_4, F'_5$, that is, we obtain

$$\langle J : \langle x_1, x_2 \rangle \rangle \cap \langle J : \langle x_1x_4 \rangle \rangle \cap \langle J : \langle x_3x_2 \rangle \rangle = \langle F_1, F_2, F_3, F'_4, F'_5 \rangle.$$
Taking \( \alpha = (1,0) \), we get \( B_\alpha = \{ x_1, x_3 \} \) as a basis for the vector space \( (S/I)_\alpha \) for \( I = I(\mathcal{H}_r(\mathbb{F}_q)) \). Adding the three points \( Y_3 = \{ [1,0,1,0],[1,0,0,0],[0,0,1,0] \} \), will increase the length by 3. Since a non-zero polynomial \( ax_1 + bx_3 \), for \( a, b \in \mathbb{F}_3 \), can have at most one extra root among these three points, the minimum distance will increase by two. Indeed, using the Coding Theory package introduced in [1], we compute parameters of the codes \( C_{\alpha,Y} \) for \( Y = \mathcal{H}_r(\mathbb{F}_q) \) to be [36,2,30] and for \( Y = \mathcal{H}_r(\mathbb{F}_q) \cup Y_3 \) to be [39,2,32] with the following commands:

\[
\begin{align*}
i11 : & \text{ alpha=\{1,0\}; Bd=flatten entries basis(alpha,coker gens gb IX);} \\
& \text{ i12 : PX=join(flatten apply(q,i-> apply (q,j-> \{i,1,1,j\}), apply(q,i->\{1,0,1,0\}));} \\
& \text{ i13 : C=evaluationCode(F,PX,Balpha);} \\
& \text{ i14 : PY=join(PX,\{\{1,0,1,0\},\{1,0,0,0\},\{0,0,1,0\}\});} \\
& \text{ i15 : C=evaluationCode(F,PY,Balpha);} \\
\end{align*}
\]

We conclude the example speculating on why the choice we made was the best possible among all \( H_r(\mathbb{F}_q) \subset Y \subset \mathbb{A}^3_r(\mathbb{F}_q) \). Since the weight \( w(c_F) \) of a codeword \( c_F = (F(P_1), \ldots, F(P_q)) \) is \( |Y| - |V_Y(F)| \) it follows that the minimum distance is

\[
\delta(C_{\alpha,Y}) = |Y| - \max\{|V_Y(F)| : F \in (S/I)_\alpha \setminus \{0\}\},
\]

where \( V_Y(f) = \{ [P] \in Y : f(P) = 0 \} \). Notice that \( ax_1 + bx_3 \) vanishes on the set \( Y_0 := V(B) \setminus Y_3 \), for every \( a, b \in \mathbb{F}_q \). Adding any subset \( Y'_0 \) of \( Y_0 \) to a set \( Y \) does not increase the length \( |Y| \) by \( |Y'_0| \) and leaves the minimum distance the same. This is because \( |V_Y(F)| \) also increases by the same amount \( |Y'_0| \) and so the difference above does not change. Finally, adding a proper subset of \( Y_3 \) does not increase the minimum distance that much, since for every proper subset of size 1 there is a polynomial vanishing on that subset. For instance, \( x_1 - x_3 \) vanishes on \( \{1,0,1,0\} \). Similarly, no polynomial can have two roots on a proper subset of size 2 and there is a polynomial with one root, so that the minimum distance increases by 1.

5.2. Weighted Projective Spaces. Let \( w_1, \ldots, w_r \) be some positive integers such that \( n = r - 1 \) of them have no nontrivial common divisor, that is we have \( \gcd(w_1, \ldots, w_1, \ldots, w_r) = 1 \), for any \( i \in [r] \). In this case, we have a row matrix \( \beta = [w_1 \cdots w_r] \) and the corresponding toric variety is denoted \( X = \mathbb{P}(w_1, \ldots, w_r) \). The semigroup \( \mathbb{N}[\beta] \) is the numerical semigroup generated by \( w_1, \ldots, w_r \) denoted also by \( \langle w_1, \ldots, w_r \rangle \) in the literature. The group \( G = \{(t^{w_1}, \ldots, t^{w_r}) : t \in \mathbb{K}^* \} \) is the torus of the affine monomial curve parameterized by \( x_i = t^{w_i} \), where \( i \in [r] \). The toric ideal \( I_{L_\beta} \) is the defining ideal of this monomial curve whose coordinate ring is the semigroup ring \( \mathbb{K}[\mathbb{N}[\beta]] = \mathbb{K}[t^{w_1}, \ldots, t^{w_r}] \) when \( \mathbb{K} = \mathbb{F}_q \).

**Proposition 5.6.** If \( X = \mathbb{P}(w_1, \ldots, w_r) \) is the weighted projective space, then its vanishing ideal \( I(\mathcal{X}(\mathbb{F}_q)) = I(\mathcal{A}_G(\mathbb{F}_q)) \).

**Proof.** As \( \mathcal{A}_G(\mathbb{F}_q) = \mathcal{X}(\mathbb{F}_q) \cup \{0\} \), we have the following equalities

\[
I(\mathcal{A}_G(\mathbb{F}_q)) = I(\mathcal{X}(\mathbb{F}_q)) \cap I(\{0\}) = I(\mathcal{X}(\mathbb{F}_q)) \cap \langle x_1, \ldots, x_r \rangle = I(\mathcal{X}(\mathbb{F}_q)).
\]

If \( w_i = 1 \), for all \( i \in [r - 2] \), but \( w_{r-1} = a \) and \( w_{r-2} = b \) are arbitrary, the vanishing ideal \( I(\mathcal{X}(\mathbb{F}_q)) \) for \( X = \mathbb{P}(1, \ldots, 1, a, b) \) is easy to compute.
Remark 5.9. Mercier and Rolland

Proof. Direct consequence of Theorem 5.7.

We recommend the paper 

\[ \text{of weighted projective spaces. In order to state some of the results scattered the} \]

\[ \text{together with Theorem 3.7 and Proposition 5.6 to give generating set s for families} \]

\[ \text{we use the set given by} \]

\[ \text{which by Theorem 3.7 come from} \]

\[ \text{Theorem 5.7. Now, we prove that they are in deed sufficient,} \]

\[ \text{in the statement of the Theorem 5.7.} \]

\[ \text{Therefore, the generators coming from} \]

\[ \text{As a particular case we single out the following.} \]

Corollary 5.8. \( I(\mathbb{P}(a, b)(\mathbb{F}_q)) \) is generated by the following binomials

\[ x_1x_2(1^{q-1})a - x_2^{q-1}, \quad x_1x_3(x_1^{q-1}b - x_3^{q-1}), \quad x_2x_3(x_2^{q-1}b - x_3^{q-1}a). \]

Proof. Direct consequence of Theorem 5.7.

Remark 5.9. Mercier and Rolland [24] has given a binomial generating set for the ideal \( I(\mathbb{P}(a, b)(\mathbb{F}_q)) \) and Theorem 5.7 generalizes this result to some weighted projective spaces. We recommend the paper [4] by Beelen, Datta and Ghorpade in order to see how they use the set given by [24] to obtain a footprint bound for the minimum distance of the corresponding code.

One can use the vast literature about numerical semigroups and their toric ideals together with Theorem 3.7 and Proposition 5.6 to give generating sets for families of weighted projective spaces. In order to state some of the results scattered the literature we recall some key concepts. For a numerical semigroup \( W \) generated by \( w_1, \ldots, w_r \), the subset of pseudo-Frobenius numbers are defined by

\[ PF(W) = \{ z \in \mathbb{Z} \mid W : z + w \in W \text{ for all } w \in W \setminus \{0\} \}. \]

The largest integer \( g(W) \notin W \) belongs to \( PF(W) \) and is called the Frobenius number of \( W \). If \( PF(W) = \{g(W)\} \), then \( W \) is called symmetric, whereas if \( PF(W) = \{g(W)/2, g(W)\} \), it is called pseudosymmetric.
It is well known that any of \( \mathbb{P}(lw_1, lw_2, w_3) \), \( \mathbb{P}(lw_1, lw_2, lw_3) \) or \( \mathbb{P}(w_1, lw_2, lw_3) \) is isomorphic to \( \mathbb{P}(w_1, w_2, w_3) \), for any positive integer \( l \), we assume that \( w_1, w_2 \) and \( w_3 \) are relatively prime to each other and \( w_1 < w_2 < w_3 \).

**Proposition 5.10.** If \( W \) is symmetric, then \( w_3 = a_{31}w_1 + a_{32}w_2 \) for some non-negative integers \( a_{31} \) and \( a_{32} \) and the vanishing ideal of \( \mathbb{P}(w_1, w_2, w_3)(\mathbb{F}_q) \) is generated by the following 4 binomials

\[
\begin{align*}
x_1x_2(x_1^{(q-1)}w_2 - x_2^{(q-1)}w_1), & \quad x_1x_3(x_1^{(q-1)}w_3 - x_3^{(q-1)}w_1), \\
x_2x_3(x_2^{(q-1)}w_3 - x_3^{(q-1)}w_2), & \quad x_1^2x_2x_3(x_3^{q-1} - x_1^{(q-1)}a_{31}x_2^{(q-1)a_{32}}).
\end{align*}
\]

If \( W \) is not symmetric, then there are \( a_1, a_2 \) and \( a_3 \) such that \( a_iw_i = a_{ij}w_j + a_{ik}w_k \), for \( \{i, j, k\} = \{1, 2, 3\} \) and the vanishing ideal of \( \mathbb{P}(w_1, w_2, w_3)(\mathbb{F}_q) \) is generated by the following 6 binomials

\[
\begin{align*}
x_1x_2(x_1^{(q-1)}w_2 - x_2^{(q-1)}w_1), & \quad x_1x_2x_3(x_1^{(q-1)}a_{31} - x_2^{(q-1)}a_{13}x_1^{(q-1)a_{31}}), \\
x_1x_3(x_1^{(q-1)}w_3 - x_3^{(q-1)}w_1), & \quad x_1x_2x_3(x_2^{(q-1)}a_{32} - x_1^{(q-1)}a_{21}x_3^{(q-1)a_{32}}), \\
x_2x_3(x_2^{(q-1)}w_3 - x_3^{(q-1)}w_2), & \quad x_1^2x_2x_3(x_3^{q-1} - x_1^{(q-1)}a_{32}x_2^{(q-1)a_{31}}).
\end{align*}
\]

**Proof.** If \( W \) is symmetric, then by [15, Theorem 3.10], \( w_3 = a_{31}w_1 + a_{32}w_2 \) for some non-negative integers \( a_{31} \) and \( a_{32} \), and the toric ideal of the semigroup \( W \) is generated by \( x_1^{a_2} - x_1^{a_3} \) and \( x_3 - x_1^{a_2}x_2^{a_3} \). When \( \varepsilon = \{1, 2, 3\} \), \( \mathbb{N}(\varepsilon) = W \), so we get the binomials \( x_1x_2x_3(x_1^{(q-1)}w_2 - x_2^{(q-1)}w_1) \) and \( x_1x_2x_3(x_3^{q-1} - x_1^{(q-1)a_{31}}x_2^{(q-1)a_{32}}) \) from here. If \( \varepsilon = \{1, 2\} \), then \( \mathbb{N}(\varepsilon) = \langle w_1, w_2 \rangle \), and so we get the binomial \( x_1x_2(x_1^{(q-1)}w_2 - x_2^{(q-1)}w_1) \). Similarly, \( \varepsilon = \{1, 3\} \) gives \( \mathbb{N}(\varepsilon) = \langle w_1, w_3 \rangle \) and the binomial \( x_1x_3(x_1^{(q-1)}w_3 - x_3^{(q-1)}w_2) \) and finally \( \varepsilon = \{2, 3\} \) gives the binomial \( x_2x_3(x_2^{(q-1)}w_3 - x_3^{(q-1)}w_2) \), completing the proof for the first case.

If \( W \) is not symmetric, then by [15, Proposition 3.2] there are positive integers \( a_1, a_2 \) and \( a_3 \) such that \( a_1w_1 = a_{ij}w_i + a_{ik}w_k \), for \( \{i, j, k\} = \{1, 2, 3\} \), satisfying \( a_{21} + a_{31} = a_{12}, a_{12} + a_{32} = a_{21}, a_{13} + a_{23} = a_{31} \), and the toric ideal is generated by

\[
\begin{align*}
g_1 &= x_1^{a_1} - x_2^{a_2}, \quad g_2 = x_2^{a_2} - x_1^{a_1}, \quad g_2 = x_3^{a_3} - x_1^{a_1}x_2^{a_2}.
\end{align*}
\]

In fact, these \( a_i \)'s are the smallest positive integers with that property. Thus, when \( \varepsilon = \{1, 2, 3\} \), \( \mathbb{N}(\varepsilon) = W \), so we get the generators

\[
\begin{align*}
x_1x_2x_3g_1(x_1^{q-1}, x_2^{q-1}, x_2^{q-1}), & \quad x_1x_2x_3g_2(x_1^{q-1}, x_2^{q-1}, x_2^{q-1}), \\
x_1x_2x_3g_3(x_1^{q-1}, x_2^{q-1}, x_2^{q-1}).
\end{align*}
\]

If \( \varepsilon = \{1, 2\} \), then \( \mathbb{N}(\varepsilon) = \langle w_1, w_2 \rangle \), and so we get \( x_1x_2(x_1^{(q-1)}w_2 - x_2^{(q-1)}w_1) \) as in the first case. Similarly, \( \varepsilon = \{1, 3\} \) gives \( \mathbb{N}(\varepsilon) = \langle w_1, w_3 \rangle \) and the binomial \( x_1x_3(x_1^{(q-1)}w_3 - x_3^{(q-1)}w_1) \) and finally \( \varepsilon = \{2, 3\} \) gives \( x_2x_3(x_2^{(q-1)}w_3 - x_3^{(q-1)}w_2) \), completing the proof for the second case.

\[\square\]

**Remark 5.11.** Let \( X = \mathbb{P}(1, 1, 2) \) and \( \mathbb{K} = \mathbb{F}_3 \). Then, the \( \mathbb{F}_3 \)-rational points are \( X(\mathbb{F}_3) = (\mathbb{F}_3^3 \setminus \{0\})/G \), where \( G = \{\lambda, \lambda^2 \} : \lambda \in \mathbb{K}^* \). However, we cannot replace \( G \) by the subgroup \( G(\mathbb{F}_4) = \{\lambda, \lambda^2 \} : \lambda \in \mathbb{F}_4^* \). For instance, the points \( [0 : 0 : 1] \) and \( [0 : 0 : 2] \) are the same in \( X(\mathbb{F}_3) \), as there is a \( \lambda \in \mathbb{K}^* \) with \( \lambda^2 = 2 \) so that \( (\lambda, \lambda^2) : (0, 0, 1) = (0, 0, 2) \). But for any \( \lambda \in \mathbb{F}_3^* \), \( \lambda^2 \) is 1 and \( [0 : 0 : 1] \) \( \neq [0 : 0 : 2] \) in \( (\mathbb{F}_3^3 \setminus \{0\})/G(\mathbb{F}_3) \). However, these points have the same vanishing ideal \( \langle x_1, x_2 \rangle \) in \( S = \mathbb{F}_3[x_1, x_2, x_3] \) in any case.
5.3. Product of Projective Spaces. The product of projective spaces is also a toric variety denoted by \( X \). The Cox ring \( S = \mathbb{F}_q[x_{1,1}, \ldots, x_{1,r_1}, \ldots, x_{k,1}, \ldots, x_{k,r_k}] \) is graded via
\[
\text{deg}(x_{1,1}) = \cdots = \text{deg}(x_{1,r_1}) = e_1, \ldots, \text{deg}(x_{k,1}) = \cdots = \text{deg}(x_{k,r_k}) = e_k,
\]
where \( e_1, \ldots, e_k \in \mathbb{Z}^k \) form the standard basis, and \( r_i = n_i + 1 \), for \( i \in [k] \). The monomial ideal is
\[
B = \langle x_{1,1}, \ldots, x_{1,r_1} \rangle \cap \cdots \cap \langle x_{k,1}, \ldots, x_{k,r_k} \rangle.
\]

**Corollary 5.12.** If \( X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) is a product of projective spaces then \( I(X(\mathbb{F}_q)) = I(\mathbb{A}_G^r(\mathbb{F}_q)) \).

**Proof.** Recall that \( X = \mathbb{A}_G^r \setminus V_G(B) \). Since \( X(\mathbb{F}_q) \) and \( \mathbb{A}_G^r(\mathbb{F}_q) \) are finite, their ideals are given by
\[
I(X(\mathbb{F}_q)) = \bigcap_{[P] \in X(\mathbb{F}_q)} I([P]) \quad \text{and} \quad I(\mathbb{A}_G^r(\mathbb{F}_q)) = \bigcap_{[P] \in \mathbb{A}_G^r(\mathbb{F}_q)} I([P]).
\]

Our aim is to prove that for any \([P] \in \mathbb{A}_G^r(\mathbb{F}_q)\) there is a point \([P'] \in X\) with \( I([P']) \subset I([P]) \) so the intersections are the same. If \([P] \in X\), then \([P'] = [P]\).

If \([P] \in V_G(B)\) with support \( \varepsilon \), then \([P] \in V_G(x_{i_0,1}, \ldots, x_{i_0,r_{i_0}})\) for some \( i_0 \in [k] \).

Then, we define the point \( P' = (p_{i,j}') \) with support \( \varepsilon' = \varepsilon \cup \{(i_0,1)\} \) in such a way that \( p_{i,j}' = p_{i,j} \) for \((i,j) \in \varepsilon\) and \( p_{i_0,1}' = 1\). Then, clearly, \( m(\varepsilon') \subset m(\varepsilon) \) and \( x_{i_0,1} \in m(\varepsilon) \setminus m(\varepsilon') \). Since \((i_0,j) \notin \varepsilon\), for all \( j \in r_{i_0} \), it follows that \( L_{\beta(\varepsilon')} = L_{\beta(\varepsilon)} \times \{0\} \) and \( \chi_{\beta}(m,0) = \chi_{\beta}(m) \) thus \( I_{\chi_{\beta},L_{\beta(\varepsilon)}}(x_{i_0,1}, \ldots, x_{i_0,r_{i_0}}) = \mathbf{I}_{\chi_{\beta},L_{\beta(\varepsilon)}} \).

By Proposition 3.4, we have \( I([P]) = m(\varepsilon) + S \cdot I_{\chi_{\beta},L_{\beta(\varepsilon)}} \). Therefore, \( I([P']) \subset I([P]) \). If we still have \([P'] \in V_G(B)\), then the same procedure will give the chain \( I([P'']) \subset I([P']) \subset I([P]) \) and continuing this way if necessary we end up with the desired point in \( X \).

**Example 5.13.** Let \( \beta = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \) and \( q = 3 \) so that \( \mathbb{F} = \mathbb{F}_3 \) and \( \mathbb{K} = \overline{\mathbb{F}}_3 \). Our toric variety is \( X = \mathbb{P}^2 \times \mathbb{P}^1 \) and its Cox ring is \( S = \mathbb{F}[x_1, \ldots, x_7] \) graded via:
\[
\text{deg}_\beta(x_1) = \text{deg}_\beta(x_2) = \text{deg}_\beta(x_3) = (1,0);
\text{deg}_\beta(x_4) = \text{deg}_\beta(x_5) = \text{deg}_\beta(x_6) = \text{deg}_\beta(x_7) = (0,1).
\]

We compute a generating set for the vanishing ideal \( I(X(\mathbb{F}_3)) = I(\mathbb{A}_G^r(\mathbb{F}_3)) \) with the following commands:

```plaintext
i1 : q=3; F = GF(q,Variable => a);
beta = matrix {{1,1,1,0,0,0,0},{0,0,0,1,1,1,1}};
i2 : r=numColumns beta; d=numRows beta;
i3 : beta = matrix {{1,1,1,0,0,0,0},{0,0,0,1,1,1,1}};
i4 : f1=x_1 f2=y_2 f3=x_3 f4=y_4 f5=y_5 f6=y_6 f7=y_7;
i5 : J=ideal(x_1 f1(z_1),x_2 f2(z_1),x_3 f3(z_1),x_4 f4(z_2),
   x_5 f5(z_2),x_6 f6(z_2),x_7 f7(z_2),y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8 y_9);
i6 : IAG=eliminate (J,for i from r to d*r-1 list R_i);
```
The final output \textbf{IAG} is the required ideal:

\[ I(A_G^7) = \langle x_6^3x_7 - x_6x_7^3, x_3^3x_7 - x_5x_7^3, x_4^3x_7 - x_4x_7^3, x_3^3x_6 - x_5x_6^3, x_4^3x_6 - x_4x_6^3, x_4^3x_5 - x_4x_5^3, x_3^3x_3 - x_2x_3^3, x_3^3x_3 - x_1x_3^3, x_1^3x_2 - x_1x_2^3 \rangle. \]

\section*{References}

[1] Taylor Ball, Eduardo Camps, Henry Chimal-Dzul, Delio Jaramillo-Velez, Hiram Lópex, Nathan Nichols, Matthew Perkins, Ivan Soprunov, German Vera-Martínez, and Gwyn Whieldon. Coding theory package for Macaulay2. \textit{J. Softw. Algebra Geom.}, 11(1):113–122, 2021.

[2] Esma Baran and Mesut Şahîn. On parameterised toric codes. \textit{Appl. Algebra Engrg. Comm. Comput.}, 2021.

[3] Esma Baran-Özkan. Vanishing ideals of parameterized subgroups in a toric variety. \textit{Turk J Math}, to appear, 2022.

[4] Peter Beelen, Mrinmoy Datta, and Sudhir R. Ghorpade. Vanishing ideals of projective spaces over finite fields and a projective footprint bound. \textit{Acta Math. Sin. (Engl. Ser.)}, 35(1):47–63, 2019.

[5] Gavin Brown and Alexander M. Kasprzyk. Seven new champion linear codes. \textit{LMS J. Comput. Math.}, 16:109–117, 2013.

[6] Gavin Brown and Alexander M. Kasprzyk. Small polygons and toric codes. \textit{J. Symbolic Comput.}, 51:55–62, 2013.

[7] Pinar Celebi-Demirarslan and Ivan Soprunov. On dual toric complete intersection codes. \textit{Finite Fields Appl.}, 33:118–136, 2015.

[8] David Cox, John Little, and Donal O’Shea. \textit{Ideals, varieties, and algorithms}. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2007. An introduction to computational algebraic geometry and commutative algebra.

[9] David A. Cox. The homogeneous coordinate ring of a toric variety. \textit{J. Algebraic Geom.}, 4(1):17–50, 1995.

[10] Eduardo Dias and Jorge Neves. Codes over a weighted torus. \textit{Finite Fields Appl.}, 33:66–79, 2015.

[11] David Eisenbud and Bernd Sturmfels. Binomial ideals. \textit{Duke Math. J.}, 84(1):1–45, 1996.

[12] D. G. Glynn and J. W. P. Hirschfeld. On the classification of geometric codes by polynomial functions. \textit{Des. Codes Cryptogr.}, 6(3):189–204, 1995.

[13] Johan P. Hansen. Toric varieties Hirzebruch surfaces and error-correcting codes. \textit{Appl. Algebra Engrg. Comm. Comput.}, 13(4):289–300, 2002.

[14] Johan P. Hansen. Secret sharing schemes with strong multiplication and a large number of players from toric varieties. In \textit{Arithmetic, geometry, cryptography and coding theory}, volume 686 of \textit{Contemp. Math.}, pages 171–185. Amer. Math. Soc., Providence, RI, 2017.

[15] Jürgen Herzog. Generators and relations of abelian semigroups and semigroup rings. \textit{Manuscripta Math.}, 3:175–193, 1970.

[16] Jürgen Herzog, Takayuki Hibi, and Hidefumi Ohsugi. \textit{Binomial ideals}, volume 279 of \textit{Graduate Texts in Mathematics}. Springer, Cham, 2018.

[17] Nathan Jacobson. \textit{Basic algebra. I}. W. H. Freeman and Company, New York, second edition, 1985.

[18] Ray Joshua and Reza Akhtar. Toric residue codes. I. \textit{Finite Fields Appl.}, 17(1):15–50, 2011.

[19] David Joyner. Toric codes over finite fields. \textit{Appl. Algebra Engrg. Comm. Comput.}, 15(1):63–79, 2004.

[20] Thomas Kahle. Decompositions of binomial ideals. \textit{J. Softw. Algebra Geom.}, 4:1–5, 2012.

[21] John Little and Hal Schenck. Toric surface codes and Minkowski sums. \textit{SIAM J. Discrete Math.}, 20(4):999–1014, 2006.

[22] John B. Little. Remarks on generalized toric codes. \textit{Finite Fields Appl.}, 24:1–14, 2013.

[23] John B. Little. Toric codes and finite geometries. \textit{Finite Fields Appl.}, 45:203–216, 2017.

[24] Dany-Jack Mercier and Robert Rolland. Polynômes homogènes qui s’annulent sur l’espace projectif $\mathbb{P}^n(F_q)$. \textit{J. Pure Appl. Algebra}, 124(1-3):227–240, 1998.

[25] Ezra Miller and Bernd Sturmfels. \textit{Combinatorial Commutative Algebra}. Cambridge Studies in Advanced Mathematics. Springer-Verlag New York, 2005.

[26] Jade Nardi. Algebraic geometric codes on minimal Hirzebruch surfaces. \textit{J. Algebra}, 535:556–597, 2019.

[27] Jade Nardi. Projective toric codes. \textit{Int. J. Number Theory}, 18(1):179–204, 2022.
[28] Jorge Neves. Regularity of the vanishing ideal over a bipartite nested ear decomposition. *J. Algebra Appl.*, 19(7):2050126, 28, 2020.

[29] Jorge Neves and Maria Vaz Pinto. Vanishing ideals over complete multipartite graphs. *J. Pure Appl. Algebra*, 218(6):1084–1094, 2014.

[30] Jorge Neves, Maria Vaz Pinto, and Rafael H. Villarreal. Joins, ears and Castelnuovo-Mumford regularity. *J. Algebra*, 560:67–88, 2020.

[31] Ignacio Ojeda Martínez de Castilla and Ramón Peidra Sánchez. Cellular binomial ideals. Primary decomposition of binomial ideals. *J. Symbolic Comput.*, 30(4):383–400, 2000.

[32] Carlos Rentería-Márquez, Aron Simis, and Rafael H. Villarreal. Algebraic methods for parameterized codes and invariants of vanishing ideals over finite fields. *Finite Fields Appl.*, 17(1):81–104, 2011.

[33] Diego Ruano. On the parameters of r-dimensional toric codes. *Finite Fields Appl.*, 13(4):962–976, 2007.

[34] Diego Ruano. On the structure of generalized toric codes. *J. Symbolic Comput.*, 44(5):499–506, 2009.

[35] Mesut Sahin. Toric codes and lattice ideals. *Finite Fields Appl.*, 52:243–260, 2018.

[36] Mesut Sahin and Ivan Soprunov. Multigraded Hilbert functions and toric complete intersection codes. *J. Algebra*, 459:446–467, 2016.

[37] Zekiye Sahin-Eser and Laura Felicia Matusevich. Decompositions of cellular binomial ideals. *J. Lond. Math. Soc. (2)*, 94(2):409–426, 2016.

[38] Ivan Soprunov. Toric complete intersection codes. *J. Symbolic Comput.*, 50:374–385, 2013.

[39] Ivan Soprunov and Jenya Soprunova. Toric surface codes and Minkowski length of polygons. *SIAM J. Discrete Math.*, 23(1):384–400, 2008/09.

[40] Ivan Soprunov and Jenya Soprunova. Bringing toric codes to the next dimension. *SIAM J. Discrete Math.*, 24(2):655–665, 2010.

[41] Azucena Tochimani and Rafael H. Villarreal. Vanishing ideals over finite fields. *Math Notes*, 105:429–438, 2019.

[42] Maria Vaz Pinto and Rafael H. Villarreal. The degree and regularity of vanishing ideals of algebraic toric sets over finite fields. *Comm. Algebra*, 41(9):3376–3396, 2013.

Department of Mathematics, Hacettepe University, Ankara, TURKEY

Email address: mesut.sahin@hacettepe.edu.tr