On the Variational Distance of Independently Repeated Experiments

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Abstract

Let $P$ and $Q$ be two probability distributions which differ only for values with probability at least $p > 0$. We show that the variational distance $\delta(P^n, Q^n)$ between the $n$-fold product distributions $P^n$ and $Q^n$ is upper bounded by $\sqrt{n/(2p)} \delta(P, Q)$, i.e., it cannot grow faster than the square root of $n$.

1 Preliminaries

Let $P$ be a probability distribution with range $Z$ and let $n \in \mathbb{N}$. We denote by $P^n$ the $n$-fold product distribution, that is,

$$P^n(z_1, \ldots, z_n) = \prod_{i=1}^{n} P(z_i)$$

for any $z_1, \ldots, z_n \in Z$. Note that $P^n$ describes $n$ independently repeated random experiments with distribution $P$.

The variational distance between two probability distributions $P$ and $Q$ with range $Z$ is defined as

$$\delta(P, Q) := \frac{1}{2} \sum_{z \in Z} |P(z) - Q(z)| .$$

Note that $\delta$ is a distance measure on the set of probability distributions with range $Z$. In particular, $\delta$ is symmetric, $\delta(P, Q) = 0$ if and only if $P = Q$, and the triangle inequality

$$\delta(P, Q) \leq \delta(P, P') + \delta(P', Q)$$

1See, e.g., [1]. $\delta(\cdot, \cdot)$ is also called statistical difference, Kolmogorov distance, or trace distance [2].
Main Result and Proof}

2.1 Upper Bounds for the Variational Distance

Let \( P \) and \( Q \) be two probability distributions with range \( \mathcal{Z} \). It is a direct consequence of the triangle inequality \( \| \) that the variational distance \( \delta(P^n, Q^n) \) between the \( n \)-fold product distributions \( P^n \) and \( Q^n \) cannot grow faster than linearly in \( n \), i.e.,

\[
\delta(P^n, Q^n) \leq n \delta(P, Q).
\] (2)

Moreover, it is easy to find examples where this inequality is almost tight. Let, e.g., \( P \) and \( Q \) be two binary distributions with range \( \mathcal{Z} = \{0, 1\} \) such that \( P(1) = 0 \) and \( Q(1) = \varepsilon \) for some \( \varepsilon > 0 \). If \( n\varepsilon \ll 1 \) then the variational distance \( \delta(P^n, Q^n) \) is roughly equal to \( n \delta(P, Q) = n\varepsilon \).

However, the upper bound \( \| \) can only be close to optimal if, for some element \( z \in \mathcal{Z} \), the relative difference \( \frac{|P(z) - Q(z)|}{P(z) + Q(z)} \) between the probabilities is large. (Note that, in the above example, this relative difference for \( z = 1 \) equals one.) Indeed, the following result states that, in all other cases, \( \delta(P^n, Q^n) \) cannot grow more than the square root of \( n \).

**Lemma 2.1.** Let \( P \) and \( Q \) be two probability distributions with range \( \mathcal{Z} \), let \( \mathcal{D} := \{ z \in \mathcal{Z} : P(z) \neq Q(z) \} \) be the subset of \( \mathcal{Z} \) where \( P \) and \( Q \) differ, and let \( \tilde{p} := \inf_{z \in \mathcal{D}} (\min(P(z), Q(z))) \). If \( \tilde{p} > 0 \) then, for any \( n \in \mathbb{N} \),

\[
\delta(P^n, Q^n) \leq \sqrt{\frac{1}{\pi \tilde{p}}} \sqrt{n + \frac{1}{\tilde{p}}} \delta(P, Q)
\]

and

\[
\delta(P^n, Q^n) \leq \sqrt{\frac{n}{2\tilde{p}}} \delta(P, Q).
\]

The first bound of Lemma 2.1 is optimal in the sense that, for any \( \tilde{p} > 0 \), there are probability distributions \( P \) and \( Q \) with minimum probability \( \tilde{p} \) such that the quotient between the left and the right hand side of the inequality approaches one for increasing \( n \) (as long as \( \delta(P^n, Q^n) \ll 1 \)). On the other hand, the constant \( \frac{1}{\pi \tilde{p}} \) in the second bound is the smallest value of \( c \) such that \( \delta(P^n, Q^n) \leq \sqrt{cn/\tilde{p}} \delta(P, Q) \) always holds.

The result of Lemma 2.1 is also complementary to a lower bound derived in [4], which states that, if the maximum probabilities of \( P \) and \( Q \) are small enough and if \( n \) is not too large, then \( \delta(P^n, Q^n) \geq \Omega(\sqrt{n}) \delta(P, Q) \).
2.2 Proof of Lemma 2.1

To prove Lemma 2.1, we consider a probability distribution $P_t$ parameterized by some real value $t$ and compute a bound on the derivative, with respect to $t$, of the variational distance $\delta(P^n_t, P^n_{t_0})$.

**Lemma 2.2.** Let $\{P_t\}_{t \in \mathbb{R}}$ be a family of probability distributions with range $\mathcal{Z}$ parameterized by $t \in \mathbb{R}$, let $t_0 \in \mathbb{R}$, and let $z_1, z_2 \in \mathcal{Z}$ such that

$$
\frac{d}{dt} P_t(z) \bigg|_{t=t_0} = \begin{cases} 
1 & \text{if } z = z_1 \\
-1 & \text{if } z = z_2
\end{cases}
$$

and $P_t(z) = P_{t_0}(z)$ for any $z \in \mathcal{Z} \setminus \{z_1, z_2\}$. If $p := P_{t_0}(z_1)$ and $p' := P_{t_0}(z_2)$ are nonzero then

$$
\frac{d}{dt} \delta(P^n_t, P^n_{t_0}) \bigg|_{t=t_0} \leq \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{p} + \frac{1}{p'}} \sqrt{n + \frac{1}{\min(p, p')}}
$$

and

$$
\frac{d}{dt} \delta(P^n_t, P^n_{t_0}) \bigg|_{t=t_0} \leq \frac{1}{2} \sqrt{\frac{1}{p} + \frac{1}{p'}} \sqrt{n} .
$$

**Proof.** Assume without loss of generality that $t_0 = 0$. Since $P_t(z)$ does not depend on $t$ for $z \in \mathcal{Z} \setminus \{z_1, z_2\}$, we have $P_t(z_1) + P_t(z_2) = p + p'$, and thus, by the definition of the variational distance,

$$
\delta(P^n_t, P^n_0) = \frac{1}{2} \sum_{k=0}^{n} q(k) \sum_{r=0}^{k} \binom{k}{r} |a_{k,r}(t)|
$$

where

$$
q(k) := \binom{n}{k} (p + p')^k (1 - p - p')^{n-k}
$$

and

$$
a_{k,r}(t) := \left( \frac{p}{p + p'} \right)^r \left( \frac{p'}{p + p'} \right)^{k-r} \left( \frac{P_t(z_1)}{p + p'} \right)^r \left( \frac{P_t(z_2)}{p + p'} \right)^{k-r} .
$$

Let $a'_{k,r}(0) := \frac{d}{dt} a_{k,r}(t) \big|_{t=0}$. Because $a_{k,r}(0) = 0$, we have

$$
\frac{d}{dt} |a_{k,r}(t)| \big|_{t=0} = |a'_{k,r}(0)| .
$$

$^2$ $\frac{d}{dt} f(t) \big|_{t=t_0}$ denotes the right derivative $\lim_{t \to t_0^+} \frac{f(t) - f(t_0)}{t - t_0}$ of the function $f$ at $t = t_0$. 

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Moreover, it follows from (11) that $a'_{k,r}(0) \geq 0$ if and only if $r \leq \bar{r} := \lfloor k \frac{p + p'}{p} \rfloor$. We thus conclude

\[
\frac{d}{dt} \delta(P^n_t, P^n_0) \bigg|_{t=0} = \sum_{k=0}^{n} q(k) \sum_{r=0}^{\bar{r}} \binom{k}{r} a'_{k,r}(0). \tag{3}
\]

Let $\alpha := \frac{p}{p + p'}$ and $\beta := \frac{p'}{p + p'}$. The last sum in the above expression can then be rewritten as

\[
\sum_{r=0}^{\bar{r}} \binom{k}{r} a'_{k,r}(0) = \frac{1}{p + p'} \sum_{r=0}^{\bar{r}} \left[ \binom{k}{r} (k - r) \alpha^r \beta^{k-r-1} - \binom{k}{r} r \alpha^{r-1} \beta^{k-r} \right]
\]

\[
= \frac{k}{p + p'} \sum_{r=0}^{\bar{r}} \binom{k-1}{r} \alpha^r \beta^{k-r-1} - \sum_{r=0}^{\bar{r}-1} \binom{k-1}{r} \alpha^r \beta^{k-r-1}
\]

\[
= \frac{k}{p + p'} \binom{k-1}{\bar{r}} \alpha^{\bar{r}} \beta^{k-\bar{r}-1}. \tag{4}
\]

With the definition $\tilde{\alpha} := \frac{\bar{r} + 1}{k+1}$, $\tilde{\beta} := \frac{(k+1) - (\bar{r} + 1)}{k+1}$, we have

\[
\binom{k-1}{\bar{r}} \alpha^{\bar{r}} \beta^{k-\bar{r}-1} = \binom{k+1}{\bar{r} + 1} \frac{(k+1)(k+1 - (\bar{r} + 1)) \alpha^{\bar{r} + 1} \beta^{(k+1) - (\bar{r} + 1)}}{\alpha \beta}
\]

\[
\leq \binom{k+1}{\bar{r} + 1} \frac{(k+1)(k+1 - (\bar{r} + 1)) \tilde{\alpha}^{\bar{r} + 1} \tilde{\beta}^{(k+1) - (\bar{r} + 1)}}{\alpha \beta}
\]

\[
\leq \sqrt{\frac{1}{2\pi(k+1)\tilde{\alpha} \tilde{\beta}}} \frac{(\bar{r} + 1)((k+1) - (\bar{r} + 1))}{k(k+1)\alpha \beta}
\]

\[
= \frac{1}{k} \sqrt{\frac{k + 1}{2\pi \alpha \beta}} \sqrt{\frac{\tilde{\alpha} \tilde{\beta}}{\alpha \beta}}. \tag{5}
\]

where the first inequality follows from (12) and the second from Lemma A.1. Using the definition of $\bar{r}$ and letting $\gamma := k\alpha - \lfloor k\alpha \rfloor$, the expression in the second square root of the last term can be bounded by

\[
\frac{\tilde{\alpha} \tilde{\beta}}{\alpha \beta} = \frac{|k\alpha| + 1}{\alpha(k+1)} \cdot \frac{(k+1) - (\lfloor k\alpha \rfloor + 1)}{\beta(k+1)} = \frac{k\alpha + (1 - \gamma)}{\alpha(k+1)} \cdot \frac{k\beta + \gamma}{\beta(k+1)}
\]

\[
= \frac{k + \frac{1-\gamma}{\alpha}}{k + 1} \cdot \frac{k + \frac{\gamma}{\beta}}{k + 1} \leq \frac{k + \frac{1-\min(\alpha, \beta)}{\alpha}}{k + 1}, \tag{6}
\]

where the last inequality follows from the fact that $\frac{1-\gamma}{\alpha} = \frac{1-\gamma}{\beta}$ and $\frac{\gamma}{\beta}$ cannot both be larger than one, since $\beta, \gamma \in [0, 1]$. Combining this with (3) and (4),
we find
\[ \sum_{r=0}^{\bar{r}} \binom{k}{r} a'_{k,r}(0) = \frac{k}{p + p'} \left( \frac{k - 1}{\bar{r}} \right) \alpha^r \beta^{k-r-1} \leq s(k), \] (7)

where
\[ s(k) := \frac{1}{p + p'} \sqrt{\frac{k + \frac{1}{\min(\alpha,\beta)}}{2\pi \alpha \beta}}. \]

Alternatively, the left hand side of (7) can be upper bounded by
\[ s'(k) := \frac{1}{p + p'} \sqrt{\frac{k}{4\alpha \beta}}. \] (8)

To see this, assume first that \( \alpha k \geq 2 \) and \( \beta k \geq 2 \). Then \( \frac{1}{\min(\alpha,\beta)} \leq \frac{k}{2} \) which implies
\[ s(k) \leq \frac{1}{p + p'} \sqrt{\frac{\frac{3}{2} k}{2\pi \alpha \beta}} < s'(k). \]

On the other hand, if \( \alpha k < 2 \) or \( \beta k < 2 \), the bound follows from a straightforward calculation using (12). (In this case, \( \bar{r} \) or \( (k - 1) - \bar{r} \) is either 0 or 1, i.e., the binomial \( \binom{k-1}{\bar{r}} \) in (7) equals 1 or \( k - 1 \)).

When (3) is combined with (7), we obtain
\[ \frac{d}{dt} \|\delta(P^n_t, P^n_0)\|_{t=0} \leq \sum_{k=0}^{n} q(k) s(k). \] (9)

Note that \( s(k) \) is a concave function in \( k \) and that \( q(k) \) are the probabilities of a binomial distribution with mean \( n(p + p') \), that is, \( \sum_{k=0}^{n} q(k) = 1 \) and \( \sum_{k=0}^{n} q(k)k = n(p + p') \). We can thus apply Jensen’s inequality to find an upper bound for the sum on the right hand side of (9), i.e.,
\[ \frac{d}{dt} \|\delta(P^n_t, P^n_0)\|_{t=0} \leq \sum_{k=0}^{n} q(k) s(k) \leq s(\sum_{k=0}^{n} q(k)k) = s(n(p + p')), \]
from which the first inequality of the lemma follows.

Similarly, because \( s'(k) \) is concave in \( k \) as well, we have
\[ \frac{d}{dt} \|\delta(P^n_t, P^n_0)\|_{t=0} \leq s'(n(p + p')), \]
which implies the second inequality of the lemma. \qed
We now use the bounds provided by Lemma 2.2 to prove our main result.

Proof of Lemma 2.1. We first prove the first inequality of the lemma for the special case where the probabilities $P$ and $Q$ only differ for two values $z_1$ and $z_2$. Assume (without loss of generality) that $P(z_1) \leq Q(z_1)$ and let $p := P(z_1)$, $p' := P(z_2)$, $q := Q(z_1)$. For $t \in [p, q]$, let $P_t$ be the distribution with range $Z$ given by

$$P_t := \frac{q-t}{q-p} P + \frac{t-p}{q-p} Q.$$ 

i.e., $P_t(z_1) = t$, $P_t(z_2) = p + p' - t$, and $P_t(z) = P(z) = Q(z)$ for any $z \in Z\{z_1, z_2\}$. We can thus apply the first inequality of Lemma 2.2 which gives

$$\frac{d}{ds} \delta(P^n_s, P^n_t) \big|_{s=t} \leq \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{t} + \frac{1}{p + p' - t}} n + \frac{1}{\min(t, p + p' - t)}\left(\frac{q - p}{q - p}\right) \sqrt{\frac{2}{\bar{p}}} \sqrt{n + \frac{1}{\bar{p}}} .$$

Using Lemma A.2 we obtain

$$\delta(P^n, Q^n) = \delta(P^n_p, P^n_q) \leq \int_p^q \frac{d}{ds} \delta(P^n_s, P^n_t) \big|_{s=t} \, dt \leq (q-p) \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\bar{p}}} \sqrt{n + \frac{1}{\bar{p}}} .$$

Since $q - p = \delta(P, Q)$, this concludes the proof of the first inequality of the lemma for the special case where the probabilities $P$ and $Q$ differ for at most two values in $Z$.

To prove the general case, we first observe that if the set $\mathcal{D}$ is infinite, then $\bar{p}$ equals zero and nothing has to be proven. On the other hand, if $\mathcal{D}$ is finite, it is easy to see that there exists a sequence $(P_i)_{i=1,\ldots,m}$ (for some $m \in \mathbb{N}$) of distributions with range $Z$ such that

- $P_1 = P$ and $P_m = Q$,
- for any $i \in \{1,\ldots,m\}$, the distributions $P_i$ and $P_{i+1}$ differ only for two elements in $\mathcal{D}$,
- $\min_{z \in \mathcal{D}} P_i(z) \geq \bar{p}$, for all $i \in \{1,\ldots,m\}$, and
- $\sum_{i=1}^{m-1} \delta(P_i, P_{i+1}) = \delta(P_1, P_m)$. 

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The general assertion then follows directly from the special case proven above and the triangle inequality (1), i.e.,

\[
\delta(P_n^m, Q_n^m) \leq \sum_{i=1}^{m-1} \delta(P_i^m, P_{i+1}^m) \leq \sum_{i=1}^{m-1} \sqrt{\frac{1}{\pi p}} \sqrt{n + \frac{1}{p}} \delta(P_i, P_{i+1}) \\
= \sqrt{\frac{1}{\pi p}} \sqrt{n + \frac{1}{p}} \delta(P, Q) .
\]

The second inequality follows by exactly the same reasoning based on the second inequality of Lemma 2.2.

\[\Box\]

A Appendix: Some Useful Identities

A.1 An Upper Bound for the Binomial Coefficient

Lemma A.1. For any \( n \geq k > 0 \),

\[
\binom{n}{k} \left( \frac{k}{n} \right)^k \left( \frac{n-k}{n} \right)^{n-k} \leq \sqrt{\frac{n}{2\pi k(n-k)}} .
\]

Proof. The assertion follows directly from Stirling’s approximation \([2]\),

\[
\sqrt{\frac{2\pi n}{e}} n^{n+1/2} e^{-n+1/(12n)} < n! < \sqrt{\frac{2\pi n}{e}} n^{n+1/2} e^{-n+1/(12n)}
\]

for \( n > 0 \).

\[\Box\]

A.2 Bounding the Variational Distance by Its Path Integral

Lemma A.2. Let \( a < b \), let \( \{P_t\}_{t \in \mathbb{R}} \) be a family of probability distributions parameterized by \( t \in [a, b] \), and let \( f(t) := \frac{d}{ds} \delta(P_s, P_t) \big|_{s=t} \) be the right derivative of the variational distance. Then

\[
\delta(P_b, P_a) \leq \int_a^b f(t)dt .
\]

Proof. Since equality holds for \( b = a \), it suffices to verify that

\[
\frac{d}{dr} \delta(P_r, P_a) \leq \frac{d}{dr} \int_a^r f(t)dt ,
\]

for any \( r \in (a, b) \). Using the triangle inequality for the variational distance, the expression on the left hand side can be bounded by

\[
\frac{d}{dr} \delta(P_r, P_a) = \lim_{\varepsilon \to 0^+} \frac{\delta(P_{r+\varepsilon}, P_a) - \delta(P_r, P_a)}{\varepsilon} \leq \lim_{\varepsilon \to 0^+} \frac{\delta(P_{r+\varepsilon}, P_r)}{\varepsilon} = f(r) .
\]
Inequality \((10)\) then follows from the second fundamental theorem of calculus, which concludes the proof.

\[ \frac{d}{dx} x^k(1-x)^{n-k} \geq 0 \iff x \leq \frac{k}{n}. \]  \hspace{1cm} (11)

As an immediate consequence of this expression we have

\[ x^k(1-x)^{n-k} \leq \left( \frac{k}{n} \right)^k \left( 1 - \frac{k}{n} \right)^{n-k}. \]  \hspace{1cm} (12)

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