Modified amplitude of the gravitational wave spectrum

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Abstract

The spectrum of thermal gravitational waves is obtained by including the high-frequency thermal gravitons created from extra-dimensional effects and is a new feature of the spectrum. The amplitude and spectral energy density of gravitational waves in a thermal vacuum state are found to be enhanced. The amplitude of the waves is modified in the frequency range \(10^{-16}–10^{8}\) Hz but the corresponding spectral energy density is less than the upper bound of various estimated results. With the addition of higher frequency thermal waves, the obtained spectral energy density of the wave in the thermal vacuum state does not exceed the upper bound put by the nucleosynthesis rate. The existence of cosmologically originated thermal gravitational waves due to extra dimension is not ruled out.

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(Some figures may appear in colour only in the online journal)

1. Introduction

One of the predictions of the general theory of relativity is the existence of gravitational waves. The sources of generation of these waves vary from the dynamics of the early universe to massive astrophysical objects such as neutron star binaries and black hole mergers etc. Thus the waves have a wide spectrum of frequencies, i.e. they vary from very low to high \(O(10^{-19}–10^{10})\)Hz. It is possible to discriminate the relic waves from other sources from the observational point of view also. The relic gravitational waves are of paramount importance in cosmology because they provide valuable information on the conditions of the very early universe. It is believed that the relic gravitational waves are mainly generated during the inflationary epoch, and the waves that amplified during the inflation are of low frequency only. Since the higher frequency waves are outside the ‘barrier’ (horizon) \[1\], the corresponding amplitudes decreased during the evolution of the universe. The features of relic gravitational waves of very high frequency range are interesting, though the energy scale of conventional

\[1\] The terminology ‘barrier’ is adopted from [1].
inflationary models does not favour it. However if extra dimensions exist (for a review, see [2] and motivation for extra dimensions [3]), then the graviton background can have a thermal spectrum [4]. According to the extra-dimensional models, the thermal gravitons with a very high frequency range can also be observed with a specific peak temperature today [4] and therefore the detection of very high frequency thermal gravitational waves is an interesting test to see the possibility of existence of extra dimensions as well. These thermal gravitational waves can contribute to the higher frequency range of the spectrum. The existence of thermal graviton background with the black body type spectrum is also discussed in [5, 6]. If the inflation were preceded by a radiation era, then there would be thermal gravitational waves at the time of inflation [6]. The generation of tensor perturbations during inflation by the stimulated emission process leads to the existence of thermal gravitational waves [7]. Direct detection of the thermal gravitons is challenging but may be possible in the near future with the 21 cm emission line of atomic hydrogen.

The inflationary scenario [8] predicts a stochastic cosmic background of gravitational waves (CGWB) [5]. The spectrum of these relic gravitational waves depends not only on the details of expansion during the inflationary era but also the subsequent stages, including the current epoch of the universe. Computation of the spectrum of the waves for matter dominated universe is usually done in decelerated expanding model [9–14]. The resulting spectrum is used for putting constrain on the detection of gravitational waves originating from sources other than early universe epoch. The result of astronomical observations on SN Ia [15, 16] shows that the universe is currently undergoing accelerated expansion indicating a non-zero cosmological constant. According to the ΛCDM concordance model, the observed acceleration of the present universe is supposed to be driven by the dark energy. Effect of the current acceleration on the nature of the spectrum and spectral energy density of the relic gravitational waves is studied [10, 17], and it is shown that the current acceleration phase of the universe does change the shape, amplitude and spectrum of the waves [17].

In this work, we consider contribution of very high frequency relic thermal gravitational waves to its spectrum and spectral energy density for the decelerated as well as accelerated universe. The focus of this work is on the spectrum of the higher frequency range of the waves due to extra-dimensional effects. The normalization of the spectrum is being done with the measured CMB anisotropy spectrum of the WMAP. The inclusion of the higher frequency relic thermal gravitational waves leads to enhancement of the spectrum. This enhancement leads to the modification of the amplitude of the spectrum in the frequency range \(10^{-16} - 10^8\) Hz as an additional feature and is possible to compare these with the sensitivity of Advanced.LIGO (Adv.LIGO), Einstein telescope (ET) and LISA missions. The corresponding spectral energy density can be compared with estimated upper bound of various studies. Also, we can check whether or not the inclusion of higher frequency thermal gravitational waves in the total spectral energy density exceeds the upper bound of the primordial nucleosynthesis rate. In this work, we use the unit \(c = ħ = k_B = 1\).

2. Gravitational wave spectrum in expanding universe

The perturbed metric for a homogeneous isotropic flat Friedmann–Robertson–Walker (FRW) universe can be written as

\[ ds^2 = S^2(\eta)(d\eta^2 - (\delta_{ij} + h_{ij})\, dx^i\, dx^j), \]

where \(S(\eta)\) is the cosmological scale factor, \(\eta\) is the conformal time and \(\delta_{ij}\) is the Kronecker delta symbol. The \(h_{ij}\) are metric perturbation fields that contain only the pure gravitational waves and are transverse-traceless, i.e. \(\nabla_i h^{ij} = 0\), \(\delta^{ij} h_{ij} = 0\).
This study mainly deals with amplitude and spectral energy density of the relic gravitational waves generated by the expanding spacetime background. Thus the perturbed matter source is therefore not taken into account. The gravitational waves are described by the linearized field equation given by
\[ \nabla_\mu (\sqrt{-g} \nabla^\mu h_{ij}(x, \eta)) = 0. \] (2)

The tensor perturbations have two independent physical degrees of freedom and are denoted as \( h^+ \) and \( h^\times \), called polarization modes. To compute the spectrum of gravitational waves \( h(x, \eta) \) in the thermal states, we express \( h^+ \) and \( h^\times \) in terms of the creation (\( a^\dagger \)) and annihilation (\( a \)) operators,
\[ h_{ij}(x, \eta) = \frac{\sqrt{16\pi} l_{pl}}{S(\eta)} \sum \int \frac{d^3k}{(2\pi)^{3/2}} \epsilon^p_{ij}(k) \frac{1}{\sqrt{2k}} [ a^\dagger_{ij} h^p_{k}(\eta) e^{i k \cdot x} + a_{ij} h^p_{k}(\eta) e^{-i k \cdot x} ], \] (3)

where \( k \) is the comoving wave number, \( k = |k| \), \( l_{pl} = \sqrt{\hbar G} \) is the Planck length and \( p = +, \times \) are polarization modes. The polarization tensor \( \epsilon^p_{ij}(k) \) is symmetric and transverse-traceless \( \epsilon^p_{ij}(k) = 0, \delta^p_{ij} \epsilon^p_{ij}(k) = 0 \) and satisfies the conditions \( \epsilon^p_{ij}(k) \epsilon^q_{ij}(k) = 2\delta^{pq} \) and \( \epsilon^p_{ij}(-k) = \epsilon^p_{ij}(k) \), the creation and annihilation operators satisfy \( [a^\dagger_{ij}, a_{ij}] = \delta_{ij} \), and the initial vacuum state is defined as
\[ a^\dagger_{ij}|0\rangle = 0, \] (4)

for each \( k \) and \( p \). The energy density of the gravitational waves in the vacuum state is
\[ \epsilon(\eta) = \frac{1}{8 \pi l^2_{pl}} h_{ij}(\eta) h_{ij}^{(0)} \]

For a fixed wave number \( k \) and a fixed polarization state \( p \), the linearized wave equation (2) gives
\[ \ddot{h}^\nu + 2 \frac{S'}{S} \dot{h}^\nu + k^2 h^\nu = 0, \] (5)

where the prime means the derivative with respect to the conformal time. Since the polarization states are the same, we here onwards denote \( h_\nu(\eta) \) without the polarization index.

Next, we rescale the field \( h_\nu(\eta) \) by taking \( h_\nu(\eta) = f_\nu(\eta)/S(\eta) \), where the mode functions \( f_\nu(\eta) \) obey the minimally coupled Klein–Gordon equation
\[ f''_\nu + \left( \frac{k^2}{S} - \frac{S'}{S} \right) f_\nu = 0. \] (6)

The general solution of the above equation is a linear combination of the Hankel function with a generic power law for the scale factor \( S = \eta^q \) given by
\[ f_\nu(\eta) = A_k \sqrt{k \eta} H^{(1)}_{-q-\frac{1}{2}}(k \eta) + B_k \sqrt{k \eta} H^{(2)}_{-q-\frac{1}{2}}(k \eta). \] (7)

For a given model of the expansion of universe, consisting of a sequence of successive scale factors with different \( q \), we can obtain an exact solution \( f_\nu(\eta) \) by matching its value and derivative at the joining points.

The approximate computation of the spectrum of gravitational waves is usually performed in two limiting cases depending upon the waves that are within or outside of the barrier. For the gravitational waves outside the barrier \( (k^2 \gg S'/S, \text{ short wave approximation}) \), the corresponding amplitude decreases as \( h_\nu \propto 1/S(\eta) \), while for the waves inside the barrier \( (k^2 \ll S'/S, \text{ long wave approximation}) \), \( h_\nu = C_k \) is simply a constant. Thus these results can be used to estimate the spectrum for the present epoch of universe.

The history of overall expansion of the universe can be modelled as the following sequence of successive epochs of power-law expansion.
The initial stage (inflationary)
\[ S(\eta) = l_0|\eta|^{1+\beta}, \quad -\infty < \eta \leq \eta_1, \]  
(8)
where \( 1 + \beta < 0, \eta < 0 \) and \( l_0 \) is a constant.

The \( z \)-stage
\[ S(\eta) = S_z(\eta - \eta_p)^{1+\beta_z}, \quad \eta_1 < \eta \leq \eta_s, \]  
(9)
where \( \beta_z + 1 > 0 \). Towards the end of inflation, during the reheating, the equation of the state of energy in the universe can be quite complicated and is rather model dependent [18]. Hence, this \( z \)-stage is introduced to allow a general reheating epoch, see for details [11].

The radiation-dominated stage
\[ S(\eta) = S_e(\eta - \eta_e)^2, \quad \eta_s \leq \eta \leq \eta_2, \]  
(10)
The matter-dominated stage
\[ S(\eta) = S_m(\eta - \eta_m)^2, \quad \eta_2 \leq \eta \leq \eta_E. \]  
(11)
where \( \eta_E \) is the time when the dark energy density \( \rho/\Lambda_1 \) is equal to the matter energy density \( \rho_m \).

Before the discovery of accelerating expansion of the universe, the current expansion used to be considered as decelerating because of the matter-dominated stage. Thus, following the matter-dominated stage, it is reasonable to add an epoch of accelerating stage, which is probably driven by either the cosmological constant, or the quintessence, or some other kind of condensate [19]. The value of redshift \( z_E \) at \( \eta_E \) is 
\[ (1 + z_E) = \frac{S(\eta_0)}{S(\eta_E)}, \]  
(12)
where \( \eta_0 \) is the present time. Since \( \rho_\Lambda \) is constant and \( \rho_m(\eta) \propto S^{-3}(\eta) \), we obtain
\[ \frac{\rho_\Lambda}{\rho_m(\eta_E)} = \frac{\rho_\Lambda}{\rho_m(\eta_0)(1 + z_E)^3} = 1. \]  
(13)

The accelerating stage (up to the present)
\[ S(\eta) = \ell_0|\eta - \eta_a|^{-1}, \quad \eta_E \leq \eta \leq \eta_0. \]  
(14)
This stage describes the accelerating expansion of the universe and is a new feature; hence its influence on the spectrum of relic gravitational waves is of interest to study. Note that the actual scale factor function \( S(\eta) \) differs from equation (14), since the matter component exists in the current universe. However, the dark energy is dominant; therefore, (14) is an approximation of the current expansion behaviour.

Given \( S(\eta) \) for the various epochs, the derivative \( S' = dS/d\eta \) and ratio \( S'/S \) follow immediately. Except for \( \beta_z \) which is imposed upon as the model parameter, there are ten constants in the expressions of \( S(\eta) \). By the continuity conditions of \( S(\eta) \) and \( S'(\eta) \) at four given joining points \( \eta_1, \eta_2, \eta_E \), and one can fix only eight constants. The remaining two constants can be fixed by the overall normalization of \( S \) and the observed Hubble constant as the expansion rate. Specifically, we put \( |\eta_0 - \eta_a| = 1 \) for the normalization of \( S \), which fixes the \( \eta_a \), and the constant \( \ell_0 \) is fixed by the following calculation:
\[ \frac{1}{H} \equiv \left( \frac{\dot{S}^2}{S} \right)_{\eta_0} = \ell_0, \]  
(15)
where \( \ell_0 \) is the Hubble radius at present.

In the expanding FRW spacetime, the physical wavelength is related to the comoving wave number as \( \lambda \equiv \frac{2\pi k_0}{k} \), and the wave number \( k_0 \) corresponding to the present Hubble
radius is $k_0 = \frac{2\pi S(n_0)}{\Omega_0} = 2\pi$. And there is another wave number $k_0 = \frac{2\pi S(n_0)}{1/\Omega} = \frac{k_0}{1+\omega}$, whose corresponding wavelength at the time $\eta_k$ is the Hubble radius $1/H$.

By matching $S$ and $\dot{S}/S$ at the joint points, one obtains

$$I_0 = \xi_0 b\xi_1^{(2+\beta)} \xi_2 \xi_1^{\frac{2+\beta}{1+\omega}},$$

(16)

where $b \equiv \|1 + \beta\|^{-(2+\beta)}$, which is defined differently from [20], $\xi_0 \equiv S(n_0)/S(n_0)$, $\xi_1 \equiv S(n_0)/S(n_0)$, $\xi_2 \equiv S(n_0)/S(n_0)$, and $\xi_1 \equiv S(n_0)/S(n_0)$. With these specifications, the functions $S(\eta)$ and $\dot{S}(\eta)/S(\eta)$ are fully determined. In particular, $S'(\eta)/S(\eta)$ rises up during the accelerating stage, instead of decreasing as in the matter-dominated stage. This causes the modifications to the spectrum of relic gravitational waves.

3. Gravitational wave spectrum in the thermal vacuum state

The power spectrum of gravitational waves is defined as

$$\int_0^\infty h^2(k, \eta) \frac{dk}{k} = \langle 0|\hat{h}^2(x, \eta)\hat{h}_I(x, \eta)|0\rangle.$$  

(17)

Substituting equation (3) in (17) and assuming that the contribution from each polarization is the same, we obtain

$$h(k, \eta) = \frac{A|\xi_1|^{2+\beta} h|\eta(\eta)|k|\eta(\eta)|.}$$

(18)

Thus once the mode function $h(\eta)$ is known, the spectrum $h(k, \eta)$ follows.

The spectrum at the present time $h(k, \eta_0)$ can be obtained, provided the initial spectrum is specified. The initial condition is considered to be the inflationary stage. Thus the initial amplitude of the spectrum is given by

$$h(k, \eta_i) = A \left(\frac{k}{k_0}\right)^{2+\beta},$$

(19)

where $A = 8\sqrt{\pi/\Omega_0}$ is a constant. The power spectrum for the primordial perturbation of energy density is $P(k) \propto |h(k, \eta_0)|^2$ and in terms of initial spectral index $n$, it is defined as $P(k) \propto k^{n-1}$. Thus the scale invariant spectral index $n = 1$ for the pure de Sitter expansion can be obtained by the relation $n = 2\beta + 5$ for $\beta = -2$.

An effective approach to deal with the thermal vacuum state is the thermo-field dynamics (TFD)[21]. In this approach, a tilde space is needed besides the usual Hilbert space, and the direct product space is made up of these two spaces. Every operator and state in the Hilbert space has the corresponding counterpart in the tilde space [21]. Therefore, a thermal vacuum state ($Tv$) can be defined as

$$|Tv\rangle = \mathcal{T}(\theta_k)(0 0\rangle,$$

(20)

where

$$\mathcal{T}(\theta_k) = \exp\left[-\theta_k(a_k a_k^\dagger - a_k^\dagger a_k^\dagger)\right].$$

(21)

is the thermal operator and $|0 0\rangle$ is the two-mode vacuum state at zero temperature. The quantity $\theta_k$ is related to the average number of the thermal particle, $\bar{n}_k = \sinh^2\theta_k$. The $\bar{n}_k$ for given temperature $T$ is provided by the Bose–Einstein distribution $\bar{n}_k = [\exp(k/T) - 1]^{-1}$, where $\bar{\omega}_k$ is the resonance frequency of the field. The $a_k$, $a_k^\dagger$ and $\bar{a}_k$, $\bar{a}_k^\dagger$, are, respectively, the annihilation and creation operators in Hilbert and tilde space, satisfy the usual commutation
relations, $[a_k, a^\dagger_{k'}] = [\tilde{a}_k, \tilde{a}^\dagger_{k'}] = \delta^3(\mathbf{k} - \mathbf{k}')$, and all other commutation relations of these operators are zero. By the appropriate action of the operator \(21\) on \(a_k\) and \(a^\dagger_k\), we obtain

\[
T a_k T = a_k \cosh \theta_k + \tilde{a}_k \sinh \theta_k, \\
T a^\dagger_k T = a^\dagger_k \cosh \theta_k + \tilde{a}^\dagger_k \sinh \theta_k.
\]  

(22)

Hence the occupation number in the thermal vacuum state can be written as

\[
\langle a^\dagger_k a_{k'} \rangle = \left( \frac{1}{e^{\frac{k}{T}} - 1} \right) \delta^3(\mathbf{k} - \mathbf{k}').
\]  

(23)

Thus, using equation (3) and equations (20)–(23) in equation (17), the power spectrum in the thermal vacuum state is obtained as

\[
h^2_T(k, \eta) = \frac{16l^2_{pl} \pi}{k^2} \left| h(\eta) \right|^2 \coth \left[ \frac{k^2}{2T} \right],
\]  

(24)

Thus in comparison with equation (19), the spectrum is expressed as

\[
h(k, \eta_i) = A \left( \frac{k}{k_0} \right)^{2+\beta} \coth^{1/2} \left[ \frac{k}{2T} \right].
\]  

(25)

The last term becomes significant when the ratio \(k/(2T)\) is less than unity. The wave number \(k\) and temperature \(T\) are comoving quantities which are related to the physical parameters at the time of inflation, see for details \([6]\). Thus an enhancement of the spectrum by a factor \(\coth^{1/2}[k/2T] = \coth^{1/2}[HS_i/2T_i]\) is expected.

It is convenient to consider the amplitude of waves in different range of wave numbers \([17]\). Thus the amplitude of the spectrum in the thermal vacuum state for different ranges is given by the following.

(i) When \(k < k_E\), the corresponding wavelength is greater than the present Hubble radius. Thus the amplitude remains as the initial one and can be written as

\[
h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{2+\beta} \coth^{1/2} \left[ \frac{k}{2T} \right].
\]  

(26)

(ii) The amplitude remains approximately the same as long as the wave remains inside the barrier but begins to decrease when it leaves the barrier by a factor \(1/S(\eta_i)\), depending on the value of the scale factor at that time. This process continues until the barrier becomes higher than \(k\) at a time \(\eta\) earlier than \(\eta_0\), so the amplitude decreases by the ratio of the scale factor at the time of leaving the barrier \(S_b\) to its value at \(\eta, S(\eta)\). This is in the range \(k_E < k < k_0\):

\[
h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{\beta-1} \coth^{1/2} \left[ \frac{k}{2T} \right] \frac{1}{(1+z_E)^3}.
\]  

(27)

Note that this range is a new feature on account of the current acceleration of the universe which is absent in the decelerating model as pointed out in \([17]\). The amplitude of the waves that left the barrier at \(S_b\) with wave numbers \(k > k_0\) has been decreased up to the present time by a factor \(S_b/S(\eta_0)\). This affects the amplitude of the present spectrum and is obtained as

\[
h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{2+\beta} \coth^{1/2} \left[ \frac{k}{2T} \right] \frac{S_b}{S(\eta_0)}.
\]  

(28)

This result can be used to obtain the spectrum of the waves in the remaining range of wave numbers.
(iii) The wave number that does not hit the barrier in the range \( k_0 \leq k \leq k_2 \) gives the amplitude as follows:

\[
 h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^\beta \frac{1}{\coth \left[ \frac{k}{2T} \right]} \frac{1}{(1+z_{E})^3};
\]  

(29)

the spectrum in this interval is different from that of the matter-dominated case by the factor \( \frac{1}{(1+z_{E})^3} \). The wave lengths of the spectrum in the range are long but smaller than the present Hubble radius.

(iv) In the range of wave number \( k_2 \leq k \leq k_s \), the amplitude is

\[
 h(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{1+\beta} \frac{1}{\left( \frac{k_0}{k_2} \right)^{\beta}} \frac{1}{(1+z_{E})^3}.
\]  

(30)

This is the interesting range from the observational point of view of Adv.LIGO, ET and LISA. Note that the temperature-dependent factor in this range is negligible; hence, the term is dropped out because of the low temperature nature of the relic waves.

(v) For the wave number range \( k_s \leq k \leq k_1 \) which is in the high-frequency case and gives the corresponding amplitude as

\[
 h_T(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{1+\beta} \frac{1}{\left( \frac{k_0}{k_2} \right)^{\beta}} \frac{1}{\coth \left[ \frac{k}{2T_s} \right]} \frac{1}{(1+z_{E})^3};
\]  

(31)

In the usual case, the temperature-dependent term can also be neglected; however, the extra-dimensional scenario predicts higher temperature for the thermal gravitational waves, hence the term again becomes significant. Therefore, the contribution from the thermal relic gravitational waves is expected to increase the amplitude of the spectrum, particularly in the higher frequency range.

It is to be noted that in (iv) the thermal contribution in \( k_2 \leq k \leq k_s \) range is negligible due to the temperature-dependent term. Similarly, the thermal effect is insignificant in the range \( k_s \leq k \leq k_1 \) also. However by taking into account the extra-dimensional effect, the spectrum of relic waves is peaked with a temperature \( T_s = 1.19 \times 10^{23} \text{ Mpc}^{-1} \) [4]. (See, appendix A for a brief discussion on \( T_s \) from extra-dimensional scenario.) Therefore, enhancement for the amplitude of the spectrum (orange lines, figures 1 and 4) in the range \( k_s \leq k \leq k_1 \) compared to \( T = 0 \) case for the accelerated as well as decelerated universe is expected. But at the same time, ignoring the thermal contribution to the amplitude of the spectrum in the range \( k_2 \leq k \leq k_s \) leads to a discontinuity at \( k_s \), see figure 1. This is evaded by fitting a new line in the range \( k_2 \leq k \leq k_s \) for the amplitude \( h \) of equation (30) as follows.

Let the amplitude of the wave in the range \( k_0 \leq k \leq k_2 \) be given by (29) and can be rewritten as

\[
 h_{1T}(k, \eta_0) = A \left( \frac{k}{k_0} \right)^\beta \frac{1}{\coth \left[ \frac{k}{2T} \right]} \frac{1}{(1+z_{E})^3};
\]  

(32)

and the amplitude in the \( k_s \leq k \leq k_1 \) is given by (31) which can also be rewritten as

\[
 h_{2T}(k, \eta_0) = A \left( \frac{k}{k_0} \right)^{1+\beta} \frac{1}{\left( \frac{k_0}{k_2} \right)^{\beta}} \frac{1}{\coth \left[ \frac{k}{2T_s} \right]} \frac{1}{(1+z_{E})^3}.
\]  

(33)

Thus the new slope for equation (30), in the range \( k_2 \leq k \leq k_s \), can be obtained by taking \( y = \log_{10}(h) \) and \( x = \log_{10}(k) \); then

\[
 \log_{10}(h) - \log_{10}(h_i) = \frac{\log_{10}(h_f) - \log_{10}(h_i)}{\log_{10}(k_f) - \log_{10}(k_i)} (\log_{10}(k) - \log_{10}(k_i)),
\]  

(34)
where the subscripts $i$ and $f$ indicate the first and last points of the straight line respectively. By putting $k_i \equiv k_2$ from equation (32) and $k_f \equiv k_s$ from equation (33) in equation (34), we obtain the amplitude

$$ h = (h_{1T})_k g(k), $$

where

$$ g(k) = \left( \frac{k}{k_2} \right)^\beta, $$

and

$$ \beta = \frac{\log_{10}(h_{1T})_k - \log_{10}(h_{1T})_k_s}{\log_{10}(k_s) - \log_{10}(k_2)} = \frac{\log_{10}\left(\left(\frac{k_s}{k_2}\right)^{1+\beta} \coth^{1/2}\left(\frac{k_s}{2T_*}\right)\right)}{\log_{10}\left(\frac{k_s}{k_2}\right)}, $$

is the slope of the line and thus we find the amplitude; for convenience we call it as ‘modified amplitude’, given by

$$ h(k, \eta_0) = A \left( \frac{k_2}{k_0} \right)^\beta \frac{1}{(1 + z_E)^3} \left( \frac{k}{k_2} \right)^\gamma. $$

When $T_*$ becomes zero, equation (37) leads to $\gamma = 1 + \beta$, and hence (30) is recovered from (38) in the range $k_2 \leq k \leq k_s$.

The overall multiplication factor $A$ in all the spectra is determined in absence of the temperature-dependent term with the CMB data of WMAP [17]. This is based on the assumption that the contributions from gravitational waves and the density perturbations

\[^2\] Here, $\coth^{1/2}\left(\frac{k_s}{2T_*}\right) = 1$. 

\[^8\]
are of the same order of magnitude, or if the CMB anisotropies at low multipole are induced by the gravitational waves, then it is possible to write $\Delta T / T \simeq h(k, \eta_0)$. The observed CMB anisotropies \cite{23} at lower multipoles are $\Delta T / T \simeq 0.37 \times 10^{-5}$ at $l \sim 2$ which correspond to the largest scale anisotropies that have been observed so far. Thus taking this to be the perturbations at the Hubble radius gives

$$h(k, \eta_0) = A \frac{1}{(1 + zE)^3} = 0.37 \times 10^{-5}. \quad (39)$$

However, there is a subtlety in the interpretation of $\Delta T / T$ at low multipoles, whose corresponding scale is very large $\sim \ell_0$. At present, the Hubble radius is $\ell_0$, and the Hubble diameter is $2\ell_0$. On the other hand, the smallest characteristic wave number is $k_E$, whose corresponding physical wave length at present is $2\pi S(\eta_0)/k_E = \ell_0(1 + zE) \simeq 1.32\ell_0$, which is within the Hubble diameter $2\ell_0$, and is theoretically observable. So, instead of equation (39), if $\Delta T / T \simeq 0.37 \times 10^{-5}$ at $l \sim 2$ were taken as the amplitude of the spectrum at $v_E$, then one would have $h_T(k_E, \eta_0) = A/(1 + zE)^{5/2} = 0.37 \times 10^{-5}$, yielding a smaller $A$ than that in equation (39) by a factor $(1 + zE)^{-1/2} \sim 2.3$ \cite{17}. The allowed values of $\beta$ and $\beta_i$ are obtained and are respectively given by $\beta = -1.9$ and $\beta_i = -0.552$ \cite{17}.

Next, we obtain the spectrum in the thermal vacuum state with the following parameters. By taking $k = 2\pi v$, $v_0 = 1.5 \times 10^{-18}$ Hz, $v_1 = 2 \times 10^{-18}$ Hz, $v_2 = 117 \times 10^{-18}$ Hz, $v_3 = 10^8$ Hz, $v_4 = 3 \times 10^{10}$ Hz (the value of $v_1$ is taken in such a way that spectral energy density does not exceed the level of $10^{-6}$, as required by the rate of primordial nucleosynthesis). The range of frequency is chosen in accordance with generation of gravitational waves that vary from early universe to various astrophysical sources, and hence the range is matching with the interest of CMB, Adv.LIGO, ET and LISA operations for detection of the gravitational waves. The spectrum is computed in the thermal vacuum state with the chosen values of parameters for the accelerated as well as decelerated model with $T = 0.001$ Mpc$^{-1}$ in the low range $k < k_2$. This temperature is considered in the context of the B mode of the CMB spectrum in the thermal state \cite{6}. And $T_0 = 1.19 \times 10^{10}$ Mpc$^{-1}$ \cite{3} for the high range $k_4 \leq k \leq k_1$ which is from the extra-dimensional scenario \cite{4}. Since we use the natural unit, the wave number and temperature that appear in the temperature-dependent term of the spectrum are computed numerically in the Mpc$^{-1}$ unit. The obtained spectra are normalized with the CMB anisotropy spectrum of WMAP data. The amplitude of the spectrum of the thermal gravitational waves is enhanced compared to its zero temperature case (vacuum case). It is observed that the spectrum for $T = 0.001$ Mpc$^{-1}$ gets maximum enhancement $\sim 1.51$ times more than the vacuum case, at $k = k_E$, and it is $\sim 20$ times for $T_0 = 1.19 \times 10^{10}$ Mpc$^{-1}$ at $k = k_4$.

The plots for the amplitude of spectrum $h_T(k, \eta_0)$ versus the frequency $v$ for $\beta = -1.9$ and $\beta_i = -0.552$ are given in figure 1. The amplitude of the spectrum gets enhanced in the frequency ranges $10^{-19}$ Hz $\leq v < 1.49 \times 10^{-17}$ Hz, and $v_4 \leq v \leq v_1$ (the lower value of this range is selected in such a way that the thermal enhancement of the spectral density does not exceed the upper bound of the nucleosynthesis rate) due to the thermal effect of gravitational waves, but for the range $1.49 \times 10^{-17}$ Hz $\leq v < v_1$ there is a suppression because of the coth$^{-1}k/2T$ term. For comparison, the amplitude of the spectra is plotted for the decelerated and accelerated universe, see figure 1.

The new enhancement of the gravitational wave spectrum due to the extra-dimensional effect (the modified amplitude, see figure 1, light green lines) can be compared with the sensitivity of Adv.LIGO, ET and LISA. The analytical expressions for the Adv.LIGO and ET

\footnote{Here, $T_0 = 0.905$ K $= 1.19 \times 10^{10}$ Mpc$^{-1}$.}
interferometers are discussed in [26]. For Adv.LIGO and ET cases, consider the root-mean-square amplitude per root Hz which is equal to

\[
\frac{h(v)}{\sqrt{v}}. \tag{40}
\]

The comparison of the sensitivity (10–10^4 Hz) curve of the ground-based interferometer Adv.LIGO [24] with the gravitational wave spectra of \( \beta = -1.9 \) for the accelerated and decelerated universe is given in figure 2. Thus it shows that the Adv.LIGO is unlikely to detect the enhancement of the spectrum from the extra-dimensional effect with its current standards but is possible with the sensitivity of ET.

Next, we compare the enhancement of the spectrum with the sensitivity (10^{-7}–10^9 Hz) of the space-based detector LISA [27]. It is assumed that LISA has one year observation time which corresponds to frequency bin \( \Delta v = 3 \times 10^{-8} \) Hz (one cycle/year) around each frequency. Hence to make a comparison with the sensitivity curve, a rescaling of the spectrum \( h(v) \) is required in equation (18) into the root mean square spectrum \( h(v, \Delta v) \) in the band \( \Delta v \), given by

\[
\frac{h(v, \Delta v) = h(v) \sqrt{\Delta v}}{v}. \tag{41}
\]

The plots of the LISA sensitivity with the modified amplitude of the spectrum are given in figure 3 for the accelerated and decelerated universe. This shows that LISA is unlikely to detect the spectrum with the new enhancement feature of the gravitational waves.
The spectral energy density parameter $\Omega_g(\nu)$ of gravitational waves is defined through the relation
$$\Omega_g(\nu) = \frac{\rho_g}{\rho_c} = \frac{\int \Omega_g(\nu) \frac{dv}{v}}{v},$$
where $\rho_g$ is the energy density of the gravitational waves and $\rho_c$ is the critical energy density. Thus
$$\Omega_g(\nu) = \frac{\pi^2}{3} h_T^2(k, \eta_0) \left( \frac{\nu}{v_0} \right)^2.$$ (42)

Since the spacetime is assumed to be spatially flat $K = 0$ with $\Omega = 1$, the fraction density of relic gravitational waves must be less than unity, $\rho_g/\rho_c < 1$. After normalization of the spectrum by using equation (39), we integrate $\int \Omega_g(\nu) \frac{dv}{v}$ from $v_0 = 10^{-10}$ Hz up to $v_1 = 3 \times 10^{10}$ Hz, with $\beta_s = -1.9$ and $\beta_s = -0.552$. The integral is evaluated for the thermal case and zero temperature case; the obtained results for the accelerated universe are

(a) $v_s \leq v \leq v_E$,
$$\frac{\rho_g}{\rho_c} = 5.8 \times 10^{-11}, \quad T = 0,$$
$$\frac{\rho_g}{\rho_c} = 8.8 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},$$

(b) $v_E \leq v \leq v_H$,
$$\frac{\rho_g}{\rho_c} = 2.3 \times 10^{-11}, \quad T = 0,$$
$$\frac{\rho_g}{\rho_c} = 3.5 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},$$

(c) $v_H \leq v \leq v_2$,
$$\frac{\rho_g}{\rho_c} = 2.4 \times 10^{-11}, \quad T = 0,$$
$$\frac{\rho_g}{\rho_c} = 3.7 \times 10^{-11}, \quad T = 0.001 \text{ Mpc}^{-1},$$
Figure 4. The spectral energy density of the gravitational waves for the accelerated (solid lines) and decelerated (dashed lines) universe.

(d) $v_2 \leq v \leq v_s$

\[
\frac{\rho_g}{\rho_c} = 8.97 \times 10^{-9}, \quad T = 0.
\]

(e) $v_s \leq v \leq v_1$

\[
\frac{\rho_g}{\rho_c} = 2.7 \times 10^{-6}, \quad T = 0.
\]

\[
\frac{\rho_g}{\rho_c} = 6.67 \times 10^{-6}, \quad T_* = 1.19 \times 10^{25} \text{ Mpc}^{-1}.
\]

It is to be noted that in (d) the thermal case is not shown because the thermal contribution in this frequency range is negligible due to the temperature-dependent term. However by taking into account the extra-dimensional effect, the upper limit of temperature of the relic waves is obtained to be $T_* = 1.19 \times 10^{25} \text{ Mpc}^{-1}$. Thus an enhancement of the spectral energy density in range $v_s \leq v \leq v_1$ compared to $T = 0$ case for the accelerated as well as decelerated universe is expected. But at the same time ignoring the thermal contribution on the spectral density in the range $v_2 \leq v \leq v_s$ leads to a discontinuity at $v_s$, see figure 4. This problem is solved by fitting a new line as discussed in the context of estimation of the amplitude of the spectrum and hence the spectral density in the range $v_2 \leq v \leq v_s$ is recomputed which gives the new value $\frac{\rho_g}{\rho_c} = 6.71 \times 10^{-7}$. This changes the slope indicating the enhancement of the spectral energy density of the gravitational waves in the range $v_2 \leq v \leq v_s$, green lines, figure 4.
Table 1. Comparison of the estimated upper bound of spectral energy density of various studies with this work. Here $\Omega_1^{(\text{dec})}$ and $\Omega_1^{(\text{acc})}$ are respectively the spectral energy density of the relic gravitational waves in the decelerated and accelerated universe of this study and $\Omega_1^{(\text{est})}$ is the estimated upper bound of various studies.

| Frequency($\nu$) (Hz) | $\Omega_1^{(\text{dec})} (\nu)$ | $\Omega_1^{(\text{acc})} (\nu)$ | $\Omega_1^{(\text{est})} (\nu)$ |
|-----------------------|----------------------------------|----------------------------------|----------------------------------|
| $10^{-9}$–$10^{-7}$   | $4.98 \times 10^{-9}$            | $1.03 \times 10^{-9}$            | $2 \times 10^{-8}$ [28]          |
| 69–156                | $34.84 \times 10^{-8}$           | $7.2 \times 10^{-8}$             | $8.4 \times 10^{-4}$ [29]         |
| 41.5–169.25           | $4.93 \times 10^{-7}$            | $1.02 \times 10^{-7}$            | $6.9 \times 10^{-4}$ [30]         |

The enhancement spectral energy density $\Omega_{g} (\nu)$ in (d) can be compared with the estimated upper bound of various studies and are given in table 1. Thus $\Omega_1^{(\text{dec})}$ and $\Omega_1^{(\text{acc})} (\nu)$ are less than the upper bound of the estimated values of the respective frequency range.

Furthermore, we see that the contribution to $\rho_{g}/\rho_{c}$ from the low-frequency range is $O(10^{-11}–10^{-10})$ while from the higher frequency range it is $O(10^{-6})$. Since the order of contribution to the total $\rho_{g}/\rho_{c}$ from the lower frequency side is very small in contrast to higher frequency side, we obtain for the accelerated universe

$$\rho_{g}/\rho_{c} \simeq 6.67 \times 10^{-6} \ \ \ \nu_a \leq \nu \leq \nu_1,$$

and is of the same order as that of the zero temperature case. However, $\rho_{g}/\rho_{c}$ of the gravitational waves with $T \neq 0$ is higher than the zero temperature case at lower frequency range $\nu_a \leq \nu \leq \nu_2$. Therefore, an enhancement for the spectral energy density in the thermal vacuum state in the frequency range $\nu_a \leq \nu \leq \nu_2$ is expected and actually it is the range of interest on the observational point of view of the relic gravitational waves. The total estimated value of $\rho_{g}/\rho_{c}$ by including the thermal relic gravitational waves in the very high frequency does not alter the upper bound of the nucleosynthesis rate. Thus the relic thermal gravitons with very high frequency range are not ruled out and will be testable with the upcoming data of various missions for detecting gravitational waves.

4. Discussion and conclusion

Gravitational waves are one of the classical predictions of Einstein’s general theory of relativity. The gravitational waves are generated during the very early evolution stages of the universe as well as from the various astrophysical objects. Therefore, frequencies of the waves vary very widely. There are many on-going experiments to detect these waves and the interested range of frequency is from $10^{-19}$ to $10^{10}$ Hz. The existence of gravitational waves with very high frequency range is not favoured by the energy scale of the conventional inflationary scenario. However, the very high frequency range gravitational waves are interesting candidates in the models with extra dimensions. The extra-dimensional theories predict the existence of thermal gravitons with the black-body-type spectrum. These relic thermal gravitational waves can also add to the spectrum of the waves; thus, the corresponding amplitude also gets enhanced. The nature of the spectrum of the waves to be observed today is dependent on the evolution history of the universe. Before the result of SN Ia observations, the current evolution of the universe was considered as matter dominated with decelerated expansion. But, according to the $\Lambda$CDM concordance model, the present universe is supposed to be driven by dark energy resulting in an accelerated phase. If this is the case, then the spectral property of the waves is studied by taking into account the current acceleration of the universe. In this work, we mainly considered the very high frequency range of relic gravitational waves in the thermal vacuum state and
obtained the spectrum for the accelerated as well as decelerated models. (The low-frequency range thermal gravitational case is considered by us without including the very high frequency thermal waves that comes from the extra-dimensional scenario; this work is in preparation. The enhancement of the lower frequency range is shown with red lines, see figures 1 and 4.) The obtained spectra are normalized with the WMAP data. It is observed that the inclusion of the very high relic thermal gravitational waves leads to a discontinuity in the amplitude of the spectrum at \( \nu_s \) (see figure 1). This is due to the fact that the temperature-dependent term is insignificant in the higher frequency side of the range \( v_2 \leq \nu \leq v_s \). To evade this problem, a new equation of line is derived and thus the amplitude gets enhanced in the range \( v_2 \leq \nu \leq v_s \). This is a new feature of the spectrum and we designate it as the ‘modified amplitude’ of the spectrum. The modified amplitude of the spectrum can be compared with the sensitivity of the Adv.LIGO, ET and LISA missions. The comparison of the Adv.LIGO sensitivity shows that the modified amplitude is unlikely to be detected with the current standards of LISA or the improved sensitivity of Adv.LIGO, whereas the proposed sensitivity of the ET is promising to verify the modified amplitude with its upcoming mission data.

The spectral energy density of the gravitational waves is estimated in the thermal vacuum state for the accelerated and decelerated universe. It is observed that the spectral energy density gets enhanced in the lower frequency range \( \mathcal{O}(10^{-11}-10^{-10}) \) and from the higher frequency range it is \( \mathcal{O}(10^{-6}) \). A comparison of the estimated upper bound of spectral energy density of various studies with this work is made. It shows that \( \Omega_{s}^{(\text{dec})} \) and \( \Omega_{s}^{(\text{acc})} \) are less than the estimated upper bound of various studies. The total estimated value of \( \rho_{s}/\rho_{c} \) by including the very high frequency thermal relic gravitational waves does not alter the upper bound of the nucleosynthesis rate. Thus the relic thermal gravitons with very high frequency range are not ruled out and are testable with the upcoming data of various missions for detecting gravitational waves.

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Appendix. Extra-dimensional scenario and thermal gravitons

Cosmology with extra dimensions has been motivated since Kaluza and Klein (KK) showed that classical electromagnetism and general relativity could be combined in a five-dimensional framework. The modern scenarios involving extra dimensions are being explored in particle physics, with most models possessing either a large volume or a large curvature. Although there exist different models of extra dimensions, there are some general features and signals common to all of them.

In the presence of \( D \) extra spatial dimensions, the (3+D+1)-dimensional action for gravity for can be written as

\[
S = \int d^4x \left[ \int d^Dy \sqrt{-g_D} \frac{R_D}{16\pi G_D} + \sqrt{-g} \mathcal{L}_m \right],
\]

where

\[
G_D = G_N \frac{m_{pl}^{2}}{m_{D}^{2+D}},
\]
and $g$ is the four-dimensional metric, $G_N$ is Newton’s constant, $g_D$, $G_N$ and $R_D$ denote the higher dimensional counterparts of the metric, Newton’s constant, and the Ricci scalar, respectively. $m_D$ is the fundamental scale of the extra-dimensional theory.

Since the gravitational interactions are not strong enough to produce a thermal gravitons at temperatures below the Planck scale ($m_{pl} \sim 1.22 \times 10^{19}$ GeV), the standard inflationary cosmology predicted the existence of the cosmic gravitational wave backgrounds which are non-thermal in nature. However if the universe contains extra dimensions that can generate the thermal gravitational waves, then its shape and amplitude of the CGWB may change significantly. This can happen when energies in the universe are higher than the fundamental scale $m_D$; the gravitational coupling strength increases significantly as the gravitational field spreads out into the full spatial volume. Instead of freezing out at $\sim O(m_{pl})$, as in 3+1 dimensions, gravitational interactions freeze-out at $\sim O(m_D)$. If the gravitational interactions become strong at an energy scale below the reheat temperature ($m_D < T_{RH}$), gravitons get the opportunity to thermalize, creating a thermal CGWB. The qualitative result, the creation of a thermal CGWB if $m_D < T_{RH}$, is unchanged by the type of extra dimensions chosen [4].

Thus, if extra dimensions do exist and the fundamental scale of those dimensions is below the reheat temperature, a relic thermal CGWB ought to exist today. Compared to the relic thermal photon background (CMB), a thermal CGWB would have the same shape, statistics and high degree of isotropy and homogeneity. The energy density ($\rho_g$) and fractional energy density ($\Omega_g$) of a thermal CGWB are

$$\rho_g = \frac{\pi^2}{15} \left( \frac{3.91}{g_*} \right)^{4/3} T_{CMB}^4,$$

$$\Omega_g = \frac{\rho_g}{\rho_c} \simeq 3.1 \times 10^{-4} (g_*)^{4/3},$$

where $\rho_c$ is the critical energy density today, $T_{CMB}$ is the present temperature of the CMB and $g_*$ is the number of relativistic degrees of freedom at the scale of $m_D$. $g_*$ is dependent on the particle content of the universe, i.e. whether (and at what scale) the universe is supersymmetric, has a KK tower, etc. Other quantities, such as the temperature ($T$), peak frequency ($\nu$), number density ($n$), and entropy density ($s$) of the thermal CGWB can be derived from the CMB if $g_*$ is known, as

$$n_g = n_{CMB} \left( \frac{3.91}{g_*} \right), \quad s_g = s_{CMB} \left( \frac{3.91}{g_*} \right)$$

$$T_g = T_{CMB} \left( \frac{3.91}{g_*} \right)^{1/3}, \quad \nu_g = \nu_{CMB} \left( \frac{3.91}{g_*} \right)^{1/3}.$$  

These quantities are not dependent on the number of extra dimensions, as the large discrepancy in size between the three large spatial dimensions and the $D$ extra dimensions suppresses those corrections by at least a factor of $\sim 10^{-29}$. If $m_D$ is just barely above the scale of the standard model, then $g_* = 106.75$. The thermal CGWB then has a temperature of 0.905 K and a peak frequency of 19 GHz [4].

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