ANALYSIS OF THE RENORMALIZATION SCHEME AMBIGUITIES IN THE QCD CORRECTIONS TO HADRONIC DECAYS OF THE TAU LEPTON†

P. A. RĄCZKA

Institute of Theoretical Physics, Warsaw University, ul. Hoża 69, PL-00-681 Warsaw, Poland

The QCD corrections to the $R_{12}^{kl}$ moment of the invariant mass distribution in hadronic decays of the $\tau$ are discussed. The next-to-next-to-leading order prediction is shown to be stable with respect to change of the renormalization scheme (RS). Formally the RS dependence is of higher order than the considered perturbative expression, but numerically it may be quite significant. In order to obtain reliable predictions one should therefore carefully investigate the possibilities to optimize the choice of the RS. Also, one should study the stability of the optimized predictions by varying the RS parameters in some a priori acceptable range. (A condition on the acceptable schemes, which is applicable to NNLO approximants, has been recently proposed by the present author.†) Such an analysis is very important in the case of the QCD corrections to the hadronic $\tau$ decays, for which the characteristic energy scale is $m_{\tau} = 1.777$ GeV. In particular, it is interesting to what extent the RS dependence affects the recent results of two collaborations, which used the hadronic $\tau$ decay data to obtain a surprisingly precise value of $\alpha_s$.

The strong interaction effects in $\tau$ may now be studied using the QCD corrections to the $R_\tau$ ratio

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau e \bar{\nu}_e)},$$

and the corrections to the $R_{12}^{kl}$ moments, defined by the relation

$$R_{12}^{kl} = \frac{1}{\Gamma_e} \int_0^{m_{\tau}^2} ds \left(1 - \frac{s}{m_{\tau}^2}\right)^k \left(\frac{s}{m_{\tau}^2}\right)^l \frac{d\Gamma_{\text{had}}}{ds},$$

where $\Gamma_e$ is the electronic width of $\tau$ and $d\Gamma_{\text{had}}/ds$ is the invariant mass distributions of the Cabbibo allowed hadronic decays of $\tau$. For $R_{12}^{kl}$ we have:

$$R_{12}^{kl} = 3 \left| V_{ud} \right|^2 S_{\text{EW}} R_0^{kl}(1 + \delta_{pt}^{kl} + \delta_m^{kl} + \delta_{SVZ}^{kl}),$$

where the factor $S_{\text{EW}} = 1.0194$ represents corrections from electroweak interactions and $R_0^{kl}$ denotes the parton model predictions. For $R_\tau$ we have:

$$R_\tau = 3 S_{\text{CKM}} S_{\text{EW}} (1 + \delta_{pt}^{\text{tot}} + \delta_m^{\text{tot}} + \delta_{SVZ}^{\text{tot}}),$$

where $S_{\text{CKM}} = (\left| V_{ud} \right|^2 + \left| V_{us} \right|^2) \approx 1$. The $\delta_{pt}$ contribution denotes the purely perturbative QCD correction, evaluated for three massless quarks. (We have $\delta_{pt}^{\text{tot}} = \delta_{pt}^{(0)}$.). The $\delta_m$ contribution denotes the correction from quark masses, which is practically negligible in the case of $R_{12}^{kl}$. For $R_\tau$ we have $\delta_m^{\text{tot}} \approx 0.009$. The $\delta_{SVZ}$ contribution is a nonperturbative QCD correction calculated using the SVZ approach

$$\delta_{SVZ} = \sum_{D=4,5,...} c_D \frac{O_D}{m_{\tau}^D}.$$

The parameters $O_D$ in Eq.(5) denote vacuum expectation values of the gauge invariant operators of dimension $D$. The $c_D$ coefficients are in principle power series in the strong coupling constant, and depend on the considered moment of the invariant mass distribution.

The interest in the $R_{12}^{kl}$ moments comes from the effort to improve the accuracy of the determination of $\alpha_s$ from the $\tau$ decay. Not unexpectedly, a major factor limiting the precision of the QCD prediction for $R_\tau$ is the uncertainty in the nonperturbative contribution. The contribution from the $D = 4$ term in the SVZ expansion for $R_\tau$ may be reliably expected to be small since $O_4$ is well

† To appear in the Proceedings of the 28th International Conference on High Energy Physics, Warsaw, Poland, 25–31 July 1996.
constrained by the sum rules phenomenology, and the relevant coefficient function starts at $O(a_s^2)$. However, the $D = 6$ contribution to $R_T$ is not suppressed, and there is little information on the value of $O_6$. It was therefore proposed to treat $O_D$ as free parameters which are to be extracted together with $\alpha_s$ from a fit to the experimental data for $R_T$ and the higher moments of the invariant mass distribution. Of particular interest here is the $R_T^{12}$ moment, because similarly to $R_T$ the $D = 4$ contribution is suppressed for this moment, and there is significant contribution from the $D = 6$ term. If in the SVZ expansion we retain only the $D = 6$ term, which appears to be a dominant source of the uncertainty in the nonperturbative sector, then a simplest self-consistent approach is to take $R_T$ and $R_T^{12}$. This is assumed in the following.

A detailed discussion of the RS dependence of $\delta pt^{12}$ was presented elsewhere, so in this note we concentrate on $\delta pt^{12}$. The QCD correction $\delta pt^{12}$ may be expressed as a contour integral in the complex energy plane with the so called Adler function under the integral:

$$\delta pt^{12} = \frac{i}{\pi} \int_C d\sigma f^{12}(\frac{\sigma}{m^2}) \delta D,V(-\sigma),$$  \hspace{1cm} (6)

where $C$ is a contour running clockwise from $\sigma = m^2 - i\epsilon$ to $\sigma = m^2 + i\epsilon$ away from the region of small $|\sigma|$. In the actual calculation we assume that $C$ is a circle $|\sigma| = m^2$. The Adler function is defined by the relation:

$$(-12\pi^2)\sigma \frac{d}{d\sigma} \Pi^{(1)}(\sigma) = 3S_{CKM}[1 + \delta D(-\sigma)] \hspace{1cm} (7)$$

where $\Pi^{(1)}$ denotes transverse part of the vector current correlator. The function $f^{12}(\sigma/m^2)$ has the form:

$$f^{12}(x) = \frac{1}{2} - \frac{70}{13}x^3 + \frac{105}{26}x^4 + \frac{126}{13}x^5 - \frac{175}{13}x^6 + \frac{60}{13}x^7.$$  \hspace{1cm} (8)

The NNLO renormalization group improved perturbative expansion for $\delta D,V$ may be written in the form:

$$\delta D,V(-\sigma) = a(-\sigma)[1 + r_1 a(-\sigma) + r_2 a^2(-\sigma)],$$  \hspace{1cm} (9)

where $a = g^2/(4\pi^2)$ denotes the running coupling constant that satisfies the NNLO renormalization group equation:

$$\sigma \frac{d}{d\sigma} = -\frac{b}{2} (1 + c_1 a + c_2 a^2).$$  \hspace{1cm} (10)

In the $\overline{MS}$ scheme we have $r_1^{\overline{MS}} = 1.63982$ and $r_2^{\overline{MS}} = 6.37101$. The renormalization group coefficients for $n_f = 3$ are $b = 4.5$, $c_1 = 16/9$ and $c_2^{\overline{MS}} = 3863/864 \approx 4.471$. By keeping the renormalization group improved expression for the Adler function under the integral and evaluating the contour integral numerically we obtain an essential improvement of the conventional perturbation expansion for $\delta pt^{12}$, resumingm to all orders some of the corrections arising from analytic continuation from spacelike to timelike momenta.

The predictions calculated in the next-to-next-to-leading order (NNLO) approximation depend on two RS parameters, which in principle may be arbitrary. Similarly as in the previous work we parametrize the freedom of choice of the RS by the parameters $r_1$ and $c_2$ — the parameter $r_2$ is determined using the RS invariant combination:

$$\rho_2 = c_2 + r_2 - c_1 r_1 - r_1^2.$$  \hspace{1cm} (11)

For the Adler function we have $\rho_2 = 5.23783$. 

![Figure 1: Contour plot of $\delta pt^{12}$ as a function of the scheme parameters $r_1$ and $c_2$, for $\Lambda^{(3)} = 325$ MeV. For technical reasons we use $c_2 - c_1 r_1$ on the vertical axis instead of $c_2$. The boundary of the region of scheme parameters satisfying the Eq. 14 is also indicated.](image-url)
To optimize the choice of the RS we use the principle of minimal sensitivity (PMS), which singles out the scheme parameters for which the finite order prediction is least sensitive to the change of RS, similarly to what we expect from the actual physical quantity. The dependence of $\delta_{pt}^{12}$ on the scheme parameters is illustrated in figure [1] for $\Lambda_{\overline{MS}}^{(3)} = 325$ MeV. (For technical reasons we use $c_2 - c_1 r_1$ on the vertical axis instead of $c_2$.) We choose as our optimized parameters $r_1 = 0$ and $c_2 = 1.5\rho_2$ — for small values of $\Lambda_{\overline{MS}}^{(3)}$ this point lies very close to the critical point, and even for large values of $\Lambda_{\overline{MS}}^{(3)}$ the RS dependence in the vicinity of this point is very small.

To investigate the stability of the predictions we use the condition proposed by the present author, based on the notion that natural renormalization schemes should not induce extensive cancellations in the expression for the RS invariant. The dependence of $\delta_{pt}^{12}$ on the scheme parameters is illustrated in figure [1] for $\Lambda_{\overline{MS}}^{(3)}$ = 325 MeV. (For technical reasons we use $c_2 - c_1 r_1$ on the vertical axis instead of $c_2$.) We choose as our optimized parameters $r_1 = 0$ and $c_2 = 1.5\rho_2$ — for small values of $\Lambda_{\overline{MS}}^{(3)}$ this point lies very close to the critical point, and even for large values of $\Lambda_{\overline{MS}}^{(3)}$ the RS dependence in the vicinity of this point is very small.

In figure [2] the NNLO PMS predictions for $\delta_{pt}^{12}$ are shown as a function of $m/\Lambda_{\overline{MS}}^{(3)}$, together with the minimal and maximal value obtained by varying the scheme parameters within the region determined by the condition (12). We see that the NNLO predictions for $\delta_{pt}^{12}$, obtained by numerically evaluating the countour integral expression (6), are free from potentially dangerous RS instabilities even for large values of $\Lambda_{\overline{MS}}^{(3)}$. This situation is similar to that encountered for $\delta_{pt}^{12}$ (6). For comparison we also show the PMS predictions obtained in the next-to-leading order (NLO). (In NLO we have $r_1^{\text{PMS}} \approx -0.64$.) We see that RS dependence of NNLO expression within the region defined by the condition (12) is smaller than the difference between NNLO and NLO PMS predictions.

In order to see how the PMS optimization affects the fits to the experimental data we first test the accuracy of the approximation in which one only retains the $O_6$ contribution in the SVZ expansion. To this end we make a fit of $\alpha_s$ and $O_6$ in the $\overline{MS}$ scheme, and compare the results with the fit performed by ALEPH, in which the $O_4$, $O_6$ and $O_8$ contributions have been taken into account in the (1.0), (1.1), (1.2) and (1.3) moments. If we take, following ALEPH, $R_\tau = 3.645 \pm 0.024$ and $D_{12}^{12} = R_{12}^{12}/R_{10}^{10} = 0.0570 \pm 0.0013$, we obtain from the fit in the $\overline{MS}$ scheme $\alpha_s(M_Z^2) = 0.1209 \pm 0.0013$ and $O_6 = -0.0010 \pm 0.0012$. This appears to be remarkably close to the values $0.121$ and $-0.0016$ obtained in the full fit by ALEPH. This gives us confidence that the “$O_6$ approximation” captures the essential features of QCD corrections in $\tau$ decays.

Performing the same fit, but using now the NNLO PMS predictions, we obtain $\alpha_s(M_Z^2) = 0.1198$ and $O_6 = -0.0011$, i.e. the value of the condensate practically does not change, but the value of the strong coupling constant is reduced by about one standard deviation of the experimental error. It is also of some interest to compare these results with the NLO PMS fit — we then obtain $\alpha_s(M_Z^2) = 0.1221$. Taking the difference of the NNLO and NLO PMS fits is perhaps the best way of estimating the accuracy of the perturbative pre-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{$\delta_{pt}^{12}$ as a function of $m/\Lambda_{\overline{MS}}^{(3)}$, obtained in the PMS scheme in NNLO and NLO (upper and lower solid curves, respectively). The dashed lines indicate variation of the prediction when scheme parameters are changed within the region satisfying the Eq. (12).}
\end{figure}
diction. We see that thus obtained uncertainty of $\alpha_s$ is of the order 0.0022, i.e. it is quite large, compared for example to the experimental uncertainty.

We may now use the optimized predictions to obtain $\alpha_s$ from the more up to date experimental data on $\tau$ decays. We take $R_\tau = 3.613 \pm 0.032$, which is a weighted average of three possible determinations involving $B_e = 0.1790 \pm 0.0017$, $B_{\mu} = 0.1744 \pm 0.0023$ and $\tau_{\tau} = (292.0 \pm 2.1) \times 10^{-15}$ sec (values according to the 1995 update of the Particle Data Group 12). We also take $D_{12} = 0.0561 \pm 0.0006$, which is a weighted average of the ALEPH 3 and CLEO 4 determinations. With these numbers we obtain $\alpha_s(M_Z^2) = 0.1177 \pm 0.0017$ ($\alpha_s(m_\tau^2) = 0.321 \pm 0.014$) and $O_6 = -0.0020 \pm 0.0005$.

Concluding, the perturbative QCD correction to the $R_\tau$ moment of the invariant mass distribution in hadronic tau decays was found to be stable with respect to change of the renormalization scheme, despite the low energy scale, provided that the contour integral expression is used, which resums some of the corrections to all orders. However, the difference between predictions in the conventionally used $\overline{MS}$ scheme and the PMS scheme was found to be phenomenologically significant. Also, the difference between the NNLO and NLO predictions in the PMS scheme was found to be significant, indicating perhaps that the uncertainty in the perturbative prediction is larger than previously expected. Finally, the optimized predictions have been used to obtain a realistic fit for $\alpha_s$ from the experimental data.

References

1. P.A. Rączka, Z. Phys. C 65, 481 (1995).
2. ALEPH Collab., D. Buskulic et al, Phys. Lett. B 307, 209 (1993).
3. P. Reeves, in Proceedings of the XXXth Rencontres de Moriond Conference “95 QCD and High Energy Hadronic Interactions,” Les Arcs, Savoie, France, 1995, edited by J. Trần Thanh Văn (EditionsFrontières, Gif-sur-Yvette, 1995), p.235.
4. CLEO Collab., T. Coan et al, Phys. Lett. B 356, 580 (1995).
5. E. Braaten, Phys. Rev. Lett. 60, 1606 (1988), ibid. 63, 577 (1989), S. Narison and A. Pich, Phys. Lett. B 211, 183 (1988),
6. F. LeDiberder and A. Pich, Phys. Lett. B 289, 165 (1992).
7. M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B 147, 385, 448, 519 (1979).
8. P.A. Rączka and A. Szymba, Z. Phys. C 70, 125 (1996).
9. S.G. Gorishny, A.L. Kataev and S.A. Larin, Phys. Lett. B 259, 144 (1991), M.A. Samuel and L.R. Surguladze, Phys. Rev. D 44, 1602 (1991).
10. A.A. Pivovarov, Z. Phys. C 53, 461 (1992), F. LeDiberder and A. Pich, Phys. Lett. B 286, 147 (1992).
11. P.M. Stevenson, Phys. Lett. B 100, 61 (1981), Phys. Rev. D 23, 2916 (1981).
12. Review of Particle Properties, Particle Data Group, L. Montanet et al, 1995 off-year partial update.