A NOTE ON INJECTIVITY OF FROBENIUS ON LOCAL COHOMOLOGY OF HYPERSURFACES

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Abstract. Let $k$ be a field of characteristic $p > 0$ such that $[k : k^p] < \infty$ and let $f \in R = k[x_0, \ldots, x_n]$ be homogeneous of degree $d$. We obtain a sharp bound on the degrees in which the Frobenius action on $H^j_m(R/fR)$ can be injective when $R/fR$ has an isolated non-F-pure point at $m$. As a corollary, we show that if $(R/fR)_m$ is not F-pure then $R/fR$ has an isolated non-F-pure point at $m$ if and only if the Frobenius action is injective in degrees $\leq -n(d-1)$.

1. Introduction

Let $k$ be a field of characteristic $p$ such that $[k : k^p] < \infty$ and let $f \in R = k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $d$. For simplicity, assume that the test ideal $\tau(f^{1-\frac{1}{p}}) = m^j$ for some $j \geq 1$, where $m = (x_0, \ldots, x_n)$. Our main theorem obtains the following sharp bound on the degrees in which the Frobenius action on $H^j_m(R/fR)$ is injective.

Theorem (Theorem 2.14). If $\tau(f^{1-\frac{1}{p}}) = m^j \subseteq m$, then the below Frobenius action is injective:

$$F : H^j_m(R/fR)_{< -n+j+d} \rightarrow H^j_m(R/fR)_{< p(-n+j+d)}.$$ 

Our assumption that $\tau(f^{1-\frac{1}{p}}) = m^j$ implies that while $(R/fR)_m$ is not F-pure, $(R/fR)_p$ is F-pure for every prime $p \subseteq m$. We say such rings have an isolated non-F-pure point at $m$. The study of F-pure rings has a long history and their theory is rich: Hochster and Roberts first defined F-pure rings and explored the relationship of F-purity to local cohomology (and the Frobenius action thereof) in [HR76]. Fedder continued this program of study, obtaining a criterion for F-purity and showing the equivalence of F-purity and F-injectivity for local Gorenstein rings of characteristic $p$ [Fed83].

A corollary to our main theorem is that when $(R/fR)_m$ is not F-pure, $R/fR$ has an isolated non-F-pure point at $m$ if and only if Frobenius acts injectively in sufficiently negative degrees. Moreover, the degree in which it must be injective depends only on the degree of $f$.

Theorem (Corollary 2.16). If $(R/fR)_m$ is not F-pure then $R/fR$ has an isolated non-F-pure point at $m$ if and only if the below Frobenius action is injective:

$$F : H^j_m(R/fR)_{\leq -n(d-1)} \rightarrow H^j_m(R/fR)_{\leq -p(n(d-1))}.$$ 

In their study of the F-pure thresholds of Calabi-Yau hypersurfaces, Bhatt and Singh proved a similar result [BS13, Theorem 3.5] under the assumption that $R/fR$ has an isolated singularity at $m$. Their methods generalize well to the setting of this paper. The relationship between isolated singularities and isolated non-F-pure points is as follows: regular rings are F-pure, so $\{\text{non-F-pure points of } R/fR \} \subseteq V(f)_{\text{sing}}$. Thus if $(R/fR)_m$ is not F-pure and has an isolated singularity, it follows that it has an isolated non-F-pure point. Interesting examples of these phenomena often arise as affine cones over smooth projective varieties.

Acknowledgements 1.1. I want to thank my advisor Wenliang Zhang for suggesting this problem to me and for useful discussions.

2. Main result

The Frobenius map on a ring $A$ of prime characteristic $p > 0$ is the ring homomorphism $F : A \rightarrow A$ given by $F(a) = a^p$. We say that $A$ is F-finite if $A$ is a finitely generated module over $F(A) = A^p$.

We fix notation: throughout, $k$ will denote an F-finite field of characteristic $p > 0$. Let $R = k[x_0, \ldots, x_n]$ be the polynomial ring in $n + 1$ variables over $k$ and $f \in R$ be homogeneous of degree $d$. Note that in this
Definition 2.1 (Test ideal, [BMS08 Definition 2.9]). The test ideal \( \tau(f^{1-p^{-1}}) \) is the smallest ideal \( a \subseteq R \) such that
\[ f^{p^{-1}} \in a^{[p^e]}. \]

Remark 2.2. Proposition 2.5 from [BMS08] gives a useful description of \( \tau(f^{1-p^{-1}}) \): let \( \{\lambda_b\}_{b \in B} \) be a basis for \( k \) over \( k^{p^e} \). The elements \( \lambda_i x_i^p = \lambda_b x_0^{i_0} \cdots x_n^{i_n} \) with \( 0 \leq i_j \leq p^e - 1 \) and \( b \in B \) form an \( \mathbb{R}^{p^e} \)-basis for \( R \), so we can express \( f^{p^{-1}} \) as an \( \mathbb{R}^{p^e} \)-linear combination
\[ f^{p^{-1}} = \sum f_{b_i}^{p^{e}} \lambda_i x_i. \]

Then the test ideal \( \tau(f^{1-p^{-1}}) \) is the ideal generated by the \( f_{b_i} \) for all \( i \) and \( b \) appearing above. That is,
\[ \tau(f^{1-p^{-1}}) = (f_{b_i} | 0 \leq i_j \leq p^e - 1; b \in B). \]

If the Frobenius map \( F : A \to A \) is pure, then we say that \( A \) is \( F \)-pure. The corresponding notion in characteristic 0 is that of log canonical (lc) points, and the set of non-lc points is obtained as the vanishing set of the non-lc ideal. Fujino, Schwede, and Takagi initiated development of the theory of non-\( F \)-pure ideals in [FST11 Section 14]. As one might expect, the vanishing locus of the non-\( F \)-pure ideal is precisely the set of primes for which \( (R/fR)_p \) fails to be \( F \)-pure. We caution the reader that the definition we give is specific to the case considered in this note; see [FST11 Definition 14.4] for the general definition.

Definition 2.3 (non-\( F \)-pure ideal; [FST11 Remark 16.2]). The non-\( F \)-pure ideal of \( f \), denoted \( \sigma(\text{div}(f)) \), is defined to be
\[ \sigma(\text{div}(f)) = \tau(f^{1-p^{-1}}) \text{ for } e \gg 0. \]

Proposition 2.4. \( \sqrt{\tau(f^{1-p^{-1}})} = \sqrt{\sigma(\text{div}(f))} \).

Proof. It follows from the definitions that \( \sigma(\text{div}(f)) \subseteq \tau(f^{1-p^{-1}}) \), so it is enough to show that if \( \tau(f^{1-p^{-1}}) \nsubseteq p \) for some prime \( p \) then \( \sigma(\text{div}(f)) \nsubseteq p \). Since \( \sigma(\text{div}(f)) \) is the non-\( F \)-pure ideal, we check that \( (R/fR)_p \) is \( F \)-pure.

By assumption, \( \tau(f^{1-p^{-1}})_p = R_p \). Since test ideals localize [BMS08 Proposition 2.13(1)] it follows that \( f^{p^{-1}} \notin (pR_p)[p^e] \). Fedder’s Criterion [Fed83 Theorem 1.12] now implies that \( (R/fR)_p \) is \( F \)-pure, and so \( \sigma(\text{div}(f)) \nsubseteq p \). \( \blacksquare \)

Definition 2.5 (isolated non-\( F \)-pure point). We say that \( R/fR \) has an isolated non-\( F \)-pure point at \( m \) if \( (R/fR)_m \) is not \( F \)-pure but \( (R/fR)_p \) is whenever \( p \nsubseteq m \).

Remark 2.6. The vanishing set \( \mathbb{V}(\sigma(\text{div}(f))) \) is precisely the set of points \( p \in \mathbb{V}(f) \) such that \( (R/fR)_p \) is not \( F \)-pure. Proposition 2.4 now says that the ideal \( \tau(f^{1-p^{-1}}) \) also defines this locus. Therefore, \( R/fR \) has an isolated non-\( F \)-pure point at \( m \) if and only if \( \sqrt{\tau(f^{1-p^{-1}})} = m \).

Definition 2.7. Let \( e_0 \in \mathbb{N}_0 \) be the least integer such that \( \tau(f^{1-p^{-1}}) \nsubseteq m^{[p^{e_0}]} \). For \( e \geq e_0 \) define
\[ M_e := \min\{\deg(g) \mid g \in (m^{[p^e]} : \tau(f^{1-p^{-1}})) \setminus m^{[p^e]} \text{ homogeneous}\}. \]
Here we adopt the convention \( \min \emptyset = \infty \).
Lemma 2.8. \( M_{e+1} - (n+1)p^{e+1} \leq M_e - (n+1)p^e \) for all \( e \geq e_0 \).

Proof. Note that \( M_e = \infty \) for \( e \geq e_0 \) if and only if \( \tau(f^{1+ \frac{1}{p}}) = R \); in this case there is no content to the lemma. Thus we assume that \( M_e < \infty \). For simplicity of notation, write \( \tau = \tau(f^{1+ \frac{1}{p}}) \). Let \( r \) be a homogeneous element of \((m^{[p^e]} : \tau) \setminus m^{[p^e]}\) with minimum degree \( M_e \). Then for each term \( t \) of every generator \( f_{j,k} \) for \( \tau \) (as in Remark 2.2), we have that \( \deg_{x_j}(rt) \geq p^e \) for some \( 0 \leq j \leq n \). Thus,

\[
\deg_{x_j}((x_0 \cdots x_n)^{p^{e+1} - p^e} r) = p^{e+1} - p^e + \deg_{x_j}(rt) \geq p^{e+1}
\]

so that \((x_0 \cdots x_n)^{p^{e+1} - p^e} r \in (m^{[p^{e+1}]} : \tau)\). Since \((m^{[p^{e+1}]} : (x_0 \cdots x_n)^{p^{e+1} - p^e}) = m^{[p^e]}\), we know

\[
(x_0 \cdots x_n)^{p^{e+1} - p^e} r \notin m^{[p^{e+1}]}.
\]

It follows that \( M_{e+1} \leq M_e - (n+1)(p^{e+1} - p^e) \).

\( \blacksquare \)

Lemma 2.9. Assume \((R/fR)_{m} \) is not F-pure. Then \( R/fR \) has an isolated non-F-pure point at \( m \) if and only if \( M_e - (n+1)p^e \) is constant for \( e \gg 0 \).

Proof. For simplicity, write \( \tau = \tau(f^{1+ \frac{1}{p}}) \). If \( \tau \subseteq m \) then \((m^{[p^e]} : \tau) \neq m^{[p^e]}\) for any \( e \), so \( M_e < \infty \) for all \( e \) in this case. Since we are assuming \((R/fR)_{m} \) is not F-pure, we conclude that \( M_e < \infty \) for all \( e \).

\( R/fR \) has an isolated non-F-pure point at \( m \) if and only if \( \sqrt{\tau} = m \), which is equivalent to \( m^\ell \subseteq \tau \) for some \( \ell \geq 1 \).

Claim: \((m^{[p^e]} : \tau) \subseteq (m^{[p^e]} : m^\ell)\) for all \( e \gg 0 \) if and only if \( m^\ell \subseteq \tau \).

Proof of claim: Let \((A,n) \) be a 0-dimensional Gorenstein local ring and let \( L \subseteq A \) be an ideal. Write \((-)^{\vee} \) for the Matlis dual \( \text{Hom}_A(-, A(n)) \) and note that \( A \cong A(n) \) since \( A \) is 0-dimensional and Gorenstein. Then

\[
(0 : L) \cong \text{Hom}_A(A/L, A)
\]

\[
\cong (A/L)^{\vee}.
\]

Now applying the Matlis dual again, we get \( A/L \cong (A/L)^{\vee \vee} \cong (0 : L)^{\vee} \) where the first isomorphism follows from finite length of \( A/L \). Let \( I, J \subseteq A \) be two ideals. If \( (0 : J) \subseteq (0 : I) \) then we have an exact sequence

\[
0 \to (0 : J) \to (0 : I)
\]

which we dualize to get

\[
A/I \to A/J \to 0.
\]

Thus, if \( A \) is a 0-dimensional Gorenstein ring and \( I, J \) are two ideals of \( A \) then \( (0 : J) \subseteq (0 : I) \) if and only if \( I \subseteq J \).

Note that \( R/m^{[p^e]} \) is a 0-dimensional Gorenstein ring for all \( e \geq 0 \). The above paragraph shows that \((m^{[p^e]} : \tau) \subseteq (m^{[p^e]} : m^\ell)\) if and only if \( m^\ell + m^{[p^e]} \subseteq \tau + m^{[p^e]} \). For \( e \gg 0 \), \( m^{[p^e]} \subseteq m^\ell \) so this last reads \( m^\ell \subseteq \tau + m^{[p^e]} \) for all \( e \gg 0 \). Therefore

\[
m^\ell \subseteq \bigcap_{e \gg 0} (\tau + m^{[p^e]}).
\]

This intersection is \( \tau \) by Krull’s intersection theorem. We conclude that \((m^{[p^e]} : \tau) \subseteq (m^{[p^e]} : m^\ell)\) for \( e \gg 0 \) if and only if \( m^\ell \subseteq \tau \).

The proof of [BS13, Lemma 3.2] shows that

\[
(m^{[p^e]} : m^\ell) = m^{[p^e]} + m^{(n+1)p^e - n- \ell} \text{ for } e \gg 0.
\]

Thus we have that \( \sqrt{\tau} = m \) if and only if \( M_e \geq (n+1)p^e - n - \ell \) for \( e \gg 0 \) and some \( \ell \geq 1 \). Lemma 2.8 shows that

\[
M_{e+1} - (n+1)p^{e+1} \leq M_e - (n+1)p^e
\]

for all \( e \geq e_0 \), so we conclude that \( R/fR \) has an isolated non-F-pure point at \( m \) if and only if

\[
-n - \ell \leq M_e - (n+1)p^e
\]

for some \( \ell \geq 1 \) and all \( e \). Since \( \{M_e - (n+1)p^e\}_{e \geq e_0} \) is a nonincreasing sequence of integers, this sequence is bounded below if and only if \( M_e - (n+1)p^e \) is constant for \( e \gg 0 \).

\( \blacksquare \)
Remark 2.10. If $\tau(f^{1+\frac{1}{p}}) = m^j$ for some $j \geq 1$ then the proof shows that in fact $M_e - (n+1)p^e = -n - j$ for all $e \geq e_0$.

Remark 2.11. We note that if $M_e < \infty$ then $M_e - (n+1)p^e + d \leq 1 + \frac{d}{p} - \frac{n+1}{p}$. Indeed, if $r \not\in m^{[p]1}$ and $\deg(r) = M_e - 1$ then $r \not\in (m^{p^e}: \tau(f^{1+\frac{1}{p}}))$. It follows that $r^p f^{p-1} \not\in m^{[p]1}$. This implies

$$p(M_e - 1) + (p - 1)d \leq (n+1)(p^{e+1} - 1).$$

Dividing both sides by $p$, we have that

$$M_e - (n+1)p^e + d \leq 1 + \frac{d - (n+1)}{p}.$$  

In particular, as long as $d \leq n+1$ or $p > d - (n+1)$ we have that $M_e - (n+1)p^e + d \leq 1$.

Definition 2.12. If $R/fR$ has an isolated non-F-pure point at $m$, define $\delta(f) = M_e - (n+1)p^e$ for $e \gg 0$.

Of major importance to our proof of the main theorem is analysis of the following diagram of short exact sequences in local cohomology. This appears as [BS13, Remark 2.2].

Remark 2.13. For $f \in R$ as above, the Frobenius map $F : R/fR \to R/fR$ fits into a diagram of short exact sequences

$$
\begin{array}{cccccc}
0 & \longrightarrow & R[-d] & \longrightarrow & R & \longrightarrow & R/fR \\
& & f & \longrightarrow & F & \longrightarrow & 0 \\
& & f^p & \longrightarrow & f & \longrightarrow & f \\
0 & \longrightarrow & R[-d] & \longrightarrow & R & \longrightarrow & R/fR \\
& & & & & & 0.
\end{array}
$$

The long exact sequence in local cohomology now gives

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^0_m(R/fR) & \longrightarrow & H^0_m(R)[-d] & \longrightarrow & H^0_m(R)[-d] \\
& & f & \longrightarrow & f & \longrightarrow & f \\
& & f^p & \longrightarrow & f^p & \longrightarrow & f \\
0 & \longrightarrow & H^0_m(R/fR) & \longrightarrow & H^0_m(R)[-d] & \longrightarrow & H^0_m(R)[-d] \\
& & & & & & 0.
\end{array}
$$

The rightmost map is injective because $R$ is regular (and so is F-pure), so the snake lemma now implies that injectivity of the map on the left is equivalent to injectivity of the middle map.

Theorem 2.14. Let $f \in R$ be homogeneous of degree $d$ and assume that $R/fR$ has an isolated non-F-pure point at $m$. Then the following Frobenius action is injective:

$$F : H^0_m(R/fR)_{\leq \delta(f) + d} \to H^0_m(R/fR)_{\leq \delta(f) + d}.$$

Proof. Writing $N = \delta(f) + d$ we have the diagram in local cohomology

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^0_m(R/fR)_{\leq N} & \longrightarrow & H^0_m(R)[-d]_{\leq N} & \longrightarrow & \cdots \\
& & f & \longrightarrow & f & \longrightarrow & \cdots \\
0 & \longrightarrow & H^0_m(R/fR)_{\leq pN} & \longrightarrow & H^0_m(R)[-d]_{\leq pN + d - 1} & \longrightarrow & \cdots.
\end{array}
$$

As remarked above, injectivity of $F$ on the left is equivalent to that of the middle map $f^{p-1}$. Assume that we have a homogeneous $0 \neq \alpha \in H^0_m(R)[-d]_{\leq N} = H^0_m(R)_{\leq \delta(f)}$ such that $f^{p-1}\alpha = 0$. We have a representation of $\alpha$ of the form

$$\alpha = \left[ \frac{g}{(x_0 \cdots x_n)^{p^e}} \right].$$
with \( g \notin m^{p^e} \) and where we may assume that the power in the bottom is \( p^e \) for some \( e \gg 0 \) by multiplying by an appropriate form of 1. Using this representation, we have

\[
  f^{p-1}F(\alpha) = 0 \iff f^{p-1}g^p \in m^{[p^{e+1}]}
\]

\[
  \iff f^{p-1} \in \left( m^{[p^{e+1}]} : g^p \right) = \left( m^{[p^e]} : g \right)^{[p]}
\]

\[
  \iff \tau(f^{1-\frac{1}{p}}) \subseteq \left( m^{[p^e]} : g \right)
\]

\[
  \iff g \in \left( m^{[p^e]} : \tau(f^{1-\frac{1}{p}}) \right).
\]

Here the equality of colon ideals in the second line follows from Kunz’s theorem \([\text{Kun}69, \text{Theorem 2.1}]\) which says Frobenius is flat if and only if \( R \) is regular, along with the fact that if \( I \to B \) is a flat ring extension then \((I :_A J)B = (IB :_B JB)\) for any ideals \( I, J \subseteq A \). Thus, \( \deg(g) \geq M_e \) and so

\[
  \deg(\alpha) = \deg(g) - (n + 1)p^e \geq M_e - (n + 1)p^e = \delta(f)
\]

This contradicts \( \deg(\alpha) < \delta(f) \). \( \blacksquare \)

**Remark 2.15.** The proof also shows that this bound is optimal: for \( e \gg 0 \) and an element \( r \in (m^{[p^e]} : \tau(f^{1-\frac{1}{p}})) \setminus m^{[p^e]} \) homogeneous of degree \( M_e \), if we take \( \alpha = [r/(x_0 \cdots x_n)^{p^e}] \) then \( \alpha \neq 0 \) but \( f^{p-1}F(\alpha) = 0 \).

**Corollary 2.16.** Let \( f \in R \) be homogeneous of degree \( d \) and assume that \((R/fR)_m \) is not F-pure. Then \( R/fR \) has an isolated non-F-pure point at \( m \) if and only if the below Frobenius action is injective:

\[
  F : H^r_m(R/fR)_{-n(d-1)} \to H^r_m(R/fR)_{-pn(d-1)}.
\]

**Proof.** Assume that \( R/fR \) has an isolated non-F-pure point at \( m \). We show that \(-n(d-1) < \delta(f) + d \). As in Remark 2.2, let \( F = \{f_{ib} | 0 \leq i_j \leq p; b \in B \} \) be a generating set for \( \tau(f^{1-\frac{1}{p}}) \). Since \( R/fR \) has an isolated non-F-pure point at \( m \), there exist \( n + 1 \) generators \( f_0, \ldots, f_n \in F \) which form a maximal regular sequence. Write \( d_i = \deg(f_i) \). The proof method of [BS13, Lemma 3.1] shows that \( m^{(\sum d_i) - n} \subseteq (f_0, \ldots, f_n) \). Indeed, let \( b = (f_0, \ldots, f_n) \). Then the Hilbert series of \( R/b \) is

\[
  P(R/b, t) = \prod_{i=0}^n \frac{1 - t^{d_i}}{1 - t}.
\]

This follows from [Eis94, Exercise 21.12(b)] together with the facts that \( P(k[x], t) = \frac{1}{1-t} \) and that \( P(M \otimes N, t) = P(M, t) \cdot P(N, t) \) whenever all quantities are defined. The degree of this polynomial is \( (\sum_{i=0}^n d_i) - (n + 1) \). It follows that there can be no monomials of degree greater than \( (\sum d_i) - (n + 1) \) in \( R/b \). This is equivalent to \( m^{(\sum d_i) - (n+1)+1} \subseteq b \).

From this we see that \( \left( m^{[p^e]} : \tau(f^{1-\frac{1}{p}}) \right) \subseteq \left( m^{[p^e]} : m^{(\sum d_i) - n} \right) \) and [BS13, Lemma 3.2] tells us that

\[
  \left( m^{[p^e]} : m^{(\sum d_i) - n} \right) = m^{[p^e]} + m^{(n+1)p^e - (\sum d_i)}.
\]

Letting \( e \gg 0 \) and \( r \in \left( m^{[p^e]} : \tau(f^{1-\frac{1}{p}}) \right) \setminus m^{[p^e]} \) be homogeneous of degree \( M_e \), the equality above shows us that

\[
  \deg(r) = M_e \geq (n + 1)p^e - (\sum d_i).
\]

By Lemma 2.9, we now conclude \( \delta(f) \geq (\sum d_i) \). Thus, \( \delta(f) + d > -(\sum d_i) + d - 1 \). Since \( d_i = \deg(f_i) \) we have that \( pd_i \leq d(p - 1) \) from which it follows that \( d_i < d - 1 \). Replacing each \( d_i \) with \( d - 1 \) we conclude

\[
  -n(d-1) < -(\sum d_i) + d - 1 < \delta(f) + d.
\]

Using the contrapositive, if \( R/fR \) does not have an isolated non-F-pure point at \( m \), then lemmas 2.8 and 2.9 tell us \( M_e - (n + 1)p^e \) \( e \geq e_0 \) is unbounded below. If \( r \in \left( m^{[p^e]} : \tau(f^{1-\frac{1}{p}}) \right) \setminus m^{[p^e]} \) has degree \( M_e \), then \( f^{p-1}F([r/(x_0 \cdots x_n)^{p^e}]) = 0 \) but \( [r/(x_0 \cdots x_n)^{p^e}] \neq 0 \). Letting \( e \gg 0 \) such that \( M_e - (n + 1)p^e < -n(d - 1) \), we see that the Frobenius action on \( H^r_m(R/fR)_{-n(d+1)p^e} \) is not injective. \( \blacksquare \)
Example 2.17. Let $f = x^2 y^2 + y^2 z^2 + z^2 x^2 \in k[x, y, z]$ with $\text{char}(k) > 2$. Then $\tau(f^{1-\frac{1}{2}}) = m$ but $f$ does not have an isolated singularity. In this case, the Bhatt-Singh result [BS13 Theorem 3.5] does not apply. Theorem 2.14 now tells us that the Frobenius action on $H^2_m(R/fR)$ is injective in degrees $\leq 0$. Note that in this case, $H^2_m(R/fR) \neq 0$ but $H^3_m(R/fR) \geq 2 = 0$ so the Frobenius action on $H^2_m(R/fR)$ is zero.

Example 2.18. We provide an example to show that $M_e - (n + 1)p^e$ does not always stabilize at the first step. Let $f = x_0^2x_1x_2x_3x_4 + x_0x_1^2x_2x_3x_4 + \cdots + x_0x_1x_2x_3x_4 + x_5^6 \in F_2[x_0, \ldots, x_5]$. Then $\tau(f^{1/2}) = (x_0, x_1, x_2, x_3, x_4^3)$. Now $M_1 = 5$, $M_2 = 16$, and we see that $M_1 - 6(2) = -7$ but $M_2 - 6(2^2) = -8$. Since $m^3 \subseteq \tau(f^{1/2})$ we have $-5 - 3 \leq M_e - 6(2^e)$ so we see that $\delta(f) = -8$.

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