Bubbling and bistability in two parameter discrete systems

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Abstract

We present a graphical analysis of the mechanisms underlying the occurrences of bubbling sequences and bistability regions in the bifurcation scenario of a special class of one dimensional two parameter maps. The main result of the analysis is that whether it is bubbling or bistability is decided by the sign of the third derivative at the inflection point of the map function.

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1. Introduction.

The studies related to onset of chaos in one-dimensional discrete systems modeled by nonlinear maps, have been quite intense and exhaustive during the last two decades. Such a system normally supports a sequence of period doublings leading to chaos. It is also possible to take it back to periodicity through a sequence of period halvings by adding perturbations or modulations to the original system[1,2]. This has, most often, been reported as a mechanism for control of chaos. In addition, there are features like tangent bifurcations, intermittency, crises etc., that occur inside the chaotic regime and are not of immediate relevance to the present work. However, if the system is sufficiently nonlinear, there are other interesting phenomena like bubble structures and bistability that have invited comparatively less attention. The simplest cases where these are realised are maps with at least two control parameters, one that controls the nonlinearity and the other which is a constant additive one. i.e., maps of the type,

\[ X_{n+1} = f(X_n, a, b) = f_1(X_n, a) + b \]  

(1)

In these maps, if \( a \) is varied for a given \( b \), the usual period doubling route to chaos is observed. But when \( a \) is kept at a point beyond the first period doubling point \( a_1 \), and \( b \) is varied, the first period doubling is followed by a period halving forming a closed loop like structure called the primary bubble in the bifurcation diagram(Fig(1.a)). If \( a \) is kept beyond the second bifurcation point \( a_2 \), and \( b \) is tuned, secondary bubbles appear on the arms of the primary bubble. Thus as we shift the map along the \( a \)-axis and drift it along the \( b \)-axis, the complete bubbling scenario develops in the different slices of the space \((X, a, b)\). This accumulates into what is known as bimodal chaos- chaos restricted or confined to the arms of the primary
bubble. This can be viewed as a separate scenario to chaos in such systems.

It has been confirmed that the Feigenbaum indices for this scenario with \( a \) as control parameter would be the same as the \( \alpha \) and \( \delta \) of the normal period doubling route to chaos[3]. However, detailed RG analysis by Oppo and Politi[4], involving the parameter \( b \) also, indicates that if \( b \) is kept at a critical value, \( b_c \), where bimodal chaos just disappears, then there is a slowing down in the convergence rate leading to an index which is \( (\delta)^{1/2} \). This has been experimentally verified in a \( CO_2 \) laser system with modulated losses[5]. The bubbling scenario is seen in the bifurcation diagrams of many nonlinear systems like coupled driven oscillators[6, 7], oscillatory chemical reactions, diode circuits, lasers[8,9], insect populations[10], cardiac cell simulations[11], coupled or modulated maps[12,13], quasi-periodically forced systems[14], DPCM transmission system[15] and traffic flow systems[16] etc. The very fact that this phenomenon appears in such a wide variety of systems makes it highly relevant to investigate and expose the common factor(s) in them i.e., the underlying basic features that make them support bubbles in their bifurcation scenario. The above mentioned continuous systems require maps with at least two parameters of type (1) to model them, the second additive parameter being the coupling strength, secondary forcing amplitude etc. We note that in all the above referred papers no specific mention is made regarding the mechanism of formation of bubbles, probably because the authors were addressing other aspects of the problem. However, there have been a number of isolated attempts to analyse the criteria for bubble formation in a few typical systems. According to Bier and Bountis[3], the two criteria are, the map must possess some symmetry and the first period doubling should occur facing the symmetry line. Later, Stone[17] makes these a little more explicit by stating
that the map should have an extending tail (with a consequent inflection point) and the inflection point should occur to the right of the critical point of the map. It is clear that this applies only to maps with one critical point. The relation of the extending tail to bubbling is briefly discussed in [18] also.

Bistability is an equally interesting and common feature associated with many non-linear systems like a ring laser[19] and a variety of electronic circuits[20]. A recent renewal of interest in such systems arises from the fact that they form ideal candidates for studies related to stochastic resonance phenomena[21]. The bistable behaviour in two parameter maps is shown in the bifurcation diagram in Fig(1.b). To the best of our knowledge, attempts to study any type of conditions for the occurrence of bistability are so far not seen reported in the literature.

Our motivation in the present work is to generalise the criteria reported earlier for bubbling and put them together with more clarity and simplicity. As a byproduct, we succeed in stating the conditions for bistability also along similar lines in systems of type (1), even though these are two mutually exclusive phenomena as far as their occurrence regime is concerned. We provide a detailed graphical analysis, which leads to a simple and comprehensible explanation for the same.

The paper is organized as follows. In section 2, the criteria for bistability and bubbling are stated followed by a brief explanation. The graphical analysis taking two simple cubic maps as examples is included in section 3 and the concluding comments in section 4.
2. The dynamics of bubbling and bistability

For the special class of maps given in (1), the basic criteria for bubbling / bistability can be stated as follows.

*The non-linearity in* \( f(X, a, b) \) *must be more than quadratic.*

This implies that, \( f'(X, a, b) \) (the prime indicating derivative with respect to \( X \)), is non-monotonic in \( X \) and there exists at least one inflection point \( X_i \), where

\[
f''(X_i, a, b) = 0.
\]  

(2)

Then we consider the following two cases.

Case (i)

\[
f'''(X_i, a, b) > 0
\]  

(3)

We define a value of \( a \) as \( a_1 \), through the relation,

\[
f'(X_i, a_1, b) = -1,
\]  

(4)

the first period doubling point of the system. For a value of \( a \) close to \( a_1 \) but greater than \( a_1 \), by adjusting the additive parameter \( b \), the system can be taken through a bubble structure in the bifurcation scenario.

Case (ii)

\[
f'''(X_i, a, b) < 0
\]  

(5)

Here \( a_1 \) is defined as the tangent bifurcation point of the system, through the relation

\[
f'(X_i, a_1, b) = +1
\]  

(6)

Then by fixing \( a \) greater than \( a_1 \), but close to \( a_1 \), and tuning \( b \), a bistability region can be produced in the system.

For case (i), the value of \( a \) chosen to be greater than \( a_1 \) makes \( f'(X_i, a, b) < -1 \) or
Moreover, conditions (2) and (3) imply that $X_i$ is a minimum for $f'$, which is concave upwards on both sides of $X_i$. Hence for a fixed point $X^*_+$ which is to the left of $X_i$, but in the immediate neighborhood, of $X_i$, $|f'(X^*_+, a, b)| < 1$, and hence will be stable. Similarly, fixed point $X^*_+$ to the right of $X_i$, but near to $X_i$, $|f'(X^*_+, a, b)| < 1$ and is stable. Now, the second parameter $b$ is simple additive for the class of maps under consideration and hence $f'$ is independent of $b$. By adjusting $b$, the fixed point can be shifted such that $f'(X^*_+, a, b)$ becomes equal to -1, the period doubling point of the map. Then $X^*_+$ will give rise to a 2-cycle with elements $X^*_1$ and $X^*_2$. Since these are in the neighborhood of $X_i$, $f'(X^*_1)$ and $f'(X^*_2)$ will be negative so that the product $f'(X^*_1)f'(X^*_2)$ is positive. With further increase of $b$, a period merging takes place for the 2-cycle, with $X^*_1$ and $X^*_2$ collapsing into $X^*_+$, which is just stable at the point where $f'(X^*_+) = -1$. Thus in the parameter window $(b_1, b_2)$, a bubble structure is formed.

The situation is exactly reversed for case (ii). Here conditions (2) and (5) makes $X_i$ a maximum of $f'$ and the falls off on both sides of $X_i$. At a value of $a > a_1$, where $a_1$ is defined by (6), $f'(X_i, a, b) > +1$. Then in the neighborhood of $X_i$, a fixed point $X^*_-$, to the left of $X_i$, can be stable since $|f'(X_i, a, b)| < 1$. Similarly $X^*_+$ on the right of $X_i$ also will be stable. By adjusting the second parameter $b$, these will be shifted to their respective tangent bifurcation points, i.e., $b_1$ where $X^*_1$ is born and $b_2$ where $X^*_-$ disappears. Then a bistability window is seen in the interval $(b_1, b_2)$.

3. Graphical analysis.

The mechanism of occurrence of bubbling and bistability explained above for maps satisfying the conditions in case (i) and case (ii) respectively can be made more transparent through a detailed graphical analysis. For this we plot the curve $C_1 = f'(X)$,
the 1-cycle fixed point curve $C_2 = f(X) - X$ and the 2-cycle curve $C_3 = f(f(X)) - X$ simultaneously as functions of $X$, for chosen values of $a$ and $b$. The zeroes of $C_2$ give the 1-cycle fixed point $X^*$ while those of $C_3$ give the elements of the 2-cycle. Their stability can be checked from the same graph, since the value of the derivative at the fixed points can be read off. We fix the value of $a$ to be greater than $a_1$, which helps to position the curve $C_1$ in the proper way. By plotting the three curves for different values of $b$, bistability regions or bubbling sequences can be traced for any given map function of type (1).

For further discussion, we consider two specific forms of maps of the cubic type, which are simple but typical examples for case (i) and (ii). They are,

$$M_1 : X_{n+1} = b - aX_n + X_n^3$$

$$M_2 : X_{n+1} = b + aX_n - X_n^3$$

For $M_1$, there are two critical points, $X_{c1} = -\sqrt{a/3}$, which is a maximum and $X_{c2} = \sqrt{a/3}$, which is a minimum. The inflection point in between occurs at $X_i = 0$ and $f''' = 6$. Hence it belongs to case (i) and $a_1$ as defined by (4) is 1. In Fig(2), the three curves mentioned above are plotted for this map at $a = 1.3$. We start from a value of $b = -1.34$, Fig(2.a) where the fixed point $X^*$ is just born via tangent bifurcation since $f'(X^*)$ here is +1, and the curves $C_2$ and $C_3$ just touches the zero line on the left of $X^*$ at $X_i$. Though $C_2$ has a zero on the right, the slope there is larger than 1 and hence it is unstable, for this value of $b$. Since $b$ is only additive, increase in the value of $b$, shifts $C_2$ upwards, resulting in a slow drift of $X^*$ from left to right. Thus as $b$ is increased to -0.7(Fig(2.b)), $f'(X^*) = -1$ and $X^*$ bifurcates into $X_1^*$ and $X_2^*$. At $b = -0.3$(Fig(2.c)), the 2-cycle is stable with $f'(X_1^*)$ and $f'(X_2^*)$, both negative and their product is positive but less than 1. Note that
the curve C3 has developed a maximum and a minimum on both sides of $X^*$, which is now unstable, cutting the zero line again at $X_1^* < X^*$ and $X_2^* > X^*$. As $b$ is further increased, they move apart. Since the value chosen is within the stability window of 2-cycle no further period doubling takes place. As $X_1^*$ crosses $X_i$, $X_1^*$ & $X_2^*$ move towards each other and merge together at $b = 0.7$ (Fig(2.d)) and coincide with the fixed point $X_i^*$. Further, $X_i^*$ disappears by a reverse tangent bifurcation at $b = 1.34$, when $f'(X_i^*)$ becomes equal to +1. Thus the above events lead to the formation of a primary bubble in the window (-0.7,0.7).

By keeping $a$ at a value beyond the second period doubling point $a_2$, of the map, the merging tendency starts only after the second period doubling and hence secondary bubbles are seen on the arms of the primary bubble. These can be repeated until at $a > a_{\infty}$, the system is taken to chaos.

Now the above analysis is repeated for map M2, which satisfies the conditions in case (ii) (Fig(3)). Here of the two critical points of the map, $X_{c1} = -\sqrt{a/3}$ is the minimum and $X_{c2} = \sqrt{a/3}$ is the maximum with a positive slope at the point of inflexion $X_i$. $a_1$ in this case is also 1. Hence in the Fig (3) $a$ is chosen to be 1.4. Fig(3.a) shows the situation for $b = -0.35$, where $f'(X_i) = -1$ and hence the 1-cycle fixed point $X_i^*$ period doubles into a 2-cycle. For lower values of $b$, we expect the full period doubling scenario since $f'$ is monotonic beyond this point (Fig(3.b)).

However, as $b$ is increased to -0.1, $f'(X_i^*) = +1$, where the other 1-cycle, $X_i^*$ to the right of $X_i$ is born by tangent bifurcation (Fig(3.c)). Note that at this point $X_i^*$ is still stable with $\left|f'(X_i^*)\right| < 1$. This continues until $b = +0.1$, where $f'(X_i^*) = +1$ and hence $X_i^*$ disappears (Fig(3.d)). The birth of $X_i^*$ is concurrent with the maximum of C2 touching the zero line ($b = b_1$) while the disappearance of $X_i^*$ occurs
as the minimum of C2 touches the zero line \( (b = b_2) \). As \( b \) is increased and C2 is moving up it is clear that the former will take place for a lower \( b \) value than the latter as the maximum of C2 occurs for \( X > X_i \) and minimum at \( X < X_i \) (slope being positive at \( X_i \)). Hence \( b_1 < b_2 \), or there is a window \((b_1, b_2)\), where bistability exists, which in our graph is \((-0.1, 0.1)\) for \( a = 1.4 \). \( X^* \) is stable beyond this point also and it period doubles as \( b \) is increased to \( b = +0.35 \) where \( f'(X^*) = -1 \). The full Feigenbaum scenario then develops for higher values of \( b \). By keeping \( a \) at higher values and tuning \( b \), the bistability can be taken to 2-cycle, 4-cycle and even chaotic regions.

The stability regions of the different types of dynamical behaviour possible for M1 can be marked out in parameter space plot in the \((a, b)\) plane (Fig(4.a)). The cone like region on the left is the stability zone of the 1-cycle fixed point (periodicity, \( p=1 \)) and it is separated from the escape region by the tangent bifurcation line on both sides. The parabola like curve inside it marks out the 2-cycle \((p=2)\) region, while the smaller parabolas indicate curves along which 4-cycles \((p=4)\) and other higher periodic cycles becomes stable until chaos is reached. The line parallel to the \( b \)-axis at a value of \( a > a_1 \), along which primary bubble is formed, is shown by the dotted line. It is clear that along this line, the system is taken from escape \( \rightarrow 1 \)-cycle \( \rightarrow 2 \)-cycle \( \rightarrow 1 \)-cycle \( \rightarrow \) escape. Similarly secondary bubbles are formed along a line drawn at \( a > a_2 \) etc. The parameter space plot for M2 is shown in Fig(4.b).

The quadrilateral like region marked as (I) beyond \( a > a_1 \) is the bistable region for 1-cycle, while quadrilateral (II) is that for 2-cycle etc. The area marked with \( p=1 \), is the stability region of 1-cycle while \( p=2 \), that for 2-cycle etc. When the system is taken along the dotted line beyond \( a_1 \), bistability is seen in the central region,
followed by period doubling bifurcations to both sides, until chaos is reached.

4. Conclusion.

Although the above discussion is confined to two simple cubic maps, the analysis is repeated for a large number of maps of type (1) chosen from a wide variety of situations covering different functional forms like exponential, trigonometric and polynomial maps. We find that the qualitative behaviour in all cases remain the same and depends only on the criteria (2)-(6). Hence the pattern of scenario detailed in this paper can be taken to be atypical as far as maps of the form (1) are concerned.

The criteria for bistability reported here are certainly novel while those for bubbling are more rigorous and general in nature compared to earlier studies. They can be used as a test to identify maps in which bistability or bubbling is possible and also to isolate the regions in the parameter space \((a, b)\) where they occur. Our main result is that whether it is bistability or bubbling is decided by the sign of the third derivative of the map function at the inflection point. If \(f'''(X_i)\) is positive, because of the concave nature of the derivative, tangent bifurcation will precede period doubling as \(b\) is increased. Hence bubbling structure is possible. Similarly when \(f'''(X_i)\) is negative, curve of \(f'\) is convex and hence period doubling precedes tangent bifurcation, leading to bistability. In case \(f'''(X_i) = 0\), higher derivatives must be considered for deciding the behaviour.

Bubbling can be looked upon as an extreme case of incomplete period doublings and the latter has been often associated with positive Schwarzian derivative[22]. But for the system under study, it is easy to check that this is always negative (independent of the form of the map function), because of properties (2), (3) and (5). In fact,
a few such maps have been reported earlier[23] though in a totally different context. The bubbling scenario in maps of the type M1, leads to bimodal chaos that is restricted to the arms of the primary bubble. Such confined chaos or even low periodic behavior prior to that, makes them better models in population dynamics of eco systems than the usual logistic type maps[24]. Attempts to extend the criteria to continuous and higher dimensional systems are underway and will be reported elsewhere.

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**Figure Captions**

Fig.1:- Bifurcation diagram showing (a) bubble structure and (b) bistable behaviour, for a fixed value of $a$, with $b$ as the control parameter.

Fig.2:- The derivative curve $C_1$, the 1-cycle solution curve $C_2$ and the 2-cycle solution curve $C_3$ plotted with the value of $a$ at 1.3 for the map $M_1$. In (a), $b = -1.34$ shows the point where the 1-cycle $X_1^*$ is just born, with $f'(X_1^*) = +1$. (b) With $b = b_1 = -0.7$, $f'(X_1^*) = -1$ hence the $X_1^*$ becomes unstable and the 2-cycle is just born. (c) $b = -0.3$, shows the elements of the stable 2-cycle with $X_1^*$ to the left and $X_2^*$ to the right of the $X_1^*$, which is unstable now and (d) $b = b_2 = +0.7$, the 1-cycle fixed point $X_1^*$ becomes stable after the merging of $X_1^*$ and $X_2^*$.

Fig.3:- Here the curves $C_1$, $C_2$ and $C_3$ for the map $M_2$ defined in (8) with $a = 1.4$ plotted. (a) At $b = -0.5$, it is clear from the figure that the 1-cycle solution is unstable and the 2-cycle is stable. (b) $b = -0.35$ gives the first period doubling point i.e., here $f'(X_1^*) = -1$. (c) At $b = b_1 = -0.1$, $f'(X_1^*) = +1$, i.e., the creation of a new fixed point $X_1^*$ by tangent bifurcation. Note that still $X_1^*$ is stable and (d) $b = b_2 = 0.1$, $f'(X_1^*)$ is +1. Hence the existing fixed point $X_1^*$ disappears. Thus $(b_1, b_2)$ gives the bistability window.

Fig.4:- Parameter space plot in $(a, b)$ plane (a) for map $M_1$ and (b) for map $M_2$. 
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