Distribution laws of smooth divisors

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Abstract

A classical result due to Deshouillers, Dress and Tenenbaum asserts that on average the distribution of the divisors of the integers follows the arcsine law. In this paper, we investigate the distribution of smooth divisors of the integers, that is, those divisors which are free of large prime factors. We show that on average these divisors are distributed according to a probability law that we will describe.

1 Introduction

Let \( n \geq 1 \) be an integer. We denote by \( P(n) \) the largest prime divisor of \( n \geq 2 \) and we set \( P(1) = 1 \). Let \( y \in [1, +\infty[ \) be a real number. Consider the set of \( y \)-smooth divisors of \( n \), that is, those divisors of \( n \) which are free of prime factors exceeding \( y \).

\[ D_{n,y} := \{ d \mid n : P(d) \leq y \}, \]

and denote by \( \tau(n, y) \) its cardinality. For each integer \( n \geq 1 \) and for each real number \( y > 1 \), we define the random variable

\[ X_{n,y} : D_{n,y} \rightarrow [0, 1], \]

which takes the values \( \log d / \log n \) with uniform probability \( 1/\tau(n, y) \) and for \( v \in [0, 1] \), its distribution function

\[ F_{n,y}(v) := P(X_{n,y} \leq v) = \frac{1}{\tau(n, y)} \sum_{d \mid n, d \leq n^v, P(n) \leq y} 1. \]

It is easy to see that the sequence \( (F_{n,y})_{n \geq 1} \) does not converge pointwise in \([0, 1]\). We consider its mean in the interval \([0, 1]\)

\[ \frac{1}{x} \sum_{n \leq x} F_{n,y}(v) = \frac{1}{x} \sum_{n \leq x} P(X_{n,y} \leq v). \tag{1} \]

The aim of this paper is to show that this mean converges to a distribution function which will be described. Deshouillers, Dress and Tenenbaum [4] studied of the analogue of this
mean by considering the set of all divisors of \( n \), that is without constraint on their prime factors. By denoting here by \( X_n \) the analogue of the random variable \( X_{n,y} \) defined on the set of all divisors of \( n \), they showed that

\[
\frac{1}{x} \sum_{n \leq x} \mathbb{P}(X_n \leq v) = \frac{2}{\pi} \arcsin(\sqrt{v}) + O \left( \frac{1}{\sqrt{\log x}} \right), \tag{2}
\]

uniformly for \( v \in [0,1] \). This arcsin law is a Dirichlet law in one dimension with parameters equal \((1/2, 1/2)\). 

Studying Couples of divisors, the authors of the present paper [7] showed that they are distributed according to a two-dimensional Dirichlet Law. The method works in higher dimension but becomes very technical. de la Bretèche and Tenenbaum [2] studied also couples of divisors by using a probabilistic model that preserves the equiprobability of the first marginal law and allows them to deduce the second marginal law. They also obtained a Dirichlet law.

Recently, Basquin [1] studied an analogue of this question of law of divisors by considering the set of divisors of smooth integers \( n \). This question is naturally related to the de Bruijn function:

\[
\Psi(x,y) := \sharp \{ n \leq x : P(n) \leq y \}.
\]

The asymptotic behavior of de Bruijn’s function is known in a large range of \( x y \)-plane. It is connected to Dickman’s function \( \rho \), which is the continuous solution in \([0, +\infty[\) to the differential-difference equation with initial condition:

\[
\begin{align*}
\rho'(w) + \rho(w-1) &= 0, \quad (w > 1) \\
\rho(w) &= 1, \quad (0 \leq w \leq 1) \\
\rho(w) &= 0, \quad (w < 0),
\end{align*}
\]

which the asymptotic behavior is well known. For example, we have

\[
\log \rho(w) = -(1 + 0(1))w \log w, \quad (w \to +\infty).
\]

Before quoting Basquin’s result and formulate the behavior of \( \Psi(x,y) \), let us introduce some notations that will be maintained throughout the rest of this paper. For \( 1 < y \leq x \), we set

\[
u := \frac{\log x}{\log y},
\]

and we denote by \((H_\epsilon)\) the subset of \( \mathbb{R}^2 \) defined by the condition

\[
x \geq x_0(\epsilon), \quad \exp \left( (\log \log x)^{\frac{\epsilon}{3} + \epsilon} \right) \leq y \leq x,
\]

where \( x_0(\epsilon) > 0 \) is a sufficiently large constant depending on \( \epsilon > 0 \). Here it is sufficient to quote the following asymptotic formula for \( \Psi(x,y) \) due to Hildebrand [6] and valid in the range \((H_\epsilon)\)

\[
\Psi(x,y) = x \rho(u) \left( 1 + O \left( \frac{\log(u + 1)}{\log y} \right) \right).
\]
Let us introduce functions $\rho_k$ for $k \in ]0, +\infty[$. Each function $\rho_k$ is the continuous solution to the differential-difference equation with initial condition:

$$
\begin{cases}
  w\rho'_k(w) + (1-k)\rho_k(w) + k\rho_k(w-1) = 0, & (w > 1) \\
  \rho_k(w) = \frac{1}{\Gamma(k)} w^{k-1}, & (0 < w \leq 1) \\
  \rho_k(w) = 0, & (w \leq 0).
\end{cases}
$$

In particular, we have $\rho_1 = \rho$. Function $\rho_k$ is the k-th fractional convolution power of $\rho$ – see Hensley’s work [5]. Its asymptotic behavior of $\rho_k$ is well known – see in particular Smida’s papers [8],[9], where we find properties of these functions and their connection to the asymptotic behavior of Dickman’s function $\rho$. In particular, we have the formula [8],

$$
\rho_k(u) = k^{u(1+O(\frac{1}{\log u}))} \rho(u), \quad (u \to +\infty).
$$

Basquin showed that

$$
\frac{1}{\Psi(x, y)} \sum_{n \leq x, P(n) \leq y} \mathbb{P}(X_n \leq v) = \frac{1}{\rho(u)} \int_0^{uv} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u-s) \, ds + O \left( \frac{\log(u+1)}{\log y} + \frac{1}{\sqrt{\log y}} \right),
$$

uniformly for $v \in [0,1]$ and $(x, y) \in (H_\epsilon)$, and he deduced that as $u \to +\infty$, the distribution function converges to the normal distribution. More precisely, he showed that

$$
\frac{1}{\rho(u)} \int_0^{uv} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u-s) \, ds = \Phi \left( u\sqrt{2\xi'(u)(v - \frac{1}{2})} \right) + O \left( \frac{1}{u} \right),
$$

where $\xi'(u) \sim 1/u$, as $u \to +\infty$ and

$$
\Phi(w) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{w} e^{-t^2} \, dt,
$$

is the normal distribution function.

## 2 Statements of results

To state our results, let us introduce the Buchstab function $\omega$. This function was discovered by Bushstab and comes from the study of the uncanceled elements in the sieve of Eratosthenes – see de Bruijn’s beautiful paper [3]. It is the unique continuous solution for $v > 1$ to the differential-difference equation, with initial condition:

$$
\begin{cases}
  v\omega'(v) + \omega(v) - \omega(v-1) = 0, & (v > 2) \\
  \omega(v) = \frac{1}{v}, & (1 \leq v \leq 2) \\
  \omega(v) = 0, & v < 1.
\end{cases}
$$
Its asymptotic behavior is known – see de Bruijn [3] and Tenenbaum’s book [10] chap. III.6). In particular for $v \geq 0$, we have
\[
\omega(v) = e^{-\gamma} + O\left(\rho(v)e^{\log^2(v+2)}\right),
\] (3)
where $\gamma$ is the Euler function and $c$ a positive constant. In the first theorem below we show the convergence of the mean of distribution functions (1) to a distribution function, in the second one we describe the limit law as $u \to +\infty$ and in the third one, we give as an example expressions of the limit law for $1 \leq u \leq 2$.

**Theorem 2.1** Uniformly for $v \in [0, 1]$ and $(x, y)$ in $(H_{\epsilon})$, we have
\[
\frac{1}{x} \sum_{n \leq x} \mathbb{P}(X_{n, y} \leq v) = \int_0^{uv} \left( \int_0^{u-s-1} \rho_\frac{x}{2}(z)\omega(u-s-z)\,dz \right) \rho_\frac{1}{2}(s)\,ds
\]
\[
+ \int_0^{uv} \rho_\frac{1}{2}(s)\rho_1(u-s)\,ds
\]
\[
+ O\left(\frac{\log(u+1)}{\log y} + \frac{1}{\sqrt{\log y}}\right).
\]
We notice that for $y = x$, that is to say $u = 1$, the formula of Theorem 2.1 is reduced to formula (2) obtained in [4]. Indeed, the first integral vanishes because $\rho_\frac{1}{2}(z) = 0$ for $z \leq 0$ and
\[
\int_0^v \rho_\frac{1}{2}(s)\rho_1(1-s)\,ds = \frac{2}{\pi} \int_0^{\sqrt{\pi}} \frac{dt}{\sqrt{1-t^2}} = \frac{2}{\pi} \arcsin(\sqrt{v}).
\]
Let us denote
\[
F(u, v) = \int_0^{uv} \left( \int_0^{u-s-1} \rho_\frac{1}{2}(z)\omega(u-s-z)\,dz \right) \rho_\frac{1}{2}(s)\,ds + \int_0^{uv} \rho_\frac{1}{2}(s)\rho_1(u-s)\,ds.
\]
We have

**Theorem 2.2** For $v \in [0, 1]$ and as $u \to +\infty$, we uniformly have
\[
F(u, v) = \frac{1}{\sqrt{e}} \int_0^{uv} \rho_\frac{1}{2}(s)\,ds + O\left(\rho_2(u)\right),
\]
where $\gamma$ is the d’Euler constant.

**Theorem 2.3** 1. For $v \in \left[0, \frac{u-1}{u}\right]$ and $1 < u \leq 2$, we have
\[
F(u, v) = \frac{2}{\pi} \arcsin(\sqrt{v}) + \frac{1}{\pi} \left(\log(u) + \log(1 - v)\right) \arcsin\left(\sqrt{\frac{uv}{u-1}}\right) - \frac{1}{2} \log(1-v).
\]
2. For \( v \in \left[ \frac{u-1}{u}, \frac{1}{u} \right] \) and \( 1 < u \leq 2 \), we have

\[
F(u, v) = \frac{2}{\pi} \arcsin(\sqrt{v}) + \frac{1}{2} \log u.
\]

3. For \( v \in \left[ \frac{1}{u}, 1 \right] \) and \( 1 < u \leq 2 \), we have

\[
F(u, v) = \frac{2}{\pi} \arcsin(\sqrt{v}) + \frac{1}{\pi} \left( \log(u) + \log(v) \right) \arcsin \left( \sqrt{\frac{u(1-v)}{u-1}} \right) - \frac{1}{2} \log(v).
\]

Let us set

\[
S(x, y, v) := \sum_{n \leq x} \mathbb{P}(X_{n,y} \leq v) = \sum_{n \leq x} \frac{1}{\tau(n, y)} \sum_{d|n, d \leq n^\varepsilon, P(d) \leq y} 1
\]

\[
= S_1(x, y, v) - S_2(x, y, v),
\]

with

\[
S_1(x, y, v) := \sum_{n \leq x} \frac{1}{\tau(n, y)} \sum_{d|n, d \leq n^\varepsilon, P(d) \leq y} 1 = \sum_{d \leq x^\varepsilon, P(d) \leq y} \sum_{m \leq x/d} \frac{1}{\tau(dm, y)}
\]

and

\[
S_2(x, y, v) := \sum_{n \leq x} \frac{1}{\tau(n, y)} \sum_{d|n, n^{\varepsilon} < d \leq x^\varepsilon, P(d) \leq y} 1.
\]

We will show that the main contribution to Theorem 2.1 comes from the estimation of \( S_1(x, y, v) \). The proof rests on the estimation

\[
\sum_{n \leq x} \frac{1}{\tau(d, y)}
\]

for \((x, y)\) in \((H_\varepsilon)\) and \(d \geq 1\), \( y \)-smooth. To study this quantity, we used the factorisation \( n = ab \) with \( a \) is \( y \)-smooth and \( b \) has all its prime factors greater than \( y \). This allowed us to use results on this topic available in the literature.

3 Proof of Theorem 2.1

3.1 Preparatory Lemmas

Let us introduce some notations which will be used in the sequel. For each fixed integer \( d \geq 1 \) we define a multiplicative function

\[
\gamma_d(n) = \frac{\tau(d)}{\tau(dn)},
\]
where \( \tau \) is the divisor function. For each prime number \( p \), we denote by \( v_p(d) \) the \( p \)-adic valuation of \( d \). We have

\[
\gamma_d(p^\alpha) = \frac{v_p(d) + 1}{v_p(d) + \alpha + 1}.
\]

we consider the Dirichlet series of \( \gamma_d(n) \)

\[
F_d(s) := \sum_{n \geq 1} \frac{\gamma_d(n)}{n^s}, \quad (\Re(s) > 1).
\]

We have \( F_d(s) = \zeta^\frac{1}{2}(s)G_d(s) \) the half-plane \( \Re(s) > 1 \), with \( \zeta \) is the Riemann zeta function and

\[
G_d(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{\frac{1}{2}} \left( \sum_{\alpha \geq 0} \frac{v_p(d) + 1}{(v_p(d) + \alpha + 1)p^{\alpha s}} \right)
\]

with

\[
B(s) := \prod_p \left( 1 - \frac{1}{p^s} \right)^{\frac{1}{2}} \left( 1 + \sum_{\alpha \geq 1} \frac{1}{(\alpha + 1)p^{\alpha s}} \right),
\]

and

\[
K_d(s) := \prod_{p \nmid d} \left( 1 + \sum_{\alpha \geq 1} \frac{\beta + 1}{(\beta + \alpha + 1)p^{\alpha s}} \right) \left( 1 + \sum_{\alpha \geq 1} \frac{1}{(\alpha + 1)p^{\alpha s}} \right)^{-1}.
\]

For each fixed integer \( d \geq 1 \), we define Dirichlet series

\[
K_d(s) := \sum_{n \geq 1} \frac{\delta_d(n)}{n^s}; \quad B(s) = \sum_{n \geq 1} \frac{b(n)}{n^s}; \quad G_d(s) = \sum_{n \geq 1} \frac{h_d(n)}{n^s}.
\]

Then we have

\[
F_d(s) = \sum_{n \geq 1} \frac{\gamma_d(n)}{n^s} = \zeta^\frac{1}{2}(s) \sum_{n \geq 1} \frac{h_d(n)}{n^s} \quad \text{(4)}
\]

and

\[
\sum_{n \geq 1} \frac{h_d(n)}{n^s} = \left( \sum_{n \geq 1} \frac{b(n)}{n^s} \right) \left( \sum_{n \geq 1} \frac{\delta_d(n)}{n^s} \right). \quad \text{(5)}
\]

**Lemma 3.1** Let \( d \geq 1 \) be a fixed integer. It exists a real number \( 0 < \eta < 1/3 \) such that the series

\[
\sum_{n \geq 1} \frac{h_d(n)}{n^s}
\]

is absolutely convergent in the half-plane \( \Re(s) = \sigma \geq 1 - \eta \) and for \( d \geq 1 \) we uniformly have

\[
\sum_{n \geq 1} \frac{h_d(n)}{n^\sigma} \ll_q \prod_{p \mid d} \left( 1 + \frac{2}{p^\sigma} \right).
\]
The result of this lemma can be deduced from a general study developed in dans [4], page 275. Indeed, we apply lemma 1 of [4], page 276, we get that the series
\[ \sum_{n \geq 1} \frac{\delta_d(n)}{n^s} = K_d(s) \]
is absolutely convergent in the half-plane \( \Re(s) = \sigma \geq 1 - \eta \) and we have
\[ \sum_{n \geq 1} \frac{|\delta_d(n)|}{n^\sigma} \ll \eta \prod_{p | d} \left( 1 + \frac{2}{p^\sigma} \right). \] (6)
The lemma 2 of [4], page 278, applies to the series
\[ \sum_{n \geq 1} \frac{b(n)}{n^s} = B(s) \]
with the exponent \( \alpha = 1/2 \) and \( \psi(n) = 1/\tau(n) \). We obtain
\[ \sum_{n \geq 1} \frac{|b(n)|}{n^\sigma} \leq \prod_p (1 + 5^{1-2\sigma}) \ll \eta. \] (7)
in the half-plane \( \Re(s) = \sigma \geq 1 - \eta \). Lemma 3.1 follows from (5), (6) and (7).

We set
\[ M_{\eta,d} := \prod_{p | d} \left( 1 + \frac{2}{p^{1-\eta}} \right). \]
From the formula (4) we define a multiplicative function \( h_d \) by the convolution identity \( \gamma_d = \tau_{\frac{1}{2}} * h_d \). From lemma 3.1 we get that its Dirichlet series satisfies the conditions (1.18) of Théorème 3 of [9], page 25, since we have
\[ \sum_{\substack{n \leq y \geq 1 \ \text{and} \ \log(n) \leq \left( \log(y) \right) \left( G_d(1) + O(\frac{\log(y) + 1}{\log(y)}) \right) \end{array} \right)} \]
\[ \rho_{\frac{1}{2}}(u) = \frac{1}{\sqrt{\pi \sqrt{u}}} = \frac{\sqrt{\log(y) \log(x)}}{\sqrt{\pi \sqrt{\log(y)}}}. \] (8)

**Lemma 3.2** 1. let be \( \eta \in ]0, \frac{1}{2}[ \) and let \( \epsilon > 0 \) be fixed. For \( d \geq 1 \) and \( (x,y) \in (H_\epsilon) \) we uniformly have
\[ \sum_{\substack{n \leq x \ \text{and} \ \log(n) \leq \left( \log(y) \right) \left( G_d(1) + O(\frac{\log(y) + 1}{\log(y)}) \right) \end{array} \right)} \]
2. For $1 < x \leq y$ and $d \geq 1$, we uniformly have

$$\sum_{n \leq x \atop P(n) \leq y} \gamma_d(n) = \frac{x}{\sqrt{\log y}} \rho_1(u) \left( G_d(1) + O \left( \frac{M_{\eta,d}}{\log x} \right) \right).$$

Basquin [2] obtained the first result of this lemma by using another convolution identity and by applying a general result of Tenenbaum et Wu [11]. We set

$$g(d) := \frac{K_d(1)}{\tau(d)} = \prod_{p \mid d} \left( \sum_{\alpha \geq 0} \frac{1}{(\beta + \alpha + 1)p^\alpha} \right) \left( \sum_{\alpha \geq 0} \frac{1}{(\alpha + 1)p^\alpha} \right)^{-1}.$$

g is a multiplicative function and for $\Re(s) > 1$, we have

$$\sum_{n \geq 1} \frac{g(n)}{n^s} = \zeta(s) \sum_{n \geq 1} \frac{\beta(n)}{n^s},$$

where $\beta$ is a multiplicative function satisfying $g = \tau_\frac{1}{2} * \beta$. We have

$$H(s) := \sum_{n \geq 1} \frac{\beta(n)}{n^s} = \prod_{p} \left( 1 - \frac{1}{p^s} \right)^{\frac{1}{2}} \left( \sum_{\alpha \geq 0} \frac{g(p^{\alpha})}{p^{\alpha s}} \right).$$

Let us note for later use that

$$G_d(1) = K_d(1)B(1) = \tau(d)g(d)B(1). \tag{9}$$

Lemme 2 of [4], page 278, applies to the series $H(s)$ with exponent $\alpha = 1/2$ and the function $\psi(n) = g(n)$. It follows that the series $H(s)$ is absolutely convergent in the half-plane $\Re(s) = \sigma \geq 1 - \eta$ and we have

$$\sum_{n \geq 1} \frac{|\beta(n)|}{n^\sigma} \ll \eta.$$

The conditions (1.18) of application of Théorème 3 of [9], page 25, are satisfied and we obtain the first result of the following lemma 3.3. The second result is an immediate consequence of lemme 3 page 282 of [4] and relation (8) above.

**Lemma 3.3**

1. Uniformly in the range $(H_s)$. We have

$$\sum_{n \leq x \atop P(n) \leq y} g(n) = H(1) \frac{x}{\sqrt{\log y}} \rho_1(u) \left( 1 + O \left( \frac{\log(u + 1)}{\log y} + \frac{1}{\sqrt{\log y}} \right) \right).$$

2. For $1 < x \leq y$, we have

$$\sum_{n \leq x} g(n) = H(1) \frac{x}{\sqrt{\log y}} \rho_1(u) \left( 1 + O \left( \frac{1}{\log x} \right) \right).$$
Remark 3.1 We set
\[ N(d) := \frac{M_{\eta,d}}{\tau(d)}. \]

The function \( N \) is multiplicative and positive and satisfies \( N(p) = 1 + O\left(\frac{1}{\sqrt{p}}\right) \) and \( N(p^\alpha) \ll 1 \). So, from Lemme 3.1 of Basquin [1], and partial summation we get
\[ \sum_{d \leq x, P(d) \leq y} \frac{N(d)}{d} \ll \eta \sqrt{\log y}, \]
uniformly in \((H_\varepsilon)\).

Lemma 3.4 We have \( B(1)H(1) = 1 \).

Proof. We have
\[ B(1) = \prod_p \left( 1 - \frac{1}{p} \right)^{\frac{1}{2}} \left( \sum_{j \geq 0} \frac{1}{(j+1)p^j} \right), \quad H(1) = \prod_p \left( 1 - \frac{1}{p} \right)^{\frac{1}{2}} \left( \sum_{\alpha \geq 0} \frac{g(p^\alpha)}{p^\alpha} \right), \]
and
\[ \sum_{\alpha \geq 0} \frac{g(p^\alpha)}{p^\alpha} = \sum_{\alpha \geq 0} \left( \sum_{j \geq 0} \frac{1}{(\alpha + j + 1)p^{\alpha+j}} \right) \left( \sum_{j \geq 0} \frac{1}{(j+1)p^j} \right)^{-1}, \]
then
\[ B(1)H(1) = \prod_p \left( 1 - \frac{1}{p} \right) \sum_{\alpha \geq 0} \left( \sum_{j \geq 0} \frac{1}{(\alpha + j + 1)p^{\alpha+j}} \right). \]

By noticing that for \( 0 < |x| < 1 \),
\[ \frac{x^{\alpha+j}}{\alpha+j+1} = \frac{1}{x} \int_0^x t^{\alpha+j} dt, \]
we get
\[ \sum_{\alpha \geq 0} \left( \sum_{j \geq 0} \frac{x^{\alpha+j}}{\alpha+j+1} \right) = \frac{1}{x} \int_0^x \frac{dt}{(1-t)^2} = \frac{1}{1-x}, \]
and then \( B(1)H(1) = 1 \). \qed

Let us denote by \( P_-(n) \) the smallest prime factors of \( n > 1 \) and we set \( P_-(1) = +\infty \). The function
\[ \Phi(x, y) := \sum_{n \leq x, P_-(n) > y} 1 \]
has been studied by de Bruijn [3]. In the lemma below we quote an asymptotic formula which is sufficient for our purpose.
Lemma 3.5 Uniformly in the range \((H_\epsilon)\) we have

\[
\Phi(x, y) = \frac{x \omega(u)}{\log y} - \frac{y}{\log y} + O\left(\frac{x}{(\log y)^2}\right).
\]

Proof. Lemma follows from Corollaire 6.14 of [10] chap. III.6 page 547 and Mertens Formula.

\qed

Lemma 3.6 Uniformly for each integer \(d\) such that \(P(d) \leq y\) and \((x, y)\) in \((H_\epsilon)\) we have

\[
\sum_{n \leq x} \frac{1}{\tau(dn, y)} = \frac{x}{\sqrt{\log y}} \frac{G_d(1)}{\tau(d)} \left( \int_0^{u-1} \rho_\frac{1}{2}(z) \omega(u - z)\,dz + \rho_\frac{1}{2}(u) \right) + O\left(\frac{M_{\eta,d}}{\tau(d)} \left( \frac{x \log(u_1)}{(\log y)\frac{3}{2}} + \frac{x}{\log y} \right) \right).
\]

Proof. We write \(n = ab\) with \(P(a) \leq y\) and \((b = 1\) or if \(p|b \Rightarrow p > y\)). Then for \(d\) \(y\)-smooth, we have

\[
\tau(dn, y) = \tau(dab, y) = \tau(da).
\]

and then

\[
T_d(x, y) := \sum_{n \leq x} \frac{1}{\tau(dn, y)} = \sum_{a \leq x, P(a) \leq y} \frac{1}{\tau(da)} \Phi\left(\frac{x}{a}, y\right),
\]

First we consider the range

\[
\exp\left(\left(\log \log x\right)^{\frac{5}{3} + \epsilon}\right) \leq y \leq \frac{x}{a}, \quad x \geq x_0(\epsilon).
\]

Write

\[
\sum_{n \leq x} \frac{1}{\tau(dn, y)} = \sum_{a \leq x, P(a) \leq y} \frac{1}{\tau(da)} \Phi\left(\frac{x}{a}, y\right) + \sum_{x/y < a \leq x, P(a) \leq y} \frac{1}{\tau(da)} := T_d(x, y) + \overline{T}_d(x, y).
\]

Lemma 3.2 gives

\[
\overline{T}_d(x, y) = \frac{G_1(d)}{\tau(d)} \frac{x}{\sqrt{\log y}} \rho_\frac{1}{2}(u) + O\left(\frac{M_{\eta,d}}{\tau(d)} \left( \frac{x \log(u_1)}{(\log y)^{\frac{3}{2}}} + \frac{x}{\log y} \right) \right).
\]

To study \(T_d(x, y)\) we apply Lemma 3.5. We obtain

\[
T_d(x, y) = \frac{x}{\log y} \sum_{a \leq x, P(a) \leq y} \frac{1}{a \tau(da)} \omega\left(u - \frac{\log a}{\log y}\right) + O\left(\frac{x}{\log^2 y} \sum_{a \leq x, P(a) \leq y} \frac{1}{a \tau(da)} \right)
\]

\[
= \frac{x}{\log y} L_1 + O\left(\frac{x}{\log^2 y} L_2 \right).
\]
Now let us study $L_1$. Partial summation and Lemma 3.2 give

$$L_1 = \frac{1}{\tau(d)} \int_{1-}^{x/y} t \omega(u - \frac{\log t}{\log y}) \left( \sum_{a \leq t, P(a) \leq y} \gamma_d(a) \right) dt$$

$$= \frac{1}{\tau(d)} \int_{1}^{x/y} \left( \sum_{a \leq t, P(a) \leq y} \gamma_d(a) \right) \left( \frac{1}{t^2} \omega(u - \frac{\log t}{\log y}) + \omega'(u - \frac{\log t}{t \log y}) \frac{1}{t} \right) dt$$

$$+ O \left( \frac{M_{n,d}}{\tau(d)} \frac{1}{\sqrt{\log y}} \right)$$

$$= \frac{1}{\sqrt{\log y}} \frac{G_d(1)}{\tau(d)} \int_{1}^{x/y} \left( \frac{1}{t} \omega(u - \frac{\log t}{\log y}) + \omega'(u - \frac{\log t}{t \log y}) \frac{1}{t} \right) \rho_{\frac{1}{2}} \left( \frac{\log t}{\log y} \right) dt$$

$$+ O \left( \frac{M_{n,d}}{\tau(d)} \int_{1}^{x/y} \left( \frac{\log \left( \frac{\log t}{\log y} \right) + 1}{\log y} + \frac{1}{\log y} \right) \omega(u - \frac{\log t}{t \log y}) \rho_{\frac{1}{2}} \left( \frac{\log t}{\log y} \right) dt \right)$$

$$+ O \left( \frac{M_{n,d}}{\tau(d)} \int_{1}^{x/y} \left( \frac{\log \left( \frac{\log t}{\log y} \right) + 1}{\log y} + \frac{1}{\log y} \right) |\omega'(u - \frac{\log t}{t \log y})| \rho_{\frac{1}{2}} \left( \frac{\log t}{\log y} \right) dt \right)$$

$$+ O \left( \frac{M_{n,d}}{\tau(d)} \frac{1}{\sqrt{\log y}} \right).$$

By change of variable $z = \frac{\log t}{\log y}$, we obtain

$$L_1 = \sqrt{\log y} \frac{G_d(1)}{\tau(d)} \int_{0}^{u-1} \rho_{\frac{1}{2}}(z) \omega(u - z) dz + O \left( \frac{M_{n,d}}{\tau(d)} \left( 1 + \frac{\log u}{(\log y)^{\frac{1}{2}}} \right) \right).$$

We estimate $L_2$ in the same way. Lemma 3.2 and by partial summation give

$$L_2 \ll \sqrt{\log y} \frac{M_{n,d}}{\tau(d)} \int_{0}^{u} \rho_{\frac{1}{2}}(z) dz \ll \sqrt{\log y} \frac{M_{n,d}}{\tau(d)}.$$ 

We get our result in the considered range from these different estimates. In the range $\frac{x}{a} < y \leq x$ we have $\Phi(x/a, y) = 1$. The result follows immediately from Lemma 3.2. \(\Box\)

**Lemma 3.7** Set $\epsilon_x = \frac{\log 2}{\log x}$. Uniformly for $(x, y) \in (H_x)$ and $0 \leq v < \epsilon_x$ we have

$$S(x, y, v) = O \left( \frac{x}{\sqrt{\log y}} \right).$$

**Proof** The condition on $v$ implies $d = 1$. By Lemma 3.6 with $d = 1$ we get

$$S(x, y, v) = \sum_{n \leq x} \frac{1}{\tau(n, y)} \sum_{d|n, d \leq n^v, P(d) \leq y} 1 \leq \sum_{n \leq x} \frac{1}{\tau(n, y)} \ll \frac{x}{\sqrt{\log y}}.$$

\(\Box\)
Lemma 3.8 Let  be \( \epsilon_x = \frac{\log 2}{\log x} \). Uniformly for \((x, y) \in (H_x)\) and \( \epsilon_x \leq v \leq 1 \), we have

\[
S_2(x, y, v) = O \left( \frac{x}{\log y} \right).
\]

Proof We have

\[
S_2(x, y, v) = \sum_{n \leq x} \frac{1}{\tau(n, y)} \sum_{d|n, u^n < d \leq x^v, \mu(d) \leq y} \frac{1}{\tau(dm, y)}.
\]

If \( 1 - \epsilon_x \leq v \leq 1 \), then \( x^{1-v} < 2 \). In this case, we have

\[
S_2(x, y, v) \leq \sum_{d \leq x, \mu(d) \leq y} \frac{x}{\tau(d)} \leq \frac{x}{\sqrt{\log y}} \left( \frac{1}{v} \frac{\log (1 - v)}{\log x} \right) \leq \frac{x}{\log y}.
\]

Suppose that \( \epsilon_x \leq v \leq 1 - \epsilon_x \). If \( y \leq \min\{x^v, x^{1-v}\} \), we apply Lemmas 3.6 and 3.3, we get

\[
S_2(x, y, v) \leq \sum_{d \leq x^v, \mu(d) \leq y} \frac{x^v}{\tau(d)} \leq \frac{x}{\log y} \left( \frac{1}{v} \frac{\log x}{\log y} \right) \leq \frac{x}{\log y}.
\]

The case \( x^v < y \leq x^{1-v} \) is similar. Last if \( y > \max\{x^{1-v}, x^v\} \), we set \( \epsilon'_x = \frac{\log 2}{\sqrt{\log x}} \) and consider three cases \( \epsilon_x \leq v \leq \epsilon'_x \), \( \epsilon'_x < v \leq 1 - \epsilon'_x \) and \( 1 - \epsilon'_x < v \leq 1 - \epsilon_x \). In each situation we have \( \frac{1}{v(1-v)} \leq \sqrt{\log x} \). By applying Lemmas 3.2 and 3.3 we get

\[
S_2(x, y, v) \leq \frac{x}{\log x} \frac{1}{\sqrt{v(1-v)}} \leq \frac{x}{\sqrt{\log x}}.
\]

□
3.2 Proof of Theorem 2.1

Taking into account Lemmas 3.7 and 3.8, it remains to estimate $S_1(x, y, v)$ for $\varepsilon_x \leq v \leq 1$. We consider two following situations: $\varepsilon_x \leq v \leq 1 - \varepsilon_x$ and $1 - \varepsilon_x < v \leq 1$. First suppose that $1 - \varepsilon_x < v \leq 1$. Dans ce cas $x/2 < x^\varepsilon$. We write

$$S_1(x, y, v) = \sum_{d \leq x/2, P(d) \leq y} \sum_{m \leq x/d} \frac{1}{\tau(dm, y)} + \sum_{x/2 < d \leq x^\varepsilon, P(d) \leq y} \sum_{m \leq x/d} \frac{1}{\tau(dm, y)}$$

$$= : \hat{S}_1 + \hat{S}_2.$$

Let us study $\hat{S}_2$. $x/2 < d \iff x/d < 2$ thus $m = 1$. We have

$$\hat{S}_2 = \sum_{x/2 < d \leq x^\varepsilon, P(d) \leq y} \frac{1}{\tau(d)} \ll \sum_{d \leq x^\varepsilon, P(d) \leq y} \frac{1}{\tau(d)} \ll \frac{x}{\sqrt{\log y}} \rho_1(u) \ll \frac{x}{\sqrt{\log y}}.$$

The evaluation of $\hat{S}_1$ is similar to the evaluation of $S_1(x, y, v)$ under complementary condition $\varepsilon_x \leq v \leq 1 - \varepsilon_x$, since $x/2 = x^{1-\varepsilon_x}$. Let us study $S_1(x, y, v)$ under the condition $\varepsilon_x \leq v \leq 1 - \varepsilon_x$. First, consider the range

$$\exp \left( (\log \log x)^{\frac{5}{3}} \right) \leq y \leq \frac{x}{d}, \quad x \geq x_0(\varepsilon).$$

We write

$$S_1(x, y, v) := \sum_{n \leq x} \frac{1}{\tau(n, y)} \sum_{d|n, d \leq x^{\varepsilon}, P(d) \leq y} \tau(dm, y)$$

$$= \sum_{d \leq x^{\varepsilon}, P(d) \leq y} T_d \left( \frac{x}{d}, y \right).$$

We apply Lemma 3.6. We obtain $S_1(x, y, v) = S_{1,1} + O(S_{1,2})$ avec, en posant

$$u_d := u - \frac{\log d}{\log y},$$

$$S_{1,1} := \frac{x}{\sqrt{\log y}} \sum_{d \leq x^{\varepsilon}, P(d) \leq y} \frac{G_d(1)}{d \tau(d)} \left( \int_0^{u_d-1} \rho_d(z) \omega(u_d - z) \, dz + \rho_d(u_d) \right),$$

and

$$S_{1,2} := x \left( \frac{1}{\log y} + \frac{\log(u + 1)}{(\log y)^2} \right) \sum_{d \leq x^{\varepsilon}, P(d) \leq y} \frac{M_{\rho_d}}{d \tau(d)} \ll x \left( \frac{1}{\sqrt{\log y}} + \frac{\log(u + 1)}{\log y} \right),$$

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by using Remark 3.1. It remains to estimate \( S_{1,1} \). From (9), we have \( \frac{G_d(1)}{r(d)} = B(1)g(d) \).

By partial summation we get

\[
S_{1,1} = \frac{x}{\sqrt{\log y}} B(1) \sum_{d \in x^v, P(d) \leq y} \frac{g(d)}{d} \left( \int_0^{u_d - 1} \rho_{\frac{1}{2}}(z) \omega(u_d - z) \, dz + \rho_{\frac{1}{2}}(u_d) \right)
\]

\[
= \frac{B(1)x^{1-v}}{\sqrt{\log y}} \left( \int_0^{(1-v)u-1} \rho_{\frac{1}{2}}(z) \omega((1-v)u - z) \, dz + \rho_{\frac{1}{2}}((1-v)u) \right) \left( \sum_{n \leq x^v, P(n) \leq y} g(n) \right)
\]

\[
- \frac{x}{\sqrt{\log y}} B(1) \int_1^{x^v} \left( \sum_{n \leq t, P(n) \leq y} g(n) \right) \left( \frac{1}{t} \left( \int_0^{u_{t-1}} \rho_{\frac{1}{2}}(z) \omega(u_t - z) \, dz + \rho_{\frac{1}{2}}(u_t) \right) \right) \, dt \equiv R_1 - R_2.
\]

From Lemmas 3.3 and 3.4

\[
R_1 \ll \frac{x}{\log y} \rho_{\frac{1}{2}}(av) \left( 1 + \rho_{\frac{1}{2}}((1-v)u) \right) \ll \frac{x}{\sqrt{\log y}}.
\]

the last upper bound follows by using the same way as in proof of Lemma 3.8. Now consider \( R_2 \). We have

\[
d \left( \frac{1}{t} \left( \int_0^{u_{t-1}} \rho_{\frac{1}{2}}(z) \omega(u_t - z) \, dz + \rho_{\frac{1}{2}}(u_t) \right) \right) = D_1 + D_2,
\]

with

\[
D_1 := - \left( \int_0^{u_{t-1}} \rho_{\frac{1}{2}}(z) \omega(u_t - z) \, dz + \rho_{\frac{1}{2}}(u_t) \right) \frac{dt}{t^2},
\]

and

\[
D_2 := \frac{1}{t} d \left( \int_0^{u_{t-1}} \rho_{\frac{1}{2}}(z) \omega(u_t - z) \, dz + \rho_{\frac{1}{2}}(u_t) \right).
\]

Write \( R_2 = R_{2,1} + R_{2,2} \) with

\[
R_{2,1} := \frac{x}{\sqrt{\log y}} B(1) \int_1^{x^v} \left( \sum_{n \leq t, P(n) \leq y} g(n) \right) D_1,
\]

and

\[
R_{2,2} := \frac{x}{\sqrt{\log y}} B(1) \int_1^{x^v} \left( \sum_{n \leq t, P(n) \leq y} g(n) \right) D_2.
\]

To evaluate \( R_{2,1} \) we apply Lemmas 3.3 and 3.4. We get

\[
R_{2,1} = - \frac{x}{\log y} \int_1^{x^v} \frac{\log t}{\log y} \left( \int_0^{u_{t-1}} \rho_{\frac{1}{2}}(z) \omega(u_t - z) \, dz + \rho_{\frac{1}{2}}(u_t) \right) \times
\]

\[
\left( 1 + O \left( \frac{\log(t + 1)}{\log y} + \frac{1}{\sqrt{\log y}} \right) \right) \frac{dt}{t}
\]

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A change of variable $s = \frac{\log t}{\log y}$ gives

$$R_{2,1} = -x \int_0^u \rho_\frac{1}{2}(s) \left( \int_0^{u-s-1} \rho_\frac{1}{2}(z) \omega(u-s-z) dz + \rho_\frac{1}{2}(u-s) \right) \times$$

$$\left( 1 + O\left( \frac{\log(s+1)}{\log y} + \frac{1}{\sqrt{\log y}} \right) \right) ds$$

$$= -x \int_0^u \rho_\frac{1}{2}(s) \left( \int_0^{u-s-1} \rho_\frac{1}{2}(z) \omega(u-s-z) dz \right) ds - x \int_0^u \rho_\frac{1}{2}(s) \rho_\frac{1}{2}(u-s) ds$$

$$+ O\left( \frac{x \log(u+1)}{\log y} + \frac{x}{\sqrt{\log y}} \right)$$

Now consider $R_{2,2}$. An easy calculation gives

$$D_2 \ll \frac{1}{t^2(u_t-1) \log y}$$

$$D_2 = -\frac{1}{t^2(u_t-1) \log y} \left( \int_0^{u_t-1} \rho_\frac{1}{2}(z) \omega(u_t-z) dz \right) dt$$

$$- \frac{1}{t^2(u_t-1) \log y} \left( \int_0^{u_t-1} z \rho'_\frac{1}{2}(z) \omega(u_t-z) dz \right) dt$$

$$- \frac{1}{t^2(u_t-1) \log y} \left( \int_0^{u_t-1} (u_t-z) \rho_\frac{1}{2}(z) \omega'(u_t-z) dz \right) dt$$

$$- \frac{1}{t^2 \log y} \rho'_\frac{1}{2}(u_t)$$

$$:= D_{2,1} + D_{2,2} + D_{2,3} + D_{2,4}.$$ 

We have

$$R_{2,2} = \frac{x}{\sqrt{\log y}} B(1) \int_1^x \left( \sum_{n \leq t, P(n) \leq y} g(n) \right) (D_{2,1} + D_{2,2} + D_{2,3} + D_{2,4}) dt = \tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3 + \tilde{D}_4$$

where

$$\tilde{D}_i = \frac{x}{\sqrt{\log y}} B(1) \int_1^x \left( \sum_{n \leq t, P(n) \leq y} g(n) \right) D_{2,i}$$

for $i \in \{1, 2, 3, 4\}$. By using the same method as previously, we get

$$\tilde{D}_1 \ll \frac{x}{\log y} \int_0^{uv} \frac{\rho_\frac{1}{2}(s)}{u-s-1} \left( \int_0^{u-s-1} \rho_\frac{1}{2}(z) \omega(u-s-z) dz \right) ds.$$
For \( s \geq u - 2 \) we have \( \omega(u - s - z) = 0 \) since \( 0 \leq z \leq u - s - 1 \leq 1 \) and for \( s \leq u - 2 \), we have \( u - s - 1 \geq 1 \). It follows that

\[
\tilde{D}_1 \ll \frac{x}{\log y} \int_0^{u-2} \rho_\frac{1}{2}(s) ds \int_0^{u-1} \rho_\frac{1}{2}(z) dz \ll \frac{x}{\log y}.
\]

The study of \( \tilde{D}_2, \tilde{D}_3 \) and \( \tilde{D}_4 \) is similar by using properties of \( \rho_\frac{1}{2}' \) and \( \omega' \). We get the same result. We omit details. Assembling these estimates we get our result in the considered range. In the complementary range \( x/d < y \leq x \), proof is similar, we omit it. \( \square \)

# 4 Proof of Theorem 2.2

## 4.1 Lemmas

We need a weak form of the following lemma

**Lemma 4.1** For each fixed integer \( N \geq 1 \) and uniformly for \( w > 1 \) we have

\[
\int_w^\infty \rho_\frac{1}{2}(t) dt = 2 \sum_{k=1}^{N} (w+k)\rho_\frac{1}{2}(w+N) + O\left(\frac{\rho_\frac{1}{2}(w)}{w^N}\right).
\]

**Proof.** Set

\[
I(w) := \int_w^\infty \rho_\frac{1}{2}(t) dt.
\]

By change of variable \( z = t + 1 \) and by using the differential equation satisfied by \( \rho_\frac{1}{2} \), we obtain

\[
I(w) = \int_{w+1}^\infty \rho_\frac{1}{2}(z-1) dz = -2 \int_{w+1}^\infty z\rho_\frac{1}{2}'(z) dz - \int_{w+1}^\infty \rho_\frac{1}{2}(z) dz = -2J(w) - I(w+1). \tag{10}
\]

with

\[
J(w) := \int_{w+1}^\infty z\rho_\frac{1}{2}'(z) dz.
\]

Integration by parts gives

\[
J(w) = -(w+1)\rho_\frac{1}{2}(w+1) - \int_{w+1}^\infty \rho_\frac{1}{2}(z) dz = -(w+1)\rho_\frac{1}{2}(w+1) - I(w+1).
\]

Substituting in (11) we obtain

\[
I(w) - I(w+1) = 2(w+1)\rho_\frac{1}{2}(w+1). \tag{11}
\]

And by iteration we obtain for every integer \( k \geq 1 \),

\[
I(w + k - 1) - I(w + k) = 2(w + k)\rho_\frac{1}{2}(w + k),
\]
Summing these inequalities, we obtain
\[ I(w) = 2(w + 1)\rho_1^2(w + 1) + 2 \sum_{k=2}^{\infty} (w + k) \rho_1^2(w + k), \]

The lemma follows from this formula and the result
\[ \rho_1^2(w + k) = O \left( \frac{\rho_1^2(w)}{w^k} \right), \]
uniformly for \( w > 1 \) and \( k > 0 \) which is a consequence of the asymptotic formula for \( \rho_1^2(w) \) (see [8]). □

**Lemma 4.2** For \( v \in [0, 1] \) and as \( u \to \infty \) we uniformly have
\[ H(u, v) := \frac{1}{e^\gamma} \int_0^{uv} \left( \int_0^{u-s-1} \rho_1^2(w)dw \right) \rho_1^2(s)ds = \frac{1}{\sqrt{e^\gamma}} \int_0^{uv} \rho_1^2(w)dw + O(u\rho(u)), \]
where \( \gamma \) is Euler’s constant.

**Proof** We have
\[ H(u, v) = \frac{1}{e^\gamma} \int_0^{uv} \left( \int_0^{\infty} \rho_1^2(w)dw - \int_{u-s-1}^{\infty} \rho_1^2(w)dw \right) \rho_1^2(s)ds \]
\[ = \frac{1}{\sqrt{e^\gamma}} \int_0^{uv} \rho_1^2(s)ds - \frac{1}{e^\gamma} \int_0^{uv} \left( \int_{u-s-1}^{\infty} \rho_1^2(w)dw \right) \rho_1^2(s)ds, \]
since by using Laplace transform, we have
\[ \int_0^{\infty} \rho_1^2(w)dw = \rho_1^2(0) = (\rho(0))^{1/2} = \sqrt{e^\gamma}. \]
We apply Lemma 4.1 in a weak version we obtain
\[ H(u, v) = \frac{1}{\sqrt{e^\gamma}} \int_0^{uv} \rho(s)ds + O \left( u \int_0^{uv} \rho_1^2(s) \rho_1^2(u-s) ds \right) \]
\[ = \frac{1}{\sqrt{e^\gamma}} \int_0^{uv} \rho(s)ds + O(u\rho(u)), \]
since
\[ \int_0^{uv} \rho_1^2(s) \rho_1^2(u-s) ds \leq \int_0^{u} \rho_1^2(s) \rho_1^2(u-s) ds = (\rho_1^2 * \rho_1^2)(u) \ll \rho(u). \] □
4.2 Proof of Theorem 2.2

We have

\[ F(u, v) = \int_0^{uv} \left( \int_0^{u-s-1} \rho_1^2(z) \omega(u-s-z) \, dz \right) \rho_1^2(s) \, ds + O(\rho(u)). \]

And

\[ \int_0^{uv} \left( \int_0^{u-s-1} \rho_1^2(z) \omega(u-s-z) \, dz \right) \rho_1^2(s) \, ds = \frac{1}{\sqrt{\gamma}} \int_0^{uv} \rho_1^2(s) \, ds - \frac{1}{\sqrt{\gamma}} \int_0^{uv} \rho_1^2(s) \left( \int_{u-s-1}^{\infty} \rho_1^2(z) \, dz \right) \]

By using formula (3) and Lemma 4.2, we get

\[
\begin{align*}
&\int_0^{uv} \left( \int_0^{u-s-1} \rho_1^2(z) \omega(u-s-z) \, dz \right) \rho_1^2(s) \, ds = \frac{1}{\sqrt{\gamma}} \int_0^{uv} \rho_1^2(s) \, ds - \frac{1}{\sqrt{\gamma}} \int_0^{uv} \rho_1^2(s) \left( \int_{u-s-1}^{\infty} \rho_1^2(z) \, dz \right) \\
&\quad + O \left( \int_0^{uv} \rho_1^2(s) \left( \int_0^{u-s-1} \rho_1^2(z) \rho(u-s-z) \, dz \right) \, ds \right) \\
&\quad = \frac{1}{\sqrt{\gamma}} \int_0^{uv} \rho_1^2(s) \, ds + O(\rho(u)) + O(\rho(u)) \\
&\quad = \frac{1}{\sqrt{\gamma}} \int_0^{uv} \rho_1^2(s) \, ds + O(\rho(u))
\end{align*}
\]

since

\[
\begin{align*}
\int_0^{uv} \rho_1^2(s) \left( \int_0^{u-s-1} \rho_1^2(z) \rho(u-s-z) \, dz \right) \, ds &\leq \int_0^{uv} \rho_1^2(s) (\rho_2 * \rho)(u-s) \, ds \\
&\leq (\rho_2 * \rho)(u) = (\rho * \rho)(u) = \rho_2(u),
\end{align*}
\]

and \( u \rho(u) \ll \rho_2(u) \).

\[
\square
\]

5 Proof of Theorem 2.3

5.1 Lemmas

Lemma 5.1 For \( 0 \leq \xi < 1 \) we have

\[
R(\xi) := \frac{1}{\pi} \int_0^{\xi} \int_0^{-s+\xi} \frac{ds' \, dz'}{\sqrt{s'} \sqrt{z'} (1-s' - z')} = -\log(1-\xi).
\]

Proof We use change of variables \((s', z') \mapsto (w, r) = (\frac{s'}{s' + z'}, s' + z')\). Then we have a \( dsdz = rdwdr \). We obtain

\[
R(\xi) = \left( \frac{1}{\pi} \int_0^{1} w^{-\frac{1}{2}}(1-w)^{-\frac{1}{2}} \, dw \right) \left( \int_0^{\xi} (1-r)^{-1} \, dr \right) = -\log(1-\xi),
\]

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since
\[
\frac{1}{\pi} \int_0^1 w^{-\frac{1}{2}} (1 - w)^{-\frac{1}{2}} \, dw = \frac{B\left(\frac{1}{2}, \frac{1}{2}\right)}{\pi} = \frac{\Gamma^2\left(\frac{1}{2}\right)}{\pi} = 1.
\]

\[\square\]

**Lemma 5.2** For \(\beta \leq 1 - \frac{1}{u}\) we have

\[
S(\beta) := \frac{1}{\pi} \int_0^\beta \left( \int_0^{-s+1-\frac{1}{u}} ds \, dz \right) \left( \int_0^{1-s-z} \sqrt{s} \sqrt{z} \, (1 - s - z) \right) = \frac{2}{\pi} \left( \log(u) + \log(1 - \beta) \right) \arcsin\left( \sqrt{\frac{u\beta}{u - 1}} \right) - \log(1 - \beta).
\]

**Proof** We write \(S(\beta) = I_1 - I_2\) with

\[
I_1 = \frac{1}{\pi} \int_0^{1 - \frac{1}{u}} \left( \int_0^{-s+1-\frac{1}{u}} ds \, dz \right) \left( \int_0^{1-s-z} \sqrt{s} \sqrt{z} \, (1 - s - z) \right), \quad I_2 = \frac{1}{\pi} \int_0^{1 - \frac{1}{u}} \left( \int_0^{-s+1-\frac{1}{u}} ds \, dz \right) \left( \int_0^{1-s-z} \sqrt{s} \sqrt{z} \, (1 - s - z) \right).
\]

From Lemma 5.1, \(I_1 = R\left(1 - \frac{1}{u}\right) = \log(u)\). Let us study \(I_2\). By change of variables \((s', z') \rightarrow (s, z) = (s' + \beta, z')\) we obtain

\[
I_2 = \frac{1}{\pi} \int_0^{1 - \frac{1}{u} - \beta} \left( \int_0^{-s'+1-\frac{1}{u} - \beta} ds' \, dz' \right) \left( \int_0^{1-s'-z'} \sqrt{s'+\beta} \sqrt{z'} \, (1 - s' - \beta - z') \right).
\]

Now we put the change of variables \((s' + \beta) = rw\) et \(s' + \beta + z' = r\) then we have \(ds' dz' = rdw dr\) and

\[
I_2 = \frac{1}{\pi} \int_0^{1 - \frac{1}{u} - \beta} \frac{dw}{\sqrt{w\sqrt{1 - w}}} \int_0^{1 - \frac{1}{u}} \frac{dr}{1 - r}.
\]

Since

\[
\int_0^{1 - \frac{1}{u}} \frac{dr}{1 - r} = \log(u) + \log(1 - \beta),
\]

then by change of variable \(t = \sqrt{w}\), we get

\[
\frac{1}{\pi} \int_0^{1 - \frac{1}{u}} \frac{dw}{\sqrt{w\sqrt{1 - w}}} = \frac{2}{\pi} \int_0^{1 - \frac{1}{u}} \frac{dt}{\sqrt{\frac{1}{u} - t^2}} = 1 - \frac{2}{\pi} \arcsin\left( \sqrt{\frac{u\beta}{u - 1}} \right).
\]

Finally,

\[
I_2 = \left( \log(u) + \log(1 - \beta) \right) \left( 1 - \frac{2}{\pi} \arcsin\left( \sqrt{\frac{u\beta}{u - 1}} \right) \right).
\]

It follows

\[
S(\beta) = \frac{2}{\pi} \left( \log(u) + \log(1 - \beta) \right) \arcsin\left( \frac{\sqrt{u\beta}}{u - 1} \right) - \log(1 - \beta).
\]

\[\square\]
Lemma 5.3 For $0 < w \leq 1$ we have
\[ \rho_{\frac{1}{2}}(w) = \frac{1}{\sqrt{\pi w}}, \]
and for $1 \leq w \leq 2$, we have
\[ \rho_{\frac{1}{2}}(w) = \frac{1}{\sqrt{\pi w}} - \frac{\log(\sqrt{w} + \sqrt{w - 1})}{\sqrt{\pi w}}. \]

Proof The first formula is the definition of $\rho_{\frac{1}{2}}$ for $0 < w \leq 1$ and the second one follows from differential equation satisfied by $\rho_{\frac{1}{2}}$ for $1 \leq w \leq 2$. □

Let us consider the integral
\[ I := \int_{0}^{uv} \left( \int_{0}^{u-s-1} \rho_{\frac{1}{2}}(z) \omega(u-s-z) \, dz \right) \rho_{\frac{1}{2}}(s) \, ds, \]
for $1 \leq u \leq 2$. We notice that on one hand for $s > u - 1$
\[ \int_{0}^{u-s-1} \rho_{\frac{1}{2}}(z) \omega(u-s-z) \, dz = 0, \]
since $\rho_{\frac{1}{2}}(z) = 0$ pour $z \leq 0$ and then we can restrict the study of the integral on $s$ at the interval $[0, M]$ where $M = \min\{u-1, uv\} \leq 1$. And on the other hand, $0 \leq z \leq u-s-1$ then $1 \leq u-s-z \leq u-s \leq u \leq 2$ and $\omega(u-s-z) = 1/(u-s-z)$. It follows
\[ I = \frac{1}{\pi} \int_{0}^{M} \left( \int_{0}^{u-s-1} \frac{dz}{\sqrt{z(u-s-z)}} \right) \frac{ds}{\sqrt{s}}. \tag{12} \]

We will give two expressions of $I$. By the change of variable $t = \sqrt{z}$ in the inner integral (13) we get
\[ I = \frac{2}{\pi} \int_{0}^{M} \log\left(\frac{\sqrt{u-s} + \sqrt{u-s-1}}{\sqrt{u-s}}\right) \frac{ds}{\sqrt{s}}. \tag{13} \]
and by the change of variables $(s', z') \mapsto (s, z) = (us', uz')$ and by putting $M' = \frac{M}{u} = \min\{1 - \frac{1}{u}, v\}$ in (13), we obtain
\[ I = \frac{1}{\pi} \int_{0}^{M'} \int_{0}^{-s'+1-s'} \frac{ds' \, dz'}{\sqrt{s'} \sqrt{z'} (1 - s' - z')} \tag{14} \]
By using notations of lemma 5.2 and the expressions (13) et (14) of $I$, we obtain
\[ I = \begin{cases} S(1 - \frac{1}{u}) & \text{si } M = u - 1, \\ S(v) & \text{si } M = uv. \end{cases} \tag{15} \]

Now we will study the integral
\[ \int_{0}^{uv} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u-s) \, ds, \quad (1 \leq u \leq 2). \]
Lemma 5.4 For \( v \in [0, \frac{u-1}{u}] \) with \( 1 \leq u \leq 2 \) we have
\[
\int_0^{uv} \rho_1(s) \rho_2(u-s) \, ds = \frac{2}{\pi} \arcsin(\sqrt{v}) - \frac{1}{2} S(v).
\]

Proof For \( v \in [0, \frac{u-1}{u}] \) we have \( uv \leq u-1 \) then \( 0 \leq s \leq uv \leq u-1 \leq 1 \) and \( 1 \leq u-s \leq 2 \). By using Lemma 5.3 we get
\[
\int_0^{uv} \rho_1(s) \rho_2(u-s) \, ds = \left[ \frac{2}{\pi} \right] \left[ \int_0^{uv} \frac{ds}{\sqrt{s} \sqrt{u-s}} - \int_0^{uv} \frac{\log(\sqrt{u-s} + \sqrt{u-u-1})}{\sqrt{s} \sqrt{u-s}} \, ds \right] := J_1 - J_2.
\]

By the change of variable \( t = \sqrt{s} \) we have
\[
J_1 = \frac{2}{\pi} \int_0^{\sqrt{v}} \frac{dt}{\sqrt{1-t^2}} = \frac{2}{\pi} \arcsin(\sqrt{v}).
\]

From (16) we have \( J_2 = \frac{1}{2} S(v) \). This completes the proof of Lemma 5.4. \( \square \)

Lemma 5.5 For \( v \in [\frac{u-1}{u}, \frac{1}{u}] \) with \( 1 \leq u \leq 2 \) we have
\[
\int_0^{uv} \rho_1(s) \rho_2(u-s) \, ds = \frac{2}{\pi} \arcsin(\sqrt{v}) - \frac{1}{2} S(1 - \frac{1}{u}).
\]

Proof For \( v \in [\frac{u-1}{u}, \frac{1}{u}] \) we have \( u-1 \leq uv \leq 1 \). We write
\[
\int_0^{uv} \rho_1(s) \rho_2(u-s) \, ds = \int_0^{u-1} \rho_1(s) \rho_2(u-s) \, ds + \int_{u-1}^{uv} \rho_1(s) \rho_2(u-s) \, ds = J_1 + J_2.
\]

Consider \( J_1 \). As in previous Lemma we have
\[
J_1 = \frac{1}{\pi} \int_0^{u-1} \frac{ds}{\sqrt{s} \sqrt{u-s}} - \frac{1}{\pi} \int_0^{u-1} \frac{\log(\sqrt{u-s} + \sqrt{u-u-1})}{\sqrt{s} \sqrt{u-s}} \, ds
\]
\[= \frac{2}{\pi} \arcsin(\sqrt{\frac{u-1}{u}}) - \frac{1}{2} S(1 - \frac{1}{u}).
\]

Now consider \( J_2 \). We have \( u-1 \leq s \leq uv \leq 1 \) et \( u(1-v) \leq u-s \leq 1 \). It follows
\[
J_2 = \frac{1}{\pi} \int_{u-1}^{uv} \frac{ds}{\sqrt{s} \sqrt{u-s}} = \frac{2}{\pi} \arcsin(\sqrt{v}) - \frac{2}{\pi} \arcsin(\sqrt{\frac{u-1}{u}}).
\]

This completes the proof of Lemma 5.5. \( \square \)
Lemma 5.6  For \( v \in \left[ \frac{1}{u}, 1 \right] \) with \( 1 \leq u \leq 2 \) we have
\[
\int_{0}^{uv} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u - s) \, ds = -\frac{3}{2} S(1 - \frac{1}{u}) + \frac{2}{\pi} \arcsin(\sqrt{v}) + S(1 - v).
\]

Proof  As \( u \leq 2 \) we have \( \frac{u - 1}{u} \leq \frac{1}{u} \leq v \leq 1 \) and then \( u - 1 \leq uv \leq u \). We write
\[
\int_{0}^{uv} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u - s) \, ds = \int_{0}^{u-1} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u - s) \, ds + \int_{u-1}^{uv} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u - s) \, ds = J_1 + J_2.
\]

\( J_1 \) has been studied in lemma 5.5. It remains to calculate \( J_2 \). As \( \frac{1}{u} \leq v \) then \( uv \geq 1 \). We write
\[
J_2 = \int_{u-1}^{1} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u - s) \, ds + \int_{1}^{uv} \rho_{\frac{1}{2}}(s) \rho_{\frac{1}{2}}(u - s) \, ds := J_{2,1} + J_{2,2}.
\]

We have
\[
J_{2,1} = \frac{1}{\pi} \int_{u-1}^{1} \frac{ds}{\sqrt{s}\sqrt{u-s}} = \frac{2}{\pi} \arcsin\left(\frac{1}{\sqrt{u}}\right) - \frac{2}{\pi} \arcsin\left(\sqrt{\frac{u-1}{u}}\right).
\]

We write
\[
J_{2,2} = \frac{1}{\pi} \int_{1}^{uv} \frac{ds}{\sqrt{s}\sqrt{u-s}} - \frac{1}{\pi} \int_{1}^{uv} \frac{\log(\sqrt{s} + \sqrt{u-s} - 1) - \log(\sqrt{s} - \sqrt{u-s} + 1)}{\sqrt{s}\sqrt{u-s}} \, ds := \hat{J}_1 - \hat{J}_2.
\]

We have
\[
\hat{J}_1 = \frac{2}{\pi} \arcsin(\sqrt{v}) - \frac{2}{\pi} \arcsin\left(\frac{1}{\sqrt{u}}\right).
\]

Now it remains to study \( \hat{J}_2 \). We put the change of variable \( s' = u - s \). We get
\[
\hat{J}_2 = \frac{1}{\pi} \int_{u(1-v)}^{u-1} \frac{\log(\sqrt{u-s'} + \sqrt{u-s'} - 1)}{\sqrt{s'}\sqrt{u-s'}} \, ds'
\]

As \( v \geq \frac{1}{u} \) we have \( u(1-v) \leq u - 1 \leq 1 \). Therefore using notations in (14) (15) and (16) and Lemmas 5.1 et 5.2 we obtain
\[
\hat{J}_2 = S(1 - \frac{1}{u}) - S(1 - v).
\]

We complete the proof by grouping different above estimates.

\[\square\]

5.2 Proof of Theorem 2.3

Theorem 2.3 follows from (15) and different Lemmas of section 5.

\[\square\]
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