Certification and Quantification of Multilevel Quantum Coherence

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Quantum coherence, present whenever a quantum system exists in a superposition of multiple classically distinguishable states, marks one of the fundamental departures from classical physics. Quantum coherence has recently been investigated rigorously within a resource-theoretic formalism. However, a finer-grained notion of multilevel coherence, which explicitly takes into account the number of superposed classical states, has remained relatively unexplored. A comprehensive analysis of multilevel coherence, which acts as the single-party analogue to multi-partite entanglement, is essential for understanding natural quantum processes as well as for gauging the performance of quantum technologies. Here we develop the theoretical and experimental groundwork for characterizing and quantifying multilevel coherence. We introduce the robustness of multilevel coherence, a bona fide quantifier which we show to be numerically computable via semidefinite programming and experimentally accessible via multilevel coherence witnesses. We further verify and lower-bound the robustness of multilevel coherence by performing a quasi-device-independent phase discrimination task, which is implemented experimentally with four-level quantum probes in a photonic setup. Our results contribute to understanding the operational relevance of genuine multilevel coherence by demonstrating the key role it plays in enhanced phase discrimination—a primitive for quantum communication and metrology—and suggest new ways to reliably and effectively test the quantum behaviour of physical systems.

Quantum coherence manifests whenever a quantum system is in a superposition of classically distinct states, such as different energy levels or spin directions. Formally, a quantum state displays coherence whenever it is described by a density matrix that is not diagonal with respect to the relevant orthogonal basis of classical states. In this sense, coherence underpins virtually all quantum phenomena, yet has only recently been characterised formally. Coherence is now recognised as a fully-fledged resource and studied in the general framework of quantum resource theories. This has led to a menagerie of possible ways to quantify coherence in a quantum system, along with an intense analysis of how coherence plays a role in fundamental physics, e.g., in quantum thermodynamics, and in operational tasks relevant to quantum technologies, including quantum algorithms and quantum metrology.

Despite a great deal of recent progress, however, the majority of current literature focuses on a rather coarse-grained description of coherence, which is ultimately insufficient to reach a complete understanding of the fundamental role of quantum superposition in the aforementioned tasks. To overcome such limitations, one needs to take into consideration the number of classical states in coherent superposition—contrasted to the simpler question of whether any non-trivial superposition exists—which gives rise to the concept of multilevel quantum coherence. Similarly to the existence of different degrees of entanglement in multi-partite systems, going well beyond the mere presence or absence of entanglement and corresponding to different capabilities in quantum technologies, one can then identify and study a rich structure for multilevel coherence.

Deciphering this structure can yield a tangible impact on many areas of physics, such as condensed matter, statistical mechanics and transfer phenomena in many-body systems. For example, for understanding the role of coherence in the function of complex biological molecules, such as those found in light harvesting, it will be crucial to differentiate between pairwise coherence among the various sites in the molecule, and genuine multilevel coherence across many sites. In quantum computation, large superpositions of computational basis states need to be generated and effective benchmarking of such devices require proper tools to certify and quantify multilevel coherence.

Recent works have presented initial approaches to...
measuring the amount of multilevel coherence [26], as well as schemes to convert it into bipartite and genuine multi-partite entanglement, enabling the fruitful use of entanglement theory tools to study coherence itself [27, 37, 38]. Nonetheless, an all-inclusive systematic framework for the characterization, certification, and quantification of multilevel coherence is still lacking.

Here we construct and present such a theoretical framework for multilevel coherence and apply it to the experimental verification and quantification of multilevel coherence in a quantum optical setting. We begin by developing a resource theory of multilevel coherence, in particular providing a new characterisation of the sets of multilevel coherence-free states (see Fig. 1) and free operations, rigorously unfolding the hierarchy of multilevel coherence. We then formalise the robustness of multilevel coherence and show that it is an efficiently computable measure, which is experimentally accessible through multilevel coherence witnesses. Using photonic four-dimensional systems we demonstrate how to quantify, witness, and bound multilevel coherence experimentally. We prove that multilevel coherence, quantified by our robustness measure, has a natural operational interpretation as a fundamental resource, in particular providing a new characterisation of multilevel coherence, requiring experimental control that is coherent across multiple levels. This exemplifies the fine-grained classification of coherence and of experimental capabilities that studying multilevel coherence provides.

I. RESULTS

A. Resource theory of multilevel coherence

We now generalize the recently formalized resource theory of coherence [11] to the notion of multilevel coherence. We remind the reader that the general structure of a resource theory contains three main ingredients, which we present below: a set of free states, which do not contain the resource, a set of free operations, which are quantum operations that cannot create the resource, and a measure of the resource.

Multilevel coherence-free quantum states. Consider a d-dimensional quantum system with Hilbert space $\mathcal{H}$, spanned by an orthonormal basis $\{|i\rangle\}_{i=1}^{d}$, with respect to which we measure quantum coherence. The choice of classical basis is typically fixed to correspond to the eigenstates of a physically relevant observable like the system Hamiltonian. Any pure state $|\psi\rangle \in \mathcal{H}$ can be written in this basis as $|\psi\rangle = \sum_{i=1}^{d} c_{i} |i\rangle$ with $\sum_{i=1}^{d} |c_{i}|^2 = 1$. The state $|\psi\rangle$ exhibits quantum coherence with respect to the basis $\{|i\rangle\}_{i=1}^{d}$ whenever at least two of the coefficients $c_{i}$ are non-zero [11]. The multilevel nature of coherence is then revealed by the number of non-zero coefficients $c_{i}$, the coherence rank $r_{C}$. We say that a state $|\psi\rangle$ has coherence rank $r_{C}(|\psi\rangle) = k$ if exactly $k$ of the coefficients $c_{i}$ are non-zero. The notion of coherence rank thereby provides a fine-grained account of the quantum coherence of $|\psi\rangle$.

To generalise multilevel coherence to mixed states $\rho \in \mathcal{D}(\mathcal{H})$, we define the sets $\mathcal{D}_k$ with $k \in \{1, \ldots, d\}$, given by all probabilistic mixtures of pure density operators $|\psi\rangle\langle\psi|$ with a coherence rank of at most $k$,

\[
\mathcal{D}_k := \text{conv}\{|\psi\rangle\langle\psi| : r_C(|\psi\rangle) \leq k\},
\]

where conv stands for convex hull. $C_{1}$ is the set of fully incoherent states, given by density matrices that are diagonal in the classical basis, while $C_{d} = \mathcal{D}(\mathcal{H})$ is the set of all states. The intermediate sets obey the strict hierarchy, $C_{1} \subset C_{2} \subset \ldots \subset C_{d}$ (see Fig. 1) and are the free states in the resource theory of multilevel coherence, e.g.
$C_k$ is the set of $(k+1)$-level coherence-free states.

For a general mixed state one defines the coherence number $n_C$ [36–38], such that a state $\rho \in \mathcal{D}(\mathcal{H})$ has a coherence number $n_C(\rho) = k$ if $\rho \in C_k$ and $\rho \notin C_{k-1}$ (for consistency, we set $C_0 = \emptyset$). This parallels the notions of Schmidt number [42] and entanglement depth [43] in entanglement theory. A state with coherence number $n_C(\rho) = k$ can be decomposed into pure states with coherence rank at most $k$, while every such decomposition must contain at least one state with coherence rank at least $k$. A state with $n_C(\rho) = k$ is said to exhibit genuine $k$-level coherence, distinguishing it from states that appear to display coherence between $k$ levels or more, yet can be prepared as mixtures of pure states with lower-level coherence, see Fig. 1b. In an experiment, the presence of multilevel coherence proves the ability to coherently manipulate a physical system across many of its levels, much in the same way that the creation of states with large entanglement depth provides a certification of the coherent control over several systems.

Note that a state may, at the same time, display large tout-court coherence, but have vanishing higher-level coherence. This is the case, for example, for a superposition of $d-1$ basis elements, like $|\psi\rangle = \sqrt{1/(d-1)} \sum_{i=2}^d |i\rangle$, which does not display $d$-level coherence despite being highly coherent. On the other hand, a pure state may be arbitrarily close to one of the elements of the incoherent basis, yet display non-zero genuine multilevel coherence for all $k$. This is the case, for example, for the state $|\phi\rangle = \sqrt{1-\epsilon}|1\rangle + \sqrt{\epsilon/(d-1)} \sum_{i=2}^d |i\rangle$ for small $\epsilon$. It should be clear that the above multi-level classification provides a much finer description of the coherence properties of quantum systems, but that it is also important to elevate such a finer qualitative classification to a finer quantitative description, as we will do in the following.

**High purity is necessary for multilevel coherence.** As indicated in Fig. 1, the set of fully incoherent states $C_1$ has zero volume within $\mathcal{D}(\mathcal{H})$ [9]. This has the important consequences that a randomly generated state will practically never be fully incoherent, and that arbitrarily small perturbations applied to a fully incoherent state will create coherence [44]. In other words, under realistic experimental conditions one cannot prepare or verify a fully incoherent state. In contrast, we show in the Supplementary Material [41] that the sets $C_k$ are always of non-zero volume for any $k \geq 2$ and thus present a rich, and experimentally meaningful hierarchy within $\mathcal{D}(\mathcal{H})$, as shown in Fig. 1a. Specifically, we have that, if

$$\|\rho - \frac{1}{d}\|_\infty \leq \frac{1}{d(2d-3)},$$

that is, if $\rho$ is close enough to the maximally mixed state $\frac{1}{d}$ in the operator norm $\| \cdot \|_\infty$, then it cannot have multilevel coherence, i.e. $\rho \in C_2$ for any reference basis. We remark that this can be considered the correspondent in coherence theory of the celebrated fact, in entanglement theory, that there is a ball of (fully) separable states surrounding the maximally mixed state [45–47].

**Multilevel coherence-free operations.** The second ingredient in the resource theory of multilevel coherence is the set of operations that do not create multilevel coherence. A general quantum operation $\Lambda$ is described by a linear completely-positive and trace-preserving (CPTP) map, whose action on a state $\rho$ can be written as $
hat{\Lambda}(\rho) = \sum_i K_i \rho K_i^\dagger$, in terms of (non-unique) Kraus operators $\{K_i\}$ with $\sum_i K_i^\dagger K_i = \mathbb{I}$ [48]. For any map $\Lambda$ and any set $S$ of states, we denote $\Lambda(S) := \{ \Lambda(\rho) : \rho \in S \}$. Generalising the formalism introduced for standard coherence [13], we refer to a CPTP map $\Lambda$ as a $k$-coherence preserving operation if it cannot increase the coherence level, i.e. $\Lambda(C_k) \subseteq C_k$. An important subset of these are the $k$-incoherent operations, which are all CPTP maps for which there exists a set of Kraus operators $\{K_i\}$ such that $K_i \rho K_i^\dagger/\text{Tr}(K_i \rho K_i^\dagger) \in C_k$ for any $\rho \in C_k$ and all $i$. Note that the (fully) incoherent operations from the resource theory of coherence correspond to $k = 1$. In the Supplementary Material [41] we further define the notion of $k$-decohering maps as those that destroy multilevel coherence.

**Measure of multilevel coherence.** The final ingredient for the resource theory of multilevel coherence is a well-defined measure. Very few quantifiers of such a resource have been suggested, and those that exist lack a clear operational interpretation [36–38]. Furthermore, many of the quantifiers of coherence, such as the intuitive $l_1$ norm of coherence, which measures the off-diagonal contribution to the density matrix, fail to capture the intricate structure of multilevel coherence, as indicated in Fig. 1b. Here we introduce the robustness of multilevel coherence (RMC) $R_{C_k}(\rho)$ as a bona-fide measure that is directly accessible experimentally and efficient to compute for any density matrix. The robustness of $(k+1)$-level coherence can be understood as the minimal amount of noise that has to be added to a state to destroy all $(k+1)$-level coherence, defined as

$$R_{C_k}(\rho) := \inf_{\tau \in \mathcal{D}(\mathcal{H})} \left\{ s \geq 0 : \rho + st \frac{1}{1+s} \in C_k \right\}.$$  

(2)

This measure generalises the recently introduced robustness of coherence [8] [9] (corresponding to $R_{C_1}(\rho)$) to provide full sensitivity to the various levels of multilevel coherence. As a special case of the general notion of robustness of a quantum resource [49–55], the quantities $R_{C_k}$ are known to be valid resource-theoretic measures [1] [28], satisfying non-negativity, convexity, and monotonicity on average with respect to stochastic free operations [3] [49] [50] [52]. The latter means for any $\rho$ that $R_{C_k}(\rho) \geq \sum_i p_i R_{C_k}(\rho_i)$ for all $k$-incoherent operations with Kraus operators $\{K_i\}$ such that $p_i = \text{Tr}(K_i \rho K_i^\dagger)$ and $\rho_i = K_i \rho K_i^\dagger/p_i$. Since (fully) incoherent operations are $k$-incoherent for any $k$, the RMC also satisfies the strict monotonicity requirement for coherence measures [1] [3], see Supplementary Material [41].
Crucially, we find that the RMC can be posed as the solution of a semidefinite program (SDP) optimization problem \cite{58-60}, see Supplementary Material \cite{41}. A variety of algorithms exist to solve SDPs efficiently \cite{59}, meaning that the RMC may be computed efficiently for any $k$—in stark contrast to the robustness of entanglement \cite{49, 50} where one has to deal with the subtleties of the characterisation of the set of separable states \cite{54}. For an arbitrary $d$-dimensional quantum state we find that

$$0 \leq R_{C_k}(\rho) \leq \frac{d}{k} - 1 \quad \forall \rho \in \mathcal{D}(\mathcal{H}),$$

since any such state can be deterministically prepared using only (fully) incoherent operations \cite{3} starting from the maximally coherent state $|\psi_d^+\rangle = d^{-1/2} \sum_{i=1}^d |i\rangle$, for which $R_{C_k}(\rho) = \frac{d}{k} - 1$ (see Supplementary Material \cite{41}).

B. Experimental verification and quantification of multilevel coherence

We now apply our theoretic framework to an experiment that produces four-dimensional quantum states with varying degree and level of coherence using the setup in Fig. 2. We use heralded single photons at a rate of $\sim 10^3$ Hz, generated via spontaneous parametric down-conversion (SPDC) in a BBO crystal, pumped at a wavelength of 410 nm. We encode quantum information in the polarization and path degrees of freedom of these photons to prepare 4-dimensional systems \cite{61} with the basis states $|0\rangle = |H\rangle_1, |1\rangle = |V\rangle_1, |2\rangle = |H\rangle_2, |3\rangle = |V\rangle_2$, where $|p\rangle_m$ denotes a state of polarization $p$ in mode $m$. This dual-encoding allows for high-precision preparation of arbitrary pure quantum states of any dimension $d \leq 4$ with an average fidelity of $\mathcal{F} = 0.997 \pm 0.002$ and purity of $P = 0.995 \pm 0.003$. An arbitrary mixed state $\rho$ can be engineered as a proper mixture, by preparing the states of a pure-state decomposition of $\rho$ for appropriate fractions of the total measurement time and tracing out the classical information about which preparation was implemented. Using the same technique, we can also subject the input states to arbitrary forms of noise.

Reversing the preparation stage of the setup allows us to implement arbitrary sharp projective measurements. Arbitrary generalized measurements \cite{48} are correspondingly implemented as proper mixtures of a projective decomposition with an average fidelity of $\mathcal{F} = 0.997 \pm 0.002$. By design, our experiment implements one measurement outcome at a time, which achieves superior precision through the use of a single fibre-coupling assembly \cite{61}, while reducing systematic bias. The whole experiment is characterized by a quantum process fidelity of $\mathcal{F}_p = 0.9956 \pm 0.0002$, limited by the interferometric contrast of $\sim 300:1$. The latter is stable over the relevant timescales of the experiment due to the inherently stable interferometric design with common mode noise rejection for all but the piezo-driven rotational degrees of freedom of the second beam displacer. All data presented here, was integrated over 20s for each outcome, which is also much faster than the observed laser drifts on the order of hours. The main source of statistical uncertainties thus comes from the Poisson-distributed counting statistics. This has been taken into account through Monte-Carlo resampling with $10^4$ runs for tomographic measurements and $10^5$ runs for all other measurements. All experimental data presented in the figures and text throughout the manuscript are based on at least $10^5$ single photon counts and contain 5σ-equivalent statistical confidence intervals, which are with high confidence normal distributed unless otherwise stated.

**Measuring multilevel coherence** To illustrate the phenomenology of multilevel coherence, we consider a family of noisy maximally coherent states

$$\rho(p) = (1 - p)\frac{I}{d} + p |\psi_d^+\rangle\langle \psi_d^+|,$$  \hspace{1cm} (4)

with $p \in [0, 1]$ and finite dimension $d$. These states interpolate between the maximally mixed state $\frac{I}{d}$ (for $p = 0$) and the maximally coherent state $|\psi_d^+\rangle = d^{-1/2} \sum_{i=1}^d |i\rangle$ (for $p = 1$). For this class of states, the RMC can be evaluated analytically to (see Supplementary Material \cite{41})

$$R_{C_k}(\rho(p)) = \max \left\{ \frac{p(d - 1) - (k - 1)}{k}, 0 \right\}. \hspace{1cm} (5)$$
In particular, this implies that $\rho(p) \in C_k$ for $p \leq \frac{k-1}{d-1}$ and $\rho(p) \notin C_k$ for $p > \frac{k-1}{d-1}$, see Supplementary Material 

The family of noisy maximally coherent states thus provides the ideal testbed for our investigation, spanning the full hierarchy of multilevel coherence, see Fig. 3. Using the setup of Fig. 2, we engineer noisy maximally coherent states $\rho(p)$ for $d = 4$ and a variety of values of $p$. We then reconstruct the experimentally prepared states using maximum likelihood quantum state tomography and compute the robustness coherence for all $k$ by evaluating the corresponding SDP, Eq. (S13) of the Supplementary Material. As illustrated in Fig. 4, this method produces very reliable results, however, it requires $d^2$ measurements and is thus experimentally infeasible already for medium-scale systems.

C. Witnessing multilevel coherence

In analogy with the parallel concept for quantum entanglement, we now introduce an efficient alternative to the tomographic approach above—multilevel coherence witnesses. Since the sets $C_k$ are convex, for any $\rho \notin C_k$ there exists a $(k+1)$-level coherence witness $W$ such that $\text{Tr}(W \rho) < 0$ and $\text{Tr}(W \sigma) \geq 0$ for all $\sigma \in C_k$. A negative expectation value for $W$ thus certifies the $(k+1)$-level coherence of $\rho$ in a single measurement.

Given any pure state $|\psi\rangle = \sum_{i=1}^{d} c_i |i\rangle \in \mathcal{H}$, one can construct a $(k+1)$-level coherence witness as

$$W_k(\psi) = \mathbb{I} - \frac{1}{\sum_{i=1}^{k} |c_i^+|^2} |\psi\rangle \langle \psi|,$$  

where $c_i^+$ are the coefficients $c_i$ rearranged into non-increasing modulus order. It is clear that $W_k(\psi)$ always reveals the $k+1$ coherence of $|\psi\rangle$ if present, since $\langle \psi| W_k(\psi) |\psi\rangle = 1 - \frac{1}{\sum_{i=1}^{k} |c_i^+|^2}$, which is negative if $|\psi\rangle \notin C_k$. For the maximally coherent state $|\psi_d^+\rangle$, we then find $W_k(\psi_d^+) = \mathbb{I} - \frac{d}{k} |\psi_d^+\rangle \langle \psi_d^+|$. 

More generally, the set $C_k$ of $(k+1)$-level coherence witnesses is obtained as the dual of the set $C_k$ and is characterised by the following theorem, proved in the Supplementary Material.

**Theorem 1.** A self-adjoint operator $W$ is in $C_k^*$ if and only if

$$P_I WP_I \geq 0 \quad \forall I \in \mathcal{P}_k,$$  

where $\mathcal{P}_k$ is the set of all the $k$-element subsets of $\{1, 2, \ldots, d\}$, and $P_I := \sum_{\{i\} \subseteq I} |i\rangle \langle i|$. 

Hence, verifying that a given self-adjoint operator $W$ is a $(k+1)$-level coherence witness requires verifying the positive semidefiniteness of all $(k \times k)$-dimensional principal sub-matrices of the matrix representation of $W$ with respect to the classical basis. Using this we recall that every SDP has a dual version, and that in the case of RMC strong duality holds, which means that the primal and dual forms of the problem are equivalent, with the latter given by

$$R_{C_k}(\rho) = \max \quad -\text{Tr}(\rho W)$$  

s.t.  

$$P_I WP_I \geq 0 \quad \forall I \in \mathcal{P}_k$$  

$$W \leq \mathbb{I}.$$  

Hence, while a lower bound on $R_{C_k}(\rho)$ can be obtained from the negative expectation value of any observable
$W \in C^k_t$ such that $W \leq I$, the dual SDP for the RMC actually computes an optimal $(k + 1)$-level coherence witness that exactly measure $R_{C_k}(\rho)$.

For the family of noisy maximally coherent state $\rho(p)$, the witness $W_k(\psi^+_d)$ of Eq. (6) turns out to be optimal, independently of the noise parameter $p$ and we calculate $\text{Tr}(W_k(\psi^+_d)\rho(p)) = \frac{1}{2}[(k - 1) - p(d - 1)]$. Figure 4 shows the absolute value of the experimentally obtained (negative) expectation values of $W_k(\psi^+_1)$ for a range of values of $p$. This demonstrates that multilevel coherence can be quantitatively witnessed in the laboratory using only a single measurement. Experimentally, however, implementing the optimal witness requires a projection onto a maximally coherent state, which is very sensitive to noise. Indeed, in our experiment we observed a small degree of beam steering by the wave plates, leading to phase uncertainty between the basis states $|0\rangle$ and $|1\rangle$. As a consequence, the witness becomes sub-optimal and only provides a lower bound on the RMC of the experimental state. In contrast, our results show that the larger number of measurements in the tomographic approach and the associated maximum likelihood reconstruction add resilience to experimental imperfections.

D. Bounding multilevel coherence

In practice, one might often neither be able to perform full tomography on a system, nor be able to measure the optimal witness. Remarkably, one can obtain a lower bound on the RMC of an experimentally prepared state $\rho$ from any set of experimental data. Specifically, the SDP in Eq. (S16) in the Supplementary Material [41] computes the minimal RMC of a state $\tau \in \mathcal{D}(\mathcal{H})$ that is consistent with a set of measured expectation values $a_i = \text{Tr}(O_i\rho)$ for $n$ observables $\{O_i\}_{i=1}^n$ to within experimental uncertainty. This is particularly appealing when one has already performed a set of (well-characterised) measurements and wishes to use these to estimate the multilevel coherence of the input state. Note that $d^2 - 1$ linearly independent observables (assuming vanishingly small errors, and not including the identity, which accounts for normalisation) are sufficient to uniquely determine the state, in which case we could use the original SDP, Eq. (6).

We experimentally estimate such lower bounds from Eq. (S16) in the Supplementary Material [41] for an increasing number of randomly chosen observables $O_i$, measured on a 4-dimensional maximally coherent state and on a noisy maximally coherent state with $p = 0.8874 \pm 0.0007$, see Fig. 5. The results show that our lower bounds become non-trivial already for a small number of observables, and converge to a sub-optimal yet highly informative value. The remaining gap of about 5% between these bounds and the tomographically estimated RMC is due to our conservative $5\sigma$ error bounds, and could be improved by incorporating maximum-likelihood or Bayesian estimation techniques, see Supplementary Material [41] for details. We also find that the number of measurements required for non-trivial bounds increases slowly with the coherence level, and the bounds saturate more quickly for states with more coherence.

E. Multilevel coherence as a resource for quantum-enhanced phase discrimination

To demonstrate the operational significance of multilevel coherence we show that it is the key resource for the following task, illustrated in Fig. 6. Suppose that a physical device can apply one of $n$ possible quantum operations $\{A_m\}_{m=1}^n$ to a quantum state $\rho$, according to the known prior probability distribution $\{p_m\}_{m=1}^n$. The output state is then subject to a single generalized measurement with elements $\{M_m\}_{m=1}^n$ satisfying $M_m \geq 0$ and $\sum_{m=1}^n M_m = I$. Our objective is to infer the label $m$ of the quantum operation that was applied.

We now consider a special case of these tasks, known as phase discrimination, which is an important primitive in quantum information processing, featuring in optimal cloning, dense coding, and error correction protocols [39, 43, 65]. Here the operations imprint a phase on the state through the transformation $U_{\phi_m}(\rho) := U_{\phi_m}\rho U_{\phi_m}^\dagger$, where $U_{\phi_m} := \exp(-iH_{\phi_m})$ is generated by the Hamiltonian $H = \sum_{j=0}^{d-1} |j\rangle \langle j|$. The probability of success for inferring the label $m$ in the task specified by

FIG. 5. Bounding multilevel coherence from arbitrary measurements. The blue, orange, and green solid lines correspond to the experimental lower bound on the robustness of multilevel coherence for $k = 2, 3, 4$, respectively for a maximally coherent state $|\psi^+_1\rangle$, while the grey solid lines are the theory prediction. These bounds are obtained from the SDP in Eq. (S16) in the Supplementary Material [41] for an increasing number of randomly chosen projective measurements, taking into account 5σ statistical uncertainties. The coloured dashed lines correspond to the lower bounds for the noisy maximally coherent state $\rho(0.8874 \pm 0.0007)$ using the same observables, with the grey dashed line being the theory prediction for this state.
For any phase discrimination task $\Theta$ and any probe state $\rho$, the probability of success $p^{\Theta}_{\text{succ}}(\rho)$ is given by
\begin{equation}
\frac{p^{\Theta}_{\text{succ}}(\rho)}{p^{\Theta}_{\text{max}}} \leq k(1 + R_{C_k}(\rho)).
\end{equation}

This theorem is proved in the Supplementary Material [41], where we also show that for the specific task $\tilde{\Theta} = \{(\frac{1}{d}, \frac{2\pi m}{d})\}_{m=1}^d$ of discriminating $d$ uniformly distributed phases and for a noisy maximally coherent probe, the bound in Eq. (10) becomes tight. This demonstrates the key role of genuine multilevel coherence as a necessary ingredient for quantum-enhanced phase discrimination, unveiling a hierarchical resource structure which goes significantly beyond previous studies only concerned with the coarse-grained description of coherence [8, 9].

Note that this provides an operational significance to the robustness of multilevel coherence in addition to its operational significance in terms of resilience of noise, which in turn can be thought of also in geometric terms.

**Quasi-device independent witnessing of multilevel coherence.** An important consequence of Eq. (10) is that when $p^{\Theta}_{\text{succ}}(\rho)/p^{\Theta}_{\text{max}} > k$, the probe state $\rho$ must have $(k + 1)$-level coherence. Consequently, the performance of an unknown state $\rho$ in any phase discrimination task $\Theta$ provides a witness of genuine multilevel quantum coherence. We remark that the success probability for an arbitrary quantum state can be evaluated without any knowledge of the devices used—neither of the one imprinting the phase, nor the final measurement. Evaluating the witness only relies on the fact that no information is imprinted on incoherent states, the witness can be evaluated without any additional knowledge of the used devices. We conclude that phase discrimination, as demonstrated in this paper, is a quasi-device-independent approach to measure multilevel coherence as quantified by the RMC.

Figure 6 shows our experimental results for quasi-device independent witnessing of multilevel coherence using the phase discrimination task $\tilde{\Theta}$ for a range of noisy maximally coherent states, also taking into ac-
count experimental imperfections when it comes to the hypothesis $U_{\phi}(\rho) = \rho$ for any $\rho \in C_1$ (see Supplementary Material [41]). As any witnessing approach, this method in general only provides lower bounds on the RMC, yet in contrast to the optimal multilevel witness measured in Fig. 4, the present approach does not rely on any knowledge of the used measurements.

II. DISCUSSION

The study of genuine multilevel coherence is pivotal, not only for fundamental questions, but also for applications ranging from transfer phenomena in many-body and complex systems to quantum technologies, including quantum metrology and quantum communication. In particular for verifying that a quantum device is working in a nonclassical regime it is crucial to certify and quantify multilevel coherence with as few assumptions as possible. Our metrological approach satisfies these criteria by making it possible to verify the preparation of large superpositions and discriminate between them, using only the ability to apply phase transformations that leave incoherent states (approximately) invariant. Notably, the goal of our phase-discrimination task is to successfully distinguish several phases in a one-shot scenario, in contrast to the task of phase estimation [66], which aims to measure an unknown phase. For the latter the figure of merit of the uncertainty of the estimate, and superpositions of the kind $(|1\rangle + |d\rangle)/\sqrt{2}$, that is, involving eigenstates of the observable that correspond to the largest gap in eigenvalues, can be argued to be optimal [67]. When dealing with phase estimation, the relevant notion is that of unspeakable coherence (or asymmetry) [4], and which eigenstates are superposed is very important. On the other hand, for the phase-discrimination task, our results show that the genuine multilevel coherence of a state like $(|1\rangle + |2\rangle + \ldots + |d\rangle)/\sqrt{d}$ is the key feature.

Our analysis of coherence rank and number, multilevel coherence witnesses, and robustness, uses and adapts notions originally studied in the context of entanglement theory [28], and hence provides further parallels between the resource theories of quantum coherence and entanglement, whose interplay is attracting substantial interest [1]. However, a notable difference between the two that we find, emphasise, and exploit, is that multilevel coherence, unlike entanglement, can be characterised and quantified via semidefinite programming, rather than general convex optimisation [5]. This highlights multilevel coherence as a remarkably powerful, yet experimentally accessible quantum resource.

Finally, our work triggers several questions to stimulate further research. These include conceptual questions regarding the exact (geometric) structure and volume of the sets $C_k$, and how sets $C_k$ and $C'_\ell$ defined with respect to different classical bases intersect. Our work starts to address these questions by establishing that $C_k$ has non-vanishing volume for any $k \geq 2$ (while it has vanishing volume for $k = 1$), and that there is a non-zero-volume ball of states around the maximally mixed state that is in $C_2$ for any choice of classical basis. From a more practical point of view, a natural question is how to best choose a finite set of observables to estimate the multilevel coherence of the state of a system, for example via the SDP in Eq. (S16) in the Supplementary Material [41]. This is particularly important when one has limited access to the system under observation, as in a biological setting [30, 68, 69]. Independently of the particular choice of observables, our work provides a plethora of readily applicable tools to facilitate the detection, classification, and quantitative estimation of quantum coherence phenomena in systems of potentially large complexity with minimum assumptions, paving the way towards a deeper understanding of their functional role.

Further theoretical investigation and experimental progress along these lines may lead to fascinating insights and advances in other branches of science where the exploitation of (multilevel) quantum coherence is of interest, potentially going as far as gravitational wave astronomy [70] and studies on the nonclassical nature of the early universe in cosmology [71, 72].

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SUPPLEMENTARY MATERIAL

Here we provide detailed derivations and proofs of all results in the main text.

1. Incoherent and $k$-incoherent operations.

Consider a (fully) incoherent operation with corresponding Kraus operators \( \{K_i\} \). This operation is also $k$-incoherent if \( r_C(\psi_i) \leq k \), where \( |\psi_i\rangle = K_i |\psi\rangle / \sqrt{p_i} \) with \( p_i = \langle \psi | K_i^\dagger K_i |\psi\rangle \), for all pure states \( |\psi\rangle \) such that \( r_C(\psi) \leq k \) and for all \( i \) and \( k \in \{2, 3, \ldots, d\} \), which means that the Kraus operators \( \{K_i\} \) together compose a $k$-incoherent operation. That this holds true is immediate, given that the Kraus operators \( K_i \) have the form \( K_i = \sum_j e^{i \phi_{ij}} \sqrt{p_{ij}} |f_i(j)\rangle \langle j| \), with \( f_i : \{1, \ldots, d\} \rightarrow \{1, \ldots, d\} \), \( \phi_{ij} \) a phase, and \( p_{ij} = (j | K_i | j) \).

$k$-decohering operation. We define a $k$-decohering map \( \Lambda \) as one that destroys multilevel coherence, more precisely, such that \( \Lambda(\mathcal{D}(\mathcal{H})) \subseteq C_k \). These operations generalise the notion of resource destroying maps \( \Lambda \) to multilevel coherence. An example of a $k$-decohering operation is the $k$-dephasing operation

\[
\Delta_k(\rho) := \frac{1}{d^{k-1}} \sum_{I \in \mathcal{P}_k} P_I \rho P_I, \quad (S1)
\]

where \( \mathcal{P}_k \) is the set of all the \( k \)-element subsets of \{1, 2, \ldots, d\}, and \( P_I := \sum_{i \in I} |i\rangle \langle i| \). Since the \( P_I \) are projectors onto \( k \)-dimensional subspaces, \( \Delta_k(\mathcal{D}(\mathcal{H})) \subseteq C_k \). The linearity and complete positivity of \( \Delta_k \) follow directly from the construction. Trace preservation is implied by the observation that \( \Delta_k \) has the alternative expression

\[
\Delta_k(\rho) = \frac{k-1}{d-1} \rho + \frac{d-k}{d-1} \Delta_1(\rho), \quad (S2)
\]

since \( \Delta_1 \) is clearly trace-preserving. That \( (S2) \) holds can be seen from the fact that, for all \( m, n \in \{1, 2, \ldots, d\} \) with \( m \neq n \),

\[
\sum_{I \in \mathcal{P}_k} P_I |m\rangle \langle m| P_I = \left( \frac{d-1}{k-1} \right) |m\rangle \langle m|,
\]

\[
\sum_{I \in \mathcal{P}_k} P_I |m\rangle \langle n| P_I = \left( \frac{k-2}{k-1} \right) |m\rangle \langle n|,
\]

with \( \binom{d-2}{k-2} \) the number of \( I \in \mathcal{P}_k \) that contain two fixed indexes. The form of Eq. \( (S2) \) then follows directly from the definition of \( \Delta_k \) in Eq. \( (S1) \).

2. Properties of the sets \( C_k \).

Using the map \( \Delta_k \) one can prove the following theorem to show that \( C_k \) has non-zero volume within \( \mathcal{D}(\mathcal{H}) \) for \( k \in \{2, \ldots, d\} \) (while the fully incoherent states \( C_1 \) have zero volume, see Fig. 1(a) in the main text). Consider the condition

\[
\rho \geq \frac{d-k}{d-1} \Delta_1(\rho), \quad (S4)
\]

and let us introduce the set \( D_k = \{\rho | \rho \in \mathcal{D}(\mathcal{H}), \rho \text{satisfies} \ (S4)\} \). One has the following.

Theorem 3. The inclusion \( D_k \subseteq \Delta_k(\mathcal{D}(\mathcal{H})) \subseteq C_k \) holds for any \( k = 1, \ldots, d \).

The set of states \( D_k \) defined by \( (S4) \) is convex. It is also easy to see that, for \( k \geq 2 \), such a set has non-zero volume. Indeed, the maximally mixed state satisfies \( (S4) \) for all \( k \), with strict inequality for \( k \geq 2 \). This implies that, for any \( k \geq 2 \), any state that is an arbitrary but small enough perturbation of the maximally mixed state will still satisfy \( (S4) \).

The proof of Theorem 3 is readily given. For \( k = 1 \), the inequality characterizes exactly the set of fully incoherent states \( C_1 \). For \( k \in \{2, \ldots, d\} \), and for as state \( \rho \) satisfying Eq. \( (S4) \), that is for \( \rho \in D_k \), consider the operator

\[
\sigma = \frac{d-1}{k-1} \rho - \frac{d-k}{k-1} \Delta_1(\rho), \quad (S5)
\]

so that \( \rho = \Delta_k(\sigma) \). We can see that \( \sigma \) has unit trace by construction, and furthermore that Eq. \( (S4) \), implies \( \sigma \geq 0 \). Thus, \( \sigma \in \mathcal{D}(\mathcal{H}) \) and \( \rho \in \Delta_k(\mathcal{D}(\mathcal{H})) \).

We now prove corollaries of Theorem 3 that provide some geometrical insight into the properties of the set \( C_k \). We observe that the support of \( \rho \) is included in the the support of \( \Delta_1(\rho) \). This means that Eq. \( (S4) \) is equivalent to

\[
(\Delta_1(\rho))^{-1/2} \rho (\Delta_1(\rho))^{-1/2} \geq \frac{d-k}{d-1} P_{\Delta_1(\rho)}, \quad (S6)
\]

where \( P_{\Delta_1(\rho)} \) is the projector onto the support of \( \Delta_1(\rho) \), and \( \Delta_1(\rho)^{-1} \) is the pseudo-inverse of \( \Delta_1(\rho) \) on its support. Since \( \rho \geq \lambda_{\min}(\rho) I \), where \( \lambda_{\min}(\rho) \) is the smallest eigenvalues of \( \rho \), inequality \( (S6) \) is implied by

\[
\lambda_{\min}(\rho) \Delta_1(\rho)^{-1} \geq \frac{d-k}{d-1} P_{\Delta_1(\rho)}.
\]

In turn, this is implied by

\[
\frac{\lambda_{\min}(\rho)}{\|\Delta_1(\rho)\|_\infty} \geq \frac{d-k}{d-1}. \quad (S7)
\]

Given that

\[
\lambda_{\min}(\rho) \geq 1/d - \|\rho - I/d\|_\infty,
\]

\[
\|\Delta_1(\rho)\|_\infty \leq 1/d + \|\Delta_1(\rho) - I/d\|_\infty,
\]

we find that

\[
\frac{1 - d\|\rho - I/d\|_\infty}{1 + d\|\Delta_1(\rho) - I/d\|_\infty} \geq \frac{d-k}{d-1}, \quad (S9)
\]
implies Eq. \((S4)\). One can impose an even stronger condition that ensures that \(\rho \in D_k\); such condition can be found by considering that \(\|\Delta_1[\rho] - I/d\|_\infty \leq \|\rho - I/d\|_\infty\), because \(\Delta_1\) is a pinching that (i) leaves \(I/d\) invariant and (ii) does not increase the operator norm. In particular, fixing \(k = 2\) we conclude that if
\[
\|\rho - I/d\|_\infty \leq \frac{1}{d(2d - 3)},
\]  
that is, if \(\rho\) is close enough to the maximally mixed state, then \(\rho \in D_k \subseteq \Delta_2(D(H)) \subseteq C_k\) for any reference basis.

**Characterising the dual sets \(C_k^*\).** Here we prove Theorem 1 of the main text, i.e. that \(W \in C_k^*\) if and only if \(P_I W P_I \geq 0\) for all \(I \in P_k\). Formally, the set \(C_k^*\) of \((k+1)\)-level coherence witnesses is obtained as the dual of the set \(C_k\) and given by
\[
C_k^* = \{ W : W = W^\dagger, \text{Tr}(W\sigma) \geq 0, \forall \sigma \in C_k \}. \tag{S11}
\]
This definition, together with the convexity of \(C_k\), implies that it is sufficient to see that
\[
\langle \psi | W | \psi \rangle \geq 0 \quad \forall | \psi \rangle \text{ such that } r_C(|\psi\rangle) \leq k, \tag{S12}
\]
if and only if \(P_I W P_I \geq 0\) for all \(I \in P_k\). This is immediate since, on the one hand, for any given \(I \in P_k\) the action of projecting with \(P_I\) on an arbitrary \(| \psi \rangle \in D(H)\) is either to return the null vector or a pure state (up to normalisation) with coherence rank not exceeding \(k\). On the other hand, for any \(| \psi \rangle\) such that \(r_C(|\psi\rangle) \leq k\), one can always find an \(I \in P_k\) such that \(P_I | \psi \rangle = | \psi \rangle\).

**3. Robustness of \((k+1)\) coherence.**

It is possible to write \(R_{C_k}(\rho)\) as the solution of the following SDP:
\[
R_{C_k}(\rho) = \min_{\sigma \in C_k} \left\{ \text{Tr}(\sum_{I \in P_k} \tilde{\sigma}_I) - 1 \right\} \quad \text{s.t.} \quad \begin{align*}
\tilde{\sigma}_I &\geq 0 \quad \forall I \in P_k \\
P_I \tilde{\sigma}_I P_I &\geq \tilde{\sigma}_I \quad \forall I \in P_k \\
\sum_{I \in P_k} \tilde{\sigma}_I &\geq \rho.
\end{align*} \tag{S13}
\]
The dual SDP is given by Eq. (8) in the main text. We now show that Eq. \((S13)\) holds. First, one may rewrite \(R_{C_k}(\rho)\) as
\[
R_{C_k}(\rho) = \inf_{\sigma \in C_k} \left\{ s \geq 0 : \rho \leq (1 + s)\sigma \right\}. \tag{S14}
\]
One then arrives to Eq. \((S13)\) by using the defying property of \(C_k\), that is, that \(\sigma \in C_k\) can be written as the convex combination of pure states with coherence rank at most \(k\). Thus, for any \(\tilde{\sigma} \geq 0\), we have that \(\frac{\tilde{\sigma}}{\text{Tr}(\tilde{\sigma})} \in C_k\) if and only if \(\tilde{\sigma} = \sum_{I \in P_k} \tilde{\sigma}_I\), such that for all \(I \in P_k\) it holds that \(P_I \tilde{\sigma}_I P_I = \tilde{\sigma}_I\) and \(\tilde{\sigma}_I \geq 0\).

We note that strong duality holds trivially since \(I \in C_k^*\).

**Robustness of multilevel coherence of the noisy maximally coherent states.** It is simple to check that for the witness \(W_k(\psi^+_d)\) in Eq. (6) in the main text we have \(\text{Tr}(W_k(\psi^+_d)\rho(p)) = \frac{1}{k}[(k - 1) - p(d - 1)]\) and \(W_k(\psi^+_d) \leq I\). One then concludes that \(R_{C_k}(\rho(p)) \geq \max \{0, -\text{Tr}(W_k(\psi^+_d)\rho(p))\} = \max \{0, \frac{1}{k}[p(d - 1) - (k - 1)]\}\). On the other hand, it can be seen that \(\rho(p) \leq (1 + s(\rho(p)))\Delta_k(\langle \psi^+_d | \psi^+_d \rangle)\) for \(s(\rho(p)) = \max \{0, \frac{1}{k}[p(d - 1) - (k - 1)]\}\). Then, from Eq. \((S14)\) we see that \(R_{C_k}(\rho(p)) \leq s(\rho(p))\) and can hence conclude that
\[
R_{C_k}(\rho(p)) = \max \left\{ 0, \frac{1}{k}[p(d - 1) - (k - 1)] \right\}. \tag{S15}
\]

**4. Bounding RMC from arbitrary measurements.**

If one has access to the expectation values \(o_i = \text{Tr}(O_i\rho)\) of a set of \(n\) observables \(\{O_i\}_{i=1}^n\) measured on an experimentally prepared state \(\rho\), one can lower bound the RMC using the SDP:
\[
R_{C_k}(\rho) \geq \min_{\tilde{\sigma}_I} \left\{ \text{Tr}(\sum_{I \in P_k} \tilde{\sigma}_I) - 1 \right\} \quad \text{s.t.} \quad \begin{align*}
\tilde{\sigma}_I &\geq 0 \quad \forall I \in P_k \\
P_I \tilde{\sigma}_I P_I &\geq \tilde{\sigma}_I \quad \forall I \in P_k \\
\sum_{I \in P_k} \tilde{\sigma}_I &\geq \rho
\end{align*} \tag{S16}
\]
where we allow for the lower and upper experimental uncertainties \(e^-_i\) and \(e^+_i\), respectively. Here we look for an optimal \(\tau \in D(H)\) that satisfies \(\sum_{I \in P_k} \tilde{\sigma}_I \geq \tau\) while being consistent with the results of the expectation values, that is \(\text{Tr}(O_i\tau) \leq o_i + e^+_i\) \(\forall i\), (S16)

Note that the SDP in Eq. \((S16)\) only requires the optimization to reproduce the measured expectation values to within the supplied error bounds. This leads to a trade-off, where smaller error bounds to the SDP lead to closer convergence to the actual value of multilevel coherence, while larger error bounds improve the stability of the estimation against statistical fluctuations. In the experiment presented in Fig. 5 of the main text, we have chosen conservative 5% error bounds, leading to a deviation about 5% between the lower bound and the tomographically estimated RMC. This could be improved by incorporating maximum-likelihood or Bayesian estimation techniques as in the case of quantum tomography.
5. Phase discrimination.

Consider the optimal \( \sigma^* \in C_k \) satisfying the optimisation in Eq. (S14) for a given state \( \rho \), i.e. such that \( \rho \leq (1 + R_{C_k}(\rho))\sigma^* \). Following from the linearity of the success probability in Eq. (9) in the main text, we see that

\[
p^{\Theta}_{\text{suc}}(\rho) \leq (1 + R_{C_k}(\rho))p^{\Theta}_{\text{suc}}(\sigma^*). \tag{S17}
\]

Now, by setting \( k = 1 \), one may also consider the optimal \( \delta^* \in C_1 \) such that \( \sigma^* \leq (1 + R_{C_1}(\sigma^*))\delta^* \), which means

\[
p^{\Theta}_{\text{suc}}(\sigma^*) \leq (1 + R_{C_1}(\sigma^*))p^{\Theta}_{\text{suc}}(\delta^*). \tag{S18}
\]

Overall then, we have

\[
p^{\Theta}_{\text{suc}}(\rho) \leq (1 + R_{C_k}(\rho))(1 + R_{C_1}(\sigma^*))p^{\Theta}_{\text{suc}}(\delta^*). \tag{S19}
\]

Since \( \sigma^* \in C_k \), we find that \( R_{C_1}(\sigma^*) \leq k - 1 \). Furthermore, we have already seen that \( p^{\Theta}_{\text{suc}}(\delta^*) \leq p_{\text{max}} \). Hence, we arrive at

\[
p^{\Theta}_{\text{suc}}(\rho) \leq (1 + R_{C_k}(\rho))kp_{\text{max}} \tag{S20}
\]

which can be rearranged to Eq. (10) in the main text.

When one considers the phase discrimination task \( \Phi \) with a probe prepared in the noisy maximally coherent state \( \rho(p) \) and optimised generalised measurements \( \Phi_m = U_{\Phi_m}(|\psi_+^{AB}\rangle \langle \psi_+^{AB}|) \), the success probability is

\[
p^{\Phi}_{\text{suc}}(\rho(p)) = \frac{1 + R_{C_1}(\rho(p))}{d} = 1 + \frac{p(d - 1)}{d}. \tag{S21}
\]

This can be input into the lower bound to \( R_{C_k}(\rho(p)) \) in Eq. (10) in the main text, for which we see that

\[
\max \left\{ \frac{1 + p(d - 1)}{k} - 1 \right\} \leq R_{C_k}(\rho(p)). \tag{S22}
\]

In fact, it can be seen from Eq. (S15) that this lower bound is tight.

6. Experimental Imperfections

Recall, that the phase discrimination task witnesses the robustness of multilevel coherence in a quasi-device independent way, relying only on the mild assumption that the device leaves incoherent states unperturbed. In practice, this assumption is not exactly satisfied, since experimental imperfections in general lead to unitaries that do not leave incoherent states exactly invariant. To take this into account, we replacing the upper bound \( p_{\text{max}} \) on the incoherent success probability with an upper bound \( p^{\Phi}_{\text{suc}}(\delta) \) for the probability of success for discriminating the elements of the ensemble \( \{p_m, \Lambda_m[\delta]\} \), where by \( \Lambda_m[\delta] \) we denote the image of a incoherent state \( \delta \) under the action of a map \( \Lambda_m \) which describes the approximate application of a phase \( \phi_m \).

In general, given an ensemble \( \{p_m, \rho_m\}_{m=1}^{n} \) of \( n \) possible states in which a system may be prepared, the optimal probability of guessing the actual state is given by

\[
\min\{p|\rho_{AB} \leq p I_A \otimes \sigma_B; \sigma \text{ a normalized state}\}.
\]

Hence, replacing the upper bound \( \rho_{AB} = \sum_m p_m |m\rangle \langle m| \otimes \rho_m \) [4]. In our case \( \rho_m = \Lambda_m[\delta] \). We will assume that the maps \( \Lambda_m \) do not modify an incoherent state \( \delta \) too much; more precisely, in terms of trace distance

\[
D(\Lambda_m[\delta], \delta) := \frac{1}{2} \|\Lambda_m[\delta] - \delta\|_1 \leq \epsilon, \text{ for all } m.
\]

This means that \( \Lambda_m[\delta] \leq \delta + \Delta_m \), with \( \Delta_m > 0 \) and \( \text{Tr}(\Delta_m) < \epsilon \), which in turn implies that \( \Lambda_m[\delta] \leq \delta + \epsilon I \). It is immediate to check that this implies \( \sum_m p_m |m\rangle \langle m| \otimes \Lambda_m[\delta] \leq (p_{\text{max}}(1 + \epsilon d))I \otimes \sigma, \) with \( \sigma = \frac{1}{1 + \epsilon} \) a normalized state.

This proves that \( p^{\Phi}_{\text{suc}}(\delta) \leq p_{\text{max}}(1 + \epsilon d) \). We can give a reasonable estimate of \( \epsilon \) in terms of the process fidelity of the experiment, using the Fuchs-van de Graaf inequality

\[
D(\xi_1, \xi_2) \leq \sqrt{1 - F^2(\xi_1, \xi_2)}, \text{ with } F(\xi_1, \xi_2) \text{ the fidelity between two states}\ [75].
\]

Thus, we arrive at the estimate

\[
p^{\Phi}_{\text{suc}}(\delta) \leq p_{\text{max}}(1 + d \sqrt{1 - F^2}).
\]

which can be substituted in place of \( p_{\text{max}} \) in Eq. (10) in the main text. In the case of our experiment with process fidelity \( F_p \approx 0.9956 \) and \( d = 4 \), this means substituting \( p_{\text{max}} \) with \( \approx 1.375 \times p_{\text{max}} \).

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