Commutators of the B-Maximal Operator and B-Maximal Commutators

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Abstract. In this paper we consider the commutator of the B-maximal operator and the B-maximal commutator associated with the Laplace-Bessel differential operator. The boundedness of the commutator of the B-maximal operator with BMO symbols on weighted Lebesgue space and weak-type inequality for the commutator of the B-maximal operator are proved.

1. Introduction

The Laplace-Bessel differential operator
\[ \Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \left( \frac{\partial^2}{\partial x_n^2} + 2\nu \frac{\partial}{\partial x_n} \right), \quad \nu > 0 \]
is known as an important operator in Fourier-Bessel harmonic analysis and applications. This operator, associated with the Bessel differential operator
\[ B_\nu = \frac{d^2}{dt^2} + \frac{2\nu}{t} \frac{d}{dt}, \quad \nu > 0 \]
has been studied many mathematicians.[2–7, 14–17, 23–27, 29, 31, 32]

Given a linear operator \( T \) acting on functions and given a function \( b \), the commutator \([T, b]\) formally defined as
\[ [T, b]f = T(bf) - bT(f). \]

The first result on commutators was obtained by Coifman, Rochberg, Weiss [12]. They showed that if \( T \) is a classical singular integral operator and \( b \in BMO \), then the commutator \([T, b]\) is bounded on \( L_p(\mathbb{R}^n) \), \( 1 < p < \infty \). Later, Chanillo [11] proved a similar result when singular integral operators are replaced by the fractional integral operators.

Coifman and Meyer [13] observed that the \( L_p \) boundedness for the commutator \([T, b]\) could be obtained from the weighted \( L_p \) estimate for \( T \) with the weight function class of Muckenhoupt \( A_p \). Later, Alvarez, Bagby, Kurtz, Perez [9] extended the idea of Coifman and Meyer and Perez [30] obtained a weak-type inequality for the commutator \([T, b]\).

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This article is dedicated to Professor Gradimir V. Milovanovic on the Occasion of his 70th anniversary
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In [28], Milman and Schonbek proved that the commutator of the classical Hardy-Littlewood maximal function \([M, b]\), defined by
\[ [M, b]f(x) = M(bf)(x) - b(x)Mf(x), \quad x \in \mathbb{R}^n \]
is bounded on \(L^p, 1 < p < \infty\) when \(b\) is in \(BMO(\mathbb{R}^n)\). Moreover, the classical maximal commutator associated with the classical translation is defined by
\[ M_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)|dy; \quad f \in L^p(\mathbb{R}^n). \]

These operators play an important role in studying the commutators of singular integral operator with \(BMO\) symbols. Alphonse [8] obtained weak type inequality for maximal commutators, and pointwise estimates of the maximal commutator and the commutator of the maximal function are proved by Agcayazi, Gogatishvili, Koca, Mustafayev [1]. Commutators have been research area many mathematicians such as Guliyev, Hasanov, Hu, Lin, Yang, Janson [18, 20, 21] and others.

In this paper, we consider the commutator \([M_b, b]\) of the Hardy-Littlewood maximal operator \(M_b\) and the \(B\)-maximal commutator associated with the Laplace-Bessel differential operator. The paper is organized as follow. Section 2 contains some basic definitions and results which are needed in this paper. Main results and its proofs are in the Section 3.

2. Preliminaries and Notations

Let \(\mathbb{R}^n_b = \{x: x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_n \geq 0\}\) and \(B(x, r) = \{y \in \mathbb{R}^n_b : |x - y| < r\}\). For a measurable set \(E \subset \mathbb{R}^n_b\) let \(|E|_b = \int x_n^2\,dx\), \(n > 0\).

Denote by \(T^\nu (y \in \mathbb{R}^n_b)\), generalized translation operator acting according to the law:
\[ T^\nu f(x) = \frac{\Gamma(\nu + 1/2)}{\Gamma(\nu)\Gamma(1/2)} \int_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_ny_n}\cos \alpha + y_n^2\right) \sin^{2\nu - 1}\alpha \,d\alpha, \]
where \(x = (x', x_n), y = (y', y_n)\) and \(x', y' \in \mathbb{R}^{n-1}\). We remark that \(T^\nu\) is closely connected with Bessel differential operator \(B_r\), see [23, 24] for details.

The weighted space \(L^p_{\nu, r}(\mathbb{R}^n_b), 1 \leq p < \infty\) consists of measurable functions on \(\mathbb{R}^n_b\) with the norm given by
\[ \|f\|_{L^p_{\nu, r}} = \left( \int_{\mathbb{R}^n_b} |f(x)|^p x_n^{2\nu}\,dx \right)^{1/p}. \]

In the case \(p = \infty\), the space \(L^{\infty}_{\nu, r}\) is defined by means of the usual modification \(\|f\|_{L^{\infty}_{\nu, r}} = \text{ess sup}\, |f(x)|, x \in \mathbb{R}^n_b\).

We denote by \(L^{\infty}_{\nu, r}(\mathbb{R}^n_b)\), locally integrable with respect to the measure \(x_n^{2\nu}\,dx\) functions defined on \(\mathbb{R}^n_b\).

Let \(1 < p < \infty\). A weight function \(w\) is said to be of Muckenhoupt class \(A_{p, r}\) if \([w]_{A_{p, r}}\) is finite, where \([w]_{A_{p, r}}\) is defined by
\[ [w]_{A_{p, r}} = \sup_{x \in \mathbb{R}^n_b, r > 0} \left\{ \frac{1}{|B(x, r)|_b} \int_{B(x, r)} w(y) y_n^{2\nu}\,dy \right\} \left( \frac{1}{|B(x, r)|_b} \int_{B(x, r)} w(y)^{-1/p - 1} y_n^{2\nu}\,dy \right)^{p-1}. \]

The Hardy-Littlewood maximal function generated by generalized translation operator, called the \(B\)-maximal function \(M_Bf\), is defined by
\[ M_B f(x) = \sup_{r > 0} \frac{1}{|B(0, r)|_b} \int_{B(0, r)} T^\nu f(x) y_n^{2\nu}\,dy, \quad x \in \mathbb{R}^n_b. \]
The operator $M_B : f \rightarrow M_B f$ is called the B-maximal operator. The boundedness of the B-maximal operator $M_B$ on $L_{p,v}$ is proved by V.Guliyev [16].

The space of functions of bounded mean oscillation associated with Laplace-Bessel differential operator is denoted by $\text{BMO}_B = \text{BMO}_B(\mathbb{R}^n_1)$ and defined by the following finite norm

$$\|f\|_{\text{BMO}_B} = \sup_{x \in \mathbb{R}^n_1, r > 0} \frac{1}{|B(0, r)|_v} \int_{B(0, r)} |f(y) - f_{B(0, r)}(x)| y_n^2 dy,$$

where $f_{B(0, r)}(x) = \frac{1}{|B(0, r)|_v} \int_{B(0, r)} T^y f(x) y_n^2 dy$.

The classical BMO space plays an important role in Fourier harmonic analysis and applications, introduced by John and Nirenberg [22] in 1961. It is easy to see that $L_\infty \subseteq \text{BMO}$. A famous example is $\log|x| \in \text{BMO}(\mathbb{R}^n_1) \setminus L_\infty(\mathbb{R}^n_1)$. BMO space turned out to be the “right” space to study instead of $L_\infty$. Many of the operators which are ill-behaved on $L_\infty$, are bounded on BMO.

Definitions of the commutator of the B-maximal operator and the B-maximal commutator associated with the Laplace-Bessel differential operator are given below.

**Definition 2.1.** Let $b$ be a measurable function defined on $\mathbb{R}^n_1$. The commutator $[M_B, b]$ of the B-maximal operator $M_B$ is defined by

$$[M_B, b] f(x) = M_B(b f)(x) - b(x) M_B f(x), \quad x \in \mathbb{R}^n_1.$$

**Definition 2.2.** Let $b \in L_{1,v}^{\text{loc}}(\mathbb{R}^n_1)$. The B-maximal commutator $M_{B,b}$ is defined by

$$M_{B,b} f(x) = \sup_{r > 0} \frac{1}{|B(0, r)|_v} \int_{B(0, r)} T^y ((b(x) - b(y)) f(x)) y_n^2 dy, \quad x \in \mathbb{R}^n_1.$$

3. Main Results

In classical theory, if $w$ and $w^{-1}$ belong to the Muckenhoupt class $A_p$, then the Hardy-Littlewood maximal operator $M$ is bounded on $L_p(w^{1\cdot}dx)$. So, Milman and Schonbek [28] prove that if $b \in \text{BMO}$, $b \geq 0$, then the commutator $[M, b]$ of the Hardy-Littlewood maximal operator is bounded on $L_{p,v}$, $1 < p < \infty$.

In Fourier-Bessel harmonic analysis, the boundedness of the Hardy-Littlewood maximal function generated by the Laplace-Bessel differential operator such that $w$ belongs to the suitable Muckenhoupt class on weighted Lebesgue space was proved by Guliyev [19]. This result is given in the next theorem.

**Theorem 3.1.** a) If $f \in L_{p,v}(w, \mathbb{R}^n_1), \quad w \in A_{p,v}(\mathbb{R}^n_1), \quad 1 < p < \infty$, then

$$\|M_B f\|_{L_{p,v}(w, \mathbb{R}^n_1)} \leq C \|f\|_{L_{p,v}(w, \mathbb{R}^n_1)}$$

where the constant $C$ depends on $p, w, v, n$.

b) If $f \in L_{1,v}(w, \mathbb{R}^n_1), \quad w \in A_{1,v}(\mathbb{R}^n_1), \quad 1 < p < \infty$, then

$$\|M_B f\|_{\text{WL}_{1,v}(w, \mathbb{R}^n_1)} \leq C \|f\|_{\text{L}_{1,v}(w, \mathbb{R}^n_1)}$$

where the constant $C$ depends on $w, v, n$. Here $\text{WL}_{1,v}(w, \mathbb{R}^n_1)$ denotes the weak-$L_{1,v}(w, \mathbb{R}^n_1)$ space.

By using similar arguments in ([28], Theorem 4.4), we get the following theorem from the Theorem 3.1.

**Theorem 3.2.** Let $f \in L_{p,v}(\mathbb{R}^n_1), \quad 1 < p < \infty$ and $b \in \text{BMO}_B$, $b \geq 0$. Then the commutator of the B-maximal operator $[M_B, b]$ is bounded on $L_{p,v}(\mathbb{R}^n_1)$, that is,

$$\|[M_B, b] f\|_{L_{p,v}} \leq \|b\|_{\text{BMO}_B} \|f\|_{L_{p,v}}.$$
The commutator of the $B$-maximal operator $[M_B, b]$ and the $B$-maximal commutator $M_{B,b}$ are essentially different from each other. However, if $b$ satisfies some conditions, then the operator $M_{B,b}$ controls $[M_B, b]$.

**Lemma 3.3.** Let $b$ is any non-negative locally integrable function defined on $\mathbb{R}_+^n$. Then

$$
\|[M_B, b]f(x)\| \leq M_{B,b}f(x)
$$

for all $f \in L^1_{\text{loc}}(\mathbb{R}_+^n)$.

**Proof.** Since $b$ is non-negative

$$
T^y \left| b(x)f(x) \right| - b(x)T^y \left| f(x) \right| = c_v \int_0^\pi \left| (bf)(x', y', \sqrt{x_n^2 - 2x_ny_n \cos \alpha + y_n^2}) \right| \sin^{2r-1} \alpha d\alpha
$$

$$
- b(x)c_v \int_0^\pi \left| f(x', y', \sqrt{x_n^2 - 2x_ny_n \cos \alpha + y_n^2}) \right| \sin^{2r-1} \alpha d\alpha
$$

$$
= c_v \int_0^\pi \left| (bf)(x', y', \sqrt{x_n^2 - 2x_ny_n \cos \alpha + y_n^2}) - |b(x)| \left| f(x', y', \sqrt{x_n^2 - 2x_ny_n \cos \alpha + y_n^2}) \right| \sin^{2r-1} \alpha d\alpha
$$

and we have

$$
\|T^y \left| b(x)f(x) \right| - b(x)T^y \left| f(x) \right| \| \leq T^y \left| (b(.) - b(x)) f(.) \right| .
$$

Since by making use of the following inequality

$$
\left| \sup_{r>0} u(r) - \sup_{r>0} v(r) \right| \leq \sup_{r>0} |u(r) - v(r)|, \quad u(r), v(r) > 0
$$

we have

$$
\|[M_B, b]f(x)\| = \|M_B(b f(x) - b(x)M_B f(x))\|
$$

$$
= \sup_{r>0} \frac{1}{|B(0, r)_v|} \int_{B(0, r)} T^y \|b(x)f(x)\|y^{2r} dy - b(x) \sup_{r>0} \frac{1}{|B(0, r)_v|} \int_{B(0, r)} T^y \|f(x)\|y^{2r} dy
$$

$$
\leq \sup_{r>0} \frac{1}{|B(0, r)_v|} \int_{B(0, r)} \left| T^y \|b(x)f(x)\| - b(x)T^y \|f(x)\| \right| y^{2r} dy
$$

$$
= M_B \left( (b(.) - b(x)) f(.) \right)(x)
$$

$$
= M_{B,b}f(x).
$$


\square

**Lemma 3.4.** Let $b \in L^1_{\text{loc}}(\mathbb{R}_+^n)$ Then

$$
\|[M_B, b]f(x)\| \leq M_{B,b}f(x) + 2b^-(x)M_B f(x)
$$

for all $f \in L^1_{\text{loc}}(\mathbb{R}_+^n)$ where $b^-(x) = \max\{-b(x), 0\}$.
Proof. Since

\[
\left| [MB, b]f(x) - [MB, |b|]f(x) \right| = \left| MB(bf)(x) - b(x)MBf(x) - MB(|b|f)(x) + |b(x)|MBf(x) \right|
\]

\[
\leq 2b^{-}(x)MBf(x)
\]

we have

\[
\left| [MB, b]f(x) \right| \leq \left| [MB, |b|]f(x) \right| + 2b^{-}(x)MBf(x)
\]

and by using Lemma 3.3, we get

\[
\left| [MB, b]f(x) \right| \leq MB(|b|f)(x) + 2b^{-}(x)MBf(x)
\]

\[
\square
\]

The weak-type inequality for the commutator of the B-maximal operator is obtained using Lemma 3.4 and the weak type \((1, 1)\) inequality for the \(B\)-maximal function. This result is the following.

**Theorem 3.5.** Let \(b \in L_{\infty}\). Then there exist a positive constant \(c_1, c_2\) such that

\[
\left| \left\{ x \in \mathbb{R}^n : |[MB, b]f(x)| > \lambda \right\} \right|_{p,v} \leq c_1 \left\| b \right\|_{L_{\infty}} \left\| f \right\|_{L_{1,v}} + \left( \frac{c_2 \left\| b \right\|_{L_{\infty}}}{\lambda} \right)^q \left\| f \right\|_{L_{q,v}}^q
\]

for all \(f \in L_{1,v} \cap L_{q,v}, 1 < q < \infty\) and for all \(\lambda > 0\).

**Proof.** For \(\lambda > 0\), by using Lemma 3.4, we have

\[
\left| \left\{ x \in \mathbb{R}^n : |[MB, b]f(x)| > \lambda \right\} \right|_{p,v} \leq \left| \left\{ x \in \mathbb{R}^n : MBbf(x) > \frac{\lambda}{2} \right\} \right|_{p,v} + \left| \left\{ x \in \mathbb{R}^n : 2b^{-}(x)MBf(x) > \frac{\lambda}{2} \right\} \right|_{p,v}
\]

\[
\leq \left| \left\{ x \in \mathbb{R}^n : MBbf(x) > \frac{\lambda}{2} \right\} \right|_{p,v} + \left| \left\{ x \in \mathbb{R}^n : 2 \left\| b \right\|_{L_{\infty}} MBf(x) > \frac{\lambda}{2} \right\} \right|_{p,v}
\]

\[
= I_1 + I_2.
\]

Since the \(B\)-maximal operator is a weak type \((1, 1)\) we have

\[
I_2 = \left| \left\{ x \in \mathbb{R}^n : 2 \left\| b \right\|_{L_{\infty}} MBf(x) > \frac{\lambda}{2} \right\} \right|_{p,v} \leq C_1 \left\| b \right\|_{L_{\infty}} \int_{\mathbb{R}^n} |f(x)|^{2/p} \, dx = C_1 \left\| b \right\|_{L_{\infty}} \left\| f \right\|_{L_{p,v}}^p.
\]
Let us estimate $I_1$. By using Hölder inequality

$$M_{\text{B},\text{f}} f(x) = \sup_{r>0} \frac{1}{|B(0,r)|_v} \int_{B(0,r)} T^y (|b(x) - b(y)| f(x)) y_n^2 \, dy$$

$$\leq \sup_{r>0} \frac{1}{|B(0,r)|_v} \int_{B(0,r)} T^y (|b(x) - b(y)| f(x)) y_n^2 \, dy$$

$$= \sup_{r>0} \frac{1}{|B(0,r)|_v} \int_{B(0,r)} c_y \int_0^\pi \left| b(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos \alpha + y_n^2}) - b(y) \right|$$

$$\times \left| f(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos \alpha + y_n^2}) \sin^{2\nu - 1} \alpha \, d\alpha \right| y_n^2 \, dy$$

$$\leq \sup_{r>0} \frac{1}{|B(0,r)|_v} \int_{B(0,r)} \left( c_y \int_0^\pi \left| b(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos \alpha + y_n^2}) - b(y) \right|^p \sin^{2\nu - 1} \alpha \, d\alpha \right)^{1/p}$$

$$\times \left( \sup_{r>0} \frac{1}{|B(0,r)|_v} \int_{B(0,r)} (T^y |b(x) - b(y)|)^{1/p} (T^y |f(x)|)^{1/q} y_n^2 \, dy \right)^{1/q}$$

$$\leq \left( \sup_{r>0} \frac{1}{|B(0,r)|_v} \int_{B(0,r)} T^n |b(x) - b(y)| y_n^2 \, dy \right)^{1/p} \left( \sup_{r>0} \frac{1}{|B(0,r)|_v} \int_{B(0,r)} T^n |f(x)| y_n^2 \, dy \right)^{1/q}$$

$$\leq c_1 \|b\|_{L^\infty} (M_{\text{B}}|f|^p)^{1/q}(x).$$

Therefore

$$I_1 = \left\{ x \in \mathbb{R}^n_+ : M_{\text{B},\text{f}} f(x) > \frac{\lambda}{2} \right\} \leq \left\{ x \in \mathbb{R}^n_+ : M_{\text{B},\text{f}} f(x) > \frac{\lambda}{2} \right\} \leq \left\{ x \in \mathbb{R}^n_+ : c_1 \|b\|_{L^\infty} (M_{\text{B}}|f|^p)^{1/q}(x) > \frac{\lambda}{2} \right\}$$

$$\leq \left( \frac{c_2 \|\|L^\infty\|_{L^\infty}}{\lambda} \right)^{q} \|f\|_{L^p}, \quad 1 < q < \infty.$$ 

Finally the desired result follows from $I_1$ and $I_2$. \qed

4. Conclusions

This paper presents the boundedness of the commutator of the B-maximal operator with BMO symbols and weak-type inequality for the commutator of the B-maximal operator on weighted Lebesgue space.

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