PROJECTIVELY EQUIVARIANT SYMBOL CALCULUS
FOR BIDIFFERENTIAL OPERATORS

FABIEN BONIVER

ABSTRACT. We prove the existence and uniqueness of a projectively equivariant symbol map, which is an isomorphism between the space of bidifferential operators acting on tensor densities over $\mathbb{R}^n$ and that of their symbols, when both are considered as modules over an imbedding of $sl(n+1, \mathbb{R})$ into polynomial vector fields.

The coefficients of the bidifferential operators are densities of an arbitrary weight. We obtain the result for all values of this weight, except for a set of critical ones, which does not contain 0. In the case of second order operators, we give explicit formulas and examine in detail the critical values.

1. Introduction

The spaces of linear differential operators acting on tensor densities over a smooth manifold $M$ naturally constitute representations of its Lie algebra of vector fields. Let $\mathcal{D}_\lambda(M)$ denote the module of differential operators acting on densities of weight $\lambda$. The classification of these modules — regarding $\lambda$ as a parameter — was recently obtained (see [2, 4, 9]).

A new and fruitful approach to this classification consists in comparing the action of vector fields on differential operators and on their symbols. This method is exposed and applied to the classification of quotient modules in [6]. One proves the existence and takes advantage of the properties of a projectively equivariant symbol map

$$\sigma_\lambda : \mathcal{D}_\lambda(\mathbb{R}^n) \to \bigoplus_{k \geq 0} \Gamma(\bigwedge^k T\mathbb{R}^n)$$

which has the following properties. It is the unique isomorphism of representations of the projective Lie algebra to preserve the principal symbol of its arguments. The latter algebra is the subalgebra of polynomial vector fields generated by

$$\frac{d}{dx^i}, x^j \frac{d}{dx^i}, \text{ and } x^j \sum_i \frac{d}{dx^i}, \quad i, j \in \{1, \ldots, n\}. \quad (1)$$

We shall denote it by $sl_{n+1}$ since it is isomorphic to $sl(n+1, \mathbb{R})$ (see for instance [3, p. 6]).

It is worth noticing that the inverse function of $\sigma_\lambda$ maps functions on the cotangent bundle of the Euclidean space that are polynomial on the fibre
which are nothing but symmetric contravariant tensor fields — onto differential operators. It may therefore be viewed as a quantization procedure (cf. 2 and references therein) that preserves the infinitesimal symmetries of the base space. It may also be used to define equivariant star-products. Two further steps are made in 1. On the one hand, the subalgebra prescribing the infinitesimal invariance is there changed to be made of conformal transformations; on the other hand, a wider class of differential operators with densities of an arbitrary weight as coefficients is studied.

This paper is intended to be a first step towards the classification of modules of bilinear bidifferential operators. It is clearly a natural approach to wonder about the existence of an equivariant symbol map for these. However, computing an analogue of $\sigma_\lambda$ using the prescription of equivariance turns out to be more intricate than in the linear case. Therefore, we have used techniques from the “conformal” case (cf. 1) and collected interesting intermediate results about the structure of the considered modules, such as their Casimir operators and the spectrum of these.

These aspects lead us towards a more algebraic study of modules of bidifferential operators over tensor densities. It is worth noticing that these have already been studied, see for instance 5.

Our paper is organized as follows. In Section 2, we recall basic definitions and choose notations. In Section 3, we first compute the Casimir operators of $sl_{n+1}$ acting on multidifferential operators and tensor fields, and then determine the spectrum of the latter when specialized to bidifferential operators. We show in Section 4 that the former has the same spectrum, which allows to define an equivariant symbol map. This is possible due to a strong assumption on the weight of the coefficients of the operators, which we shall call shift. We next weaken this assumption and give a lower bound function for the remaining critical shift values. Section 5 is devoted to an explicit computation of the symbol map restricted to second order operators. This example demonstrates that when the shift is critical, a symbol map may either not exist or not be unique.

2. Definitions

2.1. Densities. Let $\lambda$ be a real number. A density of weight $\lambda$ on $\mathbb{R}^n$ is a function $\phi : \Lambda^n T\mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ such that

$$\phi(t \omega) = |t|^\lambda \phi(\omega), \quad \forall t \in \mathbb{R} \setminus 0, \forall \omega \in \Lambda^n T\mathbb{R}^n \setminus \{0\}.$$  

Let $E^\lambda$ denote the one-dimensional vector space of densities of weight $\lambda$ on $\mathbb{R}^n$.

For the sake of simplicity, we shall also call densities the sections of $\Delta^\lambda(\mathbb{R}^n) = \mathbb{R}^n \times E^\lambda$ viewed as a fibre bundle over $\mathbb{R}^n$. 
The pull-back of $\phi$ along the flow of a vector field $X$ allows to define the Lie derivative of $\phi$ in the direction of $X$, yielding the expression:

$$L_X \phi = \sum_i X^i \partial_i \phi + \lambda \sum_i \partial_i X^i \phi.$$  (2)

We shall write $\partial_i$ for both the derivative and the unit vector along the $i$th axis.

2.2. Multidifferential operators. Let $\mathcal{M}_{\text{diff}}^p(\lambda, \mu)$ be the space of multi-linear multidifferential operators transforming $p$ densities of weights $\lambda_1, \ldots, \lambda_p$ ($\lambda$ is the multi-index of these weights) into one density of weight $\mu$.

Such an operator is said to have (total) order $r$ if it may be written

$$(f_1, \ldots, f_p) \mapsto \sum A_{\alpha_1,\ldots,\alpha_p} D^{\alpha_1} f_1 \cdots D^{\alpha_p} f_p$$  (3)

with $\sum_{j=1}^p |\alpha^j| \leq r$, where the $\alpha^j$ are multi-indices and

$$D^{\alpha^j} = \partial_{\alpha^j_1} \cdots \partial_{\alpha^j_{\ell_j}}, \quad \forall j, \quad (|\alpha^j| = k).$$

The coefficients $A_{\alpha_1,\ldots,\alpha_p}$ in (3) are densities of weight $\delta = \mu - \sum \lambda_i$.

Throughout this paper, unless otherwise stated, we shall consider $\delta, \lambda$ and $\mu$ as above. We shall moreover refer to $\delta$ as the shift value.

The space of those operators is a module over the Lie algebra $\text{Vect}(\mathbb{R}^n)$ of vector fields over $\mathbb{R}^n$. It is filtered by the order. Indeed, the Lie derivative of an $r$th order operator $T \in \mathcal{M}_{\text{diff}}^p(\lambda, \mu)$ is given by

$$L_X T = L_X \circ T - \sum_{i=1}^p \text{ith arg.} T(\ldots, \overset{i}{L_X}, \ldots),$$  (4)

which is again an $r$th order operator. The Lie derivatives of the right hand side of the latter formula are those defined by (2) with the suitable weight of densities.

2.3. Symbols. The polynomial functions to be associated to those operators, which we also name symbols, are the elements of

$$S^{(p)}_\delta = \bigoplus_{i_1, \ldots, i_p \geq 0} \Gamma(v^{i_1}(T \mathbb{R}^n) \otimes \cdots \otimes v^{i_p}(T \mathbb{R}^n) \otimes \Delta^\delta(\mathbb{R}^n)),$$  (5)

where the final term takes the shift value into account. The Leibniz rule and the Lie derivatives of vector fields and densities endow this space with a natural structure of $\text{Vect}(\mathbb{R}^n)$-module.
2.4. Projectively equivariant symbol map. The association of the tensor field
\[(\xi^1, \ldots, \xi^p) \mapsto A_{\alpha_1, \ldots, \alpha_p}(\xi^1)^{\alpha_1} \cdots (\xi^p)^{\alpha_p} \in \mathcal{S}_\delta^{(p)}\]
to the operator defined in (3) is an isomorphism of vector spaces between \(\mathcal{S}_\delta^{(p)}\) and \(\mathcal{M}_{\text{diff}}^{p}(\lambda, \mu)\). It is well known, and easy to check, that it is not equivariant with respect to \(sl_{n+1}\). Remember that we defined in (1) this subalgebra of vector fields, with respect to which we want to impose equivariance.

The projectively equivariant symbol map we are looking for is a linear bijection
\[\sigma_{\lambda, \mu}: \mathcal{M}_{\text{diff}}^{2}(\lambda, \mu) \to \mathcal{S}_\delta^{(2)}\]
that preserves the principal symbol of its arguments and such that
\[\sigma_{\lambda, \mu}(L_X T) = L_X (\sigma_{\lambda, \mu} T)\]
for all \(T \in \mathcal{M}_{\text{diff}}^{2}(\lambda, \mu)\) and \(X \in sl_{n+1}\).

2.5. Polynomial form. Though it is not equivariant, the symbolic notation (6) of a multidifferential operator is very useful to handle computations with operators that might otherwise become tedious. Moreover, one can take advantage of the following rule, "à la Fourier". For the sake of simplicity, assume that \(D\) is a differential operator, whose symbolic notation is the polynomial function \(\xi \mapsto P(\xi)\), in which \(\xi\) represents the derivatives in \(D\) that affect its argument, say \(f\). Then, if \(X\) is a vector field, the operator
\[f \mapsto D(\sum X^i \partial_i f)\]
admits the polynomial function \(\xi \mapsto P(\xi + \eta) < X, \xi >\) as its symbolic notation, if one agrees on denoting by \(\eta\) the derivatives affecting \(X\). This provides not only a convenient notation for the action of \(D\) and \(X\), but also an easy way to keep track of the applications of derivatives on products. Another example is the symbolic form of the Lie derivative of a multidifferential operator (7) below.

3. Casimir operators

In this section, we shall compute the Casimir operators \(\mathcal{C}^{op}\) of \(\mathcal{M}_{\text{diff}}^{p}(\lambda, \mu)\) and \(\mathcal{C}^{t}\) of \(\mathcal{S}_{\delta}^{(p)}\) as representations of the projective Lie algebra.

Let the derivatives of \(T \in \mathcal{M}_{\text{diff}}^{p}(\lambda, \mu)\) be represented by the covariant variables \(\xi^1, \ldots, \xi^p\). Define
\[\tau_i \zeta T = T(\ldots, \zeta + \xi^i, \ldots) - T.\]
If \( \eta \) denotes the derivatives affecting the coefficients of \( T \) and if \( \zeta \) is similarly related to a vector field \( X \), then one may write the symbolic form of the Lie derivative of \( T \) in the direction of \( X \):

\[
L_X T = \langle X, \eta \rangle T - \sum_{i=1}^{p} \xi^i T + \delta \langle X, \zeta \rangle T.
\]  
(7)

This results of a direct computation from the definition (4) of the Lie derivative on \( \mathcal{M}_{\text{diff}}^p(\lambda, \mu) \).

If \( P \) is the polynomial form of \( T \) then

\[
L_X P = \langle X, \eta \rangle P - \sum_{i=1}^{p} \langle X, \xi^i \rangle (\zeta D_{\xi^i})P + \delta \langle X, \zeta \rangle P.
\]  
(8)

To achieve the computation of the Casimir operators, we fix two bases of \( \mathfrak{sl}_{n+1} \) dual to each other with respect to some non-degenerate symmetric bilinear form (and therefore a multiple of the Killing form of the algebra). We shall consider the vector fields:

\[
\begin{align*}
e_{ij} & = -x^i \partial_j, & e^*_{ij} & = e_{ji}, \quad (i \neq j) \\
e_{ii} & = -x^i \partial_i - \sum_k x^k \partial_k \\
e_i & = -\partial_i \\
e^i & = x^i \sum_k x^k \partial_k \\
\end{align*}
\]  
(9)

for \( i, j \in \{1, \ldots, n\} \), where \( e_{ij} \) and \( e^*_{ij} \) are dual, and so on.

**Proposition 1.** The Casimir operator

\[
C^{\text{op}} = \sum_{i \neq j} L_{e_{ij}} \circ L_{e^*_{ij}} + \sum_i (L_{e_{ii}} \circ L_{e^*_i} + L_{e^*_i} \circ L_{e_i} + L_{e_i} \circ L_{e^*_i})
\]  
(10)

of \( \mathfrak{sl}_{n+1} \) acting on \( \mathcal{M}_{\text{diff}}^p(\lambda, \mu) \) equals the sum

\[
C^{\text{op}} = C^t + N_C
\]

where

\[
C^t = n(n + 1)\delta(\delta - 1)\text{id} + 2((n + 1)(1 - \delta)) \sum_k E_{\xi^k}
\]

\[
+ \sum_{k,l} \sum_{i,j} \xi^k \xi_j D_{\xi^i} D_{\xi^l} + \sum_{k,l} \sum_{i,j} \xi^k \xi^l D_{\xi^i} D_{\xi^j} D_{\xi^l}
\]

is the Casimir operator of \( \mathfrak{sl}_{n+1} \) acting on \( S_5^{(p)} \) and

\[
N_C = 2 \sum_k (E_{\xi^k} + (n + 1)\lambda_k)(\eta D_{\xi^k}),
\]

if \( \eta \) denotes the partial derivatives of the coefficients of an argument of \( C^{\text{op}} \) and \( E_{\xi^k} = \sum_i \xi^k D_{\xi^i} \).
Applying the formulas for Lie derivatives once again, we get we thus only need to collect and sum terms with constant coefficients in (10). and has therefore constant coefficients. To compute the Casimir operator, Summing these terms gives the announced expression.

\[ L_{e_i} T = -x^i \partial_i T + \sum_{k=1}^n E_x \frac{\partial}{\partial x^k} D_{\xi^k_i} T + \sum_k (E_x D_{\xi^k_i} T - \delta(n+1)T) \]

\[ L_{e_i} T = -\partial_i T \]

\[ L_{e_i^*} T = -x^i E_T + \delta(n+1)x^i T \]

where \( E = \sum_i x^i \partial_i \).

The Casimir operator intertwines \( \mathcal{M}_{\text{diff}}^p(\lambda, \mu) \) with itself as a module over \( sl_{n+1} \). It is in particular invariant with respect to the constant vector fields and has therefore constant coefficients. To compute the Casimir operator, we thus only need to collect and sum terms with constant coefficients in (10). Applying the formulas for Lie derivatives once again, we get

\[
\begin{align*}
L_{e_{ij}} \circ L_{e_{ij}} T & : \sum_{i \neq j} (\sum_k \xi^k\xi^j D_{\xi^k_i} D_{\xi^j_i} T), \\
L_{e_{ii}} \circ L_{e_{ii}} T & : (2 - 2(n+1)\delta) \sum_k E_x T \\
& + \sum_{i} \sum_{k,l} \xi^i_k \xi^l_j D_{\xi^k_i} D_{\xi^l_i} T \\
& + \sum_{i} \sum_{k,l} \xi^i_k \xi^j_l D_{\xi^k_i} D_{\xi^l_i} T + n(n+1)\delta^2 T, \\
L_{e_{ij}} \circ L_{e_{ij}} T & : \sum_k (E_x + (n+1)\lambda_k)(\eta D_{\xi^k_i}) T \\
& - \eta(n+1)\delta T + (n+1) \sum_k E_x T \\
L_{e_{ij}} \circ L_{e_{ij}} T & : \sum_k (E_x + (n+1)\lambda_k)(\eta D_{\xi^k_i}) T.
\end{align*}
\]

Summing these terms gives the announced expression. \( \square \)

### 3.1. Spectrum of \( \mathcal{C}^t \)

We shall now deal with bidifferential operators. Instead of \( \xi^1 \) and \( \xi^2 \), we shall write \( \alpha \) and \( \beta \) the covariant variables symbolizing their partial derivatives.

From now on, we shall also assume that the dimension \( n \) of the Euclidean space is at least 2. One will find a discussion of the case \( n = 1 \) in Section 4.4.

We are now going to split \( S_\lambda^{(2)} \) into spaces of homogeneous eigenvectors of \( \mathcal{C}^t \). Observe first that the Lie derivatives in direction of linear vector fields only contribute to the symbolic part of \( \mathcal{C}^{op} \).

Moreover, for any such vector field \( X \),

\[ L_X \circ L_X^* = \rho(DX) \circ \rho(DX^*), \]

where \( \rho \) denotes the natural representation of \( gl(n, \mathbb{R}) \) on \( S_\lambda^{(2)} \). Together with expression (11), this equality shows that on each space \( \mathbb{V}^k T\mathbb{R}^n \otimes \mathbb{V}^l T\mathbb{R}^n \otimes \Delta^\delta(\mathbb{R}^n) \), \( \mathcal{C}^t \) can be written

\[ \mathcal{C}_{gl(n, \mathbb{R})} + r \text{id} \quad (r \in \mathbb{N}), \]

where the symbol \( \text{id} \) represents the identity mapping.
where $C_{gl(n, \mathbb{R})}$ denotes the Casimir operator associated to the representation $\rho$ above on the space $\sqrt{k}T^*\mathbb{R}^n \otimes \sqrt{l}T^*\mathbb{R}^n$. Its irreducible $gl(n, \mathbb{R})$-submodules will therefore be made up of eigenvectors of $C_t$.

It is well known that, for $k, l \in \mathbb{N}$,

$$\sqrt{k}T^*\mathbb{R}^n \otimes \sqrt{l}T^*\mathbb{R}^n = \bigoplus_{q=0}^{\min(k, l)} V_{k, l, q}$$

where $V_{k, l, q}$ is an irreducible $gl(n, \mathbb{R})$-submodule described by a Young tableau with two lines of respective lengths $k + l - q$ and $q$. It is easy to check that

$$\vec{v}_H(k, l, q) : \alpha \otimes \beta \in \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \mapsto (\alpha_1\beta_2 - \alpha_2\beta_1)^q\alpha^{k-q}_{1}\beta^{l-q}_{1}$$

is a highest weight vector of $V_{k, l, q}$. We shall denote the space

$$\Gamma(V_{k, l, q} \otimes \Delta^\delta \mathbb{R}^n)$$

by $S_{k, l, q}$, whenever $0 \leq q \leq \min(k, l)$, and take this to be $\{0\}$ otherwise.

**Lemma 2.** The restriction of $C_t$ to $S_{k, l, q}$ equals

$$n(n + 1)\delta(\delta - 1) - 2((n + 1)\delta - n + q)(k + l) + 2(k + l)^2 + 2q(q - 1)$$

times the identity.

**Proof.** Just evaluate $C_t$ on $\vec{v}_H(k, l, q)$. \hfill \Box

Depending only on the shape of the Young tableau, this value contains $k$ and $l$ through their sum. Let us denote it by $\gamma_{k+l,q}$. Notice that $\gamma_{i,p} = \gamma_{i,p'}$ only if $p = p'$. We can thus gather eigenspaces of same eigenvalue and total tensor degree. We define

$$S_{(i,p)} = \bigoplus_{k=p}^{i-p} S_{k,i-k,p}.$$

We have just shown how each element of $S^{(2)}_\delta$ writes as a sum of homogeneous eigenvectors of $C_t$:

$$S^{(2)}_\delta = \bigoplus_{i \geq 0} \bigoplus_{0 \leq p \leq \lfloor i/2 \rfloor} S_{(i,p)},$$

if $\lfloor x \rfloor$ is the highest natural number less than or equal to $x$.

4. **Equivariant symbol map for bidifferential operators**

Our method for building a projectively equivariant symbol map relies on the comparison of the spectra of $C_t$ and $C^{op}$. For the sake of convenience, we shall first do this comparison, and show how it leads to a symbol map, under a technical hypothesis which we shall then weaken.

Let us name order the index $i$ of an eigenvalue $\gamma_{i,p}$ of $C_t$. Lemma 2 shows that the shift $\delta$ is fixed by any equality of two eigenvalues with different orders.
Definition 1. The value of the shift $\delta$ is said to be resonant if there exist $i, p, j, q \in \mathbb{N}$ satisfying $p \leq \lfloor i/2 \rfloor, q \leq \lfloor j/2 \rfloor$ and $i > j$ such that the two eigenvalues $\gamma_{i,p}$ and $\gamma_{j,q}$ of $C^l$ are equal.

A resonant value will then be denoted by $\delta_{i,p,j,q}$ for a suitable choice of the indices.

4.1. Symbol map when the shift is not resonant.

Lemma 3. Assume that $\delta$ is not resonant. Then

1. $C^l$ and $C^{op}$ have the same spectrum;
2. each eigenvector $P$ of $C^l$ belongs to some space $S_{(i,p)}$;
3. such a $P$ is the principal symbol of a unique eigenvector $Q(P)$ of $C^{op}$, which is associated to the same eigenvalue.

Proof. Let first $P \in S^{(2)}_\delta$ be such that $C^l P = \gamma_{i,p} P$. Since $\delta$ is not resonant and $p \mapsto \gamma_{i,p}$ is injective, $P \in S_{(i,p)}$.

Assume now that $P$ is an eigenvector of $C^{op}$ of eigenvalue $\gamma$ and that $P$ has total order $i$. Let us rewrite the condition

$$C^{op} P = \gamma P$$

according to the decomposition $P = \sum_{j,q} P_{j,q}$ with $P_{j,q} \in S_{(j,q)}$ and $P_j = \sum_q P_{j,q}$. We get

$$\begin{cases} \sum_{q \leq \lfloor i/2 \rfloor} (\gamma - \gamma_{i,q}) P_{i,q} = 0 \\
\sum_{q \leq \lfloor j/2 \rfloor} (\gamma - \gamma_{j,q}) P_{j,q} = N \epsilon P_{j+1} & j = 0, \ldots, i-1.
\end{cases}$$

The first equation shows that there exists one and only one $q_0$ such that $\gamma = \gamma_{i,q_0}$, and that $P_i = P_{i,q_0}$ is an eigenvector of $C^l$. The second, that $P$ is uniquely determined by $P_{i,q_0}$.

We may thus define $Q(P_{i,q_0}) = P$. The value of the map $Q$ is clearly well-defined on any eigenvector of $C^l$, hence the proof.

The correspondence $Q$ will now be considered as linearly extended to $S^{(2)}_\delta$.

Lemma 4. If $\delta$ is not resonant, then $Q$ is $sl_{n+1}$-equivariant.

Proof. Let $P \in S_{(i,p)}$. It suffices to prove that

$$Q(L_X P) = L_X (Q(P))$$

for any $X \in sl_{n+1}$. Since $L_X P$ also belongs to $S_{(i,p)}$, both members of this equality are eigenvectors of eigenvalue $\gamma_{i,p}$ of $C^{op}$. They have the same principal symbol and are thus equal.

Theorem 5. If $\delta$ is not resonant, there exists a unique $sl_{n+1}$-equivariant symbol map on $\mathcal{M}_\text{diff}^2(\lambda, \mu)$. It is a differential operator.
Proof. Taking account of the preceding lemmas, we just have to show that the map \( Q \) is such that \( Q^{-1} \) is the unique equivariant symbol map and that it is bijective.

On the one hand, any analogue of \( Q \) prolongs an eigenvector of \( C^t \) into an eigenvector of \( C^{op} \) with the same eigenvalue and principal symbols, and hence equals \( Q \).

Now, if \( Q(P) = 0 \), the principal symbol of \( P \), and therefore \( P \), equal 0. If \( D \in \mathcal{M}_{2\times2}^{2\times2}(\lambda, \mu) \) admits \( P \) as its principal symbol, then \( Q(P) = 0 \) has an order less than that of \( D \) and an evident induction allows to conclude that \( Q \) is surjective.

4.2. Critical shift values. A resonant value of the shift does not automatically prevent us from extending eigenvectors of \( C^t \) into eigenvectors of \( C^{op} \). Consider indeed \( P_{i,p} \in S_{(i,p)} \) and assume that \( \gamma_{j,q} = \gamma_{i,p} \) for some \( j < i \). The equation (14) shows that one will still be able to prolong \( P \) in an eigenvector of \( C^{op} \) if the component of \( N_C P_{j+1} \) in \( S_{(j,q)} \) vanishes.

We thus want to describe the spaces of homogeneous eigenvectors of \( C^t \) actually reached through the application of \( N_C \) on one of them.

Lemma 6. If \( k, l, q \in \mathbb{N} \) are such that \( q \leq \min(k, l) \),

\[
N_C(S_{k,l,q}) \subset S_{k-1,l,q-1} \oplus S_{k,l,q-1} \oplus S_{k-1,q-1} \oplus S_{k,l-1,q}.
\]

Hence, for \( i, p \in \mathbb{N} \) such that \( p \leq [i/2] \),

\[
N_C(S_{(i,p)}) \subset S_{(i-1,p)} \oplus S_{(i-1,p)}.
\]

Proof. Notice first that

\[
(\eta D_\alpha)\bar{v}_H(k,l,q) = q\eta_1(\alpha_2 \beta_2 - \alpha_2 \beta_1)^{q-1} \alpha_1^{k-2q} \beta_1^{l-q} \beta_2
\]

\[
- q\eta_2(\alpha_1 \beta_2 - \alpha_2 \beta_1)^{q-1} \alpha_1^{k-2q} \beta_1^{l-q+1}
\]

\[
+ (k-q)\eta_1(\alpha_1 \beta_2 - \alpha_2 \beta_1)^{q-1} \alpha_1^{k-q} \beta_1^{l-q}.
\]

The second and third terms are respectively multiples of \( \bar{v}_H(k-1,l,q-1) \) and \( \bar{v}_H(k-1,l,q) \), and the first equals

\[
\frac{1}{k+l+1-2q}(\rho(e_2 \otimes e_1)\bar{v}_H(k-1,l,q-1) - (k-q)\bar{v}_H(k-1,l,q)),
\]

where \( \rho(e_2 \otimes e_1) = \alpha_2 D_{\alpha_1} + \beta_2 D_{\beta_1} \). Therefore,

\[
(\eta D_\alpha)\bar{v}_H(k,l,q) \in S_{k-1,l,q-1} \oplus S_{k-1,l,q}.
\]

Let \( P \in S_{k,l,q} \). As

\[
(\eta D_\alpha)\rho(h \otimes \zeta)P = \langle h, \eta \rangle \langle \zeta D_\alpha \rangle P + \rho(h \otimes \zeta)(\eta D_\alpha)P,
\]

we observe that \( (\eta D_\alpha)P \) also belongs to \( S_{k-1,l,q-1} \oplus S_{k-1,l,q} \). The result then follows from the definitions of \( N_C \) and \( S_{(i,p)} \).

Remember that the value of the shift \( \delta \) is resonant if there exist \( i, p, j, q \in \mathbb{N} \) satisfying the following conditions:

1. \( p \leq [i/2] \),
2. \( q \leq \lfloor j/2 \rfloor \),
3. \( i > j \),
4. \( \gamma_{i,p} = \gamma_{j,q} \).

Lemma 3 leads us to the following definition.

**Definition 2.** A resonant value \( \delta \) is *critical* if it may be written \( \delta_{i,p;j,q} \) with \( i, p, j, q \) satisfying

5. \( 0 \leq p - q \leq i - j \)

in addition to conditions (1)–(4) above.

Before proving existence and uniqueness of a projectively equivariant sym-
bolization when \( \delta \) is resonant but not critical, we shall examine these values
somewhat more precisely.

A shift of 0 occurs in the most “natural” uses of densities: when the
weight of each considered density is 0, which means the multidifferential op-
erators act on functions, and when \( \mu = \sum \lambda_i \), which is required to consider
the natural product of densities. But 0 is a resonant value of \( \delta \), whatever
the dimension \( n > 1 \). Notice indeed that if \( \delta = 0 \) and \( k = n - 1 \), then
\( \gamma_{6+k,3} = \gamma_{5+k,0} \).

These remarks explain why we wanted to prove that zero is not a critical
value of the shift. One has the following stronger property.

**Proposition 7.** Let \( f : \mathbb{N} \setminus \{0\} \to \mathbb{Q} : i \mapsto \delta_{i,[i/2];0,0} \). Then

- \( f(1) = 1 \),
- \( f \) is increasing and
- \( \lim_{i \to +\infty} f(i) = +\infty \).

Moreover, if \( i, j, p, q \in \mathbb{N} \) satisfy conditions (1)–(5) above, then

\[ \delta_{i,p;j,q} \geq f(i). \]

**Proof.** The first three assertions are evident since

\[
\delta_{2k,k;0,0} = 1 + \frac{3(k-1)}{2(n+1)}, \quad \forall k \in \mathbb{N} \setminus \{0\}
\]

and \( \delta_{2k+1,k;0,0} = 1 + \frac{3k^2}{(2k+1)(n+1)}, \quad \forall k \in \mathbb{N} \).

To prove the stated inequality, observe first that \( \delta_{i,p;j,q} > \delta_{i,p';j,q} \) if \( p < p' \leq \lfloor i/2 \rfloor \). Indeed, one has then

\[
\delta_{i,p;j,q} - \delta_{i,p+1;j,q} = \frac{i - 2p}{i - j} > 0.
\]
Now, if $|i/2| > i - j + q$ then
\[
\delta_{i,p;j,q} \geq \delta_{i,i-j+q,j,q} \\
= \delta_{i,i-j+q,0,0} + \frac{1}{(n+1)i} (j(i-j-1)+q(2j-q)+q) \\
\geq \delta_{i,i-j+q,0,0} \\
\geq \delta_{i,|i/2|,0,0}.
\]

Else, suppose that $i = 2k + 1$ and notice that $k - j + q \geq 0$. One can write
\[
\delta_{i,p;j,q} \geq \delta_{2k+1,k;j,q}.
\]

Moreover,
\[
(\delta_{2k+1,k;j,q} - \delta_{2k+1,k,0,0})(2k+1)(2k+1-j)(n+1) = 3k^2j + 2kjq - 2kj^2 - 2kq^2 + 2kj + 2jq - j^2 + 2kq - q^2 + j + q.
\]

Denote by $a_r(k,j,q)$ the sum of the homogeneous monomials of degree $r$ in the latter expression. It is clear that $a_2(k,j,q) \geq 0$. Furthermore, if $q \leq k - 2$,
\[
a_3(k,j,q) + j = k((k-2)j + 2(k+1-j+q)(j-2q^2) + j \\
\geq k((k-2)j - 2q^2) + j \\
\geq j(k(2k-q) + 1) \\
\geq 0.
\]

Since $2q \leq j \leq 2k$, it suffices to compute explicitly the different values of $a_3(k,j,q) + j$ for $k - 1 \leq q \leq k$ to prove that $\delta_{i,p;j,q} \geq \delta_{i,|i/2|,0,0}$ for any odd natural number $i$.

One proceeds in the same way if $i = 2k$ and $k - j + q \geq 0$. \qed

**Corollary 8.** The number of critical values of the shift $\delta$ contained in any interval $[a, b]$ is finite and vanishes if $b < 1$.

We shall eventually show (see Section 5.2) that 1 is a critical value, for which there exists an equivariant symbol map if and only if $\lambda = (0, 0)$. Furthermore, it suffices to examine the case of second order operators to prove this.

4.3. **Symbol map for resonant values of $\delta$.** Let now $\delta$ be resonant but not critical and let $P \in \mathcal{S}(i,p)$. We can still define $Q(P)$, the unique eigenvector of $C^{op}$ admitting $P$ as its principal symbol and belonging to
\[
\tilde{\mathcal{S}}(i,p) = \bigoplus_{j,q: 0 \leq p-q \leq i-j} \mathcal{S}(j,q).
\]

**Proposition 9.** The map $Q$, linearly extended to $S^{(2)}_\delta$, is equivariant.
Lemma 10. Let $i, p \in \mathbb{N}$ be such that $p \leq \lfloor i/2 \rfloor$ and $X \in \mathfrak{sl}_{n+1}$. If the polynomial form of an operator $T$ belongs to $\widetilde{S}_{(i,p)}$, then so does the polynomial form of $L_X T$.

Proof of lemma 10. Assume that $P \in S_{k,l,p}$, $(k + l = i)$, is the polynomial form of $T$. As it can be seen from expressions (7) and (8), the polynomial form of $L_X T$ differs from $L_X P$ by

$$- \lambda_1 < X, \zeta > (\zeta D_\alpha)P - \lambda_2 < X, \zeta > (\zeta D_\beta)P$$

$$- \frac{1}{2} < X, \alpha > (\zeta D_\alpha)^2 P - \frac{1}{2} < X, \beta > (\zeta D_\beta)^2 P.$$

This quantity vanishes if the degree of $X$ does not exceed one. We have shown in the proof of Lemma 6 that both $< X, \zeta > (\zeta D_\alpha)P$ and $< X, \zeta > (\zeta D_\beta)P$ belong to $\widetilde{S}_{(i,p)}$ for any vector field $X \in \mathfrak{sl}_{n+1}$. When $X = \theta^*$, the last two terms equal $-k(\theta D_\alpha)P - l(\theta D_\beta)P$ and thus also belong to $\widetilde{S}_{(i,p)}$. \qed

Proof of proposition 9. Proceed as when $\delta$ is not resonant, noticing that both members of the equality

$$L_X Q(P) = Q(L_X P)$$

belong to $\widetilde{S}_{(i,p)}$ provided that $P \in S_{(i,p)}$. \qed

To prove the uniqueness of the symbol map, we must ensure that the prolongation of $P \in S_{(i,p)}$ into an eigenvector of $C^{op}$ does not get out of $\widetilde{S}_{(i,p)}$. In the statement of the next proposition, notice that tensor fields and bidifferential operators are isomorphic modules over the constant and linear vector fields.

Proposition 11. Any linear map $T : S^{(2)}_\delta \to \mathcal{M}^2_{\text{diff}}(\lambda, \mu)$ that is equivariant with respect to the constant and linear vector fields stabilizes each space $\widetilde{S}_{(i,p)}$.

Proof. A slight adaptation of the proof of theorem 5.1 of [6] shows that such a map is local. Being invariant under the action of constant vector fields, it is thus a differential operator with constant coefficients. The invariance of $T$ with respect to the linear vector fields means that the polynomial form of $T$ maps

$$X^r \otimes Y^s \quad (X, Y \in \mathbb{R}^n)$$

onto a tensor field with homogeneous component in $\vee^p T\mathbb{R}^n \otimes \vee^q T\mathbb{R}^n$ given by

$$\begin{align*}
(\alpha, \beta) \mapsto & \sum_{r_1, r_2} c_{r_1, r_2} < X, \alpha >^{r_1} < X, \beta >^{r_2} < X, \eta >^{r-(r_1+r_2)} \\
& \times < Y, \alpha >^{p-r_1} < Y, \beta >^{q-r_2} < Y, \eta >^{s-(p+q)+(r_1+r_2)}
\end{align*}$$

where $\eta$ symbolizes the derivatives of the argument of $T$. It means that the polynomial symbolization of $T$ is a linear combination of powers of the operators $\alpha D_\beta, \beta D_\alpha, \eta D_\alpha$ and $\eta D_\beta$. 


But the last two stabilize the space $\widetilde{S}_{(i,p)}$, because of the definition of this space. Furthermore, $\alpha D_\beta$ and $\beta D_\alpha$ vanish or intertwine irreducible $sl(n,\mathbb{R})$-submodules of the spaces $S_{(i,p)}$. Hence the conclusion.

4.4. The one-dimensional case. Assume that $n = 1$. The formula (10) still holds. One sees easily that the spaces

$$\bigoplus_{k+l=i} \Gamma(\bigotimes^k T\mathbb{R} \otimes \bigotimes^l T\mathbb{R} \otimes \Delta^\delta \mathbb{R})$$

are made of homogeneous eigenvectors of $C^t$ which are associated to the eigenvalue $\gamma_{i,0}$ given by Lemma 2 when $n = 1$ and $q = 0$.

The symbol map may be built as before when the shift is not resonant (i.e. when no eigenvalues $\gamma_{i,0}$ and $\gamma_{j,0}$, $(i \neq j)$, are equal). Moreover, $\gamma_{i,0} = \gamma_{j,0}$ if and only if $\delta = 1 + \frac{i+j-1}{2}$. These are the resonant values of the shift. They are all critical in the sense that iterated applications of $N_{C^t}$ on $S_{(i,0)}$ always end up in $S_{(j,0)}$ (cf. Lemma 5).

5. Second order operators

Let us now give an explicit equivariant symbol map example by carrying some computations for second order operators. We shall also observe that a critical shift value may prevent the symbol map from existing or being unique.

Examining the natural action of $sl(n,\mathbb{R})$ on highest weight vectors described in formula (13), one easily checks that

$$S_{(2,0)} = S_{2,0,0} \oplus S_{0,2,0} \oplus S_{1,1,0}$$

and that

$$S_{(2,1)} = S_{1,1,1}.$$  

This means that elements of $S_{(2,0)}$ are symmetric quadratic polynomials of $\alpha$ and $\beta$ while $S_{(2,1)}$ contains the antisymmetric ones.

Shift values for which an equality between the eigenvalues $\gamma_{2,0}$, $\gamma_{2,1}$, $\gamma_{1,0}$ and $\gamma_{0,0}$ of $C^t$ may occur are given in Figure 1. Notice that they are all critical.

| $\gamma_{2,0} = \gamma_{1,0}$ | $\gamma_{2,0} = \gamma_{0,0}$ | $\gamma_{2,1} = \gamma_{1,0} = \gamma_{0,0}$ |
|-----------------------------|-----------------------------|----------------------------------|
| $\delta = \frac{n+1}{n+1}$  | $\delta = \frac{n+2}{n+1}$  | $\delta = 1$                      |

Figure 1. Resonant (and critical) shift values for second order operators
5.1. **Generic shift values.** Assume that $\delta$ takes none of the above listed values. Using notations of the equations (14), we want first to associate to the monomial $P_2 = \alpha_i \alpha_j$ lower degree tensors $P_1$ and $P_0$ such that $P_2 + P_1 + P_0$, considered as a bidifferential operator, is an eigenvector of $C^{op}$. The sought terms $P_1$ and $P_0$ are the only solutions of the system

\[
\begin{align*}
(\gamma_{2,0} - \gamma_{1,0})P_1 &= 2((n + 1)\lambda_1 + 1)(\eta D\alpha)P_2 \\
(\gamma_{2,0} - \gamma_{0,0})P_0 &= 2(n + 1)\lambda_1(\eta D\alpha)P_1
\end{align*}
\]

and the inverse of the symbol map will thus associate to the tensor field $c \alpha_i \alpha_j$, where $c$ belongs to $\Gamma(\Delta^\delta \mathbb{R}^n)$, the operator

\[
(f, g) \mapsto c \partial_i \partial_j fg + \frac{(n + 1)\lambda_1 + 1}{(n + 1)(1 - \delta) + 2} (\partial_i fc \partial_j fg + \partial_j fc \partial_i fg) + \frac{(n + 1)\lambda_1 + 1}{(n + 1)(1 - \delta) + 2}(\partial_i fc \partial_j fg + \partial_j fc \partial_i fg) \partial_i \partial_j c fg.
\]

(15)

Exchanging $f$ and $g$ and substituting $\lambda_2$ to $\lambda_1$ in the last operator gives the one to be associated to $c \beta_i \beta_j$. Using the same notations, it is easy to compute that one should associate to the tensor field $c (\alpha_i \beta_j + \alpha_j \beta_i)$ the operator

\[
(f, g) \mapsto c (\partial_i f \partial_j g + \partial_j f \partial_i g)
\]

\[
+ \frac{(n + 1)\lambda_1}{(n + 1)(1 - \delta) + 2}(\partial_i fc \partial_j fg + \partial_j fc \partial_i fg) + \frac{(n + 1)\lambda_2}{(n + 1)(1 - \delta) + 2}(\partial_i fc \partial_j fg + \partial_j fc \partial_i fg)
\]

\[+ 2\frac{(n + 1)^2 \lambda_1 \lambda_2}{((n + 1)(1 - \delta) + 2)((n + 1)(1 - \delta) + 1)} \partial_i \partial_j c fg,
\]

(16)

and

\[
(f, g) \mapsto c (\partial_i f \partial_j g - \partial_j f \partial_i g)
\]

\[
+ \frac{\lambda_1}{1 - \delta}(\partial_i fc \partial_j fg - \partial_j fc \partial_i fg) + \frac{\lambda_2}{1 - \delta}(\partial_j fc \partial_i fg - \partial_i fc \partial_j fg)
\]

(17)

(18)

to the tensor field $c (\alpha_i \beta_j - \alpha_j \beta_i)$.

To degree 1 tensors

\[
c \alpha_i,
\]

one associates
\( (f, g) \mapsto c \partial_i f g + \frac{\lambda_1}{1 - \delta} \partial_i c f g \) (19)

and similarly, substituting \( \lambda_2 \) to \( \lambda_1 \), to degree 1 tensors in \( \beta \).

Notice that tensors of degree 0 and bidifferential operators of order 0 make up isomorphic \( \text{Vect}(\mathbb{R}^n) \)-modules.

For each generic value of \( \delta \), we have thus described a linear bijection between \( S_{(2,0)} \oplus S_{(2,1)} \oplus S_{(1,0)} \oplus S_{(0,0)} \) and the submodule of \( \mathcal{M}_\text{diff}^2(\lambda, \mu) \) made up of operators with total order not greater than 2. This bijection is the only one to be projectively equivariant and to preserve the principal symbol of its arguments.

5.2. Critical shift values. Assume that \( \delta = \frac{n+3}{n+1} \). Since \( \gamma_{2,0} = \gamma_{1,0} \), the first equation of (15) may be satisfied only if \( \lambda_1 = \frac{-1}{n+1} \); when rewritten for eigenvectors of the form \( \beta_i \beta_j \), this equation forces \( \lambda_2 = \frac{-1}{n+1} \).

Now, it may be seen from expression (17) that no eigenvector of \( \mathcal{C}^{\text{op}} \) admits \( \alpha_i \beta_j + \alpha_j \beta_i \) as its principal symbol, since this would require \( \lambda_1 \) or \( \lambda_2 \) to vanish.

We have just proved the following statement.

**Proposition 12.** For all values of \( \lambda \) and \( \mu \) such that \( \mu - \lambda_1 - \lambda_2 = \frac{n+3}{n+1} \), there exists no sl\(_{n+1}\)-equivariant symbol defined on \( \mathcal{M}_\text{diff}^2(\lambda, \mu) \).

**Proposition 13.** If \( \delta = \frac{n+2}{n+1} \), there exists a projectively equivariant symbol for bidifferential second order operators of \( \mathcal{M}_\text{diff}^2(\lambda, \mu) \) if and only if

\[
\lambda \in \{ (0, \frac{1}{n+1}), (\frac{-1}{n+1}, 0), (0, 0) \},
\]

in which case this symbol is not unique.

**Proof.** As it was done in the case \( \delta = \frac{n+3}{n+1} \), it can be checked that one of the weights \( \lambda_1 \) and \( \lambda_2 \) must vanish and the other be chosen in \( \{0, \frac{-1}{n+1}\} \). A direct computation then allows to prove that the map which associates to \( c \alpha_i \alpha_j \in S_{(2,0)} \) the bidifferential operator

\[
(f, g) \mapsto c \partial_i \partial_j f g + \frac{(n+1)\lambda_1 + 1}{(n+1)(1-\delta) + 2} (\partial_i c \partial_j f g + \partial_j c \partial_i f g) + k \partial_i \partial_j c f g
\]

is projectively equivariant for any value of the parameter \( k \). One defines similarly an operator associated to \( c \beta_i \beta_j \). This equivariance also holds for the map that associates to \( c (\alpha_i \beta_j + \alpha_j \beta_i) \in S_{(2,0)} \) the operator

\[
(f, g) \mapsto c (\partial_i f \partial_j g + \partial_j f \partial_i g) + (n+1)\lambda_1 (\partial_i c f \partial_j g + \partial_j c f \partial_i g) + (n+1)\lambda_2 (\partial_j c \partial_i f g + \partial_i c \partial_j f g) + k \partial_i \partial_j c f g,
\]
whatever the value of $k$. Moreover, any eigenvector of $C^t$ not belonging to $S(2,0)$ is the principal symbol of a unique eigenvector of $C^{op}$.

Finally, when $\delta = 1$, expression (18) shows that both $\lambda_1$ and $\lambda_2$ must vanish. Then, both maps

$$T_1 : c(\alpha_i\beta_j - \alpha_j\beta_i) \in S(2,1) \mapsto ((f,g) \mapsto \partial_i cf\partial_j g - \partial_j cf\partial_i g)$$

and

$$T_2 : c\alpha_i \mapsto ((f,g) \mapsto \partial_i c f g)$$

are $sl_{n+1}$-equivariant (cf. below). Similar maps may be defined, switching the first and second arguments. But we may release the requirement on the operators not to have an order greater than two and state the following proposition.

**Proposition 14.** If $\delta = 1$, there exists a projectively equivariant symbol for the elements of $M^2_{\text{diff}}(\lambda, \mu)$ if and only if $\lambda = (0,0)$, in which case this symbol is not unique.

**Proof.** Equation (14) shows that the prolongation of any eigenvector $P \in S(i,p)$, $(i > 2)$, of $C^t$ will be impossible only if $\gamma_{i,p} = \gamma_{j,q}$ for some $j,q$ such that $i, p, j$ and $q$ define a critical value of the shift. But then, in view of Proposition 5, $\delta = \delta_{i,p,j,q} \geq \delta_{i,\lfloor i/2 \rfloor,0,0} \geq \delta_{3,1,0,0} = \frac{n+2}{n+1}$, hence a contradiction.

5.3. A link with the linear case. We first remark that one can obtain formulas (16), (17) and (19) from the following result, which is an adaptation of [6, Prop. 4.4].

**Proposition 15.** Let $\lambda, \mu \in \mathbb{R}$ be such that $\mu - \lambda \notin \{1, \frac{n+2}{n+1}, \frac{n+3}{n+1}\}$. Denote by $D^2_{\lambda,\mu}$ the $\text{Vect}(\mathbb{R}^n)$-module of linear second order differential operators acting on $\lambda$-densities and valued in $\mu$-densities. Let also $S^{(1)}_j$ denote the module $\Gamma(\mathcal{V}^j T\mathbb{R}^n \otimes \Delta^\mu - \lambda \mathbb{R}^n)$. Then there exists a unique isomorphism of $sl_{n+1}$-modules

$$q_{\lambda,\mu} : \bigoplus_{0 \leq j \leq 2} S^{(1)}_j \rightarrow D^2_{\lambda,\mu}$$

that preserves the principal symbol of its arguments.

Let us define

$$\tau_\alpha : S^{(1)}_2 \rightarrow S^{(1)}_{2,0,0} : P \mapsto ((\alpha, \beta) \mapsto P(\alpha)),$$

$$\tau_\beta : S^{(1)}_2 \rightarrow S^{(1)}_{0,2,0} : P \mapsto ((\alpha, \beta) \mapsto P(\beta))$$

and

$$\tau_{\alpha\beta} : S^{(1)}_2 \rightarrow S^{(1)}_{1,1,0} : P \mapsto ((\alpha, \beta) \mapsto P(\alpha + \beta) - P(\alpha) - P(\beta)).$$

These are isomorphisms of $\text{Vect}(\mathbb{R}^n)$-modules. The operator (18) is then nothing but

$$(q_{\lambda_1,\mu-\lambda_2} \circ \tau_{\alpha}^{-1}(c\alpha_i\alpha_j))f \cdot g.$$
One obtains in a like manner expression (19). Similarly, operator (17) is associated to the polynomial $P = c(\alpha_i\beta_j + \alpha_j\beta_i)$ by the formula

$$(q_{\lambda_1+\lambda_2,\mu} \circ \tau_{\alpha\beta}^{-1}(P))(fg) - (q_{\lambda_1,\mu-\lambda_2} \circ \tau_{\alpha}^{-1}(c\alpha_i\alpha_j))f \cdot g - f \cdot (q_{\lambda_2,\mu-\lambda_1} \circ \tau_{\beta}^{-1}(c\beta_i\beta_j))g.$$

5.4. **Equivariant maps.** Our second remark regards the maps $T_1$ and $T_2$ defined in (20) and (21). They are in fact $\text{Vect}(\mathbb{R}^n)$-equivariant and can be interpreted as follows.

On any oriented manifold $M$ equipped with a nowhere vanishing volume form $\omega$, densities of weight 1 and volume forms make up isomorphic modules over $\text{Vect}(M)$. Any element of $\Gamma(\Lambda^2 TM \otimes \Delta^1 M)$ — the analogue of our space $S(2,1)$ — may be written $\Lambda \omega$ with $\Lambda \in \Lambda^2 TM$. Then in any chart domain where $\omega$ admits the local form $dx^1 \wedge \cdots \wedge dx^n$, $T_1$ is the local form of the globally defined operator

$$\Lambda \omega \mapsto ((f,g) \mapsto f \cdot dg \wedge d(i(\Lambda)\omega))$$

where $f$ and $g$ are smooth functions. Similarly, $T_2$ is the local form of

$$X \omega \mapsto ((f,g) \mapsto L_X \omega \cdot f \cdot g)$$

with $X \in \text{Vect}(M)$.

**Acknowledgements**

We would like to thank M. De Wilde, P. Lecomte and P. Mathonet for helpful suggestions and V. Ovsienko for his interest in this work, particularly during our stay at the C.P.T.-C.N.R.S. in Luminy.

This work was supported by a Research Fellowship of the Belgian National Fund for Scientific Research.

**References**

[1] C. Duval, P. Lecomte and V. Ovsienko, “Conformally equivariant quantization: existence and uniqueness,” Ann. Inst. Fourier (Grenoble) 49 no. 6, 1999–2029 (1999).

[2] C. Duval and V. Yu. Ovsienko, “Space of Second-Order Linear Differential Operators as a Module over the Lie Algebra of Vector Fields”, Advances in Math., 132, 316–333 (1997)

[3] D. B. Fuks, Cohomology of Infinite-Dimensional Lie Algebras, Contemporary Soviet Mathematics, Consultants Bureau, New York and London (1986)

[4] H. Gargoubi and V. Yu. Ovsienko, “Space of linear differential operators on the real line as a module over the Lie algebra of vector fields”, Internat. Math. Res. Notices, 1996, no. 5, 235–251

[5] P. Ja. Grozman, “Classification of bilinear invariant operators on tensor fields” (in Russian), Funktsional. Anal. i Prilozhen., 14 no. 2, 58–59 (1980)

[6] P. B. A. Lecomte and V. Ovsienko, “Projectively equivariant symbol calculus,” Lett. Math. Phys. 49 no. 3, 173–196 (1999)

[7] P. B. A. Lecomte, “On the cohomology of $sl(m + 1)$ acting on differential operators and $sl(m + 1)$-equivariant symbol”, Indag. Math., N. S., 11 (1), 95–114 (2000)

[8] P. B. A. Lecomte, “Classification projective des espaces d’opérateurs différentiels agissant sur les densités”, C. R. Acad. Sci. Paris, t. 328, Série 1, 287–290, (1999)
[9] P. B. A. Lecomte, P. Mathonet and E. Tousset, “Comparison of some modules of the Lie algebra of vector fields”, Indag. Math., N. S., 7 (4), 461–471 (1996)

Université de Liège, Institut de Mathématique, B37, Grande Traverse, 12, B-4000 Sart Tilman (Liège), Belgium

E-mail address: f.boniver@ulg.ac.be