LOCAL WELL-POSEDNESS FOR THE COMPRESSIBLE NAVIER–STOKES–BGK MODEL IN SOBOLEV SPACES WITH EXPONENTIAL WEIGHT

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Abstract. Sprays are complex flows constituted of dispersed particles in an underlying gas. In this paper, we are interested in the equations for moderately thick sprays consisting of the compressible Navier–Stokes equations and Boltzmann BGK equation. Here the coupling of two equations is through a friction (or drag) force which depends on the density of compressible fluid and the relative velocity between particles and fluid. For the Navier–Stokes–BGK system, we establish the existence and uniqueness of solutions in Sobolev spaces with exponential weight.

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1. INTRODUCTION

In the context of sprays consisting of dispersed particles such as droplets, dust, etc., in an underlying gas, a coupling of particles and gas was proposed in [35, 38]. In this modeling, the sprays can be classified depending on the volume fraction of the gas [19, 35], and it has a wide range of applications in the study of biotechnology, bioaerosols in medicine, chemical engineering, combustion theory, etc. We refer to [3, 15, 19, 23, 31, 35] and references therein for more physical background, modeling, and applications of the particle-fluid systems.

Among the various levels of possible descriptions depending on the physical regimes under consideration, in the present work, we concentrate on the so-called moderately thick sprays. In that modeling, the volume fraction of the gas is negligible but the inter-particle interactions, for instances, collision, breakup, coalescences, etc, are taken into account. To be more specific, we are concerned with a coupled particle-fluid system.
where the Boltzmann BGK equation is coupled with the compressible Navier–Stokes equations through a friction (or drag force):

\begin{align}
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (\rho(u - v)f) &= \mathcal{Q}, \\
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \rho - \mu \Delta_x u &= -\rho \int_{\mathbb{R}^3} (u - v)f dv,
\end{align}

subject to initial data:

\begin{equation}
(f(x, v, 0), \rho(x, 0), u(x, 0)) =:(f_0(x, v), \rho_0(x), u_0(x)), \quad (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3,
\end{equation}

where \( f = f(x, v, t) \) denotes the number density of particles of which at time \( t \in \mathbb{R}_+ \) and physical position \( x \in \mathbb{T}^3 \) with velocity \( v \in \mathbb{R}^3 \), and \( \rho = \rho(x, t) \) and \( u = u(x, t) \) are the density and the velocity of compressible fluid on \( x \in \mathbb{T}^3 \) at time \( t \in \mathbb{R}_+ \), respectively. For simplicity, we consider the viscous term \(-\mu \Delta \) with \( \mu > 0 \).

The pressure law is given by \( p(\rho) = \rho^\gamma \) with \( \gamma > 1 \) and the inter-particle interaction operator \( \mathcal{Q} = \mathcal{Q}(f) \) is given by the BGK operator \( \mathcal{Q}(f) = \nu(\rho_f)(\mathcal{M}(f) - f) \) with the collision frequency \( \nu = \nu(\rho_f) \). Here the local Maxwellian \( \mathcal{M} = \mathcal{M}(f) \) is given by

\[
\mathcal{M}(f)(x, v, t) = \frac{\rho_f(x, t)}{(2\pi T_f(x, t))^{3/2}} e^{-\frac{|v-f(x,t)|^2}{2T_f(x,t)}},
\]

where

\begin{align}
\rho_f(x, t) &:= \int_{\mathbb{R}^3} f(x, v, t) dv, \\
\rho_f(x, t) u_f(x, t) &:= \int_{\mathbb{R}^3} v f(x, v, t) dv, \\
3\rho_f(x, t) T_f(x, t) &:= \int_{\mathbb{R}^3} |v - u_f(x, t)|^2 f(x, v, t) dv,
\end{align}

and we take \( \nu(\rho_f) = \rho_f^\alpha \) with \( \alpha \in [0, 1] \), thus we would also consider the constant collision frequency. We notice that if we ignore the friction force \( \rho(u - v) \) in the kinetic equation in (1.1), then the resulting kinetic equation is the Boltzmann BGK model [4], which is one of the most widely used models for the Boltzmann equation in physics and engineering.

Over the past two decades, existence theories for coupled kinetic-fluid systems have been widely developed. For the Vlasov equation coupled with incompressible or inhomogeneous Navier–Stokes system, the global existence of weak solutions is studied in [5] [27] [30]. The local existence of strong solutions to the inhomogeneous Navier–Stokes–Vlasov system and its large-time behavior are discussed in [10]. For the compressible Euler–Vlasov system, the local classical solutions are obtained in [2].

For the incompressible Navier–Stokes–Vlasov–Fokker–Planck system, the global weak and classical solutions are constructed in [8]. The global existence of classical solutions near the global Maxwellian for the incompressible Euler–Vlasov–Fokker–Planck system and its large-time behavior are investigated in [7]. Later, this result is extended to the compressible Navier–Stokes or Euler equations in [2] [20] [28]. The global existence of weak solutions to the compressible Navier–Stokes–Vlasov–Fokker–Planck is shown in [29] [32].

However, when it comes to the moderately thick sprays case, there is little literature. The global existence of weak solutions to the incompressible or compressible Navier–Stokes–Vlasov system with a linear particle interaction operator describing the breakup phenomena is studied in [22] [35]. The local-in-time unique classical solution for the compressible Euler equations coupled with the Vlasov–Boltzmann equation with a hard-sphere type collision kernel is constructed in [31]. Recently, the Navier–Stokes–BGK system is dealt with and the global existence of weak solutions and local existence of strong solutions are obtained in [15] and [17], respectively. Apart from those existence results, we also refer to [6] [10] [11] [13] [15] [23] [24] [25] [26] [27] [33] for hydrodynamic limits, large-time behaviors, and finite-time blow-up phenomena.

In the current work, we prove the local-in-time well-posedness for the compressible Navier–Stokes–BGK (in short, NS–BGK) system [11]. We would like to mention that the BGK operator is highly nonlinear and the density-dependent friction force gives a strong coupling between particles and fluid, thus it causes significant difficulties in analysis.
1.1. Difficulties and comparison with previous works. There is little literature on the well-posedness theory for the kinetic-fluid model with fluid density-dependent friction forces. One of the main difficulties in analysis arises from the term $\rho v \cdot \nabla f$ in the kinetic equation of \eqref{1.1}. This term depends on both the particle velocity $v$ linearly and the fluid density $\rho$. To show the existence of strong solutions in Sobolev spaces, it is necessary to estimate the term $\rho v \cdot \nabla f$ and its derivatives. This implies that we need to handle the velocity growth of $f$ in the analysis. More specifically, if we estimate higher-order derivatives of $f$ with certain velocity weights of order $m \in \mathbb{N} \cup \{0\}$, then the term $\rho v \cdot \nabla f$ requires us to control the corresponding weight of order $m + 1/2$. In \cite{[14]}, the authors imposed an exponential weight $e^{|v|^2}$ on $f$ to obtain a dissipation on $f$ with the weight $|v|e^{|v|^2}$, which is enough to handle the aforementioned difficulty. However, our system \eqref{1.1} has also the local Maxwellian $M$ in the BGK operator, and this cannot be readily bounded in a Sobolev space with the weight $e^{|v|^2}$. Thus, we imposed the exponential weight $e^{(1+|v|^2)k/2}$ with $k \in (1, 2)$ on $f$ to bound the local Maxwellian in the weighted Sobolev space and simultaneously obtain a dissipation on $f$ with the weight $|v|^{k/2}e^{|v|^2}$. Here we note that the case $k = 1$ can also be handled if we impose a smallness condition on solutions, especially on the fluid density $\rho$. However, we did not include this case to deal with large initial data.

To the best of the authors’ knowledge, well-posedness for the kinetic-fluid model with fluid density-dependent drag force and nonlinear collision operator for particle interactions has not been established yet. In the absence of nonlinear collisional operators, to the best of our knowledge, there are only four papers are available on the local well-posedness theory for the Euler–Vlasov equations \cite{[2]}, the inhomogeneous Naiver–Stokes–Vlasov equations \cite{[10]}, the kinetic thermomechanical Cucker–Smale equation with the compressible Navier–Stokes system \cite{[12]}, and the Vlasov equation coupled with the compressible Navier–Stokes system with degenerate viscosity and vacuum \cite{[13]}. In \cite{[2] [12] [16]}, to handle the density-dependent friction force, the finite-speed of the propagation of the support of $f$ in velocity is significantly used, i.e., the initial data $f_0$ is compactly supported in velocity, $f(t)$ has a compact support in velocity in a compact time interval. By using that, roughly speaking, we can bound the term $\rho v \cdot \nabla f$ by $\rho \nabla u$, and also its derivatives by the derivatives of $\rho \nabla f$. Thus, the problem with velocity growth does not occur in those works.

On the other hand, in \cite{[17] [31]}, the Boltzmann or BGK collisional operator is considered in the kinetic equation, and local well-posedness theory is developed. We would like to remark that in this case, we cannot have a finite speed of propagation of the velocity-support due to the collisional operator $Q(f)$. For that reason, the polynomial, exponential, or Mittag-Leffler weight in velocity is employed for Boltzmann or BGK equation \cite{[11] [21] [34] [60]}. However, in \cite{[17] [31]}, the friction force only depends on $u - v$, but independent of the fluid density $\rho$. Thus, the term $(\rho - \bar{\rho}) v \cdot \nabla f$ does not appear, and there is no additional difficulty with the velocity growth of $f$ in those works.

1.2. Outline of our strategy. In order to handle the strong nonlinear coupling between the kinetic particles and fluids, we propose a exponential weighted solution space for $f$. To be more concrete, for $p, k \in [1, \infty)$, we denote by $L^p_k = L^p_k(\mathbb{T}^3 \times \mathbb{R}^3)$ the space of measurable functions which are weighted by $e^{(v)^k}$, where $\langle v \rangle := (1 + |v|^2)^{1/2}$, and equipped with the norm

$$\|f\|_{L^p_k} := \|e^{(v)^k}f\|_{L^p} = \left( \int_{\mathbb{T}^3 \times \mathbb{R}^3} e^{p\langle v \rangle^k} |f|^p \, dx \, dv \right)^{1/p}.$$  

The limiting case $p = \infty$ is defined by

$$\|f\|_{L^\infty_k} := \text{ess sup}_{x,v} e^{(v)^k}|f(x, v)|.$$  

For any $s \in \mathbb{N}$, $W^{s,p}_k = W^{s,p}_k(\mathbb{T}^3 \times \mathbb{R}^3)$ represents for $L^p_k$ Sobolev space of $s$-th order equipped with the norm

$$\|f\|_{W^{s,p}_k} := \left( \sum_{|\alpha| + |\beta| \leq s} \int_{\mathbb{T}^3 \times \mathbb{R}^3} e^{p\langle v \rangle^k} |\partial_\alpha^\beta f|^p \, dx \, dv \right)^{1/p}.$$  

The space $W^{s,\infty}_k = W^{s,\infty}_k(\mathbb{T}^3 \times \mathbb{R}^3)$ is analogously defined. In particular, when $p = 2$, we denote by $H^s_k = W^{s,2}_k$.

By employing the function space $H^2_k$ for $f$ with $k \in (1, 2)$, we establish the local well-posedness for the NS–BGK system \cite{[13]} (Theorem \ref{thm:1.1}). Although we also encounter the problem with the velocity growth
of \( f \), in our weighted Sobolev space, we figure out an appropriate way of using the viscous effect from the Navier–Stokes system together with the dissipative effect from the friction force, see Lemma 3.5. We would like to stress that in our strategy, neither polynomial weights nor the exponential weight with \( k \notin (1,2) \) is applicable. Note that the above function space for the kinetic equation in (1.1), BGK equation, is different from that of previous works [33–34]. Thus, we need to redevelop a theory for the existence of solutions to the BGK equation in our newly defined solution space for \( f \). In fact, we establish similar results to that of [33–34] in our solution space \( H^k_0 \) or \( W^{1,\infty}_k \) with \( k \in (1,2) \). Note that \( H^k_0 \)-regularity for \( f \) does not imply the boundedness of \( f \). On the other hand, if we assume additional boundedness assumptions on the initial data \( f_0 \), then we have \( f \in C([0,T]; H^k_0(\mathbb{T}^3 \times \mathbb{R}^3)) \cap L^\infty(0,T; W^{1,\infty}_k(\mathbb{T}^3)) \) (Theorem 1.2). In fact, this additional regularity of solutions \( f \in L^\infty(0,T; W^{1,\infty}_k(\mathbb{T}^3)) \) can simplify many computations made in Sections 2–4 below.

1.3. Main results. Before presenting our main results, we introduce several notations used throughout the paper. For simplicity, we often drop \( x \)-dependence of differential operators \( \partial_x, \nabla_x, \) and \( \Delta_x \), i.e. \( \partial_x = \partial, \nabla_x = \nabla, \) and \( \Delta_x = \Delta \). We denote by \( C \) a generic positive constant.

We then define a notion of our regular solution to the NS-BGK system (1.1).

**Definition 1.1.** For \( T \in (0, \infty) \), we say a triplet \((f, \rho, u)\) is a regular solution to system (1.1)–(1.2) on \([0, T]\) if it satisfies (1.1) in the sense of distributions with the following regularity:

(i) \( f \in C([0,T]; H^k_0(\mathbb{T}^3 \times \mathbb{R}^3)) \) with \( k \in (1,2) \),

(ii) \( \rho \in C([0,T]; H^3(\mathbb{T}^3)) \), and \( u \in C([0,T]; H^3(\mathbb{T}^3)) \cap L^2([0,T]; H^4(\mathbb{T}^3)) \).

**Theorem 1.1.** For given \( N < M \), there exists \( T^* > 0 \) only depending on \( M \) and \( N \) such that if the initial data satisfies the following conditions:

(i) \( \max \left\{ \|f_0\|^2_{H^k_0}, \frac{4\gamma}{(\gamma - 1)^2} \|\rho_0\|^{\frac{\gamma - 1}{\gamma}} - 1 \|H^3_0 + \|u_0\|^2_{H^3} \right\} < N \) and \( \inf_{x \in \mathbb{T}^3} \rho_0(x) > \delta > 0 \),

(ii) for some \( a > 0 \) and \( \varepsilon_1 > 0 \), \( f_0(x,v) \geq \varepsilon_1 e^{-(1+a)(v)^k} \) for all \((x,v) \in \mathbb{T}^3 \times \mathbb{R}^3\),

then the system (1.1)–(1.2) admits a unique regular solution on \([0, T^*]\) satisfying

\[
\max \left\{ \sup_{0 \leq t \leq T^*} \|f(t)\|^2_{H^k_0}, \sup_{0 \leq t \leq T^*} \left( \frac{4\gamma}{(\gamma - 1)^2} \|\rho^{\frac{\gamma - 1}{\gamma}}(t) - 1 \|H^3_0 + \|u(t)\|^2_{H^3} \right) \right\} < M
\]

and

\[
\inf_{(x,t) \in \mathbb{T}^3 \times [0,T^*]} \rho^{\frac{\gamma - 1}{\gamma}}(x,t) > \frac{\delta}{2}.
\]

Note that the above theorem does not give the bounded solution \( f \) to the BGK equation. In this regard, our second theorem provides more regular solution \( f \).

**Theorem 1.2.** For given \( N < M \), there exists \( T^* > 0 \) only depending on \( M \) and \( N \) such that if the initial data satisfies the following conditions:

(i) \( \max \left\{ \|f_0\|^2_{H^{1,\infty}_k}, \|f_0\|^2_{H^2_k}, \frac{4\gamma}{(\gamma - 1)^2} \|\rho_0\|^{\frac{\gamma - 1}{\gamma}} - 1 \|H^3_0 + \|u_0\|^2_{H^3} \right\} < N \) and \( \inf_{x \in \mathbb{T}^3} \rho_0(x) > \delta > 0 \),

(ii) for some \( a > 0 \) and \( \varepsilon_1^2 < N \), \( f_0(x,v) \geq \varepsilon_1 e^{-(1+a)(v)^k} \) for all \((x,v) \in \mathbb{T}^3 \times \mathbb{R}^3\),

then the system (1.1)–(1.2) admits a unique regular solution on \([0, T^*]\) satisfying

\[
\max \left\{ \sup_{0 \leq t \leq T^*} \|f(t)\|^2_{W^{1,\infty}_k}, \sup_{0 \leq t \leq T^*} \|f(t)\|^2_{H^2_k}, \sup_{0 \leq t \leq T^*} \left( \frac{4\gamma}{(\gamma - 1)^2} \|\rho^{\frac{\gamma - 1}{\gamma}}(t) - 1 \|H^3_0 + \|u(t)\|^2_{H^3} \right) \right\} < M
\]

and

\[
\inf_{(x,t) \in \mathbb{T}^3 \times [0,T^*]} \rho^{\frac{\gamma - 1}{\gamma}}(x,t) > \frac{\delta}{2}.
\]

**Remark 1.1.** Even though we only provide the well-posedness theory for the NS–BGK system (1.1) with the isentropic pressure, i.e., \( \gamma > 1 \), our framework can be applied to the case with isothermal pressure, \( \gamma = 1 \). More precisely, if the initial data satisfies the following conditions:

(i) \( \max \left\{ \|f_0\|^2_{W^{1,\infty}_k}, \|f_0\|^2_{H^2_k}, \|\log \rho_0\|^2_{H^3} + \|u_0\|^2_{H^3} \right\} < N \) and \( \inf_{x \in \mathbb{T}^3} \rho_0(x) > \delta > 0 \),

(ii) for some \( a > 0 \) and \( \varepsilon_1^2 < N \), \( f_0(x,v) \geq \varepsilon_1 e^{-(1+a)(v)^k} \) for all \((x,v) \in \mathbb{T}^3 \times \mathbb{R}^3\),
1.2. Finally, in Section 5, we discuss the existence and uniqueness of \( \mathcal{L}_k \)-are Cauchy sequences in the proposed Sobolev spaces. From which, we complete the proof of Theorem 1.1.

\[ \text{and its uniform bound estimates locally in time. In Section 4, we provide that the approximate solutions} \]

\[ \text{macroscopic quantities} \quad \rho \quad \text{behavior of solutions to the NS-BGK system} \quad (1.1) \]

\[ \text{where} \quad C > M \quad \text{This provides that one can apply almost the same argument as in} \quad [14] \quad \text{we only deal with} \quad H^2_k \text{ regularity for} \quad f, \quad \text{thus it does not imply the boundedness of} \quad f \quad \text{in three dimensions. Thus, we cannot use of} \quad \|f\|_{L^\infty} \quad \text{in estimating the local Maxwellian and other macroscopic quantities} \quad \rho_f, \quad u_f, \quad \text{and} \quad T_f. \quad \text{For that reason, the proof of Theorem 1.1 requires more delicate analyses on the} \]

\[ \text{BGK operator and the drag force in the compressible Navier–Stokes equations.} \]

Remark 1.2. In Theorems [14] and [12], if the initial data is sufficiently small in our solution space, then the life span of solutions can be extended over a certain fixed time. However, in [31], where the local well-posedness for the compressible Euler–Boltzmann system is established, no matter how small the initial data are, there is a fixed upper bound on the life span of solutions.

Remark 1.3. As one may expect, the proof of Theorem 1.2 would be easier than that of Theorem 1.1. In Theorem 1.1, we only assume that the global classical solutions \( \rho_f \) of \( (1.1) \) satisfies the following conditions:

\[ \rho_f \in L^\infty(\mathbb{R}_+; L^{3/2}(\mathbb{T}^3)), \quad \rho, \ u \in L^\infty(\mathbb{T}^3 \times \mathbb{R}_+), \]

and

\[ \inf_{(x, t) \in T^3 \times \mathbb{R}_+} \rho(x, t) > 0. \]

Then we have the exponential decay of the energy function \( \mathcal{L}(t) \) as time goes to infinity:

\[ \mathcal{L}(t) \leq C \mathcal{L}(0) e^{-Ct}, \quad \forall t > 0, \]

where \( C > 0 \) is a constant independent of \( t \). In particular, this implies

\[ m_c(t), \ v_c(t) \to \frac{1}{\rho_c + L}, \quad \text{as} \quad t \to \infty, \quad \text{where} \quad L \quad \text{denotes the bounded Lipschitz distance}. \]

1.4. Organization of the paper. The rest of this paper is organized as follows. In Section 2, we make preliminary preparations on the estimates for the local Maxwellian \( \mathcal{M}(f) \) in our proposed \( H^2_k \) space and macroscopic quantities \( T_f, \ u_f, \) and \( T_f \). Section 3 is devoted to the construction of the approximate solutions and its uniform bound estimates locally in time. In Section 4, we provide that the approximate solutions are Cauchy sequences in the proposed Sobolev spaces. From which, we complete the proof of Theorem 1.1.

Finally, in Section 5, we discuss the existence and uniqueness of \( W^{1, \infty}_k \)-solutions \( f \) which proves Theorem 1.2.
2. Preliminaries

We begin with an auxiliary lemma which will be frequently used in the rest of this paper.

Lemma 2.1. Let $d \geq 1$ and $g = g(x, v)$ be a sufficiently regular function satisfying $g \in H^2_{x,v}(\mathbb{T}^d \times \mathbb{R}^d)$. If we set

$$h(x) := \left( \int_{\mathbb{R}^d} |g(x,v)|^2 \, dv \right)^{1/2},$$

then $h \in H^1(\mathbb{T}^d)$ with

$$\|h\|_{L^2} = \|g\|_{L^2} \quad \text{and} \quad \|\nabla h\|_{L^2} \leq \|\nabla g\|_{L^2}. \tag{2.1}$$

Furthermore, if $h(x) \geq c_h > 0$ for all $x \in \mathbb{T}^d$, then there exists a constant $C > 0$ independent of $g$ such that

$$\|\nabla^2 h\|_{L^2} \leq \frac{C}{c_h} \sum_{\ell=1}^2 \|\nabla^\ell g\|_{L^2}^2 + C \|\nabla^2 g\|_{L^2}.$$

Proof. We first easily find $\|h\|_{L^2} = \|g\|_{L^2}$. By using Hölder’s inequality, we also obtain

$$|\partial h(x)| = \frac{1}{h(x)} \int_{\mathbb{R}^d} g(x,v) \partial g(x,v) \, dv \leq \left( \int_{\mathbb{R}^d} |\partial g(x,v)|^2 \, dv \right)^{1/2},$$

and thus, $\|\nabla h\|_{L^2} \leq \|\nabla g\|_{L^2}$.

For the estimate of $\|h\|_{H^1}$, we notice that for $i, j = 1, \ldots, d$,

$$\partial_{ij} h = \frac{1}{2} \frac{\partial^2 g}{h^2} \int_{\mathbb{R}^d} g(\partial_i g) \, dv + \frac{1}{h} \int_{\mathbb{R}^d} (\partial_j g)(\partial_i g) \, dv + \frac{1}{h} \int_{\mathbb{R}^d} g(\partial_{ij} g) \, dv.$$

Applying Hölder’s inequality to the above gives

$$\|\nabla^2 h\|_{L^2} \leq \frac{C}{h} \int_{\mathbb{R}^d} |\nabla g|^2 \, dv + C \left( \int_{\mathbb{R}^d} |\nabla^2 g|^2 \, dv \right)^{1/2} \leq \frac{C}{c_h} \int_{\mathbb{R}^d} |\nabla g|^2 \, dv + C \left( \int_{\mathbb{R}^d} |\nabla^2 g|^2 \, dv \right)^{1/2}.$$

Thus we have

$$\|\nabla^2 h\|_{L^2} \leq \frac{C}{c_h} \|\tilde{h}\|_{L^2}^2 + C \|\nabla^2 g\|_{L^2} \leq \frac{C}{c_h} \|\tilde{h}\|_{H^1}^2 + C \|\nabla^2 g\|_{L^2}, \tag{2.2}$$

where

$$\tilde{h} := \left( \int_{\mathbb{R}^d} |\nabla g|^2 \, dv \right)^{1/2}.$$

On the other hand, by (2.1), we get

$$\|\tilde{h}\|_{H^1}^2 \leq \sum_{\ell=1}^2 \|\nabla^\ell g\|_{L^2}^2.$$

Combining this and (2.2) concludes the desired result. \qed

In the lemma below, we show the upper bound estimates on the macroscopic quantities $\rho_f, u_f$, and $T_f$. Since its proof is rather lengthy and technical, we postpone it to Appendix A for smoothness of reading.

Lemma 2.2. Suppose that $\|f\|_{H^2} < \infty$ for $k \in (1, 2)$ and $\rho_f$ and $f$ satisfy

$$\rho_f + \left( \int_{\mathbb{R}^d} f^2 \, dv \right)^{1/2} > c_2.$$

Then there exists $C = C(c_2, k) > 0$ independent of $f$ such that

(i) $\|\rho_f\|_{H^\ell} \leq C \|f\|_{H^\ell}^2$ for $\ell = 0, 1, 2$,

(ii) $\|u_f\|_{H^2} \leq C \|f\|_{H^2}(1 + \|f\|_{H^2}^2)$, and $\|T_f\|_{H^2} \leq C \|f\|_{H^2}(1 + \|f\|_{H^2}^2)$.

Here $\rho_f, u_f$, and $T_f$ are given as in (1.3).

We then show the relation between $\rho_f$ and $T_f$, which will be used later to obtain the lower bound estimate of $T_f$, and thus the local Maxwellian $\mathcal{M}(f)$ is well-defined.
Lemma 2.3. There exists a constant $C > 0$ independent of $f$ such that

$$\rho_f \leq C \left( \int_{\mathbb{R}^3} f^2 \, dv \right)^{1/2} T_f^{3/4}.$$ 

Here $\rho_f$ and $T_f$ are given as in (1.3).

Proof. For any $R > 0$, we estimate

$$\rho_f = \int_{\mathbb{R}^3} f \, dv = \left( \int_{|u_f - v| > R} + \int_{|u_f - v| \leq R} \right) f \, dv \leq \frac{1}{R^2} \int_{\mathbb{R}^3} |u_f - v|^2 f \, dv + C_0 \left( \int_{\mathbb{R}^3} f^2 \, dv \right)^{1/2} R^{3/2},$$

where $C_0 > 0$ independent of $f$. We now choose $R > 0$ such that

$$\frac{1}{R^2} \int_{\mathbb{R}^3} |u_f - v|^2 f \, dv = C_0 \left( \int_{\mathbb{R}^3} f^2 \, dv \right)^{1/2} R^{3/2} \text{ i.e. } R = \left( \frac{3\rho_f T_f}{C_0 \left( \int_{\mathbb{R}^3} f^2 \, dv \right)^{1/2}} \right)^{2/7},$$

then

$$\rho_f \leq 2 \left( \int_{\mathbb{R}^3} |u_f - v|^2 f \, dv \right)^{3/7} \left( C_0 \left( \int_{\mathbb{R}^3} f^2 \, dv \right)^{1/2} \right)^{4/7} = 2 (3\rho_f T_f)^{3/7} \left( C_0 \left( \int_{\mathbb{R}^3} f^2 \, dv \right)^{1/2} \right)^{4/7}.$$ 

Hence we have

$$\rho_f \leq 27/4^{3/4} C_0 \left( \int_{\mathbb{R}^3} f^2 \, dv \right)^{1/2} T_f^{3/4}.$$

\[ \square \]

Remark 2.1. If $f \in L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$, we obtain

$$\rho_f \leq C \|f\|_{L^\infty} T_f^{3/2}.$$ 

This estimate is typically used in the study of existence of solutions to the BGK model, for instance see [31, 41].

We next provide the bound estimate on the local Maxwellian $M(f)$ in our solution space.

Lemma 2.4. Suppose that $\|f\|_{H^k_2} < \infty$ for $k \in (1, 2)$ and $\rho_f$, $u_f$ and $T_f$, given as in (1.3), satisfy

$$\rho_f + |u_f| + T_f < c_1, \quad \rho_f > c_2, \quad \text{and} \quad T_f > c_3^{-1}.$$ 

Then we have

$$\|M(f)\|_{H^k_2} \leq C(1 + c_3)^3 C c_1^{1/2} \|f\|_{H^k_2} (1 + \|f\|_{H^k_2}^{24}),$$

where $C$ depends only on $c_2$ and $k$.

Proof. First, we use Young’s inequality to get

$$\langle v \rangle^k \leq 2^k \left( 1 + |v|^k \right) \leq 2^k + 2^k |v - u_f|^k + |u_f|^k \leq 2^k + 2^{3k} c_1^k + 2^{3k} \left( \frac{|v - u_f|^k}{T_f} \right) \leq 2^k + 2^{3k} c_1^k + \frac{2^{3k} T_f (2k)^{3/2}}{2 - k} + \frac{|v - u_f|^2}{4T_f} \leq 2^k + 2^{3k} c_1^k + \frac{(2 - k)(16c_1^k)^{3/2}}{2} + \frac{|v - u_f|^2}{4T_f},$$

where we also used

$$|a + b|^p \leq 2^p(|a|^p + |b|^p), \quad p > 0.$$
Then we obtain
\[ e^{(v)^k} \mathcal{M}(f) \leq C e^{C (c_1 + c_2 x)} \rho f \left( \frac{1}{(2\pi T^3)^{3/2}} \right) e^{-|v-v_j|^2 / 2T^3} \leq C e^{C c_1 \frac{1}{(2\pi T^3)^{3/2}} \rho f} e^{-|v-v_j|^2 / 2T^3}, \] (2.3)
where \( C > 0 \) is a constant depending only on \( c_2 \) and \( k \) and we used \( k \in (1, 2) \). Thus, we find
\[ \| \mathcal{M}(f) \|_{L^2} \leq \| e^{(v)^k} \mathcal{M}(f) \|_{L^2} \leq C e^{C c_1 \frac{1}{(2\pi T^3)^{3/2}}} \| \rho f \|_{L^2} \leq C e^{C c_1 \frac{1}{(2\pi T^3)^{3/2}}} \| f \|_{L^2}. \]
Here \( C > 0 \) only depends on \( c_2 \) and \( k \). For first-order derivatives, we can deduce from the estimates in the previous step that for \( i, j = 1, 2, 3 \)
\[ \| \partial_i \mathcal{M}(f) \|_{L^2} \leq C \| \rho f \|_{L^2} \left( \| \partial_i \rho f \|_{L^2} + \| \rho f \|_{L^2} + \| \partial_i u_f \|_{L^2} + \| \partial_j T_f \|_{L^2} \right) \leq C \| \rho f \|_{L^2} \left( \| \partial_i u_f \|_{L^2} + \| \partial_j T_f \|_{L^2} \right) \leq C \| \rho f \|_{L^2} \left( ||f||_{H^2} + 1 \right) \| \partial_i u_f \|_{L^2} \]
due to Lemma 2.2. We also have
\[ \| \partial_{ij} \mathcal{M}(f) \|_{L^2} \leq C \| \rho f \|_{L^2} \left( \| \partial_{ij} \rho f \|_{L^2} + \| \partial_{ij} u_f \|_{L^2} + \| \partial_{ij} T_f \|_{L^2} \right) \leq C \| \rho f \|_{L^2} \left( ||f||_{H^2} + 1 \right) \| \partial_{ij} u_f \|_{L^2} \]
For second-order derivative estimates, we first have that for \( i, j = 1, 2, 3 \)
\[ \partial_{ij} \mathcal{M}(f) = \partial_{ij} \left( \partial_i \rho f - \frac{3}{2} \rho f \partial_i T_f \right) - \rho f \left( \frac{\partial_i u_f \cdot (u_f - v)}{T_f} - \left( \frac{|v-u_f|^2}{2T^2} \partial_i T_f \right) \right) e^{-|v-v_j|^2 / 2T} \]
\[ = \left[ \partial_{ij} \rho f - \frac{3}{2} \rho f \partial_i T_f \right] - \rho f \left( \frac{\partial_i u_f \cdot (u_f - v)}{T_f} - \left( \frac{|v-u_f|^2}{2T^2} \partial_i T_f \right) \right) e^{-|v-v_j|^2 / 2T} \]
\[ + \left[ \partial_{ij} \rho f - \frac{3}{2} \rho f \partial_i T_f \right] - \rho f \left( \frac{\partial_i u_f \cdot (u_f - v)}{T_f} - \left( \frac{|v-u_f|^2}{2T^2} \partial_i T_f \right) \right) e^{-|v-v_j|^2 / 2T} \]
\[ \times \left( \frac{\partial_i u_f \cdot (u_f - v)}{T_f} - \left( \frac{|v-u_f|^2}{2T^2} \partial_i T_f \right) e^{-|v-v_j|^2 / 2T} \right) \]
\[ \partial_i \partial_j \mathcal{M}(f) = \left( \frac{\partial_i (u_f)_j}{T_f} - \frac{(u_f - v)_j}{T_f} \partial_i T_f \right) \mathcal{M}(f) + \frac{(u_f - v)_j}{T_f} \partial_i \mathcal{M}(f), \]
and
\[ \partial_{v,v_j} \mathcal{M}(f) = \left( -\frac{\delta_j}{T_f} + \frac{(u_f - v)_j}{T_f} \right) \mathcal{M}(f). \]
where $\delta_{ij}$ denotes Kronecker’s delta. Thus, one gets
\[
\left\| \partial_{ij} M(f) \right\|_{L^2_k} \leq C(1 + c_3)^3 e^{C_{c_1}} \left( \left| \partial_{ij} \rho_f \right| + \left| \partial_{ij} u_f \right| + \left| \partial_{ij} T_f \right| \right) \left( e^{\frac{\left| u_f - v \right|^2}{2}} \right) \left( \left| u_f - v \right|^2 + 1 \right) \left| \partial_i \partial_j \right| T_f \left( \frac{e^{\frac{\left| u_f - v \right|^2}{2}}}{(2\pi T_f)^{3/2}} \right)_{L^2}
\]
\[
\leq C(1 + c_3)^3 e^{C_{c_1}} \left( \left| \partial_{ij} \rho_f \right|_{L^2} + \left| \partial_{ij} u_f \right|_{H^1} + \left| \partial_{ij} T_f \right|_{H^1} + \left| \partial_{ij} T_f \right|_{H^1} \right) \left( e^{C_{c_1}} \left( \left| u_f - v \right|^2 + 1 \right) \left| \partial_i \partial_j \right| T_f \left( \frac{e^{\frac{\left| u_f - v \right|^2}{2}}}{(2\pi T_f)^{3/2}} \right)_{L^2} \right)
\]
\[
\leq C(1 + c_3)^3 e^{C_{c_1}} \left( \left| \partial_{ij} \rho_f \right|_{H^2} + \left| \partial_{ij} u_f \right|_{H^2} + \left| \partial_{ij} T_f \right|_{H^2} + \left| \partial_{ij} T_f \right|_{H^2} \right) \left( e^{C_{c_1}} \left( \left| u_f - v \right|^2 + 1 \right) \left| \partial_i \partial_j \right| T_f \left( \frac{e^{\frac{\left| u_f - v \right|^2}{2}}}{(2\pi T_f)^{3/2}} \right)_{L^2} \right)
\]
\[
\leq C(1 + c_3)^3 e^{C_{c_1}} \left( \left| f \right|_{H^2_k} \left( 1 + \left| f \right|^2_{H^2_k} \right) \right)
\]
where $C$ only depends on $c_2$ and $k$. Moreover, we obtain
\[
\left\| \partial_{ij} \partial_{ij} M(f) \right\|_{L^2_k} \leq C(1 + c_3)^2 e^{C_{c_1}} \left( \left| \partial_{ij} \rho_f \right|_{H^2} + \left| \partial_{ij} u_f \right|_{L^2} + \left| \partial_{ij} T_f \right|_{L^2} \right) \leq C e^{C_{c_1} k^2} \left( 1 + \left| f \right|^2_{H^2_k} \right)
\]
and
\[
\left\| \partial_{ij} \partial_{ij} M(f) \right\|_{L^2_k} \leq C(1 + c_3)^2 e^{C_{c_1}} \left( \left| \partial_{ij} \rho_f \right|_{L^2} \leq C e^{C_{c_1} k^2} \left| f \right|_{L^2_k}.\right)
\]
Finally, we can collect all the above estimates to yield the desired result.

We can also derive the following lemma based on estimates in the previous lemma.

**Lemma 2.5.** Suppose that $f$ and $g$ satisfy (h denotes either $f$ or $g$)
\[
\rho_h + |u_h| + T_h \leq C_1 \quad \text{and} \quad \rho_h + T_h \geq C_2
\]
for some constants $C_i > 0$, $i = 1, 2$. Then we have
\[
\left\| M(f) - M(g) \right\|_{L^2_k} \leq C \left| f - g \right|_{L^2_k},
\]
where $C$ only depends on $C_i$ ($i = 1, 2$) and $k$.

**Proof.** We first observe
\[
\left| \rho_f - \rho_g \right| \leq \int_{\mathbb{R}^3} |f - g| \, dv \leq C \left( \int_{\mathbb{R}^3} e^{2(v')^k} |f - g|^2 \, dv \right)^{1/2},
\]
\[
\left| u_f - u_g \right| = \left| \frac{1}{\rho_f} \int_{\mathbb{R}^3} v f \, dv - \frac{1}{\rho_g} \int_{\mathbb{R}^3} v g \, dv \right| = \left| \frac{\rho_f - \rho_g}{\rho_g} u_f + \frac{1}{\rho_g} \int_{\mathbb{R}^3} v (f - g) \, dv \right| \leq C \left( \int_{\mathbb{R}^3} e^{2(v')^k} |f - g|^2 \, dv \right)^{1/2},
\]
for some constants $C_i > 0$, $i = 1, 2$. Then we have
\[
\left\| M(f) - M(g) \right\|_{L^2_k} \leq C \left| f - g \right|_{L^2_k},
\]
where $C$ only depends on $C_i$ ($i = 1, 2$) and $k$.
and
\[ |T_f - T_g| = \left| \frac{1}{3\rho_f} \int_{\mathbb{R}^3} |v - u_f|^2 f \, dv - \frac{1}{3\rho_g} \int_{\mathbb{R}^3} |v - u_g|^2 g \, dv \right| \]
\[ = \left| -\rho_f - \rho_g \int_{\mathbb{R}^3} (|v - u_f|^2 f - |v - u_g|^2 g) \, dv \right| \]
\[ \leq C\rho_f - \rho_g + C \int_{\mathbb{R}^3} |v|^2 |f - g| \, dv + C \int_{\mathbb{R}^3} |v||u_f - u_g|f \, dv \]
\[ + C \int_{\mathbb{R}^3} |v| |u_g||f - g| \, dv + C \int_{\mathbb{R}^3} |u_f|^2 - |u_g|^2 \, f \, dv + C \int_{\mathbb{R}^3} |u_g|^2 |f - g| \, dv \]
\[ \leq C \left( \int_{\mathbb{R}^3} e^{2(v)^k} |f - g|^2 \, dv \right)^{1/2}. \]

Moreover, we use \(|e^x - e^y| \leq \max\{e^x, e^y\}|x - y| \leq (e^x + e^y)|x - y|\) to get
\[ \left| e^{\frac{|u_f - v|^2}{2T_f}} - e^{\frac{|u_g - v|^2}{2T_g}} \right| \]
\[ \leq \left( e^{\frac{|u_f - v|^2}{2T_f}} + e^{\frac{|u_g - v|^2}{2T_g}} \right) \left| \frac{|u_f - v|^2}{2T_f} - \frac{|u_g - v|^2}{2T_g} \right| \]
\[ \leq C \left( e^{\frac{|u_f - v|^2}{2T_f}} + e^{\frac{|u_g - v|^2}{2T_g}} \right) \left( |u_f - v|^2 |T_f - T_g| + |u_f - u_g||u_f - v| + |u_g - v| \right). \]

Since
\[ \mathcal{M}(f) - \mathcal{M}(g) = \left( \frac{\rho_f - \rho_g}{2\pi T_f^{3/2}} \right) e^{\frac{|u_f - v|^2}{2T_f}} - \rho_g \left( \frac{T_f^{3/2} - T_g^{3/2}}{2\pi T_f T_g} \right) e^{\frac{|u_f - v|^2}{2T_f}} + \rho_g \left( \frac{T_f^{3/2} - T_g^{3/2}}{2\pi T_f T_g} \right) e^{\frac{|u_g - v|^2}{2T_g}}, \]
we combine the above estimates with (2.3) to obtain
\[ \left| e^{(v)^k}(\mathcal{M}(f) - \mathcal{M}(g)) \right| \]
\[ \leq C \left( e^{\frac{|u_f - v|^2}{2T_f}} + e^{\frac{|u_g - v|^2}{2T_g}} \right) \left( 1 + |u_f - v| + |u_g - v| + |u_f - v|^2 \right) \left( \int_{\mathbb{R}^3} e^{2(v)^k} |f - g|^2 \, dv \right)^{1/2}. \]

We finally integrate the above over \(T^3 \times \mathbb{R}^3\) to conclude the desired result. \(\square\)

We close this section by presenting some classical inequalities in the lemma below.

**Lemma 2.6.** For any pair of functions \(f, g \in (H^\ell \cap L^\infty)(\mathbb{R}^d)\), we obtain
\[ \|\nabla^\ell (fg)\|_{L^2} \leq C \left( \|f\|_{L^\infty} \|\nabla^\ell g\|_{L^2} + \|\nabla^\ell f\|_{L^2} \|g\|_{L^\infty} \right). \]

Furthermore, if \(\nabla f \in L^\infty(\mathbb{R}^d)\), we have
\[ \|\nabla^\ell (fg) - f\nabla^\ell g\|_{L^2} \leq C \left( \|\nabla f\|_{L^\infty} \|\nabla^{\ell-1} g\|_{L^2} + \|g\|_{L^\infty} \|\nabla^\ell f\|_{L^2} \right). \]

Here \(C > 0\) only depends on \(\ell\) and \(d\).

### 3. Approximations & Uniform Bound Estimates

In this section, we linearize the NS-BGK system (1.1) and provide the uniform bound estimates on the approximate solutions.
First, we rewrite the system to make use of the structure of symmetric hyperbolic system for the compressible Navier-Stokes equations as in (1.1) as
\begin{equation}
\begin{aligned}
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left( (1 + h) \frac{\gamma - 1}{2} (u - v) f \right) &= \rho_f^0 (\mathcal{M}(f) - f), \\
\partial_t h + u \cdot \nabla h + \frac{\gamma - 1}{2} (1 + h) \nabla \cdot u &= 0, \\
\partial_t u + u \cdot \nabla u + \frac{2\gamma}{\gamma - 1} (1 + h) \nabla h - \frac{\mu \Delta u}{(1 + h)^{\frac{\gamma - 1}{2}}} &= - \int_{\mathbb{R}^3} (u - v) f \, dx dv,
\end{aligned}
\end{equation}
where we set $1 + h := \rho^{\frac{\gamma - 1}{2}}.

3.1. **Approximate solutions.** In this subsection, we construct a sequence of approximate solutions to the reformulated system (3.1):
\begin{equation}
\begin{aligned}
\partial_t f^{n+1} + v \cdot \nabla f^{n+1} + \nabla_v \cdot \left( \rho^n (u^n - v) f^{n+1} \right) &= \rho_f^n (\mathcal{M}(f^n) - f^{n+1}), \\
\partial_t h^{n+1} + u^n \cdot \nabla h^{n+1} + \frac{\gamma - 1}{2} (1 + h^n) \nabla \cdot u^{n+1} &= 0, \\
\partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} + \frac{2\gamma}{\gamma - 1} (1 + h^n) \nabla h^{n+1} - \frac{\mu \Delta u^{n+1}}{(1 + h^n)^{\frac{\gamma - 1}{2}}} &= - \int_{\mathbb{R}^3} (u^n - v) f^n \, dv,
\end{aligned}
\end{equation}
subject to initial data and first iteration step:
\[ (f^{n+1}(x, v, 0), h^{n+1}(x, 0), u^{n+1}(x, 0)) = (f_0(x, v), h_0(x), v_0(x)) \]
and
\[ (f^0(x, v, t), h^0(x, t), u^0(x, t)) = (f_0(x, v), h_0(x), v_0(x)) \]
for $(x, v, t) \in \mathbb{T}^3 \times \mathbb{R}^3 \times (0, T)$.

For simplicity of presentation, we set
\[ X_k^n(T) := \max \left\{ \sup_{0 \leq t \leq T} \| \check{f}^n(\cdot, t) \|_{L^2_{x}}, \sup_{0 \leq t \leq T} \left( \frac{4\gamma}{(\gamma - 1)^2} \| h^n(\cdot, t) \|_{L^2} + \| u^n(\cdot, t) \|_{L^2} \right) \right\}. \]

Our goal of this subsection is to prove the following proposition.

**Proposition 3.1.** Suppose that the initial data $(f_0, \rho_0, u_0)$ satisfy the conditions in Theorem [1.1]. Then we can find $T^* > 0$ depending only on $M$ and $N$ such that system (3.2) admits the sequence of unique regular solutions $\{f^n, h^n, u^n\}_{n \in \mathbb{N}}$ on $[0, T^*]$ satisfying
\[ \sup_{n \in \mathbb{N}} \| X_k^n(T^*) < M \quad \text{and} \quad \inf_{n \in \mathbb{N}} \inf_{(x, t) \in \mathbb{T}^3 \times [0, T^*]} (1 + h^n)(x, t) > \frac{\delta}{2}. \]

For the proof of Proposition 3.1, we first investigate the fluid part $(h^{n+1}, u^{n+1})$.

**Lemma 3.1.** Let $T \in (0, \infty)$ be a fixed constant, and suppose that the initial data $(f_0, \rho_0, u_0)$ satisfy the conditions in Theorem [1.1]. If
\begin{equation}
\begin{aligned}
\max_{1 \leq m \leq n} \| X_k^m(T) \|_{L^2_{x}} < M \quad \text{and} \quad \inf_{1 \leq m \leq n} \inf_{(x, t) \in \mathbb{T}^3 \times [0, T]} (1 + h^m(x, t)) > \frac{\delta}{2},
\end{aligned}
\end{equation}
then we can find $0 < T_1 \leq T$ depending only on $M$ and $N$ such that the fluid system in (3.2) admits the unique regular solution $(h^{n+1}, u^{n+1})$ on $[0, T_1]$ satisfying
\[ \sup_{0 \leq t \leq T_1} \left( \frac{4\gamma}{(\gamma - 1)^2} \| h^{n+1}(\cdot, t) \|_{L^2} + \| u^{n+1}(\cdot, t) \|_{L^2} \right) + \frac{c_0 M}{(1 + M)^{\frac{\gamma - 1}{2}}} \int_0^{T_1} \| \nabla u^{n+1}(\cdot, s) \|_{L^2} \, ds < M \]
for some $c_0 > 0$ independent of $n$.

**Proof.** Since the proof is rather straightforward by now, we leave it in Appendix [B]. \qed

Next, we obtain the infimum estimates for the fluid density $\rho^{n+1}$. 

Lemma 3.2. Suppose that the initial data \((f_0, \rho_0, u_0)\) satisfy the conditions in Theorem 1.1. If \((3.3)\) holds, then we can find \(0 < T_2 \leq T_1\) depending only on \(M\) and \(N\) satisfying

\[
\inf_{(x,t) \in T^3 \times [0,T_2]} (1 + h^{n+1}(x,t)) > \frac{\delta}{2}.
\]

Proof. Once we define the backward characteristic flow \(\eta^{n+1}(s) = \eta^{n+1}(s; t, x)\) for \(h^{n+1}\) as

\[
\partial_s \eta^{n+1}(s) = u^n(\eta^{n+1}(s), s), \quad \eta^{n+1}(t) = x \in T^3,
\]

then we can deduce from the continuity equation in \((3.2)\) that for an \(y \in T^3\),

\[
1 + h^{n+1}(x, t) = (1 + h_0(\eta^{n+1}(0))) - \frac{\gamma - 1}{2} \int_0^t (1 + h^n)(\nabla \cdot u^{n+1})(\eta^{n+1}(s), s) \, ds
\]

\[
\geq \frac{\delta - \frac{\gamma - 1}{2}(1 + ||v^n||_{L^\infty})||\nabla \cdot u^{n+1}||_{L^\infty}}{2}.
\]

Thus, we can choose a sufficiently small constant \(0 < T_2 \leq T_1\) independent of \(n\) satisfying

\[
\inf_{(x,t) \in T^3 \times [0,T_2]} (1 + h^{n+1}(x,t)) > \frac{\delta}{2}.
\]

□

Now, it remains to estimate the kinetic density \(f^{n+1}\). Before we estimate the weighted Sobolev norms for \(f^{n+1}\), we provide preliminary estimates for characteristic flows and macroscopic quantities.

Consider both forward characteristics \(Z^{n+1}(s) = Z^{n+1}(s; 0, z) := (X^{n+1}(s; 0, z), V^{n+1}(s; 0, z))\) with \(z = (x, v)\) and backward characteristics \(\tilde{Z}^{n+1}(s) = \tilde{Z}^{n+1}(s; t, z) := (X^{n+1}(s; t, z), V^{n+1}(s; t, z))\) given by

\[
\frac{d}{ds} X^{n+1}(s) = V^{n+1}(s),
\]

\[
\frac{d}{ds} V^{n+1}(s) = \rho^n(X^{n+1}(s), s) \left( u^n(X^{n+1}(s), s) - V^{n+1}(s) \right),
\]

\[
Z^{n+1}(s) \bigg|_{s=0} = z,
\]

and

\[
\frac{d}{ds} \tilde{X}^{n+1}(s) = \tilde{V}^{n+1}(s),
\]

\[
\frac{d}{ds} \tilde{V}^{n+1}(s) = \rho^n(\tilde{X}^{n+1}(s), s) \left( u^n(\tilde{X}^{n+1}(s), s) - \tilde{V}^{n+1}(s) \right),
\]

\[
\tilde{Z}^{n+1}(s) \bigg|_{s=t} = z.
\]

We then show the growth estimate in velocity for the backward characteristic flow \((3.5)\) in the lemma below.

Lemma 3.3. Suppose that the initial data \((f_0, \rho_0, u_0)\) satisfy the conditions in Theorem 1.1. If \((3.3)\) holds, then we have

\[
|\tilde{V}^{n+1}(s)| \leq C \left( 1 + (1 + M)^{\frac{n}{n-1}} t \right) \exp \left( C(1 + M)^{\frac{n}{n-1}} t \right) (1 + |v|), \quad 0 \leq s \leq t \leq T,
\]

where \(C > 0\) is independent of \(n\) and \(T\).

Proof. From \((3.3)\) and \(\rho^n = (1 + h^n)^{\frac{n}{n-1}}\), one deduces that

\[
\tilde{X}^{n+1}(s) = x - \int_s^t \tilde{V}^{n+1}(\tau) \, d\tau,
\]

\[
\tilde{V}^{n+1}(s) = v \exp \left( - \int_s^t \rho^n(\tilde{X}^{n+1}(\tau), \tau) \, d\tau \right) - \int_s^t (\rho^n u^n)(\tilde{X}^{n+1}(\tau), \tau) \left( - \int_\tau^t \rho^n(\tilde{X}^{n+1}(r), r) \, dr \right) \, d\tau
\]

\[
\leq |v| \exp \left( C(1 + M)^{\frac{n}{n-1}} t \right) + C M (1 + M)^{\frac{1}{n-1}} t \exp(C(1 + M)^{\frac{n}{n-1}} t)
\]
where \( C \) we first notice from Lemma 2.2 that for Proof.

\[
\int \rho f^n(x, t) \, dx \leq C(1 + M) \exp \left( C(1 + M) \sqrt[4]{t} \right) \left( 1 + |v| \right).
\]

\[ \blacksquare \]

Next, we obtain the uniform boundedness of macroscopic fields associated with the kinetic equation.

**Lemma 3.4.** Suppose that the initial data \((f_0, \rho_0, u_0)\) satisfy the conditions in Theorem 1.1. If (3.3) holds, then we can find \( 0 < T_3 \leq T_2 \) depending only on \( M \) and \( N \) satisfying

\[
\begin{align*}
(i) & \quad \min \left\{ \inf_{(x,t) \in \mathbb{T}^3 \times [0,T_3]} \rho f^n(x, t), \inf_{(x,t) \in \mathbb{T}^3 \times [0,T_3]} \left( \int_{\mathbb{R}^3} (f^n)^2 \, dv \right)^{1/2} \right\} > \theta_1, \\
(ii) & \quad \inf_{(x,t) \in \mathbb{T}^3 \times [0,T_3]} T f^n(x, t) > \theta_1 (1 + M)^{-4/3}, \\
(iii) & \quad \sup_{(x,t) \in \mathbb{T}^3 \times [0,T_3]} \left( \rho f^n(x, t) + |u f^n(x, t)| + T f^n(x, t) \right) < \theta_2 (1 + M)^6,
\end{align*}
\]

where \( \theta_1 > 0 \) and \( \theta_2 > 0 \) are constants independent of \( n \) and \( T \).

**Proof.** We first notice from Lemma 2.2 that for \( i = n-1, n, \)

\[ \| \rho f_i \|_{L^\infty} \leq C \| \rho f_i \|_{H^2} \leq C \| f^n \|_{H^2} \leq C(1 + M), \]

where \( C = C(k) \) is a constant independent of \( n \) and \( T \). Then we use the backward characteristics (3.5) to obtain

\[
\frac{d}{ds} f^{n+1}(\tilde{Z}^{n+1}(s), s) = \left( 3\rho^n(\tilde{X}^{n+1}(s), s) - \rho^n(\tilde{X}^{n+1}(s), s) \right) f^{n+1}(\tilde{Z}^{n+1}(s), s) + \rho^n(\tilde{X}^{n+1}(s), s) M(f^n)(\tilde{Z}^{n+1}(s), s),
\]

and we find, for \( 0 \leq t \leq T_2, \)

\[
f^{n+1}(z, t) = f_0(\tilde{Z}^{n+1}(0)) \exp \left( \int_0^t (3\rho^n - \rho^n)(\tilde{X}^{n+1}(s), s) \, ds \right) + \int_0^t \rho^n(\tilde{X}^{n+1}(s), s) M(f^n)(\tilde{Z}^{n+1}(s), s) \exp \left( \int_s^t (3\rho^n - \rho^n)(\tilde{X}^{n+1}(\tau), \tau) \, d\tau \right) \, ds \\
\geq f_0(\tilde{Z}^{n+1}(0)) \exp \left( \int_0^t (3\rho^n - \rho^n)(\tilde{X}^{n+1}(s), s) \, ds \right) \\
\geq e^{\left( 3(\delta/2)^{1/2} - C(1 + M)^{n} \right) t} f_0(\tilde{Z}^{n+1}(0))
\]

due to \( M(f^n) \geq 0. \) From this and Lemma 3.3 we deduce that

\[
\rho f^n \geq e^{3(\delta/2)^{1/2} - C(1 + M)^{n}} \int_{\mathbb{R}^3} f_0(\tilde{Z}^{n}(0)) \, dv \\
\geq e^{3(\delta/2)^{1/2} - C(1 + M)^{n}} \int_{\mathbb{R}^3} e^{-(1+\alpha)(\tilde{Z}^{n}(0))} \, dv \\
\geq e^{3(\delta/2)^{1/2} - C(1 + M)^{n+1/2}} \int_{\mathbb{R}^3} e^{-C(1+\alpha) \left[ (1+(1+M)^{1/2} t) \exp \left( C(1 + M) \sqrt[4]{t} \right) \right] k} \, dv,
\]

Since the integrand in the last equality belongs to \( L^1(\mathbb{R}^3) \), there exists \( \theta_{11} > 0 \) and \( 0 < T_{31} \leq T_2 \) such that

\[
\inf_{(x,t) \in \mathbb{T}^3 \times [0,T_{31}]} \rho f^n(x, t) \geq \theta_{11}.
\]

For the lower bound estimate of \( T f^n \), due to Lemma 2.3 we estimate \( L^\infty \) bound on

\[
\left( \int_{\mathbb{R}^3} (f^n)^2 \, dv \right)^{1/2} =: g^n(x).
\]
For this, we use Lemma 2.31 and this requires the lower bound estimate on \( g^n \). Similarly as the above,
\[
\int_{\mathbb{R}^3} (f^n)^2 \, dv \geq e^{2 \left( \frac{3(\delta/2)^{1+1}}{\gamma^n - C(1+M)^n} \right) t} \int_{\mathbb{R}^3} (f_0(\tilde{Z}^n(0)))^2 \, dv \\
\geq e^{2 \left( \frac{3(\delta/2)^{1+1}}{\gamma^n - C(1+M)^n} \right) t} \int_{\mathbb{R}^3} e^{-2C(1+\alpha) \left[ \left( 1 + (1+M)^{1+1} \right) \exp \left( C(1+M)^{1+1} \right) \right]^{\frac{1}{\gamma^n - C(1+M)^n}}} (v)^k \, dv
\]
Similarly as before, we can find \( \theta_{12} > 0 \) and \( 0 < T_{32} \leq T_2 \) such that
\[
\inf_{(x,t) \in \mathbb{T}^3 \times [0,T_{32}]} \left( \int_{\mathbb{R}^3} (f^n)^2 \, dv \right)^{1/2} \geq \theta_{12}.
\]
Next, by Lemma 2.31 we obtain
\[
\|g^n\|_{H^2} \leq C\|f^n\|_{H^2}^3 + C\|f^n\|_{H^2}^2 \leq (1 + M),
\]
where \( C > 0 \) is independent of \( n \). Then we choose \( T_3 := \min \{ T_{31}, T_{32} \} \) and by Lemma 2.33 the lower bound of \( T_{f^n} \) can be easily obtained as
\[
\theta_{11} \leq \rho_{f^n} \leq C\|g^n\|_{L^\infty} T_{f^n}^{3/4} \leq C\|g^n\|_{H^2} T_{f^n}^{3/4} \leq C(1 + M)T_{f^n}^{3/4},
\]
i.e. \( T_{f^n} \geq \left( \frac{\theta_{11}}{C(1 + M)} \right)^{4/3} =: \theta_{13}(1 + M)^{-4/3} \).

We then choose \( \theta_1 := \min \{ \theta_{11}, \theta_{12}, \theta_{13} \} \) to obtain (i) and (ii).

By using the uniform-in-\( n \) lower bound on \( \rho_{f^n} \) and \( g^n \), together with Lemma 2.2 we also estimate the upper bounds for \( u_{f^n} \) and \( T_{f^n} \) as
\[
\|u_{f^n}\| + T_{f^n} \leq C \left( \|u_{f^n}\|_{H^2} + \|T_{f^n}\|_{H^2} \right) \leq C\|f\|_{H^2} (1 + \|f\|_{11}^2) \leq C(1 + M)^6.
\]
This completes the proof.

Now, we are ready to show the uniform bound estimates for \( f^{n+1} \) in \( H^2_{0}(\mathbb{T}^3 \times \mathbb{R}^3) \).

**Lemma 3.5.** Suppose that the initial data \( (f_0, \rho_0, u_0) \) satisfy the conditions in Theorem 1.1. If (3.3) holds, then we can find \( 0 < T_4 \leq T_3 \) depending only on \( M \) and \( N \) such that
\[
\sup_{0 \leq t \leq T_4} \|f^{n+1}(t)\|^2_{H^2_{0}} < M.
\]

**Proof.** We split the proof into three cases as follows:

- **(Step A: Zeroth-order estimates)** From the kinetic equation in (5.2), we obtain
  \[
  \frac{1}{2} \frac{d}{dt} \|f^{n+1}\|^2_{L^2_{x}} \\
  = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} e^{2(v)^k} f^{n+1} \, dx dv \\
  = 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \rho^n e^{2(v)^k} (f^{n+1})^2 \, dx dv + \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} (f^{n+1})^2 \, dx dv \\
  \leq \frac{3}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \rho^n e^{2(v)^k} (f^{n+1})^2 \, dx dv + k \int_{\mathbb{T}^3 \times \mathbb{R}^3} \rho^n (v)^{k-2} (f^{n+1})^2 \, dx dv \\
  \leq C(1 + M) \left( \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \rho^n (v)^{k-2} (f^{n+1})^2 \, dx dv \\
  + \frac{k}{2} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \rho^n (u)^{k-2} (f^{n+1})^2 \, dx dv \\
  \right).
\]
\[ + C(1 + M) \alpha e^{C(1+M)\frac{\text{d}t}{\pi T_f \rho_n}} \int_{T^3 \times R^3} \frac{\rho f_n}{(2\pi T_f \rho_n)^{3/2}} e^{-\frac{|v_n - v|^2}{4\pi T_f \rho_n}} e^{(v)^k} f^{n+1} \, dx \, dv \]

\[ \leq C(1 + M)^{\frac{3}{2} + 1} \| f^{n+1} \|_{L^2_k}^2 + C(1 + M) \alpha e^{C(1+M)\frac{\text{d}t}{\pi T_f \rho_n}} \| \rho f_n \|_{L^2} \| f^{n+1} \|_{L^2_k}, \]

where we used the estimate (2.3), \langle v \rangle^{k-2} \leq 1, and Young’s inequality. This implies

\[ \frac{d}{dt} \| f^{n+1} \|^2_{L^2_k} \leq C(1 + M)^{\frac{3}{2} + 1 + \alpha} e^{C(1+M)\frac{\text{d}t}{\pi T_f \rho_n}} \left( \| f^{n+1} \|^2_{L^2_k} + 1 \right). \tag{3.6} \]

• (Step B: First-order estimates) For \( x \)-derivatives, one gets

\[ \frac{1}{2} \frac{d}{dt} \| \partial_x f^{n+1} \|^2_{L^2_k} \]

\[ = - \int_{T^3 \times R^3} \left( v \cdot \nabla \partial_x f^{n+1} + \rho^n (u^n - v) \cdot \nabla_v \partial_x f^{n+1} + \partial_x \rho^n (u^n - v) \cdot \nabla_v f^{n+1} + \rho^n \partial_x u^n \cdot \nabla_v f^{n+1} + \partial_x \rho^n f^{n+1} - 3 \rho^n \partial_x f^{n+1} - 3 \partial_x \rho^n f^{n+1} \right) \]

\[ - \alpha \partial_x \rho_f \rho_f^{-1} (M(f^n) - f^{n+1} - \rho_f \partial_x (M(f^n) - f^{n+1})) e^{2(v)^k} \partial_x f^{n+1} \, dx \, dv \]

\[ \leq \frac{1}{2} \int_{T^3 \times R^3} |\partial_x f^{n+1}|^2 |\nabla v| (\rho^n (u^n - v) e^{2(v)^k}) \, dx \, dv + \int_{T^3 \times R^3} \partial_x \rho^n (u^n - v) \cdot \nabla_v f^{n+1} \partial_x f^{n+1} e^{2(v)^k} \, dx \, dv \]

\[ + C(1 + M)^{\frac{3}{2} + 1} \| f^{n+1} \|_{H^1_k}^2 + C(1 + M) \alpha e^{C(1+M)\frac{\text{d}t}{\pi T_f \rho_n}} \left( \| f^{n+1} \|^2_{H^1_k} + 1 \right) \]

\[ + C(1 + M)^{\alpha + 1} \| \partial_x M(f^n) \|_{L^2_k} \| \partial_x f^{n+1} \|_{L^2_k} \]

\[ \leq - \frac{k}{2} \int_{T^3 \times R^3} \rho^n (u^n - v)^2 |v| e^{2(v)^k} |\partial_x f^{n+1}| \, dx \, dv + \frac{k}{2} \int_{T^3 \times R^3} \rho^n |u^n|^2 \langle v \rangle^{k-2} e^{2(v)^k} |\partial_x f^{n+1}| \, dx \, dv \]

\[ + \int_{T^3 \times R^3} \partial_x \rho^n v \cdot \nabla_v f^{n+1} \partial_x f^{n+1} e^{2(v)^k} \, dx \, dv + C(1 + M)^{\frac{3}{2} + 17 + \alpha} e^{C(1+M)\frac{\text{d}t}{\pi T_f \rho_n}} \left( \| f^{n+1} \|^2_{H^1_k} + 1 \right), \]

where we used Lemmas 2.2 and 2.3, Young’s inequality, and

\[ \int_{T^3 \times R^3} |\partial_x \rho f^n | \rho_f^{-1} (M(f^n) + f^{n+1}) e^{2(v)^k} |\partial_x f^{n+1}| \, dx \, dv \]

\[ \leq C(1 + M)^{\alpha + 1} e^{C(1+M)\frac{\text{d}t}{\pi T_f \rho_n}} \int_{T^3 \times R^3} |\partial_x \rho f^n | \frac{\rho f_n}{(2\pi T_f \rho_n)^{3/2}} e^{-\frac{|v_n - v|^2}{4\pi T_f \rho_n}} |\partial_x f^{n+1}| e^{(v)^k} \, dx \, dv \]

\[ + C \int_{T^3 \times R^3} |\partial_x \rho f^n | |\partial_x f^{n+1}| e^{2(v)^k} \, dx \, dv \]

\[ \leq C(1 + M)^{\alpha + 1} e^{C(1+M)\frac{\text{d}t}{\pi T_f \rho_n}} |\partial_x \rho f^n |_{L^2} |\rho f_n |_{L^\infty} |\partial_x f^{n+1} |_{L^2_k} \]

\[ + C \int_{T^3} |\partial_x \rho f^n | \left( \int_{R^3} e^{2(v)^k} |\partial_x f^{n+1}|^2 \, dv \right)^{1/2} \left( \int_{R^3} e^{2(v)^k} |f^{n+1}|^2 \, dv \right)^{1/2} \]

\[ \leq C(1 + M)^{\alpha + 3} e^{C(1+M)\frac{\text{d}t}{\pi T_f \rho_n}} \| f^{n+1} \|_{H^1_k} + C |\partial_x \rho f^n |_{H^1} \left( \int_{R^3} e^{2(v)^k} |f^{n+1}|^2 \, dv \right)^{1/2} \]

\[ \leq C(1 + M)^{\alpha + 3} e^{C(1+M)\frac{\text{d}t}{\pi T_f \rho_n}} (\| f^{n+1} \|^2_{H^1_k} + 1). \]
Note that

\[
|\partial_t \rho^n v \cdot \nabla_v f^{n+1} \partial_t f^{n+1}| I_{\{|v|>1\}} \\
\leq |\partial_t \log \rho^n| |\rho^n v \cdot \nabla_v f^{n+1} \partial_t f^{n+1}| I_{\{|v|>1\}} \\
\leq \frac{1}{L} \left( (\rho^n)^{\frac{1}{2}} |\nabla_v f^{n+1}|^{\frac{1}{2}-1} |\partial_t f^{n+1}| \right)^k I_{\{|v|>1\}} + C \left( (\rho^n)^{1-\frac{1}{k}} |\partial_t \rho^n| |\nabla_v f^{n+1}|^{2-\frac{2}{k}} \right) I_{\{|v|>1\}} \tag{3.7}
\]

for any \( L > 0 \). We can use

\[
2^{-k} v^k |v|^k \leq (v)^k |v|^2, \quad |v| \geq 1, \quad k \in (1, 2) \tag{3.8}
\]

and choose \( L = 2 \frac{2-k}{k} \cdot 200 \) to obtain

\[
|\partial_t \rho^n v \cdot \nabla_v f^{n+1} \partial_t f^{n+1}| I_{\{|v|>1\}} \\
\leq \frac{1}{200} \rho^n (v)^{k-2} |v|^2 \left( \frac{|\partial_t f^{n+1}|^2}{2/k} + \frac{|\nabla_v f^{n+1}|^2}{2/(2-k)} \right) + C(1 + M)^{\frac{2-k}{k}} |\nabla_v f^{n+1}|^2.
\]

This yields

\[
|\partial_t \rho^n v \cdot (\nabla_v f^{n+1}) (\partial_t f^{n+1})| \\
\leq C(1 + M)^{\frac{2-k}{k}} |\nabla_v f^{n+1}||\partial_t f^{n+1}| I_{\{|v| \leq 1\}} + C(1 + M)^{\frac{2-k}{k}} |\nabla_v f^{n+1}|^2 I_{\{|v|>1\}} \\
+ \frac{1}{200} \rho^n (v)^{k-2} |v|^2 \left( \frac{k}{2} |\partial_t f^{n+1}|^2 + \frac{2-k}{2} |\nabla_v f^{n+1}|^2 \right) I_{\{|v|>1\}}.
\]

Thus, we obtain

\[
\frac{d}{dt} \|\partial_t f^{n+1}\|_{L^2_k}^2 \leq \frac{2-k}{200} \int_{T^3 \times R^3} \rho^n (v)^{k-2} |v|^2 e^{2(v)^k} |\nabla_v f^{n+1}|^2 \, dx \, dv \\
+ C(1 + M)^{\frac{2-k}{k}} 2^k \alpha + e^{C(1+M)^{\frac{2-k}{k}}} (\|f^{n+1}\|_{H^k}^2 + 1), \tag{3.9}
\]

where \( C > 0 \) is independent of \( n \) and \( T \). For \( v \)-derivatives,

\[
\frac{1}{2} \frac{d}{dt} \|\partial_v f^{n+1}\|_{L^2_k}^2 \\
= -\int_{T^3 \times R^3} \left( v \cdot \nabla \partial_v f^{n+1} + \partial_v f^{n+1} + \rho^n (u^n - v) \cdot \nabla \partial_v f^{n+1} \right) e^{2(v)^k} \partial_v f^{n+1} \, dx \, dv \\
\leq \frac{k}{2} \int_{T^3 \times R^3} \rho^n (v)^{k-2} |v|^2 e^{2(v)^k} |\partial_v f^{n+1}|^2 \, dx \, dv \\
+ C(1 + M)^{\frac{2-k}{k}} \|f^{n+1}\|_{H^k}^2 + C(1 + M)^{\alpha} \|\partial_v \partial_t f^{n+1}\|_{L^2_k} \|\partial_v f^{n+1}\|_{L^2_k} \\
\leq -\frac{k}{2} \int_{T^3 \times R^3} \rho^n (v)^{k-2} |v|^2 e^{2(v)^k} |\partial_v f^{n+1}|^2 \, dx \, dv \\
+ C(1 + M)^{\frac{2-k}{k}} 2^k \alpha + C(1 + M)^{18+\alpha} e^{C(1+M)^{\frac{2-k}{k}}} \|f^{n+1}\|_{H^k}^2,
\]

where we used Lemma 2.3. In other words,

\[
\frac{d}{dt} \|\partial_v f^{n+1}\|_{L^2_k}^2 \leq -\frac{k}{2} \int_{T^3 \times R^3} \rho^n (v)^{k-2} |v|^2 e^{2(v)^k} |\partial_v f^{n+1}|^2 \, dx \, dv \\
+ C(1 + M)^{\frac{2-k}{k}} 2^k \alpha + e^{C(1+M)^{\frac{2-k}{k}}} (\|f^{n+1}\|_{H^k}^2 + 1). \tag{3.10}
\]
So we combine (3.9) with (3.10) to yield
\[ \frac{d}{dt} \| f^{n+1} \|_{L^2_k}^2 \leq -\frac{9k}{10} \int_{T^3 \times \mathbb{R}^3} \rho^n \langle v \rangle^{k-2} |v|^2 e^{2(v)} \left| \nabla_v f^{n+1} \right|^2 \, dx dv \\
+ C(1 + M)^\frac{1}{2} \frac{2k^{2k-1} + 18 + 18 + \alpha e^{C(1+M)} \frac{6k}{4}}{18} (\| f^{n+1} \|_{H^k_2}^2 + 1). \]

\( (3.11) \)

- **(Step C: Second-order estimates):** We first deal with

\[ \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} |\partial_{ij} f^{n+1}|^2 dxdv + C(1 + M)^{\frac{1}{2} + \frac{1}{2}} (\| f^{n+1} \|_{H^k_2}^2 + 1) \]

\[ + C(1 + M)^{n+10} e^{C(1 + M) \frac{6k}{4}} (\| f^{n+1} \|_{H^k_2}^2 + 1) + C(1 + M)^\alpha (\| \partial_{ij} \mathcal{M}(f^n) \|_{L^2_k} + \| f^{n+1} \|_{H^k_2}^2) \| f^{n+1} \|_{H^k_2}^2 \]

\[ \leq C(1 + M)^{\frac{1}{2} + 19 + \alpha e^{C(1+M) \frac{6k}{4}}} (\| f^{n+1} \|_{H^k_2}^2 + 1) \]

\[ - \frac{k}{2} \int_{T^3 \times \mathbb{R}^3} \rho^n \langle v \rangle^{k-2} |v|^2 e^{2(v)} \left| \partial_{ij} f^{n+1} \right|^2 \, dx dv \\
+ \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} |\partial_{ij} f^{n+1}|^2 dxdv \\
+ \alpha \int_{T^3 \times \mathbb{R}^3} \rho f^n \rho f^{n-1} \mathcal{M}(f^n) \, dx dv \]

\[ =: C(1 + M)^{\frac{1}{2} + 19 + \alpha e^{C(1+M) \frac{6k}{4}}} (\| f^{n+1} \|_{H^k_2}^2 + 1) - \frac{k}{2} \int_{T^3 \times \mathbb{R}^3} \rho^n \langle v \rangle^{k-2} |v|^2 e^{2(v)} \left| \partial_{ij} f^{n+1} \right|^2 \, dx dv \\
+ \sum_{i=1}^4 l_i. \]

Here, similarly as before, due to Lemmas 2.2 and 8.4, we estimated

\[ \int_{T^3 \times \mathbb{R}^3} |\partial_{ij} f^n| \rho f^{n-1} |\partial_{ij} \mathcal{M}(f^n) + \| \partial_{ij} f^{n+1} \|_{L^2_k} \| f^{n+1} \|_{L^2_k} \]
We next estimate the terms $I_i$, $i = 1, 2, 3, 4$ as follows.

For $I_1$, we use (5.7) and (5.8) to get

$$
\left| \left( \partial_i \rho^n \nabla v \cdot \partial_j f^{n+1} + \partial_j \rho^n \nabla v \cdot \partial_i f^{n+1} \right) \cdot v \partial_i f^{n+1} \right|
\leq C(1 + M)^{\frac{1}{2}} \left( \| |v| \partial_i f^{n+1} |v| \partial_j f^{n+1} \|_{L^2} + \| |v| \partial_j f^{n+1} |v| \partial_i f^{n+1} \|_{L^2} \right)
+ C(1 + M)^{\frac{1}{2}} \left( \| |v| \partial_i f^{n+1} |v| \partial_j f^{n+1} \|_{L^2} + \| |v| \partial_j f^{n+1} |v| \partial_i f^{n+1} \|_{L^2} \right)
+ \frac{1}{200} \rho^n |v|^{k-2} |v|^2 \left( k|\partial_i f^{n+1}|^2 + \frac{2 - k}{2} (|\nabla v \partial_i f^{n+1}|^2 + |\nabla v \partial_j f^{n+1}|^2) \right) \mathbf{1}_{\{|v|>1\}}.
$$

This implies

$$
I_1 \leq C(1 + M)^{\frac{1}{2}} \left( \| |v| \partial_i f^{n+1} |v| \partial_j f^{n+1} \|_{L^2} + \frac{k}{400} \int_{T^3 \times \mathbb{R}^3} \rho^n |v|^{k-2} |v|^2 e^{2(|v|)} \| \partial_i f^{n+1} \|^2 \right) \| dv + \frac{2 - k}{200} \rho^n |v|^{k-2} |v|^2 e^{2(|v|)} \left( |\nabla v | \partial_i f^{n+1} |v| \partial_j f^{n+1} \right) \| dv.
$$

For $I_2$,

$$
I_2 = - \int_{T^3 \times \mathbb{R}^3} \partial_i \rho^n u \cdot \nabla f^{n+1} \partial_j f^{n+1} e^{2(|v|)}
+ \int_{T^3 \times \mathbb{R}^3} \partial_i \partial_j \rho^n v \cdot \nabla f^{n+1} \partial_j f^{n+1} e^{2(|v|)}
=: I_{21} + I_{22}.
$$

Here, we can handle $I_{21}$ as

$$
I_{21} \leq \| \partial_i \rho^n \|_{L^6} \| u^n \|_{L^\infty} \| (\nabla f^{n+1} e^{2(|v|)} \|_{H^1(L^2)} \| \partial_j f^{n+1} \|_{L^2}
\leq C(1 + M)^{\frac{1}{2}} \left( \| f^{n+1} \|^2 \right).
$$

due to Lemma 24.1

For $I_{22}$, we use Hölder inequality, Young’s inequality, (5.8), and Lemma 24.1 to get

$$
I_{22} \leq \int_{T^3 \times \mathbb{R}^3} \| \partial_i \rho^n \|_{L^6} \| \partial_i f^{n+1} \|_{L^\infty} \| \nabla f^{n+1} e^{2(|v|)} \|_{L^2} \| dv
+ \int_{T^3} \| \partial_i \rho^n \| \left( \int_{\mathbb{R}^3} |v|^{k} \| \partial_j f^{n+1} \|_{L^2} \right)^{\frac{1}{2}}
\times \left( \int_{\mathbb{R}^3} e^{2(|v|)} \| \nabla f^{n+1} \|_{L^2} \right)^{\frac{k-1}{k}} \| dv
\leq \| \partial_i \rho^n \|_{L^6} \| \partial_i f^{n+1} \|_{L^2} \left( \int_{\mathbb{R}^3} e^{2(|v|)} \| \nabla f^{n+1} \|_{L^2} \right)^{\frac{1}{2}}
+ C \int_{T^3} \| \partial_i \rho^n \| \left( \int_{\mathbb{R}^3} |v|^{k-2} |v|^2 e^{2(|v|)} \| \partial_j f^{n+1} \|_{L^2} \right)^{\frac{1}{2}}
\times \left( \int_{\mathbb{R}^3} e^{2(|v|)} \| \nabla f^{n+1} \|_{L^2} \right)^{\frac{k-1}{k}} \| dv
\leq C(1 + M)^{\frac{1}{2}} \left( \| f^{n+1} \|^2 \right)
$$
\[ + C \| \partial \psi \psi \| \| \left( \int_{\mathbb{R}^3} \langle \psi \rangle^{k-2} |\psi|^2 e^{2(\psi)^k} \right) \|^{\frac{1}{2}} \left. \right|_{L^2} \]
\[ \times \left\| \left( \int_{\mathbb{R}^3} \langle \psi \rangle^{k-2} |\psi|^2 e^{2(\psi)^k} \right) \|^{\frac{2-k}{2k}} \right. \left. \right|_{L^{\frac{1}{k}}} \left. \right|_{L^3} \left( \int_{\mathbb{R}^3} e^{2(\psi)^k} |\nabla \psi f^{n+1}|^2 \right)^{\frac{k-1}{k}} \right. \]
\[ \leq C(1 + M) \| f^{n+1} \|_{H^k}^{2} \]
\[ + C(1 + M) \| f^{n+1} \|_{L^1} \left( \int_{\mathbb{R}^3} e^{2(\psi)^k} |\nabla \psi f^{n+1}|^2 \right)^{\frac{k-1}{k}} \right. \]
\[ \leq C(1 + M) \| f^{n+1} \|_{H^k}^{2} \]
\[ + \frac{k}{200} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \rho^n \langle \psi \rangle^{k-2} |\psi|^2 e^{2(\psi)^k} \left( |\partial \psi f^{n+1}|^2 + |\nabla \psi f^{n+1}|^2 + |\nabla_x \nabla \psi f^{n+1}|^2 \right) \right. \]
\[ dx. \]

Thus, we have
\[ I_2 \leq C(1 + M) \| f^{n+1} \|_{H^k}^{2} \]
\[ + \frac{k}{200} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \rho^n \langle \psi \rangle^{k-2} |\psi|^2 e^{2(\psi)^k} \left( |\partial \psi f^{n+1}|^2 + |\nabla \psi f^{n+1}|^2 + |\nabla_x \nabla \psi f^{n+1}|^2 \right) \right. \]
\[ dx. \]

For \( I_3 \), similarly as before, we obtain
\[ I_3 \leq 3 \| \partial \psi \psi \| \| \partial \psi f^{n+1} \| \| f^{n+1} \|_{L^2} \left( \int_{\mathbb{R}^3} |f^{n+1}|^2 e^{2(\psi)^k} \right)^{\frac{1}{2}} \right. \]
\[ \leq C(1 + M) \| \partial \psi f^{n+1} \| \| f^{n+1} \|_{H^1} \]
\[ \leq C(1 + M) \| f^{n+1} \|_{H^k}^{2}. \]

For \( I_4 \), we use (2.3) and Lemma 3.4 to yield
\[ I_4 \leq C(1 + M) \| f^{n+1} \| \| \partial \psi f^{n+1} \| \| \partial \psi f^{n+1} \| \| f^{n+1} \|_{L^2} \right. \]
\[ + C \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left| \partial \psi f^{n+1} \right|^2 \left( \int_{\mathbb{R}^3} e^{2(\psi)^k} \right)^{\frac{1}{2}} \right. \]
\[ \leq C(1 + M) \| f^{n+1} \|_{H^k}^{2} \]
\[ + C \int_{\mathbb{T}^3} \left| \partial \psi f^{n+1} \right|^2 \left( \int_{\mathbb{R}^3} e^{2(\psi)^k} \right)^{\frac{1}{2}} \right. \]
\[ \leq I_1 + I_2, \]
where
\[ I_1^2 \leq C(1 + M)^{\alpha+3} e^{C(1 + M) \| f^{n+1} \|_{L^k}^{\frac{1}{\alpha}} \| \partial \psi f^{n+1} \|_{L^2}. \]

For \( I_2^2 \), analogously as in the proof of Lemma 3.4, we have
\[ I_2^2 \leq C \| \psi^n \| \| \partial \psi f^{n+1} \| \| f^{n+1} \|_{L^2} \]
\[ \leq C(1 + M) \| \psi^n \|_{H^k} \| \partial \psi f^{n+1} \|_{L^2} \]
\[ \leq C(1 + M) \| f^{n+1} \|_{H^k} \]
\[ + C \| f^{n+1} \|_{H^k} \]
\[ \leq C \| f^{n+1} \|_{H^k}^{2}, \]

due to Lemma 2.4 where
\[ \psi^{n+1}(x) := \left( \int_{\mathbb{R}^3} e^{2(\psi)^k} \right)^{\frac{1}{2}}. \]
This implies
\[ l_4 \leq C(1 + M)^{n+3} e^{C(1 + M)^\frac{3k}{2}} \left( \| f^{n+1} \|_{L_k^2}^3 + \| f^{n+1} \|_{H_k^2}^2 + 1 \right). \]

Thus, we gather the estimates for \( l_i \)'s to get
\[ \frac{d}{dt} \| \nabla_x f^{n+1} \|_{L_k^2}^2 \leq \frac{4k}{5} \int_{T^3 \times \mathbb{R}^3} \rho_n \langle v \rangle^{k-2} |v|^2 e^{2(v)^k} \langle \nabla_x f^{n+1} \rangle^2 \, dx \, dv + \frac{k}{5} \int_{T^3 \times \mathbb{R}^3} \rho_n \langle v \rangle^{k-2} e^{2(v)^k} (\| \nabla_x f^{n+1} \|_{L_k^2}^2 + \| \nabla_x \nabla f^{n+1} \|_{L_k^2}^2) \, dx \, dv. \quad (3.12) \]

Next, we consider
\[ \frac{1}{2} \frac{d}{dt} \| \partial_i \partial_v f^{n+1} \|_{L_k^2}^2 \]
\[ = - \int_{T^3 \times \mathbb{R}^3} v_i \cdot \nabla \partial_i \partial_v f^{n+1} + \partial_{ij} f^{n+1} + \rho_n \langle u - v \rangle \cdot \nabla_v \partial_i \partial_v f^{n+1} + \partial_i \rho_n \langle u - v \rangle \nabla_v \partial_v f^{n+1} \]
\[ + \rho_n \partial_i u_i \cdot \nabla_v \partial_v f^{n+1} - 4 \partial_i (\rho_n \partial_v f^{n+1}) - \alpha \partial_i \rho_n \rho_{f_{n-1}} \partial_v (M(f^n) - f^{n+1}) \]
\[ - \rho_n \partial_i \partial_v f^{n+1} e^{2(v)^k} \partial_i \partial_v f^{n+1} \, dx \, dv \]
\[ = \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} | \partial_i \partial_v f^{n+1} |^2 \nabla_v \cdot \langle \rho_n \langle u - v \rangle e^{2(v)^k} \rangle \, dx \, dv - \int_{T^3 \times \mathbb{R}^3} \partial_{ij} f^{n+1} \partial_i \partial_v f^{n+1} e^{2(v)^k} \, dx \, dv \]
\[ - \int_{T^3 \times \mathbb{R}^3} \partial_i \rho_n \langle u - v \rangle \cdot \nabla_v \partial_i \partial_v f^{n+1} e^{2(v)^k} \, dx \, dv \]
\[ + \int_{T^3 \times \mathbb{R}^3} \left( - \rho_n \partial_i u_i \cdot \nabla_v \partial_v f^{n+1} + 4 \partial_i (\rho_n \partial_v f^{n+1}) + \alpha \partial_i \rho_n \rho_{f_{n-1}} \partial_v (M(f^n) - f^{n+1}) \right) \]
\[ + \rho_n \partial_i \partial_v f^{n+1} e^{2(v)^k} \partial_i \partial_v f^{n+1} \, dx \, dv \]
\[ \leq - \frac{k}{2} \int_{T^3 \times \mathbb{R}^3} \rho_n \langle v \rangle^{k-2} |v|^2 e^{2(v)^k} | \partial_i \partial_v f^{n+1} |^2 \, dx \, dv \]
\[ + \int_{T^3 \times \mathbb{R}^3} \partial_i \rho_n v_1 \mathbb{1}_{\{|v|>1\}} \cdot (\nabla_v \partial_i \partial_v f^{n+1}) e^{2(v)^k} \, dx \, dv \]
\[ + C(1 + M)^{\frac{3k}{2}} + \frac{C(1 + M)^{n+2} e^{C(1 + M)^\frac{3k}{2}}}{\| f^{n+1} \|_{L_k^2}^2 + 1} \]
\[ + C(1 + M)^{2+\alpha} (\| \partial_i \partial_v f \|_{L_k^2}^2 + \| f^{n+1} \|_{H_k^2}^2) \| \partial_i \partial_v f^{n+1} \|_{L_k^2} \]
\[ \leq - \frac{2k}{5} \int_{T^3 \times \mathbb{R}^3} \rho_n \langle v \rangle^{k-2} |v|^2 e^{2(v)^k} | \partial_i \partial_v f^{n+1} |^2 \, dx \, dv \]
\[ + \frac{2 - k}{200} \int_{T^3 \times \mathbb{R}^3} \rho_n \langle v \rangle^{k-2} |v|^2 e^{2(v)^k} | \nabla_v \partial_v f^{n+1} |^2 \, dx \, dv \]
\[ + C(1 + M)^{\frac{3k}{2}} + 20 + \alpha e^{C(1 + M)^\frac{3k}{2}} \left( \| f^{n+1} \|_{L_k^2}^2 + 1 \right), \]
where we used
\[ \left| \int_{T^3 \times \mathbb{R}^3} \partial_i \rho_n v_1 \mathbb{1}_{\{|v|>1\}} \cdot \nabla_v \partial_i \partial_v f^{n+1} \partial_i \partial_v f^{n+1} e^{2(v)^k} \, dx \, dv \right| \]
\[ \leq \frac{k}{400} \int_{T^3 \times \mathbb{R}^3} \rho_n \langle v \rangle^{k-2} |v|^2 e^{2(v)^k} | \partial_i \partial_v f^{n+1} |^2 \, dx \, dv \]
\[ + \frac{2 - k}{200} \int_{T^3 \times \mathbb{R}^3} \rho_n \langle v \rangle^{k-2} |v|^2 e^{2(v)^k} | \nabla_v \partial_v f^{n+1} |^2 \, dx \, dv \]
\[ + C(1 + M)^{\frac{3k}{2}} + 20 + \alpha e^{C(1 + M)^\frac{3k}{2}} \left( \| f^{n+1} \|_{L_k^2}^2 + 1 \right). \]
Thus we can get due to (3.7) and (3.8) and
\[
\int_{T^3} |\partial_t \rho_f\|_{H^1}^2 + |\partial_t v_f\|_{H^1}^2 + |\partial_t f^{n+1}\|_{L^2}^2 \leq C(1 + M)^{\alpha + 1} \|\partial_t \rho_f\|_{H^1}^2 + \|\partial_t v_f\|_{H^1}^2 + |\partial_t f^{n+1}\|_{L^2}^2
\]

Thus we can get
\[
\frac{d}{dt} \|\nabla_x \nabla_v f^{n+1}\|_{L^2_k}^2 \leq C(1 + M)^{\frac{1}{2} - \alpha} \frac{4k}{5} \int_{T^3} \rho^n |\nabla_v f^{n+1}|^2 \|\nabla_x \nabla_v f^{n+1}\|_{L^2_k}^2 \]
\[
+ \frac{k}{10} \int_{T^3} \rho^n |\nabla_v f^{n+1}|^2 \|\nabla_x \nabla_v f^{n+1}\|_{L^2_k}^2 \]
\]

Finally, we estimate
\[
\frac{1}{2} \frac{d}{dt} \|\partial_{v,v} f^{n+1}\|_{L^2_k}^2 \leq \int_{T^3} \left( v \cdot \nabla_{v,v} f^{n+1} + \partial_i \partial_j f^{n+1} + \partial_j f^{n+1} + \rho^n (u^n - v) \cdot \nabla_v \partial_{v,v} f^{n+1} \right)
\]
\[
- 5 \rho^n \partial_{v,v} f^{n+1} - \rho^n \partial_{v,v} (M(f^n) - f^{n+1}) e^{2(nv)} \partial_{v,v} f^{n+1}
\]
\[
\leq \frac{1}{2} \int_{T^3} \left|\partial_{v,v} f^{n+1}\right|^2 \|\nabla_v \left( (\rho^n (u^n - v) e^{2(nv)} \right) \|dx dv + C(1 + M)^{\frac{1}{2} - \alpha} \|f^{n+1}\|_{H^k}^2
\]
\[
+ C(1 + M)^{\alpha} \|\partial_{v,v} M(f^n)\|_{L^2_k} \|\partial_{v,v} f^{n+1}\|_{L^2_k}
\]
\[
\leq - \frac{k}{2} \int_{T^3} \rho^n |\nabla_v f^{n+1}|^2 \|\partial_{v,v} f^{n+1}\|_{L^2_k}^2 \]
\[
+ C(1 + M)^{\frac{1}{2} - \alpha} \|f^{n+1}\|_{H^k}^2 + \|f^{n+1}\|_{H^k}^2 + 1.
\]

and this gives
\[
\frac{d}{dt} \|\nabla_x f^{n+1}\|_{L^2_k}^2 \leq - k \int_{T^3} \rho^n |\nabla_v f^{n+1}|^2 \|\nabla_x f^{n+1}\|_{L^2_k}^2 \]
\[
+ C(1 + M)^{\frac{1}{2} - \alpha} \|f^{n+1}\|_{H^k}^2 + \|f^{n+1}\|_{H^k}^2 + 1.
\]

Thus, we gather all the estimates (3.6), (3.11), (3.12), (3.13), and (3.14) to yield
\[
\frac{d}{dt} \|f^{n+1}\|_{H^k}^2 \leq C(1 + M)^{\frac{1}{2} - \alpha} \frac{4k}{5} \|f^{n+1}\|_{H^k}^2 + \|f^{n+1}\|_{H^k}^2 + 1.
\]

From the above, we can find
\[
\|f^{n+1}(\cdot,\cdot, t)\|_{H^k}^2 \leq \frac{\|f_0\|_{H^k}^2 + 1}{\left(1 - \frac{1}{2} C(1 + M)^{\frac{1}{2} - \alpha} \frac{4k}{5} + 20 + C(1 + M)^{\frac{1}{2} - \alpha} \|f^{n+1}\|_{H^k}^2 + \|f^{n+1}\|_{H^k}^2 + 1 \right)^2} - 1.
\]

Since the right hand side of the above decays to \(\|f_0\|_{H^k}^2\) as \(t \to 0\) and \(\|f_0\|_{H^k}^2 < N < M\), we can find a sufficiently small \(0 < T_4 \leq T_3\) to obtain the desired result.

**Proof of Proposition 3.1.** We now choose \(T^* := T_3\), then by strong induction, this directly concludes the uniform-in-n estimates of approximations in the desired solution space.
4. PROOF OF THEOREM 4.1

4.1. Cauchy estimates. To obtain the unique regular solution to (3.1), we investigate the following Cauchy estimates.

**Lemma 4.1.** Suppose that the initial data \((f_0, \rho_0, u_0)\) satisfy the conditions in Theorem 4.1. Then for any \(\epsilon > 0\) with \(\epsilon \in (0, k)\) we have

\[
\frac{d}{dt} \| f^{n+1} - f^n \|_{L_{k-\epsilon}^2}^2 \leq C \left( \| f^{n+1} - f^n \|_{L_{k-\epsilon}^2}^2 + \| h^n - h^{n-1} \|_{H^1} + \| u^n - u^{n-1} \|_{H^1} + \| f^n - f^{n-1} \|_{L_{k-\epsilon}^2}^2 \right)
\]

for \(t \in [0, T^*]\).

**Remark 4.1.** Instead of employing \(L_{k-\epsilon}^2 (\mathbb{T}^3 \times \mathbb{R}^3)\) space, the above estimate can be also obtained in \(L_{k,q}^2 (\mathbb{T}^3 \times \mathbb{R}^3)\) with \(q < 1\) defined by

\[
L_{k,q}^2 (\mathbb{T}^3 \times \mathbb{R}^3) := \left\{ f : \int_{\mathbb{T}^3 \times \mathbb{R}^3} e^{2q(\nu)_k} |f|^2 \, dx \, dv < \infty \right\}.
\]

**Proof.** Direct computation gives

\[
\frac{1}{2} \frac{d}{dt} \| f^{n+1} - f^n \|_{L_{k-\epsilon}^2}^2 = - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left( \nu \cdot \nabla (f^{n+1} - f^n) + \nabla \nu \cdot ((\rho^n - \rho^{n-1})(u^n - v)f^{n+1}) + \nabla \nu \cdot (\rho^{n-1}(u^n - u^n) f^{n+1}) + \nabla \nu \cdot (\rho^{n-1}(u^n - v)(f^{n+1} - f^n)) - (\rho^n - \rho^{n-1})(\mathcal{M}(f^n) - f^{n+1}) \right.
\]
\[
\left. - \rho^n_\sigma (\mathcal{M}(f^n) - \mathcal{M}(f^{n-1} + \rho^n_\sigma (f^{n+1} - f^n)) e^{2(\nu)_k} (f^{n+1} - f^n) \, dx \, dv \right.
\]
\[
= \sum_{i=1}^7 l_i.
\]

We first easily find that \(l_1 = 0\). For \(l_2\), we use the fact that

\[
\int_{\mathbb{R}^3} e^{a(\nu)_k - b(\nu)_k} \, dv < \infty
\]

for any \(a, b > 0\) to yield

\[
l_2 = 3 \int_{\mathbb{T}^3 \times \mathbb{R}^3} (\rho^n - \rho^{n-1}) f^{n+1} (f^{n+1} - f^n) e^{2(\nu)_k} \, dv
\]
\[
- \int_{\mathbb{T}^3 \times \mathbb{R}^3} (\rho^n - \rho^{n-1})(u^n - v) \cdot \nabla \nu f^{n+1} (f^{n+1} - f^n) e^{2(\nu)_k} \, dv
\]
\[
\leq C \left\langle \left( \int_{\mathbb{R}^3} (f^{n+1})^2 e^{2(\nu)_k} \, dv \right)^{1/2} \right\rangle_{H^1} \| h^n - h^{n-1} \|_{H^1} \| f^{n+1} - f^n \|_{L_{k-\epsilon}^2}
\]
\[
+ C \| u^n \|_{L^\infty} \left\langle \int_{\mathbb{R}^3} |\nabla \nu f^{n+1}| e^{2(\nu)_k} \, dv \right\rangle_{H^1} \| h^n - h^{n-1} \|_{H^1} \| f^{n+1} - f^n \|_{L_{k-\epsilon}^2}
\]
\[
+ C \left\langle \int_{\mathbb{R}^3} (\nu f^{n+1})^2 e^{2(\nu)_k} \, dv \right\rangle_{H^1} \| h^n - h^{n-1} \|_{H^1} \| f^{n+1} - f^n \|_{L_{k-\epsilon}^2}
\]
\[
\leq C \| h^n - h^{n-1} \|_{H^1} + \| f^{n+1} - f^n \|_{L_{k-\epsilon}^2}^2,
\]

where we used

\[
|\rho^n - \rho^{n-1}| = \left| (1 + h^n)^{\frac{1}{\alpha - 1}} - (1 + h^{n-1})^{\frac{1}{\alpha - 1}} \right| \leq C |h^n - h^{n-1}|.
\]

For \(l_3\), similarly as before, we obtain

\[
l_3 \leq C \| \rho^{n-1} \|_{L^\infty} \left\langle \int_{\mathbb{R}^3} |\nabla \nu f^{n+1}| e^{2(\nu)_k} \, dv \right\rangle_{H^1} \| u^n - u^{n-1} \|_{H^1} \| f^{n+1} - f^n \|_{L_{k-\epsilon}^2}
\]
\[
\leq C \| u^n - u^{n-1} \|_{H^1} + \| f^{n+1} - f^n \|_{L_{k-\epsilon}^2}^2.
\]
For $I_4$, we use integration by parts and Young's inequality to get
\[
I_4 = 3 \int_{T^3 \times R^3} \rho^{n-1}(f^{n+1} - f^n)^2 e^{2(v^{k-\varepsilon})} \, dx \, dv \\
+ \frac{1}{2} \int_{T^3 \times R^3} (f^{n+1} - f^n)^2 \nabla \cdot (\rho^{n-1}(u^n - v^{k-\varepsilon}) e^{2(v^{k-\varepsilon})}) \, dx \, dv \\
\leq C \|\rho^{n-1}\|_{L^\infty} \|f^{n+1} - f^n\|_{L^2_{k-\varepsilon}}^2 - \frac{k}{2} \int_{T^3 \times R^3} \rho^{n-1}(u^{k-2} |v| e^{2(v^{k-\varepsilon})} |f^{n+1} - f^n|^2 \, dx \, dv \\
+ \frac{k}{2} \int_{T^3 \times R^3} \rho^{n-1}(u^{k-2} |u^n|^2 e^{2(v^{k-\varepsilon})} |f^{n+1} - f^n|^2 \, dx \, dv \\
\leq C \|f^{n+1} - f^n\|_{L^2_{k-\varepsilon}}^2.
\]

For $I_5$, similarly as the estimate of $I_3$, we obtain
\[
I_5 \leq C \|\rho f^n - \rho f^{n-1}\|_{L^2} \|f^{n+1} - f^n\|_{L^2_{k-\varepsilon}}^2 \left( \left( \int_{R^3} (M(f^n)^2 + (f^{n+1})^2) e^{2(v^{k-\varepsilon})} \, dv \right)^{1/2} \right)_{L^\infty} \\
\leq C \|f^{n+1} - f^n\|^2_{L^2_{k-\varepsilon}} + C \|f^n - f^{n-1}\|^2_{L^2_{k-\varepsilon}},
\]
where we used (4.3) to get
\[
\left( \int_{R^3} M(f^n)^2 e^{2(v^{k-\varepsilon})} \, dv \right)^{1/2} \leq C \|\rho f^n\|_{L^\infty} \leq C \|\rho f^n\|_{H^2} \leq C.
\]

For $I_6$, we use Lemma 2.3 to obtain
\[
I_6 \leq C \|\rho f^{n-1}\|_{L^\infty} \|M(f^n) - M(f^{n-1})\|_{L^2_{k-\varepsilon}} \|f^{n+1} - f^n\|_{L^2_{k-\varepsilon}} \\
\leq C \|f^{n+1} - f^n\|^2_{L^2_{k-\varepsilon}} + C \|f^n - f^{n-1}\|^2_{L^2_{k-\varepsilon}}.
\]

Clearly, $I_7 \leq C \|f^{n+1} - f^n\|^2_{L^2_{k-\varepsilon}}$. Thus, we gather all the estimates for $I_i$’s and use Young’s inequality to have the desired result.

**Lemma 4.2.** Suppose that the initial data $(f_0, \rho_0, u_0)$ satisfy the conditions in Theorem 1.1. Then we have
\[
\frac{d}{dt} \|h^{n+1} - h^n\|^2_{H^1} \leq C \left( \|h^{n+1} - h^n\|^2_{H^1} + \|h^n - h^{n-1}\|^2_{H^1} + \|\nabla (u^{n+1} - u^n)\|^2_{L^2} + \|u^n - u^{n-1}\|^2_{H^1} \right) \\
+ \delta \|\nabla^2 (u^{n+1} - u^n)\|^2_{L^2}
\]
for $t \in [0, T^\ast]$, where $\delta > 0$ will be determined later.

**Proof.** We estimate
\[
\frac{1}{2} \frac{d}{dt} \|h^{n+1} - h^n\|^2_{L^2} = - \int_{T^3} (h^{n+1} - h^n) (u^n \cdot \nabla h^{n+1} - u^{n-1} \cdot \nabla h^n) \, dx \\
- \gamma + \frac{1}{2} \int_{T^3} (h^{n+1} - h^n) ((1 + h^n) \nabla \cdot (u^{n+1} - (1 + h^{n-1}) \nabla \cdot u^n) \, dx \\
= - \int_{T^3} (h^{n+1} - h^n) (u^n \cdot \nabla (h^{n+1} - h^n) + (u^n - u^{n-1}) \cdot \nabla h^n) \, dx \\
- \gamma + \frac{1}{2} \int_{T^3} (h^{n+1} - h^n) ((h^n - h^{n-1}) \nabla \cdot u^n + (1 + h^n) \nabla \cdot (u^{n+1} - u^n)) \, dx \tag{4.1}
\]
\[
\leq \frac{\|\nabla u^n\|_{L^\infty}}{2} \|h^{n+1} - h^n\|^2_{L^2} + \|\nabla h^n\|_{L^\infty} \|h^{n+1} - h^n\|_{L^2} \|u^n - u^{n-1}\|_{L^2} \\
+ C \|\nabla u^n\|_{L^\infty} \|h^{n+1} - h^n\|_{L^2} \|h^n - h^{n-1}\|_{L^2} \\
+ C \|h^{n+1} - h^n\|_{L^2} \|u^{n+1} - u^n\|_{L^2} \\
\leq C \left( \|h^{n+1} - h^n\|^2_{L^2} + \|h^n - h^{n-1}\|^2_{L^2} + \|u^n - u^{n-1}\|^2_{L^2} + \|\nabla (u^{n+1} - u^n)\|^2_{L^2} \right)
\]
where we used Young’s inequality.
We then estimate that for \( i = 1, 2, 3 \)
\[
\frac{1}{2} \frac{d}{dt} \| \partial_t (h^{n+1} - h^n) \|_{L^2}^2 \\
= - \int_{\mathbb{T}^3} \partial_t (h^{n+1} - h^n) \left( \partial_t (u^n - u^{n-1}) \cdot \nabla h^n + (u^n - u^{n-1}) \cdot \nabla \partial_t h^n \right) dx \\
- \int_{\mathbb{T}^3} \partial_t (h^{n+1} - h^n) \left( \partial_t u^n \cdot \nabla (h^{n+1} - h^n) + u^n \cdot \nabla \partial_t (h^{n+1} - h^n) \right) dx \\
- \frac{\gamma - 1}{2} \int_{\mathbb{T}^3} \partial_t (h^{n+1} - h^n) \left( \partial_t (h^{n+1} - h^n) \nabla \cdot u^n + (h^n - h^{n-1}) \nabla \cdot \partial_t u^n \right) dx \\
- \frac{\gamma - 1}{2} \int_{\mathbb{T}^3} \partial_t (h^{n+1} - h^n) \left( \partial_t h^n \nabla \cdot (u^{n+1} - u^n) + (1 + h^n) \nabla \cdot \partial_t (u^{n+1} - u^n) \right) dx
\]
\[
\leq C \left( \| \nabla h^n \|_{L^\infty} \| \partial_t (u^n - u^{n-1}) \|_{L^2} + \| \nabla \partial_t h^n \|_{H^1} \| u^n - u^{n-1} \|_{H^1} \| \partial_t (h^{n+1} - h^n) \|_{L^2} \\
+ C \left( \| \partial_t u^n \|_{L^\infty} \| \nabla (h^{n+1} - h^n) \|_{L^2} + \| \nabla u^n \|_{L^\infty} \| \partial_t (h^{n+1} - h^n) \|_{L^2} \right) \| \partial_t (h^{n+1} - h^n) \|_{L^2} \\
+ C \left( \| \nabla u^n \|_{L^\infty} \| \partial_t (h^{n+1} - h^n) \|_{L^2} + \| \nabla \cdot \partial_t u^n \|_{H^2} \| u^n - u^{n-1} \|_{H^1} \| \partial_t (h^{n+1} - h^n) \|_{L^2} \\
+ C \| \partial_t h^n \|_{L^\infty} \| \nabla (u^{n+1} - u^n) \|_{L^2} \| \partial_t (h^{n+1} - h^n) \|_{L^2} \right) \\
+ C \| \partial_t (h^{n+1} - h^n) \|_{L^2}^2 + \frac{\delta}{6} \| \nabla^2 (u^{n+1} - u^n) \|_{L^2}^2,
\]
where \( \delta > 0 \) will be determined later. Thus we have
\[
\frac{d}{dt} \| \nabla (h^{n+1} - h^n) \|_{L^2}^2 \leq C \| \nabla (h^{n+1} - h^n) \|_{L^2}^2 + C \| h^n - h^{n-1} \|_{H^1}^2 \\
+ C \| \nabla (u^{n+1} - u^n) \|_{L^2}^2 + C \| \nabla (u^n - u^{n-1}) \|_{L^2}^2 \\
+ \frac{\delta}{6} \| \nabla^2 (u^{n+1} - u^n) \|_{L^2}^2.
\]

We then combine this and (1.1) to conclude the desired result. \( \square \)

**Lemma 4.3.** Suppose that the initial data \((f_0, \rho_0, u_0)\) satisfy the conditions in Theorem 1.1. Then we have
\[
\frac{d}{dt} \| u^{n+1} - u^n \|_{H^1}^2 + \left( \frac{c_0 \mu}{(1 + M) \frac{\mu}{\beta} - \delta} \right) \| \nabla (u^{n+1} - u^n) \|_{H^1}^2 \\
\leq C \left( \| u^n - u^{n-1} \|_{H^1}^2 + \| u^{n+1} - u^n \|_{H^1}^2 + \| h^n - h^{n-1} \|_{H^1}^2 + \| h^{n+1} - h^n \|_{H^1}^2 \right) \\
+ C \| f^n - f^{n-1} \|_{L^2_{L_k = \epsilon}}^2,
\]
for \( t \in [0, T^*] \), where \( \delta > 0 \), and \( c_0 \) is a positive constant satisfying
\[
\frac{1}{(1 + h^n)^{\frac{\mu}{2} - \eta}} \geq \frac{c_0}{(1 + M)^{\frac{\mu}{2} - \eta}} \geq \frac{16 \delta}{\mu} \quad \forall n \in \mathbb{N}.
\]

**Proof.** For smoothness of reading, we postpone this proof to Appendix C. \( \square \)

4.2. **Proof of Theorem 1.1**. Now, we are ready to prove Theorem 1.1

• **(Existence):** We first gather the estimates in Lemmas 4.1, 4.3, and choose \( \delta > 0 \) small enough to yield
\[
\frac{d}{dt} \left( \| f^{n+1} - f^n \|_{L^2_{L_k = \epsilon}}^2 + \| h^{n+1} - h^n \|_{H^1}^2 + \| u^{n+1} - u^n \|_{H^1}^2 \right) \\
+ \frac{c_0 \mu}{2(1 + M) \frac{\mu}{\beta} - \delta} \| \nabla (u^{n+1} - u^n) \|_{H^1}^2 \\
\leq C \left( \| f^n - f^{n-1} \|_{L^2_{L_k = \epsilon}}^2 + \| h^{n+1} - h^n \|_{H^1}^2 + \| u^{n+1} - u^n \|_{H^1}^2 \right) \\
+ C \left( \| f^n - f^{n-1} \|_{L^2_{L_k = \epsilon}}^2 + \| h^n - h^{n-1} \|_{H^1}^2 + \| u^n - u^{n-1} \|_{H^1}^2 \right).
\]

Now we set
\[
E^{n+1}(t) := \| (f^{n+1} - f^n)(t) \|_{L^2_{L_k = \epsilon}}^2 + \| (h^{n+1} - h^n)(t) \|_{H^1}^2 + \| (u^{n+1} - u^n)(t) \|_{H^1}^2,
\]
Here we note that $E^{n+1}(0) = 0$ for any $n \in \mathbb{N}$. Then we can rewrite (4.2) as
\[
\frac{d}{dt} E^{n+1}(t) + D^{n+1}(t) \leq C (E^{n+1}(t) + E^n(t)).
\] (4.3)

Then, we sum (4.3) over $n$ to get
\[
\frac{d}{dt} \left( \sum_{r=1}^{n} E^{r+1}(t) \right) + \sum_{r=1}^{n} D^{r+1}(t) \leq C \left( \sum_{r=1}^{n} E^{r+1}(t) + CE^1(t) \right)
\leq C \left( 1 + \sum_{r=1}^{n} E^{r+1}(t) \right),
\]
where we used the uniform-in-$n$ upper bound. Thus, we integrate the above relation with respect to $t$ and use Grönwall’s lemma to obtain
\[
\sum_{r=1}^{n} E^{r+1}(t) + \int_0^t \sum_{r=1}^{n} D^{r+1}(t) \leq e^{Ct} - 1
\]
for $t \in [0, T^*)$. Since $C > 0$ is independent of $n$, the above estimate implies that the sequence $\{(f^n, h^n, u^n)\}_{n \geq 1}$ of triplets forms a Cauchy sequence in
\[
C([0, T^*]; L^2_{k-\epsilon}(\mathbb{T}^3 \times \mathbb{R}^3)) \times C([0, T^*]; H^1(\mathbb{T}^3)) \times (C([0, T^*]; H^1(\mathbb{T}^3)) \cap L^2(0, T^*; H^2(\mathbb{T}^3)))
\]
and hence it converges to
\[
(f, h, u) \in C([0, T^*]; L^2_{k-\epsilon}(\mathbb{T}^3 \times \mathbb{R}^3)) \times C([0, T^*]; H^1(\mathbb{T}^3)) \times (C([0, T^*]; H^1(\mathbb{T}^3)) \cap L^2(0, T^*; H^2(\mathbb{T}^3))).
\]
Moreover, our uniform-in-$n$ upper bound estimates imply
\[
(f^n, h^n, u^n) \rightharpoonup (f, h, u) \quad \text{in} \quad L^\infty(0, T^*; H^2_k(\mathbb{T}^3 \times \mathbb{R}^3)),
\]
\[
(h^n, u^n) \rightharpoonup (h, u) \quad \text{in} \quad L^\infty(0, T^*; H^3(\mathbb{T}^3)), \quad \text{and}
\]
\[
\nabla^4 u^n \rightharpoonup \nabla^4 u \quad \text{in} \quad L^2(\mathbb{T}^3 \times (0, T^*)).
\]

Thus, we have found a limit $(f, n, u)$ satisfying
\[
f \in L^\infty(0, T^*; H^2_k(\mathbb{T}^3 \times \mathbb{R}^3)),
\]
\[
h \in L^\infty(0, T^*; H^3(\mathbb{T}^3)), \quad \text{and}
\]
\[
u \in L^\infty(0, T^*; H^3(\mathbb{T}^3)) \cap L^2(0, T^*; D^4(\mathbb{T}^3)).
\]

Necessary arguments for the time continuity can be found in [14, 30].

\begin{itemize}
\item (Uniqueness): Suppose that we have two regular solutions $(f_1, h_1, u_1)$ and $(f_2, h_2, u_2)$ to (1.1) corresponding to the same initial data $(f_0, h_0, u_0)$. Then, we can deduce from the arguments in Lemmas [4.1, 4.3] to get
\[
\frac{d}{dt} \left( \|f_1 - f_2\|_{L^2_{k-\epsilon}}^2 + \|h_1 - h_2\|_{H^1}^2 + \|u_1 - u_2\|_{H^1}^2 \right) + \|\nabla(u_1 - u_2)\|_{H^1}^2
\leq C \left( \|f_1 - f_2\|_{L^2_{k-\epsilon}}^2 + \|h_1 - h_2\|_{H^1}^2 + \|u_1 - u_2\|_{H^1}^2 \right),
\]
and Grönwall’s lemma gives the desired result.
\end{itemize}

5. Proof of Theorem [1.2]

In this section, we provide the details of proof of Theorem [1.2]. Since the main idea of proof is similar to that of Theorem [1.1] here we only give additional required estimates for $f$. We are now interested in the existence of $W^{1,\infty}_k$-solution $f$, thus we need to estimate $\|M(f)\|_{W^{1,\infty}_k}$ parallel to Lemma [2.3].
Lemma 5.1. Suppose that $\|f\|_{W^{1,\infty}_k} < \infty$ for $k \in (1, 2)$ and $p, u$ and $T$ satisfy

$$\rho_f + |u_f| + T_f < c_1, \quad \rho_f > c_2, \quad \text{and} \quad T_f > c_3^{-1}. \tag{2.3}$$

Then we have

$$\|M(f)\|_{W^{1,\infty}_k} \leq C(1 + c_3^2)e^{C_{c_1}} \|f\|_{W^{1,\infty}_k}^2 = (1 + \|f\|_{W^{1,\infty}_k}),$$

where $C$ depends only on $c_2$ and $k$. For the derivatives, direct computation gives

$$\partial_i M(f) = \rho_f (u_f - v_f) \rho_f e^{-|v-u|^2/2T_f^2} \leq \frac{1}{2} \rho_f \int_{\mathbb{R}^3} |v - u_f|^2 \partial_i f dv + \frac{1}{\rho_f} \int_{\mathbb{R}^3} |v| \partial_i f dv \leq C(1 + |u_f|) \|f\|_{W^{1,\infty}_k},$$

and

$$\|\partial_i f\|_{W^{1,\infty}_k} \leq C \|\partial_i f\|_{L^\infty_k} + C \|\partial_i f\|_{L^\infty_k},$$

where we used

$$\frac{1}{\rho_f} \int_{\mathbb{R}^3} |u_f - v| \partial_i u_f |f dv \leq \frac{1}{\rho_f} \left( \int_{\mathbb{R}^3} |u_f - v|^2 f dv \right)^{1/2} \left( \int_{\mathbb{R}^3} |\partial_i u_f|^2 f dv \right)^{1/2} \leq \sqrt{T_f} \|\partial_i u_f\|_{L^\infty}.$$

Thus, together with (5.1) and the estimate from [17] Lemma 2.4, we can get the desired estimate. \qed

Analogously as in Section 9 we consider the approximations $\overline{v}_k^{n_0}$ and assume

$$\min_{1 \leq m \leq n} \overline{v}_k^{n_0}(T) < M \quad \text{and} \quad \inf_{1 \leq m \leq n} \inf_{(x, t) \in \mathbb{R}^3 \times [0, T]} (1 + h^m(x, t)) > \frac{\delta}{2},$$

where

$$\overline{v}_k^{n_0}(T) := \max \left\{ \sup_{0 \leq t \leq T} \|f^n(\cdot, t)\|_{W^{1,\infty}_k}, \sup_{0 \leq t \leq T} \|f^n(\cdot, t)\|_{H^1_k}, \sup_{0 \leq t \leq T} \left( \frac{4\gamma}{(\gamma - 1)^2} \|h^n(\cdot, t)\|_{H^3}^2 + \|u^n(\cdot, t)\|_{H^3}^2 \right) \right\}.$$
Lemma 5.2. Suppose that the initial data \((f_0, \rho_0, u_0)\) satisfy the conditions in Theorem 1.2. If \((3.3)\) holds, then we can find \(0 < T_5 \leq T_4\) depending only on \(M\) and \(N\) such that
\[
\sup_{0 \leq t \leq T_4} \|f^{n+1}(t)\|_{W^{1,\infty}_{1,\infty}} < M.
\]

Proof: For this, we will use the forward characteristics \((3.4)\). We separately estimate the zeroth-order and first-order estimates as follows:

• (Step A: Zeroth-order estimates) First, we have
\[
\frac{1}{2} \frac{d}{dt} \left| e^{(V^{n+1})^k} f^{n+1}(Z^{n+1}(t), t) \right|^2
\]
\[
= e^{2(V^{n+1})^k} f^{n+1}(Z^{n+1}(t), t) \left( k(V^{n+1})^k - 2 V^{n+1} \cdot \frac{dV^{n+1}}{dt} f^{n+1}(Z^{n+1}(t), t) + \frac{d}{dt} f^{n+1}(Z^{n+1}(t), t) \right)
\]
\[
= e^{2(v)^k} f^{n+1}(z, t) \left( k(v)^k - \rho^n(x, t)(u^n(x, t) - v) \cdot v f^{n+1}(z, t) + 3 \rho^n(x, t) f^{n+1}(z, t) + \rho^n f^n(x, t)(\mathcal{M}(f^n)(z, t) - f^{n+1}(z, t)) \right)_{z = Z^{n+1}(t)}
\]
\[
\leq -\frac{k}{2} \rho^n (V^{n+1})^k - 2 |V^{n+1}|^2 e^{2(V^{n+1})^k} (f^{n+1}(Z^{n+1}(t), t))^2 + C(1 + M) e^{2(V^{n+1})^k} (f^{n+1}(Z^{n+1}(t), t))^2 + C(1 + M) e^{2(V^{n+1})^k} \left[ \rho^n f^n \mathcal{M}(f^n) f^{n+1} \right] (Z^{n+1}(t), t)
\]
\[
\leq C(1 + M) \frac{1}{2} \|f^{n+1}\|_{L^2}^2 + C e^{C(1 + M)^{\frac{2k}{2}}}(1 + M)^{4 + \alpha} \|f^{n+1}\|_{L^\infty}^2
\]
\[
\leq C e^{C(1 + M)^{\frac{2k}{2}}}(1 + M)^{1 + 4 \alpha} (\|f^{n+1}\|_{L^\infty}^2 + 1),
\]
where we used Young’s inequality and Lemmas 2.4 and 3.4. We take the supremum over all possible characteristics to get
\[
\|f^{n+1}(t)\|_{L^\infty}^2 \leq \|f_0\|_{L^\infty}^2 + C e^{C(1 + M)^{\frac{2k}{2}}}(1 + M)^{1 + 4 \alpha} \int_0^t \|f^{n+1}(s)\|_{L^\infty}^2 ds
\]
\[
+ C e^{C(1 + M)^{\frac{2k}{2}}}(1 + M)^{1 + 4 \alpha} \}
\]
• (Step B: First-order estimates) We first investigate \(\partial_t f^{n+1}\). One finds
\[
\partial_t \partial_t f^{n+1} + v \cdot \nabla (\partial_t f^{n+1}) + \rho^n (u^n - v) \cdot \nabla \partial_t f^{n+1}
\]
\[
= -\partial_v \partial_v (u^n - v) \cdot \nabla f^{n+1} + \rho^n \partial_t u^n \cdot \nabla f^{n+1} + 3 \partial_t \partial_t f^{n+1} + 3 \rho^n \partial_t f^{n+1} + \partial_t (\rho^n f^n (\mathcal{M}(f^n) - f^{n+1}))
\]
Then we use the forward characteristics to get
\[
\frac{1}{2} \frac{d}{dt} \left| e^{(V^{n+1})^k} \partial_t f^{n+1}(Z^{n+1}(t), t) \right|^2
\]
\[
= e^{2(v)^k} \partial_t f^{n+1} \left( k(v)^k - \rho^n (u^n - v) \cdot v f^{n+1} - \partial_t \rho^n (u^n - v) \cdot \nabla f^{n+1} + \rho^n \partial_t u^n \cdot \nabla f^{n+1} \right.
\]
\[
+ \rho^n \partial_t f^{n+1} + 3 \partial_t \partial_t f^{n+1} + 3 \rho^n \partial_t f^{n+1} + \rho^n \partial_t f^{n+1} + \partial_t (\rho^n (\mathcal{M}(f^n) - f^{n+1})) \right)_{z = Z^{n+1}(t)}
\]
\[
\leq e^{2(V^{n+1})^k} \left[ \partial_t f^{n+1} \right]^2 \left( \frac{k}{2} \rho^n (V^{n+1})^k - 2 |V^{n+1}|^2 + \frac{k}{2} (V^{n+1})^k - 2 \rho^n |u^n|^2 \right) (Z^{n+1}(t), t)
\]
\[
+ C \|\partial_t f^{n+1}\|_{L^\infty} \left( (1 + M) \frac{1}{2} \|f^{n+1}\|_{L^\infty} + (1 + M) \|f^{n+1}\|_{L^\infty} \right)
\]
\[
+ (1 + M)^{2 \alpha} \|\partial_t f^{n+1}\|_{L^\infty} + (1 + M)^{2 \alpha} (\|\mathcal{M}(f^n)\|_{W^{1,\infty}_{1,\infty}} + \|f^{n+1}\|_{L^\infty})\right)
\]
+ e^{2(V_{n+1}^k)} V_{n+1} \cdot (\partial_t \rho^n \nabla_v f_{n+1} \partial_t f_{n+1}) (Z_{n+1}^1(t), t) \\
\leq -\frac{k}{2} \rho^n (V_{n+1})^{k-2} |V_{n+1}|^2 e^{2(V_{n+1}^k)} (\partial_t f_{n+1})^2 (Z_{n+1}^1(t), t) + C(1 + M)^{\frac{2k}{k-1}} \| f_{n+1} \|_{W_k^1}^2 \\
+ C e^{C(1 + M)^{\frac{2k}{k-1}}} (1 + M)^{4+\alpha} \| \partial_t f_{n+1} \|_{L^\infty} + e^{2(V_{n+1}^k)} V_{n+1} \cdot (\partial_t \rho^n \nabla_v f_{n+1} \partial_t f_{n+1}) (Z_{n+1}^1(t), t),

where we used \( |\partial_t f_{n+1}| \leq C \| \partial_t f^n \|_{L^\infty} \), Young’s inequality and Lemmas 2.24 and 2.33. Here, note that the last term on the right hand side of the above can be estimated as

\[
\left| e^{2(V_{n+1}^k)} V_{n+1} \cdot (\partial_t \rho^n \nabla_v f_{n+1} \partial_t f_{n+1}) (Z_{n+1}^1(t), t) \right|
\leq C(1 + M)^{\frac{2k}{k-1}} \| f_{n+1} \|_{W_k^1}^2 \left( \mathbb{I}_{\{ |V_{n+1}| \leq 1 \}} + \mathbb{I}_{\{ |V_{n+1}| > 1 \}} \right)
\leq C(1 + M)^{\frac{2k}{k-1}} \| f_{n+1} \|_{W_k^1}^2 + e^{2(V_{n+1}^k)} V_{n+1} \cdot (\partial_t \rho^n \nabla_v f_{n+1} \partial_t f_{n+1}) (Z_{n+1}^1(t), t) \mathbb{I}_{\{ |V_{n+1}| > 1 \}}.
\]

Then we use the similar arguments as (3.7) and (3.8) to obtain

\[
\frac{d}{dt} \left[ e^{2(V_{n+1}^k)} \partial_t f_{n+1}^2 (Z_{n+1}^1(t), t) \right]^2 \\
\leq -\frac{k}{2} \rho^n (V_{n+1})^{k-2} |V_{n+1}|^2 e^{2(V_{n+1}^k)} (\partial_t f_{n+1})^2 (Z_{n+1}^1(t), t) \\
+ C e^{C(1 + M)^{\frac{2k}{k-1}}} (1 + M)^{\frac{2k}{k-1} + 4+2\alpha} \left( \| f_{n+1} \|_{W_k^1}^2 + 1 \right) \\
+ \frac{(2 - k) \rho^n (V_{n+1})^{k-2} |V_{n+1}|^2 e^{2(V_{n+1}^k)}}{200} \| \nabla_v f_{n+1} \|_{L^\infty}^2 (Z_{n+1}^1(t), t) \mathbb{I}_{\{ |V_{n+1}| \geq 1 \}}.
\]

Now, we deal with \( \partial_v f_{n+1} \). From direct computation,

\[
\partial_t (\partial_v f_{n+1}) + v \cdot \nabla (\partial_v f_{n+1}) + \rho^n (u_n - v) \cdot \nabla_v (\partial_v f_{n+1})
= -\partial_t f_{n+1} + 4\rho^n f_{n+1} + \rho^n (\partial_v M(f^n) - \partial_v f_{n+1}).
\]

Then we get

\[
\frac{1}{2} \frac{d}{dt} \left[ e^{2(V_{n+1}^k)} \partial_v f_{n+1}^2 (Z_{n+1}^1(t), t) \right]^2 \\
= e^{2(V_{n+1}^k)} \left[ \partial_v f_{n+1} \left( k (V_{n+1})^{k-2} V_{n+1} \cdot \frac{d}{dt} f_{n+1} + \frac{d}{dt} \partial_v f_{n+1} \right) \right] (Z_{n+1}^1(t), t) \\
= e^{2(V_{n+1}^k)} \left[ \partial_v f_{n+1} \left( k (V_{n+1})^{k-2} \rho^n (u_n - V_{n+1}) \cdot V_{n+1} \partial_v f_{n+1} \\
- \partial_t f_{n+1} + 4\rho^n \partial_v f_{n+1} + \rho^n (\partial_v M(f^n) - \partial_v f_{n+1}) \right) \right] (Z_{n+1}^1(t), t) \\
\leq -\frac{k}{2} \rho^n (V_{n+1})^{k-2} |V_{n+1}|^2 e^{2(V_{n+1}^k)} (\partial_v f_{n+1})^2 (Z_{n+1}^1(t), t) + C(1 + M)^{\frac{2k}{k-1}} \| f_{n+1} \|_{W_k^1}^2 \\
+ C e^{C(1 + M)^{\frac{2k}{k-1}}} (1 + M)^{4+\alpha} \| \partial_v f_{n+1} \|_{L^\infty},
\]

and this gives

\[
\frac{d}{dt} \left[ e^{2(V_{n+1}^k)} \partial_v f_{n+1}^2 (Z_{n+1}^1(t), t) \right]^2 \\
\leq -k \rho^n (V_{n+1})^{k-2} |V_{n+1}|^2 e^{2(V_{n+1}^k)} (\partial_v f_{n+1})^2 (Z_{n+1}^1(t), t) \\
+ C e^{C(1 + M)^{\frac{2k}{k-1}}} (1 + M)^{\frac{2k}{k-1} + 4+\alpha} \| f_{n+1} \|_{W_k^1}^2 + 1).
\]
Thus, we combine (5.3) with (5.4) to yield
\[
\frac{d}{dt} \left( \left| e^{(V^{n+1})^k} \nabla f^{n+1}(Z^{n+1}(t), t) \right|^2 + \left| e^{(V^{n+1})^k} \nabla v f^{n+1}(Z^{n+1}(t), t) \right|^2 \right)
\leq C e^{-C(1+M) \frac{2k}{k+1}} (1 + M) \frac{2k-1}{k+1} + 4 + \alpha \int_0^t \left| f^{n+1}(\cdot, \cdot, s) \right|^2_{W^{1,\infty}_k} ds
\]
We integrate the above relation with respect to \( t \) and take the supremum over all possible characteristics to obtain
\[
\| f^{n+1}(\cdot, \cdot, t) \|_{W^{1,\infty}_k}^2 \leq \| f_0 \|_{W^{1,\infty}_k}^2 + C e^{C(1+M) \frac{2k}{k+1}} (1 + M) \frac{2k-1}{k+1} + 4 + \alpha \int_0^t \| f^{n+1}(\cdot, \cdot, s) \|_{W^{1,\infty}_k}^2 ds
\]
and hence, we can choose a sufficiently small \( 0 < T_5 \leq T_4 \) which gives the desired estimate. \( \square \)

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Appendix A. Proof of Lemma 2.2

In this appendix, we provide the details of the proof of Lemma 2.2.

For the zeroth-order estimate, we readily find
\[
\rho f \leq \int_{\mathbb{R}^3} f dv \leq C \left( \int_{\mathbb{R}^3} e^{2(v)^k} f^2 dv \right)^{1/2}, \quad |u f| \leq \frac{1}{c_2} \int_{\mathbb{R}^3} |v| f dv \leq C \left( \int_{\mathbb{R}^3} e^{2(v)^k} f^2 dv \right)^{1/2},
\]
and
\[
T_f \leq \frac{1}{3c_2} \int_{\mathbb{R}^3} |v|^2 f dv \leq C \left( \int_{\mathbb{R}^3} e^{2(v)^k} f^2 dv \right)^{1/2}.
\]
Thus,
\[
\|\rho f\|_{L^2} + \|u f\|_{L^2} + \|T_f\|_{L^2} \leq C \|f\|_{L^2_k}
\]
for some \( C > 0 \) depends only on \( c_2 \) and \( k \).

We next estimate that for \( i = 1, 2, 3 \)
\[
|\partial_i \rho f| \leq \int_{\mathbb{R}^3} |\partial_i f| dv \leq C \left( \int_{\mathbb{R}^3} e^{2(v)^k} |\partial_i f|^2 dv \right)^{1/2},
\]
\[
|\partial_i u f| \leq \frac{|\partial_i \rho f|}{\rho f} \int_{\mathbb{R}^3} v f dv + \frac{1}{\rho f} \int_{\mathbb{R}^3} |v| |\partial_i f| dv 
\leq C |\partial_i \rho f| \left( \int_{\mathbb{R}^3} e^{2(v)^k} f^2 dv \right)^{1/2} + C \left( \int_{\mathbb{R}^3} e^{2(v)^k} |\partial_i f|^2 dv \right)^{1/2},
\]
and
\[
|\partial_i T_f| \leq \frac{|\partial_i \rho f|}{3c_2} \int_{\mathbb{R}^3} |v - u f|^2 f dv + \frac{1}{\rho f} \int_{\mathbb{R}^3} |u f - v| |\partial_i f| dv + \frac{1}{\rho f} \int_{\mathbb{R}^3} |u f - v| |\partial_i u f| f dv 
\leq C |\partial_i \rho f| \int_{\mathbb{R}^3} |v|^2 f dv + C \|u f\|_{L^\infty}^2 |\partial_i \rho f| + C \|u f\|_{L^\infty}^2 \int_{\mathbb{R}^3} |\partial_i f| dv + C \int_{\mathbb{R}^3} |v|^2 |\partial_i f| dv 
\leq C \|u f\|_{L^\infty} |\partial_i u f| + |\partial_i u f| \int_{\mathbb{R}^3} |v| f dv,
\[
\leq C (|\partial_i \rho_f| + |\partial_i u_f|) \left( \int_{\mathbb{R}^3} e^{2(v_x)} f^2 \, dv \right)^{1/2} + C \left( \|u_f\|_{H^2}^2 + 1 \right) \left( \int_{\mathbb{R}^3} e^{2(v_x)} |\partial_i f|^2 \, dv \right)^{1/2} \\
+ C \|u_f\|_{H^2}^2 |\partial_i \rho_f| + C |u_f|_{H^2} |\partial_i u_f|.
\]

This together with Lemma 5.1 yields
\[
\|\partial_i \rho_f\|_{L^2} \leq C \|\partial_i f\|_{L^2} \leq C \|f\|_{H^1},
\]
\[
\|\partial_i u_f\|_{L^2} \leq C \|\partial_i \rho_f\|_{H^1} \left( \int_{\mathbb{R}^3} e^{2(v_x)} f^2 \, dv \right)^{1/2} \leq C \|\partial_i f\|_{L^2} \leq C \|f\|_{H^2},
\]
and
\[
\|\partial_i T_f\|_{L^2} \leq C \|\partial_i \rho_f\|_{H^1} \left( \int_{\mathbb{R}^3} e^{2(v_x)} f^2 \, dv \right)^{1/2} \leq C \|\partial_i f\|_{L^2} \leq C \|f\|_{H^2}.
\]
We finally obtain that for \(i, j = 1, 2, 3\)
\[
\|\partial_{ij} \rho_f\|_{L^2} \leq C \left( \int_{\mathbb{R}^3} e^{2(v_x)} |\partial_{ij} f|^2 \, dv \right)^{1/2} \leq C \|\partial_{ij} f\|_{L^2},
\]
\[
\|\partial_{ij} u_f\|_{L^2} \leq C \left( \int_{\mathbb{R}^3} e^{2(v_x)} |\partial_{ij} f|^2 \, dv \right)^{1/2} \leq C \|\partial_{ij} f\|_{L^2} \leq C \|f\|_{H^2} \left( 1 + \|f\|_{H^2}^3 \right),
\]
where we used
\[
\left( \int_{\mathbb{R}^3} e^{2(v_x)} f^2 \, dv \right)^{1/2} \leq C \|f\|_{H^2} \left( 1 + \|f\|_{H^2}^2 \right)
\]
due to Lemma 5.1. We also deduce
\[
\|\partial_{ij} T_f\|_{L^2} \leq C \left( \int_{\mathbb{R}^3} e^{2(v_x)} f^2 \, dv \right)^{1/2} \leq C \|f\|_{H^2} \left( 1 + \|f\|_{H^2}^2 \right)
\]
\[ C > 0 \text{ depends only on } c_2 \text{ and } k. \] Combining the above estimates, we conclude the desired results.

**Appendix B. Proof of Lemma 3.1**

In this appendix, we present the proof of Lemma 3.1. We separate the proof of zeroth and higher order estimates as follows:

- **(Step A: Zeroth-order estimates)** First, we have

\[
\frac{1}{2} \frac{d}{dt} \| h^{n+1} \|^2_{L^2} = - \int_{\mathbb{T}^3} (u^n \cdot \nabla h^{n+1}) h^{n+1} \, dx - \frac{\gamma - 1}{2} \int_{\mathbb{T}^3} (1 + h^n) h^{n+1} (\nabla \cdot u^{n+1}) \, dx
\]

\[
= \frac{1}{2} \int_{\mathbb{T}^3} (\nabla \cdot u^n) |h^{n+1}|^2 \, dx - \frac{\gamma - 1}{2} \int_{\mathbb{T}^3} (1 + h^n) h^{n+1} (\nabla \cdot u^{n+1}) \, dx
\]

\[
\leq C \| \nabla u^n \|_{L^\infty} \| h^{n+1} \|^2_{L^2} - \frac{\gamma - 1}{2} \int_{\mathbb{T}^3} (1 + h^n) h^{n+1} (\nabla \cdot u^{n+1}) \, dx,
\]

where \( C > 0 \) is a constant independent of \( n \) and \( T \). Using

\[
\frac{1}{(1 + h^n)^{\frac{\gamma - \nu}{\nu}}} \geq \frac{c_0}{(1 + M)^{\frac{\gamma - \nu}{\nu}}} \quad \forall n \in \mathbb{N}
\]

for some \( c_0 > 0 \) independent of \( n \), we also get

\[
\frac{1}{2} \frac{d}{dt} \| u^{n+1} \|^2_{L^2} = - \int_{\mathbb{T}^3} (u^n \cdot \nabla u^{n+1}) \cdot u^{n+1} \, dx - \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} (1 + h^n) \nabla h^{n+1} \cdot u^{n+1} \, dx
\]

\[
+ \int_{\mathbb{T}^3} \frac{\mu \Delta u^{n+1}}{(1 + h^n)^{\frac{\gamma - \nu}{\nu}}} \cdot u^{n+1} \, dx - \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u^n - v) \cdot u^{n+1} f^n \, dxdv
\]

\[
= \frac{1}{2} \int_{\mathbb{T}^3} (\nabla \cdot u^n) |u^{n+1}|^2 \, dx - \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} (\nabla h^n \cdot u^{n+1}) h^{n+1} \, dx
\]

\[
+ \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} (1 + h^n) h^{n+1} (\nabla \cdot u^{n+1}) \, dx + \frac{2\mu}{\gamma - 1} \int_{\mathbb{T}^3} \frac{(\nabla h^n \cdot \nabla u^{n+1})}{(1 + h^n)^{\frac{\gamma - \nu}{\nu}}} \cdot u^{n+1} \, dx
\]

\[
- \mu \int_{\mathbb{T}^3} \frac{|\nabla u^{n+1}|^2}{(1 + h^n)^{\frac{\gamma - \nu}{\nu}}} \, dx - \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u^n - v) \cdot u^{n+1} f^n \, dxdv
\]
\[
\begin{align*}
&\leq C\|\nabla u^n\|_{L^\infty} \|u^{n+1}\|^2_{L^2} + C\|\nabla h^n\|_{L^\infty} \|u^{n+1}\|_{L^2} \|h^{n+1}\|_{L^2} \\
&+ \frac{2\gamma}{\gamma - 1} \int_{T^3} (1 + h^n)h^{n+1}(\nabla \cdot u^{n+1}) \, dx + C\delta \frac{2\gamma + 1}{\gamma - 1} \|\nabla h^n\|_{L^\infty} \|\nabla u^{n+1}\|_{L^2} \|u^{n+1}\|_{L^2} \\
&- \frac{c_0 \mu}{(1 + M)^\frac{2\gamma + 1}{\gamma - 1}} \|\nabla u^{n+1}\|^2_{L^2} + C(1 + \|u^n\|_{L^\infty}) \|u^{n+1}\|_{L^2} \|f^n\|_{L^2} \\
&\leq C(1 + M) \left( \|u^{n+1}\|^2_{L^2} + \|h^{n+1}\|^2_{L^2} \right) + C(1 + M) \frac{2\gamma}{\gamma - 1} \|u^{n+1}\|^2_{L^2} \\
&+ \frac{2\gamma}{\gamma - 1} \int_{T^3} (1 + h^n)h^{n+1}(\nabla \cdot u^{n+1}) \, dx \\
&- \frac{c_0 \mu}{2(1 + M)^\frac{2\gamma + 1}{\gamma - 1}} \|\nabla u^{n+1}\|^2_{L^2} + C(1 + M)^2 \|u^{n+1}\|_{L^2} \\
&\leq C(1 + M)^{\frac{2\gamma}{\gamma - 1}} \left( \frac{4\gamma}{(\gamma - 1)^2} \|h^{n+1}\|^2_{L^2} + \|u^{n+1}\|^2_{L^2} \right) - \frac{c_0 \mu}{2(1 + M)^\frac{2\gamma + 1}{\gamma - 1}} \|\nabla u^{n+1}\|^2_{L^2} \\
&+ \frac{2\gamma}{\gamma - 1} \int_{T^3} (1 + h^n)h^{n+1}(\nabla \cdot u^{n+1}) \, dx + C(1 + M)^2.
\end{align*}
\]

Here \(c_0 > 0\) and \(C = C(\delta, \gamma) > 0\) are constants independent of \(n\) and \(T\).

Then we combine the above two estimates to yield
\[
\begin{align*}
\frac{d}{dt} \left( \frac{4\gamma}{(\gamma - 1)^2} \|h^{n+1}\|^2_{L^2} + \|u^{n+1}\|^2_{L^2} \right) + \frac{c_0 \mu}{(1 + M)^{\frac{2\gamma + 1}{\gamma - 1}}} \|\nabla u^{n+1}\|^2_{L^2} \\
&\leq C(1 + M)^{\frac{2\gamma}{\gamma - 1}} \left( \frac{4\gamma}{(\gamma - 1)^2} \|h^{n+1}\|^2_{L^2} + \|u^{n+1}\|^2_{L^2} \right) + C(1 + M)^2,
\end{align*}
\]

where \(C = C(\delta, \gamma) > 0\) is a constant independent of \(n\) and \(T\).

- (Step B: Higher-order estimates) For \(1 \leq \ell \leq 3\), we first estimate
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial^\ell h^{n+1}\|^2_{L^2} \\
&= - \int_{T^3} (u^n \cdot \nabla \partial^\ell h^{n+1}) \partial^\ell h^{n+1} \, dx - \int_{T^3} \partial^\ell (u^n \cdot \nabla h^{n+1}) - u^n \cdot \nabla \partial^\ell h^{n+1} \|dx \\
&- \frac{\gamma - 1}{2} \int_{T^3} (1 + h^n)(\nabla \cdot \partial^\ell u^{n+1}) \partial^\ell h^{n+1} \, dx \\
&- \frac{\gamma - 1}{2} \int_{T^3} \partial^\ell ((1 + h^n)\nabla \cdot u^{n+1}) - (1 + h^n)\nabla \cdot \partial^\ell u^{n+1} \| \partial^\ell h^{n+1} \, dx \\
&\leq C\|\nabla u^n\|_{L^\infty} \|\partial^\ell h^{n+1}\|^2_{L^2} + C\|\partial^\ell h^{n+1}\|_{L^2} \left( \|\nabla u^n\|_{L^\infty} \|\partial^\ell h^{n+1}\|_{L^2} + \|\nabla h^{n+1}\|_{L^\infty} \|\partial^\ell h^{n+1}\|_{L^2} \right) \\
&+ C\|\partial^\ell h^{n+1}\|_{L^2} \left( \|\nabla h^n\|_{L^\infty} \|\partial^\ell u^{n+1}\|_{L^2} + \|\nabla \cdot u^{n+1}\|_{L^\infty} \|\partial^\ell h^{n+1}\|_{L^2} \right) \\
&- \frac{\gamma - 1}{2} \int_{T^3} (1 + h^n)(\nabla \cdot \partial^\ell u^{n+1}) \partial^\ell h^{n+1} \, dx \\
&\leq C(1 + M) \left( \frac{4\gamma}{(\gamma - 1)^2} \|\nabla h^{n+1}\|^2_{H^2} + \|\nabla u^{n+1}\|^2_{H^2} \right) - \frac{\gamma - 1}{2} \int_{T^3} (1 + h^n)(\nabla \cdot \partial^\ell u^{n+1}) \partial^\ell h^{n+1} \, dx,
\end{align*}
\]

where we used Lemma 2.6 and \(C = C(\gamma) > 0\) is a constant independent of \(n\) and \(T\). One also obtains
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial^\ell u^{n+1}\|^2_{L^2} \\
&= - \int_{T^3} (u^n \cdot \nabla \partial^\ell u^{n+1}) \partial^\ell u^{n+1} \, dx - \int_{T^3} \partial^\ell (u^n \cdot \nabla u^{n+1}) - u^n \cdot \nabla \partial^\ell u^{n+1} \|dx \\
&- \frac{2\gamma}{\gamma - 1} \int_{T^3} \partial^\ell ((1 + h^n)\nabla h^{n+1}) \partial^\ell u^{n+1} \, dx + \mu \int_{T^3} \partial^\ell \left( \frac{\Delta u^{n+1}}{(1 + h^n)^{\frac{\gamma + 1}{\gamma - 1}}} \right) \partial^\ell u^{n+1} \, dx \\
&- \int_{T^3 \times \mathbb{R}^3} \partial^\ell ((u^n - v)f^n) \cdot \partial^\ell u^{n+1} \, dx.
\end{align*}
\]
We separately estimate $l_i$'s as follows:

(Step B-1: Estimates for $l_1$ and $l_2$) We can estimate these two terms as

$$l_1 \leq C\|\nabla u^n\|_{L^\infty} \|\partial^j u^{n+1}\|_{L^2}^2 \leq C(1 + M)\|\nabla u^{n+1}\|_{H^2}^2,$$

$$l_2 \leq C\|\partial^j u^{n+1}\|_{L^2} (\|\nabla u^n\|_{L^\infty} \|\nabla u^{n+1}\|_{L^2} + \|\nabla u^{n+1}\|_{L^\infty} \|\partial^j u^n\|_{L^2}) \leq C(1 + M)\|\nabla u^{n+1}\|_{H^2}^2,$$

where we used Lemma 2.6.

(Step B-2: Estimates for $l_3$) One uses integration by parts and Lemma 2.6 to obtain

$$l_3 = -\frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} (1 + h^n)\nabla \partial^j h^{n+1} \cdot \partial^j u^{n+1} \, dx$$

$$-\frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} [\partial^j ((1 + h^n)\nabla h^{n+1}) - (1 + h^n)\nabla \partial^j h^{n+1}] \cdot \partial^j u^{n+1} \, dx$$

$$= \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} \partial^j h^{n+1} \nabla h^n \cdot \partial^j u^{n+1} \, dx$$

$$+ \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} (1 + h^n) \partial^j h^{n+1} \nabla \cdot u^{n+1} \, dx$$

$$- \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} [\partial^j ((1 + h^n)\nabla h^{n+1}) - (1 + h^n)\nabla \partial^j h^{n+1}] \cdot \partial^j u^{n+1} \, dx$$

$$\leq C\|\nabla h^n\|_{L^\infty} \|\partial^j h^{n+1}\|_{L^2} \|\partial^j u^{n+1}\|_{L^2}$$

$$+ C\|\partial^j u^{n+1}\|_{L^2} (\|\nabla h^n\|_{L^\infty} \|\nabla h^{n+1}\|_{L^2} + \|\partial^j h^n\|_{L^2} \|\nabla h^{n+1}\|_{L^2})$$

$$+ \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} (1 + h^n) \partial^j h^{n+1} \nabla \cdot u^{n+1} \, dx$$

$$\leq C(1 + M) (\|\nabla h^{n+1}\|_{H^2}^2 + \|u^{n+1}\|_{H^2}^2) + \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} (1 + h^n) \partial^j h^{n+1} \nabla \cdot u^{n+1} \, dx.$$

Here $C = C(\gamma, \ell) > 0$ is a constant independent of $n$ and $T$.

(Step B-3: Estimates for $l_4$) We also estimate $l_4$ term by term as follows:

$$l_4 = \mu \int_{\mathbb{T}^3} \frac{\partial^j \Delta u^{n+1}}{(1 + h^n)^{\frac{\mu - 1}{\gamma}}} \cdot \partial^j u^{n+1} \, dx \mu \int_{\mathbb{T}^3} \left[ \partial^j \left( \frac{\Delta u^{n+1}}{(1 + h^n)^{\frac{\mu - 1}{\gamma}}} \right) - \frac{\partial^j \Delta u^{n+1}}{(1 + h^n)^{\frac{\mu - 1}{\gamma}}} \right] \cdot \partial^j u^{n+1} \, dx$$

$$=: l_{41} + l_{42}.$$

Before we estimate $l_{41}$, we first prove the following inequality holds: for any $j = 1, 2, 3$ and $\mu \in \mathbb{R}$,

$$\|\partial^j ((1 + h^n)^{\mu})\|_{L^2} \leq C(1 + M)^{\max(j, \mu)}, \quad \text{(B.2)}$$

where $C = C(\mu)$ is a constant independent of $n$ and $T$. For $j = 1$, we have

$$\|\partial((1 + h^n)^{\mu})\|_{L^2} = |\mu| \|\nabla h^n\|_{L^2} \|\partial h^n\|_{L^2}$$

$$\leq C \|\nabla h^n\|_{L^2} (\delta^{-1} + (1 + M)^{-1})$$

$$\leq C ((1 + M) + (1 + M)^{\mu}) \leq C(1 + M)^{\max(1, \mu)}.$$

Inductively, if (B.2) holds for $1 \leq j < 3$, then one uses Lemma 2.6 to get

$$\|\partial^{j+1}((1 + h^n)^{\mu})\|_{L^2} = |\mu| \|\partial^j((1 + h^n)^{\mu-1})\partial h^n\|_{L^2}$$

$$\leq C \|\partial^j h^n\|_{L^\infty} \|\partial^j((1 + h^n)^{\mu-1})\|_{L^2} + \|((1 + h^n)^{\mu-1})\|_{L^\infty} \|\partial^{j+1} h^n\|_{L^2}$$

$$\leq CM(1 + M)^{\max(j, \mu - 1)} + CM(\delta^{-1} + (1 + M)^{\mu - 1})$$

$$\leq C(1 + M)^{\max(j + 1, \mu)},$$

where $C = C(\mu)$. Therefore, we have

$$l_{41} \leq C(1 + M)^{\max(j + 1, \mu)} \int_{\mathbb{T}^3} (1 + h^n) \partial^j h^{n+1} \nabla \cdot u^{n+1} \, dx.$$
and this completes the proof for (B.2). For $I_{41}$,

$$I_{41} = \frac{2\mu}{\gamma - 1} \int_{\mathbb{T}^3} \left( \nabla h^n \cdot \nabla \partial^\ell u^{n+1} \right) dx - \frac{1}{(1 + h^n)^{\frac{2}{3}}} \left| \partial^\ell \nabla u^{n+1} \right|^2 dx$$

$$\leq C\mu \left\| \nabla h^n \right\|_{L^\infty} \left\| \partial^\ell \nabla u^{n+1} \right\|_{L^2} \left\| \partial^\ell u^{n+1} \right\|_{L^2} - \frac{c_0\mu}{(1 + M)^{\frac{2}{3}}} \left\| \partial^\ell \nabla u^{n+1} \right\|_{L^2}^2,$$

where $C > 0$ is constant independent of $n$ and $T$. For $I_{42}$,

$$I_{42} = \mu \sum_{r=1}^{\ell} \left( \frac{\ell}{r} \right) \int_{\mathbb{T}^3} \partial^r \left( \frac{1}{(1 + h^n)^{\frac{r}{3}}} \right) \partial^{\ell - r} \Delta u^{n+1} \cdot \partial^\ell u^{n+1} dx$$

$$\leq C\mu \left\| \nabla h^n \right\|_{L^\infty} \left\| \partial^{\ell - 1} \Delta u^{n+1} \right\|_{L^2} \left\| \partial^\ell u^{n+1} \right\|_{L^2}$$

$$+ C\mu \left\| \partial^2 \left( (1 + h^n)^{-\frac{2}{3}} \right) \right\|_{L^6} \left\| \partial^{\ell - 2} \Delta u^{n+1} \right\|_{L^3} \left\| \partial^\ell u^{n+1} \right\|_{L^2}$$

$$+ C\mu \left\| \partial^3 \left( (1 + h^n)^{-\frac{2}{3}} \right) \right\|_{L^2} \left\| \partial^{\ell - 3} \Delta u^{n+1} \right\|_{L^6} \left\| \partial^\ell u^{n+1} \right\|_{L^3}$$

$$\leq C\mu (1 + M)^3 \left( \left\| u^{n+1} \right\|_{H^3} + \left\| \nabla^4 u^{n+1} \right\|_{L^2} \right) \left\| u^{n+1} \right\|_{H^3} - \frac{c_0\mu}{(1 + M)^{\frac{2}{3}}} \left\| \partial^\ell \nabla u^{n+1} \right\|_{L^2}^2.$$

Thus, we get

$$I_4 \leq C\mu (1 + M)^3 \left( \left\| u^{n+1} \right\|_{H^3} + \left\| \nabla^4 u^{n+1} \right\|_{L^2} \right) \left\| u^{n+1} \right\|_{H^3} - \frac{c_0\mu}{(1 + M)^{\frac{2}{3}}} \left\| \partial^\ell \nabla u^{n+1} \right\|_{L^2}^2.$$

(Step B-4: Estimates for $I_5$) In this case,

$$I_5 = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left( u^n - v \right) \partial^\ell f^n \cdot \partial^\ell u^{n+1} \, dx \, dv$$

$$+ \sum_{r=1}^{\ell - 1} \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left( \frac{\ell - 1}{r} \right) \int_{\mathbb{T}^3 \times \mathbb{R}^3} \left( \partial^r u^n \right) \partial^{\ell - r - 1} f^n \cdot \partial^\ell u^{n+1} \, dx \, dv \mathbb{I}_{\{r \geq 2\}}$$

$$\leq C(1 + \left\| u^n \right\|_{L^\infty} \left\| f^n \right\|_{H^6} \left\| \partial^\ell u^{n+1} \right\|_{L^2}$$

$$+ C\sum_{r=1}^{\ell - 1} \left\| \partial^r u^n \right\|_{L^4} \left\| \partial^{\ell - r - 1} \rho_f \right\|_{L^4} \left\| \partial^\ell u^{n+1} \right\|_{L^2} \mathbb{I}_{\{r \geq 2\}}$$

$$\leq C(1 + M)^2 \left\| \partial^\ell u^{n+1} \right\|_{L^2}.$$

Here we used

$$\sum_{r=1}^{\ell - 1} \left\| \partial^r u^n \right\|_{L^4} \left\| \partial^{\ell - r - 1} \rho_f \right\|_{L^4} \leq C\sum_{r=1}^{\ell - 1} \left\| \partial^r u^n \right\|_{H^1} \left\| \partial^{\ell - r - 1} \rho_f \right\|_{H^1} \leq C(1 + M).$$

Hence, we gather all the estimates for $I_5$’s to yield

$$\frac{1}{2} \frac{d}{dt} \left\| \partial^\ell u^{n+1} \right\|_{L^2}^2 \leq C(1 + M) \left( \left\| u^{n+1} \right\|_{H^3}^2 + \left\| h^{n+1} \right\|_{H^3}^2 \right) + C\mu (1 + M)^3 \left( \left\| u^{n+1} \right\|_{H^3} + \left\| \nabla^4 u^{n+1} \right\|_{L^2} \right) \left\| u^{n+1} \right\|_{H^3}$$

$$- \frac{c_0\mu}{(1 + M)^{\frac{2}{3}}} \left\| \partial^\ell \nabla u^{n+1} \right\|_{L^2}^2 + \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} (1 + h^n) \partial^\ell h^{n+1} \nabla \cdot u^{n+1} \, dx$$

$$+ C(1 + M)^2 \left\| \partial^\ell u^{n+1} \right\|_{L^2},$$

where $C = C(\gamma, \ell) > 0$ is a constant independent of $n$ and $T$. We combine this with the estimate for $\partial^\ell h^{n+1}$ to obtain

$$\frac{d}{dt} \left( \frac{4\gamma}{(\gamma - 1)^2} \left\| \partial^\ell h^{n+1} \right\|_{L^2}^2 + \left\| \partial^\ell u^{n+1} \right\|_{L^2}^2 \right)$$

$$\leq C(1 + M)^3 \left( \frac{4\gamma}{(\gamma - 1)^2} \left\| h^{n+1} \right\|_{H^3}^2 + \left\| u^{n+1} \right\|_{H^3}^2 \right) + C\mu (1 + M)^3 \left\| \nabla^4 u^{n+1} \right\|_{L^2} \left( \left\| u^{n+1} \right\|_{H^3} + 1 \right)$$
\[-\frac{2c_0\mu}{(1+M)^{\frac{\gamma}{\gamma-1}}} \|\partial^\ell \nabla u^{n+1}\|^2_{L^2} + C(1+M)^2 \|\partial^{\ell+1} u^{n+1}\|_{L^2}.\]

Thus, we combine the above estimates for every \( \ell = 1, 2, 3 \) and use Young’s inequality to get

\[
\frac{d}{dt} \left( \frac{4\gamma}{(\gamma-1)^2} \|\nabla h^{n+1}\|^2_{H^2} + \|u^{n+1}\|^2_{H^2} \right) + \frac{c_0\mu}{(1+M)^{\frac{\gamma}{\gamma-1}}} \|\nabla^2 u^{n+1}\|^2_{H^2}
\leq C(1+M)^{\frac{\gamma-4}{\gamma-1}} \left( \frac{4\gamma}{(\gamma-1)^2} \|h^{n+1}\|^2_{H^2} + \|u^{n+1}\|^2_{H^2} \right) + C(1+M)^{\frac{\gamma-4}{\gamma-1}},
\]

where we used Young’s inequality and \( C = C(\gamma) > 0 \) is a constant independent of \( n \) and \( T \). Finally, we combine (B.1) with (B.3) to yield

\[
\frac{d}{dt} \left( \frac{4\gamma}{(\gamma-1)^2} \|h^{n+1}\|^2_{H^2} + \|u^{n+1}\|^2_{H^2} \right) + \frac{c_0\mu}{(1+M)^{\frac{\gamma}{\gamma-1}}} \|\nabla u^{n+1}\|^2_{H^3}
\leq C(1+M)^{\frac{\gamma-4}{\gamma-1}} \left( \frac{4\gamma}{(\gamma-1)^2} \|h^{n+1}\|^2_{H^2} + \|u^{n+1}\|^2_{H^2} \right) + C(1+M)^{\frac{\gamma-4}{\gamma-1}},
\]

from which, we use Grönwall’s lemma to find, for \( 0 \leq t \leq T \),

\[
\left( \frac{4\gamma}{(\gamma-1)^2} \|h^{n+1}\|^2_{H^2} + \|u^{n+1}\|^2_{H^2} \right) + \frac{c_0\mu}{(1+M)^{\frac{\gamma}{\gamma-1}}} \int_0^t e^{C(1+M)^{\frac{\gamma-4}{\gamma-1}} (t-s)} \|\nabla u^{n+1}(s)\|^2_{H^2} ds
\leq \left( \frac{4\gamma}{(\gamma-1)^2} \|h_0\|^2_{H^2} + \|u_0\|^2_{H^2} \right) e^{C(1+M)^{\frac{\gamma-4}{\gamma-1}} t} + C(1+M)^{\frac{\gamma-4}{\gamma-1}} t e^{C(1+M)^{\frac{\gamma-4}{\gamma-1}} t}.
\]

Thus, we can choose a sufficiently small \( 0 < T_1 \leq T \) such that

\[
\sup_{0 \leq t \leq T_1} \left( \frac{4\gamma}{(\gamma-1)^2} \|h^{n+1}(t)\|^2_{H^2} + \|u^{n+1}(t)\|^2_{H^2} \right) + \frac{c_0\mu}{(1+M)^{\frac{\gamma}{\gamma-1}}} \int_0^{T_1} \|\nabla u^{n+1}(s)\|^2_{H^2} ds < M,
\]

and this completes the proof.

### Appendix C. Proof of Lemma 4.3

In a straightforward manner, we get

\[
\frac{1}{2} \frac{d}{dt} \|u^{n+1} - u^n\|^2_{L^2} = - \int_{\mathbb{T}^3} (u^{n+1} - u^n) \cdot \left( u^n \cdot \nabla u^{n+1} - u^{n-1} \cdot \nabla u^n \right) dx
\]

\[
- \frac{2\gamma}{\gamma-1} \int_{\mathbb{T}^3} (u^{n+1} - u^n) \cdot ((1+h^n)\nabla h^{n+1} - (1+h^{n-1})\nabla h^n) dx
\]

\[
+ \mu \int_{\mathbb{T}^3} (u^{n+1} - u^n) \cdot \left( \frac{\Delta u^{n+1}}{(1+h^n)^{\frac{\gamma}{\gamma-1}}} - \frac{\Delta u^n}{(1+h^{n-1})^{\frac{\gamma}{\gamma-1}}} \right) dx
\]

\[
- \int_{\mathbb{T}^3 \times \mathbb{R}^3} (u^{n+1} - u^n) \cdot ((u^n - v) f^n - (u^{n-1} - v) f^{n-1}) dx dv
\]

\[=: \sum_{i=1}^4 J_i.\]

For \( J_1 \), we use Young’s inequality to have

\[
J_1 = - \int_{\mathbb{T}^3} (u^{n+1} - u^n) \cdot (u^n \cdot \nabla (u^{n+1} - u^n) + (u^n - u^{n-1}) \cdot \nabla u^n) dx
\]

\[
\leq C \|\nabla u^n\|_{L^\infty} \|u^{n+1} - u^n\|_{L^2}^2 + C \|\nabla u^n\|_{L^\infty} \|u^{n+1} - u^n\|_{L^2} \|u^n - u^{n-1}\|_{L^2}
\]

\[
\leq C(\|u^{n+1} - u^n\|_{L^2}^2 + \|u^n - u^{n-1}\|_{L^2}^2).
\]

For \( J_2 \), we use the uniform-in-\( n \) bound for \( h^n \) and Young’s inequality to get

\[
J_2 = - \frac{2\gamma}{\gamma-1} \int_{\mathbb{T}^3} (u^{n+1} - u^n) \cdot ((1+h^n)\nabla (h^{n+1} - h^n) + ((h^n - h^{n-1})\nabla h^n) dx
\]

\[
\leq C(\|u^{n+1} - u^n\|_{L^2}^2 + \|u^n - u^{n-1}\|_{L^2}^2).
\]
\[\begin{align*}
&= \frac{2\gamma}{\gamma - 1} \int_{T^3} (\nabla \cdot (u^{n+1} - u^n)) (1 + h^n)(h^{n+1} - h^n) \, dx \\
&\quad + \frac{2\gamma}{\gamma - 1} \int_{T^3} (u^{n+1} - u^n) \cdot ((h^{n+1} - h^n)\nabla h^n) \, dx \\
&\quad - \frac{2\gamma}{\gamma - 1} \int_{T^3} (u^{n+1} - u^n) \cdot ((h^n - h^{n-1})\nabla h^n) \, dx \\
&\leq C\|u^{n+1} - u^n\|_{L^2} \|h^{n+1} - h^n\|_{L^2} \|\nabla h^n\|_{L^\infty} \\
&\quad + C\|u^{n+1} - u^n\|_{L^2} \|h^n - h^{n-1}\|_{L^2} \|\nabla h^n\|_{L^\infty} \\
&\quad + C(1 + h^n\|_{L^\infty})\|h^{n+1} - h^n\|_{L^2} \|\nabla (u^{n+1} - u^n)\|_{L^2} \\
&\leq C \left(\|h^{n+1} - h^n\|_{L^2}^2 + \|u^{n+1} - u^n\|_{H^1}^2 + \|h^n - h^{n-1}\|_{L^2}^2\right). \\
\end{align*}\]

For \(J_3\), we use the upper bound of \(h^n\), Sobolev embedding and Young’s inequality to get

\[\begin{align*}
J_3 &= \mu \int_{T^3} (u^{n+1} - u^n) \cdot \left(\frac{\Delta (u^{n+1} - u^n)}{1 + h^n} - \Delta u^n \left(\frac{1}{1 + h^n} - \frac{1}{1 + h^{n-1}}\right)\right) \, dx \\
&= \frac{2\mu}{\gamma - 1} \int_{T^3} (u^{n+1} - u^n) \left(\nabla h^n, \nabla (u^{n+1} - u^n)\right) (1 + h^n)^{-\frac{2\gamma}{\gamma - 1}} \, dx \\
&\quad - \mu \int_{T^3} (u^{n+1} - u^n) \cdot \Delta u^n \left(\frac{1}{1 + h^n} - \frac{1}{1 + h^{n-1}}\right) \, dx \\
&\leq C\|\nabla h^n\|_{L^\infty} \|u^{n+1} - u^n\|_{L^2} \|\nabla (u^{n+1} - u^n)\|_{L^2} - \frac{c_0\mu}{(1 + M)^{\frac{2\gamma}{\gamma - 1}}} \|\nabla (u^{n+1} - u^n)\|_{L^2}^2 \\
&\quad + C\|u^{n+1} - u^n\|_{L^2} \|h^n - h^{n-1}\|_{L^2} \|\Delta u^n\|_{L^3} \\
&\leq C\|u^{n+1} - u^n\|_{L^2} \|\nabla (u^{n+1} - u^n)\|_{L^2} - \frac{c_0\mu}{(1 + M)^{\frac{2\gamma}{\gamma - 1}}} \|\nabla (u^{n+1} - u^n)\|_{L^2}^2 \\
&\quad + C\|u^{n+1} - u^n\|_{H^1} \|h^n - h^{n-1}\|_{L^2} \\
&\leq C\mu \left(\|u^{n+1} - u^n\|_{L^2}^2 + \|h^n - h^{n-1}\|_{L^2}^2\right) - \frac{c_0\mu}{2(1 + M)^{\frac{2\gamma}{\gamma - 1}}} \|\nabla (u^{n+1} - u^n)\|_{L^2}^2.
\end{align*}\]

For \(J_4\), we use Young’s inequality to obtain

\[\begin{align*}
J_4 &= -\int_{T^3 \times \mathbb{R}^3} (u^{n+1} - u^n) \cdot ((u^n - u^{n-1})f^n + (u^{n-1} - v)(f^n - f^{n-1})) \, dxdv \\
&\leq C\|\rho f^n\|_{L^\infty} \|u^{n+1} - u^n\|_{L^2} \|u^n - u^{n-1}\|_{L^2} + C(1 + \|u^{n} - u^{n-1}\|_{L^\infty})\|u^{n+1} - u^n\|_{L^2} \|f^n - f^{n-1}\|_{L^\infty} \\
&\leq C \left(\|u^{n+1} - u^n\|_{L^2}^2 + \|u^n - u^{n-1}\|_{L^2}^2 + \|f^n - f^{n-1}\|_{L^\infty}^2\right).
\end{align*}\]

Thus, we gather all the estimates for \(J_1\)'s to deduce

\[\begin{align*}
\frac{d}{dt}\|u^{n+1} - u^n\|_{L^2}^2 + \frac{c_0\mu}{(1 + M)^{\frac{2\gamma}{\gamma - 1}}} &\|\nabla (u^{n+1} - u^n)\|_{L^2}^2 \\
&\leq C \left(\|u^{n+1} - u^n\|_{L^2}^2 + \|h^{n+1} - h^n\|_{L^2}^2 + \|u^n - u^{n-1}\|_{L^2}^2 + \|h^n - h^{n-1}\|_{L^2}^2 + \|f^n - f^{n-1}\|_{L^\infty}^2\right). & (C.1)
\end{align*}\]

We next obtain for \(i = 1, 2, 3\)

\[\begin{align*}
\frac{1}{2} \frac{d}{dt}\|\partial_i (u^{n+1} - u^n)\|_{L^2}^2 \\
&= -\int_{T^3} \partial_i (u^{n+1} - u^n) \cdot (\partial_i u^n \cdot \nabla (u^{n+1} - u^n) + u^n \cdot \nabla \partial_i (u^{n+1} - u^n)) \, dx \\
&\quad - \int_{T^3} \partial_i (u^{n+1} - u^n) \cdot (\partial_i (u^n - u^{n-1}) \cdot \nabla u^n + (u^n - u^{n-1}) \cdot \nabla \partial_i u^n) \, dx \\
&\quad - \frac{2\gamma}{\gamma - 1} \int_{T^3} \partial_i (u^{n+1} - u^n) \cdot (\partial_i h^n \nabla (h^{n+1} - h^n) + (1 + h^n) \nabla \partial_i (h^{n+1} - h^n)) \, dx
\end{align*}\]
\[-\frac{2\gamma}{\gamma - 1} \int_{\mathbb{R}^3} \partial_i (u^{n+1} - u^n) \cdot (\partial_i (h^n - h^{n-1}) \nabla h^n + (h^n - h^{n-1}) \nabla \partial_i h^n) \, dx \]

\[+ \mu \int_{\mathbb{R}^3} \partial_i (u^{n+1} - u^n) \cdot \left( \Delta \partial_i (u^{n+1} - u^n) \left( \frac{1}{(1 + h^n)^{\frac{2}{\gamma - 1}}} + (u^{n+1} - u^n) \partial_i \left( \frac{1}{(1 + h^n)^{\frac{2}{\gamma - 1}}} \right) \right) \right) \, dx \]

\[+ \mu \int_{\mathbb{R}^3} \partial_i (u^{n+1} - u^n) \cdot \left( \Delta \partial_i u^n \left( \frac{1}{(1 + h^n)^{\frac{2}{\gamma - 1}}} - \frac{1}{(1 + h^{n-1})^{\frac{2}{\gamma - 1}}} \right) \right) \, dx \]

\[+ \mu \int_{\mathbb{R}^3} \partial_i (u^{n+1} - u^n) \cdot \left( \Delta u^n \partial_i \left( \frac{1}{(1 + h^n)^{\frac{2}{\gamma - 1}}} - \frac{1}{(1 + h^{n-1})^{\frac{2}{\gamma - 1}}} \right) \right) \, dx \]

\[- \int_{\mathbb{R}^3} \partial_i (u^{n+1} - u^n) \cdot (\partial_i (u^n - u^{n-1}) \rho_f^n - (u^n - u^{n-1}) \partial_i \rho_f^n) \, dx \]

\[- \int_{\mathbb{R}^3} \partial_i (u^{n+1} - u^n) \cdot \left( \partial_i u^{n-1} (\rho_f^n - \rho_f^{n-1}) - \int_{\mathbb{R}^3} (u^{n-1} - v) \partial_i (f^n - f^{n-1}) \, dv \right) \, dx \]

\[=: \sum_{i=1}^{16} K_i. \]

We then estimate $K_i$ as follows.

$K_1 \leq \|\partial_i u^n\|_{L^\infty} \|\nabla (u^{n+1} - u^n)\|_{L^2} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq C\|u^{n+1} - u^n\|^2_{H^1},$

$K_2 \leq \|\nabla u^n\|_{L^\infty} \|\partial_i (u^{n+1} - u^n)\|^2_{L^2}$

$\leq C\|u^{n+1} - u^n\|^2_{H^1},$

$K_3 \leq \|\nabla u^n\|_{L^\infty} \|\partial_i (u^n - u^{n-1})\|_{L^2} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq C\|u^n - u^{n-1}\|^2_{H^1} + C\|u^{n+1} - u^n\|^2_{H^1},$

$K_4 \leq \|\nabla \partial_i u^n\|_{H^1} \|u^n - u^{n-1}\|_{H^1} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq C\|u^n - u^{n-1}\|^2_{H^1} + C\|u^{n+1} - u^n\|^2_{H^1},$

$K_5 \leq \|\partial_i h^n\|_{L^\infty} \|\nabla (h^{n+1} - h^n)\|_{L^2} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq C\|h^{n+1} - h^n\|^2_{H^1} + C\|u^{n+1} - u^n\|^2_{H^1},$

$K_6 \leq \|\nabla h^n\|_{L^\infty} \|\partial_i (h^n - h^{n-1})\|_{L^2} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq C\|h^n - h^{n-1}\|^2_{H^1} + C\|u^{n+1} - u^n\|^2_{H^1},$

$K_7 \leq \|\nabla h^n\|_{L^\infty} \|\partial_i (h^n - h^{n-1})\|_{L^2} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq C\|h^n - h^{n-1}\|^2_{H^1} + C\|u^{n+1} - u^n\|^2_{H^1},$

$K_8 \leq \|\nabla \partial_i h^n\|_{H^1} \|h^n - h^{n-1}\|_{H^1} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq C\|h^n - h^{n-1}\|^2_{H^1} + C\|u^{n+1} - u^n\|^2_{H^1},$

$K_9 \leq \|\nabla h^n\|_{L^\infty} \|\nabla^2 (u^{n+1} - u^n)\|_{L^2} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq \frac{\delta}{3} \|\nabla^2 (u^{n+1} - u^n)\|^2_{L^2} + C\|u^{n+1} - u^n\|^2_{H^1},$

$K_{10} \leq \|\nabla h^n\|_{L^\infty} \|\nabla^2 (u^{n+1} - u^n)\|_{L^2} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq \frac{\delta}{3} \|\nabla^2 (u^{n+1} - u^n)\|^2_{L^2} + C\|u^{n+1} - u^n\|^2_{H^1},$

$K_{11} \leq \|\Delta \partial_i u^n\|_{L^2} \|h^n - h^{n-1}\|_{H^1} \|\partial_i (u^{n+1} - u^n)\|_{H^1}$

$\leq C\left(\|h^n - h^{n-1}\|^2_{H^1} + \|u^{n+1} - u^n\|^2_{H^1}\right) + \frac{\delta}{3} \|\nabla^2 (u^{n+1} - u^n)\|^2_{L^2},$

$K_{12} \leq \|\Delta u^n\|_{H^1} \|h^n - h^{n-1}\|_{H^1} \|\partial_i (u^{n+1} - u^n)\|_{H^1}$

$\leq C\left(\|h^n - h^{n-1}\|^2_{H^1} + \|u^{n+1} - u^n\|^2_{H^1}\right) + \frac{\delta}{3} \|\nabla^2 (u^{n+1} - u^n)\|^2_{L^2},$

$K_{13} \leq \|\rho_f\|_{L^\infty} \|\partial_i (u^n - u^{n-1})\|_{L^2} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$

$\leq C\|u^n - u^{n-1}\|^2_{H^1} + C\|u^{n+1} - u^n\|^2_{H^1},$

$K_{14} \leq \|\partial_i \rho_f\|_{H^1} \|u^n - u^{n-1}\|_{H^1} \|\partial_i (u^{n+1} - u^n)\|_{L^2}$
This together with (C.1) asserts the desired result.

For the other terms, we obtain

\[ K_6 = \frac{2\gamma}{\gamma - 1} \int_{\mathbb{T}^3} (\nabla h^n \partial_t (u^{n+1} - u^n) + (1+h^n) \nabla \cdot \partial_t (u^{n+1} - u^n)) \partial_t (h^{n+1} - h^n) \, dx \]

\[ \leq C \left( \|\nabla h^n\|_{L^\infty} \|\partial_t (u^{n+1} - u^n)\|_{L^2} + \|1+h^n\|_{L^\infty} \|\partial_t (u^{n+1} - u^n)\|_{L^2} \right) \|\partial_t (h^{n+1} - h^n)\|_{L^2} \]

\[ \leq C \|u^{n+1} - u^n\|^2_{H^1} + C \|h^{n+1} - h^n\|^2_{H^1} + \frac{\delta}{3} \|\partial_t (u^{n+1} - u^n)\|_{L^2}^2, \]

and

\[ K_9 = -\mu \int_{\mathbb{T}^3} \frac{\partial u^{n+1} - u^n)}{2(1+M)^{\gamma+1}} d\tau + \frac{2\mu}{\gamma - 1} \int_{\mathbb{T}^3} \nabla \partial_t (u^{n+1} - u^n) \partial_t (u^{n+1} - u^n) \cdot \nabla h^n \frac{1}{(1+h^n)^{\gamma+1}} d\tau \]

\[ \leq -\frac{\alpha \mu}{(1+M)^{\gamma+1}} \|\nabla \partial_t (u^{n+1} - u^n)\|_{L^2}^2 + C \|\nabla h^n\|_{L^\infty} \|\partial_t (u^{n+1} - u^n)\|_{L^2} \|\partial_t (u^{n+1} - u^n)\|_{L^2} \]

\[ \leq -\frac{\alpha \mu}{(1+M)^{\gamma+1}} \|\nabla \partial_t (u^{n+1} - u^n)\|_{L^2}^2 + C \|\partial_t (u^{n+1} - u^n)\|_{L^2}^2, \]

Combining all of the above estimates yields

\[ \frac{d}{dt} \|\nabla (u^{n+1} - u^n)\|_{L^2}^2 + \frac{\alpha \mu}{(1+M)^{\gamma+1}} \|\nabla (u^{n+1} - u^n)\|_{L^2}^2 \leq C \left( \|u^{n+1} - u^n\|^2_{H^1} + \|u^{n+1} - u^n\|^2_{H^1} + \|h^n - h^{n-1}\|^2_{H^1} + \|h^{n+1} - h^n\|^2_{H^1} \right) \]

This together with (C.1) asserts the desired result.

REFERENCES

[1] R. Alonso, J. A. Cañizo, I. Gamba, and C. Mouhot, A new approach to the creation and propagation of exponential moments in the Boltzmann equation, Commun. Partial Differ. Equ., 38, (2013), 155–169.

[2] C. Baranger and L. Desvillettes, Coupling Euler and Vlasov equations in the context of sprays: the local-in-time, classical solutions, J. Hyperbol. Differ. Equ., 3, (2006), 1–26.

[3] S. Benjelloun, L. Desvillettes, and A. Moussa, Existence theory for the kinetic-fluid coupling when small droplets are treated as part of the fluid, J. Hyperbol. Differ. Equ., 11, (2014), 109–133.

[4] P. L. Bhatnagar, E. P. Gross, and M. Krook, A model for collision processes in gases. Small amplitude process in charge d

[5] L. Boudin, L. Desvillettes, C. Grandmont and A. Moussa, Global existence of solution for the coupled Vlasov and Navier–Stokes equations, Differ. Integral Equ., 22, (2009), 1247–1271.

[6] J. A. Carrillo, Y.-P. Choi, and T. K. Karper, On the analysis of a coupled kinetic-fluid model with local alignment forces, Ann. Inst. Henri Poincaré Anal. Non Linéaire, 33, (2016), 273–307.

[7] J. A. Carrillo, R. Duan, and A. Moussa, Global classical solutions close to the equilibrium to the Vlasov–Fokker–Planck–Euler system, Kinet. Relat. Models, 4, (2011), 227–258.

[8] M. Chae, K. Kang, and J. Lee, Global existence of weak and classical solutions for the Navier–Stokes–Vlasov–Fokker–Planck equations. J. Differential Equations, 251, (2011), 9, 2431–2465.

[9] M. Chae, K. Kang, and J. Lee, Global classical solutions for a compressible fluid-particle interaction model, J. Hyperbol. Differ. Equ., 10, (2013), 537–562.

[10] Y.-P. Choi, Large-time behavior for the Vlasov/compressible Navier–Stokes equations, J. Math. Phys., 57, 071501, (2016).

[11] Y.-P. Choi, Finite-time blow-up phenomena of Vlasov/Navier–Stokes equations and related systems, J. Math. Pures Appl., 108, (2017), 991–1021.
[12] Y.-P. Choi, S.-Y. Ha, J. Jung, and J. Kim, On the coupling of kinetic thermomechanical Cucker–Smale equation and compressible viscous fluid system, J. Math. Fluid Mech., 22, (2020), 4.
[13] Y.-P. Choi and J. Jung, Asymptotic analysis for a Vlasov-Fokker-Planck/compressible Navier-Stokes system in a bounded domain, Math. Models Methods Appl. Sci., 31, (2021), 2213–2295.
[14] Y.-P. Choi and J. Jung, On regular solutions and singularity formation for Vlasov/Navier–Stokes equations with degenerate viscosities and vacuum, Kinet. Relat. Models, 15, (2022), 843–891.
[15] Y.-P. Choi and J. Jung, On the dynamics of charged particles in an incompressible flow: from kinetic-fluid to fluid-fluid models, Commun. Contemp. Math., to appear.
[16] Y.-P. Choi and B. Kwon, Global well-posedness and large-time behavior for the inhomogeneous Vlasov–Navier–Stokes equations, Nonlinearity, 28, (2015), 3309–3336.
[17] Y.-P. Choi, J. Lee, and S.-B. Yun, Strong solutions to the inhomogeneous Navier–Stokes–BGK system, Nonlinear Anal. Real World Appl., 57, (2021), 103196.
[18] Y.-P. Choi and S.-B. Yun, Global existence of weak solutions for Navier–Stokes–BGK system, Nonlinearity, 33, (2020), 1925–1955.
[19] L. Desvillettes, Some aspects of the modelling at different scales of multiphase flows, Comput. Methods Appl. Mech. Eng., 199, (2010), 1265–1267.
[20] R. Duan and S. Liu, Cauchy problem on the Vlasov–Fokker–Planck equation coupled with the compressible Euler equations through the friction force, Kinet. Relat. Models, 6, (2013), 687–700.
[21] N. Fournier, On exponential moments of the homogeneous Boltzmann equation for hard potentials without cutoff, Commun. Math. Phys., 387, (2021), 973–994.
[22] I. M. Gamba and C. Yu, Global weak solutions to compressible Navier–Stokes–Vlasov–Boltzmann systems for spray dynamics, J. Math. Fluid Mech., (2020), 22:45.
[23] T. Goudon, P.-E. Jabin and A. Vasseur, Hydrodynamic limit for the Vlasov–Navier–Stokes equations: I. Light particles regime, Indiana Univ. Math. J., 53, (2004), 1495–1515.
[24] T. Goudon, P.-E. Jabin and A. Vasseur, Hydrodynamic limit for the Vlasov–Navier–Stokes equations: II. Fine particles regime, Indiana Univ. Math. J., 53, (2004), 1517–1536.
[25] D. Han-Kwan, Large time behavior of small data solutions to the Vlasov–Navier–Stokes system on the whole space, Prob. Math. Phys., to appear.
[26] D. Han-Kwan and D. Michel, On hydrodynamic limits of the Vlasov–Navier–Stokes system, Mem. Amer. Math. Soc., to appear.
[27] D. Han-Kwan, A. Moussa and I. Moyano, Large time behavior of the Vlasov–Navier–Stokes system on the torus, Arch. Ration. Mech. Anal., 236, (2020), 1273–1323.
[28] F. Li, Y. Mu, and D. Wang, Strong solutions to the compressible Navier–Stokes–Vlasov–Fokker–Planck equations: global existence near the equilibrium and large time behavior, SIAM J. Math. Anal., 49, (2017), 984–1026.
[29] H.-L. Li and L.-Y. Shou, Global weak solutions for compressible Navier–Stokes–Vlasov–Fokker–Planck system, preprint, arXiv:2103.16333.
[30] Y. Li, R. Pan, and S. Zhu, On classical solutions for viscous polytropic fluids with degenerate viscosities and vacuum, Arch. Rational Mech. Anal., 234, (2019), 1281–1334.
[31] J. Mathiaud, Local smooth solutions of a thin spray model with collisions, Math. Models Methods Appl. Sci., 20, (2010), 191–221.
[32] A. Mellet and A. Vasseur, Global weak solutions for a Vlasov–Fokker–Planck/Navier–Stokes system of equations, Math. Models Methods Appl. Sci., 17, (2007), 1039–1063.
[33] A. Mellet and A. Vasseur, Asymptotic analysis for a Vlasov–Fokker–Planck/Navier–Stokes equations, Comm. Math. Phys., 281, (2008), 573–596.
[34] B. Perthame, M. Pulvirenti, Weighted $L^\infty$ bounds and uniqueness for the Boltzmann BGK model, Arch. Ration. Mech. Anal., 125, (1993), 289–295.
[35] P. O’Rourke, Collective drop effects on vaporising liquid sprays, PhD Thesis Princeton University, Princeton, 1981, NJ.
[36] M. Tasković, R. J. Alonso, I. M. Gamba, N. Pavlović, On Mittag-Leffler moments for the Boltzmann equation for hard potentials without cutoff, SIAM J. Math. Anal., 50, (2018), 834–869.
[37] D. Wang and C. Yu, Global weak solutions to the inhomogeneous Navier–Stokes–Vlasov equations, J. Differential Equations, 259, (2014), 3976–4008.
[38] A. Williams, Spray combustion and atomization, Phys. Fluids, 1, (1958), 541–555.
[39] L. Yao and C. Yu, Existence of global weak solutions for the Navier–Stokes–Vlasov–Boltzmann equations, J. Differential Equations, 265, (2018), 5575–5603.
[40] C. Yu, Global weak solutions to the incompressible Navier–Stokes–Vlasov equations, J. Math. Pures Appl., 100, (2013), 275–293.
[41] S.-B. Yun, Classical solutions for the ellipsoidal BGK model with fixed collision frequency, J. Differential Equations, 259, (2015), 6009–6037.
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