Atomic self-ordering in a ring cavity with counterpropagating pump fields

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Abstract – The collective dynamics of mobile scatterers and light in optical resonators generates complex behaviour. For strong transverse illumination a phase transition from homogeneous to crystalline particle order appears. In contrast, cold particles inside a single-side pumped ring cavity exhibit an instability towards bunching and collective acceleration called collective atomic recoil lasing (CARL). We demonstrate that by driving two orthogonally polarized counterpropagating modes of a ring resonator one realises both cases within one system. As a function of the two pump intensities the corresponding phase diagram exhibits regions in which either a generalized form of self-ordering towards a travelling density wave with constant centre-of-mass velocity or a CARL instability is formed. Time-dependent control of the cavity driving then allows to accelerate or slow down and trap a sufficiently dense beam of linearly polarizable particles.

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Introduction. – Ultracold particles in an optical resonator interact nonlocally via collective scattering of photons in and out of the cavity modes. Under suitable conditions this induces collective instabilities [1] or even crystallisation of the particles [2–4]. One of the earliest examples of such an instability, developed in close analogy to free-electron lasers [5], was studied in the so-called collective atomic recoil lasing (CARL) [1,6,7]. This type of instability can be realised in a single-side pumped ring cavity and it reveals a transient bunching concurrent with coherent collective backscattering of pump light for an ensemble of fast particles counterpropagating the pump field of the cavity. In an alternative geometry, considering cold particles with transverse pump in a standing-wave cavity, a phase transition from homogeneous to crystalline order was predicted [8] and experimentally verified [9,10]. Later this was identified and as well confirmed as a quantum phase transition also occurring at zero temperature [3,11].

In this paper we show that in a generalized geometry using two counterpropagating pump fields of orthogonal polarization, a very similar type of phase transition appears, where the system breaks its translational symmetry and transforms into periodic order. The geometry is related to the configuration studied in [12], where no cavity was present. It is important to note that the pump fields injected from two sides into the ring cavity do not interfere and hence do not form a prescribed optical lattice, as they have orthogonal polarization. A lattice only appears through interference of pump and backscattered light. The two fields of orthogonal polarization interact only indirectly by scattering from the same atomic density distribution [12].

This work is organized as follows: After a short presentation of the model, we study the general properties of the system and exhibit its relation to known models. In particular using a Vlasov-type approach we study the stability boundary of the homogeneous distribution. To understand the system’s behaviour in more detail, we perform specific numerical simulations in the fourth section. We reveal that self-ordered solutions with a constant centre-of-mass velocity can be realised. In addition we show that the system allows for slowing down a fast atomic or molecular beam. In the last part we derive expressions which enable to state whether our configuration settles in a self-ordered phase or a CARL instability depending on the pump parameters.

Model. – Let us consider a large ensemble of N polarizable particles within a ring cavity supporting pairs of orthogonally polarized counterpropagating modes. For simplicity, we assume them to be confined along the cavity axis and linearly polarizable with a real scalar polarizability, i.e. atoms with a ground state with zero total angular
momentum $F = 0$. For optically polarized atoms with tensor polarizabilities, the equations below would have to be adapted accordingly. In addition, for sufficiently large detuning from any optical resonance we can largely neglect mode mixing due to spontaneous Raman transitions to other Zeeman levels and thus we simply end up with an effective polarization for each field mode.

Note that this description explicitly excludes optical pumping and polarization gradient cooling as in optical molasses. When enhanced by cavity feedback this tends to localize the particles in space as shown in some earlier work [13]. While this is certainly an interesting generalization of our model, such ordering is a single-particle effect and thus fundamentally different from the collective self-ordering dynamics into a lattice structure studied in our present work below.

In a semiclassical point particle description the time evolution of the mode amplitudes $a_n$ of the cavity field $E(x) := \sum_n a_n f_n(x)$ is governed by the equations [14]

$$\dot{a}_n(t) = (i\Delta_c - \kappa) a_n(t) - i U_0 \sum_{j=1}^N E(x_j) \cdot \frac{\partial f_n(x_j)}{\partial x} + \eta_n,$$

where $2\kappa$ is the cavity linewidth and $\Delta_c := \omega_p - \omega_c$ denotes the detuning between the pump field ($\omega_p$) and the cavity modes ($\omega_c$) and $U_0$ determines the interaction strength. Physically, $U_0$ represents the optical potential depth per photon in the cavity as well as the cavity mode frequency shift per particle. In general, $U_0$ can be complex but we will restrict our treatment to real $U_0$, meaning that we only consider dispersive atom-light interactions.

We approximate the mode functions $f_n$ in the interaction zone as plane waves, so that their polarization is constant. In the following we will only consider four different modes, hence we will change notation from \{a_1, a_2, a_3, a_4\} $\rightarrow$ \{a_+, a_-, \beta_+, \beta_-\} and \{f_1, f_2, f_3, f_4\} $\rightarrow$ \{f^+_{\alpha}, f^-_{\alpha}, f^+_{\beta}, f^-_{\beta}\}, where

$$f_{\alpha,\beta}^\pm(x) = \exp(\pm ikx) e_{\alpha,\beta} \quad (2)$$

and the polarization vectors fulfill the orthogonality relation $e_\alpha \cdot e_\beta = \delta_{\alpha,\beta}$. Two counterpropagating, orthogonally polarized modes $f^+_\alpha$ and $f^-_{\beta}$ are pumped with amplitudes $\eta_\alpha \equiv \eta_+$ and $\eta_\beta \equiv \eta_-$, while the other two modes $f^-_{\alpha}$ and $f^+_{\beta}$ are only populated by scattered photons. This configuration represents only a slight change as compared to standard ring cavity cooling scheme [15–17], but constitutes a very different situation physically. As the two counterpropagating pump fields do not interfere, no prescribed optical lattice is formed and the system is inherently translation invariant. Note that imperfect mirrors in principle could lead to scattering between the two polarizations. Fortunately in a three-mirror ring cavity the two orthogonal polarization modes are sufficiently frequency shifted due to the polarization-dependent mirror reflection, so that no resonant scattering between the modes will occur.

The force on a particle within the cavity field is given by the gradient of the optical dipole potential $\phi(x) = \hbar U_0 |E(x)|^2$ associated with the local field intensity, hence $m \ddot{x}_j = -\nabla \phi(x_j)$ with particle mass $m$. We restrict our treatment to the one-dimensional motion along the cavity axis, so that $x_j$ is replaced by $x_j$. Under these assumptions, eqs. (1) and (2) lead to

$$\dot{\alpha}_+ = (i\delta - \kappa) \alpha_+ - i N U_0 \theta \alpha_- + \eta_+,$$

$$\dot{\alpha}_- = (i\delta - \kappa) \alpha_- - i N U_0 \theta^* \alpha_+,$$

$$\dot{\beta}_+ = (i\delta - \kappa) \beta_+ - i N U_0 \theta \beta_-,$$

$$\dot{\beta}_- = (i\delta - \kappa) \beta_- - i N U_0 \theta^* \beta_+ + \eta_-,$$

where $\theta = 1/N \sum_k e^{-2ikx}$ defines the orderparameter and $\delta := \Delta_c - NU_0$ is the effective cavity detuning. These equations (3) describe two independent CARL geometries with different propagation directions (cf. fig. 1) which interact via the atomic density inhomogeneities. It should be mentioned that the detuning $\delta$ and the cavity decay rate $\kappa$ are in general different for the two counterpropagating modes. However, this does not influence the basic physics discussed in this work and would add unnecessary complexity. Therefore, we assume equal detuning and decay rate for all modes in the following.

The light-induced optical potential explicitly reads

$$\phi = \hbar U_0 (\alpha^*_+ \alpha_- e^{2ikx} + \alpha_+^* \alpha_- e^{-2ikx} + \beta^*_+ \beta_- e^{2ikx} + \beta_+^* \beta_- e^{-2ikx}).$$

For very large particle numbers the numerical simulation of equations of motion can be achieved only at large computational cost. However, in the limit $N \rightarrow \infty$ the dynamics of the gas can, for sufficiently short times be reliably approximated by a Vlasov equation (in 1D) [18,19]

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{1}{m} \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial v} = 0 \quad (5)$$

for the corresponding one-body phase space distribution function $f(x,v,t)$. Such a treatment misses, however,
correlations in the density and field fluctuations which lead to cooling and heating on longer time scales [20]. Assuming periodic boundary conditions allows to restrict our treatment to the truncated phase space with $x \in (0, \lambda)$.

**Stability analysis.** — Let us now investigate the coupled dynamics of the field modes and particles as described by eqs. (3)–(5). For a spatially homogeneous distribution the system is fully translation invariant and thus $\theta = 0$. For this reason the state defined by $\alpha^0_+ = \beta^0_- = 0$ as well as by

$$f(x, v, t) = \lambda^{-1} F(v),$$

$$\alpha^0_+ = \frac{\eta_+}{\kappa - i\delta}, \quad \beta^0_- = \frac{\eta_-}{\kappa - i\delta}$$

constitutes a stationary solution of the system (3)–(5), regardless of the velocity distribution $F(v)$. Note that in the following we only consider thermal (i.e. Maxwell-Boltzmann) velocity distributions

$$F(v) = \frac{1}{\sqrt{2\pi v_T^2}} e^{-\left(\frac{v}{v_T}\right)^2}.$$  

Here we introduced the thermal velocity $v_T$ which is connected to the temperature via $mv_T^2/2 = k_B T$.

In this stationary state only forward scattering occurs without photon redistribution between the modes and thus there are no forces on the particles. Only deviations from perfect spatial homogeneity can lead to backscattering and the build-up of an optical lattice. To find out under which conditions such deviations are amplified and a subsequent phase transition to an ordered phase can occur, we perform a linear stability analysis following Landau [21]. As a result we find that the steady state (6) is unstable if and only if the dispersion relation $D(s)$ has at least one zero with a positive real part, where

$$D(s) := \delta^2 + (s + \kappa)^2 + [(s + \kappa)A - \delta S] I(s).$$

In (8) we defined the total pump parameter $S$ and the pump asymmetry $A$ according to

$$S := |\eta_+|^2 + |\eta_-|^2, \quad A := |\eta_+|^2 - |\eta_-|^2.$$  

Furthermore,

$$I(s) := \frac{NU_0^2 v_R}{\kappa^2 + \delta^2} \int_{-\infty}^{\infty} \frac{F'(v)}{s + 2kv} dv$$

with the recoil velocity $v_R = 2\hbar k/m$. One finds that for every given pump asymmetry there exists a critical total pump parameter $S_c$ such that the homogeneous state is unstable for $S > S_c$ and stable otherwise.

For equal pump intensities, i.e. $A = 0$, we recognize that (8) is almost exactly the same dispersion relation as one obtains for a transversally pumped ring cavity. The only difference is that the wave number is multiplied by a factor 2 and the transversal pump intensity is replaced by the sum of the two pump intensities $S$ [18]. Obviously there exists a close analogy between the present setup with equal pump strengths and a transversally pumped ring cavity. As in the latter case there appear stable self-organized solutions beyond an instability threshold [18], which is given by

$$S_{c=0}^A := \frac{k_B T (\kappa^2 + \delta^2)^2}{\hbar |\delta|}.$$  

The other extreme case, $A = \pm S$, corresponds to a pure CARL instability [18] in which case no self-organization can take place as shown in sect. 4 of [18].

While the dependence of the critical pump parameter on the pump asymmetry cannot be found in closed form, solving $D(0 + i\omega) = 0$ for $A(\omega)$ and $S(\omega)$ yields the stability boundary using $\omega \in \mathbb{R}$ as parameter, cf. fig. 5. The central question in the following is whether and under which conditions there occurs self-organization for nonzero pump asymmetries.

**Numerical simulation.** — To gain deeper insight into the systems behaviour we have numerically solved the Maxwell-Vlasov equations (3)–(5) for an initial condition close to homogeneous in space and a negative effective detuning $\delta$. These simulations confirm our predictions for the $A = 0$ case. Furthermore, they reveal that for sufficiently small $A$ and above threshold the system does evolve into a self-ordered state, albeit one in which the gas possesses a nonvanishing centre-of-mass velocity $v_{ph}$ constant in time. In the process of forming such a traveling wave the continuous translation symmetry is broken. As a matter of fact, the gas moves in the direction of the stronger pump beam. However, we find that, for a given $S$, as soon as $A$ exceeds a certain value, there still occurs a CARL instability resulting in a runaway centre-of-mass velocity. For an illustration of these processes see figs. 2–4. Movies of typical phase space evolutions are
During the ordering process at (a) CARL instability at centre-of-mass velocity. That the spatially periodic distribution is shifted in space. This implies that the system exhibits a certain constant (see fig. 2) centre-of-mass velocity (self-ordered, travelling-wave state with a constant phase of mass is accelerated indefinitely, or it settles in a gas either enters the CARL regime, in which the centre of mass is accelerated indefinitely, or it settles in a steady state, or it settles in a (the full movie of the time evolution can be found on-line, 396.5 kB, asym03.mp4). The parameters are chosen to be: $k v_T = 1.5 \kappa$, $N = 2 \cdot 10^5$, the effective cavity detuning is set to $\delta = -\kappa$ and $U_0 = -1/N$. (a) Spatially homogeneous distribution at $t = 0 \kappa^{-1}$. (b) During the ordering process at $t = 15 \kappa^{-1}$. (c) Self-ordered state at $t = 32 \kappa^{-1}$. If one compares panels (b) and (c) one finds that the spatially periodic distribution is shifted in space. This implies that the system exhibits a certain constant (see fig. 2) centre-of-mass velocity.

While we have started from a particle ensemble at rest up to now and found a moving gas in a steady state, one can turn the idea around and use this setup to efficiently slowing down a cold atomic or molecular beam by collective scattering, improving a similar approach which has already been presented in [16] (see the red curve in fig. 2).

**BGK waves.** – As we have seen above, in the case of instability and depending on the pump asymmetry, the gas either enters the CARL regime, in which the centre of mass is accelerated indefinitely, or it settles in a self-ordered, travelling-wave state with a constant phase velocity (i.e. centre-of-mass velocity). Let us therefore investigate this latter type of solution more closely. From eq. (5) one deduces that any nonlinear wave with phase velocity $v_{ph}$ must be of the BGK (Bernstein-Greene-Kruskal) form [22],

$$ f(x, v, t) = G \left( \frac{m(v - v_{ph})^2}{2} + \phi(x, t) \right), \quad (12) $$

where $G(\cdot)$ is an arbitrary function. Furthermore, $\phi(x, t)$ may depend on $(x, t)$ only through $x - v_{ph}t$, which implies that $\alpha_- e^{-2i k v_{ph} t}, \beta_+ e^{2i k v_{ph} t}$ as well as $\alpha_+, \beta_-$ all be independent of time. To actually find the phase velocity from the equations of motion we require $G(\cdot)$, which is obtained as the solution of an initial value problem and thus in general out of reach. Nevertheless it is possible to deduce a relationship between the phase velocity, the order parameter and the relative pump asymmetry in the form ($\Theta := N|\theta|$)

$$ \frac{A}{S} = \frac{-4 \delta (\kappa^2 + \delta^2 - U_0^2 \Theta^2) k v_{ph}}{4 (\kappa^2 + \delta^2) k^2 v_{ph}^2 + (\kappa^2 + \delta^2 - U_0^2 \Theta^2)^2 + (2 \kappa U_0 \Theta)^2} \quad (13) $$

fulfilled by any nonlinear wave solution. From (13) we find that for $\delta < 0$ the wave travels in the direction of the stronger pump beam, as long as $N|U_0| \leq \sqrt{\kappa^2 + \delta^2}$. As soon as the inequality is violated, waves with sufficiently large order parameters propagate in the opposite direction. As such waves have never been observed numerically we have reason to expect them to be dynamically unstable. Hence we stipulate that the order parameter satisfies the bound $N|U_0| |\theta| \leq \sqrt{\kappa^2 + \delta^2}$. Furthermore, eq. (13)
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Fig. 5: (Colour on-line) Phase diagram. The blue regions correspond to a stable homogeneous gas for different temperatures. From dark to light blue we have chosen $k v_T = 1.5, 50, 100 \kappa$. The dark-grey region, bounded by (15), marks the parameter regime where a warm gas transitions to a BGK state (12). The dashed black line corresponds to the bound (14).

allows to conclude that if

$$\frac{|A|}{S} > \left| \frac{\delta}{\sqrt{\kappa^2 + \delta^2}} \right|,$$  \hfill (14)

there exists no BGK-wave solution at all. This implies that if the homogeneous solution is unstable and the asymmetry exceeds the bound (14), the gas will necessarily enter the CARL regime.

Equation (13) can also be viewed as determining the necessary relative pump asymmetry $A/S$, which is needed to generate a wave with a prescribed phase velocity $v_{ph}$ and order $|\theta|$. Notice, however, that the necessary total pump strength $S$ cannot be inferred. In particular, in order to stop a beam (i.e. to achieve $v_{ph} = 0$) the pump asymmetry has to be equal to zero.

The foregoing statements exhaust the characterization of the BGK solutions without knowing $G(\cdot)$. Without going into details we state that for gas with temperature $k v_T \gg \kappa$, one finds that a BGK wave will develop as soon as $S > S_{BGK}$, where

$$S_{BGK} := S_{BGK}^{A=0} \left[ 1 + \left( \frac{\hbar A}{2 k_{B} T \sqrt{\kappa^2 + \delta^2}} \right)^2 \right].$$  \hfill (15)

The results of the stability analysis and the discussion above are summarized in fig. 5. Obviously there seems to be a sharp transition from the BGK phase to the CARL phase. This fact has also been proven to be right in numerical tests.

Conclusions and outlook. – We demonstrated that utilizing orthogonally polarized counterpropagating modes the physics of light-induced self-ordering is observable, similarly to the case of a transversely pumped ring resonator. In the case of no pump asymmetry the two setups are fully equivalent [18]. However, the system considered in this work is more versatile, because, in principle, by the choice of the pump asymmetry, ordered particle distributions with any prescribed centre-of-mass velocity can be generated. Therefore, the control of the pump intensity allows for controlling the motion of gas particles inside the ring cavity. As a consequence, a particle beam can be effectively slowed down and trapped. Note that a different loss rate for the two polarization modes, as it often appears in practice, can be easily compensated by a correspondingly enlarged pump.

Analogous physics should be present at zero temperature allowing to control and study degenerate quantum gases. For example a similar behaviour of a superfluid gas inside a transversally pumped cavity has been experimentally observed recently [23]. Interesting effects can also be expected in the case of particles in optical lattices. Here collective scattering from orthogonally polarized modes can be used to gain insight into the particle quantum statistics at minimal perturbation or to induce tailored long-range interactions.

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REFERENCES
[1] Bonifacio R., De Salvo L., Narducci L. and D’Angelo E., Phys. Rev. A, 50 (1994) 1716.
[2] Ritsch H., Domokos P., Brennecke F. and Esslinger T., Rev. Mod. Phys., 85 (2013) 553.
[3] Baumann K., Guerlin C., Brennecke F. and Esslinger T., Nature, 464 (2010) 1301.
[4] Gopalakrishnan S., Lev B. L. and Goldbart P. M., Phys. Rev. A, 82 (2010) 043612.
[5] Hoff F., Meystre P., Scully M. and Loulsegu W., Opt. Commun., 18 (1976) 413.
[6] Schmidt D., Tomczyk H., Slama S. and Zimmermann C., Phys. Rev. Lett., 112 (2014) 115302.
[7] Bux S., Tomczyk H., Schmidt D., Courtelle P. W., Piovella N. and Zimmermann C., Phys. Rev. A, 87 (2013) 023607.
[8] Domokos P. and Ritsch H., Phys. Rev. Lett., 89 (2002) 253003.
[9] Black A. T., Chan H. W. and Vuletić V., Phys. Rev. Lett., 91 (2003) 203001.
[10] Arnold K., Baden M. and Barrett M., Phys. Rev. Lett., 109 (2012) 153002.
[11] Nagy D., Konya G., Szirmai G. and Domokos P., Phys. Rev. Lett., 104 (2010) 130401.
[12] Ostermann S., Sonnleitner M. and Ritsch H., New J. Phys., 16 (2014) 043017.
[13] Gangl M. and Ritsch H., Phys. Rev. A, 64 (2001) 063414.
[14] Salzburger T. and Ritsch H., New J. Phys., 11 (2009) 055025.

[15] Gangl M. and Ritsch H., Phys. Rev. A, 61 (2000) 043405.

[16] Maes C., Asboth J. and Ritsch H., Opt. Express, 15 (2007) 6019.

[17] Klinner J., Lindholdt M., Nagorny B. and Hemmerich A., Phys. Rev. Lett., 96 (2006) 023002.

[18] Grießer T., Ritsch H., Hemmerling M. and Robb G., Eur. Phys. J. B, 58 (2010) 349.

[19] Tesio E., Robb G., Ackemann T., Firth W. and Oppo G.-L., Phys. Rev. Lett., 112 (2014) 043901.

[20] Niedenzu W., Grießer T. and Ritsch H., EPL, 96 (2011) 43001.

[21] Landau L., J. Phys., 10 (1946) 25.

[22] Bernstein I. B., Greene J. M. and Kruskal M. D., Phys. Rev., 108 (1957) 546.

[23] Keler H., Klinder J., Wolke M. and Hemmerich A., Phys. Rev. Lett., 113 (2014) 070404.