Generalized Verma modules over $\mathfrak{sl}_{n+2}$ induced from $\mathcal{U}(\mathfrak{h}_n)$-free $\mathfrak{sl}_{n+1}$-modules

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Abstract

A class of generalized Verma modules over $\mathfrak{sl}_{n+2}$ is constructed from $\mathfrak{sl}_{n+1}$-modules which are $\mathcal{U}(\mathfrak{h}_n)$-free modules of rank 1. The necessary and sufficient conditions for these $\mathfrak{sl}_{n+2}$-modules to be simple are determined. This leads to a class of new simple $\mathfrak{sl}_{n+2}$-modules.

1 Introduction

Classification of simple modules is an important step in the study of a module category over an algebra $\mathfrak{g}$. When $\mathfrak{g}$ is a nontrivial complex Lie algebra this turns out to be a very difficult problem in general, only for $\mathfrak{g} = \mathfrak{sl}_2$ (and several related small dimensional Lie algebras) there is such a classification to some extent, see [Bl, AP, Maz3]. Nevertheless, some subcategories of modules are relatively well understood, and their study has occupied many mathematicians over the last century.

The most thoroughly studied class of modules are the weight modules with finite dimensional weight spaces, which are modules on which a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts diagonally. Finite dimensional weight modules for complex finite dimensional semisimple Lie algebras were classified by Cartan already in 1913, see [Ca]. The study of infinite dimensional weight modules was started by Verma and others in the 1960’s (see [Ve]) and various results on weight modules were gathered over the years, see for example [Dj, Hu, BGG, LLZ]. This eventually led to the classification of simple weight modules over semisimple Lie algebras with finite dimensional weight spaces in 2000, see [Fe, Fu, Mat]. Some other classes of modules are also worth mentioning. These include Kostant’s Whittaker modules, on which the algebra of positive roots act locally finitely, see [Ko]. Also see [OW, BM, MZ, MW, Ni3] for some related classes of modules. Gelfand-Zetlin modules form a large family of modules parametrized by a type of triangular tableaux, see [DFO1, DFO2, Maz2, FGR]. More recently the category of $\mathcal{U}(\mathfrak{h})$-free modules was defined, its objects are modules whose restriction to the Cartan subalgebra is free of rank 1. Simple $\mathcal{U}(\mathfrak{h})$-free modules were classified for all finite-dimensional simple Lie algebras, see [Ni1, Ni1], and also for some infinite dimensional Lie algebras, see [TZ1, TZ2].

If $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi-decomposition $\mathfrak{p} = \mathfrak{a}' \oplus \mathfrak{n}$ and $V$ is an $\mathfrak{a}'$-module with $\mathfrak{n}V = 0$, one usually writes

$$M_\mathfrak{p}(V) := \text{Ind}^\mathfrak{g}_\mathfrak{p} V = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V$$

for the corresponding generalized Verma module (GVM). GVM’s have been extensively studied, especially under the condition that $V$ is a weight module, see for example [Maz1, BFL] and references therein. In 1985 McDowell studied the GVM $M_\mathfrak{p}(V)$ when $V$ is a simple non-degenerate Whittaker module, and he managed to determine necessary and sufficient conditions for $M_\mathfrak{p}(V)$ to be simple, see [McD1, McD2]. For non-weight modules
V, there are also some more theoretical results on the structure of $M_p(V)$, see [MaSt1, MiSo1, MiSo2, KMI, KM2].

The present paper is inspired by McDowell’s results. Consider $\mathfrak{sl}_{n+1}$ as a subalgebra of $\mathfrak{sl}_{n+2}$ in the top left corner. Let $p \subseteq \mathfrak{sl}_{n+2}$ be the parabolic subalgebra consisting of the standard Borel subalgebra plus $\mathfrak{sl}_{n+1}$. Let $V$ be any $\mathfrak{sl}_{n+1}$-module which is a $U(\mathfrak{h}_n)$-free module of rank 1. Using the classification in [NII] (see Lemma 5), $V$ can be extended to a $p$-module labeled $V(a; S, b, \lambda)$ for $b \in \mathbb{C}, \lambda \in \mathbb{C}, a = (a_1, \cdots, a_n, 1) \in (\mathbb{C}^*)^{n+1}, S \subseteq \{1, 2, ..., n+1\}$. In this setting our main result can be stated as follows (see Theorem 13 for all the details).

**Theorem 1.** The induced module $M_p(V(a; S, b, \lambda))$ is simple if and only if all the following conditions hold

(i). $b - \lambda + 2 \not\in \mathbb{N}$,

(ii). $nb + \lambda - 1 \not\in -\mathbb{N}$,

(iii). $1 \leq |S| \leq n$ or $(n+1)b \not\in \mathbb{Z}_+$.

Our paper is organized as follows: In Section 2 we define our setup, obtain some preliminary results on the structure of $M_p(V(a; S, b, \lambda))$, and give some technical lemmas. Section 3 is dedicated to the proof of our main theorem. We first prove that every nonzero element of $M_p(V(a; S, b, \lambda))$ can be reduced to a nonzero element of $1 \otimes V(a; S, b, \lambda)$ under the conditions for simplicity. Necessity of the conditions for simplicity is then proved by constructing explicit submodules of $M_p(V(a; S, b, \lambda))$ for every $\lambda$ and $b$ such that either $b - \lambda + 2$ or $-nb - \lambda + 1$ is a natural number.

**Remark 2.** Every annihilator of a simple module $L$ is also the annihilator of a simple highest weight module $L(\mu)$ for some $\mu \in \mathfrak{h}^*$. Moreover, a result of the paper [MaSt1] states that if $\text{Ann}(L) = \text{Ann}(L(\mu))$ for two simple $p$-modules $L$ and $L(\mu)$, then $\text{Ind}_p^q L$ is simple if and only if $\text{Ind}_p^q L(\mu)$ is. Thus an alternative way to prove the simplicity of our $M_p(V)$ above would be to find out which $\mu$’s corresponds to our $U(\mathfrak{h})$-free modules, and then use Kazhdan–Lusztig combinatorics to show that the highest weight module $\text{Ind}_p^q L(\mu)$ is simple for exactly these $\mu$. However, both of these steps seem difficult at the moment.

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## 2 Technical lemmas

Throughout the paper, we denote by $\mathbb{C}, \mathbb{N}, \mathbb{Z}_+$ the sets of all complex numbers, positive integers and nonnegative integers, respectively. For a set $S$ we define the indicator functions

$$
\delta_{s \in S} = \begin{cases} 
1 & \text{if } s \in S \\
0 & \text{if } s \not\in S
\end{cases}, \quad \delta_{s \notin S} = \begin{cases} 
0 & \text{if } s \in S \\
1 & \text{if } s \not\in S
\end{cases}.
$$

In this section we collect some of the basic definitions and establish technical results needed for studying our modules.
Let \( \mathfrak{g} \) be a semisimple complex finite dimensional Lie algebra with a fixed triangular decomposition \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \) and \( \mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+ \) a parabolic subalgebra of \( \mathfrak{g} \) with the Levi decomposition \( \mathfrak{p} = (\mathfrak{a} \oplus \mathfrak{h}_a) \oplus \mathfrak{n}_+ \), where \( \mathfrak{n} \) is nilpotent, \( \mathfrak{a}' = \mathfrak{a} \oplus \mathfrak{h}_a \) is reductive, \( \mathfrak{a} \) is semisimple and \( \mathfrak{h}_a \subset \mathfrak{h} \) is abelian and central in \( \mathfrak{a}' \). A \textit{generalized Verma module} over \( \mathfrak{g} \) is an induced module

\[
M_p(V) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V,
\]

where \( V \) is an \( \mathfrak{a}' \)-module and \( \mathfrak{n}V = 0 \).

Consider the algebra \( \mathfrak{s}l_{n+2}(\mathbb{C}) \). Denote by \( e_{i,j}(1 \leq i, j \leq n+2) \) the \((n+2) \times (n+2)\) matrix with zeros everywhere except a 1 on position \((i,j)\). For \( 1 \leq k \leq n+2 \), let

\[
h_k := e_{k,k} - \frac{1}{n+1} \sum_{i=1}^{n+1} e_{i,i}.
\]

Note that \( h_{n+1} = -h_1 - h_2 - \cdots - h_n \). Then

\[
\{e_{i,j}|1 \leq i \neq j \leq n+2\} \cup \{h_1, \ldots h_n, h_{n+2}\}
\]

is a basis for \( \mathfrak{s}l_{n+2}(\mathbb{C}) \), and

\[
\{e_{i,j}|1 \leq i \neq j \leq n+1\} \cup \{h_1, \ldots h_n\}
\]

is a basis for \( \mathfrak{s}l_{n+1}(\mathbb{C}) \). We see that \( [\mathfrak{s}l_{n+1}, h_{n+2}] = 0 \). The subspace spanned by \( \{h_1, \ldots h_n\} \) is the standard Cartan subalgebra \( \mathfrak{h}_n \) of \( \mathfrak{s}l_{n+1}(\mathbb{C}) \). With respect to the basis, the bracket operation is given by the following lemma.

**Lemma 3.** For \( 1 \leq i \neq j \leq n+2, 1 \leq i' \neq j' \leq n+2 \) and \( 1 \leq k, k' \leq n+2 \), we have

\[
[e_{i,j}, e_{i',j'}] = \delta_{j',i'}e_{i,j} - \delta_{i,j'}e_{i',j},
\]

\[
[h_k, e_{i,j}] = (\delta_{k,i} - \delta_{k,j} + \frac{1}{n+1}(\delta_{i,n+2} - \delta_{j,n+2}))e_{i,j},
\]

\[
[h_k, h_{k'}] = 0.
\]

We denote by \( \mathcal{U}(\mathfrak{g}) \) the universal enveloping algebra of a Lie algebra \( \mathfrak{g} \). From the above lemma, we can prove the following useful formulas.

**Lemma 4.** In \( \mathcal{U}(\mathfrak{s}l_{n+2}) \) we have the following relations: for \( 1 \leq i \neq k \leq n+1 \)

\[
e_{i,n+2}e_{n+2,i}^{m} = e_{i,n+2}^{m}e_{i,n+2} + me_{n+2,i}^{m-1}e_{i,k},
\]

\[
e_{i,n+2}e_{n+2,i}^{m} = e_{n+2,i}^{m}e_{i,n+2} + me_{n+2,i}^{m-1}(h_i - h_{n+2} - m + 1),
\]

\[
e_{i,k}e_{n+2,i}^{m} = e_{n+2,i}^{m}e_{i,k} - me_{n+2,i}^{m-1}e_{i,k},
\]

\[
h_ke_{n+2,i}^{m} = e_{n+2,i}^{m}(h_k + m - m + 1),
\]

\[
h_{i}e_{n+2,i}^{m} = e_{n+2,i}^{m}(h_i - \frac{mn}{n+1}),
\]

\[
h_{n+2}e_{n+2,i}^{m} = e_{n+2,i}^{m}(h_{n+2} + \frac{n+2}{n+1}).
\]

**Proof.** We prove this lemma by induction on \( m \). For \( m = 1 \), the equations are just the first two equations in Lemma 3. Now suppose that the equations hold for \( m \), then for \( 1 \leq i \neq k \leq n+1 \),

\[
e_{i,n+2}e_{n+2,i}^{m+1} = e_{i,n+2}e_{n+2,i}^{m} + me_{n+2,i}^{m-1}e_{i,k},
\]

\[
e_{i,n+2}e_{n+2,i}^{m} = (e_{n+2,i}e_{i,n+2} + me_{n+2,i}e_{i,k})e_{n+2,i}^{m}.
\]
\[ e_{i+2} = e^{m}_{n+2} + m e^{m}_{n+2} e_{i+1} + e_{i+1} e_{i+2} + e_{i+2} e_{i+1} + e_{i+1} e_{i+2} + e_{i+2} e_{i+1} \]

\[ e_{i+2} = e^{m}_{n+2} e_{i+1} + e_{i+1} e_{i+2} + e_{i+2} e_{i+1} + e_{i+1} e_{i+2} + e_{i+2} e_{i+1} \]

\[ e_{i+2} = e^{m+1}_{n+2} e_{i+1} + (m+1) e^{m}_{n+2} e_{i+1} \]

\[ e_{i+2} = e^{m+1}_{n+2} e_{i+1} + (m+1) e^{m}_{n+2} e_{i+1} \]

\[ h_{k} e_{i+2} = h_{k} e^{m}_{n+2} e_{i+1} + e^{m+1}_{n+2} e_{i+1} \]

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We recall the following fact about the classification of \( \mathcal{U}(h) \)-free \( \mathfrak{sl}_{n+1} \)-modules of rank 1.

**Lemma 5.** Let \( S \subseteq \{1, 2, \ldots, n+1\}, b \in \mathbb{C} \) and \( a = (a_{1}, \ldots, a_{n}, 1) \in (\mathbb{C}^{*})^{n+1} \). Denote by \( V(a, S, b) \) the vector space \( \mathbb{C}[h_{1}, \ldots, h_{n}] \) equipped with the \( \mathfrak{sl}_{n+1} \)-module structure

\[ h_{i} \cdot f = h_{i} f, \]

\[ h_{i} \cdot f = h_{i} f, \]
Lemma 6. For fixed \( v \) and for \( 1 \leq i \leq n + 1 \) where \( \sigma_i \) (\( 1 \leq i \leq n \)) is the algebra automorphism of \( C[h_1, \ldots, h_n] \) defined by mapping \( h_k \) to \( h_k - \delta_{k,i} \) while \( \sigma_{n+1} \) is the identity map. Then the set of \( sl_{n+1} \)-modules that are \( U(h_n) \)-free of rank 1 is

\[
\{ V(a, S, b) | S \subseteq \{ 1, 2, \ldots, n + 1 \}, b \in C, a = (a_1, \ldots, a_n, 1) \in (C^*)^{n+1} \}.
\]

Moreover, \( V(a, S, b) \) is simple if and only if \( 1 \leq |S| \leq n \) or \( (n+1)b \notin Z_+ \).

Proof. This is a reformulation of Proposition 28, Theorem 29, and Proposition 31 of [N1].

Now let \( a' = sl_{n+1}(C) + C h_{n+2} \) (where \( sl_{n+1}(C) \) is realized as the upper left subalgebra of \( sl_{n+2}(C) \)) and let \( M \) be an \( a' \)-module whose restriction to \( sl_{n+1}(C) \) is \( U(h_n) \)-free of rank 1. Since \( h_{n+2} \) is central in \( a' \), we know that \( h_{n+2} \) acts on \( M \) as a scalar \( \lambda \) and \( M \) is isomorphic to some \( V(a, S, b) \) as \( sl_{n+1} \)-modules by the above lemma. Denote this \( a' \)-module \( M \) by \( V(a, S, b, \lambda) \).

Let \( p = a' + \sum_{i=1}^{n+1} C e_{i,n+2} \). In this paper, we will study the generalized Verma module \( M_p(V(a, S, b, \lambda)) \) over \( sl_{n+2} \) induced from the simple \( a' \)-module \( V(a, S, b, \lambda) \). Clearly, as vector spaces we have

\[
M_p(V(a, S, b, \lambda)) \cong C[e_{n+2,1}, \ldots, e_{n+2,n+1}] \otimes C[h_1, \ldots, h_n].
\]

Note that \( e_{n+2,1}, \ldots, e_{n+2,n+1} \) commute. We consider elements of \( M_p(V(a, S, b, \lambda)) \) as polynomials in the \( n + 1 \) variables \( e_{n+2,1}, \ldots, e_{n+2,n+1} \) with coefficients (on the right side) in \( C[h_1, \ldots, h_n] \). To simplify notation we write \( m := (m_1, \ldots, m_{n+1}) \in Z_+^{n+1} \), and for \( 1 \leq k \leq n + 1 \) we let \( e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \) (1 in the \( k \)-th position). Let \( |m| = m_1 + \cdots + m_{n+1} \). We also define

\[
a^m := a_1^{m_1} \cdots a_{n+1}^{m_{n+1}}, \quad V(a) = (a_1, \ldots, a_{n+1}) \in (C^*)^{n+1},
\]

\[
E^m := e_{n+2,1}^{m_1} e_{n+2,2}^{m_2} \cdots e_{n+2,n+1}^{m_{n+1}}.
\]

Any nonzero element \( v \) in \( M_p(V(a, S, b, \lambda)) \) can be uniquely written as

\[
v = \sum_{k=0}^{N} \sum_{|m|=k} E^m P_m = \sum_{k=0}^{N} v_k,
\]

where \( P_m \in C[h_1, \ldots, h_n] \) and \( v_k = \sum_{|m|=k} E^m P_m \) with \( v_N \neq 0 \). We call \( N \) the degree of \( v \) and \( v_k \) is homogeneous of degree \( k \). The following lemma tells us how the elements \( e_{i,n+2}(1 \leq i \leq n + 1) \) act on an arbitrary homogeneous element in \( M_p(V(a, S, b, \lambda)) \) of degree \( N \).

Lemma 6. For fixed \( N \), let

\[
v = \sum_{|m|=N} E^m P_m,
\]

where \( P_m \in C[h_1, \ldots, h_n] \). For \( 1 \leq i \leq n + 1 \), we then have

\[
e_{i,n+2} \cdot v = \sum_{|m|=N-1} E^m \left( \sum_{k \neq i} (m_k + 1)(e_{i,k} \cdot P_{m+ek}) + (m_i + 1)(h_i - \lambda - N + 1)P_{m+e_i} \right).
\]
Proof. We compute
\[ e_{i,n+2} \cdot v = \sum_{|\overline{m}| = N} e_{i,n+2} \cdot E_{\overline{m}} P_{\overline{m}} \]
\[ = \sum_{|\overline{m}| = N} \left( e_{i,n+2} \cdot e_{i,n+2,i} (\prod_{j \neq i} e_{n+2,j})P_{\overline{m}} \right) \]
\[ = \sum_{|\overline{m}| = N} \left( e_{i,n+2} e_{i,n+2} + e_{i,n+2,i} (h_i - h_{n+2} - m_i + 1) \cdot (\prod_{j \neq i} e_{n+2,j})P_{\overline{m}} \right) \]
\[ = \sum_{|\overline{m}| = N} \left( e_{i,n+2} e_{i,n+2,i} (\prod_{j \neq i} e_{n+2,j})P_{\overline{m}} \right) + \sum_{|\overline{m}| = N} e_{i,n+2,i} (h_i - h_{n+2} - m_i + 1) \cdot (\prod_{j \neq i} e_{n+2,j})P_{\overline{m}}. \]

We compute the two sums separately:
\[ \sum_{|\overline{m}| = N} \left( e_{i,n+2} e_{i,n+2,i} (\prod_{j \neq i} e_{n+2,j})P_{\overline{m}} \right) \]
\[ = \sum_{|\overline{m}| = N} \left( e_{i,n+2,i} (\prod_{j \neq i} e_{n+2,j})P_{\overline{m}} \right) + \sum_{|\overline{m}| = N} (e_{i,n+2,i} e_{i,n+2} (\prod_{k \neq i} e_{n+2,k})P_{\overline{m}}) \]
\[ = 0 + \sum_{|\overline{m}| = N} e_{i,n+2,i} \sum_k e_{i,n+2} e_{n+2,k} (\prod_{j \neq i,k} e_{n+2,j})P_{\overline{m}} \]
\[ = \sum_{|\overline{m}| = N} e_{i,n+2,i} \sum_k (m_k e_{n+2,k} e_{i,k}) (\prod_{j \neq i,k} e_{n+2,j})P_{\overline{m}} \]
\[ = \sum_{|\overline{m}| = N} \sum_k m_k E_{\overline{m} - \epsilon_k} (e_{i,k} P_{\overline{m}}) \]
\[ = \sum_{k \neq i} \left( \sum_{|\overline{m}| = N} m_k E_{\overline{m} - \epsilon_k} (e_{i,k} P_{\overline{m}}) \right). \]

In the last step we just changed name for the summation variable \( \overline{m} \). We now introduce a new variable \( m := m - \epsilon_k \) in the inner sum. The above expression becomes
\[ \sum_{k \neq i} \sum_{|\overline{m}| = N-1} (m_k + 1) E_{\overline{m}} (e_{i,k} P_{\overline{m} + \epsilon_k}) = \sum_{|\overline{m}| = N-1} E_{\overline{m}} \left( \sum_{k \neq i} (m_k + 1)(e_{i,k} P_{\overline{m} + \epsilon_k}) \right). \]

We now turn to the second term. Note first that for \( 1 \leq i \neq j \leq n + 1 \), we have
\[ (h_i - h_{n+2}) e_{n+2,j} = e_{n+2,j} (h_i - h_{n+2} - 1). \] Using this relation many times we can calculate
\[ \sum_{|\overline{m}| = N} m_i e_{n+2,i} (h_i - h_{n+2} - m_i + 1) \cdot (\prod_{j \neq i} e_{n+2,j})P_{\overline{m}} \]
\[ = \sum_{|\overline{m}| = N} m_i e_{n+2,i} (\prod_{j \neq i} e_{n+2,j}) (h_i - h_{n+2} - m_i + 1 - \sum_j m_j)P_{\overline{m}} \]
\[ = \sum_{|\overline{m}| = N} m_i E_{\overline{m} - \epsilon_i} (h_i - \lambda - N + 1) P_{\overline{m}}. \]
By another change of variables, \( \overline{m} := m' - \epsilon_i \), this simplifies to

\[
\sum_{|m| = N-1} E^{\overline{m}} ((m_i + 1)(h_i - \lambda - N + 1)P_{\overline{m} + \epsilon_i}).
\]

Substituting any both of these into our original expression we can continue to simplify:

\[
e_{i,n+2} \cdot v = \left( \sum_{|m| = N-1} E^{\overline{m}} \left( \sum_{k \neq i} (m_k + 1)(e_{i,k} \cdot P_{\overline{m} + \epsilon_k}) \right) + \sum_{|m| = N-1} E^{\overline{m}} ((m_i + 1)(h_i - \lambda - N + 1)P_{\overline{m} + \epsilon_i}) \right) = \sum_{|m| = N-1} E^{\overline{m}} \left( \sum_{k \neq i} (m_k + 1)(e_{i,k} \cdot P_{\overline{m} + \epsilon_k}) + (m_i + 1)(h_i - \lambda - N + 1)P_{\overline{m} + \epsilon_k} \right),
\]

which alternatively can be rewritten as

\[
\sum_{|m| = N-1} E^{\overline{m}} \sum_{k=1}^{n+1} (m_k + 1) \left( (1 - \delta_{i,k})(e_{i,k} \cdot P_{\overline{m} + \epsilon_k}) + \delta_{i,k}(h_i - \lambda - N + 1)P_{\overline{m} + \epsilon_k} \right).
\]

This completes the proof. \( \Box \)

Let \( \overline{m} \in \mathbb{Z}^{n+1}_+ \) and \( N \in \mathbb{N} \) with \( N \geq |\overline{m}| \). For each \( S \subseteq \{1, 2, \ldots, n+1\} \) and \( \mathbf{a} = (a_1, \ldots, a_n, 1) \in (\mathbb{C}^*)^{n+1} \), define

\[
P_{\overline{m}}(S, \mathbf{a}) = \frac{a^{\overline{m}}}{m} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k),
\]

\[
P'_{\overline{m}}(S, \mathbf{a}) = \frac{a^{\overline{m}}}{m} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) (\delta_{n+1 \notin S} \prod_{r=1}^{N-m_{n+1}} (h_{n+1} - b - 1 + r) + \delta_{n+1 \notin S})
\]

\[
\Delta_{\overline{m}}(S, \mathbf{a}) = \frac{a^{\overline{m}}}{m} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) \prod_{r=2}^{N-m_{n+1}} (h_{n+1} - b - 2 + r) \prod_{t=1}^{m_{n+1}} (h_{n+1} + nb - t),
\]

\[
\Theta_{\overline{m}}(S, \mathbf{a}) = \frac{a^{\overline{m}}}{m} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) \prod_{k=1}^{m_{n+1}} (h_{n+1} - b - k - 1),
\]

\[
\Upsilon_{\overline{m}}(S, \mathbf{a}) = \frac{a^{\overline{m}}}{m} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) \prod_{t=1}^{m_{n+1}} (h_{n+1} + nb - t).
\]

Then we have the following lemma.
Lemma 7. Let $\mathbb{m} \in \mathbb{Z}^{n+1}$, $N \in \mathbb{N}$ with $N \geq |\mathbb{m}|$, $S \subseteq \{1, 2, \ldots, n+1\}$ and $\mathbf{a} = (a_1, \ldots, a_n, 1) \in (\mathbb{C}^*)^{n+1}$.

(1) If $n+1 \in S$, then
\[ a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\mathbb{m}+\epsilon_j}(S, \mathbf{a})) = \begin{cases} P_{\mathbb{m}}(S, \mathbf{a}), & 1 \leq j \leq n+1, j \in S, \\ (h_j - b)P_{\mathbb{m}}(S, \mathbf{a}), & 1 \leq j \leq n+1, j \notin S; \end{cases} \]
\[ a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\mathbb{m}+\epsilon_j}(S, \mathbf{a})) = \begin{cases} (h_{n+1} - b - 1)\Delta_{\mathbb{m}}(S, \mathbf{a}), & 1 \leq j \leq n, j \in S, \\ (h_j - b)(h_{n+1} - b - 1)\Delta_{\mathbb{m}}(S, \mathbf{a}), & 1 \leq j \leq n, j \notin S, \\ (h_{n+1} + nb)\Delta_{\mathbb{m}}(S, \mathbf{a}), & j = n+1. \end{cases} \]

(2) If $n+1 \notin S$, then
\[ a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\mathbb{m}+\epsilon_j}(S, \mathbf{a})) = \begin{cases} \Theta_{\mathbb{m}}(S, \mathbf{a}), & 1 \leq j \leq n, j \in S, \\ (h_j - b)\Theta_{\mathbb{m}}(S, \mathbf{a}), & 1 \leq j \leq n, j \notin S, \\ (h_{n+1} - b - 1)\Theta_{\mathbb{m}}(S, \mathbf{a}), & j = n+1; \end{cases} \]
\[ a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\mathbb{m}+\epsilon_j}(S, \mathbf{a})) = \begin{cases} \Upsilon_{\mathbb{m}}(S, \mathbf{a}), & 1 \leq j \leq n, j \in S, \\ (h_j - b)\Upsilon_{\mathbb{m}}(S, \mathbf{a}), & 1 \leq j \leq n, j \notin S, \\ (h_{n+1} + nb)\Upsilon_{\mathbb{m}}(S, \mathbf{a}), & j = n+1. \end{cases} \]

Proof. The lemma follows from direct computations.

(1) Let $n+1 \in S$. For $1 \leq j \leq n$ with $j \in S$, we have
\[ a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\mathbb{m}+\epsilon_j}(S, \mathbf{a})) = a_j^{-1}(m_j + 1)\sigma_j^{-1}\left(\frac{a_{\mathbb{m}+\epsilon_j}}{(\mathbb{m}+\epsilon_j)!}\prod_{s \in S} \prod_{k=1}^{m_s} (h_s \mathbin{-} b \mathbin{-} k) \right) \]
\[ = \frac{a_{\mathbb{m}+\epsilon_j}}{\mathbb{m}!} \prod_{s \in S} \prod_{k=1}^{m_s} (h_s \mathbin{-} b \mathbin{-} k) \]
\[ = P_{\mathbb{m}}(S, \mathbf{a}), \]
\[ a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\mathbb{m}+\epsilon_j}(S, \mathbf{a})) = a_j^{-1}(m_j + 1)\sigma_j^{-1}\left(\frac{a_{\mathbb{m}+\epsilon_j}}{(\mathbb{m}+\epsilon_j)!}\prod_{s \in S} \prod_{k=1}^{m_s} (h_s \mathbin{-} b \mathbin{-} k) \right) \]
\[ \cdot \prod_{t=1}^{m_{n+1}} (h_{n+1} + nb \mathbin{-} t + 1) \]
\[ = \frac{a_{\mathbb{m}+\epsilon_j}}{\mathbb{m}!} \prod_{s \in S} \prod_{k=1}^{m_s} (h_s \mathbin{-} b \mathbin{-} k) \prod_{r=1}^{N-m_{n+1}} (h_{n+1} \mathbin{-} b \mathbin{-} 1 + r) \prod_{t=1}^{m_{n+1}} (h_{n+1} \mathbin{-} b + nb \mathbin{-} t) \]
\[ = \frac{a_{\mathbb{m}+\epsilon_j}}{\mathbb{m}!} \prod_{s \in S} \prod_{k=1}^{m_s} (h_s \mathbin{-} b \mathbin{-} k) \prod_{r=1}^{N-m_{n+1}} (h_{n+1} \mathbin{-} b \mathbin{-} 2 + r) \prod_{t=1}^{m_{n+1}} (h_{n+1} \mathbin{-} b + nb \mathbin{-} t) \]
\[ \cdot (h_{n+1} \mathbin{-} b - 1) \prod_{s \in S} \prod_{k=1}^{m_s} (h_s \mathbin{-} b \mathbin{-} k) \prod_{r=2}^{N-m_{n+1}} (h_{n+1} \mathbin{-} b \mathbin{-} 2 + r) \prod_{t=1}^{m_{n+1}} (h_{n+1} \mathbin{-} b + nb \mathbin{-} t) \]
\[ = (h_{n+1} \mathbin{-} b - 1)\Delta_{\mathbb{m}}(S, \mathbf{a}). \]
For $1 \leq j \leq n$ with $j \notin S$, we have

$$a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{m+n+j}(S, a))$$

$$= a_j^{-1}(m_j + 1)\sigma_j^{-1}\left(\frac{a_{m+n+j}}{(m + \epsilon_j)!} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) \prod_{k=1}^{m_j+1} (h_j - b - k)\right)$$

$$= \frac{a_{m+n+j}}{m!} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) \prod_{k=1}^{m_j+1} (h_j - b - k)$$

$$= a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{m+n+j}(S, a))$$

$$= a_j^{-1}(m_j + 1)\sigma_j^{-1}\left(\frac{a_{m+n+j}}{(m + \epsilon_j)!} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) \prod_{k=1}^{m_j+1} (h_j - b - k)\right)$$

$$= \prod_{r=1}^{N-m_{n+1}} (h_{n+1} - b - 1 + r) \prod_{t=1}^{m_{n+1}} (h_{n+1} + nb - t + 1)$$

$$= \frac{a_{m+n+1}}{m!} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) \prod_{r=1}^{N-m_{n+1}} (h_{n+1} - b - 2 + r) \prod_{t=1}^{m_{n+1}} (h_{n+1} + nb - t)$$

$$= \frac{a_{m+n+1}}{m!} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) \prod_{r=1}^{N-m_{n+1}} (h_{n+1} - b - 2 + r) \prod_{t=1}^{m_{n+1}} (h_{n+1} + nb - t)$$

$$= (h_j - b)(h_{n+1} - b - 1)\Delta_{m+n}(S, a).$$

Finally, for $j = n + 1$, we have

$$a_{n+1}^{-1}(m_{n+1} + 1)\sigma_{n+1}^{-1}(P_{m+n+1}(S, a))$$

$$= a_{n+1}^{-1}(m_{n+1} + 1)\frac{a_{m+n+1}}{(m + \epsilon_{n+1})!} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k)$$

$$= \frac{a_{m+n+1}}{m!} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k)$$

$$= P_{m+n+1}(S, a),$$

$$a_{n+1}^{-1}(m_{n+1} + 1)\sigma_{n+1}^{-1}(P_{m+n+1}(S, a))$$

$$= a_{n+1}^{-1}(m_{n+1} + 1)\frac{a_{m+n+1}}{(m + \epsilon_{n+1})!} \prod_{s \notin S} \prod_{k=1}^{m_s} (h_s - b - k) \prod_{r=1}^{N-m_{n+1}-1} (h_{n+1} - b - 1 + r) \prod_{t=1}^{m_{n+1}+1} (h_{n+1} + nb - t + 1)$$
(2) Let $n + 1 \not\in S$. For $1 \leq j \leq n$ with $j \in S$, we have

$$a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{m+\epsilon_j}(S, a)) = a_j^{-1}(m_j + 1)\sigma_j^{-1}\left(\frac{a_{m+\epsilon_j}}{m + \epsilon_j}! \prod_{s \in S, k=1}^{m} (h_s - b - k) \prod_{k=1}^{m+1} (h_{n+1} - b - k)\right)$$

$$= \frac{a_{m}}{m!} \prod_{s \in S, k=1}^{m} (h_s - b - k) \prod_{k=1}^{m+1} (h_{n+1} - b - k - 1)$$

$$= \Theta_{m}(S, a),$$

$$o_j^{-1}(m_j + 1)\sigma_j^{-1}(P'_{m+\epsilon_j}(S, a))$$

$$= a_j^{-1}(m_j + 1)\sigma_j^{-1}\left(\frac{a m + \epsilon_j}{m + \epsilon_j}! \prod_{s \in S, k=1}^{m} (h_s - b - k) \prod_{k=1}^{m+1} (h_{n+1} + nb - k + 1)\right)$$

$$= \frac{a_{m}}{m!} \prod_{s \in S, k=1}^{m} (h_s - b - k) \prod_{k=1}^{m+1} (h_{n+1} + nb - k)$$

$$= T_{m}(S, a).$$

For $1 \leq j \leq n$ with $j \not\in S$, we have

$$a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{m+\epsilon_j}(S, a))$$

$$= a_j^{-1}(m_j + 1)\sigma_j^{-1}\left(\frac{a_{m+\epsilon_j}}{m + \epsilon_j}! \prod_{s \in S, k=1}^{m} (h_s - b - k) \prod_{k=1}^{m+1} (h_{j} - b - k)\right)$$

$$\cdot \prod_{k=1}^{m+1} (h_{n+1} - b - k)$$

$$= \frac{a_{m}}{m!} \prod_{s \in S, k=1}^{m} (h_s - b - k) \prod_{k=1}^{m+1} (h_{j} - b - k + 1) \prod_{k=1}^{m+1} (h_{n+1} - b - k - 1),$$

$$= \frac{a_{m}}{m!} (h_j - b) \prod_{s \in S, k=1}^{m} (h_s - b - k) \prod_{k=1}^{m+1} (h_{n+1} - b - k - 1)$$

$$= (h_j - b)\Theta_{m}(S, a),$$

$$a_j^{-1}(m_j + 1)\sigma_j^{-1}(P'_{m+\epsilon_j}(S, a))$$

$$= a_j^{-1}(m_j + 1)\sigma_j^{-1}\left(\frac{a m + \epsilon_j}{m + \epsilon_j}! \prod_{s \in S, k=1}^{m} (h_s - b - k) \prod_{k=1}^{m+1} (h_{j} - b - k)\right)$$

$$= a_j^{-1}(m_j + 1)\sigma_j^{-1}\left(\frac{a m + \epsilon_j}{m + \epsilon_j}! \prod_{s \in S, k=1}^{m} (h_s - b - k) \prod_{k=1}^{m+1} (h_{j} - b - k + 1)\right).$$
In this section, we will study the generalized Verma module $M_p(V(a, S, b, \lambda))$ over $\mathfrak{sl}_{n+2}$ for the given parameters $b \in \mathbb{C}, \lambda \in \mathbb{C}, a = (a_1, \cdots, a_n, 1) \in (\mathbb{C}^*)^{n+1}, S \subseteq \{1, 2, \ldots, n+1\}$. Indeed, we will give sufficient and necessary conditions for $M_p(V(a, S, b, \lambda))$ to be simple. Before proving our main theorem, we need several auxiliary lemmas. First, we have

Lemma 8. Let $W \subseteq M_p(V(a, S, b, \lambda))$ be a nonzero submodule. Suppose

$$v = \sum_{k=0}^{N} v_k \in W,$$

where $v_k$ is homogeneous of degree $k$, then $v_k \in W$ for all $k$.

\[ \cdot \prod_{k=1}^{m+1} (h_{n+1} + nb - k + 1) \]
\[ = \frac{a_{m+1}}{m!} \prod_{s \not\in S} \prod_{s \neq j, n+1}^{m_s} (h_s - b - k) \prod_{k=1}^{m_{j+1}} (h_j - b - k + 1) \prod_{k=1}^{m_{n+1}} (h_{n+1} + nb - k), \]
\[ = \frac{a_{m+1}}{m!} (h_j - b) \prod_{s \not\in S} \prod_{s \neq n+1}^{m_s} (h_s - b - k) \prod_{k=1}^{m_{n+1}} (h_{n+1} + nb - k) \]
\[ = (h_j - b) \mathcal{M}(S, a). \]

Now to complete the proof, it remains to verify the equations when $j = n + 1 \not\in S$.

\[ a_{n+1}^{-1}(m_{n+1} + 1)\sigma_{n+1}^{-1}(P_{m+\epsilon_{n+1}}(S, a)) \]
\[ = a_{n+1}^{-1}(m_{n+1} + 1) \frac{a_{m+\epsilon_{n+1}}}{(m + \epsilon_{n+1})!} \prod_{s \not\in S} \prod_{s \neq n+1}^{m_s} (h_s - b - k) \prod_{k=1}^{m_{n+1}} (h_{n+1} + nb - k) \]
\[ = \frac{a_{m+1}}{m!} (h_{n+1} + nb) \prod_{s \not\in S} \prod_{s \neq n+1}^{m_s} (h_s - b - k) \prod_{k=1}^{m_{n+1}} (h_{n+1} + nb - t) \]
\[ = (h_{n+1} + nb) \mathcal{M}(S, a). \]

\[
\square
\]

3 Proof of the main theorem

In this section, we will study the generalized Verma module $M_p(V(a, S, b, \lambda))$ over $\mathfrak{sl}_{n+2}$ for the given parameters $b \in \mathbb{C}, \lambda \in \mathbb{C}, a = (a_1, \cdots, a_n, 1) \in (\mathbb{C}^*)^{n+1}, S \subseteq \{1, 2, \ldots, n+1\}$. Indeed, we will give sufficient and necessary conditions for $M_p(V(a, S, b, \lambda))$ to be simple. Before proving our main theorem, we need several auxiliary lemmas. First, we have
Proof. Since

\[ h_{n+2} \cdot v_k = (\lambda + \frac{n + 2}{n + 1}) v_k, \]

we see that \( v_k \) are weight vectors of different weights with respect to \( h_{n+2} \). Hence, \( v_k \in W \) for all \( k \).

\[ \text{Lemma 9. Let } N \in \mathbb{N}, \text{ and let } v = \sum_{|m|=N} E^m P_m \text{ be a nonzero homogeneous element in } M_p(V(\mathfrak{a}, S, b, \lambda)) \text{ of degree } N \text{ where } P_m \in \mathbb{C}[h_1, \cdots, h_n]. \text{ If } e_{i,n+2} \cdot v = 0 \text{ for all } 1 \leq i \leq n+1, \]

then \( \mathcal{U}(\mathfrak{sl}_{n+2}(\mathbb{C}))v \) is a nonzero proper \( \mathfrak{sl}_{n+2}(\mathbb{C}) \)-submodule of \( M_p(V(\mathfrak{a}, S, b, \lambda)) \).

Proof. Clearly, \( \mathcal{U}(\mathfrak{sl}_{n+2}(\mathbb{C}))v \) is a nonzero submodule of \( M_p(V(\mathfrak{a}, S, b, \lambda)) \). To show it is proper, we will show that any nonzero element in \( \mathcal{U}(\mathfrak{sl}_{n+2}(\mathbb{C}))v \) has degree greater than or equal to \( N \). By the PBW Theorem we see that

\[ \mathcal{U}(\mathfrak{sl}_{n+2}(\mathbb{C}))v = \mathbb{C}[e_{n+2,1}, \cdots, e_{n+2,n+1}] \otimes \mathcal{U}(\mathfrak{sl}_{n+1}) \otimes \mathcal{U}(\mathfrak{sl}_{n+1}) \]

yielding that

\[ \mathcal{U}(\mathfrak{sl}_{n+2})v = \mathbb{C}[e_{n+2,1}, \cdots, e_{n+2,n+1}]\mathcal{U}(\mathfrak{sl}_{n+1})v. \]

So we only need to show that if the following elements are not zero, then they are homogeneous of degree \( N \): \( h_{n+2} \cdot v, h_k \cdot v(1 \leq k \leq n), e_{i,k} \cdot v(1 \leq i \neq k \leq n+1) \).

Since \( 1 \leq i \neq k \leq n+1 \), using the formulas in Lemma 4 we have

\[ h_{n+2} \cdot v = (\lambda + \frac{n + 2}{n + 1} N)v, \]

\[ h_k \cdot v = h_k \cdot \sum_{|m|=N} E^m P_m = \sum_{|m|=N} E^m (h_k - m_k + \frac{N}{n + 1}) P_m, \]

\[ e_{i,k} \cdot v = e_{i,k} \cdot \sum_{|m|=N} E^m P_m = \sum_{|m|=N} \prod_{j \neq i} e_{n+2,j} \epsilon_{n+2,i}^m \epsilon_{n+2,i}^m \]

\[ = \sum_{|m|=N} (E^m (e_{i,k} \cdot P_m) - m_i \prod_{j \neq i,k} e_{n+2,j} \epsilon_{n+2,k} \epsilon_{n+2,j}^m P_m). \]

Hence, if \( h_{n+2} \cdot v \neq 0, h_k \cdot v \neq 0, e_{i,k} \cdot v \neq 0 \), then they are homogeneous of degree \( N \). This completes the proof.

We also need some results on linear systems. Let \( \overline{h_i} = h_i - \delta_{i,n+1} \). For \( S \subseteq \{1, 2, \cdots, n+1\}, \lambda, b \in \mathbb{C} \) and \( N \in \mathbb{N} \), we define an \((n+1) \times (n+1)\) matrix \( A(\lambda, b, S, N) = (A_{ij}(\lambda, b, S, N))_{1 \leq i,j \leq n+1} \) by

\[ A_{ij}(\lambda, b, S, N) = (\delta_{i \in S} + \delta_{i \in S}(\overline{h_i} - b))(\delta_{j \in S}(\overline{h_j} - b) + \delta_{j \in S}), i \neq j, \]

\[ A_{ii}(\lambda, b, S, N) = \overline{h_i} - \lambda - N + 2. \]
Lemma 10. We have 

\[ \det A(\lambda, b, S, N) = (-nb - \lambda - N + 1)(b - \lambda - N + 2)^n. \]

Proof. Let \( Q_i(x) \) be the matrix obtained from the identity matrix by replacing the \((i, i)\) entry by \(x\). For convenience of computations we will use the fraction field \( \mathbb{C}(h_n) \) of the polynomial ring \( \mathbb{C}[h_n] \). Since \( \sum_{i=1}^{n+1} h_i = 0 \), we have 

\[ \det A(\lambda, b, S, N) = \prod_{sg \in S} \det Q_s(1) \det A(\lambda, b, S, N) \prod_{sg \in S} \det Q_s(h_s - b) \]

\[ = \det(\prod_{sg \in S} Q_s(1) A(\lambda, b, S, N) \prod_{sg \in S} Q_s(h_s - b)) \]

\[ = \begin{vmatrix} h_1 - \lambda - N + 2 & h_2 - b & \cdots & h_{n+1} - b - 1 \\ h_1 - b & h_2 - \lambda - N + 2 & \cdots & h_{n+1} - b - 1 \\ \vdots & \vdots & \ddots & \vdots \\ h_1 - b & h_2 - b & \cdots & h_{n+1} - \lambda - N + 1 \end{vmatrix} \]

\[ = (-nb - \lambda - N + 1) \begin{vmatrix} 1 & h_2 - b & \cdots & h_{n+1} - b - 1 \\ 1 & h_2 - \lambda - N + 2 & \cdots & h_{n+1} - b - 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & h_2 - b & \cdots & h_{n+1} - \lambda - N + 1 \end{vmatrix} \]

\[ = (-nb - \lambda - N + 1) \begin{vmatrix} 1 & h_2 - b & \cdots & h_{n+1} - b - 1 \\ 0 & b - \lambda - N + 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b - \lambda - N + 2 \end{vmatrix} \]

\[ = (-nb - \lambda - N + 1)(b - \lambda - N + 2)^n. \]

\[ \square \]

Lemma 11. If \( nb + \lambda + N - 1 = 0 \), then for any \( \overline{m} \in \mathbb{Z}_+^{n+1} \),

\[ (a_0^{-1}(m_1 + 1)^{-1}(P_{\overline{m}+1}(S, a)), \cdots, a_0^{-1}(m_n + 1)^{-1}(P_{\overline{m}+1}(S, a)), (m_{n+1} + 1)^{-1}(P_{\overline{m}+1}(S, a)) \]

is a solution to the linear system 

\[ A(\lambda, b, S, N)(X_1, \cdots, X_{n+1})^T = 0. \]

Proof. Following from Lemma 7, if \( n + 1 \in S \), then for \( 1 \leq i \leq n \), we have 

\[ \sum_{j=1}^{n+1} A_{ij}(\lambda, b, S, N)a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\overline{m}+1}(S, a)) \]

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Also, we have

$$\sum_{j=1}^{n+1} A_{n+1,j}(\lambda, b, S, N) a_j^{-1} (m_j + 1) \sigma_j^{-1} (P_{\overline{m_j}+1} (S, a))$$

$$= \sum_{j \in S, j \neq i} (h_j - b) P_{\overline{m}}(S, a) + \sum_{j \notin S} (h_j - b) P_{\overline{m}}(S, a) + (h_{n+1} - \lambda - N + 1) P_{\overline{m}}(S, a)$$

$$= (\sum_{j \neq i, n+1} (h_j - b) + h_{n+1} - b - 1 + h_i - \lambda - N + 2) (\delta_{i \in S} + \delta_{i \notin S} (h_i - b)) P_{\overline{m}}(S, a)$$

$$=(-nb - \lambda - N + 1) (\delta_{i \in S} + \delta_{i \notin S} (h_i - b)) P_{\overline{m}}(S, a)$$

$$= 0.$$

Now suppose that $n + 1 \notin S$. Then for $1 \leq i \leq n$,

$$\sum_{j=1}^{n+1} A_{ij}(\lambda, b, S, N) a_j^{-1} (m_j + 1) \sigma_j^{-1} (P_{\overline{m_j}+1} (S, a))$$

$$= \sum_{j \in S, j \neq i} (\delta_{i \in S} + \delta_{i \notin S} (h_i - b)) (h_j - b) \Theta_{\overline{m}}(S, a)$$

$$+ \sum_{j \notin S} (\delta_{i \in S} + \delta_{i \notin S} (h_i - b)) (h_j - b) \Theta_{\overline{m}}(S, a)$$

$$+ (\delta_{i \in S} + \delta_{i \notin S} (h_i - b)) (h_{n+1} - b - 1) \Theta_{\overline{m}}(S, a)$$

$$+ (h_i - \lambda - N + 2) (\delta_{i \in S} + \delta_{i \notin S} (h_i - b)) \Theta_{\overline{m}}(S, a)$$

$$= (\delta_{i \in S} + \delta_{i \notin S} (h_i - b)) (-nb - \lambda - N + 1) \Theta_{\overline{m}}(S, a)$$

$$= 0.$$

And

$$\sum_{j=1}^{n+1} A_{n+1,j}(\lambda, b, S, N) a_j^{-1} (m_j + 1) \sigma_j^{-1} (P_{\overline{m_j}+1} (S, a))$$

$$= \sum_{j \in S} (h_{n+1} - b - 1) (h_j - b) \Theta_{\overline{m}}(S, a) + \sum_{j \notin S} (h_{n+1} - b - 1) (h_j - b) \Theta_{\overline{m}}(S, a)$$

$$+ (h_{n+1} - \lambda - N + 1) (h_{n+1} - b - 1) \Theta_{\overline{m}}(S, a)$$
\[
(\sum_{j \neq n+1} (h_j - b) + h_{n+1} - \lambda - N + 1)(h_{n+1} - b - 1)\Theta_{\overline{m}}(S, a)
\]
\[
= (-nb - \lambda - N + 1)(h_{n+1} - b - 1)\Theta_{\overline{m}}(S, a)
\]
\[
= 0.
\]

This proves the statement in this lemma. \(\square\)

Similarly, we have

**Lemma 12.** If \(b - \lambda - N + 2 = 0\), then for any \(\overline{m} \in \mathbb{Z}_{+}^{n+1}\),

\[
(a_1^{-1}(m_1 + 1)\sigma_1^{-1}(P_{\overline{m}+\epsilon_1}(S, a)), \ldots, a_n^{-1}(m_n + 1)\sigma_n^{-1}(P_{\overline{m}+\epsilon_n}(S, a)))
\]

is a solution to the linear system

\[
A(\lambda, b, S, N)(X_1, \ldots, X_{n+1})^T = 0.
\]

**Proof.** (1) We first assume that \(n + 1 \in S\).

In this case, for \(1 \leq i \leq n\), we have

\[
\sum_{j=1}^{n+1} A_{ij}(\lambda, b, S, N) a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\overline{m}+\epsilon_j}(S, a))
\]

\[
= \sum_{j \in S \atop j \neq i, n+1} (\delta_{i \in S} + \delta_{i \in S}(h_i - b))(h_j - b)(h_{n+1} - b - 1)\Delta_{\overline{m}}(S, a)
\]

\[
+ (\delta_{i \in S} + \delta_{i \in S}(h_i - b))(h_{n+1} - b - 1)(h_{n+1} + nb)\Delta_{\overline{m}}(S, a)
\]

\[
+ \sum_{j \neq S \atop j \neq i} (\delta_{i \in S} + \delta_{i \in S}(h_i - b))(h_j - b)(h_{n+1} - b - 1)\Delta_{\overline{m}}(S, a)
\]

\[
= (\delta_{i \in S} + \delta_{i \in S}(h_i - b))(h_{n+1} - b - 1)(\sum_{j \neq i, n+1} (h_j - b) + h_i - \lambda - N + 2
\]

\[
+ h_{n+1} + nb)\Delta_{\overline{m}}(S, a)
\]

\[
= (\delta_{i \in S} + \delta_{i \in S}(h_i - b))(h_{n+1} - b - 1)(b - \lambda - N + 2)\Delta_{\overline{m}}(S, a)
\]

\[
= 0.
\]

And

\[
\sum_{j=1}^{n+1} A_{n+1,j}(\lambda, b, S, N) a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\overline{m}+\epsilon_j}(S, a))
\]

\[
= \sum_{j \in S \atop j \neq n+1} (h_j - b)(h_{n+1} - b - 1)\Delta_{\overline{m}}(S, a) + \sum_{j \notin S} (h_j - b)(h_{n+1} - b - 1)\Delta_{\overline{m}}(S, a)
\]

\[
+ (h_{n+1} - \lambda - N + 1)(h_{n+1} + nb)\Delta_{\overline{m}}(S, a)
\]

\[
= \sum_{j \neq n+1} (h_j - b)(h_{n+1} - b - 1)\Delta_{\overline{m}}(S, a) + (h_{n+1} - \lambda - N + 1)(h_{n+1} + nb)\Delta_{\overline{m}}(S, a)
\]

\[
= -(h_{n+1} + nb)(h_{n+1} - b - 1)\Delta_{\overline{m}}(S, a) + (h_{n+1} - \lambda - N + 1)(h_{n+1} + nb)\Delta_{\overline{m}}(S, a)
\]

\[
= (h_{n+1} + nb)(b - \lambda - N + 2)\Delta_{\overline{m}}(S, a)
\]

\[
= 0.
\]

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(2) Now we assume that $n + 1 \notin S$.

Now for $1 \leq i \leq n$, we have
\[
\sum_{j=1}^{n+1} A_{ij}(\lambda, b, S, N)a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\mathfrak{m}+\epsilon_j}(S, \mathfrak{a}))
= \sum_{j \in S} (\delta_{i \in S} + \delta_{i \notin S}(h_i - b))(h_j - b)\Upsilon_{\mathfrak{m}}(S, \mathfrak{a})
+ \sum_{j \notin S, j \neq i, n+1} (\delta_{i \in S} + \delta_{i \notin S}(h_i - b))(h_j - b)\Upsilon_{\mathfrak{m}}(S, \mathfrak{a})
+ \sum_{j \notin S, j \neq i, n+1} (\delta_{i \in S} + \delta_{i \notin S}(h_i - b))(h_j - b)\Upsilon_{\mathfrak{m}}(S, \mathfrak{a})
+ (h_i - \lambda - N + 2)(\delta_{i \in S} + \delta_{i \notin S}(h_i - b))\Upsilon_{\mathfrak{m}}(S, \mathfrak{a})
= (\delta_{i \in S} + \delta_{i \notin S}(h_i - b))(h_{i+1} + nb + h_i - \lambda - N + 2)\Upsilon_{\mathfrak{m}}(S, \mathfrak{a})
= (\delta_{i \in S} + \delta_{i \notin S}(h_i - b))(b - \lambda - N + 2)\Upsilon_{\mathfrak{m}}(S, \mathfrak{a}) = 0.
\]

Also, we have
\[
\sum_{j=1}^{n+1} A_{n+1,j}(\lambda, b, S, N)a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_{\mathfrak{m}+\epsilon_j}(S, \mathfrak{a}))
= \sum_{j \in S} (h_{n+1} - b - 1)(h_j - b)\Upsilon_{\mathfrak{m}}(S, \mathfrak{a})
+ \sum_{j \notin S, j \neq n+1} (h_{n+1} - b - 1)(h_j - b)\Upsilon_{\mathfrak{m}}(S, \mathfrak{a})
+ (h_{n+1} - \lambda - N + 1)(h_{n+1} + nb)\Upsilon_{\mathfrak{m}}(S, \mathfrak{a})
= (h_{n+1} + nb)(b - \lambda - N + 2)\Upsilon_{\mathfrak{m}}(S, \mathfrak{a}) = 0.
\]

Hence, the lemma is true. \(\Box\)

Now we can prove our main theorem.

**Theorem 13.** Let $b \in \mathbb{C}, \lambda \in \mathbb{C}, \mathfrak{a} = (a_1, \cdots, a_n, 1) \in (\mathbb{C}^*)^{n+1}, S \subseteq \{1, 2, \ldots, n+1\}$. Then the generalized Verma module $M^a(V(\mathfrak{a}, S, b, \lambda))$ over $\mathfrak{sl}_{n+2}(\mathbb{C})$ is simple if and only if all the following conditions hold

(i) $b - \lambda + 2 \notin \mathbb{N}$,

(ii) $nb + \lambda - 1 \notin -\mathbb{N}$,

(iii) $1 \leq |S| \leq n$ or $(n + 1)b \notin \mathbb{Z}_+$.

**Proof.** From Corollary 32 in [1] and from Lemma 5 we know that $V(\mathfrak{a}, S, b, \lambda)$ is a simple $\mathfrak{a}'$-module if and only if Condition (iii) holds. Now we assume that this condition holds.

For sufficiency we need to show: if $b - \lambda + 2 \notin \mathbb{N}$ and $nb + \lambda - 1 \notin -\mathbb{N}$, then the induced module $M^a(V(\mathfrak{a}, S, b, \lambda))$ is simple.
Suppose $W$ is a proper nonzero submodule of $M_p(V(a, S, b, \lambda))$ and fix a nonzero element $v \in S$ for which the degree $N$ of $v$ is minimal. From Lemma 8 we may assume that $v$ is homogeneous. Since we have assumed the simplicity of $V(a, S, b, \lambda)$ we have $N \geq 1$.

Let $v = \sum_{\overline{m} = N} E_{\overline{m}} P_{\overline{m}}$. By the minimality of $N$ we must have $e_{i,n+2} \cdot v = 0$ for all $1 \leq i \leq n + 1$. By Lemma 5 and Lemma 6 for any $\overline{m} = N - 1$ and any $i : 1 \leq i \leq n + 1$ we see that

$$0 = \sum_{k \neq i} (m_k + 1)(e_{i,k} \cdot P_{\overline{m}+e_k}) + (m_i + 1)(h_i - \lambda - N + 1)P_{\overline{m}+e_i},$$

$$= \sum_{k \neq i} (m_k + 1) a_i a_k^{-1} (\delta_{i \in S} + \delta_{i \not \in S} (h_i - b - 1)) (\delta_{k \in S} (h_k - b) + \delta_{k \not \in S}) \sigma_i \sigma_k^{-1} \cdot P_{\overline{m}+e_k} + (m_i + 1)(h_i - \lambda - N + 1)P_{\overline{m}+e_i}.$$ 

Hence,

$$\sum_{k \neq i} (m_k + 1) a_k^{-1} \sigma_i^{-1} ((\delta_{i \in S} + \delta_{i \not \in S} (h_i - b - 1)) (\delta_{k \in S} (h_k - b) + \delta_{k \not \in S}) \sigma_k^{-1} \cdot P_{\overline{m}+e_k} + a_i^{-1} (m_i + 1) \sigma_i^{-1} ((h_i - \lambda - N + 1)) \sigma_i^{-1} (P_{\overline{m}+e_i}) = 0, \forall |\overline{m}| = N - 1; 1 \leq i \leq n + 1.$$

Then we have

$$A(\lambda, b, S, N) \begin{pmatrix} a_1^{-1}(m_1 + 1)\sigma_1^{-1}(P_{\overline{m}+e_1}) \\ \vdots \\ a_n^{-1}(m_n + 1)\sigma_n^{-1}(P_{\overline{m}+e_n}) \\ (m_{n+1} + 1)P_{\overline{m}+e_{n+1}} \end{pmatrix} = 0 \quad \forall |\overline{m}| = N - 1.$$

When $b - \lambda + 2 \not \in \mathbb{N}$ and $nb + \lambda - 1 \not \in \mathbb{N}$, from Lemma 10 we know that

$$\det A(\lambda, b, S, N) \in \mathbb{C}^*.$$ 

Hence,

$$P_{\overline{m}+e_i} = 0, \quad \forall |\overline{m}| = N - 1; 1 \leq i \leq n + 1.$$ 

Therefore, $v = 0$. This is a contradiction. Thus, $M_p(V(a, S, b, \lambda))$ is simple.

To prove the necessity we need to show that if $nb + \lambda - 1 \in -\mathbb{N}$ or $b - \lambda + 2 \in \mathbb{N}$, then $M_p(V(a, S, b, \lambda))$ is reducible.

First we assume that $nb + \lambda + N - 1 = 0$ for some $N \in \mathbb{N}$. Take

$$v_1 = \sum_{|\overline{m}| = N} E_{\overline{m}} P_{\overline{m}}(S, a).$$

Then for any $1 \leq i \leq n + 1$, following from Lemma 9 and Lemma 11 we have

$$e_{i,n+2} \cdot v_1$$

$$= \sum_{|\overline{m}| = N - 1} E_{\overline{m}} \sum_{k=1}^{n+1} (m_k + 1) \left( (1 - \delta_{i,k})(e_{i,k} \cdot P_{\overline{m}+e_k}(S, a)) + \delta_{i,k}(h_i - \lambda - N + 1) \right)$$

$$\cdot P_{\overline{m}+e_k}(S, a)$$

$$= \sum_{|\overline{m}| = N - 1} E_{\overline{m}} \left( \sum_{k \neq i} (m_k + 1) a_i a_k^{-1} (\delta_{i \in S} + \delta_{i \not \in S} (h_i - b - 1)) (\delta_{k \in S} (h_k - b) + \delta_{k \not \in S}) \cdot \sigma_i \sigma_k^{-1} (P_{\overline{m}+e_k}(S, a)) + (m_i + 1)(h_i - \lambda - N + 1)P_{\overline{m}+e_i}(S, a) \right).$$
\[
= a_i \sum_{|m| = N-1} E_m^\sigma_i \left( \sum_{j=1}^{n+1} A_{ij}(\lambda, b, S, N)a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_m^0) \right)
\]
\[
= a_i \sum_{|m| = N-1} E_m^\sigma_i(0)
\]
\[
= 0.
\]

Hence, by Lemma 9, \(v_1\) generates a proper submodule of \(M_p(V(a, S, b, \lambda))\). Therefore \(M_p(V(a, S, b, \lambda))\) is reducible.

Now we assume that \(b - \lambda - N + 2 = 0\) for some \(N \in \mathbb{N}\). Take
\[
v_2 = \sum_{|m| = N} E_m^\sigma(S, a).
\]

From Lemma 5 and Lemma 12, for any \(1 \leq i \leq n + 1\), we have
\[
e_{i,n+2} \cdot v_2
\]
\[
= \sum_{|m| = N-1} E_m^\sigma \sum_{k=1}^{n+1} (m_k + 1) \left( (1 - \delta_{i,k})(e_{i,k} \cdot P_m^0) + \delta_{i,k}(h_i - \lambda - N + 1) \right)
\]
\[
= \sum_{|m| = N-1} E_m^\sigma \left( \sum_{k \neq i} (m_k + 1) \delta_{i,k}(h_i - \lambda - N + 1) \delta_{k \in \mathbb{S}}(S, a) \right)
\]
\[
= a_i \sum_{|m| = N-1} E_m^\sigma_i \left( \sum_{j=1}^{n+1} A_{ij}(\lambda, b, S, N)a_j^{-1}(m_j + 1)\sigma_j^{-1}(P_m^0) \right)
\]
\[
= a_i \sum_{|m| = N-1} E_m^\sigma_i(0)
\]
\[
= 0.
\]

Hence, Lemma 9 implies that \(v_2\) generates a proper submodule of \(M_p(V(a, S, b, \lambda))\). Therefore \(M_p(V(a, S, b, \lambda))\) is reducible. This completes the proof. \(\square\)

**Remark 14.** The result in Theorem 13 agrees with that in \([McD2]\) for the case of \(\mathfrak{sl}_3\) when \(U(\mathfrak{h}_1)\) is a non-degenerate Whittaker module over \(\mathfrak{sl}_2\).

**Remark 15.** The method to construct simple modules in this paper can be certainly extended to other types of simple Lie algebras. But the computations might be more complicated.

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