Abstract

In loop quantum gravity in the connection representation, the quantum configuration space $\mathcal{A}/\mathcal{G}$, which is a compact space, is much larger than the classical configuration space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations. One finds that $\mathcal{A}/\mathcal{G}$ is homeomorphic to the space $\text{Hom}(\mathcal{L}_*, G)/\text{Ad}$. We give a new, natural proof of this result, suggesting the extension of the hoop group $\mathcal{L}_*$ to a larger, compact group $\mathcal{M}(\mathcal{L}_*)$ that contains $\mathcal{L}_*$ as a dense subset. This construction is based on almost periodic functions. We introduce the Hilbert algebra $L_2(\mathcal{M}(\mathcal{L}_*))$ of $\mathcal{M}(\mathcal{L}_*)$ with respect to the Haar measure $\xi$ on $\mathcal{M}(\mathcal{L}_*)$. The measure $\xi$ is shown to be invariant under 3-diffeomorphisms. This is the first step in a proof that $L_2(\mathcal{M}(\mathcal{L}_*))$ is the appropriate Hilbert space for loop quantum gravity in the loop representation. In a subsequent paper, we will reinforce this claim by defining an extended loop transform and its inverse.
1 Introduction

Over the last years, loop quantum gravity [Rov98] has matured into a powerful candidate for a theory of quantum gravity, with predictive power no less than string theory, the other major approach to quantum gravity. There are intriguing results concerning the nature of quantum geometry [AshLew97a, AshLew97b] and black holes [ABCK98, ABK00]. With the spin network states [Bae96a, Bae96b], well defined orthonormal states of quantum geometry are at hand, and spin foams seem to lead a way to a covariant description of their evolution [BarCra98, BarCra00, Bae98].

In this paper, we consider a much more humble aspect of loop quantum gravity, namely the kinematical framework. Most of the rigorous work in loop quantum gravity has been done in the connection representation [ALMMT95], but many developments were sparked by non-rigorous work in the loop representation [RovSmo93, RovSmo95]. So for example the seminal paper by Rovelli and Smolin [RovSmo90] - in which the loop representation was defined - initiated a lot of research activity both in the loop and the connection picture. As a result, a mathematically rigorous definition of the kinematics of loop quantum gravity in the connection representation was established, using $C^*$-algebra techniques [AshIsh92, AshLew94, AshLew95, ALMMT95].

The quantum configuration space turned out to be a compact space containing the classical configuration space $\mathcal{A}/\mathcal{G}$ of Yang-Mills-theory and General Relativity in the Ashtekar formulation as a dense subspace. The Hilbert space of the unconstrained theory was found to be some $L_2(\mathcal{A}/\mathcal{G}, d\mu)$, and the measure $\mu$ was explicitly constructed. This measure is invariant under 3-diffeomorphisms. The spin network states [Bae96a, Bae96b] were found to be an orthonormal basis of $L_2(\mathcal{A}/\mathcal{G}, d\mu)$, and the diffeomorphism constraints were implemented [ALMMT95], although some questions seem to remain. Moreover, the so called loop transform, also introduced in [RovSmo90], was rigorously defined [AshLew94]. This transform translates from the connection to the loop representation. So at least in the connection representation, the kinematics of loop quantum gravity is well defined.

On the other hand, in the loop representation much of the work does not reach the same level of rigour. The states of loop quantum gravity in the
loop representation are defined rather symbolically \cite{RovSmo90, PieRov96}, as is the inner product of their Hilbert space \cite{Pie97}. There are some ideas on how to define an inverse loop transform \cite{Thi93, Thi98} that turn out to be rather involved technically.

In this and a subsequent paper we will propose a slightly different view on the kinematics of loop quantum gravity by making use of the compact group associated to the hoop group $\mathcal{L}_*$ of holonomy equivalence classes of piecewise analytic loops \cite{AshLew94}. One of the main results in the connection picture is that the quantum configuration space $\mathcal{A}/\mathcal{G}$ is homeomorphic to the space $\text{Hom}(\mathcal{L}_*, G)/\text{Ad}$ of unitary equivalence classes of homomorphisms from the hoop group $\mathcal{L}_*$ to the gauge group $G$, which is a closed connected subgroup of some $U(N)$. \text{(For loop quantum gravity, $G = SU(2)$).} We will make use of the fact that $\mathcal{L}_*$ can be embedded into a larger, compact group $\mathcal{M}(\mathcal{L}_*)$, containing $\mathcal{L}_*$ as a dense subgroup. $\mathcal{M}(\mathcal{L}_*)$ is given as the spectrum of the $C^*$-algebra of almost periodic functions on $\mathcal{L}_*$. It turns out that there is a homeomorphism $\mathcal{A}/\mathcal{G} \simeq \text{Hom}_c(\mathcal{M}(\mathcal{L}_*), G)/\text{Ad}$, too, where $\text{Hom}_c(\mathcal{M}(\mathcal{L}_*), G)/\text{Ad}$ is the space of equivalence classes of continuous representations of $\mathcal{M}(\mathcal{L}_*)$ by $G$-matrices.

This minor change in the connection representation proves to be quite useful. One considers the space $L_2(\mathcal{M}(\mathcal{L}_*))$, the Hilbert algebra of $\mathcal{M}(\mathcal{L}_*)$ with respect to the Haar measure $\xi$ on $\mathcal{M}(\mathcal{L}_*)$. The functions $t_\rho \in L_2(\mathcal{M}(\mathcal{L}_*))$, coming from the Wilson loop functions $T_\alpha$ by switching the roles of the argument and the index, are shown to be orthonormal in an appropriate normalization. Furthermore, the Haar measure $\xi$ turns out to be invariant under 3-diffeomorphisms. This suggests to regard $L_2(\mathcal{M}(\mathcal{L}_*))$ as a candidate for the Hilbert space of loop quantum gravity in the loop representation. In a subsequent paper, we will define an extended loop transform from $L_2(\mathcal{A}/\mathcal{G}, d\mu)$ to $L_2(\mathcal{M}(\mathcal{L}_*))$. The inverse transform is given canonically. In the upcoming paper, we will also present some results concerning the implementation of the diffeomorphism constraints, thus defining the kinematical framework of loop quantum gravity rigorously in both the loop and the connection representation.

The plan of the paper is as follows: in section 2, we will consider almost periodic functions and associated compact groups. In section 3, a new proof is given for the fact that the quantum configuration space $\mathcal{A}/\mathcal{G}$ of loop quantum gravity is homeomorphic to the space $\text{Hom}(\mathcal{L}_*, G)/\text{Ad}$. This is done by using the compact group $\mathcal{M}(\mathcal{L}_*)$, containing $\mathcal{L}_*$ as a dense subgroup. $\mathcal{M}(\mathcal{L}_*)$ is given as the spectrum of the $C^*$-algebra of almost periodic functions on $\mathcal{L}_*$. It turns out that there is a homeomorphism $\mathcal{A}/\mathcal{G} \simeq \text{Hom}_c(\mathcal{M}(\mathcal{L}_*), G)/\text{Ad}$, too, where $\text{Hom}_c(\mathcal{M}(\mathcal{L}_*), G)/\text{Ad}$ is the space of equivalence classes of continuous representations of $\mathcal{M}(\mathcal{L}_*)$ by $G$-matrices.

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gravity is homeomorphic to $Hom(L_*, G)/Ad$, the space of unitary equivalence classes of homomorphisms (i.e. representations) from the hoop group $L_*$ to a closed connected subgroup $G$ of $U(N)$. It is also shown that $Hom(L_*, G)/Ad$ is canonically isomorphic to $Hom_c(M(L_*), G)/Ad$, the equivalence classes of continuous representations of the compact group $M(L_*)$ associated to the hoop group $L_*$. So we get a homeomorphism

$$M/L^* \rightarrow Hom_c(M(L_*), G)/Ad,$$

suggesting the enlargement of the hoop group to its associated compact group $M(L_*)$. In section 4, we will define generalized Wilson loop functions and introduce the Hilbert algebra $L_2(M(L_*))$ of $M(L_*)$ with respect to the normalized Haar measure $\xi$ on $M(L_*).$ The generalized Wilson loop functions are shown to be orthonormal (in an appropriate normalization) and lie in the centre of this algebra. The Haar measure on $L_2(M(L_*))$ is diffeomorphism invariant. Section 5 gives a short outlook.

2 Almost periodic functions and associated compact groups

In this chapter we restate a proof known in the literature [Loo53], showing that to every topological group $H$ there exists a compact group $M(H)$, characterized by a universal property, such that there is a continuous homomorphism $\iota$ of $H$ onto a dense subset of $M(H).$ $M(H)$ is given as the spectrum of the $C^*$-algebra of almost periodic functions on $H$. Choosing $H = L_*$ in the following chapters, we will be able to shed some new light on the kinematical structure of loop quantum gravity. The compact group $M(H)$ associated to a topological group $H$ was mentioned in [AshIsh92], but not considered in the further development of the theory.

For the convenience of the reader, we will first introduce almost periodic functions:

**Definition 1** Let $H$ be a group, $E$ a non-empty set and $f : H \rightarrow E$ a function. For some fixed $s \in H$, let $f_s$ be the function on $H$ given by

$$f_s(x) = f(sx) \quad \forall x \in H.$$

$f_s$ is called the left translate of $f$ by $s$.  

4
Definition 2 A left almost periodic function on a topological group $H$ is a bounded function $f : H \rightarrow \mathbb{C}$ such that the set $S_f := \{f_s \mid s \in H\}$ of left translates of $f$ is totally bounded with respect to the supremum norm on $C_b(H)$, the space of bounded continuous complex-valued functions on $H$.

Since $C_b(H)$ equipped with the supremum norm can be regarded as a metric space, the total boundedness of $S_f$ is equivalent to the compactness of $S_f \subseteq C_b(H)$. Let $\mathcal{F}(H)$ denote the set of left almost periodic functions on $H$.

One can define right almost periodic functions in an analogous manner. Both definitions coincide, see \cite{HewRos63}, §18. From now on, we will just speak of almost periodic functions.

Next we will state three propositions on almost periodic functions with the aim of defining the compact group associated to a topological group. Proofs can be found in \cite{Loo53}, ch. 41.

**Proposition 3** $\mathcal{F}(H) \subseteq C_b(H)$ is a commutative $C^*$-algebra with unit element.

The Gelfand-Naimark theorem shows that to every commutative $C^*$-Algebra $\mathcal{F}(H)$ with unit element there exists a compact space $\mathcal{M}(H)$ such that $\mathcal{F}(H)$ is isometrically $*$-isomorphic to $C(\mathcal{M}(H))$. $\mathcal{M}(H)$ is the Gelfand spectrum of $\mathcal{F}(H)$.

**Proposition 4** There exists a continuous homomorphism $\iota$ from the group $H$ onto a dense subspace of $\mathcal{M}(H)$.

One can show \cite{Loo53} that the group operation in $H$, regarded as a subgroup of $\mathcal{M}(H)$, is uniformly continuous in the Gelfand topology on $\mathcal{M}(H)$ and hence can be extended to the whole of $\mathcal{M}(H)$. In this way, $\mathcal{M}(H)$ acquires a group structure. The central result is the following:

**Proposition 5** To every topological group $H$ there exists a compact group $\mathcal{M}(H)$ and a continuous homomorphism $\iota$ from $H$ onto a dense subgroup of $\mathcal{M}(H)$ such that for the pair $(\mathcal{M}(H), \iota)$ the following universal condition holds: a function $f : H \rightarrow \mathbb{C}$ is almost periodic if and only if there exists a continuous function $g : \mathcal{M}(H) \rightarrow \mathbb{C}$ such that the following diagram is commutative:
Since \( \mathcal{M}(H) \) is compact, \( g \) is determined by \( f \) unambiguously. We call \( \mathcal{M}(H) \) the compact group associated to \( H \).

Let \( G \) be a closed connected subgroup of some \( U(N) \). When considering loop quantum gravity, we will be interested in spaces \( \text{Hom}(H,G) \) of homomorphisms from a topological group \( H \) (which will be chosen as \( L_* \), the hoop group, or some subgroup of it) to the compact group \( G \). For these spaces \( \text{Hom}(H,G) \) of representations we have

**Lemma 6** Let \( \mathcal{M}(H) \) be the compact group associated to \( H \). Then there exists a canonical bijection between \( \text{Hom}(H,G) \) and \( \text{Hom}_c(\mathcal{M}(H),G) \), the set of all continuous homomorphisms \( \mathcal{M}(H) \to G \).

**Proof.** Let \( \varphi \in \text{Hom}(H,G) \). Then there exists a unique continuous homomorphism

\[ \mathcal{M}(\varphi) : \mathcal{M}(H) \to G \]

making the diagram

\[ \begin{array}{ccc}
\mathcal{M}(H) & \xrightarrow{\mathcal{M}(\varphi)} & G \\
\downarrow & & \downarrow \\
H & \xrightarrow{\varphi} & G
\end{array} \]

commutative. This can be seen as follows: since \( G \) is a closed subgroup of some \( U(N) \), one can consider the matrix elements

\[ \rho_{ij}(x) := \langle \rho(x)e_i, e_j \rangle, \]
where \( \{e_i\} \) is the canonical basis of the space \( \mathbb{C}^N \) the group \( G \) acts on. Let \( s \in H \). One has a left translate

\[
\rho_{ij}^s(x) = \rho_{ij}(sx) = \langle \rho(s)\rho(x)e_i, e_j \rangle.
\]

We want to show that the \( \rho_{ij} \) are almost periodic, so regard

\[
|\rho_{ij}^s(x) - \rho_{ij}^t(x)| = \left| \langle \rho(s)\rho(x)e_i, e_j \rangle - \langle \rho(t)\rho(s)e_i, e_j \rangle \right| = \left| \langle (\rho(s) - \rho(t))\rho(x)e_i, e_j \rangle \right| \leq |\rho(s) - \rho(t)| \leq |\rho(s)(I_N - \rho(s^{-1}t))| \leq |I_N - \rho(s^{-1}t)|.
\]

It follows that

\[
|\rho_{ij}^s - \rho_{ij}^t|_\infty \leq |I_N - \rho(s^{-1}t)|.
\]

Let \( \varepsilon > 0 \) and

\[
U_0 := \{g \in G \mid |I_N - g| < \varepsilon \}.
\]

Since \( G \) is compact, there exist finitely many \( x_1, ..., x_n \in H \) such that

\[
\rho(H) \subseteq \bigcup_{k=1}^n \rho(x_i)U_0.
\]

This means that for any \( x \in H \) there exists some \( k \leq n \) such that

\[
\rho(x) \in \rho(x_k)U_0 \iff \rho(x_k^{-1}x) \in U_0,
\]

so we have

\[
|\rho_{ij}^{x_k} - \rho_{ij}^s|_\infty \leq |I_N - \rho(x_k^{-1}x)| < \varepsilon,
\]

i.e., \( \rho_{ij} \) is almost periodic. It follows that \( \rho_{ij} \) can be extended to a continuous function \( \mathcal{M}(\rho_{ij}) : \mathcal{M}(H) \to \mathbb{C} \) unambiguously. Of course, one has \( \mathcal{M}(\rho_{ij}) \circ \iota = \rho_{ij} \). Define

\[
\mathcal{M}(\rho)(x)_{ij} := \mathcal{M}(\rho_{ij})(x) \quad \forall x \in \iota(H).
\]

Since \( \iota(H) \) is dense in \( \mathcal{M}(H) \), the mapping \( \mathcal{M}(\rho) : \mathcal{M}(H) \to G \) is a continuous representation. This gives a bijection \( \varphi \to \mathcal{M}(\varphi) \) between \( \text{Hom}(H,G) \) and \( \text{Hom}_c(\mathcal{M}(H),G) \). Since \( \mathcal{M}(\varphi) \) is defined by a universal property, this bijection is canonical. 

Remark 1 The bijection \( \varphi \rightarrow \mathcal{M}(\varphi) \) is natural in the category theoretical sense.

Proof. Let \( \phi : \Gamma_1 \rightarrow \Gamma_2 \) be a continuous homomorphism of topological groups. Since \( \text{Hom}(\_, G) \) is a cofunctor, one has a natural map

\[
\phi_* : \text{Hom}(\Gamma_2, G) \rightarrow \text{Hom}(\Gamma_1, G), \quad \psi \mapsto \psi \circ \phi.
\]

The universal property of associated compact groups renders the following diagram commutative:

\[
\begin{array}{ccc}
\mathcal{M}(\Gamma_1) & \xrightarrow{\mathcal{M}(\iota_2 \circ \phi)} & \mathcal{M}(\Gamma_2) \\
\downarrow \iota_1 & & \downarrow \iota_2 \\
\Gamma_1 & \xrightarrow{\phi} & \Gamma_2 \\
\downarrow \iota_1 & & \downarrow \psi \\
\mathcal{M}(\Gamma_1) & \xrightarrow{\mathcal{M}(\psi \circ \phi)} & G
\end{array}
\]

Uniqueness gives

\[
\mathcal{M}(\psi \circ \phi) = \mathcal{M}(\psi) \circ \mathcal{M}(\iota_2 \circ \phi),
\]

which means that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Hom}(\Gamma_2, G) & \xrightarrow{\mathcal{M}} & \text{Hom}(\Gamma_1, G) \\
\downarrow \phi_* & & \downarrow \psi_* \\
\text{Hom}_c(\mathcal{M}(\Gamma_2), G) & \xrightarrow{\mathcal{M}(\iota_2 \circ \phi)_*} & \text{Hom}_c(\mathcal{M}(\Gamma_1), G)
\end{array}
\]

Remark 2 Let \( \text{Hom}(\mathcal{M}(H), G) \) be the set of all homomorphisms \( \mathcal{M}(H) \rightarrow G, \varphi \in \text{Hom}(H, G) \). Then the set

\[
\{ \psi \in \text{Hom}(\mathcal{M}(H), G) \mid \psi \circ \iota = \varphi \}
\]

contains exactly one continuous homomorphism, which is \( \mathcal{M}(\varphi) \).
3 The homeomorphism between $\text{Spec}(C^*(\mathcal{H}A))$ and $\text{Hom}(\mathcal{L}_*, G)/\text{Ad}$

In this section we give a new, natural proof showing the homeomorphism between the spectrum $\mathcal{A}/\mathcal{G} := \text{Spec}(C^*(\mathcal{H}A))$ of the Ashtekar-Isham algebra $C^*(\mathcal{H}A)$ (for definitions see below) and the space $\text{Hom}(\mathcal{L}_*, G)/\text{Ad}$ of unitary equivalence classes of homomorphisms from the hoop group $\mathcal{L}_*$ to a closed connected subgroup $G$ of $U(N)$. The space $\mathcal{A}/\mathcal{G}$ of generalized connections modulo generalized gauge transformations serves as the quantum configuration space in loop quantum gravity [AshLew94]. $\mathcal{A}/\mathcal{G}$ contains the classical configuration space $\mathcal{A}/G$ of Yang-Mills theory and General Relativity in the Ashtekar formulation as a dense subset. Every element of $\mathcal{A}/\mathcal{G}$ induces an element of $\text{Hom}(\mathcal{L}_*, G)/\text{Ad}$ in a natural way: consider the mapping

$$\mathcal{A}/\mathcal{G} \rightarrow \text{Hom}(\mathcal{L}_*, G)/\text{Ad},$$

$$[A]_{\mathcal{G}} \mapsto \frac{1}{N} \text{tr}(\rho(H(\mathcal{L}, A))),$$

where $A$ is some representative of $[A]_{\mathcal{G}} \in \mathcal{A}/\mathcal{G}$, $H(\mathcal{L}, A) : \mathcal{L}_* \rightarrow G$ is the holonomy mapping with respect to the connection $A$ and $\rho : G \rightarrow M(\mathbb{C}, N)$ is some representation of $G$ by complex $(N \times N)$ matrices. The mapping is well defined since the trace of the holonomy is gauge invariant. The idea now is to show that every element of $\mathcal{A}/\mathcal{G}$, not just $\mathcal{A}/G$, can be seen as the (trace of) some homomorphism from the hoop group $\mathcal{L}_*$ to $G$. This can be done by explicitly constructing the homomorphism belonging to each $A \in \mathcal{A}/\mathcal{G}$ as in [AshIsh92] or by using projective techniques [AshLew95]. In these proofs, one has to consider Mandelstam identities, the reconstruction results by Giles [Gil81], and finite dimensional (or finitely generated) objects like tame subgroups of $\mathcal{L}_*$ and cylinder functions on them which give the interesting objects in the projective limit. A recent paper along these lines, extending some of the known results, is [AbbMan01]. While these techniques prove to be extremely useful for the further development of the theory, they are not essential to the fact that $\mathcal{A}/\mathcal{G}$ is homeomorphic to $\text{Hom}(\mathcal{L}_*, G)/\text{Ad}$, as will be shown below. In a sense, our proof is more natural than the existing ones, since it works directly with the interesting objects $\mathcal{A}/\mathcal{G}$ and $\text{Hom}(\mathcal{L}_*, G)/\text{Ad}$. But a new proof for an old result is not so exciting, in general, as long
as it does not give some new insight. Now it turns out that at some in-
termediate step in our proof it is necessary to refer to the compact group
\(\mathcal{M}(\mathcal{L}_*)\) associated to the hoop group \(\mathcal{L}_*\). While this may seem a technicality
at first, it allows a slight change of perspective when regarding the quan-
tum configuration space \(\mathcal{A}/G\): since there is a canonical bijection between
\(\text{Hom}(\mathcal{L}_*,G)/\text{Ad}\) and \(\text{Hom}_c(\mathcal{M}(\mathcal{L}_*),G)/\text{Ad}\), the unitary equivalence classes
of continuous homomorphisms from \(\mathcal{M}(\mathcal{L}_*)\) to \(G\), \(\mathcal{A}/G\) is homoemorphic to
\(\text{Hom}_c(\mathcal{M}(\mathcal{L}_*),G)/\text{Ad}\), too. This fact will be used in the following section.

Let \(G\) be a Hausdorff topological group, \(\Gamma\) a group. By \(G^\Gamma\) we denote the
space of all maps from \(\Gamma\) to \(G\).

**Theorem 7** Let \(\Gamma\) be a group, \(G\) a Hausdorff topological group. Then the
space \(\text{Hom}(\Gamma,G) \subseteq G^\Gamma\) is closed in the product topology. It follows that
\(\text{Hom}(\Gamma,G)\) is compact if \(G\) is compact.

**Proof.** Let \(H \in \overline{\text{Hom}(\Gamma,G)}\). \(H\) is a map \(\Gamma \to G\), and one has to show that
\[ H(\alpha \beta) = H(\alpha) H(\beta) \]
for all \(\alpha, \beta \in \Gamma\). Let \(U \subseteq G\) be an open neighbourhood of the neutral element
e of \(G\), \(V\) a symmetric open neighbourhood of \(e\) such that \(V^3 \subseteq U\) and \(W \subseteq V\) an open
neighbourhood of \(e\) with \(WH(\beta) \subseteq H(\beta)V\). The existence of \(W\) follows from the continuity of the multiplication in \(G\). Let \(pr_\alpha : G^\Gamma \to G\) be the projection \(f \to f(\alpha)\) to the coordinate \(\alpha\) of \(f \in G^\Gamma\). Then
\[ \mathcal{O} := pr^{-1}_\alpha(H(\alpha)W) \cap pr^{-1}_\beta(H(\beta)W) \cap pr^{-1}_{\alpha \beta}(H(\alpha \beta)W) \]
is an open neighbourhood of \(H\) in the product topology. Let \(\varphi \in \text{Hom}(\Gamma,G)\cap \mathcal{O}\). Then
\[ \varphi(\alpha) \in H(\alpha)W, \]
\[ \varphi(\beta) \in H(\beta)W, \]
\[ \varphi(\alpha \beta) \in H(\alpha \beta)W, \]
that is, there exist \(g_\alpha, g_\beta, g_{\alpha \beta} \in W\) such that
\[ \varphi(\alpha) = H(\alpha)g_\alpha, \]
\[ \varphi(\beta) = H(\beta)g_\beta, \]
\[ \varphi(\alpha \beta) = H(\alpha \beta)g_{\alpha \beta}, \]
and it follows that

$$H(\alpha \beta) = \varphi(\alpha \beta) g_{\alpha \beta}^{-1}$$

$$= \varphi(\alpha) \varphi(\beta) g_{\alpha \beta}^{-1}$$

$$= H(\alpha) g_{\alpha} H(\beta) g_{\beta} g_{\alpha \beta}^{-1}$$

$$\in H(\alpha) H(\beta) V \cdot V \cdot V$$

$$\subseteq H(\alpha) H(\beta) U,$$

where we have used $WH(\beta) \subseteq H(\beta) V$ in the fourth step. Since $U$ is an arbitrary neighbourhood of $e$, one finds that

$$H(\alpha \beta) = H(\alpha) H(\beta).$$

Lemma 8

Let $A, B$ be algebras over a field $F$. Let $E$ be a linear system of generators for $A$, that is

$$A = \text{lin} \prod_E,$$

where $\prod_E$ is the set of finite products of elements of $E$. We further assume that $A$ has a neutral element $1$ and that $1 \in E$. The map

$$\varphi : E \rightarrow B$$

can be extended to an algebra homomorphism

$$\overline{\varphi} : A \rightarrow B,$$

if and only if the following conditions hold:

1. For $a_1, \ldots, a_m, b_1, \ldots, b_n \in E$ with $a_1 \ldots a_m = b_1 \ldots b_n$ one has $\varphi(a_1) \ldots \varphi(a_m) = \varphi(b_1) \ldots \varphi(b_n)$.

2. Let $\sum_k c_k p_k = 0$ for some $p_k \in \prod_E$, $p_k = a_{k_1} \ldots a_{k_{n_k}}$, $c_k \in K$. Then

$$\sum_k c_k \varphi(a_{k_1}) \ldots \varphi(a_{k_{n_k}}) = 0.$$

Proof. If there is a homomorphic extension $\overline{\varphi} : A \rightarrow B$ of $\varphi$, the conditions (1) and (2) have to hold. On the other hand, assume that (1) and (2) are
fulfilled for some \( \varphi : \mathcal{E} \rightarrow \mathcal{B} \). Because of condition (1), \( \varphi \) can be extended to a map

\[
\varphi : \prod_{\mathcal{E}} \rightarrow \mathcal{B}
\]

by setting

\[
\varphi(a_1...a_m) := \varphi(a_1)\varphi(a_m).
\]

\( \varphi \) is defined on a linear system of generators \( \prod_{\mathcal{E}} \) of \( \mathcal{A} \) and has the property

\[
\sum_k c_kp_k = 0 \Rightarrow \sum_k c_k\varphi(p_k) = 0
\]

because of condition (2). This condition assures that \( \varphi \) can be extended unambiguously to a linear map \( \overline{\varphi} : \mathcal{A} \rightarrow \mathcal{B} \). In order to see this, consider the vector space \( \mathcal{F}(\prod_{\mathcal{E}}) \) that is freely generated by \( \prod_{\mathcal{E}} \). A concrete model of \( \mathcal{F}(\prod_{\mathcal{E}}) \) is the space of all functions \( f : \prod_{\mathcal{E}} \rightarrow \mathbb{C} \) of finite support. Any function of this type can be written as \( f = \sum_{p \in \prod_{\mathcal{E}}} c_p\chi_{\{p\}} \) with \( c_p = 0 \) for almost all \( p \in \prod_{\mathcal{E}} \), where \( \chi_{\{p\}} \) denotes the characteristic function of \( \{p\} \):

\[
\chi_{\{p\}}(q) = \delta_{pq}.
\]

We have a canonical surjective linear mapping

\[
e : \mathcal{F}(\prod_{\mathcal{E}}) \rightarrow \mathcal{A}
\]

defined by

\[
e\left( \sum_{p \in \prod_{\mathcal{E}}} c_p\chi_{\{p\}} \right) := \sum_{p \in \prod_{\mathcal{E}}} c_pp.
\]

Therefore

\[
\mathcal{A} \simeq \mathcal{F}(\prod_{\mathcal{E}})/\ker e.
\]

Define \( \varphi_{\mathcal{F}} : \mathcal{F}(\prod_{\mathcal{E}}) \rightarrow \mathcal{B} \) by

\[
\varphi_{\mathcal{F}}\left( \sum_{p} c_p\chi_{\{p\}} \right) := \sum_{p} c_p\varphi(p).
\]

The condition on \( \varphi \) implies that \( \varphi_{\mathcal{F}} \) induces a linear mapping

\[
\tilde{\varphi} : \mathcal{F}(\prod_{\mathcal{E}})/\ker e \rightarrow \mathcal{B}
\]

and, by the isomorphism above, a linear map

\[
\tilde{\varphi} : \mathcal{A} \rightarrow \mathcal{B}
\]
such that $\bar{\varphi}(a_1 \cdots a_m) = \varphi(a_1) \cdots \varphi(a_m)$ for all $a_1, \ldots, a_m \in \mathcal{E}$. Obviously, $\bar{\varphi}$ is also multiplicative:

$$\bar{\varphi}(ab) = \bar{\varphi}(a)\bar{\varphi}(b)$$

for all $a, b \in \mathcal{A}$. ■

From now on, let $G$ be a closed connected subgroup of some unitary group $U(N)$. The following result by Ashtekar and Lewandowski [AshLew94] is central to all our further considerations:

**Theorem 9 (Interpolation theorem, Ashtekar and Lewandowski.)** Let $\mathcal{L}_*$ be the hoop group of piecewise analytic hoops in the manifold $M$ with base point $\ast$, $P(M,G)$ a principal bundle over $M$ with structure group $G$. Let $\varphi : \mathcal{L}_* \to G$ be a group homomorphism. For every finite subset $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathcal{L}_*$, there exists a connection $A$ in $P(M,G)$, such that

$$\varphi(\alpha_i) = H(\alpha_i, A)$$

for all $i = 1, \ldots, n$.

**Remark 3** There is another version of this theorem: let $x$ denote the constant hoop. Given a finite subset $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathcal{L}_*$, $\alpha_i \neq x$, and an $n$-tuple $\{g_1, \ldots, g_n\} \in G^n$, one can find a connection $A$ such that

$$\forall i = 1, \ldots, n : H(\alpha_i, A) = g_i.$$

For $\alpha \in \mathcal{L}_*$, let $T_\alpha : \mathcal{A}/G \to \mathbb{C}$ be the Wilson loop function

$$T_\alpha(A) := \frac{1}{N} tr(\rho(H(\alpha, A))),$$

where $\rho$ is some matrix representation by $(N \times N)$ matrices of the structure group $G$ of the principal bundle $P(M,G)$. Subsequently, $\rho$ will be suppressed in the notation. Let $\mathcal{H}A$ be the algebra that is generated by the $T_\alpha$. $\mathcal{H}A$ is called the holonomy algebra. Due to the fact that $G \subseteq U(N)$, we have

$$T_\alpha^* = T_{\alpha^{-1}},$$

where $T_\alpha^*$ is defined by

$$T_\alpha^*(A) := \overline{T_\alpha(A)}.$$

Thus $\mathcal{H}A$ is an involutive subalgebra of the $C^*$-algebra $\mathcal{B}(\mathcal{A}/G)$ of bounded complex-valued functions on $\mathcal{A}/G$, equipped with the supremum norm.
Theorem 10 Each $\varphi \in \text{Hom}(L^*,G)$ induces a continuous algebra homomorphism $\tau_\varphi: \mathcal{H}A \to \mathbb{C}$.

Proof. We show that the map

$$\tau_\varphi: T_\alpha \mapsto \frac{1}{N} \text{tr} \varphi(\alpha)$$

fulfills the conditions (1) and (2) of Lemma 8 on the system of generators $\{T_\alpha|\alpha \in L_s\}$. Let $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ be hoops with $T_{\alpha_1} \ldots T_{\alpha_m} = T_{\beta_1} \ldots T_{\beta_n}$. According to the interpolation theorem, there exists some $A_0 \in \mathcal{A}$ such that

$$H(\alpha_i, A_0) = \varphi(\alpha_i),$$
$$H(\beta_j, A_0) = \varphi(\beta_j)$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then

$$\tau_\varphi(T_{\alpha_1}) \cdots \tau_\varphi(T_{\alpha_m}) = \frac{1}{N} \text{tr} \varphi(\alpha_1) \cdots \frac{1}{N} \text{tr} \varphi(\alpha_m)$$
$$= \frac{1}{N} \text{tr} H(\alpha_1, A_0) \cdots \frac{1}{N} \text{tr} H(\alpha_m, A_0)$$
$$= T_{\alpha_1}(A_0) \cdots T_{\alpha_m}(A_0)$$
$$= T_{\beta_1}(A_0) \cdots T_{\beta_n}(A_0)$$
$$= \frac{1}{N} \text{tr} \varphi(\beta_1) \cdots \frac{1}{N} \text{tr} \varphi(\beta_n)$$
$$= \tau_\varphi(T_{\beta_1}) \cdots \tau_\varphi(T_{\beta_n}).$$

Let $\sum_k c_k \varphi_k = 0$ for some $c_k \in \mathbb{C}$ and $p_k = T_{\alpha_{k_1}} \cdots T_{\alpha_{k_n}}$. The interpolation theorem assures the existence of some $A_0 \in \mathcal{A}$ such that

$$H(\alpha_{k_j}, A_0) = \varphi(\alpha_{k_j})$$

for all $j \leq n_k$ and all $k$. Then

$$\sum_k c_k \tau_\varphi(p_k) = \sum_k c_k \tau_\varphi(T_{\alpha_{k_1}}) \cdots \tau_\varphi(T_{\alpha_{k_n}})$$
$$= \sum_k c_k \frac{1}{N} \text{tr} \varphi(\alpha_{k_1}) \cdots \frac{1}{N} \text{tr} \varphi(\alpha_{k_n})$$
$$= \sum_k c_k \frac{1}{N} \text{tr} H(\alpha_{k_1}, A_0) \cdots \frac{1}{N} \text{tr} H(\alpha_{k_n}, A_0).$$
\[
\sum_k c_k (T_{\alpha_1} \ldots T_{\alpha_{kn_k}})(A_0) = \left( \sum_k c_k p_k \right)(A_0) = 0. 
\]

For simplicity, the algebra homomorphism \( \mathcal{H} \to \mathbb{C} \) induced by \( \tau_\varphi \) is denoted by \( \tau_\varphi \), too. From the considerations above, the continuity of \( \tau_\varphi \) can be inferred as follows:

\[
|\tau_\varphi(\sum_k c_k p_k)| = |(\sum_k c_k p_k)(A_0)| 
\leq \sup_{A \in \mathcal{A}/G} |(\sum_k c_k p_k)(A)| 
= |(\sum_k c_k p_k)|_\infty. 
\]

Let \( C^*(\mathcal{H}) \) be the closure of \( \mathcal{H} \) in \( \mathcal{B}(\mathcal{A}/G) \). \( C^*(\mathcal{H}) \) is called the Ashtekar-Isham-Algebra \footnote{AshIsh92}. Each \( \varphi \in \text{Hom}(L^*_L, G) \) gives a character of the commutative \( C^* \)-algebra \( C^*(\mathcal{H}) \), denoted by \( \tau_\varphi \) as above. We want to examine how much the map

\[
\text{Hom}(L^*_L, G) \to \text{Spec}(C^*(\mathcal{H})) =: \overline{\mathcal{A}/G} 
\]

fails to be injective. For this, we need the compact group \( \mathcal{M}(L^*_L) \) associated to the hoop group \( L^*_L \) as described in section 2.

First we will show that in our special case \( H = L^*_L \) the map \( \iota : L^*_L \to \mathcal{M}(L^*_L) \) defined in Prop. 5 is injective. It is well known that the kernel of \( \iota \) is equal to the intersection of the kernels of the continuous morphisms of \( L^*_L \) into all compact groups, see ch. 16.4 of \footnote{Dix77}. For reasons that will become clear later, we will equip \( L^*_L \) with the discrete topology. Thus the holonomy mappings \( H(\mathcal{L}, A) : L^*_L \to G \) are contained in the set of continuous morphisms of \( L^*_L \) into compact groups.

Remark 3 shows that for every \( g \in G \) and every \( \alpha \in L^*_L, \alpha \neq \alpha \), there exists a connection \( A \) such that \( H(\mathcal{L}, A) = g \). So the intersection of the kernels
of the holonomy mappings is \( \{ x \} \), and \( \iota \) is injective.

Now back to the question of how much the map
\[
\text{Hom}(\mathcal{L}_*, G) \to \text{Spec}(C^*(\mathcal{H}, \mathcal{A})),
\]
defined on generators by
\[
\tau_\varphi : T_\alpha \mapsto \frac{1}{N} \text{tr} \varphi(\alpha),
\]
fails to be injective. Let \( \varphi_1, \varphi_2 \in \text{Hom}(\mathcal{L}_*, G) \) and suppose that \( \tau_{\varphi_1} = \tau_{\varphi_2} \). Then
\[
\forall \alpha \in \mathcal{L}_* : \text{tr}(\varphi_1(\alpha)) = \text{tr}(\varphi_2(\alpha)),
\]
i.e.
\[
\forall \alpha \in \mathcal{L}_* : \text{tr}(\mathcal{M}(\varphi_1)(\iota(\alpha))) = \text{tr}(\mathcal{M}(\varphi_2)(\iota(\alpha))),
\]
and hence, because \( \mathcal{M}(\varphi_1), \mathcal{M}(\varphi_2) \) are continuous and \( \iota(\mathcal{L}_*) \) is dense in \( \mathcal{M}(\mathcal{L}_*) \):
\[
\forall \xi \in \mathcal{M}(\mathcal{L}_*) : \text{tr}(\mathcal{M}(\varphi_1)(\xi)) = \text{tr}(\mathcal{M}(\varphi_2)(\xi)).
\]
Thus, the characters of the continuous unitary representations \( \mathcal{M}(\varphi_1), \mathcal{M}(\varphi_2) : \mathcal{M}(\mathcal{L}_*) \to U(N) \) of the compact group \( \mathcal{M}(\mathcal{L}_*) \) are the same and the representations are unitarily equivalent:
\[
\exists T \in U(N) \forall \xi \in \mathcal{M}(\mathcal{L}_*) : \mathcal{M}(\varphi_2)(\xi) = T \mathcal{M}(\varphi_1)(\xi) T^*.
\]
This is equivalent to
\[
\forall \alpha \in \mathcal{L}_* : \varphi_2(\alpha) = T \varphi_1(\alpha) T^*
\]
or \( \varphi_2 = T \varphi_1 T^* \), for short. The homomorphisms \( \varphi_1, \varphi_2 \) are called unitarily equivalent, too. If \( \varphi_1, \varphi_2 \) are unitarily equivalent in this way, one has \( \tau_{\varphi_1} = \tau_{\varphi_2} \), because
\[
\tau_{\varphi_2}(T_\alpha) = \frac{1}{N} \text{tr} \varphi_2(\alpha)
= \frac{1}{N} \text{tr}(T \varphi_1(\alpha) T^*)
= \frac{1}{N} \text{tr} \varphi_1(\alpha)
= \tau_{\varphi_1}(T_\alpha)
\]
for all $\alpha \in \mathcal{L}_*$. Unitary equivalence defines an equivalence relation on $\text{Hom}(\mathcal{L}_*, G)$, which is - by the usual abuse of notation - denoted by $Ad$. The map $\varphi \mapsto \tau_{\varphi}$ induces an injective map

$$\tau : \text{Hom}(\mathcal{L}_*, G)/Ad \to \text{Spec}(C^*(\mathcal{H}A)).$$

**Lemma 11** $\text{Hom}(\mathcal{L}_*, G)/Ad$, equipped with the quotient topology defined by the canonical projection

$$\pi : \text{Hom}(\mathcal{L}_*, G) \to \text{Hom}(\mathcal{L}_*, G)/Ad,$$

is a compact space.

**Proof.** Since $\text{Hom}(\mathcal{L}_*, G)$ is a compact space (see Theorem 7), it suffices to show that $\text{Hom}(\mathcal{L}_*, G)/Ad$, equipped with the quotient topology, is a Hausdorff space. The group $U(N)$ acts on $\text{Hom}(\mathcal{L}_*, U(N))$ by conjugation. Since $U(N)$ is compact, $\text{Hom}(\mathcal{L}_*, U(N))/U(N)$ with the quotient topology is Hausdorff. The imbedding

$$j : \text{Hom}(\mathcal{L}_*, G) \to \text{Hom}(\mathcal{L}_*, U(N))$$

obviously is continuous ($G$ is some closed subgroup of $U(N)$ by definition) and induces an imbedding

$$\overline{j} : \text{Hom}(\mathcal{L}_*, G)/Ad \to \text{Hom}(\mathcal{L}_*, U(N))/U(N)$$

by the commutative diagram

$$\begin{array}{ccc}
\text{Hom}(\mathcal{L}_*, G) & \xrightarrow{j} & \text{Hom}(\mathcal{L}_*, U(N)) \\
\downarrow{\pi} & & \downarrow{\pi_{U(N)}} \\
\text{Hom}(\mathcal{L}_*, G)/Ad & \xrightarrow{\overline{j}} & \text{Hom}(\mathcal{L}_*, U(N))/U(N)
\end{array}$$

The continuity of $\overline{j}$ follows from the universal property of the quotient topology. As a direct consequence, $\text{Hom}(\mathcal{L}_*, G)/Ad$ is a Hausdorff space. $\blacksquare$
Lemma 12 Let \( \mathcal{C} \) be a commutative \( C^* \)-algebra with 1, \( \mathcal{E} \) a system of generators of \( \mathcal{C} \), i.e. \( \text{lin} \prod_{\mathcal{E}} \subseteq \mathcal{C} \) is dense in \( \mathcal{C} \). Then the Gelfand topology is the coarsest topology for which all \( \hat{e} \) (\( e \in \mathcal{E} \)) are continuous.

Proof. The Gelfand topology on \( \text{Spec}(\mathcal{C}) \) is the coarsest topology for which all Gelfand transforms \( \hat{a} : \text{Spec}(\mathcal{C}) \to \mathbb{C} \) (\( a \in \mathcal{C} \)) are continuous. Hence the weak topology on \( \text{Spec}(\mathcal{C}) \) induced by the \( \hat{e} \) (\( e \in \mathcal{E} \)), which is called the \( \mathcal{E} \)-topology, is coarser than the Gelfand topology.

On the other hand, the \( \mathcal{E} \)-topology is Hausdorff: \( \sigma_1, \sigma_2 \in \text{Spec}(\mathcal{C}) \) and

\[
\forall e \in \mathcal{E} : \hat{e}(\sigma_1) = \hat{e}(\sigma_2) \\
\iff \forall e \in \mathcal{E} : \sigma_1(e) = \sigma_2(e)
\]

implies \( \sigma_1 = \sigma_2 \). The functions \( \hat{e} \) (\( e \in \mathcal{E} \)) separate the points of \( \text{Spec}(\mathcal{C}) \), and thus the \( \mathcal{E} \)-topology is Hausdorff. Since the Gelfand topology is compact, and since to a Hausdorff topology there is no strictly finer compact topology, the \( \mathcal{E} \)-topology and the Gelfand topology are identical. \( \blacksquare \)

Lemma 13 The map

\[
\tau : \text{Hom}(\mathcal{L}_*, G)/\text{Ad} \to \text{Spec}(C^*(\mathcal{H}A))
\]

is continuous.

Proof. \( \tau \) is continuous if and only if \( \tau = \tau \circ \pi \) is continuous. Since the Wilson loop functions \( T_\alpha (\alpha \in \mathcal{L}_*) \) generate the holonomy \( C^* \)-algebra \( C^*(\mathcal{H}A) \), the Gelfand topology on \( \text{Spec}(C^*(\mathcal{H}A)) \) is the coarsest topology for which all functions

\[
\hat{T}_\alpha : \text{Spec}(C^*(\mathcal{H}A)) \to \mathbb{C}, \quad \sigma \mapsto \sigma(T_\alpha)
\]

are continuous. Thus the map \( \tau : \text{Hom}(\mathcal{L}_*, G) \to \text{Spec}(C^*(\mathcal{H}A)) \) is continuous if and only if all functions

\[
\hat{T}_\alpha \circ \tau : \text{Hom}(\mathcal{L}_*, G) \to \mathbb{C}
\]

are continuous. Now

\[
(\hat{T}_\alpha \circ \tau)(\varphi) = \tau_\varphi(T_\alpha) = \frac{1}{N} \text{tr}(\varphi(\alpha)).
\]
This simply is the composition of the projection
\[ pr_\alpha : Hom(\mathcal{L}_*, G) \to G, \quad \varphi \mapsto \varphi(\alpha) \]
and the normed trace map
\[ G \to \mathbb{C}, \quad g \mapsto \frac{1}{N} tr g, \]
and these maps are continuous. ■

Summing up, we have

**Theorem 14** The quotient \( Hom(\mathcal{L}_*, G)/Ad \) is compact and continuously embedded in \( Spec(C^*(\mathcal{H}A)) \) via the map
\[ \tau : \varphi \mod Ad \mapsto \tau_\varphi. \]

■

We now show how \( \mathcal{A}/\mathcal{G} \) can be regarded as a part of \( Hom(\mathcal{L}_*, G)/Ad \): the holonomy provides a canonical map
\[ \mathcal{A} \to Hom(\mathcal{L}_*, G), \quad A \mapsto H(_, A). \]
Assume that \( A_1 \) is gauge equivalent to \( A_2 \). Then
\[ A_2 = \phi^* A_1 \]
for some gauge transformation \( \phi : P \to G \). For the holonomy map of hoops based at \( * \) one gets
\[ H(_, A_2) = a_\phi(p_0)^{-1} H(_, A_1)a_\phi(p_0), \]
where \( p_0 \in P \) is a freely chosen but fixed point of the fiber over \( * \), serving as initial value for the horizontal lifts of hoops in \( M \). Gauge equivalent connections give unitarily equivalent homomorphisms \( \mathcal{L}_* \to G \). We have found inclusions
\[ \tau(\mathcal{A}/\mathcal{G}) \subseteq \tau(Hom(\mathcal{L}_*, G)/Ad) \subseteq Spec(C^*(\mathcal{H}A)). \]
(One has
\[ \tau(A \mod G)(T_\alpha) = \tau_{H(\alpha, A)}(T_\alpha) = \frac{1}{N} \text{tr} H(\alpha, A) = T_\alpha(A), \]
so \( \tau(A \mod G) = \varepsilon_{A \mod G} \) is the evaluation functional at \( A \mod G \).) According to a result of Rendall [Ren93], \( \tau(A/G) \) is dense in \( \text{Spec}(C^*(\mathcal{HA})) \), the space \( \tau(\text{Hom}(\mathcal{L}_s, G)/\text{Ad}) \) is compact according to the last theorem and hence closed in \( \text{Spec}(C^*(\mathcal{HA})) \). Thus we get
\[ \text{Spec}(C^*(\mathcal{HA})) = \tau(A/G) \subseteq \tau(\text{Hom}(\mathcal{L}_s, G)/\text{Ad}) = \tau(\text{Hom}(\mathcal{L}_s, G)/\text{Ad}), \]
i.e.
\[ \tau(\text{Hom}(\mathcal{L}_s, G)/\text{Ad}) = \text{Spec}(C^*(\mathcal{HA})). \]

This proves

**Theorem 15** The spectrum of the holonomy \( C^* \)-algebra \( C^*(\mathcal{HA}) \) is homeomorphic to the compact space \( \text{Hom}(\mathcal{L}_s, G)/\text{Ad} \).

This result is not new, it was already found in [AshLew94]. It is central to all further developments in loop quantum gravity [ALMNT93]. In contrast to the known proofs, we have to refer neither to the Mandelstam identities, nor the results of Giles [Gill81] nor projective techniques as first used in [AshLew93].

Since Theorem 15 states a homeomorphism between \( \text{Spec}(C^*(\mathcal{HA})) \) and the set \( \text{Hom}(\mathcal{L}_s, G)/\text{Ad} \) of all equivalence classes of homomorphisms from the hoop group \( \mathcal{L}_s \) to \( G \), it is natural to equip \( \mathcal{L}_s \) with the discrete topology.

We have seen in Lemma 6 that there is a canonical bijection
\[ \text{Hom}_c(\mathcal{M}(\mathcal{L}_s), G) \simeq \text{Hom}(\mathcal{L}_s, G), \]
which of course induces a canonical bijection

\[ \text{Hom}_c(\mathcal{M}(\mathcal{L}_*), G)/\text{Ad} \cong \text{Hom}(\mathcal{L}_*, G)/\text{Ad}. \]

The space \( \text{Hom}(\mathcal{L}_*, G) \) is equipped with the relative topology induced by the product topology on \( G^{\mathcal{L}_*} \) (see Theorem 7), and is a compact space in this topology. \( \text{Hom}(\mathcal{L}_*, G)/\text{Ad} \) is compact in the quotient topology (Lemma 11). \( \text{Hom}_c(\mathcal{M}(\mathcal{L}_*), G)/\text{Ad} \) canonically is equipped with the topology induced by the above bijection - which of course is a homeomorphism, then -, and thus becomes a compact space, too. So we have a homeomorphism

\[ \overline{\mathcal{A}/G} = \text{Spec}(C^*(\mathcal{H}\mathcal{A})) \cong \text{Hom}_c(\mathcal{M}(\mathcal{L}_*), G)/\text{Ad}. \]

4 Generalized Wilson loop functions and the Hilbert algebra \( L_2(\mathcal{M}(\mathcal{L}_*)) \)

In this section, we will introduce the notion of generalized Wilson loop functions, which are characters of representations \( \rho_h : \mathcal{M}(\mathcal{L}_*) \to SU(2) \). The ordinary Wilson loop functions simply are characters of representations \( \rho : \mathcal{L}_* \to G \). We specialize to the case \( G = SU(2) \) now. This is the gauge group needed in loop quantum gravity [Ash86, Ash87].

The elements of \( \overline{\mathcal{A}/G} = \text{Spec}(C^*(\mathcal{H}\mathcal{A})) \) can be understood as (generalized gauge equivalence classes of) generalized connections [AshIsh92]. We have seen in section 3 that to every \( h \in \overline{\mathcal{A}/G} \) there exists some unique function \( \frac{1}{2} tr \circ \rho \), where \( \rho \) is a homomorphism from \( \mathcal{L}_* \) to \( SU(2) \). Taking the trace of course corresponds to the mod \( \text{Ad} \) operation, since every equivalence class of representations is unambiguously characterized by its trace. On \( \mathcal{H}\mathcal{A} \) one has

\[ \forall \alpha \in \mathcal{L}_* : h(T_\alpha) = \frac{1}{2} tr(\rho(\alpha)). \]

Since we have found \( \text{Hom}(\mathcal{L}_*, SU(2))/\text{Ad} \cong \text{Hom}_c(\mathcal{M}(\mathcal{L}_*), SU(2))/\text{Ad} \), this means that the elements \( h \in \overline{\mathcal{A}/G} \) correspond bijectively to the functions

\[ \frac{1}{2} tr \circ \rho_h, \]

where

\[ \rho_h : \mathcal{M}(\mathcal{L}_*) \to SU(2) \]
is a continuous representation of the compact group $\mathcal{M} := \mathcal{M} (L_\ast)$. The function

$$t_h := tr \circ \rho_h$$

is the character of the representation $\rho_h$ of $\mathcal{M}$.

**Definition 16** Let $\rho : \mathcal{M} \to SU(2)$ be a continuous representation. Then

$$t_\rho := tr \circ \rho : \mathcal{M} \to \mathbb{R}$$

is called a **generalized Wilson loop function**.

Note that in contrast to some of the existing literature on Wilson loop functions, we do not include a numerical factor $\frac{1}{2}$ in this definition. Thus the trace map is not normed. For our case, $G = SU(2)$, this means that $t_\rho$ is bounded between $-2$ and $2$, not $-1$ and $1$.

But there is a much more important change: instead of regarding Wilson loop functions as functions of (gauge equivalence classes of) connections, with hoops serving as an index, we regard functions of hoops (or elements of $\mathcal{M} = \mathcal{M} (L_\ast)$, to be more precise). Here the connection, given by the corresponding element $\rho$ of $Hom_c (\mathcal{M} (L_\ast), G)$, plays the role of an index. A more suggestive notation is

$$t_A : \mathcal{M} \to \mathbb{R},$$

where $A \in \mathcal{A}/\mathcal{G}$, in contrast to

$$T_\alpha : \mathcal{A}/\mathcal{G} \to \mathbb{R}$$

(for other groups than $SU(2)$ the mappings $t_A$ and $T_\alpha$ are complex-valued, of course).

Let $d\xi$ be the normed Haar measure on $\mathcal{M}$. $L_2 (\mathcal{M}) := L_2 (\mathcal{M}, d\xi)$ is the Hilbert algebra of $\mathcal{M}$, where multiplication is given by convolution:

$$(f \ast g)(\xi) = \int_{\mathcal{M}} f(\xi \eta^{-1})g(\eta)d\eta = \int_{\mathcal{M}} f(\xi \eta)g(\eta^{-1})d\eta.$$
Since $tr(AB) = tr(BA)$, one has

$$\forall \xi, \eta \in \mathcal{M} : t_{\rho}(\xi \eta) = t_{\rho}(\eta \xi),$$

so the generalized Wilson loop functions belong to the center of the algebra $L_2(\mathcal{M})$. This holds for more general groups than $SU(2)$, but the following result is specific to two-dimensional representations, since it uses some trace identities for complex $2 \times 2$-matrices of determinant 1:

$$trC \ trD = trCD + trCD^{-1},$$

$$trC = trC^{-1}.$$

We bother to write down the next theorem and some corollaries explicitly, although results of this kind may be found in books like [HewRos70]. The proof given here is adapted to our situation.

**Theorem 17** Let $f \in L_2(\mathcal{M})$ be a symmetric function, i.e. $f(\xi) = f(\xi^{-1})$. Then

$$2t_{\rho} \ast f = \langle f, t_{\rho} \rangle t_{\rho}$$

for all generalized Wilson loop functions $t_{\rho}$ on $\mathcal{M}$.

**Proof.** One has

$$(t_{\rho} \ast f)(\xi) = \int tr \ \rho(\xi \eta^{-1})f(\eta)d\eta$$

$$= \int tr(\rho(\xi)\rho(\eta^{-1}))f(\eta)d\eta$$

$$= \int tr \ \rho(\xi) \ tr (\rho(\eta)^{-1})f(\eta)d\eta - \int tr(\rho(\xi)\rho(\eta))f(\eta)d\eta$$

$$= tr \ \rho(\xi) \int f(\eta)tr \ \rho(\eta)d\eta - \int tr(\rho(\xi)\rho(\eta))f(\eta)d\eta$$

$$= tr \ \rho(\xi) \ (f, tr \ \rho) - (t_{\rho} \ast f)(\xi)$$

$$= t_{\rho}(\xi) \langle f, t_{\rho} \rangle - (t_{\rho} \ast f)(\xi),$$

and hence

$$2t_{\rho} \ast f = \langle f, t_{\rho} \rangle t_{\rho}. $$
In the fourth step above, we used the symmetry of $f$ and $t_\rho$:

$$t_\rho(\xi) = \text{tr} \rho(\xi) = \text{tr} \rho(\xi)^{-1} = \text{tr} \rho(\xi^{-1}) = t_\rho(\xi^{-1}).$$

Remark 4 The last result can be generalized to all $f \in L_2(M)$ in the following sense:

For $f \in L_2(M)$, let $f^* \in L_2(M)$ be defined by

$$f^*(\eta) := \overline{f(\eta^{-1})}.$$ 

Then

$$\int t_\rho(\xi\eta)f(\eta)d\eta = \int t_\rho(\xi\eta)f^*(\eta^{-1})d\eta = (t_\rho \ast f^*)(\xi)$$

and one obtains the more general formula

$$t_\rho \ast (f + f^*) = \langle f, t_\rho \rangle t_\rho.$$

Writing $\tilde{f}(\eta) := f(\eta^{-1})$, this gives

$$t_\rho \ast (f + \tilde{f}) = \langle f, t_\rho \rangle t_\rho,$$

and

$$2t_\rho \ast t_\sigma = \langle t_\sigma, t_\rho \rangle t_\rho.$$

Corollary 18 For all continuous representations $\rho, \sigma : M \to SU(2)$, one has

$$2t_\rho \ast t_\sigma = \langle t_\sigma, t_\rho \rangle t_\rho.$$ 

If $t_\rho \neq t_\sigma$, i.e. if the representations $\rho$ and $\sigma$ are not equivalent, then

$$\langle t_\sigma, t_\rho \rangle = 0.$$
Proof. The first equality was shown above. The orthogonality results from the commutativity of the convolution:

\[ \langle t_\sigma, t_\rho \rangle_{t_\rho} = 2t_\rho \ast t_\sigma \]
\[ = 2t_\sigma \ast t_\rho \]
\[ = \langle t_\rho, t_\sigma \rangle_{t_\sigma} \]
\[ = \langle t_\sigma, t_\rho \rangle_{t_\sigma} \]

and hence \( \langle t_\sigma, t_\rho \rangle = 0 \) for \( t_\sigma \neq t_\rho \). \hfill \blacksquare

When \( t_\rho \) is normed by passing to

\[ e_\rho := \frac{2}{\langle t_\rho, t_\rho \rangle} t_\rho, \]

one gets

**Corollary 19** The elements \( e_\rho \) of \( L_2(\mathcal{M}) \) are idempotent with respect to the convolution:

\[ e_\rho \ast e_\rho = e_\rho. \]

If \( \rho \) and \( \sigma \) are non-equivalent representations, then \( e_\rho \) and \( e_\sigma \) are orthogonal,

\[ \langle e_\rho, e_\sigma \rangle = 0. \]

Proof. One has

\[ e_\rho \ast e_\rho = \frac{4}{\langle t_\rho, t_\rho \rangle}^2 t_\rho \ast t_\rho \]
\[ = \frac{2}{\langle t_\rho, t_\rho \rangle}^2 2t_\rho \ast t_\rho \]
\[ = \frac{2}{\langle t_\rho, t_\rho \rangle}^2 \langle t_\rho, t_\rho \rangle t_\rho \]
\[ = e_\rho. \]

\hfill \blacksquare

The generalized Wilson loop functions \( t_\rho \) are easily seen to be continuous functions \( \mathcal{M}(\mathcal{L}_s) \rightarrow \mathbb{C} \), so they come from almost periodic functions on \( \mathcal{L}_s \): consider the commutative diagram
Here \( \varphi \in \text{Hom}(\mathcal{L}_*, G) \). Lemma 6 shows the existence of a continuous homomorphism \( \mathcal{M}(\varphi) : \mathcal{M}(\mathcal{L}_*) \to G \). The trace function \( \text{tr} \) is continuous, and thus \( \text{tr} \circ \mathcal{M}(\varphi) : \mathcal{M}(\mathcal{L}_*) \to \mathbb{C} \) is continuous. It follows from Prop. 5 that the function \( \text{tr} \circ \varphi : \mathcal{L}_* \to \mathbb{C} \) is almost periodic and the function \( t_\rho = t_{\mathcal{M}(\varphi)} = \text{tr} \circ \mathcal{M}(\varphi) \) is its extension to \( \mathcal{M}(\mathcal{L}_*) \).

It seems quite suggestive to regard \( L^2(\mathcal{M}) \) as a candidate for the Hilbert space of loop quantum gravity in the loop representation. To show that this is the right choice, several issues have to be clarified. The most important is the loop transform \cite{RovSmo90,AshLew94} that translates from the connection representation to the loop representation. The loop transform, an appropriate extension of it and the inverse loop transform will be treated in a subsequent paper. Another important issue is diffeomorphism invariance. Since the ultimate goal is some form of quantum gravity, one has to regard the symmetries of the classical theory, which is General Relativity in a Hamiltonian formulation. In approaches of this kind, spacetime \( M \) is assumed to be globally hyperbolic, so that according to a theorem by Geroch \cite{HawEll73} one has

\[
M \simeq \mathbb{R} \times \Sigma,
\]

where \( \Sigma \) is a compact three-dimensional Riemannian manifold. The above isomorphism is not given canonically, but is the foliation of spacetime into space \( \Sigma \) and time \( \mathbb{R} \) as seen by some observer. Another observer corresponds to another foliation. The theory is formulated with respect to some fixed foliation. For example, the connections regarded throughout this paper are connections on a principal bundle over \( \Sigma \). Physical predictions of the theory must be independent of the choice of foliation, of course. Since we are concerned with the kinematical aspects of the theory, we do not have to consider
the full diffeomorphism group of spacetime, but merely the diffeomorphisms
\[ \phi : \Sigma \rightarrow \Sigma \]
of space. Since we have regarded piecewise analytic hoops throughout, we will only consider analytic diffeomorphisms. If \( L_2(\mathcal{M}) \) shall serve as Hilbert space for the loop representation, it is a necessary condition that its Haar measure is diffeomorphism invariant:

**Proposition 20** The Haar measure \( \xi \) on \( \mathcal{M}(\mathcal{L}_*) \) is invariant under analytic diffeomorphisms \( \phi : \Sigma \rightarrow \Sigma \).

**Proof.** Let \( \text{Diff}_{\text{an}}(\Sigma) \) be the group of analytic diffeomorphisms of \( \Sigma \). \( \text{Diff}_{\text{an}}(\Sigma) \) operates by isomorphisms on \( \mathcal{L}_* \): let \( \gamma \in \mathcal{L}_* \). Then
\[
\tilde{\phi} : \mathcal{L}_* \rightarrow \mathcal{L}_*, \quad \gamma \mapsto (\phi \circ \gamma)
\]
and \( \tilde{\phi}(\gamma \delta) = (\tilde{\phi}(\gamma))(\tilde{\phi}(\delta)) \). The following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{L}_* & \xrightarrow{\tilde{\phi}} & \mathcal{L}_* \\
| \downarrow t & & | \downarrow t \\
\mathcal{M}(\mathcal{L}_*) & \xrightarrow{\mathcal{M}(\phi)} & \mathcal{M}(\mathcal{L}_*)
\end{array}
\]

The mapping \( t \circ \tilde{\phi} : \mathcal{L}_* \rightarrow \mathcal{M}(\mathcal{L}_*) \) is a homomorphism from \( \mathcal{L}_* \) to \( \mathcal{M}(\mathcal{L}_*) \) and induces a continuous homomorphism \( \mathcal{M}(\tilde{\phi}) : \mathcal{M}(\mathcal{L}_*) \rightarrow \mathcal{M}(\mathcal{L}_*) \), which is an automorphism of \( \mathcal{M}(\mathcal{L}_*) \), of course. Hence \( \text{Diff}_{\text{an}}(\Sigma) \) induces continuous automorphisms of \( \mathcal{M}(\mathcal{L}_*) \). Let \( \phi, \psi \in \text{Diff}_{\text{an}}(\Sigma) \). Then one has \( \mathcal{M}(\phi \psi) = \mathcal{M}(\phi) \mathcal{M}(\psi) \), which gives
\[
\mathcal{M} : \text{Diff}_{\text{an}}(\Sigma) \rightarrow \text{Aut}_e(\mathcal{M}(\mathcal{L}_*)),
\phi \mapsto \mathcal{M}(\phi).
\]
As is well known, the Haar measure on a compact group is invariant under such automorphisms of the group. Let \( f \in C(\mathcal{M}(\mathcal{L}_*)) \). So we have
\[
\int_{\mathcal{M}(\mathcal{L}_*)} (f \circ \mathcal{M}(\phi))(\xi) d\xi = \int_{\mathcal{M}(\mathcal{L}_*)} f(\xi) d\xi.
\]

\[
\blacksquare
\]
5 Outlook

As already mentioned above, in a subsequent paper we will define an extended loop transform from the Hilbert space $L_2(\mathcal{A}/\mathbb{G}, d\mu)$ of loop quantum gravity in the connection representation to the Hilbert space $L_2(\mathcal{M})$. This will employ the extension of the Wilson loop functions $T_\alpha$ to $\mathcal{M}(\mathcal{L}_*)$. (Here the range of the index is extended.) The inverse extended loop transform is then given canonically, confirming the claim that $L_2(\mathcal{M})$ should be regarded as the Hilbert space of loop quantum gravity in the loop representation.

Furthermore, we will discuss some aspects of the implementation of the diffeomorphism constraints. In [AshLew94], there are remarks on the ”dual” rôles of the hoop group $\mathcal{L}_*$ and the quantum configuration space $\mathcal{A}/\mathbb{G}$. We will present some results on the action of the diffeomorphism group making this point clearer.

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