EQUIVARIANT PIERI RULE FOR THE HOMOLOGY OF THE AFFINE GRASSMANNIAN

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Abstract. An explicit rule is given for the product of the degree two class with an arbitrary Schubert class in the torus-equivariant homology of the affine Grassmannian. In addition a Pieri rule (the Schubert expansion of the product of a special Schubert class with an arbitrary one) is established for the equivariant homology of the affine Grassmannians of $SL_n$ and a similar formula is conjectured for $Sp_{2n}$ and $SO_{2n+1}$. For $SL_n$ the formula is explicit and positive. By a theorem of Peterson these compute certain products of Schubert classes in the torus-equivariant quantum cohomology of flag varieties. The $SL_n$ Pieri rule is used in our recent definition of $k$-double Schur functions and affine double Schur functions.

1. Introduction

Let $G$ be a semisimple algebraic group over $\mathbb{C}$ with a Borel subgroup $B$ and maximal torus $T$. Let $Gr_G = G((\mathbb{C}[t]))/G(\mathbb{C}[t])$ be the affine Grassmannian of $G$. The $T$-equivariant homology $H_T(Gr_G)$ and cohomology $H^T(Gr_G)$ are dual Hopf algebras over $S = H^T(pt)$ with Pontryagin and cup products respectively. Let $W_{af}$ be the minimal length cosets in $W_{af}/W$ where $W_{af}$ and $W$ are the affine and finite Weyl groups. Let $\{\xi_w \mid w \in W_{af}\}$ be the Schubert basis of $H_T(Gr_G)$. Define the equivariant Schubert homology structure constants $d_{uv}^w \in S$ by

$$\xi_u \xi_v = \sum_{w \in W_{af}} d_{uv}^w \xi_w$$

where $u, v \in W_{af}$. One interest in the polynomials $d_{uv}^w$ is the fact that they are precisely the Schubert structure constants for the $T$-equivariant quantum cohomology rings $QH^T(G/B)$ [LS2, Pet]. Due to a result of Mihalcea [Mih], they have the positivity property

$$d_{uv}^w \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I].$$

Our first main result (Theorem 6) is an “equivariant homology Chevalley formula”, which describes $d_{uv}^w$ for an arbitrary affine Grassmannian. Our second main result (Theorem 24) is an “equivariant homology Pieri formula” for $G = SL_n$, which is a manifestly positive formula for $d_{m,v}^w$ where the homology classes $\{\xi_{\sigma_m} \mid 1 \leq m \leq n - 1\}$ are the special classes that generate $H_T(Gr_{SL_n})$. In a separate work [LS3] we use this Pieri formula to define new symmetric functions, called $k$-double Schur functions and affine double Schur functions, which represent the equivariant Schubert homology and cohomology classes for $Gr_{SL_n}$.

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2. The equivariant homology of $Gr_G$

We recall Peterson’s construction \[\text{Pet}\] of the equivariant Schubert basis \(\{j_w \mid w \in W^0_{af}\}\) of $HT(Gr_G)$ using the level-zero variant of the Kostant and Kumar (graded) nil-Hecke ring \([\text{KK}]\). We also describe the equivariant localizations of Schubert cohomology classes for the affine flag ind-scheme in terms of the nilHecke ring; these are an important ingredient in our equivariant Chevalley and Pieri rules.

2.1. Peterson’s level zero affine nilHecke ring. Let $I$ and $I_{af} = I \cup \{0\}$ be the finite and affine Dynkin node sets and $(a_{ij} \mid i, j \in I_{af})$ the affine Cartan matrix.

Let $P^0_{af} = \mathbb{Z} \delta \oplus \bigoplus_{i \in I_{af}} \mathbb{Z} \Lambda_i$ be the affine weight lattice, with $\delta$ the null root and $\Lambda_i$ the affine fundamental weight. The dual lattice $P_{af}^* = \text{Hom}_\mathbb{Z}(P_{af}, \mathbb{Z})$ has dual basis \(\{d\} \cup \{\alpha_i^\vee \mid i \in I_{af}\}\) where $d$ is the degree generator and $\alpha_i^\vee$ is a simple coroot. The simple roots $\{\alpha_i \mid i \in I_{af}\} \subset P_{af}$ are defined by $\alpha_j = \delta_j + \sum_{i \in I_{af}} a_{ij} \Lambda_i$ for $j \in I_{af}$ where $(a_{ij} \mid i, j \in I_{af})$ is the affine Cartan matrix. Then $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$ for all $i, j \in I_{af}$.

Let $(a_i \mid i \in I_{af})$ (resp. $(\alpha_i^\vee \mid i \in I_{af})$) be the tuple of relatively prime positive integers giving a relation among the columns (resp. rows) of the affine Cartan matrix. Then $\delta = \sum_{i \in I_{af}} a_i \alpha_i$. Let $c = \sum_{i \in I_{af}} a_i \alpha_i^\vee \in P_{af}^*$ be the canonical central element. The level of a weight $\lambda \in P_{af}$ is defined by $\langle c, \lambda \rangle$.

There is a canonical projection $P_{af} \to P$ where $P$ is the finite weight lattice, with kernel $\mathbb{Z} \delta \oplus \mathbb{Z} \Lambda_0$. There is a section $P \to P_{af}$ of this projection whose image lies in the sublattice of $\bigoplus_{i \in I_{af}} \mathbb{Z} \Lambda_i$ consisting of level zero weights. We regard $P \subset P_{af}$ via this section.

Let $W$ and $W_{af}$ denote the finite and affine Weyl groups. Denote by $\{r_i \mid i \in I_{af}\}$ the simple generators of $W_{af}$. $W_{af}$ acts on $P_{af}$ by $r_i \cdot \lambda = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$ for $i \in I_{af}$ and $\lambda \in P_{af}$. $W_{af}$ acts on $P_{af}^*$ by $r_i \cdot \mu = \mu - \langle \mu, \alpha_i \rangle \alpha_i^\vee$ for $i \in I_{af}$ and $\mu \in P_{af}^*$. There is an isomorphism $W_{af} \cong W \ltimes Q^\vee$ where $Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee \subset P_{af}^*$ is the finite coroot lattice. The embedding $Q^\vee \to W_{af}$ is denoted $\mu \mapsto t_\mu$. The set of real affine roots is $W_{af} \cdot \{\alpha_i \mid i \in I_{af}\}$. For a real affine root $\alpha = w \cdot \alpha_i$, the associated coroot is well-defined by $\alpha^\vee = w \cdot \alpha_i^\vee$.

Let $S = \text{Sym}(P)$ be the symmetric algebra, and $Q = \text{Frac}(S)$ the fraction field. $W_{af} \cong W \ltimes Q^\vee$ acts on $P$ (and therefore on $S$ and on $Q$) by the level zero action:

$$wt_\mu \cdot \lambda = w \cdot \lambda \quad \text{for } w \in W \text{ and } \mu \in Q^\vee.$$  

Since $t_{-\theta^\vee} = r_\theta r_0$ we have

$$r_0 \cdot \lambda = r_\theta \cdot \lambda \quad \text{for } \lambda \in P.$$  

Finally, we have $\delta = \alpha_0 + \theta$ where $\theta \in P$ is the highest root. So under the projection $P_{af} \to P$, $\alpha_0 \mapsto -\theta$.

Let $QW_{af} = \bigoplus_{w \in W_{af}} Qw$ be the skew group ring, the $Q$-vector space $Q \otimes Q[W_{af}]$ with $Q$-basis $W_{af}$ and product $(p \otimes v)(q \otimes w) = p(v \cdot q) \otimes vw$ for $p, q \in Q$ and $v, w \in W_{af}$. $QW_{af}$ acts on $Q$: $q \in Q$ acts by left multiplication and $W_{af}$ acts as above.

For $i \in I_{af}$ define the element $A_i \in QW_{af}$ by

$$A_i = \alpha_i^{-1}(1 - r_i).$$
$A_i$ acts on $S$ since

$$A_i \cdot \lambda = \langle \alpha_i^\vee, \lambda \rangle$$

for $\lambda \in P$.

(7) $A_i \cdot (ss') = (A_i \cdot s)s' + (r_i \cdot s)(A_i \cdot s')$ for $s, s' \in S$.

The $A_i$ satisfy $A_i^2 = 0$ and

$$A_iA_jA_i \cdots = A_jA_iA_j \cdots$$

$m_{ij}$ times

$m_{ij}$ times

where

$$r_i r_j r_i \cdots = r_j r_i r_j \cdots$$

$m_{ij}$ times

$m_{ij}$ times

For $w \in W_{af}$ we define $A_w$ by

(8) $A_w = A_{i_1}A_{i_2} \cdots A_{i_\ell}$

where

(9) $w = r_{i_1}r_{i_2} \cdots r_{i_\ell}$ is reduced.

The level zero graded affine nilHecke ring $A$ (Peterson’s $[\text{Pet}]$ variant of the nilHecke ring of Kostant and Kumar $[\text{KK}]$ for an affine root system) is the subring of $Q_{W_{af}}$ generated by $S$ and $\{A_i \mid i \in I_{af}\}$. In $A$ we have the commutation relation

(10) $A_i \lambda = (A_i \cdot \lambda)1 + (r_i \cdot \lambda)A_i$ for $\lambda \in P$.

In particular

(11) $A = \bigoplus_{w \in W_{af}} SA_w$.

2.2. Localizations of equivariant cohomology classes. Using the relation

(12) $r_i = 1 - \alpha_i A_i$

$w \in W_{af}$ may be regarded as an element of $A$. For $v, w \in W$ define the elements $\xi^v(w) \in S$ by

(13) $w = \sum_{v \in W} (-1)^{\ell(v)} \xi^v(w) A_v$.

Using a reduced decomposition (9) for $w$ and substituting (12) for its simple reflections, one obtains the formula $[\text{AJS}] [\text{Bil}]$

(14) $\xi^v(w) = \sum_{b \in [0,1]^\ell} \left( \prod_{j=1}^\ell \alpha_i b_{ij} \right) \cdot 1$

where the sum runs over $b$ such that $\prod_{j: b_j = 1} r_{ij} = v$ is reduced and the product over $j$ is an ordered left-to-right product of operators. Each $b$ encodes a way to obtain a reduced word for $v$ as an embedded subword of the given reduced word of $w$: if $b_j = 1$ then the reflection $r_{ij}$ is included in the reduced word for $v$. Given a fixed $b$ and an index $j$ such that $b_j = 1$, the root associated to the reflection $r_{ij}$ is by definition $r_{i_1}r_{i_2} \cdots r_{j-1} \cdot \alpha_i$. The summand for $b$ is the product of the roots associated to reflections in the given embedded subword.
It is immediate that
\begin{align}
\xi^v(w) &= 0 \quad \text{unless } v \leq w \\
\xi^{id}(w) &= 1 \quad \text{for all } w.
\end{align}

The element $\xi^v(w) \in S$ has the following geometric interpretation. Let $X_{af} = G_{af}/B_{af}$ be the Kac-Moody flag ind-variety of affine type $[\text{Kum}]$. For every $v \in W_{af}$ there is a $T$-equivariant cohomology class $[X_v] \in H^T(X_{af})$ and for each $w \in W_{af}$ there is an associated $T$-fixed point (denoted $w$) in $X_{af}$ and a localization map $i_w^*: H^T(X_{af}) \to H^T(\text{pt}) \{[\text{Kum}].$

Then $\xi^v(w) = i_v^*([X_v])$. Moreover the map $H^T(X_{af}) \to H^T(W_{af}) \cong \text{Fun}(W_{af}, S)$ given by restriction of a class to the $T$-fixed subset $W_{af} \subset X_{af}$, is an injective isomorphism $\text{Fun}(W_{af}, S)$ is the $S$-algebra of functions $W_{af} \to S$ with pointwise product. The function $\xi^v \in \text{Fun}(W_{af}, S)$ is the image of $[X_v]$. The image $\Phi$ of $H^T(X_{af})$ in $\text{Fun}(W_{af}, S)$ satisfies the GKM condition $[\text{GKM1}]$ $[\text{KK}]$: For $f \in \Phi$ we have
\begin{equation}
f(w) - f(r\beta w) \in \beta S \quad \text{for all } w \in W_{af} \text{ and affine real roots } \beta.
\end{equation}

**Lemma 1.** Suppose $u, v \in W$ with $\ell(uv) = \ell(u) + \ell(v)$.
\begin{equation}
\xi^{uv}(uv) = \xi^u(u \cdot \xi^v(v)).
\end{equation}

**Lemma 2.** Suppose $v, w \in W$. Then
\begin{equation}
\xi^v(w) = (-1)^{\ell(v)} w \cdot (\xi^{v^{-1}}(w^{-1})).
\end{equation}

2.3. Peterson subalgebra and Schubert homology basis. The Peterson subalgebra of $A$ is the centralizer subalgebra $P = Z_A(S)$ of $S$ in $A$.

**Theorem 3.** $[\text{Pet}]$ There is an isomorphism $H_T(\text{Gr}_G) \to P$ of commutative Hopf algebras over $S$. For $w \in W_{af}$ let $j_w$ denote the image of $\xi_w$ in $P$. Then $j_w$ is the unique element of $P$ with the property that $j_w^w = 1$ and $j_w^x = 0$ for any $x \in W_{af} \setminus \{w\}$ where $j_w^x \in S$ are defined by
\begin{equation}
j_w = \sum_{x \in W_{af}} j_w^x A_x.
\end{equation}

Moreover, if $j_w^x \neq 0$ then $\ell(x) \geq \ell(w)$ and $j_w^x$ is a polynomial of degree $\ell(x) - \ell(w)$.

The Schubert structure constants for $H_T(\text{Gr}_G)$ are obtained as coefficients of the elements $j_w$.

**Proposition 4.** $[\text{Pet}]$ Let $u, v, w \in W^0_{af}$. Then
\begin{equation}
d^w_{uv} = \begin{cases} 
 j_u^{wv-1} & \text{if } \ell(w) = \ell(v) + \ell(wv^{-1}) \\
 0 & \text{otherwise}.
\end{cases}
\end{equation}

Due to the fact $[\text{LS2}]$ $[\text{Pet}]$ that the collections of Schubert structure constants for $H_T(\text{Gr}_G)$ and $QH^T(G/B)$ are the same and Mihalcea’s positivity theorem for equivariant quantum Schubert structure constants, we have the positivity property

**Proposition 5.** $j_w^x \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I]$ for all $w \in W^0_{af} \text{ and } x \in W_{af}$.\footnote{Using equivariance for the maximal torus $T_{af} \subset G_{af}$, the GKM condition characterizes the image of localization to torus fixed points. However after forgetting equivariance down to the smaller torus $T$, elements of $\Phi$ are characterized by additional conditions, which were determined in $[\text{GKM2}]$.}
Given \( u \in W_{af}^0 \) let \( t^u = t_\lambda \) where \( \lambda \in Q^\vee \) is such that \( t_\lambda W = uW \).

Since the translation elements act trivially on \( S \) and \( W_{af} \subset A \) via (12), we have \( t_\lambda \in \mathbb{P} \) for all \( \lambda \in Q^\vee \), so that \( t_\lambda \in \bigoplus_{v \in W_{af}^0} S_j \). For any \( w \in W_{af}^0 \), we have

\[
t^w = \sum_{w \in W_{af}^0} (-1)^{\ell(w)} \xi^u(t^w) j_v = \sum_{w \in W_{af}^0} (-1)^{\ell(w)} \xi^u(w) j_v
\]

by the definitions and Lemma 1.

Define the \( W_{af}^0 \times W_{af}^0 \)-matrices

\[
A_{wv} = (-1)^{\ell(v)} \xi^u(w)
\]

\[
B = A^{-1}.
\]

The matrix \( A \) is lower triangular by (15) and has nonzero diagonal terms, and is hence invertible over \( Q = \text{Frac}(S) \). We have

\[
j_v = \sum_{w \in W_{af}^0 \atop w \leq v} B_{wv} t^w.
\]

Taking the coefficient of \( A_x \) for \( x \in W_{af} \), we have

\[
j^x_v = (-1)^{\ell(x)} \sum_{w \in W_{af}^0 \atop w \leq v} B_{wv} \xi^x(t^w).
\]

Note that if \( \Omega \subset W_{af}^0 \) is any order ideal (downwardly closed subset) then the restriction \( A|_{\Omega \times \Omega} \) is invertible. In the sequel we choose certain such order ideals and find a formula for the inverse of this submatrix. Since the values of \( \xi^x \) are given by (14) we obtain an explicit formula for \( j^x_v \) for \( v \in \Omega \) and all \( x \in W_{af} \).

3. Equivariant Homology Chevalley Rule

**Theorem 6.** For every \( \id \neq x \in W_{af} \), \( \xi^{x^{-1}}(r_\theta) \in \theta S \) and

\[
j_{r_\theta} = \sum_{\id \neq x \in W} (\theta^{-1} \xi^{x^{-1}}(r_\theta) A_x + \xi^{x^{-1}}(r_\theta) A_{r_\theta x}).
\]

**Proof.** For \( x \neq \id \), the GKM condition (17) and (15) implies that \( \xi^{x^{-1}}(r_\theta) \in \theta S \). \( \Omega = \{ \id, r_\theta \} \subset W_{af}^0 \) is an order ideal. The matrix \( A|_{\Omega \times \Omega} \) and its inverse are given by

\[
\begin{pmatrix}
1 & 0 \\
1 & \theta
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\theta^{-1} & \theta^{-1}
\end{pmatrix}
\]

Since \( \id = t^\id \) and \( t^\theta \cdot c^\theta = t^{r_\theta} \) (as \( t^\theta \cdot c^\theta = r_\theta r_\theta \)), we have

\[
(-1)^{\ell(y)} j^y_{r_\theta} = -\theta^{-1} \xi^y(\id) + \theta^{-1} \xi^y(t^\theta).
\]

By the length condition in Theorem 3 we have

\[
(-1)^{\ell(y)} j^y_{r_\theta} = \theta^{-1} \xi^y(t^\theta) \quad \text{for } y \neq \id.
\]

By (15) \( j^y_{r_\theta} = 0 \) unless \( y \leq t^\theta \cdot c^\theta = r_\theta r_\theta \). So assume this. Suppose \( r_\theta y < y \). Write \( y = r_\theta x \). Then

\[
(-1)^{\ell(y)} \xi^y(t^\theta) = (-1)^{\ell(y)} (\alpha_0)(r_\theta \cdot \xi^x(r_\theta)) = (-1)^{\ell(x)} \theta(r_\theta \cdot \xi^x(r_\theta)) = \theta \xi^{x^{-1}}(r_\theta).
\]
If \( r_0 y > y \) then we write \( y = x \leq r_\theta \) and
\[
(-1)^{\ell(x)} \xi^x(t_{\theta^\vee}) = (-1)^{\ell(x)} r_0 \cdot \xi^x(r_\theta) = (-1)^{\ell(x)} r_\theta \cdot \xi^x(r_\theta) = \xi^{x^{-1}}(r_\theta)
\]
as required. \( \square \)

The formula \( \mathbf{(14)} \) shows that \( \xi^{x^{-1}}(r_\theta) \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I] \). The same holds for \( \theta^{-1} \xi^{x^{-1}}(r_\theta) \). Indeed,

**Lemma 7.** \( \alpha^{-1} \xi^{x}(r_\alpha) \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I] \) for any positive root \( \alpha \).

**Proof.** The reflection \( r_\alpha \) has a reduced word \( i = i_1 i_2 \cdots i_{r-1}i_r i_{r-1} \cdots i_1 \) which is symmetric. Consider the different embeddings \( j \) of reduced words of \( x \) into \( i \), as in \( \mathbf{(14)} \). If \( j \) uses the letter \( i_r \), then the corresponding term in \( \mathbf{(14)} \) has \( \theta \) as a factor. Otherwise, \( j \) uses \( i_s \) but not \( i_{s+1}, \ldots, i_r \), for some \( s \). But then there is another embedding of \( j' \) of the same reduced word of \( x \) into \( i \), which uses the other copy of the letter \( i_s \) in \( i \). The two terms in \( \mathbf{(14)} \) which correspond to \( j \) and \( j' \) contribute \( A(\beta - r_\alpha \cdot \beta) = A(\langle \alpha^\vee, \beta \rangle \alpha) \) where \( A \in \mathbb{Z}_{\geq 0}[\alpha_i \mid i \in I] \), and \( \beta \) is an inversion of \( r_\alpha \). It follows that \( \langle \alpha^\vee, \beta \rangle > 0 \). The lemma follows. \( \square \)

**Remark 8.** The polynomials \( \xi^{x^{-1}}(r_\theta) \) appearing in \( \mathbf{(25)} \) may be computed entirely in the finite Weyl group and finite weight lattice.

**Remark 9.** In \( \mathbf{[LS]} \) Proposition 2.17, we gave an expression for the non-equivariant part of \( j_{r_\theta} \), consisting of the terms \( j_{r_\theta}^x A_x \) where \( \ell(x) = 1 = \ell(r_\theta) \). This follows easily from Theorem 5 and the fact \( \mathbf{[KK]} \) that \( \xi^x(w) = \omega_i - w \cdot \omega_i \), where \( \omega_i \) is the \( i \)-th fundamental weight.

4. Alternating equivariant Pieri rule in classical types

We first establish some notation for \( G = SL_n, Sp_{2n}, \) and \( SO_{2n+1} \). Our root system conventions follow \( \mathbf{[Kac]} \).

4.1. Special classes. We give explicit generating classes for \( HT(Gr_G) \).

4.1.1. \( HT(Gr_{SL_n}) \). Define the elements
\[
\hat{\sigma}_p = r_{p-1} \cdots r_1
\]
\[
\sigma_p = r_{p-1} \cdots r_1 r_0 = \hat{\sigma}_p r_0
\]
So \( \ell(\hat{\sigma}_p) = p - 1 \) and \( \ell(\sigma_p) = p \). These elements have associated translations
\[
t_p := t_{\sigma_p^{p+1}} = t_{r_p \cdots r_2 r_1 \theta^\vee} \quad \text{for } 0 \leq p \leq n - 2.
\]

4.1.2. \( HT(Gr_{Sp_{2n}}) \). For \( 1 \leq p \leq 2n - 1 \) we define the elements \( \hat{\sigma}_p \in W \) by
\[
\hat{\sigma}_p = r_{p-1} \cdots r_2 r_1 \quad \text{for } 1 \leq p \leq n
\]
\[
\hat{\sigma}_p = r_{2n-p-1} \cdots r_{n-2} r_{n-1} \cdots r_2 r_1 \quad \text{for } n + 1 \leq p \leq 2n - 1.
\]
For \( 1 \leq p \leq 2n - 1 \) define \( \sigma_p \in W_{af}^0 \) and \( t_{p-1} \in W_{af} \) by
\[
\sigma_p = \hat{\sigma}_p r_0
\]
\[
t_{p-1} = t_{\sigma_p \theta^\vee} = t_{\hat{\sigma}_p \theta^\vee}.
\]
4.1.3. $H_T(\text{Gr}_{SO_{2n+1}})$. For $1 \leq p \leq 2n-1$ we define the elements $\hat{\sigma}_p \in W^0_{af}$ by

$$
\hat{\sigma}_p = \begin{cases}
\text{id} & \text{if } p = 1 \\
 r_p r_{p-1} \cdots r_3 r_2 & \text{if } 2 \leq p \leq n \\
 r_{2n-p} r_{2n-p+1} \cdots r_{n-1} r_n r_{n-1} \cdots r_3 r_2 & \text{if } n+1 \leq p \leq 2n-2 \\
 r_0 r_2 r_3 \cdots r_{n-1} r_n r_{n-1} \cdots r_3 r_2 & \text{if } p = 2n-1.
\end{cases}
$$

For $1 \leq p \leq 2n-1$ define $\sigma_p \in W^0_{af}$ by

$$(31) \quad \sigma_p = \hat{\sigma}_p r_0$$

For $1 \leq p \leq 2n-2$ define $t_{p-1} \in W_{af}$ by

$$(32) \quad t_{p-1} = t_{\sigma_p} = t_{\hat{\sigma}_p, \sigma^\vee}.$$

For $1 \leq p \leq 2n-1$ let $\sigma'_p$ be $\sigma_p$ but with every $r_0$ replaced by $r_1$. Then define

$$t_{2n-2} = t_{2n-1} = \sigma_{2n-1} \sigma'_{2n-1}.$$

Then we conjecture that

$$(33) \quad B_{\sigma_{2n-1}, \sigma_q} = \pm \frac{1}{\xi_{\sigma_{2n-1}}(\sigma'_q \sigma_{2n-1})} \quad \text{for } 1 \leq q \leq 2n-1$$

where $B$ is defined in ($23$). The sign is $-$ for $q \leq 2n-2$ and $+$ for $q = 2n-1$.

4.1.4. Special classes generate. Let $k' = n-1$ for $G = SL_n$ and $k' = 2n-1$ for $G = Sp_{2n}$ or $G = SO_{2n+1}$. Let $\hat{\mathbb{P}} := S[[\sigma_m | 1 \leq m \leq k']]$ be the completion of $\mathbb{P} \cong H_T(\text{Gr}_G)$ generated over $S$ by series in the special classes. It inherits the Hopf structure from $\mathbb{P}$. The Hopf structure on $\mathbb{P}$ is determined by the coproduct on the special classes.

**Proposition 10.** For $G = SL_n, Sp_{2n}, SO_{2n+1}$, $\mathbb{Q} \otimes \mathbb{P} \subset \mathbb{Q} \otimes Z \hat{\mathbb{P}}$.

**Proof.** It is known that the special classes generate the homology $H_*(\text{Gr}_G)$ non-equivariantly for $G = SL_n, Sp_{2n}, SO_{2n+1}$ see [LSS, Pom]. Furthermore, the equivariant homology Schubert structure constants $d_{uv}^w$ is a polynomial in the simple roots of degree $\ell(u) - \ell(v) - \ell(w)$, and when $\ell(u) = \ell(v) + \ell(w)$, it is equal to the non-equivariant homology Schubert structure constant. It follows easily that each equivariant Schubert class can be expressed as a formal power series in the equivariant special classes. $\square$

**Remark 11.** For $G = SL_n$ and $G = Sp_{2n}$ the special classes generate $H_*(\text{Gr}_G)$ over $\mathbb{Z}$.

4.2. The alternating equivariant affine Pieri rule. Let $k = n-1$ for $G = SL_n$, $k = 2n-1$ for $G = Sp_{2n}$, and $k = 2n-2$ for $G = SO_{2n+1}$. Our goal is to compute $j^x_m$ for $1 \leq m \leq k$; note that for $G = SO_{2n+1}$, the element $\sigma_{2n-1}$ has been treated in ($33$).

For this purpose consider the Bruhat order ideal $\Omega = \{\text{id} = \sigma_0, \sigma_1, \ldots, \sigma_k\} \in W^0_{af}$. Since $j_0 = \text{id}$, to compute $j^x_{\sigma_p}$ for $p \geq 1$ we may assume $x \neq \text{id}$ by length considerations. It suffices to invert the matrix $A$ given in ($22$) over $\mathbb{Q} \setminus \{\text{id}\}$ by length considerations. Define the matrices $M_{pm} = (-1)^m \xi_{\sigma_{2n}}(\sigma_p)$ for $1 \leq p, m \leq k$, $N_{mq} = \xi_{\sigma_{2n}}(\sigma_q r_0)$ for $1 \leq m, q \leq k$, and the diagonal matrix $D_{pq} = \delta_{pq} \xi_{\sigma_{2n}}(t_{p-1})$ for $1 \leq p, q \leq k$.

**Conjecture 12.**

$$(34) \quad MN = D.$$
Conjecture 13. For $1 \leq m \leq k$ and $x \neq \text{id}$ we have
\[
\hat{j}^x_{\sigma_m} = (-1)^{\ell(x)} \sum_{q=0}^{m-1} \xi_{\sigma_m r \theta}^q (\hat{\sigma}_{q+1} r \theta) \xi^x (t_q).
\]
In particular $\hat{j}^x_{\sigma_m} = 0$ unless $\ell(x) \geq m$ and $x \leq t_q$ for some $0 \leq q \leq m - 1$.

Conjecture 13 follows immediately from Conjecture 12: we have $M^{-1} = ND^{-1}$, and (35) follows from (24).

Theorem 14. Conjecture 12 holds for $G = SL_n$.

The proof appears in Appendix A. Examples of (34) appear in Appendix B.

5. Effective Pieri rule for $H_T(\text{Gr}_{SL_n})$

The goal of this section is to prove a formula for $\hat{j}^x_{\sigma_m}$ that is manifestly positive. In this section we work with $G = SL_n$, $W = S_n$, and $W_{af} = \tilde{S}_n$.

5.1. Simplifying (35). We first establish some notation. For $a \leq b$ write
\[
\begin{align*}
  u^b_a &= r_a r_{a+1} \cdots r_b \\
  d^b_a &= r_b r_{b-1} \cdots r_a \\
  \alpha^b_a &= \alpha_a + \alpha_{a+1} + \cdots + \alpha_b
\end{align*}
\]
for upward and downward sequences of reflections and for sums of consecutive roots. In particular we have $\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} = \alpha_1^{n-1}$.

Let $0 \leq q \leq m - 1$. We have
\[
\begin{align*}
  \xi_{\hat{\sigma}_m r \theta}^q (\hat{\sigma}_{q+1} r \theta) &= u^m_{q+1} \cdot \xi_{\hat{\sigma}_m r \theta}^q (\hat{\sigma}_m r \theta) \\
  &= (-1)^m u^m_{q+1} \cdot \xi_{\hat{\sigma}_m r \theta}^q (\hat{\sigma}_m r \theta) \\
  &= (-1)^m u^m_{q+1} \cdot \xi_{\hat{\sigma}_m r \theta}^q (\hat{\sigma}_m r \theta).
\end{align*}
\]
We also have
\[
\begin{align*}
  \xi^x (t_q) &= \xi^{q+1}_{\sigma_{q+1}} (\sigma_{q+1} \cdot \xi^q_{\sigma_1 r \theta} (\sigma_1 r \theta)) (\sigma_{q+1} r \theta) \xi^x (t_m) \\
  &= \xi^{q+1}_{\sigma_{q+1}} (\sigma_{q+1} r \theta) \cdot \xi^q_{\sigma_1 r \theta} (\sigma_1 r \theta) \xi^x (t_m) \\
  &= (-1)^{m-q-1} \xi^{q+1}_{\sigma_{q+1}} (\sigma_{q+1} \cdot \xi^q_{\sigma_1 r \theta} (\sigma_1 r \theta)) \xi^x (t_m).
\end{align*}
\]
Define
\[
D(q, m) = \xi^{q+1}_{\sigma_{q+1}} (\sigma_{q+1} \cdot \xi^q_{\sigma_1 r \theta} (\sigma_1 r \theta)) \xi^m (t_q).
\]
so that by Theorem 14
\[
\hat{j}^x_{\sigma_m} = (-1)^{\ell(x)} \sum_{q=0}^{m-1} \frac{(-1)^{q+1}}{D(q, m)} \xi^x (t_q).
\]
Explicitly we have
\[
\begin{align*}
  \xi^{q+1}_{\sigma_{q+1}} (\sigma_{q+1}) &= \alpha_q \alpha_q \cdots \alpha_q \alpha_q \\
  \xi^m (u^m_{q+1}) &= \alpha_q \alpha_q \cdots \alpha_q \alpha_q.
\end{align*}
\]
5.2. $V$’s and $\Lambda$’s. The support $\text{Supp}(b)$ of a word $b$ is the set of letters appearing in the word. For a permutation $w$, $\text{Supp}(w)$ is the support of any reduced word of $w$. A $V$ is a reduced word (for some permutation) that decreases to a minimum and increases thereafter. Special cases of $V$’s include the empty word, any increasing word and any decreasing word. A $\Lambda$ is a reduced word that increases to a maximum and decreases thereafter. A (reverse) $N$ is a reduced word consisting of a $V$ followed by a $\Lambda$, such that the support of the $V$ is less than the support of the $\Lambda$.

By abuse of language, we say a permutation is a $V$ if it admits a reduced word that is a $V$. We use similar terminology for $\Lambda$’s and $N$’s.

A permutation is connected if its support is connected (that is, is a subinterval of the integers).

**Lemma 15.** A permutation that is a $V$, admits a unique reduced word that is a $V$. Similarly for a connected $\Lambda$ or a connected $N$.

**Lemma 16.** A connected permutation is a $V$ if and only if it is a $\Lambda$, if and only if it is an $N$.

5.3. $t_q$-factorizations. For $0 \leq q \leq n - 2$, we call

$$q(q - 1) \cdots 101 \cdots (n - 1)(n - 2) \cdots q + 1$$

the standard reduced word for $t_q$. Since this word is an $N$ it follows that any $x \leq t_q$ is an $N$. We call the subwords $q(q - 1) \cdots 1$, $12 \cdots (n - 2)$ and $(n - 2) \cdots q + 1$ the left, middle, and right branches.

**Lemma 17.** If $x \in \tilde{S}_n$ admits a reduced word in which $i + 1$ precedes $i$ for some $i \in \mathbb{Z}/n\mathbb{Z}$ then $x \not\leq t_i$.

**Proof.** Suppose $x \leq t_i$. Since the standard reduced word of $t_i$ has all occurrences of $i$ preceding all occurrences of $i + 1$, it follows that $x$ has a reduced word with that property. But this property is invariant under the braid relation and the commuting relation, which connect all reduced words of $x$. \qed

Let $c(x)$ denote the number of connected components of $\text{Supp}(x)$. If $J$ and $J'$ are subsets of integers then we write $J < J' - 1$ if $\max(J) < \min(J') - 1$.

**Lemma 18.** Suppose $x \leq t_q$. Then $x$ has a unique factorization $x = v_1 \cdots v_r y_1 y_2 \cdots y_s$, called the $q$-factorization, where each $v_i, y_i$ has connected support such that

1. $\text{Supp}(v_i) < \text{Supp}(v_{i+1}) - 1$ and $\text{Supp}(y_i) < \text{Supp}(y_{i+1}) - 1$
2. $\text{Supp}(v_1 \cdots v_r) \subset [0, q]$
3. $\text{Supp}(y_1 \cdots y_s) \subset [q + 1, n - 1]$
4. Each $v_i$ is a $V$.
5. Each $y_i$ is a $\Lambda$.

We say that $v_r$ and $y_1$ touch if $q \in \text{Supp}(v_r)$ and $q + 1 \in \text{Supp}(y_1)$. We denote

$$\epsilon(x, q) = \begin{cases} 1 & \text{if } v_r \text{ and } y_1 \text{ touch} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\epsilon(x, q)$ depends only on $\text{Supp}(x)$ and $q$. 
Each $k$ in the $q$-factorization of $x \leq t_q$, is (1) in the left branch of some $v_i$, or (2) in the right branch of some $v_i$, or (3) at the bottom of a $v_i$, or (1’) in the left branch of some $y_i$, or (2’) in the right branch of some $y_i$, or (3’) at the top of a $y_i$, or finally (4) absent. We call these sets $S_1$, $S_2$, $S_3$, $S_1'$, $S_2'$, and $S_3'$. Note that $k$ can belong to both $S_1$ and $S_2$, or both $S_1'$ and $S_2'$.

For each $x$ and each $q$ such that $x \leq t_q$, we define the polynomials

$$M(x, q) = (\alpha_0^q)^{c(x)} \prod_{k \in S_2} \alpha_0^{k-1} \prod_{k \in S_1'} \alpha_0^k$$

$$L(x, q) = \prod_{k \in S_1} \alpha_0^q$$

$$R(x, q) = \prod_{k \in S_2'} (-\alpha_0^{k+1})$$

We also define $R(x, q, m) = \prod_{k \in S_2[\cap[m, n-1]} (-\alpha_0^{q+1,k})$.

**Proposition 19.** If $x \leq t_q$, then

$$\xi_x(t_q) = (\alpha_0^q)^{c(x)} M(x, q) L(x, q) R(x, q).$$

*Proof.* We compute $\xi_x(t_q)$ using (44) by computing all embeddings of reduced words of $x$ into the standard reduced word (42) of $t_q$. We refer to the $q$-factorization of $x$. Each $k \in S_1$ must embed into the left branch of the $N$, and has associated root $\alpha_k^q$. Each $k \in S_2$ embeds into the middle branch of the $N$ and has associated root $\alpha_0^{k-1}$. Each $k \in S_1'$ embeds into the middle branch of the $N$ and has associated root $\alpha_0^k$. Each $k \in S_2'$ embeds into the right branch of the $N$ and has associated root $-\alpha_0^{k+1}$. Each $k \in S_3$ is either 0 and has associated root $\alpha_0^q$, or can be embedded into the left or middle branch of the $N$, and the sum of the two associated roots for these positions is $\alpha_k^q + \alpha_0^{k-1} = \alpha_0^q$. Each $k \in S_3'$ is either $n - 1$, which has associated root $-\alpha_0^{q+1} = \alpha_0^q$, or can be embedded into the middle or right branch of the $N$, and the sum of associated roots is $\alpha_k^q - \alpha_0^{k+1} = \alpha_0^q$. Since all the various choices for embeddings of elements of $S_3$ and $S_3'$ can be varied independently, the value of $\xi_x(t_q)$ is the product of the above contributions. Each minimum of a $v_i$ and maximum of a $y_j$ contributes $\alpha_0^q$. If there is a component of $x$ which contains both $q$ and $q + 1$ (that is, if $v_r$ and $y_1$ touch) then it is unique and contributes two copies of $\alpha_0^q$. All this yields (44). \hfill \Box

5.4. **Rotations.** We now relate $\xi_x(t_q)$ with $\xi_x(t_{q'})$. Let $r_{q'}^p$ denote the transposition that exchanges the integers $p$ and $q$.

**Proposition 20.** Let $x \leq t_q$ and consider the $q$-factorization of $x$. Let $a$ be such that this reduced word of $x$ contains the decreasing subword $(q + a)(q + a - 1) \cdots (q + 1)$ but not $(q + a + 1)(q + a) \cdots (q + 1)$. If $q + 1 \notin \text{Supp}(x)$, then set $a = 1$. Then

$$\xi_x(t_{q+1}) = \xi_x(t_{q+2}) = \cdots = \xi_x(t_{q+a-1}) = 0$$

and

$$\xi_x(t_{q+a}) = M(x, q) r_{1+q}^{1+q+a} (\alpha_0^q)^{c(x)} L(x, q) R(x, q)$$

Let $y^\dagger$ denote $y$ with every $r_i$ changed to $r_{i+1}$.
Lemma 21. Let y be increasing with support in \([b, a - 1]\). Then

\[ yd^q_0 = d^q_0y^\uparrow \]

Proof of Proposition 20. We assume that \(q + 1 \in \text{Supp}(x)\), for otherwise the claim is easy.

By Lemma 17 we have \(x \not\subseteq t_{q+i}\) for \(1 \leq i \leq a - 1\). Equation (45) follows from (15). We now prove (46). The first goal is to compute the \(q\)-factorization of \(x\). Since \(x \leq t_q\) we may consider the \(q\)-factorization of \(x\). The decreasing word \((q + a - 1) \cdots (q + 2)(q + 1)\) must embed into the right hand branch, that is, \([q + 1, q + a - 1] \subset S2'\). The hypotheses imply that \(q + a \not\in S2'\). There are two cases: either \(q + a \in S1'\) or \(q + a \in S3'\) (so that \(q + a + 1 \not\in \text{Supp}(x)\)). We treat the former case, as the latter is similar: the two cases correspond to the touching and nontouching cases for the \(q + a\)-factorization of \(x\), whose existence we now demonstrate.

Suppose \(q + a \in S1'\). Then there is a \(y_1'\) with \(\text{Supp}(y_1') \subset [q + a + 1, n - 1]\) and a \(y\) with an increasing reduced word such that \(\text{Supp}(y) \subset [q + 1, q + a - 1]\) and \(y_1 = yr_{q+a}d^q_{q+1}d^{q+a-1}_0 = yd^q_{q+1}y_1'\). Suppose \(v_r\) and \(y_1\) touch. Then \(v_r' := v_ryd^q_{q+1}\) is an \(N\) and therefore a \(V\).

Moreover \(x \leq t_{q+a}\) since \(x\) has a \(q + a\)-factorization given by the \(q\)-factorization of \(x\) but with \(v_r\) and \(y_1\) replaced by \(v_r'\) and \(y_1'\) respectively. To verify that \(v_r'\) is a \(V\), by the touching assumption, \(q \in \text{Supp}(v_r)\) and we have \(v_r'y_1'd^{q+a}_{q+1} = v_r'd^{q+a}_{q+1}y_1'y^\uparrow\) which expresses \(v_r'\) in a \(V\).

Suppose \(v_r\) and \(y_1\) do not touch, that is, \(q \not\in \text{Supp}(v_r)\). We have the \(V\) given by \(v_{r+1}' = yd^q_{q+1} = d^q_{q+1}y^\uparrow\). Then \(x \leq t_{q+a}\), as \(x\) has the \(q + a\) factorization given by the \(q\)-factorization of \(x\) except that there is a new \(V\), namely, \(v_{r+1}'\) and the first \(y\) is \(y_1'\) instead of \(y_1\).

In every case we calculate that

\[
M(x, q + a) = M(x, q) \\
L(x, q + a) = \left( \prod_{k=q+2}^{q+a} \alpha_k^{q+1} \right) d^q_{q+1}L(x, q) \\
R(x, q + a) = d^q_{q+1} \left( \prod_{k=q+1}^{q+a-1} (-\alpha_k^{q+1})^{-1} \right) R(x, q) = \left( \prod_{k=q+1}^{q+a-1} \alpha_k^{q+1} \right) d^q_{q+1}R(x, q).
\]

The calculation for \(L\) and \(R\) follows from the fact that \([q + 2, q + a] \subset S1_{q+a}\), but \([q + 1, q + a - 1] \subset S2_{q+a}\). The calculation for \(M\) follows from the fact that \(\text{Supp}(y) \subset S2_q\) and \(\text{Supp}(y^\uparrow) \subset S2_{q+a}\), together with the following boundary cases:

If \(q + a + 1 \in \text{Supp}(x)\) then \(q + a \in S1_{q+a} \cap S1_{q+a}'\). Thus \(q + a\) contributes a factor of \(\alpha_0^{q+a}\) to \(M(x, q)\). This factor appears in \(M(x, q + a)\) as the factor \((\alpha_0^{q+a})^{\epsilon(x,q+a)}\), since \(\epsilon(x, q + a) = 1\).

If \(q \in \text{Supp}(x)\) one has \(\epsilon(x, q) = 1\) and \(q + 1 \in S2_{q+a}\) contributes a factor of \(\alpha_0^q\) to \(M(x, q + a)\). This factor appears in \(M(x, q)\) as the factor \((\alpha_0^q)^{\epsilon(x,q)} = \alpha_0^q\).

Using that \(\alpha_0^{q+1} = \alpha_0^q\), \(d^q_{q+1}(-\alpha_0^{q+1}) = \alpha_0^{q+a}\), and \(r_1^{q+1} = -\alpha_0^{q+a}\), the above relations between \(M(x, q)\), \(L(x, q)\), \(R(x, q)\) and their counterparts for \(q + a\), together with
Proposition 19 yield
\[ \xi^x(t_{q_j}) = (\alpha_0^{q+1})^{-1} M(x, q_j) D_{q+1}^{q+1} (-\alpha_0^{q+1}) \langle \alpha_0^q \rangle^c x \ L(x, q) \ R(x, q). \]

To obtain (46), since \( r_{1+q}^{q+1} = d_{q+1}^q u_{q+2}^q \), it suffices to show that
\[ (-\alpha_0^{q+1}) \langle \alpha_0^q \rangle^c x \ L(x, q) R(x, q) \]
is invariant under \( u_{q+2}^q \).

However it is clear that \( \alpha_0^q \) and \( L(x, q) \) are invariant, and the only part of \( R(x, q) \) that must be checked is the product \( \prod_{k \in S^2 \cap (q+1, q+a]} (-\alpha_{q+1,k}) \). However we have that \( S^2' \cap [q+1, q+a] = [q+1, q+a-1] \), and indeed the product \( \prod_{k=q+1}^{q+a} (-\alpha_{q+1,k}) \) is invariant under \( u_{q+2}^q \), as required.

Let
\[ \{ q \in [0, m-1] \mid x \leq t_q \} = \{ q_1 < q_2 < \cdots < q_p \}. \]

In light of the proof of Proposition 20 we write
\[ M(x) = M(x, q_j) \quad \text{for any } 1 \leq j \leq p. \]

Let \( \beta_i = \alpha_1^{q_i+1} \) be the root associated with the reflection \( r_{\beta_i} \) that exchanges the numbers \( 1+q_i \) and \( 1+q_i+1 \). For \( i \leq j \) we also define
\[ \beta_i^{j+1} = \beta_i + \beta_i+1 + \cdots + \beta_j = \alpha_i^{q_i+1}. \]

Let
\[ Y_i(x, m) = (\alpha_0^{q_i})^c x R(x, q_i, m) \quad \text{for } 1 \leq i \leq p \]
so that \( Y_i(x, m) = r_{\beta_i} Y_{i-1}(x, m) \).

Lemma 22.

\[ (-1)^{m-1-q_j-p+j} \frac{\xi^x(t_{q_j})}{D(q_j, m)} = \frac{M(x) Y_j(x, m)}{(\beta_1^{q_1} \beta_2^{q_2} \cdots \beta_{j-1}^{q_{j-1}})(\beta_j^{q_1+1} \beta_j^{q_2+1} \cdots \beta_j^{q_{j-1}+1})}. \]

Proof. The proof proceeds by induction on \( j \). Let \( D_j \) be the denominator of the right hand side. Suppose first that \( j = 1 \). Consider the embedding of \( x \) into \( t_{q_1} \). By the definition of \( q_1 \), it follows that \( L(x, q_1) \alpha_0^{q_1} = \xi^x q_1 (\sigma_{q_1+1}) \). By the definition of the \( q_j \), we also have \( S^2' \cap [q_1 + 1, m-1] = [q_1 + 1, m-1] \setminus \{ q_2, q_3, \ldots, q_p \} \). These considerations and Proposition 19 imply that
\[ \xi^x(t_{q_1}) = (\alpha_0^{q_1})^c x M(x) L(x, q_1) R(x, q_1) \]
\[ = (-1)^{m-1-q_1} (\alpha_0^{q_1})^c x M(x) D(q_1, m) R(x, q_1, m) \prod_{j=2}^{p} (-\alpha_0^{q_{j+1}})^{-1} \]
\[ = (-1)^{m-1-q_{j+1}} D(q_1, m) M(x) Y_1(x, m) D_1^{-1}. \]

This proves the result for \( j = 1 \). Suppose the result holds for \( 1 \leq j \leq p-1 \). We show it holds for \( j+1 \). By induction we have
\[ (\alpha_0^{q_j})^c x L(x, q_j) R(x, q_j) = \frac{D(q_j, m) Y_j(x, m)}{D_j}. \]
Proposition 20 yields
\[
\frac{\xi^x(t_{q_{j+1}})}{D(q_{j+1}, m)} = \frac{M(x)r_{\beta_j}(\alpha_i^q)^{c(x)}L(x, q_{j})R(x, q_{j})}{D(q_{j+1}, m)}
\]
\[
= \frac{M(x)}{D(q_{j+1}, m)} \frac{r_{\beta_j}D(q_j, m)Y_j(x, m)}{D_j}
\]
\[
= \frac{M(x)Y_{j+1}(x, m)}{D(q_{j+1}, m)} \frac{r_{\beta_j}D(q_j, m)}{D_j}.
\]

It remains to show
\[
(-1)^{q_{j+1}-q_j-1} \frac{D(q_{j+1}, m)}{D_{j+1}} = r_{\beta_j} \frac{D(q_j, m)}{D_j}.
\]

We have \(D(q_j, m) = \prod_{k=0}^{q_j} \alpha_k^q \prod_{k=q_j+1}^{m-1} \alpha_k^{q_j+1}\). For \(k \in [0, q_j]\) we have \(r_{\beta_j} \alpha_k^q = \alpha_k^{q_j+1}\). For \(k \in [q_j+1, q_j+1-1]\) we have \(r_{\beta_j} \alpha_k^{q_j} = -\alpha_k^{q_j+1}\), \(r_{\beta_j} \alpha_k^{q_j+1} = -\alpha_k^{q_j+1}\), and for \(k \in [q_j+1, m-1]\) we have \(r_{\beta_j} \alpha_k^{q_j+1} = \alpha_k^{q_j+1+1}\). Therefore
\[
\frac{r_{\beta_j}D(q_j, m)}{D_{j+1}} = (-1)^{q_{j+1}-q_j} \prod_{k=0}^{q_j} \alpha_k^{q_j+1} \prod_{k=q_j}^{q_j+1-1} \alpha_k^{q_j+1} \prod_{k=q_j+1}^{m-1} \alpha_k^{q_j+1+1}
\]
\[
= (-1)^{q_{j+1}-q_j} D(q_{j+1}, m).
\]

We also have \(r_{\beta_j} \beta^{-i}_{j-1} = \beta^i_j\) for \(1 \leq i \leq j - 1\) and \(r_{\beta_j} \beta^i_j = \beta^i_{j+1}\) for \(j + 1 \leq i \leq p - 1\). Therefore
\[
r_{\beta_j} D_j = \left( \prod_{i=1}^{j-1} \beta^2_i \right) (-\beta_j) \left( \prod_{i=j+1}^{p-1} \beta^i_{j+1} \right) = -D_{j+1}.
\]

Lemma 23. \(r_{\beta_j} Y_i(x, m) = Y_i(x, m)\) for \(j \geq i + 2\).

5.5. The equivariant Pieri rule. For a root \(\beta\) and \(f \in S\) define
\[
\partial_\beta f = \beta^{-1}(f - r_\beta f).
\]

Theorem 24.
\[
\beta^x = (-1)^{f(x)-m+p-1} M(x) \partial_{\beta_{p-1}} \cdots \partial_{\beta_2} \partial_{\beta_1} Y(x, m)
\]
where \(Y(x, m) = Y_1(x, m)\).

Proof. Note that if \(r_{\beta_{j+1}} Y = Y\) and \(i \leq j\) then
\[
\frac{1}{\beta_{j+1}}(1 - r_{\beta_{j+1}}) \frac{Y}{\beta^i_j \beta^i_{j+1}} = \frac{Y}{\beta^i_j \beta^i_{j+1}}.
\]
\[
\begin{align*}
\partial_{\beta_{p-1}} \cdots \partial_{\beta_2} \partial_{\beta_1} Y(x, m) \\
&= \frac{1}{\beta_{p-1}} (1 - r_{\beta_{p-1}}) \cdots \frac{1}{\beta_1} (1 - r_{\beta_1}) Y_1(x, m) \\
&= \frac{1}{\beta_{p-1}} (1 - r_{\beta_{p-1}}) \cdots \frac{1}{\beta_2} (1 - r_{\beta_2}) \left( \frac{Y_1(x, m)}{\beta_1} - \frac{Y_2(x, m)}{\beta_1} \right) \\
&= \frac{1}{\beta_{p-1}} (1 - r_{\beta_{p-1}}) \cdots \frac{1}{\beta_3} (1 - r_{\beta_3}) \left( \frac{Y_1(x, m)}{\beta_1 \beta_2^2} - \frac{Y_2(x, m)}{\beta_1 \beta_2} + \frac{Y_3(x, m)}{\beta_2^3 \beta_3} \right) \\
&= \cdots \\
&= \frac{Y_1(x, m)}{\beta_1 \beta_2^2 \cdots \beta_{p-1}^p} - \frac{Y_2(x, m)}{\beta_1 \beta_2 \beta_3 \cdots \beta_{p-1}^p} + \cdots + (-1)^j \frac{Y_j(x, m)}{\beta_1^j \cdots \beta_{j-1}^j \beta_{j+1}^j \cdots \beta_{p-1}^p} \frac{Y_{j+1}(x, m)}{\beta_{j+1}^j \cdots \beta_{p-1}^p} \\
&\quad + \cdots + (-1)^{p-1} \frac{Y_p(x, m)}{\beta_1^{p-1} \cdots \beta_{p-2}^p \beta_{p-1}^p}.
\end{align*}
\]

Thus

\[
M(x) \partial_{\beta_{p-1}} \cdots \partial_{\beta_2} \partial_{\beta_1} Y(x, m) = \sum_{j=1}^{p} (-1)^{j-1} \frac{M(x) Y_j(x, m)}{D_j} \\
= (-1)^{m-p} \sum_{j=1}^{p} (-1)^{q_j} \frac{\xi^x(t_{q_j})}{D(q_j, m)} \\
= (-1)^{m-p} \sum_{i=0}^{m-2} (-1)^{i} \frac{\xi^x(t_i)}{D(i, m)} \\
= (-1)^{m-p+1} (-1)^{\ell(x)} \frac{x}{\beta_{m-n}}
\]

by (99), as required. \(\square\)

5.6. Positivity. In this section we explain how the expression of Theorem 24 can be written explicitly as the sum of products of positive roots.

To see this we first count the gratuitous negative signs in \(M(x) = M(x, q_1)\) and \(Y(x, m) = Y_1(x, m)\). Letting \(q = q_1\), using the \(q_1\)-factorization of \(x\), and defining \(S_{2'} = S_{2'} \cap [m, n-1]\), this number is

\[
\begin{align*}
\epsilon(x, q) &= |S_2| + |S_1'| + c(x) - 1 + |S_{2'}| \\
&= |S_2| + |S_1'| + |S_3| + |S_{3'}| - 1 + |S_{2'}| \\
&= \ell(x) - 1 - |S_1| - |S_{2'} \setminus S_{2'}| \\
&= \ell(x) - 1 - q_1 - |\{q_1 + 1, m - 1\} \setminus \{q_2, q_3, \ldots, q_p\}| \\
&= \ell(x) - 1 - q_1 - (m - 1 - q_1 - (p - 1)) \\
&= \ell(x) - m + p - 1.
\end{align*}
\]
Therefore all signs cancel and we have that

\begin{equation}
    \hat{f}_x^{\tau_m} = (\alpha_{q+1}^{n-1})^{(x,q)} \prod_{k \in S_{1}} \alpha_{q+1}^{n-1} \prod_{k \in S_{1}'} \alpha_{q+1}^{n-1} \partial_{\beta_{p-1}} \cdots \partial_{\beta_{1}} (\alpha_{q+1}^{n-1})^{(x-1)} \prod_{k \in S_{2}'} \alpha_{q+1}^{k}.
\end{equation}

Let \( x_i \) be the standard basis of the finite weight lattice \( \mathbb{Z}^n \) with \( \alpha_i = x_i - x_{i+1} \). Then \( r_{\beta_j} \) acts by exchanging \( x_{q_j+1} \) and \( x_{q_j+1} \). Let us write

\[ Z = \alpha_{q+1}^{c(x-1)} \prod_{k \in S_{2}'} \alpha_{q+1}^{k} = \alpha_{q+1}^{k_1} \alpha_{q+1}^{k_2} \cdots \alpha_{q+1}^{k_d} = \prod_{i=1}^{d} (x_{q+1} - x_{k_1+1}). \]

where \( n - 1 \geq k_1 \geq k_2 \geq \cdots \geq k_d \geq m \). Note that \( q_j + 1 \leq q_p + 1 \leq m \). Since

\[ \partial_{i} \cdot (f g) = (\partial_{i} \cdot f) g + (r_{i} \cdot f)(\partial_{i} \cdot g), \]

and since \( \partial_{1} = 0 \), we have

\[ \partial_{\beta_1} Z = (\partial_{\beta_1} \cdot (x_{q+1} - x_{k_1+1}))(x_{q+1} - x_{k_2+1}) \cdots (x_{q+1} - x_{k_1}). \]

It follows that Theorem 24 yields a positive answer which can be given an explicit formula, which is determined by finding the minimum \( q \) for which \( x \leq t_q \), deriving the quantities \( c(x), \epsilon(x, q_1), S_2, S_1', S_2' \) (where only the last depends on \( m \)), and then writing down the above sum of products of positive roots.

**Example 25.** Let \( n = 8, m = 4 \), and \( x = r_0 r_4 r_5 r_7 r_4 r_2 r_1 \). The components of \( \text{Supp}(x) \) are \( [0, 2], [4, 5], \) and \( [7] \). We have \( p = 3 \) with \( (q_1, q_2, q_3) = (0, 2, 3), v_1 = t_0, y_1 = r_2 r_1, \)
\( y_2 = r_4 r_5 r_4, y_3 = 7, \epsilon(x, q_1) = 1, S_1 = S_2 = \emptyset, S_3 = \{0\}, S_1' = \{4\}, S_2' = \{1, 4\} \),
\[ S3' = \{2, 5, 7\}, \quad S2' \cap [m, n - 1] = \{4\}. \] Then \((50)\) yields
\[
j_{\tilde{\sigma}_m}^\chi = (\alpha_1^7) \alpha_2^5 \partial_{\alpha_3} \partial_{\alpha_1 + \alpha_2} (\alpha_1^7)^2 \alpha_1^4
\]
\[
= (x_1 - x_8)(x_5 - x_8) \partial_{x_3 - x_4} \partial_{x_1 - x_3} (x_1 - x_8)^2 (x_1 - x_5)
\]
\[
= (x_1 - x_8)(x_5 - x_8) \partial_{x_3 - x_4} ((x_1 - x_8)(x_1 - x_5) + (x_3 - x_8)(x_1 - x_5) + (x_3 - x_8)^2)
\]
\[
= (x_1 - x_8)(x_5 - x_8)((x_1 - x_5) + (x_3 - x_8) + (x_4 - x_8))
\]
\[
= (\alpha_1^7)(\alpha_2^5)(\alpha_1^4 + \alpha_3^7 + \alpha_4^7).
\]

**Appendix A. Proof of Theorem 14**

In this section we assume that \(G = SL_n\) and prove \((34)\).

The matrices \(M\) and \(N\) are easily seen to be lower triangular. We first check the diagonal:
\[
M_{pp}, N_{pp} = (-1)^p \xi^{\sigma_p}(\sigma_p) \xi^{\hat{\sigma}_p r_{\hat{\theta}}}(\hat{\sigma}_p r_{\hat{\theta}})
\]
\[
= \xi^{\sigma_p}(\sigma_p) (\hat{\sigma}_p r_{\hat{\theta}}) \cdot (r_{\hat{\theta}}^{-1} \hat{\sigma}_p^{-1})
\]
\[
= \xi^{\sigma_p}(\sigma_p) (\sigma_p \cdot (r_{\hat{\theta}}^{-1} \hat{\sigma}_p^{-1})
\]
\[
= \xi^{r_{\hat{\theta}}^{-1}}(t_{p-1}),
\]
by \((2), (4),\) and Lemma \(1\).

It remains to check below the diagonal. Let \(p > q\) and \(p \geq k \geq q\). We have
\[
M_{pk} = (-1)^k \xi^{\sigma_k}(\sigma_p)
\]
\[
= (-1)^k d_{k}^{p-1} \cdot \xi^{\sigma_k}(\sigma_k)
\]
\[
= (-1)^k d_{k}^{p-1} \cdot (\xi^{d_{k}^{p-1}}(d_{q}^{k-1}) \xi^{\sigma_q}(\sigma_q))
\]
\[
= (-1)^k (d_{k}^{p-1} \cdot \xi^{d_{k}^{p-1}}(d_{q}^{k-1}) \xi^{\sigma_q}(\sigma_q)).
\]

Note that the second factor is independent of \(k\). We also have
\[
N_{kq} = \xi^{\hat{\sigma}_k r_{\hat{\theta}}}(\hat{\sigma}_q r_{\hat{\theta}})
\]
\[
= u_{k}^{p-1} \cdot (\xi^{d_{k}^{p-1}}(u_{k}^{p-1} \cdot \xi^{\sigma_k r_{\hat{\theta}}}(\hat{\sigma}_q r_{\hat{\theta}})))
\]
\[
= (u_{k}^{p-1} \cdot \xi^{d_{k}^{p-1}}(u_{a}^{p-1} \cdot (u_{k}^{p-1} \cdot \xi^{\sigma_k r_{\hat{\theta}}}(\hat{\sigma}_q r_{\hat{\theta}})))
\]
with the second factor independent of \(k\). Therefore, to prove that
\[
\sum_{q \leq k \leq p} M_{pk} N_{kq} = 0
\]

it is equivalent to show that
\[
(51) \quad 0 = \sum_{q \leq k \leq p} (-1)^k (d_{k}^{p-1} \cdot \xi^{d_{k}^{p-1}}(d_{q}^{k-1}) \cdot (u_{k}^{p-1} \cdot \xi^{d_{k}^{p-1}}(u_{k}^{p-1})).
\]
The above identity can be rewritten as

$$0 = \sum_{q \leq k \leq p} (-1)^k \prod_{i=q}^{k-1} \alpha_i^{p-1} \prod_{m=k}^{p-1} \alpha_m^q.$$  

To prove this last identity, let $q'$ be such that $q < q' \leq p$. It is easy to show by descending induction on $q'$ that

$$\sum_{q' \leq k \leq p} (-1)^k \prod_{i=q}^{k-1} \alpha_i^{p-1} \prod_{m=k}^{p-1} \alpha_m^{q} = (-1)^q' \prod_{i=q+1}^{q'} \alpha_i^{p-1} \prod_{m=q'}^{p-1} \alpha_m^q.$$  

Then for $q' = q + 1$ the sum is the negative of the $k = q$ summand of (52) as required.

**APPENDIX B. EXAMPLES OF (34)**

**Example 26.** $G = SL_3$ has affine Cartan matrix

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$  

The column dependencies give the coefficients of the null root $\delta = \alpha_0 + \theta = \alpha_0 + \alpha_1 + \alpha_2$ which is set to zero due to the finite torus equivariance.

| $p$ | $\tilde{\sigma}_p$ | $\sigma_p$ | $t_{p-1}$ | $\tilde{\sigma}_p^r \theta$ |
|-----|--------------------|-------------|------------|--------------------------|
| 1   | $\text{id}$       | $r_0$      | $r_0r_1r_2r_1$ | $r_1r_2r_1$ |
| 2   | $r_1$             | $r_1r_0$   | $r_1r_0r_1r_2$ | $r_2r_1$ |

We compute the matrices

$$M = \begin{pmatrix} \alpha_1 + \alpha_2 & 0 \\ \alpha_2 & -\alpha_1 \alpha_2 \end{pmatrix} \quad N = \begin{pmatrix} \alpha_1\alpha_2(\alpha_1 + \alpha_2) & 0 \\ \alpha_2(\alpha_1 + \alpha_2) & \alpha_2(\alpha_1 + \alpha_2) \end{pmatrix}$$

$$D = \begin{pmatrix} \alpha_1\alpha_2(\alpha_1 + \alpha_2)^2 & 0 \\ 0 & -\alpha_1\alpha_2^2(\alpha_1 + \alpha_2) \end{pmatrix} \quad ND^{-1} = \begin{pmatrix} (\alpha_1 + \alpha_2)^{-1} & 0 \\ (\alpha_1(\alpha_1 + \alpha_2))^{-1} & -(\alpha_1 \alpha_2)^{-1} \end{pmatrix}$$

For $x = r_1r_2$ we compute the column vector with values $(-1)^{j}(x)\xi^x(t_j)$ for $j = 1, 2$. Acting on this column vector by $ND^{-1}$, we obtain the coefficients of $A_x$ in $j_1$ and $j_2$.

$$(-1)^{j}(x)\xi^x(t_1) = \begin{pmatrix} \alpha_2(\alpha_1 + \alpha_2) \\ \alpha_2^2 \end{pmatrix} \quad \begin{pmatrix} j_1^{x} \\ j_2^{x} \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ 0 \end{pmatrix}.$$  

Doing the same thing for $x = r_1r_0r_2$ we have

$$(-1)^{j}(x)\xi^x(t_1) = \begin{pmatrix} 0 \\ -\alpha_1 \alpha_2 \end{pmatrix} \quad \begin{pmatrix} j_1^{x} \\ j_2^{x} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix}.$$  

**Example 27.** $Sp_{2n}$ for $n = 2$ has affine Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$
We have $\delta = \alpha_0 + \theta = \alpha_0 + 2\alpha_1 + \alpha_2$.

| $p$ | $\sigma_p$ | $\sigma_p$ | $t_{p-1}$ | $\sigma_p\theta$ |
|-----|-------------|-------------|------------|-----------------|
| 1   | id          | $r_0$       | $r_0r_1r_2r_1$ | $r_1r_2r_1$ |
| 2   | $r_1$       | $r_1r_0$    | $r_1r_0r_1r_2$ | $r_2r_1$ |
| 3   | $r_2r_1$    | $r_2r_1r_0$ | $r_2r_1r_0r_1$ | $r_1$  |

We have

$$M = \begin{pmatrix} 2\alpha_1 + \alpha_2 & 0 & 0 \\ \alpha_2 & -\alpha_1\alpha_2 & 0 \\ -\alpha_2 & \alpha_2(\alpha_1 + \alpha_2) & -\alpha_2^2(\alpha_1 + \alpha_2) \end{pmatrix}$$

$$N = \begin{pmatrix} \alpha_1(\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2) & 0 & 0 \\ (\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2) & \alpha_2(\alpha_1 + \alpha_2) & 0 \\ 2\alpha_1 + \alpha_2 & \alpha_1 + \alpha_2 & \alpha_1 \end{pmatrix}$$

$$D = \begin{pmatrix} \alpha_1(\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2)^2 & 0 & 0 \\ 0 & -\alpha_1\alpha_2^2(\alpha_1 + \alpha_2) & 0 \\ 0 & 0 & -\alpha_1\alpha_2^2(\alpha_1 + \alpha_2) \end{pmatrix}$$

$$ND^{-1} = \begin{pmatrix} (2\alpha_1 + \alpha_2)^{-1} & 0 & 0 \\ (\alpha_1(\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2))^{-1} & -\alpha_1\alpha_2^{-1} & 0 \\ (\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2))^{-1} & -\alpha_2(\alpha_1 + \alpha_2)^{-1} & -\alpha_2^2(\alpha_1 + \alpha_2)^{-1} \end{pmatrix}$$

Now let $x = r_0r_1r_2$. We have

$$(-1)^{t(x)} \begin{pmatrix} \xi^x(t_1) \\ \xi^x(t_2) \\ \xi^x(t_3) \end{pmatrix} = \begin{pmatrix} (\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2)^2 \\ \alpha_2^2(\alpha_1 + \alpha_2) \\ 0 \end{pmatrix}$$

The matrix $ND^{-1}$ acting on the above column vector, gives the vector

$$\begin{pmatrix} j_{\sigma_1}^x \\ j_{\sigma_2}^x \\ j_{\sigma_3}^x \end{pmatrix} = \begin{pmatrix} (\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2) \\ 2(\alpha_1 + \alpha_2) \\ 1 \end{pmatrix}$$

Now let $x = r_1r_2r_1$. We have

$$(-1)^{t(x)} \begin{pmatrix} \xi^x(t_1) \\ \xi^x(t_2) \\ \xi^x(t_3) \end{pmatrix} = \begin{pmatrix} \alpha_1(\alpha_1 + \alpha_2)(2\alpha_1 + \alpha_2) \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} j_{\sigma_1}^x \\ j_{\sigma_2}^x \\ j_{\sigma_3}^x \end{pmatrix} = \begin{pmatrix} \alpha_1(\alpha_1 + \alpha_2) \\ (\alpha_1 + \alpha_2) \\ 1 \end{pmatrix}$$

**Example 28.** $SO_{2n+1}$ for $n = 3$ has affine Cartan matrix

$$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$
We have $\delta = \alpha_0 + \theta = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3$.

| $p$ | $\hat{\sigma}_p$ | $\sigma_p$ | $\ell_{p-1}$ | $\hat{\sigma}_p r \theta$ |
|-----|----------------|-------------|--------------|------------------|
| 1   | id             | $\sigma_1$  | $r_0^2 r_3 r_1^2 r_3^2$ | $r_2 r_3 r_1^2 r_3 r_2$ |
| 2   | $r_2$          | $r_2 \sigma_1$ | $r_2 r_0^2 r_3 r_1^2 r_3^2$ | $r_3 r_2 r_1^2 r_3 r_2$ |
| 3   | $r_3 r_2$      | $r_3 r_2 \sigma_1$ | $r_3 r_2 r_0^2 r_3 r_1^2 r_3^2$ | $r_2 r_3 r_1^2 r_3 r_2$ |
| 4   | $r_2 r_3 r_2$  | $r_2 r_3 r_2 \sigma_1$ | $r_2 r_3 r_2 r_0^2 r_3 r_1^2 r_3^2$ | $r_1 r_2 r_3 r_2$ |
| 5   | $r_0 r_2 r_3 r_2$ | $r_0 r_2 r_3 r_2 \sigma_1$ | $r_0 r_2 r_3 r_2 r_0^2 r_3 r_1^2 r_3^2$ | $r_2 r_3 r_2$ |

To save space let us write $\alpha_{ijk} := i\alpha_1 + j\alpha_2 + k\alpha_3$. We have

$$M = \begin{pmatrix} \alpha_{122} & -\alpha_{010}\alpha_{112} \\ \alpha_{112} & -\alpha_{110}\alpha_{112} \\ \alpha_{110} & -2\alpha_{100}\alpha_{112} \\ \alpha_{100} & \alpha_{100}\alpha_{111}\alpha_{112} \\ -\alpha_{100}\alpha_{110}\alpha_{011}\alpha_{112} \end{pmatrix}$$

$$N = \begin{pmatrix} \alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \\ \alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \\ 2\alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \\ \alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \\ \alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \\ \alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \end{pmatrix}$$

$D$ has diagonal entries

$$\begin{align*}
\alpha_{110}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \\
-\alpha_{100}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \\
\alpha_{100}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \\
-\alpha_{100}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012} \\
\alpha_{100}\alpha_{111}\alpha_{112}\alpha_{122}\alpha_{010}\alpha_{011}\alpha_{012}
\end{align*}$$

One may verify that $MN = D$.

**References**

[AJS] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p*, Astérisque No. 220 (1994), 321 pp.

[Bil] S. Billey, *Kostant Polynomials and the Cohomology Ring for $G/B$*, Duke Math. J. 96 (1999), 205–224.

[GKM1] M. Goresky, R. Kottwitz, R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. 131 (1998), no. 1, 25–83.

[GKM2] M. Goresky, R. Kottwitz, R. MacPherson, *Homology of affine Springer fibers in the unramified case*, Duke Math. J. 121 (2004) 509–561.

[Kac] V. Kac, *Infinite dimensional Lie algebras*. Third edition. Cambridge University Press, Cambridge, 1990.

[KK] B. Kostant, S. Kumar, *The nil Hecke ring and cohomology of $G/P$ for a Kac–Moody group G*, Adv. in Math. 62 (1986), no. 3, 187–237.

[Kum] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, 204, Birkhäuser Boston, Inc., Boston, MA, 2002. xvi+606 pp. ISBN: 0-8176-4227-7.

[LS] T. Lam and M. Shimozono, Dual graded graphs for Kac-Moody algebras. Algebra Number Theory 1 (2007), no. 4, 451–488.

[LS2] T. Lam and M. Shimozono, *Quantum cohomology of $G/P$ and homology of affine Grassmannian*, Acta. Math. 204 (2010), 49–90.

[LS3] T. Lam and M. Shimozono, k-Double Schur functions and equivariant (co)homology of the affine Grassmannian, preprint, arXiv:1105.2170.
[LSS] T. Lam, A. Schilling and M. Shimozono, Schubert polynomials for the affine Grassmannian of
the symplectic group. Math. Z. 264 (2010), no. 4, 765–811.

[Mih] L. Mihalcea, On equivariant quantum cohomology of homogeneous spaces: Chevalley formulae
and algorithms. Duke Math. J. 140 (2007), no. 2, 321–350.

[Pet] D. Peterson, Lecture Notes at MIT, 1997.

[Pon] S. Pon, Affine Stanley Symmetric Functions for Classical Groups, Ph. D. thesis, University of
California, Davis, 2010.

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