THE GLOBAL EXISTENCE AND LARGE TIME BEHAVIOR OF
SMOOTH COMPRESSIBLE FLUID IN AN INFINITELY
EXPANDING BALL, II: 3D NAVIER-STOKES EQUATIONS

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Abstract. We concern with the global existence and large time behavior of compressible fluids (including the inviscid gases, viscous gases, and Boltzmann gases) in an infinitely expanding ball. Such a problem is one of the interesting models in studying the theory of global smooth solutions to multidimensional compressible gases with time dependent boundaries and vacuum states at infinite time. Due to the conservation of mass, the fluid in the expanding ball becomes rarefied and eventually tends to a vacuum state meanwhile there are no appearances of vacuum domains in any part of the expansive ball, which is easily observed in finite time. In this paper, as the second part of our three papers, we will confirm this physical phenomenon for the compressible viscous fluids by obtaining the exact lower and upper bound on the density function.

1. Introduction. In this paper, we study the global existence and stability of a smooth compressible viscous fluid in a 3-D expanding ball. Precisely, the expanding ball is denoted by \( \Omega = \{ (t, x) : t \geq 0, |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} \leq R(t) \} \), where \( R(t) \in C^4[0, \infty) \) satisfies \( R(0) = 1, R'(0) = 0, R''(0) = 0, \) and \( R(t) = 1 + ht \) for \( t \geq 1 \) with some positive constant \( h \). From the expression of \( \Omega \), we know that the ball \( S_t = \{ x : |x| \leq R(t) \} \) at the time \( t \) is formed by pulling out an initial unit ball \( S_0 = \{ x : |x| \leq 1 \} \) with a smooth speed and acceleration (see Figure 1 below). Suppose that the movement of fluid in \( \Omega \) is described by the 3-D compressible Navier-Stokes equations:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho u_t + \rho u \cdot \nabla u + \nabla P(\rho) &= \mu \Delta u + (\mu + \lambda)\nabla \text{div} u,
\end{align*}
\]

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where $\rho \geq 0$ is the density, $u = (u_1, u_2, u_3)$ is the velocity, $\mu > 0$ and $\lambda$ are the shear and bulk viscosity coefficients, respectively, satisfying $\mu + \frac{2}{3} \lambda > 0$, and $P(\rho) = \rho^\gamma$ with $\gamma > 1$. In this setting, we will study the global existence and large time behavior of classical solution.

Figure 1. A viscous fluid in a 3-D expanding ball

For this, the following initial-boundary conditions on the equations (1)-(2) are assumed:

$$\begin{align*}
\rho(0,x) &= \rho_0(x), \quad u(0,x) = u_0(x), \quad \text{for } x \in S_0, \\
u(t,x) &= \frac{R'(t)x}{R(t)}, \quad \text{for } (t,x) \in \partial \Omega = \{(t,x) : t \geq 0, |x| = R(t)\},
\end{align*}$$

(3)

where $\rho_0(x) \in H^3(S_0)$, $u_0(x) \in H^3_0(S_0)$, and $\rho_0(x) > 0$ for $x \in S_0$. Note that $|u(t,x)| = R'(t)$ on $\partial \Omega$ by (3). This implies that the speed of gas on $\partial \Omega$ coincides with the expansion speed of the ball. For this initial-boundary value problem, we have the following theorem.

**Theorem 1.1.** Assume $\gamma > 1$ and set $s = \frac{3}{2}(\gamma - 1)$. If $\rho_0(x) \in H^3(S_0)$, $\nabla \rho_0(x) \in H^1_0(S_0)$, and $u_0(x) \in H^3_0(S_0)$, then there exist a constant $h_0 > 0$, and a small constant $\varepsilon_0 > 0$ depending only on $h_0$, such that when $\sup_{0 \leq t \leq 1, 1 \leq k \leq 4} |R^{(k)}(t)| + \|\rho_0(x) - 1\|_{H^3(S_0)} + \|u_0(x)\|_{H^3(S_0)} < \varepsilon_0$, $R(t) = 1 + ht$ for $t \geq 1$, and $0 < h < h_0$, the problem (1)-(2) with (3) admits a global solution $(\rho, u)$ in $\Omega$ satisfying

$$\begin{align*}
\rho(t,x) &\in C([0,\infty), H^3(S_t)) \cap C^1([0,\infty), H^2(S_t)), \\
u(t,x) &\in C([0,\infty), H^3(S_t)) \cap C^1([0,\infty), H^1(S_t)),
\end{align*}$$

(4) (5)

$$\frac{1}{2R^3(t)} \leq \rho(t,x) \leq \frac{3}{2R^3(t)} \quad \text{for } t \geq 1 \text{ and } (t,x) \in \Omega,$$

(6)
and
\[
\sup_{x \in S_t} \left( |R^s(t)(u(t,x) - \frac{hx}{R(t)})| + |R^s(t)\nabla(u(t,x) - \frac{hx}{R(t)})| \right) \to 0 \text{ as } t \to +\infty. \tag{7}
\]

A few remarks on the theorem are given as follows.

**Remark 1.** The assumptions \( R^0(0) = 0, \nabla \rho_0(x) \in H^1_0(S_0) \) and \( u_0(x) \in H^3_0(S_0) \) in Theorem 1.1 are used to derive \( u_t(0,x) \in H^3_0(S_0) \) from equation (2) for the compatibility of the initial velocity \( u_0(x) \) on the boundary \( \partial S_0 = \{x : |x| = 1\} \).

**Remark 2.** The solution obtained in Theorem 1.1 does not contain vacuum state in any finite time. The corresponding phenomena for the inviscid fluid governed by the Euler equations, and the rarefied gas governed by the Boltzmann equation were also confirmed in [31, 34] respectively.

**Remark 3.** Note that the energy dissipation rate is not integrable in time because the ball is pulling outwards continuously. That is, \( \int_0^t \int_{S_\tau} (\mu |\nabla u|^2 + (\mu + \lambda)(\nabla u^2) dxd\tau \) is not uniformly bounded because \( \int_t^{2t} \int_{S_\tau} \mu |\nabla u|^2 dxd\tau \sim \int_t^{2t} \frac{1}{R^2(\tau)} \times R^3(\tau) d\tau \sim t^2 \). This is different from those cases studied in the references [1, 2, 3, 6, 7, 8, 13, 14, 18, 19, 20, 24, 29] in various settings.

**Remark 4.** If the ball is pulled out rapidly, then it seems that the classical solution will blow up in finite time and vacuum may be formed from physical point of view. Hence, the assumption on the smallness of \( |R^k(\tau)| (1 \leq k \leq 4) \) for \( 0 \leq t \leq 1 \) and the initial perturbation in Theorem 1.1 is necessary for the global existence of the classical solution. Note that, for the compressible Euler flow, if the gas is rapidly expanding, then the vacuum state forming in finite time is confirmed in [4].

**Remark 5.** Note that \((\bar{\rho}(t,x), \bar{u}(t,x)) = \left( \frac{1}{R^3(t)}, \frac{hx}{R(t)} \right) \) with \( R(t) = 1 + ht \) is a special solution to (1)-(2) when the initial-boundary value condition (3) is neglected. Then in the case of small perturbation, the asymptotic stability of solution \((\rho(t,x), u(t,x)) \) to \((\bar{\rho}(t,x), \bar{u}(t,x)) \) is a direct consequence of Theorem 1.1.

Now let us review some related works. There have been extensive studies on the global existence and behavior of solutions to the Navier-Stokes equations in various settings. For results in one-dimensional case, see [9, 26, 32, 33] and the references therein. For multi-dimensional problems with constant viscosity coefficients, known results include the local existence of classical solution in [25] in the absence of vacuum, and the local existence of strong solutions in [1, 2, 3] when the initial density may vanish in some open sets. The global existence of classical solution was first obtained in [22, 23] for the initial data close to a non-vacuum state and then these results were generalized in other settings, for example in [5, 27]. Moreover, the author in [10, 11, 12, 13] studied the global existence with discontinuous initial data. Recently, for the case when the initial density may vanish in some region, under the smallness assumption on the total energy, the authors in [14] established the global existence and uniqueness of classical solutions. For large initial data with the finite total energy, the global existence of weak solutions was established by P.L. Lions in [20] (see also [6, 8, 15, 24]) for different assumptions on the adiabatic constant \( \gamma \). In addition, the authors in [7] obtained the global existence of weak solution to the compressible barotropic Navier-Stokes system in a time dependent...
domain with slip boundary condition. For the equations (1)-(2) (or including the energy equation) with the viscosity coefficients depending on the density, existence of weak and classical solutions to the initial value problem and the initial-boundary value problem have also been extensively studied, cf. [16, 17], [19], [21], [28], [32], [33] and the references therein.

Finally in the introduction, let us give some discussion on the proof of Theorem 1.1. In order to prove Theorem 1.1, we will take a coordinate transformation of the variable \((t,x)\) so that the moving domain \(\Omega\) becomes a fixed cylindrical domain \([0, +\infty) \times S_0\) under the new coordinates \((\tau, y) = (t, \frac{x}{R(t)})\). Correspondingly, the system (1)-(2) becomes a system consists of a first order hyperbolic equation for the density \(\rho(\tau, y)\) and three degenerate parabolic equations for the velocity \(u(\tau, y)\) with degenerate coefficients, cf. the system (9) in § 2. To obtain the local and global existence of the smooth solution to the degenerate system (9), we will choose new unknown functions \(\phi(\tau, y) = R^3(\tau)\rho(\tau, y)\) and \(v(\tau, y) = u(\tau, y) - R'(\tau)y\) instead of the density \(\rho\) and the velocity \(u\) so that the uniform positive lower and upper bounds of \(\phi\) can be derived. The proof is then based on a delicate weighted energy method. For obtaining the weighted energy estimate on \((\phi, v)\), some suitable weights will be chosen for the systems (10)-(11) in § 2 and (51)-(52) in § 3. In particular, for the system (51)-(52), we obtain a uniform weighted energy estimate on \((\phi, v)\) by careful estimation and involved analysis in the interior and boundary regions separately.

The rest of the paper is arranged as follows. In § 2, we prove the local existence of the solution to (1)-(2). Although its proof is standard, we still give some detail for readers’ easy reference. In § 3, we derive some uniform weighted inequalities by some subtle estimation. Based on the analysis in § 3, the uniform energy estimate will be given in § 4 and then the proof of Theorem 1.1 will be given in the last section.

The following notations will be used throughout this paper: \(\|f\|_2 = (\int_{S_0} |f|^2 dy)^{\frac{1}{2}}\), \(\int_0^t \|V(\tau)\| d\tau = \int_0^t \|V(\tau, \cdot)\| d\tau\), and \(Dg\) represents \(\partial_k g\) for any \(k = 1, 2, 3\).

2. Local existence. To study the solvability of the problem (1)-(2) with (3), we at first transform the variables \((t,x)\) such that the moving domain \(\Omega\) becomes a fixed cylindrical domain \([0, +\infty) \times S_0\). For this, set

\[
\begin{cases}
\tau = t, \\
y = \frac{x}{R(t)},
\end{cases}
\]

(8)

Then the system (1)-(2) in the coordinates \((\tau, y) \in [0, \infty) \times S_0\) takes the form of

\[
\begin{cases}
\rho_\tau - \frac{R'}{R} y \cdot \nabla \rho + \frac{1}{R} \text{div}(\rho u) = 0, \\
(\rho u)_\tau - \frac{R'}{R} y \cdot \nabla(\rho u) + \frac{1}{R} \text{div}(\rho u \otimes u) + \frac{1}{R} \nabla P(\rho) = \frac{1}{R^2} (\mu \Delta u + (\mu + \lambda)\nabla \text{div} u).
\end{cases}
\]

(9)

From now on, \(\nabla\) and \(\text{div}\) are taken with respect to the variable \(y\).

Let \(\phi = R^3\rho\) and \(v = u - R'(t)y\). Then (9) becomes

\[
\begin{align*}
\phi_\tau + \frac{1}{R} \text{div}(\phi v) &= 0, \\
\phi v_\tau + \phi R''y + \phi v \cdot \frac{1}{R} \nabla v + \frac{1}{R} \phi R'v + R^2 \nabla P(\rho) &= R(\mu \Delta u + (\mu + \lambda)\nabla \text{div} u),
\end{align*}
\]

(10)
And the initial-boundary condition (3) is given by
\[
\begin{aligned}
\phi(0, y) &= \rho_0(y), \quad v(0, y) = u_0(y), \quad y \in S_0, \\
v(\tau, y) &= 0 \quad \text{on } [0, \infty) \times \partial S_0.
\end{aligned}
\]  
(12)

We now prove the following local existence result on the problem (10)-(11) with (12).

**Theorem 2.1.** Under the assumptions of Theorem 1.1, there exists some constant \( T^* > 0 \) such that system (10)-(11) with (12) has a unique solution \((\phi, v)\) which satisfies
\[
\begin{cases}
\phi \in C([0, T^*], H^3(S_0)) \cap C^1([0, T^*], H^2(S_0)), \\
v \in C([0, T^*], H^3(S_0)) \cap C^1([0, T^*], H^2(S_0)) \cap L^2([0, T^*], H^4(S_0)).
\end{cases}
\]

Moreover, \( \phi \geq C > 0 \) holds for \((\tau, y) \in [0, T^*] \times S_0\).

**Remark 6.** Since problem (10)-(11) with (12) is equivalent to problem (1)-(2) with (3), by Theorem 2.1 we know that problem (1)-(2) with (3) has a local solution \((\rho, u)\) with \( \rho \in C([0, T^*], H^3(S_0)) \cap C^1([0, T^*], H^2(S_0)), \) \( u \in C([0, T^*], H^3(S_0)) \cap C^1([0, T^*], H^2(S_0)) \cap L^2([0, T^*], H^4(S_0)),\) and \( \rho(t, x) \geq C > 0 \) for \((t, x) \in \{(t, x) : 0 \leq t \leq T^*, |x| \leq R(t)\}\).

**Remark 7.** Without loss of generality and based on the smallness on \( h \) and the perturbation, we choose \( T^* > 1 \) so that \( \sup_{0 \leq \tau \leq T^*} (||\phi - 1||_{H^3} + ||v||_{H^3}) \) is still small when the number \( h \) and \( \sup_{0 \leq \tau \leq 1, 1 \leq k \leq 4} |R^{(k)}(t)| + ||\phi_0 - 1||_{H^3} + ||u_0||_{H^3} \) are small enough.

To prove Theorem 2.1, we first state some useful estimates on linear parabolic equations with suitable initial-boundary conditions, cf. [28].

**Lemma 2.2.** Consider the following linear parabolic equation system
\[
\begin{align*}
R^{-1}(\tau)\phi_{\tau} - (\mu \Delta u + (\mu + \lambda)\nabla \text{div} u) &= F \quad \text{in } [0, T] \times S_0, \\
u(0, y) &= u_0(y) \quad \text{for } y \in S_0, \\
u(\tau, y) &= 0 \quad \text{for } (\tau, y) \in [0, T] \times \partial S_0,
\end{align*}
\]
(13)
where \( R(\tau) \) is defined above, \( u = (u_1, u_2, u_3), \) \( F(\tau, y) = (F_1, F_2, F_3)(\tau, y), \) \( \phi(\tau, y) = \tilde{\rho}(\tau, y)R^3(\tau) \) is a given scalar function defined in \([0, T] \times S_0\) satisfying
\[
\begin{align*}
\phi &\in C([0, T], H^3), \quad \phi_{\tau} \in C([0, T], H^2), \\
c < \phi < c^{-1} &\quad \text{for } (\tau, y) \in [0, T] \times S_0 \text{ and some constant } c > 0.
\end{align*}
\]

Then we have
(i) if \( u_0 \in H^3_0 \) and \( F \in L^2([0, T], L^2) \), (13) has a unique strong solution \( u \) such that
\[
\begin{align*}
u \in C([0, T], H^3_0) \cap L^2([0, T], H^2) \quad \text{and} \quad u_{\tau} \in C([0, T], H^{-1}).
\end{align*}
\]
(ii) if \( u_0 \in H^3_0 \cap H^2, \) \( F \in L^\infty([0, T], L^2) \) and \( F_\tau \in L^2([0, T], H^{-1}), \) the solution \( u \) of (13) satisfies
\[
\begin{align*}
u \in L^\infty([0, T], H^2) \cap C([0, T], H^3_0), \quad u_{\tau} \in L^2([0, T], H^1_0) \quad \text{and} \quad u_{\tau\tau} \in L^2([0, T], H^{-1}).
\end{align*}
\]
(iii) if \( u_0 \in H^3_0 \cap H^3, \) \( F \in L^\infty([0, T], H^1) \), \( F_\tau \in L^2([0, T], L^2) \) and \( u_{\tau}(0, x) \in H^3_0 \), then the solution \( u \) of (13) satisfies
\[
\begin{align*}
u \in L^\infty([0, T], H^3) \cap C([0, T], H^3_0),
\end{align*}
\]
Moreover, if in addition $F \in L^2([0, T], H^2)$ is assumed, then
\[ u \in L^2([0, T], H^4) \text{ and } u \in C([0, T], H^3). \]

To solve (10)-(11) with (12), we will study the following linearized problem for \((\tau, y) \in [0, T] \times S_0:\)
\[
\begin{align*}
\Phi_{	au} + \frac{1}{R} \text{div}(\Phi \tilde{V}) &= 0, \quad (14) \\
\Phi V_{\tau} - RLV &= F_0(\Phi, \nabla \Phi), \quad (15) \\
\Phi(0, y) = \rho_0(y), \quad V(0, y) = u_0(y); \quad V(\tau, y) = 0 \quad \text{on} \quad [0, T] \times \partial S_0, \quad (16)
\end{align*}
\]

where
\[
LV = \mu \Delta V + (\mu + \lambda) \nabla \text{div} V,
\]
\[
F_0(\Phi, \nabla \Phi) = -\Phi R''y - \Phi \tilde{V} \cdot \frac{1}{R} \nabla \tilde{V} - \frac{1}{R} \Phi R' \tilde{V} - R^2 \nabla P(\rho), \quad \rho = \frac{\Phi}{R^3(\tau)},
\]
and \(\tilde{V}(\tau, y)\) is a given vector function defined in \([0, T] \times S_0\) such that
\[
\tilde{V} \in C([0, T], H^1_0 \cap H^3) \cap L^2([0, T], H^4), \quad \tilde{V}_\tau \in L^\infty([0, T], H^1_0) \cap L^2([0, T], H^2). \quad (17)
\]
For the convenience of statement, we assume \(T \leq 1\) at this moment.

**Lemma 2.3.** Suppose that \(\tilde{V}\) satisfies (17). Then there exists a unique solution \((\Phi, V)\) to problem (14)-(16) such that
\[
\begin{align*}
\Phi \in C([0, T], H^3), \quad \Phi_{\tau} \in C([0, T], H^2), \\
c^{-1} \leq \Phi \leq c \quad \text{for} \quad (\tau, y) \in [0, T] \times S_0 \text{ and some constant } c > 0, \\
V \in C([0, T], H^1_0 \cap H^3) \cap L^2([0, T], H^4), \quad V_{\tau} \in C([0, T], H^1_0) \cap L^2([0, T], H^2), \\
V_{\tau \tau} \in L^2([0, T], L^2).
\end{align*}
\]

**Proof.** By using the characteristics method for the first order scalar equation (14) and noting \(\tilde{V}(\tau, y) \in C([0, T], H^1_0 \cap H^3)\), one can easily obtain all the estimates of \(\Phi\) in Lemma 2.3. In addition, it is noted that \(V_{\tau}(0, y) = -R''(0)y - \tilde{V}(0, y) \cdot \nabla \tilde{V}(0, y) - \rho_0^{-1}(\nabla P(\rho_0) + Lu_0) \in H^2_0\) holds due to \(R''(0) = 0, \nabla \rho_0(y) \in H^2, u_0(y) \in H^1\) and \(\tilde{V}(0, y) \in H^1_0\), then it follows from Lemma 2.2 that all the estimates of \(V\) can be obtained.

To derive some uniform bounds on the solution \((\Phi, V)\) to (14)-(16), assume
\[
\begin{align*}
0 \leq \tau \leq T^*, \quad &\sup_{0 \leq \tau \leq T^*} \|\tilde{V}(\tau)\|_{H^1_0} + \int_0^{T^*} \|\tilde{V}(\tau)\|_{H^2}^2 d\tau \leq c_1, \\
0 \leq \tau \leq T^*, \quad &\sup_{0 \leq \tau \leq T^*} \|\tilde{V}(\tau)\|_{H^2} + \int_0^{T^*} (\|\tilde{V}(\tau)\|_{H^1_0}^2 + \|\tilde{V}(\tau)\|_{H^3}^2) d\tau \leq c_2, \\
0 \leq \tau \leq T^*, \quad &\sup_{0 \leq \tau \leq T^*} (\|\tilde{V}(\tau)\|_{H^1_0} + \|\tilde{V}(\tau)\|_{H^3}) + \int_0^{T^*} (\|\tilde{V}(\tau)\|_{H^2}^2 + \|\tilde{V}(\tau)\|_{H^4}^2) d\tau \leq c_3,
\end{align*}
\]

The proof is completed.

\[ \square \]
hold for some constants $1 < c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4$, $T^* \in (0, T)$ to be determined later. In this section, $C$ represents a generic positive constant depending only on the fixed constants $T$ and $\|R(t)\|_{L^\infty[0, T]}$. In addition, $M(\cdot)$ denotes a continuous increasing function, which maps $[1, \infty)$ into $[1, \infty)$. Under the conditions in (18), if $\Phi$ is a solution to (14), then we have

**Lemma 2.4.** If we define $T_1 = \min\{T^*, (1 + c_4)^{-1}\}$, then for $\tau \in [0, T_1]$, we have

\[
\begin{cases}
\|\Phi(\tau)\|_{H^3} \leq C_0, & \|\Phi_\tau(\tau)\|_{H^1} \leq Cc_1^2, \\
\|\Phi_\tau(\tau)\|_{H^2} \leq Cc_4^4, & \inf \Phi(\tau) \geq C^{-1} \inf \rho_0.
\end{cases}
\]

(19)

**Proof.** By equation (14) and the standard energy estimates for the hyperbolic equations, we have that, for $0 \leq \tau \leq T$,

\[
\begin{cases}
\|\Phi(\tau, x)\|_{H^3} \leq \|\rho_0\|_{H^3} \exp[\int_0^\tau \|\nabla \Phi(\tau)\|_{H^2} ds], \\
\inf \Phi(\tau) \geq \inf \rho_0 \exp[-C \int_0^\tau \|\nabla \Phi(\tau)\|_{H^2} ds].
\end{cases}
\]

(20)

Note that for $\tau \leq T^*$,

\[
\int_0^\tau \|\nabla \Phi(\tau)\|_{H^2} ds \leq \tau^{1/2} (\int_0^\tau \|\nabla \Phi(\tau)\|_{H^2} ds)^{1/2} \leq (1 + c_4)\tau^{1/2}.
\]

(21)

Substituting (21) into (20) and applying $\Phi_\tau = -\frac{1}{R} \Phi V_\tau$ yield (19).

**Lemma 2.5.** Set $T_2 = \min\{T_1, (1 + c_4^2)^{-1}, (1 + c_3^4)^{-1}\}$, where $T_1$ is given in Lemma 2.4. Then we have that the solution $V$ of (15) satisfies for $\tau \in [0, T_2]$,

\[
\|V(\tau)\|_{H^3}^2 + \int_0^\tau (\|V_\tau(\tau)\|_{H^2}^2 + \|V_\tau(\tau)\|_{L^2}^2) ds \leq M(c_0),
\]

(22)

and

\[
\|V_\tau(\tau)\|_{L^2}^2 + \|V_\tau(\tau)\|_{H^2}^2 + \int_0^\tau (\|\nabla V_\tau(\tau)\|_{L^2}^2 + \|V(\tau)\|_{H^3}^2) ds \leq M(c_1)c_2^3c_3^4.
\]

(23)

**Proof.** It follows from $\int_{S_0} (15) \cdot R^{-1} V_\tau dy$ that

\[
\int_{S_0} R^{-1} \Phi |V_\tau|^2 dy + \frac{1}{2} \frac{d}{d\tau} \int_{S_0} (\mu |\nabla V|^2 + (\mu + \lambda)(\Phi V)^2) dy 
\leq C \int_{S_0} (|\Phi V_\tau|^2 + |\Phi V_\nu V_\nu|^2 + |\Phi V V_\nu|^2 + |\nabla \rho V_\nu|^2) dy.
\]

(24)

This, together with the positivity of $R(\tau)$ and $\Phi$ (the positivity of $\Phi$ comes from Lemma 2.4), yields

\[
\|V(\tau)\|_{H^3}^2 + \int_0^\tau \|V_\tau\|_{L^2}^2 ds \leq M(c_0),
\]

(25)

where $\tau \leq T_2$.

On the other hand, we have

\[
\begin{cases}
LV = R^{-1} (\Phi V_\tau - F_0(\Phi, \nabla \Phi)), \\
V(\tau) \in H^1_0(S_0).
\end{cases}
\]

(26)
It follows from (26) and the regularity theory on the second order elliptic equations with the Dirichlet boundary condition that for $\tau \leq T_2$,
\[
\int_0^\tau \|V(s)\|_{H^2}^2 ds \leq M(c_0). \tag{27}
\]
Combining (27) and (25) yields (22).

Next we prove (23). Differentiating $(R^{-1} \times (15))$ with respect to $\tau$ yields
\[
R^{-1}\Phi V_{\tau\tau} - (LV)_{\tau} = -(R^{-1}\Phi)_{\tau} V_{\tau} + \partial_\tau (R^{-1}F_0(\Phi, \nabla \Phi)). \tag{28}
\]
Multiplying both sides of (28) by $V_{\tau}$ and integrating over $S_0$ give
\[
\frac{1}{2} \frac{d}{d\tau} \int_{S_0} R^{-1}\Phi |V_{\tau}|^2 dy + \int_{S_0} (\mu|\nabla V_{\tau}|^2 + (\mu + \lambda)(\text{div} V_{\tau})^2) dy \leq C|\int_{S_0} (R^{-1}\Phi)_{\tau} |V_{\tau}|^2 dy| + |\int_{S_0} \partial_\tau (R^{-1}F_0(\Phi, \nabla \Phi)) V_{\tau} dy|.
\tag{29}
\]
Note that we have that by Lemma 2.4 and (19),
\[
|\int_{S_0} (R^{-1}\Phi)_{\tau} |V_{\tau}|^2 dy| \leq C|\int_{S_0} \text{div}(\Phi \tilde{V})|V_{\tau}|^2 dy| + C\int_{S_0} \Phi |V_{\tau}|^2 dy,
\]
\[
\leq C|\int_{S_0} \Phi \tilde{V} \nabla(|V_{\tau}|^2) dy| + C\int_{S_0} \Phi |V_{\tau}|^2 dy \leq M(c_0)(1 + \|\tilde{V}\|_{H^2}) \|V_{\tau}\|_2^2 + \frac{\mu}{8} \|\nabla V_{\tau}\|_2^2.
\tag{30}
\]
In addition,
\[
|\int_{S_0} (R^{-1}\Phi)_{\tau} V_{\tau} dy| \leq M(c_0)\|\tilde{V}\|_{H^2}^2 + M(c_0)\|V_{\tau}\|_2^2, \tag{31}
\]
\[
|\int_{S_0} \partial_\tau (R^{-1}F_0(\Phi, \nabla \Phi)) V_{\tau} dy| \leq M(c_0)c_3^4 + M(c_0)\|V_{\tau}\|_2^2. \tag{32}
\]
Substituting (30)-(32) into (29) yields
\[
\|V_{\tau}\|_2^2 + \int_0^\tau \|\nabla V_{\tau}(s)\|_2^2 ds \leq M(c_0)(1 + c_4^3 + c_4^4 \tau + \int_0^\tau (1 + \|\tilde{V}\|_{H^2}^2) \|V_{\tau}\|_2^2 ds).
\tag{33}
\]
In addition, by (26), (22) and (18)-(19), one has
\[
\|V\|_{H^2}^2 \leq M(c_1)c_2^4c_3^5. \tag{34}
\]
Combining (33)-(34) and applying the Grönwall inequality, we obtain that for $\tau \in [0, T_2]$,
\[
\|V_{\tau}\|_2^2 + \|V\|_{H^2}^2 + \int_0^\tau \|\nabla V_{\tau}(s)\|_2^2 ds \leq M(c_1)c_2^4c_3^{\frac{3}{2}}. \tag{35}
\]
By (26), one has
\[
\|V\|_{H^3} \leq C\|R^{-1}(\Phi V_{\tau} - F_0(\Phi, \nabla \Phi))\|_{H^1} + C\|V\|_{H^2}. \tag{36}
\]
This, together with (35) and Lemma 2.4, yields for $\tau \leq T_2$
\[
\int_0^\tau \|V(s)\|_{H^3}^2 ds \leq M(c_1)c_2^4c_3^{\frac{1}{2}}. \tag{37}
\]
By (35) and (37), we complete the proof of Lemma 2.5. □
Lemma 2.6. For $\tau \leq T_2$ with $T_2$ given in Lemma 2.5, we have

$$\|\nabla \tau\|_2^2 + \|\tau\|_{H^3}^2 + \int_0^T (\|\nabla \tau (s)\|_2^2 + \|V\|_{H^2}^2 + \|V(s)\|_{H^4}^2)ds \leq M(c_1)c_3^8, \quad (38)$$

Proof. Multiplying (28) by $\nabla \tau$ and integrating over $S_0$ yield

$$\|\nabla \tau\|_2^2 + \int_0^T \int_{S_0} R^{-1}\Phi |\nabla \tau|^2 dy ds \leq M(c_1) + C \int_0^T \int_{S_0} (|\Phi| |\nabla \tau| + |F_0(\Phi, \nabla \Phi)|) |\nabla \tau| dy ds$$

$$+ C \int_0^T \int_B (|\Phi| |\nabla \tau| + |\partial_\tau (F_0(\Phi, \nabla \Phi))|) |\nabla \tau| dy ds \leq M(c_1) + C \int_0^T (|\Phi| |\nabla \tau| + |\Phi_\tau|_{H^1} |\nabla \tau|_{H^3} + |\Phi|_{L^\infty} |\nabla \tau|_{H^2} dy ds$$

$$+ C \int_0^T (\|\Phi\|_{L^\infty} |\nabla \tau|_{H^2} + |\Phi|_{L^\infty} |\nabla \tau|_{H^2} |\nabla \tau|_{L^2} dy ds.$$  

This, together with Lemma 2.4-2.5, yields for a small constant $\eta > 0$,

$$\|\nabla \tau\|_2^2 + \int_0^T \|\nabla \tau\|_2^2 ds \leq M(c_1)c_3^8 + \eta \int_0^T \|\nabla \tau\|_2^2 ds. \quad (39)$$

In addition, by (26) and Lemma 2.3, we get

$$LV = (R^{-1}\Phi)\tau + R^{-1}\Phi V_\tau - \partial_\tau (R^{-1}F_0(\Phi, \nabla \Phi)), \quad V_\tau \in H^1_0(S_0).$$

This, together with the regularity theory of the second order elliptic equations and (39) with small $\eta > 0$, yields

$$\int_0^T \|\nabla \tau\|_2^2 ds \leq M(c_1)c_3^8 \quad \text{and} \quad \int_0^T \|V\|_3^2 ds \leq M(c_1)c_3^8. \quad (40)$$

In addition, by (36), (23) and (39)-(40), direct calculation yields

$$\|V\|_{H^3}^2 \leq M(c_1)c_3^8. \quad (41)$$

Combining (39)-(41), we complete the proof of Lemma 2.6. \qed

By summarizing Lemma 2.3-2.6, we have that, for $\tau \leq T_2$,

$$\begin{align*}
\|V\|_{H^3}^2 &+ \int_0^T (\|V\|_{H^2}^2 + \|V\|_{L^2}^2)ds \leq M(c_1), \\
\|V\|_{H^3}^2 &+ \|\nabla \tau\|_2^2 + \int_0^T (\|\nabla \tau\|_2^2 + \|\tau\|_{H^3}^2)ds \leq M(c_1)c_3^3c_3^{1/2}, \\
\|\nabla \tau\|_2^2 &+ \|V\|_{H^3}^2 + \int_0^T (\|\nabla \tau\|_2^2 + \|V\|_{H^3}^2 + \|\tau\|_{H^3}^2)ds \leq M(c_1)c_3^8, \\
\|\Phi\|_{H^3}^2 &+ \|\Phi\|_{H^3}^2 < M(c_1)c_3^8.
\end{align*} \quad (42)$$

Choosing $c_1 = M(c_0), \ c_2 = M(c_1), \ c_3 = c_2^8, \ c_4 = c_2^8, \ T^* = T_2.$
Then (42) implies that for $\tau \leq T^*$

$$
\begin{align*}
&\|V\|_{H^3}^2 + \int_0^\tau (\|V\|_{H^2}^2 + \|V\|_{H^2}^2)ds \leq c_2, \\
&\|V\|_{H^2}^2 + \|V\|_{H^2}^2 + \int_0^\tau (\|V\|_{H^2}^2 + \|V\|_{H^2}^2)ds \leq c_3, \\
&\|\nabla V\|_{H^3}^2 + \|\nabla V\|_{H^3}^2 + \int_0^\tau (\|\nabla V\|_{H^2}^2 + \|V\|_{H^2}^2)ds \leq c_4, \\
&\|\Phi\|_{H^3} + \|\Phi\|_{H^2} \leq c_4.
\end{align*}
$$

(43)

We now ready to prove Theorem 2.1.

Proof of Theorem 2.1. To prove Theorem 2.1, we construct an approximate solution sequence $\{(\phi^k, v^k)\}$ for problem (10)-(11) with (12) by solving the linearized problem (14)-(16). For this, we first construct $\{(\phi^0, v^0)\}$.

Let $G \in C([0, T^*], H^1_0) \cap L^2([0, T^*], H^2)$ be the solution to the following heat equation

$$
\begin{align*}
G_\tau - \Delta G &= 0 \quad \text{in } [0, T^*] \times S_0, \\
G(0, y) &= L\tilde{V}_0, \quad G|_{[0, T^*] \times \partial S_0} = 0,
\end{align*}
$$

where $\tilde{V}_0(\tau, y) \equiv u_0(y)$, and $L\tilde{V}_0 = \mu \Delta \tilde{V}_0 + (\mu + \lambda)\nabla \text{div} \tilde{V}_0$.

Let $w \in C([0, T^*], H^1_0 \cap H^3) \cap L^2([0, T^*], H^4)$ be the solution to the problem

$$
\begin{align*}
\partial_\tau w - Lw &= G \quad \text{in } [0, T^*] \times S_0, \\
w(0, y) &= u_0(y), \quad w|_{[0, T^*] \times \partial S_0} = 0.
\end{align*}
$$

Set $\Phi^0 = \rho_0$ and $V^0(\tau, y) = w(\tau, y)$. Then we have $V^0 \in C([0, T^*], H^1_0 \cap H^3) \cap L^2([0, T^*], H^4)$. Moreover, by choosing a suitable $c_0$ in (18) and a smaller $T^*$ if needed, one can obtain

$$
\sup_{0 \leq \tau \leq T^*} (\|V^0(\tau)\|_{H^3} + \|\nabla V^0(\tau)\|_{H^2}) + \int_0^{T^*} (\|V^0(\tau)\|_{H^2} + \|V^0(\tau)\|_{H^2})d\tau \leq c_0. \tag{44}
$$

Here the choice of $c_0$ in (44) and (18) is possible because $c_0$ depends only on $\|\rho_0\|_{H^3}$ and $\|u_0\|_{H^3}$.

Let $\tilde{V} = V^0$. Then by Lemma 2.3-2.6, there exists a unique strong solution $(\Phi^1, V^1)$ to the linearized problem (14)-(16), which satisfies all the estimates in (43). Therefore, an induction argument allows us to construct the approximate solution $(\Phi^k, V^k)$ for all $k \geq 1$: assume that $V^{k-1}$ has been defined for $k \geq 1$, and let $(\Phi^k, V^k)$ be a unique solution to problem (14)-(16) with $\tilde{V} = V^{k-1}$. It follows from (43) that there exists a uniform positive constant $C$ depending on $c_0$ such that for all $k \geq 1$

$$
\sup_{0 \leq \tau \leq T^*} (\|\Phi^k(\tau)\|_{H^3} + \|\nabla \Phi^k(\tau)\|_{H^2} + \|V^k(\tau)\|_{H^3 \cap H^3}) \leq C,
$$

(45)

$$
\sup_{0 \leq \tau \leq T^*} \|\nabla V^k(\tau)\|_{H^3} + \int_0^{T^*} (\|\nabla V^k(\tau)\|_{H^2} + \|V^k(\tau)\|_{H^2})d\tau \leq C.
$$

Define

$$
\tilde{\Phi}^{k+1} = \Phi^{k+1} - \Phi^k, \quad \tilde{V}^{k+1} = V^{k+1} - V^k.
$$

Then we have

$$
\tilde{\Phi}^{k+1}_\tau + \frac{1}{R} \text{div}(\tilde{\Phi}^{k+1} V^k) + \frac{1}{R} \text{div}(\Phi^k \tilde{V}^k) = 0. \tag{46}
$$
Multiplying (46) by $\hat{\Phi}^{k+1}$ and integrating over $S_0$ yield
\[
\frac{d}{d\tau} ||\hat{\Phi}^{k+1}||^2 \leq C ||\nabla V^k||_{L^\infty} ||\hat{\Phi}^{k+1}||^2 + C ||\nabla \bar{V}^k||_2 ||\hat{\Phi}^{k+1}||_2. \tag{47}
\]

In addition, one has
\[
\bar{\Phi}^{k+1} - RL\bar{V}^{k+1} = -\bar{\Phi}^{k+1} V^k + F_0(\Phi^{k+1}, \nabla \bar{\Phi}^{k+1}) - F_0(\Phi^k, \nabla \Phi^k). \tag{48}
\]

Multiplying (48) by $\bar{V}^{k+1}$ and integrating over $S_0$, we have that, for small $\eta > 0$,
\[
\frac{d}{d\tau} \int_{S_0} \Phi^{k+1} ||\bar{\Phi}^{k+1}||^2 + \frac{d}{d\tau} \int_{S_0} \Phi^{k+1} ||\bar{\Phi}^{k+1}||^2 \leq C \eta \bar{\Phi}^{k+1} ||\bar{\Phi}^{k+1}||_2 + C \bar{\Phi}^{k+1} ||\bar{\Phi}^{k+1}||_2 + C \bar{\Phi}^{k+1} ||\nabla \bar{V}^{k+1}||_2 \tag{49}
\]
\[
\leq C \eta^{-1} ||\bar{\Phi}^{k+1}||^2 + \eta ||\nabla \bar{V}^{k+1}||^2.
\]

Combining (47) with (49) and applying the Gronwall inequality yield
\[
||\hat{\Phi}^{k+1}||^2 + ||\bar{V}^{k+1}||^2 + \int_0^{T*} ||\nabla \bar{V}^{k+1}||^2 ds \leq C \eta \exp(C \eta^{-1} T*) \int_0^{T*} ||\nabla \bar{V}^k||^2 ds. \tag{50}
\]

Choosing $\eta > 0$ and $T^*$ suitably small, then
\[
\sup_{0 \leq \tau \leq T*} \left( ||\hat{\Phi}^{k+1}(\tau)||^2 + ||\bar{V}^{k+1}(\tau)||^2 \right) + \int_0^{T*} ||\nabla \bar{V}^{k+1}(\tau)||^2 ds \leq \frac{1}{2} \int_0^{T*} ||\nabla \bar{V}^k(s)||^2 ds,
\]

which implies
\[
\sum_{k=1}^{\infty} \sup_{0 \leq \tau \leq T*} \left( ||\hat{\Phi}^{k+1}(\tau)||^2 + ||\bar{V}^{k+1}(\tau)||^2 \right) + \int_0^{T*} ||\nabla \bar{V}^{k+1}(\tau)||^2 ds \leq C \int_0^{T*} ||\nabla \bar{V}^k||^2 ds \leq C.
\]

Therefore, we conclude that the sequence $\{ (\Phi^k, V^k) \}$ converges to a limit $(\phi, v)$ in $C([0, T^*], H^2)$ and $(\phi, v)$ solves problem (10)-(11) with (12). Moreover, $(\phi, v)$ satisfies
\[
\sup_{0 \leq \tau \leq T*} \|\phi(\tau)\|_{H^3} + \|\varphi(\tau)\|_{H^2} + \|v(\tau)\|_{H_1 \cap H^3} \leq C,
\]
\[
\sup_{0 \leq \tau \leq T*} \|v_\tau\|_{H^1} + \int_0^{T*} \|v_\tau(\tau)\|_{H^2}^2 + \|v(\tau)\|_{H^3}^2 d\tau \leq C.
\]

Hence, we complete the proof of Theorem 2.1 for the local existence of the solution to problem (10)-(11) with (12). \( \square \)

3. Some uniform weighted energy estimates. In this section, we will establish some a priori energy estimates on the solution to (10)-(11) with (12). Denote by $w = \phi - 1$, $\bar{\tau} = R^2(t)$ and $s' = \frac{3}{4}(\gamma - 1)$, then it follows from (10)-(11) that for $\bar{\tau} \geq \bar{\tau}_0 \equiv (1 + h)^2$,
\[
2h \partial_\tau w + \frac{1}{\bar{\tau}} div v = f, \tag{51}
\]
\[
2h \partial_\tau v + \frac{h}{\bar{\tau}} v - L v + \frac{\gamma}{s'^2 + 1} \nabla w = g, \tag{52}
\]
\[
w(\bar{\tau}, y)|_{\bar{\tau} = \bar{\tau}_0} = w_0(y) \equiv \phi(\bar{\tau}_0, y) - 1, \quad v(\bar{\tau}, y)|_{\bar{\tau} = \bar{\tau}_0} = v_0(y) \equiv v(\bar{\tau}_0, y),
\]
\[
v = 0, \quad on \quad \bar{\tau}_0 \leq \bar{\tau} < +\infty \times \partial S_0,
\]
has

Note that Lemma 3.1

where

By the Poincare inequality, we obtain

and

We will derive a series of basic energy estimates on \((w, v)\). Set \(f_0 = -\frac{w}{\tau} \text{div} v\) and

\[
\frac{d\varphi}{d\tau} = 2h\varphi_t + \frac{1}{\tau} v \cdot \nabla \varphi.
\]

Then we have

**Lemma 3.1 (Weight \(L^2\)-estimate of \((w, v)\)).** For small \(h > 0\) and any \(t \geq \tilde{t}_0\), one has

\[
fh(\|w\|_2^2 + \|\tilde{\tau} s' v\|_2^2) + \int_{\tilde{t}_0}^t \left( \|\tilde{\tau} s' \nabla v\|_2^2 + \|\tilde{\tau} s' + 1 \frac{dw}{d\tilde{\tau}}\|_2^2 \right) d\tilde{\tau} 
\]

\[
\leq C \left( h\|w_0, v_0\|_2^2 + \int_{\tilde{t}_0}^t (|f, w| + |\tilde{\tau} s' g, v| + \|\tilde{\tau} s' + 1 f_0\|_2^2) d\tau \right).
\]

**Remark 8.** For large \(t\), (53) still holds even if the assumption on the smallness of \(h\) is removed. Indeed, this can be easily seen from (57) below.

**Proof.** By \(\int_{S_0} \gamma(51) w dy\) and \(\int_{S_0} \tilde{\tau} s' (52) \cdot v dy\), we have

\[
\gamma h \partial_{\tilde{\tau}}\|w\|_2^2 + \frac{\gamma}{\tilde{\tau}} (\text{div} v, w) = (\gamma f, w),
\]

and

\[
(2h\tilde{\tau} s' \partial_{\tilde{\tau}} v, v) + h\tilde{\tau} s' - 1 (v, v) - \tilde{\tau} s' (Lv, v) + \frac{\gamma}{\tilde{\tau}} (\nabla v, v) = (\tilde{\tau} s' g, v).
\]

Note that

\[
(2h\tilde{\tau} s' \partial_{\tilde{\tau}} v, v) + (h\tilde{\tau} s' - 1 v, v) = h \partial_{\tilde{\tau}} \|\tilde{\tau} s' v\|_2^2 + (1 - 2s') h \|\tilde{\tau} s' - \frac{1}{2} v\|_2^2,
\]

and

\[
- (\tilde{\tau} s' Lv, v) = \mu \|\tilde{\tau} s' \nabla v\|_2^2 + (\mu + \lambda) \|\tilde{\tau} s' \text{div} v\|_2^2.
\]

Then substituting these two equalities into (55) and then adding (55) to (54) yield

\[
\frac{d}{d\tilde{\tau}} \left( h\gamma \|w\|_2^2 + h\|\tilde{\tau} s' v\|_2^2 \right) + \mu \|\tilde{\tau} s' \nabla v\|_2^2 + (\mu + \lambda) \|\tilde{\tau} s' \text{div} v\|_2^2
\]

\[
= (2s' - 1) h \|\tilde{\tau} s' - \frac{1}{2} v\|_2^2 + (\gamma f, w) + (\tilde{\tau} s' g, v).
\]

By the Poincare inequality, we obtain

\[
\|\tilde{\tau} s' - \frac{1}{2} v\|_2^2 \leq C \|\tilde{\tau} s' \nabla v\|_2^2.
\]

Combining (57) with (58) yields for small \(h\),

\[
\frac{d}{d\tilde{\tau}} \left( h\gamma \|w\|_2^2 + h\|\tilde{\tau} s' v\|_2^2 \right) + C \|\tilde{\tau} s' \nabla v\|_2^2 \leq (\gamma f, w) + (\tilde{\tau} s' g, v).
\]
Integrating (58) with respect to the variable \( \bar{\tau} \) over \( (\bar{\tau}_0, t) \) yields
\[
h(||w||^2_2 + ||\bar{\tau}' v||^2_2) + \int_{\bar{\tau}_0}^{t} ||\bar{\tau}' \nabla v||^2_2 d\bar{\tau}
\leq Ch \|(w_0, v_0)||^2_2 + C \int_{\bar{\tau}_0}^{t} (||\gamma f, w|| + ||\bar{\tau}' g, v||) d\bar{\tau}.
\] (59)

In addition, one has that from (51)
\[
dw\over{d\bar{\tau}} = -\frac{1}{\bar{\tau}} \nabla v v + f_0,
\]
which gives
\[
||\bar{\tau}'^2 dw\over{d\bar{\tau}}||^2_2 \leq 2 ||\bar{\tau}' \nabla v||^2_2 + 2 ||\bar{\tau}'^2 f_0||^2_2.
\] (60)

From (59)-(60), we obtain (53). \( \square \)

**Lemma 3.2** (Weighted \( L^2 \)-estimate of \( \nabla v \)). For \( t \geq \bar{\tau}_0 \), we have
\[
h(\bar{\tau}'^2 \nabla v||^2_2 + \int_{\bar{\tau}_0}^{t} \left(||h\bar{\tau} v||^2_2 + ||h\bar{\tau}' v||^2_2\right)d\bar{\tau}
\leq \eta_1 h^2 \int_{\bar{\tau}_0}^{t} \left(\frac{\nabla w}{\bar{\tau}^2 + 2}||\nabla v||^2_2 d\bar{\tau} + \frac{Ch}{\eta_1} \|(w_0, v_0)||^2_2 + \frac{C}{\eta_1} \int_{\bar{\tau}_0}^{t} A_1 d\bar{\tau} + \int_{\bar{\tau}_0}^{t} A_2 d\bar{\tau},
\]
where \( 0 < \eta_1 < 1 \) is a small fixed constant, and
\[
A_1 = ||(\gamma f, w)|| + ||(\bar{\tau}' g, v)||,
\]
\[
A_2 = ||(\gamma f, h\bar{\tau} v)|| + ||(\bar{\tau}' g, h\bar{\tau} v)||.
\]

**Proof.** Computing \( \int_{S_0} \gamma (51) h\bar{\tau} v dy \) and \( \int_{S_0} \bar{\tau}' (52) h\bar{\tau} v dy \) yield respectively
\[
2\gamma ||h\partial_\tau w||^2_2 + \left(\frac{\gamma}{\bar{\tau}} \nabla v, h\bar{\tau} v\right) = (\gamma f, h\bar{\tau} v),
\] (62)

and
\[
2 ||h\bar{\tau}' \partial_\tau v||^2_2 + h(\bar{\tau}'^2 - 1)v, h\bar{\tau} v\right) - (\bar{\tau}'^2 (L v, h\bar{\tau} v) + (\gamma \nabla w, h\bar{\tau} v) = (\bar{\tau}'^2 g, h\bar{\tau} v). \] (63)

We then estimate the terms on the left hand sides of (62)-(63) separately. For the term \(- (\bar{\tau}'^2 L v, h\bar{\tau} v)\), we have
\[
(\bar{\tau}'^2 L v, h\bar{\tau} v) = h(\bar{\tau}'^2 \partial_\tau v||^2_2 + h(1 + \lambda) \partial_\tau ||\bar{\tau}' \nabla v||^2_2 - h\bar{\tau}' \mu ||\bar{\tau}' \nabla v||^2_2
\]
\[
- h\bar{\tau}' (\mu + \lambda) ||\bar{\tau}' \nabla v||^2_2.
\] (64)

For the terms \( h(\bar{\tau}'^2 - 1)v, h\bar{\tau} v\) and \( (\gamma \nabla v, h\bar{\tau} v)\), one has
\[
h(\bar{\tau}'^2 - 1)v, h\bar{\tau} v\ \leq \frac{1}{2} ||h\bar{\tau}' v||^2_2 + \frac{1}{2} ||\bar{\tau}' - 1 h\bar{\tau} v||^2_2,
\] (65)
\[
(\gamma \nabla v, h\bar{\tau} v) \leq \frac{\gamma}{4} ||h\bar{\tau} v||^2_2 + \gamma \frac{1}{2} ||\nabla v||^2_2.
\] (66)

For the term \( (\gamma \nabla w, h\bar{\tau} v)\), we have
\[
(\gamma \nabla w, h\bar{\tau} v) = -\partial_\tau (h\gamma \nabla w, divv) + (\bar{\gamma}) \nabla (w, hv) + (\gamma h\partial_\tau w, divv).
\]
This, together with the Hölder inequality, yields for small $\eta_1 > 0$
\[
(\gamma \frac{\partial}{\partial t} w, v_\tau) \geq -\frac{d}{dt} \left( \frac{h}{\tau} w, \text{div} v \right) - \eta_1 h^2 \| \frac{\nabla w}{\tau^{s+2}} \|^2_2 - \frac{C}{\eta_1} \| \tilde{\tau}^{s'} v \|^2_2 \\
- \frac{\gamma}{4} \| h w_\tau \|^2_2 - \gamma \| \frac{1}{\tau} \text{div} v \|^2_2.
\] (67)

Substituting (64)-(67) into (62)-(63), we obtain
\[
\gamma \| h \partial_t w \|^2_2 + \| h \tilde{\tau}^{s'} \partial_t v \|^2_2 + h \partial_t \left( \frac{\mu}{2} \| \tilde{\tau}^{s'} \nabla v \|^2_2 + \frac{\mu + \lambda}{2} \| \tilde{\tau}^{s'} \text{div} v \|^2_2 - \left( \frac{\gamma}{\tau} w, \text{div} v \right) \right)
\leq \eta_1 h^2 \| \frac{\nabla w}{\tau^{s+2}} \|^2_2 + \frac{C}{\eta_1} \| \tilde{\tau}^{s'} \nabla v \|^2_2 + C((\gamma f, h w_\tau) + (\tilde{\tau}^{s'} g, h v_\tau)).
\] (68)

Note that
\[
-\left( \frac{\gamma}{\tau} w, \text{div} v \right) \leq \frac{\mu}{4} \| \text{div} v \|^2_2 + \frac{\gamma^2}{\mu} \| w \|^2_2.
\]

Together with $\int_{\tau_0}^t \left( \frac{C}{\eta_1} \times (58) + (68) \right) d\tau$, this yields
\[
h \| \tilde{\tau}^{s'} \nabla v \|^2_2 + \int_{\tau_0}^t \left( \| h w_\tau \|^2_2 + \| h \tilde{\tau}^{s'} v_\tau \|^2_2 \right) d\tau
\leq \int_{\tau_0}^t \eta_1 h^2 \| \frac{\nabla w}{\tau^{s+2}} \|^2_2 d\tau + \frac{C}{\eta_1} \| (w_0, v_0) \|_{H^1}
+ \int_{\tau_0}^t ((\gamma f, h w_\tau) + (\tilde{\tau}^{s'} g, h v_\tau)) d\tau + \frac{C}{\eta_1} \int_{\tau_0}^t ((\gamma f, w) + (\tilde{\tau}^{s'} g, v)) d\tau,
\]
which completes the proof of Lemma 3.2.

**Lemma 3.3 (Weighted $L^2$-estimate of $(w_\tau, v_\tau)$).** For $t \geq \tau_0$, we have
\[
h \| h w_\tau \|^2_2 + h \| h \tilde{\tau}^{s'} v_\tau \|^2_2 + \int_{\tau_0}^t \| h \tilde{\tau}^{s'} \nabla v_\tau \|^2_2 d\tau
\leq C(\eta_2) h \| (w_0, v_0) \|_{H^2} + c \eta_2 h^2 \int_{\tau_0}^t \| \frac{\nabla w}{\tau^{s+2}} \|^2_2 d\tau + C(\eta_2) \int_{\tau_0}^t (A_1 + A_2 + A_3) d\tau,
\] (69)

where $A_3 = \| (\gamma h f_\tau, h w_\tau) \| + \| (\tilde{\tau}^{s'} h g_\tau, h v_\tau) \|.$

**Proof.** Computing $\int_{S_0}^t \gamma h \partial_t (51) h w_\tau dy$, we have
\[
h \gamma \frac{d}{dt} \| h w_\tau \|^2_2 + \left( \frac{\gamma}{\tau} h \text{div} v, h w_\tau \right) - \left( \frac{\gamma}{\tau^2} h \text{div} v, h w_\tau \right) = (\gamma h f_\tau, h w_\tau).
\] (70)

Computing $\int_{S_0}^t \tilde{\tau}^{2s'} h \partial_t (52) \cdot h v_\tau dy$ yields
\[
h \partial_t \| h \tilde{\tau}^{s'} v^2 \|^2_2 + (1 - 2s') h \| h \tilde{\tau}^{s'} v^2 \|^2_2 - (h^2 \tilde{\tau}^{2s'-2} v, h v_\tau)
- (h L v_\tau, \tilde{\tau}^{s'} v_\tau) + \left( \frac{\gamma}{\tau} h \nabla w_\tau, h v_\tau \right) - (2s' + 1) \left( \frac{\gamma}{\tau^2} h \nabla w, h v_\tau \right)
= (\tilde{\tau}^{s'} h g_\tau, h v_\tau).
\] (71)

Note that for small $\eta_1 > \eta_2 > 0$,
\[
-(h L v_\tau, \tilde{\tau}^{s'} h v_\tau) = \mu \| h \tilde{\tau}^{s'} \nabla v_\tau \|^2_2 + (\mu + \lambda) \| h \tilde{\tau}^{s'} \text{div} v_\tau \|^2_2,
\] (72)
\[
(2s' + 1) \left( \frac{\gamma}{\tau^2} h \nabla w, h v_\tau \right) \leq \eta_2 h^2 \| \frac{\nabla w}{\tau^{s+2}} \|^2_2 + \frac{C}{\eta_2} \| h \tilde{\tau}^{s'} v_\tau \|^2_2,
\] (73)
and

\[
\left( \frac{\gamma}{\tau^2} h^{\text{divv}}, hw_\tau \right) \leq \|hw_\tau\|^2 + \frac{s^2}{4} \|h^{\text{divv}}\|^2.
\]

Then by substituting these estimates and (72)-(73) into (71), and combining with (70), we have

\[
\frac{d}{d\tilde{\tau}} (h\|hw_\tau\|^2 + h\|h^{\text{divv}}\|^2) + \|\tilde{z}^{x'\tau} h^{\nabla v_\tau}\|^2 \\
\leq \eta_2 h^2 \left( \frac{\nabla w}{\tilde{\tau}^{x'\tau + 2}} \right)^2 + \frac{C}{\eta_2} \left( \|\tilde{z}^{x'\tau} h^{v_\tau}\|^2 + \|hw_\tau\|^2 \right) \\
+ C\|\tilde{z}^{x'} h^{\nabla v}\|^2 + C(\gamma h_f, hw_\tau) + (h^{2x'\tau} g_\tau, hv_\tau)),
\]

(74)

here we have used the fact that

\[
|h(h^{2x'-2\tau} v, hv_\tau)| \leq h\|h^{x'\tau} v\|^2 + Ch\|\tilde{z}^{x'} h^{\nabla v}\|^2.
\]

Integrating (74) over \((\tilde{\tau}_0, t)\) with respect to the variable \(\tilde{\tau}\), one has

\[
h\|hw_\tau\|^2 + \|h^{x'\tau} v\|^2 + \int_{\tilde{\tau}_0}^t \|h^{x'\tau} \nabla v_\tau\|^2 d\tilde{\tau} \\
\leq Ch\|(w_0, v_0)\|^2 + \eta_2 h^2 \int_{\tilde{\tau}_0}^t \left( \|\tilde{z}^{x'\tau} v_\tau\|^2 + \|hw_\tau\|^2 \right) d\tilde{\tau} \\
+ C \int_{\tilde{\tau}_0}^t (|\tilde{z}^{x'} h^{\nabla v}|^2 + (\gamma h_f, hv_\tau) + (\tilde{z}^{2x'\tau} h g_\tau, hv_\tau))) d\tilde{\tau}.
\]

(75)

Combining \(\frac{C}{\eta_2} \times (61) + C \times (53) + (75)\) gives

\[
h\|hw_\tau\|^2 + \|h^{x'\tau} v\|^2 + \int_{\tilde{\tau}_0}^t \|h^{x'\tau} \nabla v_\tau\|^2 d\tilde{\tau} \\
\leq Ch\|(w_0, v_0)\|^2 + \frac{Ch}{\eta_1\eta_2} \|(w_0, v_0)\|^2 + \frac{C}{\eta_1\eta_2} \int_{\tilde{\tau}_0}^t A_1 d\tilde{\tau} + \frac{C}{\eta_2} \int_{\tilde{\tau}_0}^t A_2 d\tilde{\tau} + (\frac{C}{\eta_2} \eta_1 + \eta_2) h^2 \int_{\tilde{\tau}_0}^t \left( \|\nabla w\|_{\tilde{\tau}^{x'\tau + 2}} \right)^2 d\tilde{\tau} \\
+ C \int_{\tilde{\tau}_0}^t (|\gamma h_f, hv_\tau) + (\tilde{z}^{2x'\tau} h g_\tau, hv_\tau))) d\tilde{\tau}.
\]

Set \(\eta_1 = \eta_2^2\), we then complete the proof of Lemma 3.3. 

\(\square\)

Lemma 3.4 (Weighted \(L^2\)-estimate of \(\nabla v_\tau\)). For \(t \geq \tilde{\tau}_0\) and \(0 < \eta_3 < 1\), we have

\[
h\|\tilde{z}^{x'} h^{\nabla v_\tau}\|^2 + \int_{\tilde{\tau}_0}^t \left( \|h^{2w_\tau}\|^2 + \|h^{2x'\tau} v_\tau\|^2 \right) d\tilde{\tau} \\
\leq Ch\|v_\tau(\tilde{\tau}_0)\|^2 + C(\eta_3) h\|(w_0, v_0)\|^2 + Ch^2 \int_{\tilde{\tau}_0}^t \|\nabla w\|_{\tilde{\tau}^{x'\tau + 2}}^2 d\tilde{\tau} \\
+ C(\eta_3) \int_{\tilde{\tau}_0}^t (A_1 + A_2 + A_3 + A_4) d\tilde{\tau},
\]

(76)

where

\[
A_4 = |(\gamma h_f, h^2w_\tau)| + |(\tilde{z}^{x'\tau} g_\tau, h^2v_\tau)|.
\]
Proof. Computing \( \int_{S_0} \gamma \partial_{\tau}(51) h^2 w_{\tau^2} dy \) and \( \int_{S_0} \bar{\tau}^2 s'h \partial_{\tau}(52) \cdot h^2 v_{\tau^2} dy \) yield respectively,

\[
2\gamma \|h^2 w_{\tau^2}\|_2^2 + \left( \frac{h\gamma}{\tau} \right) \text{div}_v, h^2 w_{\tau^2} - \left( \frac{h\gamma}{\tau^2} \right) \text{div}, h^2 w_{\tau^2} = (\gamma h f, h^2 w_{\tau^2}),
\]

and

\[
2\|\bar{\tau}^2 s'h v_{\tau^2}\|_2^2 + \bar{\tau}^2 s' - 1(h v, h^2 v_{\tau^2}) = - \bar{\tau}^2 s' \cdot (h L v, h^2 v_{\tau^2}) + \left( \frac{h\gamma}{\tau} \right) \nabla w, h^2 v_{\tau^2} - (2s' + 1)(\frac{h\gamma}{\tau^2} \nabla w, h^2 v_{\tau^2})
\]

\[
= (\bar{\tau}^2 s' h g, h^2 v_{\tau^2}).
\]

We now estimate the terms in (78). For the terms \( - (\bar{\tau}^2 s' h L v, h^2 v_{\tau^2}) \) and \( \left( \frac{h\gamma}{\tau} \nabla w, h^2 v_{\tau^2} \right) \), one has

\[
-h \partial_{\tau} \left( \frac{\mu}{2} \|\bar{\tau}^2 s' h v_{\tau^2}\|_2^2 + \frac{\mu + \lambda}{2} \|\bar{\tau}^2 s' \text{div} v_{\tau^2}\|_2^2 \right)
\]

\[
- s' h \mu \|\bar{\tau}^2 s' - \frac{1}{2} h^2 \nabla v_{\tau^2}\|_2^2 - s' h (\mu + \lambda) \|\bar{\tau}^2 s' - \frac{1}{2} \text{div} v_{\tau^2}\|_2^2,
\]

and

\[
\left( \frac{h\gamma}{\tau^2} \nabla w, h^2 v_{\tau^2} \right)
\]

\[
= - \frac{d}{d \tau} \left( \frac{h\gamma}{\tau} \nabla w, h^2 v_{\tau^2} \right) - \frac{\gamma}{\tau^2} (h w, h^2 \text{div} v_{\tau^2}) + \frac{\gamma}{\tau} (h^2 w_{\tau^2}, h \text{div} v_{\tau^2})
\]

\[
\leq - \frac{d}{d \tau} \left( \frac{h\gamma}{\tau^2} \nabla w, h^2 v_{\tau^2} \right) - C h (\|h w\|_2^2 + \| \frac{h}{\tau^2} \text{div} v_{\tau^2}\|_2^2) - \frac{\gamma}{2} \|h^2 w_{\tau^2}\|_2^2
\]

\[
- \frac{\gamma}{2} \| h \text{div} v_{\tau^2}\|_2^2.
\]

For the term \( - \left( \frac{h\gamma}{\tau^2} \nabla w, h^2 v_{\tau^2} \right) \), we obtain that for small \( \eta_3 > 0 \),

\[
- \left( \frac{h\gamma}{\tau^2} \nabla w, h^2 v_{\tau^2} \right)
\]

\[
= - \frac{d}{d \tau} \left( \frac{h\gamma}{\tau^2} \nabla w, h^2 v_{\tau^2} \right) - \left( \frac{2\gamma h}{\tau^3} \nabla w, h^2 v_{\tau^2} \right) + \left( \frac{h\gamma}{\tau^2} \nabla w, h^2 v_{\tau^2} \right)
\]

\[
\geq - \frac{d}{d \tau} \left( \frac{h\gamma}{\tau^2} \nabla w, h^2 v_{\tau^2} \right) - \eta_3 h \| \nabla w \|_{\tau^2 + 3}^2 - \frac{C}{\eta_3} \| \bar{\tau}^2 s' h^2 \nabla v_{\tau^2}\|_2^2 - \|h w\|_2^2.
\]

Substituting (79)-(81) into (78) and then combining with (77) yield

\[
\gamma \|h^2 w_{\tau^2}\|_2^2 + \|h^2 s' v_{\tau^2}\|_2^2 + h \partial_{\tau} \left( \frac{\mu}{2} \|\bar{\tau}^2 s' h v_{\tau^2}\|_2^2 + \frac{\mu + \lambda}{2} \|\bar{\tau}^2 s' \text{div} v_{\tau^2}\|_2^2 \right)
\]

\[
- \left( \frac{h\gamma}{\tau} w_{\tau^2}, h \text{div} v_{\tau^2} \right) - (2s' + 1) \left( \frac{h\gamma}{\tau^2} \nabla w, h v_{\tau^2} \right)
\]

\[
\leq \eta_3 h \| \nabla w \|_{\tau^2 + 3}^2 + \frac{C}{\eta_3} \|h \bar{\tau}^2 s' \nabla v_{\tau^2}\|_2^2 + C \|h v_{\tau^2}\|_2^2 + C \|h w_{\tau^2}\|_2^2
\]

\[
+ C(\gamma h f, h^2 w_{\tau^2}) + C(\bar{\tau}^2 s' h g, h^2 v_{\tau^2}).
\]
In addition,
\[
\frac{\mu}{2} \tilde{\tau}^2 h^2 \nabla v_\tau \|_2^2 + \frac{\mu + \lambda}{2} \| h \tilde{\tau}' \text{div} v_\tau \|_2^2 - \left( \frac{h \gamma}{\tau^2} w_\tau, h \text{div} v_\tau \right) - (2s' + 1) \left( \frac{h \gamma}{\tau^2} \nabla w, h v_\tau \right) \\
\geq \frac{\mu}{4} \| h \tilde{\tau}' \nabla v_\tau \|_2^2 - \left( \frac{2\gamma^2}{\mu} \| h w_\tau \|_2^2 + \frac{2(2s' + 1)\gamma^2}{\mu} \| h w_\|_2^2 \right).
\]

Thus, it follows from
\[
\int_{\tilde{\tau}_0}^t \left( 2s' \times (69) + \frac{2(2s' + 1)\gamma^2}{\mu} \times (59) \right) \text{ together with (83)}
\]

that
\[
h \| \tilde{\tau}' \| h \nabla v_\tau \|_2^2 + \int_{\tilde{\tau}_0}^t \left( \frac{\nabla w}{\tilde{\tau}'^2 + 2} + \| h^2 \tilde{\tau}' v_\tau \|_2^2 \right) d\tilde{\tau}
\]
\[
\leq Ch || v_\tau (\tilde{\tau}_0) ||_{H^1} + C(\eta_2)h|| (w_0, v_0) ||_{H^2} + C\eta_2 h^2 \int_{\tilde{\tau}_0}^t \| \nabla w \|_{\tilde{\tau}'^2 + 2}^2 d\tilde{\tau}
\]
\[
+ C(\eta_2) \int_{\tilde{\tau}_0}^t (A_1 + A_2 + A_3) d\tilde{\tau} + \eta_3 h^2 \int_{\tilde{\tau}_0}^t \| \nabla w \|_{\tilde{\tau}'^2 + 3}^2 d\tilde{\tau} + \frac{C}{\eta_3} \int_{\tilde{\tau}_0}^t \| \tilde{\tau}' \text{div} v_\tau \|_2^2 d\tilde{\tau}
\]
\[
+ C \int_{\tilde{\tau}_0}^t \left( \| h \tilde{\tau}' \text{div} v_\tau \|_2^2 + \| h w_\|_2^2 + \left( \gamma h f_\tau, h^2 w_\tau \right) \right) d\tilde{\tau}.
\]

By applying the estimates on \( \int_{\tilde{\tau}_0}^t (\| h \tilde{\tau}' \text{div} v_\tau \|_2^2 + \| h w_\|_2^2) d\tilde{\tau} \) and \( \int_{\tilde{\tau}_0}^t \left( \| h \tilde{\tau}' \text{div} v_\tau \|_2^2 + \| h w_\|_2^2 \right) d\tilde{\tau} \) obtained in Lemma 3.1.3.3, and by setting \( \eta_2 = \eta_3^2 \), we then complete the proof of (76) from (84).

Here, we point out that the property \( v \in H_0^4([0, T] \times S_0) \) plays a crucial role in deriving the lower order energy estimates in Lemma 3.1-Lemma 3.4 by using integration by parts. However, in order to derive the higher order energy estimates on \( (w, v) \), we need to cope with the fact that \( \nabla v = 0 \) may not hold on the boundary \([0, T] \times S_0 \) so that the direct integration by parts for higher order derivatives of \( v \) does not work. Motivated by the method used in [23], we will estimate \( (w, v) \) in the interior and the boundary regions separately.

We first derive the energy estimates on \( (w, v) \) in the interior region.

**Lemma 3.5** (Weighted interior energy estimates of \( (w, v) \)). For \( \delta \in (0, 1) \), define \( B_\delta = \{ y : |y| < \delta \} \). Choosing a function \( \chi_0(y) \in C_0^\infty(B_\delta) \). Then we have that for \( t \geq \tilde{\tau}_0 \) and \( k = 1, 2, 3 \),
\[
h \| \chi_0 D^k w \|_2^2 + h \| \tilde{\tau}' \chi_0 D^k v \|_2^2 + \int_{\tilde{\tau}_0}^t ( \| \tilde{\tau}' \chi_0 \nabla D^k v \|_2^2 + \| \chi_0 D^k w \|_{\tilde{\tau}'^2 + 2}^2 ) d\tilde{\tau}
\]
\[
\leq C \int_{\tilde{\tau}_0}^t \left( \| \tilde{\tau}' D^{k-1} v \|_{H^1}^2 + \| \chi_0 D^k f, D^k w \|_2^2 \right) d\tilde{\tau} + Ch \| (w_0, v_0) \|_{H^k}^2.
\]

**Proof.** Computing \( \int_{S_0} \chi_0^2(y) \gamma D^k(51) D^k w dy \) and \( \int_{S_0} \chi_0^2 \tilde{\tau}' D^k(52) D^k v dy \) yield respectively,
\[
h \gamma \frac{d}{d\tilde{\tau}} \chi_0^2 D^k w \|_2^2 + \frac{\gamma}{\tau} (\chi_0^2 \text{div} D^k v, D^k w) = (\gamma \chi_0^2 D^k f, D^k w),
\]
and
\[ h \frac{d}{dt} \|\tilde{s}' \chi_0 D^k v\|^2 + (1 - 2s')h \|\tilde{s}' - \frac{1}{2} \chi_0 D^k v\|^2 \\
- (\chi_0^2 \tilde{s}' LD^k v, D^k v) \]
\[ = \left( \chi_0^2 \tilde{s}' D^k g, D^k v \right). \tag{87} \]

Note that for small \( \eta_4 > 0 \),
\[ - (\chi_0^2 \tilde{s}' LD^k v, D^k v) \]
\[ = \mu (\tilde{s}' \nabla D^k v, \nabla (\chi_0^2 D^k v)) + (\mu + \lambda) (\tilde{s}' \text{div}(D^k v), \text{div}(\chi_0^2 D^k v)) \geq \frac{\mu}{2} \| \chi_0 \tilde{s}' \nabla D^k v \|^2 - C \| \tilde{s}' D^k v \|^2, \tag{88} \]

and
\[ \frac{\gamma}{\tau} (\chi_0 \text{div} D^k v, D^k w) + (\tilde{\gamma} \chi_0^2 \nabla D^k w, D^k v) \]
\[ = - \frac{\gamma}{\tau} (D^k w, \nabla \chi_0^2 D^k v) \leq \eta_4 \| \chi_0 D^k w \|_{\tilde{s}' + 1}^2 + \frac{C}{\eta_4} \| \tilde{s}' D^k v \|^2. \tag{89} \]

Combining (86) with (87) and then applying (88)-(89), we have
\[ \frac{d}{dt} (h \| \chi_0 D^k w \|^2 + h \|\tilde{s}' \chi_0 D^k v\|^2) + \frac{\mu}{2} \|\tilde{s}' \chi_0 \nabla D^k v\|^2 \]
\[ \leq \eta_4 \| \chi_0 D^k w \|_{\tilde{s}' + 1}^2 + \frac{C}{\eta_4} \| \tilde{s}' D^k v\|^2 + (\gamma \chi_0^2 D^k f, D^k w) + (\chi_0^2 \tilde{s}' D^k g, D^k v). \tag{90} \]

Since
\[ (\chi_0^2 \tilde{s}' D^k g, D^k v) = - \tilde{s}' (D^k \chi_0^2 D^k v) \]
\[ = - \tilde{s}' (D^k \chi_0^2 D^k v) - \tilde{s}' (D^{k-1} \chi_0^2 D^k v), \]

we obtain
\[ \| (\chi_0^2 \tilde{s}' D^k g, D^k v) \| \leq \frac{\mu}{4} \|\tilde{s}' \chi_0 D^{k+1} v\|^2 + C \| \tilde{s}' D^{k-1} g \|^2. \tag{91} \]

Substituting (91) into (90) yields
\[ \frac{d}{dt} (h \| \chi_0 D^k w \|^2 + h \|\tilde{s}' \chi_0 D^k v\|^2) + \frac{\mu}{4} \|\tilde{s}' \chi_0 \nabla D^k v\|^2 \]
\[ \leq \eta_4 \| \chi_0 D^k w \|_{\tilde{s}' + 1}^2 + \frac{C}{\eta_4} \| \tilde{s}' D^k v\|^2 + (\gamma \chi_0^2 D^k f, D^k w) + C \| \tilde{s}' D^{k-1} g \|^2. \tag{92} \]

In addition, it follows from \( \frac{1}{\tau} \times \{ (2 \mu + \lambda) \tilde{s} \times \nabla (51) + (52) \} \) that
\[ 2(2 \mu + \lambda) h \partial_\tau (\nabla w) + \frac{\gamma \nabla w}{\tilde{s}' + 1} \]
\[ = - \frac{\mu}{\tau} \left( \nabla \text{div} u - \Delta u \right) + (2 \mu + \lambda) \nabla f - \frac{1}{\tau} (2h \partial_\tau v + h \nabla v) + \frac{1}{\tau} g. \tag{93} \]
Computing \( \int_{S_0} \chi_0^2 D^k \nabla D^k w \) yields

\[
(2\mu + \lambda) h \frac{d}{dt} ||\chi_0 \nabla D^k w||_2^2 + \gamma ||\chi_0 \nabla D^k w||_{\frac{p}{p+1}}^2 \leq \frac{\mu}{T} \left( (\chi_0^2 D^k (\nabla div v - \Delta v), \nabla D^k w) \right) + C \left( (2\mu + \lambda) \chi_0 \nabla D^k f - \chi_0 \nabla D^k v_r + \frac{\partial D^k w}{\tau} - D^k g, \chi_0 \nabla D^k w) \right)
\]

\[
\leq \frac{\gamma}{2} ||\chi_0 \nabla D^k w||_2^2 + C (||\chi_0^2 D^k f, D^k w||_2) + ||\hat{\tau}' D^k v_r||_{H^1}^2 + ||\hat{\tau}' D^k g||_{L^2}^2
\]

(94)

Here we have used the fact that

\[
(\chi_0^2 D^k \Delta v, \nabla D^k w) = -\sum_{i,j} (\partial_j D^k v_i, \partial_j (\chi_0^2 \partial_i D^k w))
\]

\[
= -\sum_{i,j} (\partial_j D^k v_i, \partial_j (\chi_0^2 \partial_i D^k w)) + \sum_{i,j} (\partial_i (\chi_0^2 \partial_j D^k v_i, \partial_j D^k w))
\]

\[
+ \sum_{i,j} (\chi_0^2 \partial_i \partial_j D^k v_i, \partial_j D^k w))
\]

\[
= -\sum_{i,j} (\partial_j D^k v_i, \partial_j (\chi_0^2 \partial_i D^k w)) + \sum_{i,j} (\partial_i (\chi_0^2 \partial_j D^k v_i, \partial_j D^k w))
\]

\[
+ (\chi_0^2 \nabla div D^k v, \nabla D^k w)
\]

To derive

\[
1 \frac{T}{2} ||(\chi_0^2 D^k (\nabla div v - \Delta v), \nabla D^k w)| \leq \frac{\gamma}{2} ||\chi_0 \nabla D^k w||_{\frac{p}{p+1}}^2 + C ||\hat{\tau}' D^k v||_{H^1}^2.
\]

From (94), we have that for \( k = 1, 2, 3, \)

\[
h \frac{d}{dt} ||\chi_0 D^k w||_2^2 + ||\chi_0 D^k w||_{\frac{p}{p+1}}^2 \leq C (||\chi_0^2 D^k f, D^k w||_2) + ||\hat{\tau}' D^k v_r||_{H^1}^2 + ||\hat{\tau}' D^k v||_{H^1}^2.
\]

(95)

Adding (92) to (95) and choosing \( \eta_4 = \frac{1}{2} \) yield

\[
h \frac{d}{dt} (\gamma ||\chi_0 D^k w||_2^2 + ||\hat{\tau}' \chi_0 \nabla D^k v||_2^2) + ||\hat{\tau}' \chi_0 \nabla D^k v||_2^2 + ||\chi_0 D^k w||_{\frac{p}{p+1}}^2 \leq C (||\hat{\tau}' D^{-1} v||_{H^1}^2 + ||\chi_0^2 D^k f, D^k w||_2) + ||\hat{\tau}' D^{-1} g||_{L^2}^2
\]

(96)

Integrating (96) with respect to \( \tau \) over \( (\tau_0, T) \) yields (85).

Next we study the weighted energy estimates on \( (w, v) \) near the boundary \([0, T] \times \partial S_0\). For this, as in [30], it is convenient to use the spherical coordinates

\[
\begin{align*}
y_1 &= r \cos \theta, \\
y_2 &= r \sin \theta \cos \varphi, \\
y_3 &= r \sin \theta \sin \varphi,
\end{align*}
\]
and decompose \( v = (v_1, v_2, v_3) \) as
\[
\begin{align*}
  v_r &= \cos \theta v_1 + \sin \theta \cos \varphi v_2 + \sin \theta \sin \varphi v_3, \\
v_\theta &= -\sin \theta v_1 + \cos \theta \cos \varphi v_2 + \cos \theta \sin \varphi v_3, \\
v_\varphi &= -\sin \varphi v_2 + \cos \varphi v_3,
\end{align*}
\]
where \( \varphi \in [0, 2\pi] \) and \( \theta \in [0, \pi] \).

Set \( V = (v_r, v_\theta, v_\varphi)^T \) and \( V_T = (v_\theta, v_\varphi)^T \), and denote
\[
\begin{align*}
  \overline{\text{div}} V &= \partial_r v_r + \frac{1}{r} \partial_\theta v_\theta + \frac{1}{rsin \theta} \partial_\varphi v_\varphi, \\
  \Delta &= \partial_r^2 + \frac{1}{r^2} \partial_\theta^2 + \frac{1}{r^2 sin^2 \theta} \partial_\varphi^2, \\
  \nabla' &= \left( \frac{1}{r} \partial_\theta, \frac{1}{rsin \theta} \partial_\varphi \right)^T, \quad g_r = (g_\theta, g_\varphi)^T, \\
  \nabla' \text{div} &= \text{div} \nabla' \text{ as } \text{div} \text{ in the spherical coordinates.}
\end{align*}
\]

Lemma 3.6 (Weighted tangent energy estimates of \( (w, v) \)). For \( t \geq \tilde{t}_0, 0 < \eta < 1, \) and \( k = 1, 2, 3, \) we have
\[
\begin{align*}
  h \| \chi_1 \partial^k_T w \|_2^2 + h \| \tilde{\tau}' \chi_1 \partial^k_T V \|_2^2 + \int_{\tilde{t}_0}^t \| \tilde{\tau}' \chi_1 \nabla (\partial^k_T V) \|_2^2 d\tilde{\tau} \\
  \leq C(\eta) \int_{\tilde{t}_0}^t (\| \tilde{\tau}' L(\partial^k_T V, 0) \|_2^2 + \| (\partial^k_T f, \chi_1^2 \partial^k_T w) \|_2^2 + \| \tilde{\tau}' \partial^k_T \eta \|_2^2) d\tilde{\tau}
\end{align*}
\]
\[ + C \eta \int_{t_0}^t \| \frac{\partial \xi^k}{\partial \xi^k} \|_{L^2}^2 \, dt + C h \|(w_0, v_0)\|_{H^k}. \tag{101} \]

**Proof.** Applying $\partial \xi^k$ on both sides of (97)-(99) and using (100), we have

\[ 2h \partial \xi^k(\partial \xi^k w) + \frac{1}{\tau} \langle \partial \xi^k v, \partial \xi^k \rangle = \partial \xi^k f + \frac{1}{\tau} L(\partial \xi^k V, 0), \tag{102} \]
\[ 2h \partial \xi^k(\partial \xi^k v_r) + \frac{h}{\tau} \partial \xi^k v_r - \mu \mathbf{\Sigma}(\partial \xi^k v_r) - (\mu + \lambda) \partial_r \text{div}(\partial \xi^k V) + \frac{\gamma}{\tau \rho_{s+1}} \partial_r (\partial \xi^k w) = 0 \tag{103} \]
\[ \partial \xi^k g_r + L(\partial \xi^k V, 1), \tag{104} \]

It follows from \( \int_{S_0} \gamma \chi_1^2 (102) \partial \xi^k w \, dy \) that

\[ h \frac{d}{dt}(\gamma \| \chi_1 \partial \xi^k w \|^2_2) + \frac{\gamma}{\tau} \langle \text{div}(\partial \xi^k v_r), \chi_1^2 \partial \xi^k w \rangle = (\gamma \partial \xi^k, \chi_1^2 \partial \xi^k w) + \left( \frac{1}{\tau} \chi_1^2 L(\partial \xi^k V, 0), \partial \xi^k w \right). \tag{105} \]

In addition, computing \( \int_{S_0} \gamma \chi_1^2 (103) \partial \xi^k v_r \, dy \) and \( \int_{S_0} \gamma \chi_1^2 (104) \cdot \partial \xi^k V_r \, dy \) yields respectively,

\[ h \frac{d}{dt}(\| \partial \xi^k_v \|_2^2) + (1 - 2s') h \| \chi_1 \partial \xi^k_v \|^2_2 \]
\[ - (\gamma \chi_1^2 \partial \xi^k_v) (\mu \text{div}(\partial \xi^k v_r) + (\mu + \lambda) \partial_r \text{div}(\partial \xi^k V), \partial \xi^k v_r) + (\gamma \chi_1^2 \partial \xi^k (\partial \xi^k w), \partial \xi^k v_r) \tag{106} \]
\[ = (\partial \xi^k f_r + L(\partial \xi^k V, 1), \partial \xi^k v_r), \]

and

\[ h \frac{d}{dt}(\| \partial \xi^k V_r \|_2^2) + (1 - 2s') h \| \chi_1 \partial \xi^k V_r \|^2_2 \]
\[ - (\gamma \chi_1^2 \partial \xi^k V_r) (\mu \text{div}(\partial \xi^k v_r) + (\mu + \lambda) \text{div}(\partial \xi^k V), \partial \xi^k V_r) + (\gamma \chi_1^2 \partial \xi^k (\partial \xi^k w), \partial \xi^k V_r) \]
\[ = (\partial \xi^k g_r + L(\partial \xi^k V, 1), \partial \xi^k V_r). \tag{107} \]

Note that

\[ - (\gamma \chi_1^2 \partial \xi^k_V) (\mu \text{div}(\partial \xi^k v_r) + (\mu + \lambda) \partial_r \text{div}(\partial \xi^k V), \partial \xi^k v_r) \]
\[ = \mu \| \partial \xi^k_V \|_2^2 + \mu (\gamma \chi_1^2 \| \partial \xi^k v_r \|^2_2) + (\mu + \lambda) (\partial \xi^k (\partial \xi^k V), \partial \xi^k (\partial \xi^k v_r)), \tag{108} \]

and

\[ - (\gamma \chi_1^2 \partial \xi^k_V) (\mu \text{div}(\partial \xi^k v_r) + (\mu + \lambda) \text{div}(\partial \xi^k V), \partial \xi^k V_r) \]
\[ = \mu \| \partial \xi^k_V \|_2^2 + \mu (\gamma \chi_1^2 \| \partial \xi^k v_r \|^2_2) + (\mu + \lambda) (\partial \xi^k (\partial \xi^k V), \partial \xi^k (\partial \xi^k V_r)). \tag{109} \]
Adding (108) and (109) yields
\[
- (\chi_1^2 \frac{\partial^{2s_1}}{\partial t_1^{2s_1}} (\mu \Delta (\partial_k^2 v_r) + (\mu + \lambda) \partial_t \bar{w} (\partial_k^2 V), \partial_k^2 v_r)) \\
- (\chi_1^2 \frac{\partial^{2s_1}}{\partial t_1^{2s_1}} (\mu \Delta (\partial_k^2 v_r) + (\mu + \lambda) \partial_t \bar{w} (\partial_k^2 V), \partial_k^2 v_r)) \\
\geq \mu L \frac{h^{s_2}}{2} \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \nabla (\partial_k^2 V) \right\|_2^2 - C \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \nabla (\partial_k^2 V) \right\|_2^2.
\] (110)

Additionally, for small \( \eta \) > 0, one has
\[
\frac{\gamma}{\tau} \partial_t \partial_k^2 \bar{w} + (\chi_1^2 \frac{\partial^{2s_1}}{\partial t_1^{2s_1}} \partial_t \partial_k^2 \bar{w}, \partial_k^2 v_r) + (\chi_1 \frac{\partial^{2s_1}}{\partial t_1^{2s_1}} \nabla (\partial_k^2 V), \partial_k^2 V_t) \\
= - \frac{\gamma}{\tau} (\partial_k^2 \bar{w}, \nabla (\chi_1^2) \cdot \partial_k^2 V) \\
\leq \eta \left\| \frac{\partial_k^2 \bar{w}}{\partial t_1^{s_1}} \right\|_2^2 + C(\eta) \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \nabla (\partial_k^2 V) \right\|_2^2,
\] (111)

and
\[
\frac{\chi_1^2 \frac{\partial^{2s_1}}{\partial t_1^{2s_1}} L(\partial_k^2 V, 0), \partial_k^2 w) + \frac{\partial^{2s_2}}{\partial t_2^{2s_2}} \left( (\chi_1^2 L(\partial_k^2 V, 1), \partial_k^2 v_r) + (\chi_1^2 L(\partial_k^2 V, 1), \partial_k^2 V_t) \right) \\
\leq \mu \frac{L}{4} \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \nabla (\partial_k^2 V) \right\|_2^2 + \eta \left\| \frac{\partial_k^2 \bar{w}}{\partial t_1^{s_1}} \right\|_2^2 + C(\eta) \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} L(\partial_k^2 V, 0) \right\|_2^2.
\] (112)

Note that for small \( \eta_0 \) > 0,
\[
|\chi_1^2 \frac{\partial^{2s_1}}{\partial t_1^{2s_1}} \partial_k^2 g, \partial_k^2 V)| \\
\leq |\frac{\partial^{2s_1}}{\partial t_1^{2s_1}} \partial_k^2 - \frac{1}{\tau} \nabla (\chi_1^2) \cdot L + \frac{\partial^{2s_1}}{\partial t_1^{2s_1}} \partial_k^2 - \frac{1}{\tau} \nabla (\chi_1^2) \cdot V| + C \left( \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \nabla (\partial_k^2 V) \right\|_2^2 + C(\eta) \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \nabla (\partial_k^2 V) \right\|_2^2.
\] (113)

Combining (105), (106) and (107), together with (110)-(113), we have
\[
\frac{d}{d \tau} \left( h \left\| \chi_1 \partial_k^2 w \right\|_2^2 + h \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \nabla (\partial_k^2 V) \right\|_2^2 \right) \\
\leq \eta \left\| \frac{\partial_k^2 \bar{w}}{\partial t_1^{s_1}} \right\|_2^2 + C(\eta) \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} L(\partial_k^2 V, 0) \right\|_2^2 + C \left( \left\| \frac{\partial_k^2 f, \chi_1^2 \partial_k^2 w} {\partial t_2^{s_2}} \right\|_2^2 + C \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \partial_k^2 g \right\|_2^2.
\]

Integrating the above estimate over \( (\bar{\tau}, t) \) completes the proof of (101).  \( \square \)

We now turn to the estimates on the derivatives of \( w \) in normal direction.

**Lemma 3.7** (The first order normal derivative estimate of \( \partial_k^2 w \)). For \( t \geq \bar{\tau}, 0 < \eta < 1 \), and \( k = 0, 1, 2, \) we have
\[
h \left\| \chi_1 \partial_t \partial_k^2 w \right\|_2^2 + \int_{\bar{\tau}}^t \left( \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \nabla (\partial_k^2 V) \right\|_2^2 + \left\| \chi_1 \frac{\partial_t \partial_k^2 w}{R^{s_2 + 1}} \right\|_2^2 \right) d \tau \\
\leq Ch \left( \left\| w_0, v_0 \right\|_2^{s_2 + 1} + C \eta \int_{\bar{\tau}}^t \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \frac{\partial_k^{s_2 + 1} w}{\partial t_2^{s_2 + 1}} \right\|_2^2 d \tau \right) \\
+ \left| h \chi_1 \partial_t \partial_k^2 \chi_1 \frac{\partial_k^{s_2 + 1} w}{\partial t_2^{s_2 + 1}} \right|_2 + \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \chi_1 \frac{\partial_k^{s_2 + 1} V}{\partial t_2^{s_2 + 1}} \right\|_2^2 + \left( \chi_1 \frac{\partial_k^{s_2 + 1} \bar{w}} \partial t_2^{s_2 + 1} V + \left\| \frac{\partial^{s_2}}{\partial t_2^{s_2}} \partial_k^2 \bar{w} \right\|_2^2 \right) d \tau.
\] (114)

**Proof.** Firstly, we rewrite \( \partial_t (102) \) and (103) as
\[
2h (\partial_t \partial_k^2 w) + \frac{1}{\tau} \partial_t \partial_k^2 \partial_t v_r \\
= \frac{1}{\tau} L (\partial_k^{s_2 + 1} V, 1) + \partial_t \partial_k^2 f + \frac{1}{\tau} L (\partial_k^2 V, 1).
\] (115)
\begin{align}
2h \partial_\tau (\partial_t^k v_r) - (2\mu + \lambda) \partial_\tau \partial_t^k v_r + \frac{\gamma}{\bar{r}^2 + \epsilon} \partial_\tau (\partial_t^k w) \\
= \partial_t^k g_r + L(\partial_t^k V, 1) - \frac{h}{\bar{r}} \partial_t^k v_r + L(\partial_t^{k+2} V, 0). \tag{116}
\end{align}

Computing \( \frac{(116)}{(2\mu + \lambda)\bar{r}} + (115) \) yields
\begin{align}
2h (\partial_\tau \partial_t^k w)_{\bar{r}} + \frac{\gamma}{2\mu + \lambda} \partial_\tau \partial_t^k w \\
= \frac{1}{\bar{r}} L(\partial_t^{k+1} V, 1) + \partial_\tau \partial_t^k f + \frac{1}{\bar{r}} L(\partial_t^k V, 1) + \frac{1}{(2\mu + \lambda)\bar{r}} \left( \frac{1}{(2\mu + \lambda)\bar{r}} \right) \left( -2h \partial_\tau (\partial_t^k v_r) + \partial_t^k g_r \right) \tag{117}
\end{align}

It follows from \( \int_{S_0} \chi_1^2 (117) \partial_\tau \partial_t^k w dy \) that
\begin{align}
\frac{h}{d\bar{r}} \| \chi_1 \partial_\tau \partial_t^k w \|_2^2 + \| \chi_1 \partial_\tau \partial_t^k w \|_2^2 \\
\leq C \left( \| \bar{r} \chi_1 L(\partial_t^{k+1} V, 1) \|_2^2 + \| \bar{r} \chi_1 L(\partial_t^k V, 1) \|_2^2 \right) \tag{118}
\end{align}

Integrating (118) over \((\bar{r}_0, t)\) gives
\begin{align}
\frac{h}{d\bar{r}} \| \chi_1 \partial_\tau \partial_t^k w \|_2^2 + \int_{\bar{r}_0}^t \| \chi_1 \partial_\tau \partial_t^k w \|_2^2 \ d\bar{r} \\
\leq C h \| (w_0, u_0) \|_{H_{\bar{r}_0}^{k+1}}^2 + C \int_{\bar{r}_0}^t \left( \| \bar{r} \chi_1 L(\partial_t^{k+1} V, 1) \|_2^2 + \| \bar{r} \chi_1 L(\partial_t^k V, 1) \|_2^2 \right) \tag{119}
\end{align}

By (115), one has
\begin{align}
- (2\mu + \lambda) \partial_\tau \partial_t^k v_r \\
= - 2h \partial_\tau (\partial_t^k v_r) - \frac{\gamma}{\bar{r}^2 + \epsilon} \partial_\tau (\partial_t^k w) + \partial_t^k g_r + L(\partial_t^k V, 1) \tag{120}
\end{align}

Then estimating \( \int_{S_0} \chi_1^2 \bar{r}^{2s} (120) \partial_t^k v_r \ dy \) yields
\begin{align}
\| \chi_1 \bar{r}^{2s} \partial_\tau \partial_t^k v_r \|_2^2 \\
\leq C (\| \bar{r} \chi_1 \partial_\tau (\partial_t^k v_r) \|_2^2 + \| \bar{r} \chi_1 \partial_\tau \partial_t^k w \|_2^2 + \| \partial_\tau \partial_t^k g_r \|_2^2 \right) \tag{121}
\end{align}

On the other hand, by Lemma 3.6, (121) and the expression of \( \nabla \text{div} V \), we have that, for \( k = 0, 1, 2 \), and small \( \eta > 0 \),
\begin{align}
\int_{\bar{r}_0}^t \| \bar{r} \chi_1 \text{div}(\partial_t^k V) \|_2^2 \ d\bar{r}
\end{align}
This, together with (119), yields

\[
\leq C(\eta) \int_{\tau_0}^{t} \left( \|\chi_1 \tilde{x}' L(\partial_{t}^{k+1} V, 1)\|_2^2 + \|\partial_t^{k+1} f, \chi_1 \partial_t^{k+1} w\| + \|\chi_1 \tilde{x}' \partial_t^{k+1} g, \partial_t^{k+1} V\| \right) d\tilde{\tau}
\]

\[
+ C(\eta) \int_{\tau_0}^{t} \left( \|h \chi_1 \tilde{x}' \partial_t^{k+1} V, v\|_2^2 + \|\tilde{x}' \partial_t^{k+1} g, v\|_2^2 \right) d\tilde{\tau} + \|\chi_1 \tilde{x}' L(\partial_{t}^{k+2} V, 0)\|_2^2 d\tilde{\tau} + C(\eta) \int_{\tau_0}^{t} \|\partial_t^{k} Df, \partial_t^{k} Dv\|_2^2 d\tilde{\tau}.
\]

(122)

Thus, this together with (119), yields

\[
h\|\chi_1 \partial_t \partial_t^{k} w\|_2^2 + \int_{\tau_0}^{t} \left( \|\tilde{x}' \chi_1 \nabla \text{div} (\partial_t^{k} V)\|_2^2 + \|\chi_1 \partial_t \partial_t^{k} w\|_2^2 \right) d\tilde{\tau}
\]

\[
\leq C h\|(w_0, v_0)\|_{H^{k+1}}^2 + C(\eta) \int_{\tau_0}^{t} \|\partial_t^{k+1} w\|_2^2 d\tilde{\tau} + C(\eta) \int_{\tau_0}^{t} \|\tilde{x}' \chi_1 L(\partial_{t}^{k+1} V, 1)\|_2^2 d\tilde{\tau}
\]

\[
+ \|h \tilde{x}' \chi_1 (\partial_{t}^{k} V)_{\tilde{x}}\|_2^2 + \|\tilde{x}' \chi_1 L(\partial_{t}^{k+1} V, 1)\|_2^2 + \|\chi_1 \partial_t^{k} Df, \partial_t^{k} Dv\|_2^2 d\tilde{\tau}.
\]

Thus, the proof of Lemma 3.7 is completed. \qed

For later use in the next section, we state the following estimate that can be easily obtained by the standard regularity theory of the second order elliptic equation.

**Lemma 3.8.** For the Stokes equation in the domain \( S_0 \),

\[
\left\{ \begin{array}{l}
\text{div} u = f, \\
-\Delta u + \nabla P = g, \\
u = 0 \quad \text{on} \quad \partial S_0,
\end{array} \right.
\]

we have

\[
\|D^2 u\|_2^2 + \|DP\|_2^2 \leq C(\|f\|_{H^1}^2 + \|g\|_2^2).
\]

4. **Global energy estimates.** In this section, based on the estimates obtained in \( \S 3 \), we will establish the global energy estimates of the solution \((w, v)\) to (51)-(52). For this, we first define that for \( t_2 \geq t_1 \geq \tilde{\tau}_0 \) and \( k = 2, 3, \)

\[
N_k(t_1, t_2) = \sup_{t_1 \leq s \leq t_2} (h\|w\|_{H^{k+1}}^2 + h\|\tilde{x}' v\|_{H^{k+1}}^2 + h\|h v_{\tilde{\tau}}\|_{H^{k+1}}^2 + h\|h v_{\tilde{\tau}}\|_{H^{k+1}}^2)
\]

\[
+ \int_{t_1}^{t_2} \left( \|Dw\|_{H^{k+1}}^2 + \|h v_{\tilde{\tau}}\|_{H^{k+1}}^2 + \|\tilde{x}' Dv\|_{H^{k+1}}^2 + \|h v_{\tilde{\tau}}\|_{H^{k+1}}^2 + \|h v_{\tilde{\tau}}\|_{H^{k+1}}^2 \right) d\tilde{\tau}.
\]

To prove Theorem 1.1, we need to obtain the uniform \( H^3 \) estimates of \((w, v)\), i.e., the uniform estimates of \( N_3(\tilde{\tau}_0, t) \) for any \( t \geq \tilde{\tau}_0 \). Firstly, we will show

\[
N_3(\tilde{\tau}_0, t) \leq C \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h\|\tilde{x}' g(\tilde{\tau})\|_{H^3}^2 + C\|(w_0, v_0)\|_{H^3}^2 + C \int_{\tilde{\tau}_0}^{t} M(\tilde{\tau}) d\tilde{\tau},
\]

(123)

where

\[
M(\tilde{\tau})
= \|\tilde{x}' g(\tilde{\tau}), h v_{\tilde{\tau}}\| + \|\tilde{x}' h g_{\tilde{\tau}}, h^2 v_{\tilde{\tau}}\| + \|\tilde{x}' h g_{\tilde{\tau}}, h v_{\tilde{\tau}}\|
+ \sum_{k=0}^{3} \|D^k f, D^k w\| + \sum_{k=0}^{2} \|\tilde{x}' D^k g, D^k v\| + \|\tilde{x}' f, f_0\|_{H^3}^2
+ \|f\|_2^2 + \|\tilde{x}' g\|_{H^2}^2 + \|h g_{\tilde{\tau}}\|_2^2 + \|h f_{\tilde{\tau}}\|_2^2.
\]

(124)
and the definitions of $f, g, f_0$ and the number $s'$ are given in §3. We divide the proof of (123) into the following four steps.

**Step 1. The basic $L^2$-energy inequality of $(w, v)$**

By Lemma 3.1-3.3, we have that, for small $\eta > 0$ and $t \geq \bar{\tau}_0$,

$$h\|w\|^2_2 + h\|\hat{\tau}' v\|^2_2 + h\|\hat{\tau}' \nabla v\|^2_2 + h\|hw_{\hat{\tau}}\|^2_2 + h\|h\hat{\tau}'v_{\hat{\tau}}\|^2_2 + \int_{\bar{\tau}_0}^t (\|\hat{\tau}' \nabla v\|^2_2 + \|h\hat{\tau}'v_{\hat{\tau}}\|^2_2 + \|h\hat{\tau}' \nabla v_{\hat{\tau}}\|^2_2) d\bar{\tau}$$

$$\leq C h\|(w_0, v_0)\|^2_2 + C\eta h^2 \int_{\bar{\tau}_0}^t \|\nabla w_{\hat{\tau}}\|^2_2 d\bar{\tau} + C(\eta) \int_{\bar{\tau}_0}^t M(\bar{\tau}) d\bar{\tau},$$

where $M(\bar{\tau})$ is defined in (124).

**Step 2. The $H^1$-energy inequality of $(w, v)$**

By Lemma 3.5 with $k = 1$ and Lemma 3.1-Lemma 3.3, we have that, for small $\eta > 0$

$$h\|\chi_0 Dw\|^2_2 + h\|\hat{\tau}' \chi_0 Dw\|^2_2 + \int_{\bar{\tau}_0}^t (\|\hat{\tau}' \chi_0 \nabla Dw\|^2_2 + \|\chi_0 Dw_{\hat{\tau}}\|^2_2) d\bar{\tau}$$

$$\leq C \int_{\bar{\tau}_0}^t (\|\hat{\tau}' v\|^2_2 + \|h\hat{\tau}' v_{\hat{\tau}}\|^2_2 + \|\hat{\tau}' \nabla v\|^2_2 + M(\bar{\tau})) d\bar{\tau} + Ch\|(w_0, v_0)\|^2_2$$

$$\leq Ch\|(w_0, v_0)\|^2_2 + C\eta h^2 \int_{\bar{\tau}_0}^t \|\nabla w_{\hat{\tau}}\|^2_2 d\bar{\tau} + C(\eta) \int_{\bar{\tau}_0}^t M(\bar{\tau}) d\bar{\tau}.$$
This, together with (126), gives
\[ h\|Dw\|_2^2 + \int_{\tau_0}^t \|\tilde{s}' \nabla divv\|_2^2 d\tilde{\tau} \]
\[ \leq Ch\|(w_0, v_0)\|_{H^1}^2 + C\eta \int_{\tau_0}^t \|\nabla w_{\tilde{s}'+1}\|_2^2 + C(\eta) \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau}. \]  
(129)

Next, we rewrite (51)-(52) as
\[ \text{div}(\tilde{s}' v) = -\tilde{s}' w_{\tilde{s}'+1} + \tilde{s}' f_0, \]
and
\[ -\mu \tilde{s}' \Delta v + \nabla w_{\tilde{s}'+1} = -2h\tilde{s}' v + (\mu + \lambda) \nabla divv + \tilde{s}' g. \]

Then by Lemma 3.8 with \( u = \mu \tilde{s}' v \) and \( P = \frac{\gamma w}{\tilde{s}'+1} \), we obtain
\[ \|\tilde{s}' D^2v\|_2^2 + \|\nabla w_{\tilde{s}'+1}\|_2^2 \]
\[ \leq C\|\tilde{s}' w_{\tilde{s}'+1}\|_{H^1}^2 + C\|\tilde{s}' f_0\|_{H^1}^2 + C\|h\tilde{s}' v\|_2^2 \]
\[ + C\|\tilde{s}' v\|_2^2 + C\|\tilde{s}' \nabla divv\|_2^2 + C\|\tilde{s}' g\|_2^2. \]
(130)

From (51), we see that
\[ D(\frac{dw}{d\tilde{\tau}}) = -\frac{1}{\tilde{\tau}} Ddivv + Df_0. \]

Together with (129), this yields
\[ \int_{\tau_0}^t \|\tilde{s}' w_{\tilde{s}'+1}\|_{H^1}^2 d\tilde{\tau} \]
\[ \leq Ch\|(w_0, v_0)\|_{H^1}^2 + C\eta \int_{\tau_0}^t \|\nabla w_{\tilde{s}'+1}\|_2^2 + C(\eta) \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau}. \]
(131)

By applying Lemma 3.1 and (131), we have
\[ \int_{\tau_0}^t \|\tilde{s}' w_{\tilde{s}'+1}\|_{H^1}^2 d\tilde{\tau} \]
\[ \leq Ch\|(w_0, v_0)\|_{H^1}^2 + C\eta \int_{\tau_0}^t \|\nabla w_{\tilde{s}'+1}\|_2^2 + C(\eta) \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau}. \]
(132)

Substituting (125), (129), (132) into (130) yields
\[ \int_{\tau_0}^t \left(\|\tilde{s}' D^2v\|_2^2 + \|\nabla w_{\tilde{s}'+1}\|_2^2\right) d\tilde{\tau} \]
\[ \leq Ch\|(w_0, v_0)\|_{H^1}^2 + C\eta \int_{\tau_0}^t \|\nabla w_{\tilde{s}'+1}\|_2^2 + C(\eta) \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau}. \]

This, together with (129), derives for small \( \eta > 0 \),
\[ h\|Dw\|_2^2 + \int_{\tau_0}^t \left(\|\tilde{s}' D^2v\|_2^2 + \|\nabla w_{\tilde{s}'+1}\|_2^2\right) d\tilde{\tau} \]
\[ \leq Ch\|(w_0, v_0)\|_{H^1}^2 + C \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau}. \]  
(133)
(iii) The $H^2$-energy inequality of $(w,v)$

Firstly, we have that from (52)

$$-\mu \Delta v - (\mu + \lambda)\nabla \text{div} v = (-2hv_v - \frac{h}{\tau} v - \frac{\gamma}{\tau^{2s' + 1}} \nabla w + g).$$

By the regularity theory on the second order elliptic equation system, we obtain

$$\|\tilde{\tau}'^{s'} D^2 v\|_2^2 \leq C\tilde{\tau}^{2s'} - h v_v - \frac{h}{\tau} v - \frac{\gamma}{\tau^{2s' + 1}} \nabla w + g\|_2^2$$

$$\leq C(\|h\tilde{\tau}' v_v\|_2^2 + \|\tilde{\tau}' v\|_2^2 + \|\nabla w\|_2^2 + \|\tilde{\tau}' g\|_2^2).$$

This, together with (125) and (133), yields

$$\sup_{\tilde{\tau}_0 \leq \tau \leq \tilde{\tau}} h \|\tilde{\tau}'^{s'} D^2 v\|_2^2$$

$$\leq C h \|(w_0, v_0)\|_{H^1}^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau} + C \sup_{1 \leq \tau \leq t} h \|\tilde{\tau}' g(\tilde{\tau})\|_2^2.$$  \hspace{1cm} (134)

By Lemma 3.5 with $k = 2$, (133) and (125), we get for $t \geq \tilde{\tau}_0$,

$$h \|\chi_0 D^2 w\|_2^2 + \int_{\tilde{\tau}_0}^t \left( \|\tilde{\tau}'^{s'} \chi_0 D^3 v\|_2^2 + \|\chi_0 D^2 w_{\tau^{s' + 1}}\|_2^2 \right) d\tilde{\tau}$$

$$\leq C h \|(w_0, v_0)\|_{H^2}^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau}.$$  \hspace{1cm} (135)

By Lemma 3.6 with $k = 2$ and Lemma 3.7 with $k = 1$, one has respectively

$$h \|\chi_1 \partial^2 w\|_2^2 + \int_{\tilde{\tau}_0}^t \|\tilde{\tau}'^{s'} \chi_1 \nabla (\partial^2 V)\|_2^2 d\tilde{\tau}$$

$$\leq C h \|(w_0, v_0)\|_{H^2}^2 + C \int_{\tilde{\tau}_0}^t \|\chi_1 \partial^2 w_{\tau^{s' + 1}}\|_2^2 d\tilde{\tau} + C(\eta) \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau},$$  \hspace{1cm} (136)

and

$$h \|\chi_1 \partial_v \partial_t w\|_2^2 + \int_{\tilde{\tau}_0}^t \left( \|\tilde{\tau}'^{s'} \chi_1 \nabla \text{div}(\partial_t V)\|_2^2 + \|\chi_1 \partial_v \partial_t w_{\tau^{s' + 1}}\|_2^2 \right) d\tilde{\tau}$$

$$\leq h \|(w_0, v_0)\|_{H^2}^2 + C \int_{\tilde{\tau}_0}^t \|\chi_1 \partial_v \partial_t w_{\tau^{s' + 1}}\|_2^2 d\tilde{\tau} + C(\eta) \int_{\tilde{\tau}_0}^t \|\tilde{\tau}'^{s'} \chi_1 L(\partial^2 V, 1)\|_2^2 + M(\tilde{\tau}) d\tilde{\tau}. \hspace{1cm} (137)$$

Combining (136) and (137) gives

$$h \|\chi_1 \partial_t D w\|_2^2 + \int_{\tilde{\tau}_0}^t \|\tilde{\tau}'^{s'} \chi_1 \nabla \text{div}(\partial_t V)\|_2^2 d\tilde{\tau}$$

$$\leq C h \|(w_0, v_0)\|_{H^2}^2 + C \eta \int_{\tilde{\tau}_0}^t \|\chi_1 \partial^2 w_{\tau^{s' + 1}}\|_2^2 d\tilde{\tau} + C(\eta) \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau}. \hspace{1cm} (138)$$

And by combining (138) with (135), we obtain

$$h \|\partial_v D w\|_2^2 + \int_{\tilde{\tau}_0}^t \|\tilde{\tau}' \nabla \text{div}(\partial_v V)\|_2^2 d\tilde{\tau}$$

$$\leq C h \|(w_0, v_0)\|_{H^2}^2 + C \eta \int_{\tilde{\tau}_0}^t \|\partial_v D w_{\tau^{s' + 1}}\|_2^2 d\tilde{\tau} + C(\eta) \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau}. \hspace{1cm} (139)$$
In addition, we rewrite $\partial_t (51)$ and $\partial_t (52)$ as

$$
div(\tilde{\tau}' \partial_t v) = -\tilde{\tau}'^{+1} \partial_t (\frac{dw}{d\tilde{\tau}}) + \tilde{\tau}'^{+1} \partial_t f_0 + [div, \partial_t] \tilde{\tau}' v, \quad (140)$$

$$
- \tilde{\tau}' \mu \Delta \partial_t v + \frac{\gamma}{\tilde{\tau}'^{+1}} \tilde{\tau}' = -\tilde{\tau}' \mu(\Delta, \partial_t) [v] + [\nabla, \partial_t] \frac{\gamma}{\tilde{\tau}'^{+1}} - \tilde{\tau}' 2h \partial_t v_{\tilde{\tau}}
$$

$$- \tilde{\tau}'^{-1} h \partial_t v + \tilde{\tau}' (\mu + \lambda) \partial_t \nabla div v + \tilde{\tau}' \partial_t g. \quad (141)$$

In order to apply Lemma 3.8 to estimate $\tilde{\tau}' D^2 \partial_t v$ and $\tilde{\tau}' D \partial_t v$, we need to analyze each term on the right hand side of $(140)$-$(141)$. Firstly, from $(51)$ we have

$$
\partial_t D \frac{dw}{d\tilde{\tau}} = -\frac{1}{\tilde{\tau}} \partial_t Ddiv v + \partial_t f_0,
$$

and then by $(139)$ we obtain

$$
\int_{\tau_0}^t \left\| \tilde{\tau}'^{+1} \partial_t D \frac{dw}{d\tilde{\tau}} \right\|^2 d\tilde{\tau} \leq Ch \|(w_0, v_0)\|^2_{H^2} + C\eta \int_{\tau_0}^t \left\| \frac{\partial^2 w}{\tilde{\tau}'^{+1}} \right\|^2 d\tilde{\tau} + C (\eta) \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau}. \quad (142)
$$

This, together with $(132)$-$(133)$, yields

$$
\int_{\tau_0}^t \left\| \tilde{\tau}'^{+1} \partial_t D \frac{dw}{d\tilde{\tau}} \right\|^2 d\tilde{\tau} \leq Ch \|(w_0, v_0)\|^2_{H^2} + C\eta \int_{\tau_0}^t \left\| \frac{\partial^2 w}{\tilde{\tau}'^{+1}} \right\|^2 d\tilde{\tau} + C (\eta) \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau}. \quad (143)
$$

Then by applying Lemma 3.8 for $(140)$-$(141)$ and using $(142)$-$(143)$, we obtain

$$
\int_{\tau_0}^t \left( \left\| \tilde{\tau}' D^2 \partial_t v \right\|^2_2 + \left\| \frac{\partial_t D w}{\tilde{\tau}'^{+1}} \right\|^2_2 \right) d\tilde{\tau}
$$

$$\leq Ch \|(w_0, v_0)\|^2_{H^2} + C\eta \int_{\tau_0}^t \left\| \frac{\partial^2 w}{\tilde{\tau}'^{+1}} \right\|^2 d\tilde{\tau} + C (\eta) \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau}. \quad (144)
$$

This implies that for small $\eta > 0$,

$$
\int_{\tau_0}^t \left( \left\| \tilde{\tau}' D^2 \partial_t v \right\|^2_2 + \left\| \frac{\partial_t D w}{\tilde{\tau}'^{+1}} \right\|^2_2 \right) d\tilde{\tau} \leq Ch \|(w_0, v_0)\|^2_{H^2} + C \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau}. \quad (145)
$$

Next, we estimate $\|D^2 w\|_2^2$ and $\int_{\tau_0}^t \left\| \frac{D^2 w}{\tilde{\tau}'^{+1}} \right\|^2_2 d\tilde{\tau}$. It follows $\partial_r (117)$ with $k = 0$ that

$$
2h(\partial^2 w)_r + \frac{\gamma}{2(\mu + \lambda)} \tilde{\tau}'^{2s} \partial^2 w
$$

$$= \frac{1}{\tilde{\tau}} L (\partial^1 V, 2) + \partial^2 f + \frac{1}{\tilde{\tau}} L (V, 2) + \frac{1}{\tilde{\tau}} \left( -2h \partial_r (\partial_r v_r) + \partial_r g_r + L (V, 2) - \frac{h}{\tilde{\tau}} \partial_r v_r + L (\partial^2 V, 1) \right). \quad (145)
$$
Computing $\int_{S_0} \chi_1^2(145) \partial^2 \omega dy$ yields for small $\eta_1 > 0$,
\[
\frac{d}{dt} \| \chi_1 \phi^2 w \|_2^2 + \| \chi_1 \partial^2 \omega \|_2^2 \\
\leq C \| \tilde{\tau}^s L(\partial_t V, 2) \|_2^2 + \| \chi_1 \partial^2 \omega \|_2^2 + C |\tilde{\tau}^s L(V, 2) \|_2^2 \\
+ C |h \tilde{\tau}^s \| \partial_t (\partial_t v) \|_2^2 + C |\tilde{\tau}^s \| \partial_t g \|_2^2 + C |\tilde{\tau}^s \| \partial_t v \|_2^2 + \eta_1 \| \chi_1 \partial^2 \omega \|_2^2.
\]
By integrating (146) over $(\tilde{\tau}_0, t)$ and by Lemma 3.6-Lemma 3.7, we get
\[
h \| \chi_1 \partial^2 \omega \|_2^2 + \int_{\tilde{\tau}_0}^t \| \chi_1 \partial^2 \omega \|_2^2 d\tilde{\tau} \leq C h \|(w_0, v_0)\|_{H^2}^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau}.
\]
Combining (135), (139), (144) and (147) yields
\[
h \| D^2 w \|_2^2 + \int_{\tilde{\tau}_0}^t \| \tilde{\tau}^s D^2 \partial_t v \|_2^2 + \| \frac{D^2 w}{\tilde{\tau}^s + 1} \|_2^2 d\tilde{\tau} \leq C h \|(w_0, v_0)\|_{H^2}^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau}.
\]
By noting that $v$ satisfies the following second order elliptic equation system
\[
\begin{cases}
-\mu \Delta v - (\mu + \lambda) \nabla \text{div} v = -2h v + h \frac{d}{dt} v - \frac{\gamma}{\tilde{\tau}^s + 1} \nabla w + g, \\
v = 0 \text{ on } \partial S_0,
\end{cases}
\]
we have
\[
\|v\|_{H^3} \leq C \| -2h v - \frac{\gamma}{\tilde{\tau}^s + 1} \nabla w + g \|_{H^1}.
\]
This implies
\[
\| \tilde{\tau}^s v \|_{H^3} \leq C (\| \tilde{\tau}^s v \|_{H^1} + \| \tilde{\tau}^s v \|_{H^1} + \| \tilde{\tau}^s g \|_{H^1} + \| \nabla w \|_{H^1}).
\]
Combining (134), (148) and (151) yields
\[
h \| \tilde{\tau}^s D^2 v \|_2^2 + h \| D^2 w \|_2^2 + \int_{\tilde{\tau}_0}^t \| \tilde{\tau}^s D^2 v \|_2^2 + \| \frac{D^2 w}{\tilde{\tau}^s + 1} \|_2^2 d\tilde{\tau} \\
\leq C h \|(w_0, v_0)\|_{H^2}^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau} + C \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h \| \tilde{\tau}^s g(\tilde{\tau}) \|_2^2.
\]
Thus, by (125), (133) and (152), we have
\[
N_2(\tilde{\tau}_0, t) \leq C h \|(w_0, v_0)\|_{H^2}^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau} + C \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h \| \tilde{\tau}^s g(\tilde{\tau}) \|_2^2.
\]
(iv) The $H^3$-energy inequality of $(w, v)$

By Lemma 3.4 and (153), we have
\[
h \| \tilde{\tau}^s \nabla v \|_2^2 + \int_{\tilde{\tau}_0}^t (h \| \tilde{\tau}^s w \|_2^2 + h \| \tilde{\tau}^s g \|_2^2) d\tilde{\tau} \\
\leq C h \|(w_0, v_0)\|_{H^3}^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau} + C \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h \| \tilde{\tau}^s g(\tilde{\tau}) \|_2^2.
\]
And (151) and (153) give
\[
h \| \tilde{\tau}^s v \|_{H^3}^2 \leq C h \|(w_0, v_0)\|_{H^2}^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) d\tilde{\tau} + C \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h \| \tilde{\tau}^s g(\tilde{\tau}) \|_2^2.
\]
On the other hand, it follows from (149) that
\[-\mu\Delta v_\tau - (\mu + \lambda) \nabla d i v v_\tau = -2h v_\tau \tau - \left(\frac{h}{\tau}v_\tau\right) + g_\tau.\]  (156)

Note that
\[
\left\| -2h v_\tau \tau - \left(\frac{h}{\tau} v_\tau\right) + g_\tau \right\|_2 \leq C \left(\|h v_\tau\|_2 + \|\bar{v}_{\tau}^{-1}v_\tau\|_2 + \|\bar{v}_{\tau}^{-2}v_\tau\|_2 + \|\frac{\bar{v}_{\tau}^{\nabla w_\tau}}{\tau^{2s+1}}\|_2 + \|\frac{\nabla w_\tau}{\tau^{2s+2}}\|_2 + \|g_\tau\|_2\right). \]  (157)

Then it follows from (156)-(157) that
\[
\|h \tilde{v}_{\tau}^s D^2 v_\tau\|_2 \leq C \left(\|h^2 \tilde{v}_{\tau}^s v_\tau\|_2 + \|h \tilde{v}_{\tau}^{s-1} v_\tau\|_2 + \|h \tilde{v}_{\tau}^{s-2} v_\tau\|_2^2 + \|\tilde{v}_{\tau}^{s} g_\tau\|_2^2 + \|h \tilde{v}_{\tau}^{s} D^2 v_\tau\|_2 + \|h \tilde{v}_{\tau}^{s} v_\tau\|_2 + \|\frac{\bar{v}_{\tau}^{\nabla w_\tau}}{\tau^{2s+1}}\|_2 + \|\frac{\nabla w_\tau}{\tau^{2s+2}}\|_2\right). \]  (158)

This, together with (153)-(154), yields
\[
\int_{\tilde{\tau}_0}^t \|h \tilde{v}_{\tau}^{s} D^2 v_\tau\|_2^2 \, d\tilde{\tau} \leq C h \|v_0\|_H^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) \, d\tilde{\tau} \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h \|\tilde{v}_{\tau}^{s} g(\tilde{\tau})\|_2^2. \]  (159)

We now focus on the estimates on $D^3 w$. Firstly, by Lemma 3.5 with $k = 3$, we have the following estimate in the interior region:
\[
h \|\tilde{v}_{\tau}^{s} D^3 w\|_2^2 + \int_{\tilde{\tau}_0}^t \left(\|\tilde{v}_{\tau}^{s} \tilde{v}_{\tau} \nabla (\tilde{v}_{\tau}^{s} v_\tau)\|_2^2 + \|\tilde{v}_{\tau}^{s} \nabla \tilde{v}_{\tau} \nabla \tilde{v}_{\tau} v_\tau\|_2^2 + \|\tilde{v}_{\tau}^{s} \tilde{v}_{\tau} \nabla \tilde{v}_{\tau} v_\tau\|_2^2 + \|\tilde{v}_{\tau}^{s} \nabla \tilde{v}_{\tau} v_\tau\|_2^2\right) \, d\tilde{\tau} \leq C h \|v_0\|_H^2 + C \int_{\tilde{\tau}_0}^t M(\tilde{\tau}) \, d\tilde{\tau} + C \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h \|\tilde{v}_{\tau}^{s} g(\tilde{\tau})\|_2^2. \]  (160)

In addition, by Lemma 3.6 with $k = 3$ and Lemma 3.7 with $k = 2$, we have
\[
h \|\tilde{v}_{\tau}^{s} \chi_1 \nabla (\tilde{v}_{\tau}^{s} v_\tau)\|_2^2 + \int_{\tilde{\tau}_0}^t \left(\|\tilde{v}_{\tau}^{s} \nabla \tilde{v}_{\tau} \nabla (\tilde{v}_{\tau}^{s} v_\tau)\|_2^2 + \|\tilde{v}_{\tau}^{s} \chi_1 \nabla \tilde{v}_{\tau} \nabla \tilde{v}_{\tau} v_\tau\|_2^2 + \|\tilde{v}_{\tau}^{s} \chi_1 \nabla \tilde{v}_{\tau} v_\tau\|_2^2\right) \, d\tilde{\tau} \leq C \|v_0\|_H^2 + C \eta \int_{\tilde{\tau}_0}^t \left(\|\frac{D^3 w}{\tau^{2s+1}}\|_2^2 \, d\tilde{\tau} + \|\tilde{v}_{\tau}^{s} \nabla \tilde{v}_{\tau} v_\tau\|_2^2 + \|\tilde{v}_{\tau}^{s} \nabla \tilde{v}_{\tau} v_\tau\|_2^2 + M(\tilde{\tau})\right) \, d\tilde{\tau} \]  (161)

Combining (160) and (161) yields
\[
h \|\tilde{v}_{\tau}^{s} \nabla (\tilde{v}_{\tau}^{s} v_\tau)\|_2^2 + \int_{\tilde{\tau}_0}^t \left(\|\tilde{v}_{\tau}^{s} \nabla \tilde{v}_{\tau} \nabla (\tilde{v}_{\tau}^{s} v_\tau)\|_2^2 + \|\tilde{v}_{\tau}^{s} \chi_1 \nabla \tilde{v}_{\tau} \nabla \tilde{v}_{\tau} v_\tau\|_2^2 + \|\tilde{v}_{\tau}^{s} \chi_1 \nabla \tilde{v}_{\tau} v_\tau\|_2^2\right) \, d\tilde{\tau} \leq C h \|v_0\|_H^2 + C \eta \int_{\tilde{\tau}_0}^t \left(\|\frac{D^3 w}{\tau^{2s+1}}\|_2^2 \, d\tilde{\tau} + \|\tilde{v}_{\tau}^{s} \nabla \tilde{v}_{\tau} v_\tau\|_2^2 + \|\tilde{v}_{\tau}^{s} \nabla \tilde{v}_{\tau} v_\tau\|_2^2 + M(\tilde{\tau})\right) \, d\tilde{\tau} + C(\eta) \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h \|\tilde{v}_{\tau}^{s} g(\tilde{\tau})\|_2^2. \]  (162)
We now rewrite \( \partial^2_t (51) \) and \( \partial^2_t (52) \) as

\[
\begin{cases}
\text{div}(\hat{\tau}' \partial^2_t v) = -\hat{\tau}'^{s+1} \partial^2_t (\frac{dw}{dt}) + \hat{\tau}'^{s+1} \partial^2_t f_0 + [\text{div}, \partial^2_t] \hat{\tau}' v, \\
- \hat{\tau}' \mu \partial^2_t v + \frac{\gamma \nabla \partial^2_t w}{\hat{\tau}'^{s+1}} = -\hat{\tau}' \mu [\Delta, \partial^2_t] v + [\nabla, \partial^2_t] \frac{\gamma w}{\hat{\tau}'^{s+1}} - 2h \hat{\tau}' \partial^2_t \varepsilon + \hat{\tau}' \nabla \text{div} v + \hat{\tau}' \partial^2_t g.
\end{cases}
\]

(163)

As in (iii), in order to apply Lemma \ref{lem:3.8} to estimate \( \hat{\tau}' D^2 \partial^2_t v \) and \( \hat{\tau}' D (\frac{\partial^2_t w}{\hat{\tau}'^{s+1}}) \), we need to analyze the terms on the right hand side of (163). Firstly, from (51) we have

\[
\partial^2_t D (\frac{dw}{dt}) = -\frac{1}{\tau} \partial^2_t D \text{div} v + \partial^2_t D f_0.
\]

Together with (162), this yields

\[
\int_{\tau_0}^{t} \|\hat{\tau}'^{s+1} \partial^2_t (\frac{dw}{dt})\|_{L^2}^2 d\tilde{t} \leq C h \|(w_0, v_0)\|_{H^3} + C \eta \int_{\tau_0}^{t} \|\frac{D^3 w}{\hat{\tau}'^{s+1}}\|_{L^2}^2 d\tilde{t} + C(\eta) \int_{\tau_0}^{t} M(\tilde{t}) d\tilde{t}
\]

(164)

\[
+ C(\eta) \sup_{\tau_0 \leq \tilde{t} \leq t} h \|\hat{\tau}' g(\tilde{t})\|_2.
\]

Combining (142) and (164) yields

\[
\int_{\tau_0}^{t} \|\hat{\tau}'^{s+1} \partial^2_t (\frac{dw}{dt})\|_{H^1}^2 d\tilde{t} \leq C h \|(w_0, v_0)\|_{H^3} + C \eta \int_{\tau_0}^{t} \|\frac{D^3 w}{\hat{\tau}'^{s+1}}\|_{L^2}^2 d\tilde{t} + C(\eta) \int_{\tau_0}^{t} M(\tilde{t}) d\tilde{t}
\]

(165)

\[
+ C(\eta) \sup_{\tau_0 \leq \tilde{t} \leq t} h \|\hat{\tau}' g(\tilde{t})\|_2.
\]

It follows from direct computation that

\[
\begin{align*}
\| [\text{div}, \partial^2_t] \hat{\tau}' v \|_2^2 & \leq C \|\hat{\tau}' D^2 v\|_2^2, \\
\| \hat{\tau}' [\Delta, \partial^2_t] v \|_2^2 & \leq C \|\hat{\tau}' D^3 v\|_2^2, \\
\| [\nabla, \partial^2_t] \frac{w}{\hat{\tau}'^{s+1}} \|_2^2 & \leq C \|\frac{D^2 w}{\hat{\tau}'^{s+1}}\|_2^2, \\
\| \hat{\tau}' \partial^2_t \nabla \text{div} v \|_2^2 & \leq C (\|\hat{\tau}' \nabla \text{div} \partial^2_t v\|_2^2 + \|\hat{\tau}' D^3 v\|_2^2).
\end{align*}
\]

(166)

By Lemmas \ref{lem:3.6}-Lemma \ref{lem:3.8}, we obtain

\[
\int_{\tau_0}^{t} \left( \|\hat{\tau}' D^2 \partial^2_t v\|_2^2 + \|\frac{\partial^2_t D w}{\hat{\tau}'^{s+1}}\|_2^2 \right) d\tilde{t} \leq C h \|(w_0, v_0)\|_{H^3} + C \eta \int_{\tau_0}^{t} \|\frac{D^3 w}{\hat{\tau}'^{s+1}}\|_{L^2}^2 d\tilde{t} + C(\eta) \int_{\tau_0}^{t} M(\tilde{t}) d\tilde{t}
\]

(167)

\[
+ C(\eta) \sup_{\tau_0 \leq \tilde{t} \leq t} h \|\hat{\tau}' g(\tilde{t})\|_2.
\]

On the other hand, we rewrite \( \partial_r (97), (98) \) and (99) as

\[
2h (\partial_r w)_r + \frac{1}{\hat{\tau}} \partial^2_r v_r = \partial_r f + \frac{1}{\hat{\tau}} L(V, 1) + \frac{1}{\hat{\tau}} (\frac{1}{r} \partial^2_r \theta) + \frac{1}{r \sin \theta} \partial^2_r \psi_r.
\]

(168)
Consider $\partial^2_t [(168) + \frac{1}{\tau (2\mu + \lambda)} (169)]$. By direct computation, we have

\[
2h(\partial^3_t w)_r + \frac{\gamma}{2\mu + \lambda} \partial^3_r w = \partial^3 f + \frac{1}{\tau} L(V, 3) + \frac{1}{\tau} L(\partial^3_t \partial_r V_1, 0) + \frac{1}{\tau (2\mu + \lambda)} \left( -2h \partial_r \partial^2_t v_r + L(\partial^2_t \partial_r V_1, 0) + \partial^2_t g_r + L(V, 3) \right).
\]

Computing $\int_{\mathbb{S}^d} \chi^2 (171) \partial^3_t w \, dy$ yields for small $\eta_1 > 0$,

\[
h \frac{d}{dt} \| \chi_1 \partial^3_r w \|_2^2 + \int_{\tau_0}^t \| \chi_1 \partial^3_r w \|_2^2 + \int_{\tau_0}^t \| \chi_1 \partial^3_r w \|_2^2 \, d\tilde{\tau} \leq C \| \tilde{x}^s Dv \|_{\mathbb{S}^d} + \| (\partial^3_t \partial_r w, \partial^3_r w) \| + C \| \tilde{x}^s L(\partial^3_t \partial_r V_1, 0) \|_2^2 + C \| \tilde{x}^s \eta_1 \|_2^2.
\]

Integrating (172) over $(\tilde{\tau}, t)$ and applying Lemma 3.6-Lemma 3.7, we have

\[
h \| \chi_1 \partial^3_r w \|_2^2 + \int_{\tau_0}^t \| \chi_1 \partial^3_r w \|_2^2 \, d\tilde{\tau} \leq C \| (w_0, v_0) \|_2^2 + C \int_{\tau_0}^t \| \tilde{x}^s L(\partial^3_t \partial_r V_1, 0) \|_2^2 + M(\tilde{\tau}) \, d\tilde{\tau} + \int_{\tau_0}^t \| \tilde{x}^s g(\tilde{\tau}) \|_2^2.
\]

From the equation $\partial, \partial_r (170)$, we see that

\[-\partial^3_t \partial_r V_1 = -2h \partial_r \partial_r \partial_r V_1 + L(\partial^3_t \partial_r V_1, 0) + L(\partial^2_t \partial_r V_1, 0) - \frac{\gamma}{2\mu + \lambda} \partial^3_r w + \partial_r \partial_r g_r + L(V, 3).
\]

This implies

\[
\| \tilde{x}^s \partial^3_t \partial_r V_1 \|_2^2 \leq C \| (h \tilde{x}^s D^2 v_r) \|_2^2 + \| \tilde{x}^s \partial^2_t D^2 v_r \|_2^2 + \| \tilde{x}^s D^2 g_v \|_2^2 + \| \tilde{x}^s D^3 v_r \|_2^2 + \| \frac{\partial^3_t Dw}{2} \|_2^2.
\]

Substituting (174) into (173) and using (151), we have

\[
h \| \chi_1 \partial^3_r w \|_2^2 + \int_{\tau_0}^t \| \chi_1 \partial^3_r w \|_2^2 \, d\tilde{\tau} \leq C \| (w_0, v_0) \|_2^2 + C(\eta) \int_{\tau_0}^t M(\tilde{\tau}) d\tilde{\tau} + C \eta \int_{\tau_0}^t \| \frac{D^3 v_r}{2} \|_2^2 \, d\tilde{\tau} + \int_{\tau_0}^t \| \tilde{x}^s g(\tilde{\tau}) \|_2^2.
\]
Combining (162), (167) with (173)-(175), we obtain
\[ h\|D^3w\|^2_2 + \int_{\tilde{\tau}_0}^{t} (\|\tilde{\tau}'\|_D^4\|v\|_2^2 + \|D^3w\|_\tau^2) d\tilde{\tau} \]
\[ \leq Ch\|(w_0,v_0)\|_{H^3} + C \int_{\tilde{\tau}_0}^{t} M(\tilde{\tau}) d\tilde{\tau} + C \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h\|\tilde{\tau}'\| g(\tilde{\tau}) \|^2_2. \]

This together with (153) derives
\[ N_3(\tilde{\tau}_0, t) \leq Ch\|(w_0,v_0)\|_{H^3} + C \int_{\tilde{\tau}_0}^{t} M(\tilde{\tau}) d\tilde{\tau} + C \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} h\|\tilde{\tau}'\| g(\tilde{\tau}) \|^2_2. \] (176)

Based on the estimates in the above Step 1-4, we have the following

**Proposition 1.** For \( t \geq \tilde{\tau}_0 \), we have
\[ N_3(\tilde{\tau}_0, t) \leq Ch\|(w_0,v_0)\|_{H^3} + C \left( h^{-\frac{1}{2}} N_3^3(\tilde{\tau}_0, t) + \sum_{i=2}^{4} h^{1-i} N_3^i(\tilde{\tau}_0, t) \right). \] (177)

**Proof.** To prove (177), we need to estimate the terms on the right hand side of (176). By the definition of \( M(\tilde{\tau}) \), we only need to treat the terms such as \(|(D^3f, D^3w)|\), \(\|\tilde{\tau}^{\frac{1}{2}} f_0\|_{H^3}^2\), \(\|\tilde{\tau}^{\frac{1}{2}} g\|_{H^2}^2\), \(\|h\tilde{\tau}^{\frac{1}{2}} g\|_{L^2}^2\) and \(\|h f_\tau\|_{L^2}^2\).

For \((D^3f, D^3w)\), we have
\[ (D^3f, D^3w) = -\int_{S_0} \frac{1}{\tilde{\tau}} (D^3(\tilde{\tau} f) D^3w + D^3(v \cdot \nabla w) D^3w) dy \]
\[ = -\int_{S_0} \frac{1}{\tilde{\tau}} (D^3(\tilde{\tau} f) D^3w + [D^3, v \cdot \nabla] w D^3w + v \cdot \nabla D^3w D^3w) dy \]
\[ = -\int_{S_0} \frac{1}{\tilde{\tau}} (D^3(\tilde{\tau} f) D^3w + [D^3, v \cdot \nabla] w D^3w - \frac{1}{2} \text{div}(D^3w)^2) dy. \] (178)

We now estimate each term on the right hand side of (178). Note that
\[ \int_{S_0} \frac{1}{\tilde{\tau}} (D^3(\tilde{\tau} f) D^3w) dy \]
\[ = \frac{1}{\tilde{\tau}} \int_{S_0} (D^3(w \tilde{\tau} f) D^3w + 3D^2w Ddivw D^3w + 3Dw D^2 Ddivw D^3w + D^3div D^3w) dy. \] (179)

In addition, since
\[ \frac{1}{\tilde{\tau}} \left| \int_{S_0} (D^3 w \tilde{\tau} f) D^3w dy \right| \leq C\|\tilde{\tau}'\| \|D^3w\|_{L^\infty} \|D^3w\|_2 \]
\[ \leq C\|\tilde{\tau}'\| \|D^3w\|_{H^2} \|D^3w\|_2 \]
\[ \leq C(\|\tilde{\tau}'\|_{H^2}^2 + \|D^3w\|_{H^2}^2) \|D^3w\|_2, \]

we have
\[ \int_{\tilde{\tau}_0}^{t} \frac{1}{\tilde{\tau}} \int_{S_0} (D^3 w \tilde{\tau} f) D^3w dy d\tilde{\tau} \]
\[ \leq C \int_{\tilde{\tau}_0}^{t} \left( \|\tilde{\tau}'\|_{H^2}^2 + \|D^3w\|_{H^2}^2 \right) \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} \|D^3w(\tilde{\tau})\|_2. \]
Similarly,

\[
\frac{1}{\tau} \int_{S_0} (D^3wD\text{div}vD^3w)dy \leq C||\tilde{\tau}'D\text{div}v||_{L^\infty}||D^3w||_{L^2} \\
\leq C(||\tilde{\tau}'\text{div}v||^2_{H^3} + ||D^2w||^2_{\frac{2}{\tau' + 1}})||D^3w||_{L^2},
\]

which implies

\[
\int_{\tilde{\tau}_0}^t \frac{1}{\tau} \int_{S_0} (D^2wD\text{div}vD^3w)dy|d\tilde{\tau} \leq \text{Ch}^{-\frac{1}{2}} N_3^2(\tilde{\tau}_0, t).
\]

We also have

\[
\int_{\tilde{\tau}_0}^t \frac{1}{\tau} \int_{S_0} (DwD^2\text{div}vD^3w)dy|d\tilde{\tau} \\
\leq \text{C} \int_{\tilde{\tau}_0}^t ||\tilde{\tau}'D^2\text{div}v||_{L^2} ||D^3w||_{L^2} ||Dw||_{L^2} d\tilde{\tau} \\
\leq \text{C} \int_{\tilde{\tau}_0}^t ||\tilde{\tau}'\text{div}v||_{H^3} ||D^3w||_{L^2} ||Dw||_{H^2} d\tilde{\tau} \\
\leq \text{C} \int_{\tilde{\tau}_0}^t ((||\tilde{\tau}'\text{div}v||^2_{H^3} + ||D^2w||^2_{\frac{2}{\tau' + 1}})) d\tilde{\tau} \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} ||Dw||_{H^2} \\
\leq \text{Ch}^{-\frac{1}{2}} N_3^3(\tilde{\tau}_0, t),
\]

and

\[
\int_{\tilde{\tau}_0}^t \frac{1}{\tau} \int_{S_0} (wD^3\text{div}vD^3w)dy|d\tilde{\tau} \\
\leq \text{C} \int_{\tilde{\tau}_0}^t ||\tilde{\tau}'D^3\text{div}v||_{L^2} ||D^3w||_{L^2} |\leq ||D^3w||_{H^2} d\tilde{\tau} \\
\leq \text{C} \int_{\tilde{\tau}_0}^t (||\tilde{\tau}'\text{div}v||^2_{H^3} + ||D^2w||^2_{\frac{2}{\tau' + 1}}) d\tilde{\tau} \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} ||w||_{H^2} \\
\leq \text{Ch}^{-\frac{1}{2}} N_3^3(1, t).
\]

Substituting (180)-(183) into (179) yields

\[
\int_{\tilde{\tau}_0}^t \frac{1}{\tau} \int_{S_0} (D^3(\text{div}v)D^3w)dy|d\tilde{\tau} \leq \text{Ch}^{-\frac{1}{2}} N_3^3(\tilde{\tau}_0, t).
\]

Similarly, we can obtain

\[
\int_{\tilde{\tau}_0}^t \frac{1}{\tau} \int_{S_0} ((D^3f, v \cdot \nabla)D^3w - \frac{1}{2} \text{div}(D^3w)^2)dy|d\tilde{\tau} \leq \text{Ch}^{-\frac{1}{2}} N_3^3(\tilde{\tau}_0, t).
\]

Inserting (184)-(185) into (178), we have

\[
\int_{\tilde{\tau}_0}^t \frac{1}{\tau} (D^3f, D^3w)|d\tilde{\tau} \leq \text{Ch}^{-\frac{1}{2}} N_3^3(\tilde{\tau}_0, t).
\]

For the term $||\tilde{\tau}'+1 f_0||^2_{H^3}$, we only estimate $||\tilde{\tau}'+1 D^3f_0||^2_{L^2}$ because the rest terms in $||\tilde{\tau}'+1 f_0||^2_{H^3}$ can be more easily estimated. It follows from direct computation
that
\[
\|\tilde{\tau}^{s+1}D^3f_0\|_2^2
= \tilde{\tau}^{2s'} \int_{S_0} |D^3\omega\text{div}\nu|^2 dy + 3\tilde{\tau}^{2s'} \int_{S_0} |D^2\omega D\text{div}\nu|^2 dy
+ 3\tilde{\tau}^{2s'} \int_{S_0} |D\omega D^2\text{div}\nu|^2 dy
+ \tilde{\tau}^{2s'} \int_{S_0} |D\omega D^3\text{div}\nu|^2 dy
\leq C(\|\tilde{\tau}^{s'}\text{div}\nu\|_{L^\infty}^2 + \|\tilde{\tau}^{s'} D\text{div}\nu\|_{L^\infty}^2)\|D^2\omega\|_{H^1}^2
+ C\|\tilde{\tau}^{s'} D^2\text{div}\nu\|_{H^1}^2 (\|\omega\|_{L^\infty}^2 + \|D\omega\|_{L^\infty}^2)
\leq C\|\tilde{\tau}^{s'} Dv\|_{H^3}^2 \|\omega\|_{H^3}^2.
\]
This yields
\[
\int_{\tilde{\tau}_0}^t \|\tilde{\tau}^{s'+1}D^3f_0\|_2^2 d\tilde{\tau} \leq Ch^{-1}N_3^3(\tilde{\tau}_0, t).
\tag{187}
\]
As for \(\|\tilde{\tau}^{s'}g\|_{H^2}^2\), it suffices to estimate \(\|\tilde{\tau}^{s'} D^2g\|_{L^2}^2\). Note that
\[
g = -\frac{1}{\tilde{\tau}} v \cdot \nabla v + \frac{\gamma}{\tilde{\tau}^{s'+1}} \frac{w}{1 + w} \nabla w - \frac{1}{\tilde{\tau}^{s'+1}} \frac{1}{1 + w} \nabla P_1(w) - \frac{w}{1 + w} L v.
\tag{188}
\]
To estimate \(\|\tilde{\tau}^{s'} D^2g\|_{L^2}^2\), we need to deal with each term in the expression of \(\tilde{\tau}^{s'} D^2g\). It follows from Hölder inequality, Sobolev imbedding inequality, and direct computation that
\[
\int_{\tilde{\tau}_0}^t \|\tilde{\tau}^{s'} D^2(\frac{1}{\tilde{\tau}} v \cdot \nabla v)\|_{L^2}^2 d\tilde{\tau} \leq Ch^{-1}N_3^2(\tilde{\tau}_0, t).
\tag{189}
\]
And
\[
\|\tilde{\tau}^{s'} D^2(\frac{\gamma}{\tilde{\tau}^{s'+1}} \frac{w}{1 + w} \nabla w)\|_{L^2}^2
\leq \|\frac{\gamma}{\tilde{\tau}^{s'+1}} \frac{w}{1 + w} D^2\nabla w\|_{L^2}^2 + 2\|\frac{\gamma}{\tilde{\tau}^{s'+1}} \frac{Dw}{1 + w} D\nabla w\|_{L^2}^2
+ \|\frac{\gamma}{\tilde{\tau}^{s'+1}} \frac{2(Dw)^2 - (1 + w)D^2w}{(1 + w)^3} \nabla w\|_{L^2}^2
\leq C\|\frac{Dw}{\tilde{\tau}^{s'+1}}\|_{H^2}^2 (\|w\|_{L^\infty}^2 + \|Dw\|_{L^\infty}^2 + \|Dw\|_{L^\infty}^4)
\leq C\|\frac{Dw}{\tilde{\tau}^{s'+1}}\|_{H^2}^2 (\|w\|_{H^3}^2 + \|w\|_{H^3}^4),
\]
which implies
\[
\int_{\tilde{\tau}_0}^t \|\tilde{\tau}^{s'} D^2(\frac{\gamma}{\tilde{\tau}^{s'+1}} \frac{w}{1 + w} \nabla w)\|_{L^2}^2 d\tilde{\tau} \leq C(h^{-2}N_3^3(\tilde{\tau}_0, t) + h^{-1}N_3^3(\tilde{\tau}_0, t)).
\tag{190}
\]
Similarly, one has
\[
\int_{\tilde{\tau}_0}^t \|\tilde{\tau}^{s'} D^2(\frac{1}{\tilde{\tau}^{s'+1}} \frac{1}{1 + w} \nabla P_1(w))\|_{L^2}^2 d\tilde{\tau} \leq C\sum_{i=2}^{4} h^{-i-1}N_3^3(\tilde{\tau}_0, t).
\tag{191}
\]
For the term \(\|\tilde{\tau}^{s'} \frac{w}{1 + w} L v\|_{H^2}^2\), we have
\[
\|\tilde{\tau}^{s'} D^2(\frac{w}{1 + w} L v)\|_{L^2}^2
\leq C(\|w\|_{L^\infty}^2 + \|Dw\|_{L^\infty}^2)\|\tilde{\tau}^{s'} Dv\|_{H^2}^2 + C\|\tilde{\tau}^{s'} D^2w D^2v\|_{L^2}^2.
\]
\[ \leq C\|w\|_{H^2}^2 \|\tilde{\tau}' Dv\|_{H^2}^2 + C\|D^2w\|_2^2 \|\tilde{\tau}' D^2v\|_{L^\infty}^2 \]
\[ \leq C\|w\|_{H^2}^2 \|\tilde{\tau}' Dv\|_{H^2}^2, \]

which yields
\[ \int_{\tilde{\tau}_0}^{t} \|\tilde{\tau}' D^2\left(\frac{w}{1+w}Lv\right)\|_{L^2}^2 d\tilde{\tau} \leq C \sup_{\tilde{\tau}_0 \leq \tilde{\tau} \leq t} \|w\|_{H^2}^2 \int_{\tilde{\tau}_0}^{t} \|\tilde{\tau}' Dv\|_{H^2}^2 d\tilde{\tau} \leq Ch^{-1}N^2_3(\tilde{\tau}_0, t). \quad (192) \]

Thus, by (189)-(192) and (188), we have
\[ \int_{\tilde{\tau}_0}^{t} \|\tilde{\tau}'\|_{H^2}^2 d\tilde{\tau} \leq C \left( h^{-\frac{1}{2}}N^2_3(\tilde{\tau}_0, t) + h^{-1}N^2_3(\tilde{\tau}_0, t) \right. \]
\[ \left. + h^{-2}N^3_3(\tilde{\tau}_0, t) + \sum_{i=2}^{4} h^{1-i}N^i_3(\tilde{\tau}_0, t) \right). \quad (193) \]

Next, we estimate \(\|h\tilde{\tau}' g_{\tilde{\tau}}\|_{2}^2\). By the expression of \(g\) given in (188), we need to deal with the terms \(\tilde{\tau}'\left(\frac{1}{\tilde{\tau}} v \cdot \nabla v\right)_{\tilde{\tau}}\) and \(\tilde{\tau}'\left(\frac{\gamma w Dw}{\tilde{\tau}^{2s+1}(1+w)}\right)_{\tilde{\tau}}, \tilde{\tau}'\left(\frac{\nabla P_1(w)}{\tilde{\tau}^{2s+1}(1+w)}\right)_{\tilde{\tau}}, \tilde{\tau}'\left(\frac{w}{1+w}Lv\right)_{\tilde{\tau}},\) separately. It follows from direct computation that
\[ \|\tilde{\tau}'\left(\frac{1}{\tilde{\tau}} v \cdot \nabla v\right)_{\tilde{\tau}}\|_{2}^2 \leq C(\|\tilde{\tau}' v \cdot \nabla v\|_{2}^2 + \|\tilde{\tau}' v\|_{2}^2 + \|\tilde{\tau}' v_{\tilde{\tau}}\|_{2}^2) \]
\[ \leq (\|v\|_{L^\infty}^2 + \|Dv\|_{L^2}^2)(\|\tilde{\tau}' v\|_{2}^2 + \|\tilde{\tau}' v_{\tilde{\tau}}\|_{2}^2 + \|\tilde{\tau}' Dv_{\tilde{\tau}}\|_{2}^2) \]
\[ \leq C\|\tilde{\tau}' v\|_{H^{2}}^2 (\|\tilde{\tau}' v\|_{2}^2 + \|\tilde{\tau}' v_{\tilde{\tau}}\|_{2}^2 + \|\tilde{\tau}' Dv_{\tilde{\tau}}\|_{2}^2), \]

which leads to
\[ \int_{\tilde{\tau}_0}^{t} \|h\tilde{\tau}'\left(\frac{1}{\tilde{\tau}} v \cdot \nabla v\right)_{\tilde{\tau}}\|_{2}^2 d\tilde{\tau} \leq Ch^{-1}N^2_3(\tilde{\tau}_0, t). \quad (194) \]

And
\[ \|\tilde{\tau}'\left(\frac{\gamma w Dw}{\tilde{\tau}^{2s+1}(1+w)}\right)_{\tilde{\tau}}\|_{2}^2 \]
\[ \leq C\left(\frac{1}{\tilde{\tau}^{s+2}} \|Dw\|_{L^2} \|\tilde{\tau}\|_{L^\infty}^2 + \|\frac{Dw}{\tilde{\tau}^{s+1}(1+w)}\|_{2}^2 \right. \]
\[ \left. + \|\frac{Dw}{\tilde{\tau}^{s+1}(1+w)}\|_{2}^2 \right) \]
\[ \leq C\left(\|w\|_{L^\infty}^2 + \|Dw\|_{L^2} \|\tilde{\tau}\|_{L^\infty}^2 + \|Dw\|_{L^2} \|\tilde{\tau}\|_{L^\infty}^2 + \|Dw\|_{L^2} \|\tilde{\tau}\|_{L^\infty}^2 \right), \]

which yields
\[ \int_{\tilde{\tau}_0}^{t} \|h\tilde{\tau}'\left(\frac{\gamma w Dw}{\tilde{\tau}^{2s+1}(1+w)}\right)_{\tilde{\tau}}\|_{2}^2 d\tilde{\tau} \leq Ch^{-1}N^2_3(\tilde{\tau}_0, t). \quad (195) \]

Similarly,
\[ \int_{\tilde{\tau}_0}^{t} \|h\tilde{\tau}'\left(\frac{\nabla P_1(w)}{\tilde{\tau}^{2s+1}(1+w)}\right)_{\tilde{\tau}}\|_{2}^2 d\tilde{\tau} \leq C \sum_{i=2}^{4} h^{1-i}N^i_3(\tilde{\tau}_0, t). \quad (196) \]

In addition, we have
\[ \|\tilde{\tau}'\left(\frac{w}{1+w}Lv\right)_{\tilde{\tau}}\|_{2}^2 \]
\[ \leq C\|\tilde{\tau}'\left(\frac{w}{1+w}D^2v\right)_{\tilde{\tau}}\|_{2}^2 \]
which implies
\[
\int_{\tilde{T}_0}^t \| \tilde{h} \tilde{\tau} (\frac{w}{1 + w} \nabla w) \|_{2}^2 \, d\tilde{\tau} 
\leq C(\| w \|_{H^3}^2 + \| h w \|_{H^3}^2) \int_{\tilde{T}_0}^t (\| \tilde{\tau} D^2 v \|_{2}^2 + \| h \tilde{\tau} D^2 v \|_{2}^2) \, d\tilde{\tau} \leq C h^{-1} N_3^2 (\tilde{T}_0, t).
\]  
(198) 
Collecting (194)-(198) yields
\[
\int_{1}^t \| h \tilde{\tau} g \|_{2}^2 \, d\tilde{\tau} \leq C \sum_{i=2}^{4} h^{1-i} N_3^2 (\tilde{T}_0, t). 
\]  
(199) 
Next, we estimate the term \( \| h f \|_{H^3}^2 \). It follows from the expression of \( f \) and direct computation that
\[
\| \frac{1}{\tau} (\text{div} (vw)) \|_{2}^2 
\leq \| \frac{1}{\tau} w \text{div} v \|_{2}^2 + \| \frac{1}{\tau} w \text{div} v \|_{2}^2 + \| \frac{1}{\tau} \nabla w \cdot v \|_{2}^2 + \| \frac{1}{\tau} \nabla w \cdot v \|_{2}^2 
\leq C (\| \text{div} v \|_{L^\infty}^2 + \| v \|_{L^\infty}^2 + \| w \|_{L^\infty}^2 + \| \nabla w \|_{L^\infty} (\| w \|_{H^3}^2 + \| \text{div} v \|_{H^3}^2 
+ \| \nabla w \|_{H^3}^2 + \| v \|_{H^3}^2) 
\leq C (\| v \|_{H^3}^2 + \| w \|_{H^3}^2) (\| w \|_{H^1}^2 + \| v \|_{H^1}^2),
\]  
which yields
\[
\int_{\tilde{T}_0}^t \| h f \|_{2}^2 \, d\tilde{\tau} \leq C h^{-1} N_3^2 (\tilde{T}_0, t). 
\]  
(200) 
Finally, we estimate the term \( \sup_{\tilde{T}_0 \leq \tilde{\tau} \leq t} h \| \tilde{\tau} g \|_{H^1}^2 \). As before, we need to estimate each term in the expression of \( \tilde{\tau} g \). It follows from direct computation that
\[
\sup_{\tilde{T}_0 \leq \tilde{\tau} \leq t} \| \tilde{\tau} g \|_{2}^2 
\leq C h (\| \tilde{\tau} D \cdot \nabla v \|_{2}^2 + \| \tilde{\tau} \cdot \nabla D v \|_{2}^2) 
\leq C h \sup_{\tilde{T}_0 \leq \tilde{\tau} \leq t} (\| \tilde{\tau} D v \|_{L^\infty}^2 + \| \tilde{\tau} D v \|_{L^2}^2) 
\leq C h \sup_{\tilde{T}_0 \leq \tilde{\tau} \leq t} (\| \tilde{\tau} D v \|_{H^1}^2 \| \tilde{\tau} D v \|_{H^1}^2) 
\leq C h^{-1} N_3^2 (\tilde{T}_0, t), 
\]  
(201) 
and
\[
\begin{align*}
\| h \tilde{\tau} \gamma \|_{2}^2 & \leq C h \| \tilde{\tau} D v \|_{2}^2 
\leq C h \| \tilde{\tau} D (\frac{w}{1 + w} \nabla w) \|_{2}^2 
\leq C h \| \tilde{\tau} D (\frac{w}{1 + w} D^2 v) \|_{2}^2 
\leq C h \| \tilde{\tau} D w D^2 v \|_{2}^2 
\leq C h \| \tilde{\tau} D w D^3 v \|_{2}^2 
\leq C h \| \tilde{\tau} D w D^3 v \|_{H^3}^2 
\leq C h^{-1} N_3^2 (\tilde{T}_0, t).
\end{align*}
\]  
(202)
Similarly,
\[
h\|\tilde{\tau}'\gamma_{\tilde{\tau}^{\prime}}\frac{1}{\tilde{\tau}^{\prime+1}}D\left(\frac{1}{1+w}\nabla P_{1}(w)\right)\|_{2}^{2} \leq C \sum_{i=2}^{4} h^{1-i} N_{i}^{2}(\tilde{\tau}_{0}, t). \tag{204}\]

Then by combining (201)-(204) together with the expression of \(g\) in (188), we obtain
\[
\sup_{\tilde{\tau}_{0} \leq \tilde{\tau} \leq t} h\|\tilde{\tau}'g\|_{H^{1}}^{2} \leq C \sum_{i=2}^{4} h^{1-i} N_{i}^{2}(\tilde{\tau}_{0}, t). \tag{205}\]

Combining (186)-(187), (193), (199)-(200) and (205), we complete the proof of (177), that is, Proposition 1.

\section{The proof of Theorem 1.1.}
In this section, we will complete the proof of Theorem 1.1 based on the estimates obtained in \(\S\)2 and \(\S\)4. Set \(t' = \frac{1}{h}(\sqrt{\bar{r}} - 1)\) and \(\bar{N}_{k}(1, t') = h^{-1} N_{k}(\tilde{\tau}_{0}, t)\), that is,
\[
\bar{N}_{k}(1, t') = \sup_{1 \leq \tau \leq t'} \left(\|w\|_{H^{k}}^{2} + \|R^{s}v\|_{H^{k}}^{2} + \frac{1}{2} \|\frac{w}{R}\|_{H^{k-1}}^{2} + \frac{1}{2} \|R^{s-1}v_{r}\|_{H^{k-2}}^{2} \right)
+ 2 \int_{t'}^{t} \left(\frac{\|Dw\|_{R^{s+1}}^{2}}{R^{s+1}} + \frac{1}{2} \|R^{-\frac{1}{2}}w_{r}\|_{H^{k-1}}^{2} + \|R^{s+\frac{1}{2}}Dv\|_{H^{k}}^{2} \right)
+ \frac{1}{2} \|R^{s-\frac{3}{2}}Dv_{r}\|_{H^{k-2}}^{2} + \frac{1}{2} \|R^{s-\frac{1}{2}}v_{r}\|_{2}^{2} d\tau. \tag{206}\]

Similar to Proposition 1, we can obtain

\begin{equation}
\bar{N}_{3}(1, t') \leq C \|(w_{0}, v_{0})\|_{H^{3}}^{2} + C\bar{N}_{3}^{2}(1, t'). \tag{207}\end{equation}

Based on Proposition 2, we complete the proof of (7) by \(\bar{N}_{3}(1, t') \leq C\epsilon_{0}^{2}\) derived from (207),
\[
\|R^{s}v(\tau)\|_{H^{2}} \in L^{\infty}[1, +\infty), \quad \|R^{s+\frac{1}{2}}v(\tau)\|_{H^{2}} \in L^{1}[1, +\infty), \tag{208}\]
and
\[
\|R^{s-\frac{3}{2}}v_{r}(\tau)\|_{H^{2}} \in L^{1}[1, +\infty). \tag{209}\]

It follows from (208)-(209) that
\[
\partial_{\tau}(\|R^{s}v(\tau)\|_{H^{2}}^{2})
= 2 \sum_{k=0}^{2} (R^{s-\frac{1}{2}}D^{k}v_{r}, R^{s+\frac{1}{2}}D^{k}v) + \frac{2s\epsilon_{0}}{R} \sum_{k=0}^{2} (R^{s}D^{k}v, R^{s}D^{k}v) \in L^{1}[1, +\infty). \tag{210}\]

Together with (208), this yields \(\|R^{s}v(\tau)\|_{H^{2}} \to 0\) as \(\tau \to \infty\). And similarly, \(\|R^{s}\nabla v(\tau)\|_{H^{2}} \to 0\) as \(\tau \to \infty\). Thus, we complete the proof of (7) and the proof of Theorem 1.1.
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