A note on closedness of the sum of two closed subspaces in a Banach space

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Abstract

Let $X$ be a Banach space, and $M, N$ be two closed subspaces of $X$. We present several necessary and sufficient conditions for the closedness of $M + N$ ($M + N$ is not necessarily direct sum).

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1 Introduction

Let $X$ be a Banach space, and $M, N$ be two closed subspaces of $X$. Then, $M + N$ is not necessarily closed in $X$ even if $X$ is a Hilbert space and $M \cap N = \{0\}$ (see, e.g., [5, p.145, Exercise 9]). So, to study when $M + N$ is closed in $X$ is always an interesting problem.

For the case of $M \cap N = \{0\}$, a necessary and sufficient condition for $M + N$ being closed in $X$ is given by Kober [2] as follows:

**Theorem 1.1.** Let $X$ be a Banach space, $M, N$ be two closed subspaces of $X$ and $M \cap N = \{0\}$. Then $M + N$ is closed in $X$ if and only if there exists a constant $A > 0$ such that for all $x \in M$ and $y \in N$ we have $\|x\| \leq A \cdot \|x + y\|$.

It seems that there are seldom results concerning necessary and sufficient conditions for $M + N$ being closed in $X$ in the case of $M + N$ being not necessarily direct sum. To the best of our knowledge, the first result of a necessary and sufficient condition for $M + N$ (not necessarily direct sum) being closed in $X$ is given by Luxemburg:

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Theorem 1.2. [4, Theorem 2.5] Let $X$ be a Banach space, and $M, N$ be two closed subspaces of $X$. Then $M+N$ is closed in $X$ if and only if $T : M \times N \to X; (m, n) \mapsto m+n$ is an open mapping.

Luxemburg [4] obtain the above theorem in a more general setting. Theorem 1.2 is only one of the interesting results concerning this topic given by Luxemburg. We refer the reader to [4] for more details.

In addition, for the case of $X$ being a Banach lattice or a Hilbert space, there has been of great interest for some researchers to study if the sum of two closed subspaces of $X$ is still closed. We refer the reader to [3, 4, 7, 8] and references therein for the case of $X$ being a Banach lattice or a Fréchet space and to [1, 6] and references therein for the case of $X$ being a Hilbert space.

This short note is also devoted to this problem for the case of $X$ being a general Banach space. As one will see, we give a Kober-like theorem for the case of $M+N$ being not necessarily direct sum, and show that a necessary condition in the classical textbook [5] is also sufficient (see Remark 2.2).

2 Main results

Theorem 2.1. Let $X$ be a Banach space, and $M, N$ be two closed subspaces of $X$. Then the following assertions are equivalent:

(i) $M + N$ is closed in $X$;

(ii) $(M + N)/N$ is closed in $X/N$;

(iii) there exists a constant $K > 0$ such that for every $x \in M + N$, there is a decomposition $x = m + n$ such that

$$
\|m\| \leq K \cdot \|x\|,
$$

where $m \in M$ and $n \in N$;

(iv) $T : M \times N \to M + N; (m, n) \mapsto m + n$ is an open mapping.

Proof. ”(i) $\implies$ (ii)”. It is obvious.

”(ii) $\implies$ (iii)”. Define a mapping $\phi : (M + N)/N \to M/(M \cap N)$ by

$$
\phi(x + N) = m + (M \cap N),
$$

where $m \in M$ and $n \in N$. 

[4] Luxemburg, W. A. (1957). A general theory of existence and uniqueness of solutions of functional equations. In The Annals of Mathematics, 69(2), 139-166.

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where $x = m + n \in M + N$, $m \in M$ and $n \in N$. It is easy to see that $\phi$ is well-defined. Moreover, $\phi$ is linear and bijective. Noting that

$$
\| \phi(x + N) \| = \| m + (M \cap N) \| \geq \| m + N \| = \| x + N \|,
$$

we conclude that $\phi^{-1}$ is a bounded linear operator from $M/(M \cap N)$ to $(M + N)/N$. Since $(M + N)/N$ and $M/(M \cap N)$ are both Banach spaces, it follows from the open mapping theorem that $\phi$ is also a bounded linear operator from $(M + N)/N$ to $M/(M \cap N)$. Taking $K = \| \phi \| + 1$, the assertion (iii) follows. In fact, letting $x = m' + n' \in M + N$ and $x \neq 0$, where $m' \in M$ and $n' \in N$, we have

$$
\| m' + (M \cap N) \| = \| \phi(x + N) \| \leq \| \phi \| \cdot \| x + N \| \leq \| \phi \| \cdot \| x \| < K \| x \|.
$$

Then, there exists $y \in M \cap N$ such that

$$
\| m' + y \| < K \| x \|.
$$

Letting $m = m' + y$ and $n = n' - y$, we get $x = m + n$ and $\| m \| < K \| x \|$.

"(iii) $\implies$ (iv)". It is easy to see that

$$
\ker T = \{(x, -x) : x \in M \cap N \}.
$$

Let $\pi$ be the quotient map from $M \times N$ to $(M \times N)/\ker T$, and $\overline{T} : (M \times N)/\ker T \to M + N$ be defined as follows

$$
\overline{T}[(m, n) + \ker T] = m + n, \quad (m, n) \in M \times N.
$$

Then $\overline{T}$ is linear and bijective. For every $(m, n) \in M \times N$, by (iii), there exist $m' \in M$ and $n' \in N$ such that $m + n = m' + n'$ and

$$
\| m' \| \leq K \| m + n \|,
$$

which yields that

$$
\| m' \| + \| n' \| \leq (2K + 1) \| m + n \|.
$$

Then, we have

$$
\| \overline{T}[(m, n) + \ker T] \| = \| m + n \| \geq \frac{\| m' \| + \| n' \|}{2K + 1} \geq \frac{1}{2K + 1} \| (m, n) + \ker T \|,
$$

which means that $\overline{T}$ is an open mapping. Combining this with the fact that $\pi$ is open, we conclude that $T = \overline{T} \circ \pi$ is also open.
"(iv) \implies (i)". As noted in the Introduction, (i) is equivalent to (iv) has been shown by Luxemburg using a more general setting. Here, we give a different proof (maybe a more direct proof in the setting of Banach spaces).

Let $\pi$, $\ker T$, $\tilde{T}$ be as in the proof of "(iii) \implies (iv)". For every $(m, n) \in M \times N$ and $x \in M \cap N$, there holds

$$\|m + n\| \leq \|m + x\| + \|n - x\| = \|(m + x, n - x)\| = \|(m, n) + (x, -x)\|,$$

which yields

$$\|\tilde{T}[(m, n) + \ker T]\| = \|m + n\| \leq \inf_{x \in M \cap N} \|(m, n) + (x, -x)\| = \|(m, n) + \ker T\|,$$

i.e., $\|\tilde{T}\| \leq 1$. On the other hand, since $\pi : M \times N \to (M \times N)/\ker T$ is continuous and $T$ is an open mapping, for every open set $U \subset (M \times N)/\ker T$,

$$\tilde{T}(U) = T(\pi^{-1}(U))$$

is also an open set. Thus, $\tilde{T}$ is an open mapping, which means that $\left(\tilde{T}\right)^{-1}$ is continuous, and so bounded. Now, we conclude that as normed linear spaces, $M + N$ and $(M \times N)/\ker T$ are topological isomorphic. Then, it follows that $(M \times N)/\ker T$ is a Banach space that $M + N$ is also a Banach space. This completes the proof. \hfill \Box

**Remark 2.2.** In the classical textbook \cite{5} (see p.137, Theorem 5.20), it has been shown that (iii) is a necessary condition for (i) by using the open mapping theorem. Here, we show that (iii) is also a sufficient condition for (i). In fact, (i) is equivalent to (iii) is a Kober-like result for the case of $M + N$ being not necessarily direct sum. Moreover, we will give a direct proof of "(iii) \implies (i)" in the following. We think that it may be of interest for some readers. Here is our proof:

Let $\{x_j\}_{j=1}^\infty \subset M + N$ and $x_j \to x$ in $X$ as $j \to \infty$. Then, we can choose a subsequence $\{x_k\}$ of $\{x_j\}$ such that

$$\|x_{k+1} - x_k\| \leq \frac{1}{2^k \cdot K}, \quad k = 1, 2, \ldots.$$

By taking $x = x_2 - x_1$ in the assertion (iii), there exist $m_1 \in M$ and $n_1 \in N$ such that $x_2 - x_1 = m_1 + n_1$ and

$$\|m_1\| \leq K \cdot \|x_2 - x_1\| \leq \frac{1}{2}.$$

Similarly, by taking $x = x_3 - x_2$ in the assertion (iii), there exist $m_2 \in M$ and $n_2 \in N$ such that $x_3 - x_2 = m_2 + n_2$ and

$$\|m_2\| \leq K \cdot \|x_3 - x_2\| \leq \frac{1}{2^2}.$$
Continuing by this way, we get two sequences \(\{m_k\} \subset M\) and \(\{n_k\} \subset N\) such that

\[
x_{k+1} - x_k = m_k + n_k, \quad k = 1, 2, \ldots,
\]

and

\[
\|m_k\| \leq \frac{1}{2^k}, \quad k = 1, 2, \ldots.
\]

Then, we have \(\sum_{k=1}^{\infty} \|m_k\| < \infty\). Also, we can get \(\sum_{k=1}^{\infty} \|n_k\| < \infty\). Since \(M\) and \(N\) are both Banach spaces, there exist \(m \in M\) and \(n \in N\) such that

\[
m = \sum_{k=1}^{\infty} m_k, \quad n = \sum_{k=1}^{\infty} n_k.
\]

Recalling that \(x_k \to x\), we get

\[
x - x_1 = \sum_{k=1}^{\infty} (x_{k+1} - x_k) = m + n,
\]

which yields that \(x = x_1 + m + n \in M + N\).

**Corollary 2.3.** Let \(X\) be a Banach space, and \(M, N\) be two closed subspaces of \(X\). Then the following assertions are equivalent:

(a) \(M + N\) is closed in \(X\);

(b) \((M + N)/(M \cap N)\) is closed in \(X/(M \cap N)\).

**Proof.** Noting that \((M + N)/(M \cap N) = M/(M \cap N) + N/(M \cap N)\), it follows from Theorem 2.1 that the closeness of \((M + N)/(M \cap N)\) is equivalent to the closedness of

\[
[(M + N)/(M \cap N)]/[M/(M \cap N)].
\]

On the other hand, it is not difficult to show that \((M + N)/M\) is isometric to \([(M + N)/(M \cap N)]/[M/(M \cap N)]\), and so their closedness are equivalent. Thus, the closedness of \((M + N)/(M \cap N)\) is equivalent to the closedness of \((M + N)/M\). Again by Theorem 2.1 we complete the proof.

**Remark 2.4.** By Corollary 2.3 whenever we find an example of non-direct sum \(M + N\), which is not closed, we can get an example of direct sum \(M/(M \cap N) + N/(M \cap N)\), which is still not closed.
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