Bifurcation analysis of electrostatically actuated MEMS micro-beam

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Abstract. The aim of this paper is to investigate the effects of micro-beam stiffness and length change to the dynamic of the system. The nonlinear vibrations model of microbeam is simplified by Galerkin method and then transformed into a dynamical system. Based on the emergence of zero eigen values and the increase in the number of equilibria, the equation is analysed using normalization and the bifurcation diagram is drawn. Hopf and Pitchfork bifurcation showed by the normalized equation. The change of both parameters, stiffness and length, exhibits a codimension 2 bifurcation, Pitchfork–Hopf bifurcation. And, when we make a roundtrip around the Pitchfork–Hopf point, we meet Pitchfork bifurcation twice, a Hopf bifurcation, and a heteroclinic cycle.

1. Introduction
There is an extraordinary interest in MEMS, the abbreviation of microelectromechanical systems, technology. MEMS are micro-sized devices that combine 2 components, electrical and mechanical [10]. People are interested in MEMS because of its very small size, in micro or even nano size. Their light weight, low-energy consumption, and durability also made them more attractive [13]. The other reasons are due to its high sensitivity and easy-to-digital output [3]. Some MEMS applications, such as sensors and actuators, already sip many attentions since 1980s. The other applications of MEMS are micro accelerometers, micro-switches, blood pressure sensors, actuators, etc. [13, 11]. MEMs have two fixed electrodes which flanking a movable plate, micro-beam. Micro-beam is a narrow beam of micro dimension. Electrostatic actuation makes the micro-beam vibrating nonlinearly.

Recently, researchers have done various studies on micro-beam nonlinear vibrations using different models and approaches. The oscillation of micro-beam is one of the most interesting topics for scientists [1]. Krisnawan, K.P., 2019, in [4] study the dynamic behavior of micro-beam. He showed that the change of micro-beam stiffness can cause the appearance of Hopf bifurcation. Rezaee, M. and Sharafkhani, N., 2019, in [9] studied the nonlinear dynamic behavior of an electrostatically actuated clamped–clamped cylindrical micro-beam. They showed that the input voltage and fluid-to-beam density ratio effectively controlled the range and intensity of the lock-in regime. Li, L., et al, 2017, in [6] investigated the influence of viscoelasticity and scale effect on nonlinear dynamic behavior. They showed that viscoelasticity reduced the effective elastic modulus and making the system had a softening-type vibration. On the contrary, the scale effect increased the effective elastic modulus and making the system had a hardening-type vibration. Sadeghzadeh S and Kabiri K, 2016, in [10] using the method of higher order Hamiltonian to investigate the oscillation of an electrostatically actuated micro-beam. Zhang, S., et al, 2015, in [14] uncovers the dynamic characteristic of electrostatically
actuated shape optimized variable geometry micro-beam. The dynamic of the shape optimized micro-beam is investigated using the generalized differential quadrature method in conjunction with Levenberg-Marquardt optimization method. Yang et al, 2012, in [12] studied the effects of initial curvature and nonlinear deformation over an electrically actuated micro-beam. At the same year, Qian et al, 2012, in [8] investigate the large-amplitude vibration of electrostatically actuated micro-beam by using a homotopy analysis method. They used the Euler–Bernoulli beam theory and the Galerkin approach for the MEMS micro-beam model. Huang D et al, 2011, in [3] showed that air damping force and alternating voltage excitation have a vital role in the structure of micro-beam nonlinear vibration. Chen C et al, 2011, in [2] showed that symmetric electrostatic load can affect the dynamic stability of an electrically actuated micro-beam.

This paper is an extension of Krisnawan K.P., 2019, in analysing the micro-beam vibration. If Krisnawan use only the effects of micro-beam stiffness change, here we also consider the effect of micro-beam length to the dynamic of the micro beam vibrations. The same as Krisnawan, 2019, and Qian, 2012, the nonlinear vibrations equation of MEMS micro-beam is formed based on Euler-Bernoulli beam theory, simplified using Galerkin method, and then transformed the equation into a dynamical system. And like Zhang X et al in [15], we used the eigen values to normalize the system of equations and showed the bifurcations.

2. Problem formulation

Simplified MEMS resonator consists of 2 fixed electrodes and one movable plate of a micro-beam showed on Figure 1 [4]. The micro-beam electrode is a flexible micro-plate that can vibrate by an actuated electric force on a certain voltage.

![Micro-beam Schematics](image)

**Figure 1.** Schematics of clamped-free-clamped model of MEMS resonator

The equation of motion that governs the transverse deflection is written as in Krisnawan [4],

$$EI \frac{d^4 \omega}{dx^4} + \rho bh \frac{d^2 \omega}{dt^2} + c \frac{d \omega}{dt} - k \omega \frac{d \omega}{dt} = \left[N + \frac{Eb h}{2l} \int_0^l \left(\frac{d \omega}{dx}\right)^2 dx \right] \frac{d^2 \omega}{dx^2} + f(x, t), \quad (1)$$

where $f$ stands as the electrostatic force per unit length of the micro-beam, $\omega = \omega(x, t)$ represents the lateral displacement or the transfer deflection of the micro-beam, $x$ act as the longitudinal coordinate, $t$ is time, and the other nomenclature can be seen on Table 1. Based on the theory of plate capacitor in [7], the electrostatic force per unit length of beam is

$$f(x,t) = \frac{ebV^2}{2} \left(\frac{1}{(d - \omega)^2} - \frac{1}{(d + \omega)^2}\right) \quad (2)$$

Transforming equation (2) by Taylor expansion around $\omega = 0$, substituting the result into equation (1), and introducing a new variable $y = \frac{x}{l}$ we get

$$\frac{d^4 \omega}{dy^4} + \beta_1 \frac{d^2 \omega}{dt^2} + \beta_2 \frac{d \omega}{dt} = \left[\beta_3 + \beta_4 \int_0^l \left(\frac{d \omega}{dx}\right)^2 dx \right] \frac{d^2 \omega}{dx^2} + \beta_5 \left[\omega + \frac{2 \omega^3}{3a^2} + \frac{3 \omega^5}{5a^4} + \ldots \right] \quad (3)$$
where
\[ \beta_1 = \frac{\rho bh l^4}{EI}, \beta_2 = \frac{(c - k\omega_0)l^4}{EI}, \beta_3 = \frac{Nl^2}{EI}, \beta_4 = \frac{bh}{2l}, \beta_5 = \frac{2ebV^2 l^4}{EId^3}. \]

| \( I \) | the cross-sectional second moment of inertia (moment of inertia about the longitudinal axis of the micro-beam) |
| \( \rho \) | the mass density of the micro-beam along the x direction |
| \( b \) | the width of the micro-beam |
| \( h \) | the thickness of the micro-beam |
| \( c \) | the damping coefficient |
| \( k \) | the stiffness of the micro-beam |
| \( \omega_0 \) | The maximum lateral displacement |
| \( N \) | the axial force |
| \( l \) | the length of micro-beam |
| \( e \) | the dielectric constant of the interface (8.85 pf/m) |
| \( d \) | gap size |
| \( V \) | the actuating voltage |

3. Transformed model

Equation (3) is an advanced differential equation containing time variable and position variable. This equation will be simplified by using the Galerkin method. Suppose that equation (3) has a separate solution
\[ \omega(y,t) = \phi(y)u(t) \] (4)
where
\[ \phi(y) = A \cos(\pi y) + B \sin(\pi y), \]
where \( A \) and \( B \) are constants.

We can choose \( A \) and \( B \) such that \( A^2 + B^2 = 1 \), introduce new variables \( u_1(t) = u(t) \) and \( u_2(t) = u'_2(t) \) and substitute to equation (4). For \( |u| \) sufficiently small, the phase portrait of the result equation will be equivalent to
\[ u'_1(t) = u_2(t) \]
\[ u'_2(t) = \frac{2\beta_3 - 2\beta_5 \pi^2 - 2\beta_4 \pi^4 - 2\pi^4}{4\beta_1} u_1(t) - \frac{\beta_2}{\beta_1} u_2(t) + \frac{2\beta_5}{\beta_1} \phi^2(y) u_3^2(t). \] (5)

There are some \( y \) such that \( \phi(y) = 0 \). If \( \phi(y) = 0 \) then the result system would be the same as the linearization of system (5) around the origin, equilibrium \((0,0)\). Thus, we consider only the case when \( \phi(y) \neq 0 \). The other equilibria of equation (5) are
\[ (u_1, u_2) = \left( \pm \sqrt{\frac{2\beta_3 - 2\beta_5 \pi^2 - 2\beta_4 \pi^4 - 2\pi^4}{4\beta_1}}, 0 \right). \]

Since all parameters assumed to be positive, thus the nontrivial equilibria are provided for all \( y \) such that \( 2\beta_5 - 2\beta_3 \pi^2 - 2\beta_4 \pi^4 - 2\pi^4 > 0 \). The eigen values of system (5) over the origin are
\[ \lambda_{12} = -\beta_2 \pm \frac{\sqrt{\beta_2^2 + 4\beta_1 (2\beta_5 - 2\beta_3 \pi^2 - 2\beta_4 \pi^4 - 2\pi^4)}}{2\beta_1}. \]

For if \( E = 25\mu Pa, \ l = 1.241875 \times 10^{-8} \mu m, \ \rho = 2330 \ kg/m^2, \ b = 2.5\mu m, \ h = 0.92\mu m, \ N = 125 \ \mu Pa, \ V = 10V, \ c = 5 \times 10^{-11} \text{Ns/m}, \ d = 1.77\mu m, \) and \( \omega_0 = 2\mu m \) the eigenvalues would be double zero when \( k = 2.5 \times 10^{-5} N/m \) and \( l = 8.85 \times 10^{-5} \pi \sqrt{30} \mu m \).
Now, we introduce new parameters $\bar{l} = l - 8.85 \times 10^{-5} \pi \sqrt{30} m$ and $\bar{k} = k - 2.5 \times 10^{-5} N/m$, and substitute to system (5) to get
\begin{equation}
\begin{aligned}
u_1'(t) &= u_2(t) \\
u_2'(t) &= \alpha_1(\bar{l}) u_1(t) + \alpha_2(\bar{k}) u_2(t) + C u_3(t),
\end{aligned}
\end{equation}
where $\alpha_1(\bar{l}) = \frac{4 \times 10^{16}}{8.915 \pi \sqrt{30}} \bar{l} + \frac{8 \times 10^{10}}{2.306 \pi} \bar{l}^2 + \cdots$, $\alpha_2(\bar{k}) = \frac{8 \times 10^4}{5.359} \bar{k}$, and $C = 1.489 \times 10^{11} \phi^2(y) > 0$.

The nontrivial equilibria and the eigen values of system (6) over the origin are
\begin{equation}
(\bar{u}_1, \bar{u}_2) = \left( \pm \sqrt{-\frac{\alpha_1}{C}}, 0 \right)
\end{equation}
and
\begin{equation}
\lambda_{12}(\alpha_1, \alpha_2) = \frac{\alpha_2 \pm \sqrt{\alpha_2^2 + 4\alpha_1}}{2}.
\end{equation}

By characterizing equations (7a) and (7b), we can see that
1. If $\alpha_1 = 0$ and $\alpha_2 \neq 0$ then $\lambda_1 = 0$ and $\lambda_2 = \alpha_2$.
2. If $\alpha_1 \geq 0$ then there is only 1 equilibrium point, and if $\alpha_1 < 0$ then there are 3 equilibria.
3. If $\alpha_1 < 0$ and $\alpha_2 = 0$ then $\lambda_1 = i\sqrt{|\alpha_1|}$ and $\lambda_2 = -i\sqrt{|\alpha_1|}$.
4. If $\alpha_1 = 0$ and $\alpha_2 = 0$ then $\lambda_1 = 0$ and $\lambda_2 = 0$.
5. If $\alpha_2^2 + 4\alpha_1 \geq 0$, the eigen values are real and if $\alpha_2^2 + 4\alpha_1 < 0$, the eigen values are complex.

No. 1 and 2 indicate a pitchfork bifurcation, No. 3 indicates a Hopf bifurcation, and No. 4 indicates a codimension 2 bifurcation. There is no bifurcation when crossing the curve $\alpha_2^2 + 4\alpha_1 = 0$, but crossing the curve turns the values of eigen from real into complex or complex into real.

4. Pitchfork bifurcation

If the value of $\alpha_1$ in system (6) move from negative to positive, the value of $\lambda_1$ changes from positive to negative and the number of equilibria decreases from 3 to 1.

Lemma 1. System (6) undergoes a pitchfork bifurcation if $\alpha_1 = 0$, $\alpha_2 \neq 0$, and $C \neq 0$.

Proof.

Without loss of generality, it is assumed that $C > 0$, if it is not, the proof will be analogue. Using the manifold centre $h = a_1 u_1 + a_2 a_1 + a_3 u_1^2 + a_4 u_1 a_1 + a_5 a_1^2 + \cdots + a_9 a_1^3$ to substitute $u_2$ in system (6) we get $a_1 = a_2 = a_3 = a_5 = a_7 = a_9 = 0$, $a_4 = -\frac{1}{a_2}$, $a_6 = -\frac{c}{a_2}$, and $a_8 = \frac{1}{a_2}$. And we have the new system
\begin{equation}
u_1' = \mu u_1 - c_1 u_3^2\end{equation}
where $\mu = \frac{a_1^2 - a_1 a_2}{a_2}$ and $c_1 = \frac{c}{a_2}$. For $|\alpha_1|$ sufficiently small, equation (8) is the normal form of supercritical pitchfork if $\alpha_2 > 0$ and subcritical pitchfork if $\alpha_2 < 0$.

5. Hopf bifurcation

The eigen values of system (6) would be pure imaginer if $\alpha_2 = 0$ and $\alpha_1 < 0$. If the value of $\alpha_2$ move from negative to positive, the real values of $\lambda_1$ and $\lambda_2$ also turn from negative to positive.

Lemma 2. System (6) undergoes a Hopf bifurcation if $\alpha_2 = 0$, $\alpha_1 < 0$, and $C \neq 0$.

Proof.

Without loss of generality, it is assumed that $C > 0$, if it is not, the proof will be analogue. System (6) can be written as $u'(t) = Fu(t) + F(u(t))$ where
\begin{equation}
u(t) = \begin{bmatrix}u_1(t) \\
u_2(t)\end{bmatrix}, \quad J = \begin{bmatrix}0 & 1 \\
\alpha_1 & \alpha_2\end{bmatrix}, \quad \text{and } F(u(t)) = \begin{bmatrix}0 \\
Cu_3(t)\end{bmatrix}.
\end{equation}
If we let $\alpha_2$ as the parameter and make $\alpha_1 < 0$ fixed, one of the eigen value of matrix $J$ is
\begin{equation}
\lambda = \lambda(\alpha_2) = \frac{\alpha_2 + \sqrt{\alpha_2^2 + 4\alpha_1}}{2}
\end{equation}
and the eigen vector of matrix $J$ corresponds to $\lambda$ is $q = \begin{bmatrix}1 \\
\lambda\end{bmatrix}$. We also have $\bar{\lambda}$
as the eigen value of matrix $J^T$ and $p = \frac{1}{\sqrt{\alpha_2 + \alpha_1}} \begin{bmatrix} \bar{\lambda} - \alpha_2 \\ 1 \end{bmatrix}$ as the eigen vector of $J^T$ corresponds to eigen value $\bar{\lambda}$.

It is easy to see that $\langle p, q \rangle = 1$ and $\langle p, \bar{q} \rangle = 0$, and if $g_{kl}(\alpha_2) = \frac{\partial^{k+l}}{\partial \alpha_1 \partial \alpha_2} (p(\alpha_2), F(zq + \bar{z}\bar{q}))|_{z=0}$ we have $g_{20}(\alpha_2) = 0$, $g_{11}(\alpha_2) = 0$, $g_{21}(\alpha_2) = 24C|\alpha_1|$, thus $l_1(0) = \frac{1}{2w(0)} Re(i g_{20}g_{11} + w(0)g_{21}) = 12C|\alpha_1| > 0$ where $w(\alpha_2) = Im(\lambda)$. Based on Kuznetsov [5], system (6) has the same dynamics as system

\[
\begin{align*}
\dot{v}_1 &= \mu v_1 - v_2 + v_1(v_1^2 + v_2^2) \\
\dot{v}_2 &= v_1 + \mu v_2 + v_2(v_1^2 + v_2^2)
\end{align*}
\]

which is the normal form of subcritical Hopf bifurcation.

6. Pitchfork–Hopf bifurcation

System (6) would have pair of zero eigen values if $\alpha_1 = 0$ and $\alpha_2 = 0$. The bifurcation diagram of system (6) around the Pitchfork–Hopf point $(\alpha_1, \alpha_2) = (0,0)$ can be seen on Figure 2.

![Figure 2. Pitchfork-Hopf bifurcation diagram of system (6).](image)

Make a roundtrip near the Pitchfork–Hopf point $(\alpha_1, \alpha_2) = (0,0)$, starting from region 1, $\alpha_1 > 0$, where there is only one saddle point at the origin. Entering from region 1 to region 2 through component $F_1$ of pitchfork bifurcation curve yields 2 saddle equilibria and the origin change to unstable node. Here in region 2, we also have 2 heteroclinic orbits, both are starting from the origin and ending at the different saddle equilibria. The node on region 2 turns into an unstable focus when
we cross $C_1$ curve and get into region 3, since the eigen values change from real into complex. The heteroclinic orbits are still exists in region 3. The focus equilibrium gets its stability as we cross the component $P$ ($\alpha_1 < 0$ and $\alpha_2 = 0$) of Hopf bifurcation curve and move into region 4. An unstable limit cycle presents around the origin but both heteroclinic orbits are vanished. At curve $H$, heteroclinic orbits are showed up and become a heteroclinic cycle which orbit from the other saddle points. And, there are cycles around the unstable limit cycle. In region 5 the cycles vanished and heteroclinic orbits present starting from both saddle equilibria and ending at the origin which is a stable focus point. The eigen values change from complex into real when we move from region 5 to 6 through $C_2$ then the focus point turns into a node but still has its stability. Entering region 1 from region 6 through component $F_2$ of pitchfork bifurcation, 2 saddle equilibria are disappeared and the origin is changed into saddle.

7. Conclusion

The model of a MEMS micro-beam that is simplified by Galerkin method and then transformed into a dynamical system exhibits a codimension 2 bifurcation, Pitchfork-Hopf bifurcation. This bifurcation is appearing when Hopf meets pitchfork bifurcation. Subcritical Hopf bifurcation shows up at the negative axis of the first parameter ($\alpha_1$) and pitchfork exhibits in types of supercritical and subcritical at the second parameter ($\alpha_2$) axis. The supercritical pitchfork bifurcation appears when $\alpha_2 > 0$ and the subcritical exists when $\alpha_2 < 0$. Around the Pitchfork-Hopf points we also have a heteroclinic cycle which start from one saddle point to another saddle point and then back to the first saddle point.

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