A simple variational approach to nonlinear Maxwell equations

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CRC Preprint 2022/82, December 2022
Participating universities

Universität Stuttgart

Funded by

DFG

ISSN 2365-662X
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Abstract. We show that nonlinear Maxwell Equations in \( \mathbb{R}^3 \) admit a convenient dual variational formulation that permits to prove the existence of ground states via standard variational methods.

In this note we provide a simple variational framework for Nonlinear Maxwell Equations of the form
\[
\nabla \times \nabla \times E = f(x, E) \quad \text{in} \ \mathbb{R}^3.
\]
Here, \( E : \mathbb{R}^3 \to \mathbb{R}^3 \) stands for the electric field and the nonlinearity \( f(x, E) \in \mathbb{R}^3 \) represents the electric displacement field within the propagation medium, see [12, pp. 825-826]. In the past years, several existence results for nontrivial solutions of (1) have been found. All of them except [10, Theorem 3 (ii)] rely on variational methods so that \( f(x, E) = \partial E F(x, E) \) is assumed for some scalar-valued function \( F \). These results may be separated into two classes: the first deals with cylindrically symmetric solutions that are divergence-free. For such functions the curl-curl operator acts like the classical Laplacian so that several analytic tools like the Rellich-Kondrachov theorem can be used. Of course, the nonlinearity \( f \) then needs to be cylindrically symmetric with respect to \( x \) as well. The first contribution in this direction is due to Azzollini, Benci, D’Aprile, Fortunato [2] and their existence results have been generalized in [3, 5, 9]. The second class of papers [12, 13] by Mederski and coauthors deals with \( \mathbb{Z}^3 \)-periodic nonlinearities where a cylindrically symmetric ansatz does not make sense. The variational approach developed in these papers is very advanced and many difficulties have to be overcome to prove the existence of a nontrivial solution of (1) via some detailed analysis of generalized Palais-Smale sequences for the energy functional associated with (1). Our goal is to show that in some cases Nonlinear Maxwell Equations admit a much more convenient dual formulation that allows to prove existence results via standard methods of the calculus of variations. As a new feature, our result applies to nonlinearities \( f(x, \cdot) \) that, contrary to all other results that we are aware of, need not satisfy any symmetry assumption with respect to \( x \). On the other hand, our approach requires some decay of \( f(x, E) \) as \( |x| \to \infty \).

Our aim is to show that (1) has ground state solutions under the following assumptions:

(A) \( f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \) is a Carathéodory function with \( f(x, E) = f_0(x, |E|)|E|^{-1}E \) where \( s \mapsto s^{-1}f_0(x, s) \) is positive and increasing on \((0, \infty)\) and
\[
\frac{1}{2}f_0(x, s)s - \int_0^s f_0(x, t) \, dt \geq \Gamma(x) \min\{s^p, s^q\} \geq c f_0(x, s)s
\]
where \( 0 < c \leq \Gamma(x)(1 + |x|)^\alpha \leq C < \infty \) for \( 0 < \alpha < 3 \) and \( 2 < p < 6, \max\{2, 6 - 2\alpha\} < q < \infty \).

Here, a ground state is a nontrivial critical point of the associated energy functional
\[
I(E) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times E|^2 \, dx - \int_{\mathbb{R}^3} F(x, E) \, dx
\]
that has least energy among all nontrivial critical points. In view of assumption (A) the natural (smallest possible) function space for this functional is \( H^1(\text{curl}; \mathbb{R}^3) \cap [\Gamma^{-1/2} L^p(\mathbb{R}^3; \mathbb{R}^3) \cap \Gamma^{-1/2} L^q(\mathbb{R}^3; \mathbb{R}^3)] \). So our main result reads as follows:

Date: December 13, 2021.
2020 Mathematics Subject Classification. 35J60, 35Q61.
Theorem 1. Assume that $f$ satisfies (A). Then (1) has a ground state solution $E \in H^1(\text{curl}; \mathbb{R}^3) \cap \Gamma^{-1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) \cap \Gamma^{-1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$.

Remark 2.
(a) The result applies to the power-type nonlinearity $f(x, E) = (1 + |x|)^{-\alpha}|E|^pE$ provided that $\alpha > 1$.
(b) The assumptions on $\Gamma$ are imposed to ensure that certain powers of $\Gamma$ are Muckenhoupt weights with suitable index, see Proposition 9. The lower bound for $\Gamma$ seems to be a purely technical assumption, but this is not true. If $\Gamma$ vanishes on an open set, then ground states do not exist because of a null sequence of (concentrating) gradient field solutions of (1). The argument is the same as in Proposition 1.
(c) The existence of infinitely many other solutions is open. In the case of cylindrical symmetry this can be done by the Symmetric Mountain Pass Theorem for the functional $J$ from Remark 2 on the subspace of cylindrically symmetric functions, see [11, Proposition 1]. Whether or not cylindrical symmetry is motivated by some rearrangement principle, remains an open problem.

In the following the symbol $\lesssim$ stands for $\leq C$ for some positive number $C$, similar for $\gtrsim$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. The exponents $p'$, $q'$ denote, as usual, the Hölder conjugates given by $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{p} + \frac{1}{q} = 1$.

We fix the standard norm $\| \cdot \|_p$ on $L^p(\mathbb{R}^3; \mathbb{R}^3)$. The Fourier transform of a tempered distribution $f$ will occasionally be denoted by $\hat{f}$ or $\mathcal{F}(f)$.

1. The dual problem

We use a dual approach to prove Theorem 1. This means that instead of considering the electric field $E$ as the unknown, we treat (1) as a variational problem for $P := f(x, E)$. The main advantage is that the vector field $P$ is automatically divergence-free, which will allow us to use variational methods that almost ignore what happens on the irrotational part of the electric field $E$. The new task is to find a divergence-free vector field $P : \mathbb{R}^3 \to \mathbb{R}^3$ that solves the quasilinear equation

$$\nabla \times \nabla \times (\psi(x, P)) = P \quad \text{in } \mathbb{R}^3$$

(2)

where $\psi(x, \cdot)$ denotes the inverse of $f(x, \cdot)$. Assuming (A), the existence of such a function $\psi$ will be verified in Proposition 9. We shall moreover prove that solutions of (2) can be obtained from critical points of the energy functional

$$J(P) = \int_{\mathbb{R}^3} \Psi(x, P) \, dx - \frac{1}{2} \int_{\mathbb{R}^3} (\Delta)^{-1} P \cdot P \, dx.$$  

(3)

where $\Psi(x, \cdot)$ denotes the primitive of $\psi(x, \cdot)$ with $\Psi(x, 0) = 0$. So our aim is to provide a sufficient criterion ensuring that $J$ has a ground state solution over a suitable function space, i.e., a nontrivial critical point having least energy among all nontrivial critical points of $J$. Later, in Section 2 we will show that this ground state solution of $J$ gives rise to a ground state solution of $I$ via $E = \psi(x, P)$.

Our analysis makes use of the following assumptions:

(B) $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Carathéodory function with $\psi(x, P) = \psi_0(x, |P|)|P|^{-1}P$ where $z \mapsto \psi_0(x, z)$ is positive and increasing on $(0, \infty)$, $z \mapsto z^{-1} \psi_0(x, z)$ is decreasing on $(0, \infty)$ and, for $\Gamma$, $\alpha, p, q$ as in (A) and almost all $x \in \mathbb{R}$ and $z > 0$,

$$\int_0^z \psi_0(x, s) \, ds - \frac{1}{2} \psi_0(x, z)z \geq c\max\{\Gamma(x)^{-\alpha}z^{p'-1}, (\Gamma(x)^{-\alpha})^{q'-1}\}z \geq c\psi_0(x, z)z.$$

We will see later (Proposition 10) that assumption (B) is the natural counterpart of assumption (A). To benefit from the decay of $\Gamma$ at infinity we set up our variational approach on the divergence-free part of the reflexive Banach space $Z := \Gamma^{1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) \cap \Gamma^{1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$ with norm

$$\|P\| := \|\Gamma^{-\frac{1}{p}}P\|_{p'} + \|\Gamma^{-\frac{1}{q}}P\|_{q'}.$$  

(4)
The corresponding dual space is $Z^* = \Gamma^{-1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$ with norm
\[
\|E\|_{Z^*} := \inf_{E_1 + E_2 = E} \|1^p E_1\|_p + \|1^q E_2\|_q.
\]
(5)

We first introduce the Helmholtz Decomposition on $Z$ that decomposes a vector field on $\mathbb{R}^3$ into its divergence-free (solenoidal) part belonging to $X$ and its curl-free (irrotational) part in $Y$.

**Proposition 3.** Then there are closed subspaces $X, Y \subset Z$ such that the Helmholtz Decomposition $Z = X \oplus Y$ holds with continuous projections $\Pi : Z \to X$ and $\text{id} - \Pi : Z \to Y$ such that $E = E_1 + E_2$ with $E_1 := \Pi E, E_2 := (\text{id} - \Pi)E$ implies $\nabla \cdot E_1 = 0$ and $\nabla \times E_2 = 0$ in the distributional sense, in particular
\[
\nabla \times \nabla \times E_1 = -\Delta E_1, \quad \nabla \times \nabla \times E_2 = 0.
\]
(6)

**Proof.** For Schwartz functions $E \in S(\mathbb{R}^3; \mathbb{R}^3)$ we define $\Pi E \in S(\mathbb{R}^3; \mathbb{C}^3)$ via
\[
\overline{\Pi E}(\xi) := \hat{E}(\xi) - |\xi|^{-2}\hat{\Psi}(\xi)\xi.
\]
Then (6) follows from $\nabla \times \nabla \times E = -\Delta E + \nabla(\text{div}(E))$. We verify the boundedness of $\Pi, \text{id} - \Pi$ on $Z$. It suffices to show that all Riesz transforms $f \mapsto F^{-1}((\xi_j|\xi|^r)^{-1})$ are bounded on $\Gamma^{1/q}L^q(\mathbb{R}^3)$ where $r \in \{p, q\}$. In [14, Corollary 2.2] it is shown that the Riesz transforms are bounded on the weighted Lebesgue space $L^p_\omega(\mathbb{R}^3)$ provided that the function $\omega_\xi$ belongs to the class of Muckenhoupt weights $A_\omega$, see [14, p.1239] for a definition. We use this for $\omega_\xi(x) := \Gamma(x)^{-1/(r-1)}$ because of $\Gamma^{1/r}L^r(\mathbb{R}^3) = \omega_\xi^{-1/r}L^r(\mathbb{R}^3) = L^r_\omega(\omega_\xi)$ in the notation of [14]. From [8, Example 1.3] we get $\omega_p \in A_p, \omega_q \in A_q$ due to $0 < \alpha < 3$, which gives the boundedness of the Riesz transform as an operator on $\Gamma^{1/p}L^p(\mathbb{R}^3)$ and on $\Gamma^{1/q}L^q(\mathbb{R}^3)$, hence on $Z$. 

Since we are looking for divergence-free solutions $P$ of (2), we set up our variational approach in the space $X$ from the previous proposition. It is the subspace of divergence-free vector fields belonging to $Z = \Gamma^{1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) \cap \Gamma^{1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$. To make sense of the functional $J$ we first show that
\[
J_1 : X \to \mathbb{R}, \quad P \mapsto \int_{\mathbb{R}^3} \Psi(x, P) \, dx \quad \text{where } \Psi(x, P) := \int_0^{\|P\|} \psi_0(x, s) \, ds
\]
defines a convex $C^1$-functional.

**Proposition 4.** Assume (B). Then $J_1 \in C^1(X)$ is convex and there is $c > 0$ such that
\[
J_1'(P)[\tilde{P}] = \int_{\mathbb{R}^3} \psi(x, P) \cdot \tilde{P} \, dx, \quad J_1(P) \geq c \min\{\|P\|^{p'}, \|P\|^{q'}\} \quad \text{for } P, \tilde{P} \in X.
\]
(7)

**Proof.** Assumption (B) yields the estimates
\[
J_1(P) \lesssim \int_{\mathbb{R}^3} \int_0^{\|P\|} \max\{\Gamma(x)^{-1} s)^{p'-1}, (\Gamma(x)^{-1} s)^{q'-1}\} \, ds \, dx
\]
\[
\lesssim \int_{\mathbb{R}^3} \Gamma(x)^{1-p'}|P|^{p'} + \Gamma(x)^{1-q'}|P|^{q'} \, dx
\]
\[
= \|\Gamma^{-\frac{1}{p'}} P\|_{p'} + \|\Gamma^{-\frac{1}{q'}} P\|_{q'}.
\]
\[\boxed{\|P\|^{p'} + \|P\|^{q'} \lesssim \|\Gamma^{-\frac{1}{p'}} P\|_{p'} + \|\Gamma^{-\frac{1}{q'}} P\|_{q'}}\]
So $J_1 : X \to \mathbb{R}$ is well-defined. Similarly, one proves $J_1 \in C^1(X)$ and the formula for $J_1'(P)$ along the lines of the proof of [11, Theorem 2.19]. Since $z \mapsto \psi_0(x, z)$ is increasing by assumption (B), the function $\mathbb{R}^3 \to \mathbb{R}^3, P \mapsto \Psi(x, P)$ is convex for almost all $x \in \Omega$. This implies the convexity of $J_1$. To prove the inequality in (7) set $A(t) := tv^{p'/q'} + tv^{q'/p'}$ for $t > 0$. Then
\[
\|P\|^{p'} + \|P\|^{q'} \lesssim \|\Gamma^{-\frac{1}{p'}} P\|_{p'}^{p'} + \|\Gamma^{-\frac{1}{q'}} P\|_{q'}^{q'} + \|\Gamma^{-\frac{1}{p'}} P\|_{p'}^{p'} + \|\Gamma^{-\frac{1}{q'}} P\|_{q'}^{q'}.
\]
We denote by $X$. We have by Young’s and Hölder’s inequalities

\[
\begin{aligned}
&\geq \int_{\mathbb{R}^3} \Gamma(x)^{p'-1}|P|^{p'} + \Gamma(x)^{q'-1}|P|^{q'} \, dx \\
&+ \left( \int_{\mathbb{R}^3} \Gamma(x)^{p'-1}|P|^{p'} \, dx \right)^{\frac{q'}{p'}} + \left( \int_{\mathbb{R}^3} \Gamma(x)^{q'-1}|P|^{q'} \, dx \right)^{\frac{q'}{q}} \\
&\leq A \left( \int_{\mathbb{R}^3} \Gamma(x)^{p'-1}|P|^{p'} + \Gamma(x)^{q'-1}|P|^{q'} \, dx \right).
\end{aligned}
\]

This implies

\[
J_1(P) = \int_{\mathbb{R}^3} \Psi(x, P) \, dx
\]

\[
\geq A^{-1} \left( ||P||^{p'} + ||P||^{q'} \right)
\]

\[
\geq \min \{ ||P||^{p'} + ||P||^{q'} \}.
\]

which proves (7).

Next we show that the second part of the functional $J$ from (8) is well-behaved. To this end, we investigate the mapping properties of the linear operator \((-\Delta)^{-1} : X \rightarrow X^*\) that is given by

\[
(-\Delta)^{-1} f := \mathcal{F}^{-1} \left( |\xi|^{-2} \hat{f}(\xi) \right) = K \ast f
\]

where $K(z) = (4\pi|z|)^{-1}$. Here the convolution respectively the Fourier multiplier $|\xi|^{-2}$ act componentwise on vector fields. We denote by $X^*$ the subspace of divergence-free functions in $\mathcal{Z}^\ast$.

**Proposition 5.** Assume (B). Then the linear operator \((-\Delta)^{-1} : X \rightarrow X^*, P \mapsto (-\Delta)^{-1} P\) is compact.

**Proof.** We have by Young’s and Hölder’s inequalities

\[
||\Gamma^{\frac{1}{2}} (K1_{|z| \leq 1} \ast P)||_p \leq ||\Gamma^{\frac{1}{2}}||_{\frac{1}{2}} ||K1_{|z| \leq 1} \ast P||_p
\]

\[
\leq ||\Gamma^{\frac{1}{2}}||_{\frac{1}{2}} ||K1_{|z| \leq 1}||_{\frac{1}{2}} ||P||_{p'}
\]

\[
\leq ||\Gamma^{\frac{1}{2}}||_{\frac{1}{2}} ||K1_{|z| \leq 1}||_{\frac{1}{2}} ||\Gamma^{-\frac{1}{2}} P||_{p'}
\]

\[
\leq ||\Gamma^{\frac{1}{2}}||_{\frac{1}{2}} ||K1_{|z| \leq 1}||_{\frac{1}{2}} ||P||
\]

The prefactor is finite due to $\Gamma \in L^\infty(\mathbb{R}^3), |K(z)| \leq |z|^{-1}$ and $2 < p < 6$. Since $q > 6 - 2\alpha$ and $\alpha > 0$, we may choose $\mu > 0$ according to $\frac{1}{q} - \frac{1}{6} < \frac{1}{\mu} < \frac{\alpha}{3q}$. Then $\alpha < 3$ implies $0 < \frac{1}{\mu} < \frac{1}{q}$. We may thus define $r, s \in (1, \infty)$ via $\frac{1}{r} := \frac{2}{q} - \frac{2}{p} < \frac{1}{q}$. Then $\alpha < 3$ implies $0 < \frac{1}{\mu} < \frac{1}{q}$. We may thus define $r, s \in (1, \infty)$ via $\frac{1}{r} := \frac{2}{q} - \frac{2}{p} < \frac{1}{q}$. Then

\[
||\Gamma^{\frac{1}{2}} (K1_{|z| \geq 1} \ast P)||_q \leq ||\Gamma^{\frac{1}{2}}||_{\frac{1}{2}} ||K1_{|z| \geq 1} \ast P||_{\frac{1}{2}} ||P||_s
\]

\[
\leq ||\Gamma^{\frac{1}{2}}||_{\frac{1}{2}} ||K1_{|z| \geq 1}||_{\frac{1}{2}} ||P||_s
\]

\[
\leq ||\Gamma^{\frac{1}{2}}||_{\frac{1}{2}} ||K1_{|z| \geq 1}||_{\frac{1}{2}} ||\Gamma^{-\frac{1}{2}} P||_{q'}
\]

\[
\leq ||\Gamma^{\frac{1}{2}}||_{\frac{1}{2}} ||K1_{|z| \geq 1}||_{\frac{1}{2}} ||P||.
\]
Proof. Fix any $\chi \in C^\infty_0(\mathbb{R}^3)$ chosen such that $\chi(z) = 1$ for $|z| \leq 1$ and $\chi(z) = 0$ for $|z| \geq 2R$. The compactness of $\chi_R(-\Delta)^{-1}$ for any given $R > 0$ follows from local elliptic estimates, $p < 6$ and the Rellich-Kondrachov Theorem. Repeating the estimates from above one finds

$$\|(-\Delta)^{-1}P\|_{Z^*} \leq \left(\|\Gamma\|^2_{\infty}\|K_{\Gamma}|_{1}\|_{1}\|\bar{\psi}P\|_{2} + \|\Gamma^{1/2}\|_{\infty}\|K_{\Gamma}|_{1}\|_{1}\|P\|_{r}\right)\|P\|,$$

which proves the boundedness of the operator from $X$ to $Z^*$. Since $(-\Delta)^{-1}P$ is divergence-free (because so is $P$), we even conclude that $(-\Delta)^{-1} : X \to X^*$ is bounded. To prove the compactness, set $\chi_R = \chi(R^{-1} \cdot)$. The compactness of $\chi_R(-\Delta)^{-1}$ for any given $R > 0$ follows from local elliptic estimates, $p < 6$ and the Rellich-Kondrachov Theorem. Repeating the estimates from above one finds

$$\|(-\Delta)^{-1}P\|_{Z^*} \leq \left(\|\Gamma\|^2_{\infty}\|K_{\Gamma}|_{1}\|_{1}\|\bar{\psi}P\|_{2} + \|\Gamma^{1/2}\|_{\infty}\|K_{\Gamma}|_{1}\|_{1}\|P\|_{r}\right)\|P\|.$$

Since the prefactor goes to zero as $R \to \infty$, we infer $(-\Delta)^{-1} = \lim_{R \to \infty} \chi_R(-\Delta)^{-1}$ with respect to the operator norm. Since the operators $\chi_R(-\Delta)^{-1} : X \to X^*$ are compact for all $R > 0$, we conclude that $(-\Delta)^{-1} : X \to X^*$ is compact as well.

The previous two results imply the following:

**Proposition 6.** Assume (B). Then $J \in C^1(X)$ and

$$J'(P)[\tilde{P}] = \int_{\mathbb{R}^3} \psi(x,P) \cdot \tilde{P} \, dx - \int_{\mathbb{R}^3} (-\Delta)^{-1}P \cdot \tilde{P} \, dx$$

for all $P, \tilde{P} \in X$. In particular, $J'(P) = 0$ holds if and only if $\Pi(\psi(x,P)) = (-\Delta)^{-1}P$.

Note that the projector $\Pi$ appears in the Euler-Lagrange equation because the test functions $\tilde{P} \in X$ are arbitrary only among the divergence-free vector fields. To prove the existence of ground states, we minimize $J$ over the Nehari manifold

$$\mathcal{N} = \{ P \in X \setminus \{0\} : J'(P)[P] = 0 \} = \{ P \in X \setminus \{0\} : \int_{\mathbb{R}^3} \psi(x,P) \cdot P \, dx = \int_{\mathbb{R}^3} (-\Delta)^{-1}P \cdot P \, dx \}$$

and define the corresponding min-max-level via

$$c_\mathcal{N} := \inf_{\mathcal{N}} J.$$  \hspace{1cm} (10)

We first provide a suitable min-max characterization of this energy level with the aid of the fibering map.

**Proposition 7.** Assume (B). Then

$$c_\mathcal{N} = c \quad \text{where } c := \inf_{P \in X \setminus \{0\}} \max_{t > 0} J(tP) \in (0, \infty)$$  \hspace{1cm} (11)

and $\inf_{\mathcal{N}} J$ is attained at $P^* \in \mathcal{N}$ if and only if the infimum on the right is attained at $tP^*$ for any $t > 0$.

**Proof.** Fix any $P \in X \setminus \{0\}$ and define the fibering map

$$\gamma(t) := J(tP) = \int_{\mathbb{R}^3} \Psi(x,tP) \, dx - \frac{t^2}{2} \int_{\mathbb{R}^3} (-\Delta)^{-1}P \cdot P \, dx.$$

By definition of $\mathcal{N}$ we have $tP \in \mathcal{N}$ if and only if $J'(tP)[P] = 0$, i.e., $\gamma'(t) = 0$. We claim that for any given $P \in X \setminus \{0\}$ there is precisely one such $t$. Indeed, $\gamma'(t_1) = \gamma'(t_2) = 0$ and $t_1 > t_2 > 0$ implies

$$0 = t_1^{-1}\gamma'(t_1) - t_2^{-1}\gamma'(t_2) = \int_{\mathbb{R}^3} \left( t_1^{-1} \psi_0(x,t_1|P|) - t_2^{-1} \psi_0(x,t_2|P|) \right) |P| \, dx < 0$$

in view of the monotonicity assumption on $\psi_0$ from (B), which is a contradiction. So $\gamma$ has at most one critical point. On the other hand, $\gamma$ has at least one critical point, a global maximizer, because of $\gamma(t) > \gamma(0) = 0$
for small $t > 0$ and $\gamma(t) \to -\infty$ as $t \to \infty$. So we conclude that for any given $P \in X \setminus \{0\}$ there is a unique $t(P) > 0$ such that $t(P)P \in \mathcal{N}$. This implies $c_{\mathcal{N}} = c$ and the claimed relation between the minimizers. It remains to show $c > 0$.

For $P \in \mathcal{N}$ we have $J'(P)[P] = 0, P \neq 0$. So we get as in the proof of (5)

$$\min \{\|P\|', \|P\|\} \lesssim \int_{\mathbb{R}^3} \psi(x, P) \cdot P dx = \int_{\mathbb{R}^3} (-\Delta)^{-1} P \cdot P dx \leq \|(-\Delta)^{-1}\| \|P\|^2.$$ 

This and $p', q' < 2$ implies $\|P\| > \kappa > 0$ for some $\kappa > 0$ which is independent of $P$. As in (5) this implies (using the first inequality in assumption (B))

$$J(P) = J(P) - \frac{1}{2} J'(P)[P] = \int_{\mathbb{R}^3} \psi(x, P) - \frac{1}{2} \psi(x, P) \cdot P dx \gtrsim \min \{\|P\|', \|P\|\} \gtrsim \min \{\kappa p', \kappa q'\}.$$ 

Since this holds for all $P \in \mathcal{N}$ we get $c > 0$.

**Theorem 8.** Assume (B). Then $J \in C^1(X)$ admits a ground state.

**Proof.** We prove that the min-max level $c$ from (11) is attained at some function $P \in X \setminus \{0\}$. So let $(P_n)$ be a minimizing sequence. After rescaling we may without loss of generality assume $\int_{\mathbb{R}^3} (-\Delta)^{-1} P_n \cdot P_n dx = 1$. Then we get for any given $s > 0$

$$c + o(1) = \sup_{t > 0} J(tP_n) \geq J(sP_n) = \int_{\mathbb{R}^3} \Psi(x, sP_n) dx - \frac{s^2}{2} \text{ as } n \to \infty.$$ 

This and the second estimate in (7) show that $(P_n)$ is bounded in $X$ and we may pass to a weakly convergent subsequence that we still denote by $(P_n)$, so $P_n \rightharpoonup P_*$ in $X$. Proposition 5 implies $(-\Delta)^{-1} P_n \to (-\Delta)^{-1} P_*$ in $X^*$ and thus $\int_{\mathbb{R}^3} (-\Delta)^{-1} P_* \cdot P_* dx = 1$, whence

$$c \geq \int_{\mathbb{R}^3} \Psi(x, sP_n) dx - \frac{s^2}{2} + o(1) \\
\geq \int_{\mathbb{R}^3} \Psi(x, sP_*) dx - \frac{s^2}{2} \int_{\mathbb{R}^3} (-\Delta)^{-1} P_* \cdot P_* dx + o(1) \\
= J(sP_*) + o(1).$$ 

In the second inequality we exploited the weak lower semicontinuity (by convexity). Since the above estimate holds for all $s > 0$ we conclude that $P_*$ is a minimizer for the min-max-level $c$, so a multiple of it, say $P^* := t(P_*)P_*$, is a minimizer for $c_{\mathcal{N}}$ by Proposition 7. Using $J \in C^1(X)$ we get as in [15, Proposition 9] that the reduced functional $J(P) := \max_{t > 0} J(tP)$ is continuously differentiable away from the origin, and $J(P_*) = 0$ implies $J'(P^*) = 0$. Moreover, by construction of the Nehari manifold $\mathcal{N}$, no other nontrivial critical point of $J$ attains a smaller energy level than $P^*$, so $P^*$ is a ground state. \hfill \Box

2. **Proof of Theorem 11**

We want to prove Theorem 11 using our analysis of the dual problem of the previous section. We first show that the latter applies under the assumptions of Theorem 11.

**Proposition 9.** Assume that the function $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies (A). Then there is a function $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfying (B) such that $f(x, \cdot)^{-1} = \psi(x, \cdot)$ for almost all $x \in \mathbb{R}^3$.

**Proof.** By assumption (A) the function $z \mapsto f_0(x, z)$ is continuous and increasing on $[0, \infty)$ with $f(x, z) \to 0$ as $z \to 0$ and $f_0(x, z) \to +\infty$ as $z \to \infty$ for almost all $x \in \mathbb{R}^3$. Hence, there is an increasing inverse $\psi_0(x, \cdot) = f_0(x, \cdot)^{-1} : [0, \infty) \to [0, \infty)$. Then $f(x, \cdot)^{-1} = \psi(x, \cdot)$ where $\psi(x, P) := \psi_0(x, |P|^{-1}P)$ is Carathéodory. Moreover, $\mathbb{R}_+ \to \mathbb{R}_+, z \mapsto \frac{\psi_0(x, z)}{z}$ is decreasing because $\mathbb{R}_+ \to \mathbb{R}_+, s \mapsto \frac{f_0(x, s)}{s}$ is increasing. So it remains to verify that $\psi_0$ satisfies the estimates from (B).
To this end fix $x \in \mathbb{R}^3$ such that $f_0(x, \cdot)^{-1} = \psi_0(x, \cdot)$ holds. In the following let $s, z \in [0, \infty)$ depend on each other via $s = \psi_0(x, z), z = f_0(x, s)$. Since (A) implies $|f_0(x, s)| \sim \Gamma(x) \min\{s^{p-1}, s^{-1}\}$, we have
\[
\psi_0(x, z) = s \sim \max\{\Gamma(x)^{-1}z, (\Gamma(x)^{-1}z)^{q-1}\}. \tag{12}
\]
This shows that the second inequality in (B) hold. The Fenchel-Young identity shows that $f_0(x, \cdot)^{-1} = \psi_0(x, \cdot)$ satisfy
\[
\int_0^z \psi_0(x, t) \, dt - \frac{1}{2} \psi_0(x, z) z = \frac{1}{2} f_0(x, s) s - \int_0^s f_0(x, t) \, dt.
\]
This implies
\[
\int_0^z \psi_0(x, t) \, dt - \frac{1}{2} \psi_0(x, z) z = \frac{1}{2} s f_0(x, s) - \int_0^s f_0(x, t) \, dt \overset{(A)}{\geq} \overset{(12)}{\geq} \max\{\Gamma(x)^{-1}z^{p-1}, (\Gamma(x)^{-1}z)^{q-1}\},
\]
which finishes the proof. \hfill \Box

The above proposition shows that there is a function $\psi$ such that our variational approach of the previous section applies. In particular, a ground state of $J$ exists by Theorem \ref{thm:existence}. To show that this ground state of $J$ produces a ground state of $I$ via $E := \psi(x, P)$, we need the equivalence of the original problem and the dual problem. This is proved next.

**Lemma 10.** Assume (A). Then $I'(E) = 0$, $E \in H^1(\text{curl}; \mathbb{R}^3) \cap \Gamma^{-1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$ if and only if $J'(P) = 0$, $P \in X$ where $P, E$ are related to each other via $P = f(x, E)$, $E = \psi(x, P)$ and $\psi$ is given by Proposition \ref{prop:equivalence}.

**Proof.** Assume first $I'(E) = 0$ for $E \in H^1(\text{curl}; \mathbb{R}^3) \cap \Gamma^{-1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$. Then $P = f(x, E)$ satisfies $\nabla \times \nabla \times E = P$ and thus $\nabla \cdot P = 0$ in the distributional sense. Combining this with $|P| = |f(x, E)| \sim \Gamma(x) \min\{|E|^{p-1}, |E|^{q-1}\}$ and $E \in \Gamma^{-1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$, we infer that $P$ is divergence-free and belongs to $\Gamma^{1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) \cap \Gamma^{1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$, i.e., $P \in X$. Furthermore, $I'(E) = 0$ implies that $\nabla \times \nabla \times E = P$ holds in the weak sense. In view of $E = \psi(x, P)$ and Proposition \ref{prop:equivalence} this means that $(-\Delta)\Pi(\psi(x, P)) = P$ holds in the weak sense. This implies $|\xi|^2\mathcal{F}(-\Delta)\Pi(\psi(x, P)) = \hat{P}$ in the sense of tempered distributions and thus, by \ref{thm:existence}, $\Pi(\psi(x, P)) = (-\Delta)^{-1}P$, which is finally equivalent to $J'(P) = 0$ by Proposition \ref{prop:equivalence}.

Now assume $P \in X$ and $J'(P) = 0$. Then $X \subset \Gamma^{1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) \cap \Gamma^{1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$ and \ref{thm:existence} gives $E \in \Gamma^{-1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)$. It remains to show $E \in H^1(\text{curl}; \mathbb{R}^3)$ because $J'(P) = 0$ then implies $I'(E) = 0$ using the reverse chain of implications from above. We have
\[
\nabla \times E = \nabla \times \psi(x, P) = \nabla \times \Pi(\psi(x, P)) = \nabla \times (-\Delta)^{-1}P = F^{-1} \left( -\frac{i\xi \times \hat{P}}{|\xi|^2} \right).
\]
Standard mapping properties of Riesz transforms [17, Corollary 5.2.8] and Riesz potentials [6, Theorem 1.2.3] yield $\nabla \times E \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ provided that $P \in L^r(\mathbb{R}^3; \mathbb{R}^3)$ provided that $\frac{1}{r} - \frac{1}{2} = \frac{1}{q}$, i.e., $r = \frac{3}{2}$. So it remains to show
\[
\Gamma^{1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) \cap \Gamma^{1/q}L^q(\mathbb{R}^3; \mathbb{R}^3) \subset L^{6/5}(\mathbb{R}^3; \mathbb{R}^3). \tag{13}
\]
In fact, if $6 - 2\alpha < q \leq 6$ then this follows from
\[
\|f\|_{6/5} \leq \|\Gamma^{-1/q}f\|_{\frac{6}{q}} \|\Gamma^{1/q}\|_{\frac{6}{q} \rightarrow \infty} \lesssim \|f\|.
\]
If $q > 6$ choose $\theta \in (0, 1)$ such that $\frac{\alpha}{p} + \frac{1-\theta}{q} = \frac{5}{6}$. Then
\[
\|f\|_{6/5} \leq \|\Gamma^{-1/q}f\|^{1-\theta}_{q'} \|\Gamma^{-1/p}f\|_{p'} \|\Gamma^{1-\theta/q}f\|_{\frac{6}{q}} \lesssim \|f\|.
\]
So (13) holds and we conclude $E \in H^1(\text{curl}; \mathbb{R}^3)$. □

**Proof of Theorem 1.** For $f$ satisfying assumption (A) as in the theorem we define $\psi$ by Proposition 9. Then $\psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies assumption (B) and accordingly, by Theorem 8, the functional $J$ has a ground state solution $P^* \in X \setminus \{0\}$. By Proposition 9 $J'(E) = 0$ with $E \in H^1(\text{curl}; \mathbb{R}^3) \cap [\Gamma^{-1/p}L^p(\mathbb{R}^3; \mathbb{R}^3) + \Gamma^{-1/q}L^q(\mathbb{R}^3; \mathbb{R}^3)]$ is equivalent to $J'(P) = 0$ with $P \in X$. Then [11] Theorem 15 shows that $E^*(x) := \psi(x, P^*(x))$ defines a ground state solution for $I$, which is all we had to prove. □

**Acknowledgments**

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

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