I. INTRODUCTION

This year, we celebrate the centenary of the discovery, by Kammerlingh Onnes [1], of superconductivity which is one of the most fascinating phenomena of Solid State physics: indeed, it is so at odd from usual understanding that it took half a century to reach some acceptable microscopic picture of the phenomenon. The first clue was to understand that, in spite of their charge repulsion, two electrons can attract each other via the ion motion [2]. Although very small, this attraction, as shown by Cooper [3], can produce a two-electron bound state when acting in a region where the density of states is finite. The next step was to note that fermion pairs being boson-like particles, these two-electron bound states can condense into a collective state quite different from the “normal” electron gas. Although Schrieffer kept claiming that this condensation is quite different from a Bose-Einstein condensation of elementary bosons [4], the wave function ansatz used in the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity [5] is just the grand canonical ensemble version of all pairs being in the same state — as seen more in detail below. To support Schrieffer’s claim, it is however clear that a state reading as the product of two linear combinations of free pair states is different from two individual products because, due to the Pauli exclusion principle between fermions, one free fermion pair state is “missing in the second pair”. This Pauli blocking effect increasing with pair number, the state reading as a product of $N$ identical linear combinations of free electron pairs is for sure very different from $N$ individual products, as for $N$ elementary bosons in a Bose-Einstein condensate.

Nevertheless, even if more and more free pair states are missing when the number of correlated pair states increases — through what we called “moth-eaten effect” [6] — it is still of importance to know if the picture of superconductivity resulting from a product of identical electron pairs still is an acceptable picture of the phenomenon. This picture comes from an ansatz which has, as formidable support, the fact that it leads to easy calculations with results in agreement with experiments. Nevertheless, it is well known that wave functions are quite tricky, different ones possibly giving same values for quantities as averaged as the energy.

The fact that Cooper pairs are not elementary but composite bosons pushes us to question the widely accepted idea that they all are in the same linear combination of free-pair states, as for an elementary boson condensate, even if, clearly, some collective effect takes place in this phenomenon - as seen from the fact that the linear combination in the dense BCS limit is definitely different from the one of a single pair found by Cooper [3]. This question becomes even more relevant when considering the work done by Richardson [7] and by Gaudin [8] a few years after the BCS milestone paper on superconductivity [5]. Indeed, they succeeded to write the exact form of the $N$ electron pair eigenstates for the Hamiltonian considered by Bardeen, Cooper, Schrieffer [5] — this Hamiltonian actually being one of the very few exactly solvable model. This exact $N$-pair wave function reads as

$$B^\dagger(R_1) \cdots B^\dagger(R_N)|0\rangle,$$

where $B^\dagger(R_i)$ is the following linear combination

$$B^\dagger(R_i) = a_i^\dagger \mathbf{b}^\dagger \mathbf{c}^\dagger \mathbf{d}^\dagger \mathbf{e}^\dagger \cdots ,$$
of up and down spin electrons,

\[ B_i^\dagger (R_i) = \sum_k \frac{\omega_k}{2\epsilon_k - R_i} a_i^\dagger a_{k+}^\dagger , \] (2)

\( \omega_k \) being equal to 1 in the layer where the potential acts — called later on “potential layer”. The \( R_i \)'s are solution of \( N \) coupled equations

\[ 1 = V \sum_k \frac{\omega_k}{2\epsilon_k - R_i} + \sum_{j \neq i} \frac{2V}{R_i - R_j} . \] (3)

In view of the second term of the above equation which can be traced back to the Pauli exclusion principle [9], the \( R_i \)'s must be different. As a result, the exact wave function (1) differs from \( N \) pairs being in the same linear combination, as in the BCS ansatz.

Over the last half century [10], these Richardson-Gaudin equations stayed analytically unsolved for arbitrary pair number \( N \) and potential \( V \). In order to better understand the effect of Pauli blocking between composite bosons — which forces us to consider a fixed number of pairs feeling the potential and thus amounts to stay in the canonical ensemble — we tackled these Richardson-Gaudin equations again recently [6, 11, 12]. In contrast to traditional BCS theory which corresponds to fill half the potential layer, we considered an arbitrary filling of this layer. We first succeeded to get the \( R_i \)'s solution of these Richardson’s equations in the dilute limit of pairs, i.e., for \( N \ll N_c \) where \( N_c \) is the pair number above which single Cooper pairs would start to overlap. Later on, we found a way [13] to reach the \( N \)-pair ground state energy \( E_N \), i.e., the sum \( \sum_i R_i \), without calculating the \( R_i \)'s individually. \( E_N \) turns out to have a nicely compact \( N \)-dependence,

\[ E_N = NE_1 + \frac{N(N - 1)}{\rho} \frac{1 + \sigma}{1 - \sigma} , \] (4)

within underextensive terms in \( (N/\rho)^n \), where \( \rho \) is the density of states taken as constant in the potential layer. \( \sigma = \exp(-2/\rho V) \) while \( E_1 = 2\epsilon_{F_0} - 2\Omega \sigma/(1 - \sigma) \) is the single pair energy found by Cooper, \( \epsilon_{F_0} \) being the Fermi energy of the frozen core electrons, i.e., electrons which do not feel the potential, and \( \Omega \) the potential layer extension, of the order of twice a phonon energy.

It has been a surprise to note that this energy fully agrees with the ground state energy obtained by Bardeen, Cooper, Schrieffer, through a minimization of the Hamiltonian mean value calculated with their wave function ansatz [5] — which fundamentally takes the \( N \) pair states condensed into the same linear combination, while the exact wave function (1) corresponds to different linear combinations. Indeed, the BCS ground state energy is known to read in the weak coupling limit, \( \sigma \approx 0 \), as

\[ E_{BCS} \approx \frac{1}{2\rho} \Delta^2 , \] (5)

where, in this limit, the gap scales as \( \Delta \approx \Omega \sqrt{\sigma} \). This result agrees with Eq.(4) for half filling, i.e., \( N = \rho \Omega / 2 \), which is the pair number corresponding to a potential extending symmetrically over a phonon energy scale, on both sides of the normal electron Fermi sea. Even if cases of wave functions very different from the exact one while giving the same energy, are known to exist, the fact that the exact \( N \)-pair energy does support the BCS wave function ansatz still is rather puzzling because in this wave function, the \( N \) pairs are condensed into the same state while in the exact wave function, the \( N \) pairs all are in different states as a result of Pauli blocking.

Actually, there is another approach to superconductivity which allows reaching some microscopic understanding of this ansatz, although not yet helping to make link with Richardson-Gaudin exact wave function. It is based on the Bogoliubov transformation. This transformation allows an exact diagonalization of a part of the original BCS Hamiltonian for this Hamiltonian extended to the grand canonical ensemble. The ground state of this partial Hamiltonian just corresponds to the BCS ansatz. We are then left with showing that the remaining part of the BCS Hamiltonian brings a negligible contribution in the thermodynamical limit, i.e., when the change from canonical to grand canonical ensemble is expected to be small. In fact, such conclusions have already been reached by Boboliubov [14] through a splitting the whole Hamiltonian in two terms, a “mean-field” Hamiltonian and a perturbation. This splitting is similar to the one we here use. Bogoliubov then uses this splitting to estimate from below and from above the ground state energy of the system. Instead, we here use a perturbative approach to, in a direct way, show that the perturbative part of the Hamiltonian produce underextensive terms only, provided that the arbitrary scalars introduced in the Bogoliubov procedure are chosen properly. In Ref.[15], Bardeen and Rickayzen have used single-particle Green’s functions, in a form given by Galitskii [16], to reach similar conclusions. We also wish to mention a paper by Mattis and Lieb [17], in which it is shown that appropriate boundary and continuity conditions on the exact wave function in the large-sample limit, can lead to the equations of BCS theory obtained within the BCS ansatz.

The paper is organized as follows.

In section II, we briefly recall the BCS model Hamiltonian for superconductivity and the wave function ansatz proposed by Bardeen, Cooper, Schrieffer with emphasis on its \( N \)-pair form.
In section III, we propose a very simple presentation of the Bogoliubov approach to the BCS problem which leads to a splitting of the BCS Hamiltonian as $H_{BCS} = H_B + W$. At this stage, both $H_B$ and $W$ are not fully fixed, so that they depend on a set of arbitrary scalars. We then show how the "Bogoliubov Hamiltonian" $H_B$ can be diagonalized. This procedure enables us to find the $H_B$ energy spectrum.

In section IV, we discuss some properties of the "Bogoliubov Hamiltonian" eigenstates. In particular, we relate its ground state to the BCS wave function ansatz.

In section V, we construct a perturbation theory for the total Hamiltonian $H_{BCS}$ by treating $W$ as a small perturbation. Using this approach, we determine the optimal set of scalars introduced in the Bogoliubov procedure, which makes the $W$ contribution underextensive. The $H_B$ ground state then reduces to the BCS ansatz when these underextensive terms are neglected. This gives a strong mathematical support to this ansatz in spite of its major difference with the exact form of the BCS ground state obtained by Richardson and Gaudin.

In section VI, we concentrate on the ground state of the Bogoliubov Hamiltonian $H_B$. We consider not only the standard BCS half-filling configuration, but also arbitrary fillings of the potential layer. Although these arbitrary fillings are difficult to achieve experimentally, their analysis enables us to reach a deeper understanding of the physics of Cooper-paired states. They also allow us to establish a close link between two classical problems, namely, Cooper problem and the BCS model of superconductivity.

In Section VII, we study the $H_B$ excited states and derive the energy gap for an arbitrary filling of the potential layer. We also show the existence of two regimes, an extremely dilute and an extremely dense regime of pairs which have a gap different from the usual gap $\Delta$. These two regimes should have to be considered for a complete understanding of the BEC-BCS cross-over along the line proposed by Eagles[18] and by Leggett[19].

In Section VIII, we present some concluding comments. We, in particular, come back to the major physical discrepancy which still exists between the BCS wave function ansatz and the Richardson-Gaudin exact wave function.

II. BCS WAVE FUNCTION ANSATZ

We consider a system made of $N$ pairs of different fermions $\alpha$ and $\beta$. In the case of BCS superconductivity, these fermions are up and down spin electrons. For fermions with same mass, the free part of their Hamiltonian reads in terms of their creation operators $a_k^\dagger$ and $b_k^\dagger$ as

$$H_0 = \sum_k \epsilon_k (a_k^\dagger a_k + b_k^\dagger b_k). \quad (6)$$

In usual BCS pairing, fermion-fermion attraction is reduced to processes between zero-momentum pairs. So, the total Hamiltonian reads as $H_{BCS} = H_0 + V_{BCS}$ with

$$V_{BCS} = - \sum_{k \neq k'} V_{kk'} a_{k'}^\dagger b_{k''} - b_{k'}^\dagger a_{k''}. \quad (7)$$

In order to end with an analytically solvable problem, the $V_{kk'}$ scattering will be ultimately taken in a separable form $V_{kk'}^{\omega_k \omega_{k'}}$. However, let us first keep its general form $V_{kk'}$

We want to determine the ground state of these $N$ pairs. In order to have the lowest possible energy, they must enjoy the attractive potential as much as possible: so, these pairs must appear as products of zero-momentum free pairs with creation operator

$$B_k^\dagger = a_k^\dagger b_{-k}^\dagger. \quad (8)$$

We thus expect the ground state for $N$ pairs to read as

$$|\psi_N^{(0)}\rangle = \sum_{k_1, \ldots, k_N} F(k_1, \ldots, k_N) B_{k_1}^\dagger \cdots B_{k_N}^\dagger |0\rangle, \quad (9)$$

with $(k_1, \ldots, k_N)$ all different due to the Pauli exclusion principle.

The handling of this Pauli blocking effect exactly is definitely difficult. A smart way to overcome this difficulty is to turn to the grand canonical ensemble, with a pair number not fixed, as proposed by Bardeen, Cooper and Schrieffer through an ansatz constructed on the idea that, electron pairs being boson-like particles, they are likely to condense all into the same state

$$B^\dagger = \sum_k \varphi_k B_k^\dagger. \quad (10)$$

As a result, Bardeen, Cooper and Schrieffer proposed a wave function ansatz reading as

$$|\phi_{BCS}\rangle = \sum_{N=1}^{+\infty} \frac{1}{N!} B^\dagger N |0\rangle. \quad (11)$$

By noting that $|\phi_{BCS}\rangle$ can also be written as

$$|\phi_{BCS}\rangle = e^{B^\dagger} |0\rangle = \prod_k e^{\varphi_k B_k^\dagger} |0\rangle = \prod_k (1 + \varphi_k B_k^\dagger)|0\rangle, \quad (12)$$

since $B_k^\dagger|0\rangle = 0$ due to Pauli blocking, we end for $\varphi_k$ written as $v_k/u_k$, and $|\phi_{BCS}\rangle$ taken as
This scalar is real since \( V \) possibly treat and arbitrary scalars where the prefactor \( \Delta_k \) of the procedure. It reads \( V_{BCS} = \mathcal{E} + V_B + W \).

The two-body character of the BCS interaction now appears through \( W \) given by

\[
W = -\sum_{k,k'} V_{k'k}(B_{k'}^\dagger z_{k'} - z_{k'}^* B_{k'} - z_{k'}^* z_k).
\]

The \( z_k \) scalars will be ultimately chosen to make the \( W \) contribution to the ground state energy small.

By extracting \( W \) from \( V_{BCS} \), we are left with \( V_B \) being a one-body potential, which is the goal of the procedure. It reads

\[
V_B = -\sum_{k'} \Delta_{k'} B_{k'}^\dagger + \text{h.c.},
\]

where the prefactor \( \Delta_{k'} \) depends on \( V_{k'k} \) and the arbitrary scalars \( z_k \) through

\[
\Delta_{k'} = \sum_k V_{k'k} z_k.
\]

The scalar \( \mathcal{E} \) in Eq.(14) is then given by

\[
\mathcal{E} = \sum_{k,k'} V_{k'k} z_{k'} z_k.
\]

This scalar is real since \( V_{k'k} = V_{kk'}^* \) which follows from \( V_{BCS} = V_{BCS}^\dagger \).

While \( V_{BCS} \) conserves the particle number, \( V_B \) and \( W \) do not conserve it separately. In order to possibly treat \( W \) as a perturbation independently from \( V_B \), it is thus necessary to turn to the grand canonical ensemble. This leads us to introduce the Hamiltonian

\[
\hat{H} = H_{BCS} - \mu \hat{N},
\]

where \( \hat{N} \) is the number operator for fermions \( \alpha \) and \( \beta \),

\[
\hat{N} = \sum_k (a_k^\dagger a_k + b_k^\dagger b_k).
\]

In this grand canonical ensemble, the Hamiltonian \( \hat{H} \) then splits as

\[
\hat{H} = H_B + W,
\]

where \( H_B \) is the “Bogoliubov Hamiltonian” given by

\[
H_B = \mathcal{E} + \sum_k \left\{ \xi_k (a_k^\dagger a_k + b_k^\dagger b_{-k}) - \Delta_k B_k^\dagger - \Delta_k^* B_k \right\},
\]

in which we have set \( \xi_k = \epsilon_k - \mu \).

B. Diagonalization of the “Bogoliubov Hamiltonian”

The \( H_B \) Hamiltonian is a one-body operator. It is thus easy to diagonalize it by introducing new operators

\[
\begin{align*}
\hat{a}_k^\dagger &= x_k a_k^\dagger + y_k b_{-k} \\
\hat{b}_{-k}^\dagger &= x'_k b_{-k}^\dagger + y'_k a_k.
\end{align*}
\]

In the \( \Delta_k = 0 \) limit, i.e., in the absence of potential, the \( (x,y) \) prefactors must reduce to \( x_k = 1 = x'_k \) and \( y_k = 0 = y'_k \). If we enforce these new operators to have the same anticommutation relations as the original operators \( a_k^\dagger \) and \( b_{-k}^\dagger \), we then find that \( (x_k, y_k) \) and \( (x'_k, y'_k) \) must be such that

\[
\begin{align*}
\left[ \hat{a}_k, \hat{a}_k^\dagger \right]_+ &= 1 = |x_k|^2 + |y_k|^2, \quad (24) \\
\left[ \hat{b}_{-k}, \hat{b}_{-k}^\dagger \right]_+ &= 1 = |x'_k|^2 + |y'_k|^2, \quad (25)
\end{align*}
\]

while

\[
\left[ \hat{a}_k^\dagger, \hat{b}_{-k}^\dagger \right]_+ = 0 = y_k x'_k + x_k y'_k, \quad (26)
\]

the other three anticommutators,

\[
0 = \left[ \hat{a}_k^\dagger, \hat{a}_k^\dagger \right]_+ = \left[ \hat{b}_{-k}^\dagger, \hat{b}_{-k}^\dagger \right]_+ = \left[ \hat{a}_k, \hat{b}_{-k} \right]_+,
\]

being automatically fulfilled for the operators defined in Eqs.(23).
It will appear as convenient to rewrite Eq.(26) as
\[ \frac{y_k}{x_k} = \frac{y'_k}{x'_k} = t_k . \] (28)

When inserted into Eqs.(24,25), this gives \( 1 = |x_k|^2 \left( 1 + |t_k|^2 \right) = |x'_k|^2 \left( 1 + |t_k|^2 \right) \); so,
\[ |x_k|^2 = |x'_k|^2 . \] (29)

As a result, we also have \( |y_k|^2 = |y'_k|^2 \).

If we now write \( \tilde{a}^\dagger \tilde{a} \) as in Eq.(23), we get
\[ \tilde{a}^\dagger \tilde{a} = |x_k|^2 a^\dagger_k a_k + |y_k|^2 b^\dagger_k b_k + \left( y_k x_k^* B_k + \text{h.c.} \right) , \] (30)
and similarly for \( \tilde{b}^\dagger \tilde{b} \). By noting that \( b^\dagger - b \) and \( b^\dagger - b \) are equal due to Eq.(29), it becomes easy to see that the hamiltonian \( H_B \) given in Eq.(22) also reads
\[ H_B = \mathcal{E}_B + \sum_k E_k (\tilde{a}^\dagger \tilde{a}_k + \tilde{b}^\dagger \tilde{b}_k) , \] (32)
provided that we take \( \xi_k = E_k \left( |x_k|^2 - |y_k|^2 \right) = E_k \left( 2|x_k|^2 - 1 \right) \) and \( \Delta^*_k = -E_k \left( y_k x_k^* - y_k' x'_k \right) = -2E_k|x_k|^2 t_k \). As a result,
\[ |x_k|^2 = 1 - |y_k|^2 = \frac{1}{2} \left( 1 + \frac{\xi_k}{E_k} \right) \] (33)
\[ \Delta^*_k = -t_k \left( E_k + \xi_k \right) \]
It then follows from \( E_k \) and \( \xi_k \) being both real, that \( |t_k|^2 = |\Delta^*|^2/(E_k + \xi_k)^2 \). Since \( |t_k|^2 = |y_k|^2/|x_k|^2 \) is also equal to \( (E_k - \xi_k)/(E_k + \xi_k) \), we end with
\[ E_k = \sqrt{\xi_k^2 + |\Delta_k|^2} , \] (34)
the square root being taken with a positive sign in order for \( E_k \) to go to \( \xi_k \) in the absence of potential, i.e., when \( |\Delta_k| \to 0 \).

The scalar \( \mathcal{E}_B \) in the “Bogoliubov Hamiltonian” of Eq.(32) is then given, using Eq.(18), by
\[ \mathcal{E}_B = \sum_{k',k} V_{k'k} \tilde{z}_{k'} \tilde{z}_k + \sum \left( \xi_k - E_k \right) . \] (35)

Through these detailed calculations, we already see that the Bogoliubov procedure allows us to recover a few important results of the BCS theory, originally derived using a variational procedure based on the BCS wave function ansatz.

IV. EIGENSTATES OF THE BOGOLIUBOV HAMILTONIAN

A. Ground state

Equation (32) shows that the \( H_B \) ground state energy reduces to \( \mathcal{E}_B \) while the excited states read as products of operators \( \tilde{a}^\dagger \) and \( \tilde{b}^\dagger \) acting on the \( H_B \) ground state.

This ground state is, by construction, such that \( \tilde{a}_k \) or \( \tilde{b}_k \) acting on it gives zero. It is then easy to relate this ground state to the \( \mid \phi_{BCS} \rangle \) ansatz defined in Eq.(12). Indeed,
\[ \tilde{a}_p \mid \phi_{BCS} \rangle = \left( x_p^* a_p + y_p^* b_p \right) \left( 1 + \varphi_p^* B_p \right) \mid \phi_{BCS}(p) \rangle \]
\[ = \left( y_p^* + \varphi_p x_p \right) b_p^\dagger \mid \phi_{BCS}(p) \rangle , \] (36)
where \( \mid \phi_{BCS}(p) \rangle \) contains all the \( k \) states of the \( \mid \phi_{BCS} \rangle \) product except \( k = p \). In the same way,
\[ \tilde{b}^\dagger_p \mid \phi_{BCS} \rangle = \left( y_p^* - \varphi_p x_p \right) a_p \mid \phi_{BCS}(p) \rangle . \] (37)

We easily see that these two states reduce to zero provided that \( \varphi_p \) is chosen such that
\[ \varphi_p = \frac{y_p^*}{x_p} = \frac{y_p^*}{x_p^*} = -t_p^* , \] (38)
which always is possible due to Eq.(28). Since
\[ \langle \phi_{BCS} | \phi_{BCS} \rangle = \prod_{k} \langle 0 | (1 + \varphi_k B_k^\dagger) (1 + \varphi_k B_k) | 0 \rangle \]
\[ = \prod_{k} (1 + |\varphi_k|^2) , \] (39)
while, due to Eq.(38), \( 1 + |\varphi_k|^2 = 1 + |y_k/x_k|^2 = 1/|x_k|^2 \), the normalized ground state of the \( H_B \) Hamiltonian thus reads, within an irrelevant phase factor, as
\[ \langle \tilde{0} | \mid \psi_{BCS} \rangle = \prod_{k} |x_k| \left( 1 - \frac{y_k^*}{x_k^*} B_k^\dagger \right) | 0 \rangle \]
\[ = \prod_{k} (u_k + v_k B_k^\dagger) | 0 \rangle , \] (40)
in which we have set
\[ u_k = |x_k| = |x_k^*| \]
\[ v_k = -|x_k| \frac{y_k^*}{x_k^*} = |x'_k| \frac{y_k^*}{x_k^*} . \] (41)
From this, we get the pair number in the $H_B$ ground state through the mean value of the fermion number operator defined in Eq.(20) as

$$\bar{N} = \frac{1}{2} \langle \hat{0}|\hat{N}|\hat{0} \rangle = \sum_k |v_k|^2.$$  \hfill (42)

### B. Excited states

If we now turn to the lowest excited states of the $H_B$ Hamiltonian, we find, using Eqs.(23), that they are given by

$$\tilde{a}_p^\dagger|\tilde{0}\rangle = \left(x_p u_p - v_p y_p\right) a_p^\dagger |\tilde{0}\rangle$$

$$= \frac{|x_p|}{x_p^*} a_p^\dagger |\tilde{0}\rangle,$$  \hfill (43)

where $|\tilde{0}\rangle$ corresponds to the $|\tilde{0}\rangle$ ground state given in Eq.(40) without the $p$ state in the $k$ product. In the same way,

$$\tilde{b}_p^\dagger|\tilde{0}\rangle = \frac{|x_p'|}{x_p'^*} b_p^\dagger |\tilde{0}\rangle,$$  \hfill (44)

the excited states (43) and (44) having the same energy $E_B + E_p$.

The above equations lead us to choose $x_p$ and $x_p'$ both real, which always is possible in view of Eqs.(23). The $H_B$ excited states then take a compact form,

$$\tilde{a}_p^\dagger|\tilde{0}\rangle = a_p^\dagger |\tilde{0}\rangle,$$

$$\tilde{b}_p^\dagger|\tilde{0}\rangle = b_p^\dagger |\tilde{0}\rangle.$$  \hfill (45)

We note that the $p$ state of the $\alpha$ fermion appears through the pair operator $B_p^\dagger = a_p^\dagger b_p^\dagger$ in the ground state $|\tilde{0}\rangle$, while in the excited state $\tilde{a}_p^\dagger |\tilde{0}\rangle$, this $p$ state appears through $a_p^\dagger$ only; so, the effect of the excitation operator $\tilde{a}_p^\dagger$ is to break the $B_p^\dagger$ pair by removing its $b_p^\dagger$ component. Similarly, in $\tilde{b}_p^\dagger |\tilde{0}\rangle$, the $a_p^\dagger$ component of the $B_p^\dagger$ pair operator is removed.

Using Eqs.(28,33,41), we find

$$u_k = x_k = x'_k = \sqrt{\frac{1}{2} \left(1 + \frac{\xi_k}{E_k}\right)},$$  \hfill (46)

while $v_k$, a priori complex, is related to $\Delta_k$ through

$$v_k = -y_k^* = y'_k = -x_k t_k^* = \frac{\Delta_k}{\sqrt{2E_k(E_k + \xi_k)}}.$$  \hfill (47)

These again are standard textbook results derived through the minimization of the $H_{BCS}$ Hamiltonian mean-value calculated within the BCS ansatz (13).

### V. EIGENSTATES OF THE TOTAL BCS HAMILTONIAN

In the above procedure, we have essentially extracted the unpleasant two-body part of the original BCS Hamiltonian, through $W$. The $H_B$ eigenstate spectrum is then straightforward to find. In order for these eigenstates to have some connexion with the original BCS problem, the “perturbation” $W$ must only bring a small contribution. We do have some flexibility to reach this goal through the set of arbitrary scalars $z_k$ introduced in rewriting the BCS potential as in Eq.(14).

#### A. First-order correction

A good choice for these $z_k$ scalars clearly requires the first-order correction to the ground state energy $\langle 0|W|\tilde{0}\rangle$ to be small. Using Eqs.(23), we can rewrite the original BCS operators $(a_k^\dagger, b_k^\dagger)$ in terms of the new Bogoliubov operators $(\tilde{a}_k^\dagger, \tilde{b}_k^\dagger)$ as

$$a_k^\dagger = u_k \tilde{a}_k^\dagger + v_k \tilde{b}_{-k},$$

$$b_k^\dagger = u_k \tilde{b}_{-k}^\dagger - v_k \tilde{a}_k^\dagger.$$  \hfill (48)

Since $\tilde{a}_k |\tilde{0}\rangle = 0 = \tilde{b}_{-k} |\tilde{0}\rangle$ by construction, we easily find

$$B_k |\tilde{0}\rangle = \left(u_k \tilde{b}_{-k} - v_k \tilde{a}_k^\dagger\right)\left(u_k \tilde{a}_k + v_k \tilde{b}_{-k}\right) |\tilde{0}\rangle$$

$$= \left(u_k v_k - v_k^2 \tilde{B}_k^\dagger\right) |\tilde{0}\rangle,$$  \hfill (49)

where $\tilde{B}_k^\dagger = \tilde{a}_k^\dagger \tilde{b}_{-k}^\dagger$ creates a pair excitation. As a result, the first order correction to the $H_B$ ground state energy $E_B$ induced by the $W$ perturbation given in Eq.(15), reads as

$$\langle 0|W|\tilde{0}\rangle = -\sum_{k', k} V_{k'k}$$

$$\times \langle 0| \left(u_{k'} v_{k'} - z_{k'}^* - v_{k'}^2 \tilde{B}_{k'}^\dagger\right)$$

$$\times \left(u_k v_k - z_k - v_k^2 \tilde{B}_k^\dagger\right) |\tilde{0}\rangle.$$  \hfill (50)

Since $\langle 0|\tilde{B}_{k'} \tilde{B}_k^\dagger |\tilde{0}\rangle$ reduces to $\delta_{k', k}$, this gives

$$\langle 0|W|\tilde{0}\rangle = -\sum_k V_{kk}|v_k|^4$$

$$-\sum_{k', k} V_{k'k} \left(u_{k'} v_{k'} - z_{k'}^*\right)\left(u_k v_k - z_k\right).$$  \hfill (51)

By turning to the continuous limit, each sum over $k$ brings a sample volume $L^3$, while the potential matrix element $V_{k'k}$ depends on sample volume as $1/L^3$. As a result, the double sum over $(k', k)$ in $\langle 0|W|\tilde{0}\rangle$ is proportional to the sample volume; it
is thus extensive while the first sum is sample volume free, so that it brings an underextensive contribution to the ground state energy \( \mathcal{E}_B \) which is negligible in the thermodynamical limit. In order to neglect the whole \( \langle \hat{0}|W|\hat{0}\rangle \) contribution to the ground state energy, we are thus led to cancel the second term of Eq.(51) by choosing the scalars \( z_k \) as

\[
z_k = u_k v_k . \tag{52}\]

As a result, Eqs.(46,47) give \( \Delta_k' \), defined in Eq.(17), as

\[
\Delta_k' = \sum_k V_{k'k} z_k = \sum_k V_{k'k} u_k v_k \\
= \frac{1}{2} \sum_k V_{k'k} \frac{\Delta_k}{\sqrt{\xi_k^2 + |\Delta_k|^2}} . \tag{53}\]

To go further and get some explicit results, we must choose a particular form for the \( V_{k'k} \) potential. By taking it as separable,

\[
V_{k'k} = V \omega_{k'k} , \tag{54}\]

with \( \omega_{k}^2 = \omega_k \), the solution of Eq.(53) reduces to

\[
\Delta_k = \Delta \omega_k , \tag{55}\]

where \( \Delta \) is solution of the so-called “gap equation”:

\[
\frac{2}{V} = \sum_k \frac{\omega_k}{\sqrt{\xi_k^2 + |\Delta|^2}} . \tag{56}\]

**B. Higher-order corrections**

Within the choice Eq.(52) for the \( z_k \) scalars, we find

\[
B_{k}^\dagger - z_k^* = u_k^2 B_{k}^\dagger - v_k^* B_{k} , \\
- u_k v_k \left( \tilde{a}_{k}^\dagger \tilde{a}_{k} + \tilde{b}_{-k}^\dagger \tilde{b}_{-k} \right) . \tag{57}\]

Consequently, the \( W \) operator acting on the Bogoliubov vacuum \( |\hat{0}\rangle \) gives rise to a state which contains zero, one and two pair excitations:

\[
W|\hat{0}\rangle = - V \sum_k \omega_k |v_k|^4 |\hat{0}\rangle \\
- 2V \sum_{k'k} \omega_k u_k v_k |v_k|^2 B_{k}^\dagger |\hat{0}\rangle \\
+ V \sum_{k'k} \omega_{k'} u_{k'} v_{k'} |v_{k'}|^2 B_{k'}^\dagger B_{k}^\dagger |\hat{0}\rangle . \tag{58}\]

The second order correction to \( \mathcal{E}_B \) is given by

\[
\mathcal{E}_B^{(2)} = \langle \hat{0}|WP_\perp \frac{1}{\mathcal{E}_B - H_B} P_\perp W|\hat{0}\rangle , \tag{59}\]

where \( P_\perp \) is the projector over the subspace perpendicular to \( |\hat{0}\rangle \). Since the intermediate states are either \( B_{p}^\dagger |\hat{0}\rangle \) or \( B_{p}^\dagger B_{p}^\dagger |\hat{0}\rangle \), this second order term splits as

\[
\mathcal{E}_B^{(2)} = 4V^2 \sum_p \frac{\omega_p u_p^2 |v_p|^6}{-E_p} \\
+ \sum_{p,p'} \frac{\omega_p \omega_p |u_p|^2 |v_p|^2 + u_p^2 |v_p|^2}{-E_p - E_{p'}} . \tag{60}\]

The first term of \( \mathcal{E}_B^{(2)} \) has two \( V \)'s and one \( p \) sum only so that it depends on sample volume as \( 1/L^3 \), while the second term has two \( V \)'s but two \( p \) sums so that it is volume free. As a result, this second order term again gives a correction to the \( \mathcal{E}_B \) ground state energy smaller than \( L^3 \), i.e., underextensive and thus negligible in the thermodynamical limit.

By counting the number of potentials \( V \) and the number of \( p \) sums, it is possible to show that the higher order terms of the \( W \) expansion of the ground state energy give underextensive corrections. Consequently, within the choice of \( z_k \) scalars made in Eq.(52), the extensive part of the ground state energy for the BCS Hamiltonian in the grand canonical ensemble, indeed reduces to the scalar \( \mathcal{E}_B \) given in Eq.(35).

**VI. EXTENSIVE PART OF THE BCS GROUND STATE ENERGY**

In view of Eq.(19), the ground state energy of the BCS Hamiltonian \( H_{BCS} \) is related to \( \mathcal{E}_B \) through \( \mathcal{E}_B = \mathcal{E}_B + 2\mu N \), where \( N \) is the mean value of the fermion-pair number in the system [see Eq.(42)]. The scalar \( z_k \)'s are chosen according to Eq.(52) for \( W \) to give an underextensive contribution to the ground state energy. We moreover take the potential in the separable form of Eq.(54) to possibly reach analytical results. Using Eqs.(17,55), we then find

\[
\sum_{k'k} V_{k'k} z_{k'} z_k = V \sum_k z_k^2 \omega_k \sum_k z_k \omega_k = \frac{|\Delta|^2}{V} . \tag{61}\]

This gives the BCS ground state energy \( \mathcal{E}_B \) as

\[
\mathcal{E}_B = \mathcal{E}_B + 2\mu N = \frac{|\Delta|^2}{V} + \sum_k (\xi_k - E_k) \omega_k + 2\mu N . \tag{62}\]

Note that a factor \( \omega_k \) can be introduced in the \( k \) sum because \( \xi_k - E_k = 0 \) for \( \Delta_k = 0 \), i.e.
\( \omega_k = 0 \). This will allow us to split the \( k \) sum and calculate the two parts separately.

Due to Eqs. (42), (46) and (47), the chemical potential \( \mu \), hidden in \( \xi_k \)'s, is related to the pair number through

\[
\mathcal{N} = \sum_k \frac{\omega_k}{2} \left( 1 - \frac{\xi_k}{E_k} \right). \tag{63}
\]

### A. Standard BCS configuration

In the standard BCS configuration, the potential layer extends between \( \epsilon_{F_0} \) and \( \epsilon_{F_0} + \Omega \) in a region where the density of states can be taken as equal to a constant \( \rho \). Moreover, the potential is said to extend symmetrically with respect to the normal electron Fermi sea which corresponds to pairs filling half the potential layer, i.e., \( \mathcal{N} = \rho \Omega/2 \).

For half-filling and a constant density of states \( \rho \), Eq. (63) gives, when turning to the continuous limit

\[
\frac{\rho \Omega}{2} = \int_{\epsilon_{F_0}-\mu}^{\epsilon_{F_0}+\mu} \rho \frac{d\xi}{2} \left( 1 - \frac{\xi}{\sqrt{\xi^2 + \Delta^2}} \right). \tag{64}
\]

It is straightforward to check that this equation is fulfilled for \( \mu = \epsilon_{F_0} + \Omega/2 \).

The gap equation (56), for a chemical potential set in the middle of the potential layer, then gives

\[
\frac{2}{\mathcal{V}} = \int_{-\Omega/2}^{\Omega/2} \rho \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} = \rho \log \frac{\frac{\Omega}{2} + \sqrt{\frac{\Omega^2}{4} + \Delta^2}}{-\frac{\Omega}{2} + \sqrt{\frac{\Omega^2}{4} + \Delta^2}}. \tag{65}
\]

This leads to

\[
\Delta = \frac{\Omega e^{-1/\rho \mathcal{V}}}{1 - e^{-2/\rho \mathcal{V}}} = \frac{\Omega \sqrt{\frac{\sigma}{1 - \sigma}}}{1 - \sigma}, \tag{66}
\]

with \( \sigma = e^{-2/\rho \mathcal{V}} \).

For \( \mu = \epsilon_{F_0} + \Omega/2 \), the two sums in Eq. (62) reduce respectively to \( \sum_k \xi_k \omega_k = 0 \) and to

\[
\sum_k E_k \omega_k = \int_{-\Omega/2}^{\Omega/2} \rho \frac{d\xi}{\sqrt{\xi^2 + \Delta^2}} = \frac{\rho \Omega}{2} \sqrt{\frac{\Omega^2}{4} + \Delta^2 + \Delta^2 / \mathcal{V}}, \tag{67}
\]

as obtained through an integration by parts. If we now insert this result into Eq. (62), we end with a ground state energy in the canonical ensemble given by

\[
\tilde{\mathcal{E}}_B = 2 \mathcal{N} \epsilon_{F_0} + \mathcal{N} \Omega - \rho \Omega \sqrt{\frac{\Omega^2}{4} + \Delta^2}. \tag{68}
\]

Since \( \Delta = 0 \) for \( V = 0 \), the ground state energy in the absence of potential reduces to \( \tilde{\mathcal{E}}_B(V = 0) = 2 \mathcal{N} \epsilon_{F_0} + \mathcal{N} \Omega - \rho \Omega^2 / 4 \). So, the condensation energy induced by the attracting potential \( V \) is given by

\[
\tilde{\mathcal{E}}_B(V = 0) - \tilde{\mathcal{E}}_B = \frac{\rho \Omega}{2} \left( \sqrt{\frac{\Omega^2}{4} + \Delta^2 - \frac{\Omega}{2}} \right). \tag{69}
\]

For \( \Delta \) given by Eq. (66), \( \sqrt{1 + 4\Delta^2/\Omega^2} \) reduces to \((1 + \sigma)/(1 - \sigma)\); the condensation energy for half filling thus ends by reading as

\[
\tilde{\mathcal{E}}_B(V = 0) - \tilde{\mathcal{E}}_B = \frac{\rho \Omega}{2} \frac{\Omega \sigma}{1 - \sigma} = \frac{1}{2} \rho \Delta^2 (1 - \sigma) . \tag{70}
\]

This result fully agrees with the expression of the \( N \)-pair energy we have obtained [13] by analytically solving Richardson-Gaudin equations in the canonical ensemble, namely,

\[
E_N = N E_1 + \frac{N(N - 1)}{\rho} \frac{1 + \sigma}{1 - \sigma} \tag{71}
\]

the single pair energy obtained by Cooper being \( E_1 = 2 \epsilon_{F_0} - 2 \Omega \sigma / (1 - \sigma) \). Indeed, the above equation gives the condensation energy, i.e., the part of \( E_N \) which cancels when \( \sigma = 0 \), as

\[
E_N(V = 0) - E_N = N \left( 1 - \frac{N}{\rho \Omega} \right) \frac{2 \sigma}{1 - \sigma} \Omega . \tag{72}
\]

We can check that this condensation energy reduces to Eq. (70) for \( N = \rho \Omega / 2 \), whatever \( \sigma \), i.e., not only in the weak coupling limit, \( \sigma \approx 0 \).

### B. Arbitrary filling

We now consider an arbitrary filling of the potential layer, namely, values of \( \mathcal{N} \) not exactly equal to half the number of states between \( \epsilon_{F_0} \) and \( \epsilon_{F_0} + \Omega \). In this case, we must solve two coupled equations (56,63) for the gap \( \Delta \) and the chemical potential \( \mu \). After switching from sum to integral, the integration limits being \( \epsilon_{F_0} - \mu \) and \( \epsilon_{F_0} - \mu + \Omega \), these two coupled equations read as

\[
\frac{2}{\rho \mathcal{V}} = \ln \frac{\epsilon_{F_0} - \mu + \Omega + \sqrt{(\epsilon_{F_0} - \mu + \Omega)^2 + \Delta^2}}{\epsilon_{F_0} - \mu + \sqrt{(\epsilon_{F_0} - \mu)^2 + \Delta^2}}
\]

and

\[
\Omega - \frac{2 \mathcal{N}}{\rho} = \sqrt{(\epsilon_{F_0} - \mu + \Omega)^2 + \Delta^2} - \sqrt{(\epsilon_{F_0} - \mu)^2 + \Delta^2}. \tag{74}
\]

These equations can be solved analytically. Their solutions for \( \Delta \) and \( \mu \) read as

\[
\Delta = \sqrt{\frac{\mathcal{N}}{\rho \Omega} \left( 1 - \frac{\mathcal{N}}{\rho \Omega} \right) \frac{2 \sqrt{\sigma}}{1 - \sigma} \Omega}, \tag{75}
\]
\[ \mu - \epsilon_{F_0} = \frac{N}{\rho} \left( 1 - \frac{2N}{\rho \Omega} \right) \frac{\sigma}{1 - \sigma} \Omega. \]  

(76)

It is easy to check that this solution reduces to the values of \( \Delta \) and \( \mu \) obtained for half-filling, i.e., for \( N = \rho \Omega/2 \).

By inserting these \( \Delta \) and \( \mu \) into the BCS ground state energy \( \hat{E}_B \) given in Eq.(62), we get

\[ \hat{E}_B = 2N \epsilon_{F_0} + \frac{N^2}{\rho} - 2N \Omega \left( 1 - \frac{N}{\rho \Omega} \right) \frac{\sigma}{1 - \sigma} \Omega. \]  

(77)

Since the value of the ground state energy in the absence of potential, i.e., for \( \sigma = 0 \), reduces to the first two terms of Eqs.(77), we get the condensation energy for arbitrary filling, i.e., \( N \) not exactly equal to \( \rho \Omega/2 \), as

\[ \hat{E}_B(V = 0) - \hat{E}_B = N \left( 1 - \frac{N}{\rho \Omega} \right) \frac{2\sigma}{1 - \sigma} \Omega = \frac{\rho \Delta^2}{2(1 - \sigma)}. \]  

(78)

Again, this result fully agrees with the expression of the ground state energy derived from Richardson-Gaudin equations for arbitrary \( N \) and \( \sigma \), as given in Eq.(72).

The condensation energy, given in Eq.(78), can be seen as each of the \( N \) correlated pairs bringing its own “binding energy” to the total condensation energy, this binding energy being linearly decreased compared to one isolated Cooper pair by a factor \( \left( 1 - N/\rho \Omega \right) \) due to the Pauli exclusion principle between the electrons from which the correlated pairs are constructed.

**VII. EXCITED STATE ENERGY**

Let us end by considering excited states in the case of arbitrary filling. Their energies read \( \hat{E}_B + \hat{E}_k \) with \( \hat{E}_k \) given by Eqs. (34,55). The lowest excited state energy thus corresponds to the lowest possible value of \( \epsilon_k = (\epsilon_k - \mu)^2 \).

We first note that \( \epsilon_k \) is inside the energy layer \( (\epsilon_{F_0}, \epsilon_{F_0} + \Omega) \) by construction, while \( \mu \) does not necessarily fall inside this layer. If \( \mu \) is inside this layer, it is energetically favorable for the lowest excited state energy to have \( \epsilon_k = \mu \). This leads to the standard result

\[ (E_k)_{\text{min}} = \Delta. \]  

(79)

The difference between the minimum excited state energy and the ground state energy \( \hat{E}_B \) is then equal to \( \Delta \), commonly called “gap”.

In the standard half-filled configuration, \( \epsilon_k \) can be equal to \( \mu \) because \( \mu = \epsilon_{F_0} + \Omega/2 \) for any \( \sigma \). For arbitrary filling however, Eq.(76) shows that there are two pair numbers \( N_1 \) and \( N_2 \) which are such that \( \mu \) is either below \( \epsilon_{F_0} \) for \( N < N_1 \) or above \( \epsilon_{F_0} + \Omega \) for \( N_2 < N \). For such \( N \)'s, we cannot have \( \epsilon_k = \mu \). These pair numbers are precisely given by

\[ N_1 = \rho \Omega \frac{\sigma}{1 + \sigma}, \]  

(80)

\[ N_2 = \rho \Omega - \rho \Omega \frac{\sigma}{1 + \sigma} = \rho \Omega - N_1. \]  

(81)

Since the number of pairs possibly enjoying the potential is by construction smaller than \( \rho \Omega \), this shows that the \( N \) extensions of these two “anomalous” phases, \( 0 < N < N_1 \) and \( N_2 < N < \rho \Omega \), are very small in the weak-coupling limit, \( \rho V \ll 1 \), but exactly as wide. For \( N < N_1 \), the minimum value of \( E_k \) is reached for the \( \epsilon_k \) lower boundary \( \epsilon_{F_0} \), while, for \( N_2 < N \), it is reached for the upper boundary \( \epsilon_k = \epsilon_{F_0} + \Omega \). Using Eqs.(34,75,76), this gives \( (E_k)_{\text{min}} \) for \( 0 < N < N_1 \) as

\[ (E_k)_{\text{min}}^{(\text{dilute})} = \frac{N}{\rho} + \Omega \frac{\sigma}{1 - \sigma}. \]  

(82)

The first term in the RHS corresponds to the kinetic energy increase of \( N \) pairs under a constant density of states \( \rho \) due to an excitation at \( \epsilon_k = \epsilon_{F_0} \), while the second term is just half the binding energy of a single pair, as obtained by Cooper.

Similarly, for \( N_2 < N < \rho \Omega \), the \( E_k \) minimum is given by

\[ (E_k)_{\text{min}}^{(\text{dense})} = \left( \frac{\Omega}{\rho} - \frac{N}{\rho} \right) + \Omega \frac{\sigma}{1 - \sigma}. \]  

(83)

The first term again is the kinetic energy increase of \( N \) pairs due to an excitation at \( \epsilon_k = \epsilon_{F_0} + \Omega \), while the second term is the same as in Eq.(82).

The form of Eqs.(82,83) nicely shows the duality which exists between electrons and holes in the potential layer: bound pairs for \( N < N_1 \) are formed out of electrons, while for \( N_2 < N \) they seem formed out of holes. In order to excite the system, one has to break a pair; this explains why \( (E_k)_{\text{min}} \) contains one half of the single pair binding energy (one half only because a full “pair breaking” means adding two excitations, not just one). The first terms in Eqs.(82,83) come from kinetic energy of the excited electrons suffering Pauli blocking.

The existence of a duality between electrons and holes is also supported by the standard result of \( (E_k)_{\text{min}} \) obtained for \( N_1 < N < N_2 \), namely,

\[ (E_k)_{\text{min}} = \Delta \text{ as given in Eq.(79): the situation then is fully symmetrical with respect to the mutual exchange of electrons and holes. For } \rho V \ll 1, \text{ i.e., in the weak-coupling limit, } \rho \Omega - N_2 \text{ essentially correspond to the number of states within half the single pair binding energy. Therefore, } N < N_1 \text{ corresponds to a “superdilute” regime of pairs (or a “superdense” regime of holes), with} \]
wave functions of individual pairs overlapping only slightly, while \( N_2 < N \) corresponds to a “superdilute regime of holes” (or a “superdense” regime of electrons). Note that, with increasing \( \rho V \), the \( N_2 \) boundary approaches \( \rho \Omega / 2 \) from below, while \( N_2 \) approaches \( \rho \Omega / 2 \) symmetrically from above. Hence, an interval of \( N \)’s which correspond to the usual BCS excited state with \( (E_k)_{\min} = \Delta \) shrinks to zero when \( \rho V \to \infty \).

We wish to stress that the transitions at \( N = N_1 \) or \( N_2 \) must be smooth with respect to the ground state energy because the same expression (77) holds in the three regimes. By contrast, for the first excited states, these regimes correspond to different kinetic energies for the excitation. As easily seen from Eqs. (75, 82, 83), this induces discontinuities in the higher-order derivatives of \( (E_k)_{\min} \) with respect to \( \rho V \), on both sides of the two transitions.

The configuration considered here, with a pair number not exactly equal to half-filling but being a varying parameter, can be seen as a model for a density-induced crossover between isolated fermionic molecules when \( N \) is very small, towards a dense regime of Cooper pairs, typical of BCS superconductivity. We then expect the excitation energy in the dilute regime of pairs to be controlled by the single pair binding energy, as we find, while at higher densities, it is controlled by cooperative many-body effects. For density induced BEC-BCS crossover, see also Ref. [21].

The analysis presented in this section has similarities with the BEC-BCS crossover considered by Eagles[18] and by Leggett[19]: Eagles keeps a non-constant 3D density of states, while Leggett uses a potential without an upper cut-off at \( \epsilon_F + \Omega \), the irrelevant divergences being then cured by a “regularization” procedure. The nice aspect of the model we here consider, with a upper cut-off and a constant density of states — exact for 2D superconductors — is that it leads to analytical results. Moreover, the upper cut-off at \( \epsilon_F + \Omega \) evidences an interesting “superdense” regime of pairs for \( N \) close to complete filling \( \rho \Omega \), this regime being understood as a dilute regime of holes in the potential layer with similarities with the dilute regime of electrons when \( N \) is very small.

VIII. CONCLUDING COMMENTS

In this paper, we show that, through a proper choice of the \( z_k \) scalars appearing in the Bogoliubov Hamiltonian, the difference between the ground state energies of the BCS and the Bogoliubov Hamiltonians can be reduced to underextensive terms, negligible in the thermodynamical limit. The energy of this ground state fully agrees with the value obtained by analytically solving Richardson-Gaudin equations which give the exact eigenstates of the BCS Hamiltonian in the canonical ensemble. Actually, this agreement is rather puzzling because the ground state of the Bogoliubov Hamiltonian corresponds to the wave function ansatz proposed by Bardeen, Cooper, Schrieffer, with all pairs condensed into the same state, while the exact Richardson-Gaudin eigenstate reads as a product of correlated pairs which are by construction all different due to the Pauli exclusion principle.

At the present time, we have been able to get the \( R_i \) parameters, solutions of Richardson-Gaudin equations, analytically in the dilute limit only on the single Cooper pair scale[6, 11, 12]. These parameters, which rule the \( N \)-pair wave function, as seen from Eqs.(1) and (2), have imaginary parts which run away from the real axis when \( N \) increases. Unfortunately, we are not yet able to write the analytical form of each \( R_i \) in the dense limit. Instead, we have found a way to get the sum of these \( R_i \)’s in the dense limit, the singular imaginary parts disappearing from the sum which is real by construction since it gives the \( N \)-pair energy. The extensive term of the \( R_i \) sum we find, exactly matches the extensive part of the BCS ground state energy, as obtained through the Bogoliubov approach [see Eqs.(72) and (78)].

So, at the present time, we have two totally different forms for the ground state wave function which exactly give the same extensive part of the ground state energy: the BCS ansatz which also is the Bogoliubov Hamiltonian ground state, its projection on the \( N \)-pair subspace reading as

\[
|\psi_{BCS}\rangle = B^{\dagger N}|0\rangle ,
\]

where

\[
B^{\dagger} = \sum_k B_k^{\dagger}(v_k/u_k) \text{ with } B_k^{\dagger} = a_k^{\dagger}b_{-k}^{\dagger},
\]

and the exact Richardson-Gaudin wave function which reads as

\[
|\psi(N)\rangle = B^{\dagger}(R_1)B^{\dagger}(R_2)\cdots B^{\dagger}(R_N)|0\rangle ,
\]

where

\[
B^{\dagger}(R_i) = \sum_k B_k^{\dagger}[\omega_k/(2\epsilon_k - R_i)],
\]

the \( R_i \)'s being by construction all different. Due to Pauli blocking, it is clear that, when \( N \) increases, each of these states get more and more different from a bare juxtaposition of either \( B^{\dagger} \) or \( B^{\dagger}(R) \) correlated states. As a result, we cannot totally exclude that, when \( N \) gets very large as in the thermodynamical limit, these two states end by corresponding to the same linear combination of different \( B_k^{\dagger}B_{-k}^{\dagger} \cdots B_{-k}^{\dagger} \) products. Since a precise microscopic understanding of superconductivity goes through the knowledge of the correct ground state, to study the link between the BCS ansatz and the exact Richardson-Gaudin wave function appears to us as highly desirable. We hope that the present work will help to reopen a field commonly considered as fully understood.
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