A Generalization of
Cachazo-Douglas-Seiberg-Witten Conjecture
for Symmetric Spaces

Shrawan Kumar
Department of Mathematics
University of North Carolina
Chapel Hill, NC 27599–3250

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1 Introduction

Let $\mathfrak{g}$ be a (finite-dimensional) semisimple Lie algebra over the complex numbers $\mathbb{C}$ and let $\sigma$ be an involution (i.e., an automorphism of order 2) of $\mathfrak{g}$. Let $\mathfrak{k}$ (resp. $\mathfrak{p}$) be the $+1$ (resp. $-1$) eigenspace of $\sigma$. Then, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is a $\mathfrak{k}$-module under the adjoint action. In this paper we only consider those involutions $\sigma$ such that $\mathfrak{p}$ is an irreducible $\mathfrak{k}$-module.

We fix a $\mathfrak{g}$-invariant nondegenerate symmetric bilinear form $\langle , \rangle$ on $\mathfrak{g}$. Then, the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is an orthogonal decomposition.

Let $R := \wedge(\mathfrak{p} \oplus \mathfrak{p})$ be the exterior algebra on two copies of $\mathfrak{p}$. To distinguish, we denote the first copy of $\mathfrak{p}$ by $\mathfrak{p}_1$ and the second copy by $\mathfrak{p}_2$. It is bigraded by declaring $\mathfrak{p}_1$ (resp. $\mathfrak{p}_2$) to have bidegree $(1,0)$ (resp. $(0,1)$). Choose any basis $\{e_i\}$ of $\mathfrak{p}$ and let $\{f_i\}$ be the dual basis of $\mathfrak{p}$, i.e.,

$$\langle e_i, f_j \rangle = \delta_{i,j}.$$

Define a $\mathfrak{k}$-module map (under the adjoint action)

$$c_3 : \mathfrak{k} \to \mathfrak{p} \otimes \mathfrak{p}, \quad c_3(x) = \sum_i [x, e_i] \otimes f_i.$$
It is easy to see that $c_3$ does not depend upon the choice of the basis $\{e_i\}$. Projected onto $\Lambda^2(p)$, we get a $\mathfrak{k}$-module map $\mathfrak{k} \to \Lambda^2(p)$. This map is denoted by $c_1$ considered as a map $\mathfrak{k} \to \Lambda^2(p_1)$, and similarly for $c_2 : \mathfrak{k} \to \Lambda^2(p_2)$. We denote the image of $c_i$ by $C_i$. Let $J$ be the (bigraded) ideal of $R$ generated by $C_1 \oplus C_2 \oplus C_3$ and let us consider the quotient algebra

$$A := R/J.$$ 

The algebra $A$ is a $\mathfrak{k}$-algebra (induced from the adjoint action of $\mathfrak{k}$) and let $A^\mathfrak{k}$ be the subalgebra of $\mathfrak{k}$-invariants. The algebra $A^\mathfrak{k}$ contains the element $S := \sum e_i \otimes f_i$ in bidegree $(1,1)$.

*The aim of this paper is to understand the structure of the algebra $A^\mathfrak{k}$. In the case when $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}$ for a simple Lie algebra $\mathfrak{s}$ and $\sigma$ is the involution which switches the two factors, the study of the structure of $A^\mathfrak{k}$ was initiated by Cachazo-Douglas-Seiberg-Witten who made the following conjecture. (Observe that in this case $\mathfrak{k}$ and $\mathfrak{p}$ both can be identified with $\mathfrak{s}$ and the adjoint action of $\mathfrak{k}$ on $\mathfrak{p}$ under this identification is nothing but the adjoint action of $\mathfrak{s}$ on itself.) We will refer to this as the diagonal case.*

**Conjecture [CDSW]**

1. The subalgebra $A^\mathfrak{k}$ of $\mathfrak{k}$-invariants in $A$ is generated, as an algebra, by the element $S$.
2. $S^h = 0$.
3. $S^{h-1} \neq 0$,

where $h$ is the dual Coxeter number of $\mathfrak{k} = \mathfrak{s}$.

They proved the conjecture for $\mathfrak{s} = \mathfrak{sl}_N$ in [CDSW], and Witten proved it for $\mathfrak{s} = \mathfrak{sp}_N$ in [W]. He also proved parts (i) and (ii) of the conjecture for $\mathfrak{s} = \mathfrak{so}_N$ in [W]. Subsequently, Etingof-Kac proved the conjecture for $\mathfrak{s}$ of type $G_2$ by using the theory of abelian ideals. Kumar proved part (i) of the conjecture uniformly in [K3] using geometric and topological methods.

Returning to the general case of any involution $\sigma$, we prove the following analogous result (cf. Theorem 4.8) which is the main result of this paper.

**Theorem** Let $\sigma$ be any involution of a simple Lie algebra $\mathfrak{g}$ such that $\mathfrak{p}$ is an irreducible module under the adjoint action of $\mathfrak{k}$. Then, the subalgebra $A^\mathfrak{k}$ of $\mathfrak{k}$-invariants in $A$ is generated, as an algebra, by the element $S$.

Analogous to our proof in the diagonal case, we need to consider the algebra $B := R/(C_1 \oplus C_2)$. We show (cf. Theorem 3.1) that the subalgebra $B^\mathfrak{k}$ of $\mathfrak{k}$-invariants of $B$ is graded isomorphic with the singular cohomology
with complex coefficients $H^*(\mathcal{Y})$ of a certain finite-dimensional projective subvariety $\mathcal{Y}$ of the twisted affine Grassmannian $\mathcal{X}_\sigma$ (cf. Section 2 for the definitions of $\mathcal{X}_\sigma$ and $\mathcal{Y}$). The definition of the subvariety $\mathcal{Y}$ is motivated from the theory of abelian subspaces of $\mathfrak{p}$. The main ingredients in our proof of Theorem 3.1 are: result of Garland-Lepowsky on the Lie algebra cohomology of the nil-radical $\hat{\mathfrak{u}}_\sigma$ of a maximal parabolic subalgebra of twisted affine Kac-Moody Lie algebras; the ‘diagonal’ cohomology of $\hat{\mathfrak{u}}_\sigma$ introduced by Kostant; certain results of Han and Cellini-Frajria-Papi on abelian subspaces of $\mathfrak{p}$ and a certain deformation of the singular cohomology of $\mathcal{X}_\sigma$ introduced by Belkale-Kumar.

Having identified the algebra $B^t$ with $H^*(\mathcal{Y})$, we next use the fact that $H^*(\mathcal{X}_\sigma)$ surjects onto $H^*(\mathcal{Y})$ under the restriction map. Section 4 is devoted to study the cohomology algebra $H^*(\mathcal{X}_\sigma)$. The results here are more involved than in the diagonal case. One major difficulty arises from the fact that the fibration

$$\Omega^\sigma_1(G_o) \to \Omega^\sigma(G_o)/K_o \to G_o/K_o,$$

is nontrivial (cf. Section 4 for various notation). To complete the proof of our Theorem 4.8, we show that all but one of the generators of $H^*(\mathcal{X}_\sigma)$ go to zero under the canonical projection map $B^t \to A^t$ and the remaining one generator goes to $S$.

Finally, analogous to the Cachazo-Douglas-Seiberg-Witten Conjecture, we make the following conjecture.

\textbf{(1.3). Conjecture} $S^h = 0$ and $S^{h-1} \neq 0$ in $A^t$, where $h = h_\mathfrak{g} - h_\mathfrak{t}$ ($h_\mathfrak{g}$ being the dual Coxeter number of $\mathfrak{g}$).

Unless otherwise stated, by the cohomology $H^*(X)$ of a topological space $X$ we mean the singular cohomology $H^*(X, \mathbb{C})$ with complex coefficients.

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2 Preliminaries and Notation

(2.1) Twisted affine Lie algebras. Let $\mathfrak{g}$ be a (finite-dimensional) simple Lie algebra over $\mathbb{C}$ and let $\sigma$ be an involution of $\mathfrak{g}$. Let $\mathfrak{k} \subset \mathfrak{g}$ be the $+1$ eigenspace of $\sigma$ (which is a reductive subalgebra of $\mathfrak{g}$) and let $\mathfrak{p}$ be the $-1$ eigenspace of $\sigma$, which is a $\mathfrak{k}$-module under the adjoint action. As in the introduction, we only consider those involutions $\sigma$ such that $\mathfrak{p}$ is an irreducible $\mathfrak{k}$-module. This will be our tacit assumption on $\sigma$ throughout the paper.

Fix a Cartan subalgebra $\mathfrak{h}_\sigma$ and a Borel subalgebra $\mathfrak{b}_\sigma \supset \mathfrak{h}_\sigma$ of $\mathfrak{k}$. Let $\mathfrak{n}_\sigma$ be the nil-radical of $\mathfrak{b}_\sigma$. Associated to the pair $(\mathfrak{g}, \sigma)$ we have the twisted affine Kac-Moody Lie algebra

$\hat{\mathfrak{g}}_\sigma := \sum_{i \in \mathbb{Z}} \mathfrak{g}_i \otimes t^i \oplus \mathbb{C}c \oplus \mathbb{C}d,$

where $\mathfrak{g}_{2i} := \mathfrak{t}$ and $\mathfrak{g}_{2i+1} := \mathfrak{p}$ for any $i \in \mathbb{Z}$. The bracket in $\hat{\mathfrak{g}}_\sigma$ is defined as follows:

$$[x \otimes t^m + \lambda c + \mu d, x' \otimes t^{m'} + \lambda' c + \mu' d] =$$

$$([x, x'] \otimes t^{m+m'} + \mu m' x' \otimes t^{m'} - \mu' m x \otimes t^m) + m \delta_{m,-m'} \langle x, x' \rangle c,$$

where $\langle , \rangle$ is the normalized $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$ as in the introduction.

The Lie algebra $\hat{\mathfrak{g}}_\sigma$ is a subalgebra of the affine Kac-Moody algebra

$\hat{\mathfrak{g}} := \sum_{i \in \mathbb{Z}} \mathfrak{g} \otimes t^i \oplus \mathbb{C}c \oplus \mathbb{C}d$

with the bracket defined by the same formula as above.

We define the following subalgebras of $\hat{\mathfrak{g}}_\sigma$ called the standard Cartan, standard Borel and the standard maximal parabolic subalgebra respectively:

$\hat{\mathfrak{h}}_\sigma := \mathfrak{h}_\sigma \otimes t^0 \oplus \mathbb{C}c \oplus \mathbb{C}d,$

$\hat{\mathfrak{b}}_\sigma := \mathfrak{b}_\sigma \otimes t^0 \oplus \sum_{i > 0} \mathfrak{g}_i \otimes t^i \oplus \mathbb{C}c \oplus \mathbb{C}d,$

and

$\hat{\mathfrak{p}}_\sigma := \sum_{i \geq 0} \mathfrak{g}_i \otimes t^i \oplus \mathbb{C}c \oplus \mathbb{C}d.$
We also have the nil-radicals $\hat{n}_\sigma$ of $\hat{b}_\sigma$ and $\hat{u}_\sigma$ of $\hat{p}_\sigma$ and the Levi subalgebra $\hat{r}_\sigma$ of $\hat{p}_\sigma$ defined as follows:

\[
\hat{n}_\sigma := n_\sigma \otimes t^0 \oplus \sum_{i>0} g_i \otimes t^i,
\hat{u}_\sigma := \sum_{i>0} g_i \otimes t^i, \quad \text{and}
\hat{r}_\sigma := \mathfrak{t} \otimes t^0 \oplus \mathbb{C}c \oplus \mathbb{C}d.
\]

The evaluation at 1 gives rise to a Lie algebra homomorphism

\[
ev_1 : \hat{\mathfrak{g}}_\sigma \to \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d,
\]

where $c$ and $d$ are central in the right side.

Associated to the twisted affine Kac-Moody Lie algebra $\hat{\mathfrak{g}}_\sigma$ and its subalgebras $\hat{p}_\sigma$ and $\hat{b}_\sigma$, we have the twisted affine Kac-Moody group $\hat{\mathcal{G}}_\sigma$, the standard maximal parabolic subgroup $\hat{\mathcal{P}}_\sigma$ and the standard Borel subgroup $\hat{\mathcal{B}}_\sigma$ respectively (cf. [K2, Chapter 6]).

Let $W_\sigma$ be the (finite) Weyl group of $(\mathfrak{k}, h_\sigma)$ and let $W_\sigma$ be the (affine) Weyl group of $(\hat{\mathfrak{g}}_\sigma, h_\sigma)$. Let $\hat{\Delta}_\sigma^+ \subset (h_\sigma)^*$ be the set of positive roots of $\hat{\mathfrak{g}}_\sigma$, i.e., the set of roots for the subalgebra $\hat{n}_\sigma$ with respect to the adjoint action of $\hat{h}_\sigma$. We set $\hat{\Delta}_\sigma^- = -\hat{\Delta}_\sigma^+$. For any $w \in W_\sigma$, define

\[
\Phi(w) := \hat{\Delta}_\sigma^+ \cap w \hat{\Delta}_\sigma^- , \quad \text{and}
\hat{n}_\sigma(w) := \bigoplus_{\alpha \in \Phi(w)} (\hat{\mathfrak{g}}_\sigma)_\alpha,
\]

where $(\hat{\mathfrak{g}}_\sigma)_\alpha$ denotes the root space of $\hat{\mathfrak{g}}_\sigma$ corresponding to the root $\alpha$. Since each root in $\Phi(w)$ is real, $(\hat{\mathfrak{g}}_\sigma)_\alpha$ is one-dimensional for each $\alpha \in \Phi(w)$.

(2.2) Abelian subspaces of $\mathfrak{p}$. Let $W'_\sigma \subset W_\sigma$ be the set of minimal coset representatives in the cosets $W_\sigma/W_\sigma$.

Following [CFP], we call an element $w \in W_\sigma$ minuscule if

\[
\hat{n}_\sigma(w^{-1}) \subset \mathfrak{p} \otimes t.
\]

Let us denote the set of minuscule elements in $W_\sigma$ by $W_\sigma^\text{minu}$. Then, it is easy to see that $W_\sigma^\text{minu} \subset W'_\sigma$ and, clearly, it is a finite set.

We recall the following result from [CFP, Theorem 3.1].
Theorem. There is a bijection between $W_{\minu}$ and the set $\Xi$ of $b_{\sigma}$-stable abelian subspaces of $p$ given by $w \mapsto \text{ev}_1(\hat{n}_{\sigma}(w^{-1}))$. In particular, the cardinality $|W_{\minu}| = |\Xi|$. 

We recall the Bruhat decomposition (cf. [K2, Corollary 6.1.20]) of the projective ind-variety 

$$X_{\sigma} := G_{\sigma}/P_{\sigma} = \bigsqcup_{w \in W_{\sigma}} B_{\sigma} w P_{\sigma}/P_{\sigma},$$

where the Bruhat cell $C(w) := B_{\sigma} w P_{\sigma}/P_{\sigma}$ is isomorphic to the affine space $C^{\ell(w)}$ ($\ell(w)$ being the length of $w$ in the Coxeter group $W_{\sigma}$). Moreover, for any $w \in W_{\sigma}$, the Zariski closure 

$$C(w) = \bigsqcup_{v \in W_{\sigma}} C(v).$$

Define a subset $Y$ of $G_{\sigma}/P_{\sigma}$ by 

$$Y = \bigsqcup_{w \in W_{\minu}} C(w).$$

Then, $Y$ is a (finite-dimensional) projective subvariety of $G_{\sigma}/P_{\sigma}$. This follows from the following.

Lemma. For $w \in W_{\minu}$ and any $u \in W_{\sigma}$ such that $u \leq w$, we have $u \in W_{\minu}$.

Proof (due to P. Frajria and P. Papi). By the definition, an element $u \in W_{\sigma}$ is minuscule iff $\beta(d) = 1$ for all $\beta \in \Phi(u^{-1})$. By the $L$-shellability of the Bruhat order in $W_{\sigma}$, we can assume that $w = u s_{\alpha}$, where $\alpha \in \Delta_{\sigma}^+$ is a real root and $s_{\alpha}$ is the reflection through $\alpha$: $s_{\alpha} \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for $\lambda \in (\hat{h}_{\sigma})^\ast$. Since $u < w$, we have $w \alpha \in \Delta_{\sigma}^-$, and hence $\alpha \in \Phi(w^{-1})$. In particular, $\alpha(d) = 1$. Since $u \in W_{\sigma}$, we have $\beta(d) \neq 0$ for any $\beta \in \Phi(w^{-1})$. Thus, it suffices to prove that for any $\beta \in \Delta_{\sigma}^+$ such that $\beta(d) > 1$, we have $u \beta \in \Delta_{\sigma}^+$.

Observe that since $\beta(d) > 1$, $w \beta \in \Delta_{\sigma}^+$.

There are three cases to consider:

Case I: $s_{\alpha} \beta \in \Delta_{\sigma}^-$. 

In this case, $\langle \beta, \alpha^\vee \rangle > 0$. Thus, 

$$u \beta = w(s_{\alpha} \beta) = w(\beta - \langle \beta, \alpha^\vee \rangle \alpha) = w \beta - \langle \beta, \alpha^\vee \rangle w \alpha \in \Delta_{\sigma}^+,$$

since $w \alpha \in \Delta_{\sigma}^-$. 


\textbf{Case II:} \( s_\alpha \beta \in \hat{\Delta}^+ \) and \( s_\alpha \beta(d) \neq 1 \).

In this case, \( s_\alpha \beta \notin \Phi(w^{-1}) \), i.e., \( u \beta = ws_\alpha \beta \in \hat{\Delta}^+ \).

\textbf{Case III:} \( s_\alpha \beta \in \hat{\Delta}^\pm \) and \( s_\alpha \beta(d) = 1 \).

In this case,

\[
s_\alpha \beta(d) = \beta(d) - \langle \beta, \alpha^\vee \rangle \alpha(d)
= \beta(d) - \langle \beta, \alpha^\vee \rangle = 1, \text{ since } \alpha(d) = 1.
\]

Thus, \( \langle \beta, \alpha^\vee \rangle = \beta(d) - 1 > 0 \) (since \( \beta(d) > 1 \)) and hence \( u \beta = ws_\alpha \beta = w\beta - \langle \beta, \alpha^\vee \rangle(\alpha) \in \hat{\Delta}^\pm \), since \( \alpha \in \hat{\Delta}^- \). This proves the lemma. \( \square \)

3 Topological identification of the algebra \( B^\mathfrak{t} \)

Consider the \( \mathbb{Z}_+ \)-graded \( \mathfrak{t} \)-algebra

\[
B := \frac{\wedge(p) \otimes \wedge(p)}{(C_1 \oplus C_2)},
\]

where \( C_1 \) and \( C_2 \) are defined in the Introduction.

Following is the first main result of this paper.

\textbf{(3.1) Theorem.} \textit{The singular cohomology} \( H^*(\mathfrak{y}, \mathbb{C}) \) \textit{of} \( \mathfrak{y} \) \textit{with complex coefficients is isomorphic as a} \( \mathbb{Z}_+ \)-\textit{graded algebra with the graded algebra of} \( \mathfrak{t} \)-\textit{invariants} \( B^\mathfrak{t} \).

Before we come to the proof of the theorem, we need to recall the following results. The first theorem is a special case of a result due to Garland-Lepowsky and the second theorem is due to Han.

\textbf{(3.2) Theorem.} \textit{[K2, Theorem 3.2.7 and Identity (3.2.11.3)]} \textit{As a module for} \( \mathfrak{r}_\sigma \),

\[
H^p(\mathfrak{u}_\sigma, \mathbb{C}) \simeq \bigoplus_{\substack{w \in \mathcal{W}_\sigma, \\ \ell(w) = p}} L(w^{-1} \hat{\rho} - \hat{\rho}),
\]

where \( \hat{\rho} \) is any element of \( (\mathfrak{h}_\sigma)^* \) satisfying \( \hat{\rho}(\alpha_i^\vee) = 1 \) for all the simple coroots \( \{\alpha_0^\vee, \ldots, \alpha_i^\vee\} \subset \mathfrak{h}_\sigma \) of \( \mathfrak{g}_\sigma \) and \( L(w^{-1} \hat{\rho} - \hat{\rho}) \) denotes the irreducible \( \mathfrak{r}_\sigma \)-module with highest weight \( w^{-1} \hat{\rho} - \hat{\rho} \). Similarly, by [K2, Theorem 3.2.7],

\[
H^p(\mathfrak{u}_\sigma, \mathbb{C}) \simeq \bigoplus_{\substack{w \in \mathcal{W}_\sigma, \\ \ell(w) = p}} L(w^{-1} \hat{\rho} - \hat{\rho})^*,
\]
where \( \hat{u}_n := \sum_{i < 0} g_i \otimes t^i \).

For any \( b_\sigma \)-stable abelian subspace \( I \subset p \) of dimension \( n \), \( \land^n(I) \) is a \( b_\sigma \)-stable line in \( \land^n(p) \) and hence generates an irreducible \( k \)-submodule \( V_I \) of \( \land^n(p) \) with highest weight space \( \land^n(I) \). Thus, we get a \( k \)-module map
\[
\bigoplus_{I \in \Xi} V_I \to \land(p) \to \land(p) / \langle C_1 \rangle.
\]

If \( I \) corresponds via Theorem 2.3 to the element \( w \in W^\text{minu} \), then \( V_I \) has highest weight \( (w^{-1} \hat{\rho} - \hat{\rho})_{b_\sigma} \).

**(3.3) Theorem.** [H, Theorem 4.7] The above \( k \)-module map
\[
\bigoplus_{I \in \Xi} V_I \to \land(p) / \langle C_1 \rangle
\]
is an isomorphism. Moreover, by [P, Theorem 4.13(2)], the \( k \)-module \( \bigoplus_{I \in \Xi} V_I \) is multiplicity free.

For any \( w \in W_\sigma \), define the Schubert cohomology class \( \varepsilon^w \in H^{2w}(X_\sigma, \mathbb{Z}) \) by
\[
\varepsilon^w([C(u)]) = \delta_{w,u} \text{ for } u \in W_\sigma',
\]
where \( [C(u)] \in H_{2\ell(u)}(X_\sigma, \mathbb{Z}) \) denotes the fundamental homology class of \( C(u) \).

Following Belkale-Kumar [BK, §6], we define a new product \( \cdot_0 \) in \( H^*(X_\sigma, \mathbb{Z}) \) as follows. Express the standard cup product
\[
\varepsilon^u \cdot \varepsilon^v = \sum_{w \in W_\sigma} c_{u,v}^w \varepsilon^w.
\]
Now, define
\[
\varepsilon^u \cdot_0 \varepsilon^v = \sum_{w \in W_\sigma} c_{u,v}^w \delta_{d_{u,v,0}^w} \varepsilon^w,
\]
where
\[
d_{u,v}^w := (u^{-1} \hat{\rho} + v^{-1} \hat{\rho} - w^{-1} \hat{\rho} - \hat{\rho})(d).
\]
The product \( \cdot_0 \) descends to a product in \( H^*(Y, \mathbb{Z}) \) under the restriction map \( H^*(X_\sigma, \mathbb{Z}) \to H^*(Y, \mathbb{Z}) \).

**(3.4) Lemma.** The product \( \cdot_0 \) coincides with the standard cup product in \( H^*(Y, \mathbb{Z}) \).
Proof. For any \( w \in \mathcal{W}_\sigma \), by [K2, Corollary 1.3.22],
\[
|\Phi(w)| = \hat{\rho} - w\hat{\rho},
\]
where
\[
|\Phi(w)| := \sum_{\beta \in \Phi(w)} \beta.
\]
Thus, for any \( w \in \mathcal{W}_{\sigma_{\minu}} \), by its definition
\[
(\hat{\rho} - w^{-1}\hat{\rho})(d) = \ell(w).
\]
To prove the lemma, it suffices to show that whenever \( c_{u,v}^w \neq 0 \) for \( u, v, w \in \mathcal{W}_{\sigma_{\minu}} \), \( d_{u,v}^w = 0 \). But, \( c_{u,v}^w \neq 0 \) gives
\[
\ell(w) = \ell(u) + \ell(v).
\]
Thus,
\[
d_{u,v}^w = \left( u^{-1}\hat{\rho} - \hat{\rho} + v^{-1}\hat{\rho} - \hat{\rho} - (w^{-1}\hat{\rho} - \hat{\rho}) \right)(d)
\]
\[
= -\ell(u) - \ell(v) + \ell(w)
\]
\[
= 0 \quad \text{by (2)}.
\]

\( \square \)

Proof of Theorem 3.1. The cohomology modules \( H^p(\hat{u}_\sigma) \) and \( H^p(\hat{u}_{\sigma}) \) acquire a grading coming from the total degree of \( t \) in \( \wedge^p(\hat{u}_\sigma) \) and \( \wedge^p(\hat{u}_{\sigma}) \) respectively. This decomposes
\[
H^p(\hat{u}_\sigma) = \bigoplus_{m \in \mathbb{Z}^+} H^p_{(-m)}(\hat{u}_\sigma),
\]
where \( H^p_{(-m)}(\hat{u}_\sigma) \) denotes the space of elements of \( H^p(\hat{u}_\sigma) \) of total \( t \)-degree \(-m\). Define the diagonal cohomology
\[
H^*_D(\hat{u}_\sigma) := \bigoplus_{p \in \mathbb{Z}^+} H^p_{(-p)}(\hat{u}_\sigma),
\]
which is a subalgebra of \( H^*(\hat{u}_\sigma) \), and similarly define \( H^*_D(\hat{u}_{\sigma}) \).

Let \( \bar{\phi} : \wedge^p(p) \to H^p_{(-p)}(\hat{u}_\sigma) \) be the map induced from the map \( \bar{\phi} : \wedge^p(p) \to C^p_{(-p)}(\hat{u}_\sigma) \),
\[
\bar{\phi}(x_1 \wedge \cdots \wedge x_p)(y_1 \otimes t \wedge \cdots \wedge y_p \otimes t) = \det(\langle x_i, y_j \rangle)_{i,j},
\]

9
(for \(x_i, y_j \in \mathfrak{p}\)) by taking the cohomology class of the image. Clearly, \(\bar{\phi}(x_1 \wedge \cdots \wedge x_p)\) is a cocycle and, moreover, \(\bar{\phi}\) (and hence \(\phi\)) is surjective. It is easy to see that \(\text{Ker}(\phi|_{\wedge^2(\mathfrak{p})}) = C_1\). Now, take any \(\omega \in C_{p-1}^{-1}(\hat{u}_\sigma)\), where \(C_{p-1}^{-1}(\hat{u}_\sigma)\) denotes the space of \((p-1)\)-cochains on \(\hat{u}_\sigma\) with total \(t\)-degree \(-p\). We can write

\[
\omega = \sum_{i=1}^{N} \omega_1^i \wedge \omega_2^i,
\]

for some \(\omega_1^i \in C_{-2}^1(\hat{u}_\sigma)\) and \(\omega_2^i \in C_{-p+2}^{p-2}(\hat{u}_\sigma)\). Then,

\[
\delta \omega = \sum_{i=1}^{N} (\delta \omega_1^i) \wedge \omega_2^i,
\]

since \(\omega_2^i\) are \(\delta\)-closed, where \(\delta\) is the standard differential of the cochain complex \(C^*(\hat{u}_\sigma)\).

From this it is easy to see that \(\text{Ker} \phi = \langle C_1 \rangle\). Thus, we get a graded algebra isomorphism commuting with the \(\mathfrak{t}\)-module structures:

\[
\frac{\wedge^*(\mathfrak{p})}{\langle C_1 \rangle} \approx H_D^*(\hat{u}_\sigma).
\]

In exactly the same way, we get an isomorphism of graded algebras commuting with the \(\mathfrak{t}\)-module structures:

\[
\frac{\wedge^*(\mathfrak{p})}{\langle C_2 \rangle} \approx H_D^*(\hat{u}_\sigma^-).
\]

In particular, \(\frac{\wedge^p(\mathfrak{p})}{\langle C_1 \rangle} \cap \wedge^p(\mathfrak{p})\) is a self-dual \(\mathfrak{t}\)-module for any \(p \geq 0\).

Combining (1)–(2), we get an isomorphism (for any \(p, q \geq 0\))

\[
\left[ \frac{\wedge^p(\mathfrak{p})}{\langle C_1 \rangle} \cap \wedge^q(\mathfrak{p}) \right] \wedge \left[ \frac{\wedge^q(\mathfrak{p})}{\langle C_2 \rangle} \cap \wedge^q(\mathfrak{p}) \right] \approx \left[ H_D^p(\hat{u}_\sigma) \otimes H_D^q(\hat{u}_\sigma^-) \right] \wedge \mathfrak{t}.
\]

Since \(\frac{\wedge^*(\mathfrak{p})}{\langle C_1 \rangle}\) is multiplicity free (by Theorem 3.3) and \(\frac{\wedge^*(\mathfrak{p})}{\langle C_1 \rangle} \cap \wedge^*(\mathfrak{p})\) is self-dual for any \(p \geq 0\), the left side of (3) is nonzero only if \(p = q\). Moreover, \(c\) acts trivially on \(H_D^p(\hat{u}_\sigma) \otimes H_D^q(\hat{u}_\sigma^-)\) and \(d\) acts via the multiplication by \(q - p\). Thus, we have a graded algebra isomorphism:

\[
\left[ \frac{\wedge^*(\mathfrak{p})}{\langle C_1 \rangle} \otimes \frac{\wedge^*(\mathfrak{p})}{\langle C_2 \rangle} \right] \approx \left[ H_D^*(\hat{u}_\sigma) \otimes H_D^*(\hat{u}_\sigma^-) \right] \wedge \mathfrak{t}.
\]
By Theorem 3.2, we get

\[ H_D^p(\hat{u}_\sigma) \simeq H_D^p(\hat{u}_\sigma^-)^* \simeq \bigoplus_{w \in W_{\minu}^{\ell(w)=p}} L(w^{-1}\hat{\rho} - \hat{\rho}), \]

as \( \hat{r}_\sigma \)-modules. Combining (4)–(5), we get the isomorphism

\[ \left[ \frac{\wedge^*(p)}{(C_1)} \otimes \frac{\wedge^*(p)}{(C_2)} \right]^\ell \simeq \bigoplus_{w \in W_{\minu}^{\ell(w)=p}} \left[ L(w^{-1}\hat{\rho} - \hat{\rho}) \otimes L(w^{-1}\hat{\rho} - \hat{\rho})^* \right]^{r_{\sigma}}. \]

Now, by a similar argument to that given in [K3, Section 2.4], the proof of Theorem 3.1 follows. We omit the details. \( \square \)

## 4 Structure of the Algebra \( A^\ell \)

Let \( G \) be a connected, simply-connected complex algebraic group with Lie algebra \( g \). The involution \( \sigma \) of \( g \), of course, induces an involution of \( G \). Choose a maximal compact subgroup \( G_o \) of \( G \) which is stable under \( \sigma \) and such that the subgroup \( K_o := G_o^\sigma \) of \( \sigma \)-invariants is a maximal compact subgroup of \( K := G^\sigma \) (cf. [He]). Moreover, as is well known, \( K \) is connected and hence so is \( K_o \).

Let \( \Omega^\sigma(G_o) \) be the space of all continuous maps \( f : S^1 \to G_o \) which are \( \sigma \)-equivariant, i.e.,

\[ f(-z) = \sigma(f(z)) \quad \text{for all } z \in S^1. \]

We put the compact-open topology on \( \Omega^\sigma(G_o) \). Clearly, the subspace of constant loops can be identified with \( K_o \). Equivalently, we can view \( \Omega^\sigma(G_o) \) as the space of continuous maps \( \bar{f} : [0, 2\pi] \to G_o \) such that

\[ \bar{f}(t + \pi) = \sigma(\bar{f}(t)), \quad \text{for all } 0 \leq t \leq \pi. \]

In particular, \( \bar{f}(2\pi) = \sigma^2(\bar{f}(0)) = \bar{f}(0) \). The correspondence \( f \sim \bar{f} \) is given by \( \bar{f}(t) = f(e^{it}) \), for \( 0 \leq t \leq 2\pi \).

Consider the fibration

\[ \Omega^\sigma_1(G_o) \to \Omega^\sigma(G_o)/K_o \to G_o/K_o, \]

11
where $\gamma(fK_o) = f(1)K_o$ for $f \in \Omega^*(G_o)$ and $\Omega^*_1(G_o)$ is the subspace of $\Omega^*(G_o)$ consisting of those $f$ such that $f(1) = 1$.

Of course, $\Omega^*_1(G_o)$ can be identified with the based loop space $\Omega_1(G_o)$ of $G_o$ under $f \sim f|_{(0,1)}$.

Define the $\frak{t}$-module map $\bar{c} : \frak{t}^* \to \wedge^2(\frak{p})^*$ by $(\bar{c}f)(x \wedge y) = f([x,y])$, for $x, y \in \frak{p}$. This gives rise to a map (still denoted by)

$$\bar{c} : S(\frak{t}^*) \to \wedge(\frak{p})^*.$$ 

Consider the restriction of $\bar{c}$ to the subring of $\frak{t}$-invariants

$$c : S(\frak{t}^*)^\frak{t} \to C(\frak{g}, \frak{t}) \simeq [\wedge(\frak{p})^*]^\frak{t}.$$ 

Then, the map $c$ is the Chern-Weil homomorphism with respect to a $G_o$-invariant connection on the $G_o$-equivariant principle $K_o$-bundle $G_o \to G_o/K_o$.

Observe that since $\frak{t}$ is the $+1$ eigenspace of an involution of $\frak{g}$, the differential $\delta \equiv 0$ on $C^*(\frak{g}, \frak{t})$. Thus,

$$C^*(\frak{g}, \frak{t}) \simeq H^*(\frak{g}, \frak{t}) \simeq H^*(G_o/K_o).$$

Thus, in our case, we can think of $c$ as the map $c : S(\frak{t}^*)^\frak{t} \to H^*(\frak{g}, \frak{t}) \simeq H^*(G_o/K_o)$.

We now recall the following result due to H. Cartan on the cohomology of $G_o/K_o$ with complex coefficients (cf. [C, §10]).

(4.1) Theorem. There exists a finite-dimensional graded subspace $V \subset H^*(G_o/K_o)$ concentrated in odd degrees such that, as graded algebras,

$$H^*(G_o/K_o) \simeq \wedge(V) \otimes \text{Im} \, c.$$ 

(4.2) Corollary. Consider the map $\gamma : \Omega^*(G_o)/K_o \to G_o/K_o$ defined earlier (obtained from the evaluation at 1). Then, the induced map in cohomology

$$\gamma^* : H^*(G_o/K_o) \to H^*(\Omega^*(G_o)/K_o)$$

under the identification

$$H^*(G_o/K_o) \simeq \wedge(V) \otimes \text{Im} \, c$$

of the above theorem, satisfies

$$\gamma^*|_V \equiv 0.$$ 

In particular, $\text{Im}(\gamma^*) = \gamma^*(\text{Im} \, c)$.
Proof. This follows immediately from the fact that \( H^*(\Omega^s(G_o)/K_o) \) is concentrated in even degrees only and \( V \) lies in odd cohomological degrees. □

Let \( L^s(g) \) be the twisted loop algebra \( \bigoplus_{i \in \mathbb{Z}} g_i \otimes t^i \), i.e., \( L^s(g) \) is the space of all algebraic maps \( f: \mathbb{C}^* \to g \) satisfying \( f(-z) = \sigma(f(z)) \) for all \( z \in \mathbb{C}^* \) and the Lie algebra structure is obtained by taking the pointwise bracket. This is a subalgebra of the loop algebra

\[
L(g) := g \otimes \mathbb{C}[t, t^{-1}].
\]

Let \( L^s_1(g) \) be the kernel of the evaluation map \( L^s(g) \to g \) at 1, \( x \otimes a(t) \mapsto a(1)x \). Similarly, by \( L^s_1(G_o) \), we mean the set of algebraic maps \( f: S^1 \to G_o \) with \( f(-z) = \sigma(f(z)) \) for all \( z \in S^1 \) and \( f(1) = 1 \) (where we call a map \( f: S^1 \to G_o \) algebraic if it extends to an algebraic map \( \bar{f}: \mathbb{C}^* \to G \)).

We recall the following result from [K1, Theorem 1.6].

\[(4.3)\] Theorem. Appropriately defined, the integration map defines an algebra isomorphism in cohomology

\[
H^*(L^s(g), \mathfrak{k}) \simeq H^*(X_o).
\]

Similarly, we have an algebra isomorphism

\[
H^*(L^s_1(g)) \simeq H^*(L^s_1(G_o)),
\]

where \( L^s_1(G_o) \) is endowed with the Hausdorff topology induced from an ind-variety structure.

Analogous to the result of Garland-Raghunathan [GR], we have the following.

\[(4.4)\] Theorem. The inclusion \( L^s_1(G_o) \hookrightarrow \Omega^s_1(G_o) \) is a homotopy equivalence, where \( L^s_1(G_o) \) is endowed with the Hausdorff topology as in the previous theorem and \( \Omega^s_1(G_o) \) is equipped with the compact-open topology.

Similarly, the projective ind-variety \( X_o \) under the Hausdorff topology is homotopically equivalent with the space \( \Omega^s_1(G_o)/K_o \).

For any invariant homogeneous polynomial \( P \in S^{d+1}(g^*)^0 \) of degree \( d + 1 \) \((d \geq 1)\), define the map

\[
\phi_P : \wedge_C^{2d}(L(g)) \to \mathbb{C}
\]
by
\[ \hat{\phi}_P(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d-1}) = \frac{1}{\pi i} \int_{\theta=0}^{\pi} \Phi_P(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d-1}), \]
where \( \Phi_P : \wedge^{2d}_{\mathbb{C}}(L(\mathfrak{g})) \to \Omega^1 \) is the map defined by
\[ \Phi_P(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d-1}) := \sum_{\mu \in S_{2d}^{d+1}} \varepsilon(\mu) P(v_{\mu(0)}, [v_{\mu(1)}, v_{\mu(2)}], \]
\[ \cdots, [v_{\mu(2d-3)}, v_{\mu(2d-2)}], dv_{\mu(2d-1)}). \]

Here \( \Omega^1 \) is the space of algebraic 1-forms on \( \mathbb{C}^* \), \( d(x \otimes a(t)) = x \otimes a'(t)dt \) (for \( x \in \mathfrak{g} \) and \( a(t) \in \mathbb{C}[t, t^{-1}] \)) and in the integral \( \int_{\theta=0}^{\pi} \) we make the substitution \( t = e^{i\theta} \).

Let \( \pi_k : \mathfrak{g} \to \mathfrak{k} \) be the projection under the decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \). We similarly define \( \pi_p \). Define the \( \mathfrak{k} \)-invariant map (for any \( P \in S^{d+1}(\mathfrak{g}^*)^\mathfrak{g} \))
\[ \hat{\phi}_P : \wedge^{2d}_{\mathbb{C}}(L^\sigma(\mathfrak{g})/\mathfrak{k}) \to \mathbb{C} \]
by
\[ \hat{\phi}_P(\bar{v}_0 \wedge \cdots \wedge \bar{v}_{2d-1}) = \phi_P(v_0^\sigma \wedge \cdots \wedge v_{2d-1}^\sigma), \]
where \( \bar{v}_i := v_i + \mathfrak{k} \in L^\sigma(\mathfrak{g})/\mathfrak{k} \) and \( v_i^\sigma := v_i - v_i(1) \). Then, \( \hat{\phi}_P \) can be viewed as a cochain for the Lie algebra pair \((L^\sigma(\mathfrak{g}), \mathfrak{k})\).

\textbf{(4.5) Lemma.} Let \( P \in S^{d+1}(\mathfrak{g}^*)^\mathfrak{g} \). Then, for the differential \( \delta \) in the standard cochain complex of the pair \((L^\sigma(\mathfrak{g}), \mathfrak{k})\), \( \delta \hat{\phi}_P \) descends to a cocycle for the Lie algebra pair \((\mathfrak{g}, \mathfrak{k})\) under the evaluation map \( L^\sigma(\mathfrak{g}) \to \mathfrak{g} \) at 1.

\textit{Proof.} Observe first that, by [FT], the following diagram is commutative up to a nonzero scalar multiple (i.e., \( d \circ \beta_P = z^{-1}\Phi_P \partial \), for some \( z \in \mathbb{C}^* \)).

\begin{equation*}
\begin{array}{ccc}
\wedge^{2d+1}_{\mathbb{C}}(L(\mathfrak{g})) & \xrightarrow{\beta_P} & \Omega^0 \\
\downarrow & & \downarrow d \\
\wedge^{2d}_{\mathbb{C}}(L(\mathfrak{g})) & \xrightarrow{\Phi_P} & \Omega^1,
\end{array}
\end{equation*}

where
\[ \beta_P(v_0 \wedge \cdots \wedge v_{2d}) := \sum_{\mu \in S_{2d+1}^{d+1}} \varepsilon(\mu) P(v_{\mu(0)}, [v_{\mu(1)}, v_{\mu(2)}], \]
\[ \cdots, [v_{\mu(2d-1)}, v_{\mu(2d)}]), \]
\[ \Omega^0 \] is the space of algebraic functions on \( \mathbb{C}^* \), \( d \) is the standard deRham differential, and \( \partial \) is the standard differential in the chain complex of the Lie algebra \( L(\mathfrak{g}) \). Thus, for \( v_i \in L(\mathfrak{g}) \),

\[
(\delta \phi_P)(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d}) = \frac{1}{\pi i} \int_{\theta=0}^{\pi} \Phi_P \left( \partial (v_0 \wedge v_1 \wedge \cdots \wedge v_{2d}) \right)
= \frac{z}{\pi i} \int_{\theta=0}^{\pi} d(\beta_P(v_0 \wedge v_1 \wedge \cdots \wedge v_{2d}))
= \frac{z}{\pi i} \left( \beta_P(v_0(-1) \wedge \cdots \wedge v_{2d}(-1)) \right)
\]

(1)

We next show that for any \( v_0, \ldots, v_{2d} \in L^0(\mathfrak{g}) \),

\[
(\delta \hat{\phi}_P)(\bar{v}_0 \wedge \cdots \wedge \bar{v}_{2d}) = (\delta \phi_P)(v_0^o \wedge \cdots \wedge v_{2d}^o),
\]

where \( \bar{v}_i \) and \( v_i^o \) are defined above the statement of this lemma. For any \( x, y \in L(\mathfrak{g}) \),

\[
[x, y]^o - [x^o, y^o] = [x(1), y^o] + [x^o, y(1)].
\]

Thus,

\[
(\delta \hat{\phi}_P)(\bar{v}_0 \wedge \cdots \wedge \bar{v}_{2d}) - \delta \phi_P(\bar{v}_0^o \wedge \cdots \wedge \bar{v}_{2d}^o)
= \sum_{i<j} (-1)^{i+j} \phi_P \left( \left( [v_i, v_j]^o - [v_i^o, v_j^o] \right) \wedge \bar{v}_0^o \wedge \cdots \wedge \bar{v}_i^o \wedge \cdots \wedge \bar{v}_j^o \wedge \cdots \wedge \bar{v}_{2d}^o \right)
= \sum_{i<j} (-1)^{i+j} \phi_P \left( \left( [v_i(1), v_j^o] + [v_i^o, v_j(1)] \right) \wedge \bar{v}_0^o \wedge \cdots \wedge \bar{v}_i^o \wedge \cdots \wedge \bar{v}_j^o \wedge \cdots \wedge \bar{v}_{2d}^o \right), \text{ by (3)}
= \sum_{i<j} (-1)^{i+j} \phi_P (v_i(1) \wedge v_j^o \wedge \cdots \wedge \bar{v}_0^o \wedge \cdots \wedge \bar{v}_i^o \wedge \cdots \wedge \bar{v}_j^o \wedge \cdots \wedge \bar{v}_{2d}^o)
+ \sum_{i>j} (-1)^{i+j} \phi_P (v_i^o \wedge v_j(1) \wedge \cdots \wedge \bar{v}_0^o \wedge \cdots \wedge \bar{v}_i^o \wedge \cdots \wedge \bar{v}_j^o \wedge \cdots \wedge \bar{v}_{2d}^o)
= \sum_i (-1)^i (v_i(1) \cdot \phi_P) (v_0^o \wedge \cdots \wedge \bar{v}_i^o \wedge \cdots \wedge \bar{v}_{2d}^o)
= 0, \text{ since } \phi_P \text{ is } \mathfrak{g}-\text{invariant.}
This proves (2).

In particular, for any \( v_0 \in L^\sigma(g) \) such that \( v_0(1) = 0 \) and \( v_1, \ldots, v_{2d} \in L^\sigma(g) \), we get
\[
\hat{\delta} \hat{\phi}_P (\bar{v}_0 \wedge \cdots \wedge \bar{v}_{2d}) = \frac{z}{\pi i} \beta_P (v_0(-1) \wedge v_1(-1) \wedge \cdots \wedge v_{2d}(-1)), \quad \text{since } v_0(1) = 0
\]
\[
= 0, \quad \text{since } v_0(-1) = \sigma(v_0(1)) = 0.
\]
This proves the lemma. \( \square \)

By Identity (1) of the above lemma, the restriction \( \bar{\phi}_P \) of \( \phi_P \) to \( \wedge_{\mathbb{C}}^d(L_1^\sigma(g)) \) is a cocycle (for the Lie algebra \( L_1^\sigma(g) \)).

As is well known, \( S(g^*)^g \) is freely generated by certain homogeneous polynomials \( P_1, \ldots, P_{\ell_g} \) of degrees \( m_1 + 1, m_2 + 1, \ldots, m_{\ell_g} + 1 \) respectively, where \( \ell_g \) is the rank of \( g \) and \( m_1 < m_2 \leq \cdots \leq m_{\ell_g} \) are the exponents of \( g \).

The following result is obtained by combining [PS, Proposition 4.11.3] and Theorems 4.3 and 4.4.

\textbf{(4.6) Theorem.} The cohomology classes \( [\bar{\phi}_{P_1}], \ldots, [\bar{\phi}_{P_{\ell_g}}] \in H^*(L_1^\sigma(g)) \) freely generate the algebra
\[
H^*(L_1^\sigma(g)) \simeq H^*(L^\sigma(G_o)) \simeq H^*(\Omega^1(G_o)).
\]

Define the differential graded algebra (for short DGA)
\[
\mathcal{D} = H^*(L_1^\sigma(g)) \otimes C^*(g, \mathfrak{k})
\]
under the graded tensor product algebra structure. We define the differential \( d \) in \( \mathcal{D} \) as follows: Take \( d|_{C^*(g, \mathfrak{k})} \) as the standard differential \( \delta \) of the cochain complex \( C^*(g, \mathfrak{k}) \) of the Lie algebra pair \((g, \mathfrak{k})\) and \( d([\bar{\phi}_{P_i}]) = \delta \hat{\phi}_{P_i} \) (cf. Lemma 4.5). There is a differential graded algebra homomorphism \( \mu : \mathcal{D} \to C^*(L^\sigma(g), \mathfrak{k}) \) defined by
\[
\mu([\bar{\phi}_{P_i}]) = \hat{\phi}_{P_i}
\]
and \( \mu|_{C^*(g, \mathfrak{k})} \) is the canonical inclusion \( j : C^*(g, \mathfrak{k}) \subset C^*(L^\sigma(g), \mathfrak{k}) \) under the evaluation map at 1.

Applying the Hirsch lemma to the fibration
\[
\Omega^1(G_o) \to \Omega^d(G_o)/K_o \to G_o/K_o.
\]
(cf. [DGMS, Lemma 3.1]), and using Theorems 4.3, 4.4 and 4.6, we get the following.
(4.7) Theorem. The map \( \mu \) induces a graded algebra isomorphism in cohomology

\[
[\mu] : H^*(\mathcal{D}) \xrightarrow{\sim} H^*(X_\sigma).
\]

In particular, by Corollary 4.2, any cohomology class \([x] \in H^*(X_\sigma)\) can be represented by a cocycle \(x \in C^*(L^\sigma(\mathfrak{g}), \mathfrak{k})\) of the form

\[
x = \sum_{i=(i_1, \ldots, i_\ell) \in \mathbb{Z}_i^{\mathfrak{g}}} j(c(Q_i))(\hat{\phi}_{P_i})^{i_1} \cdots (\hat{\phi}_{P_{\ell g}})^{i_\ell},
\]

for some \(Q_i \in S(\mathfrak{t}^*)^\mathfrak{k}\), where \(c : S(\mathfrak{t}^*)^\mathfrak{k} \to C(\mathfrak{g}, \mathfrak{k})\) is the Chern-Weil homomorphism defined in the beginning of this section.

Finally, we are ready to prove the second main theorem of this paper.

(4.8) Theorem. Let \(\mathfrak{g}\) be a simple Lie algebra and let \(\sigma\) be an involution of \(\mathfrak{g}\) with \(+1\) (resp. \(-1\)) eigenspace \(\mathfrak{p}\) (resp. \(\mathfrak{p}\)). Assume that \(\mathfrak{p}\) is an irreducible \(\mathfrak{t}\)-module. Then, the algebra \(A^\mathfrak{g}\) of \(\mathfrak{g}\)-invariants of \(A\) is generated (as an algebra) by the element \(S\), where \(A\) and \(S\) are defined in the Introduction.

In particular, \((A^\mathfrak{g})^{p,q} = 0\) if \(p \neq q\).

Proof. By Theorem 3.1, the algebra \(B^\mathfrak{g}\) is graded isomorphic with the singular cohomology \(H^*(\mathcal{Y})\), where \(B := \wedge(p) \otimes \wedge(p)_{(C_1 \otimes C_2)}\). Moreover, the inclusion \(a : \mathcal{Y} \subset X_\sigma\) induces a surjection in cohomology, since \(X_\sigma\) is obtained from \(\mathcal{Y}\) by attaching real even-dimensional cells (by virtue of the Bruhat decomposition). Thus, we have

\[
H^*(X_\sigma) \xrightarrow{a^*} H^*(\mathcal{Y}) \xrightarrow{\xi} B^\mathfrak{g} \xrightarrow{\eta} A^\mathfrak{g},
\]

where \(\eta : B^\mathfrak{g} \to A^\mathfrak{g}\) is the standard quotient map.

By Theorem 4.7, any cohomology class \([x] \in H^*(X_\sigma)\) can be represented by a cocycle \(x \in C^*(L^\sigma(\mathfrak{g}), \mathfrak{k})\) of the form

\[
x = \sum_{i=(i_1, \ldots, i_\ell) \in \mathbb{Z}_i^{\mathfrak{g}}} j(c(Q_i))(\hat{\phi}_{P_i})^{i_1} \cdots (\hat{\phi}_{P_{\ell g}})^{i_\ell},
\]

for some \(Q_i \in S(\mathfrak{t}^*)^\mathfrak{k}\).

If \(Q_i\) has constant term 0, from the definition of the Chern-Weil homomorphism \(c\), it is clear that under the composite map \(\eta := \eta \circ \xi \circ a^*\), \(j(c(Q_i))\) goes to zero. Further, by an argument similar to the proof of Theorem 2.8
in [K3], we see that \( \hat{\phi}_P \) goes to zero under \( \eta \) for any \( 2 \leq i \leq \ell_g \). We briefly recall the main argument here.

For any \( \mu \in S_{2d} \) and \( P \in S^{d+1}(\mathfrak{g}^*)^\mu \), consider the linear form

\[
\hat{\phi}_{P,\mu} : \otimes_{\mathbf{C}}^{2d}(L^\sigma(\mathfrak{g})/\mathfrak{k}) \to \mathbb{C},
\]

defined by

\[
\hat{\phi}_{P,\mu}(v_0 \otimes v_1 \otimes \cdots \otimes v_{2d-1}) = \int_0^\pi P(v_{\mu(0)}, [v_{\mu(1)}, v_{\mu(2)}], \ldots, [v_{\mu(2d-3)}, v_{\mu(2d-2)}], dv_{\mu(2d-1)}),
\]

where \( v_i := v_i + \mathfrak{k} \). For the notational convenience, assume \( \mu(1) < \mu(2) \). For any fixed

\[
v_0, v_1, \ldots, v_\mu(1), \ldots, v_\mu(2), \ldots, v_{2d-1} \in L^\sigma(\mathfrak{g}),
\]

consider the restriction \( \bar{\phi}_{P,\mu} \) of the function \( \hat{\phi}_{P,\mu} \) to

\[
\bar{v}_0 \times \bar{v}_1 \times \cdots \times \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_{2p+1} \otimes t^{2p+1} \times \cdots \times \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_{2p+1} \otimes t^{2p+1} \times \cdots \times \bar{v}_{2d-1},
\]

where the two copies of \( \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_{2p+1} \otimes t^{2p+1} \) are placed in the \( \mu(1) \) and \( \mu(2) \)-th slots. Then, under the identification \( \mathfrak{g}_p \otimes t^p \cong (\mathfrak{g}_p \otimes t^p)^* \) induced from the bilinear form \( \langle , \rangle \),

\[
\bar{\phi}_{P,\mu}(v_0, v_1, \ldots, v_\mu(1), \ldots, v_\mu(2), \ldots, v_{2d-1}) = \sum_{i,j,m,n} f_i(n) \otimes f_j(m) \int_0^\pi P(v_{\mu(0)}, [e_i(n)^\sigma, e_j(m)^\sigma], [v_{\mu(3)}, v_{\mu(4)}], \ldots, [v_{\mu(2d-3)}, v_{\mu(2d-2)}], dv_{\mu(2d-1)})
\]

\[
= \sum_{i,j,m,n,k'} f_i(n) \otimes f_j(m) \int_0^\pi P(-, [e_i, e_j, e_k'], F_{k'}(n, m), -)
\]

\[
= \sum_{i,j,m,n,k'} \langle e_i, [e_j, e_k'] \rangle f_i(n) \otimes f_j(m) \int_0^\pi P(-, F_{k'}(n, m), -)
\]

\[
= \sum_{j,k',m,n} [e_j, e_{k'}](n) \otimes f_j(m) \int_0^\pi P(-, F_{k'}(n, m), -)
\]

\[
= - \sum_{j,k',m,n} [e_{k'}, e_j](n) \otimes f_j(m) \int_0^\pi P(-, F_{k'}(n, m), -),
\]
where, as in the Introduction, \( \{e_i\} \) is a basis of \( \mathfrak{p} \) and \( \{f_i\} \) is the dual basis; \( \{e'_{k'}\} \) is a basis of \( \mathfrak{k} \) and \( \{f'_{k'}\} \) is the dual basis; \( m, n \) run over the odd integers and \( F_{k'}(n, m) := f'_{k'}(n + m) - f'_{k'}(n) - f'_{k'}(m) + f'_{k'} \).

Thus, only the powers of \( \hat{\phi}_{P_1} \) contribute to the image of \( \eta \). This completes the proof of the theorem. \( \square \)

\textbf{(4.9) Remark.} It is likely that for the validity of Theorem 4.8 it is enough to assume that \( \mathfrak{g} \) is semisimple (not necessarily simple). However, we must assume that \( \mathfrak{p} \) is \( \mathfrak{k} \)-irreducible under the adjoint action since the second grade component \( (A^2)^k \) has dimension at least equal to the number of irreducible components of the \( \mathfrak{k} \)-module \( \mathfrak{p} \).
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