Spacetime Diffeomorphisms and Topological $w_\infty$ Symmetry in Two Dimensional Topological String Theory

Petr Hořava

Enrico Fermi Institute
University of Chicago
5640 South Ellis Avenue
Chicago, IL 60637, USA

ABSTRACT

This paper analyzes spacetime symmetries of topological string theory on a two dimensional torus, and points out that the spacetime geometry of the model is that of the Batalin-Vilkovisky formalism. Previously I found an infinite symmetry algebra in the absolute BRST cohomology of the model. Here I find an analog of the BV $\Delta$ operator, and show that it defines a natural semirelative BRST cohomology. In the absolute cohomology, the ghost-number-zero symmetries form the algebra of all infinitesimal spacetime diffeomorphisms, extended at non-zero ghost numbers to the algebra of all odd-symplectic diffeomorphisms on a spacetime supermanifold. In the semirelative cohomology, the symmetries are reduced to $w_\infty$ at ghost number zero, and to a topologically twisted $N=2$ $w_\infty$ superalgebra when all ghost numbers are included. I discuss deformations of the model that break parts of the spacetime symmetries while preserving the topological BRST symmetry on the worldsheet. In the absolute cohomology of the deformed model, another topological $w_\infty$ superalgebra may emerge, while the semirelative cohomology leads to a bosonic $w_\infty$ symmetry.

* e-mail address: horava@yukawa.uchicago.edu
† Robert R. McCormick Fellow; research also supported by the NSF under Grant No. PHY90-00386; the DOE under Grant No. DEFG02-90ER40560; the Czech Chart 77 Foundation; and the Czech Acad. Sci. under Grant No. 91-11045.
1. Introduction

Diffeomorphism symmetry is one of the most important yet mysterious symmetries in recent physics. It underlies classical gravity, and its full reconciliation with quantum theory still remains a challenging problem. In this context, considerable interest has been devoted recently to the conjecturally underlying phase of quantum gravity and string theory [1–10], in which all spacetime diffeomorphisms become manifest symmetries of the vacuum. (See also e.g. [11,12] for alternative approaches to the issue.) One motivation for these studies is the hope that the topological phase might be more tractable than the usual phase with local dynamics, which might be eventually obtainable by some symmetry breaking mechanism.

String theory represents one of the most promising and successful candidates for establishing links between the topological and dynamical phases of quantum gravity. In spacetime dimensions less than two, these two phases of string theory are actually equivalent [8,10]. Spacetime dimension two (or the central charge $c = 1$; see e.g. [13] for a review) represents the verge point, at which the topological symmetry (manifest for $c < 1$) is no longer obvious. The physical spectrum of two dimensional string theory consists of the massless “tachyon” mode representing the center of mass of the string, and the so-called “discrete states” [14], representing remnants of its higher excited states. The tachyon behaves like a field-theoretical degree of freedom with local dynamics, describable by an effective spacetime field theory [15]. On the other hand, the discrete modes have been conjectured to have an underlying topological origin, possibly in spacetime [14,16]. They have already proven to play a crucial role in the model, as they give rise to its ground ring structure [17] and a $w_{\infty}$ symmetry [17–20]. Hence, the dynamical phase of two dimensional string theory represents an interesting mixture of field-theoretical and non-local degrees of freedom. This non-trivial structure raises many questions, such as whether there exists an underlying phase with just global degrees of freedom, or what happens to the topological symmetry of $c < 1$ string theory as $c \to 1$ from below.
In this paper I will try to reveal some relevant features of the hypothetical topological phase of 2-D string theory, by considering a simple model: topological string theory on a two dimensional toroidal target. This program has been initiated in [21], where I have shown that – in what might be called the absolute BRST cohomology of the model – the non-zero fundamental group of the target leads to an infinite number of physical observables, and that these physical observables give rise to an infinite symmetry algebra.‡ In this paper I study the spacetime structure of the symmetry algebra, and extend the results of [21] to a semirelative BRST cohomology. Among the main results are the interpretation of the spacetime geometry of the model in terms of the Batalin-Vilkovisky (BV) geometry, as well as the existence of \( w_\infty \) and topologically twisted \( N=2 \ w_\infty \) spacetime symmetries§ in the semirelative BRST cohomology.

\( w_\infty \) algebras have played a central role in various approaches to the dynamical phase of two dimensional string theory [17–20, 27–33]. They behave as unbroken gauge symmetries, and may represent residua of some underlying stringy gauge symmetry [34]. The \( w_\infty \) symmetry may also have other far-reaching physical consequences, such as quantum coherence restoration in black hole physics via \( W \)-hair carried by stringy black holes [35–37]. Near the black hole singularity, where the black hole coset becomes effectively equivalent to a topological field theory [38], the \( w_\infty \) symmetries may get extended to a topological \( w_\infty \) [36]. In this paper we will encounter a spacetime topological \( w_\infty \) symmetry in a manifestly topological (or, more exactly, cohomological) two dimensional string theory.

The paper is organized as follows. In §2 I review some basic results of [21], such as the structure of observables and the algebra of symmetries of two dimensional topological string theory, and recall the structure of its topological ground ring. In

‡ Similar results have also been obtained in critical \( N=2 \) superstring theory by Giveon and Shapere [22]. For some earlier work on the topological torus, see [23,24]; topological aspects of \( c=1 \) were also studied in [25,26].
§ Here, as in the dynamical phase of 2D string theory [17,19], the “spacetime manifold” of the model is defined as the space parametrized by generators of the ground ring, and does not a priori coincide with the target of the sigma model. See a more detailed discussion of this issue in §3.2.
§3 I study the spacetime picture of the model. We will see that the spacetime manifold $M$ is a two dimensional torus, dual to the target. At non-zero ghost numbers, the spacetime manifold $M$ gets extended to a supermanifold $\mathcal{M}$ of dimension $(2|2)$, with a natural odd-symplectic form on it. In §3.1 and §3.2 it is shown that the bosonic part of the symmetry algebra generates all smooth local diffeomorphisms of the spacetime manifold $M$, while fermionic generators extend the symmetry algebra to all odd-symplectic diffeomorphisms on $\mathcal{M}$. The odd-symplectic form, generated on spacetime by the symmetry algebra, induces an anti-bracket structure on the space of physical observables. This provides first indications that the spacetime geometry of the model is actually that of the Batalin-Vilkovisky formalism [39–41].

The analogy between the spacetime geometry of the model and the BV geometry is further pursued in §4, where I identify the analog of the $\Delta$ operator, first as a second-order differential operator on spacetime (§4.1), and then in terms of the worldsheet CFT as a zero mode of the BRST superpartner of the energy-momentum tensor (§4.2). This $\Delta$ operator is shown to define a semirelative BRST cohomology of the model, which is analyzed from the point of view of spacetime symmetries in §4.3. In the semirelative BRST cohomology, the symmetry algebras of all spacetime diffeomorphisms and all odd-symplectic diffeomorphisms get reduced to a $w_\infty$ algebra and a topologically twisted $N=2$ $w_\infty$ superalgebra (denoted by $w_\infty^{\text{top}}$ henceforth), respectively. The existence of a topological $w_\infty$ superalgebra in spacetime has interesting consequences, such as the existence of two BRST charges: One of them is the usual BRST charge of the worldsheet topological symmetry that defines the physical states of the model by the standard BRST cohomology condition; the other one emerges in spacetime, and becomes a part of the topological $w_\infty$ superalgebra of spacetime symmetries. This interesting similarity between spacetime and worldsheet symmetries is further extended in §4.4, where it is pointed out that the model enjoys a topological $w_\infty$ algebra not only in spacetime, but also on the worldsheet. §4.5 briefly points out that the relationship between the odd-symplectic geometry and the topological $w_\infty$ symmetry may shed
some light on the issue of $W$ geometry as studied e.g. in [42].

In §5 I follow a different strategy which also reduces all spacetime diffeomorphisms to a $w_\infty$ algebra. The basic Lagrangian is deformed by BRST invariant terms, which breaks a part of the spacetime symmetry algebra as obtained in the absolute BRST cohomology. For a particular deformation, we recover the $w_\infty$ symmetry at ghost number zero, while at non-zero ghost numbers we get a new $w_\infty^{\text{top}}$ superalgebra. This new topological $w_\infty$ superalgebra is entangled in an interesting manner with the $w_\infty^{\text{top}}$ superalgebra of the semirelative cohomology: The spacetime BRST-like charge of the $w_\infty^{\text{top}}$ superalgebra of the deformed model is again the Batalin-Vilkovisky $\Delta$ operator that defines in §4 the semirelative BRST cohomology; on the other hand, the conserved charge that governs the deformation of the model is exactly the BRST-like charge of the original $w_\infty^{\text{top}}$ symmetry, as obtained in the semirelative BRST cohomology. In §5.4 the results of §4 and §5 are combined together – the BRST invariant deformation of §5.1 is shown to break the symmetries of the semirelative BRST cohomology from the topological superalgebra $w_\infty^{\text{top}}$ to a bosonic $w_\infty$.

§6 presents some summarizing remarks, as well as possible generalizations and comparison with other recent results.

2. Symmetries in 2-D Topological String Theory

In this section I review some basic results of [21], in a form suitable for the rest of the paper.

Topological strings on a two dimensional toroidal target are described in conformal coordinates on the worldsheet by the free-field Lagrangian

$$I_0 = \frac{1}{\pi} \int_\Sigma d^2z \left( \partial_z \bar{X} \partial_{\bar{z}} X - \chi_z \partial_{\bar{z}} \psi - \bar{\chi}_{\bar{z}} \partial_z \bar{\psi} \right).$$

Here $X, \bar{X}$ are complex coordinates on the target, $\psi, \bar{\psi}$ are the topological ghost fields of spin $(0, 0)$, and $\chi \equiv \chi_z dz$ and $\bar{\chi} \equiv \bar{\chi}_{\bar{z}} d\bar{z}$ are the corresponding anti-ghosts.
of spins \((1,0)\) and \((0,1)\) respectively. The theory is invariant under the BRST symmetry

\[
\begin{align*}
[Q, X] &= \psi, & [Q, \bar{X}] &= \bar{\psi}, \\
\{Q, \psi\} &= 0, & \{Q, \bar{\psi}\} &= 0, \\
\{Q, \chi_z\} &= \partial_z \bar{X}, & \{Q, \bar{\chi}_z\} &= \partial_z X.
\end{align*}
\] (2.2)

The classical Lagrangian is also invariant under the \(U(1)\) symmetry whose conserved charge is the ghost number, taking the usual values of plus one on ghost fields \(\psi\) and \(\bar{\psi}\), and minus one on anti-ghosts \(\chi\) and \(\bar{\chi}\). Here and in what follows, I have integrated out the auxiliary fields associated with the antighosts, so that the BRST charge is only nilpotent on shell.

The kinetic term in (2.1) contains an imaginary theta term, caused by the requirement that the Lagrangian be an BRST anticommutator,

\[
I_0 \equiv \{Q, \int_\Sigma \Psi\},
\] (2.3)

for a proper choice of the gauge-fixing fermion \(\Psi\). This exotic form of the kinetic term in (2.1) has several important consequences, such as the existence of an infinite number of physical states in the model. This is in accord with the fact that, if we “untwist” the topological sigma model (2.1), the resulting \(N=2\) superconformal CFT is not unitary and does not have to have a finite number of chiral primaries.

2.1. Observables and Symmetries

Already before coupling to topological gravity, the model described by (2.1) has a rich structure of BRST-invariant observables, dealt with in detail in [21]. Among them, the point-like BRST-invariant observables form a ring under operator product expansions (OPE); I refer to it as the “topological ground ring,” in analogy with the similarly defined “ground ring” of the dynamical phase of 2-D string theory.
Given a point-like observable $O^{(0)}$, one can construct conserved BRST-invariant charges on the worldsheet via the hierarchy of “descent equations” of the BRST charge:

$$\{Q, O^{(0)}\} = 0, \quad \{Q, O^{(2)}\} = dO^{(1)}, \quad \{Q, O^{(1)}\} = dO^{(0)}, \quad 0 = dO^{(2)}.$$  \hfill (2.4)  

The conserved charges are generated by loop integrals of $O^{(1)}$,

$$Q \equiv \oint_C O^{(1)},$$  \hfill (2.5)  

they are BRST invariant by the descent hierarchy (2.4), and act as symmetries on the topological ground ring. Worldsheat integrals of $O^{(2)}$,

$$W \equiv \int_{\Sigma} O^{(2)},$$  \hfill (2.6)  

which are also BRST invariant by (2.4) provided $\Sigma$ is closed, serve as possible BRST invariant deformations of the basic Lagrangian, leading to a family of topologically invariant theories.

In our case, the topological ground ring contains two sets of bosonic and two sets of fermionic observables [21], each set being parametrized by winding numbers around the two non-trivial directions on the target. Following the notation of [21] I will denote them by

* It is convenient to include into the topological ground ring all point-like physical observables, not only those of ghost number zero. In order to be more precise, we would have to distinguish between the ground ring $\mathcal{R}$ that consists of only ghost-number-zero observables, and the extended ground ring $\mathcal{R}'$, which contains point-like physical observables of all ghost numbers. I hope that no confusion will be caused if we call both $\mathcal{R}$ and $\mathcal{R}'$ the “ground ring.”
where $k_a, k_b$ is a (fixed) basis of the lattice that defines the target torus, and $\bar{X}(z)$ (resp. $X(\bar{z})$) is the left-moving (resp. right-moving) component of $X$ (resp. $\bar{X}$). Sometimes we will make use of a specific choice for $k_a, k_b$; for example, we can take $k_a = R$, $k_b = R\tau_0$ with $\text{Im} \, \tau_0 \neq 0$, so that $R$ measures the overall scale of the target, while $\tau_0$ is its modulus.

As discussed in [21] (and in [19] in the related case of the dynamical phase of 2-D strings), BRST descent equations associate a conserved charge to each point-like observable. I will denote the charges that correspond to the observables of (2.7) by

\[
L_{m,n}^a = \frac{i}{2\pi R \text{Im} \, \tau_0} \{ \bar{\tau}_0 \oint_C P_{m,n}^{(1)} - \tau_0 \oint_C Q_{m,n}^{(1)} \},
\]

\[
L_{m,n}^b = \frac{i}{2\pi R \text{Im} \, \tau_0} \{ \oint_C Q_{m,n}^{(1)} - \oint_C P_{m,n}^{(1)} \},
\]

\[
Q_{m,n} = \frac{1}{2\pi} \oint_C O_{m,n}^{(1)}, \quad G_{m,n} = \frac{1}{2\pi} \oint_C R_{m,n}^{(1)}.
\]

Explicit worldsheet expressions for the currents entering the right hand side of (2.8) can be either calculated directly from (2.4) or found in [21]. Ghost numbers of $\mathcal{L}$, $\mathcal{Q}$ and $\mathcal{G}$ are 0, -1 and +1 respectively. In (2.8) I have chosen specific linear combinations and normalizations of the charges so that their commutation relations simplify to the following form,
\[ [\mathcal{L}^a_{m,n}, \mathcal{L}^a_{p,q}] = (m - p)\mathcal{L}^a_{m+p,n+q}, \quad [\mathcal{L}^a_{m,n}, \mathcal{G}_{p,q}] = (m - p)\mathcal{G}_{m+p,n+q}, \]
\[ [\mathcal{L}^b_{m,n}, \mathcal{L}^b_{p,q}] = (n - q)\mathcal{L}^b_{m+p,n+q}, \quad [\mathcal{L}^b_{m,n}, \mathcal{G}_{p,q}] = (n - q)\mathcal{G}_{m+p,n+q}, \]
\[ [\mathcal{L}^a_{m,n}, \mathcal{L}^b_{p,q}] = n\mathcal{L}^a_{m+p,n+q} - p\mathcal{L}^b_{m+p,n+q}, \quad [\mathcal{L}^a_{m,n}, \mathcal{Q}_{p,q}] = -p\mathcal{Q}_{m+p,n+q}, \]
\[ [\mathcal{L}^b_{m,n}, \mathcal{L}^b_{p,q}] = m\mathcal{L}^b_{m+p,n+q}, \quad [\mathcal{L}^b_{m,n}, \mathcal{Q}_{p,q}] = -q\mathcal{Q}_{m+p,n+q}, \]
\[ \{\mathcal{Q}_{m,n}, \mathcal{G}_{p,q}\} = 0, \quad \{\mathcal{Q}_{m,n}, \mathcal{Q}_{p,q}\} = 0. \]

(2.9)

This is the infinite-dimensional symmetry algebra in the matter sector of the topological string theory on a two-dimensional torus, as obtained in [21].

2.2. Coupling to Topological Gravity on the Worldsheet

The symmetry algebra (2.9) is generated exclusively by the matter sector of the model. In the full-fledged topological string theory, the matter Lagrangian (2.1) has to be coupled to topological gravity on the worldsheet. This would produce an infinite tower of gravitational descendants \( \sigma^{(0)}_p \{\mathcal{O}\}, \ p = 0, 1, \ldots \), built over each observable \( \mathcal{O}^{(0)} \) of the matter sector. The gravitational descendants have essentially the structure of a direct product,

\[ \sigma^{(0)}_p \{\mathcal{O}\} \equiv \phi^p \cdot \mathcal{O}^{(0)}, \]

(2.10)

where \( \phi \) is the bosonic “ghost for ghost” [2] of the gravitational sector, carrying ghost number two. The point-like observables (2.10) enter the descent hierarchy for the total BRST charge of the coupled gravity-matter system, thus leading to conserved currents \( \sigma^{(1)}_p \{\mathcal{O}\} \) and two-forms \( \sigma^{(2)}_p \{\mathcal{O}\} \), both BRST invariant up to exterior derivative terms. If the original sigma-model observable \( \mathcal{O}^{(0)} \) carries ghost number \( q \), the ghost numbers of \( \sigma^{(0)}_p \{\mathcal{O}\} \), \( \sigma^{(1)}_p \{\mathcal{O}\} \) and \( \sigma^{(2)}_p \{\mathcal{O}\} \) are \( 2p+q \), \( 2p+q-1 \) and \( 2p+q-2 \) respectively. Since there are no point-like BRST-invariant observables of negative ghost numbers in the topological sigma model, the ghost numbers of the conserved charges

\[ \mathcal{Q}_p \{\mathcal{O}\} \equiv \oint_{\mathcal{C}} \sigma^{(1)}_p \{\mathcal{O}\} \]

(2.11)
with $p > 0$ are strictly positive, and the $Q_p\{O\}$'s do not extend the ghost-number-zero symmetry algebra. In this paper we are mainly interested in symmetries of ghost number zero, to which the gravitational sector does not contribute; hence, I will continue working on flat worldsheet and ignoring the topological gravity sector throughout.

Alternatively, we could avoid the discussion of the gravitational descendants by considering the topological torus as a string theory by itself. To get the right ghost number anomaly on the sphere, we can triple the target to a six-dimensional torus. This theory would not have to be coupled to topological gravity on the worldsheet, and may represent a very interesting example of an exactly solvable string theory.

2.3. Canonical Quantization

The presence of the imaginary theta term in the sigma model Lagrangian (2.1) leads to some subtleties in its canonical quantization on worldsheets with the Minkowski signature.

As noted above, the imaginary theta term makes the theory in the Euclidean signature non-unitary, allowing for the existence of the infinite number of physical states. In the Minkowski worldsheet signature, we will quantize the model by continuing analytically the original theory both on the worldsheet and in the spacetime. As a result of the double analytic continuation, the theta term in the Lagrangian becomes purely real. The theory, however, continues to be non-unitary, as a result of the Minkowski signature in spacetime. The reality of the theta term allows us to quantize directly by standard techniques, while the non-unitarity preserves physicality of the infinite number of winding modes.

This prescription for handling the imaginary theta term, followed by the double analytic continuation back to the Euclidean signature on the worldsheet as well as
in spacetime, leads to the following structure of zero modes,

\[ X(z, \bar{z}) = x - i \left\{ \frac{n_1 \tau_0 + n_2}{2R \text{Im} \tau_0} + R(m_1 + m_2 \tau_0) \right\} \ln z - i \left\{ \frac{n_1 \tau_0 + n_2}{2R \text{Im} \tau_0} \right\} \ln \bar{z} + \text{oscillators}, \quad n_i, m_i \in \mathbb{Z}. \]  

(2.12)

This result coincides with the expression for the zero modes as mentioned in [21], and will be useful later.

The BRST charge \( Q \) acts on (2.12) and identifies physical configurations, as those with \( n_1 = n_2 = 0 \). This physicality condition can be succinctly rewritten as the condition of holomorphicity of (2.12),

\[ \bar{\partial}_\xi X(z, \bar{z}) = 0. \]  

(2.13)

This equation, obtained here from the canonical quantization of the model, is also known from the functional integral formulation: (2.13) defines instanton configurations of the theory, and the functional integral is by standard arguments localized on the instanton moduli space.

Note also that the double Minkowski rotation that we have just used in the canonical quantization of the topological torus is quite reminiscent of the analytic continuation proposed in the high energy string scattering [43,23,34]. A possible connection between the high energy behavior of critical strings on one hand and some aspects of string theory on the topological torus on the other, has been pointed out in [23].
3. Spacetime Interpretation of the Symmetry Algebra

The commutation relations (2.9) of the symmetry algebra do not look, at the first sight, very illuminating. The main task of this section is to elucidate the structure of the symmetries in terms of spacetime geometry.

First we discuss what is meant by the spacetime manifold itself. The topological ground ring of the model is generated by two bosons of ghost number zero,

\[ a \equiv \exp\{ik_a\bar{X}(z) - i\bar{k}_aX(\bar{z})\}, \quad b \equiv \exp\{ik_b\bar{X}(z) - i\bar{k}_bX(\bar{z})\}, \quad (3.1) \]

and two fermions of ghost number one,

\[ \Theta_a \equiv \frac{i}{2R \text{Im} \tau_0}(\bar{\tau}_0\psi - \tau_0\bar{\psi}), \quad \Theta_b \equiv \frac{i}{2R \text{Im} \tau_0}(\bar{\psi} - \psi), \quad (3.2) \]

such that generic elements of the ground ring are of the form

\[ a^m b^n \Theta_a^p \Theta_b^q, \quad m, n \in \mathbb{Z}, \quad p, q \in \{0, 1\}. \quad (3.3) \]

(In (3.2), analogously as in (2.8), I have switched to the natural basis in the first cohomology group of the target, associated with the lattice basis chosen, $k_a, k_b$.)

In analogy with the framework of [17,19] the “spacetime manifold” of the model is defined as the manifold parametrized by the generators of the ground ring. The bosonic part of this manifold, denoted by $M$ from now on, is spanned by $a$ and $b$; the full supermanifold, parametrized by $a, b, \Theta_a$ and $\Theta_b$ will be denoted by $\mathcal{M}$. To some extent, this definition of spacetime may seem rather ad hoc, but I hope to offer some arguments below that the definition is in fact quite plausible. In particular, the symmetry algebra which has emerged from the BRST descent equations acquires a natural interpretation as an algebra of spacetime symmetries, if we use precisely this definition of spacetime.
3.1. Odd-Symplectic Geometry in Spacetime

By definition, the bosonic generators form a coordinate system on the spacetime manifold $\mathcal{M}$, which is extended by fermionic generators of non-zero ghost numbers to a supermanifold $\mathcal{M}$ of dimension $(2|2)$. This supermanifold can be endowed with an odd-symplectic form,

$$\omega = \frac{da}{a} \wedge d\Theta_a + \frac{db}{b} \wedge d\Theta_b.$$  \hspace{1cm} (3.4)

It has been shown in [21] that the symmetry algebra (2.9) of the topological torus, as obtained from the OPEs of the worldsheet currents, is equivalent to the algebra of all $\omega$-preserving infinitesimal diffeomorphisms, and its elements can be represented by vector fields on the spacetime supermanifold:

$$\mathcal{L}^a_{m,n} = -a^{m+1}b^n \frac{\partial}{\partial a} + a^{m}b^n \Theta_a \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right),$$

$$\mathcal{L}^b_{m,n} = -a^{m}b^{n+1} \frac{\partial}{\partial b} + a^{m}b^n \Theta_b \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right),$$

$$\mathcal{G}_{m,n} = a^{m+1}b^n \Theta_b \frac{\partial}{\partial a} - a^{m}b^{n+1} \Theta_a \frac{\partial}{\partial b} + a^{m}b^n \Theta_a \Theta_b \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right),$$

$$\mathcal{Q}_{m,n} = a^{m}b^n \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right).$$  \hspace{1cm} (3.5)

This is one of our central results: The symmetries of the topological string theory on a torus can be usefully summarized in terms of odd-symplectic geometry, which is generated naturally in spacetime.

Odd-symplectic geometry has played a central role in the geometric formulation of the BRST-BV formalism [39,40,41]. It has recently become increasingly important in string theory: It gathers much of the on-shell structure of 2-D string theory [44], is crucial for understanding prospects for a background independent formulation of open string field theory [45], and clarifies the structure of general

* I will not repeat the calculation here, and refer the reader again to [21].
non-polynomial covariant closed string field theory [46]. For a review of some algebraic structures involved, see [47].

In our case, the existence of a natural odd-symplectic form on spacetime has important consequences. Besides the usual multiplication, defined already by worldsheet OPEs, the space of physical observables carries an anti-bracket multiplication defined by the odd-symplectic form $\omega$. To avoid possible confusion and distinguish the anti-bracket from the anti-commutator, I will denote the anti-bracket by $\{ , \}_a$.

The explicit definition of $\{ , \}_a$ on two elements $f, g$ of the ground ring is

$$\{f, g\}_a = \omega^{-1}(df, dg). \tag{3.6}$$

Here $d$ is the exterior derivative on $\mathcal{M}$ defined by

$$df = \frac{\partial f}{\partial a} da + (-1)^{|f|} \left( \frac{\partial f}{\partial a} d\Theta_a + \frac{\partial f}{\partial b} db + (-1)^{|f|} \frac{\partial f}{\partial b} d\Theta_b \right), \tag{3.7}$$

$\omega^{-1}$ is the inverted odd-symplectic form,

$$\omega^{-1} \equiv a \frac{\partial}{\partial a} \wedge \frac{\partial}{\partial \Theta_a} + b \frac{\partial}{\partial b} \wedge \frac{\partial}{\partial \Theta_b}, \tag{3.8}$$

and $|f|$ denotes the ghost number of $f$. In the natural coordinate system given by $a, b, \Theta_a, \Theta_b$, the anti-bracket acquires the following form,

$$\{f, g\}_a = (-1)^{|f|} \left( a \frac{\partial g}{\partial a} \frac{\partial f}{\partial \Theta_a} + b \frac{\partial g}{\partial b} \frac{\partial f}{\partial \Theta_b} \right) + (-1)^{|f|+1} |g| \left( a \frac{\partial g}{\partial a} \frac{\partial f}{\partial \Theta_a} + b \frac{\partial g}{\partial b} \frac{\partial f}{\partial \Theta_b} \right). \tag{3.9}$$

The anti-bracket (3.6) defines on the space of physical observables the algebraic structure of what is called by mathematicians a $G$-algebra (or a Gerstenhaber algebra; cf. [48] and references therein).

Having understood the structure of the full symmetry algebra in terms of the odd-symplectic geometry in spacetime, we can also clarify the structure of the bosonic, ghost-number-zero symmetries. Indeed, the bosonic part of the symmetry
algebra (3.5), which can be obtained simply by ignoring all $\Theta_a, \Theta_b$-dependent terms in (3.5), consists of all formal diffeomorphisms of the spacetime manifold $M$:

\[
L^a_{m,n} \big|_{\Theta=0} = -a^{m+1} b^n \frac{\partial}{\partial a}, \quad L^b_{m,n} \big|_{\Theta=0} = -a^{m} b^{n+1} \frac{\partial}{\partial b}.
\]  

(3.10)

The term “formal” refers here to the fact that the coefficients of the vector fields (3.10) are formal polynomials in the bosonic ground-ring generator $s$, so that we are still in the realm of algebraic geometry. When we switch from algebraic geometry to differential geometry in the following subsection, we will find out why the anti-bracket is naturally defined on the space of physical observables, or in other words, why $\omega$ exists on $\mathcal{M}$. We will also see that the Gerstenhaber algebra of physical observables is a very special one, known to mathematicians since the sixties.

### 3.2. Spacetime versus Target

In the dynamical phase of two dimensional string theory, the relation between the target of the Liouville approach and the spacetime of the matrix model is highly nontrivial. Let us recall that a natural set of coordinates on the spacetime of the usual phase, defined again as the manifold spanned by the generators of the ground ring [17,19], is given by a time coordinate, the matrix-model eigenvalue $\lambda$, and its conjugate momentum. As has been argued e.g. in [28], the eigenvalue $\lambda$ and the target dimension may be related by a complicated integral transform.

On the other hand, in the topological theory that we are studying here, the relationship between the target (parametrized by $X$ and $\bar{X}$ of (2.1)) and the spacetime (parametrized by the bosonic generators of the ground ring, $a$ and $b$) simplifies considerably. First of all, we can see e.g. from the form of the preserved symplectic form $\omega$ that a convenient set of spacetime coordinates is given by the logarithms of $a$ and $b$,

\[
A \equiv -i \ln a, \quad B \equiv -i \ln b,
\]

(3.11)
rather than by \( a \) and \( b \) themselves. The new coordinates can also be represented in terms of the field content of the underlying CFT on the worldsheet, as specific linear combinations of \( X(\bar{z}) \) and \( \bar{X}(z) \). Making use of the results of \( \S 2.3 \) on the hamiltonian formulation, we obtain the following mode expansion for the spacetime coordinates \( A \) and \( B \):

\[
A = u + \frac{n_1 \text{Re} \tau_0 + n_2 \text{Im} \tau_0}{\text{Im} \tau_0} \tau + n_1 \sigma + \text{oscillators},
\]

\[
B = v + \frac{n_1 |\tau_0|^2 + n_2 \text{Re} \tau_0 \text{Im} \tau_0}{\text{Im} \tau_0} \tau - n_2 \sigma + \text{oscillators},
\]

\[ n_1, n_2 \in \mathbb{Z}. \] (3.12)

As we can see from this expansion, the new set of spacetime coordinates parametrize a torus, which is – as far as its metric properties are concerned – dual to the target. In the topological string theory on a torus, the spacetime and the target are thus dual to each other.

This fact has one immediate consequence: We can interpret the ground ring (3.3) as the ring of all smooth functions, and the symmetry algebra (3.5) as the algebra generating all local, smooth diffeomorphisms on the spacetime manifold – the algebraic geometry of the ground ring turned into the differential geometry of the spacetime torus. The natural emergence of all smooth spacetime functions in our topological theory can be contrasted with the dynamical phase of two dimensional string theory, where the natural functions on the spacetime quadric \([17]\) are given by polynomial functions on one hand, and by origin-supported distributions on the other. In the topological phase, spacetime geometry has been “smoothed out” when compared to the dynamical phase.

With this interpretation of the spacetime manifold, we can easily understand the structure of its fermionic extension \( \mathcal{M} \). The fermionic generators of the ground ring, \( \Theta_a \) and \( \Theta_b \), are linear combinations of worldsheet ghosts \( \psi, \bar{\psi} \) of the topological BRST symmetry, and are thus one-forms on the target. In view of the duality between the target and the spacetime manifold \( M \), \( \Theta_a \) and \( \Theta_b \) become effectively vector fields on \( M \); this fact can also be confirmed directly, by computing the transformation relations of \( \Theta_a, \Theta_b \) under the generators of spacetime diffeomorphisms,
\( \mathcal{L}_{m,n}^a, \mathcal{L}_{m,n}^b : \)

\[
\begin{aligned}
\mathcal{L}_{m,n}^a \cdot \Theta_a &= m a^m b^n \Theta_a, \\
\mathcal{L}_{m,n}^b \cdot \Theta_a &= m a^m b^n \Theta_a, \\
\mathcal{L}_{m,n}^a \cdot \Theta_b &= n a^m b^n \Theta_a, \\
\mathcal{L}_{m,n}^b \cdot \Theta_b &= n a^m b^n \Theta_b.
\end{aligned}
\]  

(3.13)

The supersymmetric extension \( \mathcal{M} \) of the spacetime manifold \( M \), parametrized by \( A, B, \Theta_a \) and \( \Theta_b \), can thus be identified with the tangent bundle to \( M \), with fibers treated as odd dimensions:

\[ \mathcal{M} \equiv \Pi(TM). \]  

(3.14)

The ground ring can be identified with the algebra of all smooth multi-vector fields on \( M \). The Lie bracket that always exists on vector fields induces an odd-symplectic form (and consequently an anti-bracket) on \( \mathcal{M} \). This explains the existence of the anti-bracket \( \{ \cdot, \cdot \} \). Together with \( \{ \cdot, \cdot \} \), the ground ring becomes the Gerstenhaber algebra of multi-vector fields on a manifold, one of the first Gerstenhaber algebras ever studied. The full symmetry algebra of the model consists of all smooth diffeomorphisms that preserve the anti-bracket.

### 3.3. Chiral BRST Cohomology and Its Spacetime Interpretation

Before pursuing further the analogy between the geometry of two dimensional topological string theory and the geometry of the BV formalism, I will briefly discuss a simple refinement of the previous results, coming from the possibility to split the BRST cohomology into chiral sectors on the worldsheet. This splitting makes a significant use of the conformal structure on the worldsheet, and most of its consequences can be lost when the Lagrangian is deformed by BRST invariant terms that do not preserve worldsheet conformal invariance.

The chiral splitting on the worldsheet causes a chiral splitting of the spacetime symmetry structure. To see this, we will consider the left-moving sector of the theory, with the chiral BRST charge given by

\[
\begin{aligned}
[Q_L, X(z)] &= \psi(z), \\
[Q_L, \bar{X}(z)] &= 0, \\
\{Q_L, \psi(z)\} &= 0, \\
\{Q_L, \chi(z)\} &= \partial_z \bar{X}(z).
\end{aligned}
\]  

(3.15)
The space of point-like physical observables of the chiral BRST charge $Q_L$ contains one bosonic series of observables with ghost number zero, and one series of fermionic observables with ghost number one, each of them parametrized by two winding numbers $m, n$. We will denote these observables by

$$O_{L;m,n}^{(0)} \equiv e^{i k_{m,n} \bar{X}(z)}, \quad P_{L;m,n}^{(0)} \equiv \psi e^{i k_{m,n} \bar{X}(z)}.$$  \hspace{1cm} (3.16)

Here $k_{m,n} \equiv mk_a + nk_b$ is again expressed with the use of a specific basis $k_a, k_b$ of the target torus.

The observables listed in (3.16) form the chiral ground ring of the theory. The chiral ground ring is generated by two bosonic observables $a_L \equiv e^{k_a \bar{X}(z)}, b_L \equiv e^{k_b \bar{X}(z)}$, and one complex fermionic observable, $\theta \equiv \psi$. BRST descent hierarchy of the chiral BRST charge $Q_L$ associates with each element of the chiral ground ring its conserved charge,

$$L_{L;m,n} \equiv \oint_C O_{L;m,n}^{(1)}, \quad Q_{L;m,n} \equiv \oint_C P_{L;m,n}^{(1)}.$$  \hspace{1cm} (3.17)

These charges act as vector fields on the space parametrized by the generators of the ground ring. We can see that the chiral symmetry algebra contains just two series of conserved charges, one bosonic of ghost number one and one fermionic of ghost number minus one.

In the full theory, the left-movers are coupled to right-movers by matching left-moving and right-moving winding numbers, which leads to identification

$$a \equiv a_L \cdot a_R, \quad b \equiv b_L \cdot b_R,$$  \hspace{1cm} (3.18)

and recovers the non-chiral structure of the spacetime manifold as studied in previous subsections.
4. Spacetime Topological Symmetry and $w_\infty$

In the usual phase of two dimensional string theory, the BRST descent equations lead – when applied to the discrete states – to symmetry algebras of a $w_\infty$ type. Thus far, in the topological theory we have studied, the ghost-number-zero symmetry algebra that naturally emerged was the algebra of all infinitesimal spacetime diffeomorphisms, $\text{Diff}_0(T^2)$. In this section (and in §5) we find a mechanism that reduces the algebra of all spacetime diffeomorphisms to a $w_\infty$ algebra, thus establishing some contact between the topological and physical theories.

4.1. More of the BV Geometry

The full symmetry algebra (2.9) of the model consists of all infinitesimal diffeomorphisms of the spacetime supermanifold that preserve the odd-symplectic form $\omega$. As we have seen, this odd-symplectic form defines an anti-bracket structure on the space of all smooth spacetime functions (i.e. on the space of physical states), by

$$\{\{f, g\}\}_a = (-1)^{|f|} \left( a \frac{\partial f}{\partial a} \frac{\partial g}{\partial \Theta_a} + b \frac{\partial f}{\partial b} \frac{\partial g}{\partial \Theta_b} \right) + (-1)^{|f|+1} |g| \left( a \frac{\partial g}{\partial a} \frac{\partial f}{\partial \Theta_a} + b \frac{\partial g}{\partial b} \frac{\partial f}{\partial \Theta_b} \right).$$

(4.1)

It is natural to wonder whether this structure can be completed to the full geometry of the BV formalism, which contains, besides the anti-bracket defined by the odd-symplectic form, a nilpotent second-order differential operator $\Delta$.

A natural candidate for the $\Delta$ operator is

$$\Delta = a \frac{\partial^2}{\partial a \partial \Theta_a} + b \frac{\partial^2}{\partial b \partial \Theta_b},$$

(4.2)

which is clearly nilpotent, and generates the anti-bracket (4.1) by

$$\{\{f, g\}\}_a = \Delta(fg) - \Delta(f) \cdot g - (-1)^{|f|} f \cdot \Delta(g).$$

(4.3)
Recalling that $\Theta_a, \Theta_b$ are essentially the (odd) coordinates along the fibres of $TM$, the $\Delta$ operator of (4.2) is nothing but the exterior derivative $d$ on the spacetime manifold, rewritten in the dual form with the use of a volume element on $M$. This is in accord with the cohomology of the nilpotent operator $\Delta$ on the extended ground ring. It turns out that the cohomology is equal to a free module over $C$, generated by the two fermionic generators of ghost number one:

$$H(\Delta) \equiv \text{Ker} \Delta / \text{Image} \Delta = C[\Theta_a, \Theta_b].$$ (4.4)

With the interpretation of $\Theta_a, \Theta_b$ as one-forms on the target, this result is exactly what one would expect from the exterior derivative on spacetime.

### 4.2. Semirelative BRST Cohomology

By construction, the model we have studied so far is topological on the worldsheet, i.e. its worldsheet Virasoro symmetries become a part of the topologically twisted $N=2$ Virasoro superalgebra. This topological algebra, whose commutation relations are

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [J_m, J_n] = m\delta_{m+n,0},$$

$$[L_m, G_n] = (m-n)G_{m+n}, \quad [J_m, G_n] = -G_{m+n},$$

$$[L_m, Q_n] = -nQ_{m+n}, \quad [J_m, Q_n] = Q_{m+n},$$

$$\{G_m, Q_n\} = L_{m+n} + nJ_{m+n} - \frac{1}{2}m(m+1)\delta_{m+n,0},$$

$$[L_m, J_n] = -nJ_{m+n} - \frac{1}{2}m(m+1)\delta_{m+n},$$ (4.5)

is realized by the following currents,

$$T_{zz}(z) \equiv \sum_m L_m \frac{z^{m+2}}{z^{m+2}} = -\partial_z X \partial_z \bar{X} + \chi_z \partial_z \psi,$$

$$Q_{zz}(z) \equiv \sum_m Q_m \frac{z^{m+1}}{z^{m+1}} = -\psi \partial_z \bar{X},$$

$$G_{zz}(z) \equiv \sum_m G_m \frac{z^{m+2}}{z^{m+2}} = -\chi_z \partial_z X,$$

$$J_{zz}(z) \equiv \sum_m J_m \frac{z^{m+1}}{z^{m+1}} = \psi \chi_z,$$ (4.6)
and analogously for the right-movers $\bar{L}_m, \bar{G}_m, \bar{Q}_m, \bar{J}_m$. The total BRST charge $Q$ of the model is given by the sum of the zero modes of the ghost-number-one scalar current, $Q \equiv Q_0 + \bar{Q}_0$. Physical states of the topological sigma model have been defined by the (absolute) BRST cohomology of $Q$, i.e. $Q|\text{phys}\rangle = 0$ and $|\text{phys}\rangle \sim |\text{phys}\rangle + |\Lambda\rangle$, with $\Lambda$ arbitrary.

Now imagine that we are interested in a direct spacetime description of the model, in terms of a string field theory. The quadratic part of the string field action is supposed to reproduce the BRST cohomology condition, $Q|\Psi\rangle = 0$, as the equation of motion for the string field $\Psi$, which is an element of the Hilbert space of the model. By simple ghost-number counting, we can see that the naive kinetic term $\langle \Psi|Q|\Psi\rangle$ of the string-field action requires an additional insertion of a ghost-number-one fermionic operator $C$. The kinetic term of the string-field action becomes

$$S_2 = \langle \Psi|CQ|\Psi\rangle.$$  \hspace{1cm} (4.8)

In critical string theory, the role of $C$ is played by the zero mode $c_0^-$ of the diffeomorphism ghosts [46]. By standard arguments, reviewed e.g. in [46,54], it is crucial for the gauge invariance of the string field action (4.8) that the physical states belong to an appropriate semirelative BRST cohomology, defined as the cohomology of $Q$ equivariant with respect to the charge $B$ conjugated to $C$. Notably, the same charge $B$ that enters the semirelative cohomology condition, enjoys another important role in the model – it makes a zero mode of the worldsheet energy-momentum tensor an exact BRST commutator:

$$\{Q, B\} = L_0 - \bar{L}_0.$$ \hspace{1cm} (4.9)

* Strictly speaking, we should treat the model as a part of a full string theory, either by coupling it to topological gravity or by introducing two other complex dimensions. In what follows, I implicitly assume that the model is a part of such a complete theory.
Whereas in topological string theory it may be intricate to identify the proper candidate for $C$ (see [54]), we have an excellent candidate for $B$: The zero mode $G_0^- \equiv G_0 - \bar{G}_0$ of the fermionic $Q$-superpartner of $T(z) - \bar{T}(\bar{z})$.

Anticipating this structure even before completing the string theory, we can define the semirelative BRST cohomology of the topological sigma model as the cohomology of $Q$, equivariant with respect to $G_0^-:\n$ $\langle \text{phys} \rangle = 0, \quad |\text{phys}\rangle \sim |\text{phys}\rangle + Q|\Lambda\rangle,$

$G_0^-|\text{phys}\rangle = 0, \quad G_0^-|\Lambda\rangle = 0.$

It is straightforward to demonstrate with the use of the explicit expressions for the worldsheet currents (4.6) that the action of $G_0^-$ on the worldsheet fields coincides with that of the searched-for $\Delta$ operator of (4.2), so we can identify

$G_0^- = \Delta.$

In §4.1 we have constructed the fermionic nilpotent operator $\Delta$ as the object that completes the odd-symplectic geometry in spacetime to the geometry of the BV formalism. Now we have found the worldsheet representation of $\Delta$ and have shown that it defines a natural semirelative BRST cohomology condition (4.10).

4.3. Spacetime Symmetries in the Semirelative BRST Cohomology

To identify the structure of symmetries in the semirelative BRST cohomology, consider first the subspace $\mathcal{K}$ of the ground ring that consists of all elements annihilated by $\Delta$, i.e. $\mathcal{K} \equiv \text{Ker} \Delta$. Explicitly, the action of $\Delta$ on the basis of the ground ring is given by

$\Delta(a^m b^n) = 0,$

$\Delta(a^m b^n \Theta_a) = ma^m b^n,$

$\Delta(a^m b^n \Theta_b) = na^m b^n,$

$\Delta(a^m b^n \Theta_a \Theta_b) = a^m b^n (m\Theta_b - n\Theta_a).$
Consequently, the space $\mathcal{K}$ of modes annihilated by $\Delta$ is spanned by

\begin{equation}
1, \ a^m b^n, \ a^m b^n (n\Theta_a + m\Theta_b), \ \Theta_a, \ \Theta_b, \ \Theta_a\Theta_b.
\end{equation}

These point-like observables produce, via the BRST descent equations, a subalgebra in the full symmetry algebra of the absolute BRST cohomology of the model. After introducing $\mathcal{D}$ by

\begin{equation}
\mathcal{D} \equiv - \mathcal{G}_{0,0},
\end{equation}

and specific linear combinations $\mathcal{W}_{m,n}, \mathcal{W}_a, \mathcal{W}_b$ of $\mathcal{L}_{m,n}^a$ and $\mathcal{L}_{m,n}^b$ by

\begin{align*}
\mathcal{W}_{m,n} &\equiv n\mathcal{L}_{m,n}^a - m\mathcal{L}_{m,n}^b; \\
\mathcal{W}_a &\equiv \mathcal{L}_{0,0}^a, \quad \mathcal{W}_b \equiv \mathcal{L}_{0,0}^b;
\end{align*}

we obtain the following commutation relations for the symmetry algebra of the semirelative cohomology,

\begin{align*}
[\mathcal{W}_{m,n}, \mathcal{W}_{p,q}] &= (mq - np)\mathcal{W}_{m+p,n+q}, & [\mathcal{W}_a, \mathcal{W}_b] &= 0, \\
[\mathcal{W}_a, \mathcal{W}_{m,n}] &= m\mathcal{W}_{m,n}, & [\mathcal{W}_b, \mathcal{W}_{m,n}] &= n\mathcal{W}_{m,n}, \\
[\mathcal{W}_{m,n}, \mathcal{Q}_{p,q}] &= (mq - np)\mathcal{Q}_{m+p,n+q}, & \{\mathcal{Q}_{m,n}, \mathcal{Q}_{p,q}\} &= 0, \\
[\mathcal{W}_a, \mathcal{Q}_{m,n}] &= m\mathcal{Q}_{m,n}, & [\mathcal{W}_b, \mathcal{Q}_{m,n}] &= n\mathcal{Q}_{m,n}, \\
\{\mathcal{D}, \mathcal{Q}_{m,n}\} &= \mathcal{W}_{m,n}, & [\mathcal{D}, \mathcal{W}_{m,n}] &= 0, \\
[\mathcal{D}, \mathcal{W}_a] &= 0, & [\mathcal{D}, \mathcal{W}_b] &= 0.
\end{align*}

(In (4.16), I have implicitly set $\mathcal{W}_{0,0} \equiv \mathcal{Q}_{0,0} \equiv 0.$)

This algebra also has a nice spacetime interpretation. The first two lines of (4.16) form precisely $w_\infty$, the algebra of all area-preserving diffeomorphisms on the toroidal spacetime. The preserved area is given by

\begin{equation}
\omega_{\text{area}} = \frac{da}{a} \wedge \frac{db}{b} \equiv dA \wedge dB,
\end{equation}
and the explicit action of $w_{\infty}$ on the spacetime manifold $M$ is realized by the following vector fields:

$$W_{m,n} = -n a^{m+1} b^n \frac{\partial}{\partial a} + m a^m b^{n+1} \frac{\partial}{\partial b},$$

$$W_a = a \frac{\partial}{\partial a}, \quad W_b = b \frac{\partial}{\partial b}.$$ (4.18)

The $W_{m,n}$ vector fields of (4.18) correspond to the area-preserving spacetime diffeomorphisms generated locally by Hamiltonians, while the remaining two generators $W_a$ and $W_b$ correspond to those diffeomorphisms that are not generated by Hamiltonians, and are in one-to-one correspondence with the generators of the first cohomology group of the spacetime. The algebra of area-preserving diffeomorphisms of a torus has been found in [49] and studied in the context of membrane physics in [50].

This establishes the existence of $w_{\infty}$ symmetries in the ghost-zero part of the semirelative BRST cohomology as defined by $G_0^-$. Now we will analyze the fermionic extension of this $w_{\infty}$ algebra by the generators of non-zero ghost numbers.

4.4. Topological $w_{\infty}$ on the Worldsheet and in the Spacetime

The whole symmetry algebra (4.16) represents a simple fermionic extension of $w_{\infty}$, and the vector-field representation of the symmetry charges acquires a non-trivial dependence on $\Theta_a, \Theta_b$. Using the definition of $W$’s, eqn. (4.15), and the vector representation of the original symmetry charges, eqn. (3.5), the vector field representation (4.18) gets completed to

$$W_{m,n} = -n a^{m+1} b^n \frac{\partial}{\partial a} + m a^m b^{n+1} \frac{\partial}{\partial b} + a^m b^n (n \Theta_a - m \Theta_b) \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right),$$

$$Q_{m,n} = a^m b^n \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right), \quad W_a = a \frac{\partial}{\partial a},$$

$$D \equiv -G_{0,0} = b \Theta_a \frac{\partial}{\partial b} - a \Theta_b \frac{\partial}{\partial a}, \quad W_b = b \frac{\partial}{\partial b}.$$ (4.19)
The fermionic generators $Q_{m,n}$ in (4.16) transform with respect to the diffeomorphism part of the algebra as modes of a two-tensor, while $D$ transforms as the zero mode of a vector. This is precisely the situation encountered in topologically twisted $N=2$ superalgebras, so we have arrived at another central result of the paper: The full spacetime symmetry algebra (4.16) in the semirelative BRST cohomology of the two dimensional topological string theory is a topologically twisted $N=2$ $w_\infty$ superalgebra (denoted by $w_{\text{top}}^\infty$ throughout the paper). $D$ is the BRST-like charge of this spacetime topological symmetry algebra, and carries ghost number one on the worldsheet. No central extension of (4.16) results from the computation of the commutation relations by the OPEs of the corresponding conserved currents on the worldsheet [21], and none is actually allowed on general algebraic grounds (cf. [33]).

It is interesting to note as an aside remark that the topological $W_\infty$ algebra of the sphere was first realized in [51] as the algebra of *worldsheet* symmetries in the theory described by our Lagrangian (2.1); indeed, the topological Virasoro symmetry (4.5) actually extends to the topological $W_\infty$ algebra. In terms of the free fields that enter the Lagrangian, the (left-moving) currents of the topological $W_{\text{top}}^\infty$ superalgebra of worldsheet symmetries are given by [51]

\[
T^j(z) \equiv \sum_m \frac{W^j_m}{z^{m+2}} = (-1)^j \frac{2^j (j+1)!}{(2j+1)!} \sum_{k=0}^{j} \binom{j}{k} \binom{j+1}{k} \partial^{j-k+1} X \partial^k \bar{X} + \partial^{j-k+1} X \partial^k \chi, \\
G^j(z) \equiv \sum_m \frac{G^j_m}{z^{m+2}} = (-1)^j \frac{2^j (j+1)!}{(2j+1)!} \sum_{k=0}^{j} \binom{j}{k} \binom{j+1}{k} \partial^{j-k+1} X \partial^k \chi, \\
Q(z) \equiv \sum_m \frac{Q_m}{z^{m+1}} = \psi \partial \bar{X}. 
\]

(4.20)

* Analogous $w_{\text{top}}^\infty$ algebras, as well as their $W_{\text{top}}^\infty$ counterparts, were first studied in a different context (and with the underlying manifold being a two-sphere) in [51].
For \( j = 0 \) we recover the topological Virasoro superalgebra of (4.5) and (4.6).

The full \( \mathcal{W}_\infty^{\text{top}} \) algebra of the worldsheet charges can be found in [51]. Here we just note that the \( \mathcal{W}_\infty^{\text{top}} \) algebra can be contracted to the corresponding \( w_\infty^{\text{top}} \) algebra, in the classical limit of the worldsheet CFT. Commutation relations of this classical \( w_\infty^{\text{top}} \) algebra are given by

\[
[W_m^j, W_n^k] = \{(k + 1)m - (j + 1)n\} W_{m+n}^{j+k}, \quad [W_m^j, Q_0] = 0,
\]

\[
[W_m^j, G_n^k] = \{(k + 1)m - (j + 1)n\} G_{m+n}^{j+k}, \quad \{G_m^j, G_n^k\} = 0,
\]

\[
\{G_m^j, Q_0\} = W_m^j, \quad \{Q_0, Q_0\} = 0,
\]

and can be contrasted with the commutation relations of the \( w_\infty^{\text{top}} \) algebra (4.16) that we have obtained in spacetime.

To conclude this aside remark on worldsheet symmetries of the model, we have seen that, as a result of [51], the worldsheet topological Virasoro algebra of the topological torus is extended to the full quantum \( \mathcal{W}_\infty^{\text{top}} \) on the worldsheet, whereas as one of the central results of the present paper, the same model also enjoys (in the semirelative BRST cohomology) a topological \( w_\infty \) symmetry in spacetime. There are obvious differences between the roles played by the worldsheet and spacetime \( w_\infty \) symmetries in our model. While the worldsheet \( \mathcal{W}_\infty^{\text{top}} \) is a symmetry of the full Hilbert space of the worldsheet CFT, the spacetime \( w_\infty^{\text{top}} \) symmetry maps physical states to physical states. The form of the currents (4.20) of the worldsheet \( \mathcal{W}_\infty^{\text{top}} \) symmetry is also very different from our expressions for the currents of the topological \( w_\infty \) symmetry in spacetime. On the other hand, in string theory we generically expect that quantum corrections from higher genera deform spacetime \( w_\infty \) symmetries to their \( \mathcal{W}_\infty \) counterparts; if this happens with the spacetime \( w_\infty \) symmetry of the topological theory studied here, it will strengthen an interesting similarity between its worldsheet and spacetime symmetries.
4.5. Semirigid $w_\infty$ Geometry in Spacetime

We have demonstrated that the spacetime symmetry algebra (4.16) of the semirelative BRST cohomology, as generated by conserved worldsheet charges, forms a $w_\infty$ analog of the topologically twisted $N=2$ Virasoro algebra. However, the usual topological Virasoro superalgebra in two dimensions contains, besides the bosonic Virasoro generators and their BRST fermionic superpartners, an infinite number of modes of the BRST current. In our case, we have obtained bosonic $w_\infty$ generators $W_{m,n}$, their superpartners $Q_{m,n}$, and the nilpotent fermionic charge $D$. This fermionic charge is the zero mode of the spacetime BRST-like current, and one might wonder why we haven’t also found the non-zero modes of the current.

The fact that just the zero mode of the spacetime BRST current emerges in the topological symmetry algebra is probably not so surprising; rather it is reminiscent of symmetries possessed by two dimensional topological gravity. Indeed, precisely the same pattern, with just the zero mode of the BRST current accompanying the bosonic symmetries and their BRST superpartners, has emerged in semirigid geometry [52], which is the suitable geometrical framework for topological gravity. Instead of leading to the full topologically twisted $N=2$ algebra, semirigid geometry ends up naturally with the algebra consisting of the topological BRST charge, the Virasoro algebra $L_m$, and the superpartners $G_m$ of the Virasoro generators $L_m$ under the BRST charge; the rest of the topologically twisted $N=2$ Virasoro superalgebra, in particular the higher modes of the BRST current, is broken.

In our case, the symmetry algebra of the topological torus induces naturally what might be called a “semirigid $w_\infty$ structure” on spacetime. In view of the relationship between the absolute and semirelative BRST cohomologies of the model, the semirigid $w_\infty$ structure on $M$ can be obtained by introducing the $\Delta$ operator and completing thus the odd-symplectic structure on $\mathcal{M}$ to the full Batalin-Vilkovisky geometry. The $\Delta$ operator induces a volume element on $M$, and allows one to identify canonically $TM$ with the cotangent bundle $T^*M$. The geometry of the $w_\infty$ symmetries can then be formulated in terms of geometrical structures on
$T^*M$, so that we find a close contact with recent results in $W$ geometry [42,4]. To see whether the relationship between the odd-symplectic and $W$ geometries can be of some further significance, it would be necessary to extend the results sketched here to surfaces of higher genera, which is however beyond the scope of this paper.

5. Spontaneous Breakdown of Spacetime Diffeomorphisms

The Lagrangian of the topological torus (2.1) has its own class of possible deformations that preserve the topological BRST invariance: any two-form that enters the descent equations for the BRST charge can be formally used as an additional term in the Lagrangian without spoiling the topological symmetry. The deformed model then realizes only a part of the original huge spacetime symmetry, and the rest of the symmetry algebra is spontaneously broken.

The basic strategy is as follows. Assume that we deform the Lagrangian by a two-form $\mathcal{O}^{(2)}$, BRST invariant up to exterior derivative of $\mathcal{O}^{(1)}$.

$$ I(\alpha) = I_0 + \alpha \int_{\Sigma} \mathcal{O}^{(2)}. \quad (5.1) $$

Here $\alpha$ is a coupling constant. The BRST charge of the original model gets also deformed, to

$$ Q(\alpha) = Q + \alpha \oint_{C} \mathcal{O}^{(1)}. \quad (5.2) $$

Physical observables of the deformed theory belong to the BRST cohomology of $Q(\alpha)$, so that they satisfy

$$ Q|_{\text{phys}} = -\alpha \oint_{C} \mathcal{O}^{(1)} |_{\text{phys}}. \quad (5.3) $$

* Here I follow closely the analogous discussion carried out for the dynamical phase of two dimensional string theory in [44].
In view of (5.3), only those symmetry charges of the undeformed model that commute with the charge $\oint O^{(1)}$ will survive, and the rest of the original symmetry algebra may be spontaneously broken. Of course, to establish this result, one must show that no other symmetry charges are generated that were not present in the the original symmetry algebra of the undeformed model.

In this section we first study deformations of the theory within the absolute BRST cohomology, and return to the semirelative BRST cohomology in §5.4. The results of the perturbative analysis of the symmetry breakdown will be rather formal, as the method ignores the analogy of the disappearance and re-appearance of some of the discrete modes as a result of the target kinematics; the proper interpretation of the results would require a wider framework which would allow us to consider all possible modes at the same time, possibly similar to the framework proposed in the context of $N=2$ superstrings in [22].

5.1. Topological Deformations and $w_{\infty}$ Diffeomorphisms

There is one natural candidate for the deforming two-form, carrying zero winding numbers as well as zero ghost number. This two-form is the top element of the descent equation for $R_{0,0}^{(0)} \equiv \psi \bar{\psi}$. Note that $R_{0,0}^{(0)}$ is a pure “homology observable” in the terminology of [21], i.e. it does not contain contributions coming from the non-zero fundamental group of the target, and would be present even if the fundamental group were zero. This class of observables in topological sigma models is much better understood than generic observables which get contributions from the non-zero fundamental group, so in this subsection we can essentially borrow results from the theory of simply-connected topological sigma models.

On-shell the two-form that will be used as a deformation of (2.1) is given by

$$R_{0,0}^{(2)}(z, \bar{z}) \equiv \partial X \wedge \bar{\partial} X.$$  \hspace{1cm} (5.4)

After being integrated over a compact worldsheet $\Sigma$, $R_{0,0}^{(2)}$ becomes BRST invariant,
and can deform the Lagrangian to

\[ I(\alpha_{0,0}) = I_0 + \alpha_{0,0} \int_{\Sigma} R_{0,0}^{(2)}(z, \bar{z}). \]  

(5.5)

The new term in the Lagrangian (unlike its analogs with non-zero winding numbers) has a natural geometrical interpretation: it essentially measures the element of the second homology group spanned by the mappings of \( \Sigma \) to the target.

In the deformed theory, the symmetry algebra is broken to the algebra that respects the new term in the Lagrangian. This condition can be usefully formulated in terms of the symmetry algebra itself: Via the topological descent equations, \( R_{0,0}^{(2)} \) defines a conserved charge

\[ G_{0,0} \equiv \frac{1}{2\pi} \oint_C R_{0,0}^{(1)}(z, \bar{z}). \]  

(5.6)

and non-zero \( \alpha_{0,0} \) deforms the BRST charge \( Q \) to

\[ Q \rightarrow Q + \alpha_{0,0} G_{0,0}. \]  

(5.7)

The unbroken subalgebra of (2.9) consists of those charges that commute with the new BRST charge of (5.7), and hence with \( G_{0,0} \).

On the ground ring, the fermionic charge given by (5.6) acts as

\[ G_{0,0} = a\Theta_b \frac{\partial}{\partial a} - b\Theta_a \frac{\partial}{\partial b}. \]  

(5.8)

Among the \( Q_{m,n} \)'s, just \( Q_{0,0} \) commutes formally with \( G_{0,0} \), but we already know from [21] that \( Q_{0,0} \equiv 0 \) identically. In the rest of the fermionic sector, all \( G_{m,n} \) survive, since they carry the top ghost number and must anticommute with one another, in particular with \( G_{0,0} \).
In the bosonic sector, we have

\[ [\mathcal{L}^a_{m,n}, \mathcal{G}_{0,0}] = m \mathcal{G}_{m,n}, \quad [\mathcal{L}^b_{m,n}, \mathcal{G}_{0,0}] = n \mathcal{G}_{m,n}, \]  

(5.9)

hence the subalgebra of bosonic charges commuting with \( \mathcal{G}_{0,0} \) consists of combinations

\[ \mathcal{W}_{m,n} \equiv n \mathcal{L}^a_{m,n} - m \mathcal{L}^b_{m,n} \]  

(5.10)

and

\[ \mathcal{W}_a \equiv \mathcal{L}^a_{0,0}, \quad \mathcal{W}_b \equiv \mathcal{L}^b_{0,0}. \]  

(5.11)

The algebra of all survivors is then

\[ \{\mathcal{W}_{m,n}, \mathcal{G}_{p,q}\} = 0, \quad \{\mathcal{G}_{m,n}, \mathcal{G}_{p,q}\} = 0, \]

(5.12)

The first two lines of (5.12) are again the \( w_\infty \) algebra of all area-preserving diffeomorphisms of the toroidal spacetime, identified in the previous section as the full algebra of ghost-number-zero symmetries in the semirelative BRST cohomology. Here we have just seen that the algebra of spacetime diffeomorphisms, which is the algebra of ghost-number-zero symmetries in the absolute BRST cohomology, can be broken spontaneously to a \( w_\infty \) subalgebra.

Now we are going to analyze the impact of the spontaneous symmetry breakdown on the full symmetry algebra of all ghost numbers. The whole algebra (5.12) is a fermionic extension of \( w_\infty \), and the symmetry charges act on the spacetime
supermanifold by the following vector fields,

\[
\mathcal{W}_{m,n} = -ma^{m+1}b^n \frac{\partial}{\partial a} + ma^{m}b^{n+1} \frac{\partial}{\partial b} + a^m b^n (n\Theta_a - m\Theta_b) \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right),
\]

\[
\mathcal{W}_a = a \frac{\partial}{\partial a}, \quad \mathcal{W}_b = b \frac{\partial}{\partial b},
\]

\[
\mathcal{G}_{m,n} = a^{m+1}b^n \Theta_b \frac{\partial}{\partial a} - a^m b^{n+1} \Theta_a \frac{\partial}{\partial b} + a^m b^n \Theta_a \Theta_b \left( m \frac{\partial}{\partial \Theta_a} + n \frac{\partial}{\partial \Theta_b} \right).
\]

Their commutation relations (5.12) are again those of a $u_{\infty}^{\text{top}}$ superalgebra. The role of the fermionic superpartners of $\mathcal{W}_{m,n}$ is now played by charges $\mathcal{G}_{m,n}$ of ghost number one, rather than by charges $\mathcal{Q}_{m,n}$ of ghost number minus one as in §4.

### 5.2. $\Delta$ as the BRST-like Charge in the Unbroken $u_{\infty}^{\text{top}}$ Algebra

We have seen above that the symmetry algebra of unbroken charges follows the general pattern of topological symmetry algebras, related to $N=2$ superalgebras by twisting. The only missing ingredient that would complete (5.12) to the topologically twisted $N=2$ $w_{\infty}$ superalgebra is the scalar fermionic supercharge, whose anticommutation relations with the fermionic generators create the bosonic ones. In the standard interpretation of topological superalgebras, this solitary fermionic charge plays the role of the BRST charge.

To be more specific, we are looking for a fermionic charge $D$ with commutation relations

\[
\{D, \mathcal{G}_{m,n}\} = \mathcal{W}_{m,n}, \quad [D, \mathcal{W}_{m,n}] = 0
\]

(5.14)

(where I implicitly set $\mathcal{W}_{0,0} \equiv 0$). It is also natural to require that $D$ commute with the two exceptional bosonic generators of the symmetry algebra,

\[
[D, \mathcal{W}_a] = 0, \quad [D, \mathcal{W}_b] = 0.
\]

(5.15)

One can easily see that even the commutation relations (5.14) themselves cannot be satisfied with $D$ a first order differential operator in $a, b, \Theta_a, \Theta_b$. Instead, they
are satisfied by $D$ being a second order differential operator,

$$D = a \frac{\partial^2}{\partial a \partial \Theta_a} + b \frac{\partial^2}{\partial b \partial \Theta_b}. \quad (5.16)$$

This is exactly the $\Delta$ operator (4.2) we encountered several times in previous sections. Consequently, the BRST-like charge $D$ of the $w_{\infty}^{\text{top}}$ superalgebra (5.12) can be identified with the BV $\Delta$ operator,

$$D \equiv \Delta, \quad (5.17)$$

which indicates an interesting relationship between the deformation of the Lagrangian studied here, and the semirelative BRST cohomology studied in §4.

Note also that the existence of the BRST-like operator $D \equiv \Delta$ allows us to construct the symmetry algebra $w_{\infty}^{\text{top}}$ of the deformed model by starting with the fermionic symmetries $G_{m,n}$ and generating the rest of $w_{\infty}^{\text{top}}$ by (anti)commutators with $\Delta$. The only symmetries that are not generated by this procedure are the non-exact diffeomorphisms $\mathcal{W}_a$ and $\mathcal{W}_b$.

### 5.3. Higher Deformations with Zero Ghost Number

Instead of the translation-invariant two-form $R_{0,0}^{(0)}$, we can use the whole hierarchy of two-forms coming from the descent equations, to deform the original Lagrangian (2.1) and obtain a model with a spontaneous breakdown of the original spacetime diffeomorphism symmetry. For any $N \in \mathbb{Z}$ we can study

$$I(\alpha_{N,N}) = I_0 + \alpha_{N,N} \int_\Sigma R_{N,N}^{(2)}(z, \bar{z}), \quad (5.18)$$

where $R_{N,N}^{(2)}$ is given by

$$R_{N,N}^{(2)} = (\partial X - i k_{N,N} \psi \bar{\chi}) \wedge (\bar{\partial} \bar{X} + i \bar{k}_{N,N} \bar{\psi} \chi) e^{i k_{N,N} X(z) - i \bar{k}_{N,N} \bar{X}(\bar{z})}, \quad (5.19)$$
with

\[ k_{m,n} = R(m + n\tau_0). \quad (5.20) \]

Note that \( I(\alpha_N,N) \) is complex; if one wants a real deformation of the Lagrangian, one can deform \( I_0 \) by real linear combinations of the deforming two-forms.

The residual bosonic symmetry generators of the deformed model are given by

\[
W_{m,n}^N \equiv (n - N)\mathcal{L}^a_{m,n} - (m - N)\mathcal{L}^b_{m,n}, \\
W_{a}^N \equiv \mathcal{L}^a_{N,N}, \\
W_{b}^N \equiv \mathcal{L}^b_{N,N},
\]

and form the following symmetry algebra:

\[
[W_{m,n}^N, W_{p,q}^N] = C_{mn,pq}^N W_{m+p,n+q}^N, \\
[W_{a}^N, W_{m,n}^N] = (N - m) W_{m+N,n+N}^N, \\
[W_{b}^N, W_{m,n}^N] = (N - n) W_{m+N,n+N}^N,
\]

with the structure constants \( C_{mn,pq}^N \) given by

\[ C_{mn,pq}^N \equiv (m - N)(q - N) - (n - N)(p - N). \quad (5.23) \]

This algebra preserves a higher two-form on the torus,

\[
\omega_{\text{area}}^N = \frac{da}{a^{N+1}} \wedge \frac{db}{b^{N+1}}. 
\]

Among the fermionic generators, \( G_{m,n} \) will again survive for all \( m, n \), and extend the bosonic symmetry algebra (5.22) to a topological \( w_{\infty} \). The relevant commutation relations are

\[
[W_{m,n}^N, G_{p,q}] = C_{mn,pq}^N G_{m+p,n+q}, \\
[W_{a}^N, G_{m,n}] = (N - m) G_{m+N,n+N}, \\
[W_{b}^N, G_{m,n}] = (N - n) G_{m+N,n+N},
\]

with \( C_{mn,pq}^N \) given again by (5.23). Note that the translational invariance in spacetime is broken for non-zero \( N \), since the generators of spacetime translations \( \mathcal{L}^a_{0,0} \) and \( \mathcal{L}^b_{0,0} \) are no longer elements of the unbroken symmetry algebra.
5.4. Symmetry Breakdown in the Semirelative BRST Cohomology

The BRST invariant two-form (5.4), used in §5.1 as a deformation of the basic Lagrangian (2.1), belongs to the semirelative BRST cohomology defined by the BV ∆ operator. In view of this, we can study the deformation by $R^{(2)}_{0,0}$ in the model with physical states defined by the semirelative cohomology condition, (4.10). The natural question is, how is the $w^\text{top}_\infty$ superalgebra of spacetime symmetries broken when the model is deformed by $R^{(2)}_{0,0}$.

To answer this question, note first that the conserved charge $G_{0,0}$ related to $R^{(2)}_{0,0}$ by the BRST descent equation, is nothing but the spacetime BRST-like charge $D$ of the semirelative $w^\text{top}_\infty$ superalgebra. In this sense, the model is deformed by its own BRST-like spacetime symmetry charge! The surviving symmetries are those that commute with this BRST-like charge, leading to a symmetry algebra that contains just the bosonic generators $W_{m,n}$ and the BRST-like charge $D \equiv -G_{0,0}$ itself; the resulting commutation relations are

\[
\begin{align*}
[W_{m,n}, W_{p,q}] &= (mq - np)W_{m+p,n+q}, & [W_a, W_b] &= 0, \\
[W_a, W_{m,n}] &= mW_{m,n}, & [W_b, W_{m,n}] &= nW_{m,n}, \\
[W_{m,n}, G_{0,0}] &= 0, & [W_a, G_{0,0}] &= 0, & [W_b, G_{0,0}] &= 0.
\end{align*}
\] (5.26)

This is the unbroken symmetry algebra in the semirelative BRST cohomology.

To conclude this section, we have seen that it is possible to deform the model of §4 by the spacetime BRST-like charge $D$ of the $w^\text{top}_\infty$ spacetime superalgebra. The only fermionic charge that survives in the semirelative BRST cohomology is decoupled from the rest of the spacetime symmetry algebra, which thus becomes the usual, bosonic $w_\infty$.  

35
6. Concluding Remarks

In this paper I have presented a model whose spacetime symmetries are closely related to all spacetime diffeomorphisms, and have found a mechanism which reduces the diffeomorphism symmetry to a $w_\infty$ algebra. This model is a two dimensional topological string theory, and the residual unbroken symmetry is quite reminiscent of the manifest $w_\infty$ symmetry of the dynamical phase of string theory in two dimensions. We have also seen that physical observables of higher ghost numbers extend the $w_\infty$ symmetry algebra to a spacetime topological $w_\infty$ superalgebra, with its own BRST-like nilpotent fermionic charge. This suggests that the theory may be cohomological not only on the worldsheet but also in spacetime, a conjecture reminiscent of some other recent observations in topological string theory [53,54].

The spacetime manifold, as defined by the generators of the ground ring, is dual to the target manifold, as defined by the field content of the sigma-model Lagrangian. This duality between the spacetime and the target suggests a mechanism capable of relating the topological phase of a physical theory to its phase with local dynamics. Indeed, the set of all point-like physical states of the topological sigma model consists entirely of ground states in each winding sector on the target. Since as we have seen the target is dual to the spacetime, these target winding modes can be (roughly) interpreted as momentum modes in spacetime, and we find local degrees of freedom that arise as topological physical states in a cohomological string theory. I believe that the lesson learned from this example might be of some wider validity in topological field theory.

The model I have studied in this paper can be extended straightforwardly to topological string theory on higher-dimensional tori of real dimension 2d. Most of the results of this paper find their natural counterparts in higher dimensions. For example, the symmetries in the absolute BRST cohomology generate all spacetime diffeomorphisms at ghost number zero, extended at non-zero ghost numbers to the algebra of all odd-symplectic diffeomorphisms on a spacetime supermanifold of
dimension (2d|2d); in the semirelative BRST cohomology, the ghost-number-zero symmetry algebra preserves a volume element on spacetime. There is, however, one important property that does not extend to higher dimensions, and makes the two-dimensional case unique: The symmetry algebra of the semirelative BRST cohomology cannot be interpreted as a topologically twisted $N=2$ superalgebra unless the spacetime dimension is two.

Another possible generalization is represented by topological string theory on higher genus Riemann surfaces, which is interesting from several points of view. First, if one expects summation over all spacetime topologies to enter the full string theory, one would face in this particular case the problem of formulating the topological sigma model on surfaces with higher genera. The topological sigma model with such targets can indeed be formulated, and is clearly not conformally invariant. This elementary fact prevents us from applying most of the methods we have used for the target torus, and would probably require much better understanding of general topological sigma models with multiply connected targets. A better understanding of this theory would be of some wider importance in various areas of physics.

Two dimensional string theory from the topological point of view has recently been studied by Mukhi and Vafa [55]. Their results are extremely interesting, and complementary to ours; using the insight of [56], the authors of [55] present a strong evidence showing that there is a hidden topological symmetry on the worldsheet of the dynamical phase of two dimensional string theory, in particular for the black hole solution. In one case of their interest, the authors of [55] have recovered the Lagrangian (2.1) that serves as the starting point of the present paper. It would be very interesting to combine their worldsheet results with the spacetime results obtained in this paper.

**Note added**

Quite recently, the importance of odd-symplectic geometry and the BV formalism in string theory and 2-D topological field theory has also been stressed by
Lian and Zuckermann; Penkava and Schwarz; and Getzler [57].

Acknowledgements

It is a pleasure to thank A. Ashtekar, J. Avan, R. Dijkgraaf, J. Distler, P. Nelson, A. Shapere, J. Stasheff, E. Verlinde, E. Witten and B. Zwiebach for stimulating discussions at various stages of the work. The author is grateful to the organizers of the École d’Été de Physique Théorique on “Gravitation and Quantizations” (July 1992) for their hospitality in Les Houches, where part of this work was done. The author is also indebted to the organizers of the JAMI conference on “Geometry and Quantum Field Theory,” Baltimore (March 1992); Spring School on String Theory and Quantum Gravity, Trieste (April 1992); and the École d’Été in Les Houches (July 1992), for the opportunity to present various aspects of these results.

REFERENCES

1. E. Witten, Commun. Math. Phys. 117 (1988) 353; 118 (1988) 411
2. E. Witten, Nucl. Phys. B340 (1990) 281
3. E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 457
4. E. Witten, in: “Strings ’90,” Proceedings of the Superstring Workshop at Texas A&M, March 1990, eds.: R. Arnowitt et al. (World Scientific, Singapore, 1991)
5. K. Li, Nucl. Phys. B354 (1991) 711, 725
6. R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 435; B352 (1991) 59; and in: “String Theory and Quantum Gravity” (World Scientific, Singapore, 1991)
7. E. Witten, Int. J. Mod. Phys. A6 (1991) 2775
8. M. Kontsevich, *Commun. Math. Phys.* **147** (1992) 1
   E. Witten, *Surveys in Diff. Geom.* **1** (1991) 243; “On the Kontsevich Model and Other Models of Two Dimensional Quantum Gravity,” IAS preprint IASSNS-HEP-91/ (June 1991); “Algebraic Geometry of Quantum Gravity in Two Dimensions,” IAS preprint IASSNS-HEP-91/74 (October 1991)

9. E. Witten, *Nucl. Phys.* **B371** (1992) 191

10. R. Dijkgraaf, “Intersection Theory, Integrable Hierarchies and Topological Field Theory,” IAS preprint IASSNS-HEP-91/91 (December 1991)

11. S.B. Giddings, *Phys. Lett.* **B268** (1991) 17

12. L. Smolin, “Recent Developments in Nonperturbative Quantum Gravity,” Syracuse preprint (February 1992)

13. I.R. Klebanov, in: “String Theory and Quantum Gravity ‘91,” Proceedings of the Trieste Spring School 1991, eds.: J.A. Harvey et al. (World Scientific, 1992)
   D. Kutasov, *ibid.*

14. A.M. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 635; “Singular States in 2D Quantum Gravity,” Princeton preprint PUPT-1289 (September 1991)

15. S.R. Das and A. Jevicki, *Mod. Phys. Lett.* **A5** (1990) 1639

16. E. Witten, *Phys. Rev.* **D44** (1991) 314

17. E. Witten, *Nucl. Phys.* **B373** (1992) 187

18. I.R. Klebanov and A.M. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 3273

19. E. Witten and B. Zwiebach, *Nucl. Phys.* **B377** (1992) 55

20. I.R. Klebanov, *Mod. Phys. Lett.* **A7** (1992) 723

21. P. Hořava, *Nucl. Phys.* **B386** (1992) 383

22. A. Giveon and A. Shapere, *Nucl. Phys.* **B386** (1992) 43

23. E. Witten, *Phys. Rev. Lett.* **61** (1988) 670
24. E. Verlinde and N.P. Warner, *Phys. Lett.* **B269** (1991) 96

25. J. Distler and C. Vafa, *Mod. Phys. Lett.* **A6** (1991) 259; “The Penner Model and D=1 String Theory,” Princeton preprint PUPT-1212, Cargèse Workshop on “Random Surfaces, Quantum Gravity and Strings,” May 1990

26. R. Dijkgraaf, G. Moore and R. Plesser, “The Partition Function of 2-D String Theory,” IAS/Yale preprint IASSNS-HEP-92/48=YCTP-P22-92 (August 1992)

27. J. Avan and A. Jevicki, *Phys. Lett.* **B266** (1991) 35; **B272** (1991) 17; *Mod. Phys. Lett.* **A7** (1992) 357

28. G. Moore and N. Seiberg, *Int. J. Mod. Phys.* **A7** (1991) 2601

29. U.H. Danielsson and D.J. Gross, *Nucl. Phys.* **B366** (1991) 3
   U.H. Danielsson, *Nucl. Phys.* **B380** (1992) 83

30. D. Minic, J. Polchinski and Z. Yang, *Nucl. Phys.* **B369** (1992) 324

31. S.R. Das, A. Dhar, G. Mandal and S.R. Wadia, *Mod. Phys. Lett.* **A7** (1992) 71, 937; *Int. J. Mod. Phys.* **A7** (1992) 5165
   A. Dhar, G. Mandal and S.R. Wadia, “Classical Fermi Fluid and Geometric Action for c = 1,” IAS/Tata preprint IASSNS-HEP-91/89=TIFR-TH-91/61 (March 1992); “Non-relativistic Fermions, Coadjoint Orbits of W∞ and String Field Theory at c = 1,” Tata preprint TIFR-TH-92/40 (June 1992)

32. I. Bars, in: “Strings ’90,” Proceedings of the Superstring Workshop at Texas A&M, March 1990, eds.: R. Arnowitt et al. (World Scientific, Singapore, 1991)
   C.N. Pope, L.J. Romans and X. Shen, *ibid.*

33. E. Sezgin, “Aspects of W∞ Symmetry,” Trieste/Texas A&M preprint IC/91/206=CTP TAMU-9/91 (1991); “Area-Preserving Diffeomorphisms, w∞ Algebras and w∞ Gravity,” Texas A&M preprint CTP-TAMU-13/92 (February 1992)
   C.N. Pope, “Lectures on W Algebras and W Gravity,” Texas A&M preprint
CTP TAMU-103/91 (December 1991)
X. Shen, “W-Infinity and String Theory,” CERN preprint CERN-TH.6404/92 (February 1992)
C.M. Hull, “Classical and Quantum W Gravity,” QMC preprint QMW/PH/92/1 (January 1992)

34. D.J. Gross, *Phys. Rev. Lett.* **60** (1988) 1229
   E. Witten, *Phil. Trans. Roy. Soc.* **A329** (1989) 345

35. J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, *Phys. Lett.* **B267** (1991) 465; **B272** (1991) 261; **B276** (1992) 56; **B278** (1992) 246; **B284** (1992) 43

36. J. Ellis, N.E. Mavromatos and D.V. Nanopoulos, *Phys. Lett.* **B288** (1992) 23

37. I. Bakas and E. Kiritsis, *Int. J. Mod. Phys.* **A7** (1992) 339
   F. Yu, *Nucl. Phys.* **B375** (1992) 173
   F. Yu and Y.-S. Wu, “An Infinite Number of Commuting Quantum $W_\infty$ Charges in the $SL(2, R)/U(1)$ Coset Model,” Utah preprint UU-HEP-92/11 (May 1992)
   T. Eguchi, H. Kanno and S.-K. Yang, “$W_\infty$ Algebra in Two-Dimensional Black Hole,” Cambridge preprint NI-92004=DAMTP 92-64 (September 1992)

38. T. Eguchi, *Mod. Phys. Lett.* **A7** (1992) 85

39. A.S. Schwarz, “Geometry of Batalin-Vilkovisky Quantization,” “Semiclassical Approximation in Batalin-Vilkovisky Formalism,” UC Davis preprints (June and October 1992)

40. E. Witten, *Mod. Phys. Lett.* **A7** (1990) 487

41. M. Henneaux, *Nucl. Phys.* **B** (*Proc. Suppl.*) **18A** (1990) 47; *Phys. Lett.* **B282** (1992) 372

42. C.M. Hull, “$W$-Geometry,” QMC preprint QMW-92-6 (November 1992)
43. D.J. Gross and P. Mende, *Phys. Lett.* **B197** (1987) 129; *Nucl. Phys.* **B303** (1988) 407

44. E. Verlinde, *Nucl. Phys.* **B381** (1992) 141

45. E. Witten, *Phys. Rev.* **D46** (1992) 5467; “Some Computations in Background Independent Off-Shell String Theory,” IAS preprint IASSNS-HEP-92/63 (October 1992)

46. B. Zwiebach, “Closed String Field Theory: Quantum Action and the B-V Master Equation,” IAS preprint IASSNS-HEP-92/41 (June 1992)

47. T. Lada and J. Stasheff, “Introduction to sh-Lie Algebras for Physicists,” North Carolina preprint UNC-MATH-92/2 (September 1992)

48. M. Gerstenhaber and S.D. Schack, “Algebraic Cohomology and Deformation Theory,” in: “Deformation Theory of Algebras and Structures and Applications,” eds: M. Hazewinkel and M. Gerstenhaber, NATO ASI Series C247 (Kluwer, 1988)

49. V.I. Arnol’d, *Ann. Inst. Fourier* **XVI** (1966) 319; “Mathematical Methods of Classical Mechanics” (Springer Verlag, 1978) App. 2.K

50. E.G. Floratos and J. Iliopoulos, *Phys. Lett.* **B201** (1988) 237

I. Antoniadis, P. Ditsas, E. Floratos and J. Iliopoulos, *Nucl. Phys.* **B300** (1988) 549

51. C.N. Pope, L.J. Romans, E. Sezgin and X. Shen, *Phys. Lett.* **B256** (1991) 191

52. J. Distler and P. Nelson, *Phys. Rev. Lett.* **66** (1991) 1955; *Nucl. Phys.* **B366** (1991) 255; “Hidden Symmetry in Topological Gravity,” Penn/Princeton preprint UPR-0463T (August 1991)

S. Govindarajan, P. Nelson and S.-J. Rey, *Nucl. Phys.* **B365** (1991) 633

S. Govindarajan, P. Nelson and E. Wong, *Commun. Math. Phys.* **147** (1992) 253
E. Wong, “Recursion Relations in Semirigid Topological Gravity,” Penn preprint UPR-0491T (November 1991)

53. S. Elitzur, A. Forge and E. Rabinovici, “On Effective Theories of Topological Strings,” CERN/Racah/SISSA preprint CERN-TH.6326 =RI/143/91/11=SISSA/158/91/EP (November 1991)

54. E. Witten, “Chern-Simons Gauge Theory as a String Theory,” IAS preprint IASSNS-HEP-92-45 (July 1992)

55. S. Mukhi and C. Vafa, “Two Dimensional Black Hole as a Topological Coset Model of c=1 String Theory,” Harvard/Tata preprint HUTP-93/A002=TIFR/TH/93-01 (January 1993)

56. M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, “Extended N=2 Superconformal Structure of Gravity and W-Gravity Coupled to Matter,” Caltech/CERN/Harvard/USC preprint CALT-68-1832=CERN-TH.6694/92=HUTP-A061/92=USC-92/021 (October 1992)

57. B.H. Lian and G.J. Zuckermann, “New Perspectives on the BRST-algebraic Structure of String Theory,” Toronto/Yale preprint (November 1992)

M. Penkava and A.S. Schwarz, “On Some Algebraic Structures Arising in String Theory,” UC Davis preprint (December 1992)

E. Getzler, “Batalin-Vilkovisky Algebras and Two-Dimensional Topological Field Theories,” MIT preprint (December 1992)