NON-PERTURBATIVE TWO-DIMENSIONAL DILATON GRAVITY

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ABSTRACT

We present a review of the canonical quantization approach to the problem of non-perturbative 2d dilaton gravity. In the case of chiral matter we describe a method for solving the constraints by constructing a Kac-Moody current algebra. For the models of interest, the relevant Kac-Moody algebras are based on $SL(2,\mathbb{R}) \otimes U(1)$ group and on an extended 2d Poincare group. As a consequence, the constraints become free-field Virasoro generators with background charges. We argue that the same happens in the non-chiral case. The problem of the corresponding BRST cohomology is discussed as well as the unitarity of the theory. One can show that the theory is unitary by choosing a physical gauge, and hence the problem of transitions from pure into mixed states is absent. Implications for the physics of black holes are discussed.

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1. Introduction

Two-dimensional dilaton gravity theories of interest can be described by an action

\[ S = S_0 + S_m \]

\[ S_0 = \int_M d^2x \sqrt{-g} e^{-2\Phi} \left[ R + \gamma (\nabla \Phi)^2 + U(\Phi) \right] \]

\[ S_m = -\frac{1}{2} \int_M d^2x \sqrt{-g} \sum_{i=1}^N (\nabla \phi_i)^2 \]  

(1.1)

where \( \Phi \) and \( \phi_i \) are scalar fields, \( \gamma \) is a constant, \( g, R \) and \( \nabla \) are determinant, scalar curvature and covariant derivative respectively, associated with a metric on the 2d manifold \( M \). For our purposes we will assume that \( M = \Sigma \times \mathbb{R} \), so that \( \Sigma = S^1 \) (a circle) or \( \Sigma = \mathbb{R} \) (a real line). We will refer to these two cases as compact and non-compact respectively.

\( S_0 \) describes the coupling of the dilaton \( \Phi \) to the metric, while \( S_m \) represents conformally coupled scalar matter. Depending on the value of the constant \( \gamma \) and the form of the potential \( U \), one can get various dilaton gravity theories. The relevant examples are:

1. **Spherical symmetry reduction of the 4d Einstein-Hilbert action** \[ S_0 = \int_M d^2x \sqrt{-g} e^{-2\Phi} \left[ R + 2(\nabla \Phi)^2 + \kappa e^{2\Phi} \right] . \] (1.2)

The reduction is performed by decomposing the 4d line element as

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^{-2\Phi(t,r)} (d\theta^2 + \sin^2 \theta d\phi^2) \] (1.3)

where \( (r, \theta, \phi) \) are the spherical coordinates, \( x^\mu = (t, r) \), while \( \kappa \) is the inverse Newton constant.

2. **Dimensional reduction of 4d dilaton gravity** \[ S_0 = \int_M d^2x \sqrt{-g} e^{-2\Phi} \left[ R + 4(\nabla \Phi)^2 + 4\lambda^2 \right] . \] (1.4)

3. **Induced 2d gravity**

\[ S_0 = \int_M d^2x \sqrt{-g} \left[ \frac{\lambda}{2} (\nabla \phi)^2 + \alpha R \phi + \Lambda \right] , \]

(1.4)

which is not of the form (1.1) but it can be related to it by a field redefinition \[ ].

From these examples it is clear that 2d dilaton gravity theories can be relevant as toy models for 4d quantum gravity and for non-critical string theory. Although
model (1) contains a black hole solution by definition, it was an interesting discovery that model (2) contains a black hole solution as well [19]. Furthermore, when (2) is coupled to scalar conformal matter, it is exactly solvable [1]. Given the fact that it is also a renormalizable field theory, this makes it an excellent toy model for the study of the black hole evaporation and backreaction. Semi-classical analysis of the model in the limit of large $N$ had raised a hope that a black hole singularity can dissappear in the quantum theory [1]. However, very soon it was shown that a singularity is present in the solution of the semi-classical equations of motion [3]. Furthermore, Hawking has given general arguments for the existence of singularity in any semi-classical approximation [4]. All this indicates that one should perform a non-perturbative, more precisely exact, quantization of the theory in order to see what really happens with the black hole.

2. Non-perturbative approaches

Non-perturbative formulation of any quantum theory of gravity has to deal with the following conceptual problems, which do not appear in ordinary field theories:

1. no background metric
2. maintaining diffeomorphism invariance
3. problem of time
4. space-time singularities.

For a more detailed discussion see [8]. The non-perturbative approaches to 2d dilaton gravity which have been studied so far are path-integral and canonical. The idea of the path-integral approach is to perform the functional integral over the metric, dilaton and matter fields exactly, and then to study the corresponding effective action and the correlation functions (for a review and references see [2, 7]). Beside its own difficulties in achieving the stated goal, it is not clear how to construct the physical Hilbert space within this approach and how to address the corresponding conceptual questions.

In the canonical approach [10, 1, 12, 13, 14] the construction of the physical Hilbert space is the primary goal, from which all other questions are answered. This is achieved from the study of the constraints, which can be derived by using the ADM (Arnowit, Deser, Misner) method [11, 15]. The ADM method takes care of the problems (1) and (2), as opposed to the method used in [13, 14], where the constraints were derived in the conformal gauge, and the quantization was based on the space of classical solutions.
Before explaining the canonical ADM formulation, we will briefly study field redefinitions, in order to arrive at the simplest possible form of the action. That in turn simplifies the constraints. Let $\psi^2 = e^{-2\Phi}$, then $S_0$ from the eq. (1.1) becomes

$$S_0 = \int_M d^2x \sqrt{-\tilde{g}} \left[ \frac{1}{2}(\nabla \psi)^2 + \frac{1}{2\gamma} R \psi^2 + \tilde{U}(\psi) \right], \quad (2.1)$$

where $\psi$ has been rescaled into $\frac{1}{\sqrt{2\gamma}} \psi$ ($\gamma \neq 0$). Then by performing a Russo-Tseytlin transformation [5]

$$\phi = \frac{1}{\gamma} \psi^2, \quad \tilde{g}_{\mu\nu} = e^{-2\rho} g_{\mu\nu}, \quad 2\rho = \frac{1}{\gamma} \psi^2 - \frac{\gamma}{2} \ln \psi \quad (2.2)$$

we get

$$S_0 = \int_M d^2x \sqrt{-\tilde{g}} \left[ \frac{1}{2}(\tilde{\nabla} \phi)^2 + \frac{1}{2} \tilde{R} \phi + V(\phi) \right], \quad (2.3)$$

where $V(\phi) = \tilde{U} e^{2\rho}$. In the dilaton gravity case $V = \frac{1}{2} \lambda e^\phi$, and hence consider

$$S_0 = \int_M d^2x \sqrt{-\tilde{g}} \left[ \frac{1}{2}(\tilde{\nabla} \phi)^2 + \alpha R \phi + \Lambda e^{3\phi} \right], \quad (2.4)$$

where $\alpha, \beta$ and $\Lambda$ are constants. The action (2.4) represents a class of solvable dilaton gravity theories, which can be seen by redefining the metric as $\tilde{g}_{\mu\nu} = e^{3\rho} g_{\mu\nu}$ [10], so that

$$S_0 = \int_M d^2x \sqrt{-\tilde{g}} \left[ \frac{1}{2}(1 - 2\alpha\beta)(\tilde{\nabla} \phi)^2 + \frac{1}{2} \tilde{R} \phi + \Lambda \right]. \quad (2.5)$$

Since eq. (2.5) represents an induced gravity action, one can use Polyakov’s results about the existence of an $SL(2, \mathbb{R})$ current algebra [20] to solve the theory. In the canonical setting this can be done by constructing gauge-independent currents, which form an $SL(2, \mathbb{R}) \otimes U(1)$ current algebra [10, 13]. However, in [10] it was not realised that for $\alpha\beta = \frac{1}{2}$, i.e. precisely in the dilaton gravity case, this current algebra degenerates into an extended 2d Poincare current algebra [12]. As we are going to show, the construction proposed in [11] for solving the theory works even in that case, after slight modifications.

### 3. Canonical formulation

Consider the following action

$$S_0 = -\int_M d^2x \sqrt{-g} \left[ \frac{\gamma}{2}(\nabla \phi)^2 + \alpha R \phi + V(\phi) + \frac{1}{2} \sum_{i=1}^{N} (\nabla \phi_i)^2 \right], \quad (3.1)$$

where $\alpha$ and $\gamma$ are constants. Note that the field redefinitions of the previous section always scaled the metric so that the form of the matter action is unchanged, because
of its conformal invariance. Canonical reformulation requires $M = \Sigma \times \mathbb{R}$, and it simplifies if we use the ADM parametrization of the metric

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + gn^2 & gn \\ gn & g \end{pmatrix},$$

where $N$ and $n$ are the lapse function and the shift vector respectively, while $g$ is a metric on $\Sigma$. By defining the canonical momenta as

$$p = \frac{\partial L}{\partial \dot{g}}, \quad \pi = \frac{\partial L}{\partial \dot{\phi}}, \quad \pi_i = \frac{\partial L}{\partial \dot{\phi}_i},$$

where $L$ is the Lagrange density of (3.1) and dots stand for $t$ derivatives, the action becomes

$$S = \int dt dx \left( p \dot{g} + \pi \dot{\phi} + \pi_i \phi_i - \frac{N}{\sqrt{g}} G_0 - n G_1 \right).$$

The constraints $G_0$ and $G_1$ are given as

$$G_0(x) = -\frac{\gamma}{2\alpha^2} (gp)^2 + \frac{1}{\alpha} gp \pi + \frac{\gamma}{2} (\phi')^2 + gV(\phi) - 2\alpha \sqrt{g} \left( \frac{\phi'}{\sqrt{g}} \right)' + \frac{1}{2} \sum_{i=1}^{N} (\pi_i^2 + (\phi'_i)^2),$$

$$G_1(x) = \pi \phi' - 2p' g - pg' + \sum_{i=1}^{N} \pi_i \phi'_i,$$

where primes stand for $x$ derivatives. The $G$'s generate the diffeomorphisms of $M$, such that $G_1$ generates the diffeomorphisms of $\Sigma$, while $G_0$ generates time-translations of $\Sigma$. A special feature of two dimensions is that

$$T_\pm = \frac{1}{2} (G_0 \pm G_1)$$

generate two commuting copies of the one-dimensional diffeomorphism algebra. When $\Sigma = S^1$ these become two commuting Virasoro algebras.

As in the 4d canonical gravity, direct quantization of the constraints (3.5) is problematic due to their non-polynomial dependence on the canonical variables. One way around this problem is to follow the strategy introduced by Ashtekar in the 4d case [9], which is to find new canonical variables such that the constraints become polynomial. This can be done by constructing first a Kac-Moody current algebra corresponding to the hidden symmetries we discussed in the previous section [10, 12]. Let us introduce the following variables

$$J^+ = -\frac{1}{g} T_- + \frac{\Lambda}{2},$$

where
\[ J^0 = gp + \delta \left( \pi + \alpha \frac{g'}{g} \right) + (\alpha + \gamma \delta)\phi' \]

\[ J^- = 4\alpha^2 g \]

\[ P_D = \frac{1}{\sqrt{2}} \left( \pi + \alpha \frac{g'}{g} + \gamma \phi' \right) . \quad (3.7) \]

When \( \gamma \neq 0 \), then for \( \delta = -\frac{\alpha}{\gamma} (J^a, P_D) \) form an \( SL(2, \mathbb{R}) \otimes U(1) \) Kac-Moody algebra

\[
\{J^0(x), J^\pm(y)\} = \pm J^\pm(x)\delta(x-y) \\
\{J^+(x), J^-(y)\} = -2\gamma J^0(x)\delta(x-y) + 4\alpha^2 \delta'(x-y) \\
\{J^0(x), J^0(y)\} = -\frac{2\alpha^2}{\gamma} \delta'(x-y) \\
\{P_D(x), P_D(y)\} = \gamma \delta'(x-y) , \quad (3.8)
\]

where all other Poisson brackets are zero. When \( \gamma = 0 \) then for \( \delta = 0 \) one gets a centrally extended Poincare Kac-Moody algebra

\[
\{J^0(x), J^\pm(y)\} = \pm J^\pm(x)\delta(x-y) \\
\{J^+(x), J^-(y)\} = 2\sqrt{2} \alpha P_D(x)\delta(x-y) + 4\alpha^2 \delta'(x-y) \\
\{J^0(x), P_D(y)\} = 2\sqrt{2} \alpha \delta'(x-y) \\
\{P_D(x), P_D(y)\} = 0 , \quad (3.9)
\]

and all other Poisson brackets are zero. The importance of this centrally extended Poincare group for the dilaton gravity was first recognized in [17].

The next step is to perform a generalized Sugawara construction, which gives a generator of 1d diffeomorphisms quadratic in the Kac-Moody currents

\[
T_G = \frac{1}{4\alpha^2} [J^+J^- - \gamma(J^0)^2] - (J^0)' + \frac{\alpha + \gamma \delta}{\sqrt{2}\alpha^2} J^0 P_D - \frac{\delta(2\alpha + \gamma \delta)}{2\alpha^2} P_D^2 + \delta \sqrt{2} P_D' . \quad (3.10)
\]

Then it is straightforward to show that

\[ T_G + T^+_M = G_1 \] , \quad (3.11)

where

\[
T^+_M = \frac{1}{2} \sum_{i=1}^{N} (P^+_i)^2 \quad , \quad P^+_i = \frac{1}{\sqrt{2}} (\pi_i \pm \phi'_i) \quad . \quad (3.12)
\]

Eq. (3.11) implies that we can replace the constraints \( G_0 = 0 \) and \( G_1 = 0 \) with

\[ J^+ - \frac{\alpha}{2} \Lambda = 0 \quad , \quad T_G + T^+_M = 0 \] . \quad (3.13)
Given the new set of constraints (3.13), one could quantize the current algebra $(J^a, P_D, P_i^+, P_i^-)$ and then try to find the physical Hilbert space by using the representation theory of the Kac-Moody algebras. However, this method works only in the case of chiral matter ($P_i^- = 0$) since

$$\{J^+(x), P_i^-(y)\} = -\frac{1}{g(x)} P_i^-(x) \delta'(x - y) .$$  \hspace{1cm} (3.14)

This is not a big restriction, since the chiral matter case is relevant for a one-sided collapse.

Instead of using the group-theoretical method, we look for a further change of variables such that we get a truly canonical variables. This approach is more convenient for addressing the problem of time and the unitarity of the theory [11]. In the $SL(2)$ case there exists a change of variables, called Wakimoto construction [18], which can be written classically as

$$J^+ = B,$$

$$J^0 = -B \Gamma + \alpha \sqrt{2} P_L,$$

$$J^- = B \Gamma^2 - 2 \alpha \sqrt{2} \Gamma P_L - 4\alpha^2 \Gamma' ,$$  \hspace{1cm} (3.15)

where the new fields satisfy

$$\{B(x), \Gamma(y)\} = \delta(x - y) , \quad \{P_L(x), P_L(y)\} = -\delta'(x - y) ,$$  \hspace{1cm} (3.16)

and all other Poisson brackets are zero. In the Poincaré case the analog of the eq. (3.16) is

$$J^+ = B,$$

$$J^0 = -B \Gamma + 2\alpha Y',$$

$$J^- = 2\sqrt{2} \alpha \Gamma P_D - 4\alpha^2 \Gamma' ,$$  \hspace{1cm} (3.17)

where the non-zero Poisson brackets are

$$\{B(x), \Gamma(y)\} = \delta(x - y) , \quad \{P_D(x), Y(y)\} = -\delta(x - y) .$$  \hspace{1cm} (3.18)

The generalized Sugawara tensor (3.10) takes the following form

$$T_G = B \Gamma - \frac{1}{2} P_0^2 + \frac{1}{2} P_1^2 + Q_0 P_0' + Q_1 P_1'$$  \hspace{1cm} (3.19)

where $P_0 = P_L$ and $P_1 = P_D$ in the $SL(2)$ case, while $P_0 = \frac{1}{\sqrt{2}}(P_D - Y')$ and $P_1 = \frac{1}{\sqrt{2}}(P_D + Y')$ in the Poincaré case. The background charges $Q_0$ and $Q_1$ are given
as \((-\sqrt{2}\alpha, -\sqrt{2}\alpha)\) in the \(SL(2)\) case or \((\sqrt{2}\alpha, -\sqrt{2}\alpha)\) in the Poincare case. Note that by performing a canonical transformation

\[
P_0 = -\tilde{P}_0, \quad P_1 = \tilde{P}_1
\]

one can transform the Poincare Sugawara tensor into the \(SL(2)\) one.

By defining a new canonical pair \((P, X)\), where

\[
P = B - \frac{1}{2}\Lambda, \quad X = \Gamma,
\]

the constraints now read

\[
P = 0
\]

and

\[
S = -\frac{1}{2}P_0^2 + \frac{1}{2}P_1^2 + Q_0P_0' + Q_1P_1' + \frac{1}{2}\sum_{i=1}^{N}(P_i^+)^2 = 0.
\]

By fixing the spatial diffeomorphism invariance as

\[
X(x) = x
\]

\(P\) is eliminated by eq. (3.22), and therefore we are left with only one constraint (eq. (3.23)) for the variables \((P_0, P_1)\) and \((\pi_i, \phi_i)\). The \(S\) constraint will play the role of the Wheeler-DeWitt equation.

In the non-chiral case \((P_i^- \neq 0)\), the constraints \(T_+\) and \(T_-\) can be recasted in the form (3.23), which follows from the study of the space of the classical solutions [13, 14]. However, the explicit canonical transformation which achieves that has not been yet constructed in the ADM formalism.

4. Quantization

In the canonical approach, there are two basic ways of quantizing a constrained system

1. quantize first and then solve the constraints (Dirac quantization),
2. solve the constraints first and then quantize (reduced phase space (RPS) quantization).

The Dirac quantization, and its variations (Gupta-Bleuler and BRST method), are often preferred to the RPS quantization because of preservation of the manifest symmetries of the theory. On the other hand, RPS quantization is easier to accomplish. In our case, we have followed so far the RPS method, and we will continue to do
so, but given the special role the constraints play in gravity, we will also explore the Dirac quantization.

In order to accomplish the Dirac quantization of the $S$ constraint, we need $\Sigma$ to be compact, since otherwise we do not know much about the representations of the 1d diffeomorphism algebra. This creates a problem for the non-compact case, i.e. the one where the black hole solutions exist, and this is usually resolved by putting the system into a large box, of length $L$.

Now we will label the vector $(P_0, P_1, P^+_i)$ as $P_I$, so that

$$P_I(x) = \frac{1}{\sqrt{L}} \left( p_I + \sum_{n \neq 0} \alpha^I_n e^{i\pi x/L} \right)$$

where $p_i = 0$. Then

$$S(x) = \frac{1}{L} \sum_n L_n e^{i\pi x/L}$$

$$L_n = \frac{i}{2} \sum_m (\alpha^I_{n-m} \alpha^I_m + i Q_I \alpha^I_n) ,$$

where $Q_i = 0$. The $L_n$’s are promoted into operators acting on a Fock space $\mathcal{F}(\alpha^I_n)$ made out of the $\alpha_n$ modes in the standard way. The $L_n$’s form a Virasoro algebra classically, but in the quantum case there is an anomaly in the algebra, in the form of the central extension term with the central charge $c = N$. This type of situation is best handled in the BRST formalism. One enlarges the original Fock space $\mathcal{F}(\alpha^I_n)$ by introducing a canonical pair of ghost fields $(b, c)$, and constructs a nilpotent operator

$$\hat{Q} = \sum_n c_{-n} (L_n - a \delta_{n,0}) + \frac{i}{2} \sum_{n,m} (n - m) : c_{-n} c_{-m} b_{n+m} : .$$

The nilpotency of $\hat{Q}$ requires

$$-Q_0^2 + Q_1^2 = 2 - N/12 \quad a = N/24 \quad ,$$

which is satisfied for $N = 24$. The physical Hilbert space $\mathcal{H}^*$ is determined as the cohomology of $\hat{Q}$

$$\mathcal{H}^* = \text{Ker } \hat{Q}/\text{Im } \hat{Q} .$$

The cohomology problem of this type has been studied extensively in the $N = 0$ case [21]. The physical Hilbert space has three sectors

$$\mathcal{H}^* = \mathcal{H}^*_{0} \oplus \mathcal{H}^*_{1} \oplus \mathcal{H}^*_{-1}$$

$$\mathcal{H}^*_{0} = \{|p_0, p_1| - p_0^2 + p_1^2 = 0\} \oplus \{\text{discrete states}\}$$

$$\mathcal{H}^*_{\pm} = \{\text{discrete states}\} .$$
where the subscripts 0, 1, −1 refer to the ghost number. The \( N \neq 0 \) cohomology problem has not been studied in detail, but one can deduce the following features. There will be only a zero-ghost sector, since the intercept \( a \neq 0 \). A basis for the physical states will be

\[
\mathcal{H}_0^* = \{ |p_0, p_1 \rangle \otimes \alpha_{n_1}^i \ldots \alpha_{n_k}^i |0 \rangle \mid -p_0^2 + p_1^2 + 2 \sum_{i=1}^k n_i = N/12, \ n_i > 0 \}, \quad (4.7)
\]
i.e. the transverse oscillator states. The exact cohomology analysis may say something about the discrete states, but we do not expect any other continuous momentum states but the ones from the eq. (4.7).

This is confirmed by the results of the RPS quantization [11]. Given the S constraint (eq. (3.23)), and the fact that \( P_0 \) and \( P_1 \) can be always represented as

\[
P_0 = \frac{1}{\sqrt{2}} (P_T - T') \quad , \quad P_1 = 2\sqrt{2} \alpha p + \frac{1}{\sqrt{2}} (P_T + T') \quad , \quad (4.8)
\]
where we have introduced an extra zero mode \( p \ (p' = 0) \) and \((P_T, T)\) is a canonical pair, one gets

\[
(2 \alpha p + P_T)(2 \alpha p + T') - 2 \alpha P_T^2 + \frac{1}{2} \sum_{i=1}^N (P_i^+)^2 = 0 \quad . \quad (4.9)
\]
In the Poincare case one has to perform the canonical transformation (3.20) first, in order to get the eq. (4.9). Now we chose \( T \) as the time variable, and at the same time we fix the diffeomorphism invariance by a gauge choice

\[
T(x, t) = t \quad . \quad (4.10)
\]
We can now solve eq. (4.9) for \( P_T \)

\[
P_T(x) = -2 \alpha p + \frac{1}{4 \alpha} e^{px} \left( k + \int^x dy e^{-py} \sum_{i=1}^N (P_i^+)^2 \right) \quad , \quad (4.11)
\]
where \( k \) is the constant of integration. Hence the independent canonical variables are \((p, q)\) and \((\pi_i, \phi_i)\), which proves our conjecture in the Dirac quantization that only the transverse mode states are physical[4].

We now proceed by specifying the Hamiltonian associated with the gauge choice (4.10), which can be determined as

\[
H = - \int_\Sigma dx P_T(x) = c_1 p + \frac{1}{4 \alpha p} \left( \sum_n \alpha_n^i \alpha_n^i + c_0 \right) \quad . \quad (4.12)
\]
\[\text{3This is true if there is no anomaly, i.e. } N = 24\]
The constants $c_0$ and $c_1$ can be determined from the boundary conditions, although $c_0$ can have a quantum contribution due to the normal ordering effects.

One gets an analogous formula to eq. (4.12) in the non-chiral case, which follows from the fact that $T_+$ and $T_-$ can be recasted in the form (3.23). Then

$$G_0 = T_+ + T_- = \frac{1}{2} \pi_I^2 + \frac{1}{2} (\phi_I')^2 + \sqrt{2} Q I \pi'_I$$

and

$$G_1 = T_+ - T_- = \pi_I \phi_I' + \sqrt{2} Q I \phi''_I.$$ 

By choosing the bosonic string light-cone gauge

$$\pi_+ = p, \quad \phi_+ = t,$$

where $V_\pm = V_0 \pm V_1$, one gets

$$H = \frac{1}{2p} \left( \sum_n (\alpha_n^- \alpha_n^+ + \tilde{\alpha}_n^- \tilde{\alpha}_n^+) + c_0 \right).$$

The unitarity of the theory follows from the fact that the expression (4.16) can be promoted into a Hermitian operator acting on the physical Hilbert space

$$\mathcal{H}^* = L^2(p) \otimes \mathcal{F}(\alpha_n^+) \otimes \mathcal{F}(\tilde{\alpha}_n^+),$$

where $L^2(p)$ is the Hilbert space of square-integrable functions of $p$, while $\tilde{\alpha}$ are the modes of $P_i^-$. Therefore one has a unitary evolution described by a Schrödinger equation

$$i \frac{\partial}{\partial t} \Psi[t, p, \phi_i(x)] = \hat{H} \Psi[t, p, \phi_i(x)],$$

$\Psi \in \mathcal{H}^*$, and hence no transitions from pure into mixed states occur in this theory.

5. Conclusions

The most straightforward conclusion is that the black holes in the quantum theory defined by eq. (4.18) do not destroy information, and a unitary S-matrix can be constructed. The authors of [14] have arrived at the similar conclusions by studying canonical quantization of the dilaton gravity in the conformal gauge. After imposing a reflecting boundary condition, they construct a S-matrix and prove its unitarity, without using the Hamiltonian, which is a more difficult but an alternative way of proving the unitarity.

The evolution of the matter is governed by a free-field Hamiltonian, which is not surprising given the fact that the exact classical solution can be written in terms of
free fields \[1\]. Actually, this classical solvability of the theory is the reason why it is possible to find a canonical transformation which makes the constraints quadratic. The non-trivial part of the Hamiltonian comes from its dependence on the zero-mode \( p \), which is the remnant of the graviton and dilaton degrees of freedom.

Although the quantum theory is unitary, it is less clear what happens to the black hole. This would require studying the operators associated with the metric and the scalar curvature. The difficulty is that \( g \) and \( R \) are complicated (non-polynomial) functions of the free-field variables, and hence it is a non-trivial task to promote them into well-defined Hermitian operators. Provided that this problem is resolved, one could study the evaporation of the black holes in the following way \[11\]. Let \( \Psi_0 \) be a physical state at \( t = 0 \) such that

\[
\langle \Psi_0 | \hat{g}(x) | \Psi_0 \rangle = f(x), \quad \langle \Psi_0 | \hat{R}(x) | \Psi_0 \rangle = h(x)
\]

\( (5.1) \)

are regular functions for every \( x \in \Sigma \). Then \( \Psi(t) = e^{-i\hat{H}t} \Psi_0 \) and for \( t > t_0 \) a horizon will form in the effective metric \( \langle \Psi(t) | \hat{g}(x) | \Psi(t) \rangle \). A density matrix \( \hat{\rho} \) could be calculated by tracing out the states which are beyond the horizon. Then one could try to find out under what conditions \( \hat{\rho} \) takes approximately the thermal form

\[
\hat{\rho} \approx \frac{1}{Z} e^{-\beta \hat{H}}
\]

\( (5.2) \)

and what are the corrections to the Hawking temperature

\[
\beta = \frac{4\pi}{\lambda} + \ldots
\]

\( (5.3) \)

One can also formulate the problem of computing the temperature corrections in the S-matrix formalism \[14\].

In spite of all these advances in formulating the exact quantum theory, we think that the conceptual problem of the space-time singularity is still unresolved. Although we managed to find observables\(^4\) which are well defined at the singularity, there will be other observables, those associated with the scalar curvature, which will not be well defined at the singularity. One way of resolving this problem \[11\] is to study

\[
R_{eff}(x, t) = \langle \Psi_0 | e^{i\hat{H}t} \hat{R}(x) e^{-i\hat{H}t} | \Psi_0 \rangle
\]

\( (5.4) \)

If it stays a regular function for every \( x \) and \( t \) and for every \( \Psi_0 \) that satisfies the conditions of eq. (5.1), then we could say that the singularity has been removed from the quantum theory. A very similar idea has been proposed in \[22\]. However, there is no a priori reason for something like this to happen, and this issue has to be a subject of further studies.

\(^4\) An observable in this context is a quantity which has weakly vanishing Poisson brackets with the constraints.
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