Analytic solutions for locally optimal designs for gamma models having linear predictor without intercept

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Abstract
The gamma model is a generalized linear model for gamma-distributed outcomes. The model is widely applied in psychology, ecology or medicine. In this paper we focus on gamma models having a linear predictor without intercept. For a specific scenario sets of locally D- and A-optimal designs are to be developed. Recently, Gaffke et al. (2018) established a complete class and an essentially complete class of designs for gamma models to obtain locally D-optimal designs. However to extend this approach to gamma model without an intercept term is complicated. To solve that further techniques have to be developed in the current work. Further, by a suitable transformation between gamma models with and without intercept optimality results may be transferred from one model to the other. Additionally by means of The General Equivalence Theorem optimality can be characterized for multiple regression by a system of polynomial inequalities which can be solved analytically or by computer algebra. By this necessary and sufficient conditions on the parameter values can be obtained for the local D-optimality of particular designs. The robustness of the derived designs with respect to misspecifications of the initial parameter values is examined by means of their local D-efficiencies.

Keywords: generalized linear model, optimal design, models without intercept, complete class, interaction.

1. Introduction
The gamma model is employed for outcomes that are non-negative, continuous, skewed and heteroscedastic specifically, when the variances are proportional to the square of the means. The gamma model with its canonical link (reciprocal) is appropriate for many real life data. In ecology and forestry, Gea-Izquierdo and Caellas (2009) mentioned that gamma models offers a great potential for many forestry applications and they used gamma models to analyze plant competition. In medical context, Grover et al. (2013) fitted a gamma model with duration of diabetes as the response variable and predictors as the rate of rise in serum creatinine (SrCr) and number of successes (number of times SrCr values exceed its normal range (1.4 mg/dl)). For a study about air pollution, Kurtoğlu and Özkale (2016) employed a gamma model to analyze nitrogen dioxide concentrations considering some weather factors (see also Chatterjee (1988), Section 8.7). In psychological studies, recently, Ng and Cribbie (2017) used a gamma model for modeling the relationship between negative automatic thoughts (NAT) and socially prescribed perfectionism (SPP).

Although, the canonical link is frequently employed in the gamma model but there is always a doubt about the suitable link function for outcomes. Therefore, a class of link functions might be employed. The common alternative links mostly come from the Box-Cox family and the power link family (see Atkinson and Woods (2015)). In fact, the family of power link functions includes the canonical link therefore it is a favorite choice for employment in this paper.

In the theory of optimal designs, the information matrix of a generalized linear model depends on the model parameters through the intensity function. Locally optimal designs can be derived through maximizing a specific

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optimality criterion at certain values of the parameters. However, although the gamma model is used in many applications, but it has no considerable attention for optimal designs. Geometric approaches were employed to derive locally D-optimal designs for a gamma model with single factor (see Ford et al., 1992), with two factors without intercept (see Burridge and Sebastiani, 1992) and for multiple factors (see Burridge and Sebastiani, 1994). Some of those results were highlighted on by Atkinson and Woods (2015). Recently, in Gaite et al., 2018 we provided analytic solutions for optimal designs for gamma models. A complete class and essentially complete class of designs were established under certain assumptions. Therefore, the complexity of deriving optimal designs is reduced and one can only look for the optimal design in those classes.

In the present paper gamma models without intercept are considered. Absence of the intercept term yields a difficulty in deriving D- and A-optimal designs. Our main goal is developing various approaches to obtain, mostly, locally D-optimal designs. This paper is organized as follows. In section 2, the proposed model, the information matrix and the locally optimal design are presented. In section 3, locally D- and A-optimal designs are derived. In section 4, a two-factor model with interaction is considered for which locally D-optimal designs are derived. The performance of some derived D-optimal designs is examined in Section 5. Finally, a brief discussion and conclusions are given in Section 6.

2. Model, information and designs

Let \( y_1, ..., y_n \) be independent gamma-distributed response variables for \( n \) experimental units, where the density is given by

\[
P(y_i; \kappa, \lambda) = \frac{y_i^{\kappa-1} e^{-\lambda y_i}}{\Gamma(\kappa)} , \kappa, \lambda, y_i > 0 , \quad (1 \leq i \leq n),
\]

The shape parameter \( \kappa \) of the gamma distribution is the same for all \( y_i \) but the expectations \( \mu_i = E(y_i) \) depend on the values \( x_i \) of a covariate \( x \). The canonical link obtained from a gamma distribution (2.1) is reciprocal (inverse)

\[
\eta_i = \kappa/\mu_i, \quad \text{where} \quad \eta_i = f^T(x_i)\beta, \quad (1 \leq i \leq n),
\]

where \( f = (f_1, ..., f_p)^T \) is a given \( \mathbb{R}^p \)-valued function on the experimental region \( X \subset \mathbb{R}^p, \nu \geq 1 \) with linearly independent component functions \( f_1, ..., f_p \), and \( \beta \in \mathbb{R}^p \) is a parameter vector (see McCullagh and Nelder, 1989, Section 2.2.4). Here, the mean-variance function is \( \nu(\mu) = \mu^2 \) and the variance of a gamma distribution is thus given by \( \text{var}(y) = \kappa^{-1}\mu^2 \) with shape parameter \( \kappa > 0 \). Therefore, the intensity function at a point \( x \in X \) (see Atkinson and Woods, 2015) is given by

\[
u(x, \beta) = \left( \text{var}(y) \frac{dn^\eta}{d\mu} \right)^{-1} = \kappa (f^T(x)\beta)^{-2}.
\]

Practically, there are various link functions that are considered to fit gamma observations. The power link family which is considered throughout presents the class of link functions as in Burridge and Sebastiani (1994), see also Atkinson and Woods (2015), Section 2.5,

\[
\eta_i = \mu_i^\rho, \quad \text{where} \quad \eta_i = f^T(x_i)\beta, \quad (1 \leq i \leq n).
\]

The exponent \( \rho \) of the power link function is a given nonzero real number. The intensity function under that family reads as

\[
u_0(x, \beta) = \kappa \rho^{-2} (f^T(x)\beta)^{-2}.
\]

Gamma-distributed responses are continuous and non-negative and therefore for a given experimental region \( X \) we assume throughout that the parameter vector \( \beta \) satisfies

\[
f^T(x)\beta > 0 \quad \text{for all} \quad x \in X.
\]

The Fisher information matrix for a single observation at a point \( x \in X \) under parameter vector \( \beta \) is given by \( \nu_0(x, \beta) f(x) f^T(x) \). Note that the positive factor \( \kappa \rho^{-2} \) is the same for all \( x \) and \( \beta \) and will not affect any design
We shall make use of approximate designs with finite support on the experimental region \( X \). The approximate design \( \xi \) on \( X \) is represented as
\[
\xi = \left\{ \begin{array}{cccc}
x_1 & x_2 & \cdots & x_m \\
\omega_1 & \omega_2 & \cdots & \omega_m 
\end{array} \right\},
\] (2.7)
where \( m \in \mathbb{N} \), \( x_1, x_2, \ldots, x_m \in X \) are pairwise distinct points and \( \omega_1, \omega_2, \ldots, \omega_m > 0 \) with \( \sum_{i=1}^{m} \omega_i = 1 \). The set \( \text{supp}(\xi) = \{ x_1, x_2, \ldots, x_m \} \) is called the support of \( \xi \) and \( \omega_1, \ldots, \omega_m \) are called the weights of \( \xi \) (see [2]). A design \( \xi \) is minimally supported if the number of support points is equal to the number of model parameters (i.e., \( m = p \)). A minimal-support design which is also called a saturated design will appear frequently in the current work. The information matrix of a design \( \xi \) at a parameter point \( \beta \) is defined by
\[
M(\xi; \beta) = \sum_{i=1}^{m} \omega_i M(x_i, \beta).
\] (2.8)

Another representation of the information matrix (2.8) can be considered by defining the \( m \times p \) design matrix \( F = [f(x_1), \ldots, f(x_m)]^T \) and the \( m \times m \) weight matrix \( V = \text{diag}(\omega_1, \omega_2, \ldots, \omega_m) \) and hence, \( M(\xi; \beta) = F^T VF \).

A locally optimal design minimizes a convex criterion function of the information matrix at a given parameter point \( \beta \). Denote by "det" and "tr" the determinant and the trace of a matrix, respectively. We will employ the popular D-criterion and the A-criterion. More precisely, a design \( \xi^* \) is said to be locally D-optimal (at \( \beta \)) if its information matrix \( M(\xi^*, \beta) \) at \( \beta \) is nonsingular and \( \det(M^{-1}(\xi^*, \beta)) = \min_{\xi} \det(M^{-1}(\xi, \beta)) \) where the minimum on the r.h.s. is taken over all designs \( \xi \) whose information matrix at \( \beta \) is nonsingular. Similarly, a design \( \xi^* \) is said to be locally A-optimal (at \( \beta \)) if its information matrix at \( \beta \) is nonsingular and \( \text{tr}(M^{-1}(\xi^*, \beta)) = \min_{\xi} \text{tr}(M^{-1}(\xi, \beta)) \) where, again, the minimum is taken over all designs \( \xi \) whose information matrix at \( \beta \) is nonsingular.

Remark 1. It is worthwhile mentioning that the set of designs for which the information matrix is nonsingular does not depend on \( \beta \) (when \( u(x, \beta) \) is strictly positive). In particular it is just the set of designs for which the information matrix is nonsingular in the corresponding ordinary regression model (ignoring the intensity \( u(x, \beta) \)). That is the singularity depends on the support points of a design \( \xi \) because its information matrix \( M(\xi; \beta) = F^T VF \) is full rank if and only if \( F \) is full rank.

Remark 2. If the experimental region is a compact set and the functions \( f(x) \) and \( u(x, \beta) \) are continuous in \( x \) then the set of all nonnegative definite information matrices is compact. Therefore, there exists a locally D- resp. A-optimal design for any given parameter point \( \beta \).

In order to verify the local optimality of a design The General Equivalence Theorem is usually employed. It provides necessary and sufficient conditions for a design to be optimal with respect to the optimality criterion, in specific D- and A-criteria and thus the optimality of a suggested design can easily be verified or disproved (see [3]). The most generic one is the celebrated Kiefer-Wolfowitz equivalence theorem under D-criterion (see [4]). In the following we obtain equivalent characterizations of locally D- and A-optimal designs.

**Theorem 2.1.** Let \( \beta \) be a given parameter point and let \( \xi^* \) be a design with nonsingular information matrix \( M(\xi^*, \beta) \).

(a) The design \( \xi^* \) is locally D-optimal (at \( \beta \)) if and only if
\[
u(x, \beta) f^T(x) M^{-1}(\xi^*, \beta) f(x) \leq p \quad \text{for all } x \in X.
\]
(b) The design \( \xi^* \) is locally A-optimal (at \( \beta \)) if and only if
\[
u(x, \beta) f^T(x) M^{-2}(\xi^*, \beta) f(x) \leq \text{tr}(M^{-1}(\xi^*, \beta)) \quad \text{for all } x \in X.
\]

**Remark 3.** The inequalities given by part (a) and part (b) of Theorem 2.1 are equations at support points of any D- or A-optimal design, respectively.
Throughout, we consider gamma models that do not explicitly involve a constant (intercept) term. More precisely, we assume that \( f_j \neq 1 \) for all \((1 \leq j \leq p)\) and thus \( f_j(\theta) = 0 \) for all \((1 \leq j \leq p)\). In particular, we restrict to a first order model with
\[
f(x) = x, \quad \text{where } x = (x_1, \ldots, x_v)^T, \quad v \geq 2,
\]
and the two-factor model with interaction
\[
f(x) = (x_1, x_2, x_1x_2)^T.
\]
Surely, condition \( \text{2.5} \), i.e., \( f^T(x)\beta > 0 \) for all \( x \in X \) implies that \( 0 \notin X \). Therefore, an experimental region as \( X = [0, \infty)^v \setminus \{0\} \) is considered. Note that this experimental region is no longer compact therefore the existence of optimal designs is not assured and has to be checked separately.

In contrast, we often consider a compact experimental region that is a \( v \)-dimensional hypercube
\[
X = [a, b]^v, \quad v \geq 2 \text{ with } a, b \in \mathbb{R} \text{ and } 0 < a < b,
\]
with vertices \( v_i, i = 1, \ldots, K, \ K = 2^v \) given by the points whose \( i \)-th coordinates are either \( a \) or \( b \) for all \( i = 1, \ldots, v \).

In [Gaffke et al. (2018)], we showed that under gamma models with regression function \( f(x) \) from (2.9) or (2.10) and experimental region \( X = [a, b]^v, v \geq 2, 0 < a < b \) the design that has support only among the vertices is at least as good as any design that has no support points from the vertices w.r.t the Loewner semi-ordering of information matrices or, more generally, of nonnegative definite \( p \times p \) matrices. That is if \( A \) and \( B \) are nonnegative definite \( p \times p \) matrices we write \( A \leq B \) if and only if \( B - A \) is nonnegative definite. The set of all designs \( \xi \) such that supp\( (\xi) \subseteq \{ v_1, \ldots, v_K \} \) is a locally essentially complete class of designs at a given \( B \). As a result, there exists a design \( \xi^* \) that is only supported by vertices of \( X \) which is locally optimal (at \( B \)) w.r.t. D- or A-criterion. On that basis, throughout, we restrict to designs whose support is a subset of the vertices of \( X \) given by a hypercube (2.11).

Remark 4. Let us denote by \( \psi(x) \) the left hand side of The Equivalence Theorems, Theorem 2.1 part (a) or part (b). Typically \( \psi(x) \) is called the sensitivity function. Actually, under non-intercept gamma models \( \psi(x) \) is invariant with respect to simultaneous scale transformation of \( x \), i.e., \( \psi(\lambda x) = \psi(x) \) for any \( \lambda > 0 \). This essentially comes from the fact that the function \( f_\beta(x) = (f^T(x)\beta)^{-1}f(x) \) is invariant with respect to simultaneous rescaling of the components of \( x \), i.e., \( f_\beta(\lambda x) = f_\beta(x) \). This property is explicitly transferred to the information matrix (2.6) since it can be represented in form \( M(x, \beta) = f_\beta(x)f_\beta^T(x) \), and hence \( M(\lambda x, \beta) = M(x, \beta) \). In fact, this property plays a main role in the solution of the forthcoming optimal designs.

3. First order gamma model

In this section we consider a gamma model with
\[
f(x) = (x_1, \ldots, x_v)^T, \quad v \geq 2, \quad x \in X,
\]
where
\[
f_\beta(x) = \frac{1}{\beta_1 x_1 + \cdots + \beta_v x_v} \begin{pmatrix} x_1 \\ \vdots \\ x_v \end{pmatrix}.
\]
Firstly let the experimental region \( X = [0, \infty)^v \setminus \{0\} \) be considered. Denote by \( e_i \) for all \((1 \leq i \leq v)\) the \( v \)-dimensional unit vectors. The parameter space is determined by condition \( \text{2.5} \), i.e., \( x^T\beta > 0 \) for all \( x \in X \) which implies that \( \beta \in (0, \infty)^v \), i.e., \( \beta_i > 0 \) for all \((1 \leq i \leq v)\). Let the induced experimental region is given by \( f_\beta(X) = \{ f_\beta(x) : x \in X \} \).

Although \( X \) is not compact but \( f_\beta(X) \) is compact. That is
\[
f_\beta(X) = \text{Conv}(f_\beta(e_i) : e_i \in X, i = 1, \ldots, v),
\]
where Conv denotes convex hull operation. That means each point \( f_\beta(x) \) for all \( x \in X \) can be written as a convex combination of \( f_\beta(e_i) \) for all \((1 \leq i \leq v)\), i.e., we obtain \( f_\beta(x) = \sum_{i=1}^v \alpha_i f_\beta(e_i) \) for some \( \alpha_i > 0 \) for all \((1 \leq i \leq v)\) such that \( \sum_{i=1}^v \alpha_i = 1 \). As a consequence, the set of all nonnegative definite information matrices is compact and a locally optimal design can be obtained (cp. Remark 2).
Theorem 3.1. Consider the experimental region $X = [0, \infty)^v \setminus \{0\}$. Let $x^*_i = e_i$ for all $(1 \leq i \leq v)$. Given a parameter point $\beta$. Then

(i) The saturated design $\xi^*$ that assigns equal weight $v^{-1}$ to the support $x^*_i$ for all $(1 \leq i \leq v)$ is locally D-optimal (at $\beta$).

(ii) The saturated design $\xi^*$ that assigns the weights $\omega^*_i = \beta_i / \sum_{i=1}^{v} \beta_i$ for all $(1 \leq i \leq v)$ to the corresponding design point $x^*_i$ for all $(1 \leq i \leq v)$ is locally A-optimal (at $\beta$).

Proof. Define the $v \times v$ design matrix $F = \text{diag}(e_i)_{i=1}^{v}$ with $v \times v$ weight matrix $V = \text{diag}(\omega^*_i / \beta^2_i)_{i=1}^{v}$. Then we have $F^TVF = \text{diag}(\omega^*_i / \beta^2_i)_{i=1}^{v}$ and $(F^TVF)^{-1} = \text{diag}(\beta^2_i / \omega^*_i)_{i=1}^{v}$ where:

For D-optimality, $\omega^*_i = v^{-1} \forall i$, $M^{-1}(\xi^*, \beta) = v \text{diag}(\beta^2_i)_{i=1}^{v}$ and $f^T(x)\text{diag}(\beta^2_i)_{i=1}^{v}f(x) = \sum_{i=1}^{v} \beta^2_i x^2_i$.

For A-optimality, $\omega^*_i = \beta_i / \sum_{i=1}^{v} \beta_i \forall i$, $M^{-1}(\xi^*, \beta) = \left( \sum_{i=1}^{v} \beta_i \right) \text{diag}(\beta^2_i)_{i=1}^{v}$, $\text{Tr}(M^{-1}(\xi^*, \beta)) = \left( \sum_{i=1}^{v} \beta_i \right)^2$, $M^{-2}(\xi^*, \beta) = \left( \sum_{i=1}^{v} \beta_i \right)^2 \text{diag}(\beta^2_i)_{i=1}^{v}$, and $f^T(x)\left( \sum_{i=1}^{v} \beta_i \right)^2 \text{diag}(\beta^2_i)_{i=1}^{v}f(x) = \left( \sum_{i=1}^{v} \beta_i \right)^2 \sum_{i=1}^{v} \beta^2_i x^2_i$.

Hence, by The Equivalence Theorem (Theorem 2.1, part (a) and part (b)) $\xi^*$ resp. $\xi^*$ is locally D- resp. A-optimal (at $\beta$) if and only if $\left( \sum_{i=1}^{v} \beta_i x_i \right)^2 \sum_{i=1}^{v} \beta^2_i x^2_i \leq 1$ for all $x \in X$ which is equivalent to $-2 \sum_{i,j=1}^{v} \beta_i \beta_j x_i x_j \leq 0$ for all $x \in X$. The latter inequality holds true by model assumptions $\beta_i > 0, x_i \geq 0$ for all $(1 \leq i \leq v)$.

**Remark 5.** The locally D-optimal designs provided by part (i) of Theorem 3.1 is robust against misspecified values of the model parameter in its parameter space $(0, \infty)^v$.

While the information matrix is invariant w.r.t. to simultaneous rescaling of the components of $x$ as it is mentioned in Remark 4 the result of Theorem 3.1 can be extended:

Corollary 3.1. Consider the experimental region $X = [0, \infty)^v \setminus \{0\}$. Given a constant vector $a = (a_1, \ldots, a_v)^T$ such that $a_i > 0$ and $a_ie_i \in X$ for all $(1 \leq i \leq v)$. Let $x^*_i = a_ie_i$ for all $(1 \leq i \leq v)$. Given a parameter point $\beta$. Then

(i) The saturated design $\xi_a^*$ that assigns equal weight $v^{-1}$ to the support $x^*_i$ (1 $\leq i \leq v$) is locally D-optimal (at $\beta$).

(ii) The saturated design $\xi_a^*$ that assigns the weights $\omega^*_i = \beta_i / \sum_{i=1}^{v} \beta_i$ for all $(1 \leq i \leq v)$ to the corresponding design point $x^*_i$ for all $(1 \leq i \leq v)$ is locally A-optimal (at $\beta$).

Actually, the derived optimal designs $\xi_a^*$ and $\xi_a^*$ are not unique at a given parameter point $\beta$. The convex combinations of locally optimal designs is optimal w.r.t. D- or A-criterion. In the following we introduce a set of locally D-optimal designs and a set of locally A-optimal designs.

Corollary 3.2. Under assumptions of Corollary 3.1. Let $\xi_a^*$ and $\xi_a^*$ are locally D- and A-optimal design at $\beta$, respectively. Let

\[
\Xi^* = \text{Conv}\{\xi_a^* : a = (a_1, \ldots, a_v)^T, a_i > 0 \forall i = 1, \ldots, v\};
\]

\[
\Xi^* = \text{Conv}\{\xi_a^* : a = (a_1, \ldots, a_v)^T, a_i > 0 \forall i = 1, \ldots, v\}.
\]

Then $\Xi^*$ is a set of locally D-optimal designs (at $\beta$) and $\Xi^*$ is a set of locally A-optimal designs (at $\beta$).

In what follows we consider a hypercube $X = [a, b]^v, v \geq 2, 0 < a < b$, as an experimental region. As given in Remark 4 we have $f_{\beta}^1(x) = f_{\beta}^1(x)$, $\lambda > 0$ and thus a transformation of a gamma model without intercept to a gamma model with intercept can be obtained if, in particular, $\lambda = x_1^{-1}$, $x_1 > 0$. This reduction is useful to determine precisely the candidate support points of a design. Another reduction might be obtained on the parameter space when, in particular, $\lambda = \beta^{-1}_1$.

Let us begin with the simplest case $v = 2$. A transformation of a two-factor model without intercept to a single-factor model with intercept is employed. Based on that D- and A-optimal designs are derived.
Theorem 3.2. Consider the experimental region $X = [a, b]^2$, $0 < a < b$. Let $x^*_1 = (a, b)^T$ and $x^*_2 = (b, a)^T$. Let $\beta = (\beta_1, \beta_2)^T$ be given such that $\beta^T x^*_i > 0$ for all $i = 1, 2$ (which is equivalent to condition $\text{(2.5)}$). Then, the unique locally $D$-optimal design $\xi^*_D(\beta)$ is the two-point design supported by $x^*_1$ and $x^*_2$ with equal weights 1/2. The unique locally $A$-optimal design $\xi^*_A(\beta)$ is the two-point design supported by $x^*_1$ and $x^*_2$ with weights $\omega^*_1 = \frac{\beta_1 + \beta_2}{(\beta_1 + \beta_2)(a + b)}$ and $\omega^*_2 = \frac{\beta_2 b + \beta_1 a}{(\beta_1 + \beta_2)(a + b)}$.

**Proof.** Since $f_\beta(x^{-1}_1 x) = f_\beta(x)$ for all $x = (x_1, x_2) \in [a, b]^2$, we write

$$
f_\beta(x) = (\beta_1 x_1 + \beta_2 x_2)^{-1} (x_1, x_2)^T = (\beta_1 + \beta_2 t)^{-1} (1, t)^T,
$$

where $t = t(x) = x_2/x_1$.

So the information matrices coincide with those from a single-factor gamma model with intercept. The range of $t = t(x)$, as $x$ ranges over $[a, b]^2$ is the interval $[(a/b), (b/a)]$. Note also that the end points $a/b$ and $b/a$ come from the unique points $x^*_1 = (a, b)^T$ and $x^*_2 = (b, a)^T$, respectively. Following the proof of Theorem 4.1 in [Gaffke et al. (2018)] yields the stated results on the locally $D$- and $A$-optimal designs in our theorem, where for local $A$-optimality we get

$$
\omega^*_1 = \frac{(\beta_1 + \beta_2) \sqrt{1 + (\frac{t}{\beta_1 + \beta_2})^2}}{(\beta_1 + \beta_2) \sqrt{1 + (\frac{t}{\beta_1 + \beta_2})^2} + (\beta_1 + \beta_2) \sqrt{1 + (\frac{t}{\beta_1 + \beta_2})^2}}
$$

and it is straightforwardly to verify that the above quantity is equal to $\frac{\beta_2 b + \beta_1 a}{(\beta_1 + \beta_2)(a + b)}$. \qed

**Remark 6.** Actually, in case $v \geq 3$ an analogous transformation of the model as in the proof of Theorem 3.2 is obvious,

$$
f_\beta(x) = (\beta_1 + \beta_2 t_1 + \beta_3 t_2 + \ldots + \beta_v t_{v-1})^{-1} (1, t_1, \ldots, t_{v-1})^T,
$$

where $t_j = t_j(x) = x_{j+1}/x_1$ for all $1 \leq j \leq v - 1$ for $x = (x_1, x_2, \ldots, x_v)^T \in [a, b]^v$, $0 < a < b$.

leading thus to a first order model with intercept employing a $(v - 1)$-dimensional factor $t = (t_1, \ldots, t_{v-1})^T$. However, its range $\{t(x) : x \in [a, b]^v\} \subseteq \mathbb{R}^{v-1}$ is not a cube but a more complicated polytope. E.g., for $v = 3$ it can be shown that

$$
\{t(x) : x \in [a, b]^3\} = \text{Conv}\left\{ \left(\begin{array}{c} a/b \\ 1 \end{array}\right), \left(\begin{array}{c} 1 \\ a/b \end{array}\right), \left(\begin{array}{c} a/b \\ a/b \end{array}\right), \left(\begin{array}{c} b/a \\ 1 \end{array}\right), \left(\begin{array}{c} 1 \\ b/a \end{array}\right), \left(\begin{array}{c} b/a \\ b/a \end{array}\right) \right\}
$$

where for each $x \in [a, b]^3$ we get $t(x) = (x_2/x_1, x_3/x_1)^T$ as it is depicted in Figure 1 for, in specific, $a = 1$ and $b = 2$. In [Gaffke et al. (2018)] we showed that the support of a design is a subset of vertices of the plyotope. One notes that for each vertex $v \in \{(a, a, a)^T, (b, b, b)^T\}$ we get $t(v) = (1, 1)^T$ which lies in the interior of the convex hull above, i.e., $(1, 1)^T$ is a proper convex combination of the vertices of the plyotope. Thus this reduction on the vertices implies that both vertices $(a, a, a)^T$ and $(b, b, b)^T$ of the hypercube $[a, b]^3$ are out of consideration as support points of any optimal design.
Let us concentrate on the experimental region $X = [1, 2]^3$. The linear predictor of a three-factor gamma model is given by $\eta(x, \beta) = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$. Assume that $\beta_2 = \beta_3 = \beta$, so the set of all parameter points under condition (2.5), i.e., $\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 > 0$ for all $x = (x_1, x_2, x_3)^T \in X$ is characterized by

$$\beta_1 \leq 0, \beta > -\beta_1 \text{ or } \beta_1 > 0, \beta > -\frac{1}{4} \beta_1$$

which is shown by Panel (a) of Figure 2.

Let the vertices of $X = [1, 2]^3$ be denoted by $v_1 = (1, 1, 1)^T$, $v_2 = (2, 1, 1)^T$, $v_3 = (1, 2, 1)^T$, $v_4 = (1, 1, 2)^T$, $v_5 = (1, 2, 2)^T$, $v_6 = (2, 1, 2)^T$, $v_7 = (2, 2, 1)^T$, $v_8 = (2, 2, 2)^T$ with intensities $u_i = u(v_i, \beta), i = 1, \ldots, 8$. Actually, the region shown in Figure 2 is the parameter space of $\beta = (\beta_1, \beta_2, \beta_3)^T$ only when $\beta_2 = \beta_3$. We aim at finding locally D-optimal designs at a given parameter point in that space. The expression “optimality subregion” will be used to refer to a subset of parameter points where saturated designs or non-saturated designs with similar support are locally D-optimal.

In the next theorem we introduce the locally D-optimal designs on respective optimality subregions. Table 1 below presents the order of the intensities in all optimality subregions and the corresponding D-optimal designs. The intensities for both vertices $v_1$ and $v_8$ are ignored due to the reduction (cp. Remark 6). It is noted that on each subregion the vertices of highest intensities perform mostly as a support of the corresponding D-optimal design. In particular, analytic solution of the locally D-optimal designs of type $\xi^*_p$ at a point $\beta$ from the subregion $-3\beta_1 < \beta < -\frac{5}{7} \beta_1$, $\beta_1 < 0$ cannot be developed so that numerical results are to be derived (cp. Remark 7).

| Subregions | Intensities order | D-optimal design |
|------------|-------------------|------------------|
| $\beta > 0, \beta_1 = 0$ | $u_2 > u_3 > u_4 > u_6 = u_7 > u_5$ | $\xi^*_1$ |
| $\beta \geq -3\beta_1, \beta_1 < 0$ | $u_2 > u_6 = u_7 \approx u_3 > u_4 > u_5$ | $\xi^*_1$ |
| $\beta > \frac{1}{2} \beta_1, \beta_1 > 0$ | $u_2 > u_3 = u_4 > u_6 = u_7 > u_5$ | $\xi^*_1$ |
| $-\frac{5}{7} \beta_1 < \beta \leq -\frac{3}{2} \beta_1, \beta_1 > 0$ | $u_5 > u_3 = u_4 > u_6 = u_7 > u_2$ | $\xi^*_2$ |
| $-\frac{5}{7} \beta_1 < \beta < \frac{1}{2} \beta_1, \beta_1 > 0$ | $u_3 = u_4 \approx u_5 > u_2 \approx u_6 = u_7$ | $\xi^*_3$ |
| $-\beta_1 < \beta \leq -\frac{5}{7} \beta_1, \beta_1 < 0$ | $u_2 > u_6 = u_7 > u_3 > u_4 > u_5$ | $\xi^*_4$ |
| $-3\beta_1 < \beta < -\frac{6}{7} \beta_1, \beta_1 < 0$ | $u_2 > u_6 > u_7 > u_3 > u_4 > u_5$ | $\xi^*_5$ |

Table 1: The order of intensity values according to subregions correspond to D-optimal designs.
Figure 2: Panel (a): The parameter space of \( \beta = (\beta_1, \beta_2, \beta_3)^T \) such that \( \beta_2 = \beta_3 = \beta \).
Panel (b): Dependence of locally D-optimal designs from Theorem 3.3 on \( \beta = (\beta_1, \beta_2, \beta_3)^T \) such that \( \beta_2 = \beta_3 = \beta \). The dashed lines are; diagonal: \( \beta = \beta_1 \), vertical: \( \beta_1 = 0 \), horizontal: \( \beta = 0 \).

**Theorem 3.3.** Consider the experimental region \( X = [1, 2]^3 \). Let a parameter point \( \beta = (\beta_1, \beta_2, \beta_3)^T \) be given such that \( \beta_2 = \beta_3 = \beta \) with \( \beta > -\beta_1 \) if \( \beta_1 \leq 0 \) or \( \beta > -\frac{1}{3} \beta_1 \) if \( \beta_1 > 0 \). Then the following designs are locally D-optimal (at \( \beta \)).

(i) If \( \beta > 0, \beta_1 = 0 \) or \( \beta \geq -3 \beta_1, \beta_1 < 0 \) or \( \beta > \frac{1}{3} \beta_1, \beta_1 > 0 \) then

\[
\xi^*_1 = \begin{pmatrix} v_2 & v_3 & v_4 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.
\]

(ii) If \( -\frac{1}{4} \beta_1 < \beta \leq -\frac{2}{3} \beta_1, \beta_1 > 0 \) then

\[
\xi^*_2 = \begin{pmatrix} v_3 & v_4 & v_5 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.
\]

(iii) If \( -\frac{5}{23} \beta_1 < \beta < \frac{1}{3} \beta_1, \beta_1 > 0 \) then

\[
\xi^*_3 = \begin{pmatrix} v_2 & v_3 & v_4 & v_5 \\ \omega_1^* & \omega_2^* & \omega_3^* & \omega_4^* \end{pmatrix},
\]

where

\[
\omega_1^* = \frac{5 + 23 \gamma}{16 (1 + 4 \gamma)}, \quad \omega_2^* = \omega_3^* = \frac{9 (1 + 3 \gamma)^2}{32 (1 + \gamma)(1 + 4 \gamma)}, \quad \omega_4^* = \frac{1 - \gamma - 20 \gamma^2}{8 (1 + \gamma)(1 + 4 \gamma)}, \quad \gamma = \frac{\beta}{\beta_1}.
\]

(iv) If \( -\beta_1 < \beta \leq -\frac{6}{5} \beta_1, \beta_1 < 0 \) then

\[
\xi^*_4 = \begin{pmatrix} v_2 & v_6 & v_7 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.
\]

**Proof.** The proof is obtained by making use of the condition of the Equivalence Theorem (Theorem 2.1, part (a)). So that we develop a system of feasible inequalities evaluated at the vertices \( v_i \) for all \( 1 \leq i \leq 8 \). For simplicity
in computations when \( \beta_1 \neq 0 \) we utilize the ratio \( \gamma = \beta / \beta_1 \) of which the range is given by \(( -\infty, -1) \cup (-\frac{1}{3}, \infty)\). It turned out that some inequalities are equivalent and thus a resulted system is reduced to an equivalent system of a few inequalities. The intersection of the set of solutions of each system with the range of \( \gamma \) leads to the optimality condition (subregion) of the corresponding optimal design. Note that \( u_1 = \beta_1^{-2}(1 + 2\gamma)^{-2}, u_2 = \beta_1^{-2}(2 + 2\gamma)^{-2}, u_3 = u_4 = \beta_1^{-2}(1 + 3\gamma)^{-2}, u_5 = \beta_1^{-2}(1 + 4\gamma)^{-2}, u_6 = u_7 = \beta_1^{-2}(2 + 3\gamma)^{-2}, u_8 = \beta_1^{-2}(2 + 4\gamma)^{-2} \).

(i) The \(3 \times 3\) design matrix \( F = [v_2, v_3, v_4]^T \) is given by

\[
F = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}
\]

with \( F^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \) and weight matrix \( V = \text{diag}(u_2, u_3, u_4) \).

Hence, the condition of The Equivalence Theorem is given by

\[
f^T(x)F^{-1}V^{-1}(F^T)^{-1}f(x) \leq (\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)^2 \quad \forall x \in \{1, 2\}^3.
\]

For case \( \beta > 0, \beta_1 = 0 \), condition (3.3) is equivalent to

\[
4(3x_1 - (x_2 + x_3))^2 + 9(3x_2 - (x_1 + x_3))^2 + (3x_3 - (x_1 + x_2))^2 \leq 16(x_2 + x_3)^2 \quad \forall x \in \{1, 2\}^3,
\]

which is independent of \( \beta \) and is satisfied by \( v_i \) for all \(1 \leq i \leq 8\) with equality holds for the support. For the other cases condition (3.3) is equivalent to

\[
(3x_1 - (x_2 + x_3))^2(2 + 2\gamma)^2 + (3x_2 - (x_1 + x_3))^2(1 + 3\gamma)^2 \leq 16(x_1 + \gamma(x_2 + x_3))^2 \quad \forall x \in \{1, 2\}^3.
\]

After some lengthy but straightforward calculations, the above inequalities reduce to

\[
15\gamma^2 + 2\gamma - 1 \geq 0 \quad \text{for } v_5
\]

\[
3\gamma^2 + 10\gamma + 3 \geq 0 \quad \text{for } v_6 \text{ or } v_7
\]

The l.h.s. of each of (3.5) and (3.6) above is a polynomial in \( \gamma \) of degree 2 and thus the sets of solutions are given by \((-\infty, -\frac{1}{3}] \cup \left[\frac{1}{3}, \infty\right)\) and \((-\infty, -\frac{1}{3}] \cup \left[-\frac{1}{9}, \infty\right)\), respectively. Note that the interior bounds are the roots of the respective polynomials. Hence, by considering the intersection of both sets with the range of \( \gamma \), the design \( \xi_5^* \) is locally D-optimal if \( \gamma \in (-\infty, -\frac{1}{3}] \cup \left[\frac{1}{2}, \infty\right) \) which is equivalent to the optimality subregion \( \beta \geq -3\beta_1, \beta_1 < 0 \) or \( \beta > \frac{1}{2}\beta_1, \beta_1 > 0 \) given in part (i) of the theorem.

(ii) The \(3 \times 3\) design matrix \( F = [v_3, v_4, v_5]^T \) is given by

\[
F = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}
\]

with \( F^{-1} = \begin{pmatrix} 2 & 2 & -3 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \) and weight matrix \( V = \text{diag}(u_3, u_4, u_5) \).

Hence, the condition of The Equivalence Theorem is equivalent to

\[
((2x_1 - x_2)^2 + (2x_1 - x_3)^2)(1 + 3\gamma)^2 + (x_3 - x_2 - 3x_1)^2(1 + 4\gamma)^2 \leq (x_1 + \gamma(x_2 + x_3))^2 \quad \forall x \in \{1, 2\}^3,
\]

and also the above inequalities reduce to

\[
69\gamma^2 + 38\gamma + 5 \leq 0 \quad \text{for } v_2.
\]

Again, the set of solutions of the polynomial determined by the l.h.s. of inequality (3.7) is given by \([-\frac{1}{3}, -\frac{23}{5}]\). By considering the intersection with the range of \( \gamma \), the design \( \xi_6^* \) is locally D-optimal if \( \gamma \in \left(-\frac{1}{4}, -\frac{5}{21}\right] \).
(iii) Consider design $\xi^n_3$. Note that $\omega^n_1 > 0$ for all $\gamma > -5/23$, $\omega^n_2 > 0$ for all $\gamma \in \mathbb{R}$ and $\omega^n_3 > 0$ for all $\gamma \in (-\frac{1}{4}, \frac{1}{2})$, and thus it is obvious that $\omega^n_1, \omega^n_2, \omega^n_3$ are positive over $(-\frac{5}{23}, \frac{1}{2})$ and $\sum_{i=1}^3 \omega^n_i = 1$. The $4 \times 3$ design matrix is given by $F = [v_2, v_3, v_4, v_5]^T$ with weight matrix $V = \text{diag}(s_2, s_3, s_4, s_5)$ where $s_i = \omega^n_i u_i, i = 2, 3, 4, 5$ and $s_3 = s_4$. The information matrix is given by

$$M(\xi^n_3, \beta) = \begin{pmatrix} 4s_2 + 2s_3 + s_5 & 2s_2 + 3s_3 + 2s_5 & 2s_2 + 3s_3 + 2s_5 \\ 2s_2 + 3s_3 + 2s_5 & s_2 + 5s_3 + 4s_5 & s_2 + 4s_3 + 4s_5 \\ 2s_2 + 3s_3 + 2s_5 & s_2 + 4s_3 + 4s_4 & s_2 + 5s_3 + 4s_5 \end{pmatrix}$$

and one calculates $\det M(\xi^n_3, \beta) = 16s_2 s_3^2 + 18s_2 s_3 s_5 + s_3^2 s_5$. Define the following quantities

$$c_1 = \frac{s_3(2s_2 + 9s_3 + 8s_5)}{16s_2 s_3^2 + 18s_2 s_3 s_5 + s_3^2 s_5}, \quad c_2 = \frac{-s_3(2s_2 + 3s_3 + 2s_5)}{16s_2 s_3^2 + 18s_2 s_3 s_5 + s_3^2 s_5},$$

$$c_3 = \frac{10s_2 s_3 + 9s_2 s_5 + s_3^2 + 3s_5}{16s_2 s_3^2 + 18s_2 s_3 s_5 + s_3^2 s_5}, \quad c_4 = \frac{-6s_2 s_3 + 9s_2 s_5 - s_3^2}{16s_2 s_3^2 + 18s_2 s_3 s_5 + s_3^2 s_5}.$$

The inverse of the information matrix is given by

$$M^{-1}(\xi^n_3, \beta) = \begin{pmatrix} c_1 & c_2 & c_2 \\ c_2 & c_3 & c_4 \\ c_2 & c_4 & c_3 \end{pmatrix}.$$ Hence, the condition of The Equivalence Theorem is equivalent to

$$c_1 x_1^2 + c_3 (x_2^2 + x_3^2) + 2c_2 (x_1 x_2 + x_1 x_3) + 2c_4 x_2 x_3 \leq 3 (x_1 + \gamma (x_2 + x_3))^2 \forall x \in \{1, 2\}^3$$

which is equivalent to the following system of inequalities

$$c_1 + 4c_2 + 2c_3 + 2c_4 \leq 3 (1 + 2\gamma)^2 \text{ for } v_1 \text{ or } v_8$$

$$4c_1 + 12c_2 + 5c_3 + 4c_4 \leq 3 (2 + 3\gamma)^2 \text{ for } v_6 \text{ or } v_7$$

However, due to the complexity of the system above we employed computer algebra using Wolfram Mathematica 11.3 (see Wolfram Research) to obtain the solution for $\gamma$.

(iv) The $3 \times 3$ design matrix $F = [v_2, v_6, v_7]$ is given by

$$F = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{with} \quad F^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & 1 & 0 \end{pmatrix} \quad \text{and weight matrix} \quad V = \text{diag}(u_2, u_6, u_7).$$

Hence, the condition of The Equivalence Theorem is equivalent to

$$\left(\frac{(x_2 - \frac{x_1}{2})^2}{2} + \frac{(x_3 - \frac{x_1}{2})^2}{2} + \frac{3x_1}{2} - x_2 - x_3\right)^2 (2 + 3\gamma)^2 \leq (x_1 + \gamma (x_2 + x_3))^2 \forall x \in \{1, 2\}^3,$$

and the above inequalities reduce to

$$90\gamma^2 + 168\gamma + 72 \leq 0 \text{ for } v_3 \text{ or } v_4 \quad (3.8)$$

$$6\gamma^2 + 16\gamma + 8 \leq 0 \text{ for } v_8 \quad (3.9)$$

In analogy to parts (i) and (ii) the sets of solutions of (3.8) and (3.9) are given by $[-1.2, -\frac{2}{3}]$ and $[-2, -\frac{2}{3}]$, respectively where the interior bounds are the roots of the respective polynomials. Hence, by considering the intersection of both sets with the range of $\gamma$, the design $\xi^n_3$ is locally D-optimal if $\gamma \in [-1.2, -1)$. □
In Panel (b) of Figure 2 the optimality subregions of $\xi^*_1$, $\xi^*_2$, $\xi^*_3$ and $\xi^*_4$ form Theorem 3.3 are depicted. Note that each design of $\xi^*_1$, $\xi^*_2$ and $\xi^*_4$ denotes a single design whereas $\xi^*_3$ determines a certain type of designs with weights depend on the parameter values. A well known form of $\xi^*_3$ is obtained at $\beta = -1/7\beta_1$ which represents the uniform design on the vertices $v_2, v_3, v_4, v_5$. Additionally, along the horizontal dashed line, i.e., $\beta = 0$, $\xi^*_4$ assigns the weights $\omega^*_1 = 5/16$, $\omega^*_2 = 9/32$, $\omega^*_4 = 1/8$ to $v_2, v_3, v_3, v_3$, respectively. For equally size of parameters, i.e., $\beta_1 = \beta$ the diagonal dashed line in Panel (b) represents a case where $\xi^*_4$ is D-optimal.

**Remark 7.** Deriving a locally D-optimal design at a given parameter point from the subregion $-3\beta_1 < \beta < -\frac{3}{2}\beta_1$, $\beta_1 < 0$ is not available analytically. Therefore, employing the multiplicative algorithm (see Yu (2010) and Harman and Trnovská (2009)) in the software package R (see R Core Team (2018)) provides numerical solutions which show that the locally D-optimal design on that subregion is of form

$$\xi^*_3 = \begin{pmatrix} v_2 & v_3 & v_4 & v_5 & v_7 \\ \omega^*_1 & \omega^*_2 & \omega^*_3 & \omega^*_4 \end{pmatrix}$$

which is supported by five vertices with weights may depend on $\beta$. The equal weights are due to the symmetry. Table 2 shows some numerical results in terms of the ratio $\gamma = \beta/\beta_1$ where $\gamma \in (-3, -\frac{3}{2})$.

| $\gamma$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_7$ |
|----------|-------|-------|-------|-------|-------|
| -2.9     | 0.3312| 0.3285| 0.3285| 0.0059| 0.0059|
| -2.5     | 0.3225| 0.3051| 0.3051| 0.0336| 0.0336|
| -2       | 0.3125| 0.2604| 0.2604| 0.0833| 0.0833|
| -1.5     | 0.3125| 0.1701| 0.1701| 0.1736| 0.1736|
| -1.23    | 0.3297| 0.0325| 0.0325| 0.3027| 0.3027|

Table 2: D-optimal designs on $X = [1, 2]^\nu$ at $\gamma \in (-3, -\frac{3}{2})$ where $\gamma = \beta/\beta_1$ and $-3\beta_1 < \beta < -\frac{3}{2}\beta_1$, $\beta_1 < 0$.

In general, for gamma models without intercept, finding optimal designs for a model with multiple factors, i.e., $\nu > 3$ is not an easy task. The optimal design given by part (i) of Theorem 3.3 might be extended for arbitrary number of factors under sufficient and necessarily condition on the parameter points:

**Theorem 3.4.** Consider the experimental region $X = [a, b]^\nu$, $\nu \geq 3$, $0 < a < b$. Let $\beta$ be a parameter point such that $f'(x)\beta > 0$ for all $x \in X$. Define $T(x) = \sum_{i=1}^\nu x_i q = \frac{a}{a-x_{\beta_{\nu+1}}} = 1$ and $c_j = (b-a)\beta_j + a\sum_{i=1}^\nu \beta_i (1 \leq j \leq \nu)$. Then the design $\xi^*$ which assigns equal weights $v^{-1}$ to the support

$$x^*_1 = (a, b, \ldots, a)^T, x^*_2 = (a, b, \ldots, a)^T, \ldots, x^*_\nu = (a, a, \ldots, b)^T$$

is locally D-optimal (at $\beta$) if and only if for all $x = (x_1, \ldots, x_\nu)^T \in [a, b]^\nu$

$$\sum_{j=1}^\nu (x_j - qT(x))^2 \leq (b-a)^2 \left( \sum_{j=1}^\nu \beta_j x_j \right)^2. \quad (3.10)$$

**Proof.** Define the $\nu \times \nu$ design matrix $F = [f(x^*_1), \ldots, f(x^*_\nu)]^T$. Thus $F = (b-a)I + a11^T$ and $F^{-1} = \left( \frac{1}{\nu} (I - q11^T) \right)$. The information matrix of $\xi^*$ is given by $M(\xi^*, \beta) = \frac{1}{\nu} F^T V F$ where $V = \text{diag}(u(x^*_j, \beta))$. Note that $u(x^*_j, \beta) = \gamma^2$.
for all \(1 \leq j \leq \nu\). Hence, the l.h.s. of the condition of the Equivalence Theorem (Theorem 2.1 part (a)) is equal to

\[
\left( \sum_{j=1}^{\nu} \beta_j x_j \right)^{-2} f^2(x) M^{-1} (\xi^*, \beta) f(x) = \nu \left( \sum_{j=1}^{\nu} \beta_j x_j \right)^{-2} f^2(x) F^{-1} V^{-1} f(x) \\
= \nu \left( (b - a) \sum_{j=1}^{\nu} \beta_j x_j \right)^{-2} \left( f^2(x) - q T(x) (\beta^T) \right) \text{diag} \left( \sum_{j=1}^{\nu} (x_j - q T(x))^2 \right) \varepsilon_j. \\
= \nu \left( (b - a) \sum_{j=1}^{\nu} \beta_j x_j \right)^{-2} \sum_{j=1}^{\nu} (x_j - q T(x))^2 \varepsilon_j^2. \\
= \nu \left( (b - a) \sum_{j=1}^{\nu} \beta_j x_j \right)^{-2} \sum_{j=1}^{\nu} (x_j - q T(x))^2 \varepsilon_j. \\
(3.11)
\]

By Equivalence Theorem design \(\xi^*\) is locally D-optimal if and only if \((3.11)\) is less than or equal to \(\nu\) for all \(x \in \{a, b\}^\nu\) leading resulting inequalities that are equivalent to assumption \((3.10)\).

\[\square\]

Note that the D-optimal design given in part (i) of Theorem 3.3 is a special case of Theorem 3.4 when \(\nu = 3\) where condition \((3.10)\) covers condition \((3.4)\) in the proof of part (i) of Theorem 3.3. Actually it can be seen that already in the general case of Theorem 3.4 the optimality condition \((3.10)\) depends only on the ratios \(\beta_j / (\sum_{j=1}^{\nu} \beta_j)\) for all \(1 \leq j \leq \nu\). Hence the scaling factor vanishes. Similarly note that already condition \((3.10)\) depends on \(a\) and \(b\) only through their ratio \(a/b\). However, assuming the model parameters are having equal size implies that the D-optimality of a design is independent of the model parameters whereas it depends on the ratio \(a/b\) as it is shown in the next corollary.

**Corollary 3.3.** Consider the experimental region \(X = \{a, b\}^\nu, \nu \geq 3, 0 < a < b\). Let \(\beta\) be a parameter point such that \(\beta_j = \beta_j' = \beta > 0 \ (1 \leq j < j' \leq \nu)\). Then the design \(\xi^*\) which assigns equal weights \(1/\nu\) to the support \(x_1^* = (b, a, \ldots, a)^T, x_2^* = (a, b, \ldots, a)^T, \ldots, x_\nu^* = (a, a, \ldots, b)^T\) is locally D-optimal (at \(\beta\)) if and only if

\[
\left( \frac{b}{a} \right)^2 \geq \frac{(\nu - 1)(\nu - 2)}{2}. \\
(3.12)
\]

**Proof.** Let \(\beta_j = \beta_j' = \beta \ (1 \leq j < j' \leq \nu)\) then condition \((3.10)\) of Theorem 3.4 reduces to

\[
\left( (\nu - 1)a^2 + b^2 \right) \left( \sum_{j=1}^{\nu} x_j \right)^2 - \left( (\nu - 1)a + b \right)^2 \sum_{j=1}^{\nu} x_j^2 \geq 0 \ \forall x \in \{a, b\}^\nu. \\
(3.13)
\]

For \(x = (x_1, \ldots, x_\nu) \in \{a, b\}^\nu, \) let \(r = r(x) = \{0, 1, \ldots, \nu\}\) denote the number of coordinates of \(x\) that are equal to \(b\). Then \(\sum_{j=1}^{\nu} x_j = (\nu - r)a^2 + rb^2\) and \(\left( \sum_{j=1}^{\nu} x_j \right)^2 = (\nu - r)a + rb)^2.\) Hence, condition \((3.13)\) is equivalent to

\[
(a - b)^2 \tau^2 + (a - b)((b + a) - 2a \tau) r + r^2(\nu - 1) \geq 0 \ \forall r \in \{0, 1, \ldots, \nu\}, \nu \geq 2 \\
(3.14)
\]

where \(\tau = \frac{(\nu - 1)a^2 + b^2}{(\nu - 1)a + b}\). The l.h.s. of inequality \((3.14)\) is a polynomial in \(r\) of degree 2 with positive leading term. The polynomial attains 0 at \(r = 1\) (\(r_1 = 1\) is a support of \(\xi^*\)) and at \(r_2 = \frac{\nu(\nu - 1)a^2}{(\nu - 1)a + b}\). Note that the polynomial is positive and increasing for all \(r > 2\) (i.e., \((3.14)\) holds true ) when \(r_2 \leq 2\) or, equivalently, \(\frac{\nu(\nu - 1)a^2}{(\nu - 1)a + b} \leq 2\) which coincides with condition \((3.12)\).

\[\square\]

**Remark 8.** Actually, condition \((3.12)\) is obviously fulfilled for \(\nu = 2\) (compare Theorem 3.2). For the case \(\nu = 3\) the bound of l.h.s. of condition \((3.12)\) is 1 and, hence always fulfilled.

4. Gamma models with interaction

In this section we are still dealing with a model without intercept. We consider a model with two factors and with an interaction term where \(f(x) = (x_1, x_2, x_1x_2)^T\) and \(\beta = (\beta_1, \beta_2, \beta_3)^T\). The experimental region is given by
\[ X = [a, b]^2, \ 0 < a < b \] and we aim at deriving a locally D-optimal design. Our approach is employing a transformation of the proposed model to a model with intercept by removing the interaction term \( x_1 x_2 \). It follows that
\[
f_\beta(x) = (\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2)^{-1} (x_1, x_2, x_1 x_2)^T = \langle \beta_1 t_2 + \beta_2 t_1 + \beta_3 \rangle^{-1} (t_2, t_1, 1)^T = f_{\beta}^* (t) \tag{4.1}
\]
where \( t = (t_1, t_2)^T, t_j = 1/x_j, j = 1, 2 \). The range of \( t = t(x) \), as \( x \) ranges over \( X = [a, b]^2 \) is a cube given by \( T = [(1/b), (1/a)]^2 \). One can rearrange the terms of (4.1) by making use of the \( 3 \times 3 \) anti-diagonal transformation matrix \( Q \). That is \( f_\beta^* (t) = (\tilde{f}(t) \tilde{\beta})^{-1} \tilde{f}(t) \) where \( \tilde{f}(t) = Q f_\beta(t) \) and \( \tilde{\beta} = (Q^T)^{-1} \beta = (\beta_1, \beta_2, \beta_3)^T \). Note that \( \tilde{f}(t) = (1, t_1, t_2)^T \).

Thus
\[
f_\beta^* (t) = (\beta_1 + \beta_2, t_1 + \beta_3)^{-1} (1, t_1, t_2)^T. \tag{4.2}
\]
Since (4.2) coincides with that for a model with intercept the D-criterion is equivariant (see Radloff and Schwabe 2016) with respect to a one-to-one transformation from \( T \) to \( Z = [0, 1]^2 \) where
\[
t_j \rightarrow z_j = \frac{1}{(1/a) - (1/b)} t_j - \frac{1/b}{(1/a) - (1/b)}, j = 1, 2. \tag{4.3}
\]
For a given transformation matrix
\[
B = \begin{pmatrix}
\frac{1}{(1/a) - (1/b)} & 0 & 0 \\
\frac{-1/(1/a) - (1/b)}{1/(1-a) - (1/b)} & 0 & 0 \\
0 & 1/(1-a) - (1/b)
\end{pmatrix}
\text{with } B^{-1} = \begin{pmatrix}
n & 0 & 0 \\
\frac{1}{b} \frac{b - \frac{1}{b}}{1/a - 1/b} & 0 \\
\frac{1}{b} & 0 & \frac{1}{a} \frac{1}{a - b}
\end{pmatrix}
\]
we have \( \tilde{\beta} = (B^T)^{-1} \tilde{\beta} = (\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2)^T \) and hence \( \tilde{\beta}_0 = \beta_3 + (1/b) (\beta_1 + \beta_2), \tilde{\beta}_1 = \beta_3 ((1/a) - (1/b)) \) and \( \tilde{\beta}_2 = \beta_3 ((1/a) - (1/b)) \). It follows that
\[
\tilde{f}_{\beta}^* (z) = B \tilde{f}_{\beta}^* (t) = (\tilde{\beta}_0 + \tilde{\beta}_1 z_1 + \tilde{\beta}_2 z_2)^{-1} (1, z_1, z_2)^T, z \in [0, 1]^2. \tag{4.4}
\]
Let \( M(x, \beta) = f_\beta(x) f_{\beta}^* (x) \), \( \tilde{M}(t, \tilde{\beta}) = \tilde{f}_{\beta}^* (t) \tilde{f}_{\beta}^* (t) \) and \( \tilde{M}(z, \tilde{\beta}) = \tilde{f}_{\beta}^* (z) \tilde{f}_{\beta}^* (z) \) be the information matrices for the models which corresponding to (4.1), (4.2) and (4.4), respectively. It is easily to observe that
\[
M(x, \beta) = Q^{-1} \tilde{M}(t, \tilde{\beta}) Q^{-1} = B^{-1} Q^{-1} \tilde{M}(z, \tilde{\beta}) Q^{-1} B^{-1},
\]
thus the derived D-optimal designs on \( X, T \) and \( Z, \tilde{Z} \) respectively are equivariant. According to the mapping of \( x \) to \( t \) in the line following (4.1) and the mapping from \( t \) to \( z \) in (4.3) each component is mapped separately: \( x_j \rightarrow t_j \rightarrow z_j \) without permuting them. Therefore, one modifies the direct one-to-one transformation \( g : X \rightarrow \tilde{Z} \) where
\[
x_j \rightarrow z_j = \frac{1}{(1/a) - (1/b)} x_j - \frac{1/b}{(1/a) - (1/b)}, j = 1, 2. \tag{4.5}
\]
Let \( \xi^*_k \) be a design defined on \( \tilde{Z} \) that assigns the weights \( \xi(x) \) to the mapped support points \( g(x), x \in \text{supp}(\xi^*_k) \). In fact, \( \xi^*_k \) on \( X \) is locally D-optimal (at \( \beta \)) if and only if \( \xi^*_k \) on \( \tilde{Z} \) is locally D-optimal (at \( \tilde{\beta} \)). It is worth noting by transformation (4.5) we obtain
\[
(b, b)^T \rightarrow (0, 0)^T, \ (b, a)^T \rightarrow (0, 1)^T,
\]
\[
(a, b)^T \rightarrow (0, 1)^T, \ (a, a)^T \rightarrow (1, 1)^T.
\]

**Corollary 4.1.** Consider \( f(x) = (x_1, x_2, x_1 x_2)^T \) on \( X = [a, b]^2, 0 < a < b \). Denote the vertices by \( v_1 = (b, b)^T, v_2 = (b, a)^T, v_3 = (a, b)^T, v_4 = (a, a)^T \). Let \( \beta = (\beta_1, \beta_2, \beta_3)^T \) be a parameter point. Then the unique locally D-optimal design \( \xi_k^* \) (at \( \beta \)) is as follows.
The parameter space which is depicted in Panel (b) of Figure 3 is characterized by different types of locally D-optimal designs. This is because D-optimality is achieved or declined by changing the values of parameters. Specifically, if $\beta_2^2 + \frac{1}{b}(\beta_1^2 + \beta_2^2) + \left(\frac{1}{b} - \frac{1}{a}\right)\beta_2^2 + \frac{2}{a}\beta_3 + \beta_2 \leq 0$ then $\xi^*$ assigns equal weights 1/3 to $v_1$, $v_2$, $v_3$. Conversely, if $\beta_2^2 + \frac{1}{b}(\beta_1^2 + \beta_2^2) + \frac{2}{a}\beta_3 + \beta_2 \leq 0$ then $\xi^*$ assigns equal weights 1/3 to $v_2$, $v_3$, $v_4$. If none of the cases (i)–(iv) applies then $\xi^*$ is supported by the four vertices

$$
\xi^* = \left(\frac{v_1}{\omega_1^*}, \frac{v_2}{\omega_2^*}, \frac{v_3}{\omega_3^*}, \frac{v_4}{\omega_4^*}\right), \text{ where } \omega_i^* > 0 \ (1 \leq i \leq 4), \sum_{i=1}^4 \omega_i^* = 1.
$$

**Proof.** The regression vector $\tilde{f}_\beta(z)$ given by (4.4) coincides with that for the two-factor gamma model with intercept on $\mathcal{Z} = [0, 1]^2$ whose intensity function is defined as $u_\beta(z) = (\tilde{\beta}_0 + \tilde{\beta}_1z_1 + \tilde{\beta}_2z_2)^{-2}$ for all $z \in \mathcal{Z}$. Denote

$$
c_1 = u_\beta((0, 0)^T) = \tilde{\beta}_0^{-2} = (\beta_3 + \frac{1}{b}(\beta_1 + \beta_2))^{-2},
$$

$$
c_2 = u_\beta((1, 0)^T) = \tilde{\beta}_0^{-2} = (\beta_3 + \frac{1}{b}(\beta_1 + \beta_2))^{-2},
$$

$$
c_3 = u_\beta((0, 1)^T) = (\tilde{\beta}_0 + \tilde{\beta}_1)^{-2} = (\beta_3 + \beta_1) + \beta_2)^{-2},
$$

$$
c_4 = u_\beta((1, 1)^T) = (\tilde{\beta}_0 + \tilde{\beta}_1 + \tilde{\beta}_2)^{-2} = (\beta_3 + \frac{1}{a}(\beta_1 + \beta_2))^2.
$$

Let $h, i, j, k \in \{1, 2, 3, 4\}$ are pairwise distinct such that $c_k = \min\{c_1, c_2, c_3, c_4\}$ then it follows from Theorem 4.2 in Gaffke et al. (2018) that $c_{k}^{-1} \geq c_{h}^{-1} + c_{i}^{-1} + c_{j}^{-1}$ then $\xi^*$ is a three-point design supported by the three vertices $v_h$, $v_i$, $v_j$, with equal weights 1/3. Hence, straightforward computations show that the condition in case (i) of the corollary is equivalent to $c_{4}^{-1} \geq c_{1}^{-1} + c_{2}^{-1} + c_{3}^{-1}$. Analogous verifying is obtained for other cases. By Remark 2 the four-point design with positive weights in case (v) applies implicitly if any of the conditions (i)–(iv) of saturated designs is satisfied at a given $\beta$. 

It is noted that, the optimality conditions (i)–(iv) provided by Corollary 4.1 depend on the values of $a$ and $b$. The D-optimality might be achieved or declined by changing the values of $a$ and $b$. To see that, more specifically, let $a = 1$ and $b = 2$, i.e., the experimental region is $X = [1, 2]^2$ and define $\gamma_1 = \beta_1/\beta_2$ and $\gamma_2 = \beta_2/\beta_3, \beta_3 \neq 0$. Here, the parameter space which is depicted in Panel (a) of Figure 3 is characterized by $\gamma_2 + \gamma_1 > -1, 2 \gamma_2 + \gamma_1 > -2$ and $\gamma_2 + 2 \gamma_1 > -2$. It is observed that from Panel (a) of Figure 3 that the design given by part (i) of Corollary 4.1 is not locally D-optimal at any parameter point belongs to the space of model parameters. In other words, condition (i), $\frac{1}{4}(\gamma_1^2 + \gamma_2^2) + \frac{1}{2}\gamma_1 \gamma_2 + \gamma_1 + \gamma_2 \leq -1$, can not be satisfied.

Let us consider another experimental region with a higher length by fixing $a = 1$ and taking $b = 4$, i.e., $X = [1, 4]^2$. The parameter space which is depicted in Panel (b) of Figure 3 is characterized by $\gamma_2 + \gamma_1 > -1, 4 \gamma_2 + \gamma_1 > -4$ and $\gamma_2 + 4 \gamma_1 > -4$. In this case all designs given by Corollary 4.1 are locally D-optimal at particular values of $\gamma_2$ and $\gamma_1$ as it is observed from the figure. It is obvious that along the diagonal dashed line ($\gamma_2 = \gamma_1$) there exist at most three different types of locally D-optimal designs.
For arbitrary values of \( a \) and \( b \), \( 0 < a < b \) let us restrict to case \( \gamma_2 = \gamma_1 = \gamma \), i.e., \( \beta_1 = \beta_2 = \beta, \beta_3 \neq 0 \) and the next corollary is immediate.

**Corollary 4.2.** Consider \( f(x) = (x_1, x_2, x_1 x_2)^T \) on an arbitrary square \( X = [a, b]^2 \), \( 0 < a < b \) in the positive quadrant. Let \( \beta_1 = \beta_2 = \beta \) and \( \beta_3 \neq 0 \). Define \( \gamma = \frac{a}{b} \). Then the D-optimal design \( \xi^* \) (at \( \beta \)) is as follows.

(i) If \( -\frac{a}{2} < \gamma \leq -\frac{ab}{3b-a} \), then \( \xi^* \) assigns equal weights 1/3 to \( v_2, v_3, v_4 \).

(ii) If \( b - 3a > 0 \) and \( \gamma \geq \frac{ab}{b-3a} \), then \( \xi^* \) assigns equal weights 1/3 to \( v_1, v_2, v_3 \).

(iii) If \( b - 3a > 0 \) and \( -\frac{ab}{3b-a} < \gamma < \frac{ab}{b-3a} \), then the design \( \xi^* \) is supported by \( v_1, v_2, v_3, v_4 \). The optimal weights are given by

\[
\omega_1^* = \frac{ab - (a - 3b)\gamma}{4b(a + 2\gamma)}, \quad \omega_2^* = \omega_3^* = \frac{(ab + (a + b)\gamma)^2}{4ab(b + 2\gamma)(a + 2\gamma)}, \quad \omega_4^* = \frac{ab - (b - 3a)\gamma}{4a(b + 2\gamma)}.
\]

**Proof.** Consider the experimental region \( X = [a, b]^2 \), \( 0 < a < b \). By assumption \( \beta_1 = \beta_2 = \beta, \beta_3 \neq 0 \) the range of \( \gamma = \frac{a}{b} \) is given by \((-a/2, \infty)\). Assumption \( b - 3a > 0 \) implies that \(-\frac{a}{2} < -\frac{ab}{3b-a} < \frac{ab}{b-3a}\). Employing Corollary 4.1 shows the following. Both conditions of parts (ii) and (iii) of Corollary 4.1 are not fulfilled by any parameter point thus the corresponding designs are not D-optimal. In contrast, the design \( \xi^* \) in (i) of Corollary 4.2 is locally D-optimal if the condition of part (v) of Corollary 4.1 holds true. That condition is equivalent to

\[
(3b^2 + 2ab - a^2)\gamma^2 + 4ab^2 + a^2b^2 \leq 0.
\]

The l.h.s. of above inequality is polynomial in \( \gamma \) of degree 2 and thus the inequality is fulfilled by \(-\frac{a}{2} < \gamma \leq -\frac{ab}{3b-a}\). Again, the design \( \xi^* \) in (ii) is locally D-optimal if the condition of part (i) of Corollary 4.1 holds true. That condition is equivalent to

\[
(3a^2 + 2ab - b^2)\gamma^2 + 4a^2b + a^2b^2 \leq 0.
\]

The l.h.s. of above inequality is polynomial in \( \gamma \) of degree 2 and thus the inequality is fulfilled by \( \gamma \geq \frac{ab}{b-3a} \) if \( b - 3a > 0 \).

The four-point design given in (iii) has positive weights on \(-\frac{ab}{3b-a} < \gamma < \frac{ab}{b-3a} \) if \( b - 3a > 0 \) and hence it is implicitly locally D-optimal by Remark 2 \( \square \)

**Remark 9.** One should note that from Corollary 4.2 when \( \beta = 0 \) the uniform design on the vertices \( v_1, v_2, v_3, v_4 \) is locally D-optimal.
5. Design efficiency

The D-optimal design for gamma models depends on a given value of the parameter $\beta$. Misspecified values may lead to a poor performance of the locally optimal design. From our results the designs are locally D-optimal at a specific subregion of the parameter space. In this section we discuss the potential benefits of the derived designs, in particular, the D-optimal designs from Theorem 3.3 for a gamma model without interaction and from Corollary 4.2 for a gamma model with interaction. Our objective is to examine the overall performance of some of the locally D-optimal designs. The overall performance of any design $\xi$ is described by its D-efficiencies, as a function of $\beta$.

$$\text{Eff}(\xi, \beta) = \left( \frac{\det M(\xi, \beta)}{\det M(\xi^*_\beta, \beta)} \right)^{1/3} \quad (5.1)$$

where $\xi^*_\beta$ denotes the locally D-optimal design at $\beta$.

**Example 1.** In the situation of Corollary 3.3 the experimental region is given by $X = [1, 2]^3$. We restrict only to the case $\beta_1 > 0, \beta_2 = \beta_3 = \beta$ and hence we utilize the ratio $\gamma = \beta/\beta_1$ with range $(-1/4, \infty)$. Our interest is in the saturated and equally weighted designs $\xi_1$ and $\xi_2$ where supp($\xi_1$) = $\{v_2, v_3, v_4\}$ and supp($\xi_2$) = $\{v_3, v_4, v_5\}$ which by Theorem 3.3 are locally D-optimal at $\gamma \geq 1/5$ and $\gamma \in (-1/4, -5/23)$, respectively. In particular, $\xi_1$ and $\xi_2$ are robust against misspecified parameter values in their respective subregions. Additionally, for $\gamma \in (-5/23, 1/5)$ we consider the locally D-optimal designs of type $\xi_3(\gamma)$ given by the theorem. Note that supp($\xi_3(\gamma)$) = $\{v_2, v_3, v_4, v_5\}$ and the weights depend on $\gamma$.

To employ (5.1) we put $\xi^*_\beta = \xi_1$ if $\gamma \geq 1/5$, $\xi^*_\beta = \xi_2$ if $\gamma \in (-1/4, -5/23]$ and $\xi^*_\beta = \xi_3(\gamma)$ if $\gamma \in (-5/23, 1/5)$. We select for examination the designs $\xi_1, \xi_2, \xi_3(-1/7)$. Moreover, as natural competitors we select various uniform designs supported by specific vertices. That is, $\xi_4$ with support $\{1, 2\}^3$ and the two half-fractional designs $\xi_5$ and $\xi_6$ supported by $\{v_1, v_5, v_6, v_7\}$ and $\{v_2, v_3, v_4, v_5\}$, respectively. Additionally, we consider $\xi_7$ which assigns uniform weights to the grid $\{1, 1.5, 2\}^3$.

In Panel (a) of Figure 4 the D-efficiencies of the designs $\xi_1, \xi_2, \xi_3(-1/7), \xi_4, \xi_5, \xi_6$ and $\xi_7$ are depicted. The efficiencies of $\xi_1$ and $\xi_2$ are, of course, equal to 1 in their optimality subregions $\gamma \in [1/5, \infty)$ and $\gamma \in (-1/4, -5/23]$, respectively. However, for $\gamma$ outside but fairly close to the respective optimality subregion both designs perform quite well; the efficiencies of $\xi_1$ and $\xi_2$ are greater than 0.80 for $-0.15 \leq \gamma < 1/5$ and $-1/4 < \gamma \leq -0.28$, respectively. However, their efficiencies decrease towards zero when $\gamma$ moves away from the respective optimality subregion. So, the overall performance of $\xi_1$ and $\xi_2$ cannot be regarded as satisfactory. The design $\xi_3(-1/7)$, though locally D-optimal only at $\gamma = -1/7$, does show a more satisfactory overall performance with efficiencies range between 0.8585 and 1. The efficiencies of the half-fractional design $\xi_4$ are greater than 0.80 only for $\gamma > -0.049$, otherwise the efficiencies decrease towards zero. The design $\xi_4$ turns out to be uniformly worse than $\xi_3(-1/7)$ and its efficiencies range between 0.5768 and 0.7615. The worst performance is shown by the designs $\xi_5$ and $\xi_7$.

**Example 2.** In the situation of Corollary 4.2 we consider the experimental region $X = [1, 4]^2$ where condition $b - 3a > 0$ is satisfied. The vertices are denoted by $v_1 = (4, 4)^T, v_2 = (4, 1)^T, v_3 = (1, 4)^T, v_4 = (1, 1)^T$. We restrict to $\beta_3 \neq 0, \beta_1 = \beta_2 = \beta$, and the range of $\gamma = \beta/\beta_3$ is $(-1/2, \infty)$. In analogy to Example 1 denote by $\xi_1$ and $\xi_2$ the saturated and equally weighted designs with support $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$, respectively. By the corollary $\xi_1$ and $\xi_2$ are locally D-optimal at $\gamma \geq 4$ and $\gamma \in (-1/2, -4/11]$, respectively. Denote by $\xi_3(\gamma)$ the design given in part (ii) of Corollary 4.2 which is locally D-optimal at $\gamma \in (-4/11, 4)$. Note that from (5.1) we put $\xi^*_\beta = \xi_1$ if $\gamma \geq 4, \xi^*_\beta = \xi_2$ if $\gamma \in (-1/2, -4/11]$ and $\xi^*_\beta = \xi_3(\gamma)$ if $\gamma \in (-4/11, 4)$. For examination we select $\xi_1, \xi_2, \xi_3(0)$. As a natural competitor we select $\xi_4$ that assigns uniform weights to the grid $\{1, 2.5, 4\}^2$. The efficiencies are depicted in Panel (b) of Figure 4. We observe that the performance of $\xi_1$ and $\xi_2$ is similar to that of the corresponding designs in Example 1. Moreover, the design $\xi_4(0)$ show a more satisfactory overall performance. The efficiencies of $\xi_4$ vary between 0.77 and 0.83 for $\gamma > -4/11$. 

16
The Equivalence Theorem as in Theorem 3.3 implicated the optimality problem to solve a system of inequalities for various link functions, for example; the Box-Cox family of link functions that is given by

\[ u(x, \lambda\beta) = (\lambda f^T(x)\beta + 1)^{-\lambda}, x \in \mathcal{X}. \]  

In the current paper we considered gamma models without intercept for which locally D- and A-optimal designs have been developed. The positivity of the expected means entails a positive linear predictor whereas absence of the intercept term requires an experimental region which is not containing the origin point \(0\). The information matrix for the non-intercept gamma model is invariant w.r.t. simultaneously scaling of \(x\) or \(\beta\). In this context, we utilized different approaches to derive the locally optimal designs. Sets of D- and A-optimal designs were derived on a non-compact experimental region. On the other hand, a transformation to models that are having intercept were employed on the case in Theorem 3.3 and thus according to Remark 6, the three-factor model without intercept requires an experimental region which is not containing the origin point. Consequently, the linear predictor is reparameterized as \(\tilde{\beta}_0 + \tilde{\beta}_1 z_1 + \tilde{\beta}_2 z_2\), where \((z_1, z_2)^T \in \mathcal{Z}\) and \(\tilde{\beta}_0 = \beta_1 + (1/2)(\beta_2 + \beta_3), \beta_1 = (3/2)\beta_2, \beta_2 = (3/2)\beta_3\).

In many applied aspects, the log-link function is considered as a main alternative to the canonical one (see Kilian et al. (2002) Wenig et al. (2009) Gregori et al. (2011) McCrone et al. (2005) Montez-Rath et al. (2006)).

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6. Conclusion

In the current paper we considered gamma models without intercept for which locally D- and A-optimal designs have been developed. The positivity of the expected means entails a positive linear predictor whereas absence of the intercept term requires an experimental region which is not containing the origin point \(0\). The information matrix for the non-intercept gamma model is invariant w.r.t. simultaneously scaling of \(x\) or \(\beta\). In this context, we utilized different approaches to derive the locally optimal designs. Sets of D- and A-optimal designs were derived on a non-compact experimental region. On the other hand, a transformation to models that are having intercept were employed on the case in Theorem 3.3 and thus according to Remark 6, the three-factor model without intercept requires an experimental region which is not containing the origin point. Consequently, the linear predictor is reparameterized as \(\tilde{\beta}_0 + \tilde{\beta}_1 z_1 + \tilde{\beta}_2 z_2\), where \((z_1, z_2)^T \in \mathcal{Z}\) and \(\tilde{\beta}_0 = \beta_1 + (1/2)(\beta_2 + \beta_3), \beta_1 = (3/2)\beta_2, \beta_2 = (3/2)\beta_3\).

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When \(\lambda \neq 0\), \(\log \mu\) is used, otherwise \(\mu = 1\). The positivity condition of the expected mean \(\mu = E(y)\) of a gamma distribution is modified to \(\lambda f^T(x)\beta > -1\) for all \(x \in \mathcal{X}\). Therefore, for a gamma model without intercept the experimental region might be considered as...
\[ X = [0, 1]^2. \] An example, consider \( f(x) = (x_1, x_2)^T \) on \( X = [0, 1]^2 \) with vertices \( v_1 = (0, 0)^T, \) \( v_2 = (1, 0)^T, \) \( v_3 = (0, 1)^T, \) \( v_4 = (1, 1)^T. \) Let \( u = u(v_2, \lambda \beta) \) for all \( 1 \leq k \leq 4 \). The Equivalence Theorem (Theorem 2.1, part (a)) approves the D-optimality of the design \( \xi^* \) which assigns equal weights \( 1/2 \) to the vertices \( v_2 \) and \( v_3 \) at the point \( \lambda \beta \). This result might be extended for a multiple-factor model as in Theorem 3.1. However, the expression \( \lambda f^T(x) \beta + 1 \) could be viewed as a linear predictor of a gamma model with known intercept. Adopting the Box-Cox family as a class of link functions for gamma models could be a topic of future research.

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