THE SUPPORT OF THE LIMIT DISTRIBUTION OF OPTIMAL RIESZ ENERGY POINTS ON SETS OF REVOLUTION IN $\mathbb{R}^3$

J. S. BRAUCHART*, D. P. HARDIN†, AND E. B. SAFF‡

Abstract. Let $A$ be a compact set in the right-half plane and $\Gamma(A)$ the set in $\mathbb{R}^3$ obtained by rotating $A$ about the vertical axis. We investigate the support of the limit distribution of minimal energy point charges on $\Gamma(A)$ that interact according to the Riesz potential $1/r^s$, $0 < s < 1$, where $r$ is the Euclidean distance between points. Potential theory yields that this limit distribution coincides with the equilibrium measure on $\Gamma(A)$ which is supported on the outer boundary of $\Gamma(A)$. We show that there are sets of revolution $\Gamma(A)$ such that the support of the equilibrium measure on $\Gamma(A)$ is not the complete outer boundary, in contrast to the Coulomb case $s = 1$. However, the support of the limit distribution on the set of revolution $\Gamma(R + A)$ as $R$ goes to infinity, is the full outer boundary for certain sets $A$, in contrast to the logarithmic case ($s = 0$).

1. Introduction

The discrete energy problem for Riesz kernels $k_s(x) := |x|^{-s}, s > 0$, on compact sets $K$ in $\mathbb{R}^3$ is concerned with finding $N$-point systems in $K$ in the most-stable equilibrium; that is, that minimize the $s$-energy

$$E_s(X_N) := \sum_{j \neq k} \frac{1}{|x_j - x_k|^s} = \sum_{k=1}^{N} \sum_{j=1 \atop j \neq k}^{N} \frac{1}{|x_j - x_k|^s}, \quad s > 0,$$

among all $N$-point sets $X_N := \{x_1, \ldots, x_N\} \subset K$, where $| \cdot |$ denotes Euclidean distance. The existence of such configurations follows from both the lower semi-continuity of the Riesz kernel $k_s$, $s > 0$, and the compactness of $K$. Even in the case that $K$ is the unit sphere in $\mathbb{R}^3$, explicit examples of such point sets are known only for a few values of $N$. For approximate physical models of configurations of minimal energy points for large $N$ on the sphere as well as toroidal surfaces, see [5, 6].

The $N$-point system $X_N$ defines a discrete measure $\mu(X_N) := (1/N) \sum_{x \in X_N} \delta_x$, by placing the charge $1/N$ at every point $x \in X_N$. In this paper we investigate the support of the limit distribution (limit in the weak-star sense as $N \to \infty$) of a

2000 Mathematics Subject Classification. Primary 11K41, 70F10, 28A78; Secondary 78A30, 52A40.

*The research of this author was supported, in part, by the U. S. National Science Foundation under grant DMS-0532154 (D. P. Hardin and E. B. Saff principal investigators).
†The research of this author was supported, in part, by the U. S. National Science Foundation under grants DMS-0505756 and DMS-0532154.
‡The research of this author was supported, in part, by the U. S. National Science Foundation under grants DMS-0532154 and DMS-0603828.
sequence of measures \( \mu(X_k^\prime) \), \( N \geq 2 \), induced by minimal energy point configurations \( X_k^\prime \) on sets of revolution \( \Gamma(A) \) in \( \mathbb{R}^3 \) obtained by revolving a compact set \( A \) in the right-half plane about the vertical axis.

If \( 0 < s < \text{dim} \Gamma(A) \) (the Hausdorff dimension of \( \Gamma(A) \)), classical potential theory for the Riesz kernel \( k_s \) (cf. [12]) can be used to study this problem. In this case, the limit distribution (as \( N \to \infty \)) of optimal \( N \)-point configurations is given by the equilibrium measure \( \mu_{s,\Gamma(A)} \) that uniquely minimizes the continuous energy

\[
T_s[\mu] := \iint k_s(x - y) \, d\mu(x) \, d\mu(y)
\]

over the class \( \mathcal{M}(\Gamma(A)) \) of (Radon) probability measures \( \mu \) supported on \( \Gamma(A) \).

(For example, when \( \Gamma(A) \) is the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) the equilibrium measure is the normalized surface area measure on \( S^2 \).)

The probability measure \( \mu_{s,\Gamma(A)} \) is characterized by the following variational principle [12, Ch. II]: For \( \Gamma(A) \) there exists a constant \( V_s = V_s(\Gamma(A)) \) such that

\[
\begin{align*}
U_s^{H_s,\Gamma(A)} &\geq V_s \quad \text{“approximately everywhere” on } \Gamma(A), \\
U_s^{H_s,\Gamma(A)} &\leq V_s \quad \text{everywhere on the support of } \mu_{s,\Gamma(A)}.
\end{align*}
\]

Here \( U_s^{H_s,\Gamma(A)} \) denotes the equilibrium potential

\[
U_s^{H_s,\Gamma(A)}(x) := \int k_s(x - y) \, d\mu_{s,\Gamma(A)}(y), \quad x \in \mathbb{R}^3.
\]

The constant \( V_s \) is the infimum of the energies of (Radon) probability measures supported on \( \Gamma(A) \), that is \( V_s = T_s[\mu_{s,\Gamma(A)}] \). The reciprocal of \( V_s \) is called the \( s \)-capacity of the set \( \Gamma(A) \), it is denoted by \( \text{cap}_s \Gamma(A) \). The term “approximately everywhere” means that the property holds everywhere with the possible exception of a set of \( s \)-capacity zero. It follows from (1.2) and (1.3) that \( U_s^{H_s,\Gamma(A)} = V_s \) approximately everywhere on the support of \( \mu_{s,\Gamma(A)} \), which provides an integral equation for the equilibrium measure on its support. Knowing this support is therefore an important step in the determination of \( \mu_{s,\Gamma(A)} \).

We remark that Fabrikant et al. [9] provide a method for finding the density \( \rho \) of a signed charge distribution for a prescribed \( k_s \)-potential distribution on certain surfaces of revolution in \( \mathbb{R}^3 \). However, their methods do not apply, for example, to the torus and, more importantly, the distribution they obtain need not be non-negative. For the analysis of charge distributions in the Coulomb case (\( s = 1 \)) on circular or ellipsoidal “slender toroidal surfaces”, see Cade [7] and Shail [15].

Several important properties of the Riesz equilibrium measure \( \mu_{s,K} \) for a compact set \( K \) of positive \( s \)-capacity are summarized in the previously cited book of Landkof. Adopting the same notation, we let \( G_\infty \) denote the unbounded connected component of the complement of \( K \). The boundary \( S \) of \( G_\infty \) is called the outer boundary of \( K \). Furthermore, let \( \tilde{K} \) be “the set of all points of \( K \) each neighborhood of which intersects \( K \) in a set of positive \( s \)-capacity” ([12, Ch. II, no. 13]). In the case \( 1 \leq s < \text{dim} K \), the First Maximum Principle yields that \( \text{supp} \mu_{s,K} \supset \tilde{S} \).

In particular, if \( s = 1 \), then \( \text{supp} \mu_{s,K} = \tilde{S} \). For \( s \leq 1 \), it follows from the superharmonicity of the kernel \( k_s \) that the equilibrium measure is concentrated on the outer boundary \( S \) of \( K \). In [11] Hardin, Saff, and Stahl proved a stronger result for the logarithmic case (limit as \( s \to 0^+ \)): For any compact set \( A \) in the interior of the right half-plane \( \mathbb{H}^+ \), the limit distribution of minimal energy point charges on \( \Gamma(A) \) that interact through a logarithmic potential \( \log(1/|x - y|) \) is supported
on its “outer-most” portion only. The “outer-most” part of a torus, for example, is
the set of revolution generated by rotating the right semi-circle about the vertical
axis. Numerical experiments (cf. [17] and Section 6) suggest that the support of
the $s$-equilibrium measure on a torus is, for sufficiently small positive $s$, likewise a
proper subset of the torus.

In this paper we provide sufficient conditions under which the support of the
equilibrium measure $\mu_{s,\Gamma(A)}$ is a proper subset of the outer boundary of $\Gamma(A)$.
More specifically we show the following.

- Using rotational symmetry, we demonstrate how to reduce the problem of
finding the support of the equilibrium measure $\mu_{s,\Gamma(A)}$ on $\Gamma(A)$ for the (sin-
gular) kernel $k_s(x) = 1/|x|^s$ to the problem of finding the support of the
equilibrium measure $\lambda_{s,A}$ on $A$ for a related kernel $K_s$ which is continu-
ous when $0 < s < 1$ and is singular when $s \geq 1$. Lemma 2.2 summarizes
properties of the kernel $K_s$. We further discuss the asymptotics of optimal
$K_s$-energy point configurations on $A$ in both the continuous and singular
cases.

- We show that there are infinite compact sets $A$ for which the support of
the equilibrium measure on $\Gamma(A)$ is all of $\Gamma(A)$ for every $0 < s < 1$. For
example, this holds for compact subsets $A$ of a horizontal or a vertical line-
segment (see Corollary 3.4).

- We construct sets of revolution $\Gamma(A)$ such that the support of the equilib-
rium measure on $\Gamma(A)$ is a proper subset of the outer boundary of $\Gamma(A)$, in
contrast to the Coulomb case $s = 1$. We demonstrate this for $0 < s < 1/3$.
(This follows from Theorem 3.7.) An example is the outer boundary of the
“washer” $\Gamma(A)$, where $A$ is the rectangle with lower left corner $1/2 - i/2$
and upper right corner $1 + i/2$ (cf. Example 3.5). We conjecture that there
exists for every $0 < s < 1$ a compact set $A$ for which $\text{supp} \mu_{s,\Gamma(A)}$ is a
proper subset of the outer boundary of $\Gamma(A)$.

- We also show that the support of the equilibrium measure for the loga-
rithmic case ($s = 0$) can differ significantly from the case $s > 0$. For
example, let $A$ be a horizontal line-segment in $\mathbb{H}^+$. Then we show that
$\text{supp} \mu_{s,\Gamma(A)} = \Gamma(A)$ for all $0 < s < 1$, while it is known that $\text{supp} \mu_{0,\Gamma(A)}$ is
the circle generated by the “right-most” point of $A$. (For further discussion,
see end of this section.)

Outline of the paper. In Section 2 we reduce the equilibrium problem to a
minimal energy problem in the plane with respect to a new kernel $K_s$ for which we
find an explicit expression.

Section 3 is devoted to the study of $\text{supp} \lambda_{s,A}$ for the kernel $K_s$. A convexity
argument (Theorem 3.1) yields that compact subsets $A$ of horizontal or vertical
line-segments are examples with $\text{supp} \lambda_{s,A} = A$ for every $0 < s < 1$ (Corollary 3.4). In contrast, we prove the existence of compact sets $A$ for which $\text{supp} \lambda_{s,A}$ is not all of the outer boundary of $A$ by using the variational inequalities for $K_s$. The essential result here is the 3-point Theorem (Theorem 3.7) which provides a sufficient condition for a point on the outer boundary to not belong to the support of the equilibrium measure corresponding to $K_s$.

In Section 4 we study the $K_s$-equilibrium measure on sets obtained by translating a given set $A \subset \mathbb{H}^+$ a distance $R$ units to the right. The asymptotic expansion of $K_s(R + z, R + w)$, $z, w \in A$, as $R$ becomes large, is given in Lemma 4.1 and it is sensitive to the order of the limit processes $s \to 0^+$ and $R \to \infty$. The relation between the energy problem for $K_s$ on $A$ and the energy problem for $K_s$ on the translate $R + A$ is discussed.

In Section 5 we study the kernel that arises as $R \to \infty$, namely

$$K_s^{(\infty)}(z, w) = -\frac{1}{1 - s} \frac{\Gamma((1 + s)/2)}{\sqrt{\pi} \Gamma(s/2)} \left| z - w \right|^{1-s}, \quad 0 < s < 1.$$  

We show that any compact subset $A$ of a line-segment $[z', z''] \subset \mathbb{H}^+$ has the property that $\text{supp} \lambda_{s,A}^{\infty} = A$ for every $0 < s < 1$, where $\lambda_{s,A}^{\infty}$ defines the equilibrium measure for this kernel, and we find an explicit expression for the equilibrium measure $\lambda_{s,A}^{\infty}$ on $A = [z', z'']$. In case that the outer boundary $S$ of $A$ is a subset of a circle $C$ we get $\text{supp} \lambda_{s,A}^{\infty} = S$ for every $0 < s < 1$; see Lemma 5.3. In particular, if $S = C$, the equilibrium measure on $A$ for the infinity kernel is simply the normalized arc-length measure on $C$.

In Section 6 we discuss the discrete Riesz $s$-energy problem on $\Gamma(A) \subset \mathbb{R}^3$ as well as the discrete $K$-energy problem on $A \subset \mathbb{H}^+$ for the kernel $K = K_s$, $\mathcal{K} = \mathcal{K}_{s(R)}$, and $\mathcal{K} = \mathcal{K}_{s(\infty)}$. We consider the potential theoretical case $0 < s < \dim \Gamma(A)$ and the hypersingular case $s \geq \dim \Gamma(A)$. In the hypersingular case the discrete energy problem becomes a weighted energy problem which allows us to use results from [4]. We find the limit distribution of minimal $K$-energy $N$-point systems, consider the separation of such optimal point configurations, and give asymptotics for the discrete minimal energy as $N \to \infty$. Also included are numerical experiments showing minimal energy point configurations on Cassinian ovals, line-segments, and circles.

An appendix to the paper provides the computations showing convexity of the kernel $K_s$ on the vertical line-segment.

2. Reduction to the plane, the kernel $K_s$

First we fix some notation. The axis of revolution is identified with the $y$-axis in $\mathbb{R}^3$. Any vertical cutting plane gives a cross-section of the set of revolution and may serve as a reference plane. Selecting a vertical cutting plane we choose one of the two closed half-planes and call it $\mathbb{H}^+$. It may be identified with the complex right half-plane. Then the set of revolution generated by $A \subset \mathbb{H}^+ := \{x + iy \mid x \geq 0, y \in \mathbb{R}\}$ is the set

$$\Gamma(A) := \{R_\phi x \mid x \in A, 0 \leq \phi < 2\pi\},$$  

where $R_\phi$ is a rotation by angle $\phi$ about the axis of revolution. The set $\Gamma(A)$ is obtained by revolving $A$ around the vertical axis. Thus, a single point $x + iy \in \mathbb{H}^+$, $x > 0$, becomes a horizontal circle with center on the vertical axis.
A Borel measure $\hat{\mu} \in \mathcal{M}(\mathbb{R}^3)$ is rotationally symmetric about the $y$-axis if
\begin{equation}
\hat{\mu}(R_\phi B) = \hat{\mu}(B)
\end{equation}
for all Borel sets $B \subset \mathbb{R}^3$ and for all rotations $R_\phi$ about the $y$-axis. (Here $R_\phi B$ denotes the pointwise rotated set $\{R_\phi x \mid x \in B\}$.) If $\hat{\mu} \in \mathcal{M}(\mathbb{R}^3)$ is rotationally symmetric about the $y$-axis, then $\hat{\mu}$ can be written as a product of two measures, the normalized Lebesgue measure on the half-open interval $[0, 2\pi)$ and a measure $\mu$ on $\mathbb{H}^+$, that is
\begin{equation}
d\hat{\mu} = \frac{d\phi}{2\pi} d\mu, \quad \mu = \hat{\mu} \circ \Gamma \in \mathcal{M}(\mathbb{H}^+).
\end{equation}
Then the energy of the (compactly supported) measure $\hat{\mu}$ can be expressed as
\begin{equation}
\mathcal{I}_s[\hat{\mu}] = \int_{\mathbb{R}^3 \times \mathbb{R}^3} k_s(x - y) \, d\hat{\mu}(x) \, d\hat{\mu}(y)
= \int_{\mathbb{H}^+ \times \mathbb{H}^+} K_s(z, w) \, d\mu(z) \, d\mu(w) =: \mathcal{J}_s[\hat{\mu}],
\end{equation}
where the kernel $K_s(z, w)$ is given by the integral
\begin{equation}
K_s(z, w) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|R_\phi z - w|^s} \, d\phi.
\end{equation}

### 2.1. The energy problem for $\mathcal{K}_s$

Let $0 < s < 1$. Let $A \subset \mathbb{H}^+$ be a compact set such that $\text{cap}_s\Gamma(A) > 0$. Then the uniqueness of the equilibrium measure $\mu_{s, \Gamma(A)}$ on $\Gamma(A)$ and the symmetry of the revolved set $\Gamma(A)$ imply that $\mu_{s, \Gamma(A)}$ is rotationally symmetric about the $y$-axis and so $d\mu_{s, \Gamma(A)} = [d\phi/(2\pi)] \, d\lambda_{s, A}$, where $\lambda_{s, A} = \mu_{s, \Gamma(A)} \circ \Gamma \in \mathcal{M}(\mathbb{H}^+)$. Furthermore, if $\nu \in \mathcal{M}(\mathbb{H}^+)$, then $d\nu := [d\phi/(2\pi)] \, d\nu$ is rotationally symmetric about the $y$-axis and so we have
\begin{equation}
\mathcal{J}_s[\lambda_{s, A}] \geq \inf_{\nu \in \mathcal{M}(\mathbb{H}^+)} \mathcal{J}_s[\nu] = \inf_{\nu \in \mathcal{M}(\mathbb{H}^+)} \mathcal{I}_s[\nu] = \mathcal{J}_s[\mu_{s, A}] = \mathcal{J}_s[\lambda_{s, A}].
\end{equation}

In the case $0 < s < 1$ the equilibrium measure on $\Gamma(A)$ is concentrated on the outer boundary of $\Gamma(A)$ (cf. [12, Ch. II, no. 13]).

**Proposition 2.1.*** Let $0 < s < \dim \Gamma(A)$. Let $A \subset \mathbb{H}^+$ be a compact set with $\text{cap}_s\Gamma(A) > 0$. Then $\lambda_{s, A} = \mu_{s, \Gamma(A)} \circ \Gamma$ uniquely minimizes $\mathcal{J}_s[\nu]$ over all measures $\nu \in \mathcal{M}(A)$. Thus, $\lambda_{s, A}$ is the equilibrium measure on $A$ for the kernel $\mathcal{K}_s$. It is supported on the outer boundary of $A$.

The $\mathcal{K}_s$-energy of a measure was defined in (2.4). The energy $V_{\mathcal{K}_s}$ of $A$ is given by
\begin{equation}
V_{\mathcal{K}_s}(A) := \inf \{ \mathcal{J}_s[\nu] \mid \nu \in \mathcal{M}(A) \}.
\end{equation}
The following relations hold:
\begin{equation}
V_{\mathcal{K}_s}(A) = \mathcal{J}_s[\lambda_{s, A}] = \mathcal{J}_s[\mu_{s, \Gamma(A)}] = V_s(\Gamma(A)).
\end{equation}
For $\nu \in \mathcal{M}(A)$, we define the $\mathcal{K}_s$-potential $W^{\nu}_s$ by
\begin{equation}
W^{\nu}_s(z) := \int_A K_s(z, w) \, d\nu(w), \quad z \in \mathbb{H}^+.
\end{equation}
Let \( \tilde{\nu} \in M(\Gamma(A)) \) be rotationally symmetric with \( d\tilde{\nu} = [d\phi/(2\pi)] d\nu \), where \( \nu = \tilde{\nu} \circ \Gamma \in M(A) \). Then the potential \( U^\nu_s \) is constant on circles \( \Gamma(\{z\}) \), \( z \in \mathbb{H}^+ \). Abusing notation, there holds the following connecting formula

\[
U^\nu_s(z) = \int_{\Gamma(A)} k_s(z - y) d\tilde{\nu}(y) = \frac{1}{2\pi} \int_{\mathbb{A}} \int_0^{2\pi} k_s(z - R_{\phi}w) d\phi d\nu(w)
\]

\[
= \int_{\mathbb{A}} \mathcal{K}_s(z, w) d\nu(w) = W^\nu_s(z), \quad z \in \mathbb{H}^+.
\]

From the properties (1.2) and (1.3) of the equilibrium potential \( U^\mu,\Gamma(A) \) we infer the variational inequalities for \( \mathcal{K}_s \) for compact sets \( A \) in the interior of \( \mathbb{H}^+ \):

\[
W^\lambda_{s,A} \geq V_{\mathcal{K}_s}(A) \quad \text{everywhere on } A,
\]

\[
W^\lambda_{s,A} \leq V_{\mathcal{K}_s}(A) \quad \text{on supp } \lambda_{s,A}.
\]

In this case we do no longer need an “approximately everywhere” exceptional set, since each point of \( A \) generates a circle in \( \mathbb{R}^3 \) with positive capacity.

### 2.2. Properties of the kernel \( \mathcal{K}_s \)

Let \( z = x + iy, w = u + iv \), where \( x, y, u, v \in \mathbb{R} \). Let \( w_* := -\overline{w} = -u + iv \) denote the reflection of \( w \) in the imaginary axis.

**Lemma 2.2.** Let \( s > 0 \). The kernel \( \mathcal{K}_s : \mathbb{H}^+ \times \mathbb{H}^+ \to \mathbb{R} \) in (2.5) has the following properties:

1. \( \mathcal{K}_s(z, w) \) is well defined for \( z \neq w \) for all \( s > 0 \).
2. \( \mathcal{K}_s \) is symmetric: \( \mathcal{K}_s(z, w) = \mathcal{K}_s(w, z) \).
3. \( \mathcal{K}_s \) is homogeneous: \( \mathcal{K}_s(rz, rw) = r^{-s}\mathcal{K}_s(z, w) \) for all \( r > 0 \).
4. \( \mathcal{K}_s \) is continuous at all points \( (z, w) \in \mathbb{H}^+ \times \mathbb{H}^+ \) with \( z \neq w \). If \( 0 < s < 1 \), then \( \mathcal{K}_s \) is continuous at \( (w, w) \) with \( \text{Re}[w] > 0 \). \( \mathcal{K}_s(z, w) \) is singular at \( z = w \) for \( s > 1 \).
5. If \( w \) is on the imaginary axis and \( s > 0 \), then \( \mathcal{K}_s(z, w) = |z - w|^{-s} \), \( z \neq w \). If \( \text{Re}[w] > 0 \), then, for \( s > 1 \), the following limit holds:

\[
|z - w|^{s-1} \mathcal{K}_s(z, w) \to \frac{\Gamma((s - 1)/2)}{\sqrt{\pi} \Gamma(s/2)} \frac{1}{|w - w_*|}, \quad \text{as } z \to w.
\]

6. \( \mathcal{K}_s(u + it, u + iv) \) decreases along vertical lines as \( |t - v| \) grows and \( \mathcal{K}_s(u + iy, u + t + iy) \) decreases along horizontal lines as \( t > 0 \) grows.
7. Let \( 0 < s < 1 \). For fixed \( w \) with \( \text{Re}[w] > 0 \), the function \( \mathcal{K}_s(z, w) \) has exactly one global maximum at \( z = w \) in \( \mathbb{H}^+ \). At \( (w, w) \) or \( (w_*, w) \), the kernel \( \mathcal{K}_s \) takes the value

\[
\mathcal{K}_s(w, w) = \mathcal{K}_s(w_*, w) = \mathcal{I}_s \left( \frac{d\phi}{2\pi} \right) |\text{Re } w|^{-s},
\]

where

\[
\mathcal{I}_s \left( \frac{d\phi}{2\pi} \right) = 2^{-s} \frac{\Gamma((1 - s)/2)}{\sqrt{\pi} \Gamma(1 - s/2)} = \frac{\Gamma(1 - s)}{[\Gamma(1 - s/2)]^2}.
\]

---

\(^1\)This follows from differentiating the integral (2.23) with respect to \( t \).
(8) The kernel $K_s$ has the following representations in terms of hypergeometric functions \cite{2} or in terms of a Legendre function \cite{1}

$$K_s(z, w) = \left| z - w_s \right|^{-s} \frac{2F_1}{|z - w_s|^2} \left( \frac{s/2, 1/2, 1 - \left| z - w \right|^2}{|z - w_s|^2} \right)$$

(2.15)

$$K_s(z, w) = \left\{ \frac{2}{|z - w_s| + |z - w|} \right\}^s \frac{2F_1}{|z - w_s|^2} \left( \frac{s/2, s/2, 1}{|z - w_s|^2} \left\{ \frac{|z - w_s| - |z - w|}{|z - w_s| + |z - w|} \right\}^2 \right)$$

(2.16)

$$K_s(z, w) = \left| z - w_s \right|^{-s/2} \left| z - w \right|^{-s/2} P_{s/2-1}^0 \left( \frac{1}{2} \frac{|z - w_s| + \frac{1}{2} |z - w|}{|z - w|} \right).$$

(Observe that the Legendre function is evaluated at values $> 1$ if $\text{Re} \ z > 0$ or $\text{Re} \ w > 0$.) For $s > 1$ one can factor out the singularity at $z = w$,

$$K_s(z, w) = \frac{|z - w_s|^{1-s}}{|z - w_s|} 2F_1 \left( 1 - \frac{s/2, 1/2, 1 - \left| z - w \right|^2}{|z - w_s|^2} \right), \quad z \neq w. \tag{2.18}$$

(9) As $s \to 0^+$ we recover the logarithmic kernel $K_0$ studied in \cite{11}:

$$\lim_{s \to 0^+} \frac{K_s(z, w) - 1}{s} = \log \frac{2}{|z - w| + |z - w_s|}. \tag{2.19}$$

(10) As $s \to 1^-$, $K_s(z, w) \to K_1(z, w)$, where

$$K_1(z, w) := \frac{2}{\pi} \frac{2}{|z - w_s| + |z - w|} 2 \mathcal{K} \left( \left\{ \frac{|z - w_s| - |z - w|}{|z - w_s| + |z - w|} \right\}^2 \right), \tag{2.20}$$

and $\mathcal{K}$ denotes the complete Elliptic integral of the first kind \cite{1}.

Remark 2.3. For the special case of the sphere the formula (2.16) reduces to the formula (4.14) in Dragnev and Saff \cite{8}.

The level sets of $K_s(\cdot, w)$, $w \in \mathbb{H}^+$ fixed, look like Cassinian ovals, cf. Figure 1. The asymptotical behavior of $K_s(R + z, R + w)$ as $R \to \infty$ is given in Lemma 4.1.

Figure 1. Level sets for $K_s(z, 1)$, $s = 1/2$.

Proof of Lemma 2.2. Let $z, w \in \mathbb{H}^+$ with $z = x + iy$ and $w = u + iv$. The relation

$$|\mathbf{R}_\phi z - w|^2 = x^2 + u^2 - 2xu \cos \phi + (y - v)^2$$

gives $1/|\mathbf{R}_\phi z - w|^4 = (E - F \cos \phi)^{-s/2}$ for the integrand in (2.5), where we define

$$E := x^2 + u^2 + (y - v)^2, \quad F := 2xu. \tag{2.21}$$

By (2.21) the kernel $K_s(z, w)$ is symmetric in $z, w$. The substitution $\phi = \psi + \pi$ yields

$$K_s(z, w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (E + F \cos \psi)^{-s/2} \, d\psi = \frac{1}{\pi} \int_{0}^{\pi} (E + F \cos \phi)^{-s/2} \, d\phi. \tag{2.22}$$

Applying the half angle formula and substituting $\phi = \psi/2$ we obtain

$$K_s(z, w) = (E + F)^{-s/2} \frac{2}{\pi} \int_{0}^{\pi/2} \left( 1 - \frac{2F}{E + F} \sin^2 \psi \right)^{-s/2} \, d\psi. \tag{2.23}$$
The integral in (2.23) resembles that of a complete elliptic integral. Indeed, for \( s = 1 \) this integral is the complete Elliptic integral of the first kind \( K(k^2) \) with elliptic modulus \( k^2 = 2F/(E + F) \). (See for example [1, 17.2.19, 17.3.1].) The transformation \( \phi = \psi - \pi \) in (2.22) gives

\[
K_s(z, w) = E^{-s/2} \frac{1}{\pi} \int_0^\pi \left( 1 - \frac{F}{E} \cos \psi \right)^{-s/2} d\psi.
\]

The integral in (2.24) is a generalization of Epstein and Hubbell’s elliptic integral. We refer to [16] for a discussion of these elliptic-type integrals.

A change of variables \( t = \sin^2 \psi \) in (2.23) yields

\[
K_s(z, w) = (E + F)^{-s/2} \frac{1}{\pi} \int_0^1 t^{1/2-1} (1-t)^{1/2-1} \left( 1 - \frac{2F}{E + F} t \right)^{-s/2} dt.
\]

Recall, that the Gauss hypergeometric series [1, 15.1.1]

\[
\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}
\]

represents the Gauss hypergeometric function \( _2F_1 \left( a, b; \frac{c}{2}; z \right) \) for all complex \( z \) within the circle of convergence, the unit circle \( |z| = 1 \). The analytic continuation in the \( z \)-plane cut along the segment \([1, \infty], [1, 15.3.1],\)

\[
\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-z t)^{-a-1} dt,
\]

\( \Re c > \Re b > 0, \)

can be used to derive a hypergeometric function representation of the kernel \( K_s(z, w) \),

\[
K_s(z, w) = (E + F)^{-s/2} \ _2F_1 \left( s/2, 1/2; \frac{2F}{E + F}; z \right).
\]

Let \( w_* := -\bar{w} = -u + iv \) denote the reflection of \( w \) in the imaginary axis. Then

\[
E - F = (x - u)^2 + (y - v)^2 = |z - w|^2,
\]

\[
E + F = (x + u)^2 + (y - v)^2 = |z - w_*|^2,
\]

and we get the relations

\[
0 \leq \frac{2F}{E + F} = \frac{|z - w_*|^2}{|z - w|^2} = \frac{4xz}{(x + u)^2 + (y - v)^2} \leq 1, \quad z, w \in \mathbb{H}^+.
\]

Substitution of (2.27) into (2.26) yields (2.15).

The hypergeometric function in (2.15) is of the form \( _2F_1 \left( a, b; \frac{c}{2}; \zeta \right) \). The quadratic transformation [1, 15.3.17] yields a more symmetrical representation (2.16).

In the argument of the hypergeometric function in (2.16) appears the expression

\[
\xi := \frac{|z - w_*| - |z - w|}{|z - w_*| + |z - w|} = \frac{|z - w_*|^2 - |z - w|^2}{(|z - w_*| + |z - w|)^2} = \frac{4 \Re z \Re w}{(|z - w_*| + |z - w|)^2}.
\]

It satisfies \( \xi^2 \leq 1 \) and equality holds for \( z = w \) or \( z = w_* \) only. Therefore we may use the series expansion of the hypergeometric function to get

\[
K_s(z, w) = \left( \frac{2}{|z - w_*| + |z - w|} \right) \sum_{\ell=0}^{\infty} \left( \frac{(s/2)\ell}{\ell!} \xi^\ell \right)^2, \quad z \neq w, w_*.
\]
If $0 < s < 1$, this series converges even for $z = w, w_s$. For $z = w, w_s$ the argument of the hypergeometric function in (2.15), (2.16) is 1. From [1, 15.1.20]

$$K_s(w, w) = K_s(w_s, w) = \frac{\Gamma(1 - s)}{\Gamma(1 - s/2)} |\text{Re} w|^{-s} = 2^{-s} \frac{\Gamma((1 - s)/2)}{\sqrt{\pi} \Gamma(1 - s/2)} |\text{Re} w|^{-s}.$$  

(The first two relations follow from (2.16), the last one from (2.15).) Note, the Re function with Legendre functions. From (2.15) and relation [1, 15.4.7] we get (2.17).

From (2.15) and relation [1, 15.3.3] we get (2.18). From (2.18) follows (2.12). If $	ext{Re}[w] = 0$, then $w = w_s$. Hence, by (2.15), $K_s(z, w) = |z - w|^{-s}, z \neq w$, for $s > 0$.

The complete Elliptic integral of the first kind $K(k^2)$ [1, 17.3.1] can be represented through a hypergeometric function [1, 17.3.9],

$$K(k^2) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - k^2 (\sin \vartheta)^2}} = \frac{\pi}{2} \frac{\Gamma(1/2, 1/2; k^2)}{\sqrt{2}}.$$  

Thus (2.20) follows from (2.16).

As $s \to 0^+$, the hypergeometric series in (2.28) reduces to 1. Thus it makes sense to consider the quotient $(K_s(z, w) - 1)/s$. Fix $z, w \in \mathbb{H}^+$ in (2.28). Let $z \neq w$.

Then

$$\frac{d}{ds} K_s(z, w) = K_s(z, w) \log \frac{2}{|z - w| + |z - w_s|} + \left( \frac{2}{|z - w_s| + |z - w|} \right)^s \frac{d}{ds} \left[ \sum_{\ell=0}^{\infty} \left( \frac{(s/2)^{\ell}}{\ell!} \xi^\ell \right)^2 \right].$$

We are only interested in $dK_s(z, w)/ds$ at $s = 0^+$. $K_s(z, w)$ becomes one at $s = 0$. The derivative in the right-most term above exists and vanishes. This follows from

$$\frac{1}{s} \left[ \sum_{\ell=0}^{\infty} \left( \frac{(s/2)^{\ell}}{\ell!} \xi^\ell \right)^2 - 1 \right] = \frac{1}{s} \left( \frac{s}{2} \xi \right)^2 \sum_{\ell=0}^{\infty} \left( \frac{1 + s/2}{(\ell + 1)!} \xi^\ell \right)^2$$

and the limit process $s \to 0^+$. By the ratio test, the infinite series on the right-hand side above is absolutely convergent for $|\xi| < 1$ (that is $z \neq w$) and $0 < s < 1$. In the case $z = w$ one uses (2.13) instead of (2.28). \qed

3. The Support of the Equilibrium Measure for the Kernel $K_s$  

By Proposition 2.1, the equilibrium measure $\lambda_{s,A}$ on $A$ for $K_s$ is supported on the outer boundary $S$ of $A$. A convexity argument yields sufficient conditions for $\text{supp} \lambda_{s,A} = S$. Recall that a function $f : [a, b] \to \mathbb{R}$ is strictly convex on $[a, b]$ if $f(\tau x + (1 - \tau)y) < \tau f(x) + (1 - \tau)f(y)$ for all $a \leq x < y \leq b$ and $0 < \tau < 1$.

**Theorem 3.1.** Let $0 < s < 1$ and $A$ be a compact set in the interior of $\mathbb{H}^+$.

(i) If $\gamma : [a, b] \to \mathbb{H}^+$, $a < b$, is a simple continuous non-closed curve covering the outer boundary $S$ of $A$, that is $S \subset \gamma^* := \{ \gamma(t) \mid a \leq t \leq b \}$, and $K_s(\gamma(t), \gamma(t))$ is a strictly convex function on the intervals $[a, t]$ and $[t, b]$ for each fixed $t \in [a, b]$, then there is some closed interval $I \subset [a, b]$ such that $\text{supp} \lambda_{s,A} = \gamma(I) \cap S$.  

(ii) If $\gamma : [a, b] \to \mathbb{H}^+$, $a < b$, is a simple continuous closed curve covering the outer boundary $S$ of $A$, then $\gamma^* := \{ \gamma(t) \mid a \leq t \leq b \}$ is a closed interval $I \subset [a, b]$, and $\text{supp} \lambda_{s,A} = \gamma(I) \cap S$.  

(iii) If $\gamma : [a, b] \to \mathbb{H}^+$, $a < b$, is a simple continuous closed curve covering the outer boundary of $A$, then $\gamma^* := \{ \gamma(t) \mid a \leq t \leq b \}$ is a closed interval $I \subset [a, b]$, and $\text{supp} \lambda_{s,A} = \gamma(I) \cap S$.  

(iv) If $\gamma : [a, b] \to \mathbb{H}^+$, $a < b$, is a simple continuous non-closed curve covering the outer boundary of $A$, then $\gamma^* := \{ \gamma(t) \mid a \leq t \leq b \}$ is a simple continuous non-closed curve covering the outer boundary of $A$, and $\text{supp} \lambda_{s,A} = \gamma(I) \cap S$.  

(v) If $\gamma : [a, b] \to \mathbb{H}^+$, $a < b$, is a simple continuous closed curve covering the outer boundary of $A$, then $\gamma^* := \{ \gamma(t) \mid a \leq t \leq b \}$ is a simple continuous closed curve covering the outer boundary of $A$, and $\text{supp} \lambda_{s,A} = \gamma(I) \cap S$.
(ii) If \( \gamma : [0, b] \to \mathbb{H}^+ \) is a simple continuous closed curve, that is \( \gamma(0) = \gamma(b) \), with \( S \subset \gamma^* \) and extended periodically by \( \gamma(t) = \gamma(t + b) \), and \( K_s(\gamma(\cdot), \gamma(\cdot)) \) is a strictly convex function on the interval \([t, t+b]\) for each fixed \( t \in [0, b] \), then \( \text{supp} \lambda_{s,A} = S \).

**Remark 3.2.** Note that \( S \) is only required to be a compact subset of \( \gamma^* \). For example, \( S \) may be a Cantor subset of \( \gamma^* \).

**Proof of Theorem 3.1.** Set \( \lambda = \lambda_{s,A} \) and \( \lambda^s = W^s \). We have \( \text{supp} \lambda \subset S \subset \gamma^* \). Suppose \( G \) is a component of the complement of \( \text{supp} \lambda \) in \( \gamma^* \). Now observe, that by our assumptions, \( G \) always corresponds to a subinterval \( I \) of one of the sets \([a, t]\), \([t, b]\) or \([t, t+b]\) for \( \gamma(t) \in \text{supp} \lambda \). Two cases are possible: (i) Both boundary points of \( G \) are in \( \text{supp} \lambda \). Then the equilibrium potential \( \lambda^s \) assumes the value \( \mathcal{J}_{\mathcal{K}_s}[\lambda] \) on the boundary of \( G \) and, due to strict convexity of \( \lambda^s \circ \gamma \) on \( I \), is strictly less than this value in the open set \( G \). Since \( \lambda^s \geq \mathcal{J}_{\mathcal{K}_s}[\lambda] \) on \( A \subset S \), no point of \( G \) is in \( A \). (ii) At least one boundary point of \( G \) is not in \( \text{supp} \lambda \). This can only happen when \( \gamma \) is a non-closed curve. Without further assumptions the convexity property alone is insufficient to show \( G \cap A = \emptyset \). From (i) follows the existence of some closed interval \( I \subset [a, b] \) such that \( \text{supp} \lambda = \gamma(I) \cap S \). If \( \gamma \) is a closed curve, then \( I = [0, b] \). \( \square \)

**Remark 3.3.** In the proof of Theorem 3.1 we use three main properties: (i) The kernel is continuous, (ii) \( \text{supp} \lambda_{s,A} \subset S \), and (iii) the equilibrium potential satisfies a variational principle. These properties also hold for \( K^{(\infty)}_s \) introduced in Section 5. Therefore, Theorem 3.1 can be applied in case of \( K^{(\infty)}_s \).

Using Theorem 3.1(i) we next show that any compact subset \( A \) of a horizontal or vertical line-segment satisfies \( \text{supp} \lambda_{s,A} = A \) for every \( 0 < s < 1 \). We contrast this with the logarithmic case, where it is still true that \( \text{supp} \lambda_{0,A} = A \) in case of a vertical line-segment [11, Cor. 1]. However, in case of a horizontal line-segment one has that \( \lambda_{0,A} \) is a unit point charge at the right-most point of \( A \) [11, Thm. 1].

**Corollary 3.4.** Suppose \( A \) is a compact subset of either (a) the horizontal line-segment \([a + ic, b + ic] \), \( 0 < a < b \), or (b) the vertical line-segment \([R + ic, R + id] \), \( R > 0, c < d \). Then \( \text{supp} \lambda_{s,A} = A \) for every \( 0 < s < 1 \).

**Proof.** For (a) consider the parametrization \( \gamma(x) = x + ic, a < x < b \). From (2.16),

\[
K_s(\gamma(x), \gamma(u)) = x^{-s} \sum_{n=0}^{\infty} \frac{(s/2)_n (s/2)_n} {(1)_n n!} u^{2n} x^{-s-2n}, \quad x > u,
\]

\[
K_s(\gamma(x), \gamma(u)) = u^{-s} \sum_{n=0}^{\infty} \frac{(s/2)_n (s/2)_n} {(1)_n n!} x^{2n} u^{-s-2n}, \quad x < u.
\]

From \((x^{-s-2n})'' > 0, n \geq 0, \) and \((x^{2n})'' > 0, n \geq 1, \) we get \( [K_s(\gamma(x), \gamma(u)))'' > 0 \) for \( x \neq u \) and for every \( 0 < s < 1 \). Termwise differentiation is justified by uniform convergence for \( |x-u| \geq \delta \). By Theorem 3.1, \( \text{supp} \lambda_{s,A} = \gamma(I) \cap A \) for some \( I = [a', b'] \subset [a, b] \). From the series representations above we observe that the kernel \( K_s(\gamma(x), \gamma(u)) \) is a strictly increasing function in \( x \) for \( x < u \) and it is a strictly decreasing function in \( x \) for \( x > u \). Hence, \( W^s_{K_s} \circ \gamma < I_s[\lambda_{s,A}] \) on \([a, b] \setminus I \).

By variational inequality (2.10), \( I = [a, b] \).
Lemma 3.6. Let \( \lambda \subset A \) for \( K_s \) is a proper subset of the outer boundary of \( A \).

Example 3.5. Let \( A \) be the rectangle with lower left corner \( 1/2 - i/2 \) and upper right corner \( 1 + i/2 \). Using Theorem 3.7(c) below with \( x = 1/2 \) and \( z' = 1 + i/2 \), it follows that \( 1/2 \notin \text{supp} \lambda_{s,A} \) for \( 0 < s < 1/3 \). Alternatively, if \( A \) is the left-half circle with radius \( 1/2 \) centered at \( 1 \), it again follows from Theorem 3.7(c) that \( 1/2 \notin \text{supp} \lambda_{s,A} \) for \( 0 < s < 1/3 \). In contrast, as \( A \) is moved to the right \( R \) units and \( R \to \infty \), we get \( \text{supp} \lambda_{s,A} = A \); see Lemma 5.3.

To prove Theorem 3.7 we use a special case of the following observation.

**Lemma 3.6.** Let \( 0 < s < 1 \). Suppose \( A \) is a compact set in the interior of \( \mathbb{H}^+ \). Let \( \lambda \) denote the unique equilibrium measure on \( A \) for \( K_s \). If

\[
K_s(z, \cdot) > \int_B K_s(\cdot, w') \, d\nu(w') \quad \text{everywhere on} \quad \text{supp} \lambda
\]

for some subset \( B \subset A \) and some probability measure \( \nu \in \mathcal{M}(B) \), then \( z \notin \text{supp} \lambda \).

**Proof.** Using (3.2) and the variational inequality (2.10), we get

\[
W_{K_s}^\lambda(z) = \int K_s(z, w) \, d\lambda(w) > \int \left( \int_B K_s(w, w') \, d\nu(w') \right) \, d\lambda(w)
= \int_B W_{K_s}^\lambda(w') \, d\nu(w') \geq J_{K_s}[\lambda] \int_B d\nu(w') = J_{K_s}[\lambda].
\]

But \( W_{K_s}^\lambda(z) > J_{K_s}[\lambda] \) implies, by the variational inequality (2.11), that \( z \notin \text{supp} \lambda \).

Let \( z = x > 0 \) and set \( B = \{ z', \overline{z'} \} \), \( z' \) in the interior of \( \mathbb{H}^+ \), \( \text{Im}[z'] \neq 0 \), and place the charge \( 1/2 \) at each point in \( B \). Then (3.2) is equivalent to the property

\[
K_s(z, \cdot) > K_s^*(\cdot, z') \quad \text{everywhere on} \quad \text{supp} \lambda
\]

where \( K_s^* \) denotes the kernel

\[
K_s^*(z, w) := |K_s(z, w) + K_s(z, \overline{w})|/2.
\]

**Theorem 3.7** (3-point Theorem). Let \( 0 < s < 1 \). Let \( x > 0 \) and \( z' \) be in the interior of \( \mathbb{H}^+ \). Let \( A \) be a compact subset of \( \{ w \in \mathbb{H}^+ : |K_s(x, w) \geq K_s(x, z') \} \) in the interior of \( \mathbb{H}^+ \) with \( x, z', \overline{z} \in A \).

(a) If \( \Delta_s := K_s(x, z') - K_s^*(z', z') > 0 \), then \( x \notin \text{supp} \lambda_{s,A} \).
(b) If \( z' = 1 + i\gamma, \gamma > 0 \), and condition

\[
4 \left( \gamma + \sqrt{1 + \gamma^2} \right) > \left( \sqrt{(1+x)^2 + \gamma^2} + \sqrt{(1-x)^2 + \gamma^2} \right)^2
\]

is satisfied, then \( \Delta_s > 0 \) (and hence, by (a), \( x \notin \text{supp}\lambda_s,A \)) for \( s > 0 \) sufficiently small.

(c) If \( x = 1/2 \) and \( z' = 1 + i/2 \), then \( \Delta_s > 0 \) (and hence, by (a), \( x \notin \text{supp}\lambda_s,A \)) for all \( 0 < s < 1/3 \). (The graph of \( \Delta_s \) is shown in Figure 2.)

Proof of Theorem 3.7. We show first (a). The function \( \mathcal{K}_s(z,\cdot) \) has a unique maximum at \( z \in \mathbb{H}^+ \) (Lemma 2.2(7)). So

\[
\mathcal{K}_s(x,w) - \mathcal{K}_s^*(w,z) \geq \mathcal{K}_s(x,w) - \mathcal{K}_s^*(z',z') \geq \mathcal{K}_s(x,z') - \mathcal{K}_s^*(z',z').
\]

The first inequality holds in \( \mathbb{H}^+ \). The last one holds on \( \{w \in \mathbb{H}^+|\mathcal{K}_s(x,w) \geq \mathcal{K}_s(x,z')\} \). Now, let \( A \) be a compact subset of \( \{w \in \mathbb{H}^+|\mathcal{K}_s(x,w) \geq \mathcal{K}_s(x,z')\} \) in the interior of \( \mathbb{H}^+ \) with \( x, z', \beta \in A \). Then

\[
W_s^*(x) \geq \left[ W_s^*(z') + W_s^*(\beta) \right] / 2 + \mathcal{K}_s(x,z') - \mathcal{K}_s^*(z',z'), \quad \nu \in M(A).
\]

This follows from (3.6) and \( \mathcal{K}_s(w,z) = \mathcal{K}_s(z,w) \). If the difference \( \Delta_s := \mathcal{K}_s(x,z') - \mathcal{K}_s^*(z',z') \) is positive, the variational inequality (2.10) implies \( W_s^*(x) > J\mathcal{K}_s[\lambda_s,A] \) Therefore, \( x \notin \text{supp}\lambda_s,A \), by variational inequality (2.11). This shows (a).

Set \( z' = \beta + i\gamma \) with \( \beta, \gamma > 0 \). Since \( \mathcal{K}_s(\rho z, \rho w) = \rho^{-s}\mathcal{K}_s(z,w), \rho > 0 \), we may fix one of the variables \( x, \beta, \) or \( \gamma \). Let \( \beta = 1 \). From (2.15), (3.4), and Lemma 2.2(7) we get

\[
\Delta_s = \left[ (1+x)^2 + \gamma^2 \right]^{-s/2} 2F_1 \left( s/2, 1/2, \frac{4x}{(1+x)^2 + \gamma^2} \right) - \frac{1}{2} 2^{-s} (1 + \gamma^2)^{-s/2} 2F_1 \left( s/2, 1/2, \frac{1}{1+\gamma^2} \right) - \frac{1}{2} 2^{-s} \Gamma((1-s)/2) \frac{\Gamma((1-s)/2)}{\sqrt{\pi \Gamma(1-s/2)}}
\]

We approximate \( \Delta_s \) by its series expansion at \( s = 0 \). From Lemma 2.2(9) and (3.4)

\[
\lim_{s \to 0^+} \frac{\Delta_s}{s} = \mathcal{K}_0(x, 1 + i\gamma) - \mathcal{K}_0^*(1 + i\gamma, 1 + i\gamma) = \frac{1}{2} \log \left( \sqrt{(1+x)^2 + \gamma^2} + \sqrt{(1-x)^2 + \gamma^2} \right) > 0
\]

which implies (b).

We show that \( \Delta_s \) (as a function in \( s \)) is strictly concave on \((0,1)\) if \( x = \gamma = 1/2 \). Using (3.4) and integral representation (2.25) we get

\[
\Delta_s := \mathcal{K}_s(x, 1 + i\gamma) - \mathcal{K}_s^*(1 + i\gamma, 1 + i\gamma) = \frac{1}{\pi} \int_0^1 t^{-1/2} (1-t)^{-1/2} g(s,t) \, dt,
\]

where

\[
g(s,t) := \left[ (1+x)^2 + \gamma^2 - 4xt \right]^{-s/2} - \frac{1}{2} \left[ 4 (1 + \gamma^2) - 4t \right]^{-s/2} - \frac{1}{2} \left[ 4 - 4t \right]^{-s/2}.
\]
Negativity of \((\partial/\partial s)^2 g(s,t)\) for all \(0 \leq t < 1\) implies that \(\Delta_s\) is strictly concave. Let \(x = \gamma = 1/2\). From \((\partial/\partial s)^2 r^{-s/2} = (1/4)F(s,r), F(s,r) := r^{-s/2}(\log r)^2\), we get

\[
4 \left( \frac{\partial}{\partial s} \right)^2 g(s,t) = F(s,5/2 - 2t) - (1/2)F(s,5 - 4t) - (1/2)F(s,4 - 4t).
\]

Negativity of the right-hand side above is equivalent with

\[
\lim_{s \to \infty} \frac{\partial}{\partial s} g(s,t)
\]

where \(\frac{\partial}{\partial s}\) is a sufficient (but not necessary) condition for \((\partial/\partial s)^2 r^{-s/2} \geq 0\) if and only if \(0 < s < s_1\) in the interval \([0,1]\) for \(s_1 = 0.341107 \ldots\). Numerical computation shows that \(\Delta_{1/3} \approx 0.0011 > 0\). This can be rigorously justified by assistance of Mathematica and use of exact arithmetic.

By Theorem 3.7(a), the positivity of \(\Delta_s\) implies \(x \notin \text{supp} \lambda_{x,A}\). By (3.7), \(\Delta_s\) depends on three parameters \(x, \gamma, \) and \(s\). See Figure 2 for a plot of the level surface \(\Delta_s = 0\). This 0-level surface is the boundary of the set of admissible configurations \((x,1/\gamma,s)\) using a three point scheme \(z = x, z' = 1 + i\gamma, \) and \(z''\). From Figure 2 we get numerical evidence that the maximum \(s\) possible for a three point approach is about 0.38.

**Figure 2.** 0-level set of \(\Delta_s\) in (3.7) cut off at \(1/\gamma = 4\) and \(\Delta_s\) for \(x = 1/2, \gamma = 1/2\).

4. **Kernel \(K_s\) in the limit \(R \to \infty\)**

We want to study the behavior of \(K_s(R + z, R + w)\) as \(R\) becomes large.

**Lemma 4.1.** Let \(0 < s < 2, s \neq 1, \) and \(z, w \in \mathbb{H}^+\). Then

\[
K_s(R + z, R + w) = \mathcal{I}_s \left( S^1; \frac{d \phi}{2\pi} \right) R^{-s} - \frac{s}{1 - s} \frac{\Gamma((1 + s)/2)}{\sqrt{\pi} \Gamma((1 + s)/2)} \frac{|z - w|^{-s}}{2R}
\]

\[
-2^{-s} s \left( \frac{\Gamma((1 - s)/2)}{\sqrt{\pi} \Gamma((1 - s)/2)} \right) \frac{\Re [z - w]}{2R} R^{-s} + \mathcal{O} \left( \frac{s}{R^2} \right), \quad R \to \infty,
\]

where \(\mathcal{I}_s(S^1; d \phi/(2\pi))\) is given in (2.14).
Remark 4.2. In the case $1 < s < 2$ the second term in (4.1) becomes the dominant term. In the special case $s = 1$ the following expansion can be shown:

\[(4.2)\]
\[
2R K_1(R + z, R + w) = \frac{6\log 2}{\pi} + \frac{2}{\pi} \log R
- \frac{2}{\pi} \log |z - w| \left[ 1 - \frac{\Re[z - w_s]}{2R} + \mathcal{O}(R^{-2}) \right] + \mathcal{O}\left( \frac{\log R}{R^2} \right), \quad R \to \infty.
\]

Proof of Lemma 4.1. Let $0 < s < 2$, $s \neq 1$. Using [1, 15.3.6], we get a representation of (2.15),

\[
\mathcal{K}_s(R + z, R + w)
= \frac{\Gamma((1 - s)/2)}{\sqrt{\pi} \Gamma((1 - s)/2)} |2R + z - w_s|^{-s} \, _2F_1\left( \frac{s/2, 1/2}{1 + s/2}; \frac{|z - w|^2}{2R + z - w_s} \right)
- \frac{2}{1 - s} \frac{\Gamma((1 + s)/2)}{\sqrt{\pi} \Gamma(s/2)} |2R + z - w_s|^{-s} \, _2F_1\left( \frac{1 - s/2, 1/2}{1 + (1 - s)/2}; \frac{|z - w|^2}{2R + z - w_s} \right),
\]

with convergent series expansions of both hypergeometric functions. The first one is of the form $1 + \mathcal{O}(sR^{-2})$, the second one is of the form $1 + \mathcal{O}(R^{-2})$. Since

\[
\left| \frac{z - w_s}{2R} \right|^{-s} = 1 - \frac{s}{2} \, \frac{\Re[z - w_s]}{R} + \mathcal{O}\left( \frac{s}{R^2} \right), \quad R \to \infty,
\]
we get

\[
\mathcal{K}_s(R + z, R + w)
= 2^{-s} \frac{\Gamma((1 - s)/2)}{\sqrt{\pi} \Gamma((1 - s)/2)} R^{-s} \left[ 1 - \frac{s}{2} \, \frac{\Re[z - w_s]}{R} + \mathcal{O}\left( \frac{s}{R^2} \right) \right] \left[ 1 + \mathcal{O}\left( \frac{s}{R^2} \right) \right]
- \frac{1}{1 - s} \frac{\Gamma((1 + s)/2)}{\sqrt{\pi} \Gamma(s/2)} |z - w|^{-1} \left[ 1 + \mathcal{O}\left( \frac{1}{R} \right) \right] \left[ 1 + \mathcal{O}\left( \frac{1}{R^2} \right) \right].
\]

We reorder the terms with respect to powers of $R$ and obtain (4.1). □

It is convenient to define the following kernels

\[(4.3)\]
\[\mathcal{K}^{(R)}(z, w) := 2R \left[ \mathcal{K}_s(R + z, R + w) - \mathcal{I}_s(\mathbb{S}^1; d\phi/(2\pi)) \right] R^{-s}, \quad 0 < s < 1,
\]
\[(4.4)\]
\[\mathcal{K}^{(R)}(z, w) := 2R \mathcal{K}_s(R + z, R + w), \quad s > 1,
\]
and

\[(4.5)\]
\[\mathcal{K}^{(\infty)}(z, w) := -\frac{2}{1 - s} \frac{\Gamma((1 + s)/2)}{\sqrt{\pi} \Gamma(s/2)} |z - w|^{-1} = \frac{\Gamma((s - 1)/2)}{\sqrt{\pi} \Gamma(s/2)} |z - w|^{-s}.
\]

Then, by (4.1),

\[(4.6)\]
\[\lim_{R \to \infty} \mathcal{K}^{(R)}(z, w) = \mathcal{K}^{(\infty)}(z, w), \quad 0 < s < 1,
\]
and, from (2.18) and [1, 15.1.20], it follows

\[(4.7)\]
\[\lim_{R \to \infty} |z - w|^{-s-1} \mathcal{K}^{(R)}(z, w) = |z - w|^{-s-1} \mathcal{K}^{(\infty)}(z, w), \quad s > 1,
\]
where in both cases the convergence is uniform on compact subsets of $\mathbb{H}^\times \times \mathbb{H}^\times$. If $s < \dim \Gamma(A)$, we let $J_{\mathcal{K}^{(R)}}[g]$ and $J_{\mathcal{K}^{(\infty)}}[g]$ denote the associated energies of the compactly supported measure $\nu \in \mathcal{M}(\mathbb{H}^\times)$. From the definition of the kernel $\mathcal{K}^{(R)}$
we see that the equilibrium measure $\lambda_{s,A}^R$ on the compact set $A \subset \mathbb{H}^+$ for the kernel $K_s^{(R)}$ is equal to the equilibrium measure $\lambda_{s,R+A}$ on $R + A$ for the kernel $K_{s}$ in the following sense: $\lambda_{s,A}^R(B) = \lambda_{A+R}(R + B)$ for a Borel set $B \subset \mathbb{H}^+$.

**Remark 4.3.** The asymptotics (4.1) holds uniformly in $0 \leq s \leq s_0 < 1$. So

$$\lim_{s \to 0^+} K_s^{(R)}(z, w)/s = K_s^{(\infty)}(z, w) + O(1/R), \quad R \to \infty.$$  

The expression $K_s^{(\infty)}(z, w):= -\text{Re}[z-w]-|z-w|$ is the $\infty$-kernel for the logarithmic case introduced in [11]. However, reversing the order of limit processes, we get

$$\lim_{{R \to \infty}} K_s^{(R)}(z, w)/s = K_s^{(\infty)}(z, w)/s.$$  

Now, in the limit $s \to 0^+$, the right-hand side above tends to $-|z-w|$.

5. The Energy Problem for the Kernel $K_{s}^{(\infty)}$

5.1. The case $0 < s < 1$. The kernel

$$K_{s}^{(\infty)}(z, w) = -2 \frac{\Gamma((1+s)/2)}{1-s \sqrt{\pi} \Gamma(s/2)} |z-w|^{1-s}, \quad 0 < s < 1,$$

falls into a class of kernels studied by Björck [3]. From his results we infer that to every compact set $A \subset \mathbb{H}^+$ and every $0 < s < 1$ there exists a unique equilibrium measure $\lambda_{s,A}^\infty$ supported on the outer boundary of $A$. ("Outer boundary" is justified by the strict superharmonicity of the infinity kernel everywhere in $\mathbb{C}$.) Let $W_{K_{s}^{(\infty)}}$ denote the potential for a measure $\mu \in \mathcal{M}(A)$ and for the kernel $K_{s}^{(\infty)}$:

$$W_{K_{s}^{(\infty)}}(z):= \int_A K_{s}^{(\infty)}(z, w) \, d\mu(w), \quad z \in \mathbb{H}^+.$$  

Then $W_{K_{s}^{(\infty)}}$ is continuous on $\mathbb{H}^+$ and from results in [3] there follows that $W_{K_{s}^{(\infty)}} \geq J_{K_{s}^{(\infty)}}[\lambda_{s,A}^\infty]$ on $A$ and equality holds on $\text{supp} \lambda_{s,A}^\infty$. We note, that $\lambda_{s,A}^\infty$ converges weak-star to $\lambda_{s,A}^\infty$ as $R \to \infty$. This follows from the weak-star compactness of $\mathcal{M}(A)$, relation (4.6), and the uniqueness of the equilibrium measure $\lambda_{s,A}^\infty$.

Suppose the curve $\gamma: [a, b] \to \mathbb{H}^+$ covers the outer boundary $S$ of $A$. Set $r_w = |\gamma(t) - w|$. Assuming $\gamma$ is twice differentiable at $t$ we have

$$\frac{d^2}{dt^2} K_{s}^{(\infty)}(\gamma(t), w) = 2 \frac{\Gamma((1+s)/2)}{\sqrt{\pi} \Gamma(s/2)} \left[ s (r_w')^2 - r_w r_w'' \right] r_w^{-s-1}. \quad (5.1)$$  

Then for fixed $w$, we have that $K_{s}^{(\infty)}(\gamma(t), w)$ is strictly convex on any interval where $s (r_w')^2 > r_w r_w''$. A sufficient condition would be $r_w'' < 0$.

In the following we give examples of compact sets $A \subset \mathbb{H}^+$ such that the support of the equilibrium measure on $A$ is given by the outer boundary of $A$.

**Lemma 5.1.** Let $A$ be a compact subset of the line-segment $[z', z'']$ in the interior of $\mathbb{H}^+$, $z'' - z' = 2re^{i\phi}$, $r > 0$, $0 \leq \phi < \pi$. Then $\text{supp} \lambda_{s,A}^\infty = A$ for all $0 < s < 1$. In particular, if $A = [z', z'']$, then

$$d \lambda_{s,A}^\infty(w) = \frac{\Gamma((1+s)/2)}{\sqrt{\pi} \Gamma(s/2)} r^{1-s} \left( r^2 - T^2 \right)^{s/2-1} dT, \quad (5.2)$$  

where $w = (z' + z'')/2 + Te^{i\phi}$, $|T| \leq r$. 

THE SUPPORT OF THE LIMIT DISTRIBUTION ... 15
Proof. W.l.o.g. consider the parametrization \( \gamma(t) = t e^{i\phi}, \ |t| \leq r. \) Then
\[
\frac{d^2}{d\beta^2} K^{(\infty)}_s(\gamma(t), \gamma(T)) = 2 s \frac{\Gamma((1+s)/2)}{\sqrt{\pi} \Gamma(s/2)} |t-T|^{-1-s} > 0, \quad 0 < s < 1.
\]
By Theorem 3.1 there exists an interval \( I \subset [-1, 1] \) such that \( \text{supp} \lambda^\infty_{s,A} = \gamma(I) \cap A. \)

Since the kernel \( K^{(\infty)}_s(\gamma(t), \gamma(T)) \) decreases as \( |t-T| \) grows, there follows that the equilibrium potential is strictly less than \( J_{K^{(\infty)}_s} [\lambda^\infty_{s,A}] \) on \( (-\infty, e^{i\phi}, \infty e^{i\phi}) \setminus \gamma(I). \) But the equilibrium potential is given by \( J_{K^{(\infty)}_s} [\lambda^\infty_{s,A}] \) on \( A. \) So, \( \text{supp} \lambda^\infty_{s,A} = A. \) Relations (5.2) follow from the constancy of the integral
\[
\int_{-r}^{r} (|t-T|^{1-s} (r^2 - T^2)^{s/2-1}) dT = \Gamma(s/2) \Gamma(1-s/2)
\]
and the fact that the \( K^{(\infty)}_s \)-potential for this measure (5.2) is strictly decreasing away from the line-segment. We used the auxiliary result Lemma 5.2.

\[\square\]

Lemma 5.2 ([14, Hilfssatz I]). Let \( -1 < \alpha < 1, \alpha \neq 0. \) Then for \( -1 \leq y \leq 1: \)
\[
\int_{-1}^{1} (1 - x^2)^{-(1+\alpha)/2} |x - y|^\alpha \, dx = \Gamma \left( \frac{1-\alpha}{2} \right) \Gamma \left( \frac{1+\alpha}{2} \right) = \frac{\pi}{\cos(\pi \alpha/2)}.
\]

Lemma 5.3. Let the outer boundary \( S \) of the compact set \( A \) be a subset of a circle \( C \) centered at \( a > 0 \) with radius \( 0 < r < a. \) Then \( \lambda^\infty_{s,A} = S \) for every \( 0 < s < 1. \) In particular, if \( S = C, \) then \( \lambda^\infty_{s,A} \) is given by the normalized arc-length measure on \( C \) and \( \lambda^\infty_{s,A} = C \) for all \( 0 < s < 1. \)

The result \( \lambda^\infty_{s,A} = S \) for \( 0 < s < 1 \) differs considerably from the logarithmic case. By [11, Thm. 4], one has \( \lambda^\infty_{s,A} = \{a + r e^{i\theta} | \ |\theta| \leq \theta \} \) for some \( \theta \in [0, \pi/3]. \)

Proof. W.l.o.g. consider the parametrization \( \gamma(\phi) = r e^{i\phi}, \ 0 \leq \phi \leq 2\pi. \) Then
\[
K^{(\infty)}_s(\gamma(\phi), \gamma(\phi')) = - \frac{2s-1}{s} \frac{\Gamma((1+s)/2)}{\sqrt{\pi} \Gamma(s/2)} \left| \frac{\phi - \phi'}{2} \right|^{s-1} r^{1-s}.
\]
By direct calculation (assisted by Mathematica)
\[
\frac{d^2}{d\phi^2} K^{(\infty)}_s(\gamma(\phi), \gamma(\phi')) = 2s-1 \frac{\Gamma((1+s)/2)}{\sqrt{\pi} \Gamma(s/2)} \left| \frac{\phi - \phi'}{2} \right|^{s} r^{1-s} > 0.
\]
Since \( \gamma \) is a simple closed continuous curve and \( \gamma(\phi) = \gamma(\phi + 2\pi), \) by Theorem 3.1, \( \text{supp} \lambda^\infty_{s,A} = S. \) In the case \( S = C, \) rotational symmetry gives \( d \lambda^\infty_{s,A} = d \phi/(2\pi). \) \[\square\]

5.2. The case \( s > 1. \) The kernel
\[
K^{(\infty)}_s(z, w) = \frac{\Gamma((s-1)/2)}{\sqrt{\pi} \Gamma(s/2)} \frac{1}{|z-w|^{s-1}}, \quad s > 1,
\]
is (up to a multiplicative constant) the Riesz-(s-1)-kernel in the plane \( \mathbb{R}^2 \) which can be identified with \( \mathbb{C}. \) If \( 1 < s < 1 + \dim A, \) then classical potential theory yields that there exists a unique equilibrium measure \( \lambda^\infty_{s,A} \) on \( A \) with \( \lambda^\infty_{s,A} \supset A, \) where \( A \) denotes the set of all points of \( A \) each neighborhood of which intersects \( A \) in a set of positive Riesz (s-1)-capacity. Examples of sets \( A \) with \( A = A \) are line-segments, circles, or more generally, any Jordan curve; discs, “washers”.

Lemma 5.4. Let \( A \) be a compact subset of \( \mathbb{C} \) with \( \dim A > 0 \) and \( s \) a real number with \( 1 < s < 1 + \dim A. \) Then \( \lambda^R_{s,A} \) converges weak-star to \( \lambda^\infty_{s,A} \) as \( R \to \infty. \)
Proof. To simplify notation we use the abbreviations \( \mathcal{K}_R = \mathcal{K}_s^{(R)} \), \( \mathcal{K}_\infty = \mathcal{K}_s^{(\infty)} \), \( \lambda_R = \lambda_{s,A}^R \), and \( \lambda_\infty = \lambda_{s,A}^{\infty} \). From the definition of \( \mathcal{K}_R \) given in (4.4) and the formula (2.18) it follows that

\[
\mathcal{K}_R(z, w) = \Omega_R(z, w)\mathcal{K}_\infty(z, w), \quad (z, w) \in \mathbb{C} \times \mathbb{C},
\]

where

\[
\Omega_R(z, w) := \frac{\mathbf{2F1}(1 - s/2, 1/2, 1 - \frac{|z-w|^2}{2R^2 - |z-w|^2})}{|1 + \frac{z-w}{2R}| \mathbf{2F1}(1 - s/2, 1/2, 1)}.
\]

We remark that \( \Omega_R \) converges uniformly to 1 on compact subsets of \( \mathbb{C} \times \mathbb{C} \) as \( R \to \infty \).

Since \( \mathcal{M}(A) \) is weak-star-compact, there exists a weak-star cluster point \( \lambda^* \) of \( \lambda_R \) as \( R \to \infty \). We will show that \( J_{\mathcal{K}_{Rk}}[\lambda] \leq J_{\mathcal{K}_{\infty}}[\lambda_\infty] \) from which the Lemma will immediately follow. Let \( R_k, k \geq 1 \), be a sequence of numbers such that \( \lim_{k \to \infty} R_k = \infty \) and \( \lambda_{R_k} \rightharpoonup \lambda^* \in \mathcal{M}(A) \) as \( k \to \infty \). Thus \( \Omega_{R_k}(\lambda_{R_k} \times \lambda_{R_k}) \rightharpoonup \lambda^* \times \lambda^* \) as \( k \to \infty \) and we have (see [12, Lemma 0.1])

\[
J_{\mathcal{K}_{R_k}}[\lambda^*] \leq \liminf_{k \to \infty} J_{\mathcal{K}_{R_k}}[\lambda_{R_k}] \leq \liminf_{k \to \infty} J_{\mathcal{K}_{R_k}}[\lambda_\infty],
\]

where the second inequality follows since \( \lambda_{R_k} \) minimizes \( J_{\mathcal{K}_{R_k}} \). Finally, since \( \Omega_R \) converges uniformly to 1 on \( A \times A \) as \( R \to \infty \), we have \( \liminf_{k \to \infty} J_{\mathcal{K}_{R_k}}[\lambda_\infty] = J_{\mathcal{K}_{\infty}}[\lambda_\infty] \) which shows that \( J_{\mathcal{K}_{\infty}}[\lambda^*] \leq J_{\mathcal{K}_{\infty}}[\lambda_\infty] \). Since \( J_{\mathcal{K}_{\infty}} \) has a unique minimizer, it follows that \( \lambda_\infty \) is the only weak-star cluster point of \( \lambda_R \) as \( R \to \infty \). \( \square \)

In the hyper-singular case \( s > 1 + \dim A = \dim \Gamma(A) \), both energy integrals \( J_{\mathcal{K}_s^{(R)}}[\nu] \) and \( J_{\mathcal{K}_s^{(\infty)}}[\nu] \) are infinite for every \( \nu \in \mathcal{M}(A) \). In Section 6.2 and 6.3 we consider the limit distribution of minimal \( \mathcal{K}_s \)-energy and \( \mathcal{K}_s^{(\infty)} \)-energy \( N \)-point systems as \( N \to \infty \) for the hyper-singular case and for sufficiently “nice” sets \( A \) (namely \( d \)-rectifiable sets).

6. DISCRETE MINIMUM ENERGY PROBLEMS ON \( A \subset \mathbb{H}^+ \)

In this section we discuss the discrete Riesz \( s \)-energy problem on \( \Gamma(A) \subset \mathbb{R}^3 \) as well as the discrete \( \mathcal{K} \)-energy problem on \( A \subset \mathbb{H}^+ \) for the kernel \( \mathcal{K} = \mathcal{K}_s \), \( \mathcal{K} = \mathcal{K}_s^{(R)} \), and \( \mathcal{K} = \mathcal{K}_s^{(\infty)} \). The \( N \)-point Riesz \( s \)-energy of \( \Gamma(A) \) is defined as

\[
\mathcal{E}_s(\Gamma(A), N) := \min \{ E_s(X_N) \},
\]

where the minimum is taken over all \( N \)-point configurations \( X_N \subset \Gamma(A) \) and \( E_s(X_N) \) is defined as in (1.1). We let \( X_N^s = X_{N,s}^* \) denote an \( N \)-point configuration in \( \Gamma(A) \) attaining this minimum.

Similarly, for an \( N \)-point configuration \( Z_N = \{ z_1, \ldots, z_N \} \subset A \), let

\[
E_\mathcal{K}(Z_N) := \sum_{j \neq k} \mathcal{K}(z_j, z_k)
\]

and let the \( N \)-point \( \mathcal{K} \)-energy of \( A \) be defined as

\[
\mathcal{E}_\mathcal{K}(A, N) := \min \{ E_\mathcal{K}(Z_N) \},
\]

over all \( N \)-point configurations \( Z_N \subset A \). This minimum is attained at a minimal \( \mathcal{K} \)-energy \( N \)-point system \( Z_N^\mathcal{K} = \{ z_1^\mathcal{K}, \ldots, z_N^\mathcal{K} \} \), that is \( \mathcal{E}_\mathcal{K}(A, N) = E_\mathcal{K}(Z_N^\mathcal{K}) \). Finally, let \( \lambda(Z_N^\mathcal{K}) := (1/N) \sum_{k=1}^N \delta_{z_k^\mathcal{K}} \). We are interested in the weak-star convergence of \( \lambda(Z_N^\mathcal{K}) \) and in the asymptotic growth of \( \mathcal{E}_\mathcal{K}(A, N) \) as \( N \to \infty \).
6.1. The potential theory case. For $0 < s < \dim \Gamma(A)$, there is a unique equilibrium measure $\lambda_{K,A}$ minimizing the $K$-energy

$$J_K[\lambda] = \int K(z, w) \, d\lambda(z) \, d\lambda(w)$$

over measures $\lambda \in \mathcal{M}(A)$. (See Proposition 2.1 for the case $K = K_s$ and Björck [3] for the case $K = K_s(\infty)$.)

**Proposition 6.1.** Suppose $A$ is an infinite compact subset in the interior of $\mathbb{H}^+$. Let $K = K_s$, $K = K_s^{(R)}$, or $K = K_s^{(\infty)}$ and $0 < s < \dim \Gamma(A)$. For $N \geq 2$, let $Z_N^*$ be a minimal $K$-energy configuration of $N$ points $\{z_1^*, \ldots, z_N^*\} \subset A$. Then $\lambda(Z_N^*)$ converges weak-star to the equilibrium measure $\lambda_{K,A}$ on $A$ as $N \to \infty$ and

$$\lim_{N \to \infty} \frac{E_K(A, N)}{N^2} = J_K[\lambda_{K,A}].$$

**Proof.** The proof follows using standard arguments as in [14], [12, pp. 160–162] and [10]. The essential ingredients of the proof are the boundedness of the sequence (6.1) and existence and uniqueness of the equilibrium measure $\lambda_{K,A}$. □

In Figure 3 we show minimal $K_s$-energy configurations for $N = 32$ points restricted to a Cassinian oval for various values of $s$ with $0 < s < 1$. A somewhat surprising result of these numerical experiments is that for fixed $N$ and for $s$ close to $1^{-}$ a rather large part of the Cassinian oval is free of points. (Note, that in the case $s = 1$, the support of the equilibrium measure $\lambda_{s,A}$ is $A$.) In Figure 4 we show minimal $K_s^{\infty}$-energy configurations for $N = 32$ points restricted to a Cassinian oval for various values of $s$ with $0 < s < 1$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Minimum $K_s$-energy configurations ($N = 32$ points) for $s = 0, 0.5, 0.75, \text{ and } 1$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Minimum $K_s^{\infty}$-energy configurations ($N = 32$ points) for $s = 0, 0.5, 0.75, \text{ and } 1$.}
\end{figure}

Numerical experiments for a circle $C$ centered at $a > 0$ with radius $r$ ($0 < r < a$) suggest that for a fixed $s$, say $s = 1/4$, the equilibrium measure $\lambda_{s,C}$ is concentrated on a proper subset of $C$. (For what we can prove, see Example 3.5.) However, the equilibrium measure $\lambda_{s,C}^R$ associated with the translate $R + C$ converges weak-star to $\lambda_{s,C}^{\infty}$ as $R \to \infty$ and we can show (see Lemma 5.3) that $\lambda_{s,C}^{\infty}$ is the uniform measure on the circle $C$ for all $0 < s < 1$. This phenomenon that the support of $\lambda_{s,C}^R$ seems to spread out as $R \to \infty$ is illustrated by considering discrete minimal $K_{s}$-energy points on the translate $R + C$ for varying values of $R$. In Figure 5 we show minimal $K_s$-energy configurations for $N = 40$ points restricted to translates $R + C$ of the unit circle $C$ centered at $a = 1$, where $R = 10^k/2$ ($k = 0, 1, \ldots, 5$).
6.2. The hypersingular case for \( d \)-rectifiable sets. Suppose \( s \geq \dim \Gamma(A) \). In this subsection we require that \( A \) be a \( d \)-rectifiable subset of the interior of \( \mathbb{H}^+ \) \((d = 1, 2)\). Recall that a set \( K \subset \mathbb{R}^p \) is \( d \)-rectifiable, \( d \leq p \), if it is the image of a bounded set \( B \) in \( \mathbb{R}^d \) with respect to a Lipschitz mapping, that is a mapping \( \phi : B \to \mathbb{R}^p \) that satisfies for some positive constant \( c \)
\[
|\phi(x) - \phi(y)| \leq c|x - y| \quad \text{for all } x, y \in B.
\]

Note that every compact subset of \( \mathbb{R}^2 \) is 2-rectifiable. Also note that if \( A \) is a \( d \)-rectifiable set in \( \mathbb{H}^+ \), then \( \Gamma(A) \) is a \((d + 1)\)-rectifiable set in \( \mathbb{R}^3 \), \( d = 1, 2 \). In order to avoid complications, we require that \( A \) is in the interior of \( \mathbb{H}^+ \). In this case \( \dim \Gamma(A) = 1 + \dim A \).

Using the properties of \( K_s \) as given in Lemma 2.2 it follows that \( K_s(z, w) = \Omega(z, w)|z - w|^{s - d} \), where \( \Omega : A \times A \to \mathbb{R} \) is continuous and positive. In the terminology of [4] then \( \Omega \) is a CPD-weight function on \( A \) (see [4] for the general definition of CPD weight function). Also, note that
\[
\Omega(w, w) = \frac{\Gamma((s - 1)/2)}{\sqrt{\pi} \Gamma(s/2)} \frac{1}{|w - w_s|}.
\]
If \( A \) is a compact set in \( \mathbb{R}^p \) and \( \Omega \) is a CPD-weight function on \( A \times A \), then for \( s \geq d \) one can define the weighted Hausdorff measure \( \mathcal{H}_d^{s, \Omega} \) on Borel sets \( B \subset \mathbb{H}^+ \) by
\[
\mathcal{H}_d^{s, \Omega}(B) := \int_{B \cap A} [\Omega(w, w)]^{-d/s} \, d\mathcal{H}_d(w).
\]

Then the following result is a corollary of Theorem 2 in [4].

**Proposition 6.2.** Let \( d = 1 \) or 2 and suppose \( A \) is a compact \( d \)-rectifiable set contained in the interior of \( \mathbb{H}^+ \) with positive \( d \)-dimensional Hausdorff measure \( \mathcal{H}_d(A) \). Let \( s > \dim \Gamma(A) \). For \( N \geq 2 \), let \( Z_N^s \) be a minimal \( K_s \)-energy configuration of \( N \) points \( \{z_1^N, \ldots, z_N^N\} \subset A \). Then the sequence \( Z_N^s \), \( N \geq 2 \), is asymptotically uniformly distributed with respect to \( \mathcal{H}_d^{s-1, \Omega} \); that is,
\[
\frac{1}{N} \sum_{k=1}^{N} \delta_{z_k^N} \Rightarrow \frac{\mathcal{H}_d^{s-1, \Omega}}{\mathcal{H}_d^{s-1, \Omega}(A)}, \quad \text{as } N \to \infty.
\]

Moreover, the minimal \( N \)-point \( K_s \)-energy satisfies
\[
\lim_{N \to \infty} \mathcal{E}_{K_s}(A, N) = \frac{C_{s-1, d}}{\mathcal{H}_d^{s-1, \Omega}(A)} \left[ \frac{1}{N^{1+(s-1)/d}} \right],
\]
where \( C_{s-1, d} \) is a positive constant which does not depend on \( A \) and \( N \).

**Remark 6.3.** The constant \( C_{s-1, d} \) is exactly the same constant which appears in the analogue of (6.5) for the non-weighted case, that is for \( \Omega(z, w) = 1 \) for all \( z, w \in A \).
It can be represented using the Riesz $s$-energy for the unit cube in $\mathbb{R}^d$ via

\[
C_{s,d} = \lim_{N \to \infty} \frac{\mathcal{E}_s([0,1]^d,N)}{N^{1+s/d}}, \quad s > d.
\]

It was shown in [13] that $C_{s,1} = 2 \zeta(s)$, where $\zeta(s)$ is the classical Riemann zeta function. However, for other values of $d$, the constant $C_{s,d}$ is as yet unknown.

In the boundary case $s = \dim \Gamma(A)$ and for a 1-rectifiable set $A$ an additional regularity condition is needed to prove a result analogous to Proposition 6.2. The following result is a corollary of Theorem 3 in [4].

**Proposition 6.4.** Let $d = 1$ or 2 and suppose $A$ is a compact $d$-rectifiable set contained in the interior of $\mathbb{H}^+$ with positive $d$-dimensional Hausdorff measure $\mathcal{H}_d(A)$. If $d = 1$, we further require that $A$ is a subset of a $C^1$-curve. Let $s = 1 + d$. For $N \geq 2$, let $Z_N^s$ be a minimal $K_{s+1}$-energy configuration of $N$ points $\{z_1^{d+1,N}, \ldots, z_N^{d+1,N}\} \subset A$. Then the sequence $Z_N^s$, $N \geq 2$, is asymptotically uniformly distributed with respect to $\mathcal{H}_d^{s,\Omega}$; that is,

\[
\frac{1}{N} \sum_{k=1}^N \delta_{z_{d+1,N}^k} \xrightarrow{\ast} \frac{\mathcal{H}_d^{s,\Omega}}{\mathcal{H}_d^{s,\Omega}(A)}, \quad \text{as } N \to \infty.
\]

Moreover, the minimal $N$-point $K_s$-energy satisfies

\[
\lim_{N \to \infty} \frac{\mathcal{E}_{K_{d+1}}(A,N)}{N^2 \log N} = \frac{\beta_d}{\mathcal{H}_d^{d,\Omega}(A)},
\]

where $\beta_d = \pi^d/2 / \Gamma(1 + d/2)$ is the volume of the unit ball in $\mathbb{R}^d$.

**Remark 6.5.** It is a consequence of Theorem 2 and 3 in [4] that minimal Riesz $s$-energy point configurations $X_N^s \subset \Gamma(A)$ are asymptotically uniformly distributed with respect to $\mathcal{H}_d s$ restricted to $\Gamma(A)$ in the hypersingular case $s \geq d + 1$. For $z \in \mathbb{H}^+$, let $\delta_z$ denote the rotationally symmetric probability measure supported on $\Gamma(\{z\})$. When $s < d + 1$ we have that both $(1/N) \sum_{x \in X_N^s} \delta_x$ and $(1/N) \sum_{z \in Z_N^s} \delta_z$ converge weak-star to $\mu_{s,\Gamma(A)}$. However, for $s > d + 1$, the discrete probability measure $(1/N) \sum_{x \in X_N^s} \delta_x$ converge weak-star to $\mathcal{H}_{s+1}$ (normalized and restricted to $\Gamma(A)$), while Proposition 6.2 implies that $(1/N) \sum_{z \in Z_N^s} \delta_z$ converges to a measure that depends on $s$. In the boundary case $s = d + 1$, the latter limit distributions are equal (cf. Proposition 6.4).

In the following we consider two examples: a line-segment in general position and a circle centered on the real axis.

**Example 6.6.** Let $A$ be the line segment with parametrization $\gamma(t) = R \cos \phi$, $|t| \leq 1$, where $R > \cos \phi$ and $0 \leq \phi < \pi$ fixed. Then $A$ is a 1-rectifiable set and the weighted Hausdorff measure $\mathcal{H}_1^{s,\Omega}$ can be explicitly calculated. Indeed, since $d \mathcal{H}_1^s(t) = dt$, we get for $s > 2$

\[
d \frac{\mathcal{H}_d^{s-1,\Omega}}{\mathcal{H}_d^{s,\Omega}(A)}(t) = \frac{(R + t \cos \phi)^{1/(s-1)} \, dt}{s - 1 (R + \cos \phi)^{s/(s-1)} - (R - \cos \phi)^{s/(s-1)}}, \quad |t| \leq 1.
\]
Note, in the case of the vertical line-segment (that is $\phi = \pi/2$), the last expression reduces to
\[
(6.9) \quad \frac{d\mathcal{H}^{s-1,\Omega}_{d}}{\mathcal{H}^{1,\Omega}_{d}(A)}(t) = \frac{1}{2} d\,t, \quad |t| \leq 1; s > 2.
\]
In Figure 6 we show minimal $K_2$ and $K_4$-energy configurations for $N = 40$ points restricted to the line-segment with $R = 3/2$ and $\phi = \pi/4$.

Example 6.7. Let $A$ be the unit circle centered at $R > 1$. Then $A$ is a 1-rectifiable set and the weighted Hausdorff measure $\mathcal{H}^{1,\Omega}_{1}$ can be explicitly calculated. Since $d\mathcal{H}_{1}(\phi) = d\phi$, one has for $s > 1$
\[
(6.10) \quad \frac{d\mathcal{H}^{s-1,\Omega}_{d}(\phi)}{\mathcal{H}^{1,\Omega}_{d}(A)} = \frac{1}{2\pi} \frac{(R + \cos \phi)^{1/(s-1)}}{R + 1} \frac{d\phi}{\sqrt{s/2}}, \quad -\pi \leq \phi \leq \pi.
\]
In Figure 6.2 we show minimal $K_{2}$ and $K_{4}$-energy configurations for $N = 40$ points restricted to the unit circle centered at $R = 3/2$.

Remark 6.8. Results similar to Proposition 6.2 and Proposition 6.4 also hold for the kernel $K_{s}^{(R)}$. In this case the diagonal of the CPD weight function becomes
\[
\Omega_{R}^{s}(w, w) := \frac{\Gamma((s - 1)/2)}{\Gamma(s/2)} \left| 1 - \frac{w - w^*}{2R} \right|^{-1}.
\]
In particular, we have the limit
\[
\lim_{R \to \infty} \Omega_{R}^{s}(z, w) = \frac{\Gamma((s - 1)/2)}{\sqrt{s/2} \Gamma(s/2)}, \quad s > \text{dim } \Gamma(A),
\]
where the convergence is uniform on compact subsets of $\mathbb{H}^+ \times \mathbb{H}^+$. Consequently,
\[
(6.11) \quad \frac{\mathcal{H}^{s-1,\Omega,R}_{d}(A)}{\mathcal{H}^{1,\Omega,R}_{d}(A)} \to \mathcal{H}|_{A}/\mathcal{H}(A), \quad \text{as } R \to \infty \text{ and } s \geq \text{dim } \Gamma(A).
\]
Here $\mathcal{H}|_{A}/\mathcal{H}(A)$ is the limit distribution of minimal $K_{s}^{\infty}$-energy $N$-point configurations as $N \to \infty$ (see next subsection).

Of interest is the question of how well minimal $K$-energy points are separated, that is, we are asking for a lower bound for the separation radius
\[
\delta(Z_{N}) := \min \{|z - w| \mid z, w \in Z_{N}, z \neq w\}
\]
of optimal $K$-energy $N$-point systems $Z_{N}$ valid for $N \geq 2$. In fact, such an estimate can be obtained on sets of arbitrary Hausdorff dimension $\alpha$. The following result is a corollary of Theorem 4 in [4].
Proposition 6.9. Let $0 < \alpha < 2$. Suppose $A$ is a compact subset in the interior of $\mathbb{H}^+$ with $\mathcal{H}_\alpha(A) > 0$. Let $K = K_\alpha$ or $K = K_R$ with $R > 0$. Then for every $s \geq \alpha$ there is a constant $c_\alpha = c_\alpha(A, \Omega, \alpha) > 0$, where $\Omega$ is the CPD-weight function associated with $K$, such that any $K$-energy minimizing configuration $Z^*_N$ on $A$ satisfies the inequality

$$\delta(Z^*_N) \geq \begin{cases} c_\alpha N^{-1/\alpha} & s > \alpha, \\ c_\alpha (N \log N)^{-1/\alpha} & s = \alpha, \end{cases} \quad N \geq 2.$$  

6.3. The hypersingular case for the kernel $K_s^\infty$. Suppose $s \geq 1 + \dim A$. The kernel $K_s^\infty$ can be written as $K_s^\infty(z, w) = \Omega^\infty(z, w)|z - w|^{-s}$, where the CPD weight function $\Omega^\infty(z, w) = \Gamma(((s - 1)/2)/\Gamma(s/2)|z - w|^{-s}$ is a positive constant. Thus, we can apply the theory developed in [4] to obtain

Proposition 6.10. Let $d = 1$ or $d = 2$. Suppose $A$ is a compact $d$-rectifiable set contained in the interior of $\mathbb{H}^+$ with positive $d$-dimensional Hausdorff measure $\mathcal{H}_d(A)$. Let $s > 1 + \dim A$. For $N \geq 2$, let $Z^*_N$ be a minimal $K_s^\infty$-energy configuration of $N$ points $\{z_1^N, \ldots, z_N^N\} \subset A$. Then the sequence $Z^*_N, N \geq 2$, is asymptotically uniformly distributed with respect to $\mathcal{H}_d$; that is,

$$\frac{1}{N} \sum_{k=1}^{N} \delta_{z_k^N} \rightarrow \frac{\mathcal{H}_d|_A}{\mathcal{H}_d(A)}, \quad \text{as } N \rightarrow \infty.$$  

Moreover, the minimal $N$-point $K_s$-energy satisfies

$$\lim_{N \rightarrow \infty} \frac{\mathcal{E}_{K_s}(A, N)}{N^{1+(s-1)/d}} = \frac{C_{s-1,d}^\infty}{\mathcal{H}_d(A)^{(s-1)/d}},$$  

where $C_{s-1,d}^\infty$ is a positive constant which does not depend on $A$ and $N$. In fact, $C_{s-1,d}^\infty = C_{s-1,d} \Gamma((s - 1)/2)/[\sqrt{\pi} \Gamma(s/2)]$ and $C_{s-1,d}$ is the same constant as in (6.6).

Remark 6.11. The first part of Proposition 6.10 holds for the boundary case $s = 1 + d$, $d = \dim A$, as well. (In the case $d = 1$ it is also required that $A$ is contained in a $C^1$-curve.) The minimal $N$-point $K_s$-energy satisfies

$$\lim_{N \rightarrow \infty} \frac{\mathcal{E}_{K_s}(A, N)}{N^2 \log N} = \frac{\beta_d}{\mathcal{H}_d(A)},$$  

where $\beta_d = \pi^{d/2}/\Gamma(1 + d/2)$ is the volume of the unit ball in $\mathbb{R}^d$.

We remark that a separation result like Proposition 6.9 can also be stated for $K = K_s^\infty$. 
Appendix A. Convexity of the $K_s$-kernel on vertical line-segments

The parameter $v$ is fixed. The second derivative of (3.1) with respect to $y$ is

$$s^{-1} \left(4R^2 + \Delta^2\right)^{4+s/2} \frac{d^2}{dy^2} K_s(R + iy, R + iv)$$

$$= - \left(4R^2 + \Delta^2\right)^2 \left[4R^2 - (1 + s) \Delta^2\right] _2 F_1 \left(\frac{1}{2}, \frac{s/2}{1}; \frac{1}{4R^2 + \Delta^2}\right)$$

$$- 2R^2 \left(4R^2 + \Delta^2\right) \left[4R^2 - (3 + 2s) \Delta^2\right] _2 F_1 \left(\frac{3}{2}, \frac{1 + s/2}{2}; \frac{4R^2}{4R^2 + \Delta^2}\right)$$

$$+ 6R^4 (2 + s) \Delta^2 _2 F_1 \left(\frac{5}{2}, \frac{2 + s/2}{3}; \frac{4R^2}{4R^2 + \Delta^2}\right),$$

where $\Delta$ denotes the difference $(y - v)$. Applying to each hypergeometric function the linear transformation [1, 15.3.3] and simplifying we get

$$s^{-1} \left(4R^2 + \Delta^2\right)^{5/2} |\Delta|^{1+s} \frac{d^2}{dy^2} K_s(R + iy, R + iv)$$

$$= (1 + s) \Delta^4 _2 F_1 \left(\frac{1}{2}, \frac{1 - s/2}{1}; \frac{4R^2}{4R^2 + \Delta^2}\right)$$

$$+ 2R^2 \left[4sR^2 + (1 + 2s) \Delta^2\right] _2 F_1 \left(\frac{1}{2}, \frac{1 - s/2}{2}; \frac{4R^2}{4R^2 + \Delta^2}\right),$$

which implies $d^2[K_s(\gamma(y), \gamma(v))]/dy^2 > 0$.

References

[1] M. Abramowitz and I.A. Stegun (eds.), Handbook of mathematical functions with formulas, graphs, and mathematical tables., Dover, 1970.

[2] G.E. Andrews, R. Askey, and R. Roy, Special functions, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999. MR MR1688958 (2000g:33001)

[3] G. Björck, Distributions of positive mass, which maximize a certain generalized energy integral., Ark. Mat. 3 (1956), 255–269.

[4] S.V. Borodachov, D.P. Hardin, and E.B. Saff, Asymptotics for discrete weighted minimal Riesz energy problems on rectifiable sets, Trans. Amer. Math. Soc., to appear.

[5] Bowick, M.; Cacciuto, A.; Nelson, D. R.; Travesset, A.; Crystalline Order on a Sphere and the Generalized Thomson Problem, Phys. Rev. Lett. 89 (2002); 185502

[6] M. Bowick, D.R. Nelson, and A. Travesset, Curvature-induced defect unbinding in toroidal geometries, Phys. Rev. E 69 (2004), 041102–041113.

[7] R. Cade, A perturbation method for solving torus problems in electrostatics, J. Inst. Math. Appl. 21 (1978), 265–284.

[8] P.D. Dragnev and E.B. Saff, Riesz spherical potentials with external fields and minimal energy points separation, Potential Anal. 26 (2007), no. 2, 139–162. MR MR2276529

[9] V. I. Fabrikant, T. S. Sankar, and M. N. S. Swamy, On the generalized potential problem for a surface of revolution, Proc. Amer. Math. Soc. 90 (1984), no. 1, 47–56.

[10] B. Farkas and B. Nagy, Transfinite diameter, Chebyshev constant and energy on locally compact spaces, cf. arXiv:0704.0859v1, 2007.

[11] D.P. Hardin, E.B. Saff, and H. Stahl, Support of the logarithmic equilibrium measure on sets of revolution in $\mathbb{R}^3$, J. Math. Phys. 48 (2007), no. 2, 022901(14).

[12] N.S. Landkof, Foundations of modern potential theory, Springer-Verlag, New York, 1972, Translated from the Russian by A. P. Dohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[13] A. Martinez-Finkelshtein, V. Maymeskul, E.A. Rakhmanov, and E.B. Saff, Asymptotics for minimal discrete Riesz energy on curves in $\mathbb{R}^4$, Canad. J. Math. 56 (2004), no. 3, 529–552.
[14] G. Pólya and G. Szegő, "Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen und räumlichen Punktmengen.," J. Reine Angew. Math. 165 (1931), 4–49.

[15] R. Shail, "Some potential problems for slender tori," J. Inst. Math. Appl. 24 (1979), no. 3, 303–325. MR 80i:65145

[16] H.M. Srivastava and R.N. Siddiqi, "A unified presentation of certain families of elliptic-type integrals related to radiation field problems," Radiat. Phys. Chem. 46 (1995), no. 3, 303–315.

[17] R. Womersley, "Visualization of Minimum Energy Points on the Torus," http://web.maths.unsw.edu.au/~rsw/Torus/, 2005.

J. S. Brauchart, D. P. Hardin and E. B. Saff: Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: Johann.Brauchart@Vanderbilt.Edu
E-mail address: Doug.Hardin@Vanderbilt.Edu
E-mail address: Edward.B.Saff@Vanderbilt.Edu