CLOSED INTERSECTING FAMILIES OF FINITE SETS
AND THEIR APPLICATIONS

KAUSHIK MAJUMDER

ABSTRACT. Paul Erdős and László Lovász established that any maximal intersecting family of $k$-sets has at most $k^k$ blocks. They introduced the problem of finding the maximum possible number of blocks in such a family. They also showed that there exists a maximal intersecting family of $k$-sets with approximately $(e - 1)k^k$ blocks. Later Péter Frankl, Katsuhiro Ota and Norihide Tokushige used a remarkable construction to prove the existence of a maximal intersecting family of $k$-sets with at least $(\frac{e}{2})^{k-1}$ blocks. In this article we introduce the notion of a closed intersecting family of $k$-sets and show that such a family can always be embedded in a maximal intersecting family of $k$-sets. Using this result we present two examples which disprove two special cases of one of the conjectures of Frankl et al. This article also provides comparatively simpler construction of maximal intersecting families of $k$-sets with at least $(\frac{e}{2})^{k-1}$ blocks.

1. INTRODUCTION

By a family we mean a family (set) of finite sets. Such a family is called intersecting if any two of its member have non empty intersection. A maximal intersecting family is an intersecting family which can not be embedded properly into any larger intersecting family. In this article, our idea is to decompose a maximal intersecting family into some suitable subfamilies and study these subfamilies to gain a better understanding of such a family. Using this idea we are able to locate a similarity between the recursive Erdős-Lovász construction in [1, Construction (c), Page 620] and a non recursive Frankl-Ota-Tokushige construction in [2, § 2, Example 1 & Example 2]. We find that each maximal intersecting family has a “core” which generates it. We call this core a closed intersecting family. In [2], Frankl et al. conjectured that the maximal intersecting family of $k$-sets constructed by them has the largest number of blocks, and it is the only such family (up to isomorphism) with these many blocks. We use the theory developed here to prove that both these conjectures are false, at least for small $k$. (See Example 4.1 and Example 4.2 below.) Before going into the technicalities let us give some notations and definitions.

1.1. Notations and Terminologies : Let $\mathcal{G}$ and $\mathcal{H}$ be two non empty families of non empty sets.

(a) $A \cup B$ denotes the union of two disjoint sets $A$ and $B$.

(b) $\mathcal{G} \cup \mathcal{H}$ denotes the union of two disjoint families $\mathcal{G}$ and $\mathcal{H}$.

(c) For any set $A$, $|A|$ will denote the cardinality of $A$.

(d) Any $B \in \mathcal{G}$ is called a block of $\mathcal{G}$.

(e) A family $\mathcal{G}$ is said to be uniform if all its blocks have the same size. If $\mathcal{G}$ is a uniform family we shall denote its common block size by $k(\mathcal{G})$.

(f) The point set of the family $\mathcal{G}$ is defined as $\bigcup_{B \in \mathcal{G}} B$ and is denoted by $P_\mathcal{G}$. Any $x \in P_\mathcal{G}$ is called a point of $\mathcal{G}$.

(g) Suppose $P_\mathcal{G}$ and $P_\mathcal{H}$ are disjoint. Then $\mathcal{G} \oplus \mathcal{H}$ denotes the collection of all sets of the form $A \cup B$, where $A \in \mathcal{G}$ and $B \in \mathcal{H}$. If $\mathcal{G}$ consists of a single $k$-set $B$, then we denote $\mathcal{G} \oplus \mathcal{H}$ by $B \oplus \mathcal{H}$. If $\mathcal{G}$ consists of a single $1$-set $\{\alpha\}$, then we denote $\mathcal{G} \oplus \mathcal{H}$ by $\alpha \oplus \mathcal{H}$.

(h) A family $\mathcal{G}$ is said to be isomorphic to the family $\mathcal{H}$ if there exists a one-to-one and onto function $\phi : P_\mathcal{G} \rightarrow P_\mathcal{H}$ such that $\phi(B) \in \mathcal{H}$ if and only if $B \in \mathcal{G}$.

(i) A blocking set of a family $\mathcal{G}$ is a set $C$ which intersects every block of $\mathcal{G}$. A transversal of $\mathcal{G}$ to be a blocking set of $\mathcal{G}$ with the smallest possible size – in case $\mathcal{G}$ has a finite blocking set. In this case we denote the common size of its transversals by $\text{tr}(\mathcal{G})$. If $\mathcal{G}$ has no finite blocking set, we may put $\text{tr}(\mathcal{G}) = \infty$. If $\text{tr}(\mathcal{G}) < \infty$, we denote the family of transversals of $\mathcal{G}$ by $G_{\mathcal{G}}$. Note that $G_{\mathcal{G}}$ is a uniform family with $k(G_{\mathcal{G}}) = \text{tr}(\mathcal{G})$.

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Definition 1.1. A family $\mathcal{F}$ is said to be a maximal intersecting family (in short MIF) if $\text{tr}(\mathcal{F}) < \infty$ and $\mathcal{F} = \mathcal{F}^\top$. We use $\text{MIF}(k)$ as a generic name for uniform MIF’s with $k(\mathcal{F}) = k$.

Erdős and Lovász established, in their landmark article [1], that any MIF($k$) has at most $k^k$ blocks. They showed by means of an example that there exists a MIF($k$) with approximately $(\varepsilon - 1)k!$ blocks. This example is constructed by a recursive procedure [1, Construction (c), Page 620] starting with the unique MIF(1). Lovász conjectured in [4], that the MIF($k$) thus constructed was the extremal one. Later in [2], an extremely elegant and complicated example was given to show that there exists a MIF($k$) with at least (approximately) $(\frac{k}{2})^{k-1}$ blocks (i.e. it has the largest number of blocks) and it disproves Lovász conjecture. In this article, we present comparatively simpler construction (see $\mathcal{G}(k, t)$ in Construction 3.1) to prove that there exists a MIF($k$) with at least (approximately) $(\frac{k}{2})^{k-1}$ blocks. (More precisely, we present an alternative proof of [2, § 2, Theorem 1], see Corollary 4.6 below). In [2], it is conjectured that the construction of Frankl et al. yields the unique MIF($k$) with the largest number of blocks. Here we show that both parts of this conjecture are false. Specifically, the uniqueness part is incorrect for $k = 4$, while the optimality part is incorrect for $k = 5$.

1.2. Organisation of this article. Section 2 of this article is the introductory part about closed intersecting families, some of its properties and examples. In Section 3 we study constructions over the cycle graph. In Section 4 it is shown that Example 4.1 and Example 4.2 are counter examples to [2, § 3, Conjecture 4] in special cases. In this final section we close this article by stating a conjecture.

2. Closed Intersecting Family of finite sets

In this section, we present results of an ab initio study on closed intersecting families.

Definition 2.1. Let $\mathcal{F}$ be a uniform family with $k(\mathcal{F}) = k$ and $\text{tr}(\mathcal{F}) = t$. $\mathcal{F}$ is said to be a closed intersecting family (in short CIF) if $\text{tr}(\mathcal{F}) \leq k(\mathcal{F}) - 1$ and $\mathcal{F} = (\mathcal{F} \cup \mathcal{F}^\top)^\top$. We use CIF($k, t$) as a generic name for CIF’s $\mathcal{F}$ with $k(\mathcal{F}) = k$ and $\text{tr}(\mathcal{F}) = t$. Note that any closed intersecting family is necessarily an intersecting family.

We have the following characterisation.

Proposition 2.2. Let $\mathcal{F}$ be an intersecting family of $k$−sets with $\text{tr}(\mathcal{F}) \leq k - 1$. Then the following statements are equivalent:

(a) Any $k$−set which is a blocking set of $\mathcal{F} \cup \mathcal{F}^\top$ is a block of $\mathcal{F}$.
(b) Any $k$−set which is a blocking set of $\mathcal{F}$ but itself is not a block of $\mathcal{F}$, then it is not a blocking set of $\mathcal{F}^\top$.
(c) $\mathcal{F} = (\mathcal{F} \cup \mathcal{F}^\top)^\top$.

Proof : Firstly we prove (a) $\Leftrightarrow$ (b) and then we prove (c) $\Leftrightarrow$ (a).

Let $C$ be a $k$−set which is a blocking set of $\mathcal{F} \cup \mathcal{F}^\top$. Suppose $C$ is a blocking set of $\mathcal{F}^\top$, then by (a) $C \in \mathcal{F}$, a contradiction. Hence $C$ is not a blocking set of $\mathcal{F}^\top$. Conversely, let $C$ be a $k$−set which is a blocking set of $\mathcal{F} \cup \mathcal{F}^\top$. Suppose $C \notin \mathcal{F}$, then by (b) $C$ is a blocking set of $\mathcal{F}^\top$, a contradiction to the assumption, so our supposition $C \notin \mathcal{F}$ was wrong. Hence $C \in \mathcal{F}$.

From (c) it implies that $\text{tr}(\mathcal{F} \cup \mathcal{F}^\top) = k$. Let $C$ be a blocking $k$−set of $\mathcal{F} \cup \mathcal{F}^\top$, then $C$ is transversal of the family $\mathcal{F} \cup \mathcal{F}^\top$. Hence $C \in \mathcal{F}$. Conversely, let $C$ be a transversal of $\mathcal{F} \cup \mathcal{F}^\top$. Suppose $|C| \leq k - 1$. Consider a set $X$, of size $k - |C|$, disjoint from $P_x$. Then $X \cup C$ is a blocking $k$−set of $\mathcal{F} \cup \mathcal{F}^\top$ and it is not a block of $\mathcal{F}$, a contradiction to (a). So $|C| = k$ and hence by (a) $C \in \mathcal{F}$, which proves (c). \hfill $\Box$

Henceforth, by closure property we refer any one of (a), (b) and (c) in our study.

Example 2.3. Let $k, t$ be positive integers with $t \leq k - 1$. All $k$−subsets of $(a + t - 1)$−set form a CIF($k, t$) and its transversals are all $t$−subsets of the point set. It has $k + t - 1$ points, $\binom{k + t - 1}{k}$ blocks and $\binom{k + t - 1}{t}$ transversals.

Example 2.4. Let $k, t$ be positive integers with $2 \leq t \leq k - 1$. Let $P$ be a $(k + t - 2)$−set. For each bi partition $(C, P \setminus C)$ of $P$ with $|C| = t - 1$, we introduce a new symbol $x_{c}$. We consider the family of all $k$−subsets of $P$ plus all $k$−sets of the form $\{x_{c}\} \cup (P \setminus C)$. It is a CIF($k, t$). Its transversals are all $t$−subsets of $P$ plus all $t$−sets of the form $\{x_{c}\} \cup C$. It has $k + t - 2 + \binom{k + t - 2}{k - 1}$ points, $\binom{k + t - 2}{k - 1} + \binom{k + t - 2}{t - 1}$ blocks and $\binom{k + t - 2}{t - 1}$ transversals.

Theorem 2.5. Let $\mathcal{F}$ be a subfamily of a MIF($k$) $\mathcal{X}$ such that $t := \text{tr}(\mathcal{F}) \leq k - 1$ and $\mathcal{X} \prec \mathcal{F} = \mathcal{A} \oplus \mathcal{F}^\top$ for some family $\mathcal{A}$. Then $\mathcal{F}$ is a CIF($k, t$) if and only if $\mathcal{A}$ is a MIF($k - t$).
2.5
Thus for each \( T \) there exists at least one \( T' \in F^T \) with \( T' \cap T = \emptyset \). Since \( A \oplus F^T \) is an intersecting family of \( k \)-sets, it follows that \( A \) is an intersecting family of \((k-1)\)-sets.

Let \( C \in A^T \). Then, for each \( T \in F^T \), \( C \cup T \) is a blocking set of \( X \) and hence \( |C \cup T| \geq k \). Thus \( |C| \geq k-t \), with equality if \( C \cup T = X \) for all \( T \in F^T \). Since each block of \( A \) is a blocking set of size \( k-t \) for \( A \), it follows that \( tr(A) = k-t \) and \( A \subseteq A^T \). Also if \( C \in A \) and \( T \in F^T \), then \( C \cup T \in X \). The argument in the previous paragraph shows that \( C \cup T \) is not a blocking set of \( F^T \) and hence \( C \cup T \notin (F \cup F^T)^T = F \). Thus \( C \cup T \in X \setminus F = A \oplus F^T \). Hence \( C \in A \). Thus \( A^T \subseteq A \) and hence \( A = A^T \). Thus \( A \) is a MIF\((k-t)\).

Conversely, suppose \( A \) is a MIF\((k-t)\). Since \( F \) is an intersecting family of \( k \)-sets, every block of \( F \) is a blocking \( k \)-set of \( F \cup F^T \) and hence \( tr(F \cup F^T) \leq k \). To show that \( F \) is a CIF\((k,t)\), it suffices to show that if \( C \) is a blocking set of size \( k \) for \( F \) which is not a block of \( F \), then \( C \) is not a blocking set of \( F^T \). If \( C \notin A \oplus F^T \), then \( C \) is not a block of \( X \) and hence there is a block \( B \in X \) disjoint from \( C \). But \( C \) is a blocking set of \( F \). So \( B \in A \oplus F^T \). Then \( B \cap P_F \) is a blocking set of \( F^T \) disjoint from \( C \), so that \( C \) is not a blocking set of \( F^T \) in this case.

On the other hand, if \( C \in A \oplus F^T \), then choose a point \( \alpha \in C \cap P_{F^T} \). (It exists since \( k = |C| > k(F^T) \).) Since \( C \) is a block of a MIF\((k)\) \( X \), there exists at least one \( B \in X \) such that \( B \cap C = \{\alpha\} \). Since \( \alpha \notin P_F \), it follows that \( B \in A \oplus F^T \) and hence \( B \cap P_F \) is a blocking set of \( F^T \) disjoint from \( C \). So \( C \) is not a blocking set of \( F^T \) in this case also.

\[ \square \]

**Corollary 2.6.** Let \( X \) be a MIF\((k)\), then there exists at least one CIF\((k, k-1)\) \( F \) and \( \alpha \in P_X \setminus P_F \) such that \( X = F \cup \alpha \oplus F^T \).

**Proof:** Let \( \alpha \in P_X \). Define \( F = \{ B \in X : \alpha \notin B \} \). Then \( F^T = \{ B \setminus \{\alpha\} : \alpha \in B \in X \} \). The conclusion follows as an application of Theorem 2.5. \( \square \)

The following theorem is a sort of converse to Theorem 2.5. Together, Theorem 2.5 and Theorem 2.7 show that closed intersecting families are the cores which may be used to obtain recursive construction of maximal intersecting families.

**Theorem 2.7.** Let \( A \) and \( F \) be a MIF\((k-t)\) and a CIF\((k,t)\) respectively where \( A \) and \( F \) have disjoint point sets. Then \( F \cup A \oplus F^T \) is a MIF\((k)\).

**Proof:** Let \( C \) be a blocking \( k \)-set of \( F \cup A \oplus F^T \). To prove \( C \in F \cup A \oplus F^T \). If \( C \in F \) we are done. So assume \( C \notin F \). By closure property of \( F \), \( C \) is not a blocking set of \( F \cup F^T \). This implies \( C \) is not a blocking set of \( F^T \). Hence at least one \( T \in F^T \) which is disjoint from \( C \). Since \( C \) is a blocking set of \( T \oplus A \) it follows that \( C \cap P_A \) is a blocking set of \( A \). So \( |C \cap P_A| \geq k-t \). Also \( C \cap P_F \) is a blocking set of \( F \), hence \( |C \cap P_F| \geq t \). But \( |C| = k \), so \( |C \cap P_A| = k-t \) and \( |C \cap P_F| = t \). Hence \( C \cap P_A \in A \) and \( C \cap P_F \in F^T \). So \( C \in A \oplus F^T \). This shows that every blocking \( k \)-set of the family \( F \cup A \oplus F^T \) is a block of that family. The converse is obvious.

\[ \square \]

**Corollary 2.8.** Let \( M(k) \) denote maximum number of blocks in a MIF\((k)\). For each integer \( k \geq t+1 \), if \( F \) is a CIF\((k, t)\) with \( b \) blocks and \( b^T \) transversals, then \( M(k) \geq b + b^T M(k-t) \).

**Proof:** We choose a MIF\((k-t)\) with \( M(k-t) \) blocks so that its point set is disjoint from \( P_F \). Call it \( A \). The result follows since by Theorem 2.7, \( F \cup A \oplus F^T \) is a MIF\((k)\) with \( b + b^T M(k-t) \) blocks. \( \square \)

### 2.1. Construction of maximal intersecting families using closed intersecting families.

**Proposition 2.9.** Let \( F \) be a CIF\((k,t)\). Suppose for each \( i \), with \( 1 \leq i \leq n \), \( A_i \) be a MIF\((k-t)\) and \( C_i \) is a subfamily of \( F^T \) with the following properties:

(a) each \( A_i \) and \( F \) have disjoint point sets;
(b) \( F^T = \bigcup_{i=1}^{n} C_i \);
(c) each \( t \)-set of \( C_i \) is a blocking set of \( F^T \cap C_i \).

Then \( F \cup \bigcup_{i=1}^{n} (A_i \oplus C_i) \) is a MIF\((k)\). Moreover, \( n \leq \tbinom{Q^T}{t} \).
Proof: Let $\mathcal{G} := \mathcal{F} \cup \bigcup_{i=1}^{n} \mathcal{A}_i \oplus \mathcal{C}_i$. Clearly it is an intersecting family of $k$-sets. Let $C$ be a blocking set of $\mathcal{G}$ with size at most $k$. To prove $C$ is a block of it. If $C \in \mathcal{F}$ we are done. So assume $C \notin \mathcal{F}$. By closure property of $\mathcal{F}$ there exists at least one $T \in \mathcal{F}^\top$ such that $C \cap P_T$ is disjoint from $T$ and $T \notin \mathcal{C}_i$ for a unique $i$. Since $C$ is a blocking set of $T \oplus \mathcal{A}_i$, hence $|C \cap P_{T_i}| \geq k - t$. Also $C \cap P_T$ is a blocking set of $\mathcal{F}$ hence $|C \cap P_T| \geq t$. This implies $|C \cap P_{A_i}| = k - t$ and $|C \cap P_T| = t$ hence $C \in \mathcal{A}_i \oplus \mathcal{C}_i$.

For the next part, by assumption (c) we observe that, for each $i$ with $1 \leq i \leq n$ there exists at least one pair $(T_i, T_i')$, where $T_i, T_i' \in \mathcal{C}_i$ with $T_i \cap T_i' = \emptyset$. Also for each $i, j$ and chosen such pairs $(T_i, T_i')$ and $(T_j, T_j')$, with $1 \leq i < j \leq n$, we have $T_i \cap T_j' \neq \emptyset$ and $T_i' \cap T_j \neq \emptyset$. Hence by using [3, § 13, Problem 32], we have $n \leq \frac{k(k - 1)}{2}$.

The proof of Theorem 2.7 is an application of Proposition 2.9. We observe that it is the case $n = 1$ in Proposition 2.9.

Proposition 2.10. Let $\mathcal{F}$ be a CIF($k, k - n$). Suppose $\mathcal{F}^\top = \bigcup_{i=1}^{n+1} \mathcal{C}_i$, where for each $i$, with $1 \leq i \leq n + 1$, the subfamily $\mathcal{C}_i$ satisfies following properties:

(a) each $\mathcal{C}_i$ is an intersecting family of $(k - n)$-sets;
(b) whenever $i \neq j$, then for each $T \in \mathcal{C}_i$ there exists at least one $T' \in \mathcal{C}_j$ such that $T \cap T' = \emptyset$.

Consider $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{n+1}$ are $(n + 1)$-parallel classes of an affine plane of order $n$ (provided it exists). This affine plane of order $n$ and $\mathcal{F}$ have disjoint point sets. Then $\mathcal{F} \cup \bigcup_{i=1}^{n+1} \mathcal{A}_i \oplus \mathcal{C}_i$ is a MIF($k$).

Proof: Let $\mathcal{G} := \mathcal{F} \cup \bigcup_{i=1}^{n+1} \mathcal{A}_i \oplus \mathcal{C}_i$. Clearly it is an intersecting family of $k$-sets. Let $P$ be the point set of this affine plane. Let $C$ be a blocking set of $\mathcal{G}$ with size at most $k$. To prove $C$ is a block of it. If $C \in \mathcal{F}$ we are done. So assume $C \notin \mathcal{F}$. By closure property of $\mathcal{F}$ there exists at least one $T \in \mathcal{F}^\top$ such that $C \cap P_T$ is disjoint from $T$. Then there exists at least one $i$, with $1 \leq i \leq n + 1$, such that $T \in \mathcal{C}_i$. But $C$ is a blocking set of $T \oplus \mathcal{A}_i$ hence $|C \cap P_T| \geq n$. Also $C \cap P_T$ is a blocking set of $\mathcal{F}$ hence $|C \cap P_T| \geq k - n$. This implies $|C \cap P_T| = n$ and $|C \cap P_T| = k - n$ and hence $C \cap P_T \notin \mathcal{F}^\top$. So by assumptions (a) and (b), $C \cap P_T \in \mathcal{C}_j$ for some $j \neq i$. Then again by assumption (b) there exists at least one $T_i \in \mathcal{C}_i$ such that $C \cap P_T$ is disjoint from $T_i$, for each $T$ with $l \neq j$. So $C \cap P_T$ is a blocking set of each such $T_i \oplus \mathcal{A}_l$. Hence $C \cap P_T$ is a line of $\mathcal{A}_j$. Therefore $C \in \mathcal{A}_j \oplus \mathcal{C}_j$.

2.2. Recursive Construction of closed intersecting families.

Theorem 2.11. Let $A$ be a MIF($l$) and let $\mathcal{F}_x, x \in P_A$, be uniform families with pairwise disjoint point sets. Suppose $k(\mathcal{F}_x) = k$ and $\text{tr}(\mathcal{F}_x) = t$ for all $x$. Put

$$
\mathcal{G} = \left\{ \biguplus_{x \in A} F_x : A \in \mathcal{A}, F_x \in \mathcal{F}_x \text{ for all } x \in A \right\}.
$$

Then we have the following.

(a) $\mathcal{G}^\top = \left\{ \biguplus_{x \in A} T_x : A \in \mathcal{A}, T_x \in \mathcal{F}_x^\top \text{ for all } x \in A \right\}$. In particular, $k(\mathcal{G}) = kl$ and $\text{tr}(\mathcal{G}) = tl$.

(b) If, further, each $\mathcal{F}_x$ is a CIF($k, t$) with $\text{tr}(\mathcal{F}_x^\top) = k$, then $\mathcal{G}$ is a CIF($kl, tl$).

Proof: If $A \in \mathcal{A}$ and $T_x \in \mathcal{F}_x^\top$ for all $x \in A$, then clearly $\biguplus_{x \in A} T_x$ is a blocking set of $\mathcal{G}$ of size $tl$. Thus, $\text{tr}(\mathcal{G}) \leq tl$. Let $B$ be a blocking set of $\mathcal{G}$ of size at most $tl$. For $x \in P_A$, put $T_x = B \cap P_{F_x}$. Let $A = \{x \in P_A : |T_x| \geq t\}$. We have

$$
\sum_{x \in P_A} |T_x| = |B| \leq tl
$$

(1)

and hence $|A| \leq l$. If $A$ is not a block of the MIF($l$) $\mathcal{A}$, then there is a block $A'$ of $\mathcal{A}$ disjoint from $A$. Hence $|T_x| \leq t - 1$ for all $x \in A'$. So, for each $x \in A'$ there is a block $F_x$ of $\mathcal{F}_x$ disjoint from $T_x$. Hence $\biguplus_{x \in A'} F_x$ is a block of $\mathcal{G}$ disjoint from $B$, a contradiction. So $A \in \mathcal{A}$ and $|A| = l$. Then (1) implies that $|T_x| = 0$ for $x \notin A$ and $|T_x| = t$ for $x \in A$. Thus, $|B| = tl$, so that $\text{tr}(\mathcal{G}) = tl$ and $B \in \mathcal{G}^\top$. Since $B = \biguplus_{x \in A} T_x \in \mathcal{G}^\top$ and $|T_x| = t$, it follows that $T_x \in \mathcal{F}_x^\top$ for all $x \in A$. This proves part (a).
Now we assume each $F_x$ is a CIF$(k, t)$. Since $A$, as well as each $F_x$, is an intersecting family it follows that $G$ is an intersecting family. Using the description of $G^T$ from part (a) and applying part (a) to the families $F_x^T$, $x \in P_A$, we see that

$$G^{\top T} = \left\{ \bigcup_{x \in A} S_x : A \in A, S_x \in F_x^T \text{ for all } x \in A \right\}.$$ 

Thus $\text{tr}(G \cup G^{\top}) \geq \text{tr}(G^{\top}) = kl$. Since all the blocks of $G$ are blocking sets of $G \cup G^{\top}$ of size $kl$, it follows that $\text{tr}(G \cup G^{\top}) = kl$ and $G \subseteq (G \cup G^{\top})^T$. Let $C$ be a transversal of $G \cup G^{\top}$. Then $C \in G^{\top T}$ and hence $C = \bigcup_{x \in A} S_x$, for some $A \in A$ and $S_x \in F_x^T$ for all $x \in A$. If we can show that $C \in G$, then we are done. Otherwise, there exists at least one $y \in A$ such that $S_y \notin F_y = (F_y \cup F_y^{T})^T$. Since $S_y \in F_y^T$ it follows that $S_y$ is not a blocking set of $F_y$. So there exists at least one $U_y \in F_y$ disjoint from $S_y$. Since $y \in A \in A$ and $A$ is a MIF$(l)$, there is a $B \in A$ such that $A \cap B = \{y\}$. For each $x \in B \setminus \{y\}$, choose arbitrary $U_x \in F_x$. Then $\bigcup_{x \in B}$ is a block of $G$ disjoint from the blocking set $C$, in contradiction. Thus $C \in G$. Hence $(G \cup G^{\top})^T \subseteq G$. Therefore $G = (G \cup G^{\top})^T$ and this proves part (b).

**Theorem 2.12.** Let $F$ and $G$ be two uniform families with disjoint point sets. Let $k(F) = k$, $k(G) = k + t$, $\text{tr}(F) = t'$ and $\text{tr}(G) = t$. Suppose $\text{tr}(G^{\top}) > t + t'$. Let $H = G \cup (F \oplus G^{\top})$. Then,

(a) $H^T = F^T \oplus G^{\top}$. In particular $k(H) = k + t$, $\text{tr}(H) = t + t'$.
(b) If, further, both $F$ and $G$ are closed intersecting families, then $H$ is a closed intersecting family.

**Proof:** Since every member of $F^T \oplus G^{\top}$ is a blocking set of $H$ of size $t + t'$, $\text{tr}(H) \leq t + t'$. Let $C$ be a transversal of $H$. Then $|C| \leq t + t'$. If $C \cap F$ is not a blocking set of $F$, then there is a block $A \in F$ disjoint from $C \cap F$. Since $|C| \leq t + t' \leq \text{tr}(G^{\top}) - 1$, there is a $B \in G^{\top}$ disjoint from $C \cap P_G$. Then $A \cup B \in H$ is disjoint from the blocking set $C$, a contradiction. Thus, $C \cap F$ is a blocking set of $F$. Clearly $C \cap P_G$ is a blocking set of $G$. Therefore $|C \cap P_G| \geq t$ and $|C \cap P_G| \geq t$. Since $|C| \leq t + t'$, it follows that $|C \cap P_G| = t$ and $|C \cap P_G| = t$. Therefore $C \cap F \subseteq F^T$ and $C \cap P_G \subseteq G^{\top}$. Thus $C \subseteq F^T \oplus G^{\top}$. This proves part (a).

Now suppose $F$ and $G$ are closed intersecting families. In particular they are intersecting families. Hence $H$ is an intersecting family. Thus the blocks of $H$ are blocking sets of $H \cup H^T$ of size $k + t$. So $\text{tr}(H \cup H^T) \leq k + t$. Let $C$ be a transversal of $H \cup H^T$. Thus $|C| \leq k + t$. If we can show that $C \in H$, then $H$ is a closed intersecting family and we are done. If $C \in G$ we are done. So suppose $C \notin G$. But $C$ is a blocking set of $G$. Since $G$ is a closed intersecting family, it follows that there is a $T \in G^{\top}$ disjoint from $C$. Since $C$ is a blocking set of $F^T \oplus G^{\top} \subseteq H$ and also of $F^T \oplus G^{\top} = H^T$, it follows that $C \cap P_F$ is a blocking set of $F \cup F^T$. Since $F$ is a closed intersecting family with $k(F) = k$, we get $|C \cap P_F| \geq k$. Also, as $C \cap P_G$ is a blocking set of $G$ and $\text{tr}(G^{\top}) = t$, $|C \cap P_G| \geq t$. But $|C| \leq k + t$. Therefore, $|C| = k + t$, $C \cap P_F \in F^T$, $C \cap P_G \in G^{\top}$. Consequently, $C \in F^T \oplus G^{\top} \subseteq H$. Hence $C \in H$. This proves part (b).

3. Constructions over the Cycle Graph

**Construction 3.1.** Let $k, t$ be positive integers with $t \leq k$. Let $X_n$, $0 \leq n \leq t - 1$, be $t$ pairwise disjoint sets with

$$|X_n| = \begin{cases} k - \left\lfloor \frac{t}{2} \right\rfloor & \text{if } 0 \leq n \leq \left\lfloor \frac{t-1}{2} \right\rfloor \\ k - \left\lceil \frac{t-1}{2} \right\rceil & \text{if } \left\lceil \frac{t-1}{2} \right\rceil + 1 \leq n \leq t - 1 \end{cases}$$

say $X_n = \{x^p_n : 0 \leq p \leq |X_n| - 1\}$. Let $F(k, t)$ be the family of all the $k$–sets of the form

$$X_n \cup \{x^{p+i}_n : 1 \leq i \leq k - |X_n|\},$$

where addition in the superscript is modulo $t$ and the sequence $\{p_n : n \geq 0\}$ defined as $p_0 = 0$ and for $n \geq 1$, $p_{n-1} \leq p_n \leq 1 + p_{n-1}$ (i.e. $p_n$ assigns only $p_{n-1} + 1 + p_{n-1}$). Let $G(k, t)$ be the family of all the $k$–sets of the form

$$X_n \cup \{x^{p+i}_n \in X_{n+i} : 1 \leq i \leq k - |X_n|\},$$

(i.e. $X_n$ is unionised with one element from each $X_{n+i}$) where addition in the superscript and subscript is modulo $t$.

Clearly, both the families $F(k, t)$ and $G(k, t)$ are examples of intersecting family of $k$–sets (since the $t$–cycle is a graph with diameter $\left\lfloor \frac{t}{2} \right\rfloor$). It is not hard to see that the family $F(k, t)$ defined above is a subfamily of $G(k, t)$. In this context, we emphasise that the family $\mathcal{G}$ of [2, § 2, Example 1 & Example 2] is not isomorphic to $F(k, t)$.
Theorem 3.2. Let $\text{tr}(G(k,t)) = t$.

Proof: We prepare a $t$–set $B$ by choosing one element from each $X_n$, with $0 \leq n \leq t-1$, then $B$ is a blocking set of $G(k,t)$. Therefore $\text{tr}(G(k,t)) \leq t$. Let $C$ be an arbitrary but fixed set of size $t$, to show $\text{tr}(G(k,t)) \geq t$, it is enough to show there exists a block of $G(k, t)$, which is disjoint from it. We divide our arguments in the following two exhaustive cases.

Case A: For each $n$, with $0 \leq n \leq t-1$, $|C \cap X_n| \leq |X_n| - 1$.

Since $|C| = t - 1$ there exists $X_n$, with $0 \leq n \leq t-1$, which is disjoint from $C$, call such an $n = n_0$. For this case we have, for each $m$, with $1 \leq m \leq k - |X_n|$, $X_m \cap C$ is non empty and choose one element namely, $x_{p_{m+1}}^m \in X_m \setminus C$. Therefore, $X_m \cup \{x_{p_{m+1}}^m \in X_{m+1} \cap C : 1 \leq i \leq k - |X_m|\}$ is the required block of $G(k, t)$, which is disjoint from $C$.

Case B: For some $n$, with $0 \leq n \leq t-1$, $C \cap X_n = X_n$. (This case will arise for $k$, with $t \leq k \leq t - 1 + \left\lfloor \frac{t}{2} \right\rfloor$.)

Since $|C| = t - 1$ so there exists at most one $n$, with $0 \leq n \leq t-1$, such that $C \cap X_n = X_n$. So

$$|C \setminus X_n| = t - 1 - |X_n| \leq \left\lfloor \frac{t}{2} \right\rfloor - 1 - k.$$

Since $k \geq t$ we have,

$$|C \setminus X_n| = t - 1 - |X_n| \leq \left\lfloor \frac{t}{2} \right\rfloor - 1.$$

So there exists at least one $m$, with $0 + 1 \leq m \leq n + \left\lfloor \frac{t}{2} \right\rfloor$, so that $(C \setminus X_n) \cap X_m$ is empty, call such an $m = m_1$. Therefore, for $1 \leq i \leq k - |X_n|$, we have $X_{m_0+i} \cap C$ is non empty and choose one element namely, $x_{p_{m+1}}^m \in X_{m_0+i} \setminus C$. Therefore, $X_m \cup \{x_{p_{m+1}}^m \in X_{m+1} \cap C : 1 \leq i \leq k - |X_m|\}$ is the required block of $G(k, t)$, which is disjoint from $C$.

Theorem 3.3. For $k \geq t + 1$, $G(k, t)$ is a CIF $(k, t)$. Moreover for each $n$, with $0 \leq n \leq t - 1$, $|X_n \cap T| = 1$, where $T$ is a transversal of $G(k, t)$.

Proof: Let $C$ be a $k$–set. If for each $n$, with $0 \leq n \leq t - 1$, $C \cap X_n \subseteq X_n$ then $X_n \setminus C$ is non empty and $T(C) := \{x_n \in X_n \setminus C : 0 \leq n \leq t - 1\}$ is a transversal of $G(k, t)$, which is disjoint from $C$. Suppose for some $n$, with $0 \leq n \leq t - 1$, $C \cap X_n = X_n$; since $|C| = k$ and $k \geq t + 1$ therefore there exists at most one such $n$, call it $n_0$. Therefore, $|C \setminus X_n| = k - |X_n|$. We observe that for each $m \neq n_0$, with $0 \leq m \leq t - 1$, $C \cap X_m \subseteq X_m$, hence $X_m \setminus C$ is non empty and choose $x_{m_0}^m \in X_m \setminus C$. If for some $m$, with $m_0 + 1 \leq m \leq n_0 + \left\lfloor \frac{t}{2} \right\rfloor$, $|X_m \setminus C| \geq 2$, then there exists $m_0$, with $m_0 + 1 \leq m_0 \leq n_0 + \left\lfloor \frac{t}{2} \right\rfloor$ such that $X_{m_0}$ is disjoint from $C$. Consequently, $X_m \cup \{x_{m_0}^m \in X_{m_0+i} \cap C : 1 \leq i \leq k - |X_m|\}$ is disjoint from $C$. So for each $m$, with $m_0 + 1 \leq m \leq n_0 + \left\lfloor \frac{t}{2} \right\rfloor$, $|X_m \setminus C| = 1$. Therefore for such case $C$ is a block of $G(k, t)$ containing $X_{m_0}$. This implies that, for an arbitrary $k$–set $C$ which is not a block of $G(k, t)$ then there exists a transversal $T(C)$ of $G(k, t)$ which is disjoint from $C$.

Let $T$ be a transversal of $G(k, t)$. Since $k \geq t + 1$ so for each $n$, with $0 \leq n \leq t - 1$, $|X_n| = k$ and $|T| = t$; then $X_n \setminus T$ is non empty and choose $x_{q_{m+1}}^m \in X_n \setminus T$. As we argued in the previous para that we have for each $n$, with $0 \leq n \leq t - 1$, $|X_n \cap T| \leq 1$. If for some $n$, with $0 \leq m \leq t - 1$, $|X_n \cap T| < 1$ i.e. $X_n$ is disjoint from $T$, then $X_m \cup \{x_{q_{m+1}}^m \in X_{m+i} \setminus T : 1 \leq i \leq k - |X_m|\}$ is disjoint from $T$, a contradiction. Hence the second part of the result follows.

Proposition 3.4. Suppose $\text{tr}(F(k,t)) = t$ for $k \geq t$. Then for $k \geq t + 1$, $F(k, t)$ is a CIF $(k, t)$.

Proof: Let $C$ be a blocking $k$–set of $F(k, t)$ but $C \notin F(k, t)$. It is enough to show that there exists at least one $T \subseteq F(k,t)$ disjoint from $C$. If for each integer $n$, with $0 \leq n \leq t - 1$, there exists at least one $x_n \in X_n$ such that $x_n \notin C$, then $\{x_n : 0 \leq n \leq t - 1\}$ is the required $T$ and we are done for this case. Suppose there exists at least one integer $n$, with $0 \leq n \leq t - 1$, such that $X_n \not\subseteq C$. Notice that for each $m$ with $m \neq n$ and $0 \leq m \leq t - 1$, there exists at least one $x_m \in X_m$ such that $x_m \not\in C$. (Suppose it is false, then there exists at least one such integer $m$ with $X_m \not\subseteq C$. This implies that $X_m \cup X_m \cap C$, a contradiction arises since $k \geq t + 1$.)

When $t = 2r - 1$, then without loss of generality we can assume $X_0 \subseteq C$. When $t = 2r$, then without loss of generality we can assume either $X_0 \subseteq C$ or $X_{1 + \left\lfloor \frac{t}{2} \right\rfloor} = X_r \subseteq C$.

Case A: Let $X_0 \not\subseteq C$.

Here $C = X_0 \cup Y$. We observe that if $Y$ is disjoint from $T_n := \{x_i^n : 0 \leq i \leq n\}$, for some $n$ with $1 \leq n \leq \left\lfloor \frac{t}{2} \right\rfloor$, then $T_n \cup \{x_i : x_i \in X_i \cap C, n + 1 \leq i \leq t - 1\}$ is the required transversal disjoint from $C$ and we are done.

So we assume that $Y \cap T_n \neq \emptyset$ for each $n$ with $1 \leq n \leq \left\lfloor \frac{t}{2} \right\rfloor$. Since $|Y| = \left\lfloor \frac{t}{2} \right\rfloor$ and $T_n, 1 \leq n \leq \left\lfloor \frac{t}{2} \right\rfloor$, is
[\frac{1}{2}]\) pairwise disjoint sets so \(Y\) intersects \(T_n\) in exactly one point. Since \(C \notin \mathbb{F}(k,t)\) so \(Y\) is not of the form \(\{x^2_i: 1 \leq i \leq \frac{1}{2}\}\). In the next para, under these assumptions on \(Y\), we produce a transversal \(T \in \mathbb{F}^T(k,t)\) which is disjoint from both \(Y\) and \(X_0\). (Consequently, such a \(T\) is disjoint from both \(C\) and \(X_0\).)

We have \(|Y \cap \{x^0_0,x^1_1\}| = 1\) suppose \(x^1_1 \in Y\) and \(x^1_{1+\epsilon_1} \notin Y\), where \(\epsilon_1 \in \{0,1\}\). Construct \(c_1 = \epsilon_1\). If \(Y\) is disjoint from \(\{x^2_1,x^3_{1+\epsilon_1}\}\), then

\[
\{x^1_{1+\epsilon_1},x^2_1,x^3_1\} \cup \{x_i: x_i \in X_i \setminus C, 3 \leq i \leq t-1\}
\]

is the required transversal and we are done. So let \(|Y \cap \{x^2_1,x^3_{1+\epsilon_1}\}| = 1\) suppose \(x^2_{1+\epsilon_2} \in Y\) and \(x^2_{1+\epsilon_1+\epsilon_2} \notin Y\), where \(\epsilon_2 \in \{0,1\}\). Construct \(c_2 = c_1 + \epsilon_2\). In general our construction procedure is as follows: suppose we constructed a sequence \(c_1, c_2, \ldots, c_m\) with the following properties.

(a) For each \(n\) with \(1 \leq n \leq m\), \(c_n = c_{n-1} + \epsilon_n\) and \(c_n \in \{0,1\}\).

(b) \(\{x_{c_n}^n: 1 \leq n \leq m\} \subseteq Y\).

(c) \(S_m := \{x^1_{1-c_1}\} \cup \{x_{n-c_{n-1}+1-c_n}^n: 2 \leq n \leq m\}\) is disjoint from \(Y\).

Now we construct \(c_{m+1}\) if necessary. If \(Y\) is disjoint from \(\{x_{c_{m+1}}^{m+1},x_{1+c_{m+1}}\}\), then

\[
S_m \cup \{x_{c_{m+1}}^{m+1},x_{1+c_{m+1}}\} \cup \{x_i: x_i \in X_i \setminus C, m+2 \leq i \leq t-1\}
\]

is the required transversal and we are done. Now let \(|Y \cap \{x_{c_{m+1}}^{m+1},x_{1+c_{m+1}}\}| = 1\), suppose \(x_{c_{m+1}+c_{m+1}}^{m+1} \in Y\) and \(x_{c_{m+1}+c_{m+1}} \notin Y\), where \(\epsilon_{m+1} \in \{0,1\}\). Construct \(c_{m+1} = c_m + \epsilon_{m+1}\). This yields \(\{x_{c_n}^n: 1 \leq n \leq m+1\} \subseteq Y\) and \(S_{m+1}\) is disjoint from \(Y\). Since \(Y\) is not of the form \(\{x^1_0\} = 1 \leq i \leq \frac{1}{2}\)\) so this sequence contains at most \(\frac{1}{2}\) terms. If this sequence contains exactly \(M\) terms, then \(Y\) is disjoint from \(\{x_{c_{M+1}}^{M+1},x_{1+c_{M+1}}\}\). Consequently,

\[
S_{M} \cup \{x_{c_{M+1}}^{M+1},x_{1+c_{M+1}}\} \cup \{x_i: x_i \in X_i \setminus C, M+2 \leq i \leq t-1\}
\]

is the required transversal.

**Case B**: Let \(X_{\lceil \frac{1}{2} \rceil} \cup Y \). This case is similar to the above case. (Precisely, we need to replace \(\frac{1}{2}\) by \(\frac{1}{2}\), \(x^p(\bullet)\) by \(x^p(\frac{1}{2}+1+(\bullet))\) and \(x(\bullet)\) by \(x(\frac{1}{2}+1+(\bullet))\).)

\[\mathbb{Q}\]

4. SOME APPLICATIONS

In this section, it is shown that Example 4.1 and Example 4.2 are counter examples to [2, § 3, Conjecture 4] in special cases. In the following examples we continue with the notation of Construction 3.1.

**Example 4.1.** It is easy to check that for \(k \geq 2\), \(tr(\mathbb{F}(k,2)) = 2\). So by Proposition 3.4 we have, for \(k \geq 3\), \(\mathbb{F}(k,2)\) is a CIF(2,2). We observe that for \(0 \leq p \leq k - 2\) and \(0 \leq q \leq k-1\), the transversals of \(\mathbb{F}(k,2)\) are \(\{x^p_0, x^p_1\}; \{x^q_0, x^q_1\}\). Hence there are \(k^2 - k + 1\) number of transversals and 2 blocks in \(\mathbb{F}(k,2)\). So if we plug in \(k = 4\) we have a CIF(4,2) with 2 blocks and 13 transversals. Let \(A\) be the unique MIF(2) isomorphic to \(\{\{a, b\}, \{b, c\}, \{a, c\}\}\) and \(P_A \cap P_{\mathbb{F}(4,2)} = \emptyset\). Therefore by Theorem 2.7, \(\mathbb{F}(4,2) \cup A \not\subseteq \mathbb{F}^T(4,2)\) is a MIF(4) with 42 blocks and 10 points. In this MIF(4) there are 3 points in 26 blocks, 5 points in 14 blocks and 2 points in 10 blocks.

**Example 4.2.** It is easy to check that for \(k \geq 3\), \(tr(\mathbb{F}(k,3)) = 3\). So by Proposition 3.4 we have, for \(k \geq 4\), \(\mathbb{F}(k,3)\) is a CIF(3,k). We observe that for \(0 \leq p, q, r \leq k-2\), the transversals of \(\mathbb{F}(k,3)\) are \(\{x^0_p, x^0_q, x^0_r\}; \{x^0_0, x^1_1, x^2_2\} \) and \(\{x^2_0, x^1_1, x^0_2\}\). Hence there are \((k-1)^3 + 3(k-1)\) number of transversals and 6 blocks in \(\mathbb{F}(k,3)\). So if we plug in \(k = 4\) and \(k = 5\) respectively, we have a CIF(4,3) and CIF(5,3) with 6 blocks and 36 & 76 transversals respectively. Let \(A\) be the unique MIF(1) (respectively, unique MIF(2) isomorphic to \(\{\{a, b\}, \{b, c\}, \{a, c\}\}\) and \(P_A \cap P_{\mathbb{F}(4,3)} = \emptyset\) (respectively, \(P_A \cap P_{\mathbb{F}(4,5)} = \emptyset\)). By Theorem 2.7, \(\mathbb{F}(4,3) \cup A \not\subseteq \mathbb{F}^T(4,3)\) is a MIF(4) with 42 blocks (respectively, \(\mathbb{F}(5,3) \cup A \not\subseteq \mathbb{F}^T(5,3)\) is a MIF(5) with 234 blocks). In this MIF(4) there are 1 point in 36 blocks, 6 points in 16 blocks and 3 points in 12 blocks.

**Remark 4.3.** Example 4.1 and Example 4.2 proves existence of two non isomorphic MIF(4) with 42 blocks. It disproves a special case (case \(k = 4\) of Conjecture 4 in [2], which claims \(M(4) = 228\).

**Remark 4.4.** Example 4.2 proves existence of a CIF(5,3) namely \(\mathbb{F}(5,3)\) with 6 blocks and 76 transversals. By Corollary 2.8 we have \(M(5) \geq 234\). It disproves a special case (case \(k = 5\) of Conjecture 4 in [2], which claims \(M(5) = 228\).}
For large positive integer $k$, any $\text{MIF}(k)$ with $\text{M}(k)$ blocks contains at least (approximately) $(e-1)k!$ blocks. Disproving this was the prime object of article [2]. Here we present an alternative simpler construction to prove $\text{M}(k)$ is at least $\left(\frac{e}{2}\right)^{k-1}$.

**Theorem 4.5.** Let $k \geq t+1$. Then

$$
\text{M}(k) \geq \begin{cases} 
(2r-1)(k-r+1)^{r-1} + (k-r+1)^{2r-1} \text{M}(k-2r+1) & \text{if } t = 2r-1 \\
2r(k-r)^{r-1} + (k-r)^{2r-1} \text{M}(k-2r) & \text{if } t = 2r.
\end{cases}
$$

(2)

**Proof:** Let $\mathcal{A}$ be a $\text{MIF}(k-t)$ with $\text{M}(k-t)$ blocks. By Theorem 3.3 and Theorem 2.7, it follows that $G(k,t) \cup \mathcal{A} \oplus G^\top(k,t)$ is a $\text{MIF}(k)$. Here we observe that any block of $G(k,t)$ is of the form

$$X_n \cup \{x_p^{n+i} \in X_{n+i} : 1 \leq i \leq k-|X_n|\},$$

where $0 \leq n \leq t-1$. It means that for each $X_n$, with $0 \leq n \leq t-1$, and for each $i$, with $1 \leq i \leq k-|X_n|$, there are $|X_{n+i}|$ number of choices for $x_{p}^{n+i}$. Therefore there are $\prod_{i=1}^{k-|X_n|} |X_{n+i}|$ choices for such blocks. Hence,

$$|G(k,t)| \geq \begin{cases} 
(2r-1)(k-r+1)^{r-1} & \text{if } t = 2r-1 \\
2r(k-r)^{r-1} & \text{if } t = 2r.
\end{cases}
$$

Also by using Theorem 3.3, we have

$$|G^\top(k,t)| = \begin{cases} 
(k-r+1)^{2r-1} & \text{if } t = 2r-1 \\
(k-r)^{2r-1} & \text{if } t = 2r.
\end{cases}
$$

Therefore the results follows from Corollary 2.8. \hfill \Box

If we plug in $t = k-1$ in (2), we have the following corollary (Theorem 1 of [2, § 2]), which shows that $\text{M}(k)$ grows like at least $(\frac{e}{2})^{k-1}$ and it disproves Lovász Conjecture.

**Corollary 4.6** (Frankl-Ota-Tokushige).

$$
\text{M}(k) \geq \begin{cases} 
\left(\frac{e}{2}+1\right)^{k-1} & \text{if } k \text{ is even} \\
\left(\frac{e}{2}+1\right)^{\frac{k-1}{2}} \left(\frac{\sqrt{e}}{2}\right)^{\frac{k+1}{2}} & \text{if } k \text{ is odd}.
\end{cases}
$$

The main problem of interest is to find a $\text{MIF}(k)$ with $\text{M}(k)$ blocks. Using Theorem 2.7, we observe that this problem actually boils down to find a $\text{CIF}(k,t) \mathcal{F}$ and a $\text{MIF}(k-t) \mathcal{A}$ so that $|\mathcal{F}| + |\mathcal{A}||\mathcal{F}^\top|$ is maximum for some suitable choice of $t \leq k-1$. So we formulate the following conjecture and close this article.

**Conjecture.** For large positive integer $k$, any $\text{MIF}(k)$ with $\text{M}(k)$ blocks contains a $\text{CIF}(k,t)$, for some $t \leq k-1$, which is isomorphic to a subfamily of $G(k,t)$.

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