FINITE POINT CONFIGURATIONS AND THE REGULAR VALUE THEOREM IN A FRACTAL SETTING

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Abstract. In this article, we study two problems concerning the size of the set of finite point configurations generated by a compact set \( E \subset \mathbb{R}^d \). The first problem concerns how the Lebesgue measure or the Hausdorff dimension of the finite point configuration set depends on that of \( E \). In particular, we show that if a planar set has dimension exceeding \( \frac{5}{4} \), then there exists a point \( x \in E \) so that for each integer \( k \geq 2 \), the set of “\( k \)-chains” has positive Lebesgue measure.

The second problem is a continuous analogue of the Erdős unit distance problem, which aims to determine the maximum number of times a point configuration with prescribed gaps can appear in \( E \). For instance, given a triangle with prescribed sides and given a sufficiently regular planar set \( E \) with Hausdorff dimension no less than \( \frac{7}{4} \), we show that the dimension of the set of vertices in \( E \) forming said triangle does not exceed \( 3 \dim_H(E) - 3 \). In addition to the Euclidean norm, we consider more general distances given by functions satisfying the so-called Phong-Stein rotational curvature condition. We also explore a number of examples to demonstrate the extent to which our results are sharp.

1. Introduction

We consider two problems concerning \( k \)-point configurations in subsets of \( \mathbb{R}^d \). The first aim is to understand how large a subset of Euclidean space must be to ensure that it contains many distinct scaled copies of a given polyhedron or another geometric shape. Upon fixing a scaling, the second problem is to determine how often a fixed shape occurs within a set of a given size.

These questions are natural analogues of some famous open questions in discrete geometry and geometric measure theory. More precisely, the first question can be viewed as an extension of the Falconer distance set problem (whose predecessor is the celebrated Erdős distinct distance problem in the discrete setting [7]), which conjectures that \( |\Delta(E)|_1 > 0 \) whenever the Hausdorff dimension of \( E \) exceeds \( \frac{d}{2} \) and remains open in all \( d \geq 2 \). Here, \( \Delta(E) \) denotes the set of distances, \( \{ |x - y| : x, y \in E \} \), and \( |\cdot|_k \) denotes the \( k \)-dimensional Lebesgue measure of a set.

The second question extends the Erdős unit distance problem to the \( k \geq 2 \) case in the continuous setting. The unit distance conjecture in the plane [7] says that if \( P \) is a planar point set with \( N \) points, then the number of pairs of points in \( P \) at a distance 1 apart is bounded above by \( C_{1, \epsilon} N^{1-\epsilon} \). These questions have attracted a great amount of attention over the decades (see for instance [2, 3, 14, 15, 30, 33, 35] and the references therein). Their study has utilized and
inspired ideas in many different fields, such as Fourier analysis (e.g. restriction
theory, decoupling) and combinatorics (e.g. polynomial method).

In order to give formal statements of the two main questions of focus, we will
need some notation. Define the chain set as follows: For \( E \subseteq \mathbb{R}^d \), \( d \geq 2 \), and for
integers \( k \geq 2 \), define the set of non-degenerate \( k \)-chains generated by \( E \) as

\[
S^k(E) := \left\{ (|x_1 - x_2|, \ldots, |x_k - x_{k+1}|) \in \mathbb{R}_+^k : x_i \in E, x_1, \ldots, x_{k+1} \right\}.
\]

Note that \( S^1(E) \) is simply the distance set of \( E \), and it will be denoted by \( \Delta(E) \) to
be consistent with classical literature. Further, we have the pinned version:

\[
S^k_p(E) := \left\{ (|x - x_1|, \ldots, |x_{k-1} - x_k|) \in \mathbb{R}_+^k : x_i \in E, x, x_1, \ldots, x_k \right\},
\]

which consists of non-degenerate \( k \)-chains in \( E \) that share a common starting place
of \( x \in E \). Similarly, if \( k = 1 \), \( S^1_p(E) \), the pinned distance set, is denoted by \( \Delta_p(E) \).

In addition to the edge-length sets generated by chains, we consider the edge-
length sets of \( k \)-trees, triangles, and other configurations with or without loops.
The edge-length set of a \( k \)-tree, denoted as \( T^k(E) \), consists of the edge lengths of
all trees, of any particular fixed shape \( T^k \), with \( k + 1 \) vertices in \( E \) and \( k \) edges.
A tree is a graph in which each vertex is connected by exactly one path (see, for
instance, [17]). We are also interested in pinned variants. \( T^k_p(E) \) will be used to
denote the set of all \( k \)-trees of a particular shape, \( T^k_p \), with a particular vertex to
be pinned (at \( x \)). Note that (pinned) chains are special examples of (pinned) trees.

In addition to the edge-length sets, we consider the vertex sets (of given point
configurations). Given any sequence of distances \( \vec{t} = (t_1, \ldots, t_k) \in \mathbb{R}_+^k \), define

\[
V S^k_t(E) := \{ (x_1, \ldots, x_{k+1}) \in E^{k+1} : |x_i - x_{i+1}| = t_i, i = 1, \ldots, k, \{x_i\} \text{ distinct} \}.
\]

as the \( k \)-chain set generated by \( E \) with prescribed distances \( \vec{t} \), where \( E^k \) denotes
the \( k \)-fold Cartesian product of \( E \). Similarly, vertex sets can be defined for the
\( k \)-tree set of a particular shape, \( VT^k(E) \), as well as more general configurations
containing loops, such as triangles, \( V Tri(E) \) (see (1.7)).

This article concerns two main questions:

1. How does the size of the edge-length set \( S^k(E), T^k(E) \), or that generated
   by other point configurations depend on the size of the set \( E \)?

2. Determine the number of times that a given \( k \)-chain, tree, or triangle with
   fixed side lengths can repeat in \( E \)? More precisely, determine the size of
   the vertex sets \( V S^k_t(E), VT^k(E), V Tri(E) \).

The notion of size or number is made formal using Lebesgue measure, Hausdorff
dimension, or Minkowski dimension.

Concerning the first question, our main contribution is establishing a method
that can serve as a bridge to extend all sufficiently good distance results to more
intricate configurations, or more generally, to extend results concerning subgraphs
to the whole graph given that they are glued together in a nice way. This method
is surprisingly simple and relies on a Fubini-like theorem. A key advantage of this
method is its flexibility to deal with much more general point configurations that
may have loops.

Regarding the second question, this seems to be the first article of its kind to
extend the unit distance question to the setting of \( k \)-point configurations in the
continuous setting (see [10, 26] for the discrete setting and [4, 24] for the \( k = 1 \) case in the continuous setting). It also appears to the the first article to examine a fractal variant of the regular value theorem for \( k \)-point configurations (see [4] for the \( k = 1 \) case). While existing techniques lend easily to results in the setting where the set \( E \) is assumed to be Ahlfors-David regular (see Remarks 1.6 and 1.21 and the references there), this article presents new techniques that extend to sets with more relaxed regularity assumptions. In particular, our techniques hold for some classic examples that fall outside the scope of AD regularity, such as the lattice example (see Example 1.7) and the train track example (see Example 1.8).

1.1. On the first question: Lebesgue measure and dimension. For \( E \subset \mathbb{R}^d \), we write \( \dim_H(E) \) to denote the Hausdorff dimension of \( E \), and we write \( \overline{\dim}_M(E) \) and \( \underline{\dim}_M(E) \) respectively, to denote the upper and lower Minkowski dimension of the set \( E \).

**Theorem 1.1.** Let \( E \subset \mathbb{R}^2 \) be a compact set satisfying \( \dim_H(E) > \frac{5}{4} \), then there exists a point \( x \in E \) such that for all integers \( k \geq 2 \), \( |T_k^x(E)|_k > 0 \). In particular, \( |S^k_x(E)|_k > 0 \).

Theorem 1.1 generalizes the work of the first listed author with Guth, Iosevich and Wang [14] where the distance set case (i.e. \( k = 1 \)) is proved. When \( k \geq 2 \), the first such result concerning the pinned chains of \( k \) distances in \( \mathbb{R}^d \) is due to Bennett, Iosevich, and the second listed author [1], where it is required that \( \dim_H(E) > \frac{d+1}{2} \). In [1], the authors demonstrate that, for each \( k \) and for each \( E \subset \mathbb{R}^d \) of dimension greater than \( \frac{d+1}{2} \), there exist an interval worth of admissible gaps (dependent only on \( k \)) for which \( E \) contains the vertices of a \( k \)-chain with side lengths in said interval. Their argument establishes continuity of the Radon-Nikodym derivative of a natural measure on \( S^k(E) \). (Also see [11] and [36], where the problem is investigated for sets of positive upper Lebesgue density). Recently, it was obtained by Liu [22] that the dimensional threshold concerning pinned \( k \)-chains can be lowered to \( \frac{4}{3} \) in the plane (in the case of \( k = 2 \), the threshold \( \frac{4}{3} \) was first achieved in [21] for the full \( k \)-chain set).

The case of trees is slightly more involved compared to chains, as the iteration procedure becomes more complicated due to the fact that a vertex may be connected to many edges. Theorem 1.1 improves the previously best known result of Iosevich and the second listed author [17] where the threshold \( \frac{3}{2} \) is obtained.

Moreover, we also study the dimension of the (pinned) tree sets and prove the following.

**Theorem 1.2.** Let \( E \subset \mathbb{R}^2 \) be a compact set satisfying \( \dim_H(E) > 1 \), then for all integers \( k \geq 2 \), we have

\[
\dim_H(T^k(E)) \geq \min \left\{ \frac{4k}{3} \dim_H(E) - \frac{2k}{3}, k \right\}.
\]

Moreover, for all \( \epsilon > 0 \), for each \( k \geq 2 \), there exists a point \( x \in E \) such that

\[
\dim_H(T^k_x(E)) \geq \min \left\{ \frac{4k}{3} \dim_H(E) - \frac{2k}{3} - \epsilon, k \right\}.
\]
Furthermore, if \( 1 < \dim_H(E) \leq \frac{5}{4} \), then for all sufficiently small \( \epsilon > 0 \), there exists an \( x \in E \) so that for all \( k \geq 2 \),
\[
\dim_H(T^k_x(E)) \geq \frac{4k}{3}(\dim_H(E) - \epsilon) - \frac{2k}{3} > \frac{2k}{3}.
\]

In particular, the result above holds for (pinned) chains. When \( k = 1 \), this was proved by Liu [23]. Note that there is a minor inaccuracy in the statement of [23, Theorem 1.1], where an \( \epsilon \) is in fact needed similarly as in (1.2) of Theorem 1.2. In fact, a slight improvement of the \( k = 1 \) case when \( \dim_H(E) \in (1, 0.1037) \) was obtained by Shmerkin [27]. Since the main contribution of our work is a method that allows one to extend the \( k = 1 \) result automatically to all \( k \geq 2 \), we omit the statement of the slight improvement of Theorem 1.2 that can be implied by [27] when \( \dim_H(E) \in (1, 0.1037) \).

In addition, we also obtain a more general version of Theorem 1.2 on the exceptional set of \( x \). For simplicity, we only state the next result for chains.

**Theorem 1.3.** Given any compact set \( E \subset \mathbb{R}^2 \) and integer \( k \geq 2 \). Suppose that \( \dim_H(E) > 1 \). Set
\[
\tau^k_0 = \tau^k_0(\dim_H(E)) = \begin{cases} 
\frac{4(k-1)}{3} \dim_H(E) + \frac{5-2k}{3}, & 1 < \dim_H(E) \leq \frac{5}{4}, \\
\frac{5}{4} - \dim_H(E), & \frac{5}{4} < \dim_H(E) \leq 2.
\end{cases}
\]
Then, for each \( \tau \in (0, \tau^k_0) \),
\[
\dim_H(\{x \in \mathbb{R}^2 : \dim_H(S^k_x(E)) < \tau\}) \leq \begin{cases} 
\max(2k + 3\tau + (1 - 4k)\dim_H(E), 2 - \dim_H(E)), & 1 < \dim_H(E) \leq \frac{5}{4}, \\
\max(5 - 3k + 3\tau - 3\dim_H(E), 2 - \dim_H(E)), & \dim_H(E) > \frac{5}{4}.
\end{cases}
\]

When \( k = 1 \), in which case \( \tau^k_0 = 1 \) and the two quantities at the end become the same, the above result also holds true and was obtained in [23]. In Theorem 1.3 by setting \( \tau \) to be equal to \( \frac{4k}{3} \dim_H(E) - \frac{2k}{3} - \epsilon \), one can immediately obtain not only Theorem 1.2 but also the fact that the exceptional set in \( E \) (consisting of bad pin points) always has lower dimension than \( E \).

The proof of Theorem 1.3 can be generalized to study the case of trees of any given shape, though the exact bound is cumbersome to state since it depends on the shape of the tree and the vertex that one chooses to pin. Given that it is unlikely that the estimate obtained here is sharp, we skip the parallel statement of Theorem 1.3 for trees. It is unclear to us whether the constraint on \( \tau \) in Theorem 1.3 can be further relaxed. In fact, it seems that the range of \( \tau \) is closely tied to the iterative nature of our proof method.

**Remark 1.4.** The machinery developed in Theorem 1.1, Theorem 1.2 and Theorem 1.3 can be used to “glue” together any variety of pinned \( k \)-point configurations (including those with loops, such as triangles) that are a priori known to exist within a compact set \( E \) in the plane. We give an example here (see Proposition 1.5) as a corollary of the method.
Proposition 1.5. Let $E \subset \mathbb{R}^d$ be a compact set and let $\mu$ denote a Frostman measure on $E$. Suppose that there exist a pair of disjoint sets $E_1, E_2 \subset E$ so that $\mu(E_i) > 0$, $i = 1, 2$, and, for some $x \in E_1$,
$$
\dim_H(\{(|x - y_1|, |y_1 - y_2|, |y_2 - x|) : y_1, y_2 \in E_1, x, y_1, y_2 \text{ distinct}\}) \geq \gamma_1 > 0,
$$
and for each $y_1 \in E_1$,
$$
\dim_H(\{(y_1, y_3, |y_3 - y_4|) : y_3, y_4 \in E_2 \text{ distinct}\}) \geq \gamma_2 > 0.
$$
Then it holds, for some $x \in E$, that
$$
\dim_H(\{(|x - y_1|, |y_1 - y_2|, |y_2 - x|, |y_1 - y_3|, |y_3 - y_4|) : y_1, y_2, y_3, y_4 \in E, x, y_i \text{ distinct}\}) \geq \gamma_1 + \gamma_2.
$$
We call the set in (1.4) a kite.

We note that similar results can be obtained for any choice of pinned point along the kite.

1.2. On the second question: Prescribed distances. We now turn to the second question, which can be viewed as an analogue of the unit distance problem. For $d, k \geq 2$, we are interested in determining the value of
$$
g_d(VS_{\ell}^k, \alpha) := \sup\{\dim_H(VS_{\ell}^k(E)) : E \text{ is a compact set in } \mathbb{R}^d, \dim_H(E) = \alpha\},
$$
where $VS_{\ell}^k(E)$ is defined in (1.1), and similarly the value when the set $VS_{\ell}^k(E)$ is replaced by the vertex set of a tree, $VT_{\ell}^k$, or a triangle $VTri_{\ell}(E)$ (see (1.7)). We will omit the dependence on $\ell$ from the notation above when the values of $\ell$ are clear from the context.

An additional motivation for our investigation arises from the regular value theorem from elementary differential geometry. The regular value theorem in elementary differential geometry says that if $\phi : X \to Y$, where $X$ is a smooth manifold of dimension $n$ and $Y$ is a smooth manifold of dimension $m < n$ with $\phi$ a submersion on the set $\phi^{-1}(y)$, for $y \in Y$ fixed, then the set
$$
\phi^{-1}(y) = \{x \in X : \phi(x) = y\}
$$
is either empty or is a $n - m$ dimensional submanifold of $X$.

A fractal variant of the regular value theorem was obtained in [4], where it was shown that, under some reasonable hypotheses on $\phi : \mathbb{E} \times \mathbb{E} \to \mathbb{R}^m$, the upper Minkowski dimension of
$$
\{(x, y) \in E \times E : \phi_l(x, y) = t_l, 1 \leq l \leq m\}
$$
does not exceed $2\alpha - m$, where $E \subset \mathbb{R}^d$ is a set of Hausdorff dimension $\alpha$.

Observe that, given any sequence of distances $t' = (t_1, \cdots, t_k) \in \mathbb{R}^k_+$, we may re-write the set $VS^k_{t'}(E)$ as

$$
(1.5) \quad VS^k_{t'}(E) = \{(x_1, \cdots, x_{k+1}) \in E^{k+1} : \phi(x_1, \cdots, x_{k+1}) = t', \}
$$

where $\phi : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^k_+$ is defined by $\phi(x_1, \cdots, x_{k+1}) = (|x_1 - x_2|, \cdots, |x_k - x_{k+1}|)$.

A direct analog of the regular value theorem would state that either $VS^k_{t'}(E)$ is empty or is a set of dimension $(k + 1)\alpha - k$. We compare this to our Theorem 1.9.

In the continuous case, similar questions for distance sets have been studied in \cite{4 5 24}, and it was observed that one needs to assume some minimal regularity for the set $E$ in order for the question to be meaningful (see \cite{24} Page 253 for a discussion). We will work with sets that are “$\delta_i$-discrete $\alpha$-regular” below, which generalizes a concept that was first introduced in \cite{19} and used in \cite{24} for the distance set $(k = 1)$ case.

Given a compact set $E \subset \mathbb{R}^d$ and $\delta > 0$, denote $E_\delta := E + B(0, \delta)$ the $\delta$-neighborhood of $E$. Let $\alpha > 0$, $E$ is said to be $\{\delta_i\}$-discrete $\alpha$-regular if there exists a sequence of positive numbers $\{\delta_i\}$ such that $\delta_i \to 0$ and

$$
(1.6) \quad |E_{\delta_i} \cap B(x, r)| \leq \left( \frac{r}{\delta_i} \right)^{\alpha} \delta_i^d
$$

for any $x \in \mathbb{R}^d$ and $r \geq \delta_i$, $\forall i$, where $| \cdot |$ denotes the Lebesgue measure. For $X, Y \in \mathbb{R}$, we use $X \leq Y$ to denote the estimate $X \leq cY$ for some constant $c > 0$.

Remark 1.6. It is not hard to see that this class of sets contains AD regular (Ahlfors-David regular) sets as special examples. Indeed, recall that if $E$ denotes an AD regular set of Hausdorff dimension $\alpha$, then $E$ supports a Borel probability measure $\mu$ so that for each $r > 0$ and for each $x \in E$, $cr^\alpha \leq \mu(B(x, r)) \leq Cr^\alpha$, for universal constants $0 < c < C$. Letting $\delta \in (0, r]$ and $x \in E$, we can write $\mu(B(x, r)) \sim r^\alpha = \delta^\alpha (\frac{r}{\delta})^\alpha$, and deduce that the number of $\delta$-balls required to cover $E \cap B(x, r)$ is approximately $(\frac{r}{\delta})^\alpha$. This holds for any set of scales $(\delta, r)$ with $\delta \leq r$.

It follows that, if $N$ denotes the number of $\delta$-balls needed to cover $E_\delta \cap B(x, r)$, then $N \sim (\frac{r}{\delta})^\alpha$. Now $|E_{\delta_i} \cap B(x, r)| \leq N \delta_i^d \leq (\frac{r}{\delta_i})^\alpha \delta_i^d$.

Moreover, the class of sets considered here includes the class of $\delta$-discrete $\alpha$-regular sets considered in \cite{24} (where (1.6) is assumed to hold for all $\delta > 0$ rather than only a sequence of scales $\{\delta_i\}$), and some examples that are not AD regular.

Example 1.7. [The lattice example] A non-trivial example of a set which is $\{\delta_i\}$-discrete $\alpha$-regular, but not AD regular is provided by a classic lattice-like construction. In more detail, we create a fractal subset of $\mathbb{R}^d$ that mimics an integer lattice as follows: For an integer $q_i$, consider the set $E_i := \bigcup_{\mathbf{x} \in \{0, 1, \ldots, q_i\}^d} \{ \mathbf{x} \in \mathbb{R}^d : |\mathbf{x} - \frac{\mathbf{x}}{q_i}| < q_i^{-d/\alpha} \}$. By taking a rapidly increasing sequence of $q_i$, and setting $E$ equal to the intersection of the sets $E_i$, one obtains a set of Hausdorff dimension $\alpha$. (For more details on this construction, see \cite{3} Theorem 8.15.) It is easy to verify that the set $E$ is not AD regular (unless $\alpha = d$) but is $\{\delta_i\}$-discrete $\alpha$-regular (by taking $\delta_i = q_i^{-d/\alpha}$).

Example 1.8. [The train track example] Another example of a set that is $\{\delta_i\}$-discrete $\alpha$-regular, but not AD regular, can be constructed from train tracks. Such an example was first studied by Katz–Tao \cite{19} and was further examined in \cite{14} in
connection with the Falconer distance problem. More precisely, consider a sequence \( \{R_i\} \) rapidly increasing to infinity, and define a set \( E = \cap E_i \) in the unit square in \( \mathbb{R}^2 \) as follows. Fix each \( R_i \), \( E_i \) is divided among several large \( R_i^{-1/2} \times 1 \) vertical rectangles, spaced by distance \( R_i^{1/2 - \frac{\alpha}{2}} \) \((1 < \alpha \leq 2)\). Within each of these large rectangles, the set \( E_i \) consists of small evenly spaced parallel horizontal rectangles with dimensions \( R_i^{-1} \times R_i^{-1/2} \). Each of these small horizontal rectangles is called a slat, and the spacing between two consecutive slats is \( R_i^{-2} \). One can verify that the set \( E \) has Hausdorff dimension \( \alpha \) and \( E \) is not AD regular. However, by taking \( \delta_i = R_i^{-1} \), one sees that \( E \) is indeed \( \{\delta_i\}\)-discrete \( \alpha \)-regular.

Going forward in our discussion, we will assume the set \( E \) to be \( \{\delta_i\}\)-discrete \( \alpha \)-regular. Without loss of generality, we will further assume \( E \) to be contained in the unit ball throughout the article, hence, the condition \( \|E_i\| \leq \delta_i^{\alpha-\alpha'} \), \( \forall i \).

In the discrete setting, similar questions for chains have been very recently explored as well, see [10] [26]. Note that in both works, the authors in fact study a slightly larger quantity than \( g_d(VS^k, \alpha) \), where an additional supremum is taken over all choices of prescribed gaps \( \bar{\ell} \). All the upper bound results we obtain in this paper also extend to this larger quantity, but we need to be more careful here for deriving the lower bound, which would depend on the particular choice of gaps \( \bar{\ell} \). In fact, as will be explained later in the paper, different choices of gaps sometimes yield very different behaviors of \( g_d(VS^k, \alpha) \).

Another distinct feature of our work, compared to the discrete setting, is that nontrivial results do exist in the \( d \geq 4 \) case; It is well known that the unit distance problem and the chain problems are trivial in the discrete case when \( d \geq 4 \), see the introduction of \[24\] [26] for more details.

1.2.1. Chains. Our first result in this direction concerns all \( d \geq 2 \).

**Theorem 1.9.** For all \( d \geq 2 \), \( k \geq 1 \), and prescribed gaps \( \bar{\ell} \),

\[
g_d(VS^k_i, \alpha) \begin{cases} (k+1)\alpha - k, & \frac{d+1}{2} \leq \alpha \leq d, \\ \leq \frac{k(d-1)\alpha}{2} + \alpha, & \alpha \leq \frac{d+1}{2}. \end{cases}
\]

Note that there always holds the trivial estimate \( g_d(VS^k_i, \alpha) \leq (k+1)\alpha \) (see a justification in Section 4.2.1), so the second estimate of Theorem 1.9 is only nontrivial when \( \alpha > \frac{d-1}{2} \).

**Remark 1.10.** To put the second listed bound in Theorem 1.9 into context, we observe an additional upper bound which is also inferior in the regime \( \alpha > \frac{d-1}{2} \). The proof techniques used in \[24\] can be used to show that, if \( \dim_H(E) = \alpha \), \( E \) is \( \{\delta_i\}\)-discrete \( \alpha \)-regular, and \( \alpha < \frac{d+1}{2} \), then the lower Minkowski dimension of the set \( \{(x,y) \in E \times E : |x-y| = 1\} \) is bounded above by \( \alpha + \frac{d-1}{2} \). In more detail, the Lebesgue measure of the \( \delta_i \)-neighborhood of \( \{(x,y) \in E \times E : |x-y| = 1\} \) is bounded above by \( \delta_i^{2d-(\alpha+\frac{d-1}{2})-\varepsilon} \), \( \forall i \). Observing that the vertex set of the \( k \)-chain set, \( VS^k_i(E) \), is contained in the set \( \{ (x^1, x^2) \in E \times E : |x^1 - x^2| = 1 \} \times E^{k-1} \), and recalling that \( |E_i| \leq \delta_i^{\alpha-\alpha'} \), \( \forall i \), we use the fact that the Hausdorff dimension is bounded above by the lower Minkowski dimension to conclude that \( g_d(VS^k_i, \alpha) \leq \alpha + \frac{d-1}{2} + (k-1)\alpha \). (A slightly better upper bound can be obtained in a similar
way, by decomposing the $k$-chain into shorter components and utilizing the upper estimate for shorter chains or distances. However, results obtained in this way will always be inferior to Theorem 1.9 when $\alpha > \frac{d-1}{2}$.

For small values of $\alpha$, different features are displayed in different ambient dimensions.

**Theorem 1.11.** Let $k \geq 1$. For all $d \geq 4$ and $\alpha \leq \lfloor \frac{d}{2} \rfloor - 1$, $g_d(VS^k_t, \alpha) = (k+1)\alpha$, for all $t$ satisfying $t_1 \leq \cdots \leq t_k$. Moreover, when $d = 2$, we have $g_2(VS^2_t, \alpha) = 2\alpha$ if $0 < \alpha \leq 1$, for all $t$.

**Remark 1.12.** The first lower bound in Theorem 1.11 relies on adapting a classic construction from the discrete setting, which utilizes orthogonal circles (see section 4.2.2 below for details). However, some relation on the $t$’s, such as $t_1 \leq \cdots \leq t_k$, is required. It is not clear, for instance, whether the set of 3-chains of gaps $(1, 2, 1)$ achieves the same lower bound (see section 4.2.2 below for details). However, it is possible to further relax the condition $t_1 \leq \cdots \leq t_k$. For instance, when $k = 2$, one can in fact show that $g_d(VS^2_t, \alpha) = (k+1)\alpha = 3\alpha$ whenever $t_2^2 < t_1^2 + t_3^2$.

When $d = 2$ and $0 < \alpha < 1$, similar deduction as in the proof of Theorem 1.11 indeed gives rise to certain upper and lower bound of $g_2(VS^2_t, \alpha)$, for general $k \geq 3$ as well. However, since these bounds are unlikely sharp, we omit the details. Similarly, one can straightforwardly extend the same method to study the case $d = 3$, $0 < \alpha < 2$, but new ideas seem to be needed in order to fully solve the problem in three dimensions.

When $k = 1$, and when $g_d(VS^k_t, \alpha)$ is defined using only those sets $E$ satisfying a stronger regularity condition (i.e. estimate 1.6 for all $\delta > 0$), Theorem 1.9 and the first result in Theorem 1.11 in the above are proved in [24]. It is straightforward to see that their argument in fact also works for sets that are $\{\delta_i\}$-discrete $\alpha$-regular. Therefore we will only prove the $k \geq 2$ cases. We also point out that, when $k = 1$ and when further restricting the sets considered to be AD regular, Theorem 1.9 (for $\alpha > \frac{d+1}{2}$) is first obtained in [4].

**Remark 1.13.** Both Theorem 1.9 and Theorem 1.11 readily extend to the case of trees. Since in most cases, the value of $g_d(VT^k_t, \alpha)$ seems to depend on the exact structure of the tree $T^k$, we omit the statement of those results for the sake of simplicity and only comment on the necessary changes for the tree case along the proofs of Theorem 1.9 and 1.11. For example, one can show that when $d \geq 2$, $k \geq 2$, and $\alpha \geq \frac{d-1}{2}$, $g_d(VT^k_t, \alpha) = (k+1)\alpha - k$. Another example is that when $d = 2$ and $0 < \alpha \leq 1$, if the $k$-tree is a star, i.e. all $k$ edges share the same vertex, then $g_2(VT^k_t, \alpha) = k\alpha$.

1.2.2. Triangles and loops. We now turn to studying finite point configurations containing closed loops. Compared to the chain or tree case, the main difficulty here is that the existence of loops in the graph prevents one from applying an iterative proof scheme. Our proof is based on a Fourier analytic approach that involves the estimate of the decay of the Fourier transform of the surface measure of a hypersurface that encodes the structure of the point configuration. For the sake of simplicity, we only study the case of triangles in this article, even though the proof strategy can be extended to more general graphs with loops. Define

$$g_d(VTri_t, \alpha) := \sup \{ \dim_H(VTri_t(E)) : E \text{ is a compact set in } \mathbb{R}^d, \dim_H(E) = \alpha \},$$
can be obtained by considering the map
\[ p : E \to \mathbb{R}^3 \] of dim \( E \) to \( \mathbb{R}^3 \). Greenleaf and Iosevich prove that if \( \min \delta_i > 3 \), then the set of triples of distances formed by triangles in \( E \) (1.8) \( \text{Tri}(E) := \{(x, y, z) \in E^3 : |x - y| = t_1, |y - z| = t_2, |z - x| = t_3\} \).

Again, we always assume that the set \( E \) is \( \{\delta_i\}\)-discrete \( \alpha \)-regular.

**Theorem 1.14.** Let \( d \geq 3 \). Then for all \( \ell \),
\[
g_d(\text{VTri}_\ell, \alpha) \leq \begin{cases} 3\alpha - 3, & 2d/3 + 1 \leq \alpha \leq d, \\ \frac{d}{2} + \frac{3\alpha}{2} - \frac{3}{2}, & 0 < \alpha \leq \frac{2d}{3} + 1. \end{cases}
\]
Moreover, if \( d = 2 \), it holds for all \( \ell \) that
\[
g_2(\text{VTri}_\ell, \alpha) \leq \begin{cases} 3\alpha - 3, & \frac{7}{4} \leq \alpha \leq 2, \\ 2\alpha - 1, & \frac{3}{2} \leq \alpha < \frac{7}{4}, \\ \alpha + \frac{1}{2}, & 1 \leq \alpha \leq \frac{3}{2}, \\ \min\{\frac{5\alpha}{3}, \frac{\alpha(2+\alpha)}{1+\alpha}\}, & 0 < \alpha \leq 1. \end{cases}
\]
Furthermore, if \( d \geq 6 \) and \( \alpha \leq \frac{d}{3} - 1 \), then \( g_d(\text{VTri}_\ell, \alpha) = 3\alpha \) whenever \( \ell = (t_1, t_2, t_3) \) forms an acute triangle.

We note that this theorem fits nicely into the current results in the field. In [12], Greenleaf and Iosevich prove that if \( E \) is a compact subset of \( \mathbb{R}^2 \) of dim \( E \) \( \geq \frac{7}{4} \), then the set of triples of distances formed by triangles in \( E \),
\[
(1.7) \quad \text{VTri}(E) := \{(x, y, z) : x, y, z \in E\},
\]
has positive 3-dimensional Lebesgue measure. The authors define a measure on the set \( \text{VTri}(E) \) and proving that its density is in \( L^2 \). In a subsequent paper using group actions and an \( L^2 \) estimate on the density, it is shown that in all dimensions \( d \geq 2 \), if \( \text{dim}_H(E) > \frac{2d+1}{3} \), then \( \text{VTri}(E) \) has positive Lebesgue measure \([13]\). In [10], it is proved that for any \( d \geq 4 \), there exists an \( \delta > 0 \) so that if \( \text{dim}_H(E) > d - \delta \), then \( E \) contains the vertices of an equilateral triangle. Theorem 1.14 above shows that, for any \( d \geq 2 \), it is possible to control the number of occurrences of such triangles through an upper bound on the Hausdorff dimension.

**Remark 1.15.** We note a simple transference mechanism between the unit distance problem and the distinct distance problem, which holds when we impose the additional regularity assumption that the set \( E \) is \( AD \)-regular. In this case, it is a straightforward exercise to verify that the results in Theorem 1.14 imply a lower bound on the set
\[
\text{dim}_H \{(|x - y|, |y - z|, |x - z|) : x, y, z \in E\}.
\]
We omit this result, however, as superior lower bounds are obtained in [34].

**Remark 1.16.** Note that one always has the trivial estimate \( g_d(\text{VTri}_\ell, \alpha) \) is no more than \( \min(3\alpha, 3d - 3) \). Indeed, the first bound follows easily from \( \text{VTri}_0(E) \subset E^3 \) and \( E \) is \( \{\delta_i\}\) discrete \( \alpha \)-regular (hence \( \text{dim}_H(E^3) = 3\alpha \)), and the second bound can be obtained by considering the map \( (x, y, z) \mapsto (x - y, y - z, z - x) \) and observing that the image set is contained in \( t_1S^{d-1} \times t_2S^{d-1} \times t_3S^{d-1} \). Therefore, the bound \( g_d(\text{VTri}_\ell, \alpha) \leq d + \frac{2\alpha}{2} - \frac{3}{2} \) is only nontrivial if \( \alpha > \frac{2d}{3} - 1 \). When \( d = 2 \), the upper bound in the theorem above when \( \alpha < \frac{7}{4} \) can be obtained from the trivial bound \( g_2(\text{VTri}_\ell, \alpha) \leq g_2(\text{VTri}_0, \alpha) \), \( \forall i = 1, 2, 3 \) and the estimate for the unit distance problem in [23, Theorem 1.3]. To see this, fix two vertices \( x \) and \( y \) at a distance \( t_1 \).
apart from each other, and observe that there are at most two choices of the third vertex $z$.

Remark 1.17. For $\alpha > d - 1$, a lower bound of $\alpha + (d - 2)$ is achievable in all dimensions. Let $A \subset \mathbb{R}^d$ so that $\dim_H(A) := \alpha \in (0,1)$. Set $E = A \bigcup \{a + S^{d-1}\}$, the Minkowski sum of $A$ and the unit sphere. Inspecting the energy integral of $E$ and observing that sum sets are Lipschitz images of Cartesian product sets, one may verify that $\dim_H(E) =: \alpha = \alpha + (d - 1)$ (see, for instance, the work of the second listed author with K. Simon: [28, 29]). Observe that, for each $x \in A,$ $(x + S^{d-1}) \cap E = x + S^{d-1}$. Now, for each $x \in A$,

\[
\{(y, z) \in E \times E : |x - y| = 1, |x - z| = 1, \text{ and } |y - z| = 1\}
\]

and this set clearly has Hausdorff dimension $(d - 1) + (d - 2)$. Restricting $x$ to $A$, it follows by Corollary 4.3 in Appendix A that

\[
\dim_H(\text{VTri}^E_1(E)) \geq \alpha + (d - 1) + (d - 2) = \alpha + (d - 2).
\]

1.2.3. Phong-Stein condition. The question of whether it is possible to replace the Euclidean distance in the above discussion (for instance (1.1)) with a more general metric was raised in [24, Page 255]. In this section, we answer this question in the affirmative not only for the unit distance problem but also for the general $k$-chain. Moreover, we show that it is possible to replace the Euclidean distance with a more general function $\phi(x, y)$ which satisfies the rotational curvature conditions introduced by Phong and Stein.

In particular, we consider $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ to be a continuous, infinitely differentiable function, which satisfies

\[
|\nabla_x \phi(x, y)| \neq 0 \quad \text{and} \quad |\nabla_y \phi(x, y)| \neq 0.
\]

Further, we assume that $\phi$ satisfies the non-vanishing Monge-Ampere determinant assumption:

\[
\text{det} \begin{pmatrix} 0 & \nabla_x \phi \\ -\nabla_y \phi^T & \nabla^2_{xy} \phi \end{pmatrix}
\]

does not vanish on the set $\{(x, y) : \phi(x, y) = t\}$, $t \neq 0$.

Remark 1.18. Examples of such functions include the dot product, $\phi(x, y) = x \cdot y$, as well as any norm, $\phi(x, y) = \|x\|_B$, generated by a smooth convex body, $B$, with non-vanishing curvature.

Given any sequence of distances $\vec{t} = (t_1, \cdots, t_k) \in \mathbb{R}^+_k$, we define

\[
\text{VS}^k_{\vec{t}, \phi}(E) := \{(x_1, \cdots, x_{k+1}) \in E^{k+1} : \phi(x_i, x_{i+1}) = t_i, i = 1, \ldots, k, \{x_i\} \text{ distinct}\}
\]

as the $(k, \phi)$-chain set generated by $E$ with prescribed gaps $\vec{t}$.

We are interested in determining the value of

\[
g_d(\text{VS}^k_{\vec{t}, \phi}, \alpha) := \sup\{\dim_H(\text{VS}^k_{\vec{t}, \phi}(E)) : E \text{ is a compact set in } \mathbb{R}^d, \dim_H(E) = \alpha\}.
\]

For ease of notation, we drop the subscript $\vec{t}$ throughout the discussion in this subsection.
We now turn to the main result of this subsection, in which we show that, in the case that the set $E$ is assumed to be $t_{\delta_i}$-discrete $\alpha$-regular (see equation (1.6) for the definition), the following analogue of Theorem 1.9 holds.

**Theorem 1.19.** Let $\alpha > 0$ and $d \geq 2$. Let $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ denote a smooth function which satisfies the gradient conditions in (1.9), as well as the curvature condition in (1.10). Then, for all $k \geq 1$,

$$g_d(VS^k_{\phi}, \alpha) \begin{cases} (k+1)\alpha - k, & \frac{d+1}{2} \leq \alpha \leq d, \\ \leq \frac{k(d-1)}{2} + \alpha, & \alpha \leq \frac{d+1}{2}. \end{cases}$$

The proof of Theorem 1.19 can be found in Section 6 and relies on merging the ideas introduced in [18] and in [24]. Equality is attained using a simple adaptation of the proof presented in Section 4.1.4. Note that there always holds the trivial estimate $g_d(VS^k_{\phi}, \alpha) \leq (k+1)\alpha$, which is inferior to the bounds in Theorem 1.19 provided $\alpha > \frac{d-1}{2}$.

**Remark 1.20.** In Appendix B, we consider an example $E$ where $\phi$ is given by a paraboloid-like surface, and we show that the upper Minkowski dimension of $VS^k_{\phi}(E)$, for this choice of $\phi$ and $E$, is bounded below by $\alpha + \frac{\alpha(d-1)k}{d+1}$.

**Remark 1.21.** In the special case that the set $E$ is AD regular (as defined in Remark 1.6), it is an immediate corollary of the work of the second listed author with Iosevich and Uriarte-Tuero [18] that $\dim_H(VS^k_{\phi}(E)) \leq (k+1)\alpha - k$, whenever $\frac{d+1}{2} < \alpha$. Similarly, if $E$ is assumed to be AD regular, then inspecting the proof of Greenleaf and Iosevich [12] and that of Iosevich and Liu [16] recovers part of Theorem 1.14 when $\alpha > \frac{1}{4}$ ($d = 2$) and when $\alpha > \frac{2d}{3} + 1$ ($d \geq 4$), respectively.

Before ending the introduction, we point out that all the upper bound estimates in Theorems 1.9, 1.11, 1.14, and 1.19 work not only for Hausdorff dimension (of the set $VS^k_{\phi}$ for instance) but also for the slightly larger lower Minkowski dimension, which is straightforward to see from the proofs.

The article is organized as follows. We study the first question in Sections 2 and 3: Theorem 1.1 is proved in Section 2, while Theorem 1.2, Theorem 1.3, and Proposition 1.5 are proved in Section 3. Sections 4, 5, 6 are devoted to the study of the second question: chains and trees (Theorems 1.9 and 1.11) are treated in Section 4, triangles (Theorem 1.14) appear in Section 5, and the general $\phi$ case (Theorem 1.19) is studied in Section 6.

2. Lebesgue Measure of Pinned Chains/ Trees: Proof of Theorem 1.1

The main ingredient of the proof is the following structure theorem, which works in all dimensions and does not require any assumption on the value of $\alpha$.

**Proposition 2.1.** Let $d \geq 2$ and $\alpha > 0$. Suppose for all pairs of compact sets $E_1, E_2 \subset \mathbb{R}^d$ with positive $\alpha$-dimensional Hausdorff measure, letting $\mu_1, \mu_2$ be Borel probability measures supported on $E_1, E_2$ respectively which satisfy $\mu_i(B(x, r)) \lesssim r^\alpha$ for $i = 1, 2$, then

$$\mu_2(G_{E_1}(E_2)) := \mu_2(\{x \in E_2 : |\Delta_x(E_1)| > 0\}) > 0,$$

where $\Delta_x(E) := \{|x - y| : y \in E\}$ denotes the pinned distance set.
Then, for all integers \( k \geq 1 \), and all pairs of compact sets \( E_1, E_2 \subset \mathbb{R}^d \) with positive \( \alpha \)-dimensional Hausdorff measure, there exists \( x \in E_2 \) such that \( |T^k_x(E_1)|_k > 0 \), for all \( k \)-trees \( T^k_v \) of any shape pinned at any vertex. In particular, \( |S^k_x(E_1)|_k > 0 \).

The notation \( G_{E_1}(E_2) \) above means the “good pins” in \( E_2 \) w.r.t. \( E_1 \).

**Proof.** We will in fact prove a stronger result: for all integers \( k \geq 1 \), all pairs of compact sets \( E_1, E_2 \subset \mathbb{R}^d \) with positive \( \alpha \)-dimensional Hausdorff measure, letting \( \mu_1, \mu_2 \) be Borel probability measures supported on \( E_1, E_2 \) respectively which satisfy 
\[
\mu_i(B(x,r)) \leq r^\alpha \quad \text{for all} \quad i=1,2,
\]
then
\[
(2.12) \quad \mu_2(G^k_{E_1}(E_2)) := \mu_2(\{x \in E_2 : |T^k_x(E_1)|_k > 0, \text{ for all } k \text{-trees } T^k_v\}) > 0.
\]

Our strategy is to prove by induction. The base case \( k = 1 \) is precisely the same as the assumption, hence is obviously true. Assume that the desired result holds for trees whose number of edges is no greater than \( k - 1 \).

Let \( E_1, E_2 \subset \mathbb{R}^d \) be two sets such that there exist probability measures \( \mu_1, \mu_2 \) supported on \( E_1, E_2 \) respectively with \( \mu_i(B(x,r)) \leq r^\alpha \) for \( i = 1,2 \). It suffices to show (2.12). In fact, fix any particular \( k \)-tree \( T^k = T^k_v \); it suffices to prove that
\[
(2.13) \quad \mu_2(G^k_{E_1,T^k}(E_2)) := \mu_2(\{x \in E_2 : |T^k_x(E_1)|_k > 0\}) > 0.
\]

Indeed, for any \( k \) fixed, there are only finitely many possibilities of shapes and vertices to be pinned, in other words, finitely many choices of \( T^k \). If (2.13) is true for some \( T^k \), then one can replace \( E_2 \) by \( G^k_{E_1,T^k}(E_2) \) and iterate the argument.

We omit the details and fix a choice of \( T^k \) (with a fixed choice of a vertex to be pinned) from now on. Moreover, by properly shrinking \( E_1, E_2 \) if needed, one can assume without loss of generality that \( E_1, E_2 \) are disjoint.

If the fixed vertex is connected to only a single edge, then one applies the induction hypothesis inside the set \( E_1 \). More precisely, one can find two subsets \( E_{1,1}, E_{1,2} \) of \( E_1 \) and measures \( \mu_{1,1} \) and \( \mu_{1,2} \) supported on \( E_{1,1}, E_{1,2} \) respectively such that \( \mu_{1,i}(B(x,r)) \leq r^\alpha \) for \( i = 1,2 \) \(^1\) and
\[
\mu_{1,2}(G^{k-1}_{E_{1,1}}(E_{1,2})) = \mu_{1,2}(\{x \in E_{1,2} : |T^{k-1}_x(E_{1,1})|_{k-1} > 0, \text{ for all } (k - 1) \text{-trees } T^{k-1}\}) > 0.
\]

Moreover, according to the assumption of the theorem, for the pair of sets \( G^{k-1}_{E_{1,1}}(E_{1,2}) \) and \( E_2 \), one must have
\[
\mu_2(G^{k-1}_{E_{1,1}}(E_{1,2})(E_2)) = \mu_2(\{x \in E_2 : |\Delta_x(G^{k-1}_{E_{1,1}}(E_{1,2}))| > 0\}) > 0.
\]

It is easy to see that (2.13) will be implied by \( G^{k-1}_{E_{1,1}}(E_{1,2})(E_2) \subset G^k_{E_1,T^k}(E_2) \). To see the inclusion, fix any \( x \in G^{k-1}_{E_{1,1}}(E_{1,2})(E_2) \). By definition, this means \( |\Delta_x(G^{k-1}_{E_{1,1}}(E_{1,2}))| > 0 \), and our goal is to prove \( \mu_2(G^{k}_{E_{1,1}}(E_{1,2})(E_2)) \). Since there is only one edge connecting to the vertex pinned at \( x \), it is straightforward to see that
\[
|T^k_x(E_1)|_k \geq \int_{t \in \Delta_x(G^{k-1}_{E_{1,1}}(E_{1,2}))} |T^{k-1}_{y_t}(y_t)(E_{1,1})|_{k-1} dt
\]
where \( y_t \) is any point in \( G^{k-1}_{E_{1,1}}(E_{1,2}) \) satisfying \( |x - y_t| = t \), and \( T^{k-1} \) denotes a \((k - 1)\)-tree of a particular shape (determined by the shape of \( T^k \)) with a particular

\(^1\)The existence of such sets is guaranteed for instance by Theorem 2.3 of [5].
vertex to be pinned. By definition of $G_{E_1,1}^{k-1}(E_{1,2})$, one has $|T_{x}^{k-1}(E_{1,1})|_{k-1} > 0$, \( \forall t \). Hence, according to the Fubini theorem and observing that \( k \)-trees produced in this way are all non-degenerate, one obtains $|T_{x}^{k}(E_{1})|_{k} > 0$.

Next, suppose that the fixed vertex is connected to at least two edges of the tree $T^{k}$; then one can decompose the tree $T^{k}$ into two sub-subtrees $T^{k_{1}}_{1}$, $T^{k_{2}}_{2}$, each containing $k_{i} \geq 1$ edges, $i = 1, 2$, so that $T^{k} = T^{k_{1}}_{1} \cup T^{k_{2}}_{2}$, and $T^{k_{1}}_{1}$, $T^{k_{2}}_{2}$ only share the vertex to be pinned. It is easy to see that $k_{1} + k_{2} = k$ and $k_{i} \leq k - 1$, $i = 1, 2$.

For sets $E_{1}, E_{2}$ satisfying the assumption, one further finds two disjoint subsets $E_{1,1}, E_{1,2} \subset E_{1}$ and probability measures $\mu_{1,1}, \mu_{1,2}$ as above. Apply the induction hypothesis to the pair $E_{1,1}, E_{2}$ first. Since $k_{1} \leq k - 1$, one has

$$\mu_{2}(G_{E_{1,1},T^{k_{1}}_{1}}^{k_{1}}(E_{2})) = \mu_{2}(|x \in E_{2} : |T_{x}^{k_{1}}(E_{1,1})|_{k_{1}} > 0|) > 0.$$  

One then applies the induction hypothesis again, this time to the pair of sets $E_{1,2}, G_{E_{1,1},T^{k_{1}}_{1}}^{k_{1}}(E_{2})$. Since $k_{2} \leq k - 1$, one obtains

$$\mu_{2}(G_{E_{1,2},T^{k_{2}}_{2}}^{k_{2}}(G_{E_{1,1},T^{k_{1}}_{1}}^{k_{1}}(E_{2}))) = \mu_{2}(|x \in G_{E_{1,1},T^{k_{1}}_{1}}^{k_{1}}(E_{2}) : |T_{x}^{k_{2}}(E_{1,2})|_{k_{2}} > 0|) > 0.$$  

(Strictly speaking, before applying the induction hypothesis in the second step above, one should have adjusted the measure $\mu_{2}$ to make it into a probability measure on the smaller set $G_{E_{1,1},T^{k_{1}}_{1}}^{k_{1}}(E_{2})$ by multiplying a constant. We omit the treatment of this issue.)

For any $x \in G_{E_{1,2},T^{k_{2}}_{2}}^{k_{2}}(G_{E_{1,1},T^{k_{1}}_{1}}^{k_{1}}(E_{2}))$, observe that

$$|T_{x}^{k}(E_{1})|_{k} \geq |T_{x}^{k_{1}}(E_{1,1})|_{k_{1}} \cdot |T_{x}^{k_{2}}(E_{1,2})|_{k_{2}} > 0.$$  

Hence,

$$\mu_{2}(G_{E_{1},T^{k}}^{k}(E_{2})) = \mu_{2}(|x \in E_{2} : |T_{x}^{k}(E_{1})|_{k} > 0|) 
\geq \mu_{2}(G_{E_{1,2},T^{k_{2}}_{2}}^{k_{2}}(G_{E_{1,1},T^{k_{1}}_{1}}^{k_{1}}(E_{2}))) > 0.$$  

The proof of Proposition 2.1 is complete. \( \square \)

Then, combined with the following Lemma 2.2, the $E_{1} = E_{2}$ case of Proposition 2.1 immediately implies Theorem 1.1. Lemma 2.2, although never stated explicitly, follows from the proof of Theorem 1.2 in [14].

**Lemma 2.2.** Let $E_{1}, E_{2} \subset \mathbb{R}^{2}$ be a pair of compact sets with positive $\alpha$-dimensional Hausdorff measure for some $\alpha > \frac{5}{4}$. Further, suppose that there exist Borel probability measures $\mu_{1}$ and $\mu_{2}$ on $E_{1}$ and $E_{2}$ respectively which satisfy $\mu_{i}(B(x, r)) \lesssim r^{\alpha}$ for $i = 1, 2$. Then

$$\mu_{2}(G_{E_{1}}^{k}(E_{2})) = \mu_{2}(|x \in E_{2} : |\Delta_{x}(E_{1})| > 0|) > 0.$$

In fact, even though the good pin point $x$ claimed in Proposition 2.1 and hence Theorem 1.1 seems to depend on $k$, one can easily find a good pin point $x$ that works well for all $k$, as claimed in Theorem 1.1. To see this, given a compact set $E$, let $E_{1}, E_{2} \subset E$ be as before, and let $G_{k} \subset E_{2}$ denote the set of “good pins” in $E_{2}$ such that $\mu_{2}(G_{k}) > 0$ and $|T_{x}^{k}(E_{1})|_{k} > 0$, \( \forall x \in G_{k}, \) for all $k$-trees. Without
loss of generality, one can assume that \( G_k \) is compact. Then, repeat the process for \( k + 1 \) with \( E_2 \) replaced by \( G_k \). One can obtain a compact good pin set \( G_{k+1} \subset G_k \). Iterate the process and let \( G = \bigcap_{k=1}^\infty G_k \subset E_2 \). By compactness, one has \( G \neq \emptyset \), and it is obvious that any point \( x \in G \) will guarantee that \( |T^k_x(E)|_k > 0 \) for all \( k \)-tree and all \( k \geq 1 \).

For the sake of completeness, we conclude this section by sketching below the proof of Lemma 2.2. We use the notation introduced in [14] below.

Proof of Lemma 2.2. The Lemma follows from the proof of the main result in [14]. We briefly outline how this works.

Let \( \alpha, E_1 \) and \( E_2 \) be as in the statement of the Lemma. Without loss of generality, assume \( E_1 \) and \( E_2 \) have distance \( \leq 1 \). By Frostman’s Lemma, each \( E_i \) supports a Borel probability measure \( \mu_i \) so that

\[
\mu_i(B(x,r)) \lesssim r^\alpha.
\]

Set \( d(x,y) = |x-y|, \) and, for \( x \) fixed and \( i \in \{1, 2\} \), denote the pushforward measure

\[
\int \psi(t)d_x^\mu_i(\mu_1) := \int E \psi(|x-y|)d\mu_i(y).
\]

Now \( d_x^\mu_i(\mu_1) \) is a probability measure on \( \Delta_x(E_i) \).

Let \( \mu_{1, \text{good}} \) be the complex measure (dependent on \( \mu_1 \)) described on page 7 of [14]. Proposition 2.1 in [14] (see page 8) implies that there exists a set \( E_2' \subset E_2 \) so that \( \mu_2(E_2') > 1 - \frac{1}{1000} \), and for each \( x \in E_2' \),

\[
\|d_x^\mu_1(\mu_1) - d_x^\mu_1(\mu_{1, \text{good}})\|_{L^1} < 1/1000
\]

and so

\[
\int |d_x^\mu_1(\mu_{1, \text{good}})| \geq 1 - \frac{1}{1000}.
\]

Proposition 2.2 in [14] implies that, for \( \mu_2 \)-almost every \( x \in E_2 \),

\[
\|d_x^\mu_1(\mu_{1, \text{good}})\|_{L^2} < \infty.
\]

Let \( E_2'' \) denote the subset of \( E_2 \) for which (2.16) holds, and set \( \tilde{E}_2 = E_2' \cap E_2'' \). Then, \( \mu_2(\tilde{E}_2) > 1 - \frac{1}{1000} \) and, following the logic on page 8 of [14], for each \( x_2 \in \tilde{E}_2 \),

\[
|\Delta_{x_2}(E_1)| > 0.
\]

\[\square\]

Remark 2.3. Just as in the case of distances, it follows by the classical lattice example that \( \frac{d}{2} \) is indeed the lowest possible threshold to ensure that \( |S^k(E)|_k > 0 \), \( \forall k \geq 2 \) where \( E \subset \mathbb{R}^d \). Consider the lattice example given in Example 1.7 which is a set of dimension \( \alpha \). It is easy to examine that \( |S^k(E)|_k = 0 \), \( \forall k \geq 1 \), whenever \( \alpha < \frac{d}{2} \). This, in particular, suggests that if the Falconer distance conjecture, which says \( |\Delta(E)| > 0 \) whenever \( \dim_H(E) > \frac{d}{2} \), is confirmed, then our method would be able to extend it to fully resolve the analogous question for chains.
3. Dimension of pinned chain/tree sets: Proof of Theorems 1.2, 1.3

3.1. Proof of Theorem 1.2. We first prove Theorem 1.2. The key observation here is that there holds a partial version of the Fubini theorem that can be used to estimate the Hausdorff dimension of a set based on the dimensions of its slices. The exact statement of the theorem is presented in Appendix A.

We begin with the following lemma, which is rephrased from [23, Theorem 1.1]. We refer the reader to [23] for its proof. In particular, the proof depends on the core idea of good and bad measures that is recalled earlier in the sketch of the proof of Lemma 2.2.

Lemma 3.1. Let \( E \subset \mathbb{R}^2 \) be a compact set with \( \dim_H(E) > 1 \) and \( \tau \in (0,1) \). Then,

\[
\dim_H \{ x \in \mathbb{R}^2 : \dim_H(\Delta_x(E)) < \tau \} \leq \max(2 + 3\tau - 3\dim_H(E), 2 - \dim_H(E)).
\]

In particular, let \( E_1 \) and \( E_2 \) be subsets in \( \mathbb{R}^2 \) with \( \dim_H(E_1) = \dim_H(E_2) > 1 \), then for all \( \epsilon > 0 \), there exists \( E'_2 \subset E_2 \) with \( \dim_H(E_2 \setminus E'_2) < \dim_H(E_2) \) so that

\[
\dim_H(\Delta_x(E_1)) \geq \min \left( \frac{4}{3} \dim_H(E_1) - \frac{2}{3} - \epsilon, 1 \right), \quad \forall x \in E'_2.
\]

Similarly as in the previous section, we have the following structural theorem that allows one to extend the dimension estimates of pinned distance sets to pinned tree sets. This theorem does not assume anything on the value of \( \alpha \).

Proposition 3.2. Let \( d \geq 2 \). Suppose for all compact sets \( E_1, E_2 \subset \mathbb{R}^d \) with \( \dim_H(E_1) = \dim_H(E_2) > \alpha_0 > 0 \), there exists \( E'_2 \subset E_2 \) with \( \dim_H(E_2 \setminus E'_2) < \dim_H(E_2) \) so that

\[
\dim_H(\Delta_x(E_1)) \geq \gamma = \gamma(\dim_H(E_1), d), \quad \forall x \in E'_2.
\]

Then, for all integers \( k \geq 1 \), and all compact sets \( E_1, E_2 \subset \mathbb{R}^d \) with \( \dim_H(E_1) = \dim_H(E_2) > \alpha_0 \), there exists \( E'_2 \subset E_2 \) with \( \dim_H(E_2 \setminus E'_2) < \dim_H(E_2) \) so that

\[
\dim_H(T^k_x(E_1)) \geq k \gamma, \quad \forall x \in E'_2,
\]

for all \( k \)-trees \( T^k_x \) of a particular shape pinned at a particular vertex. In particular, \( \dim_H(S^k_x(E_1)) \geq k \gamma, \quad \forall x \in E'_2 \).

It is easy to see that the second statement of Theorem 1.2 follows immediately from (3.18) and Proposition 3.2 with \( E_1 = E_2 \) and \( \gamma = \min \left( \frac{4}{3} \dim_H(E_1) - \frac{2}{3} - \epsilon, 1 \right) \).

In order to prove the first statement of Theorem 1.2 one simply takes a sequence \( \{ \epsilon_n \} \) that converges to 0.

Before moving on to the proof of Proposition 3.2, we momentarily take this result for granted and demonstrate the third assertion of Theorem 1.2, equation (1.3).

Given a compact set \( E \), suppose \( 1 < \dim_H(E) \leq \frac{5}{3} \). Let \( E_1, E_2 \) be two disjoint subsets of \( E \), both with the same Hausdorff dimension as \( E \). Let \( \epsilon_0 > 0 \) be a fixed small parameter such that \( \dim_H(E) - \epsilon_0 > 1 \). Then, for all \( \epsilon < \epsilon_0 \), set

\[
G_1 = \left\{ x \in E_2 : \dim_H(T^1_x(E_1)) \geq \frac{4}{3} \dim_H(E) - \frac{2}{3} - \epsilon \right\}.
\]

By Lemma 3.1 and Proposition 3.2 applied to \( E_1, E_2 \) with \( k = 1 \), \( \dim_H(E_2 \setminus G_1) < \dim_H(E) \), which in particular implies \( \dim_H(G_1) = \dim_H(E) \).
Let \( \dim_H(E) > \alpha_1 > \dim_H(E) - \frac{\epsilon}{3} \) and choose \( \widetilde{G}_1 \subset G_1 \) so that \( \widetilde{G}_1 \) is compact and \( H^{\alpha_1}(\widetilde{G}_1) > 0 \) (such a choice is possible, for instance, by Corollary 4.12 in [9]). Then in particular, \( \dim_H(\widetilde{G}_1) \geq \alpha_1 \), and for all \( x \in \widetilde{G}_1 \),

\[
\dim_H(T_x^k(E_1)) \geq \frac{4}{3} \dim_H(E) - \frac{2}{3} - \epsilon > \frac{4}{3} (\dim_H(E) - \epsilon_0) - \frac{2}{3}.
\]

Replacing \((E_1, E_2)\) with \((E_1^{(2)}, \widetilde{G}_1)\) where \( E_1^{(2)} \subset E_1 \) satisfies \( \dim_H(E_1^{(2)}) = \dim_H(\widetilde{G}_1) \) and repeating this process, one finds \( \alpha_2 \) so that \( \dim_H(\widetilde{G}_1) - \frac{\epsilon}{20} < \alpha_2 < \dim_H(\widetilde{G}_1) \) and a compact set \( \widetilde{G}_2 \subset \widetilde{G}_1 \) satisfying

1. \( \mathcal{H}^{\alpha_2}(\widetilde{G}_2) > 0 \);
2. \( \dim_H(T_2^k(E_1)) \geq \dim_H(T_2^k(E_1^{(2)})) \geq \frac{4}{3} \dim_H(E) - \epsilon_0 - \frac{\epsilon}{20}, \forall x \in \widetilde{G}_2 \).

Continuing the process, there exists a sequence \( \{\alpha_i\} \) and a sequence of nested compact sets \( \{\widetilde{G}_i\} \) satisfying

1. \( \dim_H(\widetilde{G}_{i-1}) - \frac{\epsilon}{3} \cdot 10^{-i+1} < \dim_H(\widetilde{G}_{i-1}) \);
2. \( \widetilde{G}_i \subset \widetilde{G}_{i-1}, \mathcal{H}^{\alpha_i}(\widetilde{G}_i) > 0 \);
3. \( \dim_H(T_1^k(E_1)) \geq \dim_H(T_1^k(E_1^{(i)})) \geq \frac{4}{3} \dim_H(E) - \epsilon_0 - \frac{\epsilon}{20}, \forall x \in \widetilde{G}_i \).

This implies, in particular, that there exists a point \( x \in E \) so that \( \dim_H(T_x^k(E)) \geq \frac{4k}{3} (\dim_H(E) - \epsilon_0) - \frac{2k}{3}, \) for each integer \( k \geq 1 \).

3.1.1. Proof of Proposition 3.2 For the sake of simplicity, we only prove the chain case, as the general tree case can be treated in almost the same way with a slight modification similarly to the proof of Proposition 2.1 in Section 2, which is left to the interested reader.

We prove by induction. According to the assumption, the base case \( k = 1 \) is automatically true. Now, assume that the desired result holds for \( k - 1 \). Let \( E_1, E_2 \) be the given sets in \( \mathbb{R}^d \) with Hausdorff dimension \( m \). Let \( E_{1,1}, E_{1,2} \subset E_1 \) be two subsets of \( E_1 \), so that \( \dim_H(E_{1,i}) = m, i = 1, 2 \), and the distance between them is positive. By properly shrinking \( E_1, E_2 \) without altering their dimension, one can assume without loss of generality that the distance from \( E_{1,i} \) to \( E_2 \) is also positive, \( i = 1, 2 \).

By the induction hypothesis, there exists \( E_{1,2}' \subset E_{1,2} \) so that \( \dim_H(E_{1,2}\setminus E_{1,2}') < m \) (in particular, \( \dim_H(E_{1,2}') = m \)) and

\[
\dim_H(S_{E_{1,1}}^{k-1}(E_{1,1})) \geq (k - 1)\gamma, \quad \forall y \in E_{1,2}'.
\]

Now, applying the assumption (i.e. the base case \( k = 1 \)) to the sets \( E_{1,2}' \) and \( E_2 \), one can find \( E_2' \subset E_2 \) satisfying \( \dim_H(E_2\setminus E_2') < m \), and such that

\[
\dim_H(\Delta_x(E_{1,2}')) \geq \gamma, \quad \forall x \in E_2'.
\]

Fix \( x \in E_2' \), and let \( B = \Delta_x^k(E_1) \) and \( A = \Delta_x(E_{1,2}') \). For all \( t_0 \in \Delta_x(E_{1,2}') \), let \( B_{t_0} \) denote the slice of \( B \) at \( t_0 \) in the first variable. Observe that

\[
S_{x}(E_1) = \{(x - x_1, \ldots, x_k - x_{k+1}) : x_1 \in E_{1,2}', x_2, \ldots, x_{k+1} \in E_{1,1} \text{ distinct}\}.
\]

Hence, one has for all \( t_0 \in A \) that

\[
\dim_H(B_{t_0}) \geq \dim_H(S_{E_{1,1}}^{k-1}(E_{1,1})) \geq (k - 1)\gamma,
\]
for some \( y_0 \in E' \) satisfying \(|y_0 - x| = t_0\). Then, according to the Fubini-like theorem Corollary A.3, this implies
\[
\dim_H(B) \geq (k-1)\gamma + \dim_H(A) \geq (k-1)\gamma + \gamma = k\gamma.
\]
The proof is thus complete.

**Remark 3.3.** It is easy to see that, by following the same strategy as above, one can prove other versions of the structural theorem concerning more general point configurations, such as the kite in Proposition 1.3. Indeed, for a fixed \( x_0 \in E_1 \), set \( A := A_{x_0} = \{(x_0 - y_1, |y_1 - y_2|, |y_2 - x_0|) : y_1, y_2 \in E_1\} \subset \mathbb{R}^3 \) and observe that, for each \((t_1, t_2, t_3) \in A, B_{(t_1, t_2, t_3)} \supseteq S_{y_1}(E_2)\), for some \( y_1, y_2 \in E_1 \) such that \(|x_0 - y_1, |y_1 - y_2|, |y_2 - x_0|\) = \((t_1, t_2, t_3)\). Hence, one can apply Corollary A.3 similarly as above to prove Proposition 1.3. We omit the details.

### 3.2. Proof of Theorem 1.3

We now turn to the proof of Theorem 1.3, which also relies on the Fubini-like theorem in Appendix A.

**Proof of Theorem 1.3.** This proof has a similar flavor as the above, but will make use of (3.17). It is direct to see that when \( k = 1 \), the two bounds coincide with (3.17).

Assume that the desired result holds for \( k - 1 \). More precisely, assume that for any set \( E \) with \( \dim_H(E) > 1 \), and \( \tau \in (0, \tau_k^k(\dim_H(E))) \) with
\[
\tau_k^k(\dim_H(E)) = \begin{cases} \frac{4(k-1)}{3} \dim_H(E) + \frac{5-2k}{3}, & \text{if } 1 < \dim_H(E) \leq \frac{5}{4}, \\ \frac{2}{3}, & \text{if } \frac{5}{4} < \dim_H(E) \leq 2, \\ \end{cases}
\]
there holds
\[
\dim_H(\{x \in \mathbb{R}^2 : \dim_H(S_{x}^k(E)) < \tau\}) \leq \max(\Gamma_k^k(\tau, \dim_H(E)), 2 - \dim_H(E)),
\]
where
\[
\Gamma_k^k(\tau, \dim_H(E)) := \begin{cases} 2k + 3\tau + (1 - 4k)\dim_H(E), & \text{if } 1 < \dim_H(E) \leq \frac{5}{4}, \\ 5 - 3k + 3\tau - 3\dim_H(E), & \text{if } \frac{5}{4} < \dim_H(E) \leq 2. \\ \end{cases}
\]
Our goal is to show that for all \( \tau \in (0, \tau_k^k) \)
\[
\dim_H(\{x \in \mathbb{R}^2 : \dim_H(S_{x}^k(E)) < \tau\}) \leq \max(\Gamma_k^k(\tau, \dim_H(E)), 2 - \dim_H(E)).
\]
In particular, it suffices to show that for all \( \epsilon > 0 \),
\[
\dim_H(\{x \in \mathbb{R}^2 : \dim_H(S_{x}^k(E)) < \tau\}) \leq \max(\Gamma_k^k(\tau, \dim_H(E)), 2 - \dim_H(E)) + \epsilon.
\]
First, consider the range \( \dim_H(E) \leq \frac{5}{4} \) and fix \( \tau \in (0, \frac{4(k-1)}{3} \dim_H(E) + \frac{5-2k}{3}) \).

Fix \( \epsilon > 0 \), it suffices to show that given any set \( F \subset \mathbb{R}^2 \) satisfying
\[
\dim_H(F) > \max(\Gamma_k^k(\tau, \dim_H(E)), 2 - \dim_H(E)) + \epsilon,
\]
there exists a point \( x \in F \) such that \( \dim_H(S_{x}^k(E)) \geq \tau \).

Without loss of generality, assume the distance between \( E, F \) is positive. By setting \( \max(\Gamma_k^k(\tau, \dim_H(E)), 2 - \dim_H(E)) + \epsilon = \dim_H(E) \), the induction hypothesis implies that there exists a subset \( E' \subset E \) with the same dimension as \( E \) so that
\[
\dim_H(S_{y}^k(E)) \geq \min \left( \frac{4(k-1)}{3} \dim_H(E) - \frac{2(k-1)}{3} - \frac{\epsilon}{3}, k-1 \right), \quad \forall y \in E'.
\]
Define
\[ \tau_1 := \begin{cases} \frac{2}{3} \dim_H(E) + \frac{2}{3}, & \text{if } \tau \leq \frac{4k-4}{3} \dim_H(E) - \frac{2k-2}{3}, \\ \frac{2k-2}{3} + \tau - \frac{4k-4}{3} \dim_H(E) + \frac{2}{3}, & \text{if } \tau > \frac{4k-4}{3} \dim_H(E) - \frac{2k-2}{3}. \end{cases} \]

Note that the point here is to make sure that
\[ \max(\Gamma^k(\tau, \dim_H(E)), 2 - \dim_H(E)) + \epsilon = \max(\Gamma^1(\tau_1, \dim_H(E)), 2 - \dim_H(E)). \]

Because of the bound of \( \tau \), one also has from the definition that \( \tau_1 \in (0, 1) \) (by letting \( \epsilon \) sufficiently small depending on \( \tau \)). Note that this is the only place in the proof where the bound of \( \tau \) comes into play.

Recalling the definition of \( F \), we have
\[ \dim_H(F) > \max(\Gamma^1(\tau_1, \dim_H(E')), 2 - \dim_H(E')). \]

By assumption of the theorem, there thus exists a point \( x \in F \) such that
\[ \dim_H(\Delta_x(E')) \geq \tau_1. \]

From the construction of \( E' \), for all \( y \in E' \), (3.23) holds true. Letting \( B = S^k_y(E) \) and \( A = \Delta_x(E') \), one sees that for all \( t_0 \in A \), the slice of the set \( B \) at \( t_0 \) in the first variable satisfies
\[ \dim_H(B_{t_0}) \geq \dim_H(S^k_{y_0}((E))) \]
\[ \geq \min \left( \frac{4(k-1)}{3} \dim_H(E) - \frac{2(k-1)}{3} - \epsilon, k-1 \right) \]
\[ = \frac{4(k-1)}{3} \dim_H(E) - \frac{2(k-1)}{3} - \epsilon, \]
for some \( y_0 \in E' \) satisfying \( |x - y_0| = t_0 \). Therefore, according to Corollary A.3 one obtains
\[ \dim_H(B) \geq \frac{4(k-1)}{3} \dim_H(E) - \frac{2(k-1)}{3} - \epsilon + \tau_1 \geq \tau. \]

The proof of the first case is complete.

Second, assume \( \dim_H(E) > \frac{5}{4} \) and \( \tau \in (0, k) \). Again, the goal is to show that there exists \( x \in F \) so that \( \dim_H(\Delta^k_x(E)) \geq \tau \). Same as before, the induction hypothesis implies for each \( \epsilon > 0 \) the existence of a set \( E' \subset E \) which satisfies (3.23). Since \( \dim_H(E) > \frac{5}{4} \), by taking \( \epsilon \) sufficiently small, one has in fact \( \dim_H(\Delta^k_y(E')) \geq k-1, \forall y \in E' \). Now, let
\[ \tau_1 := \begin{cases} \min\left( \frac{2}{3} \dim_H(E), 1 \right), & \text{if } \tau \leq k-1, \\ 1 - k + \tau, & \text{if } \tau > k-1. \end{cases} \]

It is straightforward to check that \( \tau_1 \in (0, 1) \) and
\[ \max(\Gamma^k(\tau, \dim_H(E)), 2 - \dim_H(E)) = \max(\Gamma^1(\tau_1, \dim_H(E)), 2 - \dim_H(E)). \]

Therefore, there is \( x \in F \) so that \( \dim_H(\Delta_x(E')) \geq \tau_1 \). Applying Corollary A.3 and arguing as above, one has
\[ \dim_H(S^k_x(E)) \geq k - 1 + \tau_1 \geq \tau, \]
which completes the proof. \( \square \)
4. Chains/trees with prescribed gaps: Proof of Theorems 1.9, 1.11

In this section we study the dimension of set of chains/trees that have prescribed gaps. Many of the upper bound estimates below extend the work of Eswarathasan–Iosevich–Taylor [4] and Oberlin–Oberlin [24], where the case \( k = 1 \), i.e. the unit distance set, was considered (for slightly more restrictive classes of sets \( E \)). On the other hand, we will also see below that sometimes the chain/tree cases display very different properties compared to the distance case. For instance, in \( \mathbb{R}^2 \), when \( 0 < \alpha \leq 1 \), the best known estimate for the unit distance set is

\[
\frac{3\alpha}{2} \leq g_2(VS_1^1, \alpha) \leq \min \left( \frac{5\alpha}{3}, \frac{\alpha(2 + \alpha)}{1 + \alpha} \right),
\]

according to [24]. However, the 2-chain set displays distinct features and we can completely determine the value of \( g_2(VS_2^2, \alpha) \) without first estimating the unit distance set.

4.1. Proof of Theorem 1.9

4.1.1. Upper bound. Let \( k \geq 2 \) and \( E \subset \mathbb{R}^d \) be a compact \( \{\delta_i\} \)-discrete \( \alpha \)-regular set that is contained in the unit ball and has Hausdorff dimension \( \alpha \). Given any \( \delta_i \), we will show that \( \forall \epsilon > 0 \),

\[
|D_k^{\delta_i}| = \left| \{(x_1, \ldots, x_{k+1}) \in E_{\delta_i}^{k+1} : t_j - 2\delta_i \leq |x_j - x_{j+1}| \leq t_j + 2\delta_i, j = 1, \ldots, k \} \right| \leq \delta_i^{(k+1)d - u(k, d, \alpha) - \epsilon},
\]

where

\[
u(k, d, \alpha) := \begin{cases} 
(k + 1)\alpha - k, & \frac{d+1}{2} \leq \alpha \leq d, \\
kd + \alpha - \frac{k}{2}, & \alpha \leq \frac{d+1}{2}.
\end{cases}
\]

Since \( D_k^{\delta_i} \) contains the \( \delta_i \)-neighborhood of \( VS_1^k(E) \), estimate (4.24) would imply that \( \dim_H(VS_1^k(E)) \leq \dim_M(VS_1^k(E)) \leq \nu(k, d, \alpha) \), hence the desired upper bound in Theorem 1.9 follows. Indeed, letting \( \gamma \) denote the lower Minkowski dimension of \( VS_1^k(E) \) and taking \( \delta_i \) sufficiently small, it follows that \( |D_k^{\delta_i}| \sim \delta_i^{(k+1)d - \gamma} \), where \( |\cdot| \) denotes the Lebesgue measure. Thus, our goal is reduced to attaining an upper bound, in terms of a power of \( \delta_i \), on \( |D_k^{\delta_i}| \). To simplify the notation, we will write \( \delta = \delta_i \) in the following.

Without loss of generality, assume that \( E = -E \). Write \( E_\delta = E + B(0, \delta) \) and define

\[
A_{\epsilon, \delta} := \{ x \in \mathbb{R}^d : t - 2\delta \leq |x| \leq t + 2\delta \}.
\]
We re-write the set $D_k^\delta$ as follows:

$$|D_k^\delta|$$

$$= \int_{E_\delta} \cdots \int_{E_\delta} \prod_{i=1}^{k} \chi_{A_{t_i, \delta}}(x_i - x_{i+1}) \, dx_1 \cdots dx_{k+1}$$

$$= \int_{E_\delta} \cdots \int_{E_\delta} \left( \prod_{i=1}^{k-1} \chi_{A_{t_i, \delta}}(x_i - x_{i+1}) \right) \chi_{E_k} \ast \chi_{A_{t_k, \delta}}(x_k) \, dx_1 \cdots dx_k$$

$$= \int_{E_\delta} \cdots \int_{E_\delta} \left( \prod_{i=1}^{k-2} \chi_{A_{t_i, \delta}}(x_i - x_{i+1}) \right) \left( f_1 \chi_{E_k} \ast \chi_{A_{t_{k-1}, \delta}}(x_{k-1}) \right) \, dx_1 \cdots dx_{k-1}$$

$$= \cdots = \langle f_k, \chi_{E_k} \rangle,$$

where we have defined $f_1 = \chi_{E_k} \ast \chi_{A_{t_k, \delta}}$, and $f_{n+1} = (f_n \chi_{E_k}) \ast \chi_{A_{t_{n+1}, \delta}}$, $\forall 2 \leq n \leq k - 1$.

The main estimate we will prove is the following $L^2$ bound:

**Lemma 4.1.** Let $f \in L^2(E_\delta)$, and $t \sim 1$, $\delta > 0$ as before. Then for all $\epsilon > 0$,

$$\int_{E_\delta} |(f \chi_{E_k}) \ast \chi_{A_{t, \delta}}(x)|^2 \, dx \leq C_\epsilon \delta^{\beta(d, \alpha) - \epsilon} \left( \int_{E_\delta} |f(x)|^2 \, dx \right)^{1/2},$$

where

$$\beta(d, \alpha) = \begin{cases} d - \alpha + 1, & \frac{d}{2} + 1 \leq \alpha \leq d, \\ \frac{d}{2}, & \frac{d}{2} \leq \alpha \leq \frac{d}{2} + 1. \end{cases}$$

Note that a special case of this estimate when $f = 1$ was obtained in [24]. We first show how Lemma 4.1 implies the desired (4.24). Applying the Cauchy-Schwarz inequality, one has

$$\langle f_k, \chi_{E_k} \rangle = \langle (f_{k-1} \chi_{E_k}) \ast \chi_{A_{t_k, \delta}} \ast \chi_{E_k} \rangle$$

$$\leq \left( \int_{R^d} |(f_{k-1} \chi_{E_k}) \ast \chi_{A_{t_k, \delta}}(x)|^2 \chi_{E_k}(x) \, dx \right)^{1/2} |E_\delta|^{1/2}$$

$$\leq C_\epsilon \delta^{\beta(d, \alpha) - \epsilon} \left( \int_{R^d} |f_{k-1}(x)|^2 \chi_{E_k}(x) \, dx \right)^{1/2} |E_\delta|^{1/2},$$

where we have applied (4.25) in the last step.

By applying (4.25) iteratively, one ultimately obtains

$$\langle f_k, \chi_{E_k} \rangle \leq \delta^{k(\beta(d, \alpha) - \epsilon)} |E_\delta| \leq \delta^{k(\beta(d, \alpha) - \epsilon) + d - \alpha},$$

where we have recalled the definition of the $\{\delta_i\}$-discrete $\alpha$-set. Estimate (4.24) thus follows immediately.

**4.1.2. Proof of Lemma 4.1** We now turn to proving Lemma 4.1.

Let $g$ be a testing function satisfying $\int |g|^2 \chi_{E_k} = 1$, then it suffices to show that

$$\int_{E_\delta} \left( |(f \chi_{E_k}) \ast \chi_{A_{t, \delta}}(x)| g(x) \right) \, dx \leq \delta^{\beta(d, \alpha) - \epsilon} \left( \int_{E_\delta} |f|^2 \right)^{1/2}.$$
Without loss of generality, assume both $f$ and $g$ are nonnegative. Let $\rho$ be a symmetric Schwartz function satisfying

$$
\chi_{B(0, C)} \leq \rho(x) \leq \sum_{j=1}^{\infty} 2^{-jd} \chi_{B(0, 2^j)}, \quad \chi_{B(0, C')} \leq |\hat{\rho}(\xi)| \leq \chi_{B(0, 2C')},
$$

and denote $\rho_\varepsilon(x) = r^{-d} \rho(\frac{x}{r})$.

Then,

$$
\left| \int [(f\chi_{E_\delta} \ast \chi_{A_\varepsilon}) (x) g(x) \chi_{E_\delta}(x) \, dx \right|
\leq \left| \langle \psi \chi_{E_\delta} \ast \chi_{A_\varepsilon} \rangle \right| 
\leq \delta \left| \langle \psi \chi_{E_\delta}, \rho_\delta \ast \rho_\delta \ast \sigma_\varepsilon \rangle \right|,
$$

where $\sigma_\varepsilon$ denotes the surface measure on the sphere in $\mathbb{R}^d$ of radius $t$ (not normalized). Recalling that $t \sim 1$ and applying Plancherel, one has that the above is bounded by

$$
\lesssim \delta \left( \int_{B(0, \frac{2C'}{r})} \left| \langle \psi \chi_{E_\delta} \ast \rho_\delta \rangle \right|^2 \frac{d\xi}{(1 + |\xi|)^{\frac{d+\alpha}{2}}} \right)^{1/2}.
$$

Hence, estimate \eqref{4.26} results from the following estimate.

**Lemma 4.2.** For $f \in L^2(E_\delta)$,

$$
\left( \int_{B(0, \frac{2C'}{r})} \left| \langle \psi \chi_{E_\delta} \ast \rho_\delta \rangle \right|^2 \frac{d\xi}{(1 + |\xi|)^{\frac{d+\alpha}{2}}} \right)^{1/2} \lesssim \delta^{\beta(d, \alpha)-\frac{1}{2}} \varepsilon \left( \int_{E_\delta} |f|^2 \right)^{1/2},
$$

where

$$
\beta(d, \alpha) := \begin{cases} 
2d - \alpha + 1, & \text{if } \frac{d+1}{2} \leq \alpha \leq d, \\
\frac{d+1}{2}, & \text{if } \alpha \leq \frac{d+1}{2}.
\end{cases}
$$

This lemma follows from a slightly simpler statement:

**Lemma 4.3.** For $f \in L^2(E_\delta)$,

$$
\left( \int_{B(0, \frac{2C'}{r})} \left| \langle \psi \chi_{E_\delta} \ast \rho_\delta \rangle \right|^2 \frac{d\xi}{(1 + |\xi|)^{d-\alpha}} \right)^{1/2} \lesssim \delta^{\frac{d-\alpha}{2}} \varepsilon \left( \int_{E_\delta} |f|^2 \right)^{1/2}.
$$

To see that Lemma 4.2 follows from Lemma 4.3, we notice that, when $\alpha \geq \frac{d+1}{2}$, $d - \alpha \leq \frac{d+1}{2}$, and the left-hand side of \eqref{4.28} is dominated (up to a constant) by the left-hand side of \eqref{4.29}. In the case that $\alpha < \frac{d+1}{2}$, for $|\xi| \leq \frac{2C'}{r}$, we observe that

$$
\frac{1}{|\xi|^{(d-1)/2}} \lesssim \delta^{\frac{d+1}{2}} |\xi|^{d-\alpha}.
$$
Now, the left-hand-side of (4.28) is bounded above by \( \delta^{\frac{1}{2}(\alpha - \frac{d}{2})} \) times the expression on the left-hand-side of (4.29), and the Lemma follows.

4.1.3. Proof of Lemma 4.3. We will focus on the demonstration of Lemma 4.3 in the rest of the subsection. For \( r \geq \delta \), we first use interpolation to show that

\[
\|f \chi_{E_{\delta}} \|_{L^2(E_{\delta})} \lesssim \frac{\delta^{\frac{1}{2}(\alpha - \frac{d}{2})}}{\delta^{\frac{1}{2} \alpha}} \|f\|_{L^2(E_{\delta})}.
\]

Observe that

\[
\|f \chi_{E_{\delta}} \|_{L^2(E_{\delta})} = \sup_{x \in E_{\delta}} \int_{E_{\delta} \cap B(x, r)} f(y) dy \lesssim \|f\|_{L^2(E_{\delta})} \|E_{\delta} \cap B(x, r)\| \lesssim \|f\|_{L^2(E_{\delta})} r^{\alpha \delta^{-\alpha}},
\]

where we have used the assumption that \( E \) is \( \{\delta\}\)-discrete \( \alpha \)-regular in the last line.

Also, observe that

\[
\|f \chi_{E_{\delta}} \|_{L^1(E_{\delta})} = \int \int f(y) \chi_{E_{\delta}}(y) \chi_{B(x, r)}(x - y) dy dx \lesssim \|f\|_{L^1(E_{\delta})} \|B(0, r)\| \lesssim \|f\|_{L^1(E_{\delta})} r^d.
\]

Interpolating these two estimates yields (4.30) (see, Riesz-Thorin interpolation theorem in [31]).

Next, by the definition of \( \rho \), for \( r \geq \delta \), we see that whenever \( \alpha < d \),

\[
\|f \chi_{E_{\delta}} \rho_r \|_{L^2} \lesssim \sum_{j=1}^{\infty} 2^{-j\frac{(d-\alpha)}{2}} \|f\|_{L^2(E_{\delta})} \|B(0, r)\|^{\alpha \delta^-\alpha} \lesssim \|f\|_{L^2(E_{\delta})} \left( \frac{\delta}{r} \right)^{\frac{d}{2} \alpha}.
\]

(The case \( \alpha = d \) is implied by the second trivial upper bound given in Remark 1.10.)

It is a direct consequence of (4.27) that, if \( |\xi| \leq \frac{\epsilon'}{r} \), then \( |\hat{\rho}_r(\xi)| \leq |\hat{\rho}(\xi)| \) (indeed, \( |\hat{\rho}_r(\xi)| = |\hat{\rho}(r \xi)| \geq \chi_{B(0, C^*)}(r \xi) = 1 \), and when \( |\xi| \leq \frac{2\epsilon'}{r} \), \( |\hat{\rho}_r(\xi)| = |\hat{\rho}(\delta \xi)| \leq \chi_{B(0, 2C^*)}(\delta \xi) \)).

It follows that

\[
\int_{\frac{\epsilon'}{r}}^{\epsilon'} \int_{\frac{\epsilon'}{r}}^{\epsilon'} |(f \chi_{E_{\delta}} \rho_r)^{\wedge}(\xi)|^2 d\xi \lesssim \int_{\frac{\epsilon'}{r}}^{\epsilon'} \int_{\frac{\epsilon'}{r}}^{\epsilon'} |(f \chi_{E_{\delta}} \rho_r)^{\wedge}(\xi)|^2 d\xi \lesssim \int_{\frac{\epsilon'}{r}}^{\epsilon'} \int_{\frac{\epsilon'}{r}}^{\epsilon'} |(f \chi_{E_{\delta}} \rho_r)^{\wedge}(\xi)|^2 d\xi \lesssim \int_{\frac{\epsilon'}{r}}^{\epsilon'} \int_{\frac{\epsilon'}{r}}^{\epsilon'} |(f \chi_{E_{\delta}} \rho_r)^{\wedge}(\xi)|^2 d\xi \lesssim \int_{\frac{\epsilon'}{r}}^{\epsilon'} \int_{\frac{\epsilon'}{r}}^{\epsilon'} |(f \chi_{E_{\delta}} \rho_r)^{\wedge}(\xi)|^2 d\xi \lesssim \int_{\frac{\epsilon'}{r}}^{\epsilon'} \int_{\frac{\epsilon'}{r}}^{\epsilon'} |(f \chi_{E_{\delta}} \rho_r)^{\wedge}(\xi)|^2 d\xi
\]

With the estimates above in tow, we turn to estimating the left-hand-side of (4.29):

\[
\left( \int_{B(0, \frac{2\epsilon'}{r})} |(f \chi_{E_{\delta}} \rho_r)^{\wedge}(\xi)|^2 \frac{d\xi}{(1 + |\xi|)^{\alpha - \alpha}} \right)^{1/2}.
\]
where the integration domain comes from the assumption that $|\hat{\rho}(\delta \xi)| \leq \chi_{B(0,2C')}(\delta \xi)$. We consider the integral over the set $\{|\xi| > C'\}$ and $\{|\xi| \leq C'\}$ separately. We have

$$
\sum_{\{1 \leq 2^j \leq \delta \}} \int_{\{2^j < |\xi| \leq 2^{j+1}\}} \left| \left( f \chi_{E_j} \ast \rho_3 \right)^\wedge (\xi) \right|^2 \frac{d\xi}{(1 + |\xi|)^{d-\alpha}}
$$

$$
\leq \sum_{\{1 \leq 2^j \leq \delta \}} \| f \|_{L^2(E_j)}^2 (\delta 2^j)^{(d-\alpha)} 2^{-j(d-\alpha)}
$$

$$
\leq \| f \|_{L^2(E_1)}^2 \delta^{d-\alpha} \log(1/\delta),
$$

where we used (4.32) with $r = 2^{-(j+1)}$.

While,

$$
\int_{\{|\xi| \leq C'\}} \left| \left( f \chi_{E_j} \ast \rho_3 \right)^\wedge (\xi) \right|^2 d\xi \leq \| (f \chi_{E_j} \ast \rho_3) \|_{L^2(E_j)}^2 \leq \| f \|_{L^2(E_1)}^2 \delta^{d-\alpha},
$$

where we used Cauchy-Schwarz in the second to last step and the definition of $\{\delta_i\}$-discrete $\alpha$-regular in the last. This concludes the proof of Lemma 4.3.

4.1.4. Lower bound. Let $k \geq 2$ and $d \geq 2$. Let $\vec{t} = (t_1, \ldots, t_k) \in \mathbb{R}_k^d$ denote arbitrary prescribed gaps. We demonstrate the existence of a $\{\delta_i\}$-discrete $\alpha$-regular set $E \subset \mathbb{R}^d$ of Hausdorff dimension $\alpha$, so that the Hausdorff dimension of $VS^k_{\vec{t}}(E)$ is at least $(k + 1)\alpha - k$. Note that, when $\alpha \geq d+1$, the upper bound and lower bound for $g_\alpha(VS^k_{\vec{t}}, \alpha)$ match. For this reason, we only state the result for this range, although the following example works for any value $\alpha \in (1, d)$.

For the sake of simplicity, we only consider the case $k = 2$ and assume $t_1 \leq t_2$. The following example is adapted from [24], where the $k = 1$ case was studied. Let $C \subset B(0, \frac{1}{4}) \subset \mathbb{R}^{d-1}$ be an AD regular set with Hausdorff dimension $\gamma \geq 0$. Define $E = C \times \{0, 4t_2\} \subset \mathbb{R}^d$ and $\alpha := \gamma + 1$. Then it is easy to see that $\dim_H(E) = \alpha$ and $E$ is $\{\delta_i\}$-discrete $\alpha$-regular. One also has

$$
VS^2_{(t_1,t_2)}(E) = \left\{ (c_1, s_1; c_2, s_2; c_3, s_3) : c_1 \in C, s_i \in [0, 4t_2] \right\},
$$

$|s_1 - s_2| = \sqrt{t_1^2 - |c_2 - c_1|^2}, |s_2 - s_3| = \sqrt{t_2^2 - |c_2 - c_3|^2}$.

With any $c_1, c_2, c_3, s_1$ fixed such that $c_1, c_2, c_3$ are distinct, at least a $s_2 \in [0, 4t_2]$ will be determined, which will in turn determine at least a $s_3 \in [0, 4t_2]$. Therefore,

$$
\dim_H(VS^2_{(t_1,t_2)}(E)) \geq \dim_H(C \times C \times C) + 1 \geq 3\gamma + 1 = 3\alpha - 2.
$$

In general, it is an easy deduction to extend this example to longer chains to show that

$$
\dim_H(VS^k_{\vec{t}}(E)) \geq \dim_H(C \times \cdots \times C) + 1 \geq (k + 1)\alpha - k.
$$

The proof of Theorem 1.9 is complete.

Note that both the above upper and lower estimates extend to general trees. Indeed, following the iterative scheme introduced in [17] for trees, one can apply the $L^2$ estimate in Lemma 4.1 to obtain the upper bound. A similar construction as in the example above will produce the matching lower bound, where one still fixes distinct $c_1, \cdots, c_{k+1}$ first together with the $s_i$ corresponding to the root of the tree. We omit the details.
4.2. Proof of Theorem [1.11]

4.2.1. Upper bound: Let \( d \geq 4, \alpha \leq \left\lfloor \frac{d}{2} \right\rfloor - 1 \), and fix \( \vec{t} \in \mathbb{R}^k \) with \( t_i \sim 1, \forall i \). It is easy to see from the assumption that \( g_d(\mathcal{V}^k_\alpha) \leq (k + 1)\alpha \).

Indeed, for any set \( E \subset \mathbb{R}^d \) that is \( \{\delta_i\}\)-discrete \( \alpha \)-regular and is contained in the unit ball, one has \( |E_{\delta_i}| \leq \delta_i^{d-\alpha} \). Hence, the \( \delta_i \)-neighborhood of \( \mathcal{V}^k_\alpha(E) \), being contained in \( E_{\delta_i} \times \cdots \times E_{\delta_i} \), has Lebesgue measure bounded by \( \delta_i^{(k+1)(d-\alpha)} \). This in particular shows that

\[
\dim_H(\mathcal{V}^k_\alpha(E)) \leq \dim_M(\mathcal{V}^k_\alpha(E)) \leq (k + 1)d - (k + 1)(d - \alpha) = (k + 1)\alpha.
\]

4.2.2. Lower bound: Let \( d \geq 4 \) and \( \alpha \leq \left\lfloor \frac{d}{2} \right\rfloor - 1 \). To see the lower bound, we consider the following example, which is inspired by the well-known orthogonal-circles example for the unit distance problem in \( d \geq 4 \) (for instance, see [20]). Assume \( d \) is even. Note that one can always reduce to this case by recalling \( g_{d+1} \geq g_d \). For the sake of simplicity, assume \( t_1, t_2, \ldots, t_k \).

We first iteratively choose sets \( K_1, \ldots, K_k \subset \mathbb{R}^\frac{d}{2} \). More precisely, let \( s_1 = t_1 \) and \( K_1 \) be an AD regular subset of \( s_1 \mathcal{S}^{\frac{d}{2}-1} \subset \mathbb{R}^\frac{d}{2} \) so that \( \dim_H(K_1) = \alpha \), where \( s_1 \mathcal{S}^{\frac{d}{2}-1} \) denotes the sphere in \( \mathbb{R}^\frac{d}{2} \) of radius \( s_1 \) centered at the origin. Let \( s_2 > 0 \) be such that \( \frac{s_2^2}{2} + \frac{s_1^2}{2} = t_2^2 \) (which is possible since \( t_2 \geq t_1 \)). Choose \( K_2 \) to be an AD regular subset of \( s_2 \mathcal{S}^{\frac{d}{2}-1} \subset \mathbb{R}^\frac{d}{2} \). In general, for \( 2 \leq i \leq k \), \( K_i \) is an AD regular subset of \( s_i \mathcal{S}^{\frac{d}{2}-1} \subset \mathbb{R}^\frac{d}{2} \) so that \( \dim_H(K_i) = \alpha \), where \( s_i > 0 \) satisfies

\[
\frac{s_{i-1}^2}{2} + \frac{s_i^2}{2} = t_i^2.
\]

It is easy to see that one indeed has \( s_i > 0 \), because from the previous step one has \( \frac{s_{i-1}^2}{2} < t_{i-1}^2 \leq t_i^2 \).

Now, define a set

\[
E = \left\{ \left( \frac{y_i}{\sqrt{2}}, 0 \right) \in \mathbb{R}^\frac{d}{2} \times \mathbb{R}^\frac{d}{2} : y_i \in K_i, \text{ for some } i \right\}
\]

\[
\cup \left\{ \left( 0, \frac{y_i}{\sqrt{2}} \right) \in \mathbb{R}^\frac{d}{2} \times \mathbb{R}^\frac{d}{2} : y_i \in K_i, \text{ for some } i \right\}.
\]

It is straightforward to check that \( E \subset \mathbb{R}^d \) is \( \{\delta_i\}\)-discrete \( \alpha \)-regular and has Hausdorff dimension \( \alpha \). We claim that \( \dim_H(\mathcal{V}^k_\alpha(E)) \geq (k + 1)\alpha \).

To see this, take the case that \( k \) is odd as an example. One observes that \( \mathcal{V}^k_\alpha(E) \) contains the following set as a subset:

\[
\left\{ \left( \frac{\tilde{y}_1}{\sqrt{2}}, 0 \right), \left( 0, \frac{\tilde{y}_1}{\sqrt{2}} \right), \left( \frac{\tilde{y}_2}{\sqrt{2}}, 0 \right), \left( 0, \frac{\tilde{y}_2}{\sqrt{2}} \right), \ldots, \left( 0, \frac{\tilde{y}_k}{\sqrt{2}} \right) : \tilde{y}_1 \in K_1, y_i \in K_i, \forall i \right\}.
\]

(In the case that \( k \) is even, one has a similar result with the last component in the set above replaced by \( \left( \frac{\tilde{y}_k}{\sqrt{2}}, 0 \right) \).) Therefore, it implies that \( \dim_H(\mathcal{V}^k_\alpha(E)) \geq (k + 1)\alpha \), and the proof of the \( d \geq 4 \) case of Theorem [1.11] is complete.

Note that one can easily make the chains obtained above non-degenerate without lowering the dimension of the chain set. Moreover, the above argument extends to general trees with lengths of edges satisfying certain conditions. However, the method fails in certain cases, for instance, the case of 3-chain of gaps \((1, 2, 1)\). In the discrete setting, a similar issue also exists. In fact, in the discrete setting, it is
known that when $d = 4$, the orthogonal-circles example is essentially the only type
of sharp example for the unit distance problem, which seems to suggest that the
lower bound for certain chains may be very difficult to obtain.

4.2.3. Upper bound: The case $d = 2$, $0 < \alpha \leq 1$, $k = 2$. In $\mathbb{R}^2$, assume that
$0 < \alpha \leq 1$, we can completely determine the value of $g_2(\mathcal{V}S_1^2, \alpha)$. Note that even
though the argument below extends to longer chains, we are unable to find matching
upper and lower bounds when $k \geq 3$.

First, we prove that $g_2(\mathcal{V}S_1^2, \alpha) \leq 2\alpha$. To see this, fix a set $E \subset \mathbb{R}^2$ that is
${\delta_1}$-discrete $\alpha$-regular and gaps $\vec{t} = (t_1, t_2) \in \mathbb{R}^2_+$. Note that for any fixed pair of
points $x_1, x_3 \in E$, the circles $S(x_1, t_1)$ and $S(x_3, t_2)$ intersect at most at two points
$p, q \in E$. Therefore, the map $(x_1, x_2, x_3) \mapsto (x_1, x_3)$ from $\mathcal{V}S_1^2(E)$ to $E \times E$ is at
most 2-to-1, hence

$$\dim_H(\mathcal{V}S_1^2(E)) \leq \dim_H(E \times E) \leq \dim_M(E \times E) \leq 2\alpha.$$  

Here, the last inequality follows from the assumption that $E$ is ${\delta_1}$-discrete $\alpha$-
regular (which implies $|E_{\delta_1}| \leq \delta_1^{2-\alpha}$, $\forall i$).

4.2.4. Lower bound: The case $d = 2$, $0 < \alpha \leq 1$, $k = 2$. Next, we prove that
$g_2(\mathcal{V}S_1^2, \alpha) \geq 2\alpha$ by constructing a sharp example, which is inspired by a construc-
tion studied in [26]. Given gaps $\vec{t} = (t_1, t_2) \in \mathbb{R}^2_+$, for $i = 1, 2$, let $E_i$ be an AD
regular subset of $t_iS^1$ so that $\dim_H(E_i) = \alpha$, where $t_iS^1$ denotes the circle of radius
$t_i$ centered at the origin. In the case that $t_1 = t_2$, choose $E_1, E_2$ that are disjoint.
Take another AD regular set $E_3$ that contains the origin with $\dim_H(E_3) = \alpha$, and
define $E = E_1 \cup E_2 \cup E_3$. It is easy to see that $\dim_H(E) = \alpha$, $E$ is ${\delta_1}$-discrete
$\alpha$-regular, and

$$\mathcal{V}S_1^2(E) \supset \{(x, 0, y) : x \in E_1, y \in E_2\}.$$  

Hence,

$$\dim_H(\mathcal{V}S_1^2(E)) \geq \dim_H(E_1) + \dim_H(E_2) = 2\alpha.$$  

The proof of this last case of Theorem [1.11] is complete.

5. Triangles with prescribed gaps: Proof of Theorem [1.14]

5.1. The large $\alpha$ case: Let $d \geq 3$ and $t_i \sim 1$. In this section, we prove the bound

$$g_d(\mathcal{V}\text{Tri}_i; \alpha) \leq \begin{cases} 3\alpha - 3, & \frac{2d}{3} + 1 \leq \alpha \leq d, \\ d + \frac{3d^2}{4} - \frac{3}{2}, & 0 < \alpha \leq \frac{2d}{3} + 1. \end{cases}$$  

The proof strategy is adapted from that of Theorem [1.9] but in the triangle case, one
cannot expect to do iteration. Instead, we will directly prove an $L^2$ bound that
is adapted to the triangle case, which involves an estimate of the Fourier transform
of the surface measure of a surface that is not the sphere anymore.
To begin with, let $E$ be a $\{\delta_i\}$-discrete $\alpha$-regular set with $\dim_H(E) = \alpha$. Without loss of generality, assume that $E = -E$. We aim to prove that for all $i$, 

\begin{equation}
\label{eq:5.33}
|D^\delta| := |\{(x, y, z) \in E^3_{\delta_i} : t_1 - 2\delta_i \leq |x - y| \leq t_1 + 2\delta_i, t_2 - 2\delta_i \leq |y - z| \leq t_2 + 2\delta_i, t_3 - 2\delta_i \leq |x - z| \leq t_3 + 2\delta_i\}| \\
\leq \delta_i^{-3\delta - \gamma(d, \alpha) - \epsilon},
\end{equation}

where 

\[ \gamma(d, \alpha) := \begin{cases} 3\alpha - 3, & \frac{2d}{3} + 1 \leq \alpha \leq d, \\
\frac{d}{2} - \frac{3\alpha}{2}, & \alpha \leq \frac{2d}{3} + 1. \end{cases} \]

Again, we denote $\delta = \delta_i$ for the sake of simplicity.

It is direct to see that 

\[ |D^\delta| = \int_{E_{\delta_i}} \int_{E_{\delta_i}} \int_{E_{\delta_i}} \chi_{A_{1i}}(y - x)\chi_{A_{2i}}(z - y)\chi_{A_{3i}}(x - z) \, dx \, dy \, dz \\
\approx \int \int \int \chi_{E_1}(z)\chi_{E_1}(z + x)\chi_{E_1}(z + y)\chi_{S_{\delta,i}}(x, y) \, dx \, dy \, dz,
\]

where $S_{\delta,i}$ denotes the $\delta$-neighborhood of the surface $S := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x| = t_1, |y| = t_2, |x - y| = t_3\}$.

Let $\rho$ be the symmetric Schwartz function on $\mathbb{R}^d$ introduced above in (1.27), and set $\rho_r(x) = r^{-d}\rho(\frac{x}{r})$. Then, one has 

\[ \chi_{S_{\delta,i}}(x, y) \lesssim \delta^3(\rho_3 \otimes \rho_3) \ast \sigma_{S_{\delta,i}}(x, y) = \delta^3 \int \rho_3(x - u)\rho_3(y - v) \, d\sigma_{S_{\delta,i}}(u, v), \]

where $\sigma_{S_{\delta,i}}(x, y)$ denotes the surface measure of $S_{\delta,i}$. Therefore, 

\[ |D^\delta| \lesssim \delta^3 \int \left( \int \chi_{E_1}(z)\chi_{E_1}(z + x)\chi_{E_1}(z + y) \, dz \right) (\rho_3 \otimes \rho_3) \ast \sigma_{S_{\delta,i}}(x, y) \, dx \, dy.
\]

With $x, y$ fixed, the inner integral is 

\[ \int \chi_{E_1}(z)\chi_{E_1}(z + x)\chi_{E_1}(z + y) \, dz \\
= \int \int \int \check{\chi}_{E_1}(\xi)\check{\chi}_{E_1}(\eta)\check{\chi}_{E_1}(\zeta)e^{i(z, z, z) + (x, y, 0)} \, d\xi \, d\eta \, d\zeta.
\]

Hence, 

\[ |D^\delta| \lesssim \delta^3 \int \int \int \check{\chi}_{E_1}(\xi)\check{\chi}_{E_1}(\eta)\check{\chi}_{E_1}(\zeta)e^{i(z, z, z) + (x, y, 0)} \, \sigma_{S_{\delta,i}}(-\xi, -\eta)\rho_3(-\xi)\rho_3(-\eta) \, d\xi \, d\eta \, d\zeta \\
= \delta^3 \int \check{\chi}_{E_1}(\xi)\check{\chi}_{E_1}(\eta)\check{\chi}_{E_1}(\zeta)e^{i(z, z, z) + (x, y, 0)} \, d\xi \, d\eta.
\]

In the last step above, we have used the observation that 

\[ e^{i(z, z, z) + (x, y, 0)} \, d\xi \, d\eta \, d\zeta = \delta_0(\xi + \eta + \zeta), \]

where $\delta_0$ denotes the Dirac $\delta$ function at the origin.

Similarly as in the proof of Theorem 1.9, we will estimate the above integral by decomposing it into single scales. At each single scale, one assumes $|\xi| + |\eta| \sim \frac{C}{\delta^3}$, 

\[ \int e^{i(z, z, z) + (x, y, 0)} \, d\xi \, d\eta \, d\zeta = \delta_0(\xi + \eta + \zeta), \]

where $\delta_0$ denotes the Dirac $\delta$ function at the origin.
where \( \frac{1}{2} \leq 2^j \leq \frac{1}{2^s} \). Observe that in this case, at least two of \(|\xi|, |\eta|, |\xi + \eta|\) are
\( \sim \frac{C}{2 \delta^s} \).

To see why it suffices to reduce to the range \( \frac{1}{2} \leq 2^j \leq \frac{1}{2^s} \), one observes that
2^j \geq \frac{1}{2} follows from \( |\xi| + |\eta| \leq \frac{C}{2 \delta} \), which is a consequence of the Fourier decay
property of \( \rho \) and \( t \to 1 \). On the other hand, 2^j \leq \frac{1}{2^s} follows from \( |\xi| + |\eta| \geq C \).
This is because the integral above over the domain \( |\xi| + |\eta| \leq C \) can be trivially bounded by
\[ |E_\delta|^{31} \sigma_{S_\delta} \|_{L^1} \leq \delta^{3(d-\alpha)} \]
which implies that the contribution of this part to \( |D^\delta| \) is bounded by \( \delta^{3d-(3\alpha-3)} \)
as desired. (Note that the case for \( \alpha < \frac{2d}{3} + 1 \) then also trivially follows since in
that range one has \( 3\alpha - 3 < d + \frac{2d}{3} - \frac{3}{2} \).)

In the following, we discuss two sub-cases for the single scale \( |\xi| + |\eta| \sim \frac{C}{2 \delta^s} \), with
the first one being the main case.

5.1.1. Case \(|\xi| \sim |\eta| \sim \frac{C}{2 \delta^s} \). By symmetry and dropping the \( \delta^3 \) for now, it suffices
to estimate
\[ \int \int_{|\xi| \sim |\eta| \sim \frac{C}{2 \delta^s}} \tilde{\chi}_{E_\delta}(\xi) \tilde{\chi}_{E_\delta}(\eta) \tilde{\chi}_{E_\delta} (\xi + \eta) \sigma_{S_\delta}(\xi, \eta) \rho_{\delta}(\xi) \rho_{\delta}(\eta) \, d\xi d\eta. \]
We will be using the following estimate, slightly generalizing Lemma 2.3 of [16]:

**Lemma 5.1.** Let \( S_\delta \) be the surface defined above, and let \( \theta \) denote the angle between
the \( t_1 \)-side and the \( t_2 \)-side of the triangle. Suppose \( |\xi| \sim |\eta| \), then
\[ |\sigma_{S_\delta}(\xi, \eta)| \leq \frac{t_2}{t_1} g_\theta(\eta) \frac{1}{|\xi|^{-\frac{d-2}{2}} \sin(\langle \xi, \eta \rangle)} \]
where \( g_\theta \in O(d) \) is some rotation by \( \theta \) and \( \langle \xi, \eta \rangle \) denotes the angle between \( \xi, \eta \).

The proof of the lemma follows exactly the same lines as [16] Lemma 2.3, where
the equilateral triangle case was discussed. We omit the proof of the general case.

Applying this lemma, one obtains
\[ \int \int_{|\xi| \sim |\eta| \sim \frac{C}{2 \delta^s}} \tilde{\chi}_{E_\delta}(\xi) \tilde{\chi}_{E_\delta}(\eta) \tilde{\chi}_{E_\delta} (\xi + \eta) \rho_{\delta}(\xi) \rho_{\delta}(\eta) \, d\xi d\eta \]
\[ \leq (2^j \delta)^{d-2} \int \int_{|\xi| \sim |\eta| \sim \frac{C}{2 \delta^s}} \tilde{\chi}_{E_\delta}(\xi) \tilde{\chi}_{E_\delta}(\eta) \tilde{\chi}_{E_\delta} (\xi + \eta) \rho_{\delta}(\xi) \rho_{\delta}(\eta) \, d\xi d\eta \]
\[ = (2^j \delta)^{d-2} \int \int_{|\xi| \sim |\eta| \sim \frac{C}{2 \delta^s}} \tilde{\chi}_{E_\delta} (\xi + \eta) \rho_{\delta}(\xi) \tilde{\chi}_{E_\delta}(\xi + \eta) \rho_{\delta}(\eta) \, d\xi d\eta \]
\[ \leq (2^j \delta)^{d-2} \int \int_{|\xi| \sim |\eta| \sim \frac{C}{2 \delta^s}} \tilde{\chi}_{E_\delta}(\xi + \eta) \rho_{\delta}(\xi) \tilde{\chi}_{E_\delta}(\eta) \rho_{\delta}(\eta) \, d\xi d\eta \]

We claim that for any fixed \( \eta \),
\[ \int |\tilde{\chi}_{E_\delta}(\xi + \eta) | \left| \xi + \frac{t_2}{t_1} g_\theta(\eta) \right|^{-\frac{d-2}{2}} (\sin(\langle \xi, \eta \rangle))^{-\frac{1}{2}} \, d\xi \leq 2^{j \frac{a-2d+1}{2}} \delta^{\frac{1-a}{2}}, \]
and similarly, when \( \xi \) is fixed,
\[ \int |\tilde{\chi}_{E_\delta}(\xi + \eta) | \left| \xi + \frac{t_2}{t_1} g_\theta(\eta) \right|^{-\frac{d-2}{2}} (\sin(\langle \xi, \eta \rangle))^{-\frac{1}{2}} \, d\eta \leq 2^{j \frac{a-2d+1}{2}} \delta^{\frac{1-a}{2}}. \]
Assume (5.35) and (5.36) for now, then by Schur’s test, one obtains

\[ (5.34) \lesssim (2^j \delta)^{d-2j} \frac{2^{j-2d+1}}{\delta} \frac{\log (\frac{1}{\delta})}{2^{j-2d+1}} \int_{|\xi| \sim \frac{C}{2^j \delta}} |\hat{\chi}E_{\xi} * \hat{\rho}_{\delta}(\xi)|^2 \, d\xi \lesssim (2^j \delta)^{d-2j} \frac{2^{j-2d+1}}{\delta} \frac{\log (\frac{1}{\delta})}{2^{j-2d+1}} (2^j \delta)^{2d-\frac{d-2}{2} \delta^{3d-3\alpha}}, \]

where the second inequality follows from the \( L^2 \) estimate used above in the proof of Theorem 1.9, more precisely from (4.31).

When \( \alpha \geq \frac{2d}{3} + 1 \), since \( 2^j \delta \leq 1 \), one has \( (2^j \delta)^{\frac{2d}{3} - \frac{d-2}{2} \delta^{3d-3\alpha}} \leq 1 \), hence the contribution of this case to \( |D^k| \) is bounded by

\[ \lesssim \delta^3 \sum_{\frac{1}{2} \leq 2^j \leq \frac{3}{2}} \delta^{3d-3\alpha} \lesssim \log (\frac{1}{\delta}) \delta^{3d-(3\alpha-3)}, \]

which matches the desired estimate. When \( \alpha < \frac{2d}{3} + 1 \), one has

\[ (5.34) \lesssim (2^j \delta)^{\frac{2d}{3} - \frac{d-2}{2} \delta^{3d-3\alpha}} = (2^j \delta)^{\frac{2d}{3} - \frac{d-2}{2} \delta^{2d-\frac{d-2}{2} \delta}}. \]

Since \( \alpha < \frac{2d}{3} + 1 \), the sum over \( j \) converges, therefore, the contribution of this case to \( |D^k| \) is bounded by

\[ \lesssim \delta^3 \delta^{2d-\frac{2d}{3} - \frac{d-2}{2}} = \delta^{3d-(d+\frac{d-2}{2})}, \]

which completes the proof of this case.

It remains to prove estimates (5.35) and (5.36). We only prove (5.35) below, as the other one can be obtained in the same way after a change of variable \( \xi \mapsto t_{\xi} \xi_i g_{-\delta}(\xi) \). By Cauchy-Schwarz, one has (5.35) bounded by

\[ \left( \int_{|\xi| \sim \frac{C}{2^j \delta}} |\hat{\chi}E_{\xi}(\xi + \eta)|^2 \, d\xi \right)^{1/2} \left( \int_{|\xi| \sim \frac{C}{2^j \delta}} \left| \frac{t_{\xi}}{t_{\xi}} g_{\theta}(\eta) \right|^{-1} (\sin(\xi, \eta))^{-(d-2)} \, d\xi \right)^{1/2}. \]

According to Lemma 2.3 of [16], and observing that the same bound holds true in the case of general triangles, the second factor is bounded by \( (\frac{C}{2^j \delta})^{\frac{d-2}{2}} \).

The first factor can be estimated similarly as above. Since \( |\xi| \sim |\eta| \sim \frac{C}{2^j \delta} \), after a change of variable, one has

\[ \int_{|\xi| \sim \frac{C}{2^j \delta}} |\hat{\chi}E_{\xi}(\xi + \eta)|^2 \, d\xi \lesssim \int_{|\xi| \lesssim \frac{C}{2^j \delta}} |\hat{\chi}E_{\xi}(\xi)|^2 \, d\xi \lesssim \int_{|\xi| \lesssim \frac{C}{2^j \delta}} |\hat{\chi}E_{\xi} * \hat{\rho}_{\delta}(\xi)|^2 \, d\xi, \]

which follows from the choice \( r \sim 2^j \delta \) and the property \( \hat{\rho}_{\delta}(\xi) = \hat{\rho}(r\xi) \gtrsim 1 \) when \( |\xi| \lesssim \frac{C}{2^j \delta} \). According to (4.31) (also see estimate (2.11) of [24]), the above is further

\[ \lesssim \|\chi E_{\xi} * \rho_{\delta}\|_2^2 \lesssim r^{\alpha-d} d^{2(d-\alpha)} = (2^j \delta)^{\frac{d-2}{2} \delta^{2d-\frac{d-2}{2}}} \delta^{3d-3\alpha}. \]

Therefore, one obtains

\[ (5.35) \lesssim 2^j \frac{2d}{3} - \frac{d-2}{2} \delta^{3d-3\alpha} = 2^j \delta^{\frac{2d}{3} - \frac{d-2}{2} \delta^{3d-3\alpha}}, \]

and the proof of the first case is complete.
5.1.2. Case $|\xi| \sim |\xi - \eta| \sim \frac{C}{\delta^2}$. The second case can be reduced to the first case above by making use of the symmetry of the surface measure $\sigma_{S_\delta}$. Such an argument in the equilateral triangle case was derived in [10] for the study of a related problem, and we only sketch the general case here for the sake of completeness.

For any $(x^0, y^0)$ such that $\Delta_{x^0y^0}$ forms a triangle of sidelengths $t_1, t_2, t_3$ (to be more specific, we assume $|x^0| = t_1$, $|y^0| = t_2$, and $|x^0 - y^0| = t_3$). One observes that

$$\int f(x, y) d\sigma_{S_\delta}(x, y) = \int_{O(d)} f(gx^0, gy^0) dg,$$

where $O(d)$ denotes the orthogonal group in $\mathbb{R}^d$ and $dg$ denotes the normalized Haar measure on $O(d)$. A direct computation shows that

$$\hat{\sigma}_{S_\delta}(\xi, \eta) = \int e^{-2\pi i (g(y^0 - x^0)(-\xi) + gy^{\delta}(\xi + \eta))} dg.$$

Observe that $\Delta_{(y^0 - x^0)00}$ is a triangle with a permutation of the original three sides. Write $\bar{s} := (t_3, t_2, t_1)$, then the point $(y^0 - x^0, y^0) \in S_{\bar{s}}$. One thus has $\hat{\sigma}_{S_{\bar{s}}}(\xi, \eta) = \hat{\sigma}_{S_{\delta}}(-\xi, \xi + \eta)$. Therefore, by a change of variable $\zeta = \xi + \eta$, one can reduce the estimate to the first case above (with $|\xi| \sim |\xi| \sim \frac{C}{\delta^2}$ and a new surface $S_{\bar{s}}$). The estimate in the first case obviously still holds for the surface $S_{\bar{s}}$, hence the proof is complete.

5.2. The large $\alpha$ case, $d = 2$. As discussed in the introduction, when $\alpha \leq \frac{7}{4}$, the upper estimate follows from the corresponding bounds for the unit distance problem, obtained in [24, Theorem 1.3]. Therefore, in this subsection, it suffices to prove that $g_2(V\text{Tri}_\delta, \alpha) \leq 3\alpha - 3$ if $\alpha \geq \frac{7}{4}$.

Note that the argument in the previous section regarding the $d \geq 3$ case still works when $d = 2$. (One does need to slightly change the argument in Section 5.1.2 where the rotation symmetry only holds after decomposing the surface $S_{\bar{s}}$ to two parts.) However, in order to obtain the upper bound $3\alpha - 3$ for $\alpha \geq \frac{7}{4}$, the argument fails to be sufficient.

Below we adapt a method originated in [12].

Recall from the previous subsection that, letting $D^3$ be as in (5.33), one has

$$|D^3| = \int_{E_1} \int_{E_3} \int_{E_2} \chi_{A_{1,2}}(y - x) \chi_{A_{2,3}}(z - y) \chi_{A_{3,1}}(x - z) dxdydz.$$

Let $\sigma_{t}$ denote the surface measure on the circle of radius $t$ in $\mathbb{R}^2$ and $\sigma^\delta_t := \sigma_t * \rho_\delta$. Then, one has that

$$|D^3| \leq \delta^3 \int \int \chi_{E_4} * \rho_\delta(x) \chi_{E_5} * \rho_\delta(y) \chi_{E_6} * \rho_\delta(z) \sigma^\delta_{t_1}(y - x) \sigma^\delta_{t_2}(z - y) \sigma^\delta_{t_3}(x - z) dxdydz.$$

In general, define

$$\Lambda^\delta_t(f_1, f_2, f_3) := \int \int \sigma^\delta_{t_1}(y - x) \sigma^\delta_{t_2}(z - y) \sigma^\delta_{t_3}(x - z) f_1(x) f_2(y) f_3(z) dxdydz.$$

Then one has $|D^3| \leq \delta^3 \Lambda^\delta_t(\chi_{E_4} * \rho_\delta, \chi_{E_5} * \rho_\delta, \chi_{E_6} * \rho_\delta)$.

Define $f^\beta_\delta(x) := \chi_{E_4} * \rho_\delta * |x|^{-2+\beta}(x)$ (initially defined for $\text{Re}(\beta) > 0$ and is extended to the complex plane by analytic continuation). Then one has $\chi_{E_4} * \rho_\delta \sim f^\beta_\delta$. In the following, we will only be interested in the range $\text{Re}(\beta) \leq \frac{1}{4}$. Note that $f^\beta_\delta \in L^2(\mathbb{R}^2)$, which follows from Plancherel and properties of $\rho_\delta$, as well as estimate (4.31) (with $f = 1$).
Now, define $F(\beta) := \Lambda^\delta(f^{-\beta}_\delta, \chi_{E_k} * \rho_\delta, f^\beta_\delta) =: \langle B(f^{-\beta}_\delta, \chi_{E_k} * \rho_\delta), f^\beta_\delta \rangle$, where the bilinear operator
\[
B(g,h) := \int \int g(x-u)h(x-v)\sigma^\delta(u)\sigma^\delta(v)(u-v) \, du \, dv.
\]

Here, we have rescaled the triangle to make it have side lengths $(a,b,1)$, where $0 < a,b < 1$. Our main estimate is the following.

**Lemma 5.2.** Let $\alpha \geq \frac{3}{4}$. Then for all $\beta$ satisfying $-\frac{1}{4} \leq \Re(\beta) \leq \frac{1}{4}$, we have $|F(\beta)| \leq \delta^{3(2-\alpha)}$.

It is easy to see that the lemma would imply that $|D^3 F(0)| \leq \delta^{3+3(2-\alpha)}$, which would imply that $\dim_H(V\text{T}ri_{\alpha}) \leq 3\alpha - 3$ when $\alpha \geq \frac{7}{4}$.

**Proof of Lemma 5.2.** Note that each input function in $F(\beta)$ has compact support, hence one has a trivial upper bound of $F(\beta)$. According to the Three Line Lemma, it suffices to check the desired bound at $\Re(\beta) = \pm \frac{1}{4}$. Furthermore, it suffices to study the $\Re(\beta) = \frac{1}{4}$ case, since by interchanging the role of the first and the third input in $F(\beta)$, the other case is symmetric.

One first observes, by taking the modulus in the definition, that $|f^\delta_\beta| \leq f^\text{Re}(\beta)_\delta$, which implies that
\[
|F(\beta)| \leq \langle B(f^{-\text{Re}(\beta)}_\delta,\chi_{E_k} * \rho_\delta), f^{\text{Re}(\beta)}_\delta \rangle \leq \|B(f^{-\text{Re}(\beta)}_\delta,\chi_{E_k} * \rho_\delta)\|_{L^1(\mathbb{R}^2)} \|f^{\text{Re}(\beta)}_\delta\|_{L^\infty(\mathbb{R}^2)}.
\]

It is proved in [12, Theorem 3.1] that
\[
B : L^2_{\beta_1}(\mathbb{R}^2) \times L^2_{\beta_2}(\mathbb{R}^2) \to L^1(\mathbb{R}^2), \quad \text{if } \beta_1 + \beta_2 = \frac{1}{2}, \beta_1, \beta_2 \geq 0,
\]
with constant independent of $\delta$. Here, $\|f^\text{Re}(\beta)_\delta\|_{L^2} := \int |\hat{f}(\xi)|^2(1 + |\xi|)^{2s} \, d\xi$. Combined with the above, this further implies when $\Re(\beta) = \frac{1}{4}$ that
\[
|F(\beta)| \leq \|f^{-\frac{1}{2}}_\delta\|_{L^2_{\beta_2}} \|\chi_{E_k} * \rho_\delta\|_{L^2} \|f^{\frac{1}{2}}_\delta\|_{L^\infty}.
\]

Then the desired bound easily follows from Lemma 5.3 and 5.4 below. \hfill \Box

**Lemma 5.3.** If $\alpha \geq \frac{7}{4}$, then $\|f^{\frac{1}{2}}_\delta\|_{L^\infty(\mathbb{R}^2)} \leq \delta^{2-\alpha}$.  

**Proof.** By assumption on $\rho_\delta$ and definition of $f^\delta_\beta$, one has
\[
\|f^\delta_\beta\|_{L^\infty(\mathbb{R}^2)} \leq \|\chi_{E_k} * |\cdot|^{-2+\frac{1}{2}}\|_{L^\infty} \|\rho_\delta\|_{L^1} \leq \sup_x \int \chi_{E_k}(y)|x-y|^{-\frac{1}{2}} \, dy.
\]

Fix any $x \in \mathbb{R}^2$, one has from the trivial estimate that
\[
\int_{|x-y| \geq 1} \chi_{E_k}(y)|x-y|^{-\frac{1}{2}} \, dy \leq |E_\delta| \leq \delta^{2-\alpha}.
\]

For the other part, we decompose the integral as
\[
\int_{|x-y| \leq 1} \chi_{E_k}(y)|x-y|^{-\frac{1}{2}} \, dy \leq \sum_{k=1}^\infty 2^{\frac{1}{2}k} \int_{|x-y| \sim 2^{-k}} \chi_{E_k}(y) \, dy.
\]

If $2^{-k} \geq \delta$, by the $\{\delta_i\}$-discrete $\alpha$-regular assumption,
\[
\int_{|x-y| \sim 2^{-k}} \chi_{E_k}(y) \, dy \leq 2^{-k\alpha} \delta^{2-\alpha}.
\]
Applying the bound in Lemma 5.3 for each fixed \( x \).

**Proof.**

By definition and Fourier inversion, one has

\[ \lambda_{E_A}(y) |y|^{-\frac{1}{2}} dy \leq \sum_{k=1}^{\log(\delta^{-1})} 2^{\frac{k}{2}} 2^{-k} \alpha \delta^{2-\alpha} + \sum_{k=\log(\delta^{-1})}^{\infty} 2^{\frac{k}{2}} 2^{-2k} \leq \delta^{2-\alpha-\epsilon}. \]

\[ \square \]

**Lemma 5.4.** If \( \alpha \geq \frac{7}{4} \), then \( \| \lambda_{E_A}^{\frac{1}{2}} \|_{L^2_{\mathbb{R}^d}((0,\infty))} \), \( \| \lambda_{E_A} * \rho_\delta \|_{L^2_{\mathbb{R}^d}((0,\infty))} \leq \delta^{2-\alpha-\epsilon} \).

**Proof.**

By definition and Fourier inversion, one has

\[ \| \lambda_{E_A} * \rho_\delta \|_{L^2_{\mathbb{R}^d}((0,\infty))} \leq \int |\hat{\lambda}_{E_A}|^2 |\xi|^{-\frac{1}{2}} d\xi \sim \int_{E_\delta} \int_{E_\delta} |x-y|^{-\frac{1}{2}} dxdy. \]

Applying the bound in Lemma 5.3 for each fixed \( x \), one has the integral on the right hand side above is

\[ \leq |E_\delta| \cdot \delta^{2-\alpha-\epsilon} \leq \delta^{2(2-\alpha)-\epsilon}. \]

The other term can be estimated similarly.

\[ \| f_{\delta}^{\frac{1}{2}} \|_{L^2_{\mathbb{R}^d}((0,\infty))} \leq \int |(f_{\delta}^{\frac{1}{2}}(\xi))^2 |\xi|^{-\frac{1}{2}} d\xi \leq \int |\hat{\lambda}_{E_A}|^2 |\xi|^{-\frac{1}{2}} d\xi. \]

Hence, the desired bound follows in the same way as above. \[ \square \]

### 5.3. The small \( \alpha \) case, \( d \geq 6 \)

In this section, we restrict to the case \( d \geq 6 \) and \( \alpha \leq \left\lfloor \frac{d}{2} \right\rfloor - 1 \), and our goal is to prove that \( g_d(V \text{Tri}_1, \alpha) = 3\alpha \), whenever \( \mathbf{t} = (t_1, t_2, t_3) \) forms an acute triangle. The upper bound is trivial, hence it suffices to find an example establishing the lower bound.

Since \( g_{d+1} \geq g_d \), it suffices to consider the case that \( d \) is an integer multiple of 3. Since the triangle is acute, there exist \( A, B, C > 0 \) satisfying

\[ A + B = t_1^2, \quad B + C = t_2^2, \quad C + A = t_3^2. \]

Let \( K_A \subset A^{1/2} S_{\mathbb{R}^d}^{d-1} \) be an AD regular set of Hausdorff dimension \( \alpha \), and similarly define \( K_B \subset B^{1/2} S_{\mathbb{R}^d}^{d-1}, K_C \subset C^{1/2} S_{\mathbb{R}^d}^{d-1} \). Define the set \( E = E_A \cup E_B \cup E_C \) where

\[ E_A := \{ (x, 0, 0) \in \mathbb{R}^{d/2} \times \mathbb{R}^{d/2} \times \mathbb{R}^{d/2} : x \in K_A \}, \]

\[ E_B := \{ (0, x, 0) \in \mathbb{R}^{d/4} \times \mathbb{R}^{d/4} \times \mathbb{R}^{d/4} : x \in K_B \}, \]

\[ E_C := \{ (0, 0, x) \in \mathbb{R}^{d/4} \times \mathbb{R}^{d/4} \times \mathbb{R}^{d/4} : x \in K_C \}. \]

Then, it is easy to see that any three points in \( E_A, E_B, E_C \) form a triangle of the given sidelength \( t_1, t_2, t_3 \). Therefore, \( \text{dim}_H(V \text{Tri}_1(E)) \geq 3\alpha \).
Lemma 6.1. With $T$ as in (6.39), and with $\phi$ satisfying (1.9) and (1.10), we have
\[
\left( \int_{E_\delta} |T(f(x))|^2 \, dx \right)^{1/2} \leq C_\delta \delta^{\beta(d, \alpha) - 1 - \epsilon} \left( \int_{E_\delta} |f(x)|^2 \, dx \right)^{1/2},
\]
where
\[
\beta(d, \alpha) - 1 := \begin{cases} 
  d - \alpha, & \frac{d+1}{2} \leq \alpha \leq d, \\
  \frac{d-1}{2}, & \alpha \leq \frac{d+1}{4}. 
\end{cases}
\]
Applying the Lemma $k$-times to the right-hand-side of (6.41), we obtain
\[ \delta^k |E_\delta|^{1/2} \delta^{k(d,\alpha)-1} |E_\delta|^{1/2} \leq \delta^{k+(d-\alpha)+k(d,\alpha)-1}, \]
which agrees with (6.37).

We rely on the following Theorem, due to Phong and Stein [25], which is stated here without proof.

**Theorem 6.2.** Let $T^\delta$ be defined as above with $\phi$ satisfying assumptions (1.9) and (1.10). Then
\[ T^\delta : L^2(\mathbb{R}^d) \to L^2_{2^{-\frac{d}{2}}} (\mathbb{R}^d) \] with constants independent of $\delta$,
where $L^2_{2^{-\frac{d}{2}}} (\mathbb{R}^d)$ denotes the Sobolev space of functions with $\gamma$ (generalized) derivatives in $L^2(\mathbb{R}^d)$.

6.1. **Proof of Lemma 6.1.** Let $g$ be a nonnegative test function in $L^2(E_\delta)$. It suffices to show that
\[ \langle T(f\chi_{E_\delta}), g\chi_{E_\delta} \rangle \leq \delta^{(d,\alpha)-1} \|f\|_{L^2(E_\delta)} \cdot \|g\|_{L^2(E_\delta)}. \]
Let $\rho$ be as in (4.27) and denote $\rho_r(x) = r^{-d} \rho \left( \frac{x}{r} \right)$. Since $f\chi_{E_\delta}(x) \leq (f\chi_{E_\delta}) * \rho_c \delta(x)$, for $f$ non-negative and continuous, where $c > 0$ is an absolute constant, we can bound the left-hand-side of this expression by
\[ \langle T((f\chi_{E_\delta}) * \rho_c \delta), (g\chi_{E_\delta}) * \rho_c \delta \rangle. \]

For ease of notation, we write this as
\[ \langle TF,G \rangle, \]
where we set $F := F_\delta = (f\chi_{E_\delta}) * \rho_\delta$, $G := G_\delta = (g\chi_{E_\delta}) * \rho_\delta$, and dropped the subscript $c$.

Let $\eta_0(\xi)$ and $\eta$ be smooth cut-off functions such that $\eta_0$ is supported in the ball $\{ |\xi| < 4 \}$, $\eta$ is supported in the annulus $1/2 \leq |\xi| \leq 4$, and $\eta_0(\xi) + \sum_j \eta(2^{-j} \xi) = 1$. Set $\eta_j(\cdot) = \eta(2^{-j} \cdot)$. For $f \in L^2(dx)$, define $\hat{P}_j f$, the classical Littlewood-Paley projection (see, for instance, [31] pages 241-243), by the relation
\[ \hat{P}_j f = \hat{f} \cdot \eta(2^{-j} \cdot). \]

Let $P_j$ and $P_k$ be Littlewood-Paley operators. Now
\[ \langle TF,G \rangle \leq \sum_{k,j=0}^\infty |\langle T(P_j F), P_k G \rangle|. \]

Applying Parseval’s identity,
\[ \sim \sum_{k,j=0}^\infty |\langle (T(P_j F))^\wedge, (P_k G)^\wedge \rangle|. \]
Since $\eta_k \sim (\eta_k)^2$, we can write
\[ \sim \sum_{k,j=0}^\infty |\langle (P_k (T(P_j F)))^\wedge, (P_k G)^\wedge \rangle|. \]
From the second term in this inner product, we see that the sum in $k$ is restricted to $2^k \leq C\frac{1}{\delta}$. Indeed, recalling that $P_k G(\xi) = (P_k (g\chi_{E_\delta} \ast \rho_\delta))^\wedge (\xi) = \eta(2^{-k}\xi)(g\chi_{E_\delta})^\wedge (\xi) \rho(\delta \xi)$, it follows that $2^k \leq \frac{1}{\delta}$. We will see below that we may also restrict to summing over $j$ so that $2^j \leq \frac{1}{\delta}$.

Applying Cauchy-Schwarz, we have

$$\langle TF, G \rangle \lesssim \sum_{j=0}^\infty \sum_{2^k \leq \frac{1}{\delta}} \| (P_k (T(P_j F))^\wedge \|_{L^2(E_\delta)} \cdot \| (P_k G)^\wedge \|_{L^2(E_\delta)}.$$  

The second term was handled in the proof of Theorem 1.9, and it can be easily deduced from Lemma 4.3 that it is bounded by $(2^k \delta)^{\frac{(d-\alpha)}{2}} \| g \|_{L^2(E_\delta)}$. Now

$$\langle TF, G \rangle \lesssim \sum_{j=0}^\infty \sum_{2^k \leq \frac{1}{\delta}} \| (P_k (T(P_j F))^\wedge \|_{L^2(E_\delta)} \cdot (2^k \delta)^{\frac{(d-\alpha)}{2}} \| g \|_{L^2(E_\delta)}.$$  

(6.43)

The remainder of this section is dedicated to bounding the first term in the summand. For $K \in \mathbb{N}$, to be determined, we handle the case when $|j - k| \leq K$ and $|j - k| > K$ separately.

**Case 1:** $|j - k| \leq K$: Write

$$\int |(P_k (T(P_j F))^\wedge (\xi)|^2 \ d\xi \sim 2^{-k(d-1)} \int_{|\xi| \leq 2^k} |(T(P_j F))^\wedge (\xi)|^2 |\xi|^{d-1} \ d\xi.$$  

Applying Theorem 6.2, we can bound the above by

$$\lesssim 2^{-k(d-1)} \int |(P_j F)^\wedge (\xi)|^2 \ d\xi$$

$$\sim 2^{-k(d-1)} \int_{|\xi| \leq 2^k} |((f\chi_{E_\delta}) \ast \rho_\delta)^\wedge (\xi)|^2 \ d\xi.$$  

Now

$$\| (P_k (T(P_j F))^\wedge \|_{L^2(E_\delta)} \lesssim 2^{-k(d-1)} \sum_{2^k \leq \frac{1}{\delta}} 2^{d(\alpha - 1)} \| f \|_{L^2(E_\delta)} \| g \|_{L^2(E_\delta)}.$$  

(6.45)

Since $|j - k| \leq K$, we can plug this expression into the sum (6.43) and get an upper bound of $\delta^{d-\alpha} \| f \|_{L^2(E_\delta)} \| g \|_{L^2(E_\delta)}$. In the regime $|j - k| \leq K$ and $\alpha \leq \frac{d+1}{2}$, setting $j = k$ and summing in $2^k \leq \frac{1}{\delta}$ yields

$$\sum_{k=0}^K 2^{d(\alpha - 1)} \left( \frac{1}{\delta} \right)^{(d-\alpha) - \left( \frac{d+1}{2} \right)} \sim \left( \frac{1}{\delta} \right)^{(d-\alpha) - \left( \frac{d+1}{2} \right)},$$

and plugging this into (6.43) yields

$$\left( \frac{1}{\delta} \right)^{(d-\alpha) - \left( \frac{d+1}{2} \right)} \delta^{\frac{d(\alpha - 1)}{2}} \| f \|_{L^2(E_\delta)} \delta^{\frac{d(\alpha - 1)}{2}} \| g \|_{L^2(E_\delta)}.$$

$$= \delta^{2(d-\alpha) - 1} \| f \|_{L^2(E_\delta)} \| g \|_{L^2(E_\delta)}.$$  

**Case 2:** $|j - k| > K$: In the case that $|j - k| > K$, we will expand the Fourier transform $(P_k [(f\chi_{E_\delta}) \ast \rho_\delta)]^\wedge (\xi)$ and examine the critical points of the phase function. We follow the proof presented in [18], with the only major change being
that we utilize our assumption that $E$ is $\{\delta_i\}$-discrete $\alpha$-regular. Using Fourier inversion, we can write

$$T(P_j [(f \chi_{E_j}) * \rho_\delta])(x) = \frac{1}{\delta} \int \psi \left( \frac{\phi(x,y) - 1}{\delta} \right) P_j(f \chi_{E_j} * \rho_\delta)(y) \psi_0(x,y) dy$$

$$= \frac{1}{\delta} \int e^{2\pi i (\phi(x,y)-1)x+\zeta y} \hat{\psi}(\delta s) (P_j(f \chi_{E_j} * \rho_\delta))^\wedge(\zeta) \, ds \, d\zeta \, dy.$$ 

Now

$$(T(P_j [(f \chi_{E_j}) * \rho_\delta]))^\wedge(\xi) \sim \int e^{2\pi i (\phi(x,y)-1)x+\zeta y} \hat{\psi}(\delta s) (P_j(f \chi_{E_j} * \rho_\delta))^\wedge(\zeta) \psi_0(x,y) \, ds \, d\zeta \, dy.$$ 

Multiplying both sides by $\eta(2^{-k})$, we see that

$$\eta(2^{-k} \xi) \cdot (T(P_j [(f \chi_{E_j}) * \rho_\delta]))^\wedge(\xi) \sim \eta(2^{-k} \xi) \int I(s,\xi,\zeta) \hat{\psi}(\delta s) (P_j(f \chi_{E_j} * \rho_\delta))^\wedge(\zeta) \, ds \, d\zeta,$$

where $I(s,\xi,\zeta) := \int e^{2\pi i (\phi(x,y)-1)s+\zeta y} \psi_0(x,y) \, dx \, dy$.

We note that $j$ is restricted to $2^j \leq C_2 \frac{1}{\delta}$. Indeed, recalling that

$$(P_j [(g \chi_{E_j}) * \rho_\delta])^\wedge(\zeta) = \eta(2^{-j} \zeta)(g \chi_{E_j})^\wedge(\zeta) \hat{\rho}(\delta \zeta),$$

it follows that $2^j \leq \frac{1}{\delta}$.

We use the following Lemma, which appears in [18] (see Lemma 2.5), to bound $|I(s,\xi,\zeta)|$.

**Lemma 6.3.** Suppose that $|\zeta| \sim 2^j$ and $|\xi| \sim 2^k$. Then there exists a $K > 0$ so that if $|j - k| > K$, then for each positive integer $M$, there exists a positive constant $c_M > 0$ so that

$$|I(s,\xi,\zeta)| \leq c_M \inf \{|s|^{-M}, 2^{-jM}, 2^{-kM}\}.$$ 

With Lemma 6.3 in tow, we return to the estimate above. Plugging in the estimate from the lemma and integrating in $s$, we have

$$|\eta(2^{-k} \xi) \cdot (T(P_j [(f \chi_{E_j}) * \rho_\delta]))^\wedge(\xi)| \leq \eta(2^{-k} \xi) \min \left\{ 2^{-j(M-1)}, 2^{-k(M-1)} \right\} \int_{|\zeta| \sim 2^j} |(f \chi_{E_j} * \rho_\delta)^\wedge(\zeta)| \, d\zeta.$$ 

Finally, applying Cauchy-Schwarz, we can bound this expression above by

$$\leq \eta(2^{-k} \xi) \min \left\{ 2^{-j(M-1)}, 2^{-k(M-1)} \right\} \frac{2^4}{\pi} \left( \int_{|\zeta| \sim 2^j} |(f \chi_{E_j} * \rho_\delta)^\wedge(\zeta)|^2 \, d\zeta \right)^{1/2}$$

$$\leq \eta(2^{-k} \xi) \min \left\{ 2^{-j(M-1)}, 2^{-k(M-1)} \right\} \frac{2^4}{\pi} \left( 2^4 \delta \right)^{\frac{(d-\alpha)}{2}} \|f\|_{L^2(E_j)}.$$ 

It follows that

$$\| (P_k (T(P_j F)))^\wedge \|_{L^2(E_j)} \leq 2^{\frac{4d}{\pi}} \min \left\{ 2^{-j(M-1)}, 2^{-k(M-1)} \right\} \frac{2^4}{\pi} \left( 2^4 \delta \right)^{\frac{(d-\alpha)}{2}} \|f\|_{L^2(E_j)},$$

and since $M$ can be taken arbitrarily large, the result follows.
Appendix A. Higher dimensional analogue of a Fubini-like theorem

Although a direct substitute of Fubini’s theorem in a fractal setting is not available, the following theorem acts as a sort of substitute; it can be found in \cite{8} (see page 72, Theorem 5.8).

Theorem A.1. Let $A$ be any subset of the $x$-axis and let $B$ be a subset of the plane. For $x \in \mathbb{R}$, let $B_x$ denote the linear set $\{y : (x, y) \in B\}$. Suppose that there exists a constant $c$ so that, for each $x \in A$, it holds that $\mathcal{H}^t(B_x) \geq c$. Then

$$\mathcal{H}^{s+t}(B) \geq bc^s \mathcal{H}^s(A),$$

where $b$ depends only on $s$ and $t$.

In this paper, we require a higher-dimensional analogue of this theorem, which to the best of our knowledge does not seem to appear in the literature. For this reason, we include the statement and outline the proof for completeness. The utility of this theorem is demonstrated by Theorem 1.2, 1.3, and Proposition 1.5.

Theorem A.2. Let $d \geq 2$ and $1 \leq k \leq (d-1)$. Let $A$ be any Borel subset of $\mathbb{R}^{d-k}$ and let $B$ be a Borel subset of $\mathbb{R}^d$. For $x \in \mathbb{R}^{d-k}$, set

$$B_x := \{(y_1, \ldots, y_k) : (x, y_1, \ldots, y_k) \in B\}.$$

Suppose that there exists some constant $c > 0$ so that, for each $x \in A$, it holds that $\mathcal{H}^t(B_x) \geq c$. Then

$$\mathcal{H}^{s+t}(B) \geq bc^s \mathcal{H}^s(A),$$

where $b$ depends only on $k$, $s$, $t$, and the ambient dimension $d$.

The following is an immediate corollary.

Corollary A.3. Let $A$, $B$, and $B_x$ as in Theorem A.2. Suppose that there exists $t \geq 0$ so that, for each $x \in A$, it holds that $\dim \mathcal{H}(B_x) \geq t$. Then

$$\dim \mathcal{H}(B) \geq t + \dim \mathcal{H}(A).$$

Rather than working with Hausdorff measures directly, we work with a comparable measure that is defined by coverings of a set by binary intervals. A binary half-open cube is a set of the form

$$[2^{-k}m_1, 2^{-k}(m_1 + 1)) \times [2^{-k}m_2, 2^{-k}(m_2 + 1)) \times \cdots \times [2^{-k}m_d, 2^{-k}(m_d + 1)),$$

where $m_1, \ldots, m_d$ are integers and $k$ is a non-negative integer. Define an outer measure on $\mathbb{R}^d$ by

$$\mathcal{M}^s(F) = \lim_{\delta \to 0} \mathcal{M}^s_\delta(F),$$

where

$$\mathcal{M}^s_\delta(F) = \inf \sum_{i=1}^{\infty} |C_i|^s,$$

where the infimum is taken over all countable $\delta$-coverings of $F$ by half-open binary cubes $\{C_i\}$ with length denoted by $|C_i|$.

Observe that, for each Borel set $F$, the measure $\mathcal{M}^s(F)$ is comparable to $\mathcal{H}^s(F)$. Indeed, since a covering of a set $F$ by binary intervals is an admissible cover in the definition of Hausdorff measures, it follows that $\mathcal{H}^s_\delta(F) \leq \mathcal{M}^s_\delta(F)$. 


Conversely, since any axes parallel box has the property that each side (call such a side $I$), is contained in the union of two consecutive half-open binary intervals of length at most $2|I|$, we have $2^{s+1}H_2^s(F) \geq M_2^s(F)$.

The following Lemma is a higher dimensional analogue of Lemma 5.7 in [8]. It is a straightforward exercise to verify that the proof there can be recycled to prove Lemma A.4.

**Lemma A.4.** Let $d \geq 1$ and $0 \leq s \leq d$. Let $A$ be any subset of $\mathbb{R}^d$, let $\{I_i\}$ be a countable $\delta$-cover of $A$ by binary cubes, and let $\{a_i\}$ be a sequence of positive numbers. Suppose $c$ is a constant such that

$$\sum_{i : x \in I_i} a_i > c$$

for all $x \in A$. Then

$$\sum_{i} a_i |I_i|^s \geq cM_2^s(A).$$

**Proof of Theorem A.2.** For simplicity of presentation, we consider the case when $A$ is a subset of the $x$ axis. We present the proof in such a way that the more general case is a natural extension. Let $A$ and $B$ be as in the statement of the theorem. Let $c > 0$ so that, for each $x \in A$, it holds that

$$H^t(B_x) \geq c.$$  

Let $\delta > 0$ and let $\{C_i\}$ be a countable covering of $B$ by cubes of diameter at most $\delta$. Denote

$$\pi(C_i) := \pi_1(C_i) := \{y_1 : (y_1, y_2, \ldots, y_d) \in C_i, \text{ for some } y_2, \ldots, y_d\},$$

and

$$(C_i)_x := C \cap \{(x, y_2, \ldots, y_d) : (y_2, \ldots, y_d) \in \mathbb{R}^{d-1}\},$$

the projection of $C_i$ onto the first coordinate and the slice of $C_i$ by an axis-parallel hyperplane determined by $x$, respectively.

The main ingredients of the proof are hypothesis (1.48) coupled with the following simple observations. Observe that, for each $x \in \mathbb{R}$, the diameters of $C_i$, $(C_i)_x$, and $\pi(C_i)$ are all comparable. (We will use $|\cdot|$ to denote the diameter of a cube.) Observe that the arbitrary covering of $B$ by cubes $\{C_i\}$ yields a natural cover of $A$ and of the sets $(B)_x$. Indeed,

$$A \subset \bigcup_{i} \pi(C_i)$$

and, for each $x \in A$,

$$B_x \subset \bigcup_{i} (C_i)_x.$$  

We begin by verifying the hypothesis of Lemma A.4 with $a_i = |C_i|^t$ and $I_i = \pi(C_i)$. By equation (1.49) and the definition of Hausdorff measure,

$$H_\delta^t(B_x) \leq \sum_{i} |(C_i)_x|^t.$$  

Set $A_\delta = \{x \in A : H_\delta^t(B_x) \geq c\}$. Now, if $x \in A_\delta$, we have

$$c \leq H_\delta^t(B_x) \leq \sum_{i} |(C_i)_x|^t \sim \sum_{i : x \in \pi(C_i)} |C_i|^t.$$
The above argument is true if we restrict to a covering of the set $B$ by binary cubes, which we will do to apply Lemma A.3. Applying Lemma A.3, we obtain the following lower bound:

$$\sum_i |C_i|^{s+t} \sim \sum_i |C_i|^{\tau} |\pi(C_i)|^s \gtrsim cM^s_\delta(A_\delta).$$

This is true for any covering of $B$ by binary cubes $\{B_i\}$, so

$$\mathcal{M}^{s+t}(B) \gtrsim cM^s_\delta(A_\delta).$$

We note that $A_\delta$ increases to $A$ as $\delta$ decreases to 0, and apply a limiting argument to complete the proof. In conclusion, we have

$$\mathcal{H}^{s+t}(B) \gtrsim c\mathcal{H}^s(A).$$

\[\square\]

**Appendix B. An example concerning the parabolic metric**

In this subsection, we give a lower bound related to Theorem 1.19 for $\alpha < \frac{d+1}{2}$. The plan is to construct a $\{\delta_i\}$-discrete $\alpha$-regular set along with a metric $\phi$ so that the distance 1 is repeated often. In particular, using the notation introduced in Section 1.2.3, we give construction of a metric $\phi$ and a set $E$ so that the upper Minkowski dimension of $VS^\alpha_{\ell t}(E)$ is bounded below by $\alpha + \frac{\alpha(d-1)k}{d+1}$. We set $\ell = 1$, but the example presented here can be modified to work for any $\ell \in \mathbb{R}^k$ with positive components. We modify the construction used in [3] to the setting of chains.

To define the set $E$, let $\{q_i\}_{i \in \mathbb{N}}$ be a sequence of positive integers such that $q_{i+1} = q_i^2$ and $q_1 = 2$. Set $L_i$ equal to the truncated and non-isotropically scaled lattice,

$$L_i := \left\{ \left( \frac{x}{q_i} \right) : x \in \mathbb{Z}^d, 0 \leq x_1, \ldots, x_{d-1} \leq q_i^{\frac{d-1}{2}} \text{ and } 0 \leq x_d \leq q_i^\frac{d}{2} \right\},$$

and set $E_i$ equal to the $q_i^{\frac{-d}{2}}$ neighborhood of $L_i$. It is known that $E = \bigcap_i E_i$ is a construction of a set of Hausdorff dimension $\alpha$ (see, for example, [3] Chapter 8, Theorem 8.15), and it is a simple calculation to check that $E$ is $\{\delta_i\}$-discrete $\alpha$-regular (with $\delta_i = q_i^{-\frac{d}{2}}$).

The function $\phi$ is defined using a variant of the incidence construction due to P. Valtr [32]. In particular, $\phi(x, y) = ||x - y||_B$, where $|| \cdot ||_B$ is the norm induced by a convex body $B$. Roughly speaking, $B$ is created by gluing two copies of a paraboloid together. Explicitly, let

$$B_U = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i \in [-1, 1], \text{ for } 1 \leq i \leq d-1, \text{ and } x_d = 1 - \left( x_1^2 + x_2^2 + \cdots + x_{d-1}^2 \right) \},$$

and

$$B_L = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i \in [-1, 1], \text{ for } 1 \leq i \leq d-1, \text{ and } x_d = -1 + x_1^2 + x_2^2 + \cdots + x_{d-1}^2 \}.$$

Now, let

$$B' = \left( B_U \cap \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_d \geq 0 \} \right) \cup \left( B_L \cap \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_d \leq 0 \} \right).$$
Finally, define $B$ to be the convex body $B'$, with the ridge at the transition between $B_U$ and $B_I$, smoothed.

The problem of calculating $\overline{\dim}_M(\mathcal{V}_s^{\phi,k}(E))$ reduces to the problem of calculating the number of balls of radius $q_i^{-\frac{d}{\alpha}}$ needed to cover $\mathcal{V}_s^{\phi,k}(L_i)$. Specifically, for each $i$ and for each $x \in L_i$,

$$N \left( \{ y \in L_i : \| x - y \|_B = 1 \} , q_i^{-\frac{d}{\alpha}} \right) \sim q_i^{-\frac{d(d-1)}{d+1}} ,$$

where $N(A, \delta)$ denotes the number of $\delta$-balls needed to cover a compact set $A$. More generally,

$$N \left( \mathcal{V}_s^{\phi,k}(L_i), q_i^{-\frac{d}{\alpha}} \right) \sim q_i^{-\frac{k(d+1)}{d+1}} ,$$

where $\mathcal{V}_s^{\phi,k}(L_i)$ denotes the vertex set of the $k$-chain with $x$, the first vertex, pinned. Thus, the number of balls of radius $q_i^{-\frac{d}{\alpha}}$ needed to cover $\mathcal{V}_s^{\phi,k}(L_i)$ and, consequently, $\mathcal{V}_s^{\phi,k}(E)$, is

$$\geq q_i^{-\frac{d}{\alpha}} \cdot q_i^{-\frac{d(d-1)}{d+1}} = (q_i^{-\frac{d}{\alpha}})^{\frac{k(d+1)}{d+1}} .$$

Here, we used the observation that, for each $i$, $E$ contains $\sim q_i^{-\frac{d}{\alpha}}$ many elements of the set of scaled lattice points, $L_i$. It follows that $\overline{\dim}_M(\mathcal{V}_s^{\phi,k}(E))$ is at least $\frac{d}{\alpha(d+1)} + \frac{d(d-1)k}{d+1}$.

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