AFFINE GEOMETRY AND FROBENIUS ALGEBRA

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Abstract. The associativity of the multiplication on a Frobenius manifold is equivalent to the WDVV equation of a symmetric cubic form in flat coordinates. Frobenius manifold could be regarded a very special type of statistical manifold. There is a natural commutative product on each tangent space of a statistical manifold. We show that it is associative, hence making it into a manifold with Frobenius algebra structure, if and only if the sectional $K$-curvature vanishes. In other words, WDVV equation is equivalent to zero sectional $K$-curvature. This gives a curvature interpretation for WDVV equation.

1. Introduction

The classical affine differential geometry studies properties of hypersurfaces in $\mathbb{R}^{n+1}$ that are invariant under affine transformations. The work of Calabi and Cheng-Yau on affine differential geometry played an important role in Yau’s proof of the Calabi conjecture.

What we are going to study in this paper is usually called information geometry or statistical geometry, which belongs to affine differential geometry in the broad sense.

A statistical manifold is defined to be a Riemannian manifold equipped with two torsion-free connections dual to each other with respect to the Riemannian metric, or equivalently defined as a Riemannian manifold with a totally symmetric $(1,2)$-tensor. It was so named because some early examples of stastitical manifolds are from families of probability distributions. Locally strictly convex equiaffine hypersurfaces in $\mathbb{R}^n$ and Hessian manifolds are examples of statistical manifolds. Many results in classical affine geometry could be studied from the point of view of statistical manifolds. If the two dual connections are both flat, we arrived at the dually flat structure introduced by Amari-Nagaoka [2] with wide applications in modern information geometry, statistics and related fields.

Hessian metric was studied by Cheng-Yau [4] as an analogue of a Kähler metric for flat affine structures. Cheng-Yau also proved existence and uniqueness theorems for Hessian metric, which played important role in mirror symmetry.

Frobenius manifold, or more generally manifold with Frobenius algebra structure on their tangent bundle, is also a statistical manifold, as they both have a symmetric $(0,3)$-tensor. But this connection seems was largely overlooked, partly because the definition of Frobenius manifold requires several very restrictive properties and mostly being studied in the complex analytic setting.

Dubrovin [7] introduced Frobenius manifolds as a geometric framework for WDVV equation in 2D topological field theory, thus unified Saito’s unfolding spaces of singularities [16] and quantum cohomology [12]. Another prominent example of Frobenius manifold is Bammnikov-Kontsevich’s construction [3] from Batalin-Vilkovisky algebras in mirror symmetry.

Dubrovin [7, p.311] remarked the similarity of the first structure connection on Frobenius manifold and the $\alpha$-connection on statistical manifold, both of which are linear deformations of Levi-Civita connection.

A statistical manifold has natural commutative product on its tangent space [5] [6] [11] [13]. Combe-Manin [6] showed that various spaces of probability distributions carry natural structures of $F$-manifolds, a weakened version of Frobenius manifold. Jiang-Tavakoli-Zhao [11] and
Nakajima-Ohmoto [13] showed that dually flat structure (with flat metric) implies Frobenius algebraic structure. Combe-Combe-Nencka [5] proved that statistical manifolds, related to exponential families and with flat structure connection have a Frobenius manifold structure.

The vanishing of WDVV equation characterized those statistical manifolds with Frobenius structure on the tangent space. In this paper, we give another characterization in terms of the sectional $K$-curvature, a concept from affine differential geometry due to Opozda [15]. More precisely, we show that the associativity holds if and only if the sectional $K$-curvature is zero. In fact, the latter condition means the commutativity of left multiplication operators. We provide alternative proofs of some results of Opozda [15] with the help of Frobenius structure. The 2-dimensional example at the end of the paper shows that at least locally, given any Riemannian metric, there is abundance of statistical structure whose tangent space has Frobenius structure.

2. Statistical structure and sectional $K$-curvature

Throughout the paper, a connection always means an affine connection on the tangent bundle. We follow the notation of [15]. Let $\nabla$ be a connection on a Riemannian manifold $(M, g)$. Then the dual connection $\nabla^*$ is defined to be the unique connection that satisfies

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

If $\nabla$ is a torsion-free connection on $(M, g)$, then $\nabla^*$ is torsion-free if and only if $\nabla g$ is symmetric as a $(0, 3)$-tensor. A triple $(M, g, \nabla)$ is called a statistical manifold if both $\nabla$ and $\nabla^*$ are torsion-free.

On a statistical manifold, the Levi-Civita connection $\hat{\nabla}$ satisfies $\hat{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$. The Amari-Chentsov tensor $T$ is the difference of Christoffel symbols of $\nabla$ and $\nabla^*$, namely

$$T_{ijk} = \Gamma^h_{ij} g_{hk}.$$  \hspace{1cm} (2.1)

Here $\Gamma_{ijk} = \Gamma^h_{ij} g_{hk}$ and similarly for $\Gamma_{ijk}$.

Lemma 2.1. [2, Theorem 6.1] On a statistical manifold $(M, g, \nabla)$, the Amari-Chentsov tensor satisfies

$$T_{ijk} = \nabla_i g_{jk}.$$  \hspace{1cm} (2.2)

In particular, the Amari-Chentsov tensor is symmetric.

Proof. We have

$$\nabla_i g_{jk} = \partial_i g_{jk} - \Gamma_{ikj} - \Gamma_{ij}.$$  \hspace{1cm} (2.2)

and

$$0 = \hat{\nabla}_i g_{jk} = \partial_i g_{jk} - \hat{\Gamma}_{ikj} - \hat{\Gamma}_{ij}.$$  \hspace{1cm} (2.2)

Since $\hat{\Gamma}_{ij} - \Gamma_{ij} = \frac{1}{2} T_{ijk}$, we have

$$\nabla_i g_{jk} = \frac{1}{2} (T_{ikj} + T_{ijk}).$$ \hspace{1cm} (2.2)

Since $\nabla_i g_{jk}$ is symmetric for $i$ and $k$, the above equation implies

$$T_{ikj} + T_{ijk} = T_{kij} + T_{kji}.$$  \hspace{1cm} (2.2)

By definition $T_{ijk}$ is symmetric for $i$ and $j$, the above equation implies

$$T_{ijk} = T_{kji}.$$  \hspace{1cm} (2.2)

Namely $T_{ijk}$ is also symmetric for $i, k$. Hence $T$ is symmetric and we get $\nabla_i g_{jk} = T_{ijk}$ from (2.2). \hspace{1cm} $\square$
Another commonly used tensor is $K_X Y$ (also denoted by $K(X,Y)$)

$$K_X Y := \nabla_X Y - \hat{\nabla}_X Y$$

Obviously $T(X, Y, Z) = -2g(K(X, Y), Z)$.

A statistical manifold can be equivalently defined as a triple $(M, g, K)$ where $K$ is a $(1, 2)$-tensor such that $g(K(X, Y), Z)$ is symmetric in $X,Y,Z$. This is because if such $K$ is given, then $\nabla = \hat{\nabla} + K$ is a torsion-free connection and

$$\nabla_X g(Y, Z) = \hat{\nabla}_X g(Y, Z) - g(K_X Y, Z) - g(Y, K_X Z) = -2g(K(X, Y), Z).$$

In particular, $\nabla g$ is symmetric, hence $\hat{\nabla}$ is torsion-free.

The Riemannian curvature tensors of $\nabla$ and $\hat{\nabla}$ satisfy [14 Proposition 4.6]

$$g(R(Z, W)X, Y) + g(\hat{R}(Z, W)Y, X) = 0.$$  \hspace{1cm} (2.3)

In particular, $\nabla$ is flat if and only if $\hat{\nabla}$ is flat. From (2.3), we get the following criterion for $R = \hat{R}$.

**Corollary 2.2.** For a statistical manifold, $R = \hat{R}$ if and only if $g(R(X, Y)Z, W)$ is skew-symmetric for $Z, W$.

We also have the following relations of $R, \hat{R}$ and $\hat{\nabla}$.

**Lemma 2.3.** [15] On a statistical manifold, the Riemannian curvature tensors of $\nabla$, $\hat{\nabla}$ and $\hat{\nabla}$ satisfy

$$R(X, Y) + \hat{R}(X, Y) = 2\hat{R}(X, Y) + 2[K_X, K_Y], \hspace{1cm} (2.4)$$

$$R(X, Y) - \hat{R}(X, Y) = 2(\hat{\nabla}_X K)_Y - 2(\hat{\nabla}_Y K)_X \hspace{1cm} (2.5)$$

**Proof.** Since $\nabla_X Y = \hat{\nabla}_X Y + K_X Y$ and $\nabla_X Y = \hat{\nabla}_X Y - K_X Y$, it is sufficient to prove

$$R(X, Y) = \hat{R}(X, Y) + (\hat{\nabla}_X K)_Y - (\hat{\nabla}_Y K)_X + [K_X, K_Y]. \hspace{1cm} (2.6)$$

We have

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$= \nabla_X (\hat{\nabla}_Y Z + K_Y Z) - \nabla_Y (\hat{\nabla}_X Z + K_X Z) - \hat{\nabla}_{[X,Y]} Z - K_{[X,Y]} Z$$

$$= \hat{\nabla}_X \hat{\nabla}_Y Z + K_X \hat{\nabla}_Y Z + \hat{\nabla}_X K_Y Z + K_X K_Y Z$$

$$- \hat{\nabla}_Y \hat{\nabla}_X Z - K_Y \hat{\nabla}_X Z - \hat{\nabla}_Y K_X Z - K_Y K_X Z - \hat{\nabla}_{[X,Y]} Z - K_{[X,Y]} Z$$

$$= (\hat{\nabla}_X \hat{\nabla}_Y Z - \hat{\nabla}_Y \hat{\nabla}_X Z - \hat{\nabla}_{[X,Y]} Z) + (\hat{\nabla}_X K_Y Z - K_Y \hat{\nabla}_X Z - K_{\hat{\nabla}_X Y} Z)$$

$$- (\hat{\nabla}_Y K_X Z - K_X \hat{\nabla}_Y Z - K_{\hat{\nabla}_Y X} Z) + (K_X K_Y Z - K_Y K_X Z)$$

$$= \hat{R}(X, Y)Z + (\hat{\nabla}_X K)_Y Z - (\hat{\nabla}_Y K)_X Z + [K_X, K_Y] Z.$$ In the next to last equation, we used the torsion-freeness $[X, Y] = \hat{\nabla}_X Y - \hat{\nabla}_Y X$. \hfill \Box$

On a statistical manifold $(M, g, K)$, we have a family of torsion-free connections

$$\nabla_X^{(\alpha)} Y = \hat{\nabla}_X Y + \alpha K_X Y, \quad \alpha \in \mathbb{R}, \hspace{1cm} (2.7)$$

called $\alpha$-connections. In the theory of Frobenius manifolds, these are called Dubrovin’s first structure connections [12] and are assumed to be flat for all $\alpha$.

**Lemma 2.4.** Assume that there exist $\beta, \gamma \in \mathbb{R}$ with $\beta + \gamma \neq 0$ such that $\nabla^{(\beta)}$ and $\nabla^{(\gamma)}$ are flat. Then $\nabla^{(\alpha)}$ are flat for all $\alpha \in \mathbb{R}$.  

Lemma 2.5. Let \( \nabla^{(\alpha)} \) be the Riemannian curvature tensor of \( \nabla \),
\[
R^{(\alpha)}(X,Y) = \frac{1+\alpha}{2} R(X,Y) + \frac{1-\alpha}{2} \overline{R}(X,Y) - (1-\alpha^2)[K_X, K_Y],
\] (2.8)
where \( R(X,Y) \) and \( \overline{R}(X,Y) \) are Riemannian curvature tensors of \( \nabla = \nabla^{(1)} \) and \( \overline{\nabla} = \nabla^{(-1)} \) respectively. When \( \alpha = 0 \), (2.8) is just (2.4).

Proof. Recall Zhang’s formula [18] of Riemannian curvature tensor of \( \nabla^{(\alpha)} \),
\[
R^{(\alpha)}(X,Y) = \frac{1+\alpha}{2} R(X,Y) + \frac{1-\alpha}{2} \overline{R}(X,Y) - (1-\alpha^2)[K_X, K_Y],
\] (2.8)
where \( R(X,Y) \) and \( \overline{R}(X,Y) \) are Riemannian curvature tensors of \( \nabla = \nabla^{(1)} \) and \( \overline{\nabla} = \nabla^{(-1)} \) respectively. When \( \alpha = 0 \), (2.8) is just (2.4).

From (2.8), we get for all \( \alpha \in \mathbb{R} \),
\[
R^{(\alpha)}(X,Y) = \alpha(R(X,Y) - \overline{R}(X,Y)).
\] (2.9)

Since \( \nabla^{(-\alpha)} \) is dual to \( \nabla^{(\alpha)} \), we have that \( \nabla^{(-\beta)} \) and \( \nabla^{(-\gamma)} \) are also flat. Taking \( \alpha = \beta \) and \( \alpha = \gamma \) in (2.9), we get \( R(X,Y) = \overline{R}(X,Y) \). Hence (2.8) becomes
\[
R^{(\alpha)}(X,Y) = R(X,Y) - (1-\alpha^2)[K_X, K_Y].
\] (2.10)

Taking \( \alpha = \beta \) and \( \alpha = \gamma \) in (2.10) and noting that \( 1-\beta^2 \neq 1-\gamma^2 \), we get \( [K_X, K_Y] = 0 \). Then (2.10) implies that \( R^{(\alpha)}(X,Y) = R(X,Y) \) for all \( \alpha \in \mathbb{R} \). So \( R^{(\alpha)}(X,Y) = 0 \) for all \( \alpha \in \mathbb{R} \).

On a statistical manifold, Opozda [15] introduced the concept of sectional \( K \)-curvature. First, Opozda noted that the \((1,3)\)-tensor \([K, K] \) given by
\[
[K, K](X,Y) := [K_X, K_Y]X = K_XK_Y Z - K_YK_X Z
\] (2.11)
satisfies the same properties as the Riemannian curvature tensor, except the second Bianchi identity.

Lemma 2.6. [15] The following equations of \([K, K] \) hold.

(i) \([K, K](X,Y) = -[K, K](Y,X) \).
(ii) \([K, K](X,Y)Z + [K, K](Y,Z)X + [K, K](Z,X)Y = 0 \).
(iii) \( g([K, K](X,Y)Z, W) = -g([K, K](X,Y)W, Z) \).
(iv) \( g([K, K](X,Y)Z, W) = g([K, K](W,Z)Y, X) \).

Proof. (i) follows from the definition. (ii) follows from (2.4), since the first Bianchi identity holds for all of \( R, \overline{R} \) and \( \hat{R} \).

For (iii), we first apply (2.4) and then (2.3) to get
\[
g([K, K](X,Y)Z, W) + g([K, K](X,Y)W, Z)
= \overline{g}(R(X,Y)Z, W) + \overline{g}(\overline{R}(X,Y)Z, W) - 2\overline{g}(\hat{R}(X,Y)Z, W)
= -\overline{g}(\overline{R}(X,Y)W, Z) + \overline{g}(\overline{R}(X,Y)Z, W) + \overline{g}(\overline{R}(X,Y)Z, W)
= 0.
\]

For (iv), we use the symmetry of \( K \) to get
\[
g([K, K](X,Y)Z, W) = g(K_XK_Y Z, W) - g(K_YK_X Z, W)
= g(K(X,W), K(Y,Z)) - g(K(Y,W), K(X,Z)).
\]

Apply the above calculation to \( g([K, K](W,Z)Y, X) \), we get the same expression. □

It follows that for any plane \( \pi \) in the tangent space \( T_p M \), we can define its sectional \( K \)-curvature
\[
k(\pi) = g([K, K](e_1,e_2)e_2,e_1), \] (2.12)
where \( e_1, e_2 \) is an orthonormal basis of \( \pi \). Note that \( k(\pi) \) is independent of the choice of the orthonormal basis.

Lemma 2.6. The following statements are equivalent (at a given point \( p \in M \)).
Remark 2.9. for $1$ to that the sectional curvatures of all planes at a point determines the tensor $[K, K]$. It is well-known that the Riemannian curvature tensor is determined by sectional curvatures.

Lemma 2.7. For a statistical manifold $(M, g, K)$, the following statements are equivalent.

(i) The sectional $K$-curvatures are constantly zero.
(ii) $[K, K] = 0$.
(iii) $R(X, Y) + \overline{R}(X, Y) = 2\hat{R}(X, Y)$.

In fact, the equivalence holds pointwise.

Proof. The equivalence of (ii) and (iii) follows from (2.4). The equivalence of (i) and (ii) amounts to that the sectional $K$-curvatures of all planes at a point determines the tensor $[K, K]$. If $S(X, Y, Z, W) = g([K, K](X, Y)W, Z)$. In fact, one can prove that if $S$ is a $(0, 4)$-tensor with properties of Lemma 2.5 and $S(X, Y, X, Y) = 0$ for all $X, Y \in T_pM$, then $S = 0$ at $p$. The argument is the same as that for the Riemannian curvature tensor.

The following lemma is the key tool used by Opozda [15] to study statistical manifolds with constant sectional $K$-curvature.

Lemma 2.8. [15] At a given point $p$ of a statistical manifold $(M, g, K)$, if the sectional $K$-curvature is equal to constant $A$, then there is an orthonormal basis $e_1, ..., e_n$ of $T_pM$ and numbers $\lambda_i$, $\mu_i$ for $1 \leq i \leq n$ such that

$$K(e_i, e_i) = \mu_1 e_1 + ... + \mu_{i-1} e_{i-1} + \lambda_i e_i$$

for $1 \leq i \leq n$ and

$$K(e_i, e_j) = \mu_i e_j$$

for $1 \leq i < j \leq n$. Moreover $\mu_i$ are determined by $\lambda_k$ and $A$ through the formula

$$\mu_i = \frac{\lambda_i - \sqrt{\lambda_i^2 - 4A_{i-1}}}{2}$$

for $1 \leq i \leq n$ and $A_0 = A$. In (2.15), we require $\lambda^2_i - 4A_{i-1} \geq 0$.

Remark 2.9. The proof is by induction. Opozda’s method to obtain the orthonormal basis $e_1, ..., e_n$ in the above lemma is as follows. Let

$$C(X, Y, Z) = g(K(X, Y), Z).$$

Then $C$ is a symmetric $(0, 3)$-tensor. Denote by $S^1$ the unit sphere in $T_pM$ and by $\Phi$ the function $\Phi(X) = C(X, X, X)$ on $S^1$. The vector $e_1 \in S^1$ is any unit vector at which $\Phi$ attains a local maximum, and $e_2 \in \{e_1\}^\perp \cap S^1$ is any unit vector at which $\Phi_{\{e_1\}^\perp \cap S^1}$ attains a local maximum, etc.

Corollary 2.10. [15] If $A = 0$ in Lemma 2.8, then there is an orthonormal basis $e_1, ..., e_n$ of $T_pM$ and numbers $\lambda_i$, $\mu_i$ for $1 \leq i \leq n$ such that

$$K(e_i, e_i) = \lambda_i e_i, \quad K(e_i, e_j) = 0$$

for $i \neq j$.

Corollary 2.10 follows immediately from Lemma 2.8. A non-inductive proof is as follows. Since $g(K_X Y, Z) = g(Y, K_X Z)$, we see that $K_X$ is self-adjoint and hence diagonalizable. Since $[K, K] = 0$, these $K_X$ commute with each other, hence they are simultaneously diagonalizable with respect to an orthonormal basis $e_1, ..., e_n$. 
Lemma 2.12. If \( K \) is symmetric, then the corresponding \((0,3)\)-tensor
\[
C(X,Y,Z) = g(K(X,Y),Z)
\]
is symmetric.

The following lemma is expected.

Lemma 2.12. If \( K \) has expression as in \((2.13)\) and \((2.14)\) with numbers \( \lambda_i, \mu_i \) satisfying \((2.15)\) and \((2.16)\), then \( K \) is totally symmetric and has constant sectional \( K \)-curvature equal to \( A \).

Proof. To show \( K \) is totally symmetric, we only need to check
\[
g(K(e_i, e_j), e_k) = g(K(e_i, e_k), e_j).
\]
Without loss of generality, we may assume \( i \leq j \).

If \( i, j, k \) are distinct, then \( i < j \),
\[
g(K(e_i, e_j), e_k) = g(\mu_i e_j, e_k) = 0,
\]
and
\[
g(K(e_i, e_k), e_j) = \begin{cases} g(\mu_i e_k, e_j) = 0 & \text{if } i < k, \\ g(\mu_k e_i, e_j) = 0 & \text{if } i > k. \end{cases}
\]

If exactly two of \( i, j, k \) are equal, we may take \( i = j \), then
\[
g(K(e_i, e_i), e_k) = g(\mu_1 e_1 + \cdots + \mu_{i-1} e_{i-1} + \lambda_i e_i, e_k) = \begin{cases} 0 & \text{if } i < k, \\ \mu_k & \text{if } i > k. \end{cases}
\]
and
\[
g(K(e_i, e_k), e_i) = \begin{cases} g(\mu_i e_k, e_i) = 0 & \text{if } i < k, \\ g(\mu_k e_i, e_i) = \mu_k & \text{if } i > k. \end{cases}
\]

In both cases, we have proved \((2.19)\).

Now we show that \( K \) has constant sectional \( K \)-curvature equal to \( A \). By Lemma 2.6, it is sufficient to prove that for all \( i, j, k, l, \)
\[
g(K(e_i, e_k), K(e_j, e_l)) - g(K(e_j, e_k), K(e_i, e_l)) = A(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}).
\]
The left-hand side of \((2.20)\) is just \( g([K, K](e_i, e_j), e_k) \).

By the symmetries of \([K, K]\) in Lemma 2.5, we could assume \( i < j, k < l, i \leq k \). Denote by \( LH \) the left-hand side and \( RH \) the righ-hand side of \((2.20)\) respectively. We treat six cases separately.

Case 1. \( i < j < k < l \).
Case 2. \( i < j = k < l \).
Case 3. \( i < k < j < l \).

In all of the above three cases, obviously \( LH = RH = 0 \).

Case 4. \( i < k < j = l \). Then \( RH = 0 \) and
\[
LH = g(\mu_i e_k, \mu_k e_k) - g(\mu_k e_i, \mu_i e_l) = 0.
\]
Case 5. \( i = k < j = l \). Then \( RH = A \). Using \( \lambda_i \mu_i - \mu_i^2 = A_{i-1} \) and \( \mu_i^2 + A_i = A_{i-1} \),
\[
LH = g(K(e_k, e_k), K(e_j, e_j)) - g(K(e_j, e_k), K(e_k, e_j))
= (\mu_1^2 + \cdots + \mu_{k-1}^2 + \lambda_k \mu_k) - \mu_k^2
= \mu_1^2 + \cdots + \mu_{k-1}^2 + A_{k-1}
= A.
\]

Case 6. \( i \leq k < l < j \). Then obviously \( LH = RH = 0 \). \( \square \)
Consider a family of probability distributions

\[ M = \{ p(x, \theta) \mid \theta \in \Theta \}, \quad \Theta \text{ is a domain in } \mathbb{R}^n \]

For each \( \theta \), \( p(x, \theta) \) is defined on a measure space \( \mathcal{X} \) and satisfies \( \int_{\mathcal{X}} p(x, \theta) dx = 1 \).

Let \( \ell_\theta(x) = \log p(x, \theta) \). Under mild conditions, \( M \) is a manifold with a Riemannian metric

\[ g_{ij}(\theta) = \mathbb{E}_\theta \left( \frac{\partial \ell_\theta(x)}{\partial \theta^i} \frac{\partial \ell_\theta(x)}{\partial \theta^j} \right) \int_{\mathcal{X}} p(x, \theta) dx, \]

called the Fisher information matrix. A statistical structure on \( M \) is given by taking

\[ C_{ijk} = \frac{1}{2} \mathbb{E}_\theta \left( \frac{\partial \ell_\theta(x)}{\partial \theta^i} \frac{\partial \ell_\theta(x)}{\partial \theta^j} \frac{\partial \ell_\theta(x)}{\partial \theta^k} \right), \quad (2.21) \]

or equivalently in terms of Amari-Chentsov tensor

\[ T_{ijk} = \mathbb{E}_\theta \left( \frac{\partial \ell_\theta(x)}{\partial \theta^i} \frac{\partial \ell_\theta(x)}{\partial \theta^j} \frac{\partial \ell_\theta(x)}{\partial \theta^k} \right). \quad (2.22) \]

**Example 2.13.** The exponential family consists of probability distributions of the form

\[ p(x, \theta) = \exp \{ Q(x) + \sum_{i=1}^n F_i(x) \theta^i - \varphi(\theta) \} \]

We know that the Fisher information matrix of exponential family is given by

\[ g_{ij}(\theta) = \frac{\partial \varphi}{\partial \theta^i \partial \theta^j}, \]

which is a Hessian metric.

Denote by \( T \) the Amari-Chentsov tensor \( \mathbf{222} \). Then \( T_{ijk}(\theta) = \partial_k (g_{ij}) = \partial_i \partial_j \varphi(\theta) \). The Christoffel symbols and curvature tensors of the \( \alpha \)-connections are given by

\[ \Gamma_{ijk}^{(\alpha)} = \frac{1 - \alpha}{2} T_{ijk}(\theta), \]
\[ R_{ijkl}^{(\alpha)} = \frac{1 - \alpha^2}{4} g^{pq} (T_{ilp} T_{jkq} - T_{ikp} T_{jql}). \]

So both \( \nabla = \nabla^{(1)} \) and \( \overline{\nabla} = \nabla^{(-1)} \) are flat. Namely the exponential family is dually flat.

Write \( S(X, Y, Z, W) = g([K, K](X, Y)W, Z) \). Then by \( \mathbf{224} \),

\[ S_{ijkl} = -\hat{R}_{ijkl} \quad (2.23) \]

Namely the sectional \( K \)-curvature \( k(\pi) \) and the sectional curvature \( \hat{k}(\pi) \) for exponential family are related by \( k(\pi) = -\hat{k}(\pi) \) for any plane \( \pi \).

If we take \( (M, g, \nabla^{(\alpha)}) \) as the statistical structure, namely the totally symmetric \( (1, 2) \)-tensor on \( M \) is now \( \alpha K \). Then \( k(\pi) = -\alpha^2 \hat{k}(\pi) \) for any plane \( \pi \). In fact, this relation holds for any Hessian manifold.

**3. Frobenius structure**

**Definition 3.1.** A Frobenius algebra \( V \) is a finite-dimensional commutative associative algebra (with unit) over \( \mathbb{R} \) (or \( \mathbb{C} \)) that satisfies either of the following two equivalent conditions:

(i) There is a non-degenerate inner product \( g \) such that

\[ g(ab, c) = g(a, bc). \]
(ii) There is a linear form $\theta : V \to \mathbb{R}$ such that

$$g(a, b) = \theta(ab)$$

is a non-degenerate inner product.

**Remark 3.2.** When $g$ is positive-definite, the Frobenius algebra is semisimple, which means that it is isomorphic to $\mathbb{R}^n$ (or $\mathbb{C}^n$) with component-wise multiplication.

A Frobenius manifold is a manifold with a smoothly varying Frobenius algebra structure on the tangent space. The non-degenerate inner product $g$ serves as a pseudo-Riemannian metric. On such manifold, we have a symmetric $(0,3)$-tensor

$$C(X, Y, Z) = g(XY, Z) = \theta(XYZ). \quad (3.1)$$

So it is a statistical manifold (with pseudo-Riemannian metric).

The full definition of a Frobenius manifold requires some additional conditions like $g$ is flat, $\hat{\nabla}C$ is symmetric and $\hat{\nabla}e$, where $e$ is unit vector field. Dubrovin’s first structure connection on a (complex) Frobenius manifold is given by

$$\nabla_X Y = \hat{\nabla}_X Y + \lambda XY,$$

where $\lambda \in \mathbb{C}$. Comparing it with the definition of $\alpha$-connection (2.7) naturally leads to the following definition of a commutative product on the tangent space of a statistical manifold $(M, g, K)$.

$$\partial_i \circ \partial_j = K(\partial_i, \partial_j), \quad (3.2)$$

where $\partial_i = \partial/\partial x^i$.

**Remark 3.3.** The definition (3.2) is essentially the same as [11, Section 4], where instead of $K$ they used the Amari-Chentsov tensor, hence differs with (3.2) by a factor $-2$.

**Proposition 3.4.** On statistical manifold $(M, g, K)$, the product (3.2) is associative if and only if $M$ has zero sectional $K$-curvature. The assertion holds pointwise.

**Proof.** By the symmetry of $K$, we have

$$\partial_i \circ (\partial_j \circ \partial_k) = K(\partial_i, K(\partial_j, \partial_k)) = K(\partial_i, K(\partial_k, \partial_j)) = K_{\partial_i}K_{\partial_k}\partial_j$$

and

$$(\partial_i \circ \partial_j) \circ \partial_k = K(K(\partial_i, \partial_j), \partial_k) = K(\partial_k, K(\partial_i, \partial_j)) = K_{\partial_k}K_{\partial_i}\partial_j.$$ 

Therefore $\partial_i \circ (\partial_j \circ \partial_k) = (\partial_i \circ \partial_j) \circ \partial_k$ if and only if $K_{\partial_i}K_{\partial_k}\partial_j - K_{\partial_k}K_{\partial_i}\partial_j = 0$. The latter is just

$$[K, K](\partial_i, \partial_j)\partial_k = 0,$$

which is equivalent to that the sectional $K$-curvatures are zero by Lemma 2.7. \hfill \square

In fact, Proposition 3.4 is also a consequence of the following purely algebraic statement.

**Proposition 3.5.** If a set $S$ is equipped with a commutative product $S \times S \to S$, then the product is associative if and only if left multiplication operators $l_a$ (where $l_a(x) = ax$) commute.

**Proof.** If the associativity holds, namely $a(bc) = (ab)c$ for all $a, b, c \in S$, then

$$a(bc) = (ab)c = (ba)c = b(ac).$$

We proved $l_a$ and $l_b$ commute.

If left multiplication operators commute, namely $a(bc) = b(ac)$ for all $a, b, c \in S$, then

$$a(bc) = a(cb) = c(ab) = (ab)c,$$

namely the product is associative. \hfill \square
Proposition 3.6. On a statistical manifold \((M, g, K)\) with zero sectional \(K\)-curvature, the product (3.2) has a unit if and only if \(K\) is non-degenerate, i.e., the map \(X \to K_X\) is a monomorphism.

Proof. By Corollary 2.10, \(K\) is non-degenerate if and only if all \(\lambda_i\) in (2.18) are nonzero. The unit \(e\) is given by

\[
e = \sum_{i=1}^{n} \frac{1}{\lambda_i} e_i.
\]

Corollary 2.10 also implies that the algebra structure on \(TM\) is semisimple. □

Remark 3.7. Propositions 3.4 and 3.6 imply that a statistical manifold \((M, g, K)\) with zero sectional \(K\)-curvature (i.e., \([K, K] = 0\)) and non-degenerate \(K\) has a semisimple Frobenius algebra structure on its tangent space.

Remark 3.8. If \(g\) is a pseudo-Riemannian metric, Proposition 3.4 still holds, but Lemma 2.8, Corollary 2.10 and Proposition 3.6 may be no longer true, since their proofs rely on the diagonalizability of \(K_X\).

On a statistical manifold \((M, g, \nabla)\), if we further assume that the affine connection \(\nabla = \hat{\nabla} + K\) is flat, then it is called a Hessian manifold. They are so named because on a Hessian manifold, the Riemannian metric \(g\) can be locally expressed as

\[
g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}.
\]

for some function \(\varphi\) on affine coordinates induced by the flat connection \(\nabla\). Since the dual connection \(\nabla\) of a Hessian manifold is also flat, they are also called dually flat manifold. We already see in Example 2.13 that the exponential family is a Hessian manifold.

Corollary 3.9. The algebra structure (3.2) on a Hessian manifold is associative if and only if the Levi-Civita connection is flat. In such case, the \(\alpha\)-connections are flat for all \(\alpha \in \mathbb{R}\).

Proof. On a Hessian manifold, we have \(R = \overline{R} = 0\). So the first assertion follows from Proposition 3.4 and (2.4). The last assertion is implied by Lemma 2.4. □

The curvature tensor of the Hessian metric (3.4) is given by (see [17])

\[
\hat{R}_{ijkl} = \frac{1}{4} g^{pq}(\varphi_{i|p}\varphi_{jq} - \varphi_{ikp}\varphi_{jql}),
\]

thus the vanishing condition of \(\hat{R}\) is the famous WDVV equation:

\[
g^{pq}(\varphi_{i|p}\varphi_{jq} - \varphi_{ikp}\varphi_{jql}) = 0.
\]

Since WDVV equation is equivalent to the associativity, we got another proof of Corollary 3.9. Although originated from string theory, WDVV equation appears in many problems of differential geometry. See [17, Section 2] for an excellent summary.

It is well-known that canonical basis exists around a semisimple point of a Frobenius manifold. The set of all semisimple points on a Frobenius manifold is open.

Proposition 3.10. [10] Around any point on a manifold with semisimple Frobenius structure, there exists a canonical basis \(u_1, \ldots, u_n\), which are local vector fields that satisfy

\[
u_i^2 = u_i, \quad u_i u_j = 0
\]

for \(i \neq j\). They are unique up to reordering.
A different proof of Proposition 3.10 was given in [15] under the setting of statistical manifold where a technical smoothing argument was used.

Denote $C(X, Y, Z) = g(XY, Z)$ on a manifold with Frobenius structure.

**Proposition 3.11.** [10] In Proposition 3.10, $\nabla C$ is symmetric if and only if
\begin{align}
g(\nabla_u u_i, u_j) &= 0 \quad (3.7) \\
g(\nabla_u u_i, u_j) &= g(\nabla_u u_j, u_i) \quad (3.8) \\
g(\nabla_u u_j, u_j) &= g(\nabla_u u_j, u_i) = g(\nabla_u u_i, u_j) \quad (3.9)
\end{align}
for distinct $i, j, k$.

In fact, when $\nabla C$ is symmetric, there exists a local orthogonal coordinate system $x^1, \ldots, x^n$ such that $u_i = \frac{\partial}{\partial x^i}$ and there is a function $\phi$ such that
\[ g = \sum_i \frac{\partial \phi}{\partial x^i} dx^i \otimes dx^i. \]

Such a metric $g$ is called Darboux-Egoroff metric.

**Corollary 3.12.** Let $(M, g, K)$ be a statistical manifold with non-degenerate $K$ and $[K, K] = 0$. Then $\nabla C = 0$ if and only if $\nabla u_i = 0$ for all $i$. In particular, $\nabla C = 0$ implies that $g$ is flat.

**Proof.** By Remark 3.7, the results of Proposition 3.11 may apply. We follow Hitchin’s argument in the proof of Proposition 3.11. Note that $C(u_i, u_j, u_k) = 0$ unless $i = j = k$.

\[ (\nabla_u C)(u_j, u_k, u_l) = u_l C(u_j, u_k, u_l) - C(\nabla_u u_j, u_k, u_l) - C(u_j, \nabla_u u_k, u_l) - C(u_j, u_k, \nabla_u u_l). \]

Assume $\nabla C = 0$. Let $i \neq j$.

Setting $k = l = i$ gives $g(\nabla_u u_j, u_i) = C(\nabla_u u_j, u_i, u_i) = 0$.

Setting $k = l = j$ and using $u_i g(u_j, u_j) = 2g(\nabla_u u_j, u_j)$ gives $g(\nabla_u u_j, u_j) = 0$.

By (3.7), $g(\nabla_u u_j, u_k) = 0$ for distinct $i, j, k$. So we proved
\[ \nabla_u u_j = 0, \quad i \neq j. \quad (3.10) \]

Setting $i = j = k = l$ gives $u_i g(u_i, u_i) = 3g(\nabla_u u_i, u_i)$. Hence $g(\nabla_u u_i, u_i) = 0$.

Setting $i = j$ and $k = l$ gives $g(\nabla_u u_i, u_j) = 0$ for $i \neq j$. So we proved
\[ \nabla_u u_i = 0, \quad \forall i. \quad (3.11) \]

By (3.10) and (3.11), we proved that $\nabla C = 0$ implies $\nabla u_i = 0$.

On the other hand, if $\nabla u_i = 0$, then $(\nabla_u C)(u_j, u_k, u_l) = 0$ unless $j = k = l$. Note that
\[ (\nabla_u C)(u_j, u_k, u_j) = u_l C(u_j, u_k, u_j) = u_i g(u_j, u_j) = 2g(\nabla_u u_j, u_j) = 0. \]

So we proved $\nabla C = 0$. □

The trace of $K$ is defined to be the vector field $E = \text{tr}_g K$, namely
\[ E^h = g^{ij} K^h_{ij}. \quad (3.12) \]

**Corollary 3.13.** Let $(M, g, K)$ be a statistical manifold with non-degenerate $K$ and $[K, K] = 0$. Then $\nabla C$ is symmetric and $\nabla E = 0$ if and only if $\nabla C = 0$ and $g(u_i, u_i)$ are constants for all $i$. 
Proof. By Corollary 2.10 there is an orthonormal basis $e_1, \ldots, e_n$ such that
$$K(e_i, e_i) = \lambda_i e_i, \quad K(e_i, e_j) = 0$$
for $i \neq j$. Then
$$u_i = \frac{1}{\lambda_i} e_i, \quad g(u_i, u_i) = \frac{1}{\lambda_i^2},$$
where $\lambda_i$ are smooth functions. Let
$$f_{ij} = g(\nabla_{u_i} u_j, u_j).$$
Assume $\nabla C$ is symmetric. Then by (3.7), (3.8) and (3.9), we get $f_{ij} = f_{ji}$ and
$$\nabla_{u_i} u_i = \lambda_i^2 f_{ii} u_i - \sum_{j \neq i} \lambda_j^2 f_{ij} u_j$$
(3.15)
$$\nabla_{u_i} u_j = \lambda_i^2 f_{ij} u_i + \lambda_j^2 f_{ij} u_j.$$ (3.16)
for $i \neq j$. From $u_i \cdot g(u_j, u_j) = 2g(\nabla_{u_i} u_j, u_j) = 2f_{ij}$, we get
$$u_i(\lambda_j^2) = -2\lambda_j^2 f_{ij}.$$ (3.17)
Since $E = \sum_{j=1}^n \lambda_j^2 u_j$, then by (3.12), (3.13) and (3.17), we get
$$\nabla u_i E = \sum_{j=1}^n u_i(\lambda_j^2) u_j + \sum_{j=1}^n \lambda_j^2 \nabla u_i u_j$$
$$= -2 \sum_{j=1}^n \lambda_j^4 f_{ij} u_j + \lambda_i^4 f_{ii} u_i - \lambda_i^2 \sum_{j \neq i} \lambda_j^2 f_{ij} u_j + \sum_{j \neq i} \lambda_j^2 \lambda_i^2 f_{ij} u_i + \sum_{j \neq i} \lambda_j^2 \lambda_i^2 f_{ij} u_j$$
$$= \left(-\lambda_i^4 f_{ii} + \sum_{j \neq i} \lambda_j^2 \lambda_i^2 f_{ij}\right) u_i - \sum_{j \neq i} \lambda_j^2 (\lambda_i^2 + \lambda_j^2) f_{ij} u_j.$$
So $\nabla u_i E = 0$ implies $f_{ij} = 0$ for all $i, j$, hence $\nabla u_i u_j = 0$ for all $i, j$. By (3.17), all $\lambda_i$ are constants.

The reverse direction of the corollary is obvious. \hfill $\Box$ 

Corollaries 3.12 and 3.13 provide alternative proofs to Theorem 4.6 and part of Corollary 4.7 in [15]. The above results showed that the behavior of $\nabla C$ or $\nabla K$ gave strong constraints to the metric $g$. See [15] for more related results.

Remark 3.14. Semisimplicity is a very important property for Frobenius manifolds. They correspond to integrable hierarchies of KdV type [8]. Opoza’s algorithm that we described in Remark 2.9 could be used to explicitly calculate canonical basis at a semisimple point of Frobenius manifold.

In [11], an invariant of statistical manifold called Yukawa term was introduced. It is defined by
$$Y = C_{ijk} C^{ijk} - C_i C^i,$$ (3.18)
where $C_i = C_{ijk} g^{jk}$. In fact, they used the Amari-Chentsov tensor, hence their Yukawa term differs with (3.13) by a factor 4.

Proposition 3.15. On a statistical manifold, if $[K, K] = 0$, then the Yukawa term $Y = 0$. In 2-dimension, the converse is also true.
Proof. By Corollary 2.10 in the orthonormal basis $e_1, \ldots, e_n$,

$$C_{iii} = C^{iii} = \lambda_i, \quad C_i = C^i = \lambda_i.$$ 

All other $C_{ijk}$ vanish. So $Y = n\lambda^2 - n\lambda^2 = 0$.

For the last assertion in 2-dimension, see (3.19) and (3.21) in Example 3.16 \(\square\)

From Proposition 3.15 the vanishing of the Yukawa term is a necessary condition for zero sectional $K$-curvature.

Example 3.16. Consider the isothermal coordinates on a 2-manifold with metric

$$g(x, y) = \varphi(x, y)(dx \otimes dx + dy \otimes dy).$$

The symmetric tensor

$$C_{ijk} = K^l_{ij}g_{kl} = \varphi K^k_{ij}$$

has four independent components

$$f_1 = C_{111}, \quad f_2 = C_{112}, \quad f_3 = C_{122}, \quad f_4 = C_{222}.$$ 

Then $[K, K] = 0$ if and only if

$$f_2^2 + f_3^2 = f_1f_3 + f_2f_4.$$ \hspace{1cm} (3.19)

The solutions consist of three families:

(i) $f_3 = 0, f_2 = f_4$,
(ii) $f_2 = f_3 = 0$,
(iii) $f_3 \neq 0, f_1 = \frac{1}{f_3}(f_2^2 + f_3^2 - f_2f_4)$.

$K$ is non-degenerate if and only if

$$\text{rank} \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \end{pmatrix} = 2.$$ \hspace{1cm} (3.20)

When both (3.19) and (3.20) are satisfied, the unit $e$ in the Frobenius algebra is

(i) $e = \frac{\varphi}{f_3^2} \partial_y$,
(ii) $e = \frac{\varphi}{f_1} \partial_x + \frac{\varphi}{f_3} \partial_y$,
(iii) $e = \frac{\varphi f_3}{f_1 f_3 - f_2^2} \partial_x - \frac{\varphi f_2}{f_1 f_3 - f_2^2} \partial_y$,

corresponding to each of the three families of solutions of (3.19).

The Yukawa term is given by

$$Y = \frac{2}{\varphi^3}(f_2^2 + f_3^2 - f_1f_3 - f_2f_4).$$ \hspace{1cm} (3.21)

Let $M$ be a statistical manifold with Frobenius structure. If $\hat{\nabla}$ is flat and $\hat{\nabla}C$ is symmetric, then by the Poincaré Lemma, there exist a potential function $F$, such that \cite[Section 3.3]{10} $C_{ijk} = F_{ijk}$, where

$$F_{ijk} = \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}$$ \hspace{1cm} (3.22)

and $x_1, \ldots, x_n$ is the flat coordinates corresponding to $\hat{\nabla}$.

Let $B$ be a flat pseudo-Riemannian metric on $M$. Denote by

$$e = \sum_{i=1}^n A_i \partial_i$$
the unit vector field, where $A_i$ are smooth functions. Then
\[ B_{ij} = C(\partial_i, \partial_j, e) = \sum_{k=1}^{n} A_k F_{ijk}. \] (3.23)

Let $F_i$ be the $n \times n$ matrix $(F_i)_{jk} = F_{ijk}$. The WDVV equations are the following equations.
\[ F_i B^{-1} F_j = F_j B^{-1} F_i \] (3.24)
for all $i, j$.

There is large amount of work solving the potential WDVV equation (3.24), we only mention a solution related to $BC_n$ root system in a recent paper of Alkadhem-Antoniou-Feigin [1].

**Theorem 3.17.** [1] Suppose numbers $r, s$ and $q$ satisfy $r = -8s - 2q(n - 2)$. Let
\[ f(z) = \frac{1}{6} z^3 - \frac{1}{4} \text{Li}_3(e^{-2z}), \]
where $\text{Li}_3$ is the trilogarithm function. Then the function
\[ F = r \sum_{i=1}^{n} f(x_i) + s \sum_{i=1}^{n} f(2x_i) + q \sum_{1 \leq i < j \leq n} \left( f(x_i + x_j) + f(x_i - x_j) \right) \] (3.25)
satisfies WDVV equations (3.24) where $B$ is determined by (3.23) and $A_k = \sinh(2x_k)$. In fact, $B$ is proportional to the identity matrix
\[ B_{ij} = h \delta_{ij}, \]
where $h(x) = r + 2q \sum_{k=1}^{n} \cosh 2x_k$.

Finally, we gave two remarks. It would be interesting to study statistical structures with constant sectional $K$ curvatures. As noted in [15], in general we don’t know whether the basis in Lemma 3.24 could be extended to be a smooth frame field, although it works for trace-free statistical structure.

We know very few results on the global aspect of statistical manifold with Frobenius structure. Hitchin [10] showed that there is a finite covering of such manifold which is parallelizable. A first example to look is the compact Hessian manifolds. There is the famous Chern conjecture [9] that a flat manifold has zero Euler characteristic, which is still open even for Hessian manifold.

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