ON OVERTWISTED, RIGHT-VEERING OPEN BOOKS

PAOLO LISCA

Abstract. We exhibit infinitely many overtwisted, right–veering, non–
destabilizable open books, thus providing infinitely many counterex-
pamples to a conjecture of Honda–Kazez–Matić. The page of all our open
books is a four–holed sphere and the underlying 3–manifolds are lens
spaces.

1. Introduction

The purpose of this note is to construct infinitely many counterexamples
to a conjecture of Honda, Kazez and Matić from [12]. For the basic notions
of contact topology not recalled below we refer the reader to [4, 6].

Let $S$ be a compact, oriented surface with boundary and $\text{Map}(S, \partial S)$
the group of orientation–preserving diffeomorphisms of $S$ which restrict
to $\partial S$ as the identity, up to isotopies fixing $\partial S$ pointwise. An open book
(a.k.a. an abstract open book) is a pair $(S, \Phi)$ where $S$ is a surface as above
and $\Phi \in \text{Map}(S, \partial S)$. Giroux [8] introduced a fundamental operation of
stabilization $(S, \Phi) \to (S', \Phi')$ on open books, and proved the existence of
a 1–1 correspondence between the set of open books modulo stabilization
and the set of contact 3–manifolds modulo isomorphism (see e.g. [5] for de-
tails). Honda, Kazez and Matić [11] showed that a contact 3–manifold is
tight if and only if it corresponds to an equivalence class of open books
$(S, \Phi)$ all of whose monodromies $\Phi$ are right–veering (in the sense of [11,
Section 2]). In [9, 11] it is also showed that every open book can be made
right–veering after a sequence of stabilizations. In [12], Honda, Kazez and
Matić proved that, when $S$ is a holed torus, the contact structure corre-
sponding to $(S, \Phi)$ is tight if and only if $\Phi$ is right–veering, and conjectured
that a non–destabilizable right–veering open book corresponds to a tight
contact 3–manifold. The Honda–Kazez–Matić conjecture was recently dis-
proved by Lekili [13], who produced a counterexample $(S, \Phi)$ with $S$ equal
to a four–holed sphere and whose underlying 3–manifold is the Poincaré
homology sphere.

We shall now describe our examples. Denote by $\delta_\gamma \in \text{Map}(S, \partial S)$ the
class of a positive Dehn twist along a simple closed curve $\gamma \subset S$.

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Theorem 1.1. Let $S$ be an oriented four–holed sphere, and $a, b, c, d, e$ the simple closed curves on $S$ shown in Figure 1. Let $h, k \geq 1$ be integers.

![Figure 1. The four–holed sphere $S$](image)

Define

$$\Phi_{h,k} := \delta_a^h \delta_b \delta_c \delta_d \delta_e^{-k-1} \in \text{Map}(S, \partial S).$$

Then,

- The underlying 3–manifold $Y(S, \Phi_{h,k})$ is the lens space $L((h+1)(2k-1)+2, (h+1)k+1)$;
- the associated contact structure $\xi(S, \Phi_{h,k})$ is overtwisted;
- $\Phi_{h,k}$ is right–veering;
- $(S, \Phi_{h,k})$ is not destabilizable.

Warning: in the above statement we adopt the convention that the lens space $L(p, q)$ is the oriented 3–manifold obtained by performing a rational surgery along an unknot in $S^3$ with coefficient $-p/q$.

We prove Theorem 1.1 in Section 2. The proof can be outlined as follows. In Proposition 2.1 we use elementary arguments to determine a contact surgery presentation for the contact 3–manifold $(Y(S, \Phi_{h,k}), \xi(S, \Phi_{h,k}))$, and in Corollary 2.2 we apply Proposition 2.1 and a few Kirby calculus moves to identify the underlying 3–manifold $Y(S, \Phi_{h,k})$. In Proposition 2.3 we appeal to calculations from [13] to deduce that the contact Ozsváth–Szabó invariant of $\xi(S, \Phi_{h,k})$ vanishes, and we conclude from the fact that $Y(S, \Phi_{h,k})$ is a lens space that $\xi(S, \Phi_{h,k})$ must be overtwisted. We show that $\Phi_{h,k}$ is right–veering in Lemma 2.4 by observing that this result follows directly from [2, Theorem 4.3], but can also be deduced imitating the proof of [13, Theorem 1.2], i.e. applying [11, Corollary 3.4]. Finally, we use results from [1, 13] to conclude that $(S, \Phi_{h,k})$ is not destabilizable.

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2. Proof of Theorem 1.1

Recall that every contact structure has a contact surgery presentation [3]. We refer the reader to [3] for the basic properties of contact surgeries, and to [14] for the use of the ‘front notation’ in contact surgery presentations, in particular for the meaning of Figure 2 below.

**Proposition 2.1.** For \( h, k \geq 1 \), the contact structure \( \xi(S, \Phi_{h,k}) \) has the contact surgery presentation given by Figure 2.

![Figure 2](image-url)

**Figure 2.** Contact surgery presentation for \( \xi(S, \Phi_{h,k}) \), \( h, k \geq 1 \).

**Proof.** Figure 3(a) represents an open book \((A, f)\), where \( A \) is an annulus and \( f \) is a positive Dehn twist along the core of \( A \). The associated contact 3–manifold is the standard contact 3–sphere \((S^3, \xi_{st})\), the annulus \( A \) can be viewed as the page of an open book decomposition of \( S^3 \), and the curve \( \kappa \) in the picture can be made Legendrian via an isotopy of the contact structure, in such a way that the contact framing on \( \kappa \) coincides with the framing induced on it by the page (see e.g. [5, Figure 11]). The knot \( \kappa \) is the unique Legendrian unknot in \((S^3, \xi_{st})\) having Thurston–Bennequin invariant \( \text{tb}(\kappa) = -1 \) and rotation number \( \text{rot}(\kappa) = 0 \). A suitable choice of orientation for \( \kappa \) uniquely specifies its negative oriented Legendrian stabilization \( \kappa_- \), which satisfies \( \text{tb}(\kappa_-) = -2 \) and \( \text{rot}(\kappa_-) = -1 \). As shown in [5], \( \kappa_- \) can be realized as sitting on the page of a Giroux stabilization \((A', f')\) of \((A, f)\). This is illustrated in Figure 3(b), assuming the orientation on \( \kappa \) was taken to be “counterclockwise” in Figure 3(a). Finally, Figure 3(c) shows an open book \((S, f'')\) obtained by Giroux stabilizing \((A', f')\) and containing both \( \kappa_- \) and \( (\kappa_-)^- \) in \( S \) (\( \kappa_- \) was also given the “counterclockwise” orientation in Figure 3(b)). Clearly \((S, f'')\) still corresponds to \((S^3, \xi_{st})\), and it is well–known that \( \kappa_- \), \( (\kappa_-)^- \) are the two Legendrian knots illustrated in Figure 2 (when oriented “clockwise” in that picture). By definition, \( \Phi_{h,k} \) is obtained by pre–composing \( f'' \) with \( k + 1 \) negative Dehn twists along parallel copies of \( \kappa_- \) and \( h \) positive Dehn twists along parallel copies of \( (\kappa_-)^- \). Moreover, if \( m \neq 0 \) is an integer, \( \frac{1}{m} \)–contact surgery along any Legendrian knot \( \lambda \) is equivalent to \( \frac{m}{|m|} \)–contact surgeries along \( |m| \) Legendrian push–offs of \( \lambda \) [3].
Since page and contact framings coincide and by e.g. [5, Theorem 5.7] positive (negative, respectively) Dehn twists correspond to \(-1\)-contact surgeries (+1–contact surgeries, respectively), it is easy to check that the resulting contact structure is given by the contact surgery presentation of Figure 2.

\(\square\)

\textbf{Corollary 2.2.} For \(h, k \geq 1\), the oriented 3–manifold underlying the open book \((S, \Phi_{h,k})\) is the lens space \(L((h + 1)(2k - 1) + 2, (h + 1)k + 1)\).

\textbf{Proof.} Using the fact that the two Legendrian unknots illustrated in Figure 2 have Thurston–Bennequin invariants \(-2\) and \(-3\), it is easy to check that the topological surgery underlying Figure 2 is given by the first (upper left) picture of Figure 4. Two +1–blowups and two inverse slam–dunks give

\[\begin{align*}
-2 + \frac{1}{k+1} & \quad \rightarrow \quad -3 - \frac{1}{h} \\
-2 & \quad \rightarrow \quad -2
\end{align*}\]

\textbf{Figure 4.} Determination of the underlying 3–manifold.

the second picture, while the third picture is obtained from the second one by sliding the \(-1\)-framed knot over the 0–framed knot and then applying
two +1–blow–downs. The last picture is obtained simply converting the $h$–framed unknot in the third picture into the string of $−2$–framed unknots via a sequence of $−1$–blowups and a final +1–blowdown. The last picture shows that the underlying 3–manifold $Y(S, \Phi_{h,k})$ is obtained by performing a rational surgery on an unknot in $S^3$ with coefficient $−p/q$, where

$$\frac{p}{q} = 2 - \frac{1}{k+1 - \frac{1}{2 - \frac{1}{\ddots - \frac{1}{2}}}} = \frac{(h + 1)(2k - 1) + 2}{(h + 1)k + 1}.$$

Therefore, according to our conventions $Y(S, \Phi_{h,k})$ can be identified with the lens space $L((h + 1)(2k - 1) + 2, (h + 1)k + 1)$.

**Proposition 2.3.** For $h, k \geq 1$, the contact structure $\xi(S, \Phi_{h,k})$ is overtwisted.

**Proof.** By [7, 10] a contact structure on a lens space is either overtwisted or Stein fillable. Moreover, Stein fillable contact structures have non–zero contact Ozsváth–Szabó invariant [15]. Finally, [13, Theorem 1.3] immediately implies that the contact invariant of $(S, \Phi_{h,k})$ vanishes, therefore $\xi(S, \Phi_{h,k})$ must be overtwisted.

**Lemma 2.4.** For $h, k \geq 1$, the diffeomorphism class $\Phi_{h,k} = \delta^h_\alpha \delta^k_\gamma \delta^i_\beta \delta^{−k−1}_e \in \text{Map}(S, \partial S)$ is right–veering.

**Proof.** The lemma follows immediately from the statement of [2, Theorem 4.3]. Alternatively, one can imitate the proof of [13, Theorem 1.2]. Indeed, applying [11, Corollary 3.4] to the monodromy $\Phi_1 = \delta^{−k−1}_e$ and a properly embedded arc $\gamma_{cd} \subset S$ disjoint from the curve $e$ and connecting the components $\partial_c$ and $\partial_d$ of $\partial S$ parallel to the curves $c$ and $d$ shows that $\Phi_2 = \delta^c_\gamma \delta^{−k−1}_e$ is right–veering with respect to $\partial_d$. Another application of the corollary to $\Phi_2$ and $\gamma_{cd}$ shows that $\Phi_3 = \delta^e_\gamma \delta^{−k−1}_e$ is right–veering with respect to $\partial_c$. Moreover, since $\delta_c$ is right–veering with respect to $\partial_e$ and the composition of right–veering diffeomorphisms is still right–veering, $\Phi_3$ is right–veering with respect to $\partial_d$ as well. Applying the corollary in the same way to $\Phi_3$ and an arc connecting the components of $\partial S$ parallel to the curves $a$ and $b$ yields the statement of the lemma.

**Proof of Theorem 1.1** Corollary 2.2, Proposition 2.3 and Lemma 2.4 establish the first three portions of the statement. We are only left to show that $(S, \Phi_{h,k})$ is not destabilizable for every $h, k \geq 1$. If $(S, \Phi_{h,k})$ were destabilizable, it would be a stabilization of an open book $(S', \Phi')$, with $S'$ a three–holed sphere and $\Phi' = \tau_1^{-a_1} \tau_2^{-a_2} \tau_3^{-a_3}$, where $a_i \in \mathbb{Z}$ and $\tau_i$ is a positive Dehn twist along a simple closed curve parallel to the $i$–th boundary components of $S'$, $i = 1, 2, 3$. By [1, Theorem 1.2], $\xi(S, \Phi_{h,k})$ is tight if and only
if \( a_i \geq 0, i = 1, 2, 3 \). Therefore, by Proposition 2.3 at least one of these exponents must be strictly negative. But the proof of [13, Theorem 1.2] shows that when one of the \( a_i \)'s is negative, any stabilization of \((S', \Phi')\) to an open book with page a four–holed sphere is not right–veering. This would contradict Lemma 2.4, therefore we conclude that \((S, \Phi_{h,k})\) cannot be destabilizable.

\[\square\]

Note: after completing the first version of this paper the author was informed of independent, unpublished work of A. Wand containing, in particular, a different proof of Proposition 2.3.

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Dipartimento di Matematica “L. Tonelli”, Largo Bruno Pontecorvo 5, Università di Pisa, 56127 Pisa, ITALY

E-mail address: lisca@dm.unipi.it