Fokker-Planck equations for Marcus stochastic differential equations driven by non-Gaussian Lévy processes

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Abstract

Marcus Stochastic differential equations are appropriate models for many engineering and scientific applications. Fokker-Planck equation describes time evolution of probability densities of stochastic dynamical systems and plays an important role in quantifying propagation and evolution of uncertainty. Fokker-Planck equation is developed in this paper for nonlinear systems driven by non-Gaussian Lévy processes and modeled by Marcus stochastic differential equations with some mild assumption on coefficients of the noise terms.

Keywords: Fokker-Planck equations, Stochastic differential equations, Marcus integral, Marcus stochastic differential equations, stochastic dynamical systems, non-Gaussian Lévy processes

1 Introduction and statement of the problem

Stochastic differential equations are often appropriate models for dynamical systems subjected to random excitations [7]. Fokker-Planck equation describes the evolution of probability density functions and is an important tool to study how uncertainties propagate and evolve in dynamical systems [7, 8]. For stochastic differential equations (SDEs) driven by Gaussian processes such as Brownian motions, there are well established formulas to obtain explicit expressions of the associated Fokker-Planck equations, regardless the SDEs are in sense of Ito or Stratonovich [7, 9]. However, Fokker-Planck equations for SDEs driven by general Lévy processes are not readily available due to the difficulty in obtaining the expressions for the adjoint operators of the infinitesimal generators associated with these SDEs [1]. For Ito SDEs driven by Lévy processes, the Fokker-Planck equations have been discussed by many authors, see [10, 9] among others. For Marcus SDEs [5, 6, 4, 1], the research is relatively few. Marcus SDEs are recently shown to be equivalent to certain SDEs that are widely used in Engineering and Physics [11, 2]. In [10], an explicit form of Fokker-Planck equations is presented for Marcus SDEs driven by Lévy processes. However, the result in [10] requires coefficient of the noise term to be nonzero, which poses as a strict restriction on its application. The objective of the current paper is to derive Fokker-planck equation for Marcus SDEs driven by non-Gaussian Levy processes without the
requirement of nonzero coefficient for the noise terms.

Lévy processes are stochastic processes with properties of independent and stationary increments, as well as stochastically continuous sample paths [1, 8]. Given a sample space $\Omega$, together with a probability measure $P$ and the corresponding mathematical expectation $E$. A Lévy process $L(t)$, taking values in $\mathbb{R}^d$, is characterized by a drift vector $b \in \mathbb{R}^d$, a $d \times d$ positive-definite covariance matrix $A$ and a measure $\nu$ defined on $\mathbb{R}^d$ and concentrated on $\mathbb{R}^d \setminus \{0\}$. In fact, this measure $\nu$ satisfies the following condition [1]

$$\int_{\mathbb{R}^d \setminus \{0\}} (\|y\|^2 \wedge 1) \nu(dy) < \infty,$$

or equivalently

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{\|y\|^2}{1 + \|y\|^2} \nu(dy) < \infty.$$  

(2)

Here $\| \cdot \|$ is the usual Euclidean norm (or length) in $\mathbb{R}^d$. This measure $\nu$ is called a Lévy jump measure for the Lévy process $L_t$. A Lévy process with the generating triplet $(b, A, \nu)$ has the Lévy-Itô decomposition

$$L(t) = bt + B(t) + \int_{\|y\|<1} y\tilde{N}(t, dy) + \int_{\|y\|\geq 1} yN(t, dy),$$

(3)

where $N(dt, dx)$ is the Poisson random measure, $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$ is the compensated Poisson random measure, and $B(t)$ is an independent $d$-dimensional Brownian motion (i.e., Wiener process) with covariance matrix $A$.

Equation (3) can be formally rewritten in a differential form as

$$dL_t = bdt + dB_t + \int_{\|y\|<1} y\tilde{N}(dt, dy) + \int_{\|y\|\geq 1} yN(dt, dy).$$

(4)

We shall consider stochastic dynamical systems described by the following SDE in sense of Marcus,

$$dX(t) = f(X(t))dt + \sigma(X(t), t) \circ dL(t),$$

(5)

where $L(t)$ is a Lévy process with the generating triplet $(b, A, \nu)$. Note that we only consider 1-dimensional case in this paper. That is to say, in (4) and (5), $X(t)$ is a scalar random variable, and $L(t)$ has a generating triplet $(b, A, \nu)$ with $b$ and $A$ being scalars, and $\nu$ being a measure defined on $\mathbb{R}$. The solution of equation (5) is interpreted as

$$X(t) = X(t) + \int_0^t f(X(s), s)ds + \int_0^t \sigma(X(s-)) \circ dL(s),$$

(6)
where "\(\circ\)" indicates Marcus integral [5][6][11] defined by
\[
\int_0^t \sigma(X_{s-}) \circ \, dL_s = \int_0^t \sigma(X_{s-}) dL(s) + \frac{1}{2} \int_0^t \sigma(X(s-)) \sigma'(X(s-)) \, d[L(s), L(s)]^c + \sum_{0 \leq s \leq t} \xi(\Delta L(s), \sigma(X(s-)), X(s-)) - X(s-) - \sigma(X(s-)) \Delta L(s),
\]
with \(\xi(r, \sigma(x), x)\) being the value at \(z = 1\) of the solution of the following ordinary differential equation:
\[
\frac{d}{dz} y(z) = rg(y(z)), \quad y(0) = x.
\]

Throughout this paper, we assume (i) the existence and uniqueness of the solution for SDE [3]: (ii) The probability density function for the solution of [3] exists and is sufficiently smooth such that the required derivative operations in the later sections can be carried out. All other assumptions given in later sections will be in addition to the above two assumptions. In this paper, we are not going to present conditions for existence and regularity of the probability density, which is out of the scope of this paper. Note that existence and regularity of probability density for solutions of SDEs driven by Lévy processes are active research topics itself. In this paper, we just focus on developing explicit forms of Fokker-Planck equations under assumptions (i) and (ii).

The later sections of the paper are organized as follows. In section 2, we derive Fokker-Planck equations under the condition \(\sigma(x) \neq 0\). The condition that \(\sigma(x) \neq 0\) is relaxed in section 3 by assuming \(\sigma(x)\) has finite zeros.

2 For \(\sigma(x) \neq 0\)

In this section, under the assumption that \(\sigma(x)\) is Lipschitz continuous, two times differential, \(\sigma(x) \neq 0\), and \(f(x)\) is differentiable, we derive Fokker-Planck equation for SDE [3] with a different approach from that in [10]. The advantage of the approach here lies in that it can be modified to be applicable in cases where \(\sigma\) has zeros.
It follows from (1), (5) and (7) that (11)
\[ dX(t) = f(X(t))dt + bσ(X(t))dt + σ(X(t))dB(t) + \frac{1}{2}\sigma(X(t))\sigma'(X(t))dt \]
+ \int_{|y|<1} [H^{-1}(H(X(s))-y) - X(s)] N(ds, dy) 
+ \int_{|y|\geq1} [H^{-1}(H(X(s))-y) - X(s)] N(ds, dy) 
+ \int_{|y|<1} [H^{-1}(H(X(s))-y) - X(s) - σ(X(s))y] ν(dy)ds. \quad (9) 

Define
\[ H(x) = \int_a^x \frac{dt}{σ(t)}, \quad (10) \]

It follows from (5) that
\[ H(ξ(ΔL(t), σ(X(t-))), X(t-)) - H(X(t-)) = ΔL(t), \quad (11) \]

It follows from (11) that
\[ ξ(ΔL(t), σ(X(t-))), X(t-)) = H^{-1}(H(X(t-)) + ΔL(t)), \quad (12) \]

where \( H^{-1} \) represents the inverse of function \( H \) as defined in (10). Substitute (12) into (9), it follows that
\[ dX(t) = f(X(t))dt + bσ(X(t))dt + σ(X(t))dB(t) \]
+ \int_{|y|<1} [H^{-1}(H(X(s))-y) - X(s)] N(ds, dy) 
+ \int_{|y|\geq1} [H^{-1}(H(X(s))-y) - X(s)] N(ds, dy) 
+ \int_{|y|<1} [H^{-1}(H(X(s))-y) - X(s) - σ(X(s))y] ν(dy)ds. \quad (13) 

By Itô’s formula (11), for \( φ(x) \in C_0^∞(R) \), it follows from (13) that
\[
φ(X(t + Δt)) - φ(X(t)) = \int_t^{t+Δt} φ'(X(s-))f(X(s-))ds + \int_t^{t+Δt} bφ'(X(s-))σ(X(s))ds \\
+ \frac{1}{2} \int_t^{t+Δt} φ''(X(s-))σ^2(X(s))ds \\
+ \int_t^{t+Δt} φ'(X(s-))σ(X(s))dB(s) + \frac{1}{2} \int_t^{t+Δt} φ''(X(s-))σ^2(X(s))ds \\
+ \int_t^{t+Δt} \int_{|y|\geq1} [φ(H^{-1}(H(X(s))-y)) - φ(X(s))] N(ds, dy) \\
+ \int_t^{t+Δt} \int_{|y|<1} [φ(H^{-1}(H(X(s))-y)) - φ(X(s))] N(ds, dy) \\
+ \int_t^{t+Δt} \int_{|y|<1} [φ(H^{-1}(H(X(s))-y)) - φ(X(s)) - φ'(x)σ(x)y] ν(dy)ds.
\quad (14)
Let \( p(x, t | X(0) = x_0) \) represent the probability density function of \( X(t) \), and for convenience, we drop off the initial condition and simply use \( p(x, t) \) instead of \( p(x, t | X(0) = x_0) \) from now on. Take expectation at both sides of (14), it follows that

\[

t - \infty \int \phi(x)p(x, t + \Delta t)dx - \infty \int \phi(x)p(x, t)dx
\]

\[
= \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \phi'(x)f(x)p(x, t)ds dx + \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \phi'(x)\sigma(x)p(x, t)ds dx
\]

\[
+ \frac{A}{2} \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \phi'(x)\sigma'(x)p(x, t)ds dx + \frac{A}{2} \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \phi''(x)\sigma^2(x)p(x, t)ds dx
\]

\[
+ \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \int_{y \geq 1} \left[ \phi(H^{-1}(H(X(t)) + y)) - \phi(x) \right] ds \nu(dy)dx
\]

\[
+ \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \int_{y < 1} \left[ \phi(H^{-1}(H(X(t)) + y)) - \phi(x) - \phi'(x)\sigma(x) y \right] ds \nu(dy)dx.
\] (15)

To derive the above equation, the following facts have been used:

\[
\mathbb{E}\{ \int_{t}^{t+\Delta t} \phi'(X(s-))\sigma(X(s))d\mathcal{B}(s) \} = 0,
\] (16)

\[
\mathbb{E}\{ \int_{t}^{t+\Delta t} \int_{y \geq 1} \left[ \phi(H^{-1}(H(X(t)) + y)) - \phi(X(s-)) \right] N(ds, dy) \} = 0,
\] (17)

and

\[
\mathbb{E}\{ \int_{t}^{t+\Delta t} \int_{y < 1} \left[ \phi(H^{-1}(H(X(t)) + y)) - \phi(X(s-)) \right] N(ds, dy) \}
\]

\[
= \int_{t}^{t+\Delta t} \int_{y \geq 1} \mathbb{E}\left[ \phi(H^{-1}(H(X(t)) + y)) - \phi(X(s-)) \right] N(ds, dy)
\]

\[
= \int_{t}^{t+\Delta t} \int_{y \geq 1} \mathbb{E}\left[ \phi(H^{-1}(H(X(t)) + y)) - \phi(X(s-)) \right] ds \nu(dy)
\]

\[
= \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \int_{y \geq 1} \left[ \phi(H^{-1}(H(x) + y)) - \phi(x) \right] ds dx \nu(dy).
\] (18)

Equation (15) can be rewritten as

\[
\int_{-\infty}^{\infty} \phi(x)p(x, t + \Delta t) - p(x, t)dx
\]

\[
= \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \phi'(x) \left[ f(x)p(x, t) + b\sigma(x)p(x, t) + A \frac{\sigma(x)\sigma'(x)p(x, t)}{2} \right] ds dx
\]

\[
+ \frac{A}{2} \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \phi''(x)\sigma^2(x)p(x, t)ds dx
\]

\[
+ \int_{t}^{t+\Delta t} \int_{-\infty}^{\infty} \int_{\mathbb{R}\setminus\{0\}} \left\{ \phi(H^{-1}(H(x) + y)) - \phi(x) - \phi'(x)\sigma(x) y I_{(-1, 1)}(y) \right\} p(x, t) \nu(dy)ds dx.
\] (19)
The interchanging order of integrals above is justified by

\[ I_A(x) = \begin{cases} 
1 & \text{for } x \in A \\
0 & \text{for } x \notin A 
\end{cases} \quad (20) \]

Divided both sides of (19) by \( \Delta t \), and take the limit by letting \( \Delta t \to 0 \), it follows that

\[
\int_{-\infty}^{\infty} \phi(x) \frac{\partial p(x,t)}{\partial t} \, dx \\
= \int_{-\infty}^{\infty} \phi'(x) \left[ f(x)p(x,t) + b\sigma(x)p(x,t) + \frac{A}{2} \sigma(x)\sigma'(x)p(x,t) \right] \, dx \\
+ \frac{A}{2} \int_{-\infty}^{\infty} \phi''(x)\sigma^2(x)p(x,t) \, dx \\
+ \int_{-\infty}^{\infty} \, dx \int_{\mathbb{R}\setminus\{0\}} \{ \phi \left( H^{-1}(H(x) + y) \right) - \phi(x) - \phi'(x)\sigma(x) y I_{(-1,1)}(y) \} p(x,t) \nu(dy) 
\]

By integration by parts, the first two integrals at the right hand side of (21) becomes

\[
\int_{-\infty}^{\infty} \phi'(x) \left[ f(x)p(x,t) + b\sigma(x)p(x,t) + \frac{A}{2} \sigma(x)\sigma'(x)p(x,t) \right] \, dx \\
= - \int_{-\infty}^{\infty} \phi(x) \frac{\partial}{\partial x} \left[ f(x)p(x,t) + b\sigma(x)p(x,t) + \frac{A}{2} \sigma(x)\sigma'(x)p(x,t) \right] \, dx, \quad (22) 
\]

and

\[
\int_{-\infty}^{\infty} \phi''(x)\sigma^2(x)p(x,t) \, dx = \int_{-\infty}^{\infty} \phi(x) \frac{\partial^2}{\partial x^2} \left[ \sigma^2(x)p(x,t) \right] \, dx, \quad (23) 
\]

respectively.

By interchanging order of integrals, the last term of (21) becomes

\[
\int_{-\infty}^{\infty} \, dx \int_{\mathbb{R}\setminus\{0\}} \{ \phi \left( H^{-1}(H(x) + y) \right) - \phi(x) - \phi'(x)\sigma(x) y I_{(-1,1)}(y) \} p(x,t) \nu(dy) \\
= \int_{\mathbb{R}\setminus\{0\}} \nu(dy) \int_{-\infty}^{\infty} \{ \phi \left( H^{-1}(H(x) + y) \right) - \phi(x) - \phi'(x)\sigma(x) y I_{(-1,1)}(y) \} p(x,t) \, dx 
\]

The interchanging order of integrals above is justified by

\[
\int_{-\infty}^{\infty} \int_{\mathbb{R}\setminus\{0\}} \left| \{ \phi \left( H^{-1}(H(x) + y) \right) - \phi(x) - \phi'(x)\sigma(x) y I_{(-1,1)}(y) \} p(x,t) \right| \, dx \, \nu(dy) \\
\leq \int_{-\infty}^{\infty} \int_{|y| < 1} \left| \{ \phi \left( H^{-1}(H(x) + y) \right) - \phi(x) - \phi'(x)\sigma(x) y I_{(-1,1)}(y) \} p(x,t) \right| \, dx \, \nu(dy) \\
+ \int_{-\infty}^{\infty} \int_{|y| \geq 1} \left| \{ \phi \left( H^{-1}(H(x) + y) \right) - \phi(x) \} p(x,t) \right| \, dx \, \nu(dy) \\
< +\infty \quad (25) 
\]
To prove the last inequality in (25), we have used (1) and the fact that \( \phi(x) \in C_0^\infty(\mathbb{R}) \).

To proceed, let us examine the integral inside the last term of (24), which can be written as

\[
\int_{-\infty}^{\infty} \left\{ \phi \left( H^{-1}(H(x) + y) \right) - \phi(x) - \phi'(x) \sigma(x) y I_{(-1,1)}(y) \right\} p(x,t) \, dx
\]

\[
= \int_{-\infty}^{\infty} \phi \left( H^{-1}(H(x) + y) \right) p(x,t) \, dx - \int_{-\infty}^{\infty} \phi(x) p(x,t) \, dx
\]

\[
- \int_{-\infty}^{\infty} \phi'(x) \sigma(x) y I_{(-1,1)}(y) p(x,t) \, dx \quad (26)
\]

Denote

\[
z = H^{-1}(H(x) + y), \quad (27)
\]

Under the condition that for \( x \in \mathbb{R}, \sigma(x) \neq 0 \) and \( \sigma(x) \) is Lipschitz, it can be shown (see Appendix I for proof) that \( H(x) \) is monotone, and for given \( y \in \mathbb{R}, H^{-1}(H(x) + y) \) maps from \((-\infty, +\infty)\) to \((-\infty, +\infty)\).

It follows (27) and (10) that

\[
x = H^{-1}(H(z) - y), \quad (28)
\]

and

\[
\frac{dx}{dz} = \frac{\sigma(H^{-1}(H(z) - y))}{\sigma(z)}. \quad (29)
\]

For the first integral at the right hand side of (26), using (28) and (29) and by the change of variable we can get

\[
\int_{-\infty}^{\infty} \phi \left( H^{-1}(H(x) + y) \right) p(x,t) \, dx
\]

\[
= \int_{-\infty}^{\infty} \phi(z) \frac{\sigma(H^{-1}(H(z) - y))}{\sigma(z)} p(H^{-1}(H(z) - y), t) \, dz
\]

\[
= \int_{-\infty}^{\infty} \phi(z) \frac{\sigma(H^{-1}(H(z) - y))}{\sigma(z)} p(H^{-1}(H(z) - y), t) \, dz, \quad (30)
\]

and for the last integral at the right hand side of (26),

\[
\int_{-\infty}^{\infty} \phi'(x) \sigma(x) y I_{(-1,1)}(y) p(x,t) \, dx = - \int_{-\infty}^{\infty} \phi(x) y I_{(-1,1)}(y) \frac{\partial}{\partial x} \left( \sigma(x) p(x,t) \right) \, dx
\]

\[
(31)
\]
By interchanging order of integrals, (33) becomes
\[
\int_{-\infty}^{\infty} \{ \phi \left( H^{-1}(H(x) + y) \right) - \phi(x) - \phi'(x) \sigma(x) y I_{(-1,1)}(y) \} p(x,t) \, dx
\]
\[
= \int_{-\infty}^{\infty} \phi(x) \left[ \frac{\sigma(H^{-1}(H(x) - y))}{\sigma(x)} p(H^{-1}(H(x) - y), t) \right.
\]
\[
- p(x,t) + y I_{(-1,1)}(y) \frac{\partial}{\partial x} (\sigma(x)p(x,t)) \] \quad \text{dx} \quad \tag{32}
\]
Substitute (32) into (24), we get
\[
\int_{-\infty}^{\infty} \nu(dy) \int_{-\infty}^{\infty} \phi(x) \left[ \frac{\sigma(H^{-1}(H(x) - y))}{\sigma(x)} p(H^{-1}(H(x) - y), t) \right.
\]
\[
- p(x,t) + y I_{(-1,1)}(y) \frac{\partial}{\partial x} (\sigma(x)p(x,t)) \] \quad \text{dx} \quad \tag{33}
\]
By interchanging order of integrals, (33) becomes
\[
\int_{-\infty}^{\infty} \int_{R \setminus \{0\}} \{ \phi \left( H^{-1}(H(x) + y) \right) - \phi(x) - \phi'(x) \sigma(x) y I_{(-1,1)}(y) \} p(x,t) \, \nu(dy)
\]
\[
= \int_{-\infty}^{\infty} \int_{R \setminus \{0\}} \phi(x) \left[ \frac{\sigma(H^{-1}(H(x) - y))}{\sigma(x)} p(H^{-1}(H(x) - y), t) \right.
\]
\[
- p(x,t) + y I_{(-1,1)}(y) \frac{\partial}{\partial x} (\sigma(x)p(x,t)) \] \quad \text{dx} \quad \tag{34}
\]
Substitute (22), (23), and (34) into (21), we get
\[
\int_{-\infty}^{\infty} \phi(x) \frac{\partial p(x,t)}{\partial t} \, dx
\]
\[
= - \int_{-\infty}^{\infty} \phi(x) \frac{\partial}{\partial x} \left[ f(x)p(x,t) + b\sigma(x)p(x,t) + A \frac{1}{2} \sigma(x)\sigma'(x)p(x,t) \right] \, dx
\]
\[
+ A \int_{-\infty}^{\infty} \phi(x) \frac{\partial^2}{\partial x^2} [\sigma^2(x)p(x,t)] \, dx
\]
\[
+ \int_{-\infty}^{\infty} dx \int_{R \setminus \{0\}} \phi(x) \left[ \frac{\sigma(H^{-1}(H(x) - y))}{\sigma(x)} p(H^{-1}(H(x) - y), t) \right.
\]
\[
- p(x,t) + y I_{(-1,1)}(y) \frac{\partial}{\partial x} (\sigma(x)p(x,t)) \] \quad \text{dx} \quad \tag{35}
\]
Since the above equation is true for any \( \phi(x) \in C_0^\infty(\mathbb{R}) \), it follows from (35) that
\[
\frac{\partial p(x,t)}{\partial t} = - \frac{\partial}{\partial x} \left[ f(x)p(x,t) + b\sigma(x)p(x,t) + A \frac{1}{2} \sigma(x)\sigma'(x)p(x,t) \right] \quad + A \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma^2(x)p(x,t)]
\]
\[
+ \int_{R \setminus \{0\}} \left[ \frac{\sigma(H^{-1}(H(x) - y))}{\sigma(x)} p(H^{-1}(H(x) - y), t) \right.
\]
\[
- p(x,t) + y I_{(-1,1)}(y) \frac{\partial}{\partial x} (\sigma(x)p(x,t)) \] \quad \text{dx} \quad \tag{36}
\]
which is the Fokker-Planck equation associated with the SDE \( \text{(5)} \) with the assumption \( \sigma(x) \neq 0 \). As expected, this equation is the same as the one given in [10].

3 For \( \sigma(x) \) has zeros

In this section, we assume \( \sigma(x) \) is Lipschitz continuous and has \( n \) zeros represented by \( x_i \) \((i = 1, 2, \cdots, n)\). For convenience, we denoted \( x_0 = -\infty \) and \( x_{n+1} = +\infty \). Without loss of generality, suppose

\[
-\infty = x_0 < x_1 < \cdots < x_n < x_{n+1} = +\infty. \tag{37}
\]

Define \( H_i \) \((i = 0, 1, \cdots, n)\) as

\[
H_i(x) = \int_{a_i}^{x} \frac{dy}{\sigma(y)} \quad \text{for} \quad x \in (x_i, x_{i+1}), \tag{38}
\]

where \( a_i \) could be any given constant satisfies \( a_i \in (x_i, x_{i+1}) \). Since \( \sigma(x) \neq 0 \) for \( x \in (x_i, x_{i+1}) \), and \( H_i \) is monotone and has inverse function. Denote \( H_i^{-1} \) as the inverse function of \( H_i \). As shown in Appendix II, with the assumption that \( \sigma(x) \) is Lipschitz continuous, one can check that \( H_i \) maps from \((x_i, x_{i+1})\) to \((-\infty, +\infty)\), and \( H_i^{-1}(H_i(x) + C) \) maps from \((x_i, x_{i+1})\) to \((x_i, x_{i+1})\) with the property

\[
\begin{cases}
\lim_{x \to x_i, +} H_i^{-1}(H_i(x) + C) = x_i, \\
\lim_{x \to x_{i+1}, -} H_i^{-1}(H_i(x) + C) = x_{i+1},
\end{cases} \tag{39}
\]

where \( C \) is a constant independent of \( x \), \( \lim_{x \to x_i, +} \) represents the right limit at \( x = x_i \), and \( \lim_{x \to x_{i+1}, -} \) the left limit at \( x = x_{i+1} \).

Given \( H_i \) defined above, equation \( \text{(12)} \) now becomes

\[
\begin{cases}
\xi(\Delta L(t), \sigma(X(t-)), X(t-)) = H_i^{-1}(H_i(X(t-)) + dL(t)) \quad \text{for} \quad X(t-) \in (x_i, x_{i+1}), \\
\xi(\Delta L(t), \sigma(X(t-)), X(t-)) = X(t-) \quad \text{for} \quad X(t-) = x_1, x_2, \cdots, x_n.
\end{cases} \tag{40}
\]

Let

\[
\tilde{H}(x, y) = \begin{cases} 
H_i^{-1}(H_i(x) + y) & \text{for} \ x \in (x_i, x_{i+1}) \\
x & \text{for} \ x = x_1, x_2, \cdots, x_n,
\end{cases} \tag{41}
\]
then \[10\] can be written as

\[
\xi(\Delta L(t), \sigma(X(t-)), X(t-)) = \tilde{H}(X(t-), L(t)).
\] (42)

Let \(\tilde{H}(x, y)\) as defined in \[11\] replace \(H^{-1}(H(x) + y)\) appeared in \[12\], and with the similar procedure to obtain \[24\], we can get

\[
\int_{-\infty}^{\infty} \phi(x) \frac{\partial p(x, t)}{\partial t} dx
\]

\[
= \int_{-\infty}^{\infty} \phi'(x) \left[ f(x)p(x, t) + b\sigma(x)p(x, t) + \frac{A}{2} \sigma(x)\sigma'(x)p(x, t) \right] dx
\]

\[
+ \frac{A}{2} \int_{-\infty}^{\infty} \phi''(x)\sigma^2(x)p(x, t) dx
\]

\[
+ \int_{-\infty}^{\infty} dx \int_{\mathbb{R} \setminus \{0\}} \left\{ \phi(\tilde{H}(x, y)) - \phi(x) - \phi'(x)\sigma(x)y I_{(-1, 1)}(y) \right\} p(x, t) \nu(dy).
\] (43)

Let us examine the integral inside the last term of \[43\] now. It follows from \[35\] and \[39\] that \(\tilde{H}(x, y)\) is continuous and invertible with respect to \(x\). Moreover, \(\tilde{H}(x, y)\) is differentiable with respect to \(x\) for \(x \in \mathbb{R}\setminus\{x_1, x_2, \ldots, x_n\}\). As shown in Appendix III, \(\tilde{H}(x, y)\) is differentiable with respect to \(x\) for \(x \in \mathbb{R}\) provided that \(\sigma(x)\) is infinitely differentiable with respect to \(x\) for all \(x \in \mathbb{R}\).

Suppose that \(\sigma(x)\) is infinitely differentiable with respect to \(x\) for all \(x \in \mathbb{R}\) such that \(\tilde{H}(x, y)\) is differentiable with respect to \(x\) for \(x \in \mathbb{R}\). Let \(z = \tilde{H}(x, y)\), then it follows from \[41\] that \(x = \tilde{H}(z, -y)\), and

\[
\frac{dx}{dz} = \begin{cases} \frac{\sigma(H(z, -y))}{\sigma(z)} & \text{for } z \in (x_i, x_{i+1}), i = 0, 1, \ldots, n, \\ \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Phi_k(x_i)y^k & \text{for } z = x_i, i = 1, \ldots, n \end{cases}
\] (44)

where \(\Phi_k(x_i)\) is defined as in \[A5\] given in Appendix III. Using \[A3\] given in Appendix III, one can show that for \(i = 1, 2, \ldots, n\),

\[
\lim_{z \to x_i} \frac{\sigma(H(z, -y))}{\sigma(z)} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Phi_k(x_i)y^k,
\] (45)

which indicates that \(\frac{dx}{dz}\) as defined in \[44\] is continuous with respect to \(z\). Denote the right hand side of \[44\] as \(\Theta(z, y)\), i.e.,

\[
\Theta(z, y) = \frac{d\tilde{H}(z, -y))}{dz} = \begin{cases} \frac{\sigma(H(z, -y))}{\sigma(z)} & \text{for } z \in (x_i, x_{i+1}), i = 0, 1, \ldots, n, \\ \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Phi_k(x_i)y^k & \text{for } z = x_i, i = 1, \ldots, n. \end{cases}
\] (46)
Compare $H^{-1}(H(x)+y)$ used in section 2 with $\tilde{H}(x, y)$ defined in this section, one can see that they are both differential with respect to $x$. Replace the role of $H^{-1}(H(x)+y)$ in section 2 by $\tilde{H}(x, y)$, and use almost the same procedure as to get (36), we can get from (43)

$$\frac{\partial p(x, t)}{\partial t} = -\frac{\partial}{\partial x}\left[f(x)p(x, t) + b\sigma(x)p(x, t) + \frac{A}{2}\sigma(x)\sigma'(x)p(x, t)\right] + \frac{A}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)p(x, t)\right] + A^2\sum_{k=1}^{\infty}\left[\Phi_k(x_i)y_k\right]\nu(dy),$$

(47)

where, according to (46), $\Theta(x, y)$ is continuous with respect to $x$ and can be expressed as

$$\Theta(x, y) = \begin{cases} \frac{\sigma(H(x), y)}{\sigma(x)} & \text{for } x \in (x_i, x_{i+1}), i = 0, 1, \ldots, n, \\ \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Phi_k(x_i)y_k & \text{for } x = x_i, i = 1, \ldots, n. \end{cases}$$

(48)

Note that (48) has a similar form as (36) with $\tilde{H}(x, -y)$ corresponding to $H^{-1}(H(x) - y)$.

4 Conclusion

In this paper, we developed explicit forms of Fokker-Planck equations for Marcus SDEs as expressed in (5). The main results are summarized as follows.

If $\sigma(x)$ has no zeros, Fokker-Planck equation associated with Marcus SDEs as expressed in (5) is expressed as in (36).

If $\sigma(x)$ is a smooth function of $x$ and has $n$ zeros represented by $x_1, x_2, \ldots, x_n$, and denoted $x_0 = -\infty$ and $x_{n+1} = \infty$, then Fokker-Planck equation is expressed as in (47).

Appendix I

In this appendix, we shall show that $H^{-1}(H(\cdot) + y)$ maps from $(-\infty, +\infty)$ to $(-\infty, +\infty)$ for any given $y \in \mathbb{R}$. It is sufficient to show that $H(x)$, as defined in (10), is monotone and maps from $(-\infty, +\infty)$ to $(-\infty, +\infty)$. 

12
Note that $\sigma(x) \neq 0$ for $x \in \mathbb{R}$, it follows from (10) that $H(x)$ is monotone. Suppose $\sigma(x) > 0$, it follows from (10) that for $x \in \mathbb{R}$,

$$H(x) = \int_a^x \frac{dt}{\sigma(t)} = \int_a^x \frac{dt}{|\sigma(t) - \sigma(a) + \sigma(a)|} \geq \int_a^x \frac{dt}{|\sigma(t) - \sigma(a)| + |\sigma(a)|} \geq \int_a^x \frac{dt}{L|t - a| + |\sigma(a)|},$$

where $L$ is the Lipshitz constant satisfying $|\sigma(t) - \sigma(a)| \leq L|t - a|$. It follows from (A1) that $H(x)$ maps from $(-\infty, +\infty)$ to $(-\infty, +\infty)$ if $\sigma(x) > 0$. For cases where $\sigma(x) < 0$, the proof is similar.

**Appendix II**

In this appendix, we shall show that $H_i^{-1}(H_i(x) + y)$ maps from $(x_i, x_{i+1})$ to $(x_i, x_{i+1})$ for any given $y \in \mathbb{R}$.

The monotonicity of $H_i(x)$ follows from the fact that $\sigma(x) \neq 0$ for $x \in (x_i, x_{i+1})$. To finish the proof, it is sufficient to show that $H_i$ maps from $(x_i, x_{i+1})$ to $(-\infty, +\infty)$, and $H^{-1}(x)$ maps from $(-\infty, +\infty)$ to $(x_i, x_{i+1})$.

Assume that $\sigma(x) > 0$ for $x \in (x_i, x_{i+1})$, it follows from (8) that for $x \in (x_i, x_{i+1})$,

$$H_i(x) = \int_{a_i}^x \frac{dt}{\sigma(t)} = \int_{a_i}^x \frac{dt}{|\sigma(t) - \sigma(a) + \sigma(a)|} \geq \int_{a_i}^x \frac{dt}{|\sigma(t) - \sigma(a)| + |\sigma(a)|} \geq \int_{a_i}^x \frac{dt}{L|t - a| + |\sigma(a)|},$$

where $L$ is the Lipshitz constant satisfying $|\sigma(t) - \sigma(a)| \leq L|t - a|$. It follows from (A2) that $H_i(x)$ maps from $(x_i, x_{i+1})$ to $(-\infty, +\infty)$ and $H^{-1}(x)$ maps from $(-\infty, +\infty)$ to $(x_i, x_{i+1})$ if $\sigma(x) > 0$. For cases where $\sigma(x) < 0$ for $x \in (x_i, x_{i+1})$, the proof is similar.

**Appendix III**

In this appendix, we shall show that $\tilde{H}(x, y)$, as defined in (11), is differentiable with respect to $x$ for $x \in \mathbb{R}$ if $\sigma(x)$ is infinitely differentiable with respect to $x \in \mathbb{R}$.

For $x \neq x_i(i = 1, 2, \cdots, n)$, the conclusion is obvious since $H_i$ is a smooth function. It remains to show the differentiability of $\tilde{H}(x, y)$ with respect to $x$ at $x = x_i(i = 1, 2, \cdots, n)$.

The continuity of $\tilde{H}(x, y)$ at $x = x_i(i = 1, 2, \cdots, n)$ follows from (59).

By Taylor expansion at $y = 0$, $\tilde{H}(x, y)$ at $x \neq x_i$ ($i = 1, 2, \cdots, n$) can be
written as
\[ \tilde{H}(x, y) = x + \sigma(x)y + \frac{1}{2} \left( \sigma(x) \left( \frac{d}{dx} \sigma(x) \right) \right)y^2 + \frac{1}{3} \left( \sigma(x) \left( \frac{d}{dx} \sigma(x) \frac{d}{dx} \sigma(x) \right) \right)y^3 + \cdots \]

(A3)

By using (A3), it is straightforward to check that
\begin{align*}
\lim_{x \to x_i^+} \frac{\tilde{H}(x, y) - x_i}{x - x_i} &= \lim_{x \to x_i^-} \frac{\tilde{H}(x, y) - x_i}{x - x_i} = \sum_{k=1}^{\infty} \frac{1}{k!} \Phi_k(x_i) y^k, \\
\end{align*}

(A4)

where \( \Phi_k(x_i) \) is defined as
\[ \Phi_k(x_i) = \lim_{x \to x_i} \left( \frac{d}{dx} \sigma(x) \left( \frac{d}{dx} \sigma(x) \cdots \frac{d}{dx} \sigma(x) \right) \right) \] \( k \)-fold.

(A5)

It follows from (A4) that \( \tilde{H}(x, y) \) is differentiable with respect to \( x \) at \( x = x_i \) \((i = 1, 2, \cdots, n)\).

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