On gravitational spherical collapse without spacetime singularity

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This note discusses possible quantum effects in the context of homogeneous flat FLRW and inhomogeneous LTB gravitational spherical collapse, which lead to bounce solutions.
I. INTRODUCTION

In this note the statement: “There is no singularity, no event horizon and no information paradox” given in the context of black holes by S. A. Hayward [1], but also in [2], is supported by applying effective equations for a flat FLRW [3, 4] homogeneous gravitational collapse, which captures quantum effects of loop quantum gravity (LQG) [5, 6]. For a discussion of quantum effects see the recent review [7]. According to [8] the presented modification of the classical equations in the homogeneous case can be interpreted by assuming that Newton’s constant is given by

\[ G_{\text{eff}} = G(1 - \rho/\rho_{\text{crit}}), \]

with the critical density \( \rho_{\text{crit}} \propto \rho_{\text{Planck}} \sim O(1/G^2\hbar) \) [8]. In more detail the presented analysis is slightly extending and motivated by the work [9–11] e.g. by also including the case of naked singularities (in the classical limit [12]). Possible bouncing solutions are discussed in the inhomogeneous LTB [13, 14] dust case [15], showing the absence of e.g. naked singularities.

Recent reviews including cosmology are e.g. [16, 17].

II. QUANTUM COLLAPSE

In order to describe gravitational collapse [18, 19] a scalar field model coupled to gravity with a scalar field \( \Phi(t) \) is assumed. The resulting energy density \( \rho(t) \) and pressure \( p(t) \) satisfy the equation of state given by

\[ p = w\rho, \quad (1 + w) = \frac{2}{3}(1 - \beta), \quad \beta < 1. \]  

(1)

The flat homogeneous FLRW metric [3, 4] is used

\[ ds^2 = -dt^2 + a^2(t)[dr^2 + r^2d\Omega^2], \]

(2)

where \( a(t) \) is the scale factor. Taking into account quantum effects to order \( \rho^2 \) in the FLRW equation, reading [3],

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{G}{3} \rho(t)(1 - \frac{\rho}{\rho_{\text{crit}}}), \]

(3)

with \( \rho_{\text{crit}} \propto O(1/G^2\hbar) \) and \( H(t) \) the Hubble parameter, both depending on \( \hbar \). As a back-reaction an ingoing negative energy flux is actually present [20]. The limit \( \rho_{\text{crit}} \gg \rho \), i.e. \( \hbar \to 0 \) implies the classical general limit (see for a summary and notation [21, 22]). The continuity equation reads

\[ \dot{\rho}(t) = -3H(t)(\rho + p) = -3H(t)(1 + w)\rho(t). \]

(4)

Eqs. 3 and 4 lead to the dependent Einstein equation

\[ \dot{H}(t) = -\frac{G}{2}(\rho + p)(1 - \frac{2\rho}{\rho_{\text{crit}}}). \]

(5)

Note \( \rho_{\text{crit}}/3 = 1/a^3 \) in [3], and in the following \( \rho_{\text{crit}}/3 = \dot{\rho} \).

The analytic solutions for \( a(t) \) and \( \rho(t) \) (in units \( G = c = 1 \)) are

\[ a(t) = a_B[(\dot{\rho} - 1)(1 - t/t_B)^2 + 1]^{-\frac{1}{2(1-\beta)}}, \]

(6)
\[
a_B = \hat{\rho}^{-\frac{1}{2(1-\beta)}},
\]

such that \(a(t = 0) = 1\) and \(a(t = t_B) = a_B \neq 0\), with

\[
t_B = \sqrt[1-\beta]{\frac{\hat{\rho} - 1}{(1-\beta)\hat{\rho}}},
\]

\(a(t)\) does not vanish as a function of \(t\), instead it bounces at \(t = t_B\) [9, 10]; see also [11, 18, 23–25]. For \(\rho_{\text{crit}} \to \infty\), the classical gravity limit [12, 26, 27] is obtained,

\[
a_{\text{cl}}(t) = (1 - t/t_s)^{\frac{1}{1-\beta}},
\]

with \(t_s = 1/(1 - \beta)\), such that

\[
t_B = t_s \sqrt[1-\beta]{\frac{\hat{\rho} - 1}{\hat{\rho}}} < t_s.
\]

(see Fig.1). Both \(a_B\) and \(t_B\) depend on \(\hbar\).

Fig.1 Schematic illustration of the scale factor: the bouncing \(a(t)\), eq.(6) (solid curve) and the classical model \(a_{\text{cl}}(t)\), eq.(9) (dashed curve). See also Fig.1 in [3] and Fig.2 in [15].

The density \(\rho(t)\) reads with \(\rho(0) = 3\),

\[
\rho(t)/3 = \frac{\hat{\rho}}{[(\hat{\rho} - 1)(1 - t/t_B)^2 + 1]},
\]

which is finite at \(t = t_B\), namely \(\rho(t_B) = \rho_{\text{crit}}\). In the classical limit it becomes

\[
\rho_{\text{cl}}(t)/3 = \frac{1}{(1 - t/t_s)^2},
\]

which is divergent at \(t = t_s = \frac{1}{1-\beta}\).

In summary the fluid is first collapsing for \(0 \leq t < t_B\), and than for \(t \geq t_B\) expanding, i.e. a bouncing situation. At \(t = t_B\), the theory becomes free according to eq.(3), i.e. \(H = 0\). The Hubble constant reads

\[
H(t) = -\frac{\sqrt{\hat{\rho}(\hat{\rho} - 1)(1 - t/t_B)}}{[(\hat{\rho} - 1)(1 - t/t_B)^2 + 1]}.
\]
III. APPARENT HORIZON

Introducing $R(r, t) = ra(t)$, the physical radius of the bouncing matter, the location of the apparent horizon is given by \[27–32\]

\[R_{ah}^2 = \frac{1}{H^2(t)}, \quad (14)\]
i.e.

\[r_{ah}^2 = \frac{1}{\dot{a}^2(t)}, \quad (15)\]
derived from the expansion

\[\Theta = \Theta_+ \Theta_- = 2(H^2(t) - \frac{1}{R^2}) = 0. \quad (16)\]

For the collapsing (expanding) phase one requires for the location

\[\Theta_+ = 0, \text{ i.e. } R_{ah} = -\frac{1}{H(t)} \quad (17)\]

versus

\[\Theta_- = 0, \text{ i.e. } R_{ah} = +\frac{1}{H(t)}, \quad (18)\]
respectively. Eq.(14) expressed in terms of $t/t_B$ reads

\[(t/t_B)_{ah} = 1 \pm \frac{\sqrt{\hat{\rho}}}{2\sqrt{\hat{\rho} - 1}} R_{ah} \mp \sqrt{\frac{\hat{\rho}}{4(\hat{\rho} - 1)}} R_{ah}^2 - \frac{1}{\hat{\rho} - 1}, \quad (19)\]

which is except for $t_B$ independent on $\beta$ and where the first (+) sign corresponds to the case \[18\].

From the expression of the vanishing square root in eq.(19) a minimum radius of the horizon in the $(t, R)$ plane is derived

\[R_{ah}^{min} = \frac{2}{\sqrt{\hat{\rho}}}, \quad (20)\]
at

\[(t/t_B)_{ah}^{min} = 1 \mp \frac{1}{\sqrt{\hat{\rho} - 1}}. \quad (21)\]

The boundary at $t_{ah}^{min}$ is evaluated by

\[R_b(t_{ah}^{min}) = r_b \left[\frac{2}{\hat{\rho}}\right]^{\frac{1}{2(1-\beta)}}\], \quad (22)\]

with $r_b$ the boundary of the fluid.

For $R_b(t_{ah}^{min}) > R_{ah}^{min}$ there are trapped regions, sketched in Fig.2, whereas for $R_b(t_{ah}^{min}) < R_{ah}^{min}$ there are no trapped regions formed in the physical region of the bouncing fluid.

In terms of the coordinate $r$ one may introduce a critical $r_c$ by $r_b = r_c$ with $R_b(t_{ah}^{min}) = R_{ah}^{min}$, which reads
\[ r_c = 2^{\frac{1-2\beta}{(1-\beta)}} \rho^{\frac{\beta}{(1-\beta)}}. \]  

Assuming the physical region of the bouncing fluid (cloud) to be given by \( r < r_b \), with a small boundary radius \( r_b < < 1 \), then for

\[ r_b < r_c \]  

there is no trapped surface formed in contrast with the condition

\[ r_b > r_c , \]  

which allows to form a trapping region (see Fig.2).

![Fig.2 Inner apparent horizon and trapped region in the collision phase. The expanding phase is obtained by reflecting the curves on the line \( t/t_B = 1 \).](image)

A numerical example is with \( \hat{\rho} = 1000 \) \([9]\) and for the classical black hole cases:
- dust \([33]\) with \( \beta = -1/2 \)
  \[ r_c = 0.5 , \]  
  i.e. no trapped region for \( r_b \leq 0.5 \).
- For the classical naked singularity case, e.g. for \( \beta = +1/2 \),
  \[ r_c = \sqrt{\hat{\rho}} > > 1 , \]  
  which is much larger than \( r_b < 1 \), and no trapped region inside the fluid is present.

The case of the presence of naked singularities in the classical limit \([12, 34, 35]\) is changed, no singularity is present for \( \beta > 0 \) in the interior region after the critical density \( \rho_{\text{crit}}(\hat{\rho}) \) is introduced.
IV. EXTERIOR METRIC

In order to obtain the full spacetime (here for $\beta < 0$), the metric eq.(2) has to be matched with the outer region $r \geq r_b$.

Convenient coordinates are the advanced Eddington-Finkelstein ones [3]. In the following only the collapsing sector $t/t_B \leq 1$ is discussed; the expanding one is obtained by symmetry with respect to the line $t/t_B = 1$.

Introducing the transformation with $R = ra(t)$,

$$adr = \frac{1}{1-RH}dR - RHdv,$$

$$dt = -\frac{1}{1-RH}dR + dv,$$

leading to

$$ds^2 = -(1 - R^2H^2)dv^2 + 2vdR + R^2d\Omega^2.$$ (30)

In order to perform the matching at the boundary $r = r_b$ it is useful to sketch it by using analytical approximations. In the following a convenient one is obtained by the limit of large negative $\beta$ ($\beta \rightarrow -\infty$), i.e. $a(t) \rightarrow 1$, but $H(t) \approx \dot{a}(t) \neq 0$ (eq.(13)), and $t_B \neq 0$. It implies $R_B(t) \approx r_b$.

The points (+) and (−) in Fig.3 are determined by eq.(19) with $R_{ah} = R_b \approx r_b$,

$$(t/t_B)_\pm \approx 1 - \frac{\sqrt{\rho r_b}}{2\sqrt{\rho - 1}} \pm \frac{\sqrt{\rho r_b^2} - 4}{2\sqrt{\rho - 1}}.$$ (31)

Fig.3 Outer apparent horizon and trapped region (see Fig.2 in [36]).
For the exterior metric the coordinates
\[ ds^2 = -F(v, R)dv^2 + 2dvdR + R^2d\Omega^2 \] (32)
are introduced, and the Hayward metric \cite{36,37} is used for illustration
\[ F(v, R) = 1 - \frac{2m(v)R^2}{R^3 + 2m(v)l^2}, \ l = \text{cst.} \] (33)
Following \cite{36} for \( m(v) \) the form
\[ 2m(v) = \text{cst} \exp\left(\frac{(v - v_0)^2}{\sigma^2}\right) \] (34)
is taken. The location of the horizons (see Fig.3) is obtained from \( \Theta_\pm = 0 \), i.e. \( F(v, R) = 0 \), or
\[ R^3 - 2m(v)R^2 + 2m(v)l^2 = 0, \] (35)
with the approximate solutions \cite{36}
\[ R_1 \approx 2m + O(l^2/2m), R_3 \approx l + O(l^2/2m). \] (36)
Matching at the boundary may be obtained by \( R_3 \approx l < r_b \) and \( R_1 \approx r_b \) (see Fig.3), i.e.
\[ v_\pm \approx v_0 \pm \sigma \sqrt{\ln\left(\frac{\text{cst}}{r_b}\right)} . \] (37)
On the boundary the \((\pm)\) points are related by solving
\[ \left(\frac{dv}{dt}\right)_b \approx \frac{1}{1 - r_bH(t)} , \] (38)
which finally fixes the parameters \( l, \text{cst}, v_0 \) in terms of \( \hat{\rho}, r_b \). For illustration one may consider the case of large \( \hat{\rho} \), with \( v \approx t \), such that
\[ \sigma \sqrt{\ln\left(\frac{\text{cst}}{r_b}\right)} \approx \frac{r_Bt_B}{2}, v_0 \approx (1 - r_b/2)t_B, \] (39)
with \( t_B \) fixed.
Fig. 4 Collapse region: sketch for matching the inner and outer regions with respect to the boundary $R_b(t)$ (see text for details).

This Fig. 4 showing matching indicates that only closed trapping horizons exist for a finite time in the discussed non-singular spacetime geometry.

V. INHOMOGENEOUS DUST

By describing inhomogeneous dust (pressure $p = 0$, $\beta = -1/2$, extending [33]) the Lemaître-Tolman-Bondi (LTB) flat metric [13, 14]

$$ds^2 = -dt^2 + R'^2 dr^2 + R^2 d\Omega^2$$

is used with $R = R(t, r)$ and $R' = \frac{\partial}{\partial r} R(t, r)$. The classical Einstein field equation reads

$$\frac{\dot{R}^2}{R^2} = \frac{\dot{a}^2(t, r)}{a^2(t, r)} = \frac{F(r)}{R^3},$$

where the mass $F(r) = r^3 M(r)$ and $R(t, r) = r\alpha(t, r)$ are introduced [38]. In comparison of this equation with eq.(33) the average density is defined by

$$\rho_{av}(t, r) = \frac{3F(r)}{R^3} = \frac{3M(r)}{a^3(t, r)}, \ G = 1.$$ (42)

Possible quantum effects to order $\rho_{av}^2$ are introduced in analogy to eq.(33) by modifying eq.(41), introducing an effective density $\rho_{\text{effective}}$ [17],

$$\frac{\dot{R}^2}{R^2} = \frac{1}{3} \rho_{\text{effective}}(t, r) = \frac{1}{3} \rho_{av}(1 - \frac{\rho_{av}}{\rho_{\text{crit}}}) =$$ (43)
This equation is solved by a non-vanishing scale factor

\[ a(t, r) = \left\{ \frac{M(r)}{\hat{\rho}} + (\hat{\rho} - M(r))(1 - t/t_B)^2 \right\}^{\frac{1}{3}}, \]

with \( a(t = 0, r) = 1 \) and which bounces at

\[ a_B(t_B, r) = \left( \frac{M(r)}{\hat{\rho}} \right)^{\frac{1}{3}} \neq 0 \]

at

\[ t_B(r) = \frac{2}{3} \sqrt{\frac{\hat{\rho} - M(r)}{\hat{\rho} M(r)}}, \]

where \( t_B \geq 0, \hat{\rho} \geq M(r) \).

Assuming \( M(r) > 0 \), but \( M'(r) < 0 \), e.g. \( M(r) \approx 1 - M_2 r^2 > 0 \), i.e. a decreasing mass with increasing radius \( r \), the bounce time \( t_B(r) \) (\( a_B(t_B, r) \)) is increasing (decreasing) with \( r \), near \( r \approx 0 \) (see Fig.5).

The crucial result is a finite density \[38\]

\[ \rho(t, r) = \frac{F'}{R^2 R'}, \]

which behaves near \( r \approx 0 \) as

\[ \rho(t, r \approx 0) \approx \frac{3M(r)}{a^3(t, r)}|_{r \approx 0} = \rho_{av}(t, r \approx 0) = \]

\[ = \frac{3\hat{\rho}}{1 + (\hat{\rho} - 1)(1 - t/t_B)^2}, \]

with \( M(r \approx 0) \approx 1. \rho \) is finite at \( t_B \), but becomes, however, singular at \( t = t_s \) in the classical limit \( \hat{\rho} \to \infty \).

The case of section I. is reproduced with \( M(r) = 1 \) and \( \beta = -1/2 \). In both cases the scale factor as a function of \( t \) does not vanish, instead it bounces at \( a = a_B \) at \( t_B \), where in the inhomogeneous dust case both functions depend on \( r \).

In the classical limit, \( \hat{\rho} \to \infty \), the radius \( R_{cl}(t, r) \) reads \[38\]

\[ R_{cl}(t, r) = r(1 - t/t_s(r))^{\frac{2}{3}} = \left[ \frac{3}{2} \sqrt{F(r)(t_s(r) - t)} \right]^{\frac{2}{3}}, \]

which vanishes at \( t_s(r) = \frac{2}{3\sqrt{M(r)}} \). Because of the \( r \) dependence there is no simultaneous collapse.

As an example of a collapse scenario: the dust has a boundary at \( r = r_b > 0 \), and \( M(0) > M(r_b) \), i.e. a maximum mass at \( r = 0 \), with \( M'(r) < 0, t' = -\frac{M'}{M^{3/2}} > 0 \). See e.g. Fig.2 in \[39\]. There is a formation of a locally naked singularity \[34, 38\]. For a review \[27\].

The homogeneous singular limit is finally obtained with \( M(r) = 1, t_s = 2/3 \) (see eq.(9)).
To generalize section II. for the inhomogeneous case the apparent horizon is defined by
\[ g^{\mu \nu} \partial_\mu \dot{R} \partial_\nu \dot{R} = 0, \]
i.e. collapsing phase: \( \dot{R}_{ah} = -1 \) and expanding phase: \( \dot{R}_{ah} = +1 \). In the following the collapsing phase up to the bouncing time \( t_B \) is considered,
\[
\dot{R}_{ah} = -\frac{r \sqrt{M(r)(\dot{\rho} - M(r))(1 - t/t_B)}}{a^2(t, r) \sqrt{\dot{\rho}}} = -1.
\]
Eq. (53) can be written for \( t = t_{ah} \) and \( r_{ah} \) as
\[
(1 - t/t_B)^2 - (r_{ah} \sqrt{M})^{3/2}(\frac{\dot{\rho}}{\dot{\rho} - M})^{1/4}(1 - t/t_B)^{3/2} + \frac{M(r)}{\dot{\rho} - M} = 0.
\]
In the classical limit \( \dot{\rho} \to \infty \) the apparent horizon is given by \[38\]
\[ t_{ah}(r) = t_s(r) - \frac{2}{3} r^3 M(r), \]
in the flat region of the marginally bound collapse. It is noted \[38\] that the collapse is simultaneous for \( t_s(r) = cst. \), i.e. \( M = 1 = cst. \), \( t_{ah} = \frac{2}{3}(1 - r^3) \), which corresponds to the Oppenheimer-Snyder collapse \[33\].

Having introduced a finite critical density \( \rho_{crit} \) as a possible result due to quantum effects the scale factor \( a(t, r) \), eq. (45), does not vanish \( a(t, r) \neq 0 \) for \( M(r) \neq 0 \), and the classical singular collapse, a black hole or a naked singularity, is replaced by a system which bounces back \( (\dot{a}(t, r) = 0 \text{ at } t_B) \) before it could reach the singularity (Fig.5).

![Fig.5 Schematic illustration of the non-vanishing scale factor \( a(t, r) \) of eq. (45) as a function of \( t \). Solid (dashed) curve for \( t > 0 \) (\( t = 0 \)).](image)

There is even no trapped region present inside the cloud bounded by \( r = r_b \), when the condition is satisfied, which generalizes the homogeneous condition in the \( r, t \) plane given in
eq. (24) for \( \beta = -1/2 \) (dust),

\[
\frac{r_{\text{ah}}^{\text{min}}}{r_b} > 1. \tag{56}
\]

To indicate in an approximate way for small values of \( r_b \) and \( r \), and \( M/\rho \simeq M(r_b)/\rho = \text{const} \) and small, it reads from eq. (54),

\[
r_{\text{ah}}^{\text{min}} \simeq \frac{2^{4/3}}{\hat{\rho}^{1/6} \sqrt{3M(r_b)}}. \tag{57}
\]

As long as \( M(r_b) < 1 \) a trapped inside region becomes more unlikely in the inhomogeneous than in the homogeneous case.

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