Non-linear shock acceleration in the presence of seed particles

Pasquale Blasi\textsuperscript{a,b,1}

\textsuperscript{a}INAF/Osservatorio Astrofisico di Arcetri, Largo E. Fermi, 5 I-50125 Firenze (Italy)
\textsuperscript{b}INFN, Sezione di Firenze

Abstract

Particles crossing repeatedly the surface of a shock wave can be energized by first order Fermi acceleration. The linear theory is successful in describing the acceleration process as long as the pressure of the accelerated particles remains negligible compared to the kinetic pressure of the incoming gas (the so-called test particle approximation). When this condition is no longer fulfilled, the shock is modified by the pressure of the accelerated particles in a nonlinear way, namely the spectrum of accelerated particles and the shock structure determine each other. In this paper we present the first description of the nonlinear regime of shock acceleration when the shock propagates in a medium where seed particles are already present. This case may apply for instance to supernova shocks propagating into the interstellar medium, where cosmic rays are in equipartition with the gas pressure. We find that the appearance of multiple solutions, previously found in alternative descriptions of the nonlinear regime, occurs also for the case of reacceleration of seed particles. Moreover, for parameters of concern for supernova shocks, the shock is likely to turn nonlinear mainly due to the presence of the pre-existing cosmic rays, rather than due to the acceleration of new particles from the thermal pool. We investigate here the onset of the nonlinear regime for the three following cases: 1) seed particles in equipartition with the gas pressure; 2) particles accelerated from the thermal pool; 3) combination of 1) and 2).

\textit{Key words:} cosmic rays, high energy, acceleration

\textsuperscript{1} E-mail: blasi@arcetri.astro.it

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1 Introduction

Diffusive acceleration at newtonian shock fronts is an extensively studied phenomenon. Detailed discussions of the current status of the investigations can be found in some recent excellent reviews [1–4]. While much is by now well understood, some issues are still subjects of much debate, for the theoretical and phenomenological implications that they may have. The most important of these is the backreaction of the accelerated particles on the shock: the violation of the test particle approximation occurs when the acceleration process becomes sufficiently efficient to generate pressures of the accelerated particles which are comparable with the incoming gas kinetic pressure. Both the spectrum of the particles and the structure of the shock are changed by this phenomenon, which is therefore intrinsically nonlinear.

At present there are three viable approaches to account for the backreaction of the particles upon the shock: one is based on the ever-improving numerical simulations [4–10] that allow a self-consistent treatment of several effects. The second approach is a fluid approach, and treats cosmic rays as a relativistic second fluid. This class of models was proposed and discussed in [11–15]. These models allow one to obtain the thermodynamics of the modified shocks, but do not provide information about the spectrum of accelerated particles.

The third approach is analytical and may be very helpful to understand the physics of the nonlinear effects in a way that sometimes is difficult to achieve through simulations, due to their intrinsic complexity.

In Ref. [16] a perturbative approach was adopted, in which the pressure of accelerated particles was treated as a small perturbation. By construction this method provides an answer only for weakly modified shocks.

An alternative approach was proposed in [17–20], based on the assumption that the diffusion of the particles is sufficiently energy dependent that different parts of the fluid are affected by particles with different average energies. The way the calculations are carried out implies a sort of separate solution of the transport equation for subrelativistic and relativistic particles, so that the two spectra must be somehow connected at $p \sim mc a\ posteriori$.

Recently, in [21–23], the effects of the non-linear backreaction of accelerated particles on the maximum achievable energy were investigated, together with the effects of geometry. The maximum energy of the particles accelerated in supernova remnants in the presence of large acceleration efficiencies was also studied in [24,25].

The need for a practical solution of the acceleration problem in the non-linear regime was recognized in [26], where a simple analytical broken-power-law
approximation of the non-linear spectra was presented.

Recently, some promising analytical solutions of the problem of non-linear shock acceleration have appeared in the literature [27–29]. These solutions seem to avoid many of the limitations of previous approaches.

Numerical simulations have been instrumental to identify the dramatic effects of the particles backreaction: they showed that even when the fraction of particles injected from the thermal gas is relatively small, the energy channelled into these few particles can be an appreciable part of the kinetic energy of the unshocked fluid, making the test particle approach unsuitable. The most visible effects of the backreaction of the accelerated particles on the shock appear in the spectrum of the accelerated particles, which shows a peculiar flattening at the highest energies. The analytical approaches reproduce well the basic features arising from nonlinear effects in shock acceleration.

While several calculations exist of the nonlinear effects in the shock acceleration of quasi-monochromatic particles injected at a shock surface, there is no description at present of how these effects appear, if they do, when the shock propagates in a medium where (pre)accelerated particles already exist. The linear theory of this phenomenon was developed by Bell [31], but has never been generalized to its nonlinear extension. In fact, the backreaction can severely affect the process of re-energization of preaccelerated particles: Bell already showed that for strong shocks the energy content of a region where cosmic rays were present could be easily enhanced by a factor $\sim 10$ at each shock passage, so that equipartition could be readily reached. In these conditions the backreaction of the accelerated particles should be expected.

We report here on the first analytical treatment of the shock acceleration in the presence of seed nonthermal particles, with the inclusion of their nonlinear backreaction on the shock. Our approach is a generalization of the analytical method introduced in [29] to describe the nonlinear shock acceleration with monochromatic injection of quasi-thermal particles. In fact, we present here also a general calculation that accounts for both thermal particles and seed nonthermal particles. This case may be of interest for the study of supernova shocks propagating through the interstellar medium (ISM) where pressure balance exists between gas and cosmic rays.

Nonlinear effects in shock acceleration of thermal particles result in the appearance of multiple solutions in certain regions of the parameter space. This behaviour resembles that of critical systems, with a bifurcation occurring when some threshold is reached. In the case of shock acceleration, it is not easy to find a way of discriminating among the multiple solutions when they appear. Nevertheless, in [30], a two fluid approach has been used to demonstrate that when three solutions appear, the one with intermediate efficiency for particle
acceleration is unstable to corrugations in the shock structure and emission of acoustic waves. Plausibility arguments may be put forward to justify that the system made of the shock plus the accelerated particles may sit at the critical point, but the author is not aware of any real proof that this is what happens. The physical parameters that play a role in this approach to criticality are the maximum momentum achievable by the particles in the acceleration process, the Mach number of the shock, and the injection efficiency, namely the fraction of thermal particles crossing the shock that are accelerated to nonthermal energies. The last of them, the injection efficiency, hides a crucial physics problem by itself, and may play an important role in establishing the level of shock modification. This efficiency parameter in reality is defined by the microphysics of the shock and should not be a free parameter of the problem. Unfortunately, our poor knowledge of such microphysics, in particular for collisionless shocks, does not allow us to establish a clear and universal connection between the injection efficiency and the macroscopic shock properties.

The paper is structured as follows: in §2 we describe the effect of nonlinearity on shock acceleration and our mathematical approach to describe it. In particular we generalize previous calculations to the case in which seed particles exist in the region where the shock is propagating; in §3 we describe the gas dynamics in the presence of a non-negligible pressure of accelerated particles. In §4 we describe our results, with particular attention for the onset of the particle backreaction and for the appearance of multiple solutions. We conclude in §5.

2 Nonlinear shock acceleration

In this section we solve the diffusion-convection equation for the cosmic rays in the most general case in which particles are injected according to some function $Q(x, p)$ and the shock propagates within a region where some cosmic ray distribution exists, that we denote as $f_\infty(p)$, spatially homogeneous before the shock crossing (upstream).

For simplicity we limit ourselves to the case of one-dimensional shocks, but the introduction of different geometrical effects is relatively simple, and in fact many of our conclusions should not be affected by geometry.

The equation that describes the diffusive transport of particles in one dimension is

$$\frac{\partial}{\partial x} \left[ D \frac{\partial f(x, p)}{\partial x} \right] - u \frac{\partial f(x, p)}{\partial x} + \frac{1}{3} \frac{du}{dx} p \frac{\partial f(x, p)}{\partial p} + Q(x, p) = 0, \quad (1)$$
where we assumed stationarity ($\partial f / \partial t = 0$). The $x$ axis is oriented from upstream to downstream, as in fig. 1. The presence of pre-existing cosmic rays is introduced as a boundary condition at upstream infinity, by requiring that $f(x = -\infty, p) = f_\infty(p)$. One should keep in mind that the common picture of a fluid whose speed is constant ($u_1$) until it hits the shock surface is not appropriate for modified shocks. In fact, in this case, the pressure of the accelerated particles may become large enough to slow down the fluid before it crosses the shock surface. Therefore in general at upstream infinity the gas flows at speed $u_0$, different from $u_1$ (fluid speed immediately upstream of the shock). The two quantities are approximately equal only when the accelerated particles do not have dynamical relevance. The injection term is taken in the form $Q(x, p) = Q_0(p)\delta(x)$. As a first step, we integrate eq. 1 around $x = 0$, from $x = 0^-$ to $x = 0^+$, denoted in fig. 1 as points “1” and “2” respectively, so that the following equation can be written:

$$\left[D \frac{\partial f}{\partial x}\right]_2 - \left[D \frac{\partial f}{\partial x}\right]_1 + \frac{1}{3} p \frac{df_0}{dp} (u_2 - u_1) + Q_0(p) = 0,$$

(2)

where $u_1$ ($u_2$) is the fluid speed immediately upstream (downstream) of the shock and $f_0$ is the particle distribution function at the shock location. By requiring that the distribution function downstream is independent of the spatial coordinate (homogeneity), we obtain $\left[D \frac{\partial f}{\partial x}\right]_2 = 0$, so that the boundary

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Fig. 1. Schematic view of the shock region.
condition at the shock can be rewritten as

$$
\left[ D \frac{\partial f}{\partial x} \right]_1 = \frac{1}{3} p \frac{df_0}{dp} (u_2 - u_1) + Q_0(p).
$$

We can now perform the integration of eq. (1) from \( x = -\infty \) to \( x = 0^- \) (point “1”), in order to take into account the boundary condition at upstream infinity. Using eq. (3) we obtain

$$
\frac{1}{3} p \frac{df_0}{dp} (u_2 - u_1) - u_1 f_0 + u_0 f_\infty + Q_0(p) + \int_{-\infty}^{0^-} dx f \frac{du}{dx} + \frac{1}{3} \int_{-\infty}^{0^-} dx p \frac{\partial f}{\partial p} = 0.
$$

We can now introduce a quantity \( u_p \) defined as

$$
u_p = u_1 - \frac{1}{f_0} \int_{-\infty}^{0^-} dx \frac{du}{dx} f(x, p), \tag{5}
$$

whose physical meaning is instrumental to understand the nonlinear backreaction of particles. The function \( u_p \) is the average fluid velocity experienced by particles with momentum \( p \) while diffusing upstream away from the shock surface. In other words, the effect of the average is that, instead of a constant speed \( u_1 \) upstream, a particle with momentum \( p \) experiences a spatially variable speed, due to the pressure of the accelerated particles that is slowing down the fluid. Since the diffusion coefficient is in general \( p \)-dependent, particles with different energies feel a different compression coefficient, higher at higher energies if, as expected, the diffusion coefficient is an increasing function of momentum.

The role of \( u_p \) can also be explained as follows: the distribution function \( f(x, p) \) at a distance \( x \) from the shock surface can be written as [28]

$$
f(x, p) = f_0(p) \exp \left[ \frac{q(p)}{3D(p)} \int_0^x dx' u(x') \right],
$$

where \( q(p) = -d \ln f_0(p)/d \ln p \) is the local slope of \( f_0(p) \) and the diffusion coefficient \( D(p) \) has been assumed independent of the location \( x \). In first approximation, we can assume that the exponential factor remains important when it is of order unity, namely when its argument is much less than unity. We can therefore introduce a distance \( x_p \), which is the distance at which the
exponential equals one. This means that
\[ u_p \approx u_1 - \int_{-x_p}^{0_-} dx \frac{du}{dx} \approx u_1 - [u_1 - u(x_p)] \approx u(x_p), \]
so that the speed \( u_p \) can be interpreted as the fluid speed at the point \( x_p \) where the particles with momentum \( p \) reverse their motion in the upstream fluid and return to the shock. Note that the diffusion coefficient enters the calculation of the distance \( x_p \) but does not enter directly the calculation of \( u_p \). In other words, different diffusion coefficients may move the point where the fluid speed is \( u_p \) closer to or farther from the shock surface, but do not affect the value of \( u_p \). This approach is similar to that introduced in [17].

With the introduction of \( u_p \), eq. (4) becomes
\[ \frac{1}{3} \frac{df_0}{dp}(u_2 - u_p) - f_0 \left[ u_p + \frac{1}{3} \frac{d}{dp} u_p \right] + u_0 f_\infty + Q_0(p) = 0, \] (6)
where we used the fact that
\[
\frac{d}{dp} \int_{-\infty}^{0_-} dx \frac{du}{dx} f = \frac{d}{dp} \left[ \frac{df_0}{dp}(u_1 - u_p) - f_0 \frac{d}{dp} u_p \right].
\]

Eq. (6) can be written in a way that resembles more the equation for shocks with no particles backreaction but in the presence of seed particles with distribution \( f_\infty(p) \) and injection function \( Q_0(p) \):
\[ p \frac{df_0}{dp} = - \frac{3}{u_p - u_2} \left\{ f_0 \left[ u_p + \frac{1}{3} \frac{d}{dp} u_p \right] - u_0 f_\infty - Q_0(p) \right\}. \] (7)

The solution of this equation can be written in the following implicit form:
\[ f_0(p) = f_0^{\text{reac}}(p) + f_0^{\text{inj}}(p) = \]
\[ \int_{p_0}^{p} \frac{d\bar{p}}{\bar{p}} \left[ u_0 f_\infty(\bar{p}) + Q_0(\bar{p}) \right] \exp \left\{ - \int_{\bar{p}}^{p} \frac{d\bar{p}'}{\bar{p}' - u_2} \left[ u_{\bar{p}'} + \frac{1}{3} \frac{d}{d\bar{p}'} u_{\bar{p}'} \right] \right\}. \] (8)

In the case of monochromatic injection with momentum \( p_{inj} \) at the shock surface, we can write
\[ Q_0(p) = \frac{\eta_{\text{gas}, 1} u_1}{4 \pi p_{inj}^2} \delta(p - p_{inj}), \] (9)
where \( n_{\text{gas},1} \) is the gas density immediately upstream \((x = 0^-)\) and \( \eta \) parametrizes the fraction of the particles crossing the shock which are going to take part in the acceleration process. The injection term in eq. (8) becomes

\[
f_{\text{inj}}^0(p) = \int_{p_0}^{p} \frac{d\tilde{p}}{\tilde{p}} \frac{3Q_0(\tilde{p})}{u_\infty - u_2} \exp \left\{ - \int_{\tilde{p}}^{p} \frac{d\tilde{p}'}{\tilde{p}'} \frac{3}{u_{\tilde{p}'} - u_2} \left[ u_{\tilde{p}'} + \frac{1}{3} \frac{d}{d\tilde{p}'} \frac{du_{\tilde{p}'}^{\prime}}{du_{\tilde{p}'}^{\prime}} \right] \right\} = \left. \frac{3R_{\text{sub}}}{R_{\text{sub}} - 1} \frac{\eta n_{\text{gas},1}}{4\pi p_{\text{inj}}^3} \exp \right\{ - \int_{p_{\text{inj}}}^{p} \frac{d\tilde{p}'}{\tilde{p}'} \frac{3}{u_{\tilde{p}'} - u_2} \left[ u_{\tilde{p}'} + \frac{1}{3} \frac{d}{d\tilde{p}'} \frac{du_{\tilde{p}'}^{\prime}}{du_{\tilde{p}'}^{\prime}} \right] \right\} \right.^{\frac{3R_{\text{tot}}}{R_{\text{tot}} U(p) - 1}}.
\]

(10)

Here we introduced the two quantities \( R_{\text{sub}} = u_1/u_2 \) and \( R_{\text{tot}} = u_0/u_2 \), which are respectively the compression factor at the gas subshock and the total compression factor between upstream infinity and downstream. The two compression factors would be equal in the test particle approximation. For a modified shock, \( R_{\text{tot}} \) can attain values much larger than \( R_{\text{sub}} \) and more in general, much larger than 4, which is the maximum value achievable for an ordinary strong non-relativistic shock. The increase of the total compression factor compared with the prediction for an ordinary shock is responsible for the peculiar flattening of the spectra of accelerated particles that represents a feature of nonlinear effects in shock acceleration. In terms of \( R_{\text{sub}} \) and \( R_{\text{tot}} \) the density immediately upstream is \( n_{\text{gas},1} = (\rho_0/m_p)R_{\text{tot}}/R_{\text{sub}} \).

In eq. (10) we can introduce a dimensionless quantity \( U(p) = u_p/u_0 \) so that

\[
f_{\text{inj}}^0(p) = \left( \frac{3R_{\text{sub}}}{R_{\text{tot}} U(p) - 1} \right) \frac{\eta n_{\text{gas},1}}{4\pi p_{\text{inj}}^3} \exp \left\{ - \int_{p_{\text{inj}}}^{p} \frac{d\tilde{p}'}{\tilde{p}'} \frac{3R_{\text{tot}} U(p')}{R_{\text{tot}} U(p') - 1} \right\}.
\]

(11)

Introducing the same formalism also for the reacceleration term in eq. (8), we obtain the general expression

\[
f_0(p) = \frac{3R_{\text{tot}}}{R_{\text{tot}} U(p) - 1} \int_{p_0}^{p} \frac{d\tilde{p}}{\tilde{p}} f_{\infty}(\tilde{p}) \exp \left\{ - \int_{\tilde{p}}^{p} \frac{d\tilde{p}'}{\tilde{p}'} \frac{3R_{\text{tot}} U(p')}{R_{\text{tot}} U(p') - 1} \right\} + \left( \frac{3R_{\text{sub}}}{R_{\text{tot}} U(p) - 1} \right) \frac{\eta n_{\text{gas},1}}{4\pi p_{\text{inj}}^3} \exp \left\{ - \int_{p_{\text{inj}}}^{p} \frac{d\tilde{p}'}{\tilde{p}'} \frac{3R_{\text{tot}} U(p')}{R_{\text{tot}} U(p') - 1} \right\}.
\]

(12)

The solution of the problem is known if the velocity field \( U(p) = u_p/u_0 \) is known. The nonlinearity of the problem reflects in the fact that \( U(p) \) is in turn a function of \( f_0 \) as it is clear from the definition of \( u_p \). In order to solve the problem we need to write the equations for the thermodynamics of the system.
including the gas, the reaccelerated cosmic rays, the cosmic rays accelerated from the thermal pool and the shock itself. We write and solve these equations in the next section.

3 The gas dynamics of modified shocks with seed particles

The velocity, density and thermodynamic properties of the fluid can be determined by the mass and momentum conservation equations, with the inclusion of the pressure of the accelerated particles and of the preexisting cosmic rays. We write these equations between a point far upstream \((x = -\infty)\), where the fluid velocity is \(u_0\) and the density is \(\rho_0 = m_{\text{gas},0}\), and the point where the fluid upstream velocity is \(u_p\) (density \(\rho_p\)). The index \(p\) denotes quantities measured at the point where the fluid velocity is \(u_p\), namely at the point \(x_p\) that can be reached only by particles with momentum \(\geq p\).

The mass conservation implies:

\[
\rho_0 u_0 = \rho_p u_p. \tag{13}
\]

Conservation of momentum reads:

\[
\rho_0 u_0^2 + P_{g,0} + P_{CR,0} = \rho_p u_p^2 + P_{g,p} + P_{CR,p}, \tag{14}
\]

where \(P_{g,0}\) and \(P_{g,p}\) are the gas pressures at the points \(x = -\infty\) and \(x = x_p\) respectively, and \(P_{CR,p}\) is the pressure in accelerated particles at the point \(x_p\) (we used the symbol \(CR\) to mean cosmic rays, in the sense of accelerated particles). The mass flow in accelerated particles has reasonably been neglected.

Our basic assumption, similar to that used in [17], is that the diffusion is \(p\)-dependent and more specifically that the diffusion coefficient \(D(p)\) is an increasing function of \(p\). Therefore the typical distance that a particle with momentum \(p\) moves away from the shock is approximately \(\Delta x \sim D(p)/u_p\), larger for high energy particles than for lower energy particles\(^2\). As a consequence, at each given point \(x_p\) only particles with momentum larger than \(p\) are able to affect appreciably the fluid. Strictly speaking the validity of the assumption depends on how strongly the diffusion coefficient depends on the momentum \(p\).

The cosmic ray pressure at \(x_p\) is the sum of two terms: one is the pressure contributed by the adiabatic compression of the cosmic rays at upstream infinity,

\(^2\) For the cases of interest, \(D(p)\) increases with \(p\) faster than \(u_p\) does, therefore \(\Delta x\) is a monotonically increasing function of \(p\).
and the second is the pressure of the particles accelerated or reaccelerated at the shock \((\tilde{P}_{CR}(p))\) and able to reach the position \(x_p\). Since only particles with momentum \(\gtrsim p\) can reach the point \(x = x_p\), we can write

\[
P_{CR,p} = P_{CR,0} \left( \frac{u_0}{u_p} \right)^{\gamma_{CR}} + \tilde{P}_{CR}(p) \simeq \]

\[
\simeq P_{CR,0}(p) \left( \frac{u_0}{u_p} \right)^{\gamma_{CR}} + \frac{4\pi}{3} \int_{p}^{p_{max}} dp \, p^3 v(p) f_0(p),
\]

where \(v(p)\) is the velocity of particles with momentum \(p\), \(p_{max}\) is the maximum momentum achievable in the specific situation under investigation, and \(\gamma_{CR}\) is the adiabatic index for the accelerated particles.

Let us consider separately the case of a strongly modified and weakly modified shock, in order to determine the best choice for \(\gamma_{CR}\). In the case of strongly modified shocks, we will show that most energy is piled up in the region \(p \sim p_{max} \gg m\), therefore in this case we can safely adopt \(\gamma_{CR} = 4/3\), appropriate for a relativistic gas. For weakly modified shocks, the accelerated particles have an approximately power law spectrum with a slope \(\alpha\). It can be shown that in this case \(\gamma_{CR} = \alpha/3\), so that the relativistic result \(\gamma_{CR} = 4/3\) still applies for \(\alpha = 4\) (strong shocks). For steeper spectra (\(\alpha > 4\)) a larger adiabatic index should be adopted, but in those cases the solution is basically independent of the choice of \(\gamma_{CR}\) because of the weak backreaction of the particles. For the purpose of carrying out our numerical calculation we will therefore always take \(\gamma_{CR} = 4/3\).

The pressure of cosmic rays at upstream infinity is simply

\[
P_{CR,0} = \frac{4\pi}{3} \int_{p_{min}}^{p_{max}} dp \, p^3 v(p) f_\infty(p),
\]

where \(p_{min}\) is some minimum momentum in the spectrum of seed particles. For simplicity, we assume that \(p_{min} = p_{inj}\), namely the minimum momentum of the seed particles coincides with the momentum at which particles are injected in the shock and are accelerated.

From eq. (14) we can see that there is a maximum distance, corresponding to the propagation of particles with momentum \(p_{max}\) such that at larger distances the fluid is unaffected by the accelerated particles and \(u_p = u_0\).

We will show later that for strongly modified shocks the integral in eq. (15) is dominated by the region \(p \sim p_{max}\). This improves even more the validity of our approximation \(P_{CR,p} = P_{CR}(> p)\). This also suggests that different choices for
the diffusion coefficient $D(p)$ may affect the value of $p_{\text{max}}$, but at fixed $p_{\text{max}}$ the spectra of the accelerated particles should not change in a significant way.

Assuming an adiabatic compression of the gas in the upstream region, we can write
\[
P_{g,p} = P_{g,0} \left( \frac{\rho_p}{\rho_0} \right)^{\gamma_g} = P_{g,0} \left( \frac{u_0}{u_p} \right)^{\gamma_g},
\]
where we used mass conservation, eq. (13). The gas pressure far upstream is $P_{g,0} = \rho_0 u_0^2/(\gamma_g M_0^2)$, where $\gamma_g$ is the ratio of specific heats for the gas ($\gamma_g = 5/3$ for an ideal gas) and $M_0$ is the Mach number of the fluid far upstream.

We introduce now a parameter $\xi_{CR}$ that quantifies the relative weight of the cosmic ray pressure at upstream infinity compared with the pressure of the gas at the same location, $\xi_{CR} = P_{CR,0}/P_{g,0}$. Using this parameter and the definition of the function $U(p)$, the equation for momentum conservation becomes
\[
\frac{dU}{dp} \left[ 1 - \frac{\gamma_{CR} \xi_{CR}}{\gamma_g M_0^2} U^{-(\gamma_{CR}+1)} - \frac{1}{M_0^2} U^{-(\gamma_g+1)} \right] + \frac{1}{\rho_0 u_0^2} \frac{d\tilde{P}_{CR}}{dp} = 0.
\]

Using the definition of $\tilde{P}_{CR}$ and multiplying by $p$, this equation becomes
\[
p \frac{dU}{dp} \left[ 1 - \frac{\gamma_{CR} \xi_{CR}}{\gamma_g M_0^2} U^{-(\gamma_{CR}+1)} - \frac{1}{M_0^2} U^{-(\gamma_g+1)} \right] = \frac{4\pi}{3\rho_0 u_0^2} p^4 v(p) f_0(p),
\]
where $f_0$ depends on $U(p)$ as written in eq. (12). Eq. (19) is therefore an integral-differential nonlinear equation for $U(p)$. The solution of this equation also provides the spectrum of the accelerated particles.

The last missing piece is the connection between $R_{\text{sub}}$ and $R_{\text{tot}}$, the two compression factors appearing in eq. (8). The compression factor at the gas shock around $x = 0$ can be written in terms of the Mach number $M_1$ of the gas immediately upstream through the well known expression
\[
R_{\text{sub}} = \frac{(\gamma_g + 1) M_1^2}{(\gamma_g - 1) M_1^2 + 2}.
\]

On the other hand, if the upstream gas evolution is adiabatic, then the Mach number at $x = 0^-$ can be written in terms of the Mach number of the fluid at upstream infinity $M_0$ as
\[
M_1^2 = M_0^2 \left( \frac{u_1}{u_0} \right)^{\gamma_g + 1} = M_0^2 \left( \frac{R_{\text{sub}}}{R_{\text{tot}}} \right)^{\gamma_g + 1},
\]

11
so that from the expression for $R_{\text{sub}}$ we obtain

$$R_{\text{tot}} = M_0^{\gamma g + 1} \left[ (\gamma g + 1) R_{\text{sub}}^{\gamma g} - (\gamma g - 1) R_{\text{sub}}^{\gamma g + 1} \right]^{\gamma g + 1}.$$ \(21\)

Now that an expression between $R_{\text{sub}}$ and $R_{\text{tot}}$ has been found, eq. (19) basically is an equation for $R_{\text{sub}}$, with the boundary condition that $U(p_{\text{max}}) = 1$. Finding the value of $R_{\text{sub}}$ (and the corresponding value for $R_{\text{tot}}$) such that $U(p_{\text{max}}) = 1$ also provides the whole function $U(p)$ and, through eq. (8), the distribution function $f_0(p)$ for the particles resulting from acceleration and reacceleration in the nonlinear regime. If the backreaction of the accelerated particles is small, the test particle solution must be recovered.

4 The nonlinearity of acceleration and reacceleration

In this section we investigate the shock modification due to the backreaction of the accelerated particles. We split the discussion in two parts: in §4.1 we consider the case of pre-existing seed particles populating the region where the shock is propagating, and re-energized by the shock. We find the conditions for the onset of the nonlinear regime in which the shock gets modified by the re-accelerated particles. In particular we show that the critical behaviour found for shock acceleration and manifesting itself through the appearance of three solutions, takes place also in this case. In §4.2 we explore the more complicated situation in which a shock accelerates a new population of particles while possibly reaccelerating a pre-existing population of seed particles.

4.1 Shocks modified by re-energized seed particles

Here we assume that a fluid is moving with speed $u_0$ in a region where the temperature is $T_0$. The Mach number is some pre-defined value $M_0$, which can be easily related to $u_0$ and $T_0$. In the region of interest we assume that a population of seed particles is present with energy per particle already higher than some injection energy necessary for the particles to feel the shock as a discontinuity. For simplicity we assume that these seed particles have a spectrum which is a power law in momentum in the form:

$$f_{\infty}(p) = f_{\infty,0} p^{-\alpha}.$$ \(22\)

In this section we explore the situation in which the shock reacceleration of a pre-existing population of cosmic rays can modify the shock structure, but
there is no acceleration of particles different from those that are already present as seed particles. This is equivalent to turn off the injection term ($\eta = 0$ in Eq. (12)). In the next section we discuss what happens when both components are present. The crucial difference between the two components is that the seed particles are by definition already above the threshold for injection, so that there is no injection efficiency that instead represents such a crucial parameter for the case of acceleration of particles extracted from the thermal pool.

Having in mind the case of shocks propagating in the interstellar medium of our Galaxy, we consider here the case in which the inflowing gas and the seed particles are in pressure equilibrium, namely $P_{g,0} = P_{CR}$. In terms of the parameter $\xi_{CR} = P_{CR}/P_{g,0}$ this implies $\xi_{CR} = 1$. The results can then be easily repeated for a generic value of $\xi_{CR}$.

Eq. (19) gives the quantity $U(p)$ as a function of the momentum $p$ for any choice of $R_{sub}$ and $R_{tot}$, while the acceptable solutions are those with the right matching conditions at $p_{max}$. For simplicity, let us assume that the $p_{max}$ in eq. (22) is the same maximum momentum that particles injected at the shock would achieve: this value only depends on the environmental conditions (energy losses of the particles) and/or on the geometry of the shock, which may allow the escape of the particles. The momentum $p_{max}$ is the same as in the distribution of the pre-existing seed particles if, for instance, the seed particles have been accelerated by a shock identical to the one we are considering. In any case, this assumption is not needed for the validity of our conclusions and may be easily relaxed, it simply serves to avoid parameter proliferation. Within this assumption, particles re-energized at the shock are simply redistributed in the momentum range between $p_{inj}$ and $p_{max}$.

The solution, namely the right pair of values for the compression parameters $R_{sub}$ and $R_{tot}$ [related through eq. (21)], is obtained when the solution corresponding to $U(p_{max}) = 1$ is selected.

In Fig. 2, we plot $U(p_{max})$ as a function of $R_{tot}$ for a shock having Mach number at infinity $M_0 = 150$. The different lines are labelled by a number representing the log$_{10}$ of $p_{max}$ in units of $mc$. In other words the parameter $p_{max}$ changes between $10^2 mc$ and $10^9 mc$. The physical solutions are found by determining the intersections of each curve with the horizontal line corresponding to $U(p_{max}) = 1$. The minimum momentum of the seed particles is taken as $10^{-3} mc$. The intersection at $R_{tot} \simeq 4$ is close to the well known linear solution, namely the solution that one would obtain in the test particle approximation. The energy channelled into the nonthermal particles for this solution is very small, and the shock remains approximately unmodified. Increasing the value of $p_{max}$, the solution moves toward slightly larger values of $R_{tot}$ (namely the shock becomes more modified). For some values of $p_{max}$, multiple solutions appear. In particular three regions of $p_{max}$ can be identified:
Fig. 2. Velocity at upstream infinity in units of $u_0$ as obtained from calculations, as a function of the total compression factor. The physical solutions are the ones with $U(p_{\text{max}}) = 1$. The plot refers to the case $M_0 = 150$ and $\xi_{CR} = 1$. The different lines are labelled by a number representing the $\log_{10}$ of $p_{\text{max}}$ in units of $mc$.

1) $p_{\text{max}} \leq p_{cr}^{(1)}$: in this region (low values of $p_{\text{max}}$) the only solution is very close to the one obtained in the test particle approximation. For the values of the parameters in Fig. 2, $p_{cr}^{(1)} \approx 10^3 mc$.

2) $p_{cr}^{(1)} \leq p_{\text{max}} \leq p_{cr}^{(2)}$: in this region (intermediate values of $p_{\text{max}}$) three solutions appear, two of which imply a strong modification of the shock, namely an appreciable part of the energy flowing through the shock is converted into energy of the accelerated particles. For the values of the parameters in Fig. 2, $p_{cr}^{(2)} \approx 10^9 mc$.

3) $p_{\text{max}} \geq p_{cr}^{(2)}$: in this region (high values of $p_{\text{max}}$) the solution becomes one again, and the shock is always strongly modified ($R_{\text{tot}} \gg 4$).

This critical behaviour appears also when one fixes the maximum momentum and uses the Mach number of the shock as the order parameter. The results are shown in Fig. 3, where Mach numbers between 10 and 500 have been considered, at fixed $p_{\text{max}} = 10^5 mc$. One can see that for Mach numbers below $\sim 100$ there is only one solution. For Mach numbers between $\sim 100$ and $\sim 500$ three solutions appear, one of which roughly corresponds to an unmodified shock. For larger Mach numbers, only this linear solution remains, and the shock is always only weakly modified. The same situation is plotted in Fig. 4, where instead of $R_{\text{tot}}$ on the x-axis there is $4 - R_{\text{sub}}$, and $R_{\text{sub}}$ is the compression
Fig. 3. Velocity at upstream infinity in units of \( u_0 \) as obtained from calculations as a function of the total compression factor. The physical solutions are the ones with \( U(p_{\text{max}}) = 1 \). The plot refers to the case \( p_{\text{max}} = 10^5 mc \) and \( \xi_{CR} = 1 \) for the Mach numbers indicated on the curves.

coefficient at the gas subshock. For unmodified shocks one expects \( 4 - R_{\text{sub}} \to 0 \). Again three regions can be identified, in the parameter \( M_0 \).

1) For \( M_0 \leq M_{cr}^{(1)} \) only one solution exists and does not necessarily correspond to unmodified shocks. In fact one can see that for \( M_0 = 10 < M_{cr}^{(1)} \sim 100 \), the compression coefficient at the subshock is \( \sim 2 \), and \( R_{\text{tot}} \sim 8 \), for the situation plotted in Figs. 3 and 4. Even smaller values of \( M_0 \) do not give a perfectly unmodified shock.

2) For \( M_{cr}^{(1)} \leq M_0 \leq M_{cr}^{(2)} \) three solutions appear, two of which imply a strong modification of the shock. For the parameters used in Figs. 3 and 4, \( M_{cr}^{(2)} \sim 500 \).

3) For \( M_0 \geq M_{cr}^{(2)} \) only the nearly unmodified solution exists.

In order to emphasize the fact that the multiple solutions actually correspond to physical solutions with very different spectral characteristics for the accelerated particles we plot in Fig. 5 the spectra for \( M_0 = 50 \) (for which there is only one solution) and \( M_0 = 150 \) (for which there are three solutions). In both cases we used \( P_{\text{max}} = 10^5 mc \). The solution corresponding to Mach number \( M_0 = 50 \) has \( R_{\text{sub}} = 2.683 \) and \( R_{\text{tot}} = 25.588 \), resulting in a total pressure of the accelerated particles \( P_{CR}/\rho_0 u_0^2 = 0.8946 \) (solid line in Fig. 5). The three
Fig. 4. The same as in Fig. 3, but as a function of $4 - R_{\text{sub}}$.

Fig. 5. Spectra of re-accelerated particles for the cases $M_0 = 50$ (single solution) and $M_0 = 150$ (three solutions). The maximum momentum is fixed to $p_{\text{max}} = 10^5 mc$.

dashed lines in Fig. 5 are for $M_0 = 150$ and represent the spectra for the three solutions. In numbers, these solutions (from top to bottom in Fig. 5) are summarized as follows:
first solution: $R_{\text{sub}} = 3.0775$, $R_{\text{tot}} = 55.607$, $P_{\text{CR}}/\rho_0 u_0^2 = 0.9611$

second solution: $R_{\text{sub}} = 3.9839$, $R_{\text{tot}} = 14.318$, $P_{\text{CR}}/\rho_0 u_0^2 = 0.734$

third solution: $R_{\text{sub}} = 3.999371$, $R_{\text{tot}} = 4.2548$, $P_{\text{CR}}/\rho_0 u_0^2 = 0.06$

In the cases of strong shock modification, the asymptotic spectra for $(p/mc) \gg 1$ are $p^2 f(p) \sim p^{-3/2}$, while the linear theory, in case of strong shocks ($M_0 \gg 1$), would predict $p^2 f(p) \sim p^{-2}$. In the region of very low energies, the spectra of the reaccelerated particles tend to zero, as known from the linear theory as well.

4.2 Modified shocks: the general case

Nonlinear effects in shock acceleration were first investigated in the case when a fraction of the thermal gas crossing the shock surface is energized to nonthermal energies. In this section we wish to apply the analytical method discussed in §2 and §3 to the most general case in which thermal particles are accelerated but seed particles may already be present in the environment.

We start our discussion with the case of acceleration of particles from the thermal distribution, in order to show that the multiple solutions already found in other analytical approaches [27,28] are also obtained by adopting the approach illustrated here. In Fig. 6, we plot the $U(p_{\text{max}})$ as a function of $R_{\text{tot}}$ for the case in which a fraction $\eta$ of the particles (as indicated in the plots) crossing the shock is actually accelerated to suprathermal energies. The Mach number is chosen to be $M_0 = 150$ and the maximum momentum is taken as $P_{\text{max}} = 10^5 mc$. One can easily see that $R_{\text{tot}} \sim 4$, the value for unmodified shocks, for small values of $\eta$ (low efficiencies), while $R_{\text{tot}}$ increases above 4 for $\eta > 10^{-4}$. Three solutions appear for intermediate efficiencies, while the solution for the accelerated particles always predicts a strongly modified shock for high efficiencies, $\eta > 3 \times 10^{-3}$. An asymptotic value of $R_{\text{tot}} \sim 60$ is achieved for the parameters used here. This example demonstrates that the critical behaviour shown to appear for high values of $\eta$ in several analytical or semi-analytical calculations is in fact also predicted by the approach proposed here. Next question to ask is however whether the presence of seed particles can change the critical behavior of shocks. In order to answer this question we consider a situation in which particles are accelerated from the thermal pool with efficiency $\eta$, and at the same time the shock propagates in a medium where the preshock pressure in seed particles equals the gas pressure. This situation is considered to resemble that of a supernova explosion in our Galaxy, where cosmic rays fill the volume remaining in quasi-equipartition with the gas. In order to test the critical structure of the shock we calculate $U(p_{\text{max}})$ as a function of the compression factor $R_{\text{tot}}$ between upstream infinity and
Fig. 6. Velocity at upstream infinity in units of $u_0$ as obtained from calculations, as a function of the total compression factor. Only particles injected at the shock from the thermal pool are considered here with efficiency $\eta$ indicated on the curves.

downstream. Our results are plotted in Fig. 7. The lines refer to the cases $\eta = 10^{-5}, 10^{-4}, 5 \times 10^{-4}, 10^{-3}, 5 \times 10^{-3}$ from bottom to top, as in Fig. 6. It is clear from this plot that the shock may be modified by the backreaction of the accelerated particles even for very low values of $\eta$, because of the presence of seed particles. In other words, the nonlinearity of the shock can well be dominated by the presence of preaccelerated particles rather than by the acceleration of particles from the thermal pool. Clearly this depends however on the values of the parameters (injection momentum, $\eta$, maximum momentum achievable, Mach number). The injection momentum, in principle, could be related to $\eta$ in order to reduce the number of free parameters of the problem. This is possible in the assumption that the particles downstream keep a thermal distribution. Simulations suggest that the particles injected in the accelerator are the ones with momentum a few times the thermal momentum of the particles downstream, so that $\eta$ can be simply calculated by integration of the Maxwell-Boltzmann (MB) distribution above this minimum momentum. Unfortunately, we do not know whether the particle distribution downstream is in fact Maxwell-Boltzmann-like. Moreover, we do not know exactly what is the threshold to impose on the injection momentum, which is a delicate issue because the number of particles taking part to the acceleration process would result from the integral of the MB distribution over its exponentially decreasing part. Therefore we preferred here to keep $\eta$ and the injection momentum as separate free parameters.
5 Discussion and Conclusions

We proposed a semi-analytical approach to show that the backreaction of particles accelerated at a shock is able to affect the shock itself, in such a way that the shock and the accelerated particles become parts of a nonlinear system. In particular, for the first time we included in this kind of calculations the seed particles that may be present in the region where the shock is propagating and that can be re-energized by the shock.

While a test-particle approach to this problem was first presented in [31], a nonlinear treatment was never investigated. In the pioneering work of Ref. [31], it was recognized that the energy of the seed particles could be enhanced by about one order of magnitude at each shock passage, and that after an infinite number of strong shocks passing through the region, the spectrum of the particles would tend to the asymptotic spectrum $E^{-3/2}$. Two comments are in order. First, the continuous increase of the cosmic ray energy due to the re-energization of seed particles leads unavoidably the shock to be modified by the nonthermal pressure, unless one starts from an uninteresting small pressure of seed particles at the beginning. Second, the fact that the spectrum becomes flatter than $E^{-2}$, which is the result for a strong non-relativistic shock, implies that most of the nonthermal energy is pushed to the highest
energies, therefore the shock again can be more easily modified. Both these
points suggest that a nonlinear treatment of shock re-acceleration is required.

In our Galaxy, cosmic rays are observed to be in rough equipartition with the
gas pressure and with magnetic fields, therefore supernova shocks or shocks
generated in other environments propagate in a medium in which the seed par-
ticles (cosmic rays) are non-negligible. In these circumstances the non-linear
effects may be very important. We showed here that in fact for some regions
of the parameter space, the shock is modified mainly by the backreaction of
the seed particles rather than by the cosmic rays accelerated at the shock from
the thermal pool. The spectra of the re-accelerated particles have also been
calculated.

The interesting phenomenon of the appearance of multiple shock solutions,
already known for the case of shock acceleration, appears also for the case of
reacceleration. This puzzling phenomenon may suggest that the shock behaves
as a self-regulating system settling on the critical point, as proposed in [27,28].
On the other hand, it is possible that the multiple solutions may be the artifact
of some of the assumptions used in the analytical approach, in particular the
request for time independent (stationary) solutions and the fact that the role of
the self-generated waves on the diffusion coefficient is not taken into account.
Further investigation, in particular in the direction of a detailed comparison
of our results with numerical simulations of shock acceleration is required in
order to unveil the physical meaning of the multiple solutions for modified
shocks.

From the phenomenological point of view, it would be certainly worth to study
the implications of nonlinear shock reacceleration on the nonthermal activity
in astrophysical environments where the effect is expected to play an important
role, in particular in the case of supernova remnants. In particular, as pointed
out by the referee, the suggested dominance of reaccelerated ambient seed
particles over freshly injected particles may have serious implications for the
spectra of secondary nuclei resulting from spallation processes.

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