General relativistic tidal heating for the Møller 58 pseudotensor

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Abstract

In his study of tidal stabilization of fully relativistic neutron stars Thorne showed that the fully relativistic expression for tidal heating is the same as in non-relativistic Newtonian theory. Furthermore, Thorne also noted that this tidal heating must be independent of how one localizes gravitational energy and is unambiguously given by that expression. Favata calculated the tidal heating for a number of classical gravitational pseudotensors including that of Møller, and obtained the result that all of them produced the same (Newtonian) value. After a re-examination of the calculation using the Møller pseudotensor we find that indeed this pseudotensor gives the desired result under the condition that the mass $M$ is a constant, while Favata considered $M$ as being time dependent, which is illegitimate since it violates the harmonic gauge condition, which he used. Moreover, we carry on to consider this Møller pseudotensor even in a black hole situation, i.e., beyond Newtonian physics.

1 Introduction

Dirac [1] elucidated that it is not possible to obtain a gravitational field energy expression that satisfies both conditions: (1) when added to other forms of energy the total energy is conserved, and (2) the energy within a definite (three-dimensional) region at a certain time is independent of the coordinate system. For the classical pseudotensors, in general, the first condition can be satisfied but the second not. In other words, localizing gravitational energy is impossible. Gravitational energy can be localized, however, there is no unique proper coordinate independent way to localize it; instead we have many expressions all reference frame dependent. Tidal heating is an empirical physical phenomenon resulting from the net work done by an external tidal field on an isolated body. The ocean tides on Earth provide a familiar example of this kind of phenomenon. However, a more dramatically example is the Jupiter-Io system, where the moon Io’s active volcanoes are the result of tidal heating [2].

In 1998 Thorne demonstrated that the expected tidal heating rate is the same both in relativistic and Newtonian gravity [3, 4]: $\dot{W} = -\frac{1}{2} I_{ij} E^{ij}$, where $\dot{W}$ refers to the work rate, the dot indicates the time derivative, $I_{ij}$ is the mass quadrupole moment of the isolated body and $E_{ij}$ is the tidal field of the external universe. Both $I_{ij}$ and $E_{ij}$ are time dependent, symmetric and trace free. Moreover, Thorne also noted that such tidal heating is independent of how one localizes the gravitational energy and is unambiguously given by a certain value. This has been verified by calculating $\dot{W}$ explicitly using various gravitational pseudotensors to represent the gravitational energy density and energy flux. In contrast, there exists a recoverable process $\dot{E}_{\text{int}} \sim \frac{d}{dt}(I_{ij} E^{ij})$ which indicates time reversible, where $E_{\text{int}}$ is the energy interaction between the isolated planet’s quadrupolar deformation and the external tidal field [4].

In 1999, Purdue used the Landau-Lifshitz pseudotensor to calculate the tidal heating and confirmed that the result agreed with the Newtonian perspective [4, 5]. Later
in 2001, Favata employed the same method to verify that the Einstein, Bergmann-Thomson and Møller pseudotensors give the same result as Purdue found. Moreover, Booth and Creighton used the quasi-local mass formalism of Brown and York to demonstrate the same result. All of them give the same value as the Newtonian perspective. Referring to the work of Purdue and Favata, this has completed the verification that the tidal heating is indeed independent of the gravitational pseudotensor.

After a re-examination of the calculation for the Møller pseudotensor, we noted that Favata misinterpreted the rate of change of the constant mass \( \dot{M} \). Here we claim that \( \dot{M} \) needs to be vanishing because of the harmonic gauge requirement (see (9) and (13) below). Nonetheless, Favata’s and our result are the same eventually (which we will define below). In the past, we thought that obtaining the energy-momentum pseudotensor through a Freud type superpotential (see (17) below) guarantees the expected tidal heating but the converse is not true. Surprisingly, the Møller pseudotensor provides a counterexample. The present paper illustrates how the relativistic tidal heating is indeed classical pseudotensor independent: including the Møller pseudotensor. Explicitly, Thorne’s assertion is correct.

Our result shows that it does give the desired value for the Møller pseudotensor. The nice property of the Møller energy-momentum complex is that the energy content of a hypersurface does not depend on the chosen spatial coordinates, while the complexes proposed by Einstein, Landau-Lifshitz, Bergmann-Thomson and Goldberg do. Perhaps this may be the reason why there were many investigators studying this energy-momentum prescription in the past couple of decades. Thus it is worthwhile to investigate the tidal heating using the Møller pseudotensor. After introducing the current quadrupole moment \( J_{ij} \) and magnetic type quadrupole moment \( B_{ij} \), we step forward to consider the black hole scenario using the Møller pseudotensor. Note that \( J_{ij} \) and \( B_{ij} \) are time dependent, symmetric and traceless.

2 Technical background

We will use \( \eta_{\mu\nu} = (-1, 1, 1, 1) \) as our spacetime signature and the geometrical units \( G = c = 1 \), where \( G \) is the Newtonian gravitational constant and \( c \) the speed of light. We adopt the convention that Greek letters indicate spacetime indices and Latin letters refer to spatial indices. In principle, the classical pseudotensors can be obtained from a rearrangement of the Einstein equation: \( G_{\mu\nu} = \kappa T_{\mu\nu} \), where the constant \( \kappa = 8\pi G/c^4 \) and \( T_{\mu\nu} \) is the material energy tensor. This is a basic requirement for pseudotensors (see Ch. 20 in [19]). One can define the gravitational energy-momentum pseudotensor in terms of a suitable superpotential \( U_{\alpha}^{[\mu\nu]} \):

\[
2\kappa\sqrt{-\text{g}}t_{\alpha}^{\mu} := \partial_\nu U_{\alpha}^{[\mu\nu]} - 2\sqrt{-\text{g}}G_{\alpha}^{\mu}.
\]

(1)

The total energy-momentum density complex can then be defined as

\[
\sqrt{-\text{g}}T_{\alpha}^{\mu} := \sqrt{-\text{g}}(T_{\alpha}^{\mu} + t_{\alpha}^{\mu}) = (2\kappa)^{-1}\partial_\nu U_{\alpha}^{[\mu\nu]},
\]

(2)

where to get the last equality we used (11) and the Einstein equation. As a consequence of the antisymmetry of the superpotential \( \partial_\mu(\sqrt{-\text{g}}T_{\alpha}^{\mu}) = 0 \). In vacuum it leads to
the energy conservation relations: $\partial_\mu(\sqrt{-g}t^\alpha_\mu) = 0$. The physical meaning of $t^0_0$ and $t^j_0$ can be interpreted as the gravitational energy density and energy flux. The energy-momentum within a spatial region $V$ can be expressed as $P_\mu = (\vec{E}, \vec{P}) = \int_V \sqrt{-g}t^\mu_0 d^3x$, the sign for the Fried superpotential can be fixed by evaluating the ADM mass [21][22]. Thus, tidal heating can be manipulated as

$$2\kappa \dot{W} = -\int_V \partial_0(\sqrt{-g}t^0_0)d^3x = \int_V \partial_j(\sqrt{-g}t^j_0)d^3x. \quad (3)$$

Using Gauss’s theorem, the last integral in (3) can be converted into a surface integral of the form:

$$2\kappa \dot{W} = \oint_{\partial V} \sqrt{-g} \dot{t}^j_0 dS_j, \quad (4)$$

where $dS_j = \hat{n}_j r^2 d\Omega$, $\hat{n}_j \equiv x_j/r$ is the unit radial normal vector and $r \equiv \sqrt{\delta_{ab}x^ax^b}$ is the distance from the body in its local asymptotic rest frame.

For the tidal heating calculation, we adopt the harmonic gauge

$$0 = \partial_\beta(\sqrt{-g}g^{\alpha\beta}) = -\sqrt{-g}\Gamma^{\alpha\beta}_\beta. \quad (5)$$

This harmonic coordinate condition provides the closest approximation to rectilinear coordinates in curved space and is suitable for studying gravitational waves [1]. Decompose the metric tensor out to 2nd order away from the Minkowski background as follows [18]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{\alpha}_\mu h_{\alpha\nu} + k_{\mu\nu}, \quad (6)$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} - k^{\mu\nu}, \quad (7)$$

where $k_{\mu\nu} = -\frac{1}{2}h_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h^2 - \frac{1}{4}\eta_{\mu\nu}h^2_{\alpha\beta}$. Here and after indices are being transversed using the Minkowski background metric. However, it might not be so simple, as in the quantity $\Gamma^{\alpha\beta}_\beta$ the index $\beta$ is raised using the spacetime metric $g^{\alpha\beta}$. Note that we classify $\eta_{\mu\nu}$ as the zeroth order, $h_{\mu\nu}$ the 1st order and $k_{\mu\nu}$ the 2nd order. The trace of $h := h_{\alpha\beta}$. The detailed expansion for the harmonic gauge quantity is

$$\Gamma^{\alpha\beta}_\beta = \partial_\beta \bar{h}^{\alpha\beta} + \mathcal{O}(h^3), \quad (8)$$

where $\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}h$. In the following we will expand things out, keeping only the relevant lowest order terms. We split the time and spatial components of this gauge quantity as follows:

$$0 = \Gamma^{00}_0 = 2\partial_0 h^{00} + \partial_c h^{0c}, \quad 0 = \Gamma^{ij}_j = \partial_0 h^{0j} + \partial_c \bar{h}^{jc}. \quad (9)$$

The related components (adopted from (8) in [18]) of the gravitational field tensors are:

$$h^{00} = \frac{2M}{r} + \frac{3}{r^5}I_{ab}x^ax^b - E_{ab}x^ax^b, \quad (10)$$

$$h^{0j} = \frac{4}{r^5}\epsilon^i_{pq}p^p_i x^q x^j + \frac{2}{3}\epsilon^i_{pq}B^p_i x^q x^j + \frac{2}{r^3} \bar{I}^i a x^a + \frac{2}{21}(5\bar{E}_{ab}x^ax^b x^j - 2\bar{E}^j_a x^a r^2), \quad (11)$$

$$h^{ij} = \eta^{ij} h^{00} + \bar{h}^{ij}, \quad (12)$$

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where $\bar{h}^{ij} = \frac{8}{3\pi} \epsilon_{pq} (i \hat{J}^{ij}) x^p x^q + \frac{1}{21} \left[ 5x(i \epsilon^{ij}) \hat{B}^q x^p x^l - r^2 \epsilon^{pq} (i \hat{B}^{ij}) x^p \right]$ and $\eta_{ij} \bar{h}^{ij} = 0$. The value of the weighting factor $\sqrt{-g} = 1 + h_{00} + O(h^2)$. From a calculation using (10) and (11), we have

$$2\partial_0 h^{00} + \partial_0 h^{0c} = 4\dot{M}r^{-1}. \quad (13)$$

According to Thorne’s argument the mass $M$ is constant in time [3] and indeed the harmonic gauge is valid at the lowest order under the requirement that $\dot{M}$ has to be vanishing. Explicitly, follows from (13): $2\partial_0 h^{00} + \partial_0 h^{0c} = 0$.

If the isolated body is absorbing an external quadrupolar field, its mass quadrupole moment and current quadrupole moment $\dot{I}_{ij} \neq 0 \neq \dot{J}_{ij}$ in general, and both of them can generate tidal work. For our tidal heating calculation purpose, we only pay attention to the lowest non-vanishing order; to that order we will get a relation of the form [6] [18]

$$\dot{W} = \partial_0 (k_1 I_{ij} E^{ij} + k_2 J_{ij} B^{ij}) + k_3 I_{ij} \dot{E}^{ij} + k_4 J_{ij} \dot{B}^{ij}, \quad (14)$$

where $k_1, k_2, k_3, k_4$ are constants. The coefficients $k_1$ and $k_2$ are related to a specific choice for the energy localization, where $\partial_0 (I_{ij} E^{ij})$ and $\partial_0 (J_{ij} B^{ij})$ are the ambiguous reversible tidal-quadrupole interaction process. We expect to get $(k_3, k_4) = (\frac{1}{2}, \frac{2}{3})$ so that $\frac{1}{2} I_{ij} \dot{E}^{ij}$ and $\frac{2}{3} J_{ij} \dot{B}^{ij}$ are the unambiguous irreversible tidal heating dissipation process that we are interested in. Therefore we only look for the tidal heating coming from the external tidal field $E_{ij}$ interacting with the evolving quadrupole moment $I_{ij}$ of an isolated body. Similarly for $J_{ij}$ and $B_{ij}$.

Here we consider the tidal heating due to the gravitational fields $E_{ij}$ and $B_{ij}$ that come from the external universe. Meanwhile there is a physical constraint that is assumed, the Laplace equation $\nabla^2 (Mr^{-1}) = 0$. This means we ignored a delta function at the origin that gives a non-vanishing Poisson equation $\nabla^2 (Mr^{-1}) \neq 0$. Moreover, based on the physical restriction of the harmonic gauge, we need the mass $M$ to be time independent, no matter if it is in the Jupiter-Io system or a black hole scenario. For the case of the Jupiter-Io system, the tidal heating of Io is in a thermal equilibrium state [23] [24]. Io is absorbing gravitational field energy from Jupiter and transfers the heat as convection. For the case of a black hole scenario, forms the extra energy absorbed from the outside universe will transfer to other forms of energy such as gravitational waves, thus the mass $M$ remains constant.

## 3 Tidal heating from the Freud superpotential

There are an infinite number of superpotentials, the Freud [8] superpotential $F_U a_{[\mu \nu]} := -\sqrt{-g} g^{\beta \gamma} \Gamma^\tau_{\beta \gamma} \delta^{\lambda \mu \nu}$ is a straightforward expression that can be used for illustrating the tidal work. Here we use it to reproduce the result of the tidal heating for the Einstein pseudotensor. We have mentioned that the energy-momentum complex can be computed as $\sqrt{-g} (E T_a^\mu) = (2\kappa)^{-1} \partial_\mu (F_U a_{[\mu \nu]}).$ As we will explain in more detail a little further below, at any point, to lowest order in Riemann normal coordinates inside matter this reduces to the source energy-momentum stress tensor $T_a^\mu = \kappa^{-1} G_a^\mu$, as one expects from the equivalence principle. In vacuum, the Einstein pseudotensor
\[2\kappa (E_t^\alpha) = \delta_\alpha^\mu (\Gamma^{\beta\gamma}_\nu \Gamma^{\nu\beta}_\lambda - \Gamma^{\beta\lambda}_\nu \Gamma^{\nu\gamma}_\lambda) + \Gamma^{\nu\beta}_\mu (\Gamma^{\mu\beta}_\alpha + \Gamma^{\beta\mu}_\alpha) + \Gamma^{\pi\alpha}_\lambda (\Gamma^{\nu\lambda}_\gamma - \Gamma^{\lambda\nu}_\gamma) - 2\Gamma^{\beta\nu}_\alpha \Gamma^{\mu\beta}_\nu. \]

(15)

In general, this pseudotensor is not symmetric. In the harmonic gauge, the gravitational energy density and energy flux \[7\] are

\[
2\kappa (E_t^0) = \frac{1}{\sqrt{-g}} 
\sum_{\alpha} \partial_\alpha \eta^{cd} h_{00,c} h_{00,d}, \quad 2\kappa (E_t^j) = \eta^{ja} h_{00,0} h_{00,a}.
\]

(16)

Using (16) in (4), we recover the known result

\[
\dot{W}^E = \frac{3}{10} \partial_0 (I_{ij} E_{ij}) - \frac{1}{2} \dot{I}_{ij} E^{ij}
\]

as Favata obtained.

To the order of concern here, it is sufficient to consider superpotentials that are linear in the connection. There are only three possible terms with suitable symmetry, one by itself is proportional to the Møller superpotential. In principle, the three parameter expression can be cast in the following way:

\[
U^{\alpha\beta}_{\mu\nu} := \sqrt{-g} \left( a_1 \delta^\tau_{\alpha} \Gamma^{\rho\lambda}_\tau \lambda + a_2 \delta^\rho_{\alpha} \Gamma^{\lambda\tau}_\rho \lambda + a_3 \Gamma^{\tau\rho}_{\alpha} \right) \delta^\mu_{\rho\tau}, \quad (17)
\]

where \(a_1, a_2, a_3\) are real. One limit that should be considered is the small region limit. Around any arbitrary point, one can introduce Riemann normal coordinates \[21, 25\] with the origin at that point, such that

\[
g_{\alpha\beta}|_0 = \eta_{\alpha\beta}, \quad g_{\alpha\beta,\mu}|_0 = 0, \quad -3 \Gamma_{\beta\mu,\nu}|_0 = R^\alpha_{\beta\mu\nu} + R^\alpha_{\mu\beta\nu}. \]

(18)

According to the equivalence principle, to lowest order the pseudotensor associated with the above superpotential should reduce to the source energy-momentum; for the superpotential (17) this requirement yields

\[
2\kappa T_{\alpha}^\mu = \frac{1}{3} \left[ (2a_1 + a_2 + 3a_3) R_{\alpha}^\mu - (2a_1 + a_2) \delta^\mu_{\alpha} R \right].
\]

(19)

In order for this to agree with the Einstein field equation, we have the following constraints \[21\]:

\[
2a_1 + a_2 = 3, \quad a_3 = 1.
\]

(20)

Now consider the linearized field equation, up to the first order \(g_{\alpha\beta} \simeq \eta_{\alpha\beta} + h_{\alpha\beta}\), we have the following

\[
2\kappa T_{\alpha}^\mu = a_1 \left[ \partial^\beta_{\alpha} \tilde{h}_{\beta \mu} - \delta^\mu_{\alpha} (\partial^\beta_{\mu} \tilde{h}^{\beta \lambda}) \right] + a_2 \partial^\mu \partial_{\beta} h_{\alpha}^\beta
\]

\[
+ \frac{1}{2} (a_3 - a_2) \partial_\alpha \partial^\mu h - \partial^\lambda \partial_{\lambda} \left( a_3 h_{\alpha}^\mu - \frac{1}{2} a_2 \delta^\mu_{\alpha} h \right).
\]

(21)

The standard approach requires that the field equation reduces to the wave equations

\[
\frac{1}{2} \partial^\lambda \partial_{\lambda} h_{\alpha}^\mu = -\kappa T_{\alpha}^\mu
\]

under the criterion that the harmonic gauge is chosen. Referring to (21), the constraints are

\[
a_1 = \text{arbitrary}, \quad a_2 = a_3.
\]

(22)

Combining (20) and (22), we have the unique solution \(a_1, a_2, a_3\) are all unity.
Here we explain what we mean by a Freud type superpotential: decompose the Freud superpotential as follows

\[ F U_\alpha^{[\mu\nu]} := -\sqrt{-g}(\eta^{\beta\sigma} - h^{\beta\sigma})\Gamma_{\beta\lambda}^\tau \delta^{\lambda\mu\nu}. \]  

(23)

Note that the linear in \( \eta \Gamma \) terms give the expected interior mass and tidal heating, while the \( h^\Gamma \) terms only alter the value \( \partial_0 (I_{ij} E^{ij}) \) or \( \partial_0 (J_{ij} B^{ij}) \) [12]. Any superpotential that agrees with the Freud superpotential to lowest order in \( h_{\mu\nu} := g_{\mu\nu} - \eta_{\mu\nu} \), is referred to as a Freud type superpotential. We know the Landau-Lifshitz (LL) superpotential can be identified as a Freud type superpotential since we can raise the \( \eta \Gamma \) terms only alter the value \( \partial_0 (I_{ij} E^{ij}) \) or \( \partial_0 (J_{ij} B^{ij}) \). Another example is the Papapetrou superpotential [12]:

\[ p U_\alpha^{[\mu\nu]} = F U_\beta^{[\mu\nu]} g^{\alpha\beta} - \sqrt{-g}(g^{\rho\tau} h^{\pi\gamma} \Gamma_{\rho\lambda}^{\sigma} + g^{\rho\tau} h^{\sigma\gamma} \Gamma_{\tau\gamma}^{\lambda}) \delta^{\mu\nu} \delta^{\lambda\alpha}. \]  

(24)

Referring to (17), to lowest order there are just three possible superpotential terms and each term has its own characteristic features.

### 3.1 The 1st term of the Freud superpotential

When \((a_1, a_2, a_3) = (1, 0, 0)\) for (17), the first term of the Freud type superpotential is \( U_\alpha^{[\mu\nu]} := \sqrt{-g} g^{\rho\lambda} \delta^{\lambda\mu\nu} \). The corresponding energy-momentum complex is

\[ (2\kappa)_1 T_\alpha^\mu = (\partial_\nu + \Gamma^\tau_{\nu\mu}) \Gamma^{\rho\lambda}_\lambda \delta^{\mu\nu}. \]  

(25)

Inside matter at the origin, the energy-momentum \( T_\alpha^\mu = (R_\alpha^{\mu} - \delta_\alpha^\mu R)/(3\kappa) \) in Riemann normal coordinates. Upon applying the harmonic gauge in vacuum, the pseudotensor \( t_\alpha^\mu = 0 \), i.e., both the energy density \( t_0^0 \) and energy flux \( t_0^\jmath \) are zero.

### 3.2 The 2nd term of the Freud superpotential

When \((a_1, a_2, a_3) = (0, 1, 0)\) for (17), the second term of the Freud type superpotential is \( U_\alpha^{[\mu\nu]} := \sqrt{-g} g^{\rho\lambda} g^{\lambda\tau} \Gamma_{\rho\lambda}^\sigma \). The associated energy-momentum complex is

\[ (2\kappa)_2 T_\alpha^\mu = \delta^{\kappa}_\nu (-R + \partial_\lambda \Gamma^{\lambda\beta}_\beta + \Gamma^{\beta\nu}_\lambda \Gamma_\lambda^{\lambda\nu}) - (\partial_\nu + \Gamma^\tau_{\nu\mu}) \Gamma^{\rho\lambda}_\lambda. \]  

(26)

Inside matter, \( T_\alpha^\mu = (R_\alpha^{\mu} - \delta_\alpha^\mu R)/(6\kappa) \) to lowest order in Riemann normal coordinates. In vacuum, using the harmonic gauge condition, this \( S \) pseudotensor can be written as

\[ (2\kappa)_2 t_\alpha^\mu = \delta^{\kappa}_\nu \Gamma^{\beta\nu}_\lambda \Gamma_\lambda^{\lambda\nu} - (\partial_\nu + \Gamma^\tau_{\nu\mu}) \Gamma^{\rho\lambda}_\lambda. \]  

(27)

The related gravitational energy density is

\[ (2\kappa)_2 t_0^0 = (1 - h_{00}) \partial_0^2 h_{00} - \frac{3}{2} (\partial_0 h_{00})^2 + h^{0c} \partial_0^2 h_{0c} - \partial_0 (h^{0c} \partial_0 h_{0c}) \]

\[ - (\partial_\lambda h_{00}) (\partial_0 h^{0c}) - (\partial_0 h^{0c}) (\partial_\lambda h_{00}) - \frac{1}{2} h^{cd} (\partial_0^2 \bar{h}_{cd}) \]

\[ + \frac{1}{2} \left[ (\partial_0 h^{0d}) (\partial_d h_{0c}) - (\partial_0 h^{0c}) (\partial_d h_{0d}) - (\partial_0 h^{0d}) (\partial_d h_{0c}) \right] \]

\[ + \frac{1}{2} \left[ (\partial_0 \bar{h}_{cd}) (\partial_d h_{0c}) + (\partial_0 \bar{h}_{de}) (\partial_d h_{0e}) \right]. \]  

(28)
and the energy flux is
\[
(2\kappa)_{2t_0}^j = -\partial_0 \partial^j h_{00} + (\partial_0 h_{00}) (\partial_0 h^{0j}) + (\partial_0 \tilde{h}^{ja}) (\partial_0 h_{00}) + h^{0j} \partial_a^2 h_{00} + h^{ij} \partial_0 h_{00} + \partial_0 \left[ 2h_{00} \partial^j h_{00} + h^{0c} \partial^j h_{0c} + \frac{1}{2} \tilde{h}^{ca} \partial^j \tilde{h}_{cd} \right].
\]  
(29)

According to (3) and (4), compute the tidal heating as follows

\[
\bar{W}_2 = \frac{1}{2\kappa} \frac{d}{dt} \left[ -\int_V \tilde{\nabla}^2 h_{00} d^3 x + \int_{\partial V} \left( 2h_{00} \partial^j h_{00} + h^{0c} \partial^j h_{0c} \right) dS_j \right]
\]

\[
= \frac{1}{2\kappa} \frac{d}{dt} \left( \bar{I}_{ij} E^{ij} + \frac{2}{3} \bar{I}_{ij} B^{ij} \right),
\]

(30)

where \( \tilde{\nabla}^2 h_{00} = 2\tilde{\nabla}^2 (M r^{-1}) = 0 \) as mentioned before. This part contributes vanishing tidal heating since it is undertaken a time reversible process. In particular, Purdue uses \( W_{\text{int}} = \frac{\gamma + 2}{10} I_{ij} E^{ij} \) to interpret different choices of energy localization by tuning the coefficient \( \gamma \). [4]

### 3.3 The 3rd term of the Freud superpotential

When \((a_1, a_2, a_3) = (0, 0, 1)\) for [17], the third term of the Freud type superpotential is \(3 U_{\alpha}^{[\mu\nu]} := \sqrt{-g} \Gamma^\rho_{\alpha \beta} \delta^\mu_{\rho \nu} \). If we multiply by a factor of 2, the Møller superpotential [10] is recovered. This is the essential part which gives the desired tidal heating value. The associated Møller pseudotensor can be obtained as

\[
2\kappa(3t_\alpha^\mu) = 2R_{\alpha}^\mu - \partial_\alpha \Gamma^\mu_{\beta \gamma} + g^{\mu \beta} \partial_\alpha \Gamma^\nu_{\beta \nu} - 2\Gamma^\beta_{\alpha \nu} \Gamma^\mu_{\beta \nu}.
\]

(31)

Inside matter this reduces to the total energy-momentum to lowest order is \(3 T_{\alpha}^\mu = R_{\alpha}^\mu / (2\kappa) \) at the origin in Riemann normal coordinates. Since this result is not compatible with Einstein’s equation, one may have doubts as to whether it is meaningful to keep calculating the tidal work? Although the Møller pseudotensor has already failed the inside matter requirement, this pseudotensor has the nice feature that its gravitational energy is coordinate system independent. In vacuum, referring to (31), the Møller pseudotensor becomes

\[
2\kappa(3t_\alpha^\mu) = -\partial_\alpha \Gamma^\mu_{\beta \gamma} + \frac{1}{2} (\partial_\alpha \partial^\mu h - h^{\beta \mu} \partial_\alpha^2 h - h^{\beta \lambda} \partial_\alpha \partial^\mu h_{\beta \lambda}) - (\partial_\alpha h^{\beta \lambda})(\partial_\beta h^\mu_{\gamma}).
\]

(32)

The associated energy density and energy flux are

\[
2\kappa(3t_0^0) = (h_{00} - 1) \partial_0^2 h_{00} + 3(\partial_0 h_{00})^2 + (\partial_c h_{00})(\partial_0 \tilde{h}^{cd}) + \frac{1}{2} \tilde{h}^{cd} \partial_0^2 \tilde{h}_{cd}
\]

\[
+ \eta^{cd} \left[ h_{0c}(\partial_d^2 h_{00} - \partial_0^2 h_{0d}) - (\partial_0 h_{0c})(\partial_0 h_{0d} + \partial_d h_{00}) \right], \tag{33}
\]

\[
2\kappa(3t_0^j) = (1 - 3h_{00}) \partial_0 \partial^j h_{00} - (\partial_0 h_{00})(\partial^j h_{00}) - h^{0j} \partial_0^2 h_{00} - \partial_c (\tilde{h}^{jc} \partial_0 h_{00}) + (\partial_0 h_{0c} - \partial_c h_{00})(\partial_0 \tilde{h}^{jc}) - (\partial_0 \tilde{h}_{cd})(\partial^c \tilde{h}^{jd})
\]

\[
+ \eta^{cd} \left[ h_{0c} \partial_0 \partial^j h_{0d} - (\partial_0 h_{0c})(\partial_d h^{0j}) \right] - \frac{1}{2} \tilde{h}_{cd} \partial_0 \partial^j \tilde{h}^{cd}.
\]

(34)

Calculate the tidal heating as follows

\[
2\kappa W_3 = \frac{d}{dt} \int_V \tilde{\nabla}^2 h_{00} d^3 x - \int_{\partial V} \left[ 2h_{00} \partial_0 \partial^j h_{00} + (\partial_0 h_{00})(\partial^j h_{00}) \right] dS_j
\]

\[
+ \int_{\partial V} \eta^{cd} \left[ h_{0c} \partial_0 \partial^j h_{0d} - (\partial_0 h_{0c})(\partial_d h^{0j}) \right] dS_j.
\]

(35)
Again, we emphasize that there is a difference between Favata’s and our understanding. The mass $M$ is time dependent in Favata’s argument, but we treated $M$ as a constant, based on the harmonic gauge constraint (see (9) and (13)).

After some simple algebra, the tidal heating for a black hole is

$$
\dot{W}_3 = \frac{1}{2} \left( I_{ij} \dot{E}^{ij} + \frac{4}{3} J_{ij} \dot{B}^{ij} \right) - \frac{2}{5} \frac{d}{dt} \left( I_{ij} E^{ij} + \frac{4}{3} J_{ij} B^{ij} \right) \\
= \frac{16}{45} M^6 \left( \dot{E}^2_{ij} + \dot{B}^2_{ij} \right) - \frac{2}{5} \frac{d}{dt} \left( I_{ij} E^{ij} + \frac{4}{3} J_{ij} B^{ij} \right),
$$  

(36)

where we used [6]:

$$
I_{ij} = \frac{32 M^6}{45} \dot{E}_{ij}, \quad J_{ij} = \frac{8 M^6}{15} \dot{B}_{ij}.
$$  

(37)

Note that $E^2_{ij}$ means $E^{ij} E_{ij}$ and similarly for $B^2_{ij}$. Only this superpotential is the substantial part which contributes the desired tidal heating. More accurately, besides failing to meet the inside matter result $2G^u \nu$, we discovered that whenever the superpotential includes this term with unit magnitude, one can guarantee that the suitable tidal heating value can be achieved. In other words, the tidal heating is (in a suitable sense) pseudotensor independent as Thorne expected and Favata intended to verify [7]. Thorne wrote: “Similarly, if, in our general relativistic analysis, we were to change our energy localization by switching from the Landau-Lifshitz pseudotensor to some other pseudotensor, or by performing a gauge change on the gravitational field, we thereby would alter $E_{\text{int}}$ but leave $W$ unchanged” (p.9 in [3]). Perhaps Thorne had assumed that all pseudotensors already had the standard form to linear order (see Ch. 20 in [19]). From our result it turns out that the Møller pseudotensor (which does not fit into the standard linear form) also fails to yield the standard tidal heating. In other words, all the Freud type classical pseudotensors yield the expected tidal heating value, which shows that Thorne’s assertion is correct [3].

The Møller superpotential is twice the 3rd Freud superpotential term, giving the consequent small region Møller result and the Io tidal heating. There two discoveries here: the 3rd Freud superpotential term gives the desired value, and the Møller superpotential does not.

4 Conclusion

Thorne argued that tidal heating is independent of how one localizes the gravitational energy and the value is unambiguous. Purdue and Favata used a number of well known pseudotensors to calculate the tidal heating and verified that Thorne’s assertion is correct for them. After a re-examination of the Møller pseudotensor, we found reasons to doubt Favata’s calculations. Substantially, a technical difference had arisen. For the mass we used $\dot{M} = 0$ which means that $M$ is a constant, while Favata treated $\dot{M} \neq 0$ which requires $M$ is a time dependent object. However, it is strictly forbidden to allow $M$ as a time dependent function according to the harmonic gauge. Moreover, our result is valid both in the Jupiter-Io type system and black hole scenario.

Here we emphasize that if a suitable gravitational energy-momentum pseudotensor fulfills the Freud type superpotential condition, this requirement ensures the expected
tidal heating. Our analysis indicates that the Freud type superpotential is sufficient but not necessary. In particular, the pseudotensor that is obtained from 1/2 the Møller superpotential gives a counterexample that succeeds in achieving the desired tidal heating, even though it has the physical handicap of failing to meet the inside matter requirement.

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