On features of the radiation from an electron moving along a helix inside a cylindrical hole in a homogeneous dielectric

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February 2, 2008

Abstract

The radiation from a charge moving along a helical trajectory inside a cylindrical hole in homogeneous dielectric medium is investigated. Prompted by availability of materials with large dielectric permittivity \( \varepsilon \) and small absorption, we discuss the features of this type of radiation for media with \( \varepsilon \gg 1 \). It is shown that there are high peaks in the angular distribution of radiation intensity at well-defined harmonics. The conditions are specified for the cavity-to-helix radii ratio, \( \rho_1/\rho_0 \), under which the angle-integrated radiation intensity on some harmonics exceeds that in the empty space. Though the amplification of radiation intensity increases with increasing \( \varepsilon \), the corresponding "resonant" values of \( \rho_1/\rho_0 \) ratio are practically independent of the dielectric permittivity of surrounding medium. It is shown that an analogous amplification of radiation takes place essentially for the same values of \( \rho_1/\rho_0 \) also for the radiation in a cylindrical waveguide with conducting walls. An explanation of this phenomenon is given.

PACS number(s): 41.60.Ap, 41.60.Bq

Keywords: Synchrotron radiation, helical undulators

1 Introduction

The radiation from a charged particle moving along a helical orbit in vacuum has been widely discussed in literature (see, for instance, \[ \text{1, 2} \] and references given therein). This type of electron motion is used in helical undulators for generating electromagnetic radiation in a narrow spectral interval at frequencies ranging from radio or millimeter waves to X-rays. The unique characteristics, such as high intensity and high collimation, have resulted in extensive applications of this radiation in a wide variety of experiments and in many disciplines. These applications motivate the importance of investigations for various mechanisms of controlling the radiation parameters. From this point of view, it is of interest to consider the influence of a medium on the spectral and angular distributions of the radiation.

In Ref. \[ 3 \] we have investigated the radiation by a charged particle moving along a helical orbit inside a dielectric cylinder immersed into a homogeneous medium. Specifically, formulae are derived for the electromagnetic fields and for the spectral-angular distribution of the radiation.
intensity in the exterior medium. It is shown that under the Cherenkov condition for dielectric permittivity of the cylinder and the velocity of the particle image on the cylinder surface, strong narrow peaks are present in the angular distribution for the number of radiated quanta. At these peaks the radiated energy exceeds the corresponding quantity for a homogeneous medium by some orders of magnitude. A special case of the relativistic motion along the direction of the cylinder axis with non-relativistic transverse velocity is discussed in detail and various regimes for the undulator parameter are considered. The electromagnetic fields generated inside the cylinder by the charge moving along a helical orbit are considered in [4]. Note that the properties of the radiation from a charged particle moving along a helical orbit in homogeneous dispersive medium are studied in [5]. In the present paper we give the results of the further investigation of the radiation features for a charge moving along a helix inside a cylindrical hole in a homogeneous medium. We also compare this results with the radiation properties in the case when the particle moves with helical trajectory inside a cylindrical waveguide with conducting walls [6, 7].

2 Radiation from a charge moving inside a cylindrical hole in a homogeneous medium

Consider a point charge \( q \) moving along the helical trajectory of radius \( \rho_0 \) inside a cylindrical hole with radius \( \rho_1 \) in a homogeneous medium with dielectric permittivity \( \varepsilon \) (for simplicity the magnetic permeability is taken to be unit). The particle velocities along the axis of the hole (drift velocity) and in the perpendicular plane we will denote by \( v_\parallel \) and \( v_\perp \), respectively. In a properly chosen cylindrical coordinate system \((\rho, \phi, z)\) the corresponding motion is described by the coordinates \( \rho = \rho_0, \phi = \omega_0 t, z = v_\parallel t \), where the \( z \)-axis coincides with the hole axis and \( \omega_0 = v_\perp/\rho_0 \) is the angular velocity of the charge. This type of motion can be produced by a uniform constant magnetic field directed along the axis of a cylinder, by a circularly polarized plane wave, or by a spatially periodic transverse magnetic field of constant absolute value and a direction that rotates as a function of the coordinate \( z \).

In [3] it have been investigated the radiation by a charged particle moving along a helical orbit inside a dielectric cylinder immersed into a homogeneous medium. Now we turn to the investigation of the features of the radiation intensity to the exterior medium at a given harmonic \( m \) in the case when the particle moves along helical trajectory inside a cylindrical hole in dielectric medium with the dielectric permittivity \( \varepsilon \gg 1 \). At large distances from the charge trajectory the dependence of elementary waves on the space-time coordinates has the form \( \exp[i\omega_m\sqrt{\varepsilon}(\rho \sin \vartheta + z \cos \vartheta - ct/\sqrt{\varepsilon})/c] \) which describes the wave with the frequency

\[
\omega_m = \frac{m\omega_0}{|1 - v_\parallel \sqrt{\varepsilon} \cos \vartheta/c|}, \tag{1}
\]

propagating at the angle \( \vartheta \) to the \( z \)-axis. The number of the radiated quanta per period of a particle rotation for the harmonic \( m \neq 0 \) is given by the formula (see [3])

\[
\frac{dN_m}{d\Omega} = \frac{q^2 m \sqrt{\varepsilon}}{\pi^2 \hbar c |1 - v_\parallel \sqrt{\varepsilon} \cos \vartheta/c|^2} \left\{ \left| D^{(1)}_m - D^{(-1)}_m \right|^2 + \left| D^{(1)}_m + D^{(-1)}_m \right|^2 \cos^2 \vartheta \right\}. \tag{2}
\]

In (2), \( d\Omega = \sin \vartheta d\vartheta d\phi \) is the solid angle element and
\[ D_m^{(p)} = \frac{v_\perp}{c \rho_1 W_{m+p}} \left[ J_{m+p}(\lambda_0 \rho_0) + \frac{p \lambda_1}{2 \alpha_m} J_{m+p}(\lambda_0 \rho_1) \frac{H_m(\lambda_1 \rho_1)}{W_{m+l}} \right] \]

\[ + \frac{v_\parallel J_m(\lambda_0 \rho_0)}{c \rho_1 W_m} \left[ \frac{p \lambda_1 J_{m+p}(\lambda_0 \rho_1) H_m(\lambda_1 \rho_1)}{\alpha_m W_{m+p} \tan \vartheta} - \tan \vartheta \right]. \]  

(3)

In these formulae \( J_m(x), H_m(x) \equiv H_m^{(1)}(x) \) are the Bessel and Hankel functions, the coefficients \( \lambda_j \) are defined by the expressions

\[ \lambda_0 = \frac{m \omega_0}{c} \sqrt{1 - \varepsilon \cos^2 \vartheta}, \quad \lambda_1 = \frac{m \omega_0}{c} \sqrt{\varepsilon \sin \vartheta}, \]  

(4)

and

\[ W_m = J_m(\lambda_0 \rho_1) \frac{\partial H_m(\lambda_1 \rho_1)}{\partial \rho_1} - H_m(\lambda_1 \rho_1) \frac{\partial J_m(\lambda_0 \rho_1)}{\partial \rho_1}, \]  

(5)

\[ \alpha_m = \frac{1}{\varepsilon_1 - 1} - \frac{\lambda_0 J_m(\lambda_0 \rho_1)}{2} \sum_{l=\pm 1} l \frac{H_{m+l}(\lambda_1 \rho_1)}{W_{m+l}}. \]  

(6)

In figure 1 we have presented the dependence of the angular density of the number of the radiated quanta per period of an electron rotation, as a function of the angle \( \vartheta \) for the harmonic \( m = 1 \). The values for the other parameters are as follows: electron energy \( E = 10 \) MeV, dielectric permittivity of the surrounding medium \( \varepsilon = 230 \), and \( \phi_0 = \pi/3 \), \( \rho_1/\rho_0 = 11.726 \). Here we have introduced the angle \( \phi_0 \) between the particle velocity and the axis \( z \):

\[ v_\perp = v \sin \phi_0, \quad v_\parallel = v \cos \phi_0. \]  

(7)

![Figure 1: The angular density of the number of the radiated quanta, \((\hbar/c^2) dN_m/d\Omega\), as a function of the angle \( \vartheta \) for the harmonic \( m = 1 \), \( E = 10 \) MeV, \( \varepsilon = 230 \), \( \phi_0 = \pi/3 \), \( \rho_1/\rho_0 = 11.726 \).](image)

In the discussion below (see equation (18)) we will introduce also the angle \( \phi_1 \) for which the velocity of the propagation for the electromagnetic waves along the cylinder axis coincides with the projection of the particle velocity on this axis. From the graph it is seen that, though there are strong narrow peaks in left part of the graph (on the location of these peaks see [3]), the
main contribution to the number of radiated quanta integrated over the angle $\vartheta$, comes from the peak near $\vartheta_0 = 1.53786$. In the reference frame moving along the z-axis by the velocity $v_\parallel$, this angle corresponds to the angle $\vartheta'_0 = \pi/2$. It is related to the angle $\phi_1$ by the formula

$$\vartheta_0 = \arccos(\cos \phi_1/\sqrt{\varepsilon}).$$

(8)

In figure 2 we give the angular density of the number of the radiated quanta on the harmonic $m = 1$ as a function of the ratio $\rho_1/\rho_0$ for $\vartheta_0 = 1.53786$. The values of the other parameters are the same as those for figure 1.

![Figure 2: The angular density of the number of the radiated quanta, $(hc/q^2)dN_m/d\Omega$, as a function of the ratio $\rho_1/\rho_0$ for $\vartheta_0 = 1.53786$. The values of the other parameters are the same as those for figure 1.](image)

Integrating $dN_m$ over the angles, we obtain the total number of the radiated quanta on a given harmonic. In figure 3 we have presented the number of radiated quanta $N_m$ for $m = 1$ as a function of the ratio $\rho_1/\rho_0$. The values of the other parameters are as follows: $E = 10$ MeV, $\phi_0 = \pi/3$, $\varepsilon = 230$. The horizontal straight line corresponds to the number of the radiated quanta in the vacuum ($\varepsilon = 1$). As it is seen from figure 3 there are peaks in the number of the radiated quanta for certain values of the ratio $\rho_1/\rho_0$. The distance between the neighboring peaks, $\Delta(\rho_1/\rho_0)_{peak}$, is approximately the same. The numerical calculations have shown that similar features take place for other values of the dielectric permittivity with the same locations of the peaks. The height of the peaks is a decreasing function of $\varepsilon$. For example, the height of the peak at $\rho_1/\rho_0 \approx 1.89$ is $3.502$ for $\varepsilon = 3$.

From the numerical calculations it follows that the amplification of the radiation takes place at once on a set of harmonics. In figure 4 we have plotted the dependence of the number of radiated quanta on the number of harmonic $m$ for the values of the parameters $\rho_1/\rho_0 = 8.595$, $E = 10$ MeV, $\phi_0 = \pi/3$, $\varepsilon = 230$. On the same graph we have also plotted the number of radiated quanta in the vacuum (dashed curve). We see the following regularity: if there are peaks in the dependence of the number of radiated quanta as a function of the ratio $\rho_1/\rho_0$ for $m = 1$, then the peaks are present also for the harmonics $m = 4l + 1, l = 1, 2, \ldots$.

3 Radiation in a cylindrical waveguide

In this section we consider the radiation intensity for a charge moving in a helical orbit inside an empty circular waveguide with radius $\rho_1$. We will assume that the waveguide walls are perfectly
Figure 3: The number of the radiated quanta, \((\hbar c/q^2)N_m\), as a function of the ratio \(\rho_1/\rho_0\) for \(m = 1\). The values of the other parameters are as follows: \(E = 10\) MeV, \(\phi_0 = \pi/3\), \(\varepsilon = 230\).

Figure 4: The number of radiated quanta, \((\hbar c/q^2)N_m\), as a function of the number of harmonic \(m\) for the values of the parameters \(\rho_1/\rho_0 = 8.595\), \(E = 10\) MeV, \(\phi_0 = \pi/3\), \(\varepsilon = 230\).
conducting. The radiation field inside the waveguide is a superposition of TM and TE waves. The corresponding frequencies are equal

$$\omega_{m,n}^{(\sigma,\pm)} = \frac{m\omega_0}{1 - v_\parallel^2/c^2} \left[ 1 \pm \frac{v_\parallel}{c} \sqrt{1 - b_{m,n}^{(\sigma)2} \left( 1 - v_\parallel^2/c^2 \right)} \right],$$

where

$$b_{m,n}^{(\sigma)} = \frac{c j_{m,n}^{(\sigma)}}{m\omega_0 \rho_1}. \quad (9)$$

In these formulae \( \sigma = 0, 1 \) correspond to TM and TE waves respectively, \( j_{m,n}^{(\sigma)} \), \( n = 1, 2, \ldots \), is the \( n \)th positive zero of the Bessel function \( J_m(x) \) (\( \sigma = 0 \)) and its derivative \( J'_m(x) \) (\( \sigma = 1 \)). We have carried out numerical calculations for the number of the quanta on a given harmonic radiated per one period of the particle orbiting:

$$N_m = \sum_{\sigma=0,1} \sum_{n=1}^{n_{\text{max}}} N_{m,n}^{(\sigma)}, \quad m = 1, 2, \ldots. \quad (10)$$

The upper limit of the summation over \( n \) in this formula is defined by the relation

$$j_{m,n}^{(\sigma)} < \frac{m\omega_0 \rho_1}{c \sqrt{1 - v_\parallel^2/c^2}} < j_{m,n}^{(\sigma)} + 1. \quad (11)$$

For separate terms in (10) one has the formulae [7]

$$N_{m,n}^{(0)} = \frac{8\pi q^2 m}{\hbar c} b_{m,n}^{(0)} J_0^2(j_{m,n}^{(0)} \rho_0/\rho_1) \sqrt{1 - b_{m,n}^{(0)2} \left( 1 - v_\parallel^2/c^2 \right)};$$

$$N_{m,n}^{(1)} = \frac{8\pi q^2}{\hbar c m \rho_0^2} b_{m,n}^{(1)} J_1^2(j_{m,n}^{(1)} \rho_0/\rho_1) \sqrt{1 - b_{m,n}^{(1)2} \left( 1 - v_\parallel^2/c^2 \right)}. \quad (12)$$

If the condition

$$j_{m,n}^{(\sigma)} \sqrt{1 - v_\parallel^2/c^2} = m(v_\perp \rho_1)/(c \rho_0)$$

takes place then the intensity for the TE waves defined by formulae [12] goes to infinity. However, under these conditions the absorption in the walls of the waveguide becomes important. We can introduce the angular variable \( \vartheta_{m,n}^{(\sigma,\pm)} \) which is related to the frequencies by the formula

$$\omega_{m,n}^{(\sigma,\pm)} = \frac{m\omega_0}{1 - v_\parallel \cos \vartheta_{m,n}^{(\sigma,\pm)}/c}. \quad (15)$$

It’s possible values are determined by the formula

$$\cos \vartheta_{m,n}^{(\sigma,\pm)} = \frac{b_{m,n}^{(\sigma)2} v_\parallel/c \pm \sqrt{1 - b_{m,n}^{(\sigma)2} \left( 1 - v_\parallel^2/c^2 \right)}}{1 + b_{m,n}^{(\sigma)2} v_\parallel^2/c^2}. \quad (16)$$

Note that the singularity in the radiation intensity (13) noted above corresponds to the values of the angular variable determined by the condition

$$\vartheta_{m,n}^{(\sigma,\pm)} = \vartheta_\perp, \quad \vartheta_\perp \equiv \arccos(v_\parallel/c). \quad (17)$$

In the reference frame moving along the direction of the axis \( z \) with the velocity \( v_\parallel \) the angle corresponding to \( \vartheta_\perp \) is equal to \( \pi/2 \).
Figure 5: The number of radiated quanta in the waveguide, \((hc/q^2)N_m\), as a function of the ratio \(\rho_1/\rho_0\). The values of the parameters are as follows: \(m = 1\), \(E = 10\) MeV, \(\phi_0 = \pi/3\).

In figure 5 we have presented the number of quanta radiated on the harmonic \(m = 1\) as a function of the ratio \(\rho_1/\rho_0\) for the values of the parameters: \(E = 10\) MeV, \(\phi_0 = \pi/3\). Note that the location of the peaks in the number of radiated quanta is determined by formula (14) with \(\sigma = 1\). Comparing the numerical results presented in figures 3 and 5 we see that the location of the peaks in the dependence of the radiation intensity of the ratio \(\rho_1/\rho_0\) is the same for the problems considered in sections 2 and 3. An explanation of this feature will be given in the next section.

4 The reason of radiation amplification

For simplicity let us consider the case when \(\rho_1/\rho_0 \gg 1\). In this case for all harmonics the hole radius is much larger than the wavelength of the radiation. We will assume that the particle is relativistic. Obviously, there is a radiation direction \(\phi_1\) for which the electromagnetic wave radiated at the point A (see figure 6) after the reflection from the wall reaches the point B at the instant of time when the particle is there. That condition is fulfilled if

\[ v_\parallel = v \cos \phi_0 = c \cos \phi_1. \]  

(18)

For certain values of the phase this can lead to the essential change in the radiation intensity. It is clear that the radiation corresponding to the angle \(\phi_1\) propagates in the exterior medium with the angle \(\vartheta_0\). As it is seen from figure 11 exactly near \(\vartheta_0\) intense radiation appears. The radiation intensity can be essentially increased if the superposition of the waves at the point B is in the same phase. This is possible if on the optical path \(2\rho_1\) an integer number of waves should be placed:

\[ \frac{2\rho_1}{\Lambda_m} + \frac{\Delta \varphi}{2\pi} = k, \]  

(19)

where \(\Delta \varphi\) is the phase change in the reflection, \(\Lambda_m\) is the wavelength in the transverse direction, and \(k = 1, 2, \ldots\). From (19) and data given in figure 2 one can conclude that the change of the phase in the reflection of the wave

\[ \Delta \varphi \approx \pi/2 \text{ at } \rho_1/\rho_0 \gg 1. \]  

(20)
By taking into account the relation (18) and formula (4), we have \( \Lambda_m = 2\pi c \sin \phi_1/(m\omega_0) \), and for the distance between two neighboring peaks one finds

\[
\Delta(\rho_1/\rho_0) \simeq \frac{\pi \tan \phi_1}{m \tan \phi_0}.
\]  

(21)

So the peaks in figures 2, 3 have to be equidistant, which indeed takes place with relative accuracy \( \approx 0.1\% \).

Now it is obvious from (19), (20) that if a resonance value for \( \rho_1/\rho_0 \) is found for some \( m \), then for the same values of the parameters the amplification of the radiation will be observed for the harmonics \( m^* = m(4l + 1) \), \( l = 0, 1, 2, \ldots \), as well (see figure 4).

![Diagram](image)

Figure 6: On the explanation of the amplification for the radiation intensity.

As it is seen from formulae (19)-(21), for the problem discussed in section 2 the resonance values for the ratio \( \rho_1/\rho_0 \) do not depend on the dielectric permittivity of the surrounding medium. For this reason, for the same values of \( \rho_1/\rho_0 \) the increase of the radiation intensity has to be observed in the waveguide as well. This is confirmed by the graphs given in figures 3 and 5.

**Acknowledgement**

The work has been supported by Grant No. 1361 from Ministry of Education and Science of the Republic of Armenia.

**References**

[1] V.A. Bordovitsyn (Ed.), Synchrotron Radiation Theory and Its Development, World Scientific, Singapore, 1999.

[2] A. Hofman, The Physics of Sinchrotron Radiation, Cambridge University Press, Cambridge, 2004.

[3] A.A. Saharian, A.S. Kotanjyan, J. Phys. A 38 (2005) 4275.

[4] A.A. Saharian, A.S. Kotanjyan, M.L. Grigoryan, J. Phys. A 40 (2007) 1405.

[5] L.A. Gevorgian, P.M. Pogosian, Izv. Akad. Nauk Arm. SSR Fiz. 19 (1994) 239.

[6] G.G. Karapetyan, Izv. Akad. Nauk Arm. SSR Fiz. 12 (1977) 186.

[7] A.S. Kotanjyan, A.A. Saharian, J. Phys. A 40 (2007) 10641.