Abstract

Many bipartite networks describe systems where a link represents a relation between a user and an item. Measuring the similarity between either users or items is the basis of memory-based collaborative filtering, a widely used method to build a recommender system with the purpose of proposing items to users. When the edges of the network are unweighted, traditional approaches allow only positive similarity values, so neglecting the possibility and the effect of two users (or two items) being very dissimilar. Here we propose a method to compute similarity that allows also negative values, the Sapling Similarity. The key idea is to look at how the information that a user is connected to an item influences our prior estimation of the probability that another user is connected to the same item: if it is reduced, then the similarity between the two users will be negative, otherwise it will be positive. Using different datasets, we show that the Sapling Similarity outperforms other similarity metrics when it is used to recommend new items to users.

Keywords: Recommender system, Collaborative filtering, Bipartite networks, Link prediction, Economic complexity, Local similarity

1. Introduction

Many complex systems can be reduced to the interaction between two classes of different objects; this is the case of several economic systems, where either countries or firms are connected to exported products or technological sectors [1, 2, 3, 4]; biological systems, where either patients or microbes are connected with diseases [5, 6]; social systems, where for instance users are connected to Facebook pages [7, 8, 9], or actors are connected to movies they participate in [10]. An effective way to represent these systems is through bipartite networks, in which links can connect only nodes belonging to the two different sets. For instance, the bipartite network representing which country exports which product is the basis of the economic complexity (EC) framework [11]. The main tool of EC is the so-called relatedness [12, 13, 14], a measure of how much a country is close to start exporting a given product. This is a key tool for institutions and policy makers, and a driver for investments [15, 16]. The traditional approach to measure the relatedness between a country $c$ and a product $p$ consists in analyzing the export basket of countries, extracting the similarity between $p$ and other products, and computing the average similarity between $p$ and the products exported by $c$ [17]. In an information system framework, this would be called an item-based collaborative filtering (CF) [18, 19].

CF is one of the most popular methods to build a recommender system whose purpose is to suggest to users those items they will probably like. The idea is to base the recommendations on the previous interactions between users and items, collected in a user-item biadjacency matrix (that in the recommender system framework is usually called rating matrix). There are two main typologies of CF: memory-based and model-based [20]. The former obtains similarity

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measures between users or items according to the user-item adjacency matrix in order to find the nearest neighbors of an user or an item, then it bases the recommendations on these neighbors. The latter makes use of machine learning and data mining methods in order to achieve high recommendation quality.

From now on we will focus on memory-based CF, that have simple and intuitive implementations and high interpretability of the results. Since our aim is not to compare different recommender systems, but to compare similarity metrics used in memory-based CF, we will not cover model-based CF in this study.

Talking about memory-based CF, there are two main approaches to produce recommendations:

- **user-based**: first you measure the similarity between users based on how similar are their links with items and then you suggest to an user an item that is popular between the neighbors users;
- **item-based**: first you measure the similarity between items and then you suggest to an user an item that is similar to those to which it is already connected.

In both the cases when one measures the similarity between two nodes there is a distinction between local and global similarity \[^{21,22}\]. The difference is that to compute a local similarity you need information only about the two involved nodes, while to compute a global similarity you need to know the whole graph structure and this makes the latter computationally demanding and hard to apply to large networks. Local similarities are usually based on co-occurrences, that is a count how many times two nodes of the same layer are linked to the same node of the other layer. In the EC framework these are largely used for both prediction and recommendation purposes \[^{17,23,24}\].

When we deal with bipartite networks it is very common to have unary links, which means that the corresponding matrix element can be either equal to 1 if present or to 0 if absent. In this case, a recommender system looks for the possible appearance of new links in a temporal network, so recommendation can be seen as a link prediction problem, in which the recommended links are the ones that more likely will appear in the future, \[^{25}\]. A key issue is that similarity metrics are positive defined, and as a consequence they neglect the information about the possible dissimilarity between objects. For instance, in the EC case, co-occurrences are clearly positive-definite. Xie et al \[^{26}\] observed that neglecting negative values in similarity metrics can be an issue when one deals with binary data (whose entries can be like (1), dislike (-1), or absence (0)); however, negative similarity should be taken into account also when we deal with unary data. Let us make an example taken from economics. Allowing only positive similarity values does not take into account the fact that, for instance, Japan is specialized in high-tech products, while Zambia has a focus on simple products like raw materials. Knowing that Japan exports a product \(p\) makes Zambia not exporting \(p\) highly probable; on the other side, if Zambia exports \(p'\), probably Japan is not interested on exporting it. So it makes sense to define a negative similarity between Zambia and Japan. In other words, we can say that Zambia and Japan are anti-correlated, so knowing that this country exports a product should have a negative effect on the recommendation of that product to Japan. It is therefore natural to distinguish the case in which two export baskets are independent (zero similarity) from a situation in which the fact that a country exports a product implies that another country does not (negative similarity).

In this paper, we propose a local similarity metric that allows also negative values, the Sapling Similarity. With a structure inspired by the functioning of decision trees, Sapling Similarity is able to identify when two nodes are anti-correlated or dissimilar, in the sense that knowing that the first is connected to a node of the other layer, reduces the probability that also the second is connected to it. As we will better discuss later, the key to distinguish anti-correlation from uncorrelation is to look at the total number of nodes of the other layer with respect to the one where the similarity is computed. We will see that our proposed similarity outperforms the other ones when one recommends products to countries both in in an user-based and an item-based collaborative filtering framework. Furthermore, we will investigate its prediction performance also in different contexts, such as the recommendation of movies and e-commerce products to users, and GPS data connecting inhabitants to points of interests in Milan (like monuments, restaurants, schools, ...). We will show that also in these very different frameworks the prediction performance of the Sapling Similarity is higher.

The rest of the paper is organized as follows: Section 2 introduces the Sapling Similarity metric giving a detailed explanation of its construction. In section 3 we describe our experimental setup, the datasets we use, the methods we compare with the Sapling Similarity, and the performance metrics we adopt in this comparison. In section 4 we present the results, providing also an insightful representations obtainable from our similarity metric. Section 5 concludes and proposes further developments.
2. Methodology

In this section we introduce the Sapling Similarity metric, illustrating its formulation and an intuitive explanation of its functioning. We start by providing the basic definitions needed and by fixing the notation.

2.1. Basic definitions

A bipartite network is defined as a graph \( G = (U, \Gamma, E) \) where \( U \) and \( \Gamma \) are two sets of nodes (called also layers), and \( E \) is the set of all the connections \((i, \alpha)\) between the nodes \( i \in U \) and \( \alpha \in \Gamma \). Let \(|U|\) be the dimension (cardinality) of the set \( U \) and \(|\Gamma|\) the dimension of the set \( \Gamma \): the bipartite network can be represented by a \(|U| \times |\Gamma|\) binary matrix \( M \) called bi-adjacency matrix and defined as:

\[
M_{\alpha\alpha} = \begin{cases} 
1 & \text{if } (i, \alpha) \in E \\
0 & \text{if } (i, \alpha) \notin E 
\end{cases}
\]  

(1)

The degree of a node \( i \in U \) or \( \alpha \in \Gamma \) is the number of links connected to it:

\[
k_i = \sum_{\alpha=1}^{|\Gamma|} M_{i\alpha} \quad k_\alpha = \sum_{i=1}^{|U|} M_{i\alpha}
\]  

(2)

The number of co-occurrences between either nodes \( i, j \in U \) or \( \alpha, \beta \in \Gamma \) is:

\[
CO_{ij} = \sum_{l=1}^{|\Gamma|} M_{i\alpha}M_{j\alpha} \quad CO_{\alpha\beta} = \sum_{l=1}^{|U|} M_{i\alpha}M_{l\beta}
\]  

(3)

Note that the CO matrices define two monopartite networks whose nodes belong to only one set.

Usually a bipartite network describes interactions between users and items (for instance, in the case of the country-exported product network we can identify the former as the user and the latter as the item). In our notation, the set \( U \) corresponds to users and the set \( \Gamma \) corresponds to items.

In the following we will focus on the Sapling Similarity between two users, and we will define \( N = |\Gamma| \) (the total number of items). The case of the similarity between items is totally equivalent, with the only changes that \( N = |U| \) (the total number of users) and the co-occurrences and degrees are referred to the nodes in the layer of the items.

2.2. The Decision Sapling

The building block of the Sapling Similarity between two users \( i \) and \( j \), is what we call the Decision Sapling, that is a decision tree with just one split. The Decision Sapling is a diagram that represents and quantifies how much the information that user \( j \) is or is not connected to an item \( \alpha \) influences our estimate of the probability that the user \( i \) is connected to \( \alpha \). In figure\[\text{[1]}\] on the left, we show a numerical example where we build the Decision Sapling of user \( i \) with respect to user \( j \); on the right we show the formulation of the generic case in terms of \( CO_{ij}, k_i, k_j \), and \( N \). A Decision Sapling is composed of three boxes and each box is divided into two areas. In the right (left) area of all boxes there are the total number and the fraction of items to which user \( i \) is (not) connected. In the lower box (the bean) these numbers are computed with respect to all \( N \) items: so on the right we have \( k_j \) (the number of items user \( i \) is connected to) and on the left we have \( N - k_j \). In the right box (right leaf) the numbers are computed by considering only the subset of items to which user \( j \) is connected: so on the right area we have the co-occurrences \( CO_{ij} \) (the number of items connected to both \( i \) and \( j \)) and on the left area we will have the number of items connected to \( j \) and not to \( i \). Note that the sum is equal to \( k_j \). In the left box (left leaf) we consider instead only the items not connected to user \( j \); so for instance in the left box we will have the number and fraction of items not connected to \( i \) nor to \( j \). Note that the fractions are always computed with respect to the total number of items in the box, so the denominator is \( N \) for the bean, \( k_j \) for the right box, and \( N - k_j \) for the left box. Let us now discuss a numerical example, reported on the left side of figure\[\text{[1]}\]. We have a total of \( N = 100 \) items; users \( i \) and \( j \) are connected to only 5 of them, with 2 common items. The a priori probability for an item to be connected with \( i \) is provided in the bean and it is equal to 5%. However, if we then look at the right leaf it emerges that by selecting the subset of items connected to \( j \) the probability that also
Figure 1: The Decision Sapling of user $i$ with respect to user $j$ is a tool to visualize and quantify how the probability that a generic item is connected to $i$ changes when one considers also the connections of $j$. On the left a numerical example, on the right the generic formulations. In the bean box at the base, we have two areas: on the right there is the number and the fraction of items user $i$ is connected to, while on the left the items user $i$ is not connected to. On the right leaf the same numbers are computed by restricting the number of items to the ones user $j$ is connected to. Finally, on the left leaf only the items to which user $j$ is not connected are considered. By comparing the fractions in the bean with the ones in the leaves one can deduce whether the similarity between $i$ and $j$ are positive or negative.

$i$ is connected to it varies from 5% to 40%. So knowing that an item is connected to $j$ increases the probability that the same item is also connected to $i$: in this case, the similarity between $i$ and $j$ has to be positive. Looking to the left leaf we see that knowing that $j$ is not connected to an item decreases the probability that $i$ is connected to it, so also from this point of view it is natural to give a positive similarity between $i$ and $j$ (the variation of the probabilities with respect to the bean is more evident in the right leaf because of the lower number of items connected to $j$).

We can modify this example to understand why, even before looking to the results of the prediction exercise, Sapling Similarity uses more features of the other similarity metrics, so providing a more comprehensive view. To the best of our knowledge, existing similarity metrics use the numerical values of $CO_{ij}$, $k_i$, and $k_j$, but not the total number of items $N$. Let us consider $N = 8$ instead of 100 in the previous example, and we leave the other numbers unchanged. In this case, the bean shows that $i$ is connected to 5 out of 8 items, which means the 62.5% of them, instead of the 5% of the previous case. If we consider the right leaf, that is the subset of items connected to $j$, the numbers do not change and so $i$ is still connected to the 40%. In this case, knowing that $j$ is connected to an item reduces the probability that also $i$ is connected to it. The similarity between $i$ and $j$ has to be negative. In conclusion, by only varying $N$ the similarity changes sign. Existing similarity metrics not only disregard the possibility of negative similarities, but also do not consider the information coming from the total dimension of the sets, since $N$ does not enter in the equations.

2.3. The Sapling Similarity: key idea

In this section we discuss how the Decision Sapling tool can be used to determine the sign and the magnitude of the similarity between two items. In figure 2 we show three examples of Decision Saplings, namely of the same user $i$ with respect to three different users $a$, $b$, and $c$. In the case (a) the Sapling Similarity $B^{spling}_{ia} = -1$ since the knowledge that $a$ is linked to an item implies that $i$ is not linked to it: in particular, $a$ and $i$ have zero co-occurrences. In the case (b) $B^{spling}_{ib} = 0$ since the fact that $b$ is linked to $a$ does not add information about the possibility that also $i$ is connected to $a$. In the case (c) $B^{spling}_{ic} = 1$ since if you know that $c$ is linked to $a$ than you know for sure that also $i$ is linked to it. In particular, here all $j$’s items co-occur with $i$. Looking at the left leaf it is easy to reach the same conclusion: for instance knowing that $a$ is not connected to a node gives certainty that $i$ is connected to it, so $B^{spling}_{ia} = -1$. In the cases in which the information provided by the items of $j$ removes all the uncertainty present in the bean the Sapling Similarity is maximally positive or negative. If considering the subset of items connected to $j$ does not add any information about $i$, the similarity is zero. In order to provide an assessment of the similarity in the
intermediate cases we make use of the Gini Impurity (GI). For each box we define:

$$GI = 1 - p_1^2 - p_0^2 = 2p_1p_0$$  \hspace{1cm} (4)$$
where $p_0$ is the fraction of zeros (the red values in figure) and $p_1$ the fraction of ones (the green values in figure). The Gini Impurity reaches its minimum value 0 when $p_0$ or $p_1$ is 0 and its maximum 0.5 when $p_0 = p_1 = 0.5$. So the lower it is the GI the higher is the polarization or certainty associated to the box. The relative variation of the Gini Impurity after the split given by user $j$ is:

$$
\Delta GI = \frac{GI^{(b)} - f(l)GI^{(l)} - f(r)GI^{(r)}}{GI^{(b)}} \hspace{1cm} (5)
$$

where $GI^{(b)}$ is the Gini Impurity of the bean, $GI^{(l)}$ the GI of the left leaf and $GI^{(r)}$ the GI of the right leaf, while $f(l)$ is the fraction of total elements in the left leaf $\frac{k_j}{N}$, and $f(r)$ the same fraction in the right leaf $\frac{k_j}{N}$. Notice that we normalize the Gini variation with $GI^{(b)}$ in order to have a $\Delta GI$ ranging between 0 and 1. $\Delta GI$ close to 0 means that considering $j$ does not vary the fractions in the leaf with respect to the bean, while $\Delta GI$ close to 1 means that the fractions after the split are more polarized than the ones in the bean, and so reduce the starting uncertainty. We point out that $\Delta GI$ is the absolute value of our similarity measure, the sign will be positive if the percentage of positive elements on the right leaf is higher than the one on the bean and negative otherwise.

2.4. The Sapling Similarity: formula

Here we finally derive the Sapling Similarity formula. In particular, we express in terms of the values that can be directly computed from the network, that is the total number of items $N$, the degrees of user $i$ ($k_i$) and $j$ ($k_j$), and the number of co-occurrences between $i$ and $j$ ($CO_{ij}$). We recall that each box is divided in a right area (which considers the items connected to $i$) and a left area (items not connected to $i$).

- **Bean:** The bean considers a total of $N$ elements, $k_i$ of which are connected to $i$; so $N - k_i$ items are not connected to $i$.

- **Right leaf:** The total number of elements considered in the right box is $k_j$. Of those, $CO_{ij}$ are connected to $i$, and the number of items connected to $j$ but not to $i$ is $k_j - CO_{ij}$.
• **Left leaf:** Here the total number is \( N - k_j \). The number of elements connected to \( i \) but not to \( j \) us \( k_i - CO_{ij} \). \( N - k_j - k_i + CO_{ij} \) items are not connected to \( i \) nor to \( j \).

So the terms in equation \( 5 \) can be written as:

\[
G^{(b)} = 2 \frac{k_i (N - k_i)}{N - N - k_i} \\
G^{(r)} = k_j \\
G^{(l)} = \frac{N - k_j}{N} \\
G^{(r)} = 2 \frac{CO_{ij} k_j - CO_{ij}}{k_j} \\
G^{(l)} = 2 \frac{k_i - CO_{ij} N - k_j - k_i + CO_{ij}}{N - k_j} \\
\]

and with a simple computation we can show that equation \( 5 \) is equivalent to:

\[
\Delta GI = 1 - f_{ij} 
\]

where:

\[
f_{ij} = \frac{CO_{ij} (1 - \frac{CO_{ij}}{k_j}) + (k_i - CO_{ij})(1 - \frac{k_i - CO_{ij}}{N - k_j})}{k_i (1 - \frac{k_i}{N})} 
\]

As already said, \( \Delta GI \) is the absolute value of our similarity measure. The sign of the Sapling Similarity is positive if the fraction of elements in the right area of the right leaf is bigger than the one on the bean, that is, if \( CO_{ij}/k_j \geq k_i/N \). If \( CO_{ij}/k_j < k_i/N \), the sign of the Sapling Similarity is negative.

**Observation.** Notice that \( \Delta GI \) is singular if \( k_i = 0, k_i = N, k_j = 0 \) or \( k_j = N \). If a node \( l \) has degree \( k_i = 0 \) then it is not connected to anything, so it is useless for our collaborative filtering and it can be deleted from the network. If \( k_i = N \) then \( l \) is connected to all items; also this case is not useful since this node does not bring useful information for our collaborative filtering.

In conclusion, the Sapling Similarity metric can be written as:

\[
B_{ij}^{sapling} = \begin{cases} 
1 - f_{ij} & \text{if } \frac{CO_{ij} N}{k_i k_j} \geq 1 \\
-1 + f_{ij} & \text{otherwise}
\end{cases} 
\]

**Lemma.** \( f_{ij} \) is symmetric.

**Proof.** we can write equation \( 7 \) as following:

\[
f_{ij} = \frac{k_i k_j (2CO_{ij} - k_i - k_j) + N(k_i k_j - CO_{ij}^2)}{k_i k_j (N - k_i) (N - k_j)} 
\]

it is trivial to prove that switching the indices \( i \) and \( j \) the result does not change (note that \( CO_{ij} \) is symmetric).

**Observation.** if \( f_{ij} \) is symmetric then also \( B_{ij}^{sapling} \) is symmetric.

3. **Experimental setup**

In this section we describe the experimental setup we adopt to show that the Sapling Similarity outperforms the traditional similarity metrics if used in a user-based or item-based collaborative filtering.
3.1. Datasets

Our first dataset is the UN-COMTRADE (comtrade.un.org) data set consisting into the export data about 169 countries and 5040 products. A common procedure in the economic complexity literature [17, 2] is to compute the Revealed Comparative Advantage (RCA) introduced by Balassa [27] defined as:

\[ RCA_{i\alpha} = \frac{E_{i\alpha}}{\sum_l \frac{\sum_l E_{i\lambda}}{\sum_l E_{\lambda \alpha}}} \] (10)

where \( E_{i\alpha} \) is the volume of product \( \alpha \) expressed in american dollars that country \( i \) exports.

Since RCA is an estimate of how much a country is competitive in the export of a product, with a threshold on the RCA values we can build a binary matrix \( M \) representing the bi-adjacency matrix of the bipartite network where a link \((i, \alpha)\) means that country \( i \) is competitive in the export of product \( \alpha \).

\[ M_{i\alpha} = \begin{cases} 1 & \text{if } RCA_{i\alpha} \geq 1 \\ 0 & \text{if } RCA_{i\alpha} < 1 \end{cases} \] (11)

Other two data we consider in our analysis are the MovieLens 1M dataset, containing the rating data of users for multiple movies from the MovieLens website (https://grouplens.org/datasets/movielens), and an Amazon dataset (http://jmcauley.ucsd.edu/data/amazon) containing the rating data of users for multiple Amazon products. The rates range from 1 to 5, but to build the matrix \( M \) we select the links with the scores above 4. After this cut we removed from the dataset the users and the movies (or the products) with degree less than 10.

Finally, the fourth dataset we consider is a bipartite network connecting users with Point Of Interest (POI) in Milan extracted from OpenStreetMap (https://www.openstreetmap.org/). The location data are provided by Cuebiq Inc. (https://www.cuebiq.com/about/data-for-good/). The link between a user and a POI represents the number of visits of the user to the corresponding place. Here we removed all the links with less than 2 visits, we considered the 5000 POI most visited and we removed all the users that visited less than 20 POI. The resulting bipartite network consists into 19064 users and 5000 POI.

Usually, the datasets used for recommender systems are affected by the data sparsity problem. In table 1 we show that our datasets have a low number of ones (present link) with respect to the zeros (missing link). In particular, the density of links ranges from the 12.5% of the export data to the 0.57% of the GPS data.

| dataset           | # users | # items | density of links |
|-------------------|---------|---------|------------------|
| Export data       | 169     | 5040    | 12.5%            |
| Movielens         | 5950    | 2811    | 3.4%             |
| Amazon            | 4881    | 2686    | 1.04%            |
| GPS data          | 19064   | 5000    | 0.57%            |

Table 1: Main properties of our four datasets.

3.2. Evaluation metrics

To recommend items to users we build a \( M_{train} \) matrix from which we compute the similarities either between the users \( (B_{user}^{(user)}) \) or between the items \( (B_{item}^{(item)}) \), then we compute the confidence values of the recommendation of item \( \alpha \) to user \( i \) through the formula [28]:

\[ S_{i\alpha} = \frac{\sum_{l=1}^{\|U\|} \frac{B_{il}^{(user)}}{|B_{il}^{(user)}|} M_{i\alpha}}{\sum_{l=1}^{\|U\|} |B_{il}^{(user)}|} \] user based (12)

\[ S_{i\alpha} = \frac{\sum_{l=1}^{\|I\|} \frac{B_{i\lambda}^{(item)}}{|B_{i\lambda}^{(item)}|} M_{i\lambda}}{\sum_{l=1}^{\|I\|} |B_{i\lambda}^{(item)}|} \] item based (13)

Finally we compute some performance metrics comparing our recommendation scores of the elements for which \( M_{train} = 0 \) with a \( M_{test} \) matrix.
3.2.1. Future exports forecast

When working with the UN-COMTRADE dataset, to reduce the noise, especially in the less developed countries, before computing the RCA values we sum up the export volumes from 1996 to 2013. From the resulting values we extract the $M_{train}$ matrix. So our recommendations are based on data till 2013 and to see how good they are we compare them with a $M_{test}$ matrix computed from the export volumes of 2018. The idea is that the better our recommendation the more in 2018 countries will export the products we have recommended.

The evaluation metrics we consider in order to evaluate the recommendations are:

- AUC-PR: The area under the precision recall curve [29];
- MAP: the mean of the average precision computed separately for each country [30];
- $P@1000$: the precision [31] of the highest 1000 recommendations;
- $mP@10$: the mean of the precision of the highest 10 recommendations computed separately for each country.

3.2.2. Recommendations on other datasets

To compare the similarity metrics in the case of the MovieLens, Amazon and GPS datasets, we use a leave-one-out approach, removing from the $M$ matrix one link for each user and then using it for the test. So what we do is to look at how good the model is to guess the link we have removed. We repeat this operation three times changing the removed link for each user and then we make an average of the performance. The evaluation metrics we consider in this case are:

- AUC-PR;
- MAP;
- HR@5: the hit ratio that is the fraction of users for which the correct answer is included in the top 5 recommendations.

Hit Ratio is a typical metric in the leave-one-out framework, while metrics such as $mP@10$ a $p@1000$ lose significance and so we do not consider them.

3.3. Other similarity metrics

We compare the Sapling Similarity with some of the most used metrics:

**Common Neighbors** [32]:

$$B_{ij}^{CN} = CO_{ij}$$ (14)

**Jaccard** [33]:

$$B_{ij}^{JA} = \frac{CO_{ij}}{k_i + k_j - CO_{ij}}$$ (15)

**Adamic/Adar** [34]:

$$B_{ij}^{AD} = \sum_l \frac{M_{il}M_{jl}}{\log(k_l)}$$ (16)

**Preferential Attachment** [35]:

$$B_{ij}^{PA} = k_ik_j$$ (17)

**Resource Allocation Index** [36]:

$$B_{ij}^{RA} = \sum_i \frac{M_{il}M_{jl}}{k_l}$$ (18)

**Cosine Similarity** [37]:

$$B_{ij}^{CS} = \frac{CO_{ij}}{\sqrt{k_ik_j}}$$ (19)
Sorensen index [38]:

\[ B_{ij}^{SO} = \frac{1}{k_i + k_j} CO_{ij} \]  

(20)

Hub depressed index [38, 17]:

\[ B_{ij}^{HDI} = \frac{1}{\max(k_i, k_j)} CO_{ij} \]  

(21)

Hub promoted index [39]:

\[ B_{ij}^{HPI} = \frac{1}{\min(k_i, k_j)} CO_{ij} \]  

(22)

Taxonomy Network [23]:

\[ B_{ij}^{TN} = \frac{1}{\max(k_i, k_j)} \sum_{l} M_{il}M_{jl} k_l \]  

(23)

Probabilistic Spreading [40]:

\[ B_{ij}^{Prob} = \frac{1}{k_j} \sum_{l} M_{il} M_{jl} k_l \]  

(24)

4. Experimental results

In this section we show the results of our recommendation exercise by comparing the performances of the different similarity metrics. We start from the case of the exports recommendation to countries and then we discuss the other datasets. We show both user-based and item-based collaborative filtering.

4.1. Recommendation of new exports to countries

In table 2, we show the results of the recommendation when a user-based and an item-based collaborative filtering are used to recommend new exports to countries. Let us start from the user-based case.

|               | User-Based | Item-Based | AUCPR | MAP  | P@1000 | mP@10 |
|---------------|------------|------------|-------|------|--------|-------|
| Common neighbors | 0.066 0.077 | 0.182 0.163 | 0.072 0.083 | 0.139 0.143 |
| Jaccard       | 0.079 0.086 | 0.187 0.171 | 0.074 0.096 | 0.156 0.188 |
| Adamic/adar   | 0.066 0.077 | 0.179 0.160 | 0.073 0.085 | 0.138 0.149 |
| Preferential attachment | 0.046 0.061 | 0.100 0.118 | 0.061 0.033 | 0.048 0.037 |
| Resource allocation | 0.066 0.077 | 0.175 0.16 | 0.074 0.089 | 0.142 0.163 |
| Cosine similarity | 0.071 0.082 | 0.173 0.164 | 0.072 0.084 | 0.149 0.140 |
| Sorensen      | 0.076 0.084 | 0.174 0.166 | 0.073 0.091 | 0.154 0.173 |
| Hub depressed index | 0.080 0.086 | 0.179 0.169 | 0.075 0.097 | 0.151 0.199 |
| Hub promoted index | 0.053 0.078 | 0.143 0.157 | 0.067 0.058 | 0.145 0.098 |
| Taxonomy network | 0.082 0.086 | 0.176 0.174 | 0.077 0.104 | 0.153 0.211 |
| Probabilistic spreading | 0.076 0.085 | 0.166 0.164 | 0.072 0.091 | 0.165 0.175 |
| Sapling Similarity | **0.090** 0.103 | **0.251** 0.225 | **0.089** 0.108 | **0.199** 0.249 |

Table 2: Performance of the recommendation of user-based (on the left) and item-based (on the right) collaborative filtering that recommends new exports to countries. In both cases the Sapling Similarity outperforms the other similarity metrics.

The worst model is Preferential Attachment. The idea behind this metric is that the more a user is popular (higher degree) the more he will be prone to create new links. However, in this case we want to compute the similarities between countries: in this case one finds that the most similar country is always Germany because it has the highest degree. For instance, this implies a suggestion to Angola to export a German technological product; this is unlikely to happen in the future.

Excluding Preferential Attachment, the worst model is Hub Promoted Index. Using this metric, the similarity between
two countries is higher if one of them has a low degree. So the number of co-occurrences between, for instance, Germany and Angola will be divided by the degree of Angola. This is not a good argument since the reason why Angola and Germany share some co-occurrences, even if they are trivially different, is that Germany exports a lot of products, so the degree of the developed country is more informative in this case. This is the reason why Hub Depressed Index performs better than Hub Promoted Index.

Resource allocation, Common Neighbors and Adamic/Adar do not take into account the degree of the countries. The result is that countries with higher degree have many co-occurrences also with countries with lower degree, so we find that less developed countries are similar to the developed ones and, as already said, this leads to unlikely recommendations.

Jaccard, Cosine Similarity, Sorensen, Probabilistic Spreading, Hub Depressed Index, and Taxonomy Network are the best performing among the classical metrics. However, all of them perform worse than our Sapling Similarity and the reason is their inability to consider negative values.

Regarding the item-based collaborative filtering, also in this case the Sapling Similarity outperforms the other metrics. This means that also when we compute the similarity between products, considering negative values is informative. To give an example of a case where two products have a negative similarity, if we look to the most different products (in the Sapling Similarity sense) with respect to Electrical capacitors; fixed, aluminium electrolytic we find Hides and skins; raw, whole, of bovine animals, Hides and skins; raw, of goats or kids, Skins; raw, of sheep or lambs. It is natural to think that a country that is specialised into electrical components like capacitors, probably is not focused on the export of hides and skins of animals.

4.2. Zambia, Saudi Arabia, and Japan: a case study

In order to provide an example of the user-based collaborative filtering, we show in figure 3 the Decision Saplings used to compute the similarity between Zambia and Saudi Arabia on the top right and Zambia and Japan on the top left. If we pick a random product and we ask if Zambia exports it, the probability is 6.4%. If we ask if Zambia exports a random product chosen from those exported by Japan, the probability drops to 3.7%. Knowing that Japan exports something reduces our confidence on the fact that also Zambia can export it since the two countries are anti-correlated.

To express this concept we need to define a negative similarity between Zambia and Japan so that in formula 12 Japan gives a negative contribution to the recommendation of a product to Zambia. The economic reason is that Japan is specialized into technological complex products, while Zambia is more focused on raw materials and does not have the capability to export the same products of Japan. If we apply the same argument to the case of Saudi Arabia and Zambia, whose Decision Sapling is represented in the top right, knowing that the former exports something does not add information about a possible export of the latter. In this sense, Zambia and Saudi Arabia are uncorrelated and their similarity should be close to zero, so that in formula 12 Saudi Arabia does not give any information on a possible export of Zambia. For the sake of completeness, the most similar countries to Zambia result to be African countries like, Tanzania, Zimbabwe, and Uganda.

With an approach based only on co-occurrences the difference between Japan, Saudi Arabia, and Zambia is practically negligible. In the bottom part of figure 3 we show how the similarity between Japan and Zambia is either higher or comparable to the one between Saudi Arabia and Zambia when we use classic similarity metrics. The only metric that is able to understand the real difference between Zambia and Japan is the Sapling Similarity because of the allowance of negative values.

4.3. Sapling Similarity Network of countries

In addition to the good performance of the Sapling Similarity when used to recommend new products to countries in a collaborative filtering, here we want to provide a visible proof of its good functioning using it to extract a network of countries, or in other words to project the country-product bipartite network into the layer of the countries. In the Sapling Similarity Network each node is a country; for each country we create a link if the linked country is among the 4 highest values in terms of Sapling Similarity. We show the result in figure 4. One can easily identify geographical clusters corresponding to the Europe and Africa regions; the clear distinction between Asiatic countries focused on mineral fuels, like Russia and Arabia, and the Asian tigers, like Singapore, South Korea, Malaysia, Thailand etc. is
far from trivial. It is also interesting to notice how Venezuela is separated from the other countries of South America and it is close to Asiatic countries related to mineral fuels.

4.4. Identification of significant links in similarity networks

All similarity metrics can be seen as a projection from the starting bipartite network into a monopartite one. The resulting network of users (or items) is very dense: the majority of the elements in the adjacency matrix are different from zero; however, many of the links are not statistically significant and must be considered noisy [41]. Indeed, in the bipartite country-product network most products share at least one co-occurrence with at least another product, and the same happens for countries. For all the similarity metrics defined in section 3.3, one co-occurrence is sufficient to have a non-zero (and positive) similarity. The purpose of this section is to show how the recommendations provided by a similarity metric improves when we filter it identifying the significant links, as done for instance by Pugliese et al. [24], who used the Configuration Model as a null model [42]. Our approach to filter the network provided by a similarity metric starts from building a collaborative filtering using export data from 1996 to 2008 that we will use to recommend new exports to countries in 2013. We do not use export data after 2013 in this step since these years will...
Figure 4: The Sapling Similarity Network (country layer). For each country only the top 4 links, in terms of the Sapling Similarity, are shown. The resulting structure reflects both geographical and industrial affinity among countries.

be used to evaluate the performance of the collaborative filtering after the filter in an out-of-sample exercise.

Considering the item-based case, once chosen a similarity metric, we select for each product the k products with the highest absolute value of it, then we put all the other elements in the B matrix equal to zero. So when computing the density of a country c around a product p through equations [13] we are involving in the computation only the products that are really similar or, in the case of the Sapling Similarity, dissimilar to p, neglecting the others. Varying the number k of products we look at how good are the recommendations provided for the exports in 2013. In figure 5 on the left we see how the value of k affect the AUC-PR of the item-based recommendations with the Sapling Similarity. As the reader can see there is a maximum at k = 45. On the right of figure 5 there is the user-based case where the maximum corresponds to k = 25. These peaks represent a proof of the importance and the goodness of the filter.

Once identified the optimal value of k for all the similarity metrics (each of them had a separate optimization), we build the collaborative filtering in the way described in section 3.2.1 using 2018 as test year, and in table 3 you can see the results.

Focusing on the user-based case, with respect to the scores in table 2 all the similarity metrics improve except for the Sapling Similarity that maintains more or less the same performance. This suggests that in the user based case, differently from the other metrics, Sapling Similarity does not need a filter. Looking to AUC-PR and MAP now Sapling Similarity is almost equivalent to Probabilistic Spreading, Taxonomy Network, Hub Depressed Index, Cosine Similarity and Jaccard, but if we look to precision@1000 and mean precision@10 Sapling Similarity remains the best
choice. In the item-based case also Sapling Similarity improves a lot and, even if it is still the best choice according to MAP, precision@1000 and mean precision@10, the gap with the other similarities is very small. So we can conclude that, without the application of filters, Sapling Similarity is the best metric at managing with a dense B matrix. If the dimension of the B matrix is relatively small, like in the user-based case (169 × 169), there is no need to apply a filter, however, if the dimension is not small, like in the item-based case (5040 × 5040), even Sapling Similarity benefits from a filter.

4.5. Other recommendations

Here we compare the recommendations provided by the models using other datasets. In table 4, you can see the results in the case of both an item-based and a user-based collaborative filtering. Independently from the type of data the Sapling Similarity outperforms the other metrics. This is a strong evidence of the better functioning of Sapling Similarity since, independently from the dataset, its superiority is evident. The different nature of our data makes this last sentence stronger: behind the reason of the links in the bipartite network of countries and exports there is the complex productive structure of a country and this does not have anything in common with the reason why inhabitants in a city visit a particular POI with respect to another or the reason why users give high rates to products or movies.

5. Conclusions and future works

This paper introduces a metric to compute similarity between either users or items in a bipartite network, the Sapling Similarity. This metric is built using concepts from information theory, with a probabilistic approach. The main novelty introduced by the Sapling Similarity is the permission of negative values, that are not conceived in the case of classical similarity metrics based on co-occurrences. Moreover, our probabilistic approach naturally requires (and introduces in the mathematical formulation) the total size N of the layer (users or items) under investigation. The idea is to look at how the probability that a user is connected to an item (or vice versa) changes if we know that this item is connected to another user. If the probability decreases, then the two users are anti-correlated and we assign a negative similarity; if the probability does not change the similarity of the two users is zero; finally if the probability increases they are positively correlated and their similarity will be positive. This criterion is expressed in mathematical terms by using the variation of Gini Impurity, following the ideas at the base of Decision Trees and Random Forest.

Similarity metrics are widely used to build a recommender system to suggest new items to users, in particular in the field of collaborative filtering. Here we show that a collaborative filtering based on the Sapling Similarity outperforms the ones based on other similarity metrics in terms of link prediction. Item-based collaborative filtering are widely used in Economic Complexity to measure the relatedness between countries and products [12], however recent studies [13, 3] show how tree-based machine learning algorithms like Random...
| Common neighbors | User-Based | Item-Based |
|------------------|-----------|-----------|
|                  | AUCPR | MAP | P@1000 | mP@10 | AUCPR | MAP | P@1000 | mP@10 |
|                  | 0.070 | 0.078 | 0.191 | 0.165 | 0.092 | 0.093 | 0.206 | 0.200 |
|                  | (+6.1%) | (+1.3%) | (+4.9%) | (+1.2%) | (+27.8%) | (+12.0%) | (+48.2%) | (+29.9%) |
| Jaccard          | 0.094 | 0.096 | 0.226 | 0.203 | 0.118 | 0.111 | 0.274 | 0.253 |
|                  | (+19.0%) | (+11.6%) | (+20.9%) | (+18.7%) | (+59.5%) | (+15.6%) | (+75.6%) | (+34.6%) |
| Adamic/adar      | 0.068 | 0.077 | 0.185 | 0.155 | 0.092 | 0.093 | 0.201 | 0.191 |
|                  | (+3.0%) | (0.0%) | (+3.4%) | (-3.1%) | (+26.0%) | (+9.4%) | (+45.7%) | (+28.2%) |
| Preferential attachment | 0.046 | 0.061 | 0.100 | 0.118 | 0.061 | 0.033 | 0.048 | 0.037 |
|                  | (0.0%) | (0.0%) | (0.0%) | (0.0%) | (0.0%) | (0.0%) | (0.0%) | (0.0%) |
| Resource allocation | 0.068 | 0.073 | 0.197 | 0.147 | 0.085 | 0.096 | 0.188 | 0.157 |
|                  | (+3.0%) | (-5.2%) | (+12.6%) | (-8.1%) | (+14.9%) | (+7.9%) | (+32.3%) | (-3.7%) |
| Cosine similarity | 0.090 | 0.093 | 0.207 | 0.205 | 0.115 | 0.108 | 0.279 | 0.248 |
|                  | (+26.8%) | (+13.4%) | (+19.7%) | (+25.0%) | (+59.7%) | (+28.6%) | (+87.2%) | (+77.1%) |
| Sorensen         | 0.093 | 0.096 | 0.214 | 0.206 | 0.117 | 0.110 | 0.267 | 0.251 |
|                  | (+22.4%) | (+14.3%) | (+23.0%) | (+24.1%) | (+60.3%) | (+20.9%) | (+73.4%) | (+45.1%) |
| Hub depressed index | 0.095 | 0.095 | 0.213 | 0.208 | 0.115 | 0.110 | 0.273 | 0.245 |
|                  | (+18.8%) | (+10.5%) | (+19.0%) | (+23.1%) | (+53.3%) | (+13.4%) | (+80.8%) | (+23.1%) |
| Hub promoted index | 0.060 | 0.083 | 0.148 | 0.189 | 0.076 | 0.069 | 0.197 | 0.157 |
|                  | (+13.2%) | (+6.4%) | (+3.5%) | (+20.4%) | (+13.4%) | (+19.0%) | (+35.9%) | (+60.2%) |
| Taxonomy network | 0.097 | 0.097 | 0.245 | 0.207 | 0.114 | 0.109 | 0.287 | 0.232 |
|                  | (+18.3%) | (+12.8%) | (+39.2%) | (+19.0%) | (+48.1%) | (+4.8%) | (+87.6%) | (+10.0%) |
| Probabilistic spreading | 0.093 | 0.100 | 0.217 | 0.200 | 0.099 | 0.106 | 0.270 | 0.234 |
|                  | (+22.4%) | (+17.6%) | (+30.7%) | (+22.0%) | (+37.5%) | (+16.5%) | (+63.6%) | (+33.7%) |
| Sapling Similarity | 0.091 | 0.101 | 0.280 | 0.224 | 0.116 | 0.112 | 0.288 | 0.255 |
|                  | (+1.1%) | (-1.9%) | (+11.6%) | (-0.4%) | (+30.3%) | (+3.7%) | (+44.7%) | (+2.4%) |

Table 3: Goodness of the recommendations of new products to countries. Before computing the density of a country around a product (or of a product around a country) some elements in the B matrix are set equal to zero through a filter in order to remove the noise given by unrelated products or countries. The percentages in the round brackets refers to the relative improvement with respect to the case where no elements in the B matrix are set equal to zero.

Forest [44] and XGBoost [45, 46] provide better results. The disadvantages of using machine learning with respect to an item-based collaborative filtering are the loss of interpretability and the high computational time required to train an algorithm (a training sample is a country with 5040 binary features, one for each product). A possible development in this direction is to use the Sapling Similarity to perform a feature selection in order to reduce the features of the training samples. This would reduce the computational time required to train the algorithms and would also increase the interpretability of the models. Note that other similarity metrics can not be used for this task, since for a machine learning algorithm the negative relations between products are key [3].

6. Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

7. Acknowledgments

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| model                          | AUC-PR | MAP   | HR@5  | AUC-PR | MAP   | HR@5  |
|-------------------------------|--------|-------|-------|--------|-------|-------|
| **Movielens**                 |        |       |       |        |       |       |
| Common neighbors              | 0.0061 | 0.0625| 0.0795| 0.0005 | 0.0297| 0.0304|
| Jaccard                       | 0.0068 | 0.0674| 0.0829| 0.0009 | 0.0981| 0.1297|
| Adamic/adar                   | 0.0062 | 0.0635| 0.0802| 0.0005 | 0.0338| 0.0388|
| Preferential attachment       | 0.0035 | 0.0462| 0.0595| 0.0004 | 0.0004| 0.0045|
| Resource allocation           | 0.0069 | 0.0675| 0.0852| 0.0006 | 0.0342| 0.0405|
| Cosine similarity             | 0.0067 | 0.0655| 0.0803| 0.0006 | 0.0388| 0.0487|
| Jaccard                       | 0.0066 | 0.0665| 0.0812| 0.0008 | 0.0965| 0.1292|
| Hub depressed index           | 0.0059 | 0.0675| 0.0829| 0.0010 | 0.0924| 0.1239|
| Hub promoted index            | 0.0053 | 0.0626| 0.0788| 0.0003 | 0.0032| 0.0025|
| Taxonomy network              | 0.0068 | 0.0730| 0.0911| 0.0012 | 0.1002| 0.1351|
| Probabilistic spreading       | 0.0083 | 0.0723| 0.0906| 0.0007 | 0.0488| 0.0615|
| Sapling Similarity            | **0.0110** | **0.0828** | **0.1047** | **0.0014** | **0.1033** | **0.1450** |
| **Amazon**                    |        |       |       |        |       |       |
| Common neighbors              | 0.0036 | 0.0376| 0.0457| 0.0006 | 0.0300| 0.0383|
| Jaccard                       | 0.0050 | 0.0455| 0.0594| 0.0008 | 0.0355| 0.0701|
| Adamic/adar                   | 0.0038 | 0.0387| 0.0475| 0.0007 | 0.0355| 0.0467|
| Preferential attachment       | 0.0010 | 0.0155| 0.0162| 0.0004 | 0.0004| 0.0154|
| Resource allocation           | 0.0045 | 0.0393| 0.0520| 0.0008 | 0.0405| 0.0539|
| Cosine similarity             | 0.0047 | 0.0439| 0.0559| 0.0007 | 0.0342| 0.0467|
| Jaccard                       | 0.0049 | 0.0451| 0.0584| 0.0007 | 0.0476| 0.0631|
| Hub depressed index           | 0.0045 | 0.0453| 0.0606| 0.0008 | 0.0553| 0.0717|
| Hub promoted index            | 0.0035 | 0.0386| 0.0459| 0.0005 | 0.0208| 0.0281|
| Taxonomy network              | 0.0057 | 0.0510| 0.0670| 0.0010 | 0.0617| 0.0805|
| Probabilistic spreading       | 0.0069 | 0.0521| 0.0707| 0.0009 | 0.0429| 0.0561|
| Sapling Similarity            | **0.0075** | **0.0586** | **0.0768** | **0.0020** | **0.0701** | **0.0936** |
| **GPS data**                  |        |       |       |        |       |       |
| Common neighbors              | 0.0154 | 0.105 | 0.140 | 0.0076 | 0.091 | 0.127 |
| Jaccard                       | 0.0164 | 0.109 | 0.146 | 0.0133 | 0.134 | 0.189 |
| Adamic/adar                   | 0.0165 | 0.109 | 0.146 | 0.0077 | 0.092 | 0.129 |
| Preferential attachment       | 0.0001 | 0.014 | 0.017 | 0.0002 | 0.0002 | 0.0005 |
| Resource allocation           | 0.0202 | 0.126 | 0.126 | 0.0079 | 0.094 | 0.132 |
| Cosine similarity             | 0.0156 | 0.106 | 0.141 | 0.0100 | 0.100 | 0.140 |
| Sorensen                      | 0.0157 | 0.128 | 0.174 | 0.0119 | 0.122 | 0.172 |
| Hub depressed index           | 0.0158 | 0.107 | 0.142 | 0.0118 | 0.138 | 0.193 |
| Hub promoted index            | 0.0154 | 0.106 | 0.141 | 0.0057 | 0.071 | 0.096 |
| Taxonomy network              | 0.0208 | 0.128 | 0.174 | 0.0142 | 0.139 | 0.196 |
| Probabilistic spreading       | 0.0209 | 0.129 | 0.175 | 0.0113 | 0.104 | 0.147 |
| Sapling Similarity            | **0.0230** | **0.135** | **0.179** | **0.0228** | **0.159** | **0.225** |

Table 4: Performance of the recommendation given by user-based (on the left) and item-based (on the right) collaborative filtering that recommends items to users with three different datasets. The Sapling Similarity always outperforms the other similarity metrics.

References

[1] Hidalgo CA, Hausmann R. The building blocks of economic complexity. Proceedings of the national academy of sciences. 2009;106(26):10570–10575.
[2] Tacchella A, Cristelli M, Caldarelli G, Gabrielli A, Pietronero L. A new metrics for countries’ fitness and products’ complexity. Scientific reports. 2012;2(1):1–7.
[3] Albora G, Zaccaria A. Machine learning to assess relatedness: the advantage of using firm-level data. arXiv preprint arXiv:220200458. 2022.
[4] Straccamore M, Pietronero L, Zaccaria A. Which will be your firm’s next technology? Comparison between machine learning and network-based algorithms. arXiv preprint arXiv:211002004. 2021.
[5] Goh KI, Cusick ME, Valle D, Childs B, Vidal M, Barabási AL. The human disease network. Proceedings of the National Academy of Sciences. 2007;104(21):8685–8690.
[46] Chen T, Guestrin C. Xgboost: A scalable tree boosting system. In: Proceedings of the 22nd acm sigkdd international conference on knowledge discovery and data mining; 2016. p. 785–794.