Almost exponential maps and integrability results for a class of horizontally regular vector fields

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Abstract

We consider a family $\mathcal{H} := \{X_1, \ldots, X_m\}$ of $C^1$ vector fields in $\mathbb{R}^n$ and we discuss the associated $\mathcal{H}$-orbits. Namely, we assume that our vector fields belong to a horizontal regularity class and we require that a suitable $s$-involutivity assumption holds. Then we show that any $\mathcal{H}$-orbit $\mathcal{O}$ is a $C^1$ immersed submanifolds and it is an integral submanifold of the distribution generated by the family of all commutators up to length $s$. Our main tool is a class of almost exponential maps of which we discuss carefully some precise first order expansions.

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1. Introduction and main results

In this paper we discuss the integrability of distributions defined by families of vector fields under a higher order horizontal regularity hypothesis and assuming an involutivity condition of order $s \in \mathbb{N}$. The central tool we exploit is given by a class of almost exponential maps which we will analyze in details assuming only low regularity on the coefficients of the vector fields.

To start the discussion, fix a family $\mathcal{H} = \{X_1, \ldots, X_m\}$ of at least Lipschitz-continuous vector fields. For any $x \in \mathbb{R}^n$ define the Sussmann’s orbit, or leaf

$$\mathcal{O}_{\mathcal{H}}^x := \left\{ e^{t_1 X_{j_1}} \cdots e^{t_p X_{j_p}} x : p \in \mathbb{N}, J := (j_1, \ldots, j_p) \in \{1, \ldots, m\}^p, t \in \Omega_{J,x} \right\}, \quad (1.1)$$

where for fixed $x \in \mathbb{R}^n$ we denote by $\Omega_{J,x} \subset \mathbb{R}^p$ the open neighborhood of the origin where the map $t \mapsto e^{t_1 X_{j_1}} \cdots e^{t_p X_{j_p}} x$ is well defined. We equip the leaf $\mathcal{O}_{\mathcal{H}}^x$ with the topology $\tau_d$ defined by the Franchi–Lanconelli distance $d$; see (2.1).

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Our purpose is to describe a regularity class of order $s \geq 2$ and a $s$-involutivity assumption that ensure that each orbit $O_{H}$ is an integral manifold of the distribution generated by the family $\mathcal{P} := \mathcal{P}_{s} := \{Y_{1}, \ldots, Y_{q}\}$ of all nested commutators of length at most $s$ constructed from the original family $\mathcal{H}$. To give coordinates on $O$ we shall use the following almost exponential maps. Fix $s \geq 2$ and denote by $\mathcal{P}$ the aforementioned family of commutators. Assign to each $Y_{j}$ the length $\ell_{j} \leq s$, just its order. Then, let

$$E_{I,x}(h) := \exp_{ap}(h_{1}Y_{i_{1}}) \cdots \exp_{ap}(h_{p}Y_{i_{p}}) x,$$

where $I = (i_{1}, \ldots, i_{p})$ is a multiindex which fixes $p$ commutators $Y_{i_{1}}, \ldots, Y_{i_{p}} \in \mathcal{P}$, $h \in \mathbb{R}^{p}$ belongs to a neighborhood of the origin and $p \in \{1, \ldots, n\}$ is suitable. See (2.15) for the definition of the approximate exponential $\exp_{ap}$. We shall use the maps in (1.2) to construct charts, developing a higher order, nonsmooth, quantitative extension of some ideas appearing in a paper by Lobry; see [Lob70]; see Theorem 3.5 and Remarks 3.6 and 3.7 below.

Here is the statement of our result. Let $\mathcal{H} = \{X_{1}, \ldots, X_{n}\}$ and let $s \geq 2$. Assume that $X_{j} := f_{j} \cdot \nabla \in C^{1}_{\text{Euc}}$ for all $j$ (here and hereafter $C^{1}_{\text{Euc}}$ refers to Euclidean regularity). Assume also that for each $p \leq s$ and $j_{1}, \ldots, j_{p} \in \{1, \ldots, m\}$, all derivatives $X_{j_{1}}^{s} \cdots X_{j_{p-1}}^{s}f_{j_{p}}$ exist and are locally Lipschitz-continuous functions with respect to distance $d$ associated to the vector fields. Here, following [MM12a], we denote by $X^{\ell}f$ the Lie derivative along the vector field $X$ of the scalar function $f$. Moreover we require that for any commutator $Y_{j} := g_{j} \cdot \nabla \in \mathcal{P}$, all maps of the form $g_{j} \circ E_{I,x}$ are continuous for all $p \in \{1, \ldots, n\}$, $I = (i_{1}, \ldots, i_{p})$ and $x \in \mathbb{R}^{n}$.[2]

Furthermore, we require the following $s$-involutivity condition. For any $X_{j} \in \mathcal{H}$ and for any $Y_{k} \in \mathcal{P}$ with maximal length $\ell_{k} = s$, at any $x \in \Omega$ where the derivative $X_{j}^{s}g_{k}(x)$ exists one can write for suitable $b' = b'(x)$

$$(\text{ad}_{X_{j}}Y_{k})_{x} := (X_{j}^{s}g_{k}(x) - Y_{k}f_{j}(x)) \cdot \nabla = \sum_{i=1}^{q} b^{i}Y_{i,x} \quad \text{with } b^{i} \text{ locally bounded.}$$  (1.3)

The class of vector fields satisfying all those assumptions will be denoted by $\mathcal{A}_{s}$; see Definition [2.5] where a more precise formulation of this assumption is described. Note that in the smooth case we have $\text{ad}_{X_{j}}Y_{k} = [X_{j}, Y_{k}]$ and ultimately (1.3) is equivalent to the Hermann condition [Her62]

$$[Y_{i}, Y_{j}] = \sum_{1 \leq k \leq q} c_{ij}^{k}Y_{k}, \quad \text{with } c_{ij}^{k} \in L^{\infty}_{\text{loc}},$$  (1.4)

which ensures that any Sussmann’s orbit $O_{\mathcal{P}}$ of the family of commutators $\mathcal{P}$ is an integral manifold of the distribution generated by $\mathcal{P}$. If furthermore $s = 1$, then $\mathcal{P} = \mathcal{H}$ and (1.3) and (1.4) are the same. Note that the appearance of operators of the form $\text{ad}_{X_{j}}Y_{k}$ is very natural in the framework of our almost exponential maps; see the non-commutative calculus formulas discussed in [MM12a] Section 3].

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1This condition is widely ensured for instance as soon as we assume that $g_{j}$ is continuous in the Euclidean topology, or at least in the Sussmann’s orbit topology defined on $O$ by the family $\mathcal{H}$; see [Sus73].
Theorem 1.1. Let $\mathcal{H} = \{X_1, \ldots, X_m\}$ be a family of vector fields of class $A_s$. Then, for any $x_0 \in \mathbb{R}^n$, the orbit $O := \mathcal{O}_H^{x_0}$ with the topology $\tau_d$ is a $C^1$ immersed submanifold of $\mathbb{R}^n$ with tangent space $T_yO = P_y$ for all $y \in O$.

Note that this result does not follow from standard ones, because the commutators $Y_j$ are not assumed to be $C^1$ in the Euclidean sense. In Example 3.14 we exhibit a family of vector fields where our theorem apply, but classical results do not. See also Remark 3.15 for some further comments. Furthermore, let us mention that if $s = 1$, i.e. $\mathcal{H} = \mathcal{P}$, then Theorem 1.1 is a consequence of the Frobenius Theorem for singular $C^1$ distributions (it is well known to experts that in such case one can prove that orbits are even $C^2$ smooth).

Note that if $s = 1$, in [MM11a] we proved a singular Frobenius-type theorem assuming only Lipschitz-continuity of the involved vector fields, generalizing part of Rampazzo’s results [Ram07] to singular distributions; in fact, in [MM11a], orbits are $C^{1,1}$.

On a technical level, the main tool we discuss is the approximate exponential $E_{t,x}$ in (1.2). Introduce the notation $p_x := \dim P_x := \dim \text{span}\{Y_1(x), \ldots, Y_q(x)\}$ for all $x \in \mathbb{R}^n$. Fix $x$, take $p := p_x$ commutators $Y_1, \ldots, Y_p$, which are linearly independent at $x$ and construct the map $E$, defined in (1.2). Then, under the hypotheses of Theorem 1.1 we shall show that if the family $\mathcal{H}$ satisfies condition $A_s$, then $E$ is a $C^1_{\text{Euc}}$, full rank map in a neighborhood of the origin $0 \in \mathbb{R}^p$, whose derivative enjoys the following remarkable expansion

$$E_s(\partial_{h_k}) = Y_{i_k}(E(h)) + \sum_{l_j = l_{i_k} + 1} a^j_k(h)Y_j(E(h)) + \sum_{i=1}^q \omega^i_k(x, h)Y_i(E(h)).$$

(1.5)

The functions $a^j_k$ and $\omega^i_k$ have a very precise rate of convergence to 0, as $h \to 0$ which will be specified in (3.22) and (3.23). Note that an expansion of $E_s(\partial_{h_k})$ can be obtained either with the Campbell–Hausdorff formula in the smooth case (see [Mor00] or [VSCC92]), or in nonsmooth situations with the techniques of [MM12b]. However, the expansions in the mentioned papers contain some remainders appearing either as formal series, or in integral form. Here we are able to express such reminders via the pointwise terms $\omega^i_k$, improving all previous results. Note also that we are improving the mentioned papers both from a regularity standpoint and because here we do not assume the Hörmander condition. At the authors’ knowledge, expansion (1.5) with precise estimates on $a^j_k$ and $\omega^i_k$ is new even in the smooth case. As a final remark, observe that Theorem 3.11 contains an explicit detailed proof of the fact that the map $E$ is $C^1$ smooth, avoiding any use of the Campbell–Hausdorff formula. Note that, even if the vector fields are smooth, such maps are not much more than $C^1$; see Remark 3.12 (ii).

The useful information one can extract from (1.5) is that $E_s(\partial_{h_k}) \in P_{E(h)}$ (note that we are interested to situations where the inclusion $P_{E(h)} \subset \mathbb{R}^n$ is strict); see Theorem 3.11 for a precise statement. Observe that, if $O \subset \mathbb{R}^p$ is a small open set containing the origin, then $E(O)$ is a $C^1$ submanifold of $\mathbb{R}^n$ and (1.5) shows that $T_{E(h)}E(O) \subset P_{E(h)}$ for all $h$. This is the starting point to prove that $\mathcal{O}_H^y$ is a integral manifold of the distribution generated by $P$. Another fact we need to prove is that the dimension of $P_y := \text{span}\{Y_j(y) : 1 \leq j \leq q\}$ is constant if $y$ belongs to a fixed orbit $\mathcal{O}_H^y$. This is obtained by means of a nonsmooth quantitative curvilinear version of the original Hermann’s argument inspired to the work of Nagel, Stein and Wainger [NSWS55] and Street [Str11].
To conclude this introduction, we give some references and motivations to study our almost exponential maps $E$. Such maps appear in [NSW85], and were used by the authors to show equivalence between different control distances; see also [VSCC92]. More recently they have revealed to be a useful tool in the companion paper [MM11b], where we shall prove Carathéodory spaces (see [MM02, FF03, Vit12]). Finally, note that the precise expansion (1.5) will be a fundamental tool in the companion paper [MM11b], where we shall prove a Poincaré inequality on orbits for a family of vector fields satisfying an integrability condition.

2. Preliminaries

Vector fields and the control distance. Consider a family of vector fields $\mathcal{H} = \{X_1, \ldots, X_m\}$ and assume that $X_j \in C^1_{\text{Euc}}(\mathbb{R}^n)$ for all $j$. Here and later $C^1_{\text{Euc}}$ means $C^1$ in the Euclidean sense. Write $X_j = f_j \cdot \nabla$, where $f_j : \mathbb{R}^n \to \mathbb{R}^n$. The vector field $X_j$, evaluated at a point $x \in \mathbb{R}^n$, will be denoted by $X_j(x)$. All the vector fields in this paper are always defined on the whole space $\mathbb{R}^n$.

Define the Franchi–Lanconelli distance \cite{FL83}

$$d(x,y) := \inf \left\{ r > 0 : y = e^{t_1Z_1} \cdots e^{t_nZ_n}x \text{ for some } \mu \in \mathbb{N} \right\}.$$  \hfill (2.1)

Here and hereafter we let $r\mathcal{H} := \{rX_1, \ldots, rX_m\}$ and $\pm \mathcal{H} := \{\pm X_1, \ldots, \pm X_m\}$. The topology associated with $d$ will be denoted with $\tau_d$. We denote instead by $d_{cc}$ the standard Carnot–Carathéodory control distance (see Fefferman–Phong \cite{FP83} and Nagel–Stein–Wainger [NSW85]). In the present paper we shall make a prevalent use of the distance $d$.

In view of the mentioned examples, we need to use the broad definition of submanifold; see \cite{Che46, KN96}. Below, if $\Sigma \subset \mathbb{R}^n$, we denote by $\tau_{\text{Euc}}|\Sigma$ the induced topology.

**Definition 2.1** (Immersed submanifold). Let $\Sigma \subset \mathbb{R}^n$ and let $\tau \supset \tau_{\text{Euc}}|\Sigma$ be a topology on $\Sigma$. We say that $\Sigma$ is a $C^k$ submanifold if $\Sigma$ is connected and for all $x \in \Sigma$ there is $\Omega \in \tau$, open neighborhood of $x$ such that $\Omega$ is a $C^k$ graph. If moreover $\tau = \tau_{\text{Euc}}|\Sigma$ then we say that $\Sigma$ is an embedded submanifold.

Horizontal regularity classes. Here we define our notion of horizontal regularity in terms of the distance $d$. Note that we do not use the control distance $d_{cc}$.

**Definition 2.2.** Let $\mathcal{H} := \{X_1, \ldots, X_m\}$ be a family of vector fields, $X_j \in C^1_{\text{Euc}}$. Let $d$ be their distance (2.1) Let $g : \mathbb{R}^n \to \mathbb{R}$. We say that $g$ is $d$-continuous, and we write $g \in C^0_\mathcal{H}(\mathbb{R}^n)$, if for all $x \in \mathbb{R}^n$, we have $g(y) \to g(x)$, as $d(y,x) \to 0$. We say that $g : \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{H}$-Lipschitz or $d$-Lipschitz in $A \subset \mathbb{R}^n$ if

$$\text{Lip}_\mathcal{H}(g; A) := \sup_{x,y \in A, x \neq y} \frac{|g(x) - g(y)|}{d(x,y)} < \infty.$$
We say that \( g \in C^1_{\text{H}}(\mathbb{R}^n) \) if the derivative \( X_j^g(x) := \lim_{t \to 0}(f(e^{tX}x) - f(x))/t \) is a d-continuous function for any \( j = 1, \ldots, m \). We say that \( g \in C^k_{\text{H}}(\mathbb{R}^n) \) if all the derivatives \( X_j^g, \ldots, X_j^k \) are d-continuous for \( p \leq k \) and \( j_1, \ldots, j_p \in \{1, \ldots, m\} \). If all the derivatives \( X_j^g, \ldots, X_j^k \) are d-Lipschitz on each \( \Omega \) bounded set in the Euclidean metric, then we say that \( g \in C^k_{\text{H}, \text{loc}}(\mathbb{R}^n) \). Finally, denote the usual Euclidean Lipschitz constant of \( g \) on \( A \subset \mathbb{R}^n \) by \( \text{Lip}_{\text{Euc}}(g; A) \).

We will usually deal with vector fields which are of class at least \( C^s_{\text{loc}} \cap C^{s-1,1}_{\text{H}, \text{loc}} \), where \( s \geq 1 \) is a suitable integer. In this case it turns out that commutators up to the order \( s \) can be defined; see Definition 2.3 In the companion paper [MM12a], we study several issues related with this definition.

**Definitions of commutator.** Our purpose now is to show that, given a family \( \mathcal{H} \) of vector fields with \( X_j \in C^{s-1,1}_{\text{H}, \text{loc}} \cap C^s_{\text{Euc}}, \) then commutators can be defined up to length \( s \).

For any \( \ell \in \mathbb{N} \), denote by \( \mathcal{W}_\ell := \{w_1 \cdots w_\ell : w_j \in \{1, \ldots, m\}\} \) the words of length \( |w| := \ell \) in the alphabet \( 1, 2, \ldots, m \). Let also \( \mathcal{S}_\ell \) be the group of permutations of \( \ell \) letters. Then for all \( \ell \geq 1 \), there are functions \( \pi_\ell : \mathcal{S}_\ell \to \{-1, 0, 1\} \) such that

\[
[A_{w_1}, [A_{w_2}, \ldots, [A_{w_{\ell-1}}, A_{w_\ell}]]] = \sum_{\sigma \in \mathcal{S}_\ell} \pi_\ell(\sigma)A_{\sigma_1(w)}A_{\sigma_2(w)} \cdots A_{\sigma_\ell(w)}, \tag{2.2}
\]

for all \( A_1, \ldots, A_m : V \to V \) linear operators on a vector space \( V \). See [MM12a] for a more formal definition and an in-depth discussion.

We are now ready to define commutators for vector fields in our regularity classes.

**Definition 2.3 (Definitions of commutator).** Given a family \( \mathcal{H} = \{X_1, \ldots, X_m\} \) of vector fields of class \( C^{s-1,1}_{\text{H}, \text{loc}} \cap C^s_{\text{Euc}}, \) define for any function \( \psi \in C^1_{\text{H}} \) the operator \( X_j^\psi(x) := \mathcal{L}_{X_j}\psi(x) \), the Lie derivative. Let also \( X_j\psi(x) := f_j(x) \cdot \nabla \psi(x) \) where \( \psi \in C^1_{\text{Euc}} \). Moreover, let

\[
f_w := \sum_{\sigma \in \mathcal{S}_\ell} \pi_\ell(\sigma)(X_{\sigma_1(w)} \cdots X_{\sigma_{\ell-1}(w)} f_{\sigma_\ell(w)}) \quad \text{for all } w \text{ with } |w| \leq s,
\]

\[
X_w \psi := [X_{w_1}, \ldots, [X_{w_{\ell-1}}, X_{w_\ell}]]\psi := f_w \cdot \nabla \psi \quad \text{for all } \psi \in C^1_{\text{Euc}} \text{ with } |w| \leq s,
\]

\[
X_w^2 \psi := \sum_{\sigma \in \mathcal{S}_\ell} \pi_\ell(\sigma)X_{\sigma_1(w)}^2 \cdots X_{\sigma_{\ell-1}(w)}^2 f_{\sigma_\ell(w)} \psi \quad \text{for all } \psi \in C^1_{\text{H}} \text{ with } |w| \leq s - 1.
\]

Finally, for any \( j \in \{1, \ldots, m\} \) and \( w \) with \( 1 \leq |w| \leq s \), let

\[
\text{ad}_{X_j} X_w \psi := (X_j^2 f_w - f_w \cdot \nabla f_j) \cdot \nabla \psi = (X_j^2 f_w - X_w f_j) \cdot \nabla \psi \quad \text{for all } \psi \in C^1_{\text{Euc}}. \tag{2.3}
\]

Non-nested commutators are precisely defined in [MM12a].

**Remark 2.4.** Let \( Z \in \pm \mathcal{H} \). If \( |w| \leq s - 1 \), then there are no problems in defining \( \text{ad}_Z X_w \). More precisely, in [MM12a] we show that \( \text{ad}_Z X_w = [Z, X_w] \). If instead \( |w| = s \), then the function \( t \mapsto f_w(e^{tZ}x) \) is Euclidean Lipschitz. In particular it is differentiable for a.e. \( t \). In other words, for any fixed \( x \in \mathbb{R}^n \), the limit \( \frac{d}{dt} f_w(e^{tZ}x) =: Z^t f_w(e^{tZ}x) \) exists for a.e. \( t \) close to 0. Therefore the pointwise derivative \( Z^t f_w(y) \) exists for almost all \( y \in \mathbb{R}^n \) and ultimately \( \text{ad}_Z X_w \) is defined almost everywhere.
• Both our definitions of commutator, $X_w$ and $X^+_w$ are well posed from an algebraic point of view, i.e. they satisfy antisymmetry and the Jacobi identity; see [MM12a].
• In [MM12a] we will also recognize that the first order operator $X_w$ agrees with $X^+_w$ against functions $\psi \in C^{s-1,1}_{\text{H},\text{loc}} \cap C^1_{\text{Euc}}$ as soon as $|w| \leq s - 1$.

The integrability class $A_s$.

**Definition 2.5** (Vector fields of class $A_s$). Let $H = \{X_1, \ldots, X_m\}$ be a family in the regularity class $C^1_{\text{Euc}} \cap C^{s-1,1}_{\text{H},\text{loc}}$. We say that the family $H$ belongs to the class $A_s$ if, fixed an open bounded set $\Omega \subset \mathbb{R}^n$, there is $C_0 > 1$ such that the following holds: for any $Z \in \pm H$, for any word $w$ with $|w| = s$, for each $x \in \Omega$ and for a.e. $t \in [-C_0^{-1}, C_0^{-1}]$, there are coefficients $b^v \in \mathbb{R}$ such that

$$\text{ad}_Z X_w(e^{tZ}x) = \sum_{1 \leq |u| \leq s} b^u X_u(e^{tZ}x) \quad \text{with}$$

$$|b^u| \leq C_0 \quad \text{for all } u \text{ with } 1 \leq |u| \leq s;$$

finally assume that if $1 \leq |w| \leq s$, for all $p \in \{1, \ldots, n\}$, for any $I \in \mathcal{I}(p, q)$, $x \in \mathbb{R}^n$, we have at any $h^*$ where $E_{I,x}$ is defined

$$f_w(E_{I,x}(h)) \rightarrow f_w(E_{I,x}(h^*)) \quad \text{as } h \rightarrow h^*.$$  

**Remark 2.6.**

• Assumption (2.6) will be used only once, in (3.25), but it is essential in order to ensure that the almost exponential maps we define later are actually $C^1_{\text{Euc}}$ smooth. It is easy to check that assumption (2.6) is satisfied as soon as $f_w : (\mathcal{O}_H, \tau_H) \rightarrow \mathbb{R}$ is continuous, where $\tau_H$ denotes the Sussmann’s orbit topology defined by the family $H$, see [Sus73]. Note that at this stage assumption (2.6) is not ensured by the $d$-Lipschitz continuity of $f_w$.
• Conditions (2.4) and (2.5) scale nicely. Namely, letting for all $r \leq 1$, $\tilde{Z} = rZ$, $\tilde{X}_w = r^{|w|} X_w$ with $|w| = s$, we have

$$\text{ad}_{\tilde{Z}} \tilde{X}_w(x) = \sum_{1 \leq |u| \leq s} \tilde{b}^u \tilde{X}_u(x) \quad \text{where } |\tilde{b}^u| \leq C_0 r \leq C_0 \quad \text{for all } u.$$  

• Let $H$ be a family of vector fields in the class $C^1_{\text{Euc}} \cap C^{s-1,1}_{\text{H},\text{loc}}$ satisfying the Hörmander bracket-generating condition of step $s$ and assume that each $f_w$ with $|w| \leq s$ is continuous in the Euclidean sense. Then $H$ satisfies $A_s$. The constant $C_0$ in (2.5) depends also on a positive lower bound on $\inf_{\Omega} |\Lambda_n(x, 1)|$, see (2.14). This case is discussed in [MM12a] Section 4.
• The pathological vector fields $X_1 = \partial_{x_1}$ and $X_2 = e^{-1/x_1^2} \partial_{x_2}$, in spite of their $C^\infty$ smoothness, do not satisfy (2.5) for any $s \in \mathbb{N}$.

Let $\Omega_0 \subset \mathbb{R}^n$ be a fixed open set, bounded in the Euclidean metric. Given a family $H$ of vector fields of class $C^1_{\text{Euc}} \cap C^{s-1,1}_{\text{H},\text{loc}}$, introduce the constant

$$L_0 := \sum_{j_1, \ldots, j_s = 1}^m \left\{ \sup_{\Omega_0} \left( |f_{j_1}| + |\nabla f_{j_1}| + \sum_{p \leq s} |X^\sharp_{j_1} \cdots X^\sharp_{j_{p-1}} f_{j_p}| \right) \right. + \text{Lip}_H(X^\sharp_{j_1} \cdots X^\sharp_{j_{s-1}} f_{j_s}; \Omega_0) \right\}.$$  

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We shall always choose points \( x \in \Omega \) and we fix a constant \( t_0 > 0 \) small enough to ensure that
\[
e^{\tau_1 Z_1} \cdots e^{\tau_N Z_N} x \in \Omega_0 \quad \text{if} \quad x \in \Omega, \ Z_j \in \mathcal{H}, \ |\tau_j| \leq t_0 \ \text{and} \ N \leq N_0, \tag{2.9}
\]
where \( N_0 \) is a suitable constant which depends on the data \( n,m \) and \( s \).

**Proposition 2.7** (measurability). Let \( \mathcal{H} \) be a family of class \( A_s \). Let \( |w| = s \) and let \( Z \in \pm \mathcal{H} \), then for any \( x \in \Omega \) we can write
\[
\text{ad}_Z X_w(e^{tZ}x) = \sum_{1 \leq |v| \leq s} b^v(t)X_w(e^{tZ}x) \quad \text{for a.e.} \ t \in (-t_0,t_0), \tag{2.10}
\]
where the functions \( t \mapsto b^v(t) \) are measurable and for a.e. \( t \) we have \( |b^v(t)| \leq C_0 \), where \( C_0 \) denotes the constant in (2.3).

**Proof.** The statement can be proved arguing as in [MM12a] Proposition 4.1. \( \square \)

**Wedge products and \( \eta \)-maximality conditions.** Following [Str11], denote by \( \mathcal{P} := \{ Y_1, \ldots, Y_q \} = \{ X_w : 1 \leq |w| \leq s \} \) the family of commutators of length at most \( s \). Let \( \ell_j \leq s \) be the length of \( Y_j \) and write \( Y_j := g_j \cdot \nabla \). Define for any \( p, \mu \in \mathbb{N} \), with \( 1 \leq p \leq \mu \), \( \mathcal{I}(p,\mu) := \{ I = (i_1, \ldots, i_p) : 1 \leq i_1 < i_2 < \cdots < i_p \leq \mu \} \). For each \( x \in \mathbb{R}^n \) define \( p_x := \dim \text{span}\{ Y_{i,x} : 1 \leq j \leq q \} \). Obviously, \( p_x \leq \min\{n,q\} \). Then for any \( p \in \{ 1, \ldots, \min\{n,q\} \} \), let
\[
Y_{I,x} := Y_{i_{1,x}} \wedge \cdots \wedge Y_{i_{p,x}} \in \bigwedge_p T_x \mathbb{R}^n \sim \bigwedge_p \mathbb{R}^n \quad \text{for all} \ I \in \mathcal{I}(p,q),
\]
and, for all \( K \in \mathcal{I}(p,n) \) and \( I \in \mathcal{I}(p,q) \)
\[
Y^K_I(x) := dx^K (Y_{i_{1},x}, \ldots, Y_{i_{p},x})(x) := \det(g_{i_a}^{k_d})_{a,b=1,\ldots,p}. \tag{2.11}
\]
Here we let \( dx^K := dx^{k_1} \wedge \cdots \wedge dx^{k_p} \) for any \( K = (k_1, \ldots, k_p) \in \mathcal{I}(p,n) \).

The family \( e^K := e_{k_1} \wedge \cdots \wedge e_{k_p} \) for any \( K \in \mathcal{I}(p,n) \), gives an orthonormal basis of \( \bigwedge_p \mathbb{R}^n \), i.e. \( \langle e^K, e^H \rangle = \delta_{K,H} \) for all \( K,H \). Then we have the orthogonal decomposition
\[
Y_I(x) = \sum_K Y^K_I(x)e^K \in \bigwedge_p \mathbb{R}^n, \quad \text{so that the number}
\]
\[
|Y_I(x)| := \left( \sum_{K \in \mathcal{I}(p,n)} Y^K_I(x)^2 \right)^{1/2} = |Y_{i_1}(x) \wedge \cdots \wedge Y_{i_p}(x)|
\]
gives the \( p \)-dimensional volume of the parallelepiped generated by \( Y_{i_1}(x), \ldots, Y_{i_p}(x) \).

Let \( I = (i_1, \ldots, i_p) \in \mathcal{I}(p,q) \) such that \( |Y_I| \neq 0 \). Consider the linear system \( \sum_{k=1}^p \xi^k Y_{i_k} = W \), for some \( W \in \text{span}\{ Y_{i_1}, \ldots, Y_{i_p} \} \). The Cramer’s rule gives the unique solution
\[
\xi^k = \frac{\langle Y_{i_k}, t^W Y_I \rangle}{|Y_I|^2} \quad \text{for each} \ k = 1, \ldots, p. \tag{2.12}
\]
where we let \( t^W Y_I := t^W(W)Y_I := Y_{i_1, \ldots, i_{k-1}} \wedge W \wedge Y_{i_{k+1}, \ldots, i_p} \).

Let \( r > 0 \). Given \( J \in \mathcal{I}(p,q) \), let \( \ell(J) := \ell_{j_1} + \cdots + \ell_{j_p} \). Introduce the vector-valued function
\[
\Lambda_p(x,r) := \langle Y^K_J(x) r^{\ell(J)} \rangle_{J \in \mathcal{I}(p,q), K \in \mathcal{I}(p,n)} =: \langle \tilde{Y}^K_J(x) \rangle_{J \in \mathcal{I}(p,q), K \in \mathcal{I}(p,n)}, \tag{2.13}
\]
where we adopt the tilde notation \( \tilde{Y}_k = r^{\ell_k} Y_k \) and its obvious generalization for wedge products. Note that \( |\Lambda_p(x,r)|^2 = \sum_{J \in \mathcal{I}(p,q)} r^{2\ell(J)} |Y_I(x)|^2 \).

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Definition 2.8 ($\eta$-maximality). Let $x \in \mathbb{R}^n$, let $I \in \mathcal{I}(p_x, q)$ and $\eta \in (0, 1)$. We say that $(I, x, r)$ is $\eta$-maximal if $|Y_I(x)|_{\tau(I)} > \eta \max_{J \in \mathcal{I}(p_x, q)} |Y_J(x)|_{\tau(J)}$.

Note that, if $(I, x, r)$ is a candidate to be $\eta$-maximal with $I \in \mathcal{I}(p, q)$, then by definition it must be $p = p_x = \dim \text{span}\{Y_j(x) : 1 \leq j \leq q\}$.

Approximate exponentials of commutators. Let $w_1, \ldots, w_\ell \in \{1, \ldots, m\}$. Given $\tau > 0$, we define, as in [NSW85], [Mor00] and especially [MM12b],

\[
C_\tau(X_{w_1}) := \exp(\tau X_{w_1}),
\]

\[
C_\tau(X_{w_1}, X_{w_2}) := \exp(-\tau X_{w_2}) \exp(-\tau X_{w_1}) \exp(\tau X_{w_2}) \exp(\tau X_{w_1}),
\]

\[
\vdots
\]

\[
C_\tau(X_{w_1}, \ldots, X_{w_\ell}) := C_\tau(X_{w_2}, \ldots, X_{w_\ell}) \exp(-\tau X_{w_1}) C_\tau(X_{w_2}, \ldots, X_{w_\ell}) \exp(\tau X_{w_1}).
\]

Then let

\[
t_{X_{w_1}w_2 \ldots w_\ell} := \exp_{ap}(tX_{w_1}w_2 \ldots w_\ell) := \begin{cases} C_{t/\tau}(X_{w_1}, \ldots, X_{w_\ell}), & \text{if } t \geq 0, \\ C_{|t|/\tau}(X_{w_1}, \ldots, X_{w_\ell})^{-1}, & \text{if } t < 0. \end{cases}
\]

(2.14)

By standard ODE theory, there is $t_0$ depending on $\ell, \Omega, \Omega_0, \sup|f_j|$ and $\sup|\nabla f_j|$ such that $\exp_{ap}(t_{X_{w_1}w_2 \ldots w_\ell}) x \in \Omega_0$ for any $x \in \Omega$ and $|t| \leq t_0$. Define, given $I = (i_1, \ldots, i_p) \in \{1, \ldots, q\}^p$, $x \in \Omega$ and $h \in \mathbb{R}^p$, with $|h| \leq C$$^{-1}$

\[
E_{I,x}(h) := \exp_{ap}(h_{i_1}Y_{i_1}) \cdots \exp_{ap}(h_{i_p}Y_{i_p})(x)
\]

\[
||h||_I := \max_{j=1, \ldots, p} |h_j|^{1/i_{i_j}} \quad \text{and} \quad Q_I(r) := \{h \in \mathbb{R}^p : ||h||_I < r\}.
\]

(2.16)

Gronwall’s inequality. We shall refer several times to the following standard fact: for all $a \geq 0, b > 0, T > 0$ and $f$ continuous on $[0, T]$,

\[
0 \leq f(t) \leq at + b \int_0^t f(\tau) d\tau \quad \forall t \in [0, T] \quad \Rightarrow \quad f(t) \leq \frac{a}{b}(e^{bt} - 1) \quad \forall t \in [0, T].
\]

(2.17)

3. Approximate exponentials and regularity of $A_s$ orbits

Let $\mathcal{H} = \{X_1, \ldots, X_m\}$ be a family of $A_s$ vector fields in $\mathbb{R}^n$. The main purpose of this section is to prove that any $\mathcal{H}$-orbit $\mathcal{O}_\mathcal{H}$ with the topology $\tau_d$ generated by the distance $d$ is a $C^1$ integral manifold of the distribution generated by $\mathcal{P}$. Recall our usual notation $\mathcal{P} := \{Y_j : 1 \leq j \leq q\}$, $P_x := \text{span}\{Y_{j,x} : 1 \leq j \leq q\}$ and $p_x := \dim P_x$.

3.1. Geometric properties of orbits

In this subsection we look at the properties of orbits $\mathcal{O}_\mathcal{H}$ for vector fields of class $A_s$. First we study how the geometric determinants $\bar{Y}_K^J$ change along a given orbit $\mathcal{O}_\mathcal{H}$. The argument we use is known, see for instance [TW03] and especially [MM12b]. However, we need to address some issues which appear due to our low regularity assumptions. Ultimately, we will show that the positive integer $p_x$ is constant as $x \in \mathcal{O}_\mathcal{H}$.
Below we shall use the following notation: given \( r > 0 \), we let \( \tilde{Y}_j = r^E Y_j =: \tilde{g}_j \cdot \nabla \) and \( \tilde{Z} = rZ \), if \( Z \in \pm \mathcal{H} \). Let also \( \tilde{Y}^{K}_j := r^{(j)} Y^{K}_j \), where the notation for \( Y^{K}_j \) has been introduced in \([2.11]\).

**Lemma 3.1.** Let \( \mathcal{H} \) be a family of vector fields of class \( \mathcal{A}_s \). Let \( p \in \{1, \ldots, q \wedge n\} \). Let \( x \in \Omega \) and \( r_0 > 0 \) so that \( B_d(x, r_0) \subset \Omega_0 \). Let \( J \in \mathcal{I}(p, q) \), \( K \in \mathcal{I}(p, n) \), \( \ell \in (0, r_0) \) and \( \tilde{Z} \in \pm r\mathcal{H} \). Then the function \([−1, 1] \ni t \mapsto \tilde{Y}^{K}_j(e^t \tilde{Z} x) \) is Lipschitz continuous and there is \( C > 1 \) depending on \( C_0 \) and \( L_0 \) in \([2.5]\) and \([2.8]\) such that

\[
\left| \frac{d}{dt} \tilde{Y}^{K}_j(e^t \tilde{Z} x) \right| \leq C|\Lambda_p(e^t \tilde{Z} x, r)| \quad \text{for a.e. } t \in (−1, 1).
\]

**Proof.** Denote \( \gamma_t := e^t \tilde{Z} x \) and let \( t, \tau \in (−1, 1) \). Then

\[
|\tilde{Y}^{K}_j(\gamma_\tau) - \tilde{Y}^{K}_j(\gamma_t)| = \left| \sum_{1 \leq \alpha \leq p} dx^K(\ldots, \tilde{Y}_{j_{\alpha-1}}(\gamma_t), \tilde{Y}_{j_\alpha}(\gamma_t), \tilde{Y}_{j_{\alpha+1}}(\gamma_t), \ldots) \right|
\]

\[
\leq C|\tau - t|,
\]

where \( C \) depends on \( L_0 \) in \([2.8]\). Then \( t \mapsto \tilde{Y}^{K}_j(\gamma_t) \) belongs to \( \text{Lip}_{\text{Euc}}(−1, 1) \). The estimate for the Lipschitz constant here is quite rough and it can be refined through a computation of the derivative. Indeed, we claim that for a.e. \( t \in (−1, 1) \) we have

\[
\frac{d}{dt} \tilde{Y}^{K}_j(\gamma_t) = \sum_{1 \leq \alpha \leq p} dx^K(\ldots, \tilde{Y}_{j_{\alpha-1}}[\tilde{Z}, \tilde{Y}_{j_\alpha}], \tilde{Y}_{j_{\alpha+1}}, \ldots, \tilde{Y}_{j_p})(\gamma_t)
\]

\[
+ \sum_{1 \leq \alpha \leq p} b^\alpha_{\gamma(t)} dx^K(\ldots, \tilde{Y}_{j_{\alpha-1}}, \tilde{Y}_\beta, \tilde{Y}_{j_{\alpha+1}}, \ldots, \tilde{Y}_{j_p})(\gamma_t)
\]

\[
+ \sum_{1 \leq \gamma \leq m} \sum_{1 \leq \beta \leq p} \partial_\gamma \tilde{f}^\beta dx^{(\gamma_1, \ldots, \gamma_{\beta-1}, \gamma, \gamma_{\beta+1}, \ldots, \gamma_p)}(\tilde{Y}_{j_1}, \ldots, \tilde{Y}_{j_p})(\gamma_t)
\]

\[
=: (A) + (B) + (C),
\]

where we wrote \( \tilde{Z} = \tilde{f} \cdot \nabla \in C^1_{\text{Euc}} \) and \( b^\alpha_{\gamma} \) are measurable functions with \( |b^\alpha_{\gamma}| \leq C_0 \). To prove \([3.1]\), observe that, if \( \ell(Y_{j_\alpha}) \leq s - 1 \), then \( t \mapsto \tilde{Y}_{j_\alpha}(\gamma_t) \) is \( C^1_{\text{Euc}}(−1, 1) \) and

\[
\lim_{\tau \to t} \frac{\tilde{Y}_{j_\alpha}(\gamma_\tau) - \tilde{Y}_{j_\alpha}(\gamma_t)}{\tau - t} = \tilde{Z}^2 g_{j_\alpha}(\gamma_t) \cdot \nabla = [\tilde{Z}, \tilde{Y}_{j_\alpha}](\gamma_t) + \tilde{Y}_{j_\alpha} \tilde{f}(\gamma_t) \cdot \nabla \quad \text{for all } t \in [−1, 1].
\]

Note that here we used \([MM12a]\) Theorem 3.1] to claim that \( \text{ad}_\tilde{Z} \tilde{Y}_{j_\alpha} = [\tilde{Z}, \tilde{Y}_{j_\alpha}] \). If instead \( \ell(Y_{j_\alpha}) = s \), then for almost any \( t \) we have

\[
\lim_{\tau \to t} \frac{\tilde{Y}_{j_\alpha}(\gamma_\tau) - \tilde{Y}_{j_\alpha}(\gamma_t)}{\tau - t} = \tilde{Z}^2 g_{j_\alpha}(\gamma_t) \cdot \nabla = \text{ad}_\tilde{Z} \tilde{Y}_{j_\alpha}(\gamma_t) + \tilde{Y}_{j_\alpha} \tilde{f}(\gamma_t) \cdot \nabla
\]

\[
+ \sum_{\beta = 1}^q b^\alpha_{\gamma}(t) \tilde{Y}_\beta(\gamma_t) + \tilde{Y}_{j_\alpha} \tilde{f}(\gamma_t) \cdot \nabla.
\]
In the first equality we used the definition of ad. Here $\tilde{Y}_{j_0}\tilde{f} := \tilde{g}_{j_0} \cdot \nabla \tilde{f}$, is well defined. In the second line we used Proposition 2.7. The term $\tilde{Y}_{j_0}\tilde{f}$, in view of Lemma A.1 gives the third line of (3.1).

Next we estimate each line of (3.1), starting with (A).

$$|(A)| \leq |d\mathbf{x}^K(\ldots,\tilde{Y}_{j_{a-1}}(\gamma(t)),[\tilde{Z},\tilde{Y}_{j_a}](\gamma(t)),\tilde{Y}_{j_{a+1}}(\gamma(t)),\ldots)| \leq C|\Lambda_p(\gamma(t),r)|,$$

for all $t \in [-1,1]$. Estimate is correct even if $\Lambda_p(\gamma(t),r) = 0$. To estimate (B), recall that $|b_0^\alpha| \leq C$. Then, for all $t \in [-1,1]$,

$$|(B)| \leq \sum_{1 \leq \alpha \leq p} \sum_{1 \leq \beta \leq q} |d\mathbf{x}^K(\ldots,\tilde{Y}_{j_{a-1}},\tilde{Y}_\beta,\tilde{Y}_{j_{a+1}},\ldots)| \leq C|\Lambda_p(\gamma(t),r)|.$$

Finally the estimate of (C) is easy and takes the form

$$|(C)| \leq \sup_{B_d(x,r)} |\nabla \tilde{f}| \max_{K \in \mathcal{L}(p,n)} |\tilde{Y}_f^K(\gamma(t))| \leq C|\Lambda_p(\gamma(t),r)| \quad \text{if } |t| \leq 1. \quad \Box$$

The previous lemma immediately implies the following proposition.

**Proposition 3.2.** Let $\mathcal{H}$ be a family in the regularity class $\mathcal{A}_\eta$. Let $x \in \Omega$, let $r \leq r_0$, where $r_0$ is small enough so that $B_d(x,r_0) \subset \Omega_0$. Let $\gamma(t) := \gamma_t$ be a piecewise integral curve of $\pm r\mathcal{H}$ with $\gamma(0) = x$. Let $p \in \{1,\ldots,q \wedge n\}$. Then we have

$$|\Lambda_p(\gamma(t),r) - \Lambda_p(x,r)| \leq |\Lambda_p(x,r)| (e^{Ct} - 1) \quad \text{for all } t \in [0,1]. \quad (3.3)$$

In particular, if $p = p_x$ and $(I,x,r)$ is $\eta$-maximal, then

$$|\tilde{Y}_J(\gamma(t)) - \tilde{Y}_J(x)| \leq \frac{Ct}{\eta} |\tilde{Y}_I(x)| \quad \text{for all } J \in \mathcal{I}(p,q) \quad t \in [0,1]. \quad (3.4)$$

Finally, if $x,y$ belong to the same orbit, then $p_x = p_y$.

**Remark 3.3.** As a consequence of the proposition and of the Cramer’s rule (2.12), if $(I,x,r)$ is $\eta$-maximal, then $(I,y,r)$ is $C^{-1}\eta$-maximal for all $y \in B_d(x,C^{-1}\eta r)$ and we may write for all such $y$ and for any $j \in \{1,\ldots,q\}$

$$\tilde{Y}_{j,y} = \sum_{k=1}^p \frac{b_j^k}{\eta} \tilde{Y}_{i_k,y}, \quad (3.5)$$

where $|b_j^f| \leq C$.

**Remark 3.4.** Proposition 3.2 shows that the oscillation of determinants $\Lambda_p$ on a ball is controlled in terms of the value of $\Lambda_p$ at the center of the ball. It is not true that the oscillation of a single vector field on a ball can be controlled by its value at the center of the ball. For instance, we can take the vector fields $X = \partial_x$ and $Y = y \partial_y + x \partial_x$. Look at the ball $B((0,y),r)$, where $0 < y \ll r$. Note that $(r,y)$ belongs to such ball, but the oscillation $|Y(0,y) - Y(r,y)| \sim r$ can not be controlled with the value $|Y(0,y)| = |y|$. 

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such property in Theorem 3.11 to show that Theorem 3.5 has an important consequence. Namely, in in Theorem 3.8, it will enable us to establish below.

\[ \text{Proof of Proposition 3.2. (See [TW03,MM12,Str11]) Let } p \in \{1, \ldots, q \land n \}. \text{ By Lemma 3.1, the map } t \mapsto \Lambda_p(\gamma_t, r) \text{ is Lipschitz. Moreover, we have for a.e. } t \in [0,1], \]
\[ \left| \frac{d}{dt} \Lambda_p(\gamma_t, r) \right| = \left| \left( \frac{d}{dt} \tilde{Y}_j^K \right)_{j \in \mathcal{I}(p,q)} \right| \leq C|\Lambda_p(\gamma_t, r)|, \]

by Lemma 3.1. Then the Gronwall’s inequality (2.17) provides immediately the required estimate (3.3). Note that this implies that if \( \Lambda_p(x, r) = 0 \), then \( \Lambda_p(\gamma_t, r) = 0 \) for all \( t \in [0,1] \). Estimate (3.4) follows immediately.

Let now \( x \) and \( y \) be a couple of points on the same leaf \( \mathcal{O}_H \). Let \( 1 \leq p \leq q \land n \) and let \( I \subset \mathbb{R} \) be an interval. Let \( I = [a,b] \) and take the piecewise integral curve of the vector fields \( X_j \) with \( \gamma(a) = x \) and \( \gamma(b) = y \). Let \( A_p := \{ t \in I : |\Lambda_p(\gamma(t))| = 0 \} \). Note that \( A_p \) is closed, because it is the zero set of the continuous function \( I \ni t \mapsto |\Lambda_p(\gamma(t))| \in \mathbb{R} \). The set \( A_p \) is also open by estimate (3.3). Therefore, either \( A_p = \emptyset \) or \( A_p = I \) and the proof is concluded.

The fact we are going to establish in the following theorem will have a key role in Subsection 3.2 when we shall study our almost exponential maps \( E \). See Remark 3.6 below.

**Theorem 3.5.** Let \( \mathcal{H} \) be a family of vector fields of class \( \mathcal{A}_s \). Let \( (I, x, r) \) be \( \eta \)-maximal where \( x \in \Omega \), \( r \leq r_0 \), \( I \in \mathcal{I}(p_x, q) \) and \( \eta \in (0,1) \). Denote \( \tilde{U}_j := r^{\theta_j}Y_j \) for \( j = 1, \ldots, p := p_x \) and \( \tilde{Z} := rZ \in \pm r\mathcal{H} \). Then there is \( C > 0 \) depending on \( L_0 \) and \( C_0 \) in (2.8) and (2.9) so that
\[ e_{s-t\tilde{Z}}(\tilde{U}_{j,e^{t\tilde{Z}}}) \in P_x \text{ for all } |t| \leq C^{-1}\eta. \]  
Moreover, if we write, for a given test function \( \psi \in C^1_{\text{Euc}}(\mathbb{R}^n) \),
\[ \tilde{U}_j(\psi e^{-t\tilde{Z}})(e^{t\tilde{Z}}x) =: \sum_{k=1}^p \left( \delta^k_j + \theta_k^j(t) \right) \tilde{U}_k \psi(x), \]

then we have
\[ |\theta_k^j(t)| \leq \frac{C|t|}{\eta} \text{ for all } j, k = 1, \ldots, p \quad |t| \leq C^{-1}\eta. \]  
Finally, for any commutator \( \tilde{Y}_h := \tilde{g}_h \cdot \nabla \), where \( h \in \{1, \ldots, q\} \), we have at any \( t \in (-C^{-1}\eta, C^{-1}\eta) \)
\[ \tilde{Y}_h(\psi e^{-t\tilde{Z}})(e^{t\tilde{Z}}x) = \sum_{k=1}^p \frac{b_k^j(t)}{\eta} \tilde{U}_k \psi(x), \]

where \( |b_k^j(t)| \leq C |t| \leq C^{-1}\eta. \)

**Remark 3.6.** The geometric interpretation of (3.6) tells that \( e_{s-t\tilde{Z}}P_{e^{t\tilde{Z}}} = P_x \), i.e. the tangent map of the \( C^1 \) diffeomorphism \( e^{-t\tilde{Z}} \) maps the (candidate) tangent bundle \( \cup_x P_x \) to the orbit \( \mathcal{O} \) to itself (we say “candidate” because we do not know yet that \( \mathcal{O} \) is a manifold). Theorem 3.5 has an important consequence. Namely, in in Theorem 3.8, it will enable us to show that integral remainders have in fact a pointwise form. Ultimately, we will apply such property in Theorem 3.11 to show that \( E_s(\partial_h) \in P_{E(h)} \).
Remark 3.7. The proof below is inspired to an argument due to Lobry; see \cite{Lob70} Lemma 1.2.1. Here we generalize such argument to a higher order, nonsmooth situation and we get more quantitative estimates. See also \cite{Lob70} and the related discussion by Balan \cite{Bal94}; see finally the paper \cite{Pel10}, for an up-to-date bibliography on the subject. Note that Lobry’s idea is also used in \cite{AS94} Lemma 5.15.

Proof of Theorem 3.5. Without loss of generality, we can work with positive values of $t$. First, we differentiate the left-hand side of (3.7). If $\ell_{ij} \leq s - 1$, then we use \cite{MM12a} Theorem 2.6-(a) and Theorem 3.1-(ii)] which give

\[ \frac{d}{dt} \tilde{U}_j(\psi e^{-t\bar{Z}})(e^{tZ}x) = \left[ \mathcal{Z}, \tilde{U}_j \right](\psi e^{-t\bar{Z}})(e^{tZ}x) = \sum_{k=1}^{p} \frac{b_k(t)}{\eta} \tilde{U}_k(\psi e^{-t\bar{Z}})(e^{tZ}x), \]  

providing that $0 < t \leq C^{-1}\eta$. Here $|b_k(t)| \leq C$. In last equality we used (3.5) with $\tilde{Y}_h = [\mathcal{Z}, \tilde{U}_j]$.

If instead $\ell_{ij} = s$, then we need first \cite{MM12a} Theorem 2.6-(b)], then (2.6) and Proposition 2.7 in the present paper. This gives for a.e. $t \in [0, C^{-1}\eta]$

\[ \frac{d}{dt} \tilde{U}_j(\psi e^{-t\bar{Z}})(e^{tZ}x) = \sum_{1 \leq h \leq q} b_h^j(t) \tilde{Y}_h(\psi e^{-t\bar{Z}})(e^{tZ}x) \]  \tag{3.5}

\[ = \sum_{1 \leq h \leq q} \sum_{1 \leq k \leq p} b_h^j(t)b_k^h(t) \frac{1}{\eta} \tilde{U}_k(\psi e^{-t\bar{Z}})(e^{tZ}x) \]  

\[ =: \sum_{1 \leq k \leq p} \frac{b_k^j(t)}{\eta} \tilde{U}_k(\psi e^{-t\bar{Z}})(e^{tZ}x) \]  

provided that $0 < t \leq C^{-1}\eta$. In this formula $b_h^j, b_k^h$ and $b_k^j$ denote measurable functions, bounded in term of the admissible constants $C_0$ and $L_0$.

By elementary ODE theory, for any fixed $\psi$, the functions $t \mapsto \tilde{U}_j(\psi e^{-t\bar{Z}})(e^{tZ}x)$ with $j = 1, \ldots, p$ are uniquely determined by their value $\tilde{U}_j(\psi)(x)$ at $t = 0$. Moreover, if we denote by $(a_j^k(t)) \in \mathbb{R}^{p \times p}$ the solution of the Cauchy problem

\[ \dot{a}(t) = b(t) \frac{a(t)}{\eta} \text{ with } a(0) = I_p \in \mathbb{R}^{p \times p}, \]  

then we can write

\[ e^{-t\bar{Z}}(\tilde{U}_j,e^{tZ}x) = \tilde{U}_j(\psi e^{-t\bar{Z}})(e^{tZ}x) = \sum_{k=1}^{p} a_k^j(t) \tilde{U}_k(\psi)(x). \]  

Then we have proved (3.10). The Cramer’s rule (2.12) confirms that the coefficients $a_k^j(t)$ are unique for each $t$.

To estimate the functions $\theta_k^j := a_k^j(t) - \delta_j^k$, where $a_k^j$ satisfy (3.12), it suffices to use estimate $|b_k^j(t)| \leq C$ if $0 < t \leq C^{-1}\eta$. The Gronwall inequality (2.17) gives $|a_k^j(t) - \delta_j^k| \leq C|t|/\eta$ for all $j,k = 1, \ldots, p$ and $0 < t \leq C^{-1}\eta$. Therefore (3.8) follows.

To obtain the proof of (3.9) it suffices to repeat the computation in (3.10) starting from $\tilde{Y}_h$ instead of $\tilde{U}_j$. This ends the proof. \qed
Under the hypotheses of Theorem 3.8 iterating the argument, we get for all \( x \in \Omega, \mu \leq N_0 \) (see (2.9)), \( j \in \{1, \ldots, p\} \) and \( Z_1, \ldots, Z_\mu \in \mathcal{H}, \)

\[
\tilde{U}_j(ye^{-t_1\tilde{Z}_1} \cdots e^{-t_\mu\tilde{Z}_\mu})(e^{t_\mu\tilde{Z}_\mu} \cdots e^{t_1\tilde{Z}_1}) = \sum_{1 \leq k \leq p} (d^k_j + \theta^k_j(t)) \tilde{U}_k \psi(x) \tag{3.14}
\]

where \( |\theta(t)| \leq C|t|/\eta \), as soon as \( \sum_{j=1}^\mu |t_j| \leq C^{-1}\eta \). Moreover, for each \( h \in \{1, \ldots, q\} \), we get, if \( x \in \Omega \), for the same values of \((t_1, \ldots, t_\mu)\) and for almost all \( \tau \in (-C^{-1}\eta, C^{-1}\eta) \),

\[
\frac{d}{dt} \tilde{Y}_h(ye^{-t_1\tilde{Z}_1} \cdots e^{-t_\mu\tilde{Z}_\mu} e^{-\tau \tilde{X}})(e^{\tau \tilde{X}} e^{t_\mu\tilde{Z}_\mu} \cdots e^{t_1\tilde{Z}_1}) = \text{ad}_\tilde{X} \tilde{Y}_h(ye^{-t_1\tilde{Z}_1} \cdots e^{-t_\mu\tilde{Z}_\mu} e^{-\tau \tilde{X}})(e^{\tau \tilde{X}} e^{t_\mu\tilde{Z}_\mu} \cdots e^{t_1\tilde{Z}_1}) = \sum_{k=1}^p \frac{b_k(x,t,\tau)}{\eta} \tilde{U}_k \psi(x),
\]

where \( |b_k(x,t,\tau)| \leq C \) for a.e. \( \tau \). Here \( X \in \mathcal{H} \). If we do not care about maximality and choose \( r = 1 \), we get, for any fixed \((t_1, \ldots, t_\mu)\) with \( \sum_j |t_j| \leq C^{-1} \) and for almost all \( \tau \) with \( |\tau| \leq C^{-1}, \)

\[
\frac{d}{dt} Y_h(ye^{-t_1\tilde{Z}_1} \cdots e^{-t_\mu\tilde{Z}_\mu} e^{-\tau \tilde{X}})(e^{\tau \tilde{X}} e^{t_\mu\tilde{Z}_\mu} \cdots e^{t_1\tilde{Z}_1}) = \text{ad}_X Y_h(ye^{-t_1\tilde{Z}_1} \cdots e^{-t_\mu\tilde{Z}_\mu} e^{-\tau \tilde{X}})(e^{\tau \tilde{X}} e^{t_\mu\tilde{Z}_\mu} \cdots e^{t_1\tilde{Z}_1}) = \sum_{1 \leq j \leq q} b_j(x,t,\tau)Y_j \psi(x),
\]

where \( |b_j(x,t,\tau)| \leq C \) for a.e. \( \tau \). Here again \( x \in \Omega \) and \( \psi \in \mathcal{C}_1^{\text{Euc}} \) is a test function. Formula (3.15) will be referred to later.

### 3.2. Derivatives of almost exponential maps and regularity of orbits

In this subsection we get several information on the derivatives of the approximate exponentials \( E_{f,t,x} \) associated with a family \( \mathcal{H} \) of \( A_v \) vector fields and we show that each orbit \( O \) with topology \( \tau_d \) is a \( C^1 \) immersed submanifold of \( \mathbb{R}^n \) with \( T_yO = P_y \) for all \( y \in O \). We will tacitly but heavily rely on the results of [MM12b] Section 3, namely on formulae

\[
\text{ad}_{X_{v_1}} \cdots \text{ad}_{X_{v_k}} X_w = X_{vw} \quad \text{for all } v, w \text{ such that } |v| + |w| = k + |w| \leq s \tag{3.16}
\]

These formulae have a key role. In the proof of Theorem 3.8 below, we shall follow the arguments of [MM12b] Theorems 3.4 and 3.5, modifying everywhere the remainders \( O_{s+1} \) in [MM12b] with our remainders defined in [MM12a]. This will give us a formula with integral remainder, see (3.17). Then, using the results of Subsection 3.1 we shall show that such integral remainder can be specified in a pointwise form.

**Theorem 3.8.** Let \( 1 \leq |w| =: \ell \leq s \), take \( x \in \Omega \) and \( t \in [0,t_0] \), where \( t_0 \) is small enough to ensure that \( C_{t_0} \in \Omega_0 \) for all \( t \in [0,t_0] \). Let \( C_t = C_t(X_{w_1}, \ldots, X_{w_\ell}) \) be the map defined in (2.14). Fix a test function \( \psi \in \mathcal{C}_1^{\text{Euc}}(\mathbb{R}^n) \). Then we have

\[
\frac{d}{dt} \psi(C_t x) = \ell t^{\ell-1} X_w \psi(C_t x) + \sum_{|v| = \ell+1} a_v t^{\ell+1} X_v \psi(C_t x) + t^s \sum_{|u| = 1} b_u(x,t) X_u \psi(C_t x),
\]
and
\[
\frac{d}{dt}\psi(C_t^{-1}x) = -\ell t^{\ell-1}X_w\psi(C_t^{-1}x) + \sum_{|v| = \ell+1}^s \overline{\pi}_v t^{v-1}X_v\psi(C_t^{-1}x) \\
+ t^s \sum_{|u|=1}^s \tau_u(x, t) X_u\psi(C_t^{-1}x).
\]

Both the sums on \(v\) are empty if \(|w| = s\). Otherwise, we have the cancellations \(\sum_{|v| = \ell+1}(a_v + \overline{\pi}_v f_v(x) = 0 \text{ for all } x \in \Omega\). The \(\text{(real)}\) coefficients \(b_u\) and \(\overline{b}_u\) are bounded in terms of the constants \(L_0\) and \(C_0\) in (2.8) and (2.5).

**Remark 3.9.** As already observed, the theorem just stated improves [MM12b] Theorem 3.5, both because we relax regularity assumptions and because we devise a pointwise form of the remainders. In particular, choosing as \(\psi\) the identity function, we see that the remainder belongs to the subspace \(P_{C_t^x} = \text{span}\{Y_{j,C_t^x} : j = 1, \ldots, q\}\) which can be a strict subspace of \(\mathbb{R}^n\).

**Proof of Theorem 3.8.** We prove the statement for \(t > 0\). By [MM12b] Theorem 3.5, we know that
\[
\frac{d}{dt}\psi(C_t^x) = \ell t^{\ell-1}X_w\psi(C_t^x) + \sum_{|v| = \ell+1}^s \overline{a}_v t^{v-1}X_v\psi(C_t^x) + O_{s+1}(t^s, \psi, C_t^x),
\]
where the numbers \(a_v\) are suitable algebraic coefficients. Note that formula (3.17) in [MM12b] is proved for smooth vector fields. Using (3.16) and changing everywhere the remainders in [MM12b] with the remainders introduced in [MM12a] Subsection 2.1, one can check that all computations fit to our setting. Therefore, we only need to deal with the integral remainders introduced and discussed in [MM12a]. Concerning such remainders, recall that
\[
O_{s+1}(t^s, \psi, C_t^x) = \text{(sum of terms like)} \int_0^t \omega(t, \tau) \frac{d}{d\tau} X_v(\psi\varphi^{-1}e^{-\tau Z})(e^{\tau Z}\varphi C_t^x)d\tau
\]
where \(|v| = s\), \(\varphi = e^{tZ_1} \cdots e^{tZ_v}\) and \(Z_1, Z_j \in \pm \mathcal{H}\). Next, by (3.15), we may write for a.e. \(\tau\)
\[
\frac{d}{d\tau} X_v(\psi\varphi^{-1}e^{-\tau Z})(e^{\tau Z}\varphi C_t^x) = \sum_{1 \leq |u| \leq s} b_u(x, t, \tau) X_u\psi(C_t^x),
\]
where for any \(t, x\) the functions \(\tau \mapsto b_u(x, t, \tau)\) are measurable and satisfy \(|b_u(t, \tau, x)| \leq C\) for a.e. \(\tau\). Therefore we get
\[
\sum_{1 \leq |u| \leq s} \int_0^t \omega(t, \tau)b_u(x, t, \tau)d\tau X_u\psi(C_t^x) =: t^s \sum_{1 \leq |u| \leq s} b_u(x, t) X_u\psi(C_t^x),
\]
where \(|b_u(x, t)| \leq C\) for all \(x \in \Omega\) and \(|t| \leq t_0\). This ends the proof. \(\square\)

Our purpose now is to study the maps
\[
E(h) := E_{l_x,r}(h) := \exp_{ap}(h^1_1 \bar{Y}_i) \cdots \exp_{ap}(h_{ap}^i \bar{Y}_{ip}) = e^{h_1 \bar{Y}_i} \cdots e^{h_{ap} \bar{Y}_{ip}} x
\]
(3.18)
where $1 \leq p \leq q$, $I \in \mathcal{I}(p,q)$, and $U_k := \tilde{Y}_{i_k}$ and $d_k := \ell_{i_k}$. We always take $x \in \Omega$ and $h$ sufficiently close to the origin so that $E(h) \in \Omega_0$, see (2.9).

Some elementary properties of $E$ are contained in the following lemma. Without loss of generality we choose $r = 1$ and $I = (1, \ldots, p)$.

**Lemma 3.10.** The map $h \mapsto e^{t_1 Y_{ap}} \cdots e^{t_p Y_{ap}} x =: E_{I,x}(h)$ satisfies for $x, x^* \in \Omega$ and $h, h^* \in B_{Euc}(C^{-1})$

$$|E_{I,x}(h) - E_{I,x}(h^*)| \leq C\left(\|h - h^*\|_I + |x - x^*|\right). \quad (3.19)$$

Moreover, for any $w$ with $1 \leq |w| \leq s$, the function $F_{X_w} : [-C^{-1}, C^{-1}] \times \Omega \to \mathbb{R}^{n \times n}$, defined as $F_{X_w}(t, x) := \nabla_x e^{tX_w}(x)$, is continuous. 

**Proof.** Observe first that, since each $Z \in \mathcal{H}$ is $C^1_{Euc}$, by the Gronwall inequality we have

$$|e^{\tau Z} y - e^{\tau_0 Z} y_0| \leq C\left(|y - y_0| + |\tau - \tau_0|\right) \quad \text{for all } y, y_0 \in \Omega, |\tau|, |\tau_0| \leq C^{-1}. \quad (3.20)$$

Next, assume first that $t > t^* \geq 0$. Write $e^{tX_w} x = e^{t_1 Z_1} \cdots e^{t_p Z_p} x$, where $Z_1, \ldots, Z_p \in \mathcal{H}$ are suitable, see (2.13), and $\tau = t^{1/\ell}$, with $\ell := |w|$. Then iterating (3.20) we get

$$|e^{tX_w} x - e^{t^* X_w} x^*| = |e^{t_1 Z_1} \cdots e^{t_p Z_p} x - e^{t^*_1 Z_1} \cdots e^{t^*_p Z_p} x^*| \leq C(|x - x^*| + |t - t^*|^{1/\ell}).$$

If instead $t > 0 > t^*$, then we get

$$|e^{tX_w} x - e^{t^* X_w} x^*| \leq |e^{tX_w} x - x| + |x^* - e^{t^* X_w} x^*| + |x - x^*| \leq C(|t|^{1/\ell} + |t^*|^{1/\ell} + |x - x^*|) \leq C(|t - t^*|^{1/\ell} + |x - x^*|).$$

This shows (3.19) for $p = 1$. Iterating one gets the general case.

Next we prove existence and continuity of the derivative $F_{X_w}$. Assume first that $t \geq 0$ and decompose $e^{tX_w} x = e^{t/\ell Z_1} \cdots e^{t/\ell Z_p} x$, where $\ell = |w|$ and $Z_1, \ldots, Z_p \in \mathcal{H}$ are suitable. Euclidean regularity of the vector fields $Z_j$ implies that the functions $(\tau, y) \mapsto F_{Z_j}(\tau, y) := \nabla_y e^{\tau Z_j} y$ are continuous if $y \in \Omega$ and $|\tau|$ is small. Therefore, the chain rule gives

$$F_{X_w}(t,x) = \nabla_x e^{tX_w}(x) = F_{Z_1}(t^{1/\ell}, e^{t/\ell Z_2} \cdots e^{t/\ell Z_p} x) F_{Z_2}(t^{1/\ell}, e^{t/\ell Z_3} \cdots (x)) \cdots F_{Z_p}(t^{1/\ell}, x).$$

Thus $F_{X_w}|_{[0,C^{-1}] \times \Omega}$ is continuous. Note that $F_{X_w}(0, x) = I_n$ for all $x$. An analogous argument shows that $F_{X_w}|_{[-C^{-1}, 0] \times \Omega}$ is continuous and concludes the proof. \hfill $\square$

At this point we may deduce the following result. See (3.18) for notation on the map $E$.

**Theorem 3.11.** Let $\mathcal{H}$ be an $A_s$ family. Let $x \in \Omega$ and let $r \in (0, r_0)$. Fix $p \in \{1, \ldots, q\}$ and $I \in \mathcal{I}(p,q)$. Then the function $E_{I,x,r}$ is $C^1$ smooth on $B_{Euc}(C^{-1})$. Moreover, for all $h \in B_{Euc}(C^{-1})$ and for any $k \in \{1, \ldots, p\}$ we have $E_k(\partial_{h_k}) \in P_{E(h)}$ and we can write

$$E_k(\partial_{h_k}) = \tilde{\partial}_{k,E(h)} + \sum_{\ell_j = d_k + 1}^s a_{ij}(h) \tilde{y}_{j,E(h)} + \sum_{i=1}^q \omega_k(x, h) \tilde{y}_{i,E(h)}, \quad (3.21)$$
where, for some $C > 1$ depending on $L_0$ and $C_0$ in (2.8) and (2.9), we have
\begin{align}
|a_j(h)| &\leq C\|h\|^{\ell-d_k} \quad \text{for all } h \in B_{\text{Euc}}(C^{-1}) \\
|\omega_i(x,h)| &\leq C\|h\|^{s+1-d_k} \quad \text{for all } h \in B_{\text{Euc}}(C^{-1}) \quad x \in \Omega.
\end{align}
(3.22) (3.23)

**Proof.** For notational simplicity we delete everywhere the tilde. In fact, the statement holds uniformly in $r \in (0,r_0)$, where $r_0$ depends on the already mentioned constants $L_0$ and $C_0$.

**Step 1.** We first prove the theorem for $p = 1$. Using the definition of $\exp_{ap}$ and Theorem 3.8 we easily obtain by a change of variable that for any commutator $Y$ of length $\ell \in \{1, \ldots, s\}$ and for all $\psi \in C_{\text{Euc}}^1$,
\begin{align}
\frac{d}{dh} \psi(e_{ap}^h Y(x)) &= Y\psi(e_{ap}^h Y(x)) + \sum_{\ell_j = \ell+1}^{s} \alpha_j(h) Y_{\ell_j} \psi(e_{ap}^h Y x) \\
&\quad + |h|^{s+1-\ell/\ell} \sum_{i=1}^{q} b_i(x,h) \psi(e_{ap}^h Y x),
\end{align}
(3.24)

for all $x \in K$ and $0 < |h| \leq C^{-1}$, where the sum is empty if $\ell = s$. If $\ell < s$, then $\alpha_j(h) = \ell^{-1} a_j h^{(\ell_j-\ell)/\ell}$ if $h > 0$, while $\alpha_j(h) = -\ell^{-1} a_j h^{(\ell_j-\ell)/\ell}$ if $h < 0$. The functions $\alpha_j$ come from the statement of Theorem 3.8. The functions $b_i(x,h)$ can be discontinuous, if we pass from $h > 0$ to $h < 0$, but we have estimate $|b_i(x,h)| \leq C$ uniformly in $x, h$.

To complete Step 1, we need to show that the function $h \mapsto \frac{d}{dh} e_{ap}^h z$ is continuous for all fixed $z \in \Omega$. Continuity at any $h \neq 0$ (say $h > 0$) follows immediately from the decomposition $e_{ap}^h Y = e^{h^{1/\ell} Z_1} \ldots e^{h^{1/\ell} Z_s}$, where $Z_j \in \pm \mathcal{H}$. We show now continuity at $h = 0$. Formula (3.24) gives $\left| \frac{d}{dh} e_{ap}^h Y z - g(e_{ap}^h Y z) \right| \leq C|h|^{1/\ell}$ (recall notation $Y =: g \cdot \nabla$).

Therefore, using the l'Hôpital's rule, we get
\begin{align}
\frac{d}{dh} \exp_{ap}^h z \big|_{h=0} &= \lim_{h \to 0} \frac{e_{ap}^h Y z - z}{h} = \lim_{h \to 0} g(e_{ap}^h Y z) + O(h^{1/\ell}) = g(z),
\end{align}
where we need the $d$-continuity of $g$. This shows existence of the derivative at $h = 0$. To see continuity, just let $h \to 0$ in (3.24).

**Step 2.** By induction on $p$, we show that $E$ is $C^1$ smooth. Assume that $(h_1, \ldots, h_{p-1}) \mapsto e_{ap}^{h_1 U_1} \ldots e_{ap}^{h_{p-1} U_{p-1}}(x)$ is $C^1$ for all choice of $U_1, \ldots, U_{p-1}$. We need to show that $(h_1, \ldots, h_p) \mapsto e_{ap}^{h_1 U_1} \ldots e_{ap}^{h_p U_p}(x)$ is $C^1$ smooth.

Let $U_1, \ldots, U_p \in \mathcal{P}$. First of all we show that the map $(h_1, \ldots, h_p) \mapsto E_s(\partial_{h_1})$ is continuous. If $h_1 \neq 0$, say $h_1 > 0$, then we decompose for suitable $Z_1, \ldots, Z_{\mu} \in \mathcal{H}$,
\begin{align}
e_{ap}^{h_1 U_1} \ldots e_{ap}^{h_p U_p} x = e^{h_1^{1/d_1} Z_1} \ldots e^{h_1^{1/d_1} Z_{\mu}} e_{ap}^{h_2 U_2} \ldots e_{ap}^{h_p U_p} x.
\end{align}

Note that by standard ODE theory, the map $(\tau_1, \ldots, \tau_{\mu}, z) \mapsto e^{\tau_1 Z_1} \ldots e^{\tau_{\mu} Z_{\mu}} z$ is $C^1$. Therefore, by means of Lemma 3.10 we have existence and continuity of $\partial_t E(h) = E_s(\partial_{h_1})$ at any point of the form $h = (h_1, h_2, \ldots, h_p)$ with $h_1 \neq 0$. 

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To discuss the case $h_1 = 0$, recall that formula (3.21) gives
\[
\left| \frac{\partial}{\partial h_1} e_{ap}^{h_1 U_1} \cdots e_{ap}^{h_p U_p} x - U_1(e_{ap}^{h_1 U_1} \cdots e_{ap}^{h_p U_p} x) \right| \leq C|h_1|^{1/d_1}.
\]
Therefore, using de l'Hôpital’s rule, for all $h = (0, h_2, \ldots, h_p) =: (0, \hat{h}_1)$, we get
\[
\partial_1 E(0, \hat{h}_1) := \lim_{h_1 \to 0} \frac{e_{ap}^{h_1 U_1} e_{ap}^{h_2 U_2} \cdots e_{ap}^{h_p U_p} x - e_{ap}^{h_2 U_2} \cdots e_{ap}^{h_p U_p} x}{h_1} = \lim_{h_1 \to 0} U_1(e_{ap}^{h_1 U_1} e_{ap}^{h_2 U_2} \cdots e_{ap}^{h_p U_p} x) + O(|h_1|^{1/d_1}) = U_1(E(0, \hat{h}_1)),
\]
where we need the $d$-continuity of $U_1$. This shows existence of $\partial_1 E(0, \hat{h}_1)$.

To show continuity of $\partial_{h_1} E$ at $h^* = (0, \hat{h}_1^*) \in B_{\text{Eucl}}(C^{-1})$, write by expansion (3.24)
\[
|\partial_1 E(h_1, \hat{h}_1) - \partial_1 E(0, \hat{h}_1^*)| = \left| U_1(E(h_1, \hat{h}_1)) + \sum_{d_1+1 \leq i_1 \leq s} \alpha_{i_1}(h_1)Y_{i_1}(E(h_1, \hat{h}_1)) + |h_1|^{(s+1-d_1)/d_1} \sum_{1 \leq i \leq q} b_i Y_i(E(h_1, \hat{h}_1)) - U_1(E(0, \hat{h}_1^*)) \right| \leq C|h_1|^{1/d_1} + |U_1(E(h_1, \hat{h}_1)) - U_1(E(0, \hat{h}_1^*))| \to 0,
\]
as $(h_1, \hat{h}_1) \to (0, \hat{h}_1^*)$, here we used assumption (2.6) for $U_1$.

To conclude Step 2, we show the continuity of $\partial_{h_k} E$ for all $2 \leq k \leq p$. Write by the chain rule
\[
\frac{\partial}{\partial h_k} E(h) = F_{U_1}(h_1, e_{ap}^{h_2 U_2} \cdots (x)) \cdots F_{U_{k-1}}(h_{k-1}, e_{ap}^{h_k U_k} \cdots (x)) \frac{\partial}{\partial h_k} e_{ap}^{h_k U_k} \cdots (x).
\]
This ends the proof, because the right-hand side depends continuously on $h_1, \ldots, h_p$, by Lemma 3.10 and the first part of Step 2.

Step 3. We show expansion (3.21) and estimates (3.22) and (3.23) for any $p$ and for all $k = 1, \ldots, p$.

Let $U_k = Y_{k}$, $d_k := \ell_k$ and $E_{(j,k)}(x) := e_{ap}^{h_j U_j} \cdots e_{ap}^{h_k U_k}(x)$ for all $1 \leq j \leq k \leq p$. We agree that $E_{(j,j-1)}$ denotes the identity function. Observe that the function $z \mapsto E_{(j,k)}(z)$ is a $C^1$ diffeomorphism for any fixed $h_j, h_{j+1}, \ldots, h_k$. Then, for $k \in \{1, \ldots, p\}$, we may use (3.21) and we get
\[
E_s(\partial_{h_k}) = U_k E_{(1,k-1)}(E_{(k,p)}(x)) + \sum_{\ell_j = d_k + 1}^s \alpha_j(h_k)Y_j E_{(1,k-1)}(E_{(k,p)}(x)) + |h_k|^{(s+1-d_k)/d_k} \sum_{i=1}^q b_i Y_i E_{(1,k-1)}(E_{(k,p)}(x)),
\]
where $b_i$ denote bounded functions and $|\alpha_j(h_k)| \leq C|h_k|^{(\ell_j-d_k)/d_k}$.
To get formula (3.21), it suffices to use a rough expansion of each term as follows. Write for $\lambda \in \{1, \ldots, p\}$ and $h_\lambda > 0$, $e_{ap}^{h_\lambda U_\lambda} = e^{-h_\lambda^{1/\ell_\lambda} Z_1} \ldots e^{-h_\lambda^{1/\ell_\lambda} Z_p}$, for suitable $Z_i \in \pm \mathcal{H}$. Then for all $j \in \{1, \ldots, q\}$ write

$$Y_j(\psi e_{ap}^{h_\lambda U_\lambda}(z)) = Y_j(\psi e^{-h_\lambda^{1/\ell_\lambda} Z_1} \ldots e^{-h_\lambda^{1/\ell_\lambda} Z_p}(z))$$

$$= Y_j(\psi e_{ap}^{h_\lambda U_\lambda} z) + \sum_{s=0}^{s-\ell_j} \text{ad}_{Z_s}^{\alpha_s} \ldots \text{ad}_{Z_1}^{\alpha_1} Y_j(\psi e_{ap}^{h_\lambda U_\lambda} z) \frac{h_\lambda^{1/\ell_\lambda}}{\alpha!}$$

$$+ O_{s+1}(|h_\lambda|^{(s+1-\ell_j)/\ell_\lambda}, \psi, e_{ap}^{h_\lambda U_\lambda} z)$$

$$= Y_j(\psi e_{ap}^{h_\lambda U_\lambda} z) + \sum_{s=0}^{s-\ell_j} c_s |h_\lambda|^{(s+1-\ell_j)/\ell_\lambda} Y_j(\psi e_{ap}^{h_\lambda U_\lambda} x)$$

$$+ |h_\lambda|^{(s+1-\ell_j)/\ell_\lambda} \sum_{i=1}^{q} b_i Y_i(\psi e_{ap}^{h_\lambda U_\lambda} x),$$

where we use the pointwise form of the remainder, see the proof of Theorem 3.8. Here $c_i$ are constants, while $b_i$ are bounded functions. The proof of (3.21) follows from (3.27) via a repeated application of this expansion. If $h_\lambda < 0$, then the terms $c_i$ and $b_i$ may change, but the argument gives the same conclusion. The proof of the theorem is concluded \( \square \)

**Remark 3.12.**

(i) Let $X_w$ be a commutator of length $|w| \leq s$. Define the function $H(t, x) := \frac{d}{dt} e^{t X_w}(x)$. Under our assumptions $A_s$ we may claim that $H(t, x)$ exists for all $(t, x)$. However, we cannot expect that the function $(t, x) \mapsto H(t, x)$ is continuous in $(-t_0, t_0) \times \Omega$. Indeed, in order to show the continuity of $H$ at a point $(0, \bar{x})$, because

$$|H(t, x) - H(0, \bar{x})| \leq |H(t, x) - H(0, x)| + |H(0, x) - H(0, \bar{x})|.$$
Proof. Let \( x_0 \in \mathbb{R}^n \) and let \( \mathcal{O} := \mathcal{O}_{\mathcal{H}}^{x_0} \) be its \( \mathcal{H} \)-orbit. We know from Remark 3.3 that \( \dim P_x = \dim P_{x_0} =: p \) is constant in \( \mathcal{O} \). For each \( x \in \mathcal{O} \) choose \( I \in \mathcal{I}(p,q) \) such that \( |Y_I(x)| \neq 0 \). By Theorem 3.11 and by the implicit function theorem, we may claim that for a suitable \( O_{I,x} \subset \mathbb{R}^p \), open neighborhood of the origin, the map \( E_{I,x} : O_{I,x} \to \mathbb{R}^n \) is a \( C^1 \) full-rank map which parametrizes a \( C^1 \) smooth, \( p \)-dimensional embedded submanifold \( E_{I,x}(O_{I,x}) \subset \mathbb{R}^n \). Note also that \( E_{I,x}(O_{I,x}) \subset \mathcal{O} \) and, by Theorem 3.11, \( T_{E_{I,x}(h)}E_{I,x}(O_{I,x}) = P_{E_{I,x}(h)} \), for all \( h \in O_{I,x} \). Let \( \mathcal{U} := \{ E_{I,x}(O) : x \in \mathcal{O}, I \in \mathcal{I}(p,q), |Y_I(x)| \neq 0 \) and \( O \subset O_{I,x} \) is a open neighborhood of the origin\}. 

We claim that the family \( \mathcal{U} \) can be used as a base for a topology \( \tau(\mathcal{U}) \) on \( \mathcal{O} \). To see that, we need to show that if the intersection of the \( p \)-dimensional submanifolds \( E_{I,x}(O) \) and \( E_{I',x'}(O') \) is nonempty, then it contains a small manifold of the form \( E_{I'',x''}(O'') \), if \( O'' \) is a sufficiently small neighborhood of the origin. Let \( \Sigma := E_{I,x}(O) \) and \( \Sigma' = E_{I',x'}(O') \) and let \( x'' \in \Sigma \cap \Sigma' \). Recall that both \( \Sigma \) and \( \Sigma' \) are embedded \( C^1 \) submanifolds of \( \mathbb{R}^n \). Let \( I' \in \mathcal{I}(p,q) \) be such that \( |Y_{I'}(x'')| \neq 0 \). Let \( O'' \subset \mathbb{R}^p \) be a small open neighborhood of the origin. For any \( h \in O'', \) the point \( E_{I',x''}(h) \) can be written as \( e^{t_1\mathcal{Z}_1} \cdots e^{t_n\mathcal{Z}_n} x \) where \( Z_j \in \mathcal{H} \) and \( \sum |Z_j| \leq C \|h\|_I \). By a repeated application of Bony’s theorem, it follows that \( E(h) \in \Sigma \), provided that \( h \) is sufficiently close to the origin. The same argument applies to \( \Sigma' \). Thus we have proved that \( \mathcal{U} \) can be used as a topology base.

A similar argument shows that any submanifold of the form \( E_{I,x}(O) \in \mathcal{U} \) contains a small ball \( B_d(x,\sigma) \). Therefore \( \tau_d \) is stronger than \( \tau(\mathcal{U}) \). The fact that \( \tau_d \) is stronger than \( \tau \) follows easily from estimate \( d(E_{I,x}(h),x) \leq C \|h\|_I \). Finally, since all paths of the form \( t \mapsto e^{t\mathcal{Z}} x \in (\mathcal{O},\tau(\mathcal{U})) \) are continuous, the orbit is connected.

The \( C^1 \) differential structure on \( \mathcal{O} \) is given by the family maps \( E_{I,x} \mid _{\mathcal{O}} \) where \( x \in \mathcal{O}, I \in \mathcal{I}(p,q) \) such that \( |Y_I(x)| \neq 0 \) and \( O \subset O_{I,x} \) is an open neighborhood of the origin.

Example 3.14. Let us consider in \( \mathbb{R}^3 \) the family \( \mathcal{H} = \{ X_1, X_2, X_3 \} \):

\[
X_1 = a(t)\partial_x \quad X_2 = xa(t)\partial_y \quad \text{and} \quad X_3 = \partial_t,
\]

where the function \( a(t) \) satisfies \( a(t) = 1 + t^3 \sin \left( \frac{1}{t} \right) \), if \( 0 < |t| < 1 \), \( a(0) = 0 \), \( a \in C^\infty(\mathbb{R} \setminus \{0\}) \) and \( \inf_{\mathbb{R}} a > 0 \). Note that \( X_j \in C^1_{\text{Euc}}(\mathbb{R}^3) \) and

\[
[X_1, X_2] = a(t)^2 \partial_y, \quad [X_1, X_3] = -ta'(t)\partial_x \quad \text{and} \quad [X_2, X_3] = -t^2a'(t)\partial_y.
\]

If \( 0 < |t| < 1 \), then

\[
\frac{d}{dt}(ta'(t)) = \frac{d}{dt}\left( 3t^3 \sin \frac{1}{t} - t^2 \cos \frac{1}{t} \right) = 9t^2 \sin \frac{1}{t} - 5t \cos \frac{1}{t} - t \sin \frac{1}{t},
\]

is discontinuous at \( t = 0 \). Therefore \( X_{13} \) and \( X_{23} \notin C^1_{\text{Euc}} \) and the \( C^1 \) singular Frobenius theorem does not apply to the family \( \mathcal{P} = \{ X_1, X_2, X_3, [X_1, X_2], [X_1, X_3], [X_2, X_3] \} \).

However, we claim that the family \( \mathcal{H} \) belongs to our class \( \mathcal{A}_2 \). To show this claim, we first prove that \( X_j \in C^{1,1}_{\mathcal{H},\text{loc}} \). To see that, it suffices to show that \( X_{33}X_3a \in C^0_{\mathcal{H}} \). But, if \( 0 < |t| < 1 \), we have

\[
X_3^2X_3a(t) = t\partial_t(ta'(t)) = 9t^3 \sin \frac{1}{t} - 5t^2 \cos \frac{1}{t} - t \sin \frac{1}{t},
\]

(3.28)
which is a continuous function up to \( t = 0 \) (note that, since \( X_3^0 a(0) = 0 \), we have \( X_3^0 X_3^0 a(0) = \lim_{t \to 0} t^{-1} (X_3^0 a(e^{tX_3}(0)) - X_3^0 a(0)) = 0 \). Since \( X_{12}, X_{13} \) and \( X_{23} \in C^0_{\text{Euc}} \), condition (2.6) is fulfilled.

Finally, we have to check the 2-involutivity, i.e. that for all \( i, j, k \) we can write \( \text{ad}_{X_i} X_{jk} = \sum_{|w| \leq 2} b^w X_w \) with \( b^w \) locally bounded. A computation shows that the nonzero terms are the following (we work with \( 0 < |t| < 1 \))

\[
- \text{ad}_{X_i} X_{23} = \text{ad}_{X_2} X_{13} = \frac{1}{2} \text{ad}_{X_3} X_{12} = ta(t)a'(t)\partial_x = \frac{ta'(t)}{a(t)} X_{12} \\
\text{ad}_{X_3} X_{13} = -t\partial_t (ta'(t))\partial_x = \frac{-t\partial_t (ta'(t))}{a(t)} X_1 \\
\text{ad}_{X_3} X_{23} = -xt\partial_t (ta'(t))\partial_x = \frac{-t\partial_t (ta'(t))}{a(t)} X_2.
\]

Since \( \inf_{\mathbb{R}} a > 0 \), one can see with the help of (3.28) that both the coefficients \( ta'(t)/a(t) \) and \(-t\partial_t (ta'(t))/a(t) \) are locally bounded. Thus, hypothesis \( A_2 \) is fulfilled and our main theorem applies.

Note finally that it is very easy to see that there are three orbits of the family \( \mathcal{H} \). Namely, \( O_1 := \{(x,y,t) : t > 0 \} \), \( O_2 = \{t = 0 \} \) and \( O_3 = \{t < 0 \} \) and they are integral manifolds of the distribution generated by the family \( \mathcal{P} \).

**Remark 3.15.** A natural question concerns sharpness of the \( C^1 \) regularity of \( \mathcal{O}_H \). It is reasonable to guess that \( C^1 \) regularity is not sharp. Actually, we do not have any example of vector fields of class \( A_s \) where the integral manifolds \( O_H \) are less than \( C^2 \). However, under our assumptions, maps \( E_{I,x} \) cannot provide more than \( C^1 \) regularity, see Remark 3.12 (ii).

A related issue concerns the regularity of the orbit \( \mathcal{O}_H \) of a generic family of \( C^1 \) (or even Lipschitz-continuous) vector fields which do not satisfy any involutivity assumptions. This would require a careful discussion of a nonsmooth version of Sussmann’s orbit theorem.

We plan to discuss such questions in a future study.

### A. Appendix

Here we prove the multilinear algebra lemma which has been used in the proof of Lemma 3.1. The same formula is proved by [Str11] Lemma 3.6], but here we exploit a slightly different argument, which does not rely on the formalism of Lie derivatives.

**Lemma A.1** (Linear algebra). Let \( p \leq n \) and let \( U_1, \ldots, U_p \) be constant vector fields in \( \mathbb{R}^n \). Let \( Z = \sum_{\alpha = 1}^{\ell - 1} \sum_{\beta = 1}^{n} f^\beta \partial_\beta \in C^1_{\text{Euc}} \). Then, for any \( (k_1, \ldots, k_p) \in \mathcal{I}(p,n) \),

\[
\sum_{\alpha = 1}^{\ell - 1} \sum_{\beta = 1}^{n} f^\beta \partial_\beta \left( U_1, \ldots, U_{\alpha - 1}, \sum_{\beta = 1}^{n} U_\alpha f^\beta \partial_\beta, U_{\alpha + 1}, \ldots, U_p \right)
= \sum_{\gamma = 1}^{\ell - 1} \sum_{\beta = 1}^{n} \partial_\gamma f^\beta \partial_\beta \left( U_1, \ldots, U_{\beta - 1}, U_{\beta + 1}, \ldots, U_p \right) \quad \text{(A.1)}
\]

Note that in the particular case \( p = n \), the right-hand side is \( \text{div}(f) \det[U_1, \ldots, U_n] \).
Proof. Recall first that if we are given $(V^\beta_\alpha)_{\alpha,\beta} \in \mathbb{R}^{p \times p}$, then the matrix $(\text{cof } V^\beta_\alpha) = \det[V_1, \ldots, V_{\alpha-1}, \partial_\beta, V_{\alpha+1}, \ldots]$. Therefore the left-hand side of (A.1) takes the form

$$\sum_{\mu=1}^{p} V^\sigma_\mu (\text{cof } V)^\mu_\sigma = (\det V) \delta_{\sigma \rho}$$

(A.2)

To prove the lemma, observe first that $dx^k_\beta (\partial_\beta) = 0$ if $\mu \in \{1, \ldots, p\}$ and $\beta \notin \{k_1, \ldots, k_p\}$. Therefore the left-hand side of (A.1) takes the form

$$\sum_{\alpha=1}^{p} dx^k_1 \wedge dx^k_\beta \left( U_1, \ldots, U_{\alpha-1}, \sum_{\beta=1}^{p} U_\alpha f^{k \beta} \partial_k, U_{\alpha+1}, \ldots, U_p \right)$$

$$= \sum_{\alpha, \beta=1, \ldots, p} U^\gamma_\alpha \partial_\gamma f^{k \beta} dx^k_1 \wedge dx^k_\beta \left( U_1, \ldots, U_{\alpha-1}, \partial_k, U_{\alpha+1}, \ldots, U_p \right)$$

$$= \sum_{\beta=1, \ldots, p} \partial_\gamma f^{k \beta} \sum_{\alpha=1}^{p} U^\gamma_\alpha \text{ cof} \begin{bmatrix} U_1^1 & \cdots & U_p^1 \\ \vdots & \ddots & \vdots \\ U_1^p & \cdots & U_p^p \end{bmatrix}_{\beta} \sum_{\gamma=1, \ldots, n} \partial_\gamma f^{k \beta} \delta_{\gamma \beta}$$

$$= \sum_{\beta=1, \ldots, p} \partial_\gamma f^{k \beta} \sum_{\gamma=1, \ldots, n} \text{ cof} \begin{bmatrix} U_1^1 & \cdots & U_p^1 \\ \vdots & \ddots & \vdots \\ U_1^p & \cdots & U_p^p \end{bmatrix}_{\beta} \sum_{\gamma=1, \ldots, n} \partial_\gamma f^{k \beta} \delta_{\gamma \beta}$$

as desired. \qed

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