ON THE CONSTRUCTION OF ODD SUN SYSTEMS

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Abstract. A $k$-cycle with a pendant edge attached to each vertex is called a $k$-sun. The existence problem for $k$-sun decompositions of $K_v$, with $k$ odd, has been solved only when $k = 3$ or 5.

In this paper, we reduce this problem to the orders $v$ in the range $2k < v < 6k$ and satisfying the obvious necessary conditions. Furthermore, we give a complete solution whenever $k$ is an odd prime.

1. Introduction

We denote by $V(\Gamma)$ and $E(\Gamma)$ the set of vertices and the list of edges of a graph $\Gamma$, respectively. Also, we denote by $\Gamma + w$ the graph obtained by adding to $\Gamma$ an independent set $W = \{\infty_i \mid 1 \leq i \leq w\}$ of $w \geq 0$ vertices each adjacent to every vertex of $\Gamma$, namely,

$$\Gamma + w := \Gamma \cup K_{V(\Gamma),W},$$

where $K_{V(\Gamma),W}$ is the complete bipartite graph with parts $V(\Gamma)$ and $W$. Denoting by $K_v$ the complete graph of order $v$, it is clear that $K_v + 1$ is isomorphic to $K_{v+1}$.

We denote by $x_1 \sim x_2 \sim \ldots \sim x_k$ the path with edges $\{x_i, x_{i+1}\}$ for $2 \leq i \leq k$. By adding the edge $\{x_1, x_k\}$ when $k \geq 3$, we obtain a cycle of length $k$ (briefly, a $k$-cycle) denoted by $(x_1, x_2, \ldots, x_k)$. A $k$-cycle with further $v - k \geq 0$ isolated vertices will be referred to as a $k$-cycle of order $v$. By adding to $(x_1, x_2, \ldots, x_k)$ an independent set of edges $\{\{x_i, x'_i\} \mid 1 \leq i \leq k\}$, we obtain the $k$-sun on $2k$ vertices (sometimes referred to as $k$-crown graph) denoted by

$$(x_1 \ x_2 \ \ldots \ x_{k-1} \ x_k \ \ x'_1 \ x'_2 \ \ldots \ x'_{k-1} \ x'_k),$$

whose edge-set is therefore $\{\{x_i, x_{i+1}\}, \{x_i, x'_i\} \mid 1 \leq i \leq k\}$, where $x_{k+1} = x_1$.

A decomposition of a graph $K$ is a set $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_t\}$ of subgraphs of $K$ whose edge-sets between them partition the edge-set of $K$; in this case, we briefly write $K = \oplus_{i=1}^t \Gamma_i$. If each $\Gamma_i$ is isomorphic to $\Gamma$, we speak of a $\Gamma$-decomposition of $K$. If $\Gamma$ is a $k$-cycle (resp., $k$-sun), we also speak of a $k$-cycle system (resp., $k$-sun system) of $K$.

In this paper we study the existence problem for $k$-sun systems of $K_v$ ($v > 1$). Clearly, for such a system to exist we must have

$$(*) \quad v \geq 2k \quad \text{and} \quad v(v - 1) \equiv 0 \pmod{4k}.$$ 

As far as we know, this problem has been completely settled only when $k = 3, 5$ [8, 10], $k = 4, 6, 8$ [12], and when $k = 10, 14$ or $2^i \geq 4$ [9]. It is important to notice

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that in [14] it is shown that condition (7) is sufficient whenever \( v \) is large enough with respect to \( k \). These results seem to suggest the following.

**Conjecture 1.** Let \( k \geq 3 \) and \( v > 1 \). There exists a \( k \)-sun system of \( K_v \) if and only if (7) holds.

Our constructions rely on the existence of \( k \)-cycle systems of \( K_v \), a problem that has been completely settled in [1, 4, 5, 11, 13]. More precisely, [4] and [11] reduce the problem to the orders \( v \) in the range \( k \leq v < 3k \), with \( v \) odd. These cases are then solved in [1, 13]. For odd \( k \), an alternative proof based on 1-rotational constructions is given in [5]. Further results on \( k \)-cycle systems of \( K_v \) with an automorphism group acting sharply transitively on all but at most one vertex can be found in [2, 6, 7, 15].

The main results of this paper focus on the case where \( k \) is odd. As in [11], we reduce the existence problem for a \( k \)-sun system of \( K_v \) to the range \( 2k < v < 6k \). More precisely, we show the following.

**Theorem 1.1.** Let \( k \geq 3 \) be an odd integer and \( v > 1 \). Conjecture [7] is true if and only if there exists a \( k \)-sun system of \( K_v \) for all \( v \) satisfying the necessary conditions in (7) with \( 2k < v < 6k \).

In Section 6, we construct \( k \)-sun systems of \( K_v \) for every odd prime \( k \) whenever \( 2k < v < 6k \) and (7) holds. Therefore, as a consequence of Theorem 1.1, we solve the existence problem for \( k \)-sun systems of \( K_v \) whenever \( k \) is an odd prime.

**Theorem 1.2.** For every odd prime \( p \) there exists a \( p \)-sun system of \( K_v \) with \( v > 1 \) if and only if \( v \geq 2p \) and \( v(v-1) \equiv 0 \pmod{4p} \).

Both results rely on the difference methods described in Section 2. These methods are used in Section 3 to construct specific \( k \)-cycle decompositions of some subgraphs of \( K_{2k} + w \), which we then use in Section 4 to build \( k \)-sun systems of \( K_{4k} + n \). This is the last ingredient we need in Section 5 to prove Theorem 1.1.

Difference methods are finally used in Section 6 to construct \( k \)-sun systems of \( K_v \) for every odd prime \( k \) whenever \( 2k < v < 6k \) and (7) holds.

## 2. Preliminaries

Henceforward, \( k \geq 3 \) is an odd integer, and \( \ell = \frac{k-1}{2} \). Also, given two integers \( a \leq b \), we denote by \([a, b]\) the interval containing the integers \( \{a, a+1, \ldots, b\} \). If \( a > b \), then \([a, b]\) is empty.

In our constructions we make extensive use of the method of partial mixed differences which we now recall but limited to the scope of this paper.

Let \( G \) be an abelian group of odd order \( n \) in additive notation, let \( W = \{\infty_u \mid 1 \leq u \leq w\} \), and denote by \( \Gamma \) a graph with vertices in \( V = (G \times [0, m-1]) \cup W \). For any permutation \( f \) of \( V \), we denote by \( f(\Gamma) \) the graph obtained by replacing each vertex of \( \Gamma \), say \( x \), with \( f(x) \). Letting \( \tau_g \), with \( g \in G \), be the permutation of \( V \) fixing each \( \infty_u \in W \) and mapping \( (x, i) \in G \times [0, m-1] \) to \((x + g, i)\), we call \( \tau_g \) the translation by \( g \) and \( \tau_g(\Gamma) \) the related translate of \( \Gamma \).

We denote by \( Orb_G(\Gamma) = \{\tau_g(\Gamma) \mid g \in G\} \) the \( G \)-orbit of \( \Gamma \), that is, the set of all distinct translates of \( \Gamma \), and by \( Dev_G(\Gamma) = \bigcup_{g \in G} \tau_g(\Gamma) \) the graph union of all translates of \( \Gamma \). Further, by \( Stab_G(\Gamma) = \{g \in G \mid \tau_g(\Gamma) = \Gamma\} \) we denote the \( G \)-stabilizer of \( \Gamma \), namely, the set of translations fixing \( \Gamma \). We recall that \( Stab_G(\Gamma) \)
Proof. Let \( g_{\tau} \) to edge of translations preserve differences, we have that \( g \) of \( D \) take \( V \) graphs with vertices in \( h \) hence \( \Delta \) \( ij \) every the spanning subgraph of \( D \) \( m \) when follows:

\[
\Delta_{ij}\Gamma = \{a_{h+1} - a_h \mid x_h = (a_h, i), x_{h+1} = (a_{h+1}, j), 1 \leq h \leq k/s\}
\]

\[
\cup \{a_h - a_{h+1} \mid x_h = (a_h, j), x_{h+1} = (a_{h+1}, i), 1 \leq h \leq k/s\};
\]

(2) if \( \Gamma = \left( \begin{array}{cccc} x_1 & x_2 & \cdots & x_k \\ x'_1 & x'_2 & \cdots & x'_k \end{array} \right) \), then

\[
\Delta_{ij}\Gamma = \Delta_{ij}(x_1, x_2, \ldots, x_k) \cup \{a'_{h} - a_h \mid x_h = (a_h, i), x'_{h} = (a'_h, j), 1 \leq h \leq k/s\}
\]

\[
\cup \{a_h - a'_{h} \mid x_h = (a_h, j), x'_h = (a'_h, i), 1 \leq h \leq k/s\}.
\]

We notice that when \( s = 1 \) we find the classic concept of list of differences. Usually, one speaks of pure or mixed differences according to whether \( i = j \) or not, and when \( m = 1 \) we simply write \( \Delta \). This concept naturally extends to a family \( F \) of graphs with vertices in \( V \) by setting \( \Delta_{ij}F = \bigcup_{\Gamma \in F} \Delta_{ij}\Gamma \). Clearly, \( \Delta_{ij}\Gamma = -\Delta_{ji}\Gamma \), hence \( \Delta_{ij}F = -\Delta_{ji}F \), for every \( i, j \in [0, m - 1] \).

We also need to define the list of neighbours of \( \infty_u \) in \( F \), that is, the multiset \( N_{\infty}(\infty_u) \) of the vertices in \( V \) adjacent to \( \infty_u \) in some graph \( \Gamma \in F \).

Finally, we introduce a special class of subgraphs of \( K_{mn} \). To this purpose, we take \( V(K_{mn}) = G \times [0, m - 1] \). Letting \( D_{ii} \subseteq G \setminus \{0\} \) for every \( 0 \leq i \leq m - 1 \), and \( D_{ij} \subseteq G \) for every \( 0 \leq i < j \leq m - 1 \), we denote by

\[
\{D_{ij} \mid 0 \leq i \leq j \leq m - 1\}
\]

the spanning subgraph of \( K_{mn} \) containing exactly the edges \( \{(g, i), (g + d, j)\} \) for every \( g \in G \), \( d \in D_{ij} \), and \( 0 \leq i \leq j \leq m - 1 \). The reader can easily check that this graph remains unchanged if we replace any set \( D_{ii} \) with \( \pm D_{ii} \).

The following result, standard in the context of difference families, provides us with a method to construct \( \Gamma \)-decompositions for subgraphs of \( K_{mn} + w \).

**Proposition 2.1.** Let \( G \) be an abelian group of odd order \( n \), let \( m \) and \( w \) be non-negative integers, and denote by \( F \) a family of k-cycles (resp., k-suns) with vertices in \( (G \times [0, m - 1]) \cup \{\infty_u \mid u \in \mathbb{Z}_w\} \) satisfying the following conditions:

1. \( \Delta_{ij}F \) has no repeated elements, for every \( 0 \leq i \leq j \leq m \);
2. \( N_{\infty}(\infty_u) = \{(g_{u, i}, i) \mid 0 \leq i < m, g_{u, i} \in G\} \) for every \( 1 \leq u \leq w \).

Then \( \bigcup_{\Gamma \in F} \text{Orb}_G(\Gamma) = \{\tau_g(\Gamma) \mid g \in G, \Gamma \in F\} \) is a k-cycle (resp., k-sun) system of \( \{\Delta_{ij}F \mid 0 \leq i \leq j \leq m - 1\} + w \).

**Proof.** Let \( F^* = \bigcup_{\Gamma \in F} \text{Orb}_G(\Gamma) \), \( K = \langle \Delta_{ij}F \mid 0 \leq i \leq j \leq m - 1\rangle \), and let \( e \) be an edge of \( K + w \). We are going to show that \( e \) belongs to exactly one graph of \( F^* \).

If \( e \in E(K) \), by recalling the definition of \( K \) we have that \( e = \{(g, i), (g + d, j)\} \) for some \( g \in G \) and \( d \in \Delta_{ij}F \), with \( 0 \leq i \leq j < m \). Hence, there is a graph \( \Gamma \in F \) such that \( d \in \Delta_{ij}\Gamma \). This means that \( \Gamma \) contains the edge \( e' = \{(g', i), (g' + d, j)\} \) for some \( g' \in G \), therefore \( e = \tau_{g-g'}(e') \in \tau_{g-g'}(\Gamma) \in F^* \). To prove that \( e \) only belongs to \( \tau_{g-g'}(\Gamma) \), let \( \Gamma' \) be any graph in \( F \) such that \( e \in \tau_{g-g'}(\Gamma'), \) for some \( x \in G \). Since translations preserve differences, we have that \( d \in \Delta_{ij}\tau_x(\Gamma') = \Delta_{ij}\Gamma' \). Considering that \( d \in \Delta_{ij}\Gamma \cap \Delta_{ij}\Gamma' \) and, by assumption, \( \Delta_{ij}F \) has no repeated elements, we
necessarily have that $\Gamma'$ has no repeated elements, and considering that $e'$ and $\tau_{-x}(\epsilon)$ are edges of $\Gamma$ that yield the same differences, then $\tau_{-x}(\epsilon) = e' = \tau_{g-x}(\epsilon)$, that is, $\tau_{g-x}(\epsilon) = \epsilon$. Since $G$ has odd order, it has no element of order 2, hence $g' - g + x = 0$, that is, $x = g - g'$, therefore $\tau_{g-g'}(\Gamma)$ is the only graph of $F^*$ containing $\epsilon$.

Similarly, we show that every edge of $(K + w) \setminus K$ belongs to exactly one graph of $F^*$. Let $\epsilon = \{\infty_u, (g, i)\}$ for some $u \in \mathbb{Z}_w$ and $(g, i) \in G \times [0, m - 1]$. By assumption, there is a graph $\Gamma \in F^*$ containing the edge $e' = \{\infty_u, (g_u, i)\}$ with $g_u, i \in G$. Hence, $\epsilon = \tau_{g-g_u}(e') \in \tau_{g-g_u}(\Gamma)$. Finally, if $\epsilon \in \tau_x(\Gamma')$ for some $x \in G$ and $\Gamma' \in F$, then $\{\infty_u, (g - x, i)\} = \tau_{-x}(\epsilon) \in \Gamma'$. Since by assumption $N_F(\infty_u)$ contains exactly one pair from $G \times \{i\}$, we necessarily have that $\Gamma = \Gamma'$ and $x = g - g_u, i$. Therefore, there is exactly one graph of $F^*$ containing $\epsilon$ and this completes the proof.

Considering that $K_{mn} = \langle D_{ij} \mid 0 \leq i \leq j \leq m - 1 \rangle$ if and only if $\pm D_{ij} = G \setminus \{0\}$ for every $i \in [0, m - 1]$, and $D_{ij} = G$ for every $0 \leq i < j \leq m - 1$, the proof of the following corollary to Proposition 2.2 is straightforward.

**Corollary 2.2.** Let $G$ be an abelian group of odd order $n$, let $m$ and $w$ be non-negative integers, and denote by $F$ a family of $k$-cycles (resp., $k$-suns) with vertices in $(G \times [0, m - 1]) \cup \{\infty_u \mid u \in \mathbb{Z}_w\}$ satisfying the following conditions:

1. $\Delta_{ij} F = \begin{cases} G \setminus \{0\} & \text{if } 0 \leq i = j \leq m - 1; \\ G & \text{if } 0 \leq i < j \leq m - 1; \end{cases}$

2. $N_F(\infty_u) = \{(g_u, i) \mid 0 \leq i < m, g_u, i \in G\}$ for every $1 \leq u \leq w$.

Then $\bigcup_{\Gamma \in F} \text{Orb}_G(\Gamma)$ is a $k$-cycle (resp., $k$-sun) system of $K_{mn} + w$.

3. **Constructing $k$-cycle systems of $\langle D_{00}, D_{01}, D_{11} \rangle + w$**

In this section, we recall and generalize some results from [11] in order to provide conditions on $D_{00}, D_{01}, D_{11} \subseteq \mathbb{Z}_k$ that guarantee the existence of a $k$-cycle system for the subgraph $\langle D_{00}, D_{01}, D_{11} \rangle + w$ of $K_{2k} + w$, where $V(K_{2k}) = \mathbb{Z}_k \times \{0, 1\}$.

We recall that every connected 4-regular Cayley graph over an abelian group has a Hamilton cycle system [3] and show the following.

**Lemma 3.1.** Let $[a, b], [c, d] \subseteq [1, \ell]$. The graph $\langle [a, b], [c, d]\rangle$ has a $k$-cycle system whenever both $[a, b]$ and $[c, d]$ satisfy the following condition: the interval has even size or contains an integer coprime with $k$.

**Proof.** The graph $\langle [a, b], [c, d]\rangle$ decomposes into $\langle [a, b], [\varnothing, \varnothing]\rangle$ and $\langle [\varnothing, \varnothing], [c, d]\rangle$. The first one is the Cayley graph $\Gamma = \text{Cay}(\mathbb{Z}_k, [a, b])$ with further $k$ isolated vertices, while the second one is isomorphic to $\langle [c, d], [\varnothing, \varnothing]\rangle$. Therefore, it is enough to show that $\Gamma$ has a $k$-cycle system.

Note that $\Gamma$ decomposes into the subgraphs $\text{Cay}(\mathbb{Z}_k, D_i)$, for $0 \leq i \leq t$, whenever the sets $D_i$ between them partition $[a, b]$. By assumption, $[a, b]$ has even size or contains an integer coprime with $k$. Therefore, we can assume that for every $i > 0$ the set $D_i$ is a pair of integers at distance 1 or 2, and $D_0$ is either empty or contains exactly one integer coprime with $k$. Clearly, $\text{Cay}(\mathbb{Z}_k, D_0)$ is either the empty graph or a $k$-cycle, and the remaining $\text{Cay}(\mathbb{Z}_k, D_i)$ are 4-regular Cayley graphs. Also, for every $i > 0$ we have that $D_i$ is a generating set of $\mathbb{Z}_k$ (since $k$ is odd and $D_i$ contains integers at distance 1 or 2), hence the graph $\text{Cay}(\mathbb{Z}_k, D_i)$ is connected. It
follows that each $\text{Cay}(Z_k, D_i)$, with $i > 0$, decomposes into two $k$-cycles, thus the assertion is proven. \hfill \Box

**Lemma 3.2.** There exists a $k$-cycle system of the graphs $\langle \{\ell\}, S \cup (S+1), \emptyset \rangle$ and $\langle \{\ell\}, (S+1) \cup (S+2), \emptyset \rangle$ whenever $S \subseteq \{2i-1 \mid 1 \leq i \leq \ell\}$.

**Proof.** The existence of a $k$-cycle system of $\Gamma = \langle \{\ell\}, S \cup (S+1), \emptyset \rangle$ has been proven in [111 Lemma 3] when $S \subseteq \{2i-1 \mid 1 \leq i \leq \ell\}$. Consider now the permutation $f$ of $\mathbb{Z}_k \times \{0,1\}$ fixing $\mathbb{Z}_k \times \{0\}$ pointwise, and mapping $(i,1)$ to $(i+1,1)$ for every $i \in \mathbb{Z}_k$. It is not difficult to check that $f(\Gamma) = \langle \{\ell\}, (S+1) \cup (S+2), \emptyset \rangle$ which is therefore isomorphic to $\Gamma$, and hence it has a $k$-cycle system. \hfill \Box

**Lemma 3.3.** Let $r, s$ and $s'$ be integers such that $1 \leq s \leq s' \leq \min\{s+1, \ell\}$, and $0 < r \neq s + s' \pmod{2}$. Also, let $D \subseteq [0,k-1]$ be a non-empty interval of size $k - (s + s' + 2r)$. Then there is a cycle $C = (x_1, x_2, \ldots, x_k)$ of $\Gamma = \langle [1+\epsilon, s + \epsilon], [1+\epsilon, s' + \epsilon] + r, [1+\epsilon, s' + \epsilon] \rangle$, for every $\epsilon \in \{0,1\}$, such that $\text{Orb}(C)$ is a $k$-cycle system of $\Gamma$. Furthermore, if $u = 0$ or $u = 1 - \epsilon = 1 \leq s - 1$, then

1. $\text{Dev}(\{x_{2u}, x_{3-u}\})$ is a $k$-cycle with vertices in $\mathbb{Z}_k \times \{0\}$;
2. $\text{Dev}(\{x_{4u}, x_{5+u}\})$ is a $k$-cycle with vertices in $\mathbb{Z}_k \times \{1\}$.

**Proof.** Set $t = k - (s + s' + 2r)$ and let $\Omega = ([1+\epsilon, s + \epsilon], [0,t-1], [1+\epsilon, s' + \epsilon] + r)$. For $i \in [0,s+s'+1]$ and $j \in [0,t+r-1]$, let $a_i$ and $b_j$ be the elements of $\mathbb{Z}_k \times \{0,1\}$ defined as follows:

$$a_i = \begin{cases} \left( -\frac{i}{2}, 0 \right) & \text{if } i \in [0,s] \text{ is even}, \\ \left( -s - \epsilon + \frac{i}{2}, 0 \right) & \text{if } i \in [1,s] \text{ is odd}, \\ a_{s+1-i} + (0,1) & \text{if } i \in [s+1,2s+1], \\ (-s' - \epsilon, 1) & \text{if } i = s + s' + 1 > 2s + 1, \end{cases}$$

$$b_j = \begin{cases} \left( \frac{j}{2}, 0 \right) & \text{if } j \in [0,t+r-2] \text{ is even}, \\ \left( t - \frac{j}{2}, 1 \right) & \text{if } j \in [1,t-1] \text{ is odd}, \\ \left( t + \left\lfloor \frac{t-j}{2} \right\rfloor, 1 \right) & \text{if } j \in [t,t+r-2] \text{ is odd}, \\ a_{s+s'+1} & \text{if } j = t + r - 1. \end{cases}$$

Since the elements $a_i$ and $b_j$ are pairwise distinct, except for $a_0 = b_0$ and $a_{s+s'+1} = b_{t+r-1}$, then the union $F$ of the following two paths is a $k$-cycle:

$$P = a_0 \sim a_1 \sim \ldots \sim a_{s+s'+1},$$
$$Q = b_0 \sim b_1 \sim \ldots \sim b_{t-1} \sim \infty_1 \sim b_t \sim \infty_2 \sim b_{t+1} \sim \ldots \sim \infty_r \sim b_{t+r-1}.$$ 

Since $\Delta_{0j}F = \Delta_{ij}P \cup \Delta_{ij}Q$, for $i, j \in \{0,1\}$, where

$$\Delta_{00}P = \pm[1+\epsilon, s + \epsilon], \quad \Delta_{01}P = \{0\}, \quad \Delta_{11}P = \pm[1+\epsilon, s' + \epsilon],$$
$$\Delta_{00}Q = \emptyset, \quad \Delta_{01}Q = [1, t-1], \quad \Delta_{11}Q = \emptyset,$$

and considering that $\text{N}_F(\infty_h) = \text{N}_Q(\infty_h) = \{b_{t+h-2}, b_{t+h-1}\}$ for every $h \in [1,r]$, Proposition 2.1 guarantees that $\text{Orb}(F)$ is a $k$-cycle system of $\Omega$. Furthermore, if $u = 0$ or $u = 1 - \epsilon = 1 \leq s - 1$, then

$$\pm(a_{s-u} - a_{s-1}) = \pm(a_{s+u+2} - a_{s+u+1}) = \pm(u + \epsilon + 1,0).$$

Since $k$ is odd, we have that $\text{Dev}(\{a_{s-u-1}, a_{s-u}\})$ and $\text{Dev}(\{a_{s+u+1}, a_{s+u+2}\})$ are $k$-cycles with vertices in $\mathbb{Z}_k \times \{0\}$ and $\mathbb{Z}_k \times \{1\}$, respectively.
If $D = [g, g + t - 1]$ is any interval of $[0, k - 1]$ of size $t$, and $f$ is the permutation of $\mathbb{Z}_k \times \{0, 1\}$ fixing $\mathbb{Z}_k \times \{0\}$ pointwise, and mapping $(i, 1)$ to $(i + g, 1)$ for every $i \in \mathbb{Z}_k$, one can check that $C = f(F)$ is the desired $k$-cycle of $\Gamma = f(\Omega)$. □

**Lemma 3.4.**

1. Let $\ell$ be odd. If $\Gamma$ is a 1-factor of $K_{2k}$, then $\Gamma + \ell$ decomposes into $k$ cycles of length $k$, each of which contains exactly one edge of $\Gamma$. Furthermore, if $\Gamma = \langle \emptyset, \{d\}, \emptyset \rangle$, then there exists a $k$-cycle $C = (c_1, c_2, \ldots, c_k)$ of $\Gamma + \ell$, with $c_1 \in \mathbb{Z}_k \times \{0\}$ and $c_2 \in \mathbb{Z}_k \times \{1\}$, such that $\text{Dev}(\{c_1, c_2\}) = \Gamma$ and $\text{Orb}(C)$ is a $k$-cycle system of $\Gamma + \ell$.

2. Let $\ell$ be even. If $\Gamma$ is a $k$-cycle of order $2k$, then $\Gamma + \ell$ decomposes into $k$ cycles of length $k$, each of which contains exactly one edge of $\Gamma$. Furthermore, if $\Gamma = \langle \{d\}, \emptyset, \emptyset \rangle$ and $d$ is coprime with $k$, then there exists a $k$-cycle $C = (c_1, c_2, \ldots, c_k)$ of $\Gamma + \ell$, with $c_1, c_2 \in \mathbb{Z}_k \times \{0\}$, such that $\text{Dev}(\{c_1, c_2\})$ is the $k$-cycle of $\Gamma$ and $\text{Orb}(C)$ is a $k$-cycle system of $\Gamma + \ell$.

**Proof.** Permuting the vertices of $K_{2k}$ if necessary, we can assume that $\Gamma$ is the 1-factor $\Gamma_0 = \langle \emptyset, \{d\}, \emptyset \rangle$ when $\ell$ is odd, and the $k$-cycle $\Gamma_1 = \langle \{d\}, \emptyset, \emptyset \rangle$ (of order $2k$) when $\ell$ is even. For $h \in \{0, 1\}$, let $C_h = (c_{h,1}, c_{h,2}, \infty_1, c_3, \infty_2, c_4, \ldots, \infty_{\ell-1}, c_{\ell+1}, \infty_{\ell})$ be the $k$-cycle of $\Gamma_h + \ell$, where

$$c_{h,1} = (0, 1-h), \quad c_{h,2} = (h, \ell, 0), \quad \text{and} \quad c_j = \begin{cases} (\frac{j-1}{\ell}, 1) & \text{if } j \in [3, \ell+1] \text{ is odd}, \\ (\frac{j}{\ell}, 0) & \text{if } j \in [4, \ell+1] \text{ is even}. \end{cases}$$

Note that the sets $\Delta_h C_h$ are empty, except for $\Delta_0 C_0 = \{0\}$ and $\Delta_0 C_1 = \{\pm \ell\}$. Also, the two neighbours of $\infty_u$ in $C_h$ belong to $\mathbb{Z}_k \times \{0\}$ and $\mathbb{Z}_k \times \{1\}$, respectively. Hence, Proposition 2.1 guarantees that $\text{Orb}(C_h)$ is a $k$-cycle system of $\Gamma_h + \ell$, for $h \in \{0, 1\}$. We finally notice that $\text{Dev}(\{c_{h,1}, c_{h,2}\}) = \Gamma_h$ (up to isolated vertices) and this completes the proof. □

The following result has been proven in [11].

**Lemma 3.5.** Let $D \subseteq [1, \ell]$. The subgraph $\langle D, \{0\}, D \rangle$ of $K_{2k}$ has a 1-factorization.

**Remark 3.6.** Considering the permutation $f$ of $\mathbb{Z}_k \times \{0, 1\}$ such that $f(i, j) = (i, 1 - j)$, and a graph $\Gamma = \langle D_0, D_1, D_2 \rangle$, we have that $f(\Gamma) = \langle D_2, -D_1, D_0 \rangle$. Therefore, all the above lemmas continue to hold when we replace $\Gamma$ by $f(\Gamma)$.

4. $k$-sun systems of $K_{4k} + n$

In this section we provide sufficient conditions for a $k$-sun system of $K_{4k} + n$ to exist, when $n \equiv 0, 1 \pmod{4}$. More precisely, we show the following.

**Theorem 4.1.** Let $k \geq 7$ be an odd integer and let $n = 0, 1 \pmod{4}$ with $2k < n < 10k$, then there exists a $k$-sun system of $K_{4k} + n$, except possibly when

- $k = 7$ and $n = 20, 21, 32, 33, 44, 45, 56, 57, 64, 65, 68, 69$,
- $k = 11$ and $n = 100, 101, 112, 113$.

To prove Theorem 4.1, we start by introducing some notions and prove some preliminary results. Let $M$ be a positive integer and take $V(K_{2^2} M) = \mathbb{Z}_2 M \times [0, 2^2 - 1]$ and $V(K_{2^2} M + w) = V(K_{2^2} M) \cup \{\infty_h \mid h \in \mathbb{Z}_w\}$, for $i \in \{1, 2\}$ and $w > 0$. 
Now assume that $w = 2u$, and let $x \mapsto \bar{x}$ be the permutation of $V(K_{4M} + 2u)$ defined as follows:

$$
\bar{x} = \begin{cases} 
(a, 2 - j) & \text{if } x = (a, j) \in \mathbb{Z}_M \times \{0, 2\}, \\
(a, 4 - j) & \text{if } x = (a, j) \in \mathbb{Z}_M \times \{1, 3\}, \\
\infty_{h + u} & \text{if } x = \infty_h.
\end{cases}
$$

For any subgraph $\Gamma$ of $K_{4M} + 2u$, we denote by $\bar{\Gamma}$ the graph (isomorphic to $\Gamma$) obtained by replacing each vertex $x$ of $\Gamma$ with $\bar{x}$.

Given a subgraph $\Gamma$ of $K_{2M} + u$, we denote by $\Gamma[2]$ the spanning subgraph of $K_{4M} + 2u$ whose edge set is

$$
E(\Gamma[2]) = \{\{x, y\}, \{x, \bar{y}\}, \{\bar{x}, \bar{y}\}, \{x, y\} \in E(\Gamma) \}.
$$

and let $\Gamma^*[2] = \Gamma[2] \oplus I$ be the graph obtained by adding to $\Gamma[2]$ the 1-factor

$$
I = \{\{x, \bar{x}\} \mid x \in \mathbb{Z}_M \times \{0, 1\}\}.
$$

Note that, up to isolated vertices, $\Gamma[2]$ is the lexicographic product of $\Gamma$ with the empty graph on two vertices.

The proof of the following elementary lemma is left to the reader.

**Lemma 4.2.** Let $\Gamma = \bigoplus_{i=1}^n \Gamma_i$ and let $w = \sum_{i=1}^n w_i$ with $w_i \geq 0$. If $\Gamma$ and the $\Gamma_i$s have the same vertex set (possibly with isolated vertices), then

1. $\Gamma + w = \bigoplus_{i=1}^n (\Gamma_i + w_i)$;
2. $\Gamma[2] = \bigoplus_{i=1}^n \Gamma_i[2]$;
3. $(\Gamma + w)[2] = \Gamma[2] + 2w$.

We start showing that if $C$ is a $k$-cycle, then $C[2]$ decomposes into two $k$-suns.

**Lemma 4.3.** Let $C = (c_1, c_2, \ldots, c_k)$ be a cycle with vertices in $(\mathbb{Z}_M \times \{0, 1\}) \cup \{\infty_h \mid h \in \mathbb{Z}_u\}$ and let $S$ be the $k$-sun defined as follows:

$$(1) \quad S = \left( \begin{array}{cccc} 
s_1 & \cdots & s_{k-1} & s_k \\
\tilde{s_2} & \cdots & \tilde{s_{k-1}} & \tilde{s_1} \end{array} \right)$$

where $s_i \in \{c_i, \overline{c_i}\}$ for every $i \in [1, k]$. Then $C[2] = S \oplus \overline{S}$.

**Proof.** It is enough to notice that $S$ contains the edges $\{s_i, s_{i+1}\}$ and $\{s_i, \overline{s_{i+1}}\}$, while $\overline{S}$ contains $\{\overline{s_i}, \overline{s_{i+1}}\}$ and $\{s_i, \overline{s_{i+1}}\}$, for every $i \in [1, k]$, where $s_{k+1} = s_1$ and $\overline{s_{k+1}} = \overline{s_1}$. \hfill \Box

**Example 4.4.** In Figure 1 we have the graph $C_7[2]$ which can be decomposed into two 7-suns $S$ and $\overline{S}$. The non-dashed edges are those of $S$, while the dashed edges are those of $\overline{S}$.

![Figure 1](image-url)
For every cycle $C = (c_1, c_2, \ldots, c_k)$ with vertices in $\mathbb{Z}_M \times \{0, 1\}$, we set

$$\sigma(C) = \left( \frac{c_1}{c_2} \cdots \frac{c_{k-1}}{c_k} \frac{c_k}{c_1} \right).$$

Clearly, $C[2] = \sigma(C) \oplus \overline{\sigma(C)}$ by Lemma 4.3.

**Lemma 4.5.** If $C = \{C_1, C_2, \ldots, C_t\}$ is a $k$-cycle system of $\Gamma + u$, where $\Gamma$ is a subgraph of $K_{2M}$, and $S_i$ is a $k$-sun obtained from $C_i$ as in Lemma 4.3, then $S = \{S_i, S'_i | i \in [1, t]\}$ is a $k$-sun system of $\Gamma[2] + 2u$. In particular, if $C = \text{Orb}(C_1)$, then $\text{Orb}(S_1) \cup \text{Orb}(S'_1)$ is a $k$-sun system of $\Gamma[2] + 2u$.

**Proof.** By assumption $\Gamma + u = \oplus_{i=1}^t C_i$, where each $C_i$ is a $k$-cycle. Also, by Lemma 4.2 we have that $\Gamma[2] + 2u = (\Gamma + u)[2] = \oplus_{i=1}^t C_i[2]$. Since $C_1[2] = S_i \oplus S'_i$ by Lemma 4.3, then $S$ is a $k$-sun system of $\Gamma[2] + 2u$.

The second part easily follows by noticing that if $C_i = \tau_g(C_1)$ for some $g \in \mathbb{Z}_M$, then $C_i[2] = \tau_g(C_1[2]) = \tau_g(S_1) \oplus \tau_g(S'_1)$. □

The following lemma describes the general method we use to construct $k$-sun systems of $K_{4k} + n$. We point out that throughout this section we take $V(K_{2k}) = \mathbb{Z}_k \times \{0, 1\}$ and $V(K_{4k}) = \mathbb{Z}_k \times [0, 3]$.

**Lemma 4.6.** Let $K_{2k} = \Gamma_1 \oplus \Gamma_2$ with $V(\Gamma_1) = V(\Gamma_2) = V(K_{2k})$. If $\Gamma_1 + w_1$ has a $k$-cycle system and $\Gamma_2[2] + w_2$ has a $k$-sun system, then $K_{4k} + (2w_1 + w_2)$ has a $k$-sun system.

**Proof.** The result follows by Lemma 4.2. In fact, noting that $K_{4k} = K_{2k}[2] \oplus I$, where $I = \{\{z, \overline{z}\} | z \in \mathbb{Z}_k \times \{0, 1\}\}$, we have that

$$K_{4k} + (2w_1 + w_2) = (\Gamma_1[2] \oplus (\Gamma_2[2] \oplus I)) + 2w_1 + w_2 =
= (\Gamma_1[2] + 2w_1) \oplus (\Gamma_2[2] + w_2) = (\Gamma_1 + w_1)[2] \oplus (\Gamma_2[2] + w_2).$$

The result then follows by Lemma 4.3. □

We are now ready to prove the main result of this section, Theorem 4.1. The case $k \equiv 1 \pmod{4}$ is proven in Theorem 4.7, while the case $k \equiv 3 \pmod{4}$ is dealt with in Theorems 4.9, 4.10, 4.11, and 4.12.

**Theorem 4.7.** If $k \equiv 1 \pmod{4}$ and $n = 0 \equiv 1 \pmod{4}$ with $2k < n < 10k$, then there exists a $k$-sun system of $K_{4k} + n$.

**Proof.** Let $n = 2(q\ell + r) + \nu$ with $1 \leq r \leq \ell$ and $\nu \in \{2, 3\}$. Note that $\ell \geq 4$ is even and $r$ is odd, since $n = 0 \equiv 1 \pmod{4}$, $9 \geq k \equiv 1 \pmod{4}$. Considering also that $2k < n < 10k$, we have that $2 \leq q \leq 10 \leq k + 2r - 1$. Furthermore, let $V(K_{4k} + n) = (\mathbb{Z}_k \times [0, 3]) \cup \{\infty_h | h \in \mathbb{Z}_{n-\nu}\} \cup \{\infty'_1, \infty'_2, \infty'_3\}$.

We start decomposing $K_{2k}$ into the following two graphs:

$$\Gamma_1 = \langle \{2, \ell\}, [k - 2r - 2, k - 1], [2, \ell - 1]\rangle$$

and $\Gamma_2 = \langle \{1\}, [0, k - 2r - 3], \{1, \ell\}\rangle$.

We notice that $\Gamma_1$ further decomposes into the following graphs:

$$\langle [2, \ell - 1], \emptyset, \emptyset\rangle, \quad \langle \emptyset, \emptyset, [2, \ell - 1]\rangle, \quad \langle [\ell], [k - 2r - 2, k - 1], \emptyset\rangle,$$

each of which decomposes into $k$-cycles by Lemmas 3.1 and 3.2, hence $\Gamma_1$ has a $k$-cycle system $\{C_1, C_2, \ldots, C_\gamma\}$, where $\gamma = k + 2r - 2$. Note that this system is
non-empty, since $1 \leq q - 1 \leq \gamma$. Without loss of generality, we can assume that each cycle $C_i$ has order $2k$ and

$$C_1 \text{ is a subgraph of } \langle [2, \ell - 1], \emptyset, \emptyset \rangle.$$  

Now set $\Omega_1 = \Gamma_1 \setminus C_1$ and $\Omega_2 = \Gamma_2 \oplus C_1$. Letting $w_1 = (q - 2)\ell = \sum_{j=2}^{\gamma} w_{1,j}$, where $w_{1,j} = \ell$ when $j < q$, and $w_{1,j} = 0$ otherwise, by Lemma 4.2 we have that $\Omega_1 + w_1 = \sum_{i=2}^{\gamma} (C_i + w_{1,i})$. Therefore, $\Omega_1 + w_1$ has a $k$-cycle system, since each $C_i + w_{1,i}$ decomposes into $k$-cycles by Lemma 4.4. Setting $w_2 = n - 2w_1 = 2(2\ell + r) + \nu$ and considering that $K_{2k} = \Gamma_1 \oplus \Gamma_2 = \Omega_1 \oplus \Omega_2$, by Lemma 4.6 it is left to show that $\Omega_2^* + w_2$ has a $k$-sun system.

Set $\Gamma_3 = C_1$, and recall that $\Omega_2^*[2] = \Omega_2[2] \oplus I = \Gamma_2[2] \oplus \Gamma_2[2] \oplus I$, where $I$ denotes the 1-factor $\{ \{ z, z \} \mid z \in \mathbb{Z}_k \times \{0, 1\} \}$ of $K_{4k}$. Hence,

$$\Omega_2^*[2] + w_2 = (\Gamma_2 + (\ell + r))[2] \oplus (\Gamma_3 + \ell)[2] \oplus (I + \nu)$$

by Lemma 4.2. Clearly, $\Gamma_2 = \Gamma_{2,1} \oplus \Gamma_{2,2}$ where $\Gamma_{2,1} = \langle \{1\}, [0, k - 2r - 3], \{1\} \rangle$ and $\Gamma_{2,2} = \langle \emptyset, \emptyset, \{ \ell \} \rangle$, hence $\Gamma_2 + (\ell + r) = (\Gamma_{2,1} + r) \oplus (\Gamma_{2,2} + \ell)$. By Lemmas 4.3 and 4.4 there exist a $k$-cycle $A = (x_1, y_1, a_2, y_2, a_3, . . . , y_q, a_1)$ of $\Gamma_{2,1} + r$ and a $k$-cycle $B = (y_1, y_2, b_3, . . . , b_k)$ of $\Gamma_{2,2} + \ell$ satisfying the following properties:

1. $\text{Orb}(A) \cup \text{Orb}(B)$ is a $k$-cycle system of $\Gamma_2 + (\ell + r)$;
2. $\text{Dev}(\{x_1, x_2\})$ is a $k$-cycle with vertices in $\mathbb{Z}_k \times \{0\}$;
3. $\text{Dev}(\{y_1, y_2\})$ and $\text{Dev}(\{a_1, a_2\})$ are $k$-cycles with vertices in $\mathbb{Z}_k \times \{1\}$.

Furthermore, denoted by $(c_1, c_2, \ldots, c_k)$ the cycle in $\Gamma_3$, Lemma 4.4 guarantees that $\Gamma_3 + \ell$ has a $k$-cycle system $\{ F_1, F_2, \ldots, F_k \}$ such that

$$F_j = (c_j, c_{j+1}, f_{j,3}, f_{j,4}, \ldots, f_{j,k}) \text{ for every } j \in [1, k] \text{ (with } c_{k+1} = c_1).$$

Let $S = \{ S_1, S_2, S_3, S_4 \}$ and $S' = \{ S_3+2j, S_4+2j \mid j \in [1, k] \}$, where

$$S_1 = \{x_1, x_2, y_1, y_2, a_3, a_4, \ldots, a_k\}, \quad S_3 = \{y_1, y_2, b_3, b_4, \ldots, b_k\},$$

$$S_{3+2j} = \{c_j, c_{j+1}, f_{j,3}, f_{j,4}, \ldots, f_{j,k}\} \text{ for } j \in [1, k], \text{ and }$$

$$S_{2i} = \overline{S_{2i-1}} \text{ for } i \in [1, k + 2].$$

By Lemma 4.5 we have that $\bigcup_{S \in S} \text{Orb}(S)$ is a $k$-sun system of $(\Gamma_2 + (\ell + r))[2]$, and $S'$ is a $k$-sun system of $(\Gamma_3 + \ell)[2]$. It follows by 3 that $\bigcup_{S \in S} \text{Orb}(S) \cup S'$ decomposes $(\Omega_2^*[2] + w_2) \setminus (I + \nu)$.

To construct a $k$-sun system of $\Omega_2^*[2] + w_2$, we first modify the $k$-suns in $S \cup S'$ by replacing some of their vertices with $\infty_1', \infty_2'$, and possibly $\infty_3'$ when $\nu = 3$. More precisely, following Table 1, we obtain $T_i$ from $S_i$ by replacing the ordered set $V_i$ of vertices of $S_i$ with $V_i'$. This yields a set $M_i$ of `missing' edges no longer covered by $T_i$ after this substitution, but replaced by those in $N_i$, namely

$$E(T_i) = (E(S_i) \setminus M_i) \cup N_i.$$

We point out that $T_{3+2j} = S_{3+2j}$, and $T_{4+2j} = S_{4+2j}$ when $\nu = 2$, for every $j \in [1, k]$. The remaining graphs $T_i$ are explicitly given below, where the elements in bold are the replaced vertices.

$$T_1 = \begin{pmatrix} x_1 & \overline{x_2} & \infty_1' & \infty_2' & y_4 & a_5 & \ldots & a_{k-1} & a_k \\ \overline{x_2} & \overline{y_2} & \omega_5 & \omega_6 & \omega_7 & \ldots & \omega_k & \overline{x_1} \end{pmatrix}.$$
where, we recall,  

\[ T_2 = \begin{cases} 
\begin{pmatrix} x_1 & x_2 & y_3 & y_4 & a_5 & \ldots & a_{k-1} & a_k \\
\infty_1 & \infty_2' & y_4 & a_5 & \ldots & a_k & x_1 \\
\infty_1 & \infty_2' & y_4 & a_5 & \ldots & a_k & x_1 \\
\infty_1 & \infty_2' & y_4 & a_5 & \ldots & a_k & x_1 \\
\end{pmatrix} & \text{if } \nu = 2, \\
\end{cases} \]

finally build the following 2 graphs:

\[ T_3 = \begin{pmatrix} y_1 & y_2 & b_3 & \ldots & b_{k-1} & b_k & y_1 \\
\infty_4 & \infty_4' & b_4 & \ldots & b_k & y_1 \\
\infty_4 & \infty_4' & b_4 & \ldots & b_k & y_1 \\
\infty_4 & \infty_4' & b_4 & \ldots & b_k & y_1 \\
\end{pmatrix}, \quad T_4 = \begin{pmatrix} y_1 & y_2 & b_3 & \ldots & b_{k-1} & b_k \\
\infty_4 & b_4 & \ldots & b_k & y_1 \\
\infty_4 & b_4 & \ldots & b_k & y_1 \\
\infty_4 & b_4 & \ldots & b_k & y_1 \\
\end{pmatrix}, \quad T_{4+2j} = \begin{pmatrix} c_j & c_j+1 & f_{j,3} & \ldots & f_{j,k-1} & f_{j,k} & c_j \\
\infty_3 & \circ_3 & \circ_3 & \ldots & \circ_3 & \circ_3 \\
\infty_3 & \circ_3 & \circ_3 & \ldots & \circ_3 & \circ_3 \\
\infty_3 & \circ_3 & \circ_3 & \ldots & \circ_3 & \circ_3 \\
\end{pmatrix} \quad \text{for every } j \in [1,k]. \]

We notice that \( \bigcup_{i=1}^{4} \text{Dev}(N_i) \cup \bigcup_{i=5}^{2k+4} N_i = \{ \{ \infty_j', 0 \} \mid j \in [1,\nu], x \in \mathbb{Z}_k \times [0,3] \}. \)

By recalling (2) and (4)-(6), it is not difficult to check that \( G_1, G_2, \ldots, G_{2\nu+1} \) are 1-suns. Furthermore,  

\[ E(G_i) \cup \bigcup_{i=1}^{4} \text{Dev}(M_i) \cup \bigcup_{i=5}^{2k+4} M_i \cup E(I), \]

where, we recall, \( I \) denotes the 1-factor \( \{ \{ z, \overline{z} \} \mid z \in \mathbb{Z}_k \times \{0,1\} \} \) of \( K_{4k} \). Therefore, \( \bigcup_{i=1}^{4} \text{orb}(T_i) \cup \{ T_5, T_6, \ldots, T_{2k+4} \} \cup \{ G_1, G_2, \ldots, G_{2\nu+1} \} \) is a k-sun system of \( \Omega_2[2]+w_2 \), and this concludes the proof. \( \square \)

**Example 4.8.** By following the proof of Theorem 1.1, we construct a k-sun system of \( K_{4k} + n \) when \((k,n) = (9,21)\); hence \((\ell,q,r,\nu) = (4,2,1,3)\).

The graphs \( \Gamma_1 = \langle [2,4], [5,8],[2,3] \rangle \) and \( \Gamma_2 = \langle \{1\}, [0,4],[1,4] \rangle \) decompose the complete graph \( K_{18} \) with vertex-set \( \mathbb{Z}_9 \times \{0,1\} \). Also \( \Gamma_1 \) decomposes into the following 9-cycles of order 18, where \( i = 0,1 \):

\[ C_{1+1} = ((0, i), (2, i), (8, i), (1, i), (3, i), (5, i), (7, i), (4, i), (6, i)), \]
\[ C_{3+1} = ((0, i), (3, i), (6, i), (8, i), (5, i), (2, i), (4, i), (1, i), (7, i)), \]
\[ C_{5+1} = ((4i, 0), (8 + 4i, 1), (1 + 4i, 0), (4i, 1), (2 + 4i, 0), (1 + 4i, 1), (3 + 4i, 0), (2 + 4i, 1), (4 + 4i, 0)), \]
\[ C_{7+1} = ((8 + 4i, 0), (5 + 4i, 1), (4i, 0), (6 + 4i, 1), (1 + 4i, 0), (7 + 4i, 1), (2 + 4i, 0), (8 + 4i, 1), (3 + 4i, 0)), \]
\[ C_9 = ((7, 0), (2, 0), (6, 0), (1, 0), (5, 0), (0, 0), (7, 1), (8, 0), (4, 1)). \]

Clearly, \( K_{18} = \Omega_1 \oplus \Omega_2 \), where \( \Omega_1 = \Gamma_1 \setminus C_1 \) and \( \Omega_2 = \Gamma_2 \oplus C_1 \).
Let $V(K_{36}) = \mathbb{Z}_9 \times [0, 3]$, and denote by $I$ the 1-factor of $K_{36}$ containing all edges of the form $\{(a, i), (a, i + 2)\}$, with $a \in \mathbb{Z}_9$ and $i \in \{0, 1\}$. Then,

$$K_{36} = K_{18}[2] \oplus I = \Omega_1[2] \oplus \Omega_2[2] \oplus I.$$  

Considering that $(\Omega_2 + 9)[2] = \Omega_2[2] + 18$, we have

$$K_{36} + 21 = \Omega_1[2] \oplus (\Omega_2[2] + 18) \oplus (I + 3) = \Omega_1[2] \oplus (\Omega_2 + 9)[2] \oplus (I + 3).$$

Since the set $\{\sigma(C_i) \mid i \in [2, 9]\}$ is a 9-sun system of $\Omega_1[2]$, it is left to build a 9-sun system of $\Omega_2[2] + 18 \oplus (I + 3)$.

We start by decomposing $\Omega_2 + 9$ into 9-cycles. Since $\Omega_2 = \Gamma_{2,1} \oplus \Gamma_{2,2} \oplus \Gamma_3$ with $\Gamma_{2,1} = \langle \{1\}, [0, 4], \{1\}\rangle$, $\Gamma_{2,2} = \langle \emptyset, \emptyset, \{4\}\rangle$ and $\Gamma_3 = C_1$, then

$$\Omega_2 + 9 = (\Gamma_{2,1} + 1) \oplus (\Gamma_{2,2} + 4) \oplus (\Gamma_3 + 4).$$

Let $A = (x_1, x_2, y_3, y_4, a_5, \ldots, a_9)$ and $B = (y_1, y_2, b_3, \ldots, b_9)$ be the 9-cycles defined as follows:

$$\begin{align*}
(x_1, x_2, y_3, y_4) &= ((0, 0), (-1, 0), (-1, 1), (0, 1)), \\
(a_5, \ldots, a_9) &= (\infty_1, (2, 0), (3, 1), (1, 0), (4, 1)), \\
(y_1, y_2) &= ((0, 1), (4, 1)), \\
(b_3, \ldots, b_9) &= (\infty_2, (1, 0), \infty_3, (1, 1), \infty_4, (0, 0), \infty_5)
\end{align*}$$

One can easily check that $\text{Orb}(A)$ (resp., $\text{Orb}(B)$) decomposes $\Gamma_{2,1} + 1$ (resp., $\Gamma_{2,2} + 4$). Also, for every edge $\{c_j, c_{j+1}\}$ of $C_1$, with $j \in [1, 9]$ and $c_{10} = c_1$, we construct the cycle $F_j = (c_j, c_{j+1}, f_{j,3}, f_{j,4}, \ldots, f_{j,9})$, where

$$\begin{align*}
(f_{j,3}, f_{j,4}, \ldots, f_{j,9}) &= (\infty_6, (1, 0), \infty_7, (1, 1), \infty_8, (0, 0), \infty_9).
\end{align*}$$

One can check that $\{F_1, F_2, \ldots, F_9\}$ is a 9-cycle system of $\Gamma_3 + 4$. Therefore, $\mathcal{U}_1 = \text{Orb}(A) \cup \text{Orb}(B) \cup \{F_1, F_2, \ldots, F_9\}$ provides a 9-cycle system of $\Omega_2 + 9$.

| $i$ | $V_i \rightarrow V'_i$ | $M_i$ | $N_i$ | $\nu$ |
|-----|----------------------|-------|-------|------|
| 1   | $(x_2, y_3) \rightarrow (\infty_1', \infty_2')$ | $\{x_1, x_2\}, \{x_2, y_3\}$ | $\{\infty_1', x_1\}, \{\infty_2', x_2\}$ | 2,3 |
| 2   | $(x_2, y_3) \rightarrow (\infty_1', \infty_2')$ | $\{y_3, y_4\}, \{y_3, y_4\}$ | $\{\infty_1', y_4\}, \{\infty_2', y_4\}$ | 2 |
| 2   | $(x_2, y_3, y_5) \rightarrow (\infty_1', \infty_2', \infty_3')$ | $\{x_2, y_3\}, \{x_2, y_3\}$ | $\{\infty_1', y_3\}, \{\infty_2', y_3\}, \{\infty_3', y_3\}$ | 3 |
| 3   | $y_2 \rightarrow \infty_1'$ | $\{y_1, y_2\}$ | $\{\infty_1', y_1\}$ | 2,3 |
| 4   | $y_2 \rightarrow \infty_1'$ | $\{y_1, y_2\}$ | $\{\infty_1', y_1\}$ | 2,3 |
| 3+2j | $\emptyset$ | $\emptyset$ | $\emptyset$ | 2 |
| 4+2j | $\emptyset$ | $\emptyset$ | $\emptyset$ | 2 |
| 4+2j | $c_{j+1} \rightarrow \infty_3'$ | $\{c_j, c_{j+1}\}$ | $\{\infty_3', c_j\}$ | 3 |

Table 1. From $S_i$ to $T_i$.  

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**ON THE CONSTRUCTION OF ODD SUN SYSTEMS**

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Since the set \( \{ C[2] \mid C \in \mathcal{U}_1 \} \) decomposes \((\Omega_2 + 9)[2]\), and each \( C[2] \) decomposes into two 9-suns, we can easily obtain a 9-sun system of \((\Omega_2 + 9)[2]\). Indeed, letting

\[
S_1 = \sigma(x_1, y_2, y_3, y_4, a_5, \ldots, a_9), \quad S_3 = \sigma(y_1, y_2, b_3, \ldots, b_9),
\]

\[
S_{3+2j} = \sigma(c_j, c_{j+1}, f_{j,3}, f_{j,4}, \ldots, f_{j,9}) \quad \text{for } j \in [1, 9], \quad \text{and}
\]

\[
S_{2i} = \overline{S_{2i-1}} \quad \text{for } i \in [1, 11],
\]

we have that \( A[2] = S_1 \oplus S_2, B[2] = S_3 \oplus S_4, \) and \( F_j[2] = S_{3+2j} \oplus S_{4+2j}, \) for every \( j \in [1, 9]. \) Therefore \( \mathcal{U}_2 = \bigcup_{i=1}^{4} \text{Orb}(S_i) \cup \{ S_5, S_6, \ldots, S_{22} \} \) is a 9-sun system of \( \Omega_2[2] + 18. \)

We finally use \( \mathcal{U}_2 \) to build a 9-sun system of \( \Omega_2[2] + 21 = (\Omega_2[2] + 18) \oplus (I + 3). \) By replacing the vertices of each \( S_i, \) as outlined in Table 1, we obtain the 9-sun \( T_i. \) The new 22 graphs, \( T_1, T_2, \ldots, T_{22}, \) are built in such a way that

\[
(a) \bigcup_{i=1}^{4} \text{Orb}(T_i) \cup \{ T_5, T_6, \ldots, T_{22} \} \text{ decomposes a subgraph } K \text{ of } \Omega_2[2] + 21;\]

\[
(b) (\Omega_2[2] + 21) \setminus K \text{ decomposes into seven 9-suns } G_1, G_2, \ldots, G_7.
\]

This way we obtain a 9-sun system of \( \Omega_2[2] + 21, \) and hence the desired 9-sun system of \( K_{36} + 21. \)

**Theorem 4.9.** Let \( k \equiv 3 \pmod{4} \geq 7 \) and \( n \equiv 0, 1 \pmod{4} \) with \( 2k < n < 10k. \)

If \( n \not\equiv 2, 3 \pmod{k-1} \) and \( \left\lfloor \frac{n-4}{k-1} \right\rfloor \) is even, then there exists a \( k \)-sun system of \( K_{4k} + n \) except possibly when \( (k, n) \in \{(7,64), (7,65)\}. \)

**Proof.** First, \( k \equiv 3 \pmod{4} \geq 7 \) implies that \( \ell \geq 3 \) is odd. Now, let \( n = 2(q\ell+r)+\nu \) with \( 1 \leq r \leq \ell \) and \( \nu \in \{2,3\}. \) Note that \( q = \left\lfloor \frac{n-4}{k-1} \right\rfloor, \) hence \( q \) is even. Also, since \( 2k < n < 10k, \) we have \( 2 \leq q \leq 10. \) By \( q \) even and \( n \equiv 0, 1 \pmod{4} \) it follows that \( r \) is odd, and \( n \not\equiv 2, 3 \pmod{k-1} \) implies that \( r \not= \ell. \) To sum up,

\[
q \text{ is even with } 2 \leq q \leq 10, \quad \text{and } r \text{ is odd with } 1 \leq r \leq \ell - 2.
\]

As in the previous theorem, let \( V(K_{4k} + n) = (\mathbb{Z}_k \times [0,3]) \cup \{ \infty_h \mid h \in \mathbb{Z}_{n-\nu} \} \cup \{ \infty_1, \infty_2, \infty_3 \}. \)

We split the proof into two cases.

Case 1) \( q \leq 2r + 4. \) We start decomposing \( K_{2k} \) into the following two graphs:

\[
\Gamma_1 = \langle \{3, \ell\}, [k-2r-2, k], [3, \ell] \rangle \quad \text{and} \quad \Gamma_2 = \langle \{1, 2\}, [1, k-2r-3], \{1, 2\} \rangle.
\]

Since \( q \leq 2r+4, \) the graph \( \Gamma_1 \) can be further decomposed into the following graphs:

\[
\Gamma_{1,1} = \langle \{ \ell\}, [k-2r+q-3, k], \emptyset \rangle, \quad \Gamma_{1,2} = \langle [3, \ell - 1], \emptyset, [3, \ell] \rangle,
\]

\[
\Gamma_{1,3} = \langle \emptyset, [k-2r-2, k-2r+q-4], \emptyset \rangle.
\]

The first two graphs have a \( k \)-cycle system by Lemmas 3.2 and 3.3 while \( \Gamma_{1,3} \) decomposes into \((q-1) \) 1-factors, say \( J_1, J_2, \ldots, J_{q-1}. \) Setting \( w_1 = (q-1)\ell, \) by Lemma 4.2 we have that:

\[
\Gamma_1 + (q-1)\ell = \oplus_{i=1}^{q-1} (J_i + \ell) \oplus (\Gamma_{1,1} \oplus \Gamma_{1,2}).
\]

Hence \( \Gamma_1 + (q-1)\ell \) has a \( k \)-cycle system since each \( J_i + \ell \) decomposes into \( k \)-cycles by Lemma 3.3.
Letting \( w_2 = n - 2w_1 = 2(\ell + r) + \nu \) and recalling that \( K_{2k} = \Gamma_1 \mathbin{\oplus} \Gamma_2 \), by Lemma 4.6 it remains to construct a \( k \)-sun system of \( \Gamma_2^*[2] + w_2 \). We start decomposing \( \Gamma_2 \) into the following graphs:

\[
\Gamma_{2,0} = \langle \{1, 2\}, \{1, k - 2r - 4\}, \{1, 2\} \rangle \quad \text{and} \quad \Gamma_{2,1} = \langle \emptyset, \{k - 2r - 3\}, \emptyset \rangle.
\]

Recalling that \( \Gamma_2^*[2] = \Gamma_2^*[2] + I \), where \( I \) denotes the 1-factor \( \{\{z, \tau\} \mid z \in \mathbb{Z}_k \times \{0, 1\} \} \) of \( K_{4k} \), by Lemma 4.2 we have that

\[
\Gamma_2^*[2] + w_2 = (\Gamma_{2,1} + \ell)[2] \mathbin{\oplus} (\Gamma_{2,0} + r)[2] \mathbin{\oplus} (I + \nu).
\]

By Lemmas 3.3 and 3.4 there exist a \( k \)-cycle \( A = (x_1, x_2, x_3, y_4, y_5, a_7, \ldots, a_k) \) of \( \Gamma_{2,0} + r \) and a \( k \)-cycle \( B = (y, x, b_3, \ldots, b_k) \) of \( \Gamma_{2,1} + \ell \), satisfying the following properties:

- \( \text{Orb}(A) \cup \text{Orb}(B) \) is a \( k \)-cycle system of \( \Gamma_2 + (\ell + r) \);
- \( \text{Dev}(\{x_1, x_2\}) \) and \( \text{Dev}(\{x_2, x_3\}) \) are \( k \)-cycles with vertices in \( \mathbb{Z}_k \times \{0\} \);
- \( \text{Dev}(\{y_4, y_5\}) \) and \( \text{Dev}(\{y_5, y_6\}) \) are \( k \)-cycles with vertices in \( \mathbb{Z}_k \times \{1\} \);
- \( x \in \mathbb{Z}_k \times \{0\} \) and \( y \in \mathbb{Z}_k \times \{1\} \).

Set \( A' = (x_1, x_2, x_3, y_4, y_5, y_6, a_7, \ldots, a_k) \) and \( B' = (y, x, b_3, \ldots, b_k) \) and let \( S = \{\sigma(A'), \sigma(A'), \sigma(B'), \sigma(B')\} \). By Lemma 4.4 we have that \( \bigcup_{S \in S} \text{Orb}(S) \) is a \( k \)-sun system of \( (\Gamma_3 + (\ell + r)) \mathbin{\oplus} 2(\ell + r) = (\Gamma_2^*[2] + w_2) \mathbin{\setminus} (I + \nu) \).

To construct a \( k \)-sun system of \( \Gamma_2^*[2] + w_2 \) we proceed as in Theorem 4.7. We modify the graphs in \( S \) and obtain four \( k \)-suns \( T_1, T_2, T_3, T_4 \) whose translates between them cover all edges incident with \( \infty_1', \infty_2' \), and possibly \( \infty_3' \) when \( \nu = 3 \). Then we construct further \( 2\nu + 1 \) \( k \)-suns \( G_1, \ldots, G_{2\nu + 1} \) to cover the missing edges. The reader can check that \( \bigcup_{i=1}^{4} \text{Orb}(T_i) \cup \{G_1, \ldots, G_{2\nu + 1}\} \) is a \( k \)-sun system of \( \Gamma_2^*[2] + w_2 \).

The graphs \( T_i \) are the following, where the elements in bold are the replaced vertices:

\[
T_1 = \begin{cases} 
(x_1 \ x_2 \ x_3 \ \infty_1' \ y_5 \ y_6 \ a_7 \ \ldots \ a_{k-1} \ a_k) \\
(\infty_1' \ y_5 \ y_4 \ y_5 \ a_7 \ \ldots \ a_k \ a_1) \\
(x_1 \ x_2 \ x_3 \ \infty_2' \ y_5 \ y_6 \ a_7 \ \ldots \ a_k \ a_1) \\
(\infty_2' \ y_5 \ y_4 \ y_5 \ a_7 \ \ldots \ a_k \ a_1) \\
(\infty_1' \ y_5 \ y_4 \ y_5 \ a_7 \ \ldots \ a_k \ a_1) \\
(x_1 \ x_2 \ x_3 \ \infty_1' \ y_5 \ y_6 \ a_7 \ \ldots \ a_k \ a_3) \\
(\infty_1' \ y_5 \ y_4 \ y_5 \ a_7 \ \ldots \ a_k \ a_3) \\
(x_1 \ x_2 \ x_3 \ \infty_1' \ y_5 \ y_6 \ a_7 \ \ldots \ a_k \ a_3) \\
(\infty_1' \ y_5 \ y_4 \ y_5 \ a_7 \ \ldots \ a_k \ a_3)
\end{cases}
\]

if \( \nu = 2 \),

if \( \nu = 3 \),

if \( \nu = 2 \),

if \( \nu = 3 \),

if \( \nu = 2 \),

if \( \nu = 3 \).
The graphs $G_i$, for $i = [1, 2\nu + 1]$, are so defined:

$$G_1 = \text{Dev}(x_1 \sim x_2 \sim x_3), \quad G_2 = \text{Dev}(y_5 \sim y_4 \sim x_3),$$

$$G_3 = \text{Dev}(\langle x_1, x_2 \rangle \oplus \{x_3, y_4\}), \quad G_4 = \text{Dev}(\langle y_5, y_4 \rangle \sim y_5),$$

$$G_5 = \text{Dev}(\langle y_5 \sim y_6 \sim y_6 \rangle), \quad G_6 = \text{Dev}(\{x_2, x_3\} \oplus \{x, y\}),$$

$$G_7 = \text{Dev}(\{x_2, x_3\} \oplus \{x, y\}).$$

Case 2) $q \geq 2r + 6$. Note that this implies $r = 1$ and $q = 8, 10$. As before $K_{2k} = \Gamma_1 \oplus \Gamma_2$ where

$$\Gamma_1 = \langle [3, \ell], \{0\}, [k - 5, k - 1, [3, \ell]\rangle \quad \text{and} \quad \Gamma_2 = \langle \{1, 2\}, [1, k - 6], \{1, 2\rangle.$$

Since $(k, n) \neq (7, 64), (7, 65)$ then $(\ell, q) \neq (3, 10)$, hence the graph $\Gamma_1$ can be decomposed into the following graphs:

$$\Gamma_{1,1} = \langle \emptyset, [k - 5, k - 1], \emptyset \rangle \quad \Gamma_{1,2} = \langle \left[3, \frac{q - 2}{2}\right], \{0\}, \left[3, \frac{q - 2}{2}\right]\rangle$$

$$\Gamma_{1,3} = \langle \left[\frac{q}{2}, \ell\right], \emptyset, \left[\frac{q}{2}, \ell\right]\rangle .$$

The graph $\Gamma_{1,1}$ decomposes into five 1-factors $J_1, \ldots, J_5$, while by Lemma 3.2 $\Gamma_{1,2}$ decomposes into $(q - 5)$ 1-factors $J'_1, \ldots, J'_{q-5}$. Letting $w_1 = q\ell$, by Lemma 4.2 we have that

$$\Gamma_1 + w_1 = (\Gamma_{1,1} + 5\ell) \oplus (\Gamma_{1,2} + (q - 5)\ell) \oplus \Gamma_{1,3} = \oplus_{i=1}^{5}(J_i + \ell) \oplus \oplus_{i=1}^{q-5}(J'_i + \ell) \oplus \Gamma_{1,3}.$$ 

By Lemmas 3.4 and 3.1 each $J_i + \ell$, each $J'_i + \ell$ and $\Gamma_{1,3}$ decompose into $k$-cycles. Hence $\Gamma_1 + q\ell$ has a $k$-cycle system. Let now $w_2 = n - 2w_1 = 2 + \nu$. Note that a $k$-sun system of $\Gamma_2[2] + w_2$ can be obtained as in Case 1, where $\Gamma_{2,1}$ is empty.

**Theorem 4.10.** Let $k \equiv 3 \mod 4 \geq 11$ and $n \equiv 3 \mod 4$ with $2k < n < 10k$. If $\left[\frac{n-4}{k-1}\right]$ is even, and $n \equiv 2, 3 \mod k - 1$, then there is a $k$-sun system of $K_{4k} + n$, except possibly when $(k, n) \in \{(11, 112), (11, 113)\}$.

**Proof.** Let $n = 2(q\ell + r) + \nu$ with $1 \leq r \leq \ell$ and $\nu \in \{2, 3\}$. Clearly, $q = \left[\frac{n-4}{k-1}\right]$, hence $q$ is even. Since $k \geq 11, 2k < n < 10k$ and $n \equiv 2, 3 \mod 2\ell$, we have that $q$ is even with $2 \leq q \leq 10$ and $r = \ell \geq 5$ is odd.

As before, let $V(K_{4k} + n) = (\mathbb{Z}_k \times [0, 3]) \cup \{\infty_h \mid h \in \mathbb{Z}_{n-k} \} \cup \{\infty'_{h}, \infty''_{h}, \infty'_{h}\}$.

We start decomposing $K_{2k}$ into the following two graphs:

$$\Gamma_1 = \langle [3, \ell], [k - 3, k], [4, \ell]\rangle, \quad \Gamma_2 = \langle \{1, 2\}, [k - 4, \{1, 2\}, \{1, 2, 3\}\rangle.$$

If $q = 2, 4$, $\Gamma_1$ can be further decomposed into

$$\Gamma_{1,1} = \langle \emptyset, [k - 3, k - 4 + q], \emptyset \rangle, \quad \Gamma_{1,2} = \langle \emptyset, [k - 3 + q, k], \{\ell\} \rangle,$$

$$\Gamma_{1,3} = \langle [3, \ell], \emptyset, [4, \ell - 1] \rangle.$$ 

The graph $\Gamma_{1,1}$ decomposes into $q$ 1-factors, say $J_1, \ldots, J_q$. Letting $w_1 = q\ell$, by Lemma 4.2 we have that

$$\Gamma_1 + w_1 = (\Gamma_{1,1} + w_1) \oplus \Gamma_{1,2} \oplus \Gamma_{1,3} = \oplus_{i=1}^{q}(J_i + \ell) \oplus \Gamma_{1,2} \oplus \Gamma_{1,3}.$$ 

Lemmas 3.4 and 3.1 guarantee that each $J_i + \ell$, $\Gamma_{1,2}$ and $\Gamma_{1,3}$ decompose into $k$-cycles, hence $\Gamma_1 + w_1$ has a $k$-cycle system. Suppose now $q \geq 6$. By
Let \((k, n) \not\in \{(11, 112), (11, 113)\}\), we have \((\ell, q) \neq (5, 10)\). In this case \(\Gamma_1\) can be further decomposed into 
\[
\Gamma_{1,1} = \langle \varnothing, [k - 3, k - 1], \varnothing \rangle, \quad \Gamma_{1,2} = \left\langle \left[\ell + 3 - \frac{q}{2}, \ell \right], \{0\}, \left[\ell + 3 - \frac{q}{2}, \ell \right] \right\rangle, 
\]
\[
\Gamma_{1,3} = \left\langle \left[3, \ell + 2 - \frac{q}{2} \right], \varnothing, \left[4, \ell + 2 - \frac{q}{2} \right] \right\rangle.
\]
The graph \(\Gamma_{1,1}\) can be decomposed into three 1-factors say \(J_1, J_2, J_3\), also by Lemma 3.5 the graph \(\Gamma_{1,2}\) can be decomposed into \((q - 3)\) 1-factors say \(J_1', \ldots, J_{q-3}'\). Set again \(w_1 = q\ell\), by Lemma 4.2 we have that 
\[
\Gamma_1 + w_1 = (\Gamma_{1,1} + 3\ell) \oplus (\Gamma_{1,2} + (q - 3)\ell) \oplus \Gamma_{1,3} = \oplus_{i=1}^{3} (J_i + \ell) \oplus \oplus_{j=1}^{q-3} (J_j' + \ell) \oplus \Gamma_{1,3}.
\]
By Lemmas 3.4 and 3.1 we have that each \(J_i + \ell\), each \(J_j' + \ell\) and \(\Gamma_{1,3}\) decompose into \(k\)-cycles, hence \(\Gamma_1 + w_1\) has a \(k\)-cycle system. Hence for any value of \(q\) we have proved that there \(\Gamma_1 + w_1\) has a \(k\)-cycle system.

Now, setting \(w_2 = n - 2w_1 = 2\ell + \nu\) and recalling that \(K_{2k} = \Gamma_1 \oplus \Gamma_2\), by Lemma 4.6 it is left to show that \(\Gamma_2^*[2] + w_2\) has a \(k\)-sun system. Let \(r_1\) and \(r_2 \geq 2\) be an odd and an even integer, respectively, such that \(r_1 + r_2 = r = \ell\). Note that \(\Gamma_2\) can be further decomposed into 
\[
\Gamma_{2,1} = \langle \{1\}, [1, k - 2r_1 - 2], \{1\} \rangle, \quad \Gamma_{2,2} = \langle \{2\}, [k - 2r_1 - 1, k - 4], \{2, 3\} \rangle.
\]
Recalling that \(\Gamma_2^*[2] = \Gamma_2[2] \oplus I\), where \(I\) denotes the 1-factor \(\{\{z, \varnothing\} \mid z \in Z_k \times \{0, 1\}\}\) of \(K_{2k}\), by Lemma 4.2 we have that 
\[
\Gamma_2^*[2] + w_2 = \oplus_{i=1}^{2} (\Gamma_{2,i} + r_1)[2] \oplus (I + \nu).
\]
By Lemma 3.3 there is a \(k\)-cycle \(A = (y_1, y_2, x_3, x_4, a_5, \ldots, a_k)\) of \(\Gamma_{2,1} + r_1\) and a \(k\)-cycle \(B = (x_1, x_2, y_3, y_4, y_5, b_6, \ldots, b_k)\) of \(\Gamma_{2,2} + r_2\) such that 
\[
(7) \quad \text{Orb}(A) \cup \text{Orb}(B) \text{ is a } k\text{-cycle system of } \Gamma_2 + \ell,
\]
\[
\text{Dev}(\{x_1, x_2\}) \text{ and } \text{Dev}(\{x_3, x_4\}) \text{ are } k\text{-cycles with vertices in } Z_k \times \{0\},
\]
\[
\text{Dev}(\{y_1, y_2\}), \text{Dev}(\{y_3, y_4\}) \text{ and } \text{Dev}(\{y_5, y_6\}) \text{ are } k\text{-cycles with vertices in } Z_k \times \{1\}.
\]
Set \(A' = (y_1, \overline{y_2}, x_3, \overline{x_4}, a_5, \ldots, a_k)\) and \(B' = (x_1, \overline{x_2}, y_3, \overline{y_4}, y_5, b_6, \ldots, b_k)\). Let \(S = \{\sigma(A'), \sigma(A'), \sigma(B'), \sigma(B')\}\), by Lemma 4.4 we have that \(\bigcup_{S \in S} \text{Orb}(S)\) is a \(k\)-sun system of \(\left(\Gamma_2 + \ell\right)[2] = \Gamma_2[2] + 2\ell = (\Gamma_2^*[2] + w_2) \setminus (I + \nu)\). To construct a \(k\)-sun system of \(\Gamma_2[2] + w_2\), we build a family \(T = \{T_1, T_2, T_3, T_4\}\) of \(k\)-suns by modifying the graphs in \(S\) so that \(\bigcup_{T \in T} \text{Orb}(T)\) covers all the edges incident with \(\varnothing'_1, \varnothing'_2\), and possibly \(\varnothing'_3\) when \(\nu = 3\). We then construct further \((2\nu + 1)\) \(k\)-suns \(G_1, G_2, \ldots, G_{2\nu+1}\) which cover the remaining edges exactly once. Hence, \(\bigcup_{T \in T} \text{Orb}(T) \cup \{G_1, G_2, \ldots, G_{2\nu+1}\}\) is a \(k\)-sun system of \(\Gamma_2^*[2] + w_2\).

The graphs \(T_1, \ldots, T_4\) and \(G_1, \ldots, G_{2\nu+1}\) are the following, where as before the elements in bold are the replaced vertices.

\[
T_1 = \left(\begin{array}{cccccc}
y_1 & \varnothing'_2 & x_3 & \overline{x_4} & a_5 & \ldots & a_{k-1} & a_k & y_1 \\
\varnothing_2 & \varnothing'_2 & x_3 & \overline{x_4} & a_5 & \ldots & a_k & \overline{y_1}
\end{array}\right),
\]
\[
T_2 = \left(\begin{array}{cccccc}
y_1 & \varnothing'_2 & x_3 & \overline{x_4} & a_5 & \ldots & a_{k-1} & a_k & y_1 \\
\varnothing_2 & \varnothing'_2 & x_3 & \overline{x_4} & a_5 & \ldots & a_k & \overline{y_1}
\end{array}\right),
\]

if \(\nu = 2\),
\[
T_3 = \left(\begin{array}{cccccc}
y_1 & \varnothing'_2 & x_3 & \overline{x_4} & a_5 & \ldots & a_{k-1} & a_k & y_1 \\
\varnothing_2 & \varnothing'_2 & x_3 & \overline{x_4} & a_5 & \ldots & a_k & \overline{y_1}
\end{array}\right),
\]

if \(\nu = 3\),
\[
T_3 = \left( \begin{array}{cccccccc}
\infty & x_1 & \infty & x_2 & y_3 & \infty & y_4 & y_5 & y_6 & \ldots & b_{k-1} & b_k & x_1 \\
\infty & y_3 & \infty & y_4 & y_5 & \infty & b_6 & b_7 & \ldots & b_k & x_1
\end{array} \right),
\]
\[
T_4 = \begin{cases}
\left( \begin{array}{cccccccc}
\infty & x_1 & \infty & x_2 & y_3 & \infty & y_4 & y_5 & y_6 & \ldots & b_{k-1} & b_k & x_1 \\
\infty & y_3 & \infty & y_4 & y_5 & \infty & b_6 & b_7 & \ldots & b_k & x_1
\end{array} \right) & \text{if } \nu = 2, \\
\left( \begin{array}{cccccccc}
\infty & x_1 & \infty & x_2 & y_3 & \infty & y_4 & y_5 & y_6 & \ldots & b_{k-1} & b_k & x_1 \\
\infty & y_3 & \infty & y_4 & y_5 & \infty & b_6 & b_7 & \ldots & b_k & x_1
\end{array} \right) & \text{if } \nu = 3.
\end{cases}
\]

\[
G_1 = \text{Dev}(y_1 \sim y_2 \sim x_3), \quad G_2 = \text{Dev}(\overline{y_2} \sim \overline{y_1} \sim y_2),
\]
\[
G_3 = \text{Dev}(y_3 \sim y_4 \sim y_3), \quad G_4 = \text{Dev}(\{x_1, x_2 \} + \{x_3, y_2\}),
\]
\[
G_5 = \begin{cases}
\text{Dev}(x_1 \sim x_2 \sim \overline{x_2}) & \text{if } \nu = 2, \\
\text{Dev}(x_1 \sim x_2 \sim \overline{y_3}) & \text{if } \nu = 3,
\end{cases}
\]
\[
G_6 = \text{Dev}(\overline{y_3} \sim \overline{x_2} \sim x_4), \quad G_7 = \text{Dev}(\overline{y_4} \sim \overline{y_3} \sim y_4).
\]

By recalling (7), it is not difficult to check that the graphs \(G_h\) are \(k\)-suns. \(\square\)

**Theorem 4.11.** Let \(k \equiv 3 \pmod{4} \geq 7\) and \(n \equiv 0, 1 \pmod{4}\) with \(2k < n < 10k\). If \(\left\lfloor \frac{n-4}{k-1} \right\rfloor\) is odd and \(n \not\equiv 0, 1 \pmod{4}\), then there is a \(k\)-sun system of \(K_{4k} + n\).

**Proof.** Let \(n = 2(q\ell + r) + \nu\) with \(1 \leq r \leq \ell\) and \(\nu \in \{2, 3\}\). Clearly, \(q = \left\lfloor \frac{n-4}{k-1} \right\rfloor\).

Also, we have that \(q \) and \(\ell \) are odd, and \(n \equiv 0, 1 \pmod{4}\); hence \(r\) is even. Furthermore, we have that \(2 \leq q \leq 10\), since by assumption \(2k < n < 10k\). Considering now the hypothesis that \(n \not\equiv 0, 1 \pmod{2\ell}\), it follows that \(r \neq \ell - 1\). To sum up,

\[
q \text{ is odd with } 3 \leq q \leq 9, \text{ and } r \text{ is even with } 2 \leq r \leq \ell - 3.
\]

As before, let \(V(K_{4k} + n) = (\mathbb{Z}_k \times [0, 3]) \cup \{\infty_j | h \in \mathbb{Z}_{n-r}\} \cup \{\infty_1', \infty_2', \infty'\}\).

We start decomposing \(K_{2k}\) into the following two graphs:

\[
\Gamma_1 = \langle [4, \ell], [k - 2r - 1, k], [3, \ell] \rangle \quad \text{and} \quad \Gamma_2 = \langle [1, 3], [1, k - 2r - 2], [1, 2] \rangle.
\]

Considering that \(3 \leq q \leq 9 \leq 2r + 5\), the graph \(\Gamma_1\) can be further decomposed into the following graphs:

\[
\Gamma_{1,1} = \langle [4, \ell], \emptyset, [3, \ell - 1] \rangle, \quad \Gamma_{1,2} = \langle \emptyset, [k - 2r - 4 + q, k], \{\ell\} \rangle,
\]

and \(\Gamma_{1,3} = \langle \emptyset, [k - 2r - 1, k - 2r - 5 + q], \emptyset \rangle\).

The first two have a \(k\)-cycle system by Lemmas 6.1 and 6.2 while \(\Gamma_{1,3}\) decomposes into \((q - 3)\) 1-factors, say \(J_1, J_2, \ldots, J_{q-3}\). Letting \(w_1 = (q - 3)\ell\), by Lemma 6.2 we have that

\[
\Gamma_1 + w_1 = \bigoplus_{i=1}^{q-3}(J_i + \ell) \oplus (\Gamma_{1,1} \oplus \Gamma_{1,2}).
\]

Therefore, \(\Gamma_1 + w_1\) has a \(k\)-cycle system, since each \(J_i + \ell\) decomposes into \(k\)-cycles by Lemma 6.4. Setting \(w_2 = n - 2w_1 = 2(3\ell + r) + \nu\) and recalling that \(K_{2k} = \Gamma_1 \oplus \Gamma_2\), by Lemma 4.6 it is left to show that \(\Gamma_2[2] + w_2\) has a \(k\)-sun system.

We start decomposing \(\Gamma_2\) into the following graphs:

\[
\Gamma_{2,0} = \langle [1, 3], [1, k - 2r - 5], [1, 2] \rangle, \quad \text{and} \quad \Gamma_{2,i} = \langle \emptyset, [k - 2r - 5 + i], \emptyset \rangle, \text{ for } 1 \leq i \leq 3.
\]
Recalling that $\Gamma'_2[2] = \Gamma_2[2] \oplus I$, where $I$ denotes the 1-factor $\{z, \overline{z}\} | z \in \mathbb{Z}_k \times \{0, 1\}$ of $K_{4k}$, by Lemma 4.2 we have that

$$\Gamma'_2[2] + w_2 = \bigoplus_{i=1}^{3} (\Gamma_{2,i} + \ell)[2] + (\Gamma_{2,0} + r)[2] \oplus (I + \nu).$$

By Lemmas 5.3 and 5.4 there exist a $k$-cycle $A = (x_1, x_2, x_3, y_4, y_5, y_6, a_7, \ldots, a_k)$ of $\Gamma_{2,0} + r$, a $k$-cycle $B_1 = (x_{1,0}, y_{1,1}, b_{1,2}, \ldots, b_{1,k-1})$ of $\Gamma_{2,1} + \ell$, and a $k$-cycle $B_i = (y_{i,0}, x_{i,1}, b_{i,2}, \ldots, b_{i,k-1})\Gamma_{2,i} + \ell$, for $2 \leq i \leq 3$, satisfying the following properties:

$$(9) \quad \text{Dev}(\{x_1, x_2\}) \text{ and } \text{Dev}(\{x_2, x_3\}) \text{ are } k\text{-cycles with vertices in } \mathbb{Z}_k \times \{0\},$$

$$(10) \quad x_{1,0}, x_{2,1}, x_{3,1} \in \mathbb{Z}_k \times \{0\}, y_{1,1}, y_{2,0}, y_{3,0} \in \mathbb{Z}_k \times \{1\};$$

$$(11) \quad \bigcup_{i=1}^{3} \text{Orb}(B_i) \cup \text{Orb}(A) \text{ is a } k\text{-cycle system of } \Gamma_2 + (3\ell + r).$$

Set $A' = (x_1, x_2, x_3, y_4, y_5, y_6, a_7, \ldots, a_k)$ and let $S = \{\sigma(A'), \overline{\sigma(A')}\} \cup \{\sigma(B_i), \overline{\sigma(B_i)} | 1 \leq i \leq 3\}$. By Lemma 4.2 we have that $\bigcup_{S \in S} \text{Orb}(S)$ is a $k$-suns system of $(\Gamma_{2} + (3\ell + r))[2] = \Gamma_{2}[2] + 2(3\ell + r) = (\Gamma_{2}^* + w_2) \setminus (I + \nu)$. To construct a $k$-sun system of $\Gamma_{2}^* + w_2$, we build a family $T = \{T_0, T_1, \ldots, T_7\}$ of $k$-sun systems by modifying the graphs in $S$ so that $\bigcup_{T \in T} \text{Orb}(T)$ covers all the edges incident with $\infty_1', \infty_2'$, and possibly $\infty_3'$ when $\nu = 3$. We then construct further $(2\nu + 1)$ $k$-suns $G_1, G_2, \ldots, G_{2\nu+1}$ which cover the remaining edges exactly once. Hence, $\bigcup_{T \in T} \text{Orb}(T) \cup \{G_1, G_2, \ldots, G_{2\nu+1}\}$ is a $k$-sun system of $\Gamma_{2}^* + w_2$. The graphs $T_0, \ldots, T_7$ and $G_1, \ldots, G_{2\nu+1}$ are the following, where as before the elements in bold are the replaced vertices:

$T_0 = \left(\begin{array}{cccccccccccc}
\frac{x_1}{\infty_1} & \frac{x_2}{\infty_1} & \frac{x_3}{\infty_1} & \frac{y_4}{\infty_1} & \frac{y_5}{\infty_1} & \frac{y_6}{\infty_1} & a_7 & \ldots & a_{k-1} & a_k \\
\frac{x_2}{\infty_2} & \frac{\infty_2}{y_4} & \frac{\infty_2}{y_5} & \frac{\infty_2}{y_6} & \frac{a_7}{a_8} & \ldots & \frac{a_k}{a_k} & x_1
\end{array}\right)$

if $\nu = 2$,

if $\nu = 3$,

if $\nu = 2$,

if $\nu = 3$,
Recalling that $\Gamma^*$, by Lemma 4.6 it is left to show that $\Gamma_2^*$ is a $k$-sun system of $K_{4k} + n$ except possibly when $(k, n) \not\in \{(11, 100), (11, 101)\}$.

**Theorem 4.12.** Let $k \equiv 3 \pmod{4} \geq 7$ and $n \equiv 0, 1 \pmod{4}$ with $2k < n < 10k$. If \( \left\lfloor \frac{n-4}{k} \right\rfloor \) is odd, and $n \equiv 0, 1 \pmod{4}$ and $\nu \in \{2, 3\}$. Reasoning as in the proof of Theorem 4.11 and considering that $n \equiv 0, 1 \pmod{2\ell}$ and $(k, n) \not\in \{(11, 100), (11, 101)\}$, we have that

\[ q \text{ is odd with } 3 \leq q \leq 9, \quad r = \ell - 1 \geq 2, \quad \text{and } (\ell, q) \neq (5, 9). \]

As before, let $V(K_{4k} + n) = (\mathbb{Z}_k \times \{0, 3\}) \cup \{\infty_h \mid h \in \mathbb{Z}_n\} \cup \{\infty^*_1, \infty^*_2, \infty^*_n\}$.

We start decomposing $K_{2k}$ into the following two graphs

\[ \Gamma_1 = \langle \{3, \ell\}, \{0\}, \{3, \ell\} \rangle, \quad \text{and } \Gamma_2 = \langle \{1, 2\}, \{1, k - 1\}, \{1, 2\} \rangle. \]

Considering (12), we can further decompose $\Gamma_1$ into the following two graphs:

\[ \Gamma_{1,1} = \langle \left[3, \frac{q+3}{2}\right], \{0\}, \left[3, \frac{q+3}{2}\right]\rangle, \quad \Gamma_{1,2} = \langle \left[\frac{q+5}{2}, \ell\right], \emptyset, \left[\frac{q+5}{2}, \ell\right]\rangle. \]

By Lemma 3.3, the graph $\Gamma_{1,1}$ decomposes into $q$ 1-factors, say $J_1, J_2, \ldots, J_q$. Letting $w_1 = q\ell$, by Lemma 4.2 we have that

\[ \Gamma_1 + w_1 = (\Gamma_{1,1} + w_1) \oplus \Gamma_{1,2} = \oplus_{i=1}^{q} (J_i + \ell) \oplus \Gamma_{1,2}. \]

Lemmas 3.3 and 3.1 guarantee that each $J_i + \ell$ and $\Gamma_{1,2}$ decompose into $k$-cycles, hence $\Gamma_1 + w_1$ has a $k$-cycle system. Let $r_1$ and $r_2$ be odd positive integers such that $r = \ell - 1 = r_1 + r_2$. Then, setting $w_2 = n - 2w_1 = 2(r_1 + r_2) + \nu$ and recalling that $K_{2k} = \Gamma_1 \oplus \Gamma_2$, by Lemma 4.6 it is left to show that $\Gamma_2^*[2] + w_2$ has a $k$-sun system.

We start decomposing $\Gamma_2$ into the following graphs:

\[ \Gamma_{2,1} = \langle \{1\}, \{1, k - 2r_1 - 2\}, \{1\} \rangle \text{ and } \Gamma_{2,2} = \langle \{2\}, \{k - 2r_1 - 1, k - 1\}, \{2\} \rangle. \]

Recalling that $\Gamma^*_2[2] = \Gamma_2[2] \oplus I$, where $I$ denotes the 1-factor $\{\{z, z\} \mid z \in \mathbb{Z}_k \times \{0, 1\}\}$ of $K_{4k}$, by Lemma 4.2 we have that

\[ \Gamma^*_2[2] + w_2 = (\Gamma_{2,1} + r_1)[2] \oplus (\Gamma_{2,2} + r_2)[2] \oplus (I + \nu). \]
By Lemma 3.3 there is a k-cycle \( A = (y_1, y_2, x_3, x_4, a_5, \ldots, a_k) \) of \( \Gamma_{2,1} + r_1 \) and \( B = (x_1, x_2, y_3, y_4, b_5, \ldots, b_k) \) of \( \Gamma_{2,2} + r_2 \) such that

\[
\text{Orb}(A) \cup \text{Orb}(B) \text{ is a k-cycle system of } \Gamma_2 + r,
\]

\[
\text{Dev}(\{x_3, x_4\}) \text{ and } \text{Dev}(\{x_1, x_2\}) \text{ are k-cycles with vertices in } \mathbb{Z}_k \times \{0\},
\]

\[
\text{Dev}(\{y_1, y_2\}) \text{ and } \text{Dev}(\{y_3, y_4\}) \text{ are k-cycles with vertices in } \mathbb{Z}_k \times \{1\}.
\]

Set \( A' = (y_1, y_2, x_3, x_4, a_5, a_6, \ldots, a_k) \), \( B' = (x_1, x_2, y_3, y_4, b_5, b_6, \ldots, b_k) \) and let \( S = \{\sigma(A'), \sigma(A'), \sigma(B'), \sigma(B')\} \). By Lemma 4.3 we have that \( \bigcup_{S \in \mathcal{S}} \text{Orb}(S) \) is a k-sun system of \( (\Gamma_2^2[2] + w_2) \setminus (I + v) \).

To construct a k-sun system of \( \Gamma_2^2[2] + w_2 \), we build a family \( \mathcal{T} = \{T_1, T_2, T_3, T_4\} \) of four k-suns, each of which is obtained from a graph in \( \mathcal{S} \) by replacing some of their vertices with \( \infty', \infty' \), and possibly \( \infty' \), and let

\[
G_1 = \text{Dev}(y_1 \sim y_2 \sim x_3), \quad G_2 = \text{Dev}(y_1 \sim y_2 \sim x_3),
\]

\[
G_3 = \text{Dev}(y_3 \sim y_4 \sim x_2), \quad G_4 = \text{Dev}(x_1 \sim x_2 \sim x_3),
\]

\[
G_5 = \begin{cases} 
\text{Dev}(y_1 \sim x_2 \sim y_3) & \text{if } \nu = 2, \\
\text{Dev}(\{y_1, x_2\} \cup \{y_3, x_2\}) & \text{if } \nu = 3,
\end{cases} \quad G_6 = \text{Dev}(y_4 \sim y_3 \sim x_2),
\]

\[
G_7 = \text{Dev}(x_4 \sim x_3 \sim y_2).
\]

By Lemma 4.3, it is not difficult to check that the graphs \( G_h \) are k-suns. \( \square \)

5. It is sufficient to solve \( 2k < v < 6k \)

In this section we show that if the necessary conditions in (4), for the existence of a k-sun system of \( K_v \), are sufficient for all \( v \) satisfying \( 2k < v < 6k \), then they are sufficient for all \( v \). In other words, we prove Theorem (11).

We start by showing how to construct k-sun systems of \( K_{g \times h} \) (i.e., the complete multipartite graph with \( g \) parts each of size \( h \)) when \( h = 4k \).

**Theorem 5.1.** For any odd integer \( k \geq 3 \) and any integer \( g \geq 3 \), there exists a k-sun system of \( K_{g \times 4k} \).
Proof. Set $V(K_{g \times 2k}) = \mathbb{Z}_{2g} \times [0, 1]$ and let $K_{g \times 4k} = K_{g \times 2k}[2]$. In [11] Theorem 2] the authors proved the existence of a $k$-cycle system of $K_{g \times 2k}$. By applying Lemma 4.5 (with $\Gamma = K_{g \times 2k}$ and $u = 0$) we obtain the existence of a $k$-sun system of $K_{g \times 4k}$. \hfill $\square$

The following result exploits Theorem 5.1 and shows how to construct $k$-sun systems of $K_{4kg+n}$, for $g \neq 2$, starting from a $k$-sun system of $K_{4k} + n$ and a $k$-sun system of either $K_n$ or $K_{4k+n}$.

**Theorem 5.2.** Let $k \geq 3$ be an odd integer and assume that both the following conditions hold:

1. there exists a $k$-sun system of either $K_n$ or $K_{4k+n}$;
2. there exists a $k$-sun system of $K_{1k} + n$.

Then there is a $k$-sun system of $K_{4kg+n}$ for all positive $g \neq 2$.

**Proof.** Suppose there exists a $k$-sun system $S_1$ of $K_n$, also, by (2), there exists a $k$-sun system $S_2$ of $K_{4k} + n$. Clearly, $S_1 \cup S_2$ is a $k$-sun system of $K_{n+4k} = K_n \oplus (K_{4k} + n)$. Hence we can suppose $g \geq 3$. Let $V$, $H$ and $G$ be sets of size $n$, $4k$ and $g$, respectively, such that $V \cap (H \times G) = \emptyset$. Let $S$ be a $k$-sun system of $K_n$ (resp., $K_{n+4k}$) with vertex set $V$ (resp., $V \cup (H \times \{x_0\})$) for some $x_0 \in G$). By assumption, for each $x \in G$, there is a $k$-sun system, say $B_x$, of $K_{4k} + n$ with vertex set $V \cup (H \times \{x\})$, where $V(K_{4k}) = H \times \{x\}$.

Also, by Theorem 5.1 there is a $k$-sun system $C$ of $K_{g \times 4k}$ whose parts are $H \times \{x\}$ with $x \in G$. Hence the $k$-suns of $B_x$ with $x \in G$ (resp., $x \in G \setminus \{x_0\}$), $S$ and $C$ form a $k$-sun system of $K_{n+4kg}$ with vertex set $V \cup (H \times G)$. \hfill $\square$

We are now ready to prove Theorem 1.1.

**Theorem 1.1.** Let $k \geq 3$ be an odd integer and $v > 1$. Conjecture [4] is true if and only if there exists a $k$-sun system of $K_v$ for all $v$ satisfying the necessary conditions in [4] with $2k < v < 6k$.

**Proof.** The existence of 3-sun systems and 5-sun systems has been solved in [10] and in [8], respectively. Hence we can suppose $k \geq 7$ and $2k < v < 6k$.

We first deal with the case where $(k, v) \neq (7, 21)$. By assumption there exists a $k$-sun system of $K_v$, which implies $v(v-1) \equiv 0 \pmod{4}$, hence Theorem 4.4 guarantees the existence of a $k$-sun system of $K_{4k} + v$. Therefore, by Theorem 5.2 there is a $k$-sun decomposition of $K_{4kg+v}$ whenever $g \neq 2$. To decompose $K_{8k+v}$ into $k$-suns, we first decompose $K_{8k+v}$ into $K_{4k} + v$ and $K_{4k} + (4k + v)$. By Theorem 5.2 (with $g = 1$), there is a $k$-sun system of $K_{4k+v}$. Furthermore, Theorem 4.4 guarantees the existence of a $k$-sun system of $K_{4k} + (4k + v)$, except possibly when $(k, 4k+v) \in \{(7, 56), (7, 57), (7, 64), (11, 100)\}$. Therefore, by Theorem 5.2 there is a $k$-sun decomposition of $K_{8k+v}$ whenever $(k, 4k+v) \notin \{(7, 56), (7, 57), (7, 64), (11, 100)\}$. For each of these for cases we construct $k$-sun systems of $K_{8k+v}$ as follows.
If $k = 7$ and $4k + v = 56$, set $V(K_{84}) = \mathbb{Z}_{83} \cup \{\infty\}$. We consider the following 7-suns

$$T_1 = \begin{pmatrix} 0 & -1 & 3 & -4 & 6 & -7 & 16 \\ 31 & 27 & 37 & 18 & 43 & 12 & 56 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & -2 & 3 & -5 & 6 & -8 & 17 \\ 32 & 27 & 38 & 19 & 44 & 12 & 58 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 0 & -3 & 3 & -6 & 6 & -9 & 18 \\ 33 & 27 & 39 & 20 & 45 & 12 & \infty \end{pmatrix}.$$

One can easily check that $\bigcup_{i=1}^{3} \text{Orb}_{\mathbb{Z}_{83}}(T_i)$ is a 7-sun system of $K_{84}$.

If $k = 7$ and $4k + v = 57$, set $V(K_{85}) = \mathbb{Z}_{85}$. Let $T_1$ and $T_2$ be defined as above, and let $T_3'$ be the graph obtained from $T_3$ replacing $\infty$ with 60. It is immediate that $\bigcup_{i=1}^{2} \text{Orb}_{\mathbb{Z}_{85}}(T_i) \cup \text{Orb}_{\mathbb{Z}_{85}}(T_3')$ is a 7-sun system of $K_{85}$.

If $k = 7$ and $4k + v = 64$, set $V(K_{92}) = (\mathbb{Z}_7 \times \mathbb{Z}_{13}) \cup \{\infty\}$. We consider the following 7-suns

$$T_1 = \begin{pmatrix} (0, 0) & (1, 1) & -(2, 1) & (3, 1) & -(4, 1) & (5, 1) & -(6, 1) \\ \infty & (-1, 1) & (2, 7) & (-3, 5) & -(3, 5) & (5, 7) & (6, 7) \end{pmatrix},$$

$$T_2 = \begin{pmatrix} (0, 0) & (1, 2) & -(2, 2) & (3, 2) & -(4, 2) & (5, 2) & -(6, 2) \\ (0, 10) & -(1, 8) & (2, 8) & (-3, 7) & -(3, 7) & (5, 8) & (6, 8) \end{pmatrix},$$

$$T_3 = \begin{pmatrix} (0, 0) & (1, 3) & -(2, 3) & (3, 3) & -(4, 3) & (5, 3) & -(6, 3) \\ (0, 12) & -(1, 9) & (2, 9) & (-3, 9) & -(3, 9) & (5, 9) & (6, 9) \end{pmatrix}.$$

$$T_4 = \text{Dev}_{\mathbb{Z}_7 \times \{0\}}((0, 0) \sim (4, 0) \sim (6, 8)), \quad T_5 = \text{Dev}_{\mathbb{Z}_7 \times \{0\}}((0, 0) \sim (6, 0) \sim (6, 8)).$$

One can easily check that $\bigcup_{i=1}^{3} \text{Orb}_{\mathbb{Z}_7 \times \mathbb{Z}_{13}}(T_i) \cup \bigcup_{i=4}^{5} \text{Orb}_{\mathbb{Z}_7 \times \{0\}}(T_i)$ is a 7-sun system of $K_{92}$.

If $k = 11$ and $4k + v = 100$, set $V(K_{144}) = (\mathbb{Z}_{11} \times \mathbb{Z}_{13}) \cup \{\infty\}$. We consider the following 11-suns

$$T_1 = \begin{pmatrix} (0, 0) & (1, 1) & -(2, 1) & (3, 1) & -(4, 1) & (5, 1) & -(6, 1) \\ \infty & (-1, 1) & (2, 7) & (-3, 7) & (4, 7) & (-5, 1) & (5, 5) \\ (7, 1) & -(8, 1) & (9, 1) & -(10, 1) \\ -(7, 7) & (8, 7) & -(9, 7) & (10, 7) \end{pmatrix},$$

$$T_2 = \begin{pmatrix} (0, 0) & (1, 2) & -(2, 2) & (3, 2) & -(4, 2) & (5, 2) & -(6, 2) \\ (0, 10) & -(1, 8) & (2, 8) & (-3, 8) & (4, 8) & (-5, 6) & (5, 7) \\ (7, 2) & -(8, 2) & (9, 2) & -(10, 2) \\ -(7, 8) & (8, 8) & -(9, 8) & (10, 8) \end{pmatrix},$$

$$T_3 = \begin{pmatrix} (0, 0) & (1, 3) & -(2, 3) & (3, 3) & -(4, 3) & (5, 3) & -(6, 3) \\ (0, 12) & -(1, 9) & (2, 9) & (-3, 9) & (4, 9) & (-5, 9) & (5, 9) \\ (7, 3) & -(8, 3) & (9, 3) & -(10, 3) \\ -(7, 9) & (8, 9) & -(9, 9) & (10, 9) \end{pmatrix},$$

$$T_4 = \text{Dev}_{\mathbb{Z}_{11} \times \{0\}}((0, 0) \sim (4, 0) \sim (6, 8)), \quad T_5 = \text{Dev}_{\mathbb{Z}_{11} \times \{0\}}((0, 0) \sim (6, 0) \sim (5, 8)), \quad T_6 = \text{Dev}_{\mathbb{Z}_{11} \times \{0\}}((0, 0) \sim (8, 0) \sim (8, 8)).$$

One can check that $\bigcup_{i=1}^{3} \text{Orb}_{\mathbb{Z}_{11} \times \mathbb{Z}_{13}}(T_i) \cup \bigcup_{i=4}^{6} \text{Orb}_{\mathbb{Z}_7 \times \{0\}}(T_i)$ is a 11-sun system of $K_{144}$. 
It is left to prove the existence of a $k$-sun system of $K_{4k+1}$ when $(k, v) = (7, 21)$ and for every $g \geq 1$. If $g = 1$, a 7-sun system of $K_{49}$ can be obtained as a particular case of the following construction. Let $p$ be a prime, $q = p^n \equiv 1 \pmod{4}$ and $r$ be a primitive root of $F_q$. Setting $S = Dev_r(0 \sim r \sim r + 1)$ where $\langle r \rangle = \{jr \mid 1 \leq j \leq p\}$, we have that $\bigcup_{j=0}^{q^n} Orb_{q_j} \circ (r^j S)$ is a $p$-sun system of $K_q$.

If $g \geq 2$, we notice that $K_{28q+21} = K_{28(q-1)+49}$. Considering the 7-sun system of $K_{49}$ just built, and recalling that by Theorem 4.1 there is a 7-sun system of $K_{28} + 49$, then Theorem 6.2 guarantees the existence of a 7-sun system of $K_{28(g-1)+49}$ whenever $g \neq 3$. When $g = 3$, a 7-sun system of $K_{105}$ is constructed as follows.

Set $V(K_{105}) = Z_7 \times Z_{15}$. Let $S_{i,j}$ and $T$ be the 7-suns defined below, where $(i, j) \in X = ([1, 3] \times [1, 7]) \setminus \{(1, 3), (1, 6)\}$, as follows:

$$S_{i,j} = \begin{pmatrix}
(0, 0) & (i, j/2) & (2i, j) & (3i, 0) & (4i, j) & (5i, 0) & (6i, j)
(i, -j/2) & (2i, 0) & (3i, 2j) & (4i, -j) & (5i, 2j) & (6i, -j) & (0, 2j)
(0, 0) & (0, 7) & (0, 2) & (0, 5) & (0, -1) & (0, 3) & (0, 1)
(2, 0) & (3, 7) & (1, 2) & (1, 8) & (1, 5) & (1, 0) & (1, 10)
\end{pmatrix},$$

$$T = \begin{pmatrix}
(0, 0) & (0, 7) & (0, 2) & (0, 5) & (0, -1) & (0, 3) & (0, 1)
(2, 0) & (3, 7) & (1, 2) & (1, 8) & (1, 5) & (1, 0) & (1, 10)
\end{pmatrix}. $$

One can check that $\bigcup_{(i,j)\in X} Orb_{[0] \times Z_{15}}(S_{i,j}) \cup Orb_{Z_7 \times Z_{15}}(T)$ is a 7-sun system of $K_{105}$. \hfill \square

6. Construction of $p$-sun systems, $p$ prime

In this section we prove Theorem 6.2. Clearly in view of Theorem 6.1 it is sufficient to construct a $p$-sun system of $K_v$ for any admissible $v$ with $2p < v < 6p$. Hence, we are going to prove the following result.

**Theorem 6.1.** Let $p$ be an odd prime and let $v(v-1) \equiv 0 \pmod{4p}$ with $2p < v < 6p$. Then there exists a $p$-sun system of $K_v$.

Since the existence of $p$-sun systems with $p = 3, 5$ has been proved in [10] and in [8], respectively, here we can assume $p \geq 7$.

It is immediate to see that by the necessary conditions for the existence of a $p$-sun system of $K_v$, it follows that $v$ lies in one of the following congruence classes modulo $4p$:

1) $v \equiv 0, 1 \pmod{4p}$;
2) $v \equiv p, 3p + 1 \pmod{4p}$ if $p \equiv 1 \pmod{4}$;
3) $v \equiv p + 1, 3p \pmod{4p}$ if $p \equiv 3 \pmod{4}$.

If $v \equiv 0, 1 \pmod{4p}$ we present a direct construction which holds more in general for $p = k$, where $k$ is an odd integer and not necessarily a prime.

**Theorem 6.2.** For any $k = 2t + 1 \geq 7$ there exists a $k$-sun system of $K_{4k+1}$ and a $k$-sun system of $K_{4k}$.

**Proof.** Let $C$ be the $k$-cycle with vertices in $Z$ so defined:

$$C = \{0, -1, 1, -2, 2, -3, 3, \ldots, 1-t, t-1, -t, 2t\}.$$

Note that the list $D_1$ of the positive differences in $Z$ of $C$ is $D_1 = [1, 2t] \cup \{3t\}$. Consider now the ordered $k$-set $D_2 = \{d_1, d_2, \ldots, d_k\}$ so defined:

$$D_2 = [2t + 1, 3t - 1] \cup [3t + 1, 4t + 2].$$
Obviously $D_1 \cup D_2 = [1,2k]$. Let $\{c_1, c_2, \ldots, c_k\}$ be the increasing order of the vertices of the cycle $C$ and set $\ell_r = c_r + d_r$ for every $r \in [1,k]$, with $r \neq \frac{k+1}{2}$ and $\ell_{t+1} = c_{t+1} - d_{t+1}$. It is not hard to see that $V = \{c_1, c_2, \ldots, c_k, \ell_1, \ell_2, \ldots, \ell_k\}$ is a set. Note also that the difference between the largest and the smallest element of $V$ is $7t + 2$. Let $S$ be the sun obtainable from $C$ by adding the pendant edges $\{c_i, \ell_i\}$ for $i \in [1,k]$. Clearly, $\Delta S = \pm(D_1 \cup D_2) = \pm[1,2k]$. So we can conclude that if we consider the vertices of $S$ as elements of $Z_{4k+1}$, the vertices are still pairwise distinct and $\Delta S = Z_{4k+1} \setminus \{0\}$. Then, by applying Corollary 2.2 (with $G = Z_{4k+1}, m = 1, w = 0$), it follows that $\text{Orb}_{Z_{4k+1}} S$ is a $k$-sun system of $K_{4k+1}$.

Now we construct a $k$-sun system of $K_{4k}$. Let $S$ be defined as above and note that $d_k = 2k$. Let $S^*$ be the sun obtained by $S$ setting $\ell_k = \infty$. It is immediate that if we consider the vertices of $S^*$ as elements of $Z_{4k-1} \cup \{\infty\}$, then Corollary 2.2 (with $G = Z_{4k-1}, m = 1, w = 1$) guarantees that $\text{Orb}_{Z_{4k-1}} S^*$ is a $k$-sun system of $K_{4k}$.

Example 6.3. Let $k = 2t+1 = 9$, hence $t = 4$. By following the proof of Theorem 6.2, we construct a 9-sun system of $K_{37}$. Taking $C = (0, -1, 1, -2, 2, -3, 3, -4, 8)$, we have that

$$\{d_1, d_2, \ldots, d_9\} = [9,11] \cup [13,18]$$

$$\{c_1, c_2, \ldots, c_9\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 8\}.$$

Hence $\{\ell_1, \ell_2, \ldots, \ell_9\} = \{5, 7, 9, 12, 14, 16, 18, 20, 26\}$ and we obtain the following 9-sun $S$ with vertices in $Z_{37}$:

$$S = \begin{pmatrix} 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 8 \\ 14 & 12 & 16 & 9 & 18 & 7 & 20 & 5 & 26 \end{pmatrix},$$

such that $\Delta S = Z_{37} \setminus \{0\}$. Therefore, $\text{Orb}_{Z_{37}} S$ is a 9-sun system of $K_{37}$.

From now on, we assume that $p$ is an odd prime number and denote by $\Sigma$ the following $p$-sun:

$$\Sigma = \begin{pmatrix} c_0 & c_1 & \ldots & c_{p-2} & c_{p-1} \\ \ell_0 & \ell_1 & \ldots & \ell_{p-2} & \ell_{p-1} \end{pmatrix}.$$

Lemma 6.4. Let $p$ be an odd prime. For any $x, y \in Z_p$ with $x \neq 0$ and any $i, j \in Z_m$ with $i \neq j$ there exists a $p$-sun $S$ such that $\Delta_{ii} S = \pm x$, $\Delta_{jj} S = y$, $\Delta_{ji} S = -y$ and $\Delta_{hk} S = \emptyset$ for any $(h,k) \in (Z_m \times Z_m) \setminus \{(i,i), (i,j), (j,i)\}$.

Proof. It is easy to see that $S = \text{Dev}_{Z_p \times \{0\}}((0,i) \sim (x,i) \sim (y + x,j))$ is the required $p$-sun.

We will call such a $p$-sun a sun of type $(i,j)$. For the following it is important to note that if $S$ is a $p$-sun of type $(i,j)$, then $|\Delta_{ii} S| = 2$, $|\Delta_{jj} S| = 0$ and $|\Delta_{ij} S| = |\Delta_{ji} S| = 1$.

The following two propositions provide us $p$-sun systems of $K_{mp+1}$ whenever $m \in \{3, 5\}$ and $p \equiv m - 2 \pmod 4$.

Proposition 6.5. Let $p \equiv 1 \pmod 4 \geq 13$ be a prime. Then there exists a $p$-sun system of $K_{3p+1}$.

Proof. We have to distinguish two cases according to the congruence of $p$ modulo 12.

Case 1. Let $p \equiv 1 \pmod 12$. 


If \( p = 13 \), we construct a 13-sun system of \( K_{40} \) as follows. Let \( S \) be the following 13-sun whose vertices are labelled with elements of \( (\mathbb{Z}_{13} \times \mathbb{Z}_4) \cup \{ \infty \} \):

\[
S = \left\{ \begin{array}{c}
\infty & (2, 1) & (4, 2) & (8, 0) & (3, 1) & (6, 2) & (12, 0) \\
(0, 2) & (4, 1) & (8, 1) & (3, 2) & (6, 0) & (12, 1) & (11, 2) \\
(11, 1) & (9, 2) & (5, 0) & (10, 1) & (7, 2) & (1, 0) \\
(9, 0) & (5, 1) & (10, 2) & (7, 0) & (1, 1) & (2, 2) \\
\end{array} \right\},
\]

We have:

\[
\Delta_{12} S = \Delta_{21} S = \pm \{2, 3, 4, 6\}, \quad \Delta_{02} S = \Delta_{20} S = \pm \{1, 4, 5, 6\},
\]

\[
\Delta_{01} S = -\Delta_{10} S = \{1, -2, \pm 3, \pm 5\}, \quad \Delta_{00} S = \Delta_{22} S = \emptyset, \quad \Delta_{11} S = \pm \{2\}.
\]

Now it remains to construct a set \( T \) of edge-disjoint 13-suns such that

\[
\Delta_{12} T = \Delta_{21} T = \{0, \pm 1, \pm 5\}, \quad \Delta_{02} T = \Delta_{20} T = \{0, \pm 2, \pm 3\},
\]

\[
\Delta_{01} T = -\Delta_{10} T = \{0, -1, 2, \pm 4, \pm 6\}, \quad \Delta_{00} T = \Delta_{22} T = \mathbb{Z}_{13}^*, \quad \Delta_{11} T = \mathbb{Z}_{13}^* \setminus \{ \pm 2 \}.
\]

In order to do this it is sufficient to take, \( T = \{T^i_{01} \mid i \in [1, 4]\} \cup \{T^i_{02} \mid i \in [1, 2]\} \cup \{T^i_{10} \mid i \in [1, 3]\} \cup \{T^i_{12} \mid i \in [1, 2]\} \cup \{T^i_{20} \mid i \in [1, 3]\} \cup \{T^i_{21} \mid i \in [1, 3]\} \),

where:

\[
T^i_{01} = \text{Dev}_{\mathbb{Z}_{13} \times \{0\}}((0, 0) \sim (x_1, 0) \sim (y_i + x_1, 1)), \text{ where } x_1 \in [1, 4], y_i \in \pm \{4, 6\},
\]

\[
T^i_{02} = \text{Dev}_{\mathbb{Z}_{13} \times \{0\}}((0, 0) \sim (x_1, 0) \sim (y_i + x_1, 2)), \text{ where } x_1 \in [5, 6], y_i \in \pm \{2\},
\]

\[
T^i_{10} = \text{Dev}_{\mathbb{Z}_{13} \times \{0\}}((0, 1) \sim (x_1, 1) \sim (y_i + x_1, 0)), \text{ where } x_1 \in [1, 3, 4], y_i \in \{0, -1, 2\},
\]

\[
T^i_{12} = \text{Dev}_{\mathbb{Z}_{13} \times \{0\}}((0, 1) \sim (x_1, 1) \sim (y_i + x_1, 2)), \text{ where } x_1 \in [5, 6], y_i \in \{1\},
\]

\[
T^i_{20} = \text{Dev}_{\mathbb{Z}_{13} \times \{0\}}((0, 2) \sim (x_1, 2) \sim (y_i + x_1, 0)), \text{ where } x_1 \in [1, 3], y_i \in \{0, \pm 3\},
\]

\[
T^i_{21} = \text{Dev}_{\mathbb{Z}_{13} \times \{0\}}((0, 2) \sim (x_1, 2) \sim (y_i + x_1, 1)), \text{ where } x_1 \in [4, 6], y_i \in \{0, \pm 5\}.
\]

We have that \( T \cup \text{Orb}_{\mathbb{Z}_{13} \times \{0\}} S \) is a 13-sun system of \( K_{40} \).

Suppose now that \( p \geq 37 \). We proceed in a very similar way to the previous case. Let \( r \) be a primitive root of \( \mathbb{Z}_p \). Consider the \( (\{\mathbb{Z}_p \times \mathbb{Z}_4\} \cup \{\infty\}) \)-labeling \( B \) of \( \Sigma \) so defined:

\[
B(c_0) = \infty, \quad B(c_i) = (r^i, i) \quad \text{for } 1 \leq i \leq p - 1
\]

\[
B(\ell_0) = (0, 2); \quad B(\ell_i) = \begin{cases} 
(r^{i+1}, i) & \text{for } i \in [1, \frac{p-1}{3}] \text{ with } i \equiv 1 \pmod{3}, \\
(r^{i-1}, i) & \text{for } i \in [\frac{p+1}{2}, \frac{3p-27}{4}] \text{ with } i \equiv 1 \pmod{3}.
\end{cases}
\]

Letting \( S = B(\Sigma) \), it is immediate that the labels of the vertices of \( S \) are pairwise distinct. Note that

\[
|\Delta_{00} S| = |\Delta_{22} S| = 0, \quad |\Delta_{11} S| = \frac{p - 9}{2}, \quad |\Delta_{01} S| = |\Delta_{10} S| = \frac{5p + 7}{12},
\]

\[
|\Delta_{ij} S| = \frac{2p - 2}{3} \quad \text{for } (i, j) \in \{(0, 2), (1, 2), (2, 0), (2, 1)\}.
\]

Hence, reasoning as in previous case, we have to construct a set \( T \) of \( p \)-suns such that if \( i \neq j \) then \( \Delta_{ij} T = \mathbb{Z}_p \times \mathbb{Z}_4 \setminus \Delta_{ij} S \) is a set and also \( \Delta_{0i} T = \mathbb{Z}_p \times \mathbb{Z}_4 \setminus \Delta_{0i} S \) is a set. In particular, this implies that for any \( T, T' \in T \) we have \( \Delta_{ij} T \cap \Delta_{ij} T' = \emptyset \) and that \( |\Delta_{0i} T| = |\Delta_{22} T| = p - 1, \quad |\Delta_{11} T| = \frac{2p - 7}{4}, \quad |\Delta_{ij} T| = \frac{p-1}{2} \quad \text{for } (i, j) \in \{(0, 2), (1, 2), (2, 0), (2, 1)\} \), and \( |\Delta_{0i} T| = |\Delta_{10} T| = \frac{2p - 7}{4} \). In order to do this it is sufficient to take \( T \) as a set consisting of \( \frac{p-1}{2} \) sums of type \( (0, 1) \), \( \frac{p-1}{12} \) sum of
type $(1, 0)$, $\frac{p+1}{3}$ suns of type $(1, 2)$, $\frac{p+2}{3}$ suns of type $(2, 0)$, $\frac{p-7}{6}$ suns of type $(2, 1)$, which exist in view of Lemma 6.4. We have that $\text{Orb}_{\mathbb{Z}_p \times \{0\}} S \cup T$ is a $p$-sun system of $K_{3p+1}$.

Case 2. Let $p \equiv 5$ (mod 12). Let $r$ be a primitive root of $\mathbb{Z}_p$. Consider the $((\mathbb{Z}_p \times \mathbb{Z}_3) \cup \{\infty\})$-labeling $B$ of $\Sigma$ so defined:

$$B(c_0) = \infty; \quad B(c_i) = (r^i, i) \quad \text{for} \quad 1 \leq i \leq p-2; \quad B(c_{p-1}) = (1, 0);$$

$$B(\ell_0) = (0, 2); \quad B(\ell_1) = (r, 2); \quad B(\ell_i) = \begin{cases} (r^{i-1}, i+1) & \text{for} \quad i \in [2, \frac{p-1}{2}] \\ (r^{i+1}, i+2) & \text{for} \quad i \in [\frac{p+1}{2}, p-3] \end{cases}$$

$$B(\ell_{p-2}) = (1, 1); \quad B(\ell_{p-1}) = (1, 2);$$

except for $\frac{p-17}{6}$ values of $i \equiv 0$ (mod 3) with $i \in [3, \frac{p-1}{2}]$ for which we set $B(\ell_i) = (r^{i-1}, i)$ and $\frac{p-5}{12}$ values of $i \equiv 0$ (mod 3) with $i \in [\frac{p+1}{2}, p-5]$ for which we set $B(\ell_i) = (r^{i+1}, i)$. Letting $S = B(\Sigma)$, it is easy to see that the labels of the vertices of $S$ are pairwise distinct. Note that

$$|\Delta_{00} S| = \frac{p-9}{2}, \quad |\Delta_{11} S| = |\Delta_{22} S| = 0, \quad |\Delta_{01} S| = |\Delta_{10} S| = \frac{p+1}{2},$$

$$|\Delta_{02} S| = |\Delta_{20} S| = \frac{7p+1}{12}, \quad |\Delta_{12} S| = |\Delta_{21} S| = \frac{2p-4}{3}.$$ 

Hence, we have to construct a set $T$ of $p$-suns such that $|\Delta_{11} T| = |\Delta_{22} T| = p-1$, $|\Delta_{00} T| = \frac{p+1}{2}$, $|\Delta_{01} T| = |\Delta_{10} T| = \frac{p-1}{2}$, $|\Delta_{02} T| = |\Delta_{20} T| = \frac{5p-1}{12}$, and $|\Delta_{12} T| = |\Delta_{21} T| = \frac{5p+1}{12}$. In order to do this it is sufficient to take $T$ as a set consisting of $\frac{p+2}{3}$ suns of type $(0, 1)$, $\frac{p-9}{12}$ suns of type $(1, 0)$, $\frac{p-7}{6}$ suns of type $(2, 0)$, and $\frac{p-5}{12}$ suns of type $(2, 1)$ which exist in view of Lemma 6.4. We have that $\text{Orb}_{\mathbb{Z}_p} S \cup T$ is a $p$-sun system of $K_{3p+1}$. □

**Proposition 6.6.** For any prime $p \equiv 3$ (mod 4) there exists a $p$-sun system of $K_{5p+1}$.

**Proof.** Set $p = 4n + 3$, and let $Y = [1, n]$ and $X = [n+1, 2n+1]$. Consider the following $((\mathbb{Z}_p \times \mathbb{Z}_5) \cup \{\infty\})$-labeling $B$ of $\Sigma$ defined as follows:

$$B(c_0) = (0, 0); \quad B(c_i) = (-1)^{i+1}(i, 1) \quad \text{for} \quad i \in [1, p-1];$$

$$B(\ell_0) = \infty; \quad B(\ell_y) = (-1)^y(y, -1) \quad \text{for} \quad y \in Y;$$

$$B(\ell_{2n+1}) = (-2n-1, 3); \quad B(\ell_{2n+2}) = (-2n-1, -3);$$

$$B(\ell_i) = (-1)^i(i, 3) \quad \text{for} \quad i \in [1, p-1] \setminus (Y \cup \{2n+1, 2n+2\}).$$

One can directly check that the vertices of $S = B(\Sigma)$ are pairwise distinct. Also, it is not hard to verify that $\Delta S$ does not have repetitions and that its complement in $(\mathbb{Z}_p \times \mathbb{Z}_5) \setminus \{(0, 0)\}$ is the set

$$D = \{ \pm(2x, 0) \mid x \in X \} \cup \{ \pm(2y, 4) \mid y \in Y \} \cup \{ \pm(0, 1) \}.$$ 

Clearly, $D$ can be partitioned into $n+1$ quadruples of the form $D_x = \{ \pm(2x, 0), \pm(r_x, s_x) \}$ with $x \in X$ and $s_x \neq 0$. Letting

$$S_x = D_{\text{ev}_{\mathbb{Z}_p \times \{0\}}}(0, 0) \sim (2x, 0) \sim (r_x + 2x, s_x)$$

for $x \in X$, it is clear that $\Delta S_x = D_x$, hence $\Delta \{S_x \mid x \in X\} = D$. Therefore, Corollary 2.2 guarantees that $\bigcup_{x \in X} \text{Orb}_{\mathbb{Z}_p \times \mathbb{Z}_5}(S_x) \cup \text{Orb}_{\mathbb{Z}_p \times \mathbb{Z}_5}(S) = \bigcup_{x \in X} \text{Orb}_{\mathbb{Z}_p \times \mathbb{Z}_5}(S_x)$ is a $p$-sun system of $K_{5p+1}$. □
Example 6.7. Here, we construct a 7-sun system of $K_{36}$ following the proof of Proposition 6.6. In this case, $Y = \{1\}$ and $X = \{2, 3\}$. Now consider the 7-sun $S$ defined below, whose vertices lie in $(\mathbb{Z}_7 \times \mathbb{Z}_9) \cup \{\infty\}$:

$$S = \begin{pmatrix}
(0, 0) & (1, 1) & -(2, 1) & (3, 1) & -(4, 1) & (5, 1) & -(6, 1) \\
\infty & -(1, -1) & (2, 3) & (-3, 3) & -(3, 3) & (5, 3) & (6, 3)
\end{pmatrix}.$$  

We have

$$\Delta S = \pm \{(1, 1), (3, 2), (5, 2), (0, 2), (2, 2), (4, 2), (6, 1), (2, 0), (4, 4), (6, -2), (1, -2), (3, 4), (5, 4)\}.$$  

Hence $\Delta S$ does not have repetitions and its complement in $(\mathbb{Z}_7 \times \mathbb{Z}_9) \setminus \{(0, 0)\}$ is the set

$$D = \pm \{(4, 0), (6, 0), (2, 4), (0, 1)\}.$$  

Now it is sufficient to take

$$S_2 = Dev_{\mathbb{Z}_7 \times \{0\}}((0, 0) \sim (4, 0) \sim (6, 4)) \quad S_3 = Dev_{\mathbb{Z}_7 \times \{0\}}((0, 0) \sim (6, 0) \sim (6, 5)).$$

One can check that $\bigcup_{x \in X} Orb_{0 \times \mathbb{Z}_9}(S_x) \cup Orb_{\mathbb{Z}_7 \times \mathbb{Z}_9}S$ is a 7-sun system of $K_{36}$.

We finally construct $p$-sun systems of $K_{mp}$ whenever $p \equiv m \pmod{4}$.  

Proposition 6.8. Let $m$ and $p$ be odd prime numbers with $m \leq p$ and $m \equiv p \pmod{4}$. Then there exists a $p$-sun system of $K_{mp}$.

Proof. For each pair $(r, s) \in \mathbb{Z}_p^* \times \mathbb{Z}_m$, let $B_{r,s} : V(\Sigma) \to \mathbb{Z}_p \times \mathbb{Z}_m$ be the labeling of the vertices of $\Sigma$ defined as follows:

$$B_{r,s}(c_0) = (0, 0),$$
$$B_{r,s}(c_i) = B_{r,s}(c_{i-1}) + \begin{cases}
(r, s) & \text{if } i \in [1, m+1] \cup \{m+3, m+5, \ldots, p-1\}, \\
(r, -s) & \text{if } i \in \{m+2, m+4, \ldots, p-2\},
\end{cases}$$
$$B_{r,s}(\ell_i) = B_{r,s}(c_i) + \begin{cases}
(r, -s) & \text{if } i \in [0, m] \cup \{m+2, m+4, \ldots, p-2\}, \\
(r, s) & \text{if } i \in \{m+1, m+3, \ldots, p-1\}.
\end{cases}$$

Since $B_{r,s}$ is injective, for every $h \in \mathbb{Z}_m$ the graph $S^h_{r,s} = \tau_{(0,h)}(B_{r,s}(\Sigma))$ is a $p$-sun.  

For $i, j \in \mathbb{Z}_m$, we also notice that $\Delta \{S^h_{r,s} \mid h \in \mathbb{Z}_m\} = \{\pm r\}$ whenever $i \neq j$. Otherwise, it is empty.

Letting $S$ be the union of the following two sets of $p$-suns:

$$\{S^h_{r,1} \mid h \in \mathbb{Z}_m, r \in [1, (p + m - 2)/4]\},$$
$$\{S^h_{r,s} \mid h \in \mathbb{Z}_m, r \in [1, (p-1)/2], s \in [2, (m-1)/2]\},$$

it is not difficult to see that for every $i, j \in \mathbb{Z}_m$

$$\Delta_{ij}S = \begin{cases}
\emptyset & \text{if } i = j, \\
\pm \left[1, \frac{p+m-2}{4}\right] & \text{if } i-j = \pm 1, \\
\mathbb{Z}_p^* & \text{otherwise.}
\end{cases}$$

It is left to construct a set $T$ of $p$-suns such that $\Delta_{ij}T = \mathbb{Z}_p \setminus \Delta_{ij}S$ whenever $i \neq j$, and $\Delta_{ii}T = \mathbb{Z}_p^* \setminus \Delta_{ii}S = \mathbb{Z}_p^*$. Therefore,

$$|\Delta_{ij}T| = \begin{cases}
p-1 & \text{if } i = j, \\
\frac{p-m-1}{2} + 1 & \text{if } i - j = \pm 1, \\
1 & \text{otherwise.}
\end{cases}$$
Proof of Theorem 6.1

We end this section by proving Theorem 6.1. One can check that \( \Delta_{ij} \) is an odd prime. In other words, the necessary conditions for the existence of a \( p \)-sun system of \( K_r \) are also sufficient whenever \( p \) is an odd prime. In other words, we end this section by proving Theorem 6.1.

\[ \Delta_{ij} \{S_{r,1}^h, S_{r,1}^i, S_{r,1}^j} \} = \{ \pm r \} \] if \( i \neq j \), otherwise it is empty.

Therefore, letting \( S = \{ S_{r,1}^h \mid h \in Z_3, r \in [1, 3] \} \), we have that \( \Delta_{ij}S \) is non-empty only when \( i \neq j \), in which case we have \( \Delta_{ij}S = \pm [1, 3] \).

Now let \( T = \{ T_{hg} \mid h \in Z_3, g \in [1, 5] \} \) where \( T_{hg} \) is the 11-sun defined as follows:

\[ T_{hg} = \{ (0, h) \sim (1, h) \sim (1, h+1) \}, \]

Note that each \( T_{hg} \) is an 11-sun of type \((h, h+1)\). Therefore we have that

\[ \Delta_{ij}T = \begin{cases} \{ \pm[1, 5] \} & \text{if } 0 \leq i = j \leq 2, \\ \{0\} \cup [4, 7] & \text{otherwise.} \end{cases} \]

By Corollary 2.2 it follows that \( S \cup T \) is an 11-sun system of \( K_{33} \).

We are now ready to show that the necessary conditions for the existence of a \( p \)-sun system of \( K_r \) are also sufficient whenever \( p \) is an odd prime. In other words, we end this section by proving Theorem 6.1.

**Proof of Theorem 6.1**

If \( p = 3, 5 \) the result can be found in [10] and in [8], respectively. For \( p \geq 7 \), the result follows from Propositions [6.5, 6.6] and [6.8].

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