Vacuum Energy in Ultralocal metrics for TT tensors with Gaussian Wave Functionals

Remo Garattini

Facoltà di Ingegneria, Università di Bergamo
Viale Marconi, 5, 24044 Dalmine (Bergamo) Italy
e-mail: Garattini@mi.infn.it

(July-1995)

Abstract

We calculate, in a class of Gauge invariant functionals, by variational methods, the difference of vacuum energy between two different backgrounds: Schwarzschild and Flat Space. We perform this evaluation in an Hamiltonian formulation of Quantum Gravity by standard "3 + 1" decomposition. After the decomposition the scalar curvature is expanded to second order with respect to the Schwarzschild metric. We evaluate this energy difference in momentum space, in the lowest possible state (regardless of any negative mode). We find a singular behaviour in the UV-limit, due to the presence of the horizon when $r = 2m$. When $r > 2m$ this singular behaviour disappears, which is in agreement with various other models presented in the literature.

I. INTRODUCTION

The problem of computing quantum corrections to a classical energy in a complicated theory such as Einstein gravity, can be approached by performing an analysis of the thermodynamical quantities that characterize the system under consideration. This analysis can be done by means of the computation of the free energy of the system at a given volume
and temperature. Defining the Euclidean action as
\[ \hat{I}[g] = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} R(g) - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} K^i_i, \] (1)
where \( R \) is the Ricci scalar of the metric \( g_{\mu\nu} \) and \( K^i_i \) is the trace of the second fundamental form, one can compute quantum corrections to the Euclidean action, of a fixed background geometry with the appropriate boundary conditions. Since we wish to study quantum fluctuations with respect to the Schwarzschild geometry, there are two types of boundary conditions that are related to the background under consideration:

a) Asymptotically Euclidean (AE),

b) Asymptotically Flat (AF).

An AE background metric is one in which the metric approaches the flat metric on \( R^4 \) outside some compact set. The boundary at infinity is topologically \( S^3 \). An AF background metric is one in which the metric approaches the flat metric \( R^3 \times S^1 \) outside some compact set. The boundary of infinity is topologically \( S^2 \times S^1 \). Anyway, we could consider a different point of view based on the Hamiltonian approach. In this framework one is able to deal with three dimensional fields configurations separated out by the time variable. The advantage of the Hamiltonian framework is that one can manage from the beginning with energy fields configurations which give directly the quantum corrections to the classical term. To do this, the first step is the separation of the three dimensional space from the time by means of the \( ADM \) variables \[4\]. In terms of these variables the line element becomes
\[ ds^2 = g_{\mu\nu} (x) dx^\mu dx^\nu = -N^2 (dx^0)^2 + g_{ij} \left( N^i dx^0 + dx^i \right) \left( N^j dx^0 + dx^j \right) = \left( -N^2 + N_i N^i \right) (dx^0)^2 + 2N_j dx^0 dx^j + g_{ij} dx^i dx^j. \] (2)

\( N \) is called the \textit{lapse function}, while \( N_i \) the \textit{shift function}. The associated matrix representation of \( g_{\mu\nu} \) is
\[ g_{\mu\nu} = \begin{pmatrix} -N^2 + N_i N^i & N_j \\ N_i & g_{ij} \end{pmatrix}, \] (3)
with the inverse given by
\[ g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N_j}{N^2} \\ \frac{N_i}{N^2} & g_{ij} - \frac{N_i N_j}{N^2} \end{pmatrix}. \] (4)

Roman indices will be raised and lowered by the induced metric on the three surface \( x^0 \).

In terms of the \( \text{ADM} \) variables, the initial action can be written as a sum of a “kinetic” and a “potential” term
\[
I = \frac{1}{16\pi G} \int dx^0 N \int dx^3 \sqrt{g} \left\{ (K_{ij}K^{ij} - K^2) + (3)R \right\},
\] (5)
where \( K_{ij} = \frac{1}{2N} (N_{ij} + N_{ji} - g_{ij,0}) \) is called the second fundamental form and “|” means covariant differentiation with respect to the 3D gravitational background, \( K \) is the trace of the second fundamental form, \( 3R \) is the scalar curvature in 3D, \( (3)\sqrt{g} \) is the invariant of the metric in 3D. In this form the time derivative is isolated and it is possible the computation of the conjugate momentum to \( g_{ij} \), that is
\[
\pi^{ij} = \frac{\delta I}{\delta \dot{g}_{ij}} = \left(-K^{ij} + (3)g^{ij}K\right) \frac{\sqrt{g}}{16\pi G}. \] (6)

By a Legendre transformation we calculate the Hamiltonian
\[
H = \int d^3 x \left\{ \frac{1}{16\pi G} \left[ (K_{ij}K^{ij} - K^2) + (3)R \right] N (3)\sqrt{g} \right\} = \int d^3 x \left\{ N \left[ \frac{16\pi G}{(3)\sqrt{g}} \left( \pi_{ij} \pi^{ij} - \frac{\pi^2}{2} \right) - (3)\sqrt{g} \frac{(3)R}{16\pi G} \right] + N_i \left( 2\pi^{ij}_i \right) \right\}. \] (7)
The first term of (7) has a quadratic structure in the momenta, suggesting, as a first approximation that we could compute quantum corrections to the energy expanding \( 3R \) in terms of the quantum field fluctuations with respect to a given background, e.g. the Schwarzschild background. Since we wish to understand the pure gravitational vacuum, we neglect the matter fields and since in our approach only the spatial part of the background comes into play our background is of the wormhole type \[10\]. Although the energy computation at quantum level is very unclear because of the constraint coming from the lapse function we will adopt the Hamiltonian approach the same and to this purpose a simple framework will
be shown in section II. The rest of the paper is structured as follows, in section III we analyze the orthogonal decomposition of the Hamiltonian both in tangent and co-tangent space, in section IV we define the gaussian wave functional for gravity in analogy with non-abelian gauge theories, in section V we give some of the basic rules to perform the functional integration and we define the Hamiltonian approximated up to second order, in section VI, we analyze the spin-2 operator or the operator acting on transverse traceless tensors, only for positive values of $E^2$. We summarize and conclude in section VII.

II. THE HAMILTONIAN ON THE SLICE

After the introduction of ADM variables, we recall that the Hamiltonian is:

$$H = \int d^3x (N\mathcal{H} + N_i\mathcal{H}^i)$$

(8)

where

$$\mathcal{H} = G_{ijkl}\pi^{ij}\pi^{kl} \left( \frac{l_p^2}{\sqrt{g}} \right) - \left( \frac{\sqrt{g}}{l_p^2} \right)^3 R \ (\text{Super Hamiltonian})$$

(9)

and

$$\mathcal{H}^i = -2\pi^{ij}_{,j} \ (\text{Super Momentum}).$$

(10)

In (9) the derivative is covariant with respect to the 3D background field, $l_p^2$ is the usual Planck mass, and $G_{ijkl}$ is the Wheeler-DeWitt (WDW) metric. If we look at $N$ and $N_i$ as fundamental objects describing the correct variables, by variational principles we obtain the usual constraint equations, that is

$$\mathcal{H} = 0, \ \mathcal{H}^i = 0 \ \text{Classical}$$

$$\mathcal{H}\Psi = 0, \ \mathcal{H}^i \Psi = 0 \ \text{Quantum}$$

(11)

The usual interpretation of these equations is that they represent constraints on the initial value problem or in other words they represent gauge invariance with respect to time and
gauge transformations. Nevertheless we have a chance to define and computing energy if we restrict on a given hypersurface, fixing the lapse function to a constant. Such a gauge choice is the most appropriate for wormhole configurations of the background geometry and in particular for the Schwarzschild wormhole. By rescaling time intervals, we obtain

\[ N = 1. \]  

(12)

Actually, this choice is compatible with the suspension constraint that one can adopt in quantum cosmology to obtain a Schrödinger-like equation, provided at the end of the calculation one assures that the gauge invariance is restored [1] [9]. Then the Hamiltonian in the time-like gauge is

\[
H = \int d^3x \mathcal{H} = \int d^3x \left[ G_{ijkl} \pi^{ij} \pi^{kl} \left( \frac{p^2}{\sqrt{g}} \right) - \left( \frac{\sqrt{g}}{p^2} \right) R \right] 
\]  

(13)

Since (13) is valid on a “fixed” hypersurface, to recover the original equation, i.e. (8), the correct procedure to perform will be a summation over all possible lapses; this means that the constraint (11) (at classical or at quantum level) will be restored after this summation.

### III. ULTRALOCAL METRICS AS A TOOL FOR DECOMPOSING TENSOR FIELDS

Instead of performing calculations in the usual WDW metric we will use a one-parameter family of supermetrics to disentangle gauge modes from physical deformations. For this reason we require an orthogonal decomposition for both \( \pi_{ij} \) and \( h_{ij} \), that is we need a metric on the space of deformations, i.e. a quadratic form on the tangent space at \( h \). The condition of ultralocality, where \( G_{ijkl} \) locally depends on \( g_{ij} \) but not on its derivatives, could be taken as a good condition for the functional measure, explicitly:

---

1 A different treatment, but close to our present approach, is based on the separation between dynamical variables and embedding parameters and can be found in Ref. [8].
\[ \langle h, k \rangle := \int_{\mathcal{M}} \sqrt{g} G^{ijkl}_{\alpha} h_{ij} (x) k_{kl} (x) \, d^3 x, \]  
(14)

where

\[ G^{ijkl}_{\alpha} = (g^{ik} g^{jl} + g^{il} g^{jk} - 2\alpha g^{ij} g^{kl}). \]  
(15)

The WDW metric, introduced in (9), is just (15) with \( \alpha = 1 \). The “inverse” metric is defined on co-tangent space and it assumes the form

\[ \langle p, q \rangle := \int_{\mathcal{M}} \sqrt{g} G^{ijkl}_{\beta} p_{ij} (x) q_{kl} (x) \, d^3 x, \]  
(16)

where

\[ G^{ijkl}_{\beta} = (g_{ik} g_{jl} + g_{il} g_{jk} - 2\beta g_{ij} g_{kl}). \]  
(17)

with \( \alpha + \beta = 3\alpha\beta \), so that

\[ G^{ijnm}_{\beta} G^{nmkl}_{\beta} = \frac{1}{2} \left( \delta^i_k \delta^j_l + \delta^i_l \delta^j_k \right). \]  
(18)

These are non-degenerate bilinear forms for \( \alpha \neq \frac{1}{3} \), for \( \alpha = \frac{1}{3} \) the metric is not invertible and becomes a projector onto the tracefree subspace, while is positive definite for \( \alpha < \frac{1}{3} \) and of mixed signature for \( \alpha > \frac{1}{3} \) with infinitely many plus as well as minus signs.

We have now the desired decomposition on the tangent space of 3-metric deformations \( h_{ij} \):

\[ h_{ij} = \frac{1}{3} h g_{ij} + (L \xi)_{ij} + h_{ij}^\perp, \]  
(19)

or, in matrix form,

\[ h = \frac{1}{3} h g + (\text{Range} L) + (\text{Ker} L)^\perp, \]  
(20)

where the operator \( L \) maps \( \xi_i \) into symmetric tracefree tensors, according to [2] [3].
\[(L \xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3} g_{ij} (\nabla \cdot \xi). \tag{21}\]

Consequently, the inversion of the metric (16) (that is (17)) guarantees us the same decomposition also in phase space (co-tangent space).

**IV. THE GAUSSIAN WAVE FUNCTIONAL**

There are some reasons to introduce a gaussian wave functional for the description of the vacuum state in gravity. Starting from the analogy between nonabelian gauge theories and gravity we illustrate how gaussian wave functional work in the former case. We define

\[\Psi [A^a_i (\vec{x})] = \mathcal{N} \exp \left\{ -\frac{1}{4} \int d^3x d^3y \delta A^a_i (\vec{x}) G^{-1}_{ij} (\vec{x}, \vec{y}) \delta A^b_j (\vec{y}) \right\} \tag{22}\]

where \(\mathcal{N}\) is a normalization factor and where

\[\delta A^a_i (\vec{x}) = A^a_i (\vec{x}) - \overline{A}^a_i (\vec{x}). \tag{23}\]

In equation (23), \(\overline{A}^a_i (\vec{x})\) is a background field which can be treated as a variational parameter together to the function \(G^{-1}_{ij} (\vec{x}, \vec{y})\) in (22).

From the definition in (22) one finds the expectation values

\[\langle \Psi | A^a_i (\vec{x}) | \Psi \rangle = \overline{A}^a_i (\vec{x}), \]

\[\langle \Psi | A^a_i (\vec{x}) A^b_j (\vec{y}) | \Psi \rangle = \overline{A}^a_i (\vec{x}) \overline{A}^b_j (\vec{y}) + \mathcal{G}^{ab}_{ij} (\vec{x}, \vec{y}), \]

\[\langle \Psi | E^a_i (\vec{x}) | \Psi \rangle = 0, \tag{24}\]

\[\langle \Psi | E^a_i (\vec{x}) E^b_j (\vec{y}) | \Psi \rangle = \frac{1}{4} \mathcal{G}^{-1ab}_{ij} (\vec{x}, \vec{y}), \]

\[\langle \Psi | B^a_i (\vec{x}) | \Psi \rangle = \overline{B}^a_i (\vec{x}) + \frac{1}{2} \epsilon_{ijk} f^{abc} \mathcal{G}^{ab}_{ij} (\vec{x}, \vec{y}), \]
where
\[ E^a_i (\vec{x}) = -i \frac{\delta}{\delta A_i^a (\vec{x})}, \quad (25) \]
and
\[ B^a_i (\vec{x}) = \varepsilon_{ijk} \left\{ \nabla_j A^a_k (\vec{x}) + \frac{1}{2} f^{abc} A^a_i (\vec{x}) A^b_j (\vec{y}) \right\}, \quad (26) \]
\( \varepsilon_{ijk} \) is the usual anti-symmetric tensor and \( f^{abc} \) are the structure constants of the gauge group, for ex. \( SU(N) \). After having experienced how the apparatus works on nonabelian gauge theories, we define a “Vacuum Trial State” for gravity, and for this purpose we recall the orthogonal decomposition (18) to look at the essential structure of the inner product between three-geometries
\[ \langle h, h \rangle := \int_M \sqrt{g} G^{ijkl}_{\alpha} h_{ij} (x) h_{kl} (x) d^3x = \int_M \sqrt{g} \left[ \left( \frac{1}{3} - \alpha \right) h^2 + (L \xi)^{ij}_{\parallel} (L \xi)_{ij} + h^{ij \perp} h_{ij} \right] \quad (27) \]
Previous formula leads us towards the definition of the “possible” trial wave functional for the gravitational ground state
\[ \Psi_{\alpha} \left[ h_{ij} (\vec{x}) \right] = N \exp \left\{ -\frac{1}{4l_p^2} \left[ \langle h K^{-1} h \rangle^\perp_{x,y} + \langle (L \xi) K^{-1} (L \xi) \rangle^\parallel_{x,y} + \langle h K^{-1} h \rangle^{Tr} \right] \right\}, \quad (28) \]
or in other terms
\[ \Psi_{\alpha} \left[ h_{ij} (\vec{x}) \right] = N \Psi_{\alpha} \left[ h^\perp_{ij} (\vec{x}) \right] \Psi_{\alpha} \left[ (L \xi)_{ij} \right] \Psi_{\alpha} \left[ \frac{1}{3} g_{ij} h (\vec{x}) \right]. \quad (29) \]
In (28) and in (29), \( h^\perp_{ij} \) is the tracefree-transverse part of the 3D quantum field, \( (L \xi)_{ij} \) is the longitudinal part and finally \( h \) is the trace part of the same field. The dependence of the functional by \( \alpha \) will not be discussed in this paper.

In (28), \( \langle \cdot, \cdot \rangle_{x,y} \) denotes space integration and \( K^{-1} \) is the inverse propagator. The main reason for a similar “Ansatz” comes not only from (27) but even to the observation that the momenta quadratic part of the Hamiltonian decouples in the same way. Even if we had to give up to (28) from the beginning, making a more general “Ansatz” about the vacuum wave
functional (and for more general we mean eqn. (22) ) one would discover that the kinetic part decouples in these three terms. For completeness, we give the analogous expectation values for TT tensors. The other components satisfy the same rules

\[ \langle \Psi | g_{ij}^\perp (\vec{x}) | \Psi \rangle = g_{ij}^\perp \],

\[ \langle \Psi | g_{ij}^\perp (\vec{x}) h_{kl}^\perp (\vec{y}) | \Psi \rangle = g_{ij}^\perp (\vec{x}) h_{kl}^\perp (\vec{y}) + K_{ijkl} \],

(30)

\[ \langle \Psi | \pi_{ij}^\perp (\vec{x}) | \Psi \rangle = 0, \]

\[ \langle \Psi | \pi_{ij}^\perp (\vec{x}) \pi_{kl}^\perp (\vec{y}) | \Psi \rangle = K^{-1}_{ijkl} \],

where \( \pi_{ij} = -i \frac{\delta}{\delta h_{ij}(x)} \) is the representation for the TT momentum.

V. ENERGY DENSITY CALCULATION IN SCHröDINGER REPRESENTATION

To calculate the energy density associated to the trial functional, we need to know the action of some basic operators on \( \Psi [h_{ij}(\vec{x})] \). The action of the operator \( h_{ij} \) on \( |\Psi\rangle = \Psi [h_{ij}(\vec{x})] \) is realized by

\[ h_{ij} (x) |\Psi\rangle = h_{ij} (x) \Psi [h_{ij}(\vec{x})]. \]

(31)

The action of the operator \( \pi_{ij} \) on \( |\Psi\rangle \), in general, is

\[ \pi_{ij} (x) |\Psi\rangle = -i \frac{\delta}{\delta h_{ij}(x)} \Psi [h_{ij}(\vec{x})]. \]

(32)

The inner product is defined by the functional integration:

\[ \langle \Psi_1 | \Psi_2 \rangle = \int \mathcal{D}h_{ij}(x) \Psi_1 \{ h_{ij} \} \Psi_2 \{ h_{kl} \}, \]

(33)

and the energy eigenstates satisfy the Schrödinger equation:

\[ \int d^3x \mathcal{H} \left\{ -i \frac{\delta}{\delta h_{ij}(x)}, h_{ij} (x) \right\} \Psi \{ h_{ij}(\vec{x}) \} = E \Psi \{ h_{ij}(\vec{x}) \}, \]

(34)
where $\mathcal{H} \left\{ -i \frac{\delta}{\delta h_{ij}(x)}, h_{ij}(x) \right\}$ is the Hamiltonian density. Instead of solving (34), which is of course impossible, we can formulate the same problem by means of a variational principle. We demand that
\[
\langle \Psi | H | \Psi \rangle = \int [Dg_{ij}^\perp(x)] \int d^3x \Psi^*_i \left\{ g_{ij}^\perp \right\} \mathcal{H} \Psi \left\{ g_{kl}^\perp \right\}
\]
be stationary against arbitrary variations of $\Psi \left\{ h_{ij}(x) \right\}$. The form of $\langle \Psi | H | \Psi \rangle$ can be computed as follows. We define normalized mean values by a straightforward modification of (30), i.e.
\[
\bar{g}_{ij}^\perp(x) = \frac{\int [Dg_{ij}^\perp(x)] \int d^3x g_{ij}^\perp(x) \Psi^*_i \left\{ g_{ij}^\perp(x) \right\} \Psi^*_i \left\{ g_{ij}^\perp(x) \right\} }{\int [Dg_{ij}^\perp(x)] \Psi \left\{ g_{ij}^\perp \right\} }.
\]
\[
\bar{g}_{ij}^\perp(x) \bar{g}_{kl}^\perp(x) + K_{ijkl}^\perp(\vec{x}, \vec{y}) = \frac{\int [Dg_{ij}^\perp(x)] \int d^3x g_{ij}^\perp(x) g_{kl}^\perp(y) \Psi^*_i \left\{ g_{ij}^\perp(x) \right\} \Psi^*_i \left\{ g_{ij}^\perp(x) \right\} }{\int [Dg_{ij}^\perp(x)] \Psi \left\{ g_{ij}^\perp \right\} }.
\]
It follows that
\[
\int [Dh_{ij}^\perp(x)] \left( g_{ij}^\perp(x) - \bar{g}_{ij}^\perp(x) \right) \Psi^*_i \left\{ g_{ij}^\perp(x) \right\} \Psi^*_i \left\{ g_{ij}^\perp(x) \right\} = 0
\]
by translation invariance of the measure
\[
\int [Dh_{ij}^\perp(x)] h_{ij}^\perp(x) \Psi^*_i \left\{ g_{ij}^\perp(x) + \bar{g}_{ij}^\perp(x) \right\} \Psi^*_i \left\{ g_{ij}^\perp(x) \right\} = 0
\]
\[
\implies \int [Dh_{ij}^\perp(x)] h_{ij}^\perp(x) \Psi^*_i \left\{ h_{ij}^\perp(x) \right\} \Psi^*_i \left\{ h_{ij}^\perp(x) \right\} = 0,
\]
and (37) becomes
\[
\int [Dh_{ij}^\perp(x)] \int d^3x h_{ij}^\perp(x) h_{kl}^\perp(y) \Psi^*_i \left\{ h_{ij}^\perp(x) \right\} \Psi^*_i \left\{ h_{ij}^\perp(x) \right\} = 0.
\]
(39)
\[
K_{ijkl}^\perp(\vec{x}, \vec{y}) \int [Dh_{ij}^\perp(x)] \Psi^*_i \left\{ h_{ij}^\perp(x) \right\} \Psi^*_i \left\{ h_{ij}^\perp(x) \right\}.
\]
Rather than applying the variational principle arbitrarily, the gaussian Ansatz is made, according to which in the beginning of this calculus one has to replace previous general formulas with
\[ \Psi_\alpha [h_{ij}(\vec{x})] = \mathcal{N} \exp \left\{ -\frac{1}{4l_p^2} \left( (g - \bar{g}) K^{-1} (g - \bar{g}) \right)_{x,y}^{\perp} + \ldots \right\}. \quad (40) \]

With this choice and with formulas (38, 39), the one loop-like Hamiltonian can be written as

\[ H^\perp = \frac{1}{4l_p^2} \int_M d^3x \sqrt{g} G^{ijkl}_{\alpha} \left[ K_{\alpha}^{-1\perp} (x, x)_{ijkl} + (\triangle_2)_{ij} \, K^\perp (x, x)_{ijkl} \right] \quad (41) \]

where the first term in square brackets comes from the kinetic part and the second comes from the expansion of \( ^3R \) up to second order in such a way to obtain a quantum harmonic oscillator equation type. The Green function \( K^\perp (x, x)_{ijkl} \) can be represented as

\[ K^\perp (x, x)_{ijkl} := \sum_N \frac{h^\perp_{ia}(x) h^\perp_{kl}(y)}{2\lambda_N(p)}, \quad (42) \]

where \( h^\perp_{ia}(x) \) are the eigenfunctions of \( \triangle^a_j \) and \( \lambda_N(p) \) are infinite variational parameters. In formula (41) we have written the Spin-2 contribution to the energy density alone; expressions like (41) exist for Spin-1 and Spin-0 terms of \( \mathcal{H} \).

**VI. THE SPECTRUM OF THE SPIN-2 OPERATOR AND THE EVALUATION OF THE ENERGY DENSITY**

The Spin-2 operator is defined by:

\[ \triangle_2 := -\triangle + 2Ric \quad (43) \]

or in components,

\[ (\triangle_2)^a_j := -\triangle^a_j + 2R^a_j \quad (44) \]

where \( \triangle \) is the curved Laplacian (Laplace-Beltrami operator) on a Schwarzschild background and

\( R^a_j \) is the mixed Ricci tensor whose components are:
\[ R^a_j = \text{diag} \left\{ \frac{-2m}{r^3}, \frac{m}{r^3}, \frac{m}{r^3} \right\}, \quad (45) \]

where \(2m = 2MG\). This operator is similar to the Lichnerowicz operator provided that we substitute the Riemann tensor with the Ricci tensor. In (43) or (44) Ricci tensor acts as a potential on the space of TT tensors; for this reason we are led to study the following eigenvalue equation

\[ \left( -\triangle \delta^a_j + 2R^a_j \right) h^i_a = E^2 h^i_j \]

where \(E^2\) is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analyzing the equation into modes of definite frequency, angular momentum and parity. In this paper we are interested to positive \(E^2\) and low lying states with \(L = M = 0\), where \(L\) is the quantum number corresponding to the square of angular momentum and \(M\) is the quantum number corresponding to the projection of the angular momentum on the \(z\)-axis. For \(L = 0\), Regge-Wheeler decomposition [7] shows that there are no odd-parity perturbations at all, therefore:

\[ h^\text{even}_{ij} = \text{diag} \left[ H(r) \left( 1 - \frac{2m}{r} \right)^{-1}, r^2 K (r), r^2 \sin^2 \vartheta K (r) \right] Y_{00} (\vartheta, \phi). \quad (47) \]

The representation (47) is very useful, because of the decoupling of the components, in fact

\[ -\triangle H (r) - \frac{4m}{r^2} H (r) = E^2 H (r) \]

\[ -\triangle K (r) + \frac{2m}{r} K (r) = E^2 K (r) \]

\[ -\triangle K (r) + \frac{2m}{r} K (r) = E^2 K (r) \]

The Laplacian in this particular geometry can be written as

\[ \triangle = \left( 1 - \frac{2m}{r} \right) \frac{d^2}{dr^2} + \left( \frac{2r - 3m}{r^2} \right) \frac{d}{dr}. \quad (49) \]

Defining reduced fields, such as:

\[ H (r) = \frac{h (r)}{r}; K (r) = \frac{k (r)}{r}, \quad (50) \]
and changing variables to

\[ x = 2m \left\{ \sqrt{\frac{r}{2m}} \sqrt{\frac{r}{2m}} - 1 + \ln \left( \sqrt{\frac{r}{2m}} + \sqrt{\frac{r}{2m}} - 1 \right) \right\}, \]

the system (48) becomes

\[ -\frac{d^2}{dx^2} h(x) - V(x) h(x) = E^2 h(x) \]

\[ -\frac{d^2}{dx^2} k(x) + V(x) k(x) = E^2 k(x) \]

(52)

where

\[ V(x) = \frac{3m}{r^3} \]

(53)

We note that the new variable is such that

\[ x \simeq r \rightarrow \infty \quad V(x) \rightarrow 0 \]

\[ x \simeq 0 \quad r \rightarrow r_0 \quad V(x) \rightarrow \frac{3m}{r_0^3} = \text{const}, \]

(54)

where \( r_0 \) is the wormhole radius, satisfying the condition \( r_0 > 2m \), strictly. The solution of (52), in both cases (flat and curved one) is a Bessel function and precisely the spherical Bessel function of the first kind for the \( L = 0 \) value of the angular momentum

\[ j_0 (pr) = p\sqrt{\frac{2}{\pi}} \frac{\sin (pr)}{pr} = \sqrt{\frac{2}{\pi}} \frac{\sin (pr)}{r} \]

(55)

The corresponding Green function for this problem will be

\[ K(x, y) = \frac{j_0(px)j_0(py)}{2\lambda} \cdot \frac{1}{4\pi} \]

(56)

Substituting (56) in (11) one gets (after normalization in spin space and after a rescaling of the fields in such a way to absorb \( l_p^2 \))

\[ E(m, \lambda) = \frac{V}{2\pi^2} \sum_{i=1}^{2} \int_{0}^{\infty} dp p^2 \left[ \lambda_i(p) + \frac{E_i^2(p, m)}{\lambda_i(p)} \right] \]

(57)
where

$$E_{1,2}^2 (p, m) = p^2 + \frac{3m}{r_0^3},$$  \hspace{1cm} (58)$$

$\lambda_i (p)$ are variational parameters corresponding to the eigenvalues for a (graviton) spin-2 particle in an external field and $V$ is the volume of the system.

By minimizing (57) with respect to $\lambda_i (p)$ one obtains $\lambda_i (p) = [E_i^2 (p, m)]^{\frac{1}{2}}$ and

$$E (m, \lambda) = \frac{V}{2\pi^2} \sum_{i=1}^{2} \int_{0}^{\infty} dp 2 \sqrt{E_i^2 (p, m)} \text{ with } p^2 > \frac{3m}{r_0^3}$$  \hspace{1cm} (59)$$

The total energy in the presence of the background is

$$E (m) = \frac{V}{2\pi^2} \frac{1}{2} \int_{0}^{\infty} dp p^2 \left( \sqrt{p^2 - c^2} + \sqrt{p^2 + c^2} \right) \text{ where } c^2 = \frac{3m}{r_0^3}$$  \hspace{1cm} (60)$$

For flat space the calculation is essentially the same with the exception of $c^2 = 0$. Therefore the equivalent of (60) in flat space is

$$E (0) = \frac{V}{2\pi^2} \frac{1}{2} \int_{0}^{\infty} dp p^2 \left( 2 \sqrt{p^2} \right)$$  \hspace{1cm} (61)$$

Now, we are in position to perform the energy difference between (60) and (61). $\Delta E (m)$ up to second order in perturbations is

$$\Delta E (m) = \frac{V}{2\pi^2} \frac{1}{2} \int_{0}^{\infty} dp p^2 \left[ \sqrt{p^2 - c^2} + \sqrt{p^2 + c^2} - 2\sqrt{p^2} \right]$$  \hspace{1cm} (62)$$

We want to evaluate the $UV$ behaviour of (62), therefore

$$\Delta E (m) = \frac{V}{2\pi^2} \frac{1}{2} \int_{0}^{\infty} dp p^3 \left[ \sqrt{1 - \left( \frac{c}{p} \right)^2} + \sqrt{1 + \left( \frac{c}{p} \right)^2} - 2 \right]$$

becomes for $p^2 >> c^2$

$$\sim \frac{V}{2\pi^2} \frac{1}{2} \int_{0}^{\infty} dp p^3 \left[ 1 - \frac{1}{3} \left( \frac{c}{p} \right)^2 - \frac{1}{8} \left( \frac{c}{p} \right)^4 + 1 + \frac{1}{2} \left( \frac{c}{p} \right)^2 - \frac{1}{8} \left( \frac{c}{p} \right)^4 + 2 \right]$$

$$= -\frac{V}{2\pi^2} \frac{c^4}{8} \int_{0}^{\infty} \frac{dp}{p}$$  \hspace{1cm} (63)$$
Introducing a cut-off one gets for the $UV$ limit
\[
\int_0^\infty \frac{dp}{p} \sim \int_0^\Lambda c \frac{dx}{x} \sim \ln \left( \frac{\Lambda}{c} \right) \tag{64}
\]
and $\Delta E(m)$ for high momenta can be estimated by the following expression
\[
\Delta E(m) \sim -\frac{V}{2\pi^2} \frac{c^4}{16} \ln \left( \frac{\Lambda^2}{c^2} \right) = -\frac{V}{2\pi^2} \left( \frac{3m}{r_0^3} \right)^2 \frac{1}{16} \ln \left( \frac{r_0^3\Lambda^2}{3m} \right). \tag{65}
\]
At this point we can compute the total energy, namely the classical contribution plus the quantum correction up to second order. Recalling the definition of asymptotic energy for an asymptotically flat background, like the Schwarzschild one
\[
E_{ADM} = \lim_{r \to \infty} \int_{\partial M} \sqrt{\hat{g}}_{ij} \left[ \hat{g}_{ik,j} - \hat{g}_{ij,k} \right] dS^k, \tag{66}
\]
where $\hat{g}_{ij}$ is the metric induced on a spacelike hypersurface $\partial M$ which has a boundary at infinity like $S^2$, one gets,
\[
M - \frac{V}{2\pi^2} \left( \frac{3m}{r_0^3} \right)^2 \frac{1}{16} \ln \left( \frac{r_0^3\Lambda^2}{3m} \right) = M - \frac{V}{2\pi^2} \left( \frac{3MG}{r_0^3} \right)^2 \frac{1}{16} \ln \left( \frac{r_0^3\Lambda^2}{3MG} \right) \tag{67}
\]
One can observe that
\[
\Delta E(m) \to \infty \text{ when } m \to 0, \text{ for } r_0 = 2m = 2GM \tag{68}
\]
and
\[
\Delta E(m) \to 0 \text{ when } m \to 0, \text{ for } r_0 \neq 2m = 2GM. \tag{69}
\]

**Remark** We would like to explain the reasons that support the results of formula (65). In that formula we introduced a particular value of the radius, which behaves as a regulator

\footnote{It is known that at one-loop level Gravity is renormalizable only in flat space. In a dimensional regularization scheme its contribution to the action is, on shell, proportional to the Euler character of the manifold that is nonzero for the Schwarzschild instanton. Although in our approach we are working with sections of the original manifold to deal with these divergences one must introduce a regulator that indeed appears in the contribution of the energy density.}
with respect to the horizon approach of the potential. The meaning of this particular value is related to the necessity of explaining the dynamical origin of black hole entropy by the entanglement entropy mechanism and by the so-called “brick wall model” \[^{[3]}\]. Indeed, the same mechanism is present when one has to regularize entropy by imposing a kind of cut-off, that in coordinate space means \( r_0 > 2m \).

VII. SUMMARY AND CONCLUSIONS

The trial wave functional approach, by means of Gaussian configurations, led to possible calculations of quantum fluctuations of the gravitational field around some fixed background geometry. In particular we have studied a spherically symmetric background and by means of Birkhoff’s theorem we can claim that our background is of the Schwarzschild type. Since we have performed this analysis without any matter contribution and recalling the definition of Ref. \[^{[10]}\] our result is valid for a Schwarzschild wormhole. However this calculation apparatus is entirely based on the possibility of explicitly breaking the invariance under reparametrisations, expressed by the gauge fixing \(^{[12]}\), leading to the conclusion that the final result seems depending on the foliation we choose to work. For this reason to restore the invariance under reparametrisations, we need to sum on every lapse function \(^{[3]}\). With the gauge choice \(^{[12]}\), the problem of defining a correct vacuum energy on every slice is well posed and the result shows us an intrinsic energy depending only on the dynamics generated by 3-surfaces.

VIII. ACKNOWLEDGMENTS

I wish to thank G. Esposito, V. Frolov, E. Gozzi, R. Parentani, D.L. Rapoport, E. Recami for helpful discussion and P. Saurgnani who gave me the technical support for the realization of the calculations. A detailed version of this procedure will be studied in a future paper \(^{[1]}\).
of this work. I also thank S. Liberati and B. Jensen who suggested me how to justify the horizon approach.

APPENDIX A: CONVENTIONS

Here we give the conventions for the metric tensor, connections and the curvature tensor:

1. Background Metric

\[ g^{11} = \frac{1}{1 - 2m/r}, \quad g^{22} = r^2, \quad g^{33} = r^2 \sin^2 \theta \]

(A1)

\[ g^{11} = 1 - \frac{2m}{r}, \quad g^{22} = r^{-2}, \quad g^{33} = r^{-2} \sin^{-2} \theta \]

2. Connection

\[ \Gamma^1_{ab} = \begin{pmatrix} -\frac{m}{r} (1 - 2m/r)^{-1} \\ - (1 - 2m/r) r \\ - (1 - 2m/r) r \sin^2 \theta \end{pmatrix} \]

(A2)

\[ \Gamma^2_{ab} = \begin{pmatrix} 0 & r^{-1} & 0 \\ r^{-1} & 0 & 0 \\ 0 & 0 & -\sin \theta \cos \theta \end{pmatrix}, \quad \Gamma^3_{ab} = \begin{pmatrix} 0 & 0 & r^{-1} \\ 0 & 0 & \cot \theta \\ r^{-1} \cot \theta & 0 \end{pmatrix} \]

3. Riemann tensor, Ricci tensor and the Scalar Curvature in 3D

\[ R_{ijm}^l = \Gamma^l_{mi,j} - \Gamma^l_{ji,m} + \Gamma^l_{ja} \Gamma^a_{mi} - \Gamma^l_{ma} \Gamma^a_{ji} \quad \text{Riemann tensor} \]

Because of the vanishing of the Weyl tensor in 3D, that is \( C_{ijm}^l = 0 \), Riemann tensor is completely determined by Ricci tensor

\[ R_{ijm} = g_{ij} R_{lm} - g_{lm} R_{ij} - g_{ij} R_{lm} + g_{im} R_{lj} \]

\[ R_{im} = R_{ilm} \quad \text{Ricci tensor} \]

\[ R = g^{ij} R_{ij} \quad \text{Scalar curvature} \]
APPENDIX B: SCALAR CURVATURE EXPANSION

In this part we give the necessary tools for the scalar curvature expansion in terms of the fluctuations of 3-surfaces around the background geometry. Metric tensor will be separated in a classical part (background) plus a quantum part i.e.

\[ g_{ij} = \bar{g}_{ij} + h_{ij} \]  

(B1)

1. Expansion of the determinant

\[
\sqrt{g_{ij}} = \exp Tr \ln \sqrt{g_{ij}} = \exp Tr \left[ \frac{1}{2} \ln \left( \bar{g}_{ij} + h_{ij} \right) \right] = \\
\exp Tr \left[ \frac{1}{2} \left( \ln \bar{g}_{ij} + \ln \left( 1 + \frac{h_{ij}}{\bar{g}_{ij}} \right) \right) \right] \simeq \\
\sqrt{\bar{g}_{ij}} \left( 1 + \frac{1}{2} h - \frac{1}{4} h^2 \right) + O(h^4) 
\]

(B2)

2. The inverse metric expansion

\[
g^{ij} = \frac{\bar{g}^i_j}{\delta_k^j + h_k} \simeq \bar{g}^{ij} - h^{ij} + h^{ik} h^k_i - h^{ik} h^l_k \bar{h}^l_i + h^{ik} h^m_l \bar{h}^m_i + O(h^5) 
\]

(B3)

3. Connection

\[
\Gamma^k_{ij} := \frac{1}{2} g^{kl} (\bar{g}_{li,j} + \bar{g}_{lj,i} - \bar{g}_{ij,l}) \\
\]

To 0th order \( \Gamma^k_{ij} = \Gamma^{k(0)}_{ij} = \frac{1}{2} g^{kl} (\bar{g}_{li,j} + \bar{g}_{lj,i} - \bar{g}_{ij,l}) \).

To 1st order \( \Gamma^k_{ij} = S^k_{ij} = \frac{1}{2} g^{kl} \left( h_{li,j} + h_{lj,i} - h_{ij,l} \right) \)

The higher order corrections to the connection are related by the formula

\[
\Gamma^{k(n)}_{ij} = -h^-_l \Gamma^{l(n-1)}_{ij} \text{ where } \Gamma^{k(1)}_{ij} = S^k_{ij} 
\]

(B4)

4. Riemann Tensor

\[
R^l_{ijm} = \Gamma^l_{mi,j} - \Gamma^l_{ji,m} + \Gamma^l_{ja} \Gamma^a_{mi} - \Gamma^l_{ma} \Gamma^a_{ji} 
\]
It is convenient to divide Riemann tensor into two terms: linear and non-linear

\[ \text{Lin} \left( R_{ijm}^l \right) = L_{ijm}^l := \Gamma_{mi,j}^l - \Gamma_{ji,m}^l \]  
(B5)

\[ N - \text{Lin} \left( R_{ijm}^l \right) = N_{ijm}^l := \Gamma_{ja}^l \Gamma_{mi}^a - \Gamma_{ma}^l \Gamma_{ji}^a \]

The higher order terms of \( L_{ijm}^l \) are simply

\[ L_{ijm}^{l(n)} := \Gamma_{mi,j}^{l(n)} - \Gamma_{ji,m}^{l(n)} \]  
(B6)

while higher orders of \( N_{ijm}^l \) are

\[ N_{ijm}^{l(n)} := \sum_{j=1}^{n-1} \left[ \Gamma_{mi}^{a(j)} \Gamma_{ja}^{l(n-j)} - \Gamma_{ma}^{l(n-j)} \Gamma_{ji}^{a(j)} \right] \]  
(B7)

5. Second order scalar curvature

Collecting together previous expansion formulas we can write the following expression for the \( ^3R \) scalar curvature expanded up to second order:

\[ \int d^3x \left[ -\frac{1}{4} h\triangle h + \frac{1}{4} h^{li} \triangle h_{li} - \frac{1}{2} h^{ij} \nabla_i \nabla_j h^l + \frac{1}{2} h \nabla_i \nabla_j h^l - \frac{1}{2} h^{ij} R_{ia} h^a_j + \frac{1}{2} h R_{ij} h^{ij} \right]. \]  
(B8)
REFERENCES

[1] L. J. Garay, Phys. Rev. D 48, 1710 (1993).

[2] P. O. Mazur and E. Mottola, Nucl. Phys. B 341, 187 (1990), D. Giulini, Phys. Rev. D 10 5630 (1995).

[3] J. W. York Jr., J. Math. Phys., 14, 4 (1973).

[4] R. Arnowitt, S. Deser, and C. W. Misner, in Gravitation: An Introduction to Current Research, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962); B. S. DeWitt, Phys. Rev. 160, 1113 (1967).

[5] A. K. Kerman and D. Vautherin, Ann. Phys., 192, 408 (1989); J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D 8, 242 (1974); R. Jackiw in Séminaire de Mathématiques Supérieures, Montréal, Québec, Canada- June 1988 - Notes by P. de Sousa Gerbert; M. Consoli and G. Preparata, Phys. Lett. B, 154, 411 (1985).

[6] G.’t Hooft, Nucl. Phys. B 256 (1985), 727; V.P. Frolov, I.Novikov, Phys. Rev. D 48 (1993), 4545.

[7] T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957)

[8] A. Kheyfets and W. A. Miller, gr-qc/9412037

[9] J.J. Halliwell, “Introductory Lectures on Quantum Cosmology”. In Jerusalem Winter School for Theoretical Physics: Quantum Cosmology and Baby Universes Vol. 7. S.Coleman, J.B. Hartle, T. Piran and S. Weinberg, eds. World Scientific, 159-243.

[10] C.W.Misner, K.S. Thorne and J.A. Wheeler, Gravitation (Freeman, San Francisco, 1973) 842; M.S. Morris and K.S. Thorne, Am. J. Phys. 56 (1988) 395.

[11] R. Garattini, in preparation.