ZERO-CYCLES OVER ZERO-DIMENSIONAL CUSPS

SHOUHEI MA

Abstract. We prove that all points of a toroidal compactification lying over 0-dimensional cusps are rationally equivalent in the integral Chow group of the toroidal compactification for Siegel modular varieties, Hilbert modular varieties and orthogonal modular varieties. This gives a generalization, and even strengthening, of the Manin-Drinfeld theorem in higher dimension from the viewpoint of algebraic cycles.

1. Introduction

In [12], [6], Manin and Drinfeld discovered that the difference of two cusps of a congruence modular curve is torsion in its Picard group. We wish to study higher dimensional analogue of the Manin-Drinfeld theorem from the viewpoint of algebraic cycles. In [11] the Manin-Drinfeld theorem was generalized in the form of rational equivalence of two cusps in the rational Chow group of the Baily-Borel compactification of some classical modular varieties. The Baily-Borel compactification is however highly singular at the boundary, and it has been recognized that the study of Chow groups of singular varieties is more difficult than for nonsingular varieties.

Toroidal compactification [1] provides an explicit desingularization of the Baily-Borel compactification when the arithmetic group is neat. Even when not smooth, its singularities are relatively mild and tractable. It should be noticed, however, that a naive analogue of the Manin-Drinfeld theorem no longer holds for the boundary divisors of toroidal compactification. In this paper we study 0-cycles over 0-dimensional cusps in the toroidal compactifications, and show that they are rationally equivalent in the integral Chow group. The viewpoint of 0-cycles thus provides one direction in which the Manin-Drinfeld theorem can be generalized and even strengthened.

To be more precise, we consider the following three series of classical modular varieties.

2010 Mathematics Subject Classification. 14C25, 11F46, 11F41, 11F55.

Key words and phrases. Manin-Drinfeld theorem, 0-cycle, toroidal compactification, Siegel modular variety, Hilbert modular variety, orthogonal modular variety.

Supported by KAKENHI 17K14158 and and 20H00112.
• Siegel modular variety $Y = \mathcal{D}/\Gamma$ attached to a symplectic lattice $\Lambda$ of rank $> 2$ and a finite-index subgroup $\Gamma$ of the symplectic group $\text{Sp}(\Lambda)$.

• Hilbert modular variety $Y = \mathbb{H}^n/\Gamma$ attached to a totally real number field $F$ of degree $n > 1$ and a finite-index subgroup $\Gamma$ of $\text{SL}_2(O_F)$.

• Orthogonal modular variety $Y = \mathcal{D}/\Gamma$ attached to an integral quadratic form $\Lambda$ of signature $(2, n)$ with $n > 1$ and a finite-index subgroup $\Gamma$ of the orthogonal group $O^+(\Lambda)$.

Let $X^{bb}$ be the Baily-Borel compactification of $Y$, and $X^{tor}$ be the toroidal compactification of $Y$ defined by a $\Gamma$-admissible collection of fans $\Sigma$. (We suppress the dependence on $\Sigma$ in the notation.) We may assume that $\Sigma$ is chosen so that $X^{tor}$ is projective (11). We have a natural morphism $\pi : X^{tor} \to X^{bb}$. Note that $X^{bb}$ has only 0-dimensional cusps in the case of Hilbert modular varieties.

Our results are as follows.

**Theorem 1.1** (Siegel case). Let $X^{tor}$ be a toroidal compactification of a Siegel modular variety. Let $P, Q \in X^{tor}$ be two points such that $\pi(P), \pi(Q)$ are 0-dimensional cusps (which may be distinct). Then $[P] = [Q]$ in the integral Chow group $\text{CH}_0(X^{tor})$ of $X^{tor}$.

**Theorem 1.2** (Hilbert case). Let $X^{tor}$ be a toroidal compactification of a Hilbert modular variety $Y$. Let $P, Q$ be two points of $X^{tor} - Y$. Then $[P] = [Q]$ in $\text{CH}_0(X^{tor})$.

**Theorem 1.3** (Orthogonal case). Let $X^{tor}$ be a toroidal compactification of an orthogonal modular variety. Suppose that either $n \geq 3$ or $\Lambda$ has Witt index 1. Let $P, Q \in X^{tor}$ be two points such that $\pi(P), \pi(Q)$ are 0-dimensional cusps. Then $[P] = [Q]$ in $\text{CH}_0(X^{tor})$.

These results are in contrast to the case of modular curves, where the difference of two cusps is nontrivial unless the modular curve has genus 0, and the calculation of its order in the Picard group has been the subject of many works on modular units and cuspidal class groups. Thus, in this aspect, the situation gets simpler in higher dimension.

The proof in the Siegel case proceeds as follows. First we prove $[P] = [Q]$ in the rational Chow group. When $\pi(P) = \pi(Q)$, this is a simple consequence of the structure of the boundary. When $\pi(P) \neq \pi(Q)$, we apply the Manin-Drinfeld theorem to a chain of interior modular curves joining $\pi(P)$ and $\pi(Q)$. Finally, we promote rational equivalence from rational to integral Chow group. This is a consequence of Roitman’s theorem (15) and Weissauer’s theorem (19), two deep theorems from different branches of Algebraic Geometry. We can say that it is this last step that makes the difference between the curve case and the higher dimensional case. At the
same time, this step leaves it somewhat mysterious how $P$ and $Q$ are equivalent in the integral Chow group.

The proof in the Hilbert case is similar. It is also possible to give a similar proof in the orthogonal case, but instead of such a repetition, we reduce the proof to the cases of Siegel 3-folds and Hilbert modular surfaces already covered in Theorems 1.1 and 1.2.

The case excluded from Theorem 1.3 is essentially that of product of two modular curves ($n = 2$, Witt index 2). In fact, Theorem 1.3 does not hold for them. This can be seen by considering their Albanese map, which is the product of the Abel-Jacobi maps of each modular curve and hence injective if both curves have genus $> 0$. In this sense, the assumption in Theorem 1.3 would be best possible.

A similar argument would be also applicable to the unitary groups $U(p, p)$, but would not work for $U(1, p)$ for two reasons. The first is the structure of the boundary: it is abelian varieties, whose $\text{CH}_0$ is infinite-dimensional by the generalized Mumford theorem [14]. The second is non-vanishing of their Albanese varieties [17]. The unitary case has dropped from [11] for these reasons.

This paper is organized as follows. In §2 we prove Theorem 1.1. In §3 we prove Theorem 1.2. In §4 we prove Theorem 1.3. In Appendix A we provide a short proof that most orthogonal modular varieties have trivial Albanese varieties (cf. [5]).

2. SIEGEL MODULAR VARIETIES

In this section we prove Theorem 1.1 (§2.2) after recollection in §2.1.

2.1. SIEGEL MODULAR VARIETIES. We first recall Siegel modular varieties and their Satake and toroidal compactifications. Let $\Lambda$ be a free abelian group of rank $2g > 2$ equipped with a nondegenerate symplectic form $\Lambda \times \Lambda \rightarrow \mathbb{Z}$. The Hermitian symmetric domain $\mathcal{D} = \mathcal{D}_\Lambda$ attached to $\Lambda$ is defined as the open locus of the Lagrangian Grassmannian consisting of $g$-dimensional isotropic subspaces $W$ of $\Lambda_\mathbb{C}$ such that the associated Hermitian form on $W$ is positive-definite. Let $\Gamma$ be a finite-index subgroup of the symplectic group $\text{Sp}(\Lambda)$. By [13], [3], $\Gamma$ is a congruence subgroup. The quotient $Y = \mathcal{D}/\Gamma$ is a quasi-projective variety, called the Siegel modular variety defined by $\Gamma$.

The Satake compactification of $Y$, denoted by $X^{\text{sb}}$ for consistency with other sections, is defined as the quotient by $\Gamma$ of the union of $\mathcal{D}$ and its rational boundary components, equipped with the Satake topology. It is normal and projective ([2]). Rational boundary components of $\mathcal{D}$ correspond to isotropic subspaces $I$ of $\Lambda_\mathbb{Q}$. In particular, when $I$ is maximal, i.e., $\dim I = g$, the corresponding boundary component is 0-dimensional, given
by the point $[I_{\mathbb{C}}]$ of the Lagrangian Grassmannian. We call (the closure of) the image of a rational boundary component in $X^{\text{bb}}$ a cusp.

A toroidal compactification $X^{\text{tor}} = X^\Sigma_I$ of $Y$ (11) is constructed by choosing a $\Gamma$-admissible collection of fans $\Sigma = (\Sigma_i)$, one for each isotropic subspace $I$ of $\Lambda_{\mathbb{Q}}$. Each fan $\Sigma_i$ is a polyhedral cone decomposition of the cone of positive semidefinite forms with rational kernel in $\text{Sym}^2 I_{\mathbb{R}}$. It defines a partial compactification $(\mathcal{D}/U(I)_{\mathbb{Z}})^{\Sigma_i}$ of $\mathcal{D}/U(I)_{\mathbb{Z}}$. Here $U(I)_{\mathbb{Z}} = \Gamma \cap U(I)_{\mathbb{Q}}$ is a lattice in $U(I)_{\mathbb{Q}} \cong \text{Sym}^2 I_{\mathbb{R}}$, the center of the unipotent part of the stabilizer $\Gamma(I)_{\mathbb{Q}}$ of $I$ in $\text{Sp}(\Lambda_{\mathbb{Q}})$. We write $\Gamma(I)_{\mathbb{Z}} = \Gamma \cap \Gamma(I)_{\mathbb{Q}}$ and $\overline{\Gamma(I)_{\mathbb{Z}}} = \Gamma(I)_{\mathbb{Z}}/U(I)_{\mathbb{Z}}$. Then $X^{\text{tor}}$ is constructed as a gluing of $Y$ and the quotients $(\mathcal{D}/U(I)_{\mathbb{Z}})^{\Sigma_i}/\overline{\Gamma(I)_{\mathbb{Z}}}$, $X^{\text{tor}}$ is a compact Moishezon space with a natural morphism $\pi : X^{\text{tor}} \to X^{\text{bb}}$. We may choose $\Sigma$ so that $X^{\text{tor}}$ is projective (11).

We are interested in the case $\dim I = g$. Let $p \in X^{\text{bb}}$ be the 0-dimensional cusp corresponding to $I$. In this case $\mathcal{D}/U(I)_{\mathbb{Z}}$ is embedded in the algebraic torus $T_I = \text{Sym}^2 I_{\mathbb{C}}/U(I)_{\mathbb{Z}}$ as the locus of symmetric forms with positive-definite imaginary part, by the tube domain realization of $\mathcal{D}$ with respect to $I$ (= isomorphism with the Siegel upper half space). The partial compactification $\mathcal{D}/U(I)_{\mathbb{Z}} \hookrightarrow (\mathcal{D}/U(I)_{\mathbb{Z}})^{\Sigma_i}$ is an open set of the torus embedding $T_I \hookrightarrow T_I^{\Sigma_i}$ defined by the fan $\Sigma_i$. If $\sigma \in \Sigma_i$ is a cone whose relative interior consists of positive-definite forms, the corresponding torus orbit $\text{orb}(\sigma) \subset T_I^{\Sigma_i} \subset T_I^{\Sigma_i}$ is contained in $(\mathcal{D}/U(I)_{\mathbb{Z}})^{\Sigma_i}$. Then the locus of boundary points of $(\mathcal{D}/U(I)_{\mathbb{Z}})^{\Sigma_i}$ which lie over the cusp $p$ consists of these torus orbits (11) p.164. This shows that

$$\pi^{-1}(p) = (\sqcup_{i} \text{orb}(\sigma)) / \overline{\Gamma(I)_{\mathbb{Z}}}.$$  

where $\sigma$ runs over cones with positive-definite relative interior, and $\sqcup_{i} \text{orb}(\sigma)$ is the union of infinitely many many toric varieties whose configuration is determined by $\Sigma_i$.

2.2. **Proof of Theorem 1.1.** We now prove Theorem 1.1. Recall that the given toroidal compactification $X^{\text{tor}} = X^\Sigma_I$ is assumed to be projective. We begin with the following reduction, which will be necessary in Step 4.

**Step 1.** We may assume that $\Gamma$ is neat and $X^{\text{tor}}$ is smooth and projective.

*Proof.* We take a neat subgroup $\Gamma'$ of $\Gamma$ of finite index. Then $\Sigma$ is also $\Gamma'$-admissible, so we have the toroidal compactification $X^{\Sigma}_{\Gamma'}$ of $\mathcal{D}/\Gamma'$ defined by $\Sigma$. Since we have a finite morphism $X^{\Sigma}_{\Gamma'} \to X^\Sigma_I$ and $X^\Sigma_I$ is projective, $X^{\Sigma}_{\Gamma'}$ is projective too. Next, by 11 Corollary III.7.6, we can take a $\Gamma'$-admissible subdivision $\Sigma'$ of $\Sigma$ such that $X^{\Sigma'}_{\Gamma'} \to X^{\Sigma}_{\Gamma'}$ is projective and $X^{\Sigma'}_{\Gamma'}$ is smooth. Then $X^{\Sigma'}_{\Gamma'}$ is also projective.

For any point of $X^{\Sigma'}_{\Gamma'}$ lying over a 0-dimensional cusp of $X^{\text{bb}}$, the points in its inverse image in $X^{\Sigma'}_{\Gamma'}$ lie over 0-dimensional cusps of $X^{\text{bb}}$. Therefore,
if we could prove Theorem 1.1 for \( X^\Sigma_{\Gamma} \), the assertion for \( X^\Sigma_{\Gamma} \) follows by pushforward. In what follows, we rewrite \((\Gamma', \Sigma')\) as \((\Gamma, \Sigma)\). \qed

Let \( P, Q \) be two points of \( X^{tor} \) such that \( p = \pi(P) \) and \( q = \pi(Q) \) are 0-dimensional cusps of \( X^{bb} \). We first show that \([P] - [Q] \in CH_0(X^{tor})\) is torsion in two steps.

**Step 2.** If \( p = q \), then \([P] = [Q]\) in \( CH_0(X^{tor})\).

*Proof.* By the description (2.1) of the \( \pi \)-fiber over \( p \), we see that \( \pi^{-1}(p) \) is a connected configuration of finitely many toric varieties. This shows that we can join \( P \) and \( Q \) by a chain of rational curves inside \( \pi^{-1}(p) \). \qed

**Step 3.** Even when \( p \neq q \), \([P] - [Q] \in CH_0(X^{tor})\) is torsion.

*Proof.* We take maximal isotropic subspaces \( I, I' \) of \( \Lambda_Q \) which correspond to \( p, q \) respectively. We first consider the case when the pairing \((I, I')\) is nondegenerate. In that case, we take splittings \( I = I_1 \oplus \cdots \oplus I_g \) and \( I' = I'_1 \oplus \cdots \oplus I'_g \) such that \((I_i, I'_j) = 0 \) if \( i \neq j \) and \((I_i, I'_i) = Q \). If we put \( \Lambda_i = I_i \oplus I'_i \), we have the orthogonal splitting \( \Lambda_Q = \Lambda_1 \oplus \cdots \oplus \Lambda_g \) as a symplectic space. Note that we have an isomorphism \( \Lambda_i \cong Q^2 \) of symplectic space which sends \( I_i, I'_i \) to \( Q(1, 0), Q(0, 1) \) respectively. This defines an embedding of the upper half plane \( \mathbb{H} = D_{Q^2} \subset \mathbb{P}^1 \) as

\[
D_{Q^2} \hookrightarrow D_{\Lambda_1} \times \cdots \times D_{\Lambda_g} \hookrightarrow D_{\Lambda}, \quad V \mapsto V \oplus \cdots \oplus V.
\]

Since \( \Gamma \) is a congruence subgroup, this induces a finite morphism \( Y(N) \to Y \) from the principal congruence modular curve \( Y(N) = \mathbb{H}/\Gamma(N) \) of some level \( N \). If \( X(N) \) is the compactification of \( Y(N) \), this extends to a morphism \( X(N) \to X^{bb} \) which sends the cusps \( 0, i\infty \) of \( X(N) \) to \( p, q \in X^{bb} \) respectively. Furthermore, since \( X(N) \) is a curve, this lifts to a morphism \( f : X(N) \to X^{tor} \) to the toroidal compactification. By the Manin-Drinfeld theorem, we have \([0] = [i\infty] \in CH_0(X(N))_Q \). Sending this equality by \( f \) and using Step 2, we find that

\[
[P] = [f(0)] = [f(i\infty)] = [Q]
\]

in \( CH_0(X^{tor})_Q \). This proves the assertion when the pairing \((I, I')\) is nondegenerate.

When \((I, I')\) is degenerate, we can find another maximal isotropic subspace \( I'' \) such that both \((I, I'')\) and \((I', I'')\) are nondegenerate. If we take a point \( R \in X^{tor} \) lying over the \( I'' \)-cusp, we can use the above case twice to see that \([P] = [R] = [Q] \in CH_0(X^{tor})_Q \). \qed

**Step 4.** We have \([P] = [Q] \) in \( CH_0(X^{tor}) \).

*Proof.* Let \( A_0(X^{tor}) \) be the subgroup of \( CH_0(X^{tor}) \) of degree 0 cycles. By the theorem of Roitman [15], the Albanese map \( A_0(X^{tor}) \to Alb(X^{tor}) \) is
injective on the torsion part of $A_0(X_{tor})$. On the other hand, by the theorem of Weissauer [19], there is no nonzero holomorphic 1-form on $X_{tor}$, hence the Albanese variety $\text{Alb}(X_{tor})$ is trivial. This implies that $A_0(X_{tor})$ is torsion-free, and so $[P] - [Q] = 0$ in $\text{CH}_0(X_{tor})$. This completes the proof of Theorem 1.1. □

Remark 2.1. A similar argument, except for the final step, and using finiteness of the Mordell-Weil groups of elliptic modular surfaces [18], yields a weaker result for boundary 1-cycles in the case of Siegel 3-folds: if $Z_1, Z_2 \subset X_{tor}$ are irreducible 1-cycles such that $\pi(Z_1), \pi(Z_2)$ are 1-dimensional cusps, then $aZ_1 \sim bZ_2 + F$ for some $a, b \in \mathbb{Z}_{>0}$ and a 1-cycle $F$ whose support is contracted by $\pi$ to points.

Remark 2.2. The situation is totally different for the boundary divisors, which tend to be linearly independent already in the level of cohomology. See [2] Proposition 4.3 for the boundary divisors over cusps of corank 1. Moreover, for $\Gamma$ neat, subdivision of fan leads to blow-up, which just adds new components to $H^2$.

Remark 2.3. In general, it is commutativity of the unipotent radical of the stabilizer of the cusp that makes Step 2 work.

3. Hilbert modular varieties

Let $F$ be a totally real number field of degree $n > 1$. The group $\text{SL}_2(F)$ acts on $\mathbb{H}^n$ through the natural embedding $\text{SL}_2(F) \hookrightarrow \text{SL}_2(\mathbb{R})^n$. Let $\Gamma$ be a finite-index subgroup of $\text{SL}_2(O_F)$. By [16], $\Gamma$ is a congruence subgroup. The quotient $Y = \mathbb{H}^n/\Gamma$ is a quasi-projective variety called the Hilbert modular variety defined by $\Gamma$. The Baily-Borel compactification $Y \hookrightarrow X^{bb}$ is obtained by adding finitely many points (0-dimensional cusps) corresponding to $\mathbb{P}^1(F)/\Gamma$, with the Satake topology ([2], [7]). A toroidal compactification $X_{tor}$ of $Y$ can be constructed by choosing a collection of suitable polyhedral cone decompositions of the cone of totally positive elements of $F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^n$, one for each cusp independently ([1], [7]). We have a natural morphism $X_{tor} \to X^{bb}$.

3.1. Proof of Theorem 1.2. The proof of Theorem 1.2 is similar (and simpler) to the case of Siegel modular varieties. Instead of repetition, we just indicate that the ingredients required in the proof have counterparts in the present case. We may assume as before that $\Gamma$ is neat and $X_{tor}$ is smooth and projective.

- The boundary divisor of $X_{tor}$ over a cusp is again a connected configuration of toric varieties.
• The diagonal embedding $\mathbb{H} \hookrightarrow \mathbb{H}^n$ gives a congruence modular curve joining the cusps $0 = (0, \ldots, 0)$ and $i\infty = (i\infty, \ldots, i\infty)$. Since $\text{SL}_2(F)$ acts transitively on $\mathbb{P}^1(F) \times \mathbb{P}^1(F) \setminus \text{diagonal}$, we can translate the diagonal embedding to obtain a congruence modular curve joining any two distinct cusps.

• There is no nonzero holomorphic 1-form on $Y$, already before compactification. (See [7] p.82 and p.18.) Therefore the Albanese variety of $X^{\text{tor}}$ is trivial.

With these ingredients, Theorem 1.2 can be proved by the same argument as in the case of Siegel modular varieties. □

**Example 3.1.** In a few examples in $n = 2$, $X^{\text{tor}}$ is a blown-up $K3$ surface, and none of the boundary curves are contracted in the minimal model $X = X^{\text{min}}$ (see [7] Chapter VII, §3 – §4). Therefore, the unique class in $\text{CH}_0(X)$ given by the boundary points coincides with the Beauville-Voisin class [4] of the $K3$ surface $X$. The fact that the span of the Beauville-Voisin class contains the image of the intersection product $\text{Pic}(X) \otimes \text{Pic}(X) \rightarrow \text{CH}_0(X)$ will provide rational equivalence between boundary 0-cycles and some interesting 0-cycles in the interior.

4. **Orthogonal modular varieties**

In this section we prove Theorem 1.3 (§4.2) after recollection in §4.1.

4.1. **Orthogonal modular varieties.** We begin by recalling orthogonal modular varieties and their Baily-Borel and toroidal compactifications. In this section we let $\Lambda$ be a free abelian group of rank $2 + n$ equipped with a nondegenerate symmetric bilinear form $\Lambda \times \Lambda \rightarrow \mathbb{Z}$ of signature $(2, n)$. Let $Q = Q_\Lambda$ be the isotropic quadric in $\mathbb{P}\Lambda_{\mathbb{C}}$ defined by $(\omega, \omega) = 0$. The Hermitian symmetric domain $D = D_\Lambda$ attached to $\Lambda$ is one of the two connected components of the open set of $Q$ defined by $(\omega, \bar{\omega}) > 0$. Let $O^+(\Lambda)$ be the index $\leq 2$ subgroup of the orthogonal group $O(\Lambda)$ preserving the component $D$. Let $\Gamma$ be a subgroup of $O^+(\Lambda)$ of finite index. (We do not assume that it is a congruence subgroup.) The quotient $Y = Y_\Gamma = D/\Gamma$ is a quasi-projective variety.

The domain $D$ has 0-dimensional and 1-dimensional rational boundary components, corresponding to 1-dimensional and 2-dimensional isotropic subspaces $I$ of $\Lambda_{\mathbb{Q}}$ respectively. When $\dim I = 1$, the corresponding boundary point is the point $[I_{\mathbb{C}}]$ of $Q$. The Baily-Borel compactification $X^{\text{bb}}$ of $Y$ is defined as the quotient by $\Gamma$ of the union of $D$ and the rational boundary components, equipped with the Satake topology. It is normal and projective ([2]).
A toroidal compactification $X^{tor}$ of $Y$ is constructed by choosing a finite collection of suitable fans $\Sigma = (\Sigma_I)$, one for each $\Gamma$-equivalence class of isotropic planes $I$ in $\Lambda_Q$ independently. No choice is required for isotropic planes: it is canonical. For each isotropic line $I$, the tube domain realization (= the linear projection from the boundary point) defines an isomorphism of $D/U(I)_\mathbb{Z}$ with an open set of the algebraic torus $T_I = U(I)_\mathbb{C}/U(I)_\mathbb{Z}$. Here $U(I)_\mathbb{Q}$ is the unipotent part of the stabilizer of $I$ in $O^+(\Lambda_Q)$ and $U(I)_\mathbb{Z} = \Gamma \cap U(I)_\mathbb{Q}$. Note that $U(I)_\mathbb{Q}$ is canonically isomorphic to the quadratic space $(I^+ \cap I) \otimes I$. Then $D/U(I)_\mathbb{Z}$ is identified with the locus in $T_I$ consisting of points whose imaginary part is in the positive cone of $U(I)_\mathbb{R}$. The fan $\Sigma_I$ gives a polyhedral cone decomposition of the extended positive cone of $U(I)_\mathbb{R}$. It defines a partial compactification $D/U(I)_\mathbb{Z} \hookrightarrow (D/U(I)_\mathbb{Z})^{\Sigma_I}$ inside the torus embedding $T_I \hookrightarrow T_I^{\Sigma_I}$ determined by $\Sigma_I$. If $\sigma \in \Sigma_I$ is a cone which is not an isotropic ray, its relative interior is contained in the positive cone, and the corresponding torus orbit $\text{orb}(\sigma)$ in $T_I^{\Sigma_I}$ is also contained in $(D/U(I)_\mathbb{Z})^{\Sigma_I}$. The union of these torus orbits $\text{orb}(\sigma)$ is the locus of boundary points of $(D/U(I)_\mathbb{Z})^{\Sigma_I}$ which lie over the $I$-cusp (11 p.164).

We also have canonical partial compactifications at 1-dimensional rational boundary components. The toroidal compactification $X^{tor}$ is then defined as a gluing of $Y$ and natural quotients of these partial compactifications (11). $X^{tor}$ is a compact Moishezon space with a morphism $\pi : X^{tor} \to X^{bb}$.

4.2. Proof of Theorem 1.3. We now prove Theorem 1.3. As before we may assume that $\Gamma$ is neat and $X^{tor}$ is smooth and projective. Let $P, Q$ be two points of $X^{tor}$ such that $p = \pi(P), q = \pi(Q)$ are 0-dimensional cusps, corresponding to isotropic lines $I_1, I_2$ of $\Lambda_Q$ respectively. When $p = q$, we have $[P] = [Q]$ in $\text{CH}_0(X^{tor})$ because $\pi^{-1}(p)$ is a connected union of finitely many toric varieties by our description of the partial compactifications. In the general case $p \neq q$, it is possible to run a similar argument as before (cf. Appendix A). But instead of such a repetition, we give another proof: we reduce the proof to the cases of Siegel 3-folds and Hilbert modular surfaces which were already proved.

We first consider the case $\Lambda$ has Witt index 2. By assumption we have $n \geq 3$. In this case we can find a subspace $\Lambda'_Q \subset \Lambda_Q$ containing $I_1, I_2$ such that $\Lambda'_Q \simeq 2U_Q \oplus (-2t)$ for some $t > 0$. Here $U_Q$ is the hyperbolic plane. We have $D_{\Lambda'_Q} \subset D_{\Lambda}$, and this induces a morphism $Y' = D_{\Lambda'/\Gamma'} \to Y$ for some arithmetic subgroup $\Gamma'$ of $SO^+(\Lambda'_Q)$. This extends to a morphism between the Baily-Borel compactifications, say $f : (X')^{bb} \to X^{bb}$, and also between toroidal compactifications, say $f : (X')^{tor} \to X^{tor}$, for a suitable choice of fans $\Sigma'$ for $\Gamma'$. Then $I_1, I_2$ give 0-dimensional cusps $p', q'$ of $(X')^{bb}$ such that $f(p') = p$ and $f(q') = q$ respectively. Now $Y'$ is isomorphic to a Siegel modular 3-fold. Therefore we can apply Theorem 1.4 to see that $[P'] = [Q']$
Send this equality by \( f_* \), we see that \([P] = [f(P')] = [f(Q')] = [Q]\) in \( \text{CH}_0(X') \). This proves Theorem 1.3 in the case \( \Lambda \) has Witt index 2.

In the case \( \Lambda \) has Witt index 1, we can instead take a subspace \( \Lambda'_Q \supset I_1, I_2 \) isometric to \( U_Q \otimes K \) with \( K \) anisotropic of signature (1, 1). Then \( D_{\Lambda'/\Gamma'} \) for a suitable \( \Gamma' < SO^+(\Lambda'_Q) \) is isomorphic to a Hilbert modular surface, and we can apply Theorem 1.2. This finishes the proof of Theorem 1.3. □

### Appendix A.

In this appendix we give a short proof of the following

**Theorem A.1.** Let \( \Lambda \) be an integral quadratic form of signature (2, \( n \)). Assume that either \( n \geq 3 \) or \( \Lambda \) has Witt index 1 with \( n = 2 \). Let \( \Gamma \) be a finite-index subgroup of \( O^+(\Lambda) \). Then there is no nonzero holomorphic 1-form on the regular locus of \( D_\Lambda/\Gamma \). In particular, the Albanese variety of a smooth projective model of \( D_\Lambda/\Gamma \) is trivial.

Triviality of the Albanese variety in this generality was first proved by Bergeron-Li-Millson-Moeglin ([5] Remark 4.1) at least for \( \Gamma \) congruence subgroups, using \( L^2 \)-cohomology and Lie algebra cohomology. Special cases were studied earlier in [10], [8]. Here we add another, relatively elementary proof. Theorem A.1 can be used to provide an alternative proof of Theorem 1.3 similar to those for Theorems 1.1 and 1.2.

(Proof of Theorem A.1). We may assume that \( \Gamma \) is neat. We first consider the case \( \Lambda \) has Witt index 2. We use induction on \( n \geq 3 \). The case \( n = 3 \) is Weissauer’s theorem [19]. Suppose that the assertion is proved for all \((\Lambda', \Gamma')\) with \( \Lambda' \) of signature (2, \( n - 1 \)) and Witt index 2.

Let \( \omega \) be a 1-form on the given \( D_\Lambda/\Gamma \). If we restrict \( \omega \) to \( D_{\Lambda'} \subset D_\Lambda \) for a subspace \( \Lambda'_Q \subset \Lambda_Q \) of signature (2, \( n - 1 \)) and Witt index 2, we see that \( \omega|_{D_{\Lambda'}} \equiv 0 \) by induction. Then we vary \( \Lambda'_Q \), say by the \( O(\Lambda_Q) \)-action. If \( p \in D_{\Lambda'} \cap D_{\Lambda''} \) for two such subspaces \( \Lambda'_Q \neq \Lambda''_Q \), then \( \omega(p) = 0 \) as a linear form on the tangent space \( T_p D_\Lambda \) because \( T_p D_\Lambda = T_p D_{\Lambda'} + T_p D_{\Lambda''} \). Therefore \( \omega \) vanishes at a dense subset of \( D_\Lambda \) and so \( \omega \equiv 0 \).

When \( \Lambda \) has Witt index 1 (then \( n \leq 4 \)), we start from \( n = 2 \). This is the case of Hilbert modular surfaces and is covered, e.g., in [7] p.82. □

### References

[1] Ash, A.; Mumford, D.; Rapoport, M.; Tai, Y. *Smooth compactifications of locally symmetric varieties*. 2nd edition. Cambridge Univ. Press, 2010.

[2] Baily, W. L., Jr.; Borel, A. *Compactification of arithmetic quotients of bounded symmetric domains*. Ann. of Math. (2) 84 (1966), 442–528.

[3] Bass, H.; Lazard, M.; Serre, J.-P. *Sous-groupes d’indices finis dans \( SL(n, \mathbb{Z}) \)*. Bull. Amer. Math. Soc. 70 (1964) 385–392.
10

[4] Beauville, A.; Voisin, C. On the Chow ring of a K3 surface. J. Alg. Geom. 13 (2004), 417–426.
[5] Bergeron, N.; Li, Z.; Millson, J.; Moeglin, C. The Noether-Lefschetz conjecture and generalizations. Invent. Math. 208 (2017), no. 2, 501–552.
[6] Drinfeld, V. G. Two theorems on modular curves. Funct. Anal. Appl. 7 (1973), no.2, 155–156.
[7] van der Geer, G. Hilbert modular surfaces. Springer, 1988.
[8] Gritsenko, V.; Hulek, K.; Sankaran, G. K. Abelianisation of orthogonal groups and the fundamental group of modular varieties. J. Algebra 322 (2009), no. 2, 463–478.
[9] Hoffman, J.W.; Weintraub, S. H. The Siegel modular variety of degree two and level three. Trans. Amer. Math. Soc. 353 (2000), 3267–3305.
[10] Kondō, S. On the Albanese variety of the moduli space of polarized K3 surfaces. Invent. Math. 91 (1988), no. 3, 587–593.
[11] Ma, S. Rational equivalence of cusps. Ann. K-Theory 5 (2020), no.3, 395–410.
[12] Manin, J. I. Parabolic points and zeta functions of modular curves. Math. USSR Izv. 6 (1972), no.1, 19–64.
[13] Mennicke, J. Zur Theorie der Siegelschen Modulgruppe. Math. Ann. 159 (1965), 115–129.
[14] Roitman, A. A. Rational equivalence of zero-dimensional cycles. Math. USSR-Sh. 18 (1974), 571–588.
[15] Roitman, A. A. The torsion of the group of 0-cycles modulo rational equivalence. Ann. of Math. (2) 111 (1980), no. 3, 553–569.
[16] Serre, J.-P. Le problème des groupes de congruence pour SL2. Ann. of Math. (2) 92 (1970), 489–527.
[17] Shimura, G. Automorphic forms and the periods of abelian varieties. J. Math. Soc. Japan 31 (1979), no. 3, 561–592.
[18] Shioda, T. On elliptic modular surfaces. J. Math. Soc. Japan 24 (1972), 20–59.
[19] Weissauer, R. Vektorwertige Siegelsche Modulformen kleinen Gewichtes. J. Reine Angew. Math. 343 (1983), 184–202.

DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, TOKYO 152-8551, JAPAN
Email address: ma@math.titech.ac.jp