We combine a pair of independent Weyl fermions to compose a Dirac fermion on the four-dimensional Euclidean lattice. The obtained Dirac operator is antihermitian and does not reproduce anomaly under the usual chiral transformation. To simulate the correct chiral anomaly, we modify the chiral transformation. We also show that chiral gauge theories can be constructed nonperturbatively with exact gauge invariance. The formulation is based on a doubler-free lattice derivative, which is a simple matrix defined as a discrete Fourier transform of momentum with antiperiodic boundary conditions. Long-range fermion hopping interactions are truncated using the Lanczos factor.

I. INTRODUCTION

In the continuum Euclidean path-integral, chiral anomaly comes from Jacobian of the fermion measure [1]. The point is that Tr$\gamma_5$ of the transformed fermion measure gives nonzero contribution because of the infinite degrees of freedom. On the other hand, lattice field theory is a framework to simulate the continuum theory with a finite number of lattice sites. Chiral anomaly cannot be reproduced on the finite lattice in the same way as the continuum theory. We should definitely distinguish finite and infinite lattice formulations and concentrate on reproducing chiral anomaly based on the finite lattice for practical numerical studies.

The species doubling problem of the lattice fermion [2–8] is closely related with chiral anomaly [9–12]. According to the Nielsen-Ninomiya theorem, a single Weyl fermion cannot exist on the lattice [8]. When formulating lattice Dirac fermion, one has almost no choice but to break chiral symmetry explicitly using Wilson terms [4]. Otherwise, one has to give up one of the assumptions of the theorem as seen in the literature.

Lüscher’s implementation of lattice chiral symmetry based on the Ginsparg-Wilson relation [12] was a breakthrough [13]. The most significant feature to be stressed is that chiral anomaly can be devised even with a finite lattice. In the Lüscher’s formulation, new chiral transformation is introduced and chiral anomaly is obtained from Jacobian of the fermion measure in a different way from the continuum theory. The lattice index theorem holds for arbitrary lattice spacing [14,13] and chiral anomaly agrees with the continuum result in the continuum limit [16–19]. The lesson to be learned is that one can modify the axial current in order to reproduce the correct chiral anomaly on the finite lattice.

In the electroweak theory, left- and right-handed fermions couple to gauge fields in different ways. This type of formulation is called chiral gauge theory. For consistent construction of chiral gauge theories, gauge symmetry needs to be maintained at the quantum level (see Ref. [20], for example). Introduction of Wilson terms has trouble because the mixing of left- and right-handed fermions complicates discussion of gauge anomaly cancelation [15,21]. SLAC fermion partially simplifies the problem because it does not use Wilson terms [3,22]. However, it has not been successful due to breakdown of locality and Lorentz invariance associated with axial currents in the continuum limit [9]. There have been discussions that deny these defects and the non-conservation of the axial current [23,24]. The defects originate in the definition of axial currents with a derivative different from the SLAC derivative. Careful and consistent treatment is necessary when discussing species doubler and chiral anomaly with SLAC fermion.

In this paper, we give a method to save the SLAC derivative. On a finite lattice, a derivative is defined as a discrete Fourier transform of momentum in a similar way to the conventional SLAC derivative. To remove doubler modes, antiperiodic boundary conditions are chosen for the derivative in real space. The doubler modes at the momentum boundary are lifted and the dispersion relation of the continuum theory is reproduced discretely. Although long-range hopping interactions appear, the pathology of SLAC fermion is avoided and the correct continuum limit is guaranteed. However, long-range interactions are not useful in practical numerical calculations because sparser Dirac operator is better for numerical efficiency. By using the Lanczos factor technique demonstrated in Ref. [25], we effectively truncate long-range hopping interactions and improve the fermion propagator. Since the proposed lattice derivative does not use Wilson terms, left- and right-handed fermions are completely independent. As a result, the Dirac operator constructed with the derivative has exact chiral symmetry. This means that the Dirac fermion does not reproduce chiral anomaly because the fermion measure of path-integral is trivially invariant under the chiral transformation on the finite lattice. To simulate the correct chiral anomaly, we modify chiral transformation using the Neuberger’s solution to the Ginsparg-Wilson relation [7]. As a result, we obtain an anomalous Ward identity for the modified chiral transformation. The zero mode of the axial current divergence is generally non-zero and gives the index theorem. The absence of chiral anomaly under the usual chiral transformation implies the existence of a single anomaly-free Weyl fermion. Chiral
gauge theories can be constructed using the single Weyl fermion as a building block.

This work is a tricky extension of SLAC fermion. In the conventional SLAC fermion, periodic boundary conditions and an infinite lattice are assumed, where the doubler remains as a singularity at the momentum boundary. On the other hand, our formulation is based on antiperiodic boundary conditions and a finite lattice and therefore free from species doubler. In addition, the axial current is defined with the modified chiral transformation and evaluated nonperturbatively. The Nielsen-Ninomiya theorem does not apply to our implementation of chiral anomaly because the modified chiral transformation is used to define a symmetry.

This paper is organized as follows. In Sec. II, the lattice derivative is defined based on the finite lattice formulation. Locality of the derivative is discussed. Long-range hopping interactions are truncated using the Lanczos factor. In Sec. III, a modified chiral transformation is introduced to reproduce the correct chiral anomaly. A zos factor. In Sec. III, a modified chiral transformation is used to define a symmetry. In Sec. IV is devoted to summary and discussions.

II. LATTICE DERIVATIVE

We define lattice derivative as a discrete Fourier transform of momentum

\[
\mathbf{n} = \frac{1}{N} \sum_{l=-N/2+1}^{N/2} ip_l e^{i2\pi n/N} = -\frac{2}{N} \sum_{l=1}^{N/2} p_l \sin \left( \frac{2\pi n}{N} \right),
\]

where \( n \) represents lattice sites and takes integer values between \(-N/2+1\) and \(N/2\). The lattice size \( N \) is a finite even number and \( p_l \) corresponds to momentum.

\[
p_l \equiv \frac{2\pi \tilde{l}}{N}, \quad \tilde{l} \equiv l - \frac{1}{2}.
\]

Antiperiodic boundary conditions have been chosen in real space, \( \mathbf{n}_{n+N} = -\mathbf{n}_n \). In Eq. (1), the summation can be carried out easily.

\[
\mathbf{n} = \frac{\pi}{N^2} \left[ \frac{N+1}{\sin \left( \frac{s \pi}{2} \right)} - \frac{N+1}{\sin^2 \left( \frac{s \pi}{2} \right)} \right],
\]

where \( s = 2\pi n/N \). The lattice derivative (1) is doubler-free and reproduces discretely the dispersion relation of the continuum theory for arbitrary lattice spacing.

In the large \( N \) limit, Eq. (1) becomes an integral

\[
\frac{1}{a} \mathbf{n} = \frac{a}{2\pi} \frac{\partial}{\partial x} \int_{-\pi/a}^{\pi/a} dk \ e^{ikx},
\]

where \( a \) is a lattice spacing and \( x = na \) is a space coordinate. In the continuum limit \( a \rightarrow 0 \), we obtain the first order derivative of the continuum theory.

\[
\lim_{a \rightarrow 0} \frac{1}{a} \mathbf{n} = a \frac{\partial}{\partial x} \delta(x).
\]

The lattice derivative (1) is local in the continuum limit.

![Figure 1](image-url)

**FIG. 1.** The lattice derivative \( \mathbf{n}(s) \equiv \mathbf{n}_n \) is plotted as a function of \( s = 2\pi n/N \) for (a) \( N = 50 \) and (b) \( N = 1000 \) in a region \(|s| \leq 2\pi \). The function \( \mathbf{n}(s) \) has antiperiodicity of \( 2\pi \), \( \mathbf{n}(s+2\pi) = -\mathbf{n}(s) \).

Figure 1 plots the lattice derivative \( \mathbf{n}(s) \equiv \mathbf{n}_n \) with \( s = 2\pi n/N \) in a range \(|s| \leq 2\pi \) for two lattice sizes \( N = 50 \) and 1000. The absolute value of \( \mathbf{n}(s) \) is large around \(|s| = 0 \) and \( 2\pi \) and small around \(|s| = \pi \). The points \(|s| = 0 \) and \( 2\pi \) are equivalent because of antiperiodicity. Therefore, \( \mathbf{n}(s) \) around \(|s| = 2\pi \) does not mean

\[
1\text{If periodic boundary conditions are chosen, there appear degenerate zero modes on the momentum boundary. Such unphysical doubler modes must be removed because they cause errors especially when fermion mass is small.}
severe nonlocality. The points $|s| = \pi$ give the most long-range hopping. As expected from Eq. (4), locality of the derivative $\nabla(s)$ is quite good when $N = 1000$. On the other hand, decay of the derivative $\nabla(s)$ is slow when the lattice size is small.

We are interested in constructing a theory with better locality for a practical purpose. In order for Eq. (1) to be a useful lattice derivative, long-range hopping interactions need to be truncated. However, truncation of hopping interactions may cause errors. We need to find a systematic way to reproduce spectra effectively with only short-range hopping interactions. For simplicity, let us consider a classical action for a free massless fermion in one-dimensional space.

$$S = a \sum_{m,n=-N/2+1}^{N/2} \frac{1}{a} \nabla_{m-n} \psi_m \psi_n. \quad (5)$$

Fourier transforms of the one-component fermions are

$$\hat{\psi}_n = \frac{1}{\sqrt{N}} \sum_{l=-N/2+1}^{N/2} e^{i2\pi l/n} \zeta_l,$$

$$\hat{\bar{\psi}}_n = \frac{1}{\sqrt{N}} \sum_{l=-N/2+1}^{N/2} e^{-i2\pi l/n} \bar{\zeta}_l,$$

where antiperiodicity is assumed

$$\psi_{n+N} = -\psi_n, \quad \bar{\psi}_{n+N} = -\bar{\psi}_n.$$

Then we have

$$S = \sum_{l,l'} \hat{\zeta}_l \hat{\psi}_{l'} \zeta_{l'}, \quad (6)$$

where

$$p_{l,l'} = -\frac{i}{N} \sum_{m,n=-N/2+1}^{N/2} \nabla_{m-n} e^{-i2\pi l/n} e^{i2\pi l'/n} / N$$

$$= p_l \delta_{l,l'}, \quad (7)$$

with the inverse transform $p_l$ of Eq. (1)

$$p_l = -2 \sum_{n=1}^{N/2-1} \nabla_n \sin \left( \frac{2\pi l n}{N} \right) + (-1)^l \nabla_{N/2}. \quad (8)$$

To obtain this, antiperiodicity of $\nabla_n$, $\psi_n$, and $\bar{\psi}_n$ has been used. Antiperiodicity in real space gives rise to periodicity in momentum space, $p_{l+N} = p_l$, $\zeta_{l+N} = \zeta_l$, and $\bar{\zeta}_{l+N} = \bar{\zeta}_l$. Some explanations would be necessary for the derivation of Eq. (8). Consider the matrix contained in Eq. (7)

$$C_{m,n} = \nabla_{m-n} e^{-i2\pi l/m + i2\pi l/n} / N,$$

which has periodicity, $C_{m+N,n} = C_{m,n+N} = C_{m,n}$. In Fig. 2, the matrix $C_{m,n}$ is shown schematically (see the upper diagram). Some examples for the indices $(m,n)$ are given. The matrix elements on the dotted lines are not contained in Eq. (7). In the upper diagram, the vertices $(N/2, -N/2)$ and $(-N/2, N/2)$ correspond to the points with $|s| = 2\pi$ in Fig. 2. Using the periodicity of the matrix, the triangles 1 and 2 can be moved to form the parallelogram (see the lower diagram). As a result, Eq. (7) can be evaluated by calculating the summation for each $|m - n|$, which draws a line segment parallel to the oblique sides of the parallelogram. The last term of Eq. (8) is a contribution of the term with $|m - n| = N/2$, which corresponds to the left oblique side of the parallelogram.

In Eq. (8), we truncate long-range terms with a parameter $N_c \leq N/2$, which represents the largest distance of fermion hopping.

FIG. 2. The matrix $C_{m,n}$ is schematically shown (the upper diagram). The numbers in brackets are examples of the indices $(m,n)$. The matrix elements on the dotted lines are not contained in Eq. (7). The two triangles 1 and 2 can be moved to form the parallelogram shape (the lower diagram).
The truncation can be implemented to Eq. (7).

\[ p_{l,l'} = -\frac{i}{N} \sum_{m,n} \nabla_{m-n} e^{-i2\pi\tilde{l}m/N + i2\pi\tilde{l}n/N}. \]

The indices \( m \) and \( n \) run from \(-N/2 + 1\) to \( N/2\). The prime symbol means that the summation is taken over \(|m-n| \leq N_c, m-n+N \leq N_c, \) and \( m-n-N \geq -N_c\). Fermion hopping has been restricted to a finite range.

Figure 3 plots \( p_l \) of Eq. (9) as a function of \( 2\pi\tilde{l}/N \) for \( N_c = 5, 10, \) and 25 with a lattice size \( N = 50\). \( N_c = 25\) gives the exact result with no truncation, which satisfies the dispersion relation of the continuum theory. If one does not mind inclusion of long-range hopping, doubler-free formulation of a single Weyl fermion is possible maintaining the correct dispersion relation. When \( N_c \) is small, there is also no doubler modes because of antiperiodic boundary conditions. Although some modes around the momentum boundary deviate from the correct dispersion, those are not so harmful because there is no genuine degeneracy with low-lying modes. However, \( p_l \) oscillates around the exact result. The oscillation tends to be large as \( N_c \) goes to small.

The small oscillation around the correct dispersion comes from the truncated terms having large \( n \)'s in Eq. (9). As shown in Ref. [25], such oscillation can be removed by introducing the Lanczos factor, which is used in Fourier analysis to cancel the Gibbs phenomenon. [26] We modify Eq. (9) as follows:

\[ p_l = \sum_{n=1}^{N_c} F_n \nabla_n \left[ (\delta_n, \frac{2\pi}{N} - 1)2\sin\left(\frac{2\pi\tilde{l}n}{N}\right) + \delta_n, \frac{2\pi}{N} (-1)^l \right]. \]

where

\[ F_n \equiv \frac{N_c + 1}{\pi n \sin \left(\frac{\pi n}{N_c + 1}\right)} \]

is the Lanczos factor. As a result, Eq. (10) becomes

\[ p_{l,l'} = -\frac{i}{N} \sum_{m,n} F_{|m-n|} \nabla_{m-n} e^{-i2\pi\tilde{l}m/N + i2\pi\tilde{l}n/N}. \]

The final form of the action is given as

\[ S = \frac{1}{\alpha} \sum_{m,n} \frac{\bar{\psi}_m}{\psi_n} F_{|m-n|} \nabla_{m-n} \psi_n. \]
Figure 4 plots $p_l$ of Eq. (11) improved with the Lanczos factor as a function of $2\pi l/N$ for $N_c = 5$, 10, and 25 with $N = 50$, which are compared with the exact result ($N_c = 25$) shown in Fig. 3. As before, there is no doubler for every $N_c$. In addition to this, the oscillation has been removed with the Lanczos factor. As $N_c$ goes to large, the deviation around the momentum boundary tends to be small. In this way, we can construct a doubler-free ultralocal derivative. If the Lanczos factor is introduced, ultralocal formulation of a single Weyl fermion is possible maintaining almost correct dispersion relation.

III. CHIRAL AND GAUGE ANOMALY

On the finite four-dimensional Euclidean lattice, consider an effective action $\Gamma[U]$

$$e^{-\Gamma[U]} = \int D\bar{\psi} D\psi e^{-S[U]},$$

where

$$S = \alpha^4 \sum_{m,n} \bar{\psi}_m \mathcal{D}_{m,n} \psi_n,$$

is a classical action for a massless Dirac fermion coupled to gauge. $\mathcal{D}$ is a Dirac operator and the Euclidean Dirac matrices satisfy $\gamma_\mu^\dagger = \gamma_\mu$ and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. Chirality is defined with $\gamma_5 \equiv \gamma_1 \gamma_2 \gamma_3 \gamma_4$. The fermion variables $\bar{\psi}_m$ and $\psi_n$ are Grassmann valued. The indices $m$ and $n$ are four-component numbers to indicate lattice sites and each component runs from $-N/2 + 1$ to $N/2$ as before. The lattice covariant derivative

$$\mathcal{D}_\mu m,n \equiv \frac{1}{\alpha} \nabla_{m,n-\mu} U_{m,n}(\mu) \prod_{\nu=1}^{4} \delta_{m,n,\nu}$$

is diagonal with respect to the space-time indices $m$ and $n$ except for the $\mu$-th ones. The Dirac operator $\mathcal{D}$ is antihermitean and satisfies $\{\mathcal{D}, \gamma_5\} = 0$. The classical action is invariant under the usual chiral transformation. The derivative $\nabla_n$ can be replaced with the truncated one in the same way as Eq. (14), if ultralocal construction is preferred. The gauge variable $U_{m,n}(\mu)$ is a product of all link variables that compose a line segment between the two sites $m$ and $n$ parallel to the $\mu$-th direction. When connecting the two sites with link variables along the $\mu$-th direction, there are two ways because of periodicity of the action. The most natural choice is the shorter path. One of the two ways is chosen depending on the distance between two sites. When $|m_\mu - n_\mu| \leq N/2$,

$$U_{m,n}(\mu) = \begin{cases} U_{m,m+\hat{\mu}} \cdots U_{n+(N-1)\hat{\mu},n} & (m_\mu > n_\mu) \\ U_{m,m-\hat{\mu}} \cdots U_{n-(N-1)\hat{\mu},n} & (m_\mu < n_\mu) \end{cases}$$

which corresponds to the hexagon sandwiched between the triangles 1 and 2 in Fig. 2. $\hat{\mu}$ is a unit vector in the $\mu$-th direction. When $|m_\mu - n_\mu| > N/2$,

$$U_{m,n}(\mu) = \begin{cases} U_{m,m+\hat{\mu}} \cdots U_{n+(N-1)\hat{\mu},n} & (m_\mu > n_\mu) \\ U_{m,m-\hat{\mu}} \cdots U_{n-(N-1)\hat{\mu},n} & (m_\mu < n_\mu) \end{cases} ,$$

which corresponds to the triangles 1 and 2 in Fig. 2 and therefore intersects the boundary. The link variables $U_{m,n+\hat{\mu}}$ are elements of a gauge group and satisfy $U_{m,n+\hat{\mu}} = U_{n,m+\hat{\mu}}$ and $U_{m,n+\hat{\mu}} U_{n,m+\hat{\mu}} = U_{m,n+\hat{\mu}}$. In the continuum limit, the lattice covariant derivative becomes

$$\lim_{\alpha \to 0} (D_\mu)_{m,n} = a^4 \delta_\mu^c (x - y) D_\mu^c ,$$

where $x = ma$ and $y = na$, and $D_\mu^c = \partial_\mu - igA_\mu$ with a parameterization $U_{m,n+\hat{\mu},m} = e^{iga A_\mu(x)}$. Eq. (17) reduces to the covariant derivative of the continuum theory in the continuum limit.

With our Dirac operator, the usual chiral transformation does not reproduce chiral anomaly because the classical action and the fermion measure is invariant. To simulate the correct chiral anomaly on the finite lattice, we introduce the modified chiral transformation with $\theta_n \ll 1$

$$\bar{\psi}_m \to \bar{\psi}_m (1 + i\theta_n(\gamma_5)_{m,n}) \psi_n,$$

where

$$(\gamma_5)_{m,n} \equiv \gamma_5 \left(1 + \frac{1}{2} \alpha G\right)_{m,n} .$$

The operator $G$ is the Neuberger’s solution [7] to the Ginsparg-Wilson relation $\gamma_5 G + G \gamma_5 = aG\gamma_5 G$ and has nothing to do with the Dirac operator (17). The modified chiral transformation is a symmetry of the effective action $\Gamma$ for arbitrary lattice spacing independent of whether the transformation is local or global. When the transformation is global $\theta_n = \theta$, it is also a symmetry of the classical action $S$ in the continuum limit $\alpha \to 0$ because the variation of the Lagrangian density induced by the transformation is proportional to lattice spacing.

$$\delta S = \frac{i}{2} \theta a^4 \sum_{m,n} \bar{\psi}_m \gamma_5 a (\mathcal{D} G - G \mathcal{D})_{m,n} \psi_n.$$  

Although the variation is classically zero in the continuum limit, the vacuum expectation value of the variation gives the index theorem for arbitrary lattice spacing.

The axial current divergence is defined as a variation of the classical action under the local chiral transformation with $\hat{\gamma}_5$.

$$(\partial_\mu J_\mu^5)_{m} = \sum_{n_1, n_2} \left[ \bar{\psi}_{n_1}(\hat{\gamma}_5)_{n_1,m} \mathcal{D}_{n_2,m} \psi_{n_2} + \bar{\psi}_{n_2} \mathcal{D}_{n_2,m} \psi_{n_1} \right] .$$
The axial current is obtained by inverting the derivative

\[(\tilde{J}_\mu^5)_n = \sum_{m}(\nabla^{-1})_{n,\mu,m} \prod_{\nu=1}^{4} \delta_{\nu,\nu,m} x \sum_{n_1,n_2} \left[ \bar{\psi}_{n_1}(\vec{\gamma}_5)_{n_1,m}(D_\mu)_{m,n_2} \psi_{n_2} + \bar{\psi}_{n_2}(D_\mu)_{n_2,m}(\vec{\gamma}_5)_{m,n_1} \psi_{n_1} \right], \tag{26}\]

which is gauge invariant. In the free theory, the currents are local in the continuum limit because the Leibniz rule holds in the limit. \(^2\) The fermion measure transforms as

\[\mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \exp\left(-2i \sum_n \theta_n A_n\right) \mathcal{D}\psi \mathcal{D}\bar{\psi}. \tag{27}\]

Chiral anomaly

\[A_n \equiv \text{tr}(\vec{\gamma}_5)_{n,n} \tag{28}\]

is a gauge-invariant quantity. As shown in Ref. [13], the index theorem holds for arbitrary lattice spacing.

\[\sum_n A_n = \text{Index}(G). \tag{29}\]

(See also Ref. [27,28].) Since the effective action \(\Gamma\) is invariant under transformation of the integration variables, the modified axial current \(\tilde{J}_\mu^5\) does not conserve.

\[\langle (\partial_\mu \tilde{J}_\mu^5)\rangle_n = -\frac{2}{a^4} A_n. \tag{30}\]

The Ward identity for the modified chiral transformation holds also for the zero mode, which gives the index theorem and corresponds to the variation under the global transformation (24). \(^3\) In the continuum limit, the chiral anomaly agrees with the continuum result [16–19]

\[\langle (\partial_\mu \tilde{J}_\mu^5) \rangle = \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu}F_{\rho\sigma}), \tag{31}\]

where \(\epsilon_{1234} = +1\) and \(F_{\mu\nu} \equiv [D_\mu^{(c)}, D_\nu^{(c)}].\) The effect of chiral anomaly can be implemented to physical quantities via the modified anomalous axial current.

Construction of anomaly-free chiral gauge theory is easy if our lattice derivative is used. In our formulation, Weyl fermions are defined with the ordinary \(\gamma_5\) (not \(\vec{\gamma}_5\)).

As a result, the fermion measure of the path integral do not depend on gauge variables. The Weyl fermions are free from gauge anomaly beforehand. Consider a classical action for a Dirac fermion

\[S = a^4 \sum_{m,n} \bar{\psi}_m \hat{D}_{m,n} \psi_n, \tag{32}\]

where only the right-handed fermion is coupled to gauge. The left-handed fermion is redundant and does not contribute to the effective action. Once the absence of gauge anomaly is confirmed, the left-handed fermion can be integrated out. The definition of the Dirac operator \(\hat{D}\) is same as Eq. (17) except that gauge is defined as a product of link variables

\[U_{n+\mu,n} = e^{i\nu_n A_{\mu,n} P_\pm}, \tag{33}\]

where \(A_{\mu,n} \equiv A_\mu^{(c)} T_\mu\) and \(P_\pm \equiv (1 \pm \gamma_5)/2.\) \(\hat{D}\) is no longer antihermitian. Under local gauge transformation, the classical action (32) is invariant. On the finite lattice, infinitesimal local gauge transformation

\[\psi' = (1 + i\theta_n T_\alpha P_+) \psi_n, \quad \bar{\psi}' = \bar{\psi}_n(1 - i\theta_n T_\alpha P_-), \tag{34}\]

does not change the fermion measure

\[\mathcal{D}\psi' \mathcal{D}\bar{\psi}' = \exp\left[-i \sum_n \theta_n \text{tr}(T_\alpha \gamma_5)\right] \mathcal{D}\psi \mathcal{D}\bar{\psi} = \mathcal{D}\psi \mathcal{D}\bar{\psi}\]

and hence the effective action. A single Weyl fermion can exist on the lattice without violating gauge symmetry. \(^4\)

IV. SUMMARY AND DISCUSSIONS

We have constructed a doubler-free covariant derivative and an anomalous Ward identity for the modified chiral symmetry on the lattice. The index theorem holds for arbitrary lattice spacing and the dependence of chiral anomaly on gauge fields agrees with the continuum result in the continuum limit. The zero mode of the anomalous Ward identity gives the index theorem. In this formulation, a single Weyl fermion can exist on the lattice maintaining gauge symmetry. Chiral gauge theories can be constructed nonperturbatively using the single anomaly-free Weyl fermion as a building block.

The proposed lattice derivative is a simple matrix that gives the correct dispersion. We have shown that introduction of the Lanczos factor enables us to construct lattice derivatives with good locality. In this case, almost the correct dispersion is reproduced except for deviation.

\(^2\)In Ref. [9] for the conventional SLAC derivative, axial currents are defined by introducing a derivative independent of the SLAC derivative, which is the cause of breaking of locality and Lorentz invariance.

\(^3\)The explicit breaking term (24) is necessary for the existence of the non-vanishing zero mode of the axial current divergence.

\(^4\)The number of Weyl fermions needs to be even when there exists Witten’s global anomaly. [29]
at high momentum. It depends on fermion mass how large the truncation parameter $N_c$ should be.

The correct chiral anomaly has been obtained without violating locality and Lorentz invariance in the continuum limit in spite of the existence of long-range hopping interactions. This is a result of a nonperturbative formulation based on the modified chiral transformation (or equivalently the modified axial current). To reproduce chiral anomaly, Wilson terms have to be introduced somewhere. We have introduced them only in the modified chiral transformation to maintain complete independence of left- and right-handed fermions in the classical action.

When calculating physical quantities dependent on chiral anomaly such as the $\eta'$ meson mass, $\gamma_5$ needs to be replaced with $\hat{\gamma}_5$ in the concerned vertex operators to include the effect of chiral anomaly. It depends on the construction of the operator $G$ how precisely the effect of chiral anomaly is implemented with finite lattice spacing. In lattice QCD, the modification of the axial current does not affect physical quantities independent of chiral anomaly because the axial current does not couple to gauge directly.

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