PARTIAL REGULARITY FOR MINIMIZERS OF SINGULAR ENERGY FUNCTIONALS, WITH APPLICATION TO LIQUID CRYSTAL MODELS

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Abstract. We study the partial regularity of minimizers for certain singular functionals in the calculus of variations, motivated by Ball and Majumdar’s recent modification of the Landau-de Gennes energy functional.

1. INTRODUCTION

In this paper we establish the partial regularity of minimizers \( u \in H^1(U; \mathbb{R}^k) \) for singular energy functionals having the form

\[
I[v] := \int_U F(v, Dv) + f(v) \, dx,
\]

where \( F \) is quasiconvex in the gradient variables and the convex function \( f \) blows up to infinity at the boundary of a bounded open set \( K \subset \mathbb{R}^k \).

As we will briefly explain later in Section 5, this sort of energy functional arises in some models in nematic liquid crystal theory recently proposed by Ball and Majumdar [B-M]. Our paper answers, at least in part, some of the regularity issues left open in [B-M], but many fascinating questions remain.

1.1. A singular variational problem. We assume hereafter that \( U \subset \mathbb{R}^n \) is bounded smooth domain and that \( K \) is a bounded, open convex subset of \( \mathbb{R}^k \).

We now introduce various assumptions concerning the nonlinearities \( f \) and \( F \) appearing in the energy functional (1.1):

(H1) Hypotheses on \( f \): The given function \( f : \mathbb{R}^k \to [0, \infty] \) is nonnegative, convex and smooth on \( K \subset \mathbb{R}^k \). We will write \( f = f(z) \).

We further require that

\[
\begin{align*}
f(z) &< \infty \quad \text{if } z \in K, \\
f(z) &= \infty \quad \text{if } z \in \mathbb{R}^k - K
\end{align*}
\]

and

\[
f(z) \to \infty \quad \text{as dist}(z, \partial K) \to 0, z \in K.
\]

(H2) Hypotheses on \( F \): We assume \( F : \mathbb{R}^k \times M_{k \times n} \to \mathbb{R} \) is given, \( M_{k \times n} \) denoting the space of real, \( k \times n \) matrices. We write \( F = F(z, P) \).

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We suppose as well that $F$ is uniformly strictly quasiconvex in the $P$ variables. This means that there exists a constant $\gamma > 0$ such that
\begin{equation}
\int_V F(z, P) + \gamma |Dw|^2 \, dx \leq \int_V F(z, P + Dw) \, dx,
\end{equation}
for each smooth bounded domain $V \subset \mathbb{R}^n$, each $z \in \mathbb{R}^k$ and $P \in \mathbb{M}^{k \times n}$, and all $w \in C^1(V; \mathbb{R}^k)$ satisfying $w = 0$ on $\partial V$. The physical significance of quasiconvexity is discussed for instance in the foundational paper [B] of Ball.

We introduce the further technical assumptions that
\begin{equation}
\begin{cases}
|D^2_P F(z, P)| \leq C, \\
\gamma |P|^2 \leq F(z, P) + C, \\
|F(z, P) - F(\hat{z}, P)| \leq C(1 + |P|^2)|z - \hat{z}|,
\end{cases}
\end{equation}
for appropriate constants $C, \gamma > 0$ and all $z, \hat{z} \in \mathbb{R}^k, P \in \mathbb{M}^{k \times n}$.

**H3 Hypothesis on admissible mappings:** We propose to minimize the functional $I[\cdot]$ over the admissible class of functions
\[ A := \{ v \in H^1(U; \mathbb{R}^k) \mid v = g \text{ on } \partial U \text{ in the trace sense} \}, \]
where the given smooth function $g : \partial U \to \mathbb{R}^k$ provides the boundary conditions. For this we need to assume
\begin{equation}
\text{there exists } u^* \in A \text{ with finite energy: } I[u^*] < \infty.
\end{equation}

Under the hypotheses (H1)-(H3), standard arguments in the calculus of variations prove the existence of a minimizer $u \in A$:
\begin{equation}
I[u] = \min_{v \in A} I[v] < \infty.
\end{equation}

The key question that we address in this paper is the regularity of $u$. Since $I[u] < \infty$, we certainly have $u \in K$ almost everywhere, but conceivably $u(x)$ lies in $\mathbb{R}^k - K$ for a dense set of points $x \in U$. We will show in Theorem 2.5 that in fact $u$ is $C^{1,\alpha}$ on a “large” open subset of $U$.

**Remark.** Our hypothesis that the second derivatives in $P$ of $F$ are bounded is restrictive for quasiconvex integrands, as most polyconvex $F$ will not satisfy this. Our partial regularity assertions are in fact valid under more general growth conditions, but to keep this paper at a reasonable length, we omit the proofs; see for instance [E1].

2. Partial regularity for a model problem

The proof of partial regularity is a fairly straightforward modification of standard, but rather complicated, variational techniques (cf. [E-G1]), with particular attention paid to the singular term involving the function $f$. The main task is to establish Theorem 2.1 asserting under various hypotheses that a certain appropriately scaled quantity $E(x_0, r)$ satisfies the decay estimate $E(x_0, r) \leq C r^{1/2} E(x_0, r)$ for small $r$. Iterating this inequality lets us pass to smaller and smaller scales, eventually to show $C^{1,\alpha}$ regularity near the point $x_0$. We prove the decay estimate by contradiction, assuming it fails on a sequence of balls, rescaling to the unit ball, and passing to limits. This procedure in effect linearizes our problem and is in fact a standard strategy for variational problems, but the full implementation forces us to confront lots of subtle details.
To keep the presentation fairly simple, we devote this section to a simplified
model where $F = F(P)$ depends only on the gradient. We therefore now consider
the energy functional
\begin{equation}
I[v] = \int_U F(Dv) + f(v) \, dx,
\end{equation}
and hereafter assume that $u \in A$ is a minimizer.

2.1. Linear approximation. Given a ball $B(x_0, r) \subset U$, we define the quantity
\begin{equation}
E(x_0, r) := r^{1/2} + \int_{B(x_0, r)} |Du - (Du)_{x_0, r}|^2 \, dx,
\end{equation}
which measures the averaged $L^2$-deviation of $Du$ over the ball from its average
value
\begin{equation}
(Du)_{x_0, r} := \int_{B(x_0, r)} Du \, dx.
\end{equation}
Later we also use the similar notation
\begin{equation}
(u)_{x_0, r} := \int_{B(x_0, r)} u \, dx.
\end{equation}
In the above formulas, the slash through the integral sign means the average over
the ball $B(x_0, r)$.

The following assertion is the key to $C^1$ partial regularity:

**Theorem 2.1.** For each $L > 0$, there exists a constant $C = C(L)$ with the property
that for each $\tau \in (0, \frac{1}{8})$ there exists $\epsilon = \epsilon(L, \tau) > 0$ such that
\begin{equation}
|[(u)_{x_0, r}], |f((u)_{x_0, r})|, |(Du)_{x_0, r}| \leq L
\end{equation}
and
\begin{equation}
E(x_0, r) \leq \epsilon
\end{equation}
imply
\begin{equation}
E(x_0, \tau r) \leq C\tau^{1/2}E(x_0, r)
\end{equation}
for each ball $B(x_0, r) \subset U$.

**Proof.** 1. We argue by contradiction. Should the theorem be false, there would exist balls \( \{B(x_m, r_m)\}_{m=1}^{\infty} \subset U \) such that
\begin{equation}
|[(u)_{x_m, r_m}], |f((u)_{x_m, r_m})|, |(Du)_{x_m, r_m}| \leq L,
\end{equation}
and
\begin{equation}
E(x_m, r_m) =: \lambda_m^2 \to 0,
\end{equation}
but
\begin{equation}
E(x_m, \tau r_m) > C_1\tau^{1/2}\lambda_m^2,
\end{equation}
for a constant $C_1$ we will select later.

2. We have from (2.2) and (2.7) that
\begin{equation}
r_m^{1/2} \leq \lambda_m^2.
\end{equation}
Also
\begin{equation}
\lambda_m^{-2}\int_{B(x_m, r_m)} |Du - (Du)_{x_m, r_m}|^2 \, dx \leq 1.
\end{equation}
We combine (2.10) with (2.6), to discover
\[ \int_{B(x_m, r_m)} |Du|^2 \, dx \leq C. \]

Put \( a_m := (u)_{x_m, r_m} \), \( A_m := (Du)_{x_m, r_m} \), and introduce the rescaled functions
\[ v_m(z) = \frac{u(x_m + r_m z) - a_m - r_m A_m z}{\lambda_m r_m} \]
for \( z \in B := B(0, 1) \). Then
\[ Dv_m(z) = \frac{Du(x_m + r_m z) - A_m}{\lambda_m}, \]
and
\[ (v_m)_B = (Dv)_B = 0. \]

Observe also that
\[ -\int_B |Dv_m(z)|^2 \, dz = \lambda_m^2 \int_{B(x_m, r_m)} |Du - (Du)_{x_m, r_m}|^2 \, dx \leq 1. \]

Since \( (v_m)_B = 0 \), Poincaré’s inequality then provides the bound
\[ \int_B |v_m|^2 \, dz \leq C. \]

Passing if necessary to a subsequence and relabelling, we may suppose that
\[ \begin{cases} v_m \to v \text{ strongly in } L^2(B; \mathbb{R}^k), \\ Dv_m \rightharpoonup Dv \text{ weakly in } L^2(B; \mathbb{M}^{k \times n}). \end{cases} \tag{2.11} \]

Also since \( |a_m|, |A_m| \leq L \), we may assume also that
\[ a_m \to a, \quad A_m \to A. \]

3. We hereafter write
\[ \mathbb{K}_\delta := \{ q \in \mathbb{K} \mid \text{dist}(q, \partial \mathbb{K}) > \delta \} \]
for small \( \delta > 0 \). According to (2.6), we have
\[ |f(a)| = \lim_{m \to \infty} |f(a_m)| \leq L. \]

It consequently follows from (1.3) that there exists \( \epsilon_0 > 0 \) such that \( a \in \mathbb{K}_{2\epsilon_0} \); and hence there exists a sufficiently large index \( M \) such that \( a_m \in \mathbb{K}_{\epsilon_0} \) for \( m > M \).

Recalling that \( u \) is a minimizer and rescaling \( B(x_m, r_m) \) to the unit ball \( B \), we see that
\[ \begin{align*}
\int_B \left[ F(A_m + \lambda_m Dv_m) + f(a_m + r_m A_m z + \lambda_m r_m v_m) \right] \, dz \\
\leq \int_B \left[ F(A_m + \lambda_m D\tilde{v}_m) + f(a_m + r_m A_m z + \lambda_m r_m \tilde{v}_m) \right] \, dz,
\end{align*} \tag{2.12} \]
provided \( \tilde{v}_m \in H^1(B; \mathbb{R}^k) \) and \( \tilde{v}_m = v_m \) on \( \partial B \). Then
\[ \int_B DF(A_m) \cdot Dv_m \, dz = \int_B DF(A_m) \cdot D\tilde{v}_m \, dz. \tag{2.13} \]

It follows that \( v_m \) is a minimizer of
\[ I_m^r[w] = \int_{B(0, r)} F_m(Dw) + \frac{1}{\lambda_m^2} f(a_m + r_m A_m z + \lambda_m r_m w) \, dz, \]
subject to its boundary conditions, for the rescaled energy density
\[ F_m(P) := \frac{F(A_m + \lambda_m P) - F(A_m) - \lambda_m DF(A_m) : P}{\lambda_m^2} \]
and \( r \in (0, 1] \). In other words,
\begin{align*}
(2.14) \quad I^m_r[v_m] & \leq I^m_r[w] \\
\text{for any } w \in H^1(B(0, r); \mathbb{R}^k) \text{ such that } w = v_m \text{ in } \partial B(0, r).
\end{align*}

4. To streamline the presentation, we sequester various intricate calculations into the proofs of two technical lemmas that follow this main proof.

According to the following Lemma 2.3 the limit \( v \) is a weak solution of the constant coefficient, uniformly elliptic system (2.26). Standard regularity theory (cf. for instance [G]) implies then that \( v \) is smooth. In particular we have the bound
\[ \max_{B(0, \frac{1}{2})} |D^2 v| \leq C \int_B |Dv|^2 \leq C. \]
Consequently
\[ \int_{B(0, \tau)} |Dv - (Dv)_{0, \tau}|^2 dx \leq C_2 \tau^2 \]
for some constant \( C_2 = C_2(L) \).

However, rescaling the inequalities (2.8) and using (2.9) gives
\[ \int_{B(0, \tau)} |Dv_m - (Dv_m)_{0, \tau}|^2 dx \geq (C_1 - 1) \tau^{1/2}. \]
But owing to the following Lemma 2.2 we have the strong convergence \( v_m \to v \) in \( H^1(B(0, \tau); \mathbb{R}^k) \). This leads to the desired contradiction, provided we take \( C_1 = C_2 + 2. \)

The previous proof invoked the following two technical lemmas.

**Lemma 2.2.** \( Dv_m \) converges strongly to \( Dv \) in \( L^2_{\text{loc}}(B; \mathbb{R}^{k \times n}) \).

**Proof.** 1. We first define a Radon measure \( \mu_m \) on \( B = B(0, 1) \) by
\[ \mu_m(A) = \int_A |Dv_m|^2 + |Dv|^2 dx, \]
for any Borel set \( A \subseteq B \). Since \( \{\mu_m(B)\}_{m=1}^\infty \) is bounded, we may assume, passing if necessary to a subsequence, that there exists a Radon measure \( \mu \) on \( B \) such that
\[ \mu_m \to \mu \text{ weakly in the sense of measures.} \]
We then also have \( \mu(B) < \infty \); whence
\[ (2.15) \quad \mu(\partial B(0, r)) = 0 \]
for all but at most countably many \( r \in (0, 1] \). Select any \( r \in (0, 1) \) such that (2.15) holds.

2. For \( R \in (r, 1) \), let \( \xi \) be a smooth cutoff function satisfying
\[ \begin{cases} 
0 \leq \xi \leq 1; \xi \equiv 1 \text{ on } B(0, r); \\
\xi \equiv 0 \text{ on } \mathbb{R}^n - B(0, R); \ |D\xi| \leq \frac{C}{R-\tau}. 
\end{cases} \]
Define \( \phi_m = (\phi^1_m, \ldots, \phi^k_m) \), where

\[
\phi^j_m(x) = \begin{cases} 
\frac{1}{r_m}, & \text{if } v^j(x) \geq \frac{1}{r_m}, \\
-\frac{1}{r_m}, & \text{if } v^j(x) \leq -\frac{1}{r_m}, \\
v^j(x), & \text{if } -\frac{1}{r_m} < v^j(x) < \frac{1}{r_m}.
\end{cases}
\]

Then

\[
|\phi_m| \leq C
\]

and so \( \lambda_m r_m |\phi_m| \leq C \lambda_m \to 0 \) uniformly. Since \( a_m \in K_{\epsilon_0} \), it follows that for \( m \) large enough

\[
a_m + r_m A_m z, a_m + r_m A_m z + \lambda_m r_m \phi_m \in K_{\epsilon_0/2}
\]

for all \( z \in B \).

Observe also that

\[
\int_B |\phi_m - \nu|^2 \, dz \leq \sum_{j=1}^k \int_{\{|r_m| > 1\}} |\nu|^2 \, dz \to 0
\]

and

\[
\int_B |D\phi_m - D\nu|^2 \, dz \leq \sum_{j=1}^k \int_{\{|r_m| > 1\}} |D\nu|^2 \, dz \to 0.
\]

Hence

\[
\phi_m \to \nu \text{ in } H^1(B; \mathbb{R}^k).
\]

3. Put

\[
\tilde{\nu}_m := \xi \phi_m + (1 - \xi)\nu_m.
\]

Then \( D\tilde{\nu}_m = \xi D\phi_m + (1-\xi)D\nu_m + (\phi_m - \nu_m)D\xi \).

We now assert that

\[
\limsup_{m \to \infty} (I^m_r[\nu_m] - I^m_r[\phi_m]) \leq 0.
\]

To see this, note that \( I^m_r[\nu_m] \leq I^m_r[\tilde{\nu}_m] \), according to (2.14). Consequently, \( 0 \geq I^m_r[\nu_m] - I^m_r[\tilde{\nu}_m] \)

\[
= I^m_r[\nu_m] - I^m_r[\phi_m] + \int_{B(0,R)-B(0,r)} F_m(D\nu_m) - F_m(D\tilde{\nu}_m) \, dz
\]

\[
+ \frac{1}{\lambda_m^2} \int_{B(0,R)-B(0,r)} f(a_m + r_m A_m z + \lambda_m r_m \nu_m)
\]

\[
- f(a_m + r_m A_m z + \lambda_m r_m \tilde{\nu}_m) \, dz;
\]

and so

\[
I^m_r[\nu_m] - I^m_r[\phi_m] \leq \int_{B(0,R)-B(0,r)} F_m(D\tilde{\nu}_m) - F_m(D\nu_m) \, dz
\]

\[
+ \frac{1}{\lambda_m^2} \int_{B(0,R)-B(0,r)} f(a_m + r_m A_m z + \lambda_m r_m \tilde{\nu}_m)
\]

\[
- f(a_m + r_m A_m z + \lambda_m r_m \nu_m) \, dz.
\]
Now
\[
\int_{B(0,R) - B(0,r)} F_m(D\nu_m) \, dz \\
= \frac{1}{\lambda_m^2} \int_{B(0,R) - B(0,r)} F(A_m + \lambda_m D\nu_m) - F(A_m) - \lambda_m DF(A_m) \cdot D\nu_m \, dz \\
= \int_{B(0,R) - B(0,r)} \int_0^1 \int_0^1 s(D\nu_m)^T \cdot D^2 F(A_m + s\lambda_m D\nu_m) D\nu_m \, dt \, ds \, dz.
\]
Likewise
\[
\frac{1}{\lambda_m^2} \int_{B(0,R) - B(0,r)} F(A_m + \lambda_m D\tilde{\nu}_m) - F(A_m) - \lambda_m DF(A_m) \cdot D\tilde{\nu}_m \, dz \\
= \int_{B(0,R) - B(0,r)} \int_0^1 \int_0^1 s(D\tilde{\nu}_m)^T \cdot D^2 F(A_m + s\lambda_m D\tilde{\nu}_m) D\tilde{\nu}_m \, dt \, ds \, dz.
\]
Combining the foregoing, we deduce that
\[
\begin{align*}
\int_{B(0,R) - B(0,r)} F_m(D\tilde{\nu}_m) - F_m(D\nu_m) \, dz \\
&\leq C \int_{B(0,R) - B(0,r)} |D\nu_m|^2 + |D\phi_m|^2 + |D\xi|^2 |\phi_m - \nu_m|^2 \, dz \\
&\leq C\mu(B(0,R) - B(0,r)) + o(1)
\end{align*}
\]
as \(m \to \infty\), where we have used (2.11) and (2.18).

We next consider the terms involving \(f\), taking particular care since \(f\) blows up at \(\partial K\). The convexity of \(f\) and (2.17) yield
\[
\frac{1}{\lambda_m^2} [f(a_m + r_mA_m z + \lambda_m r_m \nu_m) - f(a_m + r_mA_m z + \lambda_m r_m \nu_m)] \\
\leq \frac{1}{\lambda_m^2} [\xi f(a_m + r_mA_m z + \lambda_m r_m \nu_m) + (1 - \xi) f(a_m + r_mA_m z + \lambda_m r_m \nu_m) - f(a_m + r_mA_m z + \lambda_m r_m \nu_m)] \\
= \frac{1}{\lambda_m^2} [\xi f(a_m + r_mA_m z + \lambda_m r_m \phi_m) - f(a_m + r_mA_m z + \lambda_m r_m \nu_m)] \\
\leq \frac{1}{\lambda_m^2} [\xi f(a_m + r_mA_m z) + \lambda_m r_m Df(a_m + r_mA_m z) \cdot \phi_m + C|\lambda_m r_m \phi_m|^2 - f(a_m + r_mA_m z) - \lambda_m r_m Df(a_m + r_mA_m z) \cdot \nu_m)] \\
\leq C\xi\left(r_m^2 |\phi_m - \nu_m| + r_m^2 |\phi_m|^2\right),
\]
\(C\) depending upon \(\max_{z \in K^{1/2}} (|Df(z)| + |D^2 f(z)|)\).

According to (2.20), (2.19), (2.11), and (2.18), we see that
\[
\frac{r_m}{\lambda_m} |\phi_m - \nu_m| \to 0
\]
in $L^1(B)$. Furthermore, $r_m^2 |\phi_m|^2 \leq C$ and $r_m^2 |\phi_m|^2 \to 0$ almost everywhere. Hence the Dominated Convergence Theorem implies

$$
\lim_{m \to \infty} \sup \frac{1}{\lambda_m^2} \int_{B(0,R) - B(0,r)} [f(a_m + r_m A_m z + \lambda_m r_m \nu_m) 
- f(a_m + r_m A_m z + \lambda_m r_m \nu_m)] \, dz \leq 0.
$$

We recall the previous estimates $(2.20)$ and $(2.21)$, to conclude that

$$
\limsup_{m \to \infty} (I_r^m[\nu_m] - I_r^m[\phi_m]) \leq C \mu(B(0,R) - B(0,r)).
$$

Letting $R \to r$ and remembering $(2.16)$, we obtain the assertion $(2.19)$.

4. Given $0 < s < r$, we let $\rho$ be another smooth cutoff function such that

$$
\begin{cases}
0 \leq \rho \leq 1; \quad \rho \equiv 1 & \text{on } B_s; \\
\rho \equiv 0 & \text{on } \mathbb{R}^n - B(0,r), \quad |D\rho| \leq \frac{C}{r-s}.
\end{cases}
$$

Define

$$
\psi_m = \rho(\nu_m - \phi_m)
$$

and notice that $\phi_m + \psi_m = \rho \nu_m + (1 - \rho) \phi_m$.

Then

$$
I_r^m[\nu_m] - I_r^m[\phi_m] = (I_r^m[\nu_m] - I_r^m[\phi_m + \psi_m]) + (I_r^m[\phi_m + \psi_m] - I_r^m[\phi_m])
=: R_1 + R_2.
$$

Proceeding as above,

$$(2.22) \quad -R_1 = I_r^m[\rho \nu_m + (1 - \rho) \phi_m] - I_r^m[\nu_m] \leq C \mu(B(0,r) - B(0,s)) + o(1),$$

as $m \to \infty$. The term $R_2$ can be written as

$$
R_2 = \int_{B(0,r)} F_m(D\phi_m) \, dz + \int_{B(0,r)} F_m(D\psi_m) - F_m(D\phi_m) - F_m(D\psi_m) \, dz
$$

(2.23)

$$
+ \frac{1}{\lambda_m^2} \int_{B(0,r)} f(a_m + r_m A_m z + \lambda_m r_m (\phi_m + \psi_m))
- f(a_m + r_m A_m z + \lambda_m r_m \phi_m) \, dz
=: S_1 + S_2 + S_3.
$$

Now the convexity of $f$ implies

$$
-S_3 = \frac{1}{\lambda_m^2} \int_{B(0,r)} f(a_m + r_m A_m z + \lambda_m r_m \phi_m)
- f(a_m + r_m A_m z + \lambda_m r_m (\phi_m + \psi_m)) \, dz
$$

$$
\leq \frac{1}{\lambda_m^2} \int_{B(0,r)} Df(a_m + r_m A_m z + \lambda_m r_m \phi_m) \cdot (-\lambda_m r_m \psi_m) \, dz
$$

$$
\leq \frac{Cr_m}{\lambda_m} \int_{B(0,r)} |\psi_m| \, dz \leq \frac{Cr_m}{\lambda_m} \int_{B(0,r)} |\nu_m| + |\nu| \, dz
$$

$$
\leq \frac{Cr_m}{\lambda_m} \leq C \lambda_m^3 = o(1)
$$

as $m \to \infty$, according to $(2.19)$ and $(2.17)$.
The uniform strict quasiconvexity of $F$ yields
\[
S_1 = \int_{B(0,r)} F_m(D\psi_m) \, dz \\
= \frac{1}{\lambda^2} \int_{B(0,r)} (F(A_m + \lambda_m D\psi_m) - F(A_m) - \lambda_m D^2 F(A_m) \cdot D\psi_m) \, dz \\
= \frac{1}{\lambda^2} \int_{B(0,r)} (F(A_m + \lambda_m D\psi_m) - F(A_m)) \, dz \geq \gamma \int_{B(0,r)} |D\psi_m|^2 \, dz.
\]

Since
\[
F_m(P + Q) - F_m(P) - F_m(Q) = \int_0^1 (DF_m(P + tQ) - DF_m(tQ)) \, dt \cdot Q \\
= P^T (\int_0^1 \int_0^1 D^2 F_m(sP + tQ) \, ds \, dt) Q,
\]
we obtain
\[
S_2 = \int_{B(0,r)} (D\phi_m)^T G_m D\psi_m \, dz,
\]
for
\[
G_m = \int_0^1 \int_0^1 D^2 F_m(sD\phi_m + tD\psi_m) \, ds \, dt \\
= \int_0^1 \int_0^1 D^2 F(A_m + s\lambda_m D\phi_m + t\lambda_m D\psi_m) \, ds \, dt.
\]

Since $\lambda_m D\phi_m, \lambda_m D\psi_m \to 0$ strongly in $L^1$ we deduce that (up to a subsequence)
\[
\lambda_m D\phi_m(x), \lambda_m D\psi_m(x) \to 0 \text{ a.e.}
\]
Hence, recalling that $D\phi_m \to D\psi$ strongly in $L^2$ and that $D^2 F$ is bounded and continuous, we find that, using the Dominated Convergence Theorem,
\[
(D\phi_m)^T G_m \to (D\psi)^T F(A) \text{ strongly in } L^2.
\]

As $D\psi_m \to 0$ weakly in $L^2$, we therefore get that
\[
S_2 = o(1), \quad \text{as } m \to \infty.
\]

Combining the foregoing estimates on $R_1, S_1, S_2, S_3$, we eventually find that
\[
\limsup_{m \to \infty} \int_{B(0,r)} |D\psi_m|^2 \, dz \leq C\mu(B(0,r) - B(0,s)).
\]

Hence for any $0 < s < r$,
\[
(2.24) \quad \limsup_{m \to \infty} \int_{B(0,s)} |D\psi_m - D\phi_m|^2 \, dz \leq C\mu(B(0,r) - B(0,s)).
\]

Hence (2.24) and (2.18) imply
\[
\limsup_{m \to \infty} \int_{B(0,s)} |D\psi_m - D\psi|^2 \, dz \leq C\mu(B(0,r) - B(0,s)).
\]

Our sending $s \to r$ completes the proof. \(\square\)

We need one further assertion, that $\psi$ solves a linear elliptic system (and consequently is smooth). We again have to take care, as $f$ is singular.
Lemma 2.3. The function $v$ satisfies the integral identity

\[(2.25) \quad \int_{B(0,r)} (Dw)^T D^2 F(A) Dv \, dz = 0 \]

for all $w \in H^1(B(0,r) ; \mathbb{R}^k)$.

Consequently, $v$ is a weak solution of the constant coefficient elliptic system

\[(2.26) \quad \text{div} \left( D^2 F(A) Dv \right) = 0. \]

Proof. 1. First we show that for any $\varphi \in C^\infty(B; \mathbb{R}^k)$,

\[(2.27) \quad \int_{B(0,r)} (D\tilde{v})^T D^2 F(A) D\tilde{v} \, dz \geq \int_{B(0,r)} (Dv)^T D^2 F(A) Dv \, dz, \]

for $\tilde{v} = \rho \varphi + (1 - \rho)v$, where $\rho$ is a cutoff function as in Lemma 2.2.

To prove this, we set $\tilde{v}_m := \rho \varphi + (1 - \rho)v_m$. According to (2.14),

\[ I^m_r[v_m] \leq I^m_r[\tilde{v}_m]. \]

As before, the convexity of $f$ implies

\[
\frac{1}{\lambda_m^2} \left[ f(a_m + r_m A_m z + \lambda_m r_m \tilde{v}_m) - f(a_m + r_m A_m z + \lambda_m r_m v_m) \right]
\leq \frac{1}{\lambda_m^2} \rho \left[ f(a_m + r_m A_m z + \lambda_m r_m \varphi) - f(a_m + r_m A_m z + \lambda_m r_m v_m) \right]
\leq \frac{1}{\lambda_m^2} \rho \left[ f(a_m + r_m A_m z) + \lambda_m r_m Df(a_m + r_m A_m z) \cdot \varphi + C |\lambda_m r_m |^2 \right.
\left. - f(a_m + r_m A_m z) - \lambda_m r_m Df(a_m + r_m A_m z) \cdot v_m \right]
\leq C \left( \frac{r_m}{\lambda_m} |\varphi - v_m| + r_m^2 |\varphi| \right).
\]

Therefore

\[
\limsup_{m \to \infty} \frac{1}{\lambda_m^2} \int_{B(0,r)} f(a_m + r_m A_m z + \lambda_m r_m \tilde{v}_m) - f(a_m + r_m A_m z + \lambda_m r_m v_m) \, dz \leq 0.
\]

Thus

\[
\int_{B(0,r)} F_m(Dv_m) \, dz \leq \int_{B(0,r)} F_m(D\tilde{v}_m) \, dz + o(1)
\]

as $m \to \infty$. Repeating the calculations before, the above inequality is equivalent to

\[
\int_{B(0,r)} \int_0^1 \int_0^1 s(D\tilde{v}_m)^T D^2 F(A_m + ts \lambda_m Dv_m) Dv_m \, dt \, ds \, dz
\leq \int_{B(0,r)} \int_0^1 \int_0^1 s(D\tilde{v}_m)^T D^2 F(A_m + ts \lambda_m D\tilde{v}_m) D\tilde{v}_m \, dt \, ds \, dz + o(1).
\]

Lemma 2.2 shows that $Dv_m \to Dv$ in $L^2_{loc}$. Letting $m \to \infty$, we derive the inequality (2.27).

2. By approximation we see that (2.27) is still valid for $\tilde{v} = v + \lambda w$ for $w \in C_c^\infty(B(0,s))$ and $\lambda > 0$. Hence

\[
\int_{B(0,r)} (Dv + \lambda Dw)^T D^2 F(A)(Dv + \lambda Dw) \, dz \geq \int_{B(0,r)} (Dv)^T D^2 F(A) Dv \, dz.
\]
We expand the left hand side and cancel the terms that do not involve $\lambda$. Dividing by $\lambda > 0$ and then sending $\lambda \rightarrow 0$, we find that

$$\int_{B(0,r)} (Dw)^T D^2 F(A) Dw \, dz \geq 0.$$  

Replacing $w$ with $-w$, we get the reverse inequality, and so (2.25) follows. □

2.2. Iteration. We next recursively apply Theorem 2.1 on smaller and smaller concentric balls.

**Lemma 2.4.** Given $L > 0$, let $C_1 = C(2L)$ be the constant from Theorem 2.1. Then for each $\tau$ satisfying

$$0 < \tau < \min \left( \frac{1}{8}, \frac{1}{4C_1^2} \right),$$

there exists $\eta = \eta(L, \tau) > 0$ such that

$$|(u)_{x,r}|, |f((u)_{x,r})|, |(Du)_{x,r}| \leq L$$

and

$$E(x, r) \leq \eta$$

imply

$$E(x, \tau^l r) \leq C_1 \tau^{1/2} E(x, \tau^{l-1} r) \quad (l = 1, \ldots)$$

for each ball $B(x, r) \subset U$.

**Proof.** 1. We first note from hypothesis (H1) that for each $L > 0$, there exists $\epsilon_1 = \epsilon_1(L) \in (0, 1)$ such that

$$\text{if } f(a) < L \text{ and } |a - b| \leq \epsilon_1, \text{ then } f(b) < 2L.$$  

Let $\epsilon_2 = \epsilon(2L, \tau)$ be as in Theorem 2.1. Define

$$\eta = \min \left( \frac{1}{2}, \epsilon_2, \left( \frac{\epsilon_1 (1 - \tau) \tau^n}{(1 + L) C_2} \right)^{1/2}, \left( \tau^n L (1 - C_1^{1/2} \tau^{1/4}) \right)^2 \right),$$

the constant $C_2$ to be selected below.

2. We assert next that the following inequalities hold for all $l \geq 0$:

$$|(u)_{x,\tau^l r}| \leq 2L,$$

$$|f((u)_{x,\tau^l r})| \leq 2L,$$

$$|(Du)_{x,\tau^l r}| \leq 2L,$$

$$E(x, \tau^l r) \leq \eta \leq \epsilon_2.$$  

The proof is by induction, the case $l = 0$ being the hypothesis. Assume next that (2.33)-(2.36) are valid for $l = 0, 1, \ldots, p - 1$; we will show that they are also valid for $l = p$. □
Proof of (2.33). Poincaré’s inequality implies for each $l \leq p - 1$ that

\begin{equation}
| (u)_{x, \tau^{l+1} r} - (u)_{x, \tau^l r} | \leq \int_{B(x, \tau^{l+1} r)} | u - (u)_{x, \tau^l r} | dy
\end{equation}

\begin{align*}
& \leq \left( \int_{B(x, \tau^{l+1} r)} | u - (u)_{x, \tau^l r} |^2 dy \right)^{1/2} \\
& \leq \frac{1}{\tau^n} \left( \int_{B(x, \tau^l r)} | u - (u)_{x, \tau^l r} |^2 dy \right)^{1/2} \\
& \leq \frac{C \tau^l r}{\tau^n} \left( \int_{B(x, \tau^l r)} | Du |^2 dy \right)^{1/2}.
\end{align*}

The induction hypothesis then gives

\begin{align*}
\int_{B(x, \tau^l r)} | Du |^2 dy & \leq 2 \int_{B(x, \tau^l r)} | Du - (Du)_{x, \tau^l r} |^2 + | (Du)_{x, \tau^l r} |^2 dy \\
& \leq 2 (\eta + (2L)^2) \leq 10L^2.
\end{align*}

Plugging into (2.37), we find that

\begin{equation}
| (u)_{x, \tau^{l+1} r} - (u)_{x, \tau^l r} | \leq \frac{C_2 L r \tau^l}{\tau^n}.
\end{equation}

Thus

\begin{align*}
| (u)_{x, \tau r} | & \leq | (u)_{x, r} | + \sum_{l=0}^{p-1} | (u)_{x, \tau^{l+1} r} - (u)_{x, \tau^l r} | \\
& \leq L + \frac{C_2 L r}{\tau^n} \sum_{l=0}^{p-1} \tau^l \leq L \left( 1 + \frac{C_2 r}{\tau^n (1 - \tau)} \right) \leq 2L,
\end{align*}

since

\begin{equation}
r \leq E(x, r)^2 \leq \eta^2 \leq \frac{(1 - \tau) \tau^n}{C_2}.
\end{equation}

Proof of (2.34). Using (2.38), we see that

\begin{equation}
| (u)_{x, \tau r} | \leq \sum_{l=0}^{p-1} | (u)_{x, \tau^{l+1} r} - (u)_{x, \tau^l r} | \leq \frac{C_2 L r}{\tau^n (1 - \tau)} \leq \epsilon_1,
\end{equation}

the last inequality holding since

\begin{equation}
r \leq E(x, r)^2 \leq \eta^2 \leq \frac{\epsilon_1 (1 - \tau) \tau^n}{(1 + L) C_2} \leq \frac{\epsilon_1 (1 - \tau) \tau^n}{L C_2}.
\end{equation}

So (2.39) and (2.32) imply $| f((u)_{x, \tau r}) | \leq 2L$. \hfill \Box
Proof of (2.35). For \( l \leq p - 1 \), using the induction hypothesis and Lemma 2.1 we have
\[
|(Du)_{x,\tau\tau'_{l+1}} - (Du)_{x,\tau\tau'}| \leq \frac{1}{\tau^n} \left( \int_{B(x,\tau\tau')} |Du - (Du)_{x,\tau\tau'}|^2 \, dy \right)^{1/2}
\]
\[
\leq \frac{1}{\tau^n} E(x,\tau\tau')^{1/2}
\]
\[
\leq \frac{1}{\tau^n} [C_{1/2}^{1/2} \tau^{1/4}] \eta^{1/2}.
\]
Therefore
\[
|(Du)_{x,\tau\tau^{p+1}}| \leq |(Du)_{x,\tau\tau}| + \sum_{l=0}^{p-1} |(Du)_{x,\tau\tau'_{l+1}} - (Du)_{x,\tau\tau'}|
\]
\[
\leq L + \frac{1}{\tau^n} \eta^{1/2} \sum_{l=0}^{p-1} [C_{1/2}^{1/2} \tau^{1/4}]^l
\]
\[
\leq L + \frac{\eta^{1/2}}{\tau^n (1 - C_{1/2}^{1/2} \tau^{1/4})} \leq 2L,
\]
since \( \eta^{1/2} \leq \tau^n L(1 - C_{1/2}^{1/2} \tau^{1/4}) \).
\[\square\]

Proof of (2.36). The induction hypothesis and Lemma 2.1 yield
\[
E(x,\tau\tau^r) \leq (C_{1/2}^{1/2} \tau^{1/4}) \eta \leq \eta.
\]
3. Finally combining (2.33)–(2.36) and Lemma 2.1 one immediately obtains the lemma.
\[\square\]

2.3. Partial regularity. We are at last ready to state and prove our main assertion of partial regularity:

**Theorem 2.5.** There exists an open set \( U_0 \subset U \) such that
\[
|U - U_0| = 0
\]
and, for every \( \alpha \in (0, 1/4) \),
\[
u \in C^{1,\alpha}(U_0; \mathbb{R}^k).
\]

**Proof.**
1. Set
\[
U_0 := \left\{ x \in U \mid \lim_{r \to 0} (u)_x, r = u(x), \lim_{r \to 0} (Du)_{x, r} = Du(x), |u(x)| < \infty, |Du(x)| < \infty, f(u(x)) < \infty, \lim_{r \to 0} \int_{B(x, r)} |Du - (Du)_{x, r}|^2 \, dy = 0 \right\}.
\]
Then \( |U - U_0| = 0 \), since \( u \in H^1(U) \) and \( \int_U f(u) \, dx < \infty \).
2. We assert that \( U_0 \) is open and \( Du \in C^\alpha(U_0) \) for \( 0 < \alpha < 1/4 \).

For each \( x \in U_0 \), there exist \( L = L(x) \) and \( R = R(x) \in (0, \text{dist}(x, \partial U)) \) such that
\[
(2.40) \quad \begin{cases} 
|u(x, s)|, |(Du)_{x, s}| < L & \text{for all } 0 < s < R; \\
|f((u)_{x, s})| < L & \text{for all } 0 < s < R.
\end{cases}
\]
Fix $\alpha \in (0, 1/4)$. Take $\tau \in \left(0, \min \left(\frac{1}{2}, \frac{1}{4C_4} \right)\right)$ such that
\[ C_1 \tau^{1/2 - 2\alpha} < 1. \]

We then can choose $0 < r < R$ so small enough that
\begin{equation}
E(x, r) = r^{1/2} + \int_{B(x,r)} |Du - (Du)_{x,r}|^2 \, dy < \eta(L, \tau),
\end{equation}
where $\eta$ has been constructed in Lemma 2.4.

In summary, (2.40) and (2.41) imply
\begin{equation}
|u|_{x,r}, \ |f((u)_{x,r})|, \ |(Du)_{x,r}| < L, \text{ and } E(x, r) < \eta(L, \tau).
\end{equation}

Moreover, the following mappings
\[ x \mapsto (u)_{x,r}, f((u)_{x,r}), (Du)_{x,r}, E(x, r) \]
are continuous. Hence (2.42) holds for $z \in B(x, s)$ for some $s > 0$.

Applying Lemma 2.4 for any $z \in B(x, s)$
\[ E(z, \tau^{l}r) \leq (c(2L) \tau^{1/2})^l \eta(L, \tau) \leq (\tau^l r)^{2\alpha} \eta_1(L, \tau, r) \]
for $l = 1, \ldots$, where $\eta_1(L, \tau, r) = \eta(L, \tau) r^{-2\alpha}$. The previous estimate now implies (cf. for instance [G]) that $Du \in C^\alpha$ near $x$. This immediately shows that $u \in C^\alpha(U_0)$ and that $U_0$ is open.

\section{Partial regularity for the general problem}

In this section we return to the general functional
\[ I(v) = \int_U F(v, Dv) + f(v) \, dx \]
and outline the requisite modifications in the previous proof of partial regularity.

\textbf{Theorem 3.1.} Let $u$ be a minimizer of $I[\cdot]$. Then there exists an open set $U_0 \subset U$ such that
\[ |U - U_0| = 0 \]
and, for each $\alpha \in (0, 1/4)$,
\[ u \in C^{1,\alpha}(U_0; \mathbb{R}^k). \]

We start with an elementary lemma which will allow us to reduce the problem to the model problem ($F = F(P)$).

\textbf{Lemma 3.2.} Let $a_m \in \mathbb{R}^k$ and $A_m \in \mathbb{M}^{k \times n}$ be bounded and let $w_m \in H^1(B; \mathbb{R}^k)$ be bounded. Let $r_m, \lambda_m > 0$ be such that $r_m \leq \lambda_m^4 \to 0$ as $m \to \infty$.

Then, upon passing if necessary to a subsequence, we have
\begin{align*}
\int_B |F(a_m, A_m + \lambda_m Dw_m) - F(a_m + r_m A_m z + \lambda_m r_m w_m, A_m + \lambda_m Dw_m)| \, dz &= o(\lambda_m^2).
\end{align*}
Proof. 1. Using our hypothesis (H2) we deduce that
\[
\int_B |F(a_m, A_m + \lambda_m Dw_m) - F(a_m + r_m A_m z + \lambda_m r_m w_m, A_m + \lambda_m Dw_m)| dz
\]
\[
\leq C \int_B (1 + |A_m + \lambda_m Dw_m|^2) |r_m A_m z + \lambda_m r_m w_m| dz
\]
\[
\leq C \lambda_m^2 \int_B |r_m/\lambda_m + r_m w_m/\lambda_m| dz
\]
\[
+ C \lambda_m^2 \int_B |Dw_m|^2 \min\{1, r_m + r_m \lambda_m |w_m|\} dz
\]
\[
= : R_1 + R_2.
\]
It is elementary to see that $R_1 = o(\lambda_m^2)$.

2. Since $w_m$ is bounded in $H^1$, we may assume that there exists $w \in H^1$ such that $Dw_m \rightharpoonup Dw$ weakly in $L^2$. Moreover since $\|r_m \lambda_m w_m\|_{L^1(B)} \to 0$ we deduce that
\[
\lim_{m \to \infty} r_m + r_m \lambda_m |w_m|(x) = 0 \quad \text{a.e.}
\]

Fix $\epsilon > 0$. According to Ergoroff’s Theorem there exists a measurable set $A_\epsilon \subset B$ such that
\[
|B - A_\epsilon| < \epsilon \quad \text{and} \quad r_m + r_m \lambda_m |w_m| \to 0 \text{ uniformly in } A_\epsilon.
\]
We hence have
\[
R_2 \leq C_2 \lambda_m^2 \int_{A_\epsilon} |Dw_m|^2 (r_m + r_m \lambda_m |w_m|) dz + C_2 \lambda_m^2 \int_{B - A_\epsilon} |Dw_m|^2 dz
\]
\[
\leq C_2 \lambda_m^2 \|Dw_m\|_{L^2(A_\epsilon)} \|r_m + r_m \lambda_m |w_m|\|_{L^\infty(A_\epsilon)} + C_2 \lambda_m^2 \int_{B - A_\epsilon} |Dw_m|^2 dz
\]
\[
= o(\lambda_m^2) + C_2 \lambda_m^2 \int_{B - A_\epsilon} |Dw_m|^2 dz
\]
\[
= o(\lambda_m^2) + C_2 \lambda_m^2 \int_{B - A_\epsilon} |Dw|^2 dz.
\]
Letting $\epsilon \to 0$ we immediately obtain the result. \qed

We now sketch the proof of Theorem 3.1. As already said the proof is almost identical to the one of Theorem 2.5.

Proof. We first claim that Theorem 2.1 Lemma 2.2 and Lemma 2.3 still hold. Define $a_m, A_m, v_m, r_m, \lambda_m, E(x, r)$ as in Section 2. Then (compare with (2.12))
\[
\int_B F(a_m, A_m + \lambda_m Dv_m) + f(a_m + r_m A_m z + \lambda_m r_m v_m) dz
\]
\[
+ \int_B F(a_m + r_m A_m z + \lambda_m r_m v_m, A_m + \lambda_m Dv_m) - F(a_m, A_m + \lambda_m Dv_m) dz
\]
\[
\leq \int_B F(a_m, A_m + \lambda_m D\tilde{v}_m) + f(a_m + r_m A_m z + \lambda_m r_m \tilde{v}_m) dz
\]
\[
+ \int_B F(a_m + r_m A_m z + \lambda_m r_m \tilde{v}_m, A_m + \lambda_m D\tilde{v}_m) - F(a_m, A_m + \lambda_m D\tilde{v}_m) dz,
\]
provided \( \tilde{v}_m \in H^1(B) \) and \( \tilde{v}_m = v_m \) on \( \partial B \). It follows that \( v_m \) is a minimizer of

\[
I^m_r[w] = \int_{B(0,r)} F_m(Dw) + \frac{1}{\lambda^2_m} f(a_m + r_m A_m z + \lambda_m r_m w) \, dz
+ \frac{1}{\lambda^2_m} \int_{B(0,r)} F(a_m + r_m A_m z + \lambda_m r_m w_m, A_m + \lambda_m D w) - F(a_m, A_m + \lambda_m D w) \, dz,
\]

subject to its boundary conditions, for

\[
F_m(P) := \frac{F(a_m, A_m + \lambda_m P) - F(a_m, A_m) - \lambda_m DF(a_m, A_m) \cdot P}{\lambda^2_m}
\]

and \( r \in (0,1) \). In other words,

\[
I^m_r[v_m] \leq I^m_r[w]
\]

for any \( w \in H^1(B(0,r)) \) such that \( w = v_m \) in \( \partial B(0,r) \). By Lemma 3.2 we have that

\[
\frac{1}{\lambda^2_m} \int_{B(0,r)} F(a_m + r_m A_m z + \lambda_m r_m w_m, A_m + \lambda_m D w_m) - F(a_m, A_m + \lambda_m D w_m) \, dz
= o(1),
\]

as long as \( w_m \) is bounded in \( H^1(B(0,r); \mathbb{R}^k) \).

We now proceed exactly as in Section 2 to obtain the claim. We have in effect reduced the problem to the case \( F = F(P) \). Finally the end of the proof is exactly the same as in Section 2. \( \square \)

4. IMPROVED ESTIMATES FOR CONVEX \( F \)

This section is devoted to the study of improved partial regularity for minimizers when \( F \) is uniformly convex and only depends on the gradient, meaning that there exists a positive constant \( \gamma \) such that

\[
R^T D^2 F(P) R \geq \gamma |R|^2
\]

for all matrices \( P, R \in \mathbb{M}^{k \times n} \).

4.1. Second derivative estimates. We first show that the second derivatives of our minimizer exist and are locally square-integrable. This is a standard assertion in the calculus of variations when the singular term \( f \) is absent; see for instance Giaquinta [G] or [E2, Section 8.3.1].

**Theorem 4.1.** Suppose in addition to the hypotheses of Section 2 that \( F \) is uniformly convex. Then

\[
\mathbf{u} \in H^2_{loc}(U).
\]

**Proof.** 1. Fix any open set \( V \subseteq U \) and then select a cutoff function \( \xi \) satisfying

\[
\begin{cases}
0 \leq \xi \leq 1, & \xi = 1 \text{ on } V; \\
\xi = 0 & \text{near } \partial U.
\end{cases}
\]

For \( |h| > 0 \) small, let

\[
\mathbf{v} = D^{-h}_k (\xi^2 D^h_k \mathbf{u}),
\]

where \( D^h_k \mathbf{u} \) denotes the difference quotient

\[
D^h_k \mathbf{u}(x) = \frac{\mathbf{u}(x + h e_k) - \mathbf{u}(x)}{h} \quad (h \in \mathbb{R}, h \neq 0)
\]

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and $e_k$ is the unit vector in the $x_k$ direction. The explicit form of $v$ is

$$
\xi^2(x)u(x + he_k) + \xi^2(x - he_k)u(x - he_k) - (\xi^2(x) + \xi^2(x - he_k))u(x).
$$

(4.3)

2. There exists an open set $W$ such that $V \Subset W \Subset U$ and $\text{spt}(v) \subset W$ for $|h| > 0$ small enough. For small $t = t(h) > 0$, we have

$$
1 - t \frac{\xi^2(x)}{h^2} - t \frac{\xi^2(x - he_k)}{h^2} \geq 0.
$$

The convexity of $f$ consequently implies that

$$
f(u + tv) = f \left( t \frac{\xi^2(x)}{h^2} u(x + he_k) + t \frac{\xi^2(x - he_k)}{h^2} u(x - he_k) \right)
+ \left( 1 - t \frac{\xi^2(x)}{h^2} - t \frac{\xi^2(x - he_k)}{h^2} \right) f(u(x))
\leq t \frac{\xi^2(x)}{h^2} f(u(x + he_k)) + t \frac{\xi^2(x - he_k)}{h^2} f(u(x - he_k))
+ \left( 1 - t \frac{\xi^2(x)}{h^2} - t \frac{\xi^2(x - he_k)}{h^2} \right) f(u(x)).
$$

(4.2)

3. We note next that

$$
\int_U f(u + tv) \, dx \leq \int_U f(u) \, dx.
$$

(4.3)

To confirm this, observe from (4.2) that

$$
\int_U f(u + tv) - f(u) \, dx \leq \int_U \frac{\xi^2(x)}{h^2} (f(u(x + he_k)) - f(u(x))) \, dx
+ \int_U \frac{\xi^2(x - he_k)}{h^2} (f(u(x - he_k)) - f(u(x))) \, dx
= \int_U \frac{\xi^2(x)}{h^2} (f(u(x + he_k)) - f(u(x))) \, dx
+ \int_U \frac{\xi^2(y)}{h^2} (f(u(y)) - f(u(y + he_k))) \, dy
= 0.
$$

4. Since $u$ is a minimizer, we have for small $t > 0$ that

$$
0 \leq \int_U F(Du + tDv) - F(Du) \, dx + \int_U f(u + tv) - f(u) \, dx
\leq \int_U F(Du + tDv) - F(Du) \, dx,
$$

according to (4.3). Divide by $t$ and send $t \to 0^+$:

$$
0 \leq \int_U DF(Du) : Dv \, dx.
$$

(4.4)

Recalling the definition of $v$, we see that

$$
0 \leq \int_U DF(Du) : Dv \, dx = \int_U F_{\rho^s}(Du)[D_k^{-h}(\xi^2(D_k^h u^\alpha))]_x, \, dx
= - \int_U D_k^h(F_{\rho^s}(Du))[(\xi^2(D_k^h u^\alpha))]_x, \, dx.
$$
Now following a standard argument (as for instance in [E2 Section 8.3.1]), we use the uniform convexity of $F$ to bound the term
\[ \int_V |D^h Du|^2 \, dx \]
independently of $h$. This implies that $Du$ belongs to $H^1_{loc}$.

Note that in this proof we carefully avoided confronting the possibly very singular term $Df(u)$. In particular, we do not know that $Df(u)$ is integrable.

**4.2. Rate of blow-up of $f$.** If we know more about the speed of blow-up of $f$ near the boundary of $K$, then Theorem 2.5 can be improved:

**Theorem 4.2.** Assume that $F$ is uniformly convex and there exists a constant $\gamma > 0$ such that
\[ f(z) \geq \gamma \text{dist}(z, \partial K)^2, \quad (z \in K). \tag{4.5} \]
Then
\[ H^{n-2+\epsilon}(U - U_0) = 0 \tag{4.6} \]
for each $\epsilon > 0$, $\mathcal{H}^s$ denoting $s$-dimensional Hausdorff measure.

**Proof.** According to Theorem 4.1 $u \in H^2_{loc}(U)$.
1. Set
\[ U_1 := \left\{ x \in U \mid \lim_{r \to 0} \int_{B(x,r)} |Du - (Du)_{x,r}|^2 \, dy = 0 \right\}. \]
Using Poincaré’s inequality and since $D^2u \in L^2_{loc}(U)$ we have (cf. for instance [E-G2 page 77])
\[ \mathcal{H}^{n-2}(U - U_1) = \mathcal{H}^{n-2} \left( \left\{ x \in U \mid \limsup_{r \to 0} \int_{B(x,r)} |Du - (Du)_{x,r}|^2 \, dy > 0 \right\} \right) \leq \mathcal{H}^{n-2} \left( \left\{ x \in U \mid \limsup_{r \to 0} \frac{1}{rn^2} \int_{B(x,r)} |D^2u|^2 \, dy > 0 \right\} \right) = 0. \]
2. Set
\[ U_2 := \left\{ x \in U \mid \limsup_{r \to 0} (u)_{x,r} = u(x), \limsup_{r \to 0} (Du)_{x,r} = Du(x), |u(x)| < \infty, |Du(x)| < \infty \right\}. \]
We claim that $\mathcal{H}^{n-2+\epsilon}(U - U_2) = 0$ for every $\epsilon > 0$. This follows since if a function belongs to $H^1$, then the limit of its averages over balls converges to a finite limit (in fact to the function itself) except possibly for a set $E$ with capacity $\text{Cap}_2(E) = 0$ ([E-G2 page 160]), and therefore $\mathcal{H}^{n-2+\epsilon}(E) = 0$ ([E-G2 page 156]).
3. Let
\[ \Lambda = \left\{ x \in U \mid \limsup_{r \to 0} \frac{1}{rn^2} \int_{B(x,r)} f(u) \, dy > 0 \right\}. \]
Since $f(u) \in L^1(U)$, we have as before that $\mathcal{H}^{n-2}(\Lambda) = 0$; see for instance [E-G2 page 77].
3. Next define
\[ U_3 := \{ x \in U_1 \cap U_2 | f(u(x)) = \infty \}. \]
We claim that
\[(4.7) \quad U_3 \subseteq \Lambda.\]
To see this, take any \(x \in U_3 \subset U_1 \cap U_2.\) By the definition of \(U_1\) and \(U_2,\) there exists \(r, L > 0\) such that
\[(4.8) \quad \left| (Du)_{x,s} \right| \leq L, \quad \int_{B(x,s)} \left| Du - (Du)_{x,s} \right|^2 dy \leq L \text{ for all } s \leq r.\]
Then (cf. Lemma 2.4)
\[|(u)_{x,\tau^{l+1}s} - (u)_{x,\tau^ls}| \leq C\tau^ls\]
for \(l = 0, 1, \ldots;\) and hence
\[|(u)_{x,s} - u(x)| \leq \sum_{l=0}^{\infty} |(u)_{x,\tau^{l+1}s} - (u)_{x,\tau^ls}| \leq \frac{Cs}{1-\tau}.\]
As \(u(x) \in \partial K,\) the assumption (4.5) implies
\[\frac{\gamma}{\text{dist}((u)_{x,s}, \partial K)^2} \geq \frac{\gamma}{|(u)_{x,s} - u(x)|^2} \geq \frac{C}{s^2}.\]
Jensen’s inequality now implies
\[\int_{B(x,s)} f(u) dy \geq f((u)_{x,s}) \geq \frac{C}{s^2}.\]
Thus for \(s < r\) we have
\[\frac{1}{s^{n-2}} \int_{B(x,s)} f(u) dy \geq C > 0;\]
and consequently \(x \in \Lambda.\)

4. Observe finally that \(U_0 = (U_1 \cap U_2) - U_3\) and \(U - U_0 = (U - U_1) \cup (U - U_2) \cup U_3.\) Hence \(\mathcal{H}^{n-2+\epsilon}(U - U_0) = 0\) for each \(\epsilon > 0.\)

As we will discuss in Section 4, Ball-Majumdar [B-M] have introduced certain liquid crystal models for which \(f\) exhibits a logarithmic divergence near \(\partial K:\) this is much weaker than (4.5). We propose therefore to extend the previous proof to handle this case, and for motivation look at the following model problem:

**Example.** Assume \(K = B(0, 1)\) and there exists \(r \in (1/2, 1)\) such that
\[f(z) = -\log(1 - |z|), \quad \text{for } r < |z| < 1.\]
Then
\[f_{z^\alpha}(z) = \frac{z^\alpha}{|z|(1 - |z|)}\]
and
\[f_{z^\alpha z^\beta}(z) = \frac{\delta_{\alpha\beta}}{|z|(1 - |z|)} + \frac{z^\alpha z^\beta}{|z|^2(1 - |z|)^2} \left(2 - \frac{1}{|z|}\right).\]
Therefore
\[f_{z^\alpha z^\beta}(z) y^\alpha y^\beta \geq \gamma |Df(z) \cdot y|^2 \quad \text{for } r < |z| < 1,\]
where \(\gamma := 2 - \frac{1}{r}.\)
Motivated by this example, we introduce the condition that
\begin{equation}
C|y|^2 + y^T D^2 f(z)y \geq \gamma |Df(z) \cdot y|^2, \quad (z \in \mathbb{K}, y \in \mathbb{R}^k)
\end{equation}
for constants $C, \gamma > 0$.

This condition is consistent with the toy example discussed above, but we are unable to confirm that it is satisfied for the Ball-Majumdar energy. We therefore do not claim that the following has any real physical significance, although we believe it of some mathematical interest:

**Theorem 4.3.** (i) Assume that $F$ is uniformly convex and $f$ satisfies (4.9); then
\[ f(u) \in H^1_{loc}(U). \]

(ii) Therefore
\[ \mathcal{H}^{n-2+\epsilon}(U - U_0) = 0 \]
for each $\epsilon > 0$.

**Proof.** 1. Let $f^n$ be smooth, convex and everywhere defined on $\mathbb{R}^k$, such that
\[ 0 \leq f^n \leq f, \quad f^n \equiv f \text{ on } K. \]

Let $u^n$ denote the unique minimizer of
\[ I^n[v] = \int_U F(Dv) + f^n(v) \, dx \]
over the admissible class of functions $A$. The functions $\{u^n\}_{n>0}$ are uniformly bounded in $H^2(V)$ for each compactly contained subregion $V \subseteq U$ (see the proof of Theorem 4.1). As a consequence we claim that
\begin{equation}
\boxed{u^n \to u \text{ in } H^1_{loc}}
\end{equation}
as $n \to 0$. We already know that
\[ u^n \to w \text{ in } H^1_{loc} \]
for some $w \in H^1_{loc}$. It remains to show that $w = u$. Note that
\begin{equation}
\int_U f^n(u^n) + F(Du^n) \, dx \leq \int_U f^n(u^n + tv^n) + F(Du^n + tDv^n) \, dx \leq \int_U f(u) + F(Du) \, dx.
\end{equation}

We can assume that (up to a subsequence)
\begin{equation}
\boxed{u^n \to w, \quad Du^n \to Dw \text{ a.e. in } U.}
\end{equation}

Using the properties of $f^n$ and $F$ we hence deduce
\[ f^n(u^n(x)) \to f(w(x)), \quad F(Du^n(x)) \to F(Dw(x)) \text{ a.e. in } U. \]

Combining (4.11), (4.12) and Fatou’s Lemma we have
\[ \int_U f(w) + F(Dw) \, dx \leq \int_U f(u) + F(Du) \, dx. \]

Hence by uniqueness of the minimizer of $I$ we obtain $u = w$.

2. Since $u^n$ is a minimizer of $I^n$ we have
\[ \int_U f^n(u^n) - f^n(u^n + tv^n) \, dx \leq \int_U F(Du^n + tDv^n) - F(Du^n) \, dx, \]
where
\[ v^n = D_k^{-h}(\xi^2 D_k^h u^n) \]
Since $\xi (4.13)$

Recall that we may assume that and the function $\rightarrow t$

Invoking Fatou’s Lemma, we deduce that



owing to the uniform $H^2_{loc}$ estimates on $u^n$. Rewriting the left hand side, we obtain the bound



Since $f^n$ is convex, the integrand in the last expression is pointwise nonnegative.

Invoking Fatou’s Lemma, we deduce that

$$\int_U \xi^2 D^h_k (f^n_k (u^n)) D^h_k u^n,\alpha dx \leq C.$$ Since $\xi \equiv 1$ on $V$, it follows that

$$\int_V f^n_{z^n,\beta} (u^n) u^n,\alpha u^n,\beta dx \leq C.$$ (4.13)

the constant independent of $\eta$.

3. Fix a small $\delta > 0$ and let

$$A_\delta := \{ x \in U \mid u \in K_\delta \}.$$ Then (4.13) implies for $0 < \eta < \delta$ that

$$\int_{V \cap A_\delta} f^n_{z^n,\beta} (u^n) u^n,\alpha u^n,\beta dx \leq C.$$ Recall that we may assume that

$$u^n \to u, D u^n \to D u \ a.e. \ in \ U.$$ Hence

$$f^n_{z^n,\beta} (u^n) u^n,\alpha u^n,\beta \to f_{z^n,\beta} (u) u^\alpha u^\beta \ a.e. \ in \ V \cap A_\delta.$$ Since $f^n_{z^n,\beta} (u^n) u^n,\alpha u^n,\beta \geq 0$, we may invoke Fatou’s Lemma, to deduce that

$$\int_{V \cap A_\delta} f_{z^n,\beta} (u) u^\alpha u^\beta dx \leq \liminf_{\eta \to 0} \int_{V \cap A_\delta} f^n_{z^n,\beta} (u^n) u^n,\alpha u^n,\beta dx \leq C.$$ Next, let $\delta \to 0$:

$$\int_V f_{z^n,\beta} (u) u^\alpha u^\beta dx \leq C.$$ (4.14)

We combine this with our assumption (4.9), to discover

$$\int_V |D f(u) D u|^2 dx \leq \frac{1}{\gamma} \int_V (D u)^T D^2 f(u) D u + C |D u|^2 dx \leq C.$$ 4. We claim next that $D f(u) D u$ is the weak gradient of $f(u)$ in the sense of distributions. To see this, take a large number $\eta$ and define the open, convex set

$$F_\eta := \{ z \in \mathbb{R}^k \mid f(z) < \eta \}.$$ Let $\Phi_\eta$ denote the projection of $\mathbb{R}^k$ onto the closure of $F_\eta$. Then $\Phi_\eta$ is Lipschitz continuous, with Lipschitz constant equal to one. Next let

$$u_\eta := \Phi_\eta (u).$$
Then
\[ 0 \leq f(u_\eta) \leq f(u) \in L^1(V), \]
and therefore the Dominated Convergence Theorem implies that
\[ f(u_\eta) \to f(u) \quad \text{in } L^1(V). \]

Since \( \Phi_\eta \) has Lipschitz constant equal to one and since \( f \) restricted to \( F_\eta \) is smooth, we see that
\[ f(u_\eta) \in H^1(V). \]
Now if a function belongs to the Sobolev space \( H^1 \) its gradient vanishes almost everywhere on each level set. Consequently,
\[ \frac{Df(u_\eta)}{Du_\eta} = \frac{Df(u)}{Du} \quad \text{a.e. on } \{ f(u_\eta) = \eta \} = \{ u \notin F_\eta \}; \]
and hence
\[ \frac{Df(u_\eta)}{Du_\eta} = \begin{cases} \frac{Df(u)}{Du} & \text{a.e. on } \{ u \in F_\eta \}, \\ 0 & \text{a.e. on } \{ u \notin F_\eta \}. \end{cases} \]
It follows that
\[ |\frac{Df(u_\eta)}{Du_\eta}| \leq |\frac{Df(u)}{Du}| \in L^2(V), \]
according to (4.14). Since \( \frac{Df(u_\eta)}{Du_\eta} \to \frac{Df(u)}{Du} \) almost everywhere, the Dominated Convergence Theorem now implies that
\[ \frac{Df(u_\eta)}{Du_\eta} \to \frac{Df(u)}{Du} \quad \text{in } L^2(V). \]

Hence for any function \( \phi \in C^\infty(V) \) with compact support and for any \( k = 1, \ldots, n \), we have
\[ \int_V f(u) \phi_{x_k} \, dx = \lim_{\eta \to \infty} \int_V f(u_\eta) \phi_{x_k} \, dx = - \lim_{\eta \to \infty} \int_V (f(u_\eta))_{x_k} \phi \, dx = - \lim_{\eta \to \infty} \int_V f_{x_k}(u_\eta)(u_\eta^\alpha)_{x_k} \phi \, dx = - \int_V f_{x_k}(u)u_\eta^\alpha \phi \, dx. \]

Therefore \( \frac{Df(u)}{Du} \) exists in the sense of distributions and equals \( \frac{Df(u)}{Du} \in L^2(V) \).

Next, put \( (f(u))_V := \int_V f(u) \, dx < \infty \). According to Poincaré’s inequality,
\[ \int_V |f(u) - (f(u))_V|^2 \, dx \leq C \int_V |D(f(u))|^2 \, dx \leq C. \]
Thus
\[ \int_V |f(u)|^2 \, dx \leq C. \]
Hence \( f(u) \in H^1_{loc}(U) \); which in turn implies (see the proof of Theorem 4.2)
\[ \mathcal{H}^{n-2+\epsilon}(\{x \in U \mid f(u(x)) = \infty\}) = 0, \]
for each \( \epsilon > 0 \). Now proceeding similarly as in the proof of Theorem 4.2 we get that \( \mathcal{H}^{n-2+\epsilon}(U - U_0) = 0 \) for each each \( \epsilon > 0 \). \( \square \)
5. Applications

5.1. Variational models for liquid crystals. Our partial regularity theorems are motivated by some new physical models for nematic liquid crystals. This final section provides only a very brief overview of some of the relevant physical issues. The books by Virga [V] and de Gennes-Prost [dG-P] provide much more background. See also, in addition to the Ball-Majumdar paper repeatedly cited above, the research papers Ball–Zarnescu [B-Z], Fatkullin–Slastikov [F-S], Majumdar [M] and Majumdar-Zarnescu [M-Z]. The recent paper of Feireisl, Rocca, Schimperna and Zarnescu [F-R-S-Z] constructs weak solutions for nonisothermal nematic liquid crystal flows governed by the Ball-Majumdar free energy.

The nematic phase of a liquid crystal is a phase for which the molecules are free to flow but still tend to align, so as to have long range directional order locally. These long range directions are locally approximately parallel. Thus the molecule at a point $x$ has a preferred direction $n(x)$ belonging to the unit sphere $S^2$, but it can also move around a little bit in the other two directions.

There are two well-known mathematical models of nematic liquid crystals, the mean field approach and Landau-de Gennes theory.

5.2. Mean field models. In the mean field approach, the alignment of the nematic molecules at each point $x$ in space is described by a probability distribution function $\rho_x$ on the unit sphere satisfying $\rho_x(p) = \rho_x(-p)$, to model the head-to-tail symmetry. The first moment of $\rho_x$ hence vanishes:

$$\int_{S^2} p \rho_x \, d\mathcal{H}^2 = 0.$$ 

A corresponding macroscopic order parameter is

$$Q(x) = \int_{S^2} (p \otimes p - \frac{1}{3} I) \rho_x \, d\mathcal{H}^2. \tag{5.1}$$

Thus $Q$ is a symmetric, traceless $3 \times 3$ matrix whose eigenvalues $\lambda_i = \lambda_i(Q)$ are constrained by the inequalities (see e.g. [M])

$$-\frac{1}{3} \leq \lambda_i \leq \frac{2}{3} \quad (i = 1, 2, 3); \quad \sum_{i=1}^{3} \lambda_i = 0. \tag{5.2}$$

Notice that (5.2) is equivalent to

$$-\frac{1}{3} |\xi|^2 \leq (Q \xi, \xi) \leq \frac{2}{3} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^3; \quad \sum_{i=1}^{3} Q_{ii} = 0. \tag{5.3}$$

Consequently the set $\mathcal{K}$ of symmetric, traceless matrices $Q$ satisfying (5.2) is bounded, closed and convex.

5.3. Landau-de Gennes models. The Landau-de Gennes theory also describes the state of a nematic liquid crystal by the macroscopic order parameter $Q$. However, now $Q$ is only required to be symmetric, traceless $3 \times 3$ matrix; and here is no requirement of a priori bounds on the eigenvalues of $Q$ like (5.2). The corresponding Landau-de Gennes energy functional is

$$I_{LG}[Q] = \int_U f_B(Q) + F(Q, DQ) \, dx,$$
where $f_B$ is a thermotropic bulk potential. As noted in Ball-Majumdar \[B\]-\[M\], the function $f_B$ usually has the form

$$f_B(Q) = \frac{1}{2} a \text{tr} Q^2 + \frac{1}{3} b \text{tr} Q^3 + \frac{1}{4} c (\text{tr} Q^2)^2 + \cdots,$$

the coefficients $a, b, c, \ldots$ depending upon the temperature $T$. In particular, in this model there is no term that enforces the physical constraints (5.2) on the eigenvalues. The equilibrium and physically observable configurations correspond either to global or local minimizers of the Landau-de Gennes energy subject to the imposed boundary conditions. Majumdar observed in [M] that there are cases where the equilibrium order parameters can take value outside $\mathbb{K}$ even for temperatures $T$ quite close to the nematic-isotropic transition temperature.

Ball and Majumdar address this issue by defining a new bulk potential

$$f_B(Q) = T \psi(Q) - \kappa |Q|^2.$$

Here

$$\psi(Q) = \inf_{\rho \in A_Q} \int_{S^2} \rho \log \rho \, d\mathcal{H}^2,$$

where

$$A_Q = \left\{ \rho : S^2 \to \mathbb{R} \mid \rho \geq 0, \int_{S^2} \rho \, d\mathcal{H}^2 = 1, Q = \int_{S^2} (\rho \otimes \rho - \frac{1}{3} I) \, d\mathcal{H}^2 \right\}.$$

The $f$ is convex and blows up a $\partial \mathbb{K}$ (that is, whenever the eigenvalues approach the limiting values of either $-\frac{1}{3}$ or $\frac{2}{3}$ in (5.2)). Ball and Majumdar also showed that $f$ exhibits a logarithmic divergence as the eigenvalues approach either $-\frac{1}{3}$ or $\frac{2}{3}$.

**Example.** In the case of a modified Landau-de Gennes model, we set $n = 3$, $k = 5$, and we define a linear mapping $Q \mapsto q$ from the set of traceless, symmetric matrices $S^3$ to $\mathbb{R}^5$ as follows:

$$q = (Q_{ij})_{i \geq j, i+j<6}.$$

It is clear that this linear mapping is an isomorphism. Then

$$u(x) = (Q_{ij}(x))_{i \geq j, i+j<6},$$

and the bounded open convex set $\mathbb{K} \subset \mathbb{R}^5$ is

$$\mathbb{K} = \{ q \in \mathbb{R}^5 \mid -\frac{1}{3} < \lambda_i(Q) < \frac{2}{3}, \text{ for } i = 1, 2, 3 \}.$$
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