Oscillatory asymptotics for Airy kernel determinants on two intervals

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Abstract

We obtain asymptotics for Airy kernel Fredholm determinants on two intervals. We give explicit formulas for all the terms up to and including the oscillations of order 1, which are expressed in terms of elliptic \( \Theta \)-functions.

1 Introduction

The Airy point process is a well-known universal point process from the theory of random matrices and is used to model the behavior of the largest eigenvalue for a wide class of large Hermitian random matrices [5, 7, 8, 14, 22]. The Airy point process also appears in other related models. For example, it describes the transition between the solid and liquid regions in random tiling models [17] and the largest parts of Young diagrams with respect to the Plancherel measure [2, 4]. This is a determinantal point process on \( \mathbb{R} \) whose correlation kernel is given by

\[
K_{Ai}(u, v) = \frac{Ai(u)Ai'(v) - Ai'(u)Ai(v)}{u - v},
\]

where \( Ai \) denotes the Airy function. Let \( \vec{x} = (x_1, x_2, x_3) \) be such that \( x_3 < x_2 < x_1 < 0 \) and let \( F(\vec{x}) = F(x_1, x_2, x_3) \) be the probability of finding no points on \( (x_3, x_2) \cup (x_1, +\infty) \). In other words, \( F \) represents a gap probability on two disjoint intervals. It is well-known [23] that \( F(\vec{x}) \) can be written as a Fredholm determinant

\[
F(\vec{x}) = \det \left( 1 - K_{Ai}^\chi_{(x_3, x_2)} \cup (x_1, +\infty) \right),
\]

where \( K_{Ai}^\chi \) is the integral operator whose kernel is \( K_{Ai} \). The purpose of this paper is to obtain asymptotics for \( F(r\vec{x}) = F(rx_1, rx_2, rx_3) \) as \( r \to +\infty \). Such asymptotics are often referred to as large gap asymptotics.

Large gap asymptotics on a single interval have a rich history [1, 9, 24] and are now well-understood. In our setting, if we take either \( x_2 = x_3 \) and \( x_1 = x < 0 \), or \( x_2 = x_1 \) and \( x_3 = x < 0 \), then the following asymptotics hold

\[
F(r\vec{x}) = \exp \left( -\frac{|rx|^3}{12} - \frac{1}{8} \log |rx| + \zeta'(1) + \frac{1}{24} \log 2 + O(r^{-2}) \right), \quad \text{as } r \to +\infty,
\]

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where \( \zeta' \) denotes the derivative of the Riemann zeta function. This expansion was first obtained in [24], but without rigorously proving the term of order 1. Note in (1.3) that the term of order 1 is a constant (independent of \( r \)). In Theorem 1.1 below, we show that this is no longer the case when \( x_3 < x_2 < x_1 < 0 \); instead the term of order 1 presents some oscillations expressed in terms of elliptic \( \theta \)-functions.

We briefly introduce the necessary material to present our result. Let us define

\[
\sqrt{Q(z)} = \sqrt{(z-x_1)(z-x_2)(z-x_3)},
\]

(1.4)

where the principal branch is taken for each square root. Thus \( \sqrt{Q(z)} \) is analytic on \( \mathbb{C} \setminus ((-\infty, x_3] \cup [x_2, x_1]) \) and \( \sqrt{Q(z)} \sim z^{\frac{3}{2}} \) as \( z \to \infty \). Let us also define

\[
q(z) = -z^2 + \frac{x_1 + x_2 + x_3}{2} z - q_0,
\]

(1.5)

where \( q_0 \in \mathbb{R} \) is given by

\[
q_0 = \left( \int_{x_2}^{x_3} \frac{ds}{\sqrt{Q(s)}} \right)^{-1} \int_{x_3}^{x_2} s^2 - \frac{x_1 + x_2 + x_3}{2} s \sqrt{Q(s)} ds.
\]

(1.6)

The constant \( \Omega \), defined by

\[
\Omega = \int_{x_2}^{x_1} \frac{2q(x)}{|\sqrt{Q(x)}|} dx,
\]

also appears in our final result. We show in Proposition 3.1 (see also Remark 4) that \( q \) has one simple root in \( (x_3, x_2) \) and another simple root in \( (x_1, +\infty) \), so \( q(x) > 0 \) for \( x \in (x_2, x_1) \), which implies \( \Omega > 0 \). Finally, we define

\[
c_0 = \left( \int_{x_3}^{x_2} \frac{2dx}{|\sqrt{Q(x)}|} \right)^{-1} \in \mathbb{R}^+, \quad \tau = \int_{x_2}^{x_1} \frac{2i\nu dx}{|\sqrt{Q(x)}|} \in i\mathbb{R}^+.
\]

(1.7)

The \( \theta \)-function of the third kind \( \theta(z; \tau) \) associated to the constant \( \tau \) of (1.7) is given by

\[
\theta(z) = \sum_{m=-\infty}^{\infty} e^{2\pi imz} e^{\pi i m^2 \tau}.
\]

(1.8)

It is an entire function which satisfies

\[
\theta(z + 1) = \theta(z), \quad \theta(z + \tau) = e^{-2\pi i z} e^{-\pi i \tau} \theta(z), \quad \theta(-z) = \theta(z), \quad \text{for all } z \in \mathbb{C}.
\]

(1.9)

We are now ready to state our main result.

**Theorem 1.1.** Let \( x_3 < x_2 < x_1 < 0 \) be fixed. As \( r \to +\infty \), we have

\[
F(r\bar{x}) = \exp \left( cr^3 - \frac{1}{2} \log r + \log \theta(\nu) + C + O(r^{-1}) \right),
\]

(1.10)

\[
c = \frac{x_1^3 + x_2^3 + x_3^3 - (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)}{12} + \frac{q_0}{3} (x_1 + x_2 + x_3),
\]

(1.11)

where \( C = C(\bar{x}) \) is independent of \( r \) and \( \nu = \nu(r, \bar{x}) \) is given by

\[
\nu = -\frac{\Omega}{2\pi} r^2.
\]

(1.12)
Remark 1. The asymptotic formula (1.10) is not valid if \( r \to +\infty \) and simultaneously \( x_2 \to x_1 \) or \( x_3 \to x_3 \). However, it is possible to naively expand our expression for the constant \( c \) as \( x_2 \to x_1 \) or as \( x_2 \to x_3 \) and see if it agrees with the constants for the leading term of (1.3). The expression (1.11) for \( c \) is given in terms of \( q_0 \) defined in (1.6). We show in Lemma 3.3 that
\[
q_0 = \frac{x_1x_3}{2} + \mathcal{O}(x_2 - x_3), \quad \text{as } x_2 \to x_3, \tag{1.13}
\]
\[
q_0 = \frac{x_1x_3}{2} + \mathcal{O}\left(\frac{1}{\log(x_1 - x_2)}\right), \quad \text{as } x_2 \to x_1. \tag{1.14}
\]
Using these asymptotics, we obtain
\[
\lim_{x_2 \to x_1} c = -\frac{|x_3|^3}{12}, \quad \lim_{x_2 \to x_3} c = -\frac{|x_1|^3}{12},
\]
which is consistent with (1.3). Interestingly, the coefficient \(-\frac{1}{3}\) of the log \( r \) term in (1.10) is independent of \( x_1, x_2, x_3 \) and different from the coefficient \(-\frac{1}{3}\) of the log \( r \) term in (1.3).

Remark 2. In [12], Deift, Its and Zhou have obtained large gap asymptotics for the sine point process on disjoint intervals. Their final asymptotic formula is similar to our formula (1.10), in the sense that the subleading term in [12] is also logarithmic, and the third term presents oscillations also described in terms of elliptic \( \theta \)-functions. However, their constant associated to the logarithmic term is quite involved and expressed in terms of elliptic integrals, while in our case the constant is simply \(-\frac{1}{2}\).

1.1 Outline of the proof

We first follow the procedure developed by Its-Izergin-Korepin-Slavnov [16] to obtain an identity for \( \partial_x \log F(r\vec{x}) \) in terms of a certain resolvent operator. Next, we use a result of Claeys and Doeraene [6] to express this resolvent in terms of a model Riemann-Hilbert (RH) problem, whose solution is denoted \( \Psi \), see (2.18) for the exact expression. We employ the Deift/Zhou [10, 13] steepest descent method to obtain asymptotics for \( \Psi \) as \( r \to +\infty \). This method is well-established, but its application in the present situation leads to rather involved analysis. By integrating the differential identity \( \partial_r \log F(r\vec{x}) \), we have
\[
\log F(r\vec{x}) = \log F(M\vec{x}) + \int_M^r \partial_r \log F(r'\vec{x}) dr', \tag{1.15}
\]
where \( M > 0 \) is a sufficiently large constant. We obtain our main result by substituting the large \( r' \) asymptotics for \( \partial_r \log F(r'\vec{x}) \) and then performing integration with respect to \( r' \).

Our analysis of the RH problem for \( \Psi \) involves several complicated quantities related to a genus one Riemann surface, which consists of two copies of the complex plane glued together along two disjoint intervals. In particular, elliptic \( \theta \)-functions, the Abel map, see (4.11), and the inverse of the Abel map, all play an important role. The simple fact that the large \( r \) asymptotics for \( F(r\vec{x}) \) are of the form
\[
F(r\vec{x}) = \exp\left( cr^3 + c_2 \log r + c_3 \log \theta(\nu) + C + \mathcal{O}(r^{-1})\right), \tag{1.16}
\]
for certain constants \( c, c_2, c_3 \) and \( C \) requires a non-trivial amount of work. In particular, it is not straightforward to show that there are no terms proportional to \( r^\frac{3}{2} \), no oscillations of order \( \log r \), and that all oscillations of order 1 can be written in the form \( c_3 \log \theta(\nu) \). An important part of this work is devoted to understanding the algebra related to the \( \theta \)-functions and the Abel map. We obtain very
simple expressions for $c_2$ and $c_3$ only after enormous simplifications using many identities involving $\theta$-functions and the Abel map. Furthermore, the identity $c_3 = 1$ requires the use of Riemann’s bilinear identity, while $c_2 = -\frac{1}{2}$ requires the explicit evaluation of certain remarkable elliptic integrals, see Lemma 8.7. The fact that we manage to prove $c_2 = -\frac{1}{2}$ and $c_3 = 1$ can be viewed as one of the main contributions of this paper.

The integration constant $\log F(M \vec{x})$ in (1.15) is unknown and contributes to the expression $C$ of (1.10), so our method does not allow for an explicit expression for $C$. Such constants are notoriously complicated to obtain (see [18]), and we leave this for future works.

Remark 3. I. Krasovsky and T. Maroudas [19] have simultaneously and independently analyzed the large $r$ asymptotics of the Fredholm determinant $F(r \vec{x})$. Their method is based on a different differential identity than the one considered here.

1.2 Organization of the paper

First, we present the model RH problem for $\Psi$ and obtain the differential identity (2.18) in Section 2. We perform the first steps of the steepest descent analysis in Section 3. These steps are denoted by $\Psi \mapsto T \mapsto S$ and consists of constructing the so-called $g$-function and the opening of the lenses. The second main part of the steepest descent method consists of constructing approximations to $S$ in four distinct regions of the complex plane. The global parametrix $P^{(\infty)}$ approximates $S$ everywhere in the complex plane except near the points $y_j := x_j - x_3$, and the local parametrix $P^{(y_j)}$ approximates $S$ near the point $y_j$, $j = 1, 2, 3$. In Section 4, we construct $P^{(\infty)}$ explicitly in terms of elliptic $\theta$-functions. This construction is technically different from the construction in [11], but contains similar ideas. In Section 5, we use some results from [20] to build local parametrices $P^{(y_j)}$, $j = 1, 2, 3$, in terms of Bessel functions. The last step $S \mapsto R$ of the steepest descent method is done in Section 6; in particular we show that $P^{(y_j)}$ approximates $S$ in a small neighborhood of $y_j$, $j = 1, 2, 3$, for sufficiently large $r$. In Section 7, we summarize the $\theta$-function identities that are required to analyze the differential identity (2.18). In Section 8, we complete the proof of Theorem 1.1 by obtaining large $r$ asymptotics for $\partial_r \log F(r \vec{x})$ via (2.18), and subsequently large $r$ asymptotics for $\log F(r \vec{x})$ via (1.15).

2 Differential identity for $F$

In this section, we use the method of Its-Izergin-Korepin-Slavnov [16] to obtain a differential identity for $\partial_r \log F(r \vec{x})$ in terms of a model RH problem $\Psi$ obtained in [6].

A kernel $K : \mathbb{R}^2 \to \mathbb{R}$ is said to be integrable if it can be written in the form $K(x, y) = \frac{f(x)g(y)}{x - y}$, where $f(x)$ and $g(y)$ are column vectors satisfying $f^T(x)g(x) = 0$. It is well-known and easy to see from (1.1) that the Airy kernel $K^{\text{Ai}}$ is integrable. In our case, we need to consider the more involved kernel

$$K^{\text{Ai}}(x, y) = K^{\text{Ai}}(x, y)\left(\chi_{(x_1, x_2)}(y) + \chi_{(x_1, +\infty)}(y)\right).$$

This kernel is also integrable with $f$ and $g$ given by

$$f(x) = \begin{pmatrix} \text{Ai}(x) \\ \text{Ai}'(x) \end{pmatrix}, \quad g(y) = \begin{pmatrix} \text{Ai}'(y)\left(\chi_{(x_3, x_2)}(y) + \chi_{(x_1, +\infty)}(y)\right) \\ -\text{Ai}(y)\left(\chi_{(x_3, x_2)}(y) + \chi_{(x_1, +\infty)}(y)\right) \end{pmatrix}.$$

The associated integral operator $K^{\text{Ai}}$ is given by

$$K^{\text{Ai}}f(x) = \int_{-\infty}^{+\infty} K^{\text{Ai}}(x, y)\left(\chi_{(x_3, x_2)}(y) + \chi_{(x_1, +\infty)}(y)\right)f(y)dy. \quad (2.2)$$
Let us now scale the size of the intervals with a parameter \( r > 0 \), i.e. we replace \( \vec{x} \) by \( r\vec{x} \). In Proposition 2.1 below, we find an identity for \( \partial_r \log F(r\vec{x}) (= \partial_r \log \det (I - K_{r\vec{x}})) \) in terms of the resolvent operator
\[
R_{r\vec{x}} := (I - K_{r\vec{x}})^{-1} K_{r\vec{x}} = (I - K_{r\vec{x}})^{-1} - I.
\]

Next, we use a result of \([6]\) (based on \([16]\)) which relates \( R_{r\vec{x}} \) to the model RH problem \( \Psi \).

The proof of Proposition 2.1 is rather long and technical. It consists of rigorously justifying the following steps
\[
\partial_r \log (I - K_{r\vec{x}}) = -\text{Tr} [(I - K_{r\vec{x}})^{-1} \partial_r K_{r\vec{x}}] = \sum_{j=1}^{3} (-1)^{j+1} x_j \text{Tr} [(I - K_{r\vec{x}})^{-1} K_{r\vec{x}} \delta_{r\vec{x}}] = \sum_{j=1}^{3} (-1)^{j+1} x_j \lim_{u \to r\vec{x}_j} R_{r\vec{x}}(u, u), \tag{2.3}
\]
where \( \delta_{r\vec{x}_j} \) is the Dirac delta operator, \( R_{r\vec{x}} \) is the kernel of \( R_{r\vec{x}} \), and the limits \( u \to r\vec{x}_j, j = 1, 2, 3 \), are taken from the interior of \((r\vec{x}_3, r\vec{x}_2) \cup (r\vec{x}_1, +\infty)\).

**Proposition 2.1.** The following identity holds:
\[
\partial_r \log F(r\vec{x}) = \sum_{j=1}^{3} (-1)^{j+1} x_j \lim_{u \to r\vec{x}_j} R_{r\vec{x}}(u, u), \tag{2.4}
\]
where the limits \( u \to r\vec{x}_j, j = 1, 2, 3 \), are taken such that \( u \in (r\vec{x}_3, r\vec{x}_2) \cup (r\vec{x}_1, +\infty) \).

**Proof.** Let us first recall the definition of the Fredholm determinant \( f(r) := F(r\vec{x}) = \det (I - K_{r\vec{x}}) \). Let \( \mathcal{I} \) be a compact subset of \((0, +\infty)\). Set \( K^{(0)} := 1 \) and define the complex-valued function \( K^{(m)} \) for \( m \geq 1 \) by
\[
K^{(m)} (u_1, u_2, \ldots, u_m; r) = \det \begin{pmatrix} K_{r\vec{x}}(u_1, u'_1) & \cdots & K_{r\vec{x}}(u_1, u'_m) \\ \vdots & \ddots & \vdots \\ K_{r\vec{x}}(u_m, u'_1) & \cdots & K_{r\vec{x}}(u_m, u'_m) \end{pmatrix}. \tag{2.5}
\]
The Fredholm determinant \( f(r) \) and the Fredholm minor \( G(x, x'; r) \) are defined by
\[
f(r) = \sum_{m=0}^{\infty} f^{(m)}(r), \quad r \in \mathcal{I}, \tag{2.6}
\]
\[
G(x, x'; r) = \sum_{m=0}^{\infty} G^{(m)}(x, x'; r), \quad x, x' \in \mathbb{R}, \quad r \in \mathcal{I}, \tag{2.7}
\]
where \( f^{(m)} \) and \( G^{(m)} \) are defined for \( m \geq 0 \) by
\[
f^{(m)}(r) = \frac{(-1)^{m}}{m!} \int_{\mathbb{R}^m} K^{(m)} (u_1, u_2, \ldots, u_m; r) du_1 du_2 \cdots du_m, \tag{2.8}
\]
\[
G^{(m)}(x, x'; r) = \frac{(-1)^{m}}{m!} \int_{\mathbb{R}^m} K^{(m+1)} (x, u_1, u_2, \ldots, u_m; r) du_1 du_2 \cdots du_m. \tag{2.8}
\]

\(^1\) For \( m = 0 \) these definitions should be interpreted as
\[
f^{(0)}(r) = 1, \quad G^{(0)}(x, x'; r) = K^{(1)} (x; r) = K_{r\vec{x}}(x, x').
\]
In what follows we will establish convergence of the above series for \( f(r) \) and \( G(x, x'; r) \) and show that the \( r \)-derivative can be computed termwise. Since

\[
|\text{Ai}(y)|, |\text{Ai}’(y)| \leq e^{-\frac{y^2}{4}}, \quad \text{as } y \to +\infty,
\]

we see that \( K_{r\vec{x}} \) given by (2.1) (with \( \vec{x} \to r\vec{x} \)) together with (1.1) satisfies

\[
|K_{r\vec{x}}(x, y)| \leq b(y), \quad x, y \in \mathbb{R}, \quad r \in \mathcal{I},
\]

where \( b(y) = C_1 e^{-\frac{y^2}{2}} \chi_{(-\infty, +\infty)} \) for a certain constant \( C_1 > 0 \) and \( r := \max_{r \in \mathcal{I}} r \). Hadamard’s inequality for an \( m \times m \) matrix \( A \),

\[
|\det A|^2 \leq \prod_{i=1}^{m} \prod_{j=1}^{m} |A_{ij}|^2,
\]

together with the bound (2.9) gives

\[
\left|K^{(m)}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m; r)\right| \leq m^\frac{m}{2} \prod_{j=1}^{m} b(u_j’), \quad r \in \mathcal{I}.
\]

In view of (2.10), we have

\[
|f^{(m)}(r)| \leq \frac{m^\frac{m}{2} \|b\|L^1(\mathbb{R})^m}{m!}, \quad r \in \mathcal{I}, \quad m \geq 0.
\]

Using Stirling’s approximation \( m! \sim \sqrt{2\pi m} \left(\frac{m}{e}\right)^m \), we see that the series in (2.6) converges absolutely and uniformly for \( r \in \mathcal{I} \). Similarly, the estimate

\[
|G^{(m)}(x, x'; r)| \leq \frac{(m+1)^{\frac{m+1}{2}} \|b\|L^1(\mathbb{R})^m b(x’)}{m!}, \quad x, x' \in \mathbb{R}, \quad r \in \mathcal{I}, \quad m \geq 0.
\]

shows that the series (2.8) defining the Fredholm minor \( G(x, x'; r) \) converges absolutely and uniformly for \( (x, x') \) in compact subsets of \( \mathbb{R}^2 \) and for \( r \in \mathcal{I} \). Hence, \( G(x, x'; r) \) satisfies

\[
|G(x, x', r)| \leq C_2 b(x'), \quad x, x' \in \mathbb{R}, \quad r \in \mathcal{I},
\]

for a certain \( C_2 > 0 \). Expanding the determinant in (2.5) along the first column, we find, for \( m \geq 0 \),

\[
K^{(m+1)}(\vec{x}, \vec{u}_1, \ldots, \vec{u}_m; r) = K_{r\vec{x}}(x, x') K^{(m)}(u_1, \ldots, u_m, r) - \sum_{s=1}^{m} K_{r\vec{x}}(u_s, x') K^{(m)}(u_1, \ldots, u_s-1, u_{s+1}, \ldots, u_m; r).
\]

Substituting this identity into (2.8) and simplifying, we obtain

\[
G^{(m)}(x, x'; r) = f^{(m)}(r) K_{r\vec{x}}(x, x') + \int_{\mathbb{R}} G^{(m-1)}(x, x''; r) K_{r\vec{x}}(x'', x') dx'', \quad m \geq 0,
\]

where \( G^{(-1)} := 0 \). Summing this equation from \( m = 0 \) to \( m = +\infty \), we find, for \( x, x' \in \mathbb{R} \) and \( r \in \mathcal{I} \),

\[
G(x, x'; r) = f(r) K_{r\vec{x}}(x, x') + \int_{\mathbb{R}} G(x, x''; r) K_{r\vec{x}}(x'', x') dx''.
\]
If we instead expand the determinant in (2.5) along the first row, the same type of argument leads to
\[ G(x, x'; r) = f(r)K_{r\vec{x}}(x, x') + \int_{\mathbb{R}} K_{r\vec{x}'}(x, x'')G(x'', x'; r)dx''. \] (2.14b)

Since \( f(r) \) is non-zero (see [6, Remark 7]), the identities in (2.14) show that
\[ R_{r\vec{x}}(x, x') := \frac{G(x, x'; r)}{f(r)} \]
satisfies the resolvent equations
\[ R_{r\vec{x}}(x, x') - K_{r\vec{x}}(x, x') = \int_{\mathbb{R}} K_{r\vec{x}'}(x, x'')R_{r\vec{x}'}(x'', x')dx'' \]
for \( x, x' \in \mathbb{R} \). Hence, if \( R_{r\vec{x}} \) denotes the operator with kernel \( R_{r\vec{x}} \), then \( R_{r\vec{x}} \) satisfies
\[ R_{r\vec{x}} - K_{r\vec{x}} = R_{r\vec{x}}K_{r\vec{x}} = K_{r\vec{x}}R_{r\vec{x}}. \]

It follows that
\[ I + R_{r\vec{x}} = (1 - K_{r\vec{x}})^{-1}, \]
showing that \( R_{r\vec{x}} \) is the resolvent operator. We next consider the derivative \( f'(r) \) of the Fredholm determinant. Note that
\[ K^{A_i}(x, x) = A'_i(x)^2 - xA_i(x)^2 \]
is a smooth function of \( x \in \mathbb{R} \) with exponential decay as \( x \to +\infty \). We have
\[ f^{(1)}(r) = -\int_{\mathbb{R}} K_{r\vec{x}}(u_1, u_1)du_1 = -\left( \int_{rx_1}^{\infty} + \int_{rx_3}^{\infty} \right) K^{A_i}(u_1, u_1)du_1, \]
\[ f^{(2)}(r) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ K_{r\vec{x}}(u_1, u_1)K_{r\vec{x}'}(u_2, u_2) - K_{r\vec{x}'}(u_1, u_2)K_{r\vec{x}'}(u_2, u_1) \right] du_1du_2 \]
\[ = \frac{1}{2} f^{(1)}(r)^2 - \frac{1}{2} \left( \int_{rx_1}^{\infty} + \int_{rx_3}^{\infty} \right) \left( \int_{rx_1}^{\infty} + \int_{rx_3}^{\infty} \right) K^{A_i}(u_1, u_2)K^{A_i}(u_2, u_1)du_1du_2. \]

It follows that \( f^{(1)}(r) \) and \( f^{(2)}(r) \) are smooth functions of \( r \in \mathcal{I} \). Similar arguments show that \( f^{(m)}(r) \) is a smooth function of \( r \in \mathcal{I} \) for each \( m \geq 3 \). We show in a similar way as done in (2.11) that the series
\[ \sum_{m=0}^{+\infty} f^{(m)}(r) \]
converges uniformly for \( r \in \mathcal{I} \), so that \( f(r) \) is a smooth function of \( r \in \mathcal{I} \). We also see that \( G^{(m)}(x, x'; r) \), and hence also \( G(x, x'; r) \), is a smooth function of \( (x, x') \) whenever \( x, x' \in \mathbb{R} \setminus \{rx_1, rx_2, rx_3\} \). Note that the integrand of \( f^{(m)}(r) \) is a symmetric function in the variables \( u_1, \ldots, u_m \).
Therefore, differentiation of the definition (2.8) of $f^{(m)}(r)$ with respect to $r$ gives, for each $m \geq 1$,

$$
\partial_r f^{(m)}(r) = \frac{(-1)^m}{m!} \partial_r \int_{\mathbb{R}^n} \det \begin{pmatrix} K_{r x}(u_1, u_1) & \cdots & K_{r x}(u_1, u_m) \\ \vdots & \ddots & \vdots \\ K_{r x}(u_m, u_1) & \cdots & K_{r x}(u_m, u_m) \end{pmatrix} \ du_1 du_2 \cdots du_m
$$

$$
= \frac{(-1)^m}{(m-1)!} \sum_{j=1}^3 (-1)^j x_j \int_{\mathbb{R}^{m-1}} \det \begin{pmatrix} K_{r x}(u_1, u_1) & \cdots & K_{r x}(u_1, u_m) \\ \vdots & \ddots & \vdots \\ K_{r x}(u_m, u_1) & \cdots & K_{r x}(u_m, u_m) \end{pmatrix}_{u_1 = r x_j} \ du_2 \cdots du_m
$$

$$
= \frac{(-1)^m}{(m-1)!} \sum_{j=1}^3 (-1)^j x_j \lim_{u \to r x_j} \int_{\mathbb{R}^{m-1}} K^{(m)}(u, u_2, \ldots, u_m; u_1, r) \ du_2 \cdots du_m
$$

$$
= \sum_{j=1}^3 (-1)^{j+1} x_j \lim_{u \to r x_j} G^{(m-1)}(u, u; r),
$$

where the limits as $u \to r x_j$, $j = 1, 2, 3$, are taken from the interior of $(r x_3, r x_2) \cup (r x_1, +\infty)$. Summing from $m = 1$ to $m = +\infty$ and using the uniform convergence of the series for $f(r)$, $f'(r)$, and $G(x, x'; r)$ to permute $\partial_r$ and $\lim_{u \to r x}$ with the series, we conclude that

$$
f'(r) = \sum_{j=1}^3 (-1)^{j+1} x_j \lim_{u \to r x_j} G(u, u; r).
$$

The differential identity (2.4) follows after division by $f(r)$. \qed

For convenience, we define

$$
y_j = x_j - x_3, \quad j = 1, 2, 3.
$$

Using the Its-Izergin-Korepin-Slavnov method, it was shown in [6, proof of Proposition 1] that the resolvent kernel $R_{r x}(u; u)$ can be expressed in terms of a RH problem. For $u \in (r x_3, r x_2) \cup (r x_1, +\infty)$, we have

$$
R_{r x}(u, u) = \frac{1}{2 \pi i} (\Psi^{-1}\Psi')_{21}(u - x_3; x_3, \vec{y}),
$$

(2.15)

where $\vec{y} = (y_1, y_2, y_3)$ and $\Psi(\zeta; x_3, \vec{x})$ is the solution to the following RH problem.
RH problem for Ψ

(a) Ψ : C \ Γ → C^{2×2} is analytic, with

\[ \Gamma = e^{\pm \frac{2\pi i}{3}}(0, +\infty) \cup (-\infty, 0] \cup [y_2, y_1] \]  \hspace{1cm} (2.16)

and Γ oriented as in Figure 1.

(b) Ψ(z) has continuous boundary values as z ∈ Γ\{y_1, y_2, y_3\} is approached from the left (+ side) or from the right (− side) and they are related by

\[
\begin{align*}
\Psi_+(z) &= \Psi_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{for } z \in e^{\pm \frac{2\pi i}{3}}(0, +\infty), \\
\Psi_+(z) &= \Psi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } z \in (-\infty, 0), \\
\Psi_+(z) &= \Psi_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in (y_2, y_1).
\end{align*}
\]

(c) As z → ∞, we have

\[ \Psi(z) = (I + O(z^{-1})) z^\frac{1}{2} e^{i\frac{1}{3} \sum_{j=1}^3 \frac{1}{\pi i} (z - y_j^3) \sigma_3,} \]  \hspace{1cm} (2.17)

where M = (I + i\sigma_1)/\sqrt{2}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} and \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, and where principal branches of \( z^{\frac{1}{2}} \) and \( z^{\frac{2}{3}} \) are taken.

(d) Ψ(z) = O(log(z − y_j)) as z → y_j, j = 1, 2, 3.

We conclude from the definition of F (1.2) together with (2.4) and (2.15) that

\[
\partial_r \log F(r\bar{x}) = \sum_{j=1}^3 \frac{(-1)^{j+1} x_j}{2\pi i} \lim_{z \to y_j} (\Psi^{-1}_{-1}\Psi'_j)_{21} (rz; r\bar{x}_3, r\bar{y}),
\]  \hspace{1cm} (2.18)

where the limits as z → y_j, j = 1, 2, 3, are taken such that z ∈ (0, y_2) ∪ (y_1, +∞). The differential identity (2.18) expresses \( \partial_r \log F(r\bar{x}) \) in terms of Ψ, which is the solution to a RH problem. In Sections 3-6, we employ the Deift/Zhou steepest descent method [13] to obtain the asymptotics of Ψ(z), Ψ'(z) as r → +∞ for z in small neighborhoods of y_1, y_2, y_3.

3 Steepest descent for Ψ: first steps

In this section, we perform the first steps of the steepest descent method. We obtain the so-called g-function in Subsection 3.1. This function is used to normalize the RH problem at ∞ via the Ψ → T transformation of Subsection 3.2. Finally, we proceed with the opening of the lenses T → S in Subsection 3.3.

3.1 g-function

The first step of the analysis consists of normalizing the RH problem at ∞ by means of an appropriate g-function. Let us define

\[ \sqrt{R(z)} = \sqrt{(z - y_1)(z - y_2)} z \]
where the principal branch is taken for each square root. More precisely, $\sqrt{\mathcal{R}(z)}$ is analytic on $\mathbb{C} \setminus ((-\infty,0] \cup [y_2, y_1])$ and $\sqrt{\mathcal{R}(z)} \sim z^2$ as $z \to \infty$. In particular it satisfies the jumps

$$
\sqrt{\mathcal{R}(z)}_+ + \sqrt{\mathcal{R}(z)}_- = 0 \quad \text{for} \quad z \in (-\infty,0) \cup (y_2,y_1).
$$

(3.1)

We define the derivative of the $g$-function by

$$
g'(z) = \frac{p(z)}{\sqrt{\mathcal{R}(z)}},
$$

(3.2)

where

$$
p(z) := -z^2 + \frac{y_1 + y_2 - x_3}{2}z + \frac{x_3(y_1 + y_2)}{4} + \frac{(y_1 - y_2)^2}{8} - g_1
$$

(3.3)

and $g_1 \in \mathbb{R}$ is given by

$$
g_1 = \left( \int_0^{y_2} \frac{ds}{\sqrt{\mathcal{R}(s)}} \right)^{-1} \left( \int_0^{y_2} \frac{s^2 - \frac{1}{2}(y_1 + y_2 - x_3) + \frac{(y_1 - y_2)^2}{8}}{\sqrt{\mathcal{R}(s)}} ds \right). \quad (3.4)

By definition of $g_1$, we have

$$
\int_0^{y_2} g'(s)ds = 0. \quad (3.5)
$$

**Remark 4.** The functions $Q$ and $q$ defined in (1.4) and (1.5) have the relations with $\mathcal{R}$ and $p$

$$
Q(z) = \mathcal{R}(z - x_3) \quad \text{and} \quad q(z) = p(z - x_3).
$$

**Proposition 3.1.** The degree two polynomial $p$ has two simple real zeros $z_{\pm}$, with $z_- \in (0,y_2)$ and $z_+ \in (y_1, \infty)$. Moreover, we have

$$
p(y_3) < 0, \quad p(y_2) > 0, \quad p(y_1) > 0.
$$

**Proof.** The polynomial $p$ has real coefficients, thus $p(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. Then we immediately conclude from (3.5) that $p(z)$ must have two real simple zeros and that at least one of them lies on $(0,y_2)$, because $\sqrt{\mathcal{R}(z)} < 0$ for all $z \in (0,y_2)$. However it is not straightforward to prove that $p$ has one zero in $(y_1, +\infty)$, or equivalently that $p(y_1) > 0$ (since $p(z) \to -\infty$ as $z \to +\infty$). Note that $p(z)$ can be written as

$$
p(z) = -z^2 + \frac{z}{2}(y_1 + y_2 - x_3) + \alpha, \quad \text{where} \quad \alpha := \left( \int_0^{y_2} \frac{ds}{\sqrt{\mathcal{R}(s)}} \right)^{-1} \left( \int_0^{y_2} \frac{s^2 + \frac{1}{2}(y_1 + y_2 - x_3) + \frac{(y_1 - y_2)^2}{8}}{\sqrt{\mathcal{R}(s)}} ds \right).
$$

The constant $\alpha$ can be written explicitly in terms of complete elliptic integrals of the first (K) and second (E) kind, which are defined as [15, eq 8.112]

$$
K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad E(k) = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} dx.
$$

Let us define $k_* := \sqrt{\frac{x}{y_1}}$. Using [15, eq 3.131.4, 3.141.15 and 3.141.27], we obtain

$$
\int_0^{y_2} \frac{ds}{\sqrt{\mathcal{R}(s)}} = -\frac{2}{\sqrt{y_1}}K(\kappa), \quad \int_0^{y_2} s \frac{ds}{\sqrt{\mathcal{R}(s)}} = -2\sqrt{y_1}(K(\kappa) - E(\kappa)),
$$

$$
\int_0^{y_2} s^2 \frac{ds}{\sqrt{\mathcal{R}(s)}} = \frac{2}{3} \sqrt{y_1}(2(y_1 + y_2)E(\kappa) - (2y_1 + y_2)K(\kappa)).
$$
from which we deduce
\[ \alpha = \frac{y_1}{2} \left[ \frac{y_1 - y_2}{3} + x_3 - \frac{E(k_1)}{K(k_1)} \left( \frac{y_1 + y_2}{3} + x_3 \right) \right]. \tag{3.6} \]

From [21, eq 19.9.8], we have the lower bound
\[ \frac{E(k_1)}{K(k_1)} > \sqrt{1 - k_1^2} = \sqrt{1 - \frac{y_2}{y_1}} = \frac{\sqrt{x_1 - x_2}}{x_1 - x_3}. \]

We weaken this bound for our purpose. Notice
\[ 4(x_1 - x_2)(x_1 - x_3) = (x_1 + x_2 + x_3)^2 + 3x_1(x_1 - 2x_2 - 2x_3) - (x_2 - x_3)^2 < (x_1 + x_2 + x_3)^2, \]
because \( x_1 - 2x_2 - 2x_3 > 0 \) and \( x_1 < 0 \), so we have
\[ \frac{x_1 - x_2}{x_1 - x_3} = \frac{4(x_1 - x_2)^2}{4(x_1 - x_2)(x_1 - x_3)} > \frac{4(x_1 - x_2)^2}{(x_1 + x_2 + x_3)^2}. \]
The new bound is
\[ \frac{E(k_1)}{K(k_1)} > \frac{2(x_2 - x_1)}{x_1 + x_2 + x_3}, \]
from which we have
\[ p(y_1) = \frac{y_1}{2} (y_2 - y_1 - x_3) + \alpha = \frac{y_1}{6} \left[ 2(x_2 - x_1) - \frac{E(k_1)}{K(k_1)} (x_1 + x_2 + x_3) \right] > 0, \]
as desired.

We define the \( g \)-function by
\[ g(z) = \int_{y_1}^{z} g'(s)ds, \tag{3.7} \]
where the path of integration does not cross \((-\infty, y_1]\).

**Lemma 3.2.** Let \( x_3 < x_2 < x_1 < 0 \).

1. The \( g \)-function is analytic and satisfies \( g(z) = \overline{g(\overline{z})} \) for \( z \in \mathbb{C} \setminus (\infty, y_1) \).

2. The \( g \)-function satisfies the jump conditions
\[ g_+(z) + g_-(z) = 0, \quad \quad \quad \quad \quad z \in (\infty, 0) \cup (y_2, y_1), \tag{3.8} \]
\[ g_+(z) - g_-(z) = i\Omega, \quad \quad z \in (0, y_2), \tag{3.9} \]
where \( \Omega = 2i \int_{y_1}^{y_2} g_+(s)ds = -2ig_+(y_2) \in \mathbb{R}_+ \).

3. As \( z \to \infty \), we have
\[ g(z) = -\frac{2}{3} z^\frac{\omega}{2} - x_3 z^\frac{\omega}{2} + 2g_1 z^{-\frac{\omega}{2}} + \mathcal{O}(z^{-\frac{\omega}{2}}). \tag{3.10} \]
4. Let $N_0$ be a sufficiently small open neighborhood of $(-\infty, y_1]$, such that there exists $\epsilon > 0$ and $M > 0$

\[
\{ z \in \mathbb{C} : \arg z \in \left(-\pi, -\pi + \frac{\pi}{3} + \epsilon\right) \cup \left(\pi - \frac{\pi}{3} - \epsilon, \pi\right) \text{ and } |z| \geq M \} \subset N_0 \quad (3.11)
\]

We have

\[
\Re [g(z)] \geq 0, \quad \text{for all } z \in N_0 \quad (3.12)
\]

where we have equality only when $z \in (-\infty, 0] \cup [y_2, y_1]$.

Proof. 1. Analyticity follows from the definition (3.7). It is straightforward to verify from (3.2) that $g'(z) = \overline{g'(\overline{z})}$, from which we deduce

\[
g(z) = \int_{y_1}^{x} g'(s) ds = \int_{0}^{1} g'(y_1 + t(z - y_1))(z - y_1) dt = \int_{0}^{1} g'(y_1 + t(z - y_1))(z - y_1) dt = g(z).
\]

2. The jumps (3.8) follow from (3.1) and (3.5). Let $\gamma$ be a closed curve with positive orientation surrounding $[y_2, y_1]$ and not intersecting $(-\infty, 0]$. For $z \in (0, y_2)$, we use (3.8) to write

\[
g_+(z) - g_-(z) = \int_{\gamma} g'(s) ds = -2\int_{y_2}^{y_1} g_+(s) ds,
\]

and this is (3.9). It follows from Proposition 3.1 that

\[
\Omega = 2i \int_{y_2}^{y_1} \frac{g'(s) ds}{g_+(s)} = 2\int_{y_2}^{y_1} \frac{p(s) ds}{\sqrt{s}} > 0.
\]

3. From (3.2), a computation shows that

\[
g'(z) = -\sqrt{z} - \frac{x_3}{2} z^{-\frac{1}{2}} - g_1 z^{-\frac{1}{2}} + \mathcal{O}(z^{-\frac{1}{2}}), \quad \text{as } z \to \infty,
\]

and then after integration we have

\[
g(z) = -\frac{2}{3} z^{-\frac{1}{2}} - x_3 z^{-\frac{1}{2}} + g_0 + 2g_1 z^{-\frac{1}{2}} + \mathcal{O}(z^{-\frac{1}{2}}), \quad \text{as } z \to \infty, \quad (3.13)
\]

for a certain $g_0 \in \mathbb{C}$. We deduce from (3.13) that

\[
g_+(z) + g_-(z) = 2g_0 + \mathcal{O}(z^{-\frac{1}{2}}), \quad \text{as } z \to -\infty, \quad z \in \mathbb{R}^-.
\]

Comparing the above with (3.8), we conclude that $g_0 = 0$.

4. It follows from part 2 that $\Re [g(z)]$ is single valued on $\mathbb{C}$ and furthermore $\Re [g(z)] \equiv 0$ for $z \in (-\infty, 0) \cup (y_2, y_1)$. Due to Proposition 3.1, we have

\[
\Im [g_+(x)] = \Im \left[ \frac{p(x)}{\sqrt{R(x)_+}} \right] < 0, \quad \text{for } x \in (-\infty, 0) \cup (y_2, y_1).
\]

So $\Im [g_+(x)]$ decreases as $x \in (-\infty, 0) \cup (y_2, y_1)$ increases, and it follows from the Cauchy-Riemann equations that $\Re [g(x + i\epsilon)]$ increases (and is therefore positive) for sufficiently small $\epsilon > 0$. We deduce from part 1 that $\Re [g(x - i\epsilon)] > 0$ for sufficiently small $\epsilon > 0$. Another
immediate consequence of Proposition 3.1 is that $\Re[g(z)] > 0$ for all $z \in (0, y_2)$. Furthermore, from (3.7) we have

$$\Re[g(z)] \sim c(z - y_1)^\frac{1}{2}, \quad c > 0,$$

(the exact value of $c$ is unimportant here) as $z \to y_1$ so there is an open disk $D_{y_1}$ around $y_1$ such that $\Re[g(z)] \geq 0$ for all $z \in D_{y_1}$ with equality $\Re[g(z)] = 0$ if and only if $z \in D_{y_1} \cap (y_2, y_1)$. We conclude similarly that there also exists open disks $D_{y_2}$ and $D_{y_3}$ around $y_2$ and $y_3$, respectively, such that

$$\Re[g(z)] \geq 0, \quad \text{for all } z \in D_{y_2} \cup D_{y_3}$$

with equality only when $z \in (-\infty, 0] \cup [y_2, y_1]$. The fact that we can choose $\epsilon > 0$ and $M > 0$ such that (3.11) holds follows directly from (3.10).

In Lemma 3.3 below, we obtain asymptotics for $g_1$ as $y_2 \to 0$ and as $y_2 \to y_1$. This is useful to compare our results with [9], see Remark 1. Indeed, we deduce (1.13)-(1.14) from (3.15)-(3.16) by noting that

$$q_0 = g_1 + \frac{2x_1x_2 + 2x_1x_3 + 2x_2x_3 - x_1^2 - x_2^2}{8}. \quad (3.14)$$

**Lemma 3.3.** We have

$$g_1 = \frac{y_1}{4} \left( x_4 + \frac{y_1}{2} \right) + O(y_2^2), \quad \text{as } y_2 \to 0 \quad (3.15)$$

$$g_1 = -\frac{y_1(2y_1 + 3x_3)}{3\log(y_1 - y_2)} + O\left( \frac{1}{\log^2(y_1 - y_2)} \right), \quad \text{as } y_2 \to y_1. \quad (3.16)$$

**Proof.** As $y_2 \to 0$, we have

$$\int_{0}^{y_2} \frac{ds}{\sqrt{\mathcal{R}(s)}} = \int_{0}^{y_2} \frac{-ds}{\sqrt{\mathcal{R}(s)}} \frac{1}{\sqrt{y_1}} \left( 1 + \frac{s}{2y_1} + \frac{3s^2}{8y_1^2} + O\left( \frac{s^3}{y_1^3} \right) \right)$$

$$= \frac{-\pi}{\sqrt{y_1}} \left( 1 + \frac{y_2}{4y_1} + \frac{9y_2^2}{64y_1^2} + O(y_2^3) \right).$$

Similar computations show that

$$\int_{0}^{y_2} \frac{s\, ds}{\sqrt{\mathcal{R}(s)}} = -\frac{\pi y_2}{2\sqrt{y_1}} \left( 1 + \frac{3y_2}{8y_1} + O(y_2^2) \right), \quad \int_{0}^{y_2} \frac{s^2\, ds}{\sqrt{\mathcal{R}(s)}} = -\frac{3\pi y_2^2}{8\sqrt{y_1}} + O(y_2^3), \quad \text{as } y_2 \to 0,$$

from which we obtain (3.15). Now we turn to the proof of (3.16). Using (3.6), we express the constant $g_1$ explicitly in terms of complete elliptic integrals:

$$g_1 = \frac{y_1}{2} \frac{E(k_\ast)}{K(k_\ast)} \left( \frac{y_1 + y_2}{4} + x_3 \right) - \frac{x_3}{4}(y_1 - y_2) - \frac{1}{24}(y_1 - y_2)(y_1 + 3y_2).$$

We then use [21, eq 19.12.1 and 19.12.2] for the asymptotics as $y_2 \to y_1$ to obtain (3.16). \qed
3.2 Rescaling of the RH problem

Define the function \( T(z) = T(z; x_3, \vec{y}) \) as follows,

\[
T(z) = \begin{pmatrix} r^{-\frac{1}{2}} & 2ir^2 s g_1 \\ 0 & r^{\frac{1}{2}} \end{pmatrix} \Psi(rz; rx_3, r\vec{y}) e^{-r^2 s g(z) \sigma_3}.
\]  
(3.17)

The asymptotics (2.17) of \( \Psi \) then imply after a small calculation that \( T \) behaves as

\[
T(z) = (I + O(z^{-1})) z^{s} M^{-1},
\]  
(3.18)

as \( z \to \infty \), where the principal branches of the roots are chosen. Using (3.8), the jumps \( T_-(z)^{-1}T_+(z) \) for \( z \in (y_2, y_1) \) can be factorized as

\[
\begin{pmatrix} e^{-2r^2 s g_+(z)} & 1 \\ 0 & e^{-2r^2 s g_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2r^2 s g_+(z)} \end{pmatrix}.
\]  
(3.19)

3.3 Opening of the lenses

Around the interval \((y_2, y_1)\), we will split the jump contour in three parts using (3.19). This transformation is traditionally called the opening of the lenses. Let us consider lens-shaped contours \( \gamma_{2,+} \) and \( \gamma_{2,-} \) lying in the upper and lower half plane respectively, as shown in Figure 2. We define \( S \) by

\[
S(z) = T(z) \begin{cases} 
1 & z \text{ is inside the lenses around } (y_2, y_1) \text{ and } \Im[z] > 0, \\
-1 & z \text{ is inside the lenses around } (y_2, y_1) \text{ and } \Im[z] < 0, \\
1 & \text{otherwise}.
\end{cases}
\]  
(3.20)

We use Lemma 3.2, as well as the RH problem for \( \Psi \) and the definitions (3.17) of \( T \) and (3.20) of \( S \) to conclude that \( S \) satisfies the following RH problem.
RH problem for $S$

(a) $S : \mathbb{C} \setminus \Sigma_S \to \mathbb{C}^{2 \times 2}$ is analytic, with

\[
\Sigma_S = (-\infty, y_1] \cup \gamma_+ \cup \gamma_-, \quad \gamma_\pm = \gamma_2, \pm \cup \left( e^{\pm \frac{2\pi i}{3}}(0, +\infty) \right)
\]  

(3.21)

and $\Sigma_S$ oriented as in Figure 2.

(b) The jumps for $S$ are given by

\[
S_+(z) = S_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, 0) \cup (y_2, y_1),
\]

(3.22)

\[
S_+(z) = S_-(z) \begin{pmatrix} 1 \\ e^{-2\pi i g(z)} \\ 0 \end{pmatrix}, \quad z \in \gamma_+ \cup \gamma_-,
\]

(3.23)

\[
S_+(z) = S_-(z)e^{-\bar{\Omega} \frac{4}{3} \sigma_3}, \quad z \in (0, y_2).
\]

(3.24)

(c) As $z \to \infty$, we have

\[
S(z) = (I + O\left(z^{-1}\right)) z^{\frac{4}{3} \sigma_3} M^{-1}.
\]

(3.25)

(d) As $z \to y_j$, $j = 1, 2, 3$, we have

\[
S(z) = O(\log(z - y_j)).
\]

(3.26)

Deforming $\gamma_+$ and $\gamma_-$ if necessary, we assume that $\gamma_+, \gamma_- \subset N_0$ where $N_0$ has the properties described in Lemma 3.2. In particular, $\Re g(z) > 0$ for all $z \in \gamma_+ \cup \gamma_-$, so that the jumps for $S$ are exponentially close to $I$ as $r \to +\infty$ on the lenses. This convergence is uniform outside neighborhoods of $y_1, y_2, y_3$, but is not uniform as $r \to +\infty$ and simultaneously $z \to y_j$, $j \in \{1, 2, 3\}$.

4 Global parametrix

In this section we construct the global parametrix $P^{(\infty)}$. We will show in Section 6 that $P^{(\infty)}$ approximates $S$ outside of neighborhoods of $y_1, y_2, y_3$.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, y_1] \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) The jumps for $P^{(\infty)}$ are given by

\[
P^{(\infty)}_+(z) = P^{(\infty)}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, 0) \cup (y_2, y_1),
\]

(4.1)

\[
P^{(\infty)}_+(z) = P^{(\infty)}_-(z)e^{-\bar{\Omega} \frac{4}{3} \sigma_3}, \quad z \in (0, y_2).
\]

(c) As $z \to \infty$, we have

\[
P^{(\infty)}(z) = (I + O\left(z^{-1}\right)) z^{\frac{4}{3} \sigma_3} M^{-1}.
\]

(4.2)

(d) As $z \to y_j, y_2, y_1$, we have $P^{(\infty)}(z) = O((z - y_j)^{-\frac{4}{3}})$. 

Conditions (a)-(c) for the RH problem for \( P(\infty) \) are obtained from the RH problem \( S \) by ignoring the (pointwise) exponentially small jumps of \( S \). Condition (d) does not come from the RH problem for \( S \), it has been added to ensure uniqueness of the solution. The solution \( P(\infty) \) can be constructed in terms of elliptic \( \theta \)-function. This construction is technically different from the construction in [11], but contains similar ideas. Let \( X \) be the two-sheeted Riemann surface of genus one associated to \( \sqrt{\mathcal{R}(z)} \), with \( \sqrt{\mathcal{R}(z)} = \sqrt{z(z - y_2)(z - y_1)} \), and we let \( \sqrt{\mathcal{R}(z)} > 0 \) for \( z \in (y_1, +\infty) \) on the first sheet. We also define \( A \) and \( B \) cycles such that they form a canonical homology basis of \( X \). The \( A \) cycle surrounds \((0, y_2)\) with counterclockwise orientation. The upper part of the \( A \) cycle (the solid line in Figure 3) lies on the first sheet, and the lower part (the dashed line in Figure 3) lies on the second sheet. The \( B \) cycle surrounds \((−\infty, 0)\) with clockwise orientation. The unique \( A \)-normalized holomorphic one-form \( \omega \) on \( X \) is given by

\[
\omega = \frac{c_0}{\sqrt{\mathcal{R}(z)}} \frac{dz}{\sqrt{\mathcal{R}(z)}}, \quad c_0 = \left( \int_A \frac{dz}{\sqrt{\mathcal{R}(z)}} \right)^{-1}.
\]

(4.3)

By construction \( \int_A \omega = 1 \) and the lattice parameter is given by \( \tau = \int_B \omega \in i\mathbb{R}^+ \). The associated \( \theta \)-function of the third kind \( \theta(z) = \theta(z; \tau) \) is given by

\[
\theta(z) = \sum_{m=-\infty}^{\infty} e^{2\pi i m z} e^{\pi i m^2 \tau}.
\]

(4.4)

It is an entire function which satisfies

\[
\theta(z + 1) = \theta(z), \quad \theta(z + \tau) = e^{-2\pi i z} e^{-\pi i \tau} \theta(z), \quad \theta(-z) = \theta(z), \quad \text{for all} \ z \in \mathbb{C}.
\]

(4.5)

The zeros of \( \theta(z) \) are the points \( \frac{1}{2} + m_1 + \frac{1}{2} + m_2 \tau \), with \( m_1, m_2 \in \mathbb{Z} \). Let \( X_1 \) and \( X_2 \) denote the upper and lower sheet of \( X \), respectively, and consider the set

\[
\Lambda := \{ u \in \mathbb{C} : -\frac{1}{2} \leq \Re[u] \leq \frac{1}{2} \text{ and } -\frac{1}{4} \leq \Im[u] \leq \frac{1}{4} \},
\]

together with the function

\[
\varphi : (X_1 \setminus (-\infty, y_1]) \cup (X_2 \setminus (-\infty, y_1]) \to \Lambda, \quad z \mapsto \varphi(z) = \int_{y_1}^z \omega,
\]

(4.6)

where the path does not cross \((-\infty, y_1]\) and lies in the same sheet as \( z \). We denote \( X_{j,+} \) and \( X_{j,-} \) for the strict upper and lower half plane of the \( j \)-th sheet. The function \( \varphi \) maps \( X_{1,+}, X_{1,-}, X_{2,+}, X_{2,-} \) to \( \Lambda \).
into the four quadrants of $\Lambda$ as shown in Figure 4. For $z$ on the first sheet, $\varphi(z)$ has the following jumps

$$\varphi_+(z) + \varphi_-(z) = 0, \quad z \in (y_2, y_1), \quad (4.7)$$
$$\varphi_+(z) + \varphi_-(z) = 1, \quad z \in (-\infty, 0), \quad (4.8)$$
$$\varphi_+(z) - \varphi_-(z) = \tau, \quad z \in (0, y_2). \quad (4.9)$$

At the four branch points $y_1$, $y_2$, $0$, $\infty$ on the first sheet, we have

$$\varphi(y_1) = 0, \quad \varphi_\pm(y_2) = \frac{1}{2} \pm \frac{\tau}{2}, \quad \varphi(0) = \frac{1}{2}, \quad \varphi(\infty) = \frac{1}{2}. \quad (4.10)$$

see Figure 4. The Abel map is closely related to $\varphi$, so we denote it by $\varphi_A$, and is given by

$$\varphi_A : X \to \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \quad z \mapsto \varphi_A(z) := \varphi(z) \mod (\mathbb{Z} + \tau\mathbb{Z}). \quad (4.11)$$

The Abel map is a bijection with the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and is analytic on the full Riemann surface (as opposed to $\varphi$ which presents discontinuities, as can be seen in (4.7)-(4.9)). There is an explicit expression for its inverse $\varphi_A^{-1}(u)$ in terms of the Jacobi elliptic function $sn$ [21, eq 22.2.4]:

$$\varphi_A^{-1}(u) = y_1 + (y_2 - y_1) sn^2 \left( \frac{iu\sqrt{y_1}}{2c_0}, 1 - \frac{y_2}{y_1} \right).$$

We introduce the following ratio of $\theta$-functions

$$\mathcal{F}(z; \hat{d}, \nu) = \frac{\theta(z + \hat{d} + \nu)}{\theta(z + \hat{d})}, \quad z, \hat{d} \in \mathbb{C}, \quad \nu \in \mathbb{R}.$$

This function has a simple pole at $z = -\hat{d} + \frac{1}{2} + \frac{i}{2} \mod (\mathbb{Z} + \tau\mathbb{Z})$. From (4.5) and (4.7)-(4.9), for $z \in X_1$ we have

$$\mathcal{F}(\varphi_+(z); \hat{d}, \nu) = \mathcal{F}(\varphi_-(z); -\hat{d}, -\nu), \quad z \in (-\infty, 0) \cup (y_2, y_1),$$
$$\mathcal{F}(\varphi_+(z); \hat{d}, \nu) = e^{-2\pi i \nu} \mathcal{F}(\varphi_-(z); \hat{d}, \nu), \quad z \in (0, y_2).$$
Let us consider the matrix

\[
Q_1^\infty(z) := \begin{pmatrix} \mathcal{F}(\varphi(z); -\hat{d}_1, -\nu) & \mathcal{F}(\varphi(z); \hat{d}_1, \nu) \\ \mathcal{F}(\varphi(z); \hat{d}_2, -\nu) & \mathcal{F}(\varphi(z); -\hat{d}_2, \nu) \end{pmatrix}, \quad z \in \mathbb{C} \setminus (-\infty, y_1],
\]

where \( \varphi(z) \) is interpreted as the value of \( \varphi \) at the point \( z \) on the upper sheet \( X_1 \). The matrix \( Q_1^\infty \) has the jumps

\[
Q_{1,+}^\infty(z) = Q_{1,-}^\infty(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z \in (-\infty, 0) \cup (y_2, y_1),
\]

\[
Q_{1,+}^\infty(z) = Q_{1,-}^\infty(z)e^{2\pi i \sigma_3}, \quad z \in (0, y_2).
\]

Note that \( \det Q_1^\infty \) is not constant. To ensure that \( Q_1^\infty \) has no pole at \( \infty \), we assume from now that \( \hat{d}_1, \hat{d}_2 \neq \frac{\tau}{2} \) mod \( \mathbb{Z} + \tau \mathbb{Z} \). We define

\[
\beta(z) = \sqrt{\frac{z(z-y_1)}{z-y_2}}, \quad z \in \mathbb{C} \setminus ((-\infty, 0) \cup [y_2, y_1]),
\]

where the principal branch is chosen for each branch so \( \beta(z) > 0 \) for \( z > y_1 \). It can be verified that the matrix

\[
Q_2^\infty(z) := \beta(z)^{\sigma_3}M^{-1}
\]

is analytic in \( \mathbb{C} \setminus ((-\infty, 0) \cup [y_2, y_1]) \), satisfies the jumps

\[
Q_{2,+}^\infty(z) = Q_{2,-}^\infty(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z \in (-\infty, 0) \cup (y_2, y_1),
\]

with asymptotic behavior at \( \infty \) given by

\[
Q_2^\infty(z) = (I + \mathcal{O}(z^{-1}))z^{\frac{\tau}{2}}M^{-1}, \quad \text{as } z \to \infty.
\]

Therefore, the matrix

\[
Q_3^\infty(z) := \beta(z)^{\sigma_3} \sqrt{2} \begin{pmatrix} \mathcal{F}(\varphi(z); -\hat{d}_1, -\nu) & -i\mathcal{F}(\varphi(z); \hat{d}_1, \nu) \\ -i\mathcal{F}(\varphi(z); \hat{d}_2, -\nu) & \mathcal{F}(\varphi(z); -\hat{d}_2, \nu) \end{pmatrix}
\]

has the jumps

\[
Q_{3,+}^\infty(z) = Q_{3,-}^\infty(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z \in (-\infty, 0) \cup (y_2, y_1),
\]

\[
Q_{3,+}^\infty(z) = Q_{3,-}^\infty(z)e^{2\pi i \sigma_3}, \quad z \in (0, y_2).
\]

However, \( Q_3^\infty \) is not a solution to the RH problem for \( P^\infty \) because it does not have the required behavior at \( \infty \). Indeed, as \( z \to \infty \), we have

\[
Q_3^\infty(z)M^{-\frac{\tau}{2}} = \frac{1}{2} \left( \begin{array}{c} \left[ \mathcal{F}(\varphi(z); -\hat{d}_1, -\nu) + \mathcal{F}(\varphi(z); \hat{d}_1, \nu) \right] \beta(z)z^{-\frac{\tau}{4}} \\ \left[ \mathcal{F}(\varphi(z); -\hat{d}_2, -\nu) + \mathcal{F}(\varphi(z); \hat{d}_2, \nu) \right] i\beta(z)^{-1}z^{-\frac{\tau}{4}} \end{array} \right) \cdots 
\]

\[
\cdots \left[ \mathcal{F}(\varphi(z); -\hat{d}_1, -\nu) - \mathcal{F}(\varphi(z); \hat{d}_1, \nu) \right] i\beta(z)z^{\frac{\tau}{4}} \\ \left[ \mathcal{F}(\varphi(z); -\hat{d}_2, \nu) + \mathcal{F}(\varphi(z); -\hat{d}_2, \nu) \right] \beta(z)^{-1}z^{\frac{\tau}{4}} \right) = \tilde{Q} + \mathcal{O}(z^{-1}) \]

(4.17)
where the leading coefficient $\tilde{Q}$ is given by

$$\tilde{Q} = \begin{pmatrix} \mathcal{F}(\frac{1}{2}, \hat{d}_1, \nu) & 2ic_0\mathcal{F}'(\frac{1}{2}, \hat{d}_1, \nu) \\ 0 & \mathcal{F}(\frac{1}{2}, -\hat{d}_2, \nu) \end{pmatrix},$$

and where we have used the expansions

$$\varphi(z) = \frac{1}{2} - 2c_0z^{-\frac{1}{2}} + O(z^{-\frac{3}{2}}),$$

$$\mathcal{F}(\varphi(z); \hat{d}, \nu) = \mathcal{F}(\frac{1}{2}, \hat{d}, \nu) - 2c_0\mathcal{F}'(\frac{1}{2}, \hat{d}, \nu)z^{-\frac{1}{2}} + O(z^{-1}),$$

$$\mathcal{F}(\varphi(z); -\hat{d}, -\nu) = \mathcal{F}(-\varphi(z) + 1; \hat{d}, \nu) = \mathcal{F}(\frac{1}{2}, \hat{d}, \nu) + 2c_0\mathcal{F}'(\frac{1}{2}, \hat{d}, \nu)z^{-\frac{1}{2}} + O(z^{-1}),$$

as $z \to \infty$. Since $\hat{d}_1, \hat{d}_2 \neq \frac{\tau}{2}$ mod $(\mathbb{Z} + \tau\mathbb{Z})$, the quantities $\mathcal{F}(\frac{1}{2}, \hat{d}_1, \nu)$ and $\mathcal{F}(\frac{1}{2}, -\hat{d}_2, \nu)$ are well-defined. We also assume that $\Im[\hat{d}_1], \Im[\hat{d}_2] \neq \frac{1}{2}(1 + n)\Im[\tau], n \in \mathbb{Z}$, so that $\mathcal{F}(\frac{1}{2}, \hat{d}_1, \nu)$ and $\mathcal{F}(\frac{1}{2}, -\hat{d}_2, \nu)$ are both different from 0 for all values of $\nu \in \mathbb{R}$. This implies that $\tilde{Q}$ is invertible, therefore we can normalize $Q_{\alpha}^{(\infty)}$ at $\infty$. Thus we define

$$P^{(\infty)}(z) := \tilde{Q}^{-1}Q_{\alpha}^{(\infty)}(z) = \begin{pmatrix} \frac{1}{G(\frac{1}{2})} & -ic_G \beta(z)^{\gamma_3} \sqrt{2} \left( G(-\varphi(z)) & -iG(\varphi(z)) \right) \\ 0 & H(\frac{1}{2}) \end{pmatrix}$$

(4.18)

where

$$G(z) = \mathcal{F}(z; \hat{d}_1, \nu), \quad H(z) = \mathcal{F}(z; -\hat{d}_2, \nu), \quad c_G = 2c_0(\log G)'(\frac{1}{2}).$$

The above matrix $P^{(\infty)}(z)$ has the same jumps as $Q_{\alpha}^{(\infty)}$ (since the multiplication by $\tilde{Q}^{-1}$ is taken on the left), and has the prescribed asymptotic behavior as $z \to \infty$ given by (4.2). It remains to choose $\hat{d}_1$ and $\hat{d}_2$ appropriately so that $P^{(\infty)}$ has no poles. Since $\beta^{-1}$ vanishes at $z = y_2$, we choose $\hat{d}_2 = \frac{1}{2}$ so that the pole of $\mathcal{H}(\pm \varphi(z))$ also lies at $y_2$. Thus at the branch point $y_2$, we have

$$\mathcal{H}(\pm \varphi(z))\beta(z)^{-1} = O((z - y_2)^{-\frac{1}{2}}), \quad \text{as } z \to y_2. \quad (4.19)$$

Let us now inspect the first line of the right-most matrix in (4.18). Since $\beta$ vanishes at 0 and $y_1$, we have two choices for $\hat{d}_1$ (see (4.10)):

$$\hat{d}_1 = 0 \quad \text{or} \quad \hat{d}_1 = \frac{1}{2} + \frac{\tau}{2}. \quad (4.20)$$

However, the parameter $\nu$ will be such that $\nu \to -\infty$ as $r \to +\infty$, so we need $\Im[\hat{d}_1] \neq \frac{\Im[\tau]}{2}$ in order to ensure that $\tilde{Q}$ is invertible, see the discussion above (4.18). Therefore we choose $\hat{d}_1 = 0$. The choice for $\nu$ is imposed from the jumps on $(0, y_2)$, see (4.1), therefore we must have

$$\nu = -\frac{\Omega}{2\pi r^\frac{1}{2}}. \quad (4.21)$$

To summarize, the matrix-valued function $P^{(\infty)}$ given by (4.18) with $\hat{d}_1 = 0$, $\hat{d}_2 = \frac{1}{2}$ and $\nu = -\frac{\Omega}{2\pi r^\frac{1}{2}}$ satisfies the RH problem for $P^{(\infty)}$. Furthermore, we have $\mathcal{H}(z) = G(z - \frac{1}{2})$, so that (4.18) can be rewritten as

$$P^{(\infty)}(z) := \tilde{Q}^{-1}Q_{\alpha}^{(\infty)}(z) = \begin{pmatrix} \frac{1}{G(\frac{1}{2})} & -ic_G \beta(z)^{\gamma_3} \sqrt{2} \left( G(-\varphi(z)) & -iG(\varphi(z)) \right) \\ 0 & G(\frac{1}{2}) \end{pmatrix}$$

(4.22)
where
\[ G(z) = F(z; 0, \nu) = \frac{\theta(z + \nu)}{\theta(z)}. \]

The function \( G \) satisfies
\[ G(z + 1) = G(z), \quad G(z + \tau) = e^{-2\pi i \nu} G(z), \quad z \in \mathbb{C}. \quad (4.23) \]

In the following subsections, we obtain expressions for the first four terms of the asymptotics of \( P^{(\infty)}(z) \) as \( z \to y_j, \Re[z] > 0, j = 1, 2, 3 \). These coefficients will be needed in the computations of Section 6.

### 4.1 Asymptotics of \( P^{(\infty)}(z) \) as \( z \to y_1 \)

The asymptotics for \( \beta(z) \) and \( \varphi(z) \) as \( z \to y_1 \) are given by
\[ \beta(z) = \beta(y_1) (z - y_1)^{\frac{1}{4}} + \beta(y_1)^{\frac{3}{4}} (z - y_1)^{\frac{3}{2}} + O((z - y_1)^{\frac{5}{2}}), \]
\[ \beta(y_1) = \frac{y_1}{y_1 - y_2} \quad \beta(y_1)^{\frac{3}{4}} = -\frac{y_2}{4y_1(y_1 - y_2)^{\frac{3}{2}}}, \]
\[ \varphi(z) = \varphi(y_1) (z - y_1)^{\frac{1}{4}} + \varphi(y_1)^{\frac{3}{4}} (z - y_1)^{\frac{3}{2}} + O((z - y_1)^{\frac{5}{2}}), \]
\[ \varphi(y_1) = \frac{2c_0}{\sqrt{y_1y_1 - y_2}}, \quad \varphi(y_1)^{\frac{3}{4}} = -\frac{c_0(2y_1 - y_2)}{3y_1(y_1 - y_2)^{\frac{3}{2}}}. \quad (4.24) \]

The asymptotics for \( P^{(\infty)}(z) \) as \( z \to y_1 \) are given by
\[ P^{(\infty)}(z) = \sum_{j=0}^{3} (P^{(\infty)})_{y_1}^{(-\frac{j}{4})} (z - y_1)^{-\frac{j}{4} + \frac{3}{4}} + O((z - y_1)^{\frac{5}{4}}), \]
\[ (P^{(\infty)})_{y_1}^{(-\frac{j}{4})} = \frac{-G(\frac{j}{4})}{\sqrt{2}\beta(y_1)^{\frac{j}{4}} G(0)} \begin{pmatrix} c_F & ic_F \ i & -1 \end{pmatrix}, \]
\[ (P^{(\infty)})_{y_1}^{(\frac{j}{4})} = \beta(y_1)^{\frac{j}{4}} G(0) \frac{G(\frac{j}{4})}{\sqrt{2}\beta(y_1)^{\frac{j}{2}} G(0)} \begin{pmatrix} 1 & -i \ 0 & 1 \end{pmatrix}, \]
\[ (P^{(\infty)})_{y_1}^{(\frac{j}{4})} = \frac{-\varphi(y_1)^{\frac{j}{4}} G(0) + \beta(y_1)^{\frac{j}{4}} (\varphi(y_1)^{\frac{3}{4}} G(0))^{\frac{3}{4}}}{\sqrt{2}\beta(y_1)^{\frac{j}{2}} G(0)} \frac{G(\frac{j}{4})}{\varphi(y_1)^{\frac{3}{4}} G(0)} \begin{pmatrix} 1 & -i \ 0 & 1 \end{pmatrix}, \]
\[ (P^{(\infty)})_{y_1}^{(\frac{j}{4})} = \beta(y_1)^{\frac{j}{4}} G(0) + \frac{\beta(y_1)^{\frac{3}{4}} (\varphi(y_1)^{\frac{3}{4}} G(0))^{\frac{3}{4}}}{\sqrt{2}\beta(y_1)^{\frac{j}{2}} G(0)} \frac{G(\frac{j}{4})}{\varphi(y_1)^{\frac{3}{4}} G(0)} \begin{pmatrix} 1 & -i \ 0 & 1 \end{pmatrix}. \]

### 4.2 Asymptotics of \( P^{(\infty)}(z) \) as \( z \to y_2 \) from \( \Re[z] > 0 \)

We recall that \( \frac{1 + i\tau}{2} \) is a simple zero of \( z \mapsto \theta(z) \). Thus the function \( z \mapsto G(z) \) has a simple pole at \( z = \frac{1 + i\tau}{2} \). Since \( \varphi_{+}(y_2) = \frac{c}{\sqrt{2}} \), the entries on the second row of \( P^{(\infty)}(z) \) blow up as \( z \to y_2, \Re[z] > 0 \).
Let us define

\[ z \mapsto \tilde{G}(z) := G(z)(z - \frac{1}{2}) \].

(4.25)

Using the relations (4.23) and (4.25), before expanding \( p^{(\infty)}(z) \) as \( z \to y_2, \Im[z] > 0 \), we rewrite \( p^{(\infty)} \) as follows

\[
p^{(\infty)}(z) = \begin{pmatrix} \frac{1}{G(\frac{y}{2})} & -\frac{i}{\sqrt{2}} & \frac{\beta(z)\tau}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & \frac{\beta(z)\tau}{\sqrt{2}} \\ 0 & 1 & G(0) & \frac{\beta(z)\tau}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & \frac{\beta(z)\tau}{\sqrt{2}} \\ 0 & 0 & 1 & \frac{\beta(z)\tau}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & \frac{\beta(z)\tau}{\sqrt{2}} \\ 0 & 0 & 0 & 1 & \frac{\beta(z)\tau}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{2\pi i\nu} \tilde{G}(\tau - \varphi(z)) \\ -i e^{2\pi i\nu} \tilde{G}(-\varphi(z) + \frac{1}{2} + \tau) \\ \frac{\beta(z)\tau}{\sqrt{2}} \tilde{G}(\varphi(z) + \frac{1}{2}) \end{pmatrix}.
\]

The asymptotics for \( \beta(z) \) and \( \varphi(z) \) as \( z \to y_2, \Im[z] > 0 \), are given by

\[
\beta(z) = \beta(y_2) (z - y_2)^{-\frac{1}{2}} + \beta(y_2) (z - y_2)^{\frac{1}{2}} + \mathcal{O}((z - y_2)^{-\frac{3}{2}}),
\]

\[
\beta(y_2) = e^{\frac{i\pi}{4}} y_2^\frac{1}{2} (y_1 - y_2)^\frac{1}{4}, \quad \beta(y_2) = e^{\frac{i\pi}{4}} (y_1 - y_2)^\frac{1}{4}.
\]

\[
\varphi(z) = \frac{\tau}{2} + \varphi(y_2) (z - y_2)^{-\frac{1}{2}} + \varphi(y_2) (z - y_2)^{\frac{1}{2}} + \mathcal{O}((z - y_2)^{-\frac{3}{2}}),
\]

\[
\varphi(y_2) = \frac{-2i\tau}{\sqrt{y_2 y_1}}, \quad \varphi(y_2) = \frac{i\tau}{\sqrt{y_2 y_1}}.
\]

The asymptotics for \( p^{(\infty)}(z) \) as \( z \to y_2, \Im[z] > 0 \), are given by

\[
 p^{(\infty)}(z) = \sum_{j=0}^{3} \begin{pmatrix} \frac{1}{G(y_2)^{\frac{1}{2}}} & \frac{1}{G(y_2)^{\frac{1}{2}}} & \frac{1}{G(y_2)^{\frac{1}{2}}} & \frac{1}{G(y_2)^{\frac{1}{2}}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{2\pi i\nu} \tilde{G}(\tau - \varphi(z)) \\ -i e^{2\pi i\nu} \tilde{G}(-\varphi(z) + \frac{1}{2} + \tau) \\ \frac{\beta(z)\tau}{\sqrt{2}} \tilde{G}(\varphi(z) + \frac{1}{2}) \end{pmatrix}.
\]
4.3 Asymptotics of $P^{(\infty)}(z)$ as $z \to y_3 = 0$ from $\Im[z] > 0$

Using the relations (4.23) and (4.25), before expanding $P^{(\infty)}(z)$ as $z \to y_3$, $\Im[z] > 0$, we rewrite $P^{(\infty)}$ as follows

$$P^{(\infty)}(z) = \left( \frac{1}{ \gamma^{(2)}(z) } \right) \beta(z)^{\frac{1}{4_3}} \left( \frac{e^{2\pi i \nu} \tilde{G}(1 + \frac{\tau}{2} - \varphi(z))}{\sqrt{2}} - i \frac{\tilde{G}(\varphi(z))}{\varphi(z) - \frac{1 + \frac{\tau}{2}}{\psi(z) - \frac{2}{3}}} \right) \frac{1}{ \gamma^{(3)}(0) }$$

The asymptotics for $\beta(z)$ and $\varphi(z)$ as $z \to y_3 = 0$, $\Im[z] > 0$, are given by

$$\beta(z) = \beta^{(2)}(y_3) z^{\frac{1}{4_3}} + \beta^{(3)}(y_3) z^{\frac{1}{4_3}} + O(z^{\frac{1}{4_3}}),$$

$$\varphi(z) = \frac{1 + \tau}{2} + \varphi^{(2)}(y_3) z^{\frac{1}{4_3}} + \varphi^{(3)}(y_3) z^{\frac{1}{4_3}} + O(z^{\frac{1}{4_3}}),$$

The asymptotics for $P^{(\infty)}(z)$ as $z \to y_3 = 0$, $\Im[z] > 0$, are given by

$$P^{(\infty)}(z) = \sum_{j=0}^{3} \left( \frac{P^{(\infty)}(z)}{y_3^{j} \sqrt{\gamma^{(2)}(z)}} \right) z^{-\frac{j}{4_3}} + O(z^{\frac{j}{4_3}}),$$

$$P^{(\infty)}(y_3) \beta^{(2)}(z) = \frac{\beta^{(2)}(y_3) \tilde{G}(1 + \frac{\tau}{2})}{\sqrt{2} \gamma^{(2)}(z)} \left( \frac{e^{2\pi i \nu} \frac{1}{\beta^{(2)}(y_3)} \tilde{G}(\frac{1 + \frac{\tau}{2}}{\gamma^{(3)}(0)})}{\sqrt{2} \beta^{(2)}(y_3) \gamma^{(3)}(0)} \right)$$

$$P^{(\infty)}(y_3) \beta^{(3)}(z) = \frac{\beta^{(3)}(y_3) \tilde{G}(1 + \frac{\tau}{2})}{\sqrt{2} \gamma^{(2)}(z)} \left( \frac{e^{2\pi i \nu} \frac{1}{\beta^{(3)}(y_3)} \tilde{G}(\frac{1 + \frac{\tau}{2}}{\gamma^{(3)}(0)})}{\sqrt{2} \beta^{(3)}(y_3) \gamma^{(3)}(0)} \right)$$

$$P^{(\infty)}(y_3) \varphi^{(2)}(z) = \frac{\varphi^{(2)}(y_3) \tilde{G}(1 + \frac{\tau}{2})}{\sqrt{2} \gamma^{(2)}(z)} \left( \frac{e^{2\pi i \nu} \frac{1}{\varphi^{(2)}(y_3)} \tilde{G}(\frac{1 + \frac{\tau}{2}}{\gamma^{(3)}(0)})}{\sqrt{2} \varphi^{(2)}(y_3) \gamma^{(3)}(0)} \right)$$

$$P^{(\infty)}(y_3) \varphi^{(3)}(z) = \frac{\varphi^{(3)}(y_3) \tilde{G}(1 + \frac{\tau}{2})}{\sqrt{2} \gamma^{(2)}(z)} \left( \frac{e^{2\pi i \nu} \frac{1}{\varphi^{(3)}(y_3)} \tilde{G}(\frac{1 + \frac{\tau}{2}}{\gamma^{(3)}(0)})}{\sqrt{2} \varphi^{(3)}(y_3) \gamma^{(3)}(0)} \right)$$

4.4 Asymptotics of $P^{(\infty)}(z)^{-1}$ as $z \to y_k$ from $\Im[z] > 0$, $k = 1, 2, 3$

Since $\det P^{(\infty)} \equiv 1$, we have

$$P^{(\infty)}(z)^{-1} = \begin{pmatrix} P_{22}^{(\infty)}(z) & -P_{12}^{(\infty)}(z) \\ -P_{21}^{(\infty)}(z) & P_{11}^{(\infty)}(z) \end{pmatrix},$$

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and we deduce from Subsections 4.1, 4.2 and 4.3 the following asymptotic expansions

\[
P^{(\infty)}(z)^{-1} = \sum_{j=0}^{3} (P_{iv}^{j}(\infty))_{y_{k}}^{(\infty)}(z - y_{k})^{-\frac{1}{2} + \frac{j}{4}} + O((z - y_{k})^{\frac{1}{4}}), \quad k = 1, 2, 3, \]

\[
(P_{iv}^{\infty})_{y_{k}}^{(-\frac{1}{2} + \frac{j}{4})} = \left( \begin{array}{cc}
(P_{iv}^{\infty})_{y_{k},22}^{(-\frac{1}{2} + \frac{j}{4})} & -(P_{iv}^{\infty})_{y_{k},12}^{(-\frac{1}{2} + \frac{j}{4})} \\
-(P_{iv}^{\infty})_{y_{k},21}^{(-\frac{1}{2} + \frac{j}{4})} & (P_{iv}^{\infty})_{y_{k},11}^{(-\frac{1}{2} + \frac{j}{4})}
\end{array} \right), \quad k = 1, 2, 3, \quad j = 0, 1, 2, 3.
\]

5 Local parametrices

Let us choose \( \delta > 0 \) such that

\[
\delta \leq \min\{x_{1}, x_{1} - x_{2}, x_{2} - x_{3}\} / 3 \quad \text{and} \quad \mathbb{D}_{y_{j}} : = \{z \in \mathbb{C} : |z - y_{j}| < \delta\} \subset N_{0}, \quad j = 1, 2, 3,
\]

where \( N_{0} \) is as described in Lemma 3.2. The local parametrix \( P^{(y_{j})}(z) \) (for \( j \in \{1, 2, 3\} \)) is defined for \( z \in \mathbb{D}_{y_{j}} \) as the solution to a RH problem with the same jumps as \( S \) inside \( \mathbb{D}_{y_{j}} \). Furthermore, we require that \( P^{(y_{j})} \) matches with \( P^{(\infty)} \) on the boundary of \( \mathbb{D}_{y_{j}} \), in the sense that

\[
P^{(y_{j})}(z) = (I + o(1))P^{(\infty)}(z), \quad \text{as} \quad r \to +\infty \quad (5.1)
\]

uniformly for \( z \in \partial \mathbb{D}_{y_{j}} \). These constructions are standard (so we give relatively short explanations) and given in terms of an explicitly solvable model RH problem \( \Phi_{Be} \) which is expressed in terms of Bessel functions. This model RH problem was first derived in [20] and we present it in Appendix A for convenience. For an in depth discussion on how to construct a parametrix using the Bessel RH problem, we refer the reader to [3, Section 4.3].

5.1 Parametrix at \( y_{1} \)

Define the function

\[
f_{y_{1}}(z) = \frac{g^{2}(z)}{4}. \quad (5.2)
\]

This is a conformal map from \( \mathbb{D}_{y_{1}} \) to a neighborhood of 0. The expansion of \( f_{y_{1}}(z) \) as \( z \to y_{1} \) is given by

\[
f_{y_{1}}(z) = c_{y_{1}}(z - y_{1})(1 + c_{y_{1}}^{(2)}(z - y_{1}) + c_{y_{1}}^{(3)}(z - y_{1})^{2} + O((z - y_{1})^{3})), \quad (5.3)
\]

\[
c_{y_{1}} = \frac{p^{2}(y_{1})}{y_{1}(y_{1} - y_{2})}, \quad c_{y_{1}}^{(2)} = \frac{2p(y_{1})}{3p(y_{1})} - \frac{2y_{1} - y_{2}}{3(y_{1} - y_{2})},
\]

\[
c_{y_{1}}^{(3)} = \frac{1}{45} \left( 23g^{2} - 23g_{1}g_{2} + 8g_{2}^{2} + \frac{5p(y_{1})}{p(y_{1})} - \frac{14(2y_{1} - y_{2})p(y_{1})}{y_{1}(y_{1} - y_{2})p(y_{1})} + \frac{9p^{2}(y_{1})}{p(y_{1})} \right).
\]

We have freedom in the choice of \( \gamma_{\pm} \subset N_{0} \). We choose the lenses such that \( f_{y_{1}}(\gamma_{\pm} \cap \mathbb{D}_{y_{1}}) \subset e^{\frac{3\pi}{2}} \mathbb{R}^{+} \) and \( f_{y_{1}}(\gamma_{\mp} \cap \mathbb{D}_{y_{1}}) \subset e^{\frac{3\pi}{2}} \mathbb{R}^{-} \). We conclude from the RH problem for \( \Phi_{Be} \) presented in Section A that the matrix

\[
\Phi_{Be} \left( r^{3}f_{y_{1}}(z) \right) e^{-r\frac{3}{2}g(z)\sigma_{3}} \quad (5.4)
\]

has the same jumps as \( S(z) \) inside the disk \( z \in \mathbb{D}_{y_{1}} \), see (3.22)-(3.24). Let us define \( P^{(y_{1})} \) by

\[
P^{(y_{1})}(z) := E_{y_{1}}(z)\Phi_{Be} \left( r^{3}f_{y_{1}}(z) \right) e^{-r\frac{3}{2}g(z)\sigma_{3}}, \quad (5.5)
\]

\[
E_{y_{1}}(z) = P^{(\infty)}(z)M^{-1} \left( 2\pi r^{\frac{3}{2}}f_{y_{1}}(z)\frac{3}{2} \right) \quad (5.6)
\]
It can be verified from the jumps for $P^{(\infty)}$ that the prefactor $E_{y_1}(z)$ is analytic for $z \in \mathbb{D}_{y_1}$. Thus, $P^{(y_1)}(z)$ has the (exact) same jumps as $S(z)$ for $z \in \mathbb{D}_{y_1}$. Furthermore, due to (5.3) and (A.2), we have

$$P^{(y_1)}(z)P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z)\Phi_{Bc,1}P^{(\infty)}(z)^{-1}}{r^2f_{y_1}(z)^2} + O(r^{-3})$$

(5.7)

as $r \to +\infty$ uniformly for $z \in \partial\mathbb{D}_{y_1}$. In Section 8 we will need the first columns of $E_{y_1}(y_1)$ and $E'_{y_1}(y_1)$. After some computations, we find

$$E_{y_1}(y_1) = \frac{-\sqrt{2\pi\epsilon_{y_1}}G(\frac{y_1}{2})}{\beta_{y_1} G(0)} \begin{pmatrix} c_F & 0 \\ i & 0 \end{pmatrix} r^{-\frac{1}{2}},$$

(5.8)

$$E'_{y_1}(y_1) = \sqrt{2\pi\epsilon_{y_1}}r^{-\frac{1}{2}} \begin{pmatrix} \frac{e^{\frac{y_1}{2}}}{\epsilon_{y_1}} - \frac{1}{y_1}G'\left(\frac{y_1}{2}\right) - \frac{\beta_{y_1}}{2} \right)G''\left(\frac{y_1}{2}\right) \begin{pmatrix} c_F & 0 \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \ast \\ 0 & \ast \end{pmatrix} r^{-\frac{1}{2}},$$

where $\ast$ denotes unnecessary constants.

### 5.2 Parametrix at $y_2$

Define the function

$$f_{y_2}(z) = \frac{1}{4} \left( g(z) + \frac{i\Omega}{2} \right)^2,$$

(5.9)

where we take $-/+\ i$ when $z$ is above/below the real axis. This is a conformal map from $\mathbb{D}_{y_2}$ to a neighborhood of 0. Its expansion as $z \to y_2$ is given by

$$f_{y_2}(z) = c_{y_2}(z - y_2) \left( 1 + c_{y_2}^{(2)}(z - y_2) + c_{y_2}^{(3)}(z - y_2)^2 + O((z - y_2)^3) \right),$$

(5.10)

$$c_{y_2} = \frac{p^2(y_2)}{y_2(y_2 - y_1)} > 0, \quad c_{y_2}^{(2)} = \frac{2p'(y_2)}{y_2(y_2 - y_1)} = \frac{y_1 - 2y_2}{y_2}, \quad c_{y_2}^{(3)} = \frac{1}{45} \left( \frac{8y_1^2 - 23y_1y_2 + 23y_2^2}{(y_1 - y_2)^2y_2} + \frac{5p'(y_2)}{y_2} = \frac{14(y_1 - 2y_2)p'(y_2)}{(y_1 - y_2)y_2p(y_2)} + \frac{9p''(y_2)}{y_2} \right).$$

We choose the lenses $\gamma_+ \subset N_0$ and $\gamma_- \subset N_0$ such that $-f_{y_2}(\gamma_+ \cup \mathbb{D}_{y_2}) \subset e^{\pm \frac{\pi i}{4}} \mathbb{R}^+$, and we define

$$P^{(y_2)}(z) = E_{y_2}(z)\sigma_3\Phi_{Bc}(-r^2f_{y_2}(z))\sigma_3e^{-\frac{r}{2}g(z)}\sigma_3,$$

(5.11)

$$E_{y_2}(z) = P^{(\infty)}(z)e^{\pm \frac{\pi i}{4} \sigma_3 M^{-1} \left( 2\pi r z \left( -f_{y_2}(z) \right)^{\frac{1}{2}} \right)^4}.$$ 

It can be verified that $E_{y_2}(z)$ is analytic in the disk $\mathbb{D}_{y_2}$. From (5.9) and (A.2), we also conclude that

$$P^{(y_2)}(z)P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z)e^{\pm \frac{\pi i}{4} \sigma_3 M^{-1} \left( 2\pi r z \left( -f_{y_2}(z) \right)^{\frac{1}{2}} \right)^4}}{r^2f_{y_2}(z)^2} + O(r^{-3})$$

(5.12)
as \( r \to +\infty \) uniformly for \( z \in \partial \mathbb{D}_{y_2} \). Furthermore, we also verify that

\[
E_{y_2}(y_2) = e^{-\frac{\pi i}{2} \sqrt{2 \pi c_{y_2}} e^{i\pi/4}} \frac{\partial}{\partial \gamma_{y_2}} \left\{ \frac{\tilde{g}(\frac{1+i}{2})}{\gamma_{y_2}^{3/2} g(0)} \left( c_F \quad 0 \right) + \frac{\beta_{y_2}^{(-2,0)} g(\frac{1}{2})}{g(0)} \left( 1 \quad 0 \right) \right\} + \left( 0 \quad * \right) r^{-\frac{3}{2}}.
\]

(5.13)

\[E'_{y_2}(y_2) = e^{-\frac{\pi i}{2} \sqrt{2 \pi c_{y_2}} e^{i\pi/4}} \frac{1}{\gamma_{y_2}^{3/2} g(0)} \left\{ \frac{(\beta_{y_2}^{(-2,0)} + 2 \beta_{y_2}^{(-2,0)} \gamma_{y_2}^{3/2} g(\frac{1}{2}) + \frac{1}{2} \gamma_{y_2}^{3/2} \gamma_{y_2}^{3/2} \gamma_{y_2}^{3/2} g''(\frac{1}{2}) \left( 1 \quad 0 \right) + \left( 0 \quad * \right) r^{-\frac{3}{2}},
\]

as \( r \to +\infty \), where * denotes unnecessary constants.

### 5.3 Parametrix at \( y_3 = 0 \)

We define the function

\[
f_{y_3}(z) = \frac{1}{4} \left( g(z) + i \Omega \frac{z}{2} \right)^2,
\]

(5.14)

where we take the \(-/-+\) sign when \( z \) is above/below the real axis. This is a conformal map from \( \mathbb{D}_{y_3} \) to a neighborhood of 0, and the expansion of \( f_{y_3}(z) \) as \( z \to 0 \) is given by

\[
f_{y_3}(z) = c_{y_3} z(1 + c_{y_3}^{(2)} z + c_{y_3}^{(3)} z^2 + \mathcal{O}(z^3)),
\]

(5.15)

\[
c_{y_3} = \frac{\mu^2(0)}{y_1 y_2} \quad c_{y_3}^{(2)} = \frac{2 \mu'(0)}{3 \mu(0)} + \frac{y_1 + y_2}{y_1 y_2}, \quad c_{y_3}^{(3)} = \frac{1}{15} \left( \frac{8 y_1^2 + 7 y_1 y_2 + 8 y_2^2}{y_1 y_2} + \frac{5 \mu'(0)^2}{\mu(0)^2} + \frac{14 (y_1 + y_2) \mu'(0)}{y_1 y_2 \mu(0)} + \frac{9 \mu''(0)}{\mu(0)} \right).
\]

We choose the lenses \( \gamma_+ \subset N_0 \) and \( \gamma_- \subset N_0 \) such that \( f_{y_3}(\gamma_{+} \cap \mathbb{D}_{y_3}) \subset e^{\pm \frac{\pi i}{4}} \mathbb{R}^+ \), and we define

\[
P^{(y_3)}(z) = E_{y_3}(z)\Phi_{y_3}(r^3 f_{y_3}(z))e^{-r \frac{2 \phi(g(z))}{3}},
\]

(5.16)

\[
E_{y_3}(z) = P^{(\infty)}(z) e^{\pm \frac{\pi i}{4} \frac{r^3}{3} \sigma_M^{-1} \left( 2 \pi r \frac{2}{3} f_{y_3}(z) \right)^{\frac{3}{2}}}.\]

It can be verified that \( E_{y_3}(z) \) is analytic inside the disk \( \mathbb{D}_{y_3} \), and that

\[
P^{(y_3)}(z) P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z) e^{\pm \frac{\pi i}{4} \frac{r^3}{3} \sigma_M^{-1} \Phi_{y_3}} e^{\pm \frac{\pi i}{4} \frac{r^3}{3} \sigma_M^{-1}} p^{(\infty)}(z)^{-1}}{r^3 f_{y_3}(z)^{\frac{3}{2}}} + \mathcal{O}(r^{-3})
\]

(5.17)
as \( r \to +\infty \) uniformly for \( z \in \partial \mathbb{D}_{y_3} \). Furthermore, we verify that

\[
E_{y_3}(y_3) = -\sqrt{2\pi c_{y_3}} e^{\pi i \nu} r^{\frac{3}{2}} \left\{ \frac{\mathcal{G}(\frac{1}{2})}{\beta_{y_3}^{(2)} g(0)} \begin{bmatrix} c \nu & 0 \\ i & 0 \end{bmatrix} + \frac{\beta_{y_3}^{(4)} \tilde{g}(1+r)}{\varphi_{y_3}^{(4)} g(\frac{1}{2})} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} + \begin{bmatrix} 0 & 0 \\ 0 & \ast \end{bmatrix} r^{-\frac{3}{2}},
\]

(5.18)

\[
E_{y_3}'(y_3) = -\sqrt{2\pi c_{y_3}} e^{\pi i \nu} r^{\frac{3}{2}} \left\{ \frac{\left( \frac{1+\nu}{2} \right) - \beta_{y_3}^{(4)} g(\frac{1}{2})}{\beta_{y_3}^{(2)} g(0)} \begin{bmatrix} c \nu & 0 \\ i & 0 \end{bmatrix} + \frac{\beta_{y_3}^{(4)} \tilde{g}(1+r)}{\varphi_{y_3}^{(4)} g(\frac{1}{2})} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} + \begin{bmatrix} 0 & 0 \\ 0 & \ast \end{bmatrix} r^{-\frac{3}{2}}
\]

as \( r \to +\infty \), where \( \ast \) denotes unnecessary constants.

6 Small norm problem

In this section we show that for sufficiently large \( r \), \( P^{(\infty)}(z) \) approximates \( S(z) \) for \( z \in \mathbb{C} \setminus \bigcup_{j=1}^{3} \mathbb{D}_{y_j} \) and \( P(y_j) \) approximates \( S(z) \) for \( z \in \mathbb{D}_{y_j}, j = 1, 2, 3 \). We define

\[
R(z) = \begin{cases} 
S(z)P^{(y_j)}(z)^{-1}, & z \in \mathbb{D}_{y_j}, \ j = 1, 2, 3, \\
S^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus \bigcup_{j=1}^{3} \mathbb{D}_{y_j}.
\end{cases}
\]

(6.1)

Since \( P^{(y_j)}, j = 1, 2, 3 \), has the exact same jumps as \( S \) inside the disks, \( R(z) \) is analytic for \( z \in \bigcup_{j=1}^{3} \mathbb{D}_{y_j} \setminus \{y_j\} \). Furthermore, we verify from (3.26), (5.5), (5.11), (5.16) and (A.3) that \( S(z)P^{(y_j)}(z)^{-1} = \mathcal{O}(\log(z-y_j)) \) as \( z \to y_j, j = 1, 2, 3 \). This means that the singularities of \( R \) at \( y_1, y_2, y_3 \) are in fact removable and \( R \) is analytic in the full three open disks. Let \( \Sigma_R \) denote the jump contour for \( R \) which is explicitly given by

\[
\Sigma_R = \left( \gamma_+ \cup \gamma_- \cup \bigcup_{j=1}^{3} \partial \mathbb{D}_{y_j} \right) \setminus \bigcup_{j=1}^{3} \mathbb{D}_{y_j},
\]

with orientation as shown in Figure 5. In particular, note that we orient the boundaries of the disks in the clockwise direction. The jumps for \( R \) are denoted by \( J_R \),

![Figure 5: The jump contour \( \Sigma_R \).](image_url)
\[ J_R : \Sigma_R \to \mathbb{C}^{2 \times 2}, \quad z \mapsto J_R(z) := R_-(z)^{-1} R_+(z), \]

and are explicitly given by

\[ J_R(z) = \begin{cases} 
  p^{(\infty)}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\pi y_j(z)} & 1 \end{pmatrix} p^{(\infty)}(z)^{-1}, & z \in \gamma_+ \cup \gamma_- \setminus \bigcup_{j=1}^{3} \mathbb{D}_{y_j} \\
  p^{(y_j)}(z) p^{(\infty)}(z)^{-1}, & z \in \partial \mathbb{D}_{y_j}, \quad j = 1, 2, 3.
\end{cases} \tag{6.2} \]

From Lemma 3.2 and (5.7), (5.12), (5.17), as \( r \to \infty \) we have

\[ J_R(z) = \begin{cases} 
  I + O(e^{-\tilde{c}|z|^2}), & \text{uniformly for } z \in \Sigma_R \setminus \bigcup_{j=1}^{3} \partial \mathbb{D}_{y_j}, \\
  I + J_R^{(1)}(z) + O(r^{-3}), & \text{uniformly for } z \in \bigcup_{j=1}^{3} \partial \mathbb{D}_{y_j}, \tag{6.3}
\end{cases} \]

where \( \tilde{c} > 0 \) is a sufficiently small constant, and \( J_R^{(1)}(z) = O(1) \) as \( r \to +\infty \) uniformly for \( z \in \bigcup_{j=1}^{3} \partial \mathbb{D}_{y_j} \). It is important to recall that \( P^{(\infty)}(z) \) has \( r \) dependence but this is of no consequence due to the periodicity of the \( \theta \)-function in the real direction. The matrix \( J_R^{(1)}(z) \) has been computed in (5.7), (5.12), (5.17), and is given by

\[ J_R^{(1)}(z) = \begin{pmatrix} 
  \frac{p^{(\infty)}(z) \Phi_{R_+} p^{(\infty)}(z)^{-1}}{f_{y_1}(z)^{\frac{3}{2}}} & , & \text{if } z \in \partial \mathbb{D}_{y_1} \\
  \frac{p^{(\infty)}(z) e^{\frac{3}{2} \phi_3 \sigma_1 \Phi_{R_+} \sigma_3} p^{(\infty)}(z)^{-1}}{(-f_{y_2}(z)^{\frac{3}{2}})} & , & \text{if } z \in \partial \mathbb{D}_{y_2} \\
  \frac{p^{(\infty)}(z) e^{\frac{3}{2} \phi_3 \Phi_{R_+} \sigma_3}}{f_{y_3}(z)^{\frac{3}{2}}} p^{(\infty)}(z)^{-1} & , & \text{if } z \in \partial \mathbb{D}_{y_3}.
\end{pmatrix} \tag{6.4} \]

By the standard theory for RH problems [10, 11, 13], \( R(z) \) exists for sufficiently large \( r \) and satisfies

\[ R(z) = I + \frac{R^{(1)}(z)}{r^2} + O(r^{-3}), \tag{6.5} \]

as \( r \to \infty \), uniformly for \( z \in \mathbb{C} \setminus \Sigma_R \). The goal for the remainder of this section is to obtain explicit expressions for \( R^{(1)}(y_j) \), \( j = 1, 2, 3 \). Since \( R(z) \) satisfies the equation

\[ R(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{R_-(\xi) (J_R(\xi) - I)}{\xi - z} \, d\xi, \tag{6.6} \]

we deduce from (6.3) that

\[ R^{(1)}(z) = \frac{1}{2\pi i} \int_{\bigcup_{j=1}^{3} \partial \mathbb{D}_{y_j}} \frac{J_R^{(1)}(\xi)}{\xi - z} \, d\xi, \tag{6.7} \]

where we recall that the circles \( \partial \mathbb{D}_{y_j}, \quad j = 1, 2, 3 \), have clockwise orientation. From (6.4), we note that the jumps \( J_R^{(1)} \) can be analytically continued from \( \bigcup_{j=1}^{3} \partial \mathbb{D}_{y_j} \) to \( \bigcup_{j=1}^{3} (\overline{\mathbb{D}}_{y_j} \setminus \{y_j\}) \), so we can evaluate (6.7) by residue calculation. The asymptotic expansion of \( J_R^{(1)}(z) \) as \( z \to y_j, \quad j = 1, 2, 3 \), is of the form

\[ J_R^{(1)}(z) = \sum_{k=0}^{N} (J_R^{(1)})_{(y_j)}^{(k)} (z - y_j)^k + O((z - y_j)^{N+1}), \]
for any \( N \in \mathbb{N}_{>0} \), and therefore (6.7) can be rewritten as

\[
R^{(1)}(z) = \sum_{j=1}^{3} \frac{1}{z - y_j} (J_R^{(1)})_{y_j}^{(-1)}, \quad z \in \mathbb{C} \setminus \bigcup_{j=1}^{3} D_{y_j}, \quad (6.8)
\]

\[
R^{(1)}(z) = -\sum_{k=0}^{\infty} (J_R^{(1)})_{y_j}^{(k)} (z - y_j)^k + \sum_{j' \neq j}^{3} \frac{1}{z - y_{j'}} (J_R^{(1)})_{y_{j'}}^{(-1)}, \quad z \in \mathbb{D}_{y_j}, j = 1, 2, 3. \quad (6.9)
\]

In particular, we have

\[
R^{(1)}(y_j) = -(J_R^{(1)})_{y_j}^{(1)} - \sum_{j' \neq j}^{3} \frac{1}{(y_j - y_{j'})^2} (J_R^{(1)})_{y_{j'}}^{(-1)}, \quad j = 1, 2, 3. \quad (6.10)
\]

An explicit expression for \( R^{(1)}(y_j) \) requires the evaluation of the matrix-coefficients appearing in (6.10). The coefficients \((J_R^{(1)})_{y_j}^{(1)}\) are particularly hard to compute and require more coefficients in the expansions of \( P^{(\infty)}(z) \) as \( z \to y_j \) than those already computed in Subsections 4.1, 4.2 and 4.3. However, it turns out that what will be needed in Section 8 is to obtain explicit expressions for

\[
\left[ E_{y_j}(y_j)^{-1} R^{(1)}(y_j) E_{y_j}(y_j) \right]_{21}. \quad (6.11)
\]

The quantities (6.11) are much easier to evaluate than the \( R^{(1)}(y_j) \) themselves, due to heavy cancellations between \( R^{(1)}(y_j) \) and the factors \( E_{y_j}(y_j)^{\pm 1} \). In particular, (6.11) can be explicitly written in terms of the first four coefficients appearing in the expansions of \( P^{(\infty)}(z) \) as \( z \to y_j \) (these coefficients were computed in Subsections 4.1, 4.2 and 4.3). Let us define

\[
\mathcal{J}_{y_j}(z) = \left[ E_{y_j}(y_j)^{-1} J_R^{(1)}(z) E_{y_j}(y_j) \right]_{21}.
\]

As \( z \to y_{j'}, j' = 1, 2, 3 \), we have

\[
\mathcal{J}_{y_j}(z) = \sum_{k=1}^{1} (\mathcal{J}_{y_j})_{y_{j'}}^{(k)} (z - y_{j'})^k + \mathcal{O}((z - y_{j'})^2),
\]

\[
(\mathcal{J}_{y_j})_{y_{j'}}^{(k)} = \left[ E_{y_j}(y_j)^{-1} (J_R^{(1)})_{y_j}^{(k)} E_{y_j}(y_{j'}) \right]_{21}, \quad (6.12)
\]

so that (6.11) can be written as

\[
\left[ E_{y_j}(y_j)^{-1} R^{(1)}(y_j) E_{y_j}(y_j) \right]_{21} = -(\mathcal{J}_{y_j})_{y_j}^{(1)} - \sum_{j' \neq j}^{3} \frac{1}{(y_{j'} - y_j)^2} (\mathcal{J}_{y_j})_{y_{j'}}^{(-1)}. \]

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After a long computation using the expansions derived in Subsections 4.1, 4.2 and 4.3, and using the expressions for $E_{y}(y)$ given in (5.8), (5.13) and (5.18), we obtain

\[
(J_{y_{1}})^{-1} = \frac{-\sqrt{y_{1}}e^{2\pi i u}(\beta_{y_{1}}^{(4)} - \frac{1}{2})^{2}G(y^{2})^{2}}{8\sqrt{y_{1}}(\beta_{y_{1}}^{(4)})^{2}G(0)^{2}}, \quad (J_{y_{1}})^{-1} = \frac{-i\sqrt{y_{1}}e^{2\pi i u}((\beta_{y_{1}}^{(4)})^{2}G(4y^{2})^{2})}{8\sqrt{y_{1}}(\beta_{y_{1}}^{(4)})^{2}G(0)^{2}}
\]

\[
(J_{y_{2}})^{-1} = \frac{-\sqrt{y_{2}}e^{2\pi i u}(\beta_{y_{2}}^{(4)} - \frac{1}{2})^{2}G(y^{2})^{2}}{8\sqrt{y_{2}}(\beta_{y_{2}}^{(4)})^{2}G(0)^{2}}, \quad (J_{y_{2}})^{-1} = \frac{-i\sqrt{y_{2}}e^{2\pi i u}((\beta_{y_{2}}^{(4)})^{2}G(4y^{2})^{2})}{8\sqrt{y_{2}}(\beta_{y_{2}}^{(4)})^{2}G(0)^{2}}
\]

\[
(J_{y_{3}})^{-1} = \frac{-\sqrt{y_{3}}e^{2\pi i u}(\beta_{y_{3}}^{(4)} - \frac{1}{2})^{2}G(y^{2})^{2}}{8\sqrt{y_{3}}(\beta_{y_{3}}^{(4)})^{2}G(0)^{2}}, \quad (J_{y_{3}})^{-1} = \frac{-i\sqrt{y_{3}}e^{2\pi i u}((\beta_{y_{3}}^{(4)})^{2}G(4y^{2})^{2})}{8\sqrt{y_{3}}(\beta_{y_{3}}^{(4)})^{2}G(0)^{2}}
\]

The coefficient $(J_{y_{1}})^{(1)}$ requires more involved computations and is given by

\[
(J_{y_{1}})^{(1)} = \frac{i\pi^{\frac{3}{2}}}{16(\beta_{y_{1}}^{(4)})^{2}G(0)^{2}} \left\{ 3c(2)(\beta_{y_{1}}^{(4)})G(0)^{2} - 2(6(\beta_{y_{1}}^{(4)})G(0)^{2} + (\beta_{y_{1}}^{(4)})^{2}G(0)^{2} + 3G(0)G''(0)) \right\}
\]

The expressions for $(J_{y_{2}})^{(1)}$ and $(J_{y_{3}})^{(1)}$ are more complicated and significantly longer than $(J_{y_{1}})^{(1)}$, so we do not write them down. In Section 7, we derive several identities involving the $\theta$-function which allows one to simplify the formulas for $(J_{y})^{(k)}$, see proof of Proposition 8.5.

7 Some identities of $\theta$-functions

The coefficients $(P^{(\infty)})^{(\frac{4}{2} + \frac{1}{2})}$ obtained in Subsections 4.1, 4.2, 4.3, the expressions for $E_{y}(y)$ and $E_{y}(y)$ given by (5.8), (5.13), (5.18), and the matrices $(J_{y})^{(k)}$ computed in Section 6, all involve the $\theta$-function and its derivatives evaluated at the 8 points

\[
0, \quad \frac{1}{2}, \quad \frac{\tau}{2}, \quad \frac{1 + \tau}{2}, \quad \nu, \quad \nu + \frac{1}{2}, \quad \nu + \frac{1 + \tau}{2}, \quad \nu + \frac{1 + \tau}{2}.
\]

In this section, we provide a systematic way of simplifying these expressions by showing that

\[
\theta^{(j)}\left(\frac{1}{2}\right), \quad \theta^{(j)}\left(\frac{\tau}{2}\right), \quad \theta^{(j)}\left(\frac{1 + \tau}{2}\right), \quad j \geq 0
\]

can be expressed in terms of $\theta(0)$, $\theta(2)(0)$, $\theta(4)(0)$, and similarly that

\[
\theta^{(j)}\left(\nu + \frac{1}{2}\right), \quad \theta^{(j)}\left(\nu + \frac{\tau}{2}\right), \quad \theta^{(j)}\left(\nu + \frac{1 + \tau}{2}\right), \quad j \geq 0
\]

can be expressed in terms of $\theta(\nu)$, $\theta'(\nu)$, $\theta''(\nu)$, $\theta'''(\nu)$, ... We start by recalling the basic symmetries of the $\theta$-function, which can be found in (4.5). By differentiating the relations $\theta(-u) = \theta(u)$ and $\theta(\frac{1}{2} - u) = \theta(\frac{1}{2} + u)$, we immediately obtain

\[
\theta'(0) = 0, \quad \theta''(0) = 0, \quad \theta'(\frac{1}{2}) = 0, \quad \theta''(\frac{1}{2}) = 0.
\]
We also recall that \( \theta(z) \) has a simple zero at \( z = \frac{4\pi i}{\tau} \). Let us define \( \tilde{\theta} \) by
\[
\tilde{\theta}(z) = \frac{\theta(z)}{z - \frac{1 + \tau}{2}},
\]
so that \( \tilde{G} \) defined in (4.25) is given by \( \tilde{G}(z) = \frac{\theta(z + \tau)}{\theta(z)} \).
By differentiating (7.3), we obtain
\[
\tilde{\theta}'(z) = \frac{i}{2} \theta(z) \left( \frac{1 + \tau}{2} \right),
\]
so that
\[
\tilde{G}(z) = \frac{\theta(z + \tau)}{\theta(z)} = e^{2\pi i z^2} \frac{\theta(z)}{z - \frac{1 + \tau}{2}}.
\]

**Remark 5.** It can be shown by standard arguments that \( \det P^{(\tau)}(z) \equiv 1 \). This gives a non-trivial relation between the \( \theta \)-function evaluated at several points. However, it turns out that this is not enough for our needs. Proposition 7.1 below states three fundamental relations between the Abel map and the \( \theta \)-function. These identities are of central importance for us and will be used extensively in the remainder of this paper.

**Proposition 7.1.** Let \( z \mapsto \varphi_A(z) \) denote the Abel map (4.11). Then, for all \( z \) on the Riemann surface,
\[
e^{2\pi i \varphi_A(z)} \frac{\theta(\varphi_A(z) + \frac{\tau}{2})^2}{\theta(\varphi_A(z))^2} = \frac{D_1^2}{z},
\]
\[
e^{2\pi i \varphi_A(z)} \frac{\theta(\varphi_A(z) + \frac{1 + \tau}{2})^2}{\theta(\varphi_A(z))^2} = \frac{D_2^2}{(z - y_1)} + \frac{D_3^2}{(z - y_2)},
\]
where
\[
D_1 = (y_1 y_2)^{\frac{1}{2}} e^{-\frac{\pi i}{y_1}}, \quad D_2 = i \frac{y_2 e^{-\frac{\pi i}{y_1}}}{(y_1 - y_2)^{\frac{1}{2}}}, \quad D_3 = \frac{y_1^{\frac{1}{2}}}{(y_1 - y_2)^{\frac{1}{2}}}.
\]

**Proof.** An easy computation using the transformation properties (4.5) of \( \theta \) show that the three quotients
\[
e^{2\pi i u} \frac{\theta(u + \frac{\tau}{2})^2}{\theta(u)^2}, \quad e^{2\pi i u} \frac{\theta(u + \frac{1 + \tau}{2})^2}{\theta(u)^2}, \quad \frac{\theta(u + \frac{1 + \tau}{2})^2}{\theta(u)^2}
\]
are invariant under the shifts \( u \mapsto u + 1 \) and \( u \mapsto u + \tau \), so that they are well-defined on the torus \( \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \). Let us first prove (7.5a). Since \( \theta(u) \) has a simple zero at \( u = \frac{1 + \tau}{2} \) and no other zeros, it follows that the quotient \( e^{2\pi i u} \frac{\theta(u + \frac{1 + \tau}{2})^2}{\theta(u)^2} \) defines a meromorphic function on \( \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) with a double zero at \( u = \frac{1 + \tau}{2} \) and a double pole at \( u = \frac{1 - \tau}{2} \) and no other zeros or poles. Recall that
\[
\varphi_A(y_1) = 0, \quad \varphi_A(y_2) = \frac{\tau}{2}, \quad \varphi_A(y_3) = \frac{1 + \tau}{2}, \quad \varphi_A(\infty) = \frac{1}{2},
\]
this implies that the left-hand side of (7.5a) is a meromorphic function on the Riemann surface with a double zero at \( z = \infty \) and a double pole at \( z = y_3 \). Since the torus is of genus 1, it follows from the Riemann-Roch theorem that the identity (7.5a) holds for some constant \( D_1^2 \in \mathbb{C} \). Analogous arguments apply to (7.5b) and (7.5c). By evaluating the identities in (7.12) at \( y_1, \infty \) and \( \infty \), respectively, we obtain
\[
D_1 = \sqrt{y_1 \frac{\theta(\frac{\tau}{2})}{\theta(0)}} \in \mathbb{R}, \quad D_2 = i \frac{\theta(\frac{\tau}{2})}{\theta(\frac{1 + \tau}{2})} \in i\mathbb{R}, \quad D_3 = \frac{\theta(0)}{\theta(\frac{1 + \tau}{2})} \in \mathbb{R}.
\]
On the other hand, by expanding (7.5a) as $z \to y_2$ and (7.5b), (7.5c) as $z \to y_1$, we obtain

$$D_1 = \sqrt{y_2 - \frac{\tau}{2} \theta(0)}, \quad D_2 = \sqrt{y_1 \varphi(y_1)} \frac{\theta'(\frac{\tau}{2})}{\theta(0)}, \quad D_3 = \frac{\sqrt{y_1 \varphi(y_1)}}{\sqrt{y_1 - y_2} \theta(0)}. \quad (7.8)$$

Comparing (7.7) with (7.8), we obtain (7.6).

We immediately obtain the following Corollary.

**Corollary 7.2.** Let $z \mapsto \varphi_A(z)$ be the Abel map defined in (4.11). Then

$$e^{2i\pi\varphi_A(z)} \theta(\varphi_A(z) + \frac{\tau}{2}) \theta(\varphi_A(z) + \frac{1+\tau}{2}) = D_1 D_2 \sqrt{\frac{z - y_1}{(z - y_2)^2}} \quad (7.9)$$

for all $z$ on the Riemann surface, where the branch is such that $\sqrt{\frac{z - y_1}{(z - y_2)^2}} > 0$ for $z > y_1$ on the upper sheet, and

$$\frac{D_1 D_2}{D_3} = i \sqrt{y_2 e^{-\frac{\tau}{2}}}. \quad (7.10)$$

**Proof.** By multiplying (7.5a) with (7.5b), and then dividing by (7.5c), we obtain

$$\left(\frac{e^{2i\pi\varphi_A(z)} \theta(\varphi_A(z) + \frac{\tau}{2}) \theta(\varphi_A(z) + \frac{1+\tau}{2})}{\theta(\varphi_A(z)) \theta(\varphi_A(z) + \frac{1}{2})}\right)^2 = \frac{D_1^2 D_2^2}{D_3^2} \frac{z - y_1}{(z - y_2)^2}. \quad (7.11)$$

By taking the square root of (7.11), we obtain (7.9), up to a multiplicative sign that can be determined from an expansion as $z \to y_1$, see [21, Chapter 20].

By taking derivatives of (7.5a)-(7.5c), we can obtain expressions for (7.1) in terms of only $\theta(0), \theta'(0), \cdots$. We summarize the relations that will be used in Section 8 in the following Corollary.

**Corollary 7.3.** We have the relations

$$\theta\left(\frac{\tau}{2}\right) = e^{-\frac{\pi}{\tau} \varphi(\frac{y_1}{y_1} \theta(0)), \quad \theta\left(\frac{1}{2}\right) = \frac{(y_1 - y_2)^\frac{1}{2}}{y_1^2} \theta(0), \quad \theta'(\frac{\tau}{2}) = -i \pi e^{-\frac{\pi}{y_1^2} \varphi(\frac{y_1}{y_1} \theta(0),$$

as well as

$$\theta'(\frac{1+\tau}{2}) = ie^{-\frac{\pi}{\tau} \varphi(\frac{(y_1 - y_2)^\frac{1}{2}}{2c_0} \theta(0),$$

$$\theta'(\frac{\tau}{2}) = e^{-\frac{\pi}{\tau} \varphi(\frac{y_1}{y_1} \theta'(0) - \left[\tau^2 + \frac{y_1 - y_2}{4c_0^2}\right] \theta(0)),$$

$$\theta'(\frac{1+\tau}{2}) = \pi e^{-\frac{\pi}{\tau} \frac{y_1}{y_1} \frac{(y_1 - y_2)^\frac{1}{2}}{c_0} \theta(0),$$

$$\theta'(\frac{1}{2}) = \frac{(y_1 - y_2)^\frac{1}{2}}{4c_0^2 y_1^2} \left[\frac{y_2 \theta(0) + 4c_0^2 \theta''(0)}{\theta''(0)}\right),$$

$$\theta'(\frac{1}{2}) = -ie^{-\frac{\pi}{\tau} \frac{y_1}{8c_0^2} \frac{(y_1 - y_2)^\frac{1}{2}}{8c_0^2} \left[(y_1 - 2y_2) \theta(0) + 12c_0^2 \left(\pi^2 \theta(0) - \theta''(0)\right)\right).$$
Finally, we note that the relations (7.5a)-(7.5c) allows also to express (7.2) in terms of \( \theta(\nu), \theta'(\nu), \theta''(\nu), \ldots \), via the following identities

\[
\theta(\nu + \frac{7gamma}{2}) = e^{-2i\pi \nu} \frac{D^2_a}{a} \theta(\nu)^2, \\
\theta(\nu + \frac{1+gamma}{2}) = e^{-2i\pi \nu} \frac{D^2_2(a - y_1)}{a} \theta(\nu)^2, \\
\theta(\nu + \frac{1}{2}) = \frac{D^2_2(a - y_2)}{a} \theta(\nu)^2, 
\]

where \( a = \varphi^{-1}(\nu) \).

### 8 Proof of Theorem 1.1

In Section 2 we obtained the following differential identity:

\[
\partial_r \log F(r\vec{x}) = \sum_{j=1}^{3} K_{j}, \quad \text{with} \quad K_{j} := \frac{(-1)^{j+1}x_j}{2i\pi} \lim_{z \to y_j} (\Psi^{-1}(\nu) \Psi'_{y_j})_{21} (rz; rx_j, r\vec{y}), \tag{8.1}
\]

where the limits as \( z \to y_j, j = 1, 2, 3, \) are taken such that \( z \in (0, y_2) \cup (y_1, +\infty) \). In this section, we use the analysis from Sections 3-6 to obtain large \( r \) asymptotics for \( K_{j}, j = 1, 2, 3 \). These asymptotics can be simplified in two steps: first, we use the numerous identities involving the \( \theta \)-function and the Abel map of Section 7, and second, we use Riemann’s bilinear identity and some other identities for elliptic integrals. Then, we integrate these asymptotics as shown in (1.15) and prove the asymptotic formula (1.10) for \( \log F(r\vec{x}) \).

**Proposition 8.1.** Let \( \vec{x} = (x_1, x_2, x_3) \) be fixed and such that \( x_3 < x_2 < x_1 < 0 \). We have

\[
\partial_r \log F(r\vec{x}) = r^2 \sum_{j=1}^{3} (-1)^{j+1}x_j c_{y_j} + \frac{1}{2i\pi r} \sum_{j=1}^{3} (-1)^{j+1}x_j \left[ E_{y_j}(y_j)^{-1} R^{(1)}(y_j) E_{y_j}(y_j) \right]_{21} \\
+ \frac{1}{2i\pi r} \sum_{j=1}^{3} (-1)^{j+1}x_j \left[ E_{y_j}(y_j)^{-1} E_{y_j}'(y_j) \right]_{21} + O\left(r^{-\frac{3}{2}}\right) \tag{8.2}
\]

as \( r \to +\infty \), where \( c_{y_j}, j = 1, 2, 3, \) are given by (5.3), (5.10) and (5.15).

**Proof.** Using the transformations \( \Psi \to T \to S \to R \) given by (3.17), (3.20), (6.1), we see that, for \( z \) inside \( \mathbb{D}_j, j = 1, 2, 3, \) but outside the lenses, we have

\[
\Psi(rz) = \begin{pmatrix} r^{\frac{1}{4}} & -2ir^{\frac{3}{4}}g_{1} \\ 0 & r^{-\frac{1}{4}} \end{pmatrix} R(z) P(y_j)(z)e^{r\tau g(z)\sigma_3}. 
\]

Thus, for \( z \) inside \( \mathbb{D}_j \) but outside the lenses, we obtain

\[
(\Psi^{-1}\Psi')(rz) = \frac{1}{r} e^{-r\tau g(z)\sigma_3} \left\{ r^{\frac{3}{2}} g'(z)\sigma_3 + P^{(y_j)}(z)^{-1} P^{(y_j)'(z)} + P^{(y_j)}(z)^{-1} R(z) P^{(y_j)(z)} \right\} e^{r\tau g(z)\sigma_3}.
\]

Consequently, for \( j = 1, 2, 3, \) we have

\[
K_{j} = \frac{(-1)^{j+1}x_j}{2i\pi r} \lim_{z \to y_j} e^{2ir\tau g(z)} \left[ P^{(y_j)}(z)^{-1} P^{(y_j)'(z)} + P^{(y_j)}(z)^{-1} R(z) P^{(y_j)(z)} \right]_{21}, \tag{8.3}
\]
where again the limits are taken such that $z \in (0,y_2) \cup (y_1, +\infty)$, and we write $R(z)$ instead of $R_+(z)$ because $R(z)$ has no jumps on $(0,y_2) \cup (y_1, +\infty)$. The expressions (5.5), (5.11), (5.16) for $P^{(y_j)}(z)$, $j = 1, 2, 3$, are of the form

$$ P^{(y_j)}(z) = E_{y_j}(z) \Phi_{Be}(r^3f_{y_j}(z)) e^{-r^2\Phi_{Be}(z)\sigma_3}, \quad j = 1, 3 $$

$$ P^{(y_2)}(z) = E_{y_2}(z)\sigma_3 \Phi_{Be}(-r^3f_{y_2}(z))\sigma_3 e^{-r^2\Phi_{Be}(z)\sigma_3}, $$

so we can write (8.3) as

$$ K_1 = \frac{\pi}{2\pi i} \lim_{z \to y_j} \left[ \Phi_{Be}(r^3f_{y_j}(z))^{-1} E_{y_j}(z)^{-1} E'_{y_j}(z) \Phi_{Be}(r^3f_{y_j}(z)) ight] 
+ \Phi_{Be}(r^3f_{y_j}(z))^{-1} \Phi'_{Be}(r^3f_{y_j}(z)) r^3 f'_{y_j}(z) - r^2 \Phi_{Be}(z) \sigma_3 
+ \Phi_{Be}(r^3f_{y_j}(z))^{-1} E_{y_j}(z)^{-1} R(z)^{-1} R'(z) E_{y_j}(z) \Phi_{Be}(r^3f_{y_j}(z)) \right]_{21}, \quad j = 1, 3,$n

$$ K_2 = \frac{\pi}{2\pi i} \lim_{z \to y_2} \left[ \sigma_3 \Phi_{Be}(-r^3f_{y_2}(z))^{-1} \sigma_3 E_{y_2}(z)^{-1} E'_{y_2}(z) \sigma_3 \Phi_{Be}(-r^3f_{y_2}(z)) \sigma_3 
+ \sigma_3 \Phi_{Be}(-r^3f_{y_2}(z))^{-1} \Phi'_{Be}(-r^3f_{y_2}(z)) \sigma_3 (-r^3f_{y_2}(z)) - r^2 \Phi_{Be}(z) \sigma_3 
+ \sigma_3 \Phi_{Be}(-r^3f_{y_2}(z))^{-1} \sigma_3 E_{y_2}(z)^{-1} R(z)^{-1} R'(z) E_{y_2}(z) \sigma_3 \Phi_{Be}(r^3f_{y_2}(z)) \sigma_3 \right]_{21}.$$n

Note that the term $-r^2\Phi_{Be}(z)\sigma_3$ does not contribute for $K_j$, since $(\sigma_3)_{21} = 0$. As $z \to y_j$ with $z$ outside the lenses, we have $(-1)^{j+1}f_{y_j}(z) \to 0$ with $|\arg((-1)^{j+1}f_{y_j}(z))| < \frac{\pi}{4}$ for each $j = 1, 2, 3$.

Thus, using (A.7), we have

$$ \lim_{z \to y_j} \left[ \Phi_{Be}((-1)^{j+1}r^3f_{y_j}(z))^{-1} \Phi'_{Be}((-1)^{j+1}r^3f_{y_j}(z)) \right]_{21} = 2\pi i, \quad j = 1, 2, 3.$$n

Moreover, $E_{y_j}(z)$ and $R(z)$ are analytic for $z \in \mathbb{D}_{y_j}$. Hence, using (A.5)-(A.7) we have

$$ K_j = \frac{(-1)^{j+1} \pi}{2\pi i} \left\{ \left[ E_{y_j}(y_j)^{-1} E'_{y_j}(y_j) \right]_{21} + 2\pi i r^3 f'_{y_j}(y_j) + \left[ E_{y_j}(y_j)^{-1} R(y_j)^{-1} R'(y_j) E_{y_j}(y_j) \right]_{21} \right\} $$

for $j = 1, 2, 3$. We also note that $f'_{y_j}(y_j) = c_{y_j}$ for $j = 1, 2, 3$. Furthermore, by (6.5) and (6.6), we have

$$ R^{-1}(z) R'(z) = (I + O(r^{-\delta})) \left( \frac{R^{(1)}(z)}{r^{\frac{\delta}{2}}} + O(r^{-\delta}) \right), \quad \text{as} \ r \to +\infty,$n

and by (5.8), (5.13) and (5.18), we have

$$ E_{y_j}(y_j) = \begin{pmatrix} O(r^{\frac{\alpha}{2}}) \\ O(r^{\frac{\delta}{2}}) \end{pmatrix}, \quad r \to +\infty, \quad j = 1, 2, 3,$n

$$ E_{y_j}(y_j)^{-1} = \begin{pmatrix} O(r^{-\frac{\alpha}{2}}) \\ O(r^{-\frac{\delta}{2}}) \end{pmatrix}, \quad r \to +\infty, \quad j = 1, 2, 3.$$n

We infer that, for $j = 1, 2, 3$,

$$ K_j = \frac{(-1)^{j+1} \pi}{2\pi i} \left\{ \left[ E_{y_j}(y_j)^{-1} E'_{y_j}(y_j) \right]_{21} + r^2 \left[ E_{y_j}(y_j)^{-1} R^{(1)}(y_j) E_{y_j}(y_j) \right]_{21} + O(r^{-\delta}) \right\}.$$n

The proposition follows by summing from $j = 1$ to $j = 3$. \qed
8.1 Evaluation of the first term

Proposition 8.2. The first term on the right-hand side of (8.2) is given by

\[ r^2 \sum_{j=1}^{3} (-1)^{j+1} x_j c_{y_j} = c \partial_r r^3, \quad (8.4) \]

where \( c \) is defined in (1.11).

Proof. By direct computation using the explicit expressions

\[ c_{y_1} = \frac{p(y_1)^2}{y_1(y_1 - y_2)}, \quad c_{y_2} = \frac{p(y_2)^2}{y_2(y_1 - y_2)}, \quad c_{y_3} = \frac{p(y_3)^2}{y_1 y_2}, \]

and the definition of \( p(z) \) given by (3.3), we obtain

\[ r^2 \sum_{j=1}^{3} (-1)^{j+1} x_j c_{y_j} = \frac{1}{3} \left( \frac{x_1 p(y_1)^2}{y_1(y_1 - y_2)} - \frac{x_2 p(y_2)^2}{y_2(y_1 - y_2)} + \frac{x_3 p(y_3)^2}{y_1 y_2} \right) \partial_r r^3 \]

\[ = \frac{1}{24} \left( (x_1 - x_2)^2 (x_1 + x_2 - x_3) + 2x_1^3 + 8g_1(x_1 + x_2 + x_3) \right) \partial_r r^3. \quad (8.5) \]

We note that \( Q, q \) defined in (1.4), (1.5) have the relations with \( R, p \)

\[ Q(z) = R(z - x_3) \quad \text{and} \quad q(z) = p(z - x_3). \]

Therefore, \( g_1 \) given by (3.4) and \( q_0 \) given by (1.6) are related by

\[ g_1 = q_0 - \frac{2x_1 x_2 + 2x_1 x_3 + 2x_2 x_3 - x_1^2 - x_2^2}{8}. \quad (8.6) \]

We obtain (8.4) after substituting (8.6) into (8.5).

\[ \square \]

8.2 Evaluation of the second term

Proposition 8.3. The second term on the right-hand side of (8.2) is given by

\[ \frac{1}{2 \pi i r} \sum_{j=1}^{3} (-1)^{j+1} x_j \left[ E_{y'_j}(y_j)^{-1} R^{(1)r}(y_j) E_{y'_j}(y_j) \right]_{21} \]

\[ = \frac{1}{2 \pi i r} \sum_{j=1}^{3} (-1)^{j} x_j (\mathcal{J}_{y'_j})^{(1)}(y_j) \quad + \quad \frac{1}{2 \pi i r} \sum_{j=1}^{3} (-1)^{j} x_j \sum_{j' = 1}^{3} \left( \frac{1}{(y_j - y_{j'})} \mathcal{J}_{y_j}(y_{j'})^{-1} \right). \quad (8.7) \]

Proof. The claim follows after substituting the expression (6.10) for \( R^{(1)r}(y_j) \) into the left-hand side of (8.7) and by using the definition (6.12) of \( (\mathcal{J}_{y'_j})^{(1)}(y_j) \).

\[ \square \]

Proposition 8.4. The first term on the right-hand side of (8.7) is given by

\[ \frac{1}{2 \pi i r} \sum_{j=1}^{3} (-1)^{j} x_j (\mathcal{J}_{y'_j})^{(1)}(y_j) = - \frac{1}{16} \sum_{j=1}^{3} \frac{x_j p'(y_j)}{p(y_j)} \partial_r \log r. \quad (8.8) \]
Proof. This follows from involved computations which are quite long and thus will be omitted here. We use the identities of Corollary 7.3 to express \( \theta^{(j)}(\frac{1}{2}) \), \( \theta^{(j)}(\frac{1}{4}) \) and \( \theta^{(j)}(\frac{1}{2}+\frac{1}{2}) \), \( j = 0, 1, 2, 3 \), in terms of only \( \theta^{(j)}(0) \), \( j = 0, 2 \). We also use the \( j \)-th derivative with respect to \( \nu \) of (7.12a)-(7.12c), \( j = 0, 1, 2, 3 \), to express \( \theta^{(j)}(\nu+\frac{1}{2}) \), \( \theta^{(j)}(\nu+\frac{1}{2}+\frac{1}{2}) \) and \( \theta^{(j)}(\nu+\frac{1}{2}+\frac{1}{2}) \), \( j = 0, 1, 2, 3 \), in terms of only \( \theta^{(j)}(\nu) \), \( j = 0, 1, 2, 3 \). Remarkably, the final expression on the right-hand side of (8.8) does not contain the \( \theta \)-function.

\[ \theta_j = 0 \]

Definition 1. Let \( \varphi_c^{-1}(u) \) be the projection into the complex plane of \( \varphi_A^{-1}(u) \). We define \( a = a(r) \) by

\[ a := \varphi_c^{-1}(\nu) = \varphi_A^{-1}(-\frac{\Omega}{2\pi r^2}). \tag{8.9} \]

From the relations \( \varphi_c^{-1}(u) = \varphi_c^{-1}(-u) \), \( \varphi_c^{-1}(u+1) = \varphi_c^{-1}(u) \) and \( \Omega > 0 \), we conclude that the function \( r \mapsto a(r) \) is oscillatory as \( r \to +\infty \). Furthermore, we see that \( \varphi_A(x) \) is monotone for \( x \in (y_1, +\infty) \), which follows from direct inspection of (4.6). Since \( \varphi_A(y_1) = 0 \) and \( \varphi_A(\infty) = \frac{1}{2} \), we conclude that \( a(r) \) is bigger than \( y_1 \) for any \( r \) and oscillates between \( y_1 \) and \( +\infty \).

Proposition 8.5. The second term on the right-hand side of (8.7) is given by

\[ \frac{1}{2\pi ir^2} \sum_{j=1}^{3} (-1)^j x_j \sum_{j' \neq j}^{3} \frac{1}{(y_j - y_{j'})^2} (J_{y_j})(y_{j'})^{-1} = n(0) \partial_r \log r + n(-1) \partial_r \int_M \frac{d\hat{r}}{r a(\hat{r})}. \tag{8.10} \]

where \( a = a(r) \) is oscillatory and given by (8.9), \( M > 0 \) is a large constant, and \( n(-1), n(0) \) are given by

\[ n(-1) = -\frac{1}{32} \left( \frac{1}{p(y_1)} + \frac{1}{p(y_2)} + \frac{1}{p(y_3)} \right) (x_1 - x_3)(x_2 - x_3)(x_1 + x_2 + x_3), \tag{8.11} \]

\[ n(0) = -\frac{x_1 y_2 p(y_1) - x_2 y_1 p(y_2)}{16 p(y_3)(x_1 - x_2)} - \frac{x_3 p(y_3)}{16} \left( \frac{1}{y_1 p(y_1)} + \frac{1}{y_2 p(y_2)} \right). \tag{8.12} \]

Proof. By using Proposition 3.1, Corollary 7.3 together with the derivatives with respect to \( \nu \) of (7.12a)-(7.12c), we can simplify the quantities obtained at the end of Section 6 for \( (J_{y_j})(y_{j'}) \), \( 1 \leq j \neq j' \leq 3 \), as follows

\[ (J_{y_j})(y_2) = -\frac{\pi i}{8} \frac{(y_1 - y_2) y_2 p(y_1) r^2}{p(y_2)} \frac{r^2}{a}, \]

\[ (J_{y_j})(y_3) = \frac{\pi i}{8} \frac{y_1 y_2 p(y_1) r^2 (a - y_1)}{p(y_3)} \frac{r^2}{a}, \]

\[ (J_{y_2})(y_3) = -\frac{\pi i}{8} \frac{y_1 y_2 p(y_1) r^2 (a - y_2)}{p(y_3)} \frac{r^2}{a}. \]

where \( a = a(r) \) depends on \( r \) and is given by (8.9). Then, a long but straightforward computation gives

\[ \frac{1}{2\pi i r^2} \sum_{j=1}^{3} (-1)^j x_j \sum_{j' \neq j}^{3} \frac{1}{(y_j - y_{j'})^2} (J_{y_j})(y_{j'})^{-1} = n(0) \partial_r \log r + n(-1) \partial_r \int_M \frac{d\hat{r}}{r a(\hat{r})}. \]

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where \( n^{(0)} \) is given by (8.12) and where \( n^{(-1)} \) is given by

\[
n^{(-1)} = \frac{1}{16} \left( \frac{1}{p(y_1)} + \frac{1}{p(y_2)} + \frac{1}{p(y_3)} \right) \left( \frac{x_1 p(y_1) x_2 - x_3}{x_1 - x_2} - \frac{x_2 p(y_2) x_1 - x_3}{x_1 - x_2} + x_3 p(y_3) \right).
\]

Substituting the explicit expression (3.3) for \( p_j \) leads to further simplifications and we find (8.11). □

The next proposition shows that \( \int_M \frac{d\bar{r}}{r a(\bar{r})} \) is proportional to \( \log r \) and contains no oscillations of order 1.

**Proposition 8.6.** As \( r \to +\infty \), we have

\[
\int_M \frac{d\bar{r}}{r a(\bar{r})} = 2 \int_{y_1}^{+\infty} \frac{\varphi'(x)}{x} dx \log r + \bar{C} + O(r^{-1}),
\]

where \( \bar{C} \) is a constant independent of \( r \).

**Proof.** Let \( r_0 > M \) be a large constant smaller than \( r \) and such that \( a(r_0) = y_1 \). Since \( \Omega > 0 \), the values of \( r \geq r_0 \) for which \( a(r) = +\infty \) or \( a(r) = y_1 \) are given by

\[
r_j := \left( y_0 + \frac{j \pi}{\Omega} \right)^{\frac{2}{3}}, \quad j \geq 0.
\]

Let us write

\[
r = \left( y_0 + \frac{k \pi}{\Omega} + \lambda \frac{\pi}{\Omega} \right)^{\frac{2}{3}}, \quad \text{where} \quad k \in \mathbb{N} \quad \text{and} \quad 0 \leq \lambda < 2,
\]

so that

\[
\int_M \frac{d\bar{r}}{r a(\bar{r})} = \int_M \frac{d\bar{r}}{r a(\bar{r})} + \sum_{j=0}^{k-1} \left[ \int_{r_{2j}}^{r_{2j+2}} \frac{d\bar{r}}{r a(\bar{r})} + \int_{r_{2j+1}}^{r_{2j+2}} \frac{d\bar{r}}{r a(\bar{r})} \right] + \int_{r_{2k}}^{r_{2k+1}} \frac{d\bar{r}}{r a(\bar{r})}.
\]

From the change of variables \( \nu = \frac{\Omega}{2r^2} \) and using the notation \( \nu_j = -\frac{\Omega}{2r^2} r_j \), we obtain

\[
\int_{r_{2j}}^{r_{2j+2}} \frac{d\bar{r}}{r a(\bar{r})} = \frac{2}{3} \int_{r_{2j}}^{r_{2j+2}} \frac{d\nu}{\nu \varphi_C^{-1}(\nu)} = \frac{2}{3} \int_{r_{2j}}^{r_{2j+2}} \frac{1}{\nu_j} \left( 1 + O \left( \frac{\nu_j - \nu_{2j}}{\nu_j} \right) \right) \frac{d\nu}{\varphi_C^{-1}(\nu)}
\]

\[
= -4\pi \frac{\Omega^2}{3} \left( \int_{\nu_j}^{\nu_{2j+1}} \frac{d\nu}{\varphi_C^{-1}(\nu)} + O \left( \nu_{2j+1} - \nu_j \nu_j \right) \right) = -4\pi \frac{\Omega^2}{3} \left( \int_{\nu_j}^{\nu_{2j+1}} \frac{d\nu}{\varphi_C^{-1}(\nu)} + O(r_{2j}^{-1}) \right)
\]

as \( r_{2j} \to +\infty \). Noting that \( \nu_{2j+1} = \nu_{2j} + \frac{\Omega}{r_{2j}^2} \varphi_C^{-1}(\nu_{2j+1}) = +\infty \), and \( \varphi_C^{-1}(\nu) \) is strictly increasing for \( \nu \) in the oriented segment \((\nu_{2j}, \nu_{2j+1})\), we use the change of variables \( x = \varphi_C^{-1}(\nu) \) to obtain

\[
\int_{r_{2j}}^{r_{2j+2}} \frac{d\bar{r}}{r a(\bar{r})} = 4\pi \frac{\Omega^2}{3} \left( \int_{y_1}^{+\infty} \frac{\varphi'(x)}{x} dx + O(r_{2j}^{-1}) \right), \quad j = 0, \ldots, k-1.
\]

Similarly,

\[
\int_{r_{2j+1}}^{r_{2j+2}} \frac{d\bar{r}}{r a(\bar{r})} = 4\pi \frac{\Omega^2}{3} \left( \int_{y_1}^{+\infty} \frac{\varphi'(x)}{x} dx + O(r_{2j+1}^{-1}) \right), \quad j = 0, \ldots, k-1.
\]
As \( r \to +\infty \),
\[
\sum_{j=0}^{2k-1} r_j^{-2} = \frac{\Omega}{\pi} \log k + d_0 + O(k^{-1}) = \frac{3\Omega}{2\pi} \log r + d_1 + O(r^{-\frac{3}{2}}),
\]
for certain constants \( d_0, d_1, d_2 \), so we have
\[
\sum_{j=0}^{k-1} \left( \int_{r_{2j}}^{r_{2j+1}} \frac{d\tilde{r}}{\tilde{r} a(\tilde{r})} + \int_{r_{2j+1}}^{r_{2j+2}} \frac{d\tilde{r}}{\tilde{r} a(\tilde{r})} \right) = 2 \int_{y_1}^{+\infty} \frac{\varphi'(x)}{x} dx + O(r^{-1})
\]
as \( r \to +\infty \). Since \( a(\tilde{r}) \geq y_1 \) for all \( \tilde{r} \),
\[
\int_{r_{2k}}^{r_{2k+1}} \frac{d\tilde{r}}{\tilde{r} a(\tilde{r})} = O(r^{-1}), \quad \text{as } r \to +\infty,
\]
and the proof is completed. \( \square \)

By combining Propositions 8.3, 8.4, 8.5, 8.6, as \( r \to +\infty \) we have
\[
\int_r^\infty \frac{1}{2\pi i^n} \sum_{j=1}^3 (-1)^{i+1} x_j \left[ E_{y_j}(y_j)^{-1} R^{(1)}(y_j) E_{y_j}(y_j) \right]_2 d\tilde{r} = c_2 \log r + \tilde{C}_2 + O(r^{-1}), \tag{8.14}
\]
where \( M, \tilde{C}_2 \) are independent of \( r \), and \( c_2 \) is given by
\[
c_2 = -\frac{1}{16} \sum_{j=1}^3 \frac{x_j p'(y_j)}{p(y_j)} + n^{(0)} + 2n^{(-1)} \int_{y_1}^{+\infty} \frac{\varphi'(x)}{x} dx. \tag{8.15}
\]
This expression for \( c_2 \) has been obtained after considerable simplifications and does not involve the \( \theta \)-function. Surprisingly, it is possible to simplify \( c_2 \) significantly more. In fact, it holds that
\[
c_2 = \frac{1}{2}. \tag{8.16}
\]
Recall that \( g_1 \), which appears in the definition of \( p \), is a ratio of elliptic integrals, as can be seen from (3.4). We summarize the necessary relations that are needed to prove (8.16) in Lemma 8.7.

**Lemma 8.7.** We have the following identities:
\[
\int_{y_1}^{+\infty} \frac{dx}{x \sqrt{x^2 - y_2 \sqrt{x - y_1}}} = \int_0^{y_2} \frac{y - y_2}{(y - y_1) \sqrt{y \sqrt{y_2 - y} \sqrt{y_2 - y}}} \, dy, \tag{8.17}
\]
\[
\int_0^{y_2} \frac{y^2 - 2y_1 y + y_1 y_2 \, dy}{y - y_1 \sqrt{y \sqrt{y_2 - y} \sqrt{y_2 - y}}} = 0, \tag{8.18}
\]
\[
\int_0^{y_2} \frac{6y^3 - 3(3y_1 + y_2) y^2 + 2y_1 y + 2y_2 y - y_1 y_2 (y_1 - y_2) \, dy}{3(y - y_1) \sqrt{y \sqrt{y_2 - y} \sqrt{y_2 - y}}} = 0. \tag{8.19}
\]
**Proof.** Let \( f \) be a smooth and bounded function on \((y_1, +\infty)\). A direct computation using the changes of variables
\[
x = y_1 + (y_1 - y_2) \frac{y - y_3}{y_2 - y}, \quad x \in (y_1, +\infty), \quad y \in (y_3, y_2),
\]

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shows that
\[
\int_{y_1}^{+\infty} f(x) \frac{dx}{\sqrt{\sqrt{x} - y_2 \sqrt{x - y_1}}} = \int_{y_3}^{y_2} f(y_1 + (y_1 - y_2) \frac{y - y_3}{y_2 - y}) \frac{dy}{\sqrt{\sqrt{y_2} - y \sqrt{y_1 - y}}}.
\] (8.20)

Applying (8.20) to \( f(x) = \frac{1}{x} \) gives (8.17). Equations (8.18)-(8.19) are obtained by primitives, using
\[
y^2 - 2y_1 y + y_1 y_2 \quad \frac{1}{y - y_1} \sqrt{\sqrt{y_2} - y \sqrt{y_1 - y}} = \partial_y \left( \frac{2y(y - y_2)}{\sqrt{(y - y_1)(y - y_2)}} \right),
\]
\[
\frac{6y^3 - 3(3y_1 + y_2)y^2 + 2y_1(y_1 + y_2)y - y_1y_2(y_1 - y_2)}{3(y - y_1)\sqrt{\sqrt{y_2} - y \sqrt{y_1 - y}}} = \partial_y \left( \frac{2y(y - y_2)(2y - y_1 + y_2)}{3\sqrt{(y - y_1)(y - y_2)}} \right).
\]

\[\Box\]

**Proposition 8.8.** We have \( c_2 = -\frac{1}{2} \).

**Proof.** We first use (8.17) to rewrite the right-hand side of (8.15) only in terms of elliptic integral over the interval \([0,y_2]\). Then a long computation (we omit the details) using (8.18) and (8.19) gives \( c_2 = -\frac{1}{2} \). \[\Box\]

### 8.3 Evaluation of the third term

We now consider the evaluation of the third term in (8.2) given by
\[
\frac{1}{2\pi i \nu} \sum_{j=1}^{3} (-1)^{j+1} x_j \left[ E_{y_j}(y_j)^{-1} E_{y_j}'(y_j) \right]_{21}.
\] (8.21)

**Proposition 8.9 (Contribution from \( y_1 \)).** The first term in the sum (8.21) is given by
\[
\frac{x_1}{2\pi i \nu} \left[ E_{y_1}(y_1)^{-1} E_{y_1}'(y_1) \right]_{21} = \frac{8\pi c_0 x_1 p(y_1)}{3\Omega y_1(y_1 - y_2)} \frac{\partial}{\partial \nu} \log(\theta(\nu)).
\]

**Proof.** A long but direct computation using (5.8) and \( \theta'(0) = 0 \) gives
\[
\frac{x_1}{2\pi i \nu} \left[ E_{y_1}(y_1)^{-1} E_{y_1}'(y_1) \right]_{21} = -x_1 \sqrt{\sqrt{c_{y_1} \nu \ell_1}} \frac{\theta'(\nu)}{\theta(\nu)} = \frac{2c_0 \sqrt{\ell_1} x_1 p(y_1)}{y_1(y_1 - y_2)} \frac{\theta'(\nu)}{\theta(\nu)}.
\]

Recalling that \( \nu = -\frac{\Omega}{\pi \sqrt{\nu}} \), the claim follows. \[\Box\]

The second and third terms in the sum (8.21) are significantly more complicated to analyze than the first term computed in Proposition 8.9 for two reasons: 1) they require several identities derived in Section 7 (while the first term only uses \( \theta'(0) = 0 \)), and 2) the related computations are significantly longer.

**Proposition 8.10 (Contribution from \( y_2 \)).** The second term in the sum (8.21) is given by
\[
\frac{-x_2}{2\pi i \nu} \left[ E_{y_2}(y_2)^{-1} E_{y_2}'(y_2) \right]_{21} = -\frac{8\pi}{3\Omega} \frac{x_2 p(y_2) c_0}{y_2} \frac{\partial}{\partial \nu} \log(\theta(\nu)).
\]

**Proof.** The proof uses Propositions 3.1, 7.1 and Corollaries 7.2, 7.3. After long computations and considerable simplifications, we obtain
\[
\frac{-x_2}{2\pi i \nu} \left[ E_{y_2}(y_2)^{-1} E_{y_2}'(y_2) \right]_{21} = \frac{2c_0 \sqrt{\ell_2} x_2 p(y_2)}{y_1(y_1 - y_2)} \frac{\theta'(\nu)}{\theta(\nu)}.
\]

The claim follows as in Proposition 8.9 by using \( \nu = -\frac{\Omega}{\pi \sqrt{\nu}} \). \[\Box\]

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Proposition 8.11. (Contribution from $y_3$). The third term in the sum (8.21) is given by

$$\frac{x_3}{2\pi i r} \left[ E_{y_3}(y_3)^{-1} E'_{y_3}(y_3) \right]_{21} = \frac{8\pi x_3 p(y_3) c_0}{3\Omega y_1 y_2} \partial_r \log \theta(v).$$

Proof. The proof is similar to the proof of Proposition 8.10, and uses also Propositions 3.1, 7.1 and Corollaries 7.2, 7.3.

It follows from Propositions 8.9, 8.10 and 8.11 that

$$\frac{1}{2\pi i r} \sum_{j=1}^{3} (-1)^{j+1} x_j \left[ E_{y_j}(y_j)^{-1} E'_{y_j}(y_j) \right]_{21} = c_3 \partial_r \log \theta(v), \quad (8.22)$$

where the constant $c_3$ is given by

$$c_3 = \frac{8\pi c_0}{3\Omega} \left( \frac{x_1 p(y_1)}{(x_1 - x_3)(x_1 - x_2)} - \frac{x_2 p(y_2)}{(x_2 - x_3)(x_1 - x_2)} + \frac{x_3 p(y_3)}{(x_1 - x_3)(x_2 - x_3)} \right). \quad (8.23)$$

The three terms of the sum appearing on the left-hand side of (8.22) have been simplified separately in Propositions 8.9, 8.10 and 8.11, so that the expression (8.23) for $c_3$ does not involve the $\theta$-function. Remarkably, these three terms combine together and it possible to further simplify $c_3$. The tools needed to simplify $c_3$ are of a different nature than those needed to simplify $c_2$ in (8.7). In Proposition 8.12 below, we show that $c_3 = 1$ by using Riemann’s bilinear identity.

Proposition 8.12. We have $c_3 = 1$.

Proof. First, we use the explicit expression (3.3) for $p$ to note that (8.23) can be simplified into

$$c_3 = -\frac{4\pi c_0}{3\Omega} (x_1 + x_2 + x_3). \quad (8.24)$$

We consider the following one-forms on the Riemann surface $X$:

$$\omega_1 = \frac{dz}{\sqrt{R(z)}}, \quad \omega_2 = \frac{(p(z) + g_1)dz}{\sqrt{R(z)}}.$$

Let us deform the cycle $B$ of Figure 3 such that it has finite length and surrounds the interval $[y_2, y_1]$ in the positive direction. Riemann’s bilinear identity gives the relation

$$\int_A B_2 - A_2 B_1 = 2\pi i \sum_{p=y_2, y_1} \text{Res} \left( \omega_2 \int_\infty^P \omega_1; P = p \right),$$

where

$$A_1 = \int_A \omega_1 = 2 \int_0^{y_2} \frac{dx}{\sqrt{R(x)}}, \quad B_1 = \int_B \omega_1 = 2i \int_{y_2}^{y_1} \frac{dx}{\sqrt{R(x)}},$$

$$A_2 = \int_A \omega_2 = 2 \int_0^{y_2} \frac{(p(x) + g_1)dx}{\sqrt{R(x)}}, \quad B_2 = \int_B \omega_2 = 2i \int_{y_2}^{y_1} \frac{(p(x) + g_1)dx}{\sqrt{R(x)}}.$$

A simple computation shows that the Abelian differential

$$\omega_2 \int_\infty^P \omega_1$$

is given by

$$\frac{8\pi c_0}{3\Omega} (x_1 + x_2 + x_3).$$
has no residue at \( y_3, y_2, y_1 \), and has a residue at \( \infty \) given by \(-\frac{4\pi i}{3}(x_1 + x_2 + x_3)\) in the local coordinate. Therefore, we have

\[ A_1 B_2 - A_2 B_1 = -\frac{4\pi i}{3}(x_1 + x_2 + x_3). \] (8.25)

Using the definition of \( g_1, \Omega \) and \( c_0, \) it is a direct computation to verify from (8.24) that the claim \( c_3 = 1 \) is equivalent to (8.25).

Theorem 1.1 follows directly by combining Propositions 8.1, 8.2, 8.8 and 8.12, together with equations (8.14) and (8.22).

A Bessel Model RH problems

In this section, we recall the Bessel model RH problem from [20], whose solution is denoted by \( \Phi_{Be} \).

(a) \( \Phi_{Be} : \mathbb{C} \setminus \Sigma_{Be} \to \mathbb{C}^{2 \times 2} \) is analytic, where \( \Sigma_{Be} \) is shown in Figure 6.

(b) \( \Phi_{Be} \) satisfies the jump conditions

\[
\begin{align*}
\Phi_{Be,+}(z) &= \Phi_{Be,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in \mathbb{R}^-; \\
\Phi_{Be,+}(z) &= \Phi_{Be,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z \in e^{\frac{2\pi i}{3}} \mathbb{R}^+, \\
\Phi_{Be,+}(z) &= \Phi_{Be,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z \in e^{-\frac{2\pi i}{3}} \mathbb{R}^+.
\end{align*}
\] (A.1)

(c) As \( z \to \infty, z \notin \Sigma_{Be} \), we have

\[
\Phi_{Be}(z) = (2\pi z^{\frac{1}{2}})^{-\frac{2\pi}{3}} M \left(I + \frac{\Phi_{Be,1}}{z^{\frac{1}{2}}} + O(z^{-1})\right) e^{2z^{\frac{1}{2}} \pi i},
\] (A.2)

where \( \Phi_{Be,1} = \frac{1}{16} \begin{pmatrix} -1 & -2i \\ 2i & 1 \end{pmatrix} \) and \( M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \).

(d) As \( z \) tends to 0, the behavior of \( \Phi_{Be}(z) \) is

\[
\Phi_{Be}(z) = \begin{cases} 
(\mathcal{O}(1) \quad \mathcal{O}(\log z)) \\
(\mathcal{O}(\log z) \quad \mathcal{O}(\log z))
\end{cases}, \quad |\arg z| < \frac{2\pi}{3},
\] (A.3)

\[
\Phi_{Be}(z) = \begin{cases} 
\frac{I_0(2z^{\frac{1}{2}})}{2\pi i z^{\frac{1}{2}} H'_0(2z^{\frac{1}{2}})} - \frac{1}{2} K_0(2z^{\frac{1}{2}}), & |\arg z| < \frac{2\pi}{3}, \\
\frac{1}{\pi z^{\frac{1}{2}}} H_0^{(1)}(2z^{\frac{1}{2}}) & \frac{1}{\pi z^{\frac{1}{2}}} H_0^{(2)}(2z^{\frac{1}{2}}), & \frac{2\pi}{3} < |\arg z| < \pi,
\end{cases}
\] (A.4)

The unique solution to this RH problem is given by

\[
\Phi_{Be}(z) = \begin{cases} 
\left( \begin{array}{cc} I_0(2z^{\frac{1}{2}}) & \frac{1}{2} K_0(2z^{\frac{1}{2}}) \\
2\pi i z^{\frac{1}{2}} H_0'(2z^{\frac{1}{2}}) & -2\pi z^{\frac{1}{2}} K_0'(2z^{\frac{1}{2}}) \end{array} \right), & |\arg z| < \frac{2\pi}{3}, \\
\frac{1}{\pi z^{\frac{1}{2}}} H_0^{(1)}(2z^{\frac{1}{2}}) & \frac{1}{\pi z^{\frac{1}{2}}} H_0^{(2)}(2z^{\frac{1}{2}}), & \frac{2\pi}{3} < |\arg z| < \pi,
\end{cases}
\]

\[
\begin{align*}
\frac{1}{\pi z^{\frac{1}{2}}} H_0^{(1)}(2z^{\frac{1}{2}}) & \frac{1}{\pi z^{\frac{1}{2}}} H_0^{(2)}(2z^{\frac{1}{2}}), & -\pi < \arg z < \frac{-2\pi}{3},
\end{align*}
\]
where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of the first and second kind, and $I_0$ and $K_0$ are the modified Bessel functions of the first and second kind.

For certain computations, we will need a more precise expansion than (A.3). By [21, Section 10.30(i)], as $z \to 0$ we have

\[
\Phi_{Bc}(z) = \left( 1 + O(z) \right) \frac{O(\log z)}{O(1)}, \quad z \to 0, \quad |\arg z| < \frac{2\pi}{3},
\]

(A.5)

\[
\Phi_{Bc}(z)^{-1} = \left( O(1) \frac{O(\log z)}{-1 + O(z)} \right), \quad z \to 0, \quad |\arg z| < \frac{2\pi}{3},
\]

(A.6)

\[
\Phi_{Bc}(z)^{-1}\Phi_{Bc}'(z) = \left( \frac{O(\log z)}{-2\pi i + O(z)} \frac{O(z^{-1})}{O(\log z)} \right), \quad z \to 0, \quad |\arg z| < \frac{2\pi}{3}.
\]

(A.7)

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