DIFFERENTIAL IDENTITIES OF FINITE DIMENSIONAL ALGEBRAS AND POLYNOMIAL GROWTH OF THE CODIMENSIONS

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Abstract. Let $A$ be a finite dimensional algebra over a field $F$ of characteristic zero. If $L$ is a Lie algebra acting on $A$ by derivations, then such an action determines an action of its universal enveloping algebra $U(L)$. In this case we say that $A$ is an algebra with derivation or an $L$-algebra.

Here we study the differential $L$-identities of $A$ and the corresponding differential codimensions, $c_n^L(A)$, when $L$ is a finite dimensional semisimple Lie algebra. We give a complete characterization of the corresponding ideal of differential identities in case the sequence $c_n^L(A)$, $n = 1, 2, \ldots$, is polynomially bounded. Along the way we determine up to PI-equivalence the only finite dimensional $L$-algebra of almost polynomial growth.

1. Introduction

This paper deals with the differential identities of algebras over a field $F$ of characteristic zero. Recall that if $A$ is an associative algebra over $F$ and $L$ is a Lie algebra of derivations of $A$ then the action of $L$ on $A$ can be extended to the action of its universal enveloping algebra $U(L)$ and in this case, we say that $A$ is an algebra with derivations or an $L$-algebra.

When studying the polynomial identities of an $L$-algebra $A$, one is lead to consider $\text{var}^L(A)$, the $L$-variety of algebras with derivations generated by $A$, that is, the class of $L$-algebras satisfying all differential identities satisfied by $A$. Also, as in the ordinary case, for every $n \geq 1$, we consider the space $P_n^L(A)$ of multilinear differential polynomials in $n$ fixed variables modulo $\text{Id}^L(A)$, the ideal of differential identities of $A$. The dimension $c_n^L(A)$ of $P_n^L(A)$ is called the $n$th differential codimension of $A$ and the growth of $\mathcal{V} = \text{var}^L(A)$ is the growth of the sequence $c_n^L(\mathcal{V}) = c_n^L(A)$, $n = 1, 2, \ldots$. If $A$ is a finite dimensional $L$-algebra, in [6] it was proved by Gordienko that $c_n^L(A)$ is exponentially bounded and also no intermediate growth is allowed.

We say that $\mathcal{V}$ has almost polynomial growth if $c_n^L(\mathcal{V})$ cannot be bounded by any polynomial function but every proper subvariety $\mathcal{U}$ of $\mathcal{V}$ has polynomial growth, i.e., $c_n^L(\mathcal{U})$ is polynomially bounded. An example of algebra with derivations generating an $L$-variety with almost polynomial growth is $UT^2_F$, the algebra of $2 \times 2$ upper triangular matrices with $F\varepsilon$-action, where $\varepsilon$ is the inner derivation induced by $2^{-1}(e_{11} - e_{22})$ where $e_{ij}$’s are the usual matrix units (see [8]).

Our purpose here is to characterize $L$-varieties having polynomial growth and we reach our goal in the setting of varieties generated by finite dimensional $L$-algebras.
A where \( L \) is a finite dimensional semisimple Lie algebra. In this situation, we prove that \( \mathcal{V} \) has almost polynomial growth if and only if \( UT_\mathbb{Z} \notin \mathcal{V} \). As a consequence, there is only one variety with derivations generated by a finite dimensional algebra with almost polynomial growth. We also give three different characterizations of \( L \)-varieties \( \mathcal{V} \) of polynomial growth: the first one in terms of the \( L \)-exponent of \( \mathcal{V} \), the second in terms of the decomposition of the \( n \)th cocharacter of \( \mathcal{V} \) and the last one in terms of the structure of an algebra generating \( \mathcal{V} \).

We remark that our characterizations are motivated by the results obtained by Giambruno et al. in [2] concerning the graded case and also by the results obtained by Giambruno and Mishchenko in [1] concerning the involution case.

2. Preliminaries

Throughout this paper \( F \) will denote a field of characteristic zero. Let \( A \) be an associative algebra over \( F \). Recall that a derivation of \( A \) is a linear map \( \partial : A \to A \) such that

\[
\partial(xy) = \partial(x)y + x\partial(y), \quad \forall x, y \in A.
\]

In particular an inner derivation induced by \( x \in A \) is the derivation \( \text{ad } x : A \to A \) of \( A \) defined by \( (\text{ad } x)(y) = [x, y] \), for all \( y \in A \). The set of all derivations of \( A \) is a Lie algebra denoted by \( \text{Der}(A) \), and the set \( \text{ad}(A) \) of all inner derivations of \( A \) is a Lie subalgebra of \( \text{Der}(A) \).

Let \( L \) be a Lie algebra over \( F \) acting on \( A \) by derivations. If \( U(L) \) is its universal enveloping algebra, the \( L \)-action on \( A \) can be naturally extended to an \( U(L) \)-action. In this case we say that \( A \) is an algebra with derivations or an \( L \)-algebra.

Given a basis \( B = \{ h_i : i \in I \} \) of \( U(L) \), we let \( F\langle X|L \rangle \) be the free associative algebra over \( F \) with free formal generators \( x_{h_i}^j, \, i \in I, \, j \in \mathbb{N} \). We write \( x_i = x_i^1, \, 1 \in U(L) \), and then we set \( X = \{ x_1, x_2, \ldots \} \). We let \( U(L) \) act on \( F\langle X|L \rangle \) by setting

\[
h(x_{j_1}^{h_{i_1}}x_{j_2}^{h_{i_2}} \cdots x_{j_n}^{h_{i_n}}) = x_{j_1}^{h_{i_1}}x_{j_2}^{h_{i_2}} \cdots x_{j_n}^{h_{i_n}} + \cdots + x_{j_1}^{h_{i_1}}x_{j_2}^{h_{i_2}} \cdots x_{j_n}^{h_{i_n}},
\]

where \( h \in L \) and \( x_{j_1}^{h_{i_1}}x_{j_2}^{h_{i_2}} \cdots x_{j_n}^{h_{i_n}} \in F\langle X|L \rangle \). The algebra \( F\langle X|L \rangle \) is called the free associative algebra with derivations on the countable set \( X \) and its elements are called differential polynomials (see [3, 7, 10]).

A polynomial \( f(x_1, \ldots, x_n) \in F\langle X|L \rangle \) is a polynomial identity with derivation of \( A \), or a differential identity of \( A \), if \( f(a_1, \ldots, a_n) = 0 \) for all \( a_i \in A \), and, in this case, we write \( f \equiv 0 \). We denote by

\[
\text{Id}^L(A) = \{ f \in F\langle X|L \rangle : f \equiv 0 \text{ on } A \},
\]

the \( T_L \)-ideal of differential identities of \( A \), i.e., \( \text{Id}^L(A) \) is an ideal of \( F\langle X|L \rangle \) invariant under the \( U(L) \)-action. In characteristic zero \( \text{Id}^L(A) \) is completely determined by its multilinear polynomials and for every \( n \geq 1 \) we denote by

\[
P_n^L = \text{span}\{x_1^{h_{i_1}}x_2^{h_{i_2}} \cdots x_n^{h_{i_n}} : \sigma \in S_n, h_1 \in B\}
\]

the space of multilinear differential polynomials in the variables \( x_1, \ldots, x_n, \, n \geq 1 \).

The non-negative integer

\[
c_n^L(A) = \dim_F \frac{P_n^L}{P_n^L \cap \text{Id}^L(A)}, \quad n \geq 1,
\]

is called the \( n \)th differential codimension of \( A \).
Recall that the symmetric group $S_n$ acts on the left on the space $P^L_n$ as follows: for $\sigma \in S_n$, $\sigma(x^h_i) = x^{h}_{\sigma(i)}$. Since $P^L_n \cap \text{Id}^L(A)$ is stable under this $S_n$-action, the space $\frac{P^L_n}{P^L_n \cap \text{Id}^L(A)}$ is a left $S_n$-module and its character, denoted by $\chi^L_n(A)$, is called the $n$th differential cocharacter of $A$. Since char $F = 0$, we can write

$$\chi^L_n(A) = \sum_{\lambda \vdash n} m^L_{\lambda} \chi_{\lambda},$$

where $\lambda$ is a partition of $n$, $\chi_{\lambda}$ is the irreducible $S_n$-character associated to $\lambda$ and $m^L_{\lambda} \geq 0$ is the corresponding multiplicity.

We denote by $P_n$, the space of multilinear ordinary polynomials in $x_1, \ldots, x_n$ and by $\text{Id}(A)$ the $T$-ideal of the free algebra $F(X)$ of polynomial identities of $A$. We also write $c_n(A)$ for the $n$th codimension of $A$ and $\chi_n(A)$ for the $n$th cocharacter of $A$. Since the field $F$ is of characteristic zero, we have $\chi_n(A) = \sum_{\lambda \vdash n} m_{\lambda} \chi_{\lambda}$, where $m_{\lambda} \geq 0$ is the multiplicity of $\chi_{\lambda}$ in the given decomposition.

Since $U(L)$ is an algebra with unit, we can identify in a natural way $P_n$ with a subspace of $P^L_n$. Hence $P_n \subseteq P^L_n$ and $P_n \cap \text{Id}(A) = P_n \cap \text{Id}^L(A)$. As a consequence we have the following relations.

**Remark 1.** For all $n \geq 1$,

1. $c_n(A) \leq c^L_n(A)$;
2. $m_{\lambda} \leq m^L_{\lambda}$, for any $\lambda \vdash n$.

Recall that if $A$ is an $L$-algebra then the variety of algebras with derivations generated by $A$ is denoted by var$^L(A)$ and is called $L$-variety. The growth of $\mathcal{V} = \text{var}^L(A)$ is the growth of the sequence $c^L_n(\mathcal{V}) = c^L_n(A)$, $n = 1, 2, \ldots$. We say that $\mathcal{V}$ has polynomial growth if $c^L_n(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c^L_n(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

Let $A$ and $B$ be $L$-algebras. We say that $A$ is $L$-PI-equivalent to $B$, and we write $A \sim_{L^T} B$, if $\text{Id}^L(A) = \text{Id}^L(B)$. Notice that given an $L$-algebra $A$, $A$ is $L$-PI-equivalent to $B$ if and only if var$^L(A) = \text{var}^L(B)$.

In [3], Giambruno and Rizzo introduced an algebra with derivations generating a variety of almost polynomial growth. They considered $UT_2^L$ to be the algebra of $2 \times 2$ upper triangular matrices with $F$-action, where $\varepsilon$ is the inner derivation induced by $2^{-1}(e_{11} - e_{22})$, i.e.

$$\varepsilon(a) = 2^{-1}[e_{11} - e_{22}, a], \text{ for all } a \in UT_2,$$

where $e_{ij}$’s are the usual matrix units. The authors proved the following.

**Theorem 2.** [3 Theorem 5]

1. $\text{Id}^\varepsilon(UT_2) = \langle [x, y] + [x, y], x^2y, x^3 - x^\varepsilon \rangle_{T_L}$.
2. $c^\varepsilon_n(UT_2) = 2^{n-1}n - 1$.

**Theorem 3.** [3 Theorem 15] The algebra $UT_2^L$ generates a variety of algebras with derivations of almost polynomial growth.

It is easy to check that the varieties of algebras with derivations satisfy the following properties.

**Remark 4.** Let $L$ be a Lie algebra over $F$ and $A$ an $L$-algebra.
(1) If $L_1$ is a Lie algebra over $F$ such that $L \subseteq L_1$, then $\text{var}^L(A) \subseteq \text{var}^{L_1}(A)$.
(2) If $B$ is an associative $L$-algebra such that $A \subseteq B$, then $\text{var}^L(A) \subseteq \text{var}^L(B)$.

3. On finite dimensional algebras with derivations

Let $L$ be a Lie algebra over $F$ and $A$ an $L$-algebra over $F$. We recall some definitions. An ideal (subalgebra) $I$ of $A$ is an $L$-ideal (subalgebra) if it is an ideal (subalgebra) such that $I^L \subseteq I$. We denote by $J(A)$ the Jacobson radical of $A$. It is well known that $J(A)$ is an $L$-ideal of $A$ [9, Theorem 4.2].

The algebra $A$ is $L$-simple if $A^2 \neq \{0\}$ and $A$ has no non-trivial $L$-ideals.

In order to describe a Wedderburn-Malcev decomposition for algebras with derivations, we first present the structure of a semisimple $L$-algebra.

**Lemma 5.** [7, Lemma 1] Let $B$ be a finite dimensional semisimple algebra and let $L$ be a Lie algebra acting on $B$ by derivations. Then

$$B = B_1 \oplus \cdots \oplus B_m$$

where $B_i$ are $L$-simple algebras.

**Lemma 6.** [7, Lemma 7] Let $B = B_1 \oplus \cdots \oplus B_m$ be a semisimple algebra, where each $B_i$ is a simple algebra. If $L$ is a Lie algebra acting on $B$ by derivations, then $B_i$ is an $L$-ideal of $B$, for all $i = 1, \ldots, m$.

**Lemma 7.** [7, Lemma 9] If $B$ is a finite dimensional $L$-simple algebra, then $B$ is a simple algebra.

With these 3 lemmas, we have the following theorem.

**Theorem 8.** Let $A$ be a finite dimensional $L$-algebra where $L$ is a finite dimensional semisimple Lie algebra over an algebraically closed field $F$ of characteristic zero. If $J = J(A)$ is the Jacobson radical of $A$, then $A/J$ is a semisimple $L$-subalgebra of $A$ such that

$$A/J = B_1 \oplus \cdots \oplus B_m,$$

where $B_i \cong M_{n_i}(F), n_i \geq 1$, for all $i = 1, \ldots, m$.

In [7], Gordienko and Kochetov proved that if $A$ is a finite dimensional $L$-algebra, then the sequence of differential codimensions $c_n^L(A)$ is exponentially bounded. Moreover, in case $L$ is finite dimensional and semisimple, the authors proved that the limit $\lim_{n \to \infty} \sqrt[n]{c_n^L(A)}$ exists and is a positive integer. In this case, this limit is called the $L$-exponent of $A$ and is denoted by $\exp^L(A)$. In particular, we have the following.

**Theorem 9.** [7, Theorem 7] Let $A$ be a finite dimensional algebra over a field of characteristic zero. If $L$ is a finite dimensional semisimple Lie algebra acting on $A$ by derivations, then there exist constants $C_1, C_2, r_1, r_2, C_1 > 0$, and a positive integer $d$ such that

$$C_1 n^{r_1} d^m \leq c_n^L(A) \leq C_2 n^{r_2} d^m, \text{ for all } n \in \mathbb{N}.$$ 

Hence, $\exp^L(A) = d$. Moreover, if $A = A_1 \oplus \cdots \oplus A_n + J(A)$ is a Wedderburn-Malcev decomposition of $A$ as an $L$-algebra, then

$$d = \max\{\dim(A_{i_1} \oplus A_{i_2} \oplus \cdots \oplus A_{i_s}) : A_{i_1} J(A) A_{i_2} J(A) \cdots J(A) A_{i_s} \neq \{0\}\},$$

$i_r \neq i_s, 1 \leq r, s \leq n$. 


The outcome of Theorems 8 and 9 is that the exponential rate of $c_n(A)$ and $c_n^L(A)$ are the same, that is, $\exp(A) = \exp^L(A)$, where $\exp(A) = \lim_{n \to \infty} \sqrt[n]{c_n(A)}$.

As an immediate consequence of the above and Theorem 9 we have the following.

**Corollary 10.** Let $A$ be a finite dimensional $L$-algebra over a field $F$ of characteristic zero where $L$ is a finite dimensional semisimple Lie algebra. Then the following conditions are equivalent:

1. $c_n^L(A)$ is polynomially bounded;
2. $\exp^L(A) \leq 1$;
3. $c_n(A)$ is polynomially bounded;
4. $\exp(A) \leq 1$.

## 4. Varieties of polynomial growth

In this section we shall characterize the varieties of algebras with derivations of polynomial growth generated by finite dimensional algebras.

First we consider a Lie algebra $L$ and establish some terminology. If $A$ and $B$ are $L$-algebras, we say that $\varphi : A \to B$ is an $L$-isomorphism if $\varphi$ is an isomorphism of algebras such that $\varphi(a^\delta) = \varphi(a)^\delta$, for all $\delta \in L$.

Next we describe the Lie algebra of all derivation of the algebra $M_2(F)$ of $2 \times 2$ matrices over $F$. Consider the basis $\{e_{11} + e_{22}, e_{11} - e_{22}, e_{12} + e_{21}, e_{12} - e_{21}\}$ of $M_2(F)$, where the $e_{ij}$'s are the usual matrix units, and define the following inner derivations on $M_2(F)$:

$$
v(a) = 2^{-1}[e_{11} - e_{22}, a], \quad \delta(a) = 2^{-1}[e_{12} + e_{21}, a], \quad \gamma(a) = 2^{-1}[e_{12} - e_{21}, a],
$$

for all $a \in M_2(F)$.

Since any derivation of $M_2(F)$ is inner, the Lie algebra $\text{Der}(M_2(F))$ of all derivations of $M_2(F)$ is a 3-dimensional semisimple Lie algebra generated by $\{v, \delta, \gamma\}$ and it is isomorphic to the Lie algebra $\mathfrak{sl}_2$ of $2 \times 2$ traceless matrices over $F$. We recall the following theorem about the Lie algebra $\mathfrak{sl}_2$.

**Theorem 11.** [8 Proposition 8.3] If $L$ is a finite dimensional semisimple Lie algebra over an algebraically closed field $F$ of characteristic zero, then $L$ contains a Lie subalgebra isomorphic to $\mathfrak{sl}_2$.

Notice that $UT_2$ is an $F\mathfrak{sl}_2$-subalgebra of $M_2(F)$, hence $UT_2 \in \var^\mathfrak{sl}_2(M_2(F))$. Also, by using Remark 4 and the previous theorem, we have the following.

**Remark 12.** Let $A$ be an $\mathfrak{sl}_2$-algebra. For every finite dimensional semisimple Lie algebra $L$, $\var^\mathfrak{sl}_2(A) \subseteq \var^L(A)$.

Next theorem gives us a characterization of the varieties of algebras with derivations of polynomial growth in terms of the algebra $UT_2$.

**Theorem 13.** Let $L$ be a finite dimensional semisimple Lie algebra over a field $F$ of characteristic zero and let $A$ be a finite dimensional $L$-algebra over $F$. Then the sequence $c_n^L(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $UT_2 \notin \var^L(A)$.

**Proof.** First suppose that $c_n^L(A)$ is polynomially bounded. Since, by Theorem 2 the algebra $UT_2$ generates a variety of exponential growth, we have $UT_2 \notin \var^L(A)$.

Now assume $UT_2 \notin \var^L(A)$. Using an argument analogous to that used in the ordinary case (see [5] Theorem 4.1.9), we can prove that the differential codimensions do not change upon extension of the base field and so we may assume $F$ is
algebraically closed. Thus by Theorem 8 we can write

\[ A = A_1 \oplus \cdots \oplus A_m + J(A), \]

where \( A_i \cong M_{n_i}(F) \), for \( i = 1, \ldots, m \) and \( J = J(A) \) is an \( L \)-ideal of \( A \).

If \( n_i > 1 \), for some \( i \), by Remarks 1 and 12 and Theorem 11 we get

\[ \text{var}^a(L(M_2(F))) \subseteq \text{var}^L(M_2(F)) \subseteq \text{var}^L(M_{n_i}(F)). \]

This implies that \( UT_2^\varepsilon \in \text{var}^L(M_{n_i}(F)) \), as we have remarked above. Hence \( A \) contains a copy of \( UT_2^\varepsilon \), a contradiction. Thus, for every \( i \), we must have \( A_i \cong F \).

To finish the proof, we use Theorem 9 and so it is enough to guarantee that \( A_iJ A_k = 0 \), for all \( i, k \in \{1, \ldots, m\}, i \neq k \). Suppose to the contrary that there exist \( A_i, A_k, i \neq k \), such that \( A_iJ A_k \neq 0 \). If \( 1_i \) and \( 1_k \) are the unit elements of \( A_i \) and \( A_k \), respectively, it follows that there exists \( j \in J \) such that \( 1_i j 1_k \neq 0 \).

Let \( B \) be the subalgebra of \( A \) generated by \( 1_i, 1_k \) and \( 1_i j 1_k \). Clearly there is an algebra isomorphism \( \varphi \) between \( B \) and \( UT_2 \) given by \( \varphi(1_i) = e_{i1}, \varphi(1_k) = e_{22} \) and \( \varphi(1_i j 1_k) = e_{12} \). Moreover, the inner derivation \( \varepsilon \) defined on \( UT_2 \) induces the inner derivation \( \bar{\varepsilon} = \text{ad}(1_i - 1_k) \) on \( B \) and we get an \( L \)-isomorphism between the \( F\bar{\varepsilon} \)-algebra \( B \) and the \( F\varepsilon \)-algebra \( UT_2 \). It follows that \( \text{var}^L(UT_2^\varepsilon) \subseteq \text{var}^L(A) \), a contradiction. So, by Corollary 10 we conclude that \( c_2^L(A) \) is polynomially bounded.

As a consequence we have the following.

**Corollary 14.** The algebra \( UT^\varepsilon \) is the only finite dimensional algebra with derivations generating an \( L \)-variety of almost polynomial growth.

Next we shall give other characterizations of \( L \)-varieties of polynomial growth. We recall the following theorem.

**Theorem 15.** [9] Let \( A = B + J \) be an algebra over \( F \), where \( B \) is a semisimple subalgebra and \( J = J(A) \) is its Jacobson radical. Suppose \( \delta \) is a derivation of \( A \). Then \( \delta = \text{ad} a + \delta' \) where \( a \in A \) and \( \delta' \) is a derivation of \( A \) such that \( \delta'(B) = 0 \).

Next lemma will be useful to establish a structural result about \( L \)-varieties of polynomial growth.

**Lemma 16.** Let \( F \) be a field of characteristic zero, \( \bar{F} \) the algebraic closure of \( F \) and \( A \) a finite dimensional \( L \)-algebra over \( \bar{F} \), where \( L \) is a Lie algebra over \( \bar{F} \) acting on \( A \) by derivations. Suppose that \( \dim_{\bar{F}} A/J(A) \leq 1 \). Then \( A \sim_{T_L} B \) for some finite dimensional \( L \)-algebra \( B \) over \( F \) with \( \dim_{\bar{F}} B/J(B) \leq 1 \).

**Proof.** Since \( \dim_{\bar{F}} A/J(A) \leq 1 \), it follows that either \( A \cong \bar{F} + J(A) \) or \( A = J(A) \) is a nilpotent algebra. Now we take an arbitrary basis \( \{v_1, \ldots, v_p\} \) of \( J(A) \) over \( \bar{F} \) and we let \( B \) be the algebra over \( F \) generated by \( B = \{1_F, v_1, \ldots, v_p\} \) or \( B = \{v_1, \ldots, v_p\} \) according as \( A \cong \bar{F} + J(A) \) or \( A = J(A) \), respectively.

Since \( A \) is finite dimensional over \( \bar{F} \) and \( J(A) \) is a nilpotent \( L \)-ideal of \( A \), \( B \) is finite dimensional over \( F \). Also, by Theorem 15 for any \( \delta \in L \), \( \delta(1_F) \in J(A) \). Therefore, since \( J(A) \) is an \( L \)-ideal, \( B \) is an \( L \)-algebra and \( \dim_{\bar{F}} B/J(B) = \dim_{\bar{F}} A/J(A) \leq 1 \). Now notice that, as \( F \)-algebras, \( \text{Id}^L(A) \subseteq \text{Id}^L(B) \). On the other hand, if \( f \) is a multilinear differential identity of \( B \) then \( f \) vanishes on \( B \). But \( B \) is a basis of \( A \) over \( \bar{F} \). Hence \( \text{Id}^L(B) \subseteq \text{Id}^L(A) \) and \( A \sim_{T_L} B \).
Theorem 17. Let $L$ be a finite dimensional semisimple Lie algebra over a field $F$ of characteristic zero and let $A$ be a finite dimensional $L$-algebra over $F$. Then $c^L_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if $A \sim_{T_L} B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional $L$-algebras over $F$ such that $\dim B_i / J(B_i) \leq 1$, for all $i = 1, \ldots, m$.

Proof. Suppose first that $A \sim_{T_L} B$ where $B = B_1 \oplus \cdots \oplus B_m$, with $B_1, \ldots, B_m$ finite dimensional $L$-algebras over $F$ such that $\dim B_i / J(B_i) \leq 1$, for all $i = 1, \ldots, m$. Then, by Theorem 8, $c^L_n(B_i)$ is polynomially bounded, for all $i = 1, \ldots, m$, and

$$c^L_n(A) = c^L_n(B) \leq c^L_n(B_1) + \cdots + c^L_n(B_m).$$

Thus $c^L_n(A)$ is polynomially bounded.

Conversely, suppose that $c^L_n(A)$ is polynomially bounded. Assume first that $F$ is algebraically closed. By Theorem 8, we may assume that $A = A_{ss} + J$ where $A_{ss}$ is a semisimple subalgebra and $J = J(A)$ is the Jacobson radical of $A$. By Theorem 6, it follows that $A_{ss} = A_1 \oplus \cdots \oplus A_l$ with $A_1 \cong \cdots \cong A_l \cong F$ and $A_iA_k = A_iJA_k = \{0\}$, for all $1 \leq i, k \leq l$, $i \neq k$.

Set $B_1 = A_1 + J, \ldots, B_l = A_l + J$. By Theorem 11, $\delta(A_i) \subseteq J = J(B_i)$, for all $1 \leq i \leq l$, for all $\delta \in L$. Hence $B_i$ is an $L$-subalgebra of $A$, for all $1 \leq i \leq l$. We claim that

$$\text{Id}^L(A) = \text{Id}^L(B_1 \oplus \cdots \oplus B_l + J).$$

Clearly $\text{Id}^L(A) \subseteq \text{Id}^L(B_1 \oplus \cdots \oplus B_l + J)$. Now let $f \in \text{Id}^L(B_1 \oplus \cdots \oplus B_l + J)$ and suppose that $f$ is not a differential identity of $A$. We may clearly assume that $f$ is multilinear. Moreover, by choosing a basis of $A$ as the union of a basis of $B$ and a basis of $J$, it is enough to evaluate $f$ on this basis. Let $u_1, \ldots, u_l$ be elements of this basis such that $f(u_1, \ldots, u_l) \neq 0$. Since $f \in \text{Id}^L(J)$, at least one element, say $u_k$, does not belong to $J$. Then $u_k \in B_i$, for some $i$. Recalling that $A_iA_k = B_iA_k = B_iJA_k = \{0\}$, for all $i \neq k$, we must have that $u_1, \ldots, u_k \in A_i \cup J$. Thus $u_1, \ldots, u_l \in A_i + J = B_i$ and this contradicts the fact that $f$ is a differential identity of $B_i$. This prove the claim. The proof is completed by noticing that $\dim B_i / J(B_i) = 1$.

In case $F$ is arbitrary, we consider the algebra $\tilde{A} = A \otimes_F \tilde{F}$, where $\tilde{F}$ is the algebraic closure of $F$ and $\tilde{A} = A \otimes_F \tilde{F}$ is endowed with the induced $L$-action $(a \otimes \alpha)^\delta = a^\delta \otimes \alpha$, for $\delta \in L$, $a \in A$ and $\alpha \in \tilde{F}$. Clearly, over $F$, $\text{var}^L(A) = \text{var}^L(\tilde{A})$. Moreover, the differential codimensions of $A$ over $F$ coincide with the differential codimensions of $\tilde{A}$ over $\tilde{F}$. Thus, by hypothesis, it follows that the differential codimensions of $\tilde{A}$ are polynomially bounded. But then, by the first part of the proof, $\tilde{A} \sim_{TL} B_1 \oplus \cdots \oplus B_m$, where $B_1, \ldots, B_m$ are finite dimensional $L$-algebras over $F$ such that $\dim F B_i / J(B_i) \leq 1$, for all $i = 1, \ldots, m$. By Lemma 3 there exist finite dimensional $L$-algebras $C_1, \ldots, C_m$ over $F$ such that, for all $i$, $C_i \sim_{TL} B_i$ and $\dim F C_i / J(C_i) \leq 1$. It follows that $\text{Id}^L(A) = \text{Id}^L(A) = \text{Id}^L(B_1 \oplus \cdots \oplus B_m) = \text{Id}^L(C_1 \oplus \cdots \oplus C_m)$ and we are done. \hfill \Box

In next theorem, we give another characterization of $L$-varieties $V$ of polynomial growth through the behaviour of their sequences of cocharacters.

Theorem 18. Let $L$ be a finite dimensional semisimple Lie algebra over a field $F$ of characteristic zero and let $A$ be a finite dimensional $L$-algebra over $F$. Then $c^L_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if there exists a constant
such that
\[ \chi_n^L(A) = \sum_{|\lambda|-\lambda_1 < q} m_{\lambda}^L \chi_{\lambda} \]
and \( J(A)^q = \{0\} \).

**Proof.** Suppose first that \( m_{\lambda}^L = 0 \), whenever \(|\lambda| - \lambda_1 \geq q\). By Remark 1 it follows that \( n_{\lambda} = 0 \), if \(|\lambda| - \lambda_1 \geq q\). Thus, by [4, Theorem 3], we get that \( c_n(A) \) is polynomially bounded and by Corollary 10 we are done.

Conversely, suppose that \( c_n^L(A), n = 1, 2, \ldots \), is polynomially bounded. Notice that the decomposition of \( \chi_n^L(A) \) into irreducible characters does not change under extensions of the base field. This fact can be proved following word by word the proof given in [5, Theorem 4.1.9] for the ordinary case. Also if \( \overline{F} \) is the algebraic closure of \( F \) and \( J(A)^q = \{0\} \), then \( J(A \otimes_F \overline{F})^q = \{0\} \). Therefore we may assume, without loss of generality, that \( F \) is an algebraically closed field. Now the theorem can be proved following closely the proof of [4, Theorem 3] for the ordinary case, taking into account the due changes. \( \square \)

As a consequence of Theorems 13, 17, 18 and Corollary 10 we get the following theorem which gives a characterization of the \( L \)-variety generated by a finite dimensional algebras with derivations of polynomial growth.

**Theorem 19.** Let \( L \) be a finite dimensional semisimple Lie algebra over a field \( F \) of characteristic zero and let \( A \) be a finite dimensional \( L \)-algebra over \( F \). Then the following conditions are equivalent:

1. \( c_n^L(A) \leq \alpha n^t \), for some constant \( \alpha, t \);
2. \( \exp^L(A) \leq 1 \);
3. \( c_n(A) \leq \beta n^p \), for some constant \( \beta, p \);
4. \( \exp(A) \leq 1 \);
5. \( UT_2 \notin \var(L) \);
6. \( A \sim_{T_2} A_1 \oplus \cdots \oplus A_m \), with \( A_1, \ldots, A_m \) finite dimensional \( L \)-algebras over \( F \) such that \( \dim A_i/J(A_i) \leq 1 \), for all \( i = 1, \ldots, m \);
7. There exists a constant \( q \) such that
\[ \chi_n^L(A) = \sum_{|\lambda|-\lambda_1 < q} m_{\lambda}^L \chi_{\lambda} \]
and \( J(A)^q = \{0\} \).

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