Motion of test particles around a charged dilatonic black hole

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Abstract
We examine motion of test particles with various masses, electric charges and dilatonic charges in a background metric and fields of a charged dilatonic black hole.

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1 Introduction

Recently, there has been much interest in the study of charged dilatonic black holes (BHs) [1, 2, 3, 4, 5]. They have diverse connections to supergravity [1], Kaluza-Klein [2], string [3], and conformal field theories [4].

In order to study the role of dilaton, one can consider the action with an arbitrary value for the dilaton coupling [3]:

\[ S = \int d^4x \frac{\sqrt{-g}}{16\pi} [R - 2(\nabla \phi)^2 - e^{-2a\phi} F^2] \] (1)

where \( R \) is the curvature scalar and \( \phi \) is the dilaton. \( F_{\mu\nu} \) denotes the Maxwell field strength. The Newton constant is normalized to unity here.

The effective field theory of string theory corresponds to the case with \( a^2 = 1 \), while Kaluza-Klein theory (in five dimensions) requires \( a^2 = 3 \). The usual Einstein-Maxwell system can be obtained when \( a^2 \) is set to be zero.

Classical and quantum properties of charged dilatonic BHs have been investigated in various aspects by many authors [3, 6, 7, 8, 9, 10]. Some critical values for the coupling \( a \) have been found and discussed.
In this paper, we examine motion of test particles around the dilaton BH, which appears in the system governed by the ‘interpolating’ action (1). Test particles that we consider are assumed to have ‘electric’ and dilatonic charges, as well as arbitrary masses. Although the analysis will be carried out mainly for a background metric and fields of an extreme BH (defined later), the qualitative features for general cases are similar to those for this extremal case as far as we treat the motion of test particles outside BHs.

The organization of this paper is as follows. In the next section, we will present the background fields and metric for a charged dilatonic BH located at the origin. The generic motion of a test particle in the background is also studied in the section 2. In section 3, we study the possibility of a static equilibrium of the system consisting of the charged dilatonic BH and a test particle. In section 4, we treat circular motions of test particles around the BH. The scattering problem is considered in section 5. Finally, section 6 is devoted to conclusion and discussion.

2 Motion of a test particle in a charged-dilaton-BH background

The metric for a static, spherically-symmetric charged dilatonic BH can be obtained by solving field equations derived from the action (1). In the line-element form, it can be written as

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\Delta \sigma^{-2} dt^2 + \sigma^2 \{ \Delta^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \} \]  

(2)

where

\[ \Delta(r) = \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{r_-}{r} \right) \quad \text{and} \quad \sigma^2(r) = \left( 1 - \frac{r_-}{r} \right)^{2a^2/(1+a^2)}. \]  

(3)

The background configuration of the classical dilaton field and gauge potential are given by

\[ e^{2a\phi} = \sigma^2(r) \quad \text{and} \quad A = \frac{Q}{r} dt \]  

(4)

respectively.

In these expressions, \( r_+ \) and \( r_- \) are constants, which are related to the mass and charges of the BH:

\[ 2M = r_+ + \frac{1 - a^2}{1 + a^2} r_- \quad \text{and} \quad Q^2 = \frac{r_+ r_-}{1 + a^2}, \]  

(5)

where \( M \) is the black hole mass and \( Q \) is the electric charge of the BH. Without loss of generality, we can consider \( Q \) and \( a \) to take positive values. The horizon corresponds to \( r = r_+ \) for this metric. Note that the spacetime represented by (2) coincides with the Reissner-Nordstrom (RN) spacetime when \( a = 0 \).
On the other hand, the action for a point test particle, of which coordinates are denoted by $x^\mu$, is
\begin{equation}
S_{tp} = -\int dt \left[ m e^{b\phi} \sqrt{-g_{\mu\nu} \frac{\partial x^\mu}{\partial t} \frac{\partial x^\nu}{\partial t} + e A_\mu \frac{\partial x^\mu}{\partial t}} \right]
\end{equation}

where $m$, $e$ and $b$ stand for the mass, electric charge, and coupling to the dilaton field, of the test particle, respectively. $g_{\mu\nu}(x)$, $A_\mu(x)$, and $\phi(x)$ must be regarded as the classical background field (2), (4) in the present analysis. The radiations of electromagnetic, gravitational, and dilatonic (scalar) waves from the particle and the associated back reactions are therefore ignored in the present analysis.

Because of the spherical symmetry of the background fields and spacetime, the motion of the test particle is restricted in a plane which contains the centre of the charged dilatonic BH. We can take a plane defined by $\theta = \pi/2$ as such a plane. There are two constants of the motion corresponding to the Killing vectors of the spacetime. They are
\begin{align}
-m e^{b\phi} g_{\mu\nu} \frac{dt}{dr} + e A_\mu &= m \Delta(r) \sigma^{(b/a)-2}(r) \frac{dt}{dr} + \frac{e Q}{r} \equiv E \quad (7) \\
-m e^{b\phi} g_{\phi\phi} \frac{d\phi}{d\tau} &= -m \sigma^{2+(b/a)}(r)r^2 \frac{d\phi}{d\tau} \equiv -L \quad (8)
\end{align}

where
\begin{equation}
d\tau \equiv \sqrt{-g_{\mu\nu} \frac{\partial x^\mu}{\partial t} \frac{\partial x^\nu}{\partial t}} dt.
\end{equation}

The ‘on mass shell’ condition trivially derived from the action (6), i.e.,
\begin{equation}
 g^{\mu\nu}(P_{\mu} + e A_{\mu})(P_{\nu} + e A_{\nu}) + m^2 e^{2b\phi} = 0
\end{equation}

where
\begin{equation}
P_{\mu} \equiv \frac{\delta S_{tp}}{\delta(dx^\mu/dt)}
\end{equation}

can be rewritten by using the background fields and the two constants as
\begin{equation}
\left( \frac{dr}{d\tau} \right)^2 = \frac{\sigma^{-2b/a}}{m^2} \left( E - \frac{e Q}{r} \right)^2 - \Delta \sigma^{-2} \left( 1 + \frac{L^2 \sigma^{-2-(2b/a)}}{m^2 r^2} \right).
\end{equation}

We analyse the test-particle motion by using the equations (7), (8) and (12). The analyses for specific motions will be treated in the subsequent sections.

### 3 Static equilibrium

First we consider the particle motion with a constant distance $r$ from the centre of the dilatonic BH. Setting $dr/d\tau = 0$ in equation (12), we obtain
\begin{equation}
\frac{E}{m} = \frac{e Q}{mr} + \sqrt{\Delta \sigma^{-2+(2b/a)} \left( 1 + \frac{L^2 \sigma^{-2-(2b/a)}}{m^2 r^2} \right)}.
\end{equation}
The right-hand side of equation (13) can be considered as a function of \( r \). In a stationary system, \( E \) must be an extremal value. Thus the value for \( r \) that is realized in such a system is given by the solution of the equation

\[
\frac{dV}{dr} = 0
\]

where

\[
V(r) \equiv \frac{eQ}{mr} + \sqrt{\Delta \sigma - 2 + \frac{L^2 \sigma - (2b/a)}{m^2 r^2}}.
\]

In this and the next sections, we analyse mainly the cases with an extremal BH \( (r_+ = r_-) \). The qualitative features for general cases with non-zero charge are similar to the extremal case. In the extremal case, the effective potential \( V \) is simplified as

\[
V(r) = \frac{er_+}{\sqrt{1 + a^2 mr}} + \left(1 - \frac{r_+}{r}\right)^{1+ab}/(1+a^2) \sqrt{1 + \frac{L^2 m^2 r^2}{m^2 r^2}} \left(1 - \frac{r_+}{r}\right)^{-2a(a+b)/(1+a^2)}.
\]

When \( L \neq 0 \), \( r \) represents the radius of a circular orbit. This case will be treated in the next section. In the present section, we examine the possibility of static equilibrium by using the potential (16) with \( L \) set to be 0.

We can solve (13) and (15) with respect to \( m \), which takes a positive value, to obtain an allowed region in the parameter space. For \( L = 0 \) and an arbitrary charge of the BH, we obtain the following inequality:

\[
\frac{r_+(1 - r_-/r_{se})}{r_-(1 - r_+/r_{se})} \geq \frac{a^2 - 2ab - 1}{1 + a^2} \quad \text{for} \quad e \geq 0
\]

where \( r_{se} \) indicates the distance of the equilibrium point from the centre of the BH. For the extremal case, the inequality is simply reduced to \( 1 + ab > 0 \) for \( e > 0 \) (and \( 1 + ab < 0 \) for \( e < 0 \)).

From now on, we focus our attention to the case with the extreme BH. For the case with a usual, extreme RN BH, which corresponds to the case with \( a = 0 \) here, the stable balance between gravity and Coulomb force can never be attained except for the case with \( e/m = 1 \). The condition \( e/m = 1 \) is just the extremity condition for the test particle. In this case, the static forces are cancelled with each other at an arbitrary distance from the extreme RN BH. In the presence of the dilatonic force, there is a possibility that the three forces on the test particle keep the balance at a certain distance from the hole.

Taking the stability against a small shift of the distance into consideration, the equilibrium turns out to be realized in the three cases for the dilaton couplings and charges of test particles as follows:

1. \( b = a \) and \( e/m = (1 + a^2)^{1/2} \)

The test particle has like (electric) charges and its charge satisfies the extremal condition in this case [8]. The static configuration is permitted for an arbitrary distance between the extreme black hole and the test particle.
(II) $b > a$ and $0 < e/m < (1 + ab)/(1 + a^2)^{1/2}$

In this case, the dilatonic force is attractive while the Coulomb force is repulsive. The distance between the black hole and the test particle is given by

$$r_{se} = r_+ \left\{ 1 - \left( \frac{\sqrt{1 + a^2 e}}{(1 + ab)m} \right)^{(1+a^2)/(a(b-a))} \right\}^{-1}.$$  \hspace{1cm} (18)

(III) $b < -1/a$ and $e < 0$ and $|e|/m > |1 + ab|/(1 + a^2)^{1/2}$

In this case, the dilatonic force is repulsive while the Coulomb force is attractive. The distance between the black hole and the test particle is given by

$$r_{se} = r_+ \left\{ 1 - \left( \frac{\sqrt{1 + a^2 |e|}}{1 + ab|m|} \right)^{(1+a^2)/(a(b-a))} \right\}^{-1}.$$  \hspace{1cm} (19)

In the cases (II) and (III), the equilibrium distances are always outside the BH horizon ($r_{se} > r_+$), though the balances are unstable.

4 Circular motions

Let us consider the motion along a circular orbit with the radius $r$ around a maximally-charged dilatonic BH ($r_+ = r_+^*$). Since it is not pedagogical to survey all the cases in the vast range of charges of the test particle, we will choose some particular cases for the couplings and charges.

(I) $e = 0$.

We first examine the case with $e = 0$ (an electrically neutral test particle).

The effective potential becomes

$$V(r) = \left( 1 - \frac{r_+}{r} \right)^{(1+ab)/(1+a^2)} \sqrt{1 + \frac{L^2}{m^2 r^2} \left( 1 - \frac{r_+}{r} \right)^{-2a(a+b)/(1+a^2)}}.$$  \hspace{1cm} (20)

We restrict ourselves further to two cases for the extreme BH:

(i) $a = 0$. This leads to the case with an extreme RN BH. The motion of the test particle is obviously independent of the value for its dilatonic coupling. The effective potential has an extremum outside the BH if

$$\frac{L^2}{m^2} > 8r_+^2 \quad (e = 0, a = 0).$$  \hspace{1cm} (21)

The minimum radius of the circular orbit is given when $L^2/m^2 = 8r_+^2$ by

$$r_{min} = 4r_+ \quad (e = 0, a = 0)$$  \hspace{1cm} (22)

and then the energy of the test particle is given by

$$\frac{E_{min}^2}{m^2} = \frac{27}{32} \quad (e = 0, a = 0).$$  \hspace{1cm} (23)
In this case, the binding energy \((m - E)\) becomes about 8.1\% of the rest energy.

(ii) \(a = 1\). This dilaton coupling corresponds to the field theory limit of string theory. The critical behaviour is classified by the value for \(b\).

- \(b > 0\). The circular motion is possible as long as \(L^2 \neq 0\). The minimum radius of the circular orbit is given by \(r_{min} = r_+\) in the limit of \(L \to 0\). In this limit, the energy of the particle \(E\) approaches zero, i.e., the binding energy becomes 100\% of the rest energy!

- \(b = 0\). The circular motion is possible if

\[
\frac{L^2}{m^2} > \frac{r_+^2}{2} \quad (e = 0, a = 1, b = 0).
\]  

(24)

The minimum radius of the circular orbit is given when \(L^2/m^2 = r_+^2/2\) by

\[
r_{min} = r_+ \quad (e = 0, a = 1, b = 0)
\]

(25)

and then the energy of the test particle is given by

\[
\frac{E_{min}^2}{m^2} = \frac{1}{2} \quad (e = 0, a = 1, b = 0).
\]

(26)

In this case, the binding energy \((m - E)\) becomes about 29.3\% of the rest energy.

- \(-1 < b < 0\). The circular motion is possible if

\[
\frac{L^2}{m^2} > \frac{(1 - b^2)r_+^2}{2} \left( \frac{-b}{1 - b} \right)^b \quad (e = 0, a = 1, -1 < b < 0).
\]

(27)

The minimum radius of the circular orbit is given when \(L^2/m^2 = r_+^2/2\) by

\[
r_{min} = (1 - b)r_+ \quad (e = 0, a = 1, -1 < b < 0)
\]

(28)

and then the energy of the test particle is given by

\[
\frac{E_{min}^2}{m^2} = \frac{1}{2} \left( \frac{-b}{1 - b} \right)^b \quad (e = 0, a = 1, -1 < b < 0).
\]

(29)

In the limit of \(b \to -1\), the binding energy \((m - E)\) approaches zero.

- \(b \leq -1\). The circular motion is impossible in this case \((e = 0, a = 1, b \leq -1)\).

(II) \(e/m = -(1 + a^2)^{1/2}\) and \(a = b\)

In this case the effective potential becomes

\[
V(r) = -\frac{r_+}{r} + \left( 1 - \frac{r_+}{r} \right) \sqrt{1 + \frac{L^2}{m^2r^2} \left( 1 - \frac{r_+}{r} \right)^{-4a^2/(1+a^2)}}.
\]

(30)

According to the magnitude of \(a\), possible circular motions are classified:

(i) \(a^2 \geq 1\). The circular motion is possible as long as \(L^2 \neq 0\). The radius of the circular orbit approaches \(r_{min} = r_+\) in the limit of \(L \to 0\). In this limit, the
energy of the test particle $E$ approaches $-m$, i.e., the binding energy becomes 200% of the rest energy!

(i) $a^2 < 1$. The circular motion is possible if

\[
\frac{L^2}{m^2} > \frac{L_c^2}{m^2} = \frac{x^2}{x-2/(1+a^2)^2} \left( 1 - \frac{1}{x} \right) \frac{4a^2/(1+a^2)}{r_+^2}
\]

\[
(e/m = -(1+a^2)^{-1/2}, b = a < 1)
\]

where

\[
x(a) = \frac{3 - a^2 + \sqrt{(3-a^2)(1-a^2)}}{1+a^2}.
\]

The minimum radius of the circular orbit is given when $L^2 = L_c^2$ by

\[
r_{\min} = x(a)r_+ \quad (e/m = -(1+a^2)^{-1/2}, b = a < 1)
\]

and then the energy of the test particle is given by

\[
\frac{E_{\min}}{m} = \frac{[(1-a^2)/(1+a^2)][3x-(3-a^2)]/(1+a^2)}{x[x-2/(1+a^2)]}
\]

\[
(e/m = -(1+a^2)^{-1/2}, b = a < 1).
\]

The binding energy in this limiting case becomes about 13.4% for $a = 0$, while it becomes about 22.2% for $a^2 = 1/3$. In the limit of $a \to 1$, $E_{\min}$ vanishes.

5 Scattering of particles by the charged dilatonic BH

Using (8) and (12), one can find the differential equation which determines the shape of the trajectory:

\[
\left( \frac{dr}{d\varphi} \right)^2 = r^4 \left[ \frac{\sigma^4}{L^2} \left( E - \frac{eQ}{r} \right)^2 - \Delta \left( \frac{m^2\sigma^2+(2b/a)}{L^2} + \frac{1}{r^2} \right) \right].
\]

For the scattering problem, we can substitute the constants $E$ and $L$ by the value at spatial infinity as

\[
E = \frac{m}{\sqrt{1-v^2}} \quad \text{and} \quad L = \frac{mv\Delta}{\sqrt{1-v^2}}
\]

where $v$ is the velocity of the test particle at spatial infinity and $\Delta$ is the impact parameter. Using this relation, equation (35) can be rewritten as

\[
\left( \frac{dr}{d\varphi} \right)^2 = r^4 \left[ \frac{\sigma^4}{v^2d^2} \left( 1 - \frac{\sqrt{1-v^2}eQ}{mr} \right)^2 - \Delta \left( \frac{1-v^2}{v^2d^2} \right) \right].
\]
Here we examine the scattering problem in a few simple situations.

(I) ‘Small angle scattering’ of a massive test particle \((m \neq 0)\)

If the value of impact parameter is much larger than the radius of the black hole horizon and the absolute value of the potential energy is much smaller than the kinetic energy of the test particle, the scattering angle is expected to be small. In such a case, the analysis of scattering can be simplified to some extent. Equation (37) can be approximated by

\[
\left( \frac{dr}{d\varphi} \right)^2 = r^4 \left[ \frac{1}{d^2} \left( 1 + \frac{B}{r} \right) - \frac{1}{r^2} \right]
\]

(38)

where

\[
B \equiv \frac{1}{v^2} \left\{ (1 - v^2) \left( r_+ + \frac{1 + 3a^2 + 2ab}{1 + a^2} r_- \right) - \left( \frac{2\sqrt{1 - v^2}eQ}{m} + \frac{4a^2}{1 + a^2} r_- \right) \right\}
\]

(39)

is a constant, whose dimension is \([L^1]\). \(|B|/d\) should be a small, dimensionless value in the present assumption.

Equation (38) can easily be solved to obtain the scattering angle. At first order in \(B\), the scattering angle is found to be

\[
\Theta = \frac{|B|}{d} + O((B/d)^3).
\]

(40)

Then the differential cross-section for the small-angle scattering is obtained as

\[
\frac{d\sigma}{d\Omega} = \frac{d^4}{B^2} = \frac{B^2}{\Theta^2}.
\]

(41)

This is the same form as the Rutherford scattering at small angle.

Further, let us consider the low-velocity limit. In the case of \(v^2 \ll 1\), \(B\) can be simplified as

\[
B = \frac{1}{2mv^2} \left( Mm + \frac{ab}{1 + a^2} r_- eQ \right) \quad (v^2 \ll 1)
\]

(42)

where \(M\) is the mass of the charged dilatonic BH. Thus we find that in the low-velocity, or, low-energy limit the small angle scattering is regarded as the Rutherford scattering by three inverse-square-law forces, i.e., gravity, Coulomb force and dilatonic force.

(II) Low-energy scattering of an extreme BH and an extreme test particle with the same dilatonic coupling \((r_+ = r_-\) and \(a = b\)) and \(e/m = (1 + a^2)^{1/2}\)

As seen from the result of (I), the leading scattering amplitude could take a velocity-independent value if the static forces are cancelled by each other. As such a simple case, we suppose that the extreme condition is satisfied not only among the parameters of the BH but also among those of the test particle. Further we assume \(a = b\). We restrict ourselves to the low-velocity scattering for simplicity.
In this case, equation (37) can be written as
\[
\left( \frac{dr}{d\varphi} \right)^2 = r^4 \left[ \frac{1}{d^2} \left( 1 - \frac{r_+}{r} \right)^{\frac{1+5a^2}{2(1+a^2)}} - \frac{1}{r^2} \left( 1 - \frac{r_+}{r} \right)^2 \right] \quad (v^2 \ll 1). \tag{43}
\]

For \( a^2 > 1/3 \), the right-hand side of equation (43) becomes zero for a certain value for \( r \), which is larger than \( r_+ \). The value gives the minimum distance between the centre of the BH and the trajectory of the particle. Thus the particle can never be absorbed by the BH in this case for \( a^2 > 1/3 \). For \( a^2 < 1/3 \), the particle can be absorbed if the impact parameter is sufficiently small.

In general cases, one cannot express the scattering angle using analytic functions. We show the expression for the scattering angle for some cases with special value for \( a \).

(i) \( a = 0 \). In this case, the scattering angle is written by using the elliptic integral:
\[
\Theta = \frac{4}{\sqrt{(1-\beta)(\alpha-\gamma)}} F \left( \sin^{-1} \sqrt{\frac{\beta(\alpha-\gamma)}{\alpha(\beta-\gamma)}}, \sqrt{\frac{(1-\alpha)(\beta-\gamma)}{(1-\beta)(\alpha-\gamma)}} \right) - \pi \quad (a^2 = 0) \tag{44}
\]
where \( F(\varphi, k) \) is the elliptic integral of the first kind, that is:
\[
F(\varphi, k) \equiv \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}}. \tag{45}
\]
\( \alpha, \beta, \) and \( \gamma (\alpha > \beta > \gamma) \) are roots of the following algebraic equation of third degree:
\[
u^3 - \nu^2 + \left( \frac{r_+}{d} \right)^2 = 0. \tag{46}
\]
If the equation does not have real roots in the range \([0, 1]\), the trajectory of the particle ends at the horizon; the particle is swallowed by the BH. This occurs when \( d < (3\sqrt{3}/2)r_+ \).

For small angle scattering, the scattering angle is approximately given by:
\[
\Theta = \frac{r_+}{d} \quad \text{if } d \gg r_+ \quad (a^2 = 0). \tag{47}
\]

(ii) \( a^2 = 1/3 \). In this case, the scattering angle is written by:
\[
\Theta = \frac{4}{\sqrt{1 - \left( \frac{r_+}{d} \right)^2}} \sin^{-1} \sqrt{\frac{1}{2} \left( 1 + \frac{r_+}{d} \right)} - \pi \quad (a^2 = 1/3). \tag{48}
\]
Note that \( \Theta \) diverges in the limit of \( d \to r_+ \).

For small angle scattering, the scattering angle is approximately by:
\[
\Theta = \frac{2r_+}{d} \quad \text{if } d \gg r_+ \quad (a^2 = 1/3). \tag{49}
\]
(iii) $a^2 = 1$. In this case, the scattering angle is simply written by:

$$\Theta = 2 \tan^{-1} \frac{r_+}{2d} \quad (a^2 = 1). \quad (50)$$

The relation is very similar to the one for the Rutherford scattering. The differential cross section is calculated using (50):

$$\frac{d\sigma}{d\Omega} = \frac{r_+^2}{16 \sin^4(\Theta/2)} \quad (a^2 = 1). \quad (51)$$

In the present case, the small amount of the scattering angle is independent of the small value of the incident velocity, as expected.

The scattering of two extreme BHs has been analysed by using the moduli space metric [8]. It is known that the small-mass limit for one BH leads to the same description of scattering by the test-particle analysis in this case (II).

(III) Massless particle ($m = 0$)

For the massless case, we must replace (37) by

$$\left( \frac{dr}{d\varphi} \right)^2 = r^4 \left[ \frac{\sigma^4}{d^2} - \frac{\Delta}{r^2} \right]. \quad (52)$$

(This is independent of the charges of test particles ($e$ and $b$).)

We again consider the extreme BH for simplicity.

In this case, equation (52) can be written as

$$\left( \frac{dr}{d\varphi} \right)^2 = r^4 \left[ \frac{1}{d^2} \left( 1 - \frac{r_+}{r} \right)^{4a^2/(1+a^2)} - \frac{1}{r^2} \left( 1 - \frac{r_+}{r} \right)^2 \right] \quad (m = 0). \quad (53)$$

For $a^2 > 1$, the right-hand side of equation (53) becomes zero for a certain value for $r$, which is larger than $r_+$. Thus the massless particle never be absorbed by the extreme BH in this case for $a^2 > 1$. For $a^2 < 1$, the massless particle can be absorbed if the impact parameter is sufficiently small.

We show the expression for the scattering angle for some cases with special value for $a$.

(i) $a^2 = 0$. In this case, the scattering angle is written by using the elliptic integral:

$$\Theta = \frac{4}{\sqrt{(\alpha - \gamma)(\beta - \delta)}} F \left( \sin^{-1} \sqrt{\frac{\gamma(\beta - \delta)}{\beta(\gamma - \delta)}} \right) \sqrt{\frac{(\alpha - \beta)(\gamma - \delta)}{(\alpha - \gamma)(\beta - \delta)}} - \pi \quad (a^2 = 0) \quad (54)$$

where $\alpha, \beta, \gamma$ and $\delta$ ($\alpha > \beta > \gamma > \delta$) are roots of the following algebraic equation of fourth degree:

$$u^2(u - 1)^2 - \left( \frac{r_+}{d} \right)^2 = 0. \quad (55)$$

If the equation (55) does not have real roots in the range $[0, 1]$, the trajectory of the particle ends up at the horizon; the massless particle is swallowed by the BH. This occurs when $d < 4r_+$. 

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For small angle scattering, the scattering angle is approximately given by:

$$\Theta = \frac{2r_+}{d} \quad \text{if} \quad d \gg r_+ \quad (a^2 = 0). \quad (56)$$

(ii) $a^2 = 1/3$. In this case, the scattering angle is simply written by:

$$\Theta = \frac{4}{\sqrt{(1-\beta)(\alpha-\gamma)}} F \left( \sin^{-1} \sqrt{\frac{\beta(\alpha-\gamma)}{\alpha(\beta-\gamma)}} \right)^{-1} \left( 1 - \frac{\beta}{\alpha} \right)^{-1} - \pi \quad (a^2 = 1/3) \quad (57)$$

where $\alpha$, $\beta$, and $\gamma$ ($\alpha > \beta > \gamma$) are roots of the following algebraic equation of third degree:

$$u^3 - u^2 + \left( \frac{r_+}{d} \right)^2 = 0. \quad (58)$$

If the equation (58) does not have real roots in the range $[0, 1]$, the massless particle is swallowed by the BH. This occurs when $d < (3\sqrt{3}/2)r_+$. The trajectory takes the same form as that with $a = 0$ in case (II), examined previously.

(iii) $a^2 = 1$. In this case, the scattering angle is simply written by:

$$\Theta = \frac{4}{\sqrt{1 - \left( \frac{r_+}{d} \right)^2}} \sin^{-1} \sqrt{\frac{1}{2} \left( 1 + \frac{r_+}{d} \right)} - \pi \quad (m = 0, \ a^2 = 1). \quad (59)$$

The trajectory takes the same form as that with $a = 0$ in the case with $a^2 = 1/3$ in case (II), examined previously.

(iv) $a^2 = 3$. Then the massless particle is scattered in the manner of the Rutherford scattering:

$$\Theta = 2 \tan^{-1} \frac{r_+}{2d} \quad (m = 0, \ a^2 = 3). \quad (60)$$

The differential cross section is:

$$\frac{d\sigma}{d\Omega} = \frac{r_+^2}{16 \sin^2(\Theta/2)} \quad (m = 0, \ a^2 = 3). \quad (61)$$

6 Conclusions and discussions

In this paper, we have examined the motion of test particles in the background metric and fields of a charged dilatonic BH. As shown in section 3, there is the static equilibrium between a test particle and an extreme dilatonic BH in an arbitrary distance under the condition that $a = b$ and $e/m = (1 + a^2)^{1/2}$. This is similar to the RN BH case. We also found that the static equilibrium is realized at a certain distance determined by $e/m$ and the dilaton coupling $a$ for the extreme dilatonic BH. Furthermore we examined the circular motion around the extreme dilatonic BH and found that the marginally stable radius for $a = 1$ becomes smaller than for corresponding RN BH with the same mass $M$.
and that the binding energy can become large, e.g., 100% for \( a = 1 \) and \( b > 0 \), 200% for \( e/m = (1 + a^2)^{1/2} \) and \( a = b > 1 \). For the cases one can expect more energy release from the dilatonic BH than that from the RN BH if there is the relevant mechanism like the Penrose process or superradiance. In section 5 we studied the scattering of a test particle by an extreme BH. The amount of the lensing effect depends on the dilatonic coupling \( a \). The amount of the lensing effect becomes larger for \( a = 1 \), smaller for \( a^2 = 1/3 \) than that for the RN BH. These facts on the dilatonic force would be of interest from cosmological and astrophysical viewpoints, if the massless dilaton has been existed in the very early universe.

Although we have treated some restricted cases, the feature of the test-particle motion in general situations could be investigated in similar methods. One can analyse the test particle motion around the extended object of the other types \([5]\), similarly in the pedagogical manner we have examined in the present paper. Besides, it will provide a simple and effective method to reveal critical values for parameters in such models.

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