Dissipative hydrodynamics in 2+1 dimensions

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1. Introduction

A large volume of experimental data from Au+Au collisions at RHIC are successfully analysed in an ideal hydrodynamic model \(^1\). However, experimental data do show deviation from ideal behavior. The ideal fluid description works well in almost central Au+Au collisions near mid-rapidity at top RHIC energy, but gradually breaks down in more peripheral collisions, at forward rapidity, or at lower collision energies, indicating the onset of dissipative effects. To describe such deviations from ideal fluid dynamics quantitatively, requires the numerical implementation of dissipative hydrodynamics.

Though the theories of dissipative hydrodynamics \(^2\) has been known for more than 30 years, significant progress toward its numerical implementation has only been made very recently \(^3\). At the Variable Energy Cyclotron Centre, Kolkata, we have developed a numerical code (AZHYDRO-KOLKATA) to solve, the 1st order dissipative hydrodynamics in 2+1 dimension (assuming boost-invariance in the longitudinal direction) and currently extending the code to 2nd order dissipative hydrodynamics. Some of the results from 1st order viscous hydrodynamics will be presented here.

2. 1st order dissipative fluid dynamics

Relativistic dissipative hydrodynamics has been discussed in detail in ref.\(^4\). In dissipative hydrodynamics, dissipative fluxes are assumed to be small. The entropy
current is expanded in terms of dissipative fluxes. In the 1st order theory \cite{2}, the expansion contain terms linear in dissipative fluxes, explicit form of which could be obtained by satisfying the second law of thermodynamics \( \partial_\mu S^\mu \geq 0 \). 1st order theories are acausal, signal can travel faster than light. This is corrected in 2nd order theory \cite{3}, entropy expansion contains terms quadratic in dissipative fluxes. Naturally, 2nd order theories are more complex.

We consider a QGP fluid in the central rapidity region with net zero baryon density \((n_B = 0)\). We also neglect all the dissipative effects (e.g. heat conduction and bulk viscosity) other than the shear viscosity. We work in the Landau energy frame. Energy-momentum tensor, including the shear pressure tensor \( \pi^{\mu\nu} \) is written as,

\[
T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \pi^{\mu\nu}
\]

where \( u^\mu \) is the hydrodynamic 4-velocity \((u^\mu u_\mu = 1)\) and \( \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \) is the projector, orthogonal to \( u^\mu \). \( \varepsilon = u_\mu T^{\mu\nu} u^\nu \) is the energy density, \( p \) is the hydrostatic pressure. \( T^{\mu\nu} \) satisfies the conservation law,

\[
\partial_\mu T^{\mu\nu} = 0
\]

In the first order theories \cite{2,3}, the shear stress tensors are written as,

\[
\pi^{\mu\nu} = 2\eta \nabla^{\langle\mu}u^{\nu\rangle} = 2\eta \left[ \frac{1}{2} (\Delta^{\mu\sigma} \Delta^{\nu\tau} + \Delta^{\nu\sigma} \Delta^{\mu\tau}) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\sigma\tau} \right]
\]

where \( \eta \) is the shear viscosity coefficient. \( \pi^{\mu\nu} \) is symmetric \((\pi^{\mu\nu} = \pi^{\nu\mu})\), traceless \((\pi^{\mu\mu} = 0)\) and transverse to hydrodynamic velocity, \((u_\mu \pi^{\nu\mu} = 0)\). The 16-component \( \pi^{\mu\nu} \) has only 5 independent components.

Heavy ion collisions are best described in terms of proper time \( \tau = \sqrt{t^2 - z^2} \) and rapidity \( \eta_s = \frac{1}{2} \ln \frac{t+z}{t-z} \). In \((\tau, x, y, \eta_s)\) coordinates, with longitudinal boost-invariance, the hydrodynamic 4-velocity can be written as,

\[
u^\mu = (u^\tau, u^x, u^y, u^{\eta_s}) = (\gamma_\perp, \gamma_\perp v_x, \gamma_\perp v_y, 0),
\]

with \( \gamma_\perp = 1/\sqrt{1 - v_x^2 - v_y^2} \). The energy-momentum conservation equations are,

\[
\partial_\tau \bar{T}^{\tau\tau} + \partial_x (\bar{T}^{\tau\tau} \bar{v}_x) + \partial_y (\bar{T}^{\tau\tau} \bar{v}_y) = -(p + \tau^2 \pi^{\eta\eta})
\]

\[
\partial_\tau \bar{T}^{\tau x} + \partial_x (\bar{T}^{\tau x} \bar{v}_x) + \partial_y (\bar{T}^{\tau x} \bar{v}_y) = -\partial_x (\bar{p} + \bar{\pi}^{xx} - \bar{\pi}^{\tau x} v_x) - \partial_y (\bar{\pi}^{xy} - \bar{\pi}^{x \tau} v_x)
\]

\[
\partial_\tau \bar{T}^{\tau y} + \partial_x (\bar{T}^{\tau y} \bar{v}_x) + \partial_y (\bar{T}^{\tau y} \bar{v}_y) = -\partial_x (\bar{p} + \bar{\pi}^{xy} - \bar{\pi}^{\tau y} v_x) - \partial_y (\bar{\pi}^{yy} - \bar{\pi}^{y \tau} v_y)
\]

where \( \bar{v}_x = T^{\tau x}/T^{\tau\tau} \) and \( \bar{v}_y = T^{\tau y}/T^{\tau\tau} \), and we have used the notation ”tilde” to represent quantities multiplied by the factor \( \tau \), \( \bar{p} = \tau p \) and similarly \( \bar{\pi}^{ij} = \tau \pi^{ij} \). We note that unlike in ideal fluid, in viscous fluid dynamics conservation equations contain additional pressure gradients containing the dissipative fluxes. Both \( T^{\tau x} \) and \( T^{\tau y} \) components of energy-momentum tensors now evolve under the influence of additional pressure gradients.
With boost-invariance, number of independent shear stress tensor reduces to 3. \( \pi^{\tau \tau} \), \( \pi^{xx} \) and \( \pi^{yy} \) can be chosen as independent components (there could be other choices also [8]),

\[
\pi^{\tau \tau} = 2\eta \left[ \frac{\theta}{3}(\gamma^2_{\perp} - 1) + \partial_\tau \gamma_{\perp} - \frac{1}{2}D(\gamma^2_{\perp}) \right],
\]

(8)

\[
\pi^{xx} = 2\eta \left[ -\partial_x(\gamma_{\perp}v_x) - \frac{1}{2}D(\gamma^2_{\perp}v_x^2) + \frac{\theta}{3}(1 + \gamma^2_{\perp}v_x^2) \right],
\]

(9)

\[
\pi^{yy} = 2\eta \left[ -\partial_y(\gamma_{\perp}v_y) - \frac{1}{2}D(\gamma^2_{\perp}v_y^2) + \frac{\theta}{3}(1 + \gamma^2_{\perp}v_y^2) \right]
\]

(10)

where \( D = u^\mu \partial_\mu \) is the convective time derivative, and \( \theta \) is the local expansion rate,

\[
\theta = \gamma_{\perp} \tau + \partial_\tau \gamma_{\perp} + \partial_x(v_x \gamma_{\perp}) + \partial_y(v_y \gamma_{\perp}).
\]

The other shear stress-tensors required in solving Eqs. 5, 6, 7 can be obtained from the constraints satisfied by \( \pi^{\mu \nu} \), (i) \( \pi^{\mu \mu} = 0 \): tracelessness, (ii) \( u^\mu \pi^{\mu \nu} = 0 \): transverse to \( u^\mu \).

\[
\tau^2 \pi^{\eta \eta} = \pi^{\tau \tau} - \pi^{xx} - \pi^{yy}
\]

(11)

\[
2v_x \pi^{xx} = \pi^{\tau \tau} + v_x^2 \pi^{xx} - v_y^2 \pi^{yy}
\]

(12)

\[
2v_y \pi^{yy} = \pi^{\tau \tau} - v_x^2 \pi^{xx} + v_y^2 \pi^{yy}
\]

(13)

\[
2v_x v_y \pi^{xy} = \pi^{\tau \tau} - v_x^2 \pi^{xx} - v_y^2 \pi^{yy}
\]

(14)

Shear stress-tensor components contains time derivatives of velocities \( v_x \) and \( v_y \). Thus at time step \( \tau_i \) one needs the still unknown time derivatives. In 1st order theories, this problem is circumvented by calculating the time derivatives from the ideal equation of motion , \( Du^\mu = \nabla^\mu p \) and \( D\varepsilon = -(\varepsilon + p)\nabla_\mu u^\mu \).

### 3. Equation of state, viscosity coefficient and initial conditions

Through the equation of state, the macroscopic hydrodynamic models make contact with the microscopic world. We have used the equation of state, EOS-Q, developed in ref[9]. It is a two-phase equation of state. The hadronic phase of EOS-Q is modeled as a non-interacting gas of hadronic resonance. The QGP phase is modeled as that of a non-interacting quarks (u,d and s) and gluons, confined by a bag pressure B. Adjusting the Bag pressure, the two phases are matched by Maxwell construction at the critical temperature, \( T_c = 164 MeV \).

Shear viscosity coefficient \( (\eta) \) of dense nuclear (QGP or resonance hadron gas) is quite uncertain. In perturbative regime, shear viscosity of a QGP is estimated [10], \( \eta = 86.473 \frac{1}{g_{\text{quark}}} T^3 \). With entropy of QGP, \( s = 37 \frac{2\pi^2}{15} T^3 \) and \( \alpha_s \approx 0.5 \), the ratio of viscosity over the entropy, in the perturbative regime is estimated as,

\[
\left( \frac{\eta}{s} \right)_\text{pert} \approx 0.135,
\]

(15)
However, QGP produced in nuclear collisions is non-perturbative. It is strongly interacting QGP. Recently, using the ADS/CFT correspondence, shear viscosity of a strongly coupled gauze theory, N=4 SUSY YM, has been evaluated, \( \eta = \frac{\pi}{8} N_c^2 T^3 \) and the entropy is given by \( s = \frac{\pi^2}{2} N_c^2 T^3 \). Thus in the strongly coupled field theory,

\[
\left( \frac{\eta}{s} \right)_{ADS/CFT} = \frac{1}{4\pi} \approx 0.08,
\]

(16)

To demonstrate the effect of viscosity on flow and subsequent particle production, we use both the perturbative and ADS/CFT estimate of viscosity.

Solving Eqs.5-7 require initial conditions. In the present demonstrative calculations, we have used the similar initial conditions as in ref. We just mention that in ref., initial transverse energy is parameterised geometrically, with 25% hard scattering. At initial time \( \tau_i=0.6 \) fm, the central entropy density is \( s_{int}=110 \text{ fm}^{-3} \). Fluid velocities are assumed to be zero initially. In dissipative hydrodynamics, dissipative fluxes need to be specified also. We assume that by the equilibration time \( \tau_i \), the dissipative fluxes attained their longitudinal boost-invariant values. The independent components at initial time are, \( \pi_{xx} = \frac{2}{\tau_i} \eta \), \( \pi_{yy} = \frac{2}{\tau_i} \eta \) and \( \pi_{\tau \tau} = 0 \).

![Contour plots](image1)

**Fig. 1.** Contours of constant local energy density in the x-y plane at \( \tau=5.2 \) fm.

![Contour plots](image2)

**Fig. 2.** Contours of constant temperature in \( \tau-x \) plane. y=0.

### 4. Evolution of the viscous fluid

To demonstrate the effect of viscosity, with the same initial conditions, we have solved the energy-momentum conservation equations for ideal fluid and viscous fluid.

In Fig. we have shown the constant energy density contour plot in x-y plane, after an evolution of 5.2 fm. The black lines are for ideal fluid evolution. The red and blue lines are for viscous fluid with ADS/CFT (\( \eta/s=0.08 \)) and perturbative (\( \eta/s=0.135 \)) estimate of viscosity. With viscosity fluid cools slowly. Cooling gets slower as viscosity increases. To obtain an idea of transverse expansion of viscous
fluid, as opposed to ideal fluid, in Fig.2, we have shown the constant temperature contours in \( \tau - x \) plane, at a fixed value of \( y = 0 \) fm. Transverse expansion is substantially enhanced in a viscous fluid. More the viscosity, more is the transverse expansion. The plot also indicate that at late time, fluid at \( x = y = 0 \) behaves similarly to a ideal fluid.

1st order theories are acausal, signal can travel faster than light. Acausality can lead to unphysical effects like reheating early in the collisions. However, we have not found any evidence of reheating. For small viscosity, with initial conditions as required in RHIC energy Au+Au collisions, effect of causality violation is minimum.

\[
\eta/s = 0 \\
\eta/s = 0.08 \\
\eta/s = 0.135
\]

5. Particle spectra and elliptic flow

Viscosity generates entropy and particle production is enhanced. Viscosity influences the particle production by (i) changing the freeze-out surface (freeze-out surface is extended) and (ii) by introducing a correction to the equilibrium distribution function. For small departure from the equilibrium, the non-equilibrium distribution function can be approximated as,

\[
f(x, p) = f^{(0)}(x, p)[1 + \phi(x, p)],
\]

\( \phi(x, p) \) is the deviation from equilibrium distribution function \( f^{(0)} \). With shear viscosity as the only dissipative force, \( \phi(x, p) \) can be locally approximated as,

\[
\phi(x, p) = C \pi_{\mu\nu} p^{\mu} p^{\nu}; \quad C = \frac{1}{2T^2(\varepsilon + p)}
\]

completely specifying the non-equilibrium distribution function. The standard Cooper-Frye prescription for particle production from freeze-out surface \( (\Sigma_{\mu}) \) can be employed to obtain the particle spectra. Detailed expressions for \( EdN/d^3p \) with non-local equilibrium distribution function Eqs.17,18 can be found in ref.9.
In Fig. 3, $p_T$ spectra for $\pi^-$, from freeze-out surface at $T_F = 158\text{ MeV}$ is shown. Particle production is increased in viscous dynamics. We also note that effect of viscosity is more prominent at large $p_T$ than at low $p_T$. $p_T$ spectra of pions are flattened with viscosity.

We have also calculated the elliptic flow in the model. Being a ratio, elliptic flow is very sensitive to the model. Experimentally, elliptic flow saturates at large $p_T$. In Fig. 4, $p_T$ dependence of elliptic flow is shown. Elliptic flow decreases with viscosity. As viscosity increases, elliptic flow reduces. We also note that both for ADS/CFT and perturbative estimate of viscosity, elliptic flow indicate saturation at large $p_T$. The result is very encouraging, as experimentally also elliptic flow tends to saturate at large $p_T$.

6. Summary and conclusions

In a 1st order theory of dissipative hydrodynamics, we have studied the boost-invariant hydrodynamic evolution of QGP fluid with dissipation due to shear viscosity. In this model study, we have considered two values of viscosity, the ADS/CFT motivated value, $\eta/s \approx 0.08$ and perturbatively estimated viscosity, $\eta/s \approx 0.135$. Both the ideal and viscous fluids are initialised similarly. Explicit simulation of ideal and viscous fluids confirms that energy density/temperature of a viscous fluid evolves slowly than its ideal counterpart. Transverse expansion is also more in viscous dynamics. For a similar freeze-out condition freeze-out surface is extended in viscous fluid.

We have also studied the effect of viscosity on particle production. Due to viscosity particle production is enhanced, more at large $p_T$. The elliptic flow on the otherhand reduces and shows a tendency of saturation at large $p_T$.

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