Langevin description of nonequilibrium quantum fields

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We consider the non-equilibrium dynamics of a real quantum scalar field. We show the formal equivalence of the exact evolution equations for the statistical and spectral two-point functions with a fictitious Langevin process and examine the conditions under which a local Markovian dynamics is a valid approximation. In quantum field theory, the memory kernel and the noise correlator typically exhibit long time power laws and are thus highly non-local, thereby questioning the possibility of a local description. We show that despite this fact, there is a finite time range during which a local description is accurate. This requires the theory to be (effectively) weakly coupled. We illustrate the use of such a local description for studies of decoherence and entropy production in quantum field theory.

I. INTRODUCTION

Understanding the dynamics of time evolving quantum systems is a key issue in many topical areas of physics, from early-universe cosmology to high-energy nuclear collisions, condensed matter physics or ultracold atomic gases. One important line of investigation in nonequilibrium field theory concerns the issue of first principle calculations of far-from-equilibrium (quantum) dynamics. Major breakthroughs have been achieved in recent years, in particular with the use of two-particle-irreducible functional techniques [1, 2], and the field is under active development, thanks to the advent of ever faster computers and of new ideas [3–5].

An important byproduct of such investigations is that it allows one to bridge the gap between relatively simple situations where direct calculations from the basic equations of quantum field theory (QFT) can be done and more intricate cases, such as, for instance, situations with expanding and/or inhomogeneous backgrounds [6–9], or situations involving many types of fields and interactions such as lepto/baryo-genesis scenarios [8–12], where such calculations are difficult and where one often has to rely on effective descriptions. The most popular such effective approaches are kinetic descriptions [8–9, 12–20], or effective Langevin descriptions [21–28]. A key issue in this context concerns the domain of validity of such approaches.

The derivation of kinetic – e.g. Boltzmann – equations from nonequilibrium Schwinger-Dyson, or Kadanoff-Baym equations have been extensively discussed since the early works of Kadanoff and Baym [8–13, 15–17, 21–29–33]. This typically relies on a gradient expansion, which assumes a clear separation between the scale of the nonequilibrium dynamics at hand and that of the elementary excitations and processes responsible for it. The validity of the gradient expansion and of the corresponding transport equations has been studied against direct solutions of the underlying nonequilibrium Schwinger-Dyson equations in cases where the latter can be done [34–39].

Effective Langevin descriptions involve further drastic simplifications, namely the assumption of effective local damping and Gaussian white noise, resulting in a Markovian dynamics [21–28]. This assumes a clear separation of scales between the so-called “memory time”, usually identified with the short time scale of irrelevant degrees of freedom, and the “relaxation time”, which characterizes the dynamics of the relevant degrees of freedom [40–42]. Existing comparisons between given non-Markovian Langevin dynamics and their Markovian versions, see e.g. [43–51] indicate that, indeed, the better the separation of scales, i.e. the weaker the local damping, the better the Markovian approximation.

As mentioned above, memory kernels can in principle be computed in a given theory from, say, loop diagrams. Existing calculations in simple theories with scalar, fermionic and/or gauge fields show that memory kernels and noise correlators typically decay as power law in time, thereby questioning the existence of a memory time and more generally the possibility of a local, Markovian description [52–55]. Of course, there are situations where one expects a local Markovian Langevin description to be questionable, such as in presence of massless (e.g. Goldstone or gauge) excitations and/or at zero temperature [56–58]. But the absence of a clear memory time scale mentioned here also occur in more standard situations with no infrared issues and at high temperatures, where one would expect a Brownian like motion to be a
good description.

We address this issue in the present paper. We first recall the basic evolution equations for nonequilibrium two-point Green’s functions, which involve non-local memory kernels (self-energies), and show their formal equivalence with a fictitious Langevin process. We discuss the basic conditions under which the memory integral and noise kernel can be replaced by local – mass and damping – terms and argue that these conditions do not require the existence of a local limit of the kernels themselves. In order to study the validity of a local Markovian limit of the equivalent fictitious Langevin process, we focus on a simple situation where an out-of-equilibrium test field is coupled to a thermal bath with negligible back-reaction. The basic QFT equations can then be exactly solved in terms of the equilibrium spectral function of the test field. We find that there exist a characteristic time below which the actual dynamics is indeed Markovian but after which memory effects cannot be neglected, as expected on rather general grounds [55, 57]. We investigate this time scale in detail in a simple model with cubic interactions between the test and the bath fields. Finally, we discuss the consistency of the local limit directly in real time, at the level of the equations of motion.

As an illustration, we end this paper with an application of such local description to the physics of decoherence and entropy production in QFT in the so-called incomplete description picture recently advocated in [58–63].

II. THE STRATEGY

A. General setting

We consider a generic $\mathbb{Z}_2$-symmetric scalar field theory in the symmetric phase. The nonequilibrium n-point functions can be conveniently described by means of time-ordered products of field operators on a closed contour $C$ in time [66, 67]. The two-point function $G(x, y) = \langle T_C \varphi(x) \varphi(y) \rangle$ encodes both the statistical and spectral correlators $F(x, y) = \frac{1}{2} \langle \{ \varphi(x), \varphi(y) \} \rangle$ and $\rho(x, y) = i \langle [\varphi(x), \varphi(y)] \rangle$ as

$$G(x, y) = F(x, y) - \frac{i}{2} \text{sgn}_C(x^0 - y^0) \rho(x, y). \quad (1)$$

Here, the brackets $\langle \ldots \rangle$ denote an average with respect to a given density matrix describing the (quantum) state of interest. The common practice is to specify the (out-of-equilibrium) initial conditions at a given finite time $t = 0$ [1, 2]. In the present work, we choose instead to prepare the system in a Gaussian thermal state at $t \to -\infty$ and to kick it away from equilibrium at $t = 0$ by means of an appropriate external source. Both procedures are in principle equivalent. The source approach is an efficient device to let the system develop its own non-Gaussian correlations [68, 69]. The latter are of particular importance e.g. when it comes to discussing renormalization – the ultraviolet modes have to be in the correct non-trivial vacuum state [69, 70]. We shall not be concerned with these issues in this paper and adopting the source approach is only a matter of technical convenience for later calculations. Still we present it in some detail because of its potential interest also for numerical calculations.

We consider a Gaussian nonequilibrium disturbance described by a bilocal source $K$ in the action [71]:

$$S_K[\varphi] = S[\varphi] + \frac{1}{2} \int_C d^4 x d^4 y \varphi(x) K(x, y) \varphi(y), \quad (2)$$

where $S$ is the classical action of the system. The Schwinger-Dyson equation for the two-point function on the contour $C$ reads:

$$G^{-1} = G_0^{-1} - \Sigma + K, \quad \text{where} \quad iG_0^{-1}(x, y) = -[\Box + M^2] \delta_2^{(4)}(x - y) \text{is the free propagator for the field of mass } M \text{ and } \Sigma(x, y) \text{ its self-energy. Using the standard decomposition [1, 2]}

$$\Sigma(x, y) = \Sigma_0(x) \delta_2^{(4)}(x - y) + \Sigma_F(x, y) = \frac{i}{2} \text{sgn}_C(x^0 - y^0) \Sigma_\rho(x, y). \quad (3)$$

and assuming Gaussian conditions at $t \to -\infty$, the Schwinger-Dyson equation translates into the following nonlinear coupled integro-differential equations for the statistical and spectral propagators

$$\begin{align*}
\Box x + M^2(x) \bigg] F(x, y) &= -\int_{-\infty}^{x^0} d^4 z \Sigma_\rho(x, z) F(z, y) + \int_{-\infty}^{y^0} d^4 z \Sigma^K_F(z, x) \rho(z, y), \\
\Box x + M^2(x) \bigg] \rho(x, y) &= -\int_{y^0}^{x^0} d^4 z \Sigma_\rho(x, z) \rho(z, y),
\end{align*} \quad (4)$$

where $\int_a^b d^4 z \equiv \int_a^b d z^0 \int d\mathbf{z}$, $M^2(x) = M^2 + \Sigma_0(x)$ and

$$\Sigma^K_F(x, y) = \Sigma_F(x, y) - K(x, y). \quad (5)$$

For later purposes it proves convenient to rewrite Eqs. (4)-(5) by introducing the retarded and advanced propagators $G_R(x, y) = \theta(x^0 - y^0)\rho(x, y)$ and $G_A(x, y) = G_R(y, x) = -\theta(y^0 - x^0)\rho(x, y)$ and similarly for the self-energies. One gets, from Eq. (5),

$$G_R^{-1}(x, y) = \Box x + M^2(x) \bigg] \delta^{(4)}(x - y) + \Sigma_R(x, y) \quad (7)$$

and similarly for $G_A(x, y) = G_R(y, x)$. Eqs. (4)-(5) can be written [72]

$$\begin{align*}
F &= -G_R * (\Sigma_F - K) * G_A, \\
\rho &= -G_R * \Sigma_\rho * G_A, \quad (8) \quad (9)
\end{align*}$$

with $(f * g)(x, y) = \int d^4 z f(x, z) g(z, y)$. Eqs. (7)-(10) actually provide an explicit solution if the self-energies are known. In general, however, the latter are non-linear functions of the two-point functions themselves.
The source $K$ serves to prepare out-of-equilibrium conditions near $x^0 = y^0 = 0$. For instance, we shall consider an instantaneous kick at $x^0 = y^0 = 0$:

$$K(x, y) = A(x, y)\delta(x^0)\delta(y^0) + B(x, y)\delta'(x^0)\delta(y^0) + C(x, y)\left[\delta'(x^0)\delta(y^0) + \delta(x^0)\delta'(y^0)\right].$$  \hspace{1cm} (10)

Before the kick, the system has had an infinite amount of time to reach an equilibrium state with parameters (e.g. temperature) determined by the conditions (e.g. energy density) at time $t \to -\infty$:

$$F(x, y)|_{x^0, y^0 < 0} = F_{eq}(x - y).$$  \hspace{1cm} (11)

Right after the kick, we write:

$$F(x, y)|_{x^0, y^0 > 0} = F_{neq}(x, y).$$  \hspace{1cm} (12)

The functions $A$, $B$ and $C$ in (10) are related to initial conditions for the nonequilibrium statistical function $F_{neq}$. From Eq. (8), we obtain

$$F(x, y) = -\{G_R \ast \Sigma_F \ast G_A\}(x, y) + \theta(x^0)\theta(y^0)F_K(x, y),$$  \hspace{1cm} (13)

with

$$F_K(x, y) = \left\{\rho(x^0, 0)A\rho(0, y^0) + \partial_u\rho(x^0, u)B\partial_v\rho(v, y^0) - \partial_u\rho(x^0, u)C\rho(0, y^0) - \rho(x^0, 0)C\partial_v\rho(v, y^0)\right\}u = v = 0,$$  \hspace{1cm} (14)

where the dot denote a convolution with respect to spatial variables only. Using $\partial_{x^0}\rho(x, y)|y^0 = x^0 = \delta(3)(x - y)$ we obtain the relations:

$$B(x, y) = F_K(x, y)|_{x^0 = y^0 = 0},$$
$$C(x, y) = \partial_x\rho F_K(x, y)|_{x^0 = y^0 = 0},$$
$$A(x, y) = \partial_x\partial_y F_K(x, y)|_{x^0 = y^0 = 0}.$$

Somewhat more intuitive expressions of the sources can be obtained by defining

$$B_{eq}(x, y) = F(x, y)|_{x^0, y^0 \rightarrow 0-},$$
$$C_{eq}(x, y) = \partial_x\partial_y F(x, y)|_{x^0, y^0 \rightarrow 0-},$$
$$K_{eq}(x, y) = \partial_x\partial_y F(x, y)|_{x^0, y^0 \rightarrow 0}.$$  \hspace{1cm} (16)

and

$$B_{neq}(x, y) = F(x, y)|_{x^0, y^0 \rightarrow 0+},$$
$$C_{neq}(x, y) = \partial_x\partial_y F(x, y)|_{x^0, y^0 \rightarrow 0+},$$
$$K_{neq}(x, y) = \partial_x\partial_y F(x, y)|_{x^0, y^0 \rightarrow 0+}.$$  \hspace{1cm} (17)

Assuming that the function $G_R \ast \Sigma_F \ast G_A$ is continuous at $x^0 = y^0 = 0$, one has

$$B(x, y) = B_{neq}(x, y) - B_{eq}(x, y),$$
$$C(x, y) = C_{neq}(x, y) - C_{eq}(x, y),$$
$$A(x, y) = K_{neq}(x, y) - K_{eq}(x, y).$$  \hspace{1cm} (18)

### B. Equivalent Langevin process

The evolution equations (17)-(19) are formally equivalent to a fictitious Langevin process $\varphi^\xi$ driven by a random noise $\xi$ [74]:

$$[\Box_x + M^2(x)] \varphi^\xi(x) + \int d^4z \Sigma_R(x, z)\varphi^\xi(z) = \xi(x) + \Xi(x),$$  \hspace{1cm} (19)

where $\Xi$ is a fluctuating disturbance source which serves to send the system away from equilibrium at a given finite time, in the same spirit as the source $K$ above. The random variables $\xi$ and $\Xi$ are uncorrelated.

Using (17), Eq. (19) can be rewritten as $G_R^{-1} \ast \varphi^\xi = \xi + \Xi$, whose solution is formally given by:

$$\varphi^\xi(x) = \varphi_0(x) + \int d^4z G_R(x, z) [\xi(x) + \Xi(z)],$$  \hspace{1cm} (20)

where $\varphi_0$ is the general solution of the homogeneous equation $G_R^{-1} \ast \varphi_0 = 0$.

We shall denote the average over realizations of the noise $\xi$ with an overline and the average over the source $\Xi$ with double brackets. The original quantum average is identified with $(\ldots) \equiv \langle(\ldots)\rangle$. Setting $\langle(\Xi(x))\rangle = 0$ and $\langle\Xi(x)\rangle = 0$ and choosing initial conditions such that $\varphi_0(x) = 0$ guarantees that $\langle\varphi(x)\rangle = \langle(\varphi^\xi(x))\rangle = 0$. To obtain the evolution equation for the two-point function

$$F(x, y) = \langle\langle\varphi^\xi(x)\varphi^\xi(y)\rangle\rangle,$$  \hspace{1cm} (21)

we write (19) as

$$\int d^4z G_R^{-1}(x, z)\varphi^\xi(z)\varphi^\xi(y) = [\xi(x) + \Xi(x)] \varphi^\xi(y).$$  \hspace{1cm} (22)

Inserting Eq. (20) on the right-hand-side and identifying

$$\Sigma_F(x, y) = -\xi(x)(x)$$ and $K(x, y) = \langle(\Xi(x)\Xi(y))\rangle$,  \hspace{1cm} (23)

one recovers Eq. (8).

The equivalence is complete provided the sources are chosen in an appropriate way. For the case of an instantaneous kick,

$$\Xi(x) = \Upsilon(x)\delta(x^0) + \Pi(x)\delta'(x^0),$$  \hspace{1cm} (24)

one gets

$$B(x, y) = \langle(\Pi(x)\Pi(y))\rangle,$$
$$C(x, y) = \frac{1}{2}\langle(\Upsilon(x)\Pi(y) + \Pi(x)\Upsilon(y))\rangle,$$
$$A(x, y) = \langle(\Upsilon(x)\Upsilon(y))\rangle,$$

with $\langle(\Pi(x)\Pi(y) - \Pi(x)\Pi(y))\rangle = 0$.

It is worth emphasizing that, in general, the noise correlator $\Sigma_F$ and the memory kernels $\Sigma$ depend nonlinearly on the correlators $F$ and $G_R$ and are thus nonlinear functions of the field $\varphi^\xi$ and the noise $\xi$ and their derivatives. In particular, this encodes all sorts of additive and/or multiplicative noises as well as linear and/or nonlinear damping terms that one encounters in usual (mostly perturbative) derivations of Langevin equations e.g. based on the influence functional technique [23] [26].
C. Approximation strategy

We now examine sufficient conditions under which the memory integral in the exact nonlocal Langevin equation \[ \text{(19)} \] can be approximately described by a local damping term. We shall see that there is no need a priori for assuming a finite memory time. For simplicity, we shall specialize to spatially homogeneous and isotropic situations. In particular the source \( K \) and, consequently, all other two-point functions in the problem can be spatially Fourier transformed:

\[
K(x, y) = \int \frac{d^3 p}{(2\pi)^3} e^{i p \cdot (x-y)} K_p(x^0, y^0) \tag{26}
\]

and similarly for all other two-point functions. The relevant equations read

\[
\left[ \partial_t^2 + \omega_p^2(t) \right] F_p(t, t') + \int_t^\infty du \, \Sigma_p^\phi(t, u) F_p(u, t') = - \int_{-\infty}^{+\infty} du \, \Sigma_p^{F,K}(t, u) G_p^A(u, t') \tag{27}
\]

and

\[
\left[ \partial_t^2 + \omega_p^2(t) \right] G_p^R(t, t') + \int_t^\infty du \, \Sigma_p^\phi(t, u) G_p^R(u, t') = \delta(t-t'), \tag{28}
\]

where \( \omega_p(t) = \sqrt{p^2 + M^2(t)} \) and \( \Sigma_p^{F,K} = \Sigma_p^F - K_p \).

In generic theories, the memory kernel \( \Sigma_p^\phi(t, t') \) is typically a decreasing function of \( t-t' \). For instance, writing the interaction term of the field degree of freedom of interest \( \varphi_p(t) \) as \( S_{\text{int}} = \int_{t'}^t \varphi_p(t) J_p(t) \), where the current \( J_p \) involves other degrees of freedom, possibly including \( \varphi_{p'} \neq p \), one has, at lowest order in the interaction,

\[
\Sigma_p^\phi(t, t') \propto \langle [J_p(t), J_p(t')] \rangle. \tag{29}
\]

Such current-current correlator typically exhibits rapid – e.g. power law \[ 10 \] – decay at large \( t-t' \) and the memory integrals in Eqs. \[ 27 \] and \[ 28 \] are dominated by their upper bound.

As for the functions \( F_p \) and \( G_p^R \), they are generally characterized by various – possibly time-dependent – time scales: an oscillation frequency \( \epsilon_p(t) \), the damping scale in the relative time \( t-t' \) and the typical scale of the nonequilibrium evolution of equal-time correlators – the last two might actually be power laws. The most common situation – e.g. for weakly coupled (effective) theories – is that the former is the shortest time scale in the problem. We shall assume a clear separation of scales, where the oscillation frequency is always much shorter than the damping or nonequilibrium scales.

Now, if the range of integration which dominates the memory integrals is short compared to the above-mentioned damping and nonequilibrium scales, one can neglect the latter under these integrals and replace the full functions \( F_p \) and \( G_p^R \) by their short-time oscillating part around the upper bound of the integral, e.g. \[ 70 \]:

\[
F_p(u, t') \approx F_p(t, t') \cos \epsilon_p(t)(u-t) - \frac{\partial_t F_p(t, t')}{\epsilon_p(t)} \sin \epsilon_p(t)(u-t), \tag{30}
\]

with some relative error controlled by the ratios of either the damping or nonequilibrium scales over the oscillation frequency. The memory integral in \[ 27 \] can thus be approximated as

\[
\int_{-\infty}^{t} du \, \Sigma_p^\phi(t, u) F_p(u, t') \approx \delta \epsilon_p^2(t) F_p(t, t') + 2\gamma_p(t) \partial_t F_p(t, t'), \tag{31}
\]

and

\[
\delta \epsilon_p^2(t) \equiv \text{Re} \tilde{\Sigma}_p^R(t; \epsilon_p(t)), \tag{32}
\]

\[
\gamma_p(t) \equiv -\frac{\text{Im} \tilde{\Sigma}_p^R(t; \epsilon_p(t))}{2\epsilon_p(t)}, \tag{33}
\]

where we used the definition \( \Sigma_p^R(t, t') = \theta(t-t') \Sigma_p^\phi(t, t') \) and introduced the mixed time-frequency representation

\[
\Sigma_p^R(t, t') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tilde{\Sigma}_p^R(t; \omega). \tag{34}
\]

Similar manipulations hold for the memory integral in Eq. \[ 28 \].

If the above conditions are met, the equation for \( F_p \) and \( G_p^R \) thus read

\[
\left[ \partial_t^2 + 2\gamma_p(t) \partial_t + \epsilon_p^2(t) \right] F_p(t, t') = - \int_{-\infty}^{+\infty} du \, \Sigma_p^{F,K}(t, u) G_p^A(u, t') \tag{35}
\]

and

\[
\left[ \partial_t^2 + 2\gamma_p(t) \partial_t + \epsilon_p^2(t) \right] G_p^R(t, t') = \delta(t-t'), \tag{36}
\]

where we defined \( \epsilon_p^2(t) = \omega_p^2(t) + \delta \epsilon_p^2(t) \). Using similar manipulations as in the previous subsection, it is easy to check that these equations also have an equivalent Langevin process which reads:

\[
\left[ \partial_t^2 + 2\gamma_p(t) \partial_t + \epsilon_p^2(t) \right] \varphi_p^R(t) = \xi_p(t) + \Xi_p(t). \tag{37}
\]

Note that at this stage, the memory kernel has been replaced by a local damping term but the noise correlator can still be non-trivial, i.e. non-Markovian. Note also that, here, the damping rate and the noise correlator are still, in general, complicated non-linear functions of the propagators.

III. EXACTLY SOLVABLE EXAMPLE: NEAR STATIONARY SYSTEM

We now want to investigate in more details the condition under which the above local approximation is valid.
In this section we specialize to a simple situation where the out-of-equilibrium degrees of freedom weakly interact with a stationary – e.g. thermal – background with negligible backreaction. In this case, the full nonequilibrium dynamics can be exactly solved [18, 33, 44] and one can check whether the exact solution can be described by a Langevin dynamics with local damping.

A. Exact solution

The assumption of negligible backreaction means that the self-energies \( \Sigma_{F,\rho} \) are determined by the stationary background and thus are time-translation invariant: \( \Sigma_{F,\rho}^F(t, t') = \Sigma_{F,\rho}^F(t-t') \). It follows that the spectral function is also time-translation invariant: \( \rho_p(t, t') = \rho_p(t-t') \) [18]. Indeed, using the time-translation invariance of \( \Sigma_{F,\rho} \), one easily checks that the equation of motion for \( \rho_p(t, t') \), see e.g. [28], only explicitly involves the time difference \( t-t' \) and thus possesses time-translation invariant solutions. Moreover, since the equal-time commutation relations, which determine the initial conditions for \( \rho_p \), are preserved under time-evolution, the only possible solution has this symmetry. The equation of motion for \( \rho_p(t, t') \) reads:

\[
\left[ \frac{\partial_t^2}{\omega_p^2} + 1 \right] \rho_p(t) + \int_0^t d\tau \, \Sigma_p^F(t-\tau) \rho_p(\tau) = 0 ,
\]

(38)

with \( \rho(0) = 0 \) and \( \dot{\rho}(0) = 1 \). It follows that the functions \( G^R_{F,A} \) and, in turn, \( F_{eq} = -G_R \ast \Sigma_F \ast G_A \) are also time-translation invariant.

Introducing the frequency representation

\[
\rho_p(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\rho}_p(\omega)
\]

(39)

and similarly for all other time-translation invariant functions \( G^R_{F,\rho} \) and \( \Sigma_{F,\rho}^F \), one readily obtain

\[
\tilde{G}_p^F(\omega) = \frac{1}{-(\omega + i0^+)^2 + \omega_p^2 + \Sigma_p^R(\omega + i0^+) ,}
\]

(40)

as well as

\[
\tilde{F}_{eq}^p(\omega) = -i\tilde{G}_p^R(\omega) \Sigma_p^F(\omega) ,
\]

(41)

\[
\tilde{\rho}_p(\omega) = -i\tilde{G}_p^R(\omega) \Sigma_p^F(\omega) .
\]

(42)

It is useful to recall that \( \tilde{\rho}_p(\omega) = 2i \text{Im} \tilde{G}_p^R(\omega) \) and similarly for the self-energy components [27]. Notice the detailed balance relation

\[
\tilde{F}_{eq}^p(\omega) \Sigma_p^p(\omega) = \tilde{\rho}_p(\omega) \tilde{\Sigma}_p^p(\omega) ,
\]

(43)

characteristic of stationary systems. In the following we assume a thermal background to fix the ideas although the argument does not depend on this assumption. In thermal equilibrium at temperature \( T = 1/\beta \), the fluctuation-dissipation relation

\[
\tilde{F}_{eq}^p(\omega) = -i \left( n(\beta \omega) + \frac{1}{2} \right) \tilde{\rho}_p(\omega)
\]

(44)

implies

\[
\tilde{\Sigma}_p^F(\omega) = -i \left( n(\beta \omega) + \frac{1}{2} \right) \tilde{\Sigma}_p^p(\omega) ,
\]

(45)

with \( n(x) = 1/(\exp(x) - 1) \) and thus there is only one independent function, e.g. \( \tilde{\rho}_p(\omega) \).

For an instantaneous kick at \( t = t' = 0 \) as in [10]:

\[
K_p(t, t') = A_p \delta(t) \delta(t') + B_p \delta(t) \delta'(t')
\]

(46)

the complete nonequilibrium solution for \( F_p \) reads, see Eqs. (13 - 10):

\[
F_p(t, t') = F_{eq}^p(t-t') + \theta(t)\theta(t') F^K_p(t, t'),
\]

(47)

with

\[
F^K_p(t, t') = A_p \rho_p(t) \rho_p(t') + B_p \dot{\rho}_p(t) \dot{\rho}_p(t') + C_p [\dot{\rho}_p(t) \rho_p(t') + \rho_p(t) \dot{\rho}_p(t')] ,
\]

(48)

where

\[
B_p = F^K_p(t, t') \big|_{t=t'=0} ,
\]

(49)

\[
C_p = \dot{\partial}_t F^K_p(t, t') \big|_{t=t'=0} ,
\]

(49)

\[
A_p = \partial_t F^K_p(t, t') \big|_{t=t'=0} .
\]

Thus we see that, in the simple situation considered here, the nonequilibrium dynamics is fully described by the second term on the right-hand-side of Eq. (17): The momentum modes of interest get kicked away from equilibrium at \( t = 0 \) and the relaxation toward equilibrium, described by Eq. (28), is completely encoded in the (equilibrium) spectral function \( \rho_p(t) \).

B. Breit-Wigner approximation

In general, the spectral function \( \tilde{\rho}_p(\omega) \) may exhibits single-particle poles as well as branch cuts in the complex frequency plane, corresponding to multiparticle processes or Landau damping, see e.g. [52, 53, 78]. For instance, near the poles, it takes the usual Breit-Wigner form [79]

\[
\tilde{\rho}_p(\omega \rightarrow \epsilon_p) \approx \frac{Z_p}{\epsilon_p (\omega - \epsilon_p)^2 + \gamma_p^2} ,
\]

(50)

where we assume \( \gamma_p/\epsilon_p \ll 1 \). Here,

\[
Z_{p}^{-1} = 1 - \frac{d}{d\omega^2} \text{Re} \tilde{\Sigma}_p^R(\omega) \big|_{\omega=\epsilon_p}
\]

(51)

and the quasiparticle energy \( \epsilon_p \) and width \( \gamma_p \) are determined from

\[
\epsilon_p^2 = \omega_p^2 + \text{Re} \tilde{\Sigma}_p^R(\epsilon_p)
\]

(52)

and

\[
\gamma_p = -Z_p \text{Im} \tilde{\Sigma}_p^R(\epsilon_p) \frac{2\epsilon_p}{\epsilon_p} .
\]

(53)
Poles lead to exponential damping of the Fourier transform $\rho_p(t)$ at large time, whereas branch cuts result in power law behavior \(^{52, 53}\). The large-time behavior is, typically,

$$
\rho_p(t) \approx Z_p \frac{\sin \epsilon_p t}{\epsilon_p} e^{-\gamma_p |t|} + \rho_p^{\text{pow}}(t),
$$

where the last term denotes the power law contribution. Using \(^{44}\) one can obtain the corresponding large-time behavior of the equilibrium statistical function $F_p^{\text{eq}}(t-t')$:

$$
F_p^{\text{eq}}(t) \approx \kappa_p \left( \cos \epsilon_p t + \frac{2p}{\epsilon_p} \sin \epsilon_p |t| \right) e^{-\gamma_p |t|} + F_p^{\text{pow}}(t),
$$

with

$$
\kappa_p = \frac{Z_p}{\epsilon_p} \left( n (\beta \epsilon_p) + \frac{1}{2} \right).
$$

The power law contributions to both $\rho_p(t)$ and $F_p^{\text{eq}}(t)$ are due to features of the self-energies $\tilde{\Sigma}^F_p(\omega)$ and their amplitudes are thus governed by the strength of the interaction with the thermal bath, i.e. by some power of the relevant coupling constant. This is to be contrasted with the pole contributions, where the interaction strength essentially controls the decay rate $\gamma_p$ but only gives corrections to the amplitude. Thus, for weak coupling, there exist a range of time during which the exponentially decaying contributions dominate over the power laws. Only at very late time is the dynamics described by the latter.

Writing schematically the power law contributions as $\sim A/(\mu t)^\nu$, with $A$ an amplitude proportional to some positive power of the relevant coupling constant or, equivalently, of the damping rate $\gamma_p$, $\mu$ some scale of the problem at hand and $\nu$ a given exponent \(^{81}\), one finds that the power law contributions are of relative order $\gamma_p/\epsilon_p$ for times

$$
\gamma_p t \lesssim \ln \left\{ \left( \frac{\mu}{\gamma_p} \right)^\nu \frac{\gamma_p}{\epsilon_p \Lambda} \right\},
$$

In this time range, the nonequilibrium relaxation \(^{48}\) is also essentially exponential:

$$
F_p^K(t, t') \approx F_p^0(t, t') e^{-\gamma_p (t+t')},
$$

with

$$
F_p^0(t, t') = a_p^+ \cos \epsilon_p (t-t') + a_p^- \cos \epsilon_p (t+t') + a_p^0 \sin \epsilon_p (t+t'),
$$

where the constant $a_p^\pm$ and $a_p^0$ are related to initial conditions \(^{49}\)

$$
a_p^+ = \frac{Z_p^2}{2} \left( B_p \pm \frac{\gamma_p}{\epsilon_p} C_p + \gamma_p B_p \right),
$$

$$
a_p^0 = \frac{Z_p^2 C_p - \gamma_p B_p}{\epsilon_p}.
$$

Thus the complete nonequilibrium solution takes a remarkably simple form in the time range \(^{57, 81}\). In the weak coupling limit both $\gamma_p/\mu_p$ and $\epsilon_p \Lambda_p$ are small and the latter can be relatively long. In that case, the deviation from equilibrium \(^{58}\) as well as the equilibrium unequal-time correlators \(^{54}\) and \(^{55}\) have essentially completely decayed when the power law contributions start becoming important and most of the relevant dynamics is thus well described by the exponential – pole – contributions. As we now discuss, the latter can be described by an equivalent Markovian Langevin dynamics.

C. Markovian Langevin dynamics

In the time range \(^{57}\), the dynamics is dominated by the poles of the retarded Green’s function \(^{40}\). Those which are closest to the real axis give the dominant late-time contribution. Assuming that the dynamics is driven by these poles and that the Breit-Wigner approximation \(^{50}\) is justified, one can make the replacement $G_p^R(\omega) \rightarrow \hat{G}_{\text{eff}, p}(\omega)$ with \(^{82}\)

$$
\hat{G}_{\text{eff}, p}(\omega) = \frac{Z_p}{-(\omega + i \gamma_p)^2 + \Omega_p^2},
$$

where $\Omega_p^2 = \epsilon_p^2 - \gamma_p^2 \approx \epsilon_p^2$. The corresponding spectral function reads

$$
\hat{\rho}_{\text{eff}, p}(\omega) = \frac{4i Z_p \gamma_p \omega}{(\omega^2 - \epsilon_p^2)^2 + 4 \gamma_p^2 \omega^2},
$$

which reproduces Eq. \(^{50}\) near the poles. The Fourier transform reads

$$
\rho_{\text{eff}, p}(t) = Z_p \frac{\sin \epsilon_p t}{\epsilon_p} e^{-\gamma_p |t|}.
$$

The function \(^{61}\) is the frequency space representation of the retarded Green’s function’s value of the operator $Z_p^{-1} [\partial_t^2 + 2 \gamma_p \partial_t + \epsilon_p^2] \delta(t - t')$. It follows that the nonequilibrium relaxation \(^{58}\) can be described by a Langevin process with local damping:

$$
[\partial_t^2 + 2 \gamma_p \partial_t + \epsilon_p^2] \phi_p^\xi(t) = \xi_p(t) + \Xi_p(t)
$$

with appropriate sources $\Xi_p(t)$. The equilibrium state is characterized by the $\xi_p(t)$ correlator. A Gaussian white noise

$$
\overline{\xi_p(t)\xi_p(t')} = Z_p^{-1} \alpha_p \delta(t - t')
$$

gives, after simple calculations,

$$
\hat{F}_{\text{eff}, p}^{\text{eq}}(\omega) = \frac{Z_p \alpha_p}{(\omega^2 - \epsilon_p^2)^2 + 4 \gamma_p^2 \omega^2}
$$

or, equivalently,

$$
F_{\text{eff}, p}^{\text{eq}}(t) = \frac{Z_p \alpha_p}{4 \gamma_p \epsilon_p} \left( \cos \epsilon_p t + \frac{\gamma_p}{\epsilon_p} \sin \epsilon_p |t| \right) e^{-\gamma_p |t|}.
$$
Choosing
\[ \alpha_p = 4\gamma_p \epsilon_p \left( n(\beta \epsilon_p) + \frac{1}{2} \right), \tag{68} \]
one reproduces the exponential term in \[ \rho \]. Thus the simple Langevin dynamics \[ \text{(44)} \] with local damping and white (Markovian) noise related by \[ \text{(68)} \] correctly describes the full nonequilibrium dynamics of the previous subsection in the time range \[ \text{(67)}. \]

In the present case, both the local damping and the Markovian nature of the noise follow from the Breit-Wigner approximation and the assumption of a thermal bath. Indeed the Breit-Wigner ansatz \[ \text{(61)}, \text{ see also (50)}, \] which leads to local damping \( \sim \gamma_p \partial_t \), implicitly assumes that \( \Sigma_p^F(\omega) \) is sufficiently smooth on a range \( |\epsilon_p - \omega| \lesssim \gamma_p \). For the case of a thermal bath, the fluctuation-dissipation relation \[ \text{(44)} \] guarantees that \( \Sigma_p^F(\omega) \) has the same property. Assuming that the poles give the dominant contribution to \( F_{\Sigma_p}^F(t) \) amounts to neglecting the frequency dependence of \( \Sigma_p^F \) around the pole and replacing \( \Sigma_p^F(\omega) \approx \Sigma_p^F(\epsilon_p) \) in Eq. \[ \text{(41)}. \] In terms of the equivalent Langevin description, this corresponds to white noise. We see that, in principle, one may have local damping but non-local, colored noise e.g. in the case of some non-thermal background.

Finally, let us recall that the relation \[ \text{(68)} \] follows from the on-shell fluctuation-dissipation relation \[ \text{(44)} \]. Indeed, the effective Langevin dynamics corresponds to the following effective self-energy kernels:
\[ \Sigma_{\text{eff.}}^R(\omega) = \delta \epsilon_p^2 - 2i Z_p^{-1} \gamma_p \omega, \tag{69} \]
with \( \delta \epsilon_p^2 = \text{Re} \Sigma_{\text{eff.}}^R(\epsilon_p) \) and
\[ \Sigma_{\text{eff.}}^F(\omega) = -Z_p^{-1} \alpha_p = \Sigma_{\text{eff.}}^F(\epsilon_p). \tag{70} \]
In particular, one has \( \Sigma_{\text{eff.}}^p(\omega) = -4i Z_p^{-1} \gamma_p \omega \) and thus, on-shell,
\[ \Sigma_{\text{eff.}}^p(\epsilon_p) = -4i Z_p^{-1} \gamma_p \epsilon_p = \Sigma_{\text{eff.}}^F(\epsilon_p). \tag{71} \]

Eq. \[ \text{(68)} \] then directly follows from \[ \text{(44)} \] taken at \( \omega = \epsilon_p \). It generalizes the standard relation \( \alpha_p = 4\gamma_p T \) valid at high temperature \( \beta \epsilon_p \ll 1 \). Equivalently, the equilibrium correlators \[ \text{(62)} \] and \[ \text{(67)} \] also satisfy the fluctuation-dissipation relation on-shell. Here, the fluctuation-dissipation relations \[ \text{(44)} \] only hold on-shell, as a consequence of the assumption that the dynamics is governed by the poles.

To close this subsection, let us remark that the effective memory kernel and noise correlator are completely local:
\[ \Sigma_{\text{eff.}}^p(t) = -Z_p^{-1} \alpha_p \delta(t), \tag{72} \]
and
\[ \Sigma_{\text{eff.}}^R(t) = \delta \epsilon_p^2 \delta(t) + 2Z_p^{-1} \gamma_p \delta'(t). \tag{73} \]

As explained in the introduction, this seems in contradiction with actual calculations of such functions in various models, which typically give power laws in time \[ \text{(52)} \]. We present an explicit example in the next section. Then, we study in Sec. \[ \text{V} \] how the apparent contradiction is resolved.

## IV. AN EXPLICIT EXAMPLE

To illustrate the point, we consider a simple model with non-trivial dissipation \[ \text{(18)} \] where the (thermal) background is represented by a scalar field \( \chi \) of mass \( m \) interacting with the system field \( \varphi \) via a \( \varphi \chi^2 \) interaction:
\[ S = -\int d^4x \left\{ \frac{1}{2} \rho \left( \square + M^2 \right) \varphi + \frac{g}{2} \varphi^2 \chi^2 + \frac{1}{2} \chi \left( \square + m^2 \right) \chi + V(\chi) \right\}. \tag{74} \]

We assume weak enough interaction between the two fields and strong enough \( \chi \) self interactions such that back reaction can be neglected and \( \chi \) remains in thermal equilibrium at temperature \( T \). We shall compute the relevant self-energies in perturbation theory at lowest non-trivial order, assuming a free thermal gas for the \( \chi \) field. The spectral and statistical components of the space-time translation-invariant self-energy of the system field \( \Sigma(x, y) \equiv \Sigma(x - y) \) read, at one-loop,
\[ \Sigma_p(x) = -g^2 F_\chi(x) \rho_\chi(x), \tag{75} \]
\[ \Sigma_F(x) = -\frac{g^2}{2} \left( F_\chi^2(x) - \frac{1}{2} \rho_\chi^2(x) \right), \tag{76} \]
where \( F_\chi \) and \( \rho_\chi \) denote the free statistical and spectral two-point function of the field \( \chi \), respectively. Their spatial Fourier transforms read \( (\omega_p = \sqrt{p^2 + m^2}) \)
\[ F_\chi^p(t) = \left( n(\beta \omega_p) + \frac{1}{2} \right) \frac{\cos \omega_p t}{\omega_p}, \quad \rho_\chi^p(t) = \frac{\sin \omega_p t}{\omega_p}. \tag{77} \]

For simplicity, we consider only the self-energies \[ \text{(75)} - \text{(76)} \] at zero momentum in this section. The discussion is similar for non-vanishing momentum but is a bit more involved due to Landau damping effects which contribute extra features in the complex frequency plane.

Before going on, we mention a peculiarity of the present model with cubic interaction vertex: the zero mode of the statistical self-energy \( F_{p=0}^\chi(t) \) acquires a time-independent contribution and thus does not decay to zero at large times. This is most easily seen directly in the mixed-time-momentum representation:
\[ \Sigma_{p=0}(t) = -\frac{g^2}{2} \int \frac{d^3q}{(2\pi)^3} \left\{ |F_q^\chi(t)|^2 - \frac{1}{4} \left[ \rho_q^\chi(t) \right]^2 \right\}. \tag{78} \]

The constant contribution arises from the first term on the right-hand-side, using \[ \text{(77)} \] and \( 2 \cos^2 \omega_q t = \cos 2\omega_q t + 1 \). The actual value of this constant is
not important for the present discussion and we refer the interested reader to Appendix A for more details. In the following we denote by $\Sigma_{\rho=0}^F (t \to \infty) = \sigma_\infty$ (non constant contributions vanish at large time, see below). In the rest of this section, we omit the index $p = 0$ for simplicity.

The self-energy \[\Sigma F(\omega)\] read, in momentum-frequency space,\[\tilde{\Sigma}_\rho(\omega) = -i \frac{g^2}{4\pi} \omega (\omega^2 - 4m^2)^3 \sqrt{1 - \frac{4m^2}{\omega^2}} \left[ n \left( \frac{\beta \omega}{2} \right) + \frac{1}{2} \right],\] and (see Appendix A for a discussion of the $\delta(\omega)$ term)\[\tilde{\Sigma}_F(\omega) = -i \left( n(\beta \omega) + \frac{1}{2} \right) \tilde{\Sigma}_\rho(\omega) + \sigma_\infty \delta(\omega).\] In Eq. (78), one recognizes the two particle threshold at $\omega \geq 2m$ which implies that the on-shell damping rate \[\tilde{G}_R(\omega)\] is non-zero only if $M > 2m$. The square root factor is the standard threshold function for a two body decay and is responsible for the branch cut singularity in the complex frequency plane, which governs the late time power law behavior. Finally the factor $n(\beta \omega) + 1/2$ is due to Bose enhancement and yields poles at imaginary frequencies $\beta \omega_p = 4n\pi$, with $n \in \mathbb{Z}$.

The late time behavior of the Fourier transforms $\Sigma_{\rho,F}(t)$ can be obtained by standard contour integration techniques. For $m \neq 0$, the regime $tT \gg 1$ is dominated by the two particle threshold at $\omega = 2m$. We obtain, see also \[\tilde{G}_R(\omega) = \frac{g^2}{16\pi M} \sqrt{1 - \frac{4m^2}{\omega^2}} \left[ n \left( \frac{\beta M}{2} \right) + \frac{1}{2} \right].\]

The first line of Eq. (74) is reminiscent of the fluctuation-dissipation relation \[\tilde{G}_R(\omega)\] and the second line uses the identity\[\left( n(2x) + \frac{1}{2} \right) \left( n(x) + \frac{1}{2} \right) = \frac{1}{2} \left[ \left( n(x) + \frac{1}{2} \right)^2 + \frac{1}{4} \right].\]

As announced, the memory and noise kernels $\Sigma_{\rho,F}(t)$ exhibit highly non-local power law behaviors at large times. Still, as explained from rather general considerations in the previous section, there is a well-defined time regime in which the two-point correlators $F(t)$ and $\rho(t)$ can be accurately described by a Markovian Langevin dynamics with local kernels. To make the argument more precise in the present model, we now compute the explicit late-time behavior of these correlators. Again this can be done by standard contour integration techniques using Eqs. (11) and (12). The calculation is actually rather similar as the previous one for self-energies with two more pairs of complex conjugate poles in the complex frequency plane coming from the retarded and advanced propagators $G_R(\omega)$ and $G_A(\omega) = G_R(\omega)$, located at $\pm M - i\gamma$ and $\pm M + i\gamma$ [34], with

\[\gamma = \frac{i \tilde{\Sigma}_\rho(M)}{4M} = \frac{g^2}{16\pi M} \sqrt{1 - \frac{4m^2}{M^2}} \left[ n \left( \frac{\beta M}{2} \right) + \frac{1}{2} \right].\]

The late-time behavior, $Tt \gg 1$, is governed by these poles and by the branch cuts $|\omega - 2m| \geq \gamma$ along the real frequency axis. If the poles are sufficiently far from the threshold $\omega = 2m$, the pole and cut contributions are well separated. Here we assume $M \gg m$ for simplicity. We get, for the spectral function,

\[\rho(t) \approx \frac{\sin(Mt)}{M} e^{-\gamma |t|} + \rho^{\text{pow}}(t)\]

and, for the equilibrium statistical one,

\[F^{\text{eq}}(t) \approx \left( n(\beta M) + \frac{1}{2} \right) \cos \frac{M}{M} e^{-\gamma |t|} + F^{\text{pow}}(t) + F_\infty.\]

In both cases, the first term on the right-hand-side is due to the poles whereas the second ones arise from the branch cuts near threshold:

\[\rho^{\text{pow}}(t) = -\left| \tilde{G}_R(2m) \right|^2 \sigma^{\text{pow}}_\rho(t) \approx \frac{\sigma^{\text{pow}}_\rho(t)}{(M^2 - 4m^2)^2}\]

and similarly for $F^{\text{pow}}(t)$ with $\sigma^{\text{pow}}_\rho(t) \to \sigma^{\text{pow}}_F(t)$. Finally the statistical correlator does not decay to zero at large $t$ because of the constant $\sigma_\infty$ in [32].

\[F_\infty = \sigma_\infty |\tilde{G}_R(0)|^2 \approx \frac{\sigma_\infty}{M^4}.\]

In Eqs. (88)-(90), we used the exact expression

\[\left| \tilde{G}_R(\omega) \right|^2 = M^2 + \text{Re} \tilde{\Sigma}_R(\omega) - \omega^2 + \frac{1}{4} \Sigma_{\rho}(\omega),\]

as well as the fact that at the two-particle threshold $\Sigma_{\rho}(2m) = 0$, see Eq. (79).

We are now in a position to obtain a more precise estimate of the time range [57] in which a Langevin description with local memory kernel (and thus, in the present case, also a white, Markovian noise as explained in the previous section) provides a valid description in the present model. The power law contribution $\rho^{\text{pow}}$ is
at most of relative order \( \gamma/M \) as compared to the exponential one in (87) for times \( t \gtrsim t^L_\rho \) with, in the limit where \( \gamma t^L_\rho \gg 1 \),

\[
\gamma t^L_\rho \approx \ln \left\{ \left( \frac{m}{\gamma} \right)^{3/2} \frac{M}{m} \left( 1 - \frac{4 m^2}{M^2} \right) \right\},
\]

(92)

where we used (85) as well as Eq. (86) to trade the coupling \( g^2 \) for the damping rate \( \gamma \). We shall refer to \( t^L_\rho \) as the Langevin time. Demanding a similar constraint on the power law contribution in (88) leads to a slightly more constraining condition \( (t^L_F < t^L_\rho) \) for \( M > 2m \):

\[
\gamma t^L_\rho \approx \ln \left\{ \left( \frac{m}{\gamma} \right)^{3/2} \frac{M}{m} \right\}.
\]

(93)

Let us quote the relatively simpler expressions of the Langevin time \( t^L_\rho \) in the following cases. At low temperature, \( 1 \ll \beta m \ll \beta M \),

\[
\gamma t^L_\rho \approx \ln \left( \frac{m}{\gamma} \right)^{3/2} \frac{M}{m};
\]

(94)

At intermediate temperature, \( \beta m \ll 1 \ll \beta M \),

\[
\gamma t^L_\rho \approx \ln \left( \frac{m}{\gamma} \right)^{3/2} \frac{M}{T};
\]

(95)

At high temperatures, \( \beta m \ll \beta M \ll 1 \),

\[
\gamma t^L_\rho \approx \ln \left( \frac{m}{\gamma} \right)^{3/2}.
\]

(96)

It is interesting to notice that the Langevin time is governed by the ratio \( \gamma/m \) and not \( \gamma/M \).

We close this section by mentioning that in the case \( M < 2m \), the on-shell damping rate \( \gamma = 0 \) such that one has formally \( t^L_\rho,F \to \infty \) at one-loop. In this case, where dissipative processes \( \varphi \to \chi \chi \) are kinematically forbidden, the pole contributions in Eqs. (87)–(88) are not damped and the power laws never dominates, hence \( t^L_\rho,F \to \infty \). In fact, in this case damping arises at two-loop order. One has \( \gamma \propto g^2 \) and the Langevin time is parametrically reduced by a ln \( g^2 \sim \ln \sqrt{\gamma} \) contribution.

V. CONSISTENCY CHECK

We have shown an explicit example where the relevant self-energies exhibit a highly non local behavior, but where a local Langevin description still provides an accurate description in the time range (89). It is instructive, for a deeper understanding of how this comes about as well as for discussing more intricate situations where an exact solution of the non-equilibrium equations of motion is not available, to analyze this issue directly in real time,

at the level of the memory integrals in Eqs. (27)–(28), in the spirit of our discussion in Subsec. 2.C. To this aim, we shall perform the following consistency check, coming back to the general discussion of Sec. III C. We will check whether the power law decay of the memory kernel \( \Sigma^p_\rho(t) \) is sufficient for the validity of the local approximation (81).

We begin with Eq. (28) or, equivalently, Eq. (38) for the spectral function \( \rho^p_\rho(t) \). It is convenient to write the effective solution (83) as

\[
\rho^\text{eff}_\rho(t) = \frac{\sin \epsilon_p t}{\epsilon_p} e^{-\gamma_p |t|} = \frac{1}{\epsilon_p} \text{Im} \left\{ e^{-i \epsilon_p t - \gamma_p |t|} \right\},
\]

(97)

which leads, for \( t > 0 \), to:

\[
\rho^\text{eff}_\rho(t) + \gamma_p \rho^\text{eff}_\rho(t) = \text{Re} \left\{ e^{-(i \epsilon_p + \gamma_p)t} \right\}. \tag{98}
\]

Expressing \( \rho^p_\rho(t - \tau) \) in terms of \( \rho^p_\rho(t) \) and \( \partial_t \rho^p_\rho(t) = \dot{\rho}^p_\rho(t) \), we get

\[
\rho^\text{eff}_\rho(t - \tau) - \rho^\text{eff}_\rho(t) \text{Re} \left[ e^{i \tau} \right] - \frac{\dot{\rho}^\text{eff}_\rho(t) + \gamma_p \rho^\text{eff}_\rho(t)}{\epsilon_p} \text{Im} \left[ e^{i \tau} \right], \tag{99}
\]

with \( z = i \epsilon_p + \gamma_p \). Using this expression to evaluate the memory integral in Eq. (83), we obtain

\[
\int_0^t d\tau \Sigma^p_\rho(t - \tau) \rho^\text{eff}_\rho(t) \rho^\text{eff}_\rho(t - \tau) = \int_0^t d\tau \Sigma^p_\rho(\tau) \rho^\text{eff}_\rho(\tau - \tau) - \frac{\dot{\rho}^\text{eff}_\rho(t) + \gamma_p \rho^\text{eff}_\rho(t)}{\epsilon_p} \text{Im} \sigma^\gamma_\rho(t), \tag{100}
\]

where

\[
\sigma^\gamma_\rho(t) = \int_0^t d\tau \Sigma^p_\rho(\tau) e^{i \tau}. \tag{101}
\]

We are interested in the large time behavior of the integral (101). We already see that for too large time the power law suppression due to the memory kernel \( \Sigma^p_\rho(t) \) is not enough to overcome the exponential \( e^{\gamma t} \propto e^{\gamma \rho t} \) and the integral diverges exponentially, signalling the breakdown of the effective description (87). To evaluate the late time \( (\epsilon_p t \gg 1) \) behavior of the integral (101) more precisely, we introduce a time separation \( \eta \), such that \( 1/\epsilon_p < \eta \ll 1/\gamma_p \) and \( \eta \ll t \), and write

\[
\sigma^\gamma_\rho(t) \approx \int_0^\eta d\tau \Sigma^p_\rho(\tau) e^{i \epsilon_p \tau} + \int_\eta^t d\tau \Sigma^p_\rho(\tau) e^{i \tau}, \tag{102}
\]

where we neglected the exponential growth in the first term on the right-hand side. For \( \epsilon_p \eta \gg 1 \), the upper bound contribution to the latter is suppressed by the oscillating factor and one has

\[
\int_0^\eta d\tau \Sigma^p_\rho(\tau) e^{i \epsilon_p \tau} \approx \int_\eta^{\infty} d\tau \Sigma^p_\rho(\tau) e^{i \epsilon_p \tau} = \tilde{\Sigma}_\rho^R(\epsilon_p). \tag{103}
\]
This is the required contribution for Eq. (100) to reduce to the desired local description, up to \( O(\gamma_p/\epsilon_p) \) contributions, see Eq. (105) below.

To evaluate the second term on the right-hand side of (102), we use the large time, power law behavior of the memory kernel, which we parametrize as

\[
\Sigma_p(t) \sim \frac{a_p}{(\mu \rho)^\nu} \sigma(\mu t),
\]

where \( a_p \) is a dimensionful amplitude parameter, typically proportional to some positive power of the relevant coupling constant, \( \mu \) is a characteristic mass scale, \( \sigma(x) \approx 1 \) is a bounded (e.g. periodic) function and \( \nu \) a given exponent. We show in Appendix B that, for \( |zt| \to \infty \),

\[
\int_0^t d\tau \Sigma_p(\tau) e^{zt} \sim \frac{a_p}{z} \frac{e^{zt}}{(\mu t)^{\nu}} \sum_{n \geq 0} \left( \frac{-\mu}{z} \right)^n \sigma^{(n)}(\mu t),
\]

(105)

where \( \sigma^{(n)}(x) = d^n \sigma(x)/dx^n \). Recalling the definitions (32) and (33), we finally obtain, for \( \epsilon, t \gg 1 \),

\[
\text{Re} \sigma_p^z(t) = \text{Re} \bar{\Sigma}^R_p(\epsilon_p) \left[ 1 + O\left( \frac{a_p}{\epsilon_p \delta \epsilon_p (\mu t)^\nu} \right) \right]
\]

and

\[
\text{Im} \sigma_p^z(t) = \text{Im} \bar{\Sigma}^R_p(\epsilon_p) \left[ 1 + O\left( \frac{a_p}{\epsilon_p \delta \epsilon_p (\mu t)^\nu} \right) \right].
\]

(106)

(107)

As long as the last terms in brackets in the above equations can be neglected, the memory integral (100) has the desired form:

\[
\int_0^t d\tau \Sigma_p(t - \tau) \rho_p^{\text{eff}}(\tau) \approx \delta \epsilon_p^2 \rho_p^{\text{eff}}(t) + 2 \gamma_p \epsilon_p^{\text{eff}}(t),
\]

(108)

up to \( O(\gamma_p/\epsilon_p) \) corrections, with

\[
\delta \epsilon_p^2 = \text{Re} \bar{\Sigma}^R_p(\epsilon_p), \quad \gamma_p = -\frac{\text{Im} \bar{\Sigma}^R_p(\epsilon_p)}{2 \epsilon_p}.
\]

(109)

The ansatz (97) is indeed a solution of Eq. (58) with (108) and the local approximation is thus consistent within a finite time range governed by the terms in brackets in Eqs. (100)-(107). For the local description – which in the present case gives rise to an exponentially decaying solution – to make sense, the latter should be larger than \( \gamma_p^{-1} \) which typically requires a weak coupling situation. The ratio \( a_p/\gamma_p \delta \epsilon_p^2 \) is parametrically of order one in the relevant coupling, whereas the ratio \( a_p/\gamma_p \epsilon_p \) may either be of order one in the case where damping is allowed at lowest order in perturbation theory, or be enhanced by inverse powers of the coupling if damping is only possible at higher orders. Thus the most stringent restriction comes from Eq. (107) and we check that the Langevin time, for which the local description makes sense, i.e.

\[
\gamma_p t^L \approx \ln \left\{ \left( \frac{\mu}{\gamma_p} \right)^\nu \frac{\gamma_p \epsilon_p^2}{a_p} \right\},
\]

(110)

agrees with the analysis of the previous sections, see Eq. (57), or Eq. (12) for the model discussed in Sec. IV.

A similar – although more lengthy – analysis can be performed for the \( F \)-equation with the same conclusion, namely that, within the time range (110),

\[
\int_0^{\infty} d\tau \Sigma_p(t - \tau, t') \approx \delta \epsilon_p^2 F_p^{\text{eff}}(t, t') + 2 \gamma_p \partial_t F_p^{\text{eff}}(t, t'),
\]

(111)

up to corrections of relative order \( O(\gamma_p/\epsilon_p) \), where \( F_p^{\text{eff}}(t, t') \) is given by Eq. (117) with \( F_p^K(t, t') \) given by (58) and \( F_p^{\text{eq}}(t - t') \) replaced by \( F_p^{\text{eff}}(t - t') \) in (67). This completes our consistency check for the full non-equilibrium solution.

VI. APPLICATION: DECOHERENCE, ENTROPY PRODUCTION AND THERMALIZATION

We end this paper with a simple application of the previous considerations to the physics of quantum decoherence, entropy production and thermalization in quantum field theory in the context of the so-called incomplete description picture, recently advocated in [58–65]. This is based on the observation that one’s ability to measure the state of a quantum field is limited because of a restricted access to the infinite tower of \( n \)-point correlation functions: One typically has only access to low order correlators, most often to the subset of one- and two-point functions. The lack of knowledge of higher-order correlators may result, from the point of view of the observer, in effective loss of quantum purity and/or coherence and associated entropy production, even to effective thermalization [59, 60].

Let us consider the case where only the subset of independent equal-time two-point functions

\[
F_p(t) = F_p(t, t')|_{t=t'},
\]

(112)

\[
R_p(t) = \partial_t F_p(t, t')|_{t=t'},
\]

(113)

\[
K_p(t) = \partial_t \partial_t^* F_p(t, t')|_{t=t'},
\]

(114)

of a given field mode is measured at each time. From this knowledge one can reconstruct the least biased quantum state compatible with the measured correlators and infer its quantum properties. The corresponding density operator is a Gaussian in the field operators, characterized by the intrinsic – canonically invariant – occupation number

\[
n_p(t) = \frac{1}{2} \sqrt{F_p(t)K_p(t) - R_p(t)}. \]

(113)

The latter measures the quantum purity of the Gaussian state [50]:

\[
P_p(t) = \frac{1}{2n_p(t) + 1},
\]

(114)

which is equal to its maximum value 1 \( (n_p = 0) \) for a pure state. The occupation number also measures
the phase space area covered by the Gaussian state in the Wigner representation and can be related to the Gaussian entropy \[ s_p(t) = [n_p(t) + 1] \ln[n_p(t) + 1] - n_p(t) \ln n_p(t) , \] which measures the amount of missing information in the subset of measured correlators, that is the amount of information stored in unmeasured higher-order correlators.

Non-intrinsic – i.e. basis dependent – properties of the inferred Gaussian quantum state can be characterized by introducing another occupation number (note that 0 ≤ \( n_p(t) ≤ n_p \))
\[ \bar{n}_p(t) + \frac{1}{2} = \frac{K_p(t) + \epsilon_p^2 F_p(t)}{2 \epsilon_p} . \]

For instance, the squeezing, or coherence parameter
\[ g_p(t) = \sqrt{1 - \left( \frac{n_p(t) + 1/2}{\bar{n}_p(t) + 1/2} \right)^2} \]
measures the degree of quantum entanglement/coherence in the basis of semi-classical coherent states. A state with \( g_p \to 1 \) exhibits non-trivial correlations between macroscopically distant semi-classical states and thus a high degree of quantum coherence (in this basis) \[ 53 59 \].

Now to the dynamics. We consider the simple situation of Sec. \[ III \] where the field of interest is weakly coupled to a thermal bath with negligible backreaction. The exact solution of the nonequilibrium dynamics is given by Eqs. \[ (117)-(119) \], with \[ (120) \]. In the small width limit, the Langevin time \[ (57) \] is large and the power law contribution in Eqs. \[ (55) \] and \[ (58)-(60) \] can be neglected, resulting in a Markovian dynamics as explained in Subsec. \[ IIII \]. Reshuffling the various terms (hence the bar on \( F_p^0 \) below), one can write, for \( t, t' > 0 \),
\[ F_p(t, t') = F_p^eq(t - t') \left( 1 - e^{-\gamma_p |t-t'|} \right) \]
\[ + F_p^0(t, t') e^{-\gamma_p (|t|+|t'|)} , \]
(118)
where
\[ F_p^0(t, t') = \bar{a}_p^+ \cos \epsilon_p (t-t') + \bar{a}_p^- \cos \epsilon_p (t+t') + \bar{a}_p^0 \sin \epsilon_p (t+t') , \]
(119)
with, neglecting \( O(\gamma_p/\epsilon_p) \) corrections and setting \( Z_p \equiv 1 \),
\[ \bar{a}_p^\pm = \frac{1}{2} \left( F_p^0(0) \pm \frac{K_p(0)}{\epsilon_p^2} \right) , \]
\[ \bar{a}_p^0 = \frac{R_p(0)}{\epsilon_p} . \]
(120)

In the Breit-Wigner – or Markov – approximation considered here, the equal-time equilibrium two-point correlators are characterized by the on-shell equilibrium occupation number \( n_p^eq = n(\beta \epsilon_p) \):
\[ F_p^eq(0) = \frac{1}{\epsilon_p} \left( n_p^eq + \frac{1}{2} \right) , \]
(121)
with \( R_p^eq(0) = 0 \) and \( K_p^eq(0) = \epsilon_p^2 F_p^eq(0) \). It is an simple exercise to compute the various quantities \[ (113)-(117) \]. One obtains
\[ n_p(t) + \frac{1}{2} = \sqrt{ \left[ N_p^eq(t) \right]^2 + 2 N_p^eq(t) \bar{N}_p^0(t) + \left[ N_p^0(t) \right]^2} , \]
(122)
where we defined
\[ N_p^eq(t) = \left( n_p^eq + \frac{1}{2} \right) (1 - e^{-2\gamma_p t}) \],
(123)
\[ \bar{N}_p^0(t) = \left( \bar{n}_p(0) + \frac{1}{2} \right) e^{-2\gamma_p t} \]
(124)
and
\[ N_p^0(t) = \left( n_p(0) + \frac{1}{2} \right) e^{-2\gamma_p t} . \]
(125)

The occupation number \( \bar{n}_p(t) \) and the decoherence parameter \( g_p(t) \) have remarkably simple expressions:
\[ \bar{n}_p(t) = n_p^eq (1 - e^{-2\gamma_p t}) + \bar{n}_p(0) e^{-2\gamma_p t} \]
(126)
and
\[ g_p(t) = g_p(0) \frac{\bar{n}_p(0) + 1/2}{\bar{n}_p(t) + 1/2} e^{-2\gamma_p t} . \]
(127)

Note that, as expected, an initially incoherent, or thermal-like state with \( g_p(0) = 0 \) or, equivalently \( n_p(0) = \bar{n}_p(0) \), remains so: \( g_p(t) = 0 \), or \( \bar{n}_p(t) = \bar{n}_p(0) \). A particular case is \( n_p(0) = \bar{n}_p(0) = 0 \), which corresponds to a pure, vacuum-like state. Preparing such a state at time \( t = 0 \), one observes an effective loss of quantum purity due to the incomplete knowledge of higher order correlators. Indeed, in that case, the Gaussian purity \[ (114) \] reads
\[ P_p(t) = \frac{P_p^eq}{1 - (1 - P_p^eq)e^{-2\gamma_p t}} . \]
(128)
where $P_{eq} = 1/(2n_{eq}p + 1)$. This loss of purity naturally corresponds to a loss of information and thus to a growth of entropy. Fig. 1 shows the purity and the entropy as a function of $x = 2\gamma_p t$ for this case.

Let us now consider the case of an initial pure state, $n_p(0) = 0$, with a high degree of quantum coherence $g_p(0) \rightarrow 1$, i.e. $\bar{n}_p(0) \gtrsim 1$, which corresponds to a squeezed vacuum-like state. Again because information about the state of the systems spreads towards higher order correlators, one observes apparent loss of quantum purity and associated entropy growth as well as apparent loss of quantum coherence. This is illustrated in Figs. 2 and 3 for $n_{eq}p = 3$ and $n_p(0) = 1, 3, 10$.

The case $\bar{n}_p(0) = n_{eq}p$, which implies that $\ddot{n}_p(t) = -n_{eq}p$ is particular. For instance, one gets

$$n_p(t) + \frac{1}{2} = \left( n_{eq}p + \frac{1}{2} \right) \sqrt{1 - \frac{n_{eq}p(n_{eq}p + 1)}{n_{eq}p + 1/2} e^{-4\gamma_p t}},$$

and thus the Gaussian purity and entropy approach their late time equilibrium value at a rate $4\gamma_p$, twice the rate of the case $\bar{n}_p(0) \neq n_{eq}p$. This is illustrated in the insert of Fig. 2. Also the coherence factor has a purely exponential decay law

$$g_p(t) = g_p(0) e^{-2\gamma_p t},$$

shown in Fig. 3.

For the case $\bar{n}_p(0) > n_{eq}p$ purity rapidly falls off and undershoots – entropy correspondingly overshoots – its equilibrium value before exponentially approaching it at a rate $2\gamma_p$. The time $t_{ext}$ at which an extremum is reached (minimum of $P_p$ or maximum of $s_p$) decreases when $\bar{n}_p(0)$ is increased but remains bounded: $2\gamma_p t_{ext} > \ln 2$. Thus the time scale of purity loss/entropy growth is always set by $2\gamma_p$. Observe finally that cases with $\bar{n}_p(0) > n_{eq}p$ have faster loss of purity or growth of entropy than cases with $\bar{n}_p(0) < n_{eq}p$, but they have slower loss of coherence.

VII. SUMMARY AND CONCLUSION

To summarize, we have seen that the dynamics of non-equilibrium quantum fields can always be formally written as an equivalent, non-local, non-linear Langevin process with self-consistently determined memory and noise kernels. A drastic simplification is achieved when the memory integral can be approximated by a local damping term. It is often assumed that this requires the existence of a finite, short memory time scale beyond which the memory kernel essentially vanishes. The typical situation in QFT is, however, that the latter exhibit a power law behavior at large times. Still, such a power law decay, if not enough to completely localize the memory integrals, is sufficient to make the dynamics essentially local in a finite time interval – before late-time power laws come to dominate – at least if the associated local damping rate is weak enough.

We have illustrated the above points in a simple example with non-equilibrium degrees of freedom interacting with a stationary thermal bath with negligible back-reaction, where an analytical solution of the QFT problem is possible. We believe our analysis clarifies the validity of local descriptions of non-equilibrium field dynamics, as well as of related Breit-Wigner motivated approximations, employed in a wide range of problems, see e.g. [13, 19, 20, 32, 33]. Finally, we have provided a general method to work out the local limit directly at level of the equations of motion, which is useful when a complete solution is not known, for instance in non-stationary situations, or when back-reaction is important – in which case the damping rate as well as the noise kernel may depend self-consistently on the non-equilibrium correlators.

As an illustration, we have applied the above ideas to the description of quantum decoherence and entropy production in QFT in a simple situation with a stationary thermal environment along the lines of Refs [59, 60]. Interesting further applications of the methods developed here include, for instance, the study of decoherence of primordial fluctuations in inflationary scenarios [58], or the discussion of local damping descriptions in expanding geometries, which underly e.g. warm inflationary scenarios [25, 50] or the standard perturbative description of reheating in the early Universe [87].

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However, it leads to a non-vanishing contribution when multiplied by \( n(\beta \omega) + 1/2 \) to compute \( \Sigma_{p=0}^F(\omega) \) as demanded by the fluctuation dissipation relation. Indeed, using the identity

\[
\left[ n(\beta k_0) + \frac{1}{2} + n(\beta(\omega - k_0)) + \frac{1}{2} \right] \left( n(\beta \omega) + \frac{1}{2} \right) = \left( n(\beta k_0) + \frac{1}{2} \right) \left( n(\beta(\omega - k_0)) + \frac{1}{2} \right) + \frac{1}{4}. \tag{A6}
\]

which is easily demonstrated by using \( 2n(x) + 1 = \tanh^{-1}(x/2) \) and making use of standard trigonometric relations, one obtains a non-vanishing term in \( \Sigma_{p=0}^F(\omega) \):

\[
\left[- \left( n(\beta k_0) + \frac{1}{2} \right)^2 + \frac{1}{4} \right] \delta(\omega). \tag{A7}
\]

One easily checks that this term is also obtained by a direct calculation of \( \Sigma_{p=0}^F(\omega) \), without using the fluctuation dissipation relation. Finally, one checks that the final coefficient of \( \delta(\omega) \) is precisely the constant \( \sigma_\infty \) obtained by the direct calculation in the time-momentum representation described above.

**Appendix B: Asymptotic behavior of \( \sigma_p^\infty(t) \)**

This section is devoted to the asymptotic behavior of \( \int_\eta^t d\tau \Sigma_p^\rho(\tau) e^{z\tau} \) in Eq. (102) as \( |zt| \to \infty \). Integrating \( n \) times by part reads:

\[
\int_\eta^t d\tau \Sigma_p^\rho(\tau) e^{z\tau} = \sum_{k=0}^{n-1} \frac{(-1)^k}{z^{k+1}} \left( e^{z t} s_k(t) - e^{z \eta} s_k(\eta) \right)
+ (-1)^n \int_\eta^t d\tau s_n(\tau) e^{z\tau} z^n, \tag{B1}
\]

where \( s_n(t) = \frac{d^n}{d\tau^n} \Sigma_p^\rho(\tau) \). We recall that in the generic case considered here \( \Sigma_p^\rho(\tau) \) is a power law and therefore \( C^n \). Thus for any finite interval \([\eta, t], \int_\eta^t d\tau s_n(\tau) e^{z\tau} / z^n \to 0 \) as \( n \to \infty \). This allows us perform an infinite amount of integration by parts which gives

\[
\int_\eta^t d\tau \Sigma_p^\rho(\tau) e^{z\tau} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} \left( e^{zt} s_n(t) - e^{z\eta} s_n(\eta) \right). \tag{B2}
\]

Moreover, since \( \Re z > 0 \), the contribution from the lower boundary is exponentially suppressed. Using the generic ansatz (103) and the fact that \( \sigma \) is bounded, we get:

\[
\int_\eta^t d\tau \Sigma_p^\rho(\tau) e^{z\tau} \sim \frac{a_p e^{zt}}{z(\mu_p t)^\nu} \mathcal{F}(\mu_p t, \mu_p^z), \tag{B3}
\]

where

\[
\mathcal{F}(x, y) = \sum_{n \geq 0} (-y)^n \sigma^{(n)}(x) [1 + \mathcal{O}(1/x)]. \tag{B4}
\]
The above expression simplifies when the function $\sigma(x)$ is periodic and thus satisfies $\sigma^{(2p)}(x) = (-1)^p \sigma(x)$ and $\sigma^{(2p+1)}(x) = (-1)^p \sigma'(x)$. Splitting the sum into odd and even terms, one gets, for $|y| < 1$,

$$
F(x, y) = \frac{\sigma(x) - y\sigma'(x)}{1 + y^2} [1 + O(1/x)] . \quad (B5)
$$

As $|\mu_p^2/z^2| = \mu_p^2/(\epsilon_p^2 + \gamma_p^2) < 1$ we get

$$
\int_0^t d\tau \Sigma_p^\rho(\tau) e^{-\tau} \sim \frac{e^{\eta t}}{\mu_p(\mu_p')^\nu} \frac{z\sigma(\mu t) - \mu\sigma'(\mu t)}{z^2 + \mu^2} . \quad (B6)
$$

In the case where $\mu \ll \epsilon_p$ this further simplifies to

$$
\int_0^t d\tau \Sigma_p^\rho(\tau) e^{-\tau} \sim \frac{e^{\eta t}}{\epsilon_p(\mu_p')^\nu} = \frac{e^{\eta t}}{\epsilon_p} \Sigma_p(t) . \quad (B7)
$$
In general one should add the general solution of the homogeneous equation $G_{R \ast F} = 0$ but the latter typically decays with time and has an infinite time to decay away in the present set-up where initial conditions are prepared at $t \to -\infty$.

Initial conditions for the spectral function $\rho$ are fixed by equal-time commutation relations.

This (exact) equivalence has been repeatedly noticed in the literature for the case where the field of interest is linearly coupled to an environment on which it does not backreact, see e.g. [19, 40, 44]. In more general cases, the influence functional approach of Feynman and Vernon allows one to derive approximate Langevin equations [23, 40, 47]. To our knowledge, the formal but exact equivalence described in the present paper has not been pointed out before.

[75] G. D. Moore, (2002), hep-ph/0211281.

This is somewhat similar to so-called harmonic, or one-frequency ansatz employed in Refs. [23, 42, 43]. Here, the convolution with the memory kernel $\Sigma_\rho$ controls the convergence of the integral and gives better control of the approximation.

[77] It is also useful to recall that $\tilde{G}_\rho^R(-\omega) = [\tilde{G}_\rho^R(\omega)]^*$, $\tilde{\rho}_\rho^R(-\omega) = -\tilde{\rho}_\rho^R(\omega) = [\tilde{\rho}_\rho^R(\omega)]^*$, $\tilde{F}_\rho^R(-\omega) = \tilde{F}_\rho^R(\omega) = [\tilde{F}_\rho^R(\omega)]^*$ and similarly for the various self-energy components.

[78] M. Drewes, (2010), 1012.5380.

There is a similar expression near the symmetric pole $\omega \to -\epsilon_\rho$ – recall that $\tilde{\rho}_\rho^R(-\omega) = -\tilde{\rho}_\rho^R(\omega)$.

[80] All these quantities may in principle depend on the momentum $p$. We omit the $p$ index for simplicity.

[81] The solution obtained here generalizes that of Ref. [38], which corresponds to the particular nonequilibrium conditions $a_p^R = a_p^A = 0$.

[82] We choose this ansatz for convenience although the poles are slightly displaced by a negligible amount $\sim \gamma_p^2/\epsilon_\rho$ along the real axis. In particular, it exactly reproduces Eq. (50) – with no $\gamma_p^2$ correction – when $\omega \to \epsilon_\rho$.

[83] We emphasize that this constant only appears for the mode $p = 0$. Also, it is a particular feature of the present model with cubic interaction vertex. For instance, it is easy to check that there is no such constant term for the lowest non-trivial contribution to damping (a two-loop setting-sun diagram) for a theory with quartic interaction vertex.

[84] We assume the small width limit $\gamma/M \ll 1$ throughout and neglect the mass correction $\sim \text{Re}\Sigma_R(M)$ as well as the field renormalization [54] for simplicity.

[85] All these quantities may in principle depend on the momentum $p$. We omit the $p$ index for simplicity.

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