Regularity of complexified hyperbolic equations with integral conditions

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ABSTRACT
This paper considers hyperbolic wave equations with non-local in time conditions involving integrals with respect to time. It is shown that regularity of the solution can be achieved for complexified problem with integral conditions involving harmonic complex exponential weights. The paper establishes existence, uniqueness, and regularity of the solutions.

1. Introduction
The most common type of boundary conditions for evolution partial differential equations are Cauchy initial conditions. It is known that these conditions can be replaced, in some cases, by non-local conditions, for example, including integrals over time intervals. There is a significant number of works devoted to these boundary values problems.

For parabolic equations with non-local in time boundary conditions, some results and references can be found, e.g. in [1, 2]. For Schrödinger equations, some results and references can be found in, e.g. [3–6]. In [3, 4, 6], the non-local in time condition was dominated by the initial value, and the approach was based on the contraction mapping theorem. In [3, 6], the nonlocal conditions connected solutions in a finite set of times. In [4], the conditions were quite general and allowed to include integrals with respect to time. In [5], integral conditions without dominating initial value have been considered.

For hyperbolic equations with mixed highest order derivative, regularity results were obtained in [7]. In [8], systems of hyperbolic equations with integral conditions have been considered.

For hyperbolic wave equations, related results and references can be found, e.g. in [9–16]. In [9–12, 15], regularity results were obtained for hyperbolic wave equations with a variety of integral conditions with respect to state variables.
In [13, 14, 16], hyperbolic wave equations under integral conditions with respect to time have been considered. In [13, 14], the eigenfunction expansion method has been used, and the regularity result has been affected by the so-called ‘small denominators’ (‘small divisors’) problem that often causes instability of solutions for hyperbolic wave equations with non-local in time conditions. The solvability was obtained in [13, 14] for the case where the spectrum for the inputs and solutions does not contain resonance points. In [16], a regularity condition without these restrictions on the spectrum has been obtained for the hyperbolic wave equation with a Laplacian. This condition imposed certain restrictions on the kernel in the integral condition, which has vanish with a certain rate at the end of the time interval.

The paper readdresses the problem of regularity for solutions of boundary value problems for hyperbolic wave equations with non-local in time conditions. The paper suggests a complexification of boundary value problems that ensures regularity of the solutions. This complexification requires to consider integral conditions \( \int_0^T e^{i\omega t} u(t) \, dt = g \), for real nonzero \( \omega \) that can be arbitrarily small, where \( u(t) \) is the solution of the hyperbolic wave equation with real coefficients that take values in an appropriate Hilbert space. This allows to bypass, for this particular setting, the ‘small denominators’ (or ‘small divisors’) problem. We establish existence, uniqueness, and regularity of the solutions for the complexified equation. The proofs are based on the spectral expansion, similarly to the setting from [1, 5, 17, 18]. The eigenfunction expansion for the solution is presented explicitly. This allows to derive a numerical solution.

The rest of the paper is organized as follows. In Section 2, we introduce a boundary value problem with averaging over time, and we present the main result (Theorem 2.1). In Section 3, we present the proofs. Section 4 gives a numerical example of the impact of the presence of small \( \omega \neq 0 \) on the appearance of small denominators.

Section 5 presents conclusions and discusses future research.

### 1.1. Some definitions

For a Banach space \( X \), we denote the norm by \( \| \cdot \|_X \). For a Hilbert space \( X \), we denote the inner product by \( (\cdot, \cdot)_X \). We denote the Lebesgue measure and the \( \sigma \) -algebra of Lebesgue sets in \( \mathbb{R}^n \) by \( \ell_n \) and \( \mathcal{B}_n \), respectively.

Let \( \mathcal{D} \subset \mathbb{R}^n \) be a domain, and let \( H = L_2(\mathcal{D}, \mathcal{B}_n, \ell_1; \mathbb{C}) \) be the space of complex-valued functions. Let \( A : H \to H \) be a self-adjoint operator defined on an everywhere closed subset \( D(A) \) of \( H \) such that if \( u \) is a real valued function then \( Au \) is also a real valued function.

Let \( H_{BC} \) be a set of functions from \( H \) such that \( H_{BC} \) is everywhere dense in \( H \), that the set \( D(A) \cap H_{BC} \) is everywhere dense in \( H \), and that \( (u, Av)_H \) is finite and defined for \( u, v \in H_{BC} \) as a continuous extension from \( D(A) \times D(A) \).

Consider an eigenvalue problem

\[
Av = -\lambda v, \quad v \in H_{BC}.
\]

We assume that this equation is satisfied for \( v \in H_{BC} \) if

\[
(w, Av)_H = -\lambda (w, v)_H \quad \forall \, w \in H_{BC}.
\]
Assume that there exists a basis \( \{ v_k \}_{k=1}^\infty \subset H_{BC} \) in \( H \) such that
\[
(v_k, v_m)_H = 0, \quad k \neq m, \quad \| v_k \|_H = 1,
\]
and that \( v_k \) are eigenfunctions for (1), i.e.,
\[
Av_k = -\lambda_k v_k,
\]
for some \( \lambda_k \in (0, +\infty) \) such that \( \lambda_k \to +\infty \) as \( k \to +\infty \).

These assumptions imply that the operator \( A \) is self-adjoint with respect to the boundary conditions defined by choice of \( H_{BC} \).

For \( q = -1, 0, 1, 2 \), let \( H^q \) be the Hilbert spaces obtained as the closure of the set \( H_{BC} \cap D(A) \) in the norms
\[
\| u \|_{H^q} \triangleq \left( \sum_{k=1}^{\infty} \lambda_k^q (u, v_k)_H^2 \right)^{1/2},
\]
respectively. According to this definition, \( H^0 = H \).

Clearly, the bilinear forms \( (u, v)_H \), \( (v, Au)_H \) and \( (Av, Au)_H \) are well defined on \( H^{-1} \times H^1 \), \( H^1 \times H^1 \) and \( H^2 \times H^2 \), respectively, since they can be extended continuously from \( D(A) \times D(A) \).

For \( q = -1, 0, 1, 2 \), and \( r \in [1, +\infty] \), introduce the spaces
\[
C^0 \triangleq C([0, T]; H), \quad C^q \triangleq C([0, T]; H^q), \quad \mathcal{L}_r^q \triangleq L_r([0, T]; \bar{B}_1, \bar{\ell}_1; H^q),
\]
and the spaces
\[
\mathcal{W}_r^q \triangleq \left\{ u \in C^0 \cap \mathcal{L}_r^1 : \frac{du}{dr} \in C^0 \right\}
\]
considered as Banach spaces with the norms, respectively,
\[
\| u \|_{\mathcal{W}_r^q} \triangleq \| u \|_{C^0} + \| u \|_{\mathcal{L}_r^1} + \left\| \frac{du}{dr} \right\|_{C^0}.
\]

2. **Problem setting and the main result**

Let \( T > 0 \), \( \omega \in \mathbb{R} \), \( \omega \neq 0 \), \( a \in H^1 \), and \( g \in H^2 \), be given. We consider the boundary value problem
\[
\frac{d^2 u}{dt^2}(t) = Au(t), \quad t \in (0, T), \quad u(0) = a, \quad \int_0^T e^{i\omega t} u(t) \, dt = g.
\]

For \( u \in \mathcal{W}_1^1 \), we accept that Equation (8) is satisfied as an equality
\[
\frac{du}{dt}(t) - \frac{du}{dt}(s) = \int_s^t A u(r) \, dr
\]
that holds in \( H^{-1} \) for all \( s, t \) such that \( 0 \leq s \leq t \leq T \), and that conditions (4) and (4) are satisfied as equalities in \( H \).
Theorem 2.1: Assume that \( e^{2i\omega T} \neq 1 \). In this case, for any \( a \in H^1 \) and \( g \in H^2 \), there exists a unique solution \( u \in W^1_{\infty} \). Moreover, there exists \( c > 0 \) such that

\[
\|u\|_{W^1_{\infty}} \leq c(\|a\|_{H^1} + \|g\|_{H^2}) \quad \forall a \in H^1, \ g \in H^2.
\]  

(7)

Here, \( c > 0 \) depends only on \( H_{BC}, A, T, \) and \( \omega \).

By Theorem 2.1, problem (3)–(5) is well-posed in the sense of Hadamard for \( a \in H^1 \) and \( g \in H^2 \). The proof of this theorem is given below; it is based on explicit representation of the solution \( u \) for given \( g \) and \( a \) via eigenfunction expansion. It can be noted that, since \( \omega \neq 0 \) and \( \omega \in \mathbb{R} \), the solution \( u \) of problem (3)–(5) is not real valued even if both \( a \) and \( g \) are real valued.

Remark 2.1: One can reformulate the setting with complex-valued solutions as a setting with two real-valued solutions. Assume that \( u(t) = v(t) + iw(t) \), where \( v(t) = \text{Re} \ u(t) \) and \( w(t) = \text{Im} \ u(t) \). Then problem (3)–(5) can be rewritten as

\[
\frac{d^2v}{dt^2}(t) = Av(t), \quad \frac{d^2w}{dt^2}(t) = Aw(t), \quad t \in (0, T),
\]

\[
v(0) = \text{Re} \ a, \quad w(0) = \text{Im} \ a,
\]

\[
\int_0^T [\cos(\omega t)v(t) - \sin(\omega t)w(t)] \, dt = \text{Re} \ g, \quad \int_0^T [\sin(\omega t)v(t) + \cos(\omega t)w(t)] \, dt = \text{Im} \ g.
\]

Remark 2.2: It is known that real valued problem (3)–(5) with \( \omega = 0 \) is solvable for some real valued \( a \) and \( g \) but it does not feature stable solutions due to ‘small denominator’ problem arising for certain frequencies; see, e.g. in [16, p.42]. For small \( \omega \to 0 \), the part \( v \) of the solution \((v, w)\) from Remark 2.2 can be used as a stable approximation of the real valued solution of problem (3)–(5) with \( \omega = 0 \). In this case, \( \sin(\omega t)w(t) \) can be considered as some small stabilizing term.

2.1. Connection with Cauchy problem

For \( a \in H^1 \) and \( b \in H \), consider a boundary problem with the Cauchy condition

\[
\frac{d^2u}{dt^2}(t) = Au(t), \quad t \in (0, T),
\]

(8)

\[
u(0) = a,
\]

(9)

\[
\frac{du}{dt}(0) = b.
\]

(10)

For \( u \in W^1_{\infty} \), we accept that Equation (8) is satisfied as an equality (6) that holds in \( H^{-1} \) for all \( s, t \) such that \( 0 \leq s \leq t \leq T \), and that conditions (9) and (10) are satisfied as equalities in \( H^{-1} \).
**Proposition 2.1:** For any \( a \in H^1 \) and \( b \in H \), there exists a unique solution \( u \in W^1_1 \) of problem (8)–(10). This solution \( u \) is such that \( u \in W^1_\infty \). Moreover, there exists \( c > 0 \) such that

\[
\|u\|_{W^1_\infty} \leq c(\|a\|_{H^1} + \|b\|_{H}) \quad \forall a \in H^1, \ b \in H. \tag{11}
\]

Here, \( c > 0 \) depends only on \( H_{BC}, A, T, \) and \( \omega \).

The statement of Proposition 2.1 represents a minor modification of well-known results adapted to our choice of spaces; however, we provided its proof in Section 3 below for the sake of completeness.

### 3. Proofs

**Proof of Proposition 2.1.** Let \( a \) and \( b \) be expanded as

\[
a = \sum_{k=1}^{\infty} \alpha_k v_k, \quad b = \sum_{k=1}^{\infty} \beta_k v_k. \tag{12}
\]

Here, the coefficients \( \alpha_k \) are such that \( \sum_{k=1}^{+\infty} |\alpha_k|^2 = \|a\|_H < +\infty \).

We look for the solution \( u \) expanded as

\[
u(t) = \sum_{k=1}^{\infty} y_k(t)v_k, \tag{13}
\]

where \( y_k(t) \) are solutions of equations

\[
\frac{d^2 y_k}{dt^2}(t) = -\lambda_k y_k(t). \tag{14}
\]

In this case,

\[
\frac{d^2 u}{dt^2}(t) = \frac{d^2}{dt^2} \sum_{k=1}^{\infty} y_k(t)v_k = \sum_{k=1}^{\infty} (-\lambda_k y_k(t))v_k = \sum_{k=1}^{\infty} y_k(t)Av_k = Au(t).
\]

Let

\[
\theta_k \Delta = \sqrt{\lambda_k}.
\]

It can be seen that

\[
y_k(t) = C_k e^{-i\theta_k t} + D_k e^{i\theta_k t} \tag{15}
\]

for some \( C_k, D_k \in \mathbb{C} \).
The coefficients $C_k$ and $D_k$ are defined from the system
\begin{align*}
C_k + D_k &= \alpha_k, \\
- i\theta_k C_k + i\theta_k D_k &= \beta_k.
\end{align*}
(16)

This gives
\begin{align*}
D_k &= \alpha_k - C_k, \\
(\alpha_k - D_k)(-i\theta_k) + D_k i\theta_k &= 2i\theta_k D_k - \alpha_k i\theta_k = \beta_k
\end{align*}
and
\begin{align*}
-\alpha_k i\theta_k + i\theta_k D_k + D_k i\theta_k &= 2i\theta_k D_k + \alpha_k (-i\theta_k) = \beta_k.
\end{align*}

Hence
\begin{align*}
D_k &= \frac{\beta_k + \alpha_k i\theta_k}{2i\theta_k}, \\
C_k &= -\frac{\beta_k + \alpha_k i\theta_k}{2i\theta_k}.
\end{align*}
(17)

For the case of real $\alpha_k$ and $\beta_k$, we have that $C_k = \bar{D}_k$.

For $p = 0, 1$ and $q = 0, 1$ such that $p + q = 1$, we have that
\begin{align*}
\sup_{t \in [0,T]} \left\| \frac{d^p u}{dt^p} (t) \right\|_{H^q}^2 &\leq c_1 \sum_{k=1}^{\infty} \lambda_k^{p+q} |y_k(t)|^2 \\
&\leq c_2 \sum_{k=1}^{\infty} \lambda_k^{p+q} (|D_k|^2 + |C_k|^2) \\
&\leq c_3 \sum_{k=1}^{\infty} \lambda_k^{p+q} \left( |\alpha_k|^2 + \frac{|\beta_k|^2}{\lambda_k} \right) \\
&\leq c_4 (\|a\|_{H^{p+q}} + \|b\|_{H^{p+q-1}}) = c_4 (\|a\|_{H^1} + \|b\|_{H^0}).
\end{align*}
(18)

This means that estimate (11) holds.

Clearly, Equations (8)–(10) hold for the case where $a$, $b$, and $u$, are replaced by their truncated expansions
\begin{align*}
a_N &= \sum_{k=1}^{N} \alpha_k v_k, \\
b_N &= \sum_{k=1}^{N} \beta_k v_k, \\
u_N &= \sum_{k=1}^{N} y_k(t)v_k.
\end{align*}
(19)

We have that $u_N \in W^2 \cap W^1$. Estimate (18) and completeness of the Banach space $W^1$ ensures that $u_N \to u$ in $W^1$ as $N \to +\infty$. This $u$ is the solution (8)–(10), and that energy estimate (18) holds. This completes the proof of Proposition 2.1.

To proceed to the proof of Theorem 2.1, we need to adjust the approach used to the case of the integral boundary conditions.

Let $a$ and $g$ be expanded as
\begin{align*}
a &= \sum_{k=1}^{\infty} \alpha_k v_k, \\
g &= \sum_{k=1}^{\infty} \gamma_k v_k.
\end{align*}
(20)

Here, the coefficients $\alpha_k$ are such that $\sum_{k=1}^{+\infty} \lambda_k |\alpha_k|^2 = \|a\|_{H^1} < +\infty$. 

We look for the solution $u$ expanded as

$$u(t) = \sum_{k=1}^{\infty} y_k(t)v_k,$$

(21)

where $y_k(t)$ are the solutions of Equation (14) defined by (15), where $C_k$ and $D_k$ are defined from the system

$$C_k + D_k = \alpha_k,$$

$$C_k \int_0^T e^{i\omega t - it\theta_k} dt + D_k \int_0^T e^{i\omega t + it\theta_k} dt = \gamma_k.$$  

(22)

**Lemma 3.1:** Solution $(C_k, D_k)$ of system (22) exists and is uniquely defined for any $k$. Moreover, there exists $c > 0$ that depends on $T$ and $\omega$ only and such that

$$|C_k| + |D_k| \leq c(|\alpha_k| + (1 + \theta_k)|\gamma_k|)$$

(23)

for all $k$.

**Proof of Lemma 3.1.** Let

$$d_k \triangleq \int_0^T e^{i\omega t - it\theta_k} dt - \int_0^T e^{i\omega t + it\theta_k} dt.$$

In this case,

$$C_k = \alpha_k - D_k,$$

$$\alpha_k \int_0^T e^{i\omega t - it\theta_k} dt + D_k d_k = \gamma_k.$$

Suppose that we can prove that

$$\delta \triangleq \inf_k |d_k(1 + \theta_k)| > 0.$$  

(24)

In this case, $d_k \neq 0$ for all $k$, and there exists $c_p > 0$ such that, for $k \in \Lambda_1^+$,

$$|D_k| \leq c_1 |d_k|^{-1}(2|\alpha_k/(1 + \theta_k)| + |\gamma_k|) = c_1 |d_k(1 + \theta_k)|^{-1}(2|\alpha_k| + (1 + \theta_k)|\gamma_k|)$$

$$\leq \delta^{-1}(2|\alpha_k + (1 + \theta_k)|\gamma_k|).$$

This would imply the proof of the lemma.

Let us prove that (24) holds. Let

$$\Lambda_0 \triangleq \{k : \text{ either } \theta_k = \omega \text{ or } \theta_k = -\omega\},$$

$$\Lambda_1 \triangleq \{k \notin \Lambda_0 : \text{ either } e^{i\theta_k T} = e^{i\omega T} \text{ or } e^{i\theta_k T} = e^{-i\omega T}\},$$

$$\Lambda_2 \triangleq \{k : e^{i\theta_k T} \neq e^{i\omega T}, e^{i\theta_k T} \neq e^{-i\omega T}\}.$$  

(25)

Clearly, these sets are disjoint, and the set $\Lambda_0$ is either an empty set or a singleton.
The assumption of Theorem 2.1 that $e^{2i\omega T} \neq 1$ excluded the case where $e^{i\theta_k T} = e^{\omega T}$ and $e^{i\theta_k T} = e^{-i\omega T}$ simultaneously, since this would imply that $e^{i\omega T} = e^{-i\omega T}$, or $e^{2i\omega T} = 1$. Hence

$$\Lambda_0 \cup \Lambda_1 \cup \Lambda_2 = \{k = 1, 2, 3, \ldots\}. \quad \blacksquare$$

### 3.1. The case where $k \in \Lambda_0$

Let us consider first the case where $\Lambda_0 = \{k\}$ is a singleton, i.e. $\theta_k = \omega$ or $\theta_k = -\omega$.

Let us consider the case where $\omega = \theta_k$. In this case,

$$C_k = \alpha_k - D_k, \quad (\alpha_k - D_k)T + D_k\hat{d} = \gamma_k,$$

where

$$\hat{d} = \frac{1}{i\omega + i\theta_k}[e^{i\omega T + iT\theta_k} - 1] = \frac{1}{2i\omega}[e^{2i\omega T} - 1].$$

Hence $D_k(\hat{d} - T) = -\alpha_k T + \gamma_k$. We have that $\hat{d} - T \neq 0$, since, clearly, $|e^{2i\omega T}| < |1 + 2iT\omega|$. The case where $\omega = -\theta_k$ can be considered similarly.

### 3.2. The case where $k \in \Lambda_1$

Let us consider the case where $k \in \Lambda_1$. By the definitions, it follows that $\omega \neq \pm \theta_k$. In this case,

$$d_k = \frac{1}{i\omega + i\theta_k}[e^{i\omega T + iT\theta_k} - 1] - \frac{1}{i\omega - i\theta_k}[e^{i\omega T - i\theta_k T} - 1] \quad (26)$$

and

$$D_k d_k = -\frac{\alpha_k[e^{i\omega T - i\theta_k} - 1]}{i\omega - i\theta_k} + \gamma_k. \quad (27)$$

Let us consider first the case where $e^{i\theta_k T} = e^{i\omega T}$. In this case, we have that

$$d_k = \frac{1}{i\omega + i\theta_k}[e^{2i\omega T} - 1]$$

By the assumptions on $\omega$ and $T$, $e^{2i\omega T} \neq 1$. Hence

$$\inf_{k \in \Lambda_1^+} |d_k|(1 + \theta_k) > 0,$$

where $\Lambda_1^+ = \{k : e^{i\theta_k T} = e^{-i\omega T}\}$.

Using a similar approach where $e^{i\theta_k T} = e^{-i\omega T}$, we obtain that

$$\inf_{k \in \Lambda_1} |d_k|(1 + \theta_k) > 0,$$

Hence the statement of Lemma 3.1 holds for $k \in \Lambda_1$. 

3.3. The case where \( k \in \Lambda_2 \)

Let us consider the most typical case where \( k \in \Lambda_2 \), i.e.
\[
e^{i\theta_k T} \neq e^{i\omega T}, \quad e^{i\theta_k T} \neq e^{-i\omega T}.
\]

In particular, we have in this case that \( \omega \neq \theta_k \) and \( \omega \neq -\theta_k \) for all \( k \) in this case.

Let us show first that the values \( d_k(i\omega + i\theta_k) \) are separated from zero for large \( k \). We have that
\[
d_k(i\omega + i\theta_k) = e^{i\omega T - iT\theta_k} \left[ e^{2iT\theta_k} - \frac{\omega + \theta_k}{\omega - \theta_k} \right] - \frac{2\theta_k}{\theta_k - \omega}
= e^{i\omega T} \left[ e^{iT\theta_k} + e^{-iT\theta_k} \right] + \xi_k - \frac{2\theta_k}{\theta_k - \omega} = 2e^{i\omega T} \cos(2T\theta_k) + \xi_k - \frac{2\theta_k}{\theta_k - \omega},
\]
where
\[
\xi_k \triangleq e^{i\omega T - iT\theta_k} \left[ -1 - \frac{\omega + \theta_k}{\omega - \theta_k} \right] = e^{i\omega T - iT\theta_k} \left[ \frac{\theta_k + \omega}{\theta_k - \omega} - 1 \right] = e^{i\omega T - iT\theta_k} \frac{2\omega}{\theta_k - \omega}.
\]

Hence
\[
\frac{d_k}{2}(i\omega + i\theta_k) = e^{i\omega T} \left[ \cos(2T\theta_k) + \xi_k \right] - z_k,
\]
where
\[
z_k = \frac{\theta_k}{\theta_k - \omega}, \quad \zeta_k = \frac{1}{2} e^{-i\omega T} \xi_k.
\]

Clearly, we have that \( \xi_k \to 0 \) and \( z_k \to 1 \) as \( k \to +\infty \). Let \( a_k = \text{Re} \xi_k \) and \( b_k = \text{Im} \xi_k \). We have that \( \xi_k = a_k + ib_k \) and
\[
\lim_{k \to +\infty} (a_k \cos(\omega T) - b_k \sin(\omega T) - z_k) = -1.
\]

Further, we have that
\[
\text{Re} \frac{d_k}{2}(i\omega + i\theta_k) = \cos(\omega T) \cos(2T\theta_k) + a_k \cos(\omega T) - b_k \sin(\omega T) - z_k.
\]

Since \( e^{2i\omega T} \neq 1 \), it follows that \( |\cos(\omega T)| < 1 \) and \( \sup_k \cos(\omega T) \cos(2T\theta_k) < 1 \). It follows that there exists \( N > 0 \) such that
\[
\sup_{k \geq N} \text{Re} \frac{d_k}{2}(i\omega + i\theta_k) < 0.
\]

Hence
\[
\inf_{k \geq N} (|\omega| + |\theta_k|)|d_k| > 0. \quad (29)
\]

To complete the proof, it suffices to show that \( d_k \neq 0 \) for \( \theta_k \in \Lambda_2 \). We have that
\[
d_k = \frac{2f(\theta_k)}{i(\omega^2 - \theta_k^2)},
\]
where

\[ f(x) = \frac{1}{2}[(\omega - x)(e^{i\omega T + ixT} - 1) - (\omega + x)(e^{i\omega T - ixT} - 1)] \]
\[ = \frac{1}{2}(e^{i\omega T}[\omega(e^{ixT} - e^{-ixT}) - x(e^{ixT} + e^{-ixT})] + 2x) \]
\[ = e^{i\omega T}[i\omega \sin(xT) - x \cos(xT)] + x. \]

Let us show that \( f(x) \neq 0 \) for all \( x \in \{\theta_k\}_{k \in \Lambda_2} \). Suppose that \( f(x) = 0 \). In this case,

\[ \text{Im } f(x) = \omega \cos(\omega T) \sin(xT) - x \sin(\omega T) \cos(xT) = 0, \]
\[ \text{Re } f(x) = -\omega \sin(\omega T) \sin(xT) - x \cos(\omega T) \cos(xT) + x = 0. \]  

Suppose that Equation (30) hold and that \( \cos(xT) = 0 \). In this case, \( \omega \cos(\omega T) \sin(xT) = 0 \) and \( \sin(xT) \neq 0 \). This implies that \( \cos(\omega T) = 0 \) and that either \( e^{i\omega T} = e^{ixT} \) or \( e^{i\omega T} = -e^{ixT} \). Similarly, suppose that Equation (30) hold and that \( \cos(\omega T) = 0 \). In this case, \( x \sin(\omega T) \cos(xT) = 0 \) and \( \sin(\omega T) \neq 0 \). This implies that \( \cos(\omega T) = 0 \). Again, it follows either \( e^{i\omega T} = e^{ixT} \) or \( e^{i\omega T} = -e^{ixT} \). Hence \( d_k \neq 0 \) in both cases, since, as is shown above, \( d_k \neq 0 \) if \( e^{i\omega T} = e^{i\theta_k T} \) or \( e^{i\omega T} = -e^{i\theta_k T} \) for some \( k \).

Therefore, it suffices to consider the case where \( k \in \Lambda_2 \) and \( \cos(xT) \neq 0 \) and \( \cos(\omega T) \neq 0 \).

The equation for \( \text{Im } f \) in (30) gives that

\[ \frac{\sin(xT)}{\cos(xT)} = x \frac{\sin(\omega T)}{\cos(\omega T)}, \]

i.e.

\[ \tan(xT) = \frac{x}{\omega} \tan(\omega T), \quad \sin(xT) = a(\omega) x \cos(xT), \]

where \( a(\omega) \triangleq \omega^{-1} \tan(\omega T) \). Then the equation for \( \text{Re } f \) in (30) gives that

\[ -\omega \sin(\omega T)a(\omega)x \cos(xT) - x \cos(\omega T) \cos(xT) + x = 0. \]

This can be rewritten as

\[ \sin(\omega T) \tan(\omega T)x \cos(xT) + x \cos(\omega T) \cos(xT) - x = 0. \]

Hence

\[ \cos(xT)x(\sin^2(\omega T) + \cos^2(\omega T)) / \cos(\omega T) - x = 0 \]

and

\[ \cos(xT)x / \cos(\omega T) - x = 0. \]

This implies that

\[ x(1 - \cos(xT) / \cos(\omega T)) = 0. \]

This would imply that \( \cos(xT) = \cos(\omega T) \) and \( \text{Re } e^{ixT} = \text{Re } e^{i\omega T} \), and this in turn would imply that either \( e^{ixT} = e^{i\omega T} \) or \( e^{ixT} = e^{-i\omega T} \). However, this case is excluded for \( k \in \Lambda_2 \).
Therefore, \( f(\theta_k) \neq 0 \) and \( d_k \neq 0 \) for \( k \in \Lambda_2 \). Hence

\[
\inf_{k \in \Lambda_2} |d_k(1 + \theta_k)| > 0. \tag{31}
\]

Hence (24) holds. This completes the proof of Lemma 3.1.

We are now in the position to prove Theorem 2.1.

**Proof of Theorem 2.1.** Let \( u \) be defined by (21), (15) and (22). We have that, for the case of \( \gamma \) and \( a \) this \( u \) is a unique solution problem of (3)–(5) as well as problem (8)–(10) with \( b = du(0)/dt \); this can be shown similarly to the proof of Proposition 2.1. The identity is straightforward for truncated eigenfunction expansions with finite number of terms. On the next step, the extension on the case of infinite expansions is ensured by the energy estimates.

We have that

\[
\frac{du}{dt}(0) = \sum_{k=1}^{\infty} (-i\theta_k C_k + i\theta_k D_k) v_k = \sum_{k=1}^{\infty} (-i\theta_k (\alpha_k - D_k) + 2i\theta_k D_k) v_k
\]

\[
= \sum_{k=1}^{\infty} i\theta_k (-\alpha_k + 2D_k) v_k.
\]

By Lemma 3.1, it follows that \( D_k^2 \leq c_0 (\alpha_k^2 + \lambda_k \gamma_k^2) \) for some \( c_0 > 0 \) that depends on \( H_{BC} \), \( A \), \( T \), and \( \omega \). Hence

\[
\left\| \frac{du}{dt}(0) \right\|_H^2 \leq c_1 \sum_{k=1}^{\infty} \lambda_k (|C_k|^2 + |D_k|^2) \leq c_2 \sum_{k=1}^{\infty} \lambda_k [\alpha_k^2 + (1 + \theta_k)^2 \gamma_k^2] \]

\[
\leq c_3 (\|a\|_{H^1}^2 + \|g\|_{H^2}^2) \tag{32}
\]

for some \( c_1 > 0 \), \( c_2 > 0 \), and \( c_2 > 0 \), that depend on \( H_{BC} \), \( A \), \( T \), and \( \omega \). By Lemma 2.1, estimate (7) holds. This completes the proof of Theorem 2.1.

**4. A numerical example of impact of the presence of \( \omega \neq 0 \)**

Consider a toy example for the problem

\[
\begin{align*}
&u''(x, t) = u'''(x, t), \quad t \in [0, T], \ x \in (0, \pi), \\
&u(x, 0) = 0, \quad \int_0^T e^{i\omega t} u(x, t) \, dx = g(x), \quad u(0, t) = u(\pi, t) = 0,
\end{align*}
\]

where \( g \in H^2 \). This is a special case of problem (3)–(5) with \( n = 1 \), \( D = (0, \pi) \), \( A = d^2/dx^2 \). It is known that \( \lambda_k = k^2 \) and \( v_k(x) = \sin(kx) \), \( k = 1, 2, \ldots \), are the corresponding eigenvalues and eigenfunctions Respectively, \( \theta_k = k \).

For simplicity, we assume that \( \omega \notin \{1, 2, 3, \ldots \} \). In this case, \( \theta_k \in \Lambda_2 \), in the notations of the proof of Theorem 2.1. In addition, we assume that \( e^{2i\omega T} \neq 1 \). The solution constructed
in the proof of Theorem 2.1 is defined by (26) and (33), i.e.

\[
u(x, t) = \sum_{k=0}^{+\infty} y_k(t) u_k(x),
\]

where

\[
y_k(t) = C_k e^{-i\sqrt{\lambda_k} t} + D_k e^{i\sqrt{\lambda_k} t},
\]

and where \(C_k\) and \(D_k\) are defined from the system (26) and (27) such that

\[
C_k = -D_k, \quad D_k = \gamma k d_k.
\]

Let \(z(m) \triangleq \inf_{k \leq m} |d_k| (1 + \theta_k)\). As can be seen from the proof of Lemma 3.1, this values should be separated from zero to ensure regularity of solutions claimed in Theorem 2.1.

If \(\omega = 0\), then the problem of small denominators arises: for any \(T > 0\), \(z(m) \to 0\) as \(m \to +\infty\) and hence \(|D_k| \to \infty\).

In particular, for \(T = 5\), we estimated numerically that \(z(500) = 3.66 \cdot 10^{-9}\) for \(\omega = 0\) and that \(z(500) = 0.1001\) for \(\omega = 0.01\). For \(T = 10\), we estimated numerically that \(z(500) = 3.68 \cdot 10^{-9}\) for \(\omega = 0\) and that \(z(500) = 0.1998\) for \(\omega = 0.01\).

This illustrates that the impact of including even a small enough \(\omega \neq 0\) can prevent the appearance of ‘small denominators’ (small divisors).

5. Conclusions and discussion

The paper establishes solvability and regularity of a complexified boundary value problem for linear hyperbolic wave equations where a Cauchy condition is replaced by an integral condition

\[
\int_0^T e^{i\omega t} u(t) \, dt = g
\]

for the solution. It is shown that this new problem is well-posed in a wide class of solutions given the presence of the weight function \(e^{i\omega t}\), where \(\omega \in \mathbb{R} \setminus \{0\}\) can be arbitrarily small; the only condition is that \(e^{2i\omega T} \neq 1\). This leads to complex-valued solutions of the boundary value problem with this integral condition. This boundary value problem would be equivalent to a boundary value problem for a system of two real-valued hyperbolic equations, for the real and imaginary parts of the complex-valued solution, respectively, with an integral condition connecting solutions. In this case, the real part of the solution can be considered as an approximation as \(\omega \to 0\) of the solution of the real-valued solution with \(\omega = 0\). The setting considered in the paper allows many modifications and extensions. Most likely, the results can be extended on the case where the eigenvalues for \(A\) can be non-positive, and where the weight \(e^{i\omega t}\) in (5) is replaced by \(e^{rt+i\omega t}\) for \(r \in \mathbb{R}\). We leave it for future research.

So far, we have considered the case where the solutions can be expanded via the basis from the eigenfunctions. It would be interestingly to extend the result on the more general case, as it was done in [17] for wave equations with two-point conditions. We leave it for future research as well.
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