Entropy, volume growth and SRB measures for Banach space mappings

Alex Blumenthal∗ Lai-Sang Young†

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Abstract

We consider $C^2$ Fréchet differentiable mappings of Banach spaces leaving invariant compactly supported Borel probability measures, and study the relation between entropy and volume growth for a natural notion of volume defined on finite dimensional subspaces. SRB measures are characterized as exactly those measures for which entropy is equal to volume growth on unstable manifolds, equivalently the sum of positive Lyapunov exponents of the map. In addition to numerous difficulties incurred by our infinite-dimensional setting, a crucial aspect to the proof is the technical point that the volume elements induced on unstable manifolds are regular enough to permit distortion control of iterated determinant functions. The results here generalize previously known results for diffeomorphisms of finite dimensional Riemannian manifolds, and are applicable to dynamical systems defined by large classes of dissipative parabolic PDEs.

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∗Courant Institute of Mathematical Sciences, New York University, New York, USA. Email: alex.blumenthal@gmail.com.

†Courant Institute of Mathematical Sciences, New York University, New York, USA. Email: lsy@cims.nyu.edu. This research was supported in part by NSF Grant DMS-1363161.
This paper is part of a program to expand the scope of smooth ergodic theory, with a view towards making it applicable to PDEs as well as ODEs. Here, we extend to Banach space mappings an important result known in finite dimensions, namely the characterization of SRB measures as invariant measures for which entropy attains its upper bound given by the rate of unstable volume growth. The class of mappings to which our results apply includes (but is not limited to) time-\(t\) maps of semiflows generated by certain kinds of PDEs.

Orbits tend to attractors in dissipative dynamical systems. It is often assumed in the physics literature that asymptotic behaviors of “typical” orbits are captured by certain special invariant measures called SRB measures [8], in the sense that their time averages tend to space averages taken with respect to these measures. Mathematically, proving the existence of SRB measures poses nontrivial challenges, but for finite dimensional systems it has been shown that when they exist, ergodic SRB measures have the properties above, confirming that they are, to dissipative systems, what Liouville measures are to Hamiltonian systems. See [44] for a review. Without a doubt, SRB measures (named after Sinai, Ruelle and Bowen, who discovered them for uniformly hyperbolic attractors) are among the most important ideas in finite dimensional theory. This paper extends to Banach space mappings, including time-\(t\) maps of semiflows generated by certain kinds of PDEs, the characterization of SRB measures in terms of two much studied dynamical invariants, metric entropy and Lyapunov exponents.
Entropy measures the growth in randomness in the sense of information theory as a transformation is iterated, while Lyapunov exponents measure geometric instability: they give the rates at which nearby orbits diverge. Though \textit{a priori} quite different, these two ways to capture dynamical complexity are in fact closely related: For a differentiable mapping of a finite dimensional Riemannian manifold preserving a compactly supported Borel probability measure, it was shown by Ruelle \cite{ruelle1978} that entropy is always dominated by the sum of positive Lyapunov exponents (counted with multiplicity). In the case of volume preserving diffeomorphisms, Pesin showed that the two quantities above are in fact equal \cite{pesin1977}; another proof was later given by Mañé \cite{mane1978}. The ultimate results in this direction are contained in the combined works of Ledrappier et. al. \cite{ledrappier1986, ledrappier1986a, ledrappier1986b, ledrappier1986c}, which identified the SRB property of the invariant measure as both necessary and sufficient for the entropy formula to hold, and related the gap in this formula to the dimension of the invariant measure in general.

The results of \cite{ledrappier1986, ledrappier1986a, ledrappier1986b} reinforce the view that the “effective dimension” of a dynamical system is equal to its number of positive Lyapunov exponents, in that all of the dynamical complexity of a system is captured in its expanding directions. Even though it is impossible to mathematically carry out a dimension reduction procedure along the lines described, these ideas are conceptually valid. SRB measures and their entropy formula characterization are therefore especially relevant for systems with many degrees of freedom and relatively low effective dimensions, such as time-$t$ maps of semiflows defined by nonlinear dissipative parabolic PDEs. For background information on dynamical systems generated by parabolic PDEs, see e.g. \cite{temam1988}.

\textit{Technical issues associated with ergodic theory on Banach spaces}

The main results of this paper generalize \cite{ledrappier1986} and \cite{ledrappier1986a} to Banach space mappings $f$ preserving a compactly supported Borel probability measure $\mu$ with finitely many positive Lyapunov exponents. We prove that under the condition of no zero Lyapunov exponents, $\mu$ is an SRB measure if and only if the entropy of $f$ is equal to the sum of its positive Lyapunov exponents.

A number of results in nonuniform hyperbolic theory have been extended to Hilbert space mappings \cite{ruelle1978, mane1978, zhang1988, zhang1989, zhang1989a, zhang1989b, zhang1989c}, and some have further been extended to mappings of Banach space, e.g. Ruelle’s inequality for deterministic and random maps \cite{ruelle1978, zhang1989} and the absolute continuity of the stable foliation for systems with invariant cones \cite{zhang1989b}. Observe, however, that ideas surrounding the entropy formula and SRB measures have not been studied for systems on Banach spaces. A hurdle might be that in one way or another, these ideas are related to volume growth on finite dimensional (unstable) manifolds, and in Banach spaces there is no intrinsic notion of $k$-dimensional volume for $k > 1$. While one may be able to make do with a Lebesgue measure class on unstable manifolds (Haar measure is certainly well defined), we believe a systematic understanding of volume growth is conducive to understanding SRB measures and the relation between entropy and Lyapunov exponents.

It is simple enough to put a notion of volume on a fixed finite dimensional normed vector space, and one can do that – one subspace at a time – for all finite dimensional subspaces of a Banach space. But for such a notion to be useful in smooth ergodic theory, \textit{regularity} of this volume function as subspaces are varied is essential. It is well known that norms do not necessarily vary smoothly with vectors on Banach spaces; volumes and determinants are not likely to fare better. Hence it is important that the volumes we introduce are regular.
enough to support distortion estimates on unstable manifolds, as such bounds are key to many important results in hyperbolic theory. We will show that they have the regularity we need, but it is not clear that finite dimensional results involving higher regularity of determinants, e.g. [37], will carry over to Banach spaces.

In addition to the absence of an intrinsic notion of volume, another difficulty we face has to do with noninvertibility of the map \( f \), which is not onto and has arbitrarily strong contraction in some directions. Even where \( f^{-1} \) is defined, one cannot expect it to have nice properties. This leads to regularity issues for objects such as \( E^u \)-spaces the definitions of which involve backward iterations. In response to these difficulties, throughout this paper we have tried to identify differences between diffeomorphisms and maps that are not invertible, and differences between finite and infinite dimensions. We have taken special care in the treatment of volume growth, recognizing that Banach spaces do not always admit a notion of volume as nice as that on Hilbert spaces or on finite dimensional Riemannian manifolds.

The organization of this paper is as follows: The main results are stated in Section 1. Section 2 contains a discussion of volumes and determinants on finite dimensional subspaces of Banach spaces; part of this material is included for the convenience of the reader, and other parts (e.g. regularity of determinants) are new. We hope this basic material will be useful beyond the present paper. Section 3 contains a small addendum to the Multiplicative Ergodic Theorem, following up on volume growth ideas in relation to Lyapunov exponents. Sections 4 and 5 contain preparations for the proofs of our main results, such as Lyapunov charts, distortion estimates, etc. Additional technical issues and the proofs of the main results are carried out in Section 6.

1 Statement of Results

Let \((\mathcal{B}, | \cdot |)\) be a Banach space. After some preliminary work fixing a notion of volume on finite-dimensional subspaces of \( \mathcal{B} \) (Section 2), we turn to the main topic of this paper, nonuniform hyperbolic theory for Banach space mappings. We begin with some basic facts of this theory, proved under conditions (H1)–(H3) below.

**Setting for basic nonuniform hyperbolic theory.** We consider \((f, \mu)\), where \( f : \mathcal{B} \to \mathcal{B} \) is a mapping and \( \mu \) is an \( f \)-invariant Borel probability measure. The following properties are assumed:

(H1) (i) \( f \) is \( C^2 \) Fréchet differentiable and injective;
   (ii) the derivative of \( f \) at \( x \in \mathcal{B} \), denoted \( \text{df}_x \), is also injective.

(H2) (i) \( f \) leaves invariant a compact set \( \mathcal{A} \subset \mathcal{B} \), with \( f(\mathcal{A}) = \mathcal{A} \);
   (ii) \( \mu \) is supported on \( \mathcal{A} \).

(H3) We assume

\[
    l_\alpha(x) := \lim_{n \to \infty} \frac{1}{n} \log |\text{df}_{x}^{n}|_\alpha < 0 \quad \text{for } \mu - \text{a.e. } x .
\]

Here \( |\text{df}_{x}^{n}|_\alpha \) is the Kuratowski measure of noncompactness of the set \( \text{df}_{x}^{n}(\mathcal{B}) \), where \( \mathcal{B} \) is the unit ball in \( \mathcal{B} \).
Condition (H3) is discussed in more detail in Sect. 3.1. It is a relaxation of the condition that \( df_x \) is the sum of a compact operator and a contraction for each \( x \in \mathcal{A} \) (see Remark 3.2), and it implies that positive and zero Lyapunov exponents of \((f, \mu)\) are well defined and have finite multiplicity.

**Two other relevant assumptions.**

(H4) \((f, \mu)\) has no zero Lyapunov exponents.

(H5) the set \( \mathcal{A} \) in (H2) has finite box-counting dimension.

We remark that (H5) is automatically satisfied if \( f \) satisfies (H1) and (H2)(i), and \( df_x \) is the sum of a compact operator and a contraction for each \( x \in \mathcal{A} \); see [28].

For diffeomorphisms of Riemannian manifolds, one generally requires in the definition of SRB measures that the conditional measures of \( \mu \) on unstable manifolds be absolutely continuous with respect to the Riemannian measures induced on these manifolds. In Banach spaces, the notion of Riemannian volume is absent, but there is the following well defined Lebesgue measure class on any finite dimensional submanifold \( W \): For \( x \in W \), we let \( \mathcal{B}_x \) denote the tangent space to \( \mathcal{B} \) at \( x \), and choose a closed subspace \( F \) so that \( \mathcal{B}_x = E \oplus F \) where \( E \) is the subspace tangent to \( W \) at \( x \). Then on a small neighborhood \( U \) of \( x \) in \( W \), the “Lebesgue measure class” is the one that when projected to \( E \) along \( F \) gives the Haar measure class on \( E \). We state below a provisional definition of SRB measures; see Sect. 6.1 for a formal definition.

**Definition 1.1.** We say \( \mu \) is an SRB measure if (i) it has a positive Lyapunov exponent \( \mu \)-a.e. and (ii) the conditional measures of \( \mu \) on unstable manifolds are in the “Lebesgue measure class” induced on these manifolds.

Let \((f, \mu)\) be as above. We let \( h_\mu(f) \) denote the entropy with respect to \( \mu \), and let \( \lambda_1(x) > \lambda_2(x) > \cdots \), with multiplicities \( m_1(x), m_2(x), \ldots \), denote the distinct Lyapunov exponents of \((f, \mu)\) at \( x \). Write \( a^+ = \max\{a, 0\} \). Our main results are the following:

**Theorem 1.** Suppose \((f, \mu)\) satisfies (H1)–(H4) above, and assume that \( \mu \) is an SRB measure. Then

\[
h_\mu(f) = \int \sum_i m_i(x) \lambda_i^+(x) \, d\mu . \tag{1}
\]

**Theorem 2.** Suppose \((f, \mu)\) satisfies (H1)–(H5). If \( \lambda_1 > 0 \) \( \mu \)-a.e. and the entropy formula (1) holds, then \( \mu \) is an SRB measure.

The results in Theorems 1 and 2 were proved in [18],[17] in a finite dimensional context, more precisely for diffeomorphisms of compact Riemannian manifolds, and extended to Hilbert spaces in [21]. In all likelihood, the no zero exponents assumption (H4) is not necessary, but in the presence of zero Lyapunov exponents, the proofs are more elaborate and we have elected to treat that case elsewhere.

One way to understand Theorem 1 is to view the sum of positive Lyapunov exponents as representing volume growth on unstable manifolds, so that the right side of (1) tells us how
volumes on unstable manifolds are transformed, while entropy describes the transformation of the conditional measures of $\mu$. To express these ideas in a systematic way, we need a coherent notion of volume on unstable manifolds, not just a measure class (which was sufficient for purposes of defining SRB measures). We know of no previous studies of volumes on finite dimensional subspaces of Banach spaces that serve our purposes, the closest approach to these ideas being the ‘volume function’ in [23] (see also [11]), which does not arise from a genuine volume on subspaces in the usual sense. Thus we include in Section 2 a short introduction to these ideas.

With a coherent notion of induced volumes on finite dimensional subspaces in hand, we have a well defined notion of finite dimensional determinant for $df_x$, and a result to the following effect:

**Corollary 3.** The conditional densities of an SRB measure on unstable manifolds are Lipschitz and have the form

$$\frac{\rho(x)}{\rho(y)} = \prod_{i=1}^{\infty} \frac{\det(df_{f^{-i}y}|_{T_{f^{-i}y}W})}{\det(df_{f^{-i}x}|_{T_{f^{-i}x}W})}$$

for all $x, y$ on the same local unstable manifold $W$.

A precise statement of this result requires some preparation and is given in Sect. 6.5.

## 2 Volumes on Finite Dimensional Subspaces of a Banach Space

Whereas in a Hilbert space, a finite dimensional subspace is naturally an inner product space with an obvious choice of volume element, there is no such ‘obvious’ choice in a Banach space. The objective of this section is to introduce a coherent notion of volume on finite-dimensional subspaces of a Banach space, and to establish some basic properties. Definitions and basic facts of induced volumes and determinants are given in Sects. 2.1 and 2.2. Their regularity, which are the main results of this section, are proved in Sect. 2.3. As noted in the Introduction, regularity of the determinant is relevant for controlling the distortion of iterated densities on unstable manifolds.

We assume throughout that $(\mathcal{B}, |\cdot|)$ is a Banach space.

### 2.1 Relevant Banach space geometry (mostly review)

We gather in this subsection some known facts that are relevant to smooth ergodic theory on Banach spaces, casting them in a light suitable for our purposes.

#### 2.1.1 Induced volumes

Following the idea of the Busemann-Hausdorff volume in Finsler geometry [5], [38], we make the following definition.
Definition 2.1. Let $E \subset \mathcal{B}$ be a finite-dimensional subspace. We define the induced volume $m_E$ on $E$ to be the unique Haar measure on $E$ for which

$$m_E\{u \in E \mid |u| \leq 1\} = \omega_k$$

where $k = \dim E$ and $\omega_k$ is the volume of the Euclidean unit ball in $\mathbb{R}^k$.

The following are some basic properties of $m_E$ entailed by this definition.

Lemma 2.2. Let $E \subset \mathcal{B}$ be a $k$-dimensional subspace. Then $m_E$ satisfies the following.

1. For any $v \in E$ and any Borel measurable set $S \subset E$, we have $m_E(v + S) = m_E(S)$.
2. If $m'$ is any other $\sigma$-finite non-zero measure on $E$ satisfying item 1, then $m'$ and $m_E$ are equivalent with $\frac{dm'}{dm_E} \equiv c$ $m_E$-a.e. for a constant $c > 0$.
3. For any $a > 0$ and any Borel measurable set $S \subset E$, we have $m_E(aS) = a^km_E(S)$.

2.1.2 Complementation and ‘angles’

Let $\mathcal{G}(\mathcal{B})$ denote the Grassmannian of closed subspaces of $\mathcal{B}$. The topology on $\mathcal{G}(\mathcal{B})$ is the metric topology defined by the Hausdorff distance $d_H$ between unit spheres: for nontrivial subspaces $E, E' \in \mathcal{G}(\mathcal{B})$,

$$d_H(E, E') = \max\{\sup\{d(e, S_{E'}) : e \in S_E\}, \sup\{d(e', S_E) : e' \in S_{E'}\}\}$$

where $S_E = \{v \in E \mid |v| = 1\}$.

A more convenient definition, known as the aperture or gap ([14], see also [1]), is

$$\delta_a(E, E') = \max\{\sup\{d(e, E') : e \in S_E\}, \sup\{d(e', E) : e' \in S_{E'}\}\}$$

On Hilbert spaces, $\delta_a$ is a metric, and coincides with the operator norm of the difference between orthogonal projections. On Banach spaces, $\delta_a$ is not a metric, but $d_H$ and $\delta_a$ are related by the inequality $\delta_a(E, E') \leq d_H(E, E') \leq 2\delta_a(E, E')$ [14]. We will work with $d_H$ or $\delta_a$, whichever one is more convenient.

We say $E \in \mathcal{G}(\mathcal{B})$ is complemented if there exists $F \in \mathcal{G}(\mathcal{B})$ such that $\mathcal{B} = E \oplus F$, and call $F$ a complement of $E$. Observe that if $E, F \in \mathcal{G}(\mathcal{B})$ are complements, then $\pi = \pi_{E/F} : \mathcal{B} \to E$, the projection to $E$ along $F$ defined by $\pi(e + f) = e$ for $e \in E, f \in F$, is automatically bounded as an operator by the closed graph theorem. We note further that

$$|\pi_{E/F}|^{-1} = \alpha(E, F) \quad \text{where} \quad \alpha(E, F) = \inf\{|e - f| : e \in E, |e| = 1, f \in F\}.$$

Though $\alpha(\cdot, \cdot)$ is not symmetric, it satisfies $\alpha(E, F) \leq 2\alpha(F, E)$ whenever $E, F$ are complements. These quantities have the geometric connotation of ‘angle’ between $E$ and $F$.

The next lemma gives conditions under which complementation persists.

Lemma 2.3. Let $\mathcal{B} = E \oplus F$ for $E, F \in \mathcal{G}(\mathcal{B})$. If $E' \in \mathcal{G}(\mathcal{B})$ is such that $d_H(E, E') < |\pi_{E/F}|^{-1}$, then $\mathcal{B} = E' \oplus F$. 

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A proof of this result is given in the appendix.

Not every \( E \in \mathcal{G}(\mathcal{B}) \) admits a complement, but all finite dimensional subspaces do, and while there are no orthogonal complements to speak of, the following result provides a substitute that is adequate for our purposes.

**Lemma 2.4** ([42]). Every subspace \( E \subset \mathcal{B} \) of finite dimension \( k \) has a complement \( F \in \mathcal{G}(\mathcal{B}) \) with the property that \( \alpha(E, F) \geq \frac{1}{\sqrt{k}} \), equivalently, \( |\pi_{E/F}| \leq \sqrt{k} \).

The next lemma contains some estimates that are used repeatedly in Sect. 2.3. Proofs are given in the Appendix.

**Lemma 2.5.** Let \( E, E' \subset \mathcal{B} \) be subspaces with finite dimension \( k \), and let \( F \in \mathcal{G}(\mathcal{B}) \) be a complement to \( E \) with \( |\pi_{E/F}| \leq \sqrt{k} \). Suppose \( d_H(E, E') \leq \frac{1}{2\sqrt{k}} \). Then:

(a) \( \mathcal{B} = E' \oplus F \), with
\[
|\pi_{E'/F}| \leq 2\sqrt{k} \quad \text{and} \quad |\pi_{F/E'}| \leq 4\sqrt{k} \ d_H(E, E') ;
\]

(b) for any \( e \in E \) with \( |e| = 1 \),
\[
1 - 4\sqrt{k} \ d_H(E, E') \leq |\pi_{E'/F} e| \leq 1 + 4\sqrt{k} \ d_H(E, E') .
\]

### 2.1.3 Comparison with norms arising from inner products

One can leverage known results on inner product spaces by comparing \( | \cdot | \) to norms that arise from inner products. The key to this direction of thinking is John’s Theorem [4], which states that a convex body in \( \mathbb{R}^n \) is contained in a unique volume-minimizing ellipsoid. Since ellipsoids and inner products are equivalent, this result can be stated in terms of inner products. We take the liberty to state a version of John’s theorem that fits with the way it will be used in this paper.

**Theorem 2.6** (John’s Theorem). Let \( E \subset \mathcal{B} \) be a subspace of finite dimension \( k \). Then there is an inner product \( (\cdot, \cdot) \) on \( E \) and a norm \( \| \cdot \| \) arising from it such that for all \( v \in E \),
\[
\|v\| \leq |v| \leq \sqrt{k} \|v\| .
\]

The following is a direct consequence of John’s Theorem and Lemma 2.2.

**Corollary 2.7.** Let \( E \subset \mathcal{B} \) be a subspace of dimension \( k \). We let \( (\cdot, \cdot) \) and \( \| \cdot \| \) be given by Theorem 2.6. Scaling \( (\cdot, \cdot) \) by a suitable constant, one can obtain a new inner product \( (\cdot, \cdot)_E \) and norm \( \| \cdot \|_E \) on \( E \) with the property that if \( \hat{m}_E \) is the induced volume on \( E \) with respect to \( \| \cdot \|_E \), then
\[
m_E = \hat{m}_E \quad \text{and} \quad \frac{1}{\sqrt{k}} \| \cdot \|_E \leq |\cdot| \leq \sqrt{k} \| \cdot \|_E .
\]

**Proof.** Let \( \tilde{B}_E = \{ v \in E | \|v\| \leq 1 \} \), and let \( C \) be such that \( m_E \tilde{B}_E = C \omega_k \). Scale \( (\cdot, \cdot) \) so that \( \| \cdot \|_E := C^{1/k} \| \cdot \| \). We leave the rest as an exercise. \( \square \)
Remark 2.8. For purposes of this paper what matters in the results in Theorem 2.6 and Corollary 2.7 is not the bound \( \sqrt{k} \) but the fact that there is a bound that depends only on the dimension of the subspace in question. Indeed, any means of constructing an inner product on \( E \subseteq B \) would do, so long as it gives rise to a norm uniformly equivalent to the original one, with constants depending only on the dimension of \( E \).

Corollary 2.7 is used many times in the discussion to follow. It enables us to deduce quickly many results for normed vector spaces by appealing to their counterparts on inner product spaces. For example, for \( \{v_1, \ldots, v_k\} \subseteq B \), let \( P[v_1, \ldots, v_k] \) denote the parallelepiped defined by the vectors \( \{v_i\} \), i.e.,

\[
P[v_1, \ldots, v_k] = \{a_1v_1 + \cdots + a_kv_k : 0 \leq a_i \leq 1\}.
\]

Then given \( \{v_i\} \subseteq E \) and \( \lambda_i \in \mathbb{R} \), we have relations such as

\[
m_E(P[\lambda_1 v_1, \ldots, \lambda_k v_k]) = \left( \prod_{i=1}^k |\lambda_i| \right) m_E(P[v_1, \ldots, v_k])
\]

because this is true for \( \hat{m}_E \), and

\[
m_E(P[v_1, \ldots, v_k]) = \hat{m}_E(P[v_1, \ldots, v_k]) \leq \prod_{i=1}^k \|v_i\|_E \leq k^2 \prod_{i=1}^k |v_i|.
\] (2)

2.2 The determinant and its properties

Associated with the induced volumes defined in Sect. 2.1, we have, for each linear map \( A : B \to B \), a notion of determinant on finite dimensional subspaces which describes how these measures are transformed by \( A \).

Definition 2.9. Let \( A : B \to B \) be a bounded linear operator and \( E \subseteq B \) a finite-dimensional subspace. Then

\[
det(A|E) := \begin{cases} m_{A|E}(A(E)) / m_E(E) & \text{dim}(A(E)) = \text{dim}(E), \\ 0 & \text{else}, \end{cases}
\]

where \( B_E = \{v \in E : |v| \leq 1\} \).

It follows from this definition and from Lemma 2.2 that \( \det(\cdot) \) has the basic properties of the usual determinant, such as:

Lemma 2.10. Let \( E, F, G \) be subspaces of \( B \) of the same finite dimension, and let \( A, B : B \to B \) be bounded linear maps for which \( AE \subseteq F, BF \subseteq G \). Then:

1. \( m_F(A(S)) = \det(A|E) \cdot m_E(S) \) for every Borel set \( S \subseteq E \);
2. \( \det(BA|E) = \det(B|F) \cdot \det(A|E) \).

The following are further illustration of how one can leverage results for inner product spaces via John’s Theorem. The proofs are left as (easy) exercises.
Lemma 2.11. Let $A : \mathcal{B} \to \mathcal{B}$ be a bounded linear operator, and let $V, V' \subset \mathcal{B}$ be $k$-dimensional subspaces such that $A(V) = V'$. We equip $V$ and $V'$ with the inner products $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_{V'}$ in Corollary 2.7.

1. If $\{v_1, \ldots, v_k\} \subset V$ are orthonormal with respect to $(\cdot, \cdot)_V$, then it follows from (2) that
   \[
   \det(A|V) = \frac{m_V A(P[v_1, \ldots, v_k])}{m_V P[v_1, \ldots, v_k]} \leq k^{\frac{1}{2}} \prod_{i=1}^{k} |Av_i| .
   \]

2. If $\{v_1, \ldots, v_k\}$ is an orthonormal basis of $V$ corresponding to the singular value decomposition of $A|_V : V \to V'$, then
   \[
   k^{-\frac{1}{2}} \prod_{i=1}^{k} |Av_i| \leq \det(A|V) \leq k^{\frac{1}{2}} \prod_{i=1}^{k} |Av_i| .
   \]

The following is how $\det(\cdot)$ behaves with respect to splittings.

Lemma 2.12. For any $k \geq 1$ there is a constant $C_k \geq 1$ with the following property. Suppose that $V, V' \subset \mathcal{B}$ have dimension $k$ and $A : V \to V'$ is invertible. Let $V = E \oplus F, V' = E' \oplus F'$ be splittings for which $AE = E', AF = F'$. Then,
   \[
   \frac{\alpha(E', F')^q}{C_k} \leq \frac{\det(A|V)}{\det(A|E) \det(A|F)} \leq \frac{C_k}{\alpha(E, F)^q}
   \]
   where $q = \dim E$.

Proof. We let $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_{V'}$ be as above, and let $\det(A|V)$ denote the determinant with respect to these inner products. Let us take for granted the (standard) result for inner product spaces which says that
   \[
   \frac{1}{\|\pi_{E'//F'}\|_{V'}^q} \leq \frac{\det(A|V)}{\det(A|E) \det(A|F)} \leq \|\pi_{E//F}\|_{V}^q ,
   \]
   where $q = \dim E$. As noted earlier, $\det(A|V) = \hat{\det}(A|V)$. As for $\det(A|E)$, though $(\hat{m}_V)|E$ is not necessarily equal to $m_E$, they differ by a multiplicative constant depending only on $k$, so $\det(A|E)$ and $\det(A|E)$ differ in the same way, as do $\det(A|F)$ and $\det(A|F)$. Finally, as
   \[
   \frac{1}{k} |\pi_{E//F}| \leq \|\pi_{E//F}\|_{V} \leq k |\pi_{E//F}|
   \]
   and similarly for $\|\pi_{E'//F'}\|_{V'}$, the proof is complete upon relating $|\pi_{E//F}|$ to $\alpha(E, F)$.
2.3 Regularity of induced volumes and determinants

We motivate the results in this subsection as follows: Let \( W \) and \( W' \) be embedded \( k \)-dimensional submanifolds, and consider a \( C^1 \) map \( f : \mathcal{B} \to \mathcal{B} \) that maps \( W \) diffeomorphically onto \( W' \). For each \( x \in W \), let \( T_xW \) denote the tangent space to \( W \) at \( x \). From Sect. 2.1, we have an induced volume \( m_{T_xW} \) on each \( T_xW \). Under very mild regularity assumptions on \( m_E \), these volumes on the tangent spaces of \( W \) induce a \( \sigma \)-finite Borel measure \( \nu_W \) on \( W \). Analogous definitions hold for \( W' \). It follows from Sect. 2.2 that if \( f_* (\nu_W) \) is the pushforward of the measure \( \nu_W \) by \( f \), then at each \( y \in W' \),

\[
\frac{df_*(\nu_W)}{dv_W}(y) = \frac{1}{\det(df_{f^{-1}y}|T_{f^{-1}y}W)}.
\]

For reasons to become clear in the pages to follow, it is important to control these densities. This translates into regularity properties of the function \( x \mapsto \det(df_x|T_xW) \). We will tackle these questions below in a slightly more general context in preparation for the distortion estimates in Sect. 5.3.

2.3.1 Regularity of induced volumes

For linearly independent vectors \( v_1, \ldots, v_k \in \mathcal{B} \), let \( \langle v_1, \ldots, v_k \rangle \) denote the subspace spanned by \( \{v_1, \ldots, v_k\} \), and recall that \( P[v_1, \ldots, v_k] \) denotes the parallelepiped defined by the vectors \( \{v_i\} \). To simplify notation, we write \( m_{\{v_i\}} = m_{\{v_1, \ldots, v_k\}} \). We remark from the outset that there is no reason, in general, to expect the dependence of \( m_{\{v_i\}}P[v_1, \ldots, v_k] \) on \( v_1, \ldots, v_k \) to be any better than Lipschitz, as \( v \mapsto |v| \) is not differentiable in general Banach spaces. An instance of this already occurs for \( \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\} \) endowed with the norm \( \|(x, y)\| = \max\{|x|, |y|\} \), for which the breakdown of differentiability occurs along the diagonal lines \( \{(x, x)\} \) and \( \{(x, -x)\} \).

To control how far a basis \( \{v_i\} \) deviates from ‘orthogonality’, we introduce the quantity

\[
N[v_1, \ldots, v_k] = \sum_{i=1}^k |\pi(\langle v_i \rangle//\langle v_j \rangle : j \neq i)|,
\]

where \( \pi(\langle v_i \rangle//\langle v_j \rangle : j \neq i) : \langle v_1, \ldots, v_k \rangle \to \langle v_i \rangle \) is the projection operator defined earlier.

**Proposition 2.13.** For any \( k \geq 1 \) and \( \overline{N} > k \), there exist \( L = L(\overline{N}, k) > 0 \) and \( \delta = \delta(\overline{N}, k) \geq 0 \) such that the following holds. If \( \{v_i\}, \{w_i\} \) are two sets of \( k \) linearly independent unit vectors in \( \mathcal{B} \) for which \( \max_{i \leq k} |v_i - w_i| \leq \delta \) and \( N[v_1, \ldots, v_k], N[w_1, \ldots, w_k] \leq \overline{N} \), then

\[
\left| \log \frac{m_{\{v_i\}}P[v_1, \ldots, v_k]}{m_{\{w_i\}}P[w_1, \ldots, w_k]} \right| \leq L \sum_{i=1}^k |v_i - w_i|.
\]

First we prove the following lemma:

**Lemma 2.14.** For any \( k \geq 1 \), there exist \( \delta_1 > 0 \) and \( L_1 > 0 \) (depending only on \( k \)) such that the following hold. Assume

(i) \( E, E' \subset \mathcal{B} \) are two \( k \)-dimensional subspaces with \( d_H(E, E') \leq \delta_1 \),
(ii) \( F \) is a complement to \( E \) with \(|\pi_{E/F}| \leq \sqrt{k}\) (exists by Lemma \([2.4]\))

(iii) \( \{v_i\} \) is a basis of unit vectors of \( E \).

Assume \( \delta_1 \) is small enough that \( B = E' \oplus F \). Let \( v'_i := (\pi_{E'/F}v_i)/|\pi_{E'/F}v_i| \). Then

1. \( N[v'_1, \ldots, v'_k] \leq 2kN[v_1, \ldots, v_k] \);

2. \[
\log \frac{m_E P[v_1, \ldots, v_k]}{m_E P[v'_1, \ldots, v'_k]} \leq L_1 d_H(E, E')
\]

Proof. Assuming \( \delta_1 \leq \frac{1}{4\sqrt{k}} \), Lemma \([2.5]\) guarantees that \( B = E' \oplus F \) with \(|\pi_{E'/F}| \leq 2\sqrt{k}\).
Let \( v'_i \in E' \) be as in the statement. Then

\[
\frac{m_E P[v_1, \ldots, v_k]}{m_E P[v'_1, \ldots, v'_k]} = \frac{m_E P[v_1, \ldots, v_k]}{m_E (\pi_{E'/F}P[v_1, \ldots, v_k])} \cdot \frac{m_E (\pi_{E'/F}P[v_1, \ldots, v_k])}{m_E P[v'_1, \ldots, v'_k]}
\]

By Lemma \([2.2]\) the first quotient on the right side is equal to

\[
\frac{m_E B_E}{m_E (\pi_{E'/F}B_E)}
\]

In light of Lemmas \([2.2]\) and \([2.5]\) we have that

\[
(1 - 4\sqrt{k} d_H(E, E'))^k \cdot m_E B_E \leq m_E (\pi_{E'/F}B_E) \leq (1 + 4\sqrt{k} d_H(E, E'))^k \cdot m_E B_{E'}
\]

So long as \( \delta_1 \leq \frac{1}{8\sqrt{k}} \), and recalling that \(|\log(1 + z)| \leq 2|z| \) for \( z \in [-1/2, 1/2] \), it follows that

\[
\log \frac{m_E P[v_1, \ldots, v_k]}{m_E (\pi_{E'/F}P[v_1, \ldots, v_k])} \leq 8k\sqrt{k} d_H(E, E')
\]

For the second quotient on the right side of (13), observe that for each \( i \), \( \pi_{E'/F}v_i = |\pi_{E'/F}v_i| v'_i \). The same reasoning as in Sect. \([2.1.3]\) then gives

\[
m_E P[\pi_{E'/F}v_1, \ldots, \pi_{E'/F}v_k] = \left( \prod_{i=1}^{m} |\pi_{E'/F}v_i| \right) m_E P[v'_1, \ldots, v'_k]
\]

Using Lemma \([2.5]\) again to estimate the quantity in parenthesis, we obtain in a similar fashion that

\[
\log \frac{m_E (\pi_{E'/F}P[v_1, \ldots, v_k])}{m_E P[v'_1, \ldots, v'_k]} \leq 8k\sqrt{k} d_H(E, E')
\]

So, Item 2 in Lemma \([2.14]\) holds with \( L_1 = 16k\sqrt{k} \) and any \( \delta_1 \leq \frac{1}{8\sqrt{k}} \).

For Item 1 in the lemma, observe that when \( \pi_i \) is the projection onto \( v_i \) parallel to the rest of the basis and \( \pi'_i \) is the analogous for \( \{v'_i\} \), we have

\[
\pi'_i = \pi_{E'/F} \circ \pi_i \circ \pi_{E'/F}|E'
\]

so that

\[
|\pi'_i| \leq |\pi_{E'/F}| \cdot |\pi_i| \cdot |\pi_{E/F}| \leq 2\sqrt{k} \cdot |\pi_i| \cdot \sqrt{k}
\]

giving the desired bound. \( \square \)
Proof of Proposition 2.13. Let \( \{v_i\}, \{w_i\} \) be as in the statement, and denote \( \langle \{v_i\} \rangle = E, \langle \{w_i\} \rangle = E' \). We will estimate the quantity in question by

\[
\frac{m_E P[v_1, \ldots, v_k]}{m_{E'} P[v_1, \ldots, v_k]} = \frac{m_E P[v_1, \ldots, v_k]}{m_{E'} P[v_1', \ldots, v_k']} \cdot \frac{m_{E'} P[v_1', \ldots, v_k']}{m_{E'} P[w_1, \ldots, w_k]},
\]

where \( \{v_i'\} \) is as in Lemma 2.14. To apply Lemma 2.14 to the first quotient on the right side, we must show \( d_H(E, E') \leq \text{const} \cdot \sum_i |v_i - w_i| \). For \( v \in E \) with \( |v| = 1 \), we write \( v = \sum_i a_i v_i \), and let \( w = \sum_i a_i w_i \). Then

\[
d(v, E') \leq |v - w| \leq \sum_i |a_i| |v_i - w_i| \leq \left( \sum_i |a_i| \right) \cdot \max_i |v_i - w_i| \leq N[v_i, \ldots, v_k] \cdot \max_i |v_i - w_i|.
\]

Clearly, the role of \( E \) and \( E' \) can be interchanged in the above. Recalling that \( d_H \leq 2\delta_a \) where \( \delta_a \) is as in Sect. 2.1.2, we have

\[
d_H(E, E') \leq 2 \max\{ N[v_1, \ldots, v_k], N[w_1, \ldots, w_k] \} \cdot \max_i |v_i - w_i| \leq 2N \max_i |v_i - w_i|.
\]

So, as long as \( 2N \max_i |v_i - w_i| \leq \delta_1 \), where \( \delta_1 \) is as in Lemma 2.14, this lemma gives

\[
\log \frac{m_{E'} P[v_1', \ldots, v_k']}{m_{E'} P[w_1, \ldots, w_k]} \leq L_1 d_H(E, E') \leq 2N L_1 \sum_{i=1}^k |w_i - v_i|.
\]

As for the second quotient on the right side of (5), since all vectors lie in \( E' \), it is easy to see, by putting the inner product \( \langle \cdot, \cdot \rangle_{E'} \) on \( E' \) and using the regularity of \( \log \circ \det \) on \( E' \), that there is a constant \( L_1' \) (depending on \( N \)) such that

\[
\log \frac{m_{E'} P[v_1', \ldots, v_k']}{m_{E'} P[w_1, \ldots, w_k]} \leq L_1' \sum_{i=1}^k \|v_i' - w_i\|_{E'}.\]

We need to bound \( \|v_i' - w_i\|_{E'} \) by a quantity involving \( \sum_i |v_i - w_i| \). Now \( \|v_i' - w_i\|_{E'} \leq \sqrt{k} |v_i' - w_i| \) and \( |v_i' - w_i| \leq |v_i' - v_i| + |v_i - w_i| \). It remains to observe that

\[
|v_i - v_i'| \leq |v_i - \pi_{E'/F} v_i| + |\pi_{E'/F} v_i - v_i'| = |\pi_{E'/F} v_i| + |\pi_{E'/F} v_i| - 1| \leq 8 \sqrt{k} d_H(E, E') \quad \text{by Lemma 2.5}.
\]

This together with the bound on \( d_H(E, E') \) above completes the proof. \( \square \)
2.3.2 Regularity of the determinant

The following is the main result of this section.

**Proposition 2.15.** For any \( k \geq 1 \) and any \( M > 1 \) there exist \( L_2, \delta_2 > 0 \) with the following properties. If \( A_1, A_2 : \mathcal{B} \to \mathcal{B} \) are bounded linear operators and \( E_1, E_2 \subset \mathcal{B} \) are \( k \)-dimensional subspaces for which

\[
|A_j|, \ (|A_j|_{E_j})^{-1} \leq M, \ j = 1, 2, \\
|A_1 - A_2|, \ d_H(E_1, E_2) \leq \delta_2,
\]

then we have the estimate

\[
\left| \log \frac{\det(A_1|E_1)}{\det(A_2|E_2)} \right| \leq L_2(|A_1 - A_2| + d_H(E_1, E_2)). \tag{6}
\]

**Proof.** Putting the inner products from Corollary 2.7 on \( E_1 \) and \( A_1 E_1 \), we let \( \{v_i\}, \{w_i\} \) be bases for \( E_1 \) and \( A_1 E_1 \) respectively consisting of orthogonal vectors corresponding to a singular value decomposition of \( A_1|E_1 \), normalized so as to have \( |v_i| = |w_i| = 1 \), and ordered so that \( w_i = |A_1|v_i|^{-1}A_1v_i \). Taking \( \delta_2 \leq \delta_1 \) as in Lemma 2.14 and fixing a complement \( F \) to \( E_1 \) with \( |\pi_{E_1/F}| \leq \sqrt{k} \), we define

\[ v'_i = \frac{\pi_{E_2/F}v_i}{|\pi_{E_2/F}v_i|}, \quad \text{and} \quad w'_i = \frac{A_2v'_i}{|A_2v'_i|}. \]

First we argue that (with respect to the norms \(|\cdot|\)) all four of the \( N[\cdots] \) quantities so defined are bounded by some \( N \) depending only on \( k \) and \( M \): clearly, \( N[v_1, \cdots, v_k], N[w_1, \cdots, w_k] \leq k^2 \), and \( N[v'_1, \cdots, v'_k] \leq 2kN[v_1, \cdots, v_k] \) by Lemma 2.14. To bound \( N[w'_1, \cdots, w'_k] \), write \( \pi'_i \) for the parallel projection onto \( v'_i \) and \( \sigma'_i \) the parallel projection onto \( w'_i \), and observe that \( \sigma'_i \circ A_2 = A_2 \circ \pi'_i \), which yields the bound \( |\sigma'_i| \leq |A_2| \cdot |(A_2|E_2)|^{-1} \cdot |\pi'_i| \), so that \( N[w'_1, \cdots, w'_k] \leq M^2N[v'_1, \cdots, v'_k] \). This bounds all four \( N[\cdots] \) quantities by \( N = 2k^3M^2 \).

We will estimate the left side of (6) as follows:

\[
\frac{\det(A_1|E_1)}{\det(A_2|E_2)} = \frac{m_{A_1E_1}P[A_1v_1, \cdots, A_1v_k]}{m_{E_1}P[v_1, \cdots, v_k]} \cdot \left( \frac{m_{A_2E_2}P[A_2v'_1, \cdots, A_2v'_k]}{m_{E_2}P[v'_1, \cdots, v'_k]} \right)^{-1}
= \left( \prod_{i=1}^k \frac{|A_1v_i|}{|A_2v'_i|} \right) \cdot \frac{m_{A_1E_1}P[w_1, \cdots, w_k]}{m_{A_2E_2}P[w'_1, \cdots, w'_k]} \cdot \frac{m_{E_1}P[v_1, \cdots, v_k]}{m_{E_2}P[v'_1, \cdots, v'_k]},
\]

where the extraction of the parenthetical term is as discussed in Section 2.1.3.

We estimate the three factors above separately. For the first, a simple computation gives

\[
\left| \frac{|A_1v_i|}{|A_2v'_i|} - 1 \right| = \left| \frac{|A_2v'_i| - |A_1v_i|}{|A_2v'_i|} \right| \leq M|A_1v_i - A_2v'_i| \leq M|A_1 - A_2| + M^2|v_i - v'_i|.
\]

For the second and third terms, we will show that for \( \delta_2 \) small enough, \( \max_i |v_i - v'_i|, |w_i - w'_i| \leq \delta(N, k) \) with \( \delta(N, k) \) as in Proposition 2.13, so that we obtain

\[
\left| \log \frac{m_{E_2}P[v'_1, \cdots, v'_k]}{m_{E_1}P[v_1, \cdots, v_k]} \right| \leq L \sum_{i=1}^k |v_i - v'_i|, \quad \left| \log \frac{m_{A_1E_1}P[w_1, \cdots, w_k]}{m_{A_2E_2}P[w'_1, \cdots, w'_k]} \right| \leq L \sum_{i=1}^k |w_i - w'_i|,
\]

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where \( L = L(\mathbb{N}, k) \).

It now remains to control \(|v_i - v'_i|\) and \(|w_i - w'_i|\) in terms of \(|A_1 - A_2|\) and \(d_H(E_1, E_2)\). A bound on \(|v_i - v'_i|\) was given in the proof of Proposition 2.13 and it is straightforward to estimate

\[
|w_i - w'_i| \leq \frac{1}{|A_1v_i|} \left( |A_1v_i - A_2v'_i| + |A_1v_i - |A_2v'_i|| \right) \\
\leq 2M|A_1v_i - A_2v'_i| \leq 2M|A_1 - A_2| + 2M^2|v_i - v'_i|.
\]

The proof is complete.

\[\square\]

Remark 2.16. Later, when we apply Proposition 2.15 to distortion estimates, we will need to use the dependence of the constants \(\delta_2, L_2\) on the parameters \(k, M\). Keeping track of the constraints on the constants \(\delta_2, L_2\) made throughout Section 2, one can show that there exists a constant \(C_k \geq 1\), depending only on the dimension \(k \in \mathbb{N}\), such that we may take \(\delta_2 = (C_k M^{10k})^{-1}\) and \(L_2 = C_k M^{10k}\) in the conclusion to Proposition 2.15.

As we have shown, in spite of the lack of differentiability present in finite-dimensional and Hilbert spaces, the notion of determinant we have introduced in this section is at least locally Lipschitz in the sense of Proposition 2.15. This regularity is used in a crucial way in Section 5.3 when we apply Proposition 2.15 to distortion estimates.

3 Addendum to the Multiplicative Ergodic Theorem

The Multiplicative Ergodic Theorem (MET) has by now been proved a number of times. Limiting our discussion to infinite dimensions, it was proved in [36] for Hilbert space cocycles, and in [30], [40], [23] for Banach space cocycles; see also [9], [10], [11] and [3]. In Sect. 3.1, we recall a version of the MET that is adequate for our purposes, and in Sect. 3.2, we add some interpretation in terms of volume growth, following up on the ideas in the previous section. In Sect. 3.3, we discuss continuity properties of certain subspaces.

When proving Theorems 1 and 2, standard techniques will allow us to reduce to working only with ergodic measures, so to keep the exposition simple we will state and work with the MET assuming that the underlying dynamical system is ergodic.

3.1 A version of the MET for Banach space cocyles

We recall below a precise statement of the MET on Banach spaces following Thieullen [40], in a (slightly simplified) setting that is adequate for our purposes.

Standing hypotheses and notation for Section 3: Let \(X\) be a compact metric space, and let \(f : X \to X\) be a homeomorphism preserving an ergodic Borel probability measure \(\mu\) on \(X\). We consider a continuous map \(T : X \to \mathcal{B}(\mathcal{B})\) where \(\mathcal{B}(\mathcal{B})\) denotes the space of bounded linear operators on \(\mathcal{B}\), the topology on \(\mathcal{B}(\mathcal{B})\) being the operator norm topology. We will sometimes refer to the triple \((f, \mu; T)\) as a \textit{cocycle}, and write \(T^n_x = T_{f^{n-1}x} \circ \cdots \circ T_x\).
Definition 3.1. Let \( C \subset B \) be any bounded set. The Kuratowski measure of noncompactness of \( C \) is defined by

\[
\alpha(C) = \sup\{r > 0 : \text{there is a finite cover of } C \text{ by balls of radius } r\}.
\]

For \( A \in B(B) \), we denote \( |A|_\alpha = \alpha(A(B)) \), where \( B \) is the closed ball of radius 1 in \( B \). The assignment \(| \cdot |_\alpha\) is a submultiplicative seminorm for which \(|A|_\alpha \leq |A|\) for any \( A \in B(B) \) (in particular, \( A \mapsto |A|_\alpha \) is continuous as a map on \( B(B) \) with the operator norm). This and other properties of \(| \cdot |_\alpha\) can be found in [31]. Since \( x \mapsto |T^n_x|_\alpha \) is continuous for any \( n \geq 1 \), it follows from subadditivity that the limit

\[
l_\alpha = \lim_{n \to \infty} \frac{1}{n} \log |T^n_x|_\alpha \geq -\infty
\]

exists and is constant \( \mu \)-almost surely; moreover, it coincides \( \mu \)-a.s. with \( \inf_{n \geq 1} \frac{1}{n} \log |T^n_x|_\alpha \).

Remark 3.2. For \( c > 0 \), the condition that \( l_\alpha < \log c \) \( \mu \)-a.e. is implied by the following: Let \( L_c(B) = \{ A \in B(B) : A = C + K \text{, where } K \text{ is compact and } |C| < c \} \). If \( T_x \in L_c(B) \) for all \( x \in X \), then \( \sup_{x \in X} |T^n_x|_\alpha < c \), hence \( l_\alpha < \log c \) \( \mu \)-almost surely by the continuity of \( x \mapsto |T^n_x|_\alpha \) and the compactness of \( X \).

Theorem 3.3 (Multiplicative ergodic theorem [40]). In addition to the Standing Hypotheses above we assume that \( T_x \) is injective for every \( x \in X \). Then, for any \( \lambda_\alpha > l_\alpha \), there is a measurable, \( f \)-invariant set \( \Gamma \subset X \) with \( \mu(\Gamma) = 1 \) and at most finitely many real numbers

\[
\lambda_1 > \lambda_2 > \cdots > \lambda_r
\]

with \( \lambda_r > \lambda_\alpha \) for which the following properties hold. For any \( x \in \Gamma \), there is a splitting

\[
B = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_r(x) \oplus F(x)
\]

such that

(a) for each \( i = 1, 2, \ldots, r \), \( \dim E_i(x) = m_i \) is finite and constant \( \mu \)-a.s., \( T_x E_i(x) = E_i(f x) \), and for any \( v \in E_i(x) \setminus \{0\} \), we have

\[
\lambda_i = \lim_{n \to \infty} \frac{1}{n} \log \| T^n_x v \| = -\lim_{n \to \infty} \frac{1}{n} \log \| (T^n_{f^{-n}x})^{-1} v \| ;
\]

(b) the distribution \( F \) is closed and finite-codimensional, satisfies \( T_x F(x) \subset F(f x) \) and

\[
\lambda_\alpha \geq \limsup_{n \to \infty} \frac{1}{n} \log |T^n_x|_{F(x)} ;
\]

(c) the mappings \( x \mapsto E_i(x) \), \( x \mapsto F(x) \) are \( \mu \)-continuous (see Definition [3.4] below), and

(d) writing \( \pi_i(x) \) for the projection of \( B \) onto \( E_i(x) \) via the splitting at \( x \), we have

\[
\lim_{n \to \pm \infty} \frac{1}{n} \log |\pi_i(f^n x)| = 0 \text{ a.s.}
\]
Definition 3.4. Let \((X, \mu)\) be as in the beginning of Section 3.1, and let \(Z\) be a metric space. We say that a map \(\Phi : X \to Z\) is \(\mu\)-continuous if there is an increasing sequence \(\bar{K}_n, n \in \mathbb{N}\), of compact subsets of \(X\), satisfying \(\mu(\bigcup_n K_n) = 1\), for which \(\Phi|_{\bar{K}_n}\) is continuous for each \(n \in \mathbb{N}\).

Remark 3.5. When \(Z\) is separable, Lusin’s Theorem implies directly that \(\mu\)-continuity is equivalent with Borel measurability, i.e., the inverse image of a Borel subset of \(Z\) is a Borel subset of \(X\) [19]. This equivalence continues to hold for arbitrary metric spaces \(Z\) as a consequence of a deep result of Fremlin; see Theorem 4.1 in [16].

We may assume going forward that there exist Borel \(K_n \subset X, n \in \mathbb{N}\), such that

- \(\Gamma = \bigcup_n K_n\) is an \(f\)-invariant set with \(\mu(\Gamma) = 1\), and
- the mappings \(x \mapsto F(x)\) and \(x \mapsto E_i(x), 1 \leq i \leq r\), are continuous on the closure of each \(K_n\).

To see this, let \(\{\bar{K}_n^{(i)}\}\) and \(\{\bar{K}_n^{(F)}\}\) be the compact sets given by the \(\mu\)-continuity of \(x \mapsto E_i(x)\) and \(x \mapsto F(x)\) respectively, and let \(\bar{K}_n = \bigcap_{i=1}^r \bar{K}_n^{(i)} \cap \bar{K}_n^{(F)}\). It is easy to check that \(\mu(\bigcup_n \bar{K}_n) = 1\). Trimming away sets of measure 0, we obtain an invariant set as claimed.

Lemma 3.6. The \(\mu\)-continuity of \(x \mapsto E(x)\) for \(E = E_i, \) any \(i,\) or \(E = F,\) implies that the following functions are Borel measurable:

\[\begin{align*}
(i) & \quad x \mapsto |T_x|_{E(x)}|, \\
(ii) & \quad x \mapsto m(T_x|_{E(x)}) \text{ where } m(A|_V) = \min\{ |Av| : v \in V, |v| = 1 \} \text{ is the minimum norm}, \\
(iii) & \quad x \mapsto \det(T_x|E(x)) \text{ for } E = E_i.
\end{align*}\]

Proof. Items (i) and (ii) follow from the \(\mu\)-continuity of \(x \mapsto E(x)\) together with the continuity of \((A, V) \mapsto |A|_V\) and \((A, V) \mapsto m(A|_V)\) as maps on \(B(B) \times G(B)\). Now assume \(E = E_i\) with \(\dim(E_i) = m\), and let \(G_m(B)\) denote the Grassmannian of \(m\)-dimensional subspaces. Item (iii) follows from the \(\mu\)-continuity of \(x \mapsto E(x)\) together with the continuity of \((A, V) \mapsto \det(A|V)\) viewed as a map on \(B_{inj}(B) \times G_m(B), B_{inj}(B) \subset B(B)\) being the subset of injective linear operators; see Proposition 2.15. \(\square\)

3.2 Interpretation as volume growth and corollaries

We now verify for the notion of volume introduced in Section 2 that Lyapunov exponents are infinitesimal volume growth rates. The setting and notation are as in Theorem 3.3.

Proposition 3.7. For any collection of indices \(1 \leq i_1 < i_2 < \cdots < i_k \leq r\) the map \(x \mapsto \det(T_x| \bigoplus_{l=1}^k E_{i_l}(x))\) is measurable, and for \(\mu\)-a.e. \(x \in \Gamma\),

\[
\lim_{n \to \infty} \frac{1}{n} \log \det \left( T^n_x \bigoplus_{l=1}^k E_{i_l}(x) \right) = \sum_{l=1}^k m_{i_l} \lambda_{i_l}.
\]
Proof. Let us first prove the result for \( k = 1 \), writing \( \lambda, E, m \) instead of \( \lambda_{i_1}, E_{i_1}, m_i \).

Define \( \phi(x) = \log \det(df_x|E(x)) \) for \( x \in \Gamma \). By Lemma 3.6, \( x \mapsto \phi(x) \) is Borel measurable, and since \( \phi \) is bounded from above (this follows from Lemma 2.11 and that \( \sup_{x \in X} |T_x| < \infty \)), the Birkhoff Ergodic Theorem tells us that there is a constant \( \gamma \in \mathbb{R} \cup \{-\infty\} \) such that for \( \mu \)-almost every \( x \in \Gamma \),

\[
\gamma = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \phi(f^{-i} x) .
\]

It suffices to show that \( \gamma = m\lambda \): For each \( x \), choosing a basis \( \{v_1, \ldots, v_m\} \) for \( E(x) \) orthonormal with respect to \( (\cdot, \cdot)_{E(x)} \), we have \( \det(T^n_x|E(x)) \leq m^{m/2} \prod_{i=1}^{m} |T^n_x v_i| \) (see Lemma 2.11, Item 1). The growth rates of \( |T^n_x v_i| \) are given by the MET, proving \( \gamma \leq m\lambda \). Since one cannot estimate easily a lower bound for \( \det(T^n_x|E(x)) \) starting from a fixed set of vectors in \( E(x) \), we iterate \( \text{backwards} \) instead. Fixing \( \delta > 0 \) and unit vectors \( \{v_1, \ldots, v_m\} \subset E \) as above, we obtain for large enough \( n \),

\[
\det((T^n_{f^{-n}x})^{-1}|E(x)) \leq m^{m/2} \prod_{i=1}^{m} |(T^n_{f^{-n}x})^{-1} v_i| \leq m^{m/2} e^{nm(-\lambda+\delta)} ,
\]

which gives the desired lower bound for

\[
\prod_{i=1}^{n} e^{\phi(f^{-i}x)} = \det(T^n_{f^{-n}x}|E(f^{-n}x)) .
\]

Proceeding to the general case, it suffices to give a proof for \( k = 2 \), which contains the main ideas. Let \( 1 \leq i_1 < i_2 \leq r \). The bounds in Lemma 2.12 together with the result for individual \( E_i \) proven above gives

\[
(C_{m_{i_1}+m_{i_2}} |\pi_{i_1}(f^n x)|^{m_{i_1}})^{-1} \leq \frac{\det(T^n_x|E_{i_1}(x) \oplus E_{i_2}(x))}{\det(T^n_x|E_{i_1}(x)) \cdot \det(T^n_x|E_{i_2}(x))} \leq C_{m_{i_1}+m_{i_2}} |\pi_{i_1}(x)|^{m_{i_1}} .
\]

Here we have used the fact that \( |\pi_{i_1}(x)|_{E_{i_1}(x) \oplus E_{i_2}(x)} \leq |\pi_{i_1}(x)| \). The volume growth formula now follows from the single subspace case and the fact that

\[
\lim_{n \to \infty} \frac{1}{n} \log |\pi_i(f^n x)| = 0
\]

for any \( 1 \leq i \leq r \).

Let \( 1 \leq i_1 < i_2 < \cdots < i_k \leq r \). A technical fact that will be needed is the integrability of \( \log^- m(T^n_x|_{\bigoplus_{i=1}^{k}E_{i}(x)}) \), equivalently the integrability of \( \log^+ |(T^n_x|_{\bigoplus_{i=1}^{k}E_{i}(x)})^{-1}| \), which requires justification as these minimum norms can be arbitrarily small. We deduce it from Proposition 3.7.

Corollary 3.8. For any collection of indices \( 1 \leq i_1 < i_2 < \cdots < i_k \leq r \), the function \( \psi(x) = \log^+ |(T^n_x|_{\bigoplus_{i=1}^{k}E_{i}(x)})^{-1}| \in L^1(\mu) \).
Proof. Since for any \( v \in \oplus_{i=1}^{k} E_i(x) \) with \( |v| = 1 \), we have
\[
\det(T_x|_{\oplus_{i=1}^{k} E_i(x)}) \leq m^m|T_x|_{\oplus_{i=1}^{k} E_i(x)}|^{m-1}|T_x v|
\]
where \( m = \dim(\oplus_{i=1}^{k} E_i) \), it follows that
\[
\left|(T_x|_{\oplus_{i=1}^{k} E_i(x)})^{-1}\right| = \max_{v \in \oplus_{i=1}^{k} E_i(x), |v|=1} \frac{1}{|T_x v|} \leq \frac{C}{\det(T_x|_{\oplus_{i=1}^{k} E_i(x)})}
\]
where \( C > 0 \) is a constant depending only on \( \sup_{x \in X} |T_x| \) and \( m \). Our assertion follows from the fact that \( -\log \det(T_x|_{\oplus_{i=1}^{k} E_i(x)}) \in L^1(\mu) \), which we have just proved.

\[\square\]

### 3.3 Continuity of certain distributions on sets with uniform estimates

Here we discuss the continuity of the distributions \( E^i := E_1 \oplus E_2 \oplus \cdots \oplus E_i \) for any \( i \leq r \). While the results are analogous to those in finite dimensions, some of the often used arguments in finite dimensions, such as compactness of the Grassmannian of \( m \)-dimensional subspaces, are not applicable in the present setting.

Let \( i \) be fixed throughout. We let \( E^i \) be as above, and let \( \bar{\delta} > 0 \) be such that \( 3\bar{\delta} < |\lambda_i - \lambda_{i+1}| \) if \( i < r \), and \( 3\bar{\delta} < |\lambda_i - \lambda_{\alpha}| \) if \( i = r \). For \( L > 1 \), let
\[
G^i_L := \{ x \in \Gamma : |(T_{f^{-n}x}|_{E^i(f^{-n}x)})^{-1}| \leq Le^{-n(\lambda_i-\bar{\delta})} \text{ for all } n \geq 1 \}.
\]

**Lemma 3.9.** For any \( i \leq r \) and \( L > 1 \), the map \( x \mapsto E^i(x) \) is continuous with respect to the \( d_H \)-metric on \( G(B) \) as \( x \) varies over \( G^i_L \).

**Proof.** Let \( E = E^i \), and define \( \tilde{F}(x) = \oplus_{j \geq i} E_j(x) \oplus F(x) \), so that at each \( x \in \Gamma \), we have \( B = E(x) \oplus \tilde{F}(x) \). Fix \( x \in \Gamma \), and let \( x^n \in G^i_L \) be such that \( x^n \to x \). Let \( \{v^n\} \subset B \) be any sequence of unit vectors such that \( v^n \in E(x^n) \) for each \( n \), and let \( v^n = w^n_E + w^n_{\tilde{F}} \in E(x) \oplus \tilde{F}(x) \) be the decomposition with respect to the splitting \( E(x) \oplus \tilde{F}(x) \). It suffices to show that \( w^n_{\tilde{F}} \to 0 \) as \( n \to \infty \); that this is sufficient for proving \( E(x^n) \to E(x) \) follows from the fact that \( E(x) \) and \( E(x^n) \) have the same finite dimension (for instance, one could use this to show that \( \|\pi_{E(x^n)}/E(x)\|_{E(x^n)} \to 0 \) as \( n \to \infty \), hence \( \delta_n(E(x^n), E(x)) \to 0 \) as \( n \to \infty \); see Section 2.1.2). To derive a contradiction, we will assume, after passing to a subsequence, that \( |w^n_{\tilde{F}}| \geq c \) for some \( c > 0 \) for all \( n \).

The following notation will be used: \( x_{-k} = f^{-k}x \), \( x^n_{-k} = f^{-k}x^n \), and let \( v^n_{-k} \) be the unique vector in \( E(x^n_{-k}) \) such that \( T_{x^n_{-k}}^k v^n_{-k} = v^n \). We split
\[
v^n_{-k} = w^n_{-k \oplus \tilde{F}} \in E(x_{-k}) \oplus \tilde{F}(x_{-k}) .
\]

Let \( \pi_E(x) \) denote the projection onto \( E(x) \) along \( \tilde{F}(x) \). We will show that for every \( k > 0 \) large enough, there exists \( n(k) \) such that for all \( n \geq n(k) \),
\[
|\tilde{w}^{n}_{-k}| \leq (L + 2|\pi_E(x)|) e^{-k(\lambda_i-\bar{\delta})} \quad \text{and} \quad |T_{x^n_{-k}}^{k}(\tilde{w}^{n}_{-k})| \geq \frac{c}{2} . \tag{7}
\]
Now since $x \in \Gamma$, we may assume $x_{-k}$ visits infinitely often sets on which there are uniform bounds for $|T^m|_{F}$, $m = 1, 2, \ldots$, so there exists arbitrarily large $k$ for which $|T^k_{x_{-k}}|_{F(x_{-k})} \ll e^{k(\lambda - 2\delta)}$. That is clearly inconsistent with (7).

To prove (7), observe first that since $x^n \in G^i$, we have $|v_{n-k}| \leq Le^{-k(\lambda - \delta)}$, and notice that this bound is independent of $n$. Thus for each fixed $k$,\[
|T^k_{x_{-k}}v_{n-k} - v^n| = |(T^k_{x_{-k}} - T^k_{x_{-k}})(v_{n-k})| \leq |T^k_{x_{-k}} - T^k_{x_{-k}}| \cdot |v_{n-k}| \rightarrow 0 \quad \text{as } n \rightarrow \infty .
\]
In particular, $|T^k_{x_{-k}}\hat{w}_{n-k}^E - w^n,\bar{E}| \rightarrow 0$ and $|T^k_{x_{-k}}\hat{w}_{n-k}^F - w^n,\bar{F}| \rightarrow 0$ as $n \rightarrow \infty$, which implies that for all $n \geq$ some $n(k)$,\[
|T^k_{x_{-k}}\hat{w}_{n-k}^E| \leq 2|\pi_E(x)v^n| \leq 2|\pi_E(x)| \quad \text{and} \quad |T^k_{x_{-k}}\hat{w}_{n-k}^F| \geq \frac{c}{2} .
\]
Finally, as $x \in \Gamma$, $|(T^k_{x_{-k}}|_{E(x_{-k})})^{-1}| < e^{-k(\lambda - \delta)}$ holds for all large enough $k$. Thus\[
|\hat{w}_{n-k}^F| \leq |v_{n-k}| + |\hat{w}_{n-k}^E| \leq (L + 2|\pi_E(x)|) e^{-k(\lambda - \delta)} ,
\]
completing the proof of (7). \hfill \Box

In the rest of this paper, $(f, \mu)$ is assumed to satisfy Hypotheses (H1)--(H3) in Section 1. For simplicity, we first treat the ergodic case, assuming $\mu$ is ergodic from here through Sect. 5.3, removing the ergodicity assumption only in Sect. 5.4.

## 4 Preparation I: Lyapunov metrics

The goal of this section is to introduce new norms $| \cdot |_x'$ in the tangent spaces of $x \in \Gamma$ with respect to which expansions and contractions are reflected in a single time step. We also introduce a function $l$ that, roughly speaking, measures the degree to which $f$ deviates from uniform hyperbolicity.

These techniques have been used in finite dimensions and on separable Hilbert spaces (we follow more closely [19, 20] and [24]; see also the references in [2]). There is, however, the following difference: In finite dimensions, for instance, it is customary to fix a model space $\mathbb{R}^{\dim E^u} \times \mathbb{R}^{\dim E^c} \times \mathbb{R}^{\dim E^s}$ with a Euclidean inner product and to identify a neighborhood of each $x$ (with its Lyapunov metric) with a neighborhood of 0 in the model space. We do not do this here, as there is no obvious common model space for $E^s_x$, $x \in \Gamma$. Indeed there is no standard model space for infinite dimensional subspaces of Banach spaces. Instead, we will work directly on the tangent spaces $\mathcal{B}_x$ of $x$.

For completeness, we will go through the entire construction, providing complete statements of results, but will omit proofs that require no modification.

### 4.1 Adapted norms

Consider the cocycle $(f, \mu; df)$. By condition (H3), $l_\alpha < 0$. We fix an arbitrary $\lambda_\alpha \in (l_\alpha, 0)$, and apply Theorem 3.3 to obtain Lyapunov exponents $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ and a splitting of $\mathcal{B}_x$, the tangent space at $x$, into $\mathcal{B}_x = \oplus_{i=1}^r E_i(x) \oplus F(x)$ for every $x \in \Gamma$. 20
For many purposes, it is sufficient to distinguish between unstable, center and stable subspaces, defined to be

\[ E^u_x = \bigoplus_{i: \lambda_i > 0} E_i(x), \quad E^c_x = \bigoplus_{i: \lambda_i = 0} E_i(x), \quad \text{and} \quad E^s_x = \bigoplus_{i: \lambda_i < 0} E_i(x) \oplus F(x). \]

We will also write \( E^{cu}_x = E^u_x \oplus E^c_x \), and use \( \pi^u_x, \pi^c_x \) and \( \pi^s_x \) to denote the projections onto \( E^u_x, E^c_x \) and \( E^s_x \) respectively according to the splitting \( B_x = E^u_x \oplus E^c_x \oplus E^s_x \). We do not require that all these subspaces be nontrivial; in particular, in our main results, \( E^c = \{0\} \).

We now proceed to modify the norms on tangent spaces of individual points, with the aim of producing new norms with respect to which Lyapunov exponents will be reflected in a single time step. Let \( \lambda^+ = \min\{\lambda_i : \lambda_i > 0\} \) and \( \lambda^- = \max\{\lambda_0, \lambda_i : \lambda_i < 0\} \). We define \( \lambda_0 = \min\{\lambda^+, -\lambda^-\} \), fix \( \delta_0 \ll \lambda_0 \) and let \( \lambda = \lambda_0 - 2\delta_0 \). For \( n > 0 \) and \( u \in E^{cu}_x \), let us agree to use the shorthand \( df_{f^{-n}x} \) to mean \( (df_{f^{-n}x}|^E^{cu}_x)^{-1}u \). We introduce for each \( x \in \Gamma \) a new norm \( | \cdot |^p_x \) on \( B_x \) as follows:

For \( u \in E^u_x \), \( |u|^p_x = \sum_{n=0}^{\infty} \frac{|df_{f^{-n}x}u|}{e^{-n\lambda}} \),

\( v \in E^c_x \), \( |v|^p_x = \sum_{n=-\infty}^{\infty} \frac{|df_{f^{-n}x}v|}{e^{2|n|\delta_0}} \),

\( w \in E^s_x \), \( |w|^p_x = \sum_{n=0}^{\infty} \frac{|df_{f^{-n}x}w|}{e^{-n\lambda}} \),

and for \( p = u + v + w \in B_x, u \in E^u_x, v \in E^c_x, w \in E^s_x \), we define

\[ |p|_x = \max\{|u|_x, |v|_x, |w|_x\}. \quad (8) \]

To estimate how far these new norms deviate from the original ones, we let

\[ C_u(x) = \sup_{n \geq 0} \frac{\sup_{v \in E^u_x, |v| = 1} |df_{f^{-n}x}v|}{e^{-n(\lambda_0 - \delta_0)}}, \]

\[ C_c(x) = \sup_{n \in \mathbb{Z}} \frac{\sup_{v \in E^c_x, |v| = 1} |df_{f^{-n}x}v|}{e^{n|\delta_0|}}, \]

\[ C_s(x) = \sup_{n \geq 0} \frac{\sup_{v \in E^s_x, |v| = 1} |df_{f^{-n}x}v|}{e^{-n(\lambda_0 - \delta_0)}}, \]

and let

\[ C(x) = \max\{C_u(x), C_c(x), C_s(x), |\pi^u_x|, |\pi^c_x|, |\pi^s_x|\}. \]

Observe that all are finite-valued, Borel measurable functions on \( \Gamma \) (see Lemma 3.6).

The following lemma summarizes the properties of the adapted norms \( | \cdot |^p_x \). The proof is a simple computation and is omitted.

**Lemma 4.1.** The following hold for all \( x \in \Gamma \):

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1. (One-step hyperbolicity) For any $u \in E^u_x, v \in E^c_x, w \in E^s_x$, we have

\[ |df_x u'_{fx}| \geq e^\lambda |u'|_x \]
\[ e^{-2\delta_0} |v|_x \leq |df_x v|_{fx} \leq e^{2\delta_0} |v|_x \]
\[ |df_x w|_{fx} \leq e^{-\lambda} |w|_x. \]

2. The norms $|\cdot|_x'$ are related to the usual norm $|\cdot|$ by

\[ \frac{1}{3} |p| \leq |p|'_x \leq \frac{3}{1 - e^{-\delta_0}} C(x)^2 |p| . \] (9)

Identifying $B_x$ with $x + \mathcal{B}$ via the exponential map $\exp_x : B_x \to \mathcal{B}$, i.e., the map that sends $v \in B_x$ to $x + v \in \mathcal{B}$, we view $\{(B_x, |\cdot|_x') : x \in \Gamma\}$ as a collection of charts, and define the connecting maps

\[ \tilde{f}_x : B_x \to B_{fx} \quad \text{by} \quad \tilde{f}_x = \exp^{-1}_{fx} \circ f \circ \exp_x . \]

We will also use the notation $\tilde{f}_x^n = \tilde{f}_{f_{n-1}x} \circ \cdots \circ \tilde{f}_{f_1x} \circ \tilde{f}_x$. Since the derivative at 0 of $\tilde{f}_x$, written as $(d\tilde{f}_x)_0$, is the same as $df_x$, these derivatives exhibit hyperbolicity in one timestep with respect to the $|\cdot|'$ norms.

Our next task is to reduce the sizes of the domains for $\tilde{f}_x$ so that on these reduced domains, $\tilde{f}_x$ is well approximated by the linear map $(d\tilde{f}_x)_0$. Since $f$ is assumed to be $C^2$ and $\mathcal{A}$ is compact (see (H1), (H2) in Sect. 1), it is easy to see that there exist $M_0 > 0$ and $r_0 > 0$ such that $|d^2 f_x| < M_0$ for all $x \in \mathcal{B}$ with $\text{dist}(x, \mathcal{A}) < r_0$.

Below, we use the notation $\tilde{B}_x(r) = \{p \in B_x \mid |p|'_x \leq r\}$. In all statements regarding the chart maps $\tilde{f}_x$, the norms on their domain and range spaces should be understood to be $|\cdot|_x'$ and $|\cdot|'_x$ respectively. The next lemma is straightforward.

**Lemma 4.2.** Define $\tilde{l} : \Gamma \to [1, \infty)$ by

\[ \tilde{l}(x) = \max \left\{ \frac{27M_0}{1 - e^{-\delta_0}}, 1 \right\} \cdot C(fx)^2 . \] (10)

Then there exists $\delta_1 > 0$ such that for any $\delta \leq \delta_1$, the following holds for $\tilde{f}_x : \tilde{B}_x(\delta \tilde{l}(x)^{-1}) \to B_{fx}$, i.e., for $\tilde{f}_x$ restricted to the domain $\tilde{B}_x(\delta \tilde{l}(x)^{-1})$:

1. $\text{Lip}(\tilde{f}_x - (d\tilde{f}_x)_0) \leq \delta$;
2. the mapping $z \mapsto (d\tilde{f}_x)_z$ satisfies $\text{Lip} (d\tilde{f}_x) \leq \tilde{l}(x)$.

### 4.2 Measuring deviation from uniform hyperbolicity

The maneuvers in Sect. 4.1 transform the nonuniformly hyperbolic map $f$ into a family of uniformly hyperbolic local maps $\tilde{f}_x$, but it is at the expense of coordinate changes that can be unboundedly large as $x$ varies over $\Gamma$. The sizes of these coordinate changes, which we may think of as measuring how far $f$ deviates from being uniformly hyperbolic, are
incorporated into the function $\tilde{l}$ in Lemma 4.2, a function that contains two other pieces of related information: chart sizes, i.e. how quickly $f$ deviates from $df_x$ as we move away from $x$, and the regularity of $df$ as seen in these coordinates. Informally, the larger $\tilde{l}(x)$ at a point, the weaker the hyperbolicity at $x$.

The function $\tilde{l}$ is measurable and usually unbounded on $\Gamma$. We show next that it is dominated by a function that varies slowly along orbits.

**Lemma 4.3.** Given any $\delta_2 > 0$, there exists a function $l : \Gamma \to [1, \infty)$ (depending only on $\tilde{l}$ and $\delta_2$) such that for $\mu$-a.e. $x \in \Gamma$,

$$\tilde{l}(x) \leq l(x), \text{ and } l(f^\pm x) \leq e^{\delta_2}l(x). \quad (11)$$

Once this lemma is proved, we will use $l$ instead of $\tilde{l}$ in all subsequent estimates, and Lemma 4.2 clearly holds for $\tilde{f}_x$ on the domain of $\tilde{B}_x(\delta l(x)^{-1})$ for any $\delta \leq \delta_1$. An obvious advantage of having slowly varying chart sizes is that it ensures that graph transforms of functions from $E^u$ to $E^s$ are well defined (see Sect. 5.1). Another advantage of a slowly varying $l$ is, as we will see, that it ensures that estimates can deteriorate at most slow exponentially along orbits.

Lemma 4.3 is well known in finite dimensions. In the present setting, there is a subtle difference in the proof caused by the fact that $df_x|_{E^u}$ is not assumed to have a uniformly bounded inverse. This difference is exemplified by the task of finding a slowly-varying enlargement of the function $C_u$ (see Sect. 4.1). So instead of giving a full proof of Lemma 4.3, we will limit our discussion to enlarging $C_u$.

**Lemma 4.4.** Given any $\delta_2 > 0$, there exists a measurable function $C'_u : \Gamma \to [1, \infty)$ such that for $\mu$-a.e. $x \in \Gamma$,

$$C_u(x) \leq C'_u(x), \text{ and } C'_u(f^\pm x) \leq e^{\delta_2}C'_u(x).$$

We will use the following ergodic theory lemma.

**Lemma 4.5** (Lemmas 8 & 9 in [41]). Let $(X, \mathcal{F}, \mu, f)$ be an invertible measure-preserving transformation (mpt) of a probability space. Let $\phi : X \to \mathbb{R}$ be measurable and assume that either $(\phi \circ f - \phi)^+$ or $(\phi \circ f - \phi)^-$ is integrable. Then

$$\lim_{n \to \pm\infty} \frac{1}{|n|} \phi \circ f^n \to 0 \quad \text{a.s.}$$

**Proof of Lemma 4.4.** Let $\psi : \mathcal{A} \to \mathbb{R}$ be a function for which $\log \psi$ satisfies the hypotheses of Lemma 4.1 for the mpt $(\mathcal{A}, \mathcal{B}, \mu, f^{-1})$, where $\mathcal{B}$ is the Borel $\sigma$-algebra of subsets of $\mathcal{A}$. Then it will follow that the function

$$\psi'(x) := \sup_{n \in \mathbb{Z}} e^{-|n|\delta_2} \psi(f^n x)$$

is almost-surely finite valued, satisfies $\psi \leq \psi'$, and, as one can easily check, $\psi'(f^\pm x) \leq e^{\delta_2} \psi'(x)$. So it suffices to check the hypotheses of Lemma 4.5 for $\psi(x) = \log C_u(x)$. Observe
that either $C_u(x) = 1$, i.e., the supremum in the definition of $C_u(x)$ is attained at $n = 0$, or that

$$C_u(x) \leq |df_x^{-1}|_{E^u_x} \cdot \sup_{n \geq 1, v \in E^u_{f^{-1}x}, |v| = 1} e^{n(\lambda_0 - \delta_0)}|df_{f^{-1}x}^{-n-1}v|$$

$$= e^{\lambda_0 - \delta_0}|df_x^{-1}|_{E^u_x} \cdot C_u(f^{-1}x).$$

Hence

$$\log C_u(f^{-1}x) - \log C_u(x) \geq \min\{-\lambda_0 + \delta_0 - \log |df_x^{-1}|_{E^u_x}, 0\}.$$

Thus to check that $(\log C_u(f^{-1}x) - \log C_u(x))^\gamma \in L^1(\mu)$, it suffices to check

$$\log^+ |df_x^{-1}|_{E^u_x} = \log^+ |(df|_{E^u})^{-1}| \circ f^{-1}(x) \in L^1(\mu).$$

Unlike the case of finite dimensional diffeomorphisms, this requires justification, as $|(df|_{E^u})^{-1}|$ can be unboundedly large as $x$ varies over $\Gamma$. We have in fact anticipated this issue, and have proved in Corollary 3.8 that $\log^+ |(df|_{E^u})^{-1}| \in L^1(\mu).$}

The numbers $\lambda_0, \delta_0$, (hence $\lambda$) and $\delta_1$ introduced in Section 4 are fixed once and for all. We now fix $\delta_2 > 0$ with $\delta_2 \ll \lambda$, (e.g. $\delta_2 \leq \frac{1}{100\mu x} \lambda$; this will be useful later in Section 5.3), and let $l$ be given by Lemma 4.3. For $l_0 \geq 1$, let

$$\Gamma_{l_0} := \{x \in \Gamma : l(x) \leq l_0\}.$$

We will refer to sets of the form $\Gamma_{l_0}$ as uniformity sets. On such a set, one has uniform expansion and contraction estimates, uniform ‘angles’ of separation between $E^u, E^c$ and $E^s$, and uniform bounds on the extent to which the adapted norms differ from the original norms. The number $\delta \leq \delta_1$ can be chosen independently of all the quantities above, permitting us to shrink our charts as needed. Once $\delta$ is fixed, chart sizes, nonlinearities and second derivatives in charts will also be uniformly bounded for $x \in \Gamma_{l_0}$. Furthermore, Lemma 4.3 tells us that for $x \in \Gamma_{l_0}$, $f^n x \in \Gamma_{l_0 \circ |n| \delta_2}$, that is to say, the quantities above have bounds that can deteriorate at most slow exponentially along orbits.

## 5 Preparation II : Elements of Hyperbolic Theory

Unstable manifolds and how volumes on them are transformed are central ideas in this paper. In Sects. 5.1 and 5.2, we record some basic facts about continuous families (called “stacks”) of local unstable manifolds, in preparation for the definition of SRB measures in Sect. 6.1. In Sect. 5.3, we consider densities on unstable manifolds with respect to reference measures derived from the induced volumes on finite dimensional subspaces introduced earlier. We provide a detailed proof of distortion bounds as these densities are pushed forward, confirming that the notion of volume proposed is adequate for our needs.

From Section 3 through Sect. 5.3, we have operated under the assumption of ergodicity, which has simplified considerably the exposition. In Sect. 5.4, we discuss how the constructions and results given so far can be adapted for nonergodic measures.
5.1 Local unstable manifolds

Continuing to work in the Lyapunov metric, we employ the following notation: for \(\tau = u, c, s\) and \(r > 0\), we write \(\tilde{B}^r_\tau(r) = \{a \in E^r_\tau : |a|_\tau \leq r\}\) and \(\tilde{B}^u_\tau(x) = \tilde{B}^r_\tau(x) + \tilde{B}^s_\tau(x)\), so that \(\tilde{B}_x(r)\), which was defined earlier, is equal to \(\tilde{B}^u_\tau(r) + \tilde{B}^cs_\tau(r)\).

**Theorem 5.1** (Unstable Manifolds Theorem in charts). For all \(\delta > 0\) sufficiently small, there exists a unique family of continuous maps \(\{g_x : \tilde{B}^u_\delta(x) \rightarrow \tilde{B}^cs_\delta(x)\}\) such that

\[
g_x(0) = 0 \quad \text{and} \quad \tilde{f}_x(\text{graph } g_x) \supset \text{graph } g_{f_x} \quad \text{for all } x \in \Gamma.
\]

With respect to the \(|\cdot|'_x\) norms on \(\tilde{B}_x(\delta(x)^{-1})\), the family \(\{g_x\}_{x \in \Gamma}\) has the following additional properties:

1. \(g_x\) is \(C^{1+\text{Lip}}\) Fréchet differentiable, with \((dg_x)_0 = 0\);
2. Lip \(g_x \leq \frac{1}{10}\) and Lip \(dg_x \leq C\) where \(C > 0\) is independent of \(x\);
3. if \(\tilde{f}_x(u_i + g_x(u_i)) \in \tilde{B}_{f_x}(\delta(f(x)^{-1}))\) for \(u_i \in \tilde{B}^u_\delta(x)\), \(i = 1, 2\), then

\[
|\tilde{f}_x(u_1 + g_x(u_1)) - \tilde{f}_x(u_2 + g_x(u_2))|_{f_x} \geq (e^\lambda - \delta)\|u_1 + g_x(u_1) - (u_2 + g_x(u_2))\|'_x.
\]

(12)

These results are well known for finite-dimensional systems. For Hilbert space maps, stable and unstable manifolds were constructed using Lyapunov charts in [24]; the methods in that paper can be carried over without any substantive change to our Banach space setting, and we omit the proof.

We fix \(\delta' \leq \delta_1\) small enough that Theorem 5.1 holds with \(\delta \leq \delta'_1\), and write \(W^u_{\delta,x} = \text{graph}(g_x)\) where \(g_x\) is as above. It is easy to see that \(\tilde{W}^u_{\delta,x} \subset W^u_{\delta,x}\) for \(\delta < \delta'\). We let \(W^u_{\delta,x} = \exp_x \tilde{W}^u_{\delta,x}\), and call \(W^u_{\delta,x}\) a local unstable manifold at \(x\). It will be assumed implicitly in all future references to local unstable manifolds that \(\delta \leq \delta'\) where \(\delta'_1\) is as above. Note that we may shrink \(\delta\) without harm, and will do so a finite number of times in the proofs to come. The global unstable manifold at \(x\), defined to be

\[
W^u_x := \bigcup_{n \geq 0} f^n(W^u_{\delta,x}),
\]

is an immersed submanifold in \(\mathcal{B}\) (by the injectivity of \(f\) and \(df_x\); see (H1) in Section 1).

**Theorem 5.1** is proved using graph transforms, an idea we will need again later on. We state (without proof) the following known result. Let

\[
\mathcal{W}(x) = \left\{ g : \tilde{B}^u_{f_x}(\delta(f(x)^{-1})) \rightarrow E^cs_{f_x} \quad \text{s.t.} \quad g(0) = 0 \quad \text{and} \quad \text{Lip } g \leq \frac{1}{10} \right\},
\]

and for \(g \in \mathcal{W}(x)\), we let \(\Psi_x g\) denote the graph transform of \(g\) if it is defined; i.e., \(\Psi_x g : \tilde{B}^u_{f_x}(\delta(f(x)^{-1})) \rightarrow E^cs_{f_x}\) is the map with the property that

\[
\tilde{f}_x(\text{graph } g) \supset \text{graph } \Psi_x g.
\]
Lemma 5.2 (Contraction of graph transforms). The following hold for every \( x \in \Gamma \):

(i) for every \( g \in \mathcal{W}(x) \), \( \Psi_x g \) is defined and is \( \in \mathcal{W}(fx) \);

(ii) there exists a constant \( c \in (0, 1) \) such that for all \( g_1, g_2 \in \mathcal{W}(x) \),

\[
\|\Psi_x g_1 - \Psi_x g_2\|_{fx} \leq c \|g_1 - g_2\|_x
\]

where

\[
\|h\|_z = \sup_{v \in B^z(\delta l(z)^{-1}) \setminus \{0\}} \frac{|h(v)|'_z}{|v|'_z} \quad \text{for} \quad z = x, fx.
\]

We will also need the following characterization of unstable manifolds, valid only in the absence of zero exponents.

Lemma 5.3. Assume \( E^c = \{0\} \). For \( \delta \) small enough, the following hold for all \( x \in \Gamma \).

(a) \( W^u_{\delta, x} \) has the characterization

\[
W^u_{\delta, x} = \exp_x \{ z \in \tilde{B}_x(\delta l(x)^{-1}) : \forall n \in \mathbb{N}, \exists z_n \in \tilde{B}_{f^{-n}x}(\delta l(f^{-n}x)^{-1}) \text{ s.t. } \tilde{f}^{-n}_f z_n = z \};
\]

(b) for \( y \in W^u_{\delta, x} \), the tangent space \( E^u_y := T_y W^u_{\delta, x} \) to \( W^u_{\delta, x} \) at \( y \) has the characterization

\[
E^u_y = \{ v \in B_y : df^{-n}v \text{ exists for all } n \geq 1, \text{ and } \limsup_{n \to \infty} \frac{1}{n} \log |df^{-n}v| \leq -\lambda \}.
\]

The proof of Lemma 5.3 involves the so-called “backwards graph transform”, which is different in infinite dimensions because one cannot iterate backwards. We recall the definition of this transform \( \Psi^s_x \), as it will be used a number of times.

Continuing to assume \( E^c = \{0\} \), we define

\[
\mathcal{W}^s_{\frac{\pi}{2}}(x) = \left\{ h : \tilde{B}^s_x(\delta l(x)^{-1}) \to \tilde{B}^u_x(\delta l(x)^{-1}) \text{ s.t. } |h(0)|'_x \leq \frac{1}{2} \delta l(x)^{-1} \text{ and Lip } h \leq \frac{1}{10} \right\}.
\]

For \( h \in \mathcal{W}^s_{\frac{\pi}{2}}(x) \), if \( \ell : \tilde{B}^s_{f^{-1}x}(\delta l(f^{-1}x)^{-1}) \to \tilde{B}^u_{f^{-1}x}(\delta l(f^{-1}x)^{-1}) \) is a map with the property that

\[
\tilde{f}^{-1}_f(\text{graph } \ell) \subset \text{graph } h, \quad (13)
\]

then we say \( \ell \) is the graph transform of \( h \) by \( \tilde{f}^{-1} \), and write \( \ell = \Psi^s_x h \). The result in Lemma 5.3 was proved in [24] for Hilbert space maps in a context similar to ours; their proof generalizes without change to Banach spaces.

Lemma 5.4. Assume \( E^c = \{0\} \). Let \( x \in \Gamma \) and \( h \in \mathcal{W}^s_{\frac{\pi}{2}}(x) \). Then

\[\text{In the case when } (f, \mu; df) \text{ does not have zero exponents, the uniform norm}\]

\[
\|h\|_{z, \infty} = \sup_{v \in B^z(\delta l(z)^{-1})} |h(v)|'_z
\]

is often used when stating contraction estimates for the graph transform.
backward graph transforms by the linear maps $dW$ we will prove here the continuity of unstable leaves on certain measurable sets. That almost every point is contained in a "stack of unstable manifolds" will be relevant in Section 6.

5.2 Unstable stacks

Proof of Lemma 5.3. (a) Let $z \in \tilde{B}_x(\delta l(x)^{-1})$ be such that $z \not\in \text{graph } g_x$, and let $\tilde{z} \in \text{graph } g_x$ be such that $\pi_x^u z = \pi_x^u \tilde{z}$. For $n \in \mathbb{N}$, we let $z_n \in \tilde{B}_f^{-n x}(\delta l(f^{-n} x)^{-1})$ be such that $\tilde{f}_{f^{-n} x}^n z_n = z$ if such a $z_n$ exists, and let $\tilde{z}_n \in \text{graph } g_{f^{-n} x}$ be such that $\tilde{f}_{f^{-n} x}^n \tilde{z}_n = \tilde{z}$; we know $\tilde{z}_n$ exists for all $n$. We will show that $z_n$ and $\tilde{z}_n$ diverge exponentially at a rate faster than $\delta_2$, so one of them must leave the chart eventually.

We assume there exists $z_1 \in \tilde{B}_x(\delta l(f^{-1} x)^{-1})$ such that $\tilde{f}_{f^{-1} x} z_1 = z$; if no such $z_1$ exists, we are done. Let $h \in W_x^u(x) (\text{defined using } \tilde{B}_f^{-1 x}(2\delta l(f^{-1} x)^{-1}))$ be the constant map $h(v) \equiv \pi^u(z)$, and let $h_1 = \Psi^x_h$. We claim that $z_1 \in \text{graph } h_1$. If not, let $\pi_{f^{-1} x}^u z_1 = s_1 \in \tilde{B}_x(\delta l(f^{-1} x))$. By standard hyperbolic estimates,

$$|\pi_x^u(z - \tilde{f}_{f^{-1} x}(h_1(s_1) + s_1))'|_x \geq |\pi_x^u(z - \tilde{f}_{f^{-1} x}(h_1(s_1) + s_1))'|_x,$$

contradicting $z, \tilde{f}_{f^{-1} x}(h_1(s_1) + s_1) \in \text{graph } h$. This proves $z_1 \in \text{graph } h_1$. By Lemma 5.4 (ii), $|z_1 - \tilde{z}_1|_{f^{-1} x} \geq (e^{-\lambda} + \delta)^{-1}|z - \tilde{z}|_{f^{-1} x}^u$

Repeating the argument in the last paragraph with $h_1$ in the place of $h$, we obtain that either there does not exist $z_2 \in \tilde{B}_x(\delta l(f^{-2} x)^{-1})$ such that $\tilde{f}_{f^{-2} x} z_2 = z_1$, or $|z_2 - \tilde{z}_2|_{f^{-2} x} \geq (e^{-\lambda} + \delta)^{-1}|z_2 - \tilde{z}_2|_{f^{-1} x}$, providing the exponential divergence of $z_n$ and $\tilde{z}_n$ claimed.

Part (b) is proved similarly: Continuing to work in charts, we let $y = \exp_x^{-1} y$, and let $\tilde{y}_{-n}$ be such that $\tilde{f}_{f^{-n} x}(\tilde{y}_{-n}) = y$. We assume $\delta$ is small enough that Lemma 5.4 applies to backward graph transforms by the linear maps $d(\tilde{f}_{f^{-n} x})_{\tilde{y}_{-n}}$. Repeating the argument above using these graph transforms, we conclude that for $v \in B(y)$ such that $v \not\in E^u_y$, either $d_{f^{-n} y}^w v$ is not defined for some $n$, or $|d_{f^{-n} y}^w v|$ diverges exponentially as $n \to \infty$.

\[\Box\]

5.2 Unstable stacks

Notice that we have not made any assertion in Theorem 5.1 regarding the regularity of the assignment $x \mapsto g_x$. In finite dimensions and on separable Hilbert spaces, one often asserts that $W^{u}_{\delta,x}$ varies measurably with $x$. In the spirit of the discussion at the end of Sect. 3.1, we will prove here the continuity of unstable leaves on certain measurable sets. That almost every point is contained in a “stack of unstable manifolds” will be relevant in Section 6.

We first define precisely what is meant by such a stack. Consider nearby points $x, y \in \Gamma$ with $d_H(E^u_y, E^u_x), d_H(E^{cs}_y, E^{cs}_x) \ll 1$. Let $\phi_y : \text{Dom}(\phi_y) \to E^{cs}_y$, where $\text{Dom}(\phi_y) \subset E^u_y$; we let $\phi^x_y : \text{Dom}(\phi^x_y) \to E^{cs}_x$ (with $\text{Dom}(\phi^x_y) \subset E^u_x$) be the mapping for which

$$exp_y(\text{graph } \phi_y) = exp_x(\text{graph } \phi^x_y) \quad (14)$$

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if such a mapping can be uniquely defined; whether or not this can be done depends on $x$, $y$ and $\phi$. We say that $\phi_y^x$ is defined on $V \subset E^u_x$ if $\text{Dom}(\phi_y^x) \supset V$.

In discussions that involve more than one chart, it is natural to use $\| \cdot \|$ norms rather than the pointwise adapted $| \cdot |_y$ norms. We introduce the following notation: For $\tau = u, c, s$, let $B^\tau_x(r) = \{ v \in E^\tau_x : |v| \leq r \}$, and let $B_x(r) = B^u_x(r) + B^c_x(r)$ (Notice the distinction between $B^\tau_x(r)$ and $\tilde{B}^\tau_x(r)$.) Below, the space $C(B^u_x(r), E^c_x)$ of continuous functions from $B^u_x(r)$ to $E^c_x$ will be endowed with its $C^0$ norm $\| \cdot \|$ defined using the $| \cdot |$ norm on $E^c_x$.

Recall the definition of $\Gamma$, at the end of Section 4 and the definition of $K_n$ in Sect. 3.1.

**Lemma 5.5.** Let $l_0$ and $n_0 > 1$ be fixed, and fix $x_0 \in \Gamma_{l_0} \cap K_{n_0}$. For $\epsilon > 0$ and $x \in \Gamma$, we let

$$U(x, \epsilon) := \Gamma_{l_0} \cap K_{n_0} \cap \{ y : |x - y| < \epsilon \},$$

and let $g_y$ be as in Theorem 5.1. Then for any $\delta \leq \frac{1}{4} \delta_1$, we may choose $\epsilon_0 > 0$ sufficiently small so that the following holds:

(a) for every $y \in U(x_0, \epsilon_0)$, $g_{x_0}^y$ is defined on $B^u_{x_0}(\delta l_0^{-3})$, and

(b) the mapping $\Theta : U(x_0, \epsilon_0) \to C(B^u_{x_0}(\delta l_0^{-3}), E^c_{x_0})$ defined by $\Theta(y) = g_{x_0}^y$ is continuous.

We will refer to sets of the form

$$S = \bigcup_{y \in \hat{U}} \exp_{x_0}(\text{graph}(\Theta(y))),$$

where $x_0, U(x_0, \epsilon_0)$ and $\Theta$ are as in Lemma 5.5 and $\hat{U} \subset U(x_0, \epsilon_0)$ is a compact subset, as a stack of local unstable manifolds.

**Proof.** (a) We begin by giving sufficient conditions for $\phi_y^x$ to be defined on $B^u_x(2\delta l_0^{-3})$ for a given $\phi_y : \text{Dom}(\phi_y) \to E^c_x$. Identifying $B_x$ and $B_y$, the tangent spaces at $x$ and $y$, with $B + x$ and $B + y$ respectively, we define $\Xi^x_y : \text{Dom}(\phi_y) \to E^u_x$ by

$$\Xi^x_y(v) = \pi^u_x((\text{Id}_{E^c_x} + \phi_y)(v) + y - x),$$

so that formally, at least, $\phi_y^x(w) = \pi^c_x((\text{Id}_{E^c_y} + \phi_y)((\Xi^x_y)^{-1}(w)) + y - x)$. From this, we see that $\phi_y^x$ is well defined on $\Xi^x_y(\text{Dom}(\phi_y))$ if $\pi^c_x$ is invertible when restricted to the set graph($\phi_y$) + $y - x$. This is guaranteed if for all $w_1, w_2 \in \text{graph}(\phi_y)$, one has $|\pi^c_x(w_1 - w_2)| < |w_1 - w_2|$, and that is implied by

$$|\pi^c_x|_{E^c_y} + \text{Lip}(\phi_y) \cdot |\pi^c_x|_{E^c_y} < 1.$$

We bound $\text{Lip}(\phi_y)$ as follows: First we work in $B_y$, letting $\text{Lip}'(\phi_y)$ denote the Lipschitz constant of $\phi_y$ with respect to the norm $| \cdot |_y$. Assume that $d(\phi_y)_0 = 0$ and $\text{Lip}'(d(\phi_y)) < C l_0$; these properties are enjoyed by $g_y$, the graphing map for the local unstable manifold at $y$ (Theorem 5.1). Then assuming $\text{Dom}(\phi_y) \subset \tilde{B}_y^u(4\delta l_0^{-2})$, we have $\text{Lip}'(\phi_y) < 4C\delta l_0^{-1}$. Passing to the $| \cdot |$ norm, we have $\text{Lip}(\phi_y) < 12C\delta$, which we may assume is $\ll 1$.

Thus with $\epsilon_0$ small enough that $d_H(E^u_y, E^u_x)$ and $d_H(E^c_x, E^c_y)$ are sufficiently small (depending only on $n_0, l_0$), the inequality in (17) is satisfied.
Finally, let \( \text{Dom}(\phi_y) = B_y^\delta(4\delta l_0^{-3}) \subset \tilde{B}_y^\delta(4\delta l_0^{-2}) \). Shrinking \( \epsilon_0 \) if necessary so that \( |x-y| \) is sufficiently small, we have \( \Xi_y^\delta(\text{Dom}(\phi_y)) \supset B_x^\delta(2\delta l_0^{-3}) \), proving (a).

For (b), to prove the continuity of \( \Theta \) at \( x \in U = U(x_0, \epsilon_0) \), we let \( y_n \in U \) be a sequence with \( y_n \rightarrow x \) as \( n \rightarrow \infty \). That \( \Theta(y_n) \rightarrow \Theta(x) \) on \( B_x^\delta(2\delta l_0^{-3}) \) will follow once we show \( \|g^x_{y_n} - g^x_x\| \rightarrow 0 \) on \( B_x^\delta(2\delta l_0^{-3}) \). To do this, we will show that given \( \gamma > 0 \), we have \( \|g^x_{y_n} - g^x_x\| < \gamma \) for all \( n \) sufficiently large.

For \( k \in \mathbb{Z}^+ \), write \( x_{-k} = f^{-k}x \) and \( y_{-k} = f^{-k}y \). Since \( x_{-k}, y_{-k} \in \Gamma_{\log k^2} \) (Lemma 4.3), we have, by Lemma 3.9, \( E_{y_{-k}}^u \rightarrow E_{x_{-k}}^u \) as \( n \rightarrow \infty \). (We could not have concluded this from the continuity of \( E^u \) on \( K_{\gamma \delta} \) alone because \( df_x \) is not invertible.) Thus for a fixed \( k \), \( \exp_{y_{-k}}(B_{y_{-k}}^u(1)) \) converges as a family of embedded disks to \( \exp_{x_{-k}}(B_{x_{-k}}^u(1)) \), so their \( f^k \)-images converge as well. Another way to express this is as follows: Let \( \sigma_{y_{-k}} : E_{y_{-k}}^u(\delta l_0^{-2}) \rightarrow E_{x_{-k}}^{cs} \) be the function that is identically equal to 0, and let \( \phi_{y_{-k}} = \Psi_{y_{-k}} \circ \cdots \circ \Psi_{y_{-k}}(0_{y_{-k}}) \) where \( \Psi \) is the graph transform. Likewise, define \( \phi_{x_{-k}} \). Then for each fixed \( k \), \( \|\phi_{y_{-k}} - \phi_{x_{-k}}\| \rightarrow 0 \) (as mappings defined on \( B_{y_{-k}}^u(2\delta l_0^{-3}) \)) as \( n \rightarrow \infty \).

To finish, we estimate \( \|g^x_{y_n} - g^x_x\| \) by

\[
\|g^x_{y_n} - g^x_x\| \leq \|g^x_{y_{-k}} - \phi_{x_{-k}}\| + \|\phi_{y_{-k}} - \phi_{x_{-k}}\| + \|\phi_{x_{-k}} - g^x_x\|.
\]

Using Lemma 5.2 and the uniform equivalence of the \( \cdot \) and \( \cdot' \) norms on \( \Gamma_{\delta l_0} \), we have that the first and third terms above are \( < \gamma/3 \) for \( k \) large enough. Fix one such \( k \), and choose \( n \) large enough that the middle term is \( < \gamma/3 \). This gives the desired estimate.

**Remark 5.6.** Lemma 5.5 guarantees that \( \mu \)-a.e. \( x \) is contained in a stack, but observe that the involvement of the \( \mu \)-continuity set \( K_{\gamma \delta} \) is solely to guarantee control on \( E^{cs} \). That is to say, if \( V \subset \Gamma_{\delta l_0} \) is such that (17) holds for all \( x, y \in V \) (setting \( \phi_y = g^y_{y_{-k}}(B^u_{y_{-k}}(4\delta l_0^{-3})) \)), and has sufficiently small diameter (depending only on \( l_0 \), then Lemma 5.5 holds with \( V \) in the place of \( U(x_0, \epsilon_0) \). From here on we will extend the definition of stacks to include sets of the form \( \mathcal{S} = \bigcup_{y \in V} \exp_{y_{-k}}(\text{graph } \Theta(y)) \) for compact \( V \) with the properties above.

To complete the geometric picture, we will show that \( \mathcal{S} \) is homeomorphic to the product of a finite-dimensional ball with a compact set, and we will do this in the absence of zero Lyapunov exponents (the zero exponent case will require that we strengthen Lemma 5.3).

**Lemma 5.7.** Assume \( E^c = \{0\} \), and let \( \mathcal{S} \) be an unstable stack as defined in (15). Let \( \Sigma = \exp_{x_0}^{-1}(\mathcal{S}) \cap E^s_{x_0} \). Then \( \mathcal{S} \) is homeomorphic to \( \Sigma \times B^{\delta l_0^{-3}}_{\gamma \delta} \) under the mapping \( \Psi(\sigma, u) := \exp_{x_0}(u + g_\sigma(u)) \), where \( g_\sigma = \Theta(x) \) corresponds to the unique leaf of \( \mathcal{S} \) for which \( \Theta(y)(0) = \sigma \).

**Proof.** That \( g_\sigma \) is well-defined for \( \sigma \in \Sigma \) follows from the fact that distinct leaves in the unstable stack do not intersect, and that in turn is a direct consequence of Lemma 5.3.

We now check that \( \Psi \) is a homeomorphism. By compactness, it suffices to check that \( \Psi \) is a continuous bijection. To prove continuity, we define the (continuous) map \( \theta : \bar{U} \rightarrow \Sigma \) by \( \theta(y) = \Theta(y)(0) \) and the equivalence relation \( \sim \) on \( \bar{U} \) by \( x \sim y \) iff \( \Theta(x) = \Theta(y) \), i.e., if \( x \) and \( y \) fall on the same unstable leaf in \( \mathcal{S} \). As \( \Theta, \theta \) are constant on the equivalence classes of \( \sim \), they descend to continuous maps \( \hat{\Theta}, \hat{\theta} \) defined on the quotient space \( \bar{U} / \sim \). The map \( \hat{\theta} \) is a continuous bijection, hence a homeomorphism (by compactness of \( \bar{U} \)), and so the proof is complete on noting that the mapping \( \sigma \mapsto g_\sigma \) can be represented by the composition \( \hat{\Theta} \circ \hat{\theta}^{-1} \).
5.3 Induced volume on submanifolds, and distortion estimates along unstable leaves

In Section 2, we introduced $m_E$, a notion of volume induced on finite dimensional subspaces $E$ of $\mathcal{B}$. It is straightforward to extend this idea to volumes on embedded (or injectively immersed) finite dimensional submanifolds: Let $U \subset \mathbb{R}^d$ be an open set, $\phi : U \to \mathcal{B}$ a $C^1$ Fréchet embedding, and $W = \phi(U)$. For a Borel subset $V \subset W$, we define

$$
\nu_{\phi,W}(V) = \int_{\phi^{-1}V} \det(d\phi_y) \, dy
$$

where we have identified the tangent space at $y \in \mathbb{R}^d$ with $\mathbb{R}^d$ and $\det(d\phi_y)$ here is taken with respect to Euclidean volume on $\mathbb{R}^d$ and $m_{T\phi(y)W}$ on the tangent space $T_{\phi(y)}W$ to $W$ at $\phi(y)$. That $\nu_{\phi,W}$ does not depend on $\phi$ is checked in the usual way: Let $\phi' : U' \to \mathcal{B}$ be another embedding with $\phi'(U') = W$. Then $\phi' = \phi \circ (\phi^{-1} \circ \phi')$, and since $\phi^{-1} \circ \phi' : U' \to U$ is a diffeomorphism, we have, by the multiplicativity of determinants and the usual change of variables formula,

$$
\nu_{\phi',W}(V) = \int_{(\phi')^{-1}V} \det(d(\phi^{-1} \circ \phi')(y')) \, dy' = \int_{\phi^{-1}V} \det(d\phi_y) \, dy .
$$

We will denote the induced volume on $W$ by $\nu_W$ from here on, having shown that it is independent of embedding. The discussion above is easily extended to injectively immersed finite dimensional submanifolds, such as unstable manifolds.

For $x \in \Gamma$, let us abbreviate $\nu_{W_x}$ as $\nu_x$, and for $y \in W^u_x$, we will use $E^u_y$ to denote the tangent space to $W^u_x$ at $y$. Then letting $f_{y} \nu_{f^{-1}x}$ denote the pushforward of $\nu_{f^{-1}x}$ from $W^u_{f^{-1}x}$ to $W^u_x$, we have, from the discussion above,

$$
\frac{d(f_{y} \nu_{f^{-1}x})(y)}{d\nu_x}(y) = \frac{1}{\det(df_{f^{-1}x}|E^u_{f^{-1}y})} \quad \text{for} \quad y \in W^u_x . \quad (18)
$$

The distortion estimate below is crucially important for the arguments in Section 6. Note that we allow $E^c \neq \{0\}$.

Let $\delta^1_l$ be the largest $\delta$ for which Theorem 5.1 holds. Let us write $W^u_{\text{loc},x} = W^u_{\delta^1_l,x}$.

**Proposition 5.8.** For every $l \geq 1$, there is a constant $D_l$ such that the following holds for any $x \in \Gamma_l$.

(a) For all $y^1, y^2 \in W^u_{\text{loc},x}$ and all $n \geq 1$:

$$
\left| \log \frac{\det(df^n_{f^{-n}y^1}|E^u_{f^{-n}y^1})}{\det(df^n_{f^{-n}y^2}|E^u_{f^{-n}y^2})} \right| \leq D_l \left| y^1 - y^2 \right| . \quad (19)
$$

---

2This is in fact true even though $y$ is not necessarily in $\Gamma$: since $|f^{-n}x - f^{-n}y| \to 0$ exponentially as $n \to \infty$, and the tangent spaces of $f^{-n}x$ and $f^{-n}y$ to $W^u_{f^{-n}x}$ converge exponentially as well, it follows that backward time Lyapunov exponents for $y \in W^u_x$ are well defined and are identical to those at $x$, with $E^u_y$ being the tangent space to $W^u_x$ at $y$.  

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(b) For any fixed \( x' \in W^u_{\text{loc},x} \), the sequence of functions \( y \mapsto \log \Delta_N(x',y) \), where

\[
\Delta_N(x', y) := \prod_{k=1}^N \frac{\det(df_{f^{-k}x'} | E^u_{f^{-k}x'})}{\det(df_{f^{-k}y} | E^u_{f^{-k}y})}, \quad N = 1, 2, \ldots,
\]

defined for \( y \in W^u_{\text{loc},x} \) converges uniformly (at a rate depending only on \( l(x) \)) as \( N \to \infty \) to a Lipschitz continuous function with constant \( \leq D_l \) in the \( | \cdot | \) norm.

Though distortion estimates of the kind in Proposition 5.8 are standard for finite dimensional systems, they are new for mappings of infinite dimensional Banach spaces: These estimates have to do with the regularity of determinants for \( df^n \) along unstable manifolds, where determinants are defined with respect to the induced volumes introduced in this paper. Below we include a complete proof, proceeding in two steps. In the first step, formulated as Proposition 5.9 we prove a distortion estimate in charts, i.e., using Lyapunov metrics, taking advantage of the uniform expansion along unstable leaves in adapted norms. In the second step, we bring this estimate back to the usual norm \( | \cdot | \) on \( \mathcal{B} \).

Fix \( x \in \Gamma \); we introduce the following abbreviated notation. In the first step we will be working exclusively with the maps

\[
\tilde{f}_{f^{-k}x} : \tilde{B}_{f^{-k}x}(\delta'_1 l(f^{-k}x)^{-1}) \to \mathcal{B}_{f^{-(k-1)}x}, \quad k \geq 1,
\]

where the notation is as in Theorem 5.1 and the norm of interest on each \( \mathcal{B}_{f^{-k}x} \) is exclusively \( | \cdot |'_{f^{-k}x} \). As the meanings will be clear from context, we will drop the subscripts in \( \tilde{f}_{f^{-k}x} \) and \( | \cdot |'_{f^{-k}x} \), writing only \( \tilde{f} \) and \( | \cdot |' \). For a finite dimensional subspace \( E \subset \mathcal{B}_{f^{-k}x} \), \( m'_E \) will denote the volume on \( E \) induced from \( | \cdot |' \), and \( \det' \) is to be understood to be the determinant with respect to these volumes. We also write \( g = g_x \) and \( g_{-k} = g_{f^{-k}x} \), the graphing maps of \( W^u_{\text{loc},f^{-k}x} \) given by Theorem 5.1.

**Proposition 5.9.** For any \( l \geq 1 \), there is a constant \( D'_l \) with the following property. Let \( x \in \Gamma \). Then for any \( z^1, z^2 \in \text{graph } g \) with \( |z^1 - z^2|^l \leq \delta'_1 (D'_{l(x)})^{-1} \), \( i = 1, 2 \), and any \( n \geq 1 \), we have that

\[
\left| \log \frac{\det'(d\tilde{f}^n_{z^1_{n-1}} | E^1_{z^1_{n-1}})}{\det'(d\tilde{f}^n_{z^2_{n-1}} | E^2_{z^2_{n-1}})} \right| \leq D'_{l(x)} |z^1 - z^2|^l , \quad (21)
\]

where \( z^i_{n} \) is the unique point in \( \text{graph } g_{-n} \) with \( \tilde{f}^n z^i_{-n} = z^i \), and \( E^i_{z^i_{-n}} \) is the tangent space to \( \text{graph } g_{-n} \) at \( z^i_{-n} \).

**Proof of Proposition 5.9.** Consider to begin with arbitrary \( z^1, z^2 \in \text{graph } g \). Using the multiplicity of the determinant, we decompose the argument of \( \log \) in the LHS of (21) as

\[
\frac{\det'(d\tilde{f}^n_{z^1_{n-1}} | E^1_{z^1_{n-1}})}{\det'(d\tilde{f}^n_{z^2_{n-1}} | E^2_{z^2_{n-1}})} = \prod_{k=1}^n \frac{\det'(d\tilde{f}^n_{z^1_{n-k}} | E^1_{z^1_{n-k}})}{\det'(d\tilde{f}^n_{z^2_{n-k}} | E^2_{z^2_{n-k}})}, \quad (22)
\]

and bound the factors on the right side of (22) one at a time.

We will use the following slight refinement of Proposition 2.15 (see Remark 2.16).
Lemma 5.10. For $m \in \mathbb{N}$, there is a constant $C_m > 1$ with the following property. Let $X, Y$ be Banach spaces, and fix $M \geq 1$. If $A_1, A_2 : X \to Y$ are bounded linear operators and $E_1, E_2 \subset X$ are subspaces with the same finite dimension $m$ such that
\[ |A_i|, |(A_i|_{E_i})^{-1}| \leq M \quad i = 1, 2, \]
\[ |A_1 - A_2|, d_H(E_1, E_2) \leq \frac{1}{C_m M^{10m}}, \]
then
\[ \left| \log \frac{\det(A_1|_{E_1})}{\det(A_2|_{E_2})} \right| \leq C_m M^{10m} \left( |A_1 - A_2| + d_H(E_1, E_2) \right). \]

For each fixed $1 \leq k \leq n$, we apply Lemma 5.10 to $m = m_u$, $A_i = d\tilde{f}_{z_k}^i$ and $E_i = E_{z_k}^i$, for an appropriate choice of $M = M_k$. To fulfill the hypotheses of the lemma, we need the following:
\[ |d\tilde{f}_{z_k}^i|', \quad |(d\tilde{f}_{z_k}^i|_{E_{z_k}^i})^{-1}|' \leq M_k \quad \text{for } i = 1, 2, \quad (23) \]
\[ |d\tilde{f}_{z_k}^i - d\tilde{f}_{z_k}^2|', \quad d_H'(E_{z_k}^1, E_{z_k}^2) \leq C_{m_u}^{-1} M_k^{-10m_u}, \quad (24) \]
where $d_H'$ refers to the Hausdorff distance in the adapted norm $| \cdot |'$ on $B_{f-k,x}$.

First we choose $M_k$ so that (23) holds: $|d\tilde{f}_{z_k}^i|_{E_{z_k}^i})^{-1}|' \leq 1$ poses no problem, but from the way our adapted norms are defined in Sect. 4.1, we only have
\[ |d\tilde{f}_{z_k}^i|' \leq 3l(f^{-k+1}(x) \cdot |df|_{E_{z_k}^i}) \leq 3K e^{k\delta_2} l(x), \]
where $K$ is an upper bound for $|df|$ on $\{ y \in B : d(y, x) \leq r_0 \}$ (see the paragraphs preceding Lemma 4.2). So, on setting $M_k = 3K e^{k\delta_2} l(x)$, (23) is satisfied.

Next we estimate the two terms on the left side of (24):
\[ |d\tilde{f}_{z_k}^i - d\tilde{f}_{z_k}^2|' \leq l(f^{-k}(x)) (z_{-k}^1 - z_{-k}^2)' \]
\[ \leq l(f^{-k}(x)) (e^{\lambda} - \delta'_1)^{-k} |z_1 - z_2|' \]
\[ \leq l(x) \left( \frac{e^{\lambda}}{e^{\lambda} - \delta'_1} \right)^k |z_1 - z_2|' =: (*) \]
by Lemma 4.3

For $d_H'(E_{-k}^1, E_{-k}^2)$, observe that if $z_{-k}^i = g_{-k}(u_{-k}^i)$, then $E_{-k}^i = (Id + (dg_{-k})_{u_{-k}^i}) E_{u_k} f^{-k}(x)$. A simple computation (see Sect. 2.1.2) gives
\[ d_H'(E_{-k}^1, E_{-k}^2) \leq 2 |(dg_{-k})_{u_{-k}^1} - (dg_{-k})_{u_{-k}^2}|', \]
hence
\[ d_H'(E_{-k}^1, E_{-k}^2) \leq 2C l(f^{-k}(x)) |u_{-k}^1 - u_{-k}^2|' \]
by Item 2 of Theorem 5.1
\[ = 2C l(f^{-k}(x)) |z_{-k}^1 - z_{-k}^2|' \leq 2C \cdot (*) . \]
Notice that while (23) imposes a lower bound on $M_k$, (24) imposes an upper bound, namely $M_k^{10m_u} \leq (C_m \max \{2C, 1\} \cdot (\ast))^{-1}$. Both conditions can be satisfied if $|z^1 - z^2'|$ is sufficiently small, such as $|z^1 - z^2'| < \delta'_1 D'_l(x)$ where

$$D'_l \geq \delta'_1 C_m (3K)^{10m_u} \cdot \max \{2C, 1\} \cdot l^{10m_u + 1},$$

assuming, as we may, that $e^\lambda - \delta'_1 > e^{(10m_u + 1)\delta_2}$.

At last, we apply Lemma 5.10 to $z^1, z^2$ with $|z^1 - z^2'| < \delta'_1 D'_l(x)$, obtaining

$$\left| \log \frac{\det'(d\tilde{f}^n_z | E^1_z)}{\det'(d\tilde{f}^n_z | E^2_z)} \right| \leq C_m (3K e^{k\delta_2} l(x))^{10m_u} \cdot (2C + 1) l(x) \frac{(e^{\delta_2} \cdot \delta'_1)^k}{e^\lambda} \cdot |z^1 - z^2'| \leq K' \frac{(e^{(10m_u + 1)\delta_2})^k}{e^\lambda} l(x)^{10m_u + 1} |z^1 - z^2'|.$$

Reconstituting the expression (22), we obtain the estimate

$$\left| \log \frac{\det'(d\tilde{f}^n_{y^i} | E^1_{y^i})}{\det'(d\tilde{f}^n_{y^i} | E^2_{y^i})} \right| \leq K' l(x)^{10m_u + 1} |z^1 - z^2'| \leq K'' l(x)^{10m_u + 1} |z^1 - z^2'|,$$

where $K''$ is independent of $x$ and $n$. By increasing $D'_l$ once more so that $D'_l \geq K'' l^{10m_u + 1}$, the conclusion of Proposition 5.9 follows. \hfill \square

We now complete the proof of Proposition 5.8.

**Proof of Proposition 5.8** Fix $x \in \Gamma_l$. For $y^1, y^2 \in W^u_{loc, x}$, we let $z^i = \exp^{-1}_x y^i$, and write $y^i_{-k} = f^{-k} y^i$. For objects and quantities in charts, we will use the same notation as in Proposition 5.9, so for example, $y^i_{-k} = \exp_{f^{-k}_x} (u^i_{-k} + g_{-k}(u^i_{-k}))$ etc.

For part (a), we first consider $y^1, y^2 \in W^u_{loc, x}$ with $|z^1 - z^2| \leq \delta_1 D_l^{-1}$, proving that the left side of (19) is $\leq D_l |z^1 - z^2'|$ for some $D_l$ that will be enlarged a finite number of times in the course of the proof. Fixing $n \geq 1$, we compute that

$$\frac{\det'(d\tilde{f}^n_{y^1} | E^1_{y^1})}{\det'(d\tilde{f}^n_{y^2} | E^2_{y^2})} = \frac{dm_{E^1_{y^1}}/dm_{E^1_{y^2}}}{dm_{E^2_{y^1}}/dm_{E^2_{y^2}}} \times \frac{dm_{E^2_{y^1}}/dm_{E^2_{y^2}}}{dm_{E^1_{y^1}}/dm_{E^1_{y^2}}} \times \frac{\det'(d\tilde{f}^n_{z^1_{-n}} | E^1_{z^1_{-n}})}{\det'(d\tilde{f}^n_{z^2_{-n}} | E^2_{z^2_{-n}})}.$$

By Proposition 5.9 we have

$$|\log III| \leq D_l |z^1 - z^2'|.$$

It remains to estimate the terms $I$ and $II$.

For $I$, observe that if $L : E^u_x \to E^c_x$ is a linear map with $|L'| \leq 1$, then as a consequence of (8) in the definition of $|\cdot'|$ norms,

$$\frac{dm_{(Id + L)E^u_x}}{dm_{(Id + L)E^u_x}} = \frac{m_{(Id + L)E^u_x}((Id + L)O)}{m_{E^u_x}(O)}$$

(26)
for any Borel subset \( O \subset E^u_x \) of positive Haar measure. Since \( \text{Lip} g \leq 1 \) and \( E^u_y = (\text{Id} + (dg)_{u_0})E^u_x \), it follows from (26) that

\[
I = \frac{\det(\text{Id} + (dg)_{u_0}|E^u_x)}{\det(\text{Id} + (dg)_{u_0^*}|E^u_x)}.
\]

Note that all determinants involved are in the natural norm \( | \cdot | \), considered as a norm on \( \mathcal{B}_x \cong \mathcal{B} \).

We will estimate this expression for \( I \) using Lemma 5.10, applying that result with \( A_i = \text{Id} + (dg)_{u_0} \) and \( E_1 = E_2 = E^u_x \). First,

\[
|\text{Id} + (dg)_{u_0}| \leq 3l|\text{Id} + dg| = 3l,
\]

\[
|(\text{Id} + (dg)_{u_0^*}|_{E^u})^{-1}| \leq 3l|(\text{Id} + (dg)_{u_0^*}|_{E^u})^{-1}| = 3l.
\]

Here we have used again the fact that \( \text{Id} + (dg)_{u_0^*} : E^u_x \to E^u_y \) is an isometry in \( | \cdot |' \). So, for the purpose of bounding \( I \), we may take \( M \) in Lemma 5.10 to be \( M = 3l \). The only estimate needed in the analog of (24) is

\[
|(dg_0)_{u_0} - (dg)_{u_0^*}| \leq 3l|(dg)_{u_0} - (dg)_{u_0^*}|' \leq 3l \cdot Cl |u_0^* - u_0|' = 3Cl^2 |z_1 - z_2|',
\]

so it suffices to enlarge \( D_l \) to \( D_l \geq 3^{10m_u + 1} \delta_l Cm_u l^{10m_u + 2} \). Lemma 5.10 then applies to give

\[
|\log I| \leq C_{m_u}(3l)^{10m_u} \cdot 3Cl^2 |z_1 - z_2|' \leq K''m l^{10m_u + 2} |z_1 - z_2|' .
\] (27)

The estimate for \( |\log II| \) proceeds similarly, replacing \( g \) with \( g_{-n} \) and \( u_0^* \) with \( u_{-i}^* \). We leave it to the reader to check that it has the same bound as \( |\log I| \). This completes the proof of part (a) for \( y^1, y^2 \in W^u_{\text{loc},x} \) with \( |z_1 - z_2|' < \delta_1^{l}D_l^{-1} \).

For \( y^1, y^2 \in W^u_{\text{loc},x} \) for which \( |z_1 - z_2|' = |u^1 - u^2|' > \delta_1^{l}D_l^{-1} \), we insert points \( \hat{u}^1, \ldots, \hat{u}^k \) on the line segment joining \( u^1 \) and \( u^2 \) so that if \( \hat{u}^0 = u^1 \) and \( \hat{u}^{k+1} = u^2 \), then \( |\hat{u}^i - \hat{u}^{i-1}|' \leq \delta_1^{l}D_l^{-1} \). Let \( \hat{z}^i = \hat{u}^i + g(\hat{u}^i) \) and \( \hat{y}^i = \exp_x \hat{z}^i, i = 0, 1, \ldots, k + 1 \). Then the argument above gives

\[
|\log \left( \frac{\det(df_{f-n,y^1}|E^u_{f-n,y})}{\det(df_{f-n,y^2}|E^u_{f-n,y})} \right) | = |\log \left( \prod_{i=0}^{k} \frac{\det(df_{f-n,y^i}|E^u_{f-n,y^i})}{\det(df_{f-n,y^{i+1}}|E^u_{f-n,y^{i+1}})} \right) | \leq D_l(|z_1 - z^0|' + \ldots + |z_{k+1} - z^k|') = D_l |z_1 - z_2|' \leq lD_l |y^1 - y^2|' .
\]

This completes the proof of part (a).

For part (b), observe that as a consequence of (a), it will suffice to show that the sequence \( \log \Delta_N \) in (20) is uniformly Cauchy over \( y \in W^u_{\text{loc},x} \), the value of \( D_l \) having been fixed so that (a) holds. This in turn will follow from (uniform in \( y \)) bounds on

\[
|\log \prod_{k=M+1}^{N} \frac{\det(df_{f-N,y^1}|E^u_{f-N,y^1})}{\det(df_{f-N,y^2}|E^u_{f-N,y^2})} | = \frac{\det(df_{f-N-M}|E^u_{f-N-x})}{\det(df_{f-N-M}|E^u_{f-N-y})} \to \det(df_{f-N-M}|E^u_{f-N-y}) \to \det(df_{f-N-M}|E^u_{f-N-y}) \tag{28}
\]

for \( M, N \) large, \( M < N \). We leave it to the reader to check that the functions in (28) are bounded by quantities exponentially small in \( M \).
Tracing through the proof of Proposition 5.8, one sees that $D_l$ can be taken as $C l^q$, where $q \in \mathbb{N}$ and $C$ depend only on $m_u$ and are independent of $l$.

Observe that the proof of Proposition 5.8 used the Lipschitz regularity of the determinant in a crucial way. In some sense, the preceding proof used all possible regularity of the determinant available in our setting.

5.4 The nonergodic case

In Sections 3–5, up until this point, we have operated under the assumption that $(f, \mu)$ is ergodic. We now discuss the extension of these results to the nonergodic case.

The nonergodic case of the Multiplicative Ergodic Theorem reads as follows: Fix a measurable function $\lambda_\alpha > l_\alpha$. Then there is a measurable integer-valued function $r$ on $\Gamma$ such that at every $x \in \Gamma$, there are $r(x)$ Lyapunov exponents $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{r(x)}(x)$ and an associated splitting $B = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_{r(x)}(x) \oplus F(x)$ with respect to which properties (a)–(d) in Theorem 3.3 hold. Here $\dim E_i(x) = m_i(x)$ where $m_i$ are measurable functions on $\{ x \in \Gamma : r(x) \geq i \}$.

Next we define, as in Sect. 4.1, $E^\tau_x$ for $\tau = u, c, s$, and let $\lambda^+$ and let $\lambda^-$ be as before, except that they are now measurable functions that need not be bounded away from 0. For $m, n \in \{0, 1, 2, \ldots\}$ and $p, q \in \{1, 2, \ldots\}$, let

$$\Gamma(m, n; p, q) = \left\{ x \in \Gamma : \dim E^u_x = m, \quad \dim E^c_x = n; \quad \lambda^+(x) \geq \frac{1}{p}, \quad \lambda^-(x) \leq -\frac{1}{q} \right\}.$$

Then each $\Gamma(m, n; p, q)$ is either empty, or it is $f$-invariant, and $\Gamma = \bigcup_{m, n, p, q} \Gamma(m, n; p, q)$. For results that concern individual $E_i$, it will be advantageous to further subdivide $\Gamma(m, n; p, q)$ according to the dimensions of these subspaces etc. We will focus here on the extension of the results in Sections 4 and 5 to the nonergodic case, for which the decomposition into $\Gamma(m, n; p, q)$ suffices.

We claim – and leave it for the reader to check – that for these results, the proofs in Sections 4 and 5 go through verbatim provided that one restricts to one $\Gamma(m, n; p, q)$ at a time, and allow the quantities $\lambda_0, \delta_0$, hence $\lambda$, and $\delta_1, \delta_2$, hence the function $l$ and constant $\delta'_1$, to depend on $(m, n; p, q)$. Notice that when we refer to Corollary 3.8 and Lemma 3.9, the subspaces in question are $E^u$, the dimension of which is constant on $\Gamma(m, n; p, q)$ and the proofs there go through unchanged as well. Once this is checked, it will follow, for example, that for a fixed $(m, n; p, q)$, local unstable manifolds are defined for $\mu$-a.e. $x \in \Gamma(m, n; p, q)$, stacks of unstable manifolds are well defined, and $\mu$-a.e. $x \in \Gamma(m, n; p, q)$ is contained in such a stack.

Obviously, the sets $\Gamma(m, n; p, q)$ are not pairwise disjoint. If one wishes to work with pairwise disjoint $f$-invariant sets, the countable family

$$\hat{\Gamma}(m, n; p, q) = \Gamma(m, n; p, q) \setminus \Gamma(m, n; p - 1, q - 1)$$

is an alternative to ergodic decompositions.
Our proofs of Theorems 1 and 2 follow in outline [18] and [17], which contain analogous results for diffeomorphisms of finite dimensional manifolds. These proofs are conceptually as direct as can be: they relate $h_\mu(f)$, which measures the rate of information growth with respect to $\mu$, to the rate of volume growth on unstable manifolds – under the assumption that conditional measures of $\mu$ on $W^u$-manifolds are in the same measure class as the induced volumes on these manifolds. Other proofs of the entropy formula in finite dimensions, such as [32, 29], start from invariant measures with densities on the entire phase space and are less suitable for adaptation to infinite dimension.

In the last few sections we have laid the groundwork needed to extend the ideas in [18] and [17] to Banach space settings. To make feasible the idea of volume growth, we introduced a $d$-dimensional volume on unstable manifolds. To set the stage for conditional densities of invariant measures, we proved distortion estimates of iterated determinants. Some technical work remains; it is carried out in Sect. 6.2. In Sects. 6.3 and 6.4, we verify carefully that all technical issues have been addressed.

Hypotheses (H1)–(H4) are assumed throughout this section; they will not be repeated in statements of results. Notice the addition of (H4), the no zero exponents assumption, that was not present in most of the last two sections. (H5) will be introduced as needed.

6.1 Equivalent definitions of SRB measures

We begin with a formal definition of SRB measures for Banach space mappings. This definition is relatively easy to state, and is equivalent to standard definitions used in finite dimensional hyperbolic theory.

Let $\mathcal{S}$ be a compact stack of local unstable manifolds as defined in Sect. 5.2, and let $\xi^\mathcal{S}$ be the partition of $\mathcal{S}$ into unstable leaves. By Lemma 5.7 $\xi^\mathcal{S}$ is a measurable partition. Assuming $\mu(\mathcal{S}) > 0$, we let $\{\mu_{\xi^\mathcal{S}(x)}\}_{x \in \mathcal{S}}$ denote the canonical disintegration of $\mu|\mathcal{S}$ on elements of $\xi^\mathcal{S}$ (for details on canonical disintegrations, see [33, 34] and [7]). Recall that $\nu_x$ is the induced volume on $W^u_x$.

**Definition 6.1.** We say that $\mu$ is an **SRB measure** of $f$ if

(i) $f$ has a strictly positive Lyapunov exponent $\mu$-a.e., and

(ii) for any stack $\mathcal{S}$ with $\mu(\mathcal{S}) > 0$, $\mu_{\xi^\mathcal{S}(x)}$ is absolutely continuous with respect to $\nu_x$, written $\mu_{\xi^\mathcal{S}(x)} \ll \nu_x$, for $\mu$-a.e. $x \in \mathcal{S}$.

This definition was used in [43] (see also [2]), and differs **a priori** from that used in [18], [17], [19], which we now recall.

**Definition 6.2.** We say that a measurable partition $\eta$ is **subordinate to the unstable foliation** (abbreviated below as “subordinate to $W^u$”) if for $\mu$-a.e. $x$, we have

(i) $\eta(x) \subset W^u_x$,

(ii) $\eta(x)$ contains a neighborhood of $x$ in $W^u_x$, and

(iii) $\eta(x) \subset f^N(W^u_{\text{loc},f^{-N}x})$ for some $N \in \mathbb{N}$ (depending on $x$).
A proof of the existence of measurable partitions subordinate to the $W^u$ foliation was given in [18]; we will provide a sketch in Sect. 6.3. In [18, 17] and [19], SRB measures are defined in terms of partitions subordinate to $W^u$.

**Lemma 6.3.** The following are equivalent.

1. $\mu$ is an SRB measure in the sense of Definition 6.1.
2. There exists a partition $\eta$ subordinate to $W^u$ with the property that $\mu_{\eta(x)} \ll \nu_x$ for $\mu$-a.e. $x$.
3. Every partition $\eta$ subordinate to $W^u$ has the property $\mu_{\eta(x)} \ll \nu_x$ for $\mu$-a.e. $x$.

Lemma 6.3 follows by the uniqueness of the canonical disintegration and the fact that $\mu$-a.e. $x$ is contained in an unstable stack; its proof is omitted.

### 6.2 Technical issues arising from noninvertibility

Before proceeding to the proofs of our main results, we wish to dispose of some technical issues that do not present themselves in the setting of finite dimensional diffeomorphisms. These issues stem from the fact that in the course of our proofs, we will need to deal with dynamics outside of $A$. For example, to prove that $\mu$ is an SRB measure given that the entropy formula holds, we will want to compare $\mu$ to a measure with conditional densities on $W^u$-leaves, and the construction of this measure will have to proceed without a priori knowledge that $W^u$-leaves are contained in $A$.

The material in this section holds under assumptions (H1)–(H4), with no additional assumptions on $\mu$. Since the sets under consideration may not be contained in $A$, $f^{-1}$ is not necessarily defined, and certainly cannot be assumed to be continuous. As a consequence, properties that involve backward iterations, such as continuity of $y \mapsto E^u_y$, must be treated with care, and discussions of $\mu$-typical behavior do not apply.

For the rest of this subsection, we restrict ourselves to a component $\Gamma(m, n; p, q)$ for some $m, n, p, q \in \mathbb{N}$ (see Section 5.4). We let $l_0 \geq 1$ and fix an unstable stack

$$S = \bigcup_{x \in U} \exp_{x_0} \left( \text{graph } \Theta(x) \right),$$

where for each $x \in U(x_0, \varepsilon_0)$, we have $\Theta(x) : B^u_{x_0}(\delta l_0^{-3}) \to E^s_{x_0}$; all notation is as in Lemma 5.5.

**Lemma 6.4.** For all $n \in \mathbb{N}$, $f^{-n}$ is well-defined and continuous on $S$.

**Proof.** That $f^{-n}$ is well-defined on $S$ follows from Theorem 5.1 and the injectivity of $f$ on $B$ (see (H1) in Section 1); the bulk of our work is in showing that $f^{-n}|_S$ is continuous.

**Claim 6.5.** There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there is, for each $x \in \bar{U}$, a small neighborhood $V_{n,x}$ of $x$ in $\bar{U}$ such that (a) the set $f^{-n}\bar{V}_{n,x}$ obeys the criteria for possessing a compact stack $S_{n,x}$ of unstable leaves as in Remark 5.6 (here $\bar{V}_{n,x}$ is the closure of $V_{n,x}$), and (b) we have that

$$f^n(S_{n,x}) \supset \bigcup_{y \in \bar{V}_{n,x}} \exp_{x_0} \left( \text{graph } \Theta(y) \right).$$  \hfill (29)
Assuming Claim 6.5 we let \( n \geq n_0 \) and let \( \{V_{n,x}, i = 1, 2, \ldots, q\} \) be a finite subcover of \( \{V_{n,x}, x \in \tilde{U}\} \). Then \( \bigcup_{i=1}^{q} S_{n,x,i} \) is compact, and since \( f^n|_{\bigcup_{i=1}^{q} S_{n,x,i}} \) is continuous, \( f^{-n} \) is continuous on \( f^n(\bigcup_{i=1}^{q} S_{n,x,i}) \), hence \( f^{-n} \) is continuous on \( S \). For \( n < n_0 \), write \( f^{-n} = f^{n_0-n} \circ f^{-n_0} \).

It remains to prove Claim 6.5. Let us assume for the moment that we can find a neighborhood \( V_{n,x} \) satisfying (a). By Remark 5.6 the stack
\[
S_{n,x} = \bigcup_{z \in f^{-n}V_{n,x}} \exp_{f^{-n}x}(\text{graph } g^{-n}_{z}|_{D_{n,x}}),
\]
where \( D_{n,x} = B_{f^{-n}x}^{n}(\delta(e^{n\delta x}l_0)^{-3}) \), is well-defined and is comprised of continuously-varying unstable leaves.

To check (29), we relate the leaf through each point \( z \in f^{-n}V_{n,x} \) back to the leaf in the chart at \( z \), using the considerations at the beginning of the proof of Lemma 5.5 (see in particular (16)). To wit, one checks that
\[
\exp_{f^{-n}x} \text{graph } g^{-n}_{z}|_{D_{n,x}} = \exp_{x} \text{graph } g_{z}(\pi_{3}^{\sharp}(\text{graph } g^{-n}_{z}|_{D_{n,x}} + f^{-n}x - z)) \supset \exp_{x} \text{graph } g_{z} B^{-n}_{z}(\delta(e^{n\delta x}l_0)^{-3});
\]
this may require shrinking \( V_{n,x} \) so that the diameter of \( f^{-n}V_{n,x} \) is sufficiently small. Likewise, we relate the leaf in \( S \) through \( f^n z \) with the corresponding unstable manifold leaf in the chart at \( f^n z \): inspecting the proof of Lemma 5.5 we see that
\[
\exp_{x}(\text{graph } \Theta(f^{n} z)) \subset \exp_{f^{n}x} \text{graph } g_{f^{n}z} B^{-n}_{f^{n}z}(4\delta l_0^{-2}).
\]
So, to check (29), it suffices to show that for each \( z \in f^{-n}V_{n,x}, \)
\[
\tilde{f}^{n}_{z} \text{graph } g_{z} B^{-n}_{z}(\delta(e^{n\delta x}l_0)^{-3}) \supset \text{graph } g_{f^{n}z} B^{-n}_{f^{n}z}(4\delta l_0^{-2}).
\]
This follows from a graph transform argument (see Section 5.1) for \( n \geq n_0 \), where \( n_0 \) depends on \( l_0 \).

We now set about finding a neighborhood \( V_{n,x} \) satisfying (a) in Claim 6.5. We will show that there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) and any \( x \in \tilde{U} \), there is \( V_{n,x} \) such that \( |\pi_{x}^{s}|e^{\delta x}l_0 < \frac{1}{2} \), and \( |\pi_{x}^{s}|E_{x}^{u} < 2 \) for all \( z, z' \in f^{-n}V_{n,x} \); see (17). Having done so, and perhaps on shrinking \( V_{n,x} \) further, it will follow from Remark 5.6 that the stack \( S_{n,x} \) as above satisfies the conclusions of Lemma 5.5.

To control \( \pi_{z}^{s}|E_{x}^{u} \), observe that \( \pi_{z}^{s}|E_{x}^{u} \leq \pi_{z}^{s}|d_{H}(E_{x}^{u}, E_{x}^{u}) \leq 3e^{n\delta x}l_0 d_{H}(E_{x}^{u}, E_{x}^{u}). \) Since \( f^{-n} \tilde{U} \subset \Gamma_{e^{n\delta x}l_0} \) (Lemma 4.3), it follows from Lemma 3.9 that \( z \mapsto E_{x}^{u} \) is continuous on \( f^{-n} \tilde{U} \). Thus we obtain the desired bound by choosing \( V_{n,x} \) sufficiently small. To control \( \pi_{z}^{s}|E_{x}^{u} \), we let \( v \in E_{x}^{u} \) be a unit vector, and write \( v = v^{u} + v^{s} \in E_{x}^{u} \oplus E_{x}^{s} \). Since \( |v^{s}| \leq |v^{u}| + 1 \), it suffices to bound \( |v^{u}| \). Now
\[
|df_{z}^{s}(v) - df_{z}^{u}(v)| \geq |df_{z}^{u}(v^{u})| - |df_{z}^{u}(v^{s}) - df_{z}^{u}(v)|. \tag{30}
\]
Choose \( V_{n,x} \) small enough that \( \sup_{z \in f^{-n}V_{n,x}}|df_{z}^{u} - df_{z}^{u}| < 1 \), and \( n_0 \) large enough (depending only on \( l_0 \)) so that for all \( n \geq n_0 \), \( m(df_{f^{-n}y}E_{x}^{u}) \gtrsim e^{\lambda n} \) and \( |df_{f^{-n}y}E_{x}^{u}| \lesssim e^{-\lambda n} \) for all \( y \in \Gamma_{l_0} \). Then it follows from (30) and \( |v^{s}| \leq |v^{u}| + 1 \) that \( |v^{u}| \lesssim e^{-\lambda n} \). \( \square \)
We now apply Lemma 6.4 to obtain various facts about $S$.

**Lemma 6.6.** The mappings $y \mapsto E^u_y$ are continuous on $f^{-q}S$ for $q = 0, 1, 2, \ldots$.

Continuity of $E^u$ along individual unstable leaves follows from Theorem 5.1, but Lemma 6.6 asserts more than that: it asserts continuity across all the different leaves that comprise $f^{-q}S$. In the case that $S \subset \Gamma$, this result follows from Lemma 3.9 but we do not assume that. Still, we will follow the proof of Lemma 3.9 closely, supplying additional justification where needed.

**Proof.** We give the proof for $q = 0$; it will be clear that this argument will also prove the assertion for all $q \geq 1$.

Let $x, x^n \in S$ be such that $x^n \to x$. Then $x_{-k}$ and $x^n_{-k}$ are defined for all $k \in \mathbb{N}$, and for each $k$, $x^n_{-k} \to x_{-k}$ as $n \to \infty$ by Lemma 6.4. We let $E(z)$ in Lemma 3.9 be $E^u_z$. Since every $z \in S$ lies in $W^u_{\delta y}$ for some $y \in U \subset \Gamma_{t_0}$, $(df^k_{x_{-k}}|_{E^u_{x_{-k}}})^{-1}$, $k = 1, 2, \ldots$, have the uniform estimates required for $G^i_L$ in Lemma 3.9 and Lemma 5.3(b) shows that these estimates uniquely characterize $E^u_z$.

To carry out the argument in Lemma 3.9 we need to show that along the backward orbit of $x$, there are closed subspaces $\tilde{F}(x)$ and $\tilde{F}(x_{-k})$ such that (i) $\mathcal{B}_x = E^u_x \oplus \tilde{F}(x)$ and $\mathcal{B}_{x_{-k}} = E^u_{x_{-k}} \oplus \tilde{F}(x_{-k})$, (ii) $df_{x_{-k}} \tilde{F}(x_{-k}) \subset \tilde{F}(x_{-(k-1)})$, and (iii) there exist arbitrarily large $k$ for which $|df^k_{x_{-k}}|_{\tilde{F}(x_{-k})}| \leq e^{-\frac{\lambda}{2}k}$. Notice that this is needed for $x$ only, not for $x^n$.

Here is where the situations differ: For $z \in S \setminus \Gamma$, there is no intrinsically defined $E^u_z$, hence we will have to construct a surrogate sequence of subspaces $\tilde{F}(x)$ and $\tilde{F}(x_{-k})$. Identifying the tangent space $\mathcal{B}_x$ with $\mathcal{B}_y$ where $y$ is a point in $U$ with the property that $x \in W^u_{\delta y}$, we let $\tilde{F}(x) = E^s_y$, and claim that $\tilde{F}(x_{-k})$ for $k = 1, 2, \ldots$ are determined by property (ii) in the last paragraph. To justify this claim, it is convenient to work in the charts associated with the backwards orbit of $y$. Let $\tilde{\cdot}$ denote corresponding objects in charts, so that $\tilde{x}_{-k} \in \tilde{B}_f^{-k}(\delta (e^{\delta k}l_0)^{-1})$, and $\tilde{F}(\tilde{x}_{-k})$ is the subspace we seek etc. For $\delta$ small enough, we may assume, by Lemma 4.2, that $d(\tilde{f}_{-1}^k, \tilde{f}_{-1}) \tilde{x}_{-1}$ is sufficiently close to $d(\tilde{f}_{-1}^k)_0$ that the backward graph transform argument (Lemma 5.4) can be applied to give a subspace $\tilde{F}(\tilde{x}_{-1})$ such that $d(\tilde{f}_{-1}^k|_{\tilde{x}_{-1}}) \tilde{F}(\tilde{x}_{-1}) \subset \tilde{F}(\tilde{x})$. Another application of the backward graph transform gives $\tilde{F}(\tilde{x}_{-2})$ such that $d(\tilde{f}_{-2}^k|_{\tilde{x}_{-2}}) \tilde{F}(\tilde{x}_{-2}) \subset \tilde{F}(\tilde{x}_{-1})$, and so on. The situation only improves as we go along, as $\tilde{x}_{-k} \to 0$ as $k \to \infty$. Moreover, since $|d(\tilde{f}_{-k}^k|_{\tilde{x}_{-k}})\tilde{F}(\tilde{x}_{-k})|_{\tilde{f}_{-k}^k} \leq e^{-\lambda} + \delta$, property (iii) in the last paragraph follows.

We have now fully duplicated the conditions in Lemma 3.9 and the arguments there carry over verbatim to give the continuity of $E^u$ on $S$.

**Remark 6.7.** It is curious to compare Lemma 6.6 to the case of Anosov diffeomorphisms of compact (finite dimensional) manifolds, where $y \mapsto E^u_y$ is known to be Hölder continuous (and generally not more than that) as $y$ moves in stable directions, with a Hölder exponent depending on the magnitude of greatest contraction along stable directions [13]. As our map $f$ can and does contract arbitrarily strongly in stable directions, one cannot expect any control on the modulus of continuity of $y \mapsto E^u_y$.

We conclude Sect. 6.2 with two applications of Lemmas 6.4 and 6.6 to resolve some measurability issues we will encounter later on. Let $S$ be as at the beginning of this section, and let $\xi = \xi^S$ denote the (measureable) partition of $S$ into unstable leaves.
Lemma 6.8. For any Borel measurable set $B \subset S$, the map $x \mapsto \nu_x(B \cap \xi(x))$ is measurable.

Proof. Following Lemma 5.7, we have a homeomorphism $\Psi : B^u_{x_0}(\delta l^{-3}_0) \times \Sigma \to S$ where $\Sigma = \exp_{x_0}^{-1}(S) \cap E^s_{x_0}$, so it suffices to prove the corresponding result on $B^u_{x_0}(\delta l^{-3}_0) \times \Sigma$ for the family of measures $\{\nu_\sigma, \sigma \in \Sigma\}$ that are carried by $\Psi$ to $\nu_x$. More precisely, if $\Psi(B^u_{x_0}(\delta l^{-3}_0) \times \{\sigma\}) = \xi(x)$, then $\nu_\sigma$ is the measure defined by $\Psi^*(\nu_\sigma) = \nu_x$. Let $m$ denote the volume induced on the finite dimensional space $B^u_{x_0}$, and on $B^u_{x_0} \times \{\sigma\}$ via its natural identification with $B^u_{x_0}$. Then $\frac{d\nu_\sigma}{dm}(u) = \det(I + dg_\sigma(u))$ where $d$ is with respect to $m$ and the induced volume on $E^u_{\Psi(u,\sigma)}$. Observe that $(u,\sigma) \mapsto E^u_{\Psi(u,\sigma)}$ is continuous by Lemma 6.6. From this and from properties of the det-function (Proposition 2.15), we deduce that the mapping $(u,\sigma) \mapsto \tau(u,\sigma) := \frac{d\nu_\sigma}{dm}(u)$ is continuous. The desired result follows from the continuity of $\tau$.

Let $J^u(x) := \det(df_x|E^u_x)$. It follows from Proposition 5.8 (b) that on each leaf $\xi(x)$, the function

$$z \mapsto \Delta(x,z) := \prod_{i=1}^{\infty} \frac{J_u(f^{-i}x)}{J_u(f^{-i}z)}, \quad z \in \xi(x),$$

is Lipschitz-continuous and bounded from above and below. Let $q : S \to \mathbb{R}$ be given by

$$q(z) := \frac{\Delta(x,z)}{\int_{\xi(x)} \Delta(x,z) d\nu_x(z)}$$

where $z \in \xi(x)$. Observe that $q$ is well defined and independent of the choice of $x$.

Lemma 6.9. The function $q$ is continuous.

Proof. First we claim that for every $n \in \mathbb{N}$, the function $z \mapsto \det(df^u_{f^{-n}z}|E^u_{f^{-n}z})$ is continuous. That is true because (i) $f^{-n}|S$ is continuous (Lemma 6.4), (ii) $y \mapsto E^u_y$ is continuous on $f^{-n}S$ (Lemma 6.6), and (iii) $y \mapsto \det(df^u_y|E^u_y)$ is continuous on $f^{-n}S$ (Proposition 2.15 together with (ii)).

To prove the continuity of $q$, we fix, for each $z \in S$, a reference point $\bar{\sigma}(z)$ defined to be the unique point in $\xi(z) \cap \exp_{x_0}(\Sigma)$ where $\Sigma$ is as in the proof of Lemma 6.8. Define

$$z \mapsto \bar{\Delta}_n(z) := \Delta_n(\bar{\sigma}(z),z) = \prod_{i=1}^{n} \frac{J_u(f^{-i}\bar{\sigma}(z))}{J_u(f^{-i}z)}.$$

Then $z \mapsto \bar{\Delta}_n(z)$ is continuous by the argument above and the continuity of $z \mapsto \bar{\sigma}(z)$. By Proposition 5.8, the sequence $\bar{\Delta}_n$ converges uniformly to $\bar{\Delta}$ where $\bar{\Delta}(z) = \Delta(\bar{\sigma}(z),z)$. Thus $\bar{\Delta}$ is continuous on $S$.

It remains to show that $z \mapsto \int \bar{\Delta} d\nu_x$ is continuous, and that follows from the continuity of $\bar{\Delta}$ and arguments given in the proof of Lemma 6.8. 

### 6.3 Proof of entropy formula for maps with SRB measures

Recall that the distinct Lyapunov exponents of $(f,\mu)$ are denoted by $\lambda_i$ with multiplicity $m_i$. Let $h_{\mu}(f)$ denote the entropy of $f$ with respect to $\mu$, and let $a^+ := \max\{a,0\}$. This section contains the proof of Theorem 1, which we state for the reader’s convenience:
Theorem 1. Assume that $\mu$ is an SRB measure of $f$. Then
\[ h_\mu(f) = \int \sum_i m_i \lambda_i^+ \, d\mu. \]

Below we recall in outline the proof in [18] (referring the reader to [18] for detail), and point out the modifications needed to make the argument in [18] work in the present Banach space setting. We divide the proof into two parts:

(A) Construction of partitions subordinate to $W^u$. First, some notation: For a partition $\alpha$, we let $\alpha(x)$ denote the atom of $\alpha$ containing $x$. For two partitions $\alpha, \beta$, we write $\alpha \leq \beta$ if $\beta$ is a refinement of $\alpha$, and let $\alpha \vee \beta = \{ A \cap B : A \in \alpha, B \in \beta \}$, and $f^{-1}\alpha = \{ f^{-1}A : A \in \alpha \}$. Finally, we say a partition $\alpha$ is decreasing if $\alpha \leq f^{-1}\alpha$.

The following is the analog of Proposition 3.1 in [18].

Proposition 6.10. Assuming that $(f, \mu)$ has a positive Lyapunov exponent $\mu$-a.e., there is a stack of unstable manifolds $\mathcal{S}$ with $\mu(\mathcal{S}) > 0$ and a measurable partition $\eta$ on $\tilde{\mathcal{S}} := \bigcup_{n \geq 0} f^n \mathcal{S}$ with the following properties:

(a) $\eta$ is subordinate to $W^u$,

(b) it is decreasing, and

(c) for any Borel set $B \subset \mathcal{A}$, the function $x \mapsto \nu_x(\eta(x) \cap B)$ is finite-valued and measurable.

We remark that if $\eta(x)$ is to contain a neighborhood of $x$ in $W^u_x$ for $\mu$-a.e. $x$ (part of the definition of being subordinate to $W^u$), then we cannot assume $\eta(x) \subset \mathcal{A}$. This condition is used in the proof of Lemma 6.12 where we have to work with the set $\tilde{\mathcal{S}}$.

Sketch of proof. Using the notation in Sect. 5.2, we first construct a stack. The stack in the statement of this proposition will be of the form $\mathcal{S} = \mathcal{S}_r$ where
\[ \mathcal{S}_r := \bigcup_{y \in U} \exp_{x_0} \left( \text{graph} \left( \Theta(y) \bigl|_{B_{2\delta}^u(r\delta l_0^{-3})} \right) \right), \tag{31} \]
$x_0 \in \mathcal{A}$ is a point, and $r \in (0,1)$ is a number to be determined. We choose these so that $\mu(\mathcal{S}_r) > 0$ for any choice of $r$, and let $\tilde{\mathcal{S}} = \bigcup_{n \geq 0} f^n \mathcal{S}$.

The partition $\eta$ on $\tilde{\mathcal{S}}$ is constructed as follows: Let $\xi$ be the partition of $\mathcal{S}$ into unstable leaves. For $k = 0, 1, 2, \ldots$, let $\xi_k = \{ f^k(W), W \in \xi \} \cup \{ \tilde{\mathcal{S}} \setminus f^k(\mathcal{S}) \}$, and let $\eta = \bigcup_{k=0}^{\infty} \xi_k$. Since $x \in f^k(W)$ if and only if $f^k(x) \in f^{k+1}(W)$, it follows immediately that $\eta$ is decreasing.

Item (a) also follows automatically from this construction except for the requirement that $\eta(x)$ contains a neighborhood of $x$ in $W^u_x$ for $\mu$-a.e. $x$. This is done by choosing $r$ judiciously. Let
\[ \partial \mathcal{S}_r := \bigcup_{y \in U} \exp_{x_0} \left( \{ \Theta(y)(u) : u \in E^u_{x_0}, |u| = r \delta l_0^{-3} \} \right). \]

Since $\partial(\eta(x)) \subset \bigcup_{k=0}^{\infty} f^k(\partial \mathcal{S})$, to guarantee $W^u_{\epsilon(x),x} \subset \eta(x)$, it suffices to have $f^{-k}W^u_{\epsilon(x),x} \cap \partial \mathcal{S}_r = \emptyset$ for all $k \geq 0$. We choose $r$ so that $\mu(\partial \mathcal{S}_r) = 0$ and $\epsilon(x) > 0$ for $\mu$-a.e. $x$ by a Borel-Cantelli type argument. See [18] for details.

Item (c) follows from Lemma 6.8 together with standard approximation arguments (to go from $\xi$ to $\eta$).
If \((f, \mu)\) is nonergodic, it may happen that \(\mu(\tilde{S}) < 1\). It can be shown that at most a countable number of (disjoint) sets of the type \(\tilde{S}\) will cover a full \(\mu\)-measure set.

**Entropy computation.** The following are the main points in the rest of the proof in \[18\]. We first list them (as they appear in \[18\]) before commenting on modifications needed:

1. Since Ruelle’s Inequality \[35\] states that
   \[
   h_{\mu'}(f) \leq \int \sum_i m_i \lambda_i^+ d\mu'
   \]
   for any Borel probability invariant measure \(\mu'\), to prove the entropy formula it suffices to show that the reverse inequality holds for SRB measures. In particular, it suffices to show that if \(\mu\) is an SRB measure, then
   \[
   H(f^{-1}\eta|\eta) = h_{\mu}(f, \eta) = \int \sum_i m_i \lambda_i^+ d\mu = \int \log J^u_d\mu
   \]
   where \(\eta\) is the partition constructed in Part (A) and \(J^u(x) := \text{det}(df_x|E^u(x))\).

2. Let \(\nu\) be the \(\sigma\)-finite measure with the property that for any Borel subset \(K\),
   \[
   \nu(K) = \int \nu_x(\eta(x) \cap K) d\mu(x).
   \]
   By assumption, \(\mu \ll \nu\). Let \(\rho = \frac{d\mu}{d\nu}\), and let \(\mu_{\eta(x)}\) denote the disintegration of \(\mu\) on partition elements of \(\eta\). Then
   \[
   \rho(z) = \rho(fz) \cdot J_u(z) \cdot \mu_{\eta(x)}((f^{-1}\eta)(x)).
   \]

3. The main computation is the following transformation rule for \(\rho\):

   **Lemma 6.11.** For \(\mu\)-a.e. \(x\) and \(\nu_x\)-a.e. \(z \in \eta(x)\),
   \[
   \rho(z) = \rho(fz) \cdot J_u(z) \cdot \mu_{\eta(x)}((f^{-1}\eta)(x)).
   \]

4. From Lemma 6.11 one deduces easily that the information function \(I(f^{-1}\eta|\eta)\) satisfies
   \[
   I(f^{-1}\eta|\eta)(x) = \log J^u(x) + \log \frac{\rho(fx)}{\rho(x)}
   \]
   for \(\mu\)-almost every \(x\). As \(I(f^{-1}\eta|\eta) \geq 0\) and \(\log J^u \in L^1(\mu)\), it follows that \(\log - \frac{\rho(f)}{\rho} \in L^1(\mu)\). A general measure-theoretic lemma then gives \(\int \log \frac{\rho f}{\rho} d\mu = 0\). Integrating (34) gives (33).

We now comment on the modifications needed for the arguments above to carry over to Banach space mappings. For Banach space mappings, the analog of (32) is proved in \[40\], so here as well, it suffices to prove (33). But to make sense of the last equality in (33), one needs to first introduce a notion of volume on finite dimensional subspaces, so that det(\(\cdot|\cdot\))
is defined; this is done in Section 2, and the last equality in (33), which relates Lyapunov exponents to volume growth, is proved both in [3] and in Proposition 3.7.

With regard to Item 2, that $\nu$ so defined is a measure follows from part (c) of Proposition 6.10. The characterization of $\rho$ given is a purely measure-theoretic fact.

The proof of Lemma 6.11 uses only (i) the change of variables formula for induced measures on finite dimensional manifolds and (ii) the invariance of $\mu$, more precisely that $\mu(f^{-1}K) = \mu(K)$ for a countable sequence of Borel sets $K$. For the measures $\nu_x$, (i) is proved in (18). (ii) is not an issue for us as $\mu$ is supported on the compact metric space $\mathcal{M}$.

The last item is also a purely measure-theoretic fact.

This completes the proof of Theorem 1.

6.4 Entropy formula implies SRB measure

We first prove Theorem 2 under the assumption that $h_\mu(f) = h_\mu(f, \eta)$ for a partition $\eta$ of the type constructed in Proposition 6.10, leaving the justification of this assumption for later.

Lemma 6.12. Given $(f, \mu)$ with a positive Lyapunov exponent $\mu$-a.e., let $\eta$ be as in Proposition 6.10. If

$$h_\mu(f, \eta) = \int \log J^u d\mu,$$

then $\mu$ is an SRB measure whose densities on unstable manifolds satisfy (36).

Proof. Our proof follows [17] in outline. Here it is essential that the elements of $\eta$ contain open subsets of $W^u_x$ for $\mu$-a.e. $x$. Let $\mathcal{S}, \xi$ and $\tilde{\mathcal{S}}$ be as in Proposition 6.10. We discuss the ergodic case, dividing it into two main steps.

(A) Construction of a candidate SRB measure $\vartheta$. As noted in Sect. 6.2, the function $z \mapsto \Delta(x, z)$ is Lipchitz-continuous and bounded from above and below on $\xi(x)$. Since for $\mu$-a.e. $x$, there exists $n \geq 0$ such that $f^{-n}(\eta(x))$ is contained in a leaf of $\mathcal{S}$, and $f^n$ restricted to each leaf is a $C^2$ embedding (by Hypothesis (H1)), the statement above holds (with nonuniform Lipschitz bounds) for all $z \in \eta(x)$ for $\mu$-a.e. $x$. This together with the fact that $\nu_x(\eta(x)) > 0$ for $\mu$-a.e. $x$ implies that

$$p(z) := \frac{\Delta(x, z)}{\int_{\eta(x)} \Delta(x, z) d\nu_x(z)}, \quad z \in \eta(x),$$

is well defined for $\mu$-a.e. $x$.

We seek to define a probability measure $\vartheta$ on Borel subsets of $\tilde{\mathcal{S}} = \cup_{n \geq 0} f^n \mathcal{S}$ by letting

$$\vartheta(K) = \int \left( \int_{\eta(x) \cap K} p(z) d\nu_x(z) \right) d\mu(x)$$

for $K \subset \tilde{\mathcal{S}}$. That is to say, we want $\vartheta$ and $\mu$ to project to the same measure on the quotient space $\mathcal{S}/\eta$, and we want the conditional measures of $\vartheta$ on elements of $\eta$ to be given by the (normalized) densities $p(\cdot)$ (We note that $\tilde{\mathcal{S}}$ is a complete, separable metric space, being a countable union of compact sets in $\mathcal{B}$; the existence of canonical disintegrations in this
setting is proved in, e.g., [7]). To prove that \( \vartheta \) is a bona fide measure, we need to prove the measurability of \( p \), which can be deduced, via standard arguments, from the measurability of the function

\[
q(z) := \frac{\Delta(x, z)}{\int_{S(x)} \Delta(x, z) d\nu(x)}, \quad z \in S.
\]

This involves studying backward iterates not just along individual \( W^u_\log \)-leaves but across the leaves that comprise \( S \), at points that are not necessarily \( \mu \)-typical. We have treated these issues in Sect. 6.2; the continuity of \( q \) is proved in Lemma 6.9.

(B) **Proof of** \( \vartheta = \mu \). This part of the argument is identical to that in [17]; we recall it for completeness. For any \( n \in \mathbb{N} \), we have

\[
\int -\log \vartheta_{\eta(x)}((f^{-n}\eta)(x))d\mu(x) = \int \log \det(df^n_{\mu_E})(x) d\mu(x)
\]

\[
= H(f^{-n}\eta|\eta) = \int -\log \mu_{\eta(x)}((f^{-n}\eta)(x))d\mu(x).
\]

The first equality is by the change of variables formula (the same computation as in Items 3 and 4 in Part (B) of Sect. 6.3), the second is by the main assumption in Lemma 6.12, and the third is the definition of entropy.

We introduce the \( S/(f^{-n}\eta) \)-measurable function

\[
\phi_n(x) = \frac{\vartheta_{\eta(x)}((f^{-n}\eta)(x))}{\mu_{\eta(x)}((f^{-n}\eta)(x))},
\]

which is well-defined \( \mu \)-almost surely. Let \( \vartheta^{(n)} \) and \( \mu^{(n)} \) denote the restriction of \( \vartheta \) and \( \mu \) to \( \mathcal{B}_{f^{-n}\eta} \), the \( \sigma \)-algebra of measurable subsets that are unions of atoms in \( f^{-n}\eta \), and decompose \( \vartheta^{(n)} = \vartheta_{\mu}^{(n)} + \vartheta_{\mu}^{(n)} \), where \( \vartheta_{\mu}^{(n)} \ll \mu^{(n)} \) and \( \vartheta_{\mu}^{(n)} \) is mutually singular with \( \mu^{(n)} \). Observe that \( \vartheta_{\mu}^{(n)} \) can be strictly positive (this happens if a positive \( \mu \)-measure set of \( x \) has the property that \( \eta(x) \) contains one or more elements of \( f^{-n}\eta \) with \( \mu_{\eta(x)} \)-measure zero), while \( \phi_n = d\vartheta_{\mu}^{(n)}/d\mu^{(n)} \). Thus \( \int \phi_n d\mu \leq 1 \).

At the same time, it follows from the string of equalities at the beginning of the proof that \( \int \log \phi_n d\mu = 0 \). Now Jensen’s inequality tells us that

\[
\int \log \phi_n d\mu \leq \log \int \phi_n d\mu,
\]

with equality holding iff \( \phi_n \equiv \text{constant} \). So \( \phi_n \equiv 1 \), from which it follows that \( \mu \) and \( \vartheta \) coincide on \( \mathcal{B}_{f^{-n}\eta} \). The conclusion now follows from the fact that \( f^{-n}\eta \not\sim \varepsilon \), the partition into points.

It remains to address the issue of whether or not the (uncountable) partition \( \eta \) captures all the entropy of \( f \), i.e., whether \( h_{\mu}(f) = h_{\mu}(f, \eta) \). To get a handle on this, we seek a measurable partition \( \mathcal{P} \) with the properties that \( H_{\mu}(\mathcal{P}) < \infty \) and \( \eta \leq \mathcal{P}^+ := \vee_{n=0}^{\infty} f^n \mathcal{P} \). Since elements of \( \eta \) are contained \( W^u \)-leaves, and \( \mathcal{P}^+(x) = \{ y \in \mathcal{B} : f^{-n}y \in \mathcal{P}(f^{-n}x) \text{ for all } n \geq 0 \} \), the next lemma, which follows immediately from the characterization of local unstable manifolds in Lemma 5.3, is relevant:
Lemma 6.13. Suppose $\mathcal{P}$ is a partition with the property that for $\mu$-a.e. $x$ and every $n \in \mathbb{N}$,
\[ |f^{-n}y - f^{-n}x|_{f^{-n}x} \leq \delta l(f^{-n}x)^{-1} \text{ for all } y \in \mathcal{P}^+(x). \]

Then $\mathcal{P}^+(x) \subset W^{u}_{\delta,x}$.

Thus the problem is reduced to finding a finite entropy partition $\mathcal{P}$ with the property in the lemma above. In [17], such a partition was constructed by appealing to a lemma due to Mañé [29], and here lies another difference between finite and infinite dimensions: the lemma in [29] uses the finite dimensionality of the ambient manifold. Our next lemma contains a slight strengthening of this result that is adequate for our purposes; see [21] for a similar result.

Lemma 6.14 (following Lemma 2 in [29]). Let $Z$ be a compact metric space with box-counting dimension $\delta < \infty$. Let $T : Z \to Z$ be a homeomorphism, let $m$ be a Borel invariant probability on $Z$, and let $\rho : Z \to (0,1)$ be a measurable function for which $\log \rho \in L^1(m)$. Then, there exists a countable partition $\mathcal{P}$ of finite entropy such that for $m$-almost all $x \in Z$,
\[ \mathcal{P}(x) \subset B(x, \rho(x)), \]

where $B(x, r)$ is the ball of radius $r$ centered at $x$.

Proof Sketch for Lemma 6.14. Define $U_n = \{x \in Z : e^{-(n+1)} < \rho(x) \leq e^{-n}\}$ for $n \geq 1$. Since $\log \rho$ is integrable, we have that $\sum_{n=1}^{\infty} n m(U_n) < \infty$. This implies (see Lemma 1 of [29]) that
\[ \sum_{n=1}^{\infty} -m(U_n) \log m(U_n) < \infty. \]

By the definition of box-counting dimension, there exists $C > 0$ be such that for any $r > 0$, there is a finite cover of $Z$ by balls of radius $r$ with cardinality $\leq C r^{-(\delta+1)}$. It follows that there is a finite partition $\mathcal{P}_r$ of $Z$ of cardinality $\leq C r^{-(\delta+1)}$, each element of which is contained in one of these balls. Writing $r_n = e^{-(n+1)}$ for each $n \in \mathbb{N}$, we fix such a partition $\mathcal{P}_{r_n}$. The desired partition $\mathcal{P}$ is now defined as follows: Elements of $\mathcal{P}$ are of the form $A \cap U_n$, $n \geq 0$, with $A \in \mathcal{P}_{r_n}$. One then estimates
\[ \sum_{P \in \mathcal{P}, P \subset U_n} -m(P) \log m(P) \leq m(U_n) \left( \log |\mathcal{P}_{r_n}| - \log m(U_n) \right) \]

for each $n \geq 0$, where the cardinality $|\mathcal{P}_{r_n}|$ is $\leq C r_n^{-(\delta+1)}$; this decay is sufficient to show that $H_{m}(\mathcal{P}) < \infty$. 

The discussion above serves to motivate the finite box-counting assumption in Theorem 2, a complete statement of which is as follows:

Theorem 2. In addition to (H1)–(H4), we assume (H5), i.e., that the set $\mathcal{A}$ has finite box-counting dimension. If $\lambda_1 > 0$ and $(f, \mu)$ satisfies the Entropy Formula
\[ h_\mu(f) = \int \sum_i m_i \lambda_i^+ \, d\mu, \]

then $\mu$ is an SRB measure.

Theorem 2 is implied by Lemma 6.15 below and Lemma 6.12.
Lemma 6.15. Assume (H1)–(H5), and that \( \lambda_1 > 0 \) \( \mu \)-a.e. Let \( \eta \) be the partition in Lemma 6.9. Then
\[
h_\mu(f) = h_\mu(f, \eta) .
\]

Sketch of proof. This part of our proof is identical to that in [17] (see also [19]). For completeness we outline the proof of the ergodic case.

(A) Construction of an auxiliary partition \( \mathcal{P} \). The aim of this step is to produce a partition \( \mathcal{P} \) of \( \mathcal{A} \) with the property in Lemma 6.13 and with \( H_\mu(\mathcal{P}) < \infty \). Fix \( \delta > 1 \) such that \( \mu(\Gamma_{\delta}) > 0 \), and define \( N : \Gamma_{\delta} \to \mathbb{N} \) to be the first return time to \( \Gamma_{\delta} \). Extend \( N \) to all of \( \mathcal{A} \) by setting \( N|_{\mathcal{A}\setminus\Gamma_{\delta}} = 0 \). We apply Lemma 6.14 to the function \( \rho(x) = \delta_0^{-2}(e^{2\beta_2 K})^{-N(x)} \) for any \( \delta \leq \delta_1 \), where \( K \) is an upper bound on \( |df| \) for \( y \in \mathcal{B} : d(y, \mathcal{A}) \leq r_0 \); see the discussion preceding Lemma 4.2, for which log \( \rho \) is in \( L^1(\mu) \) since \( \int_{\mathcal{A}} N(x) d\mu(x) = (\mu(\Gamma_{\delta}))^{-1} \leq \infty \). It is straightforward to check that the \( \mathcal{P} \) given by Lemma 6.14 has the desired properties.

(B) Proof of \( h_\mu(f) = h_\mu(f, \eta) \). Given a small \( \varepsilon > 0 \), we let \( \mathcal{Q} = \mathcal{P} \cup \{ \mathcal{S} \cap \mathcal{A}, \mathcal{A} \setminus \mathcal{S} \} \cup \mathcal{P}_0 \) where \( \mathcal{S} \) is the stack used in the construction of \( \eta \). Then \( \mathcal{P}_0 \) is a finite partition chosen so that \( h_\mu(f, \mathcal{Q}) > h_\mu(f) - \varepsilon \). Let \( \mathcal{Q}^+ = \bigvee_{n=0}^{\infty} f^n \mathcal{Q} \), and check that by construction, \( \eta(x) \supset \mathcal{Q}^+(x) \) for \( \mu \)-a.e. \( x \). Then
\[
h_\mu(f, \eta) = \frac{1}{n} H_\mu(f^{-n}\eta|\eta) \geq \frac{1}{n} H_\mu(\vee_{j=0}^{n} f^{-j} \mathcal{Q}|\eta) - \frac{1}{n} H_\mu(\vee_{j=0}^{n} f^{-j} \mathcal{Q}|f^{-n}\eta) . \tag{35}
\]
Since \( \mathcal{Q}^+ \supset \eta \), the first term on the right side of (35) is
\[
\geq \frac{1}{n} H_\mu(\vee_{j=0}^{n} f^{-j} \mathcal{Q}|\mathcal{Q}^+) \geq h_\mu(f, \mathcal{Q}) \geq h_\mu(f) - \varepsilon .
\]
while the second term in (35) can be shown to be \( < \varepsilon \) for large \( n \) since modulo sets of \( \mu \)-measure 0, \( \vee_{j=0}^{\infty} f^{-j}\eta \) partitions \( \mathcal{A} \) into points. \( \square \)

6.5 Corollaries

We finish with the following corollaries to our main results.

Theorem 6.16. Let \( \mu \) be an SRB measure of \( f \). Let \( \eta \) be given by Proposition 6.11, and let \( \rho \) be the densities with respect to \( \nu_x \) of the conditional measures of \( \mu \) on elements of \( \eta \). Then \( y \mapsto \rho(y) \) is Lipschitz on each element of \( \eta \), and for \( \mu \)-a.e. \( x \)
\[
\frac{\rho(y)}{\rho(z)} = \prod_{i=1}^{\infty} \frac{\det(df_{f^{-i}z}|E_{f^{-2i}z})}{\det(df_{f^{-i}y}|E_{f^{-2i}y})} \quad \text{for } \nu_x \text{-a.e. } y, z \in \eta(x) . \tag{36}
\]
The conclusion above holds also if \( \mathcal{S} \) is a stack of local unstable manifolds with \( \mu(\mathcal{S}) > 0 \) and \( \eta \) is replaced by \( \xi \), the partition of \( \mathcal{S} \) into unstable leaves.

Proof. In the proof of Theorem 1, we showed that \( h_\mu(f, \eta) = \int \log J^y d\mu \). The form of the densities comes from the proof of Lemma 6.12. \( \square \)

Corollary 6.17. Let \( \mu \) be an SRB measure of \( f \). Then \( \nabla_x \subset \text{supp}(\mu) \subset \mathcal{A} \) for \( \mu \)-a.e. \( x \).

Proof. This is because \( \rho > 0 \) \( \nu_x \)-a.e. on \( \nabla_x \) for \( \mu \)-a.e. \( x \) by Theorem 6.16. \( \square \)
Appendix

Recall that the hypothesis of Lemma 2.3 are

\[(*) \quad E, E', F \in \mathcal{G}(B) \quad \text{and} \quad d_H(E, E') < |\pi_{E/F}|^{-1}.\]

Let us write \(d(\cdot, \cdot)\) instead of \(d_H(\cdot, \cdot)\) for simplicity.

**Proof of Lemma 2.3** To prove \(\mathcal{B} = E' \oplus F\), we first show \(E' \cap F = \{0\}\). If not, pick \(e' \in E' \cap F\) with \(|e'| = 1\). Since \(d(e', \mathcal{S}_E) \leq d(E, E') < |\pi_{E/F}|^{-1}\), there exists \(e \in E\) with \(|e| = 1\) such that \(|e - e'| < |\pi_{E/F}|^{-1}\). This is incompatible with

\[1 = |e| = |\pi_{E/F} e| = |\pi_{E/F} (e - e')| \leq |\pi_{E/F}| \cdot |e - e'| < 1.\]

Next, we claim that \(E' \oplus F\) is closed. It will suffice to show that there exists \(A > 0\) such that for any \(e' \in E', f \in F\), we have

\[|e'| \leq A|e' + f|. \quad (37)\]

This is known as the Kober criterion [15]. Indeed, if (37) holds and \(x_n = e'_n + f_n\) is Cauchy, then \(e'_n\) and \(f_n\) individually are Cauchy, and thus converge to some \(e' \in E', f \in F\) respectively, hence \(x_n \to x := e' + f \in E' + F\). To prove (37), pick arbitrary \(e' \in E'\) and \(f \in F\), and fix \(c > 1\) with \(cd(E, E') < |\pi_{E/F}|^{-1}\). As before, let \(e \in E\) be such that \(|e| = |e'|\) and \(|e - e'| \leq |e'| c d(E, E')\). Then

\[|e' + f| \geq |e + f| - |e - e'| \geq |e'|(|\pi_{E/F}|^{-1} - cd(E, E')) =: A^{-1}|e'|.\]

To finish, assume for the sake of contradiction that \(E' \oplus F \neq \mathcal{B}\). By Assumption (ii), there exist \(c_1 < 1 < c_2\) such that \(c_2 d(E, E')|\pi_{E/F}| < c_1\). Since \(E' \oplus F\) is closed, the Riesz Lemma [39] asserts that there exists \(x \in \mathcal{B}\) with \(|x| = 1\) such that \(|x - (e' + f)| \geq c_1\) for all \(e' \in E', f \in F\). On the other hand, since \(\mathcal{B} = E \oplus F\), we have that \(x = e + f\) for some \(e \in E, f \in F\); notice that \(|e| \leq |\pi_{E/F}|\). But there exists \(e' \in E\) with \(|e'| = |e|\) and \(|e - e'| \leq c_2|e|d(E, E')\), and for such an \(e'\),

\[|x - (e' + f)| = |e - e'| \leq |e|c_2d(E, E') < c_1,\]

contradicting our choice of \(x\).

**Lemma A.1.** Assume \((*)\). Then

(i) \[|\pi_{E'/F}| \leq \frac{|\pi_{E/F}|}{1 - |\pi_{E/F}|d(E, E')},\]

(ii) \[|\pi_{F'/E'}| \leq 2|\pi_{E'/F}|d(E, E').\]
Proof. Since $B = E \oplus F = E' \oplus F$, (i) above is equivalent to

$$\alpha(E, F) \leq d(E, E') + \alpha(E', F) \quad (38)$$

by the formula $\alpha(E, F) = |\pi_{E/F}|^{-1}$ from Sect. 2.1.2. To estimate $\alpha(E', F)$ from below, we let $e' \in E'$ with $|e'| = 1$ and $f \in F$ be arbitrary. For $c > 1$, we let $e \in E, |e| = 1$ be such that $|e - e'| \leq cd(E, E')$. Then,

$$|e' - f| \geq |e - f| - |e' - e| \geq \alpha(E, F) - cd(E, E') .$$

But $e', f$ were arbitrary and so our formula follows on taking $c \to 1$.

To prove (ii), fix $e \in E, |e| = 1$. Then for $c > 1$ arbitrarily close to 1, let $e' \in E', |e'| = 1$ be such that $|e - e'| \leq cd(E, E')$. Then

$$|\pi_{F/E'} e| = |\pi_{F/E'}(e - e')| \leq |\pi_{F/E'}| \cdot |e - e'| \leq 2|\pi_{E/F}| \cdot cd(E, E') .$$

Proof of Lemma 2.5. That $B = E' \oplus F$ follows from Lemma 2.3. The bounds in (a) are given by Lemma A.1, and (b) follows from (a).

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