Hermite–Hadamard–Fejér Type Inequalities for $p$-Convex Functions via Fractional Integrals

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Abstract

In this paper, firstly, Hermite–Hadamard–Fejér type inequalities for $p$-convex functions in fractional integral forms are built. Secondly, an integral identity and some Hermite–Hadamard–Fejér type integral inequalities for $p$-convex functions in fractional integral forms are obtained. Finally, some Hermite–Hadamard and Hermite–Hadamard–Fejér inequalities for convex, harmonically convex and $p$-convex functions are given. Many results presented here for $p$-convex functions provide extensions of others given in earlier works for convex, harmonically convex and $p$-convex functions.

Keywords Hermite–Hadamard inequalities · Hermite–Hadamard–Fejér inequalities · Fractional integrals · Convex functions · Harmonically convex functions · $p$-Convex functions

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

(1)

is well known in the literature as Hermite–Hadamard’s inequality (Hadamard 1893; Hermite 1883).

The most well-known inequalities related to the integral mean of a convex function $f$ are the Hermite–Hadamard inequalities or its weighted versions, the so-called Hermite–Hadamard–Fejér inequalities.

Fejér (1906) established the following Fejér inequality which is the weighted generalization of Hermite–Hadamard inequality (1):

$$f\left(\frac{a+b}{2}\right) \int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b w(x)dx,$$

(2)

holds, where $w : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve, and extend the inequalities (1) and (2), see Bombardelli and Varošanec (2009), Chen and Wu (2014), Dragomir and Agarwal (1998), Fang and Shi (2014), İşcan (2013, 2014c, d, 2016b, c), Mihai et al. (2015), Noor et al. (2016), Pearce and Pecaric (2000), Sarıkaya (2012) and Tseng et al. (2011).

We will now give definitions of the right-hand side and left-hand side Riemann–Liouville fractional integrals which are used throughout this paper.

Definition 1 (Kilbas et al. 2006). Let $f \in L[a,b]$. The right-hand side and left-hand side Riemann–Liouville fractional integrals $J_a^\alpha f$ and $J_b^\alpha f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by
\[ J_{a}^{x} f(x) = \frac{1}{\Gamma(x)} \int_{a}^{x} (x-t)^{x-1} f(t) dt, \; x > a \]
\[ J_{b}^{x} f(x) = \frac{1}{\Gamma(x)} \int_{x}^{b} (t-x)^{x-1} f(t) dt, \; x < b, \]
respectively, where \( \Gamma(x) \) is the Gamma function defined by \( \Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt \).

Because of the wide application of Hermite–Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite–Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite–Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see Dahmani (2010), İşcan (2014a, b, 2015), İşcan and Wu (2014), İşcan et al. (2016d), Sarıkaya et al. (2013) and Wang et al. (2012, 2013).

İşcan (2014d) gave the definition of harmonically convex function and established the following Hermite–Hadamard type inequality for harmonically convex functions as follows:

**Definition 2** Let \( I \subset \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \rightarrow \mathbb{R} \) is said to be harmonically convex if

\[
 f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{t}{x} f(y) + \left(1 - \frac{1}{x}\right) f(x),
\]
for all \( x, y \in I \) and \( t \in [0, 1] \). If the inequality in (3) is reversed, then \( f \) is said to be harmonically concave.

**Theorem 2** (İşcan 2014d). Let \( f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \), then the following inequalities hold:

\[
 f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_{a}^{b} f(x) \frac{1}{x^2} dx \leq \frac{f(a) + f(b)}{2}.
\]

Chen and Wu (2014) presented Hermite–Hadamard–Fejér inequality for harmonically convex functions as follows:

**Theorem 3** Let \( f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \) and \( w : [a, b] \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) is nonnegative, integrable and harmonically symmetric with respect to \( \frac{2ab}{a+b} \), then

\[
 f\left(\frac{2ab}{a+b}\right) \int_{a}^{b} w(x) \frac{1}{x^2} dx \leq \int_{a}^{b} f(x) w(x) \frac{1}{x^2} dx
\]
\[
 \leq \frac{f(a) + f(b)}{2} \int_{a}^{b} w(x) \frac{1}{x^2} dx.
\]

Sarıkaya et al. (2013) presented Hermite–Hadamard inequality for convex functions via fractional integrals as follows:

**Theorem 4** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L[a, b] \). If \( f \) is a convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
f\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(x + 1)}{2(b - a)^x} \left[J_{a}^{\infty} f\left(b\right) + J_{b}^{\infty} f\left(a\right)\right]
\]
\[
\leq \frac{f(a) + f(b)}{2},
\]
with \( x > 0 \).

İşcan and Wu (2014) presented Hermite–Hadamard inequality for harmonically convex functions via fractional integrals as follows:

**Theorem 5** Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a function such that \( f \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \( f \) is a harmonically convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(x + 1)}{2(b - a)^x} \left[J_{a}^{1} f\left(g\left(1/b\right)\right) + J_{b}^{1} f\left(g\left(1/a\right)\right)\right]
\]
\[
\leq \frac{f(a) + f(b)}{2},
\]
with \( x > 0 \) and \( g(x) = \frac{1}{x} \), \( x \in \left[\frac{1}{b}, \frac{1}{a}\right] \).

İşcan (2015) presented Hermite–Hadamard–Fejér inequality for convex functions via fractional integrals as follows:

**Theorem 6** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a convex function with \( a < b \) and \( f \in L[a, b] \). If \( w \) is nonnegative, integrable and symmetric to \((a + b)/2\), then the following inequalities for fractional integrals hold:

\[
f\left(\frac{a + b}{2}\right) \left[J_{a}^{\infty} w\left(b\right) + J_{b}^{\infty} w\left(a\right)\right]
\]
\[
\leq \left[J_{a}^{\infty} \left(fw\right)\left(b\right) + J_{b}^{\infty} \left(fw\right)\left(a\right)\right]
\]
\[
\leq \frac{f(a) + f(b)}{2} \left[J_{a}^{\infty} w\left(b\right) + J_{b}^{\infty} w\left(a\right)\right],
\]
with \( x > 0 \).

İşcan et al. (2016d) presented Hermite–Hadamard–Fejér inequality for harmonically convex functions via fractional integrals as follows:

**Theorem 7** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a harmonically convex function with \( a < b \) and \( f \in L[a, b] \). If \( w : [a, b] \rightarrow \mathbb{R} \) is nonnegative, integrable and harmonically symmetric with respect to \( 2ab/a + b \), then the following inequalities for fractional integrals hold:
2 Main Results

Throughout this section, \( \|w\|_\infty = \sup_{t \in [a,b]} |w(t)| \), for the continuous function \( w : [a, b] \to \mathbb{R} \).
If $p > 0$, 

\[
\begin{align*}
&f \left( \frac{[a^p + b^p]}{2} \right) \left[ J_{p^+}^p \left( (w \circ g)(b^p) \right) + J_{p^-}^p \left( (w \circ g)(a^p) \right) \right] \\
&\leq \left[ J_{p^+}^p \left( (fw \circ g)(b^p) \right) + J_{p^-}^p \left( (fw \circ g)(a^p) \right) \right] \\
&\leq \frac{f(a) + f(b)}{2} \left[ J_{p^+}^p \left( (w \circ g)(b^p) \right) + J_{p^-}^p \left( (w \circ g)(a^p) \right) \right],
\end{align*}
\]

with $g(x) = x^{1/p}, x \in [a^p, b^p]$.

(ii) If $p < 0$, 

\[
\begin{align*}
&f \left( \frac{[a^p + b^p]}{2} \right) \left[ J_{p^+}^p \left( (w \circ g)(a^p) \right) + J_{p^-}^p \left( (w \circ g)(b^p) \right) \right] \\
&\leq \left[ J_{p^+}^p \left( (fw \circ g)(a^p) \right) + J_{p^-}^p \left( (fw \circ g)(b^p) \right) \right] \\
&\leq \frac{f(a) + f(b)}{2} \left[ J_{p^+}^p \left( (w \circ g)(a^p) \right) + J_{p^-}^p \left( (w \circ g)(b^p) \right) \right],
\end{align*}
\]

with $g(x) = x^{1/p}, x \in [b^p, a^p]$.

**Proof**

(i) Let $p > 0$. Since $f : I \subset (0, \infty) \to \mathbb{R}$ is a $p$-convex function, we have, for all $x, y \in I$ (with $t = \frac{1}{2}$ in the inequality (10))

\[
f \left( \frac{[x^p + y^p]}{2} \right)^{1/p} \leq \frac{f(x) + f(y)}{2}.
\]

Choosing $x = \left[ ta^p + (1-t)b^p \right]^{1/p}$ and $y = \left[ tb^p + (1-t)a^p \right]^{1/p}$, we get

\[
f \left( \frac{[a^p + b^p]}{2} \right)^{1/p} \leq \frac{f \left( \left[ ta^p + (1-t)b^p \right]^{1/p} \right) + f \left( \left[ tb^p + (1-t)a^p \right]^{1/p} \right)}{2}.
\]

Multiplying both sides of (14) by $2t^{p-1}w \left( \left[ ta^p + (1-t)b^p \right]^{1/p} \right)$ and integrating with respect to $t$ over $[0, 1]$, using Lemma 1-i, we get

\[
f \left( \frac{[a^p + b^p]}{2} \right)^{1/p} \left[ J_{p_+}^p \left( (w \circ g)(b^p) \right) + J_{p_-}^p \left( (w \circ g)(a^p) \right) \right] \\
\leq \left[ J_{p_+}^p \left( (fw \circ g)(b^p) \right) + J_{p_-}^p \left( (fw \circ g)(a^p) \right) \right],
\]

the left hand side of (12). For the proof of the second inequality in (12), we first note that if $f$ is a $p$-convex function, then, for all $t \in [0, 1]$, it yields

\[
f \left( \left[ ta^p + (1-t)b^p \right]^{1/p} \right) + f \left( \left[ tb^p + (1-t)a^p \right]^{1/p} \right) \\
\leq \frac{f(a) + f(b)}{2}.
\]

Multiplying both sides of (15) by $2t^{p-1}w \left( \left[ ta^p + (1-t)b^p \right]^{1/p} \right)$ and integrating with respect to $t$ over $[0, 1]$, using Lemma 1-i, we get

\[
\frac{f(a) + f(b)}{2} \left[ J_{p_+}^p \left( (w \circ g)(b^p) \right) + J_{p_-}^p \left( (w \circ g)(a^p) \right) \right] \\
\leq \left[ J_{p_+}^p \left( (fw \circ g)(b^p) \right) + J_{p_-}^p \left( (fw \circ g)(a^p) \right) \right],
\]

the right hand side of (12). This completes the proof of i.

(ii) The proof is similar to i. \qed

**Remark 1** In Theorem 9, one can see the following.

(1) If one takes $p = 1$, one has (8).
(2) If one takes $p = 1$ and $w(x) = 1$, one has (6).
(3) If one takes $p = 1$ and $x = 1$, one has (2).
(4) If one takes $p = 1, x = 1$ and $w(x) = 1$, one has (1).
(5) If one takes $p = -1$, one has (9).
(6) If one takes $p = -1$ and $w(x) = 1$, one has (7).
(7) If one takes $p = -1$ and $x = 1$, one has (5).
(8) If one takes $p = -1, x = 1$ and $w(x) = 1$, one has (4).
(9) If one takes $x = 1$ and $w(x) = 1$, one has (11).

**Lemma 2** Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^o$ and $a, b \in I^o$ with $a < b$, $p \in \mathbb{R}\setminus\{0\}$ and $x > 0$. If $f' \in L[a,b]$ and $w : [a, b] \to \mathbb{R}$ is integrable and $p$-symmetric with respect to $\frac{[a^p + b^p]}{2}^{1/p}$, then the following equalities for fractional integrals hold:

(i) If $p > 0$,

\[
\frac{f(a) + f(b)}{2} \left[ J_{p_+}^p \left( (w \circ g)(b^p) \right) + J_{p_-}^p \left( (w \circ g)(a^p) \right) \right] \\
- \left[ J_{p_+}^p \left( (fw \circ g)(b^p) \right) + J_{p_-}^p \left( (fw \circ g)(a^p) \right) \right] \\
= \frac{1}{p(f(ax) + f(bx))} \int_a^b \left[ J_{p_+}^p \left( (w \circ g)(s) \right) ds \right] f \left( \left( \frac{b^p - s}{x} - s \right)^{1/p} (w \circ g)(s) ds \right) dt,
\]

with $g(x) = x^{1/p}, x \in [a^p, b^p]$.
(ii) If $p < 0$,

\[
\frac{f(a) + f(b)}{2} \left[ J_{p+}^{b}(w \circ g)(a^p) + J_{p-}^{b}(w \circ g)(b^p) \right] - \left[ J_{p+}^{a}(w \circ g)(a^p) + J_{p-}^{a}(w \circ g)(b^p) \right]
\]

\[
= \frac{1}{\Gamma(z)} \int_{a}^{b} \left[ \int_{a}^{t} (a^p - s)^{z-1}(w \circ g)(s)ds \right] \left( f \circ g \right)'(t)dt.
\]

with $g(x) = x^{1/p}, x \in [b^p, a^p]$.

**Proof**

(i) Let $p > 0$. It suffices to note that

\[
I_1 = \frac{1}{\Gamma(z)} \left( f \circ g \right)(t) \left( \int_{a}^{t} (b^p - s)^{z-1}(w \circ g)(s)ds \right) \left( f \circ g \right)'(t)dt
\]

\[
- \frac{1}{\Gamma(z)} \int_{a}^{b} (b^p - t)^{z-1}(w \circ g)(t)dt
\]

\[
= f(b) \frac{1}{\Gamma(z)} \int_{a}^{b} (b^p - s)^{z-1}(w \circ g)(s)ds
\]

\[
- \frac{1}{\Gamma(z)} \int_{a}^{b} (b^p - t)^{z-1}(w \circ g)(t)dt
\]

\[
= f(b) \int_{a}^{b} J_{p+}^{b}(w \circ g)(b^p) + J_{p-}^{b}(w \circ g)(a^p)
\]

\[
- J_{p+}^{a}(w \circ g)(b^p).
\]

(18)

and similarly

\[
I_2 = \frac{1}{\Gamma(z)} \left( f \circ g \right)(t) \left( \int_{a}^{t} (s - a^p)^{z-1}(w \circ g)(s)ds \right) \left( f \circ g \right)'(t)dt
\]

\[
+ \frac{1}{\Gamma(z)} \int_{a}^{b} (t - a^p)^{z-1}(w \circ g)(t)dt
\]

\[
= - f(a) \frac{1}{\Gamma(z)} \int_{a}^{b} (s - a^p)^{z-1}(w \circ g)(s)ds
\]

\[
+ \frac{1}{\Gamma(z)} \int_{a}^{b} (t - a^p)^{z-1}(w \circ g)(t)dt
\]

\[
= - f(a) \int_{a}^{b} J_{p+}^{a}(w \circ g)(b^p) + J_{p-}^{a}(w \circ g)(a^p)
\]

\[
+ J_{p+}^{a}(w \circ g)(b^p).
\]

(19)

A combination of (18), (19) and (20) gives (16).

This completes the proof of i.

(ii) The proof is similar to i.

**Remark 2** In Lemma 2, one can see the following.

(1) If one takes $p = 1$, one has Işcan (2015), Lemma 2.4.

(2) If one takes $p = 1$ and $w(x) = 1$, one has Sarkaya et al. (2013), Lemma 2.2.

(3) If one takes $p = 1$ and $x = 1$, one has Sarkaya (2012), Lemma 2.6.

(4) If one takes $p = 1$, $x = 1$ and $w(x) = 1$, one has Dragomir and Agarwal (1998), Lemma 2.1.

(5) If one takes $p = -1$, one has Işcan et al. (2016d), Lemma 3.

(6) If one takes $p = -1$ and $w(x) = 1$, one has Işcan and Wu (2014), Lemma 3.

(7) If one takes $p = -1$, $x = 1$ and $w(x) = 1$, one has Işcan (2014d), 2.5. Lemma.

(8) If one takes $x = 1$ and $w(x) = 1$, one has Noor et al. (2016), Lemma 2.4.

**Theorem 10** Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I$ such that $f' \in L[a, b]$, where $a, b \in I$ and $a < b$. If $|f'|$ is $p$-convex function on $[a, b]$ for $p \in \mathbb{R} \setminus \{0\}$ and $z > 0, w : [a, b] \to \mathbb{R}$ is continuous and $p$-symmetric with respect to $\left[ \frac{a^p + b^p}{2} \right]^{1/p}$, then the following inequality for fractional integrals hold:

(i) If $p > 0$,

\[
\left| f(a) + f(b) \right| \left[ J_{p+}^{b}(w \circ g)(b^p) + J_{p-}^{b}(w \circ g)(a^p) \right] - \left[ J_{p+}^{a}(w \circ g)(b^p) + J_{p-}^{a}(w \circ g)(a^p) \right]
\]

\[
\leq \frac{\|w\|_{\infty} (b^p - a^p)^{z+1}}{\Gamma(z+1)} \left[ C_1(z, p)|f'(a)| + C_2(z, p)|f'(b)| \right],
\]

where

\[
C_1(z, p) = \int_{0}^{1} \frac{[(1 - u)^z - u^z]}{p|u|^p + (1 - u)|b^p|^{(z+1)/p}} \cdot |f'(u)|du
\]

\[
= \int_{0}^{1} \frac{[(1 - u)^z - u^z]}{p|u|^p + (1 - u)|b^p|^{(z+1)/p}} \cdot \left( 1 - u \right)du
\]

with $g(x) = x^{1/p}, x \in [a^p, b^p]$.

(ii) If $p < 0$,

\[
\left| f(a) + f(b) \right| \left[ J_{p+}^{b}(w \circ g)(b^p) + J_{p-}^{b}(w \circ g)(a^p) \right] - \left[ J_{p+}^{a}(w \circ g)(b^p) + J_{p-}^{a}(w \circ g)(a^p) \right]
\]

\[
\leq \frac{\|w\|_{\infty} (b^p - a^p)^{z+1}}{\Gamma(z+1)} \left[ C_3(z, p)|f'(a)| + C_4(z, p)|f'(b)| \right],
\]

where

\[
C_3(z, p) = \int_{0}^{1} \frac{[(1 - u)^z - u^z]}{p|u|^p + (1 - u)|a^p|^{(z+1)/p}} \cdot |f'(u)|du
\]

\[
= \int_{0}^{1} \frac{[(1 - u)^z - u^z]}{p|u|^p + (1 - u)|b^p|^{(z+1)/p}} \cdot \left( 1 - u \right)du
\]
\( C_3(x, p) = \int_0^1 \frac{\left[(1-u)^x - u^x\right]}{p[au^p + (1-u)b^p]^{1-(1/p)}} \, du \quad \text{and} \quad C_4(x, p) = \int_0^1 \frac{\left[(1-u)^y - u^y\right]}{p[au^p + (1-u)b^p]^{1-(1/p)}} \, du \)

with \( g(x) = x^{1/p}, \ x \in [b^p, a^p] \).

**Proof**

(i) Let \( p > 0 \). Using Lemma 2-i, it follows that

\[
\left\| \frac{f(a) + f(b)}{2} \left[ J_{w+}^+(w \circ g)(b^p) + J_{w-}^-(w \circ g)(a^p) \right] - \left[ J_{w+}^+(w \circ g)(b^p) + J_{w-}^-(w \circ g)(a^p) \right] \right\| \leq \frac{1}{\Gamma(x+1)} \int_a^b \left( \int_0^1 \left( \int_0^{b^p-x} (w \circ g)(s) \, ds \right) \right) \left( f \circ g \right)'(t) \, dt.
\]

(21)

Since \( w \) is \( p \)-symmetric with respect to \( \left[ a^p + b^p \right] / 2 \), using Definition 4 we have \( w(x^{1/p}) = w \left( \left[ a^p + b^p - x \right]^{1/p} \right) \) for all \( x \in [a^p, b^p] \):

\[
\int_{a^p}^{b^p} (b^p-x)^{y-1}(w \circ g)(s) \, ds = \int_{a^p}^{b^p} (s-a^p)^{y-1}(w \circ g)(s) \, ds
\]

\[
= \left[ \int_{a^p}^{b^p} (s-a^p)^{y-1}(w \circ g)(s) \, ds + \int_{a^p}^{b^p} (s-a^p)^{y-1}(w \circ g)(s) \, ds \right]
\]

\[
= \int_{a^p}^{b^p} (s-a^p)^{y-1}(w \circ g)(s) \, ds
\]

\[
\leq \left\{ \begin{array}{ll}
\int_{a^p}^{b^p} (s-a^p)^{y-1}(w \circ g)(s) \, ds, & t \in \left[ a^p, \frac{a^p + b^p}{2} \right] \\
\int_{a^p}^{b^p} (s-a^p)^{y-1}(w \circ g)(s) \, ds, & t \in \left[ \frac{a^p + b^p}{2}, b^p \right].
\end{array} \right.
\]

(22)

A combination of (21) and (22) gives

\[
\left\| \frac{f(a) + f(b)}{2} \left[ J_{w+}^+(w \circ g)(b^p) + J_{w-}^-(w \circ g)(a^p) \right] - \left[ J_{w+}^+(w \circ g)(b^p) + J_{w-}^-(w \circ g)(a^p) \right] \right\| \leq \frac{1}{\Gamma(x+1)} \int_a^b \left( \int_0^1 \left( \int_0^{b^p-x} (w \circ g)(s) \, ds \right) \right) \left( f \circ g \right)'(t) \, dt.
\]

(23)

Setting \( t = u a^p + (1-u)b^p \) and \( dt = (a^p - b^p) \, du \) gives
Since $|f'|$ is $p$-convex function on $[a, b]$, we have
\[
|f'[uwp + (1 - u)b^p]| \leq uf'(a) + (1 - u)f'(b).
\]
(24)

A combination of (23) and (24) gives
\[
\frac{1}{b - a} \int_a^b w(x) dx - \frac{1}{b - a} \int_a^b f(x) w(x) dx \leq |w|_\infty \frac{1}{b - a} |C_1(1, 1)f'(a)| + C_2(1, 1)|f'(b)|.
\]

This completes the proof of i. 
(ii) The proof is similar to i.

**Remark 3** In Theorem 10, one can see the following.

(1) If one takes $p = 1$, one has Işcan (2015), Theorem 2.6. 
(2) If one takes $p = 1$ and $w(x) = 1$, one has Sarıkaya et al. (2013), Theorem 3. 
(3) If one takes $p = 1$, $\alpha = 1$ and $w(x) = 1$, one has Dragomir and Agarwal (1998), Theorem 2.2. 
(4) If one takes $\alpha = 1$ and $w(x) = 1$, one has Noor et al. (2016), Theorem 3.1. 

**Corollary 1** In Theorem 10, one can see the following.

(1) If one takes $p = 1$ and $\alpha = 1$, one has the following Hermite–Hadamard–Fejér inequality for convex functions:
\[
\frac{1}{b - a} \int_a^b w(x) dx - \frac{1}{b - a} \int_a^b f(x) w(x) dx \leq |w|_\infty \frac{1}{b - a} |C_1(1, 1)f'(a)| + C_2(1, 1)|f'(b)|.
\]

(2) If one takes $p = -1$, one has the following Hermite–Hadamard–Fejér inequality for harmonically convex functions via fractional integrals:
\[
\frac{1}{b - a} \int_a^b w(x) dx - \frac{1}{b - a} \int_a^b f(x) w(x) dx \leq |w|_\infty \frac{1}{b - a} |C_1(1, 1)f'(a)| + C_2(1, 1)|f'(b)|.
\]

(3) If one takes $p = -1$, $\alpha = 1$ and $w(x) = 1$, one has the following Hermite–Hadamard inequality for harmonically convex functions:
\[
\frac{1}{b - a} \int_a^b w(x) dx - \frac{1}{b - a} \int_a^b f(x) w(x) dx \leq |w|_\infty \frac{1}{b - a} |C_3(1, 1)f'(a)| + C_4(1, 1)|f'(b)|.
\]

**Theorem 11** Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^o$ such that $f' \in L[a, b]$, where $a, b \in I$ and $a < b$. If $|f'|^q$, $q \geq 1$, is $p$-convex function on $[a, b]$ for $p \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$, $w : [a, b] \to \mathbb{R}$ is continuous and $p$-symmetric with respect to $\frac{a^p + b^p}{2}$, then the following inequality for fractional integrals holds:

(i) If $p > 0$,
\[
\frac{1}{b - a} \int_a^b w(x) dx - \frac{1}{b - a} \int_a^b f(x) w(x) dx \leq |w|_\infty \frac{1}{b - a} \left( \frac{1}{\Gamma(x + 1)} |C_1(x, p)f'(a)| + C_2(x, p)|f'(b)| \right).
\]

(ii) If $p < 0$,
\[
\frac{1}{b - a} \int_a^b w(x) dx - \frac{1}{b - a} \int_a^b f(x) w(x) dx \leq |w|_\infty \frac{1}{b - a} \left( \frac{1}{\Gamma(x + 1)} |C_1(x, p)f'(a)| + C_2(x, p)|f'(b)| \right).
\]
In Theorem 11, one can see the following.

(3) If one takes

\[ p \]

(ii) The proof is similar to i.

\[ \frac{1}{a} \left( \frac{1}{x} \right) \frac{1}{(1-u)^{x-u}} \]

where \( C_5(x,p) \) and \( C_4(x,p) \) are the same in Theorem 10.

\[ C_6(x,p) = \int_0^1 \frac{-[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \, du \]

with \( g(x) = x^{1/p} \), \( x \in [\beta^p, \alpha^p] \).

Proof

(i) Let \( p > 0 \). Using (23), power mean inequality and the \( p \)-convexity of \( |f|^q \), it follows that

\[ \left[ \frac{f(a)+f(b)}{2} \right] ^{\frac{1}{p}} \left[ \frac{J_{\beta^p}(w \circ g)(\beta^p) + J_{\alpha^p}(w \circ g)(\alpha^p)}{2} \right] - \left[ \frac{J_{\beta^p}(fw \circ g)(\beta^p) + J_{\alpha^p}(fw \circ g)(\alpha^p)}{2} \right] \]

\[ \leq \left[ \frac{\|f\|_\infty (\beta^p - \alpha^p)^{1/p}}{2} \right] \frac{\Gamma(x+1)}{\int_0^1 \left( \frac{[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \right) \, du} \]

\[ \left[ \left( \left( \frac{1}{a} \left( \frac{1}{x} \right) \frac{1}{(1-u)^{x-u}} \right) \right) \right] ^{1/(1/p)} \]

\[ \times \left[ \left( \left( \frac{1}{a} \left( \frac{1}{x} \right) \frac{1}{(1-u)^{x-u}} \right) \right) \right] ^{1/(1/p)} \]

\[ \leq \left[ \frac{\|f\|_\infty (\beta^p - \alpha^p)^{1/p}}{2} \right] \frac{\Gamma(x+1)}{\int_0^1 \left( \frac{[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \right) \, du} \]

\[ \left( \left( \left( \frac{1}{a} \left( \frac{1}{x} \right) \frac{1}{(1-u)^{x-u}} \right) \right) \right) ^{1/(1/p)} \]

\[ \times \left[ \left( \left( \frac{1}{a} \left( \frac{1}{x} \right) \frac{1}{(1-u)^{x-u}} \right) \right) \right] ^{1/(1/p)} \]

\[ = \left[ \frac{\|f\|_\infty (\beta^p - \alpha^p)^{1/p}}{2} \right] \frac{\Gamma(x+1)}{\int_0^1 \left( \frac{[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \right) \, du} \]

This completes the proof of i.

(ii) The proof is similar to i.

Remark 4 In Theorem 11, one can see the following.

(1) If one takes \( p = 1 \), one has İşcan (2015), Theorem 2.8.

(2) If one takes \( p = 1, x = 1 \) and \( w(x) = 1 \), one has Pearce and Pecaric (2000), Theorem 1.

(3) If one takes \( p = -1, x = 1 \) and \( w(x) = 1 \), one has İşcan and Wu (2014), Theorem 5.

(4) If one takes \( p = -1, x = 1 \) and \( w(x) = 1 \), one has İşcan (2014d), 2.6. Theorem.

(5) If one takes \( x = 1 \) and \( w(x) = 1 \), one has Noor et al. (2016), Theorem 3.2.

Corollary 2 In Theorem 11, one can see the following.

(1) If one takes \( p = 1 \) and \( w(x) = 1 \), one has the following Hermite–Hadamard inequality for convex functions via fractional integrals:

\[ \left[ \frac{f(a)+f(b)}{2} \right] ^{\frac{1}{p}} \left[ \frac{\Gamma(x+1)}{\int_0^1 \left( \frac{[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \right) \, du} \right] \]

\[ \leq \left[ \frac{\|f\|_\infty (\beta^p - \alpha^p)^{1/p}}{2} \right] \frac{\Gamma(x+1)}{\int_0^1 \left( \frac{[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \right) \, du} \]

\[ \left[ \left( \left( \frac{1}{a} \left( \frac{1}{x} \right) \frac{1}{(1-u)^{x-u}} \right) \right) \right] ^{1/(1/p)} \]

\[ \left( \left( \left( \frac{1}{a} \left( \frac{1}{x} \right) \frac{1}{(1-u)^{x-u}} \right) \right) \right) ^{1/(1/p)} \]

\[ = \left[ \frac{\|f\|_\infty (\beta^p - \alpha^p)^{1/p}}{2} \right] \frac{\Gamma(x+1)}{\int_0^1 \left( \frac{[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \right) \, du} \]

This completes the proof of i.

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Corollary 2 In Theorem 11, one can see the following.

(1) If one takes \( p = 1 \) and \( w(x) = 1 \), one has the following Hermite–Hadamard inequality for convex functions via fractional integrals:

\[ \left[ \frac{f(a)+f(b)}{2} \right] ^{\frac{1}{p}} \left[ \frac{\Gamma(x+1)}{\int_0^1 \left( \frac{[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \right) \, du} \right] \]

\[ \leq \left[ \frac{\|f\|_\infty (\beta^p - \alpha^p)^{1/p}}{2} \right] \frac{\Gamma(x+1)}{\int_0^1 \left( \frac{[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \right) \, du} \]

\[ \left( \left( \left( \frac{1}{a} \left( \frac{1}{x} \right) \frac{1}{(1-u)^{x-u}} \right) \right) \right) ^{1/(1/p)} \]

\[ \left( \left( \left( \frac{1}{a} \left( \frac{1}{x} \right) \frac{1}{(1-u)^{x-u}} \right) \right) \right) ^{1/(1/p)} \]

\[ = \left[ \frac{\|f\|_\infty (\beta^p - \alpha^p)^{1/p}}{2} \right] \frac{\Gamma(x+1)}{\int_0^1 \left( \frac{[(1-u)^{x-u}]}{p[\alpha u^p + (1-u)\beta]^1/(1/p)} \right) \, du} \]

This completes the proof of i.

(ii) The proof is similar to i.
If \( p > 0 \),
\[
\frac{1}{2} \left| \frac{f(a) + f(b)}{p} \left[ J_{p,w}^a (w \circ g)(b^p) + J_{p, a}^b (w \circ g)(a^p) \right] - \left[ J_{p,w}^a (f \circ w \circ g)(b^p) + J_{p, a}^b (f \circ w \circ g)(a^p) \right] \right| \\
\leq \frac{\|w\|_{\infty} (b^p - a^p)^{2+1}}{\Gamma(x+1)} \frac{C_7(x, p, r) \left( |f'(a)|^q + |f'(b)|^q \right)^\frac{1}{q}}{2} ,
\]
where
\[
C_7(x, p, r) = \int_0^1 \left( \frac{|1 - u|^{3 - u^2}}{p[a^p + (1 - u)b^p]^{1-1/p}} \right)^r du
\]
with \( g(x) = x^{1/p}, x \in [a^p, b^p] \).

If \( p < 0 \),
\[
\frac{1}{2} \bigg| \frac{f(a) + f(b)}{p} \bigg[ J_{p,w}^a (w \circ g)(b^p) + J_{p, a}^b (w \circ g)(a^p) \bigg] - \bigg[ J_{p,w}^a (f \circ w \circ g)(b^p) + J_{p, a}^b (f \circ w \circ g)(a^p) \bigg] \bigg| \\
\leq \frac{\|w\|_{\infty} (b^p - a^p)^{2+1}}{\Gamma(x+1)} \frac{C_8(x, p, r) \left( |f'(a)|^q + |f'(b)|^q \right)^\frac{1}{q}}{2} ,
\]
where
\[
C_8(x, p, r) = \int_0^1 \left( \frac{-|1 - u|^{3 - u^2}}{p[a^p + (1 - u)b^p]^{1-1/p}} \right)^r du
\]
with \( g(x) = x^{1/p}, x \in [b^p, a^p] \).

Proof

(i) Let \( p > 0 \). Using (23), Hölder's inequality and the \( p \)-convexity of \( |f'|^q \), it follows that
\[
\frac{1}{2} \left| \frac{f(a) + f(b)}{p} \left[ J_{p,w}^a (w \circ g)(b^p) + J_{p, a}^b (w \circ g)(a^p) \right] - \left[ J_{p,w}^a (f \circ w \circ g)(b^p) + J_{p, a}^b (f \circ w \circ g)(a^p) \right] \right| \\
\leq \frac{\|w\|_{\infty} (b^p - a^p)^{2+1}}{\Gamma(x+1)} \int_0^1 \frac{|(1 - u)^{3 - u^2}|}{p[a^p + (1 - u)b^p]^{1-1/p}} \left( |a^p + (1 - u)b^p|^{1/p} \right)^{1/r} \left( |f'(a)|^q + |f'(b)|^q \right)^\frac{1}{q} du \\
\times \left( \int_0^1 \left( \frac{|(1 - u)^{3 - u^2}|}{p[a^p + (1 - u)b^p]^{1-1/p}} \right)^r \left( |a^p + (1 - u)b^p|^{1/p} \right)^{1/r} du \right)^\frac{1}{r} \\
\times \left( \int_0^1 \left( |a^p + (1 - u)b^p|^{1/p} \right)^q du \right)^\frac{1}{q} ,
\]

(ii) The proof is similar to i.

Remark 5 In Theorem 12, one can see the following.

(1) If one takes \( p = 1 \), one has İşcan (2015), Theorem 2.9-i.

(2) If one takes \( p = 1, x = 1 \) and \( w(x) = 1 \), one has Dragomir and Agarwal (1998), Theorem 2.3.

(3) If one takes \( p = 1 \) and \( x = 1 \), one has Sarker (2012), Theorem 2.8.

Corollary 3 In Theorem 12, one can see the following.

(1) If one takes \( p = 1 \) and \( w(x) = 1 \), one has the following Hermite–Hadamard inequality for convex functions via fractional integrals:
\[
\frac{f(a) + f(b)}{2} - \frac{\Gamma(x + 1)}{2(b - a)^x} \left[ J_{x,w}^a f(b) - J_{x, a}^b f(a) \right] \\
\leq \frac{b - a}{2} C_7(x, 1, r) \left( |f'(a)|^q + |f'(b)|^q \right)^\frac{1}{q} .
\]

(2) If one takes \( p = -1 \) and \( w(x) = 1 \), one has the following Hermite–Hadamard inequality for harmonically convex functions via fractional integrals:
In Theorem 9, Hermite–Hadamard–Fejér type inequalities for p-convex functions in fractional integral forms are obtained. In Corollaries 1, 2 and 3, some Hermite–Hadamard and Hermite–Hadamard–Fejér inequalities for convex, harmonically convex and p-convex functions are given. Some results presented in Remarks 3, 4 and 5 provide extensions of others given in earlier works for convex, harmonically convex and p-convex functions.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no competing interests.

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