Deconfined Thermal Phase Transitions with $Z_2$ Gauge Structures

Eun-Gook Moon
Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea
(Dated: December 17, 2018)

Introduction: Emergence of exotic excitations out of conventional electrons and spins is one of the striking characteristics of deconfined phases in strongly correlated systems [1][2]. Prime examples include Majorana fermions from localized spins in quantum spin liquids [3][6]. Physical properties of the emergent particles are investigated, topological order beyond the Landau-Ginzburg-Wilson (LGW) paradigm is introduced, and unexpected duality relations between low energy theories around quantum phase transitions are unveiled.

Previous theoretical researches have mainly focused on quantum natures of deconfined phases including unconventional quantum phase transitions [7][21]. Properties of emergent particles are investigated, topological order beyond the Landau-Ginzburg-Wilson (LGW) paradigm is introduced, and unexpected duality relations between low energy theories around quantum phase transitions are unveiled.

In this work, we consider thermal transitions associated with deconfined phases in two (2d) and three spatial dimensions (3d) and demonstrate that deconfined phases even host exotic thermal phase transitions, named deconfined thermal transitions. One seminal work was done by Wegner who showed the presence of a thermal phase transition between confined and deconfined phases with the pure $Z_2$ lattice gauge model in 3d [22]. The transition has the aspects of a 'hidden order' transition because symmetry order parameters are absent in contrast to the conventional LGW paradigm. Below, we prove the presence of a thermal phase transition between confined and deconfined phases in 2d by constructing and analyzing a lattice model. Incorporating deconfined fermions, we also show that a thermal phase transition between confined and deconfined metals may exist.

We are partly motivated by recent experiments in strongly correlated systems including exotic onset behaviors of order parameters in cuprates, iridates, and heavy fermions [23][26], which may be interpreted as phenomena beyond the LGW paradigm. Inspired by these studies, novel universality class of thermal transitions breaking $Z_2$ and $U(1)$ symmetries in deconfined phases are obtained. Evaluating all critical exponents, we provide characteristic signatures of the transitions and candidate strongly correlated systems in experiments.

Pure $Z_2$ Lattice Gauge Theory: We first recall the Wegner model [22],

$$H_W = -g \sum_i \prod_{(ab) \in \square} \sigma_{ab} \equiv -g \sum_i F_i,$$

with $Z_2$ variables $\{\sigma_{ij} = \pm 1\}$ on the link $(ij)$ between $i$ and $j$ sites. The $\square$ is for a plaquette whose position is specified by the dual site index $i^*$. The gauge flux operator $F_i^*$ is introduced with a positive $g$. The Hamiltonian $H_W$ has two phases in 3d which was shown by mapping the model to the classical Ising model with the Kramer-Wannier duality transformation. At low temperatures ($g \gg T$), $Z_2$ gauge fluxes are frozen with the perimeter law of the Wilson-loop operator, but at high temperatures ($g \ll T$), $Z_2$ gauge fluxes are proliferated with the area law of the Wilson-loop operator, which are called deconfined and confined phases respectively. Starting from the deconfined phase, it is useful to consider topological defects, $Z_2$ gauge flux loops, whose free energy is simply estimated to be $f_{3d}(l) \sim g l - T \log(l)$ with the length of the loop $l$ up to unimportant numerical factors. The transition temperature is an order of $T_c \sim g$.

The absence of a deconfined phase in 2d may be understood by a similar estimation. A topological defect is not a loop but a point in 2d, so the energy of a defect costs $2g$. The free energy of the two fluxes may be estimated as $f_{2d}(l_{2F}) \sim 4g - T \log(l_{2F})$ with the distance between...
the two fluxes, $l_{2F}$, up to numerical factors. Therefore, at any non-zero temperatures, the entropic contribution wins and the gauge fluxes prefer to be proliferated, which disallows deconfined phases. Note that the same argument applies to the absence of superconductivity in 2d gauge fluxes.

Explicit analysis on the model with a gauge choice, $\sigma^x_{i,\hat{x}} = 1$, confirms the estimation. One can define a “spin” variable $S_i \equiv \sigma^x_{i,\hat{x}+\hat{y}}$, and the flux operator becomes $F_{i} = S_i S_{i+\hat{x}}$. The unit vectors $(\hat{x}, \hat{y})$ on a square lattice are introduced. The Wegner model becomes equivalent to the decoupled set of the one dimensional (1d) Ising models,

$$H_W = -g \sum_i S_i S_{i+\hat{x}}.$$ 

and thus a thermal transition is absent at non-zero temperatures. A domain-wall in each spin chain costs finite energy, so its entropic contribution always wins proliferating domain-walls.

We consider a lattice model of the spins,

$$H_X = -g \sum_i S_i S_{i+\hat{x}} - J_r \sum_i \sum_{r=1}^{\infty} S_i S_{i+r\hat{x}}. \quad (2)$$

The term with $J_r$ describes the decoupled set of the Dyson-Ising chains. The existence of a thermal phase transitions in the Dyson-Ising model were proven for $1 < \omega \leq 2$ and the universality class is reported to be in the mean field class for $1 < \omega < 3/2$. The long-range interaction makes the domain-wall energy size-dependent, so its free energy becomes

$$f(l) \sim J_r l^{2-\omega} - T \log(l).$$

At low temperatures, the flux is frozen, formally $\langle S_i \rangle \neq 0$, and the perimeter law of a thermal deconfined phase manifests. In terms of gauge variables, the model becomes

$$H_X = -g \sum_i F_{r\hat{x}} - J_r \sum_i \sum_{r=1}^{\infty} \prod_{a=0}^{r-1} F_{r+a\hat{x}}. \quad (3)$$

We generalize the model to

$$H_Z = -\sum_i \sum_{r=1}^{\infty} J(r) \prod_{a=0}^{r-1} F_{r+a\hat{x}}. \quad (4)$$

Defining the two constants $M_0 = \sum_{r=1}^{\infty} J(r)$ and

$$K'_1 = \sum_{r=1}^{\infty} (\log \log(r+4)) r^3 J(r) \sim 0,$$

we can map $H_Z$ to the decoupled set of the Dyson-Ising chains. Applying the Dyson theorem, the existence of a deconfined thermal phase and its transition to a confined phase are proven for finite $M_0$ and $K'_1$.

We stress that the long-range interaction between the gauge fluxes is the impetus of a thermal deconfined phase in 2d. The introduction of the long-range interaction indicates the absence of the Lorentz symmetry at low energy. Also, our model breaks a rotational symmetry by picking up the $\hat{x}$ direction, yet we believe that the symmetrized model, $H_Z = (H_Z(\hat{x}) + H_Z(\hat{y})) / 2$, has the presence of the deconfined phase. More realistic spin models to realize the thermal deconfined phases with numerical analysis will be presented in future works. The Wegner model is a non-interacting theory of gauge fluxes up to the gauge constraint, and our model shows that a strongly interacting gauge flux theory realizes a deconfined phase even in 2d.

Models with deconfined fermions: We extend the model by incorporating fermions ($f_i$) and Ising spins ($s_i = \pm 1$) on the sites of a hyper-cubic lattice,

$$H_{\sigma_f} = H_Z - J_F \sum_{(i,j)} \sigma_{ij} s_i s_j - t \sum_{(i,j)} \sigma_{ij} f_i^\dagger f_j + V_f.$$ 

The $Z_2$ gauge structure is manifested by a gauge transformation, $\sigma_{ij} \rightarrow \sigma_{ij} \eta_i \eta_j$, $s_i \rightarrow \eta_i s_i$, and $f_{ia} \rightarrow f_{ia} \eta_i$ with $\eta_i = \pm 1$. A gauge invariant potential of fermions, $V_f$, is introduced. The model with $t = 0$ and $J(r) = g \delta_{r=1}$ was considered by Fradkin and Shenker, showing the equivalence between the Higgs phase of $s_i$ and the confined phase in 3d.

Quantum mechanical analysis is necessary for fermions and one can treat $\{\sigma_{ij}, s_i\}$ as static background fields. In the deconfined phases in 2d and 3d, one can safely ignore the Ising spins, and the ground state energy with the zero gauge flux $\{\sigma_0^{i,j}\}$ is obtained by diagonalizing the fermion Hamiltonian,

$$H_f(\{\sigma_0^{i,j}\}) = -t \sum_{(i,j)} \sigma_0^{i,j} f_i^\dagger f_j + V_f,$$

and filling up fermions to a chemical potential. The Hamiltonian with the two gauge fluxes $\{\sigma_2^{i,j}\}$ is

$$H_f(\{\sigma_2^{i,j}\}) = H_f(\{\sigma_0^{i,j}\}) - t \sum_{(i,j)} (\sigma_2^{i,j} - \sigma_0^{i,j}) f_i^\dagger f_j + V_f.$$

It is convenient to choose the gauge $\{\sigma_0^{i,j}\} = 1$ on every link and $\{\sigma_2^{i,j}\}$ differs from $\{\sigma_0^{i,j}\}$ only in the interconnecting line between the two fluxes. Without loss of generality, we may assume that the two fluxes are separated along the $x$ direction whose distance is $l_{2F}$. It is useful to notice that the second term on the right hand side is a perturbation to the first term and one can perform the perturbative calculation with a small parameter $l_{2F}/N_{size}$ with the system size $N_{size}$ setting a lattice constant as a unit. For simplicity, let us consider the non-interacting limit $V_f \rightarrow 0$ and diagonalize the Hamiltonian exactly, and the energy difference at the leading order is

$$E_{f_0}(\{\sigma_2^{i,j}\}) - E_{f_0}(\{\sigma_0^{i,j}\}) = \left[ \frac{t}{N_{site}} \sum_q n_F(q) (2 \cos(q) \eta_q) \right] l_{2F},$$

with the Fermi-Dirac function $n_F(q)$ in 2d. Its 3d gen-

...
eralization is straightforward. We may assign the right hand side to the tension energy between the two fluxes. It is easy to show that the second term is positive because the summation range is determined by the sign of $t$. Since the line-tension only depends on the particle number and quasi-particle dispersion relations, we believe the calculation is perturbatively safe.

Our calculations indicate that the Fermi surfaces may be a natural source to stabilize thermal deconfined phases. In 3d, the line-tension from Fermi surfaces add to the intrinsic line-tension with $g$. In 2d, the presence of the line-tension from Fermi surfaces indicates a deconfined thermal phase in 2d even for the Wegner model $(J(r) = g\delta_{r=1})$. Strictly speaking, our line-tension calculations are done at $T = 0$, yet it is tempting to conclude the line-tension survives at non-zero temperatures. Physically, one can understand that gapless excitations from the Fermi surface induce a long-range interaction between the fluxes. Note that the interaction between the fluxes has a formal similarity to the Ruderman-Kittel-Kasuya-Yoshida interaction [33] and a power counting of the interaction shows that the Fermi surfaces induce a similar type of the long-range interaction, which is desired to be checked by numerical and analytical calculations in future works.

At high temperatures, entropic contribution of topological defects dominate and a confined phase appears which may be understood as a Higgs phase of the Ising spin. Gauge-neutral fermionic operators, $c_{i\alpha} \equiv s_{i\beta} f_{\alpha}$, become good degrees of freedom, where Fermi liquids of $c$ fermions are expected. We call it a confined metal distinguished from a deconfined metal of $f$ fermions at low temperatures.

Finally, we comment on the critical theory under the fermionic fluctuations. The critical theory in 2d will be discussed in a future work, and in 3d, without fermions, the transition of the Wegner model is the dual-Ising class with the dual Ising variable $\zeta(x)$ which is a trivial representation of all symmetries. The variable is coupled to the number density of fermions at the lowest order [10], and the critical theory is

$$\begin{align*}
S_c = \int d^3x \left[ \frac{1}{2} (\nabla \zeta(x))^2 + \frac{r}{2} \zeta(x)^2 + \frac{\lambda}{4!} \zeta(x)^4 \right] \\
+ \int d^3x d\tau \left[ f^\dagger(x,\tau) \partial_\tau f(x,\tau) + H_f(x,\tau) \right] \\
+ \int d^3x d\tau (-g_2 \zeta(x)^2 - g_4 \zeta(x)^4) n_f(x,\tau) + \cdots
\end{align*}$$

with $n_f(x,\tau) \equiv f^\dagger(x,\tau) f(x,\tau)$ and the fermion Hamiltonian density, $H_f$. The tuning parameter has the temperature dependence, $r \propto T_s - T$, with the transition temperature $T_s$. In contrast to the dual Ising variable, fermions explicitly depend on imaginary time reflecting their quantum nature as usual. Defining the density fluctuation, $\delta n_f(x,\tau) \equiv n_f(x,\tau) - \bar{n}_f$ with $\bar{n}_f \equiv \frac{N_f}{N_\text{site}}$ and the total fermion number $N_f$, we integrate out the fermions. Evaluating the Yukawa term over the fermion path-integral,

$$\langle e^\frac{\mathcal{L} f}{\mathcal{L}_\text{Yukawa}} \rangle_f,$$

coupling constants are modified as

$$r \rightarrow r - 2g_2 \bar{n}_f, \quad \lambda \rightarrow \lambda - 4g_4 \bar{n}_f,$$

(see SI). We use $\delta n_f(x,\Omega_n = 0) = \int d\tau \delta n_f(x,\tau)$.

The critical exponents are for the universality classes in 3d. The first row is for the critical exponent class of the Ising model, and the second row is for the critical exponent class of the XY model. The third and fourth rows are for the critical exponent class of the Ising model and the XY model, respectively. Note that the critical exponents of the deconfined thermal phase transitions with $Z_N$ and $U(1)$ symmetries, respectively. This may be observed in angle-resolved-photo-emission-spectroscopy (ARPES) experiments. The dynamic critical exponent is expected to be $z = 1$ for the dual Ising field if there are no other dynamic channels. Thus, specific heat and ARPES have definite signatures of the transition at $T_s$ while other static experiments with charge or spin degrees of freedom are featureless.

**Symmetry Breaking Transitions:**

| Univ. class in 3d | $\alpha$ | $\beta$ | $\gamma$ | $\nu$ | $\eta$ | $\delta$ |
|------------------|---------|---------|---------|-------|-------|---------|
| $Z_2$ (Ising)    | 0.11    | 0.33    | 1.24    | 0.63  | 0.036 | 4.79    |
| $U(1)$ (XY)     | -0.015  | 0.35    | 1.32    | 0.67  | 0.038 | 4.78    |
| Mean-field      | 0       | 0.5     | 1       | 0.5   | 0     | 3       |
| DC-Z$_N$/DC-U(1) | -0.015  | 0.83    | 0.35    | 0.67  | 1.47  | 1.43    |
transitions in deconfined phases may be different from the ones in confined phases. Since fermionic fluctuations and bosonic ones with non-zero Matsubara frequencies are irrelevant to a critical theory [35], one can focus on the static component of the bosonic fluctuations.

To be specific, let us consider the case with a global symmetry \( Z_N \). One conventional way to represent \( Z_N \) is to introduce an angle variable \( (\theta_i) \) of the \( 2\pi \) periodicity with the potential term, \( V_i(\{\theta_i\}) = -u \sum_i \cos(N\theta_i) \). For \( u > 0 \), the \( N \) minimal configurations are \( \theta_i = \frac{2\pi n_i}{N} \) with \( n = 0, \ldots, N-1 \), and the order parameter is \( \Phi_i \propto (\cos(\theta_i), \sin(\theta_i)) \). The Landau theory with the \( Z_N \) symmetry is

\[
\Phi_i \propto (\cos(\theta_i), \sin(\theta_i)).
\]

The tuning parameter critical exponent (\( \nu \)) is obtained from \( \langle \bar{\phi}^2 \rangle \), which is known as \( \nu = 0.67155 \) [37]. The scaling dimension of the order parameter is known to be \( \nu = 0.67155 \) [37]. The scaling relation of the order parameter is known to be \( \bar{\phi} = 1.24 \) [37], and thus the order parameter onset exponent \( \beta = 0.83 \). With two independent exponents and the scaling relations, we find all the critical exponents summarized in Table I. Note that the negative value of \( \alpha \) indicates that the Harris criteria is valid under quenched disorder.

Few remarks are as follows. Generalization to other symmetry groups is straightforward similar to a nematic phase adjacent to a deconfined phase [38]. In 2d at \( T = 0 \), the enlarged periodicity has also been discussed in the context of quantum phase transitions in frustrated quantum magnets [39]. The quantum-classical mapping connects the DC-U(1) class with quantum XY* class [14] [39]. The quantum-classical mapping connects the DC-U(1) class with quantum XY* class, which describes a subset of spin-exchange interactions in frustrated quantum magnets.

Discussion and Conclusion: Let us consider a microscopic model of electrons and spin (c\(_{\alpha}\) and order parameters (\( \bar{\mu} \))), which we may treat as the confined degrees of freedom, due to the gauge transformation with \( \Phi \rightarrow \Phi + 2\pi \) and \( \sigma_{ij} \rightarrow -\sigma_{ij} \) for all links at i site. The confined phase \((g/T \ll 1)\) may be studied by using the high temperature expansion, and one can obtain \( F_I \) with higher order terms by selecting gauge-invariant terms.

In the deconfined phase \((g/T \gg 1)\) (or \( J_s/T \gg 1 \) in 2d), the gauge flux is frozen with \( \langle \sigma_{ij}^0 \rangle = 1 \), and the effective Hamiltonian becomes

\[
H_s = -J \sum \sigma_{ij} \cos\left(\frac{\theta_i - \theta_j}{2}\right) - u \sum \cos(2\theta_i) - g \sum F_i.
\]

The local gauge transformation includes \( \theta_i \rightarrow \theta_i + 2\pi \) and \( \sigma_{ij} \rightarrow -\sigma_{ij} \) for all links at i site. The confined phase \((g/T \ll 1)\) may be studied by using the high temperature expansion, and one can obtain \( F_I \) with higher order terms by selecting gauge-invariant terms.

Introducing the half angle variable \( \vartheta_i = \frac{\theta_i}{2} \), the Hamiltonian and order parameter are rewritten as

\[
H_s = -J \sum \cos(\vartheta_i - \vartheta_j) - u \sum \cos(2\vartheta_i)
\]

and \( \Phi_i \propto (\cos(2\vartheta_i), \sin(2\vartheta_i)) \). Note that the half angle operator \( \langle \Phi \rangle \) carries the \( Z_2 \) gauge charge, so it vanishes by definition. Thus, the \( Z_N \) symmetry breaking transition in the deconfined phase is described by the \( Z_{2N} \) clock model whose order parameter \( \langle \langle \Phi_i \rangle \rangle \) is a secondary operator of \( \vartheta_i \).

In 2d, this is precisely mapped to the recent proposal of the inverted clock model universality class with central charge one even for \( Z_2 \) symmetry breaking transitions [39]. Its critical theory in 3d may be conveniently expressed by introducing a gauge charged field, \( \bar{\Phi} = (\phi_x, \phi_y) = \rho_0 (\cos(\theta), \sin(\theta)) \) restoring the amplitude mode \( \rho_0 \),

\[
S_{DC} = \int d^3x (\nabla \bar{\Phi})^2 + r(\bar{\Phi})^2 + \frac{\lambda}{4} ((\bar{\Phi})^2)^2 - \bar{\phi}(\phi_x^2 - \phi_y^2)^N.
\]

The gauge-neutral order parameter is \( \Phi = (\phi_x^2 - \phi_y^2, 2\phi_x \phi_y) \) since \( \bar{\Phi} \rightarrow -\bar{\Phi} \) under the gauge transformation. The anisotropy term with \( \bar{u} \) is well understood in literature, which is irrelevant for \( N \geq 2 \) to the \( U(1) \) fixed point [37]. In other words, the universality classes of \( Z_N \) and \( U(1) \) symmetry breaking transitions are the same. The tuning parameter critical exponent (\( \nu \)) is obtained from \( \langle \bar{\phi}^2 \rangle \), which is known as \( \nu = 0.67155 \) [37]. The scaling relation of the order parameter is known to be \( \bar{\phi} = 1.24 \) [37], and thus the order parameter onset exponent \( \beta = 0.83 \). With two independent exponents and the scaling relations, we find all the critical exponents summarized in Table I. Note that the negative value of \( \alpha \) indicates that the Harris criteria is valid under quenched disorder.
FIG. 1. Schematic phase diagrams of deconfined thermal phase transitions in 3d. (a) Phase transition between confined and deconfined metals at $T_*$ without breaking any symmetries. At $T_*$, specific heat shows singular behavior (inset). (b) Symmetry breaking at $T_*$ in the deconfined phase below $T_*$: Its universality class may be beyond the Landau-Ginzburg-Wilson paradigm. (c) Specific heat $C_v(T) = aT + C_{sing}(T)$ for different universality classes in 3d. The constant $a$ is for background metallic contributions. The singular contribution $C_{sing}(T) \sim (T - T_*)^{-\alpha}$ characterizes the universality classes. $Z_2$ (dashed blue) and $U(1)$ (dotted green) lines are for the conventional phase transitions, and DC-$Z_2$ (red plain) is for the universality class beyond the LGW paradigm. (d) The order parameter $\langle \Phi \rangle$ onsets below $T_*$: We find $\beta = 0.83$ for the DC-$Z_2$ and $U(1)$ (1) classes have $\beta_{Z_2} = 0.33$ and $\beta_{U(1)} = 0.35$ respectively.

In conclusion, deconfined thermal phase transitions are demonstrated. We prove the existence of a thermal phase transition in 2d with $Z_2$ gauge fields and thermal phase transitions between confined and deconfined metals are illustrated. Moreover, unconventional symmetry breaking transitions in confined phases are presented. Namely, the $Z_2$ and $U(1)$ symmetry breaking transitions in 3d are in the same universality class which is impossible under the LGW paradigm. Our results may be generalized and applied to other topological phases such as exotic phases with fracton excitations. Future studies on numerical tests incorporating quantum fluctuations of the $Z_2$ gauge fields would be useful, and detailed studies on relations with microscopic models and experiments such as doped Kitaev materials and heavy fermions are highly desired.

Acknowledgement: We thank Eduardo Fradkin, Yong Baek Kim, Sungjay Lee, and Cenke Xu for invaluable discussions and critical comments. This work was supported by the POSCO Science Fellowship of POSCO TJ Park Foundation and NRF of Korea under Grant No. 2017R1C1B2009176.
Comments on $H_Z$

We make comments on the generalized model,

$$H_Z = -\sum_i \sum_{r=1}^{r-1} J(r) \prod_{a=0}^{r-1} F_{i^r+a\xi}$$

with $M_0 \equiv \sum_{r=1}^{\infty} J(r)$ and $K_r' = \sum_{r=1}^{\infty} (\log \log(r + 4))[r^3 J(r)]^{-1}$ for $J(r) \geq 0$. The model has a thermal phase transition for $1 < \omega \leq 2$. One can apply the Dyson’s theorem for $1 < \omega < 2$ but for $\omega = 2$, the existence is out of the Dyson’s theorem. Yet, the $\omega = 2$ case is also proven [30]. The constant infinite range interaction ($\omega = 0$) does not belong to the phase transition criteria because the domain-wall energy diverges in the thermodynamic limit. For $\omega > 2$, the domain-wall energy becomes finite, so the model becomes adiabatically connected to the Wegner model.

We note that the specific form of the interaction in $H_Z$ is used to prove the existence of a deconfined thermal phase and its transition. It is highly desired to find a simpler model with a short range interaction, for example

$$H_{mZ} = -g \sum_i F_i - \sum_{\langle i,j \rangle} S_{ij} S_{ij}$$

with a short range interaction $S_{ij}$, which will be discussed in future works.

Thermal phase transition of $H_Z$ with Ising matter field

Let us consider the model,

$$H = -\sum_i \sum_{r=1}^{r-1} J(r) \prod_{a=0}^{r-1} F_{i^r+a\xi} - J_{FS} \sum_{\langle i,j \rangle} \sigma_{ij} \phi_i \phi_j$$

$$= -J_r \sum_i \sum_{r=1}^{\infty} S_i S_{i^r+x^\xi} r^{\omega} - J_{FS} \sum_i (\phi_i \phi_{i+x} + S_i \phi_i \phi_{i+y}).$$

In the second line, we choose the gauge $\sigma_{ii+x} = 1$ and $\sigma_{ii+y} = S_i$, and the Hamiltonian is described by the two types of spins $S_i = \pm 1$ and $\phi_i = \pm 1$.

Starting with the presence of the phase transition with $J_{FS} = 0$, we investigate effects of $J_{FS}$ perturbatively. Let us consider the regime $J_r, T \gg J_{FS}$ where the high temperature expansion with $J_{FS}/T$ is possible. The partition function is

$$Z = \text{Tr}(e^{-H/T})$$

$$= \sum_{\{S_i\}, \{\phi_i\}} \prod_i e^{-\sum_{r=1}^{\infty} \frac{J_r}{r} S_i S_{i^r+x^\xi} e^{\frac{J_{FS}}{T} \phi_i \phi_{i+x}} e^{\frac{J_{FS}}{T} S_i \phi_i \phi_{i+y}}}$$

$$\equiv \sum_{\{S_i\}} \prod_i e^{-\sum_{r=1}^{\infty} \frac{J_r}{r} S_i S_{i^r+x^\xi} G(\{S_i\}).}$$

(8)
The function \( G(\{S_i\}) \) may be obtained by
\[
G(\{S_i\}) = \sum_{\{\phi_i\}} \prod_{\langle i, j \rangle} e^{\frac{J_{FS} \phi_i \phi_j}{T}} \left( 1 + \tanh(\frac{J_{FS}}{T}) S_i \phi_i \phi_j \right).
\]
\[
= 1 + \sum_{\sigma_i} a_{\sigma_i} S_i + \sum_{\sigma_i \sigma_j} b_{\sigma_i \sigma_j} S_i S_j S_k + \cdots
\]
up to unimportant constants. By using the Taylor expansion, one can show the coefficients have the exponential behaviors,
\[
a(r) \propto (\frac{J_{FS}}{T})^r e^{-r/\xi_{FS}}, \quad \xi_{FS}^{-1} = \log(\frac{T}{J_{FS}}),
\]
and the higher order terms also show the exponential decay, and the correlation length is tiny for \( J_{FS}/T \ll 1 \).
Assuming a second order transition for \( J_{FS} = 0 \), we may use the argument by Fradkin and Shenker [32]. Namely, the short-range interactions with \( J_{FS} \neq 0 \) are unable to destabilize the presence of the transition. We argue that the transition is perturbatively stable.

The self-consistency may be checked as follows. At the temperature regime \( J_r \gg T \gg J_{FS} \), one may set \( \langle S_i \rangle \neq 0 \) formally. By using a gauge transformation, one can make the expectation value uniform positive. Then, the effective Hamiltonian of \( \phi_i \) becomes the anisotropic Ising model whose critical temperature is determined by
\[
\sinh(\frac{2J_{FS}}{T_c}) \sinh(\frac{2J_{FS} \langle S_i \rangle}{T_c}) = 1.
\]
This demonstrates the stability of the flux frozen thermal phase for \( J_r \gg T \gg J_{FS} \). At high temperatures \( T \gg J_r \gg J_{FS} \), the flux becomes proliferated, so there is a phase transition between the flux frozen and flux proliferated phases.

Note that the matter field carries the \( Z_2 \) electric charge but not the dual \( Z_2 \) magnetic charge. Thus, the phase transition with the \( Z_2 \) electric charged particles do not qualitatively modify the thermal phase transition [9].

**Line-tension calculation**

In this section, we provide details of the line-tension calculation between the two fluxes. Without loss of generality, we assume that the two fluxes are separated along the \( x \) direction as in Fig. 2. The fermionic Hamiltonian with the two fluxes (\( \{ \sigma_{ij}^{2F} \} \)) may be rewritten as
\[
H_f(\{ \sigma_{ij}^{2F} \}) = H_f(\{ \sigma_{ij}^0 \}) - t \sum_{\langle ij \rangle} (\sigma_{ij}^{2F} - \sigma_{ij}^0) f_i^\dagger f_j. \quad (9)
\]
We fix the gauge choice here after and the zero flux notation (\( \langle \sigma_{ij}^0 = 1 \rangle \)) is used. The second term is for the two flux states separated by the length \( l_{2F} \). The notation \( \langle ij \rangle \in l_{2F} \) accounts for the interconnecting links (links with crosses). For simplicity, we set \( V_f = 0 \), which allows us to perform full analytic calculations. It is obvious that the second term on the right hand side is a perturbation to the first term. Introducing the Fourier transformation of the fermion variables,
\[
f_j = \frac{1}{\sqrt{N_{\text{site}}}} \sum_q f_q e^{iqx},
\]
the second term becomes
\[
2t \sum_{\langle ij \rangle \in l_{2F}} f_i^\dagger f_j = \frac{2t}{N_{\text{site}}} \sum_{k,q} f_k^\dagger f_q \sum_{j=1}^{l_{2F}} \cos(k x_j - q x_j + q_y).
\]
The ground state of the zero flux state \( |G\rangle = \prod |k\rangle \) may be used to determine the estimation of the energy with the two fluxes, which becomes
\[
E_0(\{ \sigma_{ij}^{2F} \}) = E_0(\{ \sigma_{ij}^0 \}) + 4g + 2t \frac{l_{2F}}{N_{\text{site}}} \sum_q n_F(q) \cos(q_y)
\]
with the Fermi-Dirac function \( n_F(q) \). Note that the positive sign of \( t \) makes the summation range \( \sum_{|q| < k_F} \) from the Fermi-Dirac function, and thus the second term is always positive. The \( k_F \) is determined by the particle number density of fermions, and a filled band vanishes the summation \( \sum_{|q| < k_F} \cos(q_y) \). If we use the conventional Sommerfeld expansion, then the free-energy of the two fluxes with \( l \) may be estimated as
\[
F_{2F} \sim \rho(k_F) t l - T \log(l), \quad (10)
\]
where \( \rho(k_F) \) is for the summation of \( n_F(q) \cos(q_y) \) which is non-zero in the presence of the Fermi surfaces. The transition temperature would be estimated as \( T_* \sim \rho(k_{2F}) t \).
Evaluation of the Yukawa coupling

The Yukawa type interaction term in the partition function is

$$\langle e^{-\int d^3x(g_2\zeta(x)z^2 + g_4\zeta(x)^4)}\delta n_f(x,\Omega_n=0)\rangle_f$$

$$= \sum_m \frac{1}{m} \langle \int d^3x(g_2\zeta(x)^2 + g_4\zeta(x)^4)\delta n_f(x,\Omega_n=0) \rangle_f^m$$

$$= 1 + \int d^3x(g_2\zeta^2(x) + g_4\zeta^4(x))\langle \delta n_f(x,\Omega_n=0) \rangle_f$$

$$+ \frac{g_2^2}{2} \int d^3x d^3y \zeta(x)^2\zeta(y)^2\langle \delta n_f(x,\Omega_n=0)\delta n_f(y,\Omega_n=0) \rangle_f + \cdots \tag{11}$$

The fermionic correlation functions may be easily obtained,

$$\langle \delta n_f(x,\Omega_n=0) \rangle_f = \int d\tau \langle \delta n_f(x,\tau) \rangle_f = 0,$$

assuming the fermionic ground state does not break translational symmetry. In the term with $g_2$, we need the contact term,

$$\langle \delta n_f(x,\Omega_n=0)\delta n_f(y,\Omega_n=0) \rangle_f = c\delta^3(x-y) + \cdots,$$

and one can determine $c$ by evaluating

$$c = \int d^3x \langle \delta n_f(x,\Omega_n=0)\delta n_f(0,\Omega_n=0) \rangle_f$$

$$= \int d^3x \int d\tau_1 \int d\tau_2 \langle \delta n_f(x,\tau_1)\delta n_f(0,\tau_2) \rangle_f = 0.$$

Thus, the corrections from thermal fermionic excitations are

$$r \to r - g_2\frac{\bar{n}_f}{T}, \quad \lambda \to \lambda - g_4\frac{\bar{n}_f}{T} \quad \tag{12}$$

There are no additional singular channels from thermal fermionic excitations. For large enough $g_2$, the self-interacting term becomes negative signaling a first order transition. Thus, the phase transition is either the dual Ising class one or a first order transition.

We also remark an extension of the the semi-quantum theory to the quantum mechanical one as

$$S_{eff} = \int d^3x d\tau \left[ \frac{1}{2} (\partial_\tau \zeta)^2 + \frac{1}{2} (\nabla \zeta)^2 + \frac{r}{2} (\zeta)^2 + \frac{\lambda}{4!} \zeta^4 ight.$$  

$$+ f_\alpha^4 (\partial_\tau + \epsilon_f(-i\nabla)) f_\alpha + g_2 \zeta^2 f_\alpha^4 f_\alpha + g_4 \zeta^4 f_\alpha^4 f_\alpha \right]$$

The boson field $\zeta(x,\tau)$ describes the long wave length fluctuations of the dual Ising field. The linear time derivative term $\zeta \partial_\tau \zeta$ vanishes, so the bare dynamical critical exponent is $z = 1$. Note that the lowest order coupling is $\zeta^2 f_\alpha^4 f_\alpha$, and thus the fermion fluctuations do not modify the dynamical critical exponent.

Implications to the hidden order phase in URu$_2$Si$_2$

Our deconfined thermal transitions in metals do not require any broken symmetries though specific heat experiments show singular temperature dependences of either continuous dual Ising or discontinuous transitions. It is tempting to apply our theories to mysterious hidden order transitions of URu$_2$Si$_2$, which has been investigated by a number of the proposed theories. In contrast to the previous theories, our transitions are intrinsically independent of symmetries, and therefore it is impossible to measure with experimental probes of broken symmetries. Note that some recent experiments, on the other hand, report rotational symmetry breaking from the tetragonal symmetry down to the orthorhombic one in URu$_2$Si$_2$ at the hidden order temperature.

We propose the presence of the two transitions, a symmetric deconfined transition at the hidden order temperature and the rotational symmetry breaking transition at lower temperature, to explain the rotational symmetry breaking. The pattern of the rotational symmetry breaking is in the $Z_2$ class from the tetragonal symmetry to the orthorhombic one, so we can use the enlarged universality class with $\alpha < 0$ and $\beta = 0.83$. The negative value of $\alpha$ indicates that it may be difficult for specific heat experiments to distinguish the two transitions. We stress that the value of $\beta = 0.83$ is much bigger than any other conventional ones in $3d$ such as one of the Ising class ($\beta_{Ising} = 0.33$) and seems to fit the magnetic torque data better. Further works on more quantitative analysis and comparison with experiments are highly desired.

We also comment on multi band effects. In the main text, we only consider the simplest case where fermions are fractionalized in a single band. A generalization to a multi-band system is straightforward. Only a part of the bands may form fractionalized particles and show deconfined thermal phase transitions, and the others remain as spectator fermions. In such a case, ARPES experiments clearly show a loss of spectral weight only on the fractionalized bands.