A Quantum Observable for the
Graph Isomorphism Problem

Mark Ettinger∗
Los Alamos National Laboratory

Peter Høyer†
BRICS‡

Abstract
Suppose we are given two graphs on $n$ vertices. We define an observable in the Hilbert space $\mathbb{C}[\langle S_n \wr S_2 \rangle^m]$ which returns the answer “yes” with certainty if the graphs are isomorphic and “no” with probability at least $1 - \frac{n!}{2^{m}}$ if the graphs are not isomorphic. We do not know if this observable is efficiently implementable.

1 Introduction
The graph isomorphism problem is to determine if two graphs $\Gamma_1, \Gamma_2$ on $n$ vertices are isomorphic. Let $\Gamma$ be the disjoint union graph of $\Gamma_1$ and $\Gamma_2$. Without loss of generality we may assume that both $\Gamma_1$ and $\Gamma_2$ are connected. In this case the automorphism group of $\Gamma$ is a subgroup of the wreath product $S_n \wr S_2$ (which is itself a subgroup of $S_{2n}$). Clearly, knowledge of a set of generators for this automorphism group is sufficient to decide the isomorphism question. This fact has resulted in the suggestion that a quantum computer may be able to efficiently find a set of generators for the automorphism group and thus solve the graph isomorphism problem. This idea originates in the hidden subgroup view of quantum algorithms [1]. The Abelian hidden subgroup problem can be solved in polynomial time and utilizes the Fourier observable or, equivalently stated, the quantum algorithm utilizes the quantum Fourier transform. We use the terminology...
“Fourier observable” to emphasize the particular point of view germane to the main result of this paper. A quantum algorithm is simply a unitary change-of-basis transformation from the computational basis to the basis of the observable. We remark that in this paper “Fourier observable” refers to the Abelian case. The difficulties of finding hidden subgroups of noncommutative groups have been explored in several papers including [1, 6]. For more information on the Abelian hidden subgroup problem, see for example the references in [1, 6].

There are several important differences between the observable presented here and the Fourier observable. The first difference is that the present observable operates on a larger Hilbert space. Recently it was shown in [2] that a hidden noncommutative group may be found in only polynomially many calls to the oracle function, although the algorithm given in [2] requires exponential time. This result was proved by showing that the tensor product states corresponding to different possible hidden subgroups are almost orthogonal in the larger Hilbert space $\mathbb{C}[G^m]$. In the present paper we work in such a Hilbert space. The second difference is that our observable reveals nothing directly about the automorphism group other than whether or not it contains an isomorphism between the two graphs. However we may then find the full automorphism group using a well known classical reduction [3]. Thirdly and finally, whereas it is known that the Fourier observable is efficiently implementable, we have not been able to demonstrate this for the observable presented below. Such an efficient implementation would result in a polynomial-time quantum algorithm for the graph isomorphism problem.

2 The Observable

Let $G = S_n \wr S_2$. Since the wreath product is a semidirect product $(S_n \times S_n) \rtimes S_2$ we write an element as a triple $(\sigma, \tau, b)$. We refer to any element of $G$ of the form $k = (g, g^{-1}, 1)$ as an involutive swap. Let $\mathcal{H} = \mathbb{C}[G^m]$. Note that $\dim(\mathcal{H}) = |G|^m = 2^m(n!)^2$. For each $k \in G$, we define a $k$-vector to be a vector of the form:

$$\frac{1}{\sqrt{2^m}}\left( (|c_1\rangle + |c_1 k\rangle) \otimes \cdots \otimes (|c_m\rangle + |c_m k\rangle) \right)$$

for some $c_1, \ldots, c_m \in G$. Define $\mathcal{H}(k)$ to be the subspace spanned by all $k$-vectors. Notice that if $v_1$ and $v_2$ are unequal $k$-vectors then they are orthogonal. Therefore $\dim(\mathcal{H}(k)) = \left( \frac{|G|}{2} \right)^m$. Let $\mathcal{H}_1 = \sum_k \mathcal{H}(k)$ be the
sum over all $n!$ involutive swaps. Notice that $\dim(\mathcal{H}_1) \leq n!(|\mathcal{G}|^2)^m$. Let $\mathcal{H}_0$ be the orthogonal complement to $\mathcal{H}_1$ in $\mathcal{H}$. Our observable is defined as $L = \lambda_0 P_0 + \lambda_1 P_1$ where $P_0$ and $P_1$ are projections onto $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively, and $\lambda_0, \lambda_1 \in \mathbb{C}$.

Let us see what this observable yields when we apply it to the states that we may easily produce, i.e., tensor products of coset states. Let $|\psi\rangle$ be a tensor product of coset states of $\mathcal{H}$, i.e.

$$|\psi\rangle = |c_1 H\rangle \otimes \cdots \otimes |c_m H\rangle,$$

where for any non-empty subset $X \subseteq G$,

$$|X\rangle = \frac{1}{\sqrt{|X|}} \sum_{x \in X} |x\rangle.$$

**Theorem 1** If $\Gamma_1$ and $\Gamma_2$ are isomorphic then $\langle \psi | P_1 | \psi \rangle = 1$.

**Proof** If $\Gamma_1$ and $\Gamma_2$ are isomorphic via the involutive swap $k$ then $k \in H$, and thus any coset state of $H$ may be written (omitting normalizations):

$$|cH\rangle = |c h_1\rangle + |c h_1 k\rangle + \cdots + |c h_m\rangle + |c h_m k\rangle.$$

It is then easy to see that tensor products of these cosets state can be written as sums of $k$-vectors. For example

$$|c_1 H\rangle \otimes |c_2 H\rangle = (|c_1 h_1\rangle + |c_1 h_1 k\rangle) \otimes (|c_2 h_1\rangle + |c_2 h_1 k\rangle) + \cdots.$$

Any sum of $k$-vectors is, by definition, in $\mathcal{H}_1$ and the result follows. $\square$

**Theorem 2** If $\Gamma_1$ and $\Gamma_2$ are not isomorphic then $\langle \psi | P_0 | \psi \rangle \geq 1 - \frac{n!}{2^m}$.

**Proof** Assume the graphs are nonisomorphic. We show $\langle \psi | P_1 | \psi \rangle \leq \frac{n!}{2^m}$. First, suppose $|\psi\rangle = |g_1\rangle \otimes \cdots \otimes |g_m\rangle = |(g_1, g_2, \ldots, g_m)\rangle$. This occurs when both graphs are rigid and $H$ is trivial. For each involutive swap $k$ there exists exactly one $k$-vector which is not orthogonal to $|\psi\rangle$ and this $k$-vector has the form:

$$\frac{1}{\sqrt{2^m}} (|(g_1, \ldots, g_m)\rangle + |(g_1 k, \ldots, g_m)\rangle + \cdots + |(g_1, \ldots, g_m k)\rangle).$$

Therefore $\langle \psi | P(k) | \psi \rangle = \frac{1}{2^m}$, where $P(k)$ is the projection onto $\mathcal{H}(k)$. This implies $\langle \psi | P_1 | \psi \rangle \leq \frac{n!}{2^m}$. For nontrivial $H$ the argument is almost identical except that since $|\psi\rangle$ is not a basis state we must sum the probability contributions over the support, resulting in identical conclusions. $\square$
3 Conclusion

We have described a quantum observable on a Hilbert space for which the logarithm of its dimension is polynomial in the number of vertices of the graphs. This observable decides the isomorphism question with high probability. However we do not know if this observable is efficiently implementable. Furthermore, we remark that Manny Knill has observed that this observable suffices to also solve the code equivalence problem. Since linear codes have canonical forms we may consider the code equivalence problem to be a hidden stabilizer problem over the same group $S_n \wr S_2$. See [3] for a discussion of the relationship of the classical complexities of graph isomorphism and code equivalence.

Finally we remark on the group $S_n \wr S_2$ with which we have been working. We could equally well work over the subgroup $G'$ which is generated by the involutive swaps. It is not difficult to show that $G'$ consists of all elements of $G$ of the form $(\sigma, \tau, b)$ where both $\sigma$ and $\tau$ are even or both are odd. Thus $G'$ has index 2 in $G$ and this allows us to work in a smaller Hilbert space.

4 Acknowledgements

We would like to thank Manny Knill and Richard Hughes, Gian-Carlo Rota and Alain Tapp for helpful discussions on this problem.

References

[1] Ettinger, Mark and Peter Høyer, “On quantum algorithms for non-commutative hidden subgroups”. To appear in Proceedings of the Sixteenth International Symposium on Theoretical Aspects in Computer Science, 1999.

[2] Ettinger, Mark, Peter Høyer and Manny Knill, “Hidden subgroups states are almost orthogonal”. In preparation, January 1999.

[3] Knill, Manny. Personal communication, November 1998.

[4] Mathon, Rudolf, “A note on the graph isomorphism problem”, Information Processing Letters, Vol. 8, March 1979, pp. 131 – 132.

[5] Petrank, Erez and Ron M. Roth, “Is code equivalence easy to decide?”, IEEE Transactions on Information Theory, Vol. 43, September 1997, pp. 1602 – 1604.
[6] RÖTTELER, Martin and Thomas BETH, “Polynomial-time solution to the hidden subgroup problem for a class of non-Abelian groups.” Available on Los Alamos e-Print archive (http://xxx.lanl.gov) as quant-ph/9812070 (December 24, 1998).