COLOR HOM-AKIVIS ALGEBRAS, COLOR HOM-LEIBNIZ ALGEBRAS AND MODULES OVER COLOR HOM-LEIBNIZ ALGEBRAS

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Abstract

In this paper we introduce color Hom-Akivis algebras and prove that the commutator of any color non-associative Hom-algebra structure map leads to a color Hom-Akivis algebra. We give various constructions of color Hom-Akivis algebras. Next we study flexible and alternative color Hom-Akivis algebras. Likewise color Hom-Akivis algebras, we introduce non-commutative color Hom-Leibniz-Poisson algebras and presente several constructions. Moreover we give the relationship between Hom-dialgebras and Hom-Leibniz-Poisson algebras; i.e. a Hom-dialgebra give rise to a Hom-Leibniz-Poisson algebra. Finally we show that twisting a color Hom-Leibniz module structure map by a color Hom-Leibniz algebra endomorphism, we get another one.

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1 Introduction

Hom-algebraic structures appeared first as a generalization of Lie algebras in [3] were the authors studied $q$-deformations of Witt and Virasoro algebras. Other interesting Hom-type algebraic structures of many classical structures were studied as Hom-associative algebras, Hom-Lie admissible algebras and more general G-Hom-associative algebras ([23]), $n$-ary Hom-Nambu-Lie algebras ([7]), Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras ([22]), Hom-alternative algebras, Hom-Malcev algebras and Hom-Jordan algebras ([28]). Hom-algebraic structures were extended to the case of $G$-graded Lie algebras by studying Hom-Lie superalgebras, Hom-Lie admissible superalgebras in [15], color Hom-Lie algebras ([2]) and color Hom-Poisson algebras ([9]). In [31], Yuan presents some constructions of quadratique color Hom-Lie algebras, and this is used to provide several examples. $T^*$-extensions and central extensions of color Hom-Lie algebras and some cohomological characterizations are established ([6]).

Akivis algebras were introduced by M. A. Akivis ([5]) as a tool in the study of some aspects of Web geometry and its connection with loop theory. These algebras were originally called "W-algebras" ([4]). Later, Hofmann and Strambach ([16]) introduce the term "Akivis algebras" for such algebraic objects.

Hom-Akivis algebras are introduced in [18], in which it is showed that the commutator of non-Hom-associative algebras lead to Hom-Akivis algebras. It is also proved that Hom-Akivis algebras can be obtain from Akivis algebras by twisting along algebra endomorphisms, and that the class of Hom-Akivis algebras is closed under self-morphisms. The connection between Hom-Akivis algebras and Hom-alternative algebras is given.

A non-commutative version of Lie algebras were introduced by Loday in [21]. The bracket of the so-called Leibniz algebras satisfies the Leibniz identity; which, combined with skew-symmetry, is a variation of the Jacobi identity. hence Lie algebras are skew-symmetric Leibniz algebras. In the Leibniz setting, the objects that play the role of associative algebras are called dialgebras, which were introduced by Loday in [20]. Dialgebras have two associative binary operations that satisfy three additional associative-type axioms. The relationship between dialgebras and Leibniz algebras are analyzed in [13]. It is proved in [26] that from any Leibniz algebra $L$ one can construct a Leibniz-Poisson algebra $A$ and the properties of $L$ are close to the properties of $A$. Leibniz algebras were extended to Hom-Leibniz algebras in [24]. A (co)homology theory of Hom-Leibniz algebras and an initial study of universal central extensions of Hom-Leibniz algebras was given in [14]. The representation of Hom-Leibniz algebras are introduced in [12] and the homology of Hom-Leibniz algebras are computed. Recently Leibniz superalgebras were studied in [1]. The Hom-dialgebras are introduced in [30] as an extention of some work of Loday ([20]). In a Hom-dialgebra, there are two binary operations that satisfy five $\alpha$-twisted associative-type axioms. Each Hom-Lie algebra can be thought of as a Hom-Leibniz algebra. Likewise, to every Hom-associative algebra is associated a Hom-dialgebra in which both binary operations are equal to the original one.

The purpose of this paper is to study color Hom-Akivis algebras (graded version of Hom-Akivis algebras ([18])), color non-commutative-Hom-Leibniz-Poisson algebras and modules over color Hom-Leibniz algebras.

Section 2 is dedicated to the background material on graded vector spaces, bicharac-
ter and non-Hom-associative algebras. In section 3, we define color Hom-Akivis algebras which is the graded version of Hom-Akivis algebras. Various properties of color Hom-Akivis algebras are studied. More precisely, we show that the commutator of color non-Hom-associative algebra structure map lead to color Hom-Akivis algebras (Theorem 3.1). Next, we provide some constructions of color Hom-Akivis algebras; on the one hand from color Akivis algebras (Proposition 3.1) and on the other hand from a given color Hom-Akivis algebras (Theorem 3.2). Flexible color Hom-Akivis algebras and Hom-alternative color Hom-Akivis algebras are also exposed (Theorem 3.3). Section 4 being devoted to the study of Color Hom-Leibniz algebras, we define in it color Hom-Leibniz algebras and give basic properties (Proposition 4.1). Next, we introduce non-commutative Hom-Leibniz-Poisson algebras (in brief color NHLP-algebras). As in the previous section on color Hom-Akivis algebras, we give several twisting of non-commutative Hom-Leibniz-Poisson algebras (Theorem 4.1). Finally we give the relationship between Hom-dialgebras and Hom-Leibniz-Poisson algebras (Theorem 4.3). In section 5, we prove that the Yau’s twisting of module Hom-algebras ([29, Lemma 2.5]) works for modules over color Hom-Leibniz algebras (Theorem 5.1) likewise it runs for modules over Hom-Lie algebras ([10]), modules over color Hom-Lie algebras ([9]) and modules over color Hom-Poisson algebras ([9]).

Throughout this paper, \(K\) denotes a field of characteristic 0 and \(G\) is an abelian group.

## 2 Preliminaries

**Definition 2.1.** A vector space \(V\) is said to be a \(G\)-graded if, there exist a family \((V_a)_{a \in G}\) of vector subspaces of \(V\) such that \(V = \bigoplus_{a \in G} V_a\). An element \(x \in V\) is said to be homogeneous of degree \(a \in G\) if \(x \in V_a\). We denote \(H(V)\) the set of all homogeneous elements in \(V\).

**Definition 2.2.** Let \(V = \bigoplus_{a \in G} V_a\) and \(V' = \bigoplus_{a \in G} V'_a\) be two \(G\)-graded vector spaces, and \(b \in G\). A linear map \(f : V \to V'\) is said to be homogeneous of degree \(b\) if \(f(V_a) \subseteq V'_{a+b}\) for all \(a \in G\). If, \(f\) is homogeneous of degree zero i.e. \(f(V_a) \subseteq V'_a\) holds for any \(a \in G\), then \(f\) is said to be even.

**Definition 2.3.** An algebra \((A, \mu)\) is said to be \(G\)-graded if its underlying vector space is \(G\)-graded and if, furthermore, \(\mu(A_a, A_b) \subseteq A_{a+b}\) for all \(a, b \in G\). Let \(A'\) be another \(G\)-graded algebra. A morphism \(f : A \to A'\) of \(G\)-graded algebras is by definition an algebra morphism from \(A\) to \(A'\) which is, in addition an even mapping.

**Definition 2.4.** Let \(G\) be an abelian group. A map \(\varepsilon : G \times G \to K^*\) is called a skew-symmetric bicharacter on \(G\) if the following identities hold: for all \(a, b, c \in G\),

(i) \(\varepsilon(a, b)\varepsilon(b, a) = 1\);

(ii) \(\varepsilon(a, b + c) = \varepsilon(a, b)\varepsilon(a, c)\);

(iii) \(\varepsilon(a + b, c) = \varepsilon(a, c)\varepsilon(b, c)\).

**Definition 2.5.** 1. A multiplicative color non-Hom-associative (i.e. non necessarily Hom-associative) algebra is a quadruple \((A, \mu, \varepsilon, \alpha_A)\) such that
(a) A is $G$-graded vector space;
(b) $\mu : A \otimes A \to A$ is an even bilinear map;
(c) $\varepsilon : G \otimes G \to K^*$ is a bicharacter;
(d) $\alpha_A$ is an endomorphism of $(A, \mu)$ (multiplicativity).

2. $(A, \mu, \alpha)$ is said to be $\varepsilon$-skew-symmetric (resp. $\varepsilon$-commutative) if, for any $x, y \in A$, $\mu(x, y) = -\varepsilon(x, y)\mu(y, x)$ (resp. $\mu(x, y) = \varepsilon(x, y)\mu(y, x)$).

3. Let $(A, \mu, \alpha_A)$ be color non-Hom-associative algebra. For any $x, y, z \in A$, the Hom-associator is defined by

$$as(x, y, z) = \mu(\mu(x, y), \alpha_A(z)) - \mu(\alpha_A, \mu(y, z)).$$

Then $(A, \mu, \varepsilon, \alpha_A)$ is said to be

(a) Hom-flexible, if $as(x, y, x) = 0$;
(b) Hom-alternative, if $as(x, y, z)$ is $\varepsilon$-skew-symmetric in $x, y, z$.

3 Color Hom-Akivis algebras

Hom-Akivis algebras was introduced in [18] as a twisted generalization of Akivis algebras. In the following, we generalize some results. We prove that the commutator of a color non-Hom-associative algebra leads to a color Hom-Akivis algebra. Some properties of color Hom-Akivis algebras are studied.

**Definition 3.1.** A color Hom-Akivis algebra is quintuple $(A, [-,-], [-,-], [-,-], \varepsilon, \alpha_A)$ consisting of a $G$-graded vector space $A$, an even $\varepsilon$-skew-symmetric bilinear map $[-,-]$, an even trilinear map $[-,-]$ and an even endomorphism $\alpha_A : A \to A$ such that for all $x, y, z \in H(A)$,

$$\oint \varepsilon(z,x)[[x,y],\alpha_A(z)] = \oint \varepsilon(z,x)[[x,y,\varepsilon(x,y)z] - \varepsilon(x,\varepsilon(y,z))].$$

If in addition, $\alpha_A$ is an endomorphism with respect to $[-,-]$ and $[-,-]$, the color Hom-Akivis algebra $A$ is said to be multiplicative.

**Theorem 3.1.** Let $(A, \mu, \varepsilon, \alpha_A)$ be a color non-Hom-associative algebra. Define the maps $[-,-] : A \otimes A \to A$ and $[-,-]_{\alpha_A} : A \otimes A \otimes A \to A$ as follows: for all $x, y, z \in A$,

$$[x, y] := \mu(x, y) - \varepsilon(x, y)\mu(y, x),$$

$$[x, y, z]_{\alpha_A} := as(x, y, z).$$

Then $(A, [-,-], [-,-]_{\alpha_A}, \varepsilon, \alpha_A)$ is a multiplicative color Hom-Akivis algebra.

**Proof.** Color Hom-Akivis identity (2) is proved by direct computation. For any homogeneous elements $x, y, z \in A$, we have

$$\varepsilon(z, x)[[x, y], \alpha_A(z)] = \varepsilon(z, x)(xy)\alpha_A(z) - \varepsilon(y, z)\alpha_A(z)(xy)$$

$$-\varepsilon(z, x)\varepsilon(x, y)(yx)\alpha_A(z) + \varepsilon(x, y)\varepsilon(y, z)\alpha_A(z)(yx).$$
Theorem 3.2. Let $A$ be a color hom-Akivis algebra, then the twisting self-map $\alpha_A$ is itself an endomorphism of $(A, [-, -], e, \alpha_A)$.

Definition 3.2. Let $(A, [-, -], e, \alpha_A)$ and $(\tilde{A}, [-, -], e, \alpha_{\tilde{A}})$ be two color hom-Akivis algebras. A morphism $f : A \to \tilde{A}$ of color hom-Akivis algebras such that

\[
\begin{align*}
[f([x, y])] &= [f(x), f(y)], \\
[f([x, y, z])] &= [f(x), f(y), f(z)].
\end{align*}
\]

Moreover, if $A$ is a color hom-Akivis algebra, then the twisting self-map $\alpha_A$ is itself an endomorphism of $(A, [-, -], e, \alpha_A)$.

The following theorem is the color version of Theorem 4.4 in [13].

**Theorem 3.2.** Let $A_{\beta^n} = (A, [-, -], [-, -], e, \alpha_A)$ be a color hom-Akivis algebra and $\beta : A \to A$ an even endomorphism. Then, for any integer $n \geq 1$, $A_{\beta^n} = (A, [-, -]_{\beta^n}, [-, -]_{\beta^n}, e, \beta^n \circ \alpha_A)$ is a color hom-Akivis algebra, where for all $x, y, z \in H(A)$,

\[
\begin{align*}
[x, y]_{\beta^n} &= \beta^n([x, y]), \\
[x, y, z]_{\beta^n} &= \beta^{2n}([x, y, z]).
\end{align*}
\]

Moreover, if $A_{\alpha_A}$ is multiplicative and $\beta$ commutes with $\alpha_A$, then $A_{\beta^n}$ is multiplicative.

**Proof.** It is clear that $[-, -]_{\beta^n}$ and $[-, -]_{\beta^n}$ are bilinear and trilinear respectively. It is also clear that the $e$-skew-symmetry of $[-, -]_{\beta^n}$ comes from that of $[-, -]$. It remains to prove the color hom-Akivis identity (2) and the multiplicativity of $A_{\beta^n}$. For any $x, y, z \in H(A)$, we have,

\[
\begin{align*}
\oint \varepsilon(z, x) ([x, y]_{\beta^n}, (\beta^n \circ \alpha_A)(z))_{\beta^n} &= \oint \varepsilon(z, x) \beta^n([x, y], (\beta^n \circ \alpha_A)(z)) \\
&= \beta^{2n} \oint \varepsilon(z, x) ([x, y, z] - \varepsilon(x, y)[y, x, z]) \\
&= \oint \varepsilon(z, x) (\beta^{2n} [x, y, z] - \varepsilon(x, y) \beta^{2n} [y, x, z]) \\
&= \oint \varepsilon(z, x) ([x, y, z]_{\beta^{2n}} - \varepsilon(x, y)[y, x, z]_{\beta^{2n}}).
\end{align*}
\]

The multiplicativity is proved similarly as in [13] Theorem 4.4.

**Corollary 3.1.** Let $(A, [-, -], [-, -], e)$ be a color Akivis algebra and $\beta : A \to A$ an even endomorphism. Then $(A, [-, -]_{\beta^n}, [-, -]_{\beta^n}, e, \alpha_A)$ is a multiplicative color hom-Akivis algebra.

Moreover, suppose that $(\tilde{A}, [-, -], [-, -], e)$ is another color Akivis algebra and $\tilde{A}$ an even endomorphism of $\tilde{A}$. If $f : A \to \tilde{A}$ is a morphism of color Akivis algebras such that $f \circ \alpha_A = \alpha_{\tilde{A}} \circ f$, then $f : (A, [-, -], [-, -], e, \alpha_A) \to (\tilde{A}, [-, -], [-, -], e, \alpha_{\tilde{A}})$ is a morphism of multiplicative color hom-Akivis algebras.
Proof. It is similar to that of [18, Corollary 4.5]. □

The above construction of color Hom-Akivis algebras is used in [18], in the case of Hom-Akivis algebras, to produce examples of Hom-Akivis algebras.

**Proposition 3.1.** Let $(A, [-, -], [-, -], ε)$ be a color Akivis algebra and $β : A → A$ an even endomorphism. Define $[-, -]_{β^n}$ by: for all $x, y, z ∈ H(A)$,

$$[x, y]_{β^n} := β([x, y]_{β^{n-1}}),$$
$$[x, y, z]_{β^n} := β^2([x, y, z]_{β^{n-1}}),$$

Then $(A, [-, -]_{β^n}, [-, -]_{β^n}, ε, α_A)$ is a multiplicative color Hom-Akivis algebra.

**Proof.** It is proved recurrently by applying Corollary 3.1. The reader may also see [18, Theorem 4.8] for the proof. □

**Definition 3.3.** A color Hom-Akivis algebra $(A, [-, -], [-, -], ε, α_A)$ is said to be

1. Hom-flexible, if $[x, y, x] = 0$ for all $x, y ∈ A$;
2. Hom-alternative, if $[-, -]$ is $ε$-alternating (i.e. $[-, -]$ is $ε$-skew-symmetric for any pair of variables).

**Theorem 3.3.** Let $(A, µ, ε, α_A)$ be a color non-Hom-associative algebra and the quintuple $(A, [-, -], as(-, -), ε, α_A)$ its associated color Hom-Akivis algebra (as in Theorem 3.1). Then we have the following:

1. if $(A, µ, ε, α_A)$ is Hom-flexible, then $(A, [-, -], as(-, -), ε, α_A)$ is Hom-flexible;
2. if $(A, µ, ε, α_A)$ is Hom-alternative, then so is $(A, [-, -], as(-, -), ε, α_A)$.

**Proposition 3.2.** Let $A_A = (A, [-, -], [-, -], ε, α_A)$ be a Hom-flexible color Hom-Akivis algebra. Then we have

$$∫ ε(z, x) = ∫ [ε(z, x) + ε(x, y)ε(y, z)] [x, y, z].$$

In particular, $A_A$ is a color Hom-Lie algebra if and only if

$$∫ [ε(z, x) + ε(x, y)ε(y, z)] [x, y, z] = 0.$$  (4)

**Proof.** By expanding the left hand side of color Hom-Akivis identity (2), and using the Hom-flexibility we get (3). □

For more details on color Hom-Lie algebras, see [2], [9], [31].

4 **Color Hom-Leibniz algebras**

In this section, we write NHLP-algebra (resp. NLP-algebra) for non-commutative Hom-Leibniz-Poisson algebra (resp. non-commutative Leibniz-Poisson algebra). Interrelation between Hom-dialgebras and NHLP-algebras are presented.
4.1 Color NHLP-algebras

This subsection is devoted to color Hom-Leibniz algebras. Color NHLP-algebras are introduced and their various twisting are given as well as examples.

Definition 4.1. A color Hom-Leibniz algebra is a quadruple \((L, \mu, \varepsilon, \alpha_L)\), consisting of a graded vector space \(L\), an even bilinear map \(\mu : L \otimes L \to L\), a bicharacter \(\varepsilon : G \otimes G \to K^*\) and an even linear space homomorphism \(\alpha_L : L \to L\) satisfying

\[
\mu(\alpha_L(x), \mu(y, z)) = \mu(\mu(x, z), \alpha(y)) + \varepsilon(x, y) \mu(\alpha(x), \mu(y, z)).
\]

(5)

A morphism of color Hom-Leibniz algebras is an even linear map which preserves the structures.

Remark 4.1. 1. When \(G\) is an abelian group with trivial grading, we obtain the following Hom-Leibniz identity (18):

\[
\mu(\alpha_L(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)) + \mu(\alpha(y), \mu(x, z)).
\]

(6)

2. The original definition of Hom-Leibniz algebras (24) is related to the identity

\[
\mu(\mu(x, y), \alpha_L(z)) = \mu(\alpha(x), \mu(y, z)) + \mu(\mu(x, z), \alpha(y))
\]

(7)

which is expressed in terms of (right) adjoint homomorphisms \(ad_{x,y} : \mu(x,y)\) of \((A, \mu, \alpha_A)\). This justifies the terms of “(right) Hom-Leibniz algebra” that could be used for the Hom-Leibniz algebras defined in (24). The dual of (6) is (7) and in this paper we consider only left color Hom-Leibniz algebras.

We have the following result.

Proposition 4.1. Let \((L, \mu, \varepsilon, \alpha_L)\) be a color Hom-Leibniz algebra. Then

\[
(x \cdot y + \varepsilon(x, y)y \cdot x)\alpha_L(z) = 0,
\]

(8)

\[
[x \cdot y, \alpha_L(z)] + \varepsilon(x, y)[\alpha_L(y), x \cdot z] = \alpha_L(x) \cdot [y, z],
\]

(9)

where we have putted \(\mu(x, y) = x \cdot y\) and \([x, y] = x \cdot y - \varepsilon(x, y)y \cdot x\).

Proof. The identity (6) implies that

\[
(x \cdot y) \cdot \alpha_L(z) = \alpha_L(x) \cdot (y \cdot z) - \varepsilon(x, y)\alpha_L(y) \cdot (x \cdot z)
\]

Likewise, interchanging \(x\) and \(y\), we have

\[
(y \cdot x) \cdot \alpha_L(z) = \alpha_L(y) \cdot (x \cdot z) - \varepsilon(y, x)\alpha_L(x) \cdot (y \cdot z)
\]

Then, multiplying the second equality by \(\varepsilon(x, y)\), and adding memberwise with the first one, we obtain (8). Next, by direct computation, we have

\[
[x \cdot y, \alpha_L(z)] + \varepsilon(x, y)[\alpha_L(y), x \cdot z] = (x \cdot y) \cdot \alpha_L(z) - \varepsilon(x, z)e(y, z)\alpha_L(z) \cdot (x \cdot y)
\]
Now we define color NHLP-algebras which is the graded and Hom-version of NLP-algebras (13).

**Definition 4.2.** A color NHLP-algebra is a G-graded vector space $P$ together with two even bilinear maps $[-, -]: P \otimes P \to P$ and $\mu: P \otimes P \to P$, a bicharacter $\varepsilon: G \otimes G \to K^*$ and $\alpha_P: P \to P$ an even linear map such that, for any $x, y, z \in P$,

\begin{align*}
\text{i)} & \quad (P, [-, -], \varepsilon, \alpha_P) \text{ is a color Hom-Leibniz-Poisson algebra i.e.} \\
\quad & \quad [\alpha_P(x), [y, z]] = [[x, y], \alpha_P(z)] + \varepsilon(x, y)[\alpha_P(y), [x, z]], \tag{10} \\
\text{ii)} & \quad (P, \mu, \varepsilon, \alpha_P) \text{ is a color Hom-associative algebra i.e.} \\
\quad & \quad \mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)) \text{ (Hom-associativity)}; \tag{11} \\
\text{iii)} & \quad \text{and the following identity holds:} \\
\quad & \quad [\alpha_P(x), \mu(y, z)] = \mu([x, y], \alpha_P(z)) + \varepsilon(x, y)\mu(\alpha_P(y), [x, z]). \tag{12}
\end{align*}

If in addition, $\alpha_P$ is an endomorphism with respect to $\mu$ and $[-, -]$, we say that $(P, \mu, [-, -], \varepsilon, \alpha_P)$ is a multiplicative color NHLP-algebra.

**Remark 4.2.** When the Hom-associative product $\mu$ is $\varepsilon$-commutative, then $(A, \mu, [-, -], \varepsilon, \alpha_A)$ is said to be a commutative color Hom-Leibniz Poisson algebra.

**Example 4.1.**

1. Any color Hom-Poisson algebra is a color NHLP-algebra algebra.

2. Any color Hom-Leibniz algebra is a color NHLP-algebra.

3. If $P$ is a Leibniz-Poisson algebra (viewed as Hom-Leibniz-Poisson algebras with trivial twisting and trivial grading), then the vector space $P \otimes P$ is a NHLP-algebra with the operations

\begin{align*}
(x_1 \otimes x_2)(y_1 \otimes y_2) &= x_1y_1 \otimes y_1y_2, \\
[x_1 \otimes x_2, y_1 \otimes y_2] &= [(x_1, x_2), y_1] \otimes y_2 + y_1 \otimes [(x_1, x_2), y_2].
\end{align*}

The proposition below is a direct consequence of the Definition 4.2.

**Proposition 4.2.** Let $(P, [-, -], \mu, \varepsilon, \alpha)$ be a color NHLP-algebra. Then $(P, [-, -], \mu^{op}, \varepsilon, \alpha)$ and $(P, k\mu, k[-, -], \alpha_P)$ are also color NHLP-algebras, with $\mu^{op}(x, y) = \mu(y, x)$ and $k \in K^*$.

It is well known in Hom-algebras setting that one can obtain Hom-algebra structures from an ordinary one and an endomorphism. The following theorem, which gives a way to construct a color NHLP-algebras from color NLP-algebras and an endomorphism, is a similar result.
Theorem 4.1. Let \((A, \mu, [-,-], \varepsilon)\) be a color NLP-algebra and \(\alpha_A : A \to A\) an even endomorphism. Then, for any integer \(n \geq 1\), \((A, \mu_{\alpha_A}^n, [-,-]_{\alpha_A}^n, \varepsilon, \alpha_A^n)\) is a multiplicative color NHLP-algebra, where for all \(x, y, z \in \mathcal{H}(A)\),

\[
\begin{align*}
\mu_{\alpha_A}^n(x, y) & := \alpha_A^n(\mu(x, y)), \\
[x, y]_{\alpha_A}^n(x, y) & := \alpha_A^n([x, y]).
\end{align*}
\]

Moreover, suppose that \((\tilde{A}, \tilde{\mu}, [-,-], \tilde{\varepsilon})\) is another color NLP-algebra and \(\tilde{\alpha}_A\) an even endomorphism of \(\tilde{A}\). If \(f : A \to \tilde{A}\) is a morphism of color NLP-algebras such that \(f \circ \alpha_A = \alpha_{\tilde{A}} \circ f\), then \(f : (A, \mu, [-,-], \varepsilon, \alpha_A) \to (\tilde{A}, \tilde{\mu}, [-,-], \tilde{\varepsilon}, \tilde{\alpha}_{\tilde{A}})\) is a morphism of multiplicative color NHLP-algebras.

Proof. We need to show that \((A, \mu_{\alpha_A}^n, [-,-]_{\alpha_A}^n, \varepsilon, \alpha_A^n)\) satisfies relations (10) and (12). We have, for any \(x, y, z \in A\),

\[
\begin{align*}
[a_A^n(x), \mu_{\alpha_A}^n(y, z)]_{\alpha_A^n} & = a_A^n([a_A^n(x), \mu_{\alpha_A}^n(y, z)]) \\
& = a_A^{2n}([x, \mu(y, z)]) \\
& = a_A^{2n}([x, y, z] + \varepsilon(x, y)[y, [x, z]]) \\
& = a_A^n([a_A^n((x, y), a_A^n(y, a_A^n(x, z)) + \varepsilon(x, y)(a_A^n(y, a_A^n(x, z)))]) \\
& = [x, y]_{\alpha_A}^n, a_A^n(z)]_{\alpha_A^n} + \varepsilon(x, y)(a_A^n(y), [x, z])_{\alpha_A^n}.
\end{align*}
\]

The compatibility condition (12) is pointed out similarly and the second assertion is proved as in the case of Theorem 3.2. \(\square\)

Example 4.2. The commutative Leibniz-Poisson algebra used in this example is given in [27]. [26]. Let \((A, [-,-])\) is a Leibniz algebra over a commutative infinite field \(K\) and let

\[
\tilde{A} = A \otimes K
\]

be a vector space with multiplications \(\cdot\) and \([-,-]\) defined as: for \(x, y \in A\) and \(a, b \in K\),

\[
(x + a) \cdot (y + b) = (bx + ay) + ab \quad \text{and} \quad \{x + a, y + b\} = [x, y].
\]

Consider the homomorphism \(\alpha_{\tilde{A}} = \alpha_A \otimes \text{Id}_K : \tilde{A} \to \tilde{A}\), then \((\tilde{A}, \cdot, [-,-], \alpha_{\tilde{A}})\) is a commutative color Hom-Leibniz-Poisson algebra (with trivial grading).

The next result gives a procedure to produce color NHLP-algebras from given one.

Theorem 4.2. Let \((P, \mu, [-,-], \varepsilon, \alpha_P)\) be a color NHLP-algebra and \(\beta : P \to P\) an even endomorphism. Then \((P, \mu_{\beta^p}, [-,-]_{\beta^p}, \varepsilon, \beta^p \circ \alpha_P)\) is a color NHLP-algebra, with

\[
\begin{align*}
\mu_{\beta^p}(x, y) & := \beta^p(\mu(x, y)), \\
[x, y]_{\beta^p}(x, y) & := \beta^p([x, y]).
\end{align*}
\]

for all \(x, y, z \in \mathcal{H}(P)\).
Proof. We need to show that \((P, \mu_{\beta^n}, [-, -]_{\beta^n}, \varepsilon, \beta^n \circ \alpha_P)\) satisfies relations (10) and (12). We have, for any \(x, y, z \in \mathcal{H}(P)\),

\[
[\beta^n \circ \alpha_P](x), [y, z]_{\beta^n} = \beta^n(\beta^n \circ \alpha_P)(x), \beta^n[y, z]
\]

Next,

\[
[\beta^n \circ \alpha_P](x), \mu_{\beta^n}(y, z)_{\beta^n} = \beta^n(\beta^n \circ \alpha_P)(x), \beta^n\mu(y, z)
\]

Then, we give a similar result for NHLP-algebras.

**Proposition 4.3.** Let \((P, \mu, [-, -], \varepsilon)\) be a color NLP-algebra and \(\beta : P \rightarrow P\) an even endomorphism. Define \(\mu_{\beta^n}\) and \([-,-]_{\beta^n}\) by

\[
\mu_{\beta^n}(x, y) := \beta(\mu_{\beta^{n-1}}(x, y)),
\]

\[
[x, y]_{\beta^n} := \beta^n([x, y]_{\beta^{n-1}}),
\]

for all \(x, y, z \in \mathcal{H}(P)\). Then \((P, \mu_{\beta^n}, [-, -]_{\beta^n}, \varepsilon, \alpha_P)\) is a multiplicative color NHLP-algebra.

### 4.2 NHLP-algebras and Dialgebras

It is well known that Hom-dialgebras give rise to Hom-Leibniz algebra (30). In this subsection, we give a similar result for NHLP-algebras.

**Definition 4.3 (30).** A Hom-dialgebra is a quadruple \((D, \cdot, \cdot, \cdot, \alpha_D)\), where \(D\) is a \(K\)-vector space, \(-, -, \cdot : D \otimes D \rightarrow D\) are bilinear maps and \(\alpha_D : D \rightarrow D\) a linear map such that the following five axioms are satisfied for \(x, y, z \in D\):

\[
(x \cdot y) + \alpha_D(z) = \alpha_D(x) \cdot (y + z),
\]

\[
\alpha_D(x) \cdot (y + z) = (x + y) \cdot \alpha_D(z) = \alpha_D(x) + (y \cdot \alpha_D(z))
\]

\[
(x + y) \cdot \alpha_D(z) = (x \cdot y) \cdot \alpha_D(z) = (x + y) \cdot \alpha_D(z)
\]

We say that a Hom-dialgebra \((D, \cdot, \cdot, \cdot, \alpha_D)\) is Hom-associative if it so for the operations \(-, -, \cdot\). So any Hom-dialgebra is Hom-associative.

The following result connects Hom-dialgebras and Hom-Leibniz-Poisson algebras.

**Theorem 4.3.** Let \((D, +, \cdot, \cdot, \alpha_D)\) be a Hom-dialgebra. Define the bilinear map \([-,-] : D \otimes D \rightarrow D\) by setting

\[
[x, y] := x \cdot y - y \cdot x.
\]

Then \((D, +, [-, -], \alpha_D)\) is a Hom-Leibniz-Poisson algebra.
\textbf{Proof.} We write down all twelve terms involved in the Hom-Leibniz identity (6):
\[
\begin{align*}
[\alpha_D(x), [y, z]] &= \alpha_D(x) \cdot (y \cdot z) - (y \cdot z) \cdot \alpha_D(x) - \alpha_D(x) \cdot (z \cdot y) + (z \cdot y) \cdot \alpha_D(x) \\
[\alpha_D(y), [x, z]] &= \alpha_D(y) \cdot (x \cdot z) - (x \cdot z) \cdot \alpha_D(y) - \alpha_D(y) \cdot (z \cdot x) + (z \cdot x) \cdot \alpha_D(y) \\
[[x, y], \alpha_D(z)] &= (x \cdot y) \cdot \alpha_D(z) - \alpha_D(z) \cdot (x \cdot y) - (y \cdot x) \cdot \alpha_D(z) + \alpha_D(z) \cdot (y \cdot x).
\end{align*}
\]
Using the Hom-dialgebra axioms in Definition 4.3 it is readily seen that (6) holds. Next,
\[
\begin{align*}
[\alpha_D(x), y \cdot z] - [x, y] \cdot \alpha_D(z) - \alpha_D(y) + [x, z] &= \alpha_D(x) \cdot (y \cdot z) - (y \cdot z) \cdot \alpha_D(x) - (x \cdot y) + \alpha_D(z) \\
&= -(x \cdot y) + \alpha_D(z) - (y \cdot x) + \alpha_D(z) \\
&= -\alpha_D(y) + (x \cdot z) + \alpha_D(y) + (z \cdot x).
\end{align*}
\]
The left hand side vanishes thanks also to the Definition 4.3 Thus the conclusion holds. \(\square\)

\section{Modules over color Hom-Leibniz algebras}

In this section we introduce modules over color Hom-Leibniz algebras and prove that the Yau’s twisting of module structure map works very well with module over color Hom-Leibniz algebras.

\textbf{Definition 5.1.} Let \(G\) be an abelian group. A Hom-module is a pair \((M, \alpha_M)\) in which \(M\) is a \(G\)-graded vector space and \(\alpha_M : M \rightarrow M\) is an even linear map.

\textbf{Definition 5.2.} Let \((L, [-, -], \varepsilon, \alpha_L)\) be a color Hom-Leibniz algebra and \((M, \alpha_M)\) a Hom-module. An \(L\)-module on \(M\) consists of even \(K\)-bilinear maps \(\mu_L : L \otimes M \rightarrow M\) and \(\mu_R : M \otimes L \rightarrow M\) such that for any \(x, y \in L\), and \(m \in M\),
\[
\begin{align*}
\alpha_M(\mu_L(x, m)) &= \mu_L(\alpha_L(x), \alpha_M(m)) \\
\alpha_M(\mu_R(m, x)) &= \mu_R(\alpha_M(m), \alpha_L(x)) \\
\mu_L([-,-] \otimes \alpha_M) &= \mu_L(\alpha_L \otimes \mu_L) - \varepsilon(x, y)\mu_L(\alpha_L \otimes \mu_L)(\tau_{LL} \otimes I_{dL}), \\
\mu_R(\alpha_M \otimes [-,-]) &= \mu_L(\alpha_L \otimes \alpha_L)(\tau_{ML} \otimes I_{dL}) + \varepsilon(x, m)\mu_L(\alpha_L \otimes \mu_R)(\tau_{ML} \otimes I_{dL}), \\
\mu_L(\alpha_L \otimes \mu_R) &= \mu_R(\mu_L \otimes \alpha_L) + \varepsilon(x, m)\mu_R(\alpha_M \otimes [-,-])(\tau_{ML} \otimes I_{dL}).
\end{align*}
\]
where \(\tau_{LL}(x \otimes y) = y \otimes x\), \(\tau_{ML}(m \otimes x) = m \otimes x\), \(\tau_{ML}(m \otimes x) = x \otimes m\).

\textbf{Remark 5.1.} The conditions (16), (17) and (18) can be written respectively as
\[
\begin{align*}
[x, y] \cdot \alpha_M(m) &= \alpha_L(x) \cdot (y \cdot m) - \varepsilon(x, y)\alpha_L(y) \cdot (x \cdot m), \\
\alpha_M(m) \star [x, y] &= (x \cdot m) \star \alpha_L(y) + \varepsilon(x, m)\alpha_L(x) \cdot (m \star y), \\
\alpha_L(x) \cdot (m \star y) &= (x \cdot m) \star \alpha_L(y) + \varepsilon(m, x)\alpha_M(m) \star [x, y],
\end{align*}
\]
where \(\mu(x \otimes y) = xy\), \(\mu_L(x \otimes m) = x \cdot m\) and \(\mu_R(m \otimes x) = m \star x\).

\textbf{Example 5.1.} Any color Hom-Leibniz algebra and any Hom-Leibniz superalgebra is a module over itself.
Theorem 5.1. Let \((L, [-, -], \varepsilon, \alpha_L)\) be a color Hom-Leibniz algebra and \((M, \mu_L, \mu_R, \alpha_M)\) a color Hom-Leibniz module. Then,

\[
\begin{align*}
\hat{\mu}_L &= \mu_L(\alpha_L^2 \otimes \text{Id}_M) : L \otimes M \to M, \\
\hat{\mu}_R &= \mu_R(\text{Id}_M \otimes \alpha_L^2) : L \otimes M \to M,
\end{align*}
\]

define another color Hom-Leibniz module structure on \(M\).

Proof. We have to point out relations (16)-(18) for \(\hat{\mu}_L\) and \(\hat{\mu}_R\). We have, respectively, any for \(x, y \in L, m \in M\),

\[
\begin{align*}
\hat{\mu}_L([-] \otimes \alpha_M)(x \otimes y \otimes m) &= \hat{\mu}_L([x, y] \otimes \alpha_M(m)) \\
&= \alpha_L^2([x, y]) \cdot \alpha_M(m) \\
&= [\alpha_L^2(x), \alpha_L^2(y)] \cdot \alpha_M(m) \\
&= \alpha_L^3(x) \cdot (\alpha_L^2(y) \cdot m) - \varepsilon(x, y) \alpha_L^3(y) \cdot (\alpha_L^2(x) \cdot m) \quad \text{(by (19))} \\
&= \hat{\mu}_L(\alpha_L \otimes \hat{\mu}_L)(x \otimes y \otimes m) \\
&\quad - \varepsilon(x, y) \hat{\mu}_L(\alpha_L \otimes \hat{\mu}_L)(\tau_{LL} \otimes \text{Id}_L)(x \otimes y \otimes m).
\end{align*}
\]

\[
\begin{align*}
\hat{\mu}_R(\alpha_M \otimes [-])(m \otimes x \otimes y) &= \hat{\mu}_R(\alpha_M(m) \otimes [x, y]) \\
&= \alpha_M(m) \ast [\alpha_L^2(x), \alpha_L^2(y)] \\
&= (\alpha_L^2(x) \cdot m) \ast \alpha_L^3(y) + \varepsilon(x, m) \alpha_M(x) \cdot (m \ast \alpha_L^2(y)) \quad \text{(by (20))} \\
&= \hat{\mu}_R(\hat{\mu}_L \otimes \alpha_L)(\tau_{ML} \otimes \text{Id}_L)(m \otimes x \otimes y) \\
&\quad + \varepsilon(x, m) \hat{\mu}_L(\alpha_L \otimes \hat{\mu}_R)(\tau_{MM} \otimes \text{Id}_L)(m \otimes x \otimes y).
\end{align*}
\]

\[
\begin{align*}
\varepsilon(m, x)\hat{\mu}_R(\alpha_M \otimes [-])&(\tau_{LM} \otimes \text{Id}_L)(x \otimes m \otimes y) \quad = \varepsilon(m, x)\hat{\mu}_R(\alpha_M \otimes [-])(x \otimes m \otimes y) \\
&= \varepsilon(m, x)\hat{\mu}_R(\alpha_M(m) \otimes [x, y]) \\
&= \varepsilon(m, x)\alpha_M(m) \ast [\alpha_L^2(x), \alpha_L^2(y)] \\
&= \alpha_L^3(x) \cdot (m \ast \alpha_L^2(y)) \\
&\quad - (\alpha_L^2(x) \cdot m) \ast \alpha_L^3(y) \quad \text{(by (21))} \\
&= \hat{\mu}_L(\alpha_L \otimes \hat{\mu}_R)(x \otimes m \otimes y) \\
&\quad - \hat{\mu}_R(\hat{\mu}_L \otimes \alpha_L)(x \otimes m \otimes y).
\end{align*}
\]

\[
\square
\]

Corollary 5.1. Let \(A_{\alpha_A^n} = (A, [-, -], \varepsilon, \alpha_A^n)\) be a multiplicative color Hom-Leibniz algebra as in Theorem 4.1 and \((M, \mu_L, \mu_R, \alpha_M)\) an \(A_{\alpha_A^n}\)-module. Then \((M, \hat{\mu}_L, \hat{\mu}_R, \alpha_M)\) is also a module over \(A_{\alpha_A^n}\).

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