A bound to kill the ramification over function fields

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Abstract

Let $k$ be a field of characteristic zero, let $X$ be a geometrically integral $k$-variety of dimension $n$ and let $K$ be its field of fractions. Under the assumption that $K$ contains all $r$th roots of unity for an integer $r$, we prove that, given an element $\alpha \in H^m(K, \mathbb{Z}/r)$, there exist $n^2$ functions $\{f_i\}_{i=1,\ldots,n^2}$ such that $\alpha$ becomes unramified in $L = K(f_1^{1/r}, \ldots, f_{n^2}^{1/r})$.

1. Introduction. Let $K$ be a field and let $\alpha \in Br K$ be an element of order $r$. In [3, 5], Saltman proved that if $K$ is the function field of a $p$-adic curve and $(r, p) = 1$, then $\alpha$ becomes trivial over an extension of $K$ of degree $r^2$. As a motivation for the question we consider in this paper, let us give a brief sketch of his arguments. Let us assume that $r$ is prime and that $K$ contains all $r$th roots of unity. In fact, one can see that this case implies the general case. We view $K$ as a function field of a regular, integral two-dimensional scheme $X$, projective over the spectrum of the ring of integers of a $p$-adic field. Saltman then proved that one can find two functions $f_1, f_2 \in K$ such that $\alpha$ becomes unramified in $L = K(f_1^{1/r}, f_2^{1/r})$ with respect to any rank one discrete valuation ring centered on $X$. This is sufficient to conclude, using the classical result that the Brauer group of a regular flat proper (relative) curve over the ring of integers of a $p$-adic field is trivial (cf. [3, 6]).

Let us consider the case of higher dimensions, that is, assume that $K$ is the field of fractions of an $n$-dimensional variety $X$, defined over a field $k$. Following Saltman’s work, given a class $\alpha \in Br K$, one may wonder if there is a bound $N$ depending only on $K$, such that we can kill the ramification of $\alpha$ with $N$ functions. Our main result (cf. theorem [3]) gives an affirmative answer $N = n^2$ for $\alpha$ of order $r$ under the assumption that $K$ contains all $r$th roots of unity. Our method also works for elements of $H^m(K, \mathbb{Z}/r)$ and not only for $m = 2$.

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2. Statement of the main result. Let $k$ be a field of characteristic zero. For $L$ a function field over $k$ containing all $r^{th}$ roots of unity we fix an isomorphism $\mu_r \cong \mathbb{Z}/r$ of $\text{Gal}(\bar{K}/K)$-modules and we write

$$H^m_{nr}(L/k, \mathbb{Z}/r) = \bigcap_{A} \ker[H^m(L, \mathbb{Z}/r) \xrightarrow{\partial_A} H^{m-1}(k_A, \mathbb{Z}/r)],$$

where $A$ runs through all discrete valuation rings of rank one with $k \subset A$ and fraction field $L$. We denote by $k_A$ the residue field of $A$ and by $\partial_A$ the residue map.

**Theorem 1.** Let $k$ be a field of characteristic zero. Let $X$ be an integral $k$-variety of dimension $n$ and let $K$ be its field of fractions. Let $r$ be an integer and assume that $K$ contains all $r^{th}$ roots of unity. Let $\alpha$ be an element of $H^m(K, \mathbb{Z}/r)$. There exist $n^2$ functions $\{f_i\}_{i=1}^{n^2}$ such that $\alpha$ becomes unramified over $L = K(f_1^{1/r}, \ldots, f_{n^2}^{1/r})$, that is, we have $\alpha_L \in H^m_{nr}(L/k, \mathbb{Z}/r)$.

We first prove two lemmas.

3. Local description. In the case of dimension two, the following statement is due to Saltman (cf. [1] 1.2).

**Lemma 2.** Let $k$ be an infinite field. Let $A$ be a local ring of a smooth $k$-variety and let $K$ be its field of fractions. Let $r$ be an integer prime to characteristic of $k$. Assume that $K$ contains all $r^{th}$ roots of unity and fix an isomorphism $\mu_r \cong \mathbb{Z}/r$ of $\text{Gal}(\bar{K}/K)$-modules. Let $\alpha$ be an element of $H^m(K, \mathbb{Z}/r)$ ramified only at $s_1, \ldots, s_h$ forming a regular subsystem of parameters of the maximal ideal of $A$. Then

$$\alpha = \alpha_0 + \sum_{\emptyset \neq I \subset \{1, \ldots, h\}} \alpha_I \cup s_I,$$

with $\alpha_0 \in H^m(A, \mathbb{Z}/r)$, $\alpha_I \in H^{m-|I|}(A, \mathbb{Z}/r)$, and $s_I = \cup_{i \in I}(s_i)$, where we denote by $(s_i)$ the class of $s_i$ in $H^1(K, \mathbb{Z}/r) \cong K^*/K^{**}$.

**Proof.** We proceed by induction on $h$ and $m$. Assume first $h = 1$. For $A$ a local ring of a smooth $k$-variety, with field of fractions $K$ and for $Y = \text{Spec } A$, we have an exact sequence due to Bloch and Ogus (cf. [1] 2.2.2)

$$0 \to H^m(A, \mathbb{Z}/r) \to H^m(K, \mathbb{Z}/r) \to \bigoplus_{x \in Y^{(1)}} H^{m-1}(k(x), \mathbb{Z}/r) \to \bigoplus_{x \in Y^{(2)}} H^{m-2}(k(x), \mathbb{Z}/r) \to \ldots$$

(1)

where the maps are induced by the residues. Denote by $K(A/s_1)$ the field of fractions of $A/s_1$. As $\alpha$ is ramified only at $s_1$, we see from the sequence (1) that $\partial_{s_1}(\alpha) \in H^{m-1}(K(A/s_1), \mathbb{Z}/r)$ is unramified. Hence, from the sequence (1) for $A/s_1$, it comes from an element of $H^{m-1}(A/s_1, \mathbb{Z}/r)$. From Levine’s conjecture (generalizing Bloch-Kato’s conjecture proved by Rost and Voevodsky), proved by Kerz [2] 1.2, any element of $H^{m-1}(A/s_1, \mathbb{Z}/r)$ is a sum of cup products of units in $A/s_1$. In particular, any element of $H^{m-1}(A/s_1, \mathbb{Z}/r)$ lifts to $A$: there exists an element $\alpha_1 \in H^{m-1}(A, \mathbb{Z}/r)$ such that $\partial_1 = \partial_{s_1}(\alpha)$. Hence $\alpha - \alpha_1 \cup (s_1)$ is unramified, so it comes from $\alpha_0 \in H^m(A, \mathbb{Z}/r)$, by (1) again.
If $m = 1$, we have $\alpha = (s)$ for $s$ a function in $K$ and the result follows from the decomposition $s = u \prod s_i^t$ with $t_i \in \mathbb{Z}$ and $u \in A^*$.

Next, we assume the assertion for $(m - 1, h - 1)$ and $(m, h - 1)$ and we prove it for $(m, h)$. From the sequence $[1]$, $\partial_{s_i}(\alpha) \in H^{m-1}(K(A/s_1), \mathbb{Z}/r)$ is ramified only at $s_2, \ldots, s_h$ where we denote by $\bar{s}_i$ the image of $s_i$ in $A/s_1$. By induction, $\partial_{s_i}(\alpha) = \bar{\alpha}_1 + \sum_{\emptyset \neq I \subseteq \{2, \ldots, h\}} \bar{\alpha}_I \cup \bar{s}_I$, where $\bar{\alpha}_1 \in H^{m-1}(A/s_1, \mathbb{Z}/r)$, $\bar{\alpha}_I \in H^{m-1-|I|}(A/s_1, \mathbb{Z}/r)$, and $\bar{s}_I = \cup_{i \in I}(\bar{s}_i)$. As before, we deduce from [2] 1.2 that all the $\bar{\alpha}_I$ and $\bar{\alpha}_1$ are sums of cup products of units in $A/s_1$ and so we can lift them to $\alpha_I$ (resp. to $\alpha_1$) on $A$. Now the element $\alpha - (\alpha_1 + \sum_{\emptyset \neq I \subseteq \{2, \ldots, h\}} \alpha_I \cup s_I) \cup (s_1)$ is ramified only at $s_2, \ldots, s_h$ and the lemma follows by induction.

4. Divisor decomposition.

Lemma 3. Let $k$ be a field of characteristic zero and let $X$ be a smooth projective $k$-variety of dimension $n$. Let $D$ be a divisor on $X$. There exists a sequence of blowing-ups $f : X' \to X$ such that the support of the total transform $f^*D$ is a simple normal crossing divisor which can be expressed a union of $n$ regular (but not necessarily connected) divisors of $X'$.

Proof. By Hironaka, we may assume that $\text{Supp}(D)$ is a simple normal crossing divisor, which means that any irreducible component of $\text{Supp}(D)$ is smooth and that the fiber product over $X$ of any $c$ components of $\text{Supp}(D)$ is smooth and of codimension $c$. Let $G = (V, E)$ be the dual graph of $D$:

- the vertices of $V$ correspond to irreducible components $D_1, \ldots, D_N$ of $D$
- the edge $(D_i, D_j)$ is in $E$ if the intersection $D_i \cap D_j$ is nonempty.

We say that we blow-up the edge $(D_i, D_j)$ if we change $X$ by the blow-up of the intersection $D_i \cap D_j$ (with reduced structure) and we change $G$ by the dual graph of the total transform of $D$, i.e. we add a vertex and corresponding edges. We write again $G = (V, E)$ for the modified graph.

We will show that after a finite sequence of blowing-ups $f : X' \to X$ of some edges we may color the vertices of $G$ in $n$ colors so that for any edge $AB \in E$ the vertices $A$ and $B$ are of different colors. Then $\text{Supp}(f^*D)$ is a simple normal crossing divisor and we have $\text{Supp}(f^*D) = \bigcup_{i=1}^n F_i$ where $F_i$ is the (disjoint) union of components of $f^*D$ such that the corresponding vertex is of the $i^{th}$ color. Hence $F_i$ are regular and the lemma follows.

If $n = 2$ we may assume, after blowing-ups of some edges, that any cycle in $G$ has even number of edges, which is sufficient to conclude.

Let us now assume that $n \geq 3$. We proceed by induction on the number $N$ of irreducible components of $D$. If $N \leq n$ the statement is clear. Assume it holds for $N$. Let $D$ be a divisor with $N + 1$ components. By the induction hypothesis, after
blowing-ups of some edges, we may assume that we may color all but the vertex
$D_{N+1}$ of $G$ in $n$ colors as desired. We have $\text{Supp}(D) = \bigcup_{i=1}^{n} F_i \cup D_{N+1}$ where $F_i$
is the union of components of $D$ of the $i^{\text{th}}$ color. If $D_{N+1}$ doesn’t intersect $F_i$
for some $i$ we color $D_{N+1}$ in $i^{\text{th}}$ color. Hence we may assume that all the intersections
$D_{N+1} \cap F_i$ are nonempty. By the same reason, we may assume that the intersection
$F_2 \cap F_3$ is nonempty. On the other hand, note that the intersection $\bigcap_i F_i \cap D_{N+1}$ is
empty as $\text{Supp}(D)$ is a simple normal crossing divisor. We proceed by the following
algorithm:

1. We first blow up all the edges $D_i D_{N+1}$ for all the components $D_j$ of $F_1$. Let us
denote $E_i$ the union of all the exceptional divisors. This union is disjoint as the
components of $F_1$ do not intersect. Note that $E_1 \cap F_2 \cap \ldots \cap F_n = \emptyset$. Otherwise,
we get a point in the intersection $\bigcap_i F_i \cap D_{N+1}$ by projection. Moreover, there
are no more edges between (the components of) $F_1$ and $D_{N+1}$ as the strict transforms
of the corresponding divisors do not intersect.

2. Next, we blow up all the edges between $F_2$ and $F_3$ and we call $E_2$ the (disjoint)
union of all new exceptional divisors. Again, we have no more edges between
$F_2$ and $F_3$ and also $E_2 \cap E_1 \cap F_3 \cap \ldots \cap F_n = \emptyset$ (or $E_2 \cap E_1$ is empty if $n = 3$).

3. If $n = 3$ we have the following picture:

Here and in what follows the punctured line (for example, $F_2 F_3$) means that
there are no edges between components of corresponding groups (e.g. no edges
between elements of $F_2 \cup F_3$).

We color (all the vertices from) $F_1$ and $D_{N+1}$ in red, $E_1$ and $E_2$ in green and
$F_2$ and $F_3$ in blue and this terminates the algorithm.

4. Assume that $n \geq 4$. We proceed until we get the group of exceptional divisors
$E_{n-1}$ and then we go to step 6. Suppose $3 \leq i \leq n - 1$ and we constructed
$E_{i-2}$ and $E_{i-1}$ but no $E_i$. Suppose there are some edges between $E_{i-2}$ and
$F_{i+1}$, otherwise we go to step 5. We blow up all these edges and we call $E_i$
the (disjoint) union of all new exceptional divisors. We get no more edges
between $E_{i-2}$ and $F_{i+1}$ and also $E_i \cap E_{i-1} \cap F_{i+2} \cap \ldots \cap F_n = \emptyset$.

5. If there are no edges between $E_{i-2}$ and $F_{i+1}$, we have the following picture:
We color $F_1$ and $D_{N+1}$ in the first color, $F_2$ and $F_3$ in the second color, $E_1$ and $F_4$ in the third, $\ldots$, $E_{i-2}$ and $F_{i+1}$ in the $i^{th}$-color, $E_{i-1}$ in color $i+1$, and, finally, $F_{i+2}, \ldots F_n$ in colors $i+2, \ldots n$ respectively.

6. At this step, we have the following picture:

![Diagram]

Moreover, $E_{n-1} \cap E_{n-2} = \emptyset$ by construction. We color $F_1$ and $D_{N+1}$ in the first color, $F_2$ and $F_3$ in the second color, $E_1$ and $F_4$ in the third, $\ldots$, $E_{n-3}$ and $F_n$ in color $n-1$, $E_{n-1}$ and $E_{n-2}$ in color $n$. This terminates the algorithm.

5. Proof of theorem [1] By resolution of singularities, we may assume that $X$ is smooth. By lemma [3] we may assume that the ramification divisor $D = \text{ram}(\alpha)$ is a simple normal crossing divisor whose support is a union of $n$ regular divisors: $\text{Supp } D = \bigcup_{i=1}^n D_i$.

For two divisors $G$ and $G'$ on $X$, with $G = \sum_{i=1}^q G_i$ where the $G_i$ are irreducible divisors, we say that $G'$ is in general position with $G$ if the support of $G'$ contains no generic point of any intersection $\bigcap_{i \in I} G_i$ for $I \subset \{1, \ldots, q\}$.

By a semilocal argument, we successively choose functions $f^j_i \in K$, $j = 1, \ldots, n$, then $f^j_i \in K$, $j = 1, \ldots, n$, and then $f^j_i \in K$, $j = 1, \ldots, n$, such that

$$\text{div}_{X}(f^j_i) = D_i + E^j_i$$

where $E^j_i$ are in general position with $D \cup \bigcup_{j' < j} \text{Supp}(E^j_i)$.

We claim that with this choice of $n^2$ functions $\alpha_L$ is unramified. Let $v$ be a discrete valuation on $L$ and let $x \in X$ be the point where the discrete valuation ring $R$ of $v$ is centered. We may assume that $x \in \text{Supp } D$, otherwise $\alpha$ is already unramified at $v$. From the construction, for any $i$, $D \cap \bigcap_{j=1}^n E^j_i = \emptyset$. Hence for any $1 \leq i \leq n$ we can find $j_i$ such that $x \notin E^j_i$, which means that the corresponding local parameter $s_i$ of $D_i$ at $x$ is an $r^{th}$ power in $K((f^j_i)^{1/r})$. Now the theorem follows from lemma [2] as any $s_I$ from the lemma is a cup product of $r^{th}$ powers on $L$. \qed
Remark 4. The bound $n^2$ is not sharp. For example, for $n = 3$ one can kill all the ramification with four functions. Let us write $\text{ram}(\alpha) = D_1 \cup D_2 \cup D_3$ as in lemma 3. As in the proof of the theorem above, we take $f_i \in K$, $i = 1, \ldots, 4$, such that

$$\text{div}(f_1) = D_1 + D_2 + D_3 + E_1;$$
$$\text{div}(f_2) = D_1 + D_2 + E_2;$$
$$\text{div}(f_3) = D_2 + D_3 + E_3;$$
$$\text{div}(f_4) = D_1 + 2D_2 + D_3 + E_4.$$  

and each $E_i$ is in general position with $\text{ram}(\alpha) \cup \bigcup_{i' < i} \text{Supp}(E_{i'})$. Let $x$ be a center of a valuation $v$ on $L = K(f_1^{1/r})_{i = 1, \ldots, 4}$. We may assume that $x \in \text{ram}(\alpha)$. It is sufficient to see that if $x \in D_i$ then a local parameter of $D_i$ at $x$ can be expressed as a product of powers of the functions $f_i$.

1. If $x \in X^{(1)}$ then $x$ lies on only one component $D_i$, which is thus defined by $f_1$.

2. If $x \in X^{(2)}$ and if $x$ lies on two components $D_i$ and $D_j$, then $\frac{f_i}{f_j}$ defines $D_1$, $\frac{f_i f_j}{f_k}$ defines $D_2$, $\frac{f_i f_j}{f_k}$ defines $D_3$. If $x$ lies on only one component $D_i$, then, by construction, $x \notin E_{i_1} \cup E_{i_2}$ for at least two indexes $1 \leq i_1 < i_2 \leq 3$. By construction, $D_i$ is then defined at $x$ by at least one among the functions $f_{i_1}$ and $f_{i_2}$.

3. Suppose that $x$ is a closed point of $X$. If $x \in D_1 \cap D_2 \cap D_3$, we use the same formulas as in the previous case. Next, suppose that $x$ lies on only two components of $\text{ram}(\alpha)$. Consider the case $x \in D_1 \cap D_2$, the other cases are similar. By construction, $x$ lies on at most one component among $E_1, E_2, E_3$. Hence we see that if $x \notin E_2 \cup E_3$ (resp. $x \notin E_1 \cup E_3$, resp. $x \notin E_1 \cup E_2$) then $D_1$ is defined by $\frac{f_i}{f_j}$ and $D_2$ is defined by $f_3$ (resp. by $\frac{f_i}{f_3}$ and by $f_3$, resp. by $\frac{f_i}{f_j}$ and by $\frac{f_i}{f_j}$).

The last case is when $x$ lies on only one component of $\text{ram}(\alpha)$. Consider the case $x \in D_1$, the other cases are similar. Then $x \notin E_1 \cap E_2 \cap E_4$ by construction. Then $D_1$ is defined by $f_1$ (resp. by $f_2$, $f_4$) if $x$ does not lie on $E_1$ (resp. on $E_2$, $E_4$).

Remark 5. By the same arguments as in the previous remark, if $r$ is prime to 2 and 3 and if $n = 3$, one can kill all the ramification with three functions $f_1, f_2, f_3$, such that

$$\text{div}(f_1) = D_1 + 3D_2 + 3D_3 + E_1;$$
$$\text{div}(f_2) = D_1 + 2D_2 + D_3 + E_2;$$
$$\text{div}(f_3) = D_1 + D_2 + 2D_3 + E_3.$$  

and each $E_i$ is in general position with $\text{ram}(\alpha) \cup \bigcup_{i' < i} \text{Supp}(E_{i'})$. 

6
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