FAMILIES OF RATIONALLLY SIMPLY CONNECTED VARIETIES
OVER SURFACES AND TORSORS FOR SEMISIMPLE GROUPS

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ABSTRACT. Under suitable hypotheses, we prove that a form of a projective
homogeneous variety \( G/P \) defined over the function field of a surface over an
algebraically closed field has a rational point. The method uses an algebro-
geometric analogue of simple connectedness replacing the unit interval by the
projective line. As a consequence, we complete the proof of Serre’s Conjecture
II in Galois cohomology for function fields over an algebraically closed field.

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1. Introduction

In this introduction we work with varieties over an algebraically closed field \( k \) of
characteristic 0, unless mentioned otherwise. In the paper \cite{GHS03} it was shown
that a family of birationally rationally connected varieties over a curve has a rational
section. A variety \( X \) is called \textit{birationally rationally connected} if a general pair of
points \((x, y) \in X \times X\) can be connected by a rational curve, i.e., if there exists an
open \( U \subset \mathbb{P}^1 \) and a morphism \( f : U \to X \) such that \( x, y \in f(U) \). This is related
to Tsen’s theorem, see \cite{Lan52}, which says that a family of hypersurfaces of degree
$d$ in $\mathbb{P}^n$ over an $r$-dimensional base variety has a section if $d' \leq n$. The relation is that a smooth hypersurface in $\mathbb{P}^n$ is birationally rationally connected if and only if $d \leq n$.

How to generalize the result of [GHS03] to families of varieties over a surface? By analogy with topology one could hope for a statement of the following type: A family of (birationally) rationally simply connected varieties over a surface has a rational section. There are two good reasons why this is too simplistic: (1) it seems hard to come up with a good notion of “rationally simply connected” varieties, and (2) Brauer-Severi schemes give counter examples.

Let $S$ be a smooth projective surface. A Brauer-Severi scheme over $S$ is a smooth projective morphism $X \to S$ all of whose geometric fibres are isomorphic to $\mathbb{P}^n$. There are plenty of examples of such Brauer-Severi schemes such that $X \to S$ does not have a rational section, because most surfaces have infinite Brauer groups, see [Gro95b], and [Gro95a, Corollaire 3.4]. Since the authors would like to think of $\mathbb{P}^n$ as a prime example of a rationally simply connected variety this seems to dash all hopes.

Not so! It turns out that in addition to a condition on the fibres of the family we need to impose additional global conditions of a cohomological nature. The goal is to find example theorems providing evidence for the principle which we have stated elsewhere:

(P) “The only obstruction to having a rational section of a family of rationally simply connected varieties over a surface should be a Brauer class on the base”.

This should not be taken too literally, e.g., the Brauer class could be an element of $H^2_{et}(S, T)$ for some torus $T$ over $S$ (or an open part of $S$); or perhaps something more general, cf. [vH03].

Let $Y$ be a smooth projective rationally connected variety. When do we call $Y$ rationally simply connected? One of the main obstacles is that the space of rational curves on $Y$ has many irreducible and even connected components; simply because there are many curve classes $\beta \in H_2(Y, \mathbb{Z})$ which can be represented by rational curves. Here is a partial list of conditions that we have considered using as a definition:

1. The Ind-scheme parametrizing maps $\mathbb{A}^1 \to Y$ is birationally rationally connected.
2. For any pair of rational curves $C_1, C_2 \subset Y$ there exists a rational surface $S \subset Y$ containing $C_1$ and $C_2$.
3. For all sufficiently positive curve classes $\beta$ there exists an irreducible component $Z_\beta$ of $\overline{\mathcal{M}}_{0,0}(Y, \beta)$ which is rationally connected.
4. A variant of (3) which considers for every $n \geq 0$ the moduli spaces of rational curves passing through $n$ general points on $X$. See the notion labeled “strongly rationally simply connected” in [dJS06].
5. The variety $Y$ is rationally simply connected by chains of free lines, i.e., $Y \to \text{Spec}(k)$ satisfies Hypothesis [LJS06]. This implies that for a general pair of points $x, y \in Y(k)$, and $n \gg 0$ the moduli space of length $n$ chains of free lines connecting $x$ to $y$ is itself birationally rationally connected (see Proposition 6.11).
There is a subtle interplay between the formulation of the property, the problem of proving a suitable variety $Y$ has the property and the problem of proving the principle (P) holds for families of varieties having the property. For example, trying to use a property such as (1) above to prove (P) one runs into the problem of not having a result like [GHS03] for families of ind-schemes.

In the study of rationally connected varieties a key technical tool is the notion of a very free rational curve. Such a curve is given by $f: \mathbb{P}^1 \to Y$ such that $f^*T_Y$ is an ample vectorbundle. Similarly, the key technical tool in the setting of rational simple connectedness is the notion of a very twisting surface. This notion was first introduced in the paper [HS05]. In the paper you are reading now, a very twisting surface is given by a morphism $f: \mathbb{P}^1 \to M_{0,1}(Y, 1)$ such that $f^*T_{\Phi}, f^*T_{\text{ev}}$ are sufficiently ample and such that $\text{ev} \circ f$ is a free curve in $Y$, see Lemma 11.6 and Definition 11.7. See [HS05], [Sta04], and [dJS06] for example usages of this notion; although we warn the reader that each time the definition is slightly different.

A smooth projective rationally connected variety which contains a very twisting surface is likely to satisfy a number of the conditions listed above. On the other hand, it turns out condition (3) above is not enough to guarantee the existence of a very twisting surface.

Let us get to the point and formulate one of the main results of this long paper.

**Corollary 1.1.** See Corollary 12.2. Let $f: X \to S$ be a morphism of nonsingular projective varieties over $k$ with $S$ a surface. Assume

1. there exists a Zariski open subset $U$ of $S$ whose complement has codimension 2 such that $X_u$ is irreducible for $u \in U(k)$,
2. there exists an invertible sheaf $\mathcal{L}$ on $f^{-1}(U)$ which is $f$-relatively ample, and
3. the geometric generic fibre $X_\eta$ of $f$ together with the base change $\mathcal{L}_\eta$ is rationally simply connected by chains of free lines (see above) and has a very twisting surface,

then there exists a rational section of $f$.

The assumption on the irreducibility of fibres is likely superfluous and can be removed in most instances, see Remark 10.2. The existence of the invertible sheaf $\mathcal{L}$ globally is what ensures the Brauer class is zero in case the Corollary is applied to a Brauer-Severi scheme over $S$. The third condition we discussed above.

Let us discuss the method of proof of the Corollary above. The first remark is that there is a natural inductive structure on the overall problem of finding sections of families of varieties. Namely, we may fibre the surface $S$ by curves over a base curve $B$. Lemma 12.3 implies that it is enough to find a rational section of $B$ into the (coarse) moduli space $\text{Sections}(X/S/B)$ of sections of $X/S/B$. Briefly, a $k$-point of $\text{Sections}(X/S/B)$ is a pair $(b, \sigma)$, where $b \in B(k)$, and $\sigma: S_b \to X_b$ is a section of the restriction of $X \to S$ to $S_b$. Applying [GHS03] to the morphism $\text{Sections}(X/S/B) \to B$ we see that it would suffice to produce an irreducible subvariety $Z \subset \text{Sections}(X/S/B)$ such that the general fibre of $Z \to B$ is rationally connected.

It turns out that the irreducible components of $\text{Sections}(X/S/B)$ usually do not have rationally connected fibres over $B$. One obstruction is the Abel map

$$\alpha_{\mathcal{L}}: \text{Sections}^\kappa(X/S/B) \to \text{Pic}^\kappa_{S/B}.$$
The image of the pair \((b, \sigma)\) is the invertible sheaf \(\sigma^*L \in \text{Pic}(S_b)\). It turns out that it is enough if the fibres of this Abel map are rationally connected (see proof of Corollary 12.2). Thus our corollary follows from the following which is the main theorem of this article.

**Theorem 1.2.** See Theorem 12.1. Let \(X \to C\) be a morphism of nonsingular projective varieties over \(k\). Let \(L\) be an invertible sheaf on \(X\). Assume the geometric generic fibre of \(X \to C\) is rationally simply connected by chains of free lines (see above), and contains a very twisting surface. In this case, for \(e \gg 0\) there exists a canonically defined irreducible component \(Z_e \subset \text{Sections}^e(X/S/B)\) so that the restriction of \(\alpha\) to \(Z_e\) has rationally connected fibres.

The proof of this theorem is long and takes up most of this paper. We have tried to split up the argument in two stages. In the first stage, Sections 2 – 10, we avoid using the existence of a very twisting surface in \(X_\eta\) using only the assumption that \(X_\eta\) is rationally simply connected by chains of free lines. This condition still implies there are many free lines in fibres of \(X \to C\). The main technical result in these sections is Proposition 8.7. Let us rephrase it here. Let \(\Sigma^e(X/C/k)\) denote the Kontsevich moduli space of stable sections of \(X \to C\) of degree \(e\), see Definition 5.1. A free section, see Definition 3.7, is a section \(s : C \to X\) such that \(s^*T_{X/C}\) is a globally generated locally free sheaf with trivial \(H^1\). A porcupine, see Definition 7.2, is roughly speaking a stable map obtained by taking a free section and gluing on a bunch of free lines (its quills). With this terminology Proposition 8.7 roughly states that given any two free sections \(s_1, s_2\) there exist extensions of them to porcupines \(p_1, p_2\) whose moduli points \([p_i] \in \Sigma^e(X/C/k)\) can be connected by a chain of rational curves whose nodes correspond to unobstructed, non-stacky points of \(\Sigma^e(X/C/k)\).

This in some sense means some huge Ind-scheme constructed out of spaces of sections of \(X/C\) is rationally connected. But, as we mentioned above, we do not know a theorem a la [GHS03] for families of Ind-schemes. And it is at this point that the existence of a very twisting surface comes in. Namely, in the second stage, Sections 11 and 12, we show how the existence of very twisting surfaces in fibres of \(X \to C\) implies a the moduli point corresponding to a general porcupine is connected by a chain of rational curves in \(\Sigma^e(X/C/k)\) to the moduli point corresponding to a free section. Combining this with the previous result we get the theorem above.

**Application I.** In an email dated Jul 8, 2005 Phillipe Gille sketched out how a theorem as above for families of homogeneous varieties (or Borel varieties) over surfaces might lead to a proof of Serre’s conjecture II for function fields of surfaces. Our method sketched above reduces this to studying sections of families of Borel varieties over curves. In the case of curves over finite fields the same method was used by G. Harder to prove Serre’s conjecture II for function fields of curves over finite fields, see for example [Har71], the references therein and [Har75]. Of course the actual details differ substantially.

In Sections 13 and 14 we show that the theory above applies to families of varieties with fibres of the form \(Z = G/P\) with \(P\) maximal parabolic. The main new results are Theorem 13.1 and Proposition 14.1. We struggled with this material for a long time, partly because we did not realize that for a general variety of type \(G/P\)
with $P$ maximal parabolic, it is not the case that tangent vectors along lines at a fixed point $p$ span the tangent space $T_pZ$. Nonetheless these varieties are rationally simply connected by chains of free lines and contain very twisting surfaces, see Corollary 14.4 and Lemmas 14.7 and 14.8. Therefore using the Corollary 1.1 stated above we immediately obtain the following corollary.

**Corollary 1.3.** Let $G$ be a linear algebraic group over $k$ and let $P \subset G$ be a standard maximal parabolic subgroup. Let $S$ be a nonsingular projective surface over $k$. Let $X \to S$ be a smooth projective morphism all of whose geometric fibres are isomorphic to $G/P$. Assume there exists an invertible sheaf $L$ on $X$ which restricts to an ample generator of $\text{Pic}(G/P)$ on a fibre. Then $X \to S$ has a rational section.

It turns out that we can reduce a more general case of a family of projective homogeneous spaces to the simple case above. Here is the most general result we obtain of this kind.

**Theorem 1.4.** See Theorem 15.5. Let $k$ be an algebraically closed field of any characteristic. Let $S$ be a quasi-projective surface over $k$. Let $X \to S$ be a projective morphism whose geometric generic fibre is of the form $G/P$ for some linear algebraic group $G$ and parabolic subgroup $P$. Assume furthermore that $\text{Pic}(X) \to \text{Pic}(X_\eta)$ is surjective. Then $X \to S$ has a rational section.

We would like to explain how this leads to a proof of Serre’s conjecture II in the special case of function fields. Namely, let $G$ be connected, semi-simple, and simply connected over $k$ algebraically closed of any characteristic. Suppose $K$ is the function field of a surface $S$ over $k$. Suppose $\xi \in H^1(K, G)$ is a Galois cohomology class. After shrinking $S$ we may assume that $\xi$ comes from an element in $H^1(S, G)$, i.e., is given by a right $G$-torsor $T$ over $S$. Consider the family

$$X = T \times^G G/B \to S$$

Since $G$ is simply connected any invertible sheaf $L_0$ on $G/B$ has a $G$-linearization and hence we can descend $O_T \boxtimes L_0$ to $X$. In other words, the assumptions of Theorem 1.4 are satisfied and we see that $X$ has a $K$-rational point. Because the fibres of $T \to X$ are $B$-torsors, this means that $\xi$ comes from an element of $H^1(K, B)$. But as $B$ is a connected solvable group over $k$ we have $H^1(K, B) = 0$ for any field $K$. We conclude $\xi$ is trivial. Thus we obtain the following special case of a conjecture of Serre, cf. [Ser02, p. 137].

**Corollary 1.5 (Serre’s Conjecture II over function fields for split groups).** Let $k$ be an algebraically closed field of any characteristic and let $K/k$ be the function field of a surface. Let $G$ be a connected, semisimple, simply connected algebraic group over $k$. Every $G$-torsor over $K$ is trivial.

In particular, suppose $G_K$ is a simple algebraic group over $K$ of type $E_8$. Note that $G_K$ is simply connected. Let $G$ be a split form of $G_K$. Note that $\text{Aut}(G) \cong \text{Inn}(G) \cong G$. We see that $G_K$ corresponds to an element of $H^1(K, G)$ and hence by Corollary 1.5 it is itself split. Combined with [CTGP01, Theorem 1.2(v)], which deals with all other types this completes the proof of Serre’s Conjecture II for function fields.

**Theorem 1.6 (Serre’s Conjecture II for function fields).** For every connected, simply connected, semisimple algebraic group $G_K$ over $K$, every $G_K$-torsor over $K$ is trivial.
We would like to point out here that lots of work has been done by lots of authors on Serre’s conjecture II. Our approach is tailored to function fields of surfaces. Perhaps some of the motivation of Serre’s Conjecture II comes from studying the relationship between cohomological dimensions of fields, the arithmetic of fields, and linear algebraic groups. We have found the paper [CTGP04] and its references a rich source of information regarding this material.

Application II. We would like to point out that Corollary 1.1 and Theorem 1.2 can be used to reprove Tsen’s theorem for families of hypersurfaces over surfaces. Whereas the proof of Tsen’s theorem can fit on a napkin, this paper has 60+ pages. However, the aim of such an investigation is to stimulate research into different generalizations of Tsen’s theorem. For example, one possible avenue for research is to study families of low degree hypersurfaces in homogeneous spaces. In any case, we briefly explain in Section 16 below how to reprove Tsen’s theorem using the main results of this paper.

2. Stacks of curves and maps from curves to a target

The correct notion when doing moduli of algebraic varieties is to use families of varieties where the total space is an algebraic space – not necessarily a scheme. Even in the case of families of curves it can happen that the total space is not locally a scheme over the base. This leads to the following definitions.

Definition 2.1. Let $S$ be a scheme.

1. A flat family of proper curves over $S$ is a morphism of algebraic spaces $\pi : C \to S$ which is proper, locally finitely presented, and flat of relative dimension 1.

2. A flat family of polarized, proper curves over $S$ is a pair $(\pi : C \to S, L)$ consisting of a flat family of proper curves over $S$ and a $\pi$-ample invertible sheaf $L$ on $C$.

3. For every morphism of schemes $u : T \to S$ and $\pi : C \to S$ as above the pullback family is the projection

$$u^*\pi = \text{pr}_T : T \times_{u,S,\pi} C \to T.$$ 

If the family is polarized by $L$ then the pullback family is polarized by the $u^*\pi$-ample invertible sheaf $(u, \text{Id}_C)^*L$.

4. For two flat families of proper curves $\pi : C \to S$ and $\pi' : C' \to S'$, a morphism $f$ from the first family to the second is given by a cartesian diagram

$$\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow & & \downarrow \\
S & \xrightarrow{id} & S'.
\end{array}$$

If $\pi : C \to S$ is polarized by $L$ and $\pi' : C' \to S'$ is polarized by $L'$ then a morphism between the polarized families is a pair $(f, \phi)$ with $f$ as above and $\phi : L \to f^*L'$ is an isomorphism.

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[1] All fibres pure dimension 1.
**Definition 2.2.** The stack of proper curves is the fibred category

\[ \text{Curves} \to \text{Schemes} \]

whose objects are families of flat proper curves \( \pi : C \to S \) and morphisms are as above. The functor \( \text{Curves} \to \text{Schemes} \) is the forgetful functor. Similarly the stack of proper, polarized curves is the fibred category

\[ \text{Curves}_{\text{pol}} \to \text{Schemes} \]

whose objects are flat families of proper, polarized curves \((\pi : C \to S, \mathcal{L})\) as in the definition above.

Denote by \((\text{Aff})\) the category of affine schemes. A technical remark is that in \([\text{LMB}00]\) stacks are defined as fibred categories over \((\text{Aff})\). So in the following propositions we state our results in a manner that is compatible with their notation. In particular when we speak of \(\text{Curves}\) over \((\text{Aff})\) we mean the restriction of the fibred category \(\text{Curves}\) to \((\text{Aff}) \subset \text{Schemes}\).

There is functor of fibred categories

\[ F : \text{Curves}_{\text{pol}} \to \text{Curves} \]

forgetting the invertible sheaves.

**Proposition 2.3.** The categories \(\text{Curves}_{\text{pol}}\) and \(\text{Curves}\) are limit preserving algebraic stacks over \((\text{Aff})\) (with the fppf topology) with finitely presented, separated diagonals. Moreover \(F\) is a smooth, surjective morphism of algebraic stacks.

**Proof.** This is folklore; here is one proof. The category \(\text{Curves}_{\text{pol}}\) is a limit preserving algebraic stack with finitely presented, separated diagonal by \([\text{Sta}06, \text{Proposition 4.2}]\). Moreover, by \([\text{Sta}06, \text{Propositions 3.2 and 3.3}]\), \(\text{Curves}\) is a limit preserving stack with representable, separated, locally finitely presented diagonal. To prove that \(\text{Curves}\) is an algebraic stack, it only remains to find a smooth cover of \(\text{Curves}\), i.e., a smooth, essentially surjective 1-morphism from an algebraic space to \(\text{Curves}\). Let \(u : X \to \text{Curves}_{\text{pol}}\) be such a smooth cover for the stack of polarized curves. We claim that \(F \circ u : X \to \text{Curves}\) is a smooth cover.

Let \((S, \pi : C \to S)\) be an object of \(\text{Curves}\). The 2-fibred product of \(F\) with the associated 1-morphism \(S \to \text{Curves}\) is the stack \(\mathcal{A}\) of \(\pi\)-ample invertible sheaves on \(C\). Consider the diagram

\[
\begin{array}{ccc}
A \times_{\text{Curves}_{\text{pol}}} X & \longrightarrow & X \\
\downarrow & & \downarrow \\
A & \longrightarrow & \text{Curves}_{\text{pol}} \\
\downarrow & & \downarrow \\
S & \longrightarrow & \text{Curves}.
\end{array}
\]

Once we have shown \(\mathcal{A}\) is an algebraic stack and \(\mathcal{A} \to S\) is smooth surjective, then the claim above follows, as well as the smoothness and surjectivity of \(F\) in the proposition.

Before we continue we remark that \(\mathcal{A}\) is an open substack of the Picard stack of \(C \to S\) by \([\text{Gro}63, \text{Théorème 4.7.1}]\). Hence, by \([\text{Sta}06, \text{Proposition 4.1}]\) (which is a
variant of \cite{LMB00} Théorème 4.6.2.1) the stack \( \mathcal{A} \) is a limit preserving algebraic stack with quasi-compact, separated diagonal. In particular, for every smooth cover \( A \to \mathcal{A} \), the induced morphism \( A \to S \) is locally finitely presented. Thus to prove \( A \to S \) is smooth, and thus that \( \mathcal{A} \) is smooth over \( S \), it suffices to prove that \( A \to S \) is formally smooth. Let \( s \) be a point of \( S \) and denote by \( C_s \) the fibre \( \pi^{-1}(s) \). By \cite{Ill71} Proposition 3.1.5, the obstructions to infinitesimal extensions of invertible sheaves on \( C_s \) live in \( \text{Ext}^2_{\mathcal{O}_{C_s}}(\mathcal{O}_{C_s}, \mathcal{O}_{C_s}) = H^2(C_s, \mathcal{O}_{C_s}) \). Since \( C_s \) is a Noetherian, 1-dimensional scheme, \( H^2(C_s, \mathcal{O}_{C_s}) \) is zero. Thus there are no obstructions to infinitesimal extensions of \( \pi \)-ample invertible sheaves on \( C_s \). Thus \( \mathcal{A} \to S \) is smooth as desired.

Next, we prove that \( \mathcal{A} \to S \) is surjective. Thus, let \( s \) be a point of \( S \). It suffices to prove there is an ample invertible sheaf on \( C_s \). This is essentially \cite{Har77} Exercise III.5.8]. Therefore \textit{Curves} is an algebraic stack.

It only remains to prove that the diagonal of \textit{Curves} is quasi-compact. Let \((\pi_i : C_i \to S), i = 1, 2\) be two families of proper flat curves over an affine scheme \( S \). We have to show that the algebraic space

\[
\text{Isom}_S(C_1, C_2)
\]

is quasi-compact. Using that \( F \) is smooth and surjective, after replacing \( S \) by a quasi-compact, smooth cover, we may assume there exist a \( \pi_i \)-ample invertible sheaves \( \mathcal{L}_i \) on \( C_i \). We may assume the Hilbert polynomial of the fibres of \( C_i \to S \) with respect to \( \mathcal{L}_i \) are constant, say \( P_i \). For every \( S \)-scheme \( T \) and every \( T \)-isomorphism \( \phi : C_{1,T} \to C_{2,T} \), the graph \( \Gamma_\phi : C_{1,T} \to (C_1 \times_S C_2)_T \) is a closed immersion. In the usual way this identifies \( \text{Isom}_S(C_1, C_2) \) as a locally closed subscheme of the Hilbert scheme of \( C_1 \times_S C_2 \) over \( S \). We will show that whenever \( T = \text{Spec}(k) \) is the spectrum of a field, the Hilbert polynomial of \( \Gamma_\phi \) with respect to \( S \)-ample invertible sheaf \( \text{pr}_1^*\mathcal{L}_1 \otimes \text{pr}_2^*\mathcal{L}_2 \) is equal to \( P_1 - P_2 - P_1(0) \). Quasi-compactness follows from the projectivity of the Hilbert scheme parametrizing closed subschemes with given Hilbert polynomial. By Snapper’s theorem (see \cite{Kle66}) the values of the Hilbert polynomial \( \chi(C_{1,T}, \mathcal{L}_{(1,T)}^{\otimes n} \otimes \phi^*\mathcal{L}_{(2,T)}^{\otimes n}) \) equal \( \chi(C_{1,T}, \mathcal{L}_{(1,T)}^{\otimes n}) + \chi(C_{1,T}, \phi^*\mathcal{L}_{(2,T)}^{\otimes n}) - \chi(C_{1,T}, \mathcal{O}) \). And of course \( \chi(C_{1,T}, \phi^*\mathcal{L}_{(2,T)}^{\otimes n}) = \chi(C_{2,T}, \mathcal{L}_{(2,T)}^{\otimes n}) \). The claim follows. □

At this point we need some definitions related to moduli of maps from curves into varieties. But since we are going to study the rational connectivity of moduli stacks of rational curves on varieties, we need to deal with moduli of morphisms from curves to stacks.

**Definition 2.4.** Let \( \mathcal{X} \) be an algebraic stack, let \( S \) be a scheme and let \( \mathcal{X} \to S \) be a morphism. A \textit{family of maps of proper curves to} \( \mathcal{X} \) \textit{over} \( S \) is a triple \((T \to S, \pi : C \to T, \zeta : C \to \mathcal{X})\) consisting of a flat family of proper curves over the \( S \)-scheme \( T \) and a 1-morphism \( \zeta \) over \( S \).

We leave it to the reader to define morphisms between families of maps of proper curves to \( \mathcal{X} \) over \( S \).

**Definition 2.5.** The \textit{stack of maps of proper curves to} \( \mathcal{X} \) is the fibred category

\[
\text{CurveMaps}(\mathcal{X}/S) \to \text{Schemes}/S
\]

whose objects are families of maps of proper curves to \( \mathcal{X} \) over \( S \), and morphisms as above.
We again have the technical remark that according to the conventions in \cite{LMB00} we should really be working with the restriction of $\text{CurveMaps}(\mathcal{X}/S)$ to the category $(\text{Aff}_S)$ of affine schemes over $S$. Denote by $\text{Curves}_S$ the restriction of $\text{Curves}$ over the category $\text{Schemes}/S$ of $S$-schemes. There is an obvious functor

\[ G : \text{CurveMaps}(\mathcal{X}/S) \to \text{Curves}_S \]

forgetting the 1-morphisms to $\mathcal{X}$.

**Proposition 2.6.** See \cite{Lie06} Proposition 2.11. Assume $S$ excellent. Suppose that $\mathcal{X} = [Z/G]$ where $Z$ is an algebraic space, $Z \to S$ is separated and of finite presentation, and where $G$ is an $S$-flat linear algebraic group scheme. The stack $\text{CurveMaps}(\mathcal{X}/S)$ is a limit preserving algebraic stack over $(\text{Aff}_S)$ with locally finitely presented, separated diagonal. And the 1-morphism $G$ is representable by limit preserving algebraic stacks.

**Proof.** The second assertion is precisely proved in \cite{Lie06} Proposition 2.11 and implies the first assertion because of Proposition 2.3. \qed

**Definition 2.7.** Let $S$ be a scheme. A flat family of proper curves over $S$, $\pi : C \to S$, is LCI if it is a local complete intersection morphism in the sense of \cite{Gro67} Définition 19.3.6.

It is true that this is equivalent to \cite{BG71} Définition VIII.1.1. See \cite{BG71} Proposition VIII.1.4.

**Proposition 2.8.** The subcategory $\text{Curves}_{\text{LCI}}$ of $\text{Curves}$ of flat families which are LCI is an open substack of $\text{Curves}$.

**Proof.** Let $\pi : C \to S$ be a flat family of proper curves, and assume for the moment that $C$ is a scheme. According to \cite{Gro67} Corollaire 19.3.8 there exists an open subscheme $U \subset S$ such that $C_s$ is LCI if and only if $s \in U$. By definition \cite{Gro67} Définition 19.3.6 the induced morphism $C_U \to U$ is LCI. It follows from \cite{Gro67} Proposition 19.3.9 that given a morphism of schemes $g : T \to S$ then inverse image $g^{-1}(U) \subset T$ has the same property for the family over $T$.

In the general case (where we do not assume that $C$ is a scheme) we use that there is a smooth cover $S' \to S$ such that the pullback family over $S'$ has a total space which is a scheme (for example by using \cite{2.3}). Hence we obtain an open $U' \subset S'$ characterized as above. Clearly $U' \times_S S' = S' \times_S U'$ and we conclude that $U'$ is the inverse image of an open $U$ of $S$. Again by definition the restriction $C_U \to U$ is LCI, and all fibres of $C \to S$ at points of $S \setminus U$ are not LCI. Clearly this shows that any base change by $T \to S$ is LCI if and only if $T \to S$ factors through $U$. \qed

**Definition 2.9.** Let $\mathcal{X}$ be an algebraic stack, let $S$ be a scheme and let $\mathcal{X} \to S$ be a morphism. The stack of maps of proper, LCI curves to $\mathcal{X}$ is the open substack $\text{CurveMaps}_{\text{LCI}}(\mathcal{X}/S)$ of $\text{CurveMaps}(\mathcal{X}/S)$ obtained as the 2-fibred product of $\text{CurveMaps}(\mathcal{X}/S) \to \text{Curves}$ and $\text{Curves}_{\text{LCI}} \to \text{Curves}$.

In order to define the determinant pushforward below we need the following lemma.

**Lemma 2.10.** Suppose we are given a diagram of morphisms of algebraic spaces

\[
\begin{array}{ccc}
C & \longrightarrow & Z \\
\downarrow & & \downarrow \\
T & \longrightarrow & S
\end{array}
\]

We then consider the determinant pushforward $\mathcal{K} : \text{Curves}_{\text{LCI}}(\mathcal{X}/S) \times \text{Curves}_{\text{LCI}}(\mathcal{X}/S) \to \text{Curves}_{\text{LCI}}(\mathcal{X}/S)$ obtained as the 2-fibred product of $\text{Curves}_{\text{LCI}}(\mathcal{X}/S) \to \text{Curves}$ and the determinant pushforward of $\text{Curves}_{\text{LCI}}(\mathcal{X}/S) \to \text{Curves}_{\text{LCI}}(\mathcal{X}/S)$.
where $C \to T$ is an LCI, flat proper family of curves, and $Z \to S$ is a smooth morphism. Then the induced morphism $C \to T \times_SZ$ is a local complete intersection morphism in the sense of [BG171, Définition VIII.1.1]. In particular, it is a perfect morphism in the sense of [BG171, Définition III.4.1].

Proof. Note that $T \times_SZ$ is smooth over $T$, and hence it suffices to prove the lemma in the case that $S = T$. By [BG171 Proposition VIII.1.6] it suffices to prove the lemma after a faithfully flat base change. Hence it is also sufficient to prove the proposition in the case that $C \to S$ is projective, by 2.3. Choose a closed immersion $C \to \mathbf{P}^N_S$ over $S$. This is regular by [BG171 Proposition VIII.1.2]. According to [BG171 Corollaire VIII.1.3] this implies that $C \to \mathbf{P}^N_S \times_SZ$ is regular, which in turn by definition implies that $C \to Z$ is a local complete intersection morphism. □

We are going to use the stack $\text{CurveMaps}(\mathcal{X}/S)$ in the case where $\mathcal{X} = Z \times_B\text{GL}_n$ for some algebraic space $Z$ separated and of finite presentation over $S$. Note that by Proposition 2.6 this is a limit preserving algebraic stack. An object of this stack is given by a datum $(T \to S, \pi : C \to T, \zeta : C \to Z, \mathcal{E})$ where $(T \to S, \pi : C \to T, \zeta : C \to Z)$ is a family of maps of proper curves to $Z$ over $S$ and where $\mathcal{E}$ is a locally free sheaf of rank $n$ over $C$. It is straightforward to spell out what the morphisms are in this stack (left to the reader). The object $(T \to S, \pi : C \to T, \zeta : C \to Z, \mathcal{E})$ belongs to $\text{CurveMaps}_{\text{LCI}}(\mathcal{X}/S)$ if and only if $\pi$ is LCI.

Assuming that $S$ is excellent, $Z$ is a scheme, $Z \to S$ is quasi-compact and smooth and $\mathcal{X} = Z \times_B\text{GL}_n$ we define a functor

$$\det(R(-)_*) : \text{CurveMaps}_{\text{LCI}}(\mathcal{X}/S) \to \text{Hom}_S(Z, \text{BG}_n) = \text{Pic}_{Z/S}$$

to the Picard stack of $Z$ over $S$. First we note that, by our discussion above both sides are limit preserving algebraic stacks over $S$. Thus we need only define the functor on fibre categories over schemes of finite type over $S$. Consider an object $(T \to S, \pi : C \to T, \zeta : C \to Z, \mathcal{E})$ of the left hand side with $T/S$ finite type. The morphism $\zeta : C \to T \times_SZ$ is a perfect morphism by Lemma 2.10. Since $\zeta$ is perfect $R\zeta_*\mathcal{E}$ is a perfect complex of bounded amplitude on $T \times_SZ$ by [BG171 Corollaire III.4.8.1]. (We leave it to the reader to check that we’ve put in enough finiteness assumptions so the Corollary applies.) The “det” construction of [KM76] associates an invertible sheaf $\det(R\zeta_*\mathcal{E})$ to the perfect complex $R\zeta_*\mathcal{E}$ on the scheme $T \times_SZ$. Also, by [KM76 Definition 4(iii)] and [BG171 Proposition IV.3.1.0], formation of $\det(R\zeta_*\mathcal{E})$ is compatible with morphisms of objects of $\text{CurveMaps}_{\text{LCI}}(\mathcal{X}/S)$.

Definition 2.11. Given $S$, $Z \to S$ and $n$ as above. The determinant pushforward $1$-morphism is the functor

$$\det(R(-)_*) : \text{CurveMaps}_{\text{LCI}}(Z \times_B\text{GL}_n/S) \to \text{Pic}_{Z/S}$$

defined above. If the Picard stack $\text{Pic}_{Z/S}$ has a coarse moduli space $\underline{\text{Pic}}_{Z/S}$ then the composite morphism

$$\text{CurveMaps}_{\text{LCI}}(Z \times_B\text{GL}_n/S) \to \underline{\text{Pic}}_{Z/S}$$

will also be called the determinant pushforward $1$-morphism.

The following is an important special case for our paper. Namely, suppose that $S = \text{Spec}(\kappa)$ is the spectrum of a field $\kappa$, and suppose that $Z = C$ is a smooth, projective, geometrically connected $\kappa$-curve of genus $g$. In this case we can “compute” the value of the determinant 1-morphism on some special points. Assume that we
Lemma 2.12. Assumptions and notation as above. There is an isomorphism
\[ \det(R_{s*}(E)) = \det(s^*E)(d_1 \cdot t_1 + \cdots + d_\delta \cdot t_\delta) \]
where \(d_i\) is the degree of the basechange of \(E\) to the connected nodal curve \(C_i\) over the field \(\kappa(t_i)\).

Proof. For a comb \(C' = s(C) \cup C_1 \cup \cdots \cup C_\delta\) as above, there is a surjection of \(O_{C'}\)-modules \(E \to E_{|s(C)}\) whose kernel we will write as \(\oplus E_{|C_i}(-t_i)\). We will use later on that \(\chi(C_i, E_{|C_i}(-t_i)) = d_i\) by Riemann-Roch on \(C_i\) over \(\kappa(t_i)\). Pushing this forward by \(\zeta\) gives a triangle of perfect complexes:

\[
R_{s*}(\oplus E_{|C_i}(-t_i)) \xrightarrow{\lambda} R_{s*}(E) \xrightarrow{\lambda} R_{s*}(E_{|s(C)})
\]

Note that \(R\zeta_*(E_{|s(C)}) = s^*(E)[0]\), and that the term \(R\zeta_*(\oplus E_{|C_i}(-t_i))\) is supported in the points \(t_i\). Hence \(\lambda\) is “good” and we may apply the “Div” construction of [KM76]. It follows that \(\det(R_{s*}(E)) \cong \det(s^*E)(-\text{Div}(\lambda))\). Using [KM76] Theorem 3(iii) and (vi), it follows that \(\text{Div}(\lambda) = -\sum d_i \cdot t_i\) (minus sign because the complex \(\mathcal{H}\) of locus citatus is our complex \(\zeta_*(\oplus E_{|C_i}(-t_i))\) shifted by 1).

We end this section with a simple semi-continuity lemma.

Lemma 2.13. Let \(S\) be an affine scheme. Let \(\pi : C \to S\) be a flat family of proper curves over \(S\). Let \(\mathcal{F}\) be a quasi-coherent sheaf on \(C\) which is locally finitely presented and flat over \(S\). For every integer \(i \geq 0\) there exists an open subscheme \(U_i\) of \(S\) such that for every \(S\)-scheme \(T\), the derived pushforward to \(T\) of the pullback of \(\mathcal{F}\) to \(T \times_S C\) is concentrated in degrees \(\leq i\) if and only if \(T\) factors through \(U_i\). Moreover, after basechange from \(S\) to \(U_i\), the formation of \(R^i\pi_*(\mathcal{F})\) is compatible with arbitrary basechange.

This also holds when \(S\) is an arbitrary scheme, algebraic space, etc., instead of an affine scheme.

Proof. This is probably true in some vast generality for morphisms of algebraic stacks, but we do not know a reference. In our case we may deduce it from the schemes case as follows. By Proposition 2.13 there exists a faithfully flat morphism of affine schemes \(S' \to S\) such that \(S' \times_S C\) is projective over \(S'\). By [LMB00] Proposition 13.1.9, the statement for the original family over \(S\) follows from the statement over \(S'\). Also by limit arguments, it suffices to consider the case when \(S'\) is of finite type. Now the result follows from [Gro63] Section 7 or [Mum70] Section 5. The general case follows from the affine case by [LMB00] Proposition 13.1.9. \(\square\)
3. Free sections

This section is a brief summary of recent work on Hom stacks together with results about free curves in this context.

Let $S$ be an algebraic space. Let $C \to S$ be a proper, flat, finitely presented algebraic stack over $S$. Let $f : \mathcal{X} \to C$ be a 1-morphism of algebraic $S$-stacks.

**Definition 3.1.** With notation as above. Let $T$ be an $S$-scheme, or an algebraic space over $S$.

1. A family of sections of $f$ over $T$ is a pair $(\tau, \theta)$ consisting of a 1-morphism of $S$-stacks $\tau : T \times_S C \to \mathcal{X}$ together with a 2-morphism $\theta : f \circ \tau \Rightarrow \text{pr}_C$, giving a 2-commutative diagram

![Diagram](image)

2. For two families of sections of $f$, say $(T', \tau', \theta')$ and $(T, \tau, \theta)$ from the first family to the second family is given by a morphism $u : T' \to T$ and a 2-morphism $\eta : \tau \circ (u \times \text{id}_C) \Rightarrow \tau'$ such that

\[
\begin{array}{c}
\begin{array}{c}
\tau' \\
\downarrow \theta'
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\tau \\
\downarrow \theta
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{pr}_C^{T' \times_S C} \\
\downarrow \text{pr}_C^{T \times_S C} \circ (u \times \text{id}_C)
\end{array}
\end{array}
\]

commutes.

**Definition 3.2.** The stack of sections of $f$ is the fibred category

\[
\text{Sections}(\mathcal{X}/C/S) \longrightarrow \text{Schemes}/S
\]

whose objects are family of sections of $f$ and whose morphisms are morphisms of families of sections of $F$.

We recall some results from the literature.

**Theorem 3.3.** Let $\mathcal{C} \to S$, $f : \mathcal{X} \to C$ be as above.

1. See [Gro62, Part IV.4.c, p. 221-19]. Assume that $C = C$ is a projective scheme over $S$ and that $X = X$ is a quasi-projective scheme over $C$ either globally over $S$, resp. fppf locally over $S$. Then Sections($\mathcal{X}/C/S$) is an algebraic space which is locally finitely presented and separated over $S$. Moreover, globally over $S$, resp. fppf locally over $S$, the connected components of Sections($\mathcal{X}/C/S$) are quasi-projective over $S$.

2. See [Ols06, Theorem 1.1]. Assume that $C$ and $X$ are separated over $S$ and have finite diagonal. Also assume that fppf locally over $S$ there is a finite, finitely presented cover of $C$ by an algebraic space. Then Sections($\mathcal{X}/C/S$) is a limit preserving algebraic stack over $S$ with separated, quasi-compact diagonal.
3. See [Lie06, Proposition 2.11]. Assume $S$ excellent and $\mathcal{C} = C$ is a proper flat family of curves. Assume that, fppf locally over $S$, we can write $\mathcal{X} = [Z/G]$ for some algebraic space $Z$ separated and of finite type over $S$ and some linear group scheme $G$ flat over $S$. Then $\text{Sections}(\mathcal{X}/\mathcal{C}/S)$ is a limit preserving algebraic stack over $S$.

**Proof.** Besides the references, here are some comments. For the first case we may realize the space of sections as an open subscheme of Hilbert scheme of $\mathcal{C} \times_S X$ over $S$. In the second and third case we think of $\text{Sections}(\mathcal{X}/\mathcal{C}/S)$ as a substack of the Hom-stack $\text{Hom}_S(\mathcal{C}, \mathcal{X})$. The references guarantee the existence of the Hom-stack as an algebraic stack. Next we consider the 2-cartesian diagram

\[
\begin{array}{c}
\text{Sections}(\mathcal{X}/\mathcal{C}/S) \\
\downarrow \\
\text{Hom}_S(\mathcal{C}, \mathcal{X})
\end{array}
\quad \xrightarrow{\sim} \quad 
\begin{array}{c}
\text{Sections}(\mathcal{X}/\mathcal{C}/S) \\
\downarrow \\
\text{Hom}_S(\mathcal{C}, \mathcal{C})
\end{array}
\]

In case 2 resp. 3 the lower horizontal arrow is representable by the result from [Ols06] resp. because of our earlier results on flat families of proper curves. □

Of course we will most often use the space of sections for a family of varieties over a curve. We formulate a set of hypothesis that is convenient for the developments later on.

**Hypothesis 3.4.** Let $\kappa$ be a field. Let $C$ be a projective, smooth, geometrically connected curve over $\kappa$. Let $f : X \to C$ be a quasi-projective morphism. Let $\mathcal{L}$ be an $f$-ample invertible sheaf on $X$. Denote the smooth locus of $X$ by $X_{\text{smooth}}$. Denote the open subset of $X_{\text{smooth}}$ on which $f$ is smooth by $X_{f, \text{smooth}}$.

According to the theorem above the space of sections $\text{Sections}(X/C/\kappa)$ is a union of quasi-projective schemes over $\kappa$ in this case. The locus $\text{Sections}(X_{f, \text{smooth}}, C/\kappa)$ is an open subscheme of $\text{Sections}(X/C/\kappa)$.

**Definition 3.5.** In the situation 3.4. For every integer $e$, the scheme of degree $e$ sections of $f$ is the open and closed subscheme $\text{Sections}^e(X/C/\kappa)$ of $\text{Sections}(X/C/\kappa)$ parametrizing sections $\tau$ of $f$ which pullback $\mathcal{L}$ to a degree $e$ invertible sheaf on $C$. The universal degree $e$ section of $f$ is denoted

\[
\sigma : \text{Sections}^e(X/C/\kappa) \times_\kappa C \longrightarrow X.
\]

The Abel map is the $\kappa$-morphism

\[
\alpha : \text{Sections}^e(X/C/\kappa) \longrightarrow \text{Pic}^e_{C/\kappa},
\]

associated to the invertible sheaf $\sigma^* \mathcal{L}$ by the universal property of the Picard scheme $\text{Pic}^e_{C/\kappa}$.

There are several variations of the notion of “free curve” as described in [Ko96, Definition II.3.1]. Of the following two notions, the weaker notion arises more often geometrically and implies the stronger notion in a precise sense.

**Definition 3.6.** In Situation 3.4. Let $T$ be a $\kappa$-scheme. Let $\tau : T \times_\kappa C \to X_{f, \text{smooth}}$ be a family of degree $e$ sections of $f$ with image in $X_{f, \text{smooth}}$.
1. Let $D$ be an effective Cartier divisor in $C$. The family $\tau$ is weakly $D$-free if the $O_T$-module homomorphism 
\[ (pr_T)_*(\tau^*T_f) \longrightarrow (pr_T)_*(\tau^*T_f \otimes_{O_{T \times_s C}} pr_C^*O_D) \]
is surjective. Here $T_f = Hom(\Omega^1_{X/C}, O_X)$.
2. Let $D$ be an effective Cartier divisor in $C$. The family $\tau$ is $D$-free if the sheaf of $O_T$-modules 
\[ R^1(pr_T)_*(\tau^*T_f \otimes_{O_{T \times_s C}} pr_C^*O_C(-D)) \]
is zero.
3. For an integer $m \geq 0$, the family is weakly $m$-free if the basechange to $\text{Spec}(\overline{\mathbb{k}})$ is weakly $D$-free for every effective, degree $m$, Cartier divisor $D$ in $C \times_{\kappa} \text{Spec}(\overline{\mathbb{k}})$.
4. For an integer $m \geq 0$, the family is $m$-free if the basechange to $\text{Spec}(\overline{\mathbb{k}})$ is $D$-free for every effective, degree $m$, Cartier divisor $D$ in $C \times_{\kappa} \text{Spec}(\overline{\mathbb{k}})$.

Whenever we talk about $D$-free, weakly $D$-free, weakly $m$-free or $m$-free sections we tacitly assume that the image of the section is contained in $X_{f, \text{smooth}}$.

**Definition 3.7.** In Situation 3.4 a section $s : C \to X$ of $f$ is called unobstructed if it is 0-free. It is called free if it is 1-free.

It it true and easy to prove that these correspond to the usual notions. Namely, if a section is unobstructed then its formal deformation space is the spectrum of a power series ring. Also, a section $s$ is free if and only if it is unobstructed and $s^*T_{X/C}$ is globally generated, which means that its deformations sweep out an open in $X$.

**Lemma 3.8.** In Situation 3.4

1. Let $D$ be an effective Cartier divisor on $C$. There is an open subscheme $U$ of Sections$^e(X/C/\kappa)$ parametrizing exactly those families whose image lies in $X_{f, \text{smooth}}$ and which are $D$-free.
2. Let $m \geq 0$ be an integer. There is an open subscheme $V$ of Sections$^e(X/C/\kappa)$ parametrizing exactly those families whose image lies in $X_{f, \text{smooth}}$ and which are $m$-free.
3. The set of $\overline{\mathbb{k}}$-points of Sections$^e(X/C/\kappa) \times \text{Sym}^m(C)$ corresponding to pairs $(\tau, D)$ such that $\tau$ is $D$-free is open.

**Proof.** The first part follows from Lemma 2.13. To see the second part consider a family $\tau : T \times_{\kappa} C \to X_{f, \text{smooth}}$. Let $T' = T \times_{\kappa} \text{Sym}^m(C)$. On $T' \times_{\kappa} C$ we have a “universal” relative Cartier divisor $D$ (of degree $m$ over $T'$). Hence we may consider the open $U \subset T' = T \times_{\kappa} \text{Sym}^m(C)$ where the relevant $R^1$ sheaf vanishes. Let $V \subset T$ be the largest open subset such that $V \times_{\kappa} \text{Sym}^m(C)$ is contained in $U$. We leave it to the reader to verify that a base change by $T'' \to T$ of $\tau$ is $m$-free if and only if $T'' \to T$ factors through $V$. The third part is similar to the others. □

**Lemma 3.9.** In situation 3.4. Let $\tau : T \times_{\kappa} C \to X_{f, \text{smooth}}$ be a family of sections, let $u : T' \to T$ be a morphism and let $\tau' = \tau \circ (u, id_C)$.

1. Let $D$ be an effective Cartier divisor in $C$. If $\tau$ is weakly $D$-free, then so is $\tau'$.
2. Let $m \geq 0$ be an integer. If $\tau$ is weakly $m$-free then so is $\tau'$.

The converse holds if $u$ is faithfully flat.
Proof. We first prove 1. The sheaf of $\mathcal{O}_T$-modules $\mathcal{G} = (\text{pr}_T)_*(\tau^* T_f \otimes \mathcal{O}_{X,C} \text{pr}_C^* \mathcal{O}_D)$ from Definition 3.10 is finite locally free (of rank the degree of $D$ times the dimension of $X$ over $C$) and its formation commutes with base change. The sheaf $F \to G$ is quasi-coherent but its formation only commutes with flat base change in general. Denote $F'$ the corresponding sheaf for $\tau'$. Assume the map $F \to G$ is surjective. Then $F' \to G'$ is surjective because $u^* F \to u^* G$ factors through $F'$. The statement about the faithfully flat case follows from this discussion as well.

To prove 2 we use the arguments of 1 for the universal situation over $T \times X$. As in the proof of Lemma 3.8 above (left to the reader).

Lemma 3.10. In situation $T \times X$, let $\tau : T \times X \to X$. If $\tau$ is $D$-free, resp. weakly $D$-free, then it is also $D'$-free, resp. weakly $D'$-free.

Proof. Consider the finite locally free sheaves of $\mathcal{O}_T$-modules defined by $\mathcal{G} = (\text{pr}_T)_*(\tau^* T_f \otimes \mathcal{O}_{X,C} \text{pr}_C^* \mathcal{O}_D)$ and $\mathcal{G'} = (\text{pr}_T)_*(\tau'^* T_f \otimes \mathcal{O}_{X,C} \text{pr}_C^* \mathcal{O}_{D'})$ that occur in Definition 3.10. It is clear that there is a surjection $\mathcal{G} \to \mathcal{G'}$. On the other hand consider the sheaves $\mathcal{H} = R^1(\text{pr}_T)_*(\tau^* T_f \otimes \mathcal{O}_{X,C} \text{pr}_C^* \mathcal{O}_C(-D))$ and $\mathcal{H'} = R^1(\text{pr}_T)_*(\tau'^* T_f \otimes \mathcal{O}_{X,C} \text{pr}_C^* \mathcal{O}_C(-D'))$. There is a surjection $\mathcal{H} \to \mathcal{H'}$ induced by the map $\mathcal{O}_C(-D) \to \mathcal{O}_C(-D')$ with finite cokernel (tensored with $\tau^* T_f$ this has a cokernel finite over $T$). In this way we deduce 1.

In order to prove 2 we work on $T' = T \times X$. There are “universal” relative effective Cartier divisors $D \subset T' \times X$ (of degree $m$) and $D' \subset T' \times X$ (of degree $m'$) with $D \subset D'$. There are sheaves of $\mathcal{O}_{T'}$-modules $\mathcal{G}$, $\mathcal{G'}$, $\mathcal{H}$, and $\mathcal{H'}$ which are variants of the sheaves above defined using the relative Cartier divisors $D$ and $D'$. If $\tau$ is $m$-free, resp. weakly $m$-free, then $\mathcal{H} = 0$, resp. $(\text{pr}_{T'})_*((\tau')^* T_f)$ surjects onto $\mathcal{G}$. The arguments just given to prove part 1 show that there are canonical surjections $\mathcal{H} \to \mathcal{H'}$ and $\mathcal{G} \to \mathcal{G'}$. Hence we deduce that $\mathcal{H'} = 0$ resp. $(\text{pr}_{T'})_*((\tau')^* T_f)$ surjects onto $\mathcal{G'}$. Then this in turn implies that $\tau$ is $m$-free, resp. weakly $m$-free.

Lemma 3.11. In situation $T \times X$, let $\tau : T \times X \to X$. If $\tau$ is $D$-free if and only if it is both 0-free and weakly $D$-free.

Proof. We first prove 1. Saying a family is 0-free is equivalent to saying it is $D'$-free where $D'$ is the empty Cartier divisor. Thus, if a family is $D$-free then it is 0-free by Lemma 3.10. The long exact sequence of higher direct images associated to the short exact sequence

$$0 \to \tau^* T_f \otimes \mathcal{O}_{X,C} \text{pr}_C^* \mathcal{O}_D(-D) \to \tau^* T_f \to \tau^* T_f \otimes \mathcal{O}_{X,C} \text{pr}_C^* \mathcal{O}_D \to 0$$

shows that

$$(\text{pr}_T)_*(\tau^* T_f) \to (\text{pr}_T)_*(\tau^* T_f \otimes \mathcal{O}_{X,C} \text{pr}_C^* \mathcal{O}_D)$$
is surjective so long as $R^1(\text{pr}_f)_*(\tau^*T_f \otimes_{\mathcal{O}_T} \mathcal{O}_C(-D))$ is the zero sheaf. Moreover, the converse holds so long as $R^1(\text{pr}_S)_*(\tau^*T_f)$ is the zero sheaf. From this it follows that if a family of sections is $D$-free, then it is weakly $D$-free and the converse holds so long as the family is 0-free.

The proof of 2 is left to the reader. (It is similar to the proofs above.)

Lemma 3.12. In situation 3.4. Let $\tau : T \times_\kappa C \to X_{f,\text{smooth}}$ be a family of sections.

1. Let $D$ be a nontrivial effective Cartier divisor of $C$. If $\deg(D) \geq 2g(C)$ and if $\tau$ is weakly $D$-free, then $\tau$ is $(\deg(D) - 2g(C))$-free (and thus $D$-free by the previous lemmas).

2. Let $m \geq 2g$ be an integer, and $m \geq 1$ if $g(C) = 0$. Every weakly $m$-free family is $m$-free.

Proof. We prove part 1. By Lemma 3.8 there is a maximal open subscheme $V$ of $T$ over which $\tau$ is $(\deg(D) - 2g(C))$-free. The goal is to prove that $V$ is all of $T$. For this, it suffices to prove that every geometric point of $S$ factors through $V$. By Lemma 3.9 the basechange of the family to any geometric point is weakly $D$-free. Thus, it suffices to prove the lemma when $\kappa$ is algebraically closed and $T = \text{Spec}(\kappa)$, i.e., $\tau$ is a section $\tau : C \to X_{f,\text{smooth}}$.

Let $m$ denote $\deg(D)$. Let us write $T = \tau^*T_f$, and let $r$ be the rank of $T$. The assumption is that the restriction map

$$H^0(C, T) \to H^0(C, T \otimes_{\mathcal{O}_C} \mathcal{O}_D)$$

is surjective. Let $p \in D$ be a point. The restriction map

$$H^0(C, T \otimes_{\mathcal{O}_C} \mathcal{O}_C(-D + p)) \to H^0(C, T \otimes_{\mathcal{O}_C} \mathcal{O}_C(-D + p))$$

is also surjective. In other words, there exist $r$ global sections of $T(-D + p)$ generating the fibre at $p$. These global sections give an injective $\mathcal{O}_C$-module homomorphism $\mathcal{O}_C^\oplus \to T(-D + p)$ which is surjective at $p$. Thus the cokernel is a torsion sheaf. Let $E \subset C$ be an effective divisor of degree $m - 2g(C)$. Twisting by $D - E - p$, this gives an injective $\mathcal{O}_C$-module homomorphism

$$\mathcal{O}_C(D - E - p)^{\oplus r} \to T(-E)$$

whose cokernel is torsion. From the long exact sequence of cohomology, there is a surjection

$$H^1(C, \mathcal{O}_C(D - E - p))^{\oplus r} \to H^1(C, T(-E)).$$

Since $\deg(D - E - p) > 2g(C) - 2$ we see that $h^1(C, \mathcal{O}_C(D - E - p))$ equals 0. Therefore $h^1(C, T(-E))$ also equals 0.

Part 2 is proved in a similar fashion. \qed

Proposition 3.13. Situation as in 3.4. Let $D \subset C$ be an effective divisor.

1. Let $\tau : T \times_\kappa C \to X_{f,\text{smooth}}$ be a family of sections. If the induced morphism $(\tau|_{T \times_\kappa D}, \text{Id}_D) : T \times_\kappa D \to X_{f,\text{smooth}} \times_\kappa D$ is smooth, then the family is weakly $D$-free.

2. Let $U$ be the open subscheme of $\text{Sections}^f(X_{f,\text{smooth}}/C/\kappa)$ over which the universal section is $D$-free (see Lemma 3.8). The morphism

$$(\sigma, \text{Id}_D) : U \times_\kappa D \to X_{f,\text{smooth}} \times_\kappa D$$
is smooth.

3. The scheme $\text{Sections}(X/C/\kappa)$ is smooth at every point which corresponds to a section that is $D$-free.

**Proof.** When $D$ is the empty divisor, both 1 and 2 are vacuous, and 3 follows from the vanishing of the obstruction space for the deformation theory. When $D$ is not empty, one argues as in [Kol96, Proposition II.3.5].

In order to state the following two propositions we need some notation. Consider the Hilbert scheme $\text{Hilb}^m_{X/\kappa}$ of finite length $m$ closed subschemes of $X$ over $\kappa$. There is an open subscheme $H^m_{X,f} \subset \text{Hilb}^m_{X/\kappa}$ parametrizing those $Z \subset X$ such that (a) $Z \subset X_{f,\text{smooth}}$, and (b) $f|_Z : Z \to C$ is a closed immersion. By construction there is a natural quasi-projective morphism $H^m_{X,f} \to \text{Sym}^m(C)$.

A point of $H^m_{X,f}$ also corresponds to a pair $(Z', \sigma)$, where $Z' \subset C$ is a closed subscheme of length $m$, and $\sigma : Z' \to X_{f,\text{smooth}}$ which is a section of $f$. And there are no obstructions to deforming a map from a zero dimensional scheme to a smooth scheme. Therefore the morphism $H^m_{X,f} \to \text{Sym}^m(C)$ is smooth.

The fibre of this morphism over $[D]$ (where $D$ is a degree $m$ effective Cartier divisor on $C$) is the space of sections of $D \times_C X_{f,\text{smooth}}$. Let us denote this fibre by $H^m_{X,f,D}$. Note that $H^m_{X,f,D}$ equals the Weil restriction of scalars $\text{Res}_{D/\kappa}^D(D \times_C X_{f,\text{smooth}})$. In fact $H^m_{X,f}$ is a Weil restriction of scalars as well, which leads to another construction. Namely $H^m_{X,f} = \text{Res}_{D/\text{Sym}^m(C)}^D(D \times_C X_{f,\text{smooth}})$, where $D \subset \text{Sym}^m(C) \times \kappa C$ is the universal degree $m$ effective Cartier divisor on $C$. We denote the evaluation map

$$ev^m : H^m_{X,f} \times_{\text{Sym}^m(C)} D \to X_{f,\text{smooth}}$$

Note that given a section $\tau$ of $f : X \to C$ which ends up in the smooth locus of $f$, we may restrict $\tau$ to any divisor $D$ and obtain a point of $H^m_{X,f,D}$. In other words there is a canonical morphism

$$\text{res} : \text{Sections}(X_{f,\text{smooth}}/C/\kappa) \times_{\kappa} \text{Sym}^m(C) \to H^m_{X,f}$$

of schemes over $\text{Sym}^m(C)$.

**Lemma 3.14.** The morphism $\text{res}$ is smooth at any point $(\tau, D)$ such that $\tau$ is $D$-free.

**Proof.** This is akin to Proposition 3.13 and proved in the same way. It also follows from part 2 of that Proposition by the fibrewise criterion of smoothness since both $\text{Sections}(X_{f,\text{smooth}}/C/\kappa) \times_{\kappa} \text{Sym}^m(C)$ and $H^m_{X,f}$ are smooth over $\text{Sym}^m(C)$ at the corresponding points.

Here are two technical results that will be used later.

**Proposition 3.15.** See [KMM92a, 1.1]. In situation 3.4 with $\text{char}(\kappa) = 0$. Let $e$ and $m \geq 0$ be integers. There exists a constructible subset $W^m_e$ of $H^m_{X,f}$ whose intersection $W^m_e \cap H^m_{X,f,D}$ with the fibre over every point $[D]$ of $\text{Sym}^m(C)$ contains a dense open subset of $H^m_{X,f,D}$ with the following property: For every $\kappa$-scheme $T$ and for every family of degree $e$ sections

$$\tau : T \times_{\kappa} C \to X_{f,\text{smooth}},$$

...
if the restriction of \( \tau \) to \( T \times_\kappa D \) is in \( W_e \cap H^m_{X,f,D} \), then \( \tau \) is weakly \( D \)-free.

**Proof.** It suffices to construct \( W^m_e \) over \( \overline{\kappa} \), and hence we may and do assume \( \kappa \) algebraically closed. Stratify Sections \( (X_f, \text{smooth}/C/\kappa) \) by a finite set of irreducible locally closed strata \( T_i \), each smooth over \( \kappa \). For each \( i \) we have the restriction morphism \( \text{res}_i : T_i \times \text{Sym}^m(C) \to H^m_{X_f} \). This is a morphism of smooth schemes of finite type over \( \text{Sym}^m(C) \). Let \( S_i \subset T_i \times \text{Sym}^m(C) \) be closed set of points where the derivative of \( \text{res}_i \)

\[
d\text{res}_i : T_i \times \text{Sym}^m(C) \to \text{res}_i^!(T_{H^m_{X,f}})
\]

is not surjective. By Sard’s theorem, or generic smoothness the image \( E_i = \text{res}_i(S_i) \subset H^m_{X,f} \) is not dense in any fibre \( H^m_{X,f,D} \). It is constructible as well by Chevalley’s theorem. Take \( W^m_e \) to be the complement of the finite union \( \bigcup E_i \). The reason this works is the following: Suppose that \( \tau : C \to X_f, \text{smooth} \) is a section such that \( \tau|_D \) is in \( W^m_e \). Let \( i \) be such that \( \tau \) corresponds to a point \( [\tau] \in T_i \). By construction of \( W^m_e \) the morphism \( T_i \to H^m_{X_f,D} \) is smooth with surjective derivative at \( \tau \).

Note that \( T_\tau(T_i) \subset H^0(C, \tau^*T_f) \), and that \( T_{\text{res}(\tau \times [D])}(H^m_{X,f,D}) = H^0(C, \tau^*T_f|_D) \). Thus the surjectivity of the derivative in terms of Zariski tangent spaces implies that \( \tau \) is weakly \( D \)-free. \( \square \)

**Lemma 3.16.** Let \( k \) be an uncountable algebraically closed field. Suppose we are given the commutative “input” diagram of varieties over \( k \):

**input:**

\[
\begin{array}{ccc}
V & \xrightarrow{\Phi} & Y \\
U \downarrow & & \downarrow \tau \uparrow \\
C & \xleftarrow{} & C \times T
\end{array}
\]

**output:**

\[
\begin{array}{ccc}
V & \xleftarrow{\Phi} & Y \\
U \downarrow & & \downarrow \tau \uparrow \\
C & \xrightarrow{} & C \times T
\end{array}
\]

We assume that \( C \) is a nonsingular projective curve, \( Y \to C \) is proper, \( U \subset Y \) is open dense, \( U \to C \) is smooth with geometrically irreducible fibres, \( V \to U \) is smooth surjective with birationally rationally connected geometric fibres. We assume that the image of \( C \times T \to Y \) meets \( U \). Let \( m > 0 \). We assume the associated rational map (see proof) from \( T \times \text{Sym}^m(C) \) to \( H^m_{X,f,C} \) is dominant. Then there exists a variety \( T' \), a dominant morphism \( T' \to T \), a compactification \( V \subset \overline{V} \) and a morphism \( C \times T' \to \overline{V} \) fitting into the commutative “output” diagram such that the image of \( C \times T' \to \overline{V} \) meets \( V \) and the associated rational map from \( T' \times \text{Sym}^m(C) \) to \( H^m_{X,f} \) is dominant.

**Remark 3.17.** This technical lemma just says that a general \( m \)-free section of \( Y \to C \) lifts to an \( m \)-free section of \( V \).

**Proof.** For \( t \in T(k) \) we denote \( \tau_t : C \to Y \) the restriction of \( \tau \) to \( C \times \{ t \} \). In other words, we think of \( \tau \) as a family of maps from \( C \) to \( Y \). Let \( W \subset \text{Sym}^m(C) \times T \) denote the set of pairs \( (D,t) \) such that \( \tau_t(D) \subset U \). This is nonempty by assumption.

The “associated rational map” is the map

\[
\text{Sym}^m(C) \times T \ni W \to H^m_{U,C}, \quad (D,t) \mapsto (\tau_t|_D : D \to U).
\]

Note that \( V \) is a nonsingular variety. We choose a smooth projective compactification \( V \subset \overline{V} \) such that \( \Phi \) extends to a morphism \( \overline{\Phi} : \overline{V} \to Y \), see [Hir64]. By Sard’s
Theorem we may shrink $U$ and assume that the fibres of $\Phi$ are projective smooth varieties over all points of $U$. In this case these fibres are rationally connected varieties by our assumption on $\Phi$. We may shrink $T$ and assume that $\tau_i(C)$ meets $U$ for all $t \in T(k)$.

The assumption on the fibres of $\Phi$ implies that for every $t \in T(k)$ we may find a section $\sigma : C \to \overline{V}$ such that $\Phi \circ \sigma = \tau_i$. See [GHS03]. It was already known before the results of [GHS03], see for example [KMM92], that one may, given any $m$ sufficiently general points $c_1, \ldots, c_m$ of the curve $C$, assume the $m$-tuple $(\sigma(c_i) \in \overline{V}_{\tau_i(c_i)})_{i=1, \ldots, m}$ is a general point of $V_{\tau_i(c_1)} \times \cdots \times V_{\tau_i(c_m)}$. This is done by the “smoothing combs” technique of [KMM92]. Namely, after attaching sufficiently many free teeth (in fibres of $\overline{V} \to Y$) to $\sigma$, the comb can be deformed to a section $\sigma$ which passes through $m$ very general points of $m$ very general fibres. (Another approach would be to use the main result of the paper [HT06].) Note that since the fibres of $\overline{V} \to U$ are smooth and rationally connected we are allowed to use that result. Of course this paper proves something much stronger!) Combined with the assumption that $\text{Sym}^m(C) \times T \subseteq W \to H_U^m$ is dominant this implies that in fact the $m$-points $\sigma(c_i)$ hit a general point of $V_{\sigma(c_1)} \times \cdots \times V_{\sigma(c_m)}$ which is the fibre of $H_U^m \to \text{Sym}^m(C)$ over the point $\sum c_i$.

The paragraph above solves the problem “pointwise”, but since the field $k$ is uncountable this implies the result of the lemma by a standard technique. Namely, the above produces lots (uncountably many) of points on the scheme $\text{Sections}(\overline{V} \times Y)/(C \times T)/(C \times T)/T$ which implies that it must have a suitable irreducible component $T'$ dominating $T$ (details left to the reader). \hfill $\square$

4. Kontsevich stable maps

Let $k$ be an algebraically closed field and let $Y$ be a quasi-projective $k$-scheme.

**Definition 4.1.** A stable map to $Y$ over $k$ is given by

1. a proper, connected, at-worst-nodal $k$-curve $C$,
2. a finite collection $(p_i)_{i \in I}$ of distinct, smooth, closed points of $C$, and
3. a $k$-morphism $h : C \to Y$.

These data have to satisfy the stability condition that the log dualizing sheaf $\omega_{C/k}(\sum_{i \in I} p_i)$ is $h$-ample.

The stability of the triple $(C, (p_i), h)$ can also be expressed in terms of the numbers of special points on $h$-contracted components of $C$ of arithmetic genus 0 and 1, or by saying that the automorphism group scheme of the triple is finite. See [Kon95].

There are many invariants of a stable map of a numerical or topological nature. One invariant is the number $n = \#I$ of marked points. Occasionally it is convenient to mark the points by some unspecified finite set $I$, but usually $I$ is simply $\{1, 2, \ldots, n\}$. Another invariant is the arithmetic genus $g$ of $C$. Yet another invariant is the curve class of $h_*(\lfloor C \rfloor)$. Often we do not need the full curve class. The following definition makes precise the notion of a “partial curve class”. We will use $\text{NS}(Y)$ to denote the Néron-Severi group of $Y$, i.e., the group of Cartier divisors up to numerical equivalence on $Y$.

**Definition 4.2.** Let $k$ be an algebraically closed field and let $Y$ be a quasi-projective $k$-scheme. A partial curve class on $Y$ is a triple $(A, i, \beta)$ consisting
of an Abelian group $A$ and homomorphisms of Abelian groups

$$i : A \to \text{NS}(Y) \text{ and } \beta : A \to \mathbb{Z}.$$  

The triple will usually be denoted by $\beta$, with $A$ being denoted $A_\beta$ and $i$ being denoted $i_\beta$. We say a stable map $(C, (p_i)_{i \in I}, h)$ belongs to the partial curve class $\beta$ or has partial curve class $\beta$ if for every element $a$ in $A$ and for every Cartier divisor $D$ on $Y$ with class $h^*D$ on $C$ equals $\beta(a)$.

**Remark 4.3.** If there exists a stable map belonging to $\beta$ then $\text{Ker}(\beta)$ contains $\text{Ker}(i)$, i.e., $\beta$ factors through $i(A) \subset \text{NS}(Y)$. From this point of view it is reasonable to restrict to cases where $A$ is a subgroup of $\text{NS}(Y)$. But it is sometimes useful to allow $i$ to be noninjective.

We will also use a relative version of stable maps and of partial curve classes.

**Definition 4.4.** Let $Y \to S$ be a quasi-projective morphism.

1. A family of stable maps to $Y/S$ is given by
   (1) an $S$-scheme $T$,
   (2) a proper, flat family of curves $\pi : C \to T$ (see Definition 2.1),
   (3) a finite collection of $T$-morphisms $(\sigma_i : T \to C)_{i \in I}$, and
   (4) a $S$-morphism $h : C \to Y$.

   These data have to satisfy the stability condition that for every algebraically closed field $k$ and morphism $t : \text{Spec}(k) \to T$, the basechange $(C_t, (\sigma_i(t))_{i \in I}, h_t)$ is a stable map to $Y_t = \text{Spec}(k) \times_t S Y$ as in Definition 2.1

2. Given two families of stable maps $(T' \to S, \pi' : C' \to T', (\sigma'_i)_{i \in I}, h')$ and $(T \to S, \pi : C \to T, (\sigma_i)_{i \in I}, h)$ a morphism from the first family to the second family is given by an $S$-morphism $u : T' \to T$, and a morphism $\phi : C' \to C$ such that $(a)$ $(\phi, u)$ forms a morphism of flat families of curves (see 2.1), $(b)$ $\sigma_i \circ u = \phi \circ \sigma'_i$, and $(c)$ $h' = h \circ \phi$.

Let $Y \to S$ be a quasi-projective morphism. Denote $\text{Pic}_{Y/S}$ the relative Picard functor, see [BLR90] Section 8.1. It is an fppf sheaf on the category of schemes over $S$. We define the relative Néron-Severi functor $\text{NS}_{Y/S}$ of $Y/S$ to be the quotient of $\text{Pic}_{Y/S}$ by the subsheaf of sections numerically equivalent to zero on all geometric fibres. Finally, we define the relative Néron-Severi group $\text{NS}(Y/S) = \text{NS}_{Y/S}(S)$ to be the sections of this sheaf over $S$.

**Definition 4.5.** A partial curve class on $Y/S$ is a triple $(A, i, \beta)$ consisting of an Abelian group $A$, and homomorphisms of Abelian groups

$$i : A \to \text{NS}(Y/S) \text{ and } \beta : A \to \mathbb{Z}.$$  

For every morphism of schemes $T \to S$ there is a pullback partial curve class on $Y_T/T$ given by $(A, i_T, \beta)$ where $i_T$ is the composition

$$A \ni i_T : \text{NS}(Y/S) \to \text{NS}(Y_T/T).$$

A family of stable maps $(T \to S, \pi : C \to T, (\sigma_i)_{i \in I}, h)$ to $Y/S$ belongs to the partial curve class $\beta$ or has partial curve class $\beta$ if for every algebraically closed field $k$ and every morphism $t : \text{Spec}(k) \to T$, the pullback $(C_t, (\sigma_i(t))_{i \in I}, h_t)$ belongs to the pullback partial curve class $(A, i_t, \beta)$. 

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Suppose that \((T \to S, \pi : C \to T, (\sigma_i)_{i \in I}, h)\) is a family of stable maps to \(Y \to S\) as in Definition 4.4. For every \(g \geq 0\) there is an open and closed subscheme of \(T\) where the geometric fibres of \(\pi\) have genus \(g\). This is true because \(R^1\pi_*\mathcal{O}_C\) is a finite locally free sheaf of \(\mathcal{O}_T\)-modules, and the genus is its rank. In addition, given a relative partial curve class \(\beta\), there is an open and closed subscheme of \(T\) where the geometric fibres \(h_t : C_t \to Y_t\) belong to the partial curve class \(\beta_t\). This is because the degree of an invertible sheaf on a proper flat family of curves is locally constant on the base. Thus the following definition makes sense.

**Definition 4.6.** Let \(Y \to S\) be quasi-projective. For every finite set \(I\) the Kontsevich stack of stable maps to \(Y / S\) is the stack
\[
\overline{M}_{g,I}(Y / S) \to \text{Schemes}/S
\]
whose objects and morphisms are as in Definition 4.4 above. When \(I\) is just \(\{1, \ldots, n\}\), this stack is denoted \(\overline{M}_{g,n}(Y / B)\). For every integer \(g \geq 0\), there is an open and closed substack \(\overline{M}_{g,1}(Y / B)\) parametrizing stable maps whose geometric fibres all have arithmetic genus \(g\). And for every partial curve class \(\beta\) on \(Y / S\), there is an open and closed substack \(\overline{M}_{g,1}(Y / B, \beta)\) parametrizing stable maps belonging to the partial curve class \(\beta\).

The following theorem is well known to the experts. We state it here for convenience, and because we do not know an exact reference.

**Theorem 4.7.** Compare [Kon95]. Let \(Y \to S\) be quasi-projective, and \(S\) excellent.

1. The stack \(\overline{M}_{g,1}(Y / S)\) is a locally finitely presented, algebraic stack over \(S\) with finite diagonal, satisfying the valuative criterion of properness over \(S\).
2. If \(S\) lives in characteristic 0 then \(\overline{M}_{g,1}(Y / S)\) is a Deligne-Mumford stack over \(S\).
3. If \(Y\) is proper over \(S\) and if \((A, i, \beta)\) is a partial curve class such that \(i(A)\) contains the class of a \(S\)-relatively ample invertible sheaf on \(Y\), then \(\overline{M}_{g,1}(Y / S, \beta)\) is proper over \(S\) with projective coarse moduli space.

**Proof.** (Sketch.) Forgetting the sections gives a 1-morphism from \(\overline{M}_{g,1}(Y / S)\) to \(\text{CurveMaps}(Y / S)\). It is easy to see that this morphism is representable and smooth. Hence, by Proposition 2.6 the stack \(\overline{M}_{g,1}(Y / S)\) is algebraic of locally of finite presentation over \(S\), with finitely presented diagonal. For the valuative criterion of properness, see [Kon95, bottom page 338]. Finiteness of the diagonal is, in view of quasi-compactness (see above), a consequence of the valuative criterion of properness for \(\text{Isom}\) between stable maps (basically trivial), combined with the fact that by definition stable maps have finite automorphism groups (gives quasi-finiteness). This proves 1. In characteristic zero a stack with finite diagonal is Deligne-Mumford, because automorphism group schemes are reduced. Finally, to see 3 note that \(\overline{M}_{g,1}(Y / S, \beta)\) is quasi-compact in this case by the argument in [Kon95, page 338]. Thus it is proper over \(S\), see [Ols05] and [Fal03 Appendix] to see why. By [KM97], the coarse moduli space exists, and by the above it is proper over \(S\). The projectivity of this coarse moduli space will not be used in the sequel; a reference is [AV02, Section 8.3]. □

**Notation 4.8.** A simple case of a partial curve class is as follows. Let \(Y \to S\) be quasi-projective and let \(\mathcal{L}\) be an \(S\)-ample invertible sheaf on \(Y\). For every integer
there is a partial curve class \( \beta = (\mathbb{Z}, 1 \mapsto [\mathcal{L}], 1 \mapsto e) \). When \( \mathcal{L} \) is understood, the corresponding Kontsevich moduli space is denoted \( \overline{M}_{g,n}(Y/S,e) \).

5. Stable sections and Abel maps

Let \( C \to S \) be a proper, flat family of curves (see 2.1) whose geometric fibres are connected smooth projective curves of some fixed genus \( g(C/S) \). The group \( \text{NS}(C/S) \) (see discussion preceding 4.5) is a free Abelian group of rank 1, simply because any two invertible sheaves on the family are numerically equivalent on geometric fibres if and only if they have the same degree on the fibres. Denote \([\text{point}]\) the element of \( \text{NS}(Y/S) \) that fppf locally on \( S \) corresponds to an invertible sheaf on \( C \) having degree 1 on the fibres.

Consider the relative partial curve class \( \beta_{C/S} = (\mathbb{Z}, 1 \mapsto [\text{point}], 1 \mapsto 1) \) on \( C/S \) (compare 4.8). For all nonnegative integers \( g \) and \( n \), we denote \( \overline{M}_{g,n}(C/S, \beta_{C/S}) \).

Loosely speaking the stable maps \( h \) arising from points of this algebraic stack have one component that maps birationally to a fibre of \( C \to S \), and all other components are contracted. If \( g = g(C/S) \), then the contracted components all have genus zero.

Next, let \( f : X \to C \) be a proper morphism. Let \( \mathcal{L} \) be an \( f \)-ample invertible sheaf on \( X \). Denote by \([\text{fibre}]\) the pullback \( f^*\text{[point]} \) in \( \text{NS}(X/S) \). For every integer \( e \), denote by \( \beta_e \) the relative partial curve class \((\mathbb{Z} \oplus \mathbb{Z}, i, \beta_e) \) with \( i(1,0) = [\text{fibre}] \), \( i(0,1) = \mathcal{L} \) and \( \beta_e(1,0) = 1, \beta_e(0,1) = e \). Using this notation, for all nonnegative integers \( g \) and \( n \), we may consider the algebraic stacks

\[
\overline{M}_{g,n}(X/S, \beta_e)
\]

introduced in the previous section. Loosely speaking a stable maps \( h \) arising from points of this algebraic stack give rise to generalized sections of \( X_s/C_s \) for some geometric point \( s \) of \( S \) having total degree \( e \) with respect to \( \mathcal{L} \).

There is an obvious way in which \( \beta_e \) and \( \beta_{C/S} \) are related. Thus by [BM96, Theorem 3.6], there are 1-morphisms of algebraic \( S \)-stacks

\[
\overline{M}_{g,n}(f) : \overline{M}_{g,n}(X/S, \beta_e) \to \overline{M}_{g,n}(C/S, 1).
\]

**Definition 5.1.** Given \( C \to S \), \( f : X \to C \) and \( \mathcal{L} \) as above. For every nonnegative integer \( n \) and every integer \( e \), the space of \( n \)-pointed, degree \( e \), stable sections of \( f \) is the algebraic stack

\[
\Sigma^e_n(X/C/S) := \overline{M}_{g(C/S),n}(X/S, \beta_e).
\]

We will also use the variant \( \Sigma^e_I(X/C/S) \) where the sections are labeled by a given finite set \( I \).

**Definition 5.2.** Compare Definitions 3.2 and 3.5 With notation as above. A family of \( n \)-pointed, degree \( e \) sections of \( X/C/S \) over \( T \to S \) is given by a family \((\tau, \theta)\) of sections of \( f \) (see Definition 5.1) together with pairwise disjoint sections \( \sigma_i : T \to T \times_S C, i = 1, \ldots, n \), such that \( \tau^*\mathcal{L} \) has degree \( e \) on the fibres of \( T \times_S C \to T \). We leave it to the reader to define morphisms of families of \( n \)-pointed, degree \( e \) sections of \( X/C/S \).
We will denote such a family as \((T \to S, \sigma_i : T \to T \times_S C, h : T \times_S C \to X)\); in other words \(\tau\) is replaced by \(h\) and we drop the notation \(\theta\) since in this case it just signifies that \(f \circ h\) equal the projection map \(T \times_S C \to S\).

Families of \(n\)-pointed, degree \(e\) sections of \(X/C/S\) naturally form an algebraic space \(\text{Sections}_n^e(X/C/S)\) locally of finite presentation over \(S\). This is because the natural 1-morphism
\[
\text{Sections}_n^e(X/C/S) \to \text{Sections}(X/C/S)
\]
is representable and smooth, combined with Theorem 3.3. There is another obvious 1-morphism, namely
\[
\text{Sections}_n^e(X/C/S) \to \Sigma_n^e(X/C/S).
\]

This 1-morphism is representable by open immersions of schemes. In this precise sense, the proper, algebraic \(S\)-stack \(\Sigma_n^e(X/C/S)\) is a “compactification” of \(\text{Sections}_n^e(X/C/S)\).

We will also use the variant \(\text{Sections}_n^e(X/C/S)\) where the sections are labeled by a given finite set \(I\).

**Notation 5.3.** Let \(n\) be a nonnegative integer. Let \(I\) be a set of cardinality \(n\). Let \(e\) be an integer. Let \(\delta\) be a nonnegative integer. Let \(J\) be a set of cardinality of \(\delta\). Let \(\mathcal{L} = (\epsilon_0, (\epsilon_j)_{j \in J})\) be a collection of integers such that \(e = \epsilon_0 + \sum_{j \in J} \epsilon_j\). Let \(L = (I_0, (I_j)_{j \in J})\) be a collection of subsets of \(I\) such that \(I = I_0 \cup \bigcup_{j \in J} I_j\) is a partition of \(I\). The triple \((J, \mathcal{L}, I)\) is called an indexing triple for \((e, I)\). In the special case that \(I = \emptyset\), so that every subset \(I_0\) and \(I_j\) is also \(\emptyset\), this is called an indexing pair for \(e\) denoted \((J, \mathcal{L})\).

**Definition 5.4.** A family of \(L\)-pointed, degree \(e\) combs in \((X/C/S, \mathcal{L})\) as above is given by the following data

1. an \(S\)-scheme \(T\)
2. an object \((\pi_0 : C_0 \to T, (p_i : T \to C_0)_{i \in I_0} \cup (q_j : T \to C_0)_{j \in J}, h_0 : C_0 \to X)\) of \(\Sigma_{\mathcal{L}_{I_0 \sqcup J}}^e(X/C/S)\) over \(T\), and
3. for \(j \in J\), an object \((\pi_j : C_j \to T, (p_i : T \to C_j)_{i \in I_j} \cup (r_j : T \to C_j), h_j : C_j \to X)\) of \(\Sigma_{\mathcal{L}_{I_j \sqcup \{j\}}}^e(X/C, \epsilon_j)\) as in Notation 1.8 over \(T\).

These data should satisfy the requirement that for every \(j \in J\), \(h_0 \circ q_j = h_j \circ r_j\) as morphisms \(T \to X\).

**Morphisms** of \(L\)-pointed, degree \(e\) combs in \(X/C/S\) are defined in the obvious way. The category of families of \(L\)-pointed, degree \(e\) combs in \(X/C/S\) with pullback diagrams as morphisms is an \(S\)-stack denoted \(\text{Comb}_L^e(X/C/S)\).

Given a family \((\zeta_0, (\zeta_j)_{j \in J})\) of combs, the family \(\zeta_0\) is the handle and the families \((\zeta_j)_{j \in J}\) are the teeth. There is a “forgetful” 1-morphism
\[
\Phi_{\text{handle}} : \text{Comb}_L^e(X/C/\kappa) \to \Sigma_{\mathcal{L}_{I_0 \sqcup J}}^e(X/C/\kappa)
\]
called the handle 1-morphism. Similarly, for every \(j \in J\) there is a 1-morphism
\[
\Phi_{\text{tooth, } j} : \text{Comb}_L^e(X/C/\kappa) \to \Sigma_{\mathcal{L}_{I_j \sqcup \{j\}}}^e(X/C, \epsilon_j)
\]
called the \(j\)th tooth 1-morphism. Moreover as constructed in [BM96, Theorem 3.6], specifically Case II on p. 18, there is also a total curve 1-morphism
\[
\Phi_{\text{total}} : \text{Comb}_L^e(X/C/S) \to \Sigma_e^e(X/C/S)
\]
obtained by attaching \( r_j(T) \) in \( C_j \) to \( q_j(T) \) in \( C_0 \) for every \( j \in J \).

By definition, \( \text{Comb}^T \Sigma_\Sigma_0(X/C/S) \) together with the handle and tooth 1-morphisms is the equalizer of the collection of marked point 1-morphisms associated to \((q_j, r_j)_{j \in J}\).

Since each of \( \Sigma_0^\ast \Sigma_0(X/C/S) \) and \( \Sigma_0^\ast \Sigma_0(X/C/S) \) is a proper, algebraic \( S \)-stack with finite diagonal, the same is true for the equalizer \( \text{Comb}^T \Sigma_\Sigma_0(X/C/S) \).

The canonical open substack, \( \text{Comb}^T \Sigma_\Sigma_0 \), is defined to be the inverse image under \( \Phi_{\text{handle}} \) of the open substack \( \Sigma^\ast_0 \Sigma(X/C/S) \).

Proposition 5.5. Given indexing triples \((J', e', I')\) and \((J'', e'', I'')\), the images of \( \Phi_{\text{total}} : \text{Comb}^T_\Sigma \Sigma(X/C/S) \to \Sigma^\ast_\Sigma(X/C/S) \) and \( \Phi_{\text{total}} : \text{Comb}^T_\Sigma \Sigma(X/C/S) \to \Sigma^\ast_\Sigma(X/C/S) \) intersect only if there exists a bijection \( \phi : J' \to J'' \) such that \( e''_{\phi(j)} = e'_j \) and \( I''_{\phi(j)} = I'_j \) for every \( j \in J' \). If such a bijection exists, the images are equal. Finally, for every algebraically closed field \( k \) over \( S \), every object of \( \Sigma^\ast_\Sigma(X/C/S) \) over \( \text{Spec} \, k \) is in the image of

\[ \Phi_{\text{total}} : \text{Comb}^T_\Sigma \Sigma(X/C/S) \to \Sigma^\ast_\Sigma(X/C/S) \]

for some collection \((J, e, I)\).

Proof. This is left to the reader. \( \square \)

Lemma 5.6. For every integer \( e \) there exists a 1-morphism over \( S \)

\[ \alpha_\Sigma : \Sigma^\ast(X/C/S) \to \text{Pic}^e_{C/S} \]

with the following property: For every indexing pair \((J, e)\) for \( e \) and every family \((T \to \ldots) \) in \( \text{Comb}^T_\Sigma \Sigma(X/C/S) \) over \( T \) as in Definition 5.4, the composition

\[ T \to \text{Comb}^T_\Sigma \Sigma(X/C/S) \to \Sigma^\ast(X/C/S) \to \text{Pic}^e_{C/S} \]

corresponds to the class of the invertible sheaf

\[ h_0^\ast(L)(\sum_{j \in J} e_j \cdot q_j(T)) \]

on \( T \times_S C \).

Proof. We use the material preceding Definition 2.11. Any flat proper family of nodal curves is LCI in the sense of Definition 2.7. Thus there is a morphism of stacks

\[ \Sigma^\ast(X/C/S) \to \text{CurvesMaps}_{\text{LCI}}(C \times B_{\mathbb{G}_m}) \]

which associates to \((T \to S, \pi : C' \to T, h : C' \to X)\) the object \((T \to S, \pi : C' \to T, f \circ h : C' \to C, h^\ast L)\). Hence we may use the construction of Definition 2.11 which gives a 1-morphism

\[ \text{CurvesMaps}_{\text{LCI}}(C \times B_{\mathbb{G}_m}) \to \text{Pic}^e_{C/S}. \]

To prove this map satisfies the property expressed in the lemma it suffices to apply Lemma 2.12. \( \square \)
6. Peaceful chains

Let $S$ be an excellent scheme. Let $f : X \to S$ be a proper, flat morphism with geometrically irreducible fibres. Let $L$ be an $f$-ample invertible sheaf on $X$. Recall (1,8) that $\overline{M}_{0,n}(X/S, 1)$ parametrizes stable $n$-pointed genus 0 degree 1 maps from connected nodal curves to fibres of $X \to S$. Over a geometric point $s : \text{Spec}(k) \to S$ such a stable map $(C/k, p_1, \ldots, p_n \in C(k), h : C \to X_s)$ is nonconstant on exactly one irreducible component $L$ of $C$ because $L$ is ample. Then, $L \cong \mathbb{P}^1_k$, the map $L \to X_s$ is birational onto its image and corresponds to a point of $\overline{M}_{0,n}(X/S, 1)$.

If $n = 0$ or $n = 1$ then $L = C$. If $n = 2$ it may happen that $C = L \cup C'$, with $p_1, p_2 \in C'(k)$ and where $C'$ is contracted. However, one gets the same algebraic stack (in the case $n = 2$) by considering moduli of triples $(L, p_1, p_2 \in L(k), h : L \to X_s)$ where $p_1$ and $p_2$ are allowed to be the same point of $L$. We will use this below without further mention. In any case it follows that $\overline{M}_{0,n}(X/S, 1)$ is actually an algebraic space. We will call $L \to X_s$ a line, and $(C/k, p_1, \ldots, p_n \in C(k), h : C \to X_s)$ an $n$-pointed line.

**Definition 6.1.** Notation as above. An $n$-pointed line is free if the morphism $L \to X_s \to X$ factors through the smooth locus of $X/S$ and the pull back of $T_{X_s}$ via the morphism $L \to X_s$ is globally generated.

We leave it to the reader to show that the locus of free lines defines an open of $\overline{M}_{0,n}(X/S, 1)$. For every integer $n$ there is an algebraic space $\text{FreeChain}_2(X/S, n)$ over $S$ parametrizing 2-pointed chains of $n$ free lines in fibres of $f$. To be precise, $\text{FreeChain}_2(X/S, n)$ is an open subset of the $n$-fold fibre product

$$\overline{M}_{0,2}(X/S, 1) \times_{\text{ev}_2, X, \text{ev}_1} \overline{M}_{0,2}(X/S, 1) \times_{\text{ev}_2, X, \text{ev}_1} \cdots \times_{\text{ev}_2, X, \text{ev}_1} \overline{M}_{0,2}(X/S, 1).$$

The points of $\text{FreeChain}_2(X/S)$ over a geometric point $s : \text{Spec}(k) \to S$ correspond to $n$-tuples of 2-pointed lines $(C_i/k, p_{i,1}, p_{i,2} \in C_i, h_i : C_i \to X_s)$ such that $h_i(p_{i,2}) = h_{i+1}(p_{i+1,1})$ for $i = 1, \ldots, n - 1$ and such that each of the 2-pointed lines is free. We will sometime denote this by $(C/k, p, q \in C(k), h : C \to X_s)$. This means that $C$ is the union of the curves $C_i$ with $p_{i,2}$ identified with $p_{i+1,1}$, with $h$ equal to $h_i$ on $C_i$ and $p = p_{i,1}$ and $q = p_{i+1,2}$. Note that there are $n + 1$ natural evaluation morphisms

$$\text{ev}_i : \text{FreeChain}_2(X/S) \to X, \quad i = 1, \ldots, n + 1$$

over $S$, namely $\text{ev}_i([((C_i/k, p_{i,1}, p_{i,2} \in C_i, h_i : C_i \to X_s)]_{i=1,\ldots,n}) = h_i(p_{i,1})$ for $i = 1, \ldots, n$ and $\text{ev}_{n+1}$ on the same object evaluates to give $h_n(p_{n,2})$. In the variant notation described above we have $\text{ev}_1((C/k, p, q \in C(k), h : C \to X_s)) = h(p)$ and $\text{ev}_{n+1}((C/k, p, q \in C(k), h : C \to X_s)) = h(q)$. Note that these morphisms map into the smooth locus of the morphism $X \to S$.

**Lemma 6.2.** With notation as above. For every positive integer $n$ and for every integer $i = 1, \ldots, n + 1$, the evaluation morphism $\text{ev}_i : \text{FreeChain}_2(X/S, n) \to X$ is a smooth morphism. In particular $\text{FreeChain}_2(X/S, n)$ is smooth over $S$.

**Proof.** We claim the morphism $\text{ev} : \overline{M}_{0,1}(X/S, 1) \to X$ is smooth on the locus of free lines. This is true because, given a free 1-pointed line $h : L \to X_s$, $p \in L(k)$ the corresponding deformation functor has obstruction space $H^1(L, h^*T_{X_s}(- p))$. This is zero as $h^*T_{X_s}$ is globally generated by assumption. A reference for the case

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2This fibre product will be denoted $\text{Chain}_2(X/C, n)$ later on.
where $S$ is a point is \cite[Proposition II.3.5]{Kol96}. Combined with a straightforward induction argument, this proves that the morphisms $\text{ev}_i$ are smooth. \hfill \Box

**Lemma 6.3.** Notation as above. Suppose that $s : \text{Spec}(k) \to S$ is a geometric point. Suppose that $(C/k, p, q \in C(k), h : C \to X_s)$ is a free 2-pointed chain of lines of length $n$. If $\text{ev}_{1, n+1} : \text{FreeChain}_2(X/S, n) \to X \times_S X$ is smooth at the corresponding point then:

1. $H^1(C, h^*T_{X_s}(-p - q)) = 0$, and
2. for any free line $h_L : L \to X_s$ passing through $h(q)$ and any point $r \in L$ the morphism $\text{ev}_{1, n+2} : \text{FreeChain}_2(X/S, n + 1) \to X \times_S X$ is smooth at the point corresponding to the chain of free lines of length $n + 1$ given by $(C \cup L, p, r \in (C \cup L)(k), h \cup h_L)$.

**Proof.** If the morphism $\text{ev}_{1, n+1}$ is smooth then in particular the map $h^*T_{X_s} \to h^*T_{X_s}|_p \oplus h^*T_{X_s}|_q$ is surjective on global sections. This easily implies the first statement. Let $C' = C \cup L$, and $h' = h \cup h_L$. The first part and the freeness of $L$ imply that $H^1(C', (h')^*T_{X_s}(-p - r))$ is zero. The second part of the lemma follows by deformation theory from this vanishing. \hfill \Box

**Definition 6.4.** In the situation above. A geometric point $x \in X_s(k)$ lying over the geometric point $s : \text{Spec}(k) \to S$ is **peaceful** if both

1. the evaluation morphism $\text{ev} : \overline{\mathcal{M}}_{0, 1}(X/S, 1) \to X$ is smooth at every point in $\text{ev}^{-1}(x)$, and
2. for all $n \gg 0$, the evaluation morphism $\text{ev}_{1, n+1} : \text{FreeChain}_2(X/S, n) \to X \times_S X$ is smooth at some point $(C/k, p, q \in C(k), h : C \to X_s)$ with $h(p) = x$.

The second condition implies that there is at least one free line through every peaceful point, and hence that a peaceful point lies in the smooth locus of $X \to S$. The first implies, upon considering the derivative of $\text{ev}$ that all lines through a peaceful point are free. Note that there may be situations where there are no peaceful points whatsoever.

**Lemma 6.5.** In the situation above. The set $X_{f, \text{pax}}$ of peaceful points of $X$ is an open subset of $X$. Furthermore, there exists a single $n_0$ such that the second condition of Definition 6.4 holds for all peaceful points and all $n \geq n_0$.

**Proof.** Denote by $Z$ the singular set of the morphism $\text{ev} : \overline{\mathcal{M}}_{0, 1}(X/S, 1) \to X$. Since $\text{ev}$ is proper, $\text{ev}(Z)$ is a closed subset of $X$. Thus the complement $U' = X \setminus \text{ev}(Z)$ is the maximal open subset of $X$ such that $\text{ev} : \text{ev}^{-1}(U') \to U'$ is smooth. For every $n$, denote by $V_n$ the open subset of $\text{FreeChain}_2(X/S, n)$ on which $\text{ev}_{1, n+1} : \text{FreeChain}_2(X/S, n) \to X \times_S X$

is smooth. The image $V'_n := \text{ev}_1(V_n)$ is open by Lemma 6.2. By Lemma 6.3 we have $V_1 \subset V_2 \subset \ldots$. Since $S$ and hence $X$ is Noetherian we find $n_0$ such that $V_{n_0} = V_{n_0+1} = \ldots$. It is clear that the set $V' := U' \cap \cup_{n \geq 1} V'_n$ is open and equals $X_{f, \text{pax}}$. \hfill \Box

**Remark 6.6.** In fact, consider the open subset $W_n \subset X \times_S X$ above which the morphism $\text{ev}_{1, n+1} : \text{FreeChain}_2(X/S, n) \to X \times_S X$ has some smooth point. This is an increasing sequence by Lemma 6.3 as well and hence equal to $W_{n_1}$ for some $n_1$. 26
Definition 6.7. In the situation above. Let \( U \) be an open subset of \( X_{f, \text{pax}} \). Let \( n > 0 \). A \( U \)-adapted free chain of \( n \) lines is a point \([h] \in \text{FreeChain}_n(X/S, n)\) such that \( \text{ev}_i([h]) \in U \) for all \( i = 1, \ldots, n + 1 \). When \( U \) is all of \( X_{f, \text{pax}} \), this is sometimes called a peaceful chain. The subset of \( \text{FreeChain}_2(X/S, n) \) parametrizing \( U \)-adapted free chains is denoted \( \text{FreeChain}_U^2(X/S, n) \). When \( U \) is all of \( X_{f, \text{pax}} \) we sometimes use the notation \( \text{PeaceChain}_2(X/S, n) \).

Without more hypotheses on our morphism \( f : X \to S \) and relatively ample invertible sheaf \( \mathcal{L} \) we cannot say much more (since after all the set of lines in fibres might be empty).

Hypothesis 6.8. Let \( S \) be an excellent scheme of characteristic 0. Let \( f : X \to S \) be a proper flat morphism with geometrically irreducible fibres. Let \( \mathcal{L} \) be \( f \)-ample. In addition we assume the following.

1. There exists an open subset \( U \) of \( X \) surjecting to \( S \) and such that the geometric fibres of the evaluation morphism \( \text{ev} : M_{0,1}(X/S, 1) \to X \) over \( U \) are nonempty, irreducible and rationally connected.

2. There exists a positive integer \( m_0 \) and an open subset \( V \) of \( X \times S \) surjecting to \( S \) and such that the geometric fibres of the evaluation morphism \( \text{ev}_{1,n+1} : \text{FreeChain}_2(X/S, m_0) \to X \times S X \) over \( V \) are nonempty, irreducible and birationally rationally connected.

Note that the second condition in particular implies that the smooth locus of \( X \to S \) intersects every fibre of \( X \to S \). Namely, the open \( V \) is contained in \( \text{Smooth}(X/S) \times S \text{Smooth}(X/S) \) by the definition of free lines. In addition, the second condition also shows that the smooth locus of each geometric fibre of \( X \to S \) is rationally chain connected by free rational curves, and hence rationally connected.

Lemma 6.9. In Situation 6.8 the peaceful points are dense in every smooth fibre of \( X \to S \).

Proof. It is well known that lines passing through a general point of a smooth projective variety are free. Hence the open \( U' \) of the proof of Lemma 6.5 has a nonempty intersection with every smooth fibre of \( f \). On the other hand, condition 2 of 6.8 implies that for every point \( s \) of \( S \) the morphism \( \text{FreeChain}_2(X/S, m_0)_s \to (X \times S X)_s \) is a dominant morphism of smooth varieties. Hence it is smooth at some point \( \xi \in \text{FreeChain}_2(X/S, m_0)_s \) by generic smoothness. The “critère de platitude par fibre” [Gro67, Theorem 11.3.10] implies that \( \text{ev}_{1,n+1} \) is flat at \( \xi \) and hence smooth as desired. Thus the open \( V'_{m_0} \) of the proof of lemma 6.5 has nonempty fibre \( V'_{m_0,s} \). We win because \( X_{f, \text{pax}} \supset U' \cap V'_{m_0} \).

Lemma 6.10. In Situation 6.8. If \( U \) is a nonempty open subset of \( X_{f, \text{pax}} \), then \( \text{FreeChain}_U^2(X/S, n) \) is an open dense subset of \( \text{FreeChain}_2(X/S, n) \). In fact, it is dense in the fibre of \( \text{FreeChain}_2(X/S, n) \to S \) over any point \( s \) for which \( U_s \neq \emptyset \). For every integer \( i = 1, \ldots, n + 1 \), each morphism \( \text{ev}_i : \text{FreeChain}_U^2(X/S, n) \to U \),

is a smooth morphism and every geometric fibre is nonempty, irreducible and birationally rationally connected.
Proof. By Lemma 6.2 the morphisms ev₁ are smooth. Thus, the preimages ev₁⁻¹(U) and are dense open subsets. Thus the finite intersection FreeChain^U_2(X/S, n) = \cap_{i=1}^{n+1}(ev₁⁻¹(U)) is also a dense open subset. Same argument for fibres. Next let x be a geometric point of U. The fibre of

\[ ev_1 : \text{FreeChain}^U_2(X/S, n) \rightarrow U \]

over x is the total space of a tower of birationally rationally connected fibrations. Indeed, the variety parametrizing choices for the first line L₁ of the chain containing x = p₁,₁ is the fibre of

\[ ev : \bar{\mathcal{M}}_{0,1}(X/S, 1) \rightarrow X \]

over x. Since ev is a proper morphism and since the geometric generic fibre of ev is rationally connected, every fibre in the smooth locus of ev is rationally connected by [KMM92] 2.4 and [Kol96] Theorem IV.3.11]. In particular, the fibre over every peaceful point x is in the smooth locus and thus is rationally connected. Next, given L₁, the variety parametrizing choices for the attachment point p₁,₂ is the intersection of L₁ with U, which is birationally rationally connected since it is open in L₁ \cong \mathbb{P}^3. Next, the variety parametrizing lines L₂ containing p₁,₂ = p₂,₁ is rationally connected for the same reason that the variety parametrizing lines L₁ is rationally connected, etc. By [GHS03], the total space of a tower of birationally rationally connected fibrations is itself birationally rationally connected. Thus the geometric fibres of ev₁ are birationally rationally connected.

The proof for the other morphisms evᵢ is similar. □

Proposition 6.11. In Situation 6.8 There exists a positive integer m₁ such that for every integer n \geq m₁ and for every nonempty open subset U of Xᵢ,n there exists an open subset Uₙ of FreeChain^U_2(X/S, n) whose intersection with every geometric fibre of

\[ ev_{₁, n+1} : \text{FreeChain}^U_2(X/S, n) \rightarrow U \times_S U \]

is nonempty, smooth, irreducible and birationally rationally connected.

Proof. Let n₀ be the integer of Lemma 6.3 Let m₀ be the integer of Hypothesis 6.8 We claim that m₁ = 2n₀ + m₀ works.

Namely, suppose that n \geq m₁. Write n = a + m₀ + b, with a \geq n₀ and b \geq n₀. Consider the opens

A = ev_{₁,a+1}(FreeChain^U_2(X/S, a)) \subset X, and B = ev₁(FreeChain^U_2(X/S, b)) \subset X.

These have nonempty fibre over every point of S over which the fibre of U \rightarrow S is nonempty by Lemma 6.10. For B you technically speaking also have to use that there is an automorphism on the space of chains which switches the start with the end of a chain. As a first approximation we let

\[ Uₙ = ev_{₁,a+1}(A) \cap ev_{₁+1,a+1}(B) \cap ev_{₁+1,a+1+m₀}(V) \subset \text{FreeChain}^U_2(X/S, n). \]

Here V \subset X \times_S X is the open of Hypothesis 6.8. Let us describe the geometric points of Uₙ. Let s : Spec(k) \rightarrow S be a geometric point of S such that Uₙ \neq \emptyset. A point \( \xi \in Uₙ(k) \) is given by the following data:

1. A point \( x \in Aₙ(k) \), a point \( y \in Bₙ(k) \), such that \( (x, y) \in Vₙ(k) \),
2. A U-adapted free chain of a lines \( \xiₐ \) with end point at x,
3. A U-adapted free chain of b lines \( \xi₏ \) with start point at y, and

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These data have to satisfy the requirements that (a) the set $s$ of integers is free (see Definition 3.7). A degree $e$ rational map $X$ is smooth over a point $x$ if it is a nonempty smooth irreducible, birationally rationally connected variety. For every pair $(x, y)$ as above and such that $X \neq \emptyset$ it is a nonempty open condition to have $(x, y) \in V_n(k)$. By the Hypothesis 6.8 the set of choices of $\xi_m$ connecting $x$ with $y$ forms an irreducible, birationally rationally connected variety. Using [CHIS03], it follows that the fibre of $U_n$ over $(x', x'')$ is smooth and birationally rationally connected.

We are not yet assured that $ev_{1,n+1} : U_n \to U \times_S U$ is smooth. Let $U'_n \subset U_n$ be the smooth locus of this morphism. We claim that $U'_n \subset U_n$ is dense in every geometric fibre $ev_{1,n+1}^{-1}((x', x''))$ we studied above, which will finish the proof of the proposition. By irreducibility of this fibre it suffices to show $U'_n$ intersects the fibre. Since $a \geq n_0$ we know by Lemma 6.3 that there is a chain $\xi_1$ as above which represents a point where $ev_{1,n+1}$ is smooth. Similarly for $\xi_0$. By openness of smoothness, we may assume the associated pair $(x, y)$ is an element of $V_n(k)$. Pick any free chain $\xi_m$ connecting $x$ and $y$ as above. We claim that the resulting chain of length $a + m_0 + b$ represents a smooth point for the morphism $ev_{1,n+1}$. The last part of the proof is similar to the proof of Lemma 6.3 and uses the results of that lemma. Namely, $\xi_m$ corresponds to a chain of lines $L_1, \ldots, L_n$ and points $p_{i,j}$, $i \in \{1, \ldots, n\}$, $j \in \{1, 2\}$, such that (where we simply write $T_X$ to denote the pullback) $H^1(L_1 \cup \ldots \cup L_n, T_X(-p_{1,1} - p_{n,2})) = 0$, $H^1(L_{a+m_0+1} \cup \ldots \cup L_n, T_X(-p_{a+m_0+1,1} - p_{n,2})) = 0$, and the lines $L_{a+1}, \ldots, L_{a+m_0}$ are free. It is easy to see that this implies that $H^1(L_1 \cup \ldots \cup L_n, T_X(-p_{1,1} - p_{n,2})) = 0$. This finishes the proof as in Lemma 6.3.

7. Porcupines

**Hypothesis 7.1.** Here $k$ is an uncountable algebraically closed field of characteristic 0. Let $C$ be a smooth, irreducible, proper curve over $k$. Let $X$ be smooth and proper over $k$. Let $f : X \to C$ be proper and flat with geometrically irreducible fibres. The invertible $O_X$-module $L$ is $f$-ample.

We are going to apply the material of the previous section to this situation. In particular, let $X_{f,pax}$ be the set of peaceful points relative to $f : X \to C$, see Lemma 6.5. Here is one more type of stable map that arises often in what follows.

**Definition 7.2.** Notation as above. Let $e$ be an integer and let $n$ be a nonnegative integer. A degree $e$ porcupine with $n$ quills over $k$ is given by the following data

1. a section $s : C \to X$ of degree $e$ with respect to $L$,
2. $n$ distinct points $q_1, \ldots, q_n \in C(k)$, and
3. $n$ 1-pointed lines $(r_i \in L_i(k), L_i \to X_{q_i})$.

These data have to satisfy the requirements that (a) the set $s^{-1}(X_{f,pax}) \neq \emptyset$ and each $q_i$ is in this open, (b) the points $r_i$ and $q_i$ get mapped to the same $k$ point of $X$, and (c) the section $s$ is free (see Definition 5.7).

It may be useful to discuss this a bit more. The first remark is that by our definitions the image of $s$ lies in the smooth locus of $X/C$ and all the lines $L_i$ are free and lie in the smooth locus as well. In other words, the union $C \cup (\bigcup L_i) \to X$ is going to be
a comb (see Definition 5.4) in the smooth locus. We will call the handle \( s : C \to X \)
the **body** of the porcupine, and we will call the teeth \( L_i \to X \) the **quills**.

**Families of degree \( e \) porcupines with \( n \) quills** are defined in the obvious manner; for example we may observe that the space of porcupines of degree \( e \) and \( n \) quills is an open substack of the moduli stack \( \text{Comb}_e^X(X/C/k) \) with \( \underline{t} = (\emptyset, \emptyset, \ldots, \emptyset) \), and \( e = (e, 1, \ldots, 1) \) of combs studied in Section 5. The parameter space for porcupines is denoted \( \text{Porcupine}^{e,n}(X/C/k) \).

Denote by

- \( \Phi_{\text{total}} : \text{Porcupine}^{e,n}(X/C/k) \to \Sigma^{e,n+1}(X/C/\kappa) \)
- \( \Phi_{\text{body}} : \text{Porcupine}^{e,n}(X/C/k) \to \text{Sections}^n_e(X/C/k) \)
- \( \Phi_{\text{quill},i} : \text{Porcupine}^{e,n}(X/C/k) \to \mathcal{M}_{0,1}(X/C,1) \)
- \( \Phi_{\text{forget}} : \text{Porcupine}^{e,n+1}(X/C/k) \to \text{Porcupine}^{e,n}(X/C/k) \)

the obvious forgetful morphisms. Note that \( \Phi_{\text{forget}} \) is smooth with irreducible fibres.

**Lemma 7.3.** In Situation 7.1. Let \( n \geq 1 \). If the geometric generic fibre of \( f \) is rationally connected then there exists an n-free section of \( f \). If the set of peaceful points is nonempty then we may assume the section meets this locus.

**Proof.** First of all, by [GHS03] there exists a section \( \tau \) of \( f \) which corresponds to free sections \( s \). By Lemma 3.12 it suffices to construct a section that is \( D \)-free for some divisor \( D \) of degree \( 2g(C) + n \). Set \( m = 2g(C) + n \). Consider the subsets \( W_e^m \) of \( H^e_{X,f} \) introduced in Proposition 3.13.

We know there exist nonempty opens \( U_{e,m} \subset H^e_{X,f} \), such that \( U_{e,m} \subset W^m_e \). Also by the proposition, a section \( \tau : C \to X_{f,\text{smooth}} \) will be \( D \)-free for some effective divisor \( D \subset C \) of degree \( m \) provided that the restriction of \( \tau \) to \( D = \sum t_i \) is in \( \bigcap t_i U_{e,m}^e \). By Lemma 3.16 (applied with \( T = \text{Spec}(k) \) and \( U = Y = C \) and \( V = X_{f,\text{smooth}} \)) we can find a section \( \tau \) with the desired properties. \( \Box \)

Before we state the next lemma, we wish to remind the reader that an irreducible scheme is necessarily nonempty. Also, see Remark 7.10 below for a more “lightweight” variant of some of the following lemmas.

**Lemma 7.4.** In Situation 7.1, assume in addition Hypothesis 6.3 for the restriction of \( f \) to some nonempty open \( S \subset C \). Let \( e_0 \) be an integer and let \( Z \) be an irreducible component of \( \Sigma^{e_0}(X/C/k) \) whose general point parametrizes a free section of \( f \). For every integer \( e \geq e_0 \) there exists a unique irreducible component \( Z_e \) of \( \Sigma^e(X/C/k) \) such that every porcupine with body in \( Z \) and with \( e - e_0 \) quills is parametrized by \( Z_e \).

**Proof.** Take any point \([s]\) of \( Z \) corresponding to a section \( s \) which is free. By Proposition 3.13 the space \( \Sigma^{e_0}(X/C/k) \) is smooth at the point \([s]\). Thus this point is contained in a unique irreducible component of \( \Sigma^{e_0}(X/C/k) \), namely \( Z \). By Lemma 3.13 free sections deform to contain a general point of \( X \). The set of peaceful points is nonempty by Lemma 6.13. Hence there are also points \([s]\) of \( Z \) which correspond to free sections \( s \) which intersect the peaceful locus. For every such section \( s \), the inverse image \( s^{-1}(X_{f,\text{pax}}) \) is a dense open subset of \( C \). Thus the variety parametrizing \((e - e_0)\)-tuples of closed points \((q_1, \ldots, q_{e - e_0})\) in \( s^{-1}(X_{f,\text{pax}}) \) is smooth and irreducible. By definition of peaceful the fibre of

\[ ev : \mathcal{M}_{0,1}(X/C,1) \to X \]
over \( s(q_i) \) is smooth and by Lemma 6.10 it is irreducible. Putting the pieces together, the variety \( W_{Z,e} \) parametrizing degree \( e \) porcupines with body in \( Z \) is smooth and irreducible. Moreover, because the body is unobstructed and because the teeth are all free, each such porcupine is unobstructed. Thus \( \Sigma^e(X/C/k) \) is smooth at every point of \( W_{Z,e} \). Thus \( W_{Z,e} \) is contained in a unique irreducible component \( Z_e \) of \( \Sigma^e(X/C/k) \).

\[ \square \]

**Lemma 7.5.** Notation and assumptions as in Lemma 7.4. Let \( e \geq e_0 \). A general point of \( Z_e \) corresponds to a section of \( X \rightarrow C \) which is free and meets \( X_{f,pax} \).

**Proof.** Another way to state this lemma is that (1) we can smooth porcupines, and (2) a smoothing of a porcupine is a porcupine. Assume \( h \) is a point corresponding to a section of \( X_{f,pax} \) and meets \( X \) of \( e \) quills. As in the proof of Lemma 7.3 above, the porcupine \( e \) is fixed and \( e \) quills attached to the body of the porcupine \( h(t) \), \( t \in T \) in pairwise distinct points of \( X_{f,pax} \). As in the proof of Lemma 7.4 there is an irreducible parameter space of choices, in other words \( T' \) is irreducible. Also \( T' \subset \Sigma^e(X/C/k) \) lies in the smooth locus, because each point of \( T' \) corresponds to a porcupine. The result follows because \( T'_1 \subset Z_e \) by definition of \( Z_e \) and \( T'_0 \) parametrizes porcupines with body \( s \).

\[ \square \]

**Lemma 7.6.** Notation and assumptions as in Lemma 7.4. For any \( e' \geq e \geq e_0 \), any porcupine of total degree \( e' \) with body in \( Z_e \) and \( e' - e \) quills corresponds to a point of \( Z_{e'} \).

**Proof.** Let \( [s] \in Z_e \) be the moduli point corresponding to a section of \( X \rightarrow C \) which meets \( X_{f,pax} \) and is free. Because \( Z_e \) is irreducible, and because of the construction of \( Z_e \), we can find an irreducible curve \( T \), a morphism \( h : T \rightarrow Z_e \), and points \( 0, 1 \in T(k) \) such that \( h(0) = [s] \) and \( h(1) \) corresponds to a point of \( W_{Z,e} \) (see proof of Lemma 7.3). After deleting finitely many points \( \neq 0,1 \) from \( T \) we may assume every point \( t \in T(k) \) which is free (see Lemma 3.8) and meets \( X_{f,pax} \). Thus every point \( h(t) \in Z_e \) corresponds to a porcupine. Consider the scheme \( T' \rightarrow T \) parametrizing choices of \( e' - e \) quills attached to the body of the porcupine \( h(t) \), \( t \in T(k) \) in pairwise distinct points of \( X_{f,pax} \). As in the proof of Lemma 7.4 there is an irreducible parameter space of choices, in other words \( T' \) is irreducible. Also \( T' \subset \Sigma^e(X/C/k) \) lies in the smooth locus, because each point of \( T' \) corresponds to a porcupine. The result follows because \( T'_1 \subset Z_{e'} \) by definition of \( Z_{e'} \) and \( T'_0 \) parametrizes porcupines with body \( s \).

\[ \square \]

**Lemma 7.7.** Notation and assumptions as in Lemma 7.4. Let \( f \) be a point corresponding to a \( 0 \)-free section \( s : C \rightarrow X \). Let \( \{h_i : C_i \rightarrow X, p_i \in C_i(k)\} \) be a finite number of \( 1 \)-pointed chains of free lines in fibres of \( f \). Assume \( h_i(p_i) \in s(C) \) and assume \( h_i(p_i) \) pairwise distinct. Then \( s(C) \cup \bigcup C_i \) defines an unobstructed nonstacky point of \( \Sigma^{e'}(X/C/k) \) which lies in \( Z_{e'} \) with \( e' = e + \sum \deg(C_i) \).
Lemma 7.8. Notation and assumptions as in Lemma 7.4. For any \( m \geq 0 \) there exist \( e \gg 0 \) such that \( Z_e \) contains a point corresponding to a section which is \( m \)-free and meets \( X_{f,pax} \).

**Proof.** Let \([s] \in Z\) be the moduli point corresponding to a section \( s : C \to X \) which is free and meets \( X_{f,pax} \). Let \( N = m + 2g(C) + 1 \). Consider pairwise distinct points \( q_1, \ldots, q_N \in C(k) \) such that \( s(q_i) \in X_{f,pax} \). Choose 2-pointed free chains of lines \( (h_i : C_i \to X, p_i, q_i \in C_i(k)) \) of some common length \( n \) such that \( s(q_i) = h(p_i) \) and \( z_i = h(q_i) \) are general points \( z_i \in X_{q_i}(k) \). We may choose these by the smoothness assumption in Definition 6.4. By Lemma 6.10 we may in fact assume all attachment points in the chains of lines are peaceful. We may also assume all the attachment points are distinct, since that condition defines an open dense subset of each fibre of \( ev_1 : \text{FreeChain} \to X \). By Lemma 7.7 any smoothing of the stable map \( s(C) \cup \cup C_i \to X \) defines a point of \( Z_e \) (for a suitable \( e \)). The smoothing will still pass through \( r \) general points of the fibres \( X_{q_i} \). Whence applying Proposition 4.15 we get that the resulting section is \( D \)-free for some divisor of degree \( N \). By Lemma 8.12 we win. \( \square \)

Lemma 7.9. Notation as in Lemma 7.4. For every \( e \geq e_0 + g(C) \), the Abel map

\[
\alpha_L|_{Z_e} : Z_e \to \text{Pic}^e_{C/k}
\]

is dominant with irreducible geometric generic fibre.

**Proof.** Let \( n \) be an integer with \( n \geq g(C) \) and let \( e = e_0 + n \). Let \( W_{Z,e} \) be as in the proof of Lemma 7.4. As in the proof of Lemma 7.4 every point of \( W_{Z,e} \) is a smooth point of \( Z_e \). Therefore to prove that

\[
\alpha_L|_{Z_e} : Z_e \to \text{Pic}^e_{C/k}
\]

is dominant with irreducible geometric generic fibre, it suffices to prove that the restriction

\[
\alpha_L|_{W_{Z,e}} : W_{Z,e} \to \text{Pic}^e_{C/k}
\]

is dominant with irreducible geometric generic fibre. And by Lemma 5.9 the morphism \( \alpha_L|_{W_{Z,e}} \) factors as follows

\[
W_{Z,e} \to Z \times C^{e-e_0} \to Z \times \text{Pic}^{e-e_0}_{C/k} \to \text{Pic}^{e_0}_{C/k} \times \text{Pic}^{e-e_0}_{C/k} \to \text{Pic}^e_{C/k}
\]
where each of the component maps is the obvious morphism. The first map has
smooth, geometrically irreducible fibres as was explained in the proof of the Lemma
above. And since $e - e_0$ is $\geq g(C)$, the second is dominant with irreducible
geometric generic fibre.

The trick is to factor the composition $\beta$ of the last two maps. Namely, it is the
composition of the map

$$(\text{Id}_Z, \beta) : Z \times_k \text{Pic}^e_{C/k} \to Z \times_k \text{Pic}^e_{C/k}, \ (s, [D]) \mapsto (s, \alpha_{\mathcal{L},Z}(s) + [D])$$

and the projection

$$\text{pr}_2 : Z \times_k \text{Pic}^e_{C/k} \to \text{Pic}^e_{C/k}.$$  

Since $(s, [D]) \mapsto (s, -\alpha_{\mathcal{L},Z}(s) + [D])$ is an inverse of $(\text{Id}_Z, \beta)$, we see $(\text{Id}_Z, \beta)$ is an
isomorphism of schemes. Moreover, considering the source as a scheme over $\text{Pic}^e_{C/k}$ via $\beta$ and considering the target as a scheme over $\text{Pic}^e_{C/k}$ via $\text{pr}_2$, it is an isomor-
phism of schemes over $\text{Pic}^e_{C/k}$. Therefore every geometric fibre of $\beta$ is isomorphic
to $Z$, which is irreducible by hypothesis. So $\beta$ is dominant with irreducible geo-
metric generic fibre. Since a composition of dominant morphisms with irreducible
geometric generic fibres is of the same sort, $\alpha_{\mathcal{L}|W_{Z,e}}$, and thus $\alpha_{\mathcal{L}|Z,e}$, are dominant
with irreducible geometric generic fibres. □

Remark 7.10. The proof of Lemmas 7.4 and 7.9 works in a more general setting.
Namely, suppose that $k, X \to C$ and $\mathcal{L}$ are as in 7.1. Redefine a porcupine tem-
porarily by replacing condition (a) with the condition: all lines on $X_{q_i}$ through $s(q_i)$
are free and they form an irreducible variety. Assume that for a general $c \in C(k)$
and a general $x \in X_c(k)$ the space of lines through $x$ is nonempty and irreducible.
Then the conclusions of Lemmas 7.4 and 7.9 hold.

8. Pencils of porcupines

Let $k, f : X \to C$ and $\mathcal{L}$ be as in Hypothesis 7.1. A ruled surface or a scroll in $X$
will be a morphism $R \to X$ such that $R \to C$ is proper smooth and all fibres are
lines in $X$. The following will be used over and over again.

Lemma 8.1. In Situation $\mathcal{L}A$ any ruled surface $R \to X$ lies in the smooth locus
of $X \to C$.

Proof. Since by assumption the total space of $X$ is smooth any section of $X \to C$
lies in the smooth locus of $X \to C$. On the other hand, weak approximation (see for
example [HT06]) holds for the ruled surface $R \to C$, a fortiori for any point $r \in R(k)$
we can find a section of $R \to C$ passing through $r$. The lemma follows. □

Let $h : C' \to X$ be a porcupine, by which we mean $C' = C \cup \bigcup L_i \to X$ as in
Definition 7.2

Definition 8.2. Notations as above. A pen for $C'$ is a ruled surface $R$ such that
$C' \to X$ factors through $R \to X$.

We will always assume a pen for $C'$ comes with a factorization. If lines in fibres of
$X \to C$ are automatically smooth (e.g. if $\mathcal{L}$ is very ample on the fibres of $X \to C$)
then $R$ and $C'$ are closed subschemes of $X$ the definition just means $C' \subset R$. 33
Lemma 8.3. Let $C_0'$ and $C_\infty'$ be porcupines whose bodies $s_0(C)$ and $s_\infty(C)$ are penned in a common ruled surface $R$. Let $(e_0, n_0)$, resp. $(e_\infty, n_\infty)$ be the numerical invariants of $C_0'$, resp. $C_\infty'$, and assume $e_0 + n_0 = e_\infty + n_\infty$. Denote the attachment points of $C_0'$, resp. of $C_\infty'$, by $q_{0,i}$, resp. $q_{\infty,j}$. The Abel images $\alpha_L([C_0'])$ and $\alpha_L([C_\infty'])$ are equal if and only if $D_0 \sim D_\infty$ in the divisor class group of $R$ where

$$D_0 := s_0(C) + \sum_{i=1}^{n_0} R_{q_{0,i}} \quad \text{and} \quad D_\infty := s_\infty(C) + \sum_{j=1}^{n_\infty} R_{q_{\infty,j}}$$

Proof. Note that $\text{Pic}(R)$ is isomorphic to $\mathbb{Z} \oplus \text{Pic}(C)$. As generator for the summand $\mathbb{Z}$ we take the class of $L$. The divisors $R_q$ on $R$ correspond to the elements $\mathcal{O}_C(q)$ in $\text{Pic}(C)$. Let $K_{rel}$ denote the relative canonical divisor of $R \to C$. The divisor $s_0(C)$ corresponds to the element $-K_R + (-1, s_0^*L)$, by a fundamental calculation left to the reader. Hence $D_0 \sim -K_r + (-1, s_0^*L(\sum q_{0,i}))$ Similar for $s_\infty$. The result now follows from Lemma 5.6.

In the above lemma, if $D_0$ and $D_\infty$ are linearly equivalent, denote by $(D_\lambda)_{\lambda \in \Pi}$ the pencil of effective Cartier divisors on $R$ spanned by $D_0$ and $D_\infty$. At this point we start constructing families of porcupines over rational curves. In order to do this we state and prove a few lemmas.

Lemma 8.4. In situation 7.1 assume the restriction of $f$ to some nonempty open $S \subset C$ satisfies Hypothesis 6.8. Suppose that $C'$ and $C''$ are porcupines whose bodies and attachment points agree, but which may have different quills. Then there exists a rational curve in the smooth nonstacky locus of $\Sigma^{e+n}(X/C/k)$ connecting the corresponding points.

Proof. A porcupine always represents a smooth nonstacky point of $\Sigma^{e+n}(X/C/k)$. The space parametrizing choice of quills given the body $s$ and the attachment points $q_1, \ldots, q_n$ is the product of $n$ fibers of the evaluation map $\overline{M}_{0,1}(X/C, 1) \to X$ at peaceful points. But these fibers are smooth projective rationally connected varieties by definition of peaceful and Lemma 6.10.

In the following lemma and below we will say that the porcupine $C''$ is an extension of $C'$ if you get $C'$ from $C''$ by deleting some of its quills. In other words, this means that $\Phi_{\text{forget}}(C'') = C'$, where $\Phi_{\text{forget}}$ is the forgetful morphism defined in Section 7.

Lemma 8.5. Same assumptions as in Lemma 8.4. Let $s_0, s_\infty : C \to X$ be sections. Assume $s_0$ and $s_\infty$ are free, and that $V = s_0^{-1}(X_{f, \text{pax}}) \cap s_\infty^{-1}(X_{f, \text{pax}})$ is not empty. Assume $s_0$ and $s_\infty$ are penned in a common ruled surface $R$. Then there exists a nonnegative integer $E$ with the following property: for all $e$ with $e \geq \min\{\text{deg}(s_0), \text{deg}(s_\infty)\}$, for any pairwise distinct points $q_0, 0, \ldots, 0, e - \text{deg}(s_0) \in C(k)$, for any pairwise distinct points $q_\infty, 0, \ldots, 0, e - \text{deg}(s_\infty) \in C(k)$ and for every extension $C''$, resp. $C'_\infty$, of $s_0$, resp. $s_\infty$, to a porcupine with quills over the points $q_0, 0, \ldots, 0, e - \text{deg}(s_0)$, resp. $q_\infty, 0, \ldots, 0, e - \text{deg}(s_\infty)$ if $s_0^*(L)(\sum q_{0,i}) \cong s_\infty^*(L)(\sum q_{\infty,i})$ as invertible sheaves on $C$, then the points $[C_0']$ and $[C_\infty']$ are connected in the stack $\Sigma(X/C/k)$ by a chain of rational curves whose nodes parametrize unobstructed, non-stacky stable sections. Moreover, the integer $E$ is bounded above by $\varphi(\text{deg}(R), \text{deg}(s_0))$ where $\varphi : \mathbb{Z}^2 \to \mathbb{Z}_{\geq 0}$ is a function such that $B \geq B' \Rightarrow \varphi(B, e) \geq \varphi(B', e)$.

Proof. We first describe our choice of $E$. For any effective Cartier divisor $D \subset C$ denote $R_D \subset R$ the corresponding divisor in $R$. Note that there exists an integer
Choose $e \geq E$. By Lemma [8.3] it suffices to prove the lemma assuming all the quills are in the surface $R$. Let $q_{i,j}$ be as in the statement of the lemma. Note that $s_0(C) + \sum R_{q_{i,j}} \sim s_\infty(C) + \sum R_{q_{\infty,j}}$ as divisors on $R$ by Lemma [8.3].

Let us connect our points $s_0(C) + \sum R_{q_{i,j}} \in \{s_0(C) + \sum R_{q_{i,j}}\}$ and $s_\infty(C) + \sum R_{q_{\infty,j}} \in \{s_0(C) + \sum R_{q_{i,j}}\}$ by a pair of general lines $\Pi_1, \Pi_2$. In other words, $\Pi_1$ and $\Pi_2$ intersect, $\Pi_1$ contains the first point and $\Pi_2$ contains the second, but otherwise the lines are general. We claim that this gives two rational curves in $\Sigma^e(X/C/k)$ that do the job. Since the curves $s_0(C) + \sum R_{q_{i,j}}$ and $s_\infty(C) + \sum R_{q_{\infty,j}}$ are nodal and by our choice of $e$ we see that all points of $\Pi_1$ and $\Pi_2$ correspond to nodal curves, hence automatically $\Pi_1$ and $\Pi_2$ correspond to families of stable sections. Finally, since the point $s_0(C) + \sum R_{q_{i,j}} \in \Pi_1$ corresponds to a smooth point of $\Sigma^e(X/C/k)$ we see that a general point of $\Pi_1$ corresponds to a smooth point as well. And since $\Pi_1 \cap \Pi_2$ is just a general point on $\Pi_1$ this node corresponds to a smooth point of $\Sigma^e(X/C/k)$ as desired.

Here is one of the key technical lemmas of this paper.

**Lemma 8.6.** In Situation [7.1], assume the restriction of $f$ to some nonempty open $S \subset C$ satisfies Hypothesis [6.3]. Let $D$ be an effective Cartier divisor of degree $2g(C) + 1$ on $C$. Suppose we have sections $s_0, s_\infty$ of $X \to C$ which are $D$-free.

Let $T_0$, resp. $T_\infty$ be the unique irreducible component of Sections($X/C/k$) of which $s_0$, resp. $s_\infty$ is a smooth point (see Proposition [5.1.3]). There exists a dense open $W \subset T_0 \times T_\infty$, an integer $n \geq 1$, integers $\epsilon_1, \ldots, \epsilon_{n+1}$, $B$ such that for every point $(\tau_0, \tau_\infty) \in W$ we have the following:

1. there exist sections $\tau_0 = \tau_1, \tau_2, \ldots, \tau_{n+1} = \tau_\infty$,
2. the degree of $\tau_i$ is $\epsilon_i$ for $i = 1, \ldots, n+1$,
3. $\tau_i^{-1}(X_{f, \text{raz}}) \neq \emptyset$ for $i = 1, \ldots, n+1$,
4. $\tau_i$ is free for $i = 1, \ldots, n+1$,
5. there exist ruled surfaces $R_i \to X$, $i = 1, \ldots, n$,
6. the degree of $R_i$ is at most $B$, and
7. $\tau_i, \tau_{i+1}$ factor through $R_i$ for $i = 1, \ldots, n$.

**Proof.** We will write $m = 2g(C) + 1$ in order to ease notation. Let $W_0 \subset T_0 \times \text{Sym}^m(C)$, be the open subset corresponding to pairs $(\tau_0, D_0)$, such that $\tau_0$ is $D_0$-free. Similarly we have $W_\infty \subset T_\infty \times \text{Sym}^m(C)$. Our assumption is that $W_0$ and

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3It is not necessary to have the same divisor for the two sections.
$W_\infty$ are not empty. Recall the spaces $H^m_{X,f}$ defined above Proposition \ref{prop:3.10} By Lemma \ref{lem:3.14} the maps

$$
res_0 : W_0 \to H^m_{X,f}, \quad (\tau_0, D_0) \mapsto (\tau_0|_{D_0} : D_0 \hookrightarrow X),
$$

$$
res_\infty : W_\infty \to H^m_{X,f}, \quad (\tau_\infty, D_\infty) \mapsto (\tau_\infty|_{D_\infty} : D_\infty \hookrightarrow X)
$$

are smooth. Let $e_0 = \deg(s_0)$ and $e_\infty = \deg(s_\infty)$.

Let $S \subset C$ be the nonempty open over which Hypothesis \ref{hyp:6.8} holds. Denote $U = X_{f,\text{pas}}$. Consider the morphism

$$
ev_{1,n+1} : \text{FreeChain}^U_2(X/C, n) \to U \times_S U \subset X \times_C X
$$

we studied in Section \ref{sec:6} By Proposition \ref{prop:6.11} there exists an $n$, and an open $V \subset \text{FreeChain}^U_2(X/C, n)$ such that $\ev_{1,n+1}|_V : V \to U \times_S U$ is surjective, smooth, with irreducible and birationally rationally connected fibres. Also, the morphisms $\ev_i|_V : V \to U$ are smooth, see Lemma \ref{lem:6.10}

Consider the following commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\ev_{1,n+1}} & U \times_S U \\
\downarrow & & \downarrow \\
X \times_C X & \xrightarrow{\tau_0 \times \tau_\infty} & C \times T_0 \times T_\infty
\end{array}
$$

By the remarks above this diagram satisfies all the assumptions of Lemma \ref{lem:3.14}.

Let us just point out that the associated rational map from $\text{Sym}^m(C) \times T_0 \times T_\infty$ to $H^m_{U \times S U} = H^m_U \times \text{Sym}^m(C) H^m_U$ is the product map

$$
\text{Sym}^m(C) \times T_0 \times T_\infty \supset W_0 \times_{\text{Sym}^m(C)} W_\infty \to H^m_U \times_{\text{Sym}^m(C)} H^m_U
$$

and by the remarks above is even smooth. We conclude that there exists a variety $T'$, a dominant morphism $T' \to T_0 \times T_\infty$, a compactification $V \subset \overline{V}$ over $X \times_C X$ and a morphism $\tau : C \times T' \to \overline{V}$ such that $C \times T' \to \overline{V} \to X \times_C X$ equals $C \times T' \to C \times T_0 \times T_\infty \to X \times_C X$, and such that the induced rational map from $\text{Sym}^m(C) \times T'$ to $H^m_U$ is dominant. See Lemma \ref{lem:3.10} and its proof for explanations. We may assume the compactification $\overline{V}$ dominates the compactification

$$
V \subset \text{FreeChain}^U_2(X/C, n) \subset \text{Chain}_2(X/C, n)
$$

by the space $\text{Chain}_2(X/C, n)$ of 2-pointed chains of (not necessarily free) lines in fibres of $X/C$. For example because in the proof of Lemma \ref{lem:3.10} the compactification $\overline{V}$ could have been chosen to dominate this compactification at the outset.

At this point, every $t \in T'(k)$ gives rise to a morphism $\tau_t : C \to \text{Chain}_2(X/C, n)$, which in turn gives rise to $n$ ruled surfaces $R_1, \ldots, R_n$ in $X$ over $C$, and $n + 1$ sections $\tau_1, \ldots, \tau_{n+1}$ of $X/C$ such that

1. $\tau_1$ corresponds to a point of $T_0$,
2. $\tau_{n+1}$ corresponds to a point of $T_\infty$,
3. $\tau_i, \tau_{i+1}$ factor through $R_i$ for $i = 1, \ldots, n$. 

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By shrinking $T'$ we may assume that both $\tau_1$ and $\tau_{n+1}$ meet $U$ and are $D$-free for some divisor $D$ of degree $m$.

Let $e_i = \deg(\tau_i)$ for $i = 2, \ldots, n$. Recall the constructible sets $W_{e_i}^m \subset H^m_{X,f}$ constructed in Proposition 3.15 in particular recall that each $W_{e_i}^m$ contains a dense open of $H^m_{X,f}$, and hence of $H^m_U$. Because the associated rational map from $\text{Sym}^m(C) \times \infty$ to $H^m_U$ is dominant, and because each $e_i : V \to U$ is smooth we see that after replacing $T'$ by a nonempty open subvariety, for each $t \in T'(k)$ the associated maps $\tau_i$ map a general divisor $D \subset C$ of degree $m$ to a point of $W_{e_i}^m$. In particular, $\tau_i$ meets $U = X_{f_{\text{pax}}}$ and there exists a divisor $D$ of degree $m$ such that $\tau_i$ is $D$-free. By Lemma 3.12 all the sections $\tau_i$ are 1-free, i.e., free. Since $T' \to T_0 \times T_\infty$ is dominant its image contains an open subset and the lemma is proved.

Let $P \subset \text{Porcupine}^{e,n}(X/C/k)$ be an irreducible component. Since the space $\text{Porcupine}^{e,n}(X/C/k)$ is smooth this is also a connected component. And since the forgetful morphisms $\Phi_{\text{forget}} : \text{Porcupine}^{e,n+m}(X/C/k) \to \text{Porcupine}^{e,n}(X/C/k)$ are smooth with geometrically irreducible fibres, we see that $P$ determines a sequence $(P_m)_{m \geq 1}$ of irreducible components, $P_m \subset \text{Porcupine}^{e,n+m}(X/C/k)$. Note that points of $P_{d-e-n}$ correspond to porcupines with total degree $d$.

**Proposition 8.7.** In Situation 7.1 assume the restriction of $f$ to some nonempty open $S \subset C$ satisfies Hypothesis 6.8. Let $D \subset C$ be an effective Cartier divisor of degree $2g(C)+1$. Let $P \subset \text{Porcupine}^{e,n+m}(X/C/k)$ be irreducible components. Assume that a general point of $P$ corresponds to a porcupine whose body is $D$-free, and similarly for $P'$. There exists an integer $E$ and for every $e \geq E$ there exists a dense open subscheme

$$U \subset P_{e-e_0-n_0} \times_{\text{Pic}^0(X/C/k)} P'_{e-e_{\text{in}}-n_{\text{in}}}$$

such that for any pair of points $(p, q) \in U$ the points $p$ and $q$ are connected in $\Sigma^e(X/C/k)$ by a chain of rational curves whose marked points and nodes parametrize unobstructed, non-sticky stable sections.

**Proof.** Let $T_0$, resp. $T_\infty$ denote the irreducible components of $\text{Sections}(X/C/k)$ to which the bodies of porcupines in $P$, resp. $P'$ belong. Let $n, e_1, \ldots, e_{n+1}$, $B$ and $W \subset T_0 \times T_\infty$ be the data found in Lemma 8.6. Note that $e_0 = e_1$ and $e_{\text{in}} = e_{n+1}$.

Set $E = \max\{e_i\} + \max\{0, \varphi(B, e_i)\} + 2g(C) + 999$ where $\varphi(\cdot, \cdot)$ is the function mentioned in Lemma 5.8. Pick $e \geq E$. The open subset $U$ will correspond to the pairs $(p, q)$ in the fibre product with $p = (s_0, q_0, i, 0, q_0) \in L_{0,i}(k), L_{0,i} \to X)$, $q = (s_{\infty}, q_{\infty}, i, r_{\infty}, q_{\infty}, i) \in L_{\infty,i}(k), L_{\infty,i} \to X)$ such that the pair $(s_0, s_{\infty}) \in W$.

Namely, let $R_1, \ldots, R_n, \tau_1, \ldots, \tau_\infty$ be as in Lemma 8.16 adapted to the pair $(s_0, s_{\infty})$. Denote $\xi = \alpha_{\xi}(p) = \alpha_{\xi}(q) \in \text{Pic}^e_{\xi/k}(k)$ the corresponding degree $e$ divisor class on $C$. For each $i = 2, \ldots, n$ it is possible to find a reduced effective divisor $Q_i \subset C$, $Q_i \subset \tau_i^{-1}(X_{f_{\text{pax}}})$ of degree $e - e_i \geq 2g(C) + 999$ such that $\tau_i^{-1}(\xi)(Q_i)$ is $\xi$. Now we will repeatedly use Lemmas 8.4 and 8.5 to find our chain of rational curves. Let $p_1$ be the point of $\text{Porcupine}^{e_0,e-e_0}(X/C/k)$ gotten from $p$ by moving all $e - e_0 = e - e_1$ quills attached to $s_0 = \tau_1$ into the ruled surface $R_1$. Then $p$ is connected to $p_1$ by a suitable chain of rational curves by Lemma 8.4. Let $p_2 \in \text{Porcupine}^{e_2,e-e_2}(X/C/k)$ be the porcupine with body $\tau_2$ and quills attached to the points of the divisor $Q_2$ lying in the ruled surface $R_1$. By Lemma 8.5 we see that $p_1$ is connected to $p_2$.
by a suitable chain of rational curves. Let \( p_3 \in \text{Porcupine}_{e_2:e_2}(X/C/k) \) be the porcupine with body \( \tau_2 \) and quills attached to the points of the divisor \( \mathcal{Q}_2 \) lying in the ruled surface \( R_2 \). By Lemma 8.4 the points \( p_2 \) and \( p_3 \) are connected by a suitable chain of rational curves. And so on and so forth.

**Corollary 8.8.** In Situation 7.1 assume in addition Hypothesis 6.8 for the restriction of \( f \) to some nonempty open \( S \subset C \). Let \( e_0, e_\infty \) be integers and let \( Z, Z' \) be irreducible components of \( \Sigma_{e_0}(X/C/k), \Sigma_{e_\infty}(X/C/k) \) whose general point parametrizes a free section of \( f \). Consider the families of components \( Z_e, Z'_e \) constructed in Lemma 7.4. Then for \( e \gg 0 \) we have \( Z_e = Z'_e \).

**Proof.** By Lemmas 7.8 and 7.6 we may assume that \( Z \) and \( Z' \) each contain a section which meets \( X_{f,\text{smooth}} \) and is \( 2g(C) + 1 \)-free. By Proposition 8.7 there exists an \( E \) and for all \( e \geq E \) there are points \( p \in Z_e \) and \( q \in Z'_e \) corresponding to porcupines with bodies in \( Z \) and \( Z' \) such that \( p \) can be connected to \( q \) by a chain of rational curves in \( \Sigma^*(X/C/k) \) whose marked points correspond to unobstructed, non-stacky stable sections. In particular such a chain lies in a unique irreducible component! \( \square \)

9. **Varieties and peaceful chains**

In this section we spell out what consequences we can draw from the work done in the previous sections for the moduli spaces of rational curves on a smooth projective variety whose moduli spaces of lines satisfy the conditions imposed in Section 6.

**Hypothesis 9.1.** Let \( k \) be an uncountable algebraically closed field of characteristic 0. Let \( Y \) be a smooth projective variety over \( k \). Let \( \mathcal{L} \) be an ample invertible sheaf on \( Y \). We assume that \( Y \rightarrow \text{Spec}(k) \) satisfies Hypothesis 6.8.

Assume that Hypothesis 9.1 holds. Let \( X = \mathbb{P}^1 \times Y \) and

\[
  f = \text{pr}_1 : X = \mathbb{P}^1 \times Y \rightarrow \mathbb{P}^1.
\]

The first thing to remark is that a morphism \( g : \mathbb{P}^1 \rightarrow Y \) gives rise to a section \( s : \mathbb{P}^1 \rightarrow X \), and conversely. Also, the section \( s \) is 0-free, i.e., unobstructed (see Definition 8.7), if and only if the rational curve \( g : \mathbb{P}^1 \rightarrow Y \) is unobstructed (at worst summands \( \mathcal{O}(-1) \) in the direct sum decomposition of \( g^*\mathcal{T}_Y \)). Similarly, \( s \) is 1-free, i.e., free (see Definition 8.7), if and only if the rational curve \( g : \mathbb{P}^1 \rightarrow Y \) is free. Arguing in this way we see that we can interpret many of the previous results for the family \( f : X \rightarrow \mathbb{P}^1 \) in terms of the moduli spaces of lines on \( Y \).

Instead of reproving everything from scratch in this setting we make a list of statements and we point out the corresponding lemmas, and propositions in the more general treatment above. We denote \( \overline{\mathcal{M}}_{0,0}(Y,e) \) the Kontsevich moduli space of degree \( e \) rational curves in \( Y \). Also, let \( Y_{\text{pax}} \) denote the set of peaceful points (w.r.t. \( Y \rightarrow \text{Spec}(k) \)), see Definition 6.4.

1. There is a sequence of irreducible components \( Z_e \subset \overline{\mathcal{M}}_{0,0}(Y,e) \), \( e \geq 1 \) uniquely characterized by the property that every comb in \( Y \) whose handle is contracted to a point of \( Y_{\text{pax}} \) and which has \( e \) teeth mapping to lines in \( Y \) is in \( Z_e \). This is the exact analogue of Lemma 7.4 starting with the fact that a constant map \( \mathbb{P}^1 \rightarrow Y \) is free!

2. A general point of \( Z_e \) corresponds to a map \( \mathbb{P}^1 \rightarrow Y \) which is free and meets \( Y_{\text{pax}} \). This is the exact analogue of Lemma 7.5.
(3) Consider a comb in \(Y\) whose handle has degree \(e\) with \(e' - e\) teeth which are lines. Such a comb is in \(Z_{e'}\) if the handle is free, is in \(Z_e\), and the attachment points map to \(Y\text{pax}\). This is the exact analogue of Lemma 7.4.

(4) Any stable map \(f : C \to Y\) in \(\overline{M}_{0,0}(Y, e')\) which has exactly one component a free curve corresponding to a point in \(Z_e\) and all other components free lines corresponds to a smooth nonstacky point of \(Z_{e'}\). This is the exact analogue of Lemma 7.7.

(5) Given any integer \(m \geq 0\) there exists \(e \gg 0\) such that the general point of \(Z_e\) corresponds to a map \(f : \mathbb{P}^1 \to Y\) with \(H^1(\mathbb{P}^1, f^*T_Y(-m)) = 0\). This is the exact analogue of Corollary 8.8 once you observe that such a \(\overline{M}_{0,0}(Y, e)\). It is perhaps not necessary, but we point out that instead of defining \(Z_e\) as in (1) above, we could just define \(Z_e\) as the unique irreducible component of \(\overline{M}_{0,0}(Y, e)\) which contains all trees of free lines. This follows from (4) (with \(e' = 0\)).

Lemma 9.2. In Situation 9.1 above. Let \(Z_{e, 1} \subset \overline{M}_{0,1}(Y, e)\) be the unique irreducible component dominating \(Z_{e} \subset \overline{M}_{0,0}(Y, e)\). The general fibre of \(ev : Z_{e, 1} \to Y\) is irreducible.

Proof. Let us temporarily denote by \(Z_{e, 1}' \subset Z_{e, 1}\) the dense open locus of stable 1-pointed maps \(\bigcup C_i \to Y\) all of whose irreducible components \(C_i \to Y\) are free and whose nodes map to points of \(Y\text{pax}\). The morphism \(ev : Z_{e, 1}' \to Y\) is smooth. Consider the locus \(W_{e, 1} \subset Z_{e, 1}'\) whose points correspond to those combs whose handles are contracted to a point of \(Y\text{pax}\) and whose teeth are (automatically) free lines and whose marked point is on the handle. By definition of the irreducible components \(Z_e\) the space \(W_{e, 1}\) is nonempty. It is smooth and \(W_{e, 1} \to Y\) is still smooth onto \(Y\text{pax}\). Clearly, Hypothesis 8.8 for \(Y \to \text{Spec}(k)\) implies the general fibre of \(W_{e, 1} \to Y\) is irreducible. From this, and the irreducibility of \(Z_{e, 1}\) it follows that the general fibre of \(Z_{e, 1} \to Y\) is irreducible, for example by [Gro67] Proposition 4.5.13. \(\square\)

Lemma 9.3. In Situation 9.1 above. Let \(Z' \subset \overline{M}_{0,0}(Y, e_0)\) and \(E\) be as in (6) above. Suppose that \(g : C \to Y\) is a genus 0 Kontsevich stable map such that \(C = \bigcup_{i=1}^A C_i \cup \bigcup_{j=1}^B L_j\) is a decomposition into irreducible components with the following properties: (1) each \(C_i \to Y\) is free and corresponds to a point of \(Z'\), (2) each \(L_i \to Y\) is a free line and a leaf of the tree, and (3) \(B \geq E - e_0\). Then \(g\) defines an unobstructed point of \(Z_{Ae_0 + B}\).

Proof. We will show there is a connected chain of curves contained in the unobstructed locus of \(\overline{M}_{0,0}(Y, 1)\) which connects \(g\) to a point of \(Z_{Ae_0 + B}\). We may first move the map a little bit such that all the nodes of \(C\) are mapped into \(Y\text{pax}\). Next, we may 1 by 1 slide all the lines \(L_i\) along the curves \(C_i\) and onto one of
the curves $C_{i_0}$ which is a leaf of the tree $\bigcup C_i$. Say this curve is $C_1$, and is attached to $C'' = \bigcup_{i \neq 1} C_i$ at the point $p \in C_1$ and $q \in C''$. Set $C' = C_1 \cup \bigcup L_i$, so $C = C' \cup_{p=q} C''$. By (3) above we see that $C' \to Y$ defines an unobstructed point of $Z_{e_0 + B}$.

We may assume (possibly moving the map a little bit) that the attachment point $p$ is mapped to a point $y \in Y$ such that the fibre of $Z_{e_0 + B, 1} \to Y$ is irreducible over this point, see Lemma 7.2. Hence we may connect the 1-pointed stable map $(C', p) \to Y$ inside the fibre of $Z_{e_0 + B, 1} \to Y$ over $y$ to a curve which is made out of a comb $C'''$ whose handle is contracted and whose teeth are free lines. Since the marked point is throughout mapped to the same $y \in Y(k)$ this connects the original stable map $C'' \cup C'' \to Y$ to a stable map $C''' \cup C'' \to Y$ where the number of lines in $C'''$ is now $e_0 + B$. We may again slide these lines over to some irreducible component $C_i$ of $C''$ and continue until all the $C_i$ are gone, and so obtain a point of $Z_{A_0 + B}$. This proves the lemma. □

The notation in the section conflicts with the notation introduced in Lemma 7.3 in case $Y$ is a fibre of a family as in Situation 7.1 such that Hypothesis 6.8 holds over a nonempty open of $C$. In that situation we will use the notation $Z_{e}(X_i)$ to denote the irreducible component defined in this section for the irreducible fibre $X_i$.

**Lemma 9.4.** In Situation 7.7 assume in addition Hypothesis 6.8 for the restriction of $f$ to some nonempty open $S \subset C$. Suppose that $s: C \to X$ is a free section, and let $Z \subset \Sigma(X/C/k)$ be the unique irreducible component containing the moduli point $[s]$. Let $Z_e$ as in Lemma 7.4. For any $t_1, \ldots, t_d \in S(k)$, for any free rational maps $s_i: \mathbb{P}^1 \to X_{t_i}$ such that $s_i(0) = s(t_i)$, if $s_i$ corresponds to $Z_{e_i}(X_{t_i})$, then the comb $C' \cup \bigcup \mathbb{P}^1 \to X/C$ defines an unobstructed point of $Z_{\deg(s) + \sum e_i}$.

**Proof.** This is very similar to the proof of Lemma 9.3 above. Namely, we first move the comb (as a comb) such that the points $s(t_i)$ are in the locus where the fibres of $Z_{e_i, 1} \to X_{t_i}$ are irreducible. then we connect the $s_i$ in these fibres to chains of lines. After this we can apply Lemma 7.4 for example. □

## 10. Families of varieties and lines in fibres

This section contains three lemmas that did not seem to fit well in other sections.

**Lemma 10.1.** In Situation 7.1 assume in addition Hypothesis 6.8 for the restriction of $f$ to some nonempty open $S \subset C$. Every geometric fibre $X_t$ of $f$ is integral and every pair of points of the smooth locus $X_t^\circ$ of the fibre is contained in (1) a chain of lines contained in $X_t^\circ$, and (2) a very free rational curve in the smooth locus.

**Proof.** Let $t \in C(k)$. Situation 7.1 requires $X_t$ to be irreducible. Since $X$ is smooth, by [GHS03], the fibre $X_t$ intersects the smooth locus of $f$. Hence $X_t$ is generically reduced and Cohen-Macaulay, and hence is reduced.

Let $x, y \in X_t(k)$ be smooth points of $X_t$. We will construct a very free rational curve in $X_t$ passing through $x$ and $y$. There exists a finite cover $C' \to C$ etale over $t$ and over the discriminant of $f: X \to C$, a point $t' \in C'(k)$ and sections $a, b: C' \to X_{t'}$ such that $a(t') = x, b(t') = y$. To construct $C'$ choose two complete intersection curves $C_x, C_y \subset X$ general apart from the condition $x \in C_x$ and $y \in C_y$.
and let \( C' \) be an irreducible component of the normalization of \( C_x \times_C C_y \) (details left to the reader).

Set \( X' = X_{C'} \) and \( f' : X' \to C' \) the base change of \( f \). By construction, all conditions of Situation 7.1 and Hypothesis 6.8 remain satisfied for \( X' \to C' \). Hence, upon replacing \( X \to C \) by \( X' \to C' \), and \( t \) by \( t' \) we may assume there are sections \( s_0, s_\infty \) such that \( s_0(t) = x \) and \( s_\infty(t) = y \). Attaching very free rational curves in fibres of \( f \) and deforming we may assume that \( s_0 \) and \( s_\infty \) are sufficiently general such that \((s_0, s_\infty) : C \to X \times_C X\) meets the open set \( V \subset X_S \times_S X_S \subset X \times_C X \) from Hypothesis 6.8.

By Chiang\textsuperscript{[5]} we can find a morphism \( \phi : C \to \text{Chain}_2(X/C, n) \) which meets \( \text{FreeChain}_2(X/S, n) \) and such that \( \text{ev}_{1,n+1} \circ \phi = (s_0, s_\infty) \). As in the proof of Lemma 8.3 this translates into a collection of \( n \) ruled surfaces \( R_1, \ldots, R_n \) in \( X/C \), \( n + 1 \) sections \( \tau_1, \ldots, \tau_{n+1} \) of \( X/C \) such that \( \tau_1 = s_0 \), \( \tau_{n+1} = s_\infty \) and such that \( \tau_i \tau_{i+1} \) factor through \( R_i \).

Note that the fibres \( R_{1,t}, R_{2,t}, \ldots, R_{n,t} \) form a chain of lines connecting \( x \) to \( y \) in the fibre \( X_t \). By Lemma 8.4 this chain lies in the smooth locus \( X_t^s \) of \( X \to C \). Thus any pair of points of \( X_t^s \) may be connected by a chain of lines in \( X_t^s \). The result follows upon applying \([\text{Kol96}] \) IV Theorem 3.9.4] for example. \( \square \)

**Remark 10.2.** The Lemmas 8.3 and 10.1 could be stated with slightly weaker hypotheses. Namely, we could remove the assumption (in 7.1) that all the fibres of \( X \to C \) are geometrically irreducible. This does not pose a problem for Lemma 8.3.

For 10.1 it means that one has to show there are no irreducible components of \( X_t \) which have multiplicity \( > 1 \). We may try to prove this by specializing the “chain of ruled surfaces” \( R_1, \ldots, R_n, \tau_1, \ldots, \tau_{n+1} \) as you move one of the two points \( x, y \) into a presumed higher multiplicity component. What might happen is that the ruled surfaces may break, and we do not see how to conclude the proof. However, if \( \mathcal{L} \) is very ample on all the fibres (an assumption that always holds in practice), then this argument works: in this case any line that meets a higher multiplicity component must be contained in it and it is easy to conclude from this.

**Lemma 10.3.** Let \( X \) be a nonsingular projective variety over an algebraically closed field \( k \) of characteristic 0. Let \( \mathcal{L} \) be an ample invertible sheaf on \( X \).

1. The locus \( T \subset X \) swept out by non free lines is a proper closed subset. Let \( D'_1, \ldots, D'_r \) be desingularizations (see [Hir64]) of the irreducible components \( D_1, \ldots, D_r \) of \( T \) which have codimension 1 in \( X \).

2. There exists a closed codimension 2 subset \( T' \subset X \) containing the singular locus of \( T \) and all codimension \( \geq 2 \) components such that any non free line not contained in \( T' \) is the image of a free rational curve on some \( D'_i \).

**Proof.** The first assertion is basic, see [Kol96] II Theorem 3.11]. For the second, first let \( T' \subset T \) be the closed subset over which \( \cup D'_i \to T \) is not an isomorphism. Any line in \( T \) not contained in \( T' \) is the image of a unique rational curve on some \( D'_i \). Finally we apply the general fact that since these curves cover \( D'_i \) the general one is free. \( \square \)

**Lemma 10.4.** Notations and assumptions as in Lemma 10.3. Let \( C \) be a nonsingular curve, and let \( f : \mathbb{P}^1 \times C \to X \) be a family of lines in \( X \). Assume that


\(\gamma = f|_{\{0\} \times C} : C \to X\) does not meet \(T\), and meets \(T\) transversally at all of its points of intersection. Then for each \(c \in C(k)\) the line \(L_c \to X\) is free.

**Proof.** For a general point \(t \in C(k)\) the line \(f_t : \mathbb{P}^1 \to X\) is free since \(f_t(0) \notin T\). Thus let \(t \in C\) be a point such that \(f_t(0) \in T\). If \(f_t(\mathbb{P}^1) \not\subset T\) we win also. Hence by Lemma 10.3 the line \(f_t : \mathbb{P}^1 \to X\) is the image of a free rational curve \(g : \mathbb{P}^1 \to D\) on the resolution \(D' = D'_t \to D = D_t\) for some \(i\). Consider the maps

\[g^*T_D \to f^*_tT_X \to f^*_tN_DX.\]

Note that the composition is zero. Since \(g^*T_D\) is globally generated (because \(g\) is free) we see the only way \(f^*_tT_X\) could have a negative summand is if \(f^*_tT_X \to f^*_tN_DX\) factors through a negative invertible sheaf on \(\mathbb{P}^1\). However, the given deformation \(f : \mathbb{P}^1 \times C \to X\) gives rise to a vector \(\theta \in H^0(\mathbb{P}^1, f^*_tT_X)\) which at the point 0 correspond to the tangent vector \(d\gamma \in T_{\gamma(0)}X\) by which assumption points out of \(D\). Thus \(f^*_tT_X \to f^*_tN_DX\) is nonzero on \(H^0\) and we win. \(\square\)

11. Perfect pens

In this section we introduce the notion of twisting ruled surfaces. Recall that ruled surfaces in \(X \to C\) w.r.t. \(\mathcal{L}\) were defined at the start of Section 8. The morphism \(R \to X\) maps into the smooth locus of \(X \to C\), see Lemma 8.4. Our first task is to show that (many) free sections lie on scrolls of free lines.

**Lemma 11.1.** In Situation 7.7 assume Hypothesis 6.8 holds for the restriction of \(f\) to some nonempty open \(S \subset C\). Let \(e_0\) be an integer and let \(Z\) be an irreducible component of \(\Sigma^{e_0}(X/C/k)\) whose general point parametrizes a free section. Then there exists a nonempty open \(U \subset Z\) such that every \(u \in U(k)\) corresponds to a section \(s\) such that (a) it is penna by a ruled surface \(R \to X\), and (b) any ruled surface \(R\) penna \(s\) has the property that all of its fibres are free lines in \(X\).

**Proof.** We are going to use Lemma 10.4. Note that \(\mathcal{L}\) may not be ample on \(X\). However, to prove the lemma we may replace \(\mathcal{L}\) by \(\mathcal{L} \otimes f^*\mathcal{N}\) for some suitable very ample sheaf \(\mathcal{N}\) on \(C\). This will allow us to assume that \(\mathcal{L}\) is ample on the total space \(X\), and it allows us to assume that any rational curve in \(X\) of degree 1 is in a fibre of \(f : X \to C\).

Let \(T' \subset T \subset X\), \(D_i\) be as in Lemma 10.3. Because \(T' \subset X\) has codimension \(\geq 2\) there is a nonempty open \(U' \subset Z\) such that every \(u \in U'(k)\) corresponds to a section \(s\) which is disjoint from \(T'\). Furthermore, we then pick a nonempty open \(U'' \subset U'\) such every \(u \in U''(k)\) corresponds to a section \(s\) which meets each irreducible component \(D_i\) transversally in smooth points, see [Ko96] II Proposition 3.7. Finally, we may, by Hypothesis 6.8 part (1), find a further nonempty open \(U \subset U''\) such that each section \(s : C \to X\) corresponding to a point of \(U\) meets the locus over which the evaluation morphism \(ev : \mathcal{M}_{0,1}(X/C, 1) \to X\) has irreducible rationally connected fibres.

Let \(s : C \to X\) correspond to a \(k\)-point of \(U\). By [GHS03] we can find a morphism \(g : C \to \mathcal{M}_{0,1}(X/C, 1)\) such that \(ev \circ g = s\). This corresponds to a ruled surface \(R \to X\) which pens \(s\). This proves (a). Next, let \(R\) be any ruled surface penning \(s\). By Lemma 8.1 it lies in the smooth locus of \(X \to C\). By Lemma 10.4 and our choice of \(s\) all fibres of \(R \to C\) are free curves in \(X\). This proves (b). \(\square\)
To define the notion of a twisting ruled surface, we introduce some notation. Let \( f : X \to C \), \( \mathcal{L} \) be as in Situation 7.1. Consider a ruled surface \( h : R \to X \). The following commutative diagram of coherent sheaves on \( R \) with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & T_{R/C} & \rightarrow & h^*T_{X/C} & \rightarrow & N_{R/X} & \rightarrow & 0 \\
& | & & \downarrow & & \downarrow & & & \\
0 & \rightarrow & T_R & \rightarrow & h^*T_X & \rightarrow & N_{R/X} & \rightarrow & 0
\end{array}
\]

defines the coherent sheaf \( N_{R/X} \). If the sheaf \( \mathcal{L} \) is relatively very ample (which is always satisfied in practice), the map \( h : R \to X \) will be a closed immersion and \( N_{R/X} \) will be a locally free sheaf. We will call the sheaf \( N_{R/X} \) the normal bundle regardless of whether \( R \to X \) is an embedding or not.

**Remark 11.2.** Here are some technical remarks for those readers who enjoy thinking about the case where \( \mathcal{L} \) is only assumed ample and not very ample on the fibres of \( X \to C \). The first is that \( N_{R/X} \) is flat over \( C \). This follows as the maps \( T_{R/C} \to h^*T_{X/C} \), \( T_R \to h^*T_X \), are injective, and \([\text{Mat80} \text{ Section (20.E)})\]. In particular it has depth \( \geq 1 \), its torsion is supported in codimension \( \geq 1 \), and any fibre meets the torsion locus in at most finitely many points. This also shows that \( N_{R/X} \mid R_i = N_{R_i/X_i} \), the normal bundle of the line \( R_i \to X_i \) (defined similarly). The second is that the deformation theory of the morphism \( R \to X \) (not the Hom-space) is given by \( H^0(R, N_{R/X}) \) (infinitesimal deformations) and \( H^1(R, N_{R/X}) \) (obstruction space). This is because the cotangent complex of \( h : R \to X \) is given by \( h^*\Omega^1_X \to \Omega^1_R \) which is quasi-isomorphic to \( h^*\Omega^1_{X/C} \to \Omega^1_{R/C} \). It follows that \( \text{Ext}^1(h^*\Omega^1_{X/C}, \mathcal{O}_R) = H^{i-1}(R, N_{R/X}) \) as usual.

**Definition 11.3.** In Situation 7.1. Let \( R \to X \) be a ruled surface in \( X \) and let \( D \) be a Cartier divisor on \( R \). For every nonnegative integer \( m \), we say \((R, D)\) is \( m\)-twisting if

1. the complete linear system \(|D|\) is basepoint free,
2. the cohomology group \( H^1(R, \mathcal{O}_R(D)) \) is 0,
3. \( D \) has relative degree 1 over \( C \),
4. the normal bundle \( N_{R/X} \) is globally generated,
5. \( H^1(R, N_{R/X}) \) equals 0, and
6. we have \( H^1(R, N_{R/X}(-D - A)) = 0 \) for every divisor \( A \) which is the pull back of any divisor on \( C \) of degree \( \leq m \).

This is a “relative” definition – it is defined with respect to the morphism \( f \). An important special case is an “absolute” variant, see Definition 11.7 below.

Suppose we are in Situation 7.1 and suppose that \((R, D)\) is an \( m\)-twisting ruled surface in \( X/C \). By assumption \(|D|\) is nonempty, base point free, and of relative degree 1. Hence a general element is smooth and defines a section \( \sigma : C \to R \). We will often say “let \((R, \sigma)\) be an \( m\)-twisting surface” to denote this situation. Having chosen \( \sigma \) we can think of \( R \to X \) as a family of stable 1-pointed lines. Let \( g = g_{(R, \sigma)} : C \to \overline{\mathcal{M}}_{0,1}(X/C, 1) \) denote the associated morphism.

**Lemma 11.4.** In the situation above:

1. The image of \( g = g_{(R, \sigma)} \) lies in the unobstructed (and hence smooth) locus of the morphism \( \text{ev} : \overline{\mathcal{M}}_{0,1}(X/C, 1) \to X \).
We have $H^1(C, g^* Tev(-A)) = 0$ for every divisor $A$ of degree $\leq m$ on $C$.

Let $\Phi : \overline{M}_{0,1}(X/C, 1) \rightarrow \overline{M}_{0,0}(X/C, 1)$ be the forgetful morphism (which is smooth). Then $g^* T_\Phi$ is globally generated, and $H^1(C, g^* T_\Phi) = 0$.

The section $h \circ \sigma : C \rightarrow X$ is free, see Definition [1.7].

Proof. The fact that $N_{R/X}$ is globally generated implies that for each $t \in C(k)$ the fibre $R_t \rightarrow X$ is free. This follows upon considering the exact sequences $0 \rightarrow T_{R_t|R_t} \rightarrow h^* T_X | R_t \rightarrow N_{R/X}| R_t \rightarrow 0$ (exact by flatness of $N_{R/X}$ over $C$), and the fact that $T_{R_t|R_t}$ is globally generated. This implies the image of $g$ is in the unobstructed locus for $ev$.

Let $\pi : R \rightarrow C$ be the structural morphism. The pull back of the relative tangent bundle $T_{ev}$ by $g$ is canonically identified with $\pi_* N_{R/X}(-\sigma)$. This is true because the normal bundle of a fibre $R_t \rightarrow X$ is an extension of the restriction $N_{R/X}| R_t$ by a rank 1 trivial sheaf on $R_t \cong \mathbb{P}^1$. Details left to the reader. The assumptions of Definition [11.3] imply that $R^1 \pi_* N_{R/X}(-\sigma) = 0$. Hence $H^1(C, g^* T_{ev}(-A)) = H^1(R, N_{R/X}(-\sigma + \pi^* A))$. Thus we get the desired vanishing from the definition of twisting surfaces.

The pull back $g^* T_\Phi$ is canonically identified with $\sigma^* \mathcal{O}_R(\sigma)$. The global generation of the sheaf $g^* T_\Phi$ is therefore a consequence of the base point freeness of $\mathcal{O}_R(\sigma)$ of Definition [11.3]. The vanishing of $H^1(C, g^* T_\Phi)$ follows on considering the long exact cohomology sequence associated to $0 \rightarrow \mathcal{O}_R \rightarrow \mathcal{O}_R(\sigma) \rightarrow \sigma_* \sigma^* \mathcal{O}_R(\sigma) \rightarrow 0$ and the vanishing of $H^1(R, \mathcal{O}_R(\sigma))$ in Definition [11.3].

Consider the exact sequence $\sigma^* T_{R/C} \rightarrow \sigma^* h^* T_{X/C} \rightarrow \sigma^* N_{R/X} \rightarrow 0$. Note that $\sigma^* T_{R/C} = \sigma^* \mathcal{O}_R(\sigma)$. By Definition [11.3] both $\sigma^* \mathcal{O}_R(\sigma)$ and $\sigma^* N_{R/X}$ are globally generated. Thus by the exact sequence $\sigma^* h^* T_{X/C}$ is globally generated. In the previous paragraph we showed that $H^1(C, \sigma^* \mathcal{O}_R(\sigma)) = 0$. There is an exact sequence $N_{R/X}(-\sigma) \rightarrow N_{R/X} \rightarrow \sigma_* \sigma^* N_{R/X} \rightarrow 0$. By Definition [11.3] $H^1(R, N_{R/X}) = 0$. The group $H^2(R, N_{R/X}(-\sigma))$ vanishes as its Serre dual $\text{Hom}_R(N_{R/X}(-\sigma), \omega_R)$ is zero (hint: consider restriction to fibres). Together these imply that $H^1(C, \sigma^* N_{R/X}) = 0$. Thus the first exact sequence of this paragraph implies that $H^1(C, \sigma^* h^* T_{X/C}) = 0$. Thus $(h \circ \sigma)^* T_{X/C}$ is globally generated with trivial $H^1$ and we conclude that $h \circ \sigma$ is 1-free, i.e., free.

The following innocuous looking lemma is why we introduce twisting surfaces. It will (eventually) show that any point of the boundary is connected by a $\mathbb{P}^1$ to a point in the interior.

Lemma 11.5. In Situation [11.1] let $(R, \sigma)$ be m-twisting with $m \geq 1$. Let $t \in C(k)$. The stable map $\sigma(C) \cup R_t \rightarrow X$ defines a nonstacky unobstructed point of $\Sigma(X/C/k)$ which is connected by a rational curve in $\Sigma(X/C/k)$ to a free section of $X \rightarrow C$.

Proof. The fact that the point is nonstacky comes from the fact that sections and lines have no automorphisms. The fact that the point is unobstructed follows from Lemma [11.4] above. Consider the linear system $|\sigma + R_t|$. Since $H^1(C, \mathcal{O}_R(\sigma)) = 0$ the map $H^0(R, \mathcal{O}_R(\sigma + R_t)) \rightarrow H^0(R_t, \mathcal{O}_{R_t}(1))$ is surjective. This implies that $|\sigma + R_t|$ is base point free. A general member of $|\sigma + R_t|$ is a section $\sigma' : C \rightarrow R$. It is trivial to show that $(R, \sigma')$ is $(m - 1)$-twisting. Hence by Lemma [11.4] we see that $\sigma'$ is free. The result is clear now by considering the pencil of curves on $R$ connecting $\sigma + R_t$ to $\sigma'$.

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It is useful to have a criterion that guarantees the existence of a twisting surface.
In particular, we would like to ensure that $g : C \to \mathcal{M}_{0,1}(X/C,1)$. We do not know a good way to do this unless $g(C) = 0$.

**Lemma 11.6.** In Situation 11.4 assume $C = \mathbb{P}^1$. Let $g : \mathbb{P}^1 \to \mathcal{M}_{0,1}(X/\mathbb{P}^1,1)$ be a section. Let $m \geq 1$. Assume we have:

1. The pull back $g^*T_R$ has degree $\geq 0$.
2. The image of $g$ is contained in the unobstructed locus of $ev$.
3. The cohomology group $H^1(\mathbb{P}^1, g^*T_{ev}(-m))$ is zero.
4. The composition $ev \circ g : \mathbb{P}^1 \to X$ is a free section of $X \to \mathbb{P}^1$.

Then the associated pair $(R, \sigma)$ is a $m$-twisting ruled surface in $X$.

**Proof.** We will use the identifications of $g^*T_R = \sigma^*\mathcal{O}_R(\sigma)$ and $g^*T_{ev} = \pi^*_s N_{R/X}(-\sigma)$ obtained in the proof of Lemma 11.4. Note that $R$ is a Hirzebruch surface, and $H^1(R, \mathcal{O}_R) = H^2(R, \mathcal{O}_R) = 0$. Combined with the fact that $\sigma^2 = \deg(g^*T_R) \geq 0$ we see that $|\sigma|$ is base point free. Also, during the course of the proof we may assume that $\sigma$ is general in its linear system on $R$. In particular this means that the section $1 \in \Gamma(R, \mathcal{O}_R(\sigma))$ is regular for the coherent sheaf $N_{R/X}$. (Note that this is automatic in the case, which always holds in practice, that $R \to X$ is a closed immersion.)

The fact that $ev \circ g : \mathbb{P}^1 \to X$ is 1-free means that $g^*ev^*T_{X/\mathbb{P}^1}$ is globally generated. The fact that $H^1(\mathbb{P}^1, g^*T_{ev}) = 0$ means that any infinitesimal deformation of the morphism $ev \circ g : \mathbb{P}^1 \to X$ can be followed by an infinitesimal deformation of $g : \mathbb{P}^1 \to \mathcal{M}_{0,1}(X/C,1)$. In terms of the pair $(R, \sigma)$ this means that the image of $\alpha : H^0(R, N_{R/X}) \to H^0(\mathbb{P}^1, \sigma^* N_{R/X})$ contains the image of $\beta : H^0(\mathbb{P}^1, (ev \circ g)^*T_{X/\mathbb{P}^1}) \to H^0(\mathbb{P}^1, \sigma^* N_{R/X})$. In this way we conclude that $N_{R/X}$ is at least globally generated over the image of $\sigma$.

This weak global generation result in particular implies that $R^1 \pi^*_s N_{R/X}(-\sigma) = 0$, and $R^1 \pi^* N_{R/X} = 0$; we can for example see this by computing the cohomology on the fibres. Thus we see that $H^1(R, N_{R/X}(-\sigma - \pi^*A)) = H^1(\mathbb{P}^1, g^*T_{ev}(-A))$. This gives us the vanishing of the cohomology group $H^1(R, N_{R/X}(-\sigma - \pi^*A))$ for any divisor $A$ of degree $\leq m$ on $\mathbb{P}^1$. We also get $H^1(R, N_{R/X}) = H^1(\mathbb{P}^1, \pi^* N_{R/X})$, and an exact sequence $0 \to \pi^*_s N_{R/X}(-\sigma) \to \pi^* N_{R/X} \to \sigma^* N_{R/X} \to 0$. The first sheaf being identified with $g^*T_{ev}$ and the second being globally generated we conclude that $H^1(R, N_{R/X}) = 0$. Note that, with $m > 0$ this argument actually also implies that $H^1(R, N_{R/X}(-R_t)) = 0$ for any $t \in \mathbb{C}(k)$. At this point, what is left, is to show that $N_{R/X}$ is globally generated. It is easy to show that a coherent sheaf on $\mathbb{P}^1$ which is globally generated at a point is globally generated. Since $m \geq 1$ the map $H^0(R, N_{R/X}) \to H^0(R_t, N_{R/X} |_{R_t})$ is surjective by the vanishing of cohomology we just established. Combined these imply that $N_{R/X}$ is globally generated. \qed

Here is the definition we promised above.

**Definition 11.7.** Let $k$ be an algebraically closed field of characteristic 0. Let $Y$ be projective smooth over $k$, and let $\mathcal{L}$ be an ample invertible sheaf on $Y$. A **scroll** on $Y$ is given by morphisms $\mathbb{P}^1 \leftarrow R \to Y$ such that (a) $R$ is a smooth projective surface, (b) all fibres $R_t$ of $R \to \mathbb{P}^1$ are nonsingular rational curves, and (c) the induced maps $R_t \to Y$ are lines in $Y$. Let $R$ be a scroll in $Y$ and suppose $D \subset R$ is
a Cartier divisor. For every nonnegative integer $m$, we say $(R, D)$ is an $m$-twisting scroll in $Y$ if the diagram

$$
\begin{array}{ccc}
\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \times Y \\
\downarrow & & \downarrow \text{pr}_1 \\
\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 
\end{array}
$$

and the divisor $D$ form an $m$-twisting ruled surface with respect to the invertible sheaf $	ext{pr}_2^* \mathcal{L}$. If $m$ is at least $2$, an $m$-twisting scroll will also be called a very twisting scroll.

The main observation of this section is that if $X \to C$ is as in Situation 7.1 and Hypothesis 6.8 holds over an open part of $C$, and if the geometric generic fibre has a very twisting surface, then $X \to C$ contains very twisting ruled surfaces.

**Lemma 11.8.** In Situation 9.1, assume that $Y$ has a very twisting scroll. Then there exist $m$-twisting scrolls $(\pi : R \to \mathbb{P}^1, h : R \to Y, \sigma)$ such that $\pi_* \mathcal{O}_R(D)$ and $\pi_* N_{R/\mathbb{P}^1, Y}$ are ample locally free sheaves on $\mathbb{P}^1$ with $m$ arbitrarily large. We can further arrange it so that $h \circ \sigma$ corresponds to an unobstructed point of one of the irreducible components $Z_\sigma$ defined in Section 9.

**Proof.** Let $(R, D)$ be an $m$-twisting scroll on $Y$. Choose a section $\sigma$ of $R$ in $|D|$ and think of the associated morphism $g : \mathbb{P}^1 \to \mathcal{M}_{0,1}(Y, 1)$ as in Lemmas 11.4 and 11.6. In fact Lemma 11.4 implies that the morphism $g$ satisfies the assumptions of Lemma 11.6. It is clear that the assumptions of Lemma 11.6 hold on an open subspace of $\text{Mor}(\mathbb{P}^1, \mathcal{M}_{0,1}(Y, 1))$. Also, a morphism $g$ satisfying those assumptions is free. Given two morphisms $g_i : \mathbb{P}^1 \to \mathcal{M}_{0,1}(Y, 1)$, $i = 1, 2$ corresponding to $m_i$-twisting surfaces with $g_i(\infty) = g_2(0)$ there exists a smoothing (see [Ko90 II Definition 1.10]) of $g_1 \cup g_2 : \mathbb{P}^1 \cup_{\infty \sim 0} \mathbb{P}^1 \to \mathcal{M}_{0,1}(Y, 1)$ whose general fibre is a morphism $g : \mathbb{P}^1 \to \mathcal{M}_{0,1}(Y, 1)$ which corresponds to an $(m_1 + m_2 - 1)$-twisting scroll.

We conclude that 3-twisting surfaces exist; by glueing 2 2-twisting surfaces which exist by the assumption on $Y$ and smoothing as above. If $(R \to \mathbb{P}^1, R \to Y, D)$ is $m$-twisting with $m \geq 3$, then $(R \to \mathbb{P}^1, R \to Y, D + R_0)$ is $(m - 1)$-twisting and has the property that $\pi_* \mathcal{O}_R(D + R_0)$ is ample.

Fix $m \geq 3$ such that an $m$-twisting surface exist. Consider a large integer $N$ and consider maps $g_i : \mathbb{P}^1 \to \mathcal{M}_{0,1}(Y, 1)$, $i = 1, \ldots, N$ each corresponding to an $m$-twisting surface $(\pi_i : R_i \to \mathbb{P}^1, h_i : R_i \to X, \sigma_i)$, and such that $g_i(\infty) = g_{i+1}(0)$ for $i = 1, \ldots, N - 1$. We may assume that all the free rational curves $h_i \circ \sigma_i$ lie in the same irreducible component $Z'$ of $\mathcal{M}_{0,0}(Y, e_0)$, for example by taking the same twisting scroll for each $i$ (with coordinate on $\mathbb{P}^1$ reversed for odd indices $i$). We may also assume that $N \geq E - e_0$ where $E$ is as in assertion 9 in Section 9. Since each $g_i$ is free we can find a smoothing of the stable map $g_1 \cup \ldots \cup g_N : \mathbb{P}^1 \cup_{\infty \sim 0} \ldots \cup_{\infty \sim 0} \mathbb{P}^1 \to \mathcal{M}_{0,1}(Y, 1)$. In fact, we may moreover assume the smoothing of $\mathbb{P}^1 \cup_{\infty \sim 0} \ldots \cup_{\infty \sim 0} \mathbb{P}^1$ has $N$ sections $z_i$ with $z_i$ limiting to a point $t'_i$ on the $i$th component of the initial chain. Let $(\pi : R \to \mathbb{P}^1, h : R \to Y, \sigma)$ correspond to a general point of the smoothing, and let $t_1, \ldots, t_N \in \mathbb{P}^1(k)$ be the values of the sections $z_i$. We claim that $(R \to \mathbb{P}^1, h : R \to Y, \sigma + \sum R_i)$ has at least two of the three desired properties.
First, since \((R, \sigma)\) is \(Nm\)-twisting we see immediately that \((R, \sigma + \sum R_{t_i})\) is \(N(m-1)\) twisting.

Second, the Cartier divisor \(\sigma + \sum R_{t_i}\) is a deformation of a divisor on the surface

\[
R_1 \cup_{R_{1, \infty} \equiv R_{2, 0}} \ldots \cup_{R_{N-1, \infty} \equiv R_{N, 0}} R_N
\]

which restricts to the divisor class of \(\sigma_i + R_{t_i}^{+}\) on every \(R_i\). By our discussion above we conclude that \(\pi_*\mathcal{O}_R(\sigma + \sum R_{t_i})\) is a deformation of a locally free sheaf on a chain of \(\mathbb{P}^1\)'s which is ample on each link, hence ample.

Third, we show that a general element of \(|\sigma + \sum R_{t_i}|\) corresponds to a point of \(Z_\epsilon\) with \(\epsilon = Ne_0 + N\). Consider the comb consisting of \(h : \sigma(\mathbb{P}^1) \cup \bigcup R_{t_i} \rightarrow Y\). It suffices to show that this defines a unobstructed point of \(\mathcal{M}_{0,0}(Y, e)\) which is in \(Z_\epsilon\). It is unobstructed because \((R, \sigma)\) is a twisting surface. By construction our comb is a “partial smoothing” of the tree of rational curves

\[
(\mathbb{P}^1 \cup R_{1, t'_i} \cup_{\infty \sim 0} \ldots \cup_{\infty \sim 0} (\mathbb{P}^1 \cup R_{N, t'_N})) \xrightarrow{h_0|\sigma_i \cup h_i|R_{t_i}^{+}} Y.
\]

Again since all fibres of twisting surfaces are free lines, and since each \(\sigma_i\) is free this stable map is unobstructed. At this point we may apply Lemma 9.3 to conclude that this stable map is in \(Z_\epsilon\).

At this point it is not yet clear that \(\pi_*\mathcal{N}_{R/X}\) is ample. Let \(\sigma'\) be a section of \(R\) representing a general element of \(\sigma + \sum R_{t_i}\). We just proved that \(\sigma'\) is a point of \(Z_\epsilon\). Let \(g' : \mathbb{P}^1 \rightarrow \mathcal{M}_{0,1}(Y, 1)\) be the morphism corresponding to the twisting surface \((R, \sigma')\). We saw above that \(g'\) corresponds to a twisting surface. Since \((g')^*T_Y\) has no \(H^1\) there are no obstructions to lifting a given deformation of \(ev \circ g' = \sigma'\) to a deformation of \(g'\). Hence we may assume that \(\sigma'\) is a general point of \(Z_\epsilon\) and in particular we may assume that \(\sigma'\) is very free, see Section 9.4.5. At this point consider the exact sequence of locally free sheaves on \(\mathbb{P}^1\)

\[
0 \rightarrow \pi_*\mathcal{N}_{R/\mathbb{P}^1 \times Y}(-\sigma') \rightarrow \pi_*\mathcal{N}_{R/\mathbb{P}^1 \times Y} \rightarrow \sigma^*\mathcal{N}_{R/\mathbb{P}^1 \times Y} \rightarrow 0.
\]

The sheaf on the left hand side is ample because \((R, \sigma')\) is \(N(m-1)\)-twisting. Combined with the surjective map \((\sigma')^*T_Y \rightarrow \sigma^*\mathcal{N}_{R/\mathbb{P}^1 \times Y}\) and the ampleness of \((\sigma')^*T_Y\) this proves the result.

In order to formulate the next lemma, let us call a very twisting scroll (as in the lemma above) \((\pi : R \rightarrow \mathbb{P}^1, h : R \rightarrow Y, \sigma)\) wonderful if \(\pi_*\mathcal{N}_{R/\mathbb{P}^1 \times Y}\) is ample, \(\pi_*\mathcal{O}_R(\sigma)\) is ample, and \(h \circ \sigma\) belongs to the irreducible component \(Z_\epsilon\).  

**Lemma 11.9.** In Situation 7.1 assume that Hypothesis 6.3 holds over a nonempty open \(S \subset C\). Assume that for some \(t \in C(k)\) the fibre \(X_t\) has a very twisting scroll. Then there exist:

1. a smooth variety \(B\) over \(k\),
2. a flat morphism \(\mathcal{L} : B \rightarrow C\),
3. a smooth projective family of surfaces \(\mathcal{R} \rightarrow B\),
4. a morphism \(\pi : \mathcal{R} \rightarrow \mathbb{P}^1\),
5. a morphism \(h : \mathcal{R} \rightarrow X\) such that \(f \circ h = \mathcal{L}\), and
6. a morphism \(\sigma : \mathbb{P}^1 \times B \rightarrow \mathcal{R}\) over \(B\) such that \(\pi \circ \sigma = pr_1\).

These data satisfy:
The first part of this lemma is just one possible formulation of what it means to have a family of wonderful very twisting surfaces. Really the lemma just claims that once there is one there are many.

**Remark 11.10.** The first part of this lemma is just one possible formulation of what it means to have a family of wonderful very twisting surfaces. Really the lemma just claims that once there is one there are many.

**Proof.** Let \((\mathbb{P}^1 \leftarrow R \to X_1, D)\) be an \(m\)-twisting scroll in a fibre. There are no obstructions to deforming the morphism \(R \to \mathbb{P}^1 \times X_1\) because \(H^1(R, N_{R/\mathbb{P}^1 \times X_1}) = 0\), see Remark 11.2. Thus there are also no obstructions to deforming the morphism \(R \to \mathbb{P}^1 \times X\) since this just adds a trivial summand to the normal bundle and \(H^1(R, \mathcal{O}_R) = 0\) (as \(R\) is a rational surface). Since the normal bundle of \(R \to \mathbb{P}^1 \times X\) is globally generated we see that we may deform \(R \to \mathbb{P}^1 \times X_1\) to a morphism \(R' \to \mathbb{P}^1 \times X_1\) with \(t\) general and \(R' \to X_1\) passing through a general point of \(X_1\). In particular there exist very twisting surfaces in fibres over \(R\) to which Lemma 11.8 applies. Thus we may now assume that \(R\) is a wonderful twisting surface.

Moreover, let \(\sigma : \mathbb{P}^1 \to R\) be a section of \(R \to \mathbb{P}^1\) such that \(D \sim \sigma\) are rationally equivalent. Pick a point \(p \in \mathbb{P}^1(k)\). Since \(m > 0\) the map \(H^0(R, N_{R/\mathbb{P}^1 \times X_1}(-\sigma)) \to H^0(R_p, N_{R/\mathbb{P}^1 \times X_1}(-\sigma(p)))\) is surjective. We see that given any infinitesimal deformation of the pointed map \((R_p, \sigma(p)) \to (X_1, h(\sigma(p)))\) (from a pointed line to \(X_1\) pointed by the image of the point) in \(X_1\) we can find an (unobstructed) infinitesimal deformation of \(R \to \mathbb{P}^1 \times X_1\) that induces it. Combined with the fact, proven above, that we can pass a deformation of \(R\) through a general point of \(X\) this shows that we can deform \(R \to X_1\) such that a given fibre of \(R \to \mathbb{P}^1\) is a general line in \(X/C\).

Finally, what is left is to show that in a family of morphisms of surfaces \(R \to \mathbb{P}^1 \times X\) the locus where the surface is a wonderful \(m\)-twisting scroll is open. This follows from semi-continuity of cohomology and can safely be left to the reader, although a very similar and more difficult case is handled in Lemma 11.11 below. □

In Situation 7.1 let \(R \to X\) be a ruled surface all of whose fibres are free lines in \(X/C\). Let \(D\) be a Cartier divisor on \(R\) of degree 1 on the fibres of \(R \to C\). Let \(t_1, \ldots, t_\delta \in C(k)\) be pairwise distinct points. Let \(\mathbb{P}^1 \leftarrow S_i \to X_{t_i}\) be scrolls. Let \(D_i\) be a Cartier divisor on \(S_i\) of degree 1 on the fibres of \(S_i \to \mathbb{P}^1\). Assume given isomorphisms \(R_{t_i} \cong S_{i,0}\) of the fibre of \(R\) over \(t_i\) with the fibre of \(S_i\) over 0 compatible with the maps into \(X_{t_i}\). Let \(C' = C \cup \bigcup \mathbb{P}^1\) be a copy of \(C\) with \(\delta\) copies of \(\mathbb{P}^1\) glued by identifying 0 in the \(i\)th copy with \(t_i \in C(k)\). Let \(R' = R \cup \bigcup S_i \to C'\) be the ruled surface over \(C'\) gotten by gluing \(R_{t_i}\) to \(S_{i,0}\) using the given isomorphisms. Let \(h' : R' \to C' \times_C X\) be the obvious morphism. Note that there is a Cartier divisor \(D'\) on \(R'\) which restricts to \(D\) on \(R\) and to \(D_i\) on \(S_i\). This Cartier divisor is unique up to rational equivalence.
In the following we are interested in smoothings of situations as above. This means that we have an irreducible variety \( T \) over \( k \), and a commutative diagram of varieties

\[
\begin{array}{ccc}
R' = R \cup \bigcup S_i & \rightarrow & C \times_{C \times T} \left( X \times T \right) \rightarrow X \times T \\
\downarrow \pi_0 & & \downarrow \pi \\
C' = C \cup \bigcup \mathbb{P}^1 & \rightarrow & C \times T \rightarrow T
\end{array}
\]

satisfying the following conditions: (1) \( C \rightarrow T \) and \( R \rightarrow T \) are flat and proper, (2) every fibre \( C_t \) of \( C \rightarrow T \) over \( t \in T(k) \) is a nodal curve of genus \( g(C) \) and \( C_t \rightarrow C \) has degree 1 (see discussion in Section \( 5 \)), (3) the morphism \( R \rightarrow C \) is smooth, (4) every fibre \( R_t \rightarrow C_t \times_C X \) is a ruled surface over every irreducible component of \( C_t \), (5) for some point \( 0 \in T(k) \) the fibre \( R_0 \rightarrow C_0 \times_C X \) is isomorphic to our map \( h' : R' \rightarrow C' \times_C X \) above, and finally (6) for some point \( t \in T(k) \) the fibre \( C_t = C \).

In addition we assume given a Cartier divisor \( D \) on \( R \) restricting to the divisor \( D' \) on \( R' = R_0 \).

**Lemma 11.11.** In the situation above.

1. If \((R,D)\) is \(m\)-twisting and \((S_i,D_i)_{i=1,\ldots,\delta}\) are \(m_i\)-twisting, with \(m_i \geq 2\). Then for \(t \in T(k)\) general the ruled surface \((R_t,D_t)\) in \(X\) is \((m+\delta)\)-twisting.

2. Let \(n_i,i=1,2,3\) be integers such that for all Cartier divisors \(A\) of degree on \(C\) we have
   
   (a) \(\deg(A) \geq n_1 \Rightarrow H^1(C,(\pi_*N_{R/X})(A)) = 0\),
   
   (b) \(\deg(A) \geq n_2 \Rightarrow H^1(C,(\pi_*N_{R/X}(-D))(A)) = 0\), and
   
   (c) \(\deg(A) \geq n_3 \Rightarrow H^1(C,(\pi_*O_R(D))(A)) = 0\).

   If \((S_i,D_i)_{i=1,\ldots,\delta}\) are wonderful very twisting scrolls and \(\delta \geq \max\{n_1+1,n_2+1,n_3\}\), then for \(t \in T(k)\) general the ruled surface \((R_t,D_t)\) in \(X\) is \((\delta-n_3)\)-twisting.

**Proof.** We will not prove the first statement. It follows by a combination of the methods of the proof of the second statement, and those in the proof of Lemma 11.8. We just remark that the increase in the twisting comes from the fact that the sheaves \((S_i \rightarrow \mathbb{P}^1)_* (N_{S_i/\mathbb{P}^1 \times X_i}(-D_i))\) are ample vector bundles on \(\mathbb{P}^1\) because \(m_i \geq 2\) combined with [Kol96, II Lemma 7.10.1].

We begin the proof of the second statement. Because the morphism \(R \rightarrow C\) is smooth we can define \(N_{R/X}\) by the short exact sequence

\[
0 \rightarrow T_{R/C} \rightarrow (R \rightarrow X)^*T_{X/C} \rightarrow N_{R/X} \rightarrow 0.
\]

As before this is a sheaf on \(R\) which is flat over \(C\). For every point \(c \in C(k)\) lying over \(t \in C(k)\) the restriction of \(N_{R/X}\) to the fibre \(R_c\) is the normal bundle of the line \(R_c \rightarrow X_t\) (compare with Remark 11.2). For all points \(c \in C(k)\) lying over 0 the sheaf \(N_{R/X}|_{R_c}\) is globally generated. Hence after replacing \(T\) by an open subset containing 0 we may assume this is true for all points \(c\).

Consider the coherent sheaves of \(O_C\)-modules \(E_1 = \pi_*O_R(D)\), \(E_2 = \pi_*N_{R/X}\) and \(E_3 = \pi_*N_{R/X}(-D)\). By the above the corresponding higher direct images are zero (as \(H^1\) of a globally generated sheaf on \(\mathbb{P}^1\) is zero). By cohomology and base change this implies \(E_i\) is a locally free sheaf on \(C\). Let \(U \subset T\) be a nonempty open such that \(C_t = C\) for all \(t \in U(k)\) (this exists because we started with a smoothing). For
$t \in U(k)$ we can and do think of $\mathcal{E}_{t,1}$ as a locally free sheaf on $C$. Let $A$ be a Cartier divisor of degree $\leq \delta - n_3$ on $C$, and let $p \in C(k)$ be any point. We would like to show that $H^1(C, \mathcal{E}_{1,1}(-p)) = 0$, $H^1(C, \mathcal{E}_{2,1}(-p)) = 0$, and $H^1(C, \mathcal{E}_{t,1}(-A)) = 0$, for $t$ general in $U(k)$. This now follows from [Ko96, II Lemma 7.10.1], the definition of $n_i$ in the lemma and the fact that $\delta \geq n$.

Namely, these vanishing imply that both $\mathcal{E}_{1,1}$ and $\mathcal{E}_{2,1}$ are globally generated and have no $H^1$, which implies that $\mathcal{O}_{R_1}(D_t)$ and $N_{R_1/X}$ are globally generated and have no $H^1$. The statement on $\mathcal{E}_{3,1}$ gives the desired vanishing of $H^1(R_1, N_{R_1/X}(-D_t - \pi_i^* A))$.  

**Proposition 11.12.** In Situation (7.1) assume Hypothesis 6.8 holds over an open part $S$ of $C$. Suppose a geometric fibre of $X \to C$ has a very twisting scroll. Then there exists an irreducible component $Z \subset \Sigma(X/C/k)$ containing a free section, such that for all $e \gg 0$ the irreducible component $Z_e$ defined in Lemma 7.4 contains a point $[s]$ such that $s = h \circ \sigma$ for some very twisting ruled surface $(h : R \to X, \sigma)$ in $X$.

**Proof.** The proof of this proposition is in two stages. In the first stage we will find some sufficiently twisting surface $(R, \sigma)$. The irreducible component $Z$ will be the unique one containing $[h \circ \sigma]$. After that, in the second stage we will find the others by glueing lines and wonderful very twisting surfaces.

Let $B, \ell : B \to C$, $R \to B$, $\pi : R \to \mathbb{P}^1$, $h : R \to X$, $\sigma : \mathbb{P}^1 \times B \to R$ be the family of wonderful very twisting scrolls in fibres of $X/C$ we found in Lemma 11.9. Let $V \subset \mathcal{X}_{0,1}(X/C, 1)$ be the nonempty open swept out by $\text{fib}_0$ as in that lemma. In addition, let $e_0$ be the degree of the rational curves $h_0 \circ \sigma_b : \mathbb{P}^1 \to X_{\mathcal{X}_{0,1}}(k)$.

By Lemma 7.3 there exists a free section $s$ of $X \to C$ meeting $X_{f, \text{pax}}$. We may deform $s$ such that it also meets the open set $ev(V)$. We may also deform $s$ such that $s$ defines a point of the open $U$ of Lemma 11.1. In the proof of Lemma 11.1 we used the main result of [GHS03] to establish the existence of a ruled surface $R$ penning $s$ all of whose fibres are free lines. Hence we may actually choose $R$ such that for some $t \in C(k)$ general the one pointed line $s(t) \in R_t$ defines a point of the open $V$. Here we use that the moduli space of lines in $X_{S/S}$ meeting $X_{f, \text{pax}}$ is irreducible by Hypothesis 6.8. Thus we may assume that $s$ is pened by a ruled surface $R$ all of whose fibres are free lines in $X/C$, by (b) of Lemma 11.1 and such that there exists an open $W \subset C$ with the property that $s(t) \in R_t \to X_t$ defines a point of $V$ for all $t \in W(k)$.

In other words, at every point of $W$ we may attach a wonderful very twisting surface out of the family $R/B$ to $R$. We reinterpret this in terms of the corresponding map $g = g_{(R, \sigma)} : C \to \overline{M}_{0,1}(X/C, 1)$. In terms of this it means that we can find arbitrarily many free maps $g_{\alpha} : \mathbb{P}^1 \to \overline{M}_{0,1}(X/C, 1)$ corresponding to wonderful very twisting surfaces in fibres that we may attach to the morphism $g$ to get combs $C \cup \bigcup_{\alpha} \mathbb{P}^1 \to \overline{M}_{0,1}(X/C, 1)$. By [Ko96] II Theorem 7.9 an arbitrarily large subcomb will smooth. Now we reinterpret this back into a smoothing of the corresponding glueing of ruled surfaces. It says, via part (2) of Lemma 11.11 that a general point of the base of this smoothing will correspond to a $(e_0 + 1)$-twisting surface $(\pi : R \to C, h : R \to X, \sigma)$. This finishes the proof of the first stage.

Let $e_1$ be the degree of $h \circ \sigma$, and let $Z \subset \Sigma^{e_1}(X/C/k)$ (as promised) be the unique irreducible component containing the free section $h \circ \sigma$. Fix an integer
\(i \in \{0, 1, \ldots , e_0 - 1\}\). Pick some effective Cartier divisor \(\Delta_i \subset C\) of degree \(i\), and pick a section \(\sigma_i\) of \(R \to \mathbb{P}^1\) which is a member of the (base point free) linear system \(|\sigma + \pi^*\Delta_i|\). Then \((R, \sigma_i)\) is \((e_0 + 1 - i)\)-twisting. Thus the associated morphism 
\[g_i : C \to \overline{\mathcal{M}}_{0,1}(X/C, 1)\]
free, see Lemma \[11.12\]. In particular after deforming \(g_i\) a bit we may assume that \(g_i\) meets the open dense subset \(V\) above.

For any \(\delta > 0\) consider pairwise distinct point \(t_1, \ldots , t_\delta \in C(k)\) such that \(g_i(t_j) \in V\) for all \(j = 1, \ldots , \delta\). In addition let \(g_{ij} : \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X_{t_j}, 1)\) be morphisms corresponding to wonderful very twisting surfaces out of our family \(R/B\) above with the property that \(g_{ij}(0) = g_i(t_j)\) (this is possible). In other words, now we have a comb \(\tilde{g}_i : C \cup \bigcup_j \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X/C, 1)\). Note that the resulting stable section \(\text{ev} \circ \tilde{g}_i\) of \(X/C\) has degree \(e_1 + i + \delta e_0\). Since both \(g_i\) and all the \(g_{ij}\) are free there exists a smoothing of this comb (for all \(\delta \geq 0\)). By part (1) of Lemma \[11.11\] a general point of the base of this smoothing corresponds to a \((e_0 + 1 - i + \delta)\)-twisting surface.

It remains to show that (for every \(i\)) the stable section \(\text{ev} \circ \tilde{g}_i\) corresponds to a point of \(Z_{e_1+i+\delta e_0}\) (since then the same will be true after smoothing the surface). This follows from Lemma \[9.4\].

**Corollary 11.13.** Assumptions and notations as in the Proposition \[11.12\] above, with \(Z, Z_e\) as in that proposition. Let \(P_{e,1}\) be the moduli space of porcupines \((s : C \to X, q \in C(k), r \in L(k), L \to X_q)\) with one quill and body \(s\) whose moduli point is in \(Z_e\). There exists a dense open \(U_{e,1} \subset P_{e,1}\) such that if \([[(s : C \to X, q \in C(k), r \in L(k), L \to X_q)] \in U_{e,1}\) then there exists a ruled surface \(R \in X\) denning \(s(C) \cup L\) (see Definition \[8.2\]) such that \((R, s)\) is very twisting.

**Proof.** Although the formulation of this corollary is cumbersome the proof is not. Namely, we already know that for all \(e \gg 0\) there exists a very twisting ruled surface \((\pi : R \to C, h : R \to X, \sigma)\) such that \(h \circ \sigma\) lies in \(Z_e\). Pick a general \(q \in C(k)\) and consider the datum \((q \in C(k), \pi : R \to C, h : R \to X, \sigma)\). After moving the ruled surface \((R, \sigma)\) a bit we may assume \(h(\sigma(q))\) is a point of \(X_f\) such that \((h \circ \sigma)(C) \cup h(R_q)\) is a porcupine. (This step is not strictly necessary for the proof.) At this point it suffices to show that any deformation of the porcupine can be followed by a deformation of the datum \((q \in C(k), \pi : R \to C, h : R \to X, \sigma)\). This is true because \(H^1(R, N_{R/X}(-\sigma - R_q)) = 0\) as \((R, \sigma)\) was assumed very twisting. \(\Box\)

12. Main theorem

Here is the main theorem of this paper.

**Theorem 12.1.** In Situation \[7.1\] assume Hypothesis \[6.8\] holds over an open part \(S\) of \(C\). Suppose a geometric fibre of \(X \to C\) has a very twisting scroll. In this case there exists a sequence \((Z_e)_{e \geq 0}\) of irreducible components \(Z_e\) of \(\Sigma^e(X/C/\kappa)\) as in Lemma \[7.4\] such that

1. a general point of \(Z_e\) parametrizes a free section of \(f\),
2. each Abel map 
\[\alpha_\xi|_{Z_e} : Z_e \to \text{Pic}^\xi C/\kappa\]
has nonempty rationally connected geometric generic fibre, and
3. for every free section \(s\) of \(f\) there exists an \(E = E(s) > 0\) such that for all \(e \geq E\), \(Z_e\) is the unique irreducible component of \(\Sigma^e(X/C/\kappa)\) containing every comb whose handle is \(s\) and whose teeth are free lines in fibres of \(f\).
Proof. We will show the sequence of irreducible components we found in Proposition \ref{1112} satisfies the conclusions of the theorem. By Lemma \ref{7.9} the irreducible components satisfy the first condition. As a second step we remark that \{Z_e\} satisfies the third part by Corollary \ref{8.8}. Also, the geometric generic fibres of α_L|Z_e are irreducible by Lemma \ref{7.9} (for large enough e).

So what is left is to show that these fibres are rationally connected. Since they are proper it is the same as showing they are birationally rationally connected. We will prove this for a general fibre rather than the geometric generic fibre, which is anyway the same thing since our field is uncountable. To show that a variety W/k is birationally rationally connected it suffices to show that for a general pair of points (p, q) ∈ W × W there exist open subsets V_i ⊂ P^1, morphisms f_i : V_i → W, for i = 1, . . . , N such that (1) each V_i nonempty open, (2) p ∈ f_1(V_1), (3) q ∈ f_N(V_N) and (4) f(V_i) ∩ f_{i+1}(V_{i+1}) contains a smooth point of W for i = 1, . . . , N − 1. See [Kol96 Theorem IV.3.10.3]. This is the criterion we will use. During the proof of the theorem we will call such a chain of rational curves a “useful chain”. We will show that there are sufficiently many rational curves available to do the job. (More technically precise and specific would be to say that we will use this criterion for the coarse moduli scheme for the stack Z_e.)

A first point is to note that since the target of α_L is an abelian variety every rational curve in Z_e is automatically contracted to a point under α_L. Hence we do not have to worry about our curves “leaving” the fibre.

Second, what does Corollary \ref{1113} say? Let E be the implied (large) constant of Corollary \ref{1113}. It says that for all e − 1 ≥ E there is a nonempty open U_{e−1,1} ⊂ Z_e of the space of porcupines of total degree 1 and one quill which are pinned by a very twisting surface. By Lemma \ref{1115} all points of U_{e−1,1} are connected by a rational curve to a point in the interior of Z_e. Now since the points of U_{e−1,1} are unobstructed, we conclude: (A) a general point of Z_e is connected by a rational point to a general point of U_{e−1,1}.

Thirdly, we repeat the previous argument several times, as follows. Consider the forgetful map U_{e−1,1} → Z_{e−1,1} × C which omits the quill but remembers the attachment point. It has rationally connected smooth projective fibres at least over a suitable open of Z_{e−1,1} × C, by Hypothesis \ref{8.8} which holds over S ⊂ C. By (A), if e − 2 ≥ E, then we can find a rational curve connecting a general point of Z_{e−1} to a general point of U_{e−2,1}, and by the previous remark we can find a rational curve in U_{e−1,1} connecting a general point of U_{e−1,1} to a point of Z_e which corresponds to a general porcupine with 2 quills. In other words, if e − 2 ≥ E, we can find a useful chain of rational curves connecting a general point of Z_e to a general porcupine with two quills.

Continuing in this manner, we see that if e − δ ≥ E, then we can connect a general point of Z_e by a useful chain of rational curves to a general porcupine with δ quills. It is mathematically more correct to say that we proved that useful chains emanating from points corresponding to porcupines whose body is in Z_{e−δ} and with δ quills sweep out an open in Z_e.

Fix e_0 an integer such that (a) e_0 > E, and (b) the general point of Z_{e_0} corresponds to a 2g(C) + 1-free section, see Lemma \ref{7.8}. Let P ⊂ Z_{e_0} be the set of points corresponding to porcupines without quills. Recall (see just above Proposition \ref{5.7})
that $P_{e-e_0}$ indicates porcupines with $e-e_0$ quills and body in $P$, in other words porcupines with body in $Z_{e_0}$ and $e-e_0$ quills. We apply Proposition 8.7 to the pair $(P, P' = P)$; let $E_1$ be the constant found in that proposition. The proposition says exactly that a general point of

$$P_{e-e_0} \times_{\alpha, \Pic^e_{\alpha \cdot, \alpha, P_{e-e_0}}}$$

can be connected by useful chains in $Z_e$. Combined with the assertion above that a general point of $Z_e$ may be connected by useful chains to a point of $P_{e-e_0}$ we win.

The impetus of a lot of the research in this paper comes from the following application of the main theorem.

**Corollary 12.2.** Let $k$ be an algebraically closed field of characteristic 0. Let $S$ be a smooth, irreducible, projective surface over $k$. Let

$$f : X \to S$$

be a flat proper morphism. Assume there exists a Zariski open subset $U$ of $S$ whose complement has codimension 2 such that the total space $f^{-1}(U)$ is smooth and such that $f : f^{-1}(U) \to U$ has geometrically irreducible fibres. Let $\mathcal{L}$ be an $f$-ample invertible sheaf on $f^{-1}(U)$. Assume in addition that the geometric generic fibre $X_{\overline{\eta}} \to \overline{\eta}$ satisfies Hypothesis [6.8] and has a very twisting surface. Then there exists a rational section of $f$.

**Proof.** Right away we may assume that $k$ is an uncountable algebraically closed field. Also, if $X_{\overline{\eta}} \to \overline{\eta}$ satisfies [6.8] and has a very twisting surface, then for some nonempty open $V \subset U$ the morphism $X_v \to V$ satisfies [6.8] and the fibres contain very twisting surfaces. In addition we may blow up $S$ (at points of $U$) and assume we have a morphism

$$b : S \to \mathbb{P}^1$$

whose general fibre is a smooth irreducible projective curve. Let $W \subset \mathbb{P}^1$ be a nonempty Zariski open such that all fibres of $b$ over $W$ are smooth, and meet the open $V$. Consider the space and moduli map

$$\Sigma^e(X/W/S/W) \to \Pic^e_{S/W}$$

introduced in Definition 5.1 and Lemma 5.6.

Let $\xi \in W$ be the generic point so its residue field $k(\xi)$ is equal to the function field $k(W)$ of $W$. Let $\overline{\xi} = \text{Spec}(k(W)) \to W$ be a geometric point over $\xi$ corresponding to an algebraic closure of $k(W)$. Note that by our choice of $W$ and the assumption of the corollary the hypotheses of the main theorem (12.1) are satisfied for the morphisms $X_w \to S_w$ for every geometric point $w$ of $W$. Let us apply this to the geometric generic point $\overline{\xi}$ of $W$. This gives a sequence of irreducible components $Z_e \subset \Sigma^e(X/W/S/W)$ for all $e \geq e_0$. Note that $Z_{e_0}$ can be defined over a finite Galois extension $\kappa(\xi) \subset L$. By construction of the sequence $\{Z_e\}_{e \geq e_0}$, we see Lemma 7.4 we see that each $Z_e$ is defined over $L$. Next, suppose that $\sigma \in \text{Gal}(L/\kappa(\xi))$. Then we similarly have a sequence $\{Z_e^\sigma\}_{e \geq e_0}$ deduced from the sequence by applying $\sigma$ to the coefficients of the equations of the original $Z_e$. On the other hand, by Corollary 5.8 we have $Z_e = Z_e^\sigma$ for all $e \gg 0$ (this also follows directly from the third
assertion of the main theorem if you think about it right). Since \( \text{Gal}(L/\kappa(\xi)) \) is finite, we conclude that for \( e \gg 0 \) the irreducible components \( Z_e \subset \Sigma^e(X/W/S/W) \) are the geometric fibres of irreducible components \( Z_{W,e} \subset \Sigma^e(X/W/S/W) \).

The conclusion of the above is that we know that for all \( e \gg 0 \) there exist irreducible components \( Z_{W,e} \subset \Sigma^e(X/W/S/W) \) such that the induced morphisms

\[
Z_{W,e} \to \text{Pic}^e_{S/W}
\]

have birationally rationally connected nonempty geometric generic fibre. But note that since \( \Sigma^e(X/W/S/W) \) is actually proper over \( W \), so is \( Z_{W,e} \) and hence its fibres over \( \text{Pic}^e_{S/W} \) are actually rationally connected (more precisely the underlying coarse moduli spaces of the fibres are rationally connected).

At this point we are done by the lemmas below. Namely, Lemma 12.3 says it is enough to construct a rational section of \( Z_{W,e} \to W \), and even such a rational section to its coarse moduli space \( Z^\text{coarse}_{W,e} \). Since the original surface \( S \) was projective, we can find for arbitrarily large \( e \) divisors \( D \) on \( S \) which give rise to sections of \( \text{Pic}^e_{S/W} \to W \). On the other hand, since \( \text{Pic}^e_{S/W} \) is smooth, Lemma 12.1 guarantees, because of the result on the geometric generic fibre above, that we can lift this up to a section of \( Z^\text{coarse}_{W,e} \to W \). Thus the proof of the corollary is finished. \( \square \)

**Lemma 12.3.** Let \( k \) be an algebraically closed field of characteristic zero. Let \( S \) be a variety over \( k \). Let \( C \to S \) be a family of smooth projective curves. Let \( X \to C \) be a proper flat morphism. Let \( \mathcal{L} \) be an invertible sheaf on \( X \) ample w.r.t. \( X \to C \). If \( \Sigma^e(X/C/S) \to S \) (see Definition 5.1) has a rational section for some \( e \), then \( X \to C \) has a rational section. The same conclusion holds if we replace \( \Sigma^e(X/C/S) \to S \) by its coarse moduli space.

**Proof.** After shrinking \( S \) we have to show that if there is a section, then \( X \to C \) has a rational section. Let \( \tau : S \to \Sigma(X/C/S) \) be a section. This corresponds to a proper flat family of curves \( C \to S \), and a morphism \( C \to X \) such that \( C \to C \) is a prestable map of degree 1. See Section 5 and the initial discussion in that section. In particular there is a nonempty Zariski open \( U \subset C \) such that \( U \to C \to X \to C \) is an open immersion. (For example, \( U \) is the open part where the morphism \( C \to C \) is flat, or it is the part you get by excising the vertical pieces of the stable maps.) This proves the first assertion of the lemma.

To prove the final assertion, after shrinking \( S \) as before, let \( t : S \to \Sigma(X/C/S)^\text{coarse} \) be a map to the coarse moduli space. After shrinking \( S \) some more we may assume there is a surjective finite etale Galois morphism \( \nu : S' \to S \) such that \( t \circ \nu \) corresponds to a morphism \( \tau' : S' \to \Sigma(X/C/S)'^\text{coarse} = \Sigma(X_{S'}/C_{S'}/S') \). By the arguments in the first part of the proof this corresponds to a section of \( X_{S'} \to C_{S'} \). To see that it descends to a rational section of \( X \to C \) it is enough to see that it is Galois invariant. This is clear because the only stackiness in the spaces of stable maps comes from the contracted components. \( \square \)

**Lemma 12.4.** Let \( \alpha : Z \to P \) be a proper morphism of varieties over an algebraically closed field \( k \) of characteristic zero. Assume the geometric generic fibre is rationally connected, and \( P \) is quasi-projective. Let \( C \subset P \) be any curve meeting the nonsingular locus of \( P \). Then there exists a rational section of \( \alpha^{-1}(C) \to C \).
Proof. (Sketch only, see also similar arguments in [GHMS05].) Curve means irreducible and reduced closed subscheme of dimension 1. Thus $C$ is generically nonsingular and meets the nonsingular locus of $P$. This means $C$ is locally at some point a complete intersection in $P$. This means that globally $C$ is an irreducible component of multiplicity 1 of a global complete intersection curve on $P$ (here we use that $P$ is quasi-projective). This means that there exists a 1-parameter flat family of complete intersections $C \subset P \times T$ parametrized by a smooth irreducible curve $T/k$ such that $C$ is an irreducible component of multiplicity 1 of a fibre $C_t$ for some $t \in T(k)$ and such that the general fibre is a smooth irreducible curve passing through a general point of $P$. In other words, by [GHS03] and the assumption of the lemma, we can find a lift of $C_{k(T)} \to P$ to a morphism $\gamma : C_{k(T)} \to X$ into $X$.

Note that if $T' \to T$ is a nonconstant morphism of nonsingular curves over $k$ and $t' \in T'(k)$ maps to $t \in T(k)$ then the pullback family $C' = C \times_T T'$ still has $C$ as a component of multiplicity one of the fibre over $t'$. Using this, we may assume that $\gamma$ is defined over $k(T)$. In other words we obtain a rational map $C \supset W \gamma^{-1} X$ lifting the morphism $C \to P$. Since $T$ is nonsingular, and since $C$ is a component of multiplicity one of the fibre of $C \to T$ we see that the generic point of $C$ corresponds to a codimension 1 point $\xi$ of $C$ whose local ring is a DVR. Because $X \to P$ is proper we see, by the valuative criterion of properness that we may assume that $\gamma$ is defined at $\xi$. This gives the desired rational map from $C$ into $X$. □

13. Special rational curves on flag varieties

In this section we show that flag varieties contain many good rational curves. In characteristic $p > 0$ there exist closed subgroup schemes $H$ of reductive groups $G$ such that $G/H$ is projective, but $H$ not reduced. To avoid dealing with these, we say $P \subset G$ is a standard parabolic subgroup if $P$ is geometrically reduced and $G/P$ is projective.

**Theorem 13.1.** Let $k$ be an algebraically closed field. Let $G$ be a connected, reductive algebraic group over $k$. Let $P$ be a standard parabolic subgroup of $G$. There exists a $k$-morphism $f : P^1 \to G/P$ such that for every standard parabolic subgroup $Q$ of $G$ containing $P$, denoting the projection by

$$\pi : G/P \to G/Q,$$

we have that the pull back of the relative tangent bundle $f^* T_\pi$ is ample on $P^1$.

**Remark 13.2.** Note that for any fixed $Q$, since the fibres of $\pi$ are rationally connected there exists a morphism $f : P^1 \to G/P$ such that $f^* T_\pi$ is ample. The importance of Theorem 13.1 is that $f$ has this property for all parabolics $Q$ simultaneously.

The purpose of this section is to prove Theorem 13.1. We use some results in the theory of reductive linear algebraic groups. It turns out that the proof in the case $X = G/B$, is somewhat easier. Thus we first reduce the general case to this case by a simple geometric argument.

**Lemma 13.3.** It suffices to prove Theorem 13.1 when $P$ is a Borel subgroup.

**Proof.** If $P$ is not a Borel, then pick any $B \subset P$ Borel contained in $P$. Let $g : P^1 \to G/B$ be a morphism as in Theorem 13.1. We claim that the composition $f$
of $g$ with the projection map $G/B \to G/P$ works. Namely, if $P \subset Q$ is another parabolic, then we have a commutative diagram

$$
\begin{array}{ccc}
G/B & \longrightarrow & G/P \\
\downarrow \tau & & \downarrow \pi \\
G/Q & \longrightarrow & G/Q
\end{array}
$$

which proves that $f^*T_\tau$ is a quotient of $g^*T_\tau$, and hence we win. \hfill \Box

We review some of the notation of [MR04, Section 2]. Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$. Choose a maximal torus $T \subset G$ and let $T \subset B$ be a Borel subgroup of $G$ containing $T$. Let $B^-$ be the opposite Borel subgroup, so that $T = B \cap B^-$. Let $U$ (resp. $U^-$) be the unipotent radical of $B$ (resp. $B^-$). Let $X = X^*(T) = \text{Hom}(T, G_m)$ and $Y = X_*(T) = \text{Hom}(G_m, T)$ be the character and cocharacter groups of $T$. Let $(,) : X \times Y \to \mathbb{Z}$ be the pairing defined by $x(y(t)) = t(x,y)$ for $t \in G_m$. The choice of $T \subset B \subset G$ determines the set of roots $\Phi \subset X$, the set of positive roots $\Phi^+$, as well as its complement $\Phi^-$. For $\alpha \in \Phi$, let $U_\alpha$ be the root subgroup corresponding to $\alpha$. Our normalization is that $\alpha \in \Phi^+ \iff U_\alpha \subset B$. We denote the simple roots $\Delta = \{\alpha_i \mid i \in I\}$. So here $I$ is implicitly defined as the index set for the simple roots. For any $J \subset I$, let $\Phi_J$ be the set of roots spanned by simple roots in $J$ and $\Phi_J^+ = \Phi_J \cap \Phi^+$. Let $P_J$ be the standard parabolic subgroup corresponding to $J$ which is characterized by $U_\alpha \subset P_J$ (resp. $\alpha \in \Phi^+$). Let $X_J = G/P_J$ be the corresponding flag variety (there will be no subscripts to $X = X^*(T)$ and so we will avoid any possible confusion). Note that $X_\emptyset = G/B$ classifies Borel subgroups of $G$ and is sometimes written as $B$.

For each $i \in I$ there is a morphism of algebraic groups

$$
h_i : SL_2, k \to G
$$

which maps the diagonal torus into $T$, such that $a \mapsto h_i \left( \begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix} \right)$ is an isomorphism $x_i : G_a \to U_{a_i}$, and such that $y_i : a \mapsto h_i \left( \begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix} \right)$ is an isomorphism $y_i : G_a \to U_{-a_i}$. It follows that $\alpha_i(h_i \left( \begin{smallmatrix} 0 & t \\ 1 & -1 \end{smallmatrix} \right))$ is equal to $t^2$. The cocharacter $G_m \to T$, $t \mapsto h_i \left( \begin{smallmatrix} 0 & t \\ t & 1 \end{smallmatrix} \right)$ is the coroot $\alpha_i^\vee$ dual to the simple root $\alpha_i$. The datum $(B, T, x_i, y_i; i \in I)$ is called an épingle or a pinning of $G$. For $i \in I$, let $s_i$ be the corresponding simple reflection in the Weyl group $W = N_G(T)/T$. The element $s_i = h_i \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right) \in N_G(T)$ is a representative of $s_i$. Then

$$
y_i(1) = x_i(1)s_i x_i(1). \quad (a)
$$

For $w \in W$, we denote by $l(w)$ its length and we set $\dot{w} = s_{i_1}s_{i_2} \cdots s_{i_n}$, where $s_{i_1}s_{i_2} \cdots s_{i_n}$ is a reduced expression of $w$. This expression is independent of the choice of the reduced expression for $w$. For $J \subset I$, we denote by $W_J$ the standard parabolic subgroup of $W$ generated by $s_j$, $j \in J$. We denote by $W^J$ (resp. $JW$) the set of minimal coset representatives in $W/W_J$ (resp. $W_J \setminus W$). We simply write $W^J \cap KW$ as $KW^J$ for $J, K \subset I$.

For any subgroup $H$ of $G$, we denote by Lie($H$) its Lie algebra. (We will think of this as the spectrum of the symmetric algebra on the cotangent space of $G$ at the identity.) Let us recall the following result (see [St90, page 26, lemma 4]).
Lemma 13.4. Let $H$ be a closed subgroup scheme of $G$. Let $X$ be a scheme of finite type over $k$ endowed with a left $G$-action. Let $f : X \to G/H$ be a $G$-equivariant morphism from $X$ to the homogeneous space $G/H$. Let $E \subset X$ be the scheme theoretic fibre $f^{-1}(H/H)$. Then $E$ inherits a left $H$-action and the map $G \times_H E \to X$ sending $(g,e)$ to $g \cdot e$ defines a $G$-equivariant isomorphism of schemes.

Let $J \subset I$ and $e = P_J/P_J \in X_J = G/P_J$. We are going to apply the lemma to the tangent bundle $T_{X_J}$ of $X_J$ and the $G$-equivariant map $f : T_{X_J} \to X_J$. Note that $f^{-1}(e) = T_{X_J,e} = \text{Lie}(G)/\text{Lie}(P_J)$ because $1 \in G$ lifts $e \in X_J$. For an element $p \in P_J$ the left multiplication $p : X_J \to X_J$ lifts to $\text{im}_p : G \to G$, $g \mapsto pgp^{-1}$ because $pgp^{-1}P_J = pgP_J$. Since $\text{im}_p$ fixes $1 \in G$ and acts by the adjoint action on $\text{Lie}(G)$ it follows that the $P_J$-action on $f^{-1}(e) = \text{Lie}(G)/\text{Lie}(P_J)$ is the adjoint action of $P_J$ on $\text{Lie}(G)/\text{Lie}(P_J)$. We define an action of $P_J$ on $G \times \text{Lie}(G)/\text{Lie}(P_J)$ by $p \cdot (g,k) = (gp^{-1}, \text{Ad}(p) \cdot k)$. Let $G \times P_J \text{Lie}(G)/\text{Lie}(P_J)$ be the quotient space. Then by the lemma, we have

$$T_{X_J} \cong G \times P_J \text{Lie}(G)/\text{Lie}(P_J).$$

In particular, when $J = \emptyset$ we get that the tangent bundle of $X_\emptyset = G/B$ is identified with $G \times_B \text{Lie}(G)/\text{Lie}(B)$.

What about the relative tangent bundles? Consider a subset $J \subset I$ as before. Then $B \subset P_J$ and we obtain a canonical morphism $p_J : X_\emptyset \to X_J$. Note that $\text{Lie}(P_J)/\text{Lie}(B)$ is a $B$-stable subspace of $\text{Lie}(G)/\text{Lie}(B)$. Again by the above lemma, we see that the vertical tangent bundle $T_{p_J} \subset T_{X_\emptyset}$ of the projection map $p_J$ is isomorphic to

$$G \times_B \text{Lie}(P_J)/\text{Lie}(B) \subset G \times_B \text{Lie}(G)/\text{Lie}(B).$$

Consider the adjoint group $\text{ad}(G)$ of $G$ together with the image $\text{ad}(T)$ of $T$. The character group $X_\alpha(\text{ad}(T))$ is the free $\mathbb{Z}$-module $\oplus_{i \in I} \mathbb{Z} \alpha_i \subset X = X_\alpha(T)$. Let $\rho^\alpha \in X_\alpha(\text{ad}(T))$ be the cocharacter such that $(\alpha_i, \rho^\alpha) = 1$ for all $i \in I$. In other words,

$$\rho^\alpha : G_m \to \text{ad}(G)$$

is a morphism into $\text{ad}(T)$ such that $\rho^\alpha(t)$ acts via multiplication by $t$ on $U_{\alpha_i}$ for all $i \in I$. From this we conclude that, for any $u \in U$ we have

$$\lim_{t \to 0} \text{im}_{\rho^\alpha(t)}(u) = 1$$

and, for any $u^{-} \in U^{-}$ we have

$$\lim_{t \to -\infty} \text{im}_{\rho^\alpha(t)}(u^{-}) = 1$$

Here, by abuse of notation, for an element $\tilde{g} \in \text{ad}(G)$ we denote $\text{im}_{\tilde{g}}$ the associated inner automorphism of $G$; in other words, $\text{im}_{\tilde{g}} = \text{im}_g$ for any choice of $g \in G$ mapping to $\tilde{g} \in \text{ad}(G)$. Since each $P_J$ contains $T$ and hence the center of $G$ we see that $G_m$ acts on $X_J = G/P_J$ via $\rho^\alpha$. This action induces a $G_m$-action on the bundles $T_{G/B}$ and $T_{p_J}$. Under the isomorphism $T_{p_J} = G \times_B \text{Lie}(P_J)/\text{Lie}(B)$ this action is given by $t \cdot (g,k) = (\text{im}_{\rho^\alpha(t)}(g), \text{Ad}_{\rho^\alpha(t)}(k))$. We leave the verification to the reader.

Let $w$ be the maximal element in $W$ and let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression of $w$. Set $u = x_{i_1}(1) s_{i_2} x_{i_3}(1) s_{i_4} \cdots x_{i_r}(1) s_{i_{r-1}}$. By [MR04, Proposition 5.2], $u \cdot B/B \in (Bw \cdot B/B) \cap (B^- \cdot B/B)$. In particular, $u \cdot B/B = u_1 \cdot B/B = u_2 \cdot B/B$ for some $u_1 \in U$ and $u_2 \in U^-$. This is the only property of $u$ that we will use.
Another way to find an element with this property is as follows: Since \( w \) is the longest element we have \( \dot{w}B\dot{w}^{-1} = B^- \). Thus finding a \( u \) as above amounts to showing that \( B\dot{w}B \cap B^- B \) is nonempty, which is clear since both are open in \( G \) (because \( BwB = wB^- B \)).

Let \( C \) be the closure of \( \{ \text{inn}_{\rho^\vee(t)}(u) : B/B ; t \in k^* \} \) in \( X_\emptyset \). In other words, \( C \) is the closure of the \( G_m \)-orbit of the point \( u_\emptyset := u \cdot B/B \) of \( X_\emptyset \) with \( G_m \) acting via \( \rho^\vee \).

Then
\[
\lim_{t \to \infty} t \cdot u_\emptyset = \lim_{t \to \infty} \text{inn}_{\rho^\vee(t)}(u_2) \cdot B/B = B/B,
\]
\[
\lim_{t \to 0} t \cdot u_\emptyset = \lim_{t \to 0} \text{inn}_{\rho^\vee(t)}(u_1) \dot{w} \cdot B/B = \dot{w} \cdot B/B.
\]

Then \( C = \{ \text{inn}_{\rho^\vee(t)}(u_\emptyset) ; t \in k^* \} \sqcup [0] \sqcup [\infty] \), where \([0] = \dot{w} \cdot P_J/P_J \) and \([\infty] = P_J/P_J \).

Let
\[ h : P^1 \to C \subset X_J \]
be the \( G_m \)-equivariant morphism with \( h(t) = t \cdot u, h(0) = [0] \), and \( h(\infty) = [\infty] \).

(Warning: \( h \) needs not be birational.)

With this notation \( h^*T_{pJ} \) is a \( G_m \)-equivariant vector bundle over \( P^1 \). We have that \( h^*T_{pJ} = \bigoplus L_l \) for some \( G_m \)-equivariant line bundles \( L_l \) over \( P^1 \). We would like to compute the degrees of the line bundles \( L_l \). For any \( G_m \)-equivariant line bundle \( L \) on \( P^1 \) we define two integers \( n(L, 0) \), and \( n(L, \infty) \) as the weight of the \( G_m \)-action on the fibre at 0, respectively \( \infty \). We leave it to the reader to prove the formula
\[ \deg L = n(L, 0) - n(L, \infty), \]
see also Remark 13.5 below. In order to compute the degrees of the \( L_l \) we compute the integers \( n(L_l, 0) \) and \( n(L_l, \infty) \). And for this in turn we compute the fibre of \( T_{pJ} \) at the points \([0] \) and \([\infty] \) as \( G_m \)-representations. In fact both points are \( T \)-invariant and hence we can think of \( T_{pJ} | [0] \) and \( T_{pJ} | [\infty] \) as representations of \( T \).

The fibre of \( T_{pJ} \) at the point \([\infty] = B/B \) is just the vector space \( \text{Lie}(P_J)/\text{Lie}(B) \) with \( T \) acting via the adjoint action. We have \( \text{Lie}(P_J) = \text{Lie}(B) \oplus \bigoplus_{\alpha \in \Phi_J \setminus \Phi^-_J} \text{Lie}(U_\alpha) \). Thus
\[ \text{Lie}(P_J)/\text{Lie}(B) \cong \bigoplus_{\alpha \in \Phi_J \setminus \Phi^-_J} \text{Lie}(U_\alpha). \]

How does \( G_m \) act on this? It acts via the coroot \( \rho^\vee \). For each \( \alpha \in \Phi^-_J \) we have \( (\alpha, \rho^\vee) < 0 \), hence all the weights are negative, in other words all the \( n(L_l, \infty) \) are negative.

The fibre of \( T_{pJ} \) at the point \([0] = \dot{w} \cdot B/B \) is the space \( \{ \dot{w} \} \times \text{Lie}(P_J)/\text{Lie}(B) \) inside \( G \times_B \text{Lie}(P_J)/\text{Lie}(B) \). For an element \( t \in T \) and an element \( k \in \text{Lie}(P_J)/\text{Lie}(B) \) we have (see above)
\[
t \cdot (\dot{w}, k) = (\text{inn}_t(\dot{w}), Ad_t(k))
= (t\dot{w}t^{-1}, Ad_t(k))
= (\dot{w}t^\alpha t^{-1}, Ad_t(k))
= (\dot{w}, Ad_{t^\alpha}(Ad_t(k)))
= (\dot{w}, Ad_{t^\alpha}(k)).
\]
Thus the weights for $T$ on $T_{p,J}|_{0^0}$ are the $w(\alpha)$, where $\alpha \in \Phi^J$. How does $G_m$ act on this? It acts via the coroot $\rho'$. For each $\alpha \in \Phi^J$ we have $(w(\alpha), \rho') > 0$, because the longest element $w$ exchanges positive and negative roots. This follows directly from $\dot{w}B\dot{w}^{-1} = B^-$. Hence all the weights are positive, in other words all the $n(L_l,0)$ are positive.

From this we conclude that all the line bundles $L_l$ have positive degree. In other words we have show that $h^*T_{p,J}$ is a positive vector bundle for all $J \subset I$. This proves Theorem 13.1.

**Remark 13.5.** The sign of the formula for the degree of an equivariant line bundle on $\mathbb{P}^1$ may appear to be wrong due to the fact that we are working with line bundles and not invertible sheaves (which is dual). The actual sign of the formula for an equivariant line bundle on $\mathbb{P}^1$ does not matter for the argument however. Namely, in either case the rest of the arguments show that all the line bundles $L_l$ have the same sign. And since we know that $G/B$ is Fano we know that the sum of all the degrees in of the line bundles in the pull back of $T_{G/B}$ has to be positive.

### 14. Rational simple connectedness of homogeneous spaces

Let $k$ be an algebraically closed field. Let $X$ and $Y$ be smooth, connected, quasi-projective $k$-schemes. Let $L$ be an ample invertible sheaf on $X$. Let $u : Y \to \mathcal{M}_{0,0}(X,1)$ be a morphism. This corresponds to the $X,C,Y,p,q$ in the following diagram of $k$-schemes

$$
\begin{array}{ccc}
\mathbb{P}^1 & \to & C \\
\sigma & \downarrow & p \\
\tau & \searrow & q \\
& & Y
\end{array}
$$

Automatically $q$ is a smooth, projective morphism whose geometric fibres are isomorphic to $\mathbb{P}^1$ which map to lines in $X$ via $p$. There is an associated 1-morphism $v : C \to \mathcal{M}_{0,1}(X,1)$.

**Proposition 14.1.** In the situation above with $\sigma : \mathbb{P}^1 \to C$ as in the displayed diagram. Assume that

1. all lines parametrized by $Y$ are free,
2. $p$ is smooth,
3. $u$ is unramified,
4. either $\tau$ is a free rational curve on $Y$, or $p \circ \sigma : \mathbb{P}^1 \to X$ is free, and
5. $\sigma^*T_p$ and $\sigma^*T_q$ are ample.

Then $X$ has a very twisting scroll.

**Proof.** We will show the pair $(R,s)$ where the scroll $R$ is defined as $R = \mathbb{P}^1 \times_{q_0\sigma,Y,q} C$ and with section $s : \mathbb{P}^1 \to R$ given by $\sigma$ satisfies all the properties of Definition 11.7. Product will mean product over Spec($k$).

We will use the following remarks frequently below. Denote the projection to $\mathbb{P}^1$ by $q_R : R \to \mathbb{P}^1$ and the projection to $C$ by $\sigma_R : R \to C$. For every coherent sheaf $\mathcal{F}$ on $C$, cohomology and base change implies the natural map

$$
\tau^*R^1q_*\mathcal{F} \to R^1q_{R,*}(\sigma^*_R\mathcal{F})
$$
is an isomorphism. If the sheaves $R^1 q_R(\mathcal{F})$ are zero (along the image of $\tau$) then

$$\tau^* q_* \mathcal{F} \rightarrow q_{R,*}(\sigma^*_R \mathcal{F})$$

is an isomorphism.

Note that $N_{R/\mathbb{P}^1 \times X}$ equals the pullback of $N_{\mathcal{C}/\mathbb{Y} \times X}$. Because $T_{X \times Y} = \text{pr}_1^* T_X \oplus \text{pr}_2^* T_Y$, and because $q$ is smooth there is a short exact sequence of locally free sheaves $0 \rightarrow T_p \rightarrow q^* T_Y \rightarrow N_{\mathcal{C}/\mathbb{Y} \times X} \rightarrow 0$ on $\mathcal{C}$. There is an associated long exact sequence

$$0 \rightarrow q_* T_p \rightarrow q_* q^* T_Y \rightarrow q_* N_{\mathcal{C}/\mathbb{Y} \times X} \rightarrow R^1 q_* T_p \rightarrow 0$$

using the fact that $R^1 q_* q^* T_Y$ is zero. We can say more: (a) $q_* q^* T_Y$ is canonically isomorphic to $T_Y$, (b) $q_* N_{\mathcal{C}/\mathbb{Y} \times X}$ is canonically isomorphic to $u^* T_{\mathcal{C}, 0, (X, 1)}$ (see Remark [112]), and (c) the sheaf homomorphism $\alpha$ above is canonically isomorphic to $d u$. By hypothesis $u$ is unramified, in other words $\alpha$ is injective. Thus $q_* T_p = 0$, and we get an isomorphism $R^1 q_* T_p = N_{\mathcal{C}/\mathcal{C}, 0, (X, 1)}$. We conclude that $R^1 q_* T_p$ is locally free, and formation of $R^1 q_* T_p$ and $q_* T_p$ commutes with arbitrary change of base. By relative duality, we get that $R^1 q_* T_p$ is dual to $q_*(\Omega_p \otimes_{\mathcal{C}} Q_\alpha)$, and that $R^1 q_*(\Omega_p \otimes_{\mathcal{C}} Q_\alpha) = 0$.

**Claim 14.2.** The sheaf $\tau^* q_* (\Omega_p \otimes_{\mathcal{C}} Q_\alpha)$ is anti-ample, and thus $\tau^* N_{\mathcal{C}/\mathcal{C}, 0, (X, 1)}$ is ample.

The surface $R$ is a Hirzebruch surface. Denote by $\mathcal{E}$ the pullback of $\Omega_p \otimes_{\mathcal{C}} Q_\alpha$ to $R$. By our remarks above the claim is equivalent to the assertion that $(q_R)_*, \mathcal{E}$ is anti-ample. The normal bundle $N_{s(\mathbb{P}^1)_*/R}$ is canonically isomorphic to $\sigma^* T_q$, which is ample by hypothesis. Therefore the divisor $s(\mathbb{P}^1)$ on $R$ moves in a basepoint free linear system. This implies the section $s$ deforms to a family $\{s_t\}_{t \in \Pi}$ of sections whose images cover a dense open subset of $R$. Of course the pullback of $\Omega_p \otimes_{\mathcal{C}} Q_\alpha$ to $R$ is a locally free sheaf $\mathcal{E}$ whose dual $\mathcal{E}^\vee$ is canonically isomorphic to the pullback of $T_p \otimes_{\mathcal{C}} T_q$. And $s^* \mathcal{E}^\vee$ equals $\sigma^* T_p \otimes_{\mathcal{C}} \sigma^* T_q$. By hypothesis, each of $\sigma^* T_p$ and $\sigma^* T_q$ is ample on $\mathbb{P}^1$. Thus $s^* \mathcal{E}^\vee$ is ample on $\mathbb{P}^1$. Since ampleness is an open condition, after shrinking $\Pi$ we may assume $s_t^* \mathcal{E}^\vee$ is ample for $t \in \Pi$. Therefore, $s_t^* \mathcal{E}$ is anti-ample.

For every $t$ there is an evaluation morphism

$$e_t : (q_R)_* \mathcal{E} \rightarrow s_t^* \mathcal{E}.$$ 

Of course, since the curves $s_t(\mathbb{P}^1)$ cover a dense open subset of $R$, the only local section of $(q_R)_* \mathcal{E}$ in the kernel of every evaluation morphism $e_t$ is the zero section. Since $(q_R)_* \mathcal{E}$ is a coherent sheaf, in fact for $N \gg 0$ and for $t_1, \ldots, t_N$ a general collection of closed points of $\Pi$, the morphism

$$(e_{t_1}, \ldots, e_{t_N}) : (q_R)_* \mathcal{E} \rightarrow \bigoplus_{i=1}^N s_{t_i}^* \mathcal{E}$$

is injective. Since every summand $s_{t_i}^* \mathcal{E}$ is anti-ample, the direct sum is anti-ample. And a locally free sheaf admitting an injective sheaf homomorphism to an anti-ample sheaf is itself anti-ample. Therefore $(q_R)_* \mathcal{E}$ is anti-ample, proving Claim [14.2].
As usual, we denote by \( ev : \overline{M}_{0,1}(X,1) \to X \) the evaluation morphism. Thus \( ev \circ v = p \) as morphisms \( C \to X \). Consider the diagram

\[
\begin{array}{ccc}
\overline{M}_{0,1}(X,1) & \xrightarrow{ev} & X \\
\downarrow q & & \\
\overline{M}_{0,0}(X,1) & \xrightarrow{\text{forget}} & \\
\end{array}
\]

whose square is cartesian. Since we assumed that all lines parametrized by \( Y \) are free it follows that \( \overline{M}_{0,1}(X,1) \) is nonsingular at every point of \( v(C) \). As \( p \) is assumed smooth, it follows that \( ev : \overline{M}_{0,1}(X,1) \to X \) is smooth at all points of \( v(C) \) (by Jacobian criterion for example). Also, the diagram shows \( v \) is unramified and \( N_C/\overline{M}_{0,1}(X,1) \) is canonically isomorphic to \( q^*N_{Y/\overline{M}_{0,0}(X,1)} \). Smoothness of \( ev \) and \( p \) implies we have a short exact sequence

\[
0 \to T_p \to v^*T_{ev} \to N_C/\overline{M}_{0,1}(X,1) \to 0
\]

By hypothesis, \( \sigma^*T_p \) is ample. Claim \([14.2]\) implies \( \tau^*N_{Y/\overline{M}_{0,0}(X,1)} = \sigma^*N_C/\overline{M}_{0,1}(X,1) \) is ample. Therefore \( (v \circ \sigma)^*T_{ev} \) is ample.

At this point we can apply Lemma \([14.0]\) to the surface \( R \to \mathbb{P}^1 \times X \) and the section \( s \), which combine to give the morphism \( g = v \circ \sigma : \mathbb{P}^1 \to \overline{M}_{0,1}(\mathbb{P}^1 \times X/\mathbb{P}^1,1) \). Condition (1) corresponds to the fact that \( s \) moves on \( R \). Condition (2) holds because the lines parametrized by \( Y \) are free. Condition (3) with \( m = 2 \) holds because, as we just saw, the sheaf \( (v \circ \sigma)^*T_{ev} \) is ample. It remains to see that the morphism \( p \circ \sigma : \mathbb{P}^1 \to X \) is free. This follows either by assumption, or if \( \tau \) is free we argue as follows: The pull back \( (p \circ \sigma)^*T_X \) is a quotient of \( \sigma^*T_C \). The sheaf \( \sigma^*T_C \) is sandwiched between \( \sigma^*T_{\overline{q}} \) and \( \sigma^*q^*T_Y \) and by assumption both are globally generated. Hence \( \sigma^*T_C \) and also \( (p \circ \sigma)^*T_X \) is globally generated.

\section*{Remark 14.3.} In the “abstract” situation of Proposition \([14.1]\) we can often deduce that the fibres of \( ev : \overline{M}_{0,1}(X,1) \to X \) are rationally connected using \([Sta04, \text{Proposition } 3.6}\) (the proof of this uses a characteristic 0 hypothesis). We will avoid this below to keep the paper more self contained; instead we will use a little more group theory.

The new result in the following corollary is the statement about very twisting scrolls.

\section*{Corollary 14.4.} Let \( k \) be an algebraically closed field. Let \( G \) be a connected semisimple group over \( k \) and let \( P \subset G \) be a standard maximal parabolic subgroup. Let \( Z = G/P \). Let \( \mathcal{L} \) be an ample generator for \( \text{Pic}(Z) \cong Z \). Then the evaluation morphism

\[
ev : \overline{M}_{0,1}(Z,1) \to Z
\]

is surjective, smooth and projective. In addition, there exists a very twisting scroll in \( Z \).

\section*{Proof.} The action of \( G \) on \( Z \) determines a sheaf homomorphism

\[
T_G \otimes_k O_Z \to T_Z
\]

which is surjective because the action is separable and \( X \) is homogeneous. Thus \( T_Z \) is globally generated. Therefore every smooth, rational curve in \( Z \) is free. This implies

\[
ev : \overline{M}_{0,1}(Z,1) \to Z
\]
is smooth. It is always projective, although at this point the space of lines on $Z$ could be empty.

Let $B$ be a Borel subgroup of $G$ contained in $Z$, and let $T$ be a maximal torus in $B$. The data $(G, B, T)$ determines a root system $(X, \Phi, Y, \Phi^\vee)$ as in Section 13 we will use the notation introduced there. As $P$ is a standard maximal parabolic subgroup, $P$ equals the parabolic subgroup $P_J$ where $J = I - \{j\}$ for an element $j$ of $I$. Thus $Z = G/P_J = X_J$ in the notation of Section 13. It is proved in \cite[§4.20]{Coh95} and \cite[Lemma 3.1]{CC98} that the subvariety

$$L := P_J \cdot P_J / P_J$$

is a line in $G/P_J$ with respect to the ample sheaf $L$.

We quickly explain how to see this in terms of the pinning we chose in Section 13. Namely, after replacing $G$ by its simply connected covering we may assume $\mathcal{L}$ corresponds to a $G$-equivariant invertible sheaf. This in turn corresponds to a character $\chi : P_J \to \mathbb{G}_m$ which generates the character group $X^*(P_J)$. Since $G$ is simply connected the lattice $Y$ is freely generated by the coroots $\alpha_i^\vee$ and hence there is an element $\beta \in X = X^*(T)$ such that $(\beta, \alpha_i^\vee) = 1$ and $(\beta, \alpha_j^\vee) = 0$ for $i \in I, i \neq j$.

This element clearly generates the character group of $P_J$ and hence the restriction of $\chi$ to the maximal torus $T$ corresponds to the element $\beta$ (up to sign). Next, observe that $h_j : SL_2 \to G$ gives a closed immersion $\overline{h}_j : SL_2/B_2 \to G/P_J$, where $B_2 \subset SL_2$ is the group of upper triangular matrices. We claim that $\text{Im}(\overline{h}_j) = L = \mathbb{P}^1$ is the desired line. (Note that $h_j$ maps into $P_J$ and hence we actually get the same line as above.) This is true because the pull back of $L$ to the image corresponds to the standard character $B_2 \to \mathbb{G}_m$ by what we said about $\chi, \beta$ above (as well as the properties of $h_j$).

The action of $G$ on $Z$ induces an action of $G$ on $\mathcal{M}_{0,0}(Z,1)$ and $\mathcal{M}_{0,1}(Z,1)$. Let $p = P_J / P_J \in G/P_J$. This is a point on our line $L$. The stabilizer of $(L, p)$ in $\mathcal{M}_{0,1}(Z,1)$ contains the Borel subgroup $B$, and thus is of the form $P_K$ for a subset $K \subset J = I - \{j\}$. Since $P_K$ is parabolic, the orbit $\mathcal{C}$ of $(L, p)$ is a projective (hence closed) $G$-orbit. The image $Y$ of $\mathcal{C}$ in $\mathcal{M}_{0,0}(Z,1)$ is also a projective $G$-orbit. Observe that $P_J$ acts transitively on the subset $\{(L, q) \mid q \in L\}$ of $\mathcal{M}_{0,1}(Z,1)$ by construction of $L$. Thus the fibre of the forgetful morphism $\overline{\mathcal{M}}_{0,1}(Z,1) \to \mathcal{M}_{0,0}(Z,1)$ over $[L]$ equals the fibre of $\mathcal{C} \to Y$ over $[L]$. By homogeneity it follows that

$$\mathcal{C} = \mathcal{M}_{0,1}(Z,1) \times \mathcal{M}_{0,0}(Z,1) Y.$$ 

So the diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{p} & Z \\
\downarrow q & & \downarrow \\
Y & & \\
\end{array}$$

is a diagram of smooth, projective morphisms where every geometric fibre of $q$ is a smooth, rational curve.

By Theorem 13.1 there exists a morphism,

$$\sigma : \mathbb{P}^1 \to \mathcal{C}$$
such that $\sigma^*T_p$ and $\sigma^*T_q$ are both ample sheaves. The morphism $p$ is smooth by homogeneity. And the morphism $u : Y \to \overline{M}_{0,0}(Z,1)$ is a closed immersion hence unramified. Thus, by Proposition 14.1, there is a very twisting scroll in $Z$. □

Remark 14.5. The variety $G/P$ can be constructed as the orbit of the highest weight vector in $\mathbb{P}(V_{\omega_j})$ where $\omega_j$ is the $j$th fundamental weight (this is what we called $\beta$ in the proof above). Under this embedding $\mathcal{L}$ is the pull back of $\mathcal{O}(1)$ and lines are lines in $\mathbb{P}(V_{\omega_j})$. In particular $\mathcal{L}$ is very ample.

Lemma 14.6. Let $k$ be an algebraically closed field of characteristic 0. Let $G$ be a linear algebraic group over $k$. Let $X, Y$ be proper $G$-varieties over $k$. Let $Y \to X$ be a $G$-equivariant morphism. Suppose that $Y$ is rationally connected and that $X$ has an open orbit whose stabilizer is a connected linear algebraic group. Then the geometric generic fibre of $Y \to X$ is rationally connected.

Proof. Let $x \in X(k)$ be a point in the open orbit. Let $H$ be the stabilizer of $x$ in $G$, which is a connected linear algebraic group by assumption. Let $F = Y_x$ be the fibre of $Y \to X$ over $x$. Consider the morphism $\Psi : G \times F \to Y$, $(g, f) \mapsto g \cdot f$. This is a dominant morphism of varieties over $k$. By construction the fibres of $\Psi$ are isomorphic to the birationally rationally connected variety $H$. It follows from the main result of [GHS03] that $G \times F$ is birationally rationally connected. Hence $F$ is rationally connected. □

Note that the Bruhat decomposition in particular implies that the variety $G/P \times G/P$ has finitely many $G$-orbits. In addition all stabilizers of points of $G/P \times G/P$ are connected, for example by [Bor91, Proposition 14.22].

Lemma 14.7. Let $k$ be an algebraically closed field of characteristic 0. Let $Z = G/P$, $\mathcal{L}$ be as in Corollary 14.4 above. Then every geometric fibre of $\text{ev} : \overline{M}_{0,1}(Z,1) \to Z$ is nonempty and rationally connected.

Proof. Because $Z$ is homogeneous, all fibres are isomorphic and it suffices to prove one fibre is rationally connected. By Lemma 14.6 it suffices to show that $\overline{M}_{0,1}(Z,1)$ is rationally connected. Clearly, it suffices to show that $\overline{M}_{0,0}(Z,1)$ is rationally connected. Since a line is determined by any pair of points it passes through, this follows from the remarks about the Bruhat decomposition immediately preceding the lemma. □

Lemma 14.8. Let $k$ be an algebraically closed field of characteristic 0. Let $Z = G/P$, $\mathcal{L}$ be as in Corollary 14.4 above. There exists a positive integer $n$ and a nonempty open subset $V$ of $Z \times Z$ such that the geometric fibres of the evaluation morphism $\text{ev}_{1,n+1} : \text{FreeChain}_2(X/k, n) \to X \times X$ over $V$ are nonempty, irreducible and birationally rationally connected. (Notation as in Section 6)

Proof. We will use without mention that every line on $Z$ is free.

We rephrase a few arguments of Campana and Kollár-Miyaoka-Mori in this particularly nice setting. Consider the diagram

$$
\overline{M}_{0,0}(Z,1) \xrightarrow{\Phi} \overline{M}_{0,1}(Z,1) \xrightarrow{\text{ev}} Z
$$
For any closed subset $T \subset Z$, the closed set $T' = \text{ev}(\Phi^{-1}(\Phi(\text{ev}^{-1}(T))))$ is the set of points which are connected by a line passing through a point in $T$. Note that if $T$ is irreducible, then so is $T'$ since by Corollary 14.4 the morphism $\text{ev}$ is smooth with irreducible fibres according to Lemma 14.7. For any point $z \in Z(k)$ consider the increasing sequence of closed subvarieties

$$T_0(z) = \{z\} \subset T_1(z) = T_0(z) \cap T_2(z) = T_1(z) \cap \ldots$$

The first is the variety of points that lie on a line passing through $z$. The second is the set of points which lie on a chain of lines of length 2 passing through $z$, etc. Let $n \geq 0$ be the first integer such that the dimension of $T_n(z)$ is maximal. Then obviously $T_{n+1}(z) = T_n(z)$. Hence $T_{2n} = T_n(z)$. We conclude that for every point $z' \in T_n(z)$ we have $T_n(z') = T_n(z)$. Consider the map $Z \to \text{Hilb}_{Z/k}, z \mapsto [T_n(z)]$. By what we just saw the fibres of this map are exactly the varieties $T_n(z)$. Since $\text{Pic}(Z) = Z$ we conclude that either $T_n(z) = \{z\}$, or $T_n(z) = Z$. The first possibility is clearly excluded and hence $T_n(z) = Z$, in other words every pair of points on $Z$ can be connected by a chain of lines of length $n$.

Thus the morphism $\text{ev} : \text{FreeChain}_2(X/k, n) \to X \times_k X$ is surjective. This is a $G$-equivariant morphism for the evident actions of $G$ on the domain and target. There exists an open orbit in $Z \times Z$, and all stabilizers are connected, see the remarks preceding Lemma 14.7. By Lemma 14.6 it suffices to prove that $\text{FreeChain}_2(X/k, n)$ is rationally connected. For $n = 1$ this was shown in the proof of Lemma 14.7. For larger $n$ it follows by induction on $n$ from Lemma 14.7 in exactly the same way as the proof of the corresponding statement of Lemma 6.10.

15. Families of projective homogeneous varieties

In this section $k$ is an algebraically closed field of any characteristic. A projective homogeneous space over $k$ or an algebraically closed extension of $k$ is a variety isomorphic to $G/P$ where $G$ is a connected, semi-simple, simply connected linear algebraic group and $P \subset G$ is a standard parabolic subgroup. (In other words we do not allow exotic, nonreduced, parabolic subgroup schemes.) Let $S$ be a variety over $k$, and let $\pi : S \to \eta$ be a geometric generic point. A split family of homogeneous spaces over $S$ will be a proper smooth morphism $X \to S$ such that $X_\pi$ is a projective homogeneous space and that $\text{Pic}(X) \to \text{Pic}(X_\pi)$ is surjective.

**Lemma 15.1.** Let $k$ be an algebraically closed field of any characteristic. Let $X \to S$ be a split family of homogeneous spaces over $S$. Then, after possibly shrinking $S$, there exists a factorization

$$X \to Y \to S$$

such that $X \to Y$ is a split family of homogeneous spaces over $Y$, such that $Y \to S$ is a split family of homogeneous spaces over $S$ and such that $\text{Pic}(Y_\eta) = Z$.

**Proof.** Choose an isomorphism $X_\pi = G/P$ for some $G$, $P$ over $\eta$. Choose a maximal parabolic $P \subset P'$ containing $P$. Let $L_0$ be an ample generator of $\text{Pic}(G/P')$, see Corollary 14.3, its proof and Remark 14.5. Let $\mathcal{L}$ be an invertible sheaf on $X$ whose restriction to $X_\pi$ is isomorphic to the pull back of $L_0$ to $X_\pi = G/P$ via the morphism $\pi : G/P \to G/P'$. Since $L_0$ is ample, some power $L_0^N$ is very ample and
globally generated. Also, \( H^0(G/P, \pi^*\mathcal{L}_N^0) = H^0(G/P', \mathcal{L}_N^0) \) as the fibres of \( \pi \) are connected projective varieties. We conclude, by cohomology and base change, that after shrinking \( S \) we may assume that \( \mathcal{L}^N \) is globally generated on \( X \).

Choose a finite collection of sections \( s_0, \ldots, s_M \) which generate the \( \kappa(\mathfrak{m}) \)-vector space \( H^0(G/P, \pi^*\mathcal{L}_N^0) = H^0(G/P', \mathcal{L}_N^0) \). Consider the morphism \( X \to \mathbb{P}^N_S \) over \( S \). Note that on the geometric generic fibre the image is the projective embedding of \( G/P' \) via the sections \( s_0, \ldots, s_M \). Hence, after possibly shrinking \( S \) and using Stein factorization, we get a factorization \( X \to Y \to S \) such that \( Y_\mathfrak{m} \cong G/P' \) with \( X_\mathfrak{m} \to Y_\mathfrak{m} \) equal to \( \pi \). The family \( Y \to S \) is split because the invertible sheaf \( \mathcal{L} \) descends to \( Y \) due to the fact that it does so on the geometric generic fibre. (Hint: just take the pushforward of \( \mathcal{L} \) under \( X \to Y \).)

The geometric generic fibre of \( X \to Y \) is isomorphic to \( P/P' \) which is a projective homogeneous variety as well (for a possibly different group). To show that \( X \to Y \) is split it suffices to show that \( \text{Pic}(G/P') \to \text{Pic}(P'/P) \) is surjective, as \( X \to S \) is split. This surjectivity can be proved using the fact that the derived group of \( P' \) is simply connected (since \( G \) is simply connected), combined with for example the results of [Pop74]. We prefer to prove it by considering the Leray spectral sequence for \( f : G/P \to G/P' \) and the sheaf \( G_m \). Namely, \( R^1f_*G_m \) is a constant sheaf with value \( \text{Pic}(P/P') \) (due to the fact that \( G/P' \) has trivial fundamental group).

Hence the obstruction to surjectivity is an element in the Brauer group \( Br(G/P') \). Since \( k \) is algebraically closed and \( G/P' \) is rational (by Bruhat decomposition) we see \( Br(G/P') = 0 \) and we win. □

Let \( k \) be an algebraically closed field of positive characteristic. Let \( R \) be a Cohen ring for \( k \) (for example the Witt ring of \( k \)). Let \( G \) be a connected, semi-simple, simply connected linear algebraic group over \( k \), and let \( P \subset G \) be a standard parabolic subgroup. According to Chevalley ([Che95], see also [Dem65] and [Spr98]) there exists a reductive group scheme \( G_R \) over \( R \) and a parabolic subgroup scheme \( P_R \subset G_R \) whose fibre over \( k \) recovers the pair \((G,P)\). Denote \( Z_R = G_R/P_R \) the smooth projective \( G_R \)-scheme over \( R \) whose special fibre is \( Z = G/P \). Since \( \text{Pic}(Z_R) \) is isomorphic to the character group of \( P_R \), and since by construction the character group of \( P_R \) equals that of \( P \), we see that \( \text{Pic}(Z_R) = \text{Pic}(Z) \). Choose an ample invertible sheaf \( \mathcal{O}(1) \) on \( Z_R \).

**Lemma 15.2.** The automorphism scheme \( \text{Aut}_R((Z_R, \mathcal{O}(1))) \) is smooth over \( R \).

**Proof.** The scheme of automorphisms \( \text{Aut}_R(Z_R) \) is smooth over \( R \) by [Dem77 Proposition 4]. Thus it suffices to show that the morphism \( \text{Aut}_R((Z_R, \mathcal{O}(1))) \to \text{Aut}_R(Z_R) \) is smooth. By deformation theory it suffices to show that \( H^1(Z, \mathcal{O}_Z) = 0 \). This follows from [Kem76 Section 6 Theorem 1].□

The following lemma follows from a general “lifting” argument that was explained to us by Ofer Gabber, Jean-Louis Colliot-Thélène and Max Lieblich. For a brief explanation, see Remark 15.4 below.

**Lemma 15.3.** Let \( k, R, G, P, G_R, P_R \) as above. Assume that \( P \) is maximal parabolic in \( G \). Let \( \Omega = \overline{K} \) be an algebraic closure of the field of fractions of \( R \). Suppose that every split family of homogeneous spaces over a surface over \( \Omega \) whose general fibre is isomorphic to \( G_\Omega/P_\Omega \) has a section. Then the same is true over \( k \).
Proof. Pick $\mathcal{O}(1)$ on $Z_R$ an ample generator of $\operatorname{Pic}(Z_R) = \operatorname{Pic}(Z) = \mathbb{Z}$. Denote $H_R = \operatorname{Aut}_R((Z_R, \mathcal{O}(1)))$. By Lemma 15.2 we see that $H_R$ is flat over $R$. At this point [dJS05a Corollary 2.3.3] applies with our $H_R$ playing the role of the group named $G_R$ in ibid., and with our $Z_R$ playing the role of the scheme called $V_R$ in ibid. □

Remark 15.4. What is involved in the proof of [dJS05a Corollary 2.3.3]? Consider a split family $X \to S/k$ of homogeneous spaces whose general fibre is isomorphic to $Z = G/P$. Pick $\mathcal{L}$ on $X$ restricting to a generator of the picard group of a general fibre. Then $\operatorname{Isom}((X, \mathcal{L}), (Z, \mathcal{O}_Z(1)))$ is a $H_k$-torsor over a nonempty affine open $U \subset S$ of the surface $S$. After shrinking $U$ we may assume $U$ is smooth over $k$. After this one lifts $U$ to $W \to \operatorname{Spec}(R)$ smooth using [Elk73]. The “lifting” result mentioned above is the statement that a $H_k$-torsor over $U$ is the restriction of a $H_R$-torsor over $W$ after possibly replacing $W$ by $W' \to W$ etale and trivial over a nonempty open $U$. This torsor in turn defines a split family of homogeneous space $X \to W$ restricting to $X_U \to U$ over $U$. It is split because the geometric generic fibre has Picard group $\mathbb{Z}$ and the ample generator exists by virtue of the fact that $H_R$ acts on $\mathcal{O}(1)$ on $Z_R$. At this point, a standard specialization argument implies the result. For details see [dJS05a].

Theorem 15.5. Let $k$ be an algebraically closed field of any characteristic. Let $S$ be a quasi-projective surface over $k$. Let $X \to S$ be a split family of projective homogeneous spaces. Then $X \to S$ has a rational section.

Proof. By Lemma 15.1 it suffices to prove the theorem in case the general fibre has Picard number $\rho = 1$, i.e., the parabolic group is maximal. By Lemma 15.3 it suffices to prove the result in the $\rho = 1$ case in characteristic zero. The case $\rho = 1$, $S$ is a projective smooth surface and $X \to S$ is smooth and projective is Corollary 1.3 which was proved in the introduction (as a consequence of the more general Corollary 12.2 using the results of Section 14).

Thus it remains to deduce the general $\rho = 1$ case in characteristic 0 from the case where the base is projective. This is done by the method of discriminant avoidance. Precise references are [dJS05b Theorem 1.3], or [dJS05a Theorem 1.0.1] (either reference contains a complete argument; the first giving a more down to earth approach than the second). Please see Remark 15.6 below for a brief synopsis of the argument. □

Remark 15.6. Let $k$ be an algebraically closed field (of any characteristic). As before let $Z = G/P$ with $P$ maximal parabolic and let $\mathcal{O}(1)$ be an ample generator of $\operatorname{Pic}(Z)$. Also, as before let $H = \operatorname{Aut}((Z, \mathcal{O}(1)))$ be the automorphism scheme of the pair. According to proof of Lemma 15.2 and the references therein, the group scheme $H$ is reductive. Let $S$ be a quasi-projective surface over $k$. Suppose that $X \to S$ is a split family of projective homogeneous spaces whose general fibre is isomorphic to $Z$. We will explain the method of [dJS05a] reducing the problem of finding a rational section of $X \to S$ to the problem in a case where the base surface is projective.

Choose an invertible sheaf $\mathcal{L}$ on $X$ whose restriction to a general fibre is an ample generator. As in Remark 15.4 there exists an open $U$ and an $H$-torsor $T$ over $U$ such that $(X_U, \mathcal{L}_U)$ is isomorphic over $U$ to $(T \times_H Z, \mathcal{O}_H(1))$. The main result of [dJS05a] implies that there exists a flat morphism $W \to \operatorname{Spec}(k[[t]])$, an open
immersion \( j : W_k \to U \), a \( H \)-torsor \( T' \) over \( W \) such that (1) \( T'_k \cong j^* T \) and (2) the generic fibre \( W_k(t(i)) \) is projective. We may use the torsor \( T' \) to obtain an extension of the pull back family \( j^* X \) to a split family \( X \to W \) of projective homogeneous spaces over all of \( W \). At this point a standard specialization argument implies that the existence of a rational section of \( X_{k(t(i))} \to W_{k(t(i))} \) implies the existence of a rational section of the original \( X \to S \).

The main result of [dJS05a] mentioned above is the following: If \( H \) is a reductive algebraic group, and \( c \geq 0 \) an integer then the algebraic stack \( B(H) = [\text{Spec}(k)/H] \) allows a morphism \( X \to B(H) \) surjective on field points such that \( X \) has a projective compactification \( X \subset \overline{X} \) whose boundary has codimension \( \geq c \) in \( \overline{X} \). This result easily implies the existence of the family of torsors as above. For details see the references above.

16. APPLICATION TO FAMILIES OF COMPLETE INTERSECTIONS

Let \( K \supset k \) be the function field of a surface over an algebraically closed field \( k \). Tsen’s theorem states that any complete intersection of type \((d_1, \ldots, d_c)\) in \( \mathbb{P}^n_K \) has a \( K \)-rational point if \( \sum d_i^2 \leq n \). See [Lan52]. An addendum is that if the inequality is violated then there are function fields \( K \) and complete intersections without rational points.

In this section we sketch a proof of Tsen’s theorem in characteristic 0 for families of hypersurfaces using the main results of this paper. As explained in the introduction this is silly, but hopefully provides a useful illustration of our methods.

First of all we may replace \( k \) by an uncountable algebraically closed overfield of characteristic zero (by standard limit techniques). Second we may assume \( d_i \geq 2 \) for all \( i \). The moduli space \( M \) of all complete intersections \( X \) of type \((d_1, \ldots, d_c)\) in \( \mathbb{P}^n \) is an open subset of a product of projective spaces \( M \subset \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_c} \).

Note that we do not require complete intersections to be nonsingular; it is then easy to see that the complement of \( M \) in \( \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_c} \) has codimension \( \geq 2 \). It is also straightforward to see that the total space of the universal family \( X \to M \) is nonsingular. It follows by standard specialization techniques that to prove Tsen’s theorem for all \( K \) and all complete intersections, it suffices to prove it for those \( X/K \) such that \( K \) corresponds to the function field of a surface \( S \subset M \) with the following properties:

1. \( S \) passes through a very general point of \( M \),
2. the closure \( \overline{S} \subset \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_c} \) differs from \( S \) by finitely many points, and
3. the restriction of \( X \) to \( S \) has smooth total space.

At this point Corollary [dJS05a] says that it is enough to show that there exists a very general complete intersection \( Y \) of type \((d_1, \ldots, d_c)\) in \( \mathbb{P}^n \) is rationally simply connected by chains of free lines, and contains a very twisting surface.

To check that \( Y \) is rationally simply connected by chains of free lines, i.e., that \( Y \to \text{Spec}(k) \) satisfies Hypothesis 6.8 we point out the following:

1. The space of lines through a general point \( y \in Y(k) \) is a nonsingular complete intersection of type \((1, 2, \ldots, d_1, 1, 2, \ldots, d_2, \ldots, 1, 2, \ldots, d_c)\) in a \( \mathbb{P}^{n-1} \). Because \( 1 + 2 + \ldots + d_1 + 1 + 2 + \ldots + d_2 + \ldots + 1 + 2 + \ldots + d_c = \sum d_i(d_i + 1)/2 \leq \sum (d_i^2 - 1) \leq n - 1 \) it is a smooth projective Fano and hence rationally connected.
For a general pair of points \( x, y \in Y(k) \) the space of pairs of lines connecting \( x \) and \( y \) is parametrized by a nonsingular complete intersection of type
\[
(1, 2, \ldots, d_1 - 1, 1, 2, \ldots, d_1 - 1, d_1, \ldots, 1, 2, \ldots, d_c - 1, 1, 2, \ldots, d_c - 1, d_c)
\]
in \( \mathbb{P}^n \). (To see this consider the equations for the locus of the intersection points of the two lines.) The sum of the degrees is equal to \( \sum d_i^2 \leq n \) and again we see that this is a smooth projective Fano variety and hence rationally connected.

To prove this argue as in [Kol96, Chapter V Section 4], especially the exercises. See also [LJS06]. This proves Hypothesis 6.8 for \( Y/k \).

As usual the hardest thing to check is the existence of a very twisting surface. A simple parameter count shows that a very twisting surface should exist on a general complete intersection with
\[
\sum d_i^2 \leq n.
\]
We do not know how to prove the existence unless \( c = 1 \). Namely, in [HS05] it was show very twisting scrolls exist if
\[
d_1^2 + d_1 + 1 \leq n.
\]
In [Sta04] Propositions 5.7 and 10.1 it is shown that very twisting scrolls exist in a general hypersurface if
\[
d_1^2 \leq n,
\]
thereby concluding our sketch of the proof of Tsen’s theorem for families of hypersurfaces.

Final remark: In the preprint [LJS06] very twisting surfaces are constructed in any smooth complete intersection in \( \mathbb{P}^n \) under the assumption that
\[
n + 1 \geq \sum (2d_i^2 - d_i).
\]
But note that these surfaces are not ruled by lines in every case; sometimes they are ruled by conics – to which the methods of this paper do not apply.

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