Status of Heavy Quark Physics on the Lattice
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The status of lattice calculations of some phenomenology of heavy quarks is presented. Emphasis is on progress made in calculating those quantities relevant to estimating parameters of the quark mixing matrix, namely leptonic decay constants, the bag parameter of neutral B mixing, and semileptonic form factors. New results from studies of quarkonia are highlighted.

1. INTRODUCTION

Lattice QCD offers the best hope for estimating the non-perturbative QCD effects in weak decays of hadrons, thereby allowing the determination of elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix from experimental data. Much of the phenomenological interest is in those weak decays which contain at least one heavy quark, but it is this case where technical issues are often most difficult and conceptual points most subtle.

There are a variety of approaches taken to simulate a heavy quark on today’s lattices. One is to use the improved actions used for light-quark calculations, taking care to keep $am$ not too large, and then to rely on an extrapolation (often guided by the result of a static calculation) in the heavy quark mass from the charm region to the bottom. A philosophically-different approach is that of non-relativistic QCD (NRQCD), namely, to integrate out the heavy quark producing an effective action which is then discretized; then $am$ must be kept large. A third approach, developed at Fermilab, seeks to produce an action which can be used for any value of $am$, with systematic errors that can be identified and controlled. An approximation to the full program is often used wherein standard relativistic actions are reinterpreted non-relativistically. Much of the debate focuses on to what extent systematic errors are controlled in this case. J. Sloan gave a nice compare-and-contrast summary of NRQCD and Fermilab-type actions at Lattice ’94 [1].

I review developments made and results announced this past year. Sec. 2 summarizes recent calculations of the leptonic decay constants, notably $f_B$, where there has been considerable progress. In Sec. 3, I present new results, and new analysis of old data, for the $B_B$ parameter of neutral B mixing. Sec. 4 outlines the successes and limitations of calculations of semileptonic form factors for B and D decays; this year has seen the initiation of some new large-scale projects. Sec. 5 highlights new developments and results of calculations, presented at this conference, of the phenomenology of quarkonia. I conclude with a qualitative overview of where we stand, and in what direction future work may be focused.

2. LEPTONIC DECAY CONSTANTS

It has been a banner year for the calculation of $f_B$, the leptonic decay constant of the B-meson. Several large groups, having invested many person-years of effort, have presented final results this year; many of these calculations have done a comprehensive job of estimating systematic errors. There appears to be a consensus among many of the groups (See Figure 1). This is encouraging, given the diversity of actions and analyses. See reference [2] for a discussion of discretization errors.

The MILC collaboration has published their final results this year [3]. They use Wilson quarks for both the light and heavy quarks, but use some aspects of the “Fermilab interpretation” [4], namely the use of non-relativistic normalization and kinetic, rather than pole mass, for the heavy quark. MILC has also found it useful to include
Figure 1. A summary of recent lattice results for $f_B$. Data and labels are taken from Table 1. The error bars with the longer feet are statistical. Systematic errors have been combined in quadrature. Also shown is a world average over the subset of results which have final and complete estimates of systematic errors.

static-Wilson results to aid in the extrapolation in the heavy quark mass. With Wilson heavy quarks in the Fermilab interpretation, there is a $O(1/M)$ error (physical, not a lattice artifact) which cannot be extrapolated away. MILC, following JLQCD [5], estimates this effect to be $\approx 2\%$ from a tree-level approximation. MILC has been very careful in estimating systematic errors; changes since their preliminary Lattice '97 results include updates of their chiral extrapolation, the method of combining errors within the quenched approximation, and some new unquenched data to estimate quenching errors.

In T. Onogi’s Lattice '97 review [6], he reported that although the MILC and JLQCD raw data agreed for common parameters, there were some residual differences from the way the scale was set from $f_\pi$. These discrepancies were discussed after last year’s symposium by members of each group, and the discrepancies have now been resolved [6].

The Fermilab (FNAL) collaboration have published their final results [7], updated since Lattice '97. They used the SW (clover action) for both heavy and light quarks, but interpreted non-relativistically; this includes the use of correctly normalized fields, an additional three-dimensional rotation for the heavy quark in the current, and the use of the kinetic mass to set the quark mass. They include the full $O(a)$ correction, but not the complete $O(aa)$ correction to the current. The advantage of their program is that they can “sit on” the $b$ mass. They state that their systematic errors are under control, are smaller than their systematic errors, and are smaller than the quenching error which they estimate to be about 10%.

The GLOK Collaboration has published their final results [7] updated since Lattice '97 (see also [8]). Their’s is a quenched, $\beta = 6.0$, calculation which uses clover light (tree-level tadpole improved) and NRQCD heavy quarks. The Hamiltonian includes all $O(1/M^2)$ corrections, the leading $O(1/M^3)$ correction, and discretization corrections at the same level. One important ingredient in the calculation was their axial current renormalization [9]. This was the first use of mass-dependent matching factors and $O(aa)$ current correction. They found mixing between lattice currents which does not vanish as $M \to \infty$ because of an $O(aa)$ lattice artifact term, which can be absorbed into an $O(aaA_{QCD})$ discretization correction. The latter is anomalously large (they found $\approx 12\%$ reduction at the $B$ mass). Furthermore, it cannot be included consistently if Wilson light quarks are used, that is, if $O(a)$ corrections are included for heavy quarks, they must be included for the light quarks as well — one is thus left with sizeable scaling violations if one uses NRQCD-heavy with Wilson-light quarks. GLOK has augmented their $\beta = 6.0$ calculation with a $\beta = 5.7$ run in order to assess scaling violations [10].

Figure 2 compares these NRQCD results of [7] with those of Fermilab [4]. For the $B$ meson sys-
tem, the actions are very similar. The results ought to agree and they do.

Figure 2. A comparison by Fermilab [4] of their results with the NRQCD results of GLOK [7].

JLQCD [11] has presented preliminary results at this conference, using actions and analysis which follow GLOK’s lead. They use quenched Wilson gauge actions at \( \beta = 5.7, 5.9 \) and 6.1, SW light quarks (with KLM normalization for the quark fields). They ran at \( \beta = 6.0 \) on a \( 16^3 \times 48 \) lattice and at \( \beta = 6.2 \) on \( 24^3 \times 48 \). They do not attempt a continuum extrapolation, so I, as they, will take their \( \beta = 6.2 \) data as their best.

The ALPHA collaboration [13] has proposed a method for systematic improvement of the Wilson action and bilinear quark operators to remove all \( O(a) \) discretization errors. The program evaluates the clover coefficient and renormalization constants (of bilinear quark operators) non-perturbatively. This year, two new projects have calculated heavy-light leptonic decay constants using this approach. APE [14] and UKQCD [15] ran at \( \beta = 6.2 \) on \( 24^3 \times 64 \) and \( 24^3 \times 48 \) lattices, respectively. Both simulate with the heavy quark near charm to keep \( O(a m) \) errors small; thus they obtain \( f_D \) by interpolation. However, APE finds that the extrapolation up to the bottom region is quite sensitive to the functional form of the fit; this inflates their systematic error. In these proceedings, UKQCD quotes only statistical errors.

Tables 1 and 2 list and Figure 1 displays selected results for heavy-light decay constants and ratios. In computing a world average (Table 3), always a notoriously difficult thing to attempt, I include only those final, not preliminary, large-scale results where modern methods have been used and a comprehensive investigation of systematic errors has been done, which includes ei-
Table 1

Results for $f_B$, $f_{B_s}$, and $f_{B_s}/f_B$ obtained in the quenched approximation. The errors are statistical, then various systematic. Results are divided into three sections. In the top section are the most recent preliminary results; thus, despite their quality they are not included in world averages. In the middle section are recent published and finalized results where all systematic errors (except quenching) have been carefully assessed; these are folded into a world average. In the last section are selected older results from pre-modern methods.

Table 3

World averages for quenched $f_B$, $f_{B_s}$, $f_D$, $f_{D_s}$, and ratios. The “unquenched” results are best guesses obtained by shifting the quenched values by the rough estimates of quenching errors from MILC [3] (the last of their errors in Tables 1 and 2). Estimates of unquenched errors assume that the MILC corrections could be off by a factor of 2.
mates of coefficients and extrapolate up from lighter masses.) The world average of $f_B$ in the quenched approximation, obtained from various actions and groups, has stabilized in last 2–3 years (world averages had been reported in the range 160–175 MeV with errors of about 25–35 MeV), and is significantly lower than several years ago (when world averages were 200(40) MeV or above). Looking back, one sees that older results were higher as several effects conspired: old static values which guided extrapolations in 1/$M_B$ were misleadingly high prior to modern variational techniques which remove excited-state contaminations, and both 1/$M_B$ corrections to the static limit and discretization errors are larger than were expected. Quenching errors currently are difficult to estimate. Unquenching may increase $f_B$ by 10–15% (MILC), maybe more, or maybe less. Lattice estimates of $f_B$ continue to be very useful phenomenologically. In the future, we’ll see continued work with improved actions and renormalization, with the emphasis shifting towards making better estimates of quenching errors.

3. NEUTRAL $B$ MIXING

The CKM matrix element $|V_{td}|$ can be extracted from the mass difference induced in neutral $B$ mixing

$$(\Delta m)_{B-\overline{B}} = f_B^2 B|V_{td}|^2 F(m_t, M_W)$$

where $F$ is a known function and the $B_B$ parameter has been used to parameterize the matrix element $\langle \overline{B} | O_L | B^0 \rangle$ of the $\Delta B = 2$ four-quark operator $O_L = \overline{q} (1 - \gamma_5) q \overline{q} (1 - \gamma_5) q$ in terms of its approximation under the factorization hypothesis (“vacuum saturation” or VSA)

$$B_B = \frac{\langle \overline{B} | O_L | B^0 \rangle}{\langle \overline{B} | O_L | B^0 \rangle_{\text{VSA}}} = \frac{\langle \overline{B} | O_L | B^0 \rangle}{8/3 f_B^2 M_B^2}$$

Since one needs a non-perturbative evaluation of the matrix element and thus the combination $f_B^2 B_B$, why calculate $f_B$ and $B_B$ separately? Reasons include (1) a quite precise value can be obtained, with an optimal choice of smearing function, from relatively few configurations, because $B_B$ can be extracted from a ratio of three- to two-point functions which are strongly correlated, (2) perturbative corrections tend to be stabilized because of cancelations in numerator and denominator, and (3) it seems as though VSA may be a surprisingly good approximation (to $\approx 10–15\%$) for the $B_B$ parameter; this is an important qualitative statement, of use to model builders, which should not be obscured by poor-statistics attempts to calculate the product $B_B f_B^2$. The $B_B$ parameter was reviewed by J. Flynn at Lattice ’96 and by A. Soni at Lattice ’95. Results from UKQCD (Lin and Lellouch) and from the Hiroshima group were reported at this conference. Other recent

|        | $f_B$/MeV | $f_{B^+}$/MeV | $f_{B^0}$/f$_B$ |
|--------|-----------|---------------|------------------|
| APE98a | 202(14)(12) | 231(11)(9) | 1.11(3) |
| UKQCD98a | 193(10) | 221(9) | 1.15(4) |
| UKQCD98b | 190(5) | 190(5) | 1.10(4) |
| MILC98 | 192(11)(6)(4) | 210(9)(3)(4) | 1.10(4)(3)(4) |
| FNAL97 | 194(13)(10) | 213(11)(11) | 1.10(4) |
| JLQCD97 | 197(2)(17) | 224(2)(19) | 1.07(4) |
| APE97 | 221(17) | 237(16) | 1.07(4) |
| LANL95 | 186(29) | 218(15) | 1.09(2)(5) |
| PCW93 | 170(30) | 170(30) | 1.09(2)(5) |
| UKQCD93 | 185(4)(12) | 212(4)(46) | 1.18(2) |
| BLS93 | 208(9)(35)(12) | 230(7)(30)(18) | 1.11(6) |

Table 2
Same as for Table 1 but for $f_B$, $f_{B^+}$, $f_{B^0}/f_B$. 

results include those from Di Pierro and Sachrajda [32], from Giménez and Reyes [33], and from Bernard, Blum and Soni [34].

This year, Lin and Lellouch [12] have calculated the $B_B$ parameter as part of larger effort to calculate $B$ parameters and decay constants for light and heavy mesons. They use the SW action, with the clover coefficient tadpole improved at tree-level, at both $\beta = 6.0$ and 6.2 to check scaling. They see little $a$-dependence. In fact, the world’s collection of results from conventional (non-static) methods (Figure 5) (the top set of results) shows consistency among groups, no $a$-dependence and agreement between Wilson and SW calculations.

![Figure 4. Heavy quark mass dependence of $B_{B_s}$. From UKQCD (Lellouch/Lin) [12].](image)

Figure 4 shows the heavy quark mass dependence of $B_{B_s}$. There is a consensus [12,30] that the extrapolated-static value is larger than that at the physical mass.

The extrapolated-static results and those obtained directly in the static approximation should agree. There are disagreements among the static calculations, some of which do not agree with extrapolated-static results (Figure 5). As with
decay constant calculations, one must fight the very poor signal-to-noise ratio inherent with using static quarks. A large number of configurations has to be generated if one uses canonical smearing techniques \([39]\). The Kentucky group obtains \([35]\) the same statistical error with an order of magnitude fewer configurations by using optimal smearing techniques that they developed \([40]\). It is notable that all groups’ raw lattice data are consistent; furthermore, the VSA holds surprisingly well for all operators (with appropriate normalization).

As a short digression, it is interesting to note that factorization seems to hold well in related static calculations. Di Pierro and Sachrajda \([32]\) need to evaluate matrix elements like \(\langle B | \bar{\gamma}_5 \gamma_\nu (1 - \gamma_5) \gamma_\rho b | B \rangle\) for their studies of inclusive decays of heavy hadrons. The operators are similar to, but different than, those for the \(B\bar{B}\) parameter calculations. For the “figure-8” contractions, they find that factorization holds surprisingly well — to \(\approx 5\%\).

For the static \(B_B\) parameter calculations, differences in renormalized quantities arise from choices made in lattice-to-continuum matching: whether products and quotients are expanded in \(\alpha\) or merely multiplied and divided makes a uncomfortably large difference.

Although all organizations of perturbation theory at one-loop are theoretically equal, some are more equal than others! The Kentucky group \([35]\) has been advocating an organization of lattice perturbation theory which uses the Lepage-Mackenzie \([11]\) choices for \(\alpha_s\) and tadpole improvement and which, by allowing explicit cancelations, is insensitive to the wave-function normalization and which does not mask the agreement of the raw data with VSA. (This also reduces the statistical errors.) Their results agree with VSA and with the extrapolated static results and can thus be used in conjunction with conventional data to interpolate to the \(B\)-meson mass, as has been done with much success for the decay constant calculations. The original clover-light–static-heavy results \([38,39]\) which disagreed with the Kentucky results and with the world’s extrapolated-static results, have recently been reanalyzed \([32,33]\) with an organization of perturbation theory which includes tadpole improvement (and corrects errors in some renormalization constants); the discrepancies have been reduced but not eliminated.

The Hiroshima group \([31]\) has begun the first calculation of the \(B_B\) parameter using NRQCD for the heavy quarks (with \(16^3 \times 48\) quenched Wilson \(\beta = 5.9\) gauge-field configurations, and SW light quarks with a tree-level tadpole-improved clover coefficient). They can see effective-mass plateaus despite using local-local correlation functions. They find large dependence on the heavy quark mass. For their preliminary calculation they rely on the static renormalization constants. It is not clear that the large mass-dependence will be mitigated upon using the correct heavy-mass dependent renormalization constants, as was the case for the decay constant \([1]\).

Relying, then, only on conventional results, I quote a world average for \(B_B(m_B)\) in Table 4. This value has been historically stable; it agrees with Soni’s \textit{Lattice ’95} estimate \([30]\). Indeed, even prehistoric estimates \([12]\) are good because any errors in normalization (e.g. non-KLM factors) cancel in the ratio. All groups agree that the light quark mass dependence of \(B_B\) is small.

Bernard, Blum and Soni (BBS) \([34]\) have recommended that the \(SU(3)\) flavor dependence be investigated by calculating directly a ratio of matrix elements giving \(\frac{M_{B_s} f_{B_s} B_{B_s}}{M_{B_d} f_{B_d} B_{B_d}}\). One would expect that many systematic errors would cancel. Lin and Lellouch \([12]\) have since calculated this

| \(B_B(m_B)\) Quenched “Unquenched” |
|-----------------|-----------------|
| \(B_B(m_b)\) | 0.86(4)(8) | 0.86(4)(8) |
| \(B_{B_s}/B_{B_d}\) | 1.00(1)(2) | 1.00(1)(2) |
| \(f_{B_d} \sqrt{f_{B_s}^n}\) | 215^{+40}_{-30} | 190^{+25}_{-25} |
| \(f_{B_s}/\sqrt{f_{B_d}^n}\) | 1.14^{+5}_{-5} | 1.14^{+7}_{-6} |

Table 4
World averages for quenched and “unquenched” quantities.
quantity using clover quarks (BBS used Wilson). Both groups find that the direct method is compatible with, but unfortunately has larger statistical errors (by a factor of 2 or 3) than, the indirect method which combines separate calculations of $B_B$ and $f_B$, for both strange and light quarks.

4. SEMILEPTONIC DECAYS

The differential decay rate for $P(Q\bar{q}) \rightarrow P'(q'\bar{q})\nu_l$ is

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2 p'^3}{24\pi^3}|V_{q'q}|^2 |f_+(q^2)|^2$$

where $f_+(q^2)$ is a vector form factor, obtained from the matrix element

$$\langle P'(p')|V_{\mu}|P(q)\rangle = \left(p + p' - q - \frac{m_P^2 - m_{P'}^2}{q^2}\right)_{\mu} f_+(q^2) + q_{\mu} \frac{m_P^2 - m_{P'}^2}{q^2} f_0(q^2)$$

Similarly, the differential decay rate for $P(Q\bar{q}) \rightarrow P^*(q'\bar{q})\nu_l$ is expressed in terms of a vector form factor $V(q^2)$ and two axial form factors $A_1(q^2)$ and $A_2(q^2)$.

Experiments are able to determine an intercept and slope for $f_+|_{(0)}$, so that it is conventional to quote theoretical predictions at $q^2 = 0$; furthermore, the form factor is often parameterized with pole dominance form $f_+(q^2) = f_+(0)/(1-q^2/m^2)$ to aid interpolation and extrapolation. Unfortunately, lattice calculations work best at small three-momenta, i.e. near zero recoil, $q_{\text{max}}^2 = (m_P - m_{P'})^2$, and so for decays such as $B \rightarrow \pi$, large extrapolations are necessary.

Semileptonic form factors were reviewed by T. Onogi at Lattice ’97 [2], by J. Flynn [26] at Lattice ’96 and by J. Simone [43] at Lattice ’95. For a very comprehensive and recent survey of lattice results, see the review by Flynn and Sachrajda [21] (on which the discussion of the present section relies heavily).

4.1. Semileptonic $D \rightarrow K, K^*$ Decays

For semileptonic $D \rightarrow K, K^*$ decays, the maximum meson recoil momenta are below 1 GeV/c so all $q^2$ can be sampled with small momentum-dependent discretization errors; thus one can interpolate to $q^2 = 0$. Furthermore, one can sit on $s$ and $c$ so the only mass-extrapolation needed is for the light quark. Thus, $D$ decays are an excellent test for the lattice; experimental determinations of $|V_{cs}|$ are already good and are likely to improve soon. Some disadvantages of the lattice calculations are that the charm and strange quark masses are non-degenerate, so there is no natural normalization condition, and that care need be taken to reduce discretization errors for the charm mass.

![Figure 6](image-url)

Figure 6. Lattice estimates of $f_+(0)$ from APE [44], UKQCD [45], LANL [46] and Wuppertal [47], compared to experiment [48].

Figure 7 compares a recent experimental estimate [48] with a collection of lattice results for $f_+(0)$, from APE [44], UKQCD [45], LANL [46] and Wuppertal [47]. (See [2] for a more complete summary including older results; those listed here...
use either the SW action, or Wilson with KLM normalization.) The agreement is quite satisfactory. The agreement for the $D \rightarrow K^*$ form factors, $V(0)$, $A_1(0)$, and $A_2(0)$, is also good, although somewhat poorer for $A_1$ and $A_2$, which depend upon correctly normalizing the lattice axial current.

All of these results are from quenched calculations without a continuum extrapolation. A new effort this year seeks to remedy this deficiency. Preliminary results from FNAL were presented by J. Simone at this conference [49]. Their quenched calculation uses the SW (clover) action interpreted non-relativistically in the “Fermilab approach”; thus they can “sit on the charm quark”. At this early stage in their project, the light quark has strange-quark mass. They see a gentle $a$-dependence of the matrix elements which bodes well for controlling the continuum extrapolation. Their goal is to precisely determine decay rates for $D \rightarrow \pi\ell\nu$ and $D \rightarrow Kl\nu$; the FOCUS experiment will soon have a high-statistics determination of $\text{BR}(D \rightarrow \pi\ell\nu)/\text{BR}(D \rightarrow Kl\nu)$.

Preliminary results from UKQCD were presented by C. Maynard at this conference [50]. Their quenched calculation on $24 \times 48$ lattices ($\beta = 6.2$) uses a $\mathcal{O}(a)$-improved SW with a non-perturbative value of clover coefficient, and simulates with a heavy quark mass near charm, and a light quark mass both near strange and close to the chiral limit. $f^+, f^0$ are simultaneously fit (to pole-dominance model) for spatial and temporal components of vector current. The statistics are quite competitive: 3–5% errors for $D \rightarrow K$ (from 216 configurations). The results are to be compared with older results which used tree-level clover-coefficients.

### 4.2. Semileptonic $B \rightarrow \pi, \rho$ Decays

Semileptonic decays of the $B$ are very important for constraining CKM matrix elements. Lattice calculations are really needed since HQET symmetry is not as helpful as for $B \rightarrow D$. Unfortunately, the large bottom mass means that simulations must be at the charm mass and then extrapolated to bottom, else care must be taken to design an action for which discretization errors are small and controlled. Furthermore, the maximum meson recoil momenta are large so all $q^2$ cannot be sampled (contrast $D$ decays) without large momentum-dependent discretization errors. Thus large extrapolation to $q^2 = 0$ are made for which one must rely on the pole dominance model.

Although heavy quark symmetry says nothing about overall normalizations, it does predict that hadronic matrix elements for light-meson decay modes such as $P \rightarrow \pi\ell\nu$, scale in the heavy quark limit, $M \rightarrow \infty$, for $q^2$ near $q^2_{max}$ (zero recoil). Form factor scaling follows: the leading mass dependence is $f^0 \sim M^{-1/2}$, $f^+ \sim M^{1/2}$ for $B \rightarrow \pi\ell\nu$, and $V \sim M^{1/2}$, $A_1 \sim M^{-1/2}$, and $A_2 \sim M^{1/2}$ and for $B \rightarrow \rho\ell\nu$. There is a consensus (see T. Onogi’s review 2 at last year’s lattice conference) that such scaling is seen in lattice simulations for $q^2_{max}$, and thus can be used to extrapolate simulated form factors to the $B$ mass.

![Figure 7. Fit to lattice results for $V$, $A_0$, $A_1$, $T_1$, $T_2$, versus $(q/\text{GeV})^2$, for $\rho$ final state. From [51].](image)

To obtain results at smaller $q^2$, one has to extrapolate aided by a pole model. UKQCD [51] has reanalyzed some older data and exploited the scaling constraints of the light-cone sum rule to guide model-dependent extrapolation to $q^2 = 0$ where the leading mass dependence for all form factors is $\sim M^{-3/2}$. (They point out that the analyses of some other groups violate these scaling relations.) Figure 7 show the results of a simultaneous fit which respects all constraints. One can see that the extrapolation to $q^2 = 0$ is very large indeed.
Figure 8. Lattice estimates of $B \rightarrow \pi f^+(0)$ from UKQCD [51], Wuppertal [47], APE [44] and ELC [52].

Figure 8 compares previous lattice estimates of $f^+$ at $q^2 = 0$ from UKQCD [51], Wuppertal [47], APE [44] and ELC [52].

Preliminary results from UKQCD were presented by C. Maynard at this conference [50], in conjunction with their semileptonic $D$ decay calculation (quenched, $24 \times 48$, $\beta = 6.2$, $O(a)$-improved SW with a non-perturbative clover coefficient, heavy quark mass near charm). At last year’s conference, we saw evidence [2] of a large violation of the soft-pion theorem which predicts the relation
\[
f^0(q_{\text{max}}^2) = f_B/f_\pi
\]
This year, UKQCD sees no such violation as demonstrated in Figure 8. The UKQCD calculation includes renormalization constants, and uses a smaller lattice spacing ($\beta = 6.2$, versus $\beta = 5.9$ for JLOCD [53] (SW action) and $\beta = 5.8$ for Hiroshima [54] (Wilson light quarks, NRQCD heavy)].

FNAL, as presented by S. Ryan at this conference, are engaged in a multi-$a$ calculation of $B$ semileptonic decays, in conjunction with their semileptonic $D$ decay calculation, with the intention of taking the chiral and continuum limits. As described in the previous subsection, the advantage of using their action is that they can sit at the bottom mass (or any other), and avoid having to extrapolate in the heavy quark mass. They verify that the heavy-mass dependence of the matrix elements is very gentle, from the charm through the bottom regime. Of course, they are subject to the same limitations as other groups for extrapolations in the light quark mass, and in $q^2$.

A novel approach [55] has been proposed to predict form factors for exclusive processes such as $B \rightarrow \pi$ by computing light-cone wave functions (not, as before, just their moments) in terms of lattice correlation functions. Although no simulation results have been presented yet, the method is intriguing because it can be used at small $q^2$ directly.

In summary, all groups agree on the $1/M$ scaling near $q_{\text{max}}^2$. ELC, APE and UKQCD use
scaling to extrapolate from charm to bottom; JLQCD, Hiroshima and FNAL can sit on the bottom. Estimates of the form factor at $q^2 = 0$ have a large extrapolation, which is model dependent. Continuum extrapolations have not been published, but are in the works. A comprehensive estimate of systematic errors, as for $f_B$, has yet to be made. One can hope that future estimates of systematic errors might follow improvements in decay constant calculations, with a lag of two or three years. It will continue to be difficult to simulate at low $q^2$, but it may not be necessary. Experimental data may soon be available at high $q^2$; if so, a comparison with accurate lattice results will be sufficient to determine $|V_{ub}|$ with impressive precision.

4.3. Semileptonic $B \rightarrow D, D^*$ Decays

Semileptonic decays $B \rightarrow D, D^*$ are used to extract the $|V_{ub}|$ CKM matrix element. Heavy quark symmetry plays a central role. The intuitive picture is well known. (See reference [56], for example). Typical momenta exchanged between heavy and light quarks are $\mathcal{O}(\Lambda_{QCD})$; light degrees of freedom cannot resolve distances $1/m_Q \ll 1/\Lambda_{QCD}$ and are blind to the mass and spin of heavy quark; thus hadronic systems which differ only in the mass and spin of the heavy quark have the same light dof configuration. The non-perturbative light dof ("brown muck") interacts with heavy-quark static color field (which extends over long distances). When an external weak current boosts the heavy quark ($m_b \rightarrow m_c$), the brown muck reacts and the form factor suppression is universal.

Heavy quark symmetry suggests rewriting the matrix elements in terms of form factors which are functions of velocity transfer $\omega = v \cdot v'$, rather than momentum transfer. Thus $\langle P' | V_{\mu} | P \rangle \propto (v + v')h_+(\omega) + a(v - v')h_-(\omega)$. Similarly, $\langle P'|P \rangle$ is expressed in terms of $h_V$ and $\langle P'|A_\mu|P \rangle$ in terms of $h_{A_1}, h_{A_2}, h_{A_3}$. Then

$$h_i(\omega) = [\alpha_i + \beta_i(m_b, m_c, \omega) + \mathcal{O}(\Lambda/m_Q)] \xi(\omega)$$

where the HQS symmetry enforces $\alpha_+, \alpha_{A_1}, A_3 = 1$ and $\alpha_{A_2} = 0$. $\xi(\omega)$ is the famous Isgur-Wise function. The experimental measurement of the differential cross sections yields $|V_{ub}|^2 \mathcal{F}(\omega)^2$ where the physical form factor, $\mathcal{F}(\omega)$, equals $\xi(\omega)$ plus perturbative and power corrections. Experimentally, the shape of $\mathcal{F}(\omega)$ is not well known, and so it is common to parameterize the extrapolation of the data back to zero recoil (where a theoretical calculation of the normalization $\mathcal{F}(1)$ would then determine $|V_{ub}|$), in terms of the slope at zero recoil, $-\rho^2$. A recent world (experimental) average is $\rho^2 = 0.75(11)$ [57]. A more precise theoretically-determined value would greatly aid the extrapolation back to zero recoil. A task for lattice calculations is then a determination of $\rho^2$, or of the corresponding slope, $-\rho^2$, of the Isgur-Wise function to which it is related by calculable corrections [61].

The first lattice calculations of the Isgur-Wise function were reviewed by Kenway at Lattice ’93. (See also the aforementioned reviews by J. Simone [43] and by Flynn & Sachrajda [21].) Since then, there have been surprisingly few lattice calculations, some of which are listed in Table 5. There also exist calculations [61] which simulate the heavy quark effective theory directly on the lattice, but renormalization is subtle, and discretization errors can be sizeable for lowest-order actions on modest-sized lattices.

|                  | $\rho^2_1$     | $\rho^2_2$     | $|V_{ub}| \times 10^2$ |
|------------------|----------------|----------------|------------------------|
| BSS [61]         | 1.42(2)(4)     | 4.45(5)(7)     |                        |
| UKQCD [62]       | 0.97(3)(4)     | 1.22(3)(4)     | 3.71(5)(7)             |
| LANL [46]        | 1.097(6)       |                |                        |

Table 5

A collection of lattice results for the slope of the Isgur-Wise function. This is then used to constrain extrapolation of the experimental data to obtain the CKM matrix element listed.
coefficient is determined non-perturbatively. The matching coefficient in the renormalized vector current $V^R_\mu = Z_V (1 + b_\gamma a m_q) [V^{\text{latt}}_\mu + \cdots]$ can be computed for degenerate transitions at zero momentum, and then compared with the ALPHA collaboration’s non-perturbative determination of $Z_V$ and $b_\gamma$. The striking agreement leads UKQCD to believe that they have good understanding of the discretization errors.

S. Hashimoto presented preliminary FNAL results for $B \to D$, using the same action as for the other semileptonic form factor calculations previously described. Recall they can simulate at the $b$ mass; currently, the light quark mass is around strange. They consider a ratio of matrix elements at zero recoil

\[ |h_\mu(1)_{B \to D}|^2 = \frac{\langle D|\bar{c} \gamma^\mu b|B\rangle \langle B|\bar{b} \gamma^\mu c|D\rangle}{\langle D|\bar{c} \gamma^\mu c|D\rangle \langle B|\bar{b} \gamma^\mu b|B\rangle} \]

for which statistical errors are small, current renormalization and systematic errors largely cancel. (This is the same ratio as first defined by Mandula and Ogilvie and then used by Draper and McNeile for lattice HQET.) Using a corresponding ratio to obtain $h_\mu$, they then construct a linear combination to obtain $F(1)$, i.e. the physical form factor at zero recoil! Previously, lattice calculations had predicted only the slope to constrain the fit to experimental data leaving $F(1)|V_{cb}|$ as the free parameter, and then relied on sum rule calculations of $F(1)$. With this development, HQET and the lattice alone can determine $|V_{cb}|$.

5. HIGHLIGHTS FROM QUARKONIA

At this conference we saw several interesting calculations of quarkonia. I will not attempt to put these results in context by making a comparison with a complete list of earlier results, due to space constraints, but rather will take this opportunity to advertise several of these calculations.

Presently, the usual approaches used in calculating properties of quarkonia on the lattice are NRQCD (expansion in $v^2$; breaks down for small masses and small lattice spacing) and the Fermilab approach (improve at $am \gg 1$; coefficients are mass-dependent; present simulations with “non-relativistic reinterpretation” of Wilson-SW action).

Problems show up in charmonium hyperfine splitting. At this conference, Shakespeare and Trottier compared the effect of $O(\alpha^3)$ versus $O(\alpha^4)$ terms, and the effect of the detailed way in which mean field is invoked (mean-link from Landau gauge versus from fourth-root-of-plaquette tadpole improvement). They see NRQCD breakdown for $aM_B^0 < 1$ (as expected) and see a clear preference for using the Landau prescription. It should be noted that this is not an indictment of NRQCD, nor of mean-field improvement. Calculations of bottomonium are well under control. For charmonium, the effect of higher order terms is as expected, except that the prefactors are a factor of two or so higher than one would predict from dimensional analysis.

T. Klassen showed us the first results from a long-term program which provides an alternative to NRQCD or Fermilab: Symanzik improvement on anisotropic $\xi = \alpha_s / \alpha_t \gg 1$ lattices. His quark action shares some formal similarities with that of Fermilab. It has on-shell $O(\alpha)$ classically-improvement plus a non-perturbative $O(\alpha)$ improvement which requires a tuning of the Wilson parameter $r(m, \xi)$. However, the coefficients of the electric and magnetic terms in the action are mass-independent (contrast Fermilab). For his first calculation, he uses mean-field (Landau) to estimate these coefficients, and obtains $r(m, \xi)$ from charmonium dispersion. His continuum extrapolation is well-behaved. He sees spectacular agreement with experiment for the charmonium $P$-state splitting.

R. Horgan presented preliminary results of a long-term project using an anisotropic lattice for NRQCD. Their goal is a high statistics calculation for bottomonium and for hybrid mesons. Early results are encouraging and are in good agreement with those from isotropic lattices.

K. Hornbostel presented a new analysis from the NRQCD collaboration of the bottom (and charm) mass. They used a tadpole-improved NRQCD action including next-to-leading order relativistic and discretization corrections, computed spin-averaged $1P-1S$ splittings to determine the scale $a_t$, tuned the bare mass so the
kinetic mass from the dispersion relation agreed with experiment, perturbatively related the bare mass to the $\overline{MS}$ mass (using a Lepage-Mackenzie $q^*$), and ran the $\overline{MS}$ mass to its own scale. They obtain $4.28(3)(3)$ GeV from their $\beta = 6.0$ data (which has the best statistics).

Bali and Boyle [71] have done some nice work in which they measure the potential on the lattice, and then parameterize it to solve the Schrödinger equation on (in)finite lattices. They then vary the lattice volume and quark masses to deduce from their model qualitative features seen in the full simulations; namely, non-negligible finite-size effects, a non-trivial dependence on quark mass, and splittings strongly dependent on the Coulomb coupling.

I would also like to advertise a calculation of the heavy-quark leptonic width using NRQCD by Jones and Woloshyn [72], a new estimate of $\alpha_s$ from the SESAM collaboration [73], and a study by Fingberg [74] of the effect of the gluon condensate on the low-lying quarkonium spectrum. Penmanen [75] presented results from a study of four-quark systems, while Morningstar [76] presented final results for the presence of gluonic excitations in the presence of a static quark-antiquark pair.

6. CONCLUSIONS

The use of improved actions, techniques, analysis and computer power in recent years has been most fruitful. Lattice QCD has made very useful and competitive predictions of many phenomenological quantities crucial for the confrontation of theory and experiment.

It has been a banner year for the calculation of heavy-light leptonic decay constants such as $f_B$. There are now reliable estimates of systematic errors for the quenched calculation, and there is good agreement among groups using a variety of methods. In the near future, we should see if the final analysis (including a continuum extrapolation) from conventionally-interpreted SW actions (which use a non-perturbative estimate of the clover coefficient) gives results which concur with the clustering of several results obtained from non-relativistic interpretation. If so, attention must turn to the remaining formidable issue of more reliably estimating the effects of the quenched approximation. With new actions and methods, more attention should be paid to the heavy-light spectrum, where current estimates of the sensitive hyperfine splitting are too small.

Estimates of the $B_B$ parameter from neutral $B$ mixing seem rather robust; since it is determined from a ratio of correlation functions, systematic errors and statistical fluctuations largely cancel. More modern estimates using Wilson or SW actions agree with older results. The vacuum saturation, or factorization, approximation appears to hold at the 15% level. Earlier variance among results using the static approximation for the heavy quark are better understood, and somewhat reduced. Simulations with an NRQCD heavy quark were presented this year, but need a difficult renormalization calculation for completeness.

For semileptonic decays, we need need more complete estimates of systematic errors (continuum limit, variety of actions). It would be nice to see more groups calculating these form factors. The kinematics of $D \to K, K^*$ provide a good place to test actions, techniques and analysis. For $B \to \pi, \rho$, the kinematics demand a large, model-dependent, extrapolation in $q^2$. (One must also beware discretization errors due to the large bottom mass; the two philosophies used are the same as for leptonic decay calculations: either attempt to control errors with a non-relativistic interpretation of the action, or simulate around the charm mass, and boldly extrapolate.) It is unlikely that the large $q^2$ extrapolation can be avoided in the near future with conventional approaches. It may not be necessary. We expect our experimental colleagues, using increased luminosities available around the turn of the millennium, to be successful in measuring the form factors at high $q^2$. Then a direct comparison of theory and lattice results should provide a good estimate of $|V_{ub}|$.

The power of heavy-quark-symmetry applied to $B \to D$ semileptonic decays has reduced the phenomenologist’s dependence on lattice estimates, but the lattice can still make fruitful contributions by providing a reliable estimate of the slope and intercept of the form factor at zero recoil. New calculations using relativistic heavy quarks have been presented at this conference; it is wel-
come to see renewed interest in these form factors. We expect NRQCD estimates to be available soon and anticipate that within a year or two these, relativistic estimates, and estimates from directly simulating HQET, will all be reconciled as has been the case for leptonic decay constants.

Calculations of quarkonia continue to be a good forum for testing new actions and techniques and for predicting parameters of the standard model; we saw updated estimates of $\alpha_s$ and $m_b$ at this conference. Bottomonium calculations are in good shape; NRQCD calculations in particular are quite precise. The convergence of the expansion in inverse mass is poor for charmomium, however, and so this is a good place to try new improved actions. We saw that anisotropic actions look very promising.

Calculations of heavy quark physics on the lattice have reached a new level of maturity in the last few years, and have provided the wider community with reliable non-perturbative estimates of several quantities of phenomenological interest. But there are still many quantities for which it is difficult to be as precise. Meanwhile, new approaches and techniques mean that the lattice calculations will continue to interesting and important for years to come. Onward and upward!

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