$T-$modules and Manin-Mumford Conjecture

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In this work we propose a strategy to study the Manin-Mumford Conjecture in the $T-$module case, based on the methods recently presented by J. Pila and U. Zannier. In our first paragraph we recall the history of this conjecture in its classic version in order to describe our objectives and our work. We then shall present in the second paragraph an analogous statement of Manin-Mumford Conjecture for abelian and uniformizable $T-$modules, giving the definitions and the preliminary results necessary to develop our technique. We shall give in particular a version of the Implicit Function Theorem for analytical sets provided by a non-archimedean topology, we introduce some basic tools of Non-Archimedean Analysis and we use them in order to prove a result of density of regular points (which we define in this new context). We finally give some results which allow to find a construction analogous to the one that has been build up by J. Pila and U. Zannier in their work. In the third paragraph we present an upper bound estimate of the number of torsion points in a $T$-module, contained in an algebraic variety, following the same path traced by J. Pila and J. Wilkie. Their work has been developed from that of E. Bombieri and J. Pila, on whom the strategy of J. Pila and U. Zannier is based. In the fourth paragraph we explain the use that we do of such an estimate with the perspective to finally prove our version of Manin-Mumford Conjecture for abelian and uniformizable $T-$modules. In the fifth and last paragraph we shall analyze the formulation of our conjecture by some specific example.
1 Introduction

The classic version of the Manin-Mumford conjecture\(^1\), which treats the situation of an elliptic curve embedded into its Jacobian variety has been proved (see [R1]) and later generalized (see [R2]) to the following statement:

**Theorem 1.** Let \( X \) be an algebraic sub-variety into an abelian variety \( A \) defined over a number field. If \( X \) contains a Zariski-dense set of torsion points, then \( X \) is the translate of some abelian sub-variety of \( A \) by a torsion point (a torsion class).

The weak version proved in [PZ] is a consequence of Theorem 1:

**Theorem 2.** Under the same hypothesis of Theorem 1, if \( X \) does not contain any torsion class of \( A \) with dimension > 0, then \( X \) contains at most finitely many torsion points.

The strategy of J. Pila and U. Zannier is based on the idea to translate the analysis of the torsion points of a given abelian variety \( A \) which are contained in \( X \) as in Theorem 2, to a typical problem of Diophantine Geometry, computing these points as rational points contained in a given compact analytic set depending on \( X \) and \( A \). It is actually known that it exists in fact a surjective complex analytic group-isomorphism:

\[ \Theta : \mathbb{C}^g \to A; \]

where \( g = \dim_{\mathbb{C}}(A) \), having its kernel \( \Lambda = \text{Ker}(\Theta) \) which is a \( \mathbb{Z} \)-lattice of rank \( 2g \) so, up to the vector space isomorphism \( \mathbb{C} \cong \mathbb{R}^2 \), one can sees a torsion point of \( A \) as a rational point of the compact set \((\mathbb{R}/\mathbb{Z})^{2g}\). Calling \( Z := \Theta^{-1}(X)/\Lambda \), we have that \( Z \) is a compact real analytic set. The study of torsion points of \( A \) in \( X \) became so the study of the rational points of \( Z \) as a subset of \((\mathbb{R}/\mathbb{Z})^{2g}\). Such a strategy is founded on two previous results. One of them is an estimate provided by J. Pila and J. Wilkie (see [PW]) on the number of rational points with denominator \( T \in \mathbb{N} \setminus \{0\} \) contained in a bounded area of a real analytic variety as \( Z \). Such an estimate has been obtained using o-minimality methods and it is the generalization at higher dimensions of analogous previous results of E. Bombieri and J. Pila on curves and surfaces (see [PZ] for more details) and apply only for points in the trascendent part of \( Z \), in other words, outside the union of all the connected real semi-algebraic sets\(^2\) of positive dimension of \( \mathbb{R}^{2g} \) in \( Z \) (the

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\(^1\)Historically, the first version of this conjecture has been proved by M. Laurent over \( \mathbb{G}_m^n \), see [Lau].

\(^2\)See [Shiota], pages 51, 100.
so-called algebraic part of $Z$). This estimate for such points takes then the shape $\ll T^\epsilon$, for any $\epsilon > 0$. The other result is another estimate for the set of $T$--torsion points provided by D. Masser in 1984 (see [Mas]). Such an estimate is on the degree of these points, and takes the shape $\gg T^\delta$ for a convenient $\delta > 0$. As an algebraic set containing a point also contains all the conjugates of this point, this implies an analogous estimate for the number of the $T$--torsion points and consequently the finiteness of such a set restricted to the transcendental part of $Z$. The work of J. Pila and U. Zannier consists in showing that the algebraic part of $Z$ actually coincides with the union of all the torsion classes contained in $Z$, proving so Theorem 2.

Our idea is to follow such a path in order to prove an analogue of Theorem 2 for a particular class of $T$--modules, which are an analogue in characteristic $p > 0$ and in modules framework of classical abelian varieties. Such objects were introduced for the first time in 1986 by G. Anderson (see [A]) and present very different properties with rapport to an abelian variety (for example, they are not compact), even if a lot of similarities between the two categories may be appreciated. We already have a first transposition of Theorem 1 (which is stronger than Theorem 2) to $T$--modules framework, due to T. Scanlon, who recently (2002) proved (see Theorem 4 or [Sc]) a na"ive transposition of Theorem 1 for the very special case of a power of a Drinfeld module (the dimension 1 case of a $T$--module). He founds his proof on Model Theory. We propose to extend this study to a more ample class of $T$--modules. Our work is divided in three steps.

1. We firstly state an analogous of Theorem 2 for a particular class of $T$--modules allowing a construction similar with that of J. Pila and U. Zannier. Such a statement is not as we’ll see the obvious transposition of Theorem 2 on sub-$T$--modules and shall ask one more condition. We could actually show with explicit counter-examples that other more na"ive formulations are false. We will analyze the reason of such a difference with the T. Scanlon’s result using a Theorem of A. Thiery (see [T]).

2. We provide, on $T$--modules of the family we choose, a construction similar with the one given by J. Pila and U. Zannier using the isomorphism $\Theta$ previously described. Our main tool will be the exponential function, an analogue of $\Theta$ introduced by G. Anderson, having anyway some important different properties (for example, is not always surjective).

3. We show an analogue of J. Pila’s and J. Wilkie’s estimate for an affi-
**noid space** (the ultrametric version of an analytic set, see the third paragraph) in the analogous set \( Z \) we will construct in our new situation. We base our method essentially on Non-Archimedean Analysis techniques, in order to manage the very particular topology of the spaces that we will have to study. At this step we did not still use any technique based on Model Theory or o-minimality.

Our main Theorem will be the following one. Let \( q \) some power of a prime number \( p \) and let \( A = \mathbb{F}_q[T] \) be the ring of polynomials with coefficients in the finite field \( \mathbb{F}_q \). We call \( k = \mathbb{F}_q(T) \) and \( k_\infty = \mathbb{F}_q((1/T)) \). We also call \( |a(T)|_{1/T} = q^{\deg_T(a(T))} \) for every \( a(T) \in A \setminus \{0\} \).

**Theorem 3.** Let \( W \subset k^n_\infty \) be a \( k_\infty \)-entire sub-set of \( k^n_\infty \) (for some \( n \in \mathbb{N} \setminus \{0,1\} \)) analytically parametrizable (see Definition 13) over \( k_\infty \), and let \( W^{alg.} \) be its algebraic part. For each real number \( \epsilon > 0 \), it exists \( c = c(W, \epsilon) > 0 \) such that, for each \( a(T) \in A \setminus \{0\} \), one has:

\[
|\{ \mathbf{z} \in (W \setminus W^{alg.})(k), \ a(T)\mathbf{z} \in A^n \}| \leq c |a(T)|_{1/T}^\epsilon.
\]

## 2 Preliminaries

Our study is focused on a geometrical problem on \( T \)-modules, which are the obvious extension of Drinfeld modules in positive dimension. We shall take the following definition, presented for the first time in the work of G. Anderson ([A]).

We agree to note \( A := \mathbb{F}_q[T] \), where \( q \) is a power of the prime number \( p \), \( k := \text{Frac}(A) \) and \( \mathcal{C} := (k_\infty)_\infty \), where \( \infty \) is the corresponding place in \( 1/T \) in \( A \). If \( K \) is a field and \( n, m \) are two positive entire numbers, the notation \( K^{n,m} \) will also indicate the ring of matrices defined on \( K \), having \( n \) lines and \( m \) columns. We use \( \tau \) to indicate the Frobenius automorphism in the following form:

\[
\tau : z \mapsto z^q, \quad \forall z \in \mathcal{C}.
\]

**Definition 1.** A \( T \)-module \( \mathcal{A} = (\mathbb{G}_a^m, \Phi) \) of degree \( \tilde{d} \) and dimension \( m \) defined on a field \( \mathcal{F} \subset \overline{\mathbb{F}_q} \) is the algebraic group \( \mathbb{G}_a^m \), defined on \( \mathcal{F} \), having the structure of \( A \)-module given by the \( \mathbb{F}_q \)-algebras homomorphism:

\[
\Phi : A \to \mathcal{F}^{n,n}\{\tau\}
\]

\[
T \mapsto \sum_{i=0}^{\tilde{d}} a_i(T)\tau^i;
\]
where $a_0$ (the **differential** of $\Phi(T)$, that we call $d\Phi(T)$ as a linear map acting over $C^m$) is under this form:

$$a_0 = TI_m + N;$$

where $N$ is a nilpotent matrix, and $a_0 \neq 0$. This shows moreover that $\Phi$ is injective, as in the case of a Drinfeld module (which is just a $T$-module having dimension 1).

**Definition 2.** A $T$-module with dimension 1 is called a **Drinfeld module**. A very interesting case of a Drinfeld module of rank 1 (for a Drinfeld module, rank and degree coincide) is that of the **Carlitz module**:

$$C(T)(\tau) := T + \tau.$$

**Definition 3.** The set of torsion points in $A$ is:

$$A_{\text{tors.}} := \{\bar{x} \in A, \exists a(T) \in A \setminus \{0\}, \Phi(a)(\bar{x}) = \bar{0}\}.$$  

**Definition 4.** A sub-$T$-module $B$ of a $T$-module $A$ is a reduced connected algebraic sub-group of $(A, +)$ such that $\Phi(T)(B) \subset B$.

**Remark 1.** We shall call from now **dimension** of a sub-$T$-module $B$ of $A$, the dimension of $B$ as an algebraic variety. We remark that the dimension of any non-trivial sub-$T$-module $B < A$ (in other words, such that $B \neq A, 0$) is strictly minor than the dimension of $A$. Thus, a sub-$T$-module of a $T$-module is not in general a $T$-module.

A result of I. Barsotti, cfr. [Ba], Theorem 3.3, allows us to explain why any sub-$T$-module of dimension $d$ of some $T$-module $A = (G^m_a, \Phi)$ is isomorphic to $G^d_a$ as an algebraic group, if it is absolutely irreducible inside $C$. It is indeed possible to show that such an algebraic group has to be necessarily the locus of zeros of a certain number of additive polynomials, being then a regular algebraic variety (see Theorem 11). As the hypotheses of the Theorem indicated in the reference are respected, it follows that such an algebraic group is isomorphic to the direct product $G^t_a \times G^s_m$, for some $t$ and $s$ convenient. Because it is easy to show that no power of $G_m$ could be isomorphic to any subgroup of $G^m_a$ we have that $t = d$ and $s = 0$.

**Definition 5.** Let $A = (G^m_a, \Phi)$ be a $T$-module defined on the field $\mathcal{F} \subset \overline{k}$. We note:

$$\text{Hom}_\mathcal{F}(A, \mathbb{G}_a);$$

the group of the $\mathbb{F}_q$-additive group homomorphisms from $A$ to $\mathbb{G}_a$, defined on $\mathcal{F}$. We call $\text{Hom}_\mathcal{F}(A, \mathbb{G}_a)$ the $T$-**motive** associated to $A$ (see [Goss], paragraph 5.4, for more details).
As a consequence of such a definition one can prove that:
\[ \text{Hom}_F(\mathcal{A}, \mathbb{G}_a) \simeq \mathcal{F}\{\tau\}^m. \]

**Definition 6.** Calling \( F \subset \mathbb{k} \) the field generated on \( k \) by the entries of the coefficient matrices of \( \Phi(T) \), the **rank** of a \( T \)-module \( \mathcal{A} \) is the rank on \( \mathcal{F}[T] \) of the \( \mathcal{F}[T] \)-module \( \text{Hom}_F(\mathcal{A}, \mathbb{G}_a) \). We say that a \( T \)-module \( \mathcal{A} \) is **abelian** if \( \text{Hom}_F(\mathcal{A}, \mathbb{G}_a) \) has finite rank.

One can prove (cfr. [Goss], Theorem 5.4.10) that if \( \mathcal{A} \) is abelian, \( \text{Hom}_F(\mathcal{A}, \mathbb{G}_a) \) is also free as a \( \mathcal{F}[T] \)-module.

**Remark 2.** The functor \( \text{Hom}_F(\cdot, \mathbb{G}_a) \) gives rise to an anti-equivalence between the categories of abelian \( T \)-modules and free and finite-rank \( T \)-motives (cfr. [Goss], Theorem 5.4.11). These two categories are both abelian.

One can see that in particular each sub-\( T \)-module as well as the quotients of every abelian \( T \)-module are still abelian. The canonical embedding:
\[ i : \mathcal{B} \hookrightarrow \mathcal{A}; \]
induces the morphism:
\[ i^* : \text{Hom}_F(\mathcal{A}, \mathbb{G}_a) \rightarrow \text{Hom}_F(\mathcal{B}, \mathbb{G}_a); \]
trivially. We also know that, if \( m_\mathcal{A} := \dim_C(\mathcal{A}) \) and \( m_\mathcal{B} := \dim_C(\mathcal{B}) \), one thus has that \( m_\mathcal{B} < m_\mathcal{A} \) if \( \mathcal{B} \neq \mathcal{A} \). As \( \mathcal{B} \simeq \mathbb{G}_a^{m_\mathcal{B}} \) and \( \mathcal{A} \simeq \mathbb{G}_a^{m_\mathcal{A}} \), we can see \( \text{Hom}_F(\mathcal{B}, \mathbb{G}_a) \subset \mathcal{F}\{\tau\}^{m_\mathcal{B}} \) acting on \( \mathbb{G}_a^{m_\mathcal{A}} \) as a set of elements inside \( \mathcal{F}\{\tau\}^{m_\mathcal{A}} \) in the obvious way, and this action could be easily extended to \( \mathcal{A} \). Therefore, one can see that \( i^* \) is surjective. By the same way we remark that the quotient homomorphism:
\[ j : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}; \]
induces in the dual category the following one:
\[ j^* : \text{Hom}_F(\mathcal{A}/\mathcal{B}, \mathbb{G}_a) \rightarrow \text{Hom}_F(\mathcal{A}, \mathbb{G}_a); \]
which is injective. As a \( T \)-module is abelian if and only if its \( T \)-motive is finitely generated on \( \mathcal{F}[T] \) the result immediately follows. We also recall that if a \( T \)-module is abelian, then it is free, so the others properties defining an abelian category follow.
**Definition 7.** Let $\mathcal{A}$ be an abelian $T$–module. The **exponential function** of $\mathcal{A}$ is the following morphism:

$$\overline{\tau} : \text{Lie}(\mathcal{A}) \to \mathcal{A};$$

such that, for each $\overline{z} \in \text{Lie}(\mathcal{A})$, we have that:

$$\overline{\tau}(d\Phi(T)\overline{z}) = \Phi(T)(\overline{\tau}(\overline{z}));$$

as described in [Goss], Definition 5.9.7. It is known (see [Goss], section 5) that such a morphism is $\mathbb{F}_q$–linear, a local homeomorphism and (see Definition 11) $\mathcal{F}$–entire too.

One can proves (cfr. [D], page 70), that if $\mathcal{B} < \mathcal{A}$ and then $\text{Lie}(\mathcal{B}) \subseteq \text{Lie}(\mathcal{A})$, the restriction of $\overline{\tau}_\mathcal{A}$ to $\text{Lie}(\mathcal{B})$ is precisely the exponential function $\overline{\tau}_\mathcal{B}$ associated to $\mathcal{B}$. If $\mathcal{A}$ is abelian, if we note:

$$\Lambda := \text{Ker}(\overline{\tau});$$

this kernel is an $\mathcal{A}$–lattice inside $\text{Lie}(\mathcal{A})$ and its $\mathcal{A}$–rank is minor or equal to the rank of $\mathcal{A}$, cfr. [Goss], Lemma 5.9.12.

**Lemma 1.** The following properties are equivalent for an abelian $T$–module $\mathcal{A} = (\mathbb{G}_m^n, \Phi)$. We call $\rho(\Lambda_\mathcal{A})$ the $\mathcal{A}$–rank of the lattice associated to $\mathcal{A}$ as the kernel of the exponential function $\overline{\tau} : \text{Lie}(\mathcal{A}) \to \mathcal{A}$. We also call $\rho(\mathcal{A})$ the rank of $\mathcal{A}$.

1. $\rho(\Lambda_\mathcal{A}) = \rho(\mathcal{A})$;
2. The exponential function $\overline{\tau} : \text{Lie}(\mathcal{A}) \to \mathcal{A}$ is surjective.

**Proof.** See [Goss], Theorem 5.9.14.

**Definition 8.** Let $\mathcal{A} = (\mathbb{G}_m^n, \Phi)$ be an abelian $T$–module. If it respects the two equivalent conditions of the Lemma 1 it is called **uniformizable**.

**Remark 3.** If $\mathcal{B}$ is a sub-$T$–module of the abelian $T$–module $\mathcal{A}$:

$$\overline{\tau}^{-1}(\mathcal{B}) = \text{Lie}(\mathcal{B}).$$

**Proof.** We remark that each $T$–module is a smooth algebraic variety. Its dimension is the same of its tangent space tangent (in any point), as an affine space on $\mathbb{C}$. If this $T$–module is the sub-$T$–module $\mathcal{B}$ of $\mathcal{A}$, such a tangent space is canonically isomorphic to $\text{Lie}(\mathcal{B})$. As $\overline{\tau}$ is defined on $\text{Lie}(\mathcal{A})$, it is clear that $\overline{\tau}^{-1}(\mathcal{A}) = \text{Lie}(\mathcal{A})$. Concerning $\mathcal{B}$, we just can say,
on the contrary, that $H := \bar{\tau}^{-1}(B) \subset \text{Lie}(A)$ is such that $H \supset \text{Lie}(B)$. By [D], Definition 2.1.12, we know that $B \simeq \mathbb{G}_a^{\dim(Lie(B))}$. On the other hand, the exponential function $\tau$ is a local homeomorphism on the same non-archimedean topology given on $A$ and $\text{Lie}(A)$, and this induces by restriction to $H$ a local homeomorphism between $H$ and $\tau(H) = B$. It is therefore possible to construct a local homeomorphism between $H$ and $\text{Lie}(B)$, and this implies that these two objects, looked as affine spaces, have the same dimension on $C$. It follows that $H = \text{Lie}(B)$. We put in evidence the fact that we didn’t need the surjectivity hypothesis on $\tau$ in this proof, nor on its restriction to $\text{Lie}(B)$.

\begin{remark}
The sub-$T$–modules of an abelian, uniformizable $T$–module are uniformizable.
\end{remark}

\begin{proof}
Let $B$ be a sub-$T$–module of some abelian and uniformizable $T$–module $A$. We have already seen that $\bar{\tau}^{-1}(B) = \text{Lie}(B)$. As $\tau$ is the exponential function defined on $\text{Lie}(A)$ (and not just only on $\text{Lie}(B)$) if its restriction to $\text{Lie}(B)$ wouldn’t be surjective, it would exists an element $x \in B$ such that $\bar{\tau}^{-1}(x) \cap \text{Lie}(B) = \emptyset$, when $\bar{\tau}^{-1}(x) \subset \text{Lie}(A)$. As $\bar{\tau}^{-1}(x) \subset \bar{\tau}^{-1}(B) = \text{Lie}(B)$ we easily get a contradiction.

\end{proof}

2.1 A conjecture

An analogous formulation of Theorem 1 has been proved by T. Scanlon in \cite{Se} for all $T$–modules which are a power of some given Drinfeld module, basing such a proof on Model Theory, extending so a previous result of L. Denis (see \cite{Den2}), where the same conclusion was proven on a power of a Drinfeld module, under an additional hypothesis on this one, which will be fully described after (see hypothesis 3). The Scanlon’s result is the following one.

\begin{theorem}
Let $A := (\mathbb{D}^m, \Phi)$ be a power of some given Drinfeld module $(\mathbb{D}, \Phi)$ defined over a field $F$. Let $X$ be an irreducible algebraic sub-variety of $A$. If $X(F)_{\text{tors}}$ is Zariski-dense in $X$, then $X$ is the translate of some sub-$T$–module of $A$ by a torsion point.
\end{theorem}

A $T$–module of such a form is easily abelian and uniformizable. This leads us to try to prove a similar result, using the new techniques introduced in \cite{PZ} for an abelian and uniformizable $T$–module $A$. We’ll firstly remark here that a naïf transposition of Scanlon’s Theorem in this more general case is not necessarily true. We’re about to use the following notion:
Definition 9. Let $A = (\mathbb{G}_m^m, \Phi)$ be a $T-$module with dimension $m$. We say that it is simple if it does not admit any non-trivial sub-$T-$module (in other words, different from itself or 0). Let $a(T) \in A \setminus \mathbb{F}_q$. We call a reduced, connected algebraic sub-group of $A$ a sub-$a(T)-$module if it is a sub-$\mathbb{F}_q[a(T)]-$module with rapport to the action of $\Phi$.

We consider the $T-$module with dimension 2 defined by the tensor power $C \otimes^2 = (\mathbb{G}_m^2, \Phi)$ of the Carlitz module $C$, introduced by G. Anderson and D. Thakur in [AT]. We suppose that $q = 2$. Such a $T-$module is then under the following form:

$$\Phi(T) \begin{pmatrix} X \\ Y \end{pmatrix} (\tau) = \begin{pmatrix} T & 1 \\ 0 & T \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tau.$$ 

We can then show that:

$$\Phi(T^2) \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} T^2X + X^2 \\ T^2Y + (T + T^2)X^2 + Y^2 \end{pmatrix}.$$ 

The algebraic sub-group $0 \times \mathbb{G}_a$ of $C \otimes^2$ is then a sub-$T^2-$module, but not a sub-$T-$module. In general, the tensor power $C \otimes^m$ of the Carlitz module $C$ is always a simple, abelian and uniformizable $T-$module (see [Yu], Proposition 1.2), but one can prove that it owns sometimes a lot of non-trivial sub-$T^j-$modules ($j$ depending on $m$ and $q$) so it provides infinitely many counter-examples to a naïf generalization of Theorem 2 as each one of these sub-$T^j-$modules is in general an algebraic sub-variety of $C \otimes^m$ containing infinitely many torsion points. For example, $C \otimes^2$ is a simple $T-$module containing the algebraic sub-variety $0 \times \mathbb{G}_a$, which is in particular a sub-$T^2-$module and contains infinitely many torsion points.

We extend therefore the class of algebraic sub-modules of $A$ which are necessary in order to find out a plausible formulation of Manin-Mumford’s conjecture in our more general case.

Definition 10. Let $A = (\mathbb{G}_a^m, \Phi)$ be a general $T-$module. Let $B$ be a sub-ring of $A$. We call sub-$B-$module of $A$ any reduced connected algebraic sub-group $B$ of $\mathbb{G}_a^m$ such that:

$$\Phi(a(T))(\mathcal{B}) \subseteq \mathcal{B}, \ \forall a(T) \in B.$$ 

We say that a sub-set of $A$ is a torsion class if it is under the following form:

$$\overline{\mathcal{X}} + \mathcal{B};$$ 

where $\overline{\mathcal{X}} \in A_{\text{tors}}$, and it exists $B$ a sub-ring of $A$ such that $\mathcal{B}$ is a sub-$B-$module of $A$. 

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Theorem 5. Let \( A = (G_m^a, \Phi) \) be an abelian uniformizable \( T \)-module with dimension \( m \). It exists then a number \( j(A) \in \mathbb{N} \setminus \{0\} \) only depending on \( A \) such that for each \( a(T) \in A \setminus \mathbb{F}_q \) we have that every sub-\( a(T) \)-module of \( A \) is a sub-\( T^{j(A)} \)-module (in other words, for each \( B \) sub-ring of \( A \), every sub-\( B \)-module of \( A \) is a sub-\( T^{j(A)} \)-module).

Proof. Let \( N \) be the nilpotent matrix associated to the differential \( d\Phi(T) \) of \( A \) introduced in Definition 1. Let \( n(A) \in \mathbb{N} \setminus \{0\} \) be its order. Let \( B \) be a sub-\( a(T) \)-module of \( A \) for some \( a(T) \in A \setminus \mathbb{F}_q \). It is then a reduced connected algebraic sub-group of \( G_m^a \). Then, in order to prove the statement it will be sufficient to show that \( \text{Lie}(B) \) is stabilized by the action of \( d\Phi(T^{j(A)}) \) for some convenient \( j(A) \in \mathbb{N} \setminus \{0\} \). We choose therefore:

\[ j(A) \geq p^{n(A)}. \]

It is easy to see then that:

\[ d\Phi(T^{j(A)}) = T^{j(A)}I_m; \]

which stabilizes every vector sub-space of \( \text{Lie}(A) \) over \( \mathbb{C} \). So in particular, it stabilizes \( \text{Lie}(B) \) too.

We remark that we just proved the existence of such a number \( j(A) \in \mathbb{N} \setminus \{0\} \), but we did not found actually its minimal possible value. We can anyway remark that such a value is 1 in the case of \( A \) is a power of a Drinfeld module, according with Scanlon’s Theorem. We show this using the following Thiery’s Theorem (see [T]).

Theorem 6. Let \( \mathbb{D}^m = (G_m^a, \Phi^m) \) the \( m \)-th power of a Drinfeld module \( \mathbb{D} = (G_a, \Phi) \) of which we call \( \mathcal{F} \) its coefficients field. It exists a bijective correspondance between the family of all sub-\( T \)-modules of \( \mathbb{D}^m \) and the family of the vector sub-spaces of \( \text{Lie}(\mathbb{D}^m) \) which are \( \text{End}_\mathcal{F}(\Phi) \)-rational. This correspondance is given by the exponential function:

\[ \tau : \text{Lie}(\mathbb{D}^m) \to \mathbb{D}^m; \]

\[ V \mapsto \tau(V). \]

Moreover, we have that the dimension of any sub-\( T \)-module of \( \mathbb{D}^m \) and of the \( \text{End}_\mathcal{F}(\Phi) \)-rational vector sub-space of \( \text{Lie}(\mathbb{D}^m) \) corresponding to it are the same.

Proof. See [T], Theorem at page 33.
Let \( B \) be therefore a sub-\( a(T) \)-module of \( D^m \). We assume that the Drinfeld module \( D \) has rank \( d \). We define:

\[
T' := a(T);
\]

and:

\[
\Psi(T') := \Phi(a(T));
\]

\( B \) is therefore a \( T' \)-module which is the \( m \)-th power of \( s \mathbb{F}_q[T'] \)-Drinfeld module of rank \( d \deg_T(a(T)) \). By Theorem 6 it is fully described by \( s = m - \dim(B) \) linear equations under the following form:

\[
\sum_{j=1}^m P_{ij}(\tau) X_j = 0;
\]

where \( P_{ij}(\tau) \in \text{End}_\mathbb{F}(\Psi) \) for each \( i = 1, \ldots, s \) and each \( j = 1, \ldots, m \). As \( \Phi(T) \in \text{End}_\mathbb{F}(\Psi) \) also (in fact, \( \Phi(T) \circ \Psi(T') = \Phi(T) \circ \Phi(a(T)) = \Phi(a(T)) \circ \Phi(T) = \Psi(T') \circ \Phi(T) \)) and \( \text{End}_\mathbb{F}(\Psi) \) is a commutative ring (because \( D \) has characteristic 0) one sees that \( B \) is actually a sub-\( -T \)-module of \( D^m \) as it is stabilized by the action of \( \Phi(T) \).

Each sub-\( a(T) \)-module of \( D^m \) (\( m \)-th power of a Drinfeld module) is then for each \( a(T) \in A \setminus \mathbb{F}_q \) a sub-\( -T \)-module too.

If \( A \) is absolutely simple (this means that it does not contain any non-trivial sub-\( -T \)-module for each \( j \in \mathbb{N} \setminus \{0\} \)) we fix \( j(A) = 1 \).

We also remark that this study on powers of some Drinfeld module provides an example of a class of \( -T \)-modules \( A \) such that for each \( i \in \mathbb{N} \setminus \{0\} \) we have that \( j(A^i) = j(A) \). One may wonder if it is a general phenomenon (we hope to come again later on such a question).

This would leads us to a more general formulation of Manin-Mumford conjecture on abelian and uniformizable \( -T \)-modules, where we propose to show that, up to a finite number, the torsion points of \( A \) could be shared in a finite number of sub-\( -T^j(A) \)-modules.

Such a modification is however not yet sufficient to completely exclude other counter-examples. The nice one which follows has been suggested by Laurent Denis. We consider the following case of a product of two non-isogeneous Drinfeld modules. We assume that \( q = 2 \). Let \( C(T)(\tau) = T + \tau \)
the Carlitz module \( \mathbb{D}_1 \), where \( \tau \) is the Frobenius automorphism over \( \overline{\mathbb{F}_2} \). We define the Drinfeld module \( \mathbb{D}_2 \) as it follows:

\[
C_{(2)}(T)(\tau) := T + (T^{1/2} + T)\tau + \tau^2.
\]

One can see then that for each \( z \in \mathbb{C} \) we have that:

\[
C_{(2)}(T)(\sqrt{z}) = \sqrt{C(T^2)(z)}.
\]

The product:

\[
\Phi(T) \begin{pmatrix} X \\ Y \end{pmatrix} := \begin{pmatrix} C(T)(X) \\ C_{(2)}(T)(Y) \end{pmatrix};
\]

is then a \( T \)-module as in Definition 1 and for each \( z \in \mathbb{C}_{\text{tors}} \) we have that \((z, z^{1/2}) \in (\mathbb{D}_1 \times \mathbb{D}_2)_{\text{tors}}\). The algebraic variety \( X = Y^2 \) contains then all these torsion points and, as \( C(T^j)(Y^2) \neq (C_{(2)}(T^j)(Y))^2 \) for each \( j \in \mathbb{N} \setminus \{0\} \), it is not stabilized by the action of \( \Phi(T^j) \). It can not be then a sub-\( T^j \)-module of \((\mathbb{D}_1 \times \mathbb{D}_2, \Phi)\) for any \( j \in \mathbb{N} \setminus \{0\} \). As this variety has \( C \)-dimension 1, it can not admit non-trivial sub-\( T^j \)-modules either. We remark that the same counter-example may be repeated identically for any \( q \), replacing the Carlitz module \( C \) by a generic Drinfeld module \( \mathbb{D}_1 = (\mathbb{G}_a, \Phi_1) \) and \( C_{(2)} \) by the Drinfeld module \( \mathbb{D}_2 = (\mathbb{G}_a, \Phi_2) \) obtained as the \( 1/q^s \)-th root of the coefficients of \( \Phi_1(T^q^s) \), which would define an infinite class of bad cases.

We remark that on the contrary the hypothesis that the Drinfeld modules of such a product would be two-by-two isogeneous makes an algebraic variety on a similar form stabilized by the action of \( \Phi(T) \). In fact, assuming that \( m = 2 \), if \( P(\tau) \in \text{Hom}_F(\Phi_1, \Phi_2) \setminus \{0\} \) is an isogeny between the two Drinfeld modules, \( \Phi_2 \) will be then ”\( P(\tau) \)-th root” of \( \Phi_1 \). In other words, \( \Phi_1(T)(\tau) \circ P(\tau) = P(\tau) \circ \Phi_2(T)(\tau) \) and the torsion points of \( \Phi_1(T) \times \Phi_2(T) \) are exactly all those which takes the shape \((P(\tau)(z), z)\), where \( z \) is a torsion point with rapport to \( \Phi_2 \). The algebraic variety \( X = P(\tau)(Y) \) which contains all the torsion points is then stabilized by the action of \( \Phi_1(T) \times \Phi_2(T) \).

We consider then the \( T \)-module \( \mathcal{A} := (\mathbb{D}_1 \times \ldots \times \mathbb{D}_m, \Phi) \), where the Drinfeld modules \((\mathbb{D}_1, \Phi_1), \ldots, (\mathbb{D}_m, \Phi_m)\) are not isogeneous two-by-two. By Definition 1, we remark that if the coefficient matrix \( a_d \) is invertible, all these Drinfeld modules necessarily have the same rank \( d \), which is impossible by the hypothesis. An analogue of the situation that we treated in the previous example could never be repeated for no choice of \( \Phi \) or \( q \) because \( \Phi(q^s) \) (the Drinfeld module obtained replacing the coefficients of its \( q^s \)-th iteration \( \Phi(T^q^s) \)
by their $q^e$-th root) has degree $q^{ed}$. The torsion points of $\mathcal{A}$ will not be contained in an algebraic variety as the one we saw in the previous example.

If we consider on the contrary a product of $m$ Drinfeld modules isogenous, as they have same rank we obtain a $T-$module which verifies the condition to have its leading coefficient invertible. There is this time a relation between the torsion points of the Drinfeld modules. We assume to simplify that $m = 2$. Our $T-$module is then the product $D_1 \times D_2$ of two Drinfeld modules $D_1 = (\mathbb{G}_a, \Phi_1)$ and $D_2 = (\mathbb{G}_a, \Phi_2)$ isogenous. If $P(\tau) \in \text{Hom}_F(\Phi_1, \Phi_2) \setminus \{0\}$ is an isogeny between the two Drinfeld modules, $\Phi_2$ will then be a "$P(\tau)$-th root" of $\Phi_1$. In other words, $\Phi_1(T)(\tau) \circ P(\tau) = P(\tau) \circ \Phi_2(T)(\tau)$ and the torsion points of $\Phi_1(T) \times \Phi_2(T)$ contain all those under the form $(P(\tau)(z), z)$, where $z$ is a torsion point with rapport to $\Phi_2$. The algebraic variety $X = P(\tau)(Y)$ which contains all these torsion points is then easily stabilized by the action of $\Phi_1(T) \times \Phi_2(T)$. This shows then that an algebraic link between the torsion points of some $T-$module is not necessarily an obstacle to the fact that this one satisfies this version of Manin-Mumford conjecture in $T-$modules case.

In such cases when we consider a $T-$module which is a product of finitely many Drinfeld modules having the same rank, nothing seems to be an obstacle to a result of Manin-Mumford type. In such situations the leading coefficient of this $T-$module is invertible. As such a direct hypothesis of this coefficient would let us lose a lot of good cases, as for example those of a tensor power of the Carlitz module, we ask more generally that it exists a number $i \in \mathbb{N} \setminus \{0\}$ such that the leading coefficient $a'_{id}$ of the $i-$th iterated $\Phi(T^i)(\tau)$ of $\Phi(T)(\tau)$ is invertible. As it is easy to see, this does not involve any change in the argument. On the other hand, we see for example that a tensor power of a Carlitz module respects such an hypothesis. We can also remark that this hypothesis does actually implies the condition on the $T-$module to be abelian, as we show in the following Theorem.

**Theorem 7.** Let $\mathcal{A} = (\mathbb{G}_a^m, \Phi)$ be a $T-$module defined over the field $\mathcal{F} \subset \overline{k}$, such that it exists a number $i \in \mathbb{N} \setminus \{0\}$ such that the leading coefficient of $\Phi(T^i) \in \mathcal{F}^{m, m}\{\tau\}$ is an invertible matrix. Therefore, $\mathcal{A}$ is abelian.

**Proof.** By the canonical isomorphism:

$$\text{Hom}_\mathcal{F}(\mathcal{A}, \mathbb{G}_a) \simeq \mathcal{F}\{\tau\}^m;$$

we consider a morphism $f \in \text{Hom}_\mathcal{F}(\mathcal{A}, \mathbb{G}_a)$ as an element $f(\tau) \in \mathcal{F}\{\tau\}^m$. As the leading coefficient of $\Phi(T^i)$ is invertible and the Ore algebra $\mathcal{F}\{\tau\}$ is a
(non commutative) ring endowed of the right division algorithm (see [Goss], Proposition 1.6.2), it is possible to divise on right by \( \Phi(T^i) \) each element of \( \mathcal{F}\{\tau\}^m \), knowing that the coefficients of such an object are vectors in \( \mathcal{F}^m \) while those of \( \Phi(T^i) \) are matrices in \( \mathcal{F}^{m,m} \). In fact, an invertible matrix in \( \mathcal{F}^{m,m} \) also divise (on right) each element of \( \mathcal{F}^m \), and this fact allows the euclidean division. The algebra \( \mathcal{F}\{\tau\}^m \) could then be shared in \( mt \tilde{d} \) (where \( \tilde{d} \) is the degree of \( \Phi(T) \) as an additive polynomial in \( \tau \)) division classes modulo \( \Phi(T^i) \), which is equivalent to say that \( \text{Hom}_\mathcal{F}(\mathcal{A}, \mathbb{G}_a) \simeq \mathcal{F}\{\tau\}^m \) is generated by \( mt \tilde{d} \) elements as a \( \mathcal{F}[T] \)-module.

We state then a \( T \)-modules version of Manin-Mumford conjecture as it follows.

**Conjecture 1.** Let \( \mathcal{A} = (\mathbb{G}_a^n, \Phi) \) be a uniformizable \( T \)-module of dimension \( m \) and rank \( d \), such that it exists \( i \in \mathbb{N} \setminus \{0\} \) such that the leading coefficient matrix of \( \Phi(T^i) \) is invertible. It exists then \( j(\mathcal{A}) \in \mathbb{N} \setminus \{0\} \) such that we have the following property. Let \( X \) be an irreducible algebraic sub-variety of \( \mathcal{A} \), such that \( \text{dim}_\mathbb{C}(X) > 0 \), defined over \( \overline{k} \). If \( X \) does not contain torsion classes with dimension \( > 0 \), \( X \) contains at most finitely many torsion points of \( \mathcal{A} \).

### 2.2 Implicit Function Theorem

It is known that the real analytic classic case of the Implicit Function Theorem, which analyse the zero locus of a certain number of smooth real functions under the form:

\[
f : \mathbb{R}^n \to \mathbb{R};
\]

needs in a key passage the total order relation that one has on \( \mathbb{R} \) by the usual absolute value. As we don’t have such a relation on non-archimedean fields anymore, we’ll give a new formulation of this Theorem in the ultrametric case.

**Definition 11.** Let \( n, i \in \mathbb{N} \setminus \{0\} \), we define:

\[
\Lambda_n(i) := \{ (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n, \sum_{j=1}^n \mu_j = i \}.
\]

For each \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n \), we call:

\[
|\mu| := \sum_{j=1}^n \mu_j.
\]

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Let $K$ be some non-archimedean complete discrete field and let $L \subseteq K$ be a sub-field of $K$. Given $\mu = (\mu_1, ..., \mu_n) \in \Lambda_n(i)$ and a vector $\mathbf{z} = (z_1, ..., z_n) \in K^n$, we define:

$$\mathbf{z}^\mu = \prod_{j=1}^n z_j^{\mu_j}.$$ 

Let $U \subset K^n$ be an open subset of $K^n$ in the non-archimedean topology induced by the one we’ve chosen on $K$. A function:

$$f : U \to K;$$

is said $L$–analytic if and only if:

$$f(\mathbf{z}) = \sum_{i \geq 0} \sum_{\mu \in \Lambda_n(i)} a_\mu \mathbf{z}^\mu, \ \forall \mathbf{z} \in U;$$

where $a_\mu \in L$ for every $\mu \in \Lambda_n(i)$ and every $i \geq 0$. In particular, the formal power series on $\mathbf{z}$ is convergent on every $\mathbf{z} \in U$ to $f(\mathbf{z})$. Let $\mathbf{z}_0 \in U(L) = L^n \cap U$. The following change of variables:

$$\mathbf{z} \mapsto \mathbf{z} + \mathbf{z}_0;$$

induces:

$$f(\mathbf{z}) = \sum_{i \geq 0} \sum_{\mu \in \Lambda_n(i)} a_\mu \mathbf{z}^\mu = \sum_{i \geq 0} \sum_{\mu \in \Lambda_n(i)} b_\mu (\mathbf{z} - \mathbf{z}_0)^\mu;$$

where $b_\mu \in L$ are some appropriate coefficients. Such coefficients are called the hyperderivatives $\frac{\partial f(\mathbf{z}_0)}{\partial \mathbf{z}^\mu}$ of $f$ in $\mathbf{z}_0$ on the direction given by the vector $\mu$. We also call $L$–analytic function on $U$ any vector $L$–analytic function on the following form:

$$f : U \to K^m;$$

for each $m \in \mathbb{N} \setminus \{0\}$. We call $f$ an $L$–entire function if $U = K^n$. An $L$–entire function is in other words a function under this form:

$$f : K^n \to K;$$

$$\mathbf{z} \mapsto \sum_{i \geq 0} \sum_{\mu \in \Lambda_n(i)} a_\mu \mathbf{z}^\mu;$$

convergent to $f(\mathbf{z})$ in each point $\mathbf{z} \in K^n$.

Remark 5. Let $\mathcal{A} = (\mathbb{A}^m, \Phi)$ be an abelian uniformizable $T$–module, defined on $\mathcal{F} \subset \mathbb{F}$. We define the field $k_{\infty}(\Phi) := k_\infty \mathcal{F}$, finite extension of $k_{\infty}$. The associated exponential function:

$$\mathbf{e} : \text{Lie}(\mathcal{A}) \to \mathcal{A};$$
is therefore \( F \)-entire and in particular \( k_{\infty}(\Phi) \)-entire, by [Goss], Lemma 5.9.3.

We recall the following property, whom the proof is a mechanical application of the previous definitions.

**Proposition 1.** Let \((K,v)\) be a non-archimedean valuation field. A formal power series:

\[ \sum_{i \geq 0} a_i z^i; \]

defined on \( K \) converge on the polydisc \( B_r(K) := \{ x \in K, |x|_v \leq r \} \), for \( r > 0 \), if and only if:

\[ \lim_{i \to +\infty} |a_i|_v r^i = 0. \]

In higher dimension, a series:

\[ \sum_{i \geq 0} \sum_{\mu(i) \in \Lambda_n(i)} a_{\mu(i)} z^\mu; \]

is convergent in the polydisc \( B^c_r(K) \) if and only if:

\[ \lim_{i \to +\infty} |a_{\mu(i)}|_v r^i = 0; \]

where \( \mu(i) \) is the multi-index such that the sum of its \( n \) components is \( i \).

**Definition 12.** Let \( K \) be a complete field with rapport to a certain valuation and let \( L \subseteq K \) be a sub-field of \( K \). Let \( U \subseteq K^n \) be an open subset of \( K^n \) in the topology induced by the \( 1/T \)-adic absolute value. A subset \( Z \subseteq U \) is \( L \)-analytic inside \( U \) if it exists a finite set of \( L \)-analytic functions \( \{f_1, ..., f_r\} \) defined on \( U \) and taking their values inside \( K \), such that:

\[ Z = \{ \zeta \in U, f_i(\zeta) = 0, \forall i = 1, ..., r \}. \]

We’ll say that \( Z \) is \( L \)-entire if it exists a finite set of \( L \)-entire functions \( \{f_1, ..., f_r\} \) defined on \( K^n \) and taking values inside \( K \), such that:

\[ Z = \{ \zeta \in K^n, f_i(\zeta) = 0, \forall i = 1, ..., r \}. \]

**Theorem 8.** Let \( K \) be a complete field, containing \( k_{\infty} \) and contained in \( \mathbb{C} \). Let \( F: K^n \to K \) be a \( K \)-analytic map on a certain open set of \( K^n \), where \( n > 1 \). Let \( Z = Z(F) \) be the zero locus of \( F \) in \( K^n \). We assume that \( Z \neq \emptyset \). Let \( \zeta_0 \in Z \) and let \( \zeta_0^\prime \) be its projection on its \( n-1 \) first components. Assume that:

\[ \frac{\partial F(\zeta_0)}{\partial z_n} \neq 0. \]
It exists, then, an open neighborhood \( U_{z_0} \subset K^n \) of \( z_0 \), and we call \( U_{z_0}^* \subset K^{n-1} \) its projection on the first \( n-1 \) component (which obviously contains \( z_0^* \)), and an analytic function:

\[
f : U_{z_0}^* \to K;
\]
such that, for each \( z \in U_{z_0} \), expressing this one as:

\[
z = (z^*, z_n) \in K^{n-1} \times K;
\]
we have:

\[
z_n = f(z^*).
\]

**Proof.** A first step consists in proving a formal Inverse Function Theorem, and so a formal Implicit Function Theorem. We may assume without loss of generality that \( z_0 = 0 \). It follows that \( F(0) = 0 \). We express, in the neighborhood \( U \) of \( 0 \), \( F \) under the following form:

\[
F(z) = \sum_{i \geq 1} \sum_{\mu \in \Lambda_n(i)} a_\mu z^\mu.
\]

We define:

\[
G \in (K[[z_1, \ldots, z_n]])^n;
\]
which describes a \( K \)-analytic function:

\[
G : K^n \to K^n;
\]
such that:

\[
G(z) = (z^*, F(z));
\]
where \( z^* \) is the projection of \( z \) on its \( n-1 \) first components in \( K^{n-1} \). We’re looking for an inverse function:

\[
H : K^n \to K^n;
\]
annihilating itself at \( \overline{0} \) with order 1, having the same form than \( G \), such that:

\[
H \circ G = G \circ H = 1.
\]
In other words, we ask that, for each \( \overline{z} \) formal vector with \( n \) components, we shall have:

\[
(G \circ H)(\overline{z}) = (z^*, (F \circ H)(\overline{z})) = \overline{z}.
\]
The formal series \( H \) may be expressed then as it follows:

\[
H(\overline{z}) = (z^*, h(\overline{z}));
\]
with:

\[ h : K^n \to K; \]

such that:

\[ F(H(\bar{u})) = F(\bar{u}, h(\bar{u})) = u_n; \]

and at the same time such that:

\[ h(G(\bar{z})) = h(\bar{z}, F(\bar{z})) = z_n. \]

We remark that:

\[ h(G(0)) = 0; \]

so we have the annihilation at 0 of \( H \), as \( G(0) = 0 \), because 0 is a zero of \( F \). It follows that:

\[ F(H(0)) = 0; \]

also. We then express \( h \) under this form:

\[ h(u) = \sum_{i \geq 1} \sum_{\eta \in \Lambda_n(i)} b_{i,\eta} \bar{u}_i^\eta. \]

Under the condition:

\[ F(u^n, h(u)) = u_n; \]

we then obtain:

\[ \sum_{i \geq 1} \sum_{\eta \in \Lambda_n(i)} a_{i,\eta} H(\bar{u})^\eta = \sum_{j=1}^{n-1} a_j u_j + a_n h(\bar{u}) + \sum_{i \geq 2} \sum_{\eta \in \Lambda_n(i)} a_{\eta}(\bar{u}, h(\bar{u}))^\eta = u_n. \]

We develop the passages until the order 3 case in order to give an idea of the methods we use to compute the coefficients \( b_{i,\eta} \) of \( h \). We have:

\[ F(H(\bar{u})) = \sum_{i=1}^{n-1} a_i u_i + a_n \left( \sum_{i=1}^{n} b_i u_i + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} u_i u_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} b_{i,j,k} u_i u_j u_k + \ldots \right) + \]

\[ + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{i,j} u_i u_j + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{i,n} u_i \left( \sum_{i=1}^{n} b_i u_i + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} u_i u_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} b_{i,j,k} u_i u_j u_k + \ldots \right) + \]

\[ + a_{n,n} \left( \sum_{i=1}^{n} b_i u_i + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} u_i u_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} b_{i,j,k} u_i u_j u_k + \ldots \right)^2 + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} a_{i,j,k} u_i u_j u_k + \]

\[ + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{i,j,n} u_i u_j \left( \sum_{i=1}^{n} b_i u_i + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} u_i u_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} b_{i,j,k} u_i u_j u_k + \ldots \right) + \]

\[ + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{i,j,n} u_i u_j \left( \sum_{i=1}^{n} b_i u_i + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{i,j} u_i u_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} b_{i,j,k} u_i u_j u_k + \ldots \right) + \]

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We associate now to each monomial $\mu$ its corresponding coefficient within the formal equality:

$$F(H(\pi)) - u_n = 0.$$ 

Such an association is the following:

| $u_i$           | $a_n b_i + a_i$ | $\forall i = 1, \ldots, n - 1$ |
|-----------------|-----------------|---------------------------------|
| $u_i u_j$       | $a_n b_{i,j} + a_{i,j}$ | $\forall i, j = 1, \ldots, n - 1$ |
| $u_i u_n$       | $a_n b_{i,n} + a_{i,n} b_n + 2a_{n,n} b_i b_n$ | $\forall i = 1, \ldots, n - 1$ |
| $u_n^2$         | $a_n b_{i,n}^2 + a_{n,n} b_n^2$ |                                    |
| $u_i u_j u_k$   | $a_n b_{i,j} + a_{i,j} + a_{i,n} b_i + a_{j,n} b_j + 2a_{n,n} b_i b_n + a_{i,j,k} + a_{i,n} b_{i,k} + a_{j,n} b_{j,k} + 2(a_{i,n} b_{j,k} + a_{j,n} b_{i,k} + a_{k,n} b_{i,j}) + 3a_{n,n} b_i b_j b_k$ | $\forall i, j, k = 1, \ldots, n - 1$ |
| $u_i u_j u_n$   | $a_n b_{i,j,n} + a_{i,j} + a_{i,n} b_i + a_{j,n} b_j + 2a_{n,n} b_i b_j + a_{i,j,n} b_n + a_{j,n,n} b_i b_n + 2(a_{i,n} b_i b_n + a_{j,n} b_j b_n) + 3a_{n,n} b_i b_j b_n$ | $\forall i, j = 1, \ldots, n - 1$ |
| $u_i u_n^2$     | $a_n b_{i,n} + 2a_{n,n} b_i b_n + a_{i,n} b_{i,n}^2 + 3a_{n,n} b_i b_n + 3a_{n,n} b_n^2$ | $\forall i = 1, \ldots, n - 1$ |
| $u_n^3$         | $a_n b_{n,n} + 2a_{n,n} b_{n,n} + 3a_{n,n} b_n^2$ |                                    |
| ...             | ...             | ...                             |

Now we introduce a new notation for the same general multiple index $\mu \in \mathbb{N}^n$ as an expression as a formal series:

$$E(\mathcal{X}) := \sum_{i \geq 1} \sum_{\mu \in \Lambda_n(i)} c_\mu \mathcal{X}^\mu;$$

may be written also as it follows:

$$E(\mathcal{X}) := \sum_{i=1}^n c_i X_i + \sum_{i_1=1}^n \sum_{i_2=1}^n c_{i_1,i_2} X_{i_1} X_{i_2} + \ldots + \sum_{i_1=1}^n \ldots \sum_{i_k=1}^n c_{i_1,\ldots,i_k} \prod_{h=1}^k X_{i_h} + \ldots$$

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In fact we remark that in this case, even if the formal series is still the same one, the order in which monomials are added each other may change, and the indexes of coefficients may not have a bounded length, in reverse of what one had using the previous notation. We call \( i(\mu) \in \mathbb{N} \setminus \{0\} \) the length \( k \) of the multiple index \( (i_1, ..., i_k) \) in this second notation, corresponding to \( \mu \) in the first one. In other words, such that:

\[
\mathcal{X}^\mu = \prod_{h=1}^{k} X_{i_h}.
\]

We remark that each one of such new multiple indexes \( (i_1, ..., i_k) \) is always such that \( i_1 \leq ... \leq i_k \) by the fact that all products are commutative. The annihilation of all coefficients of the formal series \( F(H(\pi)) - u_n \) leads us to the following system, which we call S1:

\[
\left\{
\begin{array}{l}
a_n b_n = 1 \\
a_n b_i = a_i \\
a_n b_\mu = -P_\mu(\{a_\eta\}, \{b_\mu'\}) \\
\forall |\mu| \geq 2, |\eta| \leq |\mu|, i(\mu') < i(\mu);
\end{array}
\right.
\]

where \( P_\mu \) is for each \( \mu \in \Lambda_n(i), i \geq 2 \), a polynomial with coefficients in \( \mathbb{F}_q \) and such that the sign of each one of its monomials is always positive. Such a system may be solved inductively on the value of \( i(\mu) \). We firstly remark that:

\[
b_n = \frac{1}{a_n};
\]

and that:

\[
b_i = \frac{a_i}{a_n}, \quad \forall i = 1, ..., n - 1.
\]

Such solutions exist because hypothesis ask the condition that \( a_n \neq 0 \). Replacing these solutions in the equations of the form \( a_n b_\mu = -P_\mu(\{a_\eta\}, b_i) \), where \( i(\mu) = 2 \) and \( i = 1, ..., n \), we obtain \( b_\mu \). In general, we replace the solution \( b_\mu' \) in the equation \( a_n b_\mu = -P_\mu(\{a_\eta\}, \{b_\mu'\}) \) for each \( 1 \leq i(\mu') < i(\mu) \).

Repeating the same passages at each step we finally obtain the formal series \( h \) (and therefore \( H \)) we desired. We also remark that \( \frac{\partial}{\partial u_n} h(0) \neq 0 \), as \( b_n \neq 0 \). So, \( h \) respects the same hypothesis, as a formal series, than \( F \). It exists then, repeating the same argument for \( H \), a vector of formal series:

\[
H_1 : K^n \to K^n;
\]

such that:

\[
H \circ H_1 = 1.
\]
Therefore:

\[ G = G \circ H \circ H_1 = H_1; \]

and this proves that:

\[ G \circ H = H \circ G = 1. \]

We finally remark that such a \( H \) is unique, as it has been founded as the only solution of the system \( S_1 \) we described before. If we restrict \( h \) to the projection of \( K^n \) on its \( n - 1 \) first components we obtain a formal series:

\[ \tilde{h} : K^{n-1} \to K; \]

such that:

\[ \tilde{h}(\mathbf{v}) := h(\mathbf{v}, 0); \]

and this also proves the formal Implicit Function Theorem, as:

\[ F(\mathbf{v}, \tilde{h}(\mathbf{v})) = 0; \]

as a formal identity.

The second step is now to show that \( \tilde{h} \) is a convergent series on an open neighborhood of \( \mathbf{0} \), not singular (in other words, different from \( \{\mathbf{0}\} \)). In other words, that the formal series \( \tilde{h} \) has a positive convergence radius. We define then a bounding series:

\[ \mathcal{F} : \mathbb{R}^n \to \mathbb{R}; \]

such that:

\[ \mathcal{F}(X_1, \ldots, X_n) := -\sum_{i=1}^{n-1} A_i X_i + A_n X_n - \sum_{i \geq 2} \sum_{\mu \in \Lambda_n(i)} A_{\mu} \overline{X}^\mu; \]

where \( A_\mu \geq 0 \) for each \( \mu \in \Lambda_n(i), i \geq 1 \), and that:

\[ |a_\mu|^{1/T} \leq A_\mu \ \forall \mu \in \Lambda_n(i), i \geq 2; \]

\[ A_i = |a_i|^{1/T} \ \forall i = 1, \ldots, n. \]

In particular, \( \mathcal{F}(\mathbf{0}) = 0 \) and, as \( A_n = |a_n|^{1/T} \neq 0 \), \( \mathcal{F} \) satisfies the same hypothesis on regularity at \( \mathbf{0} \) than \( F \), but over \( \mathbb{R}^n \). We also define:

\[ \mathcal{G} : \mathbb{R}^n \to \mathbb{R}^n; \]

\[ \mathcal{G}(\mathbf{X}) := (\mathbf{X}, \mathcal{F}(\mathbf{X})). \]
As we treat formal series, the topological structure of fields which is induced by the chosen valuation does not play any role, so we can repeat exactly the same passages as in the case of \( F \) and \( G \), finding then an inverse formal series \( H \) of \( G \) as the series \( H \) was previously with rapport to \( G \). It will take so the following shape:

\[
\overline{H}(y_1, \ldots, y_n) := (Y^*, \overline{h}(Y)) := (y_1, \ldots, y_{n-1}, \sum_{i=1}^{n} B_i y_i + \sum_{i \geq 2} \sum_{\mu \in \Lambda_n(i)} B_{\mu} \overline{Y}^\mu).
\]

We apply the definitions in order to repeat the previous computations in this new situation to express the coefficients of \( h \) in function of those of \( F \).

We obtain then:

\[
\overline{F}(\overline{H}(Y)) = -\sum_{i=1}^{n-1} A_i y_i + A_n (\sum_{i=1}^{n} B_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} y_i y_j + \ldots) +
\]

\[
- \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} A_{ij} y_i y_j - \sum_{i=1}^{n-1} A_{in} y_i (\sum_{i=1}^{n} B_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} y_i y_j + \ldots) +
\]

\[
- A_{nn} (\sum_{i=1}^{n} B_i y_i + \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} y_i y_j - \ldots)^2 - \ldots = y_n.
\]

The coefficients \( B_{\mu} \) of \( \overline{H} \) are then obtained, repeating the same passages already done to find \( h \), as solutions of the following system, which we call \( S_2 \), and that we construct again in the same way than \( S_1 \), where the terms \( P_{\mu} \) are the same polynomials with coefficients in \( \mathbb{F}_q \) that we introduced previously:

\[
\begin{cases}
A_n B_n = 1 \\
-A_i + A_n B_i = 0 & \forall i = 1, \ldots, n - 1 \\
A_n B_{\mu} = P_{\mu}([A_\eta], \{B_{\mu'}\}) & \forall |\mu| \geq 2, |\eta| \leq |\mu|, i(\mu') < i(\mu);
\end{cases}
\]

We finally obtain:

\[
|a_i b_n|^{1/T} = |b_i|^{1/T} \forall i = 1, \ldots, n - 1.
\]

Donc:

\[
|b_i|^{1/T} = \frac{A_i}{A_n} \forall i = 1, \ldots, n - 1.
\]

At the same time:

\[
B_i = \frac{A_i}{A_n} = \frac{|a_i|^{1/T}}{|a_n|^{1/T}} = |b_i|^{1/T} \forall i = 1, \ldots, n - 1. \quad (1)
\]
On the other hand:

\[ B_n = \frac{1}{A_n} = \frac{1}{|a_n|^{1/T}} = |b_n|^{1/T}. \]

By system S1 we have:

\[ |b_{\mu}|^{1/T} = \frac{|P_{\mu}([\{A_{\eta}\}, \{b_{\mu'}\}]^{1/T})}{|a_n|^{1/T}} \leq \frac{P_{\mu}([\{A_{\eta}\}, \{\{b_{\mu'}\}^{1/T}\}])}{A_n}; \]

where \( i(\mu) \geq 2, |\eta| \leq |\mu| \) and \( i(\mu') < i(\mu) \), because the polynomial \( P_{\mu} \) is always such that the sign of each one of its monomials is always positive, recalling that \( |a_{\eta}|^{1/T} \leq A_{\eta} \) for each \( i(\eta) > 1 \), when \( |a_i|^{1/T} = A_i \) for each \( i = 1, \ldots, n \). At the same time the system S2 implies that:

\[ B_{\mu} = \frac{P_{\mu}([\{A_{\eta}\}, \{B_{\mu'}\}])}{A_n}; \]

for each \( \mu \in \Lambda_n(i), \ i \geq 2, \ |\eta| \leq |\mu|, \ i(\mu') < i(\mu) \).

We apply induction on the value of \( i(\mu) \) using (1) at (2). One can then see that:

\[ |b_{\mu}|^{1/T} \leq B_{\mu}; \]

for each \( \mu \in \Delta_n[i], \ i \geq 2 \). The formal series \( \overline{h} \) is, then, a bounding series of \( h \).

Let \( r > 0 \) be a real number strictly less than the convergence radius of \( F \) at \( \overline{0} \). We then have that:

\[ \lim_{|\mu| \to +\infty} |a_{\mu}|^{1/TR|\mu|} = 0. \]

So, it exists:

\[ M := \max_{|\mu| \geq 1} \{|a_{\mu}|^{1/TR|\mu|}\} > 0. \]

Defining:

\[ A_i := |a_i|^{1/T} \ \forall i = 1, \ldots, n; \]

\[ A_{\mu} := \frac{M}{r^{|\mu|}} \ \forall |\mu| \geq 2; \]

we have that:

\[ F(X) = -\sum_{i=1}^{n-1} A_i X_i + A_n X_n - \sum_{i \geq 2} \sum_{\mu \in \Lambda_n(i)} A_{\mu} X^\mu = \]

\[ = -\sum_{i=1}^{n-1} A_i X_i + A_n X_n - \sum_{i \geq 2} \frac{M}{r^i} \sum_{\mu \in \Lambda_n(i)} X^\mu. \]
Let:
\[ ||.||_\infty : \mathbb{R}^n \to \mathbb{R}; \]
be the norm on \( \mathbb{R} \) given as it follows:
\[ ||(X_1, \ldots, X_n)||_\infty := \max_{i=1,\ldots,n} \{|X_i|\}; \]

we have:
\[ |\mathbf{X}^\mu| \leq ||\mathbf{X}||_{\infty}^|\mu|\].

For each \( \mathbf{X} \in \mathbb{R}^n \) such that \( ||\mathbf{X}||_{\infty} < r \), the value \( \mathcal{F}(\mathbf{X}) \), expressed by the previous series is, then, defined by the absolute convergence of such a series. More precisely, we are in the following situation:

\[
- \sum_{i=1}^{n-1} A_i X_i + A_n X_n - \sum_{i \geq 2} \sum_{j} \frac{M}{r^i} \sum_{\mu \in \Lambda_n(i)} X^\mu = - \sum_{i=1}^{n-1} A_i X_i + A_n X_n - M \sum_{i=1}^{n} \sum_{j \geq 2} \left( \frac{X_i}{r} \right)^j + 
\sum_{h=1}^{(n)} \left( \sum_{j \geq 1} \left( \frac{X_{i_1(\mu)}}{r} \right)^j \right) \left( \sum_{j \geq 1} \left( \frac{X_{i_2(\mu)}}{r} \right)^j \right) + \ldots + \sum_{h=1}^{(n)} \prod_{i(h)=1}^{n-1} \left( \sum_{j \geq 1} \left( \frac{X_i(\mu)}{r} \right)^j \right) + \prod_{i=1}^{n} \left( \sum_{j \geq 1} \left( \frac{X_i}{r} \right)^j \right). 
\]

The condition \( ||\mathbf{X}||_{\infty} < r \) implies the absolute convergence of geometric series contained in the previous inequality, which may be represented then as it follows:

\[
- \sum_{i=1}^{n-1} A_i X_i + A_n X_n - \sum_{i \geq 2} \sum_{j} \frac{M}{r^i} \sum_{\mu \in \Lambda_n(i)} X^\mu = - \sum_{i=1}^{n-1} A_i X_i + A_n X_n - M \sum_{i=1}^{n} \sum_{j \geq 2} \left( \frac{X_i}{r} \right)^j + 
\sum_{h=1}^{(n)} \left( \sum_{j \geq 1} \left( \frac{X_{i_1(\mu)}}{r} \right)^j \right) \left( \sum_{j \geq 1} \left( \frac{X_{i_2(\mu)}}{r} \right)^j \right) + \ldots + \sum_{h=1}^{(n)} \prod_{i(h)=1}^{n-1} \left( \sum_{j \geq 1} \left( \frac{X_i(\mu)}{r} \right)^j \right) + \prod_{i=1}^{n} \left( \frac{X_i}{r} \right). 
\]

We remark that the denominators are always different from 0 as \( ||\mathbf{X}||_{\infty} < r \). Imposing then the condition defining \( \mathcal{F}\mathcal{F}(\mathcal{F}(\mathbf{Y})) \):

\[ \mathcal{F}(\mathcal{F}(\mathcal{F}(\mathbf{Y}))) = Y_n; \]

we’ll obtain a rational equation of degree 2 in \( X_n = \mathcal{H}(\mathbf{Y}) \), of the form:

\[
- \sum_{i=1}^{n-1} A_i Y_i + A_n X_n - M(\alpha_1 + \frac{X^2}{1 - X_n} + \alpha_2 + \frac{X_n}{1 - X_n} + \ldots + \alpha_{2n-2} + \alpha_{2n-1} \frac{X_n}{1 - X_n} + \alpha_{2n} \frac{X_n}{1 - X_n}) = Y_n; 
\]

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where \( \alpha_1, ..., \alpha_{2n} \) are polynomials in \( \frac{Y_i}{r} \) and \( \frac{Y_i}{r}/(1 - \frac{Y_i}{r}) \) with \( i = 1, ..., n - 1 \) with degree \( \geq 1 \). More precisely, looking at the previous expression of \( F(X) \) such rational expressions of \( Y_1, ..., Y_{n-1} \) are under the following form:

\[
\alpha_1 := \sum_{i=1}^{n-1} \frac{(\frac{Y_i}{r})^2}{1 - (\frac{Y_i}{r})};
\]

\[
\alpha_2 := \sum_{h=1}^{(n-1)/2} \left( \frac{Y_{1(h)}/r}{1 - \frac{Y_{1(h)}/r}{r}} \right) \left( \frac{Y_{2(h)}/r}{1 - \frac{Y_{2(h)}/r}{r}} \right); 
\]

\[
\alpha_3 := \sum_{i=1}^{n-1} \frac{Y_i}{1 - \frac{Y_i}{r}}; 
\]

\[
\ldots 
\]

\[
\alpha_{2n} := \prod_{i=1}^{n-1} \frac{Y_i}{1 - \frac{Y_i}{r}}. 
\]

We remark in particular that \( \alpha_3 \) annihilates at \( \mathcal{O} \) with multiplicity 1 when all \( \alpha_i \)'s such that \( i \neq 3 \) always have an annihilation order at \( \mathcal{O} \) which is \( \geq 2 \).

Multiplying each term by \( 1 - (X_n/r) \) we obtain:

\[
(1 - X_n/r)A_nX_n - MX_n^2/r^2 - M(1 - X_n/r)(\alpha_1 + \sum_{i=1}^{n-1} \alpha_{2i}) + 
\]

\[
-M(\sum_{i=1}^{n-1} \alpha_{2i+1} + \alpha_{2n})X_n/r - (1 - X_n/r)(\sum_{i=1}^{n-1} A_iY_i + Y_n) = 0. 
\]

Donc:

\[
(A_n/r + M/r^2)X_n^2 + (1/r(M(\sum_{i=1}^{n-1} \alpha_{2i+1} + \alpha_{2n} - \alpha_1 - \sum_{i=1}^{n-1} \alpha_{2i}) - \sum_{i=1}^{n-1} A_iY_i - Y_n) - A_n)X_n + 
\]

\[
+ M(\alpha_1 + \sum_{i=1}^{n-1} \alpha_{2i}) + \sum_{i=1}^{n-1} A_iY_i + Y_n = 0. 
\]

Approximating to the order 1 the left part of the last equality, for \( ||Y||_{\infty} \) close to 0 the terms \( \alpha_1, ..., \alpha_{2n} \) become irrelevant except of \( \alpha_3 \):

\[
(A_n/r + M/r^2)X_n^2 + (1/r(- \sum_{i=1}^{n-1} A_iY_i - Y_n) - A_n + M\alpha_3)X_n + \sum_{i=1}^{n-1} A_iY_i + Y_n = 0. 
\]
Always approximating to the order 1 for $||Y||_\infty \to 0$ the discriminant:

$$\Delta = A_n^2 - 2/r (- \sum_{i=1}^{n-1} A_i Y_i - Y_n + M \sum_{i=1}^{n-1} \frac{Y_i/r}{1 - Y_i/r}) + 4(A_n/r + M/r^2)(- \sum_{i=1}^{n-1} A_i Y_i - Y_n).$$

This last expression is a function of $R^n$ in $\mathbb{R}$, continuous in a convenient open neighborhood of $\overline{U}$, contained in the open sphere with center $\overline{U}$ and radius 1 in $\mathbb{R}^n$. As $\Delta(\overline{0}) = A_n^2 > 0$, it exists then a sufficiently small open neighborhood of $\overline{U}$ such that $\Delta(Y) > 0$, for each $Y$ belonging to it. It exists then, $Y \neq 0$ in $\mathbb{R}^n$ such that:

$$X_n = h(Y) \in \mathbb{R};$$

as a finite real number. The bounding series $\overline{h}$ of $h$ has then a convergence radius not 0 at $\overline{0}$ pas nul and so the same is true for $h$ and so, finally, for $\overline{h}$.

\[\text{Corollary 1. Let } \overline{F} : K^{n+m} \to K^m \text{ be a vector of analytic functions on some open set of } K^n, \text{ such that its Jacobian matrix } J_{\overline{z}_0}(\overline{F}) \text{ at some point } \overline{z}_0 \in Z(\overline{F}) \text{ is such that its rows are linearly independent (in other words, its rank is } m). \text{ Up to a permutation of the columns we can divide such a matrix in two blocks as it follows:}

$$J_{\overline{z}_0}(\overline{F}) = (J_{n,m}(\overline{F}(\overline{z}_0)) | J_{m,m}(\overline{F}(\overline{z}_0)));$$

with $J_{m,m}(\overline{F}(\overline{z}_0))$ a square invertible matrix. It exist then an open neighborood $U_{\overline{z}_0} \times V_{\overline{z}_0} \subset K^n \times K^m$ and a vector of analytic functions:

$$\overline{f} : U_{\overline{z}_0} \to V_{\overline{z}_0};$$

such that for each $\overline{z}_* \in U_{\overline{z}_0}$, we have that:

$$\overline{F}(\overline{z}_*, \overline{f}(\overline{z}_*)) = 0.$$  

\[\text{Proof. As the rank of the Jacobian matrix is } m, \text{ up to permuting the columns (in other words, the variables of the vector space), we may assume that the sub-matrix corresponding to the block of the } m \text{ last columns (on the right) having order } m \times m, \text{ is invertible. Call } J_{m,m}(\overline{F}(\overline{z}_0)) \text{ such a matrix. Now we remark that the Gauss' algorithm on square matrices is still valable on any base field, so in our situation too. It exists then an invertible matrix } P \in K^{m,m} \text{ such that:}

$$PJ_{m,m}(\overline{F}(\overline{z}_0));$$

$$26$$
is upper triangular. Up to composing on the left with the linear map represented by $P$ we may then suppose without loss of generality that the analytic function $\mathcal{F}$ is such that the matrix $J_{m,m}(\mathcal{F}(z_0))$ is upper triangular with no zero terms on its diagonal. In fact, if we compose on the left $\mathcal{F}$ by the linear map that we have introduced and we note:

$$
\begin{pmatrix}
F_1(\overline{z}) \\
\vdots \\
F_m(\overline{z})
\end{pmatrix}
= 
\begin{pmatrix}
G_1(\overline{z}) \\
\vdots \\
G_m(\overline{z})
\end{pmatrix};
$$

we have that:

$$
\begin{pmatrix}
F_1(\overline{z}) \\
\vdots \\
F_m(\overline{z})
\end{pmatrix}
= 
P^{-1}
\begin{pmatrix}
G_1(\overline{z}) \\
\vdots \\
G_m(\overline{z})
\end{pmatrix}.
$$

If it exist an open neighborhood $U_{z_0} \times V_{z_0} \subset K^n \times K^m$ of $z_0$ and an analytic function:

$$
\mathcal{F} : U_{z_0} \rightarrow V_{z_0},
$$

such that we have the following functional identity:

$$
\mathcal{G}(z_1, \ldots, z_n, \mathcal{F}(z_1, \ldots, z_n)) = 0;
$$

we see that:

$$
\mathcal{F}(z_1, \ldots, z_n, \mathcal{F}(z_1, \ldots, z_n)) = 0;
$$

also, for each $\overline{z} \in U_{z_0}$. We then suppose without loss of generality that $J_{m,m}(\mathcal{F}(\overline{z}_0))$ is upper triangular. It follows that:

$$
\frac{\partial}{\partial z_{n+i}} F_j(\overline{z}_0) = 0 \text{ } \forall i < j.
$$

Theorem 8 applied to $F_1, \ldots, F_m : K^{n+m} \rightarrow K$ now says that for each $F_i$, $i = 1, \ldots, m$, it exists an open neighborhood $D^i_{z_0}$ of $z_0$ in $K^{n+m}$ in which all points of our analytic set may be expressed as it follows:

$$(\overline{z}_*, z_{n+1}, \ldots, z_{n+i-1}, f_i(\overline{z}^*), z_{n+i+1}, \ldots, z_{n+m});
$$

where, if we call $\overline{z}^*$ the projection of any point of $K^{n+m}$ on all its components which are different from the $(n+i)$-th one:

$$
f_i : W^i_{z_0} \rightarrow K;
$$

is an analytic function defined on the open neighborhood $W^i_{z_0} \subset K^{n} \times K^{m-1}$ of $\overline{z}^*_0$, projection of $D^i_{z_0}$ on all its components which are different from the $(n+i)$-th one and this $f_i$ is such that, for each $\overline{z}^* = (\overline{z}_*, z_{n+1}, \ldots, z_{n+i-1}, z_{n+i+1}, \ldots, z_{n+m}) \in W^i_{z_0}$:

$$
F_i(\overline{z}_*, z_{n+1}, \ldots, f_i(\overline{z}^*), \ldots, z_{n+m}) = 0;
$$

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with $\overline{z}$, which is the projection of $\overline{z}$ on its $n$ first components. We then have the following functional identity:

$$
\begin{align*}
F_1(\overline{z}, f_1(\overline{z}^1), z_{n+2}, \ldots, z_{n+m}) &= 0 \\
F_2(\overline{z}, z_{n+1}, f_2(\overline{z}^2), \ldots, z_{n+m}) &= 0 \\
&\vdots \\
F_m(\overline{z}, z_{n+1}, \ldots, z_{n+m-1}, f_m(\overline{z}^m)) &= 0
\end{align*}
$$

with functions $f_i, i = 1, \ldots, m$ defined on $W^i_{z_0}, i = 1, \ldots, m$ (projection on all components except the $(n + i)$–th one of a different $D^i_{z_0}, i = 1, \ldots, m$) in $n + m - 1$ variables under the following form:

$$
\begin{align*}
z_{n+1} &= f_1(\overline{z}, z_{n+2}, \ldots, z_{n+m}) \\
&\vdots \\
z_{n+m} &= f_m(\overline{z}, z_{n+1}, \ldots, z_{n+m-1})
\end{align*}
$$

Let $W_{z_0} \subset K^{n+m}$ be the intersection of all $D^i_{z_0}$s. For each $\overline{z} \in W_{z_0}$ the condition $F(\overline{z}) = 0$ is equivalent to the following one:

$$
\begin{align*}
z_{n+1} &= f_1(\overline{z}, f_2(\overline{z}^2), z_{n+3}, \ldots, z_{n+m}) \\
z_{n+2} &= f_2(\overline{z}, z_{n+1}, f_3(\overline{z}^3), \ldots, z_{n+m}) \\
&\vdots \\
z_{n+m} &= f_m(\overline{z}, f_1(\overline{z}^1), \ldots, z_{n+m-1})
\end{align*}
$$

Calling:

$$
\begin{align*}
g_1(\overline{z}) := z_{n+1} - f_1(\overline{z}, f_2(\overline{z}^2), \ldots, z_{n+m}) \\
&\vdots \\
g_m(\overline{z}) := z_{n+m} - f_m(\overline{z}, f_1(\overline{z}^1), \ldots, z_{n+m-1})
\end{align*}
$$

we obtain that in the open neighborhood $W_{z_0}$ of $\overline{z}_0$ the zero locus of the vector of analytic functions $F$ may be expressed in the following equivalent fashion:

$$
\begin{align*}
g_1(\overline{z}, z_{n+1}, z_{n+3}, \ldots, z_{n+m}) &= 0 \\
&\vdots \\
g_m(\overline{z}, z_{n+2}, z_{n+3}, \ldots, z_{n+m}) &= 0
\end{align*}
$$

where the functions $g_1, \ldots, g_m$ are analytic. We have then brought ourselves locally at $\overline{z}_0$ to a system under the following form:

$$
\overline{g}(\overline{z}) = \overline{0};
$$

where all the $m$ analytic functions indicated by the vector $\overline{g}$ at $\overline{z} \in W_{z_0} \subset K^{n+m}$ may be expressed in $n + m - 1$ variables. It remains to prove that the condition on the Jacobian matrix does not change at all. Chain rule on
hyperderivatives (see [Teich]) allows then to say by (6), taking the hyper-derivative on \(z_{n+1}, \ldots, z_{n+m}\) of each equation of this system (remembering the fact that the hyperderivative in \(z_{n+i}\) of the \(i\)-th equation of this one is identically 0), that:

\[
\begin{align*}
\partial_{n+1} F_1 (\overline{z}_0) & \partial_{n+2} f_1 (\overline{z}_0^1) + \partial_{n+2} F_1 (\overline{z}_0) = 0 \\
\vdots & \\
\partial_{n+1} F_1 (\overline{z}_0) & \partial_{n+m} f_1 (\overline{z}_0^m) + \partial_{n+m} F_1 (\overline{z}_0) = 0 \\
\vdots & \\
\partial_{n+m} F_m (\overline{z}_0) & \partial_{n+1} f_m (\overline{z}_0^{m*}) + \partial_{n+1} F_m (\overline{z}_0) = 0 \\
\vdots & \\
\partial_{n+m} F_m (\overline{z}_0) & \partial_{n+1} f_m (\overline{z}_0^m) + \partial_{n+1} F_m (\overline{z}_0) = 0
\end{align*}
\]

The condition on the diagonal of the matrix \(J_{n0}(\overline{F})\) leads us to express:

\[
\begin{align*}
\partial_{n+i} f_1 (\overline{z}_0^1) & = -\frac{\partial_{n+i} F_1 (\overline{z}_0)}{\partial_{n+i} F_1 (\overline{z}_0)} \quad \forall i \neq 1 \\
\vdots & \\
\partial_{n+i} f_m (\overline{z}_0^m) & = -\frac{\partial_{n+i} F_m (\overline{z}_0)}{\partial_{n+i} F_m (\overline{z}_0)} \quad \forall i \neq m
\end{align*}
\]

The sub-matrix \(J_{m,m}(\overline{g}(\overline{z}_0))\) of the Jacobian matrix \(J_{n0}(\overline{g})\) takes on the other hand the following form:

\[
\partial_{n+j} f_j (\overline{z}_0) = 1 - \partial_{n+i} f_i (\overline{z}_0^*) \partial_{n+i} f_{i+1} (\overline{z}_0^{i+1*});
\]

\[
\partial_{n+i+1} g_i (\overline{z}_0) = 0;
\]

\[
\partial_{n+i} g_j (\overline{z}_0) = -\partial_{n+i} f_j (\overline{z}_0^*) - \partial_{n+j+1} f_j (\overline{z}_0^*) \partial_{n+i} f_{j+1} (\overline{z}_0^{j+1*}) \quad \forall i \neq j, j + 1;
\]

where all indexes \(i\) and \(j\) represent the unique integers between 1 and \(m\) into their congruence class modulo \(m\). As:

\[
\partial_{n+i} F_j (\overline{z}_0) = 0 \quad \forall i < j;
\]

the conditions that we found previously allow us to say that:

\[
\partial_{n+i} f_j (\overline{z}_0^j) = 0 \quad \forall i < j.
\]

The square matrix \(J_{m,m}(\overline{g}(\overline{z}_0))\) defined before in the same way than \(J_{m,m}(\overline{F}(\overline{z}_0))\) is consequently such that:

\[
J_{m,m}(\overline{g}(\overline{z}_0)) = \begin{pmatrix}
1 & 0 & \partial_{n+3} g_1 (\overline{z}_0) & \cdots & \partial_{n+m} g_1 (\overline{z}_0) \\
0 & 1 & 0 & \cdots & \partial_{n+m} g_2 (\overline{z}_0) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}.
\]
We can see that it is an invertible and upper triangular matrix.

We can finally reduce then to assume without loss of generality that each equation $F_i(z) = 0$ for each $i = 1, \ldots, m$ implies just $n + m - 1$ variables and more precisely, that it may be written as it follows:

$$F_i(z^*) = 0;$$

for each $i = 1, \ldots, m$. Repeating then the same passages $m - 1$ times again we obtain that inside $W_{z_0}$ the solutions of the system take all the following shape:

$$\begin{cases} z_{n+1} = h_1(z^*) \\ \vdots \\ z_{n+m} = h_m(z^*) \end{cases}$$

In particular, $W_{z_0} = U_{z_0} \times V_{z_0} \subset K^n \times K^m$ and the vector of analytic functions $\overline{h} := (h_1, \ldots, h_m)$ we’ve built is such that:

$$\overline{h} : U_{z_0} \rightarrow V_{z_0};$$

and consequently results to be the implicit function we were searching for.

\[\square\]

**Definition 13.** Let $X \subset K^n$ be a $K$–entire sub-set of $K^n$ for some $n \in \mathbb{N} \setminus \{0\}$. We’ll say that $X$ is **analytically parametrizable** on $K$ if it exists $d \in \mathbb{N} \setminus \{0\}$ such that $d < n$ and a family $R$ of $K$–analytic functions as it follows:

$$f : B^d_1(K) \rightarrow X;$$

such that:

$$X \subseteq \bigcup_{f \in R} f(B^d_1(K)).$$

We’ll call such a family an **analytical cover** over $K$ of $X$.

### 2.3 Analytic sets

Unlike what happens on number fields, strong topological properties of a non-archimedean valuation field don’t allow us to extend an analytic function defined on some open subset, to an analytic function defined on a bigger one. In fact, in non-archimedean context the non-empty intersection of two polydiscs is necessarily one of the two, which does not allow analytic functions provide a means of linking between two different regions of their same definition domaine. This is a decisive obstacle to any attempt to build bundles on varieties defined on non-archimedean valuation fields and therefore
to use such functions in order to describe global properties of such a variety. We’re about to adapt here classic tools used to study $K$–analytic subsets of $K^n$ (given some $n \in \mathbb{N} \setminus \{0\}$ on $K$ any non-archimedean valuation complete field) on our particular case, where $K$ is a complete $1/T$–adic field, finite extension of $k_{\infty}$ contained in $C$. More precisely, we’re about to introduce **affinoid spaces**, a mathematical object specifically thought (initially by J. Tate) to locally analyse the behaviour and the properties of zero locuses of a certain number of $K$–analytic functions on their definition domaine.

**Definition 14.** Let $K$ be an algebraic extension of $k_{\infty}$, complete and contained inside $C$. Let $n \in \mathbb{N} \setminus \{0\}$. We consider the following set:

$$T_n(K) := \{ \sum_{i \geq 0} \sum_{\mu \in \Lambda_n(i)} a_{\mu} z^\mu \in K[[z_1, \ldots, z_n]], \lim_{|\mu| \to +\infty} a_{\mu} = 0 \};$$

we remark that it is an algebra, that we call **free affinoid algebra**. We also remark également that it consists on the ring of all formal series de l’anneau having their coefficients in $K$ and variables $z_1, \ldots, z_n$, convergent on the unit polydisc (with “border”) $B^1_n(C) \subset C^n$. Let $I \subseteq T_n(K)$ some ideal in such an algebra. We call:

$$\tilde{X} := \text{Sp}(T_n(K)/I);$$

the maximal spectrum of the quotient of such an algebra by $I$. We say that $\tilde{X}$ is an *affinoid space* and the quotient algebra $T_n(K)/I$ is a *Tate algebra*. We also say that $\tilde{X}$ is **irreductible** if $I$ is a prime ideal of $T_n(K)$.

**Theorem 9.**

1. Every Tate algebra is Noetherian.

2. $T_n(K)$ is UFD and its Krull dimension is $n$.

3. For each ideal $I \subseteq T_n(K)$ it exists unique a number $d \in \mathbb{N} \setminus \{0\}$ and a finite injective $K$–algebras morphism:

$$T_d(K) \hookrightarrow T_n(K)/I;$$

the Krull dimension of $T_n(K)/I$ is $d$.

4. For each maximal ideal $\mathcal{M}$ in $T_n(K)$ the quotient field $T_n(K)/\mathcal{M}$ is a finite extension of $K$.

*Proof.* See [V-F], Theorem 3.2.1, page 48. □
We remark that Theorem 9 is the convergent power series on $B^1_n(K)$ analogue of the Base Transcendent Theorem in Algebraic Geometry. Moreover, as $T_n(K)$ is a Noetherian algebra, $I$ is contained only in a finite number $r$ of prime ideals $P_1, ..., P_r$, minimal between those containing $I$ in $T_n(K)$. Let $\overline{K} \subset C$ be the algebraic closure of $K$ contained in $C$. Let $\mathcal{M} \in Sp(T_n(K))$. By Theorem 9 point 4 we have that $T_n(K)/\mathcal{M}$ is a finite extension of $K$, contained in $\overline{K}$. Let $\Gamma := Aut(\overline{K}/K)$ be the automorphism group of $\overline{K}$ over $K$. We define:

$$\chi : B^1_n(\overline{K}) \rightarrow Sp(T_n(K));$$

$$\overline{z}_0 \mapsto \mathcal{M}_{\overline{z}_0} := \{ f \in T_n(K), \ f(\overline{z}_0) = 0 \}.$$  

By [BGR], Section 7.1.1, Proposition 1, page 260, we know that $\chi$ is surjective because every maximal ideal $\mathcal{M} \in Sp(T_n(K))$ induces a morphism on this form:

$$\varphi_{\mathcal{M}} : T_n(K) \rightarrow B^1_n(\overline{K});$$

$$f \mapsto [f]_{\mathcal{M}};$$

where $[f]_{\mathcal{M}}$ is the residue class of $f$ modulo $\mathcal{M}$ in $T_n(K)$, which corresponds thereby to a point in $B^1_n(\overline{K})$. The kernel of $\varphi_{\mathcal{M}}$ is $\mathcal{M}$. In particular, $\mathcal{M} = \mathcal{M}_{\overline{z}_0}$ where $\overline{z}_0 = (\varphi(z_1), ..., \varphi(z_n))$ and if $\mathcal{M} \in Sp(T_n(K))$ we know that $\chi^{-1}(\mathcal{M}) = \chi^{-1}(\mathcal{M}_{\overline{z}_0}) = Orb_\Gamma(\overline{z}_0)$. In other words, $\chi$ induces a bijective correspondance between $Sp(T_n(K))$ and the conjugacy classes over $K$ of points of $B^1_n(\overline{K})$. Therefore, if $K$ is algebraically closed, such a bijection is still true between $B^1_n(K)$ and $Sp(T_n(K))$. In the other case we have that the restriction of $\chi$ to the set $B^1_n(\overline{K})$ of $K$-rational points of $B^1_n(\overline{K})$ is injective.

If we call:

$$Sp^*(T_n(K)) := \chi(B^1_n(K));$$

we define:

$$X := Sp^*(T_n(K)/I) = \{ \mathcal{M} \in Sp^*(T_n(K)), \ \mathcal{M} \supseteq I \}.$$  

As we said before $\chi$ induces an embedding of $K$-rational points of $B^1_n(\overline{K})$ in $Sp(T_n(K))$, and this implies that we can identify $X$ with some $K$-analytic sub-set of $B^1_n(K)$.

We have an affinoid spaces version of Hilbert’s Nullstellensatz as it follows:

$$\mathcal{I}(\tilde{X}) := \bigcap_{\mathcal{M} \in Sp(T_n(K)), \ \mathcal{M} \supseteq I} \mathcal{M} = \sqrt{I};$$

$32$
where $\tilde{X} = Sp(T_n(K)/I)$. See [BGR], Section 7.1.2, Theorem 3. Then, $\tilde{X} = Sp(T_n(K)/I(\tilde{X}))$ and $X = Sp^*(T_n(K)/I(\tilde{X}))$. Let:

$$I(X) := I(\tilde{X}).$$

We also define:

$$O(X) := T_n(K)/I(X).$$

If $I \subset T_n(K)$ is an ideal of $T_n(K)$:

$$Sp^*(T_n(K)/I) = Sp^*(T_n(K)/I(X)).$$

If actually a maximal ideal contains an ideal, it contains its radical too. We remark that we’re not allowed to univocally associate an affinoid space to some $K$–analytic sub-set of $B_n^1(K)$. Two different affinoid spaces may actually intersect each other in the same $K$–rational points inside $B_n^1(K)$ (corresponding to maximal ideals contained into the set $\chi(B_n^1(K))$ that we introduced before). For this reason we give here the following definition.

**Definition 15.** A $K$–analytic space in $B_n^1(K)$ is the couple $(I, X)$ consisting in the ideal $I \subseteq T_n(K)$ and the $K$–analytic set $X$, which is the subset of $B_n^1(K)$ consisting of all zeroes of each minimal generating set of $I$, under the following equivalence relation:

$$(I, X) \sim (I', X) \iff \sqrt{I} = \sqrt{I'} \subseteq T_n(K).$$

Let $(I, X)$ be a $K$–analytic space into $B_n^1(K)$. We say that a $K$–analytic space $(J, Y)$ into $B_n^1(K)$ is a $K$–analytic subspace of $(I, X)$ if and only if $Y \subseteq X$ and $J \supseteq I$.

We remark that in general two ideals $I, I' \subseteq T_n(K)$ such that $\sqrt{I} \neq \sqrt{I'}$ may define the same $K$–analytic sub-set $X$ into $B_n^1(K)$, which may have then, a priori, completely different properties (for example dimension or regular points, which we’ll introduce after) depending on the chosen ideal $I \subseteq T_n(K)$. We’re about to construct then the following bijection between the family of $K$–analytic spaces into $B_n^1(K)$ and that of affinoid spaces on $K$:

$$(I, X) \leftrightarrow \tilde{X};$$

where $X$ is the $K$–analytic sub-set into $B_n^1(K)$ which consists in the zero locus of some ideal $I \subseteq T_n(K)$, that we call $I(X) := \sqrt{I}$, and $\tilde{X}$ is the affinoid space defined by the ideal $I(\tilde{X}) := I(X)$ as we said before. More precisely, $X$ and $\tilde{X}$ are defined by the ideal $I$ as it follows:

$$X = Sp^*(T_n(K)/I) = Sp^*(T_n(K)/I(X)) \quad \text{and} \quad \tilde{X} = Sp(T_n(K)/I) = Sp(T_n(K)/I(\tilde{X})).$$
As such objects are univocally defined by \( \mathcal{I}(X) = \mathcal{I}(\tilde{X}) \) and as all ideals we will trait will always be under such a form (radical ideals of ideals \( I \subseteq T_n(K) \) defining \( K \)-analytic sets we’re interested in) we will directly note with the same letter \( X \) the \( K \)-analytic space associated to each \( K \)-analytic sub-set of \( B^u_n(K) \) because the ideal \( I \) defining this one will be always initially fixed. By this way, if \( \tilde{Y} \subseteq \tilde{X} \) an affinoid sub-space of \( \tilde{X} \), it will biunivocally corresponds to a reduced ideal \( \mathcal{I}(\tilde{Y}) \supseteq \mathcal{I}(\tilde{X}) \) such that the restriction of \( \tilde{Y} \) to \( B^u_n(K) \) could be the \( K \)-analytic set \( Y = sp^*(T_n(K)/\mathcal{I}(\tilde{Y})) \) to whom one associates as previously the affinoid space \( \tilde{Y} \). Once we have chosen this method of assigning to an affinoid space a \( K \)-analytic space in \( B^u_n(K) \) we can directly work on affinoid spaces using algebraic methods and tools developed on these ones, remembering that they correspond to \( K \)-analytic spaces in \( B^u_n(K) \) in which we are ultimately interested.

### 2.4 Notion of local dimension

Let now \( X \subseteq B^u_n(K) \) be a \( K \)-analytic space into \( B^u_n(K) \) at which we associate as explained before the affinoid space \( \tilde{X} = sp(\mathcal{O}(X)) \) such that \( X = B^u_n(K) \cap \tilde{X} \). We say that \( X \) is **irreducible** if \( \mathcal{I}(X) \) is a prime ideal of \( T_n(K) \). If \( X \) is the zero locus in \( K^n \) of \( f_1, \ldots, f_s \), \( K \)-entires functions defined on \( K^n \), taking their values in \( K \), we have in particular that \( f_1, \ldots, f_s \in T_n(K) \). We call \( \mathcal{I}(X) = \sqrt{(f_1, \ldots, f_r)} \subseteq T_n(K) \) and \( \mathcal{O}(X) = T_n(K)/\mathcal{I}(X) \). We suppose that:

\[
B^u_n(K) \cap X \neq \emptyset.
\]

It follows that:

\[
X \cap B^u_n(K) = sp^*(T_n(K)/(f_1, \ldots, f_s)) = sp^*(\mathcal{O}(X)) \subset sp(\mathcal{O}(X)).
\]

We observe that, as \( T_n(K) \) is a Noetherian algebra, \( \mathcal{I}(X) \) is contained only in a finite number \( r \) of minimal prime ideals \( P_1, \ldots, P_r \) of \( T_n(K) \). We call:

\[
B^u_n(K) \cap X_i := sp^*(T_n(K)/P_i) \subseteq B^u_n(K) \cap X, \forall i = 1, \ldots, r;
\]

**irreducible components** of \( B^u_n(K) \cap X \). In particular we remark that the analogue of Hilbert’s Nullstellensatz we’ve stated for affinoid spaces allows us to biunivocally associate \( K \)-analytic irreducible sub-spaces of \( B^u_n(K) \cap X \) to prime ideals of \( T_n(K) \) containing \( \mathcal{I}(X) \), as these ones correspond to irreducible affinoid sub-spaces of the affinoid space \( \tilde{X} \) we associated to \( X \) as explained before. If we call **freedom degree** of some \( K \)-analytic space \( X \) in \( B^u_n(K) \) the number \( d \in \mathbb{N} \) such that \( T_d(K) \) embeds itself as explained in Theorem 9 point 3 inside \( \mathcal{O}(X) \), this one is actually the Krull dimension
of the Tate algebra $\mathcal{O}(X)$, which we call $\dim(\mathcal{O}(X))$. For each $\overline{z}_0 \in X$ we call then the **dimension** of $X$ in $\overline{z}_0$:

$$\dim_{\overline{z}_0}(X) := \dim(\mathcal{O}(X)_{\mathcal{M}_{\overline{z}_0}}).$$

We also call **dimension** of $X$:

$$\dim(X) := \sup_{\overline{z}_0 \in X} \{\dim_{\overline{z}_0}(X)\}.$$ 

**Remark 6.** Let $X \subset B^n_d(K)$ be an irreducible $K$–analytic space into $B^n_d(K)$ and let $\mathcal{M} \in Sp^*(\mathcal{O}(X))$, we therefore have that:

$$\dim(\mathcal{O}(X)) = \dim(\mathcal{O}(X)_{\mathcal{M}}).$$

In other words, the dimension of $X$ irreducible is the local dimension of $X$ in each point.

**Proof.** By Theorem 9 point 3, if $d = \dim(\mathcal{O}(X))$ we then have that $\mathcal{O}(X)$ is an integer extension of $T_d(K)$. We remark that by our hypotheses $T_d(K)$ and $\mathcal{O}(X)$ are integer domains and that $T_d(K)$ is also integrally closed in its fraction field (as a consequence of the fact that $T_d(K)$ is a UDF, by Theorem 9 point 2). As a consequence, we know (see [At-Mac], Lemma 11.26, page 125) that:

$$\dim(\mathcal{O}(X)_{\mathcal{M}}) = \dim(T_d(K)_{\mathcal{M} \cap T_d(K)}).$$

By Theorem 9:

$$\dim(T_d(K)) = d;$$

and we can then reduce to show such a property in case of $X = B^n_d(K)$ without lost of generality. We also know that it exists a two-way correspondance between $Sp^*(T_d(K))$ and points of $B^n_d(K)$. Let $\overline{z}_0 \in B^n_d(K)$ be the corresponding point to the maximal ideal $\mathcal{M} \in Sp^*(T_d(K))$. Under translation we can then suppose $\overline{z}_0 = \overline{0}$ and:

$$\mathcal{M} = (z_1, \ldots, z_d).$$

We now see that:

$$\dim(T_d(K)_{\mathcal{M}}) = d.$$

Prime ideals of $T_d(K)_{\mathcal{M}}$ are actually in a two-way correspondance with those of $T_d(K)$ which are contained in $\mathcal{M}$. This also shows that:

$$\dim(T_d(K)_{\mathcal{M}}) \leq \dim(T_d(K)).$$
On the other hand the following prime ideals chain:

\[
(z_1, \ldots, z_d) \supset (z_1, \ldots, z_{d-1}) \supset \ldots \supset (z_1);
\]

shows us that \(d \leq \dim(T_d(K)_M)\) as they are all contained inside \(\mathcal{M}\). As we know that \(d = \dim(T_d(K))\) by Theorem 9 point 2, it follows that:

\[
\dim(T_d(K)) \leq \dim(T_d(K)_M).
\]

Let now be an affinoid space \(\tilde{X}\) defined over \(K\). Let \(\mathcal{O}(\tilde{X}) = T_n(K)/I\) be the Tate algebra associated. Every maximal ideal \(\mathcal{M} \in \tilde{X} = \text{Sp}(\mathcal{O}(X))\) corresponds to a maximal ideal of \(T_n(K)\) containing \(I\). Let \(f \in T_n(K)\). We define \(f(M) \in K\) as the residue class modulo \(M\) of \(f\) in \(T_n(K)/M\), which is a finite extension of \(K\) contained in \(K\).

**Definition 16.** Let \(\tilde{X}\) be an affinoid space defined over \(K\). We consider the family of rational sets \(U \subseteq \tilde{X}\), given in [V-P], such that for each element \(U\) of such a family it exists \(s \in \mathbb{N}\), \(f_0, \ldots, f_s \in \mathcal{O}(\tilde{X})\) such that \((f_0, \ldots, f_s) = \mathcal{O}(\tilde{X})\) and:

\[
U = R_{\tilde{X}}(f_0, \ldots, f_s) := \{M \in \tilde{X}, |f_0(M)|_{1/T} \geq |f_i(M)|_{1/T}, \forall i = 1, \ldots, s\}.
\]

We also say that such a covering of \(U\) is admissible if it consists of rational sets in \(U\) (and one can also prove that such sets are actually rational in \(\tilde{X}\), see [V-F], Lemma 4.1.3) and it admits a finite sub-covering. In the same book it is proven that this family of sub-sets of \(\tilde{X}\) taken with such coverings is a \(G\)-topology (or Grothendieck topology) (see [V-F], Lemma 4.1.3) that we call \(G\). This allows us to construct the presheaf:

\[
U \mapsto \mathcal{O}(U) := \mathcal{O}(\tilde{X}) \ll (z_{n+1}, \ldots, z_{n+s}) / (f_1 - z_{n+1}f_0, \ldots, f_s - z_{n+s}f_0);
\]

of analytic functions or regular functions on \(U\). A Theorem proven by J. Tate (see [V-F], Theorem 4.2.2) allows to show that such a presheaf is actually a sheaf. We call \((\tilde{X}, G, \mathcal{O}_{\tilde{X}})\) a rigid analytic space endowed with a sheaf \(\mathcal{O}_{\tilde{X}}\) of regular functions on the \(G\)-topology \(G\). For each \(\pi \in \tilde{X}\) we call \(\mathcal{O}_{\tilde{X}, \pi}\) the ring of germs of analytic functions associated to \(\mathcal{O}_{\tilde{X}}\) on \(\pi\). We remark that such a ring is local and its unique maximal ideal consists in equivalence classes of sequences which are compatible in \(\lim_{\to(U \ni x)} \mathcal{O}(U)\) and definitely zero on \(\pi\) (see [V-F], Definition 4.5.6). We call dimension of \(\tilde{X}\) in \(\pi\):

\[
\dim_{\pi}(\tilde{X}) := \dim(\mathcal{O}_{\tilde{X}, \pi}).
\]
We also call:
\[\text{dim}(\tilde{X}) := \sup_{z \in X}\text{dim}(\tilde{X})\].

We remark that if \(X\) is the \(K\)-analytic space of \(B^n_1(K)\) associated to \(\tilde{X}\) as previously explained, we have:
\[\text{dim}_{\pi}(\tilde{X}) = \text{dim}_\pi(X), \ \forall \pi \in X.\]

We have then for a general \(K\) and \(L \subseteq K\) the following inclusions of families:
\[
\{L - \text{algebraic varieties in } K^n\} \subset \\
\subset \{L - \text{entire sub-sets in } K^n\} \subset \\
\subset \{L - \text{analytic sub-spaces in } K^n\} \subset \\
\subset \{K - \text{rational points of a rigid analytic space analytic defined over } L\}. 
\]

During the following arguments we will try to adapt to our situation a well-known Theorem in Algebraic Geometry which establishes a relation between the notion of local dimension of a variety on a chosen point (seen as the dimension of the tangent space on this point, space which is the kernel of the Jacobian matrix that one associates to this variety in such a point) and the Krull dimension of the ring of regular functions, localised in this point by the maximal ideal associated. An analogue of such a result is still known on affinoid spaces defined on a separable field (see [V-F], Theorem 3.6.3).

Let \(A\) be a local Noetherian ring and let \(\mathcal{M}\) its unique maximal ideal. We say that it is regular if:
\[\left[\mathcal{M}/\mathcal{M}^2 : A/\mathcal{M}\right] = \text{dim}(A).\]

Let \(X \subset B^n_1(K)\) be a \(K\)-analytic space. If \(\mathcal{O}(X) = T_n(K)/\mathcal{I}(X)\) (where \(\mathcal{I}(X) = (f_1, ..., f_s)\)) we also define the following \(\mathcal{O}(X)\)-module of (hyper)differential forms:
\[
\Omega_{\mathcal{O}(X)/K} := \mathcal{O}(X) \otimes_{T_n(K)} \left(\sum_{i=1}^n T_n(K)dz_i / \sum_{i=1}^s T_n(K)df_i\right); 
\]
where the (hyper)differential on \(z_1, ..., z_n\) is the one which is trivially induced by the (hyper)derivative. If \(\mathcal{M} \in Sp(\mathcal{O}(X))\) is associated to some point \(\pi_0 \in X\) we call cotangent space \(X\) on \(\pi_0\) the vector \(\mathcal{O}(X)_{\mathcal{M}}/\mathcal{M}_{\mathcal{M}}\)-space.
\[ M_\mathcal{M}/M_\mathcal{M}^2. \] As \( T_n(K) \) is Noetherian, \( M_\mathcal{M} \) is a finitely generated \( \mathcal{O}(X)_\mathcal{M} \)-module. By Nakayama’s Lemma we have therefore:

\[
\dim_{\mathcal{O}(X)_\mathcal{M}/M_\mathcal{M}}(M_\mathcal{M}/M_\mathcal{M}^2) \geq \dim(\mathcal{O}(X)_\mathcal{M}). \tag{6}
\]

This Lemma (see [At-Mac], page 21) applied to the finitely generated \( \mathcal{O}(X)_\mathcal{M} \)-module \( M_\mathcal{M} \), also seen as an ideal of \( \mathcal{O}(X)_\mathcal{M} \) (contained in the Jacobson radical as such a ring is local) and to the sub-\( \mathcal{O}(X)_\mathcal{M} \)-module of \( M_\mathcal{M} \) consisting of representatives of elements of some \( \mathcal{O}(X)_\mathcal{M}/M_\mathcal{M} \)-base of \( M_\mathcal{M}/M_\mathcal{M}^2 \) implies actually that:

\[
\dim_{\mathcal{O}(X)_\mathcal{M}/M_\mathcal{M}}(M_\mathcal{M}/M_\mathcal{M}^2) = \min\{i \in \mathbb{N}, \ M_\mathcal{M} = (f_1, ..., f_i), \ f_1, ..., f_i \in \mathcal{O}(X)_\mathcal{M}\};
\]

while on the other hand classic properties of Commutative Algebra (see [Ash], Chapter 5, Proposition 5.4.1) show that:

\[
\dim(\mathcal{O}(X)_\mathcal{M}) = \min\{i \in \mathbb{N}, \ M_\mathcal{M} = \sqrt{(f_1, ..., f_i)}, \ f_1, ..., f_i \in \mathcal{O}(X)_\mathcal{M}\};
\]

because \( \mathcal{O}(X)_\mathcal{M} \) is a local Noetherian ring.

**Theorem 10.** Let \( X \subset B^n_1(K) \) be a \( K \)-analytic set, where \( K \) is a perfect, complete non-archimedean valued field contained in \( \mathcal{C} \). Let \( \mathcal{M} = M_{\bar{z}_0} \in \text{Sp}^\ast(T_n(K)) \) be the maximal ideal containing \( I(X) \) associated to \( \bar{z}_0 \in X \) as explained before. The following properties are then equivalent.

1. The local ring \( \mathcal{O}(X)_{\mathcal{M}_{\bar{z}_0}} \) is regular.
2. \([\Omega_{\mathcal{O}(X)/K}/\mathcal{M}\Omega_{\mathcal{O}(X)/K} : \mathcal{O}(X)/\mathcal{M}] = \dim(\mathcal{O}(X)_\mathcal{M}).\]
3. The Jacobian matrix \( J_{\bar{z}_0}(I(X)) \) given on \( \bar{z}_0 \) by generators \( f_1, ..., f_s \) of \( I(X) \) seen as functions in \( n \) variables \( z_1, ..., z_n \) has rank \( n - \dim(\mathcal{O}(X)_{\mathcal{M}_{\bar{z}_0}}). \)

**Proof.** See [V-F] Theorem 3.6.3.

Let \( X \) a \( K \)-entire sub-set inside \( K^n \) such that \( B^n_1(K) \cap X \) is an irreducible affinoid space on \( K \), where \( K \) is a field respecting hypotheses of Theorem 10. If \( I(X) \) is generated by a minimal system of \( r \) generators we have that:

\[
r \geq n - \dim(B^n_1(K) \cap X). \tag{3}
\]

\[\footnotesize\begin{align*}
\text{We remark that by composition laws of hyperderivatives of analytic functions it is easy to see that the rank of the Jacobian matrix on \( \bar{z}_0 \) given by a generating system of the same ideal \( I(X) \) is independent by the choice of such a system.}\end{align*}\]
By Theorem 9 point 3, we actually have that \( \dim(B^n_1(K) \cap X) \) is the freedom degree of \( B^n_1(K) \cap X \), which implies that \( \dim(B^n_1(K) \cap X) \geq n - r \). By Theorem 10 we have that:

\[
\rho(J_{\bar{z}_0}(\mathcal{I}(X))) \leq n - \dim(B^n_1(K) \cap X) \leq r, \quad \forall z_0 \in B^n_1(K) \cap X.
\]

(We say that \( B^n_1(K) \cap X \) is a complete intersection if \( r = n - \dim(B^n_1(K) \cap X) \)).

We suppose from now that \( K \) is not perfect. Let \( X \) a \( K \)-analytic sub-space contained inside \( B^n_1(K) \). We define the locus of \textbf{regular points} of \( X \) as it follows:

\[
X_{\text{reg.}} := \{ z_0 \in X, \ \rho(J_{\bar{z}_0}(\mathcal{I}(X))) = n - \dim(O(X)_{M_{\bar{z}_0}}) \}.
\]

We define on the other hand the locus of \textbf{singular points} of \( X \) as it follows:

\[
X_{\text{sing.}} := X \setminus X_{\text{reg.}}.
\]

We remark that the definition generally given of regular points of some rigid analytic space (which in our situation is that one associated to the affinoid space \( \tilde{X} \) defined by \( \mathcal{I}(X) \)) is the following:

\[
\tilde{X}_{\text{reg.}} := \{ \bar{z}_0 \in \tilde{X}, \ \rho(J_{\bar{z}_0}(\mathcal{I}(X))) = n - \dim(O_{\tilde{X},\bar{z}_0}) \}.
\]

This definition applied to \( K \)-rational points of \( \tilde{X} \) is not in contradiction with the previous one because one can prove (see [V-F], Proposition 4.6.1) that:

\[
\overline{O_{\tilde{X},\bar{z}_0}} \cong \hat{O}_{\tilde{X},\bar{z}_0};
\]

where for each local Noetherian ring \( R \) we define as \( \hat{R} \) its completion by rapport with its unique maximal ideal, and by [At-Mac], Corollary 11.19, for each such \( R \) we have that \( \dim(R) = \dim(\hat{R}) \).

**Remark 7.** It is really important to remark that it is impossible to give a definition of global dimension or regular point for a \( K \)-entire set \( X \) contained inside \( K^n \). The algebra of \( K \)-entire functions under the form \( f : K^n \rightarrow K \) is not Noetherian in general and it is impossible to associate to \( X \) an ideal \( \mathcal{I}(X) \) as we did for \( K \)-analytic sub-sets \( B^n_1(K) \cap X \) inside \( B^n_1(K) \) in order to repeat on \( X \) the same passages we’ve developped for \( B^n_1(K) \cap X \). As we’ve already remarked, the non-archimedean structure of the topology induced on \( K^n \) by the absolue \( 1/T \)-adic value does not allow to construct a sheaf on \( K^n \) using \( K \)-analytic functions on open sets \( U \) of \( K^n \), which force us to a local study of properties of \( X \) seeing each intersection of
this one with a unit polydisc inside $K^n$ after having associated it an ideal of $T_n(K)$ as previously explained. Such $K-$analytic spaces constructed by $K-$analytic sub-sets of $X$ that we’ve defined here may not have any relation each other and so they may have completely different properties.

2.5 Density of regular points

Definition 17. Let $K_1 \subset K_2$ a field extension. Let $X$ be an irreducible affinoid space inside $B^n_1(K_1)$. We say that it is absolutely irreducible in $K_2$ if the prime ideal $\mathcal{I}(X)$ of $T_n(K_1)$ associated to $X$ as it follows:

$$\mathcal{I}(X) := \{ f \in T_n(K_1), \ f(\overline{z}) = 0, \ \forall \overline{z} \in X \};$$

is such that the ideal $\mathcal{I}(X)T_n(K_2)$ of $T_n(K_2)$ remains prime.

We remark that if $K_2$ is an algebraic extension of $K_1$ every irreducible affinoid space $X$ inside $B^n_1(K_1)$ decomposes in a finite number of absolutely irreducible components on $K_2$ after replacing $K_1$ by a finite extension in $K_2$.  

Theorem 11. Let $X$ be an affinoid space inside $B^n_1(K)$, where $K$ is a $1/T-$adic complete valued field. We assume that $X$ is absolutely irreducible in the perfect closure of $K$ in $\mathbb{C}$, that we call $K$. Regular points of $X$ are dense in $X$ with rapport to the induced $1/T-$adic topology.

Proof. We firstly prove the Theorem supposing that $K = \mathbb{K}$. By Remark 6:

$$\dim_{\mathbb{K}}(X) = \dim(X);$$

for every $\overline{z} \in X$, as $X$ is irreducible. Therefore we have:

$$\overline{z} \in X_{\text{sing.}} \iff \rho(J_{\mathbb{K}}(\mathcal{I}(X))) < n - \dim(X).$$

Such a condition is equivalent to say that each minor of $J_{\mathbb{K}}(\mathcal{I}(X))$ having order $n - \dim(X)$ is 0. As $\dim(X)$ does not depend to point $\overline{z} \in X$ such a condition is equivalent to require that $\overline{z}$ is contained into the $K-$analytic sub-space of $X$ that we define adding to the chosen minimal system of generators of $\mathcal{I}(X)$ the annihilating condition on minors of $J_{\mathbb{K}}(\mathcal{I}(X))$ that we have explained before. It follows that $X_{\text{sing.}}$ is an affinoid sub-space of $X$. We show that this inclusion is strict. Let $d := \dim(X)$. Theorem 9 implies the existence of an entire embedding as it follows:

$$T_d(\mathbb{K}) \hookrightarrow T_n(\mathbb{K})/\mathcal{I}(X) = \mathcal{O}(X).$$
In other words, \( T_d(\mathbb{K}) \subset \mathcal{O}(X) \) is an entire ring extension. By \cite{At-Mac} Corollary 5.8 this induces a surjective finite morphism on maximal spectra, in the following way: 

\[
f : Sp^*(\mathcal{O}(X)) = X \to B_d^4(\mathbb{K}).
\]

By Theorem 10 and as \( \mathbb{K} \) is a perfect field we can say that all points \( \overline{z}_0 \in X_{reg.} \) correspond to maximal ideals \( \mathcal{M}_{\overline{z}_0} \in Sp^*(\mathcal{O}(X)) \) such that:

\[
\left[ \mathcal{M}_{\overline{z}_0}/\mathcal{M}_{\overline{z}_0}^2 : \mathcal{O}(X)_{\mathcal{M}_{\overline{z}_0}}/\mathcal{M}_{\overline{z}_0}\mathcal{M}_{\overline{z}_0} \right] = d.
\]

One can show (see \cite{Col-Maz} Lemma 1.2.2) that this implies that \( X_{sing.} \) is contained in the set of ramification points of \( f \). In fact, let \( I(X) = (f_1, ..., f_r) \subset T_n(\mathbb{K}) \) and let \( \mathcal{M}_{\overline{z}_0} \in Sp^*(\mathcal{O}(X)) \) be a maximal ideal having restriction to \( T_d(\mathbb{K}) \) corresponding to the point \( \overline{z}_0 := (z_{0,1}, ..., z_{0,d}) \in B_d^4(\mathbb{K}) \). It corresponds then to some point \( \overline{z}_0 \in X \subset B_d^4(\mathbb{K}) \) which is a \( \mathbb{K} \)-rational solution of the following system:

\[
\begin{align*}
    f_1(\overline{z}) &= 0 \\
    \vdots & \\
    f_r(\overline{z}) &= 0 \\
    z_1 &= z_{0,1} \\
    \vdots & \\
    z_d &= z_{0,d}
\end{align*}
\]

Let \( m_{\overline{z}_0} \in Sp(T_d(\mathbb{K})) \) be the maximal ideal of \( T_d(\mathbb{K}) \) associated to the point \( \overline{z}_0 \). It follows that:

\[
\mathcal{O}(X)_{\mathcal{M}_{\overline{z}_0}}/\mathcal{M}_{\overline{z}_0}\mathcal{M}_{\overline{z}_0} = \mathbb{K} \ll \overline{z}_0 \gg = \mathbb{K} = \mathbb{K} \ll \overline{z}_0^* \gg = T_d(\mathbb{K})_{m_{\overline{z}_0}}/m_{\overline{z}_0}m_{\overline{z}_0};
\]

because \( \overline{z}_0 \) and \( \overline{z}_0^* \) are \( \mathbb{K} \)-rational points. On the other hand, \( B_d^4(\mathbb{K}) \) is such that all its points are regular, so \( \overline{z}_0 \) is regular too. It follows that \( \overline{z}_0 \) is a ramification point over \( \overline{z}_0 \) if and only if:

\[
\left[ \mathcal{M}_{\overline{z}_0}/\mathcal{M}_{\overline{z}_0}^2 : \mathbb{K} \right] > \left[ m_{\overline{z}_0}m_{\overline{z}_0}^2 : \mathbb{K} \right] = d.
\]

The dimension of the cotangent space that one associates to \( T_d(\mathbb{K}) \) on \( \overline{z}_0 \) is actually \( d \) by Theorem 10 because \( \mathbb{K} \) is a perfect field and, as we already saw previously, \( B_d^4(\mathbb{K}) \) is an affinoid space not containing singular points. We consider the following induced morphism on spectra:

\[
g : Spec \ \mathcal{O}(X) \to Spec \ T_d(\mathbb{K}).
\]
The prime ideal 0 of $T_d(K)$ corresponds to the generic point $\eta$ of $B^1_d(K)$ (see [Hart] Example 2.3.3). This morphism could be restricted to $g^{-1}(\eta)$ and so it induces a finite morphism between the corresponding rings localized in 0, which corresponds to a finite field extension as $O(X)$ is an integer domain and so 0 is a prime ideal of this one. In fact $T_d(K)_\eta = T_d(K)_0$ is the quotient field of $T_d(K)$ and $O(X)_\eta = O(X)_0$ the one of $O(X)$. We call these two fields:

$$\mathbb{K}\{\{z_1, ..., z_d\}\} := T_d(K)_0;$$
$$\mathbb{K}(X) := O(X)_0.$$

We show that such a field extension:

$$\mathbb{K}\{\{z_1, ..., z_d\}\} \subseteq \mathbb{K}(X);$$

is separable. We know by Theorem 9 point 3 that it exist $n - d$ elements $z_{d+1}, ..., z_n \in \mathbb{K}(X)$ generating the field extension $\mathbb{K}\{\{z_1, ..., z_d\}\} \subseteq \mathbb{K}(X) = \mathbb{K}\{\{z_1, ..., z_d, z_{d+1}, ..., z_n\}\}$. As $z_{d+1}$ is algebraic over $\mathbb{K}\{\{z_1, ..., z_d\}\}$ it exists an irreducible polynomial $F(X) \in \mathbb{K}\{\{z_1, ..., z_d\}\}[X] \setminus \{0\}$ such that:

$$F(z_{d+1}) = 0.$$

It exists then an irreducible element $f \in T_{d+1}(K) \setminus \{0\}$ such that:

$$f(z_1, ..., z_d, z_{d+1}) = 0.$$

Therefore it must exist at least one variable $z_j$, for some $j$ between 1 and $d + 1$, such that the partial hyperderivative of $f(z_1, ..., z_{d+1})$ in $z_j$ is not 0. In the other case, as $\mathbb{K}$ is a perfect field it exist actually a number $s \in \mathbb{N} \setminus \{0\}$ and an element $g(z_1, ..., z_{d+1}) \in T_{d+1}(K) \setminus \{0\}$ such that:

$$f(z_1, ..., z_{d+1}) = g(z_1, ..., z_{d+1})^s.$$

And this is in contradiction with the hypothesis that $f(z_1, ..., z_{d+1})$ is irreducible. As the variable $z_j$ appears necessarily into the expression of $f(z_1, ..., z_{d+1})$ it follows that elements $z_1, ..., z_{j-1}, z_{j+1}, ..., z_{d+1}$ are free over $\mathbb{K}$. This means that $\mathbb{K} \ll z_1, ..., z_{j-1}, z_{j+1}, ..., z_{d+1} \gg = T_d(\mathbb{K})$. In fact $z_j$ is not free over $\mathbb{K}\{\{z_1, ..., z_{j-1}, z_{j+1}, ..., z_{d+1}\}\}$ and if these elements $z_1, ..., z_{j-1}, z_{j+1}, ..., z_{d+1}$ were not free over $\mathbb{K}$ we could deduce that the field $\mathbb{K}\{\{z_1, ..., z_{d+1}\}\}$ has a freedom degree over $\mathbb{K}$ strictly minor than $d$, and this would contradicts the hypothesis on $X$. As:

$$\mathbb{K}\{\{z_1, ..., z_{j-1}, z_{j+1}, ..., z_{d+1}\}\}(z_j) = \mathbb{K}\{\{z_1, ..., z_d\}\}(z_{d+1});$$

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we can modify the initial choice of $d$ elements $\mathbb{K}$—algebraically independents between the generators $z_1, ..., z_n$ of $\mathbb{K}(X)$ over $\mathbb{K}$. Theorem 9 point 3 does not force in fact to a univocal choice. We have that $\mathbb{K}(X)$ is a finite extension of $\mathbb{K}\{\{z_1, ..., z_j\} \cup \{z_{j+1}, ..., z_{d+1}\}\}$. We then change eventually the indexation of such generators in order to substitute $z_j$ by $z_{d+1}$ without loss of generality. It finally follows that $z_{d+1}$ is separable over $\mathbb{K}\{\{z_1, ..., z_d\}\}$ as its minimal polynomial $f(X_1, ..., X_{d+1})$ associated to $z_{d+1}$ in $X_{d+1}$ (and so, after the change of the indexation which did substitute $z_j$ by $z_{d+1}$, it is now separable in $X_{d+1}$). Now, $z_{d+2}$ is algebraic over $\mathbb{K}\{\{z_1, ..., z_d\}\}$ too. With analogous arguments we show that we can suppose without loss of generality that it is also separable over $\mathbb{K}\{\{z_1, ..., z_d\}\}$. Doing again the same passages for $z_{d+3}, ..., z_n$ we can finally say without loss of generality that the field extension:

$$\mathbb{K}\{\{z_1, ..., z_d\}\} \subseteq \mathbb{K}(X);$$

is then separable. Such a field extension admits so a discriminant ideal generated by the one of the corresponding field extension and it can’t be 0. It follows that $X_{\text{sing}}$ is a strict affinoid sub-space of $X$. Therefore, $\dim(X_{\text{sing}}) < \dim(X)$. We suppose now that $X$ is defined over $K$ and we consider the affinoid space $X(K)$ inside $K^n$. If this one is irreducible we have that:

$$\dim_K(X(\mathbb{K})) = \dim_K(X).$$

In fact, let $\mathcal{I}(X(\mathbb{K})) \subset T_n(\mathbb{K})$ be the ideal associated to $X(\mathbb{K})$. As the following ring extension:

$$T_n(K) \subset T_n(\mathbb{K});$$

is entire and $\mathcal{I}(X(\mathbb{K}))$ is a prime ideal of $T_n(K)$ it follows that:

$$\mathcal{I}(X(\mathbb{K})) = (\mathcal{I}(X(\mathbb{K}))T_n(\mathbb{K})) \cap T_n(K);$$

see [At-Mac] Theorem 5.10. We can then construct a natural embedding in the following form:

$$T_n(K)/\mathcal{I}(X(\mathbb{K})) \leftrightarrow T_n(\mathbb{K})/\mathcal{I}(X(\mathbb{K}))T_n(\mathbb{K}).$$

Such an injection is still an entire morphism as one can see. The properties of entire ring extensions imply then that:

$$\dim_K(T_n(K)/\mathcal{I}(X(\mathbb{K}))) = \dim_K(T_n(\mathbb{K})/\mathcal{I}(X(\mathbb{K}))T_n(\mathbb{K}));$$

(see [At-Mac] Corollary 5.9 and Theorem 5.11). The affinoid spaces version of Nullstellensatz implies now that:

$$\dim_K(T_n(\mathbb{K})/\mathcal{I}(X(\mathbb{K})))T_n(\mathbb{K})) = \dim_K(T_n(K)/\mathcal{I}(X));$$

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and this implies the equality of dimensions of these two affinoid spaces. We remark in particular with the same arguments that:

$$\dim_K(X'(K)) = \dim_K(X')$$

for each irreducible affinoid sub-space $X'$ of $X$. We suppose now that $X(K)$ is absolutely irreducible inside $\mathbb{K}$. Let $X$ be the affinoid space contained in $\mathbb{K}^n$, defined over $\mathbb{K}$ by the generators of the ideal $\mathcal{I}(X(K))$ that we have previously introduced, and let $\mathcal{I}(X)$ the ideal associated to $X$ as explained before. As $X(K)$ is absolutely irreducible in $\mathbb{K}$ the ideal $\mathcal{I}(X(K))T_n(\mathbb{K})$ is still a prime ideal of $T_n(\mathbb{K})$. We have in particular that:

$$\mathcal{I}(X) = \mathcal{I}(X(K))T_n(\mathbb{K}).$$

It follows that for each point $\zeta_0 \in X(K)$ the Jacobian matrix $J_{\zeta_0}(\mathcal{I}(X(K)))$ of $X(K)$ at $\zeta_0$ is also the Jacobian matrix $J_{\zeta_0}(\mathcal{I}(X))$ of $X$. In particular:

$$\rho_K(J_{\zeta_0}(\mathcal{I}(X(K)))) \geq \rho_K(J_{\zeta_0}(\mathcal{I}(X))), \quad \forall \zeta_0 \in X(K);$$

and this implies that:

$$X(K)_{\text{sing.}} = X_{\text{sing.}}(K).$$

In fact we’re talking about the same points of $X(K)$ which respect the same algebraic equations induced on $\mathbb{K}^n$ by the annihilation of all minors with order $n - \dim(X)$ of the same Jacobian matrix. It may exist some points $\zeta_0 \in X(K)$ such that the rank of $J_{\zeta_0}(\mathcal{I}(X(K)))$ over $\mathbb{K}$ is strictly higher than the rank over $\mathbb{K}$ but the minimal generators of $\mathcal{I}(X(K))$ do not necessarily remain minimal in $\mathbb{K}[X_1, ..., X_n]$; in such a case, this would actually imply that the number of rows of the matrix $J_{\zeta_0}(\mathcal{I}(X))$ would be $> n - \dim(X)$. The previous arguments allow us to say finally that:

$$\dim_K(X(K)_{\text{sing.}}) = \dim_K(X_{\text{sing.}}(K)) = \dim_K(X_{\text{sing.}}) < \dim_K(X) = \dim_K(X(K)).$$

$$\blacksquare$$

2.6 The tangent space

During this last preliminary sub-section we’re about to use all notions previously introduced in order to prove finally the fundamental conclusions which would allow us to apply (on the tangent space $\text{Lie}(\mathcal{A})$ of a $T$–module $\mathcal{A}$ respecting the hypothesis that we’ll introduce into the next section) the methods of J. Pila and J. Wilkie to our particular situation.
Lemma 2. Let $\mathcal{A} = (\mathbb{G}_m, \Phi)$ be an abelian uniformizable $T$-module having dimension $m$ and rank $d$ defined over the field $F \subset \mathbb{K}$. The exponential function $\exp$ defined on $\mathcal{A}$, with lattice $\Lambda$, induces then the following $\mathcal{A}$-module isomorphisms:

$$\mathcal{A}(\mathbb{C}) \simeq \mathbb{C}^m / \Lambda \simeq (k_\infty / A)^d \oplus \text{Lib};$$

where $\text{Lib}$ is the free part of $\mathbb{C}^m / \Lambda$ as $k_\infty$-space.

Proof. The exponential function is a surjective additive group morphism from $\mathbb{C}^m$ to $\mathcal{A}$ whose kernel is $\Lambda$. The Factorization Lemma gives us then the group isomorphism:

$$\mathcal{A}(\mathbb{C}) \simeq \mathbb{C}^m / \Lambda;$$

easily verifying the fact to be also an $\mathcal{A}$-module isomorphism. On the other hand $\mathbb{C}^m$ is a $k_\infty$-space of infinite dimension and its quotient by $\Lambda$ is the direct sum of a torsion part with dimension $d$ and a free part with infinite dimension. If, then, $\overline{z} \in Lie(\mathcal{A}) \simeq \mathbb{C}^m$, we’ll have:

$$\overline{z} = (z_1, \ldots, z_d, z') \in \bigoplus_{i=1}^{d} \mathbb{w}_i k_\infty \oplus \text{Lib};$$

where $\Lambda = <\overline{w}_1, \ldots, \overline{w}_d >_A$. As $\overline{w}_1, \ldots, \overline{w}_d$ are $k_\infty$-linearly independent one can choose them as a base over $k_\infty$ for the torsion sub-space, completing them to a base of $\mathbb{C}^m$ over $k_\infty$. Up to a similar choice, we have then an infinite-dimension $k_\infty$-vector spaces isomorphism:

$$\phi : \mathbb{C}^m \to \mathbb{C}^m;$$

fixing the base of $\text{Lib}$ and such that:

$$\phi(\overline{w}_i) := \overline{v}_i;$$

for each $i = 1, \ldots, d$, where $\overline{v}_1, \ldots, \overline{v}_d$ are the vectors of the canonical base of $k_\infty^d$. In other words the isomorphism $\phi$ acts as it follows:

$$\phi : \bigoplus_{i=1}^{d} k_\infty \overline{w}_i \oplus \text{Lib} \to k_\infty^d \oplus \text{Lib}.$$ 

The homomorphism $\pi_{A^d} : k_\infty^d \oplus \text{Lib} \to (k_\infty / A)^d \times \{0\}$ induces, after suitable quotient of $k_\infty^d \oplus \text{Lib}$ by $A^d \times \{0\}$, the following composed homomorphism:

$$\pi_{A^d} \circ \phi : \bigoplus_{i=1}^{d} k_\infty \overline{w}_i \oplus \text{Lib} \to (k_\infty / A)^d \oplus \text{Lib}.$$ 

whose the kernel is clearly $<\overline{w}_1, \ldots, \overline{w}_d >_A \simeq \Lambda$. This induces the expected isomorphisms. \qed
Let $X$ be an irreducible algebraic sub-variety, defined over $\overline{k}$, of a $T$–module $\mathcal{A}$. We define $K_X \subset \overline{k}$ the field generated by a system of fixed polynomials defining $X$, which is then a finite extension of $k$. Let:

$$K := K_X \mathcal{F}.$$ 

As $K \subset \overline{k_\infty}$, it is a valued field with rapport to a valuation univocally induced by the $1/T$–adic one of $k_\infty$ on its algebraic closure. We define then $\hat{K}$ as the completion of $K$ with rapport to this valuation. It follows that $\hat{K}$ is a finite extension of $k_\infty$. We consider the following projection maps:

$$\Pi_i : \mathcal{C}^m \to \mathcal{C}, \ \forall i = 1, \ldots, m;$$

on each component of $\mathcal{C}^m$ respectively. We define:

$$\Lambda_i := \Pi_i(\Lambda);$$

for each $i = 1, \ldots, m$. They are $m$ $A$–lattices into $\mathcal{C}$ such that:

$$\Lambda = \bigoplus_{i=1}^{m} \Lambda_i.$$ 

We define the following field:

$$K_\infty := \hat{K}(\Lambda_1, \ldots, \Lambda_m);$$

which is the completion of $\hat{K}(\Lambda_1, \ldots, \Lambda_m)$ with rapport to the unique valuation induced by that of $\hat{K}$. We remark that:

$$\mathcal{P}(K_\infty^m) \subset K_\infty^m.$$ 

**Theorem 12.** Let $\mathcal{A}$ be a $T$–module with dimension $m$ and rank $d$, and $X$ be an algebraic sub-variety of $\mathcal{A}$ as in the hypothesis of Conjecture 1. Let $\mathcal{P} : \text{Lie}(\mathcal{A}) \to \mathcal{A}$ be the exponential function associated to $\mathcal{A}$. Let:

$$Y := \mathcal{P}^{-1}(X) \subset \text{Lie}(\mathcal{A})(\mathcal{C}).$$

Therefore, up to extending $K_\infty$ to a finite extension $L$ and calling $n := [L : k_\infty]$, we have the following properties:

1. $B_1^m(L) \cap Y(L) \neq \emptyset$;
2. $Y$ is a $L$–entire sub-set of $\text{Lie}(\mathcal{A})(\mathcal{C})$;
3. \( Y(L) \) is a \( L \)-entire sub-set of \( \text{Lie}(A)(L) \);

4. \( B^m_1(L) \cap Y(L)_{\text{reg.}} \) is dense in \( B^m_1(L) \cap Y(L) \);

5. The isomorphism \( \phi \) introduced in the proof of Lemma 2 restricts to the \( L \)-rational points of \( C^m \) to a \( k_\infty \)-vector space isomorphism (that we’ll keep on calling \( \phi \)) under the following form:

\[
\phi : L^m \to k_{\infty}^m.
\]

We have then the following decomposition:

\[
A(L) \simeq L^m/\Lambda \simeq (k_\infty/A)^d \bigoplus \text{Lib}(L).
\]

Moreover, if \( Y(L) \) is an \( L \)-entire sub-set of \( L \) it follows that \( Y'(k_\infty) := \phi(Y(L)) \) is a \( k_\infty \)-entire sub-set of \( k_{\infty}^m \).

Proof. 1. As the base field \( K_X \) of \( X \) is contained in \( K \) and so in \( L \) we have that \( X \) is defined over \( L \). Up to extend \( L \) to a finite extension we have that \( B^m_1(L) \cap X(L) \neq \emptyset \). Now we recall that the exponential function is \( K \)-entire under the following form:

\[
e(z) = \sum_{i \geq 0} B_i z^q_i;
\]

for each \( z \in \text{Lie}(A)(C) \), and at the same time, a local homeomorphism. For each \( \overline{z}_0 \in X(L) \) it exists then an open neighborhood \( V_{\overline{z}_0} \subset A(C) \) of it, a point \( \overline{z}_0 \in e^{-1}(\overline{z}_0) \), an open neighborhood \( U_{\overline{z}_0} \subset \text{Lie}(A)(C) \) of \( \overline{z} \) and a \( C \)-analytic function under the following form:

\[
\overline{\log}_{\overline{z}_0} : V_{\overline{z}_0} \to U_{\overline{z}_0};
\]

that is an homeomorphism between these two open sets. The explicit construction of such a function and the proof that it is convergent over some \( U_{\overline{z}_0} \) enough small are exactly the same ones that one can do studying an Implicit Function Theorem version on general archimedean fields, as in Cartan’s style, in order to prove the existence of the inverse function (see [I], Theorem 2.1.1). We remark now that if we choose \( \overline{w}_0 = \overline{0} \in A \) and \( \overline{z}_0 = \overline{0} \in \text{Lie}(A) \), the logarithm function:

\[
\overline{\log}_{\overline{w}} : V_{\overline{w}} \to U_{\overline{w}};
\]

is \( K \)-analytic, by the construction of its formal expression in power series:

\[
\overline{\log}_{\overline{w}}(\overline{w}) = \sum_{i \geq 0} A_i \overline{w}^q_i;
\]
we easily realise that the matrices $A_i$ have their entries into $K$ and then into $L$. As $\bar{0} \in V_0 \cap B^m_1(L) \neq \emptyset$ we can assume without loss of generality that $V_0$ is a polydisc with radius $\leq 1$ containing $\bar{0}$ up to restrict the logarithmic function $\log \tau$ to some polydisc contained into $V_0$ and containing $\bar{0}$. It follows that:

$$\log \tau (V_0 \cap X(L)) \subset Y(L).$$

If $V_0 \cap X(L) \neq \emptyset$ we’re done. Assume then that $\tau \in A \setminus V_0$ for each $\tau \in X(L)$. Let $\pi \in \bar{\tau}^{-1}(\tau)$. Let $D_0^\pi$ a polydisc with radius $\delta_0 > 0$ containing $\bar{0}$ and contained in $U_0$. As the function $\log \tau$ is continue we have that $\pi(D_0^\pi)$ is an open set in $V_0$ and it contains consequently il contient a polydisc $D_0$ with radius $\epsilon_0 > 0$ containing $\bar{0}$. For each $\tau \in D_0^\pi$, we have that $|\tau|_{\infty} \leq \delta_0$. By Definition 1 we know that it exists a number $s \in N \setminus \{0\}$ such that the differential $d\Phi(T^s)$ is a diagonal matrix under the form $T^s I_m$. Let $a \in N$ be the smallest entire number such that:

$$|T^{-a} \pi \tau|_{1/T} \leq \delta_0.$$

As $\pi := T^{-a} \pi \tau \in D_0^\pi$, we have that $\log \tau (\tau) \in D_0^\pi(L)$, up to extend one more time $L$ adding all solutions of the following polynomial equation:

$$\Phi(T^a) (\tau) - \tau = \bar{0}.$$ 

As the $T$–module has finite rank, we know that this equation only has a finite number of solutions. As:

$$\tau \in \bar{\tau}^{-1}(\tau);$$

up to a finite extension of $L$ we consequently have:

$$B^m_1(L) \cap Y(L) \neq \emptyset.$$

2. We know that $X$ is defined as the zeroes locus of a finite number $r$ of polynomials $P_1, \ldots, P_r$ with coefficients in $K_X$ and so in $L$. Up to the isomorphism $\text{Lie}(A)(C) \cong C^m$ we see that $Y$ is a $L$–entire sub-set of $C^m$. In fact, each $P_i$ ($i = 1, \ldots, r$) represents a $L$–entire function acting from $C^m$ to itself; the exponential function is on the other hand a $K$–entire function too, acting from $C^m$ to itself by [Goss], Lemma 5.9.3. $Y$ is so the zeroes locus into $C^m$ of the $r$ $L$–entire functions $f_1 := P_1 \circ \tau, \ldots, f_r := P_r \circ \tau$. It is therefore a $L$–entire sub-set of $C^m$. 48
3. We remark that $Y(L)$ is a $L$-entire sub-set of $Lie(A)(L)$ as $Y$ is the zeroes locus of a finite number of $L$-entire functions taking their values into $L$ if restricted to $L^m$. We remark anyway that such an argument, apparently trivial, does not remain necessarily true in case of $f_i$ is a $L$-analytic function only. In fact, for each $z_0 \in V$ it would exists an open neighborhood $U_{z_0} \subset C^m$ such that $f_i(z)$ would be a power series (with coefficients in $L$) of $z - z_0$ for every $z \in U_{z_0}$, but as $L$ is not dense in $C$, the intersection of such a neighborhood with $L^m$ may be empty. Even if $Y(L)$ was not empty, it wouldn’t be necessarily $L$-analytic.

4. We assume that the $L$-analytic space $B^m_1(L) \cap Y(L)$ is irreducible but not absolutely irreducible in the perfect closure $\mathbb{K}$ of $L$ in $C$. As the extension $L \subseteq \mathbb{K}$ is purely inseparable the prime ideal $\mathcal{I}(Y(L))$ is a primary ideal of $T_m(\mathbb{K})$ and its radical into $T_m(\mathbb{K})$ is $\mathcal{I}(Y(\mathbb{K}))$. If $f \in \mathcal{I}(Y(\mathbb{K}))$ it is in particular an element of $T_m(\mathbb{K})$, purely inseparable over $T_m(L)$. It exists then a number $n \in \mathbb{N} \setminus \{0\}$ such that $f^{p^n} \in T_m(\mathbb{K})$. On the other hand every monic polynomial $P(X) \in T_m(L)[X]$ such that $P(f) = 0$ has its degree divisible by a power of $p$. It exists also a smallest number $n' \in \mathbb{N} \setminus \{0\}$ such that $f^{p^n'} \in \mathcal{I}(Y(L))$. So, $P(X) = X^{n'} - f^{p^{n'}} \in T_m(L)[X]$ is such that $P(f) = 0$ and consequently it exists a number $n \in \mathbb{N} \setminus \{0\}$ such that $n' = p^n$. Every finite minimal system of generators of $\mathcal{I}(Y(\mathbb{K}))$ is then a finite set of $p^n$-th roots (for suitable $h \in \mathbb{N} \setminus \{0\}$) of the same number of elements of $\mathcal{I}(Y(L))$. If we call:

$$\mathcal{I}(Y(\mathbb{K})) = (g_1, ..., g_r) \subset T_m(\mathbb{K});$$

we obtain that:

$$\mathcal{I}(Y(L)) = (f_1, ..., f_r) = (g_1^{p^{n_1}}, ..., g_r^{p^{n_r}}) = \mathcal{I}(Y(\mathbb{K}))^{p^n};$$

where $p^n := \max_{i=1,...,r}\{p^{n_i}\}$. The coefficients of the generators $g_1, ..., g_r$ of $\mathcal{I}(Y(\mathbb{K}))$ are therefore contained in a suitable extension of $L$ of degree $\leq p^n$ in $\mathbb{K}$. We can then assume, up to a finite extension of $L$ that $B^m_1(L) \cap Y(L)$ is absolutely irreducible in $\mathbb{K}$. Such an affinoid space respects the hypothesis of Theorem 11 and so $B^m_1(L) \cap Y(L)_{reg.}$ is dense in $B^m_1(L) \cap Y(L)$.

5. Each function $f : C^m \to C$ defining $Y$ is under the following form:

$$f(z) = \sum_{j \geq 0} \sum_{\mu \in \Lambda_m(j)} a_{\mu} z^\mu \quad \forall z \in C^m;$$

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where all $a_{\mu} \in L$. We introduce the following notation in order to describe a generic element $\tau \in L^m$. Let $\tau_{1,1}, ..., \tau_{n,m}$ be the canonical base of $k_{\infty}^{nm}$ over $k_{\infty}$, and let the following new indexation that we introduce on double indexes $(i, j)$, $i = 1, ..., m$, $j = 1, ..., n$:

$$(i, j) < (i, j + 1) < (i + 1, j).$$

We remark that:

$$[k_{\infty} : k_{\infty}^p] = p,$$

the result of M. F. Becker and S. MacLane [B-M] leads us to remark that it exists a primitive element $\alpha \in L$ such that $L = k_{\infty}(\alpha)$. So, let $1, \alpha, ..., \alpha^{n-1}$ be a base of $L$ over $k_{\infty}$. Let $\tau_1, ..., \tau_m$ the canonical base of $L^m$ over $L$. We define the adapted basis $\tau_{1,1}, ..., \tau_{m,n}$ of $L^m$ over $k_{\infty}$ as it follows:

$$\tau_{i,j} := \alpha^{j-1} \tau_i, \quad \forall i = 1, ..., m, \quad \forall j = 1, ..., n.$$

It is easy to see that it’s actually a set of $k_{\infty}$—linearly independent vectors of $L^m$. We define the following isomorphism between $L^m$ and $k_{\infty}^{nm}$:

$$\psi : L^m \rightarrow k_{\infty}^{nm};$$

$$\tau_{i,j} \mapsto \tau_{i,j};$$

which induces a bijection between the adapted basis of $L^m$ over $k_{\infty}$ and the canonical base of $k_{\infty}^{nm}$ over $k_{\infty}$. As $\omega_1, ..., \omega_d$ are $k_{\infty}$—linearly independent vectors of $L^m$ too, they could be completed (up to a new indexation) to a base of $L^m$ over $k_{\infty}$ adding $nm - d$ elements of the adapted basis of $L^m$ over $k_{\infty}$. Using in fact the new indexation we introduced previously on double indexes $i = 1, ..., m$, $j = 1, ..., n$ we enumerate the elements of the adapted basis of $L^m$ over $k_{\infty}$ using just one index, and this allows to describe the completion of the vectors $\omega_1, ..., \omega_d$ to a base of $L^m$ over $k_{\infty}$ as it follows:

$$\omega_1, ..., \omega_d, \tau_{d+1}, ..., \tau_{nm}.$$
of $\psi(\text{Lib.}(L)) = \bigoplus_{i=d+1}^{nm} k_\infty$ over $k_\infty$, after the following decomposition of $L^m$:

$$L^m = \bigoplus_{i=1}^{d} k_\infty \overline{w}_i \oplus \text{Lib.}(L);$$

induced by the one which follows from Lemma 2 by the isomorphism $\psi$. We consider the isomorphism $\phi : C^m \to C^m$ we’ve introduced in the proof of Lemma 2 choosing some suitable completion to an infinite base of $C^m$ over $k_\infty$ of the vectors $\overline{w}_1, \ldots, \overline{w}_d$ which restrict to $L^m$, to the completion of these vectors by the adapted basis of $L^m$ over $k_\infty$ previously described. The $k_\infty$-vector space isomorphism $\phi$ that we’ve defined respects then the following property:

$$\phi = \mathcal{L} \circ \psi.$$

It is therefore such that for each $z \in L^m$ expressed under the following form:

$$z = \sum_{i=1}^{d} w_i \overline{w}_i + \sum_{i=d+1}^{nm} w_i \overline{u}_i;$$

we have that:

$$\phi(z) = \overline{w} = (w_1, \ldots, w_{nm}).$$

We have that:

$$f(z) = \sum_{h \geq 0} \sum_{\mu \in \Lambda_m(h)} a_\mu \left( \sum_{i=1}^{m} \sum_{j=1}^{n} w_{i,j} \overline{w}_{i,j} \right)^\mu = 0, \ \forall z \in Y(L);$$

where the coefficients $w_{i,j}$ of this linear combination of the elements of the adapted basis of $L^m$ over $k_\infty$ are in $k_\infty$. In particular, if one takes the restriction to $L^m$ of the isomorphism $\psi$ that we previously described, we have that:

$$\psi(z) = \overline{w} := (w_{1,1}, \ldots, w_{1,n}, \ldots, w_{m,1}, \ldots, w_{m,n}).$$

For each $j \geq 0$, $\mu = (r_1, \ldots, r_m) \in \Lambda_m(j)$, each term $\overline{w}^\mu$ could be expressed under the following form:

$$\left( \sum_{i=1}^{m} \sum_{j=1}^{n} w_{i,j} \alpha^{j-1} \overline{w}_i \right)^\mu = \prod_{h=1}^{m} \left( \sum_{j=1}^{n} w_{h,j} \alpha^{j-1} \right)^{r_h} = \prod_{h=1}^{m} \sum_{s_1 + \ldots + s_n = r_h} \left( \sum_{j=1}^{n} w_{h,j} \alpha^{s_j (j-1)} \right)^{s_h}.$$
It follows that:

\[(f \circ \psi^{-1})(\varpi) = \sum_{j \geq 0} \sum_{\mu=(r_1, \ldots, r_m) \in \Lambda_m(j)} a_{\mu} \prod_{h=1}^{m} \sum_{s_1 + \ldots + s_n = r_h} \left( \prod_{j=1}^{n} w_{h,j}^{s_j} \right)^{r_h} \alpha_{s_1, \ldots, s_n}. \]

Let:

\[h : k^{nm} \to L;\]

the following \(L\)-analytic function:

\[h := f \circ \psi^{-1}.\]

It is then possible to express \(f(\varpi)\) in the new following form:

\[h(\varpi) = \sum_{i \geq 0} \sum_{\eta \in \Lambda_{nm}(i)} c_{\eta} \varpi^{i};\]

where each coefficient \(c_{\eta} \in L\) could be expressed as it follows. For each \(i \geq 0\), if \(\mu = (r_1, \ldots, r_m) \in \Lambda_m(i)\), for each \(r_h, h = 1, \ldots, m\) we define as before \((s_1(h), \ldots, s_n(h)) \in \Lambda_n(r_h)\). We define then:

\[\eta(\mu) := (s_j(h))_{j=1, \ldots, n, h=1, \ldots, m} \in \Lambda_{nm}(i);\]

laying out its components following the previous criterion on double indexes \((h, j)\). For each \(\eta \in \Lambda_{nm}(i)\) and each \(i \geq 0\) we define \(\mathcal{I}(\eta) := \{\mu \in \Lambda_m(i), \eta(\mu) = \eta\}\). We call:

\[\vec{\alpha} := (\alpha, \ldots, \alpha) \in L^{nm}.\]

Let \(\eta(\mu)\) be as we’ve defined it before. We have then:

\[\eta(\mu) := (s_1(1), \ldots, s_n(1), \ldots, s_1(m), \ldots, s_n(m)).\]

We call:

\[\tilde{\eta}(\mu) := (0, s_2(1), 2s_3(1), \ldots, (n-1)s_n(1), \ldots, 0, s_2(m), 2s_3(m), \ldots, (n-1)s_n(m)) \in \mathbb{N}^{nm}.\]

We then have the following property:

\[c_{\eta} = \sum_{\mu \in \mathcal{I}(\eta)} a_{\mu} \prod_{h=1}^{m} \sum_{s_1 + \ldots + s_n = r_h} \left( \prod_{j=1}^{n} w_{h,j}^{s_j} \right)^{r_h} \alpha_{s_1, \ldots, s_n}.\]

It follows that:

\[|c_{\eta}|_{1/T} \leq \max_{\mu \in \mathcal{I}(\eta)} |a_{\mu}|_{1/T} |\alpha|_{1/T}^{m(n-1)|\eta|};\]
which converges to 0 when $|\eta|$ approaches to infinity because $f : L^m \rightarrow L$ is an entire function. We also remark that it is possible to express each coefficient $c_\eta \in L$ for each $\eta \in \Lambda_{nm}(i)$ and each $i \geq 0$ as a linear combination over $k_\infty$ with rapport to the base $1, \alpha, \ldots, \alpha^{n-1}$ of $L$ over $k_\infty$. We obtain so that it exist unique $b_{\eta,1}, \ldots, b_{\eta,n} \in k_\infty$ such that:

$$c_\eta = \sum_{j=1}^{n} b_{\eta,j} \alpha^{j-1};$$

for each $\eta \in \mathbb{N}^{nm}$. It exist too the following formal power series:

$$h_1, \ldots, h_n \in k_\infty[[\overline{w}]];$$

$$h_j(\overline{w}) := \sum_{i \geq 0} \sum_{\eta \in \Lambda_{nm}(i)} b_{\eta,j} \overline{w}^i, \forall j = 1, \ldots, n;$$

such that:

$$h(\overline{w}) = \sum_{j=1}^{n} h_j(\overline{w}) \alpha^{j-1}.$$

Now we show that $h_1, \ldots, h_n$ are convergent with infinite radius.

In fact, using the result of K. Mahler [Mah], page 491, it exists $\gamma > 0$ such that for each $c_\eta$ coefficient of $h$, we have that:

$$|c_\eta|/T \geq \gamma \max_{j=1, \ldots, n} \{|b_{\eta,j}|/T|\alpha^{j-1}|/T\};$$

Up to modify the value of $\gamma$ we can assume that:

$$|c_\eta|/T \geq \gamma \max_{j=1, \ldots, n} \{|b_{\eta,j}|/T\};$$

for each coefficient $c_\eta$ of $h$. The convergence of the series $h(\psi(\overline{z}))$ implies then that of $h_j(\psi(\overline{z}))$ for each $j = 1, \ldots, n$. Now let $\mathcal{L} : k^{nm}_\infty \rightarrow k^{nm}_\infty$ the isomorphism over $k_\infty$ we’ve defined before, which maps the set $\{\psi(\overline{w}_1), \ldots, \psi(\overline{w}_d)\}$ of linearly independent elements over $k_\infty$ to the first $d$ elements $\{\overline{v}_1, \ldots, \overline{v}_d\}$ of the canonical base of $k^{nm}_\infty$ over $k_\infty$ arranged under the criterion we’ve introduced previously on double indexes. It acts over $k^{nm}_\infty$ as an invertible matrix of $k^{nm,nm}_\infty$. Then, one sees immediately that it consists in a $k_\infty$–entire function from $k^{nm}_\infty$ to itself. We saw that:

$$\phi = \mathcal{L} \circ \psi.$$

As we did show that the following function:

$$f \circ \psi^{-1} : k^{nm}_\infty \rightarrow L;$$

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could be expressed as it follows:

\[ f \circ \psi^{-1} = \sum_{j=1}^{n} \alpha^{j-1} h_j; \]

where \( h_1, \ldots, h_j \) are \( k_\infty \)-entire functions from \( k_{nm}^\infty \) to \( k_\infty \) and as the composition between entire functions remains an entire function, it follows that the function:

\[ h \circ \mathcal{L}^{-1} = f \circ \psi^{-1} \circ \mathcal{L}^{-1} = f \circ \phi^{-1}; \]

is still a linear combination of the base \( \{1, \alpha, \ldots, \alpha^{n-1}\} \) of \( L \) over \( k_\infty \) by \( k_\infty \)-analytic functions:

\[ \tilde{h}_1 := h_1 \circ \mathcal{L}^{-1}, \ldots, \tilde{h}_n := h_n \circ \mathcal{L}^{-1}; \]

from \( k_{nm}^\infty \) to \( k_\infty \). In other words:

\[ f \circ \phi^{-1} = \sum_{j=1}^{n} \alpha^{j-1} \tilde{h}_j. \]

Calling:

\[ Y' := \{ \overline{w} \in C_{mn}, \tilde{h}_{i,j}(\overline{w}) = 0, \ \forall i = 1, \ldots, r, \ \forall j = 1, \ldots, n \}; \]

where:

\[ Y(L) = \{ \overline{y} \in L^m, \ f_i(\overline{y}) = \sum_{j=1}^{n} \alpha^{j-1} \tilde{h}_j(\phi(\overline{y})) = 0, \ \forall i = 1, \ldots, r \}; \]

we have that \( Y'(k_\infty) = \phi(Y(L)) \) and that \( Y'(k_\infty) \) is a \( k_\infty \)-entire sub-set of \( \phi(L_{\text{Lie}}(A)(L)) \simeq k_{nm}^\infty \).

Let \( \alpha \in L \) be the primitive element of the field extension \( k_\infty \subseteq L \) that we’ve introduced in Theorem 12 point 5. We choose \( a(T), b(T) \in k_\infty \) tels que \( |a(T)\alpha|_{1/T} < 1 \) and \( |b(T)|_{1/T} = 1 \). We call:

\[ \beta := a(T)\alpha + b(T). \]

As:

\[ \alpha \in k_\infty(\beta); \]

it follows that \( \beta \) is a primitive element of the field extension \( k_\infty \subseteq L \) also. We can then replace \( \alpha \) by \( \beta \) and, therefore, assume without loss of generality that:

\[ |\alpha|_{1/T} = 1. \]
Definition 18. Let be \( \bar{z} = (z_1, ..., z_m) \in L^m \). We express this point univocally as it follows:

\[ \bar{z} = (z_1, ..., z_m) = \left( \sum_{j=1}^{n} \alpha^{j-1}w_{1,j}, ..., \sum_{j=1}^{n} \alpha^{j-1}w_{m,j} \right); \]

where:

\[ w_{1,1}, ..., w_{m,n} \in k_\infty. \]

We define the following norm on \( L^m \) seeing this one as a vector space over \( k_\infty \):

\[ F_\alpha(\bar{z}) := ||(w_{1,1}, ..., w_{m,n})||_\infty = \max\{|w_{1,1}|_{1/T}, ..., |w_{m,n}|_{1/T}\}. \]

We immediately remark that it is actually a norm of \( L^m \) over \( k_\infty \). Let \( r > 0 \) be a positive real number. We define the following sub-set of \( L^m \):

\[ B_{m,r,\alpha}(L) := \{ \bar{z} \in L^m, \ F_\alpha(\bar{z}) \leq r \}. \]

Remark 8. The norm \( F_\alpha \) over \( L^m \) is equivalent to the previous one \( ||.|||_\infty \) of \( L^m \) seen as a vector space over \( L \) with rapport to the canonical base:

\[ ||\bar{z}||_\infty := \max\{|z_1|_{1/T}, ..., |z_m|_{1/T}\}. \]

Proof. By [BGR] Corollary 2.1.9/4 page 78 it will be sufficient to show that it exist \( r_1, r_2 \in \mathbb{R}_{>0} \) such that:

\[ r_1 F_\alpha(\bar{z}) \leq ||\bar{z}||_\infty \leq r_2 F_\alpha(\bar{z}), \ \forall \bar{z} \in L^m. \]

As the \( 1/T \)–adic absolute value is non-archimedean we can immediately remark that:

\[ r_2 = 1. \]

The other inequality follows from a result of K. Mahler, see [Mah] page 491, that we’ve already used in the proof of Theorem 12 point 5 and which implies that:

\[ r_1 = \gamma; \]

where the constant \( \gamma > 0 \) depends on \( \alpha \) and it is exactly what one obtains by K. Mahler’s result using \( F_\alpha \) as a norm of \( L^m \) as explained in the statement we indicate.

It follows that \( B_{r,\alpha}(L) \) (as a sub-set of \( L^m \)) is also an open set with rapport to the \( 1/T \)–adic topology we’ve used until now. In particular the isomorphisms \( \phi : L^m \to k_\infty^m \) and \( \psi : L^m \to k_\infty^m \) of vector spaces over \( k_\infty \) that we’ve introduced in the proof of Theorem 12 point 5 are homeomorphisms.
Remark 9. Let $X$ be an $L$–analytic sub-set of $L^m$ and let $z_0 \in X$ as in the hypothesis of Corollary 1, assuming that $X$ is the zero locus of a finite number $m'$ of $L$–analytic functions defined over an open set $U$ of $L^m$ and defining the vector:

$$\mathcal{F} : U \to L^{m'};$$

of functions (not depending by their indexation), such that $m' < m$. So, it exists an open neighborhood of $z_0$ in $L^m$ under the form $B_{1,\alpha}^{m-m'}(L) \times V_{z_0}$ and a $L$–analytic function on the following shape:

$$f_{z_0} : B_{1,\alpha}^{m-m'}(L) \to V_{z_0} \subset L^{m-m'} \times L^{m'};$$

such that for every $z^* \in B_{1,\alpha}^{m-m'}(L)$ the following property is true:

$$\mathcal{F}(z^*, f_{z_0}(z^*)) = 0, \forall z^* \in B_{1,\alpha}^{m-m'}(L).$$

Proof. By Corollary 1 it exists an open neighborhood $U_{z_0} \times V_{z_0} \subset L^{m-m'} \times L^{m'}$ of $z_0$ and a $L$–analytic function $f : U_{z_0} \to V_{z_0}$ such that:

$$\mathcal{F}(z^*, f(z^*)) = 0, \forall z^* \in U_{z_0}.$$ 

Up to compose $f$ with a translation in $L^{m-m'}$ we assume that $z_0 = 0$. We choose an open set $B_{r,\alpha}^{m-m'}(L) \subset U_{z_0}$ with $r \in |k_\infty|_{1/T}$. The restriction of $f$ to $B_{r,\alpha}^{m-m'}(L)$ remains a $L$–analytic function on $B_{r,\alpha}^{m-m'}(L)$. We compose now $f$ with a linear map over $L$ under the following form:

$$t : B_{1,\alpha}^{m-m'}(L) \to B_{r,\alpha}^{m-m'}(L);$$

$$z^* \mapsto c z^*;$$

where $c \in k_\infty$ is such that $|c|_{1/T} = r$. This gives us a $L$–analytic function as it follows:

$$f \circ t : B_{1,\alpha}^{m-m'}(L) \to V_0$$

such that:

$$\mathcal{F}(z^*, (f \circ t)(z^*)) = 0, \forall z^* \in B_{1,\alpha}^{m-m'}(L).$$

Up to compose again $f \circ t$ by the translation:

$$z \mapsto z + z_0;$$

we obtain a neighborhood of $z_0$ in $L^{m-m'} \times L^{m'}$ and a $L$–analytic function as we expected. \qed
Definition 19. Let $X$ be an analytic sub-set of $L^m$, defined over an open set $U$ of $L^m$. We say that $X$ is \textbf{analytically $\alpha$-parametrizable} if it exist a number $d(X) \in \mathbb{N} \setminus \{0\}$ and a family $\mathcal{R}$ of $L$–analytic functions as it follows:

$$f : B_{1,\alpha}^{d(X)}(L) \to X;$$

such that:

$$X \subseteq \bigcup_{f \in \mathcal{R}} f(B_{1,\alpha}^{d(X)}(L)).$$

We call such a family a \textbf{$\alpha$–analytical cover} of $X$ over $L$.

Theorem 13. 1. Let $Y(L)$ defined as in Theorem 12. If $B_m^0(L) \cap Y(L)_{\text{reg.}}$ is dense in $B_m^0(L) \cap Y(L)$, then $B_m^0(L) \cap Y(L)$ is analytically $\alpha$–parametrizable over $L$.

2. Let $Y(L)$ and $Y'(k_{\infty})$ defined as in Theorem 12. Therefore, $B_m^0(k_{\infty}) \cap Y'(k_{\infty})$ is analytically parametrizable over $k_{\infty}$.

Proof. 1. Let $\mathcal{I}(Y(L)) = (f_1, ..., f_r) \subset T_m(L)$ be the prime ideal associated to $B_m^0(L) \cap Y(L)$. Let $\overline{y}_0 \in B_m^0(L) \cap Y(L)_{\text{reg.}}$. Let $d := \dim_L(B_m^0(L) \cap Y(L)) = \dim_K(B_m^0(K) \cap Y(K))$. The equality follows from the passages of the proof of Theorem 11. The hypothesis that $B_m^0(L) \cap Y(L)$ is absolutely irreducible in $K$ implies that $r \geq m - d$. Up to a different indexation of the generators $f_1, ..., f_r$ of $\mathcal{I}(Y(L))$ it follows that:

$$\rho_L(J_{\mathcal{I}_0}(f_1, ..., f_{m-d})) = \rho_K(J_{\mathcal{I}_0}(f_1, ..., f_{m-d})) = m - d.$$ 

We call:

$$Z(f_1, ..., f_{m-d}) := \{ \overline{y} \in B_m^0(L), \ f_1(\overline{y}) = ... = f_{m-d}(\overline{y}) = 0 \}.$$

It follows that $B_m^0(L) \cap Y(L) \subseteq Z(f_1, ..., f_{m-d})$. By Corollary 1, Theorem 11 and Remark 9 it exists then a $\alpha$–analytical cover $\mathcal{R}$ of $B_m^0(L) \cap Y(L)$ given by $L$–analytic functions under the following form:

$$f : B_{1,\alpha}^d(L) \to Z(f_1, ..., f_{m-d}).$$

We obtain in particular that:

$$B_m^0(L) \cap Y(L) \subseteq \bigcup_{f \in \mathcal{R}} f(B_{1,\alpha}^d(L)).$$

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Let $T \subseteq Z(f_1, ..., f_{m-d})$ the only irreducible component of $Z(f_1, ..., f_{m-d})$ to be absolutely irreducible in $\mathbb{K}$, and containing $B_1^m(L) \cap Y(L)$. Let $d' := \dim_L(T)$. Therefore $d' \geq d$. Now:

$$(f_1, ..., f_{m-d}) \subseteq \mathcal{I}(T) = (g_1, ..., g_s).$$

Then for each $f_i$, where $i = 1, ..., m - d$, it exist $a_{i,1}, ..., a_{i,s} \in T_m(L)$ such that:

$$f_i = \sum_{j=1}^{s} a_{i,j} g_j.$$

As $\overline{y}_0 \in T$ it follows that:

$$J_{\overline{y}_0}(f_i) = \sum_{j=1}^{s} a_{i,j} J_{\overline{y}_0}(g_j), \ \forall i = 1, ..., r.$$

If:

$$\rho_L(J_{\overline{y}_0}(f_1, ..., f_{m-d})) = m - d;$$

we have that:

$$\rho_L(J_{\overline{y}_0}(\mathcal{I}(T))) \geq m - d.$$

As $T$ is absolutely irreducible in $\mathbb{K}$ we also have that:

$$\rho_L(J_{\overline{y}_0}(\mathcal{I}(T))) \leq m - d'.$$

It follows that $d' = d$. So:

$$B_1^m(L) \cap Y(L) = T.$$

Now if $S$ is another irreducible component of $Z(f_1, ..., f_{m-d})$:

$$\dim_L(S \cap T) < \dim_L(S), \dim_L(T).$$

In fact, let $R \subseteq S \cap T$ be an irreducible component of $S \cap T$ such that $\dim_L(R) = \dim_L(S \cap T)$. As:

$$S \cap T \nsubseteq S, T;$$

it follows that:

$$\dim_L(R) < \dim(S), \dim_L(T).$$

Up to some affinoid sub-space of $B_1^m(L) \cap Y(L)$ of dimension $< d$ we can then suppose that $\overline{y}_0$ is just contained in only one irreducible component of $Z(f_1, ..., f_{m-d})$ which is $B_1^m(L) \cap Y(L)$. It follows that
$B_1^m(L) \cap Y(L)$ contains a dense sub-set of points $\overline{x}_0 \in B_1^m(L) \cap Y(L)_{\text{reg.}}$ such that for each of them it exists an $L -$analytic function $f$ defined on some convenient open neighborhood $V_{\overline{x}_0}$ of $\overline{x}_0$ such that $V_{\overline{x}_0} \setminus \{\overline{x}_0\} \neq \emptyset$ and such that $f(V_{\overline{x}_0})$ (which is a sub-set of $Z(f_1, ..., f_{m-d})$) is all contained in the irreducible component $B_1^m(L) \cap Y(L)$ of $Z(f_1, ..., f_{m-d})$. By Remark 9 we can then suppose without loss of generality that each $L -$analytic function $f \in \mathcal{R}$ is such that:

$$f : B_{1,\alpha}^d(L) \to B_1^m(L) \cap Y(L).$$

2. Let $f_1, ..., f_r$ be as in the previous point. The isomorphism $\phi$ over $k_\infty$ between $L^m$ and $k_{nm}^m$ is such that:

$$Y'(k_\infty) = \{\overline{w} \in k_{nm}^m, \ (f_i \circ \phi^{-1})(\overline{w}) = 0, \ \forall i = 1, ..., r\}.$$  

In the proof of Theorem 12 point 5 we actually associated to each $f_i$, for every $i = 1, ..., r$, the $n \ k_\infty -$entire functions $\tilde{h}_{i,1}, ..., \tilde{h}_{i,n}$ such that:

$$(f_i \circ \phi^{-1}) = \sum_{j=1}^{n} \alpha_j^{-1} \tilde{h}_{i,j}, \ \forall i = 1, ..., r.$$  

We recall to have defined in particular the following linear map over $k_\infty$:

$$\psi : L^m \to k_{nm}^m;$$

sending the adapted basis $\{\alpha_j^{-1}\overline{\tau}_j\}$ of $L^m$ over $k_\infty$ to the canonical one of $k_{nm}^m$ over $k_\infty$, and the following isomorphism of $k_\infty -$vector spaces over $k_\infty$:

$$\mathcal{L} : k_{nm}^m \to k_{nm}^m;$$

which sends the images $\psi(\overline{\tau}_1), ..., \psi(\overline{\tau}_d)$ by $\psi$ of the $d$ previously fixed periods $\overline{\tau}_1, ..., \overline{\tau}_d$ of $\Lambda$ to the first $d$ elements $\overline{v}_1, ..., \overline{v}_d$ of the canonical base of $k_{nm}^m$ over $k_\infty$ following the indexation criterion of double indexes that we introduced in the proof of Theorem 12 point 5. The $k_\infty -$linear map $\psi$ between $L^m$ and $k_{nm}^m$ is an isometry between these two vector spaces over $k_\infty$ with rapport to the $1/T -$adic absolute value on $k_{nm}^m$ and the norm $F_\alpha$ we’ve introduced in Definition 18 over $L^m$. We then obtain that:

$$\psi^{-1}(B_{1,\alpha}^m(k_\infty)) = B_{1,\alpha}^m(L).$$

Now dim$_L(Y(L))$ is also the constant value such that the $\alpha -$analytic parametrization $\mathcal{R}$ of $B_1^m(L) \cap Y(L)$ following from point 1 is such
that for each $f \in \mathcal{R}$ such a $L$–analytic function takes the following shape:

$$f : B_{1,\alpha}^{\dim_L(Y(L))}(L) \to B_1^m(L) \cap Y(L).$$

Let $d(L)(Y) := \dim_L(Y(L))$. Let $d(Y(k_\infty)) := nd_L(Y)$. We then define the linear map $\psi$ over $k_\infty$ as before, but this time between $L^{d_L(Y)}$ and $k_\infty^{d(Y(k_\infty))}$. The isomorphism $\psi : L^{d_L(Y)} \to k_\infty^{nd_L(Y)}$ of $k_\infty$–vector spaces we’ve defined is also a bijection between $B_{1,\alpha}^{d_L(Y)}(L)$ and $B_1^{d(Y(k_\infty))}(k_\infty)$ and it is such that for each $f \in \mathcal{R}$ the following function:

$$g : B_1^{d(Y(k_\infty))}(k_\infty) \to B_1^{nm}(k_\infty) \cap Y'(k_\infty);$$

such that:

$$g := \phi \circ f \circ \psi^{-1};$$

is $k_\infty$–analytic. It is in fact easy to remark that $\psi^{-1}$ is a $L$–analytic function and therefore $f \circ \psi^{-1}$ is a $L$–analytic function too. In particular, it takes the following form:

$$(f \circ \psi^{-1})(\overline{w}) = \sum_{i \geq 0} \sum_{\mu \in \Lambda_m(i)} a_\mu \psi^{-1}(\overline{w})^\mu.$$

Repeating the same passages of the proof of Theorem 12 point 5 we can express $f \circ \psi^{-1}$ as it follows:

$$(f \circ \psi^{-1})(\overline{w}) = \sum_{j=1}^n \alpha_j^{-1} h_j(\overline{w});$$

where $h_1(\overline{w}), ..., h_n(\overline{w})$ are $n$ $k_\infty$–analytic functions from $B_1^{d(Y'(k_\infty))}(k_\infty)$ to $k_\infty^m$. It is really important to remark that in the proof of Theorem 12 point 5 we obtained such a result on $h_1, ..., h_n$ which were $k_\infty$–entire over $k_\infty^m$ and taking their values in $k_\infty$ as the hypothesis of this Theorem ask that $f$ is a $L$–entire function over $L^m$, and this is not anymore true in this new situation. Such an hypothesis was anyway mandatory as the powers of the $1/T$–adic absolute value of the primitive element $\alpha$ of the finite field extension $k_\infty \subseteq L$ were not bounded. As we showed that it is possible to assume without loss of generality that:

$$|\alpha|_{1/T} = 1;$$

the same argument remains true if $f$ is some $L$–analytic function defined on $B_{1,\alpha}^{d_L(Y)}(L)$. The $k_\infty$–analytic functions $h_1, ..., h_n$ from $B_1^{d(Y'(k_\infty))}(k_\infty)$
to \( k^m \) we’ve just introduced are in particular \( n \) vectors of \( m \) \( k_{\infty} \)-analytic functions from \( B_1^{d(Y'(k_{\infty}))}(k_{\infty}) \) to \( k_{\infty} \) taking the following shape:

\[
h_j(\overline{w}) := (h_{1,j}(\overline{w}), ..., h_{m,j}(\overline{w})), \quad \forall j = 1, ..., n.
\]

It follows that:

\[
(\psi \circ f \circ \psi^{-1})(\overline{w}) = (h_{1,1}(\overline{w}), ..., h_{m,n}(\overline{w})), \quad \forall \overline{w} \in B_1^{d(Y'(k_{\infty}))}(k_{\infty}).
\]

The function \( \psi \circ f \circ \psi^{-1} \) we’ve defined is then \( k_{\infty} \)-analytic over \( B_1^{d(Y'(k_{\infty}))}(k_{\infty}) \). As we showed that \( \mathcal{L} \) is a \( k_{\infty} \)-entire function from \( k_{nm} \) to itself and \( \phi = \mathcal{L} \circ \psi \) we finally obtain that \( g \) is a \( k_{\infty} \)-analytic function from \( B_1^{d(Y'(k_{\infty}))}(k_{\infty}) \) to \( B_1^{nm}(k_{\infty}) \cap Y'(k_{\infty}) \). The following family:

\[
\mathcal{R'} := \{ g = \phi \circ f \circ \psi^{-1}, \quad \forall f \in \mathcal{R} \};
\]

is is then a \( k_{\infty} \)-analytic cover of \( Y'(k_{\infty}) \) taking the form induced by Corollary 1.

\[\square\]

We define the trivial projection functions of \( \phi(\text{Lie}(\mathcal{A})(L)) \) on its two factors which we deduce from those given by the isomorphism introduced in Theorem 12:

\[
\pi_1 : \phi(\text{Lie}(\mathcal{A})(L)) \twoheadrightarrow k^d_{\infty};
\]

\[
\pi_2 : \phi(\text{Lie}(\mathcal{A})(L)) \twoheadrightarrow \text{Lib}(L).
\]

In other words, \( \pi_1 \) is the projection of \( \phi(\text{Lie}(\mathcal{A})(L)) = k_{nm} \) on its \( d \) first components and \( \pi_2 \) is its projection on the \( nm - d \) last components. All topological argument we’ll do on \( Y(L) \) will remain valable on \( Y'(k_{\infty}) \) as it will be unchanged by the linear isomorphisms \( \mathcal{L} \) and \( \phi \). We call the image of \( \pi_1 \) as a \( k_{\infty} \)-vector sub-space of \( k_{nm} \) as it follows:

\[
\Pi := \pi_1(k_{\infty}^{nm}) \simeq k^d_{\infty}.
\]

## 3 Torsion points and varieties

We present a first step within a strategy to prove Conjecture 1, based on the ideas contained in J. Pila’s and J. Wilkie’s work [PW]. We essentially translate the problem in the language of analytic sets in \( \mathcal{C}^m \), which are periodic with rapport to the lattice \( \Lambda \) and subject to the action of \( A \) by usual multiplication. All \( T \)-modules treated here will always assumed to be abelian and uniformizable.
J. Pila’s and U. Zannier’s method used in [PZ] to prove a weaker version of Manin-Mumford conjecture is based on a reductio ad absurdum strategy. Let $A$ be an abelian variety and let $X$ be an algebraic sub-variety of $A$ which does not contain any non-trivial torsion class of $A$. The main Theorem contained in [PW] provides an higher bound estimate of the number of $N$–torsion points (where $N \in \mathbb{N} \setminus \{0\}$) of $A$ which are contained in $X$ in function of their order of torsion $N$. On the other hand a Corollary of D. Masser’s result ([Mas], Corollary page 156) provides a lower bound estimate of the same number, always in function of $N$, in a way such that if the torsion points of $A$ which are contained in $X$ are infinitely many, approaching to infinity with their order of torsion, the two previous inequalities become impossible at the same time.

We propose to repeat such an argument for abelian and uniformizable $T$–modules. In this paragraph we’ll give an higher bound estimate of the number of $a(T)$–torsion points (where $a(T)$ is contained in $A \setminus \{0\}$) of some $T$–module $A$, which are contained in an algebraic sub-variety of this one, which respects the hypothesis of Conjecture 1, in function of $|a(T)|_{1/T}$. Such a result would be our analogue (see Theorem 14) to the main Theorem showed by J. Pila and J. Wilkie in [PW] for abelian varieties. We will adapt their methods to our situation.

One can associate to each abelian variety of dimension $m$ a tangent space isomorphic to $\mathbb{C}^m$ by a topological covering induced by abelian functions, which projects such a space into the variety, in an analogous fashion with the behavior of the exponential function we’ve introduced before on $T$–modules. The method that come out from the work of J. Pila and U. Zannier is based on the bijective correspondance between the abelian sub-varieties of a given abelian variety and their tangent spaces, which one knows to be exactly the sub-spaces of the tangent space of the variety initially given, such that the kernel of the covering (which is a $\mathbb{Z}$–lattice of rank $2m$) intersects them in a way which determines a lattice having maximal rank.

Such a construction is not anymore true in case of $T$–modules. In fact the correspondance between abelian sub-varieties and sub-spaces of the tangent space, is proved (not trivially) by the consequences of Grothendieck’s GAGA Theory (of which it exists an analogue in Rigid Analytic Geometry that could be applied to our situation) and Chow’s Theorem, which on the contrary can not be applied to our situation in cause of the fact that $T$–modules are not compact as topological spaces. In our case, we develop...
the following study.

By Lemma 2 we can see the exponential function as a \( A \)-modules morphism, \( L \)-entire under the following form:

\[
\varphi : \text{Lie}(A)(C) \to A(C).
\]

In Lemma 2 we described an isomorphism:

\[
A(C) \simeq (k_\infty/A)^d \bigoplus \text{Lib};
\]

associating the trivial projections \( \pi_1 \) and \( \pi_2 \) of \( A(C) \) on its two components which we call respectively torsion part and free part, up to compose the exponential function \( \varphi \) by the isomorphism \( \phi \) that we’ve introduced in Theorem 12. Restricting these isomorphisms and these projections to \( L \)-rational points of \( A \) we obtain that:

\[
A(L) \simeq (k_\infty/A)^d \bigoplus \text{Lib}(L).
\]

We remark that \( \pi_2(\Lambda) = \emptyset \), which implies that \( \Lambda \subset \pi_1(\phi(\text{Lie}(A)(L))) = \Pi \). In particular, the torsion part of \( \text{Lie}(A)(C) \) introduced in Lemma 2 coincides (point by point) with \( \Pi \). We will identify often from now \( \Lambda \) and \( \pi_1(\phi(\Lambda)) = A^d \).

**Lemma 3.** Each torsion class \( \varphi + B \) of some \( T \)-module \( A \) with dimension \( m \) and lattice \( \Lambda \), where \( \varphi \) and \( B \) are respectively a torsion point and a sub-\( T \)-module of \( A \), corresponds to an affine sub-space \( H \) of \( C^m \) as it follows:

\[
\varphi(\varphi + H) = \varphi + B;
\]

where \( \varphi \in \varphi^{-1}(\varphi) \) and \( \varphi(H) = B \). Moreover, \( \Lambda \cap H \) has a rank which is strictly minor than that of \( \Lambda \). Therefore, we get not only \( \dim_C(H) < m \), but also \( \dim_{k_\infty}(\phi(H(L)) \cap \Pi) < d \).

**Proof.** To each sub-\( T \)-module \( B \) of \( A \) we associate its (unique) tangent space \( H = \varphi^{-1}(B) = \text{Lie}(B) \) as explained in Remark 3, remembering that we assume that the exponential function is surjective. If \( \varphi \) is a torsion

\footnote{We put in evidence that the correspondence between sub-\( T \)-modules of \( A \) and vector sub-spaces of \( \text{Lie}(A) \) induced by the exponential function is not bijective. If \( H \) is a vector sub-space of \( \text{Lie}(A) \), invariant by the action of the differential of \( \Phi, \varphi(H) \) is not necessarily a sub-\( T \)-module of \( A \). It is in fact an additive group stabilized by the action of \( \Phi \), but not an algebraic sub-variety a priori.}
point of $A$ we choose an element $y \in e^{-1}(x)$ in $Lie(A)$. The fact that the exponential function is $F_q$–additive implies that:

$$e(y + H) = x + B.$$ 

Now we show the property an the associated lattices. We have that:

$$\rho(\Lambda \cap H) = \rho(\phi(\Lambda) \cap \phi(H(L))) \leq \rho(\Lambda);$$

in an obvious fashion as $\phi(\Lambda) \cap \phi(H(L)) \subset \phi(\Lambda)$. If the two ranks was equal, the two lattices would be generated by the same number of periods, which would be then in the two cases a $k_\infty$–base of $\phi(H(L)) \cap \Pi$. If it exists then $\lambda \in \phi(\Lambda) \setminus \phi(H(L))$, it would be a $A$–linear combination of the periods. But as these ones are still a base of $\phi(H(L)) \cap \Pi$ and, then, $\phi(\Lambda) = \phi(\Lambda) \cap \phi(H(L))$. As the $T$–modules are assumed to be uniformizable they are identified by the surjective exponential function, up to isomorphism, with their lattices, which would imply that $A = B$, which is a contradiction. 

\[ \square \]

### 3.1 An higher bound estimate for the torsion points

Let $X$ be an algebraic sub-variety of the $T$–module $A$, as in the hypothesis of Conjecture 1. Let then:

$$Y = e^{-1}(X);$$

be as in Theorem 12. By this Theorem we know that $Y'(k_\infty)$ is a non-empty and $k_\infty$–entire sub-set of $k_\infty^{mn} = \phi(Lie(A)(L))$. We also define the following set:

$$Z := Y'(k_\infty) \cap \Pi \subset k_\infty^d \times \{0\}.$$ 

We remark that $Z$ is a $k_\infty$–entire sub-set of $k_\infty^d$ as it is the intersection of two $k_\infty$–entire sets.

We know that:

$$\mathcal{A}(C)_{\text{tors.}} = \bigcup_{a(T) \in A \setminus \{0\}} \mathcal{A}(C)[a(T)] = \bigcup_{a(T) \in A \setminus \{0\}} \{\pi \in \mathcal{A}(C), \Phi(a(T))(\pi) = 0\}.$$ 

We define then, for each $a(T) \in A \setminus \{0\}$:

$$Y[a(T)] := \{\overline{y} \in Y, a(T)\overline{y} \in \Lambda\};$$

et:

$$Y_{\text{tors.}} := e^{-1}(\mathcal{A}(C)_{\text{tors.}}) \cap Y = \bigcup_{a(T) \in A \setminus \{0\}} Y[a(T)] = \bigcup_{a(T) \in A \setminus \{0\}} \{\overline{y} \in Y, a(T)\overline{y} \in \Lambda\}. $$

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Lemma 4. We define:
\[ \Lambda \subseteq Z \cap \Lambda. \]

\[ Z/\Lambda \] is therefore compact.

Proof. We remark that:
\[ k_\infty/A \simeq \frac{1}{T} F_q[[\frac{1}{T}]]. \]

As \( k_\infty \) is a local field, every open (and at the same time closed) disc is compact, so in particular the following one too:
\[ \{ x \in k_\infty, v_1/T(x) > 0 \} \simeq \frac{1}{T} F_q[[\frac{1}{T}]]. \]

The topological quotient space \( k_\infty/A \) is then compact. In particular, the product \( (k_\infty/A)^d \) is compact too. The isomorphism described in Theorem 12 leads us to see \( Z/\Lambda \) as a sub-set of \( (k_\infty/A)^d \). As the topology induced on \( k_\infty^d \) by the \( 1/T \)−adic valuation is metric, a sub-set of \( k_\infty^d \) is closed with rapport to this one if and only if it is sequentially closed. We consider a convergent sequence in \( k_\infty^d \) contained in \( Z \). Let \( z_0 \) the limit of such a sequence. Without loss of generality we may assume that such a sequence is contained in an open bounded set \( V \) into \( Z \) where \( Z \) could be expressed as a zero locus of finitely many \( k_\infty \)−entire functions \( f_1, \ldots, f_r \). We also assume up to a translation that \( 0 \in V \). Let then be, for each \( j = 1, \ldots, r \), \( f_j(\bar{x}) = \sum_{i \geq 0} \sum_{\mu \in \Lambda_d(i)} a_\mu \bar{x}^i \) the expression of \( f_j : k_\infty^d \rightarrow k_\infty \) on an open disc \( B_R \) de \( 0 \in k_\infty^d \), with radius \( R > 0 \) which the intersection with \( Z \) contains \( V \). For each \( \epsilon > 0 \) it exists then a point \( \bar{x}_n \) taking part of such a sequence, such that \( ||\bar{x}_0 - \bar{x}_n||_\infty < \epsilon \) (we recall that for each \( \bar{x} = (z_1, \ldots, z_d) \in k_\infty^d \) where \( d > 1 \) we have defined \( ||\bar{x}||_\infty = \max_{i=1,\ldots,d}(|z_i|_1/T) \)). We call from now \( f := f_j \) as the argument remains the same for each \( j \) between 1 and \( r \). Therefore:
\[ ||f(\bar{x}_0) - f(\bar{x}_n)||_1/T = \sum_{i \geq 1} \sum_{\mu \in \Lambda_d(i)} a_\mu (\bar{x}_0^i - \bar{x}_n^i) \leq \max_{i \geq 1, \mu \in \Lambda_d(i)} \{|a_\mu|_1/T ||\bar{x}_0^i - \bar{x}_n^i||_1/T\}. \]

We remark that such a finite bound exists by the convergence hypothesis of the series. We have that:
\[ ||\bar{x}_0^i - \bar{x}_n^i||_1/T < \epsilon \sum_{|\mu| + |\mu| = |\mu| - 1} ||\bar{x}_0^i \bar{x}_n^\mu||_1/T < \epsilon R^{|\mu| - 1}; \]
for each $\mu \in \mathbb{N}^d$, $|\mu| \geq 1$. The convergence hypothesis of $f$ on $V$ implies that:

$$\lim_{|a| \to +\infty} a_{\mu} R^{|\mu|} = 0;$$

up to eventually replace $R$ by some number $0 < R' < R$. Therefore:

$$|f(z_0) - f(z_n)|_{1/T} < \epsilon \max\{|a_{\mu}| R^{|\mu|}\} < \epsilon M;$$

for a certain $M > 0$ only depending on $f$. Then, it exists $M > 0$ only depending on $f$ such that for every $\epsilon > 0$ we have that:

$$|f(z_0)|_{1/T} = |f(z_0) - f(z_n)|_{1/T} < \epsilon M;$$

which shows that $f(z_0) = 0$. The set $Z$ is then closed, which implies that $Z/\Lambda Z$ is closed too in $(k_\infty/A)^d$ with rapport to the quotient topology. As this last one is compact we find out the same property for $Z/\Lambda Z$. 

Let $a(T) \in A \setminus \{0\}$ and $S$ a general sub-set of $Lie(A)(C)$. We call:

$$S[a(T)]:= \overline{\mathbb{C}^{-1}(A[a(T)])} \cap S.$$ 

We remind to have identified previously $\Lambda$ with $A^d$ in $k_\infty^d$ by the projection:

$$\pi_1 \circ \phi : Lie(A)(C) \rightarrow k_\infty^d.$$ 

We also saw that the image set of such a projection is exactly the torsion part of $Lie(A(C))$ which follows from the decomposition described in Lemma 2. Up to identify for simplicity of notation $S$ with $\phi(S)$ we consequently have that:

$$\pi_1(S[a(T)]) = \{z \in \pi_1(S)(k), a(T)z \in A^d \simeq \Lambda\} =$$

$$= \{z = (\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_d}{\beta_d}) \in \pi_1(S)(k), PPCM(\{\beta_i\}_{i=1,\ldots,d})a(T)\};$$

$$\pi_2(S[a(T)]) = \{z \in \pi_2(S), a(T)z = 0\} = \{0\} \text{ ou } \emptyset.$$ 

We deduce then that:

$$S[a(T)] = \emptyset \text{ if } \overline{0} \notin \pi_2(S);$$

while if $S[a(T)] \neq \emptyset$ (in other words, if $\overline{0} \in \pi_2(S[a(T)]))$, we have the following identification of sets:

$$S[a(T)] \simeq \{z = (\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_d}{\beta_d}) \in \pi_1(S)(k), PPCM(\{\beta_i\}_{i=1,\ldots,d})a(T)\} \times \{\overline{0}\}.$$
We will use such a description in order to study the number of torsion points contained in \( X \), assuming that \( X \) does not contain no torsion classes. In fact, Lemma 2 and Lemma 3 allow us to reduce ourselves to study such a problem in \( \text{Lie}(\mathcal{A}) \) where the set corresponding to \( X \) is \( Y \). We remark that we can assume from now \( \emptyset \in X \), as the opposite case would lead us to the trivial situation where \( Y_{\text{tors.}} = \emptyset \).

Let \( X \) be an algebraic sub-variety of the abelian and uniformizable \( T \)-module \( \mathcal{A} \), respecting the hypothesis of Conjecture 1. The study of the \( a(T) \)-torsion points of \( \mathcal{A} \) contained in \( X \) is then equivalent to the study of the points of \( Y[a(T)] \subset \text{Lie}(\mathcal{A}) \) and, by the identification of the torsion part of \( \text{Lie}(\mathcal{A}((L))) \) with \( \pi_1(\phi(\text{Lie}(\mathcal{A}((L)))) = k_1^d \) which follows from Theorem 12 point 5, it is also equivalent to study the points of \( Y(L)[a(T)] \). As in this Theorem we have shown that:

\[
Y(L)[a(T)] \simeq Y'(k_\infty)[a(T)] = \{ \overline{z} = (\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_d}{\beta_d}) \in k^d, \text{PPCM}(\{\beta_i\}_{i=1}^{d})|a(T)\} \times \{ \overline{0} \};
\]

the set \( Y(L)[a(T)] \) is finally in bijection with the following one:

\[
Z[a(T)] := \{ \overline{z} = (\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_d}{\beta_d}) \in k^d, \text{PPCM}(\{\beta_i\}_{i=1}^{d})|a(T)\}.
\]

In particular, we have that:

\[
Y'(k_\infty)[a(T)] = Z[a(T)] \times \{ \overline{0} \}.
\]

We also define the following set:

\[
Z_{\text{tors.}} := \bigcup_{a(T) \in \mathcal{A} \setminus \{0\}} Z[a(T)].
\]

**Remark 10.** Let \( X \) be an algebraic sub-variety of a \( T \)-module \( \mathcal{A} \) which respects the hypothesis of Conjecture 1. The set of \( a(T) \)-torsion points of \( \mathcal{A} \) which are contained in \( X \) is then in bijection with the following one:

\[
Z(k, [a(T)]) := \{ (\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_d}{\beta_d}) \in Z[a(T)], \forall i = 1, \ldots, d, |\alpha_i|_{1/T} \leq |a(T)|_{1/T} \}.
\]

**Proof.** In fact, the points \((\alpha_1/\beta_1, \ldots, \alpha_d/\beta_d)\) of \( Z[a(T)] \) (which are \( k \)-rational) are by the definition of this last one such that \( \beta_i|a(T) \) for each \( i = 1, \ldots, d \).

Now, as \( A \) is an euclidean ring with rapport to the \( 1/T \)-adic valuation, each \( \alpha_i \in A \) such that \( |\alpha_i|_{1/T} \leq |a(T)|_{1/T} \) can be expressed univocally in the form \( \alpha_i = k(T)a(T) + r(T) \), where \( k(T) \in A \setminus \{0\} \) and \( r(T) \in A \) such that \( |r(T)|_{1/T} < |a(T)|_{1/T} \). The division by the element \( \beta_i|a(T) \) gives us
then \( \alpha_i/\beta_i = (k(T)a(T)/\beta_i) + (r(T)/\beta_i) \) which is in the form \( r(T)/\beta_i + \lambda \) with \( \lambda \in A \) and \( |r(T)|_{1/T} \leq |\beta_i|_{1/T} \leq |a(T)|_{1/T} \). By the identification of \( X \) with \( Y/(\Lambda \cap Y) \) which follows from Lemma 2 and that of \( Y(L)[a(T)] \) by \( Y'(k_\infty)[a(T)] \) which follows from Theorem 12 point 5, we can finally identify the \( a(T)-\)torsion points of \( X \) by those of \( Y'(k_\infty)(k,a(T))/\{0\} \). The set of \( a(T)-\)torsion points of the algebraic sub-variety \( X \) of \( A_\mathbb{C} \) is not identified, then, with \( Z[a(T)] \) but with \( Z(k,a(T)) \), which allows us to reduce ourselves to a particular sub-set of \( Z \) which is compact by Lemma 4. We assume then that \( |\alpha_i|_{1/T} < |a(T)|_{1/T} \) without loss of generality. 

We define, given \( S \) a generic set in \( \pi_1(\phi(Lie(A))) \) and \( a(T) \in A \) of degree \( \delta_a \) in \( T \):

\[
S(k,a(T)) := \{ \vec{z} \in S(k), H(\vec{z}) \leq |a(T)|_{1/T} \};
\]

where \( \vec{H} \) is a function defined as it follows:

\[
\vec{H} : k^d \to q^\mathbb{Z};
\]

\[
(z_1, \ldots, z_d) \mapsto \max_{i=1,\ldots,d} \{ H(z_i) \};
\]

where \( H \) is the absolute height defined on the \( k \)-rational points of \( k_\infty^d \) (see [Silv], page 202). We remark that:

\[
Z(k,a(T)) \times \{0\} \subset Y'(k_\infty)(k,a(T)).
\]

As the definition of the absolute height \( H \) over \( k \) is such that:

\[
H(z^{-1}) = H(z);
\]

for each point \( z = \frac{\alpha}{\beta} \in k \), where \( \alpha, \beta \in A \setminus \{0\} \) are relatively primes, defining for each \( \vec{z} = (z_1, \ldots, z_d) \in k^d \setminus \{0\} \) its inverse element as it follows:

\[
\vec{z}^{-1} := (z_1^{-1}, \ldots, z_d^{-1});
\]

we remark that:

\[
\vec{H}(\vec{z}^{-1}) = \vec{H}(\vec{z});
\]

for each \( \vec{z} \in k^d \setminus \{0\} \).

**Remark 11.** We have:

\[
S(k,[a(T)]) \subset S(k,a(T));
\]

for each sub-set \( S \) of \( \phi(Lie(A_\mathbb{C})) \) and each \( a(T) \in A \setminus \{0\} \) as described before.
Proof. In fact, up to identify $S$ with $\pi_1(S)$, if $\overline{z} \in S(k, a(T))$ then $\overline{z} = (\overline{\alpha_1}, ..., \overline{\alpha_d})$, where $\beta_i \neq 0$, $\text{PGCD}(\alpha_i, \beta_i) = 1$, $\beta_i|a(T)$, for each $i = 1, ..., d$. As we showed in Remark 10 that it is possible to assume without loss of generality that $|\alpha_i|_{1/T} < |a(T)|_{1/T}$ for each $i = 1, ..., d$, it follows that $\overline{H}(\overline{\alpha_1}, ..., \overline{\alpha_d}) \leq |a(T)|_{1/T}$. □

Definition 20. We call:

$$N(S, [a(T)]) := |S(k, [a(T)])| \quad \text{and} \quad N(S, a(T)) := |S(k, a(T))|.$$ 

We then have the following inequalities:

$$N(Z, [a(T)]) \leq N(Z, a(T)) \leq N(Y'(k_\infty), a(T)).$$

We now give the definition of a semi-algebraic set in the case of a non-archimedean complete field, using the definition given in [Sh], page 51, 100 in the case of the field of real numbers $\mathbb{R}$. As on a non-archimedean field the order relation given by the valuation is not of total order, we will modify the usual definition (which is essentially that of the set of the solutions of polynomial inequalities) as it follows, replacing in particular such a set of solutions by the intersection of the algebraic variety with a general polydisc.

Definition 21. Let $K$ be a complete valued field and let $n \in \mathbb{N} \setminus \{0\}$. A semi-algebraic set in $K^n$ is the intersection of an algebraic variety in $K^n$ with a polydisc, which is the cartesian product of discs in $K$ with radius eventually infinite. In the case where the radius is infinite for all the discs of this product, we have the particular case in which the semi-algebraic set is actually an algebraic variety. If $S$ is a sub-set of $K^n$, we call a semi-algebraic set in $S$ each intersection of $S$ with a semi-algebraic set of $K^n$.

Definition 22. Let $Y'$ be defined as in Theorem 12 point 5. If $\mathcal{A}$ and $\mathcal{B}$ are two $T$-modules, the notation $\mathcal{B} \leq_{j(\mathcal{A})} \mathcal{A}$ means that $\mathcal{B}$ is a sub-$T^{j(\mathcal{A})}$-module of $\mathcal{A}$, where $j(\mathcal{A}) \in \mathbb{N} \setminus \{0\}$ is defined in Theorem 5. The notation $\mathcal{B} <_{j(\mathcal{A})} \mathcal{A}$ means that this inclusion is strict. For each strict sub-$T^{j(\mathcal{A})}$-module $\mathcal{B} <_{j(\mathcal{A})} \mathcal{A}$, we define $H_B := \phi(T^{-1}(B)(L))$ and:

$$S(\mathcal{A}) := \{ \mathcal{B} <_{j(\mathcal{A})} \mathcal{A}, \ \exists \overline{y} \in Y'(k_\infty)_{\text{tors}}, \ \overline{y} + H_B \subset Y'(k_\infty) \};$$

and, for each $\mathcal{B} \in S(\mathcal{A})$:

$$Y'(k_\infty)_{\text{tors},}(\mathcal{B}) := \{ \overline{y} \in Y'(k_\infty)_{\text{tors}}, \ \overline{y} + H_B \subset Y'(k_\infty) \}.$$  

1. We call:

$$Y'(k_\infty)^{\text{lc}} := \bigcup_{\mathcal{B} \in S(\mathcal{A})} \bigcup_{\overline{y} \in Y'(k_\infty)_{\text{tors},}(\mathcal{B})} (\overline{y} + H_B) \subset Y'(k_\infty).$$

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2. We call $Y'(k_{\infty})^r_a$ the union of all connected algebraic sub-varieties in $Y'(k_{\infty})$ such that $\dim k_{\infty} > 0$.

3. We call $Y'(k_{\infty})^{\text{alg.}}$, or algebraic part over $k_{\infty}$ of $Y'(k_{\infty})$, the union of all the semi-$k_{\infty}$–algebraic sub-sets of $Y'(k_{\infty})$ of $k_{\infty}$–dimension > 0.

We remark that:

$$Y'(k_{\infty})^{t.c.} \subset Y'(k_{\infty})^r_a \subset Y'(k_{\infty})^{\text{alg.}}.$$ 

It follows that the hypothesis of Conjecture 1 for which the algebraic sub-variety $X$ of $A$ does not contain no torsion classes is equivalent to say that:

$$Y'(k_{\infty})^{t.c.} = \emptyset;$$

which is implied by the following condition:

$$Y'(k_{\infty})^{\text{alg.}} = \emptyset.$$

The identification of these two sets is part of our projects based on the present work (see paragraph 2.4) and consists in an essential point of the strategy we propose to develop in order to finally prove Conjecture 1. Such a strategy, which is the analogue in abelian and uniformizable $T$–modules case of that explored by J. Pila and U. Zannier in [PZ] in the case of abelian varieties defined over number fields, shows some important difficulties, which we will analyse in paragraph 2.4. The following Theorem is the analogue of the result of J. Pila and J. Wilkie (see [PW]) in our particular situation, where the application to the set $Y'(k_{\infty})$ defined in Theorem 12 point 5 will provide a first step in order to prove Conjecture 1 following the ideas of J. Pila and U. Zannier:

**Theorem 14.** Let $W \subset k_{\infty}^{nm}$ be a $k_{\infty}$–entire sub-set of $k_{\infty}^{nm}$ analytically parametrizable over $k_{\infty}$. For each real number $\epsilon > 0$, it exists $c = c(W, \epsilon) > 0$ such that, for each $a(T) \in A \setminus \{0\}$, one has:

$$N(W \setminus W^{\text{alg.}}, a(T)) \leq c|a(T)|_{1/T}^{\epsilon}.$$ 

We want to apply Theorem 14 to the set $B_1^{nm}(k_{\infty}) \cap Y'(k_{\infty})$. We use Theorem 13 to do it. As we will see after we can reduce ourselves to assume without loss of generality that $W$ is compact and contained into the polydisc $B_1^{nm}(k_{\infty})$. In any case we can locally repeat the same argument for each translated of $B_1^{nm}(k_{\infty})$ which intersects $Y'(k_{\infty})$ (knowing that the dimension of each affinoid space is not always the same but in any case it is $< nm$). We’ll be able then to apply Theorem 14 to the set $Y'(k_{\infty})$. In order to show Theorem 14, we begin to prove intermediate results.
Lemma 5. Let $h, d, \delta \in \mathbb{N} \setminus \{0\}$. Let $D := D_d(\delta) := |\{\mu \in \mathbb{N}^d, \sum_{i=1}^d \mu_i \leq \delta\}|$ and $B := B(h, d, \delta) := \sum_{\beta=0}^b L_h(\beta) \beta + (D_d(\delta) - \sum_{\beta=0}^b L_h(\beta)(b+1)),$ where $L_h(\beta) := |\{\mu \in \mathbb{N}^h, \sum_{i=1}^h \mu_i = \beta\}|$ and $b$ is the only natural number (see [P]) such that $D_h(b) \leq D_d(\delta) \leq D_h(b + 1).$ Let:

$$\Phi_1, \ldots, \Phi_D : k_\infty^h \to k_\infty;$$

be some analytic functions. For each compact and convex sub-set $J$ of $k_\infty^h$, it exists a real number:

$$c = c(J, \Phi_1, \ldots, \Phi_D) > 0;$$

such that, for each $U \subset k_\infty^h$ polydisc with radius $r \leq 1$, and tous $\underline{z}_1, \ldots, \underline{z}_D \in J \cap U$:

$$|\det(\Phi_i(\underline{z}_j))|_{1/T} \leq cr^B.$$

Proof. Let $\underline{z}_0 \in J \cap U$ different from $\underline{z}_1, \ldots, \underline{z}_D$. If $a, b \in \mathbb{N} \setminus \{0\}$, we recall the notations that we’ve introduced in Definition 11:

$$\Lambda_a(b) := \{ (\mu_1, \ldots, \mu_a) \in \mathbb{N}^a, \sum_{i=1}^a \mu_i = b \};$$

$$\Delta_a(b) := \bigcup_{i=1}^b \Lambda_a(i).$$

We call, then, $D_a(b) := |\Delta_a(b)|$ and $L_a(b) := |\Lambda_a(b)|$. We know (see [P]) that it exists a unique $b \in \mathbb{N} \setminus \{0\}$ such that:

$$D_h(b) \leq D_d(\delta) \leq D_h(b + 1).$$

As the functions $\Phi_1, \ldots, \Phi_D$ are analytic on $B^h_1(k_\infty)$, it exists $\zeta \in B_{\underline{z}_i, \underline{z}_0} := B(\underline{z}_0, |\underline{z}_i - \underline{z}_0|_{1/T})$ such that, for each $i, j$ between 1 and $D$:

$$\Phi_i(\underline{z}_j) = \sum_{\mu \in \Delta_h(b)} \frac{\partial^\mu \Phi_i(\underline{z}_0)}{\partial \underline{z}_0^\mu} (\underline{z}_j - \underline{z}_0)^\mu + \sum_{\mu \in \Delta_h(b+1)} \frac{\partial^\mu \Phi_i(\underline{z}_0)}{\partial \underline{z}_0^\mu} (\underline{z}_j - \underline{z}_0)^\mu.$$

For each $l$ between 1 and $D$ we consider the sub-matrices $l \times l$ in $(\Phi_i(\underline{z}_j))_{i,j}$. Such an expression is then a $k_\infty$–linear combination of $L_h(\beta)$ vectors of $k^l_\infty$. Varying $i$ and $j$ in a sub-set of $l$ elements of $\{1, \ldots, D\}$, we obtain a sub-matrix in $k^{l \times l}_\infty$. If $l > L_h(\beta)$, its $l$ columns will be necessarily $k_\infty$–linearly dependent. And its determinant will be 0. We can then compute:

$$|\det(\Phi_i(\underline{z}_j))|_{1/T} = |\det(\sum_{\beta=0}^b \sum_{\mu \in \Delta_h(\beta)} \frac{\partial^\mu \Phi_i(\underline{z}_0)}{\partial \underline{z}_0^\mu} (\underline{z}_j - \underline{z}_0)^\mu) + \ldots|.$$
+ \sum_{\mu \in \Delta_h(b+1)} \frac{\partial^{\mu} \Phi_i(\zeta)}{\partial \zeta^\mu} (\overline{z}_j - \overline{z}_0)^\mu |_{1/T};

developping with rapport to the rows (or the columns) until the sub-matrices having order at most, respectively, $L_h(\beta)$, for $\beta = 0, \ldots, b + 1$. If we call, then:

$$c = c(J, \Phi_1, \ldots, \Phi_D) := \max \max_{i=1, \ldots, D} \max_{\beta=0, \ldots, b+1} \max_{\mu \in \Delta_h(b+1)} \left\{ \left| \frac{\partial^{\mu} \Phi_i(\zeta)}{\partial \zeta^\mu} \right|_{1/T} \right\}; \quad (8)$$

knowing that $||\overline{z}_j - \overline{z}_0||_{\infty} \leq r$ for each $j = 1, \ldots, D$ it follows that:

$$|\det(\Phi_i(\overline{z}_j))|_{1/T} \leq cr^B,$$

where:

$$B = B(h, d, \delta) := \sum_{\beta=0}^b L_h(\beta) \beta + (D_d(\delta) - \sum_{\beta=0}^b L_h(\beta))(b + 1).$$

**Proposition 2.** Let $h < d$ and $\delta \in \mathbb{N} \setminus \{0\}$. It exists a real number $\epsilon = \epsilon(h, d, \delta) > 0$ such that, for each analytic function:

$$\Phi : B^h_1(k_\infty) \to k^d_\infty,$$

if:

$$S := \Phi(B^h_1(k_\infty));$$

and $a(T) \in A$ such that $|a(T)|_{1/T} \geq 1$, it exists a real number $C = C(h, d, \delta, B^h_1(k_\infty), \Phi) > 0$, such that the set $S(k, a(T))$ is contained in the union of at most $C|a(T)|_{1/T}^\delta$ hypersurfaces in $k^d_\infty$ having degree at most $\delta$. Moreover, if $\delta$ approaches to $+\infty$, then $\epsilon$ converges to 0.

**Proof.** We call (taking back the notations we’ve introduced in the proof of Lemma 5):

$$V = V(h, d, \delta) := \sum_{\beta=0}^d L_h(\beta) \beta;$$

$$\epsilon = \epsilon(h, d, \delta) := \frac{hV}{B}.$$

When $h$ and $d$ are fixed we can remark that if $\delta$ approaches to $+\infty$, then $\epsilon$ approaches to 0. Let $U \subset B^h_1(k_\infty)$ a polydisc having radius $r \in q^2$ such
that \( r \leq 1 \). Let \( \vec{z}_1, \ldots, \vec{z}_D \in U \cap \Phi^{-1}(S(k, a(T))) \), where \( D := D_d(\delta) \), are not necessarily different. Expressing:

\[ \Phi := (\Phi_1, \ldots, \Phi_d); \]

with:

\[ \Phi_1, \ldots, \Phi_d : B^h_1(k_\infty) \to k_\infty; \]

analytic functions, we restrict these ones to \( U \). For each \( \mu \in \Delta_d(\delta) \), we define (with the powers notation introduced in definition 11):

\[ \Phi_\mu := (\Phi_1, \ldots, \Phi_d)^\mu; \]

which gives us \( D \) analytic functions:

\[ \Phi_\mu : U \to k_\infty. \]

We remark, then, that every polydisc in \( k_\infty^D \), so \( B^h_1(k_\infty) \) in particular, is convex and compact with rapport to the \( 1/T \)-adic topology in \( k_\infty^h \). In fact, if \( x, y \in B_1(k_\infty) \), the minimal polydisc \( B_{x,y} \) containing the two is contained in \( B_1(k_\infty) \) as, given two non-disjoint balls, one of these is contained in the other one. On the other hand, as the \( 1/T \)-adic valuation is discrete and \( \mathbb{F}_q \cong \mathbb{F}_q[[1/T]]/(1/T) \) is a finite field, \( k_\infty \) is a local field and so its balls are compact. As a polydisc is a finite product of compact and convex sets in \( k_\infty^h \), it is necessarily still compact and convex with rapport to the product topology. The polydisc \( B^h_1(k_\infty) \) can then play the role of the set \( J \) in Lemma 5. As a consequence of this one it exists \( c(\{\Phi_\mu\}_{\mu \in \Delta_d(\delta)}) > 0 \) such that:

\[ |\det(\Phi_\mu(\vec{z}_j))|_{1/T} \leq cr^B. \]

Now, as \( \vec{z}_1, \ldots, \vec{z}_D \in \Phi^{-1}(S(k, a(T))) \):

\[ \det(a(T)\Phi_\mu(\vec{z}_j)) = a(T)^V \det(\Phi_\mu(\vec{z}_j)) \in A. \]

If we choose then:

\[ r < (c|a(T)|_{1/T}^V)^{-1/B}; \]

we have:

\[ |\det(\Phi_\mu(\vec{z}_j))|_{1/T} \leq cr^B < c((c|a(T)|_{1/T}^V)^{-1/B})^B = |a(T)|_{1/T}^{-V}. \]

So, \( |a(T)^V \det(\Phi_\mu(\vec{z}_j))|_{1/T} < 1 \) and this element is in \( A \). Therefore:

\[ \det(\Phi_\mu(\vec{z}_j)) = 0; \]
for all \( z_1, \ldots, z_D \in U \cap \Phi^{-1}(S(k, a(T))) \). We show that such an annihilation is equivalent to the existence of an hypersurface defined over \( k_\infty \) with degree \( \leq \delta \) which contains \( D \) points \( \Phi(z_j) \in k_\infty^d \), for every \( j = 1, \ldots, D \). If, in fact:

\[
f(X_1, \ldots, X_d) = \sum_{\mu \in \Delta_d(\delta)} a_\mu(X_1, \ldots, X_d)^\mu = 0;
\]

is the equation of an hypersurface (such that its degree is trivially \( \leq \delta \)), which is defined over \( k_\infty \) and contains \( \Phi(z_j) \in k_\infty^d \), for every \( j = 1, \ldots, D \). If, in fact:

\[
f(X_1, \ldots, X_d) := \det \left( \begin{array}{c} A_{\mu,j}^I (X_1, \ldots, X_d)^\mu \\ \mu \in I \cup \{ \mu^* \} \end{array} \right).
\]

We see that \( f(X_1, \ldots, X_d) \) is a polynomial defined over \( k_\infty \) with degree \( \leq \delta \). It follows that:

\[
f(\mathcal{P}_j) = \det \left( A_{\mu,j}^I \mathcal{P}_j^\mu \right) = 0;
\]

for each \( \mu \in I \cup \{ \mu^* \} \), \( j = 1, \ldots, D \). In fact if \( j \in J \) the matrix:

\[
\left( \begin{array}{c} A_{\mu,j}^I \\ \mathcal{P}_j^\mu \end{array} \right) \in k_{\infty}^{l+1,l+1};
\]

has two rows equal, while if \( j \notin J \) the determinant must be 0 as the rank of the matrix at the beginning was \( l \). We conclude that \( \Phi(U) \cap S(k, a(T)) \) is
contained, for $r < (c|a(T)|^{V}_{1/T})^{-1/B}$, in an hypersurface with degree at most $\delta$. We choose then $r$ the highest element in $q^{\mathbb{N}}$ to be $\leq (\xi^{|a(T)|^{V}_{1/T}})^{-1/B}$.

Now we recall that the $1/T$–adic topology makes of every cover by balls of some convex of $k_\infty$ a partition. As $r \in q^{\mathbb{N}}$ it follows that $-\log r \in \mathbb{N}\{0\}$.

We know that:

$$B^1_1(k_\infty) = \{ z \in k_\infty, v_{1/T}(z) \geq 0 \} = \mathbb{F}_q[[1/T]];$$

$$B^1_r(k_\infty) = \{ z \in k_\infty, v_{1/T}(z) \geq -\log r \} = (1/T^{-\log r})\mathbb{F}_q[[1/T]].$$

It follows that:

$$|B^1_1(k_\infty)/B^1_r(k_\infty)| = q^{-\log r} = 1/r.$$ (9)

In particular, the number of polydiscs $U$ with radius $r$ which cover $B^1_1(k_\infty)$ is:

$$|B^1_1(k_\infty)/B^1_r(k_\infty)|^h = r^{-h} \leq (\frac{c}{2}|a(T)|^{V}_{1/T})^{h/B} = C|a(T)|^{V}_{1/T};$$

où $C := (\xi^h)^{h/B} > 0$. This proves the statement.

By hypothesis we have an analytic cover of $W$ over $k_\infty$ which naturally restricts itself to an analytic cover of $W \setminus W_{\text{alg}}$. We then apply Proposition 2 to such an analytic cover of $W \setminus W_{\text{alg}}$ over $k_\infty$.

**Proof.** We remark that:

$$W = W_1 \cup W_2;$$

où:

$$W_1 := W \cap B^{nm}_1(k_\infty);$$

and:

$$W_2 := W \setminus W_1.$$ We remark that $W_1 \neq \emptyset$ as we can assume without loss of generality that $\overline{U} \in X$ and so that $\overline{U} \in W$, as we previously saw. As $W_1$ and $W_2$ are subsets of $W$, if we have an analytic cover of $W$ of open sets isomorphic to $B^{d(W)}_1(k_\infty)$, we will have the same in particular for $W_1$ and $W_2$. We have that $W_1$ is compact but this is not the case in general for $W_2$. We consider the following **inversion function**:

$$\frac{1}{\cdot} : W_2 \rightarrow k^{nm}_\infty;$$

$$\overline{\tau} \mapsto \overline{\tau}^{-1};$$

where $\overline{\tau}^{-1}$ is defined, as before, in the obvious fashion. We call $W_2^{-1}$ the image of $W_2$ by this function. We remark that it is a bijective function,
analytic in the two directions between $W_2$ and $W_2^{-1}$. The statement (8) makes the set of points $\mathfrak{s} \in k_\infty^{nm}$ such that $\widetilde{H}(\mathfrak{s}) \leq |a(T)|_{1/T}$ stabilized by the action of the inversion function. To prove Theorem 14 on $W_2^{-1}$ is then equivalent to prove it on $W_2$. As $W_2^{-1}$ is compact we can reduce ourselves to separately treats two compact sets, provided that Theorem 14 on $W_1$ and $W_2$ implies the same result on $W$.

Then let $A, B, C \subset k_\infty^{nm}$ such that $C = A \cup B$. We can show that:

$$A^{alg.} \cup B^{alg.} \subset C^{alg.}.$$  

We assume that Theorem 14 is true on $A$ and $B$. So, for each $\epsilon > 0$, it exist $c_A(\epsilon), c_B(\epsilon) > 0$ such that:

$$N(A \setminus A^{alg.}, |a(T)|_{1/T}) \leq c_A(\epsilon)|a(T)|_{1/T};$$

$$N(B \setminus B^{alg.}, |a(T)|_{1/T}) \leq c_B(\epsilon)|a(T)|_{1/T}.$$

Alors:

$$N(C \setminus C^{alg.}, |a(T)|_{1/T}) \leq N((A \cup B) \setminus (A^{alg.} \cup B^{alg.}), |a(T)|_{1/T}) \leq$$

$$\leq N(A \setminus A^{alg.}, |a(T)|_{1/T}) + N(B \setminus B^{alg.}, |a(T)|_{1/T}) \leq c_C(\epsilon)|a(T)|_{1/T}; \quad (10)$$

defining:

$$c_C(\epsilon) := c_A(\epsilon) + c_B(\epsilon).$$

We can then limit ourselves to $W_1$ from $W$ and assume that $W$ is compact without loss of generality.

As $W$ is compact, we can assume that each open analytic cover $\mathcal{R}$ of $W$ following from Corollary 1 is finite. We call then $N_\mathcal{R}$ the number of analytic functions under the following form:

$$\Phi_i : B_1(^d(W, \mathcal{R})_{k_\infty}) \to W, \quad \forall i = 1, ..., N_\mathcal{R};$$

such that:

$$W \subseteq \bigcup_{i=1}^{N_\mathcal{R}} \Phi_i(B_1(^d(W, \mathcal{R})_{k_\infty})).$$

For each $i = 1, ..., N_\mathcal{R}$ and for each $\delta > 0$ it exists $\epsilon(\delta) > 0$ and a constant $c(\Phi_i, \delta) > 0$ such that $\Phi_i(B_1(^d(W, \mathcal{R})_{k_\infty}))(k, a(T))$ is contained in the union of at most $c(\Phi_i, \delta)|a(T)|_{1/T}^{\epsilon(\delta)}$ hypersurfaces with degree $\leq \delta$.  

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Now, given an analytic cover $\mathcal{R} := \{\Phi_1, \ldots, \Phi_{N_\mathcal{R}}\}$ of $W$ we define:

$$M_\mathcal{R}(\delta) := \max_{i=1,\ldots,N_\mathcal{R}} \{c(\Phi_i, \delta)\} \geq 0;$$

where $c(\Phi_i, \delta) > 0$ is the constant that one associates to each analytic function $\Phi_i$ taking part of $\mathcal{R}$ and to the choice of $\delta > 0$ that have been initially done, which had been defined in (9). Calling $\mathcal{H}(W)$ the family of all the analytic covers of $W$ induced by Corollary 1 we have that it exists a constant $C(W, \delta) \geq 1$ just depending on $W$ and on $\delta$, defined as it follows:

$$C(W, \delta) := \inf_{\mathcal{R} \in \mathcal{H}(W)} \{M_\mathcal{R}\} + 1.$$

Such a constant exists because each non-empty set of $\mathbb{R}$ which admits a lower bound (which in this case is 0) admits an infimum. This one is then such that it exists at least a family $\mathcal{R} = \{\Phi_1, \ldots, \Phi_{N_\mathcal{R}}\} \in \mathcal{H}(W)$ such that the constant $c(\Phi_i, \delta)$ defined as in (9) for each $i = 1, \ldots, N_\mathcal{R}$ is such that:

$$c(\Phi_i, \delta) \leq C(W, \delta);$$

for every $i = 1, \ldots, N_\mathcal{R}$. We choose then $\mathcal{R}$ calling $N := N_\mathcal{R}$ and $d(W) := d(W, \mathcal{R})$. For such a $\mathcal{R}$ we then have that $W(k, a(T)) \subseteq \bigcup_{i=1}^{N} \Phi_i(B_1^{d(W)}(k_\infty))(k, a(T))$ and consequently that $W(k, a(T))$ is contained in the union of at most $K(W, \delta)|a(T)|^{t(\delta)}_1$ hypersurfaces with degree at most $\delta$, where $K(W, \delta) := NC(W, \delta)$.

Let $\epsilon > 0$. We then choose $\delta > 0$ enough high for that $\epsilon(\delta) \leq \epsilon/2$, following the notations of Proposition 2. We have that the set of the hypersurfaces in $k_\infty^{nm}$ with coefficients in $k_\infty$ and degree $\leq \delta$ is in bijection with the projective space $\mathbb{P}_{\nu(\delta)}(k_\infty)$, for a convenient $\nu(\delta) \in \mathbb{N}$ just depending on $\delta$. We define:

$$T := W \times \mathbb{P}_{\nu(\delta)}(k_\infty).$$

As $k_\infty$ is a local field the same arguments we did in characteristic 0 may be repeated to show that $\mathbb{P}_{\nu(\delta)}(k_\infty)$ and so $T$ is compact. If $t \in \mathbb{P}_{\nu(\delta)}(k_\infty)$ we call $H_t$ the hypersurface associated to him. Let:

$$S := \{(z, t) \in T, z \in H_t\}.$$

For each $t \in \mathbb{P}_{\nu(\delta)}(k_\infty)$ we call fibre of $t$ in a sub-set $T'$ of $T$ the following set:

$$T'_t := \{z \in W, (z, t) \in T'\}.$$

Each fibre in $S$ is an entire set as it is the intersection between an entire set and an algebraic set. We want to prove Theorem 14 as a consequence of
another stament. We actually show that for each $S' \subset S$ it exists a constant $C(S', \epsilon) > 0,$ just depending on $S'$ (and so, depending on $W$ if $S' = S$) such that $N(S'_t \setminus S'_t^{alg}, a(T)) \leq c(S', \epsilon)|a(T)|_{1/T}^{t/2}$ for each $t \in S'$. We prove such a statement by induction on the dimension of the fibres in $S$ assuming for simplicity that $S' = S$, the proof being the same in all the other cases. Let $h$ be the dimension of $S_t$ for a fixed $t \in \mathbb{P}_{\nu(\delta)}(k_\infty)$. If $h = 0$, $S_t$ is a discrete and compact set, so it is finite. In this case we have that $S_t^{alg} = \emptyset$, which proves the statement with $c(S_t, \epsilon) = |S_t|$, for all values of $\epsilon$. Let then be $h > 0$. Let:

$$A := \{ (\overline{z}, t) \in S, \operatorname{dim}(W \cap H_t) \leq h - 1 \}$$
$$B := \{ (\overline{z}, t) \in S, \operatorname{dim}(W \cap H_t) = h \}.$$

It follows that:

$$S = A \cup B.$$

For every $t \in \mathbb{P}_{\nu(\delta)}(k_\infty)$ we have that:

$$A_t^{alg.} \cup B_t^{alg.} \subset (A \cup B)_t^{alg.}.$$

For each $t \in \mathbb{P}_{\nu(\delta)}(k_\infty)$ we have that:

$$\operatorname{dim}(A_t) \leq h - 1.$$

The induction hypothesis implies that:

$$N(A_t \setminus A_t, a(T)) \leq c(A_t, \epsilon)|a(T)|_{1/T}^{t/2};$$

for some convenient constant $c(A_t, \epsilon) > 0$. The intersection of an hypersurface $H_t$ with degree $\leq \delta$ intersecting $W$ in points where the dimension of such an intersection is locally $\leq h - 1$ contains then at most $c(A_t, \epsilon)|a(T)|_{1/T}^{t/2}$ points of $W(k, a(T))$. On the other hand, we have that:

$$\operatorname{dim}(B_t) = h.$$

If then $\overline{z} \in W \cap H_t$, it exists an open neighborhood $U_{\overline{z}} \subset k_\infty^h$ of $\overline{z}$ such that:

$$U_{\overline{z}} \subset W \cap H_t.$$

So in particular $U_{\overline{z}} \subset H_t$. As $H_t$ is an algebraic set it follows that:

$$B_t^{alg.} = B_t.$$

The Theorem is so trivially true for $B$. By (11) we have for each $t \in \mathbb{P}_{\nu(\delta)}(k_\infty)$ the following estimate:

$$N(W \cap H_t \setminus (W \cap H_t)^{alg.}, a(T)) \leq c(W, \epsilon)|a(T)|_{1/T}^{t/2};$$

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for a certain constant $C(W, \epsilon) := C(S, \epsilon) > 0$. We now remark that for each hypersurface $H_t$ with degree $\leq \delta$ we have that:

$$(W \cap H_t)^{\text{alg.}} = W^{\text{alg.}} \cap H_t;$$

because $H_t$ is an algebraic set. It follows that:

$$W \setminus W^{\text{alg.}} = \bigcup_{t \in \mathbb{P}_v(k_\infty)} ((W \setminus W^{\text{alg.}}) \cap H_t) = \bigcup_{t \in \mathbb{P}_v(k_\infty)} ((W \cap H_t) \setminus (W \cap H_t)^{\text{alg.}}).$$

The compactness of $W$ and Proposition 2 imply that it exists a constant $K(W, \epsilon) > 0$ such that $W(k, a(T))$ is contained in the union of at most $K(W, \epsilon)|a(T)|^{\delta/2}/|T|$ hypersurfaces with degree $\leq \delta$. Let $S \subset \mathbb{P}_v(k_\infty)$ be a set having the same cardinality, which represents such a family of hypersurfaces. It follows that:

$$N(W \setminus W^{\text{alg.}}, a(T)) \leq \sum_{t \in S} N(S_t \setminus S_t^{\text{alg.}}, a(T)) \leq C(W, \epsilon)K(W, \epsilon)|a(T)|^{\delta/2}/|T|.$$

Calling:

$$c(W, \epsilon) := C(W, \epsilon)K(W, \epsilon);$$

we prove Theorem 14. 

\[\Box\]

4 Conclusions

The method presented in J. Pila’s and U. Zannier’s work foresees that, as we already said at the Introduction, one could show two more results in order to combine them with Theorem 14 in order to finally prove Conjecture 1. More precisely,

1. Show that $Y'(k_\infty)^{\text{alg.}} = Y'(k_\infty)^{L_c}.$

2. Show that it exists a real number $\rho > 0$ just depending on the choice of $A$ such that:

$$\forall a(T) \in A \setminus \{0\}, \ \forall \pi \in A[a(T)] \setminus \{0\}, D := [k(\pi): k] \geq |a(T)|^{\rho}/|T|.$$ 

We currently work on such objectives using trascendence techniques and the ideas contained in D. Masser’s work ”Small values of the quadratic part of the Néron-Tate height on an abelian variety” (see [Mas]), that we try to adapt to the compact analytic set $Z$ we’ve previously defined.
5 Appendix: searching for a minimal $j(\mathcal{A})$

We would like to investigate here if it would be possible to explicitly compute the minimal value of $j(\mathcal{A})$ associated to some given $T$–module $\mathcal{A}$ which is abelian and uniformizable. We’re about to examine different cases founded on the particularly interesting situation of a tensor product of the Carlitz module. It is in general a $T$–module $C^\otimes m = (\mathbb{G}_m^m, \Phi)$ such that:

\[
\Phi(T)(\tau) := \begin{pmatrix}
T & 1 & 0 & \cdots & 0 \\
T & 1 & \cdots & 0 \\
\ddots & \ddots & \ddots & \ddots & 0 \\
\ddots & \ddots & 1 & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & T \\
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\end{pmatrix} \tau.
\]

One can actually prove that it is abelian and uniformizable. We begin here to study the case of $m = 2$.

Let $C^\otimes 2 = (\mathbb{G}_m^2, \Phi)$ be such a tensor power of the Carlitz module $C$, and let $q = 2$. As we previously saw, the differential of $\Phi(T^2)$ is easily seen to be $T^2I_2$, where $I_2$ is the identity matrix with order $2 \times 2$. We say that a polynomial $a(T) \in A$ is even if and only if it takes the following shape:

\[a(T) = a_0 + a_1T^2 + \ldots + a_rT^{2r}.
\]

It follows that $d\Phi(a(T)) = a(T)I_2$ when $a(T)$ is even. Now, if $b(T) \in A \setminus \{0\}$ is not even it takes necessarily the following shape:

\[b(T) = Ta(T) + r(T);
\]

where $a(T)$ and $r(T)$ are even. Remark that the differential of $\Phi(Ta(T)) = \Phi(T) \circ \Phi(a(T))$ is, by the properties of the exponential function and the description of $\Phi(T)$ that we did before, under the following form:

\[
\begin{pmatrix}
T & 1 \\
0 & T \\
\end{pmatrix}
\begin{pmatrix}
a(T) & 0 \\
0 & a(T) \\
\end{pmatrix};
\]

so adding the differential of $r(T)$, which is $r(T)I_2$, we conclude that the differential of $b(T)$ is an upper triangular matrix whose the diagonal terms are equals to $b(T)$. As $b(T)$ is not even (so it can’t be 0) it follows that the only vector sub-space of $\text{Lie}(C^\otimes 2)$ stabilized by the action of this matrix is $\mathcal{C} \times 0$. Now we assume that it exists a sub-$b(T)$–module $\mathcal{B}$ of $C^\otimes 2$ different from $\mathcal{C} \times 0$. Therefore:

\[\Phi(b(T))(\mathcal{B}) \subseteq \mathcal{B};\]
and this implies that \( \text{Lie}(B) \) is stabilized by the action of the differential of \( \Phi(b(T)) \), and this implies, by the previous passages, that \( \text{Lie}(B) = C \times 0 \), which finally implies that such a \( B \) different from \( \tau(C \times 0) \) can not exist.

We find out then that \( \tau(C \times 0) \) is the only possible sub-\( b(T) \)–module of \( C^{\otimes 2} \) where \( b(T) \) is a general even polynomial of \( A \setminus \{0\} \). As \( T \) is not even into \( A \setminus \{0\} \) the only non-trivial sub-\( T \)–module of \( C^{\otimes 2} \) must necessarily be \( \tau(C \times 0) \). But we now that \( C^{\otimes 2} \) is simple, which implies that \( \tau(C \times 0) \) is not stabilized by the action of \( T \), or that it is not an algebraic sub-variety of \( G_a^n \). Now:

\[
\Phi(T) \circ \tau(C \times 0) = \tau(TI_2(C \times 0)) \subset \tau(C \times 0);
\]

which implies that \( \tau(C \times 0) \) is stabilized by the action of \( T \). It finally turns out that it can not be an algebraic variety in \( G_a^n \). So it can not be a sub-\( b(T) \)–module either for no \( b(T) \in A \setminus \mathbb{F}_q \). We conclude that \( C^{\otimes 2} \) does not admit any sub-\( b(T) \)–module with \( b(T) \) not even. The only possible sub-\( a(T) \)–modules of \( C^{\otimes 2} \) are consequently those such that \( a(T) \) is even. This suggests us that in this case one may have that:

\[
j(C^{\otimes 2}) = 2.
\]

We try now to extend our interest on the general case of \( m \geq 2 \). Let \( m \in \mathbb{N} \setminus \{0\} \). A direct computation shows that the action of \( \Phi(T^h) \) over \( G_a^m \) is, for each \( h = 1, ..., m \) and for \( l = m - h \), on the following form:

\[
\Phi(T^h) \left( \begin{array}{c} X_1 \\ \vdots \\ X_m \end{array} \right) = \left( \begin{array}{c} \sum_{i=0}^{h} \binom{h}{i} T^{h-i} X_{i+1} \\ \sum_{i=0}^{h} \binom{h}{i} T^{h-i} X_{i+2} \\ \vdots \\ \sum_{i=0}^{h} \binom{h}{i} T^{h-i} X_{i+l} \\ \sum_{i=0}^{h-1} \binom{h}{i} T^{h-i} X_{i+l+1} + X_1^q \\ \sum_{i=0}^{h-2} \binom{h}{i} T^{h-i} X_{i+l+2} + f_1(X_1^q, X_2^q) \\ \vdots \\ T^h X_m + f_{h-1}(X_1^q, X_2^q, \ldots, X_h^q) \end{array} \right);
\]

where \( f_1, ..., f_{h-1} \) are linear polynomials with coefficients in \( A \) and variables \( X_1, ..., X_h \). We note that if \( h = m \) the first component of this vector is \( \sum_{i=0}^{m-1} \binom{m}{i} T^{m-i} X_{i+1} + X_1^q \). If \( m \) is divisible by the characteristic \( p \) of \( k \), let
$p^\alpha$ be the maximal power of $p$ dividing $m$. We then have that:

$$
\Phi(T^p) \left( \begin{array}{c} X_1 \\ \vdots \\ X_m \end{array} \right) = \left( \begin{array}{c} T^p X_1 + X_{p^\beta + 1} \\ T^p X_2 + X_{p^\beta + 1} \\ \vdots \\ T^p X_{p^\beta + 1} + X_{2p^\beta + 1} \\ \vdots \\ T^p X_{m-p^\beta + 1} + X_1^q \\ \vdots \\ T^p X_m + f_{p^\beta -1}(X_1^q, \ldots, X_{p^\beta -1}^q) \end{array} \right), \quad \forall 1 \leq \beta \leq \alpha.
$$

For each $1 \leq \beta \leq \alpha$ we have that the algebraic sub-group:

$$
\{(X_1, \ldots, X_m) \in \mathbb{G}_a^m, \ X_{sp^\beta + 1} = 0, \ \forall s = 0, \ldots, m/p^\beta\} = (0 \times \mathbb{G}_a^{p^\beta - 1})^{m/p^\beta};
$$

of $\mathbb{G}_a^m$ is stabilized by the action of $T^p$, but not by that of $T^{p^\beta}$ in general if $\beta > 1$, and so it is a sub-$T^p$-module of $C^{\otimes m}$ which is not a sub-$T^{p^\beta}$-module. We remark on the other hand that if $m = p^\alpha$ every algebraic group of the form $0^h \times \mathbb{G}_a^k$, where $m = h + k$, is a sub-$T^p$-module of $C^{\otimes m}$.

On the other hand the computations seem to suggest the existence of sub-$T^p$-modules of $C^{\otimes m}$, where $j > m$, which are not stable by the action of $T^i$ for at least a number $0 < i \leq m$, is not very probable. On the other hand, we can also remark that if $h$ is not a power of $p$ and $p^\beta$ is the maximal power of $p$ dividing $h$, for some convenient number $\beta \in \mathbb{N}$, we have that:

$$
p \left( \binom{h}{i} \right), \ \forall i \neq sp^\beta, \ \forall s = 0, \ldots, h/p^\beta;
$$

$$
\left( \frac{h}{sp^\beta} \right) \equiv \left( \frac{h/p^\beta}{s} \right) \mod (p), \ \forall s = 0, \ldots, h/p^\beta.
$$

It follows that:

$$
\Phi(T^h) \left( \begin{array}{c} X_1 \\ \vdots \\ X_m \end{array} \right) = \left( \begin{array}{c} T^h X_1 + h/p^\beta T^{h-p^\beta} X_{p^\beta + 1} + \cdots + h/p^\beta T^p X_{h-p^\beta + 1} + X_{h+1} \\ \vdots \\ T^h X_2 + h/p^\beta T^{h-p^\beta} X_{p^\beta + 2} + \cdots + h/p^\beta T^p X_{h-p^\beta + 2} + X_{h+2} \\ \vdots \\ T^h X_{h-p^\beta + 1} + h/p^\beta T^{h-p^\beta} X_{h+1} + \cdots + h/p^\beta T^p X_m + X_1^q \\ \vdots \\ T^h X_m + f_{h-1}(X_1^q, \ldots, X_{h-1}^q) \end{array} \right).
$$

We can then remark that each sub-$T^h$-module of $C^{\otimes m}$ which is a product of powers of 0 and $\mathbb{G}_a$, takes the following shape:

$$
\{(X_1, \ldots, X_m) \in \mathbb{G}_a^m, \ X_1 = X_{p^\beta + 1} = \cdots = X_{h+1} = \cdots = X_{m-p^\beta + 1} = 0\};
$$

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and it is also sub-$T^{p^g}$-module. This finally suggests us that $j(C^\otimes m) = p^\alpha$. The case of $p \nmid m$ seems on the contrary suggest the absence of sub-$T^j$-modules of $C^\otimes m$ for each $j \in \mathbb{N} \setminus \{0\}$, which leads us to suppose that in such a situation one has that $j(C^\otimes m) = 1$. It would be also interesting to establish in such a case if $j((C^\otimes m)^h) = j(C^\otimes m) = 1$ for each $h \in \mathbb{N} \setminus \{0\}$, as $(C^\otimes m)^h$ is not simple for no $h > 1$.

It is very interesting to remark that for each $a(T) \in A \setminus \{0\}$ and each sub-$T^j$-module $B$ of $C^\otimes m$ taking one of the forms we’ve previously studied (product of powers of 0 and $\mathbb{G}_a$), for a certain $j \in \mathbb{N} \setminus \{0,1\}$, the sub-set $\Phi(a(T))(B)$ de $\mathbb{G}_a^m$ is still an algebraic group and in particular a sub-$T^j$-module of $C^\otimes m$. Which provides a class of sub-$T^j$-modules of $C^\otimes m$, turned out by the previous families, which is not necessarily made up by products of powers of 0 and $\mathbb{G}_a$. An example is given in case when $p = m = 2$ by:

$$\Phi(T)(0 \times \mathbb{G}_a) = \{(X,Y) \in \mathbb{G}_a^2, Y = TX\}.$$ 

We remark now that each vector sub-space $H$ of $\text{Lie}(C^\otimes m)$ which is stabilized by the action of the differential $d\Phi(a(T))$, for some given $a(T) \in A \setminus \{0\}$, is such that $\overline{e}(H)$, which is not in general an algebraic sub-variety of $\mathbb{G}_a^m$, is stabilized by the action of $\Phi(a(T))$. Given a sub-$a(T)$-module $B$ of $C^\otimes m$, it is then possible to verify whether it is also a sub-$b(T)$-module for some $b(T) \in A \setminus \{0, a(T)\}$, and it turns out that this is true if and only if $d\Phi(b(T))\text{Lie}(B) \subseteq \text{Lie}(B)$. We have that:

$$d\Phi(T) = TI_m + N;$$

where $N \in \mathbb{F}_q^{m,m}$ is such that $N^m = 0$. The development of the power of the binomial:

$$d\Phi(T)^j = (TI_m + N)^j;$$

shows that for each $j > m$:

$$d\Phi(T^j) \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = d\Phi(T)^j \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} = \begin{pmatrix} T^jz_1 + \binom{j}{1}T^{j-1}z_2 + \cdots + \binom{j}{m-1}T^{j-m+1}z_m \\ T^jz_2 + \binom{j}{1}T^{j-1}z_3 + \cdots + \binom{j}{m-2}T^{j-m+2}z_m \\ \vdots \\ T^jz_m \end{pmatrix}.$$ 

A quick computation shows then that the following condition:

$$\binom{j}{i} \equiv \binom{m}{i} \mod (p), \ \forall i = 1, ..., m - 1;$$

is sufficient for that each sub-$T^j$-module of $C^\otimes m$ would also be a sub-$T^m$-module.
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