ABSTRACT. We study the Berezin-Toeplitz quantization using as quantum space the space of eigenstates of the renormalized Bochner Laplacian corresponding to eigenvalues localized near the origin on a symplectic manifold. We show that this quantization has the correct semiclassical behavior and construct the corresponding star-product.

1. INTRODUCTION

Quantization is a recipe in physics for passing from a classical system to a quantum system by obeying certain natural rules. By a classical system we understand a classical phase space (a symplectic manifold) and classical observables (smooth functions). The quantum system consists of a quantum space (a Hilbert space of functions or sections of a bundle) and quantum observables (bounded linear operators on the quantum space). The quantum system should reduce to the classical one as the size of the objects gets large, that is, as the “Planck constant”, which, heuristically, corresponds to the magnitude where the quantum phenomena become relevant, tends to zero. This is the so-called classical limit.

The original concept of quantization goes back to Weyl, von Neumann, and Dirac. In the geometric quantization introduced by Kostant [14] and Souriau [25] the quantum space is the Hilbert space of square integrable holomorphic sections of a prequantum line bundle (see also [1, 6, 9]). Berezin-Toeplitz quantization is a particularly efficient version of the geometric quantization theory. Toeplitz operators and more

L. I. was supported by grants from Région Ile-de-France.
W. L. partially supported by National Nature Science Foundation in China (No. 11401232).
X. M. partially supported by NNSFC 11528103, ANR-14-CE25-0012-01 and funded through the Institutional Strategy of the University of Cologne within the German Excellence Initiative.
G. M. partially supported by DFG funded project SFB TRR 191.
generally Toeplitz structures were introduced in geometric quantization by Berezin \[2\] and Boutet de Monvel-Guillemin \[5\]. Using the analysis of Toeplitz structures \[5\], Bordemann-Meinrenken-Schlichenmaier \[4\] and Schlichenmaier \[23\] showed that the Berezin-Toeplitz quantization on a compact Kähler manifold satisfies the correspondence principle asymptotically and introduced the Berezin-Toeplitz star product (cf. (1.18) and \(1.19\)) when \(E = \mathbb{C}\).

In order to generalize the Berezin-Toeplitz quantization to arbitrary symplectic manifolds one has to find a substitute for the space of holomorphic sections of tensor powers of the prequantum line bundle. A natural candidate is the kernel of the Dirac operator, since it has similar features to the space of holomorphic sections in the Kähler case, especially the asymptotics of the kernels of the orthogonal projection on both spaces \[8\]. The Berezin-Toeplitz quantization with quantum space the kernel of the Dirac operator was carried over by Ma-Marinescu \[20\].

Another appealing candidate is the space of eigenstates of the renormalized Bochner Laplacian \[10, 18, 19\] corresponding to eigenvalues localized near the origin, cf. (1.8), (1.9). In this paper we construct the Berezin-Toeplitz quantization for these spaces and show that it has the correct semiclassical behavior. The difference between this case and \((1.9)\). In this paper we construct the Berezin-Toeplitz quantization for these spaces and show that it has the correct semiclassical behavior. The difference between this case and the quantization by the kernel of the Dirac operator comes from the possible presence of eigenvalues localized near the origin but different from zero. In this situation the analysis becomes more difficult.

Let us note also that Charles \[7\] proposed recently another approach to quantization of symplectic manifolds and Hsiao-Marinescu \[11\] constructed a Berezin-Toeplitz quantization for eigenstates of small eigenvalues in the case of complex manifolds.

The readers are referred to the monograph \[18\] (also \[16\]) for a comprehensive study of the (generalized) Bergman kernel, Berezin-Toeplitz quantization and its applications.

Let us describe the setting and results in detail. Let \((X, \omega)\) be a compact symplectic manifold of real dimension \(2n\). Let \((L, h^L)\) be a Hermitian line bundle on \(X\), and let \(\nabla^L\) be a Hermitian connection on \((L, h^L)\) with the curvature \(R^L = (\nabla^L)^2\). Let \((E, h^E)\) be a Hermitian vector bundle with Hermitian connection \(\nabla^E\). We will assume throughout the paper that \(L\) is a line bundle satisfying the pre-quantization condition
\[
\frac{\sqrt{-1}}{2\pi} R^L = \omega.
\]

We choose an almost complex structure \(J\) such that \(\omega\) is \(J\)-invariant. The almost complex structure \(J\) induces a splitting \(TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X\), where \(T^{(1,0)}X\) and \(T^{(0,1)}X\) are the eigenbundles of \(J\) corresponding to the eigenvalues \(\sqrt{-1}\) and \(-\sqrt{-1}\), respectively.

Let \(g^{TX}\) be a \(J\)-invariant Riemannian metric on \(TX\). Let \(dv_X\) be the Riemannian volume form of \((X, g^{TX})\). The \(L^2\)-Hermitian product on the space \(\mathcal{C}^\infty(X, L^p \otimes E)\) of smooth sections of \(L^p \otimes E\) on \(X\), with \(L^p := L^{\otimes p}\), is given by
\[
\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle dv_X(x).
\]

Let \(\nabla^{TX}\) be the Levi-Civita connection on \((X, g^{TX})\), and let \(\nabla^{Lp\otimes E}\) be the connection on \(L^p \otimes E\) induced by \(\nabla^L\) and \(\nabla^E\). Let \(\{e_k\}\) be a local orthonormal frame of \((TX, g^{TX})\). The Bochner Laplacian acting on \(\mathcal{C}^\infty(X, L^p \otimes E)\) is given by
\[
\Delta^{L^p\otimes E} = -\sum_k \left[ (\nabla_{e_k}^{L^p\otimes E})^2 - \nabla_{\nabla_{e_k}^{TX} e_k}^{L^p\otimes E} \right].
\]
Let \( \Phi \in \mathcal{C}^\infty(X, \text{End}(E)) \) be Hermitian (i.e., self-adjoint with respect to \( \hbar^E \)). The renormalized Bochner Laplacian is defined by

\[
\Delta_{p, \Phi} = \Delta^{L^p \otimes E} - \tau_p + \Phi,
\]

with

\[
\tau = \frac{\sqrt{2}}{2} \sum_k R^L(e_k, J e_k).
\]

Set

\[
\mu_0 = \inf_{u \in T_x^{(1,0)}X, x \in X} R^L(u, \overline{u})/|u|^2_{L^2(T_x X)}.
\]

By \([10], [17]\) Corollary 1.2, \([18]\) Theorem 8.3.1] there exists \( C_L > 0 \) independent of \( p \) such that

\[
\text{Spec}(\Delta_{p, \Phi}) \subset [-C_L, C_L] \cup [2\mu_0 p - C_L, +\infty),
\]

where \( \text{Spec}(A) \) denotes the spectrum of the operator \( A \). Since \( \Delta_{p, \Phi} \) is an elliptic operator on a compact manifold, it has discrete spectrum and its eigensections are smooth. Let

\[
\mathcal{H}_p := \bigoplus_{\lambda \in [-C_L, C_L]} \ker(\Delta_{p, \Phi} - \lambda) \subset \mathcal{C}^\infty(X, L^p \otimes E)
\]

be the direct sum of eigenspaces of \( \Delta_{p, \Phi} \) corresponding to the eigenvalues lying in \([ -C_L, C_L]\).

In mathematical physics terms, the operator \( \Delta_{p, \Phi} \) is a semiclassical Schrödinger operator and the space \( \mathcal{H}_p \) is the space of its bound states as \( p \to \infty \). The space \( \mathcal{H}_p \) proves to be an appropriate replacement for the space of holomorphic sections \( H^0(X, L^p \otimes E) \) from the Kähler case. Indeed, if \((X, \omega)\) is Kähler, then \( \mathcal{H}_p = H^0(X, L^p \otimes E) \) for \( p \) large enough. Moreover, for an arbitrary compact prequantized symplectic manifold \((X, \omega)\) as above, the dimension of the space \( \mathcal{H}_p \) is given for \( p \) large enough as in the Kähler case by the Riemann-Roch-Hirzebruch formula, see \([17]\) Corollary 1.2, \([18]\) Theorem 8.3.1,

\[
d_p := \dim \mathcal{H}_p = \langle \text{Td}(X) \text{ch}(L^p \otimes E), [X] \rangle \sim p^n (\text{rank } E) \text{vol}_\omega(X), \ p \gg 1.
\]

Another striking similarity is the fact that the kernel of the orthogonal projection on \( \mathcal{H}_p \) has an asymptotic expansion analogous to the Bergman kernel expansion for Kähler manifolds, see \([18][19]\). We will use the asymptotic expansion of \([18][19]\) together with the approach of \([20]\) to Berezin-Toeplitz quantization in order to derive the properties of Toeplitz operators modeled on the projection on \( \mathcal{H}_p \).

Let \( P_{\mathcal{H}_p} \) be the orthogonal projection from \( \mathcal{C}^\infty(X, L^p \otimes E) \) onto \( \mathcal{H}_p \). The kernel \( P_{\mathcal{H}_p}(x, x') \) of \( P_{\mathcal{H}_p} \), with respect to \( d\nu_X(x') \), is called a generalized Bergman kernel \([19]\). Note that \( P_{\mathcal{H}_p}(x, x') \in (L^p \otimes E)_x \otimes (L^p \times E)_{x'} \). For a smooth section \( f \in \mathcal{C}^\infty(X, \text{End}(E)) \) of the bundle \( \text{End}(E) \), we define the Berezin-Toeplitz quantization of \( f \) by

\[
T_{f, p} = P_{\mathcal{H}_p} f P_{\mathcal{H}_p} \in \text{End}(L^2(X, L^p \otimes E)),
\]

where we denote for simplicity by \( f \) the endomorphism of \( L^2(X, L^p \otimes E) \) induced by \( f \), namely, \( s \mapsto fs \), with \( (fs)(x) = f(x)(s(x)) \), for \( s \in L^2(X, L^p \otimes E) \) and \( x \in X \).
**Definition 1.1.** A Toeplitz operator is a sequence \( \{ T_p \} = \{ T_p \}_{p \in \mathbb{N}} \) of linear operators
\[
T_p : L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)
\]
with the following properties:
(i) For any \( p \in \mathbb{N} \), we have
\[
T_p = P_{H_p} T_p P_{H_p};
\]
(ii) There exist a sequence \( g_l \in \mathcal{C}^\infty(X, \text{End}(E)) \) such that for all \( k \geq 0 \) there exists \( C_k > 0 \) with
\[
\| T_p - P_{H_p} \left( \sum_{l=0}^{k} p^{-l} g_l \right) P_{H_p} \| \leq C_k p^{-k-1},
\]
where \( \| \cdot \| \) denotes the operator norm on the space of the bounded operators.
If each \( T_p \) is self-adjoint, then \( \{ T_p \}_{p \in \mathbb{N}} \) is called self-adjoint.

We express (1.13) symbolically by
\[
T_p = \sum_{l=0}^{k} p^{-l} T_{g_l,p} + O(p^{-k-1}).
\]
If (1.13) holds for any \( k \in \mathbb{N} \), then we write
\[
T_p = \sum_{l=0}^{\infty} p^{-l} T_{g_l,p} + O(p^{-\infty}).
\]

The main result of this paper is as follows.

**Theorem 1.2.** Let \( (X, J, \omega) \) be a compact symplectic manifold, \( (L, h^L, \nabla^L), (E, h^E, \nabla^E) \) be Hermitian vector bundles as above, and \( g^{TX} \) be an \( J \)-compatible Riemannian metric on \( TX \).
Let \( f, g \in \mathcal{C}^\infty(X, \text{End}(E)) \). Then the product of the Toeplitz operators \( T_{f,p} \) and \( T_{g,p} \) is a Toeplitz operator, more precisely, it admits the asymptotic expansion in the sense of (1.15):
\[
T_{f,p} T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + O(p^{-\infty}),
\]
where \( C_r \) are bidifferential operators, \( C_r(f, g) \in \mathcal{C}^\infty(X, \text{End}(E)) \) and \( C_0(f, g) = fg \).
If \( f, g \in \mathcal{C}^\infty(X) \), then we have
\[
C_1(f, g) - C_1(g, f) = \sqrt{-1} \{ f, g \} \text{Id}_E,
\]
where \( \{ \cdot, \cdot \} \) is the Poisson bracket on \( (X, 2\pi\omega) \), and therefore the correspondence principle holds asymptotically,
\[
[T_{f,g}, T_{g,p}] = \frac{\sqrt{-1}}{p} T_{\{f,g\},p} + O(p^{-2}), \quad p \to \infty.
\]

**Corollary 1.3.** Let \( f, g \in \mathcal{C}^\infty(X, \text{End}(E)) \). Set
\[
f * g := \sum_{k=0}^{\infty} C_k(f,g) h^k \in \mathcal{C}^\infty(X, \text{End}(E))[h]
\]
where \( C_r(f, g) \) are determined by (1.16). Then (1.19) defines an associative star-product on \( \mathcal{C}^\infty(X, \text{End}(E)) \).
Theorem 1.4. For any $f \in \mathcal{C}^\infty(X, \text{End}(E))$ the operator norm of $T_{f, p}$ satisfies

$$\lim_{p \to 0} \|T_{f, p}\| = \|f\|_\infty := \sup_{0 \neq u \in E_x, x \in X} |f(x)(u)|_{h^E} / |u|_{h^E}.$$  

In the special case when the Riemannian metric $g^{TX}$ is associated with $\omega$ we can even calculate $C_1(f, g)$, not only the difference $C_1(f, g) - C_1(g, f)$ from (1.17). To state the result, let

$$\langle \nabla^E \rangle^{1,0} : \mathcal{C}^\infty(X, \text{End}(E)) \to \mathcal{C}^\infty(X, T^*(1,0)X \otimes \text{End}(E)),$$

$$\langle \nabla^E \rangle^{0,1} : \mathcal{C}^\infty(X, \text{End}(E)) \to \mathcal{C}^\infty(X, T^*(0,1)X \otimes \text{End}(E)).$$

be the $(1,0)$-component and $(0,1)$-component respectively of the connection $\nabla^E$, and let $\langle \cdot, \cdot \rangle$ denote the pairing induced by $g^{TX}$ on $T^*X \otimes \text{End}(E)$ with values in $\text{End}(E)$.

Following an argument of [12], we get the last result of this paper.

Theorem 1.5. If $g^{TX}(< \cdot, \cdot> = \omega(\cdot, J \cdot)$, then for any $f, g \in \mathcal{C}^\infty(X, \text{End}(E))$, the coefficient $C_1(f, g) \in \mathcal{C}^\infty(X, \text{End}(E))$ defined in (1.16) is given by

$$C_1(f, g) = \langle (\nabla^E)^{1,0} f, (\nabla^E)^{0,1} g \rangle.$$

Note that this formula is clearly compatible with the formula (1.17) for the Poisson bracket in the case $f, g \in \mathcal{C}^\infty(X)$. Note also that (1.22) is a direct generalization of the formula [22, (0.20)] for Kähler manifolds.

We completed this paper a while ago. Recently Kordyukov informed us about his preprint [13] in which the Berezin-Toeplitz quantization by eigenstates of the Bochner-Laplacian is reconsidered.

The organization of the paper is as follows. In Section 2 we recall the asymptotic expansion of the generalized Bergman kernel obtained in [15]. In Section 3 we obtain the asymptotic expansion of the kernel of a Toeplitz operator. In Section 4 we show that the asymptotic expansion is also a sufficient condition for a family to be Toeplitz. In Section 5 we conclude that the set of Toeplitz operators forms an algebra. In Section 6 we prove Theorem 1.5.

2. The Asymptotic Expansion of the Generalized Bergman Kernel

Let $a^X$ be the injectivity radius of $(X, g^{TX})$. Let $d(x, y)$ denote the Riemannian distance from $x$ to $y$ on $(X, g^{TX})$. By [18, Proposition 8.3.5] (cf. also [15, (1.11)]), we have the following far off-diagonal behavior of the generalized Bergman kernel.

Proposition 2.1. For any $b > 0$ and any $k, l \in \mathbb{N}$ and $0 < \theta < 1$, there exists $C_{b,k,l,\theta} > 0$ such that

$$P_{h^p}(x, x') \big|_{\mathcal{C}^k(X \times X)} \leq C_{b,k,l,\theta} p^{-l}, \text{ for } d(x, x') > bp^{-\theta};$$

here the $\mathcal{C}^k$-norm is induced by $\nabla^L, \nabla^E, h^L, h^E$ and $g^{TX}$.

Let $\varepsilon \in (0, a^X/4)$ be fixed. We denote by $B^X(x, \varepsilon)$ and $B^{TX}(X, 0, \varepsilon)$ the open balls in $X$ and $T_xX$ with center $x$ and radius $\varepsilon$, respectively. We identify $B^{TX}(x, \varepsilon)$ with $B^X(x, \varepsilon)$ by using the exponential map of $(X, g^{TX})$.

Let $x_0 \in X$. For $Z \in B^{T_0X}(0, \varepsilon)$ we identify $(L_Z, h^L_Z), (E_Z, h^E_Z)$ and $(L^p \otimes E)_Z$ to $(L_{x_0}, h^L_{x_0}), (E_{x_0}, h^E_{x_0})$ and $(L^p \otimes E)_{x_0}$ by parallel transport with respect to the connections
\(\nabla^L, \nabla^E\) and \(\nabla^{L\otimes E}\) along the curve \(\gamma_Z : [0, 1] \ni u \to \exp^X_{x_0}(uZ)\). This is the basic trivialization we use in this paper.

Using this trivialization we identify \(f \in \mathcal{C}^\infty(X, \text{End}(E))\) to a family \(\{f_{x_0}\}_{x_0 \in X}\) where \(f_{x_0}\) is the function \(f\) in normal coordinates near \(x_0\), i.e., \(f_{x_0} : B^{x_0}(0, \varepsilon) \to \text{End}(E_{x_0})\), \(f_{x_0}(Z) = f \circ \exp^X_{x_0}(Z)\). In general, for functions expressed in normal coordinates centered at \(x_0 \in X\) we will add a subscript \(x_0\) to indicate the base point \(x_0\).

Similarly, \(P_{\mathcal{H}_r}(x, x')\) induces in terms of the basic trivialization a smooth section

\[
(Z, Z') \mapsto P_{\mathcal{H}_r, x_0}(Z, Z')
\]

of \(\pi^*\text{End}(E)\) over \(\{(Z, Z') \in TX \times_X TX : |Z|, |Z'| < \varepsilon\}\), which depends smoothly on \(x_0\).

Here \(\pi : TX \times_X TX \to X\) is the natural projection from the fibered product \(TX \times_X TX\) on \(X\) and we identify a section \(S \in \mathcal{C}^\infty(TX \times_X TX, \pi^*\text{End}(E))\) with the family \((S_x)_{x \in X}\), where \(S_x = S|_{\pi^{-1}(x)}\).

Let \(dv_{TX}\) be the Riemannian volume form on \((T_{x_0}X, g^{T_{x_0}X})\). Let \(\kappa_{x_0}(Z)\) be the smooth positive function defined by the equation

\[
(2.2) \quad dv_X(Z) = \kappa_{x_0}(Z)dv_{TX}(Z), \quad \kappa_{x_0}(0) = 1,
\]

where the subscript \(x_0\) of \(\kappa_{x_0}(Z)\) indicates the base point \(x_0 \in X\).

We identify the 2-form \(R^L\) with the Hermitian matrix \(\hat{R}^L \in \text{End}(T^{(1,0)}X)\) such that for any \(W, Y \in T^{(1,0)}X\),

\[
(2.3) \quad \hat{R}^L(W, Y) = \langle \hat{R}^L W, Y \rangle.
\]

Choose \(\{\omega_j\}_{j=1}^n\) an orthonormal basis of \(T^{(1,0)}_{x_0}X\) such that

\[
(2.4) \quad \hat{R}^L(x_0) = \text{diag}(a_1, \ldots, a_n) \in \text{End}(T^{(1,0)}_{x_0}X).
\]

We fix an orthonormal basis of \(T_{x_0}X\) given by \(e_{2j-1} = \frac{1}{\sqrt{2}}(\omega_j + \overline{\omega_j})\) and \(e_{2j} = \frac{1}{\sqrt{2}}(\omega_j - \overline{\omega_j})\). Then \(z_j = Z_{2j-1} + \sqrt{-1}Z_{2j}\) is a complex coordinate of \(Z \in \mathbb{R}^{2n} \simeq (T_{x_0}X, J)\).

By [15] Theorem 2.1 and [20] Theorem 1.18, we obtain the following version of the off diagonal expansion of the generalized Bergman kernel.

**Theorem 2.2.** For any \(x_0 \in X\) and \(\ell \in \mathbb{N}\), there exist polynomials \(J_{r,x_0}(Z, Z') \in \text{End}(E_{x_0})\) in \(Z, Z'\) with the same parity as \(\ell\) and with \(\deg J_{r,x_0} \leq 3\ell\) such that by setting

\[
(2.5) \quad \mathcal{P}_{r,x_0}(Z, Z') = J_{r,x_0}(Z, Z') \mathcal{P}(Z, Z'), \quad J_{0,x_0}(Z, Z') = \text{Id}_{E_{x_0}},
\]

with

\[
(2.6) \quad \mathcal{P}(Z, Z') = \exp \left[ -\frac{1}{4} \sum_{j=1}^n a_j \left( |z_j|^2 + |z'_j|^2 - 2z_jz'_j \right) \right],
\]

the following statement holds: for any \(b > 0\) and \(k_0, m, m' \in \mathbb{N}\), there exists \(C_{b,k_0,m,m'} > 0\) such that for \(|\alpha| + |\alpha'| \leq m'\) and any \(|Z|, |Z'| < b\) with

\[
(2.7) \quad \theta = \frac{1}{2(2n + 8 + 2k_0 + 3m' + 2m)}.
\]
we have
\begin{equation}
(2.8)
\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^\alpha} \left( p^{-n} P_{\mathcal{H}_p, x_0} (Z, Z') - \sum_{r=0}^{k} \mathcal{F}_{r, x_0} (\sqrt{p} Z, \sqrt{p} Z') \kappa_{x_0}^{-1/2} (Z) \kappa_{x_0}^{-1/2} (Z') p^{-\frac{r}{2}} \right) \right| \leq C_{b, k_0, m'} p^{-\frac{k}{2} - 1},
\end{equation}
where \( k = k_0 + m' + 2 \) and \( \mathcal{C}^m (X) \) is the \( \mathcal{C}^m \) norm for the parameter \( x_0 \in X \).

In particular when \( m' = 0 \), the following statement holds: for any \( b > 0 \) and \( k, m \in \mathbb{N} \), there exists \( C > 0 \) such that for any \( |Z|, |Z'| < bp^{-\frac{1}{2} + \theta_2} \) with
\begin{equation}
(2.9)
\theta_2 = \frac{1}{4(n + k + m + 2)},
\end{equation}
we have
\begin{equation}
(2.10)
\left| p^{-n} P_{\mathcal{H}_p, x_0} (Z, Z') - \sum_{r=0}^{k} \mathcal{F}_{r, x_0} (\sqrt{p} Z, \sqrt{p} Z') \kappa_{x_0}^{-1/2} (Z) \kappa_{x_0}^{-1/2} (Z') p^{-\frac{r}{2}} \right| \leq C p^{-k/2}.
\end{equation}
Note that the more expansion term in \( (2.10) \), the smaller of the expansion domain for the variables \( Z \) and \( Z' \). This serves as the main ingredient for the generalized Bergman kernel case.

By [20, Lemma 2.2], for any polynomials \( F, G \in \mathbb{C}[Z, Z'] \) there exist \( \mathcal{K} [F, G] \in \mathbb{C}[Z, Z'] \) such that
\begin{equation}
(2.11)
\left( (F \mathcal{P}) \circ (G \mathcal{P}) \right) (Z, Z') = \mathcal{K} [F, G] (Z, Z') \mathcal{P} (Z, Z').
\end{equation}

3. ASYMPTOTIC EXPANSION OF TOEPLITZ OPERATORS

For \( f \in \mathcal{C}^\infty (X, \text{End}(E)) \) we define the Toeplitz operator \( T_{f, p} \) on \( L^2 (X, L^p \otimes E) \) by
\begin{equation}
(1.10)
T_{f, p} (x, x') = \int_{X} P_{\mathcal{H}_p} (x, x'') f (x'') P_{\mathcal{H}_p} (x'', x') d\nu_X (x'').
\end{equation}
Note that if \( f \in \mathcal{C}^\infty (X, \text{End}(E)) \) is self-adjoint, i.e., \( f (x) = f (x)^* \) for all \( x \in X \), then the operator \( T_{f, p} \) is self-adjoint.

We examine now the asymptotic expansion of the kernel of Toeplitz operators \( T_{f, p} \). Outside the diagonal of \( X \times X \), we have the following analogue of [20, Lemma 4.2].

**Lemma 3.1.** Let \( \theta \in (0, 1) \) and \( f \in \mathcal{C}^\infty (X, \text{End}(E)) \) fixed. For any \( b > 0 \) and \( k, l \in \mathbb{N} \), there exists \( C_{b, k, l} > 0 \) such that
\begin{equation}
(3.2)
\left| T_{f, p} (x, x') \right|_{\mathcal{C}^k (X \times X)} \leq C_{b, k, l} p^{-l},
\end{equation}
for all \( p \geq 1 \) and all \( (x, x') \in X \times X \) with \( d (x, x') > bp^{-\theta} \), where the \( \mathcal{C}^k \)-norm is induced by \( \nabla^L, \nabla^E, h^L, h^E \) and \( g^{TX} \).

**Proof.** From Proposition [2.1] and (2.10), we know that for any \( k \in \mathbb{N} \) there exist \( C_k > 0 \) and \( M_k > 0 \) such that for all \( (x, x') \in X \times X \),
\begin{equation}
(3.3)
\left| P_{\mathcal{H}_p} (x, x') \right|_{\mathcal{C}^k (X \times X)} \leq C_k p^{M_k}.
\end{equation}
We split the integral in (3.1) in a sum of two integrals as follows:

\[(3.4)\quad T_{f,p}(x, x') = \left( \int_{B^X(x, \frac{b}{2}p^{-\theta})} + \int_{X \setminus B^X(x, \frac{b}{2}p^{-\theta})} \right) P_{\mathcal{H}_p}(x, x'') f(x'') P_{\mathcal{H}_p}(x'', x') dv_X(x'').\]

Assume that \(d(x, x') > bp^{-\theta}\). Then

\[(3.5)\quad d(x'', x') > \frac{b}{2}p^{-\theta} \quad \text{for} \quad x'' \in B^X(x, \frac{b}{2}p^{-\theta}),\]

\[(3.6)\quad d(x, x'') > \frac{b}{2}p^{-\theta} \quad \text{for} \quad x'' \in X \setminus B^X(x, \frac{b}{2}p^{-\theta}).\]

Now from (2.1) and (3.3)-(3.5), we get (3.2). The proof of Lemma 3.1 is complete. \(\square\)

We concentrate next on a neighborhood of the diagonal of \(X \times X\) in order to obtain the asymptotic expansion of the kernel \(T_{f,p}(x, x')\).

Let \(\{\Xi_p\}_{p \in \mathbb{N}}\) be a sequence of linear operators \(\Xi_p : L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)\) with smooth kernel \(\Xi_p(x, y)\) with respect to \(dv_X(y)\). Recall that \(\pi : TX \times_X TX \to X\) is the natural projection. Under our trivialization, \(\Xi_p(x, y)\) induces a smooth section \(\Xi_{p,0}(Z, Z')\) of \(\pi^* (\text{End}(E))\) over \(TX \times_X TX\) with \(Z, Z' \in T_{x_0}X, |Z|, |Z'| < aX\). Recall also that \(\mathcal{P}_{x_0} = \mathcal{P}\) was defined by (2.6).

Consider the following condition for \(\{\Xi_p\}_{p \in \mathbb{N}}\).

\textbf{Condition A.} There exists a family \(\{Q_{r,x_0}\}_{r \in \mathbb{N}, x_0 \in X}\) such that

(a) \(Q_{r,x_0} \in \text{End}(E_{x_0})|Z, Z'|\);

(b) \(\{Q_{r,x_0}\}_{r \in \mathbb{N}, x_0 \in X}\) is smooth with respect to the parameter \(x_0 \in X\) and there exist \(b_1, b_0 \in \mathbb{N}\) such that \(\deg(Q_r) \leq b_1 r + b_0\);

(c) for any \(k, m \in \mathbb{N}\), there exists \(\theta_{k,m} \in (0, 1/2)\) such that for any \(b > 0\), there exist \(C_{b,k,m} > 0\) such that for every \(x_0 \in X, Z, Z' \in T_{x_0}X\) with \(|Z|, |Z'| < b p^{-\frac{1}{2} + \theta_{k,m}}\) and \(p \in \mathbb{N}^*\), the following estimate holds:

\[(3.6)\quad \left| p^{-n} \Xi_{p,x_0}(Z, Z') \kappa_{x_0}^{1/2}(Z) \kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k} (Q_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} \right|_{\mathcal{E}^m(x)} \leq C_{b,k,m} p^{-k/2}.\]

\textbf{Notation A.} For any \(k, m \in \mathbb{N}\) we write the equation (3.6) for \(|Z|, |Z'| < b p^{-\frac{1}{2} + \theta_{k,m}}\) as

\[(3.7)\quad p^{-n} \Xi_{p,x_0}(Z, Z') \equiv \sum_{r=0}^{k} (Q_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathcal{O}_m(p^{-k/2}).\]

\textbf{Remark 3.2.} By Theorem 2.2, (2.9) and (2.10), we have

\[(3.8)\quad p^{-n} P_{\mathcal{H}_p,x_0}(Z, Z') \equiv \sum_{r=0}^{k} (J_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r/2} + \mathcal{O}_m(p^{-k/2}),\]

in the sense of Notation A with

\[(3.9)\quad \theta_{k,m} = \frac{1}{4(n + k + m + 2)} \quad \text{for} \quad k, m \in \mathbb{N},\]

where \(J_{r,x_0}(Z, Z') \in \text{End}(E_{x_0})\) are the polynomials in \(Z, Z'\) defined in (2.5). Note that \(J_{r,x_0}(Z, Z')\) has the same parity as \(r\) and \(\deg J_{r,x_0} \leq 3r, J_{0,x_0} = \text{Id}_{E_{x_0}}\).
The following result is about the near diagonal asymptotic expansion of the kernel $T_{f,p}(x,x')$. It is a version of [20, Lemma 4.6] in our situation.

**Lemma 3.3.** Let $f \in C^\infty(X, \text{End}(E))$. There exists a family $\{Q_{r,x_0}(f)\}_{r \in \mathbb{N}, x_0 \in X}$ such that

(a) $Q_{r,x_0}(f) \in \text{End}(E_{x_0})[Z, Z']$ are polynomials with the same parity as $r$;

(b) $\{Q_{r,x_0}(f)\}_{r \in \mathbb{N}, x_0 \in X}$ is smooth with respect to $x_0 \in X$, and $\deg Q_{r,x_0} \leq 3r$;

(c) for any $k_0, m \in \mathbb{N}$, we have

$$p^{-n}T_{f,p,x_0}(Z, Z') \approx \sum_{r=0}^{k_0} (Q_{r,x_0}(f) \mathcal{R}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + O_m(p^{-k_0/2}),$$

in the sense of Notation A with

$$\theta_{k_0,m} = \frac{1}{4(n+k+m+2)}$$

for some $k \geq k_0$, and $Q_{r,x_0}(f)$ are expressed by

$$Q_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathcal{K}[J_{r_1,x_0}, \partial_{p}\mathcal{R}_{x_0}(0)\frac{Z^\alpha}{\alpha!}J_{r_2,x_0}].$$

Especially,

$$Q_{0,x_0}(f) = f(x_0) \text{Id}_{E_{x_0}}.$$

**Proof.** For $k_0, m \in \mathbb{N}$ fixed, let $k \geq k_0$ to be determined. Set

$$\theta_2 = \frac{1}{4(n+k+m+2)}, \quad \theta_1 = 1 - 2\theta_2.$$

By (2.10), we have for any $|Z|, |Z'| < 2bp^{-\frac{1}{2}+\theta_2} = 2bp^{-\theta_1/2}$,

$$\left|p^{-n}P_{H_{p,x_0}}(Z, Z') - \sum_{r=0}^{k} \mathcal{R}_{r,x_0}(\sqrt{p}Z, \sqrt{p}Z')\kappa_{x_0}^{-1/2}(Z)\kappa_{x_0}^{-1/2}(Z')p^{-r/2}\right|_{C^m(X)} \leq C_k, p^{-k/2}.$$

For $|Z|, |Z'| < \frac{bp^{-1/2+\theta_2}}{2} = \frac{bp^{-\theta_1/2}}{2}$, we get from (3.11) that

$$T_{f,p,x_0}(Z, Z') = \int_X P_{H_{p,x_0}}(Z, y) f(y) P_{H_{p,x_0}}(y, Z')dv_X(y).$$

We split the integral into integrals over $B^X(x, bp^{-\theta_1/2})$ and $X \setminus B^X(x, bp^{-\theta_1/2})$. We have

$$d(y, \exp_{x_0} Z) \geq d(y, x_0) - |Z| > \frac{bp^{-\theta_1/2}}{2} \text{ on } X \setminus B^X(x, bp^{-\theta_1/2}),$$

since on this set $d(y, x_0) > bp^{-\theta_1/2}$ holds. By Proposition 2.1 for $\theta_1$ in (3.14), (3.3) and (3.17) we have for $|Z|, |Z'| < \frac{bp^{-\theta_1/2}}{2}$,

$$T_{f,p,x_0}(Z, Z') = \int_{|Z'| < bp^{-\frac{\theta_1}{2}}} P_{H_{p,x_0}}(Z, Z')f_{x_0}(Z')P_{H_{p,x_0}}(Z', Z')\kappa_{x_0}(Z')dv_{TX}(Z') + \mathcal{O}_m(p^{-\infty})$$
Then

(3.19)

\[ p^{-n}T_{f,p,x_0}(Z, Z') = \sum_{|a| \leq k} \frac{\partial^a f_{x_0}(0) Z^n}{\alpha!} + O(|Z|^{k+1}) \]

and

(3.20)

\[ f_{x_0}(Z) = \sum_{|a| \leq k} \frac{\partial^a f_{x_0}(0) Z^n}{\alpha!} + O(|Z|^{k+1}) \]

Combining the asymptotic expansion (3.19) and (3.20), to obtain the asymptotic expansion (3.15), we need to consider \( I_{r_1,|a|,r_2}(T_{x_0}X)(Z, Z') \) defined by

(3.21)

\[ p^{-n}T_{f,p,x_0}(Z, Z') \kappa_{x_0}^{1/2}(Z) \kappa_{x_0}^{-1/2}(Z') \]

(3.22)

\[ = \sum_{r_1,|a|,r_2 \leq k} I_{r_1,|a|,r_2}(B_{T_{x_0}X}(0, a))(Z, Z') + I_1(Z, Z') + I_2(Z, Z') + I_3(Z, Z') + \mathcal{O}_m(p^{-\infty}) \]

with

(3.23)

\[ I_1(Z, Z') = \int_{|Z'| < b^{-\theta_1/2}} p^{-n} P_{x_0}(Z, Z') \kappa_{x_0}^{1/2}(Z) \kappa_{x_0}^{-1/2}(Z') \]

and

(3.24)

\[ I_2(Z, Z') = \int_{|Z'| < b^{-\theta_2/2}} \sum_{r_1 \leq k} (J_{r_1,x_0} \phi_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-r_1/2} \]

\[ \times f_{x_0}(Z') \kappa_{x_0}^{1/2}(Z') \kappa_{x_0}^{-1/2}(Z') \]
By (3.15) and the fact that

\[
(3.28)
\]

\[
(3.27)
\]

\[
(3.26)
\]

\[
(3.25)
\]

\[
(3.24)
\]

\[
(3.23)
\]

\[
(3.22)
\]

\[
(3.21)
\]

\[
(3.20)
\]

\[
(3.19)
\]

\[
(3.18)
\]

\[
(3.17)
\]

\[
(3.16)
\]

\[
(3.15)
\]

\[
(3.14)
\]

\[
(3.13)
\]

\[
(3.12)
\]

\[
(3.11)
\]

\[
(3.10)
\]

In fact, by (3.3), there exists \( C_0 > 0 \) and \( M_0 > 0 \) such that for all \((x, x') \in X \times X\),

\[
|P_{H_p}(x, x')|_{\mathcal{E}^m(X)} \leq C_0 p^{M_0}.
\]

Combining (3.15), (3.23) and (3.26) yields

\[
|I_1(Z, Z')|_{\mathcal{E}^m(X)} \leq C p^{-\frac{3}{2}+M_0}.
\]

By (3.20), (3.26) and the fact that \( \deg J_r \leq 3r \),

\[
|I_2(Z, Z')|_{\mathcal{E}^m(X)} \leq C (1 + \sqrt{p|Z|})^{3k} \cdot p^{\frac{3}{2}+3\theta_2} \cdot p^{M_0}
\]

\[
\leq C p^{-\frac{3}{2}+3k\theta_2+M_0}.
\]

By (3.15) and the fact that \( \deg J_r \leq 3r \), we have for \(|Z|, |Z'| < \frac{1}{2}p^{-\theta_1/2}\),

\[
|I_3(Z, Z')|_{\mathcal{E}^m(X)} \leq C (1 + \sqrt{p|Z|})^{3k} p^{-\frac{3}{2}}.
\]

From (3.27)–(3.29), choose \( k > k_0 \) big enough such that

\[
k + 1 - 6k\theta_2 - 2M_0 = k \left( 1 - \frac{3}{4(n+k+m+2)} \right) - 2M_0 + 1 > k_0.
\]

Then the claim (3.25) holds. By (3.22) and (3.25),

\[
|p^{-n} T_{f_{\rho}, x_0}(Z, Z')^{k_{1/2}}(Z)^{k_{1/2}}(Z') - \sum_{r_1, r_2 < k} I_{r_1, |a|, r_2}(B^{T_{x_0} X}(0, bp^{-\theta_1/2}))(Z, Z')|_{\mathcal{E}^m(X)} \leq C p^{-k_0/2}.
\]

Note by (2.6),

\[
|\mathcal{P}(Z, Z')| = e^{-\frac{1}{4} \sum_{j} a_j |z_j - z'_j|^2} \leq e^{-\frac{1}{4} \mu_0 |Z - Z'|^2}.
\]

Then

\[
|\mathcal{P}(\sqrt{p}Z, \sqrt{p}Z')| = e^{-\frac{1}{4} \sum_{j} a_j |z_j - z'_j|^2} \leq e^{-\frac{1}{4} \mu_0 |Z - Z'|^2}.
\]

By (3.33) and the fact that \( \deg J_r \leq 3r \), we obtain

\[
|I_{r_1, |a|, r_2}(T_{x_0} X \setminus B^{T_{x_0} X}(0, bp^{-\theta_1/2}))(Z, Z')|_{\mathcal{E}^m(X)} \leq C p^n \int_{|Z''| > bp^{-\theta_1/2}} (1 + \sqrt{p}|Z| + \sqrt{p}|Z''|)^{3r_1} (1 + \sqrt{p}|Z| + \sqrt{p}|Z''|)^{3r_2} \times (\sqrt{p}|Z''|)^{|a|} \exp \left[ -\frac{\mu_0}{2} \sqrt{p}|Z - Z''| - \frac{\mu_0}{2} \sqrt{p}|Z'' - Z'| \right] d_{TX}(Z'').
\]
Note that for any $|Z|, |Z'| < \frac{b}{2}p^{-\theta_1/2}$ and $|Z''| > b p^{-\theta_1/2}$, we have
\begin{equation}
|Z| < |Z''|, |Z'| < |Z''|, |Z - Z''| \geq \frac{1}{2}|Z''|, |Z' - Z''| \geq \frac{1}{2}|Z''|.
\end{equation}
Substituting (3.35) into (3.34) yields for any $|Z|, |Z'| < \frac{b}{2}p^{-\theta_1/2}$,
\begin{equation}
|I_{r_1,|a|,r_2}(T_{x_0} X \setminus B_{T_{x_0}} X (0, bp^{-\theta_1/2}))(Z, Z')|_{\mathcal{C}^m(X)}
\leq C p^n \int_{|Z''| > bp^{-\theta_1/2}} (1 + \sqrt{p}|Z''|)^{3(r_1 + r_2)} \sqrt{p} |Z''| |\alpha| e^{-\frac{4p}{2} \sqrt{p} |Z''|} d_{TVX}(Z'').
\end{equation}
Combining (3.31) and (3.36), we obtain
\begin{equation}
p^{-n} T_{f,p,x_0}(Z, Z') \kappa^{1/2}(Z) \kappa^{1/2}(Z') - \sum_{r_1, |a|, r_2 \leq k} I_{r_1, |a|, r_2}(T_{x_0} X)(Z, Z') \bigg|_{\mathcal{C}^m(X)} \leq C p^{-\frac{k_0}{2}}.
\end{equation}
Clearly,
\begin{equation}
\sum_{r_1, |a|, r_2 \leq k} I_{r_1, |a|, r_2}(T_{x_0} X)(Z, Z')
= \left( \sum_{r_1 + |a| + r_2 \leq k_0} + \sum_{r_1 + |a| + r_2 = k_0 + 1} \right) I_{r_1, |a|, r_2}(T_{x_0} X)(Z, Z').
\end{equation}
By (2.11) and (3.21),
\begin{equation}
I_{r_1, |a|, r_2}(T_{x_0} X)(Z, Z')
= p^{-(r_1 + |a| + r_2)/2} \left( \mathcal{K} \left[ J_{r_1, x_0}, \frac{\partial^{|a|} f_{x_0}}{\partial |Z|^{|a|}}(0) \frac{Z^{|a|}}{|a|!} J_{r_2, x_0} \right] \mathcal{P} \right) (\sqrt{p} Z, \sqrt{p} Z').
\end{equation}
In view of (3.37)-(3.39), to finish the proof of Lemma 3.3, it suffices to prove that the $\mathcal{C}^m$ norm with respect to the parameter $x_0 \in X$ of the term
\begin{equation}
\sum_{r_1 + |a| + r_2 = k_0 + 1} 3k I_{r_1, |a|, r_2}(T_{x_0} X)(Z, Z'), \text{ for } |Z|, |Z'| < \frac{b}{2} p^{-\theta_1/2},
\end{equation}
is controlled by $C p^{-k_0/2}$ for large $k$.

Estimating $I_{r_1, |a|, r_2}(T_{x_0} X)(Z, Z')$ for $|Z|, |Z'| < \frac{b}{2} p^{-\theta_1/2}$, using (3.33), (3.39) and the fact that $\deg J_r \leq 3r$, we obtain
\begin{equation}
|I_{r_1, |a|, r_2}(T_{x_0} X)(Z, Z')|_{\mathcal{C}^m(X)}
\leq C p^{-s/2} (1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^{3s} \exp \left[ -\frac{p}{4} \mu_0 |Z - Z'|^2 \right]
\leq C p^{-\frac{1}{2} p^{\frac{1}{4} \mu_0} - 3s} = C p^{-s(1-6\theta_2)/2},
\end{equation}
with \( s = r_1 + |\alpha| + r_2 \). If \( s > k_0 \), then

\[
(3.42) \quad s(1 - 6\theta_2) \geq (k_0 + 1) \left( 1 - \frac{6}{4(n + k + m + 2)} \right).
\]

Choose \( k \) big enough such that

\[
(3.43) \quad (k_0 + 1) \left( 1 - \frac{6}{4(n + k + m + 2)} \right) > k_0.
\]

Then

\[
(3.44) \quad \left| I_{r_1,|\alpha|,r_2}(T_{x_0}X)(Z, Z') \right|_{\mathcal{E}^m(X)} \leq C p^{-k_0/2} \text{ for } r_1 + |\alpha| + r_2 = s > k_0.
\]

To sum up, we have proved the following statement: for fixed \( k_0 \), choose \( k > k_0 \) such that (3.30) and (3.43) hold. Set

\[
(3.45) \quad \theta_2 = \frac{1}{4(n + k + m + 2)}, \quad \theta_1 = 1 - 2\theta_2.
\]

Then for any \( |Z|, |Z'| < \frac{k}{2}p^{-\theta_1/2} \), we have

\[
(3.46) \quad \left| p^{-n}T_{f,p,x_0}(Z, Z')r_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k_0} (Q_{r,x_0}(f) \mathcal{P}_{x_0})(\sqrt{pZ}, \sqrt{pZ'})p^{-r/2} \right|_{\mathcal{E}^m(X)} \leq C p^{-k_0/2},
\]

where \( Q_{r,x_0}(f) \) is given by (3.12). This completes the proof of Lemma 3.3.

**Remark 3.4.** Let \( \Xi \) be a sequence of operators satisfying Condition A and assume that \( \Xi_p = P_{\mathcal{H}_p}\Xi_p P_{\mathcal{H}_p} \) for all \( p \in \mathbb{N} \). Applying the proof of Lemma 3.3 by splitting integrals and studying different integration regions, we deduce by Theorem 2.2 and (3.6):

For any \( k, m, m' \in \mathbb{N} \), there exist \( \theta_{k,m,m'} \in (0, 1/2) \) such that for any \( b > 0 \), there exist \( C > 0 \) such that for every \( x_0 \in X \), \( Z, Z' \in T_{x_0}X \) with \( |Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k,m,m'}} \) and \( p \in \mathbb{N}^* \), \( |\alpha| + |\alpha'| \leq m' \), we have

\[
(3.47) \quad \left| \frac{\partial^{2|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} (p^{-n}\Xi_{x_0}(Z, Z')r_{x_0}^{1/2}(Z)\kappa_{x_0}^{1/2}(Z') - \sum_{r=0}^{k} (Q_{r,x_0}(f) \mathcal{P}_{x_0})(\sqrt{pZ}, \sqrt{pZ'})p^{-r/2} \right|_{\mathcal{E}^m(X)} \leq C p^{-(k-m')/2}.
\]

In fact, by \( \Xi_p = P_{\mathcal{H}_p}\Xi_p P_{\mathcal{H}_p} \) for \( |Z|, |Z'| < bp^{-\frac{1}{2}+\theta_{k,m,m'}} \), we have the analogue of (3.16):

\[
(3.48) \quad \Xi_{x_0}(Z, Z') = \int_X P_{\mathcal{H}_{p,x_0}}(Z, y)\Xi_p(y, Z')P_{\mathcal{H}_p}(y, Z')dv_X(y).
\]

Then the estimate (3.47) follows from Theorem 2.2, (3.6), (3.15) and (3.48) in the same manner as (3.46) follows from (3.15), (3.16) and (3.20).
4. A CRITERION FOR TOEPLITZ OPERATORS

In this section we prove a useful criterion which ensures that a given family of bounded linear operators is a Toeplitz operator.

**Theorem 4.1.** Let \( \{T_p : L^2(X, L^p \otimes E) \to L^2(X, L^p \otimes E)\} \) be a family of bounded linear operators which satisfies the following three conditions:

(i) For any \( p \in \mathbb{N} \), \( P_{H_p} T_p P_{H_p} = T_p \).

(ii) For any \( b > 0 \), \( l \in \mathbb{N} \) and \( 0 < \theta < 1 \), there exists \( C_{b,l,\theta} > 0 \) such that for all \( p \geq 1 \) and all \( (x,x') \in X \times X \) with \( d(x,x') > bp^{-\theta/2} \),

\[
|T_p(x,x')| \leq C_{b,l,\theta}p^{-l}.
\]

(iii) There exists a family of polynomials \( \{\mathcal{Q}_{r,x_0} \in \text{End}(E_{x_0})[Z, Z']\} \) \( x_0 \in X \) such that

(a) each \( \mathcal{Q}_{r,x_0} \) has the same parity as \( r \) and there exist \( b_1, b_0 \in \mathbb{N} \) such that \( \deg \mathcal{Q}_r \leq b_1 r + b_0 \),

(b) the family is smooth in \( x_0 \) \( \in X \) and

(c) for any \( k_0, m \in \mathbb{N} \), there exists \( \theta_{k_0,m} \in (0,1/2) \) such that for any \( b > 0 \), \( p \in \mathbb{N}^* \), \( x_0 \in X \) and every \( Z, Z' \in T_{x_0}X \) with \( |Z|, |Z'| < bp^{-1/2} + \theta_{k_0,m} \), we have

\[
p^{-n}T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k_0} (\mathcal{Q}_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}(p^{-k_0/2}),
\]

in the sense of Notation A for \( k_0, m, \theta_{k_0,m} \).

Then \( \{T_p\} \) is a Toeplitz operator.

**Remark 4.2.** By Lemmas 3.1 and 3.3 and by (1.12), (1.13) and the Sobolev inequality (cf. [3, (4.14)]), it follows that every Toeplitz operator in the sense of Definition 1.1 verifies the Conditions (i), (ii) and (iii) of Theorem 4.1.

We start the proof of Theorem 4.1. Let \( T_p^* \) be the adjoint of \( T_p \). By writing

\[
T_p = \frac{1}{2}(T_p + T_p^*) + \frac{1}{2\sqrt{-1}}(T_p - T_p^*),
\]

we may and will assume from now on that \( T_p \) is self-adjoint.

We will define inductively the sequence \( \{g_l \}_{l \geq 0} \), \( g_l \in \mathcal{C}^\infty(X, \text{End}(E)) \) such that

\[
T_p = \sum_{l=0}^{q} P_{H_p} g_l p^{-l} P_{H_p} + \mathcal{O}(p^{-q-1}) \quad \text{for all } q \geq 0.
\]

Moreover, we can make these \( g_l \)'s to be self-adjoint.

Let us start with the case \( q = 0 \) of (4.4). For an arbitrary but fixed \( x_0 \in X \), we set

\[
g_0(x_0) = \mathcal{Q}_{0,x_0}(0,0) \in \text{End}(E_{x_0}).
\]

We will show that

\[
p^{-n}(T_p - T_{g_0,p})_{x_0}(Z, Z') \cong \mathcal{O}_n(p^{-1}),
\]

which implies the case \( q = 0 \) of (4.4), namely,

\[
T_p = P_{H_p} g_0 P_{H_p} + \mathcal{O}(p^{-1}).
\]

The proof of (4.6)–(4.7) will be done in Proposition 4.3 and Proposition 4.9.
Proposition 4.3. In the conditions of Theorem 4.1 we have
\[ \mathcal{D}_{0,x_0}(Z, Z') = \mathcal{D}_{0,x_0}(0, 0) \in \text{End}(E_{x_0}) \]
for all \( x_0 \in X \) and all \( Z, Z' \in T_{x_0}X \).

Proof. The proof is divided in the series of Lemmas 4.4–4.8. Our first observation is as follows.

Lemma 4.4. \( \mathcal{D}_{0,x_0} \in \text{End}(E_{x_0})[Z, Z'] \) and \( \mathcal{D}_{0,x_0} \) is a polynomial in \( z, \bar{z}' \).

Proof. By (4.2), for \( k_0 = 2 \) there exists \( \theta_3 \in (0, 1/2) \) such that for any \( b > 0 \) and every \( Z, Z' \in T_{x_0}X \) with \( |Z|, |Z'| < bp^{-\frac{1}{3}+\theta_3} \), we have
\[ p^{-n}T_{p,x_0}(Z, Z') \equiv \sum_{r=0}^{2} (\mathcal{D}_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}_m(p^{-1}). \]

By (3.8),
\[ p^{-n}P_{H_{p,x_0}}(Z, Z') \equiv \sum_{r=0}^{2} (J_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}_m(p^{-1}), \]
in the sense of Notation A with \( \theta_4 = 1/4(n + m + 4) \). Combining (4.9) and (4.10), modelled the way we get (3.46) from (3.15) and (3.20), we obtain that there exists \( k \geq 2 \) large enough and
\[ \theta_m = \frac{1}{4(n + k + m + 2)} \in (0, 1/2) \]
such that for every \( Z, Z' \in T_{x_0}X \) with \( |Z|, |Z'| < bp^{-\frac{1}{3}+\theta_m} \),
\[ p^{-n}(P_{H_{p,T_{p}}P_{H_{p}}})_{x_0}(Z, Z') \]
\[ \approx \sum_{r=0}^{2} \sum_{r_1 + r_2 + r_3 = r} \left( J_{r_1,x_0} \mathcal{P}_{x_0} \circ (\mathcal{D}_{r_2,x_0} \mathcal{P}_{x_0}) \circ (J_{r_3,x_0} \mathcal{P}_{x_0}) \right)(\sqrt{p}Z, \sqrt{p}Z')p^{-r/2} + \mathcal{O}_m(p^{-1}). \]

Since \( P_{H_{p,T_{p}}P_{H_{p}}} = T_{p} \), we deduce from (4.9) and (4.12) that
\[ \mathcal{D}_{0,x_0} \mathcal{P}_{x_0} = \mathcal{P}_{x_0} \circ (\mathcal{D}_{0,x_0} \mathcal{P}_{x_0}) \circ \mathcal{P}_{x_0}. \]

By [20, (8)] and (4.13), we obtain
\[ \mathcal{D}_{0,x_0} \in \text{End}(E_{x_0})[z, \bar{z}]. \]

The proof of Lemma 4.4 is complete. \( \square \)

For simplicity we denote in the rest of the proof \( F_x = \mathcal{D}_{0,x} \in \text{End}(E_x) \). Let \( F_x = \sum_{i \geq 0} F_{x}^{(i)} \) be the decomposition of \( F_x \) in homogeneous polynomials \( F_{x}^{(i)} \) of degree \( i \). We will show \( F_{x}^{(i)} \) vanish identically for \( i > 0 \), that is
\[ F_{x}^{(i)}(z, \bar{z}') = 0 \] for all \( i > 0 \) and \( z, \bar{z}' \in \mathbb{C} \).

The first step is to prove
\[ F_{x}^{(i)}(0, \bar{z}') = 0 \] for all \( i > 0 \) and \( z' \in \mathbb{C} \).
Since $T_p$ is self-adjoint, then we have

\[
(4.17) \quad F^{(i)}_z(z, z') = (F^{(i)}_z(z', z))^*.
\]

Consider $0 < \theta_{k_0, m} < 1$ as in hypothesis (iii)(c) of Theorem \[4.1\] For $Z' \in \mathbb{R}^{2n} \simeq T_x X$ with $|Z'| < \varepsilon p^{-\frac{1}{4} + \theta_{k_0, m}}$ and $y = \exp^X_x(Z')$. Set

\[
F^{(i)}(x, y) = F^{(i)}_x(0, z') \in \text{End}(E_x),
\]

\[
(4.18) \quad \tilde{F}^{(i)}(x, y) = (F^{(i)}(y, x))^* \in \text{End}(E_y).
\]

$F^i$ and $\tilde{F}^{(i)}$ define smooth sections on a neighborhood of the diagonal of $X \times X$. Clearly, the $\tilde{F}(x, y)$’s need not be polynomials in $z$ and $z'$.

Since we wish to define global operators induced by these kernels, we use a cut-off function in the neighborhood of the diagonal. Pick a smooth function $\eta \in \mathcal{C}^\infty(\mathbb{R})$ such that

\[
(4.19) \quad \eta(u) = 1 \text{ for } |u| \leq \varepsilon/2 \quad \text{and} \quad \eta(u) = 0 \text{ for } |u| \geq \varepsilon.
\]

We denote by $F^{(i)}P_{\mathcal{H}_p}$ and $P_{\mathcal{H}_p}\tilde{F}^{(i)}$ the operators defined by the kernels

\[
(4.20) \quad \eta(d(x, y))F^{(i)}(x, y)P_{\mathcal{H}_p}(x, y) \quad \text{and} \quad \eta(d(x, y))P_{\mathcal{H}_p}(x, y)\tilde{F}^{(i)}(x, y)
\]

with respect to $dv_X(y)$. Set

\[
(4.21) \quad \mathcal{T}_p = T_p - \sum_{s \leq \deg F_z} (F^{(i)}P_{\mathcal{H}_p})p^{\frac{s}{2}}.
\]

The operators $\mathcal{T}_p$ extends naturally to bounded operators on $L^2(X, L^p \otimes E)$.

From (4.2) and (4.21) we deduce that for any $k_0, m \in \mathbb{N}$, there exist $\theta_{k_0, m} \in (0, 1/2)$ such that for any $|Z'| < \varepsilon p^{-\frac{1}{4} + \theta_{k_0, m}}$, we have the following expansion in the normal coordinates around $x_0 \in X$ (which has to be understood in the sense of (3.7)):

\[
(4.22) \quad p^{-n}\mathcal{T}_{p,x_0}(0, Z') \tilde{\simeq} \sum_{r=1}^{k_0} (R_{r,x_0}(0, \sqrt{p}Z'))p^{-r/2} + \mathcal{O}_m(p^{-k_0/2}),
\]

for some polynomials $R_{r,x_0}$ of the same parity as $r$. For simplicity let us define similarly to (4.19) the kernel

\[
(4.23) \quad R_{r,p}(x, y) = p^n(R_{r,x}(0, \sqrt{p}Z'))k_x^{-1/2}(Z')\eta(d(x, y)),
\]

where $y = \exp^X_x(Z')$, and denote by $R_{r,p}$ the operator defined by this kernel.

**Lemma 4.5.** There exists $C > 0$ such that for every $p \geq 1$ and $s \in L^2(X, L^p \otimes E)$, we have

\[
(4.24) \quad \left\| \mathcal{T}_p s \right\|_{L^2} \leq C p^{-1/2} \left\| s \right\|_{L^2},
\]

\[
\left\| \mathcal{T}_p^* s \right\|_{L^2} \leq C p^{-1/2} \left\| s \right\|_{L^2}.
\]

**Proof.** In order to use (4.22) we write

\[
(4.25) \quad \left\| \mathcal{T}_p s \right\|_{L^2} \leq \left\| \left( \mathcal{T}_p - \sum_{r=1}^{k_0} p^{-r/2}R_{r,p} \right) s \right\|_{L^2} + \left\| \sum_{r=1}^{k_0} p^{-r/2}R_{r,p}s \right\|_{L^2}.
\]
By the Cauchy-Schwarz inequality we have
\[
\left\| \left( \mathcal{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} \right) s \right\|_{L^2}^2 
\leq \int_X \left( \int_X \left| \left( \mathcal{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} \right) (x, y) \right| dv_X(y) \right) 
\times \left( \int_X \left| \left( \mathcal{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} \right) (x, y)\right| \|s(y)\|^2 dv_X(y) \right) dv_X(x).
\] (4.26)

By (2.1), (4.1), (4.20), (4.21) and (4.23), we obtain uniformly in \( x \in X \),
\[
\int_X \left( \left( \mathcal{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} \right) (x, y) \right) \|s(y)\|^2 dv_X(y)
\leq \int_{B^n(x, \frac{\varepsilon}{2} p^{-\frac{1}{2}} + \theta_{k_0,m})} \left( \left( \mathcal{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} \right) (x, y) \right) \|s(y)\|^2 dv_X(y)
+ O(p^{-\infty}) \int_{X \setminus B^n(x, \frac{\varepsilon}{2} p^{-\frac{1}{2}} + \theta_{k_0,m})} \|s(y)\|^2 dv_X(y),
\] (4.27)
where \( \theta_{k,m} \) is given by (4.22). By (3.6) and (4.22) we obtain
\[
\int_{B^n(x, \frac{\varepsilon}{2} p^{-\frac{1}{2}} + \theta_{k_0,m})} \left( \left( \mathcal{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} \right) (x, y) \right) \|s(y)\|^2 dv_X(y)
= O(p^{-1}) \int_{B^n(x, \frac{\varepsilon}{2} p^{-\frac{1}{2}} + \theta_{k_0,m})} \|s(y)\|^2 dv_X(y).
\] (4.28)

In the same vein (by splitting the integral region as above) we obtain
\[
\int_X \left( \left( \mathcal{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} \right) (x, y) \right) dv_X(y) = O(p^{-1}) + O(p^{-\infty}).
\] (4.29)
Combining (4.26)–(4.29) yields
\[
\left\| \left( \mathcal{T}_p - \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} \right) s \right\|_{L^2} \leq C p^{-1} \|s\|_{L^2}.
\] (4.30)
A similar proof as for (4.30) delivers for \( s \in L^2(X, L^p \otimes E) \),
\[
\| R_{r,p} s \|_{L^2} \leq C \|s\|_{L^2},
\] (4.31)
which implies
\[
\left\| \sum_{r=1}^{k_0} p^{-r/2} R_{r,p} s \right\|_{L^2} \leq C p^{-1/2} \|s\|_{L^2} \text{ for } s \in L^2(X, L^p \otimes E),
\] (4.32)
for some constant \( C > 0 \). Relations (4.30) and (4.32) entail the first inequality of (4.24), which is equivalent to the second of (4.24), by taking the adjoint. This completes the proof of Lemma 4.5. □
Let us consider the Taylor development of $\tilde{F}^{(i)}$ in normal coordinates around $x$ with $y = \exp_x^X(Z')$:

$$\tilde{F}^{(i)}(x, y) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha \tilde{F}^{(i)}(x, 0)}{\partial Z^\alpha} (x, 0) \left( \frac{\sqrt{p} Z'}{\alpha!} \right) p^{-|\alpha|/2} + O(|Z'|^{k+1}).$$

(4.33)

The next step of the proof of Proposition 4.3 is the following.

**Lemma 4.6.** For every $j > 0$, we have

$$\frac{\partial^\alpha \tilde{F}^{(i)}(x, 0)}{\partial Z^\alpha} = 0 \quad \text{for} \quad i - |\alpha| \geq j > 0.$$  

(4.34)

*Proof.* The definition (4.21) of $\mathcal{F}_p$ shows that

$$\mathcal{F}_p^* = T_p - \sum_{j \leq \deg F_x} p^{j/2} (P_{\mathcal{H}_p} \tilde{F}^{(i)}).$$

(4.35)

Let us develop the sum on the right-hand side. Consider the Taylor development (4.33) with the expansion (3.8) of the Bergman kernel we obtain

$$p^{-n} (P_{\mathcal{H}_p} \tilde{F}^{(i)})_{x_{0}} (0, Z') \kappa^{1/2}(Z') - \sum_{r, |\alpha| \leq k} \left( J_{r, x_{0}} \mathcal{P}_{x_{0}} \right)(0, \sqrt{p} Z') \frac{\partial^\alpha \tilde{F}^{(i)}}{\partial Z^\alpha} (x_{0}, 0) \left( \frac{\sqrt{p} Z'}{\alpha!} \right) p^{-|\alpha|/2}$$

$$= \left[ p^{-n} P_{\mathcal{H}_p,x_{0}} (0, Z') \kappa^{1/2}(Z') - \sum_{r \leq k} \left( J_{r, x_{0}} \mathcal{P}_{x_{0}} \right)(0, \sqrt{p} Z') p^{-r/2} \right] \tilde{F}^{(i)}(0, Z')$$

$$+ \sum_{r \leq k} \left( J_{r, x_{0}} \mathcal{P}_{x_{0}} \right)(0, \sqrt{p} Z') p^{-r/2} \left[ \tilde{F}^{(i)}(0, Z') - \sum_{|\alpha| \leq k} \frac{\partial^\alpha \tilde{F}^{(i)}}{\partial Z^\alpha} (x_{0}, 0) \left( \frac{\sqrt{p} Z'}{\alpha!} \right) \right].$$

By (3.8), (3.33), (4.33) and $\deg J_{r, x_{0}} \leq 3r$, we obtain for $k \geq \deg F_x + 1$ and $m \in \mathbb{N}$, there exist $\theta_{k,m} \in (0, 1)$ such that for any $Z' \in T_{x_{0}}X$ with $|Z'| \leq b p^{-\frac{r}{2}} + \theta_{k,m}$, we have

$$p^{-n} \sum_{i} \left( P_{\mathcal{H}_p} \tilde{F}^{(i)} \right)_{x_{0}} (0, Z') p^{i/2}$$

$$\approx \sum_{i} \sum_{|\alpha|, r \leq k} \left( J_{r, x_{0}} \mathcal{P}_{x_{0}} \right)(0, \sqrt{p} Z') \frac{\partial^\alpha \tilde{F}^{(i)}}{\partial Z^\alpha} (x_{0}, 0) \left( \frac{\sqrt{p} Z'}{\alpha!} \right) p^{(i-|\alpha|-r)/2} + O_m(p^{(deg F-k)/2}).$$

(4.37)

Having in mind the second inequality of (4.24), this is only possible if for every $j > 0$ the coefficients of $p^{j/2}$ on the right-hand side of (4.37) vanish. Thus, we have for every $j > 0$:

$$\sum_{l \leq \deg F_x} \sum_{|\alpha|+r=l-j} \left( J_{r, x_{0}} \right)(0, \sqrt{p} Z') \frac{\partial^\alpha \tilde{F}^{(i)}}{\partial Z^\alpha} (x_{0}, 0) \left( \frac{\sqrt{p} Z'}{\alpha!} \right) = 0.$$  

(4.38)

From (4.38), we will prove by recurrence that for any $j > 0$, (4.34) holds. As the first step of the recurrence let us take $j = \deg F_x$ in (4.38). Since $J_{0, x_{0}} = \text{Id}_{E_{x_{0}}}$, then we get immediately $\tilde{F}^{(\deg F_x)}(x_{0}, 0) = 0$. Hence (4.34) holds for $j = \deg F_x$. Assume that (4.34)
holds for \( j > j_0 > 0 \). Then for \( j = j_0 \), the coefficient with \( r > 0 \) in (4.38) is zero. Since \( J_{0,x_0} = \text{Id}_{E_{x_0}} \), then (4.38) reads

\[
\sum_{\alpha} \frac{\partial^m \tilde{F}^{(j_0 + |\alpha|)}(x_0,0)}{\partial Z^\alpha}(0,0) \left( \sqrt{\tilde{p} Z'} \right)^\alpha = 0,
\]

which entails (4.34) for \( j = j_0 \). The proof of (4.34) is complete. □

**Lemma 4.7.** For \( i > 0 \), we have

\[
\frac{\partial^m F^{(i)}(0,0)}{\partial z^\alpha} = 0, \quad |\alpha| \leq i.
\]

Therefore, \( F^{(i)}(0, z') = 0 \) for all \( i > 0 \) and \( z' \in \mathbb{C} \), i.e., (4.16) holds true. Moreover,

\[
F^{(i)}(z,0) = 0 \quad \text{for all } i > 0 \text{ and all } z \in \mathbb{C}.
\]

**Proof.** Let us start with some preliminary observations. In view of (4.24), (4.34) and (4.37), a comparison the coefficient of \( p^0 \) in (4.9) and (4.35) yields

\[
\tilde{F}^{(i)}(x, Z') = F^{(i)}(0, z') + O(|Z'|^{i+1}).
\]

Using the definition (4.18) of \( \tilde{F}^{(i)}(x, Z') \) and taking the adjoint of (4.42), we get

\[
F^{(i)}(Z', x) = (F^{(i)}(0, z'))^* + O(|Z'|^{i+1}),
\]

which implies

\[
\left. \frac{\partial^m}{\partial z^\alpha} F^{(i)}(\cdot, x) \right|_x = \left( \frac{\partial^m}{\partial z^\alpha} F^{(i)}(0, z') \right)^* \quad \text{for } |\alpha| \leq i.
\]

In order to prove the Lemma it suffice to show that

\[
\left. \frac{\partial^m}{\partial z^\alpha} F^{(i)}(\cdot, z) \right|_x = 0 \quad \text{for } |\alpha| \leq i.
\]

We prove this by induction over \( |\alpha| \). For \( |\alpha| = 0 \), it is obvious that \( F^{(i)}(0, x) = 0 \), since \( F^{(i)}(0, x) \) is a homogeneous polynomial of degree \( i > 0 \). For the induction step, let \( j_X : X \to X \times X \) be the diagonal injection. By Lemma 4.4 and the definition (4.18) of \( F^{(i)}(x, y) \),

\[
\frac{\partial}{\partial z_j} F^{(i)}(x,y) = 0 \text{ near } j_X(X),
\]

where \( y = \exp \tilde{X}(Z') \). Assume now that \( \alpha \in \mathbb{N}^n \) and (4.45) holds for \( |\alpha| - 1 \). Consider \( j \) with \( \alpha_j > 0 \) and set

\[
\alpha' = (\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_n).
\]

Taking the derivative of (4.18) and using the induction hypothesis and (4.46), we have

\[
\left. \frac{\partial^m}{\partial z^\alpha} F^{(i)}(\cdot, x) \right|_x = \left. \frac{\partial}{\partial z_j} j_X^* \left( \frac{\partial^m}{\partial z^\alpha} F^{(i)} \right) \right|_x - \left. \frac{\partial^m}{\partial z^\alpha} \frac{\partial}{\partial z_j} F^{(i)}(\cdot, \cdot) \right|_{(0,0)} = 0.
\]

Thus, (4.40) is proved. The identity (4.16) follows too, since it is equivalent to (4.40). Furthermore, (4.41) follows from (4.16) and (4.17). This finishes the proof of Lemma 4.7. □

**Lemma 4.8.** We have \( F^{(i)}(z, z') = 0 \) for all \( i > 0 \) and \( z, z' \in \mathbb{C}^n \).
Proposition 4.9. We have
\begin{equation}
\frac{1}{\sqrt{p}} P_{\mathcal{H}_p}(\nabla_{X,T}^p) P_{\mathcal{H}_p} \text{ with } X \in \mathcal{C}_c^\infty(X,TX), X(x_0) = \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_{\xi_j}}.
\end{equation}

By Remark 3.4, the leading term of its asymptotic expansion as in (3.7) is
\begin{equation}
\left(\frac{\partial}{\partial z_j} F_{x_0}\right)(0, \zeta) = \left(\frac{\partial}{\partial \xi_j} F_{x_0}\right)(0, \zeta') = 0.
\end{equation}

By (4.16) and (4.41), \((\frac{\partial}{\partial z_j} F_{x_0})(z, \zeta)\) is an odd polynomial in \(z, \zeta\) whose constant term vanishes. We reiterate the argument from (4.21)-(4.44) by replacing the operator \(T_p\) with the operator \((4.49)\); we get for \(i > 0\),
\begin{equation}
\frac{\partial}{\partial z_j} F_{x_0}(0, \zeta) = 0.
\end{equation}

By (4.17) and (4.51),
\begin{equation}
\frac{\partial}{\partial z_j} F_{x_0}(z, 0) = 0.
\end{equation}

By continuing this process, we show that for all \(i > 0\), \(\alpha \in \mathbb{Z}^n\), \(z, z' \in \mathbb{C}^n\),
\begin{equation}
\frac{\partial^\alpha}{\partial z^\alpha} F_{x_0}(0, \zeta) = \frac{\partial^\alpha}{\partial \zeta^\alpha} F_{x_0}(z, 0) = 0.
\end{equation}

Thus, Lemma 4.8 is proved and (4.15) holds true. Lemma 4.8 finishes the proof of Proposition 4.3.

We come now to the proof of the first induction step leading to (4.4).

Proposition 4.9. We have
\begin{equation}
p^{-n}(T_p - T_{g_0,p})x_0(Z, Z') \cong O_m(p^{-1})
\end{equation}
in the sense of Notation A. Consequently,
\begin{equation}
T_p = P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p} + O(p^{-1}),
\end{equation}
i.e., relation (4.7) holds true in the sense of (1.14).

Proof. Let us compare the asymptotic expansion of \(T_p\) and \(T_{g_0,p} = P_{\mathcal{H}_p} g_0 P_{\mathcal{H}_p}\). Using the Notation A, the expansion (3.10) (for \(k_0 = 2\)) reads for \(\theta_m = 1/8(n + k + m + 2),\)
\begin{equation}
p^{-n}T_{g_0,p,x_0}(Z, Z') \cong (g_0(x_0)P_{x_0} + Q_{1,x_0}(g_0)P_{x_0}^{-1/2} + Q_{2,x_0}(g_0)P_{x_0}^{-1})(\sqrt{P}Z, \sqrt{P}Z') + O_m(p^{-1}),
\end{equation}

since \(Q_{0,x_0}(g_0) = g_0(x_0)\) \(\text{Id}_{P_{x_0}}\) by (3.13). The expansion (4.2) (also for \(k_0 = 2\)) takes the form for \(\theta_m\) in (4.2),
\begin{equation}
p^{-n}T_{p,x_0}(Z, Z') \cong (g_0(x_0)P_{x_0} + Q_{1,x_0}P_{x_0}^{-1/2} + Q_{2,x_0}P_{x_0}^{-1})(\sqrt{P}Z, \sqrt{P}Z') + O_m(p^{-1}),
\end{equation}

where we have used Proposition 4.3 and the definition (4.5) of \(g_0\). Thus subtracting (4.56) from (4.57) we obtain for some \(\theta_m \in (0, 1/2),\)
\begin{equation}
p^{-n}(T_p - T_{g_0,p})x_0(Z, Z') \cong ((Q_{2,x_0} - Q_{1,x_0}(g_0))P_{x_0})(\sqrt{P}Z, \sqrt{P}Z')p^{-1/2} + O_m(p^{-1}).
\end{equation}
Thus, it suffices to prove the following result.

**Lemma 4.10.**

\[ F_{1,x} := \mathcal{Q}_{1,x} - Q_{1,x}(g_0) \equiv 0. \]

**Proof.** We note first that \( F_{1,x} \) is an odd polynomial in \( z \) and \( z' \); we verify this statement as in Lemma 4.4. Thus the constant term of \( F_{1,x} \) vanishes. To show that the rest of the term vanish, we consider the decomposition yields the desired conclusion.

Due to Lemma 3.1 and Theorem 4.1 (ii), the first two conditions are easily verified. To prove the third, just subtract the asymptotics satisfies the hypotheses of Theorem 4.1. Due to Lemma 3.1 and Theorem 4.1 (ii), the proof of Theorem 4.1.

□

The proof of (4.60) is similar to that of (4.15). Namely, we define as in (4.18) the operator \( F_1(i) \), by replacing \( F_{x}(i)(0, z') \) by \( F_{1,x}(0, z') \), and we set (analogue to (4.21))

\[ \mathcal{F}_{p,1} = T_p - P_{\mathcal{H}_p}g_0P_{\mathcal{H}_p} - \sum_{i \in \text{deg} F_i} (F_1(i)P_{\mathcal{H}_p})p^{i-1/2}. \]

Due to (3.10) and (4.2), there exist polynomials \( \tilde{R}_{r,x_0} \in \mathbb{C}[Z, Z'] \) of the same parity as \( r \) such that the following expansion in the normal coordinates around \( x_0 \in X \) holds for any \( k_0 \geq 2 \):

\[ p^{-n}\mathcal{F}_{p,1,x_0}(0, Z') \equiv \sum_{r=2}^{k_0} (\tilde{R}_{r,x_0}P_{x_0})(0, \sqrt{p}Z')p^{-r/2} + O_m(p^{-k_0/2}), \]

in the sense of Condition A with \( \theta_{k_0,m} \) the minimum of \( \theta_{k_0,m} \) in (3.10) and \( \theta_{k_0,m} \) in (4.2). This is an analogue of (4.22). Now we can repeat with obvious modifications the proof of (4.15) and obtain the analogue of (4.15) with \( F_z \) replaced by \( F_{1,x} \). This completes the proof of Lemma 4.10.

□

Lemma 4.10 and the expansion (4.58) imply immediately Proposition 4.9.

**Proof of Theorem 4.1.** Proposition 4.9 shows that the asymptotic expansion (4.4) of \( T_p \) holds for \( q = 0 \). Moreover, if \( T_p \) is self-adjoint, then from (4.56), (4.57), \( g_0 \) is also self-adjoint. We show inductively that (4.4) holds for every \( q \in \mathbb{N} \). To prove (4.4) for \( q = 1 \) let us consider the operator \( p(T_p - P_{\mathcal{H}_p}g_0P_{\mathcal{H}_p}) \). We have to show now that \( p(T_p - P_{\mathcal{H}_p}g_0P_{\mathcal{H}_p}) \) satisfies the hypotheses of Theorem 4.1. Due to Lemma 3.1 and Theorem 4.1 (ii), the first two conditions are easily verified. To prove the third, just subtract the asymptotics of \( T_{p,x_0}(Z, Z') \) (given by (4.21)) and \( T_{g_0,p,x_0}(Z, Z') \) (given by (3.10)). Taking into account Proposition 4.3 and (4.59), the coefficients of \( p^0 \) and \( p^{-1/2} \) in the difference vanish, which yields the desired conclusion.

Propositions 4.3 and 4.9 applied to \( p(T_p - P_{\mathcal{H}_p}g_0P_{\mathcal{H}_p}) \) yield \( g_1 \in \mathcal{C}^\infty(X, \text{End}(E)) \) such that (4.4) holds true for \( q = 1 \).

We continue in this way the induction process to get (4.4) for any \( q \). This completes the proof of Theorem 4.1.

□

5. Algebra of Toeplitz Operators

The Poisson bracket \( \{ \cdot, \cdot \} \) on \( (X, 2\pi\omega) \) is defined as follows. For \( f, g \in \mathcal{C}^\infty(X) \), let \( \xi_f \) be the Hamiltonian vector field generated by \( f \), which is defined by \( 2\pi i\xi_f = df \). Then

\[ \{ f, g \} = \xi_f(df). \]
Proof of Theorem 1.2 First, it is obvious that \( P_{\mathcal{H}_p} T_{f,p} T_{g,p} P_{\mathcal{H}_p} = T_{f,p} T_{g,p} \). To prove (4.1), note that from Lemma 3.1 and (3.10), we know that for any \( k \in \mathbb{N} \) there exist \( C_k > 0 \) and \( M_k > 0 \) such that for all \( (x, x') \in X \times X \),

\[
|T_{f,p}(x, x')|_{\mathcal{B}(X \times X)} \leq C_k p^{M_k}.
\]

For any \( b > 0 \) and \( 0 < \theta < 1 \), if \( d(x, x') > bp^{-\theta/2} \), then

\[
T_{f,p} T_{g,p}(x, x') = \left( \int_{B^X(x, \frac{r}{2}p^{-\theta/2})} + \int_{X \setminus B^X(x, \frac{r}{2}p^{-\theta/2})} \right) T_{f,p}(x, x'') T_{g,p}(x'', x') dv_X(x'').
\]

Then (4.1) follows from (3.2), (3.5), (5.2) and (5.3). Like (3.18), for \( |Z|, |Z'| < \frac{b}{2}p^{-\theta/2} \), we have

\[
(T_{f,p} T_{g,p})_{\theta_0}(Z, Z') = \int |Z|^r p^{-n} (T_{f,p} T_{g,p})_{\theta_0}(Z, Z') dv_{\mathcal{T}X}(Z''),
\]

with

\[
Q_{r,x_0}(f, g) = \sum_{r_1 + r_2 = r} \mathcal{K}[Q_{r_1,x_0}(f), Q_{r_2,x_0}(g)].
\]

Thus, \( T_{f,p} T_{g,p} \) is a Toeplitz operator by Theorem 4.1. Moreover, it follows from Theorem 4.1 and Lemma 3.3 that \( g_t = C_t(f, g) \), where \( C_t \) are bidifferential operators.

The rest of the proof of Theorem 1.2 is exactly the same as that of [20, Theorem 1.1] and we omit it here. This finishes the proof of Theorem 1.2. \( \square \)

Proof of Theorem 1.4 Take a point \( x_0 \in X \) and \( u_0 \in E_{x_0} \) with \( |u_0|_{E_0} = 1 \) such that \( |f(x_0)(u_0)| = \|f\|_{\infty} \). Recall that we trivialized the bundles \( L, E \) in normal coordinates near \( x_0 \), and \( e_L \) is the unit frame of \( L \) which trivializes \( L \). Moreover, in this normal coordinates, \( u_0 \) is a trivial section of \( E \). Considering the sequence of sections \( S_{x_0} = p^{-n/2} P_{\mathcal{H}_p}(e_{L}^{\otimes p} \otimes u_0) \), we have by (3.3),

\[
\|T_{f,p} S_{x_0} - f(x_0) S_{x_0}\|_{L^2} \leq \frac{C}{\sqrt{p}} \|S_{x_0}\|_{L^2},
\]

which immediately implies (1.20). \( \square \)

6. PROOF OF THEOREM 1.5

In this section, we show how to adapt the results of [12] in order to prove Theorem 1.5 which provides formula (1.22) for the coefficient \( C_1(f, g) \) in the Toeplitz expansion (1.16) of the product \( T_{f,p} T_{g,p} \) of two Toeplitz operators (cf. Theorem 1.2), when \( g^{TX}(\cdot, \cdot) = \omega(\cdot, \cdot) \).

Let us introduce the main tools needed for this computation. It is shown in [19, Theorem 1.4] that the restriction on \( B^X(x, \epsilon) \) of the operator \( \Delta_{p,\Phi} \) defined in (1.4) is
equal, through the trivializations given in Section 2 and after a convenient rescaling in
\( \sqrt{t} := 1/t \), to an operator \( \mathcal{L}_t \) on \( B_{\rho_{x}}(0, \varepsilon) \) satisfying
\[
\mathcal{L}_t = \mathcal{L}_0 + \sum_{r=1}^{m} t^r \mathcal{O}_r + \mathcal{O}(t^{m+1}),
\]
for any \( m \in \mathbb{N} \), where \( \{\mathcal{O}_r\}_{r \in \mathbb{N}} \) is a family of differential operators of order equal or
less than 2, with coefficients explicitly computable in terms of local data, and where the differential operator \( \mathcal{O}(t^{m+1}) \) has its coefficients and their derivatives up to order \( k \)
dominated by \( C_k t^{m+1} \) for any \( k \in \mathbb{N} \).

Moreover, the differential operator \( \mathcal{L}_0 \) acts on the scalar part of smooth functions on
\( \mathbb{R}^{2n} \) with values in \( E_x \), has a positive discrete spectrum on \( L^2(\mathbb{R}^{2n}) \), and the kernel of the
orthogonal projection \( \mathcal{P} \) on \( \text{Ker}(\mathcal{L}_0) \) is given by (2.6) with \( a_j = 2\pi \) therein. We write
\( \mathcal{P}^\bot = \text{Id} - \mathcal{P} \), and then define an operator \( (\mathcal{L}_0)^{-1} \mathcal{P}^\bot \) by inverting the eigenvalues of
\( \mathcal{L}_0 \) on \( \text{Ker}(\mathcal{L}_0)^\bot \).

As shown in [19], there is a direct method to compute the family \( \{\mathcal{F}_{r,x}(Z, Z')\}_{r \in \mathbb{N}} \)
defined in Theorem 2.2 using (6.1). The following lemma, which has been established
in [19] Theorem 1.4, (1.111)], gives the first three elements of this family.

**Lemma 6.1.** For all \( r \in \mathbb{N} \), let \( \mathcal{F}_{r,x} \) be the operator associated with the kernel \( \mathcal{F}_{r,x}(Z, Z') \),
and let the differential operators \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) be as in (6.1). Then the following formulas hold:
\[
\mathcal{F}_{0,x} = \mathcal{P},
\]
\[
\mathcal{F}_{1,x} = -(\mathcal{L}_0)^{-1} \mathcal{P}^\bot \mathcal{O}_1 \mathcal{P} - \mathcal{P} \mathcal{O}_1 (\mathcal{L}_0)^{-1} \mathcal{P}^\bot,
\]
\[
\mathcal{F}_{2,x} = (\mathcal{L}_0)^{-1} \mathcal{P}^\bot \mathcal{O}_1 (\mathcal{L}_0)^{-1} \mathcal{P}^\bot \mathcal{O}_1 \mathcal{P} - (\mathcal{L}_0)^{-1} \mathcal{P}^\bot \mathcal{O}_2 \mathcal{P}
\]
\[
+ \mathcal{P} \mathcal{O}_1 (\mathcal{L}_0)^{-1} \mathcal{P}^\bot \mathcal{O}_1 (\mathcal{L}_0)^{-1} \mathcal{P}^\bot - \mathcal{P} \mathcal{O}_2 (\mathcal{L}_0)^{-1} \mathcal{P}^\bot
\]
\[
+ (\mathcal{L}_0)^{-1} \mathcal{P}^\bot \mathcal{O}_1 \mathcal{P} \mathcal{O}_1 (\mathcal{L}_0)^{-1} \mathcal{P}^\bot - \mathcal{P} \mathcal{O}_1 (\mathcal{L}_0)^{-2} \mathcal{P}^\bot \mathcal{O}_1 \mathcal{P}.
\]

Moreover, \( \mathcal{O}_1 \) commutes with any \( A \in \text{End}(E_x) \), and we have the formula
\[
\mathcal{P} \mathcal{O}_1 \mathcal{P} = 0.
\]
In particular, \( \mathcal{F}_{0,x} \) and \( \mathcal{F}_{1,x} \) commute with any \( A \in \text{End}(E_x) \).

**Lemma 6.2.** The following formulas hold:
\[
(\mathcal{P} \mathcal{O}_1 (\mathcal{L}_0)^{-1} \mathcal{P}^\bot)(0, Z') = 0,
\]
\[
(\mathcal{P} \mathcal{O}_1 (\mathcal{L}_0)^{-1} \mathcal{P}^\bot)(Z, 0) = 0,
\]
\[
((\mathcal{L}_0)^{-1} \mathcal{P}^\bot \mathcal{O}_1 \mathcal{P})(Z, 0) = 0.
\]

The result of Lemma 6.2 is a simple computation using [19] (1.98),(1.99),(2.25]). In fact, all the kernels associated to the situation in this paper are the scalar part of the
kernels of the corresponding situation in [12]. Lemma 6.2 is then an expression of the
fact that the corresponding formulas in [12] Lemma 3.3] have no scalar parts.
Now note that by the proof of Theorem 4.1 at the end of Section 5, we still have the following formula:

\[ C_1(f, g)(x) = Q_2(f, g)(0, 0) - Q_2(fg)(0, 0), \]

where the coefficients \( Q_2(fg) \) and \( Q_2(f, g) \) has been defined in Lemma 3.3 and (5.6) respectively. Note that the formula (6.7) is actually simpler than the one given in [20], as we only need to consider its scalar part.

Note that the reasoning of Section 5 to deduce (5.6) from the proof of Lemma 3.3 says that the composition of kernels is still computable in the same way than in [20, § 4.3], so that the kernel calculus as described in [12, § 2.3] still works in this context.

Using Lemma 6.1, Lemma 6.2 and the Sections 4 and 5, it is then straightforward to adapt the arguments of [12, § 2.2]. Thanks to Lemma 6.2, the proofs of [12 Proposition 3.2, Proposition 3.3] are even easier. We thus get [12, Theorem 1.1] in the context described in the introduction, which is exactly Theorem 1.5.

REFERENCES

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization*, Part I, Lett. Math. Phys. 1 (1977), 521–530; Part II and III, Ann. Phys. 111 (1978), 61–110, 111–151.

[2] F. A. Berezin, *Quantization*, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1116–1175.

[3] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004 corrected reprint of the 1992 original.

[4] M. Bordemann, E. Meinrenken and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and \( \text{gl}(n) \rightarrow \infty \) limits*, Comm. Math. Phys. vol. 165, (1994), 281–296.

[5] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Mathematics Studies, vol. 99, Princeton University Press, Princeton, NJ, 1981.

[6] M. Cahen, S. Gutt and J. Rawnsley, *Quantization of Kähler manifolds. I*, J. Geom. Phys. 7 1990, no. 1, 45–62.

[7] L. Charles, *Quantization of compact symplectic manifolds*, J. Geom. Anal. 26 (2016), no. 4, 2664–2710.

[8] X. Dai, K. Liu and X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. 72 (2006) 1-41.

[9] B. V. Fedosov, *Deformation quantization and index theory*, Mathematical Topics, 9, Akademie Verlag, Berlin, 1996.

[10] V. Guillemin and A. Uribe, *The Laplace operator on the \( n \)-th tensor power of a line bundle: eigenvalues which are bounded uniformly in \( n \), Asymptotic Anal. 1 (1988), 105–113.

[11] C.-Y. Hsiao and G. Marinescu, *Berezin-Toeplitz quantization for lower energy forms*, to appear in Comm. Partial Differential Equations, arXiv:1411.6654.

[12] L. Ioos, *On the composition of Berezin-Toeplitz operators on symplectic manifolds*, arXiv:1703.05688.

[13] Y. A. Kordyukov, *On asymptotic expansions of generalized Bergman kernels on symplectic manifolds*, arXiv:1703.04107.

[14] B. Kostant, *Quantization and unitary representations. I. Prequantization*, Lectures in modern analysis and applications, III, Springer, Berlin, 1970, pp. 87–208. Lecture Notes in Math., Vol. 170.

[15] W. Lu, X. Ma and G. Marinescu, *Donaldson’s Q-operators for symplectic manifolds*, to appear in Sci. China Math., 60 (2017), arXiv:1703.05276.

[16] X. Ma, *Geometric quantization on Kähler and symplectic manifolds*, in: Proceedings of the International Congress of Mathematicians, vol. II, pp. 785-810, Hindustan Book Agency, New Delhi, 2010.

[17] X. Ma and G. Marinescu, *The Spin^c Dirac operator on high tensor powers of a line bundle*, Math. Z. 240 (2002), no. 3, 651-664.

[18] X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, vol. 254, Birkhäuser Boston, Inc., Boston, MA, 2007.
[19] X. Ma and G. Marinescu, Generalized Bergman kernels on symplectic manifolds, Adv. Math. 217 (2008), no. 4, 1756–1815.
[20] X. Ma and G. Marinescu, Toeplitz operators on symplectic manifolds, J. Geom. Anal. 18 (2008), 565–611.
[21] X. Ma and G. Marinescu, Berezin-Toeplitz Quantization and its kernel expansion, Trav. Math. 19 (2011), 125–166.
[22] X. Ma and G. Marinescu, Berezin-Toeplitz quantization on Kähler manifolds, J. Reine Angew. Math. 662 (2002), 1–56.
[23] M. Schlichenmaier, Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization, Conference Moshé Flato 1999, Vol. II (Dijon), Math. Phys. Stud., vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 289–306.
[24] M. Schlichenmaier, Berezin-Toeplitz Quantization for Compact Kähler Manifolds. A Review of Results, Adv. Math. Phys. 2010, Art. ID 927280, 38 pp.
[25] J.-M. Souriau, Structure des systèmes dynamiques, Maîtrises de mathématiques, Dunod, Paris, 1970.
[26] H. Xu, An explicit formula for the Berezin star product, Lett. Math. Phys. 101 (2012), no. 3, 239–264.

Institut de Mathématiques de Jussieu–Paris Rive Gauche, UFR de Mathématiques, Université Paris Diderot - Paris 7, Case 7012, 75205 Paris Cedex 13, France
E-mail address: louis.ioos@imj-prg.fr

School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan, 430074, People’s Republic of China
E-mail address: wlu@hust.edu.cn

Institut de Mathématiques de Jussieu–Paris Rive Gauche, UFR de Mathématiques, Université Paris Diderot - Paris 7, Case 7012, 75205 Paris Cedex 13, France
E-mail address: xiaonan.ma@imj-prg.fr

Universität zu Köln, Mathematisches Institut, Weyertal 86-90, 50931 Köln, Germany
Institute of Mathematics 'Simion Stoilow', Romanian Academy, Bucharest, Romania
E-mail address: gmarines@math.uni-koeln.de