Gluing Manifolds in the Cahiers Topos

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Abstract

We show that there is a fully faithful embedding of the category of manifolds with corners into the Cahiers topos, one of the premier models for Synthetic Differential Geometry. This embedding is shown to have a number of nice properties, such as preservation of open covers and transverse fibre products.

We develop a theory for gluing manifolds with corners in the Cahiers topos. In this setting, the result of gluing together manifolds with corners along a common face is shown to coincide with a pushout along an infinitesimally thickened face. Our theory is designed with a view toward future applications in Field Theory within the context of Synthetic Differential Geometry.

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Introduction

There has been a great deal of activity in the intersection of quantum field theory and category theory in recent years. A driving impetus has been the quest to understand and elucidate the Topological Quantum Field Theories (TQFTs) of Atiyah–Lurie–Segal. The Atiyah–Lurie–Segal TQFTs describe the quantum behaviour of topological field theories by encoding everything in the rich language of higher category theory. A central rôle is played by the higher bordism categories, which encapsulate the basic structure of TQFTs via the gluing together of manifolds with corners along boundary components.

For all the success of TQFTs there are still some fundamental unanswered questions. Foremost of these is the precise manner in which a classical (topological) field theory gives rise to a quantum one. An important advance in this direction was made recently by Cattaneo, Mnëv and Reshetikhin [2] who developed a framework for classical field theories on manifolds with corners. The Cattaneo–Mnëv–Reshetikhin (CMR) formalism incorporates a mechanism for gluing together field theories defined on manifolds with corners in a manner related to that of Atiyah–Lurie–Segal TQFTs. However, there is one key difference between the two situations. Whereas gluing in TQFTs is modelled via compositions of higher morphisms in higher bordism categories, spaces of fields in the CMR formalism are glued together via fibre products. The upshot is that gluing in the classical theory has a completely 1-categorical description, at odds with the higher framework required for TQFTs.

This paper represents a first step in the efforts of the author to understand the structure underlying the CMR formalism. Viewing spaces of fields as sheaves or stacks depending contravariantly on the manifold under consideration, the fact that spaces of fields glue via fibre products leads us to try to understand gluing of manifolds via pushouts. This is then the main aim of the paper: to develop a setting in which manifolds with corners can be glued together via pushout diagrams. Independently of the field-theoretic motivation this is a natural question to ask since the usual interpretation of pushouts in category theory is indeed as a mechanism for gluing.

In order to fulfill our aim, we are quickly led outside the category of manifolds with corners to larger, better-behaved categories. Indeed, the reader can be convinced without much difficulty that the naïve attempt to glue manifolds by pushouts is doomed to failure (or see [10, §3.1]). What is missing in the naïve approach is the data of collar neighbourhoods, which are required to produce a smooth structure on the glued space. It is well-known that once collars are provided, we can glue with abandon and, moreover, that the isomorphism class of the resulting glued manifold is independent of the choice of collar.

The underlying idea of the paper is to use pushouts to glue along infinitesimally thin collars. In order to incorporate the correct notion of infinitesimal we pass to the realm of Synthetic Differential Geometry, specifically the Cahiers topos of Dubuc [3]. This allows us to bring the full power of synthetic reasoning to bear. However in order to work in the synthetic framework, we first need to show that manifolds with corners embed into the Cahiers topos. Once this is done, we establish our main result: the embedding of manifolds with corners into the Cahiers topos sends collar gluings to pushouts. We interpret this as saying that the embedding preserves manifold gluings, where gluing is interpreted in the natural sense: by collars on the one hand, and categorically via pushouts on the other.

We begin in §1 by recalling the requisite material from the differential geometry of manifolds with corners and from Synthetic Differential Geometry, in particular the definition of the Cahiers topos. In §2 we extend the work of Kock [6] and Reyes [9] by proving that manifolds with corners embed into the Cahiers topos. We prove various nice properties of this embedding. In §3 we develop specific notions of infinitesimal thickenings that are necessary to understand gluing in the Cahiers topos. The proof of our main results on gluing, Theorems 4.9 and 4.10 are contained in §4.

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1 Recollections

We begin by recalling some of the underlying material that we shall need in the rest of the paper. This section can safely be skimmed on a first reading and referred back to for details as necessary.

1.1 Manifolds with Corners

We now recollect some of the key notions from the differential topology of manifolds with corners. In what follows, we denote by $I := [0, \infty) \subset \mathbb{R}$ the non-negative half line.

**Definition 1.1.** For $M$ a paracompact Hausdorff topological space, an $n$-dimensional chart on $M$ with corners is a pair $(U, \phi)$ where $U$ is an open subset of $\mathbb{R}^n_m := \mathbb{R}^{n-m} \times I^m$ for some $0 \leq m \leq n$ and $\phi : U \to M$ is a homeomorphism with a non-empty open set $\phi(U) \subset M$.

For $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$ a continuous map $f : A \to B$ is smooth if it extends to a smooth map between open neighbourhoods of $A$ and $B$. In the case $m = n$, $f$ is a diffeomorphism if it is a smooth homeomorphism with smooth inverse.

Let $(U, \phi), (V, \varphi)$ be $n$-dimensional charts on $M$. We say that these charts are compatible if

$$\varphi^{-1} \circ \phi : \phi^{-1} (\phi(U) \cap \varphi(V)) \longrightarrow \varphi^{-1} (\phi(U) \cap \varphi(V))$$

is a diffeomorphism. An $n$-dimensional atlas for $M$ is a collection of $n$-dimensional charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ that are pairwise compatible and cover $M$: that is $M = \cup_\alpha \phi_\alpha(U_\alpha)$. A maximal atlas is an atlas that is not a proper subset of any other. We finally define an $n$-dimensional manifold with corners $M$ as a paracompact Hausdorff space equipped with a maximal atlas.

A (weakly) smooth map $f : M \to N$ between manifolds with corners is a continuous map $f$ such that for every choice of charts $(U, \phi), (V, \varphi)$ on $M, N$ respectively we have that

$$\varphi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1} (\varphi(V)) \longrightarrow V$$

is a smooth map between subsets of euclidean space. Manifolds with corners and (weakly) smooth maps between them assemble into a category $\text{Man}_{\text{crn}}$. There is an obvious fully faithful inclusion

$$\text{Man} \hookrightarrow \text{Man}_{\text{crn}}$$

of the category of manifolds without boundary.

In order to streamline terminology in what follows, we pose the

**Definition 1.2.** A basic manifold with corners $X$ is one that is diffeomorphic to a manifold with corners of the form $\mathbb{H}_n^m = \mathbb{R}^{n-m} \times I^m$ for some $n \geq 0$ and $0 \leq m \leq n$.

We note that every manifold with corners has an open covering by basics. Moreover, by a straightforward adaptation of the usual result on existence of good covers (c.f. [1, Theorem 5.1], for example) we have

**Lemma 1.3.** Let $M$ be any manifold with corners. Then there is an open cover

$$M = \bigcup_{\alpha \in A} U_\alpha$$

by basics such that each pairwise intersection $U_\alpha \cap U_\beta = U_\alpha \times_M U_\beta$ is either empty or basic.

Manifolds with corners are naturally stratified spaces, with the stratification arising from the corner structure:

\[\text{Here we are following Joyce [5], who distinguishes between weakly smooth maps as defined here and smooth maps that satisfy an additional condition over the boundary.}\]
Definition 1.4. For $x \in U$, with $U \subset \mathbb{H}^n_m$ open, we define the depth of $x = (x^1, \ldots, x^n) \in \mathbb{H}^n_m$ to be $\text{depth}_U(x) = k$; the number of coordinates $x^i$ that are zero. For $x \in M$, with $M$ a manifold with corners, we define the depth of $x$ to be $\text{depth}_M(x) := \text{depth}_U(x)$ for any choice of chart $U$ of $M$ at $x$. This is clearly independent of the choice of chart $U$.

For each $0 \leq k \leq n$ we define the depth $k$ stratum of $M$ as

$$\mathbb{S}^k(M) := \{ x \in M \mid \text{depth}_M(x) = k \}.$$ 

Each $\mathbb{S}^k(M)$ is naturally an $(n - k)$-manifold without boundary.

For a manifold with boundary (or more generally a manifold with corners) $M$ there are a few related but distinct definitions of the boundary $\partial M$ of $M$. The author’s personal preference is Joyce’s notion \cite{Joyce}, though we shall circumvent the issue of the definition of $\partial M$ entirely by instead working with the set

$$\overline{\mathbb{S}^0(M)} = \bigcup_{i=1}^{n} \mathbb{S}^i(M)$$

in lieu of the boundary.

Finally, we recall a key notion that allows us to facilitate gluing of manifolds with corners along boundary components.

Definition 1.5. Let $\Sigma$ be a manifold (possibly with corners) with a submanifold inclusion $\Sigma \hookrightarrow M$ such that $\Sigma \subset \overline{\mathbb{S}^0(M)}$. Then a collar of $\Sigma$ (in $M$) is a diffeomorphism $f: U \to \Sigma \times I$ of manifolds with corners for some open neighbourhood $U$ of $\Sigma$ in $M$ such that $f(x) = (x, 0)$ for all $x \in \Sigma$.

Finally, we remark that not every $\Sigma$ has a collar and the existence of a collar implies, by the Tubular Neighbourhood Theorem, that the normal bundle $N_{\Sigma \hookrightarrow M}$ of the inclusion $\Sigma \hookrightarrow M$ is necessarily trivial.

1.2 SDG, Well-adapted Models and the Cahiers Topos

Synthetic Differential Geometry (SDG) is an axiomatic formulation of differential geometry in topos theory. The topoi modelling SDG—the smooth topos—are categories whose objects are viewed as generalised smooth spaces, for which a notion of infinitesimal object exists that allows the internal formulation of the usual objects of differential geometry (i.e. vector fields, forms, jets, etc.).

Every smooth topos $S$ is required by definition to have a smooth line object—an internal unital algebra object $\mathbb{R}$, playing the rôle of the real line $\mathbb{R}$. Moreover if $W$ is a Weil algebra over $\mathbb{R}$, that is $W$ is an $\mathbb{R}$-algebra with a split augmentation map $\pi: W \to \mathbb{R}$ with nilpotent kernel, then denoting $\text{Spec}_\mathbb{R}W := \mathbb{R}-\text{alg}(W, \mathbb{R}) \in S$ we also require that

- the endofunctor $(-)^{\text{Spec}_\mathbb{R}W}: S \to S$ has a right adjoint $\mathbb{L}$ and
- the canonical morphism $W \to \mathbb{R}^{\text{Spec}_\mathbb{R}W}$ (induced by evaluation) is an isomorphism in $S$.

Of particular interest are those models $S$ for SDG that are well-adapted, meaning that the usual theory of differentiable manifolds is contained in $S$. This is formalised by requiring a full and faithful functor

$$\iota: \text{Man} \rightarrow S$$

such that $\mathbb{R} = \iota(\mathbb{R})$ and moreover such that

- $\iota$ preserves transverse fibre products in $\text{Man}$ (i.e. sends transverse fibre products to pullbacks); and
- $S$ is a category of sheaves such that $\iota$ sends open covers of manifolds to covering families.

\footnote{The right adjoints postulated by this axiom are often referred to as amazing right adjoints, and objects $A$ for which $(-)^{A}$ admits a right adjoint are called atomic. Indeed, the existence of such amazing right adjoints is remarkable—in the topos $\text{Set}$ only the terminal object $\ast$ has this property.}
Some good references for SDG are [6, 7, 8].

In this paper we shall work primarily with one particular well-adapted model: the Cahiers topos \( \mathcal{C} \) introduced by Dubuc in [3]. The Cahiers topos \( \mathcal{C} \) is built out of a subcategory of \( C^\infty \)-algebras. We recall that a \( C^\infty \)-algebra is a ring \( A \) over \( \mathbb{R} \) with the additional structure of an \( n \)-ary operation

\[
\Phi_f : A \times \cdots \times A \to A
\]

n times

corresponding to each smooth function \( f : \mathbb{R}^n \to \mathbb{R} \), satisfying a natural compatibility relation. Alternatively, \( C^\infty \)-algebras can be defined as algebras for the Lawvere theory \( C^\infty \) with

\[
C^\infty(n, m) := \text{Man}(\mathbb{R}^n, \mathbb{R}^m)
\]

(cf. [4, 5]). The primary examples of \( C^\infty \)-algebras are those of the form \( C^\infty(M) \), for \( M \) a manifold (with or without corners); the free \( C^\infty \)-algebra on \( n \) generators is \( C^\infty(\mathbb{R}^n) \). An important fact is that any Weil algebra over \( \mathbb{R} \) is canonically a \( C^\infty \)-algebra and, denoting by \( \otimes \) the coproduct of \( C^\infty \)-algebras we have that \( A \otimes W = A \otimes_{\mathbb{R}} W \) (the usual tensor product of \( \mathbb{R} \)-algebras) for any \( C^\infty \)-algebra \( A \). Due to the Hadamard lemma, for any \( C^\infty \)-algebra \( A \) and ring-theoretic ideal \( I \subset A \), the quotient ring \( A/I \) inherits a canonical \( C^\infty \)-algebra structure making the quotient map a morphism of \( C^\infty \)-algebras. \( C^\infty \)-algebras together with the obvious notion of morphism between them assemble into a category \( C^\infty \)-alg. Our main references for the theory of \( C^\infty \)-algebras are [4, 6, 8]; in particular the reader is referred to [3, p. 44] for the specialised notions of near-point determined, closed and germ-determined \( C^\infty \)-algebras.

We are now able to recall the definition of the Cahiers topos. We denote by \( \text{FCartSp} \) the category of formal Cartesian spaces, defined as the opposite category to the full subcategory of \( C^\infty \)-alg on objects of the form \( C^\infty(\mathbb{R}^n) \otimes W \), with \( W \) a Weil algebra over \( \mathbb{R} \). The object of \( \text{FCartSp} \) corresponding to the algebra \( A = C^\infty(\mathbb{R}^n) \otimes W \) is denoted \( \ell A = \mathbb{R}^n \times \ell W \). Following Dubuc [3], we equip \( \text{FCartSp} \) with the Grothendieck topology in which a covering is a collection of morphisms in \( \text{FCartSp} \) of the form

\[
\{ \rho_\alpha \times \text{id} : U_\alpha \times \ell W \to U \times \ell W \}
\]

where \( \{ \rho_\alpha : U_\alpha \to U \} \) is a smooth open cover of \( U \) in the usual sense. We finally define the Cahiers topos \( \mathcal{C} \) as the category of sheaves on \( \text{FCartSp} \) for this Grothendieck topology. As mentioned above, the Cahiers topos is a well-adapted model for SDG, with the embedding \( \text{Man} \hookrightarrow \mathcal{C} \) given by sending the manifold \( M \) to the sheaf

\[
\ell A \mapsto \text{C^\infty-alg}(C^\infty(M), A)
\]

that is “represented from outside” by \( M \).

We conclude this section by collecting some useful facts about \( C^\infty \)-algebras that are needed in the sequel.

Proposition 1.6 (Milnor’s exercise). Let \( M \) be a manifold with corners. Any map of \( C^\infty \)-algebras \( C^\infty(M) \to \mathbb{R} \) is of the form \( ev_x : f \mapsto f(x) \) for a unique \( x \in M \).

Proof. This is proved in [3] for manifolds without boundary. The case for manifolds with corners is a straightforward extension of this result. \( \square \)

Proposition 1.7. Let \( \pi : A \to \mathbb{R} \) be any finitely generated pointed local \( C^\infty \)-algebra and \( M \) a manifold with corners. Then any morphism \( \phi : C^\infty(M) \to A \) of \( C^\infty \)-algebras factors through the ring of germs \( C^\infty_\pi(M) \) for a uniquely-determined \( x \in M \).

Proof. By Milnor’s exercise \( \pi \circ \phi = ev_x \) for a uniquely determined \( x \in M \). Now we take \( f \in C^\infty(M) \) such that \( f \equiv 0 \) on an open neighbourhood \( U \) of \( x \) in \( M \). Choose \( g \in C^\infty(M) \) such that \( g(x) = 1 \) and \( \left| g \right|_{M \setminus U} \equiv 0 \), so that \( f \cdot g = 0 \). Applying \( \phi \) we have \( \phi(f) \cdot \phi(g) = 0 \) in \( A \), however since \( A \) is local \( \phi(g) \) is invertible whence \( \phi(f) = 0 \). \( \square \)

Corollary 1.8. Let \( \pi : W \to \mathbb{R} \) be a Weil algebra and \( M \in \text{Man}_{\text{cr\-}} \). Then any morphism \( \phi : C^\infty(M) \to W \) is of the form \( \phi(f) = f(x) + \Phi(f) \) for a unique \( x \in M \) and \( \Phi \) depending only on the germ at \( x \in M \).
2 Embedding Manifolds with Corners in the Cahiers Topos

In order to address the question of gluing manifolds with corners in the Cahiers topos, we must first check that manifolds with corners can be made sense of in this context. Our focus in the present section is the elucidation and proof of

**Theorem 2.1.** There is an embedding of manifolds with corners into the Cahiers topos

\[ \mathit{ι}: \text{Man}_{\text{crn}} \rightarrow \mathcal{C} \]

that is full and faithful, preserves transverse fibre products, sends open covers to effective epimorphisms and sends Weil prolongations to exponentials.

Note that the last point regarding Weil prolongations (see §2.2) is particularly important since exponential objects play a key rôlë in SDG, for example in the constructions of the tangent, cotangent and jet bundles.

The results of this section are already been shown in some special cases. Indeed, the case of manifolds without boundary is the heart of the theory of well-adapted models for SDG. Embedding manifolds with boundary into the Cahiers topos has already been considered by Kock [6, §III.9] and Reyes [9]. As a natural extension of these results, our proof of Theorem 2.1 follows a similar line of reasoning.

2.1 The Embedding

The embedding of manifolds with corners into the Cahiers topos is defined by a straightforward extension of the embedding \( \text{Man} \hookrightarrow \mathcal{C} \) described in §1.2. Firstly, we note that a manifold with corners \( M \) determines a presheaf on \( \text{FCartSp}^{\text{op}} \) via

\[ \mathcal{M}: \ell A \mapsto C^\infty_{\text{alg}}(C^\infty(M), A). \]

In fact, \( \mathcal{M} \) is already a sheaf: to see this we choose a manifold without corners \( X \) admitting a submanifold inclusion \( M \hookrightarrow X \) locally modelled on the inclusions \( \mathbb{B}^n_m \hookrightarrow \mathbb{R}^n \). By the embedding \( \text{Man} \hookrightarrow \mathcal{C} \), the manifold \( X \) determines a sheaf \( \underline{X}: \ell A \mapsto C^\infty_{\text{alg}}(C^\infty(X), A) \) in \( \mathcal{C} \).

In this setting, we can write \( C^\infty(M) = C^\infty(X)/m_M^0 \), with \( m_M^0 \) the ideal of functions vanishing on \( M \). Now if \( \{\rho_\alpha: A \to A_\alpha\}_{\alpha \in A} \) is a co-covering family in \( \text{FCartSp}^{\text{op}} \), we take any compatible family of maps of \( C^\infty \)-algebras \( \{\phi_\alpha: C^\infty(M) \to A_\alpha\}_{\alpha \in A} \). To show that \( \underline{M} \) is a sheaf, we must demonstrate that there is a unique \( \phi: C^\infty(M) \to A \) such that \( \rho_\alpha \circ \phi = \phi_\alpha \) for all \( \alpha \in A \).

We remark that the pullback to the inclusion \( M \hookrightarrow X \) may be presented as the quotient map

\[ \pi: C^\infty(X) \twoheadrightarrow C^\infty(X)/m_M^0 = C^\infty(M) \]

so that we have a compatible family \( \{\phi_\alpha \circ \pi: C^\infty(X) \to A_\alpha\} \). As \( \underline{X} \) is a sheaf, we have a uniquely determined \( \psi: C^\infty(X) \to A \) such that \( \rho_\alpha \circ \psi = \phi_\alpha \circ \pi \). It now suffices to show that \( \psi \) factors through \( \pi \). To see this, let \( f \in m_M^0 \) so that \( (\rho_\alpha \circ \psi)(f) = (\phi_\alpha \circ \pi)(f) = 0 \) and we define \( \zeta: C^\infty(\mathbb{R}) \to A \) by \( \zeta(x \mapsto x) = \psi(f) \) so that \( \rho_\alpha \circ \zeta = 0 \) for all \( \alpha \in A \). As \( \mathbb{R} = \underline{\mathbb{R}} \) is a sheaf, we must have \( \zeta = 0 \). In summary, we have the

**Lemma 2.2.** There is a functor \( \mathit{ι}: \text{Man}_{\text{crn}} \rightarrow \mathcal{C} \) given by

\[ M \mapsto \mathcal{M} := (\ell A \mapsto C^\infty_{\text{alg}}(C^\infty(M), A)). \]

In the rest of this section, we prove various properties of the functor \( \mathit{ι} \). The first such property is related to transverse fibre products of manifolds with corners. While there seems to be no agreement in the literature as to even the definition of smooth maps between manifolds with corners, Joyce has developed a theory singularly suited to categorical considerations. In [5], Joyce defines a transversality theory for manifolds with corners extending the usual theory. Moreover he proves that all transverse fibre products exist in \( \text{Man}_{\text{crn}} \) [5, Theorem 6.4]. Motivated by \( C^\infty \)-algebraic considerations in [4], Joyce gives the following
Theorem 2.3. The functor \( C^\infty: \text{Man}_{\text{crn}} \to C^\infty\text{-alg} \) sends transverse fibre products in \( \text{Man}_{\text{crn}} \) to pushouts. That is, if

\[
\begin{array}{ccc}
W & \to & Y \\
\downarrow & & \downarrow \\
X & \to & Z
\end{array}
\]

is a transverse pullback in \( \text{Man}_{\text{crn}} \) then \( C^\infty(W) = C^\infty(X) \otimes_{C^\infty(Z)} C^\infty(Y) \) in \( C^\infty\text{-alg} \).

From this we immediately obtain

Corollary 2.4. The functor \( i: \text{Man}_{\text{crn}} \to \mathcal{C} \) sends transverse fibre products to pullbacks, in particular it preserves products.

Our next result clarifies how open coverings behave under the functor \( i \).

Proposition 2.5. The functor \( i: \text{Man}_{\text{crn}} \to \mathcal{C} \) sends basic open covers to effective epimorphisms.

Proof. Take \( M \) a manifold with corners and choose a manifold \( X \subset M \) without corners such that the inclusion \( M \hookrightarrow X \) is locally modelled on the inclusions \( \mathbb{H}^n_m \hookrightarrow \mathbb{R}^n \). We may suppose without loss of generality that there is a cover \( U := \{ U_\alpha \}_{\alpha \in A} \) of \( M \) with each \( U_\alpha \) basic that extends to a cover \( U' := \{ U'_\alpha \}_{\alpha \in A} \) of \( X \) by basics.

The embedding \( \text{Man} \to \mathcal{C} \) preserves open covers (cf. [3] Théorème 4.10], whence we have the following isomorphism in \( \mathcal{C} \)

\[
X = \text{colim} \left( \prod_{\alpha, \beta \in A} U'_\alpha \times_{X} U'_\beta \xrightarrow{\text{transverse}} \prod_{\alpha \in A} U'_\alpha \right).
\]

But now \( i \) preserves transverse fibre products so that \( M = M \times_X X \) and since all colimits in \( \mathcal{C} \) are stable under base change (as \( \mathcal{C} \) is a topos)

\[
M = M \times_X \left( \text{colim} \left( \prod_{\alpha, \beta \in A} U'_\alpha \times_{X} U'_\beta \xrightarrow{\text{transverse}} \prod_{\alpha \in A} U'_\alpha \right) \right)
\]

where we have used \( M \times_X (U'_\alpha \times_X U'_\beta) = (M \times_X U'_\alpha) \times_X (M \times_X U'_\beta) \). Using once again that \( i \) preserves transverse fibre products, we have that \( M \times_X U'_\alpha = U'_\alpha \). The result now follows from the observation that \( U_\alpha \times_X U_\beta = U_\alpha \cap U_\beta = U_\alpha \times_M U_\beta \).

Since any manifold with corners admits an open covering by basics, we immediately obtain

Corollary 2.6. The functor \( i: \text{Man}_{\text{crn}} \to \mathcal{C} \) sends open covers to effective epimorphisms.

Finally, we show that \( i \) is fully faithful, following Kock’s argument for in the case of manifolds with boundary [5] §III.9.

Theorem 2.7. The functor \( i: \text{Man}_{\text{crn}} \to \mathcal{C} \) is fully faithful.

Proof. To see that \( i \) is faithful, we consider the global sections functor \( \Gamma: \mathcal{C} \to \text{Set} \) sending \( X \mapsto X(*) \). We recall that the terminal object in \( \mathcal{C} \) is \( *=\ell R \). If \( M \in \text{Man}_{\text{crn}} \) by Milnor’s exercise

\[
M(*) = M(\ell R) = C^\infty\text{-alg}(C^\infty(M), R) = |M|,
\]

the underlying set of the manifold \( M \), from which faithfulness follows.

To show fullness, let \( M, N \in \text{Man}_{\text{crn}} \) and let \( F: M \to N \) be any map in \( \mathcal{C} \). Applying \( \Gamma \) yields a map of point-sets \( \Gamma(F): |M| \to |N| \), which we firstly wish to show is smooth. It suffices to show the composite of \( \Gamma(F) \) with any
smooth $h: N \to \mathbb{R}$ is smooth. Smoothness at interior points of $M$ is clear (by considering open submanifolds of $M$ without boundary) so we consider the case $x \in \mathbb{S}^1(M)$.

Choose a manifold $X \supset M$ without boundary, so that smoothness of $\Gamma(f)$ at $x$ means that it can be extended to a smooth function on some open $U \subset X$ containing $x$. Without loss of generality we may now take $M = \mathbb{H}_m^n$ and $x = (p, 0)$, with $p \in \mathbb{R}^{n-m}$. Consider the map

$$s: \mathbb{R}^n \longrightarrow \mathbb{H}_m^n,$$

$$\quad (p, q_1, \ldots, q_m) \longmapsto (p, q_1^2, \ldots, q_m^2),$$

then $h \circ \Gamma(F) \circ s$ is smooth as it comes from a map $\mathbb{R}^n = \mathbb{R}^n \to \mathbb{R} = \mathbb{R}$ in $\mathcal{C}$. By a result of Schwarz [11] there is a smooth function $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$f(p, q_1^2, \ldots, q_m^2) = (h \circ \Gamma(F) \circ s)(p, q_1, \ldots, q_m) = (h \circ \Gamma(F) \circ s)(p, q_1^2, \ldots, q_m^2).$$

Since every $(z_1, \ldots, z_m) \in \mathbb{P}^m$ is of the form $(q_1^2, \ldots, q_m^2)$, we have that $h \circ \Gamma(F)$ and $f$ agree on $\mathbb{R}^n \times \mathbb{P}^k$ so that $h \circ \Gamma(F)$ is indeed smooth at $x$.

We now have that the map $\Gamma(F): M \to N$ is smooth, giving

$$F' := \Gamma(F): M \longrightarrow N$$

in $\mathcal{C}$. It remains to show that $F' = F$, noting that clearly $\Gamma(F') = \Gamma(F)$. For this, we embed $N$ inside a boundaryless manifold $X$ and then embed $X$ inside some $\mathbb{R}^N$, invoking Whitney’s embedding theorem. These embeddings are equivalently described by surjective maps of $C^\infty$–algebras

$$C^\infty(\mathbb{R}^N) \longrightarrow C^\infty(X) \longrightarrow C^\infty(N)$$

so are sent to monic maps by $\iota$. We can therefore reduce to the case $\mathcal{N} = \mathbb{R}^N$ and, since $\mathbb{R}^N = \mathbb{R}^N$, ultimately to the case $\mathcal{N} = \mathbb{R}$. Faithfulness now follows from the Lemma below by taking $g = F - F'$.

Lemma 2.8 (cf. [6][9]). Let $M$ be a manifold with corners and suppose $g: M \to \mathbb{R}$ is such that $\Gamma(g): M \to \mathbb{R}$ is the zero map. Then $g$ is the zero map.

Proof. It is enough to check that for every $\ell A \in \text{FCartSp}$ the composite $\ell A \to M \to \mathbb{R}$ is zero. The composite is induced by a $C^\infty$–algebra map $\phi: C^\infty(\mathbb{R}) \to A$, so since $C^\infty(\mathbb{R})$ is free on one generator we must check that $\phi(x \mapsto x) = 0$. Note that $A$ is necessarily near-point determined, so embeds into a product of Weil algebras [8] §4. It therefore suffices to check that for any Weil algebra $\pi: W \to \mathbb{R}$ the composite

$$
\begin{array}{ccc}
C^\infty(\mathbb{R}) & \xrightarrow{\phi} & A \\
\downarrow & & \downarrow \\
C^\infty(M) & \longrightarrow & W
\end{array}
$$

sends $\text{id}_\mathbb{R}$ to zero. By Corollary [13] the map $\varphi: C^\infty(M) \to W$ of the above diagram factors through the ring of germs $C^\infty_x(\mathbb{R})$ for some uniquely determined $x \in M$. We can now replace $M$ with $\mathbb{H}_m^n$ and $x$ with $(p, 0)$ for some $p \in \mathbb{R}^{n-m}$ without loss of generality.

Noting that any function $f \in m_0^0 \subset C^\infty(\mathbb{R}^n)$ is necessarily flat at $(p, 0)$, for any $k = 1, 2, \ldots$ and choice of $1 \leq i \leq m$ we can write

$$f(p, q_1, \ldots, q_m) = q_i^{k+1} \cdot g(p, q_1, \ldots, q_m)$$

using Hadamard’s lemma. We therefore have $m_0^0 \subset I := \sum_{i=1}^m (q_i^{k+1})$, giving rise to a surjection

$$C^\infty(\mathbb{R}^m) = C^\infty(\mathbb{R}^m)/m_0^0 \longrightarrow C^\infty(\mathbb{R}^n)/I = C^\infty(\mathbb{R}^{n-m}) \otimes \mathbb{R}[Y]/(Y^{k+1}) \quad m \text{ times}$$

on $\mathbb{R}$.

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for any \( k \leq 1 \). Due to the finite nilpotence degree of \( W \), the map \( \varphi : C^\infty(\mathbb{H}_n^m) \to W \) factors through one of the above surjections for some \( k \). Phrasing this all in \( \mathcal{E} \) now gives

\[
\varphi : \ell W \to R^{n-m} \times D_k^m \to R^{n-m} \times H^m,
\]

where \( D_k = \text{Spec}_R R[Y]/(Y^{k+1}) = \ell (R[Y]/(Y^{k+1})) \) and \( H = \mathbb{R} = \mathbb{R}_1^1 \).

To complete the proof, we need only check that \( g \) vanishes on \( R^{n-m} \times D_k^m \). Using the squaring map \( \sq \) gives

\[
R^n \to R^{n-m} \times H^m \to R
\]

which corresponds to some smooth map \( G : \mathbb{R}^n \to \mathbb{R} \). Applying \( \Gamma \), it follows that \( G = 0 \). It follows that the composition

\[
R^{n-m} \times D_k^m \overset{id \times sq}{\longrightarrow} R^{n-m} \times D_k^m \overset{g}{\longrightarrow} R
\]

is also zero, with \( sq : D_{2k+1} \to D_k \) the squaring map induced by \( R[Y]/(Y^{k+1}) \to R[Z]/(Z^{2k+2}) \), \( Y \to Z^2 \). Taking exponential adjoints gives

\[
R^{n-m} \longrightarrow R^{D_k^m} \to R^{D_{2k+1}}
\]

in which the composition is also zero. The last map, using the axioms of SDG, is equivalent to the \( k \)-fold coproduct of the map of Weil algebras \( R[Y]/(Y^{k+1}) \to R[Z]/(Z^{2k+2}) \), which is clearly monic. It follows that \( R^{n-m} \to R^{D_k^m} \) is zero, so taking exponential adjoints finally gives that \( g : R^{n-m} \times D_k^m \to R \) is also zero. \( \square \)

2.2 Weil Prolongations

Weil prolongations play a central rôle in \( C^\infty \)-algebraic models of SDG, such as the Cahiers topos. The idea is that differential-geometric constructions are realised in SDG via exponential objects: given a Weil algebra \( W \) over the smooth line \( \mathbb{R} \) (in some fixed model \( \mathcal{S} \)) together with a microlinear space \( M \in \mathcal{S} \), the \( M^W \) models a space of jets of \( M \). In the case that \( \mathcal{S} \) is a well-adapted model and \( M = \mathcal{M} \) is in the image of the embedding \( \text{Man} \hookrightarrow \mathcal{S} \), it is natural to require that the notions of jets in \( \text{Man} \) and jets internal to \( \mathcal{S} \) coincide. This is the subject of the present section; we shall show that the embedding of manifolds with corners into the Cahiers topos behaves well with respect to jets.

There is a jet space associated to any manifold with corners \( M \) and Weil algebra \( \pi : W \to \mathbb{R} \). The construction of this space—the \( W \)-prolongation of \( M \)—generalises the algebraic principle that underlies Milnor’s exercise. The \( W \)-prolongation of \( M \) is defined as

\[
W^M := C^\infty\text{-alg}(C^\infty(M), W).
\]

We observe that \( \pi \) induces a map of sets \( W^M \to M \). Following Dubuc and Reyes [3, 9] we shall show that \( W^M \) has the structure of a manifold with corners fibred over \( M \) and that \( W^M = M^W \) in the Cahiers topos.

Our first result in this direction studies the local behaviour of Weil prolongations.

**Proposition 2.9.** For any Weil algebra \( \pi : W \to \mathbb{R} \), a morphism of \( C^\infty \)-algebras \( \phi : C^\infty(\mathbb{R}^n) \to W \) factors through the quotient \( \rho : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{H}^m_n) \) iff \( \pi \circ \phi = ev_p \) for some \( p \in \mathbb{H}^m_n \).

**Proof.** First note that \( \pi \circ \phi = ev_p \) for a unique \( p \in \mathbb{R}^n \) and clearly if \( \phi \) factors through \( \rho \) then \( p \in \mathbb{H}^m_n \).

Conversely, if \( p \in \mathbb{H}^m_n \) we must verify that for all \( f \in m^p_{\mathbb{H}^m_n} \) we have \( \phi(f) = 0 \). Let \( k \) the nilpotence degree of \( W \) (that is \( (\ker \pi)^{k+1} = 0 \)) and take the Taylor expansion of \( f \) at \( p \) with Hadamard remainder:

\[
f(x) = \sum_{|I| \leq k} \frac{1}{I!} (x - p)^I \frac{\partial^I f}{\partial^I x}(p) + \sum_{|J| = k+1} (x - p)^J h_J(x),
\]

summing over multi-indices \( I \) and \( J \). Since \( f \) is necessarily flat at \( p \), the first term vanishes so that applying \( \phi \) gives \( \phi(f) = 0 \) since \( \phi(x_i - p_i) \in \ker \pi \) for each \( i = 1, \ldots, n \). \( \square \)

\[\text{For readers unfamiliar with the jargon of SDG, it suffices to think of microlinear spaces as a notion for manifolds internal to SDG.}\]
Corollary 2.10. For any open subset $U \subset \mathbb{R}^n_m$ and Weil algebra $\pi: W \to \mathbb{R}$ there is a canonical bijection between $W^U$ and the manifold with corners

$$(\pi^n)^{-1}(U) = \{(x_0 + \tilde{x}_0, \ldots, x_n + \tilde{x}_n) \mid (x_0, \ldots, x_n) \in U\}.$$ 

Therefore $W^U$ is canonically an $n(k+1)$-dimensional manifold with corners, where $k+1$ is the linear dimension of $W$ over $\mathbb{R}$. In particular we have canonical identifications $W^\mathbb{R}^n_m = \mathbb{R}^{n(k+1)}$.

Proof. The result is immediate from the Corollary 1.8 and the above since $\mathcal{C}^\infty(\mathbb{R}^n)$ is the free $\mathcal{C}^\infty$-algebra on $n$ generators.

Corollary 2.11. For any Weil algebra $\pi: W \to \mathbb{R}$ the diagram

\[
\begin{array}{ccc}
W^\mathbb{R}^n_m & \to & W^\mathbb{R}^n \\
\downarrow & & \downarrow \\
\mathbb{R}^n_m & \to & \mathbb{R}^n \\
\end{array}
\]

is a pullback square for all $n$ and $0 \leq m \leq n$, with $\mathbb{R}^n_m \hookrightarrow \mathbb{R}^n$ the canonical inclusion.

We now consider the case of a manifold with corners $M$ with an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ by basics. By Corollary 2.10 we know that each $W^U_\alpha$ is a basic manifold with corners. To show that $W^M$ is a manifold with corners, it remains only to show that the $W^U_\alpha$ cover $W^M$, to wit

Proposition 2.12. For any manifold with corners $M$ and covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of $M$ by basics, the family $\{W^U_\alpha\}_{\alpha \in A}$ gives a covering of $W^M$ by basics. In particular, the $W$-prolongation of $M$ is a manifold with corners (with topology and smooth structure induced from the $W^U_\alpha$).

Proof. We must show that the maps $W^U_\alpha \to W^M$ induced by the inclusions $U_\alpha \hookrightarrow M$ are themselves inclusions and that the family $\{W^U_\alpha \to W^M\}_{\alpha \in A}$ is jointly surjective.

Firstly, we note that for each $\alpha \in A$ we can find a smooth function $\chi_{M \setminus U_\alpha}$ vanishing precisely on $M \setminus U_\alpha$ from which it follows that $\mathcal{C}^\infty(U_\alpha) = C^\infty(M)\chi_{M \setminus U_\alpha}^{-1}$\footnote{This is where paracompactness is needed, cf. [8] §1.2.}. Letting $L: C^\infty(M) \to C^\infty(U_\alpha)$ denote the localisation, we consider the diagram

\[
\begin{array}{ccc}
C^\infty(M) & \xrightarrow{L} & C^\infty(U_\alpha) \\
\phi_1 \downarrow & & \phi_2 \downarrow \\
W & = & W \\
\end{array}
\]

such that $\phi_1 \circ L = \phi_2 \circ L$. It follows immediately that $\phi_1 = \phi_2$, demonstrating that $W^U_\alpha \to W^M$ is indeed an inclusion.

To see that the $W^U_\alpha \to W^M$ are jointly surjective, we must show that an arbitrary $\phi: C^\infty(M) \to W$ factors through $C^\infty(U_\alpha)$ for some $\alpha$. But this is immediate from Corollary 1.8 since the $U_\alpha$ cover $M$.

Corollary 2.13. If $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is an open covering of $M \in \text{Man}_{\text{corn}}$ not necessarily by basics, then $\{W^U_\alpha\}_{\alpha \in A}$ is an open covering of $W^M$.

Proof. We choose an open covering of each $U_\alpha$ by basics and invoke the Proposition.

We now turn to the question of the compatibility of jet notions internal and external to $\mathcal{C}$. We first show that for each basic, $W$-prolongations are sent to exponentials in $\mathcal{C}$ and then we glue to obtain the global case. The following is an extension of a result of Reyes [8].
Proposition 2.14. For any algebra $A \in \text{FCartSp}^{\text{op}}$ and choices of $n \geq 0$, $0 \leq m \leq n$, there is a canonical bijection
\[ \{ C^\infty(W^n) \rightarrow A \} \cong \{ C^\infty(\mathbb{H}^n_m) \rightarrow A \otimes W \} \]
that is natural in $A$.

Proof. Let $k+1$ be the linear dimension of $W$ over $\mathbb{R}$, then by Corollary 2.10 we have $C^\infty(W^{n+m}_m) = C^\infty(\mathbb{H}^{n(k+1)}_m)$. We first consider the case $m = 0$. Choosing a basis $\{ 1, \eta_1, \ldots, \eta_k \}$ for $W$ over $\mathbb{R}$, we have that a map
\[ \phi: C^\infty(\mathbb{R}^n) \rightarrow A \otimes W = A[\eta_1, \ldots, \eta_k] \]
is determined by the $n$ elements
\[ \alpha_i := \phi(\pi_i: \mathbb{R} \rightarrow x_i) = a_{i0} + a_{i1}\eta_1 + \cdots + a_{ik}\eta_k. \]  

(2)

Similarly, a map
\[ \psi: C^\infty(W^n) = C^\infty(\mathbb{R}^{n(k+1)}) \rightarrow A \]
is determined by the $n(k+1)$ elements
\[ \beta_{ij} := \psi\left( \pi_i^j: (\mathbb{R}_1, \ldots, \mathbb{R}_n) \rightarrow \mathbb{R} \rightarrow x_i^j \right) \in A. \]

The required bijection is then given by setting $a_{ij} = \beta_{ij}$; this bijection is clearly natural in $A$.

For the general case, we consider $\phi: C^\infty(\mathbb{R}^n) \rightarrow A \otimes W = A[\eta_1, \ldots, \eta_k]$ and write $A = C^\infty(\mathbb{R}^p)/I$, noting that $I$ is necessarily a closed ideal. Using the notation of [2], we claim that the map $\phi$ factors through the quotient map $\rho: C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{H}^{n+m}_m)$ iff the following condition holds:

for every $n - m + 1 \leq i \leq n$ and any lift $\tilde{a}_i$ of $a_{i0}$ to $C^\infty(\mathbb{R}^p)$ we have $\tilde{a}_i(x) \geq 0$ for every $x \in \Sigma(I)$. (3)

Recall that for an ideal $I \subset C^\infty(\mathbb{R}^p)$, $\Sigma(I) := \{ x \in \mathbb{R}^p \mid f(x) = 0 \forall f \in I \}$ is its vanishing set.

Proof of claim. The map $\phi$ factors through $\rho$ iff $\phi(f) = 0$ for every $f \in m^0_I$, which is readily seen to be equivalent to the condition
\[ f(\tilde{a}_1, \ldots, \tilde{a}_n) \in I[\eta_1, \ldots, \eta_k] \]
for any choice of lifts $\tilde{a}_i = \tilde{a}_{i0} + \alpha_{i1}\eta_1 + \cdots + \alpha_{ik}\eta_k$ of the $\alpha_i = \phi(\mathbb{R} \rightarrow x_i)$. In particular, if $h \in m^0_I \subset C^\infty(\mathbb{R})$, we require
\[ h(\tilde{a}_i) \in I[\eta_1, \ldots, \eta_k] \]
for each $n - m + 1 \leq i \leq n$. Let $\chi_1$ be a smooth function on $\mathbb{R}$ vanishing precisely on $I$, then $\chi_1 \in m^0_I$ so that for any $x \in \Sigma(I)$ we have
\[ 0 = \chi_1(\tilde{a}_i(x)) = \chi_1(\tilde{a}_{i0}(x)) + \text{terms linear in the } \eta_i's. \]

Composing with the map $A[\eta_1, \ldots, \eta_k] \rightarrow A$, the condition $\chi_1(\tilde{a}_{i0}(x)) = 0$ is equivalent to $\tilde{a}_{i0}(x) \geq 0$.

Conversely, suppose that the condition (3) holds and take any $f \in m^0_I$. For any $x \in \Sigma(I)$ we have
\[ f(\tilde{a}_1(x), \ldots, \tilde{a}_n(x)) = f(\tilde{a}_{i0}(x), \ldots, \tilde{a}_{n0}(x)) + \sum_{i=1}^n \sum_{j=0}^k g_{ij} \cdot \tilde{a}_{ij} \cdot \eta_j \]
where the $g_{ij}$ are necessarily of the form $g_{ij} = \frac{\partial^{ij}}{\partial x^1 \partial x^n}(\tilde{a}_{i0}(x), \ldots, \tilde{a}_{n0}(x))$ for some multi-indices $I$. But then $f(\tilde{a}_1(x), \ldots, \tilde{a}_n(x)) = 0$ since $f$ is flat at $(\tilde{a}_{i0}(x), \ldots, \tilde{a}_{n0}(x)) \in \mathbb{H}^n_m$, from which it is clear that $f(\tilde{a}_1, \ldots, \tilde{a}_n) \in I[\eta_1, \ldots, \eta_k]$ since $I$ is a closed ideal. \(\Box\)

The general case of the Proposition now follows easily by applying the claim twice. \(\Box\)
Corollary 2.15. For any basic manifold with corners \( U = \mathbb{H}_m \) and Weil algebra \( W \) we have
\[
W_U \cong U^{\ell W}
\]
in \( \mathcal{C} \).

Proof. This follows immediately from the above Proposition and the Yoneda lemma. We have
\[
W_U(\ell A) = \{ C^\infty(WU) \to A \} = \{ C^\infty(U) \to A \otimes W \}
\]
\[
= \{ \ell A \times \ell W \to U \} = \{ \ell A \to U^{\ell W} \} = U^{\ell W}(\ell A)
\]
for every \( \ell A \in \text{FCartSp} \).

Drawing all of the results of this section together, we finally obtain

Theorem 2.16. The embedding \( \text{Man}_{\text{crn}} \to \mathcal{C} \) sends Weil prolongations to exponentials. That is, for every manifold with corners \( M \) and Weil algebra \( W \), we have
\[
W_M \cong M^{\ell W}
\]
in \( \mathcal{C} \).

Proof. We begin by choosing a good open cover \( U = \{ U_\alpha \}_{\alpha \in A} \) for \( M \) as per Lemma 1.3, noting that Corollary 2.10 and Proposition 2.12 together imply that \( \{ WU_\alpha \}_{\alpha \in A} \) is a good open cover of \( W M \). We write
\[
U_{\alpha \beta} := U_\alpha \times_M U_\beta,
\]
and observe that by Theorem 2.3
\[
W_{U_{\alpha \beta}} = W_{U_\alpha} \times_{W M} W_{U_\beta}.
\]

Now by Corollary 2.15 we have
\[
W M = \text{colim} \left( \prod_{\alpha, \beta \in A} WU_{\alpha \beta} \longrightarrow \prod_{\alpha \in A} WU_\alpha \right)
\]
\[
= \text{colim} \left( \prod_{\alpha, \beta \in A} U^{\ell W}_{\alpha \beta} \longrightarrow \prod_{\alpha \in A} U^{\ell W}_\alpha \right)
\]
\[
= \text{colim} \left( \prod_{\alpha, \beta \in A} U_{\alpha \beta} \longrightarrow \prod_{\alpha \in A} U_\alpha \right)^{\ell W} = M^{\ell W}
\]
since \( (-)^{\ell W} \) has a right adjoint (recall that this is one of the axioms of SDG), so commutes with all colimits.

We remark finally that for a Weil algebra \( \pi : W \to \mathbb{R} \) of linear dimension \( k + 1 \) over \( \mathbb{R} \), the map \( W M \to M \) induced by \( \pi \) exhibits \( W M \) as a locally trivial fibration with typical fibre \( \mathbb{R}^{kn} \), with \( n = \dim M \).

3 Halos, Collared Halos and Myopia

Having so far realised manifolds with corners in the Cahiers topos, in this section we set up the machinery needed to tackle the problem of gluing. We introduce various notions of thickening a manifold with corners that will be used extensively in §4 below.
3.1 Halos

The first thickening notion that we introduce for manifolds with corners allows for a systematic smoothing of the boundary structure via halos. The terminology is adapted from a related notion appearing in [10].

**Definition 3.1.** Let $M$ be a manifold with corners. A fat haloing of $M$ (or simply a fat halo of $M$) is a manifold $X$ without corners, together with a smooth inclusion $M \hookrightarrow X$ locally modelled on the inclusions $\mathbb{H}^n_m \hookrightarrow \mathbb{R}^n$. We denote the set of fat halos of $M$ by $\mathcal{H}(M)$. Note that if $\partial M = \emptyset$ then clearly $\mathcal{H}(M) = \{M\}$.

The set $\mathcal{H}(M)$ is partially ordered, with partial order given by diagrams of smooth inclusions:

$$X \leq X' \iff M \hookrightarrow X \hookrightarrow X'$$

**Lemma 3.2.** The set of fat halos $\mathcal{H}(M)$ is codirected, that is every pair $X, X' \in \mathcal{H}(M)$ has a lower bound.

**Proof.** For any $X, X' \in \mathcal{H}(M)$ choose charts such that both inclusions $M \hookrightarrow X, M \hookrightarrow X'$ are locally modelled on $\mathbb{H}^n_m \hookrightarrow \mathbb{R}^n$. By restricting as necessary, from these local models we obtain a manifold $X''$ without corners containing $M$ such that $X'' \leq X$ and $X'' \leq X'$. \hfill \Box

For any manifold with corners $M$, we now define the $C^\infty$-algebra of functions on the halo of $M$ by

$$C^\infty([M]) := \colim_{X \in \mathcal{H}(M)} C^\infty(X) = \colim_{\mathcal{H}(M)/\mathcal{P}} C^\infty(X).$$

We abuse notation here by viewing $C^\infty([M])$ as the algebra of functions on some manifold $[M]$ even though $C^\infty([M])$ does not arise in this way. The following result gives a more concrete description of $C^\infty([M])$:

**Proposition 3.3.** For any choice of $X \in \mathcal{H}(M)$ there is an isomorphism

$$C^\infty([M]) \cong C^\infty(X)/m^0_M,$$

where $m^0_M \subset C^\infty(X)$ is the ideal of smooth functions vanishing on some open neighbourhood of $M$ in $X$.

**Proof.** Let $\phi: C^\infty([M]) \rightarrow A$ be any map of $C^\infty$-algebras. By definition, $\phi: C^\infty([M]) \rightarrow A$ is equivalently a compatible family of maps

$$\{\phi_Z: C^\infty(Z) \rightarrow A\}_{Z \in \mathcal{H}(M)}.$$

In particular, we have a map $\psi = \phi_X: C^\infty(X) \rightarrow A$ and moreover, as $C^\infty(X)$ is finitely generated, we may take $A = C^\infty(\mathbb{R}^n)/I$ without loss of generality. Choosing a closed embedding $X \hookrightarrow \mathbb{R}^m$ for some $m$ sufficiently large, we have a presentation $C^\infty(X) = C^\infty(\mathbb{R}^m)/m^0_X$. Then the map $\psi$ is equivalent to a smooth function $\Psi: \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $\Psi^*(f) \in I$ for all $f \in m^0_X$.

Now if $f \in m^0_X \subset C^\infty(X)$ vanishes on the neighbourhood $U$ of $M$ in $X$, we choose any fat halo $Y \in \mathcal{H}(M)$ such that $M \subset Y \subset U \subset X$. Since the family $\{\phi_Z\}_{Z \in \mathcal{H}(M)}$ is compatible, we have a commutative diagram

$$\begin{CD}
C^\infty(X) @>{\psi = \phi_X}>> A \\
C^\infty(Y) @>{\phi_Y}>> A
\end{CD}$$

with the vertical arrow given by the pullback to the inclusion $Y \hookrightarrow X$ or, equivalently, given by the projection

$$C^\infty(X) = C^\infty(\mathbb{R}^m)/m^0_X \rightarrow C^\infty(\mathbb{R}^m)/m^0_Y = C^\infty(Y).$$
since \( m^0_X \subset m^0_Y \subset C^\infty(\mathbb{R}^m) \). However, the function \( f \) vanishes on \( Y \) so by commutativity of \( \Box \) we must have \( \psi(f) = 0 \). In particular, we have that \( \psi \) descends to a map

\[
\zeta : C^\infty(X)/m^0_X \longrightarrow A.
\]

Conversely, given such a map \( \zeta \) we associate a compatible family \( \{ \phi_Z : C^\infty(Z) \longrightarrow A \}_{Z \in \mathcal{H}(M)} \) as follows. Firstly, if \( Y \leq X \) we define \( \phi_Y : C^\infty(Y) \to A \) via the composition

\[
C^\infty(Y) \longrightarrow C^\infty(X)/m^0_M \overset{\zeta}{\longrightarrow} A.
\]

The first map in this expression is the map

\[
C^\infty(Y) = C^\infty(\mathbb{R}^m)/m^0_Y \longrightarrow C^\infty(X)/m^0_M = C^\infty(\mathbb{R}^m)/\tilde{m}^0_M
\]

induced by the inclusion of ideals \( m^0_Y \subset \tilde{m}^0_M \), with \( \tilde{m}^0_M \) the ideal of smooth functions on \( \mathbb{R}^m \) vanishing on any locally closed subset of \( \mathbb{R}^m \) of the form \( U \cap X \) for \( U \supset M \) open. For an arbitrary fat halo \( Z \in \mathcal{H}(M) \), choose a lower bound \( Y \) for \( X \) and \( Z \) and define \( \phi_Z \) by the composition with the pullback map

\[
C^\infty(Z) \longrightarrow C^\infty(Y) \overset{\phi_Y}{\longrightarrow} A,
\]

where we note that the construction of \( \phi_Z \) is independent of choice of lower bound \( Y \).

We have thus established a bijection

\[
C^\infty\text{-alg}(C^\infty(X)/m^0_M, A) = \varprojlim_{\mathcal{H}(M)} C^\infty\text{-alg}(C^\infty(Z), A) = : C^\infty\text{-alg}(C^\infty([M]), A)
\]

for all \( C^\infty \)-algebras \( A \), which is the desired result. \( \Box \)

**Corollary 3.4.** The functor \( [\,] : \text{FCartSp}^{\text{op}} \to \text{Set} \) defined by

\[
[M] : \ell A \mapsto C^\infty\text{-alg}(C^\infty([M]), A)
\]

is a sheaf.

**Proof.** The proof is a straightforward adaptation of the argument immediately preceding Lemma 2.2. \( \Box \)

Finally we note that by construction there is a canonical map \( C^\infty([M]) \to C^\infty(M) \) inducing a canonical map of sheaves \( M \to [M] \).

**Lemma 3.5.** For every manifold with corners \( M \), the canonical map of sheaves \( M \to [M] \) is a monomorphism.

**Proof.** The claim is that for every \( \ell A \in \text{FCartSp} \) the map

\[
M(\ell A) = C^\infty\text{-alg}(C^\infty(M), A) \longrightarrow [M](\ell A) = C^\infty\text{-alg}(C^\infty([M]), A)
\]

is injective. But this follows immediately from Proposition 3.3 since

\[
C^\infty([M]) = C^\infty(X)/m^0_M \longrightarrow C^\infty(X)/m^0_M = C^\infty(M)
\]

is surjective. \( \Box \)
3.2 Collared Halos

In order to be able to glue together manifolds with corners along a common boundary component, the usual procedure is to first choose a collar. The rôle of the collar is to facilitate gluing; once a collar is chosen the gluing procedure is completely analogous to gluing schemes together along an open subscheme in algebraic geometry. This procedure is independent of choices of collar up to diffeomorphism.

To mimic this story in the Cahiers topos, we shall need to appropriately define collars in our setting. We proceed as follows:

Definition 3.6. Let \( \Sigma \) be a manifold with corners. A collared fat halo of \( \Sigma \) is a manifold without corners \( V \) such that

\[
\Sigma \cong \Sigma \times \{0\} \hookrightarrow V \hookrightarrow X \times \mathbb{R}
\]

for some \( X \in \mathcal{H}(\Sigma) \). The set of collared fat halos of \( \Sigma \) is denoted \( \mathcal{H}^c(\Sigma) \).

As for \( \mathcal{H}(\Sigma) \), the set \( \Sigma \) has a partial order arising from open embeddings. Likewise, the proof of Lemma 3.1 carries over directly to give the following

Lemma 3.7. The set of collared fat halos \( \mathcal{H}^c(\Sigma) \) is codirected.

As before, for any manifold with corners \( \Sigma \) we define the \( C^\infty \)-algebra of functions on the collared halo of \( \Sigma \) by

\[
C^\infty(\llbracket \Sigma \rrbracket) := \operatorname{colim}_{\mathcal{H}^c(\Sigma)^{op}} C^\infty(V).
\]

The proofs of Proposition 3.3 and Corollary 3.4 carry over mutatis mutandis to give

Proposition 3.8. For any choice of \( V \in \mathcal{H}^c(\Sigma) \) there is an isomorphism

\[
C^\infty(\llbracket \Sigma \rrbracket) \cong C^\infty(V)/m^g_{\Sigma \times \{0\}},
\]

where \( m^g_{\Sigma \times \{0\}} \subset C^\infty(V) \) is the ideal of smooth functions vanishing on some open neighbourhood of \( \Sigma \times \{0\} \) in \( V \).

Corollary 3.9. The functor \( \llbracket \Sigma \rrbracket \colon \text{FCartSp}^{op} \to \text{Set} \) defined by

\[
\llbracket \Sigma \rrbracket \colon \mathcal{L} A \mapsto C^\infty(\llbracket \Sigma \rrbracket), A
\]

is a sheaf.

3.3 The Infinitesimal Collar

We now consider a slightly different approach to collarings of boundary faces. So far we have considered thickenings of manifolds with corners in the Cahiers topos in the guise of halos and collared halos. These constructions are exhibited as taking a limit over a codirected family of finite or fat thickenings and therefore have a natural interpretation in terms of germs of (fat) thickenings.

Alternatively, as is often done in algebraic geometry we can consider instead an infinitesimal thickening. Our approach shall, as ever, be \( C^\infty \)-algebraic in nature and reflects a sort of \( C^\infty \)-algebraic version of the formal schemes of algebraic geometry. The underlying principle is that for any manifold with corners \( \Sigma \) we view the \( C^\infty \)-algebra of formal power series \( C^\infty(\Sigma)[[\epsilon]] \) as a function on \( \Sigma \) together with all jets in the direction of an arbitrarily thin collar. This principle is justified by the following adaptation of Borel’s Theorem (cf. [8, Theorem I.1.3]—the proof provided there carries over directly)

Proposition 3.10. For any choice of open neighbourhood \( U \) of \( 0 \) in \( \mathbb{R} \), taking the Taylor series at \( 0 \) in the \( \mathbb{R} \) direction gives an isomorphism

\[
C^\infty(\Sigma \times U)/m^\infty_{\Sigma \times \{0\}} \xrightarrow{\cong} C^\infty(\Sigma)[[\epsilon]]
\]

where \( m^\infty_{\Sigma \times \{0\}} \) is the ideal of functions that are flat on \( \Sigma \times \{0\} \).
In light of the above Proposition, we may view \( C^\infty(\Sigma)[\epsilon] \) as the algebra of functions on the infinitesimal collar of \( \Sigma \). Moreover, we can consider the functor
\[
\Sigma^{[\epsilon]} : \ell A \to C^\infty(\Sigma)[\epsilon],
\]
on \( \text{FCartSp}^{\text{op}} \) as representing the infinitesimal collar. Indeed

**Corollary 3.11.** The presheaf \( \Sigma^{[\epsilon]} \in [\text{FCartSp}^{\text{op}}, \text{Set}] \) is a sheaf.

**Proof.** As in the proof of Corollary 3.4.

In order to fully understand the story of gluing in the Cahiers topos, it will be important to understand the relationship between the infinitesimal collar \( \Sigma^{[\epsilon]} \) and the collared halo \( \Sigma \). As a first step in this direction, we have the following result, analogous to Lemma 3.5.

**Lemma 3.12.** There is a canonical monomorphism of sheaves \( \Sigma^{[\epsilon]} \to \Sigma \) in \( \mathcal{C} \).

**Proof.** We choose presentations
\[
C^\infty(\Sigma) = C^\infty(V)/m_\Sigma^2 \text{ and } C^\infty(\Sigma)[\epsilon] = C^\infty(\Sigma \times U)/m_{\Sigma \times \{0\}}^\infty
\]
for \( V \in \mathcal{J}(\Sigma) \) and a neighbourhood \( U \) of \( 0 \) in \( \mathcal{R} \) such that \( \Sigma \times U = (\Sigma \times \mathcal{R}) \cap V \). We therefore have an inclusion \( \Sigma \times U \to V \) inducing a surjective map \( \xi : C^\infty(V) \to C^\infty(\Sigma \times U) \). Moreover it is clear that \( \xi, m_\Sigma^2 := \{ \xi(f) \mid f \in m_\Sigma^2 \} \subset m_{\Sigma \times \{0\}}^\infty \), so we have a chain of surjections
\[
C^\infty(V)/m_\Sigma^2 \to C^\infty(\Sigma \times U)/\xi m_\Sigma^2 \to C^\infty(\Sigma \times U)/m_{\Sigma \times \{0\}}^\infty.
\]
The composite surjection \( C^\infty(\Sigma) \to C^\infty(\Sigma)[\epsilon] \) can be described more concretely as restricting a germ of functions \([f] \in C^\infty(\Sigma)\) to \( \Sigma \times \{0\} \) together with all of its derivatives in the transverse direction:
\[
[f] \mapsto \sum_{n=0}^\infty \frac{1}{n!} (\partial^n f)|_{\Sigma \times \{0\}} \epsilon^n,
\]
which is manifestly independent of choices. That the canonical induced map of sheaves
\[
\Sigma^{[\epsilon]}(\ell A) = C^\infty(\Sigma)[\epsilon], A \to C^\infty(\Sigma)[\epsilon], A = \Sigma(\ell A)
\]
is injective for all \( \ell A \in \text{FCartSp} \) now follows immediately from surjectivity.

**3.4 The Myopia of \text{FCartSp}**

A key step in understanding the gluing procedure in the Cahiers topos is the realisation that formal Cartesian spaces are _myopic_. Borrowing a phrase from Lavendhomme [7], we mean that the basic building blocks of \( \mathcal{C} \), the formal Cartesian spaces, are "so short-sighted" that they can't tell the difference between a manifold and its halo, or between an infinitesimal thickening and a fat thickening.

**Proposition 3.13.** For every manifold with corners \( M \), the monomorphism of sheaves \( \underline{M} \to [\underline{M}] \) of Lemma 3.5 is an isomorphism.

**Proof.** It suffices to show that for every \( \ell A \in \text{FCartSp} \) the map \( \underline{M}(\ell A) \to [\underline{M}] (\ell A) \) is surjective.

We first consider the case \( \ell A = \mathbb{R}^n \) for some \( n \). Then any \( \phi \in [\underline{M}] (\ell A) \) is equivalently a compatible family of maps \( \{ \phi_X : C^\infty(X) \to C^\infty(\mathbb{R}^n) \}_{X \in \mathcal{P}(M)} \) or, equivalently, a compatible family of smooth maps of manifolds
\[
X \in \mathcal{P}(M).
\]
Now suppose that for some \( X \in \mathcal{H}(M) \) there is some \( x \in X \setminus M \) in the image of \( \Phi_X \). But then we can find some \( X' \leq X \in \mathcal{H}(M) \) such that \( x \notin X' \), contradicting compatibility of the family \( \mathbb{g} \). This shows that each \( \Phi_X \) factors through a map \( \Phi: \mathbb{R}^n \to M \), so that every \( \phi \in \overline{\mathcal{M}}(\mathbb{R}^n) \) is in the image of \( \overline{\mathcal{M}} \to \overline{\mathcal{M}} \).

We consider now the general case \( \ell A = \mathbb{R}^n \times \ell W \) for some Weil algebra \( \pi: W \to \mathbb{R} \). Then

\[
\overline{\mathcal{M}}(\ell A) = C^\infty(\mathbb{R}^n, A) = \lim_{X \in \mathcal{H}(M)} C^\infty(X, \mathbb{R}^n) \\
= \lim_{X \in \mathcal{H}(M)} C^\infty(X, \mathbb{R}^n \otimes W) \\
= \lim_{X \in \mathcal{H}(M)} C^\infty(W X, \mathbb{R}^n).
\]

As above, the last term is equivalent to a compatible family of smooth maps of manifolds

\[
\{ \Phi_X: \mathbb{R}^n \to W X \}_{X \in \mathcal{H}(M)}.
\]

We observe now that \( \pi: W \to \mathbb{R} \) induces a map \( W X \to X \) for each \( X \), giving a compatible family of maps \( \Phi_X: \mathbb{R}^n \to X \), which all factor through some \( \Phi' : \mathbb{R}^n \to M \) by the above.

We now recall that for any \( X \in \mathcal{H}(M) \) the inclusion \( M \hookrightarrow X \) is locally modelled on \( \mathbb{H}^n_m \hookrightarrow \mathbb{R}^n \). Invoking Corollary \ref{cor:cartesian lokal}, we have a pullback square

\[
\begin{array}{ccc}
WM & \to & WX \\
\downarrow & & \downarrow \Phi
\\
M & \to & X
\end{array}
\]

for every \( X \in \mathcal{H}(M) \). From this we have that the \( \Phi_X \) and \( \Phi' \) together determine a smooth map \( \Phi: \mathbb{R}^n \to WM \) covering \( \Phi' \). In particular, each \( \Phi_X \) factors through \( \Phi \) so we have finally that every \( \phi \in \overline{\mathcal{M}}(\mathbb{R}^n \times \ell W) \) is in the image of \( \overline{\mathcal{M}} \to \overline{\mathcal{M}} \). This completes the proof.

\[\tag*{\Box}\]

**Corollary 3.14.** For any manifold with corners \( M = \lim_{X \in \mathcal{H}(M)} X \) in \( \mathcal{E} \).

**Proof.** This follows immediately from

\[
\overline{\mathcal{M}}(\ell A) := C^\infty(\mathbb{R}^n, A) = \lim_{X \in \mathcal{H}(M)} C^\infty(X, A) = \lim_{X \in \mathcal{H}(M)} \overline{\mathcal{M}}(\ell A)
\]

for all \( \ell A \in \text{FCartSp} \), since limits of sheaves are levelwise. \[\tag*{\Box}\]

As further evidence of the short-sightedness of formal Cartesian spaces, we have the following result

**Proposition 3.15.** For every manifold with corners \( \Sigma \), the monomorphism of sheaves \( \Sigma^{[\delta]} \to \overline{\Sigma} \) of Lemma \ref{lemma:cartesian lokal} is an isomorphism.

**Proof.** As above, it suffices to check that for every \( \ell A \in \text{FCartSp} \) the map \( \Sigma^{[\delta]}(\ell A) \to \overline{\Sigma}(\ell A) \) is surjective.

We first consider the case \( \ell A = \mathbb{R}^n \). Choose any \( \phi \in \overline{\Sigma}(\mathbb{R}^n) \), noting that this is equivalent to a compatible family of smooth maps \( \{ \Phi_V : \mathbb{R}^n \to V \}_{V \in \mathcal{E}(\Sigma)} \). As in Proposition \ref{prop:cartesian lokal} we can show that every \( \Phi_V \) factors through a map \( \Phi: \mathbb{R}^n \to \Sigma \cong \Sigma \times \{0\} \). In this case, \( \phi \) is in fact in the image of \( \Sigma \to \overline{\Sigma} \), so by the commutative diagram

\[
\begin{array}{ccc}
\Sigma(\mathbb{R}^n) & \to & \overline{\Sigma}(\mathbb{R}^n) \\
\downarrow & & \downarrow \Phi
\\n\Sigma^{[\delta]}(\mathbb{R}^n) & \to & \overline{\Sigma}(\mathbb{R}^n)
\end{array}
\]
we have the result for each formal Cartesian space of the form \( \ell A = \mathbb{R}^n \).

It remains to consider the general case \( \ell A = \mathbb{R}^n \times \ell W \) for some well algebra \( \pi : W \to \mathbb{R} \). We choose any \( \phi \in \prod_\ell (\ell A) \), which is equivalently a compatible family of smooth maps \( \{ \Phi_V : \mathbb{R}^n \to W \}_{V \in \mathcal{C}(\Sigma)} \) as per the proof of Proposition 3.13. Our by-now-standard argument shows that the compatibility of the \( \Phi_V \) forces them to all factor through a smooth map \( \Phi : \mathbb{R}^n \to W(\Sigma \times \mathbb{R}) \) such that the composite

\[
\mathbb{R}^n \xrightarrow{\Phi} W(\Sigma \times \mathbb{R}) \xrightarrow{\pi} \Sigma \times \mathbb{R}
\]

has image contained in \( \Sigma \times \{0\} \). Then \( \Phi \) is equivalently a map of \( C^\infty \)-algebras \( C^\infty(\Sigma \times \mathbb{R}) \to C^\infty(\mathbb{R}^n) \) which, by Theorem 2.16, may be equivalently formulated as a map of \( C^\infty \)-algebras \( \varphi : C^\infty(\Sigma \times \mathbb{R}) \to C^\infty(\mathbb{R}^n) \otimes W \). The condition on the image of \( \pi \circ \Phi \) translates into the requirement that the diagram

\[
\begin{array}{ccc}
C^\infty(\Sigma \times \mathbb{R}) & \xrightarrow{\varphi} & C^\infty(\mathbb{R}^n) \otimes W \\
\downarrow & & \downarrow id \otimes \pi \\
C^\infty(\Sigma \times \{0\}) & \xrightarrow{\varphi} & C^\infty(\mathbb{R}^n)
\end{array}
\]

(6)

commutes. Recall that \( C^\infty(\Sigma \times \mathbb{R}) = C^\infty(\Sigma) \otimes C^\infty(\mathbb{R}) \) (the coproduct of \( C^\infty \)-algebras) so that \( \varphi = \varpi \otimes \rho \). In particular, the map \( \rho \) is determined by the image \( \rho(x \mapsto x) = f \in C^\infty(\mathbb{R}^n) \otimes W \) as \( C^\infty(\mathbb{R}) \) is free on the generator \( x \mapsto x \). The commutativity of (6) forces \( f \) to be nilpotent, so if \( k \) is the nilpotence degree of \( W \) we can factor \( \varphi \) as

\[
C^\infty(\Sigma \times \mathbb{R}) \to C^\infty(\Sigma) \otimes C^\infty(\mathbb{R})/(x^{k+1}) = C^\infty(\Sigma)[\epsilon]/(\epsilon^{k+1}) \to C^\infty(\mathbb{R}^n) \otimes W.
\]

We have thus shown that the original map \( \phi : C^\infty(\prod_\ell) \to C^\infty(\mathbb{R}^n) \otimes W \) factors as

\[
\begin{array}{ccc}
C^\infty(\Sigma)[\epsilon] & \xrightarrow{\phi} & C^\infty(\mathbb{R}^n) \otimes W \\
\downarrow & & \downarrow id \\
C^\infty(\prod_\ell) & \xrightarrow{\phi} & C^\infty(\mathbb{R}^n) \otimes W
\end{array}
\]

so that every \( \phi \in \prod_\ell(\ell A) \) is in the image of the monomorphism \( \prod_{[\ell]} \to \prod_\ell \).

**Corollary 3.16.** Let \( \Sigma \) be any manifold with corners, then \( \Sigma^{[\ell]} = \lim_{V \in \mathcal{C}(\Sigma)} V \) in \( \mathcal{C} \).

**Proof.** This follows immediately from

\[
\prod_\ell(\ell A) := C^\infty\text{-alg}(C^\infty(\prod_\ell), A) = \lim_{V \in \mathcal{C}(\Sigma)} C^\infty\text{-alg}(C^\infty(V), A) = \lim_{V \in \mathcal{C}(\Sigma)} V(\ell A).
\]

\[\square\]

### 3.5 The Twisted Case

The notions of thickening considered in §3.2 and §3.3 above make use of a thickening in a transversal direction. So far, this has been done by thickening in the direction of a trivial line bundle. However, in order to obtain more general gluing results, it is necessary to twist this picture by performing thickenings in arbitrary line bundles.

Let \( \Sigma \) be a manifold with corners, and \( F \to \Sigma \) be a line bundle. For every fat halo \( X \in \mathcal{H}(\Sigma) \) we presuppose a choice of line bundle \( F_X \to X \) extending \( F \to \Sigma \); such a choice can always be made and is not canonical, however everything that follows is easily seen to be independent of the chosen extensions.
\textbf{Definition 3.17.} An \textit{F-collared fat halo of }$\Sigma$\textit{ is a manifold without corners }$V$ \textit{that is an open neighbourhood of the zero section of }$F_X \to X$ \textit{over }$\Sigma$ \textit{for some }$X \in \mathcal{H}(\Sigma)$\textit{. The set of }$F$\textit{-collared fat halos of }$\Sigma$\textit{ is denoted }$\mathcal{H}_F(\Sigma)$\textit{.}

As in the untwisted case, $\mathcal{H}_F(\Sigma)$ is co-directed and so we can define

$$C^\infty(\square_F) := \colim_{\mathcal{H}_F(\Sigma)^{op}} C^\infty(V).$$

Analogues of Proposition 3.8 and Corollary 3.9 hold in the twisted case, in particular we have a sheaf $\square_F$ in $\mathcal{C}$ representing the $F$-twisted halo of $\Sigma$.

Examining the proofs of Lemma 3.12 and Proposition 3.13 we see that they carry over more or less exactly to give a canonical isomorphism of sheaves $\Sigma^{[\alpha]} \to \square_F$. Explicitly this is given at the level of $C^\infty$-algebras by taking a Taylor series expansion in the fibre direction at the zero section of $F \to \Sigma$. In summary, for every line bundle $F \to \Sigma$ we have canonical isomorphisms of sheaves

$$\square_F = \lim_{V \in \mathcal{H}(\Sigma)} V \leftarrow \Sigma^{[\alpha]} \longrightarrow \square = \lim_{V \in \mathcal{H}(\Sigma)} V.$$ (7)

This is explained heuristically by saying that the infinitesimal collar is so thin that it can’t detect twisting.

\section{4 Gluing}

Having gathered the necessary results in the previous two sections, we now turn to the crux of the matter at hand: gluing manifolds in the Cahiers topos. After collecting some well-known results on gluing manifolds with boundary, we work up to the main result of this paper, Theorem 4.10 which shows that gluing is realised in the Cahiers topos via pushouts of the form

$$M \leftarrow \Sigma^{[\alpha]} \longrightarrow N.$$ The natural interpretation is that such a pushout is sewing together the manifolds $M$ and $N$ along a common boundary component $\Sigma$, making sure that all higher order jet data coincides and thereby resulting in a smooth manifold.

\subsection{4.1 Gluing Manifolds with Boundary}

We briefly recall some of the folklore regarding gluing manifolds with corners. The following is all standard—a good reference is [10 §3.1].

\textbf{Definition 4.1.} For $M$ a manifold with corners, a \textit{connected face} of $M$ is the closure of a connected component of $S^{1}(M)$. A \textit{face} of $M$ is a disjoint union of connected faces.

Let $M$ and $N$ be manifolds with corners and $\Sigma$ a manifold with corners together with inclusions $f: \Sigma \to M$ and $g: \Sigma \to N$ realising $\Sigma$ as a face of each. The quintuple $(M, N, \Sigma, f, g)$ is called a \textit{face identification} and denoted $M^{\Sigma} N$.

\textbf{Proposition 4.2.} Let $M^{\Sigma} N$ be a face identification. Then $M \cup_{\Sigma} N$ is a topological manifold with corners. Moreover, given collars $f': \Sigma \times I \to M$ and $g': \Sigma \times I \to N$ there exists a canonical smooth structure on $M \cup_{\Sigma} N$ compatible with the inclusions $M \hookrightarrow M \cup_{\Sigma} N$ and $N \hookrightarrow M \cup_{\Sigma} N$.

\textbf{Proof.} The fact that $M \cup_{\Sigma} N$ is a topological manifold with corners is checked locally. The smooth structure on $M \cup_{\Sigma} N$ arises from the fact the manifolds with corners $M \setminus \Sigma, N \setminus \Sigma$ and $\Sigma \times R$ give an open cover, where the map $\Sigma \times R \hookrightarrow M \cup_{\Sigma} N$ is given by the union of the collars. \hfill \Box

\textbf{Theorem 4.3 (cf. [10]).} Let $M^{\Sigma} N$ be a face identification such that $\Sigma$ admits collars in $M$ and $N$, but that these are not fixed. Then $M \cup_{\Sigma} N$ is unique up to diffeomorphism fixing $\Sigma$ and equal to the identity outside a neighbourhood of $\Sigma$.  

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4.2 Trinities

Our conception of gluing in the Cahiers topos uses a limiting process on thickenings, exploiting the myopia of FCartSp. In order to glue from the data of a face identification $M^\Sigma N$, we need to thicken each manifold involved in a way compatible with the rest. We therefore pose the

**Definition 4.4.** A fat trinity associated to the face identification $M^\Sigma N$ admitting collars is a triple

$$ (X, V, Y) \in \mathcal{H}(M) \times \mathcal{H}_c(\Sigma) \times \mathcal{H}(N) $$

admitting inclusions $X \leftarrow V \leftarrow Y$ extending $M \leftarrow \Sigma \leftarrow N$ such that the pushout

$$ X \cup_V Y := \bigcup_Y X $$

exists in the category of manifolds. We denote the set of fat trinities of $M^\Sigma N$ by $\mathcal{T}(M^\Sigma N)$.

**Lemma 4.5.** If $\Sigma$ admits collars in $M$ and $N$ then $\mathcal{T}(M^\Sigma N)$ is non-empty.

**Proof.** Let $X \in \mathcal{H}(M)$, $Y \in \mathcal{H}(N)$ be arbitrary fat halos of $M$ and $N$ respectively. It is not difficult to see that there is some fat halo $Z$ of $\Sigma$ fitting into the diagram of submanifold inclusions

$$ \Sigma \xrightarrow{f} X \xleftarrow{g} Z \xrightarrow{h} N \xrightarrow{i} Y $$

We observe that $Z$ is codimension one in both $X$ and $Y$, so that by the Tubular Neighbourhood Theorem we can find open neighbourhoods $U_X$ and $U_Y$ of $Z$ in $X$ and $Y$ respectively that are, by our assumption on collars which forces the normal bundles $N_X Z$ and $N_Y Z$ to be trivial, diffeomorphic to open submanifolds of $Z \times \mathbb{R}$. In particular, $U_X, U_Y \in \mathcal{H}_c(\Sigma)$. By shrinking $X, Y, U_X, U_Y$ and $Z$ as necessary, we may assume that $V = U_X = U_Y$ and that $X, Y$ and $V$ are such that we have a diagram of open submanifold inclusions

$$ \xymatrix{ V & X \ar[l] & S \ar[u] \ar[l] \\ \ar[r] & Y \ar[u] } $$

for some manifold $S$. In this case, the pushout $X \bigcup_V Y$ exists in Man and is given by the open submanifold $X \cup_V Y$ of $S$. \qed

In fact, examining the proof above we see that the Lemma can be strengthened slightly.

**Definition 4.6.** The face identification $M^\Sigma N$ is gluable if the normal bundles $N_M \Sigma$ and $N_N \Sigma$ are isomorphic.

We can define fat trinities associated to any gluable face identification $M^\Sigma N$ simply by replacing $\mathcal{H}_c(\Sigma)$ by the twisted versions $\mathcal{H}^{\Sigma}(\Sigma)$ in the definition. Then exactly as before we have

**Lemma 4.7.** If the face identification $M^\Sigma N$ is such that the normal bundles $N_M \Sigma$ and $N_N \Sigma$ are isomorphic then $\mathcal{T}(M^\Sigma N)$ is non-empty.

There is a partial order on $\mathcal{T}(M^\Sigma N)$ arising from diagrams of inclusions:

$$ (X, V, Y) \preceq (X', V', Y') \iff V \preceq V' $$

$$ \begin{array}{ccc}
X & \xrightarrow{D} & X' \\
\downarrow & & \downarrow \\
V & \xleftarrow{V} & V' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{Y} & Y'
\end{array} $$

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noting that if \((X, V, Y) \leq (X', V', Y')\) then there is an open submanifold inclusion \(X \cup_V Y \hookrightarrow X' \cup_{V'} Y'\). As was the case for \(\mathcal{H}(M)\) and \(\mathcal{K}(\Sigma)\), we have the

**Lemma 4.8.** When non-empty, the set \(\mathcal{T}(M^SN)\) is codirected.

### 4.3 Gluing in the Cahiers Topos

We have now developed all the machinery to prove our main results. In this section, we prove two important theorems. The first characterises gluing via a limit over fat trinites associated to a face identification. The second theorem shows that this limit reduces to a pushout representing gluing along an infinitesimally thickened boundary component.

**Theorem 4.9.** For every gluable face identification \(M^SN\) we have

\[
M \cup_{\Sigma} N = \lim_{\mathcal{T}(M^SN)} \big{\prod}_{\mathcal{V}} X
\]

in \(\mathcal{C}\).

**Proof.** We first recall that for any fat trinity \((X, V, Y)\) the pushout

\[
X \cup_{V} Y := X \big{\prod}_{\mathcal{V}} Y
\]

exists in the category of manifolds. We claim that

\[
X \cup_{V} Y = X \big{\prod}_{\mathcal{V}} Y \text{ in } \mathcal{C}.
\]

To see this, we choose coverings

- \(\mathcal{V} := \{U_\alpha\}_{\alpha \in A}\) of \(V\);
- \(\mathcal{X} := \{U_\beta\}_{\beta \in B}\) of \(X\) extending \(\mathcal{V}\) in the sense that \(\mathcal{V} \subset \mathcal{X}\) and \(\mathcal{X} \setminus \mathcal{V}\) is an open cover of \(X \setminus \mathcal{V}\); and
- \(\mathcal{Y} := \{U_\gamma\}_{\gamma \in C}\) of \(Y\) extending \(\mathcal{V}\) in the sense that \(\mathcal{V} \subset \mathcal{Y}\) and \(\mathcal{Y} \setminus \mathcal{V}\) is an open cover of \(Y \setminus \mathcal{V}\).

We define categories

\[
\mathcal{D} := \left\{ \begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array} \right\} \text{ and } \mathcal{E} := \left\{ \begin{array}{c}
\alpha \\
\beta
\end{array} \right\}
\]

and consider the functor \(F : \mathcal{E} \times \mathcal{D} \to \mathcal{C}\) defined via the diagram

We write \(F_E : \mathcal{D} \to [\mathcal{E}, \mathcal{C}]\) and \(F_D : \mathcal{E} \to [\mathcal{D}, \mathcal{C}]\) for the induced functors and observe that since the embedding \(\text{Man} \to \mathcal{C}\) preserves open covers

\[
\text{colim}_{\mathcal{E}} F_D = \big{\prod}_{\mathcal{V}} Y \text{ whence } \text{colim}_{\mathcal{D}} \left(\text{colim}_{\mathcal{E}} F_D\right) = X \big{\prod}_{\mathcal{V}} Y.
\]
On the other hand, taking \( \text{colim}_D \) gives the diagram

\[
\begin{pmatrix}
\bigcap_{\beta, \beta' \in B} U_{\beta} \times X \cap U_{\beta'} \\
\bigcap_{\alpha, \alpha' \in A} U_{\alpha} \times Y \cap U_{\alpha'} \\
\bigcap_{\gamma, \gamma' \in C} U_{\gamma} \times Y \cap U_{\gamma'}
\end{pmatrix} \rightarrow \begin{pmatrix}
\bigcap_{\beta \in B} U_{\beta} \\
\bigcap_{\alpha \in A} U_{\alpha} \cap \bigcap_{\gamma \in C} U_{\gamma}
\end{pmatrix}
\]

which simplifies to

\[
\text{colim}_D F_{\ell} = \prod_{\alpha, \beta \in B \cup A} U_{\alpha} \times X \cup U_{\beta} \quad \text{so that} \quad \text{colim}_D \left( \text{colim}_\ell F_{\ell} \right) = X \cup_Y Y.
\]

The claim then follows from the commutativity of small colimits:

\[
X \prod_{\underline{\gamma}} Y = \text{colim}_D \left( \text{colim}_\ell F_{\ell} \right) = \text{colim}_\ell \left( \text{colim}_D F_{\ell} \right) = X \cup_Y Y.
\]

We observe now that for any \((X, V, Y) \in \mathcal{T}(M^{\Sigma} N)\) there is a smooth inclusion

\[
M \cup_{\Sigma} N \hookrightarrow X \cup_Y Y
\]

locally modelled on \(\mathbb{H}^n_m \hookrightarrow \mathbb{R}^n\). This shows that \(X \cup_Y Y \in \mathcal{T}(M \cup_{\Sigma} N)\), giving a map of codirected sets

\[
\Xi : \mathcal{T}(M^{\Sigma} N) \longrightarrow \mathcal{T}(M \cup_{\Sigma} N).
\]

It is straightforward to check that the image of \(\Xi\) is dense, in the sense that for any fat halo \(Z\) of \(M \cup_{\Sigma} N\) we can find an element of the image of \(\Xi\) that is smaller than \(Z\). It follows from abstract nonsense that

\[
\lim_{\mathcal{T}(M^{\Sigma} N)} X \prod_{\underline{\gamma}} Y = \lim_{\mathcal{T}(M \cup_{\Sigma} N)} Z = M \cup_{\Sigma} N,
\]

invoking Proposition 3.13.

Our main result on gluing now follows by examining the limit over fat trinities.

**Theorem 4.10.** For every gluable face identification \(M^{\Sigma} N\)

\[
M \cup_{\Sigma} N = M \prod_{\Sigma[\alpha]} N.
\]

**Proof.** By the previous Theorem we have

\[
M \cup_{\Sigma} N = \lim_{\mathcal{T}(M^{\Sigma} N)} X \prod_{\underline{\gamma}} Y
\]

and since \(\mathcal{T}(M^{\Sigma} N)\) is codirected, \(\lim_{\mathcal{T}(M^{\Sigma} N)} \) is cofiltered. As cofiltered limits commute with all small colimits

\[
M \cup_{\Sigma} N = \lim_{\mathcal{T}(M^{\Sigma} N)} \left( \prod_{\underline{\gamma}} Y \right) = \left( \lim_{\mathcal{T}(M^{\Sigma} N)} X \right) \prod_{\underline{\gamma}} \left( \lim_{\mathcal{T}(M^{\Sigma} N)} Y \right).
\]
As in the proof of Theorem 4.9 above, we have dense maps of codirected sets

\[ \Xi_M : \mathcal{T}(M^2 N) \rightarrow \mathcal{H}(M); \quad (X, V, Y) \mapsto X \]

\[ \Xi_N : \mathcal{T}(M^2 N) \rightarrow \mathcal{H}(N); \quad (X, V, Y) \mapsto Y \]

\[ \Xi_{\Sigma} : \mathcal{T}(M^2 N) \rightarrow \mathcal{H}_c F(\Sigma); \quad (X, V, Y) \mapsto V \]

where \( F = N_M \Sigma = N_N \Sigma \), so that

\[ \lim \mathcal{T}(M^2 N) X = \lim \mathcal{H}(M) X, \quad \lim \mathcal{T}(M^2 N) Y = \lim \mathcal{H}(N) Y \quad \text{and} \quad \lim \mathcal{T}(M^2 N) V = \lim \mathcal{H}_c F(\Sigma) V. \]

Combining this with (7), (8) and Corollary 3.114 completes the proof.

Conclusions and Outlook

In this paper we have shown that there is an embedding of manifolds with corners into the Cahiers topos that sends boundary gluings to pushouts. In addition to providing an aesthetically pleasing categorical interpretation of boundary gluings, this result is an important step toward understanding classical field theory (in the vein of [2]) in the context of Synthetic Differential Geometry.

An interesting and indeed natural question is whether the results of this paper hold in larger well-adapted models for SDG, such as the Dubuc topos of finitely-generated germ-determined ideals (cf. [3], in which it is called \( \mathcal{G} \)). We remark that the results of §2 up until Proposition 2.14 immediately carry through to this setting, whereas many of the arguments of §3 and §4 depend crucially on the nature of the Cahiers topos. From our point of view, the question of whether or not a well-adapted model admits an embedding of \( \text{Man}_{\text{crn}} \) that sends boundary gluings to pushouts is indeed asking how well-adapted that model is—we could claim that the Cahiers topos is therefore a very well-adapted model.

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