Existence of dust ion acoustic solitary wave and double layer solution at $M = M_c$

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The Sagdeev potential technique has been considered to find the necessary condition for the existence as well as the polarity of dust ion acoustic solitary structures at $M = M_c$, in an unmagnetized nonthermal plasma consisting of negatively charged dust grains, adiabatic positive ions and nonthermal electrons, where $M_c$ is the smallest possible value of the Mach number $M$. Depending on the analytical theory, a compositional parameter space showing the nature of existence of solitary structures at $M = M_c$ has been presented. The compositional parameter space clearly indicates the upper bounds of the nonthermal parameter ($\beta_1$) associated with the nonthermal distribution of electrons for the existence of negative and positive potential solitons at $M = M_c$. The compositional parameter space also shows that the present system does not support the coexistence of both negative and positive solitons at $M = M_c$. A critical value of the nonthermal parameter ($\beta_1 = \beta_c$), which acts as sink of negative solitons and the source of positive solitons at $M = M_c$, has been found. However, no solitary wave is possible at $\beta_1 = \beta_c$. The compositional parameter space again shows that the present system supports the negative potential double layer along the curve $\beta_1 = \beta_{1c}$ in $\mu \beta_1$-parametric plane provided that $\mu \geq \mu_r$, i.e., for each $\mu$ lying within the interval $\mu_r \leq \mu < 1$, there exists one and only one $\beta_{1c}$ of $\beta_1$ such that one can get negative potential double layer at $M = M_c$ when $\beta_1 = \beta_{1c}$, where $\mu$ is the ratio of unperturbed number density of nonthermal electrons to that of ions. From the compositional parameter space, it has been observed that the negative potential solitons at $M = M_c$ are separated by the curve $\beta_1 = \beta_{1c}$. A finite jump between amplitudes of negative potential solitons separated by the curve $\beta_1 = \beta_{1c}$ has been observed. The effect of the parameters on the amplitude of the solitary structures have been discussed.
I. INTRODUCTION

In the last few decades dusty plasma attracts the researchers most, not only because of different types of wave propagation in dusty plasma due to their involvement in the study of astrophysical and space environments but also for its great variety of new phenomena associated with waves and instabilities.\textsuperscript{1–9} Dusty plasmas essentially consists of fairly massive, charged particulates in addition to electrons and ions in the usual two component plasma. Acoustic wave modes in dusty plasma have received a great deal of attention since the last decade. Depending on different time scales, there can exists two or more acoustic waves in a typical dusty plasma. Dust Acoustic (DA) and Dust Ion-Acoustic (DIA) waves are two such acoustic waves in a plasma containing electrons, ions, and charged dust grains.

Shukla and Silin\textsuperscript{3} were the first to show that due to the quasi neutrality condition \( n_{e0} + n_{d0} Z_d = n_{i0} \) and the strong inequality \( n_{e0} \ll n_{i0} \) (\( n_{e0}, n_{i0}, \) and \( n_d \) are, respectively, the number density of electrons, ions, and dust particle, where \( Z_d \) is the number of electrons residing on the dust grain surface), a dusty plasma (with negatively charged static dust grains) supports low-frequency DIA waves with phase velocity much smaller (larger) than electron (ion) thermal velocity. In case of long wavelength limit the dispersion relation of DIA wave is similar to that of IA wave for a plasma with \( n_{e0} = n_{i0} \) and \( T_i \ll T_e \), where \( T_i(T_e) \) is the average ion (electron) temperature. Due to the usual dusty plasma approximations (\( n_{e0} \ll n_{i0} \) and \( T_i \simeq T_e \)), a dusty plasma cannot support the usual Ion-Acoustic (IA) waves, but the DIA waves of Shukla and Silin\textsuperscript{3} can. Thus DIA waves are basically IA waves, modified by the presence of heavy dust particulates. The theoretical prediction of Shukla and Silin\textsuperscript{3} was supported by a number of laboratory experiments.\textsuperscript{10–12} The linear properties of DIA waves in dusty plasma are now well understood.\textsuperscript{10,13–15}

Due to a balance between the wave dispersion and nonlinearity, there exists the possibility of DIA solitary waves (DIASWs) in a dusty plasma. The DIASWs studied using the reductive perturbation technique which gives Korteweg-de Vries (KdV) equation or Korteweg-de Vries Burgers (KdVB) equation as nonlinear evolution equations.\textsuperscript{15} Bharuthram and Shukla\textsuperscript{16} studied the nonlinear waves associated with DIA waves. Mamun and Shukla\textsuperscript{17} have investigated the condition for existence of positive and negative potential DIASWs, and examined how the ion-fluid temperature modifies the amplitude of DIASWs. In presence of negatively charged dust grains, one can have DIASWs with negative potentials, when
the electron number density is less than one third of the positive ion number density. Verheest et al. have shown that in the dust-modified ion acoustic regime, negative structures can also be generated, beside positive potential soliton for electrons with a polytropic index $\gamma_e \neq 1$, as well as for Boltzmann electrons. The effect of ion-fluid temperature on DIASWs structures have been investigated by Sayed and Mamun in a dusty plasma containing adiabatic ion-fluid, Boltzmann electrons, and static dust particles. In the above mentioned works the nonlinear properties of DIASWs investigated for $M > M_c$.

In most of the earlier works, Maxwellian velocity distribution function for lighter species of particles has been used to study DIASWs and DIA double layers (DIADLs). However, the dusty plasma with nonthermally/suprathermally distributed electrons observed in a number of heliospheric environments. Therefore, it is of considerable importance to study nonlinear wave structures in a dusty plasma in which lighter species (electrons) are nonthermally/suprathermally distributed. Recently Baluku et al. investigated DIASWs in an unmagnetized dusty plasma consisting of cold dust particles and kappa distributed electrons using both small and arbitrary amplitude techniques.

In the present investigation we have considered the problem of existence as well as the polarity of DIASWs and DIADLs in a nonthermal dusty plasma consisting of negatively charged dust grains, adiabatic positive ions and nonthermal electrons at the smallest possible value of the Mach number $M$. The Sagdeev potential approach has been considered to investigate the existence of solitary wave and double layer at $M = M_c$, where $M_c$ is the lower bound of the Mach number $M$, i.e., solitary wave and/or double layer solutions of the energy integral start to exist for $M > M_c$. In most of the earlier works, solitary wave and/or double layer solutions have been investigated for $M > M_c$. However, some recent investigations have shown that finite amplitude solitary wave can exist at $M = M_c$ in the parameter regime where solitons of both polarities exist. The numerical observations of the solitary wave solution of the energy integral at $M = M_c$, motivate us to investigate the existence of DIASW and DIADL. However, we have approach in a different way without considering the parameter regime of the system for $M > M_c$. Three basic parameters of the present dusty plasma system are $\mu$, $\alpha$ and $\beta_1$, which are respectively the ratio of unperturbed number density of nonthermal electrons to that of ions, the ratio of average temperature of ions to that of nonthermal electrons, a parameter associated with the nonthermal distribution of electrons. Nonthermal distribution of electrons becomes isothermal one if $\beta_1 = 0$ and
consequently, for isothermal electron species, the present dusty plasma contains only two basic parameters $\mu$ and $\alpha$. It has been found that solitary wave or double layer do not exist for $0 < \mu < \mu_p$. For isothermal distributed electrons, PPSWs exist for $\mu_p \leq \mu < \mu_c$, whereas for all $\mu$ lies within $\mu_c < \mu \leq \mu_T$, there exist NPSWs. At $\mu = \mu_c$, no solitary wave or double layer solution is possible. However, there do not exist any double layer solution for isothermal electrons. For nonthermal distributed electrons, we have found two critical values of $\beta_1$, viz., $\beta_{1a}$ and $\beta_c$ are the upper bounds of $\beta_1$ for the existence of PPSWs and NPSWs, respectively. However, the coexistence of both NPSWs and PPSWs are not possible in the present system. Again, at $\beta_1 = \beta_c$, no solitary wave or double layer solution is possible. We have also found the existence of negative potential double layer (NPDNL) in a portion of the solution space. Depending on the nature of solutions of DIA waves, we have $\mu_p$, $\mu_c$, and $\mu_T$ such that the entire parametric space can be delimited into three subintervals of $\mu$, viz., $\mu_p \leq \mu < \mu_c$, $\mu_c < \mu < \mu_r$, and $\mu_r \leq \mu \leq \mu_T$. For each $\mu$ in $\mu_r \leq \mu \leq \mu_T$, there exists a value $\beta_{1c}$ of $\beta_1$, such that one can found the existence of NPDNL. There is an interesting phenomena occur around the point $\beta_1 = \beta_{1c}$. There exists a finite jump in amplitude of NPSWs at $\beta_1 = \beta_{1c}$, the jump being the amplitude of the double layer exists at $\beta_1 = \beta_{1c}$. The amplitude of NPSW decreases with increasing $\beta_1$ and ultimately, demolished at $\beta_1 = \beta_c$. On the other hand, PPSW creates from $\beta_1 = \beta_c$ and the amplitude of PPSW increases with $\beta_1$, having maximum amplitude at $\beta_1 = \beta_{1a}$. Throughout the investigation we have considered $\mu_T$ as the upper bound of $\mu$, which is strictly less than 1, since for $\mu \to 1$, the effect of charged dust on DIA wave is negligible.

The present paper is organized as follows: In Sec. II, the basic equations are given. The energy integral and the Sagdeev potential has been constructed in Sec. III. Physical interpretation of the energy integral for the existence of solitary wave and double layer solutions is also given in this section. In Sec. IV, an analytical approach has been presented to determine the polarity of the solitary waves. The analytical approach has been verified numerically in Sec. V. Finally, a brief conclusion has been given in Sec. VI.

II. BASIC EQUATIONS

The governing equations describing the nonlinear behavior of DIA waves, propagating along $x$-axis, in collisionless, unmagnetized dusty plasma consisting of negatively charged
immobile dust grains, adiabatic positive ions and nonthermal electrons are the following:

\[ n_{i,t} + (n_i u_i)_x = 0, \]  \( (1) \)

\[ u_{i,t} + u_i u_{i,x} = -\phi_x - \alpha n_i^{-1} p_{i,x}, \]  \( (2) \)

\[ p_{i,t} + u_i p_{i,x} + \gamma p_i u_{i,x} = 0, \]  \( (3) \)

\[ \phi_{xx} = \left( \frac{n_e}{n_i} \right) n_e - n_i + (Z_d n_d)/n_i, \]  \( (4) \)

where the parameter \( \alpha = T_i/T_e \).

Here we have used the notation \( \psi_q \) or \( (\psi_q)_q \) for \( \partial \psi_q / \partial q \) and \( n_i, n_e, u_i, p_i, \phi, x \) and \( t \) are, respectively, the ion number density, electron number density, ion velocity, ion pressure, electrostatic potential, spatial variable and time, and they have been normalized by \( n_{i0} \) (unperturbed ion number density), \( n_{e0} \) (unperturbed electron number density), \( c_i = \sqrt{(K_B T_e)/m_i} \) (ion-acoustic speed), \( n_{i0} K_B T_i, K_B T_e/e, \lambda_{Dem} = \sqrt{(K_B T_e)/(4\pi n_{i0} e^2)} \) (Debye length), and \( \omega_{pi}^{-1} = \sqrt{m_i/(4\pi n_{i0} e^2)} \) (ion plasma period). Here \( \gamma = 3 \) is the adiabatic index, \( K_B \) is the Boltzmann constant, \( T_i \) and \( T_e \) are, respectively, the average temperatures of ions and electrons, \( m_i \) is the mass of an ion, \( n_d \) is the dust number density, \( Z_d \) is the number of negative unit charges residing on dust grain surface, and \( e \) is the charge of an electron.

The above equations are supplemented by nonthermally distributed electrons as prescribed by Cairns et al for the electron species. Actually, in a number of heliospheric environments, dusty plasma contains nonthermally distributed ions or electrons. Therefore, it is of considerable importance to study DIASWs and DIADLs in dusty plasma in which electrons are nonthermally distributed. Nonthermal distribution of any lighter species of particles (as prescribed by Cairns et al for the electron species) can be regarded as population of Boltzmann distributed particles together with a population of energetic particles. This can also be regarded as a modified Boltzmannian distribution, which has the property that the number of particles in phase space in the neighbourhood of the point \( v = 0 \) is much smaller than the number of particles in phase space in the neighbourhood of the point \( v = 0 \) for the case of Boltzmann distribution, where \( v \) is the velocity of the particle in phase space. This type of velocity distribution is often termed as Cairns distribution and was considered by many authors in various studies of different collective processes in plasmas and dusty
plasmas\textsuperscript{30-38}. Under the above mentioned normalization of the dependent and independent variables, the normalized number density of nonthermal Cairns distributed electrons can be written as

\[ n_e = (1 - \beta_1 \phi + \beta_1 \phi^2) e^\phi, \]  

(5)

where

\[ \beta_1 = \frac{4\alpha_1}{1 + 3\alpha_1}, \]  

(6)

with \( \alpha_1 \geq 0 \). Here \( \beta_1 \) and consequently \( \alpha_1 \) are the parameters associated with nonthermal distribution of electrons. From (6) and inequality \( \alpha_1 \geq 0 \), it can be easily checked that the nonthermal parameter \( \beta_1 \) is restricted by the following inequality: \( 0 \leq \beta_1 < 4/3 \). However we cannot take the whole region of \( \beta_1 \) (\( 0 \leq \beta_1 < 4/3 \)). Plotting the nonthermal velocity distribution of electrons against phase velocity (\( v \)) of electrons in phase space, it can be easily shown that the number of electrons in phase space in the neighbourhood of the point \( v = 0 \) decreases with increasing \( \beta_1 \) and the number of electrons in phase space in the neighbourhood of the point \( v = 0 \) is almost zero when \( \beta_1 \to 4/3 \). Therefore, for increasing values of \( \beta_1 \) distribution function develops wings, which become stronger as \( \beta_1 \) increases, and at the same time the center density in phase space drops, the latter as a result of the normalization of the area under the integral. So, we should not take values of \( \beta_1 > 4/7 \) since that stage might stretch the credibility of the Cairns model too far\textsuperscript{38}. So here we consider the effective range of \( \beta_1 \) as follows: \( 0 \leq \beta_1 \leq \beta_{1T} \), where \( \beta_{1T} = 4/7 \approx 0.571429 \). When \( \beta_1 = 0 \), nonthermal distribution of electrons becomes isothermal one.

Now introducing a new parameter \( \mu = n_{e0}/n_{i0} \), the charge neutrality condition,

\[ Z_d n_{d0} + n_{e0} = n_{i0}, \]  

(7)

can be written as

\[ Z_d n_{d0} \quad n_{i0} = 1 - \mu, \]  

(8)

so that the Poisson equation (4) can be written as

\[ \phi_{xx} = \mu n_e - n_i + (1 - \mu). \]  

(9)

We note from (8) that \( 1 - \mu \) must be greater than zero, i.e., \( 0 < \mu < 1 \). When \( \mu \to 1 \), the effect of negatively charged dust grains on DIA wave is negligible and so, we restrict \( \mu \) by the inequality \( 0 < \mu \leq \mu_T \), where \( \mu_T \) is strictly less than 1.
III. ENERGY INTEGRAL AND THE SAGDEEV POTENTIAL

To study the arbitrary amplitude time independent DIA solitary structures we make all the dependent variables depend only on a single variable $\xi = x - Mt$, where the Mach number $M$ is normalized by $c_i$. Thus in the steady state, (1)-(3) and (9) can be written as

\[-M n_i, \xi + (n_i u_i)_{\xi} = 0,\]  
\[-M u_i, \xi + u_i u_i, \xi = -\phi_{\xi} - an_i^{-1} p_i, \xi,\]  
\[-M p_i, \xi + u_i p_i, \xi + 3p_i u_i, \xi = 0,\]  
\[\phi_{\xi \xi} = \mu n_e - n_i + (1 - \mu).\]  

Using the boundary conditions,

\[n_i \to 1, p_i \to 1, u_i \to 0, \phi \to 0, \phi_{\xi} \to 0 \text{ as } |\xi| \to \infty,\]  

and solving (10) - (12), we get a quadratic equation for $n_i^2$ and the solution of this quadratic equation can be put in the following form:

\[n_i^2 = \frac{1}{6\alpha} \left( \sqrt{\Psi(M)} - \phi - \sqrt{\Phi(M)} - \phi \right)^2,\]  

where

\[\Psi(M) = \frac{(M - \sqrt{3\alpha})^2}{2}, \Phi(M) = \frac{(M + \sqrt{3\alpha})^2}{2}.\]  

From the expression of $n_i^2$ as given by (15) we see that $n_i^2$ is nonnegative if and only if $\sqrt{\Psi(M)} - \phi$ and $\sqrt{\Phi(M)} - \phi$ both are real. Again $\sqrt{\Psi(M)} - \phi$ and $\sqrt{\Phi(M)} - \phi$ both are real if and only if $\Psi(M) - \phi$ and $\Phi(M) - \phi$ both are real and nonnegative. But $\Psi(M) - \phi$ and $\Phi(M) - \phi$ both are real and nonnegative if and only if $\phi \leq \Psi(M)$ and $\phi \leq \Phi(M)$, respectively, and finally the conditions $\phi \leq \Psi(M)$ and $\phi \leq \Phi(M)$ hold simultaneously if and only if $\phi \leq \text{Min}\{\Psi(M), \Phi(M)\}$. Now from (16) we see that $\Psi(M) \leq \Phi(M) \iff \text{Min}\{\Psi(M), \Phi(M)\} = \Psi(M)$ and consequently (15) gives theoretically valid expression of $n_i^2$ if and only if $\phi \leq \Psi(M)$. Therefore, for the present problem we are restricted by the inequality: $\phi \leq \Psi(M) = (M - \sqrt{3\alpha})^2/2$. But numerically it is not possible to get the correct value of $n_i$ from (15) for small value of $\alpha$ because the denominator of the right hand
side of (15) is \(6\alpha\). To remove \(\alpha\) from the denominator of the right hand side of (15) we first of all note that \((\sqrt{\Psi(M)} - \phi + \sqrt{\Phi(M)} - \phi)^2 > 0\) for all \(\phi \leq \Psi(M)\) and consequently we can multiply the numerator and the denominator of the right hand side of (15) by \((\sqrt{\Psi(M)} - \phi + \sqrt{\Phi(M)} - \phi)^2\) and we get the following expression of \(n_i\).

\[
 n_i = \frac{\sqrt{2M}}{\sqrt{\Psi(M)} - \phi + \sqrt{\Phi(M)} - \phi}.
\]  

(17)

From (17), we see that this equation gives both theoretically and also numerically correct expression of \(n_i\) even when \(\alpha = 0\) if \(\phi \leq \Psi(M)\). Now if we put \(\alpha = 0\) in (17), we get the same expression of \(n_i\) for the cold dust fluid with \(\Phi(M) = \Psi(M) = M^2\).

Now integrating (13) with respect to \(\phi\) and using the boundary conditions (14), we get the following equation known as energy integral with \(V(\phi) \equiv V(M, \phi, \beta_1, \mu, \alpha)\) as Sagdeev potential or pseudo-potential.

\[
 \frac{1}{2} \left( \frac{d\phi}{d\xi} \right)^2 + V(M, \phi, \beta_1, \mu, \alpha) = 0, 
\]

(18)

where

\[
 V(M, \phi, \beta_1, \mu, \alpha) = V_i - \mu V_e - (1 - \mu)\phi, 
\]

(19)

\[
 V_e = (1 + 3\beta_1 - 3\beta_1\phi + \beta_1\phi^2)e^\phi - (1 + 3\beta_1), 
\]

(20)

\[
 V_i = M^2 + \alpha - n_i(M^2 + 3\alpha - 2\phi - 2\alpha n_i^2). 
\]

(21)

Although \(V(\phi)\) is a function of \(\phi, M, \beta_1, \mu\) and \(\alpha\), but for simplicity, we can omit any argument from \(V(M, \phi, \beta_1, \mu, \alpha)\) when no particular emphasis is put upon it. For example, when we write \(V(\phi) = V(M, \phi)\), it is meant that the arguments \(\beta_1, \mu\) and \(\alpha\) assume fixed values in their physically admissible range and for the fixed values of \(\beta_1, \mu\) and \(\alpha\), nature of solitary structures as obtained from the energy integral (18) depends only on \(M\).

The energy integral (18) can be regarded as the one-dimensional motion of a particle of unit mass whose position is \(\phi\) at time \(\xi\) with velocity \(d\phi/d\xi\) in a potential well \(V(\phi)\). The first term of the energy integral can be regarded as the kinetic energy of a particle of unit mass at position \(\phi\) and time \(\xi\) whereas \(V(\phi)\) is the potential energy of the same particle at that instant. Since kinetic energy is always a non-negative quantity, \(V(\phi) \leq 0\) for the entire motion, i.e., zero is the maximum value for \(V(\phi)\). Again from (18), we find
\[
\frac{d^2\phi}{d\xi^2} + V'(\phi) = 0, \text{ i.e. the force acting on the particle at the position } \phi \text{ is } -V'(\phi).
\]
Suppose, \(V(0) = V'(0) = 0\), therefore, the particle is in equilibrium at \(\phi = 0\) because the velocity as well as the force acting on the particle at \(\phi = 0\) are simultaneously equal to zero. Now if \(\phi = 0\) can be made an unstable position of equilibrium, the energy integral can be interpreted as the motion of an oscillatory particle if \(V(\phi_m) = 0\) for some \(\phi_m \neq 0\), i.e., if the particle is slightly displaced from its unstable position of equilibrium then it moves away from its unstable position of equilibrium and it continues its motion until its velocity is equal to zero, i.e., until \(\phi\) takes the value \(\phi_m\). Now the force acting on the particle of unit mass at position \(\phi = \phi_m\) is \(-V'(\phi_m)\). For \(\phi_m < 0\), the force acting on the particle at the point \(\phi = \phi_m\) is directed towards the point \(\phi = 0\) if \(-V'(\phi_m) > 0\), i.e., if \(V'(\phi_m) < 0\).

On the other hand, for \(\phi_m > 0\), the force acting on the particle at the point \(\phi = \phi_m\) is directed towards the point \(\phi = 0\) if \(-V'(\phi_m) < 0\), i.e., if \(V'(\phi_m) > 0\). Therefore, if \(V'(\phi_m) > 0\) (for the positive potential side) or if \(V'(\phi_m) < 0\) (for the negative potential side) then the particle reflects back again to \(\phi = 0\). Again, if \(V(\phi_m) = V'(\phi_m) = 0\) then the velocity \(d\phi/d\xi\) as well as the force \(d^2\phi/d\xi^2\) both are simultaneously equal to zero at \(\phi = \phi_m\). Consequently, if the particle is slightly displaced from its unstable position of equilibrium (\(\phi = 0\)) it moves away from \(\phi = 0\) and it continues its motion until the velocity is equal to zero, i.e., until \(\phi\) takes the value \(\phi = \phi_m\). However it cannot be reflected back again at \(\phi = 0\) as the velocity and the force acting on the particle at \(\phi = \phi_m\) vanish simultaneously. Actually, if \(V'(\phi_m) > 0\) (for \(\phi_m > 0\)) or if \(V'(\phi_m) < 0\) (for \(\phi_m < 0\)) the particle takes an infinite long time to move away from the unstable position of equilibrium. After that it continues its motion until \(\phi\) takes the value \(\phi_m\) and again it takes an infinite long time to come back its unstable position of equilibrium. Therefore, for the existence of a PPSW (NPSW) solution of the energy integral (18), we must have the following: 

(a) \(\phi = 0\) is the position of unstable equilibrium of the particle, 
(b) \(V(\phi_m) = 0, V'(\phi_m) > 0 (V'(\phi_m) < 0)\) for some \(\phi_m > 0 (\phi_m < 0)\), which is nothing but the condition for oscillation of the particle within the interval \(\min\{0, \phi_m\} < \phi < \max\{0, \phi_m\}\) and 
(c) \(V(\phi) < 0\) for all \(0 < \phi < \phi_m (\phi_m < \phi < 0)\), which is the condition to define the energy integral (18) within the interval \(\min\{0, \phi_m\} < \phi < \max\{0, \phi_m\}\). For the existence of a PPDL (NPDL) solution of the energy integral (18), the conditions (a) and (c) remain unchanged but here (b) has been modified in such a way that the particle cannot be reflected again at \(\phi = 0\), i.e., the condition (b) assumes the following form: 
\(V(\phi_m) = V'(\phi_m) = 0, V''(\phi_m) < 0\) for some \(\phi_m > 0 (\phi_m < 0)\).
The above discussions for the existence of solitary waves and double layers are valid if \( \phi = 0 \) is an unstable position of equilibrium, i.e., if \( V''(0) < 0 \) along with \( V(0) = V'(0) = 0 \). In other words, \( \phi = 0 \) can be made an unstable position of equilibrium if the potential energy of the said particle attains its maximum value at \( \phi = 0 \). Now, the condition \( V''(0) < 0 \) gives a lower bound \( M_c \) of \( M \), i.e., \( V''(0) < 0 \Leftrightarrow M > M_c, \) \( V''(0) > 0 \Leftrightarrow M < M_c, \) and \( V''(0) = 0 \Leftrightarrow M = M_c \Leftrightarrow V''(M_c,0) = 0 \). This \( M_c \) is, in general, a function of the parameters involved in the system, or a constant. Therefore, if \( M < M_c \), the potential energy of the said particle attains its minimum value at \( \phi = 0 \), and consequently, \( \phi = 0 \) is the position of stable equilibrium of the particle, and in this case, it is impossible to make any oscillation of the particle even when it is slightly displaced from its position of stable equilibrium, and consequently there is no question of existence of solitary waves or double layers for \( M < M_c \). In other words, for the position of unstable equilibrium of the particle at \( \phi = 0 \), i.e., for \( V''(0) < 0(\Leftrightarrow M > M_c) \), the function \( V(\phi) \) must be convex within a neighborhood of \( \phi = 0 \) and in this case both type of solitary waves (negative or positive potential) may exist if other conditions are fulfilled. Now suppose that \( V''(M_c,0) = 0 \) and also \( V''(M_c,0) = 0 \), then if \( V'''(M_c,0) < 0 \), the potential energy of the said particle attains its maximum value at \( \phi = 0 \) and consequently, \( \phi = 0 \) is the position of unstable equilibrium. On the other hand if \( V''(M_c,0) = 0, V'''(M_c,0) = 0, \) and \( V''''(M_c,0) > 0, \) the potential energy of the said particle attains its minimum value at \( \phi = 0 \) and consequently, \( \phi = 0 \) is the position of stable equilibrium of the particle and in this case there is no question of existence of solitary wave solution and/or double layer solution of the energy integral (18).

However if \( V(M_c,0) = V'(M_c,0) = V''(M_c,0) = 0 \) and \( V'''(M_c,0) < 0 \), then with the help of simple mathematics, it can be shown that \( V(M_c,\phi) \) is a concave function in \((-\phi_e,0)\) and \( V(M_c,\phi) \) is a convex function in \((0,\phi_e)\), where \( \phi_e \) is strictly positive quantity but can be made sufficiently small. The outline of proof of this result is given in Appendix. Therefore, if a particle, placed at \( \phi = 0 \), be slightly displaced towards the positive potential side it falls within the interval \((0,\phi_e)\) and due to the convexity of the function \( V(M_c,\phi) \) within the interval \((0,\phi_e)\), it moves away from \( \phi = 0 \) and it continues its motion until its velocity is equal to zero, i.e., until \( \phi \) takes the value \( \phi = \phi_m > 0 \), where \( V(M_c,\phi_m) = 0 \) and in this case if \( V'(M_c,\phi_m) > 0 \) \( (V'(M_c,\phi_m) = 0 \) and \( V'''(M_c,\phi_m) < 0) \), one may get a PPSW (PPDL) as a solution of the energy integral (18). On the other hand, if the same particle is slightly displaced from \( \phi = 0 \) towards the negative potential side, it falls within
the interval \((-\phi_e, 0)\) and due to the concavity of the function \(V(M_c, \phi)\) within the interval \((-\phi_e, 0)\), it moves towards the point \(\phi = 0\) and consequently, there does not exist any solitary wave or double layer solution in the negative potential side. By similar arguments, we can conclude that if \(V(M, 0) = V'(M, 0) = V''(M_c, 0) = 0\) and \(V'''(M_c, 0) > 0\) then one may obtain a NPSW (NPDL) as a solution of the energy integral \([18]\), whereas there does not exist any PPSW or PPDL. At this point, it is important to note that we can not expect the coexistence of both negative and positive potential solitary waves at \(M = M_c\). Therefore, the necessary conditions for the existence of PPSW or PPDL at \(M = M_c\) are \(V(M_c, 0) = V'(M_c, 0) = V''(M_c, 0) = 0\) and \(V'''(M_c, 0) < 0\) whereas the necessary conditions for the existence of NPSW or NPDL at \(M = M_c\) are \(V(M_c, 0) = V'(M_c, 0) = V''(M_c, 0) = 0\) and \(V'''(M_c, 0) > 0\).

IV. POLARITY OF SOLITONS AT \(M = M_c\)

For the present problem, it can be easily checked that \(V(M, 0) = V'(M, 0) = 0\) for any value of \(M\) and also for any values of the parameters \(\beta_1, \mu\) and \(\alpha\), whereas the condition \(V''(M, 0) < 0\) gives \(M > M_c\), where

\[
M_c^2 = 3\alpha + \frac{1}{\mu(1 - \beta_1)}. \tag{22}
\]

From \((22)\), we see that for \(M_c\) to be real and positive, we must have \(\mu > 0\) and \(0 \leq \beta_1 < 1\). As the effective range of \(\beta_1\) is \(0 \leq \beta_1 \leq \beta_{1T}\), where \(\beta_{1T} = 4/7 \approx 0.571429\), \(M_c\) is well-defined as a real positive quantity for all \(0 < \mu \leq \mu_T\) and \(0 \leq \beta_1 \leq \beta_{1T}\).

From the discussions of the previous section we have seen that to determine the polarity of solitary wave or double layer at \(M = M_c\), the function \(V''(M_c, 0)\) plays an important role and for the present problem that function is given by

\[
V''(M_c, 0) = \mu \{12\alpha\beta_n^3\mu^2 + 3\beta_n^2\mu - 1\}, \tag{23}
\]

where

\[
\beta_n = 1 - \beta_1. \tag{24}
\]

Now we can write \((23)\) in the following convenient form:

\[
V''(M_c, 0) = \begin{cases} 
12\alpha\beta_n^3\mu(\mu + \mu_*)(\mu - \mu_c) & \text{when } \alpha \neq 0, \\
3\beta_n^2\mu(\mu - \mu_c) & \text{when } \alpha = 0,
\end{cases} \tag{25}
\]
where

\[
\mu_* = \mu_*(\alpha, \beta_1) = \frac{1 + b(\alpha, \beta_1)}{8\alpha\beta_n},
\]

(26)

\[
\mu_c = \mu_c(\alpha, \beta_1) = \frac{2}{3\beta_n^2[1 + b(\alpha, \beta_1)]},
\]

(27)

\[
b = b(\alpha, \beta_1) = \sqrt{1 + \frac{16\alpha}{3\beta_n}}.
\]

(28)

Now as 0 ≤ β_1 < 1, from (21), we get 0 ≤ 1 − β_n < 1 ⇔ 0 ≥ −1 + β_n > −1 ⇔ 1 ≥ β_n > 0 ⇔ 0 < β_n ≤ 1, consequently, from (28), we see that b ≥ 1 for all 0 < β_n ≤ 1 and for all 0 ≤ α ≤ 1 and the sign of equality holds for α = 0 and for all 0 < β_n ≤ 1, i.e., b = 1 only when α = 0 and for any admissible value of β_n. Therefore, from (26), we see that \(\mu_*\) is well defined as strictly positive real number for all 0 < β_n ≤ 1 and for all 0 < α ≤ 1 whereas from (27), we see that \(\mu_c\) is well defined as strictly positive real number for all 0 < β_n ≤ 1 and for all 0 ≤ α ≤ 1. Again we note that \(\mu_*\) occurs in the expression of \(V^{'''}(M_c, 0)\) when \(\alpha \neq 0\), i.e., when 0 < α ≤ 1, therefore, in any case (\(\alpha \neq 0\) or \(\alpha = 0\)) sign of \(V^{'''}(M_c, 0)\) depends only on the factor \(\mu - \mu_c\), more specifically, \(V^{'''}(M_c, 0) < 0\) or \(V^{'''}(M_c, 0) > 0\) according to \(\mu < \mu_c\) or \(\mu > \mu_c\). As \(\mu\) is restricted by the inequality 0 < \(\mu \leq \mu_T < 1\) for the case of negatively charged dust particles, then the condition \(\mu < \mu_c \Leftrightarrow V^{'''}(M_c, 0) < 0\) is always true if \(\mu_c \geq 1\). But if 0 < \(\mu_c < \mu_T < 1\) then \(V^{'''}(M_c, 0) < 0\) or \(V^{'''}(M_c, 0) > 0\) according to \(\mu < \mu_c\) or \(\mu > \mu_c\) and when \(\mu = \mu_c\), \(V^{'''}(M_c, 0) = 0\). In this case, i.e., when \(V^{'''}(M_c, 0) = 0\), for solitary wave solution at \(M = M_c\), it is necessary that \(V^{'''}(M_c, 0) < 0\). So, now our problem, is to find the sign of \(V^{'''}(M_c, 0)\) when \(V^{'''}(M_c, 0) = 0\), i.e., when \(\mu = \mu_c\). When \(\alpha \neq 0\), with the help of simple algebra, we get the following expression of \(V^{'''}(M_c, 0)\) at \(\mu = \mu_c\)

\[
V^{'''}(M_c, 0) = \frac{2[(3\beta_n - 2)^2(b + 1)^2 + 5(b^2 - 1) + 4b]}{9\beta_n^3(b + 1)^3},
\]

(29)

whereas for \(\alpha = 0\), the expression of \(V^{'''}(M_c, 0)\) at \(\mu = \mu_c\) is given by

\[
V^{'''}(M_c, 0) = \frac{(3\beta_n - 2)^2 + 1}{9\beta_n^3}.
\]

(30)

As \(\beta_n > 0\) for all 0 ≤ β_1 < 1, from (30), we see that \(V^{'''}(M_c, 0) > 0\) for \(\alpha = 0\). Again as \(b > 1\) for \(\alpha \neq 0\), i.e., for all 0 < α ≤ 1 and for all 0 < β_n ≤ 1, from (30), we see that \(V^{'''}(M_c, 0) > 0\) for \(\alpha \neq 0\). Therefore, in any case, \(\alpha = 0\) or \(\alpha \neq 0\), for the present problem, we see that for
any admissible values of parameters of the system $V''(M_0, 0) > 0$ when $\mu = \mu_c$, i.e., when $V'''(M_0, 0) = 0$. Therefore, there does not exist any solitary wave solution of the energy integral (18) at $M = M_c$ when $V'''(M_c, 0) = 0$.

Again, differentiating (25) with respect to $\beta_1$, we get

$$\frac{\partial}{\partial \beta_1}(V''(M_c, 0)) = \begin{cases} -36\alpha^2 \beta_n (\beta_n + \frac{1}{6\alpha^2}) & \text{when } \alpha \neq 0, \\ -6\mu^2 \beta_n & \text{when } \alpha = 0. \end{cases} \quad (31)$$

Now as $\beta_n > 0$ for all $0 \leq \beta_1 < 1$ and $\mu > 0$, from (31), we see that $(\partial/\partial \beta_1)(V''(M_c, 0)) < 0$ for any given value of $\alpha$, and consequently, $V''(M_c, 0)$ is strictly decreasing function of $\beta_1$ for all $0 \leq \beta_1 < 1$. Again, it can be easily checked that $V''(M_c, 0) > 0$ for $\beta_1 = 0$ if $\mu > \mu_c(\alpha, 0)$ whereas $V''(M_c, 0) < 0$ for $\beta_1 = 0$ if $\mu < \mu_c(\alpha, 0)$. Again, it can be easily checked that $\lim_{\beta_1 \to 1-0} V''(M_c, 0) = -\mu < 0$, where $\beta_1 \to 1-0 \iff \beta_1 < 1$ and $\beta_1$ approaches to 1, i.e., $\beta_1$ approaches to 1 from the left side of 1. Therefore, if $\mu > \mu_c(\alpha, 0)$, $V''(M_c, 0)$ strictly decreases with increasing $\beta_1$ for all $0 \leq \beta_1 < 1$ starting from a positive value and ending with a negative value whereas if $\mu < \mu_c(\alpha, 0)$, $V''(M_c, 0)$ strictly decreases with increasing $\beta_1$ for all $0 \leq \beta_1 < 1$ starting from a negative value and ending with a negative value. So, $V''(M_c, 0)$ intersects the axis of $\beta_1$ at $\beta_1 = \beta_c(0 \leq \beta_c < 1)$ only if $\mu > \mu_c(\alpha, 0)$ and consequently, we have the following conclusions regarding the sign of $V''(M_c, 0)$.

a. If $\mu > \mu_c(\alpha, 0)$ and $\beta_c < \beta_{1T}$, we get $V''(M_c, 0) > 0$ for all $0 \leq \beta_1 < \beta_c$, whereas $V''(M_c, 0) < 0$ for all $\beta_c < \beta_1 \leq \beta_{1T}$ and $V''(M_c, 0) = 0$ at $\beta_1 = \beta_c$.

b. If $\mu > \mu_c(\alpha, 0)$ and $\beta_c > \beta_{1T}$, we get $V''(M_c, 0) > 0$ for all $0 \leq \beta_1 \leq \beta_{1T}$.

c. If $\mu < \mu_c(\alpha, 0)$, we get $V''(M_c, 0) < 0$ for all $0 \leq \beta_1 \leq \beta_{1T}$.

From the above discussions, it is interesting to note that when $\beta_1 = \beta_c$ then $\mu = \mu_c$, since for $\beta_1 = \beta_c$, we have $V''(M_c, 0) = 0$ and the equation $V''(M_c, 0) = 0$ has only one solution $\mu = \mu_c$ within the admissible range of $\mu$. On the other hand when $\mu = \mu_c$ then $\beta_1 = \beta_c$, since for $\mu = \mu_c$, we have $V''(M_c, 0) = 0$ and the equation $V''(M_c, 0) = 0$ has only one solution $\beta_1 = \beta_c$ within the admissible range of $\beta_1$. Therefore, we can conclude that when $\mu = \mu_c$ then $\beta_1 = \beta_c$ and conversely, when $\beta_1 = \beta_c$ then $\mu = \mu_c$ and consequently, at $\beta_1 = \beta_c$, we have $V''(M_c, 0) > 0$, i.e., there does not exist any solitary wave solution and/or double layer solution of the energy integral at $M = M_c$ when $\beta_1 = \beta_c(\iff \mu = \mu_c)$. 

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So far, we have found that if $\mu < \mu_c$, then only PPSWs are possible at $M = M_c$ for all $0 \leq \beta_1 \leq \beta_{1T}$. On the other hand, if $\mu > \mu_c$ and $\beta_c < \beta_{1T}$, then there exist NPSW for all $0 \leq \beta_1 < \beta_c$ and PPSW for all $\beta_c < \beta_1 \leq \beta_{1T}$. However, there do not exist any solitary wave or double layer solution at $\beta_1 = \beta_c$. Now, the question is whether $\beta_{1T}$ is really the upper bound of $\beta_1$ for the existence of PPSWs.

For this purpose, consider the existence of PPSW at $M = M_c$ for some $\beta_1 > \beta_c$ and for some fixed $\mu > \mu_p$. Therefore, there exists a $\phi_m > 0$ such that

$$V(M_c, \phi) < 0 \quad \text{for all} \quad 0 < \phi < \phi_m, \quad V(M_c, \phi_m) = 0 \quad \text{and} \quad V'(M_c, \phi_m) > 0.$$  \hspace{1cm} (32)

Now as $V(M_c, \phi)$ is real for $\phi \leq \Psi(M_c)$ we must have $\phi_m \leq \Psi(M_c)$, otherwise $V(M_c, \phi_m)$ is not a real quantity. Therefore,

$$V(M_c, \phi) < 0 \quad \text{for all} \quad 0 < \phi < (\phi_m \leq \Psi(M_c)), \quad \text{defines a large amplitude PPSW, which is in conformity with (32)}.$$  \hspace{1cm} (33)

Again let $\beta_{1a}$ be the maximum value of $\beta_1$ up to which solitary wave solution can exist at $M = M_c$. As $\Psi(M)$ increases with $M$ and $M_c$ increases with $\beta_1$ then we have $\Psi(M_c) \leq \Psi(M_c)|_{\beta_1 = \beta_{1a}}$. Therefore,

$$V(M_c, \phi) < 0 \quad \text{for all} \quad 0 < \phi < (\phi_m \leq \Psi(M_c)) \leq \Psi(M_c)|_{\beta_1 = \beta_{1a}},$$

defines the largest amplitude solitary waves if

$$V(M_c, \Psi(M_c)|_{\beta_1 = \beta_{1a}} = 0 \text{ and } V'(M_c, \phi)|_{\phi = \Psi(M_c), \beta_1 = \beta_{1a}} > 0.$$  \hspace{1cm} (34)

Therefore, for the existence of the PPSWs, $\beta_1$ is restricted by the following inequality: $\beta_c < \beta_1 \leq \beta_{1a}$, where $\beta_{1a}$ is the largest positive root of the equation $V(M_c, \Psi(M_c)|_{\beta_1 = \beta_{1a}} = 0$ and $V'(M_c, \phi)|_{\phi = \Psi(M_c), \beta_1 = \beta_{1a}} > 0$ for all $\beta_1 \leq \beta_{1a}$. Thus for $\mu > \mu_c$, PPSW exists in $\beta_c < \beta_1 \leq \beta_p$, where $\beta_p = \min \{\beta_{1a}, \beta_{1T}\}$.

V. NUMERICAL DISCUSSIONS

With the help of the above discussions, we have investigated the entire system numerically and we have found the following solitary wave solution at $M = M_c$: (i) There exists a value
μ_p of μ such that for 0 < μ < μ_p there do not exist any solitary wave or double layer at \( M = M_c \) for any value of \( \beta_1 \) in 0 ≤ \( \beta_1 \) ≤ \( \beta_{1T} \). In fact, for any value of μ lies within 0 < μ < μ_p, there do not exist any non-zero \( \phi \) such that \( V(M_c, \phi) = 0 \); (ii) For any value of μ lies within \( \mu_p \leq \mu < \mu_c \), there exist only PPSWs for all 0 ≤ \( \beta_1 \) ≤ \( \beta_{1a} \); (iii) In \( \mu_c < \mu < \mu_r \), NPSWs exist for 0 ≤ \( \beta_1 < \beta_c \) and PPSWs exist in \( \beta_c < \beta_1 \leq \beta_{1a} \); (iv) For all μ lies within \( \mu_r \leq \mu \leq \mu_T \), NPSWs exist in both 0 ≤ \( \beta_1 < \beta_{1c} \) and \( \beta_{1c} < \beta_1 < \beta_c \), whereas PPSWs exist for all \( \beta_c < \beta_1 \leq \beta_{1a} \). For any point on the curve \( \beta_1 = \beta_{1c} \) in \( \mu_r \leq \mu \leq \mu_T \), one can always find a PPDL solution. These findings can be summarized in FIG. 1. This figure gives the exact scenario of the nature of solutions of the system.

It looks that the solution space is corresponds to nonthermal distributed electrons. However, if we move along \( \beta_1 = 0 \) axis, we can find different intervals of μ on the basis of the polarity of solitary wave solutions in presence of isothermal electrons. In this case, there does not exist any solitary wave or double layer for 0 < μ < \( \mu_p \). Solitary wave starts to exist for \( \mu \geq \mu_p \). Only PPSWs can be found for all μ lies within the interval \( \mu_p \leq \mu < \mu_c \), whereas for μ lies within \( \mu_c < \mu \leq \mu_T \), there exist only NPSWs. We have taken two values of μ, one lies in \( \mu_p \leq \mu < \mu_c \) and the other in \( \mu_c < \mu \leq \mu_T \) and draw \( V(M_c, \phi) \) vs. \( \phi \) in FIG. 2. FIGS. 2(a) and 2(b) verifies the solution space for isothermal electrons. At \( \mu = \mu_c \), there does not exist any solitary wave solution, the reasons of which being described in our theoretical section. The amplitude of PPSW decreases with increasing μ in \( \mu_p \leq \mu < \mu_c \) and ultimately the PPSWs demolished at \( \mu = \mu_c \). On the other hand, NPSW arrises in the frame after μ crosses \( \mu_c \) and the amplitude of NPSW gradually increases with μ in \( \mu_c < \mu \leq \mu_T \). Thus one may assume that the point \( \mu = \mu_c \) acts as a source for NPSWs and as well as a sink for PPSWs. It is to be noted that double layer solution cannot be found for Maxwellian electrons.

Now we are to verify the solution space for nonthermal electrons. For this purpose, we have taken a value of μ from each of the subintervals \( \mu_p \leq \mu < \mu_c \), \( \mu_c < \mu < \mu_r \), and \( \mu_r \leq \mu \leq \mu_T \) and draw \( V(M_c, \phi) \) against \( \phi \) in FIGS. 3-5 for different values of \( \beta_1 \). In FIG. 3, we have taken two values of \( \beta_1 \) lies within 0 ≤ \( \beta_1 \) ≤ \( \beta_{1a} \) and both the curves verify the existence of PPSW. FIG. 4 shows that for a value of μ lies within \( \mu_c < \mu < \mu_r \), there exist NPSWs for all 0 ≤ \( \beta_1 < \beta_c \), whereas only PPSWs exist for all \( \beta_1 \) lies within \( \beta_c < \beta_1 \leq \beta_{1a} \). Again, (a) and (b) of FIG. 5 show the existence of NPSWs for all 0 ≤ \( \beta_1 < \beta_c \) and PPSWs for all \( \beta_1 \) lies within \( \beta_c < \beta_1 \leq \beta_{1a} \), respectively. FIG. 5(c) shows the existence of NPDL
solution at \( \beta_1 = \beta_{1c} \) for each \( \mu \) lies within \( \mu_r \leq \mu \leq \mu_T \).

A closer look at the solution space given by FIG. 1 reveals an interesting feature near the curve \( \beta_1 = \beta_{1c} \). We have seen that NPSWs exist for both \( \beta_1 \leq \beta_{1c} \) and \( \beta_1 \geq \beta_{1c} \) (at least in a neighborhood of \( \beta_1 = \beta_{1c} \)) and a NPDL exists at \( \beta_1 = \beta_{1c} \). The immediate question is whether the characteristic of NPSWs in \( 0 \leq \beta_1 < \beta_{1c} \) and \( \beta_{1c} < \beta_1 < \beta_c \) are same. In search of the answer, we draw FIG. 6. In FIGS. (a) and (b), \( V(M_c, \phi) \) is plotted against \( \phi \) for three different values of \( \beta_1 \), viz., \( \beta_{1c}, \beta_{1c} - 0.0005 \) and \( \beta_{1c} + 0.0005 \). FIG. (b) shows that at \( \beta_1 = \beta_{1c} \), there exists a NPDL whereas, for \( \beta_1 = \beta_{1c} + 0.0005 \), there is a negative potential soliton. However the curve corresponding to \( \beta_1 = \beta_{1c} - 0.0005 \) has no real root of \( \phi \) in the neighborhood of \( \phi = \phi_{dl} \), where \( \phi_{dl} \) is the amplitude of the NPDL occurs at \( \beta_1 = \beta_{1c} \). We notice from figure (a) that \( V(M_c, \phi) \) again vanishes far beyond \( \phi = \phi_{dl} \) for \( \beta_1 = \beta_{1c}, \beta_1 = \beta_{1c} + 0.0005 \), and \( \beta_1 = \beta_{1c} - 0.0005 \). However, the curves corresponding to \( \beta_1 = \beta_{1c} \) and \( \beta_1 = \beta_{1c} + 0.0005 \) are unable to produce any solitary wave solution but the curve corresponding to \( \beta_1 = \beta_{1c} - 0.0005 \) gives a NPSW. The amplitude of this solitary wave is much higher than that of \( \beta_{1c} + 0.0005 \). Thus there is a jump in amplitude of solitary waves and this phenomena has also been observed in some recent works. Therefore, we can say that there exists two types of NPSW, where the first type solitary wave is restricted by \( 0 \leq \beta_1 < \beta_{1c} \) and the second type solitary waves exist for all \( \beta_{1c} < \beta_1 < \beta_c \). Moreover, there is a jump in amplitude between two types of solitary waves corresponding to the nonthermal parameter \( \beta_1 = \beta_{1c} - \epsilon \) and \( \beta_1 = \beta_{1c} + \epsilon \), where \( \epsilon \) is a sufficiently small positive quantity. In FIG. 7 variation in amplitude of NPSWs have been shown for values of \( \beta_1 \) lies within \( \beta_{1c} < \beta_1 < \beta_c \) whenever \( \mu_r \leq \mu \leq \mu_T \). This figure shows that the potential drop is maximum at \( \beta_1 = \beta_{1c} \) and this is reason behind the occurrence of NPDL solution at \( \beta_1 = \beta_{1c} \). This figure also shows that amplitude of NPSW decreases with increasing \( \beta_1 \) and ultimately, these NPSWs demolished at \( \beta_1 = \beta_c \). FIG. 8 verifies the fact that there exist NPDL solution at \( \beta_1 = \beta_{1c} \) for values of \( \mu \) in \( \mu_r \leq \mu \leq \mu_T \). This figure also shows that the amplitude of NPDLs at \( M = M_c \) decreases with increasing \( \mu \) in \( \mu_r \leq \mu \leq \mu_T \). Again FIG. 1 shows that \( \beta_{1c} \) is an increasing function of \( \mu \) and hence the amplitude of NPDLs at \( M = M_c \) decreases with increasing \( \beta_1 \) as well.

In FIG. 9 the variation in amplitude of NPSW and PPSW have been shown for values of \( \beta_1 \) lies within \( 0 \leq \beta_1 < \beta_c \) and \( \beta_c < \beta_1 \leq \beta_{1a} \) for \( \mu_c < \mu < \mu_r \). This figure shows that amplitude of NPSW decreases with increasing \( \beta_1 \) and ultimately, these NPSWs demolished at \( \beta_1 = \beta_c \),
whereas, PPSWs start to occur beyond $\beta_1 = \beta_c$ and amplitude of PPSW increases with $\beta_1$ in $\beta_c < \beta_1 \leq \beta_{1a}$ having maximum amplitude at $\beta_1 = \beta_{1a}$. Therefore, we can conclude that $\beta_1 = \beta_c$ acts like a sink for NPSWs and source for PPSWs. In fact the same phenomena occurs in $\mu_r \leq \mu \leq \mu_T$ when $\beta_1$ lies within $\beta_{1c} < \beta_1 < \beta_c$ and $\beta_c < \beta_1 \leq \beta_{1a}$.

The entire numerical investigation depends on the solution space or compositional parametric space given by FIG. 1. So for compactness of our investigation, it is customary to show that the solution space given by FIG. 1 is unique. To show that this solution space is unique we have to show that the values $\mu_p$, $\mu_c$, and $\mu_r$ maintain the same orderings for any value of $\alpha$ as they are in FIG. 1. To do this, we have numerically investigated these values of $\mu$ and obtain FIG. 10. This figure shows that for any value of $\alpha$, we always have $\mu_p < \mu_c < \mu_r$. Therefore, for any value of $\alpha$, the nature of solutions follow FIG. 1.

VI. CONCLUSIONS

The existence of dust ion acoustic solitary wave and double layer at the lowest possible Mach number $M_c$ have been investigated in an unmagnetized nonthermal plasma consisting of negatively charged dust grains, adiabatic positive ions and nonthermal electrons. An analytical approach has been constructed to find the polarity of the solitary wave or double layer. In the analytical approach a function $V'''(M_c, \phi = 0)$ has been constructed, the real root $\beta_c$ of which is a critical value of the nonthermal parameter $\beta_1$. Solitary waves exist for $\beta_1 < \beta_c$ as well as $\beta_1 > \beta_c$ and the polarity of the solitons have been determined by the sign of $V'''(M_c, \phi = 0)$. This critical value $\beta_c$ of $\beta_1$ acts as sink of negative solitons and the source of positive solitons. However, it is impossible to find any solitary wave at $\beta_1 = \beta_c$.

We have found the upper bounds of the nonthermal parameter for the existence of negative and positive potential solitary waves. We have also found the existence of negative potential double layer at some point of the nonthermal parameter and observe a jump in amplitude of negative solitons at that point, the jump being the amplitude of the double layer formed thereat. Depending on these bounds or critical values of the nonthermal parameter, we have constructed a compositional parameter space for the system. In this solution space, we have delimit the admissible range of $\mu$ into three disjoint subintervals $\mu_p \leq \mu < \mu_c$, $\mu_c < \mu < \mu_r$, and $\mu_r \leq \mu \leq \mu_T$, by means of three critical points $\mu_p$, $\mu_c$, and $\mu_r$ of $\mu$. However, there do not exists any solution in $0 < \mu < \mu_p$. From this solution space we
have also found the delimitation of $\mu$ for isothermal electrons. In this case, positive solitons exist for $\mu_p \leq \mu < \mu_c$, whereas only positive solitons are possible in $\mu_c < \mu \leq \mu_T$. It is impossible to find any solitons at $\mu = \mu_c$. We have not found any double layer solution for isothermal distributed electrons. Here, we have considered $\mu_T$ as the upper bound of $\mu$, which is strictly less than 1, since for $\mu \to 1$, the effect of charged dust on DIA wave is negligible. The amplitude of negative potential solitary waves decreases with increasing $\beta_1$ while the amplitude of positive potential solitary waves increases with $\beta_1$.

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FIG. 1. A graphical presentation of different solitary structures have been given with respect to different subintervals of $\mu$ within the admissible interval of the nonthermal parameter $\beta_1$. In this solution space ‘P’ stands for the existence of PPSW and ‘N’ stands for the existence of NPSW. For any given value of $\alpha$, NPSWs exist in $0 \leq \beta_1 < \beta_c$ and PPSWs exist in $\beta_c < \beta_1 \leq \beta_{1a}$. At any point on the curve $\beta_1 = \beta_{1c}$, one can always find a NPDL at $M = M_c$. This solution space has been drawn for $\alpha = 0.9$. We have found $\mu_p \approx 0.14$, $\mu_c \approx 0.2$, and $\mu_r \approx 0.44$. 
FIG. 2. $V(M_c, \phi)$ is plotted against $\phi$ for isothermal distribution of electrons with values of $\mu$ lies within the intervals $\mu_p \leq \mu < \mu_c$ (figure (a)) and $\mu_c < \mu \leq \mu_T$ (figure (b)). For $\alpha = 0.9$, we have found $\mu_p \approx 0.145$ and $\mu_c \approx 0.195$.

FIG. 3. $V(M_c, \phi)$ is plotted against $\phi$ for nonthermal distribution of electrons for values of $\beta_1$ lies within $0 \leq \beta_1 \leq \beta_{1a}$ with values of $\mu$ lies within the intervals $\mu_p \leq \mu < \mu_c$. 
FIG. 4. $V(M_c, \phi)$ is plotted against $\phi$ for two different values of $\beta_1$ lies within $0 \leq \beta_1 < \beta_c$ (figure (a)) and $\beta_c < \beta_1 \leq \beta_{1T}$ (figure (b)) with values of $\mu$ lies within the intervals $\mu_c < \mu < \mu_r$. For $\alpha = 0.9$, we have found $\mu_c \approx 0.195$, $\mu_r \approx 0.432$ and for $\mu = 0.3$, we have found $\beta_c \approx 0.223$.

FIG. 5. In (a) and (b), $V(M_c, \phi)$ is plotted against $\phi$ for values of $\beta_1$ lies within $0 \leq \beta_1 < \beta_c$ and $\beta_c < \beta_1 \leq \beta_{1T}$, respectively with values of $\mu = 0.5$ lies within the intervals $\mu_c < \mu < \mu_r$. In (c), the same has been plotted at $\beta_1 = \beta_{1c}$. For $\alpha = 0.9$ and for $\mu = 0.5$, we have found $\beta_c \approx 0.427$ and $\beta_{1a} \approx 0.511$. 
FIG. 6. $V(M_c, \phi)$ is plotted against $\phi$ for three different values of $\beta_1$, viz., $\beta_{1c}[-\cdot-], \beta_{1c}-0.0005[\cdot\cdot\cdot]$ and $\beta_{1c}+0.0005[-\cdot\cdot\cdot]$ in (a) and (b). In (b) we have shown the region of $\phi$ from $\phi = -2$ to $\phi = 0$, whereas in (a) a region of $\phi$ from $\phi = -11.57$ to $\phi = -11.54$ has been shown. The NPDL profile corresponding to $\beta_1 = \beta_{1c}$ has been shown in (c), whereas the profiles of NPSWs corresponding to $\beta_1 = \beta_{1c} - 0.0005$ and $\beta_1 = \beta_{1c} + 0.0005$ have been drawn in (d). Profiles in (d) shown a jump in amplitude of solitary wave by going from $\beta_1 = \beta_{1c} - 0.0005$ to $\beta_1 = \beta_{1c} + 0.0005$. In all these four figures we have used $\alpha = 0.9, \mu = 0.5$. 
FIG. 7. Variation in amplitude of NPSWs have been shown for values of $\beta_1$ lies within $\beta_{1c} < \beta_1 < \beta_c$. This figure shows that amplitude of NPSW decreases with increasing $\beta_1$ and ultimately, these NPSWs demolished at $\beta_1 = \beta_c$. For $\alpha = 0.9$ and $\mu = 0.5$, we have found $\beta_{1c} \approx 0.369$ and $\beta_c \approx 0.427$.

FIG. 8. $V(M_c, \phi)$ is plotted against $\phi$ and $\beta_1 = \beta_{1c}$ for four different values of $\mu$, viz., $\mu = 0.44$ (- - -), $\mu = 0.45$ (···), $\mu = 0.5$ (- - -) and $\mu = 0.6$ (——) and respective $\beta_{1c}$’s lies in the neighborhood of $\beta_1 = 0.31001$, $\beta_1 = 0.32203$, $\beta_1 = 0.36917$ and $\beta_1 = 0.43519$. 
FIG. 9. Variation in amplitude of NPSW and PPSW have been shown for values of $\beta_1$ lies within $0 \leq \beta_1 < \beta_c$ and $\beta_c < \beta_1 \leq \beta_{1a}$. This figure shows that amplitude of NPSW decreases with increasing $\beta_1$ and ultimately, these NPSWs demolished at $\beta_1 = \beta_c$, whereas, PPSWs start to occur beyond $\beta_1 = \beta_c$ and amplitude of PPSW increases with $\beta_1$ in $\beta_c < \beta_1 \leq \beta_{1a}$ having maximum amplitude at $\beta_1 = \beta_{1a}$. For $\alpha = 0.9$ and $\mu = 0.3$, we have found $\beta_c \approx 0.223$ and $\beta_{1a} \approx 0.33$.

FIG. 10. $\mu_p$, $\mu_r$ and $\mu_c$ are plotted against $\alpha$. From this figure we see that for any value of $\alpha$, we have $\mu_p < \mu_c < \mu_r$. 

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