SOME GENERALIZATIONS OF THE ALUTHGE TRANSFORM OF OPERATORS AND THEIR CONSEQUENCES

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Abstract. Let $A = U|A|$ be the polar decomposition of $A$. The Aluthge transform of the operator $A$, denoted by $\tilde{A}$, is defined as $\tilde{A} = |A|^1/2 U|A|^{1/2}$. In this paper, first we generalize the definition of Aluthge transform for non-negative continuous functions $f,g$ such that $f(x)g(x) = x$ ($x \geq 0$). Then, by using of this definition, we get some numerical radius inequalities. Among other inequalities, it is shown that if $A$ is bounded linear operator on a complex Hilbert space $\mathcal{H}$, then

$$h\left( w(A) \right) \leq \frac{1}{4} \left( h\left( g^2(|A|) \right) + h\left( f^2(|A|) \right) \right) + \frac{1}{2} h\left( w\left( \tilde{A}_{f,g} \right) \right),$$

where $f,g$ are non-negative continuous functions such that $f(x)g(x) = x$ ($x \geq 0$), $h$ is a non-negative non-decreasing convex function on $[0, \infty)$ and $\tilde{A}_{f,g} = f(|A|)Ug(|A|)$.

1. Introduction

Let $\mathbb{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. In the case when $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $M_n$ of all $n \times n$ matrices with entries in the complex field. For an operator $A \in \mathbb{B}(\mathcal{H})$, let $A = U|A|$ ($U$ is a partial isometry with $\ker U = \text{rng}|A|^{1/2}$) be the polar decomposition of $A$. The Aluthge transform of the operator $A$, denoted by $\tilde{A}$, is defined as $\tilde{A} = |A|^{1/2} U|A|^{1/2}$. In [12], Okubo introduced a more general notion called $t$-Aluthge transform which has later been studied also in detail. This is defined for any $0 < t \leq 1$ by $\tilde{A}_t = |A|^{t/2} U|A|^{1-t}$. Clearly, for $t = \frac{1}{2}$ we obtain the usual Aluthge transform. As for the case $t = 1$, the operator $\tilde{A}_1 = |A|U$ is called the Duggal transform of $A \in \mathbb{B}(\mathcal{H})$. For $A \in \mathbb{B}(\mathcal{H})$, we generalize the Aluthge transform of the operator $A$ to the form

$$\tilde{A}_{f,g} = f(|A|)Ug(|A|),$$

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in which \( f, g \) are non-negative continuous functions such that \( f(x)g(x) = x \ (x \geq 0) \).

The numerical radius of \( A \in \mathbb{B}(\mathcal{H}) \) is defined by

\[
w(A) := \sup\{|\langle Ax, x \rangle| : x \in \mathcal{H}, \|x\| = 1\}.
\]

It is well known that \( w(\cdot) \) defines a norm on \( \mathbb{B}(\mathcal{H}) \), which is equivalent to the usual operator norm \( \| \cdot \| \). In fact, for any \( A \in \mathbb{B}(\mathcal{H}) \), \( \frac{1}{2}\|A\| \leq w(A) \leq \|A\| \); see [6]. Let \( r(\cdot) \) denote to the spectral radius. It is well known that for every operator \( A \in \mathbb{B}(\mathcal{H}) \), we have \( r(A) \leq w(A) \). An important inequality for \( \omega(A) \) is the power inequality stating that \( \omega(A^n) \leq \omega(A)^n \ (n = 1, 2, \cdots) \). The quantity \( w(A) \) is useful in studying perturbation, convergence and approximation problems as well as integrative method, etc. For more information see [3, 7, 8, 9] and references therein.

Let \( A, B, C, D \in \mathbb{B}(\mathcal{H}) \). The operator matrices

\[
\begin{bmatrix}
A & 0 \\
0 & D
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & B \\
C & 0
\end{bmatrix}
\]

are called the diagonal and off-diagonal parts of the operator matrix

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

respectively.

In [11], it has been shown that if \( A \) is an operator in \( \mathbb{B}(\mathcal{H}) \), then

\[
w(A) \leq \frac{1}{2} \left( \|A\| + \|A^2\|^{\frac{1}{2}} \right). \tag{1.1}
\]

Several refinements and generalizations of inequality (1.1) have been given; see [1, 4, 14, 15]. Yamazaki [15] showed that for \( A \in \mathbb{B}(\mathcal{H}) \) and \( t \in [0, 1] \) we have

\[
w(A) \leq \frac{1}{2} \left( \|A\| + w(\tilde{A}_t) \right). \tag{1.2}
\]

Davidson and Power [5] proved that if \( A \) and \( B \) are positive operators in \( \mathbb{B}(\mathcal{H}) \), then

\[
\|A + B\| \leq \max\{\|A\|, \|B\|\} + \|AB\|^{\frac{1}{2}}. \tag{1.3}
\]

Inequality (1.3) has been generalized in [2, 13]. In [13], the author extended this inequality to the form

\[
\|A + B^*\| \leq \max\{\|A\|, \|B\|\} + \frac{1}{2} \left( \|A\|\|B^*\|^{1-t} + \|A^*\|^{1-t}\|B\| \right), \tag{1.4}
\]

in which \( A, B \in \mathbb{B}(\mathcal{H}) \) and \( t \in [0, 1] \).

In this paper, by applying the generalized Aluthge transform of operators, we establish some inequalities involving the numerical radius. In particular, we extend inequality (1.2) and (1.4) for two non-negative continuous functions. We also show some upper bounds for the numerical radius of \( 2 \times 2 \) operators matrices.
2. MAIN RESULTS

To prove our numerical radius inequalities, we need several known lemmas.

**Lemma 2.1.** [1, Theorem 2.2] Let $X, Y, S, T \in \mathcal{B}(\mathcal{H})$. Then

$$r(XY + ST) \leq \frac{1}{2} (w(YX) + w(TS)) + \frac{1}{2} \sqrt{(w(YX) - w(TS))^2 + 4\|YS\|\|TX\|}.$$

**Lemma 2.2.** [15, 11] Let $A \in \mathcal{B}(\mathcal{H})$. Then

(a) $w(A) = \max_{\theta \in \mathbb{R}} \|\text{Re} (e^{i\theta} A)\|$.

(b) $w \left( \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right) = \frac{1}{2} \|A\|$.

**Polarization identity:** For all $x, y \in \mathcal{H}$, we have

$$\langle x, y \rangle = \frac{1}{4} \sum_{k=0}^{3} \|x + i^k y\|^2 i^k.$$

Now, we are ready to present our first result. The following theorem shows a generalization of inequality (1.2).

**Theorem 2.3.** Let $A \in \mathcal{B}(\mathcal{H})$ and $f, g$ be two non-negative continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ ($x \geq 0$). Then, for all non-negative non-decreasing convex function $h$ on $[0, \infty)$, we have

$$h(w(A)) \leq \frac{1}{4} \|h \left( g^2 (|A|) \right) + h \left( f^2 (|A|) \right) \| + \frac{1}{2} h \left( w \left( \tilde{A}_{f,g} \right) \right).$$
Proof. Let $x$ be any unit vector. Then

\[
\Re \langle e^{i\theta} Ax, x \rangle = \Re \langle e^{i\theta} U |A|x, x \rangle \\
= \Re \langle e^{i\theta} Ug (|A|) f (|A|) x, x \rangle \\
= \Re \langle e^{i\theta} f (|A|) x, g (|A|) U^* x \rangle \\
= \frac{1}{4} \left\| (e^{i\theta} f (|A|) + g (|A|) U^*) x \right\|^2 - \frac{1}{4} \left\| (e^{i\theta} f (|A|) - g (|A|) U^*) x \right\|^2 \\
\text{(by polarization identity)} \\
\leq \frac{1}{4} \left\| (e^{i\theta} f (|A|) + g (|A|) U^*) x \right\|^2 \\
\leq \frac{1}{4} \left\| (e^{i\theta} f (|A|) + g (|A|) U^*) \right\|^2 \\
= \frac{1}{4} \left\| (e^{i\theta} f (|A|) + g (|A|) U^*) (e^{-i\theta} f (|A|) + U g (|A|)) \right\| \\
= \frac{1}{4} \left\| g^2 (|A|) + f^2 (|A|) + e^{i\theta} \tilde{A}_{f,g} + e^{-i\theta} (\tilde{A}_{f,g})^* \right\| \\
\leq \frac{1}{4} \left\| g^2 (|A|) + f^2 (|A|) \right\| + \frac{1}{4} \left\| e^{i\theta} \tilde{A}_{f,g} + e^{-i\theta} (\tilde{A}_{f,g})^* \right\| \\
= \frac{1}{4} \left\| g^2 (|A|) + f^2 (|A|) \right\| + \frac{1}{2} \left\| \Re (e^{i\theta} \tilde{A}_{f,g}) \right\| \\
\leq \frac{1}{4} \left\| g^2 (|A|) + f^2 (|A|) \right\| + \frac{1}{2} \left( \tilde{A}_{f,g} \right).
\]

Now, taking the supremum over all unit vector $x \in \mathcal{H}$ and applying Lemma 2.2 in the above inequality produces

\[
w (A) \leq \frac{1}{4} \left\| g^2 (|A|) + f^2 (|A|) \right\| + \frac{1}{2} w \left( \tilde{A}_{f,g} \right).
\]
Therefore,
\[
\begin{align*}
  h(w(A)) & \leq h \left( \frac{1}{4} \left\| g^2(|A|) + f^2(|A|) \right\| + \frac{1}{2} w \left( \tilde{A}_{f,g} \right) \right) \\
  & = h \left( \frac{1}{2} \left\| g^2(|A|) + f^2(|A|) \right\| + \frac{1}{2} w \left( \tilde{A}_{f,g} \right) \right) \\
  & \leq \frac{1}{2} h \left( \frac{\left\| g^2(|A|) + f^2(|A|) \right\|}{2} \right) + \frac{1}{2} h \left( w \left( \tilde{A}_{f,g} \right) \right) \\
  & \leq \frac{1}{4} \left\| h \left( g^2(|A|) \right) + h \left( f^2(|A|) \right) \right\| + \frac{1}{2} h \left( w \left( \tilde{A}_{f,g} \right) \right) \\
  & \leq \frac{1}{4} \left\| h \left( g^2(|A|) \right) + h \left( f^2(|A|) \right) \right\| + \frac{1}{2} h \left( w \left( \tilde{A}_{f,g} \right) \right)
\end{align*}
\]
(by the convexity of \( h \))

Then, for all non-negative non-decreasing convex function \( h \) on \([0, \infty)\) and all \( t \in [0, 1] \), we have
\[
  h(w(A)) \leq \frac{1}{4} \left\| h \left( |A|^{2t} \right) + h \left( |A|^{2(1-t)} \right) \right\| + \frac{1}{2} h \left( w \left( \tilde{A}_t \right) \right).
\]  
(2.1)

**Corollary 2.5.** Let \( A \in \mathbb{B}(\mathcal{H}) \). Then, for all \( t \in [0, 1] \) and \( r \geq 1 \), we have
\[
  w^r(A) \leq \frac{1}{4} \left\| |A|^{2r} + |A|^{2(1-t)r} \right\| + \frac{1}{2} w^r \left( \tilde{A}_t \right).
\]

In particular,
\[
  w^r(A) \leq \frac{1}{2} \left( \|A\|^r + w^r \left( \tilde{A} \right) \right).
\]

**Proof.** The first inequality follows from inequality (2.1) for the function \( h(x) = x^r \) (\( r \geq 1 \)). For the particular case, it is enough to put \( t = \frac{1}{2} \). \( \square \)

**Theorem 2.6.** Let \( A, B \in \mathbb{B}(\mathcal{H}) \), \( f, g \) be two non-negative continuous functions on \([0, \infty)\) such that \( f(x)g(x) = x \) (\( x \geq 0 \)) and \( r \geq 1 \). Then
\[
  w^r \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \frac{1}{4} \max \left( \left\| g^{2r}(|A|) + f^{2r}(|A|) \right\|, \left\| g^{2r}(|B|) + f^{2r}(|B|) \right\| \right)
\]
\[
  + \frac{1}{4} \left( \left\| f(|B|)g(|A^*|) \right\|^r + \left\| f(|A|)g(|B^*|) \right\|^r \right).
\]
Proof. Let $A = U|A|$ and $B = V|B|$ be the polar decompositions of $A$ and $B$, respectively and let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. It follows from the polar the composition of $T = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \begin{bmatrix} |B| & 0 \\ 0 & |A| \end{bmatrix}$ that

$$\tilde{T}_{f,g} = f(|T|) \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} g(|T|) = \begin{bmatrix} f(|B|) & 0 \\ 0 & f(|A|) \end{bmatrix} \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \begin{bmatrix} g(|B|) & 0 \\ 0 & g(|A|) \end{bmatrix} = \begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ f(|A|)Vg(|B|) & 0 \end{bmatrix}. $$

Using $|A^*|^2 = AA^* = U|A|^2U^*$ and $|B^*|^2 = BB^* = V|B|^2V^*$ we have $g(|A|) = U^*g(|A^*|)U$ and $g(|B|) = V^*g(|B^*|)V$ for every non-negative continuous function $g$ on $[0, \infty)$. Therefore,

$$w(\tilde{T}_{f,g}) = w\left( \begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ f(|A|)Vg(|B|) & 0 \end{bmatrix} \right) \leq w\left( \begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix} \right) + w\left( \begin{bmatrix} 0 & 0 \\ f(|A|)Vg(|B|) & 0 \end{bmatrix} \right) = w\left( \begin{bmatrix} 0 & f(|B|)Ug(|A|) \\ 0 & 0 \end{bmatrix} \right) + w\left( \begin{bmatrix} 0 & f(|A|)Vg(|B|) \\ 0 & 0 \end{bmatrix} \right) = \frac{1}{2}\|f(|B|)Ug(|A|)\| + \frac{1}{2}\|f(|A|)Vg(|B|)\| \quad \text{(by Lemma 2.1(b))}$$

$$= \frac{1}{2}\|f(|B|)UU^*g(|A^*|)U\| + \frac{1}{2}\|f(|A|)VV^*g(|B^*|)V\| \leq \frac{1}{2}\|f(|B|)g(|A^*|)\| + \frac{1}{2}\|f(|A|)g(|B^*|)\|, \quad \text{(2.2)}$$
where $U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is unitary. Applying Theorem 2.3 and inequality (2.2), we have

$$w^r(T) \leq \frac{1}{4} \left( \| g^{2r}(|T|) + f^{2r}(|T|) \| + \frac{1}{2} \left( w^r(\tilde{T}_{f,g}) \right) \right)$$

$$\leq \frac{1}{4} \max \left( \| g^{2r}(|A|) + f^{2r}(|A|) \| , \| g^{2r}(|B|) + f^{2r}(|B|) \| \right)$$

$$+ \frac{1}{2} \left( \| f(|B|)g(|A^*|) \| + \| f(|A|)g(|B^*|) \| \right)^{r}$$

$$\leq \frac{1}{4} \max \left( \| g^{2r}(|A|) + f^{2r}(|A|) \| , \| g^{2r}(|B|) + f^{2r}(|B|) \| \right)$$

$$+ \frac{1}{4} \| f(|B|)g(|A^*|) \|^{r} + \frac{1}{4} \| f(|A|)g(|B^*|) \|^{r}$$

(by the convexity $h(x) = x^r$).

\begin{proof}
Applying the power inequality of the numerical radius, we have

$$w^r(AB) \leq \frac{1}{4} \max \left( \left\| |A|^{2tr} + |A|^{2(1-t)r} \right\| , \left\| |B|^{2tr} + |B|^{2(1-t)r} \right\| \right)$$

$$+ \frac{1}{4} \left( \left\| |A|^t |B^*|^{1-t} \right\|^{r} + \left\| |B|^t |A^*|^{1-t} \right\|^{r} \right)$$

\end{proof}

Corollary 2.7. Let $A, B \in B(\mathcal{H})$. Then, for all $t \in [0, 1]$ and $r \geq 1$, we have

$$w^r(AB) \leq \frac{1}{4} \max \left( \left\| |A|^{2tr} + |A|^{2(1-t)r} \right\| , \left\| |B|^{2tr} + |B|^{2(1-t)r} \right\| \right)$$

$$+ \frac{1}{4} \left( \left\| |A|^t |B^*|^{1-t} \right\|^{r} + \left\| |B|^t |A^*|^{1-t} \right\|^{r} \right)$$

Proof. Applying the power inequality of the numerical radius, we have

$$w^r(AB) \leq \max \left( w^r(AB) , w^r(BA) \right)$$

$$= w^r \left( \begin{bmatrix} AB & 0 \\ 0 & BA \end{bmatrix} \right)$$

$$= w^r \left( \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \right)^2$$

$$\leq w^r \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right)$$

$$\leq \frac{1}{4} \max \left( \left\| |A|^{2tr} + |A|^{2(1-t)r} \right\| , \left\| |B|^{2tr} + |B|^{2(1-t)r} \right\| \right)$$

$$+ \frac{1}{4} \left( \left\| |A|^t |B^*|^{1-t} \right\|^{r} + \left\| |B|^t |A^*|^{1-t} \right\|^{r} \right)$$

(by Theorem 2.6).
Corollary 2.8. Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be positive operators. Then, for all \( t \in [0, 1] \) and \( r \geq 1 \), we have

\[
\left\| A^{\frac{1}{2}}B^{\frac{1}{2}} \right\|^r \leq \frac{1}{4} \max \left( \left\| A^{tr} + A^{(1-t)r} \right\|, \left\| B^{tr} + B^{2(1-t)r} \right\| \right) \\
+ \frac{1}{4} \left( \left\| A^t B^{1-t} \right\|^r + \left\| B^t A^{1-t} \right\|^r \right).
\]

Proof. Since the spectral radius of any operator is dominated by its numerical radius, then \( r_{AB}^{\frac{1}{2}} \leq w_{AB}^{\frac{1}{2}} \). Applying a commutativity property of the spectral radius, we get

\[
r_{AB}^{\frac{1}{2}} = r_{A^{\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}}}^{\frac{1}{2}}(AB) = \left( A^{\frac{1}{2}}B^{\frac{1}{2}}B^{\frac{1}{2}}A^{\frac{1}{2}} \right)^{\frac{1}{2}}(AB) = \left( A^{\frac{1}{2}}B^{\frac{1}{2}} \left( A^{\frac{1}{2}}B^{\frac{1}{2}} \right)^* \right)^{\frac{1}{2}} \\
= \left\| A^{\frac{1}{2}}B^{\frac{1}{2}} \left( A^{\frac{1}{2}}B^{\frac{1}{2}} \right)^* \right\|^{\frac{1}{2}} \\
= \left\| A^{\frac{1}{2}}B^{\frac{1}{2}} \right\|^r.
\]

(2.3)

Now, the result follows from Corollary 2.7.

An important special case of Theorem 2.6, which refines inequality (1.4) can be stated as follows.

Corollary 2.9. Let \( A, B \in \mathbb{B}(\mathcal{H}) \) and \( r \geq 1 \). Then

\[
\left\| A + B \right\|^r \leq \frac{1}{2^{2-r}} \max \left( \left\| A \right\|^{2tr} + \left\| A \right\|^{2(1-t)r}, \left\| B \right\|^{2tr} + \left\| B \right\|^{2(1-t)r} \right) \\
+ \frac{1}{2^{2-r}} \left( \left\| A^t \right\| \left\| B \right\|^{1-t} \right)^r + \left\| B^t \right\| \left\| A \right\|^{1-t} \right)^r.
\]

In particular, if \( A \) and \( B \) are normal, then

\[
\left\| A + B \right\|^r \leq \frac{1}{2^{1-r}} \max \left( \left\| A \right\|^r, \left\| B \right\|^r \right) + \frac{1}{2^{1-r}} \left\| AB \right\|^r.
\]
Proof. Applying Lemma 2.2 and Theorem 2.3, we have
\[
\| A + B^* \|^r = \| T + T^* \|^r
\]
\[
\leq 2^r \max_{\theta \in \mathbb{R}} \| \text{Re} (e^{i\theta} T) \|^r
\]
\[
= 2^r w^r (T)
\]
\[
\leq \frac{2^r}{4} \left( \| |A|^{2r} + |B|^{2(1-t)r} \|, \| |B|^{2r} + |B|^{2(1-t)r} \| \right)
\]
\[
+ \frac{2^r}{4} \left( \| |A|^t |B^*|^{1-t} \|^r + \| |B|^t |A^*|^{1-t} \|^r \right)
\]
(by Theorem 2.6),

where \( T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \). Now, the desired result follows by replacing \( B \) by \( B^* \). For the particular case, since \( A \) and \( B \) are normal, then \( |B^*| = |B| \) and \( |A^*| = |A| \). Applying equality (2.3) for the operators \(|A|^\frac{1}{2}\) and \(|B|^\frac{1}{2}\), we have
\[
\| |A|^\frac{1}{2} |B|^\frac{1}{2} \|^r = r^\frac{r}{2} (|A| |B|)
\]
\[
\leq \| |A| |B| \| \frac{1}{2}
\]
\[
= \| U^* AB^* V \| \frac{1}{2}
\]
\[
= \| AB^* \| \frac{1}{2},
\]

where \( A = U|A| \) and \( B = V |B| \) are the polar decompositions of the operators \( A \) and \( B \). This completes the proof of the corollary. \( \square \)

In the next result, we show another generalization of inequality (1.2).

**Theorem 2.10.** Let \( A \in \mathbb{B}(\mathcal{H}) \) and \( f, g, h \) be non-negative non-decreasing continuous functions on \([0, \infty)\) such that \( f(x)g(x) = x \ (x \geq 0) \). Then
\[
h \left( w(A) \right) \leq \frac{1}{2} \left( h \left( \tilde{A}_{f,g} \right) + \| h(|A|) \| \right).
\]

**Proof.** Let \( A = U|A| \) be the polar decomposition of \( A \). Then for every \( \theta \in \mathbb{R} \), we have
\[
\| \text{Re} (e^{i\theta} A) \| = r \left( \text{Re} (e^{i\theta} A) \right)
\]
\[
= \frac{1}{2} r \left( e^{i\theta} A + e^{-i\theta} A^* \right)
\]
\[
= \frac{1}{2} r \left( e^{i\theta} U|A| + e^{-i\theta} |A| U^* \right)
\]
\[
= \frac{1}{2} r \left( e^{i\theta} U g(|A|) f(|A|) + e^{-i\theta} f(|A|) g(|A|) U^* \right).
\]

(2.4)
Now, if we put \( X = e^{i\theta} Ug(|A|), \ Y = f(|A|), \ S = e^{-i\theta} f(|A|) \) and \( T = g(|A|)U^* \) in Lemma 2.1, then we get

\[
\begin{align*}
& r(e^{i\theta} Ug(|A|)f(|A|) + e^{-i\theta} f(|A|)g(|A|)U^*) \\
\leq & \frac{1}{2} \left( w(f(|A|)Ug(|A|)) + w(g(|A|)U^*f(|A|)) \right) \\
& + \frac{1}{2} \sqrt{4\|e^{-i\theta} f(|A|)g(|A|)\|\|g(|A|)U^*e^{i\theta} U f(|A|)\|} \\
& \quad \text{(by Lemma 2.1)} \\
\leq & w(f(|A|)Ug(|A|)) + \sqrt{f(|A|)\|f(|A|)\||g(|A|)\|g(|A|)\|} \\
= & w(f(|A|)Ug(|A|)) + \sqrt{f(|A|)\|g(|A|)f(|A|)\|} \\
& \quad \text{(by the functional calculus)} \\
= & w(f(|A|)Ug(|A|)) + \sqrt{|A||A|} \\
= & w\left( \tilde{A}_{f,g} \right) + |A|. \tag{2.5}
\end{align*}
\]

Using inequalities (2.4), (2.5) and Lemma 2.2 we get

\[
\omega(A) = \max_{\theta \in \mathbb{R}} \| \text{Re} (e^{i\theta} A) \| \leq \frac{1}{2} \left( w\left( \tilde{A}_{f,g} \right) + |A| \right).
\]

Hence

\[
\begin{align*}
& h\left( w(A) \right) \leq h\left( \frac{1}{2} \left[ w\left( \tilde{A}_{f,g} \right) + |A| \right] \right) \\
& \quad \text{(by the monotonicity of } h) \\
& \leq \frac{1}{2} h \left( w\left( \tilde{A}_{f,g} \right) \right) + \frac{1}{2} h \left( |A| \right) \\
& \quad \text{(by the convexity of } h) \\
& = \frac{1}{2} h \left( w\left( \tilde{A}_{f,g} \right) \right) + \frac{1}{2} \left| h(|A|) \right|,
\end{align*}
\]

as required. \( \Box \)

**Another proof for Theorem 2.3:** We can obtain Theorem 2.3 from Theorem 2.10. To see this, first note that by the hypotheses of Theorem 2.3 we have

\[
\begin{align*}
h(|A|) &= h(g(|A|)f(|A|)) \\
&\leq h\left( \frac{g^2(|A|) + f^2(|A|)}{2} \right) \quad \text{(by the arithmetic-geometric inequality)} \\
&\leq \frac{1}{2} \left( h\left( g^2(|A|) \right) + h\left( f^2(|A|) \right) \right) \quad \text{(by the convexity of } h). \tag{2.6}
\end{align*}
\]
Hence, using Theorem 2.10 and inequality (2.6) we get
\[
    h(w(A)) \leq \frac{1}{2} \left[ h\left(w\left(\tilde{A}_{f,g}\right)\right) + \|h(|A|)\| \right] \\
    \leq \frac{1}{2} \left[ h\left(w\left(\tilde{A}_{f,g}\right)\right) + \frac{1}{2} \left\| h\left(g^2(|A|)\right) + h\left(f^2(|A|)\right) \right\| \right] \\
    = \frac{1}{2} h\left(w\left(\tilde{A}_{f,g}\right)\right) + \frac{1}{4} \left\| h\left(g^2(|A|)\right) + h\left(f^2(|A|)\right) \right\| .
\]

Remark 2.11. For the special case \( f(x) = x^t \) and \( g = x^{1-t} \) (\( t \in [0,1] \)), we obtain the inequality (1.2)
\[
    w(A) \leq \frac{1}{2} \left( w\left(\tilde{A}_t\right) + \|A\| \right),
\]
where \( A \in B(\mathcal{H}) \) and \( t \in [0,1] \).

Using Theorem 2.10, we get the following result.

**Corollary 2.12.** Let \( A, B \in B(\mathcal{H}) \) and \( f, g \) be two non-negative non-decreasing continuous functions such that \( f(x)g(x) = x \) (\( x \geq 0 \)). Then
\[
    2w^r \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \max\{\|A\|^r, \|B\|^r\} + \frac{1}{2} \left( \|f(|B|)g(|A^*|)\|^r + \|f(|A|)g(|B^*|)\|^r \right),
\]
where \( r \geq 1 \).

**Proof.** Using Theorem 2.10 and inequality (2.2), we have
\[
    2w^r \left( \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) \leq \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\|^r + w^r \left( \tilde{T}_{f,g} \right) \\
    = \max\{\|A\|^r, \|B\|^r\} + \left( \frac{1}{2} \left[ \|f(|B|)g(|A^*|)\| + \|f(|A|)g(|B^*|)\| \right] \right)^r \\
    \leq \max\{\|A\|^r, \|B\|^r\} + \frac{1}{2} \left( \|f(|B|)g(|A^*|)\|^r + \|f(|A|)g(|B^*|)\|^r \right)
\]
and the proof is complete. \( \square \)

**Corollary 2.13.** Let \( A, B \in B(\mathcal{H}) \) and \( f, g \) be two non-negative non-decreasing continuous functions on \( [0, \infty) \) such that \( f(x)g(x) = x \) (\( x \geq 0 \)). Then
\[
    \|A + B\| \leq \max\{\|A\|, \|B\|\} + \frac{1}{2} \left( \|f(|B|)g(|A|)\| + \|f(|A^*|)g(|B^*|)\| \right).
\]
Proof. Let \( T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \). Then

\[
\|A + B^*\| = \|T + T^*\|
\leq 2\max_{\theta \in \mathbb{R}} \|\text{Re} (e^{i\theta}T)\|
= w(T) \quad \text{(by Lemma 2.2)}
\leq \max\{\|A\|, \|B\|\} + \frac{1}{2} \left(\|f(|B|)g(|A^*|)\| + \|f(|A|)g(|B^*|)\|\right) \quad \text{(by Theorem 2.10)}.
\]

If we replace \( B \) by \( B^* \), then we get the desired result. \( \square \)

In the last results, we present some upper bounds for operator matrices. For this purpose, we need the following lemma.

**Lemma 2.14.** \[10, \text{Theorem 1}\] Let \( A \in \mathbb{B}(\mathcal{H}) \) and \( x, y \in \mathcal{H} \) be any vectors. If \( f, g \) are non-negative continuous functions on \([0, \infty)\) which are satisfying the relation \( f(x)g(x) = x \) \((x \geq 0)\), then

\[
|\langle Ax, y \rangle|^2 \leq \langle f^2(|A|)x, x \rangle \langle g^2(|A^*|)y, y \rangle.
\]

**Theorem 2.15.** Let \( A, B, C, D \in \mathbb{B}(\mathcal{H}) \) and \( f_i, g_i \) \((1 \leq i \leq 4)\) be non-negative continuous functions such that \( f_i(x)g_i(x) = x \) \((1 \leq i \leq 4)\) for all \( x \in [0, \infty) \). Then

\[
\omega \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \max \left\{ \|f_1^2(|A|) + g_2^2(|B^*|) + f_3^2(|C|)\|^{\frac{1}{2}}, \|g_4(|D^*|)\| \right\}
+ \max \left\{ \|g_1(|A^*|)\|, \|f_2^2(|B|) + g_3^2(|C^*|) + f_4^2(|D|)\|^{\frac{1}{2}} \right\}.
\]

Proof. Let \( T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) and \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) be a unit vector \( \text{(i.e., } \|x_1\|^2 + \|x_2\|^2 = 1\). \) Then
Let

\[ |\langle Tx, x \rangle| = \left| \left\langle \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \]

\[ = \left| \left\langle \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \right| \]

\[ = |\langle Ax_1, x_1 \rangle + \langle Bx_2, x_1 \rangle + \langle Cx_1, x_2 \rangle + \langle Dx_2, x_2 \rangle| \]

\[ \leq |\langle Ax_1, x_1 \rangle| + |\langle Bx_2, x_1 \rangle| + |\langle Cx_1, x_2 \rangle| + |\langle Dx_2, x_2 \rangle| \]

\[ \left( f_2^2(|A|x_1, x_1) \right)^{\frac{1}{2}} \left( g_3^2(|A^*|)x_1, x_1 \right)^{\frac{1}{2}} + \left( f_2^2(|B|x_2, x_2) \right)^{\frac{1}{2}} \left( g_2^2(|B^*|)x_1, x_1 \right)^{\frac{1}{2}} \]

\[ + \left( f_2^2(|C|x_1, x_1) \right)^{\frac{1}{2}} \left( g_2^2(|C^*|)x_2, x_2 \right)^{\frac{1}{2}} + \left( f_2^2(|D|x_2, x_2) \right)^{\frac{1}{2}} \left( g_1^2(|D^*|)x_2, x_2 \right)^{\frac{1}{2}} \]

\[ \leq \left( (f_2^2(|A|x_1, x_1) + g_3^2(|B^*|)x_1, x_1) + \left( f_2^2(|C|x_1, x_1) + g_2^2(|C^*|)x_2, x_2 \right) \right)^{\frac{1}{2}} \]

(by the Cauchy-Schwarz inequality)

\[ + \left( g_2^2(|A^*|)x_1, x_1 \right) + \left( f_2^2(|B|)x_2, x_2 \right) + \left( g_3^2(|C^*|)x_2, x_2 \right) + \left( f_2^2(|D|)x_1, x_1 \right) \right)^{\frac{1}{2}} \]

\[ = \left( \left( f_2^2(|A|) + g_3^2(|B^*|) + f_3^2(|C|) \right)x_1, x_1 \right) + \left( g_2^2(|B^*|) + f_2^2(|D|) \right)x_2, x_2 \right) \}

\[ + \left( g_2^2(|A^*|)x_1, x_1 \right) \right)^{\frac{1}{2}} \]

\[ \leq \left( f_2^2(|A|) + g_2^2(|B^*|) + f_3^2(|C|) \right) \|x_1\|^2 + \left( g_2^2(|D^*|) \right) \|x_2\|^2 + \left( g_1^2(|A^*|) \right) \|x_1\|^2 \]

Let

\[ \alpha = \|f_2^2(|A|) + g_2^2(|B^*|) + f_3^2(|C|) \|, \quad \beta = \|g_2^2(|D^*|) \|, \quad \mu = \|f_2^2(|B|) + g_3^2(|C^*|) + f_4^2(|D|) \| \]

and \( \lambda = \|g_1^2(|A^*|) \| \).

It follows from

\[ \max_{\|x_1\|^2 + \|x_2\|^2 = 1} \left( \alpha \|x_1\|^2 + \beta \|x_2\|^2 \right) = \max_{\theta \in [0, 2\pi]} \left( \alpha \sin^2 \theta + \beta \cos^2 \theta \right) = \max \{\alpha, \beta\} \]

and

\[ \max_{\|x_1\|^2 + \|x_2\|^2 = 1} \left( \lambda \|x_1\|^2 + \mu \|x_2\|^2 \right) = \max_{\theta \in [0, 2\pi]} \left( \lambda \sin^2 \theta + \mu \cos^2 \theta \right) = \max \{\lambda, \mu\} \]
that

\[ |\langle Tx, x \rangle| \leq \left( \| f_1^2(|A|) + g_2^2(|B^*|) + f_2^2(|C|) \| \| x_1 \|^2 + \| g_2^2(|D^*|) \| \| x_2 \|^2 \right)^{\frac{1}{2}} \]

\[ + \left( \| f_3^2(|B|) + g_3^2(|C^*|) + f_4^2(|D|) \| \| x_2 \|^2 + \| g_2^2(|A^*|) \| \| x_1 \|^2 \right)^{\frac{1}{2}} \]

\[ \leq \max \left\{ \| f_1^2(|A|) + g_2^2(|B^*|) + f_2^2(|C|) \|^\frac{1}{2}, \| g_2^2(|D^*|) \|^\frac{1}{2} \right\} \]

\[ + \max \left\{ \| g_2^2(|A^*|) \|^\frac{1}{2}, \| f_2^2(|B|) + g_3^2(|C^*|) + f_4^2(|D|) \|^\frac{1}{2} \right\} \]

\[ = \max \left\{ \| f_1^2(|A|) + g_2^2(|B^*|) + f_2^2(|C|) \|^\frac{1}{2}, \| g_4(|D^*|) \| \right\} \]

\[ + \max \left\{ \| g_1(|A^*|) \|, \| f_2^2(|B|) + g_3^2(|C^*|) + f_4^2(|D|) \|^\frac{1}{2} \right\}. \]

Taking the supremum over all unit vectors \( x \) we get the desired result. \( \square \)

**Corollary 2.16.** Let \( A, B, C, D \in \mathbb{B}(\mathcal{H}) \). Then

\[
\omega \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) \leq \max \left\{ \| |A|^{2\alpha} + |B^*|^{2\gamma} + |C|^{2\mu} \|^\frac{1}{2}, \| |D^*| \| \right\} \]

\[ + \max \left\{ \| |A^*|^{\beta} \|, \| |B|^{2\zeta} + |C^*|^{2\nu} + |D|^{\kappa} \| \right\}, \]

where \( \alpha + \beta = \gamma + \zeta = \mu + \nu = \omega + \kappa = 1 \). In particular,

\[
\omega \left( \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \right) \leq \max \left\{ \| |A^*|^{\beta} \|, \| |B|^{\zeta} \| \right\} + \| |A|^{2\alpha} + |B^*|^{2\gamma} \|^\frac{1}{2}, \]

in which \( \alpha + \beta = \gamma + \zeta = 1 \).

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