Analytically solvable $2 \times 2$ PT-symmetry dynamics from su(1,1)-symmetry problems

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A protocol for explicitly constructing the exact time-evolution operators generated by $2 \times 2$ time-dependent PT-symmetry Hamiltonians is reported. Its mathematical applicability is illustrated with the help of appropriate examples. The physical relevance of the proposed approach within gain-loss system scenarios, like two-coupled wave-guides, is discussed in detail.

I. INTRODUCTION

Several decades ago Feshbach described the dynamics of a quantum physical system coupled with a continuum of states, revealing the potentiality of non-Hermitian matrices in Physics [1]. Such an effective approach allowed to bring to light and understand a lot of interesting and peculiar physical features of this kind of open systems [2]. In this respect, to investigate the mathematical properties of these non-Hermitian matrices shed light on the coupling between the system and the surrounding environment [2]. Such matrices are now called pseudo-Hermitian matrices [3], that is, matrices possessing either an entire real spectrum (called in this case quasi-Hermitian matrices) or complex-conjugated eigenvalues [3].

The quasi-Hermitian operators commutating with the operator $PT$ ($P$ and $T$ accomplishing the inversion of spatial and time variable, respectively) have been called PT-symmetry Hamiltonians [4]. Their discovery has paved the way towards new (non-Hermitian) Quantum Physics [5]. From an experimental point of view, indeed, PT-symmetry can be established by controllably and appropriately incorporating gain and loss in the system. Its physical realizability and related interesting applications appear in several fields: photonics [6], optics, electronics, microwaves mechanics, acoustics and atomic systems ([5] and references therein).

This work deals with physical systems living in a two-dimensional Hilbert space and describable by $2 \times 2$ time-dependent quasi-Hermitian PT-symmetry matrices. Despite its apparent simplicity, the two-level case goes beyond a simple toy model. A wide variety of problems may be traced back to that of a two-state dynamics effectively simulating the most relevant changes happening in complicated quantum-mechanical systems, e.g. in nuclear magnetic resonance [7], quantum information processing [8] and polarization optics [9]. In the PT-symmetry cases, investigated in this paper, the two-level model describes a general sink-source or gain-loss system. It proves to be very useful for the comprehension of basic theoretical concepts [10] and of many experimental results, e.g. [11] and [12]. Other physical systems, e.g. coupled waveguides [13–15], are exactly described by a two-dimensional PT-symmetry Hamiltonian; in such cases, like in other photonic structures [16], the dynamics of the physical system is governed by a Schrödinger-like equation where the time variable is substituted with the spatial one (the propagation direction).

The matrix representation of a sink-source system may be cast as follows

$$\hat{H} = \begin{pmatrix} re^{-i\theta} & \gamma \\ \gamma & re^{i\theta} \end{pmatrix},$$

(1)

where $r$ and $\theta$ are real. The diagonal entry $re^{-i\theta}$ ($re^{i\theta}$) describes the energy time evolution in the sink-source [10]. The off-diagonal parameter $\gamma$ may be interpreted as the coupling existing between the one-state sink and the one-state source [10].

It has been highlighted [10] that this two-box model is able to capture in its parameter space the passage from a condition where the exchange of energy between the two boxes takes place enabling the system to reach equilibrium to the opposite physical situation. The existence of such a radical change has been experimentally realized by parametrically changing the coupling constant $\gamma$ [10–12, 14, 15] and may be interpreted as an evidence of the transition from unbroken to broken PT-symmetry phase.

In this paper we wish to deal with time-dependent PT-symmetry Hamiltonian problems. To this end we introduce a prescribed time-variation of the coupling constant $\gamma$, turning the Hamiltonian model (1) into

$$\hat{H}(t) = \begin{pmatrix} re^{-i\theta} & \gamma(t) \\ \gamma(t) & re^{i\theta} \end{pmatrix},$$

(2)

Investigating this kind of problems may find applications whenever one is interested in controlling a gain-loss physical system, representable by the Hamiltonian (2). To find the time-evolution operator generated by $\hat{H}(t)$ is, generally speaking, a challenging problem strongly dependent on the off-diagonal time-dependent element. For this reason we search a general mathematical protocol providing examples of time-dependent coupling parameters leading to the prediction of the exact quantum dynamics of the corresponding physical system. To this end, in the following, strategically, we construct a tool aimed at solving the general dynamical problem generated by the class of $2 \times 2$ quasi-Hermitian su(1,1) Hamiltonians. This procedure is immediately exploitable for treating dynamical problems characterized by PT-symmetry since
they constitute a sub-class of the more general class of bi-dimensional su(1,1) dynamical problems [17].

In this respect it is worth noting that an approach aimed at individuating exactly solvable two-dimensional su(2) dynamical problems (generally describing a two-level system subjected to time-dependent external driving forces), has been recently presented [18]. Such a method allowed to find several remarkable exact solutions for $2 \times 2$ dynamical systems [19, 20] as well as deep connections with mathematical issues [21]. Moreover, it has proved very useful also for the exact solution of more complex dynamical problems, as interacting qudit systems [22–25].

On the basis of the result [18] and taking account of the ‘affinity’ existing between the SU(2) and the SU(1,1) symmetry-groups (both are sub-groups of the more general SL(2, C) group), we extend the constructive method, successful for su(2) problems, to the su(1,1) case. Thus, what is reported in the following possesses a twofold interest. We identify classes of exactly solvable dynamical problems governed by time-dependent bi-dimensional su(1,1) Hamiltonians, which is physically relevant in its own within the framework of pseudo- and quasi-Hermitian matrices [17]. Moreover, through this general procedure, we reach the main goal of this paper, getting analytically solvable PT-symmetry dynamics with direct physical meaning thanks to the application to the general sink-source model and the gain-loss wave-guide scenarios.

The paper is organized as follows. In Sec. II the class of $2 \times 2$ su(1,1)-symmetry matrices is introduced as well as the generalized von Neumann-Liouville equation related to statistically interpretable dynamics of gain-loss systems described by non-Hermitian Hamiltonians. Within this context, a parametric solution of the Schrödinger equation and the related evolution operator $U(t)$ generated by the Hamiltonian (5) as a solution of the Schrödinger equation $iU(t) = H(t)U(t)$ ($\hbar = 1$) is an element of SU(1,1), and is parametrized in terms of Cayley-Klein parameters according to

$$U(t) = \begin{pmatrix} a(t) & -b(t) \\ -b^*(t) & a^*(t) \end{pmatrix}. $$

The entries $a(t)$ and $b(t)$ are complex-valued functions and satisfy the relation $\det[U] = |a(t)|^2 - |b(t)|^2 = 1$. This implies that $|a(t)|$ and $|b(t)|$ can be greater than one, unlike what occurs in the SU(2) model [27]. Then they do not have a direct probabilistic physical meaning. The probability interpretation must be reconsidered in the SU(1,1) symmetric model since we are addressing a Hamiltonian model with a non-euclidean metric. To solve this issue it is useful to consider a new non linear equation of motion proposed for the time evolution of the density matrix of a physical system described by a non-Hermitian Hamiltonians $H(t) = K(t) - i\Gamma(t)$ [28], namely

$$\rho(t) = -i[K(t), \rho(t)] - \{\Gamma(t), \rho(t)\} + 2\rho(t)\text{Tr}[\rho(t)\Gamma(t)], $$

where $K(t)$ and $\Gamma(t)$ are self-adjoint operators. The general solution of Eq. (7) can be expressed, in terms of the evolution operator $U$ that satisfies the Schrödinger equation $iU(t) = H(t)U(t)$, as

$$\rho(t) = \frac{U(t)\rho(0)U^*(t)}{\text{Tr}[U(t)\rho(0)U^*(t)]},$$

A null-trace su(1,1) matrix that depends on a real parameter $t$, given by a linear combination, with real $t$-dependent coefficients, of the three generators $\hat{K}^0$, $\hat{K}^1$ and $\hat{K}^2$ of the su(1,1) algebra, in terms of Pauli matrices, reads

$$H(t) = \Omega(t)\hat{\sigma}^+ + i\omega_\gamma(t)\hat{\sigma}^- - i\omega_x(t)\hat{\sigma}^y. $$

In the canonical basis $\{\{\pm\}, \{\varphi\}\}$ (where $\varphi \in [\pm \frac{\pi}{2}]$), the matrix representation of such an operator has the form

$$H(t) = \begin{pmatrix} \Omega(t) & -\omega_\gamma(t) \\ \omega_\gamma(t) & \Omega(t) \end{pmatrix}, $$

where $\Omega(t) = \omega_x(t) - i\omega_\gamma(t) = |\omega(t)|e^{i\phi_\omega(t)}$ is a real (generally complex) function of $t$. Comparing (5) with (1), it is easy to see that the subclass of PT-symmetric su(1,1) Hamiltonians may be obtained by putting $\phi_\omega = \pm \pi/2$ or equivalently by $\omega_\gamma = 0$ (up to a unitary transformation consisting in a rotation of $\pi/2$ with respect to the $\gamma$ direction, disregarding the constant term in the Hamiltonian (1) that has no relevance in the dynamics). In the su(1,1) case, thus, the coupling parameter $\gamma$ between the sink and the source is played by the parameter $\Omega$.

The su(1,1) matrices are pseudo-Hermitian Hamiltonians [3], since there exists one linear hermitian (non-singular) matrix $\eta = \{+, +\} \{-, -\}$ such that $H^\dagger(t) = \eta H(t)\eta^{-1}$. Moreover, in the region of the Hamiltonian parameter space defined by $|\omega|^2 < \Omega^2$ the su(1,1) Hamiltonian possesses a real spectrum, implying that in such a parameter region it is quasi-Hermitian [17]. The nonunitary evolution operator $U(t)$ generated by the Hamiltonian (5) as a solution of the Schrödinger equation $iU(t) = H(t)U(t)$ ($\hbar = 1$) is an element of SU(1,1), and is parametrized in terms of Cayley-Klein parameters according to

$$A.2 \times 2$ su(1,1)-symmetry matrices, generalized von-Neumann-Liouville equation and exact solution of the related dynamical problem
and it possesses ‘good’ physical properties that allow possible physical interpretations of the mathematical results [28, 29]. Such an approach has been used to interpret experimental results reported in [30].

A. Parametric Solutions of the su(1,1) Dynamical Problem

It is important to observe that the dynamical problem described by Eq. (8) can be reduced to the solution of the Schrödinger equation $i\dot{U} = HU$, giving rise to a system of linear differential equations which may be put in the form

$$\Omega = i[\dot{a}a^* - \dot{b}b^*], \quad \omega = i[-\dot{a}b + \dot{b}a]. \quad (9)$$

Hereafter we omit the explicit time-dependences, if not necessary. Following the approach reported in Ref. [18], with appropriate changes to the class of $2 \times 2$ su(1,1) matrices (see Appendix A), the two entries $a$ and $b$ defining the time evolution matrix $U$ in Eq. (6) may be represented as follows

$$a = \cosh[\Lambda_\phi] \exp\left[ i\left(\frac{\phi_0 - \phi_0(0)}{2} - \frac{\Theta}{2} - \mathcal{R}\right)\right], \quad (10a)$$

$$b = -i \sinh[\Lambda_\phi] \exp\left[ i\left(\frac{\phi_0 + \phi_0(0)}{2} - \frac{\Theta}{2} + \mathcal{R}\right)\right], \quad (10b)$$

where

$$\Lambda_\phi = \int_0^t |\omega| \cos[\Theta] \, t', \quad (11a)$$

$$\mathcal{R} = \int_0^t |\omega| \sin[\Theta] \, t'. \quad (11b)$$

$\Theta$ is an arbitrary, real, time-dependent function such that $\Theta(0) = 0$ and $\Omega$ and $\omega$ appearing in $H$ are related through the following relation

$$\dot{\Theta} + 2 |\omega| \sin[\Theta] \coth[2\Lambda_\phi] = 2\Omega + \dot{\phi}_0. \quad (12)$$

Equation (12) practically serves as a recipe in the sense that, choosing at will the function $\Theta(0) = 0$, it determines $\Omega$ and $\omega$ in terms of $\Theta$ and $\omega$ making the dynamical problem (9) exactly solvable.

We underline that Eq. (12) provides a possible time variation of the parameter $\gamma(t)$ in Eq. (1) so that the dynamical problem may be exactly solved. Indeed, we recall that, by setting $\phi_0(t) = \pm \pi/2$ in our framework, $\Omega$ and $|\omega|$ play the role of $\gamma$ and $|\sin \theta|$, respectively (the sign of the Hamiltonian parameter $\sin \theta$, depending on the value of $\theta$, is taken into account by the appropriate choice of $\phi_0$). Summing up, the constraint $|\sin \theta| = \text{const}$ (meaning $|\omega(t)| = \text{const}$), leads, through $\Theta$, to a $t$-dependent expression of $\gamma$ making solvable the related dynamical problem.

We emphasize also that Eqs. (10) and (12) are parametric solutions of the Schrödinger equation and this means that they are valid and may be reinterpreted also for problems whose dynamics is ruled by a Schrödinger-like equation. In guided wave optics, for example, a space-dependent Schrödinger equation appears [14, 15]; in such a case the space variable represents the propagation direction of the waves in the guides and a space-dependent coupling may be reached by varying the distance between the guides [16]. Thus, e.g. for the physical systems studied in Refs. [14, 15], we may interpret Eq. (12) as a prescription how to vary over space the coupling between the guides in such a way to have an exactly solvable system of space-dependent differential equations for the amplitudes of the waves propagating in the guides. Given an initial condition of the amplitudes, the latter may be written at a certain space point in terms of the solutions in Eqs. (10), provided that the time variable is appropriately substituted by the spatial one.

III. EXACT PT-SYMMETRY EXAMPLES

In the following examples we consider $PT$-symmetry cases, that is, we set $\phi_0 = \pi/2$.

A. Example 1

If we choose such a parameter $\Theta$ that, given $|\omega|$, satisfies

$$\int_0^\tau |\omega| \cos[\Theta] \, t' = \frac{1}{2} \arcsinh|\kappa|, \quad \kappa = 2 \int_0^\tau |\omega| \, t', \quad (13)$$

from Eqs. (10) we get

$$a = \sqrt{1 + \kappa^2 + 1} \exp\left[ -i \left(\frac{\Theta}{2} + \mathcal{R}\right)\right], \quad (14a)$$

$$b = -\sqrt{1 + \kappa^2 - 1} \exp\left[ -i \left(\frac{\Theta}{2} - \mathcal{R}\right)\right], \quad (14b)$$

where

$$\mathcal{R} = \frac{\arcsinh|\kappa|}{2}. \quad (15)$$

In this instance, we actually have $\Theta = \arctan|\kappa|$ and the relation between the Hamiltonian parameters reads

$$\Omega = \frac{|\omega|}{2} \left[2 + \frac{1}{1 + \kappa^2}\right]. \quad (16)$$

In the framework of gain-loss models we may interpret this relation as a prescription how to vary over time the coupling parameter $\rho$ in Eq. (1) between the sink and the source in such a way that the dynamical problem is solved by Eqs. (14). Supposing $|\sin \theta| = \text{const}$ (in this case $0 < \theta < \pi/2$, in view of the choice $\phi_0 = \pi/2$; for $\pi/2 < \theta < \pi$ we have to set $\phi_0 = -\pi/2$), we may write Eq. (16) as

$$\gamma(t) = \frac{|\sin \theta|}{2} \left[2 + \frac{1}{1 + \tau^2}\right], \quad (17)$$

with $\tau = r \sin(\theta) \, t$. The parameter $\gamma$ is plotted in Fig. 1 as a function of the dimensionless parameter $\tau$. 
If we suppose our system initially prepared in the state \( \rho_0 = |−\rangle \langle −| \), the probability of finding it in the opposite state, according to Eq. (8), is

\[
P^+ = \rho_{11} = \frac{|b|^2}{1 + 2|b|^2},
\]

where \( \rho_{11} \) is the (1,1)-element of the matrix \( \rho = U \rho_0 U^\dagger / \text{Tr} \{ U \rho_0 U^\dagger \} \), solution of the equation (7). The plot of \( P^+ \) is reported in Fig. 1b against the dimensionless time \( \tau \). We note that the value reached asymptotically by the transition probability is 1/2. By Eq. (18) we see, indeed, that 1/2 is the maximum value reachable by the transition probability, precisely when \( |b| \gg 1 \).

A direct physical application of current interest may be found in the framework of coupled waveguides. In Ref. [14], for example, a PT-symmetry physical scenario of optical beam propagation is investigated. Under appropriate conditions, the optical-field dynamics of the two coupled wave-guides is described by the following space-dependent Schrödinger-like equation

\[
\frac{d}{dz} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} i \epsilon & -\gamma \\ -\gamma & -i \epsilon \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix},
\]

where \( E_{1,2} \) are the field amplitudes in the first and the second guide, respectively, \( \epsilon \) is the effective gain coefficient, \( \gamma \) is the coupling constant and \( z \) represents the one-dimensional location of the signals in the guides. In our case, we suppose a spatial configuration of the wave-guides such that the coupling parameter exhibits a spatial-dependence \( \gamma = \gamma(z) \). It is easy to see that, if we apply a unitary transformation accomplishing \( \hat{\sigma}_c \to \hat{\sigma}_c, \hat{\sigma}_c \to -\hat{\sigma}_c \), the PT-symmetry ‘Hamiltonian’ governing the dynamics of the optical system becomes of the form (5). In this instance, then, Eq. (16), reading

\[
\gamma(\epsilon, z) = \frac{\epsilon}{2} \left[ 2 + \frac{1}{1 + (\epsilon z)^2} \right],
\]

gives us a prescription how to spatially vary the coupling parameter between the two guides, in terms of the constant gain parameter, so that the system (19) may be analytically solved.

In such a case the probability \( P^+ \) represents the possibility to transfer the signal in the second wave-guide when it is initially injected in the first one. The space-dependence of the coupling parameter we are discussing has, then, the effect of asymptotically transferring half part of the initial optical signal from the first channel to the second one.

**B. Example 2**

By putting instead

\[
\int_0^\infty |\omega| \cos[\Theta] \, d\omega' = \arcsin[k/2]
\]

it is easy to verify that we get exactly the same expression for \( \Theta \) written in Eq. (15), with \( \kappa \) replaced by \( \kappa/2 \), while \( \Re \) results

\[
\Re = \frac{1}{2} \arctan[k/2].
\]

In this case we have

\[
a = \sqrt{1 + \frac{k^2}{4}} \exp \left[ -i \left( \frac{\Theta}{2} + \Re \right) \right],
\]

\[
b = \frac{k}{2} \exp \left[ -i \left( \frac{\Theta}{2} - \Re \right) \right],
\]

and the necessary condition that the Hamiltonian parameters must satisfy results

\[
\Omega = |\omega|.
\]

We see that this second choice turns out to be a trivial case. However, two interesting observations may be developed. Firstly, it is worth noticing that, developing the same calculation within the framework of su(1,1) matrices, Eq. (24) becomes \( 2 \Omega + \phi_0 = 2 |\omega| \), which, in turn, is a particular case of the more general solvability condition \( 2 \Omega + \phi_0 = 2 \nu |\omega| \) found in [17] for the class of su(1,1) matrices. It is possible to show that, within the class of su(1,1) matrices, this more general relation may be derived through our approach as reported in Appendix B. Secondly, the last example shows how a slight change in the choice of the function \( \Theta \) might lead us to a significantly different scenario and then to a substantially different dynamics of the physical system. This fact, then, underlines the potentiality of the method [18], here proposed for su(1,1) matrices, in identifying exactly solvable scenarios of possible experimental interest.

Before closing this section, we emphasize that in the examples discussed before we were able to find the closed form of all the necessary quantities in order to solve the dynamical problem. However, if we were interested only in specific physical observables, it might happen that we are requested to find the explicit form of only few quantities. For example, if we were interested in the study of the transition probability \( P^+ \) or in the knowledge of

\[
\langle \hat{\sigma}^\dagger \hat{\sigma} \rangle = \text{Tr} \{ \rho \hat{\sigma}^\dagger \hat{\sigma} \} = -1/(|a|^2 + |b|^2),
\]

\[
\langle \hat{\sigma}_0^\dagger \hat{\sigma}_0 \rangle = \sqrt{\frac{\kappa^2}{1 + \kappa^2}} \cos[\phi_0(t) - \Theta(t) - \pi/2].
\]
for \( \rho(0) = \ket{-}\bra{-} \), it is sufficient to analytically write the expression of \( |a\rangle \) and \( |b\rangle \) (\( \Theta \) is chosen at will). In this way, we are not obliged to solve analytically the expression of the integral \( \mathcal{R} \) (Eq. (11b)) involved in the exponentials in Eqs. (14b) and (23b), which results in some cases very hard to solve. Thus, this fact means that, concentrating only on specific physical quantities, the choices of \( \Theta \) we may perform and then the classes of exactly solvable models we may identify become wider and wider. For example, if we choose \( |\omega\rangle = |\omega_0\rangle \cos^2(\tau) \) and \( \Theta = \tau \) with \( \tau = |\omega_0| \tau \), although we made relatively simple choices, we are not able to find the analytical expression of the integral \( \mathcal{R} \) in Eq. (11b). However, we may derive the exact form of the transition probability \( P_+^\tau \) in Eq. (18) plotted in Fig. 2a and, through Eq. (12), we may write the exact time-dependence of \( \Omega \) plotted in Fig. 2b for \( \phi_0 = 0 \).

![Figure 2](https://example.com/figure2.png)

Figure 2: (Color online) Time dependence of a) the transition probability \( P_+^\tau \) and b) the parameter \( \Omega \) against the dimensionless time \( \tau = |\omega_0| \tau \) for \( |\omega\rangle = |\omega_0\rangle \cos^2(\tau) \), \( \Theta = \tau \) and \( \phi_0 = 0 \).

IV. CONCLUSIONS

In conclusion, getting inspired by previous results in other contexts [18], we have developed a protocol through which we found a parametrization for the solutions of the dynamical problem related to two-dimensional time-dependent quasi-Hermitian su(1,1) Hamiltonians. Such a result turns out to be of physical interest at the light of the fact that \( 2 \times 2 \) PT-symmetry Hamiltonians, describing sink-source or gain-loss systems [10], are a special sub-class of the su(1,1) matrices [17].

This fact allows us to interpret transparently the solvability condition [Eq. (12)] of the dynamical problem from a physical point of view. Such a relation may be read as the prescription how to vary over time the coupling between the sink and the source in order to controllably drive the dynamics of the whole gain-loss system. Moreover, we brought to light also the relevance of the result in guided wave optics scenarios [13–16]. In such cases, the dynamical problem is converted in a space-dependent one obeying to a space-dependent Schrödinger-like equation. Thus, our solutions, provided that the time variable is substituted with the spatial one, are still valid, getting prescriptions how to vary the coupling (space distance) between the wave-guides as to have an analytically solvable model. The applicability and versatility of our method have been showed with two illustrative examples and by deriving in new terms a previous result recently reported in literature [17].

Finally, we emphasize that the parametrized solutions of a \( 2 \times 2 \) su(1,1) dynamical problem here reported possess a more general value. Indeed, as it happens in the su(2) dynamical case, a higher dimensional su(1,1) dynamical problem may be reduced to the \( 2 \times 2 \) one and its solution may be written in terms of the two parameters \( a \) and \( b \) defining the Cayley-Klein parametrization of the evolution operator in Eq. (6) [31].

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Appendix A

In the following we apply the method reported in ref. [18] to the class of \( 2 \times 2 \) su(1,1) matrices. The Schrödinger equation \( i\dot{U} = HU \), with \( H \) and \( U \) defined in Eq. (5) and (6), respectively, gives rise to a system of linear differential equations which may be put in the form

\[
\Omega = i[\dot{a}a^* - \dot{b}b^*], \quad \omega = i[-\dot{a}b + \dot{b}a].
\] (A1)

Let us introduce the following function

\[
X = \int_0^t \frac{\Omega}{\dot{a}^2} dt'.
\] (A2)

By such a position and by the second equation in (A1), we may write respectively

\[
\omega = a^2X, \quad b = -i\dot{a}X.
\] (A3)

Thus, the relation which the two functions \( a \) and \( b \) have to satisfy reads \( |a|^2|1 - |X|^2| = 1 \), implying \( |X|^2 \leq 1 \). In this way, the first equation in (9) becomes a closed integral-differential equation for \( a \), namely

\[
\dot{a} = \left(-i\Omega + \frac{XX^*}{1 - |X|^2}\right)a,
\] (A4)

which is solved to yield

\[
a = \frac{1}{|1 - |X|^2|^{1/2}} \exp \left[-i \int_0^t \Omega dt' + i \int_0^t \frac{\text{Im}[XX^*]}{1 - |X|^2} dt' \right].
\] (A5)

Let us consider an arbitrary complex function \( X \) in the form

\[
X = A \exp[i\phi], \quad A(0) = 0,
\] (A6)
with $\phi$ and $A$ real functions of time. The latter must satisfy the condition $A^2 \leq 1$ due to the condition $|X|^2 \leq 1$. The function $\phi(t)$ has to satisfy

$$\dot{\phi} = \frac{A}{A} \tan[\Theta],$$  \hfill (A7)

where $\Theta$ is a real function of time $t$, defined by

$$\Theta = \phi_0 + \phi + 2 \int_0^t \Omega \, dt' - 2 \int_0^t \frac{\phi}{1-A^2} \, dt' - 2\phi(0).$$  \hfill (A8)

By Eq. (A7) we derive $A^2 + \dot{\phi}^2 A^2 = A^2 (1 + \tan^2[\Theta]) = |\omega|^2 (1 - |X|^2)$, implying

$$A = \tanh[\Lambda_\theta], \quad \dot{\phi} = \frac{2|\omega|\sin[\Theta]}{\sinh[2\Lambda_\theta]},$$  \hfill (A9)

with

$$\Lambda_\theta = \int_0^t |\omega| \cos[\Theta] \, dt',$$  \hfill (A10)

so that we have to put $\sin[\Theta(0)] = 0$, which is assured by assuming $\phi_0(0) = \phi(0)$, if we want to keep parameters well behaved. We see that the function $A$ in Eq. (A9) satisfies the condition $A^2 \leq 1$. From the equations in (A9) we find that $\Theta$ must satisfy the following integral-differential equation

$$\Theta + 2|\omega|\sin[\Theta] \coth[2\Lambda_\theta] = 2\Omega + \phi_0.$$  \hfill (A11)

The solutions $a$ and $b$ may be written as

$$a = \cosh[\Lambda_\omega] \exp[i \left( \frac{\phi_\omega - \phi_\omega(0)}{2} - \frac{\Theta}{2} - \mathscr{R} \right)],$$  \hfill (A12a)
$$b = -i \sinh[\Lambda_\omega] \exp[i \left( \frac{\phi_\omega + \phi_\omega(0)}{2} - \frac{\Theta}{2} + \mathscr{R} \right)],$$  \hfill (A12b)

with

$$\mathscr{R} = \int_0^t |\omega| \sin[\Theta] \, dt',$$  \hfill (A13)

and putting $\Theta(0) = 0$.

### Appendix B

To recover the solvability condition $\Omega(t) + \phi_\omega(t)/2 = v|\omega(t)|$, let us assume the function $X$ as

$$X_v = v^{-1} \sin[\phi_v] \exp\left[i(\phi_v + \phi_\omega^0)\right], \quad \phi_v(0) = 0,$$  \hfill (B1)

with $\phi_\omega^0 = \text{const.}$, and then it is necessary that $\sin^2\phi_v(t) < v^2$. Under this assumption and according to the general theory, the transverse field has to be put

$$\omega = \frac{v\phi_v}{v^2 - \sin^2\phi_v}$$

$$\times \exp\left\{-2i \int_0^t \Omega \, dt' + 2i \int_0^t \frac{v^2\phi_v}{v^2 - \sin^2\phi_v} \, dt' + i\phi_\omega^0\right\}.$$  \hfill (B2)

Assuming $v$ and $\dot{\phi}$ positive we may write

$$|\omega| = \frac{v\phi_v}{v^2 - \sin^2\phi_v}$$  \hfill (B3)

and from Eq. (B2) it is possible to derive

$$2\Omega + \phi_\omega = 2v|\omega|,$$  \hfill (B4)

being nothing but the relation we were looking for, got through an other approach in Ref. [17] where its physical reason has been brought to light. This relation is valid for the dynamical regimes $v < 1$ and $v \geq 1$ with $v \geq 0$ and the consistency of the procedure leads to the following expression for $\phi_v$

$$\phi_v = \arctan \left[ \frac{v}{\sqrt{v^2 - 1}} \tan \left( \sqrt{v^2 - 1} \int_0^t |\omega| \, dt' \right) \right].$$  \hfill (B5)

For this case the solutions $\alpha_v$ and $b_v$ can be constructed and acquire different expressions depending on the value of $v$. In the regime $v > 1$, we get

$$\alpha_v = \left[ \cos[\Lambda_v] - i \frac{v}{\sqrt{v^2 - 1}} \sin[\Lambda_v] \right] \exp\left\{ i \frac{\phi_\omega - \phi_\omega^0}{2} \right\},$$  \hfill (B6a)
$$b_v = -i \sin[\Lambda_v] \exp\left\{ i \frac{\phi_\omega + \phi_\omega^0}{2} \right\},$$  \hfill (B6b)

while, in the regime $0 \leq v < 1$, we have

$$\alpha_v = \left[ \cosh[\Lambda_v'] - i \frac{v}{\sqrt{1-v^2}} \sinh[\Lambda_v'] \right] \exp\left\{ i \frac{\phi_\omega - \phi_\omega^0}{2} \right\},$$  \hfill (B7a)
$$b_v = -i \sinh[\Lambda_v'(t)] \exp\left\{ i \frac{\phi_\omega + \phi_\omega^0}{2} \right\},$$  \hfill (B7b)

with

$$\Lambda_v = \sqrt{v^2 - 1} \int_0^t |\omega| \, dt', \quad \Lambda_v' = \sqrt{1-v^2} \int_0^t |\omega| \, dt'.$$  \hfill (B8)

In Fig. 3 we see the manifestation of the two different regimes

![Figure 3: (Color online) Time dependence of the transition probability $P^\pm$ as a function of $\tau = |\omega|\tau$ for the case (B7b) with $v = 1/\sqrt{2}$ (red dot-dashed line) and (B6) with $v = \sqrt{2}$ (blue dotted line). The upper full line corresponds to $P^+ = 1/2$.](image)

for $P^+$, namely oscillatory for $v > 1$ and asymptotic for $v \leq 1$. 
It is possible to show [17] that such different regimes are in general not related to the reality or complexity of the spectrum. Indeed, in the special case $\Omega = \Omega_0$, $\omega = |\omega_0|$ and $\phi_0 = \phi_0$ the Hamiltonian possesses a time-independent spectrum ($E_{\pm} = \sqrt{\Omega_0^2 - |\omega_0|^2}$). It is possible to show that we may have a $\nu$-based passage from a real to a complex spectrum, keeping the same dynamical regime and, vice versa, it may occur a $\nu$-based change of dynamical regime while keeping the reality (or complexity) of the spectrum. Only for $\phi_0 = 0$, that is for $PT$-symmetry Hamiltonians and, more in general, for time-independent $su(1,1)$ matrices, we have the coincidence between the two $\nu$-dependent effects, which agree with previous results [29]. This suggests the fact that the physical feature consisting in the mismatch between the spectrum and the dynamical regime $\nu$-based change, is peculiar of $su(1,1)$ time-dependent pseudo-Hermitian Hamiltonians.

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