Geometric Algebras and Extensors

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Abstract

This is the first paper in a series (of four) designed to show how to use geometric algebras of multivectors and extensors to a novel presentation of some topics of differential geometry which are important for a deeper understanding of geometrical theories of the gravitational field. In this first paper we introduce the key algebraic tools for the development of our program, namely the euclidean geometrical algebra of multivectors $\mathcal{C}\ell(V, G_E)$ and the theory of its deformations leading to metric geometric algebras $\mathcal{C}\ell(V, G)$ and some special types of extensors. Those tools permit obtaining, the remarkable golden formula relating calculations in $\mathcal{C}\ell(V, G)$ with easier ones in $\mathcal{C}\ell(V, G_E)$ (e.g., a noticeable relation between the Hodge star operators associated to $G$ and $G_E$). Several useful examples are worked in details for the purpose of transmitting the “tricks of the trade”.

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1 Introduction

This is the first paper in series of four, designed to show how Clifford (geometric) algebra methods can be conveniently used in the study of differential geometry and geometrical theories of the gravitational field. It dispenses the use of fiber bundle theory and is indeed a very powerful and economic tool for performing sophisticated calculations. This first paper deal with the algebraic aspects of the theory, namely Clifford algebras and the theory of extensors. Our presentation is self contained and serves besides the purpose of fixing conventions also the one of introducing a series of “tricks of the trade” (not found easily elsewhere) necessary for quickly and efficient computations.

The other three papers develop in a systematic way a theory of multivector and extensor fields [2] and their use in the differential geometry of manifolds of arbitrary topology [8, 9]. There are many novelties in our presentation of differential geometry, in particular the way we introduce the concept of deformation of geometric structures, which is discussed in detail in [8] and which permit us [8, 9] to relate some well distinct geometric structures on a given manifold. Moreover, the method permit also to solve problems in one given geometry in terms of an eventually simple one, and here is a place where our theory may be a useful one for the study of geometrical theories of the gravitational field.

The main issues discussed in the present paper are the constructions of the Euclidean and the metric geometric (or Clifford) algebras of multivectors, denoted $\mathcal{C}l(V, G_E)$ and $\mathcal{C}l(V, G)$ which can be associated to a real vector space $V$ of dimension $n$ once we equip $V$ respectively with an Euclidean ($G_E$) and an arbitrary ($G$) non degenerated metric of signature $(p, q)$ with $p + q = n$. The Euclidean geometrical algebra is the key tool for performing almost all calculations of the following papers. It plays in our theory a role analogous to matrix algebra in standard presentations of linear algebra. Metric geometric algebras are introduced as deformations of the Euclidean geometric algebra. This is conveniently explored with the introduction of the concept of a deformation extensor associated to a given metric extensor and and by proving the remarkable golden formula. Extensors are a new kind of geometrical objects which play a crucial role in the theory presented in this series and in what follows the basics of their theory is described. These objects have been apparently introduced by Hestenes and Sobczyk in [4] and some applications of the concept appears in [5], but a rigorous theory was lacking.
It is important to observe that in [4] the preliminaries of the geometric calculus have been applied to the study of the differential geometry of vector manifolds. However, as admitted in [14] there are some problems with such approach. In contrast, our formulation applies to manifolds of an arbitrary topology and is free of the problems that paved the construction in [4]. On the following papers dealing with differential geometry, the concept of a canonical vector space associated to an open set of an arbitrary manifold is introduced and the main ingredients of differential geometry, like connections and their torsion and curvature extensors are introduced using the Euclidean geometric calculus in the canonical vector space. After that metric extensors are introduced and the concept of deformed geometries relative to a given geometry is introduced. An intrinsic Cartan calculus is developed together with some other topics that are ready to be used in geometrical theories of the gravitational field, as, we hope, the reader will convince himself consulting the other papers in the series.

As to the explicit contents of the present paper they are summarized as follows. Section 2 is dedicated to the construction of the Euclidean and metric geometric algebras. The concepts of multivectors (homogeneous and non homogeneous ones) and their exterior algebra is briefly recalled. Next we introduce the scalar product of multivectors and the concepts of right and left contractions and interior algebras, and then give a definition of a general real Clifford (or geometrical) algebra $\mathcal{C}(V, G)$ of multivectors associated to a pair $(V, G)$. We fix an Euclidean metric $G_E$ in $V$ and use $\mathcal{C}(V, G_E)$, the Euclidean geometric algebra as our basic tool for performing calculations, and study some relations between $\mathcal{C}(V, G)$ and $\mathcal{C}(V, G_E)$. Section 3 is dedicated to an introduction to the theory of extensors, with emphasis on the properties of some special kind of extensors that will be used in our approach to differential geometry in the next papers of the series. Of special interest is the golden formula which permit us to make calculations related to $\mathcal{C}(V, G)$ using the simple algebra $\mathcal{C}(V, G_E)$ once we determine a gauge extensor $h$ related to the metric extensor $g$ (which determines $G$) and the relation between Hodge (star) extensors associated to $G_E$ and $G$. In Section 4 we present our conclusions.

1More details on the theory of extensors may be found in [2].

2As additional references to several aspects of the theory of the theory of Clifford algebras, that eventually may help the interested reader, we quote [1, 3, 7, 10, 11, 12, 13]
2 Geometric Algebras of Multivectors

2.1 Multivectors

Let $V$ be an $n$-dimensional vector space over the real field $\mathbb{R}$, $T(V)$ the tensor algebra of $V$ and $\wedge V$ the exterior algebra of $V$. The elements of $\wedge V$ are called multivectors.

We recall that the exterior algebra $\bigwedge V$ is a $2^n$-dimensional associative algebra with unity. In addition, it is a $\mathbb{Z}$-graded algebra, i.e.,

$$\bigwedge V = \bigoplus_{r=0}^{n} \bigwedge^r V,$$

and

$$\bigwedge^r V \wedge \bigwedge^s V^* \subseteq \bigwedge^{r+s} V,$$

for $r, s \geq 0$, where $\bigwedge^r V$ is the chosen $\binom{n}{r}$-dimensional subspace of homogeneous $r$-vectors. We have moreover the following identifications $\bigwedge^0 V = \mathbb{R}$; $\bigwedge^1 V = V$; and of course, $\bigwedge^r V = \{0\}$ if $r > n$. If $A \in \bigwedge^r V$ for some fixed $r$ ($r = 0, \ldots, n$), then $A$ is said to be homogeneous. For any homogeneous multivectors $A_p \in \bigwedge^p V$, $B_q \in \bigwedge^q V$ we define here their exterior product by

$$A_p \wedge B_q = \frac{(p+q)!}{p!q!} A_p \otimes B_q.$$

(1)

where $A : T_k(V) \to \bigwedge^k V$ is the usual antisymmetrization operator. Of course, we have:

$$A_p \wedge B_q = (-1)^{pq} B_q \wedge A_p.$$

(2)

For each $k = 0, 1, \ldots, n$ the linear mapping $\langle \rangle_k : \bigwedge V \to \bigwedge^k V$ such that $X = X_0 \oplus X_1 \oplus \ldots \oplus X_n := X_0 + X_1 + \ldots + X_n, X_k \in \bigwedge^k V, k = 0, 1, \ldots, n$, then

$$\langle X \rangle_k = X_k$$

(3)

is called the $k$-component projection operator because $\langle X \rangle_k$ is just the $k$-component of $X$.

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3 If the exterior algebra is defined as the quotient algebra $T(V)/J$, where $J$ is the bilateral ideal generated by elements of the form $u \otimes v + v \otimes u$, $u, v \in V$ then the exterior product $A_p \wedge B_q$ is given by $A_p \wedge B_q = A(A_p \otimes B_q)$. Details on the relation between $\wedge$ and $\otimes$ may be found in [3].
For each \( k = 0, 1, \ldots, n \) any multivector \( X \) such that \( \langle X \rangle_j \neq 0 \), for \( k \neq j \), is of course a homogeneous multivector of degree \( k \), or for short, a \( k \)-homogeneous multivector. It should be noticed that 0 is an homogeneous multivector of any degree 0, 1, \ldots, \( n \).

Let \( \{ e_j \} \) be a basis of \( V \), and \( \{ \varepsilon^j \} \) be its dual basis for \( V^* \), i.e., \( \varepsilon^j(e_i) = \delta^j_i \).

Now, let us take \( t \in T^k V \) with \( k \geq 1 \). Such a contravariant \( k \)-tensor \( t \) can be expanded onto the \( k \)-tensor basis \( \{ e_{j_1} \otimes \ldots \otimes e_{j_k} \} \) with \( j_1, \ldots, j_k = 1, \ldots, n \), by the well-known formula

\[
t = t^{j_1 \ldots j_k} e_{j_1} \otimes \ldots \otimes e_{j_k},
\]

where \( t^{j_1 \ldots j_k} = t(\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) \) are the so-called \( j_1 \ldots j_k \)-contravariant components of \( t \) with respect to \( \{ e_{j_1} \otimes \ldots \otimes e_{j_k} \} \).

Using the definition of the antisymmetrization operator \( \mathcal{A} \) it follows (non-trivially) a remarkable identity which holds for the basis 1-forms \( \varepsilon^1, \ldots, \varepsilon^n \) belonging to \( \{ \varepsilon^j \} \). It is

\[
\mathcal{A} t(\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) = \frac{1}{k!} \delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} t(\varepsilon^{i_1}, \ldots, \varepsilon^{i_k}),
\]

where \( \delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} \) is the so-called generalized Kronecker symbol of order \( k \),

\[
\delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} = \det \begin{bmatrix} \delta^j_{i_1} & \ldots & \delta^j_{i_k} \\ \ldots & \ldots & \ldots \\ \delta^j_{i_k} & \ldots & \delta^j_{i_1} \end{bmatrix} \quad \text{with} \quad i_1, \ldots, i_k \text{ and } j_1, \ldots, j_k \text{ running from } 1 \text{ to } n.
\]

Let us take \( X \in \bigwedge^k V \) with \( k \geq 2 \). By definition \( X \in T^k V \) and is completely skew-symmetric, hence, it must be \( X = \mathcal{A} X \). Then, by using Eq.\((5)\) we get a combinatorial identity which relates the \( i_1 \ldots i_k \)-components to the \( j_1 \ldots j_k \)-components for \( X \). It is

\[
X^{j_1 \ldots j_k} = \frac{1}{k!} \delta^{j_1 \ldots j_k}_{i_1 \ldots i_k} X^{i_1 \ldots i_k}.
\]

From Eq.\((1)\) by using a well-known property of the antisymmetrization operator, namely: \( \mathcal{A}(\mathcal{A} t \otimes u) = \mathcal{A}(t \otimes \mathcal{A} u) = \mathcal{A}(t \otimes u) \), we have the following formula for expressing simple \( k \)-vectors in terms of the tensor products of \( k \) vectors. It is

\[
v_1 \wedge \ldots \wedge v_k = \varepsilon^{i_1 \ldots i_k} v_{i_1} \otimes \ldots \otimes v_{i_k}.
\]
If \( \omega^1, \ldots, \omega^k \in V^* \), then
\[
v_1 \wedge \ldots \wedge v_k(\omega^1, \ldots, \omega^k) = \epsilon^{i_1 \ldots i_k} \omega_1(v_{i_1}) \ldots \omega_k(v_{i_k}). \tag{9}
\]

Eq. (8) implies (non-trivially) a remarkable identity which holds for the basis vectors \( e_1, \ldots, e_n \) belonging to any basis \( \{e_j\} \) of \( V \). It is
\[
e_{i_1} \wedge \ldots \wedge e_{i_k} = \delta_{i_1 \ldots i_k} e_{j_1} \otimes \ldots \otimes e_{j_k}. \tag{10}
\]

Once again let us take \( X_k \in \bigwedge^k V \) with \( k \geq 2 \). Since \( X_k \in T^k V \) and is completely skew-symmetric, the use of Eq. (7) and Eq. (10) in Eq. (4) allows us to obtain the expansion formula
\[
X_k = \frac{1}{k!} X_{i_1 \ldots i_k} e_{i_1} \wedge \ldots \wedge e_{i_k}, \tag{11}
\]
where \( X_k(\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) = X_{i_1 \ldots i_k} \).

From this we see that a basis for the \( 2^n \)-dimensional vector space of the algebra \( \bigwedge V \) is the set \( \{e_J\} \) where \( J \) are collective indices, denoting specifically \( \{1, e_i, \frac{1}{2!}e_i \wedge e_{i_2}, \ldots, \frac{1}{n!} e_i \wedge e_{i_2} \ldots \wedge e_{i_n}\} \), \( i_1, \ldots, i_n = 0, 1, 2, \ldots, n \). Then for a general multivector \( X \in \bigwedge V \) we have the expansion formula,
\[
X = s + v^i e_i + \frac{1}{2!} b^{ij} e_i \wedge e_j + \frac{1}{3!} t^{ijk} e_i \wedge e_j \wedge e_k + \ldots + pe_1 \wedge \ldots \wedge e_n, \tag{12}
\]
with \( s, v^i, b^{ij}, t^{ijk}, \ldots, p \in \mathbb{R} \).

We recall moreover that the exterior product of \( X, Y \in \bigwedge V \), namely \( X \wedge Y \in \bigwedge V \), is given by
\[
X \wedge Y = \sum_{k=0}^{n} \sum_{j=0}^{k} \langle X \rangle_j \wedge \langle Y \rangle_{k-j}. \tag{13}
\]

2.2 Metric Structure

Let us equip \( V \) with a metric tensor, \( G : V \times V \to \mathbb{R} \). As usual we write
\[
G(v, w) \equiv v \cdot w, \tag{14}
\]
and call \( v \cdot w \) the scalar product of the vectors \( v, w \in V \).

The pair \( (V, G) \) is called a metric structure for \( V \). Sometimes, \( V \) is said to be a scalar product vector space.
Let \( \{e_k\} \) be any basis of \( V \), and \( \{\varepsilon^k\} \) be its dual basis for \( V^* \). Let \( G_{jk} = G(e_j, e_k) \), since \( G \) is non-degenerate, it follows that \( \det [G_{jk}] \neq 0 \). Then, there exists the \( jk \)-entries for the inverse matrix of \( [G_{jk}] \), namely \( G^{jk} \), i.e., \( G^{ks}G_{sj} = G_{js}G^{sk} = \delta^j_j \).

We introduce the scalar product of 1-forms \( \omega, \sigma \in V^* \) by

\[
\omega \cdot \sigma = G^{jk} \omega(e_j)\sigma(e_k).
\]  

It should be noticed that the real number given by Eq.(15) does not depend on the choice of \( \{e_k\} \).

Now, we can define the so-called reciprocal bases \( \{e^k\} \) and \( \{\varepsilon_k\} \) of the bases \( \{e_k\} \) and \( \{\varepsilon^k\} \). Associated to \( \{e_k\} \) we introduce the well-defined basis \( \{e^k\} \) by

\[
e^k = G^{ks}e_s, \text{ for each } k = 1, \ldots, n. \tag{16}
\]

Such \( e^1, \ldots, e^n \in V \) are the unique basis vectors for \( V \) which satisfy

\[
e^k \cdot e_j = \delta^k_j. \tag{17}
\]

Associated to \( \{\varepsilon^k\} \), we can also introduce a well-defined basis \( \{\varepsilon_k\} \) by

\[
\varepsilon_k = G_{ks}\varepsilon^s, \text{ for each } k = 1, \ldots, n. \tag{18}
\]

Such \( \varepsilon_1, \ldots, \varepsilon_n \in V^* \) are the unique basis 1-forms for \( V^* \) which satisfy

\[
\varepsilon_j \cdot \varepsilon^k = \delta^k_j. \tag{19}
\]

The bases \( \{e^k\} \) and \( \{\varepsilon_k\} \) are respectively called the reciprocal bases of \( \{e_k\} \) and \( \{\varepsilon^k\} \) (relative to the metric tensor \( G \)).

Note that \( \{\varepsilon_k\} \) is the dual basis of \( \{e^k\} \), i.e.,

\[
\varepsilon_k(e^l) = \delta^l_k, \tag{20}
\]

an immediate consequence of Eqs.(18) and (16).

From Eqs.(16),(17), (18) and (19), taking into account Eq.(15), we easily get that

\[
\varepsilon_j \cdot \varepsilon_k = \delta^j_j \cdot \delta^k_k, \tag{21}
\]

\[
e^j \cdot e^k = G^{jk} = \varepsilon^j \cdot \varepsilon^k. \tag{22}
\]
Using Eq. (17) we get two expansion formulas for \( v \in V \)

\[
v = v \cdot e^k e_k = v \cdot e_k e^k.
\]  

(23)

Using Eq. (19) we have that for all \( \omega \in V^* \)

\[
\omega = \omega \cdot \varepsilon^k e_k = \omega \cdot e^k \varepsilon_k.
\]

(24)

Let us take \( X \in \bigwedge^k V \) with \( k \geq 2 \). By following analogous steps to those which allowed us to get Eq. (11) we can now obtain another expansion formula for \( k \)-vectors, namely

\[
X = \frac{1}{k!} X_{j_1 \ldots j_k} e^{j_1} \wedge \ldots \wedge e^{j_k},
\]

(25)

where \( X_{j_1 \ldots j_k} = X(\varepsilon_{j_1}, \ldots, \varepsilon_{j_k}) \) are the so-called \( j_1 \ldots j_k \)-covariant components of \( X \) (with respect to the \( k \)-tensor basis \( \{e^{j_1} \otimes \ldots \otimes e^{j_k}\} \) with \( j_1, \ldots, j_k = 1, \ldots, n \)).

Next, we will obtain a relation between the \( i_1 \ldots i_k \)-covariant components of \( X \) and the \( j_1 \ldots j_k \)-contravariant components of \( X \). A straightforward calculation yields

\[
X(\varepsilon_{i_1} \ldots \varepsilon_{i_k}) = X(e_{i_1} \cdot e_{s_1} \varepsilon_{s_1}, \ldots, e_{i_k} \cdot e_{s_k} \varepsilon_{s_k})
= X(\varepsilon_{s_1}, \ldots, \varepsilon_{s_k})(e_{i_1} \cdot e_{s_1}) \ldots (e_{i_k} \cdot e_{s_k})
= \frac{1}{k!} X(\varepsilon_{j_1}, \ldots, \varepsilon_{j_k}) \delta_{j_1 \ldots j_k}^{s_1 \ldots s_k} (e_{i_1} \cdot e_{s_1}) \ldots (e_{i_k} \cdot e_{s_k}),
\]

hence,

\[
X_{i_1 \ldots i_k} = \frac{1}{k!} X_{j_1 \ldots j_k} \text{Det} \begin{bmatrix}
e_{i_1} \cdot e_{j_1} & \ldots & e_{i_1} \cdot e_{j_k} \\
\ldots & \ldots & \ldots \\
e_{i_k} \cdot e_{j_1} & \ldots & e_{i_k} \cdot e_{j_k}
\end{bmatrix}.
\]

Finally we recall that we can take also as a basis the \( 2^n \)-dimensional vector space of the algebra \( \bigwedge V \) the set \( \{e^J\} \) where \( J \) are collective indices, denoting specifically \( \{1, e^i, \frac{1}{2!} e^i \wedge e^j, \ldots, \frac{1}{n!} e^i \wedge e^j \wedge \ldots \wedge e^n\} \), \( i_1, \ldots, i_n = 0, 1, 2, \ldots, n \). Then for a general multivector \( X \in \bigwedge V \) we have the expansion formula,

\[
X = s + v_i e^i + \frac{1}{2!} b_{ij} e^i \wedge e^j + \frac{1}{3!} t_{ijk} e^i \wedge e^j \wedge e^k + \ldots + p e^1 \wedge \ldots \wedge e^n,
\]

(26)

with \( s, v_i, b_{ij}, t_{ijk}, \ldots, p \in \mathbb{R} \).
2.3 Scalar Product for $\Lambda^p V$

Once a metric structure $(V, G)$ has been given we can equip $\Lambda^p V$ with a scalar product of $p$-vectors. $\Lambda V$ can then be endowed with a scalar product of multivectors. This is done as follows.

The scalar product of $X_p, Y_p \in \Lambda^p V$, namely $X_p \cdot Y_p \in \mathbb{R}$, is defined by the axioms:

Ax-i For all $\alpha, \beta \in \mathbb{R}$:
\[
\alpha \cdot \beta = \alpha \beta \quad \text{(real product of } \alpha \text{ and } \beta). \tag{27}
\]

Ax-ii For all $X_p, Y_p \in \Lambda^p V$, with $p \geq 1$:
\[
X_p \cdot Y_p = \frac{1}{p!} X_p(\varepsilon^{i_1}, \ldots, \varepsilon^{i_p})Y_p(\varepsilon_{i_1}, \ldots, \varepsilon_{i_p}),
\]
\[
= \frac{1}{p!} X_p(\varepsilon_{i_1}, \ldots, \varepsilon_{i_p})Y_p(\varepsilon^{i_1}, \ldots, \varepsilon^{i_p}), \tag{28}
\]

where $\{\varepsilon_i\}$ is the reciprocal basis of $\{\varepsilon^i\}$, as defined by Eq.(18).

It is a well-defined scalar product on $\Lambda^p V$, since it is symmetric, satisfies the distributive laws, has the mixed associativity property and is non-degenerate i.e., if $X_p \cdot Y_p = 0$ for all $Y_p$, then $X_p = 0$. For the special case of vectors Eq.(28), of course, reduces to
\[
v \cdot w = \varepsilon^i(v)\varepsilon_i(w) = \varepsilon_i(v)\varepsilon^i(w), \tag{29}
\]
i.e., $G = \varepsilon^i \otimes \varepsilon_i = \varepsilon_i \otimes \varepsilon^i$.

The well-known formula for the scalar product of simple $k$-vectors can be easily deduced from Eq.(28). It is:
\[
(v_1 \wedge \ldots \wedge v_k) \cdot (w_1 \wedge \ldots \wedge w_k) = \text{Det} \begin{bmatrix} v_1 \cdot w_1 & \ldots & v_1 \cdot w_k \\ \ldots & \ldots & \ldots \\ v_k \cdot w_1 & \ldots & v_k \cdot w_k \end{bmatrix}. \tag{30}
\]

Now, we can generalize Eq.(23) in order to get the expected expansion formulas for $k$-vectors. For all $X \in \Lambda^k V$ it holds two expansion formulas
\[
X = \frac{1}{k!} X \cdot (e^{j_1} \wedge \ldots \wedge e^{j_k})(e_{j_1} \wedge \ldots \wedge e_{j_k}) = \frac{1}{k!} X \cdot (e_{j_1} \wedge \ldots \wedge e_{j_k})(e^{j_1} \wedge \ldots \wedge e^{j_k}). \tag{31}
\]
and of course, $X_{j_1\ldots j_k} = X \cdot (e^{j_1} \wedge \ldots \wedge e^{j_k})$. Analogously, we can prove that $X_{j_1\ldots j_k} = X \cdot (e_{j_1} \wedge \ldots \wedge e_{j_k})$. 

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2.3.1 Scalar Product of Multivectors

The scalar product of $X, Y \in \bigwedge V$, namely $X \cdot Y \in \mathbb{R}$, is defined by

$$X \cdot Y = \sum_{k=0}^{n} \langle X \rangle_k \cdot \langle Y \rangle_k. \quad (32)$$

By using Eqs. (27) and (28) we can easily note that Eq. (32) can still be written as

$$X \cdot Y = X_0 Y_0 + \sum_{k=1}^{n} \frac{1}{k!} X_k(\varepsilon^{i_1}, \ldots, \varepsilon^{i_k}) Y_k(\varepsilon_{i_1}, \ldots, \varepsilon_{i_k})$$

$$= X_0 Y_0 + \sum_{k=1}^{n} \frac{1}{k!} X_k(\varepsilon_{i_1}, \ldots, \varepsilon_{i_k}) Y_k(\varepsilon^{i_1}, \ldots, \varepsilon^{i_k}). \quad (33)$$

It is important to observe that the operation defined by Eq. (32) is indeed a well-defined scalar product on $\bigwedge V$, since it is symmetric, satisfies the distributive laws, has the mixed associative property and is not degenerate, i.e., if $X \cdot Y = 0$ for all $Y$, then $X = 0$.

2.4 Involutions

We recall that the main involution (or grade involution) denoted by $^\wedge : \bigwedge V \to \bigwedge V$ satisfies: (i) if $\alpha \in \mathbb{R}, \hat{\alpha} = \alpha$; (ii) if $a_1 \wedge \ldots \wedge a_k \in \bigwedge^k V$, $k \geq 1$,

$$a_1 \wedge \ldots \wedge a_k \hat{\wedge} = (-1)^k a_1 \wedge \ldots \wedge a_k$$

(iii) if $a, b \in \mathbb{R}$ and $\sigma, \tau \in \bigwedge^k V$ then

$$a \sigma + b \tau \hat{\wedge} = a \hat{\sigma} + b \hat{\tau}$$

(iv) if $\tau = \sum_k \tau_k, \tau_k \in \bigwedge^k V$ then

$$\hat{\tau} = \sum_k \hat{\tau}_k. \quad (34)$$

We recall also that the reversion operator is the anti-automorphism $\tilde{} : \bigwedge V \ni \tau \mapsto \tilde{\tau} \in \bigwedge V$ such that if $\tau = \sum_k \langle \tau \rangle_k, \langle \tau \rangle_k \in \bigwedge^k V$ then:

(i) if $\alpha \in \mathbb{R}, \tilde{\alpha} = \alpha$; (ii) if $a_1 \wedge \ldots \wedge a_k \in \bigwedge^k V$, $k \geq 1, (a_1 \wedge \ldots \wedge a_k)^\tilde{\wedge} = a_k \wedge \ldots \wedge a_1$; (iii) if $a, b \in \mathbb{R}$ and $\sigma, \tau \in \bigwedge^k V$ then $(a \sigma + b \tau)^\tilde{\wedge} = a \tilde{\sigma} + b \tilde{\tau}$; (iv)
if \( \tau = \sum \tau_k, \tau_k \in k \wedge V \) then
\[
\tilde{\tau} = \sum_{k=0}^{n} \tilde{\tau}_k,
\]
(35)
where \( \tilde{\tau} \) is called the reverse of \( \tau \).

Finally, we recall that the composition of the graded evolution with the reversion operator, denoted by the symbol \( - \) is called by some authors the 
\textit{conjugation} and, \( \tilde{\tau} \) is said to be the \textit{conjugate} of \( \tau \). We have \( \tilde{\tau} = (\hat{\tau})^{-} = (\hat{\tau})^\sim \).

\subsection{2.5 Contracted Products}

The left contracted product of \( X_p \in \bigwedge^p V \) and \( Y_q \in \bigwedge^q V \) with \( 0 \leq p \leq q \leq n \), namely \( X_p \cdot Y_q \in \bigwedge^{q-p} V \), is defined by for all \( X_p \in \bigwedge^p V \) and \( Y_q \in \bigwedge^q V \) with \( p \leq q \) by:
\[
X_p \cdot Y_q = \frac{1}{(q-p)!}(\tilde{X}_p \wedge e^{i_1} \wedge \ldots \wedge e^{i_{q-p}}) \cdot Y_e e^{i_1} \wedge \ldots \wedge e^{i_{q-p}}.
\]
(36)
It is clear that all \( X_p, Y_p \in \bigwedge^p V \) it holds
\[
X_p \cdot Y_p = \tilde{X}_p \cdot e = X_p \cdot \tilde{Y}_p.
\]
(37)
The right contracted product of \( X_p \in \bigwedge^p V \) and \( Y_q \in \bigwedge^q V \) with \( n \geq p \geq q \geq 0 \), namely \( X_p \cdot Y_q \in \bigwedge^{p-q} V \), is defined for all \( X_p \in \bigwedge^p V \) and \( Y_q \in \bigwedge^q V \) with \( p > q \) by
\[
X_p \cdot Y_q = \frac{1}{(p-q)!}X_p \cdot (e^{i_1} \wedge \ldots \wedge e^{i_{q-p}} \wedge \tilde{Y}_q)e^{i_1} \wedge \ldots \wedge e^{i_{p-q}}
\]
(38)
Of course, for all \( X_p, Y_p \in \bigwedge^p V \) we have
\[
X_p \cdot Y_p = \tilde{X}_p \cdot Y_p = X_p \cdot \tilde{Y}_p.
\]
(39)
It should be noticed that the \((q-p)\)-vector defined by Eq. (36) and the \((p-q)\)-vector defined by Eq. (38) do not depend on the choice of the reciprocal bases \( \{e_i\} \) and \( \{e^i\} \) used for calculating them.
Let us take $X_p \in \bigwedge^p V$ and $Y_q \in \bigwedge^q V$ with $p \leq q$. For all $Z_{q-p} \in \bigwedge^{q-p} V$ the following identity holds

$$(X_p \cdot Y_q) \cdot Z_{q-p} = Y_q \cdot (\widehat{X_p} \wedge Z_{q-p}).$$

(40)

For $p < q$ Eq.(40) follows directly from Eq.(36) and Eq.(31). But, for $p = q$ it trivially follows by taking into account Eq.(37), etc.

Let us take $X_p \in \bigwedge^p V$ and $Y_q \in \bigwedge^q V$ with $p \geq q$. For all $Z_{p-q} \in \bigwedge^{p-q} V$ the following identity holds

$$(X_p \cdot Y_q) \cdot Z_{p-q} = X_p \cdot (Z_{p-q} \wedge \widehat{Y_q}).$$

(41)

For $p > q$ Eq.(41) follows directly from Eq.(38) and Eq.(31). For $p = q$ it follows from Eq.(39).

We recall moreover that for any $V_p, W_p \in \bigwedge^p V$ and $X_q, Y_q \in \bigwedge^q V$ with $p \leq q$ we have $(V_p + W_p) \cdot X_q = V_p \cdot X_q + W_p \cdot X_q$, and $V_p \cdot (X_q + Y_q) = V_p \cdot X_q + V_p \cdot Y_q$ and also for any $V_p, W_p \in \bigwedge^p V$ and $X_q, Y_q \in \bigwedge^q V$ with $p \geq q$ we have $(V_p + W_p) \cdot X_q = V_p \cdot X_q + W_p \cdot X_q$, and $V_p \cdot (X_q + Y_q) = V_p \cdot X_q + V_p \cdot Y_q$. More important, we have for any $X_p \in \bigwedge^p V$ and $Y_q \in \bigwedge^q V$ with $p \leq q$

$$X_p \cdot Y_q = (-1)^{p(q-p)} Y_q \cdot X_p.$$  

(42)

which follows by using Eq.(40), Eq.(41) and that $X_p \wedge Y_q = (-1)^{pq} Y_q \wedge X_p$. 

Indeed, we have that $(X_p \cdot Y_q) \cdot Z_{q-p} = Y_q \cdot (\widehat{X_p} \wedge Z_{q-p}) = (-1)^{p(q-p)} Y_q \cdot (Z_{q-p} \wedge \widehat{X_p}) = (-1)^{p(q-p)} (Y_q \cdot \widehat{X_p}) \cdot Z_{q-p}$, hence, by non-degeneracy of the scalar product, the required result follows.

### 2.5.1 Contracted Product of Nonhomogeneous Multivectors

The left and right contracted products of $X, Y \in \bigwedge V$, namely $X \cdot Y \in \bigwedge V$ and $X \cdot Y \in \bigwedge V$, are defined by

$$X \cdot Y = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \langle \langle X \rangle_j \cdot \langle Y \rangle_{k+j} \rangle_k.$$  

(43)

$$X \cdot Y = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \langle \langle X \rangle_{k+j} \cdot \langle Y \rangle_j \rangle_k.$$  

(44)
We finalize this section presenting two noticeable formulas involving the contracted products and the scalar product, and two other remarkable formulas relating the contracted products to the exterior product and scalar product. They appear frequently in calculations.

For any \( X, Y, Z \in \bigwedge V \)

\[
(X \circ Y) \cdot Z = Y \cdot (\tilde{X} \wedge Z),
\]
\[
\quad (X \wedge Y) \cdot Z = X \cdot (Z \wedge \tilde{Y}). \tag{45}
\]

For any \( X, Y, Z \in \bigwedge V \)

\[
X \circ (Y \circ Z) = (X \wedge Y) \cdot Z, \tag{47}
\]
\[
(X \wedge Y) \circ Z = X \cdot (Y \wedge Z). \tag{48}
\]

### 2.6 Clifford Product and \( \mathcal{C}\ell(V, G) \)

The two interior algebras together with the exterior algebra allow us to define a *Clifford product* of multivectors which is also an internal law on \( \bigwedge V \). The Clifford product of \( X, Y \in \bigwedge V \), denoted by juxtaposition \( XY \in \bigwedge V \), is defined by the following axioms:

**Ax-ci** For all \( \alpha \in \mathbb{R} \) and \( X \in \bigwedge V \):

\[
\alpha X \text{ is the scalar multiplication of } X \text{ by } \alpha. \tag{49}
\]

**Ax-cii** For all \( v \in V \) and \( X \in \bigwedge V \):

\[
vX = v \circ X + v \wedge X, \tag{50}
\]
\[
Xv = X \wedge v + X \wedge v. \tag{51}
\]

**Ax-ciii** For all \( X, Y, Z \in \bigwedge V \):

\[
(XY)Z = X(YZ). \tag{52}
\]

The Clifford product is distributive and associative. \( \bigwedge V \) endowed with this Clifford product is an associative algebra which will be called the geometric algebra of multivectors associated to a metric structure \( (V, G) \). It will be denoted by \( \mathcal{C}\ell(V, G) \). Using the above axioms we can derive a general formula.

\[\text{For a proof see } \cite{3}\]
for the Clifford product of two arbitrary multivectors \( A = \sum_r \oplus A_r, B = \sum_s \oplus B_s \in \mathcal{C}(V, G) \). We have
\[
AB = \sum_{r,s} \oplus A_r B_s ,
\]
\[
A_r B_s = \langle A_r B_s \rangle_{r-s} + \langle A_r B_s \rangle_{r-s+2} + ... + \langle A_r B_s \rangle_{r+s}
\]

To continue we introduce one more convention. We denote by \( X_p \ast Y_q \) either \( \wedge \), or \( \cdot \), or \( \langle \rangle \), or \( (\langle) \) or \( (\rangle) \) or \( (Clifford\ product) \).

### 2.7 Euclidean and Metric Geometric Algebras

#### 2.7.1 \( \mathcal{C}(V, G_E) \)

Let us equip \( V \) with an arbitrary (but fixed once for all) Euclidean metric \( G_E \). \( V \) endowed with an Euclidean metric \( G_E \), i.e., \( (V, G_E) \), is called an Euclidean metric structure for \( V \). Sometimes, \( (V, G_E) \) is said to be an Euclidean space.

Associated to \( (V, G_E) \) an Euclidean scalar product of vectors \( v, w \in V \) is given by
\[
v \cdot_{G_E} w = G_E(v, w).
\]
(53)

We introduce also an Euclidean scalar product of \( p \)-vectors \( X_p, Y_p \in \bigwedge^p V \) and Euclidean scalar product of multivectors \( X, Y \in \bigwedge V \), namely \( X_p \cdot_{G_E} Y_p \in \mathbb{R} \) and \( X \cdot_{G_E} Y \in \mathbb{R} \), using respectively the Eqs. (27) and (28), and Eq. (32). The Clifford algebra associated to the pair \( (V, G_E) \) will be denoted \( \mathcal{C}(V, G_E) \) and called Euclidean geometric algebra. It will play a role in our theory a role analogous to the one of matrix algebra in standard presentations of linear algebra.

#### 2.7.2 \( \mathcal{C}(V, G) \)

Let us take any metric tensor \( G \) on the vector space \( V \). Associated to the metric structure \( (V, G) \) a scalar product of vectors \( v, w \in V \) is represented by
\[
v \cdot_G w = G(v, w).
\]
(54)

Of course, the corresponding scalar product of \( p \)-vectors \( X_p, Y_p \in \bigwedge^p V \) and scalar product of multivectors \( X, Y \in \bigwedge V \), namely \( X_p \cdot_G Y_p \in \mathbb{R} \) and \( X \cdot_G Y \in \mathbb{R} \),...
\( \mathbb{R} \), are defined respectively by Eqs. (27) and (28) and Eq. (32). The Clifford algebra associated to the pair \((V, G)\) will be denoted \( \mathcal{Cl}(V, G) \) and called metric geometric algebra.

We will find a relationship between \((V, G)\) and \((V, G_E)\), thereby showing how an arbitrary \(G\)-scalar product on \( \bigwedge^p V \) and \( \bigwedge V \) is related to a \(G_E\)-scalar products on \( \bigwedge^p V \) and \( \bigwedge V \). This starting point which permits us to relate \( \mathcal{Cl}(V, G) \) with \( \mathcal{Cl}(V, G_E) \) is the concept of a metric operator \( g \) (but a convenient algorithm needs the concept of deformation extenders to be introduced in Section 3) which we now introduce.

### 2.7.3 Enter \( g \)

To continue, choose once and for all a fiducial Euclidean metric structure \((V, G_E)\). We now recall that for any metric tensor \( G \) there exists an unique linear operator \( g : V \to V \), such that for all \( v, w \in V \)

\[
    v \cdot_G w = g(v) \cdot_{G_E} w. \tag{55}
\]

Such \( g \) is given by

\[
    g(v) = (v \cdot_G e_k) e^k_{G_E} = (v \cdot_G e^k) e_k, \tag{56}
\]

where \( \{e_k\} \) is any basis of \( V \), and \( \{e^k\} \) is its reciprocal basis with respect to \((V, G_E)\), i.e., \( e_k \cdot_G e^l = \delta^l_k \). Note that the vector \( g(v) \) does not depend on the basis \( \{e_k\} \) chosen for calculating it.

We now show that \( g(v) \) given by Eq. (56) satisfies Eq. (55). Using Eq. (23) we have,

\[
    g(v) \cdot_{G_E} w = (v \cdot_G e_k) e^k_{G_E} \cdot_{G_E} w = (v \cdot_G (e^k \cdot_{G_E} w)e_k) = v \cdot_G w.
\]

Now, suppose that there is some \( g' \) which satisfies Eq. (55), i.e., \( v \cdot_G w = g'(v) \cdot_{G_E} w. \) Then, using once again Eq. (23) we have

\[
    g'(v) = (g'(v) \cdot_{G_E} e_k) e^k_{G_E} = (v \cdot_G e_k) e^k = g(v),
\]

i.e., \( g' = g \). So the existence and the uniqueness of such a linear operator \( g \) are proved.
Since $G$ is a symmetric covariant 2-tensor over $V$, i.e., $G(v, w) = G(w, v) \forall v, w \in V$, it follows from Eq.(55) that $g$ is an adjoint symmetric linear operator with respect to $(V, G_E)$, i.e.,

$$
\begin{align*}
  g(v) \cdot w &= v \cdot g(w),

d \quad (57)
\end{align*}
$$

The property expressed by Eq.(57) may be coded as $g = g^{\dagger(G_E)}$.

Since $G$ is a non-degenerate covariant 2-tensor over $V$ (i.e., if $G(v, w) = 0 \forall w \in V$, then $v = 0$) it follows that $g$ is a non-singular (invertible) linear operator. Its inverse linear operator is given by the noticeable formula

$$
\begin{align*}
  g^{-1}(v) &= G^{jk}(v \cdot e_j)e_k,

d \quad (58)
\end{align*}
$$

where $G^{jk}$ are the $jk$-entries of the inverse matrix of $[G_{jk}]$ with $G_{jk} \equiv G(e_j, e_k)$. Note that the vector $g^{-1}(v)$ does not depend on the basis $\{e_k\}$ chosen for its calculation.

We must prove that indeed $g^{-1} \circ g = g \circ g^{-1} = i_V$, where $i_V$ is the identity function for $V$.

By using Eq.(58), Eq.(55), Eq.(16) for $(V, G)$ and Eq.(23) for $(V, G_E)$, we immediately have that $g^{-1} \circ g(v) = v$, i.e., $g^{-1} \circ g = i_V$. Also, $g \circ g^{-1}(v) = v$, i.e., $g \circ g^{-1} = i_V$.

It should be remarked that such $g$ only depends on the choice of the fiducial Euclidean structure $(V, G_E)$. However, $g$ codifies all the geometric information contained in $G$. Such $g$ will be called the metric operator for $G$.

Now, we show how the scalar product $X_p \cdot G_Y$ is related to the scalar product $X_p \cdot G^{-1}_E Y$. For any simple $k$-vectors $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V$ and $w_1 \wedge \ldots \wedge w_k \in \bigwedge^k V$ it holds

$$
\begin{align*}
  (v_1 \wedge \ldots \wedge v_k) \cdot (w_1 \wedge \ldots \wedge w_k) &= (g(v_1) \wedge \ldots \wedge g(v_k)) \cdot (w_1 \wedge \ldots \wedge w_k),

d \quad (59)
\end{align*}
$$

To show Eq.(59) we will use Eq.(30) for $(V, G)$ and $(V, G_E)$. By using Eq.(55) and a fundamental property of any outermorphism, a straightforward calculation yields

$$
\begin{align*}
  (v_1 \wedge \ldots \wedge v_k) \cdot (w_1 \wedge \ldots \wedge w_k) &= (g(v_1) \wedge \ldots \wedge g(v_k)) \cdot (w_1 \wedge \ldots \wedge w_k)
\end{align*}
$$
For any \( k \)-vectors \( X_k, Y_k \in \bigwedge^k V \) it holds
\[
X_k \cdot Y_k = g(X_k) \cdot Y_k.
\] (60)

where if \( X_k = \frac{1}{k!}X_{i_1...i_k}e_{i_1} \wedge ... \wedge e_{i_k}, \) \( g(X_k) = \frac{1}{k!}X_{i_1...i_k}g(e_{i_1}) \wedge ... \wedge g(e_{i_k}). \)

The operator \( g \) is called the exterior power extension of \( g \) and the general properties of a general exterior power operator are discussed in Section 3.3.

Next, we show how the scalar product \( X \cdot Y \) is related to the scalar product \( X \cdot g_Y \).

We also recall that for any multivectors \( X, Y \in \bigwedge V \) it holds
\[
X \cdot Y = g(X) \cdot Y.
\] (61)

The \( G \)-contracted products are related to the \( G_E \)-contracted products by two noticeable formulas.

For any \( X, Y \in \bigwedge V \)
\[
X \lhd Y = g(X) \lhd_Y, \tag{62}
\]
\[
X \rhd Y = X \rhd_{g_E} g(Y). \tag{63}
\]

The Clifford algebra associated to the pair \((V, G)\) will be denoted \( \mathcal{C} \ell(V, G) \) or \( \mathcal{C} \ell(V, g) \) and called metric geometric algebra. We shall use also, in what follows, the notation \( X \cdot g_Y \) meaning \( X \cdot Y \).

3 Theory of Extensors

In this section we recall some basic notions of the theory of extensors thus completing the presentation of the algebraic notions necessary for the remaining papers of the series\(^5\).

\(^5\)We recall that extensors are a particular case of a more important concept, namely of multivector functions of multivector variables. These objects and still more general ones called multivector functionals are used, e.g., in the Lagrangian formulation of the theory of multivector and extensor fields. Their theory may be found in [3][13].
3.1 General $k$-Extensors

Let $\bigwedge^1 V, \ldots, \bigwedge^k V$ be $k$ subspaces of $\bigwedge V$ such that each of them is any sum of homogeneous subspaces of $\bigwedge V$, and $\bigwedge^0 V$ is either any sum of homogeneous subspaces of $\bigwedge V$ or even the trivial subspace consisting of the null vector $\{0\}$. A multilinear mapping from the cartesian product $\bigwedge^1 V \times \cdots \times \bigwedge^k V$ to $\bigwedge^0 V$ will be called a general $k$-extensor over $V$, i.e.,

$$t : \bigwedge^1 V \times \cdots \times \bigwedge^k V \rightarrow \bigwedge^0 V$$

such that for any $\alpha_j, \alpha'_j \in \mathbb{R}$ and $X_j, X'_j \in \bigwedge^0 V,$

$$t(\ldots, \alpha_j X_j + \alpha'_j X'_j, \ldots) = \alpha_j t(\ldots, X_j, \ldots) + \alpha'_j t(\ldots, X'_j, \ldots),$$

(64)

for each $j$ with $1 \leq j \leq k$.

It should be noticed that the linear operators on $V, \bigwedge^p V$ or $\bigwedge V$ which appear in ordinary linear algebra are particular cases of 1-extensors over $V$. Note also that a covariant $k$-tensor over $V$ is just a $k$-extensor over $V$. On this way, the concept of general $k$-extensor generalizes and unifies both of the concepts of linear operator and of covariant $k$-tensor. These mathematical objects are of the same nature!

The set of general $k$-extensors over $V$, denoted by $k$-ext($\bigwedge^1 V, \ldots, \bigwedge^k V; \bigwedge^0 V$), has a natural structure of real vector space. Its dimension is clearly given by

$$\dim k\text{-ext}(\bigwedge^1 V, \ldots, \bigwedge^k V; \bigwedge^0 V) = \dim \bigwedge^1 V \cdots \dim \bigwedge^k V \dim \bigwedge^0 V.$$  (65)

We shall need to consider only some particular cases of these general $k$-extensors over $V$. So, special names and notations will be given for them.

We will equip $V$ with an arbitrary (but fixed once and for all) Euclidean metric $G_E$, and denote the scalar product of multivectors $X, Y \in \bigwedge V$ with respect to the Euclidean metric structure $(V, G_E)$, $X \cdot Y$ instead of the more detailed notation $X \cdot Y_{G_E}$.

3.1.1 $(p, q)$-Extensors

Let $\{e_j\}$ be any basis for $V$, and $\{e^j\}$ be its Euclidean reciprocal basis for $V$, i.e., $e_j \cdot e^k = \delta^k_j$. Let $p$ and $q$ be two integer numbers with $0 \leq p, q \leq n$. A linear mapping which sends $p$-vectors to $q$-vectors will be called a $(p, q)$-extensor over $V$. The space of these objects, namely 1-ext($\bigwedge^p V; \bigwedge^q V$), will

---

6We can have, e.g., $\bigwedge^1 V = \bigwedge^3 V \oplus \bigwedge^5 V$, $\bigwedge^2 V = \bigwedge^1 V \oplus \bigwedge^3 V \oplus \bigwedge^n$, etc.
be denoted by $\text{ext}_p^q(V)$ for short. By using Eq.\(\text{(65)}\) we get

$$\dim \text{ext}_p^q(V) = \binom{n}{p} \binom{n}{q}. \quad (66)$$

For instance, we see that the $(1,1)$-extensors over $V$ are just the well-known linear operators on $V$.

To continue we recall that an Euclidean scalar product $\cdot$ has been introduced in order to be possible to define the Euclidean Clifford Algebra $\mathbb{C}(V, G_E)$ which, as we hope it is clear at this point, is the basic instrument in our calculations. This suggests the introduction of the following basis for the space of extensors.

Let $\varepsilon_{j_1..j_p;k_1..k_q} \in \text{ext}_p^q(V)$ be $(\binom{n}{p}) \binom{n}{q}$ extensors such that

$$\varepsilon_{j_1..j_p;k_1..k_q}(X) = (e_{j_1} \land \ldots \land e_{j_p}) \cdot X e_{k_1} \land \ldots \land e_{k_q}. \quad (67)$$

We show now that they define a $(p, q)$-extensor basis for $\text{ext}_p^q(V)$.

Indeed, the extensors given by Eq.\(\text{(67)}\) are linearly independent, and for each $t \in \text{ext}_p^q(V)$ there exist $(\binom{n}{p}) \binom{n}{q}$ real numbers, say $t_{j_1..j_p;k_1..k_q}$, given by

$$t_{j_1..j_p;k_1..k_q} = t(e_{j_1} \land \ldots \land e_{j_p}) \cdot (e_{k_1} \land \ldots \land e_{k_q}) \quad (68)$$

such that

$$t = \frac{1}{p!q!} \sum_{j_1..j_p;k_1..k_q} t_{j_1..j_p;k_1..k_q} \varepsilon_{j_1..j_p;k_1..k_q}. \quad (69)$$

Such $t_{j_1..j_p;k_1..k_q}$ will be called the $j_1 \ldots j_p; k_1 \ldots k_q$-th covariant components of $t$ with respect to the $(p, q)$-extensor basis $\{\varepsilon_{j_1..j_p;k_1..k_q}\}$.

Of course, there are still other kinds of $(p, q)$-extensor bases for $\text{ext}_p^q(V)$ besides the one given by Eq.\(\text{(67)}\) which can be constructed from the vector bases $\{e_j\}$ and $\{e^j\}$. The total number of these different kinds of $(p, q)$-extensor bases for $\text{ext}_p^q(V)$ are $2^{p+q}$.

Now, if we take the basis $(p, q)$-extensors $\varepsilon_{j_1..j_p;k_1..k_q}$ and the real numbers $t_{j_1..j_p;k_1..k_q}$ defined by

$$\varepsilon_{j_1..j_p;k_1..k_q}(X) = (e_{j_1} \land \ldots \land e_{j_p}) \cdot X e_{k_1} \land \ldots \land e_{k_q}, \quad (70)$$

$$t_{j_1..j_p;k_1..k_q} = t(e_{j_1} \land \ldots \land e_{j_p}) \cdot (e_{k_1} \land \ldots \land e_{k_q}), \quad (71)$$

we get an expansion formula for $t \in \text{ext}_p^q(V)$ analogous to that given by Eq.\(\text{(155)}\), i.e.,

$$t = \frac{1}{p!q!} \sum_{j_1..j_p;k_1..k_q} t_{j_1..j_p;k_1..k_q} \varepsilon_{j_1..j_p;k_1..k_q}. \quad (72)$$
Such $t^{j_1\ldots j_p;k_1\ldots k_q}$ are called the $j_1\ldots j_p;k_1\ldots k_q$-th contravariant components of $t$ with respect to the $(p,q)$-extensor basis $\{\varepsilon^{j_1\ldots j_p;k_1\ldots k_q}\}$.

### 3.1.2 Extensors

A linear mapping which sends multivectors to multivectors will be simply called an extensor over $V$. They are the linear operators on $\Lambda V$. For the space of extensors over $V$, namely $1$-$\text{ext}(\Lambda V;\Lambda V)$, we will use the short notation $\text{ext}(V)$. By using Eq. (65) we get

$$\dim \text{ext}(V) = 2^n2^n.$$  

(73)

For instance, we will see that the so-called Hodge star operator is just a well-defined extensor over $V$ which can be thought as an exterior direct sum of $(p, n-p)$-extensor over $V$.

There are $2^n2^n$ extensors over $V$, namely $\varepsilon^{J;K}$, given by

$$\varepsilon^{J;K}(X) = (e^J \cdot X)e^K$$

(74)

which can be used to introduce an extensor basis for $\text{ext}(V)$.

In fact they are linearly independent, and for each $t \in \text{ext}(V)$ there exist $2^n2^n$ real numbers, say $t_{J;K}$, given by

$$t_{J;K} = t(e_J) \cdot e_K$$

(75)

such that

$$t = \sum_J \sum_K \frac{1}{\nu(J)!} \frac{1}{\nu(K)!} t_{J;K} \varepsilon^{J;K},$$

(76)

where we define $\nu(J) = 0, 1, 2, \ldots$ for $J = \emptyset, j_1j_2, \ldots$, where all index $j_1, j_2, \ldots$ runs from 1 to $n$. Such $t_{J;K}$ will be called the $J;K$-th covariant components of $t$ with respect to the extensor bases $\{\varepsilon^{J;K}\}$.

We notice that exactly $(2^{n+1} - 1)^2$ extensor bases for $\text{ext}(V)$ can be constructed from the basis vectors $\{e_j\}$ and $\{e^j\}$. For instance, whenever the basis extensors $\varepsilon_{J;K}$ and the real numbers $t^{J;K}$ defined by

$$\varepsilon_{J;K}(X) = (e_J \cdot X)e_K,$$

$$t^{J;K} = t(e^J) \cdot e_K$$

(77)

(78)

---

7The extended (or exterior power) of $t \in \text{ext}^1(V)$ as defined in Section 3.3. is just an extensor over $V$, i.e., $\mathcal{L} \in \text{ext}(V)$.

8Recall once again that $J$ and $K$ are colective indices, $e_J = 1, e_{j_1}, e_{j_1} \wedge e_{j_2}, \ldots(e^J = 1, e^{j_1}, e^{j_1} \wedge e^{j_2}, \ldots)$. 

21
are used, an expansion formula for \( t \in \text{ext}(V) \) analogous to that given by Eq.\((76)\) can be obtained, i.e.,

\[
t = \sum_{J} \sum_{K} \frac{1}{\nu(J)!} \frac{1}{\nu(K)!} t^{J,K} \varepsilon_{J,K}.
\]  

(79)

Such \( t^{J,K} \) are called the \( J; K \)-th contravariant components of \( t \) with respect to the extensor bases \( \{ \varepsilon_{J,K} \} \).

3.1.3 Elementary \( k \)-Extensors

A multilinear mapping which takes \( k \)-uple of vectors into \( q \)-vectors will be called an elementary \( k \)-extensor over \( V \) of degree \( q \). The space of these objects, namely \( k\text{-ext}(V, \ldots, V; \wedge^q V) \), will be denoted by \( k\text{-ext}^q(V) \). It is easy to verify (using Eq.\((65)\)) that

\[
\dim k\text{-ext}^q(V) = n^k \left( \frac{n}{q} \right).  
\]  

(80)

It should be noticed that an elementary \( k \)-extensor over \( V \) of degree 0 is just a covariant \( k \)-tensor over \( V \), i.e., \( k\text{-ext}^0(V) \equiv T_k(V) \). It is easily realized that \( 1\text{-ext}^q(V) \equiv \text{ext}^1_q(V) \).

The elementary \( k \)-extensors of degrees 0, 1, 2, \ldots are sometimes said to be scalar, vector, bivector, \ldots elementary \( k \)-extensors.

The \( n^k \left( \frac{n}{q} \right) \) elementary \( k \)-extensors of degree \( q \) belonging to \( k\text{-ext}^q(V) \), namely \( \varepsilon_{j_1, \ldots, j_k;k_1 \ldots k_q} \), given by

\[
\varepsilon_{j_1, \ldots, j_k;k_1 \ldots k_q}(v_1, \ldots, v_k) = (v_1 \cdot e^{j_1}) \ldots (v_k \cdot e^{j_k}) e^{k_1} \wedge \ldots \wedge e^{k_q}
\]  

(81)

define elementary basis vectors, (i.e., \( k \)-extensor of degree \( q \)) for \( k\text{-ext}^q(V) \).

In fact they are linearly independent, and for all \( t \in k\text{-ext}^q(V) \) there are \( n^k \left( \frac{n}{q} \right) \) real numbers, say \( t_{j_1, \ldots, j_k;k_1 \ldots k_q} \), given by

\[
t_{j_1, \ldots, j_k;k_1 \ldots k_q} = t(e_{j_1}, \ldots, e_{j_k}) \cdot (e_{k_1} \wedge \ldots \wedge e_{k_q})
\]  

(82)

such that

\[
t = \frac{1}{q!} t_{j_1, \ldots, j_k;k_1 \ldots k_q} \varepsilon_{j_1, \ldots, j_k;k_1 \ldots k_q}.
\]  

(83)

Such \( t_{j_1, \ldots, j_k;k_1 \ldots k_q} \) will be called the \( j_1, \ldots, j_k; k_1 \ldots k_q \)-th covariant components of \( t \) with respect to the basis \( \{ \varepsilon_{j_1, \ldots, j_k;k_1 \ldots k_q} \} \).
We notice that exactly $2^{k+q}$ elementary $k$-extensors of degree $q$ bases for $k$-$ext^q(V)$ can be constructed from the vector bases $\{e_j\}$ and $\{e^i\}$. For instance, we may define $\varepsilon_{j_1,\ldots,j_k;1\ldots q}$ (the basis elementary $k$-extensor of degree $q$) and the real numbers $t_{j_1,\ldots,j_k; 1\ldots q}$ by

\[
\varepsilon_{j_1,\ldots,j_k; 1\ldots q}(v_1, \ldots, v_k) = (v_1 \cdot e_{j_1}) \ldots (v_k \cdot e_{j_k}) e_{k_1} \wedge \ldots \wedge e_{k_q},
\]

(84)

\[
t_{j_1,\ldots,j_k; 1\ldots q} = t(e^{j_1}, \ldots, e^{j_k}) \cdot (e_{k_1} \wedge \ldots \wedge e_{k_q}).
\]

(85)

Then, we also have other expansion formulas for $t \in k$-$ext^q(V)$ besides that given by Eq.(83), e.g.,

\[
t = \frac{1}{q!} t_{j_1,\ldots,j_k; 1\ldots q} \varepsilon_{j_1,\ldots,j_k; 1\ldots q}.
\]

(86)

Such $t_{j_1,\ldots,j_k; 1\ldots q}$ are called the $j_1, \ldots, j_k; 1\ldots q$-th contravariant components of $t$ with respect to the basis $\{\varepsilon_{j_1,\ldots,j_k; 1\ldots q}\}$.

### 3.2 Projectors

Let $\Lambda^\circ V$ be either any sum of homogeneous subspaces of $\Lambda V$ or the trivial subspace $\{0\}$. Associated to $\Lambda^\circ V$, a noticeable extensor from $\Lambda V$ to $\Lambda^\circ V$, namely $\langle \rangle_{\Lambda^\circ V}$, can defined by

\[
\langle X \rangle_{\Lambda^\circ V} = \begin{cases} 
\langle X \rangle_{p_1} + \cdots + \langle X \rangle_{p_v}, & \text{if } \Lambda^\circ V = \Lambda^{p_1} V \oplus \cdots \oplus \Lambda^{p_v} V \\
0, & \text{if } \Lambda^\circ V = \{0\}
\end{cases}.
\]

(87)

Such $\langle \rangle_{\Lambda^\circ V} \in 1$-$ext(\Lambda V; \Lambda^\circ V)$ will be called the $\Lambda^\circ V$-projector extensor.

We notice that if $\Lambda^\circ V$ is any homogeneous subspace of $\Lambda V$, i.e., $\Lambda^\circ V = \Lambda^p V$, then the projector extensor is reduced to the so-called $p$-part operator, i.e., $\langle \rangle_{\Lambda^\circ V} = \langle \rangle_{p}$.

We now summarize the fundamental properties for the $\Lambda^\circ V$-projector extensors.

Let $\Lambda_1^\circ V$ and $\Lambda_2^\circ V$ be two subspaces of $\Lambda V$. If each of them is either any sum of homogeneous subspaces of $\Lambda V$ or the trivial subspace $\{0\}$, then

\[
\langle \langle X \rangle_{\Lambda_1^\circ V} \rangle_{\Lambda_2^\circ V} = \langle X \rangle_{\Lambda_1^\circ V \cap \Lambda_2^\circ V}
\]

(88)

\[
\langle X \rangle_{\Lambda_1^\circ V} + \langle X \rangle_{\Lambda_2^\circ V} = \langle X \rangle_{\Lambda_1^\circ V \cup \Lambda_2^\circ V}.
\]

(89)

\footnote{Note that for such a subspace $\Lambda^\circ V$ there are $\nu$ integers $p_1, \ldots, p_{\nu}$ $(0 \leq p_1 < \cdots < p_{\nu} \leq n)$ such that $\Lambda_1^\circ V = \Lambda^{p_1} V \oplus \cdots \oplus \Lambda^{p_{\nu}} V$.}
Let $\bigwedge^\circ V$ be either any sum of homogeneous subspaces of $\bigwedge V$ or the trivial subspace $\{0\}$. Then, it holds

$$\langle X \rangle_{\bigwedge^\circ V} \cdot Y = X \cdot \langle Y \rangle_{\bigwedge^\circ V} .$$

(90)

We see that the concept of $\bigwedge^\circ V$-projector extensor is just a natural generalization of the concept of $p$-part operator.

3.3 Exterior Power Extension Operator

Let $\{e_j\}$ be any basis for $V$, and $\{\varepsilon^j\}$ be its dual basis for $V^*$. As we know, $\{\varepsilon^j\}$ is the unique 1-form basis associated to the vector basis $\{e_j\}$ such that $\varepsilon^j(e_i) = \delta^j_i$. The linear mapping $\text{ext}_1^1(V) \ni t \mapsto t \in \text{ext}(V)$ such that for any $X \in \bigwedge V$ and $X = X_0 + \sum_{k=1}^n X_k$, then

$$t(X) = X_0 + \sum_{k=1}^n \frac{1}{k!} X_k (\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k})$$

(91)

will be called the exterior power extension operator, or extension operator for short. We call $t$ the extended of $t$. It is the well-known outermorphism of $t$ in ordinary linear algebra.

The extension operator is well-defined since it does not depend on the choice of $\{e_j\}$.

We summarize now the basic properties satisfied by the extension operator.

**e1** The extension operator is grade-preserving, i.e.,

$$\text{if } X \in \bigwedge^p V, \text{ then } t(X) \in \bigwedge^p V.$$  

(92)

It is an obvious result which follows from Eq.(91).

**e2** For any $\alpha \in \mathbb{R}$, $v \in V$ and $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V$,

$$t(\alpha) = \alpha,$$

(93)

$$t(v) = t(v),$$

(94)

$$t(v_1 \wedge \ldots \wedge v_k) = t(v_1) \wedge \ldots \wedge t(v_k).$$

(95)
For any $X, Y \in \bigwedge V$, 
\[ t(X \wedge Y) = t(X) \wedge t(Y). \] (96)

Eq.(96) an immediate result which follows from Eq.(95).

We emphasize that the three fundamental properties as given by Eq.(93), Eq.(94) and Eq.(96) together are completely equivalent to the extension procedure as defined by Eq.(91).

We present next some important properties for the extension operator.

**e4** Let us take $s, t \in \text{ext}_{1}(V)$. Then, the following result holds
\[ s \circ t = s \circ t. \] (97)

It is enough to present the proofs for scalars and simple $k$-vectors.

For $\alpha \in \mathbb{R}$, by using Eq.(93) we get
\[ s \circ t(\alpha) = \alpha = s(\alpha) = s(t(\alpha)) = s \circ t(\alpha). \]

For a simple $k$-vector $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V$, by using Eq.(95) we get
\[ s \circ t(v_1 \wedge \ldots \wedge v_k) = s \circ t(v_1) \wedge \ldots \wedge s \circ t(v_k) = s(t(v_1)) \wedge \ldots \wedge s(t(v_k)) 
= s(t(v_1)) \wedge \ldots \wedge t(v_k) = s \circ t(v_1) \wedge \ldots \wedge v_k), 
= s \circ t(v_1) \wedge \ldots \wedge v_k).

Next, we can easily generalize to multivectors due to the linearity of extensors. It yields
\[ s \circ t(X) = s \circ t(X). \]

**e5** Let us take $t \in \text{ext}_{1}(V)$ with inverse $t^{-1} \in \text{ext}_{1}(V)$, i.e., $t^{-1} \circ t = t \circ t^{-1} = i_V$. Then, $(t^{-1}) \in \text{ext}(V)$ is the inverse of $t \in \text{ext}(V)$, i.e.,
\[ (t^{-1})^{-1} = (t^{-1}). \] (98)

Indeed, by using Eq.(97) and the obvious property $i_V = i_{\bigwedge V}$, we have that
\[ t^{-1} \circ t = t \circ t^{-1} = i_V \Rightarrow (t^{-1}) \circ t = t \circ (t^{-1}) = i_{\bigwedge V}, \]
which means that the inverse of the extended of $t$ equals the extended of the inverse of $t$.

In accordance with the above corollary we use in what follows a more simple notation $t^{-1}$ to denote both $(t)^{-1}$ and $(t^{-1})$.  

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Let \( \{ e_j \} \) be any basis for \( V \), and \( \{ e^j \} \) its Euclidean reciprocal basis for \( V \), i.e., \( e_j \cdot e^k = \delta^k_j \). There are two interesting and useful formulas for calculating the extended of \( t \in ext_1^1(V) \), i.e.,

\[
\tilde{t}(X) = 1 \cdot X + \sum_{k=1}^{n} \frac{1}{k!} (e^{j_1} \wedge \ldots \wedge e^{j_k}) \cdot X t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k})
\]

\[
= 1 \cdot X + \sum_{k=1}^{n} \frac{1}{k!} (e_{j_1} \wedge \ldots \wedge e_{j_k}) \cdot X t(e^{j_1}) \wedge \ldots \wedge t(e^{j_k}).
\]

### 3.4 Standard Adjoint Operator

Let \( \wedge_1^o V \) and \( \wedge_2^o V \) be two subspaces of \( \wedge V \) such that each of them is any sum of homogeneous subspaces of \( \wedge V \). Let \( \{ e_j \} \) and \( \{ e^j \} \) be two Euclidean reciprocal bases to each other for \( V \), i.e., \( e_j \cdot e^k = \delta^k_j \).

We call standard adjoint operator of \( t \) the linear mapping \( 1-\text{ext}(\wedge_1^o V; \wedge_2^o V) \ni t \to t^\dagger \in 1-\text{ext}(\wedge_2^o V; \wedge_1^o V) \) such that for any \( Y \in \wedge_2^o V \):

\[
t^\dagger(Y) = t((1)_{\wedge_1^o V}) \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} t((e^{j_1} \wedge \ldots \wedge e^{j_k})_{\wedge_1^o V}) \cdot Y e_{j_1} \wedge \ldots \wedge e_{j_k}
\]

\[
= t((1)_{\wedge_1^o V}) \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} t((e_{j_1} \wedge \ldots \wedge e_{j_k})_{\wedge_2^o V}) \cdot Y e^{j_1} \wedge \ldots \wedge e^{j_k}.
\]

Using a more compact notation by employing the collective index \( J \) we can write

\[
t^\dagger(Y) = \sum_{J} \frac{1}{\nu(J)!} t((e^J)_{\wedge_2^o V}) \cdot Y e_J
\]

\[
= \sum_{J} \frac{1}{\nu(J)!} t((e_J)_{\wedge_1^o V}) \cdot Y e^J,
\]

We call \( t^\dagger \) the standard adjoint of \( t \). It should be noticed the use of the \( \wedge_1^o V \)-projector extensor.

The standard adjoint operator is well-defined since the sums appearing in each one of the above places do not depend on the choice of \( \{ e_j \} \).
Let us take \( X \in \bigwedge_1^\circ V \) and \( Y \in \bigwedge_2^\circ V \). A straightforward calculation yields
\[
X \cdot t^\dagger(Y) = \sum_j \frac{1}{\nu(J)!} t(\langle e^J \rangle_{\bigwedge_1^\circ V}) \cdot Y \cdot (X \cdot e^J)
\]
\[
= t\left( \sum_j \frac{1}{\nu(J)!} \langle (X \cdot e^J) e^J \rangle_{\bigwedge_1^\circ V} \right) \cdot Y
\]
\[
= t(\langle (X)e^J \rangle_{\bigwedge_1^\circ V}) \cdot Y,
\]
i.e.,
\[
X \cdot t^\dagger(Y) = t(X) \cdot Y.
\] (105)

It is a generalization of the well-known property which holds for linear operators.

Let us take \( t \in 1\text{-}ext(\bigwedge_1^\circ V; \bigwedge_2^\circ V) \) and \( u \in 1\text{-}ext(\bigwedge_3^\circ V; \bigwedge_4^\circ V) \). We can note that \( u \circ t \in 1\text{-}ext(\bigwedge_1^\circ V; \bigwedge_3^\circ V) \) and \( t^\dagger \circ u^\dagger \in 1\text{-}ext(\bigwedge_3^\circ V; \bigwedge_1^\circ V) \). Then, let us take \( X \in \bigwedge_1^\circ V \) and \( Z \in \bigwedge_3^\circ V \), by using Eq.(105) we have that
\[
X \cdot (u \circ t)^\dagger(Z) = (u \circ t)(X) \cdot Z = t(X) \cdot u^\dagger(Z) = X \cdot (t^\dagger \circ u^\dagger)(Z).
\]

Hence, we get
\[
(u \circ t)^\dagger = t^\dagger \circ u^\dagger.
\] (106)

Let us take \( t \in 1\text{-}ext(\bigwedge_1^\circ V; \bigwedge_3^\circ V) \) with inverse \( t^{-1} \in 1\text{-}ext(\bigwedge_3^\circ V; \bigwedge_1^\circ V) \), i.e., \( t^{-1} \circ t = t \circ t^{-1} = i_{\bigwedge^\circ V} \), where \( i_{\bigwedge^\circ V} \in 1\text{-}ext(\bigwedge_1^\circ V; \bigwedge_3^\circ V) \) is the so-called identity function for \( \bigwedge^\circ V \). By using Eq.(106) and the obvious property \( i_{\bigwedge^\circ V} = i^\dagger_{\bigwedge^\circ V} \), we have that
\[
t^{-1} \circ t = t \circ t^{-1} = i_{\bigwedge^\circ V} \Rightarrow t^\dagger \circ (t^{-1})^\dagger = (t^{-1})^\dagger \circ t^\dagger = i_{\bigwedge^\circ V},
\]

hence,
\[
(t^\dagger)^{-1} = (t^{-1})^\dagger,
\] (107)
i.e., the inverse of the adjoint of \( t \) equals the adjoint of the inverse of \( t \).

In accordance with the above corollary it is possible to use a more simple symbol, say \( t^* \), to denote both of \( (t^\dagger)^{-1} \) and \( (t^{-1})^\dagger \).

Let us take \( t \in ext_1^1(V) \). We note that \( t \in ext(V) \) and \( (t^\dagger) \in ext(V) \). A

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straightforward calculation by using Eqs. (99) and (100) yields

\[
(t^\dagger)(Y) = 1 \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} (e^{j_1} \wedge \ldots e^{j_k}) \cdot Y t^\dagger(e_{j_1}) \wedge \ldots t^\dagger(e_{j_k})
\]

\[
= 1 \cdot Y + \frac{1}{k!} (e^{j_1} \wedge \ldots e^{j_k}) \cdot Y t^\dagger(e_{j_1}) \cdot e_{p_1} e^{p_1} \wedge \ldots t^\dagger(e_{j_k}) \cdot e_{p_k} e^{p_k}
\]

\[
= 1 \cdot Y + \frac{1}{k!} (e_{j_1} \cdot t(e_{p_1}) e^{j_1} \wedge \ldots e_{j_k} \cdot t(e_{p_k}) e^{j_k}) \cdot Y e^{p_1} \wedge \ldots e^{p_k}
\]

\[
= t(1) \cdot Y + \frac{1}{k!} t(e_{p_1} \wedge \ldots e_{p_k}) \cdot Y e^{p_1} \wedge \ldots e^{p_k}
\]

\[
= (t^\dagger)(Y).
\]

Hence, we get

\[
(t^\dagger) = (t)\dagger.
\]

This means that the extension operator commutes with the adjoint operator. In accordance with the above property we may use a more simple notation \( t^\dagger \) to denote without ambiguities both \( (t^\dagger) \) and \( (t)\dagger \).

### 3.5 Standard Generalization Operator

Let \( \{e_k\} \) be any basis for \( V \), and \( \{e^k\} \) be its Euclidean reciprocal basis for \( V \), as we know, \( e_k \cdot e^l = \delta^l_k \).

The linear mapping \( ext^\dagger_1(V) \ni t \mapsto \tilde{t} \in ext(V) \) such that for any \( X \in \bigwedge V \)

\[
\tilde{t}(X) = t(e^k) \wedge (e_k \perp X) = t(e_k) \wedge (e^k \perp X)
\]

will be called the generalization operator. We call \( \tilde{t} \) the generalized of \( t \).

The generalization operator is well-defined since it does not depend on the choice of \( \{e_k\} \).

We present now some important properties which are satisfied by the generalization operator.

\( g1 \) The generalization operator is grade-preserving, i.e.,

\[
\text{if } X \in \bigwedge^k V, \text{ then } \tilde{t}(X) \in \bigwedge^k V.
\]
The grade involution $\hat{\sim} \in \text{ext}(V)$, reversion $\widetilde{\sim} \in \text{ext}(V)$, and conjugation $\cdot \in \text{ext}(V)$ commute with the generalization operator, i.e.,

$$t(\hat{X}) = \hat{\sim}(t(X)), \quad (111)$$
$$t(\widetilde{X}) = \widetilde{\sim}(t(X)), \quad (112)$$
$$t(X) = \sim(t(X)). \quad (113)$$

They are immediate consequences of the grade-preserving property.

For any $\alpha \in \mathbb{R}$, $v \in V$ and $X, Y \in \bigwedge V$ it holds

$$\sim(\alpha) = 0, \quad (114)$$
$$\sim(v) = t(v), \quad (115)$$
$$\sim(X \wedge Y) = \sim(X) \wedge Y + X \wedge \sim(Y). \quad (116)$$

We can show that the basic properties given by Eq. (114), Eq. (115) and Eq. (116) together are completely equivalent to the generalization procedure as defined by Eq. (109).

The generalization operator commutes with the adjoint operator, i.e.,

$$(\sim t)^\dagger = (t^\dagger) \sim, \quad (117)$$

or put it on another way, the adjoint of the generalized of $t$ is just the generalized of the adjoint of $t$.

The proof of this result is a straightforward calculation which uses Eq. (105) and the multivector identities: $X \cdot (a \wedge Y) = (a \cdot X) \wedge Y$ and $X \cdot (a \cdot Y) = (a \wedge X) \cdot Y$, with $a \in V$ and $X, Y \in \bigwedge V$. Indeed,

$$(\sim t)^\dagger(X) \cdot Y = X \cdot \sim(t(Y))$$
$$= (e_j \wedge (t(e^j) \cdot X)) \cdot Y = (e_j \wedge (t(e^j) \cdot e^k e_k \cdot X)) \cdot Y$$
$$= (e^j \cdot t^\dagger(e^k) e_j \wedge (e_k \cdot X)) \cdot Y = (t^\dagger(e^k) \wedge (e_k \cdot X)) \cdot Y$$
$$= (t^\dagger)(X) \cdot Y.$$
Hence, by the non-degeneracy property of the Euclidean scalar product, the required result follows.

In agreement with the above property we use in what follows a more simple symbol, $t^\dagger$ to denote both $(t^\dagger)^\dagger$ or $(t^\dagger)$.

**g5** The symmetric (skew-symmetric) part of the generalized of $t$ is just the generalized of the symmetric (skew-symmetric) part of $t$, i.e.,

$$ (t)_\pm = (t_\pm)^\sim. \quad (118) $$

This property follows immediately from Eq.$(117)$.

We see also that it is possible to use a more simple notation, $t^\sim_\pm$ to denote $(t^\sim)_\pm$ or $(t_\pm)^\sim$.

**g6** The skew-symmetric part of the generalized of $t$ can be factorized by the noticeable formula $^{10}$$

$$ t^\sim_\sim (X) = \frac{1}{2} \text{biv}[t] \times X, \quad (119) $$

where $\text{biv}[t] \equiv t(e^k) \wedge e^k$ is a characteristic invariant of $t$, called the bivector of $t$.

We prove this result recalling Eq.$(118)$, the well-known identity $t_\sim(a) = \frac{1}{2} \text{biv}[t] \times a$ and the multivector identity $B \times X = (B \times e^k) \wedge (e_k \wedge X)$, with $B \in \bigwedge^2 V$ and $X \in \bigwedge V$. We have that

$$ t_\sim(X) = t_\sim(e^k) \wedge (e_k \wedge X) = (\frac{1}{2} \text{biv}[t] \times e^k) \wedge (e_k \wedge X) = \frac{1}{2} \text{biv}[t] \times X. $$

**g7** A noticeable formula holds for the skew-symmetric part of the generalized of $t$. For all $X, Y \in \bigwedge V$

$$ t_\sim(X * Y) = t_\sim(X) * Y + X * t_\sim(Y), \quad (120) $$

where $*$ is any product either $(\wedge), (\cdot), (\wedge, \cdot)$ or (Clifford product).

In order to prove this property we must use Eq.$(119)$ and the multivector identity $B \times (X * Y) = (B \times X) * Y + X * (B \times Y)$, with $B \in \bigwedge^2 V$ and

$^{10}$Recall that $X \times Y \equiv \frac{1}{2}(XY - YX)$. 

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\(X, Y \in \bigwedge V\). By taking into account Eq.(114) we can see that the following property for the Euclidean scalar product of multivectors holds

\[t \sim \sim (X) \cdot Y + X \cdot t \sim \sim (Y) = 0.\] (121)

It is consistent with the well-known property: the adjoint of a skew-symmetric extensor equals minus the extensor!

### 3.6 Determinant

We define the determinant\(^{11}\) of \(t \in \text{ext}^1_1(V)\) as the mapping \(\det : \text{ext}^1_1(V) \ni t \mapsto \det[t] \in \bigwedge^0 V \equiv \mathbb{R}\), such that for all non-zero pseudoscalar \(I \in \bigwedge^n V\)

\[t(I) = \det[t] I.\] (122)

As can easily the value of \(\det[t]\) does not depend on the choice of the pseudo-scalar \(I\). We recall that we choose the symbol \(\det[t]\) for the determinant of a \((1, 1)\)-extensor in order to not confuse this concept with the concept of the determinant of a square matrix (see below).

We present now some of the most important properties satisfied by the \(\det[t]\).

**d1** Let \(t\) and \(u\) be two \((1, 1)\)-extensors. It holds

\[\det[u \circ t] = \det[u] \det[t].\] (123)

Indeed, take a non-zero pseudoscalar \(I \in \bigwedge^n V\). By using Eq.(97) and Eq.(122) we can write that

\[
\begin{align*}
\det[u \circ t] I &= u \circ t(I) = u \circ t(I) = u(t(I)) \\
&= u(\det[t] I) = \det[t] u(I),
\end{align*}
\]

\[= \det[t] \det[u] I.\]

**d2** Let us take \(t \in \text{ext}^1_1(V)\) with inverse \(t^{-1} \in \text{ext}^1_1(V)\). It holds

\[\det[t^{-1}] = (\det[t])^{-1}.\] (124)

\(^{11}\)The concept of determinant of a \((1,1)\)-extensor is related, but distinct from the well known determinant of a square matrix. For details the reader is invited to consult [13].
Indeed, by using Eq. (123) and the obvious property \( \det[i_V] = 1 \), we have that

\[
t^{-1} \circ t = t \circ t^{-1} = i_V \Rightarrow \det[t^{-1}] \det[t] = \det[t] \det[t^{-1}] = 1,
\]

which means that the determinant of the inverse equals the inverse of the determinant.

Due to the above corollary it is often convenient to use the short notation \( \det^{-1}[t] \) for both \( \det[t^{-1}] \) and \( (\det[t])^{-1} \).

Let us take \( t \in \text{ext}^1_1(V) \). It holds

\[
\det[t^\dagger] = \det[t].
\]

Indeed, take a non-zero \( I \in \bigwedge^n V \). Then, by using Eq. (122) and Eq. (105) we have that

\[
\det[t^\dagger]I \cdot I = t^\dagger(I) \cdot I = I \cdot t(I) = I \cdot \det[t]I = \det[t]I \cdot I,
\]

whence, the expected result follows.

Let \( \{e_j\} \) be any basis for \( V \), and \( \{e^j\} \) be its Euclidean reciprocal basis for \( V \), i.e., \( e_j \cdot e^k = \delta^k_j \). There are two interesting and useful formulas for calculating \( \det[t] \), i.e.,

\[
\det[t] = t(e_1 \wedge \ldots \wedge e_n) \cdot (e^1 \wedge \ldots \wedge e^n),
\]

\[
= t(e^1 \wedge \ldots \wedge e^n) \cdot (e_1 \wedge \ldots \wedge e_n).
\]

They follow from Eq. (122) by using \( (e_1 \wedge \ldots \wedge e_n) \cdot (e^1 \wedge \ldots \wedge e^n) = 1 \) which is an immediate consequence of the formula for the Euclidean scalar product of simple \( k \)-vectors and the reciprocity property of \( \{e_k\} \) and \( \{e^k\} \).

Each of Eq. (126) and Eq. (127) is completely equivalent to the definition of determinant given by Eq. (122).

We will end this section presenting an useful formula for the inversion of a non-singular \((1,1)\)-extensor.

Let us take \( t \in \text{ext}^1_1(V) \). If \( t \) is non-singular, i.e., \( \det[t] \neq 0 \), then there exists its inverse \( t^{-1} \in \text{ext}^1_1(V) \) which is given by

\[
t^{-1}(v) = \det^{-1}[t]t^\dagger(vI)I^{-1},
\]

where \( I \in \bigwedge^n V \) is any non-zero pseudoscalar.

To show Eq. (128) we must prove that \( t^{-1} \) given by the above formula satisfies both of conditions \( t^{-1} \circ t = i_V \) and \( t \circ t^{-1} = i_V \).
Let $I \in \bigwedge^n V$ be a non-zero pseudoscalar. Take $v \in V$, by using the extensor identities\footnote{These extensor identities follow directly from the fundamental identity $X \cdot (t(Y)I) = t(t^!(X)I) \cdot Y$ with $X, Y \in \bigwedge V$. For the first one: take $X = v, Y = I$ and use $(t^!)^! = t$, eq.\eqref{eq122} and $\det(t) = \det[t]$. For the second one: take $X = vI, Y = I^{-1}$ and use eq.\eqref{eq122}.} $t^!(t(v)I)I^{-1} = t(t^!(vI)I^{-1}) = \det[t]v$, we have that
\[
t^{-1} \circ t(v) = t^{-1}(t(v)) = \det^{-1}[t]I I^{-1} = \det^{-1}[t] \det[t]v = i_V(v).
\]
And
\[
t \circ t^{-1}(v) = t(t^{-1}(v)) = \det^{-1}[t]t(t^!(vI)I^{-1}) = \det^{-1}[t] \det[t]v = i_V(v).
\]

Finally we clarify the following. Let $T \in T^2(V)$ be such that for any $u, v \in V$ we have $T(u, v) = t(u) \cdot v$. Then, given an arbitrary basis $\{e_i\}$ of $V$ we have $T(e_i, e_j) := T_{ij} = t(e_i) \cdot e_j$. The relation between the determinant of the matrix $[T_{ij}]$, denoted $\Det[T_{ij}]$ and $\det[t]$ is then given by:
\[
\Det[T_{ij}] = \det[t] (e_1 \wedge ... \wedge e_n) \cdot (e_1 \wedge ... \wedge e_n) .
\]
So, in general unless $\{e_i\}$ is an Euclidean orthonormal basis we have that $\Det[T_{ij}] \neq \det[t]$.

### 3.7 Metric and Gauge Extensors

As we know from Section 2.8 whenever $V$ is endowed with another metric $G$ (besides $G_E$) there exists an unique linear operator $g$ such that the $G$-scalar product of $X, Y \in \bigwedge V$, namely $X \cdot_g Y$, is given by
\[
X \cdot_g Y = g(X) \cdot Y .
\]

Of course, from the definition of $(p, q)$-extensors (Section 3.1.1) we immediately realize that the linear operator $g$ is a $(1,1)$-extensor. It is called the pseudo-orthogonal) metric extensor for $G$.

#### 3.7.1 Metric Extensors

We now recall that such $g \in ext^1_1(V)$ is symmetric and non-degenerate, and has signature $(p, q)$, i.e., $g = g^!, \det[g] \neq 0$, $g$ has $p$ positive and $q$ negative ($p + q = n$) eigenvalues.
Let \( \{b_j\} \) be any orthonormal basis for \( V \) with respect to \((V, G_E)\), i.e., \( b_j \cdot b_k = \delta_{jk} \).

Once the Clifford algebra \( C\ell(V, G_E) \) has been given we are able to construct exactly \( n \) (pseudo Euclidean) metric extensors with signature \((1, n-1)\). The eigenvectors \( b_1, \ldots, b_n \).

Indeed, associated to \( \{b_j\} \) we introduce the \((1,1)\)-extensors \( \eta_{b_1}, \ldots, \eta_{b_n} \) defined by
\[
\eta_{b_j}(v) = b_j v b_j,
\]
for each \( j = 1, \ldots, n \). They obviously satisfy
\[
\eta_{b_j}(b_k) = \begin{cases} b_k, & k = j \\ -b_k, & k \neq j \end{cases}.
\]

This means that \( b_j \) is an eigenvector of \( \eta_{b_j} \) with the eigenvalue \(+1\), and the \( n-1 \) basis vectors \( b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n \) all are eigenvectors of \( \eta_{b_j} \) with the same eigenvalue \(-1\).

As we can easily see, any two of these \((1,1)\)-extensors commutates, i.e.,
\[
\eta_{b_j} \circ \eta_{b_k} = \eta_{b_k} \circ \eta_{b_j}, \text{ for } j \neq k.
\]

Moreover, they are symmetric and non-degenerate, and pseudo Euclidean orthogonal, i.e.,
\[
\eta_{b_j}^\dagger = \eta_{b_j},
\]
\[
\det[\eta_{b_j}] = (-1)^{n-1}
\]
\[
\eta_{b_j}^\dagger = \eta_{b_j}^{-1}.
\]

Therefore, they all are pseudo Euclidean orthogonal metric extensors with signature \((1, n-1)\).

The extended of \( \eta_{b_j} \) is given by
\[
\overline{\eta_{b_j}}(X) = b_j X b_j,
\]

### 3.7.2 Constructing Metrics of Signature \((p, n-p)\)

We can now construct a pseudo orthogonal metric operator with signature \((p, n-p)\) and whose orthonormal eigenvectors are just the basis vectors \( b_1, \ldots, b_n \). It is defined by
\[
\eta_b = (-1)^{p+1} \eta_{b_1} \circ \cdots \circ \eta_{b_p}.
\]

\(^{13}\text{As well-known, the eigenvalues of any orthogonal symmetric operator are } \pm 1.\)
\[ \eta_b(a) = (-1)^{p+1} b_1 \ldots b_p a b_p \ldots b_1. \] (139)

We have that
\[ \eta_b(b_k) = \begin{cases} b_k, & k = 1, \ldots, p \\ -b_k, & k = p + 1, \ldots, n \end{cases} \] (140)

which means that \( b_1, \ldots, b_p \) are eigenvectors of \( \eta_b \) with the same eigenvalue \(+1\), and \( b_{p+1}, \ldots, b_n \) are eigenvectors of \( \eta_b \) with the same eigenvalue \(-1\).

The extensor \( \eta_b \) is symmetric and non-degenerate, and orthogonal, i.e., \( \eta_b^\dagger = \eta_b \), \( \det[\eta_b] = (-1)^{n-p} \) and \( \eta_b^\dagger = \eta_b^{-1} \) and thus \( \eta_b \) is a pseudo orthogonal metric extensor with signature \((p, n-p)\).

The extended of \( \eta_b \) is obviously given by
\[ \eta_b(X) = (-1)^{p+1} b_1 \ldots b_p X b_p \ldots b_1. \] (141)

What is the most general pseudo-orthogonal metric extensor with signature \((p, n-p)\)?

To find the answer, let \( \eta \) be any pseudo-orthogonal metric extensor with signature \((p, n-p)\). The symmetry of \( \eta \) implies the existence of exactly \( n \) Euclidean orthonormal eigenvectors \( u_1, \ldots, u_n \) for \( \eta \) which form just a basis for \( V \). Since \( \eta \) is pseudo-orthogonal and its signature is \((p, n-p)\), it follows that the eigenvalues of \( \eta \) are equal \( \pm 1 \) and the eigenvalues equation for \( \eta \) can be written (re-ordering \( u_1, \ldots, u_n \) if necessary) as
\[ \eta(u_k) = \begin{cases} u_k, & k = 1, \ldots, p \\ -u_k, & k = p + 1, \ldots, n \end{cases}. \]

Now, due to the orthonormality of both \( \{b_k\} \) and \( \{u_k\} \), there must be an orthogonal \((1,1)\)-extensor \( \Theta \) such that \( \Theta(b_k) = u_k \), for each \( k = 1, \ldots, n \), i.e., for all \( a \in V : \Theta(a) = \sum_{j=1}^n (a \cdot b_j) u_j \).

Then, we can write
\[
\Theta \circ \eta_b \circ \Theta^\dagger(u_k) = \Theta \circ \eta_b(b_k) = \Theta\left( \begin{cases} b_k, & k = 1, \ldots, p \\ -b_k, & k = p + 1, \ldots, n \end{cases} \right) = \begin{cases} u_k, & k = 1, \ldots, p \\ -u_k, & k = p + 1, \ldots, n \end{cases} = \eta(u_k),
\]

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for each \( k = 1, \ldots, n \). Thus, we have

\[ \eta = \Theta \circ \eta_b \circ \Theta^\dagger. \]  

By putting Eq.(138) into Eq.(142) we get

\[ \eta = (-1)^{p+1} \eta_1 \circ \cdots \circ \eta_p, \]  

where each of \( \eta_j \equiv \Theta \circ \eta_{b_j} \circ \Theta^\dagger \) is an Euclidean orthogonal metric extensor with signature \((1, n-p)\).

But, by using the vector identity 

\[ abc = (a \cdot b)c - (a \cdot c)b + (b \cdot c)a + a \wedge b \wedge c, \]

with \( a, b, c \in V \), we can prove that

\[ \eta_j(v) = \Theta(b_j)v\Theta(b_j), \]  

for each \( j = 1, \ldots, p \).

Now, by using Eq.(144) we can write Eq.(143) in the remarkable form

\[ \eta(v) = (-1)^{p+1} \Theta(b_1 \ldots b_p)v\Theta(b_p \ldots b_1). \]  

### 3.7.3 Metric Adjoint Extensor

Given a metric extensor \( g \) for \( V \), to each \( t \in 1\text{-}ext(\bigwedge^1 V; \bigwedge^2 V) \) we can assign \( t^{\dagger(g)} \in 1\text{-}ext(\bigwedge^2 V; \bigwedge^1 V) \) defined as follows:

\[ t^{\dagger(g)} = g^{-1} \circ t^\dagger \circ g. \]  

It will be called the metric adjoint of \( t \).

As we can easily see, \( t^{\dagger(g)} \) is the unique extensor from \( \bigwedge^2 V \) to \( \bigwedge^1 V \) which satisfies the fundamental property

\[ X \cdot t^{\dagger(g)}(Y) = t(X) \cdot Y, \]  

for all \( X \in \bigwedge^1 V \) and \( Y \in \bigwedge^2 V \) and where we recall the symbol \( \cdot \) refers to the scalar product determined by a general (nondegenrated) metric \( G \) or equivalently its associated extensor \( g \) according to Eq.(130).

The noticeable property given by Eq.(147) is the metric version of the fundamental property given by Eq.(105).
Lorentz Extensor  A $(1, 1)$-extensor over $V$, namely $\Lambda$, is said to be $\eta$-orthogonal if and only if for all $v, w \in V$

$$\Lambda(v) \cdot \Lambda(w) = v \cdot w. \quad (148)$$

Recalling the non-degeneracy of the $\eta$-scalar product, Eq.(148) implies that

$$\Lambda^t(\eta) = \Lambda^{-1}. \quad (149)$$

Or, by taking into account Eq.(146), we can still write

$$\Lambda^t \circ \eta \circ \Lambda = \eta. \quad (150)$$

We emphasize that the $\eta$-scalar product condition given by Eq.(148) is logically equivalent to each of Eq.(149) and Eq.(150).

Sometimes, when the signature of the $\eta$-orthogonal $(1, 1)$-extensor is $(1, n-1)$, the extensor $\Lambda$ is called a Lorentz extensor.

### 3.7.4 Gauge Extensors

Let $g$ and $\eta$ be pseudo-orthogonal metric extensors, of the same signature $(p, n-p)$. Then, we can show that there exists a non-singular $(1, 1)$-extensor $h$ such that

$$g = h^t \circ \eta \circ h. \quad (151)$$

Such $h$ is given by

$$h = d_\sigma \circ d \sqrt{|\lambda|} \circ \Theta_{uv}, \quad (152)$$

where $d_\sigma$ is a pseudo-orthogonal metric extensor, $d \sqrt{|\lambda|}$ is a metric extensor, and $\Theta_{uv}$ is a pseudo-orthogonal operator which are defined by

$$d_\sigma(a) = \sum_{j=1}^{n} \sigma_j (a \cdot u_j) u_j \quad (153)$$

$$d \sqrt{|\lambda|}(a) = \sum_{j=1}^{n} \sqrt{|\lambda_j|} (a \cdot u_j) u_j \quad (154)$$

$$\Theta_{uv}(a) = \sum_{j=1}^{n} (a \cdot u_j) v_j, \quad (155)$$

where $\sigma_1, \ldots, \sigma_n$ are real numbers with $\sigma_1^2 = \cdots = \sigma_n^2 = 1$, $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $g$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are respectively the orthonormal eigenvectors of $\eta$ and $g$. For a proof see, e.g., [13].
3.7.5  Golden Formula

Let \( h \) be any gauge extensor for \( g \), i.e., \( g = h^\dagger \circ \eta \circ h \), where \( \eta \) is a pseudo-orthogonal metric extensor with the same signature as \( g \). Let \( * \) mean either \( \wedge \) (exterior product), \( \cdot \) (\( g \)-scalar product), \( \cup \), \( \cap \) (\( g \)-contracted products) or \( \rtimes \), \( \ltimes \) (\( g \)-Clifford product). And analogously for \( \eta \).

The \( g \)-metric products \( * \) and the \( \eta \)-metric products are related by a remarkable formula, called in what follows the golden formula. For all \( X, Y \in \bigwedge V \)

\[
h_g(X * Y) = [h_g(X) * \eta g h_g(Y)],
\]

(156)

where \( h \) denotes the extended \([3]\) of \( h \). A proof of the golden formula can be found in, e.g., [13].

3.8  Hodge Extensors

3.8.1  Standard Hodge Extensor

Let \{\( e_j \)\} and \{\( e^j \)\} be two Euclidean reciprocal bases to each other for \( V \), i.e., \( e_j \cdot e^k = \delta^k_j \). Associated to them we define a non-zero pseudoscalar

\[
\tau = \sqrt{e_\Lambda \cdot e_\Lambda e^\Lambda},
\]

(157)

where \( e_\Lambda \equiv e_1 \wedge \ldots \wedge e_n \in \bigwedge^n V \) and \( e^\Lambda \equiv e^1 \wedge \ldots \wedge e^n \in \bigwedge^n V \). Note that \( e_\Lambda \cdot e_\Lambda > 0 \), since an Euclidean scalar product is positive definite. Such \( \tau \) will be called a standard volume pseudoscalar for \( V \).

The standard volume pseudoscalar has the fundamental property

\[
\tau \cdot \tau = \tau \rtimes \bar{\tau} = \tau \bar{\tau} = 1,
\]

(158)

which follows from the obvious result \( e_\Lambda \cdot e^\Lambda = 1 \).

From Eq. (158), we can easily get an expansion formula for pseudoscalars of \( \bigwedge^n V \), i.e.,

\[
I = (I \cdot \tau)\tau.
\]

(159)

The extensor \( * \in ext(V) \) which is defined by \( * : \bigwedge V \to \bigwedge V \) such that

\[
* X = \bar{X} \rtimes \tau = \bar{X} \bar{\tau},
\]

(160)

will be called a standard Hodge extensor on \( V \).
It should be noticed that

\[ \text{if } X \in \bigwedge^p V, \text{ then } \star X \in \bigwedge^{n-p} V. \]  

That means that \( \star \) can be also defined as a \((p,n-p)\)-extensor over \( V \).

The extensor over \( V \), namely \( \star^{-1} \), which is given by \( \star^{-1} : \bigwedge V \rightarrow \bigwedge V \) such that

\[ \star^{-1} X = \tau \hat{X} = \tau \tilde{X} \]  

is the inverse extensor of \( \star \). We have immediately that for any \( X \in \bigwedge V \)

\[ \star^{-1} \circ \star X = \tau \tilde{X} = X, \quad \star \circ \star^{-1} X = X \tau \tau = X, \quad \text{i.e., } \star^{-1} \circ \star = \star \circ \star^{-1} = i_{\bigwedge V}, \]

where \( i_{\bigwedge V} \in \text{ext}(V) \) is the so-called identity function for \( \bigwedge V \).

We recall now some identities that we shall use in the next papers of this series.

(i) Let for \( X, Y \in \bigwedge V \). We have

\[ (\star X) \cdot (\star Y) = X \cdot Y, \]  

which means that the standard Hodge extensor preserves the Euclidean scalar product.

(ii) Let us take \( X, Y \in \bigwedge^p V \). By using Eq.(159) together with the multivector identity \((X \wedge Y) \cdot Z = Y \cdot (X \wedge Z)\), and Eq.(163) we get

\[ X \wedge (\star Y) = (X \cdot Y) \tau. \]  

This noticeable identity is completely equivalent to the definition of the standard Hodge extensor given by Eq.(160) and indeed it is analogous to the one used to define the Hodge dual of form fields in texts dealing with the Cartan calculus of differential forms.

(iii) Let \( X \in \bigwedge^p V \) and \( Y \in \bigwedge^{n-p} V \). By using the multivector identity \((X \wedge Y) \cdot Z = Y \cdot (X \wedge Z)\) and Eq.(159) we get

\[ (\star X) \cdot Y \tau = X \wedge Y. \]  

3.8.2 Metric Hodge Extensor

Our objective in this section is to find a formula which permit us to related the Hodge dual operators associated to two distinct metrics. Let then, \( g \) be a metric extensor on \( V \) with signature \((p,q)\), i.e., \( g \in \text{ext}^1_1(V) \) such that \( g = g^\dagger \)
and \( \det[g] \neq 0 \), and it has \( p \) positive and \( q \) negative eigenvalues. Associated to \( \{ e_j \} \) and \( \{ e^j \} \) we can define another non-zero pseudoscalar

\[
\tau_g = \sqrt{|e^\wedge_g \cdot e^\wedge|} = \sqrt{|\det[g]|} \tau.
\]  

(166)

It will be called a metric volume pseudoscalar for \( V \). It has the fundamental property

\[
\tau_g \cdot \tau = \tau \perp \tau = \tau \tau = (-1)^q.
\]  

(167)

Eq.(167) follows from Eq.(158) by taking into account the definition of determinant of a \((1,1)\)-extensor, and recalling that \( \text{sgn}(\det[g]) = (-1)^q \).

An expansion formula for pseudoscalars of \( \wedge^n V \) can be also obtained from Eq.(167), i.e.,

\[
I = (-1)^q(I \cdot \tau) \tau.
\]  

(168)

The extensor \( \star_g \in \text{ext}(V) \) which is defined by \( \star_g : \wedge V \to \wedge V \) such that

\[
\star_g X = \tilde{X} \perp \tau = \tilde{X} \tau
\]  

(169)

will be called a metric Hodge extensor on \( V \). It should be noticed that in general we need to use of both the \( g \) and \( g^{-1} \) metric Clifford algebras.

We see that

\[
\text{if } X \in \bigwedge^p V, \text{ then } \star_g X \in \bigwedge^{n-p} V.
\]  

(170)

It means that \( \star_g \in \text{ext}(V) \) can also be defined as \( \star_g \in \text{ext}^{n-p}(V) \).

The extensor over \( V \), namely \( \star_g^{-1} \), which is given by \( \star_g : \wedge V \to \wedge V \) such that

\[
\star_g^{-1} X = (-1)^q \tau \perp \tilde{X} = (-1)^q \tau \tilde{X}
\]  

(171)

is the inverse extensor of \( \star \).

Let us take \( X \in \bigwedge V \). By using Eq.(167), we have indeed that

\[
\star^{-1} \circ \star_g \circ X = (-1)^q \tau \perp \tilde{X} = X, \text{ and } \star \circ \star^{-1} X = (-1)^q \tau \perp \tilde{X} = X, \text{ i.e.,}
\]

\[
\star^{-1} \circ \star = \star \circ \star^{-1} = i_{\bigwedge V}.
\]
Take $X, Y \in \bigwedge V$. The identity $(X g^{-1} A) \cdot Y = X \cdot (Y g^{-1} \tilde{A})$ and Eq.(167) yield

$$(X \circ Y) \cdot (Y \circ Y) = (-1)^q X \cdot Y. \quad (172)$$

Take $X, Y \in \bigwedge^p V$. Eq.(168), the identity $(X \wedge Y) \cdot Z = Y \cdot (\tilde{X} \downarrow Z)$ and Eq.(172) allow us to obtain

$$X \wedge (Y \circ Y) = (X \cdot Y) \circ g. \quad (173)$$

This remarkable property is completely equivalent to the definition of the metric Hodge extensor given by Eq.(169).

Take $X \in \bigwedge^p V$ and $Y \in \bigwedge^{n-p} V$. The use of identity $(X \downarrow Y) \cdot Z = Y \cdot (\tilde{X} \wedge Z)$ and Eq.(168) yield

$$(X \circ Y) \cdot Y \circ g = (-1)^q X \wedge Y. \quad (174)$$

It might as well be asked what is the relationship between the standard and metric Hodge extensors as defined above by Eq.(160) and Eq.(169).

Take $X \in \bigwedge V$. By using Eq.(166), the multivector identity for an invertible $(1,1)$-extensor $t^{-1}(X) \downarrow Y = t^\dagger(X \downarrow t^*)(Y)$, and the definition of determinant of a $(1,1)$-extensor we have that

$$X = g^{-1} \tilde{X} \sqrt{|\det| g|} \tau = \sqrt{|\det| g|} g(\tilde{X} \downarrow g^{-1}(\tau))$$

$$= \frac{|\det| g|}{\det| g|} g(\tilde{X} \downarrow \tau) = \frac{\text{sgn}(\det| g|)}{\sqrt{|\det| g|}} g \circ \star(X),$$

i.e.,

$$g \circ \star = (-1)^q g \circ \star. \quad (175)$$

Eq.(175) is then the formula which relates the metric Hodge extensor $\star$ with the standard Hodge extensor $\star$.

We already know that for any metric extensor $g \in ext^1_1(V)$ there exists a non-singular $(1,1)$-extensor $h \in ext^1_1(V)$ (the gauge extensor for $g$) such that

$$g = h^\dagger \circ \eta \circ h. \quad (176)$$
where \( \eta \in \text{ext}^1(V) \) is a pseudo-orthogonal metric extensor with the same signature as \( g \).

We now obtain a noticeable formula which relates the \( g \)-metric Hodge extensor with the \( \eta \)-metric Hodge extensor.

As we know, the \( g \) and \( g^{-1} \) contracted products \( \lrcorner_g \) and \( \lrcorner_g^{-1} \) are related to the \( \eta \)-contracted product \( \lrcorner_\eta \) (recall that \( \eta = \eta^{-1} \)) by the following *golden* formulas

\[
\begin{align*}
\hat{h}(X \lrcorner g Y) &= \hat{h}(X) \lrcorner_\eta \hat{h}(Y), \\
\hat{h}^*(X \lrcorner g^{-1} Y) &= \hat{h}^*(X) \lrcorner_\eta \hat{h}^*(Y).
\end{align*}
\]

(177)

(178)

Now, take \( X \in \bigwedge V \). By using Eq.(178), Eq.(166), the definition of determinant of a \( (1,1) \)-extensor, Eq.(176) and the obvious equation \( \tau = \tau \) we have that

\[
\star_X = \hat{h}^*(\hat{X} \lrcorner_\eta \hat{h}(\tau)) = \sqrt{|\text{det}[g]|} |\text{det}[h^*]| \hat{h}^*(\hat{X} \lrcorner_\eta \hat{h}^*(\tau)) = |\text{det}[h^*]| \hat{h}^*(\hat{X} \lrcorner_\eta \hat{h}^*(\tau)) = sgn(\text{det}[h]) \hat{h} \circ \star_\eta \circ \hat{h}^*(X),
\]

i.e.,

\[
\star_X = sgn(\text{det}[h]) \hat{h} \circ \star_\eta \circ \hat{h}^*.
\]

(179)

This formula which relates the \( g \)-metric Hodge extensor \( \star \) with the \( \eta \)-metric Hodge extensor \( \star_\eta \) will play an important role in the applications we have in mind.

### 3.9 Some Useful Bases

For future reference we end this paper introducing some related bases that are useful in the developments that we have in mind in this series.

(i) Let \( \{e_k\} \) be any basis for \( V \), and \( \{e^k\} \) be its Euclidean reciprocal basis for \( V \), i.e., \( e_k \cdot e^l = \delta^l_k \). Let us take a non-singular \( (1,1) \)-extensor \( \lambda \). Then, it is easily seen that the \( n \) vectors \( \lambda(e_1), \ldots, \lambda(e_n) \in V \) and the \( n \) vectors

\[14]\text{Recall that } \lambda^* = (\lambda^{-1})^\dagger = (\lambda^\dagger)^{-1}.\]
$\lambda^*(e^1), \ldots, \lambda^*(e^n) \in V$ define two well-defined Euclidean reciprocal bases for $V$, i.e.,

$$\lambda(e_k) \cdot \lambda^*(e^i) = \delta^i_k. \quad (180)$$

The bases $\{\lambda(e_k)\}$ and $\{\lambda^*(e^k)\}$ are conveniently said to be a $\lambda$-deformation of the bases $\{e_k\}$ and $\{e^k\}$. Sometimes, the first ones are named as the $\lambda$-deformed bases of the second ones.

(ii) Let $h$ be a gauge extensor for $g$, and $\eta$ be a pseudo-orthogonal metric extensor with the same signature as $g$. According to Eq. (151), the $g$-scalar product and $g^{-1}$-scalar product are related to the $\eta$-scalar product by the following formulas

$$X \cdot Y_g = h(X) \cdot \eta h(Y), \quad X \cdot Y_{g^{-1}} = h^*(X) \cdot \eta h^*(Y). \quad (181)$$

The $\eta$-deformed bases $\{h(e_k)\}$ and $\{h^*(e^k)\}$ satisfy the noticeable properties

$$h(e_j) \cdot h(e_k) = g, \quad h^*(e^j) \cdot h^*(e^k) = g^{jk}. \quad (182)$$

The bases $\{h(e_k)\}$ and $\{h^*(e^k)\}$ are called the gauge bases associated with $\{e_k\}$ and $\{e^k\}$.

(iii) Let $u_1, \ldots, u_n$ be the $n$ Euclidean orthonormal eigenvectors of $\eta$, i.e., the eigenvalues equation for $\eta$ can be written (reordering $u_1, \ldots, u_n$ if necessary) as

$$\eta(u_k) = \begin{cases} u_k, & k = 1, \ldots, p \\ -u_k, & k = p + 1, \ldots, n \end{cases},$$

and $u_j \cdot u_k = \delta_{jk}$.

The $h^{-1}$-deformed bases $\{h^{-1}(u_k)\}$ and $\{h^\dagger(u_k)\}$ satisfy the remarkable properties

$$h^{-1}(u_j) \cdot h^{-1}(u_k) = \eta_{jk}, \quad h^\dagger(u_j) \cdot h^\dagger(u_k) = \eta_{jk}, \quad (183)$$

where

$$\eta_{jk} \equiv \eta(u_j) \cdot u_k = \begin{cases} 1, & j = k = 1, \ldots, p \\ -1, & j = k = p + 1, \ldots, n \\ 0, & j \equiv k \end{cases}.$$
4 Conclusions

This paper, the first in a series of four presents the theory of geometric (Clifford) algebras and the theory of extensors. Our presentation has been devised in order to provide a powerful computational tool which permits its efficient application in the study of differential geometry (in the next papers of the series) in a natural and simple way. Besides a thoughtful presentation of the concepts, we detailed many calculations in order to provide the “tricks of the trade” to readers interested in applications\textsuperscript{16}. Among the results obtained worth to quote here, a distinction goes to the theory of deformation of the Euclidean Clifford algebra $\mathcal{C}\ell(V, G_E)$ through the use of metric and gauge extensors which permits to generate all other Clifford algebras $\mathcal{C}\ell(V, G)$ (where $G$ is a metric of signature $(p, q)$ with $p + q = n = \dim V$ and $G_E$ is an Euclidean metric in $V$) and derivation of the golden formula, essential for reducing all calculations in any Clifford algebra $\mathcal{C}\ell(V, G)$ to the ones in $\mathcal{C}\ell(V, G_E)$, thus providing many useful formulas as, e.g., a remarkable relation between Hodge (star) operators associated to $G$ and $G_E$.

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\textsuperscript{16}We advise that someone interested in aspects of the general theory of Clifford algebras not covered here we strongly recommend the excellent textbooks \cite{Ablamowicz:2004, Moya:2007, Fernandez:2001}.
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