ON THREE-WAVE INTERACTION SCHRÖDINGER SYSTEMS
WITH QUADRATIC NONLINEARITIES: GLOBAL WELL-POSEDNESS AND STANDING WAVES

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Abstract. Reported here are results concerning the global well-posedness in
the energy space and existence and stability of standing-wave solutions for 1-
dimensional three-component systems of nonlinear Schrödinger equations with
quadratic nonlinearities. For two particular systems we are interested in, the
global well-posedness is established in view of the a priori bounds for the local
solutions. The standing waves are explicitly obtained and their spectral
stability is studied in the context of Hamiltonian systems. For more general
Hamiltonian systems, the existence of standing waves is accomplished with a
variational approach based on the Mountain Pass Theorem. Uniqueness results
are also provided in some very particular cases.

1. Introduction. The propagation of optical beams in a nonlinear dispersive medi-
un with quadratic response has attracted the attention of a broad community of
physicists in recent years. Particularly because for very short pulses, the nonlinear
cubic Schrödinger (NLS) equation, which is good enough to describe long-distance
propagation, should be corrected to include additional terms that take into ac-
count effects as higher-order dispersion and Raman scattering (see [15] for a brief
explanation). As a result, models generalizing the cubic NLS equation should be
derived. An important processes in such a direction is the so-called multistep cascading
mechanism. In particular, multistep cascading can be achieved by second-order
nonlinear processes such as second harmonic generation (SHG) and
sum-frequency mixing (SFM) (see [18]). These procedures has gained attention due to their po-
tential ability to produce stronger nonlinearities.

Here, we are mostly interested in two particular models: three-step cascading
models. For the first one, we consider the fundamental beam with frequency \( \omega \)
entering a nonlinear medium with a quadratic response. Let \( A \) and \( B \) be two
orthogonal polarization components of the fundamental wave. Also, denote by \( S \) and
\( T \) two orthogonal polarizations of the second harmonic wave with frequency \( 2\omega \). Ac-
cording to [16] (see also [15]) a multistep cascading process consists of the following.
The fundamental harmonic wave generates the second harmonic wave \( S \) via SHG
process. Then by a down-conversion process the orthogonal fundamental wave \( B \) is

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generated and, finally, the fundamental wave \( A \) can be reconstructed. The reduced amplitude equations derived in the slowly varying envelope approximation with the assumption of zero absorption of all interacting waves is given, in dimensionless variables, by the following coupled nonlinear Schrödinger system

\[
\begin{align*}
2i\psi_t + \psi_{xx} - \beta \psi + \frac{1}{2}(u^2 + v^2) &= 0, \\
i\psi_t + \psi_{xx} - \beta_1 \psi + \chi \overline{\psi} &= 0, \\
iu_t + u_{xx} - u + \overline{\psi} &= 0,
\end{align*}
\]  

(1)

where \( x, t \in \mathbb{R} \), \( w = w(x,t) \), \( v = v(x,t) \), and \( u = u(x,t) \) are complex-valued functions representing the \( S \), \( B \), and \( A \) polarizations, respectively. The positive real constants \( \beta \) and \( \beta_1 \) are dimensionless parameters that characterize the nonlinear phase matching between the parametrically interacting waves. The parameter \( \chi > 0 \) depends on the type of phase matching and can take any positive value. Here and throughout, \( \overline{\psi} \) denotes the complex conjugate of \( \psi \).

For the second model, the procedure is similar but now, as a second step, the third-harmonic wave is generated. The reduced amplitude equations is now given by (see [15])

\[
\begin{align*}
2i\psi_t + \psi_{xx} - \psi + \overline{\psi} + \psi u &= 0, \\
2i\psi_t + \psi_{xx} - \beta \psi + \frac{1}{2}w^2 + \overline{w}u &= 0, \\
3iu_t + u_{xx} - \beta_1 u + \chi \overline{w} &= 0,
\end{align*}
\]  

(2)

where again \( w = w(x,t) \), \( v = v(x,t) \), and \( u = u(x,t) \) are complex-valued functions representing (in dimensionless variables) the complex electric fields envelopes of the fundamental harmonic, second harmonic, and third harmonic, respectively. The constants \( \beta \), \( \beta_1 \), and \( \chi \) are, as before, dimensionless parameters and has similar physical meaning. The system (2) presents a fundamental model for three-wave multistep cascading solitons in the absence of walk-off (see [15]).

From the mathematical point of view, the study of nonlinear quadratic Schrödinger systems with two or three-wave interaction, in one or higher dimension, has attracted the attention of many researchers in recent years (see [1, 3, 4, 5, 9, 17, 22, 23, 26, 33, 34], and references therein). Since such systems appear in physical applications, the topics of study are mainly focused on local and global well-posedness of the Cauchy problem, existence and nonlinear/spectral stability of standing or traveling waves, asymptotic behaviour of global solutions and blow up. In general, even when a result is known for the corresponding scalar equation, in view of the wave interactions, the extension of such a result for systems demands extra efforts.

Our first concern in this paper is to establish the global well-posedness of (1) and (2) in \( L^2 \) and in \( H^1 \) (the usual \( L^2 \)-based Sobolev space of order 1), the energy space. The local-in-time existence of solutions follows from the standard contraction mapping principle, applied to the equivalent integral formulation. The global well-posedness in \( L^2 \) follows directly in view of the conservation of the total power. In a similar fashion, the global well-posedness in \( H^1 \) follows from de conservation of the energy.

Our second concern consists in establishing the existence of standing-wave solutions. Here, we will pay particular attention to the case \( \chi = 1 \). This case remains physically important because according to [15], when sum-frequency mixing is due to the third-order down-conversion frequency mixing process one should take \( \chi = 1 \). The mathematical reason for this is that under such an assumption the systems can be viewed as Hamiltonian systems.
Standing waves or optical solitons are, roughly speaking, waves that propagate without changing their profile. For system (1) they are special solutions of the form
\[
\begin{align*}
    w(x,t) &= e^{2i\gamma t} \psi(x), \\
    v(x,t) &= e^{i\gamma t} \phi(x), \\
    u(x,t) &= e^{i\gamma t} \varphi(x),
\end{align*}
\] (3)
and for system (2) they are of the form
\[
\begin{align*}
    w(x,t) &= e^{i\gamma t} \psi(x), \\
    v(x,t) &= e^{2i\gamma t} \phi(x), \\
    u(x,t) &= e^{3i\gamma t} \varphi(x),
\end{align*}
\] (4)
where \(\gamma\) is a real parameter and \(\psi, \phi, \) and \(\varphi\) are real-valued functions decaying to zero as \(x \to \pm \infty\). Substituting (3) into (1), we obtain the following nonlinear system of ordinary differential equations
\[
\begin{align*}
    \psi'' - (4\gamma + \beta) \psi + \frac{1}{2}(\varphi^2 + \phi^2) &= 0, \\
    \phi'' - (\gamma + \beta_1) \phi + \psi \varphi &= 0, \\
    \varphi'' - (\gamma + 1) \varphi + \psi \phi &= 0.
\end{align*}
\] (5)
Hence, (1) possesses standing waves of the form (3), if one shows that (5) admits at least one solution decaying to zero at infinity. We are first interested in proportional waves. Under suitable assumptions (see Section 3), we can show that (5) admits a solution with
\[
\begin{align*}
    \phi &= a \psi, \\
    \varphi &= b \psi,
\end{align*}
\] (6)
and
\[
\psi(x) = \frac{3(\gamma + 1)}{2} \sech^2 \left( \frac{\sqrt{\gamma + 1}}{2} x \right). \quad (7)
\]
Similarly, substituting (4) in (2), we see that \(\psi, \phi, \) and \(\varphi\) must satisfy the system
\[
\begin{align*}
    \psi'' - (\gamma + 1) \psi + \psi \phi + \phi \varphi &= 0, \\
    \phi'' - (4\gamma + \beta) \phi + \frac{1}{2} \psi^2 + \psi \varphi &= 0, \\
    \varphi'' - (9\gamma + \beta_1) \varphi + \phi \psi &= 0.
\end{align*}
\] (8)
Also in this situation, assuming suitable balance between the parameters, we are able to show that (8) has a solution fulfilling relation (6) with
\[
\psi(x) = \frac{3(\gamma + 1)}{2a(b + 1)} \sech^2 \left( \frac{\sqrt{\gamma + 1}}{2} x \right). \quad (9)
\]
With the above standing waves in hand, their stability is at issue. Our main goal here is to show that the standing waves (3) and (4) are spectrally stable. The approach employed here was established by Grillakis \[6,7\] (see also \[24, 25, 32\] where the theory was already been used for coupled systems) and, roughly speaking, it consists in counting the number of real eigenvalues of the linearized problem around the standing wave. We emphasize that for both systems, we are taking the advantage of their Hamiltonian structures.

It is to be observed that the above proportional standing wave solutions are obtained in a very special regime. Next, we will be concerned with non-proportional waves. To this end, after renaming variables, we note that (5) and (8) may be written in the form
\[
\begin{align*}
    w'' - \alpha_2 w + F_w(w,v,u) &= 0, \\
    v'' - \alpha v + F_v(w,v,u) &= 0, \\
    u'' - \alpha_1 u + F_u(w,v,u) &= 0,
\end{align*}
\] (10)
with \(F(w,v,u) = \frac{1}{2} w(u^2 + v^2)\) and \(F(w,v,u) = u w v + \frac{1}{2} w^2 v\), respectively. This permits us writing such systems in a variational form. The method we use here
allows us to consider more general quadratic nonlinearities. So, we can see (5) and (8) as particular cases of the more general system (10).

From this point on, we consider system (10) and assume the following concerning the nonlinearity $F$:

(F) the function $F$ has the form

$$F(w, v, u) = \sum_{0 \leq i, j, k \leq 3, i+j+k=3} a_{ijk} w^i v^j u^k,$$

with $a_{ijk} \geq 0$, and

$$\nabla F(w, v, u) \neq 0,$$ for $w \neq 0, v \neq 0, u \neq 0$.

**Remark 1.** Of course the functions $F(w, v, u) = \frac{1}{2}w(u^2 + v^2)$ and $F(w, v, u) = wvu + \frac{1}{2}w^2v$ satisfy the assumptions in (F).

Under assumption (F), we follow close the arguments in [34], which employs the Mountain Pass Theorem, to prove that the variational functional associated with (10) has at least one nontrivial critical point in the functional space $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R})$. From elliptic regularity, such a critical point turns out to be a smooth solution of (10).

Besides this introduction, the paper is organized as follows. In Section 2 we observe that the local Cauchy problems associated with (1) and (2) can be solved in a standard way and then establish the global well-posedness. In Sections 3 and 4 we show the existence and stability of the proportional standing waves. In Section 5, we introduce the basic framework to study (10) by employing a variational approach and prove our main theorem concerning non-proportional standing waves. Finally, in Section 6, it is proved some uniqueness results for the particular systems (5) and (8).

**Notation.** Let us give some notation. By $C$ we denote several constants that may vary from line to line. By $L^p(\mathbb{R})$, $1 \leq p \leq \infty$ we denote the standard Lebesgue space with norm $\| \cdot \|_{L^p}$. For $s \in \mathbb{R}$, we let $H^s(\mathbb{R})$ be the usual $L^2$-based Sobolev space of order $s$. To simplify the notation, we set

$$H^s = H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^s(\mathbb{R}), \quad s \in \mathbb{R},$$

and

$$L^p = L^p(\mathbb{R}) \times L^p(\mathbb{R}) \times L^p(\mathbb{R}), \quad 1 \leq p \leq \infty,$$

endowed with their standard norms. In particular, for real-valued functions, in $H^1$ it is defined the inner product

$$((f_1, f_2, f_3), (g_1, g_2, g_3))_{H^1} = \int_{\mathbb{R}} (f'_1 g'_1 + f'_2 g'_2 + f'_3 g'_3 + f_1 g_1 + f_2 g_2 + f_3 g_3) dx.$$

Thus, $H^1$ becomes a real Hilbert space with induced norm

$$\|(f_1, f_2, f_3)\|_{H^1}^2 = \int_{\mathbb{R}} (f'^2_1 + f'^2_2 + f'^2_3 + f_1^2 + f_2^2 + f_3^2) dx.$$

It should be clear that in Section 2 the functions are taken to be complex-valued. We still use the above notations with the standard modifications and believe this will not cause a confusion. The product of a 3-by-3 matrix $A$ by a column vector $(v_1, v_2, v_3)^T$ will be denote by $A(v_1, v_2, v_3)$. 
2. Well-posedness results. This section is devoted to establish the global well-posedness of systems (1) and (2). Attention is mainly turned to the energy spaces $L^2$ and $H^1$, because these are the natural spaces to study the soliton solutions.

2.1. Local and global well-posedness in $L^2$. Our first result establishes the local and global well-posedness in $L^2$. More precisely,

**Theorem 2.1.** Assume that $(w_0, v_0, u_0) \in L^2$. Then, there is a unique global solution $(w, v, u)$ of (1) (or (2)) such that $(w(0), v(0), u(0)) = (w_0, v_0, u_0)$ and, for any $T > 0$, $(w, v, u) \in C([0, T]; L^2)$. In addition, for any $T < \infty$,

$$w, v, u \in C([0, T]; L^2(\mathbb{R})) \cap L^{12}([0, T]; L^3(\mathbb{R})) := Y,$$

and the map data-solution is Lipschitz from a neighborhood of $(w_0, v_0, u_0)$ in $L^2$ to $Y \times Y \times Y$.

**Proof.** As usual, first we need to show the local well-posedness, that is, the solution exists at least on a time interval, say, $[0, T]$. This is quite standard by now. Indeed, the existence of a unique local solution with

$$w, v, u \in C([0, T]; L^2(\mathbb{R})) \cap L^{12}([0, T]; L^3(\mathbb{R})),$$

and $T$ depending only upon the $L^2$ norm of the initial data follows from an application of the contraction mapping principle, applied to the equivalent integral formulation, taking into account the well-known Strichartz estimates. This is a consequence of the fact that quadratic nonlinearities are “subcritical” in one dimension (see, for instance, [20, Chapter 5]). The global existence then follows because (1) and (2) conserve the total power given, respectively, by

$$\mathcal{M}(t) = \int_{\mathbb{R}} \left( |u|^2 + \frac{|v|^2}{\lambda} + 4|w|^2 \right) dx$$

and

$$\widetilde{\mathcal{M}}(t) = \int_{\mathbb{R}} \left( \frac{9}{\lambda} |u|^2 + 4|v|^2 + |w|^2 \right) dx.$$ (12) (13)

This completes the proof of the theorem. \qed

2.2. Local and global well-posedness in $H^1$. This subsection is devoted to study local and global well-posedness in $H^1$. As we show below, with no much efforts we can indeed establish the local well-posedness in $H^s$, $s > 1/2$. So, we start with the following result.

**Lemma 2.2.** Assume that $(w_0, v_0, u_0) \in H^s$, $s > 1/2$. Then, there are $T > 0$, depending only on the norm of the initial data, and a unique local solution $(w, v, u)$ of (1) (or (2)) such that $(w(0), v(0), u(0)) = (w_0, v_0, u_0)$ and $(w, v, u) \in C([0, T]; H^s)$. In addition, the map data-solution is Lipschitz from a neighborhood of $(w_0, v_0, u_0)$ in $H^s$ to $Z \times Z \times Z$, where

$$Z := C([0, T]; H^s(\mathbb{R})).$$

**Proof.** In this case, the existence of a local solution

$$(w, v, u) \in C([0, T]; H^s(\mathbb{R})), \quad \text{follows directly from the contraction mapping principle, applied to the equivalent integral formulation, just taking into consideration that } H^s(\mathbb{R}) \text{ is a Banach algebra for } s > 1/2. \text{ Of course, the existence time } T \text{ depends on the } H^s \text{ norm of the initial data.} \quad \Box$$
Remark 2. As is well known, in addition to the conclusions of Lemma 2.2, there is a blow up alternative, that is, there exists $0 < T^* \leq \infty$ such that the solution given in Lemma 2.2 is defined in $[0, T^*)$ and if $T^* < \infty$ then
\[
\lim_{t \to T^*} \|(w(t), v(t), u(t))\|_{H^1} = +\infty.
\]
Concerning global well-posedness, our result writes as follows.

Theorem 2.3. Assume $\chi > 0$ and $(w_0, v_0, u_0) \in H^1$. Then the local solution obtained in Lemma 2.2 can be extended to any time interval of the form $[0, T], T > 0$.

Proof. In order to extend the solution globally by using the standard extension principle, it suffices to establish a uniform bound for the local solution.

Initially, we turn our attention to system (1). In what follows we proceed formally and deduce some suitable quantities. This procedure can be made rigorous by taking sufficient regular solutions and then passing to the limit. Multiplying the first equation in (1) by $\bar{w}_t$, summing with its complex conjugate, and integrating on $\mathbb{R}$, we deduce
\[
\frac{d}{dt} \int_{\mathbb{R}} (|w_x|^2 + \beta |w|^2) dx = \frac{1}{2} \int_{\mathbb{R}} (|\bar{w}_t|^2 + |w|^2) + w_t (\bar{w}^2 + \bar{v}^2) dx. \tag{14}
\]
Similarly,
\[
\frac{d}{dt} \int_{\mathbb{R}} (|v_x|^2 + \beta_1 |v|^2) dx = \chi \int_{\mathbb{R}} (\bar{v}_t \bar{w} + v \bar{v}) dx \tag{15}
\]
and
\[
\frac{d}{dt} \int_{\mathbb{R}} (|u_x|^2 + |u|^2) dx = \int_{\mathbb{R}} (\bar{u}_t \bar{w} + u \bar{u}) dx. \tag{16}
\]
Now let us define the quantity,
\[
E(t) = \int_{\mathbb{R}} \left( |u_x|^2 + |w_x|^2 + |u|^2 + \beta |w|^2 + \frac{1}{\chi} (|v_x|^2 + \beta_1 |v|^2) \right.
\]
\[
- Re(u^2 \bar{w}) - Re(v^2 \bar{v}) \left. \right) dx. \tag{17}
\]
By using (14)–(16), it is easily seen that
\[
\frac{dE}{dt} = 0. \tag{18}
\]
This means that $E$ is a conserved quantity of (1), that is, $E(t) = E(0)$ as long as the local solution exists. This is enough to get the uniform bound of the solutions.

Indeed, since $\beta, \beta_1, \chi > 0$, the quantity
\[
\int_{\mathbb{R}} \left( (|u_x|^2 + |w_x|^2 + |u|^2 + \beta |w|^2 + \frac{1}{\chi} (|v_x|^2 + \beta_1 |v|^2)) dx \right.
\]
is an equivalent norm in $H^1$. Hence,
\[
\|(w, v, u)\|^2_{H^1} = E(0) + Re \int_{\mathbb{R}} (u^2 \bar{w} + v^2 \bar{v}) dx
\]
\[
\leq E(0) + \|w\|_{L^\infty} (\|u\|^2_{L^2} + \|v\|^2_{L^2})
\]
\[
\leq E(0) + C \|w\|_{H^1} (\|u\|^2_{L^2} + \|v\|^2_{L^2}) \tag{19}
\]
\[
\leq E(0) + \frac{1}{2} \|(w, v, u)\|^2_{H^1} + \frac{C^2}{2} (\|u\|^2_{L^2} + \|v\|^2_{L^2})^2,
\]
where we used Sobolev and Young’s inequalities. Combining (19) with (12), we easily see that $\|(w, v, u)\|_{H^1}$ is bounded by a universal constant.
Next, we turn attention to system (2). Define,
\[ \tilde{E}(t) = \int_{\mathbb{R}} \left( |v_x|^2 + |w_x|^2 + \beta |v|^2 + |w|^2 + \frac{1}{\chi} (|u_x|^2 + \beta_1 |u|^2) - \text{Re}(w^2 v) - 2 \text{Re}(w v u) \right) dx. \]  
(20)

Following the same steps as before, we deduce
\[ \frac{d\tilde{E}}{dt} = 0. \]
Again, we see that \( \tilde{E} \) is conserved and, consequently, the norm \( \| (w, v, u) \|_{H^1} \) is uniformly bounded, in any finite-length interval, as long as the local solution exists. The proof of Theorem 2.3 is thus completed.

3. **Spectral stability of standing waves for system (1).** In this section, we will first prove that under suitable conditions on the parameters, system (5) possesses an explicit solution. The spectral stability/instability of such a solution then becomes the main issue of study.

In order to solve (5), we look for proportional solutions \( \phi = a \psi \) and \( \varphi = b \psi \), where \( a \) and \( b \) are real constants. By assuming \( 4 \gamma + \beta = \gamma + \beta_1 = \gamma + 1 \), that is, \( \beta_1 = 1 \) and \( \gamma = (1 - \beta)/3 \), we see that (5) reduces to
\[ \begin{cases} 
\psi'' - (\gamma + 1) \psi + \frac{1}{2} (a^2 + b^2) \psi^2 = 0, \\
\psi'' - (\gamma + 1) \psi + \psi^2 = 0.
\end{cases} \]  
(21)

Finally, by assuming that \( a \) and \( b \) belong to the circle of radius \( \sqrt{2} \), that is,
\[ a^2 + b^2 = 2, \]  
(22)

system (21) reduces to the single equation
\[ - \psi'' + (\gamma + 1) \psi - \psi^2 = 0. \]  
(23)
For \( \gamma + 1 > 0 \), which implies \( \beta < 4 \), (23) has the solution
\[ \psi(x) = \frac{3(\gamma + 1)}{2} \text{sech}^2 \left( \frac{\sqrt{\gamma + 1}}{2} x \right). \]  
(24)

Next, we intent to study the spectral stability of the above solution. As we will see below, our approach is based on the Hamiltonian structure of (1). In order to write (1) as a real Hamiltonian system we set \( w = P + iQ, \) \( v = M + iN, \) and \( u = R + iS. \) Separating real and imaginary parts, we see that it reads as
\[ \frac{dU}{dt}(t) = JE'(U(t)), \]  
(25)
where \( U = (P, M, R, Q, N, S) \), \( J \) is the skew-symmetric matrix
\[ J = \begin{pmatrix}
0 & 0 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}, \]  
(26)
and $E$ is the energy functional (17). In (25), $E'$ denotes the Fréchet derivative of $E$. To simplify notation we set

$$\Psi := (\psi, a\psi, b\psi, 0, 0, 0) \equiv (\psi, \phi, \varphi, 0, 0, 0).$$ \hfill (27)

We now define the constrained energy $G$ by

$$G(U) = E(U) + \gamma M(U).$$ \hfill (28)

In view of (5), it is easily seen that $\Psi$ is a critical point of $G$, that is, $G'(\Psi) = 0$. As a result, it is expected that the stability of the waves in question is determined by the Hessian $G''(\Psi)$.

Next we observe that $G''(\Psi)$ is given by a 6-by-6 matrix operator, more precisely,

$$G''(\Psi) := L = \left( \begin{array}{cc} L_R & 0 \\ 0 & L_I \end{array} \right),$$ \hfill (29)

where

$$L_R = \left( \begin{array}{ccc} -\partial^2_x + \beta + 4\gamma & -\phi & -\varphi \\ -\phi & -\partial^2_x + \gamma + 1 & 0 \\ -\varphi & 0 & -\partial^2_x + \gamma + 1 \end{array} \right)$$ \hfill (30)

and

$$L_I = \left( \begin{array}{ccc} -\partial^2_x + \beta + 4\gamma & -\phi & -\varphi \\ -\phi & -\partial^2_x + \gamma + 1 & 0 \\ -\varphi & 0 & -\partial^2_x + \gamma + 1 \end{array} \right).$$ \hfill (31)

3.1. Spectral analysis. Here we will study the spectrum of the operator $L$. Due to the interaction between the components, the spectrum of matrix-type operators are, in general, hard to be described in details (see [29] for additional discussions). However, by using a diagonalization argument we are able to count the number of nonpositive eigenvalues of $L$. First of all, by noting that $L$ is a “diagonal” operator, a complex number $\lambda$ is an eigenvalue of $L$ if and only if $\lambda$ is an eigenvalue of either $L_R$ or $L_I$. Thus, it suffices to study the spectrum of the operators $L_R$ and $L_I$.

3.1.1. The spectrum of $L_R$. By recalling that $\beta + 4\gamma = \gamma + 1$ and $\delta = a\psi$, $\varphi = b\psi$, we see that

$$L_R = \left( \begin{array}{ccc} -\partial^2_x + \gamma + 1 & 0 & 0 \\ 0 & -\partial^2_x + \gamma + 1 & 0 \\ 0 & 0 & -\partial^2_x + \gamma + 1 \end{array} \right) + \psi \left( \begin{array}{ccc} 0 & -a & -b \\ -a & -1 & 0 \\ -b & 0 & -1 \end{array} \right).$$ \hfill (32)

It then suffices to diagonalize the constant-coefficient matrix in the second term of (32), which we shall call $E_R$. Since $E_R$ is a symmetric matrix, it is equivalent to a diagonal matrix for which the principal diagonal entries are the eigenvalues of $E_R$. It is easy to see that $E_R$ has $r_1 = 1$, $r_2 = -1$, and $r_3 = -2$ as eigenvalues. Let $A_R$ be the matrix whose columns are normalized eigenvectors associated with the eigenvalues $r_1$, $r_2$, and $r_3$, respectively. More precisely,

$$A_R = \left( \begin{array}{ccc} -\sqrt{2} & 0 & 1 \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \\ 2\sqrt{3} & \sqrt{3} & \sqrt{3} \\ b\sqrt{2} & a & b \\ 2\sqrt{3} & \sqrt{2} & \sqrt{3} \end{array} \right).$$ \hfill (33)
Note that
\[ A_R^{-1} = A_R^* = A_R^T. \] (34)

With the matrix \( A_R \) in hand, we are able to diagonalize the operator \( L_R \) by defining
\[ L_{RD} := A_R^{-1} L_R A_R, \]
that is,
\[
L_{RD} = \begin{pmatrix}
-\partial_x^2 + \gamma + 1 + \psi & 0 & 0 \\
-\partial_x^2 + \gamma + 1 - \psi & 0 & 0 \\
0 & -\partial_x^2 + \gamma + 1 - 2\psi & 0
\end{pmatrix}. \] (35)

In the sequel we shall study the spectrum of \( L_{RD} \). Since \( L_{RD} \) is a diagonal operator, it is sufficient to describe the spectrum of the operators appearing in the diagonal entries. In what follows we set
\[
L_j := -\partial_x^2 + \gamma + 1 - j\psi, \quad j = -2, -1, 1, 2. \] (36)

Since \( L_j \) is a compact perturbation of \(-\partial_x^2 + \gamma + 1\), Weyl’s theorem implies that the essential spectrum of \( L_j \) is the interval \([\gamma + 1, \infty)\).

**Lemma 3.1.** Let \( \psi \) be the function in (24).

(i) The operator \( L_2 \) defined in \( L^2(\mathbb{R}) \) with domain \( H^2(\mathbb{R}) \) has a unique negative eigenvalue, which is simple. Zero is a simple eigenvalue with associated eigenfunction \( \psi' \). Moreover, the rest of the spectrum is positive and bounded away from zero.

(ii) The operator \( L_1 \) defined in \( L^2(\mathbb{R}) \) with domain \( H^2(\mathbb{R}) \) has no negative eigenvalues. Zero is a simple eigenvalue with associated eigenfunction \( \psi \). Moreover, the rest of the spectrum is positive and bounded away from zero.

(iii) The operator \( L_{-1} \) defined in \( L^2(\mathbb{R}) \) with domain \( H^2(\mathbb{R}) \) is strictly positive.

**Proof.** The proof is essentially an application of the Sturm-Liouville theory combined with the comparison theorem. Indeed, by taking the derivative with respect to \( x \) in (23), we immediately see that zero is an eigenvalue of \( L_2 \) with associated eigenfunction \( \psi' \). Since \( \psi' \) has exactly one zero on the whole line, it follows that the eigenvalue zero is the second one. This proves part (i).

Also from (23), we promptly see that zero is the first eigenvalue of \( L_1 \) with eigenfunction \( \psi \). Finally, to establish part (iii) it suffices to apply the comparison theorem taking into account part (ii) and the fact that \( \psi > -\psi \). \[ \square \]

As an immediate consequence we have.

**Corollary 1.** Let \( \psi \) be the function in (24). The operator \( L_{RD} \) defined in \( L^2 \) with domain \( H^2 \) has a unique negative eigenvalue, which is simple. Zero is a double eigenvalue with associated eigenfunctions \((0, \psi, 0)\) and \((0, 0, \psi')\). Moreover, the rest of the spectrum is positive, bounded away from zero and the essential spectrum is the interval \([\gamma + 1, +\infty)\).

**Corollary 2.** Let \( \psi \) be the function in (24). The operator \( L_R \) defined in \( L^2 \) with domain \( H^2 \) has a unique negative eigenvalue, which is simple. Zero is a double eigenvalue with associated eigenfunctions \((0, -b\psi, a\psi) \equiv (0, -\varphi, \phi)\) and \((\psi', a\psi', b\psi') \equiv (\psi', \phi', \varphi')\). Moreover, the rest of the spectrum is positive and bounded away from zero.

**Proof.** This is a consequence of the facts that if \( L_{RD} u = \lambda u \) then \( L_R A_R u = \lambda A_R u \) and, conversely, if \( L_R u = \lambda u \) then \( L_{DR} A_R^{-1} u = \lambda A_R^{-1} u \). \[ \square \]
3.1.2. The spectrum of $\mathcal{L}_I$. Here we will study the spectrum of $\mathcal{L}_I$. As above, there exists a matrix $A_I$, satisfying $A_I^{-1} = A_0^* = A_0^T$, such that the operator $\mathcal{L}_ID := A_I^{-1}\mathcal{L}_IA_I$ is diagonal. More precisely,

$$\mathcal{L}_ID = \begin{pmatrix}
-\partial_x^2 + \gamma + 1 + \psi & 0 & 0 \\
0 & -\partial_x^2 + \gamma + 1 + 2\psi & 0 \\
0 & 0 & -\partial_x^2 + \gamma + 1 - \psi
\end{pmatrix}.$$  \hfill (37)

The operators in the diagonal of $\mathcal{L}_ID$, except $\mathcal{L}_-\mathcal{I}$, were studied in Lemma 3.1.

Lemma 3.2. Let $\psi$ be the function in (24). The operator $\mathcal{L}_-\mathcal{I}$ defined in $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$ is strictly positive.

Proof. Because $2\psi > -\psi$, the proof is a consequence of the comparison theorem and Lemma 3.1 (ii). \hfill \Box

Corollary 3. Let $\psi$ be the function in (24). The operator $\mathcal{L}_ID$ defined in $L^2$ with domain $H^2(\mathbb{R})$ has no negative eigenvalues. Zero is a simple eigenvalue with associated eigenvector $(0,0,\psi)$. Moreover, the rest of the spectrum is positive, bounded away from zero and the essential spectrum is the interval $[\gamma + 1, +\infty)$.

Corollary 4. Let $\psi$ be the function in (24). The operator $\mathcal{L}_I$ defined in $L^2$ with domain $H^2(\mathbb{R})$ has no negative eigenvalues. Zero is a simple eigenvalue with associated eigenvectors $(2\psi,a\psi,b\psi) \equiv (2\psi,\phi,\varphi)$. Moreover, the rest of the spectrum is positive and bounded away from zero.

3.2. Spectral stability. To start with, we observe that if $(w,v,u)$ is a solution of (1) so is $(e^{2is}w,e^{is}v,e^{is}u)$, for any $s \in \mathbb{R}$. We denote this symmetry by $T_p(s)$. In terms of the real coordinate $U$, this action is represented by

$$T_p(s)U = \begin{pmatrix}
\cos 2s & 0 & 0 & -\sin 2s & 0 & 0 \\
0 & \cos s & 0 & 0 & -\sin s & 0 \\
0 & 0 & \cos s & 0 & 0 & -\sin s \\
\sin 2s & 0 & 0 & \cos 2s & 0 & 0 \\
0 & \sin s & 0 & 0 & \cos s & 0 \\
0 & 0 & \sin s & 0 & 0 & \cos s
\end{pmatrix} \begin{pmatrix}
P \\
M \\
R \\
Q \\
N \\
S
\end{pmatrix}.$$  \hfill (38)

Next, we define $V = V(t)$ by

$$V = T_p(-\gamma t)U - \Psi.$$

By using the group properties of $T_p(s)$ together with the fact that $\Psi$ is a critical point of the functional $G$, it is easy to see, from Taylor’s expansion and (25), that $V(t)$ satisfies

$$\frac{dV}{dt} = J\mathcal{L}_\gamma V + O(||V||^2),$$  \hfill (39)

where $J$ and $\mathcal{L}_\gamma$ are the operators defined in (26) and (29), respectively. The precise definition of stability we are interested in is given below.

Definition 3.3 (Spectral Stability). The standing wave solution (3) is called spectrally unstable if the operator $J\mathcal{L}_\gamma$ has at least one eigenvalue $\lambda$ such that $Re(\lambda) > 0$. Also, it is spectrally stable if all eigenvalues are zero or purely imaginary.

First of all, due to the Hamiltonian structure of the problem in hand, we point out that $J\mathcal{L}_\gamma$ has finitely many eigenvalues with strictly positive real part (see, for instance, Lemma 5.6 and Theorem 5.8 in [8]). Thus, studying spectral stability
amounts in locating the eigenvalues off the imaginary axis. The theory for counting such eigenvalues has gained substantially attention in recent decades (see, for instance, [6, 7, 8, 11, 13, 14, 27], and references therein).

Our main theorem in this section reads as follows.

**Theorem 3.4.** Let \( \Psi \) be as in (27). Then the standing wave (3), with \( \gamma = (1-\beta)/3 \), is spectrally stable

**Proof.** In order to show \( \Psi \) is spectrally stable we need to prove that \( J_L \gamma \) has no eigenvalues with strictly positive real part. According to Corollary 4, the number of negative eigenvalues of \( L_I \) is zero. Hence, we deduce that the unstable eigenvalues of \( J_L \gamma \), that is, those with a strictly positive real part, may occur only as real positive eigenvalues (see e.g. [6] or [27]). Thus our task is to show when \( J_L \gamma \) has or not one positive real eigenvalue. We will use the theory put forward in [7]. Define

\[
Y := \left[ \ker(L_R) \cup \ker(L_I) \right]^\perp, \\
\tilde{L}_R := \text{restriction of } L_R \text{ on } Y, \\
\tilde{L}_I^{-1} := \text{restriction of } L_I^{-1} \text{ on } Y,
\]

where the restriction operators are understood to act from \( Y \) to \( Y \). With these notation, Theorem 2.6 in [7] states that \( J_L \gamma \) has exactly

\[
\max\{ n(\tilde{L}_R), n(\tilde{L}_I^{-1}) \} - d(\mathcal{C}(\tilde{L}_R) \cap \mathcal{C}(\tilde{L}_I^{-1})) 
\]

\( \pm \) pairs of real eigenvalues, where \( \mathcal{C}(L) = \{ y \in Y; \langle Ly, y \rangle < 0 \} \) denotes the negative cone of the operator \( L \), \( d(\mathcal{C}(L)) \) denotes the dimension of a maximal linear subspace that is contained in \( \mathcal{C}(L) \) and \( n(L) \) denotes the number of negative eigenvalues. Thus, we need to decide if the number in (40) is positive or zero. If it is zero we have spectral stability while if it is positive we have spectral instability. Because \( n(L_I) = 0 \) (see Corollary 4), we deduce that it reduces to \( n(\tilde{L}_R) \). As a consequence, we have spectral stability if \( n(\tilde{L}_R) = 0 \) (and spectral instability if \( n(\tilde{L}_R) > 0 \)).

To study the quantity \( n(\tilde{L}_R) \) we make use of information on the spectrum of \( L_R \), paying particular attention to how projection onto the subspace \( Y \) might affect the negative index. As \( L_R \) is self-adjoint, its negative eigenspace is orthogonal to its kernel, and hence is influenced only by the projection off the kernel of \( L_I \). Consequently, we are faced the problem of locating the negative eigenvalues of a constrained self-adjoint operator. Here, we will take the advantage of the theory put forward in [11] (see also [12]). In fact, we will use the following result.

**Proposition 1.** Let \( L : D(L) \subset X \to X \) be a self-adjoint operator on the Hilbert \( X \) with dense domain \( D(L) \). Suppose that \( S \subset \ker(L)^\perp \) is an \( m \)-dimensional subspace. Let \( \{ s_1, \ldots, s_m \} \) be a basis of \( S \) and define the Hermitian matrix \( D \) by

\[
D_{ij} = \langle L^{-1}s_i, s_j \rangle_X.
\]

If \( D \) is nonsingular then

\[
n(L_{S^\perp}) = n(L) - n(D),
\]

where \( n(L_{S^\perp}) \) is the number of negative eigenvalues of \( L \) when it is constrained to act on \( S^\perp \) and \( n(D) \) is the number of negative eigenvalues of \( D \).
Now, turning back to our problem, we apply Proposition 1 with $\mathcal{L} = \mathcal{L}_R$ and $S = \ker(\mathcal{L}_1)^\perp$ to obtain

$$n(\mathcal{L}_R) = n(\mathcal{L}_R) - n(D) = 1 - n((\mathcal{L}_R^{-1}\Phi, \Phi)_{L^2}),$$

where $\Phi = (2\psi, \phi, \varphi)$ is a basis of $\ker(\mathcal{L}_1)$. Consequently, we have spectral stability if $(\mathcal{L}_R^{-1}\Phi, \Phi)_{L^2} < 0$ (and spectral instability if $(\mathcal{L}_R^{-1}\Phi, \Phi)_{L^2} > 0$). By recalling that $\mathcal{L}_R = A_R\mathcal{L}_{RD}A_R^{-1}$ and using (34), we have

$$D = (\mathcal{L}_R^{-1}\Phi, \Phi)_{L^2} = (A_R\mathcal{L}_{RD}A_R^{-1}\Phi, \Phi)_{L^2} = (\mathcal{L}_{RD}^{-1}\Phi, A_R^{-1}\Phi)_{L^2}. \quad (41)$$

But using (22) we easily see that

$$A_R^{-1}\Phi = \frac{1}{\sqrt{3}} \begin{pmatrix} -\sqrt{2}\psi \\ 0 \\ 4\psi \end{pmatrix} \equiv \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}.$$

Thus, from (41) and (35), we infer

$$D = (\mathcal{L}_R^{-1}f_1, f_1)_{L^2} + (\mathcal{L}_R^{-1}f_3, f_3)_{L^2}. \quad (42)$$

Let $\zeta_1$ and $\zeta_3$ be such that $\mathcal{L}_1\zeta_1 = f_1$ and $\mathcal{L}_3\zeta_3 = f_3$. From the Fredholm alternative it follows that such functions do exist and are unique. Taking into account the explicit expression of $\mathcal{L}_2$, it is not difficult to see that

$$\zeta_3(x) = -\frac{1}{3(\gamma + 1)}(4\psi(x) + 2\sqrt{3}x\psi'(x))$$

and (solving the integral for $\psi$)

$$(\mathcal{L}_2^{-1}f_3, f_3)_{L^2} = (\zeta_3, f_3)_{L^2} = \sqrt{\gamma + 1} \left( 8 - \frac{32}{\sqrt{3}} \right). \quad (43)$$

Next, let us inspect the quantity $\mathcal{L}_1^{-1}\zeta_1 = (\zeta_1, f_1)_{L^2}$. Although we are not able to give the exact value, we can obtain a suitable estimate and compare it with $\mathcal{L}_2^{-1}f_3, f_3)_{L^2}$. Indeed, by definition, $\zeta_1$ satisfies

$$-\zeta_1'' + (\gamma + 1)\zeta_1 + \psi(x)\zeta_1 = -\frac{\sqrt{2}}{\sqrt{3}}\psi(x). \quad (44)$$

Integrating both side of (44) on $\mathbb{R}$ and taking into account that $\zeta_1$ together with its first derivative must go to zero at infinity, we obtain

$$\int_{\mathbb{R}} \psi(x)\zeta_1(x) \, dx = -\frac{\sqrt{2}}{\sqrt{3}} \int_{\mathbb{R}} \psi(x) \, dx - (\gamma + 1) \int_{\mathbb{R}} \zeta_1(x) \, dx. \quad (45)$$

Now, since the right-hand side of (44) is negative, the maximum principle implies that $\zeta_1$ must be negative. So,

$$\int_{\mathbb{R}} \psi(x)\zeta_1(x) \, dx \geq -\frac{\sqrt{2}}{\sqrt{3}} \int_{\mathbb{R}} \psi(x) \, dx,$$

and

$$\langle \zeta_1, f_1 \rangle_{L^2} = -\frac{\sqrt{2}}{\sqrt{3}} \int_{\mathbb{R}} \psi(x)\zeta_1(x) \, dx \leq \frac{2}{3} \int_{\mathbb{R}} \psi(x) \, dx = 4\sqrt{\gamma + 1}. \quad (46)$$

Finally, from (42), (43), and (46), we deduce that

$$D \leq \sqrt{\gamma + 1} \left( 8 - \frac{32}{\sqrt{3}} \right) + 4\sqrt{\gamma + 1} = \sqrt{\gamma + 1} \left( 12 - \frac{32}{\sqrt{3}} \right) < 0.$$

---

1 From Corollaries 2 and 4 we have $S \perp \ker(\mathcal{L}_R)$.
This completes the proof of the Theorem.

**Remark 3.** Note we have solved system (5) only for $\gamma = (1 - \beta)/3$. Thus, in particular, we do not have the existence of a smooth curve of critical points of $G$, say, $\gamma \mapsto \Psi_\gamma$, for $\gamma$ in some open interval. The existence of such a curve plays a crucial role in many situations in order to determine the spectral/orbital stability of standing waves for Hamiltonian systems (see e.g., \[8, 13, 14, 27\] and subsequent results). Note also that the kernel of $\mathcal{L}_\gamma$ is 3-dimensional. Hence, here we cannot directly apply the abstract criterion established in these cited works to prove Theorem 3.4 or the orbital stability.

4. **Spectral stability of standing waves for system (2).** In this section we will show the existence and spectral stability of proportional standing wave solutions for (2). Since the analysis is similar to that in Section 3 we only give the main steps.

Substituting (4) in (2), we see that $\psi$, $\phi$, and $\varphi$ must satisfy system (8). Assume that

$$4\gamma + \beta = \gamma + 1 = 9\gamma + \beta_1 > 0 \quad (47)$$

and $\phi = a\psi, \varphi = b\psi$, where $a$ and $b$ are nonzero real constants. Note that (47) implies that $\beta$ and $\beta_1$ must satisfy the relation $8(1 - \beta) = 3(1 - \beta_1)$. In this case, we must have

$$\gamma = \frac{1 - \beta}{3} = \frac{1 - \beta_1}{8}. \quad (48)$$

With these assumptions, system (8) reduces to

$$\begin{cases}
\psi'' - (\gamma + 1)\psi + a(b + 1)\psi^2 = 0, \\
\psi'' - (\gamma + 1)\psi + \frac{1}{a} \left(\frac{1}{2} + b\right)\psi^2 = 0, \\
\psi'' - (\gamma + 1)\psi + \frac{b}{a}\psi^2 = 0.
\end{cases} \quad (49)$$

By assuming that $b$ and $a$ are given by the relations

$$b^2 + b - 1 = 0, \quad a^2 = b \left(\frac{1}{2} + b\right), \quad (50)$$

we see that system (49) reduces to the single equation

$$\psi'' - (\gamma + 1)\psi + a(b + 1)\psi^2 = 0,$$

which has the solution

$$\psi(x) = \frac{3(\gamma + 1)}{2a(b + 1)} \sech^2 \left(\frac{\sqrt{\gamma + 1}}{2} x\right), \quad \gamma + 1 > 0. \quad (51)$$

**Remark 4.** Note that each one of the equations in (50) has two real roots. In particular, we have

$$b = \frac{-1 \pm \sqrt{5}}{2} \quad \text{and} \quad a = \pm \sqrt{1 - \frac{b}{2}}.$$ 

Contrary to the analysis for system (1), where $a$ and $b$ may belong to the circle $a^2 + b^2 = 2$, here there are only two possibilities for $b$ and two possibilities for $a$ (according to the sign of $b$). In addition, in the present case, the function $\psi$ may be negative.
We now intent to investigate the spectral stability of (4). By writing \( w = P + iQ, \)
\( v = M + iN, \) \( u = R + iS, \) and separating real and imaginary parts, we see that (2)
writes as
\[
\frac{d}{dt} U(t) = \tilde{J} \tilde{E}'(U(t)),
\]
where \( U = (P, M, R, Q, N, S), \) \( \tilde{J} \) is a skew-symmetric matrix, and \( \tilde{E} \) is the energy
functional in (20).

As before, we define \( \tilde{G}(U) = \tilde{E}(U) + \gamma \tilde{M}(U) \) and set
\[
\Psi := (\psi, a\psi, b\psi, 0, 0, 0) \equiv (\psi, \phi, \varphi, 0, 0, 0).
\]
(52)

Hence the Hessian of \( \tilde{G} \) at \( \Psi \) takes the form
\[
\tilde{G}''(\Psi) := \begin{bmatrix}
\gamma \\
T_{\gamma} & 0 \\
0 & T_{\gamma}
\end{bmatrix},
\]
(53)
where
\[
\begin{align*}
T_{\gamma} &= \begin{bmatrix}
-\partial^2_x + \gamma + 1 + \phi & -\psi - \varphi & -\phi \\
-\psi - \varphi & -\partial^2_x + \beta + 4\gamma & -\psi \\
-\phi & -\psi & -\partial^2_x + \beta_1 + 9\gamma
\end{bmatrix}, \\
T_I &= \begin{bmatrix}
-\partial^2_x + \gamma + 1 + \phi & -\psi + \varphi & -\phi \\
-\psi + \varphi & -\partial^2_x + \beta + 4\gamma & -\psi \\
-\phi & -\psi & -\partial^2_x + \beta_1 + 9\gamma
\end{bmatrix}.
\end{align*}
\]
(54)
(55)

4.1. Spectral analysis. We will now study the spectrum of the operator \( T_{\gamma} \). As
discussed in Section 3, it suffices to study the spectra of \( T_R \) and \( T_I \).

4.1.1. The spectrum of \( T_R \). The idea is to use diagonalization again. Set
\[
v_1 = (-b, 0, 1), \quad v_2 = (b + 1, -2a(b + 1), 1), \quad v_3 = (b + 1, a(b + 1), 1),
\]
and define
\[
A_R = \begin{bmatrix}
\frac{b}{|v_1|} & \frac{b + 1}{|v_2|} & \frac{b + 1}{|v_3|} \\
0 & \frac{2a(b + 1)}{|v_2|} & \frac{a(b + 1)}{|v_3|} \\
\frac{1}{|v_1|} & \frac{1}{|v_2|} & \frac{1}{|v_3|}
\end{bmatrix},
\]
(56)
where \( |\cdot| \) stands for the Euclidean norm in \( \mathbb{R}^3 \). It is easy to see that
\[
|v_1| = \sqrt{b^2 + 1}, \quad |v_2| = \sqrt{3b + 9}, \quad \text{and} \quad |v_3| = \frac{\sqrt{3b + 9}}{\sqrt{2}}.
\]
(57)

By defining \( T_{RD} := A_R^{-1}T_RA_R \), we see that
\[
T_{RD} = \begin{bmatrix}
-\partial^2_x + \gamma + 1 + ab\psi & 0 & 0 \\
0 & -\partial^2_x + \gamma + 1 + a(b + 1)\psi & 0 \\
0 & 0 & -\partial^2_x + \gamma + 1 - 2a(b + 1)\psi
\end{bmatrix}.
\]
(58)

The spectrum of \( T_{RD} \) is studied next.

**Lemma 4.1.** Let \( \psi \) be the function in (51).
(i) The operator
\[ T_2 := -\partial_x^2 + \gamma + 1 - 2a(b + 1)\psi \]
defined in \( L^2(\mathbb{R}) \) with domain \( H^2(\mathbb{R}) \) has a unique negative eigenvalue, which is simple. Zero is a simple eigenvalue with associated eigenfunction \( \psi' \). Moreover, the rest of the spectrum is positive, bounded away from zero and the essential spectrum is the interval \( [\gamma + 1, +\infty) \).

(ii) The operators
\[ T_0 = -\partial_x^2 + \gamma + 1 + ab\psi \quad \text{and} \quad T_1 = -\partial_x^2 + \gamma + 1 + a(b + 1)\psi \]
defined in \( L^2(\mathbb{R}) \) with domain \( H^2(\mathbb{R}) \) are strictly positive. The essential spectrum is the interval \( [\gamma + 1, +\infty) \).

Proof. The proof is similar to that of Lemma 3.1. For the second part, observe that
\[ T_0 = -\partial_x^2 + \gamma + 1 + \frac{b}{b+1} \left( 3(\gamma + 1) + \frac{1}{2} \right) \text{sech}^2 \left( \frac{\sqrt{\gamma + 1}}{2} x \right) \]
and
\[ T_1 = -\partial_x^2 + \gamma + 1 + \frac{3(\gamma + 1)}{2} \text{sech}^2 \left( \frac{\sqrt{\gamma + 1}}{2} x \right) \].
Therefore, the potential functions of \( T_0 \) and \( T_1 \) are positive (even for \( b \) negative).

As a consequence of the above result, the following facts become clear.

Corollary 5. Let \( \psi \) be the function in (51). The operator \( T_{RD} \) defined in \( L^2 \) with domain \( H^2 \) has a unique negative eigenvalue, which is simple. Zero is a simple eigenvalue with associated eigenfunction \( (0, 0, \psi') \). Moreover, the rest of the spectrum is positive, bounded away from zero and the essential spectrum is the interval \( [\gamma + 1, +\infty) \).

Corollary 6. Let \( \psi \) be the function in (51). The operator \( T_R \) defined in \( L^2 \) with domain \( H^2 \) has a unique negative eigenvalue, which is simple. Zero is a simple eigenvalue with associated eigenfunction \( A_R(0, 0, \psi') \equiv (\psi', \phi', \varphi') \). Moreover, the rest of the spectrum is positive and bounded away from zero.

4.1.2. The spectrum of \( T_I \). Here also there exists an orthogonal matrix \( A_I \) such that \( T_{ID} := A_I^{-1} T_R A_I \) is diagonal, more precisely,
\[ T_{ID} = \begin{pmatrix} -\partial_x^2 + \gamma + 1 + \tilde{r}_1 \psi & 0 & 0 \\ 0 & -\partial_x^2 + \gamma + 1 + \tilde{r}_2 \psi & 0 \\ 0 & 0 & -\partial_x^2 + \gamma + 1 - a(b + 1)\psi \end{pmatrix}, \]
where
\[ \tilde{r}_{1,2} = \frac{a + a(b + 1) \pm \sqrt{\Delta}}{2}, \quad \Delta := a^2(1 + (b + 1))^2 + 4(2 - 5b) = \frac{23 - 36b}{2} > 0. \]

In view of the constants \( \tilde{r}_{1,2} \) appearing in (59), the spectrum of \( T_{ID} \) may change according to the sign of \( b \) (differently from the operator \( T_{RD} \)).

Corollary 7. Assume \( b > 0 \) and let \( \psi \) be the function in (51). The operator \( T_{ID} \) defined in \( L^2 \) with domain \( H^2 \) has no negative eigenvalues. Zero is a simple eigenvalue with associated eigenfunction \( (0, 0, \psi) \). Moreover, the rest of the spectrum is positive, bounded away from zero and the essential spectrum is the interval \( [\gamma + 1, +\infty) \).
Proof. As in the proof of Lemma 3.1 we already know that zero is the principal eigenvalue of $\widehat{T}_1 := -\partial_x^2 + \gamma + 1 - a(b+1)\psi$. Thus, it is sufficient to show that $\widehat{T}_2 := -\partial_x^2 + \gamma + 1 + \bar{r}_j\psi$, $j = 1, 2$ are positive. Let $\bar{r}_3 = -a(b+1)$.

**Case 1.** $a > 0$. Here, it is clear that $\bar{r}_3 < 0$ and $\bar{r}_1 > 0$. Also, since $2 - 5b < 0$,

$$\bar{s}_2 = \frac{a(b+2) - \sqrt{a^2(b+2)^2 + 4(2-5b)}}{2} > \frac{a(b+2) - \sqrt{a^2(b+2)^2}}{2} = 0$$

Thus, $\bar{r}_3\psi < \bar{r}_1\psi$ and $\bar{r}_3\psi < \bar{s}_2\psi$. The comparison theorem then yields the desired result.

**Case 2.** $a < 0$. In this case, $\bar{r}_3 > 0$, $\bar{s}_2 < 0$ and $\psi < 0$. In addition,

$$\bar{s}_1 < \frac{a(b+2)}{2} - |a(b+2)| = 0.$$

Consequently, we also obtain $\bar{r}_3\psi < \bar{s}_1\psi$ and $\bar{r}_3\psi < \bar{s}_2\psi$. The proof is thus completed.

Remark 5. Corollary 7 is not true if $b < 0$. Assume, for instance that $a > 0$. A simple inspection reveals that $\bar{r}_1 > \bar{r}_3$. Since $\psi < 0$ we then have $\bar{r}_1\psi < \bar{r}_3\psi$. The comparison theorem implies that $T_\gamma$ has at least one negative eigenvalue.

**Corollary 8.** Assume $b > 0$ and let $\psi$ be the function in (51). The operator $T_I$ defined in $L^2$ with domain $H^2$ has no negative eigenvalues. Zero is a simple eigenvalue with associated eigenfunction $A_I(0,0,\psi) \equiv (\psi, 2\phi, 3\varphi)$ and the rest of the spectrum is positive and bounded away from zero.

4.2. **Spectral stability.** This subsection is intended to study the spectral stability of the standing wave (4) with $\psi$, $\phi$, and $\varphi$ as above. The calculations run in much the way as those in Section 3. So, we give only the main steps. To start, we define $V = T_p(-\gamma t)U - \Psi$, where $\Psi$ is as in (52) and $T_p(s)$ is similar to the transformation (38) with the modification that now if $(w, v, u)$ is a solution of (2) so is $(e^{is}w, e^{2is}v, e^{3is}u)$, for any $s \in \mathbb{R}$. Upon linearization, we get

$$\frac{dV}{dt} = \widetilde{J}_\gamma V.$$  \hspace{1cm} (60)

Our main theorem in this section reads as follows.

**Theorem 4.2.** Let $\Psi$ be as in (52) and assume $b > 0$. Then the standing wave solution (4), with $\gamma = (1-\beta)/3$, is spectrally stable in the sense of Definition 3.3, that is, the operator $\widetilde{J}_\gamma$ has no eigenvalues with positive real part.

Proof. As in the proof of Theorem 3.4 we can define $\widehat{T}_R$ and $\widehat{T}_I$. In addition, $\widetilde{J}_\gamma$ has exactly

$$\max\{n(\widehat{T}_R), n(\widehat{T}_I^{-1})\} - d(C(\widehat{T}_R) \cap C(\widehat{T}_I^{-1}))$$

positive eigenvalues. By reasoning as in the proof of Theorem 3.4, we will have proved the theorem if we show that

$$D := (\langle \widehat{T}_R^{-1}\Phi, \Phi \rangle_{L^2} - \langle \widehat{T}_R^{-1}A_R^{-1}\Phi, A_R^{-1}\Phi \rangle_{L^2} < 0$$

where now $\Phi = (\psi, 2\phi, 3\varphi)$ is a basis of $\ker(T_I)$. In view of the definition of $A_R^{-1}$, we obtain $A_R^{-1}\Phi = (f_1, f_2, f_3)^T$, where to simplify we set $f_j = k_j\psi$, $j = 1, 2, 3$.

$$k_1 := \frac{2}{2b + 1} \sqrt{b^2 + 1} > 0,$$

$$k_2 := \left(\frac{1}{6b + 3} - \frac{2}{3b + 3} + \frac{b^2}{6b + 3}\right) \sqrt{3b + 9} < 0,$$
and
\[ k_3 := \left( \frac{2}{6b+3} + \frac{2}{3b+3} + \frac{2b^2}{2b+1} \right) \frac{\sqrt{3b+9}}{\sqrt{2}} > 0. \]

Thus, we can write
\[ D = \langle T_0^{-1} f_1, f_1 \rangle_{L^2} + \langle T_1^{-1} f_2, f_2 \rangle_{L^2} + \langle T_2^{-1} f_3, f_3 \rangle_{L^2}. \]

Let \( \zeta_1, \zeta_2, \) and \( \zeta_3 \) be such that \( T_0 \zeta_1 = f_1, T_1 \zeta_2 = f_2, \) and \( T_2 \zeta_3 = f_3. \) Taking into account the explicit expression of \( T_2 \) we now see that
\[ \zeta_3(x) = -\frac{k_3}{\gamma + 1} \left( \psi(x) + \frac{1}{2} x \psi'(x) \right) \]
and
\[ \langle T_2^{-1} f_3, f_3 \rangle_{L^2} = \langle \zeta_3, f_3 \rangle_{L^2} = -\sqrt{\gamma + 1} \frac{9k_3^2}{2a^2(b+1)^2}. \]

Let us assume \( a > 0 \) (if \( a < 0 \) a similar analysis with standard modifications hold). Since, \( \psi > 0, k_2 < 0, \) and \( \zeta_2 \) satisfies
\[ -\zeta''_2 + (\gamma + 1) \zeta_2 + ab \psi(x) \zeta_2 = k_2 \psi, \]
the maximum principle implies that \( \zeta_2 < 0. \) Hence, integrating the last equality, we obtain
\[ \langle T_1^{-1} f_2, f_2 \rangle_{L^2} = \langle \zeta_2, f_2 \rangle_{L^2} \leq \frac{k_2^2}{a(b+1)} \int_R \psi(x) dx = \sqrt{\gamma + 1} \frac{6k_2^2}{a^2(b+1)^2}. \]

Also, since \( k_1 > 0 \) and \( \zeta_1 \) satisfies
\[ -\zeta''_1 + (\gamma + 1) \zeta_1 + ab \psi(x) \zeta_1 = k_1 \psi, \]
another application of the maximum principle gives \( \zeta_1 > 0. \) So, after integrating we see that
\[ \langle T_0^{-1} f_1, f_1 \rangle_{L^2} = \langle \zeta_1, f_1 \rangle_{L^2} \leq \frac{k_1^2}{ab} \int_R \psi(x) dx = \sqrt{\gamma + 1} \frac{6k_1^2}{a^2b(b+1)}. \]

By replacing (63)–(65) into (62),
\[ D \leq \frac{\sqrt{\gamma + 1}}{a^2} \left( \frac{6k_1^2}{b+2} + \frac{6k_2^2}{b+2} - \frac{9k_3^2}{2(b+2)} \right), \]
where we used that \( b(b+1) = 1 \) and \((b+1)^2 = b+2.\) After squaring \( k_3, \) we obtain
\[ D \leq -\frac{\sqrt{\gamma + 1}}{a^2} \left( \frac{25b^7 + 125b^6 + 217b^5 + 283b^4 + 223b^3 + 27b^2 - 38b - 18}{(b+1)^2(b+1)^2(b+2)} \right). \]

Hence, \( D < 0 \) if the quantity \( D_0 := 25b^7 + 125b^6 + 217b^5 + 283b^4 + 223b^3 + 27b^2 - 38b - 18 \) is positive. But, using that \( b^2 = 1-b, \) we easily see that \( D_0 = 126 - 58b, \) which is obviously positive. The proof of the theorem is thus completed.

5. Standing waves for quadratic nonlinearities. The goal of this section is to show the existence of decaying to zero solutions for systems of the form
\[
\begin{align*}
    u'' - \alpha_2 w + F_u(w, v, u) &= 0, \\
    v'' - \alpha_1 v + F_v(w, v, u) &= 0, \\
    u'' - \alpha_1 u + F_u(w, v, u) &= 0,
\end{align*}
\]
where \( \alpha, \alpha_1, \) and \( \alpha_2 \) are positive real numbers. Our main theorem reads as follows.

**Theorem 5.1.** Suppose \( \alpha, \alpha_1, \alpha_2 > 0 \) and assume that (F) holds. Then, system (66) has at least one nontrivial solution which is homoclinic to the origin.
As we said, the argument to show Theorem 5.1 is based on the Mountain Pass Theorem and some ideas from the concentration-compactness method. Our approach is inspired by [34], where the authors proved a similar result for the two-component system

\[
\begin{align*}
\begin{cases}
  w'' - \alpha_2 w + \frac{1}{2} u^2 = 0, \\
u'' - \alpha_1 u + wu = 0.
  
\end{cases}
\tag{67}
\end{align*}
\]

Before proceeding, we point out that for arbitrary values of $\alpha_1, \alpha_2 > 0$, the existence of nontrivial solutions for (67), which are homoclinic to the origin, was established in [34] (see also [21, 32], and [33] for the existence and stability of soliton and multisoliton solutions). Uniqueness of positive symmetric solutions for (67) was proved in [22]. Also, existence and nonlinear stability of periodic pulses (with respect to periodic perturbations) were addressed in [1].

The rest of this section is then devoted to prove Theorem 5.1. We start with some preliminaries.

5.1. Preliminary results. For $U = (w, v, u) \in H^1$ we define the functional

\[
I(U) = \frac{1}{2} \int_\mathbb{R} \left( w'^2 + v'^2 + u'^2 + \alpha_2 w^2 + \alpha v^2 + \alpha_1 u^2 - 2F(w, v, u) \right) dx.
\tag{68}
\]

It is easy to see that $I \in C^1(H^1, \mathbb{R})$ and for any $U = (w, v, u) \in H^1$, $V = (v_1, v_2, v_3) \in H^1$ the Fréchet derivative of $I$ at $U$ applied at $V$ is given by

\[
I'(U)V = \int_\mathbb{R} \left[ w'v'_1 + v'v'_2 + u'v'_3 + \alpha_2 wv_1 + \alpha v v_2 + \alpha_1 u v_3 - (F_w(U)v_1 + F_v(U)v_2 + F_u(U)v_3) \right] dx.
\tag{69}
\]

At this point, it should be noted that critical points of $I$ are weak solutions of system (10) that decay to zero as $x \to \pm \infty$. Moreover, as is well known (see e.g. [28]), such weak solutions are indeed $C^2$ classical solutions. Thus, in order to prove Theorem 5.1 we only have to prove that $I$ has at least one nontrivial critical point in $H^1$.

Associated with the functional $I$ is an equivalent norm in $H^1$ defined as

\[
\|(f_1, f_2, f_3)\|_I^2 = \int_\mathbb{R} \left( f_1'^2 + f_2'^2 + f_3'^2 + \alpha_2 f_1^2 + \alpha f_2^2 + \alpha_1 f_3^2 \right) dx.
\tag{70}
\]

In particular there exist constants $C_1, C_2 > 0$ satisfying

\[
C_1 \|U\|^2_{H^1} \leq \|U\|^2_I \leq C_2 \|U\|^2_{H^1},
\tag{71}
\]

for any $U \in H^1$. The inner product associated with the norm $\| \cdot \|_I$ is

\[
\langle (f_1, f_2, f_3), (g_1, g_2, g_3) \rangle_I = \int_\mathbb{R} \left( f_1'g_1' + f_2'g_2' + f_3'g_3' + \alpha_2 f_1 g_1 + \alpha f_2 g_2 + \alpha_1 f_3 g_3 \right) dx.
\]

Lemma 5.2. Under the hypothesis (F), we have

\[
\left| \int_\mathbb{R} F(w, v, u) dx \right| \leq 20C^* \|(w, v, u)\|^3_{H^1}, \quad \text{for all } (w, v, u) \in H^1,
\]

where $C^* = \max\{a_{ijk}; \ 0 \leq i, j, k \leq 3, i + j + k = 3\}$. 

Proof. By Young’s inequality, we promptly deduce
\[ |\int_{\mathbb{R}} F(w,v,u)dx| \leq |\int_{\mathbb{R}} |F(w,v,u)|dx \]
\[ \leq 10C^*\|\langle w,v,u \rangle\|_{L^\infty}^2\|\langle w,v,u \rangle\|_{H^1}. \]
The conclusion then follow in view of the Sobolev embedding \( \|U\|_{L^\infty} \leq 2\|U\|_{H^1} \) (see Lemma 2.1 in [34]).

Now we recall the Mountain Pass Theorem (see e.g. [30]).

**Theorem 5.3** (Mountain Pass Theorem). Let \( E \) be a real Banach space and \( I \in C^1(E, \mathbb{R}) \). Assume that

(H1) there exist \( a, \rho > 0 \) such that
\[ I|_{\partial B_{\rho}(0)} \geq I(0) + a, \]
where \( B_{\rho}(0) \) denotes the ball of radius \( \rho \) in \( E \) centred at the origin.

(H2) There exists \( e \in E \setminus B_{\rho}(0) \) such that
\[ I(e) \leq I(0). \]

Then, there exists a sequence \( \{u_n\} \subset E \) satisfying
\[ I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\| \to 0, \quad \text{as } n \to \infty, \]
where
\[ c = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)), \]
with \( \Gamma = \{g \in C([0,1], E); g(0) = 0, g(1) = e\} \).

5.2. **Proof of Theorem 5.1.** In this subsection we will show Theorem 5.1. First we prove that the assumptions of Theorem 5.3 holds for the functional \( I \) defined in (68).

**Lemma 5.4.** Let \( I \in C^1(H^1, \mathbb{R}) \) be the functional in (68). Then assumptions (H1) and (H2) in Theorem 5.3 hold.

**Proof.** Let \( U = (w,v,u) \in H^1 \). From Lemma 5.2 and (71), we get
\[ I(U) = \frac{1}{2}\|U\|_{H^1}^2 - \int_{\mathbb{R}} F(U)dx \]
\[ \geq \frac{C_1}{2}\|U\|_{H^1}^2 \left( 1 - \frac{40C^*}{C_1} \|U\|_{H^1} \right). \]
Because the function \( t \mapsto \frac{C_1}{2}t^2(1 - \frac{40C^*}{C_1}t) \) is positive on the open interval \((0, \frac{C_1}{40C^*})\), if we take \( \rho := \frac{C_1}{80C^*} \) then for \( \|U\|_{H^1} = \rho \), we have
\[ I(U) \geq \frac{C_1}{4} \left( \frac{C_1}{80C^*} \right)^2 =: a > 0. \]
Since \( I(0) = 0 \), this shows that (H1) is valid.

Now take any smooth function \( U_0 = (w_0, v_0, u_0) \in H^1 \) satisfying \( w_0 > 0, v_0 > 0, u_0 > 0 \). From the assumption on the coefficients in (F) it follows that \( \int_{\mathbb{R}} F(U_0)dx > 0 \). Also, note that for any positive constant \( A \), \( F(AU_0) = A^3F(U_0) \). As a consequence,
\[ \lim_{A \to \infty} \frac{I(AU_0)}{A^2} = \lim_{A \to \infty} \left( \frac{1}{2}\|U_0\|_{H^1}^2 - A \int_{\mathbb{R}} F(w_0, v_0, u_0)dx \right) = -\infty. \]
This obviously implies that
\[ \lim_{A \to \infty} I(AU_0) = -\infty. \]
By taking \( A_0 > 0 \) large enough such that \( I(A_0U_0) \leq -1 \) and \( A_0\|U_0\|_{H^1} > \rho \), where \( \rho \) is as above, we see that \( e := A_0U_0 \in H^1 \setminus B_\rho(0) \) and \( I(e) \leq -1 < 0 = I(0) \). This completes the proof of the lemma.

**Remark 6.** We have used that the coefficients \( a_{ijk} \) are nonnegative in order to show Lemma 5.4. This assumption can be weakened if we assume the existence of \( U_0 \in H^1 \) satisfying \( \int_\mathbb{R} F(U_0)dx > 0 \).

As a consequence of the above lemma and Theorem 5.3, we obtain a sequence \( \{U_n\} \subset H^1 \) satisfying
\[ I(U_n) \to c \quad \text{and} \quad \|I'(U_n)\| \to 0, \quad \text{as} \quad n \to \infty, \tag{72} \]
where
\[ c = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)), \]
with \( \Gamma = \{g \in C([0,1],H^1); \ g(0) = 0, g(1) = e\} \) and \( e = A_0U_0 \) is given in the proof of Lemma 5.4.

The idea now is to prove that the above sequence converges (in some sense) to a nontrivial critical point of the functional \( I \).

**Lemma 5.5.** Let \( I \) be the functional in (68). Then, any sequence \( \{U_n\} \subset H^1 \) satisfying (72) must be bounded.

**Proof.** The proof is similar to that of Lemma 2.4 in [34]. So we omit the details. \( \square \)

**Lemma 5.6.** Let \( I \) be the functional in (68) and assume that \( U_n = (w_n,v_n,u_n) \in H^1 \) satisfies (72). Then, there exists a positive constant \( d \) such that for \( n \) sufficiently large,
\[ \sup_{x \in \mathbb{R}} |w_n(x)| + \sup_{x \in \mathbb{R}} |v_n(x)| + \sup_{x \in \mathbb{R}} |u_n(x)| \geq d. \]

**Proof.** From Lemma 5.5 there exists a constat \( B > 0 \) such that \( \|U_n\|_{H^1} \leq B \), for all \( n \in \mathbb{N} \). From assumption (F),
\[ 2I(U_n) - I'(U_n)U_n = \frac{1}{3} \int_\mathbb{R} (F_w(U_n)w_n + F_v(U_n)v_n + F_u(U_n)u_n)dx. \tag{73} \]
Thus, taking the limit, as \( n \to \infty \), in (73), in view of (72), we get
\[ 6c = \lim_{n \to \infty} \int_\mathbb{R} (F_w(U_n)w_n + F_v(U_n)v_n + F_u(U_n)u_n)dx. \]
In particular, taking into account that \( c > 0 \), for \( n \) large enough,
\[ 3c \leq \int_\mathbb{R} |F_w(U_n)w_n + F_v(U_n)v_n + F_u(U_n)u_n|dx. \tag{74} \]
Note that, after applying Young’s inequality, there exists a constant $C > 0$ (depending only on $a_{ij,k}$) such that
\[
\int_{\mathbb{R}} |F_w(U_n)w_n| \, dx \leq \sup_{x \in \mathbb{R}} |w_n(x)| \int_{\mathbb{R}} |F_w(U_n)| \, dx
\]
\[
\leq C \sup_{x \in \mathbb{R}} |w_n(x)| (\|w_n\|_{L^2(\mathbb{R})}^2 + \|v_n\|_{L^2(\mathbb{R})}^2 + \|u_n\|_{L^2(\mathbb{R})}^2)
\]
\[
\leq C \sup_{x \in \mathbb{R}} |w_n(x)| \|U_n\|_{H^1}^2
\]
\[
\leq CB^2 \sup_{x \in \mathbb{R}} |w_n(x)|.
\]
Similarly, we obtain
\[
\int_{\mathbb{R}} |F_v(U_n)v_n| \, dx \leq CB^2 \sup_{x \in \mathbb{R}} |v_n(x)|, \quad \int_{\mathbb{R}} |F_u(U_n)u_n| \, dx \leq CB^2 \sup_{x \in \mathbb{R}} |u_n(x)|.
\]
It follows from (74) that
\[
3c \leq CB^2 \left( \sup_{x \in \mathbb{R}} |w_n(x)| + \sup_{x \in \mathbb{R}} |v_n(x)| + \sup_{x \in \mathbb{R}} |u_n(x)| \right)
\]
and we may take $d := 3c/(CB^2)$. This completes the proof. \(\Box\)

With Lemma 5.6 in hand we are able to show that the sequence $U_n = (w_n, v_n, u_n) \in H^1$ satisfying (72) obtained with the help of the Mountain Pass Theorem converges (up to a translation) to a nontrivial critical point of $I$. Indeed, from Lemma 5.6, we may assume, without loss of generality, that for all $n \in \mathbb{N}$,
\[
\|U_n\|_{L^\infty} = \sup_{x \in \mathbb{R}} |w_n(x)| + \sup_{x \in \mathbb{R}} |v_n(x)| + \sup_{x \in \mathbb{R}} |u_n(x)| \geq d.
\]
As a consequence, for any $n \in \mathbb{N}$, we may choose $x_n \in \mathbb{R}$ satisfying
\[
|U_n(x_n)| \geq d. \quad (75)
\]
Since the norm in $H^1$ is invariant by translations, the sequence $\{U_n(\cdot + x_n)\}$ is also bounded in $H^1$. Taking into account that $H^1$ is a Hilbert space, up to a subsequence, $\{U_n(\cdot + x_n)\}$ converges weakly to a function $\tilde{U} = (\tilde{w}, \tilde{v}, \tilde{u}) \in H^1$. By denoting $U_n(\cdot + x_n) = \tilde{U}_n$, this means that
\[
(\tilde{U}_n, V) \to (\tilde{U}, V), \quad \text{for all } V \in H^1 \quad (76)
\]
or equivalently,
\[
J(\tilde{U}_n) \to J(\tilde{U}), \quad \text{for all } J \in (H^1)^*. \quad (77)
\]
Here $(\cdot, \cdot)_H$ denotes either $(\cdot, \cdot)_H$ or $(\cdot, \cdot)_I$ and $(H^1)^*$ is the dual space of $H^1$.

Let us now prove that $\tilde{U}$ is a nontrivial critical point of $I$. Indeed, if $J$ is the functional given by $J(U) = U(0)$, it is clear that $J \in (H^1)^*$ and in view of (77),
\[
|\tilde{U}_n(0)| = |J(\tilde{U}_n)| \to |J(\tilde{U})| = |\tilde{U}(0)|.
\]
But from (75),
\[
|\tilde{U}_n(0)| = |U_n(x_n)| \geq d.
\]
This implies that $|\tilde{U}(0)| \geq d$, that is, $\tilde{U}$ is nontrivial.

It remains to show that
\[
J'(\tilde{U})V = 0, \quad \text{for all } V \in H^1. \quad (78)
\]
Observe that, for any \( V = (v_1, v_2, v_3) \in H^1 \),
\[
I'(\bar{U}_n) V - I'(\tilde{U}) V = (\bar{U}_n, V)_I - (\tilde{U}, V)_I
+ \int_{\mathbb{R}} [F_w(\bar{U}) v_1 + F_v(\bar{U}) v_2 + F_u(\bar{U}) v_3 - F_w(\bar{U}_n) v_1 - F_v(\bar{U}_n) v_2 - F_u(\bar{U}_n) v_3] dx.
\]
From (76),
\[
\lim_{n \to \infty} \left( (\bar{U}_n, V)_I - (\tilde{U}, V)_I \right) = 0.
\]
Since \( I \) is invariant by translations, we also have \( I(\bar{U}_n) \to c \) and \( \|I'(\bar{U}_n)\| \to 0 \). Therefore,
\[
|I'(\bar{U}_n) V| \leq \|I'(\bar{U}_n)\|\|V\|_{H^1} \to 0.
\]
Hence, to prove (78) it suffices that
\[
\lim_{n \to \infty} \int_{\mathbb{R}} [F_w(\bar{U}) v_1 + F_v(\bar{U}) v_2 + F_u(\bar{U}) v_3 - F_w(\bar{U}_n) v_1 - F_v(\bar{U}_n) v_2 - F_u(\bar{U}_n) v_3] dx = 0.
\]
This is the content of the next lemma.

**Lemma 5.7.** Under the above notation,

\begin{align*}
(i) \quad & \lim_{n \to \infty} \int_{\mathbb{R}} [F_w(\bar{U}) v_1 - F_w(\bar{U}_n) v_1] dx = 0, \\
(ii) \quad & \lim_{n \to \infty} \int_{\mathbb{R}} [F_v(\bar{U}) v_2 - F_v(\bar{U}_n) v_2] dx = 0, \\
(iii) \quad & \lim_{n \to \infty} \int_{\mathbb{R}} [F_u(\bar{U}) v_3 - F_u(\bar{U}_n) v_3] dx = 0.
\end{align*}

**Proof.** Note that if \( \bar{U}_n = (\bar{w}_n, \bar{v}_n, \bar{u}_n) \), then
\[
F_w(\bar{U}) - F_w(\bar{U}_n)
= 3a_{300}(\bar{w}^2 - \bar{w}_n^2) + 2a_{210}(\bar{w}\bar{v} - \bar{w}_n\bar{v}_n) + 2a_{201}(\bar{w}\bar{u} - \bar{w}_n\bar{u}_n)
+ a_{111}(\bar{u}\bar{v} - \bar{u}_n\bar{v}_n) + a_{120}(\bar{v}^2 - \bar{v}_n^2) + a_{102}(\bar{u}^2 - \bar{u}_n^2).
\]
Thus, (i) is reduced to proving the following: if \( (f_n, g_n) \to (f, g) \) weakly in \( H^1(\mathbb{R}) \times H^1(\mathbb{R}) \) and \( \phi \in H^1(\mathbb{R}) \) is arbitrary, then
\[
\lim_{n \to \infty} \int_{\mathbb{R}} (f_n g_n \phi - f g \phi) dx = 0.
\]
But, this statement is exactly what was proved in [34, Lemma 2.6] and part (i) is proved. Parts (ii) and (iii) are proved similarly. \( \square \)

The proof of Theorem 5.1 is then completed.

**Remark 7.** It would be nice to consider a Schrödinger system with a general nonlinearity as in (66) and study the spectral stability of the solutions provided by Theorem 5.1.

6. **Uniqueness.** In this section we establish some uniqueness results for systems

\[
\begin{align*}
\tag{79}
\begin{cases}
w'' - \alpha_2 w + \frac{1}{2}(u^2 + v^2) = 0, \\
v'' - \alpha v + wv = 0, \\
\alpha_1 u + wu = 0,
\end{cases}
\end{align*}
\]
and
\[
\begin{cases}
  w'' - \alpha_2 w + wv + vu = 0, \\
  v'' - \alpha v + \frac{1}{2} w^2 + wu = 0, \\
  u'' - \alpha_1 u + wv = 0.
\end{cases}
\] (80)

Of course the two systems appear when one takes \( F(w, v, u) = \frac{1}{2} w(u^2 + v^2) \) and \( F(w, v, u) = wvu + \frac{1}{2} w^2 v \) in (10), which are the nonlinearities of main interest in this manuscript.

As is well known, for \( \alpha_2 > 0 \), equation \( w'' - \alpha_2 w + w^2 = 0 \) has a unique (up to translation) localized solution, which is given by (7) (see e.g. [2], [19]). In the meanwhile, the scenario for systems is completely different and uniqueness questions turns out to be an interesting and challenging issue (see e.g. [22], [31] and references therein).

To start with, we note that uniqueness does not hold for (79). Indeed, if \( (w, v, u) \) is a solution of (79) so are \( (w, -v, u) \), \( (w, v, -u) \) and \( (w, -v, -u) \). Hence, if we are interested in uniqueness, we need to restrict the class where solutions are allowed. Let us start by establishing that the unique positive solutions of (79) with \( \alpha = \alpha_1 = \alpha_2 \) is the one we have obtained in Section 3. Here and hereafter, by a positive solution we mean a solution of the form \( (w, v, u) \) with \( w > 0 \), \( v > 0 \), and \( u > 0 \).

**Proposition 2.** Suppose \( \alpha_2 > 0 \). Let \( (w, v, u) \) be a solution of
\[
\begin{cases}
  w'' - \alpha_2 w + \frac{1}{2}(u^2 + v^2) = 0, \\
  v'' - \alpha v + wv = 0, \\
  u'' - \alpha_2 u + wu = 0,
\end{cases}
\] (81)
provided by Theorem 5.1. Assume \( w > 0 \) and \( v > 0 \). If \( u \) does not change sign on \( \mathbb{R} \) then
\[
v = aw, \quad u = bw,
\]
where \( a^2 + b^2 = 2 \).

**Proof.** Let \( \mathcal{L} \) denote the linear operator \( -\partial_x^2 + \alpha_2 - w \). From (81) we see that \( \mathcal{L}v = \mathcal{L}u = 0 \). Since \( v \) (and \( u \)) does not change sign it follows that 0 is the principal eigenvalue of \( \mathcal{L} \); hence, from the Sturm-Liouville theory, it must be a simple eigenvalue. As a consequence, there exists a nonzero constant \( \lambda \) such that \( u = \lambda v \). This implies that (81) reduces to
\[
\begin{cases}
  -w'' + \alpha_2 w - \frac{1}{2}(\lambda^2 + 1)v^2 = 0, \\
  -v'' + \alpha v - wv = 0.
\end{cases}
\] (82)

Now we use some ideas employed in [22]. Define the positive real number \( \beta \) by
\[
\beta^2 = \frac{1}{2}(\lambda^2 + 1).
\]
Let \( z = v - (1/\beta)w \). Multiplying the first equation in (82) by \( 1/\beta \) and subtracting from the second one, we obtain
\[-z'' + \alpha_2 z + \beta vz = 0.
\]
Multiplying the above equation by \( z \) and integrating over \( \mathbb{R} \), we deduce the identity
\[
\int_{\mathbb{R}} (z'^2 + \alpha_2 z^2 + \beta vz^2)dx = 0.
\]
Since \( v > 0 \), it must be the case that \( z \equiv 0 \), that is, \( v = (1/\beta)w \). This also implies that \( u = (1/\beta)w \). Finally, note that
\[
\left( \frac{1}{\beta} \right)^2 + \left( \frac{\lambda}{\beta} \right)^2 = \left( \frac{1}{\beta} \right)^2 (1 + \lambda^2) = 2.
\]
This completes the proof.

The above result shows that (81) possesses a unique positive solution. A natural question now presents itself: does system (79) admit a unique positive solution? For Schrödinger type system with two components this question was addressed for instance in [10, 22], and [31]. In particular, in [22] the author proved the uniqueness of nontrivial positive and symmetric solution for (67). In our case, the situation is different. Indeed, the result to follow shows that (79) does not admit a positive solution.

Proposition 3. Suppose \( \alpha > 0, \alpha_1 > 0 \), and \( \alpha_2 > 0 \). Let \((w,v,u)\) be a solution of
\[
\begin{align*}
    w'' - \alpha_2 w + \frac{1}{2}(u^2 + v^2) &= 0, \\
    v'' - \alpha v + wv &= 0, \\
    u'' - \alpha_1 u + wu &= 0,
\end{align*}
\]
provided by Theorem 5.1. Assume that \( w > 0 \) and \( u, v \) are nonzero. If \( \alpha \neq \alpha_1 \) then either \( v \) or \( u \) change sign on \( \mathbb{R} \).

Proof. Let \( L \) denote the operator \(-\partial_x^2 + \alpha - w\). From (83), 0 is an eigenvalue of \( L \) with associated eigenfunction \( v \). Moreover, from the last equation in (83), we have
\[
L u = -(\alpha_1 - \alpha)u.
\]
If \( \alpha_1 - \alpha > 0 \) then, from (84), \( u \) is an eigenfunction associated with a negative eigenvalue. The Sturm-Liouville theory then implies that \( v \) must change sign. If \( \alpha_1 - \alpha < 0 \) then \( u \) is an eigenfunction associated with a positive eigenvalue. In this case, \( u \) must change sign. \( \Box \)

Remark 8. If we assume that \( u \equiv 0 \) (or \( v \equiv 0 \)) then uniqueness of positive symmetric solutions follows from Theorem 1.3 in [22].

Next, we prove that the unique positive solution of (80) is the one given in Section 4.

Proposition 4. Assume \( \alpha_2 > 0 \). Let \((w,v,u)\) be a solution of
\[
\begin{align*}
    w'' - \alpha_2 w + wv + vu &= 0, \\
    v'' - \alpha_2 v + \frac{1}{2}w^2 + wu &= 0, \\
    u'' - \alpha_2 u + wv &= 0.
\end{align*}
\]
Assume that \((w,v,u)\) is a positive solution. Then,
\[
v = aw, \quad u = bw,
\]
where \( b \) and \( a \) are the positive roots of \( b^2 + b - 1 = 0 \) and \( a^2 = b^2 + b/2 \), respectively.

Proof. Define \( z = v - aw \) and \( z_1 = u - bw \). Multiply the first equation in (85) by \( b \) and subtract from the third one to obtain
\[
-z''_1 + \alpha_2 z_1 - wv + bwv + bwv = 0.
\]
Using that $b - 1 = -b^2$, we deduce

$$-wv + bwv + bwv = -b^2 wv + bw(u - bw) = bwz_1.$$ 

Thus, from (86),

$$-z_1'' + \alpha_2 z_1 + bwz_1 = 0.$$ 

Multiplying this last equation by $z_1$, integrating on $\mathbb{R}$, and taking into account that $b > 0$, we get that $z_1 \equiv 0$.

Next, multiplying the first equation in (85) by $a$ and subtracting from the second one, we infer that

$$-z'' + \alpha_2 z - \left(\frac{1}{2} + b\right) w^2 + a(b + 1)wv = 0,$$

where we used that $u = bw$. By using that $1/2 + b = a^2/b$ and $b + 1 = 1/b$, we can write the last equation as

$$-z'' + \alpha_2 z + \frac{a}{b} wz = 0,$$

Since $a/b > 0$, this implies, as before, that $z \equiv 0$.

This all together, imply that $u$ and $v$ must be multiple of $w$, that is, $u = bw$ and $v = aw$. Substituting this in (85) and using the relations between $a$ and $b$, we see that it reduces to the single equation $w'' - \alpha_2 w + a(b + 1)w^2 = 0$. Since this equation has a unique solution, the proof is completed.

**Remark 9.** The uniqueness of positive solutions for (80) is not ruled out. Actually, we believe that such solutions are unique. However, we are unable to prove this result. It seems that the techniques employed in [10, 21, 22, 31] does not apply for three-component systems.

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