The maximum mutual information between the output of a discrete symmetric channel and several classes of Boolean functions of its input

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Abstract

We prove the Courtade-Kumar conjecture, for several classes of $n$-dimensional Boolean functions, for all $n \geq 2$ and for all values of the error probability of the binary symmetric channel, $0 \leq p \leq \frac{1}{2}$. This conjecture states that the mutual information between any Boolean function of an $n$-dimensional vector of independent and identically distributed inputs to a memoryless binary symmetric channel and the corresponding vector of outputs is upper-bounded by $1 - H(p)$, where $H(p)$ represents the binary entropy function. That is, let $X = [X_1 \ldots X_n]$ be a vector of independent and identically distributed Bernoulli($\frac{1}{2}$) random variables, which are the input to a memoryless binary symmetric channel, with the error probability in the interval $0 \leq p \leq \frac{1}{2}$, and $Y = [Y_1 \ldots Y_n]$ the corresponding output. Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be an $n$-dimensional Boolean function. Then, $MI(f(X), Y) \leq 1 - H(p)$. Our proof employs Karamata’s theorem, concepts from probability theory, transformations of random variables and vectors and algebraic manipulations.

Index Terms

Boolean function, mutual information, Karamata’s theorem, memoryless discrete symmetric channel, binary entropy function

I. INTRODUCTION

Boolean functions represent a fundamental mathematical formalism used to analyse and provide solutions to a wide range of problems in digital circuit design, theoretical computer science, logic, combinatorics, game theory, reliability theory, artificial intelligence, cryptography, coding theory [1]. More recently, Boolean networks have been successfully employed in the modelling and the analysis of complex biological systems, such as gene regulatory networks [2], [3]. In the effort to understand the organizational principles of such complex systems, several information-theoretic studies of Boolean networks have been carried out [4], [5], [6]. In information theory, a recent conjecture, termed the Courtade-Kumar conjecture, was stated in [7], involving the mutual information between any Boolean function of $n$ independent and identically distributed inputs to a memoryless binary symmetric channel and the $n$ outputs of the channel. Several proofs have appeared in the literature, for particular cases of this conjecture, but the most general case has remained unsolved. We bring further contributions to this effort. We prove the Courtade-Kumar conjecture [7], for several classes of Boolean functions, for all dimensions, $\forall n \geq 2$, and for all error probabilities of the memoryless binary symmetric channel, $\forall 0 \leq p \leq \frac{1}{2}$. We state our result as Theorem 1.

Our paper is structured as follows: we start the introductory section with our contributions, followed by the prior results that have been obtained so far in the literature, in the effort to solve the Courtade-Kumar conjecture. We also mention several generalizations of this conjecture. In the beginning of Section II we introduce the mathematical notation we used throughout this article. We continue this section with the description of the fundamental mathematical concepts from the hypothesis of this conjecture and the ones we used for its proof: the binary symmetric channel, the mutual information, concepts from probability theory and transformations of random variables and Karamata’s theorem [8]. The essence of this paper, the proof of the Courtade-Kumar conjecture for several classes of Boolean functions, for any dimension $n \geq 2$ and any error probability $0 \leq p \leq \frac{1}{2}$, is given in Section III. We present the conclusions of this study in Section IV.

A. Our contributions

Theorem 1: Let $X_i$ be a Bernoulli random variable, with the probability of success $q_X = \frac{1}{2}$ and the input to a discrete memoryless binary symmetric channel, without feedback and with error probability $0 \leq p \leq \frac{1}{2}$. Let $Y_i$ be the output of such a channel, when $X_i$ is given as its input. Let $X = [X_1 X_2 \ldots X_n]$ be an $n$-dimensional random vector of such $X_i$ i.i.d. Bernoulli random variables and $Y = [Y_1 Y_2 \ldots Y_n]$ the result of sending $X$ through the binary symmetric channel. Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be an $n$-dimensional Boolean function, which has any of the following properties:

1) For any $X^{(i)} \in \{0,1\}^n$

\[
\begin{align*}
&f(X^{(i)}) = 1 \\
&f(X) = 0, \forall X \in \{0,1\}^n, X \neq X^{(i)};
\end{align*}
\]

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2) For any $X^{(i)} \in \{0, 1\}^n$
\[
\begin{align*}
  f(X^{(i)}) &= 0 \\
  f(X) &= 1, \forall X \in \{0, 1\}^n, X \neq X^{(i)};
\end{align*}
\] (2)

3) $X^{(i)} = [X_r X^{(i)}_{n-r}], \forall X^{(i)}_{n-r} \in \{0, 1\}^{n-r}$, that is $i \in \{1, 2, \ldots, 2^{n-r}\}$, $\forall r \in \{1, 2, \ldots, n-1\}$
\[
\begin{align*}
  f(X^{(i)}) &= 1 \\
  f(X) &= 0, \forall X \in \{0, 1\}^n, X \neq X^{(i)};
\end{align*}
\] (3)

4) $X^{(i)} = [X_r X^{(i)}_{n-r}], \forall X^{(i)}_{n-r} \in \{0, 1\}^{n-r}$, that is $i \in \{1, 2, \ldots, 2^{n-r}\}$, $\forall r \in \{1, 2, \ldots, n-1\}$
\[
\begin{align*}
  f(X^{(i)}) &= 0 \\
  f(X) &= 1, \forall X \in \{0, 1\}^n, X \neq X^{(i)}.
\end{align*}
\] (4)

Let $\text{MI}(f(X), Y)$ denote the mutual information between the result of such a Boolean function of the input vector to the binary symmetric channel, $X$, and its output, $Y$. Let $H(p)$ denote the binary entropy function. Then, the following inequality holds
\[
\text{MI}(f(X), Y) \leq 1 - H(p), \forall n \geq 2, \forall 0 \leq p \leq \frac{1}{2}.
\] (5)

The novelty of our work lies in applying Karamata’s theorem to prove the Courtade-Kumar conjecture, for the classes 1 and 2 of Boolean functions, for all dimensions, $\forall n \geq 2$, and all error probabilities, $\forall 0 \leq p \leq \frac{1}{2}$. In addition, we transform the conjecture for the classes 3 and 4 of Boolean functions into a form similar to the previous case, such that the mutual information inequality follows directly from that result. In order to apply Karamata’s theorem, we need to write the desired inequality, $\text{MI}(Y, Z) \leq 1 - H(p)$, as two sums, one on the left-hand side of the inequality and the other on the right-hand side. Firstly, we write the joint probability mass values of $Y$ and $Z$ as functions of the error probability $p$. Secondly, we use a factoring of the left-hand sums and a simplification with elements from the sums of the right-hand side of the inequality. By these algebraic manipulations, we transform the mutual information from its definition, into a simpler algebraic expression, similar to the form found in Karamata’s theorem. We prove that all the conditions in Karamata’s theorem are satisfied. Then, this transformed expression follows as a direct result.

B. Prior work related to the Courtade-Kumar conjecture

In the article [7], the authors introduce the Courtade-Kumar conjecture that gives the upper bound on the mutual information between a Boolean function of a random vector of inputs to a memoryless binary symmetric channel and the vector of the outputs. The mutual information is computed between a Boolean function of $n$ independent and identically distributed (i.i.d.) Bernoulli random variables, with success probability, $q = \frac{1}{2}$, and the output of a memoryless binary symmetric channel, with error probability, $0 \leq p \leq \frac{1}{2}$, where $H(p)$ denotes the binary entropy function.

Definition 1 (Courtade-Kumar conjecture [7]): Let $X = [X_1 X_2 \ldots X_n]$ be a vector of $n$ i.i.d. Bernoulli random variables, with success probability $q = \frac{1}{2}$. Let $Y = [Y_1 Y_2 \ldots Y_n]$ be the vector of outputs, when $X$ is given as an input random vector to a memoryless binary symmetric channel, with error probability $0 \leq p \leq \frac{1}{2}$. Let $f$ be an $n$-dimensional Boolean function, $f : \{0, 1\}^n \to \{0, 1\}$. Then, for any Boolean function $f$ and any $0 \leq p \leq \frac{1}{2}$, the following bound holds
\[
\text{MI}(f(X), Y) \leq 1 - H(p).
\] (6)

As preliminary steps in proving the Courtade-Kumar conjecture, the authors state other conjectures and prove several weaker theorems. In Theorem 1 of [7], they prove that, if $f(X)$ is an equiprobable Boolean function, then the following inequality holds
\[
\sum_{i=1}^{n} \text{MI}(f(X), Y_i) \leq 1 - H(p).
\] (7)

A Boolean function $f$ is termed equiprobable, if the probability of the function being equal to 1 is equal to the probability of the function being equal to 0, for any combination of the input values. Both of these probabilities are equal to $\frac{1}{2}$, as the function can have only two values, 0 and 1. Balanced functions, equiprobable functions and functions with the expectation equal to $\frac{1}{2}$ are equivalent descriptions for such Boolean functions.

The authors give two more conjectures, formulated as Conjecture 2 and 3, which, if proven, would facilitate the proof of the Courtade-Kumar conjecture. In Definition 1 and 2, the authors introduce the concept of lex functions, which are a subset of Boolean functions.
The authors numerically verify that Conjecture 2 holds, for all values of $n$ and for all $p$ in the interval $[0, \frac{1}{2}]$, using increments of 0.001.

In [9], the author relates problems in financial investments to the rate-distortion theory and derives upper bounds on functions describing such investments, which involve the maximization of the mutual information between various random variables describing such processes. We would like to point out that this reference is incorrect in the articles [7], [9], where it is cited as the best known bound on the mutual information under study in the Courtade-Kumar conjecture

$$\text{MI}(f(X), Y) \leq (1 - 2 \cdot p)^2. \quad (9)$$

From a mathematical point of view, the problem studied in [Ch 3, Th. 3, Th. 4, Th. 5, [9]] is different from the one in the Courtade-Kumar conjecture [7]. In [9], the mathematical model is a cascade of two binary symmetric channels that form a Markov chain, whereas, in [7], it is a binary symmetric channel and a transformation of its input random vector by a Boolean function. The author of [9] proves that the derivative of the maximum mutual information between the input to the first binary symmetric channel and the output of the last binary symmetric channel, subject to some constraints, can be found in [Ch 3, Corollary 1, [9]] is upper bounded by $(1 - 2 \cdot p)^2$, where $1 - p$ is the error probability of the last channel. Unless a proof is presented that relates the mutual information from the Courtade-Kumar conjecture, to the one studied in [Ch 3, Th. 3, Th. 4, Th. 5, [9]], we cannot draw the conclusion of [9]. The results shown in the PhD dissertation [9] have been published in [10].

Using Fourier analysis for Boolean functions, the authors of [6] investigate the mutual information between a Boolean function $f$ of $n$ i.i.d. inputs, defined as $X = [X_1, X_2, \ldots, X_n]$ and one of the inputs, $X_i$, that is $\text{MI}(f(X), X_i)$. They show that this mutual information between a function $f$ that produces an output with fixed mean, $\mu = \mathbb{E}[f(X)]$, and one input variable, $X_i$, is maximized, if the function $f$ is canalizing in the variable $X_i$. A canalizing $n$-dimensional function represents a Boolean function, for which, whenever one of the $n$ input variables has a particular value, the output of the function will have a certain value, corresponding to this input, regardless of the combination of the values of the other $n - 1$ input variables [6].

The authors of [6] prove this theorem in the case when the input binary vector $X$ is uniformly distributed and in the case when it is product distributed, with some constraints on the canalizing input and the canalizing value of the function. If the mean $\mu$ of the output produced by the function $f$ is not fixed, then the dictatorship function is the maximizing function of this mutual information, in the case when the input binary vector $X$ is uniformly distributed and in the case when it is product distributed. The dictatorship function is an $n$-dimensional Boolean function, such that $f(X) = f(X_1, \ldots, X_n) = X_i$ or $f(X) = f(X_1, \ldots, X_n) = \overline{X_i}$.

The authors also investigated the mutual information, $\text{MI}(f(X), X_T)$, in the case of several inputs, defined as $X_T = \{X_i : i \in T\}$, with $|T| \leq n$, where the symbol $|\cdot|$ denotes the cardinality of a set. They found that, when the input binary vector $X$ is product distributed and the output of the function has a fixed expectation, $\mu = \mathbb{E}[f(X)]$, the mutual information $\text{MI}(f(X), X_T)$ is maximized when the function $f$ is jointly canalizing in the set $T$. More recent results include the following articles. The authors of [11] employ Fourier analysis and the hypercontractivity theorem to prove the bound stated in their Theorem 1, in the case of balanced Boolean functions and $p$ in the range $\frac{1}{2} \cdot \left(1 - \frac{1}{\sqrt{3}}\right) \leq p \leq \frac{1}{2}$:

$$\text{MI}(f(X), Y) \leq \frac{\log(e) - 9}{2} \cdot (1 - 2 \cdot p)^2 + 9 \cdot \left(1 - \frac{\log(e)}{2}\right) \cdot (1 - 2 \cdot p)^4. \quad (10)$$

In Corollary 1, they prove that the Courtade-Kumar conjecture holds for the dictatorship function, as a special case of equi-probable Boolean functions, when $p \to \frac{1}{2}$. This region is termed the noise interval $p \in [\frac{1}{2}, 1 - \frac{\sqrt{2}}{2}, \frac{1}{2}, 2^{-n}]$, where $\frac{1}{2} - p$ is defined as $\frac{1}{2} - 2^{-n}$. Related to this result, in Theorem 1.15, the author of [12] proves that the Courtade-Kumar conjecture holds for high noise, that is $\text{MI}(f(X), Y) \leq 1 - H(p)$ holds for any Boolean function and for any noise $\epsilon \geq 0$, such that $(1 - 2 \cdot e)^2 \leq \delta$ and $\frac{1}{2} - \frac{\sqrt{2}}{2} \leq \epsilon \leq \frac{1}{2} + \frac{\sqrt{2}}{2}$, where $\delta > 0$ is a constant of small value. The author of [12] provides an improved version of Theorem 1 derived by Wyner and Ziv in [13], known as Mrs. Gerber’s Lemma, which was employed in [9], for the proof of Theorem 4. This strengthening of Mrs. Gerber’s Lemma is used in the proof of the Courtade-Kumar conjecture for high noise [12].
We mention here several studies of generalizations of the Courtade-Kumar conjecture. An extension of the Courtade-Kumar conjecture to two \( n \)-dimensional Boolean functions, is hypothesized to hold in [14], as Conjecture 3. It states that, for any Boolean functions \( f, g : \{0, 1\}^n \to \{0, 1\} \), the mutual information \( \text{MI}(f(X), g(Y)) \leq 1 - H(p) \). For several specific cases of the joint probability mass function of the binary random variables \( f(X) \) and \( g(Y) \), the authors analytically prove another conjecture, termed Conjecture 4, which implies Conjecture 3. A similar form of Conjecture 4 of [14] is analytically proved in [15], in a more general context than that of the results of [14]. In section V of [15], the authors prove that the mutual information \( \text{MI}(B, \tilde{B}) \leq 1 - H(p) \), for Boolean functions, \( B = f(X) \) and \( \tilde{B} = g(Y) \), an estimator of \( Y \), with fixed mean \( \mathbb{E}(B) = \mathbb{E}(\tilde{B}) = a \) and \( \mathbb{P}(B = \tilde{B} = 0) \geq a^2 \). Conjecture 3 of [14] is proved to hold in [16]. The Courtade-Kumar conjecture is generalized to continuous random variables in the preprint [17]. Here, the aim is to maximize \( \text{MI}(f(X), Y) \), where the function \( f \) takes as input \( n \)-dimensional real vectors and produces as output values from the set \( \{0, 1\} \). The authors investigate two cases: when \( X \) and \( Y \) are \( n \)-dimensional correlated Gaussian random vectors and when \( X \) and \( Y \) are correlated random vectors from the unit sphere.

II. MATHEMATICAL BACKGROUND

A. Mathematical notations and symbols

Throughout this article, we use the following mathematical notations and symbols:

- \( X_i \) denotes a discrete random variable, with ensemble \( \mathcal{X}_i \).
- \( X \) denotes a discrete \( n \)-dimensional random vector, \( X = [X_1 \ X_2 \ldots X_n] \), with ensemble \( \mathcal{X} \).
- \( \text{MI}(X_i, Y_i) \) represents the mutual information between the random variables \( X_i \) and \( Y_i \).
- \( \mathbb{P}(X_i = 0) \) is the probability that the discrete random variable \( X_i \) is equal to 0.
- \( \mathbb{P}(Y_i = 0|X_i = 0) \) is the conditional probability that the discrete random variable \( Y_i \) is equal to 0, given that the discrete random variable \( X_i \) is equal to 0.
- \( p_{X_i}(x_i) \) is the probability at the value \( X_i = x_i \). We may omit the index \( X_i \), \( p(x_i) \) to refer to the same quantity. To avoid confusion, we use the index whenever probability mass functions for different random variables or vectors appear in the same derivations.
- \( p_{Y_i|X_i}(y_i|x_i) \) is the conditional probability at the value \( Y_i = y_i \), given that \( X_i = x_i \). Similarly, we use the index whenever conditional probability mass functions for different random variables or vectors appear in the same derivations.
- \( H(p) = -p \cdot \log p - (1-p) \cdot \log (1-p) \) denotes the binary entropy function, for a Bernoulli random variable, with the probability of success \( 0 \leq p \leq \frac{1}{2} \).
- \( \log(\cdot) \) denotes the base 2 logarithm.

B. The binary symmetric channel

Definition 4 (Binary symmetric channel): The binary symmetric channel is defined as having the input and output modelled as Bernoulli random variables with success probabilities, \( q_X \) and \( q_Y \): \( X \in \mathcal{X}_i = \{0, 1\} \), \( X \sim \text{Bernoulli}(q_X) \) and \( Y \in \mathcal{Y}_j = \{0, 1\} \), \( Y \sim \text{Bernoulli}(q_Y) \) [Ch 7 of [18]]. In our problem, \( q_X = \frac{1}{2} \). The conditional probabilities describing the relationship between the input and output random variables are as follows:

\[
\begin{align*}
p_{Y|X}(0|0) &= \mathbb{P}(Y = 0|X = 0) = 1 - p \\
p_{Y|X}(0|1) &= \mathbb{P}(Y = 0|X = 1) = p \\
p_{Y|X}(1|0) &= \mathbb{P}(Y = 1|X = 0) = p \\
p_{Y|X}(1|1) &= \mathbb{P}(Y = 1|X = 1) = 1 - p.
\end{align*}
\]  
(11)

The probability of error is denoted as \( p \) and is in the range \( 0 \leq p \leq \frac{1}{2} \). This channel is characterized as memoryless and without feedback: when the binary symmetric channel is used with consecutive inputs, \( \forall i = 1 : n \), it has no memory, that is \( p(y_i|x_1, \ldots, x_i, y_1, \ldots, y_{i-1}) = p(y_i|x_i) \) and no feedback, that is \( p(x_i|x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1}) = p(x_i|x_1, \ldots, x_{i-1}) \) [Ch 7 of [18]]. For completeness, in Appendix A using these two properties, we prove by induction the known result [Ch 7 of [18]] that

\[
\begin{align*}
p(x_{k+1}, x_k, \ldots, x_1, y_{k+1}, y_k, \ldots, y_1) &= \prod_{i=1}^{k+1} p(x_i, y_i), \forall k = 1, n - 1, \\
p(y_{k+1}, y_k, \ldots, y_1|x_{k+1}, x_k, \ldots, x_1) &= \prod_{i=1}^{k+1} p(y_i|x_i), \forall k = 1, n - 1.
\end{align*}
\]  
(12)
C. The mutual information

Definition 5 (Mutual information): Let \(X\) and \(Y\) be two discrete random vectors, with the joint probability mass function denoted by \(p_{XY}(x, y)\) and their marginal probability mass functions denoted by \(p_X(x)\) and \(p_Y(y)\). Then, the mutual information between \(X\) and \(Y\) is defined as \([13, 19, 20]\)

\[
MI(X, Y) = \sum_x \sum_y p_{XY}(x, y) \cdot \log \frac{p_{XY}(x, y)}{p_X(x) \cdot p_Y(y)}.
\]

(13)

D. Probability theory and transformations of random variables

Given two events, \(A\) and \(B\), the following fundamental results are known from probability theory [Ch 1 section 3 of [21]]:

\[
\mathbb{P}(A|B) = \frac{\mathbb{P}(A, B)}{\mathbb{P}(B)},
\]

\[
\mathbb{P}(B|A) = \frac{\mathbb{P}(A, B)}{\mathbb{P}(A)}.
\]

(14)

Using these equations, we obtain the joint probability mass functions of the input and the output of the binary symmetric channel as:

\[
p_{XY}(0, 0) = \mathbb{P}(X = 0, Y = 0) = \mathbb{P}(Y = 0|X = 0) \cdot \mathbb{P}(X = 0) = \frac{1}{2}(1 - p)
\]

\[
p_{XY}(0, 1) = \mathbb{P}(X = 0, Y = 1) = \mathbb{P}(Y = 1|X = 0) \cdot \mathbb{P}(X = 0) = \frac{1}{2}p
\]

\[
p_{XY}(1, 0) = \mathbb{P}(X = 1, Y = 0) = \mathbb{P}(Y = 0|X = 1) \cdot \mathbb{P}(X = 1) = \frac{1}{2}p
\]

\[
p_{XY}(1, 1) = \mathbb{P}(X = 1, Y = 1) = \mathbb{P}(Y = 1|X = 1) \cdot \mathbb{P}(X = 1) = \frac{1}{2}(1 - p).
\]

(15)

The remainder of this section pertains to transformations of random variables. We present the definition of how to obtain the joint probability mass function of a random variable \(Z\) and the random vector \(Y\), when \(Z\) is any function of a random vector \(X\), that is, \(Z = f(X)\) [Ch 5, section 6 of [21]]. The joint probability mass function \(p_{XY}(x, y)\) is known.

Definition 6 (Probability mass function of transformations of random vectors): Let \(X\) be an \(n\)-dimensional discrete random vector, \(X = [X_1 \ X_2 \ldots \ X_n]\), \(Z\) a discrete random variable and \(f\) an \(n\)-dimensional function, such that \(Z = f(X)\). Then, the probability mass function of \(Z\), \(p_Z(z)\), is obtained from the probability mass function of \(X\), \(p_X(x)\), as

\[
p_Z(z) = \sum_{\{x \in E_X, f(x) = z\}} p_X(x).
\]

(16)

Let \(X, Y\) be two \(n\)-dimensional discrete random vectors, \(Z\) a discrete random variable and an \(n\)-dimensional function \(f\), such that \(Z = f(X)\). Let \(T, U\) be two random vectors and \(g\) be a multidimensional function, such that

\[
g(X, Y) = \begin{bmatrix} g_1(X, Y) \\ g_2(X, Y) \\ g_3(X, Y) \end{bmatrix}
\]

\[
\begin{bmatrix} T \\ U \\ Z \end{bmatrix} = \begin{bmatrix} g_1(X, Y) = Y \\ g_2(X, Y) = X \\ g_3(X, Y) = f(X) \end{bmatrix}
\]

(17)

Then, the random vector \(\begin{bmatrix} T \\ U \\ Z \end{bmatrix}\) is the transformed random vector \(\begin{bmatrix} X \\ Y \end{bmatrix}\), by the function \(g\). Its joint probability mass function is equal to

\[
p_{TZU}(t, u, z) = \sum_{x \in E_X} \sum_{y \in E_Y} p_{XY}(x, y) = \sum_{u \in E_U} p_{TU}(t, u) \cdot p_{UZ}(u, z)
\]

\[
\Rightarrow p_{YZ}(y, z) = p_{TZ}(t, z) = \sum_{u \in E_U} p_{TU}(t, u)
\]

(18)
E. Karamata’s inequality

Karamata’s inequality [3], also known as the Hardy-Littlewood-Pólya inequality [22], Th. 108, page 89, represents the main element of our proof of the Courtade-Kumar conjecture for certain classes of Boolean functions. A generalized version of this inequality can be found in [23], although we do not employ this extension in our proofs.

**Theorem 2:** Let \( g : \mathbb{R} \to \mathbb{R} \) be a convex function and \( x = [x_1 \, x_2 \ldots x_n] \) and \( y = [y_1 \, y_2 \ldots y_n] \) be two vectors, such that the following conditions hold

1. \( y_1 \geq y_2 \geq \ldots \geq y_n \) and \( x_1 \geq x_2 \geq \ldots \geq x_n \)
2. \( \sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} x_i, \forall k \in \{1, 2, \ldots, n-1\} \)
3. \( \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i \).

They represent the definition of majorization of \( y \), by \( x \). Then, the above conditions are necessary and sufficient for the following inequality to hold

\[
\sum_{i=1}^{n} g(y_i) \leq \sum_{i=1}^{n} g(x_i).
\]

III. **Proof of the Courtade-Kumar Conjecture, for Several Classes of \( n \)-Dimensional Boolean Functions, \( \forall n \geq 2 \) and \( \forall 0 \leq p \leq \frac{1}{2} \)**

We prove the Courtade-Kumar conjecture for four classes of Boolean functions of \( n \) variables, in the general case, that is \( \forall n \geq 2 \) and \( \forall 0 \leq p \leq \frac{1}{2} \), as stated in [1].

**Proof:**

\[
\text{MI}(Y, Z) = \sum_{y} \sum_{z} p_{YZ}(y, z) \cdot \log \frac{p_{YZ}(y, z)}{p_{Y}(y) \cdot p_{Z}(z)}.
\]

We need to write the joint probability mass values, \( p_{YZ}(y, z) \), and the marginal probability mass values, \( p_{Y}(y) \) and \( p_{Z}(z) \), as functions of the error probability \( p \).

A. **Computing the probability mass function values**

**Lemma 1:** For any \( k \in \{1, 2, \ldots, n\} \), let \( Y = [y_1 \, y_2 \ldots y_k] \in \{0, 1\}^k \) be fixed and \( X^{(i)} = [x^{(i)}_1 \, x^{(i)}_2 \ldots x^{(i)}_k] \in \{0, 1\}^k \) range over all the \( 2^k \) possible values. Then, the following identity holds

\[
\sum_{i=1}^{2^k} p(Y, X^{(i)}) = \frac{1}{2^k},
\]

**Proof:** \( X^{(i)} \) ranges from \([0 \, 0 \ldots 0]\) to \([1 \, 1 \ldots 1]\). For any fixed \( Y \), there is one \( X^{(i)} \), such that \( X^{(i)} = Y \). There are \( \binom{k}{1} \) number of vectors \( X^{(i)} \) that differ from \( Y \) in one position. There are \( \binom{k}{j} \) number of vectors \( X^{(i)} \) that differ from \( Y \) in \( j \) positions. As a result, the summation of the joint probabilities becomes

\[
\sum_{i=1}^{2^k} p(Y, X^{(i)}) = \sum_{i=1}^{2^k} \prod_{j=1}^{k} p(y_j, x^{(i)}_j)
\]

\[
= \frac{(1 - p)^k}{2^k} + \binom{k}{1} \cdot \frac{(1 - p)^{k-1} \cdot p}{2^k} + \ldots + \binom{k}{r} \cdot \frac{(1 - p)^{k-r} \cdot p^r}{2^k} + \ldots + \frac{p^k}{2^k}
\]

\[
= \frac{(1 - p + p)^k}{2^k} = \frac{1}{2^k}.
\]
With the definitions of the section [1-2] we know that

\[
\mathbf{p}_{\mathbf{TUZ}}(\mathbf{t}, \mathbf{u}, \mathbf{z}) = \sum_{\mathbf{x} \in \mathbf{x}} \sum_{\mathbf{y} \in \mathbf{E}_{\mathbf{Y}, g_1(\mathbf{x}, \mathbf{y})} = g_2(\mathbf{x}, \mathbf{y}) = g_3(\mathbf{x}, \mathbf{y}) = z} \mathbf{p}_{\mathbf{XY}}(\mathbf{x}, \mathbf{y})
\]

\Rightarrow \mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, z) = \mathbf{p}_{\mathbf{TZ}}(\mathbf{t}, z) = \sum_{\mathbf{u} \in \mathbf{U}} \mathbf{p}_{\mathbf{TUZ}}(\mathbf{t}, \mathbf{u}, \mathbf{z})

\Rightarrow \mathbf{p}_{\mathbf{TUZ}}(\mathbf{t}, \mathbf{u}, 0) = \mathbf{p}_{\mathbf{XY}}(\mathbf{u}, \mathbf{t})$, if \(0 = f(\mathbf{u})\), or \(\mathbf{p}_{\mathbf{TUZ}}(\mathbf{t}, \mathbf{u}, 0) = 0\), if \(0 \neq f(\mathbf{u})\)

\[
\mathbf{p}_{\mathbf{TUZ}}(\mathbf{t}, \mathbf{u}, 1) = \mathbf{p}_{\mathbf{XY}}(\mathbf{u}, \mathbf{t})$, if \(1 = f(\mathbf{u})\), or \(\mathbf{p}_{\mathbf{TUZ}}(\mathbf{t}, \mathbf{u}, 0) = 0\), if \(1 \neq f(\mathbf{u})\)

\Rightarrow \mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, 0) = \mathbf{p}_{\mathbf{TZ}}(\mathbf{t}, 0) = \sum_{\mathbf{u} \in \mathbf{U}} \mathbf{p}_{\mathbf{TUZ}}(\mathbf{t}, \mathbf{u}, 0) = \sum_{\mathbf{u} \in \mathbf{U}} \mathbf{p}_{\mathbf{XY}}(\mathbf{u}, \mathbf{t}) \Rightarrow \mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, 0) = \sum_{\mathbf{u} \in \mathbf{U}, 0 = f(\mathbf{u})} \mathbf{p}_{\mathbf{XY}}(\mathbf{x}, \mathbf{y})

\mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, 1) = \mathbf{p}_{\mathbf{TZ}}(\mathbf{t}, 1) = \sum_{\mathbf{u} \in \mathbf{U}} \mathbf{p}_{\mathbf{TUZ}}(\mathbf{t}, \mathbf{u}, 1) = \sum_{\mathbf{u} \in \mathbf{U}, 1 = f(\mathbf{u})} \mathbf{p}_{\mathbf{XY}}(\mathbf{x}, \mathbf{y}).

Let \(\mathbf{X} = [x_1 \ x_2 \ldots x_n]\) be the input vector to the binary symmetric channel and \(\mathbf{Y} = [y_1 \ y_2 \ldots y_n]\) be the output of the binary symmetric channel. According to Appendix \(\Delta\) we have that

\[
\mathbf{p}_{\mathbf{XY}}(x, y) = \prod_{i=1}^{n} p(x_i, y_i).
\]

(24)

For any pair of vectors, \((\mathbf{X}^{(i)}, \mathbf{Y}^{(j)}), \forall i, j \in \{1, 2, \ldots, 2^n\}\), let \(N_{ij}^{(1)}\) denote the number of positions on which the elements of the vectors \(\mathbf{X}^{(i)}\) and \(\mathbf{Y}^{(j)}\) are identical and let \(N_{ij}^{(2)}\) denote the number of positions on which the elements of the vectors \(\mathbf{X}^{(i)}\) and \(\mathbf{Y}^{(j)}\) are different. Then, we have that

\[
\mathbf{p}_{\mathbf{XY}}(\mathbf{X}^{(i)}, \mathbf{Y}^{(j)}) = \prod_{k=1}^{n} p(\mathbf{x}_{k}^{(i)}, \mathbf{y}_{k}^{(j)}) = \frac{(1 - p)^{N_{ij}^{(1)}} \cdot p^{N_{ij}^{(2)}}}{2^n} \text{ and } N_{ij}^{(1)} + N_{ij}^{(2)} = 2^n.
\]

(25)

Let \(N_0\) denote the number of elements of the output table of the Boolean function \(f\) that are equal to 0. Let \(N_1\) denote the number of elements of the output table of the Boolean function \(f\) that are equal to 1. We mention that there is no relationship between \(N_1\) and \(N_{ij}^{(1)}\), defined earlier.

\[
\text{Let } \{x_i^{(0)}\}, \forall i \in \{1, 2, \ldots N_0\} \text{ and } \{x_k^{(1)}\}, \forall k \in \{1, 2, \ldots N_1\}, \text{ such that } f(x_i^{(0)}) = 0 \text{ and } f(x_k^{(1)}) = 1
\]

\[
\Rightarrow \mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, 0) = \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_1^{(0)}, \mathbf{y}) + \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_2^{(0)}, \mathbf{y}) + \ldots + \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_{N_0}^{(0)}, \mathbf{y}), \forall \mathbf{y} \in \mathbf{E}_{\mathbf{Y}} = \{0, 1\}^n
\]

\[
\mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, 1) = \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_1^{(1)}, \mathbf{y}) + \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_2^{(1)}, \mathbf{y}) + \ldots + \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_{N_1}^{(1)}, \mathbf{y}), \forall \mathbf{y} \in \mathbf{E}_{\mathbf{Y}} = \{0, 1\}^n.
\]

(26)

B. Boolean functions from the classes 1 and 2 of Theorem [7]

Let \(f: \{0, 1\}^n \rightarrow \{0, 1\}\) be an \(n\)-dimensional Boolean function, such that, for any input \(\mathbf{X}^{(i)} \in \{0, 1\}^n\),

\[
\begin{cases}
  f(\mathbf{X}^{(i)}) = 1 \\
  f(\mathbf{X}) = 0, \forall \mathbf{X} \in \{0, 1\}^n, \mathbf{X} \neq \mathbf{X}^{(i)};
\end{cases}
\]

(27)

The output of such Boolean functions has only one element equal to 1 and the rest are equal to 0, that is \(N_1 = 1\) and \(N_0 = 2^n - 1\), \(\forall k \geq 2\).

\[
\Rightarrow \mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, 1) = \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_1^{(1)}, \mathbf{y}), \forall \mathbf{y} \in \mathbf{E}_{\mathbf{Y}} = \{0, 1\}^n
\]

\[
\mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, 0) = \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_1^{(0)}, \mathbf{y}) + \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_2^{(0)}, \mathbf{y}) + \ldots + \mathbf{p}_{\mathbf{XY}}(\mathbf{x}_{2^n-1}^{(0)}, \mathbf{y}), \forall \mathbf{y} \in \mathbf{E}_{\mathbf{Y}} = \{0, 1\}^n
\]

\[
= \mathbf{p}_{\mathbf{Y}}(\mathbf{y}) - \mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, 1) = \frac{1}{2^n} - \mathbf{p}_{\mathbf{YZ}}(\mathbf{y}, 1), \forall \mathbf{y} \in \mathbf{E}_{\mathbf{Y}} = \{0, 1\}^n.
\]

(28)

For any \(\mathbf{x}_1^{(1)} \in \{0, 1\}^n\), there exists: one vector, that is \(m_0 = 1, \mathbf{y}_{i_0} \in \{0, 1\}^n\), such that \(\mathbf{y}_{i_0} = \mathbf{x}_1^{(1)}\), a number \(m_1 = \binom{n}{1}\) of the vectors \((\mathbf{y}_{i_1}), \forall i_1 \in \{m_0 + 1, m_0 + 2, \ldots, m_0 + m_1\}\), such that \((\mathbf{y}_{i_1})\) differ from \(\mathbf{x}_1^{(1)}\) in one position and a number \(m_k = \binom{n}{k}\) of the vectors \((\mathbf{y}_{i_k}), \forall i_k \in \{(m_0 + \ldots + m_{k-1}) + 1, (m_0 + \ldots + m_{k-1}) + 2, \ldots, (m_0 + \ldots + m_{k-1}) + m_k\}\), such that \((\mathbf{y}_{i_k})\) differ from \(\mathbf{x}_1^{(1)}\) in \(k\) positions, \(\forall k \in \{0, 1, 2, \ldots, n\}\). As a result, we obtain
$p_{YZ}(y_i, 1) = \frac{(1-p)^n}{2^n}$, $p_{YZ}(y_i, 0) = \frac{1}{2^n} - p_{YZ}(y_i, 1), i_0 = m_0 = \binom{n}{0} = 1$

$p_{YZ}(y_1, 1) = \frac{(1-p)^{n-1} \cdot p}{2^n}$, $p_{YZ}(y_1, 0) = \frac{1}{2^n} - p_{YZ}(y_1, 1)$

$\forall i_1 \in \{m_0 + 1, m_0 + 2, \ldots, m_0 + m_1\}, m_1 = \binom{n}{1}$

$\vdots$

$p_{YZ}(y_k, 1) = \frac{(1-p)^{n-k} \cdot p^k}{2^n}$, $p_{YZ}(y_k, 0) = \frac{1}{2^n} - p_{YZ}(y_k, 1)$

$\forall i_k \in \{(m_0 + \ldots + m_{k-1}) + 1, (m_0 + \ldots + m_{k-1}) + 2, \ldots, (m_0 + \ldots + m_{k-1}) + m_k\}, m_k = \binom{n}{k}$

$\vdots$

$p_{YZ}(y_n, 1) = \frac{p^n}{2^n}$, $p_{YZ}(y_n, 0) = \frac{1}{2^n} - p_{YZ}(y_n, 1)$

$\forall i_n \in \{(m_0 + \ldots + m_{n-1}) + 1, (m_0 + \ldots + m_{n-1}) + 2, \ldots, (m_0 + \ldots + m_{n-1}) + m_n\}, m_n = \binom{n}{n} = 1$

$p_Z(1) = \sum_{i=1}^{2^n} p_{YZ}(y_i, 1) = \frac{(1-p)^n + \binom{n}{1} \cdot (1-p)^{n-1} \cdot p + \ldots + \binom{n}{k} \cdot (1-p)^{n-k} \cdot p^k + \ldots + \binom{n}{n} \cdot p^n}{2^n} = \frac{1}{2^n}$

$p_Z(0) = 1 - p_Z(1) = \frac{2^n - 1}{2^n}$.

As a result, for any $y_1 \in \{0, 1\}^n$, that is for any Boolean function that has only a value of 1 in its output table and the rest of the values are 0, we have that joint probability mass function values for $z = 1$, that is $p_{YZ}(y, 1), \forall y \in \{0, 1\}^n$, belong to the same set of values. The same conclusion applies for $z = 0$ and $p_{YZ}(y, 0), \forall y \in \{0, 1\}^n$. For any $y \in \{0, 1\}^n$, we also have that $p_Z(1) = \frac{1}{2^n}$ and $p_Z(0) = \frac{2^n - 1}{2^n}$. These results yield that the mutual information is identical, for Boolean functions that have only a value of 1 in their output table and the rest of the values are 0, regardless of the position of the values of 1 in the output table of the functions.

$MI_1(Y, Z) = \sum_{y} \sum_{z} p_{YZ}(y, z) \cdot \log \frac{p_{YZ}(y, z)}{p_Y(y) \cdot p_Z(z)}$

$= \sum_{y} p_{YZ}(y, 0) \cdot \log \frac{p_{YZ}(y, 0)}{p_Y(y) \cdot p_Z(0)} + p_{YZ}(y, 1) \cdot \log \frac{p_{YZ}(y, 1)}{p_Y(y) \cdot p_Z(1)}$

$= \sum_{y} p_{YZ}(y, 0) \cdot \log \frac{p_{YZ}(y, 0)}{\frac{1}{2^n}} + p_{YZ}(y, 1) \cdot \log \frac{p_{YZ}(y, 1)}{\frac{1}{2^n}}$

$= \sum_{y} 2^n \cdot [p_{YZ}(y, 0) + p_{YZ}(y, 1)] + p_{YZ}(y, 0) \cdot \log \frac{p_{YZ}(y, 0)}{2^n - 1} + p_{YZ}(y, 1) \cdot \log [p_{YZ}(y, 1)]$

$= 2^n + \sum_{y} (2^n - 1) \cdot \frac{p_{YZ}(y, 0)}{2^n - 1} \cdot \log \frac{p_{YZ}(y, 0)}{2^n - 1} + p_{YZ}(y, 1) \cdot \log [p_{YZ}(y, 1)]$.

Let $Q = \{q_i\}, P = \{p_i\}$ and $W = \{w_i\}, \forall i \in \{1, 2, \ldots, 2^n\}$, such that:

$q_i = p_{YZ}(y_i, 1) = \frac{(1-p)^{n-k} \cdot p^k}{2^n}, \forall i_k \in \{(m_j + 1, m_j + 2, \ldots, m_j + m_k) = \binom{n}{k}, \forall k \in \{0, 1, \ldots, n\}$

$w_i = \frac{1 - (1-p)^{n-k} \cdot p^k}{2^n - 1}$

$p_i = p_{YZ}(y_i, 0) = \frac{1}{2^n - 1} - p_{YZ}(y_i, 1) = \frac{1 - (1-p)^{n-k} \cdot p^k}{2^n - 1} = w_i \cdot \frac{2^n}{2^n - 1}$

$Q = \left[ \begin{array}{cccc}
q_1 = \frac{(1-p)^n}{2^n} & q_2 = \frac{(1-p)^{n-1} \cdot p}{2^n} & \cdots & q_k = \frac{(1-p)^{n-k} \cdot p^k}{2^n} & \cdots & q_{2^n} = \frac{p^n}{2^n}
\end{array} \right]_{m_0, m_1, \ldots, m_k, \ldots, m_{2^n}}$
\[ w_i = \begin{cases} 
\frac{1 - (1-p)^n}{2^n} & \text{if } m_i = 0 \\
\frac{1 - (1-p)^{n-1} \cdot p}{2^n} & \text{if } m_i = 1 \\
\vdots \\
\frac{1 - (1-p)^{n-k} \cdot p^k}{2^n} & \text{if } m_i = k \\
\frac{1 - p^n}{2^n} & \text{if } m_i = n-1
\end{cases} \]

\Rightarrow \text{MI}_1(Y, Z) = 2n + \sum_{i=1}^{2^n} (2^n - 1) \cdot p_i \cdot \log p_i + \sum_{i=1}^{2^n} q_i \cdot \log q_i.

(31)

The binomial theorem \[23\] states that

\[ \forall x, y \in \mathbb{R}, (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^{n-k} \cdot y^k; \text{ if } x = y = 1 \Rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k} \geq 1 + n. \]

(32)

\[ \sum_{i=1}^{2^n} w_i = \frac{1 - (1-p)^n}{2^n - 1} + \binom{n}{1} \cdot \frac{1 - (1-p)^{n-1} \cdot p}{2^n - 1} + \ldots + \binom{n}{k} \cdot \frac{1 - (1-p)^{n-k} \cdot p^k}{2^n - 1} + \ldots + \frac{1 - p^n}{2^n - 1} = \]

\[ = \left[ \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{k} + \ldots + \binom{n}{n} \right] - 1 = \frac{2^n - 1}{2^n - 1} = 1 \Rightarrow \sum_{i=1}^{2^n} w_i = 1. \]

(33)

\[ \frac{(1-p)^{n-k} \cdot p^k}{2^n} \cdot \log \frac{(1-p)^{n-k} \cdot p^k}{2^n} = (-n) \cdot \frac{(1-p)^{n-k} \cdot p^k}{2^n} + (n-k) \cdot \frac{(1-p)^{n-k} \cdot p^k}{2^n} \cdot \log (1-p) + \]

\[ + k \cdot \frac{(1-p)^{n-k} \cdot p^k}{2^n} \cdot \log p. \]

(34)

\[ \sum_{i=1}^{2^n} q_i \cdot \log q_i = \frac{(1-p)^n}{2^n} \cdot \log \frac{(1-p)^n}{2^n} + \binom{n}{1} \cdot \frac{(1-p)^{n-1} \cdot p}{2^n} \cdot \log \frac{(1-p)^{n-1} \cdot p}{2^n} + \ldots + \]

\[ + \binom{n}{k} \cdot \frac{(1-p)^{n-k} \cdot p^k}{2^n} \cdot \log \frac{(1-p)^{n-k} \cdot p^k}{2^n} + \ldots + \binom{n}{n} \cdot \frac{p^n}{2^n} \cdot \log \frac{p^n}{2^n} = \]

\[ = -\frac{n}{2^n} \cdot \left[ (1-p)^n + \binom{n}{1} \cdot (1-p)^{n-1} \cdot p + \ldots + \binom{n}{k} \cdot (1-p)^{n-k} \cdot p^k + \ldots + p^n \right] + \]

\[ + \frac{1}{2^n} \cdot \left[ n \cdot (1-p)^n + (n-1) \cdot \binom{n}{1} \cdot (1-p)^{n-1} \cdot p + \ldots + (n-k) \cdot \binom{n}{k} \cdot (1-p)^{n-k} \cdot p^k + \ldots \right. \]

\[ + \binom{n}{n-1} \cdot (1-p)^{n-1} \cdot p + \ldots + \binom{n}{1} \cdot (1-p)^{n-1} \cdot p + \ldots + \binom{n}{n} \cdot (1-p)^n \cdot p^n \]

\[ + k \cdot \binom{n}{k} \cdot (1-p)^{n-k} \cdot p^k + \ldots + n \cdot p^n \left. \right] \cdot \log p. \]

(35)

\[ (n-k) \cdot \binom{n}{k} = (n-k) \cdot \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1)!}{k! \cdot (n-k-1)!} = \frac{n \cdot (n-1)!}{k! \cdot (n-1-k)!} = n \cdot \binom{n-1}{k} \]

(36)

\[ \sum_{i=1}^{2^n} q_i \cdot \log q_i = -\frac{n}{2^n} \cdot (1-p)^n + \frac{n}{2^n} \cdot (1-p) \cdot \left[ (1-p)^{n-1} + \binom{n-1}{1} \cdot (1-p)^{n-2} \cdot p + \ldots + \binom{n-1}{k} \cdot (1-p)^{n-1-k} \cdot p^k + \ldots + p^{n-1} \right] \cdot \log p \]

\[ = -\frac{n}{2^n} + \frac{n}{2^n} \cdot (1-p) \cdot (1-p)^{n-1} \cdot \log (1-p) + \frac{n}{2^n} \cdot (1-p)^{n-1} \cdot p \cdot \log p \]

\[ = -\frac{n}{2^n} + \frac{n}{2^n} \cdot (1-p) \cdot \log (1-p) + \frac{n}{2^n} \cdot p \cdot \log p \]

(37)
We want to prove that

$$\text{MI}_1(Y, Z) = 2n + \sum_{i=1}^{2^n} q_i \cdot \log q_i + \sum_{i=1}^{2^n} (2^n - 1) \cdot p_i \cdot \log p_i \leq 1 - H(p)$$

$$\iff \sum_{i=1}^{2^n} (2^n - 1) \cdot p_i \cdot \log p_i \leq 1 - 2 \cdot n + \frac{n}{2^n} + \left( \frac{n}{2^n} - 1 \right) \cdot H(p).$$ (38)

$$\sum_{i=1}^{2^n} (2^n - 1) \cdot p_i \cdot \log p_i = \sum_{i=1}^{2^n} (2^n - 1) \cdot \frac{w_i}{2^n} \cdot \log \frac{w_i}{2^n} =$$

$$= (2^n - 1) \cdot \frac{-n}{2^n} \sum_{i=1}^{2^n} w_i + \frac{1}{2^n} \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i =$$

$$= -n \cdot \frac{(2^n - 1)}{2^n} + \frac{1}{2^n} \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i.$$ (39)

Let \( a = \frac{1 - p}{2^n - 1} \) and \( b = \frac{p}{2^n - 1} \). We want to prove that

$$- n \cdot \frac{(2^n - 1)}{2^n} + \frac{1}{2^n} \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i \leq 1 - 2 \cdot n + \frac{n}{2^n} + \left( \frac{n}{2^n} - 1 \right) \cdot H(p)$$

$$\iff (-n) \cdot (2^n - 1) + \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i \leq 2^n - 2 \cdot n \cdot 2^n + n + (n - 2^n) \cdot H(p)$$

$$\iff \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i \leq 2^n - n \cdot 2^n + (n - 2^n) \cdot H(p)$$

$$\iff \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i \leq (-2^n) \cdot (n - 1) + (2^n - n) \cdot [(1 - p) \cdot \log (1 - p) + p \cdot \log p]$$

$$\iff \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i \leq (-2^n) \cdot (n - 1) + (2^n - n) \cdot \left[ (n - 1) + (1 - p) \cdot \log \frac{1 - p}{2^n - 1} + p \cdot \log \frac{p}{2^n - 1} \right]$$

$$\iff \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i \leq (-2^n) \cdot (n - 1) + (2^n - n) \cdot \left[ (1 - p) \cdot \log \frac{1 - p}{2^n - 1} + p \cdot \log \frac{p}{2^n - 1} \right]$$

$$\iff \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i \leq (-n) \cdot (n - 1) + (2^n - n) \cdot 2^{n-1} \cdot (a \cdot \log a + b \cdot \log b).$$ (40)

The number of elements on the left side of the inequality is equal to \( 2^n \cdot (2^n - 1) \). The number of elements on the right side of the inequality is equal to \( 2 \cdot 2^{n-1} \cdot (2^n - n) \). We need to transform the element \((-n) \cdot (n - 1)\), from the right side of the inequality, into a sum of the type \( x \cdot \log x \), such that the number of elements on the right side of the inequality equals that of the left side. That is, we need \( 2^n \cdot (2^n - 1) - 2 \cdot 2^{n-1} \cdot (2^n - n) = (n - 1) \cdot 2^n \) elements. We need to divide \((-n) \cdot (n - 1)\) into a sum that has \((n - 1) \cdot 2^n\) elements. That is, we need to find \( x \), such that

\[(n - 1) \cdot 2^n \cdot x \cdot \log x = (-n) \cdot (n - 1)\]

$$\iff x \cdot \log x = \frac{-n}{2^n}$$

$$\iff x = \frac{1}{2^n}.$$ (41)

The right hand side sequence has three distinct elements ordered as

\[ a = \frac{1 - p}{2^n - 1} \geq c = \frac{1}{2^n} \geq b = \frac{p}{2^n - 1}. \] (42)

The left hand side sequence has the elements ordered as

\[ w_{2^n} = \frac{1 - p^n}{2^n - 1} \geq w_{2^n-1} = \frac{1 - (1 - p) \cdot p^{n-1}}{2^n - 1} \geq \ldots \geq w_i = \frac{1 - (1 - p)^{n-k} \cdot p^k}{2^n - 1} \geq \ldots \geq w_1 = \frac{1 - (1 - p)^n}{2^n - 1}. \] (43)
Let $X = [x_1, x_2 \ldots x_{2^n \cdot (2^n - 1)}]$ and $Y = [y_1, y_2 \ldots y_{2^n \cdot (2^n - 1)}]$ be equal to

$$
X = \begin{bmatrix}
\underbrace{a \ a \ a \ldots a}_{2^{n-1}(2^n-n)} & \underbrace{c \ c \ c \ldots c}_{2^n \cdot (n-1)} & \underbrace{b \ b \ b \ldots b}_{2^{n-1}(2^n-n)} \\
\text{elements} & \text{elements} & \text{elements}
\end{bmatrix}, \\
Y = \begin{bmatrix}
\underbrace{w_2 \ w_2 \ w_2 \ldots w_2}_{2^n-1} & \underbrace{w_{2^n-1} \ldots w_1}_{2^n-1} \\
\text{elements} & \text{elements}
\end{bmatrix}.
$$

(44)

$\Rightarrow X$ and $Y$ are in descending order, which satisfies the first condition of Karamata’s theorem.

Let $g : \mathbb{R}_+ \to \mathbb{R}$, $g(x) = x \cdot \log x$. Let $\log_x \cdot \cdot \cdot$ denote the natural logarithm.

$$
g'(x) = \frac{1}{\log_e 2} \left( \log_e x + x \cdot \frac{1}{x} \right) = \log_x x + \frac{1}{\log_e 2} \Rightarrow g''(x) = \frac{1}{\log_e 2} \cdot \frac{1}{x} \geq 0, \forall x \geq 0 \Rightarrow g$ is a convex function.

(45)

We want to prove that

$$
\sum_{i=1}^{2^n} (2^n - 1) \cdot w_i \cdot \log w_i \leq (-n) \cdot (n-1) + (2^n - n) \cdot 2^{n-1} \cdot (a \cdot \log a + b \cdot \log b)
$$

(46)

$$
\Leftrightarrow \sum_{i=1}^{2^n} y_i \cdot \log y_i \leq \sum_{i=1}^{2^n} x_i \cdot \log x_i \\
\Leftrightarrow \sum_{i=1}^{2^n} g(y_i) \leq \sum_{i=1}^{2^n} g(x_i).
$$

(47)

1) We prove that $w_{2^n} \leq a$:

$$
\frac{w_{2^n} \leq a}{2^n - 1} \Rightarrow \frac{1 - p^n}{2^n - 1} \leq \frac{1}{2^n - 1} \\
\Leftrightarrow 2^n - 2^{n-1} \cdot p^n \leq 2^n - 1 - (2^n - 1) \cdot p \\
\Leftrightarrow (2^n - 1) \cdot p - 2^{n-1} \cdot p^n \leq 2^n - 1 - 1.
$$

(48)

We will prove the last inequality using the functions $f$, defined as

$$
f(x) : \left[0, \frac{1}{2} \right] \to \mathbb{R}_+, f(x) = (2^n - 1) \cdot x - 2^{n-1} \cdot x^n
$$

$$
\Rightarrow f(0) = 0, f \left( \frac{1}{2} \right) = 2^{n-1} - 1 \\
\Rightarrow f'(x) = 2^n - 1 - 2^{n-1} \cdot n \cdot x^{n-1}.
$$

(49)

We need to prove that the function $f$ is increasing and that its maximum point is equal to $x^* = \frac{1}{2}$, which yields $f(x^*) = 2^{n-1} - 1$.

$$
f'(x) = 0 \Rightarrow x^* \text{ is the order } n-1 \text{ root of this equation, } (x^*)^{n-1} = \frac{2^n - 1}{2^{n-1} \cdot n}
$$

We proved that $2^n \geq 1 + n$, as $\sum_{i=1}^{2^n} \Rightarrow (x^*)^{n-1} = \frac{2^n - 1}{2^{n-1} \cdot n} \geq \frac{1}{2^{n-1}} \Leftrightarrow x^* \geq \frac{1}{2}$

(50)

$$
f'(x) = 2^n - 1 - 2^{n-1} \cdot n \cdot x^{n-1} \geq n - 2^{n-1} \cdot n \cdot \frac{1}{2^{n-1}} = 0 \Rightarrow f'(x) \geq 0 \Rightarrow \text{ the function } f \text{ is increasing}
$$

(51)

The results of $\sum_{i=1}^{2^n} \text{ and } \sum_{i=1}^{2^n}$ yield $f(x) \leq f \left( \frac{1}{2} \right), \forall x \in \left[0, \frac{1}{2} \right]$ \Rightarrow $(2^n - 1) \cdot p - 2^{n-1} \cdot p^n \leq 2^n - 1 - 1

\Rightarrow w_{2^n} \leq a.

(52)

Let $SL_k$ and $SR_k, \forall k \in \{1, 2, \ldots 2^n \cdot (2^n - 1)\}$, denote the partial sums computed with the elements of the left-hand sequence of the inequality (46) and with the right-hand one, respectively. We need to prove that
increasing, given by that fact that

\[2^n - 1 \leq 2^n - (2^n - n)\]
\[\iff 2^n - 1 \leq 2^{n-1} \cdot (2^n - 1) - 2^{n-1} \cdot (n - 1)\]

\[\iff 2^n - 1 \cdot (n - 1) \leq (2^{n-1} - 1) \cdot (2^n - 1)\]
\[\iff 2^{n-1} \cdot \frac{(n - 1)}{2^{n-1} - 1} \leq (2^n - 1)\]

We proved that \(2^n \geq 1 + n\), as \((52)\), \(\Rightarrow 2^n - 1 \geq 1 + n - 1 \Rightarrow \frac{n - 1}{2^{n-1} - 1} \leq 1 \Rightarrow 2^n - 1 \cdot \frac{n - 1}{2^{n-1} - 1} \leq 2^n - 1\)

\(1 \leq 2^n - 1 \Rightarrow 2^n + 1 \leq 2 \cdot 2^n \Rightarrow 2^n - 1 \leq 2^n - 1 \Rightarrow 2^n - 1 \Rightarrow \frac{(n - 1)}{2^{n-1} - 1} \leq (2^n - 1) \Rightarrow 2^n - 1 \leq 2^n - 1 \cdot (2^n - n). \quad (53)\)

Let \(K = 2^{n-1} \cdot (2^n - n)\)
\(2^n - 1 \leq 2^n \cdot (2^n - n)\) and \(w_k \leq w_{2^n} \leq a, \forall k \in \{2^n, 2^n - 1, \ldots, 1\}\)
\(\Rightarrow SL_k = \sum_{j=1}^{k} y_j \leq SR_k = \sum_{j=1}^{k} x_j = k \cdot x_1 = k \cdot a, \forall k \in \{1, 2, \ldots, K\}. \quad (54)\)

2) We prove that \(2 \cdot w_{2^n} \leq a + c:\)
\[2 \cdot w_{2^n} \leq a + c\]
\[\iff 2 \cdot \frac{1 - p^n}{2^n - 1} - \frac{1 - p}{2^n - 1} + \frac{1}{2^n} \leq \frac{1 - p}{2^n - 1} + \frac{1}{2^n}\]
\[\iff 2 \cdot \frac{1 - p^n}{2^n - 1} + \frac{p}{2^n} \leq \frac{3}{2^n} \quad (55)\]

We will prove the last inequality using the functions \(f\), defined as
\[f(x) : \left[0, \frac{1}{2}\right] \rightarrow \mathbb{R}^+, f(x) = 2 \cdot \frac{1 - x^n}{2^n - 1} + \frac{x}{2^n - 1}\]
\[f(0) = \frac{2}{2^n - 1}\] and \(f \left(\frac{1}{2}\right) = \frac{3}{2^n}\)
\[f'(x) = \frac{-2 \cdot n \cdot x^{n-1}}{2^n - 1} + \frac{1}{2^n - 1}. \quad (56)\]

We will show that \(x_1 = \frac{1}{2}\) is the maximum point of \(f(x)\), which yields the inequality \((55)\). Let \(x^*\) be a critical point of the function \(f\). Then, we have that
\[f'(x^*) = 0 \Rightarrow \frac{2 \cdot n \cdot (x^*)^{n-1}}{2^n - 1} = \frac{1}{2^n - 1}\]

According to the binomial theorem \((24)\), \(\forall x, y \in \mathbb{R}, (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot y^k\)

If \(x = y = 1 \Rightarrow 2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{k} + \ldots + \binom{n}{n-1} + \binom{n}{n} \geq 2 \cdot n + 1 \Rightarrow \frac{2 \cdot n}{2^n - 1} \leq 1\)

\(\forall 0 \leq x \leq \frac{1}{2} \Rightarrow \frac{2 \cdot n \cdot x^{n-1}}{2^n - 1} \leq \frac{1}{2^n - 1} \Rightarrow x \leq x^*\) and \(f'(x) \geq 0, \forall 0 \leq x \leq \frac{1}{2}. \quad (57)\)

That is, the critical point of the function \(f\) lies outside the interval on which it is defined. In addition, the function \(f\) is increasing, given by that fact that \(f'(x) \geq 0, \forall 0 \leq x \leq \frac{1}{2}\). These results yield
\[f(x) \leq f \left(\frac{1}{2}\right), \forall 0 \leq x \leq \frac{1}{2}\]
\[\Rightarrow 2 \cdot \frac{1 - p^n}{2^n - 1} + \frac{p}{2^n} \leq \frac{3}{2^n}\]
\[\Rightarrow 2 \cdot w_{2^n} \leq a + c. \quad (58)\]
3) We verify that the inequalities involving the partial sums from Karamata’s theorem hold: Here, we use the inequality $2 \cdot w_{2^n} \leq a + c$, which we proved at the previous step.

Previously, in this article, we denoted $X = [x_1 \ x_2 \ldots \ x_{2^n - (2^n - 1)}]$ and $Y = [y_1 \ y_2 \ldots \ y_{2^n - (2^n - 1)}]$, with the property that

$$
X = \begin{bmatrix}
a & a & a & \ldots & a & c & c & \ldots & c & b & b & \ldots & b \\
2^{2n-1} & 2^{2n-1} & \cdots & 2^{2n-1} & 2^{2n-1} & 2^{2n-1} & \cdots & 2^{2n-1} & 2^{2n-1} \end{bmatrix}, \quad Y = \begin{bmatrix}
w_1 & w_2 & w_3 & \ldots & w_{2^n} \\
2^{2n-1} & 2^{2n-1} & \cdots & 2^{2n-1} & 2^{2n-1} \end{bmatrix}.
$$

We need to verify that $K = 2^{2n-1} \cdot (2^n - n) \geq 2^n \cdot (n - 1), \forall n \geq 2$, in order to apply the inequality $2 \cdot w_{2^n} \leq a + c$.

$$
2^{2n-1} \cdot (2^n - n) \geq 2^n \cdot (n - 1) \\
\iff 2^n \cdot (n - 1) \geq 2^{2n-1} \cdot n \\
\iff 2^n + 2 \geq 3 \cdot n.
$$

As [32], we proved that $2^n \geq n + 1 \Rightarrow 2^{n-2} \geq n - 2 + 1 \Rightarrow 4 \cdot 2^{n-2} \geq 2 \cdot 2^{n-2} \geq 4 \cdot (n - 1) + 2 = 4 \cdot n - 2 \geq 3 \cdot n, \forall n \geq 2$.

$$
2^n + 2 \geq 3 \cdot n \Rightarrow 2^{2n-1} \cdot (2^n - n) \geq 2^n \cdot (n - 1). \quad (60)
$$

1) If $n = 2 \Rightarrow K = 2 \cdot (4 - 2) = 4 = 2^n \cdot (n - 1)$. In this case, the number of elements equal to $a$ is the same as the number of elements equal to $c$.

As [54], we proved that $SL_k = \sum_{j=1}^{k} y_j \leq SR_k = \sum_{j=1}^{k} x_j = k \cdot x_1 = k \cdot a, \forall k \in \{1, 2, 3, 4\}$.

$$
2 \cdot w_4 \leq a + c \Rightarrow w_j + w_k \leq a + c, \forall j, k \in \{4, 3, 2, 1\} \\
\Rightarrow w_j + y_k \leq a + c, \forall j, k \in \{12, 11, \ldots, 1\}.
$$

$$
SL_{K+i} = w_4 + w_4 + w_4 + w_4 + w_4 + y_{K+i} \quad \text{and} \quad SR_{K+i} = 4 \cdot a + i \cdot c, \forall i \in \{1, 2, 3, 4\}.
$$

$$
\Rightarrow SL_{K+i} \leq 4 \cdot a + i \cdot c = SR_{K+i}, \forall i \in \{1, 2, 3, 4\}.
$$

(61)

2) If $n \geq 3$, we need to verify that $K - 2^n \cdot (n - 1) \geq 1$.

$$
K - 2^n \cdot (n - 1) \geq 1 \\
\iff 2^n \cdot (n - 1) \geq 2^n \cdot (n - 1) + 1 \\
\iff 2^{n-1} \cdot n \cdot 2^n - n + 2^n + \frac{1}{2^{n-1}} \\
\iff 2^n + 2 \geq 3 \cdot n + 1 \geq 3 \cdot n + \frac{1}{2^{n-1}}.
$$

We need to prove that $2^n + 1 \geq 3 \cdot n$.

As [32], we proved that $2^n \geq n + 1 \Rightarrow 2^{n-2} \geq n - 2 + 1 \Rightarrow 2^n \geq 4 \cdot n - 4 \Rightarrow 2^n + 1 \geq 4 \cdot n - 3 \geq 3 \cdot n, \forall n \geq 3$.

$$
2^n + 2 \geq 3 \cdot n + 1 \geq 3 \cdot n + \frac{1}{2^{n-1}} \Rightarrow K - 2^n \cdot (n - 1) \geq 1, \forall n \geq 3.
$$

(62)

As [54], we proved that $SL_k = \sum_{j=1}^{k} y_j \leq SR_k = \sum_{j=1}^{k} x_j = k \cdot x_1 = k \cdot a, \forall k \in \{1, 2, \ldots, K\}$.

$$
2 \cdot w_{2^n} \leq a + c \Rightarrow w_j + w_k \leq a + c, \forall j, k \in \{2^n, 2^n - 1, \ldots, 1\} \\
\Rightarrow y_j + y_k \leq x_1 + x_{K+i}, \forall j, k \in \{2^n, (2^n - 1), 2^n \cdot (n - 1)\}.
$$

$$
2^{n-1} \cdot (2^n - n) \geq 2^n \cdot (n - 1) \Rightarrow K - i \geq 1, \forall i \in \{1, 2, \ldots, 2^n \cdot (n - 1)\}.
$$

$$
SL_{K+i} = SL_{K+i} + y_{K+i+1} + \ldots + y_{K+i} + y_{K+i+1} + \ldots + y_{K+i} \\
= SL_{K+i} + (y_{K+i+1} + y_{K+i+1}) + \ldots + (y_{K+i} + y_{K+i}) \\
\Rightarrow SL_{K+i} \leq SL_{K+i} + i \cdot (x_1 + x_{K+i}) = SL_{K+i}, \forall i \in \{1, 2, \ldots, 2^n \cdot (n - 1)\}.
$$

(63)
4) We prove that $w_1 \geq b$:

$$w_1 \geq b \iff \frac{1 - (1 - p)^n}{2^n - 1} \geq \frac{p}{2^n - 1} \iff 2^{n-1} \geq 2^{n-1} \cdot (1 - p)^n + (2^n - 1) \cdot p.$$  

(64)

We prove the last inequality using the functions $f$, defined as

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{2} & \end{cases}, f(x) = 2^{n-1} \cdot (1 - x)^n + (2^n - 1) \cdot x$$

$$f(0) = f \left( \frac{1}{2} \right) = 2^{n-1}$$

$$f'(x) = 2^{n-1} \cdot n \cdot (1 - x)^{n-1} \cdot (-1) + (2^n - 1)$$

$$f'(x) = 0 \iff (1 - x)^{n-1} = \frac{2^n - 1}{n \cdot 2^{n-1}},$$

with the solution to this equation denoted as $x^*$.

As [32], we proved that $2^n \geq 1 + n \Rightarrow \frac{2^n - 1}{n \cdot 2^{n-1}} \geq \frac{1}{2n-1} \Rightarrow (1 - x)^{n-1} \geq \frac{1}{2n-1}$, which is true $\forall x \in \left[0 \frac{1}{2}\right] \Rightarrow x^* \in \left[0 \frac{1}{2}\right]$

$$f''(x) = n \cdot (n - 1) \cdot 2^{n-1} \cdot (1 - x)^{n-2} \geq 0, \forall x \in \left[0 \frac{1}{2}\right] \Rightarrow f$$

is a convex function and $x^*$ is a minimum point

$$\Rightarrow x_1 = 0 \text{ and } x_2 = \frac{1}{2} \text{ are maximum points and } f(0) = f \left( \frac{1}{2} \right) = 2^{n-1}$$

$$\Rightarrow 2^{n-1} \geq f(x), \forall x \in \left[0 \frac{1}{2}\right]$$

$$\Rightarrow 2^{n-1} \geq 2^{n-1} \cdot (1 - p)^n + (2^n - 1) \cdot p \Rightarrow w_1 \geq b.$$  

(65)

5) We verify that the final inequalities involving the partial sums from Karamata’s theorem hold: Here, we use the inequality $w_1 \geq b$, which we proved at the previous step.

$$SL_{2^n \cdot (2^n - 1)} = \sum_{i=1}^{2^n} (2^n - 1) \cdot w_i = 2^n - 1$$

$$SR_{2^n \cdot (2^n - 1)} = (n - 1) \cdot 2^n \cdot \frac{1}{2n} + (2^n - n) \cdot 2^{n-1} \cdot \frac{1 - p}{2^{n-1}} + (2^n - n) \cdot 2^{n-1} \cdot \frac{p}{2^{n-1}} = n - 1 + 2^n - n = 2^n - 1$$

$$\Rightarrow SL_{2^n \cdot (2^n - 1)} = SR_{2^n \cdot (2^n - 1)}.$$  

(66)

From [53], we have that $2^n - 1 \leq 2^{n-1} \cdot (2^n - n)$, which represents the total number of elements equal to $b$. The partial sum inequalities hold only for $2^n - 1$ elements equal to $b$. We need to determine that the remaining number of elements equal to $b$, satisfies the inequalities involving the partial sums from Karamata’s theorem. They are denoted as $\{SL_{2^n \cdot (2^n - 1)} - 2, \ldots, SL_{2^n \cdot (2^n - 1)} - 2^{n-1} \cdot 2^{n-1} \cdot (2^n - n)\}$ and $\{SR_{2^n \cdot (2^n - 1)} - 2, \ldots, SR_{2^n \cdot (2^n - 1)} - 2^{n-1} \cdot 2^{n-1} \cdot (2^n - n)\}$. There are $2^{n-1} \cdot (2^n - n)$ right-hand partial sums, which contain the elements equal to $b$.

Let $M = 2^n \cdot (2^n - 1) - (2^n - 1)$. We proved that

$$w_i \geq w_1 \geq b, \forall i \in \{2^n, 2^n - 1, \ldots, 1\}$$

and

$$SL_k = \sum_{j=1}^{k} y_j \leq SR_k = \sum_{j=1}^{k} x_j, \forall k \in \{2^n \cdot (2^n - 1), 2^n \cdot (2^n - 1) - 1, \ldots, M\}.$$  

(68)
The sums above are well defined, because $M - i \geq 1$, $\forall i \in \{1, 2, \ldots, 2^{n-1} \cdot (2^n - n) - (2^n - 1)\}$.

\[ \forall i \in \{1, 2, \ldots, 2^{n-1} \cdot (2^n - n) - (2^n - 1)\} \Rightarrow M - i \geq [2^n \cdot (2^n - 1) - (2^n - 1)] - [2^{n-1} \cdot (2^n - n) - (2^n - 1)] \]
\[ \Rightarrow M - i \geq 2^n \cdot (2^n - 1) - 2^{n-1} \cdot (2^n - n). \]

We need to prove that $2^n \cdot (2^n - 1) - 2^{n-1} \cdot (2^n - n) \geq 1$
\[ \Rightarrow 2^n \cdot (2^n - 1) \geq 2^{n-1} \cdot (2^n - n) + 1 \]
\[ \Rightarrow 2 \cdot (2^n - 1) \geq 2^n - n + \frac{1}{2^{n-1}} \]
\[ \Rightarrow 2 + n \geq 2 + \frac{1}{2^{n-1}} \]
\[ \Rightarrow 2^n + n \geq 2 + \frac{1}{2^{n-1}} \]
\[ \Rightarrow 2^n \cdot (2^n - 1) - 2^{n-1} \cdot (2^n - n) \geq 1, \forall n \geq 2 \]
\[ \Rightarrow M - i \geq 1, \forall n \geq 2, i \in \{1, 2, \ldots, 2^{n-1} \cdot (2^n - n) - (2^n - 1)\}. \quad (69) \]

The first partial sum that does not contain an element equal to $b$ is given by $i = 2^{n-1} \cdot (2^n - n) - (2^n - 1) \Rightarrow M - i = 2^n \cdot (2^n - 1) - 2^{n-1} \cdot (2^n - n) = K + 2^n \cdot (n - 1)$. As a result of the above derivations, $\text{SL}_{K + 2^n \cdot (n - 1)} \leq \text{SR}_{K + 2^n \cdot (n - 1)}$, which we also proved, as the inequality \[63\]. In conclusion, all the conditions in Karamata’s theorem are satisfied. This yields
\[ \sum_{i=1}^{2^n \cdot (2^n-1)} g(y_i) \leq \sum_{i=1}^{2^n \cdot (2^n-1)} g(x_i) \]
\[ \Rightarrow \text{MI}_1(Y, Z) \leq 1 - H(p). \quad (70) \]

We will now prove that the same result holds, for Boolean functions that have one element equal to 0 in their output table and the rest are equal to 1, that is $N_1 = 2^n - 1$ and $N_0 = 1$. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be an $n$-dimensional Boolean function, such that, for any input $X^{(i)} \in \{0, 1\}^n$,
\[ \begin{cases} f(X^{(i)}) = 0 \\ f(X) = 1, \forall X \in \{0, 1\}^n, X \neq X^{(i)}; \end{cases} \]
\[ \Rightarrow \text{PYZ}(Y, 0) = \text{PXY}(x_1^{(0)}, Y), \forall Y \in \mathcal{E}_Y = \{0, 1\}^n \]
\[ \text{PYZ}(Y, 1) = \text{PXY}(x_1^{(1)}, Y) + \text{PXY}(x_2^{(1)}, Y) + \ldots + \text{PXY}(x_{2^n-1}^{(1)}, Y), \forall Y \in \mathcal{E}_Y = \{0, 1\}^n \]
\[ = \text{PY}(Y) - \text{PYZ}(Y, 0) = \frac{1}{2^n} - \text{PYZ}(Y, 0), \forall Y \in \mathcal{E}_Y = \{0, 1\}^n. \quad (72) \]

For any $x_1^{(0)} \in \{0, 1\}^n$, there exists: one vector, that is $m_0 = 1$, $y_{i_0} \in \{0, 1\}^n$, such that $y_{i_0} = x_1^{(0)}$, a number $m_1 = \binom{n}{1}$ of the vectors $(y_{i_1})$, $\forall i_1 \in \{m_0 + 1, m_0 + 2, \ldots, m_0 + m_1\}$, such that $(y_{i_1})$ differ from $x_1^{(0)}$ in one position and a number $m_k = \binom{n}{k}$ of the vectors $(y_{i_k})$, $\forall i_k \in \{(m_0 + \ldots + m_{k-1}) + 1, (m_0 + \ldots + m_{k-1}) + 2, \ldots, (m_0 + \ldots + m_{k-1}) + m_k\}$, such that $(y_{i_k})$ differ from $x_1^{(0)}$ in $k$ positions, $\forall k \in \{0, 1, 2, \ldots n\}$. As a result, we obtain
\[ \text{PYZ}(y_{i_0}, 0) = \frac{(1-p)^n}{2^n}, \text{PYZ}(y_{i_0}, 1) = \frac{1}{2^n} - \text{PYZ}(y_{i_0}, 0), i_0 = m_0 = \binom{n}{0} = 1 \]
\[ \text{PYZ}(y_{i_1}, 0) = \frac{(1-p)^{n-1} \cdot p}{2^n}, \text{PYZ}(y_{i_1}, 1) = \frac{1}{2^n} - \text{PYZ}(y_{i_1}, 0) \]
\[ \forall i_1 \in \{m_0 + 1, m_0 + 2, \ldots, m_0 + m_1\}, m_1 = \binom{n}{1} \]
\[ \vdots \]
\[ \text{PYZ}(y_{i_k}, 0) = \frac{(1-p)^{n-k} \cdot p^k}{2^n}, \text{PYZ}(y_{i_k}, 1) = \frac{1}{2^n} - \text{PYZ}(y_{i_k}, 0) \]
\[ \forall i_k \in \{(m_0 + \ldots + m_{k-1}) + 1, (m_0 + \ldots + m_{k-1}) + 2, \ldots, (m_0 + \ldots + m_{k-1}) + m_k\}, m_k = \binom{n}{k} \]
\[ \vdots \]
\[ \text{PYZ}(y_{i_n}, 0) = \frac{p^n}{2^n}, \text{PYZ}(y_{i_n}, 1) = \frac{1}{2^n} - \text{PYZ}(y_{i_n}, 0) \]
According to Lemma 1, we also have that the Boolean function has values of $i$ for any $x_i^{(0)}$, such that $x_i^{(0)} = 1$ and $p_{YZ}(y, 0)$, $\forall y \in \{0, 1\}^n$. Let the vectors $Y^{(k)} = [Y^{(k)}_r, Y^{(k)}_{n-r}]$, $\forall k \in \{1, 2, \ldots, 2^n\}$, and $X^{(i)} = [X_r, X^{(i)}_{n-r}]$, $\forall i \in \{1, 2, \ldots, 2^n\}$, such that $X^{(i)} \in \{X_r, 0 \ldots 0, \ldots, X_r, 0 \ldots 1 \ldots 1\}$. The output table of the Boolean function has $N_1 = 2^{n-r}$ number of ones, such that these values correspond to the vector of inputs $X^{(i)} \in \{X_r, 0 \ldots 0, \ldots, X_r, 0 \ldots 0, \ldots, X_r, 1 \ldots 1\}$, where $X_r$ is fixed. The rest of the output values are zeros.

From the properties of the binary symmetric channel (Appendix A) (check this again), we can write that

$$p(Y^{(k)}, X^{(i)}) = p(Y^{(k)}_r, X_r) \cdot p(Y^{(i)}_{n-r}, X^{(i)}_{n-r}), \forall k \in \{1, 2, \ldots, 2^n\}.$$  

According to Lemma [2] and

$$\sum_{i=1}^{2^n} p(Y^{(k)}_{n-r}, X^{(i)}_{n-r}) = \frac{1}{2^{n-r}}.$$  

Let $q_k = p_{YZ}(Y^{(k)}_r, 1)$

From [26] $p_{YZ}(Y^{(k)}_r, 1) = p(Y^{(k)}_r, X^{(i)}) = \sum_{i=1}^{2^n} p(Y^{(k)}_r, X_r) \cdot p(Y^{(k)}_{n-r}, X^{(i)}_{n-r}) = \frac{p(Y^{(k)}_r, X_r)}{2^{n-r}}$

$$p_k = p_{YZ}(Y^{(k)}_r, 0) = p(Y^{(k)}_r) - p_{YZ}(Y^{(k)}_r, 1) = \frac{1}{2^n} - q_k, \forall k \in \{1, 2, \ldots, 2^n\}.$$  

For any $k \in \{1, 2, \ldots, 2^n\}$, the total number of $Y^{(k)} = [Y^{(k)}_r, Y^{(k)}_{n-r}]$ that have the same $Y^{(k)}_r$ is equal to $N_1 = 2^{n-r}$. This produces a number of $N_1 = 2^{n-r}$ identical probability mass values, $q_k = \frac{p(Y^{(k)}_r, X_r)}{2^{n-r}}$ and $N_1 = 2^{n-r}$ identical probability mass values, $p_k = \frac{1}{2^n} - q_k$. Let the vectors $v = [v_1 v_2 \ldots v_{2^n}]$ and $t = [t_1 t_2 \ldots t_{2^n}]$ denote the distinct values of the vectors $q = [q_1 q_2 \ldots q_{2^n}]$ and $v = [p_1 p_2 \ldots p_{2^n}]$, respectively.
\[
\begin{align*}
\text{MI}_1(Y, Z) &= \sum_y pYZ(y, 0) \cdot \log \frac{pYZ(y, 0)}{pY(y) \cdot pZ(0)} + pYZ(y, 1) \cdot \log \frac{pYZ(y, 1)}{pY(y) \cdot pZ(1)} \\
&= \sum_y pYZ(y, 0) \cdot \log \left(\frac{pY\left(\frac{y}{2}\right)}{2^n}\right) + pYZ(y, 1) \cdot \log \left(\frac{pY\left(\frac{y}{2}\right)}{2^n}\right) \\
&= 2n + \sum_{k=1}^{2^n} p_k \cdot \log \frac{p_k}{2^n - 2^{n-r}} + q_k \cdot \log \frac{q_k}{2^{n-r}} \\
&= 2n + 2^{n-r} \cdot \sum_{i=1}^{2^r} t_i \cdot \log \frac{t_i}{2^n - 2^{n-r}} + v_i \cdot \log \frac{v_i}{2^{n-r}}.
\end{align*}
\]

For any \(X_r \in \{0, 1\}^r\) fixed, there exists: one vector, that is \(m_0 = 1, Y_r^{(i_0)} \in \{0, 1\}^r\), such that \(Y_r^{(i_0)} = X_r\), a number \(m_1 = \binom{r}{j}\) of the vectors \((Y_r^{(i_1)}), \forall i_j \in \{m_0 + 1, m_0 + 2, \ldots, m_0 + m_1\}\), such that \((Y_r^{(i_1)})\) differ from \(X_r\) in one position and a number \(m_j = \binom{r}{j}\) of the vectors \((Y_r^{(i_j)}), \forall i_j \in \{m_0 + \ldots + m_{j-1} + 1, (m_0 + \ldots + m_{j-1} + 2, \ldots, (m_0 + \ldots + m_{j-1} + m_j\},\) such that \((Y_r^{(i_j)})\) differ from \(X_r\) in \(j\) positions, \(\forall j \in \{0, 1, 2, \ldots, r\}\). As a result, we obtain

\[
p(Y_r^{(i_j)}, X_r) = \frac{(1 - p)^{r-j} \cdot p^j}{2^n}, \forall i_j \in \{(m_0 + \ldots + m_{j-1} + 1, (m_0 + \ldots + m_{j-1} + 2, \ldots, (m_0 + \ldots + m_{j-1} + m_j, \\
m_j = \binom{r}{j}, \forall j \in \{0, 1, 2, \ldots, r\}.
\]

\[
\Rightarrow v_i = \frac{(1 - p)^{r-j} \cdot p^j}{2^n}, \forall i_j \in \{(m_0 + \ldots + m_{j-1} + 1, (m_0 + \ldots + m_{j-1} + 2, \ldots, (m_0 + \ldots + m_{j-1} + m_j, \\
m_j = \binom{r}{j}, \forall j \in \{0, 1, 2, \ldots, r\}.
\]

\[
\Rightarrow t_i = \frac{1 - (1 - p)^{r-j} \cdot p^j}{2^n}, \forall i_j \in \{(m_0 + \ldots + m_{j-1} + 1, (m_0 + \ldots + m_{j-1} + 2, \ldots, (m_0 + \ldots + m_{j-1} + m_j, \\
m_j = \binom{r}{j}, \forall j \in \{0, 1, 2, \ldots, r\}.
\]

\[
\Rightarrow \text{MI}_1(Y, Z) = 2n + \sum_{i=1}^{2^r} (2^{n-r} \cdot t_i) \cdot \log \frac{(2^{n-r} \cdot t_i)}{2^{2(n-r)} \cdot (2^{r} - 1)} + (2^{n-r} \cdot v_i) \cdot \log \frac{2^{n-r} \cdot v_i}{2^{2(n-r)}} \\
= 2r + \sum_{i=1}^{2^r} (2^{n-r} \cdot t_i) \cdot \log \frac{(2^{n-r} \cdot t_i)}{2^{2(n-r)} - 1} + (2^{n-r} \cdot v_i) \cdot \log \frac{(2^{n-r} \cdot v_i) \leq 1 - H(p)}{2^{2(n-r)}}.
\]

The last inequality represents the result proved for Boolean functions from the classes 1 and 2, with \(n = r\). Equality is obtained for \(r = 1\), that is for the Boolean function termed the dictatorship function.

If \(r = 1 \Rightarrow N_1 = 2^{n-1}, N_0 = 2^{n-1}, v = \left[\frac{1 - p}{2^n}, \frac{p}{2^n}\right]\) and \(t = \left[\frac{p}{2^n}, \frac{1 - p}{2^n}\right]\)

\[
\Rightarrow \text{MI}_1(Y, Z) = 2 + \sum_{i=1}^{2^n} (2^{n-1} \cdot t_i) \cdot \log (2^{n-1} \cdot t_i) + (2^{n-1} \cdot v_i) \cdot \log (2^{n-1} \cdot v_i) \\
= 2 + \frac{p}{2^n} \cdot \log \frac{p}{2^n} + \frac{1 - p}{2^n} \cdot \log \frac{1 - p}{2^n} + \frac{1 - p}{2^n} \cdot \log \frac{1 - p}{2^n} + \frac{p}{2^n} \cdot \log \frac{p}{2^n} = 2 - 1 - H(p) = 1 - H(p) \\
\Rightarrow \text{MI}_1(Y, Z) = 1 - H(p).
\]

We will now prove that the same result holds, for Boolean functions that have \(N_0 = 2^{n-r}\) elements equal to 0 in their output table and the rest are equal to 1, that is \(N_1 = 2^n - 2^{n-r} = 2^{n-r} \cdot (2^r - 1), \forall r \in \{1, 2, \ldots, n-1\}\). These Boolean functions satisfy an additional condition: the 0 values from the output table correspond to the input vectors \(X^{(i)} = [X_r \quad X_{r-1}^{(i)}] \in \{[X_r \ 0 \ 0 \ldots 0 \ 0], [X_r \ 0 \ 0 \ldots 0 \ 1], \ldots, [X_r \ 1 \ 1 \ldots 1]\}\), where \(X_r\) is fixed, \(\forall i \in \{1, 2, \ldots, 2^{n-r}\}\).

Let \(q_k = pYZ(Y^{(k)}, 0), p_k = pYZ(Y^{(k)}, 1) = pY(Y^{(k)}) - pYZ(Y^{(k)}, 0), \forall k \in \{1, 2, \ldots, 2^n\}\).
\[
\text{MI}_0(Y, Z) = \sum_y p_{YZ}(y, 0) \cdot \log \frac{p_{YZ}(y, 0)}{p_Y(y) \cdot p_Z(0)} + p_{YZ}(y, 1) \cdot \log \frac{p_{YZ}(y, 1)}{p_Y(y) \cdot p_Z(1)} \\
= \sum_y p_{YZ}(y, 0) \cdot \log \frac{p_{YZ}(y, 0)}{\left(\frac{1}{2}\right)^n \cdot 2^n} + p_{YZ}(y, 1) \cdot \log \frac{p_{YZ}(y, 1)}{\left(\frac{1}{2}\right)^n \cdot 2^n} \\
= 2n + \sum_{k=1}^{2^n} q_k \cdot \log \frac{q_k}{2^{n-r}} + p_k \cdot \log \frac{p_k}{2^{n-2n-r}}. 
\]

The derivations regarding \( q_k \) and \( p_k \) are identical to the case \( N_1 = 2^{n-r} \). As a result,

\[
\text{MI}_0(Y, Z) = \text{MI}_1(Y, Z) \leq 1 - H(p). 
\] (86)

IV. Conclusions

In this study, we proved the Courtade-Kumar conjecture, for several classes of Boolean functions, for all dimensions, \( \forall n \geq 2 \), and for all values of the error probability, \( \forall 0 \leq p \leq \frac{1}{2} \). We transformed the problem from information-theoretic terms into an algebraic expression, composed of sums of a convex function, evaluated at different points. We provided an algebraic proof using Karamata’s theorem as our main tool. Probability theory, transformations of random variables and vectors and algebraic manipulations are additional important elements. Our proof differs entirely from the other proofs from the literature that have made the most progress towards solving the Courtade-Kumar conjecture: [11], [12], [16]. We bring further improvement in the effort to solve this conjecture in its most general form. Our novelty lies in showing that, for several classes of Boolean functions, provided they satisfied certain conditions, yielded the same mutual information. This property, if found for the other classes of Boolean functions, can simplify the conjecture in the most general case.

We have tried to apply Karamata’s theorem to other classes of Boolean functions, in order to solve the conjecture in its most general form. However, we have been unsuccessful in both applying the theorem directly to the mutual information inequality and in finding a suitable algebraic transformation of the original inequality into an expression that can be proved with Karamata’s theorem.

APPENDIX A

Properties of the Binary Symmetric Channel

Using the properties of no memory and no feedback, we prove, by induction, the following equations describing the conditional and joint mass functions, for several inputs to the binary symmetric channel,

\[
p(y_{k+1}, y_k, \ldots, y_1 | x_{k+1}, x_k, \ldots, x_1) = \prod_{i=1}^{k+1} p(y_i | x_i), \forall k = 1, n - 1, \\
p(x_{k+1}, x_k, \ldots, x_1, y_{k+1}, y_k, \ldots, y_1) = \prod_{i=1}^{k+1} p(x_i, y_i), \forall k = 1, n - 1. 
\] (87)

**Proof:**

Step 1: Verify that the identity holds for \( k = 1 \) and \( k = 2 \).

\( k = 1 \)

\[
p(y_1 | x_1) = p(y_1 | x_1). 
\] (88)

This statement is true.

\( k = 2 \)

\[
p(y_2, y_1 | x_2, x_1) = \frac{p(x_1, x_2, y_1, y_2)}{p(x_1, x_2)} = \frac{p(y_2 | x_2, x_1, y_1) \cdot p(x_2, x_1, y_1)}{p(x_1, x_2)}. 
\] (89)
We use the property that the channel has no memory, that is,
\[ p(y_2|x_2, x_1, y_1) = p(y_2|x_2), \]
and the property that it has no feedback, that is,
\[ p(x_2|x_1, y_1) = p(x_2|x_1). \]

\[ \Rightarrow p(y_2, y_1|x_2, x_1) = \frac{p(y_2|x_2) \cdot p(x_2|x_1, y_1) \cdot p(x_1, y_1)}{p(x_1) p(x_2)} = \frac{p(y_2|x_2) \cdot p(x_2|x_1) \cdot p(y_1|x_1)}{p(x_2)}. \] (90)

From the fact that \( X_1, X_2 \) are i.i.d
\[ p(x_2|x_1) = \frac{p(x_1, x_2)}{p(x_1)} = \frac{p(x_1) \cdot p(x_2)}{p(x_1)} = p(x_2). \] (91)

\[ \Rightarrow p(y_2, y_1|x_2, x_1) = p(y_2|x_2) \cdot p(y_1|x_1). \] (92)

Step 2: \( \forall 1 \leq k \leq n - 1 \), assume that the equation
\[ p(y_k, y_{k-1}, \ldots, y_1|x_k, x_{k-1}, \ldots, x_1) = \prod_{i=1}^{k} p(y_i|x_i) \] (93)
holds and prove that this implies that the equation
\[ p(y_{k+1}, y_k, \ldots, y_1|x_{k+1}, x_k, \ldots, x_1) = \prod_{i=1}^{k+1} p(y_i|x_i) \] (94)
holds.

\[ p(y_{k+1}, y_k, \ldots, y_1|x_{k+1}, x_k, \ldots, x_1) = \frac{p(y_{k+1}, y_k, \ldots, y_1, x_{k+1}, x_k, \ldots, x_1)}{p(x_{k+1}, x_k, \ldots, x_1)} \]
\[ = \frac{p(y_{k+1}|x_{k+1}, x_k, \ldots, x_1, y_k, \ldots, y_1) \cdot p(x_{k+1}, x_k, \ldots, x_1, y_k, \ldots, y_1)}{p(x_{k+1}, x_k, \ldots, x_1)}. \] (95)

We use the property that the channel has no memory, that is
\[ p(y_{k+1}|x_{k+1}, x_k, \ldots, x_1, y_k, \ldots, y_1) = p(y_{k+1}|x_{k+1}) \] (96)
and the property that it has no feedback, that is
\[ p(x_{k+1}|x_k, \ldots, x_1, y_k, \ldots, y_1) = p(x_{k+1}|x_k, \ldots, x_1). \] (97)

\[ \Rightarrow p(y_{k+1}, y_k, \ldots, y_1|x_{k+1}, x_k, \ldots, x_1) = \frac{p(y_{k+1}|x_{k+1}) \cdot p(x_{k+1}, x_k, \ldots, x_1, y_k, \ldots, y_1)}{p(x_{k+1}) \cdot p(x_k, \ldots, x_1)} \]
\[ = \frac{p(y_{k+1}|x_{k+1}) \cdot p(x_{k+1}|x_k, \ldots, x_1, y_k, \ldots, y_1) \cdot p(x_k, \ldots, x_1, y_k, \ldots, y_1)}{p(x_{k+1}) \cdot p(x_k, \ldots, x_1)} \]
\[ = \frac{p(y_{k+1}|x_{k+1}) \cdot p(x_{k+1}|x_k, \ldots, x_1) \cdot p(y_k, \ldots, y_1|x_k, \ldots, x_1)}{p(x_{k+1})}. \] (98)

From the fact that \( X_1, X_2, \ldots, X_{k+1} \) are i.i.d
\[ p(x_{k+1}|x_k, \ldots, x_1) = \frac{p(x_{k+1}, x_k, \ldots, x_1)}{p(x_k, \ldots, x_1)} = \frac{p(x_{k+1}) \cdot \prod_{i=1}^{k} p(x_i)}{\prod_{i=1}^{k} p(x_i)} = p(x_{k+1}). \] (99)

\[ \Rightarrow p(y_{k+1}, y_k, \ldots, y_1|x_{k+1}, x_k, \ldots, x_1) = p(y_{k+1}|x_{k+1}) \cdot p(y_k, \ldots, y_1|x_k, \ldots, x_1). \] (100)

Then, from our assumption that
\[ p(y_k, \ldots, y_1|x_k, \ldots, x_1) = \prod_{i=1}^{k} p(y_i|x_i), \]
\[ \Rightarrow p(y_{k+1}, y_k, \ldots, y_1|x_{k+1}, x_k, \ldots, x_1) = \prod_{i=1}^{k+1} p(y_i|x_i). \] (101)
\[ p(x_{k+1}, x_k, \ldots, x_1, y_{k+1}, y_k, \ldots, y_1) = \prod_{i=1}^{k+1} p(y_i | x_i) \prod_{j=1}^{k+1} p(x_j) \]

\[ = \prod_{i=1}^{k+1} p(y_i | x_i) \cdot p(x_i) \]

\[ = \prod_{i=1}^{k+1} p(x_i, y_i). \]