Cosmological Power Spectrum in Noncommutative Spacetime

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Abstract

We compute the cosmological power spectrum in a noncommutative space-time using a canonical approach. The power spectrum is computed at leading order in the noncommutative parameter $\theta^{\mu\nu}$ and in the spatial separation $(x' - x)^i$. We obtain an anisotropic dipolar imaginary primordial power spectrum of the form which was recently anticipated in the literature on the basis of the observed dipole modulation in CMBR data.

1 Introduction

A remarkable prediction of quantum gravity is that space-time may be noncommutative. The basic idea is that in order to probe short distances we require higher energies. However at sufficiently high energy we shall necessarily form black holes and hence lose precision about space-time coordinates. This idea imposes some uncertainty relationships among different coordinates which can be derived by proposing that these coordinates do not commute \cite{1–6}. It has been argued that this noncommutativity of coordinates might have interesting implications for cosmology \cite{7–16}. In particular the power spectrum generated during inflation could be modified. The modified power spectrum can lead \cite{17–19} to the observed hemispherical anisotropy \cite{20–28} and may also lead to signatures of non-Gaussianity \cite{15,16}.

The power spectrum obtained in \cite{15,16} has been related to the hemispherical anisotropy in \cite{17,19}. The hemispherical anisotropy is parametrized in terms of the phenomenological dipole modulation model \cite{29,32}. These papers \cite{17,19} argue that the power spectrum obtained in \cite{15,16} is not acceptable since it produces imaginary correlations among temperature spherical harmonic coefficients, $a_{lm}$’s, while they should be real. In their calculation the authors assume that the transfer function which relates the power spectrum in the early Universe is approximately the same as that assumed in commutative space times. This is purely an assumption and requires a detailed investigation. If we assume that this somewhat primitive calculation \cite{17,19} carries a grain of truth, the consequences are mind-boggling. It literally implies that the shortest distance,
perhaps Planck scale physics, associated with the noncommutativity of space-time, may currently be probed at the largest distance scales in the Universe. Furthermore anisotropies or inhomogeneities at very early times may be observable today as anisotropies on the largest distance scales \([35, 36]\) and might be responsible for some of the observed anisotropies in the Universe \([37–43]\) besides the hemispherical anisotropy \([24, 25]\).

The power spectrum in \([15, 16]\) is obtained by assuming that all products in noncommutative space-time must be taken to be star products. This is a reasonable assumption since a star product implements the basic commutation relation among different coordinates, given by \([1, 3–6]\),

\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} . \tag{1}
\]

Here the parameter, \(\theta_{\mu\nu}\), is antisymmetric in nature and the coordinate functions, \(\hat{x}_\mu(x)\), depend on the choice of coordinate system. Different choices will lead to different models of noncommutative space-time. The authors \([15, 16]\) consider a scalar field theory in a background expanding Universe. The coordinates \(\hat{x}_\mu\) are taken to be the comoving coordinates. They compute the two point correlations of the scalar field, \(\phi\), by assuming that their product can be taken to be the star product. Although it is well known that we can write the action functional in noncommutative space-times by converting all operator products into star products, yet it does not follow that while computing correlation functions we should only consider star products of different fields. We are simply interested in determining the correlation between two fields at different positions. Hence it appears reasonable to simply take their ordinary product while computing the correlation.

In the current paper we adopt a canonical approach to compute the power spectrum. We directly compute the expectation value of the ordinary product of two field operators with space-like separation. Hence we compute the correlation function,

\[
\Delta(x, x') = \langle 0 | \phi(x, t) \phi(x', t) | 0 \rangle . \tag{2}
\]

It is convenient to express this in terms of the symmetric and antisymmetric part as follows

\[
\Delta(x, x') = \frac{1}{2} (\Delta_+(x, x') + \Delta_-(x, x')) \tag{3}
\]

where,

\[
\Delta_+(x, x') = \langle 0 | \{\phi(x, t), \phi(x', t)\} | 0 \rangle
\]

\[
\Delta_-(x, x') = \langle 0 | [\phi(x, t), \phi(x', t)] | 0 \rangle . \tag{4}
\]

In a commutative space-time two field operators at space-like separation commute with one another and hence the second term, \(\Delta_-(x, x')\), vanishes. In \([15, 16]\), it has been argued that the star product of two scalar fields does not commute even for space-like separations. In the present paper we show that this is true even for ordinary product.

## 2 Commutation Relations of the Scalar Field

Let \(\phi(x, t)\) be a scalar field. In a noncommutative space-time we expect that its equal time commutation relations with other operators will acquire additional
corrections in comparison to the standard field theory. Here we assume that the additional corrections involve a power series in the spatial separations \((\mathbf{x}' - \mathbf{x})\).

Hence, in the short distance limit, the dominant correction is proportional to \((\mathbf{x}' - \mathbf{x})^3\). The field at position \((t, \mathbf{x}')\) can be obtained by the standard translation operator, \(P_i\). Hence we obtain,

\[
\phi(\mathbf{x}', t) = \exp\left[i(x' - x)^i P_i^0\right] \phi(\mathbf{x}, t) \exp\left[-i(x' - x)^i P_i^0\right],
\]

where the momentum operator \(P_i^0\) can be obtained by using Noether’s theorem and is given as

\[
P_i^0 = -\int \pi(\mathbf{x}, t) \partial_j \phi(\mathbf{x}, t) d^3 \mathbf{x}.
\]

In noncommutative space-time the ordinary product between the fields in the action is replaced by the \(\ast\)-product. Since the momentum operator is derived directly by variation of the action, we can replace products of fields by star products in this operator also. Hence the momentum can be written as

\[
P_j = -\int \pi(\mathbf{x}, t) \partial_j \phi(\mathbf{x}, t) d^3 \mathbf{x}.
\]

Here the arrow pointing left (right) means that the derivative acts on the fields appearing on the left (right) of the operator. Expanding in the leading order of the noncommutative parameter \(\theta^{\mu\nu}\), we can write the modified momentum as

\[
\tilde{P}_j = P_j^0 + i \delta P_j
\]

where the unperturbed part of the momentum is

\[
P_j^0 = -\int \pi(\mathbf{x}, t) \partial_j \phi(\mathbf{x}, t) d^3 \mathbf{x}
\]

and the perturbed part is

\[
\delta P_j = -\frac{1}{2} \int \partial_\mu \pi^{\mu\nu} \partial_\nu \partial_j \phi(\mathbf{x}, t) d^3 \mathbf{x}.
\]

Substituting the modified momentum \(\tilde{P}_j\), we can write,

\[
\phi(\mathbf{x}', t) = \exp\left[i(x' - x)^i (P_i^0 + i \delta P_i)\right] \phi(\mathbf{x}, t) \exp\left[-i(x' - x)^i (P_i^0 + i \delta P_i)\right],
\]

to a leading order in the parameter \(\theta^{\mu\nu}\). Using the Zassenhaus formula the field \(\phi(\mathbf{x}', t)\) can be expanded as,

\[
\phi(\mathbf{x}', t) \simeq \exp[-(x' - x)^i \delta P_i] \phi_0(\mathbf{x}', t) \exp[(x' - x)^i \delta P_i],
\]

where \(\phi_0(\mathbf{x}', t)\) is the field ignoring the corrections due to noncommutativity of space-time and is given by,

\[
\phi_0(\mathbf{x}', t) = \exp\left[i(x' - x)^i P_i^0\right] \phi(\mathbf{x}, t) \exp\left[-i(x' - x)^i P_i^0\right].
\]

The field \(\phi_0(\mathbf{x}', t)\) commutes with \(\phi(\mathbf{x}, t)\), because \(\phi(\mathbf{x}, t) = \phi_0(\mathbf{x}, t)\), that is, the field at the position \((\mathbf{x}, t)\) does not get corrections due to non-commutativity of space-time. In the linear order of the \((x' - x)^i\), the scalar field \(\phi(\mathbf{x}', t)\) is

\[
\phi(\mathbf{x}', t) = \phi_0(\mathbf{x}', t) + (x' - x)^i [\phi_0(\mathbf{x}', t), \delta P_i].
\]

We next compute the commutator of the fields at spatial separations for flat space and expanding Universe.
2.1 Flat space

The scalar field for the flat space is

$$\phi(x, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left( a_k e^{ik \cdot x} + a_k^* e^{-ik \cdot x} \right)$$

(15)

where the four vectors are

$$k_\mu = (-\omega_k, k), \quad x^\mu = (t, x)$$

(16)

and similarly the conjugate momentum \(\pi(x, t)\) is

$$\pi(x, t) = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} \left( a_k e^{ik \cdot x} - a_k^* e^{-ik \cdot x} \right).$$

(17)

Here we consider only the terms proportional to \(\lambda_l = \theta^0\). A direct calculation yields,

$$(x - x')^j [\phi(x, t), [\phi_0(x', t), \delta P_j]] = -(x' - x)^j \lambda_l \int d^3p p_l \cos (p \cdot (x - x'))$$

(18)

This leads to a non-zero value of \(\Delta_-(x, x')\), given by,

$$\Delta_-(x, x') = -(x' - x)^j \lambda_l \int d^3p p_l \cos (p \cdot (x - x'))$$

(19)

It is remarkable that we obtain a result in real space precisely of the type anticipated in [17, 19]. In Fourier space it will lead to an imaginary dipolar power spectrum [17, 19], which may be related to the hemispherical anisotropy.

We next make some comments about this formula. We have earlier made an expansion in powers of \((x' - x)^j\). Hence in order to be consistent, we should set \((x - x')\) inside the integrand equal to zero. However this produces a divergent result. We may regulate the final integral by imposing an ultraviolet cutoff in order to obtain a finite result. A regulator of the order of Planck mass is reasonable since it is not possible to probe energy scale above this value.

Alternatively we may determine the form of the additional terms that are generated. The main point is that in Eq. 18 we obtain a dipolar power which in position space is real. It appears plausible that all the remaining terms linear in \(\lambda_l\) are of the same form. Let us consider the next term which arises from the commutator \([iP_{0i}, \delta P_j]\) in the Zassenhaus formula. We now need the commutator,

$$(x' - x)^l (x' - x)^j [\phi(x, t), [\phi_0(x', t), [iP_{0i}, \delta P_j]]].$$

In comparison to the earlier commutator, this involves one additional commutator of fields which will yield an additional factor of the imaginary number \(i\). The operator \(iP_{0i}\) will not yield an additional power of \(i\). We also see that overall the terms \(iP_{0i}\) and \(\delta P_j\) will lead to three factors of spatial momentum, \(p_i, p_j\) and \(p_l\) where \(l\) is the index corresponding to \(\lambda_l\), as in Eq. 18. Now from the exponential factors in the expansion of the field operators, Eq. 15 only the sine term will contribute since the coefficient of this term contains an odd number of momentum factors. Hence this will contribute another factor of \(i\).
which will cancel with the $i$ due to the additional commutator. This would finally yield a real result of the form similar to the right hand side of Eq. 19 with the integrand replaced by $(x' - x)^3 p_j p_k \sin(p \cdot (x - x'))$. Similarly we expect that contributions from additional terms in the Zassenhaus formula may lead to an integrand in Eq. 19 of the form $p_j p_k f(p \cdot (x - x'))$ where $f(x)$ is an even function of $x$. However a formal proof is required which is not available so far.

Another issue is that we have used the standard expression for the commutator of $\phi(x, t)$ with $\pi(x', t)$. This commutator may itself involve corrections due to noncommutativity of space-time which must be included. However these corrections are of order $(x' - x)$. Due to the additional power of $(x' - x)^2$ in the exponent in Eq. 14 we expect them to yield a contribution of order $(x' - x)^3$ to $\Delta^2$. We are justified in ignoring these corrections as long as we retain terms only linear in $(x' - x)$. In any case a computation of such corrections would be useful.

2.2 Expanding Universe

We next consider the case of an expanding, de Sitter Universe. In this case the form of the scalar field and the corresponding conjugate momenta becomes:

$$\phi(x, t) = \int \frac{d^3k}{(2\pi)^3} \left( a_k e^{ik \cdot x} \zeta_k(t) + a_k^\dagger e^{-ik \cdot x} \zeta_k^*(t) \right)$$

(20)

$$\Pi(x, t) = \int \frac{d^3k}{(2\pi)^3} \left( a_k e^{ik \cdot x} \dot{\zeta}_k(t) + a_k^\dagger e^{-ik \cdot x} \dot{\zeta}_k^*(t) \right)$$

(21)

where $\zeta_k = \frac{ik}{a_\eta}$ and the mode function $a_k = \frac{e^{-ik \cdot x}}{\sqrt{2k}} \left( 1 - \frac{1}{k^2} \right)$. Now the commutation between the fields yields

$$(x' - x)^3 [\phi(x, t), [\phi_0(x', t), \delta P_j]] = \frac{1}{2} (x' - x)^3 \int \frac{d^3p}{(2\pi)^3} p_j p_k \lambda^3 (z + z^*)$$

(22)

where

$$z = \exp(i p \cdot (x - x')) \zeta_p \left[ \zeta_p^\dagger (\zeta_p^\dagger \zeta_p - \zeta_p \zeta_p^\dagger) + 2 \zeta_p^\dagger (\zeta_p^\dagger \zeta_p - \zeta_p \zeta_p^\dagger) \right].$$

(23)

We are interested in the dominant term in the limit $\eta \to 0$, corresponding to the end of inflation. In this limit, we obtain,

$$(x' - x)^3 [\phi(x, t), [\phi_0(x', t), \delta P_j]] = (x' - x)^3 \lambda^3 \int \frac{d^3p}{(2\pi)^3} p_j p_k \cos(p \cdot (x - x')) (-H^0 \eta^0),$$

(24)

where $H$ is the Hubble’s constant which is constant during inflation and $-1/H \sim a_\eta$. Hence we obtain a non-zero result. In this case also the integral shows an ultraviolet divergence in the limit, $|x - x'| \to 0$, which can be remedied by choosing appropriate ultraviolet cutoff. The comments made in the case of flat space-time after Eq. 19 apply in this case also. With these qualifications we do not set the term $(x - x')$ to be zero inside the integrand. Since there are additional corrections to this, which, as in the case of flat space-time, may be
of the same form, we may treat the result in Eq. 24 as a model. The resulting
power spectrum can be written as,
\[ \Delta_n(x, x') = (x' - x)^3 \lambda^3 \int \frac{d^3p}{(2\pi)^3} P_j p_i \cos(p \cdot (x - x')) (-H^6 \eta^6). \] (25)

We next determine the power spectrum in Fourier space. The power spectrum in Eq. 25 may be compared with the anisotropic power spectrum proposed in [19],
\[ F(R) = f_1(R) + \lambda \cdot R f_2(R), \] (26)
where \( \lambda \) represents the preferred direction, \( R = (x - x') \) and \( f_1 \) and \( f_2 \) depend only on \( |R| \). The second term in this formula is clearly of the form obtained in Eq. 25. In Fourier space the power spectrum is given by, (here \( K = (k + k')/2 \))
\[ \langle \delta(k)\delta^*(k') \rangle = \delta^3(k - k') \int \frac{d^3R}{(2\pi)^3} e^{iK \cdot R} F(R). \] (27)

We focus on the anisotropic part of the power generated by the underlying noncommutativity of space-time. Using the power spectrum given in Eq. 25, we obtain,
\[ \langle \delta(k)\delta^*(k') \rangle_{\text{aniso}} = -\frac{H^6 \eta^6 \lambda^3}{2} \delta^3(k - k') \]
\[ \times \int \frac{d^3p}{(2\pi)^3} \frac{d^3R}{(2\pi)^3} p_j p_l R_j \left( e^{i(K+p) \cdot R} + e^{i(K-p) \cdot R} \right). \] (28)

This gives us two integrals which can be proved to be the same by appropriate transformations viz. \( A = K + p \) and \( B = K - p \). Considering any of the integrals and then performing an integration over the \( R \) variable, the integral in Eq. 28 transforms to:
\[ \int \frac{d^3A}{(2\pi)^3} (A_i - K_i) (A_j - K_j) (-i) \frac{\partial}{\partial A_j} \delta^3(A) = \frac{-4iK_j}{(2\pi)^3} \] (29)

Finally, using \( K = (k + k')/2 \) and due to the presence of \( \delta(k - k') \), we obtain,
\[ \langle \delta(k)\delta^*(k') \rangle_{\text{aniso}} = i \frac{4H^6 \eta^6 K_i \lambda^3}{(2\pi)^3} \delta^3(k - k') \equiv i \frac{4H^6 \eta^6 k_i \lambda^3}{(2\pi)^3} \delta^3(k - k'). \] (30)

Hence Eq. 27 can be written as
\[ \langle \delta(k)\delta^*(k') \rangle = \delta^3(k - k') [P_{\text{iso}}(k) + i(\lambda \cdot \hat{k}) P_{\text{aniso}}(k)], \] (31)
with
\[ P_{\text{aniso}}(k) = \frac{4kH^6 \eta^6}{(2\pi)^3}. \] (32)

Hence we obtain a power spectrum of the form which was anticipated in [17,19]. This should be regarded as a model since we have only computed one term and presented some justification that the remaining terms would have a similar form. However their precise functional form is unknown so far. Furthermore a formal proof is required that all the possible terms that contribute must have form similar to the term we obtained. We could have alternatively set \((x - x') \) in the integrand in Eq. 25 to zero and computed the resulting integral after imposing an ultraviolet cutoff. In this case the power spectrum in Fourier space would be singular since it is linear in \((x - x') \) in position space.
3 Conclusion

We have derived the direction dependent power spectrum in the noncommutative space-time framework. The power spectrum has been obtained to a leading order in the noncommutativity parameter, $\theta^0$. Furthermore we have kept only the leading term in the Zassenhaus expansion formula. Hence the result is valid only to leading order in the spatial separations, $(x' - x)^i$. However in our analysis we keep the full form of the power, keeping all powers of $(x - x')$ which arise in the computed term. We have argued that if we compute terms to all orders in $(x' - x)^i$ but leading order in $\theta^0$ the final result may still be similar although the precise functional form would change. In any case, we find that the power spectrum in Fourier space shows a dipolar imaginary structure exactly as anticipated in [17, 19]. Such a structure is required in order that it yield an acceptable CMB temperature anisotropy pattern. It has been argued that this might provide an explanation of the observed hemispherical anisotropy [20–28] or equivalently the dipole modulation [29–34] of CMB temperature. However due to the assumptions made in our calculations our final result can only be regarded as a suggestive model. More work is required in order to determine the power spectrum which is valid to all orders in $(x - x')$ given an underlying model of non-commutative field theory.

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