TRIVALENT EXPANDERS, (Δ − Y)-TRANSFORMATION, AND HYPERBOLIC SURFACES

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Abstract. We construct a new family of trivalent expanders tessellating hyperbolic surfaces with large isometry groups. These graphs are obtained from a family of Cayley graphs of nilpotent groups via (Δ − Y)-transformations. We compare this family with Platonic graphs and their associated hyperbolic surfaces and see that they are generally very different with only one hyperbolic surface in the intersection. Moreover, we study combinatorial, topological and spectral properties of our trivalent graphs and their associated hyperbolic surfaces.

1. Introduction and statement of results

In this article, we consider a family of surface tessellations with interesting spectral gap properties. More precisely, we construct a family of trivalent graphs $T_k$ ($k \geq 2$), which tessellate closed hyperbolic surfaces $S_k$ with large isometry groups, growing linear on genus. The faces of these tessellations are regular hyperbolic $2^\left\lfloor \log_2 k \right\rfloor + 2$-gons and interior angles $2\pi/3$. The graphs $T_k$ are (Δ − Y)-transformations of Cayley graphs of increasing 2-groups $G_k$, and they form a family of expanders (see Theorem 1.1 below for more details).

Using the Brooks-Burger transfer principle, the expansion properties of the trivalent graphs $T_k \subset S_k$ translate into a uniform lower bound on the smallest positive Laplace eigenvalue $\lambda_1(S_k)$ of the surfaces $S_k$ (see Corollary 1.3). Prominent explicit examples where such an interplay between groups, graphs and compact and non-compact hyperbolic surfaces has been utilized, are finite quotients of $\text{PSL}(2, \mathbb{Z})$ and co-compact arithmetic lattices in $\text{PSL}(2, \mathbb{R})$ (see, e.g., Buser [10], Brooks [6], and Lubotzky [16] and the references therein). While many finite quotients of these examples are simple, all our finite groups $G_k$ are nilpotent and very different in nature. We also show that, while our graph $T_2$ in the...
surface $S_2$ is dual to the Platonic graph $\Pi_8$ (as defined below), there is no direct relation between our family of graphs $T_k$ and these Platonic graphs $\Pi_N$ from $k \geq 3$ onwards. This fact is stated in Proposition 1.4.

Another way to associate hyperbolic surfaces to the trivalent graphs $T_k$ is to replace the vertices by hyperbolic regular $Y$-pieces and to glue their boundary cycles in accordance with the edges between the vertices of $T_k$, as explained in [9]. We denote the so obtained hyperbolic surfaces by $\hat{S}_k$. For these surfaces, Buser’s results [9] yield a uniform lower bound on $\lambda_1(\hat{S}_k)$ (see Corollary 1.6).

Let us first give a brief overview over the construction of $T_k$ and its spectral properties, and then go into further details. We start with a sequence of 2-groups $G_k$, following the construction in [18, Section 2]. Then we consider 6-valent Cayley graphs $X_k$ of these groups and apply $(\Delta - Y)$-transformations in all triangles of $X_k$, to finally obtain the trivalent graphs $T_k$. The $(\Delta - Y)$-transformations are standard operations to simplify electrical circuits, and were also used in [2] in connection with Colin de Verdière’s graph parameter. We have the following relations between the spectra of the adjacency operators of the graphs $X_k$ and $T_k$ which are proved in Section 3. Note that the spectra of the graphs $T_k$ are symmetric around the origin, since these graphs are bipartite.

**Theorem 1.1.** Let $k \geq 2$. Then every eigenvalue $\lambda \in [-3, 3]$ of $T_k$ gives rise to an eigenvalue $\mu = \lambda^2 - 3 \in [-3, 6]$ on $X_k$. In particular, there exists a positive constant $C < 6$ such that

(i) the graphs $X_k$ are 6-valent expanders with spectrum in $[-3, C] \cup \{6\}$,

(ii) the bipartite graphs $T_k$ are trivalent expanders with spectrum in $[-\sqrt{C + 3}, \sqrt{C + 3}] \cup \{\pm 3\}$.

The finite groups $G_k$ are constructed as follows. We start with the infinite group $\tilde{G}$ of seven generators and seven relations:

$$ (1) \quad \tilde{G} = \langle x_0, \ldots, x_6 \mid x_i x_{i+1} x_{i+3} \text{ for } i = 0, \ldots, 6 \rangle, $$

where the indices are taken modulo 7. As explained in [12, Thm 3.4], this group acts simply transitively on the vertices of a thick Euclidean building of type $\tilde{A}_2$. Let $S = \{x_0^{\pm 1}, x_1^{\pm 1}, x_3^{\pm 1}\}$. We consider the index two subgroup $G \leq \tilde{G}$, generated by $S$. (Note that $x_3 = x_1^{-1}x_0^{-1}$.) We use a representation of the group $G$ by infinite (finite band) upper triangular Toeplitz matrices with special periodicity properties. Let $G^k$ be the finite index normal subgroup of elements in $G$ whose matrices have vanishing first $k$ upper diagonals. The groups $G_k$ are then the
finite quotients $G/G^k$. The details of this construction are given in Section 2.1.

The finite width conjecture in [18] asks whether the groups $G_k$ have another purely abstract group theoretical description via the lower exponent-2 series

$$G = P_0(G) \geq P_1(G) \geq P_2(G) \geq \cdots,$$

with

$$P_k(G) = [P_{k-1}(G), G]P_{k-1}(G)^2$$

for $k \geq 1$: namely, do we have $G^k = P_k(G)$ for $k \geq 1$ (see [18, Conj. 1])? We denote the order of $G_k$ by $2^{N_k}$. We know for $k \geq 1$ that (see [18, Cor. 2.3])

$$N_k \geq 8\lfloor k/3 \rfloor + 3 \cdot (k \mod 3) - 1,$$

and the finite width conjecture would imply that (3) holds with equality. MAGMA computations confirm this conjecture for all indices up to $k = 100$. Henceforth, we use the same notation for the elements $x_0, x_1, x_3$ in $G$ and their images in the quotients $G_k$, for simplicity.

The graphs $T_k$ can be naturally embedded as tessellations into both closed hyperbolic surfaces $S_k$ and complete non-compact finite area hyperbolic surfaces $S_k^\infty$. The edges of the tessellation are geodesic arcs and the vertices are their end points. The details of these embeddings are presented in Section 2.2. The following proposition describes the combinatorial properties of the tessellations $T_k \subset S_k$. The proof is given in Section 2.5.

**Proposition 1.2.** Let $k \geq 2$. Then the generators $x_0, x_1, x_3$ of $G_k$ have all the same order $2^{n_k}$ with

$$n_k = \lfloor \log_2 k \rfloor + 1.$$

Let $|G_k| = 2^{N_k}$ (with $N_k$ estimated from below in (3)) and $V_k, E_k$ and $F_k$ denote the sets of vertices, edges, and faces of the tessellation $T_k \subset S_k$, respectively. Then the isometry group of $S_k$ has order $\geq 2^{N_k}$, we have

$$|V_k| = 2^{N_k+1}, \quad |E_k| = 3 \cdot 2^{N_k} \quad \text{and} \quad |F_k| = 3 \cdot 2^{N_k-n_k},$$

and all faces of $T_k \subset S_k$ are regular hyperbolic $2^{n_k+1}$-gons with interior angles $2\pi/3$. Moreover, the genus of $S_k$ is given by

$$g(S_k) = 1 + 2^{N_k-n_k-1}(2^{n_k} - 3).$$
Since the hyperbolic surfaces $S_k$ are closed, the spectrum of the Laplace operator on $S_k$ (with multiplicities) is a non-decreasing sequence of real numbers $\lambda_k(S_k)$, tending to infinity. In this enumeration, we have $\lambda_0(S_k) = 0$ and the corresponding eigenfunction is constant. The following result is a consequence of our Theorem 1.1 above and the Brooks-Burger transfer principle. The details of the proof are given in Section 5.1. Note that the proof provides an explicit estimate of the constant $C_1$ in Corollary 1.3 in terms of the constant $C$ given in Theorem 1.1.

**Corollary 1.3.** There is a positive constant $C_1 > 0$ such that we have for the compact hyperbolic surfaces $S_k$ ($k \geq 2$),

$$\lambda_1(S_k) \geq C_1.$$ 

In Section 5.2, we explain that the surfaces $S_k$ are the conformal compactifications of the non-compact surfaces $S_k^\infty$. Moreover, the Cheeger constants of the graphs $T_k$ have a uniform lower positive bound since they are an expander family. These facts imply via the Theorems 3.3 and 4.2 of [7] that the Cheeger constants of both families $S_k$ and $S_k^\infty$ have also uniform lower positive bounds.

It is instructive to compare our tessellations $T_k \subset S_k$ to the well studied tessellations of hyperbolic surfaces by Platonic graphs $\Pi_N$. It turns out that both families agree in one tessellation (up to duality) but are, otherwise, very different. Let us first define the graphs $\Pi_N$. Let $N$ be a positive integer $\geq 2$. The vertices of $\Pi_N$ are equivalence classes $[\lambda, \mu] = \{\pm (\lambda, \mu)\}$ with

$$\{(\lambda, \mu) \in \mathbb{Z}_N \times \mathbb{Z}_N \mid \gcd(\lambda, \mu, N) = 1\}.$$ 

Two vertices $[\lambda, \mu]$ and $[\nu, \omega]$ are connected by an edge if and only if

$$\det \begin{pmatrix} \lambda & \nu \\ \mu & \omega \end{pmatrix} = \lambda \omega - \mu \nu = \pm 1.$$ 

Note that every vertex of $\Pi_N$ has degree $N$. These graphs can also be viewed as triangulations of finite area surfaces $\mathbb{H}^2/\Gamma(N)$ by ideal hyperbolic triangles, where $\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ denotes the upper half plane with hyperbolic metric $dx^2 + dy^2/y^2$ and

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}) \mid \gamma \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

is the principal congruence subgroup of the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$. The vertices of this triangulation are the cusps of $\mathcal{S}^\infty(\Pi_N) \equiv \mathbb{H}^2/\Gamma(N)$.

As mentioned above, $\Pi_8$ is isomorphic to the dual of $T_2$ in $S_2$. Since the valence of the dual graph $T_k^*$ is a power of 2, any isomorphism of
\( T^*_k \) with a Platonic graph \( \Pi_N \) would imply \( N = 2^{n_k + 1} \) with \( n_k \) given in (4). However, this leads to a contradiction for all \( k \geq 3 \). The next proposition summarizes these results. The proof is given in Section 4.

**Proposition 1.4.** The Platonic graph \( \Pi_8 \) is isomorphic to the dual of \( T_2 \) in the unique compact genus 5 hyperbolic surface \( S_2 \) with maximal automorphism group of order 192. For \( k \geq 3 \), there is no graph isomorphism between \( T^*_k \) and \( \Pi_N \), for any \( N \geq 2 \).

It is easily checked via Euler’s polyhedral formula that any triangulation \( X \) of a compact oriented surface \( S \) satisfies

\[
|E(X)| = 3(|V(X)| - 2) + 6g(S),
\]
i.e., the number of edges |\( E(X) \)| of every triangulation \( X \) with at least two vertices is \( \geq 6g(S) \). Therefore, the ratio

\[
\frac{6g(S)}{|E(X)|} \leq 1
\]
measures the non-flatness of such a triangulation, i.e., how efficiently the edges of \( X \) are chosen to generate a surface of high genus. Note that, for every \( k \geq 2 \), the dual graph \( T^*_k \) can be viewed as a triangulation of \( S_k \) and that the number of edges of \( T_k \) and \( T^*_k \) coincide. Then we have the following asymptotic result, proved in Section 2.5.

**Proposition 1.5.** We have

\[
\lim_{k \to \infty} \frac{6g(S_k)}{|E(T^*_k)|} = 1,
\]
where \( E(T^*_k) \) denotes the set of edges of \( T^*_k \).

The other above-mentioned family \( \hat{S}_k \) of surfaces associated to the graphs \( T_k \) can be viewed as tubes around \( T_k \) with specific hyperbolic metrics. The surfaces \( \hat{S}_k \) form an infinite family of coverings. It follows from [9] that their smallest positive Laplace eigenvalue has also a uniform positive lower bound.

**Corollary 1.6.** The compact hyperbolic surfaces \( \hat{S}_k \) (\( k \geq 2 \)) have genus

\[
1 + |V_k|/2 \text{ and isometry groups of order } \geq |V_k|/2.
\]
They form a tower of coverings

\[
\cdots \to \hat{S}_{k+1} \to \hat{S}_k \to \hat{S}_{k-1} \to \cdots,
\]
where all the covering indices are powers of 2. There is a positive constant \( C_2 > 0 \) such that we have, for all \( k \),

\[
\lambda_1(\hat{S}_k) \geq C_2.
\]
Corollary 1.6 is proved in Section 5.3. There is a well-known classical result by Randol [19] which is, in some sense, complementary to this corollary, namely, there exist finite coverings $\tilde{S}$ of every closed hyperbolic surface $S$ with arbitrarily small first positive Laplace eigenvalue.

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2. Combinatorial properties of the tessellations $T_k \subset S_k$

Let $\tilde{G}$ be the group defined in (1). It was shown in [18, Section 2] that the subgroup $G$, generated by $x_0, x_1$, is an index two subgroup of $\tilde{G}$. (Note that our group $\tilde{G}$ is denoted in [18] by $\Gamma$, which is reserved for $\text{PSL}(2, \mathbb{Z})$ in this paper.) $G$ is explicitly given by $G = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle$ with

\begin{align*}
    r_1(x_0, x_1) &= (x_1x_0)^3x_1^{-3}x_0^{-3}, \\
    r_2(x_0, x_1) &= x_1x_0^{-1}x_1^{-1}x_0^{-3}x_1^{-2}x_0^{-1}x_1x_0x_1, \\
    r_3(x_0, x_1) &= x_3^{-1}x_1x_0x_1x_2^{-1}x_0x_1x_0.
\end{align*}

2.1. A faithful matrix representation of $G$. Let us first recall the faithful representation of $G$ by infinite upper triangular Toeplitz matrices, given in [18] and based on representations introduced in [12]. In fact, every $x \in G$ has a representation of the form

$$
    x = \begin{pmatrix}
        1 & a_{11} & a_{21} & \cdots & a_{k1} & 0 & 0 & \cdots & \cdots \\
        0 & 1 & a_{12} & a_{22} & \cdots & a_{k2} & 0 & \cdots & \cdots \\
        0 & 0 & 1 & a_{13} & a_{23} & \cdots & a_{k3} & 0 & \cdots \\
        \vdots & \vdots & 0 & 1 & a_{11} & a_{21} & \cdots & a_{k1} & \cdots \\
        \vdots & \vdots & \vdots & \vdots & 1 & a_{12} & a_{22} & \cdots & \cdots \\
        \vdots & \vdots & \vdots & \vdots & \vdots & 1 & a_{13} & a_{23} & \cdots \\
        \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
    \end{pmatrix},
$$

where each element $a_{ij}$ is in the set $M(3, \mathbb{F}_2)$ of $(3 \times 3)$-matrices with entries in $\mathbb{F}_2$, and 0 and 1 stand for the zero and identity matrix in $M(3, \mathbb{F}_2)$. Note the periodic pattern in the upper diagonals of the matrix, i.e., the $j$-th upper diagonal is uniquely determined by the
first three entries \( a_j = (a_{j1}, a_{j2}, a_{j3}) \), which can be understood as a \((3 \times 9)\)-matrix with values in \( \mathbb{F}_2 \). We use the short-hand notation \( M_0(a_1, a_2, \ldots, a_k) \) for the matrix in \( (8) \). If the first \( l \) upper diagonals in \( (8) \) vanish, we also write \( M_l(a_{l+1}, \ldots, a_k) \). Let \( G^k \) be the subgroup of all elements \( x \in G \) with vanishing first \( k \) upper diagonals in their matrix representation. It follows from the structure of these matrices that \( G^k \) is normal and that the quotient group \( G_k = G/G^k \) is a 2-group, i.e., nilpotent.

Recall that we use the same notation for the generators \( x_0, x_1, x_3 \) of \( G \) and their images in the quotient \( G_k \). We will see later that the faces of the tessellation \( T_k \subset S_k \) are determined by the orders of these generators in \( G_k \). We will now determine these orders. Let

\[
\begin{align*}
\alpha_0 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}, & \beta_0 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \\
\alpha_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, & \beta_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \\
\alpha_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}, & \beta_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

Then we have \( x_i = M_0(\alpha_i, \ldots) \) for \( i = 0, 1, 3 \), and we obtain the following fact about the leading diagonal of their 2-powers.

**Lemma 2.1.** We have for \( i \in \{0, 1, 3\} \) and \( l \geq 0 \):

\[
x_i^{2^l} = \begin{cases} M_{2^{l-1}}(\alpha_i, \ldots), & \text{if } l \text{ is even}, \\
M_{2^{l-1}}(\beta_i, \ldots), & \text{if } l \text{ is odd}. \end{cases}
\]

This implies, in particular, for \( k \in \mathbb{N} \) that the order of \( x_i \) in \( G_k \) is \( 2^{n_k} \) with \( n_k \) given in \( (4) \).

**Proof.** Since \( G_k \) is a 2-group, \( \text{ord}_{G_k}(x_i) \) has to be a power of 2. The formulas \( (11) \) follow via a straightforward calculation using Prop. 2.5 in [18]. This implies that \( \text{ord}_{G_k}(x_i) = 2^l \) if and only if \( 2^{l-1} \leq k < 2^l \), i.e., \( l = \lceil \log_2 k \rceil + 1 = n_k \).

\[\square\]

2.2. The surfaces \( S_k \) and \( S_k^\infty \) via covering theory. Let \( X_k \) be the Cayley graph \( \text{Cay}(G_k, S) \). We will now explain how to construct the closed hyperbolic surfaces \( S_k \): We start with an orbifold \( S_0 \) by gluing together two compact hyperbolic triangles \( T_1, T_2 \subset \mathbb{H}^2 \) with angles \( \pi/\text{ord}_{G_k}(x_0), \pi/\text{ord}_{G_k}(x_1) \) and \( \pi/\text{ord}_{G_k}(x_3) \) along their corresponding sides. Both triangles are equilateral since, by Lemma 2.1, we have
ord_{G_1}(x_0) = \text{ord}_{G_1}(x_1) = \text{ord}_{G_1}(x_3) = 2^{n_k}. It is useful to think of the two triangles \( T_1 \) and \( T_2 \) in \( S_0 \) to be coloured black and white, respectively. Let \( P_0, P_1, P_2 \in S_0 \) be the singular points (i.e., the identified vertices of the triangles \( T_1 \) and \( T_2 \) in \( S_0 \)) and \( Q \in S_0 \) be the center of the white triangle \( T_1 \). Note that \( S_0 \{ P_0, P_1, P_2 \} \) carries a hyperbolic metric induced by the triangles \( T_1, T_2 \). Choose a geometric basis \( \gamma_0, \gamma_1, \gamma_2 \) of the fundamental group \( \pi_1(S_0 \{ P_0, P_1, P_2 \}, Q) \) such that \( \gamma_i \) is a simple loop (starting and ending at \( Q \)) around the singular point \( P_1 \in S_0 \) and \( \gamma_0 \gamma_1 \gamma_2 = e \). Note that \( \pi_1(S_0 \{ P_0, P_1, P_2 \}, Q) \) is a free group in the generators \( \gamma_0, \gamma_1 \). The surjective homomorphism

\[
\Psi : \pi_1(S_0 \{ P_0, P_1, P_2 \}, Q) \to G_k,
\]
given by \( \Psi(\gamma_0) = x_0, \Psi(\gamma_1) = x_1 \) and \( \Psi(\gamma_2) = x_3 \), induces a branched covering \( \pi : S_k \to S_0 \), by Riemann’s existence theorem (see [21] Thms. 4.27 and 4.32] or [4, (17)]) with all ramification indices equals \( 2^{n_k} \) and, therefore, the closed surface \( S_k \) carries a hyperbolic metric such that the restriction

\[
\pi : S_k \{ P_0, P_1, P_2 \} \to S_0 \{ P_0, P_1, P_2 \}
\]
is a Riemannian covering. The surface \( S_k \) is a Belyi surface since it is a branched covering over \( S_0 \) ramified at the three points \( P_0, P_1, P_2 \). Moreover, \( S_k \) is tessellated by \( 2|G_k| = 2^{n_k+1} \) equilateral hyperbolic triangles, half of them black and the others white. Hurwitz’s formula yields

\[
g(S_k) = 1 + \frac{1 - \mu_k}{2}|G_k|,
\]
where

\[
(12) \quad \mu_k = \frac{1}{\text{ord}(x_0)} + \frac{1}{\text{ord}(x_1)} + \frac{1}{\text{ord}(x_3)} = \frac{3}{2^{n_k}}.
\]
Recall that the orders \( 2^{n_k} \) of \( x_i \) (\( i = 0, 1, 3 \)) were given in Lemma [21]. In the case \( k = 2 \) we have \( |G_2| = 32 \) and \( \text{ord}(x_0) = \text{ord}(x_1) = \text{ord}(x_3) = 4 \), which leads to

\[
g(S_2) = 1 + \frac{1}{8} \cdot 32 = 5.
\]

\( G_k \) acts simply transitive on the black triangles of \( S_k \). Let \( V = \pi^{-1}(Q) \) and \( V_{\text{white}}, V_{\text{black}} \subset V \) be the sets of centers of white and black triangles, respectively. Choose a reference point \( Q_0 \in V_{\text{white}} \), and identify the vertices of the Cayley graph \( X_k = \text{Cay}(G_k, S) \) with the points in \( V_{\text{white}} \) by \( G_k \ni h \mapsto hQ_0 \in V_{\text{white}} \). Then two adjacent vertices in \( X_k \) are the centers of two white triangles which share a black triangle as their common neighbour. The corresponding edge in \( X_k \) can then be represented by the minimal geodesic passing through these three
triangles and connecting these two vertices. Moreover, $G_k$ acts on the surface $S_k$ by isometries and we have $S_0 = S_k / G_k$, i.e., the isometry group of $S_k$ has order $\geq |G_k| = 2^{N_k}$.

We could also start the process by gluing together two ideal hyperbolic triangles $T^\infty_1$ and $T^\infty_2$, coloured black and white, along their corresponding edges. Each edge of $T^\infty_i$ ($i = 1, 2$) has a unique intersection point (tick mark) with the incircle of the triangle, and these tick-marks of corresponding edges of $T^\infty_1$ and $T^\infty_2$ are identified under the gluing. The resulting surface $S^\infty_0$ is topologically a 3-punctured sphere, carrying a complete hyperbolic metric of finite volume, and the same arguments as above then lead to an embedding of the graphs $X_k$ into complete non-compact finite area hyperbolic surfaces $S^\infty_k$, triangulated by ideal black and white triangles, where the vertices of $X_k$ correspond to the centers of the white ideal triangles in $S^\infty_k$, and we have $S^\infty_0 = S^\infty_k / G_k$.

2.3. From the Cayley graphs $X_k$ to the trivalent graphs $T_k$. For the transition from $X_k$ to $T_k$ we use the $(\Delta - Y)$-transformation. In this transformation, we add a new vertex $v$ for every combinatorial triangle of the original graph, remove the three edges of this triangle and replace them by three edges connecting $v$ with the vertices of this triangle. We apply this rule to our graph $X_k$ and obtain a graph $T_k$, which we can view again as an embedding in $S_k$ with the following properties: The vertex set of $T_k$ coincides with $V$, and there is an
edge (minimal geodesic segment) connecting every black/white vertex in \( V \) with the vertices in the three neighbouring white/black triangles. The best way to illustrate this transformation is to present it in the universal covering of the surface \( S_k \), i.e., the Poincaré unit disc \( D^2 \) (see Figure 1, the new vertices replacing every triangle are green). Note that \( T_k \) has twice as many vertices as \( X_k \), which shows that the isometry group of the above compact surface \( S_k \) has order \( \geq |G_k| = |V_k|/2 \), where \( V_k \) denotes the vertex set of \( T_k \). Moreover, \( T_k \) is the dual of the triangulation of \( S_k \) by the above-mentioned compact black and white triangles.

2.4. A direct construction of \( S_k \) and \( S_k^\infty \) from \( T_k \). There is another method to obtain the hyperbolic surfaces \( S_k \) and \( S_k^\infty \) using the construction in [7, Section 4] (see also [17, Chapter 1]). The start data are our trivalent graphs \( T_k \) with a suitable orientation.

**Definition 2.2.** An orientation \( O \) on a trivalent graph \( T \) is a choice, at each vertex \( v \) of \( T \), of a cyclic ordering of the three edges emanating from it.

Let us first introduce an orientation \( O_k \) on \( T_k \). We start with the Cayley graph \( X_k \) and orient its edges such that they only carry the Cayley graph labels \( x_0, x_1 \) and \( x_3 \), and not their inverses (see the Figure 1 on the left). Every triangle in \( X_k \) forms then an oriented cycle with consecutive labels \( x_0, x_1, x_3 \). This orientation induces an orientation on the new green vertex in \( T_k \) corresponding to this triangle, as illustrated by the oriented green circular arcs in the Figure 1 on the right. A blue vertex \( v \) of \( T_k \) stems from a vertex of \( X_k \), and we can give the labels 0, 1, 3 to the three edges in \( T_k \) emanating from \( v \), agreeing with the label of the edge in the corresponding triangle of \( X_k \) not adjacent to \( v \) (see again Figure 1 for illustration). The orientation of the three edges emanating from \( v \) in \( T_k \) (illustrated by an oriented blue circular arc) is then given by the cyclic ordering 0, 3, 1.

Now we follow the explanations in [17, Sections 1.1-1.4] closely. Let \( T \subset D^2 \) be an oriented compact equilateral hyperbolic triangle with interior angles \( \pi/2^n \). We refer to the mid-points of the sides of \( T \) as tick-marks. The orientation of \( T \subset D^2 \) induces a cyclic order on these tick-marks. Connect the center of \( T \) with the three tick-marks by geodesic arcs and assume that these arcs are coloured red. Then we paste a copy of \( T \subset D^2 \) on each vertex \( v \) of \( T_k \) such that its center agrees with \( v \), its tick-marks agree with the mid-points of the edges of \( T_k \) emanating from \( v \), and that the cyclic orders of these edges and of the tick-marks agree. Observe that, even though the mid-points of
adjacent sides of triangles meet up at mid-points of edges of $T_k$, we have not yet identified the sides of these triangles. This identification is made in such a way that the orientations of adjacent triangles match up. The resulting hyperbolic surface $S_k$ carries then a global orientation and the union of the red geodesic arcs from their mid-points to their tick-marks in the triangles provide an embedding of the graph $T_k$ into this surface such that the faces are regular $2n_k$-gons.

The complete non-compact finite area hyperbolic surfaces $S_k^\infty$ are obtained in the same way by starting instead with an oriented ideal hyperbolic triangle $\mathcal{T} \subset \mathbb{D}^2$ with tick-marks. As explained at the end of [17, Section 1.4], the cusps of $S_k^\infty$ are then in bijection with the left-hand-turn pathes in $(T_k, O_k)$. This alternative construction of $S_k$ and $S_k^\infty$ is later useful in Section 5.2.

2.5. Proofs of Propositions 1.2 and 1.5.

Proof of Proposition 1.2. The orders of $x_0, x_1, x_3 \in G_k$ were given in Lemma 2.1. It was explained in Section 2.2 that the isometry group of $S_k$ has order $\geq |G_k| = 2^{N_k}$, and in Section 2.3 that $|V_k| = 2|G_k| = 2^{N_k+1}$. Lemma 2.1 implies that the faces of the triangulation $T_k \subset S_k$ are regular $2^{n_k+1}$-gons, which yields $2|E_k| = 3|V_k| = 2^{n_k+1}|F_k|$, proving (5). The genus $g(S_k)$ can be derived either from Hurwitz’s formula (11) or from the Euler characteristic $\chi(S_k) = |V_k| - |E_k| + |F_k|$. This finishes the proof of Proposition 1.2. □

Remark 2.3. The number of cusps of the surface $S_k^\infty$ agrees with the number of faces of the tessellation $T_k \subset S_k$. For example, the genus five surface $S_5$ mentioned in Section 2.2 is tessellated into 24 octagons, and the surface $S_5^\infty$ has, therefore, 24 cusps.

Proof of Proposition 1.5. We conclude from (11), $|V_k| = 2|G_k|$, $|E(T_k^*)| = |E_k|$, and the trivalence of $T_k$ that

$$6g(S(T_k)) \frac{|E(T_k^*)|}{|E(T_k)|} = 6 \frac{1 + (1 - \mu_k)|V_k|/4}{3|V_k|/2}.$$ 

Note that $|V_k| = 2|G_k| = 2^{N_k+1} \to \infty$ because of (3), which implies that

$$\lim_{k \to \infty} \frac{6g(S_k)}{|E(T_k^*)|} = 1 - \lim_{k \to \infty} \mu_k.$$ 

Recall from (12) and (4) that $\mu_k = 3/2^{n_k} \to 0$ as $k \to \infty$, finishing the proof of (7). □
3. Spectral properties of the graphs $X_k$ and $T_k$

In this section, we establish the expander properties of $X_k$ and $T_k$ and relations between their eigenfunctions and eigenvalues, which proves Theorem 1.1. We also investigate Ramanujan properties of these families of graphs.

3.1. Precise relations between eigenfunctions and eigenvalues.

As before, let $\tilde{G}$ be the group defined in (1) and $G$ be the index two subgroup generated by $S$. Then both groups $\tilde{G}$ and $G$ have Kazhdan property (T) (see [18, Section 3]). Using [16, Prop. 3.3.1], we conclude that the Cayley graphs $X_k = \text{Cay}(G_k, S)$ are expanders.

The adjacency operator $A_X$, acting on functions on the vertices of a graph $X$, is defined as

$$A_X f(v) = \sum_{w \sim v} f(w),$$

where $w \sim v$ means that the vertices $v$ and $w$ are adjacent. It is easy to see that the eigenvalues of the adjacency operator of a finite $n$-regular graph lie in the interval $[-n, n]$.

Recall that the set $V(X_k)$ of vertices of our 6-valent graph $X_k$ is a subset of the vertex set $V_k$ of the trivalent graph $T_k$. We have the following relations between the eigenfunctions of the adjacency operators on $X_k$ and $T_k$.

**Theorem 3.1.**

(a) Every eigenfunction $F$ on $T_k$ to an eigenvalue $\lambda \in [-3, 3]$ gives rise to an eigenfunction $f$ to the eigenvalue $\mu = \lambda^2 - 3 \in [-3, 6]$ on $X_k$ (with $f(v) = F(v)$ for all $v \in V(X_k)$).

(b) Every eigenfunction $f$ on $X_k$ to an eigenvalue $\mu \in [-6, 6] - \{-3\}$ gives rise to two eigenfunctions $F_\pm$ to the eigenvalues $\pm \sqrt{\mu + 3}$ on $T_k$ with

$$F_\pm(v) = \begin{cases} f(v) & \text{if } v \in V(X_k), \\ \pm \sqrt{\mu + 3} \sum_{w \sim v} f(w) & \text{if } v \in V_k - V(X_k). \end{cases}$$

(c) An eigenfunction $f$ on $X_k$ to the eigenvalue $-3$ gives rise to an eigenfunction $F$ to the eigenvalue 0 of $T_k$ with

$$F(v) = \begin{cases} f(v) & \text{if } v \in V(X_k), \\ 0 & \text{if } v \in V_k - V(X_k). \end{cases}$$

if and only if we have, for all triangles $\Delta \subset V(X_k)$,

$$\sum_{v \in \Delta} f(v) = 0.$$
In the following proof, we use ∼ for adjacency in $T_k$ and ∼$_{X_k}$ for adjacency in $X_k$. Moreover, $d_{T_k}$ denotes the combinatorial distance function on the vertex set $V_k$ of $T_k$.

Proof. (a) Let $f$ and $F$ be two functions on $X_k$ and $T_k$, related by $f(v) = F(v)$ for all $v \in V(X_k)$. Then

$$A_{X_k}f(v) = \sum_{w \sim_{X_k} v} f(w) = \sum_{d_{T_k}(w,v)=2} F(w) = (A_{T_k})^2 F(v) - 3F(v),$$

which can also be written as $A_{X_k} = (A_{T_k})^2 - 3$. This implies immediately the connection between the eigenfunctions and eigenvalues.

(b) Let $A_{X_k}f = \mu f$ and $F_\pm$ be defined as in the theorem. Let $\lambda = \pm \sqrt{\mu + 3}$. Then we have for $v \in V(X_k)$:

$$A_{T_k}F_\pm(v) = \sum_{w \sim v} F_\pm(w) = \frac{1}{\lambda} \sum_{w \sim v} \sum_{x \sim w} F_\pm(x)$$

$$= \frac{1}{\lambda} \left( \sum_{w \sim_{X_k} v} f(w) + 3f(v) \right) = \frac{\mu + 3}{\lambda} f(v) = \lambda F_\pm(v),$$

and for $v \in V_k - V(X_k)$:

$$A_{T_k}F_\pm(v) = \sum_{w \sim v} F_\pm(w) = \lambda \left( \frac{1}{\lambda} \sum_{w \sim v} f(w) \right) = \lambda F_\pm(v).$$

Note that $1/\lambda$ is well defined since $\mu \neq -3$ and, therefore, $\lambda = \pm \sqrt{\mu + 3} \neq 0$.

(c) In the case of $\mu = -3$ we have $\lambda = 0$, and

$$A_{T_k}F(v) = \sum_{w \sim v} F(w) = 0$$

holds trivially for $v \in V(X_k)$. For all vertices $v \in V_k - V(X_k)$, the conditions

$$0 = A_{T_k}F(v) = \sum_{w \sim v} f(v)$$

translate into the condition that the summation of $f$ over the vertices of every triangle in $X_k$ must vanish. □

An immediate consequence of Theorem 3.1 is that the expander property of the family $X_k$ carries over to the graphs $T_k$ (with the spectral bounds given in Theorem 1.1). Moreover, the spectrum of $X_k$ cannot contain eigenvalues in the interval $[-6,-3)$, since this would lead to non-real eigenvalues of $T_k$. Therefore, these arguments also complete the proof of Theorem 1.1.
Remark 3.2. It would be interesting to find an explicit value for the constant $C > 0$ in Theorem 1.1. This would be possible if we were able to estimate the Kazdhan constant of the index two subgroup $G$ of $\tilde{G}$ (with respect to some choice of generators). While the Kazhdan constant of $\tilde{G}$ with respect to the standard set of seven generators was explicitly computed in [13], it seems to be a difficult and challenging question to obtain an explicit estimate for the Kazhdan constant of the subgroup $G$.

3.2. Ramanujan properties. Recall that a finite $n$-regular graph $X$ is Ramanujan if all non-trivial eigenvalues $\lambda \neq \pm n$ lie in the interval $[-2\sqrt{n-1}, 2\sqrt{n-1}]$. Since the 6-regular graphs $X_k$ are Cayley graphs of quotients of the group $G$ with property (T), not all of these graphs can be Ramanujan (see [16, Prop. 4.5.7]). MAGMA computations provide the following numerical results:

| graph | number of vertices | largest non-trivial eigenvalue |
|-------|--------------------|--------------------------------|
| $X_2$ | 32                 | 2.828427...                   |
| $X_3$ | 128                | 4.340172...                   |
| $X_4$ | 1024               | 4.475244...                   |
| $X_5$ | 8192               | 5.160252...                   |

This implies that only $X_2$ and $X_3$ are Ramanujan; their largest non-trivial eigenvalue needs to be $< 2\sqrt{5} = 4.472135...$, which is no longer true for $k = 4$. Moreover, since $X_{k+1}$ is a lift of $X_k$, the spectrum of $X_k$ is contained in the spectrum of $X_{k+1}$.

Next, we consider Ramanujan properties of the graphs $T_k$. Theorem 3.1 implies that $T_k$ is Ramanujan if and only if the largest non-trivial eigenvalue of $X_k$ is $\leq 5$. Therefore, the above numerical results imply that only $T_2$, $T_3$, $T_4$ are Ramanujan. Here are the numerical results for the largest non-trivial eigenvalues $\lambda_1(T_k)$ of the first $T_k$'s:

| graph | number of vertices | $\lambda_1(T_k)$ |
|-------|--------------------|-----------------|
| $T_2$ | 64                 | 2.414213...     |
| $T_3$ | 256                | 2.709275...     |
| $T_4$ | 2048               | 2.734089...     |
| $T_5$ | 16384              | 2.856615...     |

Note that the spectrum of $T_k$ is symmetric around the origin since the graphs $T_k$ are bipartite. The spectral gap of $T_k$ is defined as the
value $\sigma(T_k) = 3 - \lambda_1(T_k)$ and can be described variationally as

$$\sigma(T_k) = \inf \left\{ \frac{\sum_{\{v,w\} \in E_k} (F(v) - F(w))^2}{\sum_{v \in V_k} (F(v))^2} \bigg| \sum_{v \in V_k} F(v) = 0 \right\}.$$  

4. Comparison with Platonic graphs

4.1. Basics about Platonic graphs. We first recall a few important facts about the Platonic graphs $\Pi_N$ and the surfaces $S^\infty(\Pi_N) = \mathbb{H}^2/\Gamma(N)$. For more details see, e.g., [14]. Let $\mathcal{F}$ be the Farey tessellation of the hyperbolic upper half plane $\mathbb{H}^2$, and let $\Omega(\mathcal{F})$ be the set of oriented geodesics in $\mathcal{F}$. Recall that the Farey tessellation is a triangulation of $\mathbb{H}^2$ with vertices on the line at infinity $\mathbb{R} \cup \{\infty\}$, namely, the subset of extended rationals $\mathbb{Q} \cup \{\infty\}$. Two rational vertices with reduced forms $a/c$ and $b/d$ are joined by an edge, a geodesic of $\mathbb{H}^2$, if and only if $ad - bc = \pm 1$ (see [14, Fig. 1] for an illustration of the Farey tessellation). The group of conformal transformations of $\mathbb{H}^2$ that leave $\mathcal{F}$ invariant is the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$, which acts transitively on $\Omega(\mathcal{F})$. The principal congruence subgroups $\Gamma(N)$ are the normal subgroups given in (6).

It is well known (see, e.g., [14]) that $\mathcal{F}/\Gamma(N)$ and $\Pi_N$ are isomorphic, and $\mathcal{F}/\Gamma(N)$ is a triangulation of the surface $S^\infty(\Pi_N) = \mathbb{H}^2/\Gamma(N)$ by ideal triangles (the vertices are, in fact, the cusps of $S^\infty(\Pi_N)$). The tessellation $\Pi_N \subset S^\infty(\Pi_N)$ can be interpreted as a map $\mathcal{M}_N$ in the sense of Jones/Singerman [15]. The group $\text{Aut}(\mathcal{M}_N)$ of automorphisms of $\mathcal{M}_N$ is the group of orientation preserving isometries of $S^\infty(\Pi_N)$ preserving the triangulation. As $\Gamma(N)$ is normal in $\Gamma$, we have that the map $\mathcal{M}_N$ is regular; meaning that $\text{Aut}(\mathcal{M}_N)$ acts transitively on the set of directed edges of $\Pi_N$ (see [15 Thm 6.3]). Moreover, by [15 Thm 3.8],

$$\text{Aut}(\mathcal{M}_N) \cong \Gamma/\Gamma(N) \cong \text{PSL}(2, \mathbb{Z}_N).$$

(Note that in the case of a prime power $N = p^r$, $\text{PSL}(2, \mathbb{Z}_N)$ is the group defined over the ring $\mathbb{Z}_N = \mathbb{Z}/(N\mathbb{Z})$ and not over the field $\mathbb{F}_q$ with $q = p^r$ elements.) Let $N \geq 7$. Noticing that all vertices of $\Pi_N$ have degree $N$, we obtain a smooth compact surface $S(\Pi_N)$ by substituting every ideal triangle in $\Pi_N \subset S^\infty(\Pi_N)$ by a compact equilateral hyperbolic triangle with interior angles $2\pi/N$, and glueing them along their edges in the same way as the ideal triangles of $S^\infty(\Pi_N)$. The group of orientation preserving isometries of $S(\Pi_N)$ preserving this triangulation is, again, isomorphic to $\text{PSL}(2, \mathbb{Z}_N)$. Hence, the automorphism group of the triangulation $\Pi_8 \subset S(\Pi_8)$ is $\text{PSL}(2, \mathbb{Z}_8)$ of order 192. This
implies that $\mathcal{S}(\Pi_8)$ is the unique compact hyperbolic surface of genus 5 with maximal automorphism group (see [3]).

4.2. **Duality between $T_2$ and $\Pi_8$ in $S_2$.** The $\Pi_8$-triangulation of $\mathcal{S}(\Pi_8)$ is illustrated in Figure 2; the black-white pattern on the triangles is a first test whether this triangulation can be isomorphic to the $T_2$-triangulation of $S_2$. (The $\Pi_N$-triangulations for $3 \leq N \leq 7$ can be found in Figs. 3 and 4 of [14].) $\text{PSL}(2, \mathbb{Z}_8)$ acts simply transitively on the directed edges of this triangulation. Consider now a refinement of this triangulation by subdividing each $(\pi/4, \pi/4, \pi/4)$-triangle into six $(\pi/2, \pi/3, \pi/8)$-triangles. It is easily checked that the smaller $(\pi/2, \pi/3, \pi/8)$-triangles admit also a black-white colouring such that

![Figure 2. The Platonic graph $\Pi_8$: Each triangle corresponds to a hyperbolic $(\pi/4, \pi/4, \pi/4)$-triangle of the tessellation of $\mathcal{S}(\Pi_8)$. The edges along the boundary path are pairwise glued to obtain $\mathcal{S}(\Pi_8)$.](image-url)
the neighbours of all smaller black triangles are white triangles and vice versa (see Figure 3). Each black \((\pi/2, \pi/3, \pi/8)\)-triangle is in 1-1 correspondence to a half-edge of \(\Pi_8\) which, in turn, can be identified with a directed edge of \(\Pi_8\). Consequently, the orientation preserving isometries of the surface \(S(\Pi_8)\) corresponding to the elements in \(\text{PSL}(2, \mathbb{Z}_8)\) act simply transitively on the black \((\pi/2, \pi/3, \pi/8)\)-triangles. In fact, \(\text{PSL}(2, \mathbb{Z}_8)\) can be interpreted as a quotient of the triangle group \(\Delta^+(2, 3, 8)\), namely,

\[
\text{PSL}(2, \mathbb{Z}_8) \cong \langle x^2, y^3, z^8 \mid xyz, (xz^2xz^5)^2 \rangle,
\]

where \(x, y, z\) correspond to rotations by \(\pi/2, 2\pi/3, \pi/4\) about the three vertices of a given \((\pi/2, \pi/3, \pi/8)\)-triangle.

\[
MAGMA \text{ computations show that } \text{PSL}(2, \mathbb{Z}_8) \text{ has a unique normal subgroup } N \text{ of index } 6, \text{ generated by the elements } X = x^{-1}z^2x, Y = y^{-1}z^2y \text{ and } Z = z^2, \text{ which is isomorphic to the triangle group quotient } \Delta^+(4, 4, 4)/P_2(\Delta^+(4, 4, 4)) \text{ via the explicit isomorphism }
\]

\[
X \mapsto \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad Z \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},
\]

where \(P_2(\Delta^+(4, 4, 4))\) is a group in the lower exponent-2 series of \(\Delta^+(4, 4, 4)\) and was defined in (2). Note that the matrices in (14), viewed as elements in \(\text{PSL}(2, \mathbb{Z})\), generate a group acting simply transitively on the black triangles of the Farey tessellation in \(\mathbb{H}^2\), as illustrated in Figure 4. The images of a black triangle \(\mathcal{T}\) with vertices 0, 1, \(\infty\) under
\( \{X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}\} \) are the six black triangles each sharing a common white triangle with \( T \).

**Figure 4.** The action of the elements \( X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1} \in PSL(2, \mathbb{Z}) \) on a triangle \( T \) with vertices 0, 1, \( \infty \) of the Farey tessellation.

MAGMA computations also show that we have the explicit isomorphism

\[ N = \langle X, Y, Z \rangle \cong G_2 = \langle x_0, x_1, x_3 \rangle, \]

given by \( X \mapsto x_0, Y \mapsto x_1, Z \mapsto x_3 \). The normal group \( N \triangleleft PSL(2, \mathbb{Z}_8) \) is of order 32 and the quotient \( S_0 = \mathcal{S}(\Pi_8)/N \) is an orbifold consisting of two hyperbolic \((\pi/4, \pi/4, \pi/4)\)-triangles (one of them black and the other white). We conclude from the explicit isomorphism \( N \cong G_2 \) that the covering procedure discussed in Section 2.2 leads to isometric surfaces \( \mathcal{S}(\Pi_8) \cong S_2 \), and that the tessellation \( \Pi_8 \subset \mathcal{S}(\Pi_8) \) is dual to the tessellation \( T_2 \subset \mathcal{S}(T_2) \) via this isometry of surfaces. This confirms the first statement of Proposition 1.4.

**Remark 4.1.** For \( k = 2 \), the \((\Delta - Y)\)-transformation \( X_2 \to T_2 \) has a group theoretical interpretation. There exists a group extension \( \widetilde{G_2} \) of \( G_2 \) by \( \mathbb{Z}_2 \), generated by involutions \( A, B, C \) satisfying \( X = AB \), \( Y = BC \) and \( Z = CA \), and \( T_2 \) is the Cayley graph of \( \widetilde{G_2} \) with respect to the generators \( A, B, C \). This group theoretic interpretation of the \((\Delta - Y)\)-transformation fails for \( k \geq 5 \). In fact, the group

\( T = \langle A, B, C \mid A^2, B^2, C^2, r_1(AB, BC), r_2(AB, BC), r_3(AB, BC) \rangle \)

with \( r_1, r_2, r_3 \) given in (8) is finite and of order 6144. If the introduction of the above involutions \( A, B, C \) would lead to a group extension \( \widetilde{G_k} \), then \( \widetilde{G_k} \) would have to be of order \( 2|G_k| \) and a quotient of \( T \) and,
therefore, of order $\leq 6144$. However, we have $2|G_5| = 16384$ in contradiction to the second condition. Thus we do not obtain a Cayley graph representation of the graphs $T_k$ for $k \geq 5$ via this procedure.

4.3. **Non-duality of $T_k \subset S_k$ and Platonic graphs for $k \geq 3$.** Let $V(\Pi_N)$ denote the vertex set of $\Pi_N$. Then we have

$$|V(\Pi_N)| = \frac{N^2}{2} \prod_{p|N} \left(1 - \frac{1}{p^2}\right),$$

where the product runs over all primes $p$ dividing $N$. This formula can be found in [14, p. 441], where $\Pi_N$ is viewed as a triangular map and denoted by $\mathcal{M}_3(N)$. An isomorphism $T_k^* \cong \Pi_N$ leads to $N = 2^{n_k+1}$, since all vertices of $T_k^*$ have degree $2^{n_k+1}$ (see Proposition 1.2) and all vertices of $\Pi_N$ have degree $N$. In this case, the formula for the number of vertices of $\Pi_N$ simplifies to

$$|V(\Pi_{2^{n_k+1}})| = \frac{2^{2n_k+2}}{2} \left(1 - \frac{1}{4}\right) = 3 \cdot 2^{2n_k-1}.$$

On the other hand, if $V(T_k^*)$ denotes the vertex set of $T_k^*$, we conclude from Proposition 1.2

$$|V(T_k^*)| = 3 \cdot 2^{N_k-n_k},$$

Hence, an isomorphism $T_k^* \cong \Pi_N$ leads to the identity $2n_k-1 = N_k-n_k$, i.e.,

$$3 \lfloor \log_2 k \rfloor + 3 = 3n_k = N_k + 1 \geq 8[k/3] + 3 \cdot (k \mod 3),$$

by (4) and (3). But one easily checks that this inequality holds only for $k = 1, 2$. (In the case $k = 1$, we have $\Pi_4 = T_1^*$, since $T_1$ is combinatorially the cube and $\Pi_4$ is the octagon.) This shows that the graph family $\Pi_N$ cannot contain any of the dual graphs $T_k^*$, for indices $k \geq 3$. This finishes the proof of Proposition 1.4.

5. **Spectral properties of the closed hyperbolic surfaces $S_k$ and $\hat{S}_k$**

5.1. **A lower eigenvalue estimate for the surfaces $S_k$.** The goal of this section is to prove Corollary 1.3. Let us start with lower estimates on the first nontrivial Neumann eigenvalues of special bounded sets $S \subset \mathbb{H}^2$ with piecewise smooth boundaries, characterized by

(15) $$\nu(S) := \inf \left\{ \int_S \|\nabla f\|^2 \bigg| f \in C^\infty(S) \text{ with } \int_S f^2 = 1 \text{ and } \int_S f = 0 \right\}.$$
Lemma 5.1. For \( k \geq 2 \), let \( Q(k) \subset \mathbb{H}^2 \) be a quadrilateral obtained as the union of two adjacent equilateral hyperbolic triangles with interior angles \( \pi/2^k \). Then we have for all \( k \geq 2 \),
\[
\nu(Q(k)) > \frac{1}{4}.
\]

Proof. The result follows via Cheeger’s inequality (see, e.g., [11, Thm. 8.3.3]), once it is shown that
\[
\frac{1}{4} \leq h(Q(k)) = \inf \frac{\ell(\gamma)}{\min\{\text{area } A, \text{area } A'\}},
\]
where \( \gamma \) runs through all curves decomposing \( Q(k) \) into two connected, relatively open subsets \( A \) and \( A' \). Then we must have one of the following cases: (i) \( \gamma \) is a closed curve, (ii) \( \gamma \) is an arc with both endpoints on the same side of \( Q(k) \), (iii) \( \gamma \) is an arc having endpoints in adjacent sides of \( Q(k) \), and (iv) \( \gamma \) is an arc having endpoints in opposite sides of \( Q(k) \).

![Figure 5. The quadrilateral Q(k) as union of the two triangles ΔABC and ΔBCD.](image)

Due to [11] p.220, the only remaining case to consider is case (iv). Let \( \alpha = \pi/2^k \) and \( Q(k) = \Delta ABC \cup \Delta BCD \), as illustrated in Figure 5. We assume the endpoints of \( \gamma \) lie on the opposite sides \( AC \) and \( BD \), whose bi-infinite geodesic extensions are \( c \) and \( c' \). Then we have obviously \( \ell(\gamma) \geq d(c, c') \). If \( h \) denotes the length of the heights of \( \Delta ABC \) and \( \Delta BCD \), then we conclude from [11] Thm. 2.2.2 that
\[
\cosh(h) = \frac{\cos(\alpha)}{\sin(\alpha/2)}.
\]

It is easy to see from \( 3\alpha/2 < \pi/2 \) that if \( B_t \) denotes the point on \( c \) at distance \( t = d(B, B_t) \) from \( B \) in the direction of \( F \), then \( t \to d(B, B_t) \)
is initially strictly monotone decreasing. The same holds for the point \( C_t \in c' \) at distance \( t \) from \( C \) in the direction of \( E \). As a consequence, the unique minimal distance points \( X \in c \) and \( X' \in c' \), i.e., \( d(X, X') = d(c, c') \), must lie in the segments \( BF \) and \( EC \), respectively, and the quadrilateral \( □X X' CF \) is a trirectangle. Then we conclude from [11, Thm. 2.3.1] that

\[
\cosh(d(X, X')) = \cosh(h) \sin(3\alpha/2).
\]

Combining this with (16), we obtain, using \( \alpha \leq \pi/4 \):

\[
\cosh(\ell(\gamma)) \geq \frac{\cos(\alpha) \sin(3\alpha/2)}{\sin(\alpha/2)} = \cos(\alpha)(1 + 2\cos(\alpha)) \geq \frac{1 + \sqrt{2}}{\sqrt{2}},
\]

i.e., \( \ell(\gamma) \geq 1.12838 \ldots \). Since area \( \triangle ABC = \pi - 3\alpha \leq \pi \), we conclude that

\[
\frac{\ell(\gamma)}{\min\{\text{area } A, \text{area } A'\}} \geq \frac{1.12838}{\pi} > \frac{1}{4},
\]

finishing the proof of \( \nu(Q(k)) > 1/4 \). \( \square \)

Let \( S = S_k \). Note that \( S \) comes with a tessellation \( T_k = (V_k, E_k, F_k) \) by regular polygons. To bound the first nontrivial eigenvalue \( \lambda_1(S) \) of \( S \) from below by the spectral gap \( \sigma(T_k) \) of the tessellation \( T_k = (V_k, E_k, F_k) \), we employ the Brooks-Burger transfer principle and follow closely the arguments given in [5], which are a variant of Burger’s arguments [8]. For the reader’s convenience, we present them here.

**Proof of Corollary 1.3.** Let \( S = S_k \) and \( f \in C^\infty(S) \) be the normalized eigenfunction to the eigenvalue \( \lambda_1 = \lambda_1(S) \), i.e.,

\[
\Delta f = \lambda_1 f \quad \text{and} \quad \int_S f^2 = 1.
\]

Then \( f \) is orthogonal to the constant function, i.e., \( \int_S f = 0 \) and satisfies \( \int_S ||\text{grad } f||^2 = \lambda_1 \).

Now we associate to \( f \) a corresponding function \( F \) on the set of vertices \( V_k \) of the tessellation \( T_k \subset S \). For every vertex \( v \in V_k \), there is an equilateral hyperbolic triangle \( T(v) \subset S \) with interior angles \( \pi/2^n k \) of the dual tessellation containing \( v \). In fact, we have

\[
T(v) = \{ z \in S \mid d(z, v) \leq d(z, w) \text{ for all } w \in V_k \}.
\]

The function \( F : V_k \rightarrow \mathbb{R} \) is now defined as follows

\[
F(v) = \frac{V}{V} \int_{T(v)} f,
\]

where \( V = \text{area}(T(v)) \), and we have \( V \sum_{v \in V_k} F(v) = \int_S f = 0 \).
Our next goal is to compare the Rayleigh quotients of $f$ and $F$. The characterisation (15) implies that we have the following Poincaré inequalities for adjacent vertices $v, w \in V_k$:

$$\int_{\mathcal{T}(v)} \left( f - F(v) \right)^2 < \frac{1}{\nu(\mathcal{T}(v))} \int_{\mathcal{T}(v)} \|\text{grad } f\|^2,$$

$$\int_{\mathcal{T}(v) \cup \mathcal{T}(w)} \left( f - \frac{F(v) + F(w)}{2} \right)^2 < \frac{1}{\nu(\mathcal{T}(v) \cup \mathcal{T}(w))} \int_{\mathcal{T}(v) \cup \mathcal{T}(w)} \|\text{grad } f\|^2.$$

Using [11, (8.4.1)] and Lemma 5.1, we conclude that both inequalities above hold by choosing the same coefficient $1/\nu$ with $\nu = 1/4$ at the right hand sides.

Using

$$\frac{1}{2} \left( \frac{F(v) - F(w)}{2} \right)^2 \leq \left( f(z) - \frac{F(v) + F(w)}{2} \right)^2 + (f(z) - F(v))^2$$

and the Poincaré inequalities, we obtain

$$V \sum_{\{v, w\} \in E_k} (F(v) - F(w))^2$$

$$= \int_{\mathcal{T}(v)} \frac{1}{2} \left( \frac{F(v) - F(w)}{2} \right)^2 + \int_{\mathcal{T}(w)} \frac{1}{2} \left( \frac{F(v) - F(w)}{2} \right)^2$$

$$< \frac{2}{\nu} \int_{\mathcal{T}(v) \cup \mathcal{T}(w)} \|\text{grad } f\|^2,$$

leading to

$$(17) \quad V \sum_{\{v, w\} \in E_k} (F(v) - F(w))^2 < \frac{8D}{\nu} \int_S \|\text{grad } f\|^2 = 8D \frac{\lambda_1}{\nu}$$

after summation over the edges, and using the fact that $T_k$ has vertex degree $D = 3$. On the other hand, using the first Poincaré inequality again we have

$$V(F(v))^2 = \int_{\mathcal{F}(v)} f^2 - \int_{\mathcal{F}(v)} (f - F(v))^2 > \int_{\mathcal{F}(v)} f^2 - \frac{1}{\nu} \int_{\mathcal{F}(v)} \|\text{grad } f\|^2,$$

and, after summing over the vertices of $T_k$,

$$(18) \quad V \sum_{v \in V_k} (F(v))^2 > 1 - \frac{\lambda_1}{\nu}.$$
Combining (17) and (18), we have either $\lambda_1 \geq 1/4 = \nu$ or

$$\sigma(T_k) \leq \frac{\sum_{\{v,w\} \in E_k} (F(v) - F(w))^2}{\sum_{v \in V_k} (F(v))^2} < \frac{8D\lambda_1}{\nu} \frac{1}{1 - \lambda_1/\nu},$$

which implies, in either case,

$$\lambda_1 > \nu \left( \frac{\sigma(T_k)}{8D + \sigma(T_k)} \right) = \frac{1}{4} \left( \frac{\sigma(T_k)}{24 + \sigma(T_k)} \right).$$

The Corollary follows now directly from Theorem 1.1(ii) since $\sigma(T_k) = 3 - \sqrt{C} + 3 > 0$ with the constant $C \in (0, 6)$ given in the theorem. □

5.2. Conformal compactifications. We start with the definition of a conformal compactification of a hyperbolic surface as given, e.g., in [6].

**Definition 5.2.** Let $S^\infty$ be a complete non-compact finite area hyperbolic surface with $k$ cusps. The conformal compactification of $S^\infty$ is the unique closed Riemann surface $S$ with $k$ points $\{p_1, \ldots, p_k\} \subset S$ such that $S^\infty$ is conformally equivalent to $S \setminus \{p_1, \ldots, p_k\}$.

The conformal compactification of $S^\infty$ can be constructed as follows: Choose disjoint neighbourhoods of the cusps of $S^\infty$ such that every such neighbourhood is conformally equivalent to a punctured disk. Fill in each disk the missing point and choose the unique conformal structure of this disk, and you obtain a compact surface $S$ conformally equivalent to $S \setminus \{p_1, \ldots, p_k\}$. For more details and many explicit examples, we refer the readers to [6, 7, 17].

It is not always the case that the conformal compactification $S$ carries a hyperbolic metric. For example, the conformal compactification of the hyperbolic $S_0^\infty$ in Section 2.2, obtained by glueing together two ideal hyperbolic triangles $T_1^\infty$ and $T_2^\infty$, is the Riemann sphere which only carries a metric of positive constant curvature. A sufficient condition to guarantee that the conformal compactification $S$ carries a hyperbolic metric which, in an appropriate sense, is even geometrically close to the hyperbolic structure of $S^\infty$, is the so-called large cusp condition (see, e.g., [6, Def. 2.1] or [17, Def. 2.2.1]):

**Definition 5.3.** A hyperbolic surface $S^\infty$ has cusps of length $\geq L$ if, for every cusp of $S^\infty$, there exists a closed horocycle of length $\geq L$ about it, and if all these horocycles are disjoint.

In fact, it can be shown (see [17, Thm. 2.3.1]) that if all cusps of $S^\infty$ have length bigger than $2\pi$, then its conformal compactification carries a hyperbolic metric. Following arguments explained in detail in [17], we
show that our closed surfaces $S_k$ are the conformal compactifications of the surfaces $S_k^\infty$.

**Proposition 5.4.** The closed surfaces $S_k$ of Section 2 are the conformal compactifications of the complete non-compact finite area hyperbolic surfaces $S_k^\infty$.

**Proof.** Let $k \geq 2$ be fixed. Recall that the group $G_k$ acts on $S_k^\infty$ by isometries and that $S_k^\infty$ has a geodesic triangulation by (black and white) ideal triangles. We noted at the end of Section 2.4 that the cusps of $S_k^\infty$ are in bijection with left-hand-turn paths in $(T_k, O_k)$ of length $2^{n_k+1} \geq 8 > 2\pi$ (with $n_k$ given in (4)). This implies that the large cusp condition (with cusp lengths increasing in $k$) is satisfied and the conformal compactification $S_{\text{comp}}$ of $S_k^\infty$ carries a hyperbolic metric with a corresponding geodesic triangulation. Moreover, any isometry of $S_k^\infty$ induces a corresponding isometry of $S_{\text{comp}}$. For more details on these facts see, e.g., [17, pp. 13]. Therefore $G_k$ acts also on $S_{\text{comp}}$ by isometries, and $S_{\text{comp}}/G_k$ is an orbifold with a triangulation consisting of two compact triangles whose vertices are the singular points. The orders of the generators $x_0, x_1, x_3$ of $G_k$ determine the angles of these two triangles uniquely, and we conclude that $S_{\text{comp}}/G_k$ is isometric to the orbifold $S_0 = S_k/G_k$ introduced in Section 2.2. This isometry lifts then to a corresponding isometry between $S_k$ and $S_{\text{comp}}$. □

We like to mention the following consequence of Proposition 5.4.

**Corollary 5.5.** There are uniform lower positive bounds for the Cheeger constants of the families of surfaces $S_k$ and $S_k^\infty$.

**Proof.** From a classical result by Tanner [20] or Alon-Milman [1] we know that

$$\frac{\sigma(T_k)}{2} \leq h(T_k) := \inf_E \frac{\#(E)}{\min\{\#(A), \#(A')\}},$$

where $E \subset E_k$ runs through all collection of edges such that $T_k \setminus E_k$ disconnects into two components with disjoint vertex sets $A \subset V_k$ and $A' \subset V_k$. This implies together with Theorem 1.1(ii) that the combinatorial Cheeger constants have the following uniform positive lower bound

$$h_0 = \frac{3 - \sqrt{C+3}}{2} \leq h(T_k).$$

Moreover, it follows from Theorem 4.2 in [7] that there are constants $c_h, C_h > 0$ such that we have the following relation between the respective Cheeger constants

$$c_h h(T_k) \leq h(S_k^\infty) \leq C_h h(T_k).$$
From these facts we conclude that
\begin{equation}
(20) \quad c_h h_0 \leq h(S_k^\infty).
\end{equation}
Moreover, it follows from Theorem 3.3(a) in [7], Proposition 5.4 above, and the increasing cusp length properties of our surfaces $S_k^\infty$ that, for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that
\begin{equation}
(21) \quad \frac{1}{1 + \epsilon} h(S_k) \leq h(S_k^\infty) \leq (1 + \epsilon) h(S_k)
\end{equation}
for all $k \geq k_0$. Combining, finally, (20) and (21) proves the corollary.

**Remark 5.6.** We could have proved Corollary 1.3 alternatively by combining Corollary 5.5 and Cheeger’s inequality for surfaces, but the proof given in Section 5.1 is much more direct.

### 5.3. A lower eigenvalue estimate for the surfaces $\hat{S}_k$.

It only remains to prove Corollary 1.6 from the Introduction. The explicit construction of the surfaces $\hat{S}_k$ is explained in Buser [9, Section 3.2].

**Proof of Corollary 1.6.** The identity $2g - 2 = |V_k|$ between the genus of the surface $\hat{S}_k$ and the number of vertices of the trivalent graph $T_k$ is easily checked. Moreover, every automorphism of the graph $T_k$ induces an isometry on $\hat{S}_k$. Since the graphs $T_k$ form a power of coverings with powers of 2 as covering indices, the same holds true for the associated surfaces $\hat{S}_k$. We know from [9, (4.1)] that
\[
\lambda_1(\hat{S}_k) \geq \frac{1}{144\pi^2} h(T_k).
\]
Combining this with (19) leads to
\[
\lambda_1(\hat{S}_k) \geq \frac{3 - \sqrt{C - 3}}{288\pi^2}
\]
with the constant $C > 0$ from Theorem 1.1. This finishes the proof of Corollary 1.6. \hfill $\square$

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