Rigidity of Secondary Characteristic Classes

Jerry M. Lodder

1 Introduction

In this paper we study the variability and rigidity of secondary characteristic classes which arise from flat connections $\theta$ on a differentiable manifold $M$. In particular we consider $\theta$ as a Lie-algebra valued one-form on $M$, and study the characteristic map

$$\phi_\theta : H^*_\text{Lie}(g) \to H^*_\text{dR}(M),$$

where $H^*_\text{Lie}$ denotes Lie algebra cohomology (for a Lie algebra $g$), and $H^*_\text{dR}$ denotes de Rham cohomology. An element $\alpha \in H^*_\text{Lie}(g)$ is called variable if there exists a one-parameter family of flat connections $\theta_t$ with

$$\frac{d}{dt} \left[ \phi_{\theta_t}(\alpha) \right]_{t=0} \neq 0,$$

otherwise $\alpha$ is called rigid. For example, the universal Godbillon-Vey invariant for codimension $k$ foliations is known to be a variable class. Letting $H^*_L$ denote Loday’s Leibniz cohomology $[3]$ $[4]$ $[5]$, we prove that if $H^*_L(g) = 0$ for $n \geq 1$, then all classes in $H^*_\text{Lie}(g)$ are rigid. This result imparts a geometric meaning to Leibniz cohomology.

Moreover, in the case of codimension one foliations (with trivial normal bundle), there is a flat $W_1$-valued connection on $M$, where $W_1$ is the Lie algebra of formal vector fields on $\mathbb{R}^1$. From $[8]$,

$$H^*_L(W_1) \simeq \Lambda(\alpha) \otimes T(\zeta),$$

where $\Lambda(\alpha)$ is the exterior algebra on the Godbillon-Vey invariant (in dimension three) and $T(\zeta)$ denotes the tensor algebra on a four-dimensional
class. When $M$ is provided with a one-parameter family of such foliations, we compute the image of a characteristic map

$$HL^4(W_1) \to H^4_{\text{dR}}(M).$$

The image in de Rham cohomology is independent of the choices made when constructing the $W_1$ connections, and involves the time derivative (derivative with respect to $t$) of the Bott connection.

## 2 The Characteristic Map and Rigidity

Let $M$ be a differentiable ($C^\infty$) manifold with flat connection $\theta$. We consider the formulation of $\theta$ as a Lie-algebra valued one-form

$$\theta : TM \to g,$$

where $TM$ denotes the tangent bundle of $M$ and $g$ is a Lie algebra. This section describes a characteristic map

$$\phi_\theta : H^*_\text{Lie}(g; \mathbb{R}) \to H^*_\text{dR}(M)$$

from Lie algebra cohomology (with $\mathbb{R}$ coefficients) to the de Rham cohomology of $M$. In the case of a topological Lie algebra, then $H^*_\text{Lie}$ is understood as continuous cohomology, computed using continuous cochains.

Let $\Omega^k(g; \mathbb{R})$ be the $\mathbb{R}$-vector space of skew-symmetric (continuous) cochains

$$\alpha : g^\otimes k \to \mathbb{R}.$$ 

For comparison with Leibniz cohomology (and to establish our sign conventions), we write the coboundary for Lie algebra cohomology

$$d : \Omega^k(g; \mathbb{R}) \to \Omega^{k+1}(g; \mathbb{R})$$

as

$$d(\alpha)(g_1 \otimes g_2 \otimes \ldots \otimes g_{k+1}) =$$

$$\sum_{1 \leq i < j \leq k} (-1)^{j+1} \alpha(g_1 \otimes \ldots \hat{g}_i \otimes [g_i, g_j] \otimes g_{i+1} \ldots \hat{g}_j \ldots \otimes g_{k+1}), \quad (2.1)$$

2
where each $g_i \in \mathfrak{g}$. For a differentiable manifold $M$, let

$$\Omega^k(M) := \Omega^k(M; \mathbb{R})$$

denote the $\mathbb{R}$-vector space of $k$-forms on $M$. Then the de Rham coboundary

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

$$\omega \mapsto d\omega$$

has a global formulation as

$$d\omega(X_1 \otimes X_2 \otimes \ldots \otimes X_{k+1}) =$$

$$\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1 \otimes \ldots \hat{X}_i \ldots \otimes X_{k+1})) +$$

$$\sum_{1 \leq i < j \leq k+1} (-1)^{j+1} \omega(X_1 \otimes \ldots \otimes X_{i-1} \otimes [X_i, X_j] \otimes X_{i+1} \otimes$$

$$\ldots \hat{X}_j \ldots \otimes X_{k+1}),$$

where each $X_i \in \chi(M)$, the Lie algebra of smooth $(C^\infty)$ vector fields on $M$.

With the above sign conventions for the coboundary maps, the connection $\theta \in \Omega^1(M; \mathfrak{g})$ is flat if and only if the Maurer-Cartan equation holds,

$$d\theta = -\frac{1}{2} [\theta, \theta],$$

where $d\theta \in \Omega^2(M; \mathfrak{g})$ is given by

$$(d\theta)(X_1 \otimes X_2) = X_1(\theta(X_2)) - X_2(\theta(X_1)) - \theta([X_1, X_2]).$$

Recall that for a differentiable function $f : M \to \mathfrak{g}$, $X(f)$ denotes the partial derivatives of the component functions of $f$ with respect to $X$. If $e_1, e_2, e_3, \ldots$ is a basis for $\mathfrak{g}$, then

$$f = f_1 e_1 + f_2 e_2 + f_3 e_3 + \cdots,$$

where each $f_i : M \to \mathbb{R}$ is differentiable. Thus,

$$X(f) = X(f_1)e_1 + X(f_2)e_2 + X(f_3)e_3 + \cdots.$$
Also, the symbol $[\theta, \theta]$ denotes the composition

$$\Omega^1(M; g) \otimes \Omega^1(M; g) \xrightarrow{\theta \otimes \theta} \Omega^2(M; g \otimes g) \xrightarrow{[\cdot, \cdot]} \Omega^2(M; g)$$
given by

$$[\theta, \theta](X_1 \otimes X_2) = [\theta(X_1) \otimes X_2 - \theta(X_2) \otimes \theta(X_1)]$$

$$= 2[\theta(X_1), \theta(X_2)].$$

Consider the following map [2, p. 234]

$$\phi : \Omega^*(g; \mathbb{R}) \to \Omega^*(M; \mathbb{R})$$
given on a cochain $\alpha \in \Omega^k(g; \mathbb{R})$ by

$$\phi(\alpha)(X_1 \otimes X_2 \otimes \ldots \otimes X_k) = \alpha(\theta(X_1) \otimes \theta(X_2) \otimes \ldots \otimes \theta(X_k)).$$

For the case $k = 1$, it can be easily seen from (2.3) that $\phi$ is a map of cochain complexes. In particular,

$$d(\phi(\alpha))(X_1 \otimes X_2)$$

$$= X_1(\alpha(\theta(X_2))) - X_2(\alpha(\theta(X_1))) - \alpha([X_1, X_2])$$

$$= \alpha\left(X_1(\theta(X_2)) - X_2(\theta(X_1)) - \theta([X_1, X_2])\right)$$

$$= \alpha(-[\theta(X_1), \theta(X_2)])$$

$$= \phi(d\alpha)(X_1 \otimes X_2).$$

In [2, p. 235] it is proven that in general

$$d\phi = \phi d.$$

The induced map

$$\phi_\theta : H^*_{\text{Lie}}(g) \to H^*_{\text{dR}}(M)$$
is called the characteristic map.

A specific example of a flat connection studied in this paper arises from the theory of foliations. Let $\mathcal{F}$ be a $C^\infty$ codimension one foliation on $M$ with trivial normal bundle. Given a choice of a trivialization, then a determining one-form $\omega_0$ is defined for the foliation by $\omega_0(v) = 0$ is $v$ is tangent to a leaf and $\omega_0(\eta) = 1$ if $\eta$ is a unit vector of the normal bundle with positive
orientation. Letting \( d \) denote the de Rham coboundary, a sequence of one-forms \( \omega_1, \omega_2, \omega_3, \ldots \) can be defined so that

\[
d\omega_0 = \omega_0 \wedge \omega_1 \\
d\omega_1 = \omega_0 \wedge \omega_2 \\
d\omega_2 = \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2 \\
d\omega_k = \sum_{i=0}^{[k/2]} \frac{k - 2i + 1}{k + 1} \binom{k + 1}{i} (\omega_i \wedge \omega_{k+1-i}).
\]

(2.4)

Consider the topological Lie algebra of formal vector fields

\[
W_1 = \left\{ \sum_{k=0}^{\infty} c_k x^k \frac{d}{dx} \mid c_k \in \mathbb{R} \right\}
\]

in the \( \mathcal{M} \)-adic topology, where \( \mathcal{M} \) is the maximal ideal of \( \mathbb{R}[[x]] \) given by those series with zero constant term. Then a \( W_1 \)-valued one-form is defined on \( M \) by

\[
\theta_F(v) = \sum_{k=0}^{\infty} \omega_k(v) x^k \frac{d}{dx},
\]

(2.5)

where \( v \in TM \). From (2.4), it can be proven [2, p. 231] that \( \theta_F \) is a flat connection on \( M \). The resulting homomorphism

\[
\phi_{\theta_F} : H^*_\text{Lie}(W_1) \to H^*_\text{dR}(M)
\]

(2.6)

is the classical characteristic map in foliation theory. (A similar construction exists for foliations of any codimension with trivial normal bundle.)

We wish to study a one-parameter variation of a flat structure

\[
\theta_t : TM \to g, \quad t \in \mathbb{R}, \quad \theta_0 = \theta,
\]

(2.7)

which depends smoothly on the parameter \( t \). Such a structure may arise from a one-parameter variation of a foliation.

**Definition 2.1.** A Lie algebra cohomology class \( \alpha \in H^*_\text{Lie}(g) \) is called variable (for \( \theta \)) if there exists a family \( \theta_t \) such that

\[
\frac{d}{dt} \left[ \phi_{\theta_t}(\alpha) \right]_{t=0} \neq 0.
\]

Otherwise, \( \alpha \) is called rigid.
By work of Thurston, the universal Godbillon-Vey invariant $\alpha \in H^3_{\text{Lie}}(W_1)$ is a variable class \[12\]. One of the goals of this paper is to prove that if the Leibniz cohomology of $g$ vanishes, i.e., $HL^n(g) = 0$, $n \geq 1$, then all characteristic classes in $H^*_\text{Lie}(g)$ are rigid. In the remainder of this section we restate the definition of rigidity in terms of a known condition concerning $H^*_\text{Lie}(g; g')$, the Lie algebra cohomology of $g$ with coefficients in the coadjoint representation

$$g' = \text{Hom}_R^c(g, R),$$

($c$ denotes continuous maps).

First introduce the current algebra $\tilde{g} = C^\infty(R, g)$ of differentiable maps from $R$ to $g$. Then given $\theta_t$ as in (2.7), there is a flat $\tilde{g}$ connection on $M$

$$\Theta : TM \to \tilde{g}$$

$$\Theta(v)(t) = \theta_t(v), \quad v \in TM,$$

and a characteristic map

$$\phi_\Theta : H^*_\text{Lie}(\tilde{g}) \to H^*_\text{dR}(M).$$

Using an idea of D. Fuks [3], define a “time derivative” map on cochains

$$D : \Omega^q(g) \to \Omega^q(\tilde{g})$$

by

$$D(\alpha)(\varphi_1, \varphi_2, \ldots, \varphi_q) = \sum_{i=1}^q \alpha \left( \varphi_1(0), \ldots, \varphi_{i-1}(0), \frac{d}{dt}\left[\varphi_i(t)\right]_{t=0} \varphi_{i+1}(0), \ldots, \varphi_q(0) \right),$$

where $\alpha \in \Omega^q(g)$, $(\varphi_1, \varphi_2, \ldots, \varphi_q) \in (\tilde{g})^\otimes q$. Then $D$ is a map of cochain complexes, and there is an induced map

$$D^* : H^*_\text{Lie}(g) \to H^*_\text{Lie}(\tilde{g}).$$

It follows that given $\alpha \in H^*_\text{Lie}(g)$, we have

$$\phi_\Theta \circ D^*(\alpha) = \frac{d}{dt} \left[\phi_{\theta_t}(\alpha)\right]_{t=0}.$$
Recall that $\mathfrak{g}' = \text{Hom}_R(\mathfrak{g}, R)$ is a left $\mathfrak{g}$-module with

$$(g\gamma)(h) = \gamma([h, g]),$$

where $g, h \in \mathfrak{g}$ and $\gamma \in \mathfrak{g}'$. Then $D^*$ can be factored as $\Phi^* \circ V^*$ [2 p. 244], where

$$V^* : H^q_{\text{Lie}}(\mathfrak{g}) \to H^{q-1}(\mathfrak{g}; \mathfrak{g}'), \quad q \geq 1,$$

$$\Phi^* : H^{q-1}(\mathfrak{g}; \mathfrak{g}') \to H^q_{\text{Lie}}(\tilde{\mathfrak{g}}), \quad q \geq 1,$$

are induced by

$$\text{Var} : \Omega^q(g; R) \to \Omega^{q-1}(g; \mathfrak{g}'),$$

$$\Phi : \Omega^{q-1}(g; \mathfrak{g}') \to \Omega^q(\tilde{\mathfrak{g}}; R)$$

$$\begin{align*}
(\text{Var})(\alpha)(g_1, g_2, \ldots, g_{q-1})(g_0) &= (-1)^{q-1}\alpha(g_0, g_1, \ldots, g_{q-1}) \\
\Phi(\gamma)(\varphi_1, \varphi_2, \ldots, \varphi_q) &= \\
&= \sum_{i=1}^q (-1)^{q-i} \gamma(\varphi_1(0), \ldots, \hat{\varphi}_i(0), \ldots, \varphi_q(0))(\varphi'_i(0)),
\end{align*}$$

where $\alpha \in \Omega^q(g; R)$, $g_i \in \mathfrak{g}$, $\gamma \in \Omega^{q-1}(g; \mathfrak{g}')$, $\varphi_i \in \tilde{\mathfrak{g}}$. We then have a commutative diagram

$$\begin{array}{ccc}
H^*_\text{Lie}(\mathfrak{g}) & \xrightarrow{D^*} & H^*_\text{Lie}(\tilde{\mathfrak{g}}) \\
\downarrow V^* & & \uparrow \Phi^* \\
H^{q-1}_{\text{Lie}}(\mathfrak{g}; \mathfrak{g}') & \xrightarrow{=} & H^{q-1}_{\text{Lie}}(\mathfrak{g}; \mathfrak{g}').
\end{array}$$

Lemma 2.2. If $H_{\text{Lie}}^{n-1}(\mathfrak{g}; \mathfrak{g}') = 0$ for $n \geq 1$, then all characteristic classes in $H_{\text{Lie}}^*(\mathfrak{g})$ are rigid.

Proof. This follows from equation (2.10), diagram (2.13) and the definition of rigidity (definition (2.1)).

In the next section we prove that if $HL^n(\mathfrak{g}) = 0$ for $n \geq 1$, then $H_{\text{Lie}}^{n-1}(\mathfrak{g}; \mathfrak{g}') = 0$ for $n \geq 1$.  

7
3 Leibniz Cohomology

Still considering \( g \) to be a Lie algebra (over \( \mathbb{R} \)), recall that the Leibniz cohomology of \( g \) with trivial coefficients,

\[
HL^*(g; \mathbb{R}) := HL^*(g),
\]

is the homology of the cochain complex \([8]\)

\[
\mathbb{R} \xrightarrow{0} C^1(g) \xrightarrow{d} C^2(g) \xrightarrow{d} \cdots \xrightarrow{d} C^k(g) \xrightarrow{d} C^{k+1}(g) \xrightarrow{d} \cdots , \tag{3.1}
\]

where \( C^k(g) = \text{Hom}^c_{\mathbb{R}}(g^\otimes k, \mathbb{R}) \), and for \( \alpha \in C^k(g) \), \( d\alpha \) is given in equation \([2.1]\). Keep in mind that for Leibniz cohomology, the cochains are not necessarily skew-symmetric.

In this section we prove the following:

**Theorem 3.1.** If \( HL^n(g; \mathbb{R}) = 0 \) for \( n \geq 1 \), then \( H_{\text{Lie}}^{n-1}(g; g') = 0 \) for \( n \geq 1 \), where \( g' = \text{Hom}^c_{\mathbb{R}}(g; \mathbb{R}) \).

The proof involves a spectral sequence similar to the Pirashvili spectral sequence \([11]\), except tailored to the specific algebraic relation between \( HL^*(g) \) and \( H_{\text{Lie}}^{*}(g; g') \). Recall that the projection to the exterior power \( g^\otimes q \to g^\wedge q \) induces a homomorphism

\[
H_{\text{Lie}}^*(g) \to HL^*(g).
\]

Letting \( C^*_\text{rel}(g)[2] = C^*(g)/\Omega^*(g) \), we have a long exact sequence

\[
\cdots \to H^q_{\text{Lie}}(g) \to HL^q(g) \to H_{\text{rel}}^{q-2}(g) \to H_{\text{Lie}}^{q+1}(g) \to \cdots .
\]

The Pirashvili spectral sequence arises from a filtration of \( C^*_\text{rel}(g)[2] \) and converges to \( H^*_\text{rel}(g) \).

Consider now the map of cochain complexes

\[
i : \Omega^{q-1}(g; g') \to C^q(g)
\]

given by

\[
(i(\beta))(g_0 \otimes g_1 \otimes \ldots \otimes g_{q-1}) = (-1)^{q-1} \beta(g_1 \otimes g_2 \otimes \ldots \otimes g_{q-1})(g_0),
\]
where $\beta \in \Omega^{q-1}(g; g')$ and $g_i \in g$ for $i = 0, 1, 2, \ldots, q - 1$. Letting

$$C^*_\mathrm{RG}(g)[2] = C^*(g)/i[\Omega^{q-1}(g; g')]$$

we also have a long exact sequence

$$
\begin{align*}
0 & \rightarrow H^0_\mathrm{Lie}(g; g') \rightarrow HL^1(g) \rightarrow 0 \\
& \rightarrow H^1_\mathrm{Lie}(g; g') \rightarrow HL^2(g) \rightarrow H^0_{\mathrm{RG}}(g) \rightarrow H^2_\mathrm{Lie}(g; g') \\
& \cdots \rightarrow H^{q-1}_\mathrm{Lie}(g; g') \rightarrow HL^q(g) \rightarrow H^{q-2}_{\mathrm{RG}}(g) \rightarrow H^q_\mathrm{Lie}(g; g') \rightarrow \cdots.
\end{align*}
$$

The filtration for the Pirashvili spectral sequence \cite{9} \cite{11} can be immediately applied to yield a decreasing filtration \{ $F^s_*$ \}$_{s \geq 0}$ for $C^*_{\mathrm{RG}}(g)[2]$. We use the same grading as in \cite{9} \cite{11}, which becomes $F^0_* = C^*_{\mathrm{RG}}[2]$, and for $s \geq 1$,

$$F^s_* = A/B$$

$A = \{ f \in C^*(g) \mid f$ is skew-symmetric in the last $(s + 1)$ tensor factors $\}$

$B = i[\Omega^{q-1}(g; g')]$.

Then as in \cite{9}, each $F^s_*$ is a subcomplex of $C^*_{\mathrm{RG}}(g)$, and

$$F^0_* \supseteq F^1_* \supseteq F^2_* \supseteq \cdots \supseteq F^s_* \supseteq F^{s+1}_* \supseteq \cdots.$$

To identify the $E^2$ term of the resulting spectral sequence, consider $\mathrm{coker}(\mathrm{Var})$, where $\mathrm{Var}$ is defined in equation (2.12). Letting

$$CR^*(g)[1] = \Omega^{q-1}(g; g')/\mathrm{Var}[\Omega^*(g)],$$

there is a short exact sequence

$$0 \rightarrow \Omega^*(g) \overset{\mathrm{Var}}{\rightarrow} \Omega^{q-1}(g; g') \rightarrow CR^*(g)[1] \rightarrow 0,$$

and an associated long exact sequence

$$\cdots \rightarrow H^q_\mathrm{Lie}(g) \rightarrow H^{q-1}_\mathrm{Lie}(g; g') \rightarrow HR^{q-2}(g) \rightarrow H^{q+1}_\mathrm{Lie}(g) \rightarrow \cdots.$$

**Theorem 3.2.** The filtration $F^s_*$ of $C^*_{\mathrm{RG}}(g)[2]$ yields a spectral sequence converging to $H^*_\mathrm{RG}(g)$ with

$$E^{s,0}_2 = 0, \quad s = 0, 1, 2, \ldots,$$

$$E^{s,n}_2 \simeq HL^n(g) \otimes HR^s(g), \quad n = 1, 2, 3, \ldots, \quad s = 0, 1, 2, \ldots,$$

where $\otimes$ denotes the completed tensor product.
Proof. The proof follows from the identification of the $E^2$ term in \[9\] or \[11\].
Also, note that 
\[E_{s,0}^s = (F^s/F^{s+1})_0, \quad F_0^s = A/B,\]
where 
\[A = \{ f \in C^{s+2}(\g) \mid f \text{ is alternating in the last } (s + 1) \text{ factors} \}\]
\[B = i[\Omega^{s+1}(\g; \g')].\]
Then $A = B$, $F_0^s = 0$, and $E_{0,0}^{s,0} = 0$. \hfill \Box

**Theorem 3.3.** If $HL^n(\g) = 0$ for $n \geq 1$, then $H_{\text{Lie}}^{n-1}(\g; \g') = 0$ for $n \geq 1$.

**Proof.** If $HL^n(\g) = 0$ for $n \geq 1$, then from theorem (3.2), the $E_2$ term for the spectral sequence converging to $H_{\text{RC}}^*(\g)$ is zero. Thus, $H_{\text{RC}}^n(\g) = 0$ for $n \geq 0$. The result now follows from long exact sequence (3.2). \hfill \Box

**Theorem 3.4.** If $HL^n(\g) = 0$ for $n \geq 1$, and $\theta_t$ is a one-parameter family of flat $\g$-connections on $M$, then all characteristic classes in $H_{\text{Lie}}^*(\g)$ are rigid.

**Proof.** The theorem follows from lemma (2.2) and theorem (3.3). \hfill \Box

By checking dimensions in theorem (3.2), exact sequence (3.2), and diagram (2.13), we have:

**Corollary 3.5.** If $HL^n(\g) = 0$ for $1 \leq n \leq p$, then all characteristic classes in $H_{\text{Lie}}^n(\g)$ are rigid for $1 \leq n \leq p$.

We close this section with two observations, one concerning a theorem of P. Ntolo on the vanishing of $HL^*(\g)$ for $\g$ semi-simple, the other concerning the highly nontrivial nature of $HL^*(W_1)$.

**Theorem 3.6.** ([10]) If $\g$ is a semi-simple Lie algebra (over $\mathbb{R}$), then 
\[HL^n(\g) = 0 \quad \text{for } n \geq 1.\]

By contrast, the Leibniz cohomology of formal vector fields, $HL^*(W_1)$, contains many non-zero classes which do not appear in $H_{\text{Lie}}^*(W_1)$ \[9\]. In the next section we compute the image of a characteristic map 
\[HL^4(W_1) \to H^4_{\text{dR}}(M),\]
where $M$ supports a family of codimension one foliations.
4 Foliations

Letting \( W_1 \) denote the Lie algebra of formal vector fields defined in section two, recall that [2, p. 101]

\[
H_{\text{Lie}}^{q-1}(W_1; W'_1) \simeq \mathbb{R}
\]

for \( q = 3 \) and \( q = 4 \), and zero otherwise. The generator of the class for \( q = 3 \) is the universal Godbillon-Vey invariant, called \( \alpha \) in this paper, and we denote the generator of the class for \( q = 4 \) by \( \zeta \). From [2], the map

\[
H_{\text{Lie}}^{q-1}(W_1; W'_1) \to H^*(W_1)
\]

given in exact sequence (3.2) is injective. As dual Leibniz algebras [3], we have [4]

\[
H^*(W_1) \simeq \Lambda(\alpha) \otimes T(\zeta),
\]

where \( \Lambda(\alpha) \) is the exterior algebra on \( \alpha \), and \( T(\zeta) \) denotes the tensor algebra on \( \zeta \).

Let \( M \) be a \( C^\infty \) manifold with a one-parameter family \( \mathcal{F}_t \) of codimension one foliations having trivial normal bundles. Let \( \omega_i(t) \) be the corresponding one-forms given in equation (2.4) considered as differentiable functions of \( t \). Recall the definitions of \( \Phi^* \) and \( \phi_\Theta \) given in equations (2.11) and (2.8) respectively. In this section we prove the following:

**Theorem 4.1.** Let \( M \) and \( \mathcal{F}_t \) be given as above. Then the composition

\[
H^4(W_1) \simeq H^3_{\text{Lie}}(W_1; W'_1) \xrightarrow{\Phi^*} H^4_{\text{Lie}}(\tilde{W}_1) \xrightarrow{\phi_\Theta} H^4_{dR}(M)
\]

sends \( \zeta \) to the de Rham cohomology class of

\[
c(\zeta) := \omega'_1(0) \wedge \omega_0(0) \wedge \omega_1(0) \wedge \omega_2(0).
\]

Moreover, the cohomology class of \( c(\zeta) \) does not depend on the choice\(^\ddagger\) of \( \omega_0(t) \) or \( \omega_1(t) \).

\(^\ddagger\)Since \( c(\zeta) \) may also be written as \( -\omega'_1(0) \wedge \omega_1(0) \wedge d\omega_1(0) \), where \( d \) denotes the de Rham coboundary, it is not necessary to show that the class of \( c(\zeta) \) is independent of the choice of \( \omega_2(t) \).
Proof. We first compute $c(\zeta)$ on the level of cochains. Consider the vector space basis $\{\beta_i\}_{i \geq 0}$ of $\text{Hom}_R^c(W_1, \mathbb{R})$ given by

$$\beta_i \left( \frac{x^j}{j!} \frac{d}{dx} \right) = \delta_{ij}.$$  

From [9], the class of $\zeta$ in $HL^4(W_1)$ is represented by the cochain

$$\beta_1 \otimes (\beta_0 \wedge \beta_1 \wedge \beta_2).$$

The cochain map

$$i : \Omega^3(W_1; W'_1) \to C^4(W_1)$$

inducing the isomorphism

$$H^3_{\text{Lie}}(W_1; W'_1) \xrightarrow{\cong} HL^4(W_1)$$

satisfies

$$i((\beta_0 \wedge \beta_1 \wedge \beta_2) \otimes \beta_1) = -\beta_1 \otimes (\beta_0 \wedge \beta_1 \wedge \beta_2).$$

Also, it is know that

$$(\beta_0 \wedge \beta_1 \wedge \beta_2) \otimes \beta_1$$

generates $H^3_{\text{Lie}}(W_1; W'_1)$ (as an $\mathbb{R}$ vector space).

Let $v_1, v_2, v_3, v_4 \in T_p(M)$. From the definition of $\Phi$ and $\phi_\Theta$, the image of $\zeta$ in $\Omega^4(M)$ is the 4-form which sends $v_1 \otimes v_2 \otimes v_3 \otimes v_4$ to

$$-(\beta_1 \otimes \beta_0 \wedge \beta_1 \wedge \beta_2) \left( - A'_1 \otimes A_2 \otimes A_3 \otimes A_4 + A'_2 \otimes A_1 \otimes A_3 \otimes A_4 \\
- A'_3 \otimes A_1 \otimes A_2 \otimes A_4 + A'_4 \otimes A_1 \otimes A_2 \otimes A_3 \right),$$

where

$$A_i = \sum_{n \geq 0} \omega_n(0)(v_i) \frac{x^n}{n!} \frac{d}{dx}, \quad i = 1, 2, 3, 4,$$

$$A'_i = \sum_{n \geq 0} \omega'_n(0)(v_i) \frac{x^n}{n!} \frac{d}{dx}, \quad i = 1, 2, 3, 4.$$

By the definition of the $\beta_i$'s, the image of $\zeta$ is thus

$$(\omega'_1(0) \wedge \omega_0(0) \wedge \omega_1(0) \wedge \omega_2(0))(v_1 \otimes v_2 \otimes v_3 \otimes v_4).$$
To show that the de Rham cohomology class of $c(\zeta)$ does not depend on the choice of $\omega_0(t)$, consider the one-form

$$u_0(t) = f \cdot \omega_0(t),$$

where $f : M \to \mathbf{R}$ is a $C^\infty$ function with $f(p) \neq 0$ for all $p \in M$. Letting $d$ denote the de Rham coboundary, we have from equation (2.4)

$$d\omega_1(t) = \omega_0(t) \wedge \omega_2(t).$$

Then

$$\omega'_0(0) \wedge \omega_0(0) \wedge \omega_1(0) \wedge \omega_2(0) = -\omega'_0(0) \wedge \omega_1(0) \wedge d\omega_1(0).$$

Also,

$$du_0(t) = u_0(t) \wedge \left( -\frac{df}{f} + \omega_1(t) \right)$$

$$u_1(t) = -\frac{df}{f} + \omega_1(t)$$

$$u'_1(t) \wedge u_1(t) \wedge du_1(t) = \omega'_1(t) \wedge \omega_1(t) \wedge d\omega_1(t)$$

$$+ \frac{df}{f} \wedge \omega'_1(t) \wedge d\omega_1(t).$$

It follows that

$$u'_1(0) \wedge u_1(0) \wedge du_1(0) = \omega'_1(0) \wedge \omega_1(0) \wedge d\omega_1(0)$$

$$+ d\left( \log(|f|) \omega'_1(0) \wedge d\omega_1(0) \right).$$

Compare with Ghys [3]. Of course,

$$d(\omega'_1(0)) = \omega'_0(0) \wedge \omega_2(0) + \omega_0(0) \wedge \omega'_2(0).$$

To show that the cohomology class of $c(\zeta)$ does not depend on the choice of $\omega_1(t)$, consider the one-forms

$$u(t) = \omega_1(t) + f \cdot \omega_0(t),$$

where $g : M \to \mathbf{R}$ is a $C^\infty$ function (which may have zeroes on $M$). Then

$$u'(0) \wedge u(0) \wedge du(0) = \omega'_1(0) \wedge \omega_1(0) \wedge d\omega_1(0)$$

$$+ g \cdot \omega'_0(0) \wedge \omega_1(0) \wedge d\omega_1(0)$$

$$+ \omega'_1(0) \wedge \omega_1(0) \wedge dg \wedge \omega_0(0)$$

$$+ g \cdot \omega'_0(0) \wedge \omega_1(0) \wedge dg \wedge \omega_0(0).$$
It can be checked that
\[ u'(0) \wedge u(0) \wedge du(0) = \omega'(0) \wedge \omega(0) \wedge d\omega(0) + d(A), \]
\[ A = g \cdot \omega'_0(0) \wedge d\omega'_1(0) - dg \wedge \omega'_0(0) \wedge \omega'_1(0) - \frac{1}{2} g^2 \cdot \omega'_0(0) \wedge d\omega_0(0). \]

The paper is closed by noting that the current algebra \( \tilde{g} \) is a Leibniz algebra in the sense of Loday with the Leibniz bracket of \( \varphi_1, \varphi_2 \in \tilde{g} \) given by
\[ \langle \varphi_1(t), \varphi_2(t) \rangle = [\varphi_1(t), \varphi'_2(0)]_{\text{Lie}}, \]
where \([ , ]_{\text{Lie}}\) is the usual Lie bracket on \( \tilde{g} \), and \( \varphi'_2(0) \) is the constant path at \( \varphi'_2(0) \). The Leibniz bracket is not necessarily skew-symmetric,
\[ \langle \varphi_1(t), \varphi_2(t) \rangle \neq -\langle \varphi_2(t), \varphi_1(t) \rangle, \]
but satisfies the following version of the Jacobi identity
\[ \langle \varphi_1(t), \langle \varphi_2(t), \varphi_3(t) \rangle \rangle = \langle \langle \varphi_1(t), \varphi_2(t) \rangle, \varphi_3(t) \rangle - \langle \langle \varphi_1(t), \varphi_3(t) \rangle, \varphi_2(t) \rangle, \]
which is the defining relation for a Leibniz algebra. Also see [1] and [5].

**ACKNOWLEDGEMENTS**
The author would like to express his gratitude to the Institut des Hautes Études Scientifiques for their support while this paper was being written.

**References**

[1] Balavoine, D., “Eléments de carré nul dans les algèbres de Lie graduées,” *Comptes Rendus Acad. Sci.*, Série I, 319, (1994), 783–788.

[2] Fuks, D.B., *Cohomology of Infinite-Dimensional Lie Algebras*, (A.B. Sosinskii translator), Consultants Bureau, New York, 1986.

[3] Ghys, E., “L’invariant de Godbillon-Vey,” *Séminaire Bourbaki*, 706, (1988–89).

[4] Godbillon, C., Vey, J., “Un invariant des feuilletages de codimension 1,” *Comptes Rendus Acad. Sci.*, Série A, 273, (1971), 92–95.
[5] Loday, J.-L., “Une version non commutative des algèbres de Lie: les algèbres de Leibniz,” *L’Enseignement Math.*, 39, (1993), 269–293.

[6] Loday, J.-L., “La Renaissance des Opérades,” *Séminaire Bourbaki*, 792 (1994–95).

[7] Loday, J.-L., “Overview on Leibniz Algebras, Dialgebras and Their Homology,” *Fields Institute Communications*, 17, (1997), 91–102.

[8] Loday, J.-L., Pirashvili, T., “Universal Enveloping Algebras of Leibniz Algebras and (Co)-homology,” *Math. Annalen*, 296 (1993), 139–158.

[9] Lodder, J.M., “Leibniz Cohomology for Differentiable Manifolds,” *Annales Inst. Fourier, Grenoble*, 48, 1 (1998), 73–95.

[10] Ntolo, P., “Homologie de Leibniz d’algèbres de Lie semi-simples,” *Comptes Rendus Acad. Sci.*, Série I, 318, (1994), 707–710.

[11] Pirashvili, T., “On Leibniz Homology,” *Annales Inst. Fourier, Grenoble*, 44, 2, (1994), 401–411.

[12] Thurston, W., “Noncobordant Foliations of $S^3$,” *Bulletin of the American Math. Soc.*, Vol. 78, No. 4, (1972), 511–514.

Math Sciences, Department 3MB
New Mexico State University
Las Cruces, NM 88003

e-mail: jlodder@nmsu.edu