Quantum kinetic theory for spin transport: general formalism for collisional effects

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Abstract

We systematically derive the collision term for the axial kinetic theory, a quantum kinetic theory delineating the coupled dynamics of the vector/axial charges and spin transport carried by the massive spin-1/2 fermions traversing a medium. We employ the Wigner-function approach and propose a consistent power-counting scheme where the axial-charge distribution function, a non-conserved quantity for massive particles, is accounted as the first-order quantity in the $\hbar$ expansion, while the vector-charge distribution function the zeroth-order quantity. Among the terms generally expressed with the fermion self-energies, we identify the crucial terms which are responsible for the spin-diffusion effect and the quantum effects inducing the spin polarization. We also confirm that the obtained collisional axial kinetic theory smoothly reduces to the chiral kinetic theory in the massless limit. Our general expression also reduces to a particularly simple form when the spin frame vector is fixed to the specific Lorentz frame, i.e., the rest frame of a massive fermion. As an application to the weakly coupled quark-gluon plasma at high temperature, we compute the spin-diffusion term for massive quarks up to the leading logarithmic order.
I. INTRODUCTION

The observations of global polarization of $\Lambda$ hyperons in heavy ion collisions have triggered intensive studies for the spin polarization of relativistic fermions [1, 2]. In particular, the experimental measurements have been motivated by theoretical proposals upon the polarization led by non-head-on scattering of hard partons [3] and by the thermal vorticity from the statistical model in equilibrium [4]. Although the simulations based on the modified Cooper-Frye formula for spin polarization [5, 6] have shown remarkable agreements with global polarization of $\Lambda$ [7, 8], more recent observations for local polarization caused the new tension between experiments and theories [9, 10]. Further phenomenological studies have alluded to an essential role for non-equilibrium corrections [11, 12], where the spin polarization is no longer just dictated by thermal vorticity and equilibrium distribution functions for $\Lambda$. Also, the feed-down effects are analyzed in Refs. [13, 14], which show only minor corrections upon the local polarization. It is hence imperative to understand the dynamics of the spin polarization for not only hadrons but also quarks (and even gluons) in quark gluon plasmas (QGP). In general, the current studies in theory for the dynamical spin polarization in heavy ion physics may be divided into two directions: One is to construct the so-called hydrodynamics of spin as a macroscopic effective theory incorporating spin as a hydrodynamic variable obeying angular momentum conservation [15–17]. Alternatively, non-equilibrium spin transport can be studied through quantum kinetic theory as a microscopic theory having a direct connection to the underlying quantum field theory.

In fact, the construction for quantum kinetic theory of massless fermions, known as chiral kinetic theory (CKT), was initiated by Refs. [18, 19] from Berry connection and by Refs. [20, 21] from the Wigner-function approach based on quantum field theory with the motivation to explore non-equilibrium transport in chiral matter beyond the renown anomalous phenomena in equilibrium such as the chiral magnetic/vortical effects [22–24]. There have been also plenty of followup studies for extension and applications [25–42]. In particular, the issue for Lorentz covariance associated with the side-jump phenomenon was addressed in Refs. [26, 27] and further refined in Refs. [30] from the Wigner-function approach with systematic inclusion of background electromagnetic fields and collisions in CKT. The side-jump effect is also shown to contribute to antisymmetric part of the canonical energy momentum tensor responsible for angular-momentum transfer between spin and orbital angular momentum in chiral fluids [43] (see also Ref. [44] for a related study), which further manifests the origin of side jumps in connection to spin-orbit interaction. Overall, the CKT can be regarded as a modified Boltzmann(Vlasov) equation involving quantum corrections such as the chiral anomaly, magnetic-moment coupling, and spin-orbit interaction. Moreover, the CKT has also been recently applied to the study of the $\Lambda$ polarization in heavy ion collisions [45, 46].

However, in order to consistently investigate the spin transport of $\Lambda$ or strange quarks as the seed for $\Lambda$ in QGP before hadronization, it is inevitable to consider the mass corrections. Unlike massless fermions of which the spin orientation is enslaved to the momentum direction, the spin is now a new dynamical degree of freedom coupled with the charge transport for massive fermions. There have been recent studies on the construction of free-streaming quantum kinetic theory delineating entangled dynamics between vector/axial charges and the spin polarization for massive fermions [47–51]. In particular, the axial kinetic theory in Ref. [50] obtained from the Wigner-function approach up to $O(h)$, which involves a scalar kinetic equation (SKE) and an axial-vector kinetic equation (AKE), successfully cover both the massless and massive cases.
Nevertheless, the collisional effects as an key factor for the spin polarization, have not been incorporated therein. On the other hand, in Ref. [52], the spin-diffusion term at $\mathcal{O}(\hbar^0)$ in collisions was computed in perturbative QCD (see also Refs. [53, 54] for other studies of collisions). Nonetheless, the inclusion of $\mathcal{O}(\hbar)$ corrections in collisions, which is responsible for the spin polarization, has not been obtained and the combination with the free-streaming part of quantum kinetic theory into a consistent kinetic theory for spin transport has not been achieved. Therefore, we would like to generalize the axial kinetic theory to systematically include collisions in the present work.

In this paper, by employing a particular power-counting scheme, we construct the quantum kinetic theory with collisions and background electromagnetic fields for spin transport of spin-1/2 fermions with arbitrary mass. Our power counting entails that the axial-charge distribution function $f_A$ is at $\mathcal{O}(\hbar)$ as opposed to the vector-charge distribution function $f_V$ at $\mathcal{O}(\hbar^0)$ in the $\hbar$ expansion applied in the Wigner-function formalism. However, this power counting may be applicable in most of physical systems such as heavy ion collisions. The quantum kinetic theory we obtain could be regarded as an “approximated” axial kinetic theory with collisions, for which the free-streaming part has been established in Ref. [50]. In our approach, we solve for Wigner functions and derive the kinetic equations from Kadanoff-Baym equations with collisions characterized by the self-energy based on our power-counting scheme. As an application, we further evaluate the spin-diffusion term in collisions for weakly-coupled QGP. To give a quick view and simple explanation, here we summarize our findings in a short list:

1. In our power counting, the leading-order SKE remains the same as a classical Boltzmann equation, whereas the AKE governing the dynamics of spin polarization characterized by an axial-vector component $A^\mu$ in Wigner functions can be written as $\Box^{(n)} A^\mu = \hat{\mathcal{C}}^{\mu}_{cl} + \hbar \hat{\mathcal{C}}^{(n)\mu}_Q$. Here $\Box^{(n)} A^\mu$ denotes the free-streaming AKE, while $\hat{\mathcal{C}}^{\mu}_{cl}$ and $\hbar \hat{\mathcal{C}}^{(n)\mu}_Q$ correspond to the “classical” and “quantum” parts of collisions. Such separation is explicitly shown in a generic spacetime-dependent frame and in the rest frame for massive fermions as well.

2. It is found $\hat{\mathcal{C}}^{\mu}_{cl}$ and $\hbar \hat{\mathcal{C}}^{(n)\mu}_Q$ are proportional to $f_A$ and $f_V$, respectively. Consequently, $\hat{\mathcal{C}}^{\mu}_{cl}$ serves as a spin-diffusion term, which vanishes when $f_A = 0$. On the contrary, the “quantum” correction $\hbar \hat{\mathcal{C}}^{(n)\mu}_Q$, which survives when $f_A = 0$, is dubbed as the spin-polarization term and responsible for polarizing spin via the intertwined dynamics of vector-charge transport due to spin-orbit interaction.

3. Similar to the free-streaming case [50], for a generic spacetime-dependent frame, $\hbar \hat{\mathcal{C}}^{(n)\mu}_Q$ can be separated into the term proportional to the four momentum and to the mass, which establishes a smooth connection to the CKT with collisions [30, 31] and manifests spin enslavement by chirality in the massless limit.

4. The spin diffusion term $\hat{\mathcal{C}}^{\mu}_{cl}$ for massive quarks in weakly-coupled QGP is obtained up to the leading logarithmic order, which incorporates nonlinear terms in distribution functions as a consequence of quantum statistics for fermions. It turns out that even the spin diffusion is affected by entangled dynamics between $f_V$ and $f_A$ as opposed to the previous study with only the linearized collision terms in distribution functions [52].

The paper is organized as follows: In Sec. II, we briefly review the Wigner-function approach and derive the master equations led by Kadanoff-Baym equations with collisions in the proposed
power-counting scheme. Also, the perturbative solutions for Wigner functions are found. In Sec. III, we then derive the SKE and AKE in an approximated axial kinetic theory with collisions and background electromagnetic fields as a generic formalism for studying quantum transport of fermions. Subsequently, in Sec. IV, we further investigate the spin-diffusion term in weakly-coupled QGP by utilizing our formalism. Eventually, we make concluding remarks and outlook in Sec. V. For references, we present most of details for computations and critical steps for derivations in Appendices.

II. WIGNER-FUNCTION APPROACH WITH COLLISIONS

A. Generalities

We shall start with the Wigner transformation applied to quantum expectation values of correlation functions of fermionic fields,

\[ S_{\alpha\beta}^{<}(q,X) = \int d^{4}Ye^{iqa\cdot}Y\tilde{S}_{\alpha\beta}^{<}\{(x,y), (1) \]

where \(X = (x + y)/2\) and \(Y = x - y\) and we work in the Minkowski spacetime with the mostly negative spacetime metric. Here, \(\tilde{S}_{\alpha\beta}^{\leq}(x,y) = \langle \bar{\psi}_{\beta}(x)U(y,x)\psi_{\alpha}(x)\rangle\) and \(\tilde{S}_{\alpha\beta}^{>}(x,y) = \langle \psi_{\alpha}(x)U^{\dagger}(x,y)\bar{\psi}_{\beta}(y)\rangle\) are lessor and greater propagators, respectively. To maintain the gauge invariance, we also insert the gauge link, e.g., \(U(y,x) = \exp\left(-i \int_{x}^{y}dz A_{\rho}(z)\right)\) for QED with \(A_{\rho}\) denoting the \(U(1)\) electromagnetic gauge field. Note that \(q^{\mu}\) thus represents the kinetic momentum. Hereafter, we focus on \(S^{<}(q,X)\) and suppress the indices of spinors. After the Wigner transformation, the lessor propagator obeys the Kadanoff-Baym equations derived from the Schwinger-Dyson equation,

\[ (\Pi - m)S^{<} + \gamma^{\mu}i\frac{\hbar}{2}\nabla_{\mu}S^{<} = \frac{i\hbar}{2}\left(\Sigma^{<} \star S^{>} - \Sigma^{>} \star S^{<}\right), \]

\[ S^{<}(\Pi - m) - i\frac{\hbar}{2}\nabla_{\mu}S^{<} \gamma^{\mu} = -\frac{i\hbar}{2}\left(S^{>} \star \Sigma^{<} - S^{<} \star \Sigma^{>}\right), \]

where \(\Sigma^{<}(>)\) represents the lessor (greater) self-energy. Since we only focus on the scattering process, here we drop the real parts of the retarded and advanced self-energies and of the retarded propagators. See Ref. [30] for the same setup to derive the equations for Weyl fermions. The symbol of \(\star\) represents the Moyal product incorporating higher-order corrections in \(\hbar\). The star product of two functions \(A(q,X)\) and \(B(q,X)\) are expanded as

\[ A \star B = AB + \frac{i\hbar}{2}\{A,B\}_{\text{P.B.}} + \mathcal{O}(\hbar^{2}),\]

where we define the Poisson bracket as \(\{A,B\}_{\text{P.B.}} \equiv (\partial_{\mu}A)(\partial_{\mu}B) - (\partial_{\mu}A)(\partial_{\mu}B)\). The sum and difference of Eq. (2) read

\[ \{(\Pi - m), S^{<}\} + \frac{i\hbar}{2}\left(\{\gamma^{\mu}, \nabla_{\mu}S^{<}\} - [\Sigma^{<}, S^{>}], + [\Sigma^{>, S^{<}}],_{\star}\right) = 0, \]

\[ [(\Pi - m), S^{<}] + \frac{i\hbar}{2}\left(\{\gamma^{\mu}, \nabla_{\mu}S^{<}\} - \{\Sigma^{<}, S^{>}\}, + \{\Sigma^{>, S^{<}}\},_{\star}\right) = 0, \]
Here, we introduced \( \{F, G\} \equiv FG + GF, [F, G] \equiv FG - GF, \{F, G\}^* \equiv F \times G + G \times F \) and \([F, G]^* \equiv F \times G - G \times F\), where \(F\) and \(G\) are arbitrary matrix-valued functions.

The notations and conventions in the above equations are as follows. First, the derivative operators are given as \[\nabla_\mu = \partial_\mu + j_0(\Box) F_{\nu \mu} \partial_\nu^\gamma, \quad \Pi_\mu = q_\mu + \frac{\hbar}{2} j_1(\Box) F_{\nu \mu} \partial_\nu^\gamma, \quad \Box = \frac{\hbar}{2} \partial_\mu \partial_\mu^\gamma. \quad (5)\]

We will hereafter use \(\partial_\mu \equiv \partial/\partial X^\mu\) for convenience. Here \(j_0(\Box), j_1(\Box)\) are spherical Bessel functions and \(\partial_\rho\) in \(\Box\) only act on the field strength \(F_{\nu \mu}\) when having spacetime-dependent background fields. Making the \(\hbar\) expansion, which corresponds to the gradient expansion for \(\partial_\mu \ll q_\mu\), one finds
\[
\nabla_\mu = \partial_\mu + F_{\nu \mu} \partial_\nu^\gamma - \frac{\hbar^2}{24} (\partial_\rho \partial_\lambda F_{\nu \mu}) \partial_\rho^\gamma \partial_\lambda^\nu + \mathcal{O}(\hbar^4),
\]
\[
\Pi_\mu = q_\mu + \frac{\hbar^2}{12} (\partial_\rho F_{\nu \mu}) \partial_\rho^\gamma \partial_\nu^\gamma + \mathcal{O}(\hbar^4). \quad (6)
\]

By using the complete basis for the Clifford algebra \([55, 56]\), we may decompose the Wigner functions into
\[
S^\gamma = S + i\mathcal{P} \gamma^5 + V_\mu \gamma^\mu + A_\mu \gamma^5 \gamma^\mu + \frac{S^\mu \sigma^\mu}{2} \sigma^{\mu \nu},
\]
\[
S^\gamma \bar{S} = \bar{S} + i\bar{\mathcal{P}} \gamma^5 + \bar{V}_\mu \gamma^\mu + \bar{A}_\mu \gamma^5 \gamma^\mu + \frac{\bar{S}^\mu \sigma^\mu}{2} \sigma^{\mu \nu}, \quad (7)
\]
where \(\sigma^{\mu \nu} = i[\gamma^\mu, \gamma^\nu]/2\) and \(\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3\). We shall then focus on \(V^\mu\) and \(A^\mu\), which give rise to the vector-charge and axial-charge currents through \(J^\mu_\nu = 4 \int_q V^\mu\) and \(J^5_\mu = 4 \int_q A^\mu\), where \(\int_q \equiv \int d^4q/(2\pi)^4\). In fact, from field theory, the axial-charge current can be regarded as a spin current for fermions. One may also establish a direct connection between \(A^\mu\) and the momentum spectrum of spin polarization from the modified Cooper-Frye formula \([5]\). See e.g., Refs. \([6, 43]\) and Appendix B for references. It is worthwhile to note that the axial-charge currents engendered by magnetic fields and vorticity in equilibrium, known as the chiral separation effect and axial vortical effect, could be thus pertinent to the spin polarization particularly in the case with mass corrections \([57-59]\). See also Refs. \([60-63]\) for axial-charge currents triggered by electric fields.

Similarly, it is useful to carry out the same spinor-basis decomposition for the self-energies,
\[
\Sigma^\gamma = \Sigma_S + i\Sigma_P \gamma^5 + \Sigma_V_\mu \gamma^\mu + \Sigma_A_\mu \gamma^5 \gamma^\mu + \frac{\Sigma_{T^\mu \nu}}{2} \sigma^{\mu \nu},
\]
\[
\Sigma^\gamma \bar{\Sigma} = \bar{\Sigma}_S + i\bar{\Sigma}_P \gamma^5 + \bar{\Sigma}_V_\mu \gamma^\mu + \bar{\Sigma}_A_\mu \gamma^5 \gamma^\mu + \frac{\bar{\Sigma}_{T^\mu \nu}}{2} \sigma^{\mu \nu}. \quad (8)
\]

From the Kadanoff-Baym equations and decomposition of the Wigner functions and of the self-energies, one can derive the master equations leading to the derivation of axial kinetic theory. See Appendix A for the derivation of the master equations in general forms, while further simplification is needed for practical purposes as will be carried out in the following.
B. Setting the power-counting scheme

Thus far, we have not specified the orders of $\mathcal{V}^\mu$ and $A^\mu$ in the $\hbar$ expansion. In general, they could be both comparable and different in magnitudes. In most of practical situations such as in heavy ion collisions, the axial-charge current is usually smaller than the vector-charge current since the spin polarization is basically generated by quantum effects. This observation motivates us to introduce the power counting such that $V^\mu \sim \mathcal{O}(\hbar^0)$ and $A^\mu \sim \mathcal{O}(\hbar)$. These assignments lead to $S \sim \mathcal{O}(\hbar^0)$ and $S^{\mu\nu} \sim \mathcal{O}(\hbar)$ in light of Eq. (A20) and Eq. (A17e). Also, in the free case, from Eq. (A21), we find $\mathcal{P} \sim \mathcal{O}(\hbar^2)$. Consequently, similar power counting will be applied to $\Sigma^{<\cdot>}$. From Eq. (A23a), it is clear that the ordinary Boltzmann equation at $\mathcal{O}(\hbar^0)$ comprises only $\Sigma_S$ and $\Sigma_{V^\mu}$ in collisions when $V^\mu \sim \mathcal{O}(\hbar^0)$ and $A^\mu \sim \mathcal{O}(\hbar)$, which implies $\Sigma_S \sim \mathcal{O}(\hbar^0)$ and $\Sigma_{V^\mu} \sim \mathcal{O}(\hbar^0)$ and so do $\Sigma_S$ and $\Sigma_{V^\mu}$ responsible for the “classical” collision term. On the contrary, other components in $\Sigma^{<\cdot>}$ come from quantum corrections. In light of Eq. (A23d), one finds that at most $\Sigma_A^\mu \sim \mathcal{O}(\hbar)$ and $\Sigma_P^{\mu\nu} \sim \mathcal{O}(\hbar)$ to balance the orders of free-streaming and collision parts, which is also consistent with Eq. (A21).

Physically, the “classical” Boltzmann equation only incorporates the vector-charge conservation. In order to have nonzero $\Sigma_A^\mu$ or $\Sigma_P^{\mu\nu}$, either the scattered fermion or gluon should carry nonzero chirality imbalance or spin (angular-momentum), which has to come from quantum corrections at least at $\mathcal{O}(\hbar)$ from our power counting has implied the suppression of spin currents compared to the vector-charge currents. Albeit there exists no explicit restriction for $\Sigma_P$ from generic master equations, it is expected that the presence of $\Sigma_P$ has to be induced by nonzero pseudo-scalar condensate, which should be at $\mathcal{O}(\hbar^2)$ from the consistency with the anomaly equation (mass correction upon the chiral anomaly). Nonetheless, due to the lack of a rigorous proof for the order of $\Sigma_P$ from Kadanoff-Baym equations, we will naively take $\Sigma_P \sim \mathcal{O}(\hbar^0)$ for completeness. We hence conclude that $\Sigma_S \sim \mathcal{O}(\hbar^0)$ and $\Sigma_{V^\mu} \sim \mathcal{O}(\hbar^0)$, while $\Sigma_A^\mu \sim \mathcal{O}(\hbar)$, $\Sigma_P^{\mu\nu} \sim \mathcal{O}(\hbar)$, and $\Sigma_P \sim \mathcal{O}(\hbar^0)$ in our power counting. All components certainly can also have higher-order corrections in $\hbar$ depending on the details of collisions for different systems.

C. Master equations and constrains

By implementing the power counting and the results shown in Appendix A, the master equations obtained from the Kadanoff-Baym equations at “the leading order $\mathcal{O}(\hbar^0)$” now read

\begin{align*}
\mathcal{D}^\mu \mathcal{V}_\mu &= -\frac{1}{m} q_\mu \Sigma_S \mathcal{V}^\mu, & (9a) \\
q^{[\mu} \mathcal{V}^{\nu]} &= 0, & (9b) \\
(q^2 - m^2) \mathcal{V}_\mu &= 0, & (9c) \\
q \cdot A &= -\frac{\hbar}{2m} q_\mu \Sigma_P \mathcal{V}^\mu, & (9d) \\
(q^2 - m^2) A^\mu &= \frac{\hbar}{2} \epsilon^{\alpha\beta\gamma\delta} q_\alpha D_\beta \mathcal{V}_\gamma, & (9e) \\
qu \cdot \mathcal{D} A_\mu - F_{\mu\nu} A^\nu + \frac{\hbar}{2m} q_\mu [\Sigma_P (q \cdot \mathcal{V})]_{\text{P.B.}} - \frac{h m}{2} [\Sigma_P \mathcal{V}_\mu]_{\text{P.B.}} &= \frac{\hbar}{4} \epsilon_{\mu\nu\rho\sigma} [\mathcal{D}^\nu, \mathcal{D}^\rho] \mathcal{V}^\sigma - m (\Sigma_S A_\mu - \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \Sigma_P^{\nu\rho} \mathcal{V}^\sigma) - q_\alpha \Sigma_A^\alpha \mathcal{V}_\alpha + q_\mu \Sigma_A^{\mu\alpha} \mathcal{V}_\alpha, & (9f)
\end{align*}
where we introduced shorthand notations \( \mathcal{D}^\mu \mathcal{M} \equiv \Delta^\mu \mathcal{M} + \hat{\Sigma}_V^\mu \mathcal{M} \) and \( \hat{X}Y \equiv \hat{X}Y - XY \) which goes like \( \hat{\Sigma}_V^\mu \nu = \hat{\Sigma}_V^\mu \nu - \Sigma_V^\mu \nu \) (see Appendix A). In addition, \([\Sigma_P(q \cdot V)]_{\text{P.B.}} = \{\Sigma_P,q \cdot V\}_{\text{P.B.}} - \{\Sigma_P,q \cdot \hat{V}\}_{\text{P.B.},a} \), \([\Sigma_P V^\mu]_{\text{P.B.}} = \{\Sigma_P V^\mu\}_{\text{P.B.}} - \{\Sigma_P \nu \Sigma_P^\mu\}_{\text{P.B.},a} \), and \( \epsilon_{\mu \nu \rho \sigma} \) is the totally antisymmetric tensor with \( \epsilon_{0123} = -1 \). Note that we have to keep the \( \mathcal{O}(h) \) terms linear to \( V^\mu \) in Eqs. (9e) and (9f) since \( A^\mu \sim \mathcal{O}(h) \). In addition, we consider weakly coupled systems and thus drop the \( \mathcal{O}(\Sigma^2) \) terms. Moreover, one can further implement

\[
\frac{\hbar}{2m} q_\mu \{\Sigma_P(q \cdot V)\}_{\text{P.B.}} - \frac{\hbar m}{2} [\Sigma_P V^\mu]_{\text{P.B.}} = \frac{\hbar}{2m} (\partial_\mu \Sigma_P) q \cdot V
\]

to simplify Eq. (9f).

From Eqs. (9b) and (9c), one immediately obtains the leading-order solution for the vector part,

\[
\mathcal{V}^\mu = 2\pi \delta(q^2 - m^2) q^\mu f_V,
\]

where \( f_V(q,X) \) denotes the vector-charge distribution function. For completeness, we should, in principle, multiply \( \mathcal{V}^\mu \) by the sign function for energy to include antiparticles. For notational simplicity, we will mostly focus on just the positive-energy solution throughout the paper. For \( \tilde{V}^\mu \), we simply replace \( f_V \) by \( \tilde{f}_V = 1 - f_V \) as the vector-charge distribution function for an outgoing fermion. We may also easily show that Eq. (9g) is a redundant equation, which can be derived from Eqs. (9a)-(9c). Plugging the leading-order solution for \( \mathcal{V}^\mu \) into Eq. (9a), one acquires the SKE as the usual Boltzmann equation

\[
\delta(q^2 - m^2) \left( q \cdot D f_V + q_\mu \Sigma^\mu_V f_V + m \Sigma^\mu_S f_V \right) + \mathcal{O}(h) = 0.
\]

For the axial part \( A^\mu \), we have to solve Eqs. (9d) and (9e) up to \( \mathcal{O}(h) \). With the leading-order solution (11) constrained above, one however finds that the collisional term vanishes on the left-hand side of Eq. (9e). This means that the collisional effects do not modify the dispersion relation for \( A^\mu \) at the leading order. Nevertheless, the same as the collisionless case [50], we are not able to uniquely determine the magnetization-current term at the next-to-leading order from the master equations. In the absence of a background field, one can determine the corresponding term by explicitly solving the free Dirac equation [50]. In addition, we refer to the correspondence between the cases with and without a background field in the massless limit, where the latter is given by the former with the simple replacement of the partial derivative by the derivative operator with a background field. Based on those observations, we could generalize the free-theory form to the case with background fields and collisions.

Namely, we “postulate”\(^1\)

\[
A^\mu = 2\pi \left[ \delta(q^2 - m^2) \left( a^\mu f_A + h S^\mu_{m(n)} D^\nu f_V + \frac{h}{2m} q^\mu \Sigma^\mu_P f_V \right) + h F^{\mu\nu} q_\rho \delta(q^2 - m^2) f_V \right],
\]

\(^1\)The generalization of the magnetization-current term with collisions here is, however, rather natural since \( \Sigma^\mu_V \) and \( \Sigma^\mu_P \) are only vectors at \( \mathcal{O}(h^0) \) in self-energies, which can then be coupled to \( S^\mu_{m(n)} \).
where $C_p[f_V] = -\sum_p f_V$ and

$$S^\mu_{m(n)} = \frac{\epsilon^{\mu\nu\alpha\beta} q_\alpha n_\beta}{2a \cdot n} = \frac{\epsilon^{\mu\nu\alpha\beta} q_\alpha n_\beta}{2(q \cdot n + m)},$$

(14)

which is generalization of the $A^\mu$ found in Ref. [50] with the replacement of $\Delta_\nu f_V$ by $D_\nu f_V$. Here, $a^\mu$ represents a spin four-vector satisfying $q \cdot a = q^2 - m^2$ and $f_A$ denotes the axial-charge distribution function.

At $\mathcal{O}(\hbar)$, the dispersion relation is modified by, e.g., the magnetic-moment coupling from the last term in Eq. (13) with $\delta'(q^2 - m^2) = d\delta(q^2 - m^2)/dq^2$. On the other hand, the $S^\mu_{m(n)}$ term corresponds to the so-called magnetization-current term led by spin-orbit interaction, which depends on a frame vector $n^\mu$ specifying the spin basis. The presence of such a term implies the frame dependence of $f_V$ because of the frame invariance of full $A^\mu$. In the massless limit, $a^\mu = q^\mu$ according to the spin enslavement specified by the helicity. The expression in Eq. (13) then agrees with the solution directly solved from Kadanoff-Baym equations of Weyl fermions [30]. Furthermore, for the solution of $\tilde{A}^\mu$, we simply replace $f_A$ and $f_V$ in Eq. (13) by $\tilde{f}_A$ and $\tilde{f}_V$, respectively. Nonetheless, unlike $\tilde{f}_V = 1 - f_V$, we have $\tilde{f}_A = -f_A$ due to its origin from the expectation value of the off-diagonal components of the fermionic density operator in spinor space (see Ref. [50] for a detailed definition of $f_A$ in field theory). As a consequence of our power counting, Eq. (13) corresponds to the leading-order solution for $A^\mu$ starting at $\mathcal{O}(\hbar)$. Albeit the power counting we apply, we will still dub the terms with $\hbar$ prefactors as the “quantum corrections” at $\mathcal{O}(\hbar)$ or the “$\mathcal{O}(\hbar)$ terms” throughout this paper for convenience.

III. APPROXIMATED AXIAL KINETIC THEORY WITH COLLISIONS

A. Axial kinetic equation with general frame vector and its massless limit

We now utilize Eqs. (9f) and (13) to derive the AKE. Given that the collisionless part up to $\mathcal{O}(\hbar)$ has been obtained in Ref. [50], we only need to further work out the collisional part. From Eq. (9f), by using

$$\epsilon_{\mu\nu\sigma\rho}[D^\nu, D^\rho]\mathcal{V}^\sigma = \epsilon_{\mu\nu\sigma\rho} \left( [\Delta^\nu, \Delta^\rho] \mathcal{V}^\sigma + 2(\Delta^\nu \Sigma^\rho) \mathcal{V}^\sigma \right)$$

(15)

and the solution in Eq. (13), the AKE with collisions then takes the form, for $n^\mu = n^\mu(X)$,

$$\Box^{(n)} A^\mu = G_1^{(n)\mu} + \hbar G_2^{(n)\mu}.$$  

(16)

Here the free-streaming part is given by [50]

$$\Box^{(n)} A^\mu = \delta(q^2 - m^2) \left( q \cdot \Delta(a^\mu f_A) + F^\mu a_\nu f_A \right) + \hbar q^\mu \left\{ \delta(q^2 - m^2) \left[ (\partial_\alpha S_{m(n)}^{\alpha\nu}) \Delta_\nu + \frac{S_{m(n)}^{\alpha\nu} F_{\alpha\beta n^\beta \Delta_\nu}}{q \cdot n + m} \right] + \frac{S_{m(n)}^{\alpha\nu} (\partial_\alpha F_{\beta\nu}) q^2}{q \cdot n + m} \right\} f_V$$

$$+ \hbar m \left[ \frac{\delta(q^2 - m^2) \epsilon^{\mu\nu\alpha\beta}}{2(q \cdot n + m)} \left( m(\partial_\alpha n_\beta) \Delta_\nu + (mn_\beta + q_\beta) \frac{(F_{\alpha\rho n^\rho - \partial_\alpha (q \cdot n)) \Delta_\nu}{q \cdot n + m} \right) \right],$$

8
\[-(\partial_\nu F_{\rho\alpha})\partial_{q}\right) \right] + \delta'(q^2 - m^2) \left( \frac{mn_\beta + q_\beta}{q \cdot n + m} q \cdot \Delta \right) f_V. \tag{17}\]

Taking \( \hbar = 0 \), the equation above reproduces the so-called Bargmann-Michel-Telegdi (BMT) equation as a classical kinetic equation for spin transport \cite{64}. On the other hand, the collision terms are

\[
\hat{\mathcal{C}}_1^{(n)\mu} = \delta(q^2 - m^2) \left[ -a^\mu q_\nu \Sigma_\nu f_A - m^2 \Sigma_\mu f_V + q^\mu q_\nu \Sigma_\nu f_V - m \left( a^\mu \Sigma_A f_A - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} q_\nu \Sigma_{T\rho\sigma} f_V \right) \right], \tag{18}\]

and

\[
\hat{\mathcal{C}}_2^{(n)\mu} = \frac{\delta(q^2 - m^2)}{2} \left[ \epsilon^{\mu\nu\rho\sigma} q_\nu (\Delta_\rho \Sigma_{V\sigma}) f_V - m (\partial^\mu \Sigma_{\nu}) f_V - 2 \Sigma_{V\nu} f_V \left( q \cdot \Delta S_{(m(n))}^{\mu\nu} - F_{\lambda\mu} S_{(m(n))}^{\lambda\nu} \right) \right] \]
\[-2 S_{m(n)}^{\mu\nu} \left( q \cdot \Sigma_\nu f_V + m \Sigma_\mu (\Delta_\nu f_V) + (q \cdot \Delta \Sigma_\nu f_V) f_V \right) + \frac{1}{m} q^\mu (q \cdot \Delta \Sigma_\mu f_V) f_V \]
\[-S_{m(n)}^{\mu\nu} \Delta_\nu \left( \delta(q^2 - m^2) q \cdot \Delta f_V \right) - \delta(q^2 - m^2) (\Delta_\nu S_{m(n)}^{\mu\nu}) q \cdot \Delta f_V \]
\[-F^{\mu\nu} q_\mu \delta'(q^2 - m^2) (q \cdot \Sigma_\nu f_V + m \Sigma_\mu f_V) \tag{19}\]

Note that \( \hat{\mathcal{C}}_1^{(n)\mu} \) implicitly contains the \( \mathcal{O}(\hbar) \) terms from, e.g., \( \Sigma_\alpha \) and \( \Sigma_{T\rho\sigma} \). By analogy with the collisionless case, we further decompose \( \hat{\mathcal{C}}_2^{(n)\mu} \) into the piece proportional to \( q^\mu \), which survives in the massless limit and reproduces the collision term in CKT, and another piece proportional to \( m \), which stems from the purely finite-mass correction. Carrying out complicated yet straightforward calculations as shown in Appendix C, we can rewrite \( \hat{\mathcal{C}}_2^{(n)\mu} \) as

\[
\hat{\mathcal{C}}_2^{(n)\mu} = q^\mu \hat{\mathcal{C}}_{q2} + m \hat{\mathcal{C}}_{m2}, \tag{20}\]

where

\[
\hat{\mathcal{C}}_{q2} = \delta(q^2 - m^2) \left[ -S_{m(n)}^{\rho\nu} (\Delta_\rho \Sigma_{V\nu}) f_V - \Sigma_{V\nu} f_V \left( \partial_\alpha S_{m(n)}^{\rho\nu} + \frac{S_{m(n)}^{\rho\nu} F_{\rho\sigma} n^\sigma}{q \cdot n + m} \right) \right] \]
\[+ \frac{1}{2m} (q \cdot \Delta \Sigma_\nu f_V) \right] - \delta'(q^2 - m^2) S_{m(n)}^{\rho\nu} F_{\rho\nu} C_V[f_V], \tag{21}\]

and

\[
\hat{\mathcal{C}}_{m2} = \delta(q^2 - m^2) \left[ (\Delta_\nu S_{m(n)}^{\mu\nu}) C_S[f_V] - S_{m(n)}^{\mu\nu} (\Delta_\nu \Sigma_\nu f_V) + \frac{\epsilon^{\mu\nu\rho\sigma} (q_\rho + mn_\rho)}{2(q \cdot n + m)} (\Delta_\sigma \Sigma_{V\nu}) f_V \right] \]
\[-\frac{1}{2} (\partial^\mu \Sigma_{\nu}) f_V - \frac{\epsilon^{\mu\nu\alpha\beta} \Sigma_{\nu} f_V}{2(q \cdot n + m)} \left( m \partial_\alpha n_\beta - (q_\beta + mn_\beta) (\partial_\alpha q \cdot n + n^\rho F_{\rho\alpha}) q \cdot n + m \right) \right] \]
\[+ \delta'(q^2 - m^2) \left[ \frac{(q_\rho + mn_\rho) \tilde{F}_{\rho\mu}}{q \cdot n + m} C_V[f_V] - 2 S_{m(n)}^{\mu\nu} q^\lambda F_{\lambda\nu} C_S[f_V] + \tilde{F}_{\mu\nu} q_\mu \tilde{C}_S[f_V] \right]. \tag{22}\]
Here, $C_V[f_V] \equiv -q_\mu \Sigma^\mu_{V'} f_V$ and $C_S[f_V] \equiv -\Sigma_S f_V$. It is clear to see that $\hat{C}^{(n)}_q$ reduces to the $O(h)$ correction in the collision term of the CKT found in Refs. [30, 31] by taking $m = 0$ \(^2\). Combining with $\hat{C}_1^{(n)\mu}$ and collisionless part, one finds that the CKT with collisions is smoothly reproduced by AKE in the massless limit when $m = 0$ and $a^\mu = q^\mu$.

### B. Rest-frame expression

Notably, when focusing on massive fermions, we can set the frame vector at their rest frame $n^\mu = n^\mu_r(q) = q^\mu/m$ to simplify both the Wigner functions and AKE. This frame choice is rather different from the previous one when $n^\mu(X)$ only depends on spacetime coordinates. In such a case, the magnetization-current term in $A^\mu$ vanishes and the $A^\mu$ reduces to

\[ A^\mu = 2\pi \left[ \delta(q^2 - m^2) a^\mu f_A + \hbar \tilde{F}^{\mu\nu} q_\nu \delta'(q^2 - m^2) f_V + \frac{\hbar \delta(q^2 - m^2)}{2m} q^\mu C_P[f_V] \right]. \] (23)

Accordingly, the AKE becomes

\[ \Box^{(n_r)} A^\mu = \hat{C}_1^{(n_r)\mu} + \hbar \hat{C}_2^{(n_r)\mu}, \] (24)

where $\hat{C}_1^{(n_r)\mu}$ is the same as Eq. (18) by taking $n^\mu = n^\mu_r$, while

\[ \Box^{(n_r)} A^\mu = \delta(q^2 - m^2) \left( q \cdot \Delta (a^\mu f_A) + F^{\mu\nu} a_\nu f_A - \frac{1}{2} \hbar e^{\mu\nu\rho\sigma} q_\rho (\partial_\sigma F_{\beta\nu}) \partial_\beta f_V \right) \]

\[ + \hbar \tilde{F}^{\mu\nu} q_\nu \delta'(q^2 - m^2) q \cdot \Delta f_V \] (25)

and

\[ \hat{C}_2^{(n_r)\mu} = \frac{\delta(q^2 - m^2)}{2} \left( \epsilon^{\mu\nu\rho\sigma} q_\nu (\Delta_\rho \Sigma_\Sigma_{\nu\sigma} f_V) + \frac{1}{m} q^\mu (q \cdot \Delta \Sigma_{\nu\sigma} f_V - m(\partial^\nu \Sigma_\nu f_V) f_V \right) \]

\[ - \tilde{F}^{\mu\nu} q_\nu \delta'(q^2 - m^2) (q \cdot \Sigma_{\nu\sigma} f_V + m \Sigma_S f_V) \] (26)

In general, when considering also the $\hbar$ corrections in $\mathcal{V}^\mu$, such a frame choice is only valid when $m$ is much larger than the gradient and electromagnetic scales in the system. The magnetization-current term in $\mathcal{V}^\mu$ explicitly reveals the breakdown for the choice of a rest frame away from the aforementioned regime. Although we do not explicitly include the $O(\hbar)$ term in $\mathcal{V}^\mu$ based on our power counting, it is still essential to be aware of the valid regime for the frame choice $n^\mu = n^\mu_r$. In heavy-ion phenomenology, one may assume the validity is held, which could be somewhat applicable for the spin transport of strange quarks in QGP. For up and down quarks or other applications, it is inevitable to maintain the general frame vector $n^\mu = n^\mu(X)$. Note that the quantum correction on collisions in Eq. (26) is also present in Eq. (19), we may regard other terms in Eq. (19) as the $O(\Delta^\mu/m)$ corrections on top of Eq. (26).

\(^2\) In Refs. [30, 31], $\Sigma_\rho$ and $\Sigma_{\rho}$ are set to zero. As argued previously, such terms are actually expected to be at higher order and vanishing in the massless limit.
C. $h$ sorting with the present order counting

In order to explicitly disentangle the (classical) spin-diffusion and (quantum) spin-polarization parts in collisions, we have to retrieve the $h$ terms in $\Sigma_{V\rho}$, $\Sigma_{S}$, $\Sigma_{A\rho}$ and $\Sigma_{T\rho\sigma}$. That is, we have to further make the decomposition $\Sigma_{V\rho} = \Sigma_{V\rho}^{cl} + h\Sigma_{V\rho}^{Q(n)}$, $\Sigma_{S} = \Sigma_{S}^{cl} + h\Sigma_{S}^{Q(n)}$, $\Sigma_{A\rho} = \Sigma_{A\rho}^{cl} + h\Sigma_{A\rho}^{Q(n)}$ and $\Sigma_{T\rho\sigma} = \Sigma_{T\rho\sigma}^{cl} + h\Sigma_{T\rho\sigma}^{Q}$, where their explicit forms depend on the details of collisions in systems. Nonetheless, $h\Sigma_{V\rho}^{Q(n)}$ and $h\Sigma_{S}^{Q(n)}$ are coupled to $f_{A}$ in $\hat{C}_{1}^{(n)\mu}$, which actually contribute to $O(h^{2})$ corrections from our power counting. Consequently, we only have to retain the quantum corrections from $h\Sigma_{A\rho}^{Q(n)}$ and $h\Sigma_{T\rho\sigma}^{Q}$. One can then rewrite the collision term in the AKE as

$$\hat{C}_{1}^{(n)\mu} + h\hat{C}_{2}^{(n)\mu} = \hat{C}_{cl}^{\mu} + h\hat{C}_{Q}^{\mu},$$

where

$$\hat{C}_{cl}^{\mu} = \delta(q^{2} - m^{2}) \left[ q^{\mu}q_{\alpha}\Sigma_{A}^{cl}\tilde{f}_{V} - m^{2}\Sigma_{A}^{cl}f_{V} - a^{\mu}(q_{\alpha}\Sigma_{V}^{cl}f_{A} + m\Sigma_{S}^{cl}f_{A}) + m\frac{\epsilon^{\mu\nu\rho\sigma}}{2}q_{\nu}\Sigma_{T\rho\sigma}^{cl}f_{V} \right]$$

and

$$h\hat{C}_{Q}^{(n)\mu} = hq^{\mu}\left( \tilde{c}_{q_{2}}^{(n)} + \delta(q^{2} - m^{2})q_{\nu}\Sigma_{A}^{Q(n)\nu}f_{V} \right) + hm\left[ \tilde{c}_{m_{2}}^{(n)\mu} + \delta(q^{2} - m^{2})\left( \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}q_{\nu}\Sigma_{T\rho\sigma}^{Q(n)}f_{V} - m\Sigma_{A}^{Q(n)\mu}f_{V} \right) \right].$$

Analogously, in the rest frame, it is found

$$h\hat{C}_{Q}^{(n_{r})\mu} = h\hat{C}_{2}^{(n_{r})\mu} + h\delta(q^{2} - m^{2}) \left[ q^{\mu}q_{\alpha}\Sigma_{A}^{Q(n_{r})\alpha}\tilde{f}_{V} - m^{2}\Sigma_{A}^{Q(n_{r})\nu}f_{V} + m\frac{\epsilon^{\mu\nu\rho\sigma}}{2}q_{\nu}\Sigma_{T\rho\sigma}^{Q(n_{r})}f_{V} \right].$$

Note that the classical part $\hat{C}_{cl}^{\mu}$ is explicitly frame independent. Now, all the $O(h)$ terms are collected into $h\hat{C}_{Q}^{(n)\mu}$. Despite complication, one immediately finds that $\hat{C}_{cl}^{\mu}$ is proportional to $a^{\mu}f_{A}$; such a term hence results in the diffusion of spin. On the contrary, $\hat{C}_{Q}^{(n)\mu}$ is instead proportional to $f_{V}$ and $\tilde{f}_{V}$. Even when initial spin ($\sim a^{\mu}f_{A}$) is zero, such a term can lead to the spin polarization from the entangled vector-charge transport.

IV. SPIN DIFFUSION OF QUARKS IN WEAKLY COUPLED QGP

A. Scattering between massive fermions and a medium

We now apply the formalism established in the previous sections to investigate the collision term for massive quarks traversing weakly-coupled QGP in relativistic heavy ion collisions. For simplicity, we will just focus on the spin-diffusion term such as $\hat{C}_{cl}^{\mu}$ in the AKE and leave the $h\hat{C}_{Q}^{(n)\mu}$ for future study. Recently, a related study for spin diffusion has been presented in Ref. [52] with a different approach. We will mostly follow the theoretical setup therein. In the following, we call fermions quarks and gauge bosons gluons interchangeably, and include the color-group factors. Here, the color degrees of freedom do not play crucial roles (like in the color conductivity), and the same computation holds for QED with simple replacements of the relevant degrees of
freedom. Furthermore, we consider the massive quarks with quark mass much greater than the scale of thermal mass in QGP and accordingly neglect the Compton scattering with gluons as the subleading effects analogous to the study of heavy-quark transport in heavy ion collisions (See e.g. Ref. [65] and the same approximation in Ref. [52]).

The gluon-exchange processes between a massive fermion and the medium constitutes are written down as

$$\Sigma^{(>)}(q, X) = \lambda_c \int \gamma^\mu S^{(>)}(q', X)\gamma^\nu G^{(>)}_{\mu\nu}(q - q', X),$$  \hspace{1cm} (31)

where $$\lambda_c$$ denotes an overall coefficient including the coupling and we drop $$\mathcal{O}(\hbar^2)$$ and higher-order correction in our order counting. The gluon propagator $$G^{(>)}_{\mu\nu}$$ contains the information of the spectral functions which depend on the scatterers. Here, we assume a dilute population of the massive fermions in the medium and neglect the contributions of the massive scatterers, and retain the massless-fermion and gluon scatterers. Having assumed the weakly coupled system, we focus on the the lowest-order contributions in the coupling constant $$g_c$$, i.e., the 2-to-2 scatterings between a massive quark and a massless quark/gluon.

Inserting Eq. (7), one can straightforwardly decompose the gamma structures in Eq. (31) as

$$\chi^{(>)}_{q\mu
u} = \gamma^\mu S^{(>)}(q')\gamma^\nu$$

Thus, contracted with the gluon propagator, we have

$$\chi^{(>)}_{q\mu
u}G^{(>)}_{\mu\nu} = (S\gamma^\mu + i\tilde{S}\gamma^\mu G^{(>)}_{\mu\nu}) + i\gamma^\nu \left( -\tilde{S}\gamma^\mu G^{(>)}_{\mu\nu} - i\tilde{S}\alpha\beta G^{(>)}_{\mu\nu} \frac{\epsilon^{\mu\alpha\beta\gamma}}{2} \right)$$

$$+ \gamma^\mu (\tilde{V}^\mu (G^{(>)}_{\mu\nu} - \tilde{V}^\mu G^{(>)}_{\mu\nu} - \epsilon_{\mu\sigma\rho\nu}\tilde{A}^\sigma G^{(>)}_{\mu\nu}))$$

$$+ \gamma^\mu (\tilde{A}^\nu (G^{(>)}_{\mu\nu} - \tilde{A}^\nu G^{(>)}_{\mu\nu} + \epsilon_{\mu\sigma\rho\nu}\tilde{V}^\sigma G^{(>)}_{\mu\nu}))$$

$$+ \frac{1}{2} \epsilon_{\sigma\rho\sigma} \left( 2\tilde{S}\rho G^{(>)}_{\sigma\mu} - \tilde{S}G^{(>)}_{\rho\mu} - 2i\tilde{S}G^{(>)}_{\rho\mu} - i\tilde{P}\epsilon_{\mu\sigma\rho} G^{(>)}_{\sigma\mu} \right),$$  \hspace{1cm} (33)

where $$A_{\mu B_{\nu}} = A_{\mu}B_{\nu} + A_{\nu}B_{\mu}$$. We consider gluon propagators $$G^{(>)}_{\mu\nu}$$ symmetric in the Lorentz indices, and then find that the imaginary terms in the above vanish in the contractions:

$$\Sigma^{(>)}(q) = \lambda_c \int \left[ S\gamma^\mu - i\tilde{P}G^{(>)}_{\mu\nu}\gamma^\nu + \gamma^\mu (2\tilde{V}^\mu G^{(>)}_{\mu\nu} - \tilde{V}_\mu G^{(>)}_{\mu\nu}) $$

$$+ \gamma^\mu (2\tilde{A}^\nu G^{(>)}_{\mu\nu} - \tilde{A}_\nu G^{(>)}_{\mu\nu} + \epsilon_{\mu\sigma\rho\nu}\tilde{V}^\sigma G^{(>)}_{\mu\nu})) + \frac{\epsilon_{\sigma\rho\sigma}}{2} \left( 2\tilde{S}\rho G^{(>)}_{\sigma\mu} - \tilde{S}G^{(>)}_{\rho\mu} - 2i\tilde{S}G^{(>)}_{\rho\mu} - i\tilde{P}\epsilon_{\mu\sigma\rho} G^{(>)}_{\sigma\mu} \right) \right],$$  \hspace{1cm} (34)

where $$A_{[\mu B_{\nu}] = A_{\mu}B_{\nu} - A_{\nu}B_{\mu}}$$. In general, $$G^{(>)}_{\mu\nu}$$ possibly contains anti-symmetric components led by scatterings with spin-polarized scatterers in the medium. For example, when considering the
scattering with massless quarks, such anti-symmetric components can arise from the side-jump terms, whereas $\chi^\mu_\nu G^\mu_\nu$ should still remain real. Such quantum corrections from a polarized medium will not be considered in the present work.

One can now read out the corresponding terms up to $\mathcal{O}(h)$ between Eqs. (8) and (34) as

$$\dot{\Sigma}_S = \frac{\lambda_c}{m} \int_{q'} S_q G^\alpha_{\gamma} = \frac{\lambda_c}{m} \int_{q'} q' \cdot \bar{V}_q q_\alpha$$

$$\dot{\Sigma}_P = \frac{\lambda_c}{m} \int_{q'} (\bar{P} q') G^\alpha_{\gamma},$$

$$\dot{\Sigma}_{V\mu} = \frac{\lambda_c}{m} \int_{q'} \bar{V}_q a\gamma (2G^\alpha_{\mu} - G^\gamma_{\mu} \eta_{\alpha\mu}),$$

$$\dot{\Sigma}_{A\mu} = \frac{\lambda_c}{m} \int_{q'} \bar{A}_q a\gamma (-2G^\alpha_{\mu} + G^\gamma_{\mu} \eta_{\alpha\mu}),$$

$$\dot{\Sigma}_{T\mu\nu} = \frac{\lambda_c}{m} \int_{q'} (2\bar{S}_{a\gamma} G_{\nu} - \bar{S}_{a\gamma} G^\alpha_{\nu}) = \frac{\lambda_c}{m} \int_{q'} (2\bar{S}_{a\gamma} G_{\nu} - \bar{S}_{a\gamma} G^\alpha_{\nu}).$$

The rightmost sides are obtain by using Eqs. (A19)-(A21) up to the linear orders in $\Sigma$’s and $\bar{h}$. In the present case, we confirm that $\dot{\Sigma}_P$ is at $\mathcal{O}(h^2)$ as anticipated earlier.

Now, given explicit forms of $G^\mu_\nu$, $\bar{V}$, and $\bar{A}$, we can directly evaluate $\dot{\Sigma}_{\mu}$ from Eq. (35) and $\Sigma_{<\mu}$ in the same fashion. Inserting these expressions into the collision terms in the SKE (12), we have

$$(q \cdot \dot{\Sigma}_V + m \dot{\Sigma}_S) = \frac{\lambda_c}{m} \int_{q'} 2p \delta(q' - q) (2q^\mu G^\nu_{\mu} - p \cdot q G^\nu_{\mu}) \bar{f}_V q',$$  \hspace{1cm} (36)

where $p^\mu = q^\mu - q'^\mu$. The other term $(q \cdot \dot{\Sigma}_V + m \dot{\Sigma}_S)$ takes a similar form. In addition, by making the decompositions $\Sigma_{A\rho} = \dot{\Sigma}_{A\rho} + \hbar \dot{\Sigma}_{A\rho}^{(n)}$ and $\Sigma_{T\rho\sigma} = \dot{\Sigma}_{T\rho\sigma} + \hbar \dot{\Sigma}_{T\rho\sigma}^{(n)}$, we identify the classical and quantum parts. From the classical part in Eq. (13), Eqs. (35d) and (35e) yield

$$\dot{\Sigma}_{A\rho} = \frac{\lambda_c}{m} \int_{q'} 2p \delta(q' - q) (2a_\rho G^\nu_{\mu} - 2a_\rho G^\nu_{\mu}) \bar{f}_A q',$$

$$\dot{\Sigma}_{T\rho\sigma} = -\frac{\lambda_c}{m} \int_{q'} 2p \delta(q' - q) (2G^\nu_{\mu} \epsilon_{\mu\nu}\eta_{\alpha\beta} + G^\nu_{\mu} \epsilon_{\rho\gamma}\eta_{\alpha\beta}) q'^\alpha a_\rho \bar{f}_A q'.$$

Those terms are further investigated with a specific gluon propagator provided by the hard-thermal loop approximation in the next section. The quantum parts are also identified in the same way. However, computation of those quantum corrections with specific gluon propagators are left as open issues. Note also that we have dropped possible antisymmetric parts of the gluon propagator in Eq. (34).

**B. Weakly coupled QGP and hard-thermal-loop approximation**

Although Eqs. (35a)-(35e) work for even non-equilibrium media, we now focus on the spin transport in equilibrium QGP as a concrete example. In such a case, we can make further
simplification for the self-energy in Eq. (31). For simplicity, we will only evaluate the spin-diffusion terms in the SKE and AKE, while the calculation for the spin-polarization term in the AKE is more involved, which will be presented in the followup work. We will implement the hard-thermal-loop (HTL) approximation, which allows us to derive the leading-logarithmic result in weakly coupled QCD as in the derivation shown in Ref. [52] from a distinct approach. Recall that \( p^\mu = (q - q')^\mu \) and \( g_c \) denotes the coupling constant for strong interaction. We then apply the cut-gluon propagator in the Coulomb gauge and the fluid-rest frame, which gives rise to

\[
G^<_{\mu \nu}(q, q') \approx g_{\text{cgp}} \left[ \rho_L(p) P^C_{\mu \nu} + \rho_T(p) p_{\mu \nu}^T \right] \tag{39}
\]

with

\[
P^C_{\mu \nu} \equiv u_\mu u_\nu, \quad P^T_{\mu \nu} \equiv -\Theta_{\mu \alpha} \Theta_{\nu \beta} \left( \eta^{\alpha \beta} + \frac{p^\alpha p^\beta}{|p|^2} \right) = -\left( \Theta_{\mu \nu} + \frac{p_{\perp \mu} p_{\perp \nu}}{|p|^2} \right) \tag{40}
\]

where \( g_{\text{cgp}} \equiv g_{0p} = 1/(e^{\beta p^0} - 1) \), \( g_{\text{cgt}} \equiv 1 + g_{0p} \) and \( \Theta_{\mu \nu} \equiv \eta_{\mu \nu} - u^\mu u^\nu \). Here, \( u^\mu \) and \( \beta = 1/T \) denote the fluid four velocity and the inverse of temperature in local equilibrium, respectively.

We have introduced notations:

\[
V^0 \equiv V \cdot u, \quad V^\mu_\perp \equiv V^\mu - V^0 u^\mu = \Theta^{\mu \nu} V_\nu, \tag{41a}
\]

\[
V^i \equiv V^i_\perp, \quad \hat{V}^i \equiv \hat{V}^i_\perp \equiv V^i / |V|, \tag{41b}
\]

for an arbitrary vector \( V^\mu \). Then, we have \( V^\perp \cdot k_\perp = -V \cdot k \) and, especially, \( V^2_\perp = -|V|^2 \) when \( k^\mu = V^\mu \).

In our setup, the HTL approximation is more precisely applied to \( g_c T \ll |p^\mu| \ll T \). On the other hand, \( \rho_L/T(p) \) correspond to the HTL gluon spectral densities, which take explicit forms as (e.g., see Ref. [66])

\[
\rho_L(p) \approx \frac{\pi m_D^2 p_0}{|p|^5}, \quad \rho_T(p) \approx \frac{\pi m_D^2 p_0}{2|p|^5 \left( 1 - \left( \frac{m^2}{|p|^2} \right) \right)}, \tag{42}
\]

where \( m_D \sim g_c T \) corresponds to the Debye mass. The explicit form from the gluons in \( SU(N_c) \) color group and the \( N_f \)-flavored massless quarks is given by

\[
m_D^2 = \frac{g_c^2 T^2 (2N_c + N_f)}{6}, \quad \lambda_c = g_c^2 C_2(F) = \frac{g_c^2 (N_c^2 - 1)}{2N_c}. \tag{43}
\]

Moreover, one should keep in mind the relation

\[
G^>_{\mu \nu}(p) = (1 + g_{0p}^{-1}) G^<_{\mu \nu}(p) \tag{44}
\]

from detailed balance. In light of the theoretical frameworks constructed in the previous section, we may now write down the SKE and AKE in the HTL approximation. It is interesting that one can linearize the kinetic equations in terms of the distribution functions by taking \( g_{0p}^{-1} \to 0 \) and \( G^>_{\mu \nu}(p) \approx G^<_{\mu \nu}(p) \). However, in the practical calculation, we have to at least approximate \( g_{0p} \approx T/p_0 - 1/2 + p_0/(12T) + \mathcal{O}(p_0^3/T^3) \) to keep all relevant terms contributing to the leading
logarithmic order. In the following, we will append subindices to \( f_{V/A} \) and \( a^\mu \) for specifying their momentum dependence.

For the SKE in Eq. (12), it is found

\[
0 = \delta(q^2 - m^2) \left[ q \cdot \Delta f_{Vq} + \lambda_c \int_p Q_1(q, p) \{(1 + g_{0p}^{-1}) \tilde{f}_{Vq} f_{Vq} - \tilde{f}_{Vq} f_{Vq'} \} \right],
\]

(45)

where

\[
Q_1(q, p) = 2\pi \delta(q^2 - m^2) \left( 2q^\mu G^{<\mu}_\nu q^\nu - p \cdot q' G^{<\mu}_\nu \right).
\]

(46)

By using \( \tilde{f}_{Vq} = 1 - f_{Vq} \), we may further rewrite Eq. (45) into

\[
0 = \delta(q^2 - m^2) \left[ q \cdot \Delta f_{Vq} + \lambda_c \int_p \left\{ Q_1(f_{Vq} - f_{Vq-p}) + \tilde{Q}_1 f_{Vq}(1 - f_{Vq'}) \right\} \right],
\]

(47)

where \( \tilde{Q}_1 = g_{0p}^{-1} Q_1 \). One can then further approximate \( f_{Vq} - f_{Vq-p} \approx p^\mu \partial_{q^\mu} f_{Vq} + \mathcal{O}(|p|/|q|) \) assuming the small-momentum transfer \( |p| \ll |q| \).

Next, we can simplify the AKE with the same approximation. Inserting the rightmost sides of Eq. (35) into Eq. (28), we obtain

\[
\delta(q^2 - m^2) \left[ q \cdot \Delta \tilde{a}_q^\mu + F^{\mu\nu} \tilde{a}_{q\nu} - \tilde{C}_c^{\mu} \right] = 0,
\]

(48a)

\[
\tilde{C}_c^{\mu} = \lambda_c \int_p \left\{ -Q_1 \tilde{a}_q^\mu - \tilde{Q}_1 (1 - f_{Vq-p}) \tilde{a}_q^\mu + Q_2^{\mu\nu} \tilde{a}_{q-p\nu} + \tilde{Q}_2^{\mu\nu} f_{Vq} \tilde{a}_{q-p\nu} \right\},
\]

(48b)

where \( \tilde{a}_q^\mu \equiv a^\mu f_{AQ}^{\mu} \), \( \tilde{Q}_2^{\mu\nu} = g_{0p}^{-1} Q_2^{\mu\nu} \), and

\[
Q_2^{\mu\nu} = -2\pi \delta(q^2 - m^2) \left[ \left( p^\mu q^\nu - \eta^{\mu\nu} q \cdot p \right) G^{<\nu}_\rho q^\rho - 2 \left( p^\mu G^{<\nu}_\rho q^\rho - q \cdot p G^{<\nu}_\rho \right) \right. \\
\left. + 2 \left( q^\nu G^{<\rho\sigma} q^\rho - \eta^{\nu\rho} q^\sigma G^{<\rho\sigma} \right) \right] .
\]

(49)

Here we also use \( \tilde{f}_{AQ} = -f_{AQ} \). One can similarly approximate \( \tilde{a}_{q-p\nu} \approx \tilde{a}_{q\nu} - p^\beta \partial_{q^\beta} \tilde{a}_{q\nu} \). Such a diffusion term in collisions of the AKE has also been constructed in Ref. [52] from a distinct approach, in which different parameterization of the spin vector is applied. Nonetheless, it may be more practical to adopt our parameterization for the spin vector, which has a direct connection to the axial-charge current equivalent to the spin polarization through the Wigner functions and the combination with the free-steaming part of the AKE. Note also that Eqs. (47) and (48) contain nonlinear terms in distribution functions. While those nonlinear terms are not included in Ref. [52], they are imperative to preserve the quantum statistics for fermions. For example, as will be shown, \( f_{Vq} \) follows the Fermi-Dirac distribution instead of just the Boltzmann distribution in equilibrium with the vanishing collision term in the SKE. In addition, as shown in Eq. (48), the nonlinear terms further reveal the entangled dynamics between the vector/axial charges and spin diffusion.
C. SKE and AKE with diffusion effects in the leading-log approximation

We now explicitly compute the collision terms in axial kinetic theory with the HTL approximation. Notations have been introduced in Eq. (41). The basic strategy is to collect all the terms up to $\mathcal{O}(|p|^{-3})$ in the integrand. When combined with the integral measure, they give rise to the leading logarithmic results in $\alpha$ with the cut-offs provided by the HTL resummation. Moreover, we will consider the onshell kinetic equations. We hence take $f_{Vq} = f_{Vq}(q, X)$ as just a function of $q$ and $X$ by using $q_0 = E_q = \sqrt{|q|^2 + m^2}$ for fermions (here we neglect anti-fermions) in the Wigner functions. Similarly, we take $u \cdot \bar{a}_q = -q_\perp \cdot \bar{a}_q/q_0$ for $\bar{a}_{qu} = \bar{a}_{qu}(q, X)$.

1. Results

The computations for the diffusion terms in SKE and AKE are complicated yet straightforward. We hence present the details of computations for Eqs. (45) and (48) in Appendices D and E, respectively. In the following, we just summarize the final results. Up to the leading logarithmic order in $\alpha_c$, the SKE takes the form

$$0 = \delta(q^2 - m^2) \left[ q \cdot \Delta - \kappa_{LL} \left\{ 2(1 - f_{Vq}) + S^{(1)}_q \partial_{q^\perp} \bar{a}_q + S^{(2)} q^\alpha \partial_{q^\perp} \bar{a}_q + S^{(3)} \eta^\alpha \partial_{q^\perp} \bar{a}_q \right\} \right] f_{Vq},$$

(50)

where we denote the coefficient of the leading log result $\kappa_{LL} \equiv [g_c^2 C_2 (F) m_0^2 / (8\pi)] \ln(1/g_c)$ and the Minkowski metric $\eta^{\alpha \beta}$. We also introduce the four “velocity” $v^\mu = (v^0, v^i)$ $\equiv q^\mu/m$, which has the normalization $v^\mu v_\mu = 1$ under the delta function, and then the rapidity $\eta_q \equiv 2^{-1} \ln[(E_q + |q|) / (E_q - |q|)] = 2^{-1} \ln[(v^0 + |v|) / (v^0 - |v|)]$. The coefficient in each term is given as

$$S^{(1)} = \frac{m T \theta_0}{|v|^2} (1 - 2 f_{Vq}), \quad S^{(2)} = \frac{m T \theta_0}{2 |v|^3} \left( \frac{|v|^2 \eta_\bar{a}}{v_0^3} + \frac{3 \theta_1}{v_0} \right), \quad S^{(3)} = \frac{m T}{2} \left( \frac{v_0^3 \theta_3}{|v|^3} - 3 v_0 \right),$$

(51)

where $\theta_v \equiv |v| - v^0 \eta_q$. Note that Eq. (50) agrees with the result in Ref. [52] except for additional nonlinear terms in $f_{Vq}$ coming from Fermi-Dirac statistics.

Carrying out similar yet more sophisticated computations for Eq. (48), we also derive the AKE with the spin-diffusion term up to leading-logarithmic order. Combining with the classical free-streaming part dictated by the BMT equation, the AKE reads

$$0 = \delta(q^2 - m^2) \left[ q \cdot \Delta \bar{a}_q + F^{\mu \nu} \bar{a}_{\nu q} - \frac{\kappa_{LL} T}{E_q} \left( \bar{a}_q \dot{Q}^{(1)}_c + u^\mu \dot{Q}^{(2)}_c + \dot{Q}^{(3)}_c \right) \right. \left. + \dot{Q}^{(4)}_c q^\perp \partial_{q^\perp} \bar{a}_{\nu q} + \dot{Q}^{(5)}_c q^\perp \partial_{q^\perp} \bar{a}^\mu_q + \dot{Q}^{(6)}_c \eta^\alpha \partial_{q^\perp} \partial_{q^\perp} \bar{a}_q^\mu + \dot{Q}^{(7)}_c q^\perp \partial_{q^\perp} \partial_{q^\perp} \bar{a}_q^\mu \right],$$

(52)
\[ \dot{\mathcal{Q}}_{\text{cl}}^{(1)} = \frac{2m}{T} \left[ \left( v_0(1 - 2f_{Vq}) - \frac{T}{m} \right) - \frac{v_0^3}{|v|^2} m \theta_{-1} \tilde{q}_\perp^\rho \partial_{\perp q}^\rho f_{Vq} \right], \]  
\[ \dot{\mathcal{Q}}_{\text{cl}}^{(2)} = m \frac{v_0}{|v|^3} \left( (\theta_1 - |v|^3) \partial_{q_\perp} \tilde{a}_q^\nu + (3\theta_1 + |v|^3) \tilde{q}_\perp^\rho \partial_{q_\perp} \tilde{a}_q^\nu \right) \]
\[ + m \frac{v_0}{|v|^2 T} \left( v_0^3 (1 - 2f_{Vq}) \theta_{-1} - \frac{2T}{m} (\theta_{-1} + 2|v|^3) \right) \tilde{q}_\perp \cdot \tilde{a}_q, \]  
\[ \dot{\mathcal{Q}}_{\text{cl}}^{(3)} = m \frac{v_0^2 \theta_{-1} \partial_{q_\perp} \tilde{a}_q^\nu + (3\theta_1 + |v|^3) \tilde{q}_\perp^\rho \partial_{q_\perp} \tilde{a}_q^\nu}{|v|^2 T} \]
\[ + m \frac{v_0}{|v|^2} \theta_{-1} \left( v_0(1 - 2f_{Vq}) - \frac{2T}{m} \right) \tilde{q}_\perp \cdot \tilde{a}_q, \]  
\[ \dot{\mathcal{Q}}_{\text{cl}}^{(4)} = -2m|v|, \]  
\[ \dot{\mathcal{Q}}_{\text{cl}}^{(5)} = \frac{m^2 v_0^3}{2|v|^3} \theta_{-1} (1 - 2f_{Vq}), \]  
\[ \dot{\mathcal{Q}}_{\text{cl}}^{(6)} = -\frac{m^2 v_0^3}{2|v|^3} \left( 3|v|^3 - v_0^2 \theta_{-3} \right), \]  
\[ \dot{\mathcal{Q}}_{\text{cl}}^{(7)} = \frac{m^2 v_0^3}{2|v|^3} (3v_0 \theta_1 + \eta_0 |v|^2). \]

2. Nonrelativistic limit

In the non-relativistic limit \((m \gg T, |q|)\), we have \(v_0 \sim 1 + |v|^2/2\) and \(\theta_n \sim -(3n + 2)|v|^3/6\). We immediately find that the SKE and AKE reduce to similar forms,

\[ 0 = \left( \partial_0 + \frac{q_\perp^\nu \partial_{\perp q}^\rho}{m} \right) f_{Vq} + \frac{2}{3} \kappa_{\text{LL}} \left[ T g^{\mu \rho} \partial_{q_\perp} \partial_{q_\perp} - \frac{1}{m} \left( 3(1 - f_{Vq}) + (1 - 2f_{Vq}) q_\perp^\rho \partial_{q_\perp} \right) \right] f_{Vq}, \]  
\[ 0 = \left( \partial_0 + \frac{q_\perp^\nu \partial_{\perp q}^\rho}{m} \right) \tilde{a}_q^\mu + \frac{2}{3} \kappa_{\text{LL}} \left[ T g^{\mu \rho} \partial_{q_\perp} \partial_{q_\perp} \tilde{a}_q^\mu + \frac{1}{m} \left\{ \left( 2(q_\perp^\nu \partial_{q_\perp} f_{Vq}) - 3(1 - 2f_{Vq}) \right) \tilde{a}_q^\mu \right\} \right], \]

\[ + 2T(\partial_{q_\perp} \tilde{a}_q^\nu) u^\mu - (1 - 2f_{Vq}) q_\perp^\rho \partial_{q_\perp} \tilde{a}_q^\mu \]

where we further retain the terms up to \(O(1/m)\). It turns out that the orientation of spin for heavy quarks is fixed yet the “spin” (axial-charge) density characterized by \(a_{Vq}\) undergoes the diffusive process same as the vector-charge density led by \(f_{Vq}\) when \(m \to \infty\). Nonetheless, the modification upon the spin orientation by, e.g., the fluid velocity could emerge at higher orders suppressed by the mass of heavy quarks. Note that the Compton scattering, neglected in this work, also give rise to \(1/m\) corrections, that, however, do not come with the logarithm enhancement \(\sim \log(1/g_c)\). Therefore, the above \(1/m\) corrections provide the consistent results within the leading-log approximation.
3. Consistency checks

In thermal equilibrium, the vector-charge distribution function takes the Fermi-Dirac form 
\[ f_{Vq} = 1/(e^{(E_q - \mu)/T} + 1) \] such that
\[
\partial_{\perp q} f_{qV} = f_{Vq}(1 - f_{Vq}) \frac{q_{\perp\beta}}{E_q T},
\]
\[
\partial_{\perp q} \partial_{\perp q} f_{qV} = f_{Vq}(1 - 3f_{Vq} + 2f_{Vq}^2) \frac{q_{\perp\alpha}q_{\perp\beta}}{E_q^2 T^2} + f_{Vq}(1 - f_{Vq}) \left( \Theta_{\alpha\beta} + \frac{q_{\perp\alpha}q_{\perp\beta}}{E_q^2} \right),
\]
and hence
\[
\hat{q}_{\perp}^\beta \partial_{\perp q} f_{qV} = -f_{Vq}(1 - f_{Vq}) \frac{\lvert q \rvert}{E_q T},
\]
\[
\hat{q}_{\perp}^\alpha \partial_{\perp q} \partial_{\perp q} f_{qV} = -f_{Vq}(1 - 3f_{Vq} + 2f_{Vq}^2) \frac{\lvert q \rvert^2}{E_q^2 T^2} + f_{Vq}(1 - f_{Vq}) \left( 3 - \frac{\lvert q \rvert^2}{E_q^2} \right),
\]
\[
\hat{q}_{\perp}^\alpha \hat{q}_{\perp}^\beta \partial_{\perp q} \partial_{\perp q} f_{qV} = f_{Vq}(1 - 3f_{Vq} + 2f_{Vq}^2) \frac{\lvert q \rvert^2}{E_q^2 T^2} - \frac{m^2 f_{Vq}(1 - f_{Vq})}{E_q^3 T^2}.
\]

Using Eqs. (55) and (56), one can explicitly show that the Fermi-Dirac distribution satisfies the SKE in Eq. (50).

As for the AKE, although each coefficient \( \hat{Q}_{ci}^{(i)} \) takes a complicated form, we can make a cross check with the SKE in the massless limit. Generically, to consider the spin diffusion for massless or light quarks, it is inevitable to further incorporate the gluon Compton scattering. Nevertheless, taking the massless limit here is just to scrutinize the consistency of our results. In the massless limit, \( \hat{a}_q^\mu = q^\mu f_A \) and thus
\[
\partial_{\perp q} \hat{a}_q^\mu = (\Theta_{\perp q}^\mu - u^\mu q_{\perp\beta}) f_A + q^\mu \partial_{\perp q} f_A, \quad \partial_{\perp q} \hat{a}_q^\beta = 3f_A + q_{\perp\alpha} \partial_{\perp q}^\alpha f_A,
\]
\[
\hat{q}_{\perp}^\beta \partial_{\perp q} \hat{a}_q^\mu = (\hat{q}_{\perp}^\mu + u^\mu) f_A + q^\mu \hat{q}_{\perp}^\beta \partial_{\perp q} f_A,
\]
\[
\partial_{\perp q} \partial_{\perp q} \hat{a}_q^\mu = (\Theta_{\perp q}^\mu - u^\mu q_{\perp\beta}) \partial_{\perp q} f_A + (\Theta_{\perp q}^\alpha - u^\alpha q_{\perp\beta}) \partial_{\perp q}^\beta f_A + q^\mu \partial_{\perp q} \partial_{\perp q} f_A - \frac{u^\mu}{\lvert q \rvert} (\Theta_{\perp q}^\alpha + q_{\perp\alpha} q_{\perp\beta}) f_A,
\]
\[
\partial_{\perp q} \partial_{\perp q} \hat{a}_q^\beta = 2(\partial_{\perp q} - u^\mu q_{\perp\alpha} \partial_{\perp q}^\alpha) f_A + q^\mu \partial_{\perp q} \partial_{\perp q} f_A - \frac{2u^\mu}{\lvert q \rvert} f_A,
\]
where \( \partial_{\perp q} \hat{q}_{\perp q} = (\Theta_{\perp q} + q_{\perp\alpha} q_{\perp\beta}) / \lvert q \rvert \). One can show that Eq. (52) then reduces to
\[
0 = q^\mu \delta(q^2) \left[ q \cdot \Delta f_{Aq} - \kappa_{LL} \left( 2f_{Aq}(1 - 2f_{Vq} - q_{\perp}^\alpha \partial_{\perp q} f_{Vq}) \right) + (1 - 2f_{Vq}) q_{\perp}^\alpha \partial_{\perp q} f_{Aq} - \lvert q \rvert T f_{q} q_{\perp}^\alpha \partial_{\perp q} f_{Aq} \right].
\]

We may check the consistency with the SKE in the massless limit by dropping the nonlinear terms since the \( f_A \) contributions are ignored in the SKE based on our power counting, which could cause discrepancies from the chirality-mixing terms. By further taking \( f_{Vq} = (f_{Rq} + f_{Lq})/2 \)
and \( f_{Aq} = f_{Rq} - f_{Lq} \), one finds that Eq. (58) is consistent with the SKE in Eq. (50) in the massless limit. Both the linearized Eqs. (58) and (50) result in

\[
0 = \delta(q^2 - m^2) \left[ q \cdot \Delta - \kappa_{LL} \left( 2 + q_\perp^\beta \partial_{q_\perp^\beta} - |q| T \eta^{\alpha\beta} \partial_{q_\alpha} \partial_{q_\beta} \right) \right] f_{R/Lq}.
\] (59)

Such a remarkable check should support the correctness of Eq. (52). The check is also performed in Ref. [52].

V. CONCLUDING REMARKS AND OUTLOOK

In this paper, we have derived the (approximated) axial kinetic theory with background fields and collisions in the cases when the vector charge is more dominant than the axial charge (or more precisely the spin current) as natural conditions in most of physical systems. It is found that the AKE as a kinetic equation dictating the spin transport not only embraces the spin-diffusion term but also quantum corrections responsible for spin polarization, which reveals nontrivial entanglement of vector/axial-charge and spin transport through collisions. We also show how our formalism can reproduce the spin-diffusion term up to the leading-logarithmic order in weakly coupled QGP for massive quarks as a practical application.

Although we have explicitly evaluated the spin-diffusion term for massive quarks, which are sufficiently heavy for dropping the gluon Compton scattering, the quantum correction could be calculated in a similar fashion as the follow-up work. Moreover, when considering the spin transport for light quarks, it is inevitable to further incorporate the Compton scattering even for just spin diffusion. On the other hand, in rotating QGP, it is expected that both quarks and gluons are polarized, which could be treated as quantum corrections in the self energies. Recently, there have been some relevant studies for the quantum corrections upon polarized photons [67, 68]. Albeit the validity of our formalism is held even in the presence of such corrections, it could be challenging to systematically include the polarization of scattered gluons or even other quarks and to obtain an analytic form of the collision term. Nevertheless, to understand the dynamical evolution of the spin polarization for peculiarly strange quarks associated with the local polarization of \( \Lambda \) hyperons, it might be essential to carry out the aforementioned studies in the future.

On the other hand, our formalism is rather generic, which may have potential applications not only in heavy ion collisions but also other physical systems. For instance, it is proposed in Refs. [69–71] that the electron and neutrino transport with anomalous effects led by chirality imbalance (axial charge) could influence the macroscopic hydrodynamic evolution of matter in core collapse supernovae. However, the chirality imbalance of electrons produced by the electron capture process may be compensated by elastic electron scattering with the effect of nonzero electron mass [72–74]. Our formalism could be applied to track the axial-charge evolution in such a scenario.

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Appendix A: Derivation of the master equations

Based on the spinor decomposition of the propagators (7) and the self-energies (8), we perform the decomposition of the Kadanoff-Baym equations (4). One may use some useful relations such as

\[ \gamma^\mu \gamma^\alpha \gamma^\beta = \eta^{\mu\alpha} \gamma^\beta - \eta^{\mu\beta} \gamma^\alpha + \eta^{\alpha\beta} \gamma^\mu - i \epsilon^{\mu\alpha\beta\lambda} \gamma^5 \gamma^\lambda, \quad \gamma^5 \sigma_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\kappa\sigma} \gamma^{\kappa\sigma}. \]

Here, \( \epsilon_{\mu\nu\rho\sigma} \) is the totally antisymmetric tensor with \( \epsilon_{0123} = -1 \). The Kadanoff-Baym equations (4) contain the commutators \( [\gamma^\mu, \nabla_\mu S^<] \) and \( [(\Pi - m), S^<] \) and their counterparts with the anticommutation relations. They can be decomposed with the following relations

\[ [\gamma^\mu, S^<] = -2A^\mu \gamma^5 + 2iS^\mu_\alpha \gamma^\alpha - 2iP \gamma^5 \gamma^\alpha - 2i\gamma^\nu \sigma^{\mu\nu}, \quad \{\gamma^\mu, S^<\} = 2\gamma^\mu + 2S\gamma^\mu + \epsilon_{\mu\nu\rho\sigma} S^{\nu\rho} \gamma^5 \gamma^\alpha - \epsilon_{\mu\alpha\beta} A^\nu \sigma^{\alpha\beta}. \] (A1a, b)

Based on the decompositions above, we now have

\[ \frac{i}{2} [\gamma^\mu, \nabla_\mu S^<] = -i \nabla_\mu A^\mu \gamma^5 - \nabla^\nu S_{\nu\mu} \gamma^\mu + \nabla_\mu P \gamma^5 \gamma^\mu + \nabla_\mu \gamma^\nu \sigma^{\mu\nu}, \] (A1c)
\[ \frac{i}{2} \{\gamma^\mu, \nabla_\mu S^<\} = i \nabla^\nu \gamma^\mu + i \nabla_\mu S^\nu_\gamma - \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \nabla_\sigma S_{\nu\rho} \gamma^5 \gamma^\mu - \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \nabla^\rho A^\sigma \sigma^{\mu\nu}, \] (A1d)
\[ [(\Pi - m), S^<] = -2\Pi^\nu A^\nu \gamma^5 + 2i\Pi^\nu S_{\nu\mu} \gamma^\mu - 2i\Pi^\nu P \gamma^5 \gamma^\mu - 2i\Pi^\mu \gamma^\nu \sigma^{\mu\nu}, \] (A1e)
\[ \{(\Pi - m), S^<\} = 2(\Pi^\nu \gamma^\mu - mS) - 2imP \gamma^5 + 2(\Pi^\mu S - m\gamma^\mu) \gamma^\nu - (\epsilon_{\mu\nu\rho\sigma} \Pi_\sigma S_{\nu\rho} + 2mA^\nu) \gamma^5 \gamma^\mu - (mS_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} \Pi^\rho A^\sigma) \sigma^{\mu\nu}. \] (A1f)

Assuming the following decomposition \( G(q, X) = \hat{K}^\mu G_\mu(q, X) \) and \( F(q, X) = \hat{Q}^\mu F_\mu(q, X) \) with \( \hat{K}^\mu \) and \( \hat{Q}^\mu \) being arbitrary matrices, it is found

\[ \{G, F\}_* = \frac{1}{2} \{\hat{K}^\mu, \hat{Q}^\nu\} \{G_\mu(q, X), F_\nu(q, X)\}_* + \frac{1}{2} [\hat{K}^\mu, \hat{Q}^\nu][G_\mu(q, X), F_\nu(q, X)]_*, \]
\[ [G, F]_* = \frac{1}{2} [\hat{K}^\mu, \hat{Q}^\nu] \{G_\mu(q, X), F_\nu(q, X)\}_* + \frac{1}{2} \{\hat{K}^\mu, \hat{Q}^\nu\}[G_\mu(q, X), F_\nu(q, X)]_*. \] (A2)
Accordingly, the self-energy parts can be decomposed in the same and straightforward way as

$$\{\Sigma^>, S_<\} = i\left(\{\bar{\Sigma}_V^\alpha, A^\alpha\} + \{\Sigma^\alpha_A, V^\alpha\}\right) i\gamma^5$$

$$+ i\left(\{\Sigma_P, A^\alpha\} + \{\Sigma^\mu, S^\mu\} - \{\Sigma^\alpha_A, P\} + \{\bar{\Sigma}_{T^\beta_A}, V^\beta\}\right) \gamma^\alpha$$

$$+ i\left(\{\Sigma_P, V^\alpha\} - \{\Sigma^\alpha_V, P\} + \{\Sigma^\alpha_A, S^\mu\} + \{\bar{\Sigma}_{T^\beta_A}, A^\beta\}\right) \gamma^5 \gamma^\alpha$$

$$+ i\left(-\{\bar{\Sigma}_V[a], V^\beta\} + \{\Sigma_A[a], A^\beta\} - \{\bar{\Sigma}_{T^\mu[a]}, S^\mu\}\right) \frac{\sigma^{\alpha\beta}}{2}$$

$$+ \{\Sigma_S, S\} + \{\Sigma_P, P\} + \{\Sigma_V, V^\mu\} - \{\Sigma^\alpha_A, A^\mu\} + \frac{1}{2} \{\Sigma^\mu_T, S^{\mu\nu}\}$$

$$+ \left(\{\Sigma_S, P\} + \{\Sigma_P, S\} + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta}\{\Sigma^\mu_T, S^{\alpha\beta}\}\left) i\gamma^5\right.$$  \hspace{1cm} (A3)

and

$$\{\Sigma^>, S_<\} = \{\Sigma_S, S\} - \{\Sigma_P, P\} + \{\Sigma^\alpha_V, V^\mu\} - \{\Sigma^\alpha_A, A^\mu\} + \frac{1}{2} \{\Sigma^\mu_T, S^{\mu\nu}\}$$

$$+ \left(\{\Sigma_S, P\} + \{\Sigma_P, S\} + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta}\{\Sigma^\mu_T, S^{\alpha\beta}\}\right) i\gamma^5$$

$$+ \left(\{\Sigma_S, V^\alpha\} + \{\Sigma^\alpha_V, S\} + \frac{1}{2} \epsilon_{\mu\nu\lambda\alpha}(\{\Sigma^\mu_A, S^{\nu\lambda}\} + \{\Sigma^\mu_T, A^\lambda\})\right) \gamma^\alpha$$

$$+ \left(\{\Sigma_S, A^\alpha\} + \{\Sigma^\alpha_A, S\} + \frac{1}{2} \epsilon_{\mu\nu\lambda\alpha}(\{\Sigma^\mu_V, S^{\nu\lambda}\} + \{\Sigma^\mu_T, V^\lambda\})\right) \gamma^5 \gamma^\alpha$$

$$+ \left(\{\Sigma_S, S^\alpha\} + \{\Sigma_T^\alpha_A, S\} + \epsilon_{\mu\nu\alpha\beta}(-\{\Sigma^\mu_V, A^\nu\} + \{\Sigma^\mu_A, V^\nu\})\right)$$

$$- \frac{1}{2} \epsilon_{\mu\nu\alpha\beta}(\{\Sigma_P, S^{\mu\nu}\} + \{\Sigma^\mu_T, P\})\frac{1}{2} \sigma^{\alpha\beta}$$

$$+ i\left(\{\Sigma^\alpha_V, A\} - \{\Sigma^\alpha_A, V\}\right) i\gamma^5$$

$$+ i\left(\{\Sigma_P, A\} + \{\Sigma^\alpha, S\} - \{\Sigma^\alpha_A, P\} + \{\Sigma_T^\alpha, V^\beta\}\right) \gamma^\alpha$$

$$+ i\left(\{\Sigma_P, V\} + \{\Sigma^\alpha_A, S\} - \{\Sigma^\alpha_V, P\} + \{\Sigma_T^\alpha, A^\beta\}\right) \gamma^5 \gamma^\alpha$$

$$+ i\left(-\{\Sigma^\alpha_V, V\} + \{\Sigma^\alpha_A, A\} - \{\Sigma^\alpha_T, S^\mu\}\right) \frac{1}{2} \sigma^{\alpha\beta},$$

where we defined the antisymmetrization $T_{\mu\nu} = T_{\mu\nu} - T_{\nu\mu}$. Plugging those decompositions back to the Kadanoff-Baym equations (4), we find

$$0 = \{\hat{\Pi} - m, S^<\} + \frac{i\hbar}{2} \left(\gamma^\mu, \nabla_\mu S^<\right) - \{\Sigma^<, S^>_\} + \{\Sigma^>, S^<_\}$$  \hspace{1cm} (A5)

$$= \mathcal{H}_S + \mathcal{H}_S \gamma^5 + \mathcal{H}_{V^\mu} \gamma^\mu + \mathcal{H}_{S^\mu} \gamma^\mu \gamma^5 + \mathcal{H}_{T^\mu\nu} \frac{1}{2} \sigma^{\mu\nu},$$

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\[ 0 = \left[ (\mathcal{H} - m), S^c \right] + \frac{i\hbar}{2} \left( \{ \gamma^\mu, \nabla_\mu S^c \} - \{ \Sigma^c, S^c \} \right) \tag{A6} \]

\[ = \mathcal{D}_S + \mathcal{D}_5 i\gamma^5 + \mathcal{D}_V \gamma^\mu + \mathcal{D}_{S_5} \gamma^\mu + \mathcal{D}_{T_\mu} \frac{1}{2} \gamma^\mu, \]

where

\[ \mathcal{H}_S = 2\Pi^\mu \mathcal{V}_\mu - 2m\mathcal{S} \]

\[ + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{S} \right] - \left[ \Sigma_P, \mathcal{P} \right] + \left[ \Sigma_V, \mathcal{V}_\mu \right] - \left[ \Sigma_A, \mathcal{A}^\mu \right] + \frac{1}{2} \left[ \Sigma_{T_\mu}, \mathcal{S}^{\mu\nu} \right] \right), \tag{A7} \]

\[ \mathcal{H}_5 = -2m\mathcal{P} - \hbar \mathcal{A}^\mu + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{P} \right] + \left[ \Sigma_P, \mathcal{S} \right] + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} \left[ \Sigma^{\mu\nu}_{T}, \mathcal{S}^{\alpha\beta} \right] \right), \tag{A8} \]

\[ \mathcal{H}_V = 2\Pi^\alpha \mathcal{S} - 2m\mathcal{V}_\alpha - \hbar \mathcal{D}_\alpha \mathcal{P} - \frac{\hbar}{2} \left( \left[ \Sigma_P, \mathcal{A} \right] - \left[ \Sigma_{A\alpha}, \mathcal{P} \right] + \left[ \Sigma_{T_\mu}, \mathcal{V}_\mu \right] \right) \]

\[ + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{V}_\alpha \right] + \left[ \Sigma_{V_\alpha}, \mathcal{S} \right] + \frac{1}{2} \epsilon_{\mu\nu\alpha\lambda} \left[ \Sigma^{\mu}_{T}, \mathcal{S}^{\nu\lambda} \right] + \left[ \Sigma^{\mu}_{T}, \mathcal{V}_\lambda \right] \right), \tag{A9} \]

\[ \mathcal{H}_{5\alpha} = -\epsilon_{\alpha\beta\rho} \Pi^\rho \mathcal{S}^{\mu\nu} - 2m\mathcal{A}_\alpha + \hbar \mathcal{D}_\alpha \mathcal{P} - \frac{\hbar}{2} \left( \left[ \Sigma_P, \mathcal{A}_\beta \right] - \left[ \Sigma_{A\beta}, \mathcal{P} \right] + \left[ \Sigma_{T_\mu}, \mathcal{A}_\mu \right] \right) \]

\[ + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{A}_\beta \right] + \left[ \Sigma_{A\beta}, \mathcal{S} \right] + \frac{1}{2} \epsilon_{\mu\nu\alpha\lambda} \left[ \Sigma^{\mu}_{T}, \mathcal{S}^{\nu\lambda} \right] + \left[ \Sigma^{\mu}_{T}, \mathcal{V}_\lambda \right] \right), \tag{A10} \]

\[ \mathcal{H}_{T\alpha\beta} = -2m\mathcal{S}_{\alpha\beta} - 2\epsilon_{\alpha\beta\rho} \Pi^\rho \mathcal{A}^\mu - \hbar \mathcal{D}_{[\alpha} \mathcal{V}_{\beta]} - \frac{\hbar}{2} \left( \left[ \Sigma_{A[\alpha}, \mathcal{A}_{\beta]} \right] - \left[ \Sigma_{T[\alpha}, \mathcal{A}_{\beta]} \right] \right) \]

\[ + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{S}_{\alpha\beta} \right] + \left[ \Sigma_{T\alpha\beta}, \mathcal{S} \right] + \epsilon_{\mu\nu\alpha\beta} \left( - \left[ \Sigma_V, \mathcal{A}^\nu \right] + \left[ \Sigma_{A}^{\mu\nu}, \mathcal{V}^\nu \right] \right) \right) \]

\[ - \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \left[ \Sigma^{\mu}_{T}, \mathcal{S}^{\nu\alpha\beta} \right], \tag{A11} \]

and

\[ \mathcal{D}_S = \hbar \mathcal{D}_\mu \mathcal{V}_\mu \]

\[ + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{V} \right] - \left[ \Sigma_P, \mathcal{P} \right] - \left[ \Sigma_A, \mathcal{A}^\mu \right] + \frac{1}{2} \left[ \Sigma_{T_\mu}, \mathcal{V}_\mu \right] \right), \tag{A12} \]

\[ \mathcal{D}_5 = 2i\Pi^\alpha \mathcal{A}_\alpha \]

\[ + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{A} \right] + \left[ \Sigma_P, \mathcal{S} \right] + \frac{1}{4} \epsilon_{\mu\nu\alpha\lambda} \left[ \Sigma^{\mu}_{T}, \mathcal{S}^{\nu\lambda} \right] \right) \]

\[ - \frac{\hbar}{2} \left( \left[ \Sigma_{V_\alpha}, \mathcal{A}_\alpha \right] - \left[ \Sigma_{A\alpha}, \mathcal{V}_\alpha \right] \right), \tag{A13} \]

\[ \mathcal{D}_V = 2i\Pi^\alpha \mathcal{S}_{\alpha}\mathcal{V}_\alpha \]

\[ + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{V}_\alpha \right] + \frac{1}{2} \epsilon_{\mu\nu\alpha\lambda} \left( \left[ \Sigma_{A}^{\mu\nu}, \mathcal{V}_\lambda \right] + \left[ \Sigma^{\mu}_{T}, \mathcal{A} \right] \right) \right) \]

\[ - \frac{\hbar}{2} \left( \left[ \Sigma_P, \mathcal{A}_\alpha \right] + \left[ \Sigma_P, \mathcal{S}_{\alpha\mu} \right] - \left[ \Sigma_A, \mathcal{P} \right] + \left[ \Sigma_{T\alpha\beta}, \mathcal{V}_{\beta} \right] \right), \tag{A14} \]

\[ \mathcal{D}_{5\alpha} = -2i\Pi_\alpha \mathcal{P} - \frac{i\hbar}{2} \epsilon_{\alpha\beta\rho} \mathcal{D}^\rho \mathcal{S}^{\mu\nu} \]

\[ + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{A}_\alpha \right] + \left[ \Sigma_A, \mathcal{S} \right] + \frac{1}{2} \epsilon_{\alpha\beta\rho} \left[ \Sigma_{T_\mu}, \mathcal{V}_\rho \right] \right) \]

\[ - \frac{\hbar}{2} \left( \left[ \Sigma_P, \mathcal{V}_\alpha \right] + \left[ \Sigma_P, \mathcal{S}_{\beta\alpha} \right] - \left[ \Sigma_V, \mathcal{P} \right] + \left[ \Sigma_{T\alpha\beta}, \mathcal{A}_\beta \right] \right), \tag{A15} \]

\[ \mathcal{D}_{T\alpha\beta} = -2i\Pi_\alpha \mathcal{V}_\beta \]

\[ + \frac{i\hbar}{2} \left( \left[ \Sigma_S, \mathcal{S}_{\alpha\beta} \right] + \left[ \Sigma_{T\alpha\beta}, \mathcal{S} \right] + \epsilon_{\alpha\beta\mu\nu} \left[ \Sigma_{A}^{\mu\nu}, \mathcal{V}_\nu \right] \right) \]

\[ - \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} \left( \left[ \Sigma_P, \mathcal{S}^{\mu\nu} \right] - \left[ \Sigma_{T_\mu, T_\nu} \right] \right) - \frac{\hbar}{2} \left( \left[ \Sigma_V, \mathcal{V}_\beta \right] - \left[ \Sigma_{A[\alpha}, \mathcal{A}_{\beta]} \right] + \left[ \Sigma_{T_\mu[\alpha}, \mathcal{S}^{\mu\nu}_{\beta]} \right] \right). \tag{A16} \]
Here we introduce a shorthand notation $\hat{XY} = XY - XY$, where $X$ and $Y$ are the coefficients of the Clifford decomposition of the propagators and self-energies, e.g., $\{\Sigma_{V\mu}, \Sigma_{V\nu}\} = \{\Sigma_{V\mu}, \Sigma_{V\nu}\} - \{\Sigma_{V\mu}, \Sigma_{V\nu}\}$. We also introduced $\hat{D}_\mu M = \nabla_\mu M + \{\Sigma_{V\mu}, M\}/2$. $\mathcal{K}_S = \mathcal{K}_5 = \mathcal{K}_{V\alpha} = \mathcal{K}_{5\alpha} = \mathcal{D}_{Ta\beta} = \mathcal{D}_S = \mathcal{D}_5 = \mathcal{D}_{V\alpha} = \mathcal{D}_{5\alpha} = \mathcal{D}_{Ta\beta} = 0$ give the set of full quantum master equations.

They reduce to

$$mS = P^\mu V_\mu - \frac{\hbar^2}{4} [\Sigma S - \Sigma P + \Sigma V_\mu \Sigma_{V\mu} + \Sigma_{Ap} \Sigma_{Ap} + \Sigma_{V\mu} \Sigma_{V\mu}] + \frac{1}{2} \Sigma T_{\mu \nu} S^\mu_\nu\{P.B. + O(h^3), (A17a)$$

$$mP = -\frac{\hbar}{2} (\hat{D}_\mu \Sigma_{V\mu} - \Sigma_{Ap} \Sigma_{Ap}) - \frac{\hbar^2}{4} [\Sigma S + \Sigma P + \Sigma V_\mu \Sigma^\mu_\nu + \Sigma_{P\alpha} \Sigma_{P\alpha}] + \frac{1}{2} \Sigma T_{\mu \nu} S^\mu_\nu\{P.B. + O(h^3), (A17b)$$

$$2P_{\alpha} + h \hat{D}_\nu S_{\nu\alpha} - 2mV_\alpha - h(\Sigma P A_\alpha - \Sigma_{Ap} - \Sigma T_{\mu \alpha} S_\mu)$$

$$= \frac{\hbar^2}{2} \Sigma S\Sigma_{Ap} + \Sigma_{Ap} S + \frac{1}{2} \Sigma T_{\mu \nu} S^\mu_\nu S^\nu_\mu\{P.B. + O(h^3), (A17c)$$

$$h\hat{D}_\mu P - \epsilon_{\alpha \beta \rho \sigma} P_{\rho \sigma} S_{\alpha \beta} - 2m A_\alpha - h(\Sigma P A_\alpha + \Sigma_{Ap} S_{\alpha \mu} - \Sigma T_{\mu \alpha} A_\mu)$$

$$= \frac{\hbar^2}{2} \Sigma S A_\alpha + \Sigma_{Ap} - 2m A_\alpha + \frac{1}{2} \Sigma T_{\mu \nu} S^\mu_\nu S^\nu_\mu\{P.B. + O(h^3), (A17d)$$

$$mS_{\alpha \beta} + \epsilon_{\alpha \beta \rho \sigma} P_{\rho \sigma} A_\sigma - \frac{\hbar}{2} (\hat{D}_{[\alpha} A_{\beta]} - \Sigma A_{[\alpha} A_{\beta]} + \Sigma T_{[\rho \mu] S^\rho_\mu S^\mu_{\beta]}\}$$

$$= -\frac{\hbar^2}{4} \Sigma S S_{\alpha \beta} + \Sigma T_{\alpha \beta} S - \epsilon_{\mu \nu \alpha \beta} S_{\mu \nu} A_\mu + \epsilon_{\mu \nu \rho \sigma} S_{\rho \sigma} A_\mu - \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} (\Sigma P S_{\mu \nu} + \Sigma T_{\mu \nu} S^\mu_\nu)\{P.B. + O(h^3), (A17e)$$

and

$$\hat{D}^\mu V_\mu = -\Sigma S + \Sigma P + \Sigma_{Ap} - \frac{1}{2} \Sigma T_{\mu \nu} S^\mu_\nu + O(h^3), (A18a)$$

$$2P_{\nu} + h \hat{D}_\nu S_{\nu \alpha} + h(\Sigma S V_\alpha + \Sigma_{Ap} S_{\alpha \mu} - \Sigma T_{\mu \alpha} S_{\alpha \mu})$$

$$= \frac{\hbar^2}{2} \Sigma S A_\alpha + \Sigma_{Ap} + \Sigma T_{\mu \alpha} S_{\mu \alpha}\{P.B. + O(h^3), (A18b)$$

$$2P_{\alpha} + \frac{\hbar}{2} \epsilon_{\alpha \beta \rho \sigma} (\hat{D}^\rho S_{\rho \sigma} + \Sigma T_{\sigma \nu} V_\nu) - h(\Sigma S A_\alpha + \Sigma_{Ap} S_{\alpha \mu})$$

$$= 1 - \frac{\hbar^2}{2} [\Sigma P A_\alpha + \Sigma_{Ap} S_{\alpha \mu} + \Sigma T_{\mu \alpha} A_\mu] + O(h^3), (A18d)$$

$$\Pi_{(\alpha V_\beta)} + \frac{\hbar}{2} \epsilon_{\alpha \beta \mu \nu} (\hat{D}^\nu A_\mu - \Sigma_{\nu \mu} A_\beta) - \frac{\hbar}{2} [\Sigma S_{\alpha \beta} + \Sigma T_{\alpha \beta} S_{\alpha \beta}] + h(\Sigma P S_{\mu \nu} + \Sigma T_{\mu \nu} S^\mu_\nu)\{P.B. + O(h^3), (A18e)$$

where $[AB]_{P.B} = \{A(q, X), B(q, X)\}_{P.B.}$ is a shorthand notation for the Poisson bracket. Equations (A17a), (A17b) and (A17e) can be used to eliminate $S$, $P$, and $S_{\mu \nu}$ from the other equations. Since $S_{\mu \nu}$ is contracted with $\Sigma T_{\mu \nu}$ and $\Sigma T_{\mu \nu}$ on the right-hand side, one may not express $S_{\mu \nu}$ as an explicit function of $V_\mu$ and $A_\mu$ only. However, assuming that the interaction is sufficiently
weak, we may drop the nonlinear terms in the self-energy. Within this assumption, we may rewrite Eqs. (A17a), (A17b) and (A17e) as

\[
S_{\alpha\beta} \approx -\frac{1}{m} \epsilon_{\alpha\beta\rho\sigma} \Pi^\rho A^\sigma + \frac{\hbar}{2m} \left\{ D_{[\alpha} V_{\beta]} - \Sigma_{A}[A_{\beta}] + \frac{q_\mu}{m} \epsilon_{\mu\rho\sigma} \left[ q_{T\beta} A_{\rho} \right] \right\} \\
+ \frac{\hbar^2}{4m^2} \left( \Sigma_{T[\mu}[A_{\nu]} - \Sigma_{T[\mu}[\nabla_{\nu} A] \right] - \frac{\hbar^2}{4m} \epsilon_{\alpha\beta\mu\nu} \Sigma_S(q^\nu A^\mu) + \frac{1}{m} \Sigma_{T\alpha\beta}(q \cdot V) \\
- \epsilon_{\mu\nu\alpha\beta} \Sigma^\mu_{V} A^\nu + \epsilon_{\mu\nu\alpha\beta} \Sigma^\mu_{A} A^\nu - \frac{1}{m} \Sigma_{P}(q_{\alpha} A_{\beta}) \right]_{P.B.} + O(\hbar^3). \tag{A19}
\]

\[
S = \frac{\Pi^\mu}{m} V_\mu - \frac{\hbar^2}{4m} \left[ \frac{\Sigma_S(q \cdot V)}{m} \right] + \frac{1}{2m} \epsilon_{\mu\nu\alpha\beta} \Sigma_{T}[q_{\alpha} A_{\beta}] \right\]_{P.B.} + O(\hbar^3) \tag{A20}
\]

and

\[
P = -\frac{\hbar}{2m} \left( D_{[\mu} A^{\mu} - \Sigma_{A}[\nabla_{\nu} V_{\nu}] \right) - \frac{\hbar^2}{4m} \left[ \frac{\Sigma_{P}(q \cdot V)}{m} \right] + \frac{1}{2m} \epsilon_{\mu\nu\alpha\beta} \Sigma_{T}(q_{\mu} A_{\nu}) \right]_{P.B.} + O(\hbar^3). \tag{A21}
\]

Inserting Eqs. (A20), (A21) and (A19) and maintaining the linear terms in the self-energies and the explicit \( \hbar \) dependence up to \( O(\hbar) \) (or \( O(\hbar^2) \) for involving at least the next-leading-order corrections), the master equations (A17c) and (A17d) up to \( O(\hbar) \) are reduced as

\[
q^\mu(q \cdot V) - m^2 V_\mu \approx \frac{\hbar}{2} \left[ m(\Sigma_{P}[V_{\mu}] + \Sigma_{T}[V_{\alpha}]) - 2(q_{\lambda} A_{\beta} + \frac{1}{2} \epsilon_{\mu\alpha\beta} q_{\alpha} \Sigma_{V} A_{\beta}) - \epsilon_{\mu\alpha\beta} q_{\alpha} \Sigma_{V} A_{\beta} \right]. \tag{A22a}
\]

\[
q^\mu A_\mu - q^\mu q \cdot A - m^2 A^\mu \approx \frac{\hbar}{2} \left[ m(\Sigma_{P}[V_{\mu}] + \Sigma_{T}[A_{\alpha}]) + \epsilon_{\mu\alpha\beta} q_{\alpha} D_{\beta} A_{\gamma} - 2q_{\mu} \Sigma_{V} A_{\beta} A_{\gamma} - \frac{1}{m} \epsilon_{\alpha\beta\gamma} \epsilon_{\lambda\rho\sigma} q^\alpha \Sigma_{T}[V_{\rho}] A_{\sigma} \right]. \tag{A22b}
\]

Similarly, the master equations (A18a)-(A18e) up to \( O(\hbar) \) are reduced as

\[
D^\mu V_\mu \approx \Sigma_{A}[A_{\mu}] - \frac{1}{2m} \left[ 2q_{\mu} \Sigma_{S}[V_{\mu}] - \epsilon_{\mu\alpha\beta} \Sigma_{T}[\nabla_{\alpha} A_{\beta}] + m(\Sigma_{P}[\nabla \cdot A] + \Sigma_{T}[\nabla_{\mu} V_{\mu}] \right] \tag{A23a}
\]

\[
q \cdot A \approx -\frac{\hbar}{2m} q_{\mu} (\Sigma_{P}[V_{\mu}] + \Sigma_{T}[A_{\mu}]), \tag{A23b}
\]

\[
D_{\mu}(q \cdot V) - q^\nu D_{[\nu} V_{\mu]} \approx -m \Sigma_{S}[V_{\mu}] + \frac{1}{m} \epsilon_{\alpha\beta\gamma} q_{\mu} \epsilon_{\nu\rho\sigma} \left( \Sigma_{T}[\nabla_{\nu} A_{\rho}] + \frac{1}{2m} \epsilon_{\mu\nu\rho\sigma} \left( m \Sigma_{T}[\nabla \cdot A] - \hbar(q^\nu A_{\rho}) \partial_{q\rho} A_{\sigma} + \hbar \Sigma_{A} \nabla \cdot \nabla \right) \right) \tag{A23c}
\]

\[
F_{\mu\nu} A^{\nu} - q \cdot D_{\mu} A_{\nu} - \frac{\hbar}{4} \epsilon_{\mu\nu\rho\sigma} \left( D_{\rho}^{\prime} + D_{\rho}^{\prime} \right) V_{\nu} - \frac{\hbar}{2m} q_{\mu} \Sigma_{P}[q \cdot V_{\nu}] + \frac{1}{2m} \Sigma_{T}[q_{\mu} A_{\nu}] \right]_{P.B.} \tag{A23d}
\]

\[
q^\nu A_{\mu} - \frac{1}{m} \epsilon_{\mu\nu\rho\sigma} \Sigma_{T}[q_{\rho} A_{\sigma}] = -\frac{1}{m} \epsilon_{\alpha\beta\gamma} \sigma_{T}[q_{\mu} A_{\nu}] A_{\beta} \tag{A23e}
\]
where we used the following relation
\[ 2\epsilon_{\nu\mu\rho}(\Pi^\nu\Pi^\rho)\mathcal{A}^\sigma = \frac{h^2}{6} \epsilon_{\nu\mu\rho}(\partial^\rho F^{3\nu} + \partial^\nu F^{\rho\mu})\partial_\eta \mathcal{A}^\sigma = \frac{h^2}{2} \epsilon_{\nu\mu\rho}(\partial^\rho F^{3\nu})\partial_\eta \mathcal{A}^\sigma. \]  
(A24)

The master equations (A22) and (A23) are in general the full master equations with collisional effects up to \( \mathcal{O}(\hbar) \). In our power counting \( \mathcal{V}^\mu \sim \mathcal{O}(\hbar^0) \) and \( \mathcal{A}^\mu \sim \mathcal{O}(\hbar) \), these equations reduce to Eqs. (9a)-(9g).

**Appendix B: Angular-momentum decomposition for fermions and spin polarization**

We briefly review the derivation of the gauge-invariant angular-momentum decomposition for fermions proposed in Ref. [75] as a covariant version of the Ji’s decomposition [76]. Such a derivation is more explicitly shown in the review paper [77] (See also Ref. [78]). Here we just summarize the derivation therein. The starting point is the Belinfante angular momentum for “on-shell” fermions,
\[ M_B^{\mu\nu} = \frac{x^\nu}{4} \bar{\psi} \left( \gamma^{\mu} \slashed{D}^{\nu} + \gamma^{\nu} \slashed{D}^{\mu} \right) \psi - (\nu \leftrightarrow \rho), \]
(B1)

where \( \bar{\psi} \slashed{D} \psi = \bar{\psi}(\slashed{D}_\mu - \slashed{D}^{\mu}) \psi \) and \( D_\mu = \partial_\mu + i e A_\mu \) denotes the covariant derivative. One may rewrite \( M_B^{\mu\nu} \) as
\[ M_B^{\mu\nu} = \frac{x^\nu}{2} \bar{\psi} \gamma^{\mu} \slashed{D}^{\rho} \psi - \frac{x^\nu}{4} \bar{\psi} \left( \gamma^{\mu} \slashed{D}^{\rho} - \gamma^{\rho} \slashed{D}^{\mu} \right) \psi - (\nu \leftrightarrow \rho). \]
(B2)

Next, one has to employ the relations,
\[ \sigma^{\mu\nu} \slashed{D} = \gamma^{\nu} \slashed{D}^{\mu} - \gamma^{\mu} \slashed{D}^{\nu} - ie^{\mu\nu\rho\sigma} \gamma_\rho \gamma_5 \slashed{D}_\sigma, \]
\[ i \slashed{D} \sigma^{\mu\nu} = \slashed{D}_\nu \gamma^\mu - \slashed{D}_\mu \gamma^\nu - ie^{\mu\nu\rho\sigma} \gamma^\rho \gamma_5 \slashed{D}_\sigma. \]
(B3)

YH: I put the negative sign in front of \( \epsilon \). For example, \( \sigma^{01} \gamma^2 = i \gamma^0 \gamma^1 \gamma^2 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^5 = - \gamma_5 \gamma_3 = - \gamma_3 \gamma_5 = - e^{0123} \gamma_5 \gamma_3 \), with \( e^{\mu\nu\rho\sigma} = i[\gamma^\mu, \gamma^\nu]/2 \) and the equations of motion \( i \slashed{D} \psi = m \psi \) and \( i \bar{\psi} \slashed{D} = -m \bar{\psi} \), to obtain a useful identity,
\[ \bar{\psi} \left( \gamma^{\mu} \slashed{D}^{\rho} - \gamma^{\rho} \slashed{D}^{\mu} \right) \psi = -e^{\rho\mu\alpha\beta} \partial_\alpha \left( \bar{\psi} \gamma_5 \gamma_3 \psi \right). \]
(B4)

By using the identity, it is found
\[ M_B^{\mu\nu} = \frac{i}{2} \bar{\psi} \gamma^\mu \left( x^\nu \slashed{D}^{\rho} - x^\rho \slashed{D}^{\nu} \right) \psi - \frac{1}{2} e^{\mu\nu\rho\beta} \bar{\psi} \gamma_\beta \gamma_5 \psi \frac{1}{4} \partial_\alpha \left( \left( x^\nu e^{\rho\alpha\beta} - x^\rho e^{\nu\alpha\beta} \right) \bar{\psi} \gamma_5 \gamma_3 \psi \right) \]
and thus yields the gauge-invariant angular-momentum decomposition by dropping the surface term above,
\[ M_B^{\mu\nu} = \frac{i}{2} \bar{\psi} \gamma^\mu \left( x^\nu \slashed{D}^{\rho} - x^\rho \slashed{D}^{\nu} \right) \psi - \frac{1}{2} e^{\mu\nu\rho\beta} \bar{\psi} \gamma_\beta \gamma_5 \psi, \]
(B5)

where the first term above is regarded as the orbital angular momentum and the second term proportional to an axial-charge current is responsible for spin. This gauge-invariant decomposition introduced in Ref. [75] is defined as the “canonical angular momentum” in Ref. [43] though it actually agrees with the usual canonical angular momentum obtained from the Noether’s theorem only in the absence of gauge fields.
Appendix C: Decomposition of the collision term in AKE

Implementing the leading-order SKE, the following term in \( \mathcal{C}_2^{(n)\mu} \) can be also written as

\[
- S_{m(n)}^{\mu\nu} \Delta_\nu \left( \delta(q^2 - m^2) q \cdot \Delta f_V \right) = - S_{m(n)}^{\mu\nu} \left[ \delta(q^2 - m^2) \Delta_\nu + 2 \lambda_\nu \delta'(q^2 - m^2) \right] (C_V[f_V] + m C_S[f_V]), \tag{C1}
\]

where \( C_V[f_V] = q \cdot \Sigma_V \bar{f}_V - q \cdot \Sigma_V f_V \) and \( C_S[f_V] = \Sigma_S \bar{f}_V - \Sigma_S f_V \). Moreover, one may rewrite

\[
(q \cdot \Sigma_V + m \Sigma_S) \Delta_\nu \bar{f}_V - (q \cdot \Sigma_V + m \Sigma_S) \Delta_\nu f_V = \Delta_\nu C_V[f_V] - q^\rho (\bar{f}_V \Delta_\nu \Sigma_{V,\rho} - f_V \Delta_\nu \Sigma_{V,\rho}) - F^\rho_\nu (\Sigma_{V,\rho} \bar{f}_V - \Sigma_{V,\rho} f_V) + m (\Sigma_S \Delta_\nu \bar{f}_V - \Sigma_S \Delta_\nu f_V). \tag{C2}
\]

Then we may re-express \( \mathcal{C}_2^{(n)\mu} \) as

\[
\mathcal{C}_2^{(n)\mu} = \frac{\delta(q^2 - m^2)}{2} \left\{ \epsilon^{\mu\nu\rho\sigma} q_\nu (\Delta_\rho \bar{f}_V - \Delta_\rho f_V) - 2 S_{m(n)}^{\mu\nu} \left( m (\Sigma_S \Delta_\nu \bar{f}_V - \Sigma_S \Delta_\nu f_V) - F^\nu_\nu (\Sigma_{V,\rho} \bar{f}_V - \Sigma_{V,\rho} f_V) \right) + 2 (\Sigma_V \bar{f}_V - \Sigma_V f_V) \left( q \cdot \Delta S_{m(n)}^{\mu\nu} - F^\mu_\nu \lambda^\nu_{m(n)} \right) \right\} - m \delta(q^2 - m^2) S_{m(n)}^{\mu\nu} \Delta_\nu C_S[f_V] - 2 S_{m(n)}^{\mu\nu} \delta'(q^2 - m^2) (C_V[f_V] + m C_S[f_V]) - \delta(q^2 - m^2) (\Delta_\nu S_{m(n)}^{\mu\nu}) \times (C_V[f_V] + m C_S[f_V]) + \delta(q^2 - m^2) \left[ q^\mu (f_V q \cdot \Delta \Sigma_P - \bar{f}_V q \cdot \Delta \Sigma_P) - m^2 ((\partial^\mu \Sigma_P) f_V - (\partial^\mu \Sigma_P) \bar{f}_V) \right]. \tag{C3}
\]

The Schouton identity gives rise to

\[
S_{m(n)}^{\mu\nu} q^\alpha = q^\mu S_{m(n)}^{\nu\alpha} + q^\nu S_{m(n)}^{\mu\alpha} + \frac{\epsilon^{\mu\nu\rho\sigma} q_\rho}{2} - \frac{\epsilon^{\mu\nu\rho\alpha}}{2(q \cdot n + m)} (m q_\rho + q^2 n_\rho). \tag{C4}
\]

We thus find

\[
2 S_{m(n)}^{\mu\nu} \left( \bar{f}_V (q \cdot \Delta \Sigma_{V,\nu}) - f_V (q \cdot \Delta \Sigma_{V,\nu}) \right) = 2 \left[ \left( q^\mu S_{m(n)}^{\nu\alpha} + q^\nu S_{m(n)}^{\mu\alpha} \right) (\bar{f}_V \Delta_\rho \Sigma_{V,\nu} - f_V \Delta_\rho \Sigma_{V,\nu}) - \epsilon^{\mu\nu\rho\alpha} q_\rho ((\Delta_\rho \bar{f}_V) f_V - (\Delta_\rho f_V) \bar{f}_V) \right] - \frac{\epsilon^{\mu\nu\rho\alpha} (m q_\rho + q^2 n_\rho)}{(q \cdot n + m)} (\bar{f}_V \Delta_\sigma \Sigma_{V,\nu} - f_V \Delta_\sigma \Sigma_{V,\nu}). \tag{C5}
\]
and $\hat{C}_2^{(n)\mu}$ becomes

$$
\hat{C}_2^{(n)\mu} = \frac{\delta(q^2 - m^2)}{2} \left( 2q^\mu S_{m(n)}^{\rho\sigma}(\bar{f}_V \Delta_\rho \Sigma_\nu - f_V \Delta_\rho \bar{\Sigma}_\nu) + 2S_{m(n)}^{\rho\sigma}(m(\Sigma_\delta \Delta_\rho \bar{f}_V - \bar{\Sigma}_\delta \Delta_\rho f_V) \\
-F_\rho^\mu(\Sigma_\nu \rho f_V - \Sigma_\nu f_V) - \frac{\epsilon^{\mu\rho\sigma\mu}(q_\rho + mn_\rho)}{(q \cdot n + m)}(\bar{f}_V \Delta_\rho \Sigma_\nu - f_V \Delta_\rho \bar{\Sigma}_\nu) + 2(\Sigma_\nu \rho f_V - \Sigma_\nu f_V) \\
\times\left(q \cdot \Delta \Sigma_{m(n)}^{\rho\sigma} - F_{\rho\sigma} S_{m(n)}^{\lambda\nu}\right) - 2S_{m(n)}^{\mu\nu}q^\lambda F_{\lambda\mu}\delta(q^2 - m^2)(C_V[f_V] + mC_S[f_V]) - \delta(q^2 - m^2) \\
\times\left[ (\Delta_\nu S_{m(n)}^{\rho\sigma})(C_V[f_V] + mC_S[f_V]) + mS_{m(n)}^{\rho\sigma}\Delta_\nu C_S[f_V] \right] \\
+ \bar{F}_{\mu\nu} q_\rho \delta(q^2 - m^2)(C_V[f_V] + mC_S[f_V]) + \frac{\delta(q^2 - m^2)}{2m} \left[ q^\mu (f_V \rho \cdot \Delta \Sigma P - \bar{f}_V q \cdot \Delta \Sigma P) \\
- m^2 ((\partial^\mu \Sigma P) f_V - (\partial^\mu \Sigma P) \bar{f}_V) \right]. \quad (C6)
$$

Next, applying the Schouten identity again, it is found

$$
\delta(q^2 - m^2)q \cdot \Delta \Sigma_{m(n)}^{\mu\nu}
\begin{align*}
&= \delta(q^2 - m^2) \left[ q^\mu \Delta_\alpha S_{m(n)}^{\alpha\nu} + q^\nu \Delta_\alpha S_{m(n)}^{\alpha\mu} + q^\rho \epsilon^{\mu\rho\alpha\sigma} \Delta_\alpha \frac{q_\rho n_\sigma}{2(q \cdot n + m)} + q^\sigma \epsilon^{\mu\rho\sigma\alpha} \Delta_\alpha \frac{q_\rho n_\sigma}{2(q \cdot n + m)} \right] \\
&= \delta(q^2 - m^2) \left[ q^\mu \Delta_\alpha S_{m(n)}^{\alpha\nu} + q^\nu \Delta_\alpha S_{m(n)}^{\alpha\mu} + m^2 \Delta_\alpha \frac{\epsilon^{\mu\rho\alpha\sigma} n_\sigma}{2(q \cdot n + m)} + q^\rho F_{\rho\alpha} \epsilon^{\mu\rho\alpha\sigma} n_\sigma \right] \\
&\quad - m \Delta_\alpha \frac{\epsilon^{\mu\rho\alpha\sigma} q_\rho}{2(q \cdot n + m)} - \frac{\epsilon^{\mu\rho\alpha\sigma} F_{\rho\alpha} n_\sigma q_\rho}{2(q \cdot n + m)} \right]. \quad (C7)
\end{align*}
$$

and

$$
-S_{m(n)}^{\lambda\nu} F_{\lambda}^\mu = -\frac{q_\rho n_\sigma}{2(q \cdot n + m)} (\epsilon^{\lambda\mu\rho\sigma} F_{\lambda}^\nu + \epsilon^{\lambda\mu\nu\sigma} F_{\lambda}^\rho + \epsilon^{\lambda\nu\rho\sigma} F_{\lambda}^\mu), \quad (C8)
$$

which lead to

$$
\delta(q^2 - m^2) \left( q \cdot \Delta \Sigma_{m(n)}^{\mu\nu} - S_{m(n)}^{\lambda\nu} F_{\lambda}^\mu \right)
\begin{align*}
&= \delta(q^2 - m^2) \left[ q^\mu \Delta_\alpha S_{m(n)}^{\alpha\nu} + \bar{F}_{\mu\nu} + q^\nu \Delta_\alpha S_{m(n)}^{\alpha\mu} - m \Delta_\alpha \frac{\epsilon^{\mu\rho\alpha\sigma} (q_\rho + mn_\rho)}{2(q \cdot n + m)} + S_{m(n)}^{\rho\mu} F_{\nu}^\rho \\
&\quad + \frac{q^\rho F_{\rho\alpha} \epsilon^{\mu\rho\alpha\sigma} n_\sigma}{(q \cdot n + m)} \right]. \quad (C9)
\end{align*}
$$
We hence obtain
\[
\hat{C}_2^{(n)\mu} = \frac{\delta(q^2 - m^2)}{2} \left\{ 2q^\mu S_{\mu(n)}^{\rho\nu}(\bar{f}_V \Delta_\rho \Sigma_{\nu} - f_V \Delta_\rho \bar{\Sigma}_{\nu}) + 2S_{\mu(n)}^{\rho\nu} m(\Sigma_S \Delta_\nu \bar{f}_V - \bar{\Sigma}_S \Delta_\nu f_V) 
- \frac{\epsilon_{\mu\rho\sigma\nu} m(q_{\rho} + mn_{\rho})}{(q \cdot n + m)} (\bar{f}_V \Delta_\sigma \Sigma_{\nu} - f_V \Delta_\sigma \bar{\Sigma}_{\nu}) + 2(\bar{f}_V \Sigma_{\nu} - f_V \Sigma_{\nu}) \left[ q^\mu \Delta_\alpha S_{\mu(n)}^{\nu\alpha} + \bar{F}^{\mu\nu} 
+ \frac{q^\rho F_{\rho\sigma} \epsilon_{\mu\rho\sigma\nu} n_\sigma}{(q \cdot n + m)} - m \Delta_\alpha \frac{\epsilon_{\mu\rho\sigma\nu} (q_{\rho} + m n_{\rho})}{2(q \cdot n + m)} \right] \right\} - 2S_{\mu(n)}^{\rho\nu} q^\lambda \delta'(q^2 - m^2) C_V [f_V] + \delta'(q^2 - m^2) \times \bar{F}^{\mu\nu} q_\nu (C_V [f_V] + m C_S [f_V])
- m \left( 2S_{\mu(n)}^{\rho\nu} q^\lambda F_{\lambda\nu} \delta'(q^2 - m^2) + \delta(q^2 - m^2) (\Delta_\nu S_{\mu(n)}^{\rho\nu} + S_{\mu(n)}^{\rho\nu} \Delta_\nu) \right) C_S [f_V] + \frac{\delta(q^2 - m^2)}{2m} \left[ q^\mu (f_V q \cdot \Delta \Sigma_P - \bar{f}_V q \cdot \Delta \Sigma_P) - m^2 (\partial_{\nu} \bar{\Sigma}_{\mu} f_V - (\partial_{\nu} \Sigma_P) \bar{f}_V) \right]. \tag{C10}
\]

On the other hand, one finds
\[
-2S_{\mu(n)}^{\rho\nu} \delta'(q^2 - m^2) q^\lambda F_{\lambda\nu} = -2\delta'(q^2 - m^2) \left( q^\mu S_{\mu(n)}^{\rho\nu} + q^\nu S_{\mu(n)}^{\rho\mu} \right) F_{\rho\nu} - 2 \bar{F}^{\mu\nu} q_\rho \delta'(q^2 - m^2) + \frac{2\delta(q^2 - m^2) \bar{F}^{\mu\rho}}{(q \cdot n + m) m(q_{\rho} + m n_{\rho})} - \frac{2\delta(q^2 - m^2)}{q \cdot n + m} \bar{F}^{\mu\rho} n_\rho, \tag{C11}
\]
which yields
\[
-2S_{\mu(n)}^{\rho\nu} \delta'(q^2 - m^2) q^\lambda F_{\lambda\nu} = -\delta'(q^2 - m^2) \left( q^\mu S_{\mu(n)}^{\rho\nu} F_{\rho\nu} + \bar{F}^{\mu\rho} q_\rho - \frac{m(q_{\rho} + m n_{\rho}) \bar{F}^{\mu\rho}}{(q \cdot n + m)} \right) - \frac{\delta(q^2 - m^2)}{q \cdot n + m} \bar{F}^{\mu\rho} n_\rho. \tag{C12}
\]
In addition, one can show
\[
\frac{q^\rho F_{\rho\sigma} \epsilon_{\mu\rho\sigma\nu} n_\sigma}{(q \cdot n + m)} = \frac{1}{q \cdot n + m} (-q^\mu \bar{F}^{\nu\sigma} n_\sigma + \bar{F}^{\mu\sigma} n_\sigma q_\nu + m \bar{F}^{\mu\nu}) - \bar{F}^{\mu\nu}. \tag{C13}
\]
Accordingly, \(\hat{C}_2^{(n)\mu}\) takes the form
\[
\hat{C}_2^{(n)\mu} = \frac{\delta(q^2 - m^2)}{2} \left\{ 2q^\mu S_{\mu(n)}^{\rho\nu}(\bar{f}_V \Delta_\rho \Sigma_{\nu} - f_V \Delta_\rho \bar{\Sigma}_{\nu}) + 2S_{\mu(n)}^{\rho\nu} m(\Sigma_S \Delta_\nu \bar{f}_V - \bar{\Sigma}_S \Delta_\nu f_V) 
- \frac{\epsilon_{\mu\rho\sigma\nu} m(q_{\rho} + m n_{\rho})}{(q \cdot n + m)} (\bar{f}_V \Delta_\sigma \Sigma_{\nu} - f_V \Delta_\sigma \bar{\Sigma}_{\nu}) + 2(\bar{f}_V \Sigma_{\nu} - f_V \Sigma_{\nu}) \left[ q^\mu \Delta_\alpha S_{\mu(n)}^{\nu\alpha} + \bar{F}^{\mu\nu} 
- \frac{q^\rho \bar{F}^{\mu\rho} n_\sigma}{(q \cdot n + m)} + \frac{m \bar{F}^{\mu\nu}}{(q \cdot n + m)} - m \Delta_\alpha \frac{\epsilon_{\mu\rho\sigma\nu} (q_{\rho} + m n_{\rho})}{2(q \cdot n + m)} \right] \right\} + \frac{\delta(q^2 - m^2)}{2m} \left[ q^\mu (f_V q \cdot \Delta \Sigma_P - \bar{f}_V q \cdot \Delta \Sigma_P) - m^2 (\partial_{\nu} \bar{\Sigma}_{\mu} f_V - (\partial_{\nu} \Sigma_P) \bar{f}_V) \right].
\]
\[-\delta'(q^2 - m^2) \left( q^\mu S_{m(n)}^{\mu \nu} F_{\rho \nu} - \frac{m(q_\rho + mn_\rho) \tilde{F}^{\rho \mu}}{(q \cdot n + m)} \right) C_V[f_V] - m \left( 2S_{m(n)}^{\mu \nu} q^\lambda F_{\lambda \nu} - \tilde{F}^{\mu \nu} q_\nu \right) \]
\times \delta'(q^2 - m^2) + \delta(q^2 - m^2) \left( \Delta_{\nu} S_{m(n)}^{\mu \nu} \right) + \delta(q^2 - m^2) S_{m(n)}^{\mu \nu} \Delta_{\nu} \right) C_S[f_V], \tag{C14}
\]

which can be further rearranged as

\[
\hat{\mathcal{C}}_2^{(n)}_\mu = q^\mu \left\{ \delta(q^2 - m^2) \left[ S_{m(n)}^{\rho \nu}(\bar{f}_V \Delta_\rho \Sigma_{\nu \nu} - f_V \Delta_\rho \Sigma_{\nu \nu}) + (f_V \Sigma_{\nu \nu} - f_V \Sigma_{\nu \nu}) \left( \partial_\alpha S_{m(n)}^{\rho \nu} + \frac{S_{m(n)}^{\rho \nu} F_{\rho \sigma} n^\sigma}{q \cdot n + m} \right) \right] \right. \\
+ \frac{1}{2m} \left( f_V q \cdot \Delta_\rho P - \bar{f}_V q \cdot \Delta_\rho P \right) \left. \right] - \delta'(q^2 - m^2) S_{m(n)}^{\rho \nu} F_{\rho \nu} C_V[f_V] \right. \\
+ m \left\{ \delta(q^2 - m^2) \left[ S_{m(n)}^{\rho \nu}(\Sigma_S \Delta_\rho \Sigma_{\nu \nu} - \Sigma_S \Delta_\rho f_V) - \Delta_{\nu} \left( S_{m(n)}^{\rho \nu} C_S[f_V] \right) - \frac{\epsilon^{\mu \rho \sigma}(q_\rho + mn_\rho)}{2(q \cdot n + m)} \right] \right. \\
\times (\tilde{f}_V \Delta_\sigma \Sigma_{\nu \nu} - f_V \Delta_\sigma \Sigma_{\nu \nu}) + (\bar{f}_V \Sigma_{\nu \nu} - f_V \Sigma_{\nu \nu}) \left( \frac{\tilde{F}^{\mu \nu}}{(q \cdot n + m)} - \epsilon^{\rho \mu \sigma} \Delta_\alpha \frac{(q_\rho + mn_\rho)}{2(q \cdot n + m)} \right) \\
- \frac{1}{2} \left( (\partial^\mu \Sigma_\rho P f_V - (\partial^\mu \Sigma_\rho P) \bar{f}_V \right) \\
\left. \right] + \delta'(q^2 - m^2) \left[ \frac{(q_\rho + mn_\rho) \tilde{F}^{\rho \nu}}{(q \cdot n + m)} C_V[f_V] - (2S_{m(n)}^{\mu \nu} q^\lambda F_{\lambda \nu} - \tilde{F}^{\mu \nu} q_\nu) C_S[f_V] \right] \right\}, \tag{C15}
\]

where we utilize

\[
\Delta_\alpha S_{m(n)}^{\rho \nu} = \frac{\tilde{F}^{\nu \sigma} n_\sigma}{(q \cdot n + m)} = \partial_\alpha S_{m(n)}^{\rho \nu} + \frac{S_{m(n)}^{\rho \nu} F_{\rho \sigma} n^\sigma}{q \cdot n + m}. \tag{C16}
\]

**Appendix D: Derivation of the spin diffusion for SKE**

From Eqs. (39) and (46), we find

\[
Q_1(p) = 2\pi \delta(q^2 - m^2) \left( 2q^\mu G^{< \mu}_{\mu \nu} q^\nu - p \cdot q' G^{< \mu}_{\mu \nu} \right) \\
= 2\pi \delta(q^2 - m^2) g_{0p} \frac{p_0}{T} \left( 2\hat{\rho}_T (|q|^2 (1 - z^2) - p_0 q_0 + |q||p|z) + \hat{\rho}_L (2q_0^2 - p_0 q_0 - |q||p|z) \right), \tag{D1}
\]

where we defined \( \hat{\rho}_T/L \equiv \rho_T/L T/p_0 \) and \( z \equiv \hat{p} \cdot \hat{q} \). We also used the transversality \( p_\mu P_T^{\mu \nu} = 0 \) and the on-shell conditions \( m^2 = q^2 = p^2 + q^2 + 2p \cdot q' \) and \( m^2 = q^2 = p^2 + q^2 - 2p \cdot q \) that lead to relations \( 2p \cdot q = p^2 = -2p \cdot q' \).

The on-shell condition can be also arranged as

\[
q^2 - m^2 = p^2 - 2q \cdot p = (p^0 - q^0)^2 + (2|q||p|z - p^2 - q_0^2) \tag{D2}
\]
Therefore, the delta function is nonzero when the momenta satisfy the conditions
\[ p^0 = q^0 \pm \sqrt{p^2 - 2|q||p|z + q_0^2} \sim q^0 \pm \left( q^0 - \frac{|q||p|z}{q_0} + \frac{p^2(q_0^2 - q^2z^2)}{2q_0^3} \right) \] (D3)
where the square root is expanded for a small momentum transfer $|p|$. We take the positive-energy solution $q^0 = q^0 - p^0 > 0$ for particles from the lower sign. Defining $\tilde{p} \equiv |q||p|z/q_0 - p^2(q_0^2 - q^2z^2)/(2q_0^3)$, we have\(^3\)
\[ \delta(q^2 - m^2) \sim \frac{1}{2q^0} \left( 1 + \frac{|q||p|z}{q_0^2} + O(|p|^2) \right) \delta(p^0 - \tilde{p}). \] (D4)

Based on the dispersion relation above, we can approximate the spectral function as
\[ \hat{\rho}_T(p) \to \frac{\pi T m_D^2}{2|p|^5} I_T(z) \left( 1 - \frac{|q||p|z}{q_0^2} + O(|p|^2) \right), \] (D5)
where $I_T(z) \equiv [1 - (|q|z/q_0)^2]^{-1}$. Here, the presence of the delta function $\delta(p^0 - \tilde{p})$ is assumed on the right-hand side of arrow. On the other hand, we have $\hat{\rho}_L(p) = \pi T m_D^2/|p|^5$.

Note that the difference between the distribution functions provides positive powers in the small momentum transfer limit:
\[ f_{Vq} - f_{Vq-p} \approx p^0\partial_{q^0}f_{Vq} - \frac{p^2p^3}{2}\partial_{q^\alpha}\partial_{q^\beta}f_{Vq}, \] (D6)
where $p^0 \sim O(|p|^1)$. Therefore, we have found the momentum dependences
\[ \hat{\rho}_{L,T} \sim |p|^{-5}, \quad f_{Vq} - f_{Vq-p} \sim |p|, \quad d^3p \sim d|p||p|^2. \] (D7)
This order counting suggests that one should maintain the other factors up to $O(|p|)$ to get the leading-log result. Therefore, we expand the remaining factor as
\[ \delta(q^2 - m^2)g_0 \frac{p_0}{T} \approx \frac{1}{2|q_0|} \left[ 1 + \frac{|p||q|z}{q_0} \left( 1 - \frac{q_0}{2T} \right) + O(|p|^2) \right] \delta(p^0 - \tilde{p}). \] (D8)
where we used $g_0 \approx T/p_0 - 1/2$.

Combining all pieces together, it is found
\[ \int_{p} Q_1(f_{Vq} - f_{Vq-p}) \approx \frac{\pi T m_D^2}{2|q_0|} \int_{p} \frac{1}{|p|^5}(2\pi)\delta \left( p_0 - \frac{p \cdot q}{q_0} \right) \left[ 1 - \frac{|p||q|z}{2q_0T} \right] \left[ 2q_0^2 + |q|^2(1 - z^2)I_T(z) \right] (f_{Vq} - f_{Vq-p}). \] (D9)

In Eq. (D6), we apply the following decomposition of $p_{\perp}^\beta$ in terms of the transverse and longitudinal components with respect to $q_{\perp}^\mu$:
\[ p_{\perp}^\mu = q_{\perp}^\mu|p|z + \hat{\Theta}_q^{\alpha\beta}p_\nu, \quad \hat{\Theta}_q^{\alpha\beta} = \Theta^{\alpha\beta} + q_{\perp}^{\alpha}q_{\perp}^{\beta}, \] (D10)
\(^3\) It turns out that the $O(|p|^2)$ term in the Delta function does not contribute to the leading logarithmic order.
where \( q_α \hat{Θ}^{αβ} = 0 \). Moreover, the distribution function \( f_{Vq} \) in general can be a function of \( q_0 \) before implementing the on-shell condition. It is in fact more convenient to keep the off-shell form when \( F^{μν} ≠ 0 \) due to the presence of \( δ'(q^2 - m^2) \) term in the AKE. Nonetheless, in the absence of background fields, it is more practical and convenient to write down the on-shell kinetic equations. We hence take \( f_{Vq} = f_{Vq}(q, X) \) as just a function of \( q \) and \( X \) by using \( q_0 = E_q = \sqrt{|q|^2 + m^2} \) for fermions (here we neglect anti-fermions) in the Wigner functions. Accordingly, all the terms proportional to \( ∂_{q_0} f_{Vq} \) can be dropped.

Applying the replacement

\[
pαpβ → p_0 α uα + p_0 |p|z(uαqβ + uβqα) + |p|^2 (qαqβz^2 - \frac{Θ^{αβ}}{2}(1 - z^2))
\]

(D11)

for the integrand and carrying out the integration, Eq. (D9) results in

\[
\begin{align*}
\delta(q^2 - m^2) & \int p Q_1(f_{Vq} - f_{Vq-p}) \\
& \approx -\frac{πm_2^2\delta(q^2 - m^2)}{2q_0} \int_{m_D}^T \frac{d|p|}{2(2π^2)|p|} \int_{-1}^1 dz \left( \frac{|q|^2(1 - z^2)I_T(z)}{2} + q_0^2 \right) \left[ \frac{|q|^2}{q_0} \hat{q}_⊥ β + \left( \hat{q}_⊥ q_β z^2 \right) - \frac{Θ^{αβ}}{2}(1 - z^2) \right] Θ^{αβ} T \partial_{q_α} f_{Vq} \\
& = -\frac{m_2^2\delta(q^2 - m^2)}{8πq_0} \ln(1/g_c) \left[ q_0 \left( j^T_1 = \frac{q_0}{|q|^2}(j^2_2 + j^L_1)q_β \partial_{q_β} - T \left( J^T_0 - \frac{4q_0^2}{|q|^2}j^L_1 + \frac{3q_0^4}{|q|^4}j^2_2 \right) + q_0^2(j^L_0 - j^L_1) \right) Θ^{αβ} T \partial_{q_α} \partial_{q_β} \right] f_{Vq},
\end{align*}
\]

(D12)

where we take \( u^α \partial_{q^α} f_{Vq} = 0 \) and

\[
\begin{align*}
\begin{align*}
J^T_0 & \equiv \int_{-1}^1 dz \frac{1}{2(1 - \frac{|q|^2z^2}{q_0^2})} = \frac{q_0}{|q|} n_q, \quad J^T_1 & \equiv \int_{-1}^1 dz \frac{(\frac{|q|z}{q_0})^2}{2(1 - \frac{|q|^2z^2}{q_0^2})} = \frac{q_0}{|q|} n_q - 1,
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
J^2_2 & \equiv \int_{-1}^1 dz \frac{\frac{|q|z}{q_0}}{2(1 - \frac{|q|^2z^2}{q_0^2})} = \frac{q_0}{|q|} n_q - \frac{|q|^2}{3q_0^2} - 1,
\end{align*}
\end{align*}
\]

(D13)

where \( n_q = 2^{-1} \ln \left[ (q_0 + |q|)/(q_0 - |q|) \right] \), and

\[
\begin{align*}
\begin{align*}
J^L_0 & \equiv \int_{-1}^1 dz = 2, \quad J^L_1 & \equiv \int_{-1}^1 z^2 dz = \frac{2}{3},
\end{align*}
\end{align*}
\]

(D14)
By inputting explicit forms of $j_{i}^{L/T}$ into (D12), we eventually obtain

$$
\delta(q^2 - m^2) \int_{p} Q_{1}(f_{Vq} - f_{Vq-p})
\approx -\frac{m_{D}^{2}\delta(q^2 - m^2)}{8\pi\ln(1/g_{c})}\left[\frac{E_{q}^{2}}{|q|^{2}}\left(1 - \frac{m_{D}^{2}g_{q}}{2|q|^{2}}\right)q_{\perp}^{2} + \frac{m^{2}T}{2|q|^{3}}\left(1 - \frac{3E_{q}^{2}}{|q|^{2}}\right)\eta_{q} + 3E_{q}\right)q_{\perp}^{a}\partial_{q_{\perp}^{a}}
\right]
$$

where we take $q_{0} = E_{q}$.

Subsequently, we should also calculate the $\tilde{Q}_1 f_{Vq}$ term in Eq. (47). While the same counting (D7) is applied to this case, there is not the order-one suppression from the difference $f_{Vq} - f_{Vq-p}$. Nevertheless, a factor of $g_{q}^{-1}$ provides the same order of suppression instead. Therefore, maintaining the linear terms from the other factors as in the computation above, we find

$$
\delta(q^2 - m^2) \int_{p} \tilde{Q}_{1} f_{Vq}
\approx -\frac{\pi m_{D}^{2}\delta(q^2 - m^2)}{2q_{0}} \int_{m_{D}}^{T} \frac{d|p|}{(2\pi)^{2}|p|} \int_{-1}^{1} dz \left(\frac{|q|^{2}f_{T}(1 - z^{2})}{2} + q_{0}^{2}\right) \frac{1}{q_{0}} \left(1 - \frac{|q|^{2}z^{2}}{q_{0}^{2}}\right) f_{Vq}
$$

which yields

$$
\delta(q^2 - m^2) \int_{p} \tilde{Q}_{1} f_{Vq}(1 - f_{Vq-p})
\approx -\frac{m_{D}^{2}\delta(q^2 - m^2)}{4\pi} \ln(1/g_{c}) \left[f_{Vq}(1 - f_{Vq}) - \frac{q_{0}^{2}}{|q|^{2}} \left(1 - \frac{\eta_{q}m_{D}^{2}}{q_{0}|q|}\right) f_{Vq}q_{\perp}^{a}\partial_{q_{\perp}^{a}}f_{Vq}\right].
$$

Accordingly, the SKE results in the form given in Eq. (50).

**Appendix E: Derivation of the spin diffusion for AKE**

To evaluate the collision term in AKE, we implement the same strategy as in the SKE.

1. **Half of the terms: $[Q_{2}^{\mu\nu}\tilde{a}_{\nu} - Q_{1}\tilde{a}_{\mu}]$**

For convenience, we first apply the following decompositions to two of the four terms in Eq. (48):

$$
Q_{2}^{\mu\nu}\tilde{a}_{\nu} - Q_{1}\tilde{a}_{\mu} = 2\pi\delta(q^2 - m^2)\left(Q_{2}^{\mu} + Q_{1}^{\mu} + Q_{2}^{\mu}\right),
$$

(E1)
where

\[
Q_{\text{cl}(a)}^\mu \equiv 2(q_{\rho}^\mu G_{\rho}^{<q})(\delta_\mu^\nu - \delta_\nu^\mu) + G_{\rho}^{<p}(p \cdot q_\mu ) + p \cdot q \delta_\mu^\nu = g_0 \frac{p_0}{T} \left( \dot{Q}_{\text{cl}(a)}^{T \mu} + \dot{Q}_{\text{cl}(a)}^{L \mu} \right),
\]

(E2a)

\[
Q_{\text{cl}(b)}^\mu \equiv - (G_{\rho}^{<p} p_\mu + 2q_\mu G_{\rho}^{<p}) q \cdot \delta_\mu^\nu = g_0 \frac{p_0}{T} \left( \dot{Q}_{\text{cl}(b)}^{T \mu} + \dot{Q}_{\text{cl}(b)}^{L \mu} \right),
\]

(E2b)

\[
Q_{\text{cl}(c)}^\mu \equiv 2(p^\mu G_{\rho}^{<q} q^\rho - q \cdot p G_{\rho}^{<q}) \delta_\mu^\nu = g_0 \frac{p_0}{T} \left( \dot{Q}_{\text{cl}(c)}^{T \mu} + \dot{Q}_{\text{cl}(c)}^{L \mu} \right).
\]

(E2c)

The remaining two terms will be addressed in Sec. E2. We utilized \( q \cdot a_q = (q^2 - m^2) f_{Aq} \) from the on-shell condition. On the rightmost side, the terms proportional to the transverse and longitudinal spectral functions are denoted as \( \dot{Q}_{\text{cl}}^{T \mu} \) and \( \dot{Q}_{\text{cl}}^{L \mu} \), respectively. Note that, similar to \( f_{Vq} \mu \) in the SKE, the expansion of \( a_{q-p} \) give rise to the factors of \( p \) for the small momentum transfer limit. Recalling the order counting in Eq. (D7), we thus maintain the terms in \( \dot{Q}_{\text{cl}}^{T,L \mu} \) up to \( \mathcal{O}(|p|^2) \) in the following. For \( Q_{\text{cl}(a)}^\mu \), we have

\[
\dot{Q}_{\text{cl}(a)}^{T \mu} = 2 \hat{\rho}_T \left[ |q|^2 (1 - z^2) (\delta_\mu^\nu - \delta_\nu^\mu) - \left( (p_0 q_\mu - |q||z|)(\delta_\mu^\nu - \delta_\nu^\mu) - (p_\mu^2 - |q|^2) \delta_\mu^\nu \right) \right] \\
vq \approx 2 \hat{\rho}_T \left[ \left( |q|^2 (1 - z^2) - p_\mu^2 \right) \delta_\mu^\nu \right] \\
+ |q|^2 (1 - z^2) \frac{p_\rho p_\beta}{2} \partial_{\rho} \partial_{\beta} \delta_\mu^\nu + \mathcal{O}(|p|^3),
\]

(E3a)

\[
\dot{Q}_{\text{cl}(a)}^{L \mu} = \hat{\rho}_L \left[ 2(q_\mu^2 - p_\mu q_\mu)(\delta_\mu^\nu - \delta_\nu^\mu) + \left( (p_0 q_\mu - |q||z|)(\delta_\mu^\nu - \delta_\nu^\mu) - (p_\mu^2 - |q|^2) \delta_\mu^\nu \right) \right] \\
vq \approx \hat{\rho}_L \left[ \left( |q|^2 (1 - z^2) - p_\mu^2 \right) \delta_\mu^\nu \right] \\
+ q_\mu^2 (p_\rho p_\beta \partial_{\rho} \delta_{\beta} \delta_\mu^\nu + \mathcal{O}(|p|^3)).
\]

(E3b)

As for \( Q_{\text{cl}(b)}^\mu \), we have

\[
\dot{Q}_{\text{cl}(b)}^{T \mu} = 2 \hat{\rho}_T \left( p_\mu^2 + q_\mu^2 - |q||p_\mu^\mu||^2 \right) p \cdot \delta_\mu^\nu - q_\mu^2 (p_\mu^2 - q_\mu^2) q \cdot \delta_\mu^\nu + \mathcal{O}(|p|^3),
\]

(E4a)

Here, we drop the terms proportional to \( q \cdot a_q = (q^2 - m^2) f_{Aq} \) based on the on-shell condition. Finally, for \( Q_{\text{cl}(c)}^\mu \), we have

\[
\dot{Q}_{\text{cl}(c)}^{T \mu} = 2 \hat{\rho}_T \left( p_\mu^2 |q|^2 z^2 - q_\mu^2 \right) + (q_0 p_0 - |q||z|)(\Theta_{\mu}^\nu + \tilde{p}_\mu^\nu) \delta_\mu^\nu \\
\approx 2 \hat{\rho}_T \left( (p_0 u_\mu + p_\mu^2) |q|^2 z^2 - q_\mu^2 \right) + (q_0 p_0 - |q||z|)(\Theta_{\mu}^\nu + \tilde{p}_\mu^\nu) \delta_\mu^\nu \\
\times \left( a_{\mu q} - \rho_\nu \partial_{\mu} \delta_\nu^\nu + \mathcal{O}(|p|^2) \right),
\]

(E5a)

\[
\dot{Q}_{\text{cl}(c)}^{L \mu} = \tilde{\rho}_L \left( 2 q_\mu^2 p_\mu^2 - 2 q_\mu^2 |q|^2 u_\mu u_\nu \right) \delta_\mu^\nu \\
\approx \tilde{\rho}_L \left( 2 q_\mu^2 p_\mu^2 + 2 |q|^2 |z|^2 u_\mu u_\nu \right) \delta_\mu^\nu + \mathcal{O}(|p|^2).
\]

(E5b)
In the following, we evaluate those terms one by one. As in the case of SKE, we first retrieve the integrals for $|p|$ that give rise to the logarithm in the following subsections. The angle integrals, with respect to $z$, will be performed in Sec. E.3 afterwards.

a. Evaluating $Q^\mu_{cl(a)}$

Combining with Eqs. (D8) and (D5), we find

$$
\delta(q^2 - m^2) \int_p 2\pi \delta((q - p)^2 - m^2) Q^\mu_{cl(a)} \\
\approx \frac{\pi T m_T^2 \delta(q^2 - m^2)}{2 q_0} \int_{m_D}^T \frac{d|p|}{(2\pi)^2|p|} \int_{-1}^1 dz \left[ \left( \frac{q_0 z^2}{T} + \frac{I_T z^2 (1 - z^2)|q|^2}{2 q_0 T} \right) q^\beta \partial_{q^\beta} q^\alpha - \left( \frac{q_0^2 + I_T |q|^2 (1 - z^2)}{2} \right) \left( (1 - z^2) \eta^\alpha^\beta + (1 - 3 z^2) \hat{q}_1^\alpha q_1^\beta \right) \frac{\partial_{q^\alpha} \partial_{q^\beta} q_0}{2} \right] \hat{a}_q^\mu,
$$

(E6)

where we take $u^\alpha \partial_{q^\alpha} \hat{a}_q^\mu = 0$ by imposing $q_0 = E_q = \sqrt{|q|^2 + m^2}$ upon $\hat{a}_q^\mu$ in advance and utilize

$$
\left( \hat{q}_1^\alpha q_1^\beta z^2 - \frac{\hat{\Theta}^\alpha^\beta}{2} (1 - z^2) \right) = -\frac{\Theta^\alpha^\beta}{2} (1 - z^2) - \frac{(1 - 3 z^2)}{2} \hat{q}_1^\alpha \hat{q}_1^\beta.
$$

(E7)

b. Evaluating $Q^\mu_{cl(b)}$

Next, we evaluate the term proportional to $Q^\mu_{cl(b)}$:

$$
\delta(q^2 - m^2) \int_p 2\pi \delta((q - p)^2 - m^2) Q^\mu_{cl(b)} \approx \frac{\pi T m_T^2 \delta(q^2 - m^2)}{2 q_0} \int \frac{d^3 p}{(2\pi)^3 |p|^5} (\hat{K}^{T\mu}_{cl(b)} + \hat{K}^{L\mu}_{cl(b)}),
$$

(E8)

where

$$
\hat{K}^{T\mu}_{cl(b)} = I_T \left[ \left| p \right| (q_+^\mu - \hat{p}_+^\mu) |q| z \left( \frac{|q|}{q_0} u^\nu z + \hat{p}_+^\nu \right) + \frac{|p|^2}{2 q_0^2 T} \left( |q|^2 z^2 (u^\nu ((T - q_0) q_+^\mu + 2 q_0 T u^\mu) + \hat{p}_+^\mu \hat{p}_+^\nu q_0^2) + q_0^2 T (2 \hat{p}_+^\mu \hat{p}_+^\nu q_0 - q_+^\nu u^\nu) + \left| q \right| q_0^2 z (3 \hat{p}_+^\mu T u^\nu - \hat{p}_+^\nu q_+^\mu + 2 \hat{p}_+^\nu T u^\mu) + \left| q \right|^3 \hat{p}_+^\mu u^\nu z^3 (q_0 - T) \right) \right] \hat{a}_q u^\nu,
$$

(E9a)

$$
\hat{K}^{L\mu}_{cl(b)} = -2 \hat{a}_q u^\nu \left| p \right| (\hat{p}_+^\nu q_0 + |q| u^\nu z) + \frac{|p|^2}{2 q_0^2 T} \left( u^\mu (u^\nu (|q|^2 z^2 (q_0 - 2 T) + q_0^2 T) + \hat{p}_+^\nu |q| q_0 z (q_0 - T)) - \hat{p}_+^\nu q_0 T (|q| u^\nu z + \hat{p}_+^\nu q_0) \right) \hat{a}_q + 2 q_0 u^\mu \left( \frac{|q|}{q_0} u^\nu z + \hat{p}_+^\nu \right) \left( \frac{|q|}{q_0} u^\nu z + \hat{p}_+^\nu \right) \left| p \right|^2 \partial_{q^\nu} \hat{a}_q + O(|p|^3).
$$

(E9b)
It is clear that \( \mathcal{O}(|p|) \) terms in \( \tilde{K}_{cl}^{T/L}\mu \) can be dropped by symmetry of the integration. We should further employ the useful decomposition and replacements in Eqs. (D10), (D11), and the following relation in the integral,

\[
z p_\perp \bar{p}_\perp p_\perp^0 = z|p|^3(q_\perp^0 z + \hat{p}_T^\perp)(q_\perp^0 z + \hat{p}_T^\perp)(q_\perp^0 z + \hat{p}_T^\perp)
\]

\[
\rightarrow |p|^3 \left( z^4 q_\perp^{\nu} q_\perp^{\mu} q_\perp^0 - \frac{z^2(1-z^2)}{2} \left( q_\perp^{\mu} \hat{\Theta}_q^{\mu\nu} + q_\perp^{\nu} \hat{\Theta}_q^{\mu\nu} + q_\perp^{\mu} \hat{\Theta}_q^{\nu\mu} \right) \right),
\]

(E10)

where \( \hat{p}_T = \hat{\Theta}_q^{\mu\nu} \hat{p}_{\perp\nu} \) and we drop terms with odd powers of \( z \). Dropping the vanishing terms by symmetry in Eq. (E8), we obtain

\[
\delta(q^2 - m^2) \int_p 2\pi \delta((q - p)^2 - m^2)Q_{cl}^\mu
\]

\[
\approx \frac{\pi T m_p^2 \delta(q^2 - m^2)}{2q_0} \int_{m_D}^T \frac{d|p|}{(2\pi)^2 |p|} \int_{-1}^1 dz \left( \mathcal{K}_{cl}^{T\mu} + \mathcal{K}_{cl}^{L\mu} \right),
\]

(E11)

where

\[
\mathcal{K}_{cl}^{T\mu} = I_T \left\{ u^{\mu} z^2 \left( \frac{q_\perp^\nu}{q_0} + \frac{|q|^2 u^{\nu}}{q_0^2} \right) - \frac{q_\perp^{\mu}}{4q_0^2 T} \left( q_0^2 q_\perp^\nu \left( 3|q|^2 z^2(1-z^2) + 2q_0 T(1-3z^2) \right) + 2u^{\nu}|q| \left( z^2(1-z^2)|q|^2(q_0 - T) + q_0^2 T(1-3z^2) \right) - \frac{\Theta^{\mu\nu}}{4}(1-z^2) \left( 2 + \frac{|q|^2 z^2}{q_0 T} \right) \right) \right. \]

\[
+ \frac{1}{2} \left[ q_\perp^{\mu}(1-z^2) \left( q_\perp^{\nu} \hat{\Theta}_q^{\mu\nu}(1-5z^2) + (1-z^2) \Theta^{\mu\nu} - 3 \frac{q_\perp^{\mu}}{q_0 T} u^{\nu} z^2 \right) \right.

\[
- z^2(1-z^2) \left( \Theta^{\mu\nu} q_\perp^{\nu} + \Theta^{\nu\mu} \left( q_\perp^{\nu} + \frac{|q|^2}{q_0} u^{\nu} \right) \right) \right] \partial_{q_\perp^{\nu}} \bar{a}_{\nu
\] (E12)

and

\[
\mathcal{K}_{cl}^{L\mu} = u^{\mu} \left[ u \cdot \bar{a}_q \left( 1 + \frac{|q|^2 z^2(q_0 - 2T)}{q_0^2 T} \right) + \hat{q}_\perp \cdot \bar{a}_q \left( \frac{|q|^2 z^2(q_0 - T)}{q_0^2 T} \right) \right] + \hat{q}_\perp \cdot \bar{a}_q \left( \frac{1-3z^2}{2} \right)
\]

\[
- u \cdot \bar{a}_q \left( \frac{|q|^2 z^2}{q_0} \right) + \Theta^{\mu\nu} \bar{a}_q \left( \frac{1-z^2}{2} \right) + u^{\mu} \left( 2z^2 q_\perp^{\mu} u^{\nu} - \hat{q}_\perp^{\nu} \hat{q}_\perp^{\nu} q_0 (1-3z^2) - \Theta^{\mu\nu} q_0 (1-z^2) \right) \partial_{q_\perp^{\nu}} \bar{a}_{\nu
\] (E13)

Since \( \delta(q^2 - m^2)q \cdot \bar{a}_q = \delta(q^2 - m^2)(E_q u \cdot \bar{a}_q + q_\perp \cdot \bar{a}_q) / q_0 = 0 \), we can use \( u \cdot \bar{a}_q = -q_\perp \cdot \bar{a}_q / q_0 \) to further simplify \( \mathcal{K}_{cl}^{T/L\mu} \), which results in

\[
\mathcal{K}_{cl}^{T\mu} = I_T \left\{ \hat{q}_\perp \cdot \bar{a}_q \left[ u^{\mu} m_2 \frac{|q|^2 z^2}{q_0^2 T} - \frac{q_0}{2T} \left( z^2(1-z^2) \left( 1 + \frac{q_0}{2T} \right) + \frac{m^2}{q_0 T} \left( 1 - \frac{q_0}{T} \right) \right) + \frac{m^2}{q_0^2 T} \left( 1 - 5z^2 + 2z^4 \right) \right] \right.

\[
+ \frac{q_0}{2T} z^2(1-z^2) \right] \right) - \frac{\Theta^{\mu\nu} \bar{a}_q}{2z^2(1-z^2)} \left( 1 + \frac{|q|^2 z^2}{2q_0 T} \right) + \frac{1}{2} \left( 1-z^2 \right) \hat{q}_\perp^{\nu} \left( |q|^2 \hat{q}_\perp^{\nu} (1-5z^2) \right
\]
\[ + |q| \Theta^{\alpha \nu} (1 - z^2) - \frac{3 |q|^2 \hat{z}_\perp^2}{q_0} \hat{\rho}^\mu \Theta^\alpha \nu \partial_{\hat{\rho}^\mu} - \left( |q| \hat{q}_\perp^\nu + \frac{|q|^2}{q_0} u^\nu \right) z^2 \Theta^\rho \mu \partial_{z^2} \tilde{a}_{q\mu} \right \}, \]

and

\[ \mathcal{K}^{L\mu}_{\text{cl}(b)} = u^\mu \left[ \frac{m^2 |q|^2 z^2}{q_0^2 T} - \frac{|q|}{q_0} \left( 1 - \frac{|q|^2 z^2}{q_0^2} + \frac{m^2 z^2}{q_0^2} \right) \right] \hat{q}_\perp \cdot \hat{a}_q + \hat{q}_\perp^\mu \left( \frac{1 - z^2}{2} - \frac{m^2 z^2}{q_0^2} \right) \hat{q}_\perp \cdot \hat{a}_q \]

\[ + \Theta^{\alpha \nu} \tilde{a}_{\nu} \left( 1 - z^2 \right) \frac{2}{2} + u^\mu \left( 2 z^2 \hat{q}_\perp^\nu u^\nu - \hat{q}_\perp^\nu u^\nu (1 - 3 z^2) - \Theta^{\alpha \nu} (1 - z^2) \right) \partial_{q^\nu} \tilde{a}_{q\nu}. \]

\[ (E15) \]

c. Evaluating \( Q^{\mu}_{\text{cl}(c)} \)

In the same way, we should cope with the other term

\[ \delta (q^2 - m^2) \int p \delta ((q - p)^2 - m^2) Q^{\mu}_{\text{cl}(c)} \approx \frac{\pi T m T_\rho^2 \delta (q^2 - m^2)}{2 q_0} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{|p|^5} \left( \tilde{K}^{T\mu}_{\text{cl}(c)} + \tilde{K}^{L\mu}_{\text{cl}(c)} \right), \]

\[ (E16) \]

where

\[ \tilde{K}^{T\mu}_{\text{cl}(c)} = I_T \left[ - |p| (q'_\perp - \hat{p}_\perp^\nu |q| z) \left( \frac{|q|}{q_0} u^\mu z + \hat{p}_\perp^\mu \right) + \frac{|p|^2}{2 q_0^2 T} \left( |q| q_0 z (q'_\perp - \hat{p}_\perp^\nu |q| z) (\hat{p}_\perp^\mu q_0 + |q| u^\mu z) \right. \right. \]

\[ - T (q_0^2 - |q|^2 z^2) (\hat{p}_\perp^\mu \hat{p}_\perp^\nu q_0 + \hat{p}_\perp^\nu |q| u^\mu z + \Theta^{\alpha \nu} q_0 - \hat{q}_\perp^\nu u^\nu) \right) \]

\[ + |p|^2 (q'_\perp - \hat{p}_\perp^\nu |q| z) \left( \frac{|q|}{q_0} u^\mu z + \hat{p}_\perp^\mu \right) \left( \frac{|q|}{q_0} u^\mu z + \hat{p}_\perp^\mu \right) \partial_{q^\nu} + O(|p|^3) \right] \tilde{a}_{q\nu}, \]

\[ (E17) \]

and

\[ \tilde{K}^{L\mu}_{\text{cl}(c)} = \frac{2 |p| (\hat{p}_\perp^\mu q_0 + |q| u^\mu z) u \cdot \tilde{a}_q - \frac{|p|^2}{q_0^2 T} |q| z (q_0 - 2 T) (\hat{p}_\perp^\mu q_0 + |q| u^\mu z) u \cdot \tilde{a}_q } \]

\[ - 2 |p|^2 (q_0 \hat{p}_\perp^\mu + |q| z u^\mu) \left( \frac{|q|}{q_0} u^\mu z + \hat{p}_\perp^\mu \right) \partial_{q^\nu} u \cdot \tilde{a}_q + O(|p|^3). \]

\[ (E18) \]

As above, the \( O(|p|) \) terms in \( \tilde{K}^{T/L\mu}_{\text{cl}(c)} \) do not contribute to the integral. Now, by employing Eqs. (D10), (D11), and (E10), we find

\[ \delta (q^2 - m^2) \int p \delta ((q - p)^2 - m^2) Q^{\mu}_{\text{cl}(c)} \approx \frac{\pi T m^2 T_\rho^2 \delta (q^2 - m^2)}{2 q_0} \int_{m T}^T \frac{d|p|}{(2\pi)^2 |p|} \int_1^1 dz \left( \mathcal{K}^{T\mu}_{\text{cl}(c)} + \mathcal{K}^{L\mu}_{\text{cl}(c)} \right), \]

\[ (E19) \]
where

\[
K_{c_{\text{cl(c)}}}^{T\mu} = I_T \left[ \frac{u^\mu q^\nu}{2q_0} (1 - z^2) \left( 1 + \left| \frac{q_0}{q_0 T} (q_0 - T) \right| \right) - \frac{\Theta^\mu^\nu}{4} \left( 1 + z^2 \right) \left( 1 - \left| \frac{q_0}{q_0 T} (q_0 - T) \right| \right) - \left| \frac{q_0^2}{q_0 T} (1 - z^2) \right| \right] \\
+ \frac{\bar{q}_1^\mu \bar{q}_1^\nu}{4} \left( 1 - 3z^2 \right) \left( 1 - \left| \frac{q_0}{q_0 T} \right| \right) + \frac{3}{2} \left| \frac{q_0^2}{q_0 T} \right| \left( 1 - z^2 \right) + \frac{u^\mu |q| z^2 (1 - z^2)}{2q_0} \left( 3\bar{q}_1^\mu \bar{q}_1^\nu + |q| \Theta^\mu^\nu \right) \partial_{q_1^\nu} \\
+ \frac{(1 - z^2)}{2} \left( z^2 \Theta^\mu^\nu q_1^\nu + \bar{q}_1^\nu \left[ (5z^2 - 1)\bar{q}_1^\rho \bar{q}_1^\rho + z^2 \Theta^\mu^\nu \right] \right) - (1 - z^2) q_1^\nu \Theta^\mu^\nu \partial_{q_1^\nu} \right] \tilde{a}_{q^\nu} \quad \text{(E20a)}
\]

\[
K_{c_{\text{cl(c)}}}^{L\mu} = -u \cdot \tilde{a}_q \frac{(q_0 - 2T)|q| z^2}{q_0 T} \left( \frac{|q| T}{q_0} u^\mu + \bar{q}_1^\mu \right) + q_0 \left( \bar{q}_1^\mu \bar{q}_1^\nu (1 - 3z^2) + \Theta^\mu^\nu (1 - z^2) \right) \\
- 2q_1^\mu q_0^{-1} u^\nu z^2 \partial_{q_1^\nu} (u \cdot \tilde{a}_q). \quad \text{(E20b)}
\]

2. Rest of the terms: [\(Q_2^{\mu^\nu} f_{Vq} \tilde{a}_{q^\nu} - Q_1 (1 - f_{Vq}) \tilde{a}_q^\nu\)]

As we have evaluated two of the four terms in Eq. (48), the remaining two terms are given as

\[
C_2 \equiv \delta(q^2 - m^2) \int q^{-1} \left[ \frac{Q_2^{\mu^\nu} f_{Vq} \tilde{a}_{q^\nu} - Q_1 (1 - f_{Vq}) \tilde{a}_q^\nu} \right]. \quad \text{(E21)}
\]

While \(Q_1\) has been given in the computation of the SKE, we now have the tensor part

\[
Q_2^{\mu^\nu} \tilde{a}_{q^\nu} = -2\pi \delta(q^2 - m^2) \rho_T \left\{ p^\nu \left( \bar{q}_1^\nu \left( 1 + \left| \frac{q_0}{q_0 T} \right| \right) - \frac{\Theta^\mu^\nu}{4} \left( 1 + z^2 \right) \left( 1 - \left| \frac{q_0}{q_0 T} \right| \right) - \left| \frac{q_0^2}{q_0 T} (1 - z^2) \right| \right) \right\} \tilde{a}_{q^\nu} \\
- \bar{q}_1^\nu \rho_T \left( \rho_T \left( 1 + \left| \frac{q_0}{q_0 T} \right| \right) - \frac{\Theta^\mu^\nu}{4} \left( 1 + z^2 \right) \left( 1 - \left| \frac{q_0}{q_0 T} \right| \right) - \left| \frac{q_0^2}{q_0 T} (1 - z^2) \right| \right) \right\} \\
+ 2u^\nu \rho_T \left( s \cdot u \right) \rho_T \left( 1 - \frac{z^2}{2} \right) - 2q_1^\nu q_0 \rho_T \left( s \cdot u \right) \rho_T \left( 1 - \frac{z^2}{2} \right) \right\} \tilde{a}_{q^\nu} - 2q_1^\nu q_0 \rho_T \left( s \cdot u \right) \rho_T \left( 1 - \frac{z^2}{2} \right) \right\} \tilde{a}_{q^\nu} \quad \text{(E22)}
\]

Combining the two terms, it is found

\[
C_2 \approx \frac{\pi m_T^2 \delta(q^2 - m^2)}{2q_0} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{|p|^5} \left( \tilde{K}_{c_{\text{cl(c)}}}^{T\mu} + \tilde{K}_{c_{\text{cl(c)}}}^{L\mu} \right), \quad \text{(E23)}
\]

where the integrand is given as

\[
\tilde{K}_{c_{\text{cl(c)}}}^{T\mu} = -I_T \left[ \frac{|q|^3 |p| q_0^2}{q_0 T} (1 - z^2) (1 - 2f_{Vq}) \tilde{a}_q^\nu + \frac{|q|^2 |p|^2}{q_0^2 T} \left( \tilde{a}_q^\nu (z^2 - 1) \left( q_0^2 (1 - 2z |q| \tilde{p}_1^\nu \partial_q q_{Vq}) - 2f_{Vq} \right) + (2f_{Vq} - 1) |q|^2 z^2) + 2f_{Vq} q_0 z \left( \tilde{a}_q^\nu (\tilde{p}_1^\mu \tilde{q}_1^\nu q_0 + \tilde{p}_1^\nu |q| u^\nu z^2 - \tilde{p}_1^\mu (q_0^2 z^2) \right) \right. \\
- \tilde{q}_1^\nu |q| u^\nu z + \tilde{q}_1^\nu |q| u^\nu z \right) - |q| \tilde{p}_1^\nu (\partial_q q_{Vq}) \tilde{a}_q^\nu \left( z^2 - 1 \right) \right) + O(|p|^3), \quad \text{(E24a)}
\]
\[ \tilde{\mathcal{K}}^{L\mu}_{\text{cl}(d)} = - \left[ \frac{2|q||p|q_0 z}{T} (1 - 2 f_{Vq}) \tilde{a}_\mu^\nu + \frac{|p|^2}{q_0^2 T} \left( \tilde{a}_\mu^\nu \left( q_0^3 (2z|q|p_+ (\partial_{q^\perp} f_{Vq}) + 2f_{Vq} - 1) + (1 - 2 f_{Vq})|q|^2 z^2 \right) \right) \right] + \mathcal{O}(|p|^3). \] (E24b)

Here, we utilize \( u \cdot \tilde{a}_q = -q_\perp \cdot \tilde{a}_q / q_0 \) in computations as above. Carrying out the integration, we obtain

\[ C_2 \approx \frac{m_D^2 T \delta(q^2 - m^2)}{8 \pi E_q} \ln(1/g_c) \left( \tilde{a}_\mu \tilde{Q}_{\text{cl}}^{(1)} + u^\mu \tilde{Q}_{\text{cl}}^{(2)} + q_\perp \tilde{Q}_{\text{cl}}^{(3)} + \tilde{Q}_{\text{cl}}^{(4)} q_\perp \partial_{\mu} \tilde{a}_{q\nu} + \tilde{Q}_{\text{cl}}^{(5)} q_\perp \partial_{\mu} \tilde{a}_{q\nu} + \tilde{Q}_{\text{cl}}^{(6)} q_\perp \partial_{\mu} \tilde{a}_{q\nu} + \tilde{Q}_{\text{cl}}^{(7)} q_\perp \partial_{\mu} \tilde{a}_{q\nu} \right), \] (E25)

where

\[ \tilde{Q}_{\text{cl}}^{(1)} = \frac{1}{|q|q_0 T} \left[ |q|q_0^2 \left( (1 - 2 f_{Vq}) (j_0^L + j_1^T) + j_2^T \right) - 2|q|(q_\perp \partial_{q^\perp} f_{Vq}) (j_1^L + j_1^T) \right] - (2 f_{Vq} - 1)|q|^3 (j_0^T - j_1^T) + 2(q_\perp \partial_{q^\perp} f_{Vq}) j_2^T q_0^4, \] (E26a)

\[ \tilde{Q}_{\text{cl}}^{(2)} = -2 \left( (j_1^L + j_1^T)|q|^2 - j_2^T q_0^2 \right) \frac{f_{Vq}}{|q|T} \tilde{a}_\mu \tilde{a}_q, \] (E26b)

\[ \tilde{Q}_{\text{cl}}^{(3)} = -2 \left( (j_1^L + j_1^T)|q|^2 - j_2^T q_0^2 \right) \frac{f_{Vq}}{q_0 T} \tilde{a}_\mu \tilde{a}_q, \] (E26c)

\[ \tilde{Q}_{\text{cl}}^{(5)} = 2 \left( q_0^2 q_2^T - |q|(j_1^L + j_1^T) \right) \frac{f_{Vq}}{T} q_0, \] (E26d)

\[ \tilde{Q}_{\text{cl}}^{(4)} = \tilde{Q}_{\text{cl}}^{(6)} = \tilde{Q}_{\text{cl}}^{(7)} = 0. \] (E26e)

3. Assembling all the pieces

Combining Eqs. (E6), (E11), and (E20a) and carrying out the integration, we acquire

\[ C_1 \equiv \delta(q^2 - m^2) \int_p (Q_2^{\mu\nu} \tilde{a}_{q^\perp} - Q_1 \tilde{a}^\mu_q) \approx \frac{m_D^2 T \delta(q^2 - m^2)}{8 \pi E_q} \ln(1/g_c) \left( \tilde{a}_\mu \tilde{Q}_{\text{cl}}^{(1)} + u^\mu \tilde{Q}_{\text{cl}}^{(2)} + q_\perp \tilde{Q}_{\text{cl}}^{(3)} + \tilde{Q}_{\text{cl}}^{(4)} q_\perp \partial_{\mu} \tilde{a}_{q\nu} + \tilde{Q}_{\text{cl}}^{(5)} q_\perp \partial_{\mu} \tilde{a}_{q\nu} + \tilde{Q}_{\text{cl}}^{(6)} q_\perp \partial_{\mu} \tilde{a}_{q\nu} + \tilde{Q}_{\text{cl}}^{(7)} q_\perp \partial_{\mu} \tilde{a}_{q\nu} \right), \] (E27)
where

\[ Q_{cl}^{(1)} = \frac{1}{2} \left( j_1^T + 3j_1^T - j_0^L - 3j_0^T + \frac{q_0^2}{|q|^2} (j_1^T - j_2) \right), \]  
\[ Q_{cl}^{(2)} = -\frac{1}{2} \left[ 2q_0 (\partial_{q_1} \tilde{a}_q) \left( j_0^L - (j_1^L + j_1^T) + j_2^T \frac{q_0^2}{|q|^2} \right) + 2q_0 \tilde{q}_1^\rho (\partial_{q_1} \tilde{a}_{q\nu}) \left( j_0^L - 3(j_1^L + j_1^T) + 3j_2^T \frac{q_0^2}{|q|^2} \right) \right. \\
\left. - \tilde{q}_1^\rho \tilde{a}_{q\nu} \frac{|q|}{q_0} \left( j_1^T \frac{(2q_0 - T)}{T} - 3j_0^L - j_0^T - 2j_1^T \frac{q_0 (m^2 - |q|^2)}{|q|^2 T} + j_1^T \frac{m^2 (2q_0 + 3T)}{|q|^2} \right) + j_2^T \frac{q_0^2}{|q|^2} \left( T - 2q_0 \right) \right], \]  
\[ Q_{cl}^{(3)} = -\frac{1}{2} \left[ \frac{2}{|q|^2} \left( \frac{q_0^2}{|q|^2} j_1^T - |q|^2 j_0^T \right) \left( \partial_{q_1} \tilde{a}_q' + \tilde{q}_1^\rho \left( \partial_{q_1} \tilde{a}_{q\nu} \right) \right) \left( j_0^L |q|^2 + 3j_2^T q_0^2 - 3|q|^2 (j_1^L + j_1^T) \right) \right. \\
\left. - \tilde{q}_1^\rho \tilde{a}_{q\nu} \frac{|q|^2}{q_0} \left( j_0^L - j_0^T \right) - m^2 q_0^2 (j_1^L + j_1^T) + 3m^2 q_0 j_1^T + j_1^L \frac{|q|^2}{q_0^2 T} (2q_0 - 3T) - 2j_1^T \frac{m^4}{q_0^2 |q|^2} \right. \\
\left. + 5j_1^T \frac{m^2}{|q|^2} + 2j_1^T \frac{|q|^2}{q_0^2 T} (q_0 - 2T) + 2j_2^T \frac{m^2 q_0}{|q|^2 T} - 6j_2^T \frac{m^2}{|q|^2 T} - 2j_2^T \frac{q_0}{q_0^2} (2q_0 + 3T) + 3j_2^T \frac{q_0^2}{|q|^2} \right]. \]

\[ Q_{cl}^{(4)} = |q| (j_1^L + j_1^T - j_0^L - j_0^T) + \frac{q_0^2}{|q|^2} (j_1^T - j_2), \]  
\[ Q_{cl}^{(5)} = \frac{q_0^2}{|q|^2} \left( (j_1^L + j_1^T) |q|^2 - j_2^T q_0^2 \right), \]  
\[ Q_{cl}^{(6)} = -\frac{1}{2} \left( j_0^L |q|^2 + (j_0^L - j_0^T - 2j_1^T) q_0^2 + j_2^T \frac{q_0^4}{|q|^2} \right), \]  
\[ Q_{cl}^{(7)} = -\frac{1}{2} \left( j_0^L |q|^2 + (j_0^L - 3j_1^L - 4j_1^T) q_0^2 + 3j_2^T \frac{q_0^4}{|q|^2} \right). \]

Note that we have implemented the following relations in computations,

\[ \Theta_{\mu\nu} \tilde{a}_{q\nu} = \tilde{a}_q' - (\tilde{a}_q \cdot u) u^\mu = \tilde{a}_q' + u^\mu \frac{|q|}{E_q} \tilde{q}_1 \cdot \tilde{a}_q, \]
\[ \partial_{q_{1\rho}} (\tilde{a}_q \cdot u) = - \partial_{q_{1\rho}} \left( \frac{q_{1\rho} \cdot \tilde{a}_q}{E_q} \right) = - \frac{1}{E_q} \Theta_{\mu\nu} \tilde{a}_{q\nu} + \frac{|q|^2}{E_q^2} \tilde{q}_1 \cdot \tilde{a}_q + q_{1\rho} \partial_{q_{1\rho}} \tilde{a}_{q\nu}, \]
\[ \Theta_{\mu\nu} q_{1\beta} \partial_{q_{\beta\rho}} \tilde{a}_{q\nu} = q_{1\beta} \partial_{q_{\beta\rho}} \tilde{a}_q - u^\rho q_{1\beta} \partial_{q_{\beta\rho}} (\tilde{a}_q \cdot u) = q_{1\beta} \partial_{q_{\beta\rho}} \tilde{a}_q + \frac{u^\rho}{E_q} \left( m^2 q_{1\beta} + q_{1\rho} q_{1\beta} \partial_{q_{\beta\rho}} \right) \tilde{a}_q. \]

By inputting exact expressions of \( j_1^{T/L} \) from Eqs. (D13) and (D14), it is found

\[ Q_{cl}^{(1)} = -2, \]  
\[ \text{(E29a)} \]
\[ Q^{(2)}_{cl} = \left( \frac{q_0^2}{|q|^2} \frac{2|q|}{q_0} \left( 2 + \frac{m^2}{|q|^2} \right) - \frac{\eta_q m^2}{|q|^2 T} (q_0 - 2T) \right) \hat{q}_\perp \cdot \hat{a}_q, \]  
(E29b)

\[ Q^{(3)}_{cl} = \left( \frac{q_0^2}{|q|^2 T} - \frac{\eta_q m^2}{q_0 |q|} \right) \hat{q}_\perp \cdot \hat{a}_q + \left( \frac{g_0 (m^2 + |q|^2)}{|q|^2} - \frac{3\eta_q m^2 q_0}{|q|^3} \right) \hat{q}_\perp \hat{q}_\perp \hat{a}_q, \]  
(E29c)

\[ Q^{(4)}_{cl} = -2|q|, \]  
(E29d)

\[ Q^{(5)}_{cl} = \frac{g_0}{|q|^2} \left( 1 - \frac{\eta_q m^2}{q_0 |q|} \right), \]  
(E29e)

\[ Q^{(6)}_{cl} = -\frac{g_0}{2} \left( 3 - \frac{g_0}{|q|^2} + \frac{\eta_q m^4}{q_0 |q|^3} \right), \]  
(E29f)

\[ Q^{(7)}_{cl} = \frac{m^2 q_0}{2|q|^3} (3|q|q_0 + \eta_q (|q|^2 - 3q_0^2)). \]  
(E29g)

and

\[ \hat{Q}^{(1)}_{cl} = \frac{2g_0}{|q|^2 T} \left[ |q| (1 - 2f_{Vq}) + (\hat{q}_\perp \hat{q}_\perp f_{Vq}) \left( \frac{m^2 q_0}{|q|} \eta_q - q_0^2 \right) \right], \]  
(E30a)

\[ \hat{Q}^{(2)}_{cl} = \frac{m^2 q_0 q_0}{\eta_q - q_0^2} \frac{f_{Vq}}{|q|^2 T} \hat{q}_\perp \cdot \hat{a}_q, \]  
(E30b)

\[ \hat{Q}^{(3)}_{cl} = \frac{2}{|q|} \hat{q}_\perp \left( \frac{m^2 q_0 q_0}{\eta_q - q_0^2} \frac{f_{Vq}}{q_0 T} \hat{q}_\perp \cdot \hat{a}_q, \right) \]  
(E30c)

\[ \hat{Q}^{(5)}_{cl} = \frac{2}{|q|} \hat{q}_\perp \left( \frac{2m^2 q_0 q_0}{\eta_q - q_0^2} \frac{g_0 f_{Vq}}{q_0 T} \right), \]  
(E30d)

\[ \hat{Q}^{(4)}_{cl} = \hat{Q}^{(6)}_{cl} = \hat{Q}^{(7)}_{cl} = 0. \]  
(E30e)

Finally, the sum of all the pieces \( C_1 + C_2 \) from Eqs. (E25) and (E27) provide the collision term of the AKE:

\[ \lambda^{-1}_c \delta(q^2 - m^2) \hat{c}^{\mu}_{cl} \approx \frac{m^2 g_0 \delta(q^2 - m^2) T}{8\pi E_q} \ln(1/g_c) \left( \hat{a}_q \hat{Q}^{(i)}_{cl} + u^{\mu} \hat{Q}^{(i)}_{cl} + \hat{q}_\perp \hat{Q}^{(i)}_{cl} + \hat{q}_\perp \hat{q}_\perp \hat{a}_q \right), \]  
(E31)

The coefficients \( \hat{Q}^{(i)}_{cl} \equiv \hat{Q}^{(i)}_{cl} + Q^{(i)}_{cl} \) (\( i = 1, \ldots, 7 \)) are shown in Eqs. (53a)-(53g).

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