Research Article

Precise Asymptotics in the Law of the Iterated Logarithm under Sublinear Expectations

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By an inequality of partial sum and uniform convergence of the central limit theorem under sublinear expectations, we establish precise asymptotics in the law of the iterated logarithm for independent and identically distributed random variables under sublinear expectations.

1. Introduction

Motivated by the work of g-expectation of Peng [1], Peng [2, 3] initiated the concept of the sublinear expectation space, which is a powerful tool to model the uncertainty of probability and distribution. We could consider sublinear expectation as an extension of the classical linear expectation. Peng [2, 3] constructed the basic framework, investigated basic properties, and proved the law of large number and central limit theorem under sublinear expectations. Motivated by the seminal work of Peng [2, 3], more and more limit theorems under sublinear expectation space have been established, which generalize the corresponding fundamental, important limit theorems in probability and statistics. Zhang [4–6] proved the exponential inequalities and Rosenthal’s inequalities and obtained an extension of the central limit theorem and Donsker’s invariance principle under sublinear expectations. Wu [7] established precise asymptotics for complete integral convergence under sublinear expectations. Yu and Wu [8] studied Marcinkiewicz-type complete convergence for weighted sums under sublinear expectations. Wu and Jiang [9] obtained a strong law of large numbers and Chover’s law of the iterated logarithm under sublinear expectations. Ma and Wu [10] studied the limiting behavior of weighted sums of extended negatively dependent random variables under sublinear expectations. Xu and Zhang [11, 12] studied three series theorem for independent random variables and the law of logarithm for arrays of random variables under sublinear expectations. Chen [13] proved strong laws of large numbers for sublinear expectations. For more results about limit theorems under sublinear expectations, the interested reader could refer to the studies of Hu et al. [14], Fang et al. [15], Kuczmaszewska [16], Wang and Wu [17], Hu and Yang [18], Zhang [19], and references therein.

Precise asymptotics in the law of the iterated logarithm is one of the fundamental problems in probability theory. Many related results have been derived in the probabilistic space. Their results can be found in the work of Gut and Spătaru [20]; Zhang [21]; Xiao et al. [22]; Huang et al. [23]; Jiang and Yang [24]; Wu and Wen [25]; Xu et al. [26]; Xu [27, 28]; and Xu [29]. However, in sublinear expectations, due to the uncertainty of sublinear expectation and related capacity, the precise asymptotics in the law of the iterated logarithm under sublinear expectations have not been reported. Motivated by the work of Wu [7], Xiao et al. [22], Xu et al. [26], and Xu [29], we try to investigate precise asymptotics in the law of the iterated logarithm under sublinear expectations. The aim of this paper is to prove the precise asymptotics in the law of the iterated logarithm for independent, identically distributed random variables under sublinear expectations. The main contribution of this paper...
is that we prove an useful inequality under sublinear expect-
ations in Lemma 1, and we extend the results of Xiao et al. [22], Xu et al. [26], and Xu [29] to those of the sublinear 
eq 0n expectation spaces. Our results may have the potential ap-
lications in finance or engineering fields (cf. Wu [7], Peng [3], Zhang [19], and references therein). Our basic idea in
this paper comes from that of Wu [7], Xiao et al. [22], Xu et al. [26], Xu [29], Špătaru [30], and Fuk and Nagaev [31]. In
conclusion, our results combined with the work of Wu [7] imply heuristically that many results about precise asym-
ptotics in the law of the iterated logarithm in probability spaces may still hold under sublinear expectations.

The rest of this paper is organized as follows: in Section 2, we
summarize necessary basic notions, concepts, and rel-
vant properties and give necessary lemmas under sublinear
expectations. In Section 3, we give our main results, The-
orems 1 and 2, whose proofs are presented in Sections 4 and
5, respectively.

2. Preliminaries

We use notations similar to those of Peng [3]. Let \((\Omega, \mathcal{F})\) be a
given measurable space. Let \(\mathcal{H}\) be a subset of all random
variables on \((\Omega, \mathcal{F})\) such that \(I_1 \in \mathcal{H}\), where \(A \in \mathcal{F}\), and if
\(X_1, \ldots, X_n \in \mathcal{H}\), then \(\varphi(X_1, \ldots, X_n) \in \mathcal{H}\) for each \(\varphi \in C_{\text{Lip}}(\mathbb{R}^n)\), where \(C_{\text{Lip}}(\mathbb{R}^n)\) denotes the linear space of
(local Lipschitz) function \(\varphi\) satisfying

\[
|\varphi(x) - \varphi(y)| \leq C (1 + |x|^m + |y|^m) (|x - y|), \quad \forall x, y \in \mathbb{R}^n,
\]

for some \(C > 0, m \in \mathbb{N}\), depending on \(\varphi\). We regard \(\mathcal{H}\) as the
space of random variables.

**Definition 1.** A sublinear expectation \(\mathbb{E}\) on \(\mathcal{H}\) is a func-
tional \(\mathbb{E}: \mathcal{H} \to \mathbb{R} = [-\infty, \infty]\) satisfying the following properties:
for all \(X, Y \in \mathcal{H}\), we have the following:

(a) Monotonicity: if \(X \geq Y\), then \(\mathbb{E}[X] \geq \mathbb{E}[Y]\)

(b) Constant preserving: \(\mathbb{E}[c] = c, \forall c \in \mathbb{R}\)

(c) Positive homogeneity: \(\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0\)

(d) Subadditivity: \(\mathbb{E}[X + Y] \geq \mathbb{E}[X] + \mathbb{E}[Y]\) whenever \(\mathbb{E}[X] + \mathbb{E}[Y]\)

is not of the form \(-\infty - \infty \text{ or } -\infty + \infty\)

A set function \(V: \mathcal{H} \to [0, 1]\) is called a capacity if it
satisfies the following:

(a) \(V(\varnothing) = 0, \ V(\Omega) = 1\)

(b) \(V(A) \leq V(B), A \subseteq B, A, B \in \mathcal{H}\)

A capacity \(V\) is said to be subadditive if it satisfies
\(V(A + B) \leq V(A) + V(B), A, B \in \mathcal{H}\).

In this paper, given a sublinear expectation space
\((\Omega, \mathcal{F}, \mathbb{E})\), we define a capacity: \(\mathbb{V}(A) = \inf \{\mathbb{E}[\xi]\}
\)

\(\xi \leq A, \xi \in \mathcal{H}\), \(\forall A \in \mathcal{F}\) (see Zhang [4]). Clearly, \(\mathbb{V}\) is a
subadditive capacity. We also define the Choquet expecta-
tions \(C_{\mathbb{V}}\) by

\[
C_{\mathbb{V}}(X) = \int_0^\infty \mathbb{V}(X > x)dx + \int_{-\infty}^0 (\mathbb{V}(X > x) - 1)dx.
\]

A sublinear expectation \(\mathbb{E}: \mathcal{H} \to \mathbb{R}\) is said to be contin-
uous if it satisfies the following:

(a) Lower continuity: \(\mathbb{E}[X_n] \to \mathbb{E}[X]\), if \(0 \leq X_n \to X, \ X_n \in \mathcal{H}\)

(b) Upper continuity: \(\mathbb{E}[X_n] \to \mathbb{E}[X]\), if \(0 \leq X_n \to X, \ X_n \in \mathcal{H}\)

A capacity \(V: \mathcal{H} \to [0, 1]\) is said to be continuous capacity
if it satisfies the following:

(1) Lower continuity: \(V(A_n) \to V(A), \ A_n \uparrow A, \ A_n, A \in \mathcal{H}\)

(2) Upper continuity: \(V(A_n) \to V(A), \ A_n \downarrow A, \ A_n, A \in \mathcal{H}\)

Assume that \(X = (X_1, \ldots, X_m), X_i \in \mathcal{H}\), and
\(Y = (Y_1, \ldots, Y_n), Y_j \in \mathcal{H}\), are two random variables on
\((\Omega_1, \mathcal{F}_1, \mathbb{E}_1)\) and \((\Omega_2, \mathcal{F}_2, \mathbb{E}_2)\). They are said to be identi-
cally distributed if

\[
\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{\text{Lip}}(\mathbb{R}^n),
\]

whenever the sublinear expectations are finite. \(\{X_{n_{1}}\}\)

is said to be identically distributed if for each \(i \geq 1, X_i\)

and \(X_1\) are identically distributed.

For \(0 \leq \alpha^2 \leq \beta^2 < \infty\), a random variable \(\xi\) under a sub-
linear expectation space \((\Omega, \mathcal{F}, \mathbb{E})\) is called a G-normal
\(\mathcal{N}(\mathcal{F}, \mathbb{E})\) distributed random variable, if for any \(\varphi \in C_{\text{Lip}}(\mathbb{R}^n)\), \(u(x,t) = \mathbb{E}[\varphi(X + \sqrt{t} \xi)] (x \in \mathbb{R}, t \geq 0)\) is the
unique viscosity solution of the following heat equation:

\[
\partial_t u - G(\partial_x^2 u) = 0,
\]

\(u(0,x) = \varphi(x),\)

where \(G(a) = (\alpha^2 a^2 - \beta^2 a^2)/2\).

In the rest of this paper, let \((X, X_n, n \geq 1)\) be a sequence of
i.i.d. random variables under sublinear expectation space
\((\Omega, \mathcal{F}, \mathbb{E})\) with \(\mathbb{E}[X] = \mathbb{E}(-X) = 0, \mathbb{E}(X^2) = \mathbb{E}(X^2)\), \(\mathbb{E}(X^2) = \mathbb{E}(X^2)\), \(\mathbb{E}(-X^2) = \mathbb{E}(-X^2)\), \(\mathbb{E}(-X^2) = \mathbb{E}(-X^2)\).

We denote by \(\xi\) a G-normal-distributed random variable
with \(\mathbb{E}(\xi) = \mathbb{E}(-\xi) = 0, \mathbb{E}(\xi^2) = \mathbb{E}(\xi^2)\), \(\mathbb{E}(\xi^2) = \mathbb{E}(\xi^2)\). We de-
note by \(C\) a positive constant which may vary from line to
line.

To prove our results, we need the following lemmas.

**Lemma 1.** Suppose \(\mathbb{E}[X]^\alpha < \infty, 1 < \alpha \leq 2\). Then, for \(x, y > 0\),

\[
\mathbb{E}[S_n > x] \leq 2n\mathbb{E}[|X| > y] + \frac{e\mathbb{E}[X]^\alpha}{n\mathbb{E}[X]^\alpha + xy^{\alpha-1}}^{xy}.
\]
Proof. We borrow the proofs from those of Theorem 2 by Fuk and Nagaev [31], and Lemma 2 by Spătaru [30]. Let

$$\bar{X}_i = \begin{cases} X_i & \text{for } |X_i| \leq y, \\ 0 & \text{for } |X_i| > y, \end{cases} \quad i = 1, \ldots, n,$$

and

$$\bar{S}_n = \sum_{i=1}^n \bar{X}_i.$$  \hspace{1cm} (6)

Therefore, by the subadditivity property of $\mathbb{V}(\cdot)$,

$$\mathbb{V}[S_n \geq x] \leq \mathbb{V}[\bar{S}_n \neq S_n] + \mathbb{V}[\bar{S}_n \geq x].$$  \hspace{1cm} (7)

By Markov’s inequality under sublinear expectations, for any positive $h$,

$$\mathbb{V}[\bar{S}_n \geq x] \leq e^{-hx} \mathbb{E}(e^{\bar{S}_n}).$$  \hspace{1cm} (8)

From this and (7), it follows that

$$\mathbb{V}[S_n \geq x] \leq \sum_{i=1}^n \mathbb{V}[|X_i| \geq y] + e^{-hx} \mathbb{E}(e^{\bar{S}_n})$$

$$= n \mathbb{V}[|X| \geq y] + e^{-hx} \mathbb{E}(e^{\bar{S}_n}).$$  \hspace{1cm} (9)

Application of the monotonicity of $u^{-c}(e^{hu} - 1 - hu)$ for $u \leq y$ and $u^{-a}(e^{hu} - 1 - hu)$ for $u > 0$ and the subadditivity property of sublinear expectations yields

$$e^{-hx} \mathbb{E}(e^{\bar{S}_n}) \leq \exp \left( \frac{x}{y} - \frac{x - n \mathbb{E}(|X|^{a} I_{|X|\leq y})}{y^a} \mathbb{E}(|X|^a) \frac{\mathbb{V}(S_n \geq x)}{n \mathbb{E}(|X|^a) + x y^{a-1}} \right).$$  \hspace{1cm} (13)

Since $\mathbb{E}(X) = \mathbb{E}(-X) = 0$, by Proposition 3.6 in the study of Peng [3] and Definition 1, we see that

$$\mathbb{E}(X I_{|X|\leq y}) = \mathbb{E}(-X I_{|X|\leq y}) \leq \mathbb{E}(|X| I_{|X|\leq y}) \leq \frac{1}{y^a} \mathbb{E}(|X|^a) \leq \frac{1}{y^a} \mathbb{E}(|X|^a).$$  \hspace{1cm} (14)

Therefore,

$$\mathbb{E}(X I_{|X|\leq y}) \geq \frac{y}{n \mathbb{E}(X I_{|X|\leq y})} \geq \frac{x}{y}.$$  \hspace{1cm} (15)

Combining this with (13) and (9), we conclude that

$$\mathbb{V}[S_n \geq x] \leq n \mathbb{V}[|X| > y] + n^{x/y} \left( \frac{e \mathbb{E}(|X|^a)}{n \mathbb{E}(|X|^a) + x y^{a-1}} \right)^{x/y}.$$  \hspace{1cm} (16)

Combining (16) with the inequality derived from it with $-X$ and $-X_k$ in place of $X$ and $X_k$, respectively, leads to (5). \hspace{1cm} \square

Remark 1. (see Lemma 2 in [7]). For any $X \in \mathcal{H}$, we have

$$C_n(X^2) < \infty \iff \int_1^\infty x \mathbb{V}(|X| > x) dx < \infty.$$  \hspace{1cm} (17)

Lemma 2 (see Lemma 5 in [7]). Assume that $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables with $\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0$ and $\lim_{n \to \infty} \mathbb{E}(X^2 - c)^+ = 0$. Write $\bar{X}_n^2 = \mathbb{E}[X_1^2]$ and $\mathbf{g} = \sqrt{\mathbb{E}[X_1^2]}$. Suppose that $\mathbb{E}$ is continuous and set $\Delta_n(x) = \mathbb{V}(|S_n|/\sqrt{n} \geq x) - \mathbb{V}(|\xi| \geq x), \xi \sim \mathcal{N}(0, [\mathbf{g}^2, \mathbf{g}^2])$ under $\mathbb{E}$. Then,

$$\Delta_n = \sup_{x \in \mathbb{R}} |\Delta_n(x)| \to 0, \text{ as } n \to \infty.$$  \hspace{1cm} (18)

3. Main Results

The following are our main results.

Theorem 1. For $b, d > 0$, we have

$$\lim_{n \to \infty} \frac{1}{b} \sum_{n=3}^{\infty} \frac{\log \log n}{n \log n} \mathbb{V}[|S_n| \geq \varepsilon \sqrt{n} (\log \log n)^{b-1}] = \frac{C_n(|\xi|^{b/d})}{b}.$$  \hspace{1cm} (19)
Theorem 2. For \( d > 0 \), we have
\[
\lim_{\epsilon \downarrow 0} \frac{1}{-\log \epsilon} \sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n} \mathbb{V}\{ |S_n| \geq \epsilon \sqrt{n} (\log \log n)^d \} = \frac{1}{d}
\]

In the following two sections, for \( M \geq 3 \) and \( 0 < \epsilon < 1 \), set \( b(\epsilon) = \left\lfloor \exp\{\exp[M\epsilon^{-1/d}]\} \right\rfloor \).

4. Proof of Theorem 1

Proposition 1. For \( b, d > 0 \), we have
\[
\lim_{\epsilon \downarrow 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{|\xi| \geq \epsilon (\log \log n)^d\} = C_v\{(|\xi|^{b/d}) \}
\]

Proof. By Lemma 2 and Toeplitz’s lemma,
\[
\lim_{\epsilon \downarrow 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{|\xi| \geq \epsilon (\log \log n)^d\}
\leq \lim_{\epsilon \downarrow 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{|\xi| \geq \epsilon (\log \log n)^d\}
\]

The proof is complete.

Proposition 2. For \( b, d > 0 \), we have
\[
\lim_{\epsilon \downarrow 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{|\xi| \geq \epsilon \sqrt{n} (\log \log n)^d\} = 0.
\]

Proof. We could obtain that
\[
\lim_{\epsilon \downarrow 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{|\xi| \geq \epsilon \sqrt{n} (\log \log n)^d\}
\leq C \lim_{\epsilon \downarrow 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{|\xi| \geq \epsilon \sqrt{n} (\log \log n)^d\}
\]

The proof is complete.

Remark 2. By the proof of (24) and (25) in the study by Wu [7], \( C_v\{(|\xi|^{b/d}) \) is finite for any \( b, d > 0 \).

Proposition 3. For \( b, d > 0 \), we have
\[
\lim_{M \rightarrow \infty} \limsup_{\epsilon \downarrow 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{|\xi| \geq \epsilon \sqrt{n} (\log \log n)^d\} = 0.
\]

Proof. We could obtain that
\[
\lim_{M \rightarrow \infty} \limsup_{\epsilon \downarrow 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{|\xi| \geq \epsilon \sqrt{n} (\log \log n)^d\}
\leq C \limsup_{\epsilon \downarrow 0} \epsilon^{b/d} \sum_{n=3}^{\infty} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{|\xi| \geq \epsilon \sqrt{n} (\log \log n)^d\}
\]

Note that \( \int_{M^d} \mathbb{V}\{|\xi| \geq t\} \, dt \) is integrable:
\[
\int_{M^d} \mathbb{V}\{|\xi| \geq t\} \, dt = 0,
\]
as \( M \rightarrow \infty \). Proposition 3 is established.
Proposition 4. For \( b, d > 0 \), we have

\[
\lim_{M \to \infty} \limsup_{\varepsilon \downarrow 0} \varepsilon^{b/d} \sum_{n > b(\varepsilon)} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{ |S_n| \geq \varepsilon \sqrt{n} (\log \log n)^d \} = 0. \tag{28}
\]

Proof. When \( 0 < b < 2d \), by Markov’s inequality under sublinear expectations, we have

\[
\limsup_{\varepsilon \downarrow 0} \varepsilon^{b/d} \sum_{n > b(\varepsilon)} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{ |S_n| \geq \varepsilon \sqrt{n} (\log \log n)^d \} \leq C \limsup_{\varepsilon \downarrow 0} \varepsilon^{b/d-2} \sum_{n > b(\varepsilon)} \frac{(\log \log n)^{b-1-2d}}{n^2 \log n} \mathbb{E}[S_n^2] = C \limsup_{\varepsilon \downarrow 0} \varepsilon^{b/d-2} (\log \log (b(\varepsilon)))^{b-2d} \leq C M^{b-2d} \to 0, \quad \text{as} \ M \to \infty. \tag{29}
\]

For \( b \geq 2d \), by Lemma 1, we see that

\[
\sum_{n > b(\varepsilon)} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{ |S_n| \geq \varepsilon \sqrt{n} (\log \log n)^d \} \leq \sum_{n > b(\varepsilon)} \frac{(\log \log n)^{b-1}}{n \log n} \mathbb{V}\{ |X| \geq \varepsilon \sqrt{n} (\log \log n)^d / T \} + C \sum_{n > b(\varepsilon)} \frac{(\log \log n)^{b-1}}{n \log n} \frac{1}{(\log \log n)^2 dT} \varepsilon^{2d} = L_1 + L_2, \tag{30}
\]

where \( T \) is a positive constant to be specified later. On the one hand, we obtain that

\[
\limsup_{\varepsilon \downarrow 0} \varepsilon^{b/d} L_2 \leq \limsup_{\varepsilon \downarrow 0} \varepsilon^{b/d-2T} \sum_{n > b(\varepsilon)} \frac{(\log \log n)^{b-1-2d}}{n \log n} \mathbb{V}\{ |X| > \varepsilon \sqrt{n} (\log \log n)^d \} \leq \limsup_{\varepsilon \downarrow 0} C \varepsilon^{b/d-2T} (\log \log b(\varepsilon))^{b-2d} \to \text{CM}^{b-2d} T \to 0, \quad \text{as} \ M \to \infty, \tag{31}
\]

for any \( T > b/(2d) \). On the other hand, for \( L_1 \), without loss of generality, set \( T = 1 \). By the countable subadditivity property of sublinear expectations and the fact that \((\log \log x)^{b-1}/\log x \to 0\), as \( x > b(\varepsilon) \to \infty \), we obtain that

\[
L_1 = \sum_{n > b(\varepsilon)} \frac{(\log \log n)^{b-1}}{\log n} \mathbb{V}\{ |X| > \varepsilon \sqrt{n} (\log \log n)^d \} \leq C \int_{x > b(\varepsilon)} \frac{\log \log x}{\log x} \mathbb{V}\{ |X| > \varepsilon \sqrt{x} (\log \log x)^d \} dx \leq C \int_{x > b(\varepsilon)} \mathbb{V}\{ |X|^2 > \varepsilon^2 x (\log \log x)^2 d \} dx \leq C \varepsilon^{-2} \int_{x > b(\varepsilon) M^{2d}} \mathbb{V}\{ |X|^2 > y \} dy. \tag{32}
\]

Hence, for \( b \geq 2d \), we have

\[
\limsup_{\varepsilon \downarrow 0} \varepsilon^{b/d} L_1 \leq C \varepsilon^{b/d-2} \int_{x > b(\varepsilon) M^{2d}} \mathbb{V}\{ |X|^2 > y \} dy = 0. \tag{33}
\]

Thus, (28) holds for each \( b, d > 0 \). Now, by Proposition 1–4 and the triangle inequality, \( \forall \beta > 0, \exists M > 0 \), which is sufficiently large, such that
\[
\lim_{\epsilon, \delta \to 0} \epsilon \sum_{n=3}^{\infty} \frac{(\log n)^{b-1}}{n \log n} \mathbb{P}[|S_n| \geq \epsilon \sqrt{n} (\log \log n)^d] \\
\leq \lim_{\epsilon, \delta \to 0} \epsilon \sum_{n=3}^{\infty} \frac{(\log n)^{b-1}}{n \log n} \mathbb{P}\left[|\xi| \geq \epsilon (\log \log n)^d\right] \\
+ \lim_{\epsilon, \delta \to 0} \epsilon \sum_{n \geq b(\epsilon)} \frac{(\log n)^{b-1}}{n \log n} \mathbb{P}\left[|S_n| \geq \epsilon \sqrt{n} (\log \log n)^d\right] - \mathbb{P}\left[|\xi| \geq \epsilon (\log \log n)^d\right] \\
+ \limsup_{\epsilon, \delta \to 0} \epsilon \sum_{n > b(\epsilon)} \frac{(\log n)^{b-1}}{n \log n} \mathbb{P}[|S_n| \geq \epsilon \sqrt{n} (\log \log n)^d] \\
+ \limsup_{\epsilon, \delta \to 0} \epsilon \sum_{n > b(\epsilon)} \frac{(\log n)^{b-1}}{n \log n} \mathbb{P}[|S_n| \geq \epsilon \sqrt{n} (\log \log n)^d] \\
= \frac{C_V(\epsilon, \beta)}{b} + \beta, \\
\tag{34}
\]

We derive Theorem 1 from the arbitrariness of \( \beta > 0 \). \( \Box \)

5. Proof of Theorem 2

Proposition 5. For \( d > 0 \), we have
\[
\lim_{\epsilon, \delta \to 0} \frac{1}{-\log \epsilon} \sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n} \mathbb{P}[|\xi| \geq \epsilon (\log \log n)^d] = \frac{1}{d} \\
\tag{35}
\]

Proof. We claim that

Indeed, by Lemma 4 in the study by Wu [7], \( \forall \alpha > 0, \exists \delta > 0 \), such that \( \forall t < \delta < 1, \mathbb{P}[|\xi| > t] > 1 - \alpha d \). Therefore,
Proposition 7. For $d > 0$, we have

\[
\lim_{\varepsilon \to 0} \frac{1}{d} \sum_{n > b(\varepsilon)} \frac{1}{n \log n \log \log n} \log \left( \frac{\log(n \log n)}{\varepsilon} \right) = 0.
\]

Proof. By Markov inequality under sublinear expectations, we see that

\[
\lim_{\varepsilon \to 0} \frac{1}{d} \sum_{n > b(\varepsilon)} \frac{1}{n \log n \log \log n} \log \left( \frac{\log(n \log n)}{\varepsilon} \right) = 0.
\]
Thus this proves Proposition 7.

Proposition 8. For $d > 0$, we have

\[
\lim_{\varepsilon \downarrow 0} \frac{1}{-\log \varepsilon} \sum_{n > b(\varepsilon)} \frac{1}{n \log n \log \log n} \mathbb{P}\left( |S_n| \geq \varepsilon \sqrt{n} (\log \log n)^d \right) = 0.
\]  

Proof. By Markov inequality under sublinear expectations, we deduce that

\[
\begin{align*}
&\lim_{\varepsilon \downarrow 0} \frac{1}{-\log \varepsilon} \sum_{n > b(\varepsilon)} \frac{1}{n \log n \log \log n} \mathbb{P}\left( |S_n| \geq \varepsilon \sqrt{n} (\log \log n)^d \right) \\
&\leq \lim_{\varepsilon \downarrow 0} \frac{1}{-\varepsilon \log \varepsilon} \sum_{n > b(\varepsilon)} \frac{1}{n \log n (\log \log n)^{1+2d}} E\left[ S_n^2 \right] \\
&\leq C \lim_{\varepsilon \downarrow 0} \frac{(\log \log (b(\varepsilon)))^{-2d}}{-\varepsilon \log \varepsilon} \\
&\leq C \lim_{\varepsilon \downarrow 0} \frac{M^{-2d}}{-\log \varepsilon} = 0.
\end{align*}
\]

The proof is complete.

Finally, similar to the proof of Theorem 1, by the triangle inequality and Propositions 5–8, we finish the proof of Theorem 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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