HOLOMORPHIC KOSZUL-BRYLINSKI HOMOLOGY

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Abstract. In this note, we study the Koszul-Brylinski homology of holomorphic Poisson manifolds. We show that it is isomorphic to the cohomology of a certain smooth complex Lie algebroid with values in the Evens-Lu-Weinstein duality module. As a consequence, we prove that the Evens-Lu-Weinstein pairing on Koszul-Brylinski homology is nondegenerate. Finally we compute the Koszul-Brylinski homology for Poisson structures on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \).

1. Introduction

In [2], Brylinski introduced a homology theory for Poisson manifolds, which is nowadays called Koszul-Brylinski homology. Evens, Lu & Weinstein [7] and Xu [15] proved independently that, for unimodular Poisson manifolds, the Koszul-Brylinski homology is (up to a change of degree) isomorphic to the Lichnerowicz-Poisson cohomology [10]. And Evens, Lu & Weinstein introduced a pairing on Koszul-Brylinski homology groups. In this note, we study the Koszul-Brylinski homology of holomorphic Poisson manifolds. Koszul-Brylinski homology is defined as the hypercohomology of the complex of sheaves

\[
\cdots \to \Omega_X^{i+1} \xrightarrow{\partial_\pi} \Omega_X^i \xrightarrow{\partial_\pi} \Omega_X^{i-1} \xrightarrow{\partial_\pi} \cdots ,
\]

where \( \partial_\pi = i_\pi \circ \partial - \partial \circ i_\pi \). As is explained in [9], any holomorphic Poisson manifold gives rise to a holomorphic Lie algebroid structure \((T_X)^*_\pi\) on the holomorphic vector bundle \( (T_X)^*\), which in turn induces a complex Lie algebroid structure \( T_X^{0,1} \simeq (T_X^{1,0})^*\) on the complex vector bundle \( T_X^{0,1} \oplus (T_X^{1,0})^*\). We show that the cohomology of this complex Lie algebroid with values in the Evens-Lu-Weinstein duality module is isomorphic to the Koszul-Brylinski homology. As a consequence, we prove that the Evens-Lu-Weinstein pairing on Koszul-Brylinski homology is nondegenerate. We also introduce the Euler characteristic for the Koszul-Brylinski homology of a Poisson manifold and show that it coincides with the signed Euler characteristic of the manifold. Finally we compute the Koszul-Brylinski homology for Poisson structures on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \). We refer the reader to the works of Etingof & Ginzburg [6] and Pichereau [13] for more on the Koszul-Brylinski homology of algebraic Poisson varieties.

2. Holomorphic Lie algebroid cohomology

Let \( A \) be a holomorphic Lie algebroid over a complex manifold \( X \); i.e. \( A \to X \) is a holomorphic vector bundle whose sheaf of holomorphic sections \( \mathcal{A} \) is endowed with a
Lie bracket $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, and there exists a holomorphic bundle map $A \xrightarrow{a} T_X$, called anchor, which induces a morphism of sheaves of $\mathcal{O}_X$-modules $A \xrightarrow{a} \Theta_X$ such that
\[
a([s_1, s_2]) = [a(s_1), a(s_2)], \quad \forall s_1, s_2 \in \mathcal{A};
\]
\[
[s_1, f s_2] = (a(s_1)f)s_2 + f[s_1, s_2], \quad \forall s_1, s_2 \in \mathcal{A}, \ f \in \mathcal{O}_X.
\]
This holomorphic Lie algebroid structure gives rise to a complex of sheaves:
\[
\cdots \xrightarrow{d_A} \Omega^{k-1}_A \xrightarrow{d_A} \Omega^k_A \xrightarrow{d_A} \Omega^{k+1}_A \xrightarrow{d_A} \cdots,
\]
where $\Omega^k_A$ stands for the sheaf of holomorphic sections of the holomorphic vector bundle $\wedge^k A^*$, and $d_A$ is given by the usual Cartan formula. By definition \cite{7,9}, the holomorphic Lie algebroid cohomology of $A$ (with trivial coefficients) is the hypercohomology of this complex of sheaves:
\[
H^*(A, \mathbb{C}) := \mathbb{H}^*(X, \Omega^*_A).
\]
A holomorphic vector bundle $E \to X$ (with sheaf of holomorphic functions $\mathcal{E}$) is said to be a module over the holomorphic Lie algebroid $A$, if there is a morphism of sheaves (of $\mathbb{C}$-modules)
\[
\mathcal{A} \otimes \mathcal{E} \to \mathcal{E} : V \otimes s \mapsto \nabla_V s
\]
such that, for any open subset $U \subset X$, the relations
\[
\nabla_{fV}s = f \nabla_V s
\]
\[
\nabla_V(fs) = (\rho(V)f)s + f \nabla_V s
\]
\[
\nabla_V \nabla_W s - \nabla_W \nabla_V s = \nabla_{[V,W]} s
\]
are satisfied $\forall f \in \mathcal{O}_X(U)$, $\forall V, W \in \mathcal{A}(U)$ and $\forall s \in \mathcal{E}(U)$. Such a morphism $\nabla$ is called a representation of $A$ on $E$. Given an $A$-module $E \to X$, one can form the complex of sheaves
\[
\cdots \xrightarrow{d^\mathcal{E}_A} \Omega^{k-1}_A \otimes \mathcal{O}_X \mathcal{E} \xrightarrow{d^\mathcal{E}_A} \Omega^k_A \otimes \mathcal{O}_X \mathcal{E} \xrightarrow{d^\mathcal{E}_A} \Omega^{k+1}_A \otimes \mathcal{O}_X \mathcal{E} \xrightarrow{d^\mathcal{E}_A} \cdots
\]
By definition, the Lie algebroid cohomology of $A$ with values in $E$ is the hypercohomology of this complex of sheaves:
\[
H^*(A, E) := \mathbb{H}^*(X, \Omega^*_A \otimes \mathcal{O}_X \mathcal{E}).
\]
Given a holomorphic Lie algebroid $A$ with anchor $a$, we define $a^{1,0} = \frac{1-iJ}{2} \circ a : A \to T_{\mathbb{C}}X$. Here $J$ stands for the almost complex structure $J : T_X \to T_X$ of the complex manifold $X$. Of course, for any holomorphic function $f \in \mathcal{O}_X(U)$, we have $a^{1,0}(V)f = a(V)f$, for all $V \in \Gamma(U, A)$. Now regard $A$ as a complex vector bundle. The Lie bracket, which was defined so far only on the sheaf of holomorphic sections of $A$, extends naturally to all smooth sections through the Leibniz rule
\[
[s_1, fs_2] = (a^{1,0}(s_1)f)s_2 + f[s_1, s_2], \quad \forall s_1, s_2 \in \Gamma(A), \ f \in C^\infty(X, \mathbb{C}),
\]
with $a^{1,0}$ substituted to $a$. We use the symbol $A^{1,0}$ to denote the resulting complex Lie algebroid structure on $A$ \cite{9}. 

Now recall that the complex vector bundle $T_{X}^{0,1}$ is endowed with a canonical complex Lie algebroid structure whose Lie bracket is completely determined by the relation $[\partial_{\bar{\tau}}, \partial_{\bar{\tau}}] = 0$ and the anchor, which is simply the injection $T_{X}^{0,1} \hookrightarrow T_{X} \otimes \mathbb{C}$.

**Proposition 2.1** ([9 Theorems 4.2 and 4.8]). If $A$ is a holomorphic vector bundle with anchor $\alpha$ over a complex manifold $X$, there exists a unique complex Lie algebroid structure on the complex vector bundle $T_{X}^{0,1} \oplus A^{1,0}$ with anchor $\alpha^{\infty}(X^{0,1} + \xi) = X^{0,1} + a^{1,0}(\xi)$ such that $[\Theta_{X}, A] = 0$ and both $T_{X}^{0,1}$ and $A^{1,0}$ are Lie subalgebroids.

This complex Lie algebroid is denoted $T_{X}^{0,1} \bowtie A^{1,0}$. The pair $(T_{X}^{0,1}, A^{1,0})$ is an example of matched pair [9,11,12].

**Theorem 2.2** ([9 Lemma 4.16 and Theorem 4.19]). Let $A \to X$ be a holomorphic Lie algebroid and $E \to X$ a complex vector bundle. Then $E$ is a module over the holomorphic Lie algebroid $A$ if, and only if, $E$ is a module over the complex Lie algebroid $T_{X}^{0,1} \bowtie A^{1,0}$. Moreover, we have

$$H^{*}(A, E) \cong H^{*}(T_{X}^{0,1} \bowtie A^{1,0}, E).$$

Note that the complex Lie algebroid $T_{X}^{0,1} \bowtie A^{1,0}$ is an elliptic Lie algebroid in the sense of Block [1]. That is, $\mathcal{R} \circ a^{\infty}$ is surjective. Therefore, when $X$ is compact, the cohomology groups $H^{*}(T_{X}^{0,1} \bowtie A^{1,0}, E)$ are finite dimensional and we can consider the Euler characteristic

$$\chi(A, E) = \sum_{i} (-1)^{i} \dim H^{i}(A, E). \quad (4)$$

**Proposition 2.3.** Let $A \to X$ be a holomorphic Lie algebroid and $E$ an $A$-module. Assume that $X$ is compact. Then

$$\chi(A, E) = \sum_{i} (-1)^{i} \chi(X, \wedge^{i} A^{*} \otimes E),$$

where $\chi(X, \wedge^{i} A^{*} \otimes E)$ is the Euler characteristic of the holomorphic bundle $\wedge^{i} A^{*} \otimes E$.

**Proof.** By definition, $H^{*}(A, E)$ is isomorphic to the hypercohomology $\mathbb{H}^{*}(X, \Omega_{A}^{*} \otimes_{\mathcal{O}_{X}} E)$ of the complex of sheaves [3], which, according to Theorem 2.2, is computed by the total cohomology $H^{n}(T_{X}^{0,1} \bowtie A^{1,0}, E)$ of the double complex
where \( \Omega^{i,j}_X = \Gamma(\wedge^i T_{X,1} \otimes \wedge^j T_{X,0}) \) and \( \mathcal{A}^{k,l} = \Gamma(\wedge^k (A^{1,0})^* \otimes \wedge^l (A^{0,1})^* \otimes E) \).

Set \( C^{p,q} = \Omega^{p,q}_X \otimes C^\infty \mathcal{A}^{q,0} \) and \( C^n = \bigoplus_{p+q=n} C^{p,q} \). The spectral sequence induced by the filtration \( F_q(C^n) = \bigoplus_{p+q \geq q} C^{p,q} \) of \( C^\bullet \) starts with \( E_0^{p,q} = C^{p,q} \), \( d_0^{p,q} = 0 \) and \( E_1^{p,q} = H^p(C^{*q}, \overline{\partial}) \), and converges to \( H^n(T_{X,1} \otimes A^{1,0}, E) \).

Since the Euler characteristic of \( E_r^{p,q} \) does not change from one sheet to the next, we have

\[
\chi(A, E) = \sum_n (-1)^n \dim H^n(A, E) = \sum_n (-1)^n \dim H^n(T_{X,1} \otimes A^{1,0}, E) = \sum_n (-1)^n \dim \left( \bigoplus_{p+q=n} E_{\infty}^{p,q} \right) = \sum_n (-1)^n \dim \left( \bigoplus_{p+q=n} E_1^{p,q} \right) = \sum_n (-1)^n \left( \bigoplus_{p+q=n} \dim H^p(C^{*q}, \overline{\partial}) \right) = \sum_q (-1)^q \left( \sum_p (-1)^p \dim H^p(C^{*q}, \overline{\partial}) \right) = \sum_q (-1)^q \chi(X, \wedge^q A^* \otimes E). \quad \square
\]

3. Holomorphic Poisson manifolds

A holomorphic Poisson manifold is a complex manifold \( X \) whose sheaf of holomorphic functions \( \mathcal{O}_X \) is a sheaf of Poisson algebras. By a sheaf of Poisson algebras over \( X \), we mean that, for each open subset \( U \subset X \), the ring \( \mathcal{O}_X(U) \) is endowed with a Poisson bracket such that all restriction maps \( \mathcal{O}_X(U) \to \mathcal{O}_X(V) \) (for arbitrary open subsets \( V \subset V \subset X \)) are morphisms of Poisson algebras. Moreover, given an open subset \( U \subset X \), an open covering \( \{U_i\}_{i \in I} \) of \( U \), and a pair of functions \( f, g \in \mathcal{O}_X(U) \), the local data \( \{f|_{U_i}, g|_{U_i}\} (i \in I) \) glue up and give \( \{f|_U, g|_U\} \) if they coincide on the overlaps \( U_i \cap U_j \). On a given complex manifold \( X \), the holomorphic Poisson structures are in one-to-one correspondence with the sections \( \pi \in \Gamma(\wedge^2 T_{X,1}^0) \) such that \( \overline{\partial} \pi = 0 \) and \([\pi, \pi] = 0 \). The Poisson bracket on functions and the bivector field are related by the formula \( \pi(\partial f, \partial g) = \{f, g\} \), where \( f, g \in \mathcal{O}_X \).

Given a holomorphic Poisson bracket

\[
\mathcal{O}_X \otimes \mathcal{C} \mathcal{O}_X \to \mathcal{O}_X : (f, g) \mapsto \{f, g\},
\]

the formula

\[
[f_1, dg_1, f_2, dg_2] = f_1 X_{g_1}(f_2) \, dg_2 - f_2 X_{g_2}(f_1) \, dg_1 + f_1 f_2 \, d\{g_1, g_2\}, \quad (5)
\]
where \( f_1, f_2, g_1, g_2 \in \mathcal{O}_X \), defines a Lie bracket on \( \Omega_X \). Here \( X_f \in \Theta_X \) denotes the derivation
\[
X_f : \mathcal{O}_X \to \mathcal{O}_X : g \mapsto \{f, g\}
\]
of \( \mathcal{O}_X \) associated to the holomorphic function \( f \in \mathcal{O}_X \). Since \( \Gamma((T_{X}^{1,0})^*) = C^\infty(X, \mathbb{C})\Omega_X \), the bracket on \( \Omega_X \) extends to \( \Gamma((T_{X}^{1,0})^*) \) by the Leibniz rule:
\[
[f dz_k, gdz_l] = f X_{z_k}(g)dz_l - gX_{z_l}(f)dz_k + fg d\{z_k, z_l\},
\]
for all \( f, g \in C^\infty(X, \mathbb{C}) \). If the bivector field associated to the Poisson bracket on \( \mathcal{O}_X \)
is \( \pi \in \Theta_X^2 \subseteq \Gamma(\wedge^2 T_{X}^{1,0}) \), then the Lie bracket is given by
\[
[\alpha, \beta] = L_{\pi^*\alpha}\beta - L_{\pi^*\beta}\alpha - \partial(\pi(\alpha, \beta)), \quad \forall \alpha, \beta \in \Gamma((T_{X}^{1,0})^*).
\]

Once its sheaf of sections \( \Omega_X \) has been endowed with this Lie bracket, the cotangent bundle \( (T_X)^* \) becomes a holomorphic Lie algebroid with anchor map \( \pi^* : (T_X)^* \to T_X \), which we refer to by the symbol \( (T_X)^*_\pi \). By Proposition 2.1, we can associate to it the complex Lie algebroid \( T_{X}^{0,1} \bowtie (T_{X}^{1,0})^*_\pi \).

The complex Lie algebroid structure on \( T_{X}^{0,1} \bowtie (T_{X}^{1,0})^*_\pi \) is characterized as follows: the anchor is \( \text{id}_{T_{X}^{0,1}} \oplus \pi^* : T_{X}^{0,1} \oplus (T_{X}^{1,0})^* \to T_X \otimes \mathbb{C} \), and the Lie bracket on \( \Gamma(T_{X}^{0,1}) \bowtie (T_{X}^{1,0})^*_\pi \) satisfies \( [\overline{\mathcal{O}}_X, \Omega_X] = 0 \), coincides with the Lie bracket of vector fields on \( \overline{\mathcal{O}}_X \) and with the bracket defined by (5) on \( \Omega_X \).

4. **Holomorphic Koszul-Brylinski homology**

Let \( \Theta_X^k \) and \( \Omega_X^k \) denote the sheaves of holomorphic sections of \( \wedge^k T_X \) and \( \wedge^k (T_X)^* \), respectively.

The Koszul-Brylinski operator \( \partial_\pi : \Omega_X^k \to \Omega_X^{k-1} \) is defined as \( \partial_\pi := \iota_\pi \partial - \partial \iota_\pi \), where \( \partial : \Omega_X^k \to \Omega_X^{k+1} \) is the holomorphic exterior differential (i.e. the Dolbeault operator) and \( \iota_\pi : \Omega_X^k \to \Omega_X^{k-2} \) is the contraction with the holomorphic Poisson bivector field \( \pi \). The operator \( \partial_\pi \) satisfies \( \partial_\pi^2 = 0 \), \( \partial_\pi d + d \partial_\pi = 0 \), and
\[
\partial_\pi(\alpha \wedge \beta) = \partial_\pi \alpha \wedge \beta + (-1)^k \alpha \wedge \partial_\pi \beta + (-1)^k [\alpha, \beta], \quad \forall \alpha \in \Omega_X^k, \beta \in \Omega_X^k.
\]

**Definition 4.1.** Let \( (X, \pi) \) be a holomorphic Poisson manifold. Its Koszul-Brylinski homology is the hypercohomology of the complex of sheaves
\[
\cdots \xrightarrow{\partial_\pi} \Omega_X^{k+1} \xrightarrow{\partial_\pi} \Omega_X^k \xrightarrow{\partial_\pi} \Omega_X^{k-1} \xrightarrow{\partial_\pi} \cdots
\]
which is denoted \( H_k(X, \pi) \).

**Remark 4.2.** If \( \pi = 0 \), we have \( H_k(X, \pi) \cong \bigoplus_{j-i=n-k} H^j(X, \Omega_X^k) \).

As was pointed out earlier, a holomorphic Poisson manifold \( (X, \pi) \) automatically gives rise to a holomorphic Lie algebroid structure \( (T_X)^*_\pi \). The Lichnerowicz-Poisson cohomology \( H^*(X, \pi; E) \) of \( (X, \pi) \) with coefficients in a \( (T_X)^*_\pi \)-module \( E \) is defined to be the Lie algebroid cohomology of \( (T_X)^*_\pi \) with coefficients in the module \( E \), i.e. the hypercohomology of the complex of sheaves
\[
\cdots \xrightarrow{d_\pi^k} \Theta_X^{k-1} \otimes \mathcal{O}_X \mathcal{E} \xrightarrow{d_\pi^k} \Theta_X^k \otimes \mathcal{O}_X \mathcal{E} \xrightarrow{d_\pi^k} \Theta_X^{k+1} \otimes \mathcal{O}_X \mathcal{E} \xrightarrow{d_\pi^k} \cdots
\]
In particular, when $E$ is the trivial module $X \times \mathbb{C} \to X$, the associated differential complex is
\[ \cdots \to d_\pi^{k+1} \Theta_X \to \Theta_X^k \to \Theta_X^{k+1} \to \cdots. \]
One has $d_\pi V = [\pi, V]$. The hypercohomology of this complex of sheaves is the holomorphic Lichnerowicz-Poisson cohomology $H^*(X, \pi)$ of the holomorphic Poisson manifold $(X, \pi)$ [9].

Assuming $X$ compact, let
\[ \chi^{LP}(X, \pi; E) = \sum_i(-1)^i \dim H^i(X, \pi; E) \] (7)
be the Euler characteristic of the Lichnerowicz-Poisson cohomology $H^*(X, \pi; E)$.

**Proposition 4.3.** If $(X, \pi)$ is a compact holomorphic Poisson manifold, then
\[ \chi^{LP}(X, \pi; E) = \sum_i(-1)^i \chi(X, \wedge^i T_X \otimes E), \]
where $\chi(X, \wedge^i T_X \otimes E)$ stands for the usual Euler characteristic of the holomorphic bundle $\wedge^i T_X \otimes E$.

**Proof.** By definition, we have $H^k(X, \pi; E) = H^k((T_X)_\pi^*, \wedge^n(T_X)^*)$, whence
\[ \chi^{LP}(X, \pi; E) = \chi((T_X)_\pi^*, \wedge^n(T_X)^*). \]
Therefore, it suffices to apply Proposition 2.3 to the Lie algebroid $A = (T_X)_\pi^*$ and its module $E = \wedge^n(T_X)^*$ to conclude. \qed

A result of Evens, Lu & Weinstein (transposed to the holomorphic setting) asserts that, if $A \to X$ is a holomorphic Lie algebroid with $\dim_C X = n$ and $\text{rk}_C A = r$, the holomorphic vector bundle $Q_A = \wedge^r A \otimes \wedge^n(T_X)^*$ is naturally a module over $A$. When the holomorphic Lie algebroid $A$ is the cotangent bundle $(T_X)_\pi^*$ of a holomorphic Poisson manifold $(X, \pi)$, we have $Q_A = \wedge^n(T_X)^* \otimes \wedge^n(T_X)^*$. Its square root $\sqrt{Q_A} = \wedge^n(T_X)^*$ is also an $A$-module; the representation is the map
\[ \Omega_X \otimes \Omega^n_X \to \Omega^n_X : \alpha \otimes \omega \mapsto \nabla_\alpha \omega \]
such that $\nabla_d \omega = L_X f \omega$, for all $f \in \mathcal{O}_X$ and $\omega \in \Omega^n_X$. Here $\Omega_X$ and $\Omega^n_X$ are the sheaves of holomorphic sections of $(T_X)^*$ and $\wedge^n(T_X)^*$ respectively. Hence, we obtain the complex of sheaves
\[ \cdots \to d_\pi^* \Theta_X^{k-1} \otimes \mathcal{O}_X \to \Theta_X^k \otimes \mathcal{O}_X \to \Theta_X^{k+1} \otimes \mathcal{O}_X \to \cdots. \] (8)

An argument of Evens, Lu & Weinstein (see [7, Equation (22)]) adapted to the holomorphic context shows that the isomorphism of sheaves of $\mathcal{O}_X$-modules
\[ \tau : \Theta_X^k \otimes \mathcal{O}_X \to \Omega_X^{n-k} : X \otimes \alpha \mapsto \iota_X \alpha \]
is in fact an isomorphism between the complexes of sheaves (8) and (9):

\[
\begin{array}{ccccccccc}
\Omega^n_X & \rightarrow & \cdots & \rightarrow & \Theta^k_X \otimes \mathcal{O}_X & \Omega^n_X \\
\downarrow \text{id} & & & & \downarrow \tau & & & & \downarrow \tau \\
\Omega^n_X & \rightarrow & \cdots & \rightarrow & \Theta^{n-k}_X \otimes \mathcal{O}_X & \Theta^{k+1}_X \otimes \mathcal{O}_X & \Omega^n_X \\
\end{array}
\]

This isomorphism of complexes of sheaves induces an isomorphism of the corresponding sheaf cohomologies. Thus we obtain the following theorem, which is a holomorphic analogue of a result of Evens, Lu & Weinstein [7, Corollary 4.6].

**Theorem 4.4.** For any holomorphic Poisson manifold \((X, \pi)\), the chain map \(\tau\) induces an isomorphism

\[
H^k(X, \pi; \wedge^n(T_X)^*) \xrightarrow{\cong} H_{2n-k}(X, \pi).
\]

Assume that \((X, \pi)\) is a compact holomorphic Poisson manifold. Let

\[
\chi_{KB}(X, \pi) = \sum_i (-1)^i \dim H_i(X, \pi)
\]

be the Euler characteristic of the Koszul-Brylinski homology.

**Theorem 4.5.** For a compact holomorphic Poisson manifold \((X, \pi)\), we have

\[
\chi_{KB}(X, \pi) = (-1)^n \chi(X),
\]

where \(\chi(X)\) denotes the standard Euler characteristic of \(X\).

**Proof.** We have

\[
\chi_{KB}(X, \pi) = \chi^{LP}(X, \pi; \wedge^n(T_X)^*)
\]

by Theorem 1.4

\[
= \sum_i (-1)^i \chi(X, \wedge^i T_X \otimes \wedge^n(T_X)^*)
\]

by Proposition 4.3

\[
= (-1)^n \sum_j (-1)^j \chi(X, \wedge^j(T_X)^*)
\]

\[
= (-1)^n \chi(T_X, \mathcal{C})
\]

by Proposition 2.3

Of course, since \(\Omega_X^\bullet \xrightarrow{\partial} \Omega_X^{\bullet+1}\) and \(\Gamma(\wedge^\bullet(T_X \otimes \mathcal{C})^*) \xrightarrow{d} \Gamma(\wedge^{\bullet+1}(T_X \otimes \mathcal{C})^*)\) are two acyclic resolutions of the locally constant sheaf \(\mathcal{C}\) over \(X\), we have

\[
\chi(T_X, \mathcal{C}) = \sum_i (-1)^i \dim H^i(\Gamma(\Omega_X^\bullet \xrightarrow{\partial} \Gamma(\Omega_X^{\bullet+1})))
\]

\[
= \sum_i (-1)^i \dim H^i(\Gamma(\wedge^\bullet(T_X \otimes \mathcal{C})^*) \xrightarrow{d} \Gamma(\wedge^{\bullet+1}(T_X \otimes \mathcal{C})^*)) = \chi(X). \quad \square
\]

**Definition 4.6.** A holomorphic Poisson manifold \((X, \pi)\) is said to be unimodular if \(\wedge^n(T_X)^*\) is isomorphic, as a \((T_X)^*_\pi\)-module, to the trivial module \(\mathcal{C}\).
The notion of modular class was introduced independently by Brylinski & Zuckerman [3] for holomorphic Poisson manifolds, and by Weinstein [14] for real Poisson manifolds. For the relation between Calabi-Yau algebras and unimodular Poisson structures, see [4].

From the definition, it is clear that a holomorphic Poisson manifold \((X, \pi)\) is unimodular if and only if there exists a global holomorphic section \(\omega \in \Omega^n_X\) such that the vector field \(H \in \Theta_X\) defined by

\[
\nabla_d f \omega = L_X f \omega = H(f) \cdot \omega \quad (f \in \mathcal{O}_X)
\]

is a holomorphic Hamiltonian vector field.

**Proposition 4.7.** For a unimodular holomorphic Poisson manifold \((X, \pi)\), the chain map \(\tau\) induces an isomorphism

\[
H^k(X, \pi) \cong H_{2n-k}(X, \pi).
\]

5. **Koszul-Brylinski double complex**

In this section, we describe a double complex computing the Koszul-Brylinski homology.

**Theorem 5.1.** The Koszul-Brylinski homology of a holomorphic Poisson manifold \((X, \pi)\) is isomorphic to the total cohomology of the double complex

\[
\cdots \to \Omega_X^{-n+k+1} \xrightarrow{\delta_{n-k+1}} \Omega_X^{-n+k} \oplus \Omega_X^{-n+k+1} \xrightarrow{\partial_{n-k+1}} \Omega_X^{-n+k-1} \to \cdots
\]

**Proof.** According to Theorem 2.2, we have

\[
H_*(X, \pi) \cong H^*(T_X^{0,1} \otimes (T_X^{1,0})^*, \wedge^n (T_X^{1,0})^*).
\]

The r.h.s. is the Lie algebroid cohomology of \(T_X^{0,1} \otimes (T_X^{1,0})^*\) with coefficients in the module \(\wedge^n (T_X^{1,0})^*\). Moreover, the representation of the complex Lie algebroid \(T_X^{0,1} \otimes (T_X^{1,0})^*_\pi\) on \(\wedge^n (T_X^{1,0})^*\) is the map

\[
\Gamma(T_X^{0,1}) \otimes \Gamma(\wedge^n (T_X^{1,0})^*) \to \Gamma(\wedge^n (T_X^{1,0})^*) : (X + \xi, \omega) \mapsto \nabla_{X + \xi} \omega
\]

defined by

\[
\nabla_{X + \xi}(f \ dz_1 \wedge \cdots \wedge dz_n) = \frac{\partial f}{\partial x_\xi} \ dz_1 \wedge \cdots \wedge dz_n
\]

\[
\nabla_{X + \xi}(f \ dz_1 \wedge \cdots \wedge dz_n) = L_{X + \xi}(f \ dz_1 \wedge \cdots \wedge dz_n)
\]
(for all \( f \in C^\infty(X, \mathbb{C}) \)).

Consider the complex

\[
\Gamma \left( \wedge^m (T^0_{X} \oplus (T^1_{X})^*) \otimes \wedge^n (T^1_{X})^* \right) \xrightarrow{\partial^\nabla} \Gamma \left( \wedge^{m+1} (T^0_{X} \oplus (T^1_{X})^*) \otimes \wedge^n (T^1_{X})^* \right)
\]

Set \( C^{k,l} = \wedge^k (T^0_{X})^* \otimes \wedge^l T^1_{X} \otimes \wedge^n (T^1_{X})^* \) so that

\[
\wedge^m (T^0_{X} \oplus (T^1_{X})^*) \otimes \wedge^n (T^1_{X})^* = \bigoplus_{k+l=m} C^{k,l}.
\]

Since \( A := T^0_{X} \) and \( B := (T^1_{X})^* \) are complex Lie subalgebroids of \( T^0_{X} \otimes (T^1_{X})^* \), one has

\[
d^\nabla \Gamma(C^{k,l}) \subset \Gamma(C^{k+1,l} \oplus C^{k,l+1}).
\]

Composing \( d^\nabla \) with the natural projections on each of the direct summands, we get the commutative diagram

\[
\begin{array}{ccc}
\Gamma(C^{k,l}) & \xrightarrow{\partial^\nabla_A} & \Gamma(C^{k+1,l}) \\
\downarrow & & \downarrow \\
\Gamma(C^{k+1,l}) & \xrightarrow{-(-1)^k \partial^\nabla_B} & \Gamma(C^{k+1,l} \oplus C^{k,l+1}) & \rightarrow & \Gamma(C^{k,l+1}),
\end{array}
\]

where the operators \( \partial^\nabla_A \) and \( \partial^\nabla_B \) are given by

\[
(\partial^\nabla_A \alpha)(A_0, \ldots, A_k, B_1, \ldots, B_l) = \sum_{i=0}^{k} (-1)^i \left( \nabla_{A_i}(\alpha(A_0, \ldots, \hat{A_i}, \ldots, A_k, B_1, \ldots, B_l)) \\
- \sum_{j=1}^{l} \alpha(A_0, \ldots, \hat{A_i}, \ldots, A_k, B_1, \ldots, \text{pr}_B[A_i, B_j], \ldots, B_l) \\
+ \sum_{i<j} (-1)^{i+j} \alpha([A_i, A_j], A_0, \ldots, \hat{A_i}, \ldots, \hat{A_j}, \ldots, A_k, B_1, \ldots, B_l) \right)
\]

and

\[
(\partial^\nabla_B \alpha)(A_1, \ldots, A_k, B_0, \ldots, B_l) = \sum_{i=0}^{l} (-1)^i \left( \nabla_{B_i}(\alpha(A_1, \ldots, A_k, B_0, \ldots, \hat{B_i}, \ldots, B_l)) \\
- \sum_{j=1}^{k} \alpha(A_1, \ldots, \hat{B_i}, \ldots, A_k, B_0, \ldots, \text{pr}_A[B_i, A_j], \ldots, B_l) \\
+ \sum_{i<j} (-1)^{i+j} \alpha(A_1, \ldots, A_k, [B_i, B_j], B_0, \ldots, \hat{B_i}, \ldots, \hat{B_j}, \ldots, B_l) \right)
\]

for all \( \alpha \in \Gamma(\wedge^k A^* \otimes \wedge^l B^*) \), \( A_0, \ldots, A_k \in \Gamma(A) \) and \( B_0, \ldots, B_k \in \Gamma(B) \). Here \( \text{pr}_B[A_i, B_j] \) denotes the \( B \)-component of \( [A_i, B_j] \in A \otimes B \) and \( \text{pr}_A[B_i, A_j] \) the \( A \)-component of \( [B_i, A_j] \).
Since $d^\nabla = \partial_A^\nabla + (-1)^k \partial_B^\nabla$, it follows from $(d^\nabla)^2 = 0$ that $(\partial_A^\nabla)^2 = 0$, $(\partial_B^\nabla)^2 = 0$ and $\partial_A^\nabla \circ \partial_B^\nabla = \partial_B^\nabla \circ \partial_A^\nabla$. Thus the complex $\Gamma(C_{k,l})$ is the total complex of the double complex

$$
\begin{array}{ccc}
\Gamma(C_{k,l}) & \xrightarrow{\partial_B^\nabla} & \Gamma(C_{k,l+1}) \\
\downarrow{\partial_A^\nabla} & & \downarrow{\partial_A^\nabla} \\
\Gamma(C_{k+1,l}) & \xrightarrow{\partial_B^\nabla} & \Gamma(C_{k+1,l+1})
\end{array}
$$

Hence it follows that $H^*(T^{0,1}_X, (T^{1,0}_X)_0, \wedge^n(T^{1,0}_X)^*)$ is isomorphic to the total cohomology of the double complex

$$
\begin{array}{ccc}
\Gamma(\wedge^i(T^{0,1}_X)^* \otimes \wedge^j(T^{1,0}_X) \otimes \wedge^n(T^{1,0}_X)^*) & \xrightarrow{\partial_B^\nabla} & \Gamma(\wedge^i(T^{0,1}_X)^* \otimes \wedge^{j+1}(T^{1,0}_X) \otimes \wedge^n(T^{1,0}_X)^*) \\
\downarrow{\partial_A^\nabla} & & \downarrow{\partial_A^\nabla} \\
\Gamma(\wedge^{i+1}(T^{0,1}_X)^* \otimes \wedge^j(T^{1,0}_X) \otimes \wedge^n(T^{1,0}_X)^*) & \xrightarrow{\partial_B^\nabla} & \Gamma(\wedge^{i+1}(T^{0,1}_X)^* \otimes \wedge^{j+1}(T^{1,0}_X) \otimes \wedge^n(T^{1,0}_X)^*)
\end{array}
$$

By $\tau$ we denote the natural contraction map

$$
\tau : \Gamma((\wedge^i(T^{0,1}_X)^* \otimes \wedge^j(T^{1,0}_X))^* \otimes \wedge^n(T^{1,0}_X)^*) \to \Omega^{n-j,i},
$$

which is an isomorphism of $C^\infty(X, \mathbb{C})$-modules.

Take a local holomorphic chart $(U; z_1, \ldots, z_n)$ of $X$, and set

$$
b = \partial_{z_{j_1}} \wedge \cdots \wedge \partial_{z_{j_l}};
$$

$$
\omega = dz_1 \wedge \cdots \wedge dz_n.
$$

Because of (9), we have

$$
\tau d^n_{\nabla}(b \otimes \omega) = (-1)^{l+1} \partial_{\pi} \tau (b \otimes \omega).
$$

Lemma 5.2. For all $f \in C^\infty(X, \mathbb{C})$, $b \in \Theta^k_X$ and $\mu \in \Omega^l_X$, we have:

$$
d_{\pi}(fb) = -(\pi^{\sharp} \partial f) \wedge b + f(d_{\pi} b),
$$

$$
\partial_{\pi}(f \mu) = (\pi^{\sharp} \partial f) \cup \mu + f(\partial_{\pi} \mu).
$$

As a consequence, we have

**Proposition 5.3.**

$$
\tau \circ \partial_A^\nabla = \overline{\partial} \circ \tau
$$

$$
\tau \circ \partial_B^\nabla = (-1)^{k+1} \partial_{\pi} \circ \tau
$$

*Proof. The first relation (16) is a simple consequence of the definition (12) of $\partial_A^\nabla$, while the second (17) follows from (13), (15) and Lemma 5.2.*

Now the conclusion of the theorem follows immediately.
6. Evens-Lu-Weinstein duality

We recall a remarkable duality construction due to Evens, Lu & Weinstein [7].

Consider a compact complex (and therefore orientable) manifold $X$ with $\dim_\mathbb{C} X = n$, a complex Lie algebroid $B$ over $X$ with $\text{rk}_\mathbb{C} B = r$ and a module $E$ over $B$. The complex dual $E^*$ is also a module over $B$. We will use the symbol $\nabla$ to denote the representations of $B$ on both $E$ and $E^*$.

The complex vector bundle $Q_B = \wedge^n B \otimes \wedge^{2n}(T_X \otimes \mathbb{C})^*$ is a module over the complex Lie algebroid $B$ with representation $D : \Gamma(Q_B) \to \Gamma(B^* \otimes Q_B)$ [7] given by

$$D_b(X \otimes \mu) = [b, X] \otimes \mu + X \otimes L_{\rho(b)}\mu,$$

for all $b \in \Gamma(B)$, $X \in \Gamma(\wedge^n B)$ and $\mu \in \Gamma(\wedge^{2n}(T_X \otimes \mathbb{C})^*)$.

By $H^*(B, E)$ and $H^*(B, E^* \otimes Q_B)$, we denote the Lie algebroid cohomology of $B$ with coefficients in $E$ and $E^* \otimes Q_B$, respectively. We use the notation $d_B^\nabla$ to denote their coboundary differential operators in both cases. Let $\Xi$ be the isomorphism of vector bundles:

$$\Xi : \wedge^r B^* \otimes (\wedge^r B \otimes \wedge^{2n}(T_X \otimes \mathbb{C})^*) \to \wedge^{2n}(T_X \otimes \mathbb{C})^*: \xi \otimes (X \otimes \mu) \mapsto (\xi \cup X) \mu.$$

The following lemma can be verified by a direct computation.

**Lemma 6.1.** We have

$$\Xi \circ d_B^\nabla(\xi \otimes (X \otimes \mu)) = (-1)^{r-1}d(\rho(\xi \cup X) \cup \mu),$$

for any $\xi \otimes (X \otimes \mu) \in \Gamma(\wedge^{r-1} B^* \otimes Q_B)$

Consider the bilinear map

$$\langle \cdot, \cdot \rangle : \Gamma(\wedge^k B^* \otimes E) \otimes \Gamma(\wedge^{r-k} B^* \otimes E^* \otimes Q_B) \to \Gamma(\wedge^{2n}(T_X \otimes \mathbb{C})^*)$$

defined by

$$\langle \xi_1 \otimes e, \xi_2 \otimes e \otimes (X \otimes \mu) \rangle = \epsilon(e) \cdot (\xi_1 \wedge \xi_2)(X) \cdot \mu.$$

**Lemma 6.2.** If $\xi_1 \otimes e \in \Gamma(\wedge^{k-1} B^* \otimes E)$ and $\xi_2 \otimes e \otimes (X \otimes \mu) \in \Gamma(\wedge^{r-k} B^* \otimes E^* \otimes Q_B)$, then

$$\langle d_B^\nabla(\xi_1 \otimes e), \xi_2 \otimes e \otimes (X \otimes \mu) \rangle + (-1)^{r-1} \langle \xi_1 \otimes e, d_B^\nabla(\xi_2 \otimes e \otimes (X \otimes \mu)) \rangle = \Xi \circ d_B^\nabla(\epsilon(e) \cdot \xi \otimes (X \otimes \mu)) = (-1)^{r-1}d(\epsilon(e) \cdot \rho(\xi \cup X) \cup \mu).$$

Therefore, by Stokes’ theorem, the pairing

$$\langle \alpha, \beta \rangle = \int_X \langle \alpha, \beta \rangle$$

where $\alpha \in \Gamma(\wedge^k B^* \otimes E)$ and $\beta \in \Gamma(\wedge^{r-k} B^* \otimes E^* \otimes Q_B)$ satisfies

$$\langle d_B^\nabla(\alpha), \beta \rangle + (-1)^{r-1}\langle \alpha, d_B^\nabla(\beta) \rangle = 0$$
(where \( \alpha \in \Gamma(\wedge^{k-1} B^* \otimes E) \) and \( \beta \in \Gamma(\wedge^{r-k} B^* \otimes E^* \otimes Q_B) \)) and thus induces a pairing at the cohomology level \([7]\):
\[
\langle \cdot, \cdot \rangle : H^k(B, E) \otimes H^{r-k}(B, E^* \otimes Q_B) \to \mathbb{C}.
\] (18)

The following is due to Block \([1]\).

**Proposition 6.3.** If \( B \) is an elliptic Lie algebroid, the cohomology pairing \((18)\) is perfect.

Given a holomorphic Poisson manifold \((X, \pi)\), we can take \( B = T^0_X \bowtie (T^1_X)_\pi^* \). Then \( Q^+_B = \wedge^n(T^1_X)_\pi^* \) and, taking \( E = Q^+_B \), we have \( E = \wedge^n(T^1_X)_\pi^* \) and \( E^* \otimes Q_B = \wedge^n(T^1_X)_\pi^* \).

In this particular case, we get the cohomology pairing
\[
\langle \cdot, \cdot \rangle : H^k(T^0_X \bowtie (T^1_X)_\pi^*, \wedge^n(T^1_X)_\pi^*) \otimes H^{2n-k}(T^0_X \bowtie (T^1_X)_\pi^*, \wedge^n(T^1_X)_\pi^*) \to \mathbb{C},
\]

If we identify the cochain group \( \bigoplus_{k,l} C^{k,l} \) with \( \bigoplus_{p,q} \Omega^{p,q}_X \) via the contraction map \( \tau \) (see Equation \((14)\)), then a straightforward (though lengthy) computation shows that, on the cochain level, the above cohomology pairing is given by
\[
\Omega^{i,j}_X \otimes \Omega^{k,l}_X \to \mathbb{C} : \zeta \otimes \eta \mapsto \int_X (\zeta \wedge \eta)^{top}.
\]

We have proved the following theorem.

**Theorem 6.4.** Let \((X, \pi)\) be a compact holomorphic Poisson manifold. The pairing
\[
\langle \cdot, \cdot \rangle : H_{2n-k}(X, \pi) \otimes H_k(X, \pi) \to \mathbb{C} : \left[ \zeta \right] \otimes \left[ \eta \right] \mapsto \int_X (\zeta \wedge \eta)^{top}
\]
(where \( \zeta, \eta \in \bigoplus_{k,l} \Omega^{k,l}_X \)) is nondegenerate.

**Remark 6.5.** When \( X \) is a compact complex manifold considered as a zero Poisson manifold, then \( H_k(X, \pi) \cong \bigoplus_{j-i=n-k} H^{i,j}(X) \). The above theorem easily follows from Serre duality.

7. Examples

The purpose of this section is the computation of the Koszul-Brylinski Poisson homology of all Poisson structures with which \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) can be endowed.

From now on, \( X \) will denote the complex manifold \( \mathbb{CP}^1 \times \mathbb{CP}^1 \). Since \( X \) is 2-dimensional, all holomorphic bivector fields on it are automatically Poisson tensors. Thus the Poisson tensors on \( X \) form the complex vector space \( H^0(X, \wedge^2 T_X) \), which is known to be 9-dimensional. Here is a more explicit description of \( H^0(X, \wedge^2 T_X) \).
Proposition 7.1. Let $P^{2,2}$ denote the 9-dimensional vector space of all bihomogeneous polynomials on $\mathbb{C}^2 \times \mathbb{C}^2$ of bidegree $(2,2)$. Given any $p \in P^{2,2}$, there exists a unique holomorphic bivector field $\pi_p$ on $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ such that, in an affine chart $(z_1, z_2) \mapsto ([1 : z_1], [1 : z_2])$ of $\mathbb{CP}^1 \times \mathbb{CP}^1$, we have

$$\pi_p = q(z_1, z_2) \partial_{z_2} \wedge \partial_{z_1},$$

where $q(z_1, z_2) = p((1, z_1), (1, z_2))$. The map

$$p \in P^{2,2} \mapsto \pi_p \in H^0(X, \wedge^2 T_X)$$

is an isomorphism of complex vector spaces. As a consequence, the space of all holomorphic Poisson bivector fields is a 9-dimensional vector space over $\mathbb{C}$.

Theorem 7.2. For any holomorphic Poisson bivector field $\pi$ on $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, we have

$$H_0(X, \pi) = 0, \quad H_1(X, \pi) = 0, \quad H_2(X, \pi) \cong \mathbb{C}^1, \quad H_3(X, \pi) = 0, \quad H_4(X, \pi) = 0.$$ 

Proof. Let us first assume that $\pi = 0$. In this case, $H_k(X, \pi) = \bigoplus_{j-i=2-k} H^{i,j}(X)$ and we obtain

$$H_0(X, \pi) = H^{0,2}(X) = 0, \quad H_1(X, \pi) = H^{1,2}(X) \oplus H^{0,1}(X) = 0,$$

$$H_3(X, \pi) = H^{2,0}(X) = 0, \quad H_4(X, \pi) = H^{2,1}(X) \oplus H^{1,0}(X) = 0,$$

and

$$H_2(X, \pi) = H^{0,0}(X) \oplus H^{1,1}(X) \oplus H^{2,2}(X) \cong \mathbb{C}^1.$$ 

Now let us assume that $\pi \neq 0$. By definition, $H_0(X, \pi)$ consists of those $\alpha \in \Omega^2_X$ such that $\overline{\partial} \alpha = 0$ and $\partial_{\pi} \alpha = 0$. The first condition means that $\alpha$ is a holomorphic 2-form on $X$. Since $H^0(X, \Omega^2_X) = H^{0,2}(X) = H^{0,2}(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$, it follows that $H_0(X, \pi) = 0$.

We now proceed to compute $H_1(X, \pi)$. Assume that $\theta + \omega \in \Omega^{1,0} \oplus \Omega^{2,1}$ is a Koszul-Brylinski 1-cycle. That is, $\partial_{\pi} \theta + \overline{\partial} \omega + \partial \omega = 0$. Hence it follows that $\partial_{\pi} \theta = 0$, $\overline{\partial} \theta + \partial_{\pi} \omega = 0$ and $\overline{\partial} \omega = 0$. Since $H^{1,2}(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$, there exists $\beta \in \Omega^2_X$ with $\omega = \overline{\partial} \beta$. On the other hand, from $\partial_{\pi} \theta = 0$, it follows that $i_{\pi} \partial \theta = 0$ since $\partial_{\pi} \theta = [\overline{\partial}, i_{\pi}] \theta = -i_{\pi} \partial \theta$. Therefore, $\partial \theta$ vanishes at those points where $\pi$ does not vanish. By Proposition 7.1, $\pi$ is nonzero on a dense subset of $\mathbb{CP}^1 \times \mathbb{CP}^1$. Thus, we have $\partial \theta = 0$, which implies that $\theta = \partial \alpha$ for some $\alpha \in \Omega^0_X$, since $H^{1,0}(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$. It follows that

$$0 = \overline{\partial} \theta + \partial_{\pi} \omega = \overline{\partial} \partial \alpha + \partial_{\pi} \overline{\partial} \beta = \overline{\partial}(-\partial \alpha - \partial_{\pi} \beta).$$

Since $H^{0,1}(\mathbb{CP}^1 \times \mathbb{CP}^1) = 0$, we have $\partial \alpha + \partial_{\pi} \beta = 0$. Thus $\theta + \omega = (\overline{\partial} - \partial_{\pi}) \beta$ from which we conclude that $H_1(X, \pi) = 0$.

By Evens-Lu-Weinstein duality, we have

$$H_3(X, \pi) \cong H_1(X, \pi) = 0 \quad \text{and} \quad H_4(X, \pi) \cong H_0(X, \pi) = 0.$$ 

Moreover, according to Theorem 4.5,

$$\chi_{KB}(X, \pi) = \chi(X) = \chi(\mathbb{CP}^1) + \chi(\mathbb{CP}^1) = 4.$$

Thus we have $H_2(X, \pi) \cong \mathbb{C}^4$. This concludes the proof. \qed
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