WEYL MODULES AND WEYL FUNCTORS
FOR HYPER–MAP ALGEBRAS

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Abstract. We investigate the representations of the hyperalgebras associated to the map algebras $\mathfrak{g} \otimes \mathcal{A}$, where $\mathfrak{g}$ is any finite-dimensional complex simple Lie algebra and $\mathcal{A}$ is any associative commutative unitary algebra with a multiplicatively closed basis. We consider the natural definition of the local and global Weyl modules, and the Weyl functor for these algebras. Under certain conditions, we prove that these modules satisfy certain universal properties, and we also give conditions for the local or global Weyl modules to be finite-dimensional or finitely generated, respectively.

Introduction

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathcal{X}$ be an affine scheme (for instance, an algebraic variety), both defined over a field $K$, often assumed to be algebraically closed and with characteristic zero. The Map algebra is the Lie algebra of regular maps from $\mathcal{X}$ to $\mathfrak{g}$. Denoting by $\mathcal{A}$ the coordinate ring of $\mathcal{X}$, the map algebra can also be realized as the Lie algebra $\mathfrak{g} \otimes \mathcal{A}$. These algebras generalize the loop and current algebras, which play an important role in the theory of affine Kac-Moody Lie algebras. The representation theory of the map algebras is an extremely active area of research. Recently, there has been an intensive study of the finite-dimensional representation theory of the map algebras $\mathfrak{g} \otimes \mathcal{A}$, where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over the complex numbers and $\mathcal{A}$ is an associative commutative unitary algebra.

Parallel to this, the hyperalgebras in positive characteristic are constructed by considering an integral form of the universal enveloping algebra of a Lie algebra and then tensoring this form over $\mathbb{Z}$ with an arbitrary field $\mathbb{F}$, which we must assume is algebraically closed. In the case of a complex simple Lie algebra $\mathfrak{g}$, Kostant [31] constructed such an integral form of $U(\mathfrak{g})$. The corresponding hyperalgebra is usually denoted by $U_\mathbb{F}(\mathfrak{g})$. The affine analogues of Kostant’s form were constructed by Garland [26] in the non-twisted affine Kac-Moody algebra case and by Mitzman [33] in a more general way for all affine Kac-Moody algebras. Suitable integral forms for the map algebras were formulated by the second author in [10].

The global Weyl Modules introduced via generators and relations in the context of affine Lie algebras in [18] are parameterized by a dominant integral weight $\lambda$ of a complex semisimple Lie algebra $\mathfrak{g}$, denoted by $W(\lambda)$. They are infinite-dimensional if $\lambda \neq 0$ and $W(\lambda)$ is also a right module for a polynomial algebra $\mathcal{A}_\lambda$ specifically constructed. Once the global Weyl modules were defined, the local Weyl modules were obtained by tensoring the global Weyl modules with irreducible modules for $\mathcal{A}_\lambda$ or, equivalently, they can be given via generators and relations. After that, in [19] a more general case was considered where $\mathcal{A}$ was the coordinate ring of an algebraic variety and partial results analogous to those in [18] were obtained. Finally, in [12] (motivated by [14]), the authors took a categorical approach to these modules for the map algebras and it was shown that there is a natural definition of the local and

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global Weyl modules via homological properties. This leads to the so called Weyl functor from the category of left modules of a commutative algebra to the category of modules for a Lie algebra, similarly to [18]. Such a functor was not considered in [19]. An advantage of this approach (considering the commutative algebra $A$) is the study of the structure of the global Weyl modules $W(\lambda)$, which are free $A$-modules (see [18] for $\mathfrak{sl}_2$, [17] for $\mathfrak{sl}_{r+1}$, [23] for simply laced algebras, and the general case by passing to the quantum setting using [30] and [4]).

Furthermore, in a zero-characteristic base field, the Weyl modules were widely considered in several contexts: for twisted affine Lie algebras in [24, 15], for the general context of equivariant map algebras in [25, 22], for the Lie superalgebras in [11, 19, 20] as well as some considerations involving Levi subalgebras in [21], and for the hyperspecial current algebras in [15]. There is an application to invariant theory in [17] and a few others in [16, 34].

The corresponding hyperalgebras in the (twisted and un-twisted) affine case and their finite-dimensional representations, with an emphasis on the local Weyl modules, were studied by Moura and Jakelic in [28] and by Moura and the first author in [8].

In these papers the authors considered the positive characteristic analogues of local Weyl modules and explored the universal properties of these modules. Recently, the multicurrent and multiloop cases were considered by the authors in [6]. However, there is no mention of global Weyl modules in all these papers. The authors only considered the local Weyl modules defined via generators and relations.

Basically, many of the results in characteristic zero remain valid in positive characteristic, but, unfortunately, it is unknown the adaptation of all these constructions from the aforementioned papers (except those already in the hyperalgebra context) to the positive characteristic setting, for instance a few results in [18, 19, 17, 23] and [12, Sections 5 and 6]. This difficulty already appeared with many other results first proved in characteristic zero and then generalized to positive characteristic setting by using very different tools as in the pairs of papers [18, 28], [13, 8], and [23, 7].

The results of this work show that the Weyl functor and the Weyl modules defined in this paper satisfy properties similar to the ones satisfied by Weyl functors defined in the characteristic zero setting.

Section 1 is mostly dedicated to reviewing all preliminaries on Lie algebras, details of the construction of hyperalgebras, some useful straightening identities, as well as fixing the basic notation of the paper. Section 2 is dedicated to reviewing the relevant facts about the finite-dimensional modules for hyperalgebras $U_F(g)$ and to define the main categories of objects for this paper. We define the Weyl functor and global and local Weyl modules in this section. Our main results on the structure of these modules are in Subsection 2.5.

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1. Preliminaries

Throughout this work, \( \mathbb{C} \) denotes the set of complex numbers and \( \mathbb{Z}, \mathbb{Z}_+ \) and \( \mathbb{N} \) are the sets of integers, non-negative integers, and positive integers respectively.

1.1. Simple Lie algebras. Let \( g \) be a finite-dimensional complex simple Lie algebra and \( I \) the set of vertices of the associated Dynkin diagram. Fix a Cartan subalgebra \( h \) of \( g \) and let \( R \) denote the corresponding set of roots. Let \( \{ \alpha_i : i \in I \} \) (respectively, \( \{ \omega_i : i \in I \} \)) denote the simple roots (respectively, fundamental weights). Set \( Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i, \quad Q^+ = \bigoplus_{i \in I} \mathbb{Z}_+ \alpha_i, \quad P = \bigoplus_{i \in I} \mathbb{Z} \omega_i, \quad P^+ = \bigoplus_{i \in I} \mathbb{Z}_+ \omega_i, \) and \( R^+ = R \cap Q^+ \). We denote the Weyl group of \( g \) by \( W \) and its longest element by \( w_0 \).

Let \( C := \{ x^\pm_i, h_i : \alpha \in R^+, \ i \in I \} \) be a Chevalley basis of \( g \) and set

\[ x^\pm_i := x^\pm_{\alpha_i}, \quad h_\alpha := [x^+_\alpha, x^-_{\alpha}], \quad \text{and} \quad h_i := h_{\alpha_i}. \]

For each \( \alpha \in R^+ \), the subalgebra of \( g \) spanned by \( \{ x^\pm_{\alpha_i}, h_\alpha \} \) is naturally isomorphic to \( \mathfrak{sl}_2 \). We have a triangular decomposition \( g = n^- \oplus h \oplus n^+ \) with

\[ n^\pm := \bigoplus_{\alpha \in R^+} \mathbb{C} x^\pm_{\alpha}. \]

1.2. Map algebras. Fix \( A \) a commutative \( \mathbb{C} \)-associative algebra with unity over \( \mathbb{C} \) and suppose that \( A \) has a multiplicatively closed basis \( B \) (see [3] for details).

The map algebra of a Lie algebra \( a \) over \( \mathbb{C} \) is the \( \mathbb{C} \)-vector space \( a \otimes A := a \otimes A \), with Lie bracket given by linearly extending the bracket

\[ [g \otimes a, g' \otimes b] = [g, g']_a \otimes ab, \quad g, g' \in a, \quad a, b \in A, \]

where \([.,.]_a\) is the Lie bracket of \( a \).

The Lie algebra \( a \) can be embedded into \( a \otimes A \) as \( a \otimes 1 \) and, if \( s \) is a subalgebra of \( a \), then \( s \otimes A =: s \otimes A \) is naturally a subalgebra of \( a \otimes A \). In particular, we have a decomposition

\[ a \otimes A = n^+_A \oplus h_A \oplus n^-_A \]

where \( h_A \) is an abelian subalgebra of \( g_A \).

Remark 1.2.1. In the case that \( A \) is the coordinate ring of some algebraic variety \( X \), it follows that \( a \otimes A \) is isomorphic to the Lie algebra of regular maps \( X \to a \).
1.3. Universal enveloping algebras. For a Lie algebra \( \mathfrak{a} \), the corresponding universal enveloping algebra of \( \mathfrak{a} \) will be denoted by \( U(\mathfrak{a}) \). The Poincaré–Birkhoff–Witt (PBW) Theorem implies that

\[
U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)
\]

\[
U(\mathfrak{g}_A) \cong U(\mathfrak{n}^-_A) \otimes U(\mathfrak{h}_A) \otimes U(\mathfrak{n}^+_A).
\]

The assignments \( \Delta : \mathfrak{g}_A \to U(\mathfrak{g}_A) \otimes U(\mathfrak{g}_A) \) where \( x \mapsto x \otimes 1 + 1 \otimes x \), \( S : \mathfrak{g}_A \to \mathfrak{g}_A \) where \( x \mapsto -x \), and \( \epsilon : \mathfrak{g}_A \to \mathbb{C} \) where \( x \mapsto 0 \), can be uniquely extended so that \( U(\mathfrak{g}_A) \) becomes a Hopf algebra with comultiplication \( \Delta \), antipode \( S \), and counit \( \epsilon \).

The augmentation ideal (i.e., the kernel of \( \epsilon \)) for a Hopf algebra \( H \) is denoted \( H^0 \).

1.4. Integral forms and hyperalgebras. Given \( \alpha \in R \) and \( a \in A \setminus \{0\} \), consider the power series with coefficients in \( U(\mathfrak{h}_A) \) given by

\[
\Lambda_{\alpha,a}(u) = \sum_{r=0}^{\infty} \Lambda_{\alpha,a,r} u^r = \exp \left( -\sum_{s=1}^{\infty} \frac{h_a \otimes a^s}{s} u^s \right).
\]

Note that \( \Lambda_{\alpha,f,r} \) is a polynomial in \( (h_a \otimes f^j) \) for \( j \in \{1, \ldots, r\} \). For \( i \in I \), we simply write \( \Lambda_{I,a,r} \) in place of \( \Lambda_{\alpha,a,r} \).

The following lemma shows that elements of the form \( \Lambda_{\alpha,a,k,r}, k \geq 2 \), are linear combinations of products of elements of the form \( \Lambda_{\alpha,a,s} \).

**Lemma 1.4.1.** Let \( a \in B, \alpha \in R \), and \( r, k \in \mathbb{N} \). Then

\[
\Lambda_{\alpha,a,k,r} = k \Lambda_{\alpha,a,kr} + \sum_{(s,n)} m_{s,n} \Lambda_{\alpha,a,s_1} \cdots \Lambda_{\alpha,a,s_l}
\]

where \( m_{s,n} \in \mathbb{Z} \) and the sum is over all \( s, n \in \mathbb{N}^l \) for some \( l \in \mathbb{N} \) such that \( s_i \neq s_j \), \( l \sum n_j s_j > 1 \), and \( \sum n_j s_j = r k \).

**Proof.** The lemma is proven for \( A = \mathbb{C}[t, t^{-1}] \) and \( a = t \) as part of [25, Lemma 5.11]. We extend this to the current setting by replacing \( t \) with \( a \). \( \square \)

The pure tensors in \( C \otimes B \) form a basis for \( \mathfrak{g}_A \), which we shall denote by \( \mathbb{B} \). Given an order on \( \mathbb{B} \) and a PBW monomial according to this order, we construct an ordered monomial in the elements of

\[
\mathcal{M}(A) = \left\{ (x_\alpha^\pm \otimes b)^{(k)}, \Lambda_{i,c,r}, \left( h_i \otimes 1 \right)_k \mid \alpha \in R^+, i \in I, b, c \in B, c \neq 1, k, r \in \mathbb{N} \right\},
\]

where

\[
(x_\alpha^\pm \otimes b)^{(k)} = \frac{(x_\alpha^\pm \otimes b)^k}{k!}
\]

and

\[
(h_i)(h_i - 1) \ldots (h_i - k + 1) = \frac{(h_i \otimes 1)^k}{k!},
\]

through the correspondence

\[
(x_\alpha^\pm \otimes b)^k \leftrightarrow (x_\alpha^\pm \otimes b)^{(k)}, \quad (h_i \otimes 1)^k \leftrightarrow \left( h_i \otimes 1 \right)_k, \quad \text{and} \quad (h_i \otimes c)^r \leftrightarrow \Lambda_{i,c,r}.
\]

By using a similar correspondence we consider monomials in \( U(\mathfrak{g}) \) formed by elements of

\[
\mathcal{M} = \left\{ (x_\alpha^\pm)^{(k)}, (h_i)_k \mid \alpha \in R^+, i \in I, k \in \mathbb{N} \right\}.
\]

It is clear that we have a natural inclusion \( \mathcal{M} \subset \mathcal{M}(A) \) and the set of ordered monomials thus constructed are bases of \( U(\mathfrak{g}) \) and \( U(\mathfrak{g}_A) \), respectively.
Let $U_Z(g) \subseteq U(g)$ and $U_Z(g_A) \subseteq U(g_A)$ be the $Z$–subalgebras generated respectively by 
\[ \{(x_\alpha^+)^{(k)} \mid \alpha \in R^+, k \in \mathbb{N}\} \quad \text{and} \quad \{(x_\alpha^+ \otimes b)^{(k)} \mid \alpha \in R^+, b \in B, k \in \mathbb{N}\}. \]

We have the following fundamental theorem which was proved in [31] for $U_Z(g)$ and in [2] for $U_Z(g_A)$ (see also [26, 33] for other specific cases of these theorems).

**Theorem 1.4.2.** The subalgebra $U_Z(g)$ (resp., $U_Z(g_A)$) is a free $Z$-module and the set of all the ordered monomials constructed from $M$ (resp., $M(A)$) is a $Z$-basis of $U_Z(g)$ (resp., $U_Z(g_A)$).

Furthermore, if $a \in \{g, n^\pm, h, g_A, h_A\}$ and we set
\[ U_Z(a) := U(a) \cap U_Z(g) \]
we have
\[ \mathbb{C} \otimes_Z U_Z(a) \cong U(a). \]
Therefore, $U_Z(g_A)$ and $U_Z(g)$ are integral forms of $U(g_A)$ and $U(g)$, respectively.

**Remark 1.4.3.** $U_Z(a)$ is a free $Z$-module spanned by monomials formed by elements of $M \cap U(a)$ for the appropriate $M \in \{M, M(A)\}$. Notice that $U_Z(g) = U(g) \cap U_Z(g_A)$ which allows us to regard $U_Z(g)$ as a $Z$-subalgebra of $U_Z(g_A)$.

Given a field $F$, the $F$–hyperalgebra of $a$ is defined by
\[ U_F(a) := F \otimes_Z U_Z(a). \]

**Notation:** we will keep denoting by $x$ the image of the element $x \in U_Z(a)$ in $U_F(a)$.

The PBW Theorem gives the following isomorphisms
\[ U_F(g) \cong U_F(n^-)U_F(h)U_F(n^+) \quad \text{(1.4.1)} \]
\[ U_F(g_A) \cong U_F(n^-_A)U_F(h_A)U_F(n^+_A) \quad \text{(1.4.2)} \]
and the Hopf algebra structure on $U(g_A)$ induces a natural Hopf algebra structure over $Z$ on $U_Z(g_A)$ and this in turn induces a Hopf algebra structure on $U_F(g_A)$.

We refer to the $F$–hyperalgebra $U_F(g_A)$ as a hyper-map algebra of $g$ and $A$ over $F$.

**Remark 1.4.4.** Recall that, if the characteristic of $F$ is zero, the algebra $U_F(g_A)$ is isomorphic to $U((g_A)_F)$, where $(g_A)_F := F \otimes_Z (g_A)_Z$ and $(g_A)_Z$ is the $Z$-span of $B$. But, if $F$ has positive characteristic, we have an algebra homomorphism $U((g_A)_F) \to U_F(g_A)$ that fails to be injective and to be surjective.

1.5. **Straightening identities.** The next lemmas are essential tools in the proofs of Theorem 1.4.2 and it is also crucial in the study of finite-dimensional representations of hyper-map algebras.

**Definition 1.5.1.** Given $a, b \in A$ and $\alpha \in R^+$, we define the following series with coefficients in $U(g_A)$:
\[ X_{\alpha,a,b}(u) = \sum_{j=0}^{\infty} (x_\alpha^- \otimes a^jb^{j+1}) u^{j+1}. \]
Then, we set \( X_{\alpha,a,b}(u) \) \( _n \) \( \in U(g_A) \) to be the coefficient of $u^n$ in $X_{\alpha,a,b}(u)$. 
The next lemma was originally proved in the $U(g \otimes \mathbb{C}[t, t^{-1}])$ setting in [26, Lemma 7.5] and in [10, Lemma 5.4] for a general map algebra $U(g_A)$. The first presentation in the context of hyperalgebras was in [28, Lemma 1.4].

**Lemma 1.5.2.** Let $\alpha \in R^+, a, b \in A$ and $r, s \in \mathbb{N}$ with $s \geq r \geq 1$. Then,

$$(x^+_{\alpha} \otimes a)^{(r)}(x^-_{\alpha} \otimes b)^{(s)} \equiv (-1)^r \sum_{j=0}^{r} \left( (X^-_{\alpha,a,b}(u))^{(s-r)} \right)_{s-r+j} \Lambda_{\alpha,ab,r-j} \mod U_F(g_A)U(n^+_A).$$

Given $\alpha \in R^+, a \in A$ and $k \geq 0$, define the degree of $(x^+_a \otimes a)^{(k)}$ to be $k$. For a monomial of the form $(x^+_a \otimes a_1)^{(k_1)} \cdots (x^+_a \otimes a_l)^{(k_l)}$, where $a_1, \ldots, a_l \in A$, $k_1, \ldots, k_l \in \mathbb{Z}_+$, and the choice of $\pm$ fixed, define its degree to be $k_1 + \cdots + k_l$.

**Lemma 1.5.3.** Let $\alpha, \beta \in R^+, i \in I$, $a, b \in A$, $k, l \in \mathbb{N}$.

1. $(x^+_\alpha \otimes a)^{(k)}(x^-_{\beta} \otimes b)^{(l)}$ is in the $\mathbb{Z}$-span of elements $(x^+_\beta \otimes b)^{(l)}(x^-_{\alpha} \otimes a)^{(k)}$ and other monomials of degree strictly smaller than $k + l$.

2. $$(x^+_\alpha)^{(l)}(x^-_{\alpha})^{(k)} = \sum_{m=0}^{\min\{k,l\}} (x^-_{\alpha})^{(k-m)}(h_{\alpha} - k - l + 2m)_{m} (x^+_\alpha)^{(l-m)}.$$  

3. $$(h_i)_{l}(x^+_\alpha \otimes a)^{(k)} = (x^-_{\alpha} \otimes a)^{(k)}(h_i \pm k \alpha(h_i))_l.$$  

4. $$(x^+_a \otimes a)^{(k)}(x^-_{a} \otimes a)^{(l)} = \binom{k + l}{k}(x^-_{a} \otimes a)^{(k+l)}.$$  

5. $$\Lambda_{\alpha,a,r}(x^-_{\alpha} \otimes b)^{(k)} = \sum_{s=0}^{r} \left( \sum_{j \geq 0} (j + 1)(x^-_{\alpha} \otimes a^j b)u^j \right)_{r-s}^{(k)} \Lambda_{\alpha,a,s}.$$  

**Proof.** The item (1) can be deduced from [2, Equation (4.1.6)] and the item (2) can be deduced from [27, Lemma 26.2]. (3) is proved by induction on $k + l$ and (4) is easily established. (5) is a reformulation of Proposition 4.1.2 (4.1.5) in [2] (also, see [26] (8.12) and [33, Lemma 4.3.4 (iii)] for these formulas in the context of affine Kac-Moody algebras).  

### 2. The categories of modules

Let $\mathbb{F}$ be an algebraically closed field. The symbol $\otimes$ denotes the tensor product of $\mathbb{F}$-vector spaces.

The following subsections, except [2.1 and 2.7] are the positive characteristic partial counterpart of [12, Sections 3 and 4].
2.1. Finite-dimensional modules for hyperalgebras $U_F(g)$. We now review the finite-dimensional representation theory of $U_F(g)$ and refer to [28] Section 2] for a more detailed review. This is motivation for what we will develop in the next sections.

Let $V$ be a $U_F(g)$-module. A nonzero vector $v \in V$ is called a weight vector if there exists $\mu \in U_F(h)^*$ such that $hv = \mu(h)v$ for all $h \in U_F(h)$. The subspace consisting of all weight vectors of weight $\mu$ is called weight space of weight $\mu$, which we denote by $V_\mu$. When $V_\mu \neq 0$, $\mu$ is called a weight of $V$ and $\text{wt}(V) = \{\mu \in U_F(h)^* : V_\mu \neq 0\}$ is called the set of weights of $V$. If $V = \bigoplus_{\mu \in U_F(h)^*} V_\mu$, then $V$ is said to be a weight module.

Moreover, when $v \in V$ is a weight vector and $(x_\alpha^+)^{(k)}v = 0$ for all $\alpha \in R^+, k > 0$, then $v$ is called a highest-weight vector. When $V$ is generated by a highest-weight vector, it is said to be a highest-weight module.

Notice that we have an inclusion $P \hookrightarrow U_F(h)^*$ determined by associating to $\mu \in P$ the functional (which we keep denoting $\mu$) given by

$$\mu \left( \begin{pmatrix} h_i \\ k \end{pmatrix} \right) := \left( \begin{pmatrix} \mu(h_i) \\ k \end{pmatrix} \right) \quad \text{and} \quad \mu(xy) := \mu(x)\mu(y) \quad \text{for all} \quad i \in I, k \geq 0, x, y \in U_F(h).$$

In particular, this inclusion provides a partial order $\leq$ on $U_F(h)^*$ defined by $\mu \leq \lambda$ if $\lambda - \mu \in Q^+$ and we have

$$(x_\alpha^+)^{(k)}V_\mu \subseteq V_{\mu+\alpha} \quad \text{for all} \quad \alpha \in R^+, \mu \in U_F(h)^*, k > 0. \quad (2.1.1)$$

**Theorem 2.1.1.** Let $V$ be a $U_F(g)$-module.

1. If $V$ is finite-dimensional, then $V$ is a weight-module, $\text{wt}(V) \subseteq P$, and $\dim V_\mu = \dim V_{\mu+\alpha}$ for all $\sigma \in W, \mu \in U_F(h)^*$.

2. If $V$ is a highest-weight module of highest weight $\lambda$, then $\dim(V_\lambda) = 1$ and $V_\mu \neq 0$ only if $\sigma(h_\lambda) \leq \mu \leq \lambda$. Moreover, $V$ has a unique maximal proper submodule and a unique irreducible quotient. In particular, $V$ is indecomposable.

3. For each $\lambda \in P^+$, the $U_F(g)$-module $W_F(\lambda)$ given by the quotient of $U_F(g)$ by the left ideal $I_F(\lambda)$ generated by $U_F(n^+)^0$, $h - \lambda(h)$ and $(x_\alpha^-)^{(k)}$, for all $h \in U_F(h)$, $\alpha \in R^+$, $k > \lambda(h_\alpha)$, is nonzero and finite-dimensional. Moreover, every finite-dimensional highest-weight module of highest weight $\lambda$ is a quotient of $W_F(\lambda)$.

4. If $V$ is finite-dimensional and also irreducible, then there exists a unique $\lambda \in P^+$ such that $V$ is isomorphic to the irreducible quotient $V_F(\lambda)$ of $W_F(\lambda)$. If the characteristic of $F$ is zero, then $W_F(\lambda)$ is irreducible.

5. For each $\lambda \in P^+$, $\text{ch}(W_F(\lambda))$ is given by the Weyl character formula. In particular, $\mu \in \text{wt}(W_F(\lambda))$ if, and only if, $\sigma\mu \leq \lambda$ for all $\sigma \in W$. \hfill \Box

The module $W_F(\lambda)$ defined in Theorem 2.1.1(5) is called the Weyl module of highest weight $\lambda$.

**Remark 2.1.2.** The notions of lowest-weight vectors and modules are similar and are obtained by replacing $(x_\alpha^+)^{(k)}$ by $(x_\alpha^-)^{(k)}$.

2.2. The category $\mathcal{I}_F$ of $U_F(g_A)$-modules and a weight-bounded subcategory. Let $V$ be a $U_F(g)$-module. It is said that $V$ is locally finite-dimensional if any element of $V$ lies in a finite-dimensional $U_F(g)$-submodule of $V$. In other words, $V$ is isomorphic to a direct sum of irreducible finite-dimensional $U_F(g)$-modules.
Let $\mathcal{F}$ be the full subcategory of the category of $U_{\mathcal{F}}(\mathfrak{g}_A)$-modules which are locally finite-dimensional $U_{\mathcal{F}}(\mathfrak{g})$-modules. Clearly, $\mathcal{F}$ is an abelian category that is closed under tensor products. We shall abuse notation and write $V \in \mathcal{F}$ to mean $V$ is an object of $\mathcal{I}_F$.

In the rest of this paper we shall use the following elementary result without mention:

**Lemma 2.2.1.** Let $V \in \mathcal{F}$.

1. If $V_\lambda \neq 0$ and $\text{wt}(V) \subset \lambda - Q^+$, then $\lambda \in P^+$ and $U_{\mathcal{F}}(\mathfrak{n}_A^+)\lambda V_\lambda = (x_\alpha^-)^{(s)} V_\lambda = 0$, for all $\alpha \in R^+$, $s > \lambda(h_i)$. If, additionally, $V = U_{\mathcal{F}}(\mathfrak{g}_A)V_\lambda$ and $\dim V_\lambda = 1$, then $V$ has a unique irreducible quotient.

2. If $V = U_{\mathcal{F}}(\mathfrak{g}_A)V_\lambda$ and $U_{\mathcal{F}}(\mathfrak{n}_A^+)\lambda V_\lambda = 0$, then $\text{wt}(V) \subset \lambda - Q^+$.

3. If $V$ is irreducible and finite-dimensional, then there exists $\lambda \in \text{wt}(V)$ such that $\dim V_\lambda = 1$ and $\text{wt}(V) \subset \lambda - Q^+$.

Given a left $U_{\mathcal{F}}(\mathfrak{g})$-module $V$, by regarding $U_{\mathcal{F}}(\mathfrak{g}_A)$ as a right $U_{\mathcal{F}}(\mathfrak{g})$-module via right multiplication, set

$$P(V) := U_{\mathcal{F}}(\mathfrak{g}_A) \otimes_{U_{\mathcal{F}}(\mathfrak{g})} V.$$ 

Then, $P(V)$ is a left $U_{\mathcal{F}}(\mathfrak{g}_A)$-module by left multiplication and we have an isomorphism of $\mathcal{F}$-vector spaces

$$P(V) \cong U_{\mathcal{F}}(\mathfrak{g}_A) \otimes V. \quad (2.2.1)$$

where $\mathcal{A}_+$ is a fixed vector space complement to the subspace $\mathcal{C}$ of $\mathcal{A}$.

**Proposition 2.2.2.** Let $V$ be a locally finite-dimensional $U_{\mathcal{F}}(\mathfrak{g})$-module. Then,

1. $P(V) \in \mathcal{F}$.

2. If $V \in \mathcal{F}$, then the map $P(V) \to V$ given by $u \otimes v \to uv$ is a surjective morphism of objects in $\mathcal{F}$.

3. If $V$ is projective in the category of locally finite-dimensional $U_{\mathcal{F}}(\mathfrak{g})$-modules, then $P(V)$ is projective in $\mathcal{F}$.

4. For any $\lambda \in P^+$, we have $P(W_\mathcal{F}(\lambda))$ generated as a $U_{\mathcal{F}}(\mathfrak{g}_A)$-module by the element $p_\lambda := 1 \otimes v_\lambda$ with defining relations

$$U_{\mathcal{F}}(\mathfrak{n}_A^+)p_\lambda = 0, \quad h p_\lambda = \lambda(h)p_\lambda, \quad (x_\alpha^-)^{(s)}p_\lambda = 0,$$

for all $h \in U_{\mathcal{F}}(\mathfrak{h})$, $\alpha \in R^+$, $s, k \in \mathbb{Z}_+$, $s > \lambda(h_i)$. \hfill $\square$

The proof of the above proposition is analogous to its characteristic zero counterpart [12, Proposition 3].

**Remark 2.2.3.** In characteristic zero $P(V)$ is always projective since all locally finite-dimensional $\mathfrak{g}$-modules are completely reducible and, hence, projective in the category of locally finite-dimensional $\mathfrak{g}$-modules.

Given $\nu \in P^+$ and $V \in \mathcal{F}$, let $V^\nu \in \mathcal{F}$ be the unique maximal $U_{\mathcal{F}}(\mathfrak{g}_A)$-quotient of $V$ satisfying

$$\text{wt}(V^\nu) \subset \nu - Q^+, \quad (2.2.2)$$

or, equivalently,

$$V^\nu = V/\sum_{\mu \neq \nu} U_{\mathcal{F}}(\mathfrak{g}_A)V_\mu.$$

Notice that any morphism $\pi : V \to V'$ of objects in $\mathcal{F}$ induces a morphism $\pi^\nu : V^\nu \to (V')^\nu$.

Let $\mathcal{F}'$ be the full subcategory of objects $V \in \mathcal{F}$ such that $V = V^\nu$. It follows from the finite-dimensional representation theory of hyperalgebras that

$$V \in \mathcal{F}' \implies \#\text{wt}(V) < \infty, \quad (2.2.3)$$
since the weights of \( V \) are bounded above and it is a locally finite-dimensional module. This implies that \( V \) is a direct sum of finite-dimensional simple \( U_\mathfrak{F}(\mathfrak{g}) \)-modules.

The next result is immediate.

**Lemma 2.2.4.** Let \( \nu \in P^+ \).

1. \( \mathcal{I}_F^\nu \) is an abelian category, but not a tensor subcategory of \( \mathcal{I}_F \).
2. If \( V \) is projective in \( \mathcal{I}_F^\nu \), then \( P(V)^\nu \) is projective in \( \mathcal{I}_F^\nu \)

\( \square \)

**Remark 2.2.5.** In the characteristic zero setting, part (2) is true for any \( V \), cf. [12] Corollary 3.2.

2.3. The global Weyl modules. We are now able to define the main object of study for this paper. It is a natural extension of the definition in the characteristic zero setting to the hyperalgebras:

**Definition 2.3.1.** The *global Weyl module* of weight \( \lambda \in P^+ \) for \( U_\mathfrak{F}(\mathfrak{g}_A) \) is defined as

\[
\mathbf{W}_\mathfrak{F}(\lambda) := P(W_\mathfrak{F}(\lambda))^\lambda,
\]

Let \( w_\lambda \) be the image of \( p_\lambda \) in \( \mathbf{W}_\mathfrak{F}(\lambda) \). The following proposition is essentially an immediate consequence of Proposition 2.2.2 and provides a definition of \( \mathbf{W}_\mathfrak{F}(\lambda) \) via generators and relations.

**Remark 2.3.2.** The original definition given in [18] in the characteristic zero context was via generator and relations.

**Proposition 2.3.3.** For \( \lambda \in P^+ \), the module \( \mathbf{W}_\mathfrak{F}(\lambda) \) is generated by \( w_\lambda \) with the defining relations:

\[
U_\mathfrak{F}(n^+_A)^0 w_\lambda = 0, \quad hw_\lambda = \lambda(h)w_\lambda, \quad (x^-_\alpha)^{\langle s \rangle} w_\lambda = 0, \quad h \in U_\mathfrak{F}(\mathfrak{h}), \quad \alpha \in R^+, \quad s, k \in \mathbb{Z}_+, \quad s > \lambda(h_i).
\]

\((2.3.1)\)

**Proof.** The relation \( U_\mathfrak{F}(n^+_A)^0 w_\lambda = 0 \) follows directly from the fact that \( \mathbf{W}_\mathfrak{F}(\lambda) \subseteq \lambda - Q^+ \). The other relations are valid since they are already satisfied by \( p_\lambda \). It remains to see that these are in fact all the relations we have for \( \mathbf{W}_\mathfrak{F}(\lambda) \): let \( W' \) be the module generated by an element \( w_\lambda \) with the giving relations. By Proposition 2.2.2 \( W' \) is a quotient of \( P(W_\mathfrak{F}(\lambda)) \). Further, \( \text{wt}(W') \subseteq \lambda - Q^+ \) and it implies that \( W' \) satisfies (2.2.2). Finally, the maximality of \( \mathbf{W}_\mathfrak{F}(\lambda) \) implies that \( W' \) a quotient of \( \mathbf{W}_\mathfrak{F}(\lambda) \).

\( \square \)

2.4. The Weyl functor. Consider the annihilator of \( w_\lambda \) in the \( U_\mathfrak{F}(\mathfrak{g}_A) \)-module \( \mathbf{W}_\mathfrak{F}(\lambda) \), i.e.

\[
\text{Ann}_{U_\mathfrak{F}(\mathfrak{g}_A)}(w_\lambda) = \{ u \in U_\mathfrak{F}(\mathfrak{g}_A) : uw_\lambda = 0 \}
\]

and set

\[
\text{Ann}_{U_\mathfrak{F}(\mathfrak{h}_A)}(w_\lambda) := \text{Ann}_{U_\mathfrak{F}(\mathfrak{g}_A)}(w_\lambda) \cap U_\mathfrak{F}(\mathfrak{h}_A).
\]

Clearly \( \text{Ann}_{U_\mathfrak{F}(\mathfrak{h}_A)}(w_\lambda) \) is an ideal of \( U_\mathfrak{F}(\mathfrak{h}_A) \) and we set

\[
\mathcal{A}_\mathfrak{F}^\lambda := \frac{U_\mathfrak{F}(\mathfrak{h}_A)}{\text{Ann}_{U_\mathfrak{F}(\mathfrak{h}_A)}(w_\lambda)}.
\]

Now we regard \( \mathbf{W}_\mathfrak{F}(\lambda) \) as a right module for \( U_\mathfrak{F}(\mathfrak{h}_A) \) as follows:

\[
(uw_\lambda) \cdot x := ux \cdot w_\lambda, \quad \forall u \in U_\mathfrak{F}(\mathfrak{g}_A), \quad x \in U_\mathfrak{F}(\mathfrak{h}_A).
\]
To see that this action is well defined, one must prove that:

\[ U_F(n^+_A)x_{\lambda} = 0, \quad (h - \lambda(h))x_{\lambda} = 0, \quad (x^{-}_\alpha)^{(s)}x_{\lambda} = 0, \]

for all \( h \in U_F(h), \ alpha \in \mathbb{R}^+, x \in U_F(h_A), \) and \( s, k \in \mathbb{Z}_+, s > \lambda(h_i). \) However, the validity of the first two equalities are obvious and the third one follows from \((x^+_\alpha)^{(s)}(x_{\lambda}) = 0\) for all \( s > 0 \) and \( W_F(\lambda) \in \mathcal{I}_F. \)

Moreover, for all \( \mu \in P, \) the subspaces \( W_F(\lambda)_\mu \) are \( U_F(h_A)-\)modules for both the left and right actions and

\[ \text{Ann}_{U_F(h_A)}(w_\lambda) = \{ u \in U_F(h_A) : w_\lambda u = 0 = uw_\lambda \} = \{ u \in U_F(h_A) : W_F(\lambda)u = 0 \}. \]

Then \( W_F(\lambda) \) is a \( (U_F(\mathfrak{g}_A), \mathcal{A}_F^\lambda)-\)bimodule and each subspace \( W_F(\lambda)_\mu \) is a right \( \mathcal{A}_F^\lambda-\)module. In particular, \( W_F(\lambda)_\lambda \) is a \( \mathcal{A}_F^\lambda-\)bimodule and we have an isomorphism of bimodules

\[ W_F(\lambda)_\lambda \cong_{\mathcal{A}_F^\lambda} \mathcal{A}_F^\lambda. \]

**Definition 2.4.1.** Let \( \text{mod} \mathcal{A}_F^\lambda \) be the category of left \( \mathcal{A}_F^\lambda-\)modules and let \( W_\lambda^\lambda : \text{mod} \mathcal{A}_F^\lambda \rightarrow \mathcal{I}_F^\lambda \) be the right exact functor given by

\[ W_\lambda^\lambda(M) = W_F(\lambda) \otimes \mathcal{A}_F^\lambda M \quad \text{and} \quad W_\lambda^\lambda f = 1 \otimes f, \]

where \( M \in \text{mod} \mathcal{A}_F^\lambda \) and \( f \in \text{Hom}_{\mathcal{A}_F^\lambda}(M, M') \) for some \( M' \in \text{mod} \mathcal{A}_F^\lambda. \) We call this functor the Weyl functor.

Once \( W_F(\lambda) \in \mathcal{I}_F, \) the \( U_F(\mathfrak{g})\)-action on \( W_\lambda^\lambda(M) \) is also locally finite and so \( W_\lambda^\lambda(M) \in \mathcal{I}_F^\lambda. \)

The preceding discussion also shows that

\[ W_\lambda^\lambda(\mathcal{A}_F^\lambda) \cong_{U_F(h_A)} W_F(\lambda) \quad \text{and} \quad W_\lambda^\lambda(M)_\mu \cong_{\mathcal{A}_F^\lambda} W_F(\lambda)_\mu \otimes \mathcal{A}_F^\lambda M, \]

for all \( \mu \in P, \) \( M \in \text{mod} \mathcal{A}_F^\lambda. \)

**Lemma 2.4.2.** For all \( \lambda \in P^+ \) and \( V \in \mathcal{I}_F^\lambda \) we have \( \text{Ann}_{U_F(h_A)}(w_\lambda)V_\lambda = 0. \)

**Proof.** This is immediate from Lemma 2.2.1 and Proposition 2.3.3. \( \square \)

From this lemma, the left action of \( U_F(h_A) \) on \( V_\lambda \) induces a left action of \( \mathcal{A}_F^\lambda \) on \( V_\lambda. \) We denote the resulting \( \mathcal{A}_F^\lambda-\)module by \( R_\lambda^\lambda(V). \) Further, given \( f \in \text{Hom}_{\mathcal{I}_F^\lambda}(V, V') \) the restriction \( f_\lambda : V_\lambda \rightarrow V'_\lambda \) is a morphism of \( \mathcal{A}_F^\lambda-\)modules and the rules

\[ V \rightarrow R_\lambda^\lambda(V) \quad \text{and} \quad f \rightarrow R_\lambda^\lambda(f) = f_\lambda \]

define the functor

\[ R_\lambda^\lambda : \mathcal{I}_F^\lambda \rightarrow \text{mod} \mathcal{A}_F^\lambda \]

that is exact, since restricting \( f \) to a weight space is exact.

The next theorem establishes that the Weyl functors satisfy properties similar to the ones satisfied by the Weyl functors defined in the non-hyper setting. Particularly, item (4) gives a categorical definition of \( W_\lambda^\lambda(M). \) Its proof is identical to the proof in the characteristic zero setting, cf. [12 §3.7] and [24 §4].

**Theorem 2.4.3.** Let \( \lambda \in P^+. \)

1. Given \( M \in \text{mod} \mathcal{A}_F^\lambda, \) as left \( \mathcal{A}_F^\lambda-\)modules, we have \( R_\lambda^\lambda W_\lambda^\lambda(M) \cong M. \) In particular, we have an isomorphism of functors \( \text{id}_{\mathcal{A}_F^\lambda} \cong R_\lambda^\lambda W_\lambda^\lambda. \)
(2) Let \( V \in \mathcal{I}_\lambda^\Phi \). There exists a canonical map of \( U_\Phi(\mathfrak{g}, A) \)-modules \( \eta_V : \mathcal{W}_\Phi^\lambda R_\Phi^\lambda(V) \to V \) such that \( \eta : \mathcal{W}_\Phi^\lambda R_\Phi^\lambda \to \text{id}_{\mathcal{I}_\lambda^\Phi} \) is a natural transformation of functors and \( R_\Phi^\lambda \) is a right adjoint to \( \mathcal{W}_\Phi^\lambda \).

(3) The functor \( \mathcal{W}_\Phi^\lambda \) maps projective objects to projective objects.

(4) Let \( V \in \mathcal{I}_\lambda^\Phi \). \( V \cong \mathcal{W}_\Phi^\lambda R_\Phi^\lambda(V) \) if, and only if, for all \( U \in \mathcal{I}_\lambda^\Phi \) with \( U_\lambda = 0 \), we have
\[
\text{Hom}_{\mathcal{I}_\lambda^\Phi}(V, U) = \text{Ext}^1_{\mathcal{I}_\lambda^\Phi}(V, U) = 0.
\]

(5) The functor \( \mathcal{W}_\Phi^\lambda \) is exact if, and only if, for all \( U \in \mathcal{I}_\lambda^\Phi \) with \( U_\lambda = 0 \) we have
\[
\text{Ext}^2_{\mathcal{I}_\lambda^\Phi}(\mathcal{W}_\Phi^\lambda(M), U) = 0 \quad \forall \ M \in \text{mod} \mathcal{A}_\lambda^\Phi.
\]

2.5. Finite-dimensional and finitely generated modules. The first part of next theorem was proved in [15] in the characteristic zero setting. The present context uses a similar approach to that in [6, 8, 28].

**Theorem 2.5.1.** Let \( \lambda \in P^+ \) and assume \( A \) is finitely generated.

1. \( \mathcal{A}_\Phi^\lambda \) is a finitely generated \( \mathcal{F} \)-algebra.
2. \( \mathcal{W}_\Phi(\lambda) \) is a finitely generated right \( \mathcal{A}_\Phi^\lambda \)-module.
3. If \( M \in \text{mod} \mathcal{A}_\Phi^\lambda \) is a finitely generated (resp. finite-dimensional) then \( \mathcal{W}_\Phi^\lambda(M) \) is a finitely generated (resp. finite-dimensional) \( U_\Phi(\mathfrak{g}, A) \)-module.

**Proof.** For part (1), let \( w_\lambda \) a generator for \( \mathcal{W}_\Phi(\lambda) \). Recall that \( U_\Phi(\mathfrak{h}, A) \) is commutative and its elements are polynomials in \( \Lambda_{i,a,r} \) and \( (h_i^a) \) where \( i \in I, \ a \in A, \) and \( k \in \mathbb{Z}_+ \). Now, since \( (h_i^a)w_\lambda = (\lambda(h_i))w_\lambda \) for all \( i \in I \) and \( k \in \mathbb{Z}_+ \), we see that \( (h_i^a)w_\lambda \) is zero when \( k > \lambda(h_i) \).

Thus, to prove that \( \mathcal{A}_\Phi^\lambda \) is a finitely generated \( \mathcal{F} \)-algebra, it suffices to prove that \( \Lambda_{i,a,r}w_\lambda = 0 \) for all but finitely many \( r \in \mathbb{Z}_+ \). This is immediate, since we have \( \Lambda_{i,a,r}w_\lambda = 0 \) for \( r > \lambda(h_i) \) by Proposition 2.3.3 and Lemma 1.5.2.

For part (2), let \( \{a_1, \ldots, a_m\} \) be a generating set for \( A \). Given \( s = (s_1, \ldots, s_m) \in \mathbb{Z}_+^m \), define \( a^s := a_1^{s_1} \cdots a_m^{s_m} \) and \( x^s := x^{a^s} \).

Using the decomposition (1.4.2), we conclude that the elements of the set
\[
S = \left\{ \left( x_{\beta_1, b_1} \right)^{(n_1)} \cdots \left( x_{\beta_\ell, b_\ell} \right)^{(n_\ell)} w_\lambda \mid \beta_i \in \mathbb{R}^+, \ b_i \in \mathbb{Z}_+^m, \ \ell, n_i \in \mathbb{N}, \ i \in \{1, \ldots, \ell\} \right\}
\]
generate \( \mathcal{W}_\Phi(\lambda) \) as a right \( \mathcal{A}_\Phi^\lambda \)-module.

In order to prove the statement, it suffices to prove that \( \mathcal{W}_\Phi(\lambda) \) is spanned by the elements
\[
\left( x_{\beta_1, s_1} \right)^{(k_1)} \cdots \left( x_{\beta_\ell, s_\ell} \right)^{(k_\ell)} w_\lambda,
\]
with \( s_1, \ldots, s_\ell \in \mathbb{Z}_+^m \) such that \( \max(s_j) < \lambda(h_{\beta_j}) \) for all \( j \in \{1, \ldots, r\} \), \( \beta_1, \ldots, \beta_\ell \in \mathbb{R}^+ \) and \( \sum_j j_k \beta_j \leq \lambda - \mu_0 A \). This last condition comes from the definition of the module \( \mathcal{W}_\Phi(\lambda) \), (cf. Sections 3.0 - 3.2).

Let \( \mathcal{R}_\lambda = \mathbb{R}^+ \times \mathbb{Z}_+^m \times \mathbb{Z}_+ \) and consider \( \Xi \) as the set of functions \( \xi : \mathbb{N} \to \mathcal{R}_\lambda \) such that \( j \mapsto \xi_j = (\beta_j, s_j, k_j) \) such that \( k_j = 0 \) for all \( j \) sufficiently large. Let \( \Xi' \) be the subset of \( \Xi \) consisting of the elements \( \xi \) such that \( \max(s_j) < \lambda(h_{\beta_j}) \) for all \( j \).

Given \( \xi \in \Xi \) we associate an element \( v_\xi \in \mathcal{W}_\Phi(\lambda) \) as follows
\[
v_\xi := \left( x_{\beta_1, s_1} \right)^{(k_1)} \cdots \left( x_{\beta_\ell, s_\ell} \right)^{(k_\ell)} w_\lambda.
\]
Let $\mathcal{S}$ be the $\mathbb{Z}$-span of all vectors associated to elements belonging to $\mathcal{F}$ and by $\mathcal{S} \cdot A^\Lambda_F$ the set obtained by the right action of $A^\Lambda_F$ on the elements of $\mathcal{S}$. Define the degree of $\xi$ to be $d(\xi) := \sum_j k_j$ and the maximal exponent of $\xi$ to be $e(\xi) := \max\{k_j\}$. Notice that $e(\xi) \leq d(\xi)$ and $d(\xi) \neq 0$ implies $e(\xi) \neq 0$. Since there is nothing to be proved when $d(\xi) = 0$, we assume from now on that $d(\xi) > 0$.

Let $\Xi_{d,e}$ be the subset of $\Xi$ consisting of those $\xi$ satisfying $d(\xi) = d$ and $e(\xi) = e$, and set $\Xi_d = \bigcup_{1 \leq e \leq d} \Xi_{d,e}$.

We prove by induction on $d$ and sub-induction on $e$ that if $\xi \in \Xi_{d,e}$ is such that there exists $j \in \mathbb{N}$ with $\max(s_j) \geq \lambda(h_{\beta_j})$, then $v_\xi \in \mathcal{S} \cdot A^\Lambda_F$.

More precisely, given $0 < e < d \in \mathbb{N}$, we assume, by the induction hypothesis, that this statement is true for every $\xi$ which belongs either to $\Xi_{d,e'}$ with $e' < e$ or to $\Xi_d$ with $d' < d$.

We split the proof according to two cases: $e = d$ and $e < d$.

**Step 1.** When $e = d$, it follows that $v_\xi = (x^-_\alpha \otimes a) w_\lambda$ for some $\alpha \in R^+$ and $a \in A$.

**Step 1.1.** If $e = 1$, then $v_\xi = (x^-_\alpha \otimes a) w_\lambda$ for some $a \in A$ and $\alpha \in R^+$. In order to prove this case, we first deal with the generators of $A$ and then we deal with an arbitrary element in $A$.

By Lemma 2.5.2 for each $a_i \in \{a_1, \ldots, a_m\}$ and $s \in \mathbb{N}$, with $s \geq \lambda(h_\alpha)$, we obtain:

$$
0 = (-1)^s (x^+_\alpha \otimes a_i)^{(s)} (x^-_\alpha \otimes 1)^{(s+1)} w_\lambda \\
= \sum_{j=0}^s (x^-_\alpha \otimes a^j_i) \Lambda_{\alpha,a_i,s-j} w_\lambda \\
= \sum_{j=0}^s \left( (x^-_\alpha \otimes a^j_i) w_\lambda \right) \Lambda_{\alpha,a_i,s-j}.
$$

(2.5.1)

So

$$
(x^-_\alpha \otimes a^1_i) w_\lambda = \sum_{j=0}^{s-1} \left( (x^-_\alpha \otimes a^j_i) w_\lambda \right) (-\Lambda_{\alpha,a_i,s-j}).
$$

(2.5.2)

Thus, in particular,

$$
(x^-_\alpha \otimes a^\lambda_{h_\alpha}) w_\lambda \in \text{span}\left\{ (x^-_\alpha \otimes a^j_i) w_\lambda A^\Lambda_F \mid 0 \leq j < \lambda(h_\alpha) \right\}
$$

and an induction on $s$ gives

$$
(x^-_\alpha \otimes a^s_i) w_\lambda \in \text{span}\left\{ (x^-_\alpha \otimes a^j_i) w_\lambda A^\Lambda_F \mid 0 \leq j < \lambda(h_\alpha) \right\},
$$

(2.5.3)

for each $i \in \{1, \ldots, m\}$.

Now, let $a_k \in \{a_1, \ldots, a_m\}$ such that $a_k \neq a_i$.

By applying $(h_\alpha \otimes a^\lambda_k)$ to equation (2.5.1) we also see that

$$
(x^-_\alpha \otimes a^\lambda_k a^s_i) w_\lambda \in \text{span}\left\{ (x^-_\alpha \otimes a^j_i) w_\lambda A^\Lambda_F, (x^-_\alpha \otimes a^\lambda_k a^j_i) w_\lambda A^\Lambda_F \mid 0 \leq j < \lambda(h_\alpha) \right\}.
$$

If $r, s < \lambda(h_\alpha)$, then $(x^-_\alpha \otimes a^r_k a^s_i) w_\lambda \in \mathcal{S} \cdot A^\Lambda_F$. Otherwise, if $s < \lambda(h_\alpha)$ and $r \geq \lambda(h_\alpha)$, by applying $h_\alpha \otimes a^r_k$ to equation (2.5.1) (with $i = k$) we obtain:

$$
0 = (h \otimes a^r_k) \sum_{j=0}^r (x^-_\alpha \otimes a^j_k) w_\lambda \Lambda_{\alpha,a_k,r-j}
$$

(2.5.1)
\[ \sum_{j=0}^{r} (x_\alpha^{-} \otimes a_i^j_k) w_\lambda(h \otimes a_i^j_k) \Lambda_{\alpha, a_i, r-j} - 2 \sum_{j=0}^{r} (x_\alpha^{-} \otimes a_i^j_k) w_\lambda \Lambda_{\alpha, a_i, r-j}. \]

So,

\[ (x_\alpha^{-} \otimes a_i^j_k) w_\lambda = \frac{1}{2} \sum_{j=0}^{r} (x_\alpha^{-} \otimes a_i^j_k) w_\lambda(h \otimes a_i^j_k) \Lambda_{\alpha, a_i, r-j} - \sum_{j=0}^{r-1} (x_\alpha^{-} \otimes a_i^j_k) w_\lambda \Lambda_{\alpha, a_i, r-j}. \]

The first sum is in \( S \cdot A^2_r \) by (2.5.3) and the second is in \( S \cdot A^2_r \) by a further induction on \( r \). Thus, \( (x_\alpha^{-} \otimes a^r_i) w_\lambda \in S \cdot A^2_r \) for all \( r, s \in \mathbb{N} \) and \( i, k \in \{1, \ldots, m\} \). Hence, by repeating this argument for all generators of \( A \), we conclude that \( (x_\alpha^{-} \otimes a) w_\lambda \in S \cdot A^2_r \), for all \( a \in A \) and \( \alpha \in R^+ \).

**Step 1.2** If \( e > 1 \), then \( v_x = (x_\alpha^{-} \otimes a^r_i) w_\lambda \) for some \( \alpha \in R^+ \) and \( r = (r_1, \ldots, r_m) \in \mathbb{Z}_+^m \). In this case, we first deal with the case \( r \) is a multiple of a canonical element \( e_i \in \mathbb{Z}_+^m \). So, suppose \( r = r_i e_i \) for some \( i \in \{1, \ldots, m\} \), that means

\[ (x_\alpha^{-} \otimes a^r_i) w_\lambda = (x_\alpha^{-} \otimes a^{r_i}_i) w_\lambda. \]

If \( r_i < \lambda(h_\alpha) \), there is nothing to prove. Otherwise, we first note that the defining relations of \( \mathbf{W}_F(\lambda) \) and Lemma 1.5.2 imply that

\[ 0 = (-1)^{e r_i} (x_\alpha^{-} \otimes a_i^{-})^{(e r_i)} (x_\alpha^{-} \otimes 1)^{(e r_i + e)} w_\lambda \]
\[ = \sum_{j=0}^{e r_i} \left( \left( x_\alpha^{-} \otimes a_i^{-} \right) (u)^{(e r_i)} \right) \Lambda_{\alpha, a_i, e r_i - j} w_\lambda \]
\[ = \sum_{j=0}^{e r_i} \left( \left( \sum_{z=0}^{\infty} (x_\alpha^{-} \otimes a_i^{-}) u^{z+1} \right)^{(e r_i)} \right) \Lambda_{\alpha, a_i, e r_i - j} w_\lambda \]
\[ = \sum_{j=0}^{e r_i} \left( \left( \sum_{z=0}^{\infty} (x_\alpha^{-} \otimes a_i^{-}) u^{z+1} \right)^{(e)} \right) \Lambda_{\alpha, a_i, e r_i - j} w_\lambda \]
\[ = \sum_{j=0}^{e r_i} \left( \sum_{t \in \mathbb{Z}_+ \atop 0 \leq t \leq r_i} (x_\alpha^{-} \otimes a_i^{-})^{(e)} w_\lambda \right) \Lambda_{\alpha, a_i, e r_i - j} + \lambda w_\lambda \]
\[ = (x_\alpha^{-} \otimes a_i^{-})^{(e)} w_\lambda + \sum_{j=0}^{e r_i - 1} \left( \sum_{t \in \mathbb{Z}_+ \atop 0 \leq t < r_i} (x_\alpha^{-} \otimes a_i^{-})^{(e)} w_\lambda \right) \Lambda_{\alpha, a_i, e r_i - j} + \lambda w_\lambda \]
where $Xw_\lambda$ belongs to the $A^2_\mathbb{F}$-span of vectors $v_\phi$ with $\phi \in \Xi_{e,e'}$, for $e' < e$. Thus,

$$(x_\alpha^\sim \otimes a_i^r_i)^{(e)}{w_\lambda} = - \sum_{j=0}^{\epsilon r_i - 1} \left( \sum_{t \in \mathbb{Z}_+ \atop 0 \leq t < r_i \atop t = j} (x_\alpha^\sim \otimes a_i^t_i)^{(e)}{w_\lambda} \right) \Lambda_{\alpha,a_i^r_i,e \epsilon r_i - j} + Xw_\lambda,$$

the term $Xw_\lambda$ fits the induction hypothesis, and by a further induction on $r_i \geq \lambda(h_\alpha)$ we conclude that $(x_\alpha^\sim \otimes a_i^r_i)^{(e)}{w_\lambda} \in S \cdot A^2_\mathbb{F}^\lambda$.

**Step 1.3** Still in the case $e > 1$, we know consider the arbitrary case of $v_\xi = (x_\alpha^\sim \otimes a^r_i)^{(e)}{w_\lambda}$ for some $\alpha \in R^+$ and $r = (r_1, \ldots, r_m) \in \mathbb{Z}_+^m$, with $r$ not a multiple of a canonical element in $\mathbb{Z}_+^m$. In this case, if $r_i < \lambda(h_\alpha)$ for $i = 1, \ldots, m$, there is nothing to prove. Otherwise, we have $r_i \geq \lambda(h_\alpha)$ for some $i \in \{1, \ldots, m\}$. Set $r' = r - r_i e_i$.

From Step 1.2, we have

$$0 = (x_\alpha^\sim \otimes a_i^r_i)^{(e)}{w_\lambda} + \sum_{j=0}^{\epsilon r_i - 1} \left( \sum_{t \in \mathbb{Z}_+ \atop 0 \leq t < r_i \atop t = j} (x_\alpha^\sim \otimes a_i^t_i)^{(e)}{w_\lambda} \right) \Lambda_{\alpha,a_i^r_i,e \epsilon r_i - j} + Xw_\lambda$$

where $Xw_\lambda$ belongs to the $A^2_\mathbb{F}$-span of vectors $v_\phi$ with $\phi \in \Xi_{e,e'}$, for $e' < e$. Thus, by applying $\Lambda_{\alpha,a^r_i,e}$ to this equation, from Lemma [1.5.3][5], with $k = r = e, a = a^r_i$ and $b = a_i^r_i$, we get

$$0 = \Lambda_{\alpha,a^r_i,e}(x_\alpha^\sim \otimes a_i^r_i)^{(e)}{w_\lambda} + \sum_{j=0}^{\epsilon r_i - 1} \left( \sum_{t \in \mathbb{Z}_+ \atop 0 \leq t < r_i \atop t = j} (x_\alpha^\sim \otimes a_i^t_i)^{(e)}{w_\lambda} \right) \Lambda_{\alpha,a_i^r_i,e \epsilon r_i - j} + Xw_\lambda$$

$$= 2 (x_\alpha^\sim \otimes a^r_i)^{(e)}{w_\lambda} + \sum_{s=1}^{e-1} \left( \sum_{j=0}^{(s+1)} (x_\alpha^\sim \otimes a^{r_j} a_i^r_i)^{(e)} {w_\lambda} \Lambda_{\alpha,a^r_i,e} \right)_{e-s}$$

$$+ \sum_{j=0}^{\epsilon r_i - 1} \sum_{t \in \mathbb{Z}_+ \atop 0 \leq t < r_i \atop t = j} \left( \sum_{k \geq 0} (x_\alpha^\sim \otimes a^{r_k} a_i^r_i)^{(e)} {w_\lambda} \Lambda_{\alpha,a_i^r_i,e \epsilon r_i - j} \Lambda_{\alpha,a^r_i,e} \right)_{e-s}$$

$$+ \sum_{j=0}^{\epsilon r_i - 1} \sum_{t \in \mathbb{Z}_+ \atop 0 \leq t < r_i \atop t = j} \left( x_\alpha^\sim \otimes a_i^r_i \Lambda_{\alpha,a_i^r_i,e \epsilon r_i - j} \Lambda_{\alpha,a^r_i,e} \right)_{e-s}$$

$$+ \sum_{j=0}^{\epsilon r_i - 1} \sum_{t \in \mathbb{Z}_+ \atop 0 \leq t < r_i \atop t = j} \left( x_\alpha^\sim \otimes a_i^r_i \Lambda_{\alpha,a_i^r_i,e \epsilon r_i - j} \Lambda_{\alpha,a^r_i,e} \right)_{e-s}$$

$$+ \sum_{j=0}^{\epsilon r_i - 1} \sum_{t \in \mathbb{Z}_+ \atop 0 \leq t < r_i \atop t = j} \left( x_\alpha^\sim \otimes a_i^r_i \Lambda_{\alpha,a_i^r_i,e \epsilon r_i - j} \Lambda_{\alpha,a^r_i,e} \right)_{e-s}$$
Simple modules in \( A \)

Since it is immediate from Theorem 2.4.3 that \( V \) is a quotient of \( W \) it follows that if \( U \) is a proper submodule of \( V \), then \( \dim V < \dim W \). The remaining case \( e = e(\xi) < d(\xi) = d \) is handled with a simple repeated application of Lemma 1.5.5 and the induction hypothesis completes the proof of part (2).

Finally, part (3) is immediate from part (2), Definition 2.4.1 and (2.2.3).

\[ \square \]

2.6. Simple modules in \( T_{\mathbb{F}}^\lambda \) and local Weyl modules. Let \( \text{irr}(\text{mod} \ A_{\mathbb{F}}^\lambda) \) be the set of irreducible representations of \( A_{\mathbb{F}}^\lambda \). Since \( A_{\mathbb{F}}^\lambda \) is a commutative finitely generated algebra it follows that if \( M \in \text{irr}(\text{mod} \ A_{\mathbb{F}}^\lambda) \) then \( \dim M = 1 \). By Theorem 2.5.1 we see that

\[ \dim W_{\mathbb{F}}(M) < \infty \text{ and } R_{\mathbb{F}}^\mu W_{\mathbb{F}}(M) = M, \text{ for all } M \in \text{irr}(\text{mod} \ A_{\mathbb{F}}^\lambda). \]

**Definition 2.6.1.** The \( U_{\mathbb{F}}(g_A) \)-module \( W_{\mathbb{F}}^\lambda(M) \), where \( M \) is an irreducible object of \( \text{mod} \ A_{\mathbb{F}}^\lambda \), is called local Weyl module.

Consider \( V_{\mathbb{F}}^\mu(M) \) as the unique irreducible quotient of \( W_{\mathbb{F}}^\mu(M) \) (see Lemma 2.2.1). The next result shows that any irreducible module in \( T_{\mathbb{F}}^\lambda \) is isomorphic to \( V_{\mathbb{F}}^\mu(M) \) for some \( \mu \in P^+ \).

**Proposition 2.6.2.** Let \( \lambda \in P^+ \) and assume that \( V \in T_{\mathbb{F}}^\lambda \) is irreducible.

1. There exists \( \mu \in P^+ \cap (\lambda - Q^+) \) such that \( \text{wt} V \subset \mu - Q^+ \) and \( \dim V_\mu = 1 \). In particular, \( V \) is the unique irreducible quotient of \( W_{\mathbb{F}}^\mu R_{\mathbb{F}}^\mu(V) \) and \( \dim V < \infty \).

2. If \( V' \in T_{\mathbb{F}} \) we have \( V \cong V' \) as \( U_{\mathbb{F}}(g_A) \)-modules if, and only if, \( R_{\mathbb{F}}^\mu(V) \cong R_{\mathbb{F}}^\mu(V') \) as \( A_{\mathbb{F}}^\mu \)-modules.

**Proof.** Since \( V \in T_{\mathbb{F}}^\lambda \), it follows that there exists \( \mu \in \lambda - Q^+ \) with \( V_\mu \neq 0 \) and \( U_{\mathbb{F}}(n_A^+)V_\mu = 0 \). It is immediate from Theorem 2.4.3 that \( V \) is a quotient of \( W_{\mathbb{F}}^\mu R_{\mathbb{F}}^\mu(V) \). If \( S_\mu \) is a nonzero proper \( U_{\mathbb{F}}(h_A) \)-submodule of \( V_\mu \), then \( U_{\mathbb{F}}(g_A)S_\mu \) is a proper submodule of \( V \), a contradiction.
Hence, $R_\pi^\mu(V)$ is an irreducible $A_\pi^\mu$-module which implies that $\dim V_\mu = 1$. Theorem 2.5.1 now implies that $\dim W_\pi^\mu R_\pi^\mu(V) < \infty$ and $\dim V < \infty$. The proof that $V$ is the unique irreducible quotient of $W_\pi^\mu R_\pi^\mu(V)$ is standard once we have $R_\pi^\mu W_\pi^\mu R_\pi^\mu(V) \cong V_\mu$. The final statement of the lemma is now deduced from the first part. $\square$

2.7. On endomorphisms of Global Weyl modules. The global Weyl modules are the natural objects to play a role similar to that of the Verma modules in the study of the representations of $U_\pi(\mathfrak{g} \otimes A)$. One of the fundamental properties of Verma modules is that the space of morphisms between two Verma modules is either zero or one-dimensional. Searching for the analogue of this property for global Weyl modules, the situation is more complicated and partial results are obtained under certain restrictions on $\mathfrak{g}$, $\lambda$, and $A$, even in characteristic zero, cf. [5]. We have this first similar (small) advance, whose proof follows that of [5, Lemma 1.11].

Proposition 2.7.1. For $\lambda \in P^+$, $a \in A_\lambda^\mu$ the assignment $w_\lambda \mapsto w_\lambda a$ extends to a homomorphism $W_\pi(\lambda) \to W_\pi(\lambda)$ of $U_\pi(\mathfrak{g} \otimes A)$-modules and we have

$$\text{Hom}_{U_\pi(\mathfrak{g} \otimes A)}(W_\pi(\lambda), W_\pi(\lambda)) \cong A_\lambda^\mu.$$

Proof. From the $(U_\pi(\mathfrak{g} \otimes A), A_\lambda^\mu)$-bimodule structure of $W_\pi(\lambda)$, we can see that $w_\lambda a$ satisfies the defining relations of $W_\pi(\lambda)$ which yields the first statement of the proposition. For the second, let $\pi : W_\pi(\lambda) \to W_\pi(\lambda)$ be a nonzero $U_\pi(\mathfrak{g} \otimes A)$-module map. Since $W_\pi(\lambda)_\lambda = U(\tilde{\mathfrak{h}}_A) w_\lambda$, there exists $u_\pi \in U(\tilde{\mathfrak{h}}_A)$ such that $\pi(w_\lambda) = u_\pi w_\lambda$. Since $\pi$ is nonzero, the image $\tilde{u}_\pi$ of $u_\pi$ in $A_\lambda^\mu$ is nonzero. Thus, we obtain a well-defined map $\text{Hom}_{U_\pi(\mathfrak{g} \otimes A)}(W_\pi(\lambda), W_\pi(\lambda)) \to A_\lambda^\mu$ given by $\pi \mapsto \tilde{u}_\pi$, which is an isomorphism of right $A_\lambda^\mu$-modules. $\square$

Future projects. on this subject could include the decomposition of the global Weyl modules as $\mathfrak{g}$-modules, the freeness of global Weyl modules as $A_\pi^\mu$-modules, the structure of the Weyl modules for fundamental weights, the independence of the choice of the irreducible module for $A_\pi^\mu$ to construct the local Weyl modules, the structure of the algebras $A_\lambda^\mu$ and their irreducible representations, in a similar fashion to [12, Sections 6 and 7], which was inspired by [11, 32, 35].

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