Long time Evolution of Quantum Averages
Near Stationary Points

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We construct explicit expressions for quantum averages in coherent states for a Hamiltonian of degree 4 with a hyperbolic stagnation point. These expressions are valid for all times and “collapse” (i.e., become infinite) along a discrete sequence of times. We compute quantum corrections compared to classical expressions. These corrections become significant over a time period of order \( C \log \frac{1}{\hbar} \).

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The central result of this paper is the exactly solvable evolution of quantum averages in coherent state for a Hamiltonian of degree 4 containing a hyperbolic point. We tend to think about this Hamiltonian

\[ H(a^\dagger, a) = i\omega(a^\dagger a^2 - a a^\dagger) + \mu(a^\dagger a^2 - a a^\dagger)^2 \]

as a model one giving a useful insight into the global in time evolution of quantum averages for a more general Taylor expansion around a hyperbolic point, where the explicit expression is hard to obtain. As far as we know this is the first explicit computation in the presence of a hyperbolic point (apart from the quadratic case which is classical). General properties of spreading of a quantum wave packet were considered in [1]. In [2] the quantum energy levels were computed, e.g., near a nondegenerate local maximum of a double-well potential (this corresponds to a hyperbolic point of the Hamiltonian).

There are two qualitative conclusions from the solutions we obtained below. First, the quantum corrections near a hyperbolic point become of order 1 over a logarithmic time \( C \log \frac{1}{\hbar} \). This time has appeared in [3]. (See also [4], [5].) Second, it turns out, for such a Hamiltonian quantum averages in coherent states do become infinite along a certain discrete sequence of times. For example, given the observable \( \hat{x}^2 \), these singularities occur at \( t = \frac{\pi}{32\mu \hbar} + \ell \frac{1}{160\mu h}, \quad \ell = 0, \pm 1, \pm 2, \ldots \). It is natural to call this phenomenon the “collapse of quantum averages.”

The paper is organized as follows. We start with reviewing the general equation describing the evolution of quantum averages in Sections 1, 2. In Section 3 we discuss the case of an elliptic point, which has already appeared in the literature [6], [7]. In Section 4 we derive the explicit expressions for the evolution of quantum averages for the case of a hyperbolic point. We discuss the collapse phenomenon in Section 5 and quantum corrections in Section 6.
1.

We consider a time independent polynomial Hamiltonian

\[ H(a_1^\dagger, \ldots, a_N^\dagger, a_1, \ldots, a_N) = \sum_{\ell, s} H_{\ell s} a_1^{\ell_1} \cdots a_N^{\ell_N} a_1^{s_1} \cdots a_N^{s_N} \]

\[ \ell = (\ell_1, \ldots, \ell_N) \in \mathbb{Z}_+^N, \quad s = (s_1, \ldots, s_N) \in \mathbb{Z}_+^N. \]  

Here the creation and annihilation operators are defined as follows

\[ a_k^\dagger = \frac{1}{\sqrt{2}} \left( x_k - \hbar \frac{\partial}{\partial x_k} \right), \quad a_k = \frac{1}{\sqrt{2}} \left( x_k + \hbar \frac{\partial}{\partial x_k} \right), \quad k = 1, \ldots, N; \]

\[ [a_k^\dagger, a_\ell] = -\delta_{k\ell}\hbar \quad \text{for} \quad k, \ell = 1, \ldots, N. \]  

The condition

\[ H_{\ell s} = H_{st}^* \quad s, \ell \in \mathbb{Z}_+^N \]

ensures that the operator (1.1) is symmetric. Let

\[ H(\alpha_1^\ast, \ldots, \alpha_N^\ast, \alpha_1, \ldots, \alpha_N) = \sum_{\ell, s} H_{\ell s} \alpha_1^{\ell_1} \cdots \alpha_N^{\ell_N} \alpha_1^{s_1} \cdots \alpha_N^{s_N}, \quad \alpha \in \mathbb{C}^N; \]

then \( H \) is the Wick quantization of \( \mathcal{H} \). Let the total degree of \( \mathcal{H} \) be \( d \).

We remind the definition of the Poisson vectors and the set of coherent states [8] (see also [9]) that we need below.

Let for \( \alpha \in \mathbb{C}^N \) the Poisson vector \( \Phi_\alpha \) be defined as follows:

\[ \Phi_\alpha(x) = (\pi\hbar)^{-N/4} \exp \left\{ -\frac{1}{2\hbar}(x^2 - 2\sqrt{2}x \cdot \alpha + \alpha^2) \right\}, \quad x \in \mathbb{R}^n. \]

The coherent state is the normalized Poisson vector:

\[ |\alpha\rangle = \exp \left( -\frac{|\alpha|^2}{2\hbar} \right) |\Phi_\alpha\rangle. \]

For an operator-valued function \( F(t) \), the Heisenberg equation describes the evolution of an observable

\[ \dot{F} = \frac{i}{\hbar}[H, F]. \]

The corresponding averages are defined as follows:

\[ f(\alpha^\ast, \alpha, t) = \langle \alpha | F(t) | \alpha \rangle, \quad \alpha \in \mathbb{C}^N. \]

2.

In our recent paper [10] we derived a general equation for the averages, extending the earlier results [11], [7]. Here we recall the form of the equation for \( f(\alpha^\ast, \alpha, t) \) (see eqn. (1.10) in [10]):

\[ \frac{\partial}{\partial t} f(\alpha^\ast, \alpha, t) = \]

\[ = \frac{i}{\hbar} \sum_{\ell,s \in \mathbb{Z}_+^N} \frac{1}{\ell!} \left\{ \left( \frac{\partial}{\partial \alpha} \right)^\ell \mathcal{H}(\alpha^\ast, \alpha) \left( \hbar \frac{\partial}{\partial \alpha^\ast} \right)^\ell - \left( \frac{\partial}{\partial \alpha^\ast} \right)^\ell \mathcal{H}(\alpha^\ast, \alpha) \left( \hbar \frac{\partial}{\partial \alpha} \right)^\ell \right\} f(\alpha^\ast, \alpha, t). \]
Here \( r! = f_1! \ldots r_N! \); \((\frac{\partial}{\partial \alpha_j})^r = (\frac{\partial}{\partial \alpha_j})^{r_1} \cdots (\frac{\partial}{\partial \alpha_j})^{r_N} \); \((\frac{\partial}{\partial \alpha_j})^r = (\frac{\partial}{\partial \alpha_j})^{r_1} \cdots (\frac{\partial}{\partial \alpha_j})^{r_N} \).

Clearly, only the terms with \( 1 \leq r_1 + \cdots + r_N \leq d \) contribute to the sum in (2.1). We make one general remark about the initial value problem for (2.1) such that

\[ f(\alpha^*, \alpha, 0) = a^m \alpha^q ; \quad m, q \in \mathbb{Z}_+^N. \]  

(2.2)

It often happens that the coefficients in the Hamiltonian contain certain parameters of nonlinearity, for example,

\[ H_{\ell,s} = e^{(\ell+s)} |G_{\ell,s}| \]

(2.3)

where \( \mu_j \) are real, \( j = 2, \ldots, d \); \( |\ell + s| = \ell_1 + \cdots + \ell_N + s_1 + \cdots s_N \). It is of interest to ask how the solution \( f(\alpha^*, \alpha, t) \) depends on these parameters (and on \( h \)). From the general form of the equation (2.1) it is reasonable to expect that the solution to (2.1), (2.2) is of the form

\[ f(\alpha^*, \alpha, t) = \frac{m + s}{h^{1/4}} F \left( \frac{\alpha^*}{h^{1/4}}, \alpha, \mu_2, h^{1/2}, \mu_3, h^{1/4}, \ldots, h^{1/4} \mu_d, t \right). \]  

(2.4)

The explicit form of \( F \) for a general Hamiltonian is not easy to determine. In what follows we discuss two examples of nonlinear systems where such a computation can be made.

In the sequel we will compare solutions to (2.1)–(2.2) with the solution to Liouville’s’ equation of classical mechanics

\[ \frac{\partial}{\partial t} f_{cl}(\alpha^*, \alpha, t) = i \sum_{j=1}^{N} \left( \frac{\partial}{\partial \alpha_j} H(\alpha^*, \alpha) \frac{\partial}{\partial \alpha_j^*} - \frac{\partial}{\partial \alpha_j^*} H(\alpha^*, \alpha) \frac{\partial}{\partial \alpha_j} \right) f_{cl}(\alpha^*, \alpha, t), \]

(2.5)

\[ f_{cl}(\alpha^*, \alpha, 0) = a^m \alpha^q, \quad m, q \in \mathbb{Z}_+^N. \]  

(2.6)

Sometimes more general initial datum ought to be considered.

3. Elliptic point (See [7], [6].)

Since this case has been already discussed in literature, we can be brief. Let \( N = 1 \),

\[ H(a^t, a) = \omega |a|^2 + \mu |a|^4, \]

(3.1)
i.e., \( H(a^t, a) = \omega a^t a + \mu a^2 a^2 \). The equation (2.1) takes the form

\[ \frac{\partial}{\partial t} f(\alpha^*, \alpha, t) = i(\omega + 2\mu |\alpha|^2) \left( \frac{\alpha^*}{\partial \alpha^*} - \frac{\alpha}{\partial \alpha} \right) f(\alpha^*, \alpha, t) + i\mu \left( a^2 \left( \frac{\partial}{\partial \alpha^*} \right)^2 - \alpha^2 \left( \frac{\partial}{\partial \alpha} \right)^2 \right) f(\alpha^*, \alpha, t). \]  

(3.2)

The solution to the equation (3.2) with the initial condition

\[ f(\alpha^*, \alpha, 0) = a^m \alpha^q \]

(3.3)

has the following form

\[ f(\alpha^*, \alpha, t) = a^m \alpha^q e^{i(\omega(t - q(1 - q^2)) + \mu h t (m(1 - q(1 - q^2)) \frac{|\alpha|^2}{h^2} \]  

(3.4)

The solution to Liouville’s equation (2.5), (2.6) for the same Hamiltonian (3.1) is

\[ f_{cl}(\alpha^*, \alpha, t) = a^m \alpha^q e^{i(\omega + 2\mu |\alpha|^2)(m - q)t}. \]  

(3.5)
Assuming $|\mu t| \ll 1; m \neq q$ both of order 1, we get

$$f(\alpha^*, \alpha, t) = f_d(\alpha^*, \alpha, t)(1 + i\mu t(m - 1) - q(q - 1) + O(|\mu t|))$$

\[
\cdot \exp(-2\mu^2 t^2|\alpha|^2(m - q)^2 + O(\mu^3 t^3|\alpha|^2))
\]

In particular by the time $\frac{1}{|\mu||\alpha|\sqrt{\hbar}}$ quantum corrections are of the same order of magnitude as the classical solution.

### 4. Hyperbolic point

Let $N = 1$,

$$H(a^d, a) = i\omega(a^{+2} - a^2) + \mu(a^{+2} - a)^2. \quad (4.1)$$

In this section we give an explicit solution to (2.1). The operator (4.1) corresponds to the Wick Hamiltonian

$$\mathcal{H}(\alpha^*, \alpha) = i\omega(\alpha^{+2} - \alpha^2) + \mu(\alpha^{+2} - \alpha^2)^2 - 4\mu \alpha^* \alpha - 2\mu \hbar^2. \quad (4.2)$$

this Hamiltonian contains hyperbolic point at the origin with Lyapunov exponents

$$\lambda_{\pm} = \pm 2\sqrt{\omega^2 - \mu^2 \hbar^2}, \quad |\mu|\hbar < |\omega|. \quad (4.3)$$

For simplicity we consider the initial condition for (2.1) of the form

$$f(\alpha^*, \alpha, 0) = \langle \alpha|\hat{x}^n|\alpha \rangle, \quad n = 1, 2, 3, \ldots. \quad (4.4)$$

This corresponds to quantum averages in coherent states of the $n$-th power of the coordinate operator.

An arbitrary polynomial in $\alpha^*, \alpha$ can be handled by the same method, but the resulting expression is of more involved combinatorial structure.

It will be convenient to introduce the coordinate and the momentum operator

$$\hat{x} = \frac{1}{\sqrt{2}}(a^d + a), \quad \hat{p} = \frac{i}{\sqrt{2}}(a^d - a) \quad (4.5)$$

and their evolution according to Heisenberg’s equation (1.6)

$$X(t) = e^{\frac{i}{\hbar}Ht}\hat{x}e^{-\frac{i}{\hbar}Ht}, \quad (4.6)$$

$$P(t) = e^{\frac{i}{\hbar}Ht}\hat{p}e^{-\frac{i}{\hbar}Ht}. \quad (4.7)$$

Then $X(0) = \hat{x}$, $P(0) = \hat{p}$, $[X(t), P(t)] = i\hbar$. The operator $H$ can be expressed using (4.1), (4.5) as

$$H = 2\omega\hat{x}\hat{p} - i\omega \hbar + \mu(2i\hat{x}\hat{p} + \hbar)^2. \quad (4.8)$$

Since $[H, \hat{x}\hat{p}] = 0$, it follows that

$$X(t) \quad P(t) = \hat{x}\hat{p}, \quad t \in \mathbb{R}. \quad (4.9)$$

This obvious remark will be crucial for our computation. We have from (1.6), (4.8), (4.9)

$$\dot{X}(t) = (2\omega - 8\mu X(t)P(t))X(t) = (2\omega - 8\mu \hat{x}\hat{p})X(t) \quad (4.10)$$

$$\dot{P}(t) = (-2\omega - 8\mu \hbar + 8\mu X(t)P(t))P(t) = (-2\omega - 8\mu \hbar + 8\mu \hat{x}\hat{p})P(t). \quad (4.11)$$
Using (4.10), (4.11) and induction in \( n \), we get
\[
\frac{d}{dt} X^n(t) = (2n\omega - 4\mu \hbar n(n-1) - 8\mu \hbar \dot{x}) X^n(t) \quad (4.12)
\]
\[
\frac{d}{dt} P^n(t) = (-2n\omega - 4\mu \hbar n(n+1) + 8\mu \hbar \dot{x}) P^n(t) \quad (4.13)
\]

From (4.12), formally
\[
X^n(t) = e^{(2n\omega - 4\mu \hbar n(n-1) - 8\mu \hbar \dot{x})t} x^n
\]
\[
= e^{(2n\omega - 4\mu \hbar n(n-1))t} e^{8\mu \hbar t \dot{x}} x^n . \quad (4.14)
\]

For any entire function \( G(x) \), \( x \in \mathbb{C} \) we have
\[
\left( e^{ix \frac{d}{dt}} G \right) (x) = G(e^{it} x) . \quad (4.15)
\]

But the coherent state \( |\alpha \rangle \) is analytic in \( x \), therefore using (4.14), (4.15)
\[
f(\alpha^*, \alpha, t) = \langle \alpha | X^n(t) | \alpha \rangle
\]
\[
= e^{-\frac{|\alpha|^2}{2}} e^{2\omega nt + 4\mu \hbar n(n+1)t} \int \Phi^*_\alpha(x) x^n \Phi_\alpha(e^{8\mu \hbar t \dot{x}}) \, dx \quad (4.16)
\]
\[
= \frac{1}{(\pi \hbar)^{1/2}} e^{-\frac{|\alpha + \alpha^*|^2}{2\hbar}} e^{2\omega nt + 4\mu \hbar n(n+1)} \int_{-\infty}^{\infty} x^n e^{-\frac{1}{8\hbar}(1 + e^{16\mu \hbar nt}) x^2 - 2\sqrt{2}(\alpha^* + \alpha e^{8\mu \hbar nt}) x} \, dx . \quad (4.17)
\]

We compute the integral in the right side of (4.16) assuming
\[
t \neq \frac{\pi}{16\mu \hbar} + \ell \frac{\pi}{8\mu \hbar} , \quad \ell \in \mathbb{Z} .
\]

The condition (4.17) will be discussed in detail in the next section; here we just note that (4.17) implies \( \text{Re}(1 + e^{16\mu \hbar nt}) > 0 \) and the integral in (4.16) is absolutely convergent. Changing variables in (4.16) we arrive at the expression
\[
f(\alpha^*, \alpha, t) = \frac{(2\hbar)^{2+n}}{(\pi \hbar)^{1/2}} e^{-\frac{(\alpha + \alpha^*)^2}{2\hbar}} e^{2\omega nt} \left( \frac{e^{4\mu \hbar n t}}{(1 + e^{16\mu \hbar n t})^{1/2}} \right)^{n+1} \quad (4.18)
\]
\[
e^{-\frac{4\mu \hbar n t}{2\hbar \cos 8\mu \hbar n t}} x^n \left( x + \frac{\alpha^* + \alpha e^{8\mu \hbar n t}}{\sqrt{2}(1 + e^{16\mu \hbar n t})^{1/2}} \right)^n e^{-x^2} \, dx .
\]

In (4.18) we choose the branch of the square root so that \( \text{Re} \sqrt{1 + e^{16\mu \hbar n t}} > 0 \). The expression \( e^{4\mu \hbar n t} / \sqrt{1 + e^{16\mu \hbar n t}} \) because of this condition ought to be interpreted as follows:
\[
\frac{e^{4\mu \hbar n t}}{\sqrt{1 + e^{16\mu \hbar n t}}} = \frac{1}{\sqrt{2 \cos 8\mu \hbar n t}} \text{def} \frac{1}{\sqrt{2}} \left[ \frac{1 + e^{8\mu \hbar n t}}{\cos 8\mu \hbar n t} \right]^{1/2} . \quad (4.19)
\]

The square bracket in (4.19) stands for the entire part of a real number. In other words, passing through every point (4.17) from the left to the right contributes a factor of \( e^{\pi i/4} \) to this expression.
We discuss the limit of this expression as

\[ \lim_{n \to \infty} e^{2\omega nt} \frac{(\alpha^* e^{-4in\mu t} + \alpha e^{4in\mu t})^{n-2k}}{(2\cos 8n\mu t)^{\frac{n}{2} - k}} \int_{-\infty}^{\infty} x^{2k} e^{-x^2} dx \]

The identity (4.20) is the central result of this paper.

The explicit form of the equation (2.1) for the Hamiltonian (4.2) is as follows:

\[
\frac{\partial}{\partial t} f = i [-2i\omega \alpha - 4\mu (\alpha^2 - \alpha^2) \alpha - 4\mu \alpha^*] \frac{\partial}{\partial \alpha^*} f
- i [2i\omega \alpha^* + 4\mu (\alpha^2 - \alpha^2) \alpha^* - 4\mu \alpha] \frac{\partial}{\partial \alpha} f
+ ih [-i\omega - 2\mu (\alpha^2 - 3\alpha^2)] \left( \frac{\partial}{\partial \alpha^*} \right)^2 f
- ih [i\omega + 2\mu (3\alpha^2 - \alpha^2)] \left( \frac{\partial}{\partial \alpha} \right)^2 f
+ 4ih^2 \mu \alpha \left( \frac{\partial}{\partial \alpha^*} \right)^3 f - 4ih^2 \mu \alpha^* \left( \frac{\partial}{\partial \alpha} \right)^3 f
+ ih^3 \mu \left( \frac{\partial}{\partial \alpha^*} \right)^4 f - ih^3 \mu \left( \frac{\partial}{\partial \alpha} \right)^4 f
\]

(4.21)

Verifying that the expression (4.20) satisfies (4.21) directly, e.g., for \( n = 1 \) is a very long computation, but it does indeed.

5. Collapse of quantum averages

To simplify matters we take \( n = 2 \). We have using (4.20)

\[
f(\alpha^*, \alpha, t) = \frac{1}{2} e^{-\frac{(\alpha + \alpha^*)^2}{2k}} e^{2\omega t} \frac{(\alpha^* e^{-8i\mu t} + \alpha e^{8i\mu t})^{n-2k}}{(2\cos 8n\mu t)^{\frac{n}{2} - k}} \left[ \frac{(\alpha^* e^{-8i\mu t} + \alpha e^{8i\mu t})^2}{\cos 16\mu t} + \hbar \right].
\]

We discuss the limit of this expression as \( t \to \frac{\pi}{32\mu \hbar} \). For simplicity let \( \alpha = i \). Since

\[
(\alpha^* e^{-8i\mu t} + \alpha e^{8i\mu t})^2 = \left( -i \left( \frac{1 - i}{\sqrt{2}} \right) + i \left( \frac{1 + i}{\sqrt{2}} \right) \right)^2 = 2 \quad \text{at} \quad t = \frac{\pi}{32\mu \hbar},
\]

we have

\[
f(-i, i, t) \to \infty \quad \text{as} \quad t \to \frac{\pi}{32\mu \hbar} - 0.
\]
But
\[ f(\alpha^*, \alpha, t) = \|\hat{x}e^{-\hat{H}t}|\alpha\rangle\|^2_{L^2(\mathbb{R})}. \]

The phenomenon we observed is the quantum evolution of coherent states for certain \( \alpha \) in our explicitly integrable system may take them out of the domain of the self-adjoint unbounded operator \( \hat{x} \) in \( L^2(\mathbb{R}) \). This happens along a discrete sequence
\[ t_\ell = \frac{\pi}{32\mu \hbar} + \ell \cdot \frac{\pi}{16\mu \hbar}, \quad \ell = 0, \pm 1, \pm 2, \ldots. \]

It is natural to describe it as collapse of certain quantum averages that occurs at these times \( t_\ell \).

6. Quantum corrections in presence of a hyperbolic point

For simplicity we set \( n = 1 \) in (4.20), i.e.,
\[ f(\alpha^*, \alpha, t) = \langle \alpha | X(t) | \alpha \rangle = \frac{1}{\sqrt{2}} e^{-\frac{(\alpha^*+\alpha)^2}{2\hbar^2}} \frac{e^{2\omega t}}{(\cos 8\mu \hbar t)^{3/2}} \frac{1}{\sqrt{2}} (\alpha^* e^{-4i\mu \hbar t} + \alpha e^{4i\mu \hbar t}). \] (6.1)

We assume as in Section 3 that \( |\mu \hbar| \ll 1 \). A simple computation leads to the result
\[ f(\alpha^*, \alpha, t) = \frac{1}{\sqrt{2}} e^{2\omega t + 4i\mu \hbar (\alpha^2 - \alpha^*2)} (1 + \mathcal{O}(\mu^2 \hbar^2 t^2)) \]
\[ \times \left( (\alpha^* + \alpha) + 4i\mu \hbar (\alpha - \alpha^*) + \mathcal{O}(|\alpha|^2 \mu^2 \hbar^2 t^2) \right) \] \[ \times \exp \left( 16\mu^2 \hbar^2 |\alpha|^2 + \mathcal{O}(|\alpha|^2 |\mu|^3 \hbar^3 t^3) \right). \] (6.2)

It is of interest to note that
\[ f_{cl}(\alpha^*, \alpha, t) = \frac{1}{\sqrt{2}} e^{2\omega t + 4i\mu \hbar (\alpha^2 - \alpha^*2)} (\alpha^* + \alpha). \]

Over the time of order \( \frac{1}{|\mu| |\alpha| \sqrt{\hbar}} \) the quantum corrections are of the same order of magnitude as classical solutions. To measure the deviation of the quantum average from its classical value the most natural quantity is dispersion. We will give estimates for
\[ D(\alpha^*, \alpha, t) = \langle \alpha | X^2(t) | \alpha \rangle - \langle \alpha | X(t) | \alpha \rangle^2 \]
which is the dispersion of \( X(t) \) in coherent state. It is a purely quantum quantity; the corresponding classical quantity made out of solution to the Liouville’s equation of classical mechanics vanishes identically. To present the result in a simpler way we assume
\[ |\mu \hbar| \ll 1, \quad |\alpha|^2 \gg \hbar, \quad \mu^2 \hbar^2 |\alpha| \ll 1. \] (6.3)

The main term in the dispersion \( D(\alpha^*, \alpha, t) \) under these assumptions is as follows:
\[ D(\alpha^*, \alpha, t) \approx e^{4\omega t - 8i\mu \hbar (\alpha^2 - \alpha^*2)} \left( \frac{1}{2} \hbar + 4i\mu \hbar (\alpha^2 - \alpha^*2) + 16(\alpha^* + \alpha)^2 \mu^2 \hbar^2 |\alpha|^2 \right). \] (6.4)

If we replace (6.3) by
\[ |\mu \hbar| \ll 1, \quad |\alpha|^2 \gg \hbar, \quad \mu^2 \hbar^2 |\alpha|^2 \gg 1, \] (6.5)
we get the expression

\[ D(\alpha^*, \alpha, t) \approx \frac{1}{2} e^{4\omega t + 8i\mu(\alpha^2 - \alpha^*2)}(\alpha^* + \alpha)^2 \exp(64\mu^2\hbar^2|\alpha|^2) . \] (6.6)

Finally, if

\[ |\mu\hbar t| \ll 1 , \quad |\alpha|^2 \gg \hbar , \quad 64\mu^2\hbar^2|\alpha|^2 \approx 1 \] (6.7)

we get

\[ D(\alpha^*, \alpha, t) \approx \frac{1}{2} e^{4\omega t + 8i\mu(\alpha^2 - \alpha^*2)}(\alpha^* + \alpha)^2 \left[ \exp(64\mu^2\hbar^2|\alpha|^2) - \exp(32\mu^2\hbar^2|\alpha|^2) \right] . \] (6.8)

It is clear from these expressions that over a logarithmically small time \( C \log \frac{1}{\hbar} \) the quantum dispersion becomes at least of order 1.

**Conclusions**

For the polynomial Hamiltonian of degree 4 in \( a^\dagger, a \) with a hyperbolic point at the origin we here obtained the explicit formulae for quantum averages in coherent states valid for all times. Quantum corrections for averages become significant already on a logarithmically small time \((\hbar)\). The quantum averages for simple observables, such as \( \hat{x}^n \), blow up along a discrete sequence of times for this Hamiltonian.

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