Reexamination of optimal quantum state estimation of pure states

A. Hayashi, T. Hashimoto, and M. Horibe
Department of Applied Physics
Fukui University, Fukui 910-8507, Japan

A direct derivation is given for the optimal mean fidelity of quantum state estimation of a \(d\)-dimensional unknown pure state with its \(N\) copies given as input, which was first obtained by M. Hayashi in terms of an infinite set of covariant positive operator valued measures (POVM’s) and by Bruß and Macchiavello establishing a connection to optimal quantum cloning. An explicit condition for POVM measurement operators for optimal estimators is obtained, by which we construct optimal estimators with finite POVM using exact quadratures on a hypersphere. These finite optimal estimators are not generally universal, where universality means the fidelity is independent of input states. However, any optimal estimator with finite POVM for \(M(> N)\) copies is universal if it is used for \(N\) copies as input.

I. INTRODUCTION

One of the essential differences between quantum theory and classical theories, from the information theoretical point of view, is that an unknown quantum state cannot be copied exactly, which was formulated as the no-cloning theorem by Wootters and Zurek [1].

Suppose we are given \(N\) identically prepared copies of an unknown state \(\rho\) on a \(d\)-dimensional space \(\mathcal{H}_d\) and try to estimate the state \(\rho\) as precisely as possible by some measurement. Since we cannot increase the number of copies by cloning the given unknown state, our performance surely depends on \(N\), the number of copies given to us at the beginning. This is the problem of quantum state estimation, which has been studied since long ago [2, 3].

Two important points in formulating quantum state estimation are a priori distribution of the input states and a figure of merit to be optimized. In this paper we assume that the state \(\rho\) is pure and completely unknown in the sense that the state \(\rho\) is distributed over all pure states in a unitary invariant way. As a figure of merit we take the fidelity defined as \(\text{tr} \left[ \rho \rho' \right] = |\langle \phi | \phi' \rangle|^2\), where \(\rho = |\phi\rangle \langle \phi|\) is the given input pure state and \(\rho' = |\phi'\rangle \langle \phi'|\) is the output pure state as a guess for \(\rho\).

The optimal mean fidelity in the case of qubit \((d = 2)\) pure state estimation was found to be \((N + 1)/(N + 2)\) by Massar and Popescu [4]. They also pointed out that the optimal value of the mean fidelity is achieved by a joint measurement on the combined system of \(N\) copies but not realized by repeated separate measurements on each copy. Conceptually this unexpected result should be taken seriously since the input is a simple uncorrelated \(N\)-fold tensor product \(\rho^{\otimes N}\), though the improvement of joint measurement over separate measurement is relatively small (see also [5]). An algorithm for constructing an optimal and finite positive operator valued measure (POVM) has been given in [6]. Bagan et al. also discussed optimal and finite POVM’s in two-dimensional case with a different approach [7].

Using the framework of covariant measurements [8], Masahito Hayashi studied the estimation problem in more general settings: in general dimensions \(d\) and for a family of covariant error functions [9]. He showed that the error is minimized by the unique infinite covariant set of POVM’s in both Bayesian and minimax approaches, provided that the error function is a monotone increasing function of \(\text{tr} \left[ \rho \rho' \right]\). As for the mean fidelity he found the optimal value to be \((N + 1)/(N + d)\). Bruß and Macchiavello also obtained the optimal mean fidelity by establishing a connection between optimal state estimation and optimal quantum cloning [10], the latter of which is another problem directly related to the no-cloning theorem.

In optimal quantum cloning we are given \(N\) identically prepared copies of quantum state \(\rho\) on \(\mathcal{H}_d\) and try to produce a density matrix \(R_\rho\) on \(\mathcal{H}_d^{\otimes M}\) in an approximation of \(\rho^{\otimes M}\) as exactly as possible [10]. There are two kinds of figures of merit for approximate cloning. In the many-particle test the full fidelity \(\text{tr} \left[ \rho^{\otimes M} R_\rho \right]\) is used for a figure of merit, whereas the one-particle reduced fidelity \(\text{tr} \left[ \rho R_\rho \right]\) is employed in the single-particle test.

The general formula for the optimal many-particle fidelity as a function of \(d\), \(N\), and \(M\) in the case of pure states was derived by Werner [12]. It was also shown that the optimal fidelity is attained by the unique cloner. This unique optimal cloner was later shown to be also optimal with respect to the single-particle fidelity [13].

The connection established by Bruß and Macchiavello [10] is the following (see also [11]). For given \(N\) copies of a pure state, first employ the optimal cloner to produce infinite number of the best approximate copies, by which we can estimate the approximate copy as precisely as we want. On the other hand applying the optimal estimator to the input first, we obtain the best approximate estimation of the input by which we can produce infinitely many copies of the same quality. Thus they identified the optimal single particle fidelity in the large \(M\) limit with the optimal mean fidelity of quantum state estimation.

For experimental implementation of POVM measurement, it is desirable that the number of outcomes of POVM
measurement is finite. However, finite optimal POVM’s for state estimation have been constructed only in the two dimensional case (qubit). For general dimensions, the optimal fidelity was derived, but finite optimal POVM’s have not been discussed so far. In this paper we will show that one can construct finite optimal POVM’s in general dimensions. We also show that the finite optimal POVM may be chosen to be universal, where universality means the fidelity is independent of input states. These are the main results in this paper.

In Sec. II, we will first give a direct derivation of the optimal mean fidelity of quantum state estimation of a d-dimensional unknown pure state with its N copies given as input. Our main concern is whether the optimal fidelity can be achieved by a finite set of POVM’s. Therefore we do not assume the covariance of measurement, since the covariance implies an infinite set of POVM’s, when input states are specified by a set of continuous parameters as in the case considered in this paper. We also avoid employing the optimal single-particle fidelity of approximate cloning, which is not straightforward to obtain. In Sec. III, we will study an explicit condition for POVM operators for optimal estimators. Using exact quadratures on a hypersphere, we establish the existence of a finite set of POVM’s of optimal covariance implies an infinite set of POVM’s, when input states are specified by a set of continuous parameters as in

II. OPTIMAL MEAN FIDELITY

Suppose we are given N identically prepared copies of a randomly selected pure state $\rho = |\phi\rangle\langle\phi|$ on a d-dimensional complex Hilbert space $\mathcal{H}_d$ and try to estimate the state $\rho$ as precisely as possible by some POVM measurement $\{E_a\}_{a=1}^A$ on $\rho^\otimes N$. Since all inputs belong to the totally symmetric subspace of $\mathcal{H}_d^\otimes N$, the completeness relation of the POVM can be written as $\sum_{a=1}^A E_a = S_N$, where $S_N$ is the projector onto this totally symmetric subspace. With the outcome of the measurement labeled with "$a$", we infer that the state was a prespecified pure state $\rho_a = |\phi_a\rangle\langle\phi_a|$. Our task is to maximize the following mean fidelity:

$$F(N, d) = \sum_{a=1}^A \langle \text{tr} [E_a \rho^\otimes N] \rangle \text{tr} [\rho_a \rho] \right), (1)$$

with respect to our strategy, the set of $\{E_a, \rho_a\}_{a=1}^A$. In the above equation $\langle \cdots \rangle$ means an average over the input state $\rho$.

We assume the input state is distributed over all pure states on $\mathcal{H}_d$ in a unitary invariant way. First let us fix an orthonormal basis $\{|i\rangle, (i = 1, \cdots, d)\}$ in $\mathcal{H}_d$ and write a pure state $|\phi\rangle$ as $|\phi\rangle = \sum_{i=1}^d c_i |i\rangle$, where the coefficients $c_i$ satisfy the normalization condition $\sum_{i=1}^d c_i^* c_i = 1$. We assume that 2d-dimensional real vector $(\Re c_i, \Im c_i)_{i=1, \cdots, d}$ is uniformly distributed on $(2d-1)$-dimensional hypersphere. It is clear that the distribution defined above is independent of the reference basis. More precisely let $\{|\tilde{i}\rangle, (i = 1, \cdots, d)\}$ be another orthonormal basis. Then for any function $f$, the following can be easily shown:

$$\langle f \left( \sum_{i=1}^d c_i |i\rangle \right) \rangle = \langle f \left( \sum_{i=1}^d c_i |\tilde{i}\rangle \right) \rangle .$$

As shown in the appendix, the average of a product of the same number of $c$’s and $c^*$’s is given by

$$\langle c_{i_1} c_{j_1}^* c_{j_2} c_{j_2}^* \cdots c_{j_l} c_{j_l}^* \rangle = \frac{(d-1)!}{(d+l-1)!} \text{ (sum of all contractions between $i$’s and $j$’s)} . (3)$$

Using this formula Eq. (3) and writing a density operator for a pure state as $\rho = |\phi\rangle\langle\phi| = \sum_{i=1}^d c_i c_i^* |i\rangle\langle j|$, we obtain the following useful relation for the average of an N-fold tensor product of identical pure density matrices:

$$\langle \rho^\otimes N \rangle = \frac{S_N}{d_N} , (4)$$

where the sum of all permutation operators divided by $N!$ is identified with $S_N$ and $d_N$ is the dimension of the totally symmetric subspace, which is given by $d_N = \text{tr} [S_N] = N + d - 1 C_{d-1}$.

It should be noted that the relation Eq. (4) is a consequence of the unitary invariance of the distribution of $\rho$, which can be seen in the following way. For any unitary $U$ on $\mathcal{H}_d$ we have $\langle U^\otimes N \rho^\otimes N U^+ \otimes N \rangle = \langle \rho^\otimes N \rangle$, implying that the operator $\langle \rho^\otimes N \rangle$ on the totally symmetric subspace of $\mathcal{H}_d^\otimes N$ commutes with $U^\otimes N$ for any $U$. Shur’s lemma
then requires that $⟨\rho^\otimes N⟩$ be proportional to $S_N$, since $U^\otimes N$ acts on the totally symmetric space irreducibly. The proportional coefficient turns out to be $1/d_N$ by a trace argument. Thus we obtain the formula of Eq.(4).

Going back to the mean fidelity Eq.(1), we first rewrite it as

$$F(N, d) = \sum_{a=1}^A \langle \text{tr} \left[ E_a \rho^\otimes (N+1) \rho_a(N+1) \right] \rangle,$$

where the trace is taken over a total of $N + 1$ subsystems and the operator $\rho_a(N + 1)$ should be understood to act on the $(N + 1)$th subsystem only; namely, for a single-particle operator $\Omega$ we use the following notation: $\Omega(n) \equiv 1^\otimes (n-1) \otimes \Omega \otimes 1^\otimes (N+1-n)$. Using the formula Eq.(4), we perform the integration over $\rho$ to obtain

$$F(N, d) = \frac{1}{d_{N+1}} \sum_{a=1}^A \text{tr} [E_a S_{N+1} \rho_a(N + 1)].$$

By tracing out the $(N + 1)$th subsystem in the above equation, we finally obtain

$$F(N, d) = \frac{1}{(N + 1)d_{N+1}} \sum_{a=1}^A \text{tr} \left[ E_a \left( 1 + \sum_{n=1}^N \rho_a(n) \right) \right],$$

where we used the following relation which holds for any single-particle operator $\Omega$:

$$\text{tr}_{N+1} [S_{N+1} \Omega(n + 1)] = \frac{1}{N + 1} S_N \left( 1 + \sum_{n=1}^N \Omega(n) \right).$$

Now it is easy to obtain an upper bound for the mean fidelity. Since $\rho_a(n) \leq 1$, we have $\text{tr} [E_a \rho_a(n)] \leq \text{tr} [E_a]$. Applying this inequality to Eq.(7) and using the completeness of the POVM, we immediately find

$$F(N, d) \leq \frac{d_N}{d_{N+1}} = \frac{N + 1}{N + d}.$$

Equality in Eq.(9) holds if and only if $\text{tr} [E_a \rho_a(n)] = \text{tr} [E_a]$ for $n = 1, \cdots, N$, implying that $E_a$ is supported by the intersection of supports of $\rho_a(n)$, namely, $E_a$ is proportional to $\rho_a^\otimes N$.

Let us write $E_a$ as

$$E_a = d_N w_a \rho_a^\otimes N,$$

where $w_a$ is a positive coefficient and the common factor $d_N$ is introduced for later convenience. The completeness of the POVM implies $d_N \sum_{a=1}^A w_a \rho_a^\otimes N = S_N$. Recalling the formula Eq.(4), we conclude that the necessary and sufficient condition for the POVM that achieves the upper bound Eq.(9) is given by

$$\sum_{a=1}^A w_a \rho_a^\otimes N = \langle \rho^\otimes N \rangle.$$

The right-hand side in this equation is the average of $\rho^\otimes N$ defined as a continuous integration over a hypersphere, whereas the left-hand side is the sum of a finite number of sample density operators $\rho_a^\otimes N$ with positive weights; namely, a continuous integration is replaced by a finite sum in Eq.(11), which is a standard technique of numerical integrations (quadrature). Though a quadrature is in general an approximation, it may be exact for a certain class of functions. For example, the quadrature with a trapezoidal rule is exact for any linear functions. In this sense the condition (11) means a quadrature on the hyper-sphere that is exact for $\rho^\otimes N$. Since quadratures with those properties exist as explicitly shown in the next section, we conclude that the optimal value of the mean fidelity is given by

$$F_{\text{optimal}}(N, d) = \frac{N + 1}{N + d}.$$
III. FINITE SET OF POVM AND UNIVERSALITY

In this section we first show that we can construct a finite set of POVM’s that achieves the upper bound of the mean fidelity Eq. 11 or equivalently there exists a finite set \{w_a, \rho_a\}_{a=1}^A which satisfies Eq. 11. We write |\phi_a\rangle = \sum_{i=1}^d c_i^a |i\rangle \) for \rho_a = |\phi_a\rangle \langle \phi_a |. In terms of the expansion coefficient \(c_i^a\), Eq. 11 is equivalent to

\[ \sum_{a=1}^A w_a c_i^a c_j^a * c_i^a c_j^a * \cdots c_i^a c_j^N = (c_i c_j^* c_i^* c_j^* \cdots c_i^* c_j^* N) . \] (13)

Equation 11 imposes no condition on products of different numbers of \(c\)'s and \(c^*\)'s. To make the subsequent argument simpler, however, we assume that they are zero like their exact average value. Then it suffices to show that there exists a quadrature with positive weights on the hypersphere \(S^{2d-1}\) that is exact for any polynomial of degree \(2N\).

For a point \(\chi\) on \(S^{m-1}, m = 2d\), we write its polar coordinate parametrization as

\[
\begin{align*}
\chi_1 &= \cos \theta_1, \\
\chi_2 &= \sin \theta_1 \cos \theta_2, \\
\chi_3 &= \sin \theta_1 \sin \theta_2 \cos \theta_3, \\
&\vdots \\
\chi_{m-2} &= \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{m-2}, \\
\chi_{m-1} &= \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{m-2} \cos \phi, \\
\chi_m &= \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{m-2} \sin \phi,
\end{align*}
\] (14)

with the range of angle variables \(0 \leq \theta_1 \leq \pi\) and \(0 \leq \phi \leq 2\pi\). The integration measure is given by the standard form:

\[
\int_0^\pi d\theta_1 \sin^{m-2} \theta_1 \int_0^\pi d\theta_2 \sin^{m-3} \theta_2 \cdots \int_0^\pi d\theta_{m-2} \sin \theta_{m-2} \int_0^{2\pi} d\phi. \] (15)

Let us consider the integral of \(\chi_1^\nu_1 \chi_2^\nu_2 \cdots \chi_m^\nu_m\) with non-negative integers \(\nu\) which add up to \(2N\). For each single integration we construct an \(n\)-point quadrature of the type \(\int dx f(x) = \sum_{\kappa=1}^n \omega_\kappa f(x_\kappa)\) with positive weights \(\omega_\kappa\) that is exact for the functions under consideration. We start with the \(\phi\) integration:

\[
\int_0^{2\pi} d\phi \cos^{\nu_1^\nu_{m-1}} \phi \sin^{\nu_m} \phi. \] (16)

In the integrand we have the \((\nu_1 + \nu_m)\)th power of \(e^{i\phi}\) or \(e^{-i\phi}\) at most. Therefore a simple trapezoidal rule, \(\omega_\kappa = 2\pi/n\) and \(\phi_\kappa = 2\pi \kappa/n\), gives exact results provided that \(\nu_1 + \nu_m \leq 2N < n\). We should remember that this integral vanishes unless both \(\nu_1 - 1\) and \(\nu_m\) are even.

Next we consider the \(\theta_1\) integral

\[
I_{m-2} = \int_0^{\pi} d\theta \cos^{\nu_1} \theta \sin^{\nu_1 + 1} \theta, \] (17)

where the subscript ”\(m-2\)” of the variable \(\theta\) is omitted. Note that we can assume \(\nu_1 + \nu_m\) is even since otherwise the whole integral is zero by the \(\phi\) integration alone, which is exact, whatever wrong results other integrations produce. When \(\nu_1 + \nu_m\) is even, by setting \(\cos \theta = x\) we obtain

\[
I_{m-2} = \int_{-1}^1 dx x^{\nu_1 - 2} (1 - x^2)^{\nu_1 + \nu_m}/2. \] (18)

Since the integrand is a polynomial of degree \(\nu_1 - 2\) + \(\nu_1 + \nu_m\), we can use, for example, the Gauss-Legendre quadrature formula of weights \(w_\kappa\) and points \(x_\kappa\) (see [14] for example). In terms of variable \(\theta\) the rule, \(\omega_\kappa = w_\kappa / \sin \theta_\kappa\) and \(\theta_\kappa = \cos^{-1} x_\kappa\), gives the exact result provided that \(\nu_1 - 1 + \nu_1 + \nu_m \leq 2N < 2n\). Let us note that \(I_{m-2} = 0\) unless \(\nu_1 - 2\) is even.

We must examine one more integral, the \(\theta_1\) integral:

\[
I_{m-3} = \int_0^{\pi} d\theta \cos^{\nu_1} \theta \sin^{\nu_1 + 1} \theta. \] (19)
By the same reason as in the case of $I_{m-2}$ we can assume that $\nu_{m-2} + \nu_{m-1} + \nu_m$ is even. Then the integration range can be enlarged to $[0, 2\pi]$ so that an argument similar to that in the $\phi$ integration applies. It turns out that the rule $\omega_\kappa = \pi/n$ and $\theta_\kappa = 2\pi(2\kappa - 1)/n$ is exact if $\nu_{m-3} + \nu_{m-2} + \nu_{m-1} + \nu_m + 2 \leq 2(N+1) < 2n$. If we changed the integration variable by $\cos \theta = x$, this rule would correspond to the Gauss-Tschebyscheff quadrature $[14]$. The integral $I_{m-3}$ does not vanish only if $\nu_{m-3}$ is even.

It is clear that a similar argument also holds for remaining integrals; namely, for the integral $I_{m-i}$ the Gauss-Legendre (-Tschebyscheff) type of quadrature can be used when $i$ is even (odd). The weights $w_a$ of the whole integral are positive since they are given by the product of weights $\omega$'s of the single integrations. Thus we can conclude that there exists a finite set $\{w_a, \rho_a\}_{a=1}^A$ which satisfies Eq. (11).

Now that we have shown there exists a finite set of POVM's that achieves the optimal mean fidelity of Eq. (12), we study universality of finite optimal estimators. By universality we mean that the fidelity is independent of input states. The unaveraged fidelity of an optimal estimator for input $\rho^\otimes N$ can be written as

$$\sum_{a=1}^A \text{tr} \left[ E_a \rho^\otimes N \right] \text{tr} [\rho_a \rho] = d_N \sum_{a=1}^A w_a \text{tr} \left[ \rho_a^\otimes(N+1) \rho^\otimes(N+1) \right].$$

(20)

From this equation we find the fidelity is independent of $\rho$ if and only if $\sum_{a=1}^A w_a \rho_a^\otimes(N+1) = S_{N+1}/d_{N+1}$, namely

$$\sum_{a=1}^A w_a \rho_a^\otimes(N+1) = \left\langle \rho^\otimes(N+1) \right\rangle.$$  

(21)

This is a stronger condition than condition (11) that is required for optimal estimators for $N$ copies. Therefore optimal estimators are not generally universal, but any optimal estimators for $M (> N)$ copies are universal if it is used to estimate $N$ copies of an unknown state.

A closely related question to this issue is the following. Suppose that for given $N$ copies of an unknown pure state we first produce $M (> N)$ copies by the optimal cloner and then estimate the resulting state by the optimal estimator for $M$ copies. What is the fidelity of this apparently detourlike two-step estimation procedure? Using the unique optimal cloner from $N$ to $M$ copies given by [12],

$$T(\rho^\otimes N) = \frac{d_N}{d_M} S_M \left( \rho^\otimes N \otimes 1^\otimes(M-N) \right) S_M,$$

(22)

and an optimal estimator $\{E_a, \rho_a\}_{a=1}^A$ for $M$ copies, we find that the unaveraged fidelity of the two-step estimation is optimal and universal:

$$\sum_{a=1}^A \text{tr} \left[ E_a T(\rho^\otimes N) \right] \text{tr} [\rho_a \rho] = d_N \sum_{a=1}^A w_a \text{tr} \left[ \rho_a^\otimes N \rho^\otimes N \right] \text{tr} [\rho_a \rho] = F_{\text{optimal}}(N, d).$$

(23)

IV. CONCLUDING REMARKS

In this paper we gave a direct derivation of the optimal mean fidelity of pure state estimation in a way we find simpler than the original ones [5, 8]. In order to show the existence of the optimal estimator with a finite POVM, we avoided the assumption of covariance of the measurement and the use of connection to the optimal cloning fidelity. As a figure of merit we used the mean fidelity. It should be noted that the optimal fidelity is not changed if we take the infimum of fidelity for a figure of merit. As shown in the preceding section, in this minimax approach the condition for finite optimal estimators for $N$ copies is Eq. (24), which gives the universal fidelity, instead of Eq. (11).

We showed how to construct a finite set of POVM's for optimal estimators by the use of an exact quadrature on a hypersphere. We expressed the integration on hypersphere as a multiple of single integrations, for each of which we constructed an exact quadrature rule. But this procedure does not generally give the minimal set of points on the hypersphere, or equivalently the minimal set of POVM’s. In the case of qubit the minimal set of POVM’s has been studied for several values of $N$ [15]. The weight $w_a$ obtained by our procedure depends on "$a". It may be desirable to have a constant weight since $w_a$ is equal to the probability of finding outcome "$a" for the random input considered in this paper; $w_a = \langle \text{tr} \left[ E_a \rho^\otimes N \right] \rangle$. Exact quadratures with a constant weight for polynomials of degree $t$ on the hypersphere are called spherical $t$-designs. There is an existence result for all values of $t$ in any dimension [16], but explicit examples are in general not straightforward to construct.
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APPENDIX A

In this appendix we sketch a derivation of the formula Eq.(3) based on a generating function for the readers convenience. In the text we considered the average of a function $f$ of a normalized complex vector $c = (c_1, c_2, \ldots, c_d)$ and its conjugate $c^+$:

$$< f(c, c^+) > = \frac{\int dcdc^+ f(c, c^+)}{\int dcdc^+},$$

where

$$\int dcdc^+ = \int_{-\infty}^{\infty} \prod_{i=1}^{d} d(Re c_i) d(Im c_i) \delta(c^+ c - 1).$$

(A2)

It is convenient to introduce a generating function $G$ of $\lambda = (\lambda_1, \ldots, \lambda_d)$ and its conjugate $\lambda^+$:

$$G(\lambda, \lambda^+) = \int dcdc^+ e^{i(\lambda^+ c + c^+ \lambda)},$$

(A3)

so that the average $\langle c_{i_1} c_{j_1}^* c_{i_2} c_{j_2}^* \cdots c_{i_l} c_{j_l}^* \rangle$ is calculated as

$$\langle c_{i_1} c_{j_1}^* c_{i_2} c_{j_2}^* \cdots c_{i_l} c_{j_l}^* \rangle = \left[ \frac{\partial}{\partial \lambda_{i_1}^*} \frac{\partial}{\partial \lambda_{j_1}^*} \cdots \frac{\partial}{\partial \lambda_{i_l}^*} \frac{\partial}{\partial \lambda_{j_l}^*} G(\lambda, \lambda^+) \right]_{\lambda = 0} / G(0).$$

(A4)

First we express the $\delta$ function in the form of a Fourier transform as $\delta(c^+ c - 1) = \frac{1}{2\pi} \int d\omega e^{i\omega(c^+ c - 1)}$. Then the integration over $c$ and $c^+$ can be performed by a Gauss integral. The result is

$$G(\lambda, \lambda^+) = \frac{(i\pi)^d}{2\pi} \int d\omega \frac{1}{(\omega + i\epsilon)^d} e^{-i\frac{\lambda^+ \lambda}{\omega + i\epsilon}} e^{-i(\omega + i\epsilon)},$$

(A5)

where $\epsilon$ is a small positive constant, which should go to zero in the end. Expanding $e^{-i\frac{\lambda^+ \lambda}{\omega + i\epsilon}}$ and performing the $\omega$ integration by a complex contour integral, we obtain

$$G(\lambda, \lambda^+) = \pi^d \sum_{n=0}^{\infty} \frac{1}{n!(n + d - 1)!}(-\lambda^+ \lambda)^n.$$  

(A6)

Now the formula of Eq.(3) can be shown as follows:

$$\langle c_{i_1} c_{j_1}^* c_{i_2} c_{j_2}^* \cdots c_{i_l} c_{j_l}^* \rangle = \frac{(d-1)!}{l!(l+d-1)!} \frac{\partial}{\partial \lambda_{i_1}^*} \frac{\partial}{\partial \lambda_{j_1}^*} \cdots \frac{\partial}{\partial \lambda_{i_l}^*} \frac{\partial}{\partial \lambda_{j_l}^*} (\lambda^+ \lambda)^l$$

$$= \frac{(d-1)!}{l!(l+d-1)!} \text{ (sum of all contractions between } i's \text{ and } j's).$$

(A7)

It is also easy to see that the average of any product of different number of $c$’s and $c^*$’s vanishes.

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