Indirect exchange interaction between magnetic impurities in one-dimensional gapped helical states

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Abstract
We investigate theoretically indirect exchange interaction between magnetic impurities mediated by one-dimensional gapped helical states. Such states, containing massive Dirac fermions, may be realized on the edge of a two-dimensional topological insulator when time-reversal symmetry is weakly broken. We find that the indirect exchange interaction consists of Heisenberg, Dzyaloshinsky–Moriya, in-plane and out-of-plane Ising terms. These terms decay exponentially when the Fermi level lies inside the bandgap whereas the Dzyaloshinsky–Moriya term has smallest amplitude. Outside the bandgap, the massive helical states modify oscillatory behaviors of the range functions so that the period of oscillations decreases near the edge of band in terms of energy gap or Fermi energy. In addition, the out-of-plane Ising term vanishes in the case of zero-gap structure. Also, the oscillation amplitude of out-of-plane Ising term increases versus energy gap but it decreases as a function of Fermi energy. While the oscillation amplitudes of other components remain constant as functions of energy gap and Fermi energy. Analytical results are also obtained for subgap and overgap regimes. Furthermore, the effects of electron–electron interactions are analyzed.

Keywords: indirect exchange interaction, one-dimension, magnetic impurity, gapped helical states, topological insulator

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum nature of phenomena becomes more pronounced in low dimensional systems [1]. Especially, in recent years much attention in condensed matter physics has been paid to two-dimensional systems [2] due to the synthesis of graphene [3] and related materials [4] with Dirac-like dispersion relation. Materials with linearly dispersing spectrum have attracted considerable amount of attention due to providing new opportunities for both fundamental aspects and potential applications. Moreover, Weyl semimetals have been predicted [5] and reported [6] that are three-dimensional analogs of graphene. One-dimensional (1D) Dirac materials may also provide a promising alternative to studying exotic characteristics of chiral quantum states. They may raise an interesting perspective in a variety of contexts. In particular, since features of indirect exchange interactions between magnetic impurities depend on both the dimensionality and the band dispersion of host materials, one may expect an interesting behavior in 1D Dirac materials.

The study of indirect exchange interactions between magnetic impurities in one-, two- and three-dimensional materials with parabolic band structure goes back to the seminal works by Ruderman, Kittel, Kasuya, and Yosida (RKKY) [7–9]. This interaction, known also as RKKY interaction, has been investigated in new systems with Dirac-like band structure, for instance, graphene [10–12] and phosphorene [13, 14] in two dimensions and Weyl/Dirac semimetals [15–18] in three dimensions. Moreover, exotic spin textures due to
and at the edge of two-dimensional [26–30] topological insulators have been predicted as another example of non-trivial RKKY interaction. Another work has discussed RKKY interaction near the helical edge by taking into account of both bulk and edge modes [31]. Furthermore, it has been shown that magnetic exchange interaction is adjustable by a vertical bias in thick films [32] or a thin slab of topological insulators [32, 33]. Recently, the RKKY interaction mediated by surface states [34] and helical Majorana edge states [35, 36] in a topological superconductor was also investigated.

Several research have been performed on magnetic topological insulators [37] elucidating magnetic properties of promising topological insulator candidate materials such as HgTe quantum wells [38], Bi2Se3 [39], and Sb2Te3 [40]. Experimentally, the chemical doping of topological insulators has been realized with transition metal ions [37] and rare Earth ions [41]. In the former case, impurity magnetic moments are ferromagnetically ordered resulting in massive Dirac Fermions in Mn-doped [42] and Fe-deposited [43] Bi2Se3 revealing quantum anomalous Hall effect [44]. It has been shown that out-of-plane magnetic ordering can be observed in Bi2Se3[2−x]Te3−x [45] and (Bi0.5Sb0.5)Te3 [46, 47] doped with magnetic Cr. More recently, using synchrotron-based x-ray techniques, it has been demonstrated that the topological insulator surface would host magnetic ordering while the bulk would not in Cr-doped Bi2Se3 thin film [48]. On the other hand, large magnetic moments can be introduced into the topological insulators via rare Earth Gd dopants with 7 unpaired 4f-spins. Bulk antiferromagnetic ordering and Dirac gap opening have been reported for Gd-doped topological insulators with stoichiometry Bi1.09Gd0.06Sb0.94Te3 [49]. Also, a large effective magnetization can be induced in Dy-doped Bi2Te3 thin film by proximity coupling to a ferromagnetic insulator [50] or to Cr-doped Sb2Te3 layer [51, 52].

Theoretically, on the other hand, it has been revealed that the conductance properties of topological boundary states depend on the magnetic impurity coupling [53, 54]. The coupling of helical edge states to spin impurities or nuclear in topological insulator [55, 56] leads to the Anderson localization [57] and anisotropic nuclear spin–spin interaction [58] with spiral spin ordering [56]. It has also been demonstrated that Mn-doped InAs/GaSb [59] develops a ferromagnetic ordering at low enough temperature. Therefore, the effect of ferromagnetism generated by the magnetic impurities would gap out the boundary states spectrum [42, 43, 59, 60] and topological edge states become spin-polarized [61]. Although, there exist systems with 1D Dirac dispersion [38, 62], their magnetic properties have not been demonstrated fully in the absence of time-reversal symmetry. So, it deserves to investigate indirect interaction between magnetic moments in host systems supporting 1D Dirac dispersion with spin polarized states.

The purpose of this research is to study indirect exchange interactions between magnetic impurities mediated by 1D helical carriers of a two-dimensional topological insulator with weakly broken time-reversal symmetry. Here, we focus on the role of gapped spectrum of helical edge states. We find that unlike the usual 1D quantum wire case [63], the RKKY exchange interaction in our system is strongly anisotropic and includes the Heisenberg, Dzyaloshinsky–Moriya, and Ising terms which is somewhat similar to gapless spin–orbit-coupled quantum wire [64] and gapful spin–orbit-coupled carbon nanotubes [65]. However, in contrast, here, the Ising terms are comprised of in-plane and out-of-plane interactions which have the same form as the case of three-dimensional topological insulators in the presence of bandgap [19]. The corresponding range functions display an oscillatory (exponentially) decay with a spatial separation of magnetic moments if chemical potential resides in the band (bandgap). Furthermore, all the range functions are significantly affected by the energy gap, in particular, near the band edge leading to decreasing of oscillation period as a function of energy gap or Fermi energy. Moreover, the Heisenberg, Dzyaloshinsky–Moriya, and in-plane Ising terms have invariant oscillation amplitude independent of Fermi energy position with respect to energy gap. The oscillation amplitude of out-of-plane Ising term increases as a function of energy gap whereas it decreases as a function of Fermi energy. For both subgap and over gap regimes analytical expressions are presented for asymptotic behaviors of the range functions. Also, the effect of electron–electron interactions on the range functions are investigated.

The paper is organized as follows. In section 2, the model is introduced and the resulting indirect exchange interaction between magnetic impurities, including range functions, are calculated. In section 3, the range functions are evaluated numerically and asymptotic expressions for them are obtained analytically. The effect electron–electron interactions on the range functions are analyzed in section 4. Section 5 is devoted to summarize and discussion. Some detail calculations are presented in appendix A.

2. Model and theory

The setup we consider throughout the paper is schematically depicted in figure 1(a). It is composed of a two-dimensional topological insulator with 1D helical edge state and two magnetic impurities located on the edges of the topological insulator in the presence of Zeeman exchange field. The helical edge states of two-dimensional topological insulators can be thought of as 1D Dirac states. Also, the inclusion of exchange field which is perpendicular to the surface gaps out the metallic edge states at the boundaries due violating time-reversal symmetry. The Hamiltonian describing the 1D gapped Dirac states is [60, 66, 67],

$$H_0(k) = v_k \sigma_x + \Delta \sigma_y,$$

and the corresponding edge spectrum reads

$$\epsilon_{\lambda} = \lambda \sqrt{(v_k)^2 + \Delta^2},$$

where the Pauli matrices $\sigma_{x,y}$ represent the physical spin of the electron, $k$ is a wave vector along the 1D channel, the band index $\lambda = \pm$, and $v_k$ is Fermi velocity. Since the spin
quantization axis of edge states is along the $x$ direction, the exchange field induced by out-of-plane spin polarization opens up gap $2\Delta$ in the edge spectrum [see figure 1(b)]. This lifts two-fold degeneracy of the so-called Dirac point. Note also that in the topological insulators due to presence of strong spin–orbit interaction, the electron spin is no longer a good quantum number originating from the off-diagonal elements of the Bernevig–Hughes–Zhang (BHZ) Hamiltonian [38].

Two localized magnetic impurities with moments $S_j (j = 1, 2)$ can be coupled to itinerant spin-polarized Dirac fermions with spin density $s(R_j) = \delta(r - R_j)\sigma$ at position $R_j$. This coupling can be modeled by

$$H_{\text{int}} = -J \sum_{j=1,2} S_j \cdot s(R_j),$$

where $J$ denotes coupling strength of magnetic impurities with the host Dirac fermions. Using the second-order perturbation theory and treating $H_{\text{int}}$ as a perturbation, the indirect exchange interaction between the two magnetic impurities mediated by host carriers can be obtained by [7–9]

$$H_{\text{RKKY}} = -\frac{J^2}{\pi} \text{Tr} \left[ \int_{-\infty}^{\infty} d\epsilon \text{Im} \left\{ (S_1 \cdot \sigma) G^0(R, \epsilon^+) (S_2 \cdot \sigma) G^0(-R, \epsilon^+) \right\} \right],$$

(4)

where $\text{Tr}$ stands for trace over spin degree of freedom, $\text{Im}$ is imaginary part, $\epsilon_i$ is the Fermi energy measured from the Dirac point. $G^0(R, \epsilon^+)$ denotes the Green’s function matrix in real-space with $R = R_2 - R_1$ being a distance between the two magnetic centers and $\epsilon^+ = \epsilon + i\eta$ with $\eta \to 0^+$. The real-space Green’s function can be written as,

$$G^0(k, \epsilon^+) = \frac{\epsilon^+ - H_0(k)}{(\epsilon^+_0)^2 - (\epsilon^+)^2 - (\Delta)^2},$$

(6)

where $G^0(k, \epsilon^+)$ is the momentum space Green’s function associated with equation (1). Substituting equation (6) into equation (5) and using the residue theorem, we derive a closed form expression for the real-space Green’s function as

$$G^0(\pm R, \epsilon^+) = -\frac{i e^{i\epsilon^+ R}}{2\eta \sqrt{\epsilon^+}} (\epsilon^+ \sigma_0 \pm \sqrt{\epsilon^+} \sigma_x + \Delta \sigma_y),$$

(7)

where $\alpha = (\epsilon^+)^2 - (\Delta)^2$ and $\sigma_0$ is the unit matrix. Combining the result in equation (7) with equation (4), yields a formula for $H_{\text{RKKY}}$ as follows:

$$H_{\text{RKKY}} = F_1(S_1 \cdot S_2 + F_2(R, \epsilon^+) | S_1 \times S_2 |)$$

$$+ F_3(R, \epsilon^+) | S_1 |^2 + F_4(R, \epsilon^+) | S_2 |^2,$$

(8)

where the range functions are

$$F_1 = \text{Im} \int_{-\infty}^{\epsilon^+} d\epsilon f(\alpha),$$

(9)

$$F_2 = -\text{Im} \int_{-\infty}^{\epsilon^+} d\epsilon \frac{i\epsilon}{\sqrt{\alpha}} f(\alpha),$$

(10)

$$F_3 = -F_1,$$

(11)

$$F_4 = \text{Im} \int_{-\infty}^{\epsilon^+} d\epsilon \frac{\Delta(2i\sqrt{\alpha} + \Delta)}{4\alpha} f(\alpha),$$

(12)

with $f(\alpha) = \frac{2\alpha}{\alpha^2} e^{2i\pi R}$. As one can see from equation (8), the indirect exchange interaction consists of the Heisenberg, Dzyaloshinsky–Moriya, and two component Ising interactions whose range functions are given by $F_1$, $F_2$, $F_3$, and $F_4$, respectively. The Heisenberg, $x$- and $y$-component Ising interactions favor collinear magnetic spin alignment while the Dzyaloshinsky–Moriya imposes in-plane non-collinear magnetic spin orientation. Notably, the Heisenberg, Dzyaloshinsky–Moriya and $x$-component of Ising terms come from helical nature of the edge states and their range functions are modified due to gap opening [see equations (9)–(11)]. Interestingly, equation (12) implies that the out-of-plane coupling term, as shown in the present study, is non negligible when compared to non-interacting [26] and interacting [56] gapless helical cases. This originates from the partial mixing of opposite spin states of helical states due to gap term. It should also be noted that if $\Delta = 0$, then $F_4 \to 0$, recovering the previously obtained results [26]. Furthermore, in the gap region, the integrands of equations (9)–(12) become purely real functions, whereby all the range functions $F_1, F_2, F_3$, and $F_4$ suppress. The suppression of indirect exchange interaction results actually from the lack of available states for itinerant carriers. However, when the chemical potential lies within the gap region then, for sufficiently small energy gap, the indirect exchange interaction can be mediated by virtual interband transitions of electrons though Bloembergen–Rowland mechanism [68, 69].

3. Analytical and numerical results

In what follows, due to particle–hole symmetry, without loss of generality, we assume that $\epsilon_i > 0$. In general, although there are no explicit expressions for the integrals of range functions [equations (9)–(12)], it would be possible to estimate analytical statements in some limiting cases. In the small gap regime,
i.e., $\epsilon_I \gg \Delta$, equations (9)–(12) can be expanded up to lowest order in $\Delta$ as

$$F_1 \approx - \frac{j^2}{\pi v_F} \left[ \frac{1}{2R} \cos(\gamma) + \frac{\Delta^2 R}{v_F^2} \right],$$

(13)

$$F_2 \approx \frac{j^2}{2\pi v_F} \left[ \frac{1}{R} \sin(\gamma) - \frac{\Delta^2}{v_F^2} \cos(\gamma) \right],$$

(14)

$$F_3 = -F_1,$$

(15)

$$F_4 \approx \frac{j^2}{2\pi v_F} \Delta \cos(\gamma),$$

(16)

where $\gamma = \frac{\pi v_F}{\epsilon_I}$ and $\cos(\gamma)$ is cosine integral function [70]. We note that the first term of range functions of the Heisenberg, Dzyaloshinsky–Moriya, and in-plane Ising interactions, having dominant contribution, decay as $R^{-1}$ reminiscing the range function behavior of usual 1D electron gas. Also, exploiting the asymptotic form of cosine integral function [70],

$$\cos(\gamma) \approx 1 - \frac{\sin(\gamma)}{x},$$

(17)

for long distance limit, i.e., $x, R \gg 1$, we identify that the spatial dependence of $y$-component of Ising interaction falls off similar to the other interactions. As can be seen from equations (13)–(16), for $F_1, F_2$, and $F_3$, due to gap term, the second term is proportional to $\Delta^2$, while the first (or dominant) term in the asymptotic expansion of $F_4$ is proportional to $\Delta$. Furthermore, the sign of second terms in $F_1, F_2$, and $F_3$ is not identical.

In the subgap limit, i.e., $\epsilon_I \ll \Delta$, for long distance case, on the other hand, using steepest descent method [71], we can determine leading-order asymptotic approximations to equations (9)–(12) as,

$$F_1 \approx - \frac{j^2}{2\pi v_F} \sqrt{\frac{\Delta}{\pi R}} e^{-\frac{\Delta}{\epsilon_I}},$$

(18)

$$F_2 \approx \frac{j^2 \eta_I}{\pi v_F} \Delta e^{-\frac{\Delta}{\epsilon_I}},$$

(19)

$$F_3 = -F_1,$$

(20)

$$F_4 \approx \frac{j^2}{8\pi v_F} \sqrt{\frac{\Delta}{\pi R}} e^{-\frac{\Delta}{\epsilon_I}}.$$

(21)

Note that, in this regime, these range functions have an exponentially decaying behavior determined by the energy gap and Fermi velocity. The amplitude of $F_1, F_3$, and $F_4$ depends on $\Delta$ and $R$ as $\sim R^{-1}$ independent of Fermi energy. In contrast, the amplitude of $F_2$ is proportional to $\frac{\epsilon_I}{\Delta}$ independent of $R$. This indicates that, the Dzyaloshinsky–Moriya interaction is zero at charge neutrality point, $\epsilon_I = 0$. Importantly, the per-factor $\eta_I$ in the amplitude of $F_2$ makes the Dzyaloshinsky–Moriya interaction vanishingly small even when $\epsilon_I \neq 0$. This implies that the perpendicular uniform polarization induced by $\Delta$ has a substantial detrimental effect on the in-plane non-collinear magnetic spin alignment in the absence of free charge carriers.

Generally, the behaviors of the range functions $F_1, F_2, F_3,$ and $F_4$, presented in equations (9)–(12), can be explored by numerical evaluation. We take $\epsilon_I v_F^{-1}$ as a unit of energy, the lattice constant $a$ as a length unit. Equations (9)–(12) are plotted as a function of $R$ in figure 2 for different values of $\Delta$ with $\epsilon_I = 4v_F a^{-1}$. As shown in the main panels of figure 2, for $\epsilon_I > \Delta$ the oscillations of all the range functions are damped by increasing $R$ and the period of oscillations is increased with the increase of $\Delta$. In the meanwhile, the amplitude of all the range functions decreases with the increment of band gap, interestingly, except for the case of out-of-plane Ising term that increases. These features can be interpreted as follows. The opening of gap reduces the Fermi surface so that the Fermi momentum $k_F = \sqrt{\epsilon_I^2 - \Delta^2}/v_F$ decrease. Because the spacial periodicity of the range functions depend on $k_F^{-1}$, the periods of the oscillations increase. At the same time, the magnetization-induced gap decreases density of states but it provides spin-polarized states. As such, $F_1, F_2,$ and $F_3$, which depend on the density of states, are smeared out. While the out-of-plane Ising term, which can be directly related to the magnetization, is promoted. On the other hand, for $\epsilon_I < \Delta$, as shown in the insets of figure 2, all the range functions decay exponentially with the distance of two magnetic impurities. We also observe that the rate of exponential decay becomes faster with increasing $\Delta$ arising from the suppression of excitations of massive carriers across the gap. $F_2$ takes infinitesimally small values compared to the others. So, the spiral ordering of Dzyaloshinsky–Moriya term can be suppressed by the gap term. Because, the gap opening destroys the helicity and the spiral ordering originates from the helicity, thus, $\Delta$ destroys the Dzyaloshinsky–Moriya interaction in the subgap regime.

The dependence of range functions, equations (9)–(12), on Fermi energy for various values of $\Delta$ with $R/a = 1$ is depicted in figure 3. As $\epsilon_I$ increases, the absolute values of both $F_1$
and $F_3$ increase from small values until reach a maximum at $\epsilon_f = \Delta$ [see figures 3(a) and (c)]. Also, the magnitudes of Dzyaloshinsky–Moriya and out-of-plane Ising interactions are smaller than those of the other range functions for $\epsilon_f < \Delta$ [see figures 3(b) and (d)]. This results in dominating the ferromagnetic coupling in the subgap regime [19]. The range function of out-of-plane Ising term shows a sharp dip at $\epsilon_f = \Delta$ which can be attributed to the high spin-polarized density of states available at the band edge. All the range functions exhibit oscillatory behavior for $\epsilon_f > \Delta$ except that the envelope of $F_3$ is damped simultaneously. Also, $F_1$, $F_2$, and $F_3$ demonstrate constant amplitudes for different values of $\Delta$ while the amplitude of oscillations of $F_3$ increases as a function of $\Delta$.

The gap dependence of $F$'s for different values of $\epsilon_f$ is illustrated in figure 4. One can see that except for $F_3$ which is zero at $\Delta = 0$ [see figure 4(d)] the other range functions can take various values depending on both Fermi energy and $R$ [see figures 4(a), (b), and (c)]. By increasing $\Delta$ the values of $F$'s change with small modulations at first. Then, as $\Delta$ increases further the range functions begin to oscillate in $\Delta \lesssim \epsilon_f$. As already mentioned above, from figure 4 it is also visible that the oscillation amplitude of $y$-component of Ising term is enhanced by increasing bandgap while the amplitude of oscillations of the other range functions remains unchanged even for different values of $\epsilon_f$. Note, interestingly, that the range function of Dzyaloshinsky–Moriya interaction is identically zero for $\epsilon_f = 0$ irrespective of $\Delta$, as illustrated in figure 4(b).

In both figures 3 and 4, the period of oscillatory part of the range functions decreases by approaching to the band edge and, as a result, takes smallest values near band edge. When the Fermi energy is so small that $\Delta > \epsilon_f$ the range functions tend to zero rapidly. This is a result of vanishing of Fermi surface. Consequently, in the helical system, the magnetization effect on the indirect exchange interaction becomes more pronounced when the Fermi energy lies in the band but close the band edge. Under such condition, magnetic moments tend to be aligned along the magnetization direction. On the contrary, by going away from the band edge such that $\epsilon_f \gg \Delta$, the behavior of indirect exchange interaction approaches to the usual helical system despite the presence of the magnetic field. Finally, it is worthwhile noting that the above-presented numerical behavior is in agreement with the analytical one quite well in the two limits of $\epsilon_f \gg \Delta$ and $\epsilon_f \ll \Delta$.

4. Effects of electron–electron interaction

The RKKY interaction in interacting quantum wires without [72, 73] and with [74, 75] spin–orbit interaction has been studied before. These studies have also been extended to spin–orbit-coupled helical liquids [29]. Now, let us consider the effects of electron–electron interactions on RKKY interactions in the 1D massive helical states. Due to absence of Fermi points in the subgap regime, we restrict ourself to the $\epsilon_f > \Delta$ case involving Tomonaga–Luttinger liquid model [1]. The gap $\Delta$ bends the dispersion near the band edge, as can be seen from figure 1(b), so it renormalizes the Fermi velocity as $\nu_f' = \nu_f\sqrt{1 - (\Delta/\epsilon_f)^2}$. We linearize [38] the massive helical states of equation (1) on one of the system edges around Fermi points at momenta $\pm k_f = \pm \sqrt{\epsilon_f^2 - \Delta^2}/\epsilon_f$ to obtain the low-energy Hamiltonian as

$$H_0 = -iv_f' \left[ \psi^+ \chi_r(x) \partial_x \psi^- (x) - \psi^- \chi_r(x) \partial_x \psi^+ (x) \right], \quad (22)$$

and the low-energy electron field operator as

$$\psi(x) = \sum_r \chi_r (r k_f) \psi_r (x) e^{i k_r x}, \quad (23)$$

where the operators $\psi_r (x)$ with $r = +, -$ are the annihilation fields for right- and left-moving fermions at coordinate point $x$ along the edge. The spinor $\chi_r (k)$ in spin space is given by
with \( \theta_k = \frac{1}{2} \tan^{-1}(-\Delta/k_B T) \) being \( k \)-dependent spin-rotation angle implying that the gap term mixes the spin states. The fermionic-field operators can be expressed in terms of bosonic fields \( \phi_i(x) \) as \cite{1}

\[
\psi_i(x) = \frac{1}{\sqrt{2\pi\alpha_0}} e^{i\sqrt{2\pi}\phi_i(x)},
\]

where \( \alpha_0 \) is a short-distance cutoff, and \( \eta_i \) are Klein factors.

The Hamiltonian of electron–electron interactions having dominant contribution is given by

\[
\mathcal{H}_{e-e} = g_2\psi_1^\dagger(x)\psi_-(x)\psi_1^\dagger(x)\psi_+(x) + \frac{g_4}{2}\left[(\psi_1^\dagger(x)\psi_-(x))^2 + (\psi_1^\dagger(x)\psi_+(x))^2\right],
\]

where \( g_{2,4} \) are interaction constants. We treat the total Hamiltonian \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{e-e} \) in the bosonized form yielding,

\[
\mathcal{H} = \frac{v}{2} \left[ \frac{1}{K} (\partial_\tau \Phi)^2 + K(\partial_\tau \Theta)^2 \right],
\]

with bosonic fields

\[
\Phi = \phi_+ + \phi_-, \quad \Theta = \phi_+ - \phi_-.
\]

Here, we have introduced the speed of collective excitations \( v \) and Luttinger parameter \( K \) as

\[
v = \sqrt{\left(\nu \frac{g_4}{2\pi} \right)^2 - \left(\frac{g_2}{2\pi}\right)^2},
\]

\[
K = \frac{2\pi
\nu \frac{g_4}{2\pi} + g_4 - g_2}{2\nu \frac{g_4}{2\pi} + g_4 - g_2}
\]

depending on the gap and Fermi energy through \( \nu \). For non-interacting case \( K = 1 \) while \( 0 < K < 1 \) (\( K > 1 \)) corresponds to repulsive (attractive) electronic interactions. Also, we have assumed that the interactions are weak enough to ignore backscattering potential \cite{76, 77}. Note, however, that it has been shown that elastic spin–flip backscattering can be occurred by coupling of electrons to nuclear spins through the hyperfine interaction causing a significant edge resistance at low enough temperatures \cite{56}.

In terms of the electron field operators, the spin density can be defined as \( \mathcal{S}(x) = \psi_i^\dagger(x)\psi_i(x) \). Using the identity

\[
\chi_{i}(r_k)\sigma \chi_{i'}(r'_{k'}) = \frac{r_{i,i'}^2 \cos(2\theta_{k})}{-r_{i,i'} + \delta_{i,i'} \cos(2\theta_{k}) + \delta_{i,i'} \sin(2\theta_{k})},
\]

and equation (25), the components of spin density \( s(x) \) can be determined in the bosonized language \cite{72, 73} as

\[
s_i = \frac{1}{\sqrt{\pi}} \cos(2\theta_{k}) \partial_\tau \Theta,
\]

Having obtained the components of spin density and following the procedure in reference \cite{1}, the imaginary-time spin–spin correlation functions

\[
S^{ij}(\tau, R) = \langle s^i(\tau, R)s^j(0, 0) \rangle,
\]

can be determined yielding nonzero components as

\[
S^{xx}(\tau, R) = \frac{\cos(2\theta_{k})}{2\pi^2 K} X(\tau, R),
\]

\[
S^{xy}(\tau, R) = \frac{\sin^2(2\theta_{k})k}{2\pi^2 a_0^2} X(\tau, R) + \frac{\cos(2k_{F}\tau)}{2\pi^2 a_0^2} Y(\tau, R),
\]

\[
S^{yx}(\tau, R) = \frac{\cos(2\theta_{k})\cos(2k_{F}\tau)}{2\pi^2 a_0^2} Y(\tau, R),
\]

\[
S^{yy}(\tau, R) = -S^{xy}(\tau, R) = \frac{\cos(2\theta_{k})\sin(2k_{F}\tau)}{2\pi^2 a_0^2} Y(\tau, R),
\]

with

\[
X(\tau, R) = \left[ \left( |v| \tau + a_0 \right)^2 - R^2 \right]^{-1/2},
\]

\[
Y(\tau, R) = \left[ \left( \frac{a_0}{|v| \tau + a_0} \right)^2 + R^2 \right]^{1/2}.
\]

By integrating \( S^{ij}(\tau, R) \) over \( \tau \), i.e.,

\[
\kappa^{ab}(R) = \int_{-\infty}^{\infty} d\tau S^{ab}(\tau, R),
\]

one gets the spin susceptibility. Plugging \( \kappa^{ab}(R) \) into the following general formula for RKKY interaction

\[
H_{RKKY} = -J \sum_{a,b=x,y,z} \kappa^{ab}(R)S_a^iS_b^j,
\]

we obtain the similar magnetic spin structure as equation (8). The modified range functions for \( a_0 \ll |x| \ll \nu/\k_B T \) can be expressed as

\[
F_1 = \frac{J^2 \nu(K)}{2\pi R^2 K} \cos^2(2\theta_{k}) \cos(2k_{F}R),
\]

\[
F_2 = \frac{J^2 \nu(K)}{2\pi R^2 K} \cos(2\theta_{k}) \sin(2k_{F}R),
\]

\[
F_3 = -F_1,
\]

\[
F_4 = \frac{J^2 \nu(K)}{2\pi R^2 K} \sin^2(2\theta_{k}) \cos(2k_{F}R),
\]

where \( \nu(K) = \frac{2K^{1/2-1/4}(K-1/2)}{\Gamma(\nu(K))} \) with \( \Gamma(p) \) being Gamma function. Consequently, we observed that since the spin basis of Hamiltonian (22) is the same as Hamiltonian (1) so the spin texture structure of equation (8) remains unchanged for interacting electrons in the host Luttinger liquid and only the range
functions are modified. This modification affects both the power-law decay, characterizing by exponent $2K - 1$, and the magnitude of range functions. Note that the above equations are valid in the range $1/2 < K \leq 1$ below which RKKY-Kondo competition becomes important [30]. Moreover, for non-interacting case, i.e., $K = 1$, and $\epsilon_f \gg \Delta$, the dominant terms of equations (44)–(47) can be obtained, being the same as those for equations (13)–(16), by expanding up to lowest order in $\Delta$. (see the appendix A).

5. Summary and discussion

The indirect exchange interaction between magnetic impurities is investigated in the edge of time-reversal broken two-dimensional topological insulators. The resulting interaction is composed of Heisenberg, Dzyaloshinsky–Moriya, in-plane and out-of-plane Ising terms. In the some limiting cases, the analytical results are approximately extracted for the range functions supporting the numerical results. The range functions fall off with a power-law decay $R^{-1}$ and exponentially in the over gap and subgap regimes, respectively, at large distance. As the energy gap increases, the rate of spatial exponential decay and the spatial periodicity of the range functions increase. The range functions of these interactions are significantly influenced through gapped band structure providing small periods near the band edge as a function of Fermi energy or energy gap.

We found that if the Fermi energy lies inside the bandgap then the range function of Dzyaloshinsky–Moriya interaction gets much smaller values than those of other interactions. In this regime, the ferromagnetic ordering is favorable in the system. Also, the amplitude of range function of out-of-plane Ising interaction increases as the gap increases for finite Fermi surface while the amplitudes of the other interactions do not change. This indicates that the spin-polarized states favor the out-of-plane Ising term when the Fermi energy resides near the band edge in the over gap case.

For $\epsilon_f > \Delta$ case, where the system supports metallic ground state, we took into account electron–electron interactions. Using bosonization technique, the components of spin density are determined. Then, the spin–spin correlation, spin susceptibility, and, subsequently, the range functions are calculated analytically. It is shown that the magnitude and power-law decay of the range functions change depending on the collective excitations velocity and Luttinger parameter. Also, the analytical results of the interacting case are equivalent to those for the non-interacting case under conditions $\epsilon_f \gg \Delta$ and $K = 1$.

It is worthwhile noting that by combining density functional theory calculations with complementary experimental techniques, it has been revealed that transition metal dopants affect not only the magnetic state of host material, but also the electronic structure [78]. However, using magnetic insulator, such as EuS, EuO, and EuSe, which is proximity coupled to the system can induce considerable Zeeman exchange field with gap size 9 meV [79, 80]. Also, it has been demonstrated that the fabricated MnBi$_2$Se$_4$/Bi$_2$Se$_3$ heterostructure exhibits ferromagnetism up to room temperature accompanied by a Dirac gap opening of $\sim$100 meV [81]. In contrast, the surface state gap varies between several tens to a hundred meV [37] in magnetically doped topological insulators. The gap formation has been controversial depending on the type of impurity and the impurity distance from the surface [78]. However, the bandgap has been estimated to be as high as 10 meV for Mn-doped InAs/GaSb [59, 60] which can be suppressed by non-magnetic impurities [60]. For a EuS–Bi$_2$Se$_3$, the exchange coupling can be estimated to be of the order of 10 eV $A^2$ [79]. Using the values of lattice constant $a = 4.19$ Å and Fermi velocity $v_f = 4.48 \times 10^5$ m s$^{-1}$ of Bi$_2$Se$_3$, we find that the range functions are of the order of 10 meV for $R = 1$ nm.

We believe that the ferromagnetic ordering of impurities which is dominated in the subgap regime supports the bandgap. Therefore, the renormalized spectrum will impact the interactions between impurities promoting ferromagnetic ordering as long as the Fermi surface resides within the bandgap. Finally, we remark that our setup can be implemented by source-probe measurement method [82] for low dimensional systems and can be served as an alternative approach in investigating magnetic properties of edge states.

Appendix A. Mathematical details

In this section, we show that if $K = 1$, the dominant approximated expressions of the range functions of section 4 coincide with those for section 3 in the over gap regime. We insert $\Delta K = \Delta K_1$, $\epsilon_f > \Delta$, and Fermi velocity $v_f = 4.48 \times 10^5$ m s$^{-1}$ of Bi$_2$Se$_3$, we find that the range functions are of the order of 10 meV for $R = 1$ nm.

We believe that the ferromagnetic ordering of impurities which is dominated in the subgap regime supports the bandgap. Therefore, the renormalized spectrum will impact the interactions between impurities promoting ferromagnetic ordering as long as the Fermi surface resides within the bandgap. Finally, we remark that our setup can be implemented by source-probe measurement method [82] for low dimensional systems and can be served as an alternative approach in investigating magnetic properties of edge states.

### Appendix A. Mathematical details

In this section, we show that if $K = 1$, the dominant approximated expressions of the range functions of section 4 coincide with those for section 3 in the over gap regime. We insert

$$ k_i = \frac{1}{v_f} \sqrt{\epsilon_f^2 - \Delta^2}, \quad (A.1) $$

into factors $\cos(2k_iR)$ and $\sin(2k_iR)$ in equations (44)–(47). Expanding these functions into a series and retaining the terms up to second order of the parameter $\Delta$, we have

$$ \cos(2k_iR) \approx \cos(\gamma) + \frac{\Delta^2 R}{v_f \epsilon_f} \sin(\gamma), \quad (A.2) $$

$$ \sin(2k_iR) \approx \sin(\gamma) - \frac{\Delta^2 R}{v_f \epsilon_f} \cos(\gamma), \quad (A.3) $$

with $\gamma = \frac{2 R \epsilon_f}{v_f}$. From $\theta_k = 1/2 \tan^{-1}(-\Delta/(k_i v_f))$ and equation (A.1), it easy to obtain

$$ \sin^2(2\theta_k) = \frac{\Delta^2}{\epsilon_f^2}, \quad (A.4) $$

$$ \cos^2(2\theta_k) = 1 - \frac{\Delta^2}{\epsilon_f^2}, \quad (A.5) $$

$$ \cos(2\theta_k) \approx 1 - \frac{\Delta^2}{2\epsilon_f}. \quad (A.6) $$

Also, the function $\nu(K)$ in equations (44)–(47) for $K = 1$ in the regime $\epsilon_f \gg \Delta$ can be approximated by

$$ \nu(1) \approx \frac{1}{v_f} \left( 1 + \frac{\Delta^2}{2\epsilon_f} \right). \quad (A.7) $$

Therefore, substituting equations (A.2)–(A.7) into equations (44)–(47) and retaining the terms up to second order of
For comparison, by taking into account the dominant term of equation \((17)\), which is the first one, equations \((13)–(16)\) will be matched the equations above.

\[
\Delta, \text{ the dominant terms of range functions can be expressed as }
\]
\[
F_1 \approx -\frac{j^2}{2\pi v} \left[ \frac{1}{R} \cos(\gamma) + \frac{\Delta^2}{v\epsilon} \sin(\gamma) \right], \quad (A.8)
\]
\[
F_2 \approx -\frac{j^2}{2\pi v} \left[ \frac{1}{R} \sin(\gamma) - \frac{\Delta^2}{v\epsilon} \cos(\gamma) \right], \quad (A.9)
\]
\[
F_3 = -F_1, \quad (A.10)
\]
\[
F_4 \approx -\frac{j^2}{4\pi v\epsilon} \frac{\Delta}{\epsilon} \frac{\sin(\gamma)}. \quad (A.11)
\]

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