Relaxation in a Fuzzy Dark Matter Halo. II. Self-consistent Kinetic Equations

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Abstract

Fuzzy dark matter (FDM) is composed of ultra-light bosons having a de Broglie wavelength that is comparable to the size of the stellar component of galaxies at typical galactic velocities. FDM behaves like cold dark matter on large scales. However, on the scale of the de Broglie wavelength, an FDM halo exhibits density fluctuations that lead to relaxation, a process similar to the two-body relaxation that occurs in classical gravitational N-body systems and is described by the Fokker–Planck equation. We derive the FDM analog of that kinetic equation, which describes the evolution of the velocity distribution in a spatially homogeneous FDM halo. We show that the evolution of the velocity distribution predicted by our kinetic equation matches numerical solutions of the coupled Schrödinger–Poisson equations. We also determine the dielectric function and the dispersion relation for linear waves in an FDM halo.

Unified Astronomy Thesaurus concepts: Dark matter (353); Galaxy kinematics (602)

1. Introduction

Cosmological models based on cold dark matter (CDM) explain most features of the cosmic microwave background, large-scale structure, and other cosmological phenomena. However, CDM has been less successful in predicting the properties of small-scale structure, such as the abundance of dwarf galaxies and the dark matter density profiles near the centers of galaxies (e.g., Weinberg et al. 2015; Bullock & Boylan-Kolchin 2017; Del Popolo & Le Delliou 2017). This shortcoming may reflect either our limited understanding of baryonic physics on these scales or deviations of the behavior of dark matter from the predictions of CDM.

Fuzzy dark matter (FDM) is dark matter composed of bosons with mass $m_b \approx 10^{-21} \text{ to } 10^{-22} \text{ eV}$, so small that the de Broglie wavelength

$$\lambda = \frac{h}{m_b v} = 1.20 \text{ kpc} \times \left( \frac{10^{-22} \text{ eV}}{m_b} \right) \left( \frac{100 \text{ km s}^{-1}}{v} \right)$$

is comparable to galaxy scales at a typical galaxy velocity $v$ (see, e.g., Hu et al. 2000; Marsh 2016; Hui et al. 2017; Hui 2021). FDM behaves like CDM on scales much larger than the de Broglie wavelength and thus preserves the success of CDM in explaining the properties of large-scale structure and the cosmic microwave background. However, on small scales FDM behaves quite differently from CDM. In particular, the wavelike nature of FDM suppresses small-scale structure, and one of the original motivations for introducing FDM was the hope that it would be more successful than CDM in explaining the properties of the dark matter distribution on scales less than a few kpc. The boson mass required to do so is $m_b \approx 3 \times 10^{-21} \text{ eV}$.

The FDM hypothesis generates many testable predictions (see, e.g., Grin et al. 2019; Niemeyer 2020, for recent reviews). The most serious tension is with constraints from the $\text{Ly}_\alpha$ forest power spectrum (e.g., Iršič et al. 2017; Kobayashi et al. 2017; Nori et al. 2019) which typically require $m_b \gtrsim 10^{-20} \text{ to } 10^{-22} \text{ eV}$ in standard models of structure formation.

FDM suppresses the early formation of low-mass galaxies and measurements of the galaxy luminosity function at high redshift constrain the boson mass to be at least a few times $10^{-22} \text{ eV}$ (e.g., Bozek et al. 2015; Schive et al. 2016; Corasaniti et al. 2017; Ni et al. 2019). The 21 cm absorption signal detected by the EDGES experiment implies copious star formation as early as redshift $z \approx 20$ and thereby also sets a strong constraint on the boson mass (Lidz & Hui 2018; Schneider 2018; Nebrin et al. 2019), but this observational result remains controversial. A recent analysis of the abundance of Milky Way satellite galaxies constrains the boson mass to $m_b \gtrsim 3 \times 10^{-22} \text{ eV}$ (Nadler et al. 2021), but this result depends on semi-analytic models for dwarf galaxy formation.

Additional potential constraints arise from the rotation curves of disk galaxies (Bar et al. 2019); the inspiral of massive objects due to dynamical friction, either supermassive black holes (Hui et al. 2017) or globular clusters (Lancaster et al. 2020; Bar et al. 2021), and the dynamical effects of the dense soliton that is expected to form at the center of massive galaxies (Bar et al. 2019; Davoudiasl & Denton 2019; Desjacques & Nusser 2019; Davies & Mocz 2020). The fluctuations in the FDM density that we describe below contribute to and can even dominate the heating and disruption of galactic stellar discs (Church et al. 2019), star clusters (Marsh & Niemeyer 2019), and stellar streams (Amorisco & Loeb 2018), and can excite Brownian motion of the central supermassive black hole in a galaxy (e.g., El-Zant et al. 2020b). Finally, the superradiance instability may generate dense boson clouds around rapidly rotating black holes that have detectable effects (e.g., Baumann et al. 2019).

FDM presents a rich set of novel physical phenomena that are not present in CDM. In particular, FDM exhibits density fluctuations on the scale of the de Broglie wavelength that arise from interference patterns. These fluctuations never damp, in contrast to the fluctuations in the density of CDM that arise from incomplete phase mixing. The gravitational field from these fluctuations scatters both condensed baryonic objects—stars, globular clusters, black holes, etc.—and the FDM waves themselves.
Hui et al. (2017) argued, and Bar-Or et al. (2019; hereafter Paper I) showed explicitly, that scattering of condensed objects by FDM fluctuations can be analyzed by treating the FDM fluctuations as quasiparticles with an effective mass of the order of the mass contained within the typical angular de Broglie wavelength, \( \lambda_d = h/(m_0, \sigma) \) with \( \sigma \) being the one-dimensional velocity dispersion in the galaxy (see also El-Zant et al. 2020a, 2020b). In particular, Paper I computed the diffusion coefficients originating from a homogeneous FDM background, which can be used in the classical Fokker–Planck (FP) equation to describe the evolution of the distribution function (DF) of a population of stars or other condensed objects.

The goal of this paper is to describe the evolution of the FDM distribution function itself due to scattering by these same fluctuations, by deriving the appropriate wave version of the FP equation. We shall focus on systems that are homogeneous on large scales after averaging over the fluctuations, although our results can be applied to inhomogeneous systems so long as the system size is large compared to the de Broglie wavelength. Relaxation of the FDM distribution may lead to the formation of a central soliton or Bose–Einstein condensate, but we do not study the formation of the condensate here.

The relaxation time of a test particle orbiting in a stellar system of density \( \rho_0 \) and velocity dispersion \( \sigma \), composed of classical particles of mass \( m \), is (Binney & Tremaine 2008; Equation (7.106))

\[
t_{\text{relax}} \approx 0.34 \frac{\sigma^3}{G^2 m \rho_0 \ln \Lambda},
\]

where \( \ln \Lambda \) is the Coulomb logarithm, with \( \Lambda \approx R/b \) where \( R \) is the size of the system and \( b \) is the larger of \( Gm/\sigma^2 \) and the size of the test particle. In an FDM halo the effective mass is \( m_{\text{eff}} \approx \rho_0 \lambda_d^3 \) so the relaxation of the FDM halo takes place on a timescale of

\[
t_{\text{relax}} \approx \frac{m_0^3 \sigma^6}{G^2 \rho_0^2 h^3 \ln (R/\lambda_d)}.
\]

Our aim is to place this approximate result on a solid quantitative foundation.

Many of our results have appeared already in the literature in several contexts: weak turbulence, the nonlinear Schrödinger equation, quantum plasmas, etc. (see, e.g., Levkov et al. 2018). Nevertheless, we have found it simpler and more transparent to provide self-contained derivations.

The present paper is organized as follows. In Section 2, we discuss the relaxation of particles and waves. In particular, we present a closed kinetic equation describing the self-consistent relaxation of a homogeneous FDM halo under the effects of its self-generated fluctuations. In Section 3, we present a first heuristic derivation of that kinetic equation relying on the Boltzmann–Nordheim–Uehling–Uhlenbeck (BNUU) equation. In Section 4, we revisit that same derivation, this time starting from a quasi-linear expansion of the coupled Schrödinger–Poisson system. In Section 5, we discuss the linear stability of the FDM halo, and in Section 6 we present some applications of this generalized kinetic equation. Finally, we conclude in Section 7.

## 2. Relaxation and Kinetic Equations

We shall work with an infinite halo that is homogeneous in an ensemble-averaged sense and characterized by a mean density \( \rho_0 \). We also invoke the Jeans swindle, that is, we ignore any acceleration due to \( \rho_0 \) (Binney & Tremaine 2008, Section 5.2.2), and focus only on the effects due to the density perturbations. We define the halo DF \( F_0(v) \) such that \( F_0(v) dv dv \) is the ensemble-averaged mass of halo particles in the phase space element \( dv dv \). Thus, the mean density and one-dimensional velocity dispersion are given by

\[
\rho_0 = \int dv F_0(v); \quad 3\rho_0 \sigma^2 = \int dv v^2 F_0(v).
\]

In order to highlight the connections between the classical and wave cases, we will successively consider the case of the relaxation of classical particles induced by a classical halo (Section 2.1), followed by the relaxation of classical particles induced by a fuzzy halo (Section 2.2), and finally the relaxation of a fuzzy halo induced by itself (Section 2.3).

### 2.1. Relaxation of Particles by Particles

First we review relaxation in a halo composed of classical particles of mass \( m \). Let \( F(v) \) be the DF of a population of point-like classical test objects of mass \( m \). The evolution of \( F(v) \) due to interactions between the test objects and the background halo particles is described by the classical Landau equation (Landau 1936; Lifshitz & Pitaevskii 1981; Chavanis 2013), which reads

\[
\frac{\partial F(v)}{\partial t} = 2G^2 \ln \Lambda \frac{\partial}{\partial v} \int dv' u_{ij}(v - v') \times \left[ mF(v') \frac{\partial F(v)}{\partial v_j} - m \frac{\partial F_b(v')}{\partial v_j} F(v) \right],
\]

where

\[
u_{ij}(v) = \int d\hat{k} \hat{k}_i \delta_D(\hat{k} \cdot v) = \frac{\partial^2 \nu}{\partial v_i \partial v_j} = \frac{\pi v^2 \delta_{ij} - v_i v_j}{v^3},
\]

is the collision kernel. Here \( \hat{k} = \hat{k}/k \) is a unit vector along \( \hat{k} \) and summation over repeated Cartesian indices is assumed. The Coulomb logarithm reads \( \ln \Lambda = \ln(k_{\text{max}}/k_{\text{min}}) \), with \( k_{\text{max}} \) and \( k_{\text{min}} \) the maximum and minimum wavenumbers that contribute to the relaxation. Typically \( k_{\text{min}}^{-1} \) is approximately the size of the halo or the radius of the orbit of the test object, and \( k_{\text{max}} \approx \sigma^2/[G(m + m_\text{r})] \) corresponds to the scale associated with strong deflections in two-body encounters. The first term in Equation (5) represents diffusion and is independent of the mass of the test objects, while the second term represents dynamical friction (also called friction due to polarization), and is independent of the mass of the halo particles at fixed halo density. This equation is equivalent to the standard FP equation of Chandrasekhar (1942).

### 2.2. Relaxation of Particles by Waves

Let us now assume that the diffusion of the classical test particles is sourced by a background fuzzy halo composed of ultra-light particles of individual mass \( m_{\text{r}} \), which must therefore be treated as waves.
As shown in Paper I, the Landau equation for the evolution of classical test particles becomes

\[
\frac{\partial F(v)}{\partial t} = 2G^2\ln\Lambda \frac{\partial}{\partial v_j} \int \! dv' u_{ij}(v - v')
\times \left\{ \left[ m_b + \frac{\hbar^2}{m_b} F_b(v') \right] \frac{\partial F(v)}{\partial v_j} - m_i \frac{\partial F_i(v')}{\partial v_j} F(v) \right\}.
\]

(7)

Here \( h = 2\pi\hbar \) and the factor \( k_{\text{max}} \) in the Coulomb logarithm is modified to \( k_{\text{max}} \sim \min\{m_m\sigma/h, \alpha^2/[G(m_t + m_b)]\} \). The similarity to Equation (5) is striking. The main difference arises in the diffusion term where the mass of the background classical particles \( m \) is now replaced with \([m_b + (h/m_b)^2] F_b(v')\).

2.3. Relaxation of Waves by Waves

The Landau equation can also be generalized to describe the self-consistent evolution of a halo DF composed of ultra-light fuzzy particles. In this case, Equation (7) becomes

\[
\frac{\partial F_b(v)}{\partial t} = 2G^2\ln\Lambda \frac{\partial}{\partial v_j} \int \! dv' u_{ij}(v - v')
\times \left\{ \left[ m_b + \frac{\hbar^2}{m_b} F_b(v') \right] \frac{\partial F_b(v)}{\partial v_j} - m_b \frac{\partial F_b(v')}{\partial v_j} F_b(v) \right\}.
\]

(8)

Once again, the similarity to Equation (7) is striking. The main difference arises in the friction component, where \( m_b \), the mass of the test particle, is now replaced with \([m_b + (h/m_b)^2] F_b(v')\) (see also Lancaster et al. 2020 for an extensive discussion of dynamical friction in FDM halos). Deriving Equation (8) self-consistently is one of the main goals of the present paper. We give a physically motivated derivation of this result relying on the BNUU equation in Section 3, and a more rigorous derivation starting from the quasi-linear expansion of the Schrödinger–Poisson system in Section 4. Before that, let us first discuss some of the main properties of Equation (8).

2.4. Some Properties of the Landau Equation

In this section, we briefly review some of the main dynamical properties of Equation (8) and its solutions, in particular the relaxation time (Equation (18)) and the associated Bose–Einstein steady state (Equation (24)).

First, we note that Equation (8), like Equations (5) and (7), is a flux-conservative equation, that is, it has the form \( \partial F_b/\partial t = -\partial F_b/\partial v_j \), with \( F_i \) being the mass flux in direction \( v_i \). Moreover, Equation (8) can be written as an FP equation:

\[
\frac{\partial F_b(v)}{\partial t} = -\frac{\partial}{\partial v_j} [D_{ij}(v) F_b(v)] + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_j} [D_{ij}(v) F_b(v)],
\]

(9)

where the flux is

\[
F_i = -D_i(v) F_b(v) + \frac{1}{2} \frac{\partial}{\partial v_j} [D_{ij}(v) F_b(v)].
\]

(10)

The first- and second-order diffusion coefficients are given by

\[
D_i(v) = D_i^c(v) + D_i^b(v); \quad D_i(v) = D_i^c(v) + D_i^b(v).
\]

(11)

Here, \( D_i^c(v) \) and \( D_i^b(v) \) are the classical diffusion coefficients, while \( D_i^b \) and \( D_i^b \) ("b" for boson) capture the contributions associated with wave interference. The first-order or "drift" coefficients read

\[
D_i^c(v) = 4G^2m_b\ln\Lambda \int \! dv' u_{ij}(v - v') \frac{\partial F_b(v')}{\partial v_j},
\]

\[
D_i^b(v) = 2G^2\frac{\hbar^2}{m_b} \ln\Lambda \int \! dv' u_{ij}(v - v')
\times \left[ \frac{\partial F_b(v')}{\partial v_j} + \frac{\partial F_b(v')}{\partial v_j} \right],
\]

(12)

while the second-order or "diffusion" coefficients are

\[
D_i^b(v) = 4G^2m_b\ln\Lambda \int \! dv' u_{ij}(v - v') F_b(v'),
\]

\[
D_i^b(v) = 4G^2\frac{\hbar^2}{m_b} \ln\Lambda \int \! dv' u_{ij}(v - v') F_b(v').
\]

(13)

We can rewrite these coefficients in terms of Rosenbluth potentials (Rosenbluth et al. 1957). We use the relations

\[
u_{ij}(v - v') = \pi \frac{\partial^2}{\partial v_i \partial v_j} |v - v'|;
\]

\[
\frac{\partial}{\partial v_i} u_{ij}(v - v') = 2\pi \frac{\partial}{\partial v_i} \frac{1}{|v - v'|}.
\]

(14)

Equation (12) then becomes

\[
D_i^c(v) = 8\pi G^2 m_b \ln\Lambda \frac{\partial}{\partial v_i} \int \! dv' F_b(v') \left[ \frac{1}{v - v'} \right],
\]

\[
D_i^b(v) = 4\pi G^2 \frac{\hbar^2}{m_b} \ln\Lambda \frac{\partial}{\partial v_i} \int \! dv' F_b(v') \left[ \frac{1}{v - v'} \right]
+ F_b(v) \frac{\partial}{\partial v_i} \int \! dv' F_b(v') \left[ \frac{1}{v - v'} \right]
+ F_b(v) \frac{\partial}{\partial v_i} \int \! dv' F_b(v') \left[ \frac{1}{v - v'} \right],
\]

(15)
while Equation (13) becomes

\[
D_{ij}^b(v) = 4\pi G^2 m_b \ln \Lambda \frac{\partial^2}{\partial v_i \partial v_j} \int dv' \left| v - v' \right| F_b(v'),
\]

\[
D_{ij}^m(v) = 4\pi G^2 \frac{m^3}{\rho_0} \ln \Lambda \frac{\partial^2}{\partial v_i \partial v_j} \int dv' \left| v - v' \right| F_m^2(v'),
\]

\[
= 4\pi G^2 m_{eff} \ln \Lambda \frac{\partial^2}{\partial v_i \partial v_j} \int dv' \left| v - v' \right| F_{eff}(v').
\]

(16)

In these expressions, we introduced in particular an effective DF, \( F_{eff}(v) \), and an effective mass, \( m_{eff} \), through

\[
F_{eff}(v) = \frac{\int dv' F_b(v') F_b^2(v')}{\int dv' F_b^2(v')}; \quad m_{eff} = \frac{\hbar^3}{m_b} \frac{\int dv F_b^3(v)}{\int dv F_b(v)}.
\]

As already discussed in Paper I and recovered in Equation (16), the fuzzy diffusion term, \( D_{ij}^b(v) \), is identical to the classical one, \( D_{ij}^m(v) \), except that the mass of the particle and the DF are replaced by their effective counterparts \( m_{eff} \) and \( F_{eff} \). In particular, when the underlying DF is Maxwellian, these diffusion coefficients are the same as those in a halo of classical particles with mass \( m_{eff} = \rho_0 (\lambda_c / \sqrt{4\pi})^3 \) and effective velocity dispersion \( \sigma_{eff} = \sigma / \sqrt{2} \). As a result, in the fuzzy case, the relaxation time from Equation (2) becomes

\[
t_{\text{relax}} \simeq 0.34 \frac{\sigma_{eff}^3}{G^2 m_{eff} \rho_0 \ln \Lambda} = 0.34 \frac{m^3 b \sigma^6}{G^2(2\pi)^{3/2} h^3 F_0 \ln \Lambda}.
\]

(18)

Since the diffusion coefficient in the wave–wave Landau equation (the first term in Equation (8)) is the same as the diffusion coefficient in the particle-wave Landau equation (first term in Equation (7)), the wave–wave relaxation time is the same as the wave-particle relaxation time.

When the DF is isotropic, i.e., when \( F_b(v) \) depends only on \( v = |v| \), Equation (8) can be rewritten in an even simpler form. To do so, we rely once again on the properties of the Rosenbluth potentials; see Equations (L22) and (L23) of Binney & Tremaine (2008). In particular, we can write

\[
\int dv' u_{ij}(v - v')
= \begin{cases} 4\pi^2 \frac{\delta_{ij}}{v} - \frac{v_i v_j}{v^3} - \frac{v_i v_j}{v^3} & \text{for } v' < v, \\
\frac{8\pi^2 (v^2 v_j)}{3 v} & \text{for } v < v', 
\end{cases}
\]

for \( v' < v \), \( \delta_{ij} \) is the Kronecker delta symbol,

as well as

\[
\int dv' u_{ij}(v - v') v_j' = \begin{cases} 8\pi^2 \frac{v^2 v_j}{3 v} & \text{for } v' < v, \\
\frac{8\pi^2 v^3}{3 v} & \text{for } v < v', 
\end{cases}
\]

for \( v < v' \),

where we recall that the sum over \( j \) is implied in the second equation. In order to get the last relation, we used the fact that \( u_{ij}(v)v_j = 0 \) in conjunction with Equation (19).

Then, after some lengthy manipulations, Equation (8) becomes

\[
\frac{\partial F_b(v)}{\partial t} = \frac{16\pi^2 G^2 \ln \Lambda}{3 \rho_0} \frac{1}{v^2} \frac{d}{dv} \left[ \frac{1}{v} \frac{d F_b(v)}{dv} \int_0^v dv' v'^4 F_b(v') \right] \frac{m_b + \frac{\hbar^3}{m_b} F_b(v')}{m_b} + \frac{\hbar^3}{m_b} F_b(v') \int_0^v dv' v'^2 F_b(v')
\]

+ \frac{3F_b(v)}{m_b} \left[ m_b + \frac{\hbar^3}{m_b} F_b(v') \int_0^v dv' v'^2 F_b(v') \right].
\]

(21)

In the limit of high phase space density, i.e., many particles per unit phase–space cell of volume \( h^3 \), or

\[
\frac{\hbar^3}{m_b} F_b \gg 1, \quad \text{or} \quad m_{eff} \simeq \rho_0 \left( \frac{h}{m_b \sigma} \right)^3 \gg m_b,
\]

we can reduce Equations (8) and (21) to

\[
\frac{\partial F_b(v)}{\partial t} = 2G^2 \ln \Lambda \frac{h^3}{m_b} \frac{\partial}{\partial v} \int dv' u_{ij}(v - v') \frac{\partial F_b(v)}{\partial v_j} - \frac{3F_b^2(v)}{m_b} \frac{\partial F_b^2(v)}{\partial v_j}
\]

\[
\times \left[ F_b(v') \frac{\partial F_b(v)}{\partial v_j} - F_b(v) \frac{\partial F_b^2(v)}{\partial v_j} \right] \int_0^v dv' v'^4 F_b^2(v')
\]

\[
= \frac{16\pi^2 G^2 \ln \Lambda}{3 \rho_0} \frac{1}{v^2} \frac{d}{dv} \left[ \frac{1}{v} \frac{d F_b(v)}{dv} \int_0^v dv' v'^4 F_b^2(v') \right] \frac{m_b + \frac{\hbar^3}{m_b} F_b(v')}{m_b} + \frac{\hbar^3}{m_b} F_b(v') \int_0^v dv' v'^2 F_b(v')
\]

\[
+ \frac{3F_b^2(v)}{m_b} \int_0^v dv' v'^2 F_b(v').
\]

(23)

The second of these is equivalent to Equation (S23) in Levkov et al. (2018).

The generic steady state of Equation (21) is the Bose–Einstein DF,

\[
F_b(v) = \frac{m_b^4}{h^3} \left( 1 - \frac{1}{e^{\frac{v^2}{\beta}} + 1} \right).
\]

(24)

In that expression, the inverse temperature \( \beta \) and the fugacity \( z \) are determined from the conservation of mass and kinetic energy per unit volume, following Equation (4). In practice, one has

\[
\rho_0 = \frac{m_b^4 (2\pi)^{3/2}}{h^3 \beta^{3/2}} \text{Li}_3(z); \quad \rho_0 \sigma^2 = \frac{m_b^4 (2\pi)^{3/2}}{h^3 \beta^{3/2}} \text{Li}_{5/2}(z),
\]

(25)

with \( \text{Li}_n(z) = \sum_{k=1}^{\infty} z^k / k^n \) the polylogarithm. Combining these equations to eliminate \( \beta \), we can write

\[
\sigma^3 = \frac{h^3 \rho_0}{m_b^4 (2\pi)^{3/2} \text{Li}_{5/2}(z)}.
\]

(26)
which has a minimum value, $\sigma_c$, reached at $z = 1$,
\[
\sigma_c^2 = \frac{\zeta^{3/2}(5/2)}{\zeta^{2}(3/2)\rho_0} \frac{h^3}{m_b^3(2\pi)^{3/2}}\rho_0^{-3/2},
\]
\[
\simeq 1.09 \times 100 \text{ km s}^{-1} \text{ pc}^{-1} \text{ eV}^{-2},
\]
with $\zeta(x) = \text{Li}_1(x)$ the Riemann zeta function. The corresponding value of $\beta$ is then given by
\[
\beta_c = \frac{\zeta(5/2)}{\zeta(3/2)} \sigma_c^2 \frac{m_b}{\rho_0^{3/2}} \frac{2}{2\pi n^2},
\]
\[
\text{Systems with an initial velocity dispersion below $\sigma_c$ cannot occupy the steady state given by Equation (24). In this case the steady state is given by}
\[
F_b(v) = \frac{m_b^4}{h^3} \frac{1}{v^3n^3 - 1} + \rho_s \delta_D(v),
\]
where the last term is the Bose–Einstein condensate or soliton. In this expression, the inverse temperature follows from the second of Equations (25) and reads
\[
\beta^{3/2} = \frac{m_b^4(2\pi)^{3/2}}{h^3\rho_0\sigma_c^2} \zeta(5/2).
\]
Finally, the mass density in the soliton is determined from the conservation of mass and energy. As such, it is given by
\[
\rho_s = \left(1 - \frac{\beta^{3/2}}{\beta^{3/2}}\right) \rho_0 / \left(1 - \frac{\sigma_c^6/5}{\sigma_c^{6/5}}\right) \rho_0.
\]

For the typical boson masses $m_b = 10^{-21} - 10^{-22} \text{ eV}$ suggested for FDM, $\sigma_c$ is much larger than galaxy velocity dispersions and the thermal equilibrium state is always a condensate. In principle, the entire FDM halo should eventually either escape to infinity or collapse into a condensate, although in a cosmological context continued infall of new material would keep the condensate perturbed away from strict thermal equilibrium. As stated in Section 1, we do not consider the formation of the condensate or soliton in this paper.

3. Deriving the Kinetic Equation from the Boltzmann–Nordheim–Uehling–Uhlenbeck (BNUU) Equation

In this section we give a (relatively) simple derivation of Equation (8). The BNUU equation (Nordheim 1928; Uehling & Uehling 1933; Erdős et al. 2004; Chavanis 2004) is a heuristic generalization of the Boltzmann equation to quantum systems. For a homogeneous system, the BNUU equation reads
\[
\frac{\partial f(p_1)}{\partial t} = \int dp_2 dp_3 dp_4 \ S^{(4)}(p_1, p_2, p_3, p_4) \times \{ f(p_3)f(p_4)[1 + e^{h f(p_2)}][1 + e^{h f(p_3)}] - f(p_1)f(p_2)[1 + e^{h f(p_3)}][1 + e^{h f(p_2)}]\},
\]
with $\epsilon = 0$ for a classical system, $+1$ for bosons, and $-1$ for fermions. Here $p$ is the momentum and $f(p) dp$ is the ensemble-averaged number of particles in a phase space volume element $dp$ (the dependence of the DF $f(p)$ on time is not shown explicitly). The function $S^{(4)}$ describes the rate at which particles with momenta $p_1$ and $p_2$ are scattered to momenta $p_3$ and $p_4$. More precisely, $S^{(4)}(p_1, p_2, p_3, p_4) f(p_1)f(p_2) dp_1 dp_2 dp_3 dp_4$ is the rate per unit volume at which particles are scattered from the momentum-space volumes $dp_1$ and $dp_2$ into the volumes $dp_3$ and $dp_4$. For $\epsilon = 0$, Equation (32) reduces to the classical Boltzmann equation, and for $\epsilon = -1$ factors such as $1 - h^2 f(p_3)$ ensure that no particles are scattered into states that are fully occupied according to the Pauli principle.

Since momentum is conserved in collisions, the function $S^{(4)}$ must contain a factor $\delta\delta(p_3 + p_4 - p_1 - p_2)$ and we use this to carry out the integral over $p_3$. Then we replace $p_3$ by the momentum transfer $q = p_3 - p_1$ to obtain
\[
\frac{\partial f(p_1)}{\partial t} = \int dp_2 dq \ S(p_1 + \frac{1}{2} q, p_2 - \frac{1}{2} q) \times \{ f(p_1 + \frac{1}{2} q)f(p_2 - \frac{1}{2} q)[1 + e^{h f(p_1)}] - f(p_1)f(p_2)[1 + e^{h f(p_1 + q)}][1 + e^{h f(p_2 - q)}]\}.
\]
Here we have rewritten $S^{(4)}$ as a function of three variables, $S(p_1 + \frac{1}{2} q, p_2 - \frac{1}{2} q) \equiv S^{(3)}(p_1, p_2; p_1 + q, p_2 - q) \delta\delta(p_1 + q - p_2)$.

For gravitational scattering between particles of mass $m_b$, $S$ is given by (e.g., Goodman 1983)
\[
S_{\text{grav}}(a; b; q) = \frac{4G^2 m_b^5}{q^4} \delta_D[(b - a) \cdot q].
\]
In that expression, the delta function ensures that the relative momentum is conserved in the collision, $|p_1 - p_2| = |p_1 - p_2|$, and the factor $q^{-4}$ reflects the angular dependence of the Coulomb differential scattering cross-section, $|\sin^2(\theta/2)|^{-4}$, where $\theta$ is the scattering angle.

We now assume that $q$ is small (equivalent to the FP approximation of weak deflections) and that $S$ varies slowly with $q$ in its first two arguments. This allows us then to expand Equation (33) to second order in $q$. We abbreviate the notation by writing $f_1 = f(p_1)$, $f_{\partial q} = \partial f/p_1$, $f_{\partial t} = \partial f/\partial t$, $f_{\partial q_{i}} = \partial f/p_{i}$, etc. We also assume summation over repeated indices. Then
\[
\partial_t f(p_1) = \int dp_2 dp_3 dp_4 \ S^{(4)}(p_1, p_2, p_3, p_4) \times \left[ (f_{i} + q_{i} f_{i, t} + \frac{1}{2} q_{i} q_{i, t f}) f_{j} - q_{i} q_{j, t f} + \frac{1}{2} q_{k} q_{m} f_{i, t m f} \right] \times \left[ (1 + e^{h f_{j}})(1 + e^{h f_{j}}) - f_{i} f_{j} (1 + e^{h f_{i}}) \right]\times \left[ (1 + e^{h f_{j}})(1 + e^{h f_{j}}) - f_{i} f_{j} (1 + e^{h f_{i}}) \right]\times \left[ (1 + e^{h f_{j}})(1 + e^{h f_{j}}) - f_{i} f_{j} (1 + e^{h f_{i}}) \right];
\]
here the function $S(p_1 + \frac{1}{2} q, p_2 - \frac{1}{2} q; q)$ was only expanded to first order in $q$, as the terms within brackets vanish at zero order in $q$. In that expression, $S$ stands for $S(p_1, p_2; q)$ which is even in $q$ by Equation (34) (or more generally by detailed balance). As a result, the terms that are first order in $q$ vanish.
when integrated. Thus, Equation (35) simplifies to
\[
\partial_t f(p_1) = \int d\rho_2\, dq_2\, q_2j \left\{ \frac{1}{2}S[\partial_{r_1},\partial_{r_2},f_2(1 + e^{h^2j_2})]ight.
\]
\[+ \partial_{r_1}f_2(1 + e^{h^2j_2})
\]
\[+ \partial_{r_2}f_2(1 + e^{h^2j_2})]\]

Integrating the term involving \(\partial_{r_2}S\) by parts gives
\[
\partial_t f(p_1) = \int d\rho_2\, dq_2\, q_2j \left\{ \frac{1}{2}S[\partial_{r_1},\partial_{r_2},f_2(1 + e^{h^2j_2})]ight.
\]
\[+ \partial_{r_1}f_2(1 + e^{h^2j_2})
\]
\[= \frac{1}{2} \partial_{r_1}\int d\rho_2\, dq_2\, q_2j[\partial_{r_1},f_2(1 + e^{h^2j_2})]
\]
\[= \partial_{r_1}f_2(1 + e^{h^2j_2})\right\}.
\]
(36)

Following Equation (34), the integral over \(q\) yields
\[
\int dq\, q_2j\, S(p_1, p_2; q) = 4G^2m_1^{5}\ln \Lambda\, u_{ij}(p_1 - p_2),
\]
(38)

where \(u_{ij}\) is the collision kernel from Equation (6), and \(\Lambda = \frac{d_{\text{max}}}{d_{\text{min}}}\), with \(d_{\text{max}}\) and \(d_{\text{min}}\), the maximum and minimum impact parameters included in the integral.

Now change variables from momentum \(p\) to velocity \(v = p/m_b\), from \(v_1\) and \(v_2\) to \(v\) and \(v'\), and from the number density in position–momentum phase space \(f(p)\) to the mass density in position–velocity space \(F_b(v) = m_b^4 f(m_b v)\). We find
\[
\frac{\partial F_b(v)}{\partial t} = 2G^2\ln \Lambda \frac{\partial}{\partial v_j} \int d\nu' u_{ij}(v - v')
\]
\[\times \left\{ \left[ m_b + e \frac{h^2}{m_b}F_b(v') \right] \frac{\partial F_b(v)}{\partial v_j}
\]
\[- \left[ 1 + e \frac{h^2}{m_b}F_b(v') \right] \frac{\partial F_b(v')}{\partial v_j} \right\}.
\]
(39)

which proves that Equation (8) follows directly from the BNUU equation. Of course, the present derivation remains heuristic because it stems from the heuristic BNUU Equation (32). In the following section, we will present a more careful derivation of Equation (8), through a detailed study of the self-consistent dynamics of the wave function of an FDM halo.

4. Deriving the Kinetic Equation from the Schrödinger–Poisson Equation

We now revisit the derivation of the kinetic Equation (8) by studying the gravitational interaction of waves \(\psi(r, t)\) that evolve according to the Schrödinger–Poisson equations (Ruffini & Bonazzola 1969)
\[
i\hbar \frac{\partial \psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m_b} \nabla^2 \psi(r, t) + m_b\Phi(r, t)\psi(r, t),
\]
(40)
\[\nabla^2 \Phi(r, t) = 4\pi G |\psi(r, t)|^2 - \rho_0.
\]
(41)

In the Poisson equation, the term \(-\rho_0\) appears because of the Jeans swindle, already described in Section 2. We also note that the wave function \(\psi(r, t)\) is normalized so that \(|\psi(r, t)|^2\) is a mass density, i.e., \(|\psi(r, t)|^2 = \rho_0\).

We introduce the rapidly fluctuating function
\[
W_d(r, v, t) = \int \frac{ds}{(2\pi)^3} e^{-i\nu \cdot s} \psi \left( r + \frac{1}{2} /s /m_b, t \right)
\]
\[\times \psi^* \left( r - \frac{1}{2} /s /m_b, t \right),
\]
(42)
which can be thought of as a quantum analog to the discrete classical phase-space DF, \(F_d(r, v, t) = m_b \sum_{i=1}^{N} \delta_{\delta}(r - r_i(t)) \delta_{\nu}(v - v_i(t))\). When ensemble-averaged, \(W = \langle W_d \rangle\) becomes the Wigner function (Wigner 1932), the quantum analog to the ensemble-averaged mean phase-space DF, \(F_b = \langle F_d \rangle\), i.e., the DF whose kinetic equation we want to derive. Although these analogies with the quantum-mechanical case are helpful, we note that here we are dealing only with classical physics, as we are in the limit of large occupation numbers (Guth et al. 2015). Planck’s constant \(\hbar\) appears only in the combination \(\hbar/m_b\) which determines the dispersion relation of the waves. The limit of the large occupation number is also the working assumption in numerical simulations (Schive et al. 2014). In this limit, \(W(r, v, t)\) can be interpreted classically as the density of bosons in the infinitesimal volume \(d^3v\) (see, e.g., Section 5.1.1 of Levkov et al. 2018). With our conventions, we also note that \(W_d\) and \(W\) satisfy the normalizations
\[
\int dv\, W_d(r, v, t) = |\psi(r, t)|^2;\quad \int dv\, W(r, v, t) = \rho_0,\]
(43)
where we used the relation \(\int dv/(2\pi)^3 e^{-i\nu \cdot s} = \delta_{\nu s}\).

The time evolution of \(W_d(r, v, t)\) follows from the Schrödinger Equation (40). More precisely, using the relation
\[
\nabla^2 \psi \left( r \pm \frac{1}{2} /s /m_b, t \right)
\]
\[= \frac{\partial}{\partial s} \left[ \nabla \psi \left( r \pm \frac{1}{2} /s /m_b, t \right) \right],
\]
(44)
to perform an integration by parts with respect to \(ds\) in Equation (42), we obtain
\[
\frac{\partial W_d(r, v, t)}{\partial t} + v \cdot \nabla W_d(r, v, t) = i\frac{m_b}{\hbar} \int \frac{ds}{(2\pi)^3} e^{-i\nu \cdot s}
\]
\[\times \left[ \Phi \left( r - \frac{1}{2} /s /m_b, t \right) - \Phi \left( r + \frac{1}{2} /s /m_b, t \right) \right]
\]
\[\times \psi \left( r + \frac{1}{2} /s /m_b, t \right) \psi^* \left( r - \frac{1}{2} /s /m_b, t \right).
\]
(45)

We can rewrite Equation (45) in the shorter form
\[
\frac{\partial W_d(r, v, t)}{\partial t} + v \cdot \nabla W_d(r, v, t) = \mathcal{D}[\Phi(r, t), W_d(r, v, t)],
\]
(46)
where the nonlinear term from the rhs reads
\[
\mathcal{D}[\Phi(r, t), W_d(r, v, t)] = i \int d\omega \int B \left\{ \phi (k, \omega) \mathcal{D}[W_d(r, v, t)] \right\}.
\]
(47)
In this expression, the Laplace–Fourier transformed potential, \( \tilde{\Phi}(k, \omega) \), is introduced with the convention

\[
\tilde{\Phi}(k, \omega) = \int \frac{dr}{(2\pi)^3} \int_0^\infty dt \, e^{-i(kr - \omega t)} \Phi(r, t);
\]

\[
\Phi(r, t) = \int dk \int_B \frac{d\omega}{2\pi} e^{i(kr - \omega t)} \tilde{\Phi}(k, \omega),
\tag{48}
\]

where the Bromwich contour, \( B \), has to pass above all the poles of the integrand, i.e., \( Im(\omega) \) has to be large enough. In Equation (47), we also defined the finite-difference operator

\[
\widehat{D}_k[W_d(v)] = \frac{m_b}{\hbar} \left[ W_d'\left( v + \frac{\hbar}{2m_b} k \right) - W_d'\left( v - \frac{\hbar}{2m_b} k \right) \right].
\tag{49}
\]

We already note that in the limit \( \hbar \to 0 \), this finite-difference operator satisfies

\[
\widehat{D}_k[W_d(r, v, t)] \to k \cdot \frac{\partial W_d(r, v, t)}{\partial v};
\]

\[
\mathbb{D}[\Phi(r, t), W_d(r, v, t)] \to \nabla \Phi(r, t) \cdot \frac{\partial W_d(r, v, t)}{\partial v};
\tag{50}
\]

so that Equation (46) reduces to the classical collisionless Boltzmann equation (Binney & Tremaine 2008)

\[
\frac{\partial W_d}{\partial t} + v \cdot \nabla W_d - \nabla \Phi \cdot \frac{\partial W_d}{\partial v} = 0.
\tag{51}
\]

Our goal now is to describe how the unavoidable fluctuations in the system arising from interference patterns lead to the relaxation of the halo’s underlying mean DF. To pursue this, we perform a quasi-linear expansion of Equation (46). This is a standard procedure in kinetic theory and we generally follow the method presented in Chavanis (2012b) in the context of the relaxation of classical discrete self-gravitating systems.

We write the function \( W_d(r, v, t) \) as a perturbation around its ensemble average,

\[
W_d = W + f.
\tag{52}
\]

Since we used the Jeans swindle in Equation (41), we note that \( \Phi(r, t) \) is already the fluctuating potential. Separating the average and fluctuating components of Equation (46), we obtain two evolution equations for the instantaneous fluctuations, \( \partial f/\partial t \), and the system’s ensemble-averaged DF, \( \partial W/\partial t \). They read

\[
\frac{\partial f}{\partial t}(r, v, t) + v \cdot \nabla f(r, v, t) = \mathbb{D}[\Phi(r, t), W(r, v, t)],
\tag{53}
\]

\[
\frac{\partial W(r, v, t)}{\partial t} + v \cdot \nabla W(r, v, t) = \mathbb{D}[\Phi(r, t), f(r, v, t)];
\tag{54}
\]

where Equation (53) has been linearized in the perturbations by neglecting the quadratic term \( \mathbb{D}[\Phi(r, t), f(r, v, t)] \) therein. These equations are valid in the weak-coupling limit where \( t_d \ll t_{\text{relax}} \). In the present case, the dynamical time \( t_d \) and the relaxation time \( t_{\text{relax}} \) are approximately

\[
t_d \simeq \max \left[ \frac{\hbar}{m_b \sigma^2}, \left( G\rho_0 \right)^{-1/2} \right];
\]

\[
t_{\text{relax}} \simeq \min \left[ \frac{m_b^3 \sigma^6}{G^2 \rho_0^3 \Gamma^3}, \frac{\sigma^3}{G^2 m_b \rho_0} \right].
\tag{55}
\]

Here the classical dynamical time is \( (G\rho_0)^{-1/2} \) and the classical relaxation time is given by Equation (2); if the wavelike nature of the particles is important the dynamical time is \( \sim \lambda_\text{v} / \sigma \), the time taken for a particle traveling at the typical speed to cross a de Broglie wavelength, and the relaxation time is given by Equation (3).

In the weak-coupling limit, we can assume that \( W \) is a (quasi-)stationary solution of the l.h.s. of Equation (54), i.e., we can assume that \( W = W(v, t) \) is homogeneous and therefore only slowly changes in time. The l.h.s. of Equation (54) is then the ensemble-averaged collisionless Boltzmann equation that describes the free streaming of particles in the absence of any potential perturbations, while its rhs is the collision term. In particular, we note that since both the potential \( \Phi(r, t) \) and the perturbed DF \( f(r, v, t) \) are quadratic in the wave function, the collision term from Equation (54) involves a product of four factors \( \psi(r, t) \) evaluated at different locations, that must subsequently be ensemble-averaged.

Following these assumptions, we can take the Laplace–Fourier transform of Equation (53), which gives

\[
\tilde{f}(k, v, \omega) = -\frac{\widehat{D}_k[W_d(v)]}{\omega - k \cdot v} \tilde{\Phi}(k, \omega) + i \frac{\tilde{f}_0(k, v)}{\omega - k \cdot v},
\tag{56}
\]

where the Fourier transform with respect to space and the Laplace transform with respect to time are defined with the conventions of Equation (48). We also assumed that \( W(v) \) can be taken as constant on the timescales over which the fluctuations evolve. In Equation (56), we also introduced

\[
\tilde{f}_0(k, v) = \int \frac{dr}{(2\pi)^3} e^{-ikr} f_0(r, v),
\tag{57}
\]

as the Fourier transform of the fluctuations of the DF at the initial time, \( f_0(r, v) = f(r, v, t = 0) \).

Using the normalizations from Equation (43), we can rewrite the Poisson Equation (41) in the simple form

\[
\nabla^2 \Phi(r, t) = 4\pi G \int dv \, f(r, v, t),
\tag{58}
\]

which becomes in Fourier–Laplace space

\[
\tilde{\Phi}(k, \omega) = -\frac{4\pi G}{k^2} \int dv \, \tilde{f}(k, v, \omega).
\tag{59}
\]

We now have at our disposal Equations (56) and (59), which jointly couple the DF and potential fluctuations, \( \tilde{f}(k, v, \omega) \) and \( \tilde{\Phi}(k, v, \omega) \). Solving these self-consistently amounts then to accounting for collective effects, i.e., accounting for the ability of the system to amplify its own self-generated perturbations. To make progress, the traditional solution is to act on both sides of Equation (56) with the same operator as in the rhs of Equation (59). One immediately obtains

\[
\tilde{\Phi}(k, \omega) = \frac{1}{\epsilon(k, \omega)} \frac{4\pi G}{ik^2} \int dv \, \frac{\tilde{f}_0(k, v)}{\omega - k \cdot v},
\tag{60}
\]
where the dielectric function is

$$
\epsilon(k, \omega) = 1 - \frac{4\pi G k^2}{\omega - k \cdot v} \int dv \, \tilde{D}_k[W(v)].
$$

As in Equation (50), one can straightforwardly obtain the classical dielectric function through the substitution \( \tilde{D}_k[W] \rightarrow k \cdot \partial W/\partial v \). As usual, the dielectric function can be rewritten using Landau’s prescription. Following our convention from Equation (48), this amounts to making the replacement \( \omega_R \rightarrow \omega_R + i0^+ \) and using the Plemelj formula

$$
\frac{1}{\omega_R + i0^+} = \mathcal{P} \left( \frac{1}{\omega_R} \right) - i\pi \delta_D(\omega_R),
$$

for \( \omega_R \in \mathbb{R} \) and with \( \mathcal{P} \) being Cauchy’s principal value. We will further discuss the properties of that dielectric function in Section 5 when investigating the linear stability of the present system.

We can now turn to Equation (54) in order to relate the long-term evolution of the mean DF, \( \partial W(v, t)/\partial t \), to the correlations of the initial DF fluctuations, \( \tilde{f}_0(k, v) \). Starting from Equation (54), we can use the definition from Equation (47) to write

$$
\frac{\partial W(v, t)}{\partial t} = i \int dk \, dk' \, e^{i(k+k')r} \tilde{D}_k \times \left[ \int dw \, dw' \, \frac{1}{2\pi} \, \frac{1}{2\pi} \, e^{-iR \cdot \omega t} \langle \tilde{f}(k, v, \omega) \tilde{f}(k', v') \rangle \right],
$$

In principle, the rhs depends on position \( r \) but we shall argue below that this dependence vanishes. Since Equation (56) has two terms, we get two contributions,

$$
\frac{\partial W(v, t)}{\partial t} = F_1(v) + F_2(v),
$$

where we introduced

$$
F_1(v) = -i \int dk \, dk' \, e^{i(k+k')r} \tilde{D}_k \times \left[ \int dw \, dw' \, \frac{1}{2\pi} \, \frac{1}{2\pi} \, e^{-iR \cdot \omega t} \langle \tilde{f}(k, v, \omega) \tilde{f}(k', v') \rangle \right],
$$

$$
F_2(v) = -i \int dk \, dk' \, e^{i(k+k')r} \tilde{D}_k \times \left[ \int dw \, dw' \, \frac{1}{2\pi} \, \frac{1}{2\pi} \, e^{-iR \cdot \omega t} \langle \tilde{f}(k, v, \omega) \tilde{f}(k', v') \rangle \right],
$$

which, as we will show, respectively capture the contributions from the drift and diffusion components. To proceed further, we rewrite these two expressions using Equation (60). After some manipulations, these two components become

$$
F_1(v) = i \int dk \, dk' \, e^{i(k+k')r} \tilde{D}_k \left[ \frac{4\pi G}{k^2} \int dw' \int \frac{d\omega \, d\omega'}{2\pi} \, e^{-i(\omega + \omega')t} \langle \tilde{f}_0(k, v) \tilde{f}_0(k', v') \rangle \right],
$$

$$
F_2(v) = i \int dk \, dk' \, e^{i(k+k')r} \tilde{D}_k \left[ \frac{1}{2\pi} \, \frac{1}{2\pi} \, e^{-iR \cdot \omega t} \langle \tilde{f}_0(k, v) \tilde{f}_0(k', v') \rangle \right],
$$

To pursue the calculation further, we must now characterize the properties of the correlations of the initial fluctuations in the system. In a homogeneous system, such correlations can only depend on the positions \( r \) and \( r' \) of the two points through their difference \( r - r' \). Moreover, we assume that the particles’ velocities are chosen independently from the ensemble-averaged DF \( F_1(v) \), so the correlation function must vanish if \( v = v' \). Therefore, we can write

$$
\langle f_0(r, v) f_0(r', v') \rangle = \delta_D(v - v') C(r - r', v),
$$

$$
\langle \tilde{f}_0(k, v) \tilde{f}_0(k', v') \rangle = \delta_D(k + k') \delta_D(v' - v) \tilde{C}(k, v),
$$

with \( \tilde{C}(k, v) \) the Fourier transform of the correlation function \( C(r - r', v) \). Since the DF \( W(v, r, t) \) is real, \( f_0(r, v) \) is also real. Therefore \( C(r, v) \) is real and from its definition it is also an even function of \( r \). Thus, \( \tilde{C}(k, v) \) is real and an even function of \( k \).

We postpone to Appendix B the explicit calculation of the function \( \tilde{C}(k, v) \). Inserting Equation (67) in Equation (66), we get

$$
F_1(v) = -i \int dk \, \tilde{D}_k \left[ \frac{4\pi G}{k^2} \tilde{C}(k, v) \right] \times \left[ \int dw \, dw' \, \frac{1}{2\pi} \, \frac{1}{2\pi} \, e^{-iR \cdot \omega t} \langle \tilde{f}_0(k, v) \tilde{f}_0(k', v') \rangle \right],
$$

$$
F_2(v) = -i \int dk \, \tilde{D}_k \left[ \frac{1}{2\pi} \, \frac{1}{2\pi} \, e^{-iR \cdot \omega t} \langle \tilde{f}_0(k, v) \tilde{f}_0(k', v') \rangle \right],
$$

where we have used the relation \( \tilde{D}_k[W(v)] = -\tilde{D}_k[W(v)] \).

If we assume that the system is initially stable, then all the inverse Laplace transforms present in Equation (68) can be
explicitly computed, as detailed in Appendix A. We get
\begin{equation}
F_1(v) = i \int dk \hat{\Delta}_k \left[ \frac{4\pi G}{k^2} \frac{\hat{C}(k, v)}{|\epsilon(k, k \cdot v)|^2} \epsilon(k, k \cdot v) \right].
\end{equation}
\begin{equation}
F_2(v) = -i \int dk \hat{\Delta}_k \left( \frac{4\pi G^2}{k^4} \hat{\Delta}_k[W(v)] \int dv' \frac{\hat{C}(k, v')}{|\epsilon(k, k \cdot v')|^2} \right.
\times \left. \left\{ \frac{1}{k \cdot (v - v')} + i\pi \delta_D[k \cdot (v - v')] \right\} \right),
\end{equation}
\begin{equation}(69)\end{equation}

The dielectric function $\epsilon(k, k \cdot v)$ is given by Equation (61) and the Plemelj formula (62),
\begin{equation}
\epsilon(k, k \cdot v) = 1 - \frac{4\pi G}{k^2} \int dv' \frac{\hat{\Delta}_k[W(v')]}{k \cdot (v - v')}
+ \frac{4\pi^2 G}{k^2} \int dv' \frac{\delta_D[k \cdot (v - v')] \hat{\Delta}_k[W(v')]}{|\epsilon(k, k \cdot v')|^2}.
\end{equation}
\begin{equation}(70)\end{equation}

Since $\hat{\Delta}_k[W(v')]$ is an odd function of $k$, the contribution of the real part of the dielectric function integrates to zero in the expression for $F_2(v)$ (this result also follows from the physical argument that $F_1(v)$ must be real). Similar arguments can be used to simplify the expression for $F_2(v)$. All in all, we get
\begin{equation}
F_1(v) = -\pi \int dk \hat{\Delta}_k \left[ \frac{4\pi G^2}{k^4} \int dv' \delta_D[k \cdot (v - v')] \hat{\Delta}_k[W(v')] \right.
\times \left. \frac{\hat{C}(k, v)}{|\epsilon(k, k \cdot v')|^2} \right],
\end{equation}
\begin{equation}
F_2(v) = \pi \int dk \hat{\Delta}_k \left[ \frac{4\pi G^2}{k^4} \int dv' \delta_D[k \cdot (v - v')] \hat{\Delta}_k[W(v')] \right.
\times \left. \frac{\hat{C}(k, v')}{|\epsilon(k, k \cdot v')|^2} \right].
\end{equation}
\begin{equation}(71)\end{equation}

Glancing back at Equation (64), we can finally rewrite the kinetic equation as
\begin{equation}
\frac{\partial W}{\partial t} = \pi \int dk \hat{\Delta}_k \left[ \frac{4\pi G^2}{k^4} \int dv' \delta_D[k \cdot (v - v')] \hat{\Delta}_k[W(v')] \right.
\times \left. \left( \hat{\Delta}_k[W(v')] \hat{C}(k, v') - \hat{\Delta}_k[W(v')] \hat{C}(k, v) \right) \right].
\end{equation}
\begin{equation}(72)\end{equation}

Equation (72) is equivalent to Equation (9) in Kadomtsev & Pogutse (1970) except that there the gravitational interactions are replaced with Coulomb interactions, paying careful attention to the change of sign of the attraction, while the finite-difference operators are replaced with their classical limits as in Equation (50). In order to finalize the calculation, it now only remains to compute explicitly the autocorrelation function, $\hat{C}(k, v)$, as defined in Equation (67).

We compute this correlation function in Appendix B.
\begin{equation}
\hat{C}(k, v) = \frac{1}{(2\pi)^3} \left[ m_b + \frac{h^3}{m_b^3} W(v) \right] W(v).
\end{equation}
\begin{equation}(73)\end{equation}
As a consequence, Equation (72) becomes
\begin{equation}
\frac{\partial W(v)}{\partial t} = 2G^2 \int dk \int dv' \hat{\Delta}_k \left\{ \delta_D[k \cdot (v - v')] \hat{\Delta}_k[W(v')] \hat{\Delta}_k[W(v')] \right\}
\times \left[ \frac{\partial W(v)}{\partial v} \frac{m_b + \frac{h^3}{m_b^3} W(v')}{|\epsilon(k, k \cdot v')|^2} \right.
\times \left. \left( \frac{h^3}{m_b^3} W(v') - \frac{h^3}{m_b^3} W(v) \right) \right].
\end{equation}
\begin{equation}(74)\end{equation}

The FP equation is based on the approximation that deflections are weak or that the momentum change due to gravitational scattering is small (see discussion preceding Equation (35)). In the present context, this corresponds to the approximation that $k$ is small compared to the characteristic scale of changes in $W(v)$. In this case we can replace the discrete derivatives in $\hat{\Delta}_k[W(v')]$ with their continuous analogs, as in the first of Equations (50). Equation (74) then becomes
\begin{equation}
\frac{\partial W(v)}{\partial t} = 2G^2 \int dk \frac{\partial}{\partial v} \int dv' \frac{\partial W(v')}{\partial v} \left[ \frac{m_b + \frac{h^3}{m_b^3} W(v')}{|\epsilon(k, k \cdot v')|^2} \right.
\times \left. \left( \frac{h^3}{m_b^3} W(v') - \frac{h^3}{m_b^3} W(v) \right) \right].
\end{equation}
\begin{equation}(75)\end{equation}

Equation (75) is the main result of this section, a Balescu–Lenard type kinetic equation describing the self-consistent relaxation of a homogeneous FDM halo. In particular, we note that this kinetic equation involves the dielectric function, $1/|\epsilon|^2$, which describes how the fluctuations are dressed by collective effects.

In Section 5 we show that this system is unstable for perturbations with a wavenumber smaller than the effective Jeans scale $\sim \min(k_j, \bar{k}_j)$, where $k_j$ is the classical Jeans wavenumber and $\bar{k}_j$ its wave analog. Assuming that the system is much smaller than the effective Jeans allows us to neglect collective effects and set $1/|\epsilon|^2 \rightarrow 1$. Equation (75) finally becomes
\begin{equation}
\frac{\partial W(v)}{\partial t} = 2G^2 \int dv' \frac{\partial W(v')}{\partial v} \left[ \frac{m_b + \frac{h^3}{m_b^3} W(v')}{|\epsilon(k, k \cdot v')|^2} \right.
\times \left. \left( \frac{h^3}{m_b^3} W(v') - \frac{h^3}{m_b^3} W(v) \right) \right].
\end{equation}
\begin{equation}(76)\end{equation}

where $u_j(v)$ is defined in Equation (6). We have therefore reached our final result, as we have recovered the kinetic Equation (8) describing the self-consistent relaxation of a
homogeneous FDM halo. Once again, this result is valid only for systems with large occupation numbers (Guth et al. 2015).

### 5. Linear Stability

As for many physical systems, the dielectric function (61) is central to understanding the dynamical behavior of an FDM halo. To illustrate the utility of this function we explore the stability properties of the halo. Throughout this section, we will assume for simplicity that the unperturbed DF is isotropic, \( F_b(v) = F_b(\mathbf{v}) \).

In the particle limit, an infinite homogeneous system is susceptible to an instability characterized by the classical Jeans wavenumber \( k_j \) (Jeans 1902; Binney & Tremaine 2008). If the DF is Maxwellian,

\[
F_b(v) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} e^{-\mathbf{v}^2/(2\sigma^2)},
\]

then

\[
k_j = \left( \frac{4\pi G \rho_0 \hbar^2}{\sigma^3} \right)^{1/2},
\]

and perturbations with \( k \leq k_j \) are unstable, while ones with \( k > k_j \) are stable.

In contrast, a halo composed of waves rather than particles that has zero velocity dispersion (i.e., the unperturbed wave function \( \psi(\mathbf{r}, t) = \text{ cst.} \)) is unstable to perturbations with wavenumber \( k < k_j \) where the Jeans wavenumber is (see, e.g., Khlopov et al. 1985; Bianchi et al. 1990; Hu et al. 2000; Chavanis 2011)

\[
k_j = 2 \left( \frac{\pi \rho_0 \hbar^2}{\sigma^2} \right)^{1/4}.
\]

In an FDM halo with non-zero velocity dispersion, the effective Jeans wavenumber can be determined from the dielectric function (61). For an isotropic DF,

\[
\epsilon(k, \omega) = 1 - \frac{4\pi G m_b}{\hbar^2} \int \frac{du}{\omega - ku} \times \left[ F_b(u + \frac{\hbar k}{2m_b}) - F_b(u - \frac{\hbar k}{2m_b}) \right],
\]

with \( F_b(u) = \int dv_1 dv_2 F_b[(v_1^2 + v_2^2 + u^2)^{1/2}] \) and \( v_1 \) and \( v_2 \) are the velocities along the two axes perpendicular to \( \mathbf{k} \). The system is linearly unstable if there exists a frequency \( \omega \) in the upper half of the complex plane and a real wavenumber \( k \) such that \( \epsilon(k, \omega) = 0 \). In order to investigate the system’s stability, we place ourselves at the limit of marginal stability, i.e., we assume that \( \text{Im}(\omega) \to 0 \). In that case, we can use the Plemelj formula from Equation (62) to rewrite Equation (80) as

\[
\epsilon(k, \omega) = 1 - \frac{4\pi G m_b}{\hbar^2} \int \frac{du}{\omega - ku} \times \left[ F_b(u + \frac{\hbar k}{2m_b}) - F_b(u - \frac{\hbar k}{2m_b}) \right]
+ \frac{4\pi G m_b}{\hbar^2} \left[ F_b(\frac{\omega}{k} + \frac{\hbar k}{2m_b}) - F_b(\frac{\omega}{k} - \frac{\hbar k}{2m_b}) \right].
\]

(81)

In order to have \( \epsilon(k, \omega) = 0 \), both the real part and the imaginary part of Equation (81) must vanish. Let us assume for simplicity that the DF \( F_b(v) \) is a monotonic decreasing function of \( v = |v| \) (i.e., we ignore two-stream instabilities). Then it is straightforward to show that \( F_b(u) \) is an even function of \( u \), monotonic decreasing for \( u > 0 \) and monotonic increasing for \( u < 0 \). As a consequence, the second term from Equation (81) vanishes if and only if \( k = 0 \) or \( \omega = 0 \). Because that first possibility is not of physical interest, we may then assume that \( \omega = 0 \) when investigating the system’s marginal stability.

For simplicity, let us now assume that the system’s unperturbed DF is Maxwellian (see Equation (77)). In that case, we can rewrite the dielectric function from Equation (80) as

\[
\epsilon(k, \omega) = 1 - \frac{4\pi G m_b \rho_0}{\sqrt{2\pi} \hbar^2} \int \frac{dv}{\omega - kv} \times \left[ e^{-|v+\hbar k/(2m_b)|^2/(2\sigma^2)} - e^{-|v-\hbar k/(2m_b)|^2/(2\sigma^2)} \right]
\]

\[
= 1 - \left( \frac{k_j}{k} \right)^3 \frac{1}{\sqrt{2\pi} \eta} \int \frac{dx}{x - \omega} \times \left[ e^{-|x-k\eta/(2\hbar)|^2} - e^{-|x-k\eta/(2\hbar)|^2} \right]
\]

\[
= 1 - \left( \frac{k_j}{k} \right)^3 \frac{1}{\eta} \left[ Z\left(\frac{\omega}{k} - \frac{k\eta}{2k_j}\right) - Z\left(\frac{\omega}{k} + \frac{k\eta}{2k_j}\right) \right]
\]

(82)

In that expression, we introduced the rescaled frequency \( \eta = \sqrt{2\pi} \frac{k_j^2}{\rho_0 \hbar^2} \), as well as the dimensionless ratio

\[
\eta = \sqrt{2\pi} \frac{k_j^2}{\rho_0 \hbar^2} = \sqrt{\frac{2\pi G \rho_0 \hbar}{\sigma^2 m_b}}
\]

\[
= 0.0315 \left( \frac{m_b}{10^{-22} \text{eV}} \right)^{-1} \left( \frac{\rho_0}{0.01\text{M}_\odot \text{pc}^{-3}} \right)^{1/2}
\]

\[
\times \left( \frac{\sigma}{100 \text{km s}^{-1}} \right)^{-2}.
\]

We note that \( \eta \ll 1 \) corresponds to the classical limit, while \( \eta \gg 1 \) is associated with the wave limit. In the last line of Equation (82), we introduced the plasma dispersion function (see, e.g., Fried & Conte 1961), defined as

\[
Z(\omega) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds \frac{e^{s^2}}{s - \omega}
\]

\[
= \frac{1}{\pi} \left[ \int_{-\infty}^{\infty} ds \frac{e^{s^2}}{s - \omega} + \pi ie^{-\omega^2} \right] \text{ if } \text{Im}(\omega) > 0,
\]

\[
= \frac{1}{\pi} \left[ \int_{-\infty}^{\infty} ds \frac{e^{s^2}}{s - \omega} + 2\pi ie^{-\omega^2} \right] \text{ if } \text{Im}(\omega) < 0.
\]

(84)

Following Equation (82), the requirement of marginal stability at \( \omega = 0 \) now leads to the implicit relation

\[
k_j^2 = \frac{k_j^3}{2\eta} \left[ Z\left(\frac{k_j \eta}{2k_j}\right) - Z\left(\frac{k_j \eta}{2k_j}\right) \right],
\]

(85)
with $k_s$ the critical wavenumber at marginal stability. For $x \in \mathbb{R}$, we can use the two expansions (Fried & Conte 1961)

$$Z(x) \simeq \begin{cases} i\sqrt{\pi} e^{-x^2} - 2x & \text{for } |x| \ll 1, \\ i\sqrt{\pi} e^{-x^2} - \frac{1}{x} & \text{for } |x| \gg 1. \end{cases}$$

Then we can approximate the rhs of Equation (85) as

$$k_c \simeq \begin{cases} k_j & \text{for } \eta \ll 1, \\ \hat{k}_j & \text{for } \eta \gg 1. \end{cases}$$

We therefore recover the known result that in the limit $\eta \ll 1$ ($k_j \ll \hat{k}_j$), the system’s stability is determined by classical physics and the critical wavenumber is $k_j$, while in the limit $\eta \gg 1$ ($k_j \gg \hat{k}_j$), stability is dominated by wave effects and the critical wavenumber is $\hat{k}_j$. This behavior is illustrated in Figure 1, where we show the critical wavenumber for a Maxwellian DF as determined from Equation (85), as a function of the ratio $k_j/\hat{k}_j$.

Chavanis (2011) derived a simple dispersion relation by assuming that the fuzzy halo is a fluid with a sound speed $c_s$. Assuming that this parameter is a proxy for the halo velocity dispersion, i.e., $c_s = \sigma$, Equation (138) of Chavanis (2011) gives the simple dispersion relation

$$k_c^2 = \frac{k_j^2}{\eta^2} \left[ \sqrt{1 + 2\eta^2} - 1 \right],$$

so that, in this model, perturbations with $k > k_c$ are stable, while ones with $k < k_c$ are unstable. The prediction of Equation (88) is also illustrated in Figure 1; it correctly recovers the transition between the particle and wave regimes as one varies the ratio $k_j/\hat{k}_j$.

In Figure 2, we illustrate the importance of the dressing of relaxation by collective effects, through the factor $1/|(\kappa(k, \omega)|^2$ as appearing in Equation (75). Here the dielectric function $\epsilon(k, \omega)$ is given by Equation (82). We note that this dressing becomes negligible (i.e., $\epsilon(k, \omega) \to 1$) as the system becomes more stable (i.e., $k \gg k_c$). Moreover, for a given value of $k/k_c$, as quantum effects become more important (i.e., $\eta \gg 1$) collective effects become negligible, $\epsilon(k, \omega) \sim \eta^{-1}$.

### 6. Numerical Application

In this final section we present some numerical simulations of a spatially homogeneous FDM system and compare them with the predictions from the kinetic theory derived above. In order to investigate the long-term relaxation of the system, we also present direct time integrations of the diffusion equation itself, in particular highlighting the unavoidable formation of the central soliton for cold enough initial conditions.

We use numerical methods similar to those of Levkov et al. (2018). Importantly, in order to evade the Jeans instability, we assume $G < 0$, which ensures that the system is stable and does not affect kinetic equations such as (8) as they only contain $G^2$. We consider a three-dimensional box of length $L$ that is discretized in $K^3$ cells. Each location on the grid is characterized by a position $r = \Delta \cdot \mathbf{n}$, with $\Delta = L/K$ and $\mathbf{n} \in \{0, \ldots, K - 1\}^3$. At each grid location, we track the local value of the wave function $\psi(t)$ as well as the gravitational potential $\Phi(t)$.

The initial conditions are set so that the wave function approximates a uniform density Maxwellian DF, as defined in Equation (77). To do so, we naturally perform the discrete Fourier expansion

$$\tilde{\psi}_k(t) = \frac{\Delta r^3}{(2\pi)^3} \sum_r \psi_r(t) e^{-ikr}; \quad \psi_r(t) = \Delta^3 \sum_k \tilde{\psi}_k(t) e^{ikr},$$

where we introduced $k = \Delta \cdot \mathbf{n}$, with $\Delta = 2\pi/L$ and $\mathbf{n} \in \{-K/2, \ldots, -1, 1, \ldots, K/2\}^3$. Each of the Fourier wavenumbers is then initialized with

$$\tilde{\psi}_k(0) = \sqrt{f_k(\mathbf{k})} \epsilon_k; \quad f_k(k) = \frac{h^3}{\Delta^3 m_0^3} F_h(v = \hbar k/m_0),$$

where $\phi_k$ is a random phase uniformly distributed in $[0, 2\pi]$ and uncorrelated on the $k$-grid, that is $\langle \phi_k \phi_{k'} \rangle \propto \delta_{kk'}$. Once the
wave function is known, we can compute its associated density $|\psi|^2$. This is subsequently used in the Poisson equation (40) to estimate the potential $\Phi$. The calculation of the potential is performed in Fourier space, using an FFT (i.e., assuming periodic boundary conditions), and further accelerated by GPU.

Once $\psi(t)$ and $\Phi(t)$ are known, we may proceed with the forward integration in time of the Schrödinger–Poisson equations. This is performed through appropriate sequences of kick and drift operators, given by

\begin{align*}
\text{Drift:} & \quad \hat{\psi}_k \rightarrow \hat{\psi}_k e^{-i\Delta \Phi_k^2/(2m_b)} \\
\text{Kick:} & \quad \psi_r \rightarrow \psi_r e^{-i\Delta m_b \Phi_r/\hbar}.
\end{align*}

The timestep $\Delta t$ and order of the integrator have to be picked carefully. In practice, we used a sixth-order explicit symplectic integrator (Yoshida 1990). Once we are able to perform numerical simulations of the system, we measure the value of $\partial F_0(v=\hbar k/m_b)/\partial t$ by fitting the function $t \rightarrow \langle \hat{\psi}_k(t) \rangle^2 \Delta \mu^2 m_b^2 /\hbar^2$ with a linear function of $t$, and performing an ensemble average over $10^3$ realizations with different initial conditions.

Having integrated the FDM dynamics, we may now compare the results with the prediction from kinetic theory. In the limit where collective effects can be neglected, the system's relaxation is described by Equation (8). In particular, for an isotropic system with a Maxwellian DF, as in Equation (77), we note that the flux generated by the classical diffusion coefficients $D^c_k$ and $D^c_k$ (see Equation (11)) vanishes exactly, so the only surviving contributions are from the wave diffusion coefficients $D^b_k$ and $D^b_k$. With this simplification we can obtain from Equation (21) the diffusion flux at $t = 0$ through

\begin{equation}
\frac{\partial F_0(v)}{\partial t} \bigg|_{t=0} = \frac{(2\pi)^{1/2} G^2 \hbar^3 v^3 \ln \Lambda}{2m_v \sigma^3} \times \left[ 18 e^{-3\sigma^2/(2\sigma^2)} + \pi^{1/2} \sigma^{-1/2} e^{-v^2/(2\sigma^2)} \text{erf}(v/\sigma) \\
- 2 L/\pi^{1/2} \sigma^{-1/2} e^{-v^2/(2\sigma^2)} \text{erf}[v/(2/\sigma^2)] \right],
\end{equation}

an exact quantitative expression of the approximate relaxation time (3).

In Figure 3, we compare the numerical simulations with the prediction from Equation (92), using an estimate $\ln \Lambda \approx 2.1$.

As illustrated in this figure, we recover a good agreement between the numerical simulations and the kinetic prediction.

In addition to integrating the Schrödinger–Poisson equations directly, we investigated the evolution of the DF itself by directly integrating forward in time the isotropic Landau Equation (21). To do so, we divide the interval $0 \leq v \leq v_{\text{max}}$ onto a regular grid. At each of the grid locations, the isotropic integrals from Equation (21) are computed using explicit second-order integration rules. Once the evolution rate, $\partial F_0(v)/\partial t$, has been determined on the velocity grid, we integrate it forward in time using a first-order explicit Euler method with a timestep given by $\Delta t = 10^{-3} \times \min\{F_0(v)/(\partial F_0(v)/\partial t)\}$.

Using that method, we show in Figure 4 that a Maxwellian DF with $\sigma = 1.3\lambda_c$ (see Equation (27)) relaxes to a Bose–Einstein steady state in a few relaxation times (as defined in Equation (18)). In that same figure, we also recover that when $\sigma = \sigma_c$, the DF develops a $F_b \propto 1/v^2$ cusp at small $v$ on finite time, so that the system is almost on the verge of forming the central soliton.

### 7. Conclusions

In this paper, we investigated the self-consistent relaxation of an FDM halo driven by the unavoidable and undamped fluctuations that it must sustain from interference patterns. The main result was presented in Equation (8), which is the appropriate generalization of the FP equation for particles to waves. We showed how this kinetic equation can be derived either from the heuristic BNUU equation (Section 3) or from the quasi-linear perturbation of the Schrödinger–Poisson system (Section 4). We showed in particular how the diffusion can be accelerated through collective effects that can dress the perturbations. The strength of the collective effects is encapsulated in the dielectric function (Equation (61)), as illustrated in Equation (75) with a Balescu–Lenard type kinetic equation. We subsequently described in Section 5 the linear stability of a homogeneous FDM halo, making clear the connection between the classical and quantum limits. Finally, we illustrated some of these results in Section 6 using tailored numerical simulations.

Of course, the present paper is only a first step toward a description of the evolution of an FDM halo. First, it is important to extend the present derivation to inhomogeneous FDM halos. So long as the typical de Broglie wavelength $\lambda_c$ is small compared to the size of the halo, a good first approximation would be to treat the FDM diffusion coefficients (15) and (16) as local diffusion coefficients in an inhomogeneous FP equation analogous to (9). For more accurate analyses, one would benefit in particular from the recent progress in the context of classical self-gravitating systems that recently led to the derivation of the inhomogeneous Balescu–Lenard equation (Heyvaerts 2010; Chavanis 2012a). As highlighted in Figure 4 where we numerically integrated the kinetic Equation (21) forward in time, FDM halos with a sufficiently small velocity dispersion—which includes all galaxy velocity dispersions if the particle mass $m_0 \ll 1$ eV—unavoidably relaxes to a Bose–Einstein condensate.
Describing the formation and growth of that condensate is essential for understanding the long-term fate of FDM halos. Finally, should FDM prove to be a viable alternative to classical CDM, the present kinetic theory needs to be implemented in a cosmological context (see, e.g., Amin & Mocz 2019).

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Appendix A
Computing the Inverse Laplace Transforms

In this section, we compute explicitly the Laplace transforms appearing in the expression for $F_2(v)$ in Equation (68). We want to evaluate

$$I(k, v) = \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{1}{\epsilon(-k, \omega') (\omega + k \cdot v) (\omega' + k \cdot v)} e^{-i(\omega + \omega')t}.$$  

(A1)

Recall that each integral is along a Bromwich contour $\mathcal{B}$, a horizontal contour that passes above all the poles of the integrands. We assume that the system is linearly stable, so the function $\omega \rightarrow 1/\epsilon(k, \omega)$ has no poles in the upper half of the complex plane. We carry out the integrals by lowering the integration contours to very negative imaginary values, so that $e^{-i(\omega + \omega')t}$ vanishes. Assuming that $t$ is large enough for the transients associated with the system’s damped modes (i.e., the contributions from the poles of the dielectric functions) to be negligible, only the contributions from the poles on the real axis $\omega = k \cdot v$ and $\omega' = -k \cdot v$ remain. Paying careful attention to the direction of integration, we note that these poles each contribute $-2\pi i$ times the associated residue. Using these arguments, we obtain

$$I(k, v) = -\frac{\epsilon(k, k \cdot v)}{|\epsilon(k, k \cdot v)|^2}. \quad (A2)$$

Here we have used the symmetry relation

$$\epsilon(-k, -\omega_R) = \epsilon^*(k, \omega_R), \quad (A3)$$

for $\omega_R \in \mathbb{R}$, which directly follows from Equation (61).

The second integral to compute appears in the expression for $F_2(v)$ in Equation (68). We must evaluate

$$J(k, v, v') = \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \frac{1}{\epsilon(k, \omega) \epsilon(-k, \omega')} \frac{e^{-i(\omega + \omega')t}}{(\omega - k \cdot v)(\omega - k \cdot v')(\omega' + k \cdot v')}. \quad (A4)$$

Using the same approach as in Equation (A1), we first perform the integral over $\omega'$, noting that it involves a single pole along the real axis at $\omega' = -k \cdot v'$, the other poles being damped. Equation (A4) then becomes

$$J(k, v, v') = -i e^{k \cdot v'} \frac{\epsilon^*(k, k \cdot v')}{\epsilon^*(k, k \cdot v')} \times \int \frac{d\omega}{2\pi} \frac{1}{\epsilon(k, \omega) (\omega - k \cdot v)(\omega - k \cdot v')} e^{-i\omega t}. \quad (A5)$$

For the remaining integral there are two poles on the real axis, which each contribute $-2\pi i$ times the associated residue. We
obtain

\[ J(k, v, v') = -\frac{e^{i(k \cdot v')}}{e^{\varphi(k \cdot v')}} \times \left[ \frac{1}{\epsilon(k, k \cdot v)} \frac{e^{-i(k \cdot v')}}{k \cdot (v - v')} \right] \]

\[ + \frac{1}{\epsilon(k, k \cdot v')} \frac{e^{-i(k \cdot v')}}{k \cdot (v' - v)} \]

\[ = \frac{1}{\epsilon(k, k \cdot v')^2} \frac{1}{k \cdot (v - v')} \times \left\{ P \left[ \frac{1}{k \cdot (v - v')} + i\pi\delta_D[k \cdot (v - v')] \right] \right\}. \tag{A6} \]

Relying on our assumption of timescale separation, i.e., the assumption that the fluctuations evolve on timescales much faster than the mean system, we can take the limit \( t \to \infty \) of Equation (A6). We then use the identity

\[ \lim_{t \to \infty} e^{ixt} = -i\pi\delta_D(x) \tag{A7} \]

to simplify Equation (A6) to

\[ J(k, v, v') = \frac{1}{\epsilon(k, k \cdot v')^2} \]

\[ \times \left\{ P \left[ \frac{1}{k \cdot (v - v')} + i\pi\delta_D[k \cdot (v - v')] \right] \right\}. \tag{A8} \]

**Appendix B**

**Computing the Correlations of the Potential Fluctuations**

In Section 4, we showed that the evolution of the FDM halo DF is sourced by the correlations of the initial fluctuations in the system. These correlations are described by the function \( \tilde{C}(k, v) \), the Fourier transform of the correlation function, as defined in Equation (67). Let us now explicitly compute this correlation.

In order to introduce that calculation, let us start by considering the classical case. In that regime, the system’s discrete DF is given by

\[ F_D(r, v, t) = m_b \sum_{i=1}^{N} \delta_D[r - r_i(t)] \delta_D[v - v_i(t)], \tag{B1} \]

where at the initial time, the phase-space positions and velocities of the particles are drawn independently from another, uniformly in space, and according to the DF \( F_b(v) \) for their velocities. Similarly to Equation (52), the instantaneous fluctuations in the system’s DF are given by \( f = F_D - F_b \). At the initial time, we can then write

\[ \langle f_0(r, v) f_0(r', v') \rangle = m_b^2 \sum_{i,j} \delta_D(r - r_i) \delta_D(v - v_i) \]

\[ \times \delta_D(r' - r_j) \delta_D(v' - v_j) - F_b(v) F_b(v'), \tag{B2} \]

where we dropped the time dependence \( (t = 0) \) to shorten the notation. As the particles are chosen independently, there are two types of terms in the double sum, depending on whether \( i = j \) or \( i \neq j \). We then get

\[ \langle f_0(r, v) f_0(r', v') \rangle = m_b^2 \delta_D(r - r') \delta_D(v - v') \]

\[ \times \sum_i \delta_D(r - r_i) \delta_D(v - v_i) \]

\[ + m_b^2 \sum_{i,j} \delta_D(r - r_i) \delta_D(v - v_i) \delta_D(r' - r_j) \delta_D(v' - v_j) \]

\[ - F_b(v) F_b(v'). \tag{B3} \]

Since \( F_b \) obeys the normalization convention \( \int dv F_b(v) = \rho_0 \), we have

\[ \langle \delta_D(r - r_i) \delta_D(v - v_i) \rangle = \frac{1}{N m_b} F_b(v). \tag{B4} \]

As a consequence, in the limit \( N \gg 1 \), the last two terms in Equation (B3) cancel and we have

\[ \langle f_0(r, v) f_0(r', v') \rangle = m_b F_b(v) \delta_D(r - r') \delta_D(v - v'). \tag{B5} \]

Following the convention from Equation (48), this can be rewritten in Fourier space as

\[ \langle \tilde{f}_0(k, v) \tilde{f}_0(k', v') \rangle \]

\[ = \frac{1}{(2\pi)^3} m_b F_b(v) \delta_D(k + k') \delta_D(v - v'), \tag{B6} \]

and the needed correlation function from Equation (67) is then

\[ \tilde{C}(k, v) = \frac{1}{(2\pi)^3} m_b F_b(v). \tag{B7} \]

We note that this correlation function is independent of \( k \), a consequence of our assumption that the initial positions of the particles are chosen independently from a homogeneous distribution.

Let us now adapt this calculation to the FDM case and compute the statistics of the persistent fluctuations present in the halo. We consider the following wave function

\[ \psi(r, t) = \int dk \ \varphi(k) e^{ik \cdot r - \omega(k)t}. \tag{B8} \]

In the limit where the potential fluctuations \( \Phi(r, t) \) vanish, this wave function is a solution of the free Schrödinger equation provided that it satisfies the dispersion relation

\[ \omega(k) = \frac{\hbar k^2}{2m_b}. \tag{B9} \]

Let us now assume that the wave function in \( k \)-space, \( \varphi(k) \), is the sum of Gaussian wavepackets of the form

\[ \varphi(k) = A \sum_{i=1}^{N} e^{i\phi_i} e^{-ik \cdot r_i - \omega(k) m_b \varepsilon^2 / 2}. \tag{B10} \]

In that expression, \( \{r_i, v_i\} \) are random positions and velocities drawn independently from the DF \( F_b(v) \) and \( \{\phi_i\} \) are independent random phases. In addition, \( \varepsilon \) is an ad hoc parameter, so that \( \varepsilon \) and \( \hbar/(2m_b\varepsilon) \) are, respectively, the initial uncertainties in the positions and velocities. This parameter will prove to be useful in managing our asymptotic developments. In Equation (B10), we also introduced the prefactor \( A \) that is tuned to satisfy the normalization condition stemming from Equation (43), namely that \( \langle |\psi(r, t)|^2 \rangle = \rho_0 \).
Let us now determine the value of $A$. Starting from Equation (B8), we write

$$
\langle |\psi(r, t)|^2 \rangle = \int dk dk' \langle \varphi(k) \varphi^a(k') \rangle e^{i(k-k')r} e^{-i(\omega(k)-\omega(k'))}. \tag{B11}
$$

The two-point correlation function of $\varphi(k)$ is

$$
\langle \varphi(k) \varphi^a(k') \rangle = A^2 \sum_{i,j} \left\{ e^{i(\Delta_0 - \Delta')} \frac{\exp(-i(k-k') \cdot r)}{2E_0} \right\}
$$

Thus, using $\Delta = k - k' = \omega(k) - \omega(k')$, we have

$$
\langle \varphi(k) \varphi^a(k') \rangle = 0
$$

To get the second line we noted that $\langle \varphi(\xi, \xi') \rangle$ is non-zero only for $i=j$. Using Equation (B4), one can write

$$
\langle \varphi(k) \varphi^a(k') \rangle = \frac{A^2}{m_b} \int dv F_\delta(v) e^{-i(k-k') \cdot r} \exp(-\frac{1}{2}\frac{m_b}{\hbar^2} \Delta) \overline{F_\delta(v)} e^{-i(k-k') \cdot r} \exp(-\frac{1}{2}\frac{m_b}{\hbar^2} \Delta).
$$

We can now use this result to pursue the simplification of Equation (B11). We have

$$
\langle |\psi(r, t)|^2 \rangle = \frac{(2\pi)^3 A^2}{m_b} \int dv F_\delta(v) e^{-2 \frac{m_b}{\hbar^2} \Delta}.
$$

Recalling from Equation (43) that $\int dv F_\delta(v) = \rho_0$, we obtain the value of $A$ as

$$
A = \frac{m_b}{2\sqrt{\pi}} A^2.
$$

Relying on the definition from Equation (42), we can now compute the rapidly fluctuating function $W_d(r, v, t)$ associated with Equation (B8). Following some cumbersome calculations, it takes the form

$$
W_d(r, v, t) = \int \frac{ds}{(2\pi)^3} e^{i\omega s} \psi(r + \frac{1}{2}/m_b, t)
$$

$$
\times \psi^*(r - \frac{1}{2}/m_b, t)
$$

$$
= \frac{2m_b}{\hbar} \int dk dk' \varphi(k) \varphi^a(k') \delta_D(k - k')
$$

$$
\times \left( k_1 + k_2 - \frac{2m_b}{\hbar} v \right) e^{i(k_1 - k_2)(r - v)}
$$

$$
= m_b \left( \frac{m_b}{\pi \hbar} \int_{ij} \frac{e^{i(\xi, \xi')}}{\Delta} \exp \left[ -2\frac{m_b}{\hbar^2} \frac{1}{2}(v_i + v_j) - v^2 \right] \right.
$$

$$
\times \exp \left[ -\frac{i}{2} \frac{m_b}{\hbar^2} \Delta \frac{1}{2}(v_i - v_j) \cdot v \right] \left[ \frac{1}{2}(v_i + v_j) - v + vr \right]
$$

$$
\times \exp \left[ -\frac{i}{2} \frac{m_b}{\hbar^2} \Delta \right].
$$

(B16)

In the second line we have used the relation $\omega(k_2) - \omega(k_1) = \frac{1}{2}(\hbar/m_b)(k_2^2 - k_1^2) = \frac{1}{2}(\hbar/m_b)(k_2 + k_1) \cdot (k_2 - k_1)$; when multiplied by the delta function in that equation this reduces to $v \cdot (k_2 - k_1)$.

In order to check these calculations, let us now follow the same method as in Equation (B12) to compute the ensemble average of Equation (B16), and therefore the Wigner function, $W(r, v, t)$. Owing to the presence of the factor $e^{i(\xi, \xi')}$ in Equation (B16), only the contributions with $i=j$ remain. All in all, we obtain

$$
W(r, v, t) = \frac{(m_b)}{(\pi \hbar)} \int dv' F_s(v') \exp \left[ -2\frac{m_b}{\hbar^2} |v' - v|^2 \right]
$$

$$
\times \exp \left[ -\frac{i}{2} \frac{m_b}{\hbar^2} \Delta \frac{1}{2}(v_i - v_j) \cdot v \right] \left[ \frac{1}{2}(v_i + v_j) - v + vr \right]
$$

$$
\times \exp \left[ -\frac{i}{2} \frac{m_b}{\hbar^2} \Delta \right].
$$

(B17)

To get the last line, we assumed that $\epsilon \gg \tilde{\lambda}_0$, with $\sigma$ the system’s typical velocity; physically, this means that the size of the wavepacket is much larger than the typical de Broglie wavelength. In that limit, we can then use the replacement

$$
\exp(-\alpha \sigma^2) \approx \frac{\pi^{3/2}}{\alpha^{3/2}} \delta_D(v/\sigma),
$$

with $\alpha^{-1} = \hbar^2/(2\epsilon^2 m_b^2 \sigma^2)$ our small parameter. Thus we have recovered in Equation (B17) the known result that the ensemble-averaged Wigner function is the same as the ensemble-averaged DF, in the limit of large occupation numbers.

Having computed the rapidly fluctuating function, $W_d(r, v, t)$, in Equation (B16), we can now find the correlation of its fluctuations at the initial time, as required by Equation (67).
Following Equation (52), we write

\[
\langle f_0(r, v) f_0(r', v') \rangle = \langle W_0(r, v) W_0(r', v') \rangle - W(r, v) W(r', v'),
\]  

(B19)

where all the functions are evaluated at the initial time. Glancing back at Equation (B16), we note that \(W_0(r, v)\) can be rewritten in the shorter form

\[
W_0(r, v) = \sum_{i,j} e^{(\phi_i - \phi_j)} g_{ij}(r, v),
\]  

(B20)

where the expression for the function \(g_{ij}(r, v)\) naturally follows from Equation (B16). As a result, the ensemble average in the rhs of Equation (B19) takes the form

\[
\langle W_0(r, v) W_0(r', v') \rangle = \sum_{i,j,k,l} e^{(\phi_i - \phi_j + \phi_k - \phi_l)} g_{ij}(r, v) g_{kl}(r', v').
\]  

(B21)

Because the wavepackets are drawn independently, the phase term \(e^{(\phi_i - \phi_j + \phi_k - \phi_l)}\) is non-zero only in three cases, namely (i) \(i = j = k = l\), (ii) \(i = j\) and \(k = l\) with \(i \neq k\), and (iii) \(i = l\) and \(j = k\) with \(i \neq k\). In these cases, the ensemble average in the rhs of Equation (B21) as

\[
\langle W_0(r, v) W_0(r', v') \rangle = N \langle g_{ii}(r, v) g_{ii}(r', v') \rangle + N^2 \langle g_{ii}(r, v) g_{kl}(r', v') \rangle + N^2 \langle g_{kl}(r, v) g_{kl}(r', v') \rangle,
\]  

(B22)

where, in the last two terms, it is understood that \([r_i, v_i]\) and \([r_k, v_k]\) are two independent sets of random variables. Let us now compute in turn each of the terms appearing in Equation (B22). We can first write

\[
N \langle g_{ii}(r, v) g_{ii}(r', v') \rangle = m_b \left( \frac{m_b}{\pi \hbar} \right)^6 \int dv_i F_b(v_i)
\times \exp \left\{ - \frac{2\pi m_b^2}{\hbar^2} |v_i - v|^2 - \frac{1}{2} |v_i - v|^2 \right\}
\times \int dr_i \exp \left\{ - \frac{1}{2\epsilon} |r_i - r|^2 + |r_i - r'|^2 \right\}
= m_b \left( \frac{m_b}{\pi \hbar} \right)^6 \exp \left\{ - \frac{2\pi m_b^2}{\hbar^2} |v - v'|^2 \right\}
\times \int dv_i F_b(v_i) \exp \left\{ - \frac{4\pi m_b^2}{\hbar^2} |v_i - \frac{1}{2} (v + v')|^2 \right\}
\times \exp \left\{ - \frac{1}{2\epsilon} |r - r'|^2 \right\}
\times \int dr_i \exp \left\{ - \frac{1}{2\epsilon} |r_i - \frac{1}{2} (r + r')|^2 \right\}
\approx m_b \delta_0(r - r') \delta_0(v - v') W(v),
\]  

(B23)

where, to get the last line, we assumed once again that \(\epsilon \gg \lambda_\sigma\), and used the asymptotic replacement from Equation (B18). The second term from Equation (B22) reads

\[
N^2 \langle g_{ii}(r, v) g_{kl}(r', v') \rangle = \left( \frac{m_b}{\pi \hbar} \right)^6 \int dr_i dr_k F_b(v_i) F_b(v_k)
\times \exp \left\{ - \frac{2\pi m_b^2}{\hbar^2} |v_i - v|^2 - \frac{1}{2} |v_i - v|^2 \right\}
\times \int (m_b/\pi \hbar)^3 dr_i \exp \left\{ - \frac{2\pi m_b^2}{\hbar^2} |v_i - v'|^2 \right\}
\times \exp \left\{ - \frac{1}{2\epsilon} |r_i - r|^2 \right\}
= W(r, v) W(r', v'),
\]  

(B24)

where we used the result from Equation (B17). The last term from Equation (B22) then reads

\[
N^2 \langle g_{kl}(r, v) g_{kl}(r', v') \rangle = \left( \frac{m_b}{\pi \hbar} \right)^6 \int dr_i dr_k dr_i dr_k F_b(v_i) F_b(v_k)
\times \exp \left\{ - \frac{2\pi m_b^2}{\hbar^2} \left( \frac{1}{2} (v_i + v_k) - v \right)^2 + \frac{1}{2} (v_i + v_k) - v \right)^2 \right\}
\times \exp \left\{ - \frac{1}{2\epsilon} \left( \frac{1}{2} (r_i + r_k) - r \right)^2 + \frac{1}{2} (r_i + r_k) - r \right)^2 \right\}
\times \exp \left\{ - \frac{i m_b}{\pi} [(r_i - r_k) \cdot (v - v') - (r - r') \cdot (v_i - v_k)] \right\}
= \left( \frac{m_b}{\pi \hbar} \right)^6 \exp \left\{ - \frac{\pi m_b^2}{\hbar^2} |v - v'|^2 \right\}
\times \int dr_i dr_k \exp \left\{ - \frac{4\pi m_b^2}{\hbar^2} \left( \frac{1}{2} (v_i + v_k) - \frac{1}{2} (v + v') \right)^2 \right\}
\times \exp \left\{ - \frac{1}{2\epsilon} \left( \frac{1}{2} (r_i + r_k) - \frac{1}{2} (r + r') \right)^2 \right\}
\times \exp \left\{ - \frac{i m_b}{\pi} [(r_i - r_k) \cdot (v - v') - (r - r') \cdot (v_i - v_k)] \right\}
\]  

(B25)

At this stage, the asymptotic formula from Equation (B18) has to be used carefully because the complex exponential from Equation (B25) is rapidly fluctuating. To clarify this calculation, let us perform the change of variables

\[
\sigma_r = \frac{1}{2} (r_i + r_k), \quad \delta_r = r_i - r_k, \quad \sigma_v = \frac{1}{2} (v_i + v_k), \quad \delta_v = v_i - v_k.
\]  

(B26)

Equation (B25) becomes

\[
N^2 \langle g_{kl}(r, v) g_{kl}(r', v') \rangle = \left( \frac{m_b}{\pi \hbar} \right)^6 \exp \left\{ - \frac{\pi m_b^2}{\hbar^2} |v - v'|^2 \right\}
\times \exp \left\{ - \frac{1}{4\epsilon} |r - r'|^2 \right\}
\times \int d\sigma_r d\delta_r F_b(\sigma_r + \frac{1}{2} \delta_r) F_b(\sigma_r - \frac{1}{2} \delta_r)
\times \exp \left\{ - \frac{4\pi m_b^2}{\hbar^2} |\sigma_r - \frac{1}{2} (v + v')|^2 \right\}
\times \exp \left\{ - \frac{i m_b}{\pi} (r - r') \cdot \delta_r \right\}
\times \int d\sigma_v d\delta_v \exp \left\{ - \frac{1}{4\epsilon} |\sigma_v - \frac{1}{2} (r + r')|^2 \right\}
\times \exp \left\{ - \frac{i m_b}{\pi} \delta_v \cdot (v - v') \right\}
\]  

(B27)

In this expression, we note that the exponential factor \(\exp\left[-\frac{1}{4\epsilon} |r - r'|^2\right]\) is nearly zero unless \(|r - r'| \lesssim \epsilon\). In that
regime, the complex exponential \( \exp[-i \frac{m_b}{h} (r - r') \cdot \delta_e] \) will average to nearly zero unless \( |\delta_e| \ll \frac{h}{m_b} \). Our assumption that \( \varepsilon \gg \lambda_p \), we conclude that the dominant contribution to the integral comes from \( |\delta_e| \ll \sigma \). We recall that the typical variance of \( F_b(r) \) is \( \sigma \), so in Equation (B27) we may perform the replacement \( F_b(\sigma_e + \frac{1}{2} \delta_e) \approx F_b(\sigma_e) \). We get
\[
N^2 \langle g_b(r, v) g_b(r', v') \rangle \approx 2^b \delta_D(r - r') \delta_D(v - v')
\]
\[
\times \int d\sigma d\sigma' \exp \left[ -\frac{1}{h} |\sigma - \sigma'|^2 \right] F_b^2(\sigma_e)
\]
\[
\times \exp \left[ -\frac{4e^2 m_b^2}{h^2} |\sigma - \sigma'|^2 \right]
\]
\[
\approx \frac{h^2}{m_b^2} \delta_D(r - r') \delta_D(v - v') W^2(v), \tag{B28}
\]
Gathering together Equations (B23), (B24), and (B27), we can now rewrite the correlation from Equation (B19) to obtain
\[
\langle f_0(r, v) f_0(r', v') \rangle = \left[ m_b + \frac{h^2}{m_b} W(v) \right] W(v) \delta_D(r - r') \delta_D(v - v'). \tag{B29}
\]
Following Equations (48) and (67), we finally obtain the needed correlation function as
\[
\hat{C}(k, v) = \frac{1}{(2\pi)^3} \left[ m_b + \frac{h^2}{m_b} W(v) \right] W(v), \tag{B30}
\]
which reduces to the classical correlation function (B7) to \( h \to 0 \).

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