Classical singularities and Semi-Poisson statistics
in quantum chaos and disordered systems

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We investigate a 1D disordered Hamiltonian with a non-analytical step-like dispersion relation whose level statistics is exactly described by Semi-Poisson statistics (SP). It is shown that this result is robust, namely, does not depend neither on the microscopic details of the potential nor on a magnetic flux but only on the type of non-analyticity. We also argue that a deterministic kicked rotator with a non-analytical step-like potential has the same spectral properties. Semi-Poisson statistics (SP), typical of pseudo-integrable billiards, has been frequently claimed to describe critical statistics, namely, the level statistics of a disordered system at the Anderson transition (AT). However we provide convincing evidence they are indeed different: each of them has its origin in a different type of classical singularities.

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The properties of a quantum particle in a random potential is one of the most intensively studied problems in condensed matter physics since the landmark paper by Anderson [1]. According to the one parameter scaling theory, in more than two dimensions, there exists a metal insulator transition for a critical amount of disorder. Unfortunately the AT in three and higher dimensions takes place in a region of strong disorder not directly accessible to current analytical techniques. Despite the lack of rigorous analytical results, it is by now well established, mainly through numerical simulations, that the AT is fully characterized by the level statistics well established, mainly through numerical simulations, of the number variance or the set of multifractals exponent $\alpha(q)$, $\beta(q)$, $\delta(q)$, and $\gamma(q)$.

The aim of this paper is to incorporate the systems described by SP into the classification introduced in [15] which relate quantum properties with singularities of the classical potential. We shall see that differences between SP and critical statistical stem from the fact that both come from different classical singularities. In order to proceed we propose a non-analytic Hamiltonian whose level statistics is exactly described by semi-Poisson statistics. It is then shown that differences between critical
and semi-Poisson statistics are due to the fact that both come from different type of singularity in the classical potential.

![Graph showing level spacing distribution for different N values](image)

**FIG. 1:** Level spacing distribution for $\gamma = \pi/2$ and $A = 10$ and different system sizes. The agreement with SP is impressive even in the tail of $P(s)$ (see Inset).

The models—We start our investigation with a generalized kicked rotor with a a step-like singularity,

$$
\mathcal{H} = \frac{p^2}{2} + V(q) \sum_n \delta(t - nT)
$$

with $V(q)$ a step function, $V(q) = \alpha; q \in [-\beta, \beta]$ and otherwise $V(q) = 0$; $\beta \in [-\pi, \pi]$ and $\alpha$ is a real number. By using the method introduced in [16] one can map the evolution matrix associated to Eq. (1) onto the following 1D Anderson model with long-range hopping,

$$
\mathcal{H}\psi_i = \epsilon_i\psi_i + \sum_j F(i - j)\psi_j
$$

where $\epsilon_i \sim \tan(\alpha^2)$ are pseudo random numbers (for $\alpha > 1$) and $F(m - n) = \int_0^\pi d\theta \tan(V(\theta))e^{-i\theta(m-n)} = \frac{A\sin(\gamma(m-n))}{m-n}$ with $A, W$ real positive constants related to $\beta, \alpha$. The case $\gamma = \pi/2$ describes the high energy limit of the interaction of a charged particle with quenched Coulomb scatterers of alternate sign.

In this paper we shall mainly investigate the Hamiltonian Eq. (2) assuming that $\epsilon_i$ is a random number extracted from a box distribution $[-W/2, W/2]$. We proceed so in order to make an accurate analysis of the level statistics necessary for a precise comparison with SP statistics. However a detailed study of the Hamiltonian Eq. (1) including the details of the multifractal spectrum and the quantum wave-packet evolution, will be published elsewhere [18].

Results—

We first state our main results:

1. For $A \gg W$ and any $\gamma$ in Eq. (2) the spectrum is scale invariant and the level statistics is exactly described by semi-Poisson statistics (these findings are in agreement with an unpublished calculations of Charles Schmit). 2. There is a transition from SP to Poisson statistics as the strength of disorder is increased. 3. In the region $A \gg W$ the eigenfunctions are multifractal but with a multifractal spectrum clearly different from the one observed at an AT.

We start by providing analytical evidence that the level statistics of Eq. (2) is described by SP statistics. We express the Hamiltonian (1) in Fourier space as,

$$
\mathcal{H} = E_k\psi_k + \sum_{k \neq k'} \hat{A}(k,k')\psi'_k
$$

where $E_k = \sum_{r} \sin(\gamma) e^{ikr}$ and $\hat{A}(k,k') = \frac{1}{N} \sum_{n} \epsilon_n e^{-in(k-k')}$. We fix $\gamma = \pi/2$ (our findings do not depend on $\gamma$), after a simple calculation we found that $E_k$ is not a smooth function (this step-like singularity is indeed the seed for the appearance of SP), $E_k = A\pi/2$ for $k < \pi$ and $E_k = -A\pi/2$ for $k > \pi$. There are thus only two possible values of the energy separated by a gap $\delta = A\pi$. Upon adding a weak ($A \gg W$) disordered potential this degeneracy is lifted and the spectrum is composed of two separate bands of size $\sim W$ around each of the bare points $-A\pi/2, A\pi/2$. Since the Hamiltonian is invariant under the transformation $A \rightarrow -A$, the spectrum must also posses that symmetry. That means that, to leading order in $A$ (neglecting $1/A$ corrections), the number of independent eigenvalues of Eq. (2) is $n/2$ instead of $n$.

We now show how this degeneracy affects the roots (eigenvalues) of the characteristic polynomial $P(t) = \det(H - tI)$. Let $P_{\text{dis}}(t) = a_0 + a_1 t + \ldots + a_n t^n$ be the characteristic polynomial associated with the disordered part of the Hamiltonian. We remark that despite of its complicated, its roots, by definition, are random numbers with a box distribution $[-W, W]$. On the other hand, in the clean case $P_{\text{clean}}(t) = (t - A)^{n/2} (t + A)^{n/2} \pi$ factors are not considered. Due to the $A \rightarrow -A$ symmetry the full (Eq. (2) case $P_{\text{full}}$ corresponds with $P_{\text{dis}}$ but replacing $t^k$ factors by a combination $(t - A)^{k_1} (t + A)^{k_2}$ with $k_1 + k_2 = k$. The roots of $P_{\text{full}}$ will be in general complicated functions of $A$. However in the limit of interest, $A \gg W \rightarrow \infty$, an analytical evaluation is possible. By setting $t = t_1 - A$ we look for roots $t_1$ of order the unity in the $A$ band. We next perform an expansion of the characteristics polynomial $P_{\text{full}}$ to leading order in $A$. Thus we keep terms $A^{n/2}$ and neglect lower powers in $A$. The resulting $P_{\text{full}}$ is given by

$$
P_{\text{full}} = t_1^{n/2} + a_n - 2t_1^{n/2} - 3 + a_n - 3t_1^{n/2} - 4 + \ldots + 2a_n/n +2t_1/n + 2a_n/n^2/(n + 2)
$$

where the coefficients $a_n$ are the same than those of $P_{\text{dis}}$ above but only $n/2$ of them appear in the full case. The eigenvalues $\epsilon'_i$ of Eq. (2) around the $A$ band are $\epsilon'_i = \epsilon_i + \beta_i$ with $\beta_i$ a root of $P_{\text{full}}$. The effect of the long range interaction is just to remove all the terms with coefficients $a_0$ to $a_{n/2}$ from the characteristic polynomial of the diagonal disordered case. The spectrum is thus that
of a pure diagonal disorder where half of the eigenvalues have been removed. The remaining eigenvalues are still symmetrically distributed (the ones with largest modulus are well approximated by $t_{\text{max}} = \pm \sqrt{\frac{n_A - 2}{3}}$) around $A$. That means, by symmetry considerations, that the removed ones must be either the odd or the even ones. This is precisely the definition of semi-Poisson statistics. In conclusion, the power-law random banded reproduces exactly the mechanism which is utilized in the very definition of semi-Poisson statistics. We finally mention that the only effect of the coefficients $3, 4, \ldots, n + 2/2$ is to renormalize the effective size of the spectrum, $\sim 2W$ for diagonal disorder and $\sim 2W/\sqrt{3}$ for the Eq. $\mathbb{E}$.

![FIG. 2: Number variance for different sizes $N$, $A$ and $\gamma$. Provided that $A \gg 1$ the number variance is given by SP statistics for any $N$, $A$ and $\gamma$. In the inset $P(s)$ is shown for $A = 10$, $N = 800$ and different $\gamma$. As shown $P(s)$ is not sensitive to the specific value of $\gamma$.](image)

The above analytical arguments have been fully corroborated by numerical calculations. By using standard diagonalization techniques we have obtained the eigenvalues of the Hamiltonian Eq. $\mathbb{E}$ for different volumes ranging from $N = 500$ to $N = 8400$. The number of different realizations of disorder is chosen such that for each $N$ the total number of eigenvalues be at least $5 \times 10^5$, in all cases $W = 1$. Eigenvalues close to the band edges (around 20%) were discarded from the statistical analysis. The eigenvalues thus obtained were unfolded (by using the splines method) with respect to the mean spectral density. We first investigate the level statistics in the region $A \gg W$ where, according to the analytical findings above, semi-Poisson statistics hold. As shown in Fig. 1, the level spacing distribution $P(s)$ (including the tail in the inset) does not depend on the system size for volumes ranging from $N = 500$ to $N = 8400$. Moreover though level repulsion $P(s) \propto s$ $s \ll 1$ is still present, the asymptotic decay of $P(s)$ (see inset) is exponential as for a insulator. All these features are spectral signatures of an AT.

The study of long range correlators as the number variance (see Fig. 2), $\Sigma^2(L) = \langle L^2 \rangle - \langle L^2 \rangle = \int_0^L \rho(x)dx - \int_0^L (L - s)R_2(s)ds$, ($\rho(x)$ is the spectral density) further confirms this point. It does not depend on the system size and its asymptotic behavior is linear $\Sigma^2(L) \sim 0.5L$ $L \gg 1$ as at the AT.

We now compare the level statistics of the Hamiltonian Eq. $\mathbb{E}$ with SP. As shown in Fig. 1 and Fig 2 (inset) we could not detect any perceptible deviation from the SP $P(s) = 4s e^{-2s}$ prediction for different $\gamma$’s and $A \gg W$. The agreement is impressive even for tail of $P(s)$ (inset). Also long-range correlators as the number variance follow the semi-Poisson prediction $\Sigma^2(L) = L/2 + (1 - e^{-4L})/8$ for different parameter values (see Fig.2). Deviation for small volumes are well known finite size effects.

A remark is in order, though the above analysis clearly show that the level statistics of our model is described by SP and share generic features of an disordered conductor at the AT there are still important quantitative differences. Level statistics at the AT depends on the dimension of the space. For instance in 3D (4D) the slope of the number variance is 0.27 (0.41), by contrast semi-Poisson predicts 0.5. Clear differences are also observed in short range correlators as $P(s)$ $\mathbb{F}$ where it has been found that, despite their similarities, critical and SP have different functional forms. The reason for such discrepancy is as follows:

As mentioned previously, level statistics at the 3D AT (critical statistics) is very accurately described by RMM $\mathbb{H}$ whose joint distribution of eigenvalues can be considered as an ensemble of free particles at finite temperature with a nontrivial statistical interaction. The statistical interaction resembles the Vandermonde determinant, and the effect of finite temperature is to suppress the correlations of distant eigenvalues. In SP this suppression is abrupt, in contrast to critical statistics, where the effect of the temperature is smooth. The reason for the differences between SP and critical statistical is thus due to the fact that the interaction among eigenvalues is not strictly restricted to nearest neighbors (SP) at the AT.

We now investigate the eigenvector properties of the Hamiltonian Eq. $\mathbb{E}$ We shall see that, though the they are to some extent multifractals, there exist important difference with respect to those at the AT. We have studied the scaling of the eigenfunction moments $P_q$ with respect to the sample size $L$. For multifrac-

![FIG. 3: $P(s)$ for $\gamma = \pi/2$, $N = 800$ and different $A$. A transition to Poisson statistics is observed in the limit $A \ll 1$.](image)
tal wavefunctions, $\langle P_q \rangle = \int d^4r |\psi(r)|^{2q} \propto L^{-D_q(q-1)}$ where the bracket stands for ensemble average and $D_q$ is a set of exponents describing the transition. Below we show the multifractal dimensions $D_q (\pm 10\% )$ for $A = 10, \gamma = \pi/2$ obtained by numerical fitting of $\langle \log P_2 \rangle$: $D_{1.5} \sim 0.36, D_2 \sim 0.30, D_{2.5} \sim 0.28, D_3 \sim 0.26, D_4 \sim 0.24, D_5 \sim 0.22, D_6 \sim 0.22$. For small $q$, $D_q$ depends clearly on $q$ however for larger $q$ the dependence is quite weak suggesting that may exist a critical $q_c$ such that for $q > q_c, D_q \in [0,1]$ is a constant. This situation would correspond with eigenstates which are truly multifractal only up to certain scale. Similar results have been recently reported for certain triangular billiards \[15\] \[16\] though in this case it was claimed that $D_q$ is constant for any $q$. We remark that at the 3D AT the eigenfunction are truly multifractal and consequently $D_q$ depends explicitly on $q$ for any $q$.

We now interpret the special properties of eigenfunctions and level statistics in the context of the original non-random Hamiltonian consisting of a kick rotor with a non-analytical step-like potential. From the above arguments it is clear that the Hamiltonian Eq. \[1\] avoids dynamical localization typical of a smooth potential due exclusively to the step-like singularity of the potential. In a recent paper we found that certain types of classical singularities induce quantum power-law localization of the corresponding eigenvectors. For the case of log singularities it was explicitly shown that the eigenvector were multifractals and the level statistics was given by critical statistics.

The results of this paper shows that SP can be considered as the level statistics associated with chaotic systems with classical step-like singularities (in $1+1$ dimensions) or with disordered systems with a step-like dispersion relation. We can thus unify different intermediate statistics (‘critical statistics’ and ‘semi-Poisson statistics’) in a broader classification based on the universal relation between classical singularities and level statistics features.

Finally we would like to discuss briefly two different issues. We have observed (see Fig. 3) that as $A$ becomes comparable to $W$ the level statistics shifts slowly toward Poisson. The level statistics in this region is still scale invariant and even for $A \ll W$ small deviations from Poisson are not negligible. This finding suggests that the model is indeed critical for all values of $\gamma$ and $A$. Another issue of interest is the robustness of our results under perturbations. We have added a flux to the Hamiltonian Eq. \[2\] in order to check whether the breaking of time reversal invariance has any impact on the level statistics. The results are negatives, we have only observed the effect of the flux in the $s \to 0$ limit of the level spacing distribution $P(s) \sim s^2$ instead $P(s) \sim s$.

In conclusion we have introduced a new class of systems with level statistics described by semi-Poisson statistics and multifractal wavefunctions. The appearance of semi-Poisson statistics has been related to a step-like singularities of the classical potential and to a singular step-like dispersion relation in a disordered system. We have discussed similarities and differences with critical statistics and claimed that both are part of a larger classification scheme. Finally we have discussed the transition to Poisson in our model and the effect of a flux on the level statistics.

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