On Limit Cycles in Supersymmetric Theories

Jean-François Fortin,*† Benjamin Grinstein,§iii Christopher W. Murphy,§iii and Andreas Stergiou§iv

*Theory Division, Department of Physics, CERN, CH-1211 Geneva 23, Switzerland
†Stanford Institute for Theoretical Physics, Department of Physics, Stanford University, Stanford, CA 94305, USA
§Department of Physics, University of California-San Diego, La Jolla, CA 92093, USA

Contrary to popular belief conformality does not require zero beta functions. This follows from the work of Jack and Osborn, and examples in non-supersymmetric theories were recently found by some of us. In this note we show that such examples are absent in unitary $\mathcal{N} = 1$ supersymmetric four-dimensional field theories. More specifically, we show to all orders in perturbation theory that the beta-function vector field of such theories does not admit limit cycles. A corollary of our result is that unitary $\mathcal{N} = 1$ supersymmetric four-dimensional theories cannot be superscale-invariant without being superconformal.
1. Introduction

In recent papers by some of us two independent methods were used to claim the existence of unitary four-dimensional quantum field theories that are scale but not conformally invariant (SFTs) [1–3]. A natural interpretation of the renormalization-group (RG) behavior of such theories is that they live on RG limit cycles with a constant “number of degrees of freedom.” Nevertheless, the work of Jack and Osborn [4] (see also [5]), which we think is widely unappreciated in the literature, has lead us to a new understanding of the conditions for conformal invariance. More specifically, it has become clear that a theory does not need to have zero beta functions in order for it to be conformal, and that the claimed examples of non-conformal scale-invariant field theories [1–3] are actually conformal.

We will not have much to say here about this new understanding—more details will be given in a forthcoming publication [6]. Our aim in the present note is to show that unitary $\mathcal{N} = 1$ supersymmetric theories in four dimensions cannot flow to a superconformal phase with nonzero beta functions. In other words, we will show that the beta-function vector field of supersymmetric theories does not admit limit cycles, in contrast to that of non-supersymmetric theories. (Let us remark here that we use “limit cycles” loosely to mean recursive flows in the beta-function vector field of a theory, that is, flows that may be cyclic or ergodic.) A corollary of this result is that there are no unitary $\mathcal{N} = 1$ supersymmetric SFTs in four dimensions.

The subject of scale without conformal invariance in unitary $\mathcal{N} = 1$ supersymmetric theories with an R-symmetry was investigated recently by Antoniadis and Buican [7]. Their treatment relies on carefully analyzing constraints in the operator content of such theories, and relies on various well motivated assumptions. A criterion is then given for a unitary supersymmetric theory to contain a superscale-invariant phase: it has to contain at least two real nonconserved dimension-two scalar singlet operators [7]. The most constraining assumption in the analysis of [7] is perhaps that an R-symmetry is required along the RG flow.

The operator content of possible supersymmetric SFTs was also studied by Nakayama [8], without the requirement of an R-symmetry. The so-called virial multiplet was constructed and its implications for scale without conformal invariance in supersymmetric theories were explored. In concrete examples difficulties were found in constructing a nontrivial virial multiplet in perturbation theory. However, relaxing the constraint of unitarity produced non-conformal scale-invariant field theories in a simple Wess–Zumino model.

With the recent work mentioned in the last two paragraphs in mind, it seems unlikely that supersymmetric theories can host a superscale-invariant phase that is not superconformal. Still, 

---

1We acknowledge helpful discussions on this point with Markus Luty, Joseph Polchinski and Riccardo Rattazzi, as well as informative correspondence with Hugh Osborn.

2For other studies of superscale and superconformal invariance see [9].
we think it is interesting to ponder the existence of supersymmetric limit cycles. Examples of limit cycles in non-supersymmetric theories are more generic than previously thought: in addition to a four-dimensional example, limit cycles in $4 - \epsilon$ dimensions have also been found \cite{1,2,10}. Thus, it is worthwhile to analyze the constraints supersymmetry imposes on such RG behavior.

The conclusion of our present note is that supersymmetry does not allow for limit cycles, and thus it does not allow for SFTs. Our method of proof, as will become clear below, is very different in spirit from that employed by Antoniadis and Buican, and by Nakayama. More specifically, in order to reach our conclusion we analyze supersymmetric theories with superspace-dependent couplings, and show that a quantity corresponding to the $S$ of \cite{4} (see also \cite{6}) is constrained to be zero by supersymmetry. The quantity $S$ is related to the frequency with which a theory traverses its putative limit cycle, and thus the fact that $S = 0$ in supersymmetry immediately shows that supersymmetric limit cycles cannot occur.

Note Added: As this work was being finalized, Nakayama added an appendix to \cite{8} where he also showed that $S$ must vanish to all orders in perturbation theory in $\mathcal{N} = 1$ supersymmetric field theories.

2. Preliminaries

In this section we give a brief review of material that is necessary for our arguments.

We are interested in four-dimensional theories that are classically scale-invariant. They are parametrized by coupling constants $g_i$. Following Jack and Osborn we promote these to spacetime-dependent couplings, $g_i(x)$. This is useful in two ways. Firstly, the couplings now act as sources for composite operators appearing in the Lagrangian. This allows us to define finite composite operators as functional derivatives of the renormalized generating functional for Green functions, $W$, with respect to the couplings. A similar method is used frequently to define the stress-energy tensor: the theory is lifted to curved space and the stress-energy tensor is obtained as a functional derivative of $W$ with respect to the metric. Secondly, it allows us to obtain a local version of the Callan–Symanzik equation, with terms involving derivatives of couplings interpreted as anomalies and thus satisfying Wess–Zumino consistency conditions \cite{11}.

In order to render this theory finite one must include all possible dimension-four counterterms consistent with diffeomorphism invariance. In addition, the counterterms may be further constrained by formal symmetries of the theory in which both quantum fields and couplings transform. Consider, for example, a theory of real scalars with bare Lagrangian

\begin{equation}
\mathcal{L}_0 = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi_0 \partial_\nu \phi_0 - \frac{1}{4!} g^0_{abcd} \phi_0^a \phi_0^b \phi_0^c \phi_0^d.
\end{equation}

This is written in terms of bare fields $\phi_0$. In the potential term the bare couplings $g^0_{abcd}$ are completely symmetric under exchange of the indices $a, b, c$ and $d$. The kinetic part of the Lagrangian
exhibits a continuous symmetry under transformations of the fields \( \delta \phi_0 = -\omega_{ab} \phi_0 b \), where \( \omega \) is in the Lie algebra of the flavor group \( G_F = SO(n_S) \). The whole Lagrangian is \( G_F \)-symmetric if we agree to transform, in addition, the couplings as

\[
\delta g^0_{abcd} = -\omega_{ae} g^0_{ebcd} - \omega_{bc} g^0_{ae} g^0_{cd} - \omega_{ce} g^0_{ab} g^0_{de} - \omega_{de} g^0_{abc} ,
\]

or \( \delta g^0_I = -(\omega g^0)_I \) for short, where, following Jack and Osborn, we use the compact notation \( I = (abcd) \). For spacetime-independent couplings the theory is renormalized by including the usual wave-function, \( \phi_0 = Z \phi \), and coupling constant, \( g^0_I = g_I + L_I(g) \), renormalization. But in the presence of spacetime-dependent coupling constants one must introduce new counterterms. Among them we are particularly interested in the counterterm of the form

\[
\mathcal{L}_{c.t.} = (\partial^\mu g_I)(N_I)_{ab} \phi_0 b \partial_\mu \phi_0 a ,
\]

(2.2)

with \( (N_I)_{ab} = -(N_I)_{ba} \), that is, in the Lie algebra of \( G_F \); see [4] for a complete account of counterterms required in the case of spacetime-dependent couplings in a curved background.

Finite operators corresponding to currents associated with generators of \( G_F \) are most readily introduced by introducing background gauge fields. We promote the Lagrangian (2.1) to

\[
\tilde{\mathcal{L}}_0 = \frac{1}{2} g^{\mu\nu} D_0 \phi_0 a D_0 \phi_0 a + \frac{1}{12} \phi_0 a R - \frac{1}{4} g^0_{abcd} \phi_0 a \phi_0 b \phi_0 c \phi_0 d ,
\]

where the covariant derivative,

\[
D_0 \phi_0 = (\partial_\mu + A_0 a) \phi_0 a ,
\]

is introduced with an eye towards including the counterterm (2.2) through the renormalization of \( A_0 a \),

\[
A_0 a = A_\mu + N_I (D_\mu g) a , \quad D_\mu = \partial_\mu + A_\mu .
\]

We have left implicit the Lie-algebra indices (so that \( N_I^T = -N_I \) and \( A_\mu^T = -A_\mu \)). Note that \( N_I \) is a function of the renormalized couplings that has an expansion in \( \epsilon \)-poles starting at order \( 1/\epsilon \). If the theory contains gauge fields and some of the scalars are charged under the gauge group \( G_g \subseteq G_F \), it is straightforward to include an additional quantum gauge field in addition to the background field \( A_\mu \).

The generating functional \( W \) is now a function of the background gauge field in addition to the metric and couplings, and finite operators are defined by functional differentiation:

\[
\langle T_{\mu\nu}(x) \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}(x)} , \quad \langle [\mathcal{O}_I(x)] \rangle = \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta g^I(x)} \quad \text{and} \quad \langle [\phi_a D_\mu \phi_0 a] \rangle = \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta A^{\mu}_{ab}(x)} .
\]

With this formalism Jack and Osborn obtain the trace-anomaly equation [4, Eq. (6.15)]

\[
T^\mu_\mu = \beta_I [\mathcal{O}_I] + \partial^\mu [(\partial_\mu \phi)^T S \phi] - ((1 + \gamma) \phi) \cdot \frac{\delta}{\delta \phi} S_0 ,
\]

3
where \( S_0 = \int d^4x \sqrt{-g} \mathcal{L}_0 \) and \( \beta_I \), and \( \gamma \) are, as usual, the beta function of the coupling \( g_I \) and the anomalous dimension of the field \( \phi \), respectively. We have specialized their result to the case of flat metric, spacetime-independent couplings, and vanishing background vector field. The last term, involving the functional derivative of the quantum action, vanishes by the equations of motion. The surprising aspect of this result is the often neglected term that involves the total divergence of the current \( [(\partial_\mu \phi)^T S \phi] \). It is defined in terms of the \( G_F \)-Lie algebra element

\[
S \equiv -g_I N^I_1,
\]

where \( N_I = \sum_{n=1}^{\infty} N^n_I / \epsilon^n \), so that \( N^I_1 \) is the residue of the simple \( \epsilon \)-pole in \( N_I \). Moreover, using the equation of motion (or the generalized symmetry under \( G_F \)) Jack and Osborn get \[4\, \text{Eq. (6.23)}\]

\[
T^\mu_\mu = (\beta_I - (Sg)_I)[\mathcal{O}_I] - ((1 + \gamma + S)\phi) \cdot \frac{\delta}{\delta \phi} S_0.
\]

This shows that a theory is conformal provided \( \beta_I - (Sg)_I = 0 \). The account above is readily generalized to the case of real scalars interacting with Weyl fermions in the presence of quantum gauge fields.

In \[6\] we used Weyl consistency conditions \[4, 5\] and perturbation theory to show that \( S \) has two important properties:

1. \( S \) vanishes at fixed points. That is, if \( \beta_I = 0 \) then \( S = 0 \).

2. On cycles, defined by \( \beta_I = (Qg)_I \) for \( Q \) in the Lie algebra of \( G_F \), one has \( S = Q \).

Perturbation theory is only needed to establish positivity of the natural metric in the space of operators, \( \chi^2_{IJ} \) in the notation of \[4\]. It follows that in a theory for which \( S = 0 \) identically there is no possibility of limit cycles, and that conformal invariance corresponds to fixed points. We will show below this is precisely the case for supersymmetric theories.

### 3. Finding Limit Cycles

In this section we review how to determine whether the beta-function vector field of a theory admits limit cycles \[2, 3, 6\], making the procedure manifestly supersymmetric whenever possible. However, we often use what is known in the non-supersymmetric case to deduce what conditions have to be satisfied in the supersymmetric case.

Consider a classically scale-invariant supersymmetric field theory in four dimensions with \( N_f \) chiral superfields of mass dimension one. Classical scale invariance implies that the theory is renormalizable. The part of the Lagrangian we are interested in is

\[
\mathcal{L} = \int d^4 \theta \bar{\Phi}_a \Phi^*_a + \left( \int d^2 \theta \frac{1}{3!} y_{abc} \Phi_b \Phi_c \Phi^*_e + \text{h.c.} \right).
\]  

3Lower case Roman letters are indices in flavor space for (anti-)chiral superfields.
There may be vector superfields in addition to the chiral superfields \( \Phi_a \) in the Kähler potential. However, we do not concern ourselves with vector superfields: their trivial flavor structure renders them unable to play a role in determining whether limit cycles exist.

The Kähler potential exhibits a continuous symmetry under transformations of the fields \( \delta \Phi_a = -\omega_{ab} \Phi_b \), where \( \omega \) is in the algebra of the “flavor” group \( G_F = SU(N_f) \). The Yukawa couplings in the superpotential break \( G_F \). This flavor symmetry can be extended to the whole Lagrangian by treating the coupling constants as spurions, non-dynamical fields that are allowed to transform under \( G_F \). More specifically, the coupling constant \( y_{abc} \) is promoted to a superspace-dependent chiral superfield of mass dimension zero,

\[
Y_{abc}(z) = y_{abc}(z) + \sqrt{2} \theta y^\psi_{abc}(z) + \theta^2 y^F_{abc}(z),
\]

where \( z^\mu = x^\mu + i \theta \sigma^\mu \bar{\theta} \). The \( y^\psi \) and \( y^F \) components of the spurion field are irrelevant and we ignore them in what follows. The Lagrangian (3.1) is manifestly \( G_F \)-symmetric if the Yukawa couplings transform as

\[
\delta Y_{abc} = - \omega_{aa'} Y_{a'bc} - \omega_{bb'} Y_{ab'c} - \omega_{cc'} Y_{abc}.
\]

The theory also possesses a spurious \( U(1) \) R-symmetry in addition to the \( G_F \) symmetry. The fields and couplings transform under the R-symmetry as

\[
\Phi \rightarrow e^{i\alpha} \Phi, \quad \Phi^\dagger \rightarrow e^{-i\alpha} \Phi^\dagger, \quad Y \rightarrow e^{-i\alpha} Y, \quad Y \rightarrow e^{i\alpha} Y.
\]

The R-symmetry is non-anomalous because the R-charge of the fermionic component of \( \Phi \) is zero.

We now look for a supersymmetric version of the new type of counterterm that is required in the presence of superspace-dependent couplings, as in (2.2). In supersymmetric theories the only candidate for this counterterm has the form

\[
\mathcal{L}_{c.t.} = \int d^4 \theta \Phi^\dagger F_{ab} \Phi_b,
\]

where \( F_{ab} \) is a function of the couplings. If the theory is to be unitary, \( F_{ab} \) must be Hermitian, \( F_{ab}(Y, \bar{Y}) = F_{ba}(\bar{Y}, Y) = F_{ba}^*(Y, \bar{Y}) \). One can readily check that one of the components of (3.3) is of the form (2.2), that is, the product of the current associated with \( G_F \) and the derivative of the couplings

\[
\mathcal{L}_{c.t.} \supset ((N_I)_{ab} \partial^\mu y_I - (N_I)_{ba} \partial^\mu y_I^*) (\phi_a^* \partial_\mu \phi_b - \partial_\mu \phi_a^* \phi_b),
\]

with \( I \) again a shorthand for contracted flavor indices. \( N \) can be expressed in terms of \( F \) as

\[
(N_I)_{ab} = \frac{\partial F_{ab}(y, y^*)}{\partial y_I}, \quad (N_I)_{ba}^* = \frac{\partial F_{ab}(y, y^*)}{\partial y_I^*}.
\]

Both \( N \) and \( F - 1 \) are functions of the renormalized couplings that have \( \epsilon \)-pole expansions starting at order \( 1/\epsilon \).
4. Absence of Limit Cycles in Supersymmetric Theories

We are finally ready to prove at the quantum level that a unitary, \( N = 1 \) supersymmetric field theory in four dimensions does not have limit cycles. Our strategy is to show that \( S \) is exactly zero in supersymmetric theories with the aforementioned qualifications. This we can show without recourse to perturbation theory. However, we are mindful that the proof in [6] that \( S = Q \) on cycles and \( S = 0 \) at fixed points does rely on perturbation theory.

The expression for \( S \) in our case is

\[
S_{ab} \equiv -\frac{1}{2} (N_1^{d})_{ab} y_I - \text{h.c.,}
\]

\[
= -\frac{1}{2} \left( y_{I} \frac{\partial F_{ab}^{1}(y, y^{*})}{\partial y_{I}} - y_{I}^{*} \frac{\partial F_{ab}^{1}(y, y^{*})}{\partial y_{I}^{*}} \right),
\]

where \( F^{1} \) is the residue of the simple \( 1/\epsilon \) pole in \( F \). The Hermitian conjugate is subtracted in (4.1), as expected since \( S \) is anti-Hermitian. The quantum action is invariant under the R-symmetry introduced in Section 3, see (3.2). Therefore

\[
F_{ab}(Y, \overline{Y}) = F_{ab}(e^{-i\alpha}Y, e^{i\alpha}\overline{Y}),
\]

or, by taking \( \alpha \) to be infinitesimal,

\[
0 = Y_{I} \frac{\partial F_{ab}(Y, \overline{Y})}{\partial Y_{I}} - \overline{Y}_{I} \frac{\partial F_{ab}(Y, \overline{Y})}{\partial \overline{Y}_{I}}.
\]

Comparing the scalar component of this equation with (4.2) shows \( S = 0 \). The theory cannot exhibit renormalization group limit cycles. Furthermore, unitarity and superscale invariance imply superconformal invariance in unitary four dimensional \( N = 1 \) supersymmetric field theories.

5. A Perturbative Proof and a Four-Loop Example

If \( S \) vanishes in supersymmetric theories non-perturbatively, the implication must also be true to all orders in perturbation theory. In this section we illustrate the vanishing of \( S \) in perturbation theory with a four-loop example. Remarkably, four-loop calculations in the Wess–Zumino model exist in the literature [12]. For a diagram containing only chiral superfields, it is a simple combinatoric exercise to convert the results of [12] to the model under consideration in this work.

In non-supersymmetric theories a scalar-propagator loop correction contributes to \( S \) if the corresponding diagram is not symmetric under \( a \leftrightarrow b \). Such diagrams first arise at the three-loop level in ordinary field theories [6]. In \( N = 1 \) supersymmetric Wess–Zumino models asymmetric diagrams arise at four loops, see e.g. Fig. 11. The four-loop contribution of the diagrams of Fig. 11
Fig. 1: Four-loop diagrams that contribute to $F$ that are asymmetric under exchange of the external legs. The lines are superfield propagators.

to $F^1$ is

$$(16\pi^2)^4 F^1_{ab} \supset 3 \left( \zeta(3) - \frac{1}{2} \zeta(4) \right) (y_{acd} y_{dkm} y_{fkl} y_{bef} y_{cjm} y_{fgi} y_{ghi} y_{cgh} y_{ijk}) + y_{acd} y_{dkm} y_{fkl} y_{cjm} y_{fgi} y_{ghi} y_{cgh} y_{cgh},$$

where $\zeta$ is the Riemann zeta function. From this expression for $F^1$ we see that $S$ vanishes by (4.2). There are at least two ways to understand this diagrammatic result.

It is obvious from the form of (4.2) that $S$ counts the difference in the number of $y$’s and $y^*$’s in $F$. The non-renormalization of the superpotential guarantees that any diagram containing an unequal number of $y$’s and $y^*$’s vanishes. Thus, the only diagrams that contribute to $F$ contain an equal number of $y$’s and $y^*$’s, and $S$ must vanish to all orders in perturbation theory. In contrast with the non-supersymmetric case, not even diagrams asymmetric under exchange of the external legs can contribute to $S$.

The second way in which our result can be understood is as follows. In non-supersymmetric theories momentum is allowed to flow into the diagram that gives $N^1$ from an external leg and out of the diagram through a coupling. If the diagram is asymmetric, then interchanging the external lines of the diagram results in a different routing of the external momentum through the diagram, and thus to a different numerical coefficient for the corresponding contribution to $N^1$. This leads to a nonzero contribution to $S$ after antisymmetrization. In the supersymmetric case, however, the coefficient of all diagrams contained in the $\theta$-expansion of an asymmetric diagram—like the one in Fig. 1—comes from the zeroth-order in $\theta$ diagram, which is calculated with no external momentum flowing into the diagram. Thus, there is no possibility of a contribution to $S$. This is true to all orders in perturbation theory.

Acknowledgments

We thank Ken Intriligator and Hugh Osborn for useful discussions. We are especially grateful to Markus Luty, Joseph Polchinski and Riccardo Rattazzi for constructive discussions. BG, CM, and AS are supported in part by the U.S. Department of Energy under contract No. DOE-FG03-97ER40546. JFF is supported by the ERC grant BSMOXFORD No. 228169.
References

[1] J.F. Fortin, B. Grinstein & A. Stergiou, “Scale without Conformal Invariance: An Example”, Phys.Lett. B704, 74 (2011), arXiv:1106.2540.

[2] J.F. Fortin, B. Grinstein & A. Stergiou, “Scale without Conformal Invariance at Three Loops”, JHEP 1208, 085 (2012), arXiv:1202.4757.

[3] J.F. Fortin, B. Grinstein & A. Stergiou, “Limit Cycles in Four Dimensions”, arXiv:1206.2921.

[4] I. Jack & H. Osborn, “Analogs for the c theorem for four-dimensional renormalizable field theories”, Nucl.Phys. B343, 647 (1990).

[5] H. Osborn, “Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories”, Nucl.Phys. B363, 486 (1991).

[6] J.F. Fortin, B. Grinstein & A. Stergiou, “Limit Cycles and Conformal Invariance”, arXiv:1208.3674v2.

[7] I. Antoniadis & M. Buican, “On R-symmetric Fixed Points and Superconformality”, Phys.Rev. D83, 105011 (2011), arXiv:1102.2294.

[8] Y. Nakayama, “Supercurrent, Supervirial and Superimprovement”, arXiv:1208.4726.

[9] S. Zheng & Y. Yu, “Is There Scale Invariance in N=1 Supersymmetric Field Theories ?”, arXiv:1103.3948. Y. Nakayama, “Comments on scale invariant but non-conformal supersymmetric field theories”, arXiv:1109.5883.

[10] J.F. Fortin, B. Grinstein & A. Stergiou, “Scale without Conformal Invariance: Theoretical Foundations”, JHEP 1207, 025 (2012), arXiv:1107.3840.

[11] J. Wess & B. Zumino, “Consequences of anomalous Ward identities”, Phys.Lett. B37, 95 (1971).

[12] L. Avdeev, S. Gorishnii, A.Y. Kamenshchik & S. Larin, “Four Loop Beta Function in the Wess-Zumino Model”, Phys.Lett. B117, 321 (1982).