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LETTER

Micromasers as quantum batteries

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Abstract

We show that a micromaser is an excellent model of quantum battery. A highly excited, pure, and effectively steady state of the cavity mode, charged by coherent qubits, can be achieved, also in the ultrastrong coupling regime of field-matter interaction. Stability of these appealing features against loss of coherence of the qubits and the effect of counter-rotating terms in the interaction Hamiltonian are also discussed.

1. Introduction

Given current trend to miniaturization of technology, with devices operating at dimensions of the order of the nanoscale, quantum effects must be taken into account. The possibility of using genuine quantum effects to improve performances of such devices compared to their classical counterparts—generically dubbed as quantum advantage—has in turn led to enormous efforts and research, to find all tasks where a quantum advantage can be obtained [1–6].

Any quantum machine needs energy to operate. Therefore, in parallel with developments on building nanodevices, topic of quantum batteries (see [7, 8], and references therein) has emerged and flourished in the last few years. A quantum battery is a quantum mechanical system suitable to store energy in some highly excited states, to be released on demand. Starting from the seminal work of Alicki and Fannes, [9], several figures of merit, to characterize the performance of a quantum battery, have been introduced and studied. These include energy storage [10–14], work extraction [9, 15], fluctuations properties [16–19] and charging power [4–6, 20–33], to name some of them.

In parallel with such theoretical developments, proposals to concretely build quantum batteries have emerged, including models realized by spin chains [21] and qubit systems charged by electromagnetic fields [22]. However, with some very recent notable exceptions [34], achievements on actual experimental realizations have been so far quite limited [35]. As for any quantum information protocol, the search for high-speed operations is vital. Such possibility is naturally offered by circuit quantum electrodynamics (QEDs), one of the most promising platforms for quantum hardware, in which one can address the so-called ultrastrong coupling (USC) regime of light–matter interaction, where the qubit–cavity interaction energy becomes comparable, or can even exceed the bare frequencies of the uncoupled systems [36, 37].

In this paper we revisit, with an eye towards features persisting in the USC limit, a very well-established model, which has been extensively implemented and studied at the experimental level: the micromaser [38–44], where a stream of qubits (two-level atoms in cavity QED [45]) sequentially interact with a cavity mode with a high-quality factor. That is, the radiation decay time is much larger than the characteristic time of the qubit–field interaction, and the overall evolution is to a good approximation coherent. We will show that micromasers can be regarded as excellent models of quantum batteries. In particular, they display excellent performances in terms of charging temporal stability and ergotropy [46].
By charging temporal stability we mean the ability of battery to keep, under time evolution and after an initial transient time, a stable value of its mean energy [5, 16, 18, 19]. In many models of quantum batteries, after an initial regime, in which energy grows since battery is evolving towards excited states, time evolution of mean energy undergoes temporal fluctuations. These fluctuations are of course unwanted, since they require a very high precision in controlling the charging time, in order to reach the target value of mean energy. We will numerically show that a micromaser can be quickly charged to reach an almost steady state, whose energy is controlled by the physical parameters defining the model. By almost steady state we mean a state which, although metastable, is characterized by very long lifetime. An almost steady state solves the problem of temporal stability, since it is constant for very long times.

By ergotropy it is meant the amount of energy that can be extracted from a battery via unitary operations [47]. Often, part of energy is locked in correlations and cannot be used. For example, when the battery state is a mixture it cannot be transformed into the ground state by means of unitary transformations. Therefore not all its energy can be extracted. On the opposite side, a pure state has ergotropy which equals its mean energy. We will show that for a micromaser the almost steady states mentioned above are approximately pure and therefore, in principle, almost all their energy can be reversibly extracted. This is a very surprising feature, since dynamics in micromasers involves a trace over qubit degrees of freedom after each interaction.

The possibility of building pure states (steady or not) in micromasers has been discussed in previous studies [39–43]. However, they relied on several assumptions: very weak coupling between qubits and field and/or highly fine-tuned values of these couplings. All these assumptions are unwanted in a quantum battery: weak coupling implies slow charging and consequently low values of charging power, while fine-tuned values require very high precision in building the battery, thus making it hard to realize. Our numerical results show that all these assumptions can be relaxed to a large extent: the battery reaches an almost pure steady state even when entering in USC and without fine-tuning.

The combination of above features promotes micromasers to status of very robust and reliable quantum batteries, thus making them as very promising models for experimental realizations. It is worth noticing that in the recent literature quantum batteries modelled by collision models (see [48] for an extensive overview) have been theoretically investigated. For example, the effect of coherence on the charging power was analyzed in [31], while the possibility of building active steady states was studied in [49]. Our paper can be viewed as the first attempt to confirm the appealing features of batteries based on collision models in an experimentally well-established setup. In addition, micromasers enjoy the rather unique feature of having steady states which are pure for practical purposes.

2. Model and figures of merit

We consider a quantum battery made up of a quantized electromagnetic field (EM) in a cavity, modelled as a harmonic oscillator. It is initially prepared in its ground state \(|0\rangle\). The charging protocol is realized via a stream of two-level systems (qubits) which sequentially interact with the battery, thus realizing a micromaser (see, figure 1 for a graphical representation). The initial state of each qubit is

\[ \rho_q = q|g\rangle\langle g| + (1 - q)|e\rangle\langle e| + c\sqrt{q(1 - q)}(|e\rangle\langle g| + |g\rangle\langle e|), \]  

where \(|g\rangle\) and \(|e\rangle\) are ground and excited state of the qubit, respectively. Parameters \(q\) and \(c\) control the degrees of population inversion and coherence, respectively. The evolution of the system is described, in interaction picture, by the Hamiltonian [44, 50]

\[ \hat{H}_I = g (\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-) + e^{i2\omega t}\hat{a}\hat{\sigma}_+ + e^{-i2\omega t}\hat{a}^\dagger\hat{\sigma}_-. \]  

where \(\omega\) is the frequency of both qubit and field (we are considering the resonant case in which they are equal, [44]); \(\hat{a}\), \(\hat{a}^\dagger\) are creation/annihilation operators for the field and \(\hat{\sigma}_+, \hat{\sigma}_-\) are raising and lowering operators for the qubit (respectively). Finally, \(g\) is the coupling constant for interaction between qubit and field and we take units such that \(\hbar = 1\). It is customary to name the first two terms in equation (2) as rotating terms and the last two terms as counter-rotating terms [44, 50].

From equation (2) we derive the time evolution operator, \(\hat{U}_I(g, \omega, \tau)\), to be

\[ \hat{U}_I(g, \omega, \tau) \equiv T \exp\left\{ -i \int_0^\tau \hat{H}_I(t)dt \right\}, \]  

where \(T\) and \(\tau\) are the time ordering operator and the interaction time between a single incoming qubit and the battery, respectively. We fix, without losing generality, \(\tau = 1\), since different values of \(\tau\) can be
always re-absorbed in a simultaneous redefinition of $g$ and $\omega$. Accordingly, $\hat{U}_1(g, \omega) \equiv \hat{U}_1(g, \omega, \tau = 1)$ is the time evolution operator.

Denoting with $\rho_B(k)$ the battery state after having interacted with $k$ qubits, we denote by $\rho(k)$ the product state of $\rho_B(k)$ and the incoming $(k + 1)$th qubit, i.e. $\rho(k) = \rho_B(k) \otimes \rho_q$. The system is then evolved by means of $\hat{U}_1(g, \omega)$, and $\rho_B(k + 1)$ is obtained as follows:

$$\rho_B(k + 1) = \text{Tr}_q \left( \hat{U}_1(g, \omega) \rho(k) \hat{U}_1^\dagger(g, \omega) \right),$$

(4)

where $\text{Tr}_q$ is the trace over qubit degrees of freedom. Although the battery is initially in a pure state $|0\rangle$, by tracing out qubit degrees of freedom it becomes mixed, in general. More concretely, one expects that the purity of $\rho_B$, $\mathcal{P}(k)$, defined as $\mathcal{P}(k) \equiv \text{Tr}(\rho_B^2(k))$, should decrease from 1 (which is the value for pure states) before the first collision to almost vanish after many collisions.

The energy stored after $k$ collisions amounts to $E(k) \equiv \text{Tr}(\hat{H}_B \rho_B(k))$, where $\hat{H}_B \equiv \omega \hat{a} \hat{a}^\dagger$ is the battery Hamiltonian and trace is taken over battery degrees of freedom. We now turn to characterize the charging performance of the battery by measuring $E(k)$ and $\mathcal{P}(k)$.

### 3. Numerical results

The time evolution operator, equation (3), gets enormously simplified when counter-rotating terms in equation (2) can be omitted. As it is known, $[44, 50]$, counter-rotating terms can be omitted when weak-coupling condition $\frac{\omega g}{\sqrt{m}} \ll 1$ is met. However, to build a battery which is fast in charging, weak-coupling is not the most relevant limit. Hence, it will be desirable to consider the case in which $g$ and $\omega$ are comparable, such that $0.1 < \frac{\omega g}{\sqrt{m}} < 1$. This case can be treated, still without the need of considering counter-rotating terms, by performing a simultaneous modulation of the frequencies of field and qubit, as first observed in $[51]$. We now assume that such a modulation has been performed and consider the celebrated Jaynes–Cummings (JC) operator $[52]$,

$$\hat{U}_1(g) = e^{-ig(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})}.$$  

(5)

Micromasers driven by JC operator have been heavily studied in literature, starting from $[39–41]$, both numerically and analytically. The most important feature of the JC operator, in our context, is the existence of trapping chambers. It was analytically shown in $[39, 40, 42]$ that when $\epsilon = 1$ $[53]$ and, more crucially, when $g$ is fine-tuned to satisfy

$$g = \frac{Q}{\sqrt{m}} \pi, \quad Q, m \in \mathbb{N},$$

(6)

with $Q$ and $\sqrt{m}$ not sharing any common integer divisors, the harmonic oscillator Hilbert space dynamically separates the states $|n\rangle$ with $n < m$ from the rest of the Hilbert space. Obviously, by choosing $Q = 2Q$ and $\tilde{m} = 4m$, equation (6) is again satisfied and another dynamically separated chamber, including states $|n\rangle$ satisfying $m < n < 4m$, appears. By increasing $Q$ it is realized that the full Hilbert space splits in chambers, dynamically separated from each other. In consequence, since the harmonic oscillator is initially prepared in

![Figure 1. A pictorial view of the micromaser quantum battery. The incoming qubits are initially prepared in a superposition of ground and excited states. When entering the cavity they interact with the EM field and they decrease their energy, leaving the cavity in a low energy state (represented with a black colored Bloch sphere). In the figure, $\omega_f$ and $\omega_i$ denote the frequencies of the qubits and the fields, respectively, which in the main text have been assumed to be equal and denoted by $\omega$.](image)
its ground state, it is possible to show that it reaches a steady state given by a macroscopic superposition of number states, \( |\psi\rangle = \sum_n f_n |n\rangle \), involving at most \( m \) number states, with \( f_n \) satisfying a recursion relation.

On the other hand, to characterize a quantum battery it is important to consider both the case in which \( g \) is not fine-tuned and to estimate the number of collisions necessary to reach such a state.

Given any value \( g \), clearly it can be written as

\[
g = \frac{Q}{\sqrt{m + \epsilon}}, \quad Q, m \in \mathbb{N}, \quad -0.5 < \epsilon \leq 0.5,
\]

i.e. it is approximated by a certain fine-tuned value, with \( \epsilon \) quantifying the error in considering such an approximation. Moreover, by increasing \( Q \) — and correspondingly \( m - \epsilon \) can be made arbitrarily small, thus suggesting that, by taking \( Q \) and \( m \) very large, an almost fine-tuned dynamics could be eventually reached. Such a picture, however, does not give any information on the number of collisions necessary to reach such an almost fine-tuned regime and, more important, on the behavior of the state during evolution. In particular, it does not provide any information of the role played on dynamics by other approximate trapping chambers, having \( Q \) and \( m \) small. These questions are crucial when dealing with a quantum battery.

To this end, we have numerically computed the micromaser time evolution, controlled by equation (4), for two non fine-tuned values of \( g \), as well as for their closest fine-tuned counterparts having \( Q = 1 \), i.e. for \( g = \pi/\sqrt{m} \). Results are reported in figure 2(a). For both fine-tuned and non fine-tuned values, system reaches, after \( \approx 30 \) collisions, an effective steady value of \( E(k) \), which remains constant up to \( 10^6 \) collisions. This state is essentially pure, as demonstrated by \( P(k) \). While these results are non surprising when \( g \) is fine-tuned, and they are in agreement with analytical results [39, 40, 42], it is remarkable that they hold in non fine-tuned cases as well. In particular, when \( g = 0.5 \) and \( Q = 1 \), \( \epsilon \approx 0.48 \) is almost maximal. Nevertheless, battery reaches an effective steady state which, for practical purposes, is stable and pure. These are wanted properties for a model of quantum battery, which our results show to be present up to, at least, \( 10^6 \) collisions. To better investigate the properties of these states, with and without fine-tuning of \( g \), we have studied the density matrix, \( \rho_{Bv} \), after \( 10^6 \) collisions. Results are reported in figure 2(b). As expected, when \( g \) is fine-tuned all non-vanishing elements of the density matrix are strictly confined in the first trapping chamber. What is non-trivial and interesting is the behavior when \( g \) is not fine-tuned: we see that the vast majority of non-vanishing elements are still confined in the first trap. However, some non-vanishing elements are actually escaping and a bubble is forming around matrix element (30, 30).

This result is confirming that in the presence of just partial trapping, dynamics turns out to be extremely slow. For many collisions the first trap, i.e. the case with \( Q = 1 \), is enough to fully control the dynamics. Furthermore, to get a reliable battery, it is important that battery charging does not depend too strongly on fine-tuned values of \( g \). Figure 1(b) (lower panel) shows that in case of non-fine-tuned \( g \) there are alternate bubbles in the system density matrix; however figure 1(a) confirms that for practical purposes first trapping state in the density matrix is sufficient as it is a long-lived trapping state, up to \( 10^6 \) collisions.

Another interesting point to investigate is the stability of the above features when the incoming qubits are not completely coherent, i.e. for \( c \neq 1 \). We discuss this case in appendix A and we summarize here our main findings. The purity of the steady state gets affected and it satisfies \( P \lesssim c \). However, its energy also increases, such that the ergotropy actually increases when \( c < 1 \). At the same time, we found that the lifetime of the meta-stable steady state is shorter when \( c < 1 \) but such a reduction becomes significant for low values of \( c \) only, \( c \approx 0.4 \) or even smaller (depending on the value of \( g \)). All in all, stability of the micromaser is robust for deviations away from \( c = 1 \) case and, up to a certain extents, taking \( c < 1 \) increases the energetic performance of the battery.

Finally, we can provide a qualitative description of how the parameters affect the value of the plateau, i.e. the energy stored in the battery steady state. From the results discussed so far, we see that \( g \) is inversely related to the energy of the battery. The reason for such a behavior is easy to understand from equation (7): large values of \( g \) (at least for \( g < \pi \)) require smaller values of \( m \) which, in turns, imply that the maximum level involved in the superposition is small. Conversely, from figure 2(a) we see that large values of \( g \) lead to faster charging, i.e. larger values of the charging power. As we discussed, larger values of \( c \) imply longer lifetime of the steady state at the price of lower values of its energy (and its ergotropy too). Finally, as intuitively reasonable, the population inversion parameter \( q \) is also inversely related to the steady state energy: larger values of \( q \) reduce the energy of the steady state, with the battery charging which become quite inefficient for \( q \approx 0.5 \). In such a case the steady state is still stable but its energy is very low. This behavior is in agreement with the analytical expressions for the coefficient \( f_n \) describing the steady state in the fine-tuned case, as derived in [40, 42].
4. Stability with counter-rotating terms

Although by performing high-amplitude, low-frequency modulations [51], the validity of the JC approximation can be pushed well beyond the standard weak-coupling limit \((g/\omega \ll 1)\), it is important to investigate the stability of the physical features just described in the presence of the counter-rotating terms in equation (2). Hence, without performing any modulations, we have studied the time evolution of the battery by means of the full time evolution operator \(\hat{U}_t(g, \omega)\) in equation (3). Importance of counter-rotating terms becomes more prominent when entering in USC regime, \(\frac{g}{\omega} \sim 0.1\). Accordingly, we have checked that counter-rotating terms do not significantly affects dynamics up to \(\frac{g}{\omega} \sim 0.05\). More interestingly, we have studied \(E(k)\) and \(P(k)\) for \(\frac{g}{\omega} = 0.1\) and larger values of \(g\), i.e. in USC regime. Results are reported in figure 2(a). We see that for \(\frac{g}{\omega} \geq 0.5\) all features described for JC evolution are lost: the system does not reach any stability with energy increasing indefinitely and purity steadily decreasing. The same is not true and turns out to be more interesting for \(\frac{g}{\omega} = 0.5\). In this case, dynamics is different for \(g = 0.5\) and 0.9, thus showing that the actual behavior is controlled not only by \(g/\omega\) but also by \(g\) itself. In particular, for \(g = 0.5\) we see that trapping dynamics is still effective after \(10^6\) collisions, although both \(E(k)\) and \(P(k)\) are negatively affected by counter-rotating terms. On the other hand, when \(g = 0.9\) we see that stability is preserved only up to \(\approx 10^4\) collisions. Finally, also in this case it is of interest to check how the...
charging is affected when $c \neq 1$. On general grounds, the same considerations already expressed for the JC case apply, with the ergotropy which is positively affected when $c < 1$ at the expenses of the lifetime of the steady state. On the other hand, when dealing with the full Hamiltonian equation (2), one has to treat with care a possible phase factor in the coherence parameter $c$. Indeed, the rotation along the $z$ axis, necessary to cancel the phase factor, translates into a time-dependent redefinition of the $\omega$ parameter $\omega \rightarrow \omega' = \omega - \frac{\theta}{T}$, with $\theta$ being the angle of the rotation. Hence, depending on the sign of $\theta$, such a redefinition can increase the effective value of $\omega$, thus stabilizing the charging, or decreasing it when $t \rightarrow 1$. In the latter case, the stability of the steady state can be heavily affected. We present an example of this behavior in appendix A, while we leave a detailed investigation of this effect for future studies. All in all, our results show that trapping properties of the JC operator are rather robust, even in USC limit and without any modulation.

5. Conclusions and outlook

We have numerically shown that a micromaser, charged by means of coherent qubits, can be thought as an excellent model of quantum battery. The system reaches an effectively steady state which is dynamically stable. More crucially, such state is essentially pure, which means that all of its energy can be extracted, in principle, via unitary operations. Finally, we have shown that our results are quite stable, even when entering in USC regime, and they are generic, i.e., they do not require fine-tuning. Combining all our results, we conclude that a micromaser, charged in presence of an external modulation and at USC, behaves as a very reliable model of quantum battery, even when modulation is not perfect and counter-rotating terms cannot be fully ignored. Micromasers have been extensively studied in literature, even at experimental level [54]. Hence, we think our results show an explicit and promising model of quantum battery.

Most of recent theoretical developments have been tailored towards systems involving many batteries and instances where collective quantum effects improve performance, i.e. examples of quantum advantage. Micromasers are single body systems but they can be combined in architectures to build many-body setups, [55, 56]. Hence, our results show that they are promising building blocks of many-body quantum batteries, an aspect that we plan to explore in near future.

On a more theoretical ground, our results show that, for practical purposes, pure steady states can be found in micromaser at USC and without any fine-tuning. It would be extremely interesting to better investigate and characterize these states and to find analytic arguments (perhaps along the lines of [57]) supporting their existence and predicting their lifetime.

Note added

Few days after the appearance on arXiv of our paper, we were contacted by the authors of [58]. They were also close to finish a complementary work dealing with harmonic oscillators, charged via collision models, thought as quantum batteries.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Charging with $c \neq 1$

In this appendix, we explore how much the stability properties discussed in the main text are robust when the condition $c = 1$ for the incoming qubit is broken. We will focus on the modulated JC case only, in which the evolution is controlled by the time evolution operator equation (3). When counter-rotating terms are at work results are qualitatively similar.
Figure 3. Charging of the micromaser quantum battery for $q = 0.25$ and $g = 0.5, 0.9$ and $c < 1$. We see that by reducing $c$ the stability of the metastable steady state is reduced to a certain extent.

Figure 4. The degree of purity of the metastable steady state, measured by $P(c)$, as a function of the coherence of the incoming qubit.

As first step, one needs to check that the presence of a metastable steady state is robust for $c \neq 1$. We have numerically tested that the lifetime of the metastable steady state is negatively affected by reducing $c$, and that the minimum value of $c$ for which a certain degree of stability survives is highly dependent on the other parameters of the system ($g$ and $q$). As an example, in figure 3 we consider two examples of charging with $c < 1$. In both cases, we observe that the stability of the plateau is still clearly visible up to $\sim 10^5$ collisions, after that the micromaser exits from the trap and starts to increase its energy. The values of $c$ at which this mechanism becomes visible depend on $g$ but in both cases they are pretty small: $c \approx 0.15$ when $g = 0.5$ and $c \approx 0.4$ for $g = 0.9$.

Having established the persistence of the plateau and the presence of an almost steady state even when $c$ is fairly small, we now move to characterize its degree of purity as a function of the coherence of the incoming qubits. As we can see in figure 4 the purity of the metastable steady state is largely independent on $g$ and it has a behavior approximately linear as a function of $c$. In particular, we see that $P(c)$ is always quite larger than $c$.

Given that, for $c \neq 1$ the metastable steady state is not completely pure, it makes sense to study the behavior of its ergotropy (denoted by $W(c)$) as a function of $c$, i.e. the amount of energy that can be extracted from it by means of unitary transformations (see, for example, [17] for an explanation of how it can be computed). To this purpose, we show in figure 5 the behavior of $W(c)$ for both $g = 0.5$ and $g = 0.9$. In both cases, we see that the ergotropy increases when reducing $c$. The reason behind this unexpected
behavior has to be found on the fact that, by reducing $c$, the energy of the steady state increases quite significantly, at a level that it compensated its lacking of purity.

As mentioned in the main text, when modulation is absent and the counter-rotating terms cannot be neglected, a particular care must be taken when dealing with $c$ being not real, i.e. for $c = \rho e^{i\alpha}$.

The reason is the following. By means of a rotation along the $z$-axis, the phase factor can be cancelled to make $c$ real. Let us denote such a rotation operator as $R(\theta) = e^{i\theta/2}$. When acting with such a rotation on the interaction Hamiltonian, equation (2), we get

$$\hat{H}_I = g(\hat{a} e^{-i\theta} \hat{\sigma}_+ + \hat{a}^\dagger e^{i\theta} \hat{\sigma}_- + e^{12\omega t} \hat{a}^\dagger e^{-i\theta} \hat{\sigma}_+ + e^{-12\omega t} e^{i2\theta} \hat{a} e^{i\theta} \hat{\sigma}_-).$$

(8)

The phase factors appearing in front of the rotating terms can be cancelled by redefining the harmonic oscillator creation/annihilation operators $\hat{a} \rightarrow e^{i\theta} \hat{a}$. However, this redefinition does not cancel the phase factors from the counter-rotating terms and the Hamiltonian can be written as

$$\hat{H}_I = g(\hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_- + e^{12\omega t} \hat{a}^\dagger \hat{\sigma}_+ + e^{-12\omega t} e^{i2\theta} \hat{a} \hat{\sigma}_-),$$

(9)

from which we see that a phase factor on $c$ is equivalent to a time-dependent redefinition of $\omega$, i.e.

$\omega \rightarrow \omega' = \omega - \frac{\theta}{2}$.

The effect of such a redefinition is easy to understand: when $\theta$ is negative, such a redefinition will effectively increase the value of $\omega$ (at any time) and will have a stabilizing effect, while the opposite will happen when $\theta$ is negative.
In figure 6 we present an explicit example of this behavior.

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