Abstract. Standard eddy viscosity models, while robust, cannot represent backscatter and have severe difficulties with complex turbulence not at statistical equilibrium. This report gives a new derivation of eddy viscosity models from an equation for the evolution of variance in a turbulent flow. The new derivation also shows how to correct eddy viscosity models. The report proves the corrected models preserve important features of the true Reynolds stresses. It gives algorithms for their discretization including a minimally invasive modular step to adapt an eddy viscosity code to the extended models. A numerical test is given with standard and over diffusive eddy viscosity models. The correction (scaled by $10^{-8}$) does exhibit intermittent backscatter.

Key words. eddy viscosity, backscatter, complex turbulence

1. Introduction. The sensitivity and richness of scales in turbulent flows motivate many approaches to modelling turbulence. Fundamentally, they involve choosing an averaging operator, separating variables into means and fluctuations, deriving non-closed equations for means and modelling the effect of fluctuations on means. One fundamental approach is statistical modeling which begins with ensemble averaging. Usually the ergodic hypothesis is invoked and ensemble averaging is converted into time averaging, leading to a collection of RANS and URANS models of increasing complexity. Local spacial averaging and increasing computing power has led to reinterpretation of early, simple statistical models as LES models (identifying the mixing length with the averaging radius). The boundaries between these (and other) approaches overlap considerably.

With increased computational resources, solving for a moderately sized velocity ensemble begins to be feasible and is required to quantify uncertainty. Thus, ensemble averaging and statistical models (without invoking ergodicity) become possible and are considered herein. Most\(^1\) statistical models are based on eddy viscosity (EV) with differences in how the eddy viscosity coefficient is determined from the flow variables. Due to the wide experience with them, their limitations are also well recognized. EV models represent only dissipative effects of the Reynolds stresses and cannot represent intermittent energy flow from turbulent fluctuations back to the mean velocity\(^2\) without ad hoc fixes, such as negative viscosities ("more than a little strange" in [29], § 5.9, p. 373). Based on an exact equation derived in Section 2 linking this energy flow to the Reynolds stresses, Section 3 shows how to correct any eddy viscosity models systematically to include this intermittent energy flow from fluctuations back to means. The proposed correction is based on a new and fundamental derivation of eddy viscosity models. Various aspects of the corrected models are explored in Sections 4, 5, 6 and 7.

\(^1\)For example, only Chapter 6 of [39] addressed non-eddy viscosity models.

\(^2\)This flow is called backscatter for local spacial averaging. It occurs also for statistical models and URANS models. For RANS models based on long time averaging the result of Chacon-Rebollo and Lewandowski, Remark 2.4 below, confirms that it does not occur in a space averaged sense.
1.1. Formulation. To begin, given an ensemble of initial conditions

\[ u(x, 0; \omega_j) = u_0(x; \omega_j), j = 1, \cdots, J, \ x \in \Omega, \]

let \( u(x, t; \omega_j), p(x, t; \omega_j) \) be associated solutions to the incompressible Navier-Stokes equations (NSE) driven by a body force \( f(x, t) \)

\[
u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f(x, t), \quad \text{and} \quad \nabla \cdot u = 0, \quad \text{in} \quad \Omega, \]

\[ u = 0 \quad \text{on} \quad \partial \Omega. \]  \hfill (1.1)

Let \( \langle \cdot \rangle \) denote ensemble averaging

\[
\langle u \rangle (x, t) := \frac{1}{J} \sum_{j=1}^{J} u(x, t; \omega_j) \quad \text{and} \quad u'(x, t; \omega_j) = u(x, t; \omega_j) - \langle u \rangle (x, t).
\]

Ensemble averaging the NSE yields the non-closed system: \( \nabla \cdot \langle u \rangle = 0 \) and

\[
\langle u \rangle_t + \langle u \rangle \cdot \nabla \langle u \rangle - \nu \Delta \langle u \rangle - \nabla \cdot R(u, u) + \nabla \langle p \rangle = f(x, t), \] \hfill (1.2)

where the Reynolds stress \( R(u, u) \), e.g., [39] §2.4, [10] §3.2, [29] §5.1, is

\[
R(u, u) := \langle u \rangle \otimes \langle u \rangle - \langle u \otimes u \rangle = - \langle u' \otimes u' \rangle.
\]

Statistical models of turbulence (based on the Boussinesq assumption) begin with ensemble averaging and replace \( R(u, u) \) by an enhanced viscous term depending only on \( \langle u \rangle \). Based on the material of Section 2, Section 3 shows that these eddy viscosity models are based on three steps.

1. The Boussinesq assumption (from [4], [33] and Theorem 2.3 below) that turbulent fluctuations (the action of \( \nabla \cdot R(u, u) \) in (1.2)) are dissipative on average in (1.2). This is followed by assuming that space-time averaged dissipativity holds pointwise in time and space.

2. The eddy viscosity hypothesize that this dissipativity aligns with the gradient or deformation tensor and thus can be represented by a viscous term with a turbulent viscosity coefficient \( \nu_T(\langle u \rangle) \), [31].

3. Model parametrization/calibration is done by fitting the turbulent viscosity coefficient \( \nu_T(\langle u \rangle) \) to flow data. Calibration is equivalent to specifying a fluctuation model for \( \nabla w' \) in terms of \( \nabla \langle u \rangle \).

The resulting eddy viscosity model (whose solution \( w(x, t) \) with pressure \( q(x, t) \) is intended to be an approximation of the true average velocity \( \langle u \rangle \) results: \( \nabla \cdot w = 0 \) and

\[
w_t + w \cdot \nabla w - \nabla \cdot ([\nu + \nu_T(w)] \nabla w) + \nabla q = f(x, t), \quad \text{in} \quad \Omega, \]

\[ w = 0 \quad \text{on} \quad \partial \Omega \quad \text{and} \quad w(x, 0) = \langle u_0 \rangle \quad \text{in} \quad \Omega. \] \hfill (EV)

EV models are the models of choice for most industrial turbulent flows and many increasingly complex parametrizations of \( \nu_T(w) \) are known, e.g., [39], [10], [26], [28], [38]. They have well recognized limitations in not modeling complex turbulence, backscatter or turbulence not at statistical equilibrium, e.g., [13], [27], [35], [37]. (The "Virtually all practical engineering computations are done with some variety of eddy viscosity formulation..." [10] §7.4.1, p. 195.)
second assumption that the dissipativity of the Reynolds stress term aligns with $\nabla \langle u \rangle$ also fails for some flows, [27], but is not the issue addressed herein.)

The correction required for eddy viscosity models to represent backscatter in non-statistically stationary turbulence, the case when the action of the fluctuations is intermittently non-dissipative, is derived and analyzed herein. Given the eddy viscosity parameterization $\nu_T(w)$, choose a re-scaling parameter $\beta > 0$ and define

$$a(w) := \sqrt{\nu^{-1} \nu_T(w)}.$$

The corrected EV model (derived in Section 2) is then $\nabla \cdot w = 0$ and

$$w_t + \beta^2 a(w) \frac{\partial}{\partial t} (a(w)w) + w \cdot \nabla w - \nabla \cdot [(\nu + \nu_T(w)) \nabla w] + \nabla q = f(x, t).$$

In Section 3 time averaged dissipativity, an important feature of the true Reynolds stresses, is proven to be preserved in (Corrected EV).

The (Corrected EV) differs from (EV) by the extra term $\beta^2 a(w) \frac{\partial}{\partial t} (a(w)w)$. This term means time discretization introduces new issues, especially in adapting legacy codes from (EV) to (Corrected EV). Section 4 shows how time discretization can be done and preserve these important model properties, including the important case of modular adaptation of a legacy code available for (EV). Phenomenology is used to obtain some insight into calibration of the re-scaling parameter $\beta$ in Section 7. Section 8 tests corrections of a standard mixing length model and the over-diffused Smagorinsky model. Even with a very small re-scaling of the correction, $\beta^2 \simeq O(10^{-8})$, the numerical test shows that the corrected model does exhibit backscatter.

2. The Boussinesq Assumption. This section first recalls the formulation of the Boussinesq assumption and its proof (for strong solutions) from [21]. The variance equation derived in its proof is essential for the model development in Section 3 onward. A new proof for general weak solutions is given in Section 3.1.

Eddy viscosity models are based on the early work of Saint-Venant and his famous student Boussinesq, [7]. As used today they are very little changed from work of Prandtl, Kolmogorov and von Karman, [11]. We thus begin with the early work. The Boussinesq assumption and the eddy viscosity hypothesis ($R(u, u)$ aligns with $\nabla \langle u \rangle$), developing earlier ideas of Saint-Venant [33], are, respectively often expressed as:

Boussinesq assumption: turbulent fluctuations are dissipative on the mean,

EV hypothesis: this dissipativity can be modeled by $-\nabla \cdot (\nu_T \nabla \langle u \rangle)$.

Boussinesq [4] justified dissipativity (and a mixing length eddy viscosity model) by an analogy with the kinetic theory of gases identifying the mean free pass with an early mixing length idea. If there is a scale separation, mathematical support exists for both, well summarized in Chapters 11 and 12 in [28], see also [5]. Boussinesq’s assumption of scale separation has been criticized since turbulent flows contain a near continuum of scales, e.g., [27], [13], [35]. The essential idea in the proof is implicit in energy equations for means and fluctuations, e.g., Doering and Gibbon [8], Chapter 3.

Proceeding formally for a moment, take the dot product of the ensemble averaged NSE, (1.2) above, with $\langle u \rangle$, integrate over the flow domain $\Omega$ and integrate by parts. This gives the energy equation for the mean flow

$$\frac{d}{dt} \frac{1}{2} \int_\Omega |\langle u \rangle|^2 dx + \int_\Omega \nu |\nabla \langle u \rangle|^2 dx + \int_\Omega R(u, u) : \nabla \langle u \rangle dx = \int_\Omega f(x, t) \cdot \langle u \rangle dx. \quad (2.1)$$
The first term, $\frac{1}{2} \int_\Omega |\langle u \rangle|^2 \, dx$, is the rate of change of kinetic energy of the mean flow. The second, $\int_\Omega \nu |\nabla \langle u \rangle|^2 \, dx$, is the rate of energy dissipation (since the term is non-negative) of the mean flow due to molecular viscosity. The third term, $\int_\Omega R(u, u) : \nabla \langle u \rangle \, dx$, is the effect of fluctuations upon the mean flow and the last term, $\int_\Omega f \cdot \langle u \rangle \, dx$, is the rate of energy input.

**Definition 2.1 (Variance).** The variance of $u$ and $\nabla u$ are, respectively,

$$V(u) := \int_\Omega \langle |u|^2 \rangle - |\langle u \rangle|^2 \, dx$$

and

$$V(\nabla u) := \int_\Omega \langle |\nabla u|^2 \rangle - |\nabla \langle u \rangle|^2 \, dx.$$ 

Variance is nonnegative and measures the size of fluctuations according to the following well known identity.

**Lemma 2.2.** We have

$$V(u) = \int_\Omega \langle |u'|^2 \rangle \, dx \geq 0 \quad \text{and} \quad V(\nabla u) = \int_\Omega \langle |\nabla u'|^2 \rangle \, dx \geq 0.$$ 

$LIM$ denotes a Banach or generalized limit and will be used for long time averages, e.g., [8], [6]. $||\cdot||, (\cdot, \cdot)$ denote the usual $L^2(\Omega)$ norm and inner product.

**Theorem 2.3 (Boussinesq).** Suppose that each realization is a strong solution of the NSE. The ensemble is generated by different initial data and $u(x, 0; \omega_j) \in L^2(\Omega)$, $f(x, t) \in L^\infty(0, \infty; L^2(\Omega))$. Then,

$$LIM_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega R(u, u) : \nabla \langle u \rangle \, dx \, dt =$$

$$= LIM_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega \nu |\nabla u'|^2 \, dx \, dt \geq 0.$$ 

In particular, for any sequence $T_i \to \infty$ for which the limit exists (including the limit infimum and supremum)

$$\lim_{T_i \to \infty} \frac{1}{T_i} \int_0^{T_i} \int_\Omega R(u, u) : \nabla \langle u \rangle \, dx \, dt \geq 0.$$ 

Furthermore, the variance of strong solutions satisfies

$$\frac{d}{dt} \frac{1}{2} V(u(t)) + \nu V(\nabla u(t)) = \int_\Omega R(u, u) : \nabla \langle u \rangle \, dx,$$

equivalently

$$\frac{1}{2} \frac{d}{dt} \langle ||u'(t)||^2 \rangle + \nu \langle ||\nabla u'||^2 \rangle = \int_\Omega R(u, u) : \nabla \langle u \rangle \, dx.$$ 

**Proof.** The proof strategy is as follows. We calculate the energy equality for $\langle \frac{1}{2}||u||^2 \rangle$ and for $\frac{1}{2}||\langle u \rangle||^2$. By subtracting, we obtain an estimate for the variance of $\nabla u$. The variance evolution equation contains one inconvenient term which is eliminated by time averaging. We begin with the standard energy equality for strong solutions of each realization for the NSE

$$\frac{d}{dt} \frac{1}{2} ||u||^2 + \nu ||\nabla u||^2 = \int_\Omega f(x, t) \cdot u \, dx.$$ 

(2.2)
Using the Poincaré-Friedrichs inequality \( ||\nabla u||^2 \geq \lambda_0 ||u||^2 \) (where \( \lambda_0 > 0 \)) and \( (f, u) \leq (\nu \lambda_0)/2 ||u||^2 + C||f||^2 \) on the RHS we have

\[
\frac{d}{dt} \frac{1}{2} ||u||^2 + \frac{\nu \lambda_0}{2} ||u||^2 \leq C||f||^2.
\]  

(2.3)

Thus, using an integrating factor, for any \( T \)

\[
||u(T)||^2 \leq ||u^0||^2 + \int_0^T e^{-\nu \lambda_0 (T-t)} ||f(t)||^2 \, dt
\]

\[
\leq ||u^0||^2 + C \sup_{0 < t < \infty} ||f||^2 < \infty
\]

for each realization. Thus, each \( ||u(T)||^2 \) is uniformly bounded in \( T \) (a well known result). Time averaging the energy equality over \([0, T]\) gives, for each realization

\[
\frac{1}{2T} ||u(T)||^2 - \frac{1}{2T} ||u^0||^2 + \frac{1}{T} \int_0^T \nu ||\nabla u||^2 \, dt = \frac{1}{T} \int_0^T \int_{\Omega} f(x, t) \cdot u \, dx \, dt
\]

(2.4)

\[
\leq \sqrt{\frac{1}{T} \int_0^T ||f||^2 \, dt} \sqrt{\frac{1}{T} \int_0^T ||u||^2 \, dt} \leq C.
\]

Since \( u(x, 0; \omega_j) \in L^\infty(0, \infty; L^2(\Omega)) \), \( f(x, t) \in L^\infty(0, \infty; L^2(\Omega)) \), this implies the uniform in \( T \) bounds:

\[
||u(T)|| \leq C < \infty \quad \text{and} \quad \frac{1}{T} \int_0^T ||\nabla u||^2 \, dt \leq C < \infty.
\]

Thus, since \( ||\langle u \rangle|| \leq \langle ||u||\rangle \) and \( ||\nabla \langle u \rangle||^2 = \langle ||\nabla u||^2 \rangle \leq \langle ||\nabla u||^2 \rangle \) (versions of the triangle inequality) we also have similar uniform in time bounds on averages

\[
||\langle u \rangle (\cdot, T)|| \leq \langle ||u(\cdot, T; \omega_j)||\rangle \leq C < \infty \quad \text{and}
\]

\[
\frac{1}{T} \int_0^T ||\nabla \langle u \rangle||^2 \, dt \leq C \frac{1}{T} \int_0^T \langle ||\nabla u||^2 \rangle \, dt \leq C < \infty.
\]

Ensemble averaging (2.4) yields:

\[
\frac{1}{T} \left\{ \left( \frac{1}{2} ||u(T)||^2 \right) - \frac{1}{2} ||u^0||^2 \right\} + \frac{1}{T} \int_0^T \langle \nu ||\nabla u(t)||^2 \rangle \, dt
\]

(2.5)

\[
= \frac{1}{T} \int_0^T \int_{\Omega} f(x, t) \cdot \langle u \rangle \, dx \, dt.
\]

Next, time average (2.1) over \([0, T]\). This gives

\[
\frac{1}{T} \left\{ \frac{1}{2} \langle ||u(T)||^2 \rangle - \frac{1}{2} \langle ||u^0||^2 \rangle \right\} + \frac{1}{T} \int_0^T \nu \langle ||\nabla \langle u \rangle(t)||^2 \rangle \, dt
\]

(2.6)

\[
+ \frac{1}{T} \int_0^T \left( \int_{\Omega} R(u, u) : \nabla \langle u \rangle \, dx \right) \, dt = \frac{1}{T} \int_0^T \left( \int_{\Omega} f(x, t) \cdot \langle u \rangle \, dx \right) \, dt.
\]

Consider the terms in the above. Since, as noted above, \( ||\langle u \rangle|| \leq C \) the first term (in braces) \( \to 0 \) as \( T \to \infty \). The second term is uniformly bounded in \( T \) as is the last term (on the RHS). Thus the third, Reynolds stress, term is also uniformly bounded.
in $T$. Thus, generalized limits of this term exist as $T \to \infty$. Subtracting (2.6) from (2.5) gives

$$\frac{1}{T} \left\{ \left\{ \frac{1}{2} |u(T)|^2 \right\} - \left\{ \frac{1}{2} |u^0|^2 \right\} - \frac{1}{2} \int_\Omega |\langle u \rangle (T)|^2 dx + \frac{1}{2} \int_\Omega |\langle u \rangle (0)|^2 dx \right\} \quad (2.7)$$

$$+ \frac{\nu}{T} \int_0^T \left( \|\nabla u(t)\|^2 - \|\nabla \langle u \rangle (t)\|^2 \right) dt$$

$$= \frac{1}{T} \int_0^T \int_\Omega R(u, u) : \nabla \langle u \rangle dx dt.$$

As $T \to \infty$ the first term (in braces) $\to 0$. Thus, we have

$$\lim_{T \to \infty} \frac{\nu}{T} \int_0^T \left( \|\nabla u(t)\|^2 - \|\nabla \langle u \rangle (t)\|^2 \right) dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega R(u, u) : \nabla \langle u \rangle dx dt.$$

The LHS integrand is the variance of $\nabla u$ so that

$$\langle \|\nabla u(t)\|^2 \rangle - \|\nabla \langle u \rangle (t)\|^2 = \langle \|\nabla u'(t)\|^2 \rangle.$$

Thus, as claimed,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_\Omega R(u, u) : \nabla \langle u \rangle dx dt$$

$$= \lim_{T \to \infty} \frac{\nu}{T} \int_0^T \langle \|\nabla u(t)\|^2 \rangle - \|\nabla \langle u \rangle (t)\|^2 \rangle dt$$

$$= \lim_{T \to \infty} \frac{\nu}{T} \int_0^T \langle \|\nabla u'(t)\|^2 \rangle dt \geq 0.$$

Note that (2.7) can be rewritten as

$$\frac{1}{2T} \left\{ V(u(T)) - V(u^0) \right\} + \frac{\nu}{T} \int_0^T V(\nabla u(t)) dt = \frac{1}{T} \int_0^T \int_\Omega R(u, u) : \nabla \langle u \rangle dx dt.$$

Multiply through by $T$ and then take $d/dT$. This establishes the claimed variance evolution equation:

$$\frac{1}{2} \frac{d}{dt} \langle \|u(t)\|^2 \rangle + \nu \langle \|\nabla u(t)\|^2 \rangle = \int_\Omega R(u, u) : \nabla \langle u \rangle dx,$$

equivalently

$$\frac{1}{2} \frac{d}{dt} \langle \|u'(t)\|^2 \rangle + \nu \langle \|\nabla u'(t)\|^2 \rangle = \int_\Omega R(u, u) : \nabla \langle u \rangle dx.$$

\[ \Box \]

Remark 2.4 (The proof of Chacon Rebollo-Lewandowski for time averages). When $f = f(x)$ and the averaging operator is the classical Reynolds averaging, given by

$$\langle v \rangle (x) := \lim_{T_i \to \infty} \frac{1}{T_i} \int_0^{T_i} v(x, t) dt,$$
dissipativity was proven in 2014 in Theorem 3.5, p. 70 of [6]. They show that for weak solutions of the 3d (likely the same proof holds in 2d) NSE, there exists a subsequence $T\rightarrow\infty$ such that the above limit exists. The average, so defined, satisfies

$$
\langle u \rangle \cdot \nabla \langle u \rangle - \nu \triangle \langle u \rangle + F + \nabla \langle p \rangle = f(x) \text{ and } \nabla \cdot \langle u \rangle = 0,
$$

where the extra term $F$ incorporates the effects of the fluctuations on the means under the limiting process. This implies that

$$
\nu \|\nabla \langle u \rangle\|^2 + (F, \langle u \rangle) = (f, \langle u \rangle).
$$

The triangle inequality and time averaging the energy inequality easily show that

$$
\nu \|\nabla \langle u \rangle\|^2 \leq \langle \nu \|\nabla u\|^2 \rangle \leq (f, \langle u \rangle).
$$

Subtracting then proves dissipativity: $(F, \langle u \rangle) \geq 0$.

2.1. The proof for weak solutions. The classical development of turbulence models is through the Reynolds stresses. This approach is very natural for strong solutions but not for weak solutions. In this section we will prove the Boussinesq hypothesis for weak solutions bypassing the Reynolds stresses by an alternate (and easier) route through the energy inequality. It is known, e.g., [15], for each $\omega_j$, weak distributional solutions exist satisfying the energy inequality

$$
\frac{d}{dt} \frac{1}{2} \|u(t)\|^2 + \nu \|\nabla u\|^2 + \int_{\Omega} D(u)dx = \int_{\Omega} f(x, t) \cdot udx.
$$

If strict inequality ("<") holds at some instants in time (rather than equality) then the NSE contain an extra dissipative mechanism other than molecular viscosity, [15]. If so, this is believed to be dissipation through pumping of energy by the nonlinearity to successively smaller scales. There has been a number of papers studying features of this additional dissipation, e.g., [9], [34]. For example, Duchon and Robert [9] proved (modulo a subsequence) that one can add an (unknown) dissipative term $(D(u)\text{ where } \int_{\Omega} D(u)dx \geq 0)$ to the NSE restoring the energy equality

$$
\frac{d}{dt} \frac{1}{2} \|u(t)\|^2 + \nu \|\nabla u\|^2 + \int_{\Omega} D(u)dx = \int_{\Omega} f(x, t) \cdot udx.
$$

(2.8)

The operator $D(u)$ can thus have the interpretation of an effect of small scales or fluctuations through the nonlinearity. Taking the ensemble average gives

$$
\frac{d}{dt} \frac{1}{2} \|\langle u(t)\rangle\|^2 + \nu \|\nabla \langle u\rangle\|^2 + \left( \int_{\Omega} D(u)dx \right) = \int_{\Omega} f \cdot \langle u \rangle dx.
$$

This equality can be algebraically rearranged by

$$
\|\langle u\rangle\|^2 = \|\langle u\rangle\|^2 + \left[\|\langle u\rangle\|^2 - \|\langle u\rangle\|^2\right] = \|\langle u\rangle\|^2 + \text{Var}(u)
$$

and

$$
\|\nabla \langle u\rangle\|^2 = \|\nabla \langle u\rangle\|^2 + \text{Var}(\nabla u) = \|\nabla \langle u\rangle\|^2 + \|\nabla \langle u\rangle\|^2.
$$

The result is a simple rearrangement of the energy equality

$$
\frac{d}{dt} \left[ \frac{1}{2} \|\langle u(t)\rangle\|^2 + \frac{1}{2} \|u(t)\|^2 \right] + \nu \|\nabla \langle u\rangle\|^2
$$

$$
+ \left( \nu \|\nabla u\|^2 + \int_{\Omega} D(u)dx \right) = \int_{\Omega} f \cdot \langle u \rangle dx.
$$

(2.9)
Comparing this energy balance with the case of strong solutions, it is easy to identify the terms arising from the fluctuations:

\[ \text{Fluctuations’ effects: } \frac{d}{dt} \left( \frac{1}{2} \langle ||u'(t)||^2 \rangle \right) + \left\langle \nu ||\nabla u'||^2 \right\rangle + \left\langle \int_{\Omega} D(u) dx \right\rangle. \]

The Boussinesq hypothesis follows by exactly the same proof from this point onwards as in the case of strong solutions.

**Theorem 2.5 (Boussinesq hypothesis for weak solutions).** For weak solutions of the NSE satisfying (2.8) we have

\[ \text{LIM}_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left( \frac{d}{dt} \left( \frac{1}{2} ||u'(t)||^2 \right) + \nu ||\nabla u'||^2 + \int_{\Omega} D(u) dx \right) dt = \]

\[ \text{LIM}_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left( \nu ||\nabla u'||^2 + \int_{\Omega} D(u) dx \right) dt \geq 0. \]

**3. Correcting EV Models for Non-equilibrium Effects.** It is widely believed that, for example, the flow behind a screen in a wind tunnel with constant inflows will "forget" its initial conditions after some time. The mean flow then becomes independent of initial perturbations and

\[ \frac{d}{dt} \langle ||u'(T)||^2 \rangle = 0 \]

for \( T \) large enough. Such a flow is called statistically steady. The assumption of statistically steady turbulence is fundamental to the statistical theory of turbulence but is as murky as the whole field. Unfortunately, for many flows it is not a reasonable assumption (“... turbulence does not adjust rapidly to imposed mean straining ...” p. 362 in [30]). This section presents a coherent method (with two realizations) for adapting eddy viscosity models to non-equilibrium turbulence.

Statistically steady state is often expressed as the case when \( \mathcal{P}/\varepsilon = 1 \) where

\[ \mathcal{P} = \text{production of turbulent kinetic energy (TKE)} = \int_{\Omega} R(u, u) : \nabla \langle u \rangle dx, \]

\[ \varepsilon = \text{dissipation of TKE} = \nu \langle ||\nabla u'||^2 \rangle = \nu V(\nabla u(t)). \]

The variance of strong solutions obeys an exact balance equation

\[ \frac{1}{2} \frac{d}{dt} \langle ||u'(t)||^2 \rangle + \nu \langle ||\nabla u'||^2 \rangle = \int_{\Omega} R(u, u) : \nabla \langle u \rangle dx \]

that reveals (i) the above definition of statistical equilibrium \( \frac{d}{dt} \langle ||u'||^2 \rangle = 0 \) is equivalent to \( \mathcal{P}/\varepsilon = 1 \), and (ii) the term \( \frac{d}{dt} \langle ||u'(t)||^2 \rangle \) omitted by assuming statistical equilibrium and must therefore be accounted for in an extension of eddy viscosity models.

This section shows that eddy viscosity models are based on three assumptions and that they are only consistent in a time averaged sense. Next we show how to take a given eddy viscosity model and extend it to be energy consistent with the NSE pointwise in time.
3.1. Eddy viscosity models for statistically steady turbulence. The ensemble averaged Navier Stokes equations, (1.2) above, involve the non-closed Reynolds stress \( R(u, u) = -\langle u' \otimes u' \rangle \). This term, which must be modeled, accounts for all effects of the fluctuations on the mean flow. The equation for the kinetic energy evolution of \( \langle u \rangle \) is

\[
\frac{d}{dt} \frac{1}{2} \| \langle u \rangle \|^2 + \nu \| \nabla \langle u \rangle \|^2 + \int_{\Omega} R(u, u) : \nabla \langle u \rangle \, dx = (f, \langle u \rangle).
\]

Thus, the effect of fluctuations on the mean flow is determined by the sign of \( RS(t) := \int_{\Omega} R(u, u) : \nabla \langle u \rangle \, dx \).

When \( RS(t) > 0 \), the effect of \( R(u, u) \) is dissipative while when \( RS(t) < 0 \), (volume averaged) statistical-backscatter\(^4\) occurs and fluctuations increase the energy in the mean flow. In Section 2 (following [21]), two key properties of this Reynolds stress term were proven: time averaged dissipativity and an equation for the evolution of variance of fluctuations:

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T RS(t) \, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T \nu \langle |\nabla u'|^2 \rangle \, dx dt \geq 0, \quad (3.1)
\]

\[
\int_{\Omega} R(u, u) : \nabla \langle u \rangle \, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \langle |u'|^2 \rangle \, dx + \nu \int_{\Omega} \langle \nabla u' : \nabla u' \rangle \, dx. \quad (3.2)
\]

Eddy viscosity models then follow from three assumptions. First, the statistical equilibrium assumption that dissipativity holds approximately at every instant in time

\[
\int_{\Omega} R(u, u) : \nabla \langle u \rangle \, dx \simeq \int_{\Omega} \nu \langle |\nabla u'|^2 \rangle \, dx. \quad (3.3)
\]

The second is that \( \nabla u' \) aligns with \( \nabla \langle u \rangle \). Third, calibration\(^5\) provides a model of the fluctuations in terms of the mean flow

\[
\text{action}(\nabla u') \simeq a(\langle u \rangle) \nabla \langle u \rangle. \quad (3.4)
\]

With these three we have

\[
\int_{\Omega} R(u, u) : \nabla \langle u \rangle \, dx \simeq \int_{\Omega} \nu a(\langle u \rangle)^2 \nabla \langle u \rangle : \nabla \langle u \rangle \, dx
\]

\[
= \int_{\Omega} -\nabla \cdot (\nu a(\langle u \rangle)^2 \nabla \langle u \rangle) \cdot v \, dx \text{ evaluated at } v = \langle u \rangle.
\]

Letting \( \nu_T(\langle u \rangle) = \nu a(\langle u \rangle)^2 \), this yields the eddy viscosity closure

\[-\nabla \cdot R(u, u) \iff -\nabla \cdot (\nu_T(\langle u \rangle) \nabla \langle u \rangle) + \text{ terms incorporated in } \nabla p.\]

\(^4\)When using ensemble averaging, we shall call flow of energy from fluctuations back to means statistical-backscatter.

\(^5\)Alternatively, the dissipation \( \langle \nu |\nabla u'|^2 \rangle \) in (3.3) is replaced in the model by \( \nu_T(\langle u \rangle) |\nabla \langle u \rangle|^2 \). Ideally, then \( \langle |\nabla u'|^2 \rangle \simeq \nu^{-1} \nu_T(\langle u \rangle) |\nabla \langle u \rangle|^2 \) so \( a(\langle u \rangle) = \sqrt{\nu^{-1} \nu_T(\langle u \rangle)} \).
### 3.2. The statistically non-steady case.

Far from equilibrium, step (3.3) omits \( \frac{1}{2} \frac{d}{dt} \int_{\Omega} \langle |u'|^2 \rangle \, dx \) in (3.2). This is the term that accounts for backscatter and other non-equilibrium effects. To model this term, \( u' \) must be expressed in terms of \( \langle u \rangle \). We present two natural approaches.

**Approach 1:** To model \( u' \) in terms of \( \langle u \rangle \), the simplest is to re-scale (by \( \beta \), Section 4) the fluctuation model (3.4), yielding

\[
\text{action}(u') \simeq \beta a(\langle u \rangle) \langle u \rangle.
\]

This assumption yields

\[
\int_{\Omega} R(u, u) : \nabla \langle u \rangle \, dx \simeq \frac{1}{2} \frac{d}{dt} \int_{\Omega} \beta^2 a(\langle u \rangle)^2 \langle u(x, t) \rangle^2 dx + \int_{\Omega} \nu a(\langle u \rangle)^2 |\nabla \langle u \rangle (x, t)\rangle^2 dx,
\]

arising from an anisotropic time derivative \( \beta^2 a(\langle u \rangle) \frac{\partial}{\partial t} (a \langle u \rangle) \langle u \rangle \) and an eddy viscosity term \( -\nabla \cdot (\nu_T(\langle u \rangle) \nabla \langle u \rangle) \) where \( \nu_T(\langle u \rangle) = \nu a(\langle u \rangle)^2 \). This gives the closure model

\[
-\nabla \cdot R(u, u) \simeq \beta^2 a(\langle u \rangle) \frac{\partial}{\partial t} (a \langle u \rangle \langle u \rangle) - \nabla \cdot (\nu_T(\langle u \rangle) \nabla \langle u \rangle)
\]

and thus we have the corrected model: \( \nabla \cdot w = 0 \) and

\[
w_t + \beta^2 a(w) \frac{\partial}{\partial t} (a(a)w) + w \cdot \nabla w + \nabla p - \nabla \cdot [(\nu + \nu_T)(w) \nabla w] = f.
\]

**Approach 2:** To model \( u' \) in terms of \( \langle u \rangle \) in an alternate way, recall the Kolmogorov-Prandtl relation for the turbulent viscosity

\[
\nu_T(\langle u \rangle) = l \sqrt{k'} \text{ where } k' = \frac{1}{2} \langle |u'|^2 \rangle \text{ and } l = \text{ quantity with units length}.
\]

Since the LHS is in terms of means while the RHS is in terms of fluctuations, this relation may be used to infer a fluctuation model. This is simplest to illustrate through an example. Consider the (simplest case of) Baldwin-Lomax model for which \( \nu_T(\langle u \rangle) = l^2 |\nabla \times \langle u \rangle| \). Since this must agree with the Kolmogorov-Prandtl relation we have \( l^2 |\nabla \times \langle u \rangle| = l \sqrt{\langle |u'|^2 \rangle} \) suggesting the fluctuation model

\[
\text{action}(u') \simeq l |\nabla \times \langle u \rangle|.
\]

This fluctuation model yields

\[
\int_{\Omega} R(u, u) : \nabla \langle u \rangle \, dx \simeq \frac{1}{2} \frac{d}{dt} \int_{\Omega} l^2 |\nabla \times \langle u \rangle (x, t)\rangle^2 dx + \int_{\Omega} l^2 |\nabla \times \langle u \rangle | |\nabla \langle u \rangle (x, t)\rangle^2 dx,
\]

arising from an anisotropic time derivative \( l^2 \frac{\partial}{\partial t} (\nabla \times \langle u \rangle) \) and an eddy viscosity term \( -\nabla \cdot (\nu_T(\langle u \rangle) \nabla \langle u \rangle) \) where \( \nu_T(\langle u \rangle) = l^2 |\nabla \times \langle u \rangle| \). This gives the closure model

\[
-\nabla \cdot R(u, u) \simeq l^2 \frac{\partial}{\partial t} (\nabla \times \langle u \rangle) - \nabla \cdot (l^2 |\nabla \times \langle u \rangle| \nabla \langle u \rangle)
\]

and thus we have the corrected model: \( \nabla \cdot w = 0 \) and

\[
w_t + l^2 \frac{\partial}{\partial t} (\nabla \times \nabla \times w) + w \cdot \nabla w + \nabla p - \nabla \cdot [(\nu + l^2 |\nabla \times w|) \nabla w] = f,
\]

where the mixing length is specified as in the Baldwin-Lomax model, [39], Section 3.4.2, p. 81 or [10], Section 6.1.2, p. 120.
4. Three Criticisms of the Corrected Models. The process of criticism and improvement is essential to progress in science. Two serious criticisms (below) of the corrected models have been conveyed to the authors and we add a third of our own. We note first that the only perfect model of turbulence is the Navier-Stokes equations and eddy viscosity models already violate the reversibility property of the true Reynolds stresses. The adjustments to EV models for non-equilibrium effects proposed in Section 3 are only a first step on a path which seems promising. Since first steps should be followed by better steps, we now present the three criticisms.

Criticism 1. The first is that the true Reynolds stresses enter in the equations for the mean velocity as a divergence of a tensor \( \nabla \cdot R(u, u) \). Over a volume element, \( \nabla \cdot R(u, u) \) acts as a distribution of a surface force. However, the corrected models replace it with

\[
\text{Approach 1: } \beta^2 a(w) \frac{\partial}{\partial t} (a(w)w) - \nabla \cdot (\nu_T(w)\nabla w),
\]
\[
\text{Approach 2: } l^2 \frac{\partial}{\partial t} (\nabla \times \nabla \times w) - \nabla \cdot (l^2 |\nabla \times w| \nabla w).
\]

The substitution in Approach 1 includes both a surface force \(-\nu_T(w)\nabla w\) and a volume force \(\beta^2 a(w) \frac{\partial}{\partial t} (a(w)w)\). The substitution in Approach 2 for the Baldwin-Lomax model does replace \(\beta^2 a(w) \frac{\partial}{\partial t} (a(w)w)\) with two surface force terms. From this point of view, Approach 2 is preferable (and yields a simpler model).

Criticism 2. The second criticism is that, due to the use of \(\partial/\partial t\) rather than the material derivative \(D/Dt\) in the new terms \(\beta^2 a(w) \frac{\partial}{\partial \tau} (a(w)w)\) and \(l^2 \frac{\partial}{\partial \tau} (\nabla \times \nabla \times w)\), the resulting models violate Galilean invariance. This may be not so serious for the numerical test given in a fixed domain. However, there are many important flow problems for which it is a serious criticism. The obvious attempted correction is to replace \(\partial/\partial t\) with \(D/Dt\) in the new terms as follows

\[
\text{Approach 1: } \beta^2 a(w) \frac{\partial}{\partial \tau} (a(w)w) \Leftrightarrow \beta^2 a(w) \big[ \frac{\partial}{\partial \tau} (a(w)w) + w \cdot \nabla (a(w)w) \big],
\]
\[
\text{Approach 2: } l^2 \frac{\partial}{\partial \tau} (\nabla \times \nabla \times w) \Leftrightarrow l^2 \big[ \frac{\partial}{\partial \tau} (\nabla \times \nabla \times w) + w \cdot \nabla (\nabla \times \nabla \times w) \big].
\]

The theory developed for the model is based on extracting information from the energy equation of the model. For the model based on Approach 1 the extra term that would be introduced by using the material derivative satisfies

\[
(\beta^2 a(w)w \cdot \nabla (a(w)w), w) = \beta^2 (w \cdot \nabla (a(w)w), a(w)w) = 0,
\]
which is exactly as needed to develop the theory. Thus, for Approach 1 Galilean invariance can be restored as above (at the cost of making time discretization more complex).

For Approach 2 the extra term’s influence in the energy balance of the model is (in the most favorable case when all boundary integrals vanish)

\[
l^2 (w \cdot \nabla (\nabla \times \nabla \times w), w) = l^2 (\omega \cdot \nabla \omega, w) \text{ where } \omega := \nabla \times w,
\]
which is non-zero in general. Thus, the theory developed for the model does not extend to the correction for lack of Galilean invariance from Approach 2. Finding the correct modification of Approach 2 is an open problem.

Criticism 3. The proposed corrections to EV models are developed to account for space aggregate statistical-backscatter only. It can occur that statistical-backscatter may occur in small subregions and be dominated by turbulent dissipation occurring through the rest of the domain. It is possible that for a well calibrated
model the calibration process gives sufficient accuracy that the prediction of spatially localized statistical-backscatter is accurate. However, there is no reason to believe it occurs from the construction of the model. We believe that analysis of the accuracy of models of the localization of backscatter (of all kinds) requires a deeper understanding of the phenomena.

4.1. Localization of Statistical Backscatter. Motivated by Criticism 3, this section shows that three statistics suffice to completely determine localization of backscatter. An equation for the space time evolution of variance is derived. From this, the primary turbulent flow statistics that influence localization of backscatter are isolated.

Definition 4.1 (Skewness and Flatness). Given ensemble \( \phi(x,t;\omega_j), j = 1, \ldots, J \), with variance \( V(\phi) \), its skewness \( S(\phi) \) and flatness \( F(\phi) \) are

\[
S(\phi) := \frac{\langle [\phi]\rangle - \langle \phi \rangle |\langle \phi \rangle|^2}{V^{3/2}(\phi)}, \text{ and } F(\phi) := \frac{\langle [\phi]^4 \rangle - |\langle \phi \rangle|^4}{V^2(\phi)}.
\]

We prove the following.

Theorem 4.2. Consider an ensemble of (sufficiently regular) solutions of the NSE. For this ensemble, the local in space and time variance satisfies the following

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} V(\phi) \right) + \nabla \cdot \left( \frac{1}{2} \nabla V(\phi) + \nabla \cdot \nabla \phi \right) - 2 \nabla \cdot \nabla (\phi) + \nabla \cdot \nabla \phi = 0.
\]

\textit{Proof.} The proof assumes sufficient regularity for the equation for the evolution of \( \frac{1}{2} |\phi|^2(x,t) \) to hold. Recall that the equation for the kinetic energy satisfied by each (sufficiently regular) solution of the NSE is

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} |\phi|^2 \right) + \nabla \cdot \left( \frac{1}{2} |\phi|^2 + p \right) - \nabla \cdot \nabla (\phi) - \nabla \cdot \nabla (\phi) = 0.
\]

Averaging (1.1) gives

\[
\frac{\partial}{\partial t} \langle \frac{1}{2} |\phi|^2 \rangle + \nabla \cdot \left( \langle \frac{1}{2} |\phi|^2 + p \rangle \right) - \nabla \cdot \nabla (\langle |\phi|^2 \rangle) + \frac{\nu}{2} |\nabla \langle |\phi|^2 \rangle|^2 = f \cdot (\phi).
\]

The mean velocities \( \langle u \rangle \) satisfy

\[
\langle u \rangle_t + \langle u \rangle \cdot \nabla \langle u \rangle - \nabla \cdot R(\langle u \rangle, \phi) + \nabla \langle p \rangle = f(x,t) \text{ and } \nabla \cdot \langle u \rangle = 0.
\]

Repeating the steps (taking the dot product with \( \langle u \rangle \) and using the same vector identities) leading to (1.1) gives

\[
\frac{\partial}{\partial t} \langle \frac{1}{2} |\phi|^2 \rangle + \nabla \cdot \left( \langle \phi \rangle \left( \frac{1}{2} |\phi|^2 + \langle p \rangle \right) \right) - \nu \Delta \langle |\phi|^2 \rangle + \frac{\nu}{2} \langle |\nabla \phi|^2 \rangle = f \cdot \langle \phi \rangle.
\]

The various cases of \( \phi, \psi \) being scalars, vectors or tensors will be clear from the context. We shall not give separate definitions for each case.
contact forces & $\frac{1}{2}S(u)V^{3/2}(u) + \text{Cov}(u,p) - \nu \nabla V(u)$ \\
body forces with zero time average & $\frac{\partial}{\partial t} \left( \frac{1}{2} V(u) \right)$ \\
persistent body forces & $\frac{\nu}{2} V(\nabla u)$ \\

Table 4.1

Subtracting (4.3) from (4.2) gives an equation for $\nabla \cdot R(u,u) \cdot \langle u \rangle$

$$
\frac{\partial}{\partial t} \left( \frac{1}{2} V(u) \right) + \nabla \cdot \left( \langle u \left[ \frac{1}{2} |u|^2 + p \right] \rangle - \langle u \rangle \left[ \frac{1}{2} |\langle u \rangle|^2 + \langle p \rangle \right] \right) \\
= -\nu \Delta V(u) + \frac{\nu}{2} V(\nabla u) = -\nabla \cdot R(u,u) \cdot \langle u \rangle.
$$

The second term can be written in terms of skewness and pressure-velocity covariance as follows, yielding the claimed formula,

$$
\frac{V^{3/2}(u)}{2} \langle \langle \frac{1}{2} |u|^2 + p \rangle \rangle - \langle u \rangle \left[ \frac{1}{2} |\langle u \rangle|^2 + \langle p \rangle \right] = \\
\nu \frac{3}{2} V^{3/2}(u) \langle \langle u \rangle \rangle^{2} - \langle u \rangle \langle \langle u \rangle \rangle^{2} + \langle up \rangle - \langle u \rangle \langle p \rangle = \frac{1}{2} V^{3/2}(u) S(u) + \text{Cov}(u,p).
$$

\[\square\]

**Remark 4.3.** The effect of the Reynolds stress on the mean flow’s kinetic energy depends only on (i) the variance of $u$ and $\nabla u$ for volume averaged backscatter and (ii) those plus the skewness of $u$ and the velocity-pressure covariances for pointwise backscatter:

$$
-\nabla \cdot R(u,u) \cdot \langle u \rangle = \frac{\partial}{\partial t} \left( \frac{1}{2} V(u) \right) + \frac{\nu}{2} V(\nabla u) \tag{4.4}
$$

$$
+ \nabla \cdot \left( \frac{1}{2} S(u) V^{3/2}(u) + \text{Cov}(u,p) - \nu \nabla V(u) \right).
$$

The terms on the RHS are a decomposition of the effects of the Reynolds stresses on the kinetic energy in the mean flow. This decomposition is shown in Table 4.1.

5. Analysis of The Corrected Smagorinsky Model. The classic example of an over-diffused model is the standard Smagorinsky model for which

$$
\nu_T(w) = (C_s l)^2 |\nabla w|.
$$

(For numerical values of $C_s$ when $l = \delta = \Delta x$ see [36] Table 1.) This has been used as a statistical model where $l =$ mixing length and more commonly as an LES model where $l = \delta = \Delta x$. Various fixes for it include van Driest damping (reducing near wall model dissipation) and Germano’s dynamic selection of $\mu = C_s^2(x,t)$. These are often successful but the latter leads to negative values of $\mu$ that model backscatter but can induce instabilities. Often these are clipped ($\mu \Leftarrow \max\{\mu, 0\}$) eliminating backscatter being represented in the model.

In this section we prove that the Smagorinsky model, corrected by Approach 1, preserves the property of the true turbulent fluctuations that on long time average
they are dissipative, Theorem 2.3. For the Smagorinsky model we have
\[ \nu_T(w) = (C_s l)^2 |\nabla w|, \] and thus \( a(w) = \nu^{-1/2} C_s l \sqrt{|\nabla w|} \).

Let \( \beta > 0 \), \( a(w) = \nu^{-1/2} C_s l \sqrt{|\nabla w|} \) and consider
\[
\begin{aligned}
w_t + \beta^2 a(w) \frac{\partial}{\partial t} (a(w) w) + w \cdot \nabla w + \nabla p - \nabla \cdot (|\nu + \nu_T(w)| \nabla w) &= f, \\
\nabla \cdot w &= 0 \text{ in } \Omega \times (0, \infty), \\
w &= 0 \text{ on } \partial \Omega, \ w(x, 0) = w_0(x), \text{ in } \Omega.
\end{aligned}
\tag{5.1}
\]

The model dissipation function for the effect of the model of the Reynolds stresses on
the kinetic energy in the mean flow will be denoted by
\[
MD(t) := \int_{\Omega} \beta^2 a(w) \frac{\partial}{\partial t} (a(w) w) \cdot w + \nu_T(w) |\nabla w|^2 dx.
\]

The numerical tests in Section 8 establish (Figure 8.2) that, even for very small \( \beta \),
the extra term does allow the sign of the model dissipation \( MD(t) \) to fluctuate. We
prove the time average of \( MD(t) \) is non-negative.

We assume
\[
f(x, t) \in L^\infty (0, \infty; L^2(\Omega)), w_0(x) \in L^2(\Omega), \]
\( w \) is a strong solution of (5.1).

**Lemma 5.1.** For \( a(w) = \nu^{-\frac{1}{2}} C_s l \sqrt{|\nabla w|} \) we have for all \( w \in W^{1,3}(\Omega) \)
\[
||a(w)w||^2 \leq C(\Omega) \nu^{-1} (C_s l)^2 ||\nabla w||^3_{L^3}.
\]

**Proof.** By Hölder’s inequality and the Poincaré-Friedrichs inequality
\[
||a(w)w||^2 = \nu^{-1} (C_s l)^2 \int_{\Omega} |\nabla w||w|^2 dx
\leq \nu^{-1} (C_s l)^2 ||w||^2_{L^4} ||\nabla w||_{L^3} \leq C(\Omega) \nu^{-1} (C_s l)^2 ||\nabla w||^3_{L^3}.
\]

We prove next that \( MD(t) \) dissipates energy in the time averaged sense.

**Theorem 5.2.** Let \( C_s > 0, \ l > 0, \ \beta > 0 \) and \( w \) be a strong solution of (5.1).
Then
\[
1 \int_0^T \left( \int_{\Omega} |\nu + \nu_T(w)| \nabla w : \nabla w dx \right) dt \leq C < \infty, \ C \text{ is independent of } T. \tag{5.3}
\]

The model also satisfies long term balance between energy input and dissipation: for
any generalized limit \( \text{LIM}_T \to \infty \)
\[
\text{LIM}_T \to \infty \frac{1}{T} \int_0^T \left( \int_{\Omega} |\nu + \nu_T(w)| |\nabla w|^2 dx \right) dt = \text{LIM}_T \to \infty \frac{1}{T} \int_0^T (f, w) dt.
\]

Further
\[
0 \leq \lim_{T \to \infty} \inf \frac{1}{T} \int_0^T MD(t) dt \leq \lim_{T \to \infty} \sup \frac{1}{T} \int_0^T MD(t) dt < \infty.
\]
Proof. Take the inner product of the corrected Smagorinsky model with \( w \). This yields
\[
\frac{1}{2} \frac{d}{dt} \left[ ||w||^2 + \beta^2 ||a(w)w||^2 \right] + \int_{\Omega} [\nu + \nu_T(w)] |\nabla w|^2 dx = (f, w). \tag{5.4}
\]
Let
\[
F := \frac{1}{2\nu} ||f||_{L^\infty(0, \infty; L^2(\Omega))} \quad \text{and} \quad y(t) := \frac{1}{2} \left[ ||w||^2 + \beta^2 ||a(w)w||^2 \right].
\]
By Lemma 3.1,
\[
2y(t) = ||w||^2 + \beta^2 ||a(w)w||^2 \leq C \left( \int_{\Omega} [\nu + \nu_T(w)] |\nabla w|^2 dx \right)
\leq C \left( \nu ||\nabla w||^2 + (C_sT)^2 ||\nabla w||_{L^3}^3 \right).
\]
Since \((f, w) \leq \frac{\nu}{2} ||\nabla w||^2 + \frac{1}{2\nu} ||f||_{L^2}^2\), \( y(t) \) thus satisfies
\[
y'(t) + \alpha y(t) \leq F(< \infty) \quad \text{for some} \quad \alpha > 0.
\]
This implies \( y(t) \in L^\infty(0, \infty) \) which is the first claimed á priori bound. This bound now implies that \((f, w)(t) \in L^\infty(0, \infty)\). Integrating (5.4) over \([0, T]\) and dividing by \( T \) gives
\[
\frac{1}{2T} \left[ ||w||^2 + \beta^2 ||a(w)w||^2 \right] (T) + \frac{1}{T} \int_0^T \left( \int_{\Omega} [\nu + \nu_T(w)] |\nabla w|^2 dx \right) dt \tag{5.5}
= \frac{1}{2T} \left[ ||w_0||^2 + \beta^2 ||a(w_0)w_0||^2 \right] + \frac{1}{T} \int_0^T (f, w) dt.
\]
Since \((f, w) \in L^\infty(0, \infty)\) this implies (5.3) holds, which implies as \( T \to \infty \) limit inferiors and superiors exist. The á priori estimates (5.2), (5.3) imply that (5.5) takes the form
\[
\mathcal{O}(\frac{1}{T}) + \frac{1}{T} \int_0^T \left( \int_{\Omega} [\nu + \nu_T(w)] |\nabla w|^2 dx \right) dt = \mathcal{O}(\frac{1}{T}) + \frac{1}{T} \int_0^T (f, w) dt. \tag{5.6}
\]
Letting \( T \to \infty \) implies long term balance between energy input and dissipation.

Consider now the time average of \( MD(t) \). We have
\[
\frac{1}{T} \int_0^T MD(t) dt = \frac{1}{T} \int_0^T \int_{\Omega} \left[ \beta^2 \frac{d}{dt} (a(w)w) \cdot (a(w)w) + \nu_T(w) |\nabla w|^2 \right] dx dt
= \frac{1}{T} \int_0^T \left[ \beta^2 \frac{d}{dt} ||a(w)w||^2 + (C_sT)^2 ||\nabla w||_{L^3}^3 \right] dt
= \frac{\beta^2}{2T} \left( ||a(w(T))w(T)||^2 - ||a(w(0))w(0)||^2 \right) + \frac{1}{T} \int_0^T (C_sT)^2 ||\nabla w||_{L^3}^3 dt.
\]
From (5.5), the right hand side equals
\[
\frac{1}{T} \int_0^T MD(t) dt = \frac{1}{T} \int_0^T (f, w) dt - \frac{1}{T} \int_0^T \nu ||\nabla w||^2 dt - \frac{1}{2T} \left[ ||w||^2 (T) - ||w_0||^2 \right],
\]

while from (5.6) we have
\[ \frac{1}{T} \int_0^T MD(t)dt = \frac{1}{T} \int_0^T (f,w)dt - \frac{1}{T} \int_0^T \nu \|\nabla w\|^2 dt = \mathcal{O}(\frac{1}{T}) - \mathcal{O}(\frac{1}{T}) + \frac{1}{T} \int_0^T (C_\delta t)^2 \|\nabla w\|^2 dt. \]

Thus the limit inferior of the time average of \( MD(t) \) exists and is non-negative. \( \square \)

6. Time Discretization of Corrected Eddy Viscosity Models. There are two natural approaches to correcting EV models for non-equilibrium effects. The second adds a linear regularization term whose time discretization seems straightforward. The first adds a nonlinear time derivative term of an unusual form whose time discretization is not transparent. This section presents three unconditionally stable, linearly implicit time discretizations of (Corrected EV).

Method 1 is first order accurate and thus simply establishes that stable time discretization of the new terms is possible. Method 2 shows how the new terms can be added by a postprocessing step to an established EV model code. First order methods are not of sufficient accuracy for most applications but present the essential ideas where their rigorous validation is clearest. Method 3 is second order accurate. All three are proven to be long time, nonlinearly energy stable and preserve the essential feature of time averaged dissipativity, Proposition 6.3. We suppress the spacial discretization to reduce notational clutter and focus on the essential time discretization of the new term. Let the timestep and associated quantities be denoted as usual by

\[ \text{timestep} = k, \quad t_n = nk, \quad f^n(x) := f(x,t_n), \]
\[ w^n(x) = \text{approximation to} \ w(x,t_n), \quad a^n := a(w^n), \quad \nu^n_T := \nu_T(w^n), n \geq 0, \]

and set \( a^{-1} = a^0 = a(w^0) \). We consider time discretization of (Corrected EV) under no-slip boundary conditions. The first method is: given \( w^0 \), find \( w^{n+1} \) satisfying

\[ \frac{w^{n+1} - w^n}{k} + \beta^2 a^n a^n w^{n+1} - a^{n-1} w^n \]
\[ + w^n \cdot \nabla w^{n+1} - \nabla \cdot ([\nu + \nu^n_T] \nabla w^{n+1}) + \nabla q^{n+1} = f^{n+1}(x) \text{ in } \Omega, \quad (\text{Method 1}) \]
\[ \nabla \cdot w^{n+1} = 0 \text{ in } \Omega, \quad \text{and } w^{n+1} = 0 \text{ on } \partial \Omega. \]

This method is linearly implicit. We shall prove in Theorem 6.1 that it is unconditionally, nonlinearly stable. In linearly implicit methods (considered herein) nonlinearities are lagged to previous time levels (or extrapolated), so there is no difficulty if \( \nu_T \) is determined by solving other nonlinear equations, such as in the \( k - \varepsilon \) model, \([6],[28]\).

Comparing the model, this method and the ensemble averaged NSE, we see that

1. True effect of fluctuations on means: \( RS(t) = \int_\Omega R(u,u) : \nabla \langle u \rangle \ dx. \)
2. Model: \( MD(t) = \int_\Omega \beta^2 a(w) \frac{\partial}{\partial t} (a(w)w \cdot w + \nu_T(w)|\nabla w|^2 \ dx. \)
3. Discrete version: \( MD^{n+1} = \int_\Omega \beta^2 a^n a^n w^{n+1} - a^{n-1} w^n \cdot w^{n+1} + \nu^n_T |\nabla w^{n+1}|^2 \ dx. \)

**THEOREM 6.1.** (Method 1) is unconditionally, nonlinearly energy stable. For any
\[ N \geq 1 \]
\[
\left( \frac{1}{2} |w^N|^2 + \frac{1}{2} \beta^2 |a^{N-1}w^N|^2 \right) + \sum_{n=0}^{N-1} \frac{1}{2} \left( |w^{n+1} - w^n|^2 + \beta^2 |a^{n}w^{n+1} - a^{n-1}w^n|^2 \right) \\
+ k \sum_{n=0}^{N-1} \int_{\Omega} [\nu + \nu_T^2] |\nabla w^{n+1}|^2 dx \\
= \left( \frac{1}{2} |w^0|^2 + \frac{1}{2} \beta^2 |a^{-1}w^0|^2 \right) + k \sum_{n=0}^{N-1} (f^{n+1}, w^{n+1}).
\]

**Proof.** Multiply through by \( k \), take the \( L^2 \) inner product of (Method 1) with \( w^{n+1} \) and use \((w^n \cdot \nabla w^{n+1}, w^{n+1}) = 0\). This gives
\[
|w^{n+1}|^2 - (w^{n+1}, w^n) + \beta^2 |a^n w^{n+1}|^2 - \beta^2 (a^{n-1} w^n, a^n w^{n+1}) \\
+ k \int_{\Omega} [\nu + \nu_T^2] |\nabla w^{n+1}|^2 dx = k \int_{\Omega} f^{n+1} \cdot w^{n+1} dx.
\]
The second and fourth terms are treated by the polarization identity
\[
(w^{n+1}, w^n) = \frac{1}{2} |w^{n+1}|^2 + \frac{1}{2} |w^n|^2 - \frac{1}{2} |w^{n+1} - w^n|^2,
\]
\[
(a^{n-1} w^n, a^n w^{n+1}) = \frac{1}{2} |a^n w^{n+1}|^2 + \frac{1}{2} |a^{n-1} w^n|^2 - \frac{1}{2} |a^n w^{n+1} - a^{n-1} w^n|^2.
\]

Collecting terms and summing from \( n = 0 \) to \( N - 1 \) finishes the proof. \( \square \)

Since Theorem 6.1 is an energy equality we can identify various effects:
- Model kinetic energy = \( \frac{1}{2} |w^N|^2 + \frac{1}{2} \beta^2 |a^{N-1}w^N|^2 \),
- Model dissipation = \( \int_{\Omega} \nu_T^2 |\nabla w^{n+1}|^2 dx \),
- Numerical diffusion = \( \frac{1}{2} (|w^{n+1} - w^n|^2 + \beta^2 |a^n w^{n+1} - a^{n-1} w^n|^2) \).

Rewriting the energy equality in an equivalent form gives
\[
\frac{1}{2k} (|w^{n+1}|^2 - |w^n|^2) + \frac{1}{2k} |w^{n+1} - w^n|^2 + \nu |\nabla w^{n+1}|^2 + \\
\left\{ \frac{\beta^2}{2k} (|a^n w^{n+1}|^2 - |a^{n-1} w^n|^2) + \frac{\beta^2}{2k} |a^n w^{n+1} - a^{n-1} w^n|^2 + \int_{\Omega} \nu_T^2 |\nabla w^{n+1}|^2 dx \right\} \\
= (f^{n+1}, w^{n+1}).
\]
The first and last line arise from the terms in the usual backward Euler discretization of the NSE. The second line (bracketed) is an equivalent form of \( MD^{n+1} \).

**Lemma 6.2.** For (Method 1) we have
\[
MD^{n+1} = \frac{\beta^2}{2k} (|a^n w^{n+1}|^2 - |a^{n-1} w^n|^2) \\
+ \frac{\beta^2}{2k} |a^n w^{n+1} - a^{n-1} w^n|^2 + \int_{\Omega} \nu_T^2 |\nabla w^{n+1}|^2 dx.
\]
Dissipativity on time average holds for all three methods with the same manipulations of their discrete energy equality. We record it here for (Method 1).

**PROPOSITION 6.3** (Time Averaged Dissipativity). Suppose \( \sup_{0 < n < \infty} ||f(t^n)|| < \infty \). Then, for \( T_N = N\Delta t, \)

\[
\lim_{T_N \to \infty} \inf_{T_N} \frac{1}{T_N} \left( \Delta t \sum_{n=0}^{N} MD^{n+1} \right) \geq 0.
\]

**Proof.** The proof is a discrete analog of the continuous case and will be omitted.

\[ \square \]

### 6.1. Modular Correction of EV Models.

Given a code that computes an approximation to the EV model

\[
w_t + w \cdot \nabla w - \nabla \cdot ([\nu + \nu_T(w)] \nabla w) + \nabla q = f(x, t). \tag{6.1}
\]

Algorithm 6.4 presents a minimally intrusive, modular, postprocessor to solve:

\[
w_t + \beta^2 a(w) \frac{\partial}{\partial t} (a(w)w) + w \cdot \nabla w - \nabla \cdot ([\nu + \nu_T(w)] \nabla w) + \nabla q = f(x, t). \tag{6.2}
\]

Precise stability analysis requires a specific choice of the algorithm used to solve (6.1). For this we select the simple, linearly implicit, backward Euler method.

**Derivation and Consistency Error.** To derive the modular postprocessor, rewrite (6.1) and (6.2) as, respectively,

\[
y'(t) = f(t, y) \quad \text{and} \quad y'(t) + \beta^2 a(y) \frac{d}{dt}(a(y)y) = f(t, y).
\]

The postprocessing given in Step 2 below suffices.

**Algorithm 6.4** (Method 2). Given \( y^n, y^{n-1}, \)

**Step 1:** Calculate \( y_{\text{temp}}^{n+1} \) by: \( \frac{y_{\text{temp}}^{n+1} - y^n}{k} = f(t_{n+1}, y_{\text{temp}}^{n+1}). \)

**Step 2:** Postprocess to obtain \( y^{n+1} \) from \( y_{\text{temp}}^{n+1} \) by:

\[
[1 + \beta^2 a(y^n)^2] y^{n+1} = y_{\text{temp}}^{n+1} + \beta^2 a(y^n)a(y^{n-1})y^n.
\]

Eliminating \( y_{\text{temp}}^{n+1} \) from Step 2 shows that \( y^{n+1} \) satisfies (with \( a^n = a(y^n) \))

\[
\frac{y^{n+1} - y^n}{k} + \beta^2 a^n \frac{a^n y^{n+1} - a^{n-1} y^n}{k} = f(t_{n+1}, y_{\text{temp}}^{n+1}).
\]

Although this is close to Method 1, \( y_{\text{temp}}^{n+1} \) not \( y^{n+1} \) occurs in the RHS. The LHS is clearly a first order approximation to \( y'(t) + \beta a(y)(a(y)y)' \). The RHS, \( f(t_{n+1}, y_{\text{temp}}^{n+1}) \), is a first order approximation to \( f(t, y) \) provided \( y_{\text{temp}}^{n+1} - y^{n+1} = O(k) \). Rearranging Step 2 gives

\[
y_{\text{temp}}^{n+1} - y^{n+1} = k \left\{ \beta^2 a(y^n) \frac{a(y^n) y^{n+1} - a(y^{n-1}) y^n}{k} \right\} = k \left( \beta^2 a(y) \frac{d}{dt}(a(y)y) \right) + O(k^2) = O(k).
\]

Thus, Algorithm 6.4 is a first order accurate approximation of \( y'(t) + \beta^2 a(y)(a(y)y)' = f(t, y) \).
Unconditional Stability of the Modular Algorithm. The utility of Algorithm 6.4 thus depends on its stability. This is now analyzed for its application to the corrected EV model. Algorithm 6.4 for (6.2) reads as follows.

**Algorithm 6.5.** For \( n \geq 0 \), given \( w^n \)

**Step 1:** Find \( w_{n+1}^{\text{temp}} \) satisfying

\[
\frac{w_{n+1}^{\text{temp}} - w^n}{k} + w^n \cdot \nabla w_{n+1}^{\text{temp}} - \nabla \cdot \left( [\nu + \nu_T^2] \nabla w_{n+1}^{\text{temp}} \right) + \nabla q_{n+1}^{\text{temp}} = f^{n+1}(x) \text{ in } \Omega, \\
\nabla \cdot w_{n+1}^{\text{temp}} = 0 \text{ in } \Omega, \text{ and } w_{n+1}^{\text{temp}} = 0 \text{ on } \partial \Omega.
\]

**Step 2:** Given \( w_{n+1}^{\text{temp}}, q_{n+1}^{\text{temp}} \), find \( w^{n+1}, q^{n+1} \) satisfying

\[
[1 + \beta^2 (a^n)^2] w^{n+1} + \nabla q^{n+1} = w_{n+1}^{\text{temp}} + \beta^2 a^n a^{-1} w^n \text{ in } \Omega \tag{6.3}
\]

\[
\nabla \cdot w^{n+1} = 0 \text{ in } \Omega, \text{ and } w^{n+1} = 0 \text{ on } \partial \Omega.
\]

We prove Algorithm 6.5 is unconditionally stable.

**Theorem 6.6.** Algorithm 6.5 is unconditionally, nonlinearly energy stable. For any \( N \geq 1 \)

\[
\left( \frac{1}{2} ||w^{N}||^2 + \frac{1}{2} \beta^2 ||a^{-1} w^{N}||^2 \right) \tag{6.3}
\]

\[+ \sum_{n=0}^{N-1} \left( \frac{1}{2} ||w^{n+1} \text{ temp} - w^{n+1}||^2 + ||w_{n+1}^{\text{temp}} - w^n||^2 \right) \tag{6.3}
\]

\[+ k \sum_{n=0}^{N-1} \left( \int _\Omega [\nu + \nu_T^2] |\nabla w_{n+1}^{\text{temp}}|^2 dx \right) \tag{6.3}
\]

\[= \left( \frac{1}{2} ||w^{0}||^2 + \frac{1}{2} \beta^2 ||a^{-1} w^{0}||^2 \right) + k \sum_{n=0}^{N-1} (f^{n+1}, w_{n+1}^{\text{temp}}). \tag{6.3}
\]

**Proof.** Take the \( L^2 \) inner product of Step 1 with \( w_{n+1}^{\text{temp}} \) and follow the proof of Theorem 6.1. This gives

\[
\frac{1}{2} ||w_{n+1}^{\text{temp}}||^2 - \frac{1}{2} ||w^n||^2 + \frac{1}{2} ||w_{n+1}^{\text{temp}} - w^n||^2 \tag{Step 1 Energy}
\]

\[+ k \int _\Omega [\nu + \nu_T^2] |\nabla w_{n+1}^{\text{temp}}|^2 dx = k(f^{n+1}, w_{n+1}^{\text{temp}}). \tag{Step 1 Energy}
\]

Consider Step 2. Taking the \( L^2 \) inner product with \( w^{n+1} \) gives

\[
||w^{n+1}||^2 + \beta^2 ||a^n w^{n+1}||^2 = (w_{n+1}^{\text{temp}}, w^{n+1}) + \beta^2 (a^{-1} w^n, a^n w^{n+1}).
\]

The two terms on the RHS are treated with the polarization identity

\[
(w_{n+1}^{\text{temp}}, w^{n+1}) = \frac{1}{2} ||w_{n+1}^{\text{temp}}||^2 + \frac{1}{2} ||w^{n+1}||^2 - \frac{1}{2} ||w_{n+1}^{\text{temp}} - w^{n+1}||^2,
\]

\[
(a^{-1} w^n, a^n w^{n+1}) = \frac{1}{2} ||a^{-1} w^n||^2 + \frac{1}{2} ||a^{n+1} w^n||^2 - \frac{1}{2} ||a^n w^{n+1} - a^{-1} w^n||^2,
\]

\[= \frac{1}{2} ||w^{n+1}||^2 + \frac{1}{2} ||w^n||^2 + \frac{1}{2} ||w_{n+1}^{\text{temp}} - w^n||^2 - \frac{1}{2} ||w_{n+1}^{\text{temp}} - w^{n+1}||^2.
\]
linear extrapolation of a variable $\phi$ to the new kinetic energy term. It shortens the notation considerably to denote the early implicit method comprised of an IMEX combination of BDF2 and AB2 adapted giving $N$ stable. For any $n \geq 3$ satisfying

$$\phi^{n+1} := 2\phi^n - \phi^{n-1}, \text{ for } \phi = w, a, \nu_T.$$  

The method is: given $w^0, w^1, w^2$ and $w^3$ (found by another method) find $w^{n+1}$ for $n \geq 3$ satisfying

$$\frac{3w^{n+1} - 4w^n + w^{n-1}}{2k} + \beta^2 a^{n+1} w^{n+1} - 4a^n w^n + a^{n-1} w^{n-1}$$

$$= w^{n+1} \cdot \nabla ( \nu + \nu_T^{n+1} \nabla w^{n+1} ) + \nabla q^{n+1} = f^{n+1}(x) \text{ in } \Omega,$$

$$\nabla \cdot w^{n+1} = 0 \text{ in } \Omega, \text{ and } w^{n+1} = 0 \text{ on } \partial \Omega.$$

**Theorem 6.7.** The method (Method 3) is unconditionally, nonlinearly, long time stable. For any $N \geq 4$

$$\frac{1}{4}(\|w^N\|^2 + \|w^{N+1}\|^2)$$

$$+ \frac{1}{4}(\beta^2\|a^N w^N\|^2 + \beta^2 \|2a^N w^N - a^{N-1} w^{N-1}\|^2)$$

$$+ \sum_{n=3}^{N-1} \left\{ \frac{1}{4} \beta^2 \|a^{n+1} w^{n+1} - 2a^n w^n + a^{n-1} w^{n-1}\|^2$$

$$+ \frac{1}{4} \|w^{n+1} - 2w^n + w^{n-1}\|^2 + k \int_\Omega [\nu + \nu_T^{n+1}] |\nabla w^{n+1}|^2 dx \right\}$$

$$= \sum_{n=3}^{N-1} k(f^{n+1}, w^{n+1}) + \frac{1}{4}(\|w^3\|^2 + \|2w^3 - w^2\|^2)$$

$$+ \frac{1}{4}(\beta^2\|a^3 w^3\|^2 + \beta^2 \|2a^3 w^3 - a^2 w^2\|^2).$$
Proof. Take the $L^2$ inner product of (Method 3) with $w^{n+1}$ and multiply through by $k$. This gives
\begin{align*}
\frac{1}{4} (\|w^{n+1}\|^2 + \|w^{n+2}\|^2) - \frac{1}{4} (\|w^n\|^2 + \|w^{n+1}\|^2) \\
+ \frac{1}{4} \left( \beta^2 \|a^{n+1}w^{n+1}\|^2 + \beta^2 \|2a^{n+1}w^{n+1} - a^{n}w^n\|^2 \right) \\
- \frac{1}{4} \left( \beta^2 \|a^{n}w^n\|^2 + \beta^2 \|2a^{n}w^n - a^{n-1}w^{n-1}\|^2 \right) \\
+ \frac{1}{4} \beta^2 \|a^{n+1}w^{n+1} - 2a^{n}w^n + a^{n-1}w^{n-1}\|^2 \\
+ \frac{1}{4} \|w^{n+1} - 2w^n + w^{n-1}\|^2 + k \int_{\Omega} [\nu^*_{n+1}] |\nabla w^{n+1}|^2 dx = k(f^{n+1}, w^{n+1}).
\end{align*}

Summing up above equality from $n = 3$ to $n = N - 1$ completes the proof. □

7. The Re-scaling Parameter $\beta$. The first approach to correcting EV models involves a re-scaling parameter $\beta$. With $a(w) = \sqrt{\nu T(w)/\nu}$, we begin with the fluctuation model
\begin{equation}
\text{action}(\nabla u') \simeq a(\langle u \rangle) \nabla \langle u \rangle. \tag{7.1}
\end{equation}

In Approach 1, a fluctuation model relating $\text{action}(u')$ to $\langle u \rangle$ is obtained by dropping $" \nabla"$ and re-scaling (by a factor $\beta << 1$) yielding
\begin{equation}
\text{action}(u') \simeq \beta \cdot a(\langle u \rangle) \langle u \rangle. \tag{7.2}
\end{equation}

**Definition 7.1.** Let $u$ denote an ensemble of realizations of the NSE with perturbed initial data. The associated turbulent intensities of $u$ and $\nabla u$ are, respectively,
\begin{align*}
I(u) := \frac{\langle ||u'||^2 \rangle}{\|\langle u \rangle\|^2} \quad \text{and} \quad I(\nabla u) := \frac{\langle ||\nabla u'||^2 \rangle}{\|\nabla \langle u \rangle\|^2}.
\end{align*}

Phenomenology often begins by postulating that for fully developed turbulence, kinetic energy is concentrated in the means ($\langle ||u'||^2 \rangle << \|\langle u \rangle\|^2$) while energy dissipation is concentrated in the fluctuations ($\langle ||\nabla u'||^2 \rangle >> \|\nabla \langle u \rangle\|^2$). Thus, on the scales where one is significant, the other is negligible. This implies that for fully developed turbulence the expected case is
\begin{equation*}
I(u) << 1 << I(\nabla u).
\end{equation*}

Beginning with the fluctuation models (7.1), (7.2) we thus have
\begin{align*}
\beta^2 \frac{\langle a(\langle u \rangle) \langle u \rangle \rangle}{\|\langle u \rangle\|^2} & \simeq \frac{\langle ||u'||^2 \rangle}{\|\langle u \rangle\|^2} = I(u)(<< 1) \quad \text{and} \\
(1 <<) I(\nabla u) = \frac{\langle ||\nabla u'||^2 \rangle}{\|\nabla \langle u \rangle\|^2} & \simeq \frac{\|a(\langle u \rangle) \nabla \langle u \rangle\|^2}{\|\nabla \langle u \rangle\|^2}.
\end{align*}

The quantities
\begin{align*}
\frac{\|a(\langle u \rangle) \langle u \rangle\|^2}{\|\langle u \rangle\|^2} \quad \text{and} \quad \frac{\|a(\langle u \rangle) \nabla \langle u \rangle\|^2}{\|\nabla \langle u \rangle\|^2}
\end{align*}

are...
both represent (squares of) weighted averages of $a(\langle u \rangle)$. If (as expected) these are of comparable magnitude, $\beta \ll 1$ is expected and we obtain the estimate

$$\beta^2 \simeq \frac{I(u)}{I(\nabla u)} \ll 1.$$  \hfill (7.3)

**Mesh Dependence.** Apparently one option is to calculate the turbulent intensities on a given mesh and use these to find the re-scaling parameter $\beta$. Unfortunately, the result is severely limited by the chosen mesh as we now develop. Suppose spacial discretization is performed by a standard, conforming finite element method based on a mesh of elements $e$ with element diameter (the local mesh width) denoted $h_e$. For meshes satisfying an angle condition eliminating nearly degenerate elements and piecewise polynomial finite element velocities $u_h$, the inverse property

$$||\nabla u_h||_{L^2(e)} \leq C_{INV} h_e^{-1} ||u_h||_{L^2(e)}$$ \hfill (7.4)

holds, where the constant depends only on local polynomial degree and mesh geometry (element angles). This implies that

$$||\nabla u_h|| \leq C_{INV} h^{-1} ||u_h|| \quad \text{where} \quad h := \min_e h_e.$$

Let $u_h$ denote an ensemble of discrete velocities computed on the same, fixed mesh as described above. Define the characteristic length-scale $L$ of the ensemble mean velocity (expected but not guaranteed to be large) by

$$L^{-1} := \frac{||\nabla \langle u \rangle||}{||\langle u \rangle||}.$$

**Proposition 7.2 (Limitations on calculated turbulence intensities).** Suppose (7.4) holds. Then

$$\beta_h := \sqrt{\frac{I(u_h)}{I(\nabla u_h)}}$$

satisfies

$$C_{INV} h \leq \beta_h \leq C_{PF} \frac{1}{L}.$$

**Proof.** We have, by rearranging,

$$\frac{I(u_h)}{I(\nabla u_h)} = \frac{\langle ||u'_h||^2 \rangle}{\langle ||\nabla u'_h||^2 \rangle} \frac{\langle ||\nabla \langle u_h \rangle||^2 \rangle}{\langle ||\langle u_h \rangle||^2 \rangle} = \frac{\langle ||u'_h||^2 \rangle}{\langle ||\nabla u'_h||^2 \rangle} L^{-2}.$$

From the inverse estimate and the Poincaré-Friedrichs inequality (noting that $C_{PF}^2 = O (\text{diameter of } \Omega)$) we have

$$C_{PF}^{-1} ||u'_h|| \leq ||\nabla u'_h|| \leq C_{INV} h^{-1} ||u'_h||.$$

Thus,

$$C_{PF}^2 \frac{\langle ||u'_h||^2 \rangle}{\langle ||\nabla u'_h||^2 \rangle} \leq 1 \leq C_{INV}^2 h^{-2} \frac{\langle ||u'_h||^2 \rangle}{\langle ||\nabla u'_h||^2 \rangle}.$$
Rearranging, it follows from the definition of $L$ that, as claimed,

$$ C_{INV}^2 h^2 L^{-2} \leq \frac{I(u_h)}{I(\nabla u_h)} = \frac{\langle ||u_h'||^2 \rangle}{\langle ||\nabla u_h'||^2 \rangle} L^{-2} \leq C_P^2 L^{-2}. $$

Since $\beta^2 \simeq \frac{I(u)}{I(\nabla u)} << 1$ suggests that the true value of $\beta$ is small, $\beta << 1$, while calculable turbulent intensities are limited by the mesh resolution, a reasonable default choice of $\beta$ may be

$$ \beta(x) = h_c(x)^2 \text{ or } \beta = (\min h_c)^2. $$

**Remark 7.3** (Associating statistical means and fluctuations with spacial scales). Some information can be obtained at the cost of blurring the line separating statistical models and LES models by associating statistical means and fluctuations respectively with large and small spacial scales. With this association, in 3d, for fully developed, homogeneous, isotropic turbulence an estimate of the quotient $I(u)/I(\nabla u)$ (and thus $\beta$) can be calculated from the K41 theory and its predicted, time-averaged, energy density distribution $E(k) \simeq \alpha \varepsilon^{2/3} k^{-5/3}$, [30]. The separation between large and small scales is determined by a length scale $\delta$. The wave numbers of the well resolved scales and unresolved scales are then, respectively, $\pi/L < k < \pi/\delta$ and $\pi/\delta < k < \pi/\eta$. $\eta = Re^{-3/4} L$ is the Kolmogorov micro-scale. Calculate

$$ I(u) \simeq \int_{\pi/\delta}^{\pi/\eta} E(k) dk \int_{\pi/L}^{\pi/\delta} E(k) dk = \left( \frac{\pi}{\eta} \right)^{-2/3} - \left( \frac{\pi}{\delta} \right)^{-2/3} - \left( \frac{\pi}{\eta} \right)^{-2/3} - \left( \frac{\pi}{\delta} \right)^{-2/3}. $$

In the limit $Re \to \infty, \eta \to 0$ and $\delta << L$

$$ I(u) \to \left( \frac{\delta}{L} \right)^{2/3} = \left( \frac{\delta}{L} \right) + H.O.T.s. $$

Similarly, we calculate for $\delta >> \eta$

$$ I(\nabla u) \simeq \int_{\pi/\delta}^{\pi/\eta} k^2 E(k) dk \int_{\pi/L}^{\pi/\delta} k^2 E(k) dk = \left( \frac{\delta}{\eta} \right)^{4/3} + H.O.T.s. $$

In particular, dropping higher order terms, $I(u) \simeq (\delta/L)^{2/3} << 1 << I(\nabla u) \simeq (\delta/\eta)^{4/3}$. Thus, to leading order, in 3d

$$ \beta \simeq \sqrt{\frac{I(u)}{I(\nabla u)}} \simeq Re^{-1/2} \left( \frac{\delta}{L} \right)^{-2/3}. $$

Phenomenology of two dimensional turbulence is more complicated. The simplest case for forced turbulence is when the model incorporates some extra mechanism to extract energy from the largest scales, energy is injected in an intermediate scale and $\delta/L$ is smaller than the injection scale but much larger than the micro-scale. Adapting the above spectral calculation to 2d gives

$$ \beta \simeq \frac{\delta}{L} \left( \ln \frac{\delta}{\eta} \right)^{-1/2}. $$


8. Numerical Explorations. Corrected models and their time discretization give a closure that is dissipative on time average. The first question that must be answered before accuracy is addressed is whether the correction incorporates statistical-backscatter, i.e., whether the model dissipation $MD(t)$ changes sign while being non-negative on time average. To test the sign of model dissipation, we preform two tests for simple models (rather than models which perform better in practical calculations). First we consider a simple mixing length model with a standard parameterization of the mixing length in the near wall region (e.g., $l = 0.41y$) and wake. The theory presented is not yet adapted to LES models. Nevertheless, in the second test we consider the Smagorinsky model. These two are chosen because they are over-diffused and among models in use likely the ones for which statistical-backscatter would be most difficult to introduce. Next, since it is believed that an inverse cascade and backscatter are much more common in the physics of 2d flow at high $Re$ than for 3d flows, we have selected a 2d test problem. (It is also one for which we have done a number of detailed simulations of the evolution of velocity ensembles in [20], [21], [22], [23]. While not directly relevant herein, this experience with velocity ensembles for this flow was useful in validation.)

Fig. 8.1: Shown above is the mesh used for the flow between two offset circles.

2D test problem: Flow Between Offset Circles. Pick

$$
\Omega = \{(x, y) : x^2 + y^2 \leq r_1^2 \text{ and } (x - c_1)^2 + (y - c_2)^2 \geq r_2^2\},
$$

$$r_1 = 1, r_2 = 0.1, c = (c_1, c_2) = (\frac{1}{2}, 0),
$$

$$f(x, y, t) = (-4y(1 - x^2 - y^2), 4x(1 - x^2 - y^2))^T,
$$
with no-slip boundary conditions on both circles. The flow (inspired by the extensive work on variants of Couette flow, [12]), driven by a counterclockwise force (with \( f \equiv 0 \) at the outer circle), rotates about \((0, 0)\) and interacts with the immersed circle. This induces a von Kármán vortex street which re-interacts with the immersed circle creating more complex structures. This flow also exhibits near wall turbulent streaks and a central polar vortex that pulsates. We discretize in space using the usual finite element method with Taylor-Hood elements, [16], using the code FreeFEM++, [17] and in time using (Method 1). It is worth mentioning that Approach 2 discussed in Section 4 is based on the Baldwin Lomax model, which is commonly used for aerospace flows (over a wing) but known to be not so good for flows with global rotation such as our test case herein because \( \text{curl} \ u \) is large everywhere rather than selective. Thus Approach 2 is not tested here.

**Test 1.** We test the following completely standard mixing length model on this problem, which addresses the issue of over damping near the boundary.

\[
\nu_T = l^2 |\tilde{S}|, \quad \text{where, with } y = \text{ distance to the nearest wall,} \\
l = \begin{cases} 
0.41 \cdot y & 0 < y < 0.2 \cdot (Re)^{-1/2}, \\
0.41 \cdot 0.2 \cdot (Re)^{-1/2} & \text{otherwise.} 
\end{cases}
\]

Here \( |\tilde{S}| = \sqrt{2 S_{ij} \tilde{S}_{ij}} \) and \( \tilde{S}_{ij} \) is the strain rate or deformation tensor. All the theory of the previous sections applies to choosing \( S \) instead of \( \nabla w \).

The numerical solutions are computed on an under-resolved, Delaunay-generated triangular mesh with 80 mesh points on the outer circle and 60 mesh points on the inner circle, providing \( 18,638 \) total degrees of freedom, refined near the inner circle (see Figure 8.1). For this mesh the shortest edge of all triangles is \( \min_e h_e = 0.0110964 \) and the longest edge \( \max_e h_e = 0.108046 \). We take

\[
a(\cdot) = \sqrt{\frac{\nu_T}{\nu}}, \quad \beta = 6 \cdot 10^{-5}, \quad \nu = 10^{-4}, \quad \Delta t = 0.01, \quad T = 10.
\]

We compute the following quantities.

\[
MD = \int_{\Omega} \beta^2 a(t^n) \left( \frac{a(t^n) w(t^{n+1}) - a(t^{n-1}) w(t^n)}{\Delta t} \right) \cdot w(t^{n+1}) dx \\
+ \nu_T \| \nabla w^{n+1} \|^2,
\]

\[
TMD = \int_{\Omega} \beta^2 a(t^n) \left( \frac{a(t^n) w(t^{n+1}) - a(t^{n-1}) w(t^n)}{\Delta t} \right) \cdot w(t^{n+1}) dx,
\]

\[
EVD = \int_{\Omega} \nu_T^2 \| \nabla w^{n+1} \|^2 dx,
\]

\[
VD = \nu \| \nabla w^{n+1} \|^2.
\]

Note that if \( \beta = 0 \) (i.e., if we were solving the usual mixing length model) we would have \( MD = EVD > 0 \). Observe in the first plot of Figure 8.2 that with \( \beta = 6 \cdot 10^{-5} \), \( MD(t) \) is on time average positive (consistent with theoretical predictions) but there are times when \( MD \) becomes negative which indicates backscatter. Thus the correction to the eddy viscosity model does have built into it the possibility of representing backscatter. Various other statistics are also plotted in Figure 8.2 including \( TMD \) which represents the effect of the new term, the eddy viscosity dissipation term \( EVD \) and the viscous dissipation term \( VD \).
Test 2. In this test, we test the well-known over-diffused Smagorinsky model (mixing length is taken to be the mesh-width)  
\[ \nu_T = (0.1 \Delta x)^2 |\tilde{S}|. \]

The choice \( C_s = 0.1 \) is common though not universal, see Table 1 in [36]. We take \( \Delta x \) to be the length of the shortest edge of all triangles. We used the same mesh and discretization parameters (except \( \beta \)) as in Test 1. Here we take \( \beta = 8 \times 10^{-5} \).

As we can see from Figure 8.3, MD becomes negative at times, which shows this corrected model is able to catch backscatter. Note this choice of mixing length results in increased damping from the eddy viscosity term (compared with Test 1) and consequently a higher value of \( \beta \) to get the model to catch backscatter. In practice calibration will be essential.

9. Conclusions. We have shown that eddy viscosity models can be adapted to non-equilibrium turbulence and incorporate statistical-backscatter without using negative turbulent viscosities. Solutions of the new models have been proven to share the property of the true Reynolds stresses of being dissipative on time average. We have also given three methods for time discretization preserving this property, including a modular correction for an eddy viscosity code. There are many important open questions including accuracy tests, existence of weak solutions to the new models, model calibration and extension to and testing for better EV models.

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Fig. 8.2: Shown above are data from the numerical simulation (corrected mixing length model) of the 2D flow between two offset cylinders with $\nu = 1/10000$, $\Delta t = 0.01$, $m = 80$, $n = 60$, $\beta = 6 \times 10^{-5}$. 
Fig. 8.3: Shown above are data from the numerical simulation (corrected Smagorinsky model) of the 2D flow between two offset cylinders with $\nu = 1/10000$, $\Delta t = 0.01$, $m = 80$, $n = 60$, $\beta = 8 \times 10^{-5}$. 

29