Regular Functions of Nilpotent Orbits and Normality of their Closures

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Abstract

For any nilpotent orbit $\mathcal{O}$ in a complex semisimple Lie algebra $\mathfrak{g}$, it is known that the normality of its Zariski closure $\mathcal{O}$ is related to the regular functions of $\mathcal{O}$. On the other hand, if $G$ is classical, Kraft and Procesi gave a combinatorial criterion on the normality of $\mathcal{O}$. In this paper, we will give an algorithm computing the multiplicities of the fundamental representations in the ring of regular functions of $\mathcal{O}$, and will relate it to the Kraft-Procesi criterion on the normality of $\mathcal{O}$.

1 Introduction

The idea of the Orbit Method, originally proposed by Kirillov, says that every (co)adjoint orbit in $\mathfrak{g}$ (or its dual $\mathfrak{g}^*$) is related to an irreducible, unitary representation of $G$. This idea is realized perfectly when $\mathfrak{g}$ is a nilpotent Lie algebra, and some generalizations are needed if $\mathfrak{g}$ is a solvable Lie algebra. However, the situation becomes much more complicated in the case of semisimple Lie algebras. One of the many difficulties arising from the semisimple case is, not all adjoint orbits in $\mathfrak{g}$ are closed. In fact, the most interesting cases stems from the orbits consisting of nilpotent elements only, which are called nilpotent orbits. It is therefore of interest to look at the algebraic and geometric structure of nilpotent orbits.

In [16], McGovern studies the structure of the ring of regular functions of all nilpotent orbits with base field $\mathbb{C}$. More precisely, for any nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$, there is a natural $G$-action on its ring of regular functions $R[\mathcal{O}]$ given by

$$g \cdot f(x) \mapsto (g \cdot f)(x) := f(g^{-1}xg)$$
Theorem 3.1 and Corollary 3.2 give the $G$-module structure $R[\mathcal{O}]$ under the above action:

**Theorem 1.** ([16][Theorem 3.1, Corollary 3.2] Let $\mathcal{O}$ be a complex nilpotent orbit in $\mathfrak{g}$, with a choice of Jacobson-Morozov triple $\{e, f, h\}$. Write $\mathfrak{g}_i$ as the $i$-eigenspace of $ad(h)$ on $\mathfrak{g}$, and $\mathfrak{q} = \sum_{i \geq 0} \mathfrak{g}_i$. Let $Q = LU$ be the Levi decomposition of the parabolic subgroup $Q$ with Lie algebra $\mathfrak{q}$. Then

$$R[\mathcal{O}] \cong \sum_i (-1)^i \text{Ind}_L^G (\wedge^i \mathfrak{g}_i)$$

as $G$-modules. Or equivalently,

$$R[\mathcal{O}] \cong \text{Ind}_T^G \left( \prod_{\alpha \in \Delta^+ : \mathfrak{g}_\alpha \subset \mathfrak{q}_{\geq 0} + \mathfrak{q}_1} (1 - e^\alpha) \right)$$

where $\Delta^+$ is the set of all positive roots of $\mathfrak{g}$.

Using Frobenius reciprocity for both expressions on the right, we can find out the multiplicity of any $G$-module $V_\lambda$ appearing in $R[\mathcal{O}]$. However, it is not easy to read off the multiplicities in practice. Motivated by the study of Dixmier algebras, McGovern further conjectured the following:

**Conjecture 1.** ([16][Conjecture 5.1] For each nilpotent orbit $\mathcal{O}$, the $G$-structure of $R[\mathcal{O}]$ can be expressed by

$$R[\mathcal{O}] \cong \sum_{w \in W_{\mathcal{O}}} c_w \text{Ind}_T^G (\mu - w \cdot \mu)$$

for a fixed character $\mu$, $c_w \in \mathbb{Q}$ and $W_{\mathcal{O}}$ is a subset of $W_G$, the Weyl group of $G$.

For instance, if $\mathcal{O}$ is the regular orbit, then $W_{\mathcal{O}} = \{Id\}$, $\mu = 0$ and $c_{Id} = 1$. If $\mathcal{O}$ is the trivial orbit, then $W_{\mathcal{O}} = W_G$, $\mu = \rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ and $c_w = (-1)^{l(w)} (l(w)$ is the length of the Weyl group element $w$ in its reduced form).

In fact, the conjecture is known to be true for all nilpotent orbits of Type A. Furthermore, Chmutova and Ostrik in [10] proved the above conjecture for some special kinds of orbits, which they called strongly Richardson. In a preprint of Barbasch [7], the above conjecture holds for all special classical nilpotent orbits (In fact it is conjectured that a similar formula
can be obtained for all classical orbits). The main observation is, $R[\mathcal{O}]$ can be seen as a spherical unipotent representation, whose global character formula is known explicitly by Barbasch and Vogan (e.g. [5], [6]). The character formula will in turn give the expression of the form in the above Conjecture. Given enough knowledge on $W_\mathcal{O}$, $\mu$ and $c_w$, the above formula is a better one in computing the $G$-module multiplicities (See Chapter 3.3), which leads to the main results in the next Chapter.

2 Statement of Main Results

2.1 Classical Nilpotent Orbits

It is well-known that all nilpotent orbits in a complex classical Lie algebra can be expressed as partitions, where the partition corresponds to the size of the Jordan blocks. Since the case of Type A is clear, we put our focus on the other classical types. Here is the classification of nilpotent orbits in type $B$, $C$ and $D$, given in [11]:

- For type $B_n$, all nilpotent orbits are identified with the partitions of $2n + 1$ in which even parts occur with even multiplicity.
- For type $C_n$, nilpotent orbits are identified with the partitions of $2n$ in which odd parts occur with even multiplicity.
- For type $D_n$, nilpotent orbits are identified with the partitions of $2n$ in which even parts occurs with even multiplicity, except for the ‘very even’ partitions - partitions with only even parts, each having even multiplicity. Each very even partitions corresponds to 2 orbits.

If we only consider $O(2n, \mathbb{C})$-conjugates of nilpotent elements in $\mathfrak{o}(2n, \mathbb{C}) = \mathfrak{so}(2n, \mathbb{C})$, the two nilpotent orbits corresponding to a very even partition will be merged into one orbit. In this case, there is at most one orbit for every partition of $2n$. From now on, we only consider $O(2n, \mathbb{C})$-conjugates of nilpotent elements in $\mathfrak{so}(2n, \mathbb{C})$. Therefore, there is only one orbit for the very even partitions of $2n$.

Note that all nilpotent orbits of classical type are characterized by partitions. And partitions are often expressed as Young diagrams whose row sizes are the sizes of the corresponding partition. In fact it is sometimes more convenient to look at the column sizes of a Young diagram. The column sizes of the Young diagram corresponding to a partition is given by the dual partition of the original partition. More precisely, let $[r_1, r_2, \ldots, r_i]$ be a partition of $n$, with $r_1 \geq r_2 \geq \cdots \geq r_i > 0$, its dual partition is given
by \((c_k, c_{k-1}, \ldots, c_1)\), where \(c_{k+1-j} = \#\{i | r_i \geq j\}\).

For example, let \(\mathcal{O} = [4, 2]\) in \(\mathfrak{sp}(6, \mathbb{C})\). Then the Young diagram corresponding to \(\mathcal{O}\) is given by

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And the dual partition of \(\mathcal{O} = (2, 2, 1, 1)\).

From now on, we will determine a nilpotent orbit by its dual partition, or equivalently the column sizes of its corresponding Young diagram. Here is a restatement of the characterization of classical nilpotent orbits:

- Any nilpotent orbit in \(\mathfrak{sp}(2m, \mathbb{C})\) can be paramatrized by a partition of \(2m\) with column sizes \((c_{2k}, c_{2k-1}, \ldots, c_0)\), where \(c_{2k} \geq c_{2k-1} \geq \cdots \geq c_0 \geq 0\) (by insisting \(c_{2k}\) to be the longest column, we put \(c_0 = 0\) if necessary), such that \(c_{2i} + c_{2i-1}\) is even for all \(i \geq 0\). (Note: \(c_{-1} = 0\))
- Any nilpotent orbit in \(\mathfrak{o}(n, \mathbb{C})\) can be paramatrized by a partition of \(n\) with column sizes \((b_{2k+1}, b_{2k}, \ldots, b_0)\), where \(b_{2k+1} \geq b_{2k} \geq \cdots \geq b_0 \geq 0\) (putting \(b_0 = 0\) if necessary), such that \(b_{2i} + b_{2i-1}\) is even for all \(i \geq 0\). (Note: \(b_{-1} = 0\))

For example, \((7, 6, 4, 4)\) is a nilpotent orbit in \(\mathfrak{o}(21, \mathbb{C})\) but it does not define a nilpotent orbit in \(\mathfrak{sp}\).

For two partitions \(\varsigma\) and \(\psi\) parametrized by their column sizes, define the join \(\varsigma \lor \psi\) by ‘combining’ the columns, so that if \(\varsigma = (c_m, \ldots, c_0)\), \(\psi = (c'_n, \ldots, c'_0)\), then \(\varsigma \lor \psi = (d_{m+n+1}, \ldots, d_0)\), where \(\{d_{m+n+1}, \ldots, d_0\} = \{c_m, \ldots, c_0, c'_n, \ldots, c'_0\}\) as sets and \(d_{m+n+1} \geq d_{m+n} \geq \cdots \geq d_0\).

## 2.2 Kraft-Constant on Normality

Given the combinatorial description of the nilpotent orbit \(\mathcal{O}\) in the classical case, we can now state the criterion on the normality of the Zariski closure of \(\mathcal{O}\), given by Kraft and Procesi:

**Theorem 2** ([13], [14]). All nilpotent orbits of Type A have normal orbit closures.

Let \(\mathcal{O} = (c_{2k}, c_{2k-1}, \ldots, c_0)\) be a nilpotent orbit in \(\mathfrak{sp}(2m, \mathbb{C})\). If there is a chain of column lengths of the form

\[ c_{2i} \neq c_{2i-1} = c_{2i-2} = \cdots = c_{2j-1} = c_{2j-2} \neq c_{2j-3} \]

then the Zariski closure of \(\mathcal{O}\) is normal.
then $\mathcal{O}$ is not normal. Similarly, the closure of a nilpotent orbit $\mathcal{P} = (b_{2k+1}, \ldots, b_0)$ in $\mathfrak{o}(n, \mathbb{C})$ is not normal if there is a chain of column lengths of the form

$$b_{2i} \neq b_{2i-1} = b_{2i-2} = \cdots = b_{2j-1} = b_{2j-2} \neq b_{2j-3}$$

(Note: $b_{2k+2} = 0$)

For instance, in $\mathfrak{sp}(2m, \mathbb{C})$, the orbit closures $(8, 6, 6, 6), (6, 6, 6)$ are normal, while $(8, 6, 6, 4)$ is not normal. In $\mathfrak{o}(n, \mathbb{C}), (8, 6, 6, 6), (8, 6, 6, 4)$ are normal, $(6, 6, 6)$ is not normal.

Apart from the combinatorial criterion of normality given in Theorem 2, there is an algebro-geometric criterion of normality. Let $\mathcal{O}$ be a $G$-orbit with Zariski closure $\overline{\mathcal{O}}$. If $\overline{\mathcal{O}} \setminus \mathcal{O}$ has codimension greater than or equal to 2, then the ring of regular functions of $\mathcal{O}$, $R[\mathcal{O}]$ is the integral closure of $R[\overline{\mathcal{O}}]$ in its field of fractions (see [3][Proposition 8.2] for instance). Note that all (real or complex) nilpotent orbits are symplectic manifolds with the Kostant-Kirillov-Souriau symplectic form, therefore they are all of even (real or complex) dimensions. In particular, the nilpotent orbit closure $\overline{\mathcal{O}}$ satisfies the hypothesis of the algebro-geometric result. So

$$R[\mathcal{O}] \cong R[\overline{\mathcal{O}}] \text{ if and only if } \overline{\mathcal{O}} \text{ is normal}$$

2.3 The Main Theorems

It is well-known that if $X$ is an algebraic variety endowed with an action of a reductive group $G$ (so $\mathcal{O}$ and $\overline{\mathcal{O}}$ are examples of $X$), then $R[X]$ can be decomposed as a direct sum of isotypic components of highest weight representations, i.e. $R[X] = \bigoplus \lambda \text{Hom}_G(V_\lambda, R[X]) \otimes V_\lambda$. It turns out that in our case when $X = \mathcal{O}$, the summands are all of finite dimensions, with $\text{dim}(\text{Hom}_G(V_\lambda, R[\mathcal{O}])) = m_\lambda$. The first Theorem of the paper is to compute the multiplicities $m_\lambda$ for fundamental representations $V_\lambda$ in $R[\mathcal{O}]$ using the formula of the form in Conjecture 1:

**Theorem 3.** Suppose $\mathcal{O} = (c_{2k}, c_{2k-1}, \ldots, c_0)$ is a special nilpotent orbit in $\mathfrak{sp}(2m, \mathbb{C})$ (which will be defined in Chapter 3). Remove all column pairs of same size, leaving the orbit $(d_{2l}, d_{2l-1}, \ldots, d_0)$, i.e.

$$\mathcal{O} = (d_{2l}, d_{2l-1}, \ldots, d_0) \vee (\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x)$$

with $d_{i+1} \neq d_i$ for all $i$. Let:

- $Y := \{\alpha_i | i = 1, \ldots, x\}$
• $Z := \{ z_j = \frac{d_{2+j} + d_{2j-1}}{2} | j = 0, \ldots, l \} \setminus \{ 0 \}$, and
• $W := Y \cup Z = \{ w_1, w_2, \ldots, w_J \}$ with $w_1 \leq w_2 \leq \cdots \leq w_J$, and $J = k + 1$
if $d_0 \neq 0$ and $J = k$ otherwise.

Then for any irreducible representations $\mu_i$ of highest weight $(1^i 0^{m-i})$, i.e. $\mu_i := \wedge^i \mathbb{C}^{2m} / \wedge^{i-2} \mathbb{C}^{2m}$ in $Sp(2m, \mathbb{C})$ (if $i - 2 < 0$, take $\wedge^{i-2} \mathbb{C}^{2m} = \text{triv}$),

$$[R[O] : \mu_i] = [\text{Ind}_{GL(W)}^{Sp(2m, \mathbb{C})} (\text{triv}) : \mu_i]$$

where $GL(W) = GL(w_1, \mathbb{C}) \times GL(w_1, \mathbb{C}) \times \cdots \times GL(w_J, \mathbb{C})$.

Suppose $\mathcal{P} = (b_{2k+1}, b_{2k}, b_{2k-1}, \ldots, b_0)$ is a special nilpotent orbit in $\mathfrak{o}(n, \mathbb{C})$. First remove all column pairs of same size, leaving the orbit $(e_{2l+1}, e_{2l}, e_{2l-1}, \ldots, e_0)$, i.e.

$$\mathcal{P} = (e_{2l+1}, e_{2l}, e_{2l-1}, \ldots, e_0) \vee (\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x)$$

with $e_{i+1} \neq e_i$ for all $i$. Let:

• $Y := \{ \alpha_i | i = 1, \ldots, x \}$
• $Z := \{ z_j = e_{2j} + e_{2j-1} | j = 0, \ldots, l \} \setminus \{ 0 \}$ (note that $e_{2l+1}$ is not used), and
• $W := Y \cup Z = \{ w_1, w_2, \ldots, w_J \}$ as before.

Then for the irreducible representations $\mu_j'$ of highest weight $(1^j 0^{n/2-j})$, i.e. $\mu_j' := \wedge^j \mathbb{C}^n$ in $O(n, \mathbb{C})$,

$$[R[\mathcal{P}] : \mu_j'] = [\text{Ind}_{O(n, \mathbb{C})}^{O(e_{2l+1}, \mathbb{C}) \times GL(W)} (\text{triv}) : \mu_j']$$

Note that the right hand side of both equations above can be computed easily using Frobenius reciprocity. We will describe an algorithm computing the multiplicities in Chapter 4. For example, let $O = (8, 6, 6, 4, 4, 2, 2)$ in $sp(32, \mathbb{C})$. Then $W = \{ 2, 4, 4, 6 \}$ and the algorithm gives multiplicities as follows:

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|---|-----|---|-----|---|-----|---|
| $[R[\mathcal{P}] : \mu_i']$ | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |

Let $\mathcal{P} = (7, 5, 3, 3, 1)$ in $\mathfrak{o}(19, \mathbb{C})$. Then the $W = \{ 3, 3 \}$. We have the multiplicities of $\mu_j'$ as follows:

| $i$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|-----|---|---|---|---|---|-----|---|-----|---|
| $[R[\mathcal{P}] : \mu_i']$ | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 | 0 | 0 |

Remarks.

1. Notice that our Theorem gives $[R[O] : \mu_{2j+1}] = 0$ for all integers $j$. This is obvious by how $G$ acts on $R[O]$: Indeed $-I \in G$ acts trivially on $R[O]$, yet $-I$ acts like $-1$ on $\wedge^{2j+1} \mathbb{C}^{2m}$. So none of such $G$-modules can appear
in $R[O]$.

2. It is conjectured that the above Theorem holds for all nilpotent orbits of classical type. However, the required tools were not published and we will only focus on the special nilpotent cases.

The second Theorem of this paper is to give a criterion on the non-normality of $\overline{O}$, by computing the fundamental representation multiplicities of $R[O]$ and $R[\overline{O}]$:

**Theorem 4.** Let $O$ be a nilpotent orbit in $\mathfrak{sp}(2m, \mathbb{C})$. Then $\overline{O}$ is not normal iff 

$$[R[O] : \mu_i] < [R[O] : \mu_i]$$

for some $i > 0$.

Let $P = (b_{2k+1}, \ldots, b_0)$ be a nilpotent orbit in $\mathfrak{o}(n, \mathbb{C})$. Then $\overline{P}$ is not normal iff 

$$[R[\overline{P}] : \mu_i'] < [R[P] : \mu_i']$$

for some $i > 0$.

On the other hand, in the paper [18], Sommers and Trapa studied all Richardson orbits $O$, and the multiplicities $[R[O] : \mu_{ad}]$ for the adjoint representation $\mu_{ad}$. By explicitly constructing regular functions on $\overline{O}$, they concluded that $[R[O] : \mu_{ad}] = [R[\overline{O}] : \mu_{ad}]$ irrespective of the normality of $\overline{O}$. Indeed, $\mu_{ad}$ is not equal to any of the $\mu_i$ in type $C$, and $\mu_{ad} = \mu_2'$ in type $B$ and $D$. We will see that if $\overline{O}$ is not normal, the discrepancy of multiplicities in Theorem 4 occurs at least at $\mu_4$ and $\mu_4'$.

### 3 Proof of Theorem 3

#### 3.1 Special Nilpotent Orbits

The notion of special nilpotent orbits was first introduced by Lusztig. One of the many properties of special nilpotent orbits, given in [4] is, for any irreducible, admissible $(\mathfrak{g}, K)$ module $X$ with integral infinitesimal character, the associated variety of $\text{Ann}(X)$ must be the closure of a special nilpotent orbit. Here is the classification of special nilpotent orbits of classical types:

- **Type A:** All nilpotent orbits are special.
- **Type B:** If $P = (b_{2k+1}, \ldots, b_0)$ as above, then either $b_{2i}$ is odd or $b_{2i} =$
\[ b_{2i-1} = 2\beta. \]

- Type C: If \( \mathcal{O} = (c_{2i}, \ldots, b_0) \) as above, then either \( c_{2i} \) is even or \( c_{2i} = c_{2i-1} = 2\gamma + 1. \)

- Type D: If \( \mathcal{P} = (b_{2i+1}, \ldots, b_0) \) as above, then either \( b_{2i} \) is even or \( b_{2i} = b_{2i-1} = 2\beta + 1. \)

3.2 The Barbasch Model on \( \mathcal{O} \)

In [7, Section 2], Barbasch constructed a \((\mathfrak{g}_C, K_C)\)-module of all special nilpotent orbits of classical type. For our purpose, we are only concerned with the Theorem below:

**Theorem 5** (Barbasch). Let \( \mathcal{O} \) be a special nilpotent orbit of classical type. Then \( R[\mathcal{O}] \) is isomorphic to the spherical special unipotent representation \( X_1 \) corresponding to \( \mathcal{O} \), defined in [3][Theorem III] and [6][Chapter 4].

**Lemma 1.** Retain the notations in the above Theorem. Then as \( G \)-modules

\[ X_1 \cong R[\mathcal{O}] - Y \]

where \( Y \) is a collection of irreducible \( G \)-modules whose growth of multiplicities (e.g. [19]) is strictly smaller than the \( GK \)-dimension of \( \mathcal{O} \).

**Proof.** From Chapter 12 of [20], it is known that \( X_1 = \text{Ind}_{G^e}^G(\chi) - Y \), where \( G^e \) is the isotropy group of the element \( e \in \mathcal{O} \), and \( \chi \) is an irreducible representation of \( G^e \). From Chapter 2 or Theorem 4.2 of [20], it is easy to see that \( \chi = \text{triv} \), and consequently \( \text{Ind}_{G^e}^G(\text{triv}) = R[\mathcal{O}] \).

Therefore it suffices to show that \( Y = 0 \). To do so, we need to consider an induced orbit \( \mathcal{O}^+ \) of \( \mathcal{O} \). The definition of an induced orbit is given in [15] as follows:

**Definition 1.** Let \( \mathfrak{p} := \mathfrak{m} \oplus \mathfrak{n} \subset \mathfrak{g}^+ \) be the Levi decomposition of a parabolic subalgebra of \( \mathfrak{g}^+ \), and \( \mathcal{O} \) is a nilpotent orbit of \( \mathfrak{m} \). We call \( \mathcal{O}^+ = \text{Ind}_{\mathfrak{m}}^{\mathfrak{g}^+}(\mathcal{O}) \) is an induced orbit from \( \mathfrak{m} \) if \( \mathcal{O}^+ \cap (\mathcal{O} + \mathfrak{n}) \) is dense in \( \mathcal{O} + \mathfrak{n} \).

For any \( e \in \mathcal{O} \), one can pick \( e^+ \in \mathcal{O}^+ \) such that \( e^+ \in e + \mathfrak{n}^+ \).

**Proposition 1** (Barbasch-Vogan). Let \( G \) be a complex Lie group of classical type. A triangular orbit is defined as follows:

- Type B: \( \Delta_p = (2p + 1, 2p - 1, 2p - 1, 2p - 3, 2p - 3, \ldots, 3, 3, 1, 1, 0) \)
- Type C: \( \Delta_p = (2p, 2p, 2p - 2, 2p - 2, \ldots, 4, 4, 2, 2, 0) \)

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• Type D: \( \Delta_p = (2p + 2p, 2p) \)

Then for any classical special nilpotent orbit \( O \subset g \), we can find suitable \( k_1, \ldots, k_r \in \mathbb{N} \); \( l_1, \ldots, l_s \in \mathbb{N} \) such that

\[
O^+ = \text{Ind}_{g^p \oplus g(l_1) \oplus \ldots \oplus g(l_s)}^g(O \oplus 0 \oplus \ldots \oplus 0) = \text{Ind}_{g^p \oplus g(l_1) \oplus \ldots \oplus g(l_s)}^g(\Delta_p \oplus 0 \oplus \ldots \oplus 0)
\]

with the properties

1. \( l_t < 2p \) for all \( t \), and
2. Writing \( A(O) \) as the fundamental group of the orbit \( O \), and \( \overline{A(O)} \) as the Lusztig quotient ([5] Chapter 4) of \( A(O) \), we have \( |A(O^+)|/|A(O)| = 2^r \).

As an example, suppose \( O = (4, 2, 2) \) in \( g = \mathfrak{sp}(8) \), then

\[
O^+ = (4, 4, 2, 2, 2, 2) = \text{Ind}_{g \oplus \mathfrak{gl}(3) \oplus \mathfrak{gl}(1)}^{g \oplus \mathfrak{gl}(2)}(\Delta_2) = \text{Ind}_{g \oplus \mathfrak{gl}(3) \oplus \mathfrak{gl}(1)}^{g \oplus \mathfrak{gl}(2)}(\Delta_2)
\]

and \( A(O) = \overline{A(O)} = \text{triv}, A(O^+) = \overline{A(O^+)} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}. \)

For any special nilpotent orbit \( O \subset g \) we study the induced nilpotent orbit \( O^+ \subset g^+ \), where \( p := g \oplus \mathfrak{gl}(k_1) \oplus \ldots \oplus \mathfrak{gl}(k_r) \). Let \( e \in O \) and \( e^+ \in O^+ \) as before, write \( H^0 \) be the identity component of any group \( H \), then there is an injective map given in [15] by:

\[
\alpha : P^{e^+}/(P^{e^+})^0 \to A(O^+) = (G^+)^{e^+}/((G^+)^{e^+})^0
\]

and a surjective map:

\[
\beta : P^{e^+}/(P^{e^+})^0 \to A(O) = M^e/(M^e)^0
\]

since all the fundamental groups are abelian in the classical cases, there is an embedding of fundamental groups \( A(O) \xrightarrow{\alpha \circ \beta^{-1}} A(O^+) \). The main observation in this section is the following:

**Proposition 2.** Retain the notation as above. Then

\[
\text{Ind}_{G \times GL(k_1) \times \ldots \times GL(k_r)}^G(X_1) = \bigoplus_i R[O^+]_{\Phi_i}
\]

where \( \text{Ind}_{A(O)}(1) = \oplus_i \Phi_i \) as representation of \( A(O^+) \), and \( R[O]_{\Phi_i} \) is the global section of the \( G^+ \)-equivariant bundle \( G^+ \times (G^+)^{e^+} \Phi_i \to G^+/(G^+)^{e^+} \cong O^+ \) by considering \( \Phi_i \) as a representation of \( (G^+)^{e^+} \) trivial on \( ((G^+)^{e^+})^0 \).
Proof. When \( A(O^+) = \overline{A(O^+)} \), this is a special case of \([7]\). More precisely, Equation (18) of \([7]\) (which relied on a Theorem in \([8]\)) says that

\[
\text{Ind}_{G \times GL(k_1) \times \cdots \times GL(k_r)}^G(X_1) = \bigoplus_{i} X^+_\Psi_i
\]

where \( X^+_\Psi_i \) are special unipotent representations corresponding to \( O^+ \), parametrized by irreducible representations \( \Psi_i \) of \( \overline{A(O^+)} \). When \( A(O^+) = \overline{A(O^+)} \), \( \Psi_i = \Phi_i \). Furthermore, since \( O^+ \) is smoothly induced \([5]\) [Chapter 8], all the character formulas of \( X^+_\Psi_i = X^+_\Phi_i \) are given in \([5]\) [Chapter 8,9, Theorem III]. They are precisely equal to that of \( R[O^+]_{\Phi_i} \).

For the general case, one can write down the character formulas of both sides of the equation, and conclude that it is equal to the right hand side without referring to \([8]\).

Example 1. We present an example of the calculation of character formulas. Suppose \( O = (4) \) be the trivial orbit in \( G = \text{Sp}(4, \mathbb{C}) \). Then \( O^+ = (4, 4, 2, 2) = \text{Ind}_{G \times GL(3) \times GL(1)}^{G_{12, \mathbb{C}}}(O) \). Here the spherical unipotent representation of \( O \) is the trivial representation with (Weyl) character formula \( X_1 = \sum_{w \in W(C_2)} (-1)^{l(w)} X(21, w(21)) \), where \( X(\lambda, \mu) \) is the \( K \)-finite part of the principal series representation \( \text{Ind}_{B}^G(C(\lambda, \mu)) \) (\([5]\) Definition 1.7(d)). Then

\[
\text{Ind}_{G \times GL(3) \times GL(1)}^{Sp(12, \mathbb{C})}(X_1) = \sum_{w \in W(C_2 \times A_2 \times A_0)} (-1)^{l(w)} \begin{pmatrix} 21 & 10 & -1 & 0 \\ 21 & 10 & -1 & 0 \\ \end{pmatrix}
\]

On the other hand, \( A(O^+) = \overline{A(O^+)} = (\mathbb{Z}/2\mathbb{Z})^2 \). The four local systems \( \{ R[O^+]_{\Phi_i} \}_{i=1, 2, 3, 4} \) are equal to

\[
\text{Ind}_{G_{12, \mathbb{C}}}^{Sp(12, \mathbb{C})}(\text{triv} \times \text{triv}), \text{Ind}_{G_{12, \mathbb{C}}}^{Sp(12, \mathbb{C})}(\text{triv} \times \text{det})
\]

\[
\text{Ind}_{G_{12, \mathbb{C}}}^{Sp(12, \mathbb{C})}(\text{det} \times \text{triv}), \text{Ind}_{G_{12, \mathbb{C}}}^{Sp(12, \mathbb{C})}(\text{det} \times \text{det})
\]

adding up the four character formulas above, we get exactly that of \( \text{Ind}_{G \times GL(3) \times GL(1)}^{G_{12, \mathbb{C}}}(X_1) \).
Proof of Theorem 5
Induce both sides of the equation in Lemma 1 and get
\[ \text{Ind}_{G^{+} \times \cdots \times \text{GL}(k_r)}^{G^{+} \times \cdots \times \text{GL}(k_r)}(X_1) = \text{Ind}_{G^{+} \times \cdots \times \text{GL}(k_r)}^{G^{+} \times \cdots \times \text{GL}(k_r)}(R[O]) - \text{Ind}(Y) \]
by Proposition 2 and Proposition 1.0.1 of [7],
\[ \oplus_i R[O^{+}]\phi_i = (\oplus_j R[O^{+}]\phi_j - Z) - \text{Ind}(Y) \]
Therefore \( Z = \text{Ind}(Y) = 0 \) and hence \( Y = 0 \). \( \square \)

3.3 Spherical Unipotent Representations
In this subsection, we explore the theory of spherical unipotent representations \( X_1 \) mentioned above. The details of the calculations are given in [6], [7] or [22]. Here are some important aspects of the character theory of \( X_1 \):

Proposition 3. Let \( O \) be a classical nilpotent orbit in \( g \). There exists \( \chi_O \in t^* \) such that
\[ X_1 \cong \sum_{i \in (\mathbb{Z}/2\mathbb{Z}) \times O} \sum_{w \in W} \frac{\text{tr}(\sigma_i(w))}{2^{s_O}} X \left( \frac{\chi_O}{w \chi_O} \right) \]
where \( \sigma_i \) are the left cell representations of \( W \) corresponding to \( \chi_O \), parametrized by elements in \((\mathbb{Z}/2\mathbb{Z}) \times O\).

Note that if we just consider the \( K_C \cong G \) representations, we have
\[ R[O] \cong X_1|_G \cong \sum_{w \in W} \sum_{i \in (\mathbb{Z}/2\mathbb{Z}) \times O} \frac{\text{tr}(\sigma_i(w))}{2^{s_O}} \text{Ind}_T^G(\chi_O - w \chi_O) \]
which is Conjecture 1. In fact, by using [5][Chapter 6], one can reduce the number of summands on the right much further. For example, suppose \( Q = (8, 6, 4, 2) \) in \( \text{sp}(20, \mathbb{C}) \). Then \( \chi = (4, 3, 2, 1, 0, 1, 2; 2, 1, 0) \). The character formula of \( X_Q \) is
\[
\frac{1}{4} \sum_{w \in W(C_4 \times D_5 \times C_2 \times D_1)} (-1)^{(w)} X \left( \begin{array}{c} 4321, 210 & 21, 0 \\ 4321, 210 & 21, 0 \end{array} \right) \\
+ \frac{1}{4} \sum_{w \in W(D_5 \times C_2 \times C_2 \times D_1)} \frac{(-1)^{(w)}}{2^{s_O}} X \left( \begin{array}{c} 43210, 21 & 21, 0 \\ 43210, 21 & 21, 0 \end{array} \right)
\]
\[ + \frac{1}{4} \sum_{w \in (C_4 \times D_2 \times D_3 \times C_0)} (-1)^{l(w)} X \begin{pmatrix} 4321, 210 \ 4321 \ 210 \ 210 \end{pmatrix} \]

\[ + \frac{1}{4} \sum_{w \in (D_5 \times C_2 \times D_3 \times C_0)} (-1)^{l(w)} X \begin{pmatrix} 43210, 21 \ 43210, 21 \ 210 \ 210 \end{pmatrix} \]

all the coefficients of the above expression are \( \pm \frac{1}{4} \).

### 3.4 Proof of Theorem 3.

**Lemma 2.** Let \( G \) be a classical Lie group and \( O \) is a nilpotent orbit in the Lie algebra \( \mathfrak{g} \). Using Conjecture 1, we can express \( R[O] \) and \( \text{Ind}_{G \times \mathbb{C}[W]}(\text{triv}) \) (equivalently \( R[P] \) and \( \text{Ind}_{O \times \mathbb{C}[W]}(\text{triv}) \)) into a finite sum \( \sum \lambda c_{\lambda} \text{Ind}_{T}^{G}(\lambda) \).

Then the coefficients of \( \text{Ind}_{T}^{G}(1k0^l) \) of both expressions are the same.

The proof of the Lemma is computational, which we omit the details and instead give an example of such computation. It is worth noting that such result is hard to be obtained using the formula in Theorem 1.

**Example 2.** Let \( G = Sp(12, \mathbb{C}) \) and \( O = (8, 4) \). Then

\[ R[O] = \frac{1}{2} \left( \sum_{w \in W(C_4 \times D_2)} (-1)^{l(w)} \text{Ind}_{T}^{G}(4321, 10 - w(4321, 10)) \right. \]

\[ \left. + \sum_{w' \in W(D_5 \times C_1)} (-1)^{l(w')} \text{Ind}_{T}^{G}(43210, 1 - w'(43210, 1)) \right] \]

To find the coefficient of \( \text{Ind}_{T}^{G}(000000) \) in the above expression, one needs to find out the \( w \in W(C_4 \times D_2) \) so that \( (4321, 10) - w(4321, 10) \) can be \( W \)-conjugated to have weight \( (000000) \) (and respectively for \( w' \in W(D_5 \times C_1) \)). Obviously this forces \( w = w' = I \), and hence

\[ R[O] = \frac{1}{2} (\text{Ind}_{T}^{G}(000000) + \text{Ind}_{T}^{G}(000000)) + \sum_{\lambda \in \mathfrak{t}} c_{\lambda} \text{Ind}_{T}^{G}(\lambda), \quad ||\lambda|| > 0 \]

\[ = \text{Ind}_{T}^{G}(000000) + \sum_{\lambda \in \mathfrak{t}} c_{\lambda} \text{Ind}_{T}^{G}(\lambda) \]

To find out the coefficients of \( \text{Ind}_{T}^{G}(110000) \), one needs to find out which \( w \in W(C_4 \times D_2) \) so that \( (4321, 10) - w(4321, 10) \) can be \( W \)-conjugated to have weight \( (110000) \) (and respectively for \( w' \in W(D_5 \times C_1) \)). The list of
all such $w(4321,10)$ and $w(43210,1)$ are given below:

| $w \lambda$ | $\lambda - w \lambda$ | $w' \lambda$ | $\lambda - w \lambda$ |
|-------------|-----------------|-------------|-----------------|
| 3421,10     | 1-10000         | 3421,10     | 1-10000         |
| 4231,10     | 01-1000         | 4231,10     | 01-1000         |
| 4312,10     | 001-100         | 43120,1     | 001-100         |
| 4321,01     | 00001-1         | 43201,1     | 0001-10         |
| 4321,0-1    | 000011          | 4320-1,1    | 000110          |

Therefore,

$$R[\mathcal{O}] = \text{Ind}_{G^T}(000000) + \frac{1}{2}((-5) + (-5)) \text{Ind}_{G^T}(110000) + \sum_{\lambda \in \mathfrak{t}} c_\lambda \text{Ind}_{G^T}(\lambda), \quad ||\lambda||^2 > 2$$

Continuing the calculations, we get

$$R[\mathcal{O}] = \text{Ind}_{G^T}(000000) - 5 \text{Ind}_{G^T}(110000) + 6 \text{Ind}_{G^T}(111100) - \text{Ind}_{G^T}(111111) + \sum_{\lambda \in \mathfrak{t}} c_\lambda \text{Ind}_{G^T}(\lambda)$$

where at least one of the entries of $\lambda$ is $> 1$.

On the other hand, $GL(W) = GL(6)$ in this case, by the Weyl character formula

$$\text{Ind}_{GL(6)}^{Sp(12,\mathbb{C})}(\text{triv}) = \sum_{w \in W(A_{\lambda})} (-1)^{(w)} \text{Ind}_{G^T}(\frac{5 3 1 - 1 - 3 - 5}{2 2 2 2 2 - w(\frac{5 3 1 - 1 - 3 - 5}{2 2 2 2 2 - 2 2 2 2 2}))$$

which also gives

$$\text{Ind}_{GL(6)}^{Sp(12,\mathbb{C})}(\text{triv}) = \text{Ind}_{G^T}(000000) - 5 \text{Ind}_{G^T}(110000) + 6 \text{Ind}_{G^T}(111100) - \text{Ind}_{G^T}(111111) + \sum_{\lambda \in \mathfrak{t}} c'_\lambda \text{Ind}_{G^T}(\lambda)$$

Now we can start proving Theorem 3. Suppose $\mu$ is the irreducible representation of $G$ with highest weight $(1^\alpha 0^\beta)$. Then

$$[R[\mathcal{O}] : \mu] = |\sum_{\lambda} c_\lambda \text{Ind}_{G^T}(\lambda) : \mu| = \sum_{\lambda} c_\lambda [\lambda : \mu]|_T$$

Note that $\mu|_T$ consists just the weights of the representation $\mu$, and all dominant weights of $\mu$ must be of the form $(1^{\alpha'} 0^{\beta'})$, $\alpha' \leq \alpha$. So

$$[R[\mathcal{O}] : \mu] = \sum_{\lambda = (1^{\alpha'} 0^{\beta'}) \alpha' \leq \alpha} c_\lambda [\lambda : \mu]|_T = \sum_{\lambda = (1^{\alpha'} 0^{\beta'}) \alpha' \leq \alpha} [c_\lambda \text{Ind}_{G^T}(\lambda) : \mu]$$
however we know that the coefficients \( \{ c_\lambda \mid \lambda = (1^{\alpha_0} 0^{\beta'}) \} \) are the same for \( R[\mathcal{O}] \) and \( \text{Ind}_{GL(W)}^{G} (\text{triv}) \) (respectively \( R[\mathcal{P}] \) and \( \text{Ind}_{O(e) \times GL(W)}^{G} (\text{triv}) \)). Hence the Theorem follows.

**Remarks.**

1. Lemma 2 computes the coefficients of \( c_\lambda \) of the expression \( R[\mathcal{O}] = \sum \lambda c_\lambda \text{Ind}_{G}^{T} (\lambda) \) when \( \lambda \) is ‘small’. In the above mentioned paper of Chmutova and Ostrik [10], they tried to find out the \( \lambda \) of greatest norm with \( c_\lambda \neq 0 \). Our calculations work perfectly well in finding out the ‘largest’ \( \lambda \) in the expression, whose results are given in the preprint [21].

2. It is worth noting that for those \( \lambda \) such that \( c_\lambda \neq 0 \), the sum of the entries of \( \lambda \) is always even. This resonates with the fact that \( R[\mathcal{O}] \) does not contain any \( G \)-module of highest weight \( (1^{2j+1} 0^{k}) \), as pointed out at the Remarks after Theorem 3.

### 4 Proof of Theorem 4

We can now state and prove another criterion of the normality of \( \overline{\mathcal{O}} \) by computing the multiplicities of irreducible finite-dimensional representations in \( X_1 \cong R[\mathcal{O}] \) (Theorem 4).

Before moving on, we state an algorithm computing the fundamental representation multiplicities of \( R[\mathcal{O}] \), using Frobenius reciprocity. This will be used for the proof of Theorem 4.

**Lemma 3.** Let \( G(n) = Sp(n, \mathbb{C}) \). Write \( GL(W) = GL(w_1) \times GL(w_2) \times \cdots \times GL(w_J) \) as before. Then

\[
[\text{Ind}_{GL(W)}^{G(n)} (\text{triv}) : \wedge^{2i+1} \mathbb{C}^n] = 0
\]

\[
[\text{Ind}_{GL(W)}^{G(n)} (\text{triv}) : \wedge^{2i} \mathbb{C}^n] = \# \{(n_1, n_2, \ldots, n_J) \in \mathbb{N}^J \mid \sum_l n_l = i, n_l \leq w_l \text{ for } 1 \leq l \leq J \}
\]

Similarly, let \( G(n) = O(n, \mathbb{C}) \) and \( GL(W) = GL(w_1) \times GL(w_2) \times \cdots \times GL(w_J) \) as before. Then

\[
[\text{Ind}_{O(e) \times GL(W)}^{G(n)} (\text{triv}) : \wedge^{2i+1} \mathbb{C}^n] = 0
\]
\[ \text{Ind}_{O(e) \times GL(W)}^G (\text{triv}) : \bigwedge^{2i} C^n = \# \{(n_1, n_2, \ldots, n_J) \in \mathbb{N}^J \mid \sum_{l} n_l = i, \ n_l \leq w_l \text{ for } 1 \leq l \leq J \} \]

**Proof.** Consider the first part of the Lemma. By Frobenius reciprocity,
\[ \text{Ind}_{GL(W)}^G (\text{triv}) : \bigwedge^j C^n = [\text{triv} : \bigwedge^j C^n]_{GL(W)}. \]
As we have seen in the last Section,
\[ \bigwedge^j C^n|_{GL(W)} = \bigoplus_{\sum_{l} n_l = 2i, \ n_l \leq 2w_l} \bigotimes_l \text{Res}_{GL(w_l)}(2w_l)^j C^{2w_l} \]
and \[ \text{Res}_{GL(w_l)}(2w_l)^j C^{w_l} : \text{triv}_{GL(w_l)} = 1 \text{ when } j_l \text{ is even and } 0 \text{ otherwise.} \]
Therefore, if \( j = 2i + 1 \) is odd, then there must be at least one summand \( j_l \) which is odd, and the first statement of the Lemma follows.

For \( j = 2i \) even,
\[ \text{Ind}_{GL(W)}^G (\text{triv}) : \bigwedge^j C^n = [\text{triv} : \bigoplus_{\sum_{l} 2n_l = 2i, \ 2n_l \leq 2w_l} \bigotimes_l \text{Res}_{GL(w_l)}(2w_l)^j C^{2w_l}] \]
hence the Lemma follows.

For the second part of the Lemma, note that
\[ \bigwedge^j C^n|_{O(e) \times GL(W)} = \bigoplus_{\sum_{l} j_l = j, \ j_0 \leq e, \ j_0 \leq 2w_l} \bigotimes_l \text{Res}_{GL(w_l)}(2w_l)^j C^{2w_l} \]
and \[ [\bigwedge^j C^n : \text{triv}_{O(e)}] = 1 \text{ iff } j_0 = 0. \] Hence the second part of the Lemma follows from the first part. \( \square \)

**Theorem 6.** Suppose \( O = (c_{2k}, \ldots, c_1, c_0) \) is a nilpotent orbit in \( \mathfrak{sp}(2m, \mathbb{C}) \) with \( W \) as defined as in Theorem 3. Let \( w_1 \leq w_2 \leq \cdots \leq w_J \) be the elements in \( W \). Define a sequence of sequences \( A_i = (\alpha_{i0}, \alpha_{i1}, \ldots) \) recursively by:
- Begin with the sequence \( A_1 = (\alpha_{10}, \alpha_{11}, \alpha_{12}, \ldots) = (1, \left( \begin{smallmatrix} J-1 \end{smallmatrix} \right), \left( \begin{smallmatrix} J \end{smallmatrix} \right), \ldots). \)
- Define the \( i \)-th sequence recursively by \( A_{i+1} = A_i - (0, \ldots, 0, \alpha_{i0}, \alpha_{i1}, \alpha_{i2}, \ldots). \)

Then
\[ [R(O) : \mu_{2i}] = \alpha_{(J+1)i} \text{ for } i \leq \frac{m}{2} \]
\[ [R(O) : \mu_{2i+1}] = 0 \text{ for all } i \]

Suppose \( P = (b_{2k+1}, \ldots, b_1, b_0) \) is a nilpotent orbit in \( \mathfrak{o}(n, \mathbb{C}) \) with \( W \) as defined as in Theorem 3. Let \( w_1 \leq w_2 \leq \cdots \leq w_J \) be the elements in \( W \).
Define a sequence of sequences $B_i = (\beta_{i0}, \beta_{i1}, \ldots)$ recursively by:
- Begin with the sequence $B_1 = (\beta_{10}, \beta_{11}, \beta_{12}, \ldots) = (1, (\frac{1}{2}), (\frac{J+2}{2}), \ldots)$.
- Define the $i$-th sequence $B_i$ recursively by $B_{i+1} = B_i - (0, \ldots, 0, \beta_{i0}, \beta_{i1}, \beta_{i2}, \ldots)$.

Then

$$[R[P]: \mu'_{2i}] = \beta_{(J+1)i} \text{ for } 2i \leq n$$

$$[R[P]: \mu'_{2i+1}] = 0 \text{ for all } i$$

Proof. Consider the case of $O(n, \mathbb{C})$ first. We will prove the Theorem by induction, namely we will show that for all $r \geq 1$,

$$\beta_{ri} = \#\{(n_1, n_2, \ldots, n_J) \in \mathbb{N}^J | \sum_l n_l = i, n_l \leq w_l \text{ for } 1 \leq l \leq r - 1\}$$

By putting $r = J + 1$, the Theorem follows from the above Lemma.

When $r = 1$, the entries $\beta_{1i} = \binom{J+i-1}{i}$ of the sequence $B_1$ records the possibilities of partitioning $i$ into $J$ spots. So the hypothesis holds. Now suppose the hypothesis holds for $r = k$, then consider the entries of the sequence $B_k$

$$\beta_{ki} = \#\{(n_1, n_2, \ldots, n_J) \in \mathbb{N}^J | \sum_l n_l = i, n_l \leq w_l \text{ for } 1 \leq l \leq k - 1\}$$

If $0 \leq i \leq w_k$, then every integer $n_l$ cannot exceed $w_k$, hence $\beta_{ki}$ is equal to $\#\{(n_1, n_2, \ldots, n_J) \in \mathbb{N}^J | \sum_l n_l = i, n_l \leq w_l \text{ for } 1 \leq l \leq k\}$. If $i > w_k$, then the set $\{(n_1, n_2, \ldots, n_J) \in \mathbb{N}^J | \sum_l n_l = i, n_l \leq w_l \text{ for } 1 \leq l \leq k - 1, n_k > w_k\}$ consists of elements of the form

$$\{(n_1', n_2', \ldots, n_J') + (0, \ldots, w_k, \ldots, 0) \mid \sum_l n_l' = i - w_k, n_l' \leq w_l \text{ for } 1 \leq l \leq k-1\}$$

which has cardinality $\beta_{k(i-w_p)}$. Therefore the cardinality of the set $\{(n_1, n_2, \ldots, n_J) \in \mathbb{N}^J | \sum_l n_l = i, n_l \leq w_l \text{ for } 1 \leq l \leq k\}$ is equal to $\beta_{ki} - \beta_{k(i-w_p)}$. These are precisely the entries of $\beta_{(k+1)i}$, and consequently the Theorem follows for $O(n, \mathbb{C})$.  

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Similarly for the case of $Sp(2m, \mathbb{C})$, we have

$$[R[\mathcal{O}] : \wedge^2 \mathbb{C}^{2m}] = \beta_{(J+1)i}$$

it therefore suffices to show that $\alpha_{(J+1)i} = \beta_{(J+1)i} - \beta_{(J+1)(i-1)}$. Again, one can use induction to show that

$$\alpha_{ri} = \beta_{ri} - \beta_{r(i-1)}$$

for all $r$. When $r = 1$ the formula above reads $\binom{J-1}{i} = \binom{J}{i} - \binom{J-1}{i}$, which is a well-known combinatorial formula. Then the above formula holds for all $r$ since the recursive construction of $A_i$ and $B_i$ are the same.

Our strategy prove Theorem 4 is to find an upper bound on $[R[\mathcal{O}] : \mu]$ for all fundamental representations $\mu$, and show that the upper bound is strictly smaller than $[R[\mathcal{O}] : \mu]$ if $\mathcal{O}$ is not normal. The upper bound we need is given in the Lemma below:

**Lemma 4.** Let $\mathcal{O} = (c_{2k}, c_{2k-1}, \ldots, c_0)$ be a nilpotent orbit in $sp(2m, \mathbb{C})$, and $\mu$ be any finite dimensional irreducible representation, then

$$[R[\mathcal{O}] : \mu] \leq [R[\mathcal{O}^\sharp] : \mu]$$

where $\mathcal{O}^\sharp = \left( \frac{c_{2k} + c_{2k-1}}{2}, \frac{c_{2k} + c_{2k-1}}{2}, \frac{c_{2k-2} + c_{2k-3}}{2}, \frac{c_{2k-2} + c_{2k-3}}{2}, \ldots, \frac{c_0 + c_1}{2}, \frac{c_0 + c_1}{2}, \frac{c_0}{2}, \frac{c_0}{2} \right)$

Let $\mathcal{P} = (b_{2k+1}, b_{2k}, \ldots, b_0)$ be a nilpotent orbit in $o(n, \mathbb{C})$, and $\mu'$ be any finite dimensional irreducible representation, then

$$[R[\mathcal{P}] : \mu'] \leq [R[\mathcal{P}^\sharp] : \mu']$$

where $\mathcal{P}^\sharp = \left( \frac{b_{2k+1}}{2}, \frac{b_{2k} + b_{2k-1}}{2}, \frac{b_{2k} + b_{2k-1}}{2}, \frac{b_{2k-2} + b_{2k-3}}{2}, \frac{b_{2k-2} + b_{2k-3}}{2}, \ldots, \frac{b_0 + b_1}{2}, \frac{b_0 + b_1}{2}, \frac{b_0}{2}, \frac{b_0}{2} \right)$

**Proof.** We only work in the case when $G = Sp(2m, \mathbb{C})$. Note that by the Kraft-Procesi criterion (Theorem 1), $\overline{\mathcal{O}^\sharp}$ is normal. Therefore the paragraphs preceding Theorem 2 says $R[\mathcal{O}^\sharp] = R[\overline{\mathcal{O}^\sharp}]$. On the other hand, note that $\overline{\mathcal{O}^\sharp} \supset \overline{\mathcal{O}}$. Consequently, we have a $G$-module surjection

$$R[\mathcal{O}^\sharp] = R[\overline{\mathcal{O}^\sharp}] \twoheadrightarrow R[\overline{\mathcal{O}}]$$

and hence $[R[\overline{\mathcal{O}}] : \mu] \leq [R[\overline{\mathcal{O}^\sharp}] : \mu]$ for any finite dimensional $G$-representations $\mu$. However, the latter term is equal to $[R[\mathcal{O}^\sharp] : \mu]$. Hence the result follows.
Using the algorithm in the beginning of this section, one can find out the multiplicities \([R[O^g] : \mu]\). And now we are in the position to prove Theorem 4:

**Proof of Theorem 4**

One direction is easy - if \(O\) is normal, then \(R[O] = R[O]\) as \(G\)-modules, hence \([R[O] : \mu_i] = [R[O] : \mu_i]\) for all \(i\).

Now suppose \(O\) is not normal, and let \(O^g\) as in last Section. Then we obtain a new set of integers \(W^g\) computing the multiplicities \([R[O^g] : \mu]\). By the Kraft-Procesi criterion (Theorem 2), the two sets of integers \(W^g\) and \(W\) are different. More precisely, writing \(d := \min\{i \mid 1 \leq i \leq J, w_i \neq w_i^g\}\), then \(w_d^g < w_d\). Using the algorithm in Theorem 6, it says \([R[O^g] : \mu_{2d+2}] < [R[O] : \mu_{2d+2}]\). Now Lemma 3 says \([R[O] : \mu_i] \leq [R[O^g] : \mu_i]\) for all \(i\), and consequently the theorem follows.

**Example 3.** Let \(O = (8, 6, 6, 4, 4, 2, 2)\) in \(sp(32, \mathbb{C})\). Following Theorem 2 (Kraft-Procesi criterion), its closure is not normal.

By Theorem 3, \(W = \{2, 4, 4, 6\}\). Now \(O^g = (7, 7, 5, 5, 3, 3, 1, 1)\), and \(W^g = \{1, 3, 5, 7\}\). We therefore have the multiplicities as follows:

| \(i\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| \([R[O] : \mu_i]\) | 1 | 0 | 3 | 0 | 5 | 0 | 7 | 0 | 8 | 0 | 8 | 0 | 7 | 0 | 5 | 0 | 2 |
| \([R[O^g] : \mu_i]\) | 1 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 12 | 0 | 13 | 0 | 12 | 0 | 8 | 0 | 3 |

The discrepancies of the two rows of numbers reflects the non-normality of \(O\).

Let \(P = (7, 5, 3, 3, 1)\) in \(o(19, \mathbb{C})\). Then \(P^g = (7, 4, 4, 2, 2)\), \(W = \{3, 3\}\) and \(W^g = \{2, 4\}\). We therefore have the multiplicities of \(\mu'_{2i}\) as follows:

| \(i\) | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|-------|---|---|---|---|---|----|----|----|----|----|
| \([R[P] : \mu'_{2i}]\) | 1 | 2 | 3 | 3 | 3 | 2 | 1 | 0 | 0 | 0 |
| \([R[P^g] : \mu'_{2i}]\) | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 | 0 | 0 |

**5 Final Remarks**

Note that in the above examples, the discrepancy occurs at \(i > 2\). Indeed, by the Kraft-Procesi criterion of non-normality, if \(w_1 \leq w_2 \leq \cdots \leq w_J\) are the elements of \(W\), and \(w_1^g \leq w_2^g \leq \cdots \leq w_J^g\) are the elements of \(W^g\), then \(d\) defined in the proof of Theorem 4 above must be greater than 0. So the discrepancy between \([R[P] : \mu'_{2i}]\) and \([R[P^g] : \mu'_{2i}]\) occurs at \(i = 2d + 2 > 2\).
This agrees with the construction in Section 4 of [18].

On the other hand, there is an upper bound of the multiplicities of \( R[\mathcal{O}] \) by that of \( R[\mathcal{O}^\ast] \) by Lemma 4. In fact, we have the following

**Conjecture 2.** Let \( \mathcal{O} \) be a classical nilpotent orbit in \( G \), and \( \mu \) is any irreducible, finite dimensional representation of \( G \). Then the multiplicities \( [R[\mathcal{O}] : \mu] \) can be computed. In particular, if \( \mu \) is an irreducible representation with highest weight consisting of 1’s and 0’s only, then

\[
[R[\mathcal{O}] : \mu] = [R[\mathcal{O}^\ast] : \mu]
\]

In [22] there is a more refined conjecture in the context of \((\mathfrak{g}_C, K_C)\)-modules. Some evidences of the conjecture can also be found there.

**References**

[1] Achar, P., *An order-reversing duality map for conjugacy classes in Lusztig’s canonical quotient*, Transform. Groups 8, 107-145 2003

[2] Achar, P. and Sommers, E., *Local systems on nilpotent orbits and weighted Dynkin diagrams*, Represent. Theory 6, 190-201 2002

[3] Anker, J-P. and Orsted, B., *Lie theory: Lie algebras and representations*, Birkhauser, 2004

[4] Barbasch, D. and Vogan, D., *Primitive Ideals and Orbital Integrals in Complex Classical Groups*, Invent. Math. 259, 153-199, 1982

[5] Barbasch, D. and Vogan, D., *Unipotent Representations of Complex Semisimple Groups*, Annals of Mathematics 121, No.1, 41-110, 1985

[6] Barbasch, D., *The Unitary Dual for Complex Classical Lie Groups*, Invent. Math. 96, 103-176, 1989

[7] Barbasch, D., *Regular Functions on Covers of Nilpotent Coadjoint Orbits* http://arxiv.org/abs/0810.0688v1, 2008

[8] Berline, N. and Vergne, M., *Fourier transforms of orbits of the coadjoint representation* Representation theory of reductive groups (Park City, Utah, 1982), 53-67, 1983
[9] Brylinski, R., *Dixmier Algebras for Classical Complex Nilpotent Orbits via Kraft-Procesi Models I*, The orbit method in geometry and physics: in honor of A.A. Kirillov, Birkhauser, 2003

[10] Chmutova, T. and Ostrik, V., *Calculating canonical distinguished involutions in the affine Weyl groups*, Experiment. Math. **11**, 99-117, 2002

[11] Collingwood, D. and McGovern, W., *Nilpotent orbits in semisimple Lie algebras*, Van Norstrand Reinhold Mathematics Series, 1993

[12] Eisenbud, D. *Commutative Algebra: with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, Springer-Verlag, 1995

[13] Kraft, H. and Procesi, C., *Closures of conjugacy classes of matrices are normal*, Invent. Math. **53**, 227-247, 1979

[14] Kraft, H. and Procesi, C., *On the Geometry of Conjugacy Classes in Classical Groups*, Comment. Math. Helvetici **57**, 539-602, 1982

[15] Lusztig, G. and Spaltenstein, N., *Induced unipotent classes*, J. London Math. Soc. (2) **19**, 41-52, 1979

[16] McGovern, W., *Rings of regular functions on nilpotent orbits and their covers*, Invent. Math. **97**, 209-217, 1989

[17] McGovern, W., *Completely Prime Maximal Ideals and Quantization*, Memoirs of the American Mathematical Society **519**, 1994

[18] Sommers, E., and Trapa, P., *The adjoint representation in rings of functions*, Representation Theory, **1**, 182-189, 1997

[19] Vogan, D., *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. **48**, 75-98 1978

[20] Vogan, D., *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), **315-388**, 1991

[21] Wong, K., *Local Systems on Classical Nilpotent Orbits and Maximal Length Elements*, [http://arxiv.org/abs/1308.2020](http://arxiv.org/abs/1308.2020), 2013

[22] Wong, K., *Dixmier Algebras on Complex Classical Nilpotent Orbits and their Representation Theories*, Ph.D. Thesis, Cornell University, 2013