Adaptive Low-Complexity Sequential Inference for Dirichlet Process Mixture Models

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Abstract

We develop a sequential low-complexity inference procedure for the Infinite Gaussian Mixture Model (IGMM) for the general case of an unknown mean and covariance. The observations are sequentially allocated to classes based on a sequential maximum a-posterior (MAP) criterion. We present an easily computed, closed form for the conditional likelihood, in which the parameters can be recursively updated as a function of the streaming data. We propose a novel adaptive design for the Dirichlet process concentration parameter at each iteration, and prove, under a simplified model, that the sequence of concentration parameters is asymptotically well-behaved. We sketch an equivalence between the steady-state performance of the algorithm and Gaussian classification. The methodology is applied to the problem of adaptive modulation recognition and obviates the need for storing a large modulation library required for traditional modulation recognition. We also numerically evaluate the bit error rate performance (BER) of the DPMM-trained classifier when used as a demodulator and show that there is critical signal-to-noise ratio (SNR) that characterizes whether successful decoding is possible.

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1 Introduction

Dirichlet process mixture models (DPMM) have been widely used for clustering data [11, 14]. Traditional finite mixture models often suffer from overfitting or underfitting of data due to possible mismatch between the model complexity and amount of data. Thus, model selection or model averaging is required to find the correct number of clusters or the model with the appropriate complexity. This may require significant computation. Bayesian nonparametric modeling are alternative approaches to parametric modeling, an example being DPMM’s which can automatically infer the actual number of clusters from the data via Bayesian inference techniques.

The use of Monte-Carlo Markov chain (MCMC) methods for Dirichlet process mixtures has made inference tractable [12]. However, these methods can exhibit slow convergence and their convergence can be tough to detect. Alternatives include variational methods [4], which are deterministic algorithms that convert inference to optimization. These approaches can take a significant computational effort even for moderate sized data sets. For large-scale data sets, there is a need for inference algorithms that are much faster and do not require multiple passes through the data. In this work, we focus on algorithms that adapt to each sample as they arrive, making them highly scalable. Wang & Dunson [16] have recently proposed a sequential maximum a-posterior (MAP) estimator for the class labels given streaming data. The algorithm is called SUGS and each iteration is composed of a selection step and a posterior update step.

An extensive literature exists on modulation recognition; a survey of the existing state-of-the-art recognition techniques can be found in [5]. Here, we propose an adaptive approach and solve an unsupervised clustering problem as a means for detecting the signal modulation and estimating its parameters. This of course breaks down for very low SNR, and we perform numerical simulations to find the critical SNR above which our approach is feasible.

The choice of concentration parameter $\alpha$ is critical for DPMM’s as it controls the number of clusters [1]. While most fast DPMM algorithms use a fixed $\alpha$ [7, 9, 10], imposing a prior distribution on $\alpha$ and sampling from it provides more flexibility, but this approach still heavily relies on experimentation and prior knowledge. Thus, many fast inference methods for Dirichlet process mixture models have been proposed that can adapt $\alpha$ to the data, including the works of Escobar & West [6] where learning of $\alpha$ is incorporated in the Gibbs sampling analysis, Blei & Jordan [4] where a Gamma prior is used in a conjugate manner directly in the variational inference algorithm. Wang & Dunson [16] extended SUGS to model uncertainty on the concentration parameter $\alpha$ in a Bayesian way directly in the sequential inference procedure. We found this approach to be unstable in the sense that too many degenerate classes are created and stability highly depends
on the domain of $\alpha$.

In this report, we propose an adaptive non-Bayesian approach for adapting $\alpha$ and call the resulting algorithm ASUGS, standing for Adaptive SUGS. While the basic idea behind ASUGS is directly related to SUGS, it utilizes a novel stable method for choosing the concentration parameter adaptively as new data comes in, which greatly improves the clustering performance. We also prove, using large-sample approximations to the conditional likelihood, that ASUGS asymptotically behaves as a Gaussian classifier. We apply ASUGS to the problem of automatic modulation recognition and use the asymptotic expression of the conditional likelihood for demodulation.

2 Sequential Framework for Dirichlet Process Mixture Model

Here, we review the SUGS framework of Wang & Dunson [16] for online clustering. Here, the nonparametric nature of the Dirichlet process manifests itself as modeling mixture models with countably infinite components. Let the observations be given by $y_i \in \mathbb{R}^d$, and $\gamma_i$ to denote the class label of the $i$th observation (a latent variable). We define the available information at time $i$ as $y^{(i)} = \{y_1, \ldots, y_i\}$ and $\gamma^{(i-1)} = \{\gamma_1, \ldots, \gamma_{i-1}\}$. The online sequential updating and greedy search (SUGS) algorithm is included below for completeness:

1. Set $\gamma_1 = 1$. Calculate $\pi(\theta_1|y_1, \gamma_1)$.
2. For $i \geq 2$:
   (a) Choose best class label for $y_i$:
   $\gamma_i \in \arg \max_{1 \leq h \leq k_{i-1}+1} P(\gamma_i = h|y^{(i)}, \gamma^{(i-1)}).$ (1)
   (b) Update the posterior distribution using $y_i, \gamma_i$:
   $\pi(\theta_{\gamma_i}|y^{(i)}, \gamma^{(i)}) \propto f(y_i|\theta_{\gamma_i})\pi(\theta_{\gamma_i}|y^{(i-1)}, \gamma^{(i-1)}).$ (2)

where $\theta_h$ are the parameters of class $h$, $f(y_i|\theta_h)$ is the observation density conditioned on class $h$ and $k_{i-1}$ is the number of classes created at time $i - 1$. The algorithm sequentially allocates observations $y_i$ to classes based on maximizing the conditional posterior probability.
2.1 Calculation of Class Posteriors

Next, we consider the calculation of the conditional posterior probability $P(\gamma_i = h| y^{(i)}, \gamma^{(i-1)})$ in (1). Define the auxiliary variables:

$$L_{i,h}(y_i) = P(\gamma_i = h, y^{(i-1)}, \gamma^{(i-1)}) = \int f(y_i|\theta_h)\pi(\theta_h|y^{(i-1)}, \gamma^{(i-1)})d\theta_h$$

$$\pi_i,h = P(\gamma_i = h|\alpha, y^{(i-1)}, \gamma^{(i-1)})$$

From Bayes’ rule, we have for $h = 1, \ldots, k_i - 1 + 1$:

$$P(\gamma_i = h| y^{(i)}, \gamma^{(i-1)}) \propto P(\gamma_i = h, y_i, y^{(i-1)}, \gamma^{(i-1)})$$

$$\propto P(y_i|\gamma_i = h, y^{(i-1)}, \gamma^{(i-1)})P(\gamma_i = h| y^{(i-1)}, \gamma^{(i-1)}) = L_{i,h}(y_i)\pi_i,h$$

2.2 Posterior Updates

At each iteration of the algorithm, after the class label $\gamma_i$ is chosen, we need to update the posterior distribution:

$$\pi(\theta_h|y^{(i-1)}, \gamma^{(i-1)})$$

From Bayes’ rule, the posterior update for the parameters within each class is:

$$\pi(\theta_h|y^{(i)}, \gamma^{(i)}) = \pi(\theta_h|y^{(i-1)}, \gamma^{(i-1)})$$

for all $h \neq \gamma_i$, and

$$\pi(\theta_{\gamma_i}|y^{(i)}, \gamma^{(i)}) \propto P(\theta_{\gamma_i}, y_i, \gamma_i, y^{(i-1)}, \gamma^{(i-1)})$$

$$\propto f(y_i|\theta_{\gamma_i})\pi(\theta_{\gamma_i}|y^{(i-1)}, \gamma^{(i-1)})$$

2.3 Probabilistic Assignment to Classes

The predictive probability of assignment of an observation $y_i$ to a class $h$ is given by:

$$\pi_{i,h} = \left\{ \begin{array}{ll} \frac{n_{i-1}(h)}{i-\alpha}, & h = 1, \ldots, k_{i-1} \\ \frac{\alpha}{i-1+\alpha}, & h = k_{i-1} + 1 \end{array} \right.$$

(3)

where the counts $n_{i-1}(h)$ are given by $\sum_{l=1}^{i-1} I(\gamma_l = h)$. The variables $\pi_{i,h}$ are used in the calculation of $P(\gamma_i = h|y^{(i)}, \gamma^{(i-1)})$. Using this notation, the class posteriors in (1) can be calculated as:

$$P(\gamma_i = h| y^{(i)}, \gamma^{(i-1)}) \propto \pi_{i,h} L_{i,h}(y_i)$$

(4)

for $h = 1, \ldots, k_{i-1} + 1$. 

4
2.4 Adaptation of Concentration Parameter $\alpha$

It is well known that the concentration parameter $\alpha$ has a strong influence on the growth of the number of classes [1]. Our experiments show that in this sequential framework, the choice of $\alpha$ is even more critical. Although the parameter $\alpha$ is handled from a fully Bayesian treatment in [16], we found that this method is very sensitive to the range of $\alpha$ being chosen. Thus, we propose an alternative method for choosing $\alpha$ sequentially as a function of data that we found works well and is stable throughout our experiments.

The idea is to start with a prior distribution on $\alpha$ that favors small $\alpha$ and shape it into a posterior distribution using the data. Let $p_0(\alpha)$ denote the prior for $\alpha$, e.g., a discretized Gamma distribution. Define $p_i(\alpha) = p(\alpha|y^{(i)}, \gamma^{(i)})$ as the posterior distribution at time $i$. Then, we have the update:

$$p_i(\alpha) \propto p(y_i, \gamma_i|y^{(i-1)}, \gamma^{(i-1)}, \alpha)p(\alpha|y^{(i-1)}, \gamma^{(i-1)})$$

$$\propto p_{i-1}(\alpha)\pi_{i,\gamma_i}(\alpha)$$ (5)

where $\pi_{i,\gamma_i}(\alpha)$ is given in (3). Once this update is made after the selection of $\gamma_i$, the $\alpha$ to be used in the next selection step is the mean of the distribution $p_i(\alpha)$, i.e., $\alpha_{i+1} = E[\alpha|y^{(i)}, \gamma^{(i)}]$. We also found that sampling from this distribution to pick an $\alpha$ yields good performance as well. Then, the selection step (see (1)) of ASUGS becomes:

$$\gamma_i \in \arg \max_h L_{i,h}(y_i)\pi_{i,h}(\alpha_i)$$

3 Unknown Mean & Covariance

We consider the general case of an unknown mean and covariance for each class. Let $T$ denote the precision (or inverse covariance) matrix. The probabilistic model for the mean and covariance matrix of each class is given as:

$$y_i|\mu, T \sim N(\cdot|\mu, T)$$

$$\mu|T \sim N(\cdot|\mu_0, c_0T)$$

$$T \sim W(\cdot|\delta_0, V_0)$$ (6)

where $N(\cdot|\mu, T)$ denote the observation density which is assumed to be multivariate normal with mean $\mu$ and precision matrix $T$. The parameters $\theta = (\mu, T) \in \Omega_1 \times \Omega_2$ follow a normal-Wishart joint distribution. The domains here are $\Omega_1 = \mathbb{R}^d$ and $\Omega_2 = S^{d+}_{++}$ is the positive definite cone. This leads to closed-form expressions for $L_{i,h}(y_i)$’s due to conjugacy [15]. For
concreteness, let us write the distributions of the model (6):

\[
f(y_i | \theta) = p(y_i | \mu, T) = \frac{\det(T)^{1/2}}{(2\pi)^{d/2}} \exp \left( -\frac{1}{2} (y_i - \mu)^T T (y_i - \mu) \right)
\]

\[
p(\mu | T) = p(\theta_1 | \Theta_2) = \frac{\det(c_0 T)^{1/2}}{(2\pi)^{d/2}} \exp \left( -\frac{c_0}{2} (\mu - \mu_0)^T T (\mu - \mu_0) \right)
\]

\[
p(T) = p(\Theta_2) = \frac{\det(V_0)^{-\delta_0}}{2^{2\delta_0} \Gamma_d(\delta_0)} \det(T)^{\delta_0 - \frac{d+1}{2}} \exp \left( -\frac{1}{2} \text{tr}(V_0^{-1} T) \right)
\]

where \( \Gamma_d(\cdot) \) is the multivariate Gamma function.

To calculate the class posteriors, we note from (4) that the conditional likelihoods of \( y_i \) given assignment to class \( h \) and the previous class assignments need to be calculated first. We derive closed-form expressions for these quantities in this section under the probabilistic model (6).

The conditional likelihood of \( y_i \) given assignment to class \( h \) and the history \((y^{(i-1)}, \gamma^{(i-1)})\) is given by:

\[
L_{i,h}(y_i) = \int f(y_i | \theta_h) \pi(\theta_h | y^{(i-1)}, \gamma^{(i-1)}) d\theta_h
\]  

(7)

We thus need to obtain an expression for the posterior distribution \( \pi(\theta_h | y^{(i-1)}, \gamma^{(i-1)}) \).

Due to the conjugacy of the distributions involved in (6), the posterior distribution \( \pi(\theta_h | y^{(i-1)}, \gamma^{(i-1)}) \) always has the form:

\[
\pi(\theta_h | y^{(i-1)}, \gamma^{(i-1)}) = N(\mu^{(i-1)}_h, \Sigma^{(i-1)}_h) \mathcal{W}(T_h | \delta^{(i-1)}_h, V^{(i-1)}_h)
\]

(8)

where \( \mu^{(i-1)}_h, \Sigma^{(i-1)}_h, \delta^{(i-1)}_h, V^{(i-1)}_h \) are hyperparameters that can be recursively computed as new samples come in. This would greatly simplify the computational complexity of the second step of the SUGS algorithm. Next, we derive the form of this recursive computation of the hyperparameters.

For simplicity of the derivation, let us consider the initial case \( y = y_1 \). Then, from Bayes’ rule:

\[
p(\theta | y) = p(\mu, T | y) = p(\mu | T, y) p(T | y)
\]

3.1 Calculation of \( p(\mu | T, y) \)

Note the factorization:

\[
p(\mu | T, y) \propto p(y | \mu, T) p(\mu | T)
\]

According to (6), we can write:

\[
y = \mu + \Sigma^{1/2} \epsilon
\]

\[
\mu = \mu_0 + \Sigma_0^{1/2} \epsilon'
\]
where $\epsilon \sim N(0, I)$, $\epsilon' \sim N(0, I)$, $\epsilon$ is independent of $\epsilon'$ and $\Sigma = T^{-1}$, $\Sigma_0 = (c_0 T)^{-1}$. From this, it follows that the conditional density $p(y|\mu, T)$ is also multivariate normal with mean $E[\mu|T, y]$ and covariance $\text{Cov}(\mu|T, y)$. Note that:

\[
E[y|T] = \mu_0 \\
\text{Cov}(y|T) = E[\text{Cov}(y|\mu, T)|T] + \text{Cov}(E[y|\mu, T]|T) = \Sigma + \Sigma_0 = (1 + c_0^{-1})T^{-1} \\
\text{Cov}(\mu, y|T) = \Sigma_0
\]

Using these facts, we obtain:

\[
E[\mu|T, y] = E[\mu|T] + \text{Cov}(\mu, y|T)\text{Cov}(y|T)^{-1}(y - E[y|T]) \\
= \mu_0 + T^{-1}((1 + c_0^{-1})T^{-1})^{-1}(y - \mu_0) \\
= \mu_0 + (1 + c_0^{-1})^{-1}(y - \mu_0) \\
\text{Cov}(\mu|T, y) = \text{Cov}(\mu|T) - \text{Cov}(\mu, y|T)\text{Cov}(y|T)^{-1}\text{Cov}(y, \mu|T) \\
= \Sigma_0 - \Sigma_0(\Sigma + \Sigma_0)^{-1}\Sigma_0^T \\
= c_0^{-1}\left(1 - \frac{c_0^{-1}}{1 + c_0^{-1}}\right)T^{-1} \\
= \frac{c_0^{-1}}{1 + c_0^{-1}}T^{-1}
\]

Thus, we have:

\[
p(\mu|T, y) = N(\mu|\mu_0 + (1 + c_0^{-1})^{-1}(y - \mu_0), (1 + c_0)T)
\]

where the conditional precision matrix becomes $(1 + c_0)T$. As a result, once the $\gamma_i$th component is chosen in the SUGS selection step, the parameter updates for the $\gamma_i$th class become:

\[
\mu_{\gamma_i}^{(i)} = \mu_{\gamma_i}^{(i-1)} + (1 + c_{\gamma_i}^{(i-1)})^{-1}(y_i - \mu_{\gamma_i}^{(i-1)}) \\
c_{\gamma_i}^{(i)} = c_{\gamma_i}^{(i-1)} + 1
\]

(9)

### 3.2 Calculation of $p(T|y)$

Next, we focus on calculating $p(T|y) = \int_{\mathbb{R}^d} p(T, \mu|y)d\mu$, where

\[
p(T, \mu|y) \propto p(y|T, \mu)p(\mu|T)p(T) \\
\propto \det(T)^{(d_0+1)/2} \det(T)^{1/2} \exp\left(-\frac{1}{2} \text{tr}(V_0^{-1}T)\right) \\
\times \exp\left(-\frac{1}{2} \left[c_0(\mu - \mu_0)^T T(\mu - \mu_0) + (y - \mu)^T T(y - \mu)\right]\right)
\]

7
Rewriting the term inside the brackets by completing the square, we obtain:

\[ c_0(\mu - \mu_0)^T T(\mu - \mu_0) + (y - \mu)^T T(y - \mu) \]

\[ = c_0\|T^{1/2} \mu - T^{1/2} \mu_0\|_2^2 + \|T^{1/2} y - T^{1/2} \mu\|_2^2 \]

\[ = (1 + c_0) \left\{ \|T^{1/2} \mu\|_2^2 - 2 \left( T^{1/2} \mu, \frac{c_0 T^{1/2} \mu_0 + T^{1/2} y}{1 + c_0} \right) \right. \]

\[ + \left. \frac{c_0\|T^{1/2} \mu_0\|_2^2 + \|T^{1/2} y\|_2^2}{1 + c_0} \right\} \]

\[ = (1 + c_0) \left\{ \|T^{1/2} \mu - \frac{c_0 T^{1/2} \mu_0 + T^{1/2} y}{1 + c_0}\|_2^2 - \frac{c_0 T^{1/2} \mu_0 + T^{1/2} y}{1 + c_0} - \frac{c_0\|T^{1/2} \mu_0\|_2^2 + \|T^{1/2} y\|_2^2}{1 + c_0} \right\} \]

Integrating out \( \mu \), we obtain:

\[ \int \exp \left( -\frac{1}{2} \left[ c_0(\mu - \mu_0)^T T(\mu - \mu_0) + (y - \mu)^T T(y - \mu) \right] \right) d\mu \]

\[ = \exp \left( -\frac{1 + c_0}{2} \left( \frac{c_0\|T^{1/2} \mu_0\|_2^2 + \|T^{1/2} y\|_2^2}{1 + c_0} - \frac{c_0 T^{1/2} \mu_0 + T^{1/2} y}{1 + c_0} \right) \right) \]

\[ \times \int \exp \left( -\frac{1}{2} \|T^{1/2} \mu - \frac{c_0 T^{1/2} \mu_0 + T^{1/2} y}{1 + c_0}\|_2^2 \right) d\mu \]

\[ \propto \det(T)^{-1/2} \exp \left( -\frac{1}{2} \frac{c_0}{1 + c_0} (y - \mu_0)^T T(y - \mu_0) \right) \]

Using this result, we obtain:

\[ p(T|y) \propto \det(T)^{(\delta_0 + 1/2) - \frac{n+1}{2}} \exp \left( -\frac{1}{2} \text{tr} \left( T \left\{ V_0^{-1} + \frac{c_0}{1 + c_0} (y - \mu_0)(y - \mu_0)^T \right\}^{-1} \right) \right) \]

As a result, the conditional density is recognized to be a Wishart distribution

\[ \mathcal{W}(T|\delta_0 + 1/2, \left\{ V_0^{-1} + \frac{c_0}{1 + c_0} (y - \mu_0)(y - \mu_0)^T \right\}^{-1}) \]

Thus, the parameter updates for the \( \gamma_i \)th class become:

\[ \delta_i^{(i)} = \delta_i^{(i-1)} + 1/2 \]

\[ V_i^{(i)} = \left( V_i^{(i-1)} \right)^{-1} + \frac{c_i^{(i-1)}}{1 + c_i^{(i-1)}} (y_i - \mu_i^{(i-1)})(y_i - \mu_i^{(i-1)})^T \]

\[ \left( 10 \right) \]

3.3 Calculation of \( L_{i,h}(y_i) \)

Now, let us return to the calculation of (7).

\[ L_{i,h}(y_i) = \int_{S_{+}^d} \int_{R^d} \mathcal{N}(y_i|\mu, T) \mathcal{N}(\mu|\mu_{h}^{(i-1)}, c_{h}^{(i-1)} T) \mathcal{W}(T|\delta_{h}^{(i-1)}, V_{h}^{(i-1)}) d\mu dT \]

\[ = \int_{S_{+}^d} \mathcal{W}(T|\delta_{h}^{(i-1)}, V_{h}^{(i-1)}) \left\{ \int_{R^d} \mathcal{N}(y_i|\mu, T) \mathcal{N}(\mu|\mu_{h}^{(i-1)}, c_{h}^{(i-1)} T) d\mu \right\} dT \]

\[ 8 \]
Evaluating the inner integral within the brackets:

\[
\int_{\mathbb{R}^d} \mathcal{N}(y_i|\mu, T)\mathcal{N}(\mu|\mu_h^{(i-1)}, c_h^{(i-1)}T) d\mu
\]

\[
\propto \det(T)^{1/2} \det((c_h^{(i-1)}T)^{1/2})
\]

\[
\times \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} \left[ c_h^{(i-1)}(\mu - \mu_h^{(i-1)})^T T(\mu - \mu_h^{(i-1)}) + (y_i - \mu)^T T(y_i - \mu) \right] \right) d\mu
\]

\[
= \det(T)^{1/2} \det((1 + c_h^{(i-1)}T)^{1/2}) \exp \left( -\frac{1}{2} \text{tr} \left( T \left\{ \frac{c_h^{(i-1)}}{1 + c_h^{(i-1)}} (y_i - \mu_h^{(i-1)})(y_i - \mu_h^{(i-1)})^T \right\} \right) \right)
\]

\[
= \left( \frac{c_h^{(i-1)}}{1 + c_h^{(i-1)}} \right)^{d/2} \det(T)^{1/2} \exp \left( -\frac{1}{2} \text{tr} \left( T \left\{ \frac{c_h^{(i-1)}}{1 + c_h^{(i-1)}} (y_i - \mu_h^{(i-1)})(y_i - \mu_h^{(i-1)})^T \right\} \right) \right)
\]
Using this closed-form expression for the inner integral, we further obtain:

\[
L_{i,h}(y_i) \propto \frac{c_h^{(i-1)}}{1 + c_h^{(i-1)}} \frac{d/2}{(1)} \int \frac{\text{det}(V_h^{(i-1)})^{-\delta_h^{(i-1)}}}{2^{d/2} \Gamma_d(\delta_h^{(i-1)})} \text{det}(T)^{\delta_h^{(i-1)} + 1/2 - \frac{d+1}{2}} \times \exp\left( -\frac{1}{2} \text{tr} \left( V_h^{(i-1)} \left\{ \frac{c_h^{(i-1)}}{1 + c_h^{(i-1)}} (y_i - \mu_h^{(i-1)})(y_i - \mu_h^{(i-1)})^T \right\} \right) \right) dT
\]

\[
\propto \frac{c_h^{(i-1)}}{1 + c_h^{(i-1)}} \frac{d/2}{(1)} \frac{\Gamma_d(\delta_h^{(i-1)} + \frac{1}{2})}{\Gamma_d(\delta_h^{(i-1)})} \times \frac{\text{det}(V_h^{(i-1)})^{-\delta_h^{(i-1)}}}{\text{det}\left(\left\{ (V_h^{(i-1)})^{-1} + \frac{c_h^{(i-1)}}{1 + c_h^{(i-1)}} (y_i - \mu_h^{(i-1)})(y_i - \mu_h^{(i-1)})^T \right\}^{-1}\right)^{-\delta_h^{(i-1)} + \frac{1}{2}}} \times \frac{\Gamma_d(\delta_h^{(i-1)} + \frac{1}{2})}{\Gamma_d(\delta_h^{(i-1)})} \times \frac{\text{det}(V_h^{(i-1)})^{1/2}}{\left(1 + \frac{c_h^{(i-1)}}{1 + c_h^{(i-1)}} (y_i - \mu_h^{(i-1)})^T V_h^{(i-1)}(y_i - \mu_h^{(i-1)})\right)^{\delta_h^{(i-1)} + \frac{1}{2}}}
\]

where we used the determinant identity \( \text{det}(I + ab^T) = 1 + b^T Ma \) in the last step.

### 4 Stability of Concentration Parameter

The update rule (5) transforms the density \( p_n(\alpha) \) into \( p_{n+1}(\alpha) \) based on whether a new class was formed or not. In this section, we derive a model for the posterior distribution \( p_n(\alpha) \) using large-sample approximations. This model allows us to find conditions under which the sequence of concentration parameters stays bounded on average, i.e., \( \mathbb{E}[\alpha_n] = O(1) \) as \( n \to \infty \).

The probability density of the \( \alpha \) parameter is updated at the \( j^{th} \) step in
the following fashion:

\[ p_{j+1}(\alpha) \propto p_j(\alpha) \cdot \begin{cases} \frac{\alpha}{\alpha+1} & \text{innovation class chosen} \\ \frac{1}{\alpha+1} & \text{otherwise} \end{cases}, \tag{13} \]

where only the \( \alpha \)-dependent factors in the update are shown. The \( \alpha \)-independent factors are absorbed by the normalization to a probability density. Choosing the innovation class pushes mass toward infinity while choosing any other class pushes mass toward zero. Thus there is a possibility that the innovation probability grows without bound in a undesired manner. One way to assess the innovation growth is to examine the fraction of innovations (choices of the innovation class) at step \( n \). Under reasonable assumptions, we will show that the mean of the innovation fraction approaches zero as the steps increase.

Assuming that the initial distribution of \( \alpha \) is \( p_0(\alpha) = \lambda e^{-\lambda \alpha} \), the distribution at step \( n \) is proportional to \( \frac{\alpha^r_n}{\alpha+1} \prod_{j=1}^n (1+\alpha/j)^{-1} e^{-\lambda \alpha} \), where \( r_n \) represents the number of choices of the innovation class. We have the following result:

**Theorem 1.**

\[
\lim_{N \to \infty} \frac{\sum_{k=1}^N \log \left(1 + \frac{\alpha}{k}\right)}{\alpha \log N} = 1. \tag{14}
\]

**Proof.** It is sufficient to establish the limit for \( \sum_{k=m}^N \log(1 + \alpha/k) \) for fixed \( m \). Choose \( m \) such that \( |\alpha| < m - 1 \) and use \( \log(1 - x) = \sum_{k=1}^\infty x^k/k \) for \( |x| < 1 \) to get

\[
\sum_{k=m}^N \log \left(1 + \frac{\alpha}{k}\right) = \sum_{k=m}^\infty \sum_{l=1}^N (-1)^{l+1} \frac{1}{l} \left(\frac{\alpha}{k}\right)^l = \sum_{l=1}^\infty (-1)^{l+1} \frac{\alpha^l}{l} \sum_{k=m}^N \frac{1}{k^l}. \tag{15}
\]

Separate Eq. 15 into two terms:

\[
\sum_{l=1}^\infty (-1)^{l+1} \frac{\alpha^l}{l} \sum_{k=m}^N \frac{1}{k} = \alpha \sum_{k=m}^N \frac{1}{k} + \sum_{l=2}^\infty (-1)^{l+1} \frac{\alpha^l}{l} \sum_{k=m}^N \frac{1}{k^l}. \tag{16}
\]

The first term is expressed in terms of the Euler-Mascheroni constant \( \gamma_e \) as

\[
\sum_{k=m}^N \frac{1}{k} = \log N - \gamma_e - \sum_{k=1}^{m-1} \frac{1}{k} + o(1). \tag{17}
\]

Thus, dividing by \( \log N \) and taking the limit \( N \to \infty \) we have a limiting value of unity. The second term of Eq. 16 is bounded. To see this, use, for \( l > 1 \),

\[
\sum_{k=m}^\infty \frac{1}{k^l} \leq \int_{m-1}^\infty \frac{dx}{x^l} = \frac{1}{(l-1)(m-1)^{(l-1)}}. \tag{18}
\]
Then the second term of Eq. 16 is bounded by
\[
\sum_{l=2}^{\infty} \sum_{k=m}^{\infty} \frac{1}{l(l-1)} (m-1)^{-(l-1)} = (m-1) \sum_{l=2}^{\infty} \frac{1}{l(l-1)} \left( \frac{\alpha}{m-1} \right)^l < \infty.
\]
(19)

The result follows since the second term, being bounded, has no effect on the limit.

Making use of Th.1, a model for \( p_n(\alpha) \) is \( \alpha r_n e^{-(\lambda + \log n)\alpha} \), suitably normalized, and denoted \( q_n(\alpha) \). The mean of \( q_n(\alpha) \) is expressed by \( \bar{\alpha}_n = (r_n + 1)/(\lambda + \log n) \). We use the mean in this form to choose class membership.

Abusing notation slightly, let \( L_k \equiv L_k(y_n) = L_{n,k}(y_n) \) denote the posterior probability of assigning \( y_n \) to class \( k \). More generally, the \( L_k \) can represent any inference probabilities based on the previous observations. Let \( L_0 \equiv L_0(y_n) = L_{n,k_{n-1}+1}(y_n) \) represent the prior on the observation \( y_n \) associated with the innovation class \( k_{n-1}+1 \). In effect, we have assigned a class 0 to innovations, which are also represented by the creation of a new class \( k_{n-1}+1 \).

To gain some insight into the frequency innovations occur, we consider several assumptions on the inference probabilities \( L_k \).

First, assume
\[
\gamma \overset{\text{def}}{=} \max_{\min_k \neq 0} \frac{L_0}{L_k} < \infty.
\]
(20)
The outer maximization occurs over all sample paths. We only need this for large \( n \) (i.e., \( n > N \)). This assumption holds for finite symbol alphabets when the assignment probabilities have nonzero floors at all symbols. Intuitively, as the number of classes increase, it becomes easier to fit the observations with the learned classes than with the prior used to form \( L_0 \).

Define, for nonzero \( k \),
\[
\begin{align*}
    s_k &\overset{\text{def}}{=} \frac{n_k}{n - 1 + \bar{\alpha}_{n-1}} L_k \\
    s_0 &\overset{\text{def}}{=} \frac{\bar{\alpha}_{n-1}}{n - 1 + \bar{\alpha}_{n-1}} L_0
\end{align*}
\]
(21)

We assume assignment is random based on the (scaled) probabilities \( s_k \), with the choice of the innovation class 0 leading to the creation of a new class \( k_{n-1}+1 \). The assignment probabilities are proportional to the \( s_k \). We have \( s_k > \gamma^{-1} L_0 n_k/(n-1+\bar{\alpha}_{n-1}) \) so that \( \sum_{k \neq 0} s_k > \gamma^{-1} L_0(n-1)/(n-1+\bar{\alpha}_{n-1}) \).

After normalization, we find that the probability of assignment to class 0 is at most \( \gamma \bar{\alpha}_{n-1}/(n-1+\gamma \bar{\alpha}_{n-1}) < \gamma \bar{\alpha}_{n-1}/(n-1) \). Let the innovation probability be denoted \( \tau_{n-1} \). We have \( \tau_{n-1} \leq \gamma \bar{\alpha}_{n-1}/(n-1) = \frac{\gamma (r_{n-1}+1)}{(n-1)(\lambda + \log n - 1)} \leq \sigma_{n-1} \), where \( \sigma_n \overset{\text{def}}{=} \min\left(\frac{\gamma (r_{n-1}+1)}{n(\lambda + \log n)}, 1\right) \).
Define, again abusing notation slightly, the mixture-model PDF

\[ \tilde{L}_0 \overset{\text{def}}{=} \sum_{k \neq 0} \frac{n_k}{n-1} L_k \]  

so that the probabilities of choosing an innovation or a previous class are proportional to \( \alpha \frac{n-1}{\tilde{L}_0} \) and \( \frac{(n-1)\tilde{L}_0}{n-1+\alpha} \), respectively. Since \( L_0 \) and \( \tilde{L}_0 \) are PDFs, we cannot have \( L_0 \leq \tilde{L}_0 \) unless \( L_0 = \tilde{L}_0 \). If, however, \( L_0 \leq \gamma \tilde{L}_0 \) for some \( \gamma \geq 1 \), then \( \sum_{k \neq 0} s_k > \gamma^{-1}L_0(n-1)/(n-1+\tilde{L}_0) \) so that \( \tau_{n-1} \leq \gamma \alpha_{n-1}/(n-1+\gamma \tilde{L}_0) < \gamma \alpha_{n-1}/(n-1) \leq \sigma_{n-1} \), as above.

If the update rule were to be modified after some iteration by replacing \( L_0 \) with \( \tilde{L}_0 \), then we could assume \( \gamma = 1 \).

The asymptotic behavior of the innovation random variable \( r_n \) is described by the random process associated with the probabilities of transition

\[
\text{prob}\{r_{n+1} = k|r_n\} = \begin{cases} 
\tau_n & k = r_n + 1 \\
1 - \tau_n & k = r_n \\
0 & k \neq r_n, r_n + 1
\end{cases}.
\]  

(23)

When, as above, the transition probability \( \tau_n \) satisfies the inequality \( \tau_n \leq \sigma_n \), the expectation of \( r_n \) is majorized, using Th.4, by the expectation of a similar random process based on the transition probability \( \sigma_n \) instead of \( \tau_n \). The latter can be described as a modification of a Polya urn process given by the following more general form which simplifies the notation a bit:

\[
\text{prob}\{r_{n+1} = r_n + 1|r_n\} = \frac{r_n + 1}{a_n}, \\
\text{prob}\{r_{n+1} = r_n|r_n\} = 1 - \frac{r_n + 1}{a_n}.
\]  

(24)

The \( a_n \) form a deterministic sequence which takes the form \( \gamma^{-1}n(\lambda + \log n) \) for our application. The asymptotic behavior of \( r_n \) and related variables is described in the following theorem.

**Theorem 2.** Assume \( r_n \) evolves as in Eq. 24 with \( a_n = (\lambda + \log n)/\gamma \), and \( \gamma \geq 1 \) for all \( n > N \) Then

\[
\frac{m_1(n)}{n} = \frac{\mathbb{E}[r_n]}{n} = O\left(\frac{\log^\gamma n}{n}\right), \\
\mathbb{E}[r_n] = O(\log^\gamma n), \\
\mathbb{E}[\alpha_n] = O(\log^{\gamma-1} n)
\]  

(25)

**Proof.** We present two proofs. The first uses a recursion relation for the characteristic function of \( r_n \).
**First Proof** We start with an initial distribution on \( r_1 \) that can take on real, nonnegative values. The values of \( a_k \) are free, but \( a_k \geq k \) seems to be required for the process to make sense. We wish to track the distribution as \( n \) increases. First, consider the conditional moment-generating function at time \( n \):

\[
E[e^{\lambda r_{n+1}}|r_n] = \frac{r_n + 1}{a_n}e^{\lambda(r_n+1)} + \left[1 - \frac{r_n + 1}{a_n}\right]e^{\lambda r_n}.
\]  

(26)

Let \( c_n(\lambda) \) denote the moment generating function of \( r_n \), i.e., \( c_n(\lambda) = E[e^{\lambda r_n}] \). Take the expectation of Eq. 26 over \( r_n \) and make use of

\[
E[r_ne^{\lambda r_n}] = \frac{\partial}{\partial \lambda} c_n(\lambda)
\]

to get

\[
c_{n+1}(\lambda) = \left(1 + \frac{e^\lambda - 1}{a_n} (1 + D)\right) c_n(\lambda),
\]  

(27)

where \( D \) denotes differentiation by \( \lambda \). This equation provides a recurrence relation using the non-commuting operators of differentiation \( D \) and function multiplication \( e^\lambda - 1 \) and is thus difficult to use to find asymptotic distributions. However, Eq. 27 is useful to evaluate recurrence relations for the moments of \( r_n \). These relations can be used to address asymptotic behavior for different weightings \( a_n \).

Using Eq. 27, the first moment of \( r_n \) becomes

\[
m_1(n+1) = \frac{1}{a_n} + \left(1 + \frac{1}{a_n}\right) m_1(n)
\]  

(28)

We use this expression to find the behavior of the fraction \( m_1(n)/n \). First,

\[
\frac{m_1(n+1)}{n+1} = \frac{1}{(n+1)a_n} + \left(1 + \frac{1}{a_n}\right) \left(1 + \frac{1}{n}\right)^{-1} \frac{m_1(n)}{n}.
\]  

(29)

We can expand this relation in the form

\[
\frac{m_1(n)}{n} = \frac{1}{na_n} + \sum_{k=1}^{n-1} \frac{1}{(k+1)a_k} \prod_{j=k+1}^{n-1} \left(1 + \frac{1}{a_j}\right) \left(1 + \frac{1}{j}\right)^{-1} + m_1(1) \prod_{j=1}^{n-1} \left(1 + \frac{1}{a_j}\right) \left(1 + \frac{1}{j}\right)^{-1}
\]  

(30)

Note that

\[
\prod_{j=k}^{n} \left(1 + \frac{1}{a_j}\right) \left(1 + \frac{1}{j}\right)^{-1} \leq \frac{4}{3} \prod_{j=k}^{n} \left(1 + \frac{1}{a_j}\right) \left(1 - \frac{1}{j}\right),
\]  

(31)

provided \( k \geq 2 \). Furthermore, with \( a_n = \gamma^{-1} n(\lambda + \log n) \),

\[
\log \left(\prod_{j=k}^{n} \left(1 + \frac{1}{a_j}\right) \left(1 - \frac{1}{j}\right)\right) \leq \sum_{j=k}^{n} \frac{\gamma}{j(\lambda + \log j)} - \frac{1}{j},
\]

(32)
using $\log(1 + x) \leq x$ when $x > -1$. Since $\sum_{n}^{\infty} 1/k \log k = O(\log \log n)$ and $\sum_{n}^{\infty} 1/k = O(\log n)$, we have $m_1(n)/n$ goes to zero as $n$ increases. The same result would not hold if $a_k = k$ since $\prod (1 - 1/k^2)$ is nonzero.

It is sufficient to consider $a_n$ of the form $a_n = \frac{\gamma}{\log n}$ since $\lambda > 0$ and larger $a_n$ have smaller asymptotics, as will be apparent from the argument below. From Eq. 30, Eq. 31 and Eq. 32, we can address the convergence rate of $m_1(n)/n$ if we investigate the behavior of

$$\beta_{kn} \overset{\text{def}}{=} \exp \left( \sum_{j=k}^{n} \frac{\gamma}{j \log j} - \frac{1}{j} \right).$$

(33)

First, note that

$$\sum_{k=1}^{n} \frac{1}{k \log k} \leq \int_{1-1}^{n} \frac{dx}{x \log x} = \int_{\log(1-1)}^{\log n} \frac{dt}{t} = \log \log n - \log \log(l - 1),$$

(34)

when $l > 2$. Next, note that

$$\int_{l}^{n+1} \frac{dx}{x} = \log(n+1) - \log l = \log n + \log(1 + 1/n) - \log l \geq \log n + \frac{1}{n+1} - \log l$$

(35)

using $x/(1 + x) \leq \log(1 + x) \leq x$ for $-1 < x < \infty$. Put Eq. 34 and Eq. 35 together to get

$$\sum_{j=k}^{n} \frac{\gamma}{j \log j} - \frac{1}{j} \leq \gamma \log \log n - \gamma \log(k - 1) + \log k$$

(36)

and hence

$$\beta_{kn} \leq \frac{k \log^2 n}{n \log^2 (k - 1)}.$$  

(37)

For fixed $k$, this means $\beta_{kn} = O(\log^2 n)$. Assuming $a_k = k \log k / \gamma$, from Eq. 30 we have

$$\frac{m_1(n)}{n} \leq \gamma \sum_{k=1}^{\log n} \frac{\beta_{k+1/n-1}}{k^2 \log k} + m_1(1)\beta_{1/n-1} + O\left(\frac{1}{n^2 \log n}\right).$$

(38)

The second term is $O(\log^2 n)$. We examine the first:

$$\sum_{k=1}^{n} \frac{\beta_{k+1/n-1}}{k^2 \log k} = O\left(\frac{\log^2 n}{n}\right),$$

(39)

since

$$\sum_{k=1}^{n} \frac{1}{k \log^2 k} \leq \int_{1-1}^{n} \frac{dx}{x \log^2 x} = \int_{\log(1-1)}^{\log n} \frac{dt}{t^2} = \frac{1}{\log(l - 1)} - \frac{1}{\log n} \leq \frac{1}{\log(l - 1)},$$

(40)
when \( l > 2 \) and since \( \log^\gamma k \log k > \log^2 k \) for sufficiently large \( k \). Thus \( \frac{m_1(n)}{n} = O\left(\frac{\log^\gamma n}{n}\right) \). It follows that \( \mathbb{E}[r_n] = O\left(\log^\gamma n\right) \) and that \( \mathbb{E}[\bar{a}_n] = O\left(\log^\gamma n\right) \).

The same conclusion applies if the recursion Eq. 28 is started at \( m_1(l) \) instead of \( m_1(1) \) by a trivial modification of the above argument.

**Second Proof** We can study the generalized Polya urn model in the slightly modified form:

\[
P(r_{n+1} = k | r_n) = \begin{cases} 
\frac{r_n}{a_n}, & \text{if } k = r_n + 1 \\
1 - \frac{r_n}{a_n}, & \text{if } k = r_n 
\end{cases}
\]

(41)

Since \( a_n \to \infty \), the asymptotic means of \( r_n \) are the same for this process as they are for Eq. 24. We require \( a_k \geq k \) for this to make sense. As before, modifications to a finite number of \( a_k \) only change constant factors in the asymptotics and do not effect the order.

Taking the conditional expectation of \( r_{n+1} \) with respect to the filtration \( \mathcal{F}_n = \sigma(r_1, \ldots, r_n) \), we obtain:

\[
\mathbb{E}[r_{n+1} | \mathcal{F}_n] = \mathbb{E}[r_{n+1} | r_n] \\
= \frac{r_n}{a_n} (r_n + 1) + (1 - \frac{r_n}{a_n})r_n \\
= r_n \left(1 - \frac{1}{a_n}\right)
\]

(42)

Next, define the adapted process \( M_n = \frac{r_n}{\beta_n} \) for a positive deterministic sequence \( \{\beta_n\} \). Dividing both sides of (42) by \( \beta_{n+1} \), we obtain:

\[
\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n \frac{\beta_n}{\beta_{n+1}} \left(1 + \frac{1}{a_n}\right)
\]

(43)

where we also define the sequence \( \{\beta_n\} \) recursively as:

\[
\beta_{n+1} = \beta_n \frac{1 + a_n^{-1}}{1 + c_n}
\]

(44)

where \( c_n \) is a sequence satisfying \( \sum_n c_n < \infty \). Let us set \( c_k = \frac{1}{k^\gamma} \). Unrolling the recursion (44), we obtain:

\[
\beta_{n+1} = \beta_1 \prod_{k=2}^n \frac{1 + a_k^{-1}}{1 + c_k}
\]

(45)

Using the large-sample asymptotics given by:

\[
\frac{\prod_{k=2}^n \left(1 + \frac{\gamma}{k \log k}\right)}{(\log n)^\gamma} \to C_1 \approx 2.1614
\]

\[
\prod_{k=2}^n \left(\frac{1}{1 + \frac{1}{k^\gamma}}\right) \to C_2 \approx 1.837
\]
we obtain
\[ \beta_n \sim C_1 C_2 (\log n)^\gamma, \] (46)
where \( C \) is a constant.

Using (44) into (43), we obtain:
\[ \mathbb{E}[M_{n+1}|\mathcal{F}_n] = (1 + c_n)M_n \]
Taking the expectation of both sides, we obtain a deterministic recursion:
\[ \mathbb{E}[M_{n+1}] = (1 + c_n)\mathbb{E}[M_n] \] (47)
Recall Lemma 2 in Sec. 2.2 from [13] restated as Lemma 1 in the appendices. Applying Lemma 1 to the sequence \( \{\mathbb{E}[M_n]\} \), we conclude from (47) that \( \mathbb{E}[M_n] \to M_\infty \) as \( n \to \infty \). Thus, we have:
\[ \mathbb{E}[r_n] = O(\beta_n) \]
Using the asymptotic approximation of \( \beta_n \) derived in (46), we obtain \( \mathbb{E}[r_n] = O((\log n)^\gamma) \), implying that the average number of classes grows slowly as a function of \( n \). Since \( \alpha_n \sim \gamma \frac{r_n}{\log n} \), we obtain:
\[ \mathbb{E}[\alpha_n] = O((\log n)^{\gamma-1}) \] (48)
Thus, the sequence of concentration parameters stays bounded on average iff \( \gamma = 1 \).

From Th.2 and Th.4 in App. B, we have the result:

**Theorem 3.** Let \( \tau_n \) be a sequence of real-valued random variables \( 0 \leq \tau_n \leq 1 \) satisfying \( \tau_n \leq \frac{a_{n+1}}{a_n} \) for \( n \geq N \), where \( a_n = \gamma^{-1}n(\lambda + \log n) \), and where the nonnegative, integer-valued random variables \( r_n \) evolve according to Eq. 23. Then
\[ \mathbb{E}[r_n] = O((\log n)^\gamma) \]
\[ \mathbb{E}[\alpha_n] = O((\log n)^{\gamma-1}) \] (49)

**Proof.** From Th.4 and Eq. 23, it is sufficient to look at the asymptotics of Eq. 24, which are evaluated in Th.2. \( \square \)

**Corollary 1.** If \( L_0 \leq \gamma \bar{L}_0 \), beginning at some iteration, then Eq. 49 holds.

**Corollary 2.** If \( \gamma \geq 1 \), then \( \alpha_n \) is bounded.
5 Steady-State Selection Rule

In this section, we derive an asymptotic expression for the selection rule (1) in order to gain insight into the steady-state performance of the algorithm.

For simplicity, we assume that on average (and in the long term) the number of data points in each class is the same. The ratio of multivariate Gamma functions in (12) can be simplified to a ratio of standard Gamma functions:

\[
\rho_d(a) := \frac{\Gamma_d(a + \frac{1}{2})}{\Gamma_d(a)} = \frac{\pi^{d/2} \prod_{j=1}^d \Gamma(a + \frac{1-j/2}{2})}{\pi^{d/2} \prod_{j=1}^d \Gamma(a + \frac{1-j/2}{2})} = \frac{\Gamma(a + \frac{1}{2})}{\Gamma(a + \frac{1-2d}{2})}
\]

Theorem 1.6 in [2] provides tight bounds on the Gamma function, i.e., for all real \(x \geq 1\), we have:

\[
x^x e^{-x} \sqrt{2\pi(x + c_0)} \leq \Gamma(x + 1) \leq x^x e^{-x} \sqrt{2\pi(x + c_1)} \quad (50)
\]

where the constants are \(c_0 \approx 0.166\) and \(c_1 \approx 0.176\). Using the bounds (50), it can be shown:

\[
c_d(a - \frac{1}{2})^{d/2} \rho_d(a) \leq \rho_d(a) \leq c_d(a - \frac{1}{2})^{d/2} \rho_d(a - \frac{1}{2}; c_0, c_1) \quad (51)
\]

where \(c_d = e^{-d/2}\) and

\[
f_d(x; a, b) := \left[ \frac{1 + \frac{b}{x}}{1 + \frac{a-d/2}{x}} \right] \left( \frac{1}{1 - \frac{d/2}{x}} \right)^{x-d/2}
\]

Since \(\lim_{x \to \infty} f_d(x; a, b) = 1\) for any constants \(a, b\), we have from (51)

\[
\lim_{a \to \infty} \frac{\rho_d(a)}{c_d(a - \frac{1}{2})^{d/2}} = 1.
\]

Under normal convergence conditions of the algorithm, all classes \(h = 1, \ldots, K\) will be correctly identified and populated with an almost-equal number of observations since all observations from each are equally likely according to a uniform prior in the generative model. Thus, the conditional class prior for each class \(h\) vanishes asymptotically assuming bounded \(\alpha\) (see Section 4), i.e.,

\[
\pi_{i,h} = \frac{n_i-1(h)}{i-1+\alpha} \sim \frac{(i-1)/K}{i-1+\alpha} \xrightarrow{i \to \infty} \frac{1}{K}.
\]

Thus, the sequential MAP criterion becomes equivalent to sequential maximum-likelihood (ML) governed by the conditional likelihood only.

The conditional likelihood becomes:

\[
L_{i,h}(y_i) \propto \frac{r_h^{(i-1)} \rho_d(\delta_h^{(i-1)}) \det(V_h^{(i-1)})^{1/2}}{(1 + r_h^{(i-1)}(y_i - \mu_h^{(i-1)})^T V_h^{(i-1)}(y_i - \mu_h^{(i-1)}))^{(i-1) + 1/2}}
\]
where $r_h^{(i-1)} = c_h^{(i-1)}/h^{(i-1)} \in [0, 1)$. According to (9), we expect $r_h^{(i-1)} \to 1$ as $i \to \infty$ since $c_h^{(i-1)} \sim \frac{i-1}{K}$. Also, we expect $c_h^{(i-1)} \sim \frac{i-1}{K}$ as $i \to \infty$ according to (10). Thus, the large-sample approximation to the conditional likelihood becomes:

$$L_i,h(y_i) \overset{i \to \infty}{\propto} \frac{(C(i-1) - \frac{1}{2})^{d/2} \det(V_h^{(i-1)})^{1/2}}{\left(1 + (y_i - \mu_h^{(i-1)})^T V_h^{(i-1)} (y_i - \mu_h^{(i-1)}) \right)^{(C(i-1) - \frac{1}{2})^{d/2} \det(V_h^{(i-1)})^{1/2}}}, \tag{52}$$

where $C$ is independent of $h$. The parameter updates (9) and (10) imply $V_h^{(i)} \to V_h$ and $\mu_h^{(i)} \to \mu_h$ as $i \to \infty$. Note that maximizing $L_i,h(y_i)$ is equivalent to minimizing the inverse of the ratio in (52). Raising this inverse ratio to the power $(C(i-1) - \frac{1}{2})^{-1}$ and taking the limit as $i \to \infty$, its denominator becomes asymptotically negligible:

$$\frac{1 + (y_i - \mu_h^{(i-1)})^T V_h^{(i-1)} (y_i - \mu_h^{(i-1)})}{(C(i-1) - \frac{1}{2})^{d(C(i-1)/K - 1)} \det(V_h)^{(1+C(i-1)/K - 1)}} \to 1 + (y_i - \mu_h)^T V_h (y_i - \mu_h)$$

where we used the fact that

$$\lim_{i \to \infty} (C(i-1) - \frac{1}{2})^{d(C(i-1)/K + 1) - 1} = \exp \left( d \lim_{i \to \infty} \frac{\log e \left( C(i-1) - \frac{1}{2} \right) }{C(i-1)/K + \frac{1}{2} } \right) = \exp \left( d \lim_{i \to \infty} \frac{1}{C(i-1)/K - 1} \right) = 1,$$

and

$$\lim_{i \to \infty} \det(V_h^{(i-1)})^{(1+C(i-1)/K - 1)} = \det(V_h)^{\lim_{i \to \infty} \left(1+C(i-1)/K - 1\right)} = 1.$$

Thus, as the number of samples grow, the decision rule (52) becomes equivalent to the weighted minimum distance problem:

$$\min_h (y_i - \mu_h)^T V_h (y_i - \mu_h)$$

This decision rule is optimal if the observations from each class $h$ are obtained from a multivariate Gaussian distribution with mean $\mu_h$ and covariance $V_h^{-1}$.

6 Improved Estimation through Averaging and Forgetting

The first few samples of the ASUGS training algorithm often cause a bias in the centroid mean and covariance for a few clusters. Once all the correct
clusters are identified and their associated parameters are approximately obtained, further improvements in the estimation of the cluster parameters can be obtained by using the next few training data to refine the ASUGS class parameter estimates using the iterative scheme:

\[ \gamma_i \in \arg \min_h (y_i - \mu_h^{(i-1)})^T V_h^{(i-1)} (y_i - \mu_h^{(i-1)}) \]  
(53)

\[ \mu_{h_i}^{(i)} = \lambda_i \mu_{h_i}^{(i-1)} + (1 - \lambda_i) y_i \]  
(54)

\[ V_{h_i}^{(i)} = (\lambda_i V_{h_i}^{(i-1)})^{-1} + (1 - \lambda_i) (y_i - \mu_{h_i}^{(i-1)})(y_i - \mu_{h_i}^{(i-1)})^T \]  
(55)

where \( \{\lambda_i\} \) is a sequence of forgetting factors. The larger \( \lambda_i \) is, the less weight is placed on the new observation \( y_i \) per (54) and (55). Thus, \( \lambda_i \) control the amount of averaging when incorporating new observations. Assuming the classification step (53) leads to the correct class, the updates (54) and (55) implement an averaged version of the sample mean and sample covariance matrix for each class by taking a convex combination of the parameter estimate from the last epoch and the contribution from the new observation \( y_i \).

The recursions (54) and (55) implements a time-varying first-order filter. We remark that the precision matrix update (55) ensures that the precision matrix is always positive definite as long as the update is initialized with a positive definite matrix. This sequential estimation algorithm is reminiscent of the k-means algorithm for unsupervised clustering [3] or Lloyd algorithm in source coding and compression used to identify optimal partitions for quantization [8].

Assuming that the assignment of observations based on (53) is correct, we focus on one class. Consider the set of time instants \( T_c = \{ t \in \mathbb{N} : \gamma_t = c \} \). Then, for all \( t \in T_c \), consider only the mean update (54) for simplicity, where \( y_i \) are random vectors in \( \mathbb{R}^d \) with mean \( \mu_c \) and covariance \( K_c \).

For the choice of constant forgetting factors \( \lambda_i = \lambda \) for all \( i \), it follows that:

\[ \mu_c^{(t)} = \lambda^t \mu_c^{(0)} + (1 - \lambda) \sum_{i=1}^t \lambda^{t-i} y_i \]

Taking the mean, it follows that \( \mathbb{E}[\mu_c^{(t)}] \rightarrow \mu_c \) as \( t \rightarrow \infty \). However, this estimator does not converge in the mean-square sense because:

\[ \mathbb{E}\|\mu_c^{(t)} - \mu_c\|_2^2 \rightarrow \text{tr}(K_c)(1 - \lambda)^2 \]  
(1 - \lambda)^2 \rightarrow \|\mu_c\|_2^2 

For adaptive forgetting factors \( \lambda_i = 1/i \), this implements the sample-mean estimator. It is well known that the sample mean estimator \( \hat{\mu}_c^{(t)} \) satisfies:

\[ \mathbb{E}\|\hat{\mu}_c^{(t)} - \mu_c\|_2^2 = \frac{\text{tr}(K_c)}{t} \rightarrow 0 \]

20
Thus, assuming the assignment of observations is always perfect, we obtain asymptotically consistent estimates of the mean and covariance of each class $c$ as the number of training samples grow to infinity, i.e., $\hat{\mu}_c^{(t)} \to \mu_c$ and $\hat{K}_c^{(t)} = (\hat{V}_c^{(t)})^{-1} \to K_c$ almost surely, by the strong law of large numbers.

An example simulation where ASUGS is followed by the refinement procedure (53)-(55) in Fig. 1 for a fixed forgetting factor $\lambda_i = \lambda = 0.95$ for all $i \geq 1$ and for adaptive $\lambda_i = 1/i$. It is evident that the refinement stage outperforms the ASUGS estimates.

![Figure 1: ASUGS training with refinement for SNR of 20dB. The estimated means of each class is shown over a set of 100 trials. The refinement improves estimation performance for the class parameters. A set of 900 training data point were used for DPMM-based clustering and training using the ASUGS algorithm and a set of 45,000 training data points were used for the refinement step.](image)

7 Simulations

In this section, we have data $y_i$ randomly (uniformly) chosen from a constellation, and we fit an infinite GMM model to this data using the algorithm derived above.

7.1 Learning 16-QAM modulation

The data is included in Figure 2. The correct number of classes for this data set is 16.

The progress and clustering performance are in Figs. 3 and 4 for a fixed value of $\alpha = 20$. This value of $\alpha$ tends to overestimate the number of classes needed to describe the data. Thus, one expects an adaptive algorithm to outperform the fixed-$\alpha$ counterpart. As a result, we consider a prior distribution discretized to the shape of a Gamma(1.2,0.5) distribution ranging
from 0 to 30. Fig. 7 shows that the posterior distribution of $\alpha$ concentrates around 2.73, which is more appropriate for this data set than 20. The number of classes is considerably less than the ones corresponding to the fixed $\alpha = 20$. In addition, Fig. 6 shows that the clusters are much more concentrated.

Figure 2: Original noisy 16-QAM constellation.

Figure 3: Progress of algorithm for $\alpha = 20$ at 20 dB SNR.
7.2 Learning 8-PSK Constellation

Here, the correct number of classes is 8. The same conclusions can be drawn for this simulation as in the 16-QAM constellation.

The progress and clustering performance are in Figs. 9 and 10 for a fixed value of $\alpha = 20$. This value of $\alpha$ tends to overestimate the number of classes needed to describe the data. Thus, one expects an adaptive algorithm to outperform the fixed-$\alpha$ counterpart. As a result, we consider a prior distribution discretized to the shape of a Gamma(1, 2, 0.5) distribution.
Figure 6: Clustering performance of algorithm for adaptive $\alpha \in (0, 30]$ at 20 dB SNR.

Figure 7: Prior and posterior distribution of concentration parameter $\alpha$ for 16-QAM constellation at 20 dB SNR.

ranging from 0 to 30. Fig. 13 shows that the posterior distribution of $\alpha$ concentrates around 1.92, which is more appropriate for this data set than 20. The number of classes formed is less than the ones corresponding to the fixed $\alpha = 20$. In addition, Fig. 12 shows that the clusters are much more concentrated.
Figure 8: Original noisy 8-PSK constellation.

Figure 9: Progress of algorithm for $\alpha = 20$ at 20 dB SNR.

7.3 Bit Error Rate Performance

We characterize the bit error rate (BER) performance of the system when the DPMM sequential learning algorithm ASUGS is used to identify and estimate the constellation parameters over a range of different training SNR’s, and is tested for a fixed test SNR using the standard minimum distance demodulator. For simplicity, we focus on the QPSK and 16-QAM modulations in Figs. 14 and 15 respectively.

There is a critical SNR $\rho$ above which the BER is reasonably small and
Figure 10: Clustering performance of algorithm for $\alpha = 20$ at 20 dB SNR.

Figure 11: Progress of algorithm for adaptive $\alpha \in (0, 30]$ at 20 dB SNR.

close to the optimal decoder BER for a large enough number of ASUGS iterations, while for any training SNR below $\rho$, no matter how many iterations of ASUGS, the BER does not improve. This establishes that this approach based on ASUGS is a high SNR technique, but works for reasonably small SNR.
Figure 12: Clustering performance of algorithm for adaptive $\alpha \in (0, 30]$ at 20 dB SNR.

Figure 13: Prior and posterior distribution of concentration parameter $\alpha$ for 8-PSK constellation at 20 dB SNR.

8 Conclusion

We have developed a fast algorithm for performing online clustering on data sets lying in arbitrarily high dimensional Euclidean spaces using Dirichlet process mixtures of Gaussians. Our adaptive method consists of a greedy selection step and on updating the parameters associated with the sufficient statistics of the posterior distribution of the class parameters using
Figure 14: Bit Error Rate (BER) performance for QPSK modulation. There is a critical SNR $\rho \approx 8$ dB above which successful demodulation is possible and below which demodulation fails.

closed-form expressions. We propose an adaptive design for the concentration parameter and prove that the sequence of concentration parameters is asymptotically well-behaved under a simplified model. When applying this methodology to modulation recognition, arbitrary constellations can be automatically recognized without any special parameter tuning.

A A Majorization Result

**Theorem 4.** Let $r_n$ and $w_n$ be bounded (at each $n$) random time series with the update laws

$$prob\{r_{n+1} = r_n + 1\} = \tau_n$$
$$prob\{r_{n+1} = r_n\} = 1 - \tau_n$$

and

$$prob\{w_{n+1} = w_n + 1\} = \sigma_n$$
$$prob\{w_{n+1} = w_n\} = 1 - \sigma_n,$$

and assume $\sigma_n \geq \tau_n$ for all $n \geq 1$ and that $w_0 = r_0$. Then $\mathbb{E}[w_n] \geq \mathbb{E}[r_n]$. 

28
Figure 15: Bit Error Rate (BER) performance for 16-QAM modulation. There is a critical SNR $\rho \approx 11$ dB above which successful demodulation is possible and below which demodulation fails.

**Proof.** We show that given

$$\text{prob}\{r_n > t\} \leq \text{prob}\{w_n > t\}$$

(58)

for a particular $n$ and all $t$, the same inequality holds for $n + 1$. We have

$$\text{prob}\{r_{n+1} > t\} = (1 - \tau_n)\text{prob}\{r_n > t\} + \tau_n\text{prob}\{r_n > t - 1\}$$

$$\leq (1 - \tau_n)\text{prob}\{w_n > t\} + \tau_n\text{prob}\{w_n > t - 1\}$$

$$\leq (1 - \sigma_n)\text{prob}\{w_n > t\} + \sigma_n\text{prob}\{w_n > t - 1\}$$

$$= \text{prob}\{w_{n+1} > t\},$$

(59)

as required. The last inequality follows since $\text{prob}\{w_n > t - 1\} \geq \text{prob}\{w_n > t\}$ and $\sigma_n \geq \tau_n$.

Assume the nonnegative random variables $r$ and $w$ satisfy the tail inequalities

$$1 - Q(t) \equiv \text{prob}\{w > t\} \geq \text{prob}\{r > t\} \equiv 1 - P(t),$$

(60)

where $Q(t)$ and $P(t)$ are the cumulative distribution functions of $w$ and $r,$
respectively. Integrating by parts,
\[
\int_0^a tdP(t) = aP(a) - \int_0^a P(t) dt \leq aP(a) - \int_0^a Q(t) dt
\]
\[
= a(P(a) - Q(a)) + \int_0^a t dQ(t). \tag{61}
\]
When \( w \) and \( r \) are bounded, we have, as a consequence of Eq. 61, \( \mathbb{E}[w] \geq \mathbb{E}[r] \).

\[ \square \]

B A Lemma

Lemma 1. [13] Let \( \{u_k\} \) be a non-negative sequence of reals and assume:
\[
u_{k+1} \leq (1 + c_k)u_k + z_k
\]
\[
c_k \geq 0, \sum_k c_k < \infty
\]
\[
z_k \geq 0, \sum_k z_k < \infty
\]
Then, \( u_k \to u_\infty \geq 0 \).

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