Feynman rules for Gauss’s law

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Abstract

I work on a set of Feynman rules that were derived in order to incorporate the constraint of Gauss’s law in the perturbation expansion of gauge field theories and calculate the interaction energy of two static sources. The constraint is implemented via a Lagrange multiplier field, $\lambda$, which, in the case of the non-Abelian theory, develops a radiatively generated effective potential term. After analysing the contributions of various solutions for $\lambda$, the confining properties and the various phases of the theory are discussed.
1 Introduction

In a previous work I investigated the possibility of expressing the constraint of Gauss’s law in the perturbative expansion of gauge field theories \( \Pi \) via a Lagrange multiplier field, \( \lambda \), in order to calculate the interaction energy for static sources, and argued for the generation of an effective potential term, of the Coleman-Weinberg type, for \( \lambda \), and its relation to questions of confinement in the non-Abelian case.

Here, I continue with the calculation of the interaction energy of two static fermionic sources due to the constant and multi-bubble solutions for \( \lambda \) from the effective action derived. I find a term that is linearly rising with respect to the separation distance in specific scales of the theory that are described and analysed in the text. The result holds in the perturbative regime, together with the usual Coulomb interaction, the only non-perturbative input being the bubble solutions. The fact that the fermionic static charges in the non-Abelian theory are not part of a gauge and Lorentz covariant current creates additional terms, whose contributions are estimated, and the limits where the various approximations hold are analysed.

In Sec. 2, I start with a description of the combinatorics for the Abelian case which do not change the theory but amount to a reshuffling of the propagators, and in Sec. 3, I treat the non-Abelian, self-interacting case. I derive an effective potential term for \( \lambda \), and I show that, in general, a non-zero value for \( \lambda \) signals confinement, via a linearly rising interaction energy. Here, I also discuss the various \( \lambda \)-configurations and their contributions. In Sec. 4, I present some numerical results and other comments that clarify the arguments of Sec. 3. In Sec. 5, I conclude with an overview, a discussion of the limitations, additional corrections and possible extensions.

2 The Abelian theory

In order to investigate the consequences of the constraint of Gauss’s law in the perturbation expansion of gauge field theories I will start with the Abelian case, including a massive fermion, with Lagrangian \( \mathcal{L} \) and action

\[
S = \int_x \mathcal{L} = \int_x -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi,
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( D_\mu = \partial_\mu + ieA_\mu \). Integrations are over \( d^4x \) and the metric conventions are \( g_{\mu\nu} = (+ - - -) \), \( \partial_\mu = (\partial_0, \partial_i) \), \( \partial^\mu = (\partial_0, -\partial_i) \).
Because the Lagrangian is independent of $\dot{A}_0 = \partial_0 A_0$, the respective equation of motion for that field, namely

$$\frac{\delta S}{\delta A_0} = 0,$$  \hspace{1cm} (2)

is not a dynamical equation, but, rather, a constraint corresponding to Gauss’s law, which will be incorporated in the perturbative expansion via a Lagrange multiplier field, $\lambda$, in the path integral

$$Z(J_\mu, \Lambda) = \int [dA_\mu][d\psi][d\bar{\psi}][d\lambda] e^{i\int \tilde{\mathcal{L}}},$$ \hspace{1cm} (3)

where

$$\tilde{\mathcal{L}} = \frac{1}{2} (\partial_0 A_i + \partial_i A_0)^2 - \frac{1}{4} F_{ij}^2$$

$$+ \bar{\psi} (i\gamma^\mu \partial_\mu - e\gamma^0 A_0 + e\gamma^i A_i - m) \psi$$

$$- \lambda (\nabla^2 A_0 + \partial_0 \partial_i A_i + e\bar{\psi} \gamma^0 \psi)$$

$$- \frac{1}{2} (\partial_0 A_0 + \partial_i A_i + \partial_0 \lambda)^2 + A_0 J_0 - A_i J_i + \lambda \Lambda.$$ \hspace{1cm} (4)

In the above equation the first and the second lines contain the original gauge and fermion terms, the third line is the constraint $\lambda \frac{\delta S}{\delta A_0}$, implemented with a gauge-invariant $\lambda$, and the last line contains the gauge-fixing term and the sources $J_\mu, \Lambda$.

I have used a special gauge-fixing condition since the associated term, which can be derived by the usual Faddeev-Popov procedure, gives the simplest set of Feynman rules. The problems of gauge independence and gauge invariance are important and will be discussed later in relation to the non-Abelian theory.

After the usual inversion procedures one obtains the following propagators:

$$G_{00} = -\frac{1}{k^2} - \frac{1}{\tilde{k}^2}.$$ \hspace{1cm} (5)

$$G_{\lambda\lambda} = -\frac{1}{\tilde{k}^2}.$$ \hspace{1cm} (6)

$$G_{0\lambda} = \frac{1}{k^2} = G_{\lambda 0}.$$ \hspace{1cm} (7)

$$G_{ii} = \frac{1}{k^2}.$$ \hspace{1cm} (8)
One can easily deduce the vertices from (4) as well as the fact that the propagators are combined in all interactions so as to reproduce all the usual QED diagrams. $G_{00}$, $G_{\lambda\lambda}$ and $G_{0\lambda}$ appear together and their sum gives the ordinary 0 − 0 propagator in Feynman gauge. For example, for two static current sources separated by a spatial distance, $r$, one obtains the usual Coulomb interaction energy from the sum of the diagrams in Fig. 1 in the static limit of $k_0 = 0$.

$$V_{\text{Coul}}(r) = 4\pi\alpha_e \int \frac{d^3k}{(2\pi)^3} \frac{e^{ikr}}{k^2} = \frac{\alpha_e}{r},$$

with $\alpha_e = e^2/4\pi$.

### 3 The non-Abelian theory

I now consider the case of a non-Abelian gauge theory with gauge group $G$ with structure constants $f_{abc}$, a massive fermion in the representation $R$ with generators $T^a$, and initial action

$$S_0 = \int \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi,$$

with $F_{\mu\nu}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf_{abc} A^b_\mu A^c_\nu$ and $D_\mu = \partial_\mu + ig A^a_\mu T^a$.

The theory is gauge invariant, with $\psi \rightarrow \omega(\theta)\psi$, $A_\mu \rightarrow \omega A_\mu \omega^{-1} + ig^{-1}(\partial_\mu \omega)\omega^{-1}$ under the local gauge transformation $\omega(\theta) = e^{-i\lambda^a(x)T^a} \in G$ (with the usual notation $A_\mu = T^a A^a_\mu$).

After imposing the constraint in

$$\tilde{S} = S_0 + \int \lambda^a \frac{\delta S_0}{\delta A_0},$$

the theory is still gauge-invariant with $\lambda \rightarrow \omega \lambda \omega^{-1}$, and can be gauge-fixed as

$$\tilde{S}_{\text{gf}} = \tilde{S} + \int x B^a G^a + \frac{1}{2} B^a B^a + \bar{c}^a \frac{\delta G^a}{\delta \theta^b} c^b,$$

with an auxiliary field, $B^a$, that can be integrated out, and the gauge-fixing condition $G^a = \partial^\mu A^a_\mu + \partial_0 \lambda^a$, similar to the Abelian case. The resulting action is BRST-invariant when the associated, nilpotent operator of the BRST symmetry, $Q$, with an infinitesimal, anticommuting parameter $\epsilon$, is also acting on $\lambda$ with $Q\lambda^a = -\epsilon \frac{1}{2} f^{abc} \lambda^b c^c$. 

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The resulting gauge field propagators are the same as the Abelian theory and diagonal in color indices. In particular, one again obtains the static Coulomb interaction from the diagrams of Fig. 1, including color indices,

\[ V_{\text{Coul}} = -\frac{4\pi C_R \alpha_s}{k^2}, \quad (13) \]

in momentum space in the singlet channel, with \( \alpha_s = \frac{g^2}{4\pi} \) and \( C_R \) the quadratic Casimir of the representation \( R \).

The incorporation of the constraint of Gauss’s law via the term \( \lambda^a \frac{\delta S_0}{\delta A^a_0} \) has the additional effect of introducing interactions between the gauge field and \( \lambda \). These are the same as the usual interactions, with one \( A_0 \) leg replaced by \( \lambda \). For example, in Fig. 2, a vertex of the non-Abelian theory is shown together with the new corresponding vertex with the same value.

The usual QCD interactions can be reproduced, with the exception that, for diagrams with external \( \lambda \) legs, the Coulomb interaction is missing in the internal propagators: in Fig. 3, this is shown for the \( A_i - A_j \) propagator, with momentum \( k \), and external, constant \( \lambda \) fields, where the missing Coulomb interaction gives a factor of \( g^2 C_2 \lambda^2 \frac{k_i k_j}{k^2} \), where \( \lambda^2 = \lambda^a \lambda^a \) and \( f^{acd} f^{bcd} = C_2 \delta^{ab} \).

This amounts to a mass term in loops like Fig. 4, where the \( \lambda - A_0 \) and \( \lambda - \lambda \) interaction cannot be inserted in the loop, and has the effect of generating a gauge invariant and gauge-independent effective potential from these terms \([1]\), which would otherwise add up to zero. It is of the Coleman-Weinberg form \([2]\),

\[ U(\lambda) = \frac{(\alpha_s C_2)^2}{4} \lambda^4 \left( \ln \frac{\lambda^2}{\mu^2} - \frac{1}{2} \right), \quad (14) \]

renormalized at a scale \( \mu \) where \( dU/d\lambda = 0 \), and appears in the effective action with the opposite sign (it is upside-down).

The auxiliary field \( \lambda \) has similar interactions with \( A_0 \), which, of course, does not develop an effective potential term because of gauge invariance. This shows in the fact that the corresponding integral expressions can be, with a suitable regularization, set to zero \([3]\). The same is true for the ghost terms that appear from the gauge-fixing condition, the terms coming from the four-vertices, and the gauge-dependent expressions for a general gauge-fixing. These will not contribute, at one-loop order, to the effective potential and the effective action derived for \( \lambda \).
Terms of the form $(D_i \lambda^a)^2$ are also generated in the effective action, since they are gauge and BRST-invariant. However, these, like the original terms of the effective action, are multiplied by wave-function renormalization terms that are of higher order in the perturbative expansion. A general such wave-function renormalization term can be written in the form $Z(\lambda) = Z_{\text{tree}} + \text{const} \cdot \alpha_s^2 \ln(\lambda^2/\mu^2)$, with calculable constants and suitable renormalization conditions so that, at $\lambda = \mu$, it retains its tree-level value \[2\]. I will assume an approximation that neglects the higher-order terms in these expansions.

These terms of the form $(D_i \lambda^a)^2$, contain expressions that are not Lorentz covariant, and are generated in the effective action, in the non-Abelian case, since the charges that appear in Gauss’s law (with $E_a^i F_{a0i} = D_i E_a^i - g \rho^a = 0$),

\[\delta S_0 \delta A_0^a - \frac{1}{2} G^2 + U(\lambda)\].

(15)

are part of a gauge but not Lorentz covariantly conserved current, $\rho^a = j^a_0$, with $j^a_\mu = \bar{\psi} \gamma_\mu T^a \psi$, that satisfies $D^\mu j_\mu = 0$, with the gauge covariant derivative in the adjoint representation, but not $\partial^\mu j_\mu = 0$. The calculation of the static interaction energy from $E(r) = \langle \Omega | T(\rho^a(r) \rho^a(0)) | \Omega \rangle$, is well-defined, however, since these terms appear only at higher order in the effective action. In fact, this is how the usual Coulomb interaction is calculated in QCD as well as in the present work, namely from the diagrams in Fig. 1. The presence of these terms at higher order does not invalidate the analysis given here.

For values of $\lambda \approx \mu$, and for slowly-varying $\lambda$, therefore, I expect this approximation to hold, and will accordingly examine the following effective action around $\lambda \approx \mu$, where $Z(\mu) \approx Z_{\text{tree}}$ for the various wave-function renormalization factors.

$$S_{\text{eff}} = \int \frac{-1}{4} F_{\mu\nu}^2 + \bar{\psi} (i\gamma^\mu D_\mu - m) \psi + \lambda^a \frac{\delta S_0}{\delta A_0^a} - \frac{1}{2} G^2 + U(\lambda).$$

(16)

In order to proceed with the calculation of the interaction energy of two static sources, one has to solve for the auxiliary field $\lambda^a$. First, we note that because of the gauge invariance of the effective action, one may rotate $\lambda$ and consider the solution in terms of a maximal torus of $SU(N)$, that is, $N - 1$ non-zero $\lambda^a$.

Also, since there may be various solutions for the auxiliary field, in order to find their respective contribution to the interaction energy, one should
examine the set of different solutions, $\lambda_\alpha(\vec{x})$ (with $\alpha$ a generic index), of the Euclidean equations for $\lambda$ (they are three-dimensional configurations, constant in Euclidean time). Then one has the expansion

$$
e^{W_E(J)} = e^{-E(r)T} = \sum_\alpha N_\alpha e^{-S_\alpha T - E_\alpha(r)T}$$

(17)

Here, $W_E(J)$ is the generating functional for the Euclidean, connected diagrams, $E(r)$ is the desired interaction energy of two static sources, separated by a distance $\vec{r}$, and $T$ is the Euclidean time. The $S_\alpha$’s are the three-dimensional Euclidean actions of the solutions $\lambda_\alpha(\vec{x})$ of the Euclidean equations of motion, and the $E_\alpha(r)$’s are the interaction energies with insertions of the $\lambda_\alpha$’s, calculated from the diagrams considered in this work. In the limit of large $T$, the solution with the smallest $S_\alpha$ is dominant and $E(r)$ is approximately equal, modulo a constant, to the corresponding $E_\alpha(r)$.

There are also additional, determinental, pre-exponential factors, $N_\alpha$, in (17) that arise from fluctuation determinants, zero modes, and diagrams of higher order, that I have not shown here, assuming that they give higher order corrections. Typically, the $N_\alpha$’s include square roots of fluctuation determinants, and if there is an instability in the initial state, or in the $\lambda_\alpha$ configuration, this shows up as one or more negative modes in the determinants and imaginary, or generally complex $N_\alpha$’s.

Solving for $\lambda$ from the three-dimensional, Euclidean effective action,

$$
\tilde{S}_{3,E}(\lambda, A_0) = \int d^3x \left( \frac{1}{2} A_0^a \nabla^2 A_0^a + \lambda^a \nabla^2 A_0^a - U(\lambda) \right)
$$

(18)

gives the basic equation for $\lambda^a$:

$$\nabla^2 \lambda^a = -\frac{\partial U}{\partial \lambda^a},$$

(19)

and the Euclidean action for a general $\lambda$-configuration is

$$\tilde{S}_{3,E}(\lambda) = -\int d^3x \ U(\lambda).$$

(20)

I considered the effective action without the gauge-fixing term here, the same conclusions are reached if one includes the gauge-fixing term. Also, I have not substituted a background configuration for $A_0$, a solution of $\nabla^2 A_0^a = -\nabla^2 \lambda^a$, in accordance with the general procedure of solving for the Lagrange
multiplier, and the treatment of $\lambda$ and the quantum fields on a different footing. I will comment more on this crucial detail later. It is related to the issues of gauge invariance and consistency of the theory; essentially, an effective potential and a vacuum expectation value can be generated for $\lambda$ but not for $A_0$, the upshot being that the background field method is not valid, and (17) is, strictly speaking, not a stationary point expansion but a sum over the different solutions for $\lambda$.

Since we are considering a maximal torus of $SU(N)$, a spherically symmetric solution of (19),

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}\right) \lambda^a = -\frac{\partial U}{\partial \lambda^a},$$

(21)
corresponds to an equation of a “particle” moving in $N - 1$ dimensions, in “potential” $U$ and “time” $r$, with a “frictional” force [4].

However, since $U$ depends on $\lambda^2$, we can consider solutions without “angular momentum” in $N - 1$ dimensions, that have only one non-zero $\lambda^a$. Their linear combinations with constant coefficients turn out to be equivalent since they have the same action and give the same interaction energy; one may consider their contributions to correspond to additional zero modes, to be included in the determinental prefactors. So, although it would be interesting to look for solutions with “angular momentum” and several non-zero $\lambda$’s, in this work I will consider solutions of (19) with only one non-zero $\lambda^a$ (I will still keep the color index, $a$, in the expressions, for definiteness, but will eventually drop it).

Then the solutions of (19) are the same as the solutions of the tunnelling problem of a quantum field theory in three Euclidean space-time dimensions (or four dimensions in high temperature) in the inverted potential $-U$ (although one should keep in mind that we are examining a different physical problem with possibly different interpretations).

The basic equation (19) admits, besides the obvious, constant solutions, $\lambda^a = 0$ and $\lambda^a = \mu$, “bubble-like” solutions, same as the instantons of the three-dimensional tunneling problem, as well as their combinations of multi-bubble solutions, centered at different points in three-dimensional space, sufficiently apart.

A single bubble solution, centered at $\vec{x} = 0$, is a three-dimensional, spherically symmetric configuration, $\lambda^a_s(\vec{x}) = \lambda^a_s(x)$, with

$$\lambda^a_s \sim \mu \text{ in a radius } R_s \sim \frac{1}{\alpha_s C_2 \mu},$$

(22)
In momentum space, they are also bubbles $\tilde{\lambda}_a^\alpha(q)$ with

$$\tilde{\lambda}_a^\alpha \sim \frac{1}{\alpha_s^3 C_2^3 \mu^2}$$

in a radius $R_s \sim \frac{1}{\alpha_s C_2^2 \mu}$.

Here and in the following calculations, I denote generally by $x, q, k$ the modulus of the corresponding three-dimensional vectors, by $\sim$ equality modulo numerical factors of order unity, and by $\tilde{f}(\vec{k})$ the three-dimensional Fourier transform of a function $f(\vec{r})$.

We still assume that we are in the perturbative regime, with $\alpha_s C_2$ less than unity, otherwise higher loop diagrams give comparable contributions; the aforementioned bubble solutions, however, and their combinations, are non-perturbative configurations, to be considered as insertions of external fields.

The first contribution to the static interaction energy between two fermionic current sources, because of a general configuration $\lambda_a^\alpha(\vec{x})$, is given by

$$E_\alpha(r) = V_{\text{Coul}}(r) + V_\alpha(r),$$

where

$$V_\alpha(r) = - \left(4\pi\right)^2 \alpha_s^2 C_R C_2 \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{d^3 \vec{q}'}{(2\pi)^3} \frac{d^3 \vec{k}}{(2\pi)^3} e^{i(\vec{q} + \vec{q}' + \vec{k}) \cdot \vec{r}} \tilde{\lambda}_\alpha^\alpha(\vec{q}) \tilde{\lambda}_\alpha^\alpha(\vec{q}') \frac{(\vec{k} - \vec{q}) \cdot (\vec{k} + \vec{q} + 2\vec{q}')}{k^2 (k + \vec{q})^2 (k + \vec{q} + \vec{q}')^2}.$$  

is the contribution of the diagram of Fig. 5 with external insertions of $\lambda_\alpha$, in the static limit of $k_0 = 0$. The Coulomb interaction term, $V_{\text{Coul}}(r)$, is always there, since it corresponds to the Coulomb interaction diagrams, with no insertions of $\lambda$-fields. There is no summation over $\alpha$ in (25), and the summation over $a$ is reduced to only one term, as explained before.

$V_\alpha(r)$ is not necessarily spherically symmetric, as is implied by the notation; this depends on the $\lambda$-configuration, which, so far, is arbitrary. For highly symmetric configurations, one expects spherical symmetry; this will be commented upon in the particular solutions discussed below.

Now, one may consider the effect of the different solutions of (19) to the static interaction energy:

First, in the trivial, $\lambda_\alpha = 0$, solution, which has zero action, there is obviously only the Coulomb interaction, $E_\alpha(r) = V_{\text{Coul}}(r) = -\frac{\alpha_s}{r}$.

From now on, I will omit the factor of $C_R$, which is common in all expressions and also dispose of the color index, $a$, as explained before.
Second, in the also trivial, constant solution, $\lambda(\vec{x}) = \mu$, with positive Euclidean action $\tilde{S}_{3,E}(\mu) = -U(\mu) V$, (where $V$ is the three-dimensional volume) and $\tilde{\lambda}(\vec{k}) = (2\pi)^3 \delta(\vec{k}) \mu$, one gets from (25) a spherically symmetric, strictly confining interaction,

$$V_\mu(r) = - (4\pi)^3 \alpha_s^2 C_2 \int \frac{d^3 k}{(2\pi)^3} \frac{\mu^2 e^{i\vec{k} \cdot \vec{x}}}{k^4} = (2\pi) \alpha_s^2 C_2 \mu^2 r.$$ (26)

This is an important result of this work. Namely, that a non-zero value of the Lagrange multiplier field, $\lambda$, although weighted with a constant factor, exponentially proportional to the Euclidean volume in (17), with the Feynman rules derived in this work, signals confinement.

A full derivation of the static interaction energy, $E(r)$, would require a complete treatment of the expansion in (17), including the various zero and negative (unstable) modes as discussed before. One may keep in mind that, as far as the Coulomb and the confining interactions are concerned, the former has no dimensionful parameters whereas the latter has only one, which may be rescaled with various numerical factors. Also, although the Lagrange multiplier field, $\lambda$, is considered to be solved for exactly and substituted back in the theory, as a usual Lagrange multiplier, one may gain some insight by considering the analogies with the three dimensional field theory tunneling problem [4]:

If there was a second minimum of $-U$ (other than 0, see Fig. 7), for larger $\lambda$ and negative $-U$, this would be the “ground state” of the theory; it would have the exponentially larger factor in (17), and the static interaction energy, $E(r)$, would be given almost exactly by the Coulomb interaction plus the confining term (26) (with the appropriate multiple of $\mu$, corresponding to the second minimum).

This is not the case, although it would certainly be interesting to examine modifications of the effective potential that admit a second minimum of $-U$, either phenomenological or from radiative, boson or fermion corrections. However, it is natural for the auxiliary field, $\lambda$, to have values where the potential $-U$ is negative, which are weighed with an exponentially larger factor in the Euclidean action.

A general configuration of large $\lambda$, constant or spatially variable, where $-U$ is negative, is not a solution of (19), but bubble and multi-bubble solutions exist in the three-dimensional theory, and the Euclidean action of each
bubble, (18) with (19), is negative, from well-known scaling arguments [4].

The \( \lambda \)-configuration, therefore, which is more interesting to examine, besides the constant solutions for \( \lambda \), is one with closely-packed bubble solutions in three-dimensional space; it has values of the field with non-zero \( \lambda \) and negative Euclidean action, and if one imagines starting with a “non-equilibrium” configuration of large \( \lambda \) field values, this configuration of bubble “condensate” is the “closest” it will most probably relax to, with the lowest Euclidean action.

I should comment again on the crucial fact that a background configuration for \( A_0 \), satisfying \( \nabla^2 A_0^a = -\nabla^2 \lambda^a \), was not substituted in the Euclidean action. This would give a positive kinetic term in the bubbles and invalidate the numerics of the above arguments; the pure Coulomb phase, \( \lambda = 0 \), would then be the one with the lowest action, and confining and other interactions would be exponentially suppressed. Also, diagrams with insertions of background \( A_0 \) fields would give gauge-dependent and non-invariant contributions, as opposed to the manifestly gauge-independent and gauge invariant results considered here. The present treatment is in spirit with our original motivation of treating \( \lambda \) and \( A_0 \) on a different footing: the former is a Lagrange multiplier for Gauss’s law, that has to be solved for and substituted back in the action, and the latter is a pure quantum field (component of a four-vector) for which a “background field” splitting as \( A_0 = A_{0,\text{background}} + A_{0,\text{quantum}} \), or a redefinition, \( A_0 = A_0 + \lambda \), is ambiguous or inconsistent. Although the usual background-field method derives the correct results for the quadratic fluctuations in quantum field theory, usually faster and more compactly than other methods, it relies on discarding the linear terms, whereas the method presented here and in [1] relies on the special combinatorics of linear terms (such as those with \( \lambda \)) and cannot be used together with the background-field method.

Also, the general idea of the calculations in this work is that \( \lambda \) can develop an effective potential term and a vacuum expectation value, whereas \( A_0 \) can not, because of gauge invariance. For the same reason, \( A_0 \) can not fully serve as a Lagrange multiplier for Gauss’s law, as it is often stated (the term \( \lambda \delta S_0 / \delta A_0 \) in (11) and (16) is gauge-invariant, but a similar term with \( A_0 \) instead of \( \lambda \) is not). The upshot of these arguments, and the insufficiency of the background field method in our case, is that (17) is, strictly speaking, not considered as a stationary point expansion, but as a sum over the different solutions for \( \lambda \).

The evaluation of (25) in the multi-bubble \( \lambda \)-configuration can be performed as follows: I will assume that the bubble solutions have a fixed radius,
$R_s$, which is the smallest distance in which they can approach without significant deformation, essentially where the value of $\lambda_s$ is a small percentage of is central value, and their Euclidean action becomes essentially constant; this distance will be kept as a parameter for the numerical approximations to be performed later. Then the centers of the multi-bubble, closely-packed solution, lie on a three-dimensional lattice, $\Lambda$, for closely packed spheres, of which there are two regular kinds in three dimensions, the face-centered cubic (fcc) and the hexagonal close-packed (hcp) lattices, as well as their permutations in the different layers of the spheres (the distance between two adjacent spheres, naturally, being $2R_s$) [5].

Then the bubble “condensate” configuration examined is

$$\lambda_\alpha(\vec{x}) = \sum_{\vec{x}_\Lambda \in \Lambda} \lambda_s(\vec{x} - \vec{x}_\Lambda)$$

(27)

and its Fourier transform

$$\tilde{\lambda}_\alpha(\vec{q}) = \tilde{\lambda}_s(\vec{q}) \sum_{\vec{x}_\Lambda \in \Lambda} e^{i\vec{q} \cdot \vec{x}_\Lambda},$$

(28)

becomes

$$\tilde{\lambda}_\alpha(\vec{q}) = (Vol_{\tilde{\Lambda}}) \sum_{\vec{q}_\Lambda \in \tilde{\Lambda}} \tilde{\lambda}_s(\vec{q}_\Lambda) \delta(\vec{q} - \vec{q}_\Lambda),$$

(29)

after using Poisson’s summation formula, where $\tilde{\Lambda}$ is the dual lattice, with the dual spacing, $(2\pi)/(2R_s)$, and $(Vol_{\tilde{\Lambda}})$ is the volume of its unit cell, which is $(2\pi)^3/(2R_s)^3$ times the determinant of the (unscaled) dual basis vectors [5].

Keeping in mind that $\tilde{\lambda}_s(\vec{q})$ is also a bubble, with a maximum value at $\vec{q} = 0$, and then decreasing in a radius $\tilde{R}_s \sim 1/R_s$, we get the main contribution of the bubble “condensate” configuration to (25) from the zeroth term in (29) and their product in (25), to be

$$V_{\alpha,0}(r) = -(4\pi)^2 \alpha_s^2 C_2 \int \frac{d^3k}{(2\pi)^3} (Vol_{\tilde{\Lambda}})^2 \tilde{\lambda}_s^2(0) \frac{e^{i\vec{k} \cdot \vec{x}}}{k^4}$$

$$= (2\pi)\alpha_s^2 C_2 (Vol_{\tilde{\Lambda}})^2 \tilde{\lambda}_s^2(0) r$$

(30)

again a confining interaction. The next contribution to (25) is given by

$$V_{\alpha,1}(r) = -(4\pi)^2 \alpha_s^2 C_2 \int \frac{d^3k}{(2\pi)^3} \sum_i \tilde{\lambda}_s(0) \tilde{\lambda}_s(q_i) e^{i(k+\vec{q}_i) \cdot \vec{x}}$$

(31)
$(Vol_{\tilde{\Lambda}})^2 \left( \frac{\vec{k} \cdot (\vec{k} + \vec{q}_i)}{k^4(k + q_i)^2} + \frac{(\vec{k} - \vec{q}_i) \cdot (\vec{k} + \vec{q}_i)}{k^2(k + \vec{q}_i)^4} \right)$, \hspace{1cm} (31)

where the summation over $i$, here, denotes the summation over the minimal vectors, $\vec{q}_i$, of the dual lattice, that are closer to the origin. In (30) and (31), I have canceled factors of $(2\pi)^3$ in the integrations with corresponding factors in $(Vol_{\tilde{\Lambda}})$.

It can be seen from (30), that a general lattice configuration of $\lambda$-bubbles gives a confining interaction term, where, instead of the factor $\mu$ in (26), one gets its “average” value of $(Vol_{\tilde{\Lambda}}) \tilde{\lambda}(0)$.

It is important to note that, although in (31) $\tilde{\lambda}_s(q_i) < \tilde{\lambda}_s(0)$, which is, in fact, a strong inequality ($<<$) for the close-packing lattice, one has to examine this term, in comparison with the previous expression, in order to see whether it affects or destroys its confining nature. This will be done numerically in the next section. Also, it may be noticed that as the original lattice spacing becomes larger (the bubbles become more sparse) the contribution of (31) becomes comparable to (30) (since $q_i$ gets closer to 0) and there is no reason to expect the confining term to still persist. Also, the overall factor of $(Vol_{\tilde{\Lambda}})^2$ becomes smaller, since the dual lattice spacing decreases, resulting in a decrease in the effective “string tension”. In fact, since most of the three-dimensional volume is then in the Coulomb phase of $\lambda = 0$, one expects the Coulomb interaction to be the dominant one. Obviously, as the bubble distance becomes larger than $2R_s$, there is no reason for the bubbles to be regularly arranged on a lattice, and the configurations of the general solutions of (19) are those of a bubble gas or liquid model.

4 Numerical results and comments

Here, I will discuss the first numerical results in the case when $g^2 = 1$, $N = 3$, $\alpha_s = 1/4\pi$. All graphs in this Section are scaled in appropriate powers of $\mu$, which is the only dimensionful scale of the problem. Also, as before, I will omit the factor of $C_R$, which is common in all expressions for the interaction energy.

In Fig. 6, I show the effective potential, $U(\lambda)$, the spherically symmetric bubble solution $\lambda_s(r)$, and its Fourier transform, $\tilde{\lambda}_s(k)$. Because of the slow convergence of the numerical integrations, the Fourier transforms in this and other Figures have been calculated at discrete points (usually at incre-
ments of $10^{-3}$, the coarsest being $10^{-2}$, at appropriate powers of $\mu$) and then extrapolated for the graphs.

In Fig. 7, I show the inverted effective potential $-U(\lambda)$ and the three-dimensional, Euclidean action, $\bar{S}_{3E}$, of the spherically symmetric bubble solution, $\lambda(x)$, as a function of $r$,

$$\bar{S}_{3E}(r) = -4\pi \int_0^r x^2 dx U(\lambda(x)). \quad (32)$$

After inspection of these two Figures, 6 and 7, I will consider the numerical results for bubble solutions in an fcc lattice, with lattice spacing, $2L_s$, with various values of $L_s$. Strictly speaking, one should start with the smallest distance, $L_s = R_s$, which will give the greatest value for the string constant, as we will see, and also has the largest value of the weight factor (the most negative Euclidean action) for fixed volume. For lack of an exact multi-bubble solution, one may try lattice configurations with various spacings between the bubbles, it seems plausible, however, that multi-bubble, closely-packed configurations, would exist when $R_s$ is somewhere between $6\mu^{-1}$ and $14\mu^{-1}$.

It should be clear from the discussion of the previous Section that lattice configurations with any value of $L_s$ give calculable contributions, as long as $L_s$ is not very small, so that they are solutions of the Euclidean equations; they are weighted by different factors of the exponential of the Euclidean action, and their confining interaction energies of $\alpha, 0$ and $\alpha, 1$ are also scaled, by virtue of the fact that $(Vol_{\Lambda}) \sim 1/(2L_s)^3$. Also, the contributions of $\alpha, 0$ should be calculated, since they may also destroy their confining properties.

In Fig. 8, I show the static interaction energy, $E(r)$, first in the constant background of $\lambda = \mu$, that is $V_{\text{Coul}}(r) + V_\mu(r)$, and then in the bubble “condensate” background of the bubbles arranged in closely-packed fcc lattice with a lattice spacing of $2L_s$, that is, $V_{\text{Coul}}(r) + V_{\alpha,0}(r) + V_{\alpha,1}(r)$. The middle graph is for $L_s = 8\mu^{-1}$ and the last for $12\mu^{-1}$.

For the evaluation, I used the fact that the dual of an fcc lattice is a body-centered cubic (bcc) lattice, with the rescaled unit volume, $(Vol_{\Lambda}^5) = 16/(2L_s)^3$, and with minimal vectors, $(\pm 1, \pm 1, \pm 1)$, with modulus $\sqrt{3}2\pi L_s [5]$.

In Fig. 9, I show the sum $V_{\alpha,0}(r) + V_{\alpha,1}(r)$, that comes from the evaluation of $\alpha, 0$ and $\alpha, 1$, without the Coulomb term, again for bubbles arranged in an fcc lattice. The first two graphs are again with the lattice spacing, $L_s$, of $8\mu^{-1}$ and $12\mu^{-1}$ (these contributions were included in the last two diagrams of the
previous Figure, together with the Coulomb term). The last two graphs are
the result of increasing $L_s$ to $24\mu^{-1}$ and $36\mu^{-1}$.

One can see that, as the lattice spacing is increased, the slope of the
confining term is decreased (mainly because $Vol_\Lambda \sim L_s^{-3}$), and there are also
effects reminiscent of “string breaking”, that distort the linear nature of the
curves. The latter arise from the contributions of (31) as explained before. I
show here the result of the calculation of (31) when the radial distance, $r$, is along an axis of the lattice. Calculations along different directions give
similar results, suggesting that the expressions have (approximate) spherical
symmetry.

It is also important to note that I show here the result for the real part of
(31). The result for the imaginary part of (31) is negligible compared to the
real part (smaller by several orders of magnitude) and becomes comparable
to the real part (although still smaller) only in the regions that show the
effects of “string breaking”. These regions become more common and can be
seen as the distance between the bubbles grows. For the first configuration
with $L_s = 8\mu^{-1}$, there was only one point, in the region examined, with a
non-negligible imaginary part, which was still much smaller than the real
part shown here, and the overall contribution of $V_{\alpha,1}$ was not sufficient to
distort $V_{\alpha,0}$ in any discernible way. In the case of $L_s = 12\mu^{-1}$ there are
three narrow places with “instabilities” and non-negligible imaginary part in
the region examined (they can be discerned in the second graph of Fig. 9),
and in the last two graphs, with $L_s$ equal to $24\mu^{-1}$ and $36\mu^{-1}$ respectively,
one can see the regions of instabilities to grow, and so do the corresponding
imaginary parts in these regions, although they are not shown here.

Finally, in Fig. 10, I show the result (the real part) of the calculation of
the interaction energy, $V_{\alpha}(r)$, via (25), for the single bubble solution, centered
at 0 (the horizontal axis is still in units of $\mu^{-1}$, the vertical here is in units of
$10^{-4}\mu$). We see that, near the origin, there is still an approximate, average,
linear rise, which was expected since the bubble solution starts off as nearly
constant for small distances [4]. The imaginary part is non-negligible in more
places, essentially in most of the spikes seen in the Figure. Then, at the two
larger spikes, the imaginary part becomes even larger than the real part,
and we see instabilities that halt the linear rise; the interaction energy starts
decreasing and goes to zero, with obvious instabilities, at larger distances.

It should be clear that the evaluation of the various interaction energies in
these, highly symmetric backgrounds, still does not give a complete evaluation of $\tilde{E}(r)$ in (17). Any small deformations from the closely-packed bubble
lattice contribute, with a Euclidean action which is only slightly greater (for finite volume) and a corresponding interaction energy which would be, apparently, still confining, but deformed with respect to those shown here; that is, these configurations do not correspond, strictly speaking, to zero modes and have to be treated accordingly. The contributions of all possible multi-bubble solutions, with the appropriate weight factors, should be included, in what resembles a bubble “liquid” or “gas” model; several arguments have been given here, however, as to the significance of the “solid”, highly-ordered, closely-packed lattice configurations (of which only the fcc lattice has been dealt with, the hcp having been deferred to future work). In principle, however, both in finite volume and in the large volume limit, one should examine all the different configurations and their contributions to the expansion. Also, one has to consider the contributions of the various translational and rotational zero modes, as well as clarify the physical significance of the various negative modes, and whether they are related to instabilities such as string breaking.

For all physical reasons, however, it seems that, both in the finite and in the large volume limit, the relevant configurations are the lattices of closely-packed bubbles. One can speculate on the fact that, in more “excited” situations, for example at high temperatures or densities, the contributions of the various terms of (17) change, and different “phases” of the bubble vacuum become more important, much like in ordinary phase transitions.

Finally, I should note that the results given in this work (with the factor of $C_R$ restored) satisfy the Casimir and large-$N$ scaling properties and requirements expected from a confining interaction [6], and that there are also arguments for the presence of a “Lüscher-type” term, which have not been examined yet. For example, for large distances, $r$, such that $k << q_i$, one gets terms in the parenthesis of (31) that scale as $\frac{1}{k^2q_i^2}$, resulting in an interaction energy proportional to

$$\alpha_s^2 C_2 C_R \tilde{\lambda_s}(0) \tilde{\lambda_s}(q_i) (Volk^2) \frac{1}{q_i^2} \frac{1}{r} \sim \frac{\tilde{\lambda_s}(q_i) C_R}{\tilde{\lambda_s}(0) C_2} \frac{1}{r}.$$  

(33)

after restoring $C_R$ and using the power counting arguments of the previous Section. It should be noted, however, that, essentially because the bubble solution is the same as the one of the tunnelling problem with an inverted potential that is unbounded below, there are finite factors that may become important in the power counting arguments, although they formally are of
order unity (we see, for example, that $\lambda_s(0) \approx 4\mu$ in the numerical solution of Fig. 6).

\section{Discussion}

In the present work I examined some of the consequences of a set of Feynman rules that were initially mentioned in \cite{1} as an expression of the constraint of Gauss’s law, and the associated effective action and stationary point bubble and multi-bubble solutions, in order to calculate the static interaction energy. There are some similarities and common features with other approaches \cite{7} and effective actions proposed. However, the treatment here is initially perturbative, having the non-perturbative input of the afore-mentioned bubble solutions. The action considered here is also BRST-invariant, leading to the expectation that the entire procedure is renormalizable.

I used a Lagrange multiplier, auxiliary field, $\lambda$, in order to impose the constraint of Gauss’s law. It is usually stated, or derived in standard quantization procedures, that $A_0$ is the Lagrange multiplier associated with this constraint. However, it is not a direct consequence of the path integral quantization with constraints that $\lambda$ becomes $A_0$, as the Lagrange multiplier that enforces the constraint that $A_0$ itself satisfies. Doing so maintains the manifest Lorentz invariance, and in the Abelian case the procedures are equivalent, but in the non-Abelian case the present method gives some additional terms, specifically for the calculation of the interaction energy for static sources.

The combinatorics described in \cite{1} and here can be interpreted as enforcing the Coulomb interaction to behave “classically”, that is, to propagate only in tree diagrams, not in loops. This is not an \textit{ad hoc} assumption, but a consequence of enforcing Gauss’s law at all orders in perturbation theory, not just the “final” or “physical” states. The fact that the Coulomb interaction is missing from loops has the effect of generating the effective potential term for $\lambda$ and the consequences described above.

The theory is still Lorentz invariant at the fundamental level, one has, however, considered static fermionic sources in order to calculate the static interaction energy. In the non-Abelian case, since these sources are part of a gauge but not Lorentz covariantly conserved current, there are additional terms that are generated at higher order in the effective action that are not Lorentz covariant. These do not spoil the calculation of the interaction energy at this order, but may contribute at higher orders. First of all, these terms are
calculable since the theory is renormalizable, and their contributions to the final result can be seen to be of higher order (at least of order $\alpha_s^4$). Also, the basic equation, (19), is not affected by these corrections, except for possible multiplicative factors close to unity, and the bubble solutions still exist. That is, the extra terms do not change the basic result of the dynamics, namely the Coulomb and the confining nature of the interaction.

The characteristic property of asymptotic freedom is not directly used in the considerations of the non-Abelian theory, although it will be important in a fuller treatment that includes the wavefunction renormalization terms, $Z(\lambda)$, and their dielectric properties.

The generation of an effective potential of the Coleman-Weinberg type, along with non-perturbative solutions that are not scale invariant like the QCD instantons, makes the present approach particularly convenient for the investigation of the question of confinement and, generally, the issue of the phases of gauge theories. The various solutions of (19) can describe the different phases involved, as described before, and one may examine the effective action at finite temperature along with the corresponding deconfinement phase transition. The fact, for example, that the high-temperature phase transition in Coleman-Weinberg models is of the first-order yields a similar expectation for the deconfinement phase transition when investigated by the present method. It is also possible to do similar calculations in the case of a spontaneously broken gauge theory and consider the relations between the Higgs, the confined and the Coulomb phases.

These problems and other extensions will be dealt with in future work.

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Figure 1: The propagators combine to reproduce the Coulomb interaction between two static sources (large blobs). Solid, wavy and curly lines denote the $A_0$, $A_i$ and $\lambda$ fields respectively.

Figure 2: Two vertices for the non-Abelian theory.
Figure 3: The modifications to the i-j propagator.

Figure 4: A diagram for the generated effective potential.

Figure 5: The diagram for the interaction energy of two static sources (large blobs) with two insertions of $\lambda$ (smaller blobs).
Figure 6: The effective potential, $U(\phi)$ (on top), the bubble profile, $\lambda_s(r)$, and its Fourier transform, $\hat{\lambda}_s(k)$. All units are appropriate powers of $\mu$. 

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Figure 7: The top graph is the inverted potential, $-U$, as a function of $\lambda$ and the bottom is the Euclidean action of the bubble solution as a function of the radial distance, $r$. 
Figure 8: I show the full result for the static interaction energy, $E(r)$, first for the constant solution $\lambda = \mu$ (on top), and then for the bubble “condensate” solution, with bubbles in an fcc lattice with lattice spacing $8\mu^{-1}$ and $12\mu^{-1}$, in an fcc lattice.
Figure 9: I show the sum $V_{\alpha,0}(r) + V_{\alpha,1}(r)$ for bubbles arranged in an fcc lattice, with increasing lattice spacing. The first two graphs correspond to $L_z$ equal to $8\mu^{-1}$ and $12\mu^{-1}$, and were also used in the previous Figure, and the last two graphs correspond to increasing $L_z$ to $24\mu^{-1}$ and $36\mu^{-1}$. 

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Figure 10: I show the result of $V_\alpha(r)$ from (25) for the single bubble solution, centered at 0 (the horizontal axis is still in units of $\mu^{-1}$, the vertical is in units of $10^{-4}\mu$).