Average value of some certain types of arithmetic functions with Piatetski-Shapiro sequences

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Abstract: In this paper, we study asymptotic behaviour of the sum \( \sum_{n \leq N} f\left(\lfloor n^c \rfloor\right) \), where \( f(n) = \sum_{d \mid n} g(d) \) under three different types of assumptions on \( g \) and \( 1 < c < 2 \).

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1 Introduction and results

The research on arithmetic functions with Piatetski-Shapiro sequences \( \{ \lfloor n^c \rfloor \} \), with real \( c \), is an extensive topic in the number theory, see, for example, [1–4, 6, 7, 9, 11–16, 19]. Very recently, Srichan [14] studied asymptotic behaviour of the sum:
\( T^c_{f_k}(N) := \sum_{n \leq N} f_k\left( \lfloor n^c \rfloor \right), \quad 1 < c < 2, \)

and established some asymptotic formulas for \( T^c_{f_k}(N) \) under assumptions on

\[ f_m(n) = \sum_{d \mid m} g(d), \]

where \( g(d) \) is a multiplicative function with \( g(d) = O(d^\varepsilon), \varepsilon > 0 \). He proved that for \( 1 < c < 3/2 \), we have

\[ T^c_{f_2}(N) = N \sum_{d=1}^{\infty} \frac{g(d)}{d^2} + O\left( N^{c/2+1/4+\varepsilon} \right), \quad (1) \]

and for \( 1 < c < 2 \), and \( m \geq 3 \),

\[ T^c_{f_m}(N) = N \sum_{d=1}^{\infty} \frac{g(d)}{d^m} + O\left( N^{c/3+1/3+\varepsilon} \right). \quad (2) \]

If \( g(n) = \mu(n) \), (1) and (2) generalize many works involving square-free and \( m \)-free integers in Piatetski-Shapiro sequences, see, for example, [6, 7, 12–16, 19].

In this paper, we shall extend this problem with other classes of arithmetic functions. In 2019, Bordellés, Dai, Heyman, Pan and Shparlinski [5] introduced the following three different types of assumptions on \( g \). Namely,

\[ \text{Type I:} \quad |g(n)| \ll \tau_k(n) \quad (n \in \mathbb{N}), \quad (3) \]

where \( \tau_k(n) := \sum_{n_1 \ldots n_k=n} 1 \) is the generalized divisor function,

\[ \text{Type II:} \quad |g(n)| \ll n^{\phi-1}(\log(en))^{-A} \quad (n \in \mathbb{N}), \quad (4) \]

where \( A > 0 \) and \( \phi = \frac{1+\sqrt{5}}{2} \approx 1.618 \) is the Golden Ratio, and

\[ \text{Type III:} \quad \sum_{n \leq x} |g(n)|^2 \ll x^\theta \quad (x \geq 1, n \in \mathbb{N}), \quad (5) \]

where \( 0 < \theta < 2 \). In [5], they studied the asymptotic behaviour of the quantity

\[ S_g(x) := \sum_{n \leq x} g\left( \left\lfloor \frac{x}{n} \right\rfloor \right), \]

where \( \lfloor t \rfloor \) is the integral part of the real number \( t \). Later, many authors have studied the functions in types (3)–(5) with integer parts of real-valued function, see, for example, [10, 17, 18, 20]. Note that the function \( g(n) \) in [14] belongs to the class of the functions in (3). Thus, it would be interesting to study the same problem in [14] under the three different types of assumptions on \( g \) in (3)–(5).

In this paper, we shall study the asymptotic behaviour of multiplicative functions

\[ f(n) := \sum_{d \mid n} g(d), \]

where \( g(d) \) is a multiplicative function satisfying (3)–(5) on Piatetski-Shapiro sequences.
We study asymptotic behaviour of the sum

$$T_c^\infty(N) := \sum_{n \leq N} f(\lfloor n^c \rfloor),$$

and prove the following theorems.

**Theorem 1.1.** For $1 < c < 3/2$, and $g$ is a multiplicative function satisfying (3), we have

$$T_c^\infty(N) = N \sum_{d=1}^\infty \frac{g(d)}{d^2} + O\left(N^{c/2+1/4}(\log N)^{k-1}\right).$$

**Remark 1.2.** Theorem 1.1 covers Srichan’s result (1) in [14].

**Theorem 1.3.** For $1 < c < (2 + \phi)/2\phi$, $\phi$ the Golden Ratio and $g$ a multiplicative function satisfying (4), we have

$$T_c^\infty(N) = N \sum_{d=1}^\infty \frac{g(d)}{d^2} + O\left(N^{1/2+c\phi/2-\phi/4}\right).$$

**Theorem 1.4.** For $1 < c < (5 + \theta)/(2 + 2\theta)$, and $g$ a multiplicative function satisfying (5), we have

$$T_c^\infty(N) = N \sum_{d=1}^\infty \frac{g(d)}{d^2} + O\left(N^{(3-\theta+2c+2c\theta)/8}\right).$$

This can easily be generalized to

$$f_m(n) = \sum_{d^m | n} g(d), \ m \geq 3,$$

instead of the case $m = 2$. We obtain the following results.

**Theorem 1.5.** For $1 < c < 2$, $m \geq 3$ and $g$ a multiplicative function satisfying (3), we have

$$T_{f_m}^\infty(N) = N \sum_{d=1}^\infty \frac{g(d)}{d^m} + O\left(N^{(c+1)/3}(\log N)^{k-1}\right).$$

**Theorem 1.6.** Let $\phi$ be the Golden Ratio and let $g$ be a multiplicative function satisfying (4). For $1 < c < 2 + \phi/2\phi$ and $2 \leq m \leq 4$, we have

$$T_{f_m}^\infty(N) = N \sum_{d=1}^\infty \frac{g(d)}{d^m} + O\left(N^{(2c\phi-\phi+m)/2m}\right).$$

For $1 < c < 2$ and $m \geq 5$, we have

$$T_{f_m}^\infty(N) = N \sum_{d=1}^\infty \frac{g(d)}{d^m} + O\left(N^{(c+1)/3}\right).$$
Theorem 1.7. Let $1 < \theta < 2$ and let $g$ be a multiplicative function satisfying (5). For $1 < c < (5 + \theta)/(2 + 2\theta)$, and $2 \leq m \leq 4$, we have

$$T_{f_m}^c(N) = N \sum_{d=1}^{\infty} \frac{g(d)}{d^m} + O\left(N^{(2m-1-\theta+2c+2c\theta)/4m}\right).$$

For $1 < c < 2$ and $m \geq 5$, we have

$$T_{f_m}^c(N) = N \sum_{d=1}^{\infty} \frac{g(d)}{d^m} + O\left(N^{(c+1)/3}\right).$$

The main ingredient in the proof of Theorems is a good estimate for the number of integers $n$ up to $x$ such that $\lfloor nc \rfloor$ belongs to an arithmetic progression in [7].

Lemma 1.8. [Theorem 2, [7]] For $1 < c < 2$, $x$ being a real number and $q$ and $a$ being two integers such that $0 \leq a < q \leq x^c$, we have

$$\sum_{n \leq x} 1 = \frac{x}{q} + O\left(\min\left(\frac{x}{q}, \frac{x^{(c+1)/3}}{q^{1/3}}\right)\right).$$

2 Proof of Theorems

Proof of Theorem 1.1. For $1 < c < 3/2$, we have

$$T_f^c(N) = \sum_{n \leq N} \sum_{d \mid \lfloor n^c \rfloor} g(d) = \sum_{d \leq N^{c/2}} g(d) \sum_{n \leq N} 1.$$

Using Lemma 1.8, we get

$$T_f^c(N) = N \sum_{d \leq N^{c/2}} \frac{g(d)}{d^2} + O\left(\sum_{d \leq N^{c/2}} \left|g(d)\right| \min\left(\frac{N^c}{d^2}, \frac{N^{(c+1)/3}}{d^{2/3}}\right)\right). \quad (6)$$

In view of the hypothesis (3) and the well-known formula

$$\sum_{n \leq N} \tau_k(n) \ll N (\log N)^{k-1}, \quad (7)$$

see [8, eq. 13.2] and the partial summation, we have

$$\sum_{d > N^{c/2}} \frac{|g(d)|}{d^2} \ll \sum_{d > N^{c/2}} \frac{\tau_k(d)}{d^2} \ll \frac{(\log N)^{k-1}}{N^{c/2}}.$$ 

Thus,

$$N \sum_{d \leq N^{c/2}} \frac{g(d)}{d^2} = N \sum_{d=1}^{\infty} \frac{g(d)}{d^2} + O\left(N^{1-c/2} (\log N)^{k-1}\right). \quad (8)$$

To bound the error term in (6), we write

$$\sum_{d \leq N^{c/2}} \left|g(d)\right| \min\left(\frac{N^c}{d^2}, \frac{N^{(c+1)/3}}{d^{2/3}}\right) = \sum_{d \leq N^{c/2-1/4}} \frac{N^{(c+1)/3} \left|g(d)\right|}{d^{2/3}} + \sum_{N^{c/2-1/4} < d \leq N^{c/2}} \frac{N^c \left|g(d)\right|}{d^2}.$$
Using again (7) and the partial summation, we have
\[
\sum_{d \leq N^{c/2-1/4}} \frac{N^{(c+1)/3}|g(d)|}{d^{2/3}} \ll N^{(c+1)/3} \sum_{d \leq N^{c/2-1/4}} \frac{\tau_k(d)}{d^{2/3}} \ll N^{c/2+1/4}(\log N)^{k-1}
\]  
(9)
and
\[
\sum_{N^{c/2-1/4} < d \leq N^{c/2}} \frac{N^c|g(d)|}{d^2} \ll N^c \sum_{N^{c/2-1/4} < d \leq N^{c/2}} \frac{\tau_k(d)}{d^2} \ll N^{c/2+1/4}(\log N)^{k-1}.
\]  
(10)
Inserting (8)–(10) in (6), Theorem 1.1 follows.

**Proof of Theorem 1.3.** Let \( \phi \) be the Golden Ratio and \( 1 < c < (2 + \phi)/2\). The proof follows closely that of Theorem 1.1 in the case \( k = 1 \) above. Since \( |g(n)| \ll n^{\theta-1}(\log(en))^{-A} \), we have
\[
\sum_{d \leq x} |g(d)| \ll x^\phi, \quad \sum_{d > x} \frac{|g(d)|}{d^2} \ll x^{\theta-2}.
\]

Thus,
\[
N \sum_{d \leq N^{c/2}} \frac{g(d)}{d^2} = N \sum_{d=1}^\infty \frac{g(d)}{d^2} + O\left(N^{1-c+\phi c/2}\right).
\]  
(11)
Using again the partial summation with the hypothesis (4), we have
\[
\sum_{d \leq N^{c/2-1/4}} \frac{N^{(c+1)/3}|g(d)|}{d^{2/3}} \ll N^{(c+1)/3} \sum_{d \leq N^{c/2-1/4}} \frac{d^{\phi-1}}{d^{2/3}} \ll N^{\phi/2+1/2-\phi/4}
\]  
(12)
and
\[
\sum_{N^{c/2-1/4} < d \leq N^{c/2}} \frac{N^c|g(d)|}{d^2} \ll N^c \sum_{N^{c/2-1/4} < d \leq N^{c/2}} \frac{d^{\phi-1}(\log N)^{-A}}{d^2} \ll N^{\phi/2+1/2-\phi/4}(\log N)^{-A}.
\]  
(13)
Then, Theorem 1.3 follows from (11)–(13).

**Proof of Theorem 1.4.** Let \( 1 < c < (5 + \theta)/(2 + 2\theta) \). The proof follows closely that of Theorem 1.1 in the case \( k \geq 2 \) above. By the Cauchy inequality and the hypothesis (5), we have
\[
\sum_{d \leq x} |g(d)| \ll x^{(1+\theta)/2}.
\]
By using the partial summation, we have
\[
\sum_{d > x} \frac{|g(d)|}{d^2} \ll x^{(\theta-3)/2}.
\]
Thus,
\[
N \sum_{d \leq N^{c/2}} \frac{g(d)}{d^2} = N \sum_{d=1}^\infty \frac{g(d)}{d^2} + O\left(N^{\phi/4-3c/4}\right).
\]  
(14)
Using again the partial summation with the hypothesis (5), we have

\[
\sum_{d \leq N^{c/2-1/4}} \frac{N^{(c+1)/3}|g(d)|}{d^{2/3}} \ll N^{c\theta/4 + c/4 + 3/8 - \theta/8} \tag{15}
\]

and

\[
\sum_{N^{c/2-1/4} < d \leq N^{c/2}} \frac{N^c|g(d)|}{d^2} \ll N^{c\theta/4 + c/4 + 3/8 - \theta/8}. \tag{16}
\]

Then, Theorem 1.4 follows from (14)–(16).

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