THE $m$-STEP SOLVABLE ANABELIAN GEOMETRY OF NUMBER FIELDS

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To the memory of Professor Michel Raynaud

Abstract. Given a number field $K$ and an integer $m \geq 0$, let $K_m$ denote the maximal $m$-step solvable Galois extension of $K$ and write $G^m_K$ for the maximal $m$-step solvable Galois group $\text{Gal}(K_m/K)$ of $K$. In this paper, we prove that the isomorphy type of $K$ is determined by the isomorphy type of $G^3_K$. Further, we prove that $K_m/K$ is determined functorially by $G^m_K$ (resp. $G^{m+4}_K$) for $m \geq 2$ (resp. $m \leq 1$). This is a substantial sharpening of a famous theorem of Neukirch and Uchida. A key step in our proof is the establishment of the so-called local theory, which in our context characterises group-theoretically the set of decomposition groups (at nonarchimedean primes) in $G^3_K$, starting from $G^2_K$.

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§0. Introduction/Main results. Let $K$, $L$ be number fields with algebraic closures $\overline{K}$, $\overline{L}$ and absolute Galois groups $G_K \overset{\text{def}}{=} \text{Gal}(\overline{K}/K)$, $G_L \overset{\text{def}}{=} \text{Gal}(\overline{L}/L)$, respectively. A celebrated theorem of Neukirch and Uchida states that (profinite) group isomorphisms between $G_K$ and $G_L$ arise functorially from field isomorphisms between $K$ and $L$. More precisely, one has the following (cf. [Neukirch1], [Uchida1]), which is the first established (birational) anabelian result of this sort (well before Grothendieck announced his anabelian programme).

Neukirch-Uchida Theorem. Let $\tau : G_K \sim G_L$ be an isomorphism of profinite groups. Then there exists a unique field isomorphism $\sigma : \overline{K} \sim \overline{L}$ such that $\tau(g) = \sigma g \sigma^{-1}$ for every $g \in G_K$. In particular, $\sigma(K) = L$ and $K$, $L$ are isomorphic.

Moreover, the theorem is still valid when one replaces $G_K$, $G_L$ by their respective maximal prosolvable quotients $G^\text{sol}_K$, $G^\text{sol}_L$ and $\overline{K}$, $\overline{L}$ by the maximal prosolvable extensions $K^\text{sol}$, $L^\text{sol}$ of $K$ in $\overline{K}$ and $L$ in $\overline{L}$, respectively (cf. [Neukirch2], [Uchida2]). Thus, the isomorphy type of the (maximal prosolvable) Galois group of a number
field determines functorially the isomorphy type of the number field. An explicit description of the isomorphy type of the (prosolvable) Galois group of a number field seems to be out of reach for the time being. This prompts the natural question:

Is it possible to prove any refinement of the Neukirch-Uchida theorem, whereby one replaces the full (prosolvable) Galois groups of number fields by some profinite quotients whose structure can be better approached/understood?

Class field theory provides a description of the maximal abelian quotient of the Galois group of a number field. The structure of the maximal \( m \)-step solvable quotient of the Galois group of a number field can be in principle approached via class field theory, say for small values of \( m \) as in (commutative) Iwasawa theory in case \( m = 2 \). In this paper we prove the following sharpening of the Neukirch-Uchida theorem. Write \( G_K^m \) and \( G_L^m \) for the maximal \( m \)-step solvable quotients of \( G_K, G_L \) and \( K_m/K, L_m/L \) for the corresponding subextensions of \( \bar{K}/K \) and \( \bar{L}/L \), respectively (cf. Notations).

**Theorem 1.** Assume that there exists an isomorphism \( \tau_3 : G_K^3 \cong G_L^3 \) of profinite groups. Then there exists a field isomorphism \( \sigma : K \cong L \).

**Theorem 2.** Let \( m \geq 0 \) be an integer and \( \tau_{m+3} : G_K^{m+3} \cong G_L^{m+3} \) an isomorphism of profinite groups.

(i) There exists a field isomorphism \( \sigma_m : K_m \cong L_m \) such that \( \tau_m(g) = \sigma_m g \sigma_m^{-1} \) for every \( g \in G_K^m \), where \( \tau_m : G_K^m \cong G_L^m \) is the isomorphism induced by \( \tau_{m+3} \). In particular, \( \sigma_m \) induces an isomorphism \( K \cong L \).

(ii) Assume \( m \geq 2 \) (resp. \( m = 1 \)). Then the isomorphism \( \sigma_m : K_m \cong L_m \) (resp. \( \sigma : K \cong L \) induced by \( \sigma_1 : K_1 \cong L_1 \)) in (i) is uniquely determined by the property that \( \tau_m(g) = \sigma_m g \sigma_m^{-1} \) for every \( g \in G_K^m \), where \( \tau_m : G_K^m \cong G_L^m \) is the isomorphism induced by \( \tau_{m+3} \).

Theorem 1 is not functorial in the sense that we do not know a priori how an isomorphism \( \sigma \) in the underlying statement relates to the isomorphism \( \tau_3 \). In this respect Theorem 1 is a sharpening of the weak version of the Neukirch-Uchida theorem originally proved in [Neukirch1]. The uniformity assertion in the Neukirch-Uchida theorem (existence and uniqueness of \( \sigma \) therein) was established by Uchida in [Uchida1], [Uchida2], using among others Neukirch’s results. Uchida also removed the assumption originally imposed by Neukirch that at least one of \( K \) and \( L \) is Galois over the prime field \( \mathbb{Q} \).

Theorem 2 above is functorial. It implies in particular that the isomorphy type of \( K \) is functorially determined by the isomorphy type of \( G_K^4 \).

As in the proof of the Neukirch-Uchida theorem, a key step in the proofs of Theorems 1 and 2 is the establishment of the so-called local theory, i.e., starting from an isomorphism of (quotients of) absolute Galois groups of number fields one establishes a one-to-one correspondence between the sets of their nonarchimedean primes and one between the corresponding decomposition groups. The latter is usually achieved via a purely group-theoretic characterisation of decomposition groups. (In the proof of the Neukirch-Uchida theorem decomposition groups are characterised group-theoretically using the method of Brauer groups.) In our context we prove the following (cf. Theorem 1.25 and Corollary 1.27 (i)).

**Theorem 3.** Let \( m \geq 2 \) (resp. \( m = 1 \)) be an integer. Then one can reconstruct group-theoretically the set of nonarchimedean primes of \( K_m \) (resp. \( K \)) and the set
of decomposition groups of $G_K^{m+2}$ at those primes starting from the profinite group $G_K^{m+2}$.

Before proving Theorem 3, we establish a certain separatedness result that enables one to recover the set of nonarchimedean primes of $K_m$ once one has recovered the set of decomposition groups in $G_K^m$. More precisely, we prove that the natural surjective map from the set of nonarchimedean primes of $K_m$ to the set of decomposition groups in $G_K^m$ is bijective if $m \geq 2$ (cf. Corollary 1.6).

To conclude the proof of Theorem 1, we use Theorem 3 and resort to a recent result of Cornelissen et al. in [Cornelissen-de Smit-Li-Marcolli-Smit]. In order to apply this result, one needs, in addition to recovering the decomposition groups in $G_K^1$ starting from $G_K^3$ (as in Theorem 3), to recover the Frobenius elements in these decomposition groups (modulo inertia groups). One of the key technical results we establish to that effect is that one can recover group-theoretically the cyclotomic character of $G_K^1$ starting from $G_K^3$ (cf. Theorem 1.26).

Concerning the proof of Theorem 2. Once the above local theory in our context is established (cf. Theorem 3), the rest of the proof of Theorem 2 is somewhat similar to that of the Neukirch-Uchida theorem with some necessary adjustments. It does not rely on any of the results in [Cornelissen-de Smit-Li-Marcolli-Smit] mentioned above.

In §4, we establish the result that, for every prime number $l$, one can recover group-theoretically the $l$-part of the cyclotomic character of $G_K$ up to twists by finite characters starting from $G_K^2$ (cf. Proposition 4.9). This result is optimal as it is not possible to obtain a similar result starting solely from $G_K^1$. Indeed, it is not possible in general to distinguish (group-theoretically) the $\mathbb{Z}_l$-quotient of $G_K^1$ corresponding to the cyclotomic $\mathbb{Z}_l$-extension of $K$ among the various $\mathbb{Z}_l$-quotients of $G_K^1$. This phenomenon seems to be one of the main reasons why the isomorphy type of $K$ is not encoded in the isomorphy type of $G_K^1$ as is well-known (see [Angelakis-Stevenhagen], for example). In order to prove this result we use the machinery and techniques of Iwasawa theory. Our proof of recovering the $l$-part of the cyclotomic character (up to twists by finite characters) relies on a careful analysis of the structure of annihilators of certain $\Lambda$-modules, where $\Lambda$ is the (multi-variable in general) Iwasawa algebra associated to the maximal pro-$l$ abelian torsion free quotient of $G_K$.

Our results are almost optimal. As mentioned above Theorem 1 does not hold if one starts with an isomorphism $\tau_1 : G_K^1 \sim G_L^1$, as is well-known. The best improvements of Theorems 1 and 2 one can hope for are the following.

Questions. 1) Does the conclusion in Theorem 1 hold if one starts with an isomorphism $\tau_2 : G_K^2 \sim G_L^2$?

2) Can one replace the $m+3$ in Theorem 2 by $m+i$, for some $0 \leq i < 3$?

Relation to other works.

- The main result in [Cornelissen-de Smit-Li-Marcolli-Smit] (mentioned above) states that two global fields are isomorphic if and only if their abelianised Galois groups are isomorphic and some extra (a priori non-Galois-theoretic) conditions hold (cf. loc. cit., Theorem 3.1). Our Theorem 3, together with the fact established in Theorem 1.26 that one can recover the cyclotomic character from $G_K^3$, implies that the information of $K$ needed to apply the main theorem of [Cornelissen-de Smit-Li-Marcolli-Smit] is group-theoretically encoded in $G_K^3$. 
The proof of our Theorem 1 follows then by applying the main theorem of loc. cit.. It would be interesting to investigate how precisely the conditions in the main theorem of loc. cit. relates to the Galois-theoretic information of \( K \) and how precisely this main theorem relates to our Theorem 1. Also, contrary to the proof of Theorem 1, our proof of Theorem 2 does not rely on the main (or any other) result in [Cornelissen-de Smit-Li-Marcolli-Smit]. In particular, Theorem 2(i) for \( m = 0 \) gives an alternative proof of Theorem 1.

- Our main Theorems 1 and 2 are bi-anabelian, i.e., with a reference to two, a priori distinct, number fields. A mono-anabelian version of Theorem 2 would be a version whereby one establishes a purely group-theoretic algorithm which starting solely from an isomorph of \( G^n_{K} \), for suitable \( n \geq 1 \), reconstructs an isomorph of the number field \( K \). In [Hoshi] Hoshi establishes such an algorithm starting from an isomorph of \( G_K \). More generally, he also establishes such an algorithm starting from an isomorph of the maximal prosolvable quotient \( G^\text{sol}_K \) of \( G_K \), under the assumption that the maximal prosolvable extension \( K^\text{sol} \) of \( K \) in \( \overline{K} \) is Galois over the prime field \( \mathbb{Q} \). The algorithm relies crucially on this assumption, as well as the Neukirch-Uchida theorem itself. In fact, Hoshi’s algorithm does not provide an alternative proof of the Neukirch-Uchida theorem but rather uses it in an essential way. In our context, one could adapt Hoshi’s arguments to show, using Theorems 2 and 3, the following (details of proof may be considered in a subsequent work).

Let \( m \geq 0 \) be an integer, and assume that \( K_m \) is Galois over \( \mathbb{Q} \). Then there exists a purely group-theoretic algorithm which starting from \( G^n_{K} \), for a suitable integer \( n \geq m \) that can be made effective, reconstructs functorially the field \( K_m \) together with the natural action of \( G^n_{K} \) on \( K_m \).

- Finally, we mention that the authors prove an (a mono-anabelian) analogue of the main results of this paper for global function fields in positive characteristics (cf. [Saïdi-Tamagawa]).

Future perspectives. The Neukirch-Uchida theorem had several deep and substantial applications in anabelian geometry and has played a prominent role in the theory for more than 40 years. We hope that Theorem 2 will have a similar impact, for example in developing a new anabelian geometry, over finitely generated fields, stemming solely from the arithmetic of \( m \)-step solvable extensions of finitely generated fields and \( m \)-step solvable arithmetic fundamental groups, for small values of \( m \).

Notations.
- Given a finite set \( H \) we write \(|H|\) for its cardinality.
- For a profinite group \( G \) let \([G,G]\) be the closed subgroup of \( G \) which is (topologically) generated by the commutator subgroup of \( G \). We write \( G^{\text{ab}} \defeq G/[G,G] \) for the maximal abelian quotient of \( G \).
- Given a profinite group \( G \), and a prime number \( l \), we write \( G^{(l)} \) for the maximal pro-\( l \) quotient of \( G \), and \( G^{(l')} \) for the maximal prime-to-\( l \) quotient of \( G \).
- Let \( G \) be a profinite group and consider the derived series
  
  \[ \ldots \subset G[i+1] \subset G[i] \subset \ldots \subset G[1] \subset G[0] = G, \]

  where \( G[i+1] = [G[i],G[i]] \), for \( i \geq 0 \), is the \( (i+1) \)-th derived subgroup which is a characteristic subgroup of \( G \). We write \( G^i \defeq G/G[i] \) and refer to it as
A number field is a finite field extension of the field of rational numbers. By definition, \( G[i] = \text{Ker}(G \twoheadrightarrow G^i) \). For \( j \geq i \geq 0 \) we write \( G[j, i] = \text{Ker}(G^j \twoheadrightarrow G^i) = G[j][i] = G[i]^{j-i} \). We write \( G^{\text{sol}} = G/(\cap_{i \geq 0} G[i]) = \lim_{i \to 0} G^i \) and refer to it as the \( (\text{maximal}) \) prosolvable quotient of \( G \).

- Given a profinite group \( G \) we write \( \text{Sub}(G) \) for the set of closed subgroups of \( G \), and \( C(G) \) for the centre of \( G \). For \( H \in \text{Sub}(G) \), we write \( N_G(H) \) for the normaliser of \( H \) in \( G \).
- Given a profinite group \( G \) and a prime number \( l \), we write \( G_l \) an \( l \)-Sylow subgroup of \( G \), which is defined up to conjugation.
- Let \( G \) be a profinite group, \( H \subset G \) a closed subgroup, and \( l \) a prime number. We say that \( H \) is \( l \)-open in \( G \) if an \( l \)-Sylow subgroup of \( H \) is open in an \( l \)-Sylow subgroup of \( G \). We say that \( G \) is \( l \)-infinite if an \( l \)-Sylow subgroup of \( G \) is infinite, or, equivalently, if \( \{1\} \) is not \( l \)-open in \( G \). We say that two subgroups \( H_1, H_2 \subset G \) are commensurable (resp. \( l \)-commensurable) if \( H_1 \cap H_2 \) is open (resp. \( l \)-open) in both \( H_1 \) and \( H_2 \). The intersection of two open (resp. \( l \)-open) subgroups of \( G \) is open (resp. is not \( l \)-open in general). Accordingly, commensurability (resp. \( l \)-commensurability) relation is (resp. is not in general) an equivalence relation on \( \text{Sub}(G) \).
- Given an abelian profinite group \( A \), we write \( \overline{A_{\text{tor}}} \) for the closure in \( A \) of the torsion subgroup \( A_{\text{tor}} \) of \( A \), and set \( A_{\text{tor}}^{l} = A/\overline{A_{\text{tor}}} \). Given a profinite group \( G \), we set \( G_{\text{ab}}^{l} = (G_{\text{ab}})^{l} \).
- Given a field \( K \), we write \( \overline{K} \) an algebraic closure of \( K \), \( K^{\text{sep}} \) for the maximal separable extension of \( K \) contained in \( \overline{K} \), and \( G_K \) for the absolute Galois group \( \text{Gal}(K^{\text{sep}}/K) \) of \( K \).
- Given a field \( K \) and an integer \( m \geq 0 \), we write \( K_m/K \) for the maximal \( m \)-step solvable subextension of \( K^{\text{sep}}/K \), which corresponds to the quotient \( G_K \twoheadrightarrow G_K^m \). By definition, we have \( G_{K_m} = G_K[m] \).
- Given a field \( K \), and \( H \subset \text{Aut}(K) \) a group of automorphisms of \( K \), we write \( K^H \subset K \) for the subfield of \( K \) which is fixed under the action of \( H \).
- A number field is a finite field extension of the field of rational numbers \( \mathbb{Q} \). For an (possibly infinite) algebraic extension \( F \) of \( \mathbb{Q} \), we write \( \text{Primes}_F \) (resp. \( \text{Primes}_F^{\text{na}} \)) for the set of primes (resp. nonarchimedean primes) of \( F \). We often identify \( \text{Primes}_F^{\text{na}} \) with the set of prime numbers. For \( Q \subset F \subset F' \subset \overline{Q} \) and \( p \in \text{Primes}_F^{\text{na}} \), we write \( p_F \in \text{Primes}_F^{\text{na}} \) for the image of \( p \) in \( \text{Primes}_F^{\text{na}} \). Further, for a set of primes \( S \subset \text{Primes}_F \), we write \( S(F') = \{ p_F \in \text{Primes}_{F'} | p_F \in S \} \).
- For \( Q \subset F \subset F' \subset F'' \subset \overline{Q} \) subfields with \( F'/F \) Galois and \( p \in \text{Primes}_F^{\text{na}} \), write \( D_p(F'/F) \subset \text{Gal}(F'/F) \) for the decomposition group (i.e. the stabiliser) of \( p_{F'} \in \text{Primes}_{F'}^{\text{na}} \) in \( \text{Gal}(F'/F) \). (We sometimes write \( D_p = D_p(F'/F) \), when no confusion arises.) Further, for \( Q \subset F \subset F' \subset \overline{Q} \) subfields with \( F'/F \) Galois, set \( \text{Dec}(F'/F) = \{ D_p(F'/F) \mid p \in \text{Primes}_{F'}^{\text{na}} \} \subset \text{Sub}(\text{Gal}(F'/F)) \). Thus, one has a canonical \( \text{Gal}(F'/F) \)-equivariant surjective map \( \text{Primes}_{F'}^{\text{na}} \twoheadrightarrow \text{Dec}(F'/F) \), \( p \mapsto D_p(F'/F) \).
- Given an algebraic extension \( K \) of \( \mathbb{Q} \) and a nonarchimedean prime \( p \) of \( K \), we write \( \kappa(p) \) for the residue field at \( p \). Further, when \( K \) is a number field, we write \( K_p \) for the completion of \( K \) at \( p \), and, in general, we write \( K_p \) for the
union of $K'_{p_K}$, for finite subextensions $K'/\mathbb{Q}$ of $K/\mathbb{Q}$. For a subfield $M \subset K$, we sometimes write $M_p$ instead of $M_{p,M}$.

- Given a number field $K$ and a nonarchimedean prime $p \in \mathfrak{P}rimes^\text{na}_K$ above a prime $p \in \mathfrak{P}rimes_{\mathbb{Q}}$, $K_p/\mathbb{Q}_p$ is a finite extension. We write $d_p$, $e_p$, $f_p$, and $N(p)$ for the local degree $[K_p : \mathbb{Q}_p]$, the ramification index of $K_p/\mathbb{Q}_p$, the residual degree $|\kappa(p) : \mathbb{F}_p|$, and the norm $|\kappa(p)|$, respectively, where $\mathcal{O}_K$ is the ring of integers of $K$. (Thus, $d_p = e_pf_p$ and $N(p) = p^{f_p}$.)

- Let $K$ be a number field, and $p$ a prime number which splits as $\langle p \rangle = \prod_{i=1}^{k} \mathfrak{p}_i^{e_i}$ in $K$. Define the splitting type of $p$ in $K$ by $(f_1, \ldots, f_k)$ ordered by $f_1 \leq f_2 \leq \cdots \leq f_k$, which is a monotone non-decreasing finite sequence of positive integers. For each monotone non-decreasing finite sequence $A$ of positive integers, write $\mathcal{P}_K(A) \subset \mathfrak{P}rimes^\text{na}_{\mathbb{Q}}$ for the set of prime numbers with splitting type $A$ in $K$. Two number fields $K_1, K_2$ are called arithmetically equivalent if $\mathcal{P}_K_1(A) = \mathcal{P}_K_2(A)$ for every such sequence $A$.

- Let $l$ be a prime number and set $\tilde{l} \overset{\text{def}}{=} l$ (resp. $\tilde{l} \overset{\text{def}}{=} 4$) for $l \neq 2$ (resp. $l = 2$). Then the multiplicative group $\mathbb{Z}_l^\times$ is canonically decomposed into the direct product $\mathbb{Z}_l^\times \cong (1 + \mathbb{Z}_l) \times (\mathbb{Z}_l^\times)_{\text{tor}}$. We denote the first projection $\mathbb{Z}_l^\times \to 1 + \mathbb{Z}_l$ by $\alpha \mapsto \overline{\alpha}$ ($\alpha \in \mathbb{Z}_l^\times$). More explicitly, we have $\overline{\alpha} = \alpha \cdot \lfloor \alpha \text{ mod } \tilde{l} \rfloor^{-1}$, where we denote the Teichmüller lift (i.e. the unique group-theoretic section of $\mathbb{Z}_l^\times \to (\mathbb{Z}/\tilde{l}\mathbb{Z})^\times$) by $\beta \mapsto \lfloor \beta \rfloor$ ($\beta \in (\mathbb{Z}/\tilde{l}\mathbb{Z})^\times$).

- Given a prime number $l$, a profinite group $G$, and a character $\chi : G \to \mathbb{Z}_l^\times$, we write $\overline{\chi} : G \to 1 + \mathbb{Z}_l$ for the character defined by $\overline{\chi}(g) = \overline{\chi(g)}$ ($g \in G$).

- Given a commutative ring $R$, an $R$-module $M$, and a subset $S = \{m_1, m_2, \ldots\}$ of $M$, we write $\langle S \rangle_R = \langle m_1, m_2, \ldots \rangle_R \subset M$ (or simply $\langle S \rangle$ if there is no risk of confusion) for the $R$-submodule of $M$ generated by $S$. Given $x \in M$ we write $\text{Ann}_R(x) \overset{\text{def}}{=} \{r \in R \mid rx = 0 \}$ for the annihilator of $x$ in $R$. We write $M_{R,\text{tor}} \overset{\text{def}}{=} \{m \in M \mid rm = 0 \text{ for some non-zero-divisor } r \in R \}$. Given $a \in R$ we write $(a) = \langle a \rangle_R \subset R$ for the principal ideal of $R$ generated by $a$. An $R$-submodule $N$ of $M$ is called $R$-cofinite if the quotient $M/N$ is a finitely generated $R$-module.

§1. The local theory.
In this section we establish the local theory necessary to prove Theorems 1 and 2. We use the notations in the Introduction.

1.1. Structure of local Galois groups.
Let $K$ be a number field, $p \in \mathfrak{P}rimes^\text{na}_K$ a nonarchimedean prime above a prime $p \in \mathfrak{P}rimes_{\mathbb{Q}}$, $\mathfrak{p}$ a prime of $\overline{K}$ above $p$, and $D_{\mathfrak{p}} \subset G_K$ the decomposition group at $\mathfrak{p}$. Thus, $D_{\mathfrak{p}} \cong \text{Gal}(\overline{K}_{\mathfrak{p}}/K_{\mathfrak{p}})$ is isomorphic to the absolute Galois group of $K_{\mathfrak{p}}$ (cf. [Neukirch-Schmidt-Wingberg], (8.1.5) Proposition). We write $D_{\mathfrak{p}} \twoheadrightarrow D_{\mathfrak{p}}^\text{tame} \twoheadrightarrow D_{\mathfrak{p}}^\text{ur}$ for the maximal tame and unramified quotients of $D_{\mathfrak{p}}$, respectively (cf. loc. cit., discussion before (7.5.2) Proposition), and set $I_{\mathfrak{p}} \overset{\text{def}}{=} \text{Ker}(D_{\mathfrak{p}} \twoheadrightarrow D_{\mathfrak{p}}^\text{ur})$ and $I_{\mathfrak{p}}^\text{tame} \overset{\text{def}}{=} \text{Ker}(D_{\mathfrak{p}}^\text{tame} \twoheadrightarrow D_{\mathfrak{p}}^\text{ur})$. For $m \geq 0$, let $\mathfrak{p}_m$ be the image $\mathfrak{p}_{K_m}$ of $\mathfrak{p}$ in $K_m$ and $D_{\mathfrak{p}_m} \subset G_{K_m}$ the decomposition group at $\mathfrak{p}_m$. Thus, we have a natural surjective homomorphism $D_{\mathfrak{p}} \twoheadrightarrow D_{\mathfrak{p}_m}$ which factors as $D_{\mathfrak{p}} \twoheadrightarrow D_{\mathfrak{p}}^m \twoheadrightarrow D_{\mathfrak{p}_m}$.

Proposition 1.1. Let $m \geq 0$ be an integer. Then the following hold.
(i) The surjective map $D^m_{l_1} \to D_{l_1}$ is an isomorphism. In particular, the natural surjective maps $\text{Gal}(K/K_p)^{ab} \to D^1_p \to D_{l_1}$ are isomorphisms.

(ii) If $m \geq 1$, then $p$ is the unique prime number $l$ such that $\log_{l} |D^m_{l_1}/tor|/l|D^m_{l_1}/tor| \geq 2$.

(iii) If $m \geq 1$, then $d_p = \log_p |D^m_{l_1}/tor|/p|D^m_{l_1}/tor| - 1$.

(iv) If $m \geq 1$, then $f_p = \log_{l_1}(1 + |(D^m_{l_1}/tor)(p^i)|)$, $e_p = d_p/f_p$, and $N(p) = p^{f_p}$.

(v) If $m \geq 1$, then there is a factorisation $D_p \to D_{l_1} \to D^m_{l_1}$.

(vi) If $m \geq 2$, then there is a factorisation $D_p \to D_{l_1} \to D_{l_1}^{\text{tame}}$, and $\text{Ker}(D_{l_1} \to D_{l_1}^{\text{tame}})$ is the maximal normal $p$-subgroup of $D_{l_1}$.

(vii) For each $2 \leq j \leq m$ (resp. $0 \leq j \leq m - 1$), the kernel of the projection $D_{l_1} \to D_{l_1}$ is pro-$p$ (resp. infinite).

(viii) If $m \geq 2$ and $l$ is a prime number, then the inflation maps $H^2(D_{l_1}^{\text{tame}}, \mathbb{F}_l(1)) \to H^2(D_{l_1}, \mathbb{F}_l(1)) \to H^2(D_{l_1}^{\text{tame}}, \mathbb{F}_l(1)) \to H^2(D_{l_1}, \mathbb{F}_l(1))$ are isomorphisms for $l \neq p$, and the inflation map $H^2(D_{l_1}, \mathbb{F}_l(1)) \to H^2(D_{l_1}^{\text{tame}}, \mathbb{F}_l(1)) \to H^2(D_{l_1}, \mathbb{F}_l(1))$ is surjective for $l = p$.

(ix) If $m \geq 2$, $D_{l_1}$ is centre free.

(x) If $m \geq 2$, $D_{l_1}$ is torsion free.

**Proof.** (i) This follows by induction on $m \geq 0$, from the fact that the natural map $D^1_{l_1} \to G^1_K$ is injective (cf. [Gras], III, 4.5 Theorem) and applying this to finite extensions of $K$ corresponding to various open subgroups of $G^1_K$.

(ii)(iii)(iv) These assertions follow immediately from local class field theory.

(v) This follows from the fact that $D^m_{l_1} \simeq \mathbb{Z}$ is abelian.

(vi) This follows from (i), the fact that $D^m_{l_1}$ is metabelian (cf. [Neukirch-Schmidt-Wingberg], (7.5.2) Proposition), and $\text{Ker}(D_{l_1} \to D_{l_1}^{\text{tame}})$ is the maximal normal pro-$p$ subgroup of $D_{l_1}$ (cf. loc. cit., (7.5.7) Corollary (i)).

(vii) Let $2 \leq j \leq m$. Then, by (vi), there is a factorisation $D_{l_1} \to D_{l_1} \to D_{l_1}^{\text{tame}}$, and $\text{Ker}(D_{l_1} \to D_{l_1})$ is a subgroup of the pro-$p$ group $\text{Ker}(D_{l_1} \to D_{l_1}^{\text{tame}})$. Thus, $\text{Ker}(D_{l_1} \to D_{l_1})$ is also pro-$p$. Next, let $0 \leq j \leq m - 1$. Then $\text{Ker}(D_{l_1} \to D_{l_1})$ contains $\text{Ker}(D_{l_1} \to D_{l_1}) = \text{Ker}(D_{l_1} \to D_{l_1}) = D_{l_1}[m - 1]$ ab, where the first equality follows from (i). So, it suffices to prove that $D_{l_1}[i]$ ab is infinite for $i \geq 1$. Let $D_{l_1} \supset I_{l_1} \to I_{l_1}^{\text{tame}}$ be the inertia and the tame inertia groups, and $I_{l_1}^{\text{wild}} = \text{Ker}(I_{l_1} \to I_{l_1}^{\text{tame}})$ the wild inertia group. By (i) and local class field theory, $\text{Im}(I_{l_1} \to G^1_K)$ is a direct product of a finitely generated pro-$p$ abelian group and a prime-to-$p$ finite cyclic group. By (v) (resp. (vi)), for $i \geq 1$ (resp. $i \geq 2$), $D_{l_1}[i] \subset I_{l_1}$ (resp. $D_{l_1}[i] \subset I_{l_1}^{\text{wild}}$). It follows from these that the image of $D_{l_1}[1] = I_{l_1} \cap D_{l_1}[1] \to I_{l_1}^{\text{tame}}$, hence $D_{l_1}[i]$ is ab, and that for $i \geq 2$, $D_{l_1}[i] \subset I_{l_1}^{\text{wild}}$, hence $D_{l_1}[i]$ is a free pro-$p$ group (cf. [Iwasawa], Theorem 2(ii)), and $D_{l_1}[i]$ is ab is a free pro-$p$ abelian group. Thus, it suffices to prove $D_{l_1}[i] \neq \{1\}$. Suppose $D_{l_1}[i] = \{1\}$. Then $D_{l_1} = D_{l_1}^{i}$ is (i-step) solvable. This is absurd, as $I_{l_1}^{\text{wild}} \subset D_{l_1}$ is a free pro-$p$ group of countable rank (cf. loc. cit.), hence is not solvable.

(viii) If $l \neq p$, this follows from the fact that $\text{Ker}(D_{l_1} \to D_{l_1}^{\text{tame}})$, hence $\text{Ker}(D_{l_1} \to D_{l_1})$ and $\text{Ker}(D_{l_1} \to D_{l_1}^{\text{tame}})$ are pro-$p$ groups. If $l = p$, consider the commutative
diagram

\[
\begin{array}{c}
H^1(D_{p_m}, \mathbb{F}_p) \otimes H^1(D_{p_m}, \mathbb{F}_p(1)) \longrightarrow H^2(D_{p_m}, \mathbb{F}_p(1)) \\
\downarrow \\
H^1(D_{\hat{p}}, \mathbb{F}_p) \otimes H^1(D_{\hat{p}}, \mathbb{F}_p(1)) \longrightarrow H^2(D_{\hat{p}}, \mathbb{F}_p(1)) (= \mathbb{F}_p)
\end{array}
\]

where the horizontal maps are cup products. By local Tate duality, the lower horizontal map gives a perfect pairing, hence is surjective (as \(H^1(D_{\hat{p}}, \mathbb{F}_p(1)) = K^\times_p/(K^\times_p)^p \neq 0\)). Thus, it suffices to show that the natural inflation maps \(H^1(D_{p_m}, \mathbb{F}_p(i)) \rightarrow H^1(D_{\hat{p}}, \mathbb{F}_p(i)), i = 0, 1\) are surjective. Let \(\chi^{\text{cycl}}(p) : D_{\hat{p}} \rightarrow \mathbb{F}_p^\times\) be the mod \(p\) cyclotomic character (which factors through \(D_{p_m}\)) and \(\Delta_{\hat{p}} \defeq \text{Im}(\chi^{\text{cycl}}(p))\). Write \(N_{p_m} \defeq \text{Ker}(D_{p_m} \rightarrow \Delta_{\hat{p}}),\) and \(N_{\hat{p}} \defeq \text{Ker}(D_{\hat{p}} \rightarrow \Delta_{\hat{p}})\). Then we have the commutative diagram

\[
\begin{array}{c}
H^1(D_{p_m}, \mathbb{F}_p(i)) \longrightarrow H^1(D_{\hat{p}}, \mathbb{F}_p(i)) \\
\downarrow \\
H^1(N_{p_m}, \mathbb{F}_p(i))^\Delta_{\hat{p}} \longrightarrow H^1(N_{\hat{p}}, \mathbb{F}_p(i))^\Delta_{\hat{p}}
\end{array}
\]

where the horizontal and vertical maps are inflation and restriction maps, respectively. Here, the vertical maps are isomorphisms since \(|\Delta_{\hat{p}}|\) is prime to \(p\) (as it divides \(p - 1\)). The lower horizontal map is also an isomorphism since \(N_{p_m}^{\text{ab}} \cong N_{\hat{p}}^{\text{ab}}\) (use (i) and \(m \geq 2\)), and both \(N_{\hat{p}}\) and \(N_{p_m}\) act trivially on \(\mathbb{F}_p(i)\). Thus, the upper horizontal map is an isomorphism, as desired.

(ix) We prove this by induction on \(m \geq 2\). If \(m = 2\), this follows from [Ladkanai], Theorem 9.3. If \(m > 2\), then it follows from the induction hypothesis for \(m - 1\) that \(C(D_m^m) \subset \text{Ker}(D_m^m \rightarrow D_m^{m-1}) \subset \text{Ker}(D_m^m \rightarrow D_m^{m-2})\). Now, applying the \(m = 2\) case to the maximal metabelian quotients of open subgroups of \(D_m^m\) obtained as the inverse image of various open subgroups of \(D_m^{m-2}\), one sees \(C(D_m^m) = \{1\}\).

(x) For each \(i \geq 1\), \((D_{\hat{p}} : D_{\hat{p}}[i])\) is divisible by \(l^\infty\) for every prime number \(l\), hence \(\text{cd}(D_{\hat{p}}[i]) \leq 1\), which implies that \(D_{\hat{p}}[i + 1, i] = D_{\hat{p}}[i]^{ab}\) is torsion free. (Observe \(D_{\hat{p}}[i]^{ab} = \prod_{l \in \text{Primes}_{\mathbb{Q}}^{\text{na}}} (D_{\hat{p}}[i](l))^{ab}\) and that for each \(l \in \text{Primes}_{\mathbb{Q}}^{\text{na}}, \text{cd}(D_{\hat{p}}[i]) \leq 1\) implies \(\text{cd}(D_{\hat{p}}[i](l)) \leq 1\), hence \(D_{\hat{p}}[i](l)\) is a free pro-\(l\) group.) Similarly, as \(|D_{\hat{p}}^{ur}|\) is divisible by \(l^\infty\) for every prime number \(l\), one has \(\text{cd}(I_{\hat{p}}) \leq 1\). This implies that \(I_{\hat{p}}^{ab}\) (which is a subquotient of \(D_{\hat{p}}^{2}\), as \(D_{\hat{p}}^{ur}\) is abelian) is torsion free. Now, let \(H\) be any finite subgroup of \(D_m^m\). As \(D_m^m \simeq \hat{\mathbb{Z}}\) is torsion free, \(H\) is contained in \(\text{Im}(I_{\hat{p}} \rightarrow D_m^m) = I_{\hat{p}}/(D_{\hat{p}}[m])\). As \(I_{\hat{p}}^{ab}\) is torsion free and \(m \geq 2\), \(H\) is contained in \(\text{Im}(I_{\hat{p}}[1] \rightarrow D_m^m) \subset \text{Im}(D_{\hat{p}}[1] \rightarrow D_m^m) = D_{\hat{p}}[m, 1]\). As \(D_{\hat{p}}[i + 1, i]\) is torsion free for \(i = 1, \ldots, m - 1\), \(H\) must be trivial, as desired. \(\square\)

1.2. Separatedness in \(C_K^m\).

Let \(K\) be a number field, \(p, p' \in \text{Primes}_{\mathbb{Q}}^{\text{na}},\) and \(D_p, D_{p'} \subset G_K\) the decomposition groups at \(p, p'\), respectively. Then it is well-known (cf. [Neukirch-Schmidt-Wingberg], (12.1.3) Corollary) that the following separatedness property holds:

\[
D_p \cap D_{p'} \neq \{1\} \iff p = p'.
\]

This does not hold as it is, if \(\overline{K}\) and \(G_K\) are replaced by \(K_m\) and \(G_K^m\), respectively (cf. Propositions 1.3 and 1.13). However, we show certain weaker separatedness properties hold in the latter situation.
Lemma 1.2. Let $G$ be a profinite group, $A$ an abelian normal closed subgroup of $G$, and $F$ an infinite closed subgroup of $G$. Then, for each open subgroup $H$ of $G$ containing $A$, there exists an open subgroup $H'$ of $H$ containing $A$, such that $\text{Im}(F \cap H' \to (H')^{ab}) \neq \{1\}$.

Proof. As $F$ is infinite and $H$ is open in $G$, $F \cap H$ is nontrivial.

Case 1. $F \cap H \not\subseteq A$, i.e. $\text{Im}(F \cap H \to G/A) \neq \{1\}$. In this case, there exists a normal open subgroup $N$ of $H$ containing $A$, such that $\Phi \overset{\text{def}}{=} \text{Im}(F \cap H \to G/N) \neq \{1\}$. Take a nontrivial abelian (e.g. cyclic) subgroup $\Phi'$ of $\Phi$, and let $H'$ be the inverse image of $\Phi'$ under $H \to H/N$. Then $H'$ is an open subgroup of $H$ containing $N \supset A$, and $\text{Im}(F \cap H' \to G/N) = \Phi'$. Now, as $\Phi'$ is abelian, the natural map $H' \to \Phi'$ factors as $H' \to (H')^{ab} \to \Phi'$. Since $\text{Im}(F \cap H' \to G/N) = \Phi' \neq \{1\}$, one has $\text{Im}(F \cap H' \to (H')^{ab}) \neq \{1\}$, a fortiori.

Case 2. $F \cap H \subseteq A$. In this case, one has

$$A = \bigcap_{A \subset H' \subseteq H \text{ open}} H' = \lim_{A \subset H' \subseteq H \text{ open}} H',$$

hence

$$A = A^{ab} = \lim_{A \subset H' \subseteq H \text{ open}} (H')^{ab}.$$ 

Since $F \cap H \subseteq A$ is nontrivial, this shows that there exists an open subgroup $H'$ of $H$ containing $A$, such that $\text{Im}(F \cap H \to (H')^{ab}) \neq \{1\}$. This completes the proof, as $F \cap H' \supset (F \cap H) \cap H' = F \cap H$. □

Proposition 1.3. Let $m \geq 1$ be an integer, $K$ a number field, $p, p' \in \mathfrak{Primes}_{K,m}^{na}$, $D_p, D_{p'} \subset G_K^m$ the decomposition groups at $p, p'$, respectively, and $\overline{p}, \overline{p'}$ the images of $p, p'$ in $\mathfrak{Primes}_{K_{m-1}}^{na}$, respectively. Then the following are equivalent.

(i) $D_p \cap D_{p'} \neq \{1\}$.

(i') $D_p \cap D_{p'} \cap G_K[m, m - 1] \neq \{1\}$.

(i'') $D_p \cap G_K[m, m - 1]$ and $D_{p'} \cap G_K[m, m - 1]$ are commensurable.

(iii) $D_p \cap G_K[m, m - 1] = D_{p'} \cap G_K[m, m - 1]$.

(ii) $\overline{p} = \overline{p'}$.

Proof. The implications (ii) $\implies$ (i) $\implies$ (i') $\implies$ (ii) and (i') $\implies$ (i) are immediate. The implication (ii') $\implies$ (i') follows from the fact that $D_p \cap G_K[m, m - 1]$ and $D_{p'} \cap G_K[m, m - 1]$ are infinite as follows from Proposition 1.1(vii). The implication (i) $\implies$ (ii) for $m = 1$ follows from [Gras], III, 4.16.7 Corollary. Thus, we prove the implication (i) $\implies$ (ii) for $m \geq 2$. For this, assume that (i) holds, and set $F \overset{\text{def}}{=} D_p \cap D_{p'}$ ($\neq \{1\}$). As $D_p$ is torsion free by Proposition 1.1(x), $F$ is infinite. Let $M/K$ be any finite subextension of $K_{m-1}/K$ and set $H \overset{\text{def}}{=} \text{Gal}(K_m/M) \subset G_K^m$. Then $H$ is an open subgroup of $G_K^m$ containing $A \overset{\text{def}}{=} G_K[m, m - 1] \subset G_K^m$. By Lemma 1.2, there exists an open subgroup $H'$ of $H$ containing $A$, such that $\text{Im}(F \cap H' \to (H')^{ab}) \neq \{1\}$. Set $M' \overset{\text{def}}{=} K_{m'}^H \subset K_{G_K^m[m, m - 1]} = K_{m-1}$ and $M'_1 \overset{\text{def}}{=} K_{m'}^{H'[1]} \subset K_m$. As $M'$ is contained in $K_{m-1}$, $M'_1$ coincides with the maximal abelian extension $(M')^{ab}$ of $M'$. Write $p_{M'_1}, p'_{M'_1}$ for the images of $p, p'$ in $\mathfrak{Primes}_{M'_1}^{na}$.
respectively, $\overline{p}_M$, $\overline{p}'_M'$ for the images of $\overline{p}, \overline{p}'$ in $\Primes_{M'}^{\text{na}}$, respectively, $\overline{p}_M$, $\overline{p}'_M$ for the images of $\overline{p}, \overline{p}'$ in $\Primes_{M}^{\text{na}}$, respectively, and $D, D' \subset (H')^{\text{ab}} = \text{Gal}(M'_1/M')$ for the decomposition groups at $p_{M'_1}, p'_{M'_1}$, respectively. Now, since

$$D \cap D' \supset \text{Im}(D_p \cap D_{p'} \cap H' \rightarrow (H')^{\text{ab}}) \supset \text{Im}(F \cap H' \rightarrow (H')^{\text{ab}}) \neq \{1\},$$

one obtains $\overline{p}_M' = \overline{p}'_M'$ by applying the implication (i) $\implies$ (ii) for $m = 1$ (which we have already established) to the number field $M'$. Thus, one has $\overline{p}_M = \overline{p}'_M$, a fortiori. As $M/K$ is an arbitrary finite subextension of $K_{m-1}/K$, this shows that $\overline{p} = \overline{p}'$, as desired. □

**Lemma 1.4.** Let $G$ be a profinite group and $H$ a closed subgroup of $G$. Let $l$ be a prime number. Consider the following conditions (i)-(v).

(i) $H$ is $l$-open in $G$ (cf. Notations).

(ii) For every $l$-Sylow subgroup $L_H$ of $H$, there exists an $l$-Sylow subgroup $L$ of $G$ containing $L_H$ such that $L_H$ is open in $L$.

(iii) For every $l$-Sylow subgroup $L_H$ of $H$ and every $l$-Sylow subgroup $L$ of $G$ containing $L_H$, $L_H$ is open in $L$.

(iv) $l^n \nmid (G : H)$ i.e. there exists an integer $N > 0$ such that for any surjective homomorphism $\pi : G \rightarrow \overline{G}$ with $\overline{G}$ finite, $l^n \nmid (\overline{G} : \overline{H})$, where $\overline{H} \overset{\text{def}}{=} \pi(H)$.

(v) For any surjective homomorphism $\pi : G \rightarrow \overline{G}$ with $\overline{G}$ almost pro-$l$ (i.e. admitting an open pro-$l$ subgroup), $\overline{H} \overset{\text{def}}{=} \pi(H)$ is open in $\overline{G}$.

Then one has (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv) $\implies$ (v).

**Proof.** Omitted. □

The main result in this subsection is the following.

**Proposition 1.5.** Let $K$ be a number field, and $\tilde{K}/K$ a (an infinite) Galois extension such that $\mathbb{Q}^{\text{ab}} \subset \tilde{K}$. Then the natural surjective map: $\Primes_{\tilde{K}^{\text{ab}}}^{\text{na}} \rightarrow \text{Dec}(\tilde{K}^{\text{ab}}/K)$ is bijective.

**Corollary 1.6.** Let $K$ be a number field and $m \geq 2$ an integer. Then the natural surjective map $\Primes_{K_m}^{\text{na}} \rightarrow \text{Dec}(K_m/K)$ is bijective.

**Proof of Corollary 1.6.** Apply Proposition 1.5 to $\tilde{K} = K_{m-1}$. □

**Corollary 1.7.** With the notations and assumptions in Proposition 1.5, the centraliser of $\text{Gal}(\tilde{K}^{\text{ab}}/K)$ in $\text{Aut}(\tilde{K}^{\text{ab}})$ is trivial, and $\text{Gal}(\tilde{K}^{\text{ab}}/K)$ is centre free. In particular, for $m \geq 2$, the centraliser of $G_{K_m}^{\text{m}}$ in $\text{Aut}(K_m)$ is trivial, and $G_{K_m}^{\text{m}}$ is centre free.

**Proof of Corollary 1.7.** By definition, the natural action (by conjugacy) on $\text{Dec}(\tilde{K}^{\text{ab}}/K)$ of the centraliser of $\text{Gal}(\tilde{K}^{\text{ab}}/K)$ in $\text{Aut}(\tilde{K}^{\text{ab}})$ is trivial, hence, by Proposition 1.5, that on $\Primes_{\tilde{K}^{\text{ab}}}^{\text{na}}$ is trivial. The first assertion follows from this, together with Lemma 1.8 below (applied to $E = F = \tilde{K}$). The second assertion follows from the first. The third and the fourth assertions follow from the first and the second, applied to $\tilde{K} = K_{m-1}$. □
Lemma 1.8. Let $E, F$ be algebraic extensions of $\mathbb{Q}$. Then the natural map

$$\text{Isom}_{\text{(fields)}}(E, F) \rightarrow \text{Isom}_{\text{(sets)}}(\text{Primes}_{E}^{na}, \text{Primes}_{F}^{na})$$

is injective.

Proof. Let $\sigma_1, \sigma_2 \in \text{Isom}_{\text{(fields)}}(E, F)$ and assume that they induce the same bijection $\text{Primes}_{E}^{na} \cong \text{Primes}_{F}^{na}$. Set $\sigma \overset{\text{def}}{=} \sigma_1\sigma_2^{-1} \in \text{Aut}(F)$ and $F_0 \overset{\text{def}}{=} F(\sigma)$. Thus, $F/F_0$ is a (possibly infinite) Galois extension with Galois group $\langle \sigma \rangle$. Let $F'_0/F_0$ be any finite subextension of $F/F_0$. Then $F'_0/F_0$ is a finite cyclic extension with Galois group $\langle \sigma'_0 \rangle$, where $\sigma'_0 \overset{\text{def}}{=} \sigma|_{F'_0}$. Further, there exists a finite subextension $F_{00}/Q$ of $F_0/Q$ (which may depend on $F'_0/F_0$) and a finite cyclic extension $F_{00}'/F_{00}$ contained in $F'_0$ such that $F_{00}' \otimes_{F_{00}} F_0 \sim F_0$. Note that $\text{Gal}(F'_0/F_{00}) = \langle \sigma'_{00} \rangle$, where $\sigma'_{00} = \sigma'_0|_{F_{00}}$.

By assumption, $\sigma$ induces the identity on $\text{Primes}_{F}^{na}$, hence $\sigma'_0$ and $\sigma'_{00}$ induces the identity on $\text{Primes}_{F_0}^{na}$ and $\text{Primes}_{F_{00}}^{na}$, respectively. Now, by Chebotarev’s density theorem, there exists $p \in \text{Primes}_{F_{00}}^{na}$ which splits completely in $F_{00}'/F_{00}$. Then $\text{Gal}(F_{00}'/F_{00}) = \langle \sigma'_{00} \rangle$ acts regularly (i.e. transitively and freely) on the set of primes in $\text{Primes}_{F_{00}}^{na}$ above $p$. As this action is also trivial, one must have $F'_0 = F_0$, hence $F'_0 = F_0$. As $F'_0/F_0$ is arbitrary, we conclude $F = F_0$, i.e. $\sigma = 1$, as desired. □

Proposition 1.5 follows immediately from the implication $(ii) \implies (i)$ in the following Proposition 1.9.

Proposition 1.9. With the notations and assumptions in Proposition 1.5, let $p, p' \in \text{Primes}_{\mathbb{Q}}^{na}$. Write $D_p \overset{\text{def}}{=} D_p(\bar{K}^{ab}/K), D_{p'} \overset{\text{def}}{=} D_{p'}(\bar{K}^{ab}/K)$. Consider the following conditions.

(i) $p = p'$.
(ii) $D_p = D_{p'}$.
(iii) $D_p, D_{p'}$ are commensurable.
(iv) For every prime number $l$, $D_p, D_{p'}$ are $l$-commensurable.
(v) For every prime number $l$ but one $l_0$, $D_p, D_{p'}$ are $l$-commensurable.
(vi) For some prime number $l$, $D_p, D_{p'}$ are $l$-commensurable.
(vii) $p_{\bar{K}} = p'_{\bar{K}}$.

Then one has $(i) \iff (ii) \iff (iii) \iff (iv) \implies (v) \implies (vi) \implies (vii)$.

Proof. Here, the implications $(i) \implies (ii) \implies (iii) \implies (iv) \implies (v) \implies (vi)$ follow immediately, while the implication $(vi) \implies (vii)$ follows from [Gras], III, (4.6.17) Corollary (applied to various finite subextensions of $\bar{K}/K$), as in the proof of Proposition 1.3, $(i) \implies (ii)$. So, we may concentrate on proving the implication $(iv) \implies (i)$. We start with some lemmas.

Lemma 1.10. Let $F$ be a number field, $F'/F$ a finite abelian extension, $q, q' \in \text{Primes}_{F'}^{na}$, with $q \neq q'$. Then there exists a subextension $F''/F$ of $F'/F$ such that $F''/F$ is cyclic of prime power order; $q_{F''} \neq q'_{F''}$; and $q_F, q'_F$ split completely in $F''/F$.

Proof of Lemma 1.10. If $q_F \neq q'_F$, we may take $F'' = F$. So, we may assume $q_F = q'_F$, which implies that there exists $\sigma \in G \overset{\text{def}}{=} \text{Gal}(F'/F)$ such that $q' = \sigma q$. As
\(F'/F\) is abelian, one has \(D \overset{\text{def}}{=} D_q(F'/F) = D_{q'}(F'/F)\). As \(q \neq q'\), one has \(\sigma \not\in D\). Let \(\sigma\) be the image of \(\sigma\) in the quotient group \(G/D\). Write \(G/D \simeq C_1 \oplus \cdots \oplus C_r\), where \(C_i (i = 1, \ldots, r)\) is a cyclic subgroup of prime power order. As \(\sigma \neq 1\), there exists at least one \(i\) such that the image of \(\sigma\) under the projection \(G/D \to C_i\) is nontrivial. Now the subextension \(F''/F\) of \(F'/F\) corresponding to the quotient \(G \to G/D \to C_i\) satisfies the conditions. □

**Lemma 1.11.** Let \(F\) be a number field, and \(q \in \mathcal{P}rimes_F\). For each integer \(m > 0\), write \(\psi_{F,q,m}\) for the natural map \(F^\times/(F^\times)^m \to F_q^\times/(F_q^\times)^m\) induced by the natural inclusion \(F^\times \hookrightarrow F_q^\times\). Then, for each pair of positive integers \(m, n\), the natural map \(\text{Ker}(\psi_{F,q,m}) \to \text{Ker}(\psi_{F,q,m})\) induced by the natural projection \(F^\times/(F^\times)^mn \to F^\times/(F^\times)^m\) is surjective.

**Proof of Lemma 1.11.** Consider the following commutative diagram

\[
\begin{array}{cccc}
F^\times/(F^\times)^n & \rightarrow & F^\times/(F^\times)^{mn} & \rightarrow & F^\times/(F^\times)^m & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
F_q^\times/(F_q^\times)^n & \rightarrow & F_q^\times/(F_q^\times)^{mn} & \rightarrow & F_q^\times/(F_q^\times)^m & \rightarrow & 1
\end{array}
\]

in which the two rows are exact. (Here, the first horizontal arrows are induced by the \(m\)-th power maps and the second horizontal arrows are the natural projections.) Observe that the vertical arrows are surjections. Indeed, \(F_q^\times/k\) is the ring of integers of \(q\), and the image of the natural inclusion \(F^\times \hookrightarrow F_q^\times\) is dense.

Now, the assertion follows from diagram-chasing. □

**Proof of Proposition 1.9, (iv) \implies (i).** Assume that (iv) holds. (Then (vii) holds, a fortiori, i.e. \(p_{\overline{K}} = p_{\overline{K}}'\).) Suppose that (i) does not hold, i.e. \(p 
eq p'\). Then there exists a finite subextension \(L/K\) of \(K_{ab}/K\) such that \(p_L \neq p'_L\). Set \(M \overset{\text{def}}{=} L \cap \overline{K}\).

Then \(M/K\) is a finite subextension of \(K/K\) and \(L/M\) is a finite extension. As \(M \subset \overline{K}\), one has \(p_M = p'_M\). Note that we may replace \(L\) (and \(M\), correspondingly) by any finite extension of \(L\) contained in \(K_{ab}\).

**Step 1.** Reduction: We may assume that \(L/M\) is abelian. Indeed, if we replace \(L\) by the Galois closure \(L_1\) of \(L/M\), and \(M_1 \overset{\text{def}}{=} L_1 \cap \overline{K}\), this holds.

**Step 2.** Reduction: We may assume that \(L/M\) is cyclic of order \(l^r\) for some prime number \(l\) and some integer \(r \geq 0\), and that \(p_M\) splits completely in \(L/M\). Indeed, this follows from Lemma 1.10, applied to \(L/M\).

**Step 3.** Reduction: We may assume that the conditions of Step 2 and the condition that \(\zeta_{l^r} \in M\) hold. Indeed, if we replace \(L\) by \(L(\zeta_{l^r})\) (and \(M\) by \(L(\zeta_{l^r}) \cap \overline{K}\)), this holds with \(r\) replaced by \(r'\) below. First, one has \(M(\zeta_{l^r}) \subset L(\zeta_{l^r}) \cap M_{ab} \subset L(\zeta_{l^r}) \cap \overline{K}\). Next, consider the commutative diagram of fields:

\[
\begin{array}{ccc}
L & \subset & L(\zeta_{l^r}) \\
(\zeta_{l^r}) & \cap & \overline{K} \\
M & \subset & M(\zeta_{l^r})
\end{array}
\]

As \(L/M\) is cyclic of order \(l^r\) and \(p_M\) splits completely in \(L/M\), we see that \(L(\zeta_{l^r})/L(\zeta_{l^r}) \cap \overline{K}\) is cyclic of order \(l^{r'}\) with \(r' \leq r\) and \(p_{L(\zeta_{l^r}) \cap \overline{K}}\) splits completely.
in $L(\zeta_{l^r}) / L(\zeta_{l^r}) \cap \widetilde{K}$. (Observe $\text{Gal}(L(\zeta_{l^r}) / (L(\zeta_{l^r}) \cap \widetilde{K})) \subset \text{Gal}(L(\zeta_{l^r}) / M(\zeta_{l^r})) \hookrightarrow \text{Gal}(L/M)$.) As $\zeta_{l^r} \in M(\zeta_{l^r}) \subset L(\zeta_{l^r}) \cap \widetilde{K}$, we are done.

Step 3 is the final reduction step and we will not replace $L$ (or $M$) any more.

Step 4. $r > 0$. Indeed, otherwise, $L = M$, which contradicts $p_L \neq p'_L$ and $p_M = p'_M$.

Step 5. $L/M$ is linearly disjoint from $M(\zeta_{l^\infty})/M$. Indeed, one has

$$M \subset L \cap M(\zeta_{l^\infty}) \subset L \cap M\mathbb{Q}^{ab} \subset L \cap \widetilde{K} = M.$$ 

Step 6. The $l$-adic cyclotomic character and $l$-commensurability. Let $\chi_{\text{cycl}}^{(l)} : G_K \to \mathbb{Z}_l^\times$ denote the cyclotomic character. As $K(\zeta_{l^\infty}) \subset K\mathbb{Q}^{ab} \subset \widetilde{K}$, $\chi_{\text{cycl}}^{(l)}$ factors through $\text{Gal}(\widetilde{K}/K)$, hence, a fortiori, through $\text{Gal}(\widetilde{K}^{ab}/K)$. By abuse, write $\chi_{\text{cycl}}^{(l)}$ again for the $l$-adic character of $\text{Gal}(\widetilde{K}^{ab}/K)$ induced by $\chi_{\text{cycl}}$. Since the number of $l$-power of unity in each finite extension of $M_p$ is finite (as $M_p$ is a finite extension of $\mathbb{Q}_p$, where $p$ is the characteristic of $\kappa(p)$), $\chi_{\text{cycl}}^{(l)}(D_p(\widetilde{K}^{ab}/M)) \subset \mathbb{Z}_l^\times$ is open. Further, as $D_p$ and $D_p'$ are $l$-commensurable, $\chi_{\text{cycl}}(D_p(\widetilde{K}^{ab}/M) \cap D_p'(\widetilde{K}^{ab}/M)) \subset \mathbb{Z}_l^\times$ is open. So, we may take $\tilde{\sigma} \in D_p(\widetilde{K}^{ab}/M) \cap D_p'(\widetilde{K}^{ab}/M)$ such that $\chi_{\text{cycl}}^{(l)}(\tilde{\sigma}) \neq 1$, and $\chi_{\text{cycl}}^{(l)}(\tilde{\sigma}) = 1 + u_0 l^{r_0}$ with $r_0 \geq 0$ and $u_0 \in \mathbb{Z}_l^\times$. (As $\zeta_{l^r} \in M$, one has $r_0 \geq r > 0$.)

Step 7. Kummer theory. As $L/M$ is a cyclic extension of order $l^r$ and $\zeta_{l^r} \in M$, there exists $f \in M^\times \setminus (M^\times)^l$ such that $L = M(f^{1/l^r})$. As $p_M$ splits completely in $L/M$, one has $f \in (M_p^\times)^{l^r}$. Then the image $f \mod (M^\times)^{l^r}$ of $f$ in $M^\times/(M^\times)^{l^r}$ lies in the kernel of the natural map $\psi_{M_p,M,l^r} : M^\times/(M^\times)^{l^r} \to M_p^\times/(M_p^\times)^{l^r}$. By Lemma 1.11, there exists an element $\gamma$ of the kernel of $\psi_{M_p,M,l^r} : M^\times/(M^\times)^{l^r+r_0} \to M_p^\times/(M_p^\times)^{l^r+r_0}$ which maps to $f \mod (M^\times)^{l^r}$ under the projection $M^\times/(M^\times)^{l^r+r_0} \to M^\times/(M^\times)^{l^r}$. Take any $g \in M^\times$ whose image in $M^\times/(M^\times)^{l^r+r_0}$ is $\gamma$.

As $f \mod (M^\times)^{l^r} = g \mod (M^\times)^{l^r} \in M^\times/(M^\times)^{l^r}$, one has $L = M(f^{1/l^r}) = M(g^{1/l^r})$. Set $M' \stackrel{\text{def}}{=} M(\zeta_{l^{r+r_0}})$, $L' \stackrel{\text{def}}{=} LM' = M'(f^{1/l^r}) = M'(g^{1/l^r})$, and $L'' \stackrel{\text{def}}{=} M'(g^{1/l^r+r_0})$. Then one has $M' \subset M\mathbb{Q}^{ab} \subset \widetilde{K}$. By Step 5, $L'/M'$ is cyclic of order $l^r$, hence $g \in (M')^\times \setminus ((M')^\times)^l$. This, together with the fact that $\zeta_{l^{r+r_0}} \in M'$, implies $L''/M'$ is cyclic of order $l^{r+r_0}$. Further, as $g \in M^\times$, the extension $L'/M'$ is Galois. By the choice of $g$, the image of $g$ in $M_p^\times/(M_p^\times)^{l^r+r_0}$ is trivial, hence the image of $g$ in $(M_p')^\times/((M_p')^\times)^{l^{r+r_0}}$ is trivial, a fortiori. Thus, $p_M$ splits completely in $L''/M'$.

Step 8. End of proof. Consider the following commutative diagram

$$1 \to \text{Gal}(L''/M') \to \text{Gal}(L''/M) \to \text{Gal}(M'/M) \to 1$$

$$1 \to D_p(L''/M') \to D_p(L''/M) \to D_p(M'/M) \to 1$$

in which the two rows are exact. Here, as $p_M$ splits completely in $L''/M'$, $D_p(L''/M') = 1$. Hence the projection $\text{Gal}(L''/M) \to \text{Gal}(M'/M)$ induces an isomorphism $D_p(L''/M) \cong D_p(M'/M)$. Also, one has a natural injection $\text{Gal}(M'/M) \hookrightarrow (\mathbb{Z}/l^{r+r_0}\mathbb{Z})^\times$ induced by the modulo $l^{r+r_0}$ cyclotomic character, and an isomorphism $\text{Gal}(L''/M') \cong$
There exists a Galois module $\mathbb{Z}/l^{r+\tau_0}\mathbb{Z}(1)$ as $\text{Gal}(M'/M)$-modules. (Indeed, observe the commutative diagram

\[
\begin{array}{ccc}
M^\times / (M^\times)^{l^{r+\tau_0}} & \ni & H^1(G_M, \mathbb{Z}/l^{r+\tau_0}\mathbb{Z}(1)) \\
\downarrow & & \downarrow \\
((M')^\times / ((M')^\times)^{l^{r+\tau_0}})_{\text{Gal}(M'/M)} & \ni & H^1(G_{M'}, \mathbb{Z}/l^{r+\tau_0}\mathbb{Z}(1))_{\text{Gal}(M'/M)} \\
& \text{Hom}_{\text{Gal}(M'/M)}(\sigma_{ab}, \mathbb{Z}/l^{r+\tau_0}\mathbb{Z}(1)) & \ni \\
& \text{Isom}_{\text{Gal}(M'/M)}(\text{Gal}(L''/M'), \mathbb{Z}/l^{r+\tau_0}\mathbb{Z}(1)) & \ni \\
\end{array}
\]

in which the horizontal isomorphisms arise from Kummer theory, the vertical arrows are induced by the natural inclusion $M \hookrightarrow M'$, and the equality follows from the fact that $\zeta_{l^{r+\tau_0}} \in M'$. Here, the image of $g$ mod $(M^\times)^{l^{r+\tau_0}} \in (M^\times)^{l^{r+\tau_0}}$ in $H^1(G_{M'}, \mathbb{Z}/l^{r+\tau_0}\mathbb{Z}(1))_{\text{Gal}(M'/M)}$ lies in $\text{Isom}_{\text{Gal}(M'/M)}(\text{Gal}(L''/M'), \mathbb{Z}/l^{r+\tau_0}\mathbb{Z}(1))$, hence gives a $\text{Gal}(M'/M)$-equivariant isomorphism $\text{Gal}(L''/M') \cong \mathbb{Z}/l^{r+\tau_0}\mathbb{Z}(1)$.

Define $\sigma \in D_p(L''/M') \cap D_p'(L''/M')$ to be the image of $\bar{\sigma} \in D_p(\tilde{K}_{ab}/M) \cap D_p'(\tilde{K}_{ab}/M)$ under the natural surjective homomorphism $\text{Gal}(\tilde{K}_{ab}/M) \to \text{Gal}(L''/M')$. Note that the image of $\sigma$ in $\text{Gal}(M'/M) \hookrightarrow (\mathbb{Z}/l^{r+\tau_0}\mathbb{Z})^\times = (\mathbb{Z}/l^{r+\tau_0}\mathbb{Z})^\times$ is $1 + u_0 l^{\tau_0}$ mod $l^{r+\tau_0}$, which acts on $\text{Gal}(L''/M') \cong \mathbb{Z}/l^{r+\tau_0}\mathbb{Z}$ by multiplication.

Next, one has $p_{L''} \neq p'_{L''}$ (as $p_L \neq p'_{L''}$) and $p_{M'} = p'_{M'}$ (as $M' \subset \tilde{K}$). Accordingly, there exists $\tau \in \text{Gal}(L''/M') \setminus \{1\}$ such that $p'_{L''} = \tau p_{L''}$, hence $D_p'(L''/M') = \tau D_p(L''/M') \tau^{-1}$. In particular, as $\alpha \in D_p'(L''/M')$, $\tau^{-1} \sigma \tau \in \tau^{-1} D_p(L''/M') \tau = D_p(L''/M')$. Thus, $\sigma, \tau^{-1} \sigma \tau \in D_p(L''/M')$.

As the image of $\tau \in \text{Gal}(L''/M')$ in $\text{Gal}(M'/M)$ is trivial, the images of $\sigma, \tau^{-1} \sigma \tau \in \text{Gal}(L''/M')$ in $\text{Gal}(M'/M)$ coincide with each other. But as the projection $\text{Gal}(L''/M') \to \text{Gal}(M'/M)$ induces an isomorphism $D_p(L''/M') \cong D_p(M'/M)$, we conclude $\sigma = \tau^{-1} \sigma \tau$, hence

$$\tau \in \text{Gal}(L''/M')^{(\sigma)} \cong (\mathbb{Z}/l^{r+\tau_0}\mathbb{Z})^{(1 + u_0 l^{\tau_0})} = l^r \mathbb{Z}/l^{r+\tau_0}\mathbb{Z} = l^r (\mathbb{Z}/l^{r+\tau_0}\mathbb{Z}),$$

i.e., $\tau \in \text{Gal}(L''/M')^{l^r} = \text{Gal}(L''/L')$. This implies $p'_{L'} = \tau p_{L'} = p_{L'}$, a contradiction. $\square$

In Proposition 1.9, the implication $(v) \implies (i)$ fails in general. (For example, it fails in the case $\tilde{K} = K_{m-1}$ treated in Corollary 1.6.) More precisely, we show Proposition 1.13 below. Let $P_0$ be a subset of $\mathfrak{Primes}_Q^{\text{na}}$.

**Definition 1.12.** (i) We say that a profinite group is $P_0$-perfect, if it admits no nontrivial pro-$l$ abelian quotient for any $l \in P_0$.

(ii) We say that a field $F \subset \overline{Q}$ is $P_0$-perfect, if $\text{Gal}(\overline{Q}/F)$ is $P_0$-perfect.

**Proposition 1.13.** Assume $P_0 \subsetneq \mathfrak{Primes}_Q^{\text{na}}$. Then, with the assumptions in Proposition 1.5, the following are equivalent.

(i) For each pair $p, p' \in \mathfrak{Primes}_K^{\text{na}}$ such that $D_p, D_{p'}$ are $l$-commensurable for every $l \in \mathfrak{Primes}_Q^{\text{na}} \setminus P_0$, one has $p = p'$.

(ii) $\tilde{K}$ is $P_0$-perfect.
Proof. The implication (ii) \(\implies\) (i) just follows from the proof of Proposition 1.9, (iv) \(\implies\) (i). (The assumption \(P_0 \subsetneq \Primes^a_Q\) is used to ensure (vii) there, via (vi) there.) More precisely, the only new input is the observation that, under the assumption that (ii) holds, \(l\) in Step 2 there automatically belongs to \(\Primes^a_Q \setminus P_0\). Indeed, otherwise, i.e., if \(l \in P_0\), one must have \(L \subset \tilde{K}\), since \(L/M\) is cyclic of \(l\)-power order, \(M \subset \tilde{K}\), and \(\tilde{K}\) is \(P_0\)-perfect.

To prove the implication (i) \(\implies\) (ii), suppose that \(\tilde{K}\) is not \(P_0\)-perfect. Then there exists \(l_0 \in P_0\) such that \(\tilde{K}\) admits a finite cyclic extension \(\tilde{L}\) of degree \(l_0\), which then descends to a finite cyclic extension \(L/M\) of degree \(l_0\) over some field \(M\) with \(K \subset M \subset \tilde{K}\) and \([M : K] < \infty\). (Thus, in particular, \(\tilde{L} = L\tilde{K}\).) By Chebotarev’s density theorem, \(\mathfrak{p}_M\) splits completely in \(L/M\) for some \(\mathfrak{p} \in P_{\tilde{K}ab}\).

We fix such \(\mathfrak{p}\).

Next, as \(D_p(\tilde{K}ab/M)\) is a quotient of \(G_{M_p}\), it is prosolvable. So, there exists an \(l_0\)-Hall subgroup \(D\) of \(D_p(\tilde{K}ab/M)\), that is, \(D\) is pro-prime-to-\(l_0\) and \(D_p(\tilde{K}ab/M) : D\) is a (possibly infinite) power of \(l_0\).

Now, consider the exact sequence of profinite abelian groups

\[
1 \to \Gal(\tilde{K}ab/L) \to \Gal(\tilde{K}ab/K) \to \Gal(\tilde{L}/\tilde{K}) \to 1,
\]

which induces an exact sequence of pro-\(l_0\) abelian groups

\[
1 \to \Gal(\tilde{K}ab/L_{l_0}) \to \Gal(\tilde{K}ab/K_{l_0}) \to \Gal(\tilde{L}/\tilde{K}) \to 1,
\]

where \(()_{l_0}\) refers to the \(l_0\)-Sylow subgroups. As \(\Gal(\tilde{L}/\tilde{K}) = \Gal(L\tilde{K}/\tilde{K}) \cong \Gal(L/M)\), the Galois group \(\Gal(\tilde{K}ab/M)\) acts (by conjugation) trivially on \(\Gal(\tilde{L}/\tilde{K})\), hence, a fortiori, so does \(D \subset D_p(\tilde{K}ab/M) \subset \Gal(\tilde{K}ab/M)\). But as \(D\) is a pro-prime-to-\(l_0\) group, the sequence obtained by taking the \(D\)-fixed parts

\[
1 \to (\Gal(\tilde{K}ab/L)_{l_0})^D \to (\Gal(\tilde{K}ab/K)_{l_0})^D \to \Gal(\tilde{L}/\tilde{K})^D = \Gal(\tilde{L}/\tilde{K}) \to 1
\]

remains exact. Thus, a generator of \(\Gal(\tilde{L}/\tilde{K}) \simeq \mathbb{Z}/l_0\mathbb{Z}\) lifts to an element \(\tau\) of \((\Gal(\tilde{K}ab/K)_{l_0})^D \subset \Gal(\tilde{K}ab/K)^D\). As the image of \(\tau\) in \(\Gal(L/M)\) is nontrivial and \(\mathfrak{p}_M\) splits completely in \(L/M\), it follows that \(\tau \notin D_p(\tilde{K}ab/M)\), or, equivalently, 

\[\mathfrak{p}' \overset{\text{def}}{=} \tau \mathfrak{p} \neq \mathfrak{p}.\]

On the other hand, as \(D\) fixes (i.e. commutes with) \(\tau\), one has \(\tau D\tau^{-1} = D\). Note that \(D\) (resp. \(\tau D\tau^{-1} = D\)) is an \(l_0\)-Hall subgroup of \(D_p(\tilde{K}ab/M)\) (resp. \(D_p(\tilde{K}ab/M)\tau^{-1} = D_{\mathfrak{p}'}(\tilde{K}ab/M)\)). Namely, \(D\) is an \(l_0\)-Hall subgroup of both \(D_p(\tilde{K}ab/M)\) and \(D_{\mathfrak{p}'}(\tilde{K}ab/M)\), which implies that \(D_p(\tilde{K}ab/M)\) and \(D_{\mathfrak{p}'}(\tilde{K}ab/M)\) are \(l\)-commensurable for every \(l \in \Primes^a_Q \setminus \{l_0\}\), hence, in particular, for every \(l \in \Primes^a_Q \setminus P_0\). This contradicts (i), as \(\mathfrak{p}' \neq \mathfrak{p}\). \(\square\)

1.3. Correspondence of decomposition groups.

Finally, we establish local theory in our context. More precisely, starting from an isomorphism of the maximal \((m + 2)\)-step solvable Galois groups of number fields, we establish a one-to-one correspondence between the sets of their nonarchimedean primes and a one-to-one correspondence between the corresponding decomposition groups (cf. Corollary 1.27). This is achieved via a purely group-theoretic characterisation of decomposition groups (cf. Theorem 1.25).

As in the proof of the Neukirch-Uchida theorem, our proof is based on the following local-global principle.

\[\]
**Proposition 1.14.** Let $K$ be a number field, and $l$ a prime number. For a prime $p \in \text{Primes}_K$, let $D_p \subset G_K$ be a decomposition group at $p$ ($D_p$ is only defined up to conjugation). Then the following hold.

(i) If $p \notin \text{Primes}^n_K$, then $H^2(D_p, \mathbb{F}_l(1)) \xrightarrow{\sim} \mathbb{F}_l$.

(ii) If $l$ is odd or $K$ is totally imaginary, then there exists a natural injective homomorphism

$$\psi_K : H^2(G_K, \mathbb{F}_l(1)) \to \bigoplus_p H^2(D_p, \mathbb{F}_l(1)),$$

where the direct sum is over all nonarchimedean primes $p \in \text{Primes}^n_K$.

*Proof.* Assertions (i) and (ii) are well-known (cf. [Neukirch-Schmidt-Wingberg], (8.1.5) Proposition, (7.1.8) Theorem (ii), (9.1.10) Corollary, and the fact that $H^2(D_p, \mathbb{F}_l(1)) = 0$ if $p$ is archimedean and either $l \neq 2$ or $K$ is totally imaginary). \[Q.E.D.\]

**Corollary 1.15.** Let $L$ be an algebraic extension of $\mathbb{Q}$, and $l$ a prime number. For a prime $p \in \text{Primes}_L$, let $D_p \subset G_L$ be a decomposition group at $p$ ($D_p$ is only defined up to conjugation). Then the following hold.

(i) If $p \notin \text{Primes}^n_L$, then $H^2(D_p, \mathbb{F}_l(1)) \xrightarrow{\sim} \mathbb{F}_l$ with $\epsilon \leq 1$. Further, $\epsilon = 1$ if and only if $D_p$ is $l$-open in $D_p$, where $p \notin \text{Primes}$ is the characteristic of $\kappa(p)$.

(ii) If $l$ is odd or $L$ is totally imaginary, then there exists a natural injective homomorphism

$$\psi_L : H^2(G_L, \mathbb{F}_l(1)) \to \prod_p H^2(D_p, \mathbb{F}_l(1)),$$

where the product is over all nonarchimedean primes $p \in \text{Primes}^n_L$.

*Proof.* The first half of assertion (i) and assertion (ii) are reduced to assertion (i) and assertion (ii) of Proposition 1.14, respectively, by taking the inductive limits. The second half of assertion (i) also follows from assertion (i) of Proposition 1.14, and the fact that for open subgroups $D' \subset D \subset D_p$ ($\xrightarrow{\sim} G_{Q_p}$), the restriction map from $H^2(D, \mathbb{F}_l(1)) = \mathbb{F}_l$ to $H^2(D', \mathbb{F}_l(1)) = \mathbb{F}_l$ is the $(D : D')$-multiplication. \[Q.E.D.\]

The following lemma will be of important use later.

**Lemma 1.16.** Let $G$ be a profinite group, $m \geq 0$ an integer and $N$ a finite discrete $G^m$-module. For a closed subgroup $F \subset G^m$, write $\tilde{F}$ (resp. $\tilde{F}_1$) for the inverse image of $F$ in $G$ (resp. in $G^{m+1}$). Then the natural map $H^2(F, N) \xrightarrow{\text{def}} \text{Im}(H^2(F, N) \xrightarrow{\text{inf}} H^2(\tilde{F}_1, N)) \xrightarrow{\text{inf}} H^2(\tilde{F}, N)$ is injective.

*Proof.* We have the following commutative diagram

$$
\begin{array}{ccc}
H^1(G[m], N) & \xrightarrow{F} & H^2(F, N) \\
\uparrow & & \uparrow \\
H^1(G[m + 1, m], N) & \xrightarrow{F} & H^2(F, N)
\end{array}
$$

where the horizontal sequences are the exact sequences arising from the Hochschild-Serre spectral sequences, the vertical maps are inflation maps, $G[m] = \text{Ker}(G \to G^m) = \text{Ker}(\tilde{F} \to F)$, and $G[m + 1, m] = \text{Ker}(G^{m+1} \to G^m) = \text{Ker}(\tilde{F}_1 \to F)$. The left vertical map is an isomorphism since $G[m]^{ab} \xrightarrow{\sim} G[m + 1, m]$ and both $G[m]$
and \( G[m + 1, m] \) act trivially on \( N \). The middle vertical map is an isomorphism since it is the identity. Now, our assertion follows by an easy diagram chasing. \( \Box \)

The following result is the main step towards establishing the desired local theory in our context.

**Proposition 1.17.** Let \( K \) be a number field, and \( l \) a prime number. Let \( F \subset G_K^m \) be a closed subgroup. (We use the notations in Lemma 1.16 for \( G = G_K^m \) and \( F \).)

Set \( L \overset{\text{def}}{=} (K_m)^F \), and assume that either \( l \) is odd or \( L \) is totally imaginary. For each nonarchimedean prime \( \tilde{p} \in \text{Primes}^n_L \), write \( \tilde{F}_{\tilde{p}} \subset F = G_L \) for a decomposition group at \( \tilde{p} \) (defined up to conjugation in \( \tilde{F} \)). Then there exists a natural injective homomorphism

\[
\mathcal{H}^2(F, F(1)) \to \prod_{\tilde{p}} H^2(\tilde{F}_{\tilde{p}}, F(1)),
\]

where the product is over all nonarchimedean primes \( \tilde{p} \in \text{Primes}^n_L \).

**Proof.** We have natural maps

\[
\mathcal{H}^2(F, F(1)) \to H^2(\tilde{F}, F(1)) \to \prod_{\tilde{p}} H^2(\tilde{F}_{\tilde{p}}, F(1))
\]

where the first map is an inflation map and the second map is a product of restriction maps. The first map is injective by Lemma 1.16 and the second map is injective by Corollary 1.15 (ii). Thus, the composite is also injective, as desired. \( \Box \)

**Definition 1.18.** Let \( l \) be a prime number. Given a profinite group \( F \), we say that \( F \) is an \( l \)-decomposition like group if there exists an exact sequence \( 1 \to F_1 \to F \to F_2 \to 1 \) where \( F_1, F_2 \) are free pro-\( l \)-of rank 1 (i.e., isomorphic to \( \mathbb{Z}_l \)).

**Lemma 1.19.** Let \( F \) be an \( l \)-decomposition like group and \( D \) a closed subgroup of \( F \). Consider the following conditions:

(i) \( D = F \).
(ii) The restriction map \( H^2(F, \mathbb{F}_l) \to H^2(D, \mathbb{F}_l) \) is an isomorphism.
(iii) The restriction map \( H^2(F, \mathbb{F}_l) \to H^2(D, \mathbb{F}_l) \) is nontrivial.
(iv) \( D \) is open in \( F \).
(v) \( D \) is an \( l \)-decomposition like group.
(vi) \( H^2(D, \mathbb{F}_l) \simeq \mathbb{F}_l \).
(vii) \( H^2(D, \mathbb{F}_l) \neq 0 \).
(viii) \( \text{cd}_l(D) = 2 \).

Then one has \( (i) \iff (ii) \iff (iii) \iff (iv) \iff (v) \iff (vi) \iff (vii) \iff (viii) \).

**Proof.** The implications (i) \( \iff \) (ii), (i) \( \iff \) (iv), (vi) \( \iff \) (vii) are trivial. We prove (v) \( \implies \) (vi). As \( D \) is an \( l \)-decomposition like group, there exists an exact sequence \( 1 \to D_1 \to D \to D_2 \to 1 \) with \( D_1, D_2 \simeq \mathbb{Z}_l \). Then by the Hochschild-Serre spectral sequence and the fact that \( \text{cd}_l(\mathbb{Z}_l) = 1 \) and \( H^1(\mathbb{Z}_l, \mathbb{F}_l) \simeq \mathbb{F}_l \), one has \( H^2(D, \mathbb{F}_l) \to H^1(D_2, H^1(D_1, \mathbb{F}_l)) \simeq H^1(D_2, \mathbb{F}_l) \simeq \mathbb{F}_l \), as desired. (For the second isomorphism, observe that the isomorphism \( H^1(D_1, \mathbb{F}_l) \simeq \mathbb{F}_l \) is automatically \( D_2 \)-equivariant, since any homomorphism from the pro-\( l \) group \( D_2 \) to \( \mathbb{F}_l^\times \) is trivial.) In particular, by applying (v) \( \implies \) (v) to \( D = F \), one has \( H^2(F, \mathbb{F}_l) \simeq \mathbb{F}_l \). Thus, the implication (ii) \( \implies \) (iii) follows.
Next, as $F$ is an $l$-decomposition like group, there exists an exact sequence $1 \to F_1 \to F \to F_2 	o 1$ with $F_1, F_2 \cong \mathbb{Z}_l$. Set $D_1 \overset{\text{def}}{=} D \cap F_1$ and $D_2 \overset{\text{def}}{=} \text{Im}(D \to F_2)$. Then, if the restriction map from $H^2(F, \mathbb{F}_l) \to H^1(D_2, H^1(F_1, \mathbb{F}_l)))$ to $H^2(D, \mathbb{F}_l) \to H^1(D_2, H^1(D_1, \mathbb{F}_l)))$ is nontrivial, one must have $D_1 = F_1$ and $D_2 = F_2$, hence $D = F$. This shows (iii) $\implies$ (i). Next, if $D$ is open in $F$, then $D_1$ and $D_2$ must be open in $F_1$ and $F_2$, respectively. As any open subgroup of $\mathbb{Z}_l$ is isomorphic to $\mathbb{Z}_l$, this shows (iv) $\implies$ (v). Further, as $H^2(D, \mathbb{F}_l) \to H^1(D_2, H^1(D_1, \mathbb{F}_l)))$, (vii) implies $D_1 \neq 1$ and $D_2 \neq 1$. As any nontrivial closed subgroup of $\mathbb{Z}_l$ is open, this shows (vii) $\implies$ (iv). Finally, as $D$ is an extension of $\to F_2 \cong \mathbb{Z}_l$ by $D_1 \to F_1 \cong \mathbb{Z}_l$, one has $\text{cd}(D) \leq 2$. Thus, (vii) $\iff$ (viii) follows (cf. [Neukirch-Schmidt-Wingberg], (3.3.2) Proposition). This completes the proof. \)

**Definition 1.20.** Let $m \geq 2$ be an integer, $F \subset G_{K}^m$ a closed subgroup, and $l$ a prime number. Then we say that $F$ satisfies condition $(\ast l)$ if the following two conditions hold.

$(\ast l)(a)$ $F$ is an $l$-decomposition like group (cf. Definition 1.18).

$(\ast l)(b)$ With the notations in Definition 1.20, $H^2(F, \mathbb{F}_l) \neq 0$.

**Remark 1.21.** We use the notations in Definition 1.20.

(i) Assume that $F$ satisfies condition $(\ast l)(a)$. Then the $F$-module $\mathbb{F}_1(1)$ is isomorphic to the trivial module $\mathbb{F}_1$, as any homomorphism from the pro-$l$ group $F$ to $\mathbb{F}_1^\times$ is trivial. This, together with Lemma 1.19, (i) $\implies$ (vi), shows that condition $(\ast l)(b)$ is equivalent to saying that $H^2(F, \mathbb{F}_1(1)) \cong \mathbb{F}_l$.

(ii) Condition $(\ast l)$ is group-theoretic in the following sense: One can detect purely group-theoretically whether or not a closed subgroup $F \subset G_{K}^m$ satisfies $(\ast l)$, if we start from (the isomorphy type of) $G_{K}^{m+1}$.

**Proposition 1.22.** Let $m \geq 2$ be an integer, $F \subset G_{K}^m$ a closed subgroup, and $l$ a prime number. Then $F$ satisfies condition $(\ast l)$ if and only if $F$ is an open subgroup of an $l$-Sylow subgroup of the decomposition group $D_p \subset G_{K}^m$ at some $p \in \text{Primes}_{K,m}^{na}$ with residue characteristic $\neq l$. Further, then the image $\overline{p} \in \text{Primes}_{K,m-1}^{na}$ of $p$ is uniquely determined by $F$.

**Proof.** First, we prove the ‘only if’ part of the first assertion. So, assume that $F$ is an open subgroup of an $l$-Sylow subgroup of the decomposition group $D_p \subset G_{K}^m$ at some $p \in \text{Primes}_{K,m}^{na}$ with residue characteristic $p \neq l$. By Proposition 1.1(vi), $F$ is isomorphically mapped onto an open subgroup $\overline{F}$ of an $l$-Sylow subgroup of $D_p^{tame}$ (where $\overline{p}$ is any element of $\text{Primes}_{\overline{F}}$ above $p$) by the surjection $D_p \to D_p^{tame}$. Note that there exists an exact sequence

$$1 \to I_p^{tame} \to D_p^{tame} \to D_p^{ur} \to 1$$

with $D_p^{ur} \cong \mathbb{Z}_l$ and $I_p^{tame} \cong \mathbb{Z}_l^{(p')}$. This implies that $F$ ($\cong \overline{F}$) satisfies $(\ast l)(a)$. Again by Proposition 1.1(vi), there exists a closed subgroup $F'$ of $D_p$ which is isomorphically mapped onto $F$ by the surjection $D_p \to D_p'$. The isomorphism $F' \cong F$ factors as $F' \hookrightarrow \overline{F} \to F$, where $F' \hookrightarrow \overline{F}$ is the natural inclusion and $\overline{F} \to F$ is the natural surjection induced by the surjection $G_K \to G_{K}^m$. Accordingly, the isomorphism $H^2(F, \mathbb{F}_l) \cong H^2(F', \mathbb{F}_l)$ induced by the isomorphism $F' \cong F$ is the composite of the inflation map $H^2(F, \mathbb{F}_l) \to H^2(\overline{F}, \mathbb{F}_l)$ and the restriction map
$H^2(\widetilde{F}, \mathbb{F}_l) \to H^2(F', \mathbb{F}_l)$. In particular, the inflation map $H^2(F, \mathbb{F}_l) \to H^2(\widetilde{F}, \mathbb{F}_l)$ is injective, which implies $H^2(F, \mathbb{F}_l) \cong H^2(\widetilde{F}, \mathbb{F}_l)$. Now, by the fact that $F$ satisfies condition $(*_1)(a)$, together with Lemma 1.19, $(i) \implies (vi)$, one has $H^2(F, \mathbb{F}_l) (\cong H^2(\widetilde{F}, \mathbb{F}_l)) \cong \mathbb{F}_l \neq 0$. Thus, $F$ satisfies $(*_1)$.

Next, we prove the ‘if’ part of the first assertion. So, assume that $F$ satisfies condition $(*_1)$. Let $L \overset{def}{=} (K_m)^F$. We claim that $L$ is totally imaginary. Indeed, otherwise, one has an embedding $L \hookrightarrow \mathbb{R}$, which extends to an embedding $K_m \hookrightarrow \mathbb{C}$. To these embeddings, a homomorphism $\text{Gal}(\mathbb{C}/\mathbb{R}) \to G_K^m$ is associated. As $K_m \supset \mathbb{Q}_m \supset \mathbb{Q}_1 \supset \mathbb{Q}(\sqrt{-1})$, this homomorphism is injective. This is absurd, since $F$ is torsion free as it satisfies $(*_1)(a)$. Together with this, Proposition 1.17 and Remark 1.21(i) imply that we have an injective homomorphism

$$H^2(F, \mathbb{F}_l) \to \prod_{\bar{p}} H^2(\widetilde{F}_{\bar{p}}, \mathbb{F}_l),$$

where the product is over all nonarchimedean primes $\bar{p} \in \mathcal{P}_{\text{Primes}}^n$. This map is nontrivial since $F$ satisfies condition $(*_1)(b)$. Thus, there exists $\bar{p} \in \mathcal{P}_{\text{Primes}}^n$ such that the map $H^2(F, \mathbb{F}_l) \to H^2(\widetilde{F}_{\bar{p}}, \mathbb{F}_l)$ is nontrivial. In particular, the map $H^2(F, \mathbb{F}_l) \to H^2(\widetilde{F}_{\bar{p}}, \mathbb{F}_l)$ and the group $H^2(\widetilde{F}_{\bar{p}}, \mathbb{F}_l(1)) (\cong H^2(\widetilde{F}_{\bar{p}}, \mathbb{F}_l))$ are nontrivial, where $\widetilde{F}_{\bar{p}} \overset{def}{=} \text{Im}(\widetilde{F}_{\bar{p}} \to G_K^m)$ is a decomposition subgroup of $F = \text{Gal}(K_m/L)$ at $\bar{p}$. The former nontriviality, together with Lemma 1.19, $(iii) \implies (i)$, implies that $F = F_{\bar{p}}$. The latter nontriviality, together with Corollary 1.15(i), implies that $\widetilde{F}_{\bar{p}}$ is $l$-open in a decomposition subgroup of $G_{\mathbb{Q}}$ at $p$, where $p$ is the residue characteristic of $\bar{p}$. In particular, $F = F_{\bar{p}}$ is $l$-open in a decomposition subgroup $D_{\bar{p}}$ of $G_K^m$ at $p$ (where $p \in \mathcal{P}_{\text{Primes}}^m$ stands for the image of $\bar{p}$), as desired. Next, take a finite abelian extension $K'$ of $K_{\bar{p}}$ with $[K' : \mathbb{Q}_p] > 1$, which corresponds to an open subgroup $H$ of $D_{\bar{p}}$ containing $D_{\bar{p}}[1]$ (cf. Proposition 1.1(i)). As $F \cap H$ is $l$-open in $H$, the natural map $(F \cap H)^{ab} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \to (H^ab \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$ must be surjective. Here, by Lemma 1.19, $(iv) \implies (v)$, $F \cap H$ is an $l$-decomposition like group, hence (topologically) generated by 2 elements and $\dim_{\mathbb{Q}_l}((F \cap H)^{ab} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) \leq 2$. On the other hand, by local class field theory, together with the fact that $m \geq 2$, one has $\dim(H^ab \otimes_{\mathbb{Z}_l} \mathbb{Q}_l) = 1$ (resp. $[K' : \mathbb{Q}_p] + 1 > 2$) if $l \neq p$ (resp. $l = p$). Therefore, one must have $l \neq p$, as desired.

Finally, the second assertion follows immediately from Proposition 1.3. This completes the proof. □

Let $m \geq 2$ be an integer. For a prime number $l$, write

$$\mathcal{D}_{m,l} \overset{def}{=} \{F \in \text{Sub}(G_K^m) \mid F \text{ satisfies } (*_1)\}.$$  

On the set $\mathcal{D}_{m,l}$ we define a relation $\approx$ as follows. If $F, F' \in \mathcal{D}_{m,l}$ then $F \approx F'$ if and only if for any open subgroup $H \subset G_K^m$ containing $G_K[m, m-1]$, the images of $F \cap H$ and $F' \cap H$ in $H^1 = H^{ab}$ are commensurable. It is easy to see that $\approx$ is an equivalence relation on the set $\mathcal{D}_{m,l}$, and we write $\mathcal{D}_{m,l} \overset{def}{=} \mathcal{D}_{m,l}/\approx$ for the corresponding set of equivalence classes.

**Proposition 1.23.** We use the above notations. Then there exists a natural map $\phi_{m,l} : \mathcal{D}_{m,l} \to \mathcal{P}_{\text{Primes}}^{n_{K,m}}$ with the following properties.
(i) The map \( \phi_{m,l} \) is injective, and induces a bijection \( D_{m,l} \overset{\sim}{\rightarrow} \Primes_{K_{m-1}}^{\mathrm{na},(l')} \), where \( \Primes_{K_{m-1}}^{\mathrm{na},(l')} \subset \Primes_{K_{m-1}}^{\mathrm{na}} \) denotes the set of nonarchimedean primes of \( K_{m-1} \) whose image in \( \Primes_{\mathbb{Q}}^{\mathrm{na}} \) is distinct from \( l \).

(ii) The map \( \phi_{m,l} \) is \( G_{K}^{m} \)-equivariant with respect to the natural actions of \( G_{K}^{m} \) on \( D_{m,l} \) and \( \Primes_{K_{m-1}}^{\mathrm{na}} \). In particular, the action of \( G_{K}^{m} \) on \( D_{m,l} \) factors through \( G_{K}^{m} \).

(iii) Let \( a \in D_{m,l} \) and \( \overline{p} \triangleq \phi_{m,l}(a) \). Then the stabiliser \( \St(a) \triangleq \{ g \in G_{K}^{m-1} \mid g \cdot a = a \} \) coincides with the decomposition group \( D_{\overline{p}} \subset G_{K}^{m-1} \) at \( \overline{p} \).

**Proof.** First, Proposition 1.22 implies that there exists a well-defined surjection \( \tilde{\phi}_{m,l} : \tilde{D}_{m,l} \to \Primes_{K_{m-1}}^{\mathrm{na},(l')} \), where the union is over all prime numbers \( m \). Let \( F,F' \in \tilde{D}_{m,l} \) and assume \( F \approx F' \). By Proposition 1.22, there exist \( p,p' \in \Primes_{K_{m-1}}^{\mathrm{na},(l')} \), such that \( F \) and \( F' \) are open subgroups of \( l \)-Sylow subgroups of \( D_{p} \) and \( D_{p'} \), respectively. Let \( M/K \) be any finite subextension of \( K_{m-1}/K \), which corresponds to an open subgroup \( H \subset G_{K}^{m} \) containing \( G_{K}[m,m-1] \). Write \( \overline{p},\overline{p}' \in \Primes_{K_{m-1}}^{\mathrm{na}} \) and \( p_{M},p'_{M} \in \Primes_{K_{m-1}}^{\mathrm{na}} \), such that \( F \approx F' \), and \( F \cap H \) and \( F' \cap H \) are \( \overline{p} \)-infinite, hence nontrivial (cf. Proposition 1.1(v)). Now, by Proposition 1.3 (i) \( \Rightarrow \) (ii), one has \( p_{M} = p'_{M} \). As \( M/K \) is an arbitrary finite subextension of \( K_{m-1}/K \), this shows \( \overline{p} = \overline{p}' \), as desired.

Next, we prove assertion (i). Assume \( F,F' \in \tilde{D}_{m,l} \) have the same image \( \overline{p} \in \Primes_{K_{m-1}}^{\mathrm{na},(l')} \) via the above map \( \tilde{D}_{m,l} \to \Primes_{K_{m-1}}^{\mathrm{na}} \). This means that there exist \( p,p' \in \Primes_{K_{m-1}}^{\mathrm{na}} \) above the prime \( l \) (which is not above the prime \( l \in \Primes_{\mathbb{Q}}^{\mathrm{na}} \)), with decomposition groups \( D_{p},D_{p'} \subset G_{K}^{m} \), such that \( F,F' \) are \( l \)-open subgroups of \( D_{p},D_{p'} \), respectively (cf. Proposition 1.22). We show \( F \approx F' \). Let \( H \subset G_{K}^{m} \) be any open subgroup containing \( G_{K}[m,m-1] \), and \( M \triangleq K_{m}^{H} \subset K_{m-1} \). Let \( p_{M} \) be the image of \( \overline{p} \) in \( M \). The images of \( F \cap H \) and \( F' \cap H \) in \( H^{1} = G_{M}^{1} \) are \( l \)-open in the decomposition group \( D_{p_{M}} \subset G_{M}^{1} = G_{M}^{1} \) at \( p_{M} \), hence both open in the \( l \)-Sylow subgroup \( D_{p_{M},l} \) of \( D_{p_{M}} \). Thus, they are \( \overline{p} \)-commensurable, and \( F \approx F' \). The second assertion in (i) follows by considering for a nonarchimedean prime \( \overline{p} \in \Primes_{K_{m-1}}^{\mathrm{na}} \) of residue characteristic \( \neq l \), a prime \( p \in \Primes_{K_{m-1}}^{\mathrm{na}} \), and an \( l \)-Sylow subgroup of the decomposition group \( D_{p} \subset G_{K}^{m} \) at \( p \), which satisfies condition (**) Further, \( G_{K}^{m} \) acts naturally on \( \tilde{D}_{m,l} \) via the action on its subgroups by conjugation and \( \tilde{D}_{m,l} \to \Primes_{K_{m-1}}^{\mathrm{na},(l')} \) is clearly \( G_{K}^{m} \)-equivariant. Assertion (ii) follows from this, together with assertion (i). Assertion (iii) follows from assertions (i) and (ii).

Let \( m \geq 2 \) be an integer. For each prime number \( l \) set (cf. Proposition 1.23(iii))

\[
\St(D_{m,l}) \triangleq \{ \St(a) \mid a \in D_{m,l} \} \subset \Sub(G_{K}^{m-1}),
\]

and set

\[
\St(D_{m}) \triangleq \bigcup_{l} \St(D_{m,l}),
\]

where the union is over all prime numbers \( l \).
Proposition 1.24. We use the above notations. Let \( m \geq 2 \) be an integer, and \( l_1 \neq l_2 \) prime numbers. Then one has

\[
\text{Dec}(K_{m-1}/K) = \text{St}(\mathcal{D}_m) = \text{St}(\mathcal{D}_{m,l_1}) \cup \text{St}(\mathcal{D}_{m,l_2})
\]

in \( \text{Sub}(G^m_K) \). In particular, the subset \( \text{Dec}(K_{m-1}/K) \subset \text{Sub}(G^m_K) \) can be recovered group-theoretically from \( G^{m+1}_K \).

Proof. The first assertion follows from the various definitions and Proposition 1.23. The second assertion follows from the first and the fact that for each prime number \( l \), the \( G^{m-1}_K \)-set \( \mathcal{D}_{m,l} \), hence the subset \( \text{St}(\mathcal{D}_{m,l}) \subset \text{Sub}(G^{m-1}_K) \), can be recovered group-theoretically from \( G^{m+1}_K \) (cf. Remark 1.21(ii)). \( \square \)

We have a natural surjective map \( \mathfrak{Primes}^{\text{na}}_{K_{m-1}} \xrightarrow{d_{m-1}} \text{Dec}(K_{m-1}/K) \). The map \( d_{m-1} \) is bijective if \( m \geq 3 \) (cf. Corollary 1.6). Further, if \( m = 2 \), then the surjective map \( \mathfrak{Primes}^{\text{na}}_{K_1} \xrightarrow{d_1} \text{Dec}(K_1/K) \) induces a natural bijective map \( \mathfrak{Primes}^{\text{na}}_{K_2} \xrightarrow{d} \text{Dec}(K_1/K) \) (cf. Proposition 1.3). In summary, as a consequence of Proposition 1.24, we have the following. (Here we make a renumbering by replacing \( m - 1 \) with \( m \)).

Theorem 1.25. Let \( m \geq 2 \) (resp. \( m = 1 \)) be an integer. The \( G^m_K \)-set \( \text{Dec}(K_m/K) \) (resp. the set \( \text{Dec}(K_1/K) \) ), hence the \( G^m_K \)-set \( \mathfrak{Primes}^{\text{na}}_{K_m} \) (resp. the set \( \mathfrak{Primes}^{\text{na}}_{K_2} \) ), can be recovered group-theoretically from \( G^{m+2}_K \).

For an integer \( m \geq 1 \), let \( \chi_{\text{cycl}} : G^m_K \to \mathbb{Z}^\times \) be the cyclotomic character.

Theorem 1.26. Let \( m \geq 3 \) be an integer. Then \( \chi_{\text{cycl}} \) can be recovered group-theoretically from \( G^m_K \).

Proof. We may assume \( m = 3 \), and show that for each prime number \( l \), the \( l \)-part of the cyclotomic character \( \chi^{(l)}_{\text{cycl}} : G^2_K \to \mathbb{Z}^\times_l \) can be recovered group-theoretically from \( G^3_K \). We claim that the following group-theoretic characterisation of \( \chi^{(l)}_{\text{cycl}} \) holds: Let \( \chi : G^2_K \to \mathbb{Z}^\times_l \) be a character. Then, \( \chi = \chi^{(l)}_{\text{cycl}} \) if and only if for every \( F \in \widehat{\mathcal{D}}_{2,l} \), every \( h \in F \cap G_K[2,1] \) and every \( g \in N_{G^2_K}(F \cap G_K[2,1]) \), one has \( ghg^{-1} = h^{\chi(g)} \). (Recall that the definition of the set \( \mathcal{D}_{2,l} \) involves \( G^3_K \)).

To prove this claim, we first determine the normaliser \( N_{G^2_K}(F \cap G_K[2,1]) \) for each \( F \in \widehat{\mathcal{D}}_{2,l} \). By Proposition 1.22, \( F \) is an open subgroup of an \( l \)-Sylow subgroup of the decomposition group \( D_p \subset G^2_K \) at some \( p \in \mathfrak{Primes}^{\text{na}}_{K_2} \) with residue characteristic \( \neq l \), and the image \( \overline{p} \in \mathfrak{Primes}^{\text{na}}_{K_3} \) of \( p \) is uniquely determined by \( F \). Then one has \( N_{G^2_K}(F \cap G_K[2,1]) = D_p \cdot G_K[2,1] = \pi^{-1}(D_{\overline{p}}) \), where \( \pi \) denotes the projection \( G^2_K \to G^1_K \). Indeed, write \( D_p \supset I_p \to I_p^{\text{tame}} \) for the inertia and the tame inertia groups. By Proposition 1.1(v), \( D_p \cap G_K[2,1] \subset I_p \), hence \( F \cap G_K[2,1] \subset (D_p \cap G_K[2,1])_l \sim (D_p \cap G_K[2,1])^{(l)}_p \to I_p^{(l)} \sim (I_p^{\text{tame}})^{(l)} \sim \mathbb{Z}_l \), where \( (D_p \cap G_K[2,1])_l \) stands for the \( l \)-Sylow subgroup of the profinite abelian group \( D_p \cap G_K[2,1] \), and the last isomorphism follows from Proposition 1.1(vi). As \( F \) is an open subgroup of an \( l \)-Sylow subgroup of the decomposition group \( D_p \subset G^2_K \), \( F \cap G_K[2,1] \) is open in \( (D_p \cap G_K[2,1])_l \). By Proposition 1.1(i), \( D_p \sim G^{ab}_{K_p} \), where
$p_0$ is the image of $p$ in $\text{Primes}_{K_2}^{na}$, hence, in particular, (by local class field theory) the $l$-Sylow group of $D_{\overline{p}}$ is isomorphic to the direct product of a finite cyclic group of $l$-power order (that is the $l$-Sylow subgroup of $\kappa(p_0)\times$) and $\mathbb{Z}_l$. It follows from this that the inclusion $(D_p \cap G_K[2,1])^{(l)} \hookrightarrow I_p^{(l)} \simeq (\mathbb{Z}_l)$. Now, since $D_p \cap G_K[2,1] = \text{Ker}(D_p \rightarrow G_K[2,1])$ is normal in $D_p$ and $F \cap G_K[2,1] = ((D_p \cap G_K[2,1]))^{(l)}(D_p \cap G_K[2,1])$ is characteristic in $D_p \cap G_K[2,1], F \cap G_K[2,1]$ is normal in $D_p$, hence $D_p \subset N_{G_K}^g(F \cap G_K[2,1])$. On the other hand, since $G_K[2,1] = G_K[2,1]^{ab}$ is abelian, one has $G_K[2,1] \subset N_{G_K}^g(F \cap G_K[2,1])$. Thus, $D_p \cdot G_K[2,1] \subset N_{G_K}^g(F \cap G_K[2,1])$. Conversely, let $g \in N_{G_K}^g(F \cap G_K[2,1])$. Then $D_p \supset F \cap G_K[2,1] = g(F \cap G_K[2,1])$ implies $D_p g^{-1} = D_{gp}$.

As the inclusion $F \cap G_K[2,1] \hookrightarrow I_p^{(l)} \simeq (\mathbb{Z}_l)$ is open, $F \cap G_K[2,1]$ is nontrivial. Therefore, by Proposition 1.3, $(i) \implies (ii)$, one has $\overline{\mathbb{F}} = \overline{\mathbb{F}}(\pi(g))$, which implies $\pi(g) \in D_{\overline{p}}$. Thus, $g \in \pi^{-1}(D_{\overline{p}})$.

Finally, we prove the ‘if’ part of the claim. For this, it suffices to show that $G_K^1$ is (topologically) generated by $N_{G_K}^g(F \cap G_K[2,1])$ for $F \in D_{\overline{2}}$. As in the preceding arguments, one has $F \subset D_p$ for some $p \in \text{Primes}_{K_2}$ and $N_{G_K}^g(F \cap G_K[2,1]) = D_p \cdot G_K[2,1]$. So, it suffices to prove that $G_K^1$ is generated by $D_{\overline{p}}$, where $\overline{p}$ is the image of $p$ in $\text{Primes}_{K_1}^{na}$. This follows from Chebotarëv’s density theorem, together with (the surjectivity in) Proposition 1.23(i), as desired. \(\square\)

**Corollary 1.27.** Let $m \geq 2$ (resp. $m = 1$) be an integer, $K, L$ number fields, and $\tau_{m+2} : G_K^{m+2} \simeq G_L^{m+2}$ an isomorphism of profinite groups. Let $\tau_m : G_K^m \simeq G_L^m$ be the isomorphism of profinite groups induced by $\tau_{m+2}$.

(i) There exists a unique bijection $\phi_m : \text{Primes}_{K_m}^{na} \simeq \text{Primes}_{L_m}^{na}$ (resp. $\phi : \text{Primes}_{K_m}^{na} \simeq \text{Primes}_{L_m}^{na}$) such that the following diagram commutes

\[
\begin{array}{ccc}
\text{Dec}(K_m/K) & \overset{\tau_m}{\longrightarrow} & \text{Dec}(L_m/L) \\
d_m \uparrow & & \uparrow d_m \\
\text{Primes}_{K_m}^{na} & \overset{\phi_m}{\longrightarrow} & \text{Primes}_{L_m}^{na}
\end{array}
\]

(resp.

\[
\begin{array}{ccc}
\text{Dec}(K_1/K) & \overset{\tau_1}{\longrightarrow} & \text{Dec}(L_1/L) \\
d \uparrow & & \uparrow d \\
\text{Primes}_{K_1}^{na} & \overset{\phi}{\longrightarrow} & \text{Primes}_{L_1}^{na}
\end{array}
\]

where $\tau_m$ is induced by $\tau_m : G_K^m \simeq G_L^m$ (i.e. $\tau_m(D) \overset{\text{def}}{=} \tau_m(D)$), the vertical maps are the natural (bijective) ones (cf. the paragraph before Theorem 1.25). For
$m \geq 2$, $\phi_m$ is compatible with $\tau_m$ and the natural actions of $G_K^m \cdot G_L^m$ on $\text{Primes}_{K_{m}}^{na}$, $\text{Primes}_{L_{m}}^{na}$, respectively, hence, in particular, $\phi_m$ induces a unique bijection $\phi : \text{Primes}_{K}^{na} \to \text{Primes}_{L}^{na}$.

(ii) The bijection $\phi : \text{Primes}_{K}^{na} \to \text{Primes}_{L}^{na}$ fits into the following commutative diagram:

$$\begin{array}{c}
\text{Primes}_{K}^{na} \\
\downarrow \\
\text{Primes}_{Q}^{na} \\
\downarrow \\
\text{Primes}_{Q}^{na} \over\sim \text{Primes}_{Q}^{na}
\end{array}$$

where the vertical maps are the natural ones ($p \mapsto (p \mapsto (\kappa(p)))$).

(iii) Let $p \in \text{Primes}_{K}^{na}$ and $q \overset{\text{def}}{=} \phi(p) \in \text{Primes}_{L}^{na}$. Then $d_p = d_q$, $e_p = e_q$, $f_p = f_q$, and $N(p) = N(q)$. In particular, $K$ and $L$ are arithmetically equivalent (cf. Notations).

(iv) Let $p \in \text{Primes}_{K_{m}}^{na}$ (resp. $p \in \text{Primes}_{K}^{na}$) and $q \overset{\text{def}}{=} \phi_m(p) \in \text{Primes}_{L_{m}}^{na}$ (resp. $q \overset{\text{def}}{=} \phi(p) \in \text{Primes}_{L}^{na}$). Let $D_p, D_q$ be the decomposition subgroups of $G_K^m, G_L^m$, respectively, corresponding to $p, q$, respectively, $I_p, I_q$ the inertia subgroups of $D_p, D_q$, respectively, and $\text{Frob}_p \in D_p/I_p, \text{Frob}_q \in D_q/I_q$ the Frobenius elements. Then the isomorphism $D_p \overset{\sim}{\to} D_q$ induced by $\tau_m$ (cf. (i)) restricts to an isomorphism $I_p \overset{\sim}{\to} I_q$. Further, the induced isomorphism $D_p/I_p \overset{\sim}{\to} D_q/I_q$ maps $\text{Frob}_p$ to $\text{Frob}_q$.

(v) Assume $m \geq 2$. Let $H$ be an open subgroup of $G_K^m$ and $K'/K$ (resp. $L'/L$) the finite subextension of $K_m/K$ (resp. $L_m/L$) corresponding to $H \subset G_K^m$ (resp. $\tau_m(H) \subset G_L^m$). Let $\phi' : \text{Primes}_{K}^{na} \to \text{Primes}_{L}^{na}$ be the bijection induced by the bijection $\phi_m : \text{Primes}_{K_{m}}^{na} \to \text{Primes}_{L_{m}}^{na}$ (cf. (i)). Let $p' \in \text{Primes}_{K}^{na}$, and $q' \overset{\text{def}}{=} \phi'(p') \in \text{Primes}_{L}^{na}$. Then $d_{p'} = d_{q'}$, $e_{p'} = e_{q'}$, $f_{p'} = f_{q'}$, and $N(p') = N(q')$. In particular, $K'$ and $L'$ are arithmetically equivalent.

Proof. (i) This is a precise reformulation of Theorem 1.25, hence follows from Proposition 1.24, together with Corollary 1.6 and Proposition 1.3.

(ii) This follows from (i) and Proposition 1.1 (ii).

(iii) This follows from (i), (ii) and Proposition 1.1 (ii)(iii)(iv).

(iv) By (ii) and Theorem 1.26, the following diagram commutes

$$\begin{array}{c}
G_K^{m+2} \\
\downarrow \chi_{\text{cycl}}^{(p')} \\
(\hat{\mathbb{Z}}^{(p')})^x \\
\chi_{\text{cycl}}^{(p')} \\
\downarrow \chi_{\text{cycl}}^{(p')} \\
G_L^{m+2}
\end{array}$$

where $p$ is the characteristic of $\kappa(p)$ and $\chi_{\text{cycl}}^{(p')}$ stands for the prime-to-$p$ part of the cyclotomic character $\chi_{\text{cycl}}$. Now, since $\text{Ker}(\chi_{\text{cycl}}^{(p')}|D_p) = I_p$ and $\text{Ker}(\chi_{\text{cycl}}^{(p')}|D_q) = I_q$, the first assertion follows, and $\chi_{\text{cycl}}^{(p')}|D_p$ (resp. $\chi_{\text{cycl}}^{(p')}|D_q$) induces an injective map $D_p/I_p \hookrightarrow (\hat{\mathbb{Z}}^{(p')})^x$ (resp. $D_q/I_q \hookrightarrow (\hat{\mathbb{Z}}^{(p')})^x$), which is again denoted by $\chi_{\text{cycl}}^{(p')}$. Since $\text{Frob}_p \in D_p/I_p$ (resp. $\text{Frob}_q \in D_q/I_q$) is characterised by $\chi_{\text{cycl}}^{(p')}|\text{Frob}_p = p^{f_p}$ (resp. $\chi_{\text{cycl}}^{(p')}|\text{Frob}_q = p^{f_q}$), the second assertion follows from (ii) and (iii).
Then \( q \) and \( p \) (resp. \( \tilde{q} \)) are below (resp. is above) \( q' \) (cf. (i) and (ii)). By (i) and (iv), one has \( \tau_m : D_p \xrightarrow{\sim} D_{\tilde{q}} \) and \( \tau_m : I_p \xrightarrow{\sim} I_{\tilde{q}} \), hence

\[
d_{p'} = (D_p : D_p \cap H)d_{\phi} = (D_{\tilde{q}} : D_{\tilde{q}} \cap \tau_m(H))d_{\tilde{q}} = d_{q'},
\]
\[
e_{p'} = (I_p : I_p \cap H)e_{\phi} = (I_{\tilde{q}} : I_{\tilde{q}} \cap \tau_m(H))e_{\tilde{q}} = e_{q'},
\]
\[
f_{p'} = d_{p'}/e_{p'} = d_{q'}/e_{q'} = f_{q'},
\]
\[
N(p') = p_{f_{p'}} = p_{f_{q'}} = N(q'),
\]
where the first and the second formulae follow from (iii), the third formula follows form the first and the second, and the last formula follows from the third. This finishes the proof of Corollary 1.27. □

§2. Proof of Theorem 1.

For a number field \( K \) let \( I_K \) be the multiplicative monoid freely generated by the elements of \( \mathfrak{Primes}_K^{na} \) (cf. [Cornelissen-de Smit-Li-Marcolli-Smit]), endowed with the norm function defined by \( p \mapsto N(p) \) (\( p \in \mathfrak{Primes}_K^{na} \)). Let \( K, L \) be number fields and \( \tau : G_K^3 \xrightarrow{\sim} G_L^3 \) an isomorphism of profinite groups. We show \( K \) and \( L \) are isomorphic. The bijection \( \phi : \mathfrak{Primes}_K^{na} \xrightarrow{\sim} \mathfrak{Primes}_L^{na} \) in Corollary 1.27 (i) induces a norm-preserving bijection \( \tilde{\phi} : I_K \xrightarrow{\sim} I_L \) (cf. Corollary 1.27 (iii)). Let \( N \subset G_K^1 \) be an open subgroup corresponding to the extension \( K'/K \) with \( K' \overset{\text{def}}{=} (K_1)^N \), and \( M \overset{\text{def}}{=} \tau_1(N) \) which corresponds to the extension \( L'/L \) with \( L' \overset{\text{def}}{=} (L_1)^M \). Let \( p \in \mathfrak{Primes}_K^{na} \) and \( q \overset{\text{def}}{=} \phi(p) \). Then \( p \) (resp. \( q \)) is unramified in the extension \( K'/K \) (resp. \( L'/L \)) if and only if the image of \( I_p \) (resp. \( I_q \)) in \( \text{Gal}(K'/K) \) (resp. \( \text{Gal}(L'/L) \)) is trivial. In particular, \( p \) is unramified in \( K'/K \) if and only if \( q \) is unramified in \( L'/L \), and the isomorphism \( \text{Gal}(K'/K) \xrightarrow{\sim} \text{Gal}(L'/L) \) induced by \( \tau_1 : G_K^1 \xrightarrow{\sim} G_L^1 \) maps the image of \( \text{Frob}_p \) to that of \( \text{Frob}_q \) (cf. Corollary 1.27 (iv)). Thus, by the main theorem of [Cornelissen-de Smit-Li-Marcolli-Smit], there exists a field isomorphism \( \sigma : K \xrightarrow{\sim} L \) (cf. loc. cit. Theorem 3.1, the equivalence (ii) \( \iff \) (iv)). This finishes the proof of Theorem 1. □

§3. Proof of Theorem 2.

Let \( m \geq 0 \) be an integer, \( K, L \) number fields, and \( \tau_{m+3} : G_K^{m+3} \xrightarrow{\sim} G_L^{m+3} \) an isomorphism of profinite groups. We show the existence of a field isomorphism \( \sigma_m : K_m \xrightarrow{\sim} L_m \) such that \( \tau_m(g) = \sigma_m g \sigma_m^{-1} \) for every \( g \in G_m^m \), where \( \tau_m : G_K^m \xrightarrow{\sim} G_L^m \) is the isomorphism induced by \( \tau_{m+3} \). We follow Uchida’s method in [Uchida2]. Let \( K'/K \) be a finite Galois subextension of \( K_m/K \) with Galois group \( H \), corresponding to a normal open subgroup \( U \subset G_K^m \), and \( V \overset{\text{def}}{=} \tau_m(U) \) corresponding to a finite Galois subextension \( L'/L \) of \( L_m/L \) with Galois group \( J \). The isomorphism \( \tau_m \) induces naturally an isomorphism \( \tau : H \xrightarrow{\sim} J \). Let \( T(K') \) be the (finite) set of field isomorphisms \( \sigma : K' \xrightarrow{\sim} L' \) such that \( \tau(h) = \sigma h \sigma^{-1} \) for every \( h \in H \). It is easy to see that \( \{T(K')\}_{K'/K} \) forms a projective system, and the projective limit \( \varprojlim T(K') \) consists of isomorphisms \( \sigma_m : K_m \xrightarrow{\sim} L_m \) satisfying the condition in \( K'/K \) Theorem 2. Further, if \( T(K') \neq \emptyset \) for any \( K' \) as above, then the projective limit \( \varprojlim T(K') \) over all such finite sets \( T(K') \) would be nonempty.
Now, the same proof as in [Uchida2] shows that \( T(K') \neq \emptyset \). More precisely, the proof in loc. cit. applies as it is by noting the following. First, one needs to know that \( K' \) and \( L' \) above are arithmetically equivalent, which in our case follows from Corollary 1.27(v). Second, one needs to know that certain finite abelian extensions of \( K' \) and \( L' \) introduced (and denoted by \( \prod_{j=0}^m M_{1,j} \) and \( \prod_{j=0}^m M_{2,j} \)) in loc. cit. are arithmetically equivalent. Since these extensions are contained in \( K_{m+1} \) and \( L_{m+1} \), respectively, and correspond to each other via \( \tau_{m+1} : G_K^{m+1} \cong G_L^{m+1} \), this follows again from Corollary 1.27(v) (applied to \( m + 1 \) and \( m + 3 = (m + 1) + 2 \) instead of \( m \) and \( m + 2 \)). The rest of the proof that \( T(K') \neq \emptyset \) is the same as in loc. cit. This finishes the proof of (i).

Next, assume \( m \geq 1 \) and let \( \sigma_{m,i} : K_m \rightarrow L_m \) be isomorphisms for \( i = 1, 2 \) such that \( \tau_m(g) = \sigma_{m,i} g \sigma_{m,i}^{-1} (i = 1, 2) \). In particular, for each \( i = 1, 2 \), the isomorphism \( \sigma_{m,i} : K_m \rightarrow L_m \) induces an isomorphism \( \sigma_i : K \rightarrow L \). Also, for each \( i = 1, 2 \), the isomorphism \( \sigma_{m,i} : K_m \rightarrow L_m \) induces a bijection \( \phi_{m,i} : \text{Primes}(K_m)^{na} \rightarrow \text{Primes}(L_m)^{na} \), and, for every \( p \in \text{Primes}(K_m)^{na} \), one has

\[
D_{\phi_{m,i}(p)} = \sigma_{m,i} D_{p} \sigma_{m,i}^{-1} = \tau_m(D_{p}).
\]

Thus, if \( m \geq 2 \) (resp. \( m = 1 \)), the bijections \( \phi_{m,i} : \text{Primes}(K_m)^{na} \rightarrow \text{Primes}(L_m)^{na} \) (resp. \( \phi_i : \text{Primes}(K)^{na} \rightarrow \text{Primes}(L)^{na} \) induced by \( \sigma_i : K \rightarrow L \)) for \( i = 1, 2 \) must coincide with each other (cf. Corollary 1.6 (resp. Proposition 1.3)). This, together with Lemma 1.8, finishes the proof of (ii). This finishes the proof of Theorem 2. \( \square \)

§4. Appendix. Recovering the cyclotomic character from \( G_K^2 \).

In this section we use the notations in the Introduction. Fix a prime number \( l \), and set \( \tilde{l} \overset{\text{def}}{=} l \) (resp. \( 4 \)) for \( l \neq 2 \) (resp. \( l = 2 \)) (cf. Notations). In Theorem 1.26, we proved that the cyclotomic character \( \chi_{\text{cycl}} \) (hence its \( l \)-part \( \chi_{\text{cycl}}^{(l)} \)) can be recovered group-theoretically from \( G_K^m \), if \( m \geq 3 \). In this section, we prove that \( \chi_{\text{cycl}}^{(l)} \) can be recovered group-theoretically from \( G_K^2 \), up to twists by finite characters.

For an integer \( m \geq 1 \) let \( G_K^m \rightarrow \Gamma \) be the maximal quotient of \( G_K^m \) which is pro-\( l \) abelian and torsion free. Thus, \( \Gamma \) (depends only on \( G_K^m \) and) is (non-canonically) isomorphic to \( \mathbb{Z}_l^r \) for some integer \( r \) with \( r_2 + 1 \leq r \leq [K : \mathbb{Q}] \), where \( r_2 \) is the number of complex primes of \( K \); Leopoldt’s conjecture predicts the equality \( r = r_2 + 1 \) (cf. [Neukirch-Schmidt-Wingberg], Chapter X, §3). Write \( K^{(\infty)} / K \) for the corresponding (infinite) subextension of \( \overline{\mathbb{K}} / K \) with Galois group \( \Gamma \). (Note that \( K^{(\infty)} \subset K_1 \).) Write \( K^{(n)} / K \) for the (finite) subextension of \( K^{(\infty)} / K \) corresponding to the subgroup \( \Gamma(n) \overset{\text{def}}{=} \{ \gamma^n \mid \gamma \in \Gamma \} \subset \Gamma \), \( n \geq 0 \) an integer. We denote by \( \Lambda = \mathbb{Z}_l[[\Gamma]] \) the associated complete group ring (cf. loc. cit. Chapter V, §2). It is known that given a set of free generators \( \{ \gamma_1, \ldots, \gamma_r \} \) of \( \Gamma \), we have an isomorphism \( \Lambda \cong \mathbb{Z}_l[[T_1, \ldots, T_r]] \), \( \gamma_i \mapsto 1 + T_i \) (\( 1 \leq i \leq r \)). (See [Neukirch-Schmidt-Wingberg], (5.3.5) Proposition, in case \( r = 1 \). The general case is similar.) Slightly more generally, let \( \mathcal{O} / \mathbb{Z}_l \) be a finite extension of (complete) discrete valuation rings. Then we denote by \( \Lambda_{\mathcal{O}} = \mathcal{O}[[\Gamma]] = \Lambda \otimes_{\mathbb{Z}_l} \mathcal{O} \) the associated complete group ring over \( \mathcal{O} \) (cf. loc. cit.). Consider the exact sequence \( 1 \rightarrow H \rightarrow G_K^2 \rightarrow \Gamma \rightarrow 0 \) where \( H \overset{\text{def}}{=} \text{Ker}(G_K^2 \rightarrow \Gamma) \). By pushing out this sequence by the projection \( H \rightarrow P \overset{\text{def}}{=} (H^{(l)})^{ab} \) we obtain an exact sequence \( 1 \rightarrow P \rightarrow Q \rightarrow \Gamma \rightarrow 1 \). Write \( K'/K \) for the corresponding subextension of \( \overline{\mathbb{K}} / K \) with Galois group \( Q \). (Note that \( K' \subset K_2 \).)
Thus, $K'/K^{(\infty)}$ is the maximal abelian pro-$l$ extension of $K^{(\infty)}$. Note that the quotient $G^2_K \to Q$ can be reconstructed group-theoretically from $G^2_K$ by its very definition. Let $\Sigma \overset{\text{def}}{=} \{ l, \infty \}(K)$ be the finite set of primes of $K$ consisting of the nonarchimedean primes above $l$ and the primes at infinity, and $S \supset \Sigma$ a finite set of primes of $K$. Write $P \to P_{\Sigma}$ (resp. $P \to P_S$) for the quotient corresponding to the maximal subextension $K^{(\infty)},\Sigma/K^{(\infty)}$ (resp. $K^{(\infty),S}/K^{(\infty)}$) of $K'/K^{(\infty)}$ which is unramified outside $\Sigma(K^{(\infty)})$ (resp. $S(K^{(\infty)})$). Thus, $K' = \bigcup_S K^{(\infty),S}$, where the union is over all finite sets $S'$ of primes of $K$ containing $\Sigma$. Note that $P_S$ (resp. $P$) has a natural structure of $\Lambda$-module of which $\text{Ker}(\overline{\chi}_s)$ is to show that $\gamma_p$ splits completely in $\overline{\chi}_s$ for the decomposition group of $\Gamma$ at $p$ (which is well-defined since $\Gamma$ is abelian). Then $\Gamma_p$ is canonically generated by the Frobenius element $\gamma_p$ at $p$ (i.e. the image of $Frob_p$ in $\Gamma$) and isomorphic to $\ZZ_l$. Note that no nonarchimedean prime of $K$ splits completely in $K^{(\infty)}$, and the extension $K^{(\infty)}/K$ is unramified outside $\Sigma$.

Let $\chi^{(l)}_{\text{cycl}} : G^1_K \to \ZZ_l^{\times}$ be the $l$-part of the cyclotomic character, then $\overline{\chi^{(l)}_{\text{cycl}}}$ (cf. Notations) factors as $G^1_K \to \Gamma \twoheadrightarrow 1 + l\ZZ_l$. We will refer to the induced character $\overline{w} : \Gamma \to 1 + l\ZZ_l \subset \ZZ_l^{\times}$ as the cyclotomic character of $\Gamma$. Thus, the goal of this section is to show that $w$ can be recovered group-theoretically from $G^2_K$ (cf. Proposition 4.9).

**Proposition 4.1.** Let $S \supset \Sigma$ be a finite set of primes of $K$. Then there exists a canonical exact sequence of $\Lambda$-modules

$$0 \to \bigoplus_{p \in S \setminus \Sigma} \text{Ind}^\Gamma_p \ZZ_l(1) \to P_S \to P_{\Sigma} \to 0$$

(where for an integer $N > 0$, $\mu_N$ stands for the group of $N$-th roots of unity), and passing to the projective limit over all finite sets $S \supset \Sigma$ we obtain a canonical exact sequence of $\Lambda$-modules

$$(4.1) \quad 0 \to \prod_{p \in \text{Primes}_K \setminus \Sigma} \text{Ind}^\Gamma_p \ZZ_l(1) \to P \to P_{\Sigma} \to 0$$

Further, for each $p \in \text{Primes}_K \setminus \Sigma$ with $\mu_l \subset \kappa(p)$, the $\Lambda$-module $\text{Ind}^\Gamma_p \ZZ_l(1)$ is isomorphic to $\Lambda/ \langle \gamma_p \rangle \cong \QQ_l/\ZZ_l$, where $\epsilon_p = 1$ (resp. $\epsilon_p = -1$) if $\mu_l \subset \kappa(p)$ (resp. $\mu_l \not\subset \kappa(p)$). (In particular, $\epsilon_p = 1$ if $l \neq 2$.)

**Proof.** We follow the arguments in [Neukirch-Schmidt-Wingberg], proof of (11.3.5) Theorem. First, the weak Leopoldt conjecture holds for the extension $K^{(\infty)}/K$. In other words, let $K_S/K$ (resp. $K_{\Sigma}/K$) be the maximal subextension of $\overline{K}/K$ which is unramified outside $S$ (resp. $\Sigma$), then $H^2(\text{Gal}(K_{\Sigma}/K^{(\infty)}), \QQ_l/\ZZ_l) = 0$ (cf. [Nguyen-Quang-Do], Corollaire 2.9). Thus, we have an exact sequence of cohomology groups with $\QQ_l/\ZZ_l$-coefficients

$$0 \to H^1(\text{Gal}(K_{\Sigma}/K^{(\infty)})) \to H^1(\text{Gal}(K_S/K_{\Sigma})) \to H^1(\text{Gal}(K_S/K_{\Sigma}))^{\text{Gal}(K_{\Sigma}/K^{(\infty)})} \to 0,$$

or dually, using (10.5.4) Corollary in [Neukirch-Schmidt-Wingberg],

$$0 \to \lim_{\longrightarrow} \bigoplus_{p(n) \in S \setminus \Sigma(K^{(n)})} (I_{p(n)}^{(l)})_{G_{K^{(n)}}} \to P_S \to P_{\Sigma} \to 0,$$
where $I_{p^{(n)}}$ is the inertia subgroup in $G_{(K^{(n)})_{p^{(n)}}}$, and $(I_{p^{(n)}})_{G_{(K^{(n)})_{p^{(n)}}}}$ is the coinvariant of the $G_{(K^{(n)})_{p^{(n)}}}$-module $I_{p^{(n)}}$. Further,

\[
\lim_{n} \bigoplus_{p^{(n)} \in S \setminus \Sigma(K^{(n)})} (I_{p^{(n)}})_{G_{(K^{(n)})_{p^{(n)}}}} = \lim_{n} \bigoplus_{p \in S \setminus \Sigma} \text{Ind} \left( \text{Gal}(K^{(n)}/K) \to (I_{p})_{G_{(K^{(n)})_{p^{(n)}}}} \right) = \bigoplus_{p \in S \setminus \Sigma} \text{Ind}^{\Gamma_{p}}(I_{p})_{G_{(K^{(n)})_{p^{(n)}}}}
\]

where in the second and third terms (resp. in the fourth term) $p^{(n)}$ (resp. $p^{(\infty)}$) stands for a fixed prime in $\mathcal{P} \text{rimes}_{K^{(n)}}$ (resp. $\mathcal{P} \text{rimes}_{K^{(\infty)}}$) above $p$. It follows from the various definitions (cf. loc. cit., the proof of (11.3.5) Theorem) that one has

\[
(I_{p})_{G_{(K^{(\infty)})_{p^{(\infty)}}}} = \begin{cases} 
0, & \mu_{l} \not\in \kappa(p), \\
I_{p} \simeq \mathbb{Z}_{l}(1), & \mu_{l} \subset \kappa(p).
\end{cases}
\]

Thus, the first exact sequence in Proposition 4.1 is obtained, and the exact sequence (4.1) is obtained by passing to the projective limit. The last assertion follows immediately from the various definitions. (Observe that $\chi_{\text{cycl}}^{(l)}(\text{Frob}_{p}) = \epsilon_{p}w(\gamma_{p})$ holds for $p$ with $\mu_{l} \subset \kappa(p)$.) □

For $p \in \mathcal{P} \text{rimes}_{K} \setminus \Sigma$ with $\mu_{l} \subset \kappa(p)$, we write $J_{p} \overset{\text{def}}{=} \text{Ind}_{\Gamma_{p}}^{p} \mathbb{Z}_{l}(1)$, and $J \overset{\text{def}}{=} \prod_{p \in \mathcal{P} \text{rimes}_{K} \setminus \Sigma, \mu_{l} \subset \kappa(p)} J_{p}$. Thus, we have the exact sequence

\[
(4.1) \quad 0 \to J \to P \to P_{\Sigma} \to 0.
\]

Write $\Gamma^{\text{prim}} \overset{\text{def}}{=} \Gamma \setminus \Gamma(1)$, and let $\{\gamma_{1}, \ldots, \gamma_{r}\}$ be a set of free generators of $\Gamma$. Let $\gamma \in \Gamma \setminus \{1\}$, then one may write $\gamma = \prod_{i=1}^{r} \gamma_{i}^{\alpha_{i,\gamma}}$ with $\alpha_{i,\gamma} \in \mathbb{Z}_{l}$. Write $\gamma_{\Sigma}$ for the subgroup $\langle \gamma \rangle$ of $\Gamma$ (topologically) generated by $\gamma$. Then $(\Gamma/\Gamma_{\gamma})_{\text{tor}}$ is finite cyclic of order $l^{m_{\gamma}}$ for some $m_{\gamma} \geq 0$. There exists a unique element $\tilde{\gamma}_{l^{m_{\gamma}}} = \gamma$. For $p \in \mathcal{P} \text{rimes}_{K} \setminus \Sigma$ with $\mu_{l} \subset \kappa(p)$, we write $\alpha_{i,p}$ and $m_{p}$ instead of $\alpha_{i,\gamma}$ and $m_{\gamma}$, respectively. The following lemma will be useful.

Lemma 4.2. With the above notations, the following (i)-(vii) are equivalent.

(i) $(\Gamma/\Gamma_{\gamma})_{\text{tor}} = 0$ (i.e. $m_{\gamma} = 0$).

(ii) $\gamma$ is a member of a set of free generators of $\Gamma$.

(iii) The image of $\gamma$ under the map $\Gamma \to \Gamma \otimes_{\mathbb{Z}_{l}} \mathbb{F}_{l}$ is nontrivial.

(iv) $\alpha_{i,\gamma} \in \mathbb{Z}_{l}^{\times}$ for some $1 \leq i \leq r$.

(v) $\gamma \in \Gamma^{\text{prim}}$.

(vi) For every $\alpha \in 1 + l\mathbb{Z}_{l}$, $\gamma - \alpha$ is a prime element of $\Lambda$.

(vii) For every finite extension $\mathcal{O}/\mathbb{Z}_{l}$ of discrete valuation rings and every $\alpha \in 1 + m$, where $m$ is the maximal ideal of $\mathcal{O}$, $\gamma - \alpha$ is a prime element of $\Lambda_{\mathcal{O}}$.

Proof. Easy. □
Lemma 4.3. Let \( \mathcal{O}/\mathbb{Z}_l \) be a finite extension of (complete) discrete valuation rings and \( m \subset \mathcal{O} \) the maximal ideal. Let \( \gamma \in \Gamma, \gamma' \in \Gamma_{\text{prim}}, \) and \( \alpha, \alpha' \in 1 + m. \) Then \( \gamma' - \alpha' \) divides \( \gamma - \alpha \) in \( \Lambda_{\mathcal{O}} \) (i.e. \( \gamma - \alpha \in (\gamma' - \alpha')_{\Lambda_{\mathcal{O}}} \)) if and only if there exists \( \nu \in \mathbb{Z}_l \) such that \( \gamma = (\gamma')^\nu \) and \( \alpha = (\alpha')^\nu. \) In particular, \( (\gamma' - \alpha') \mid (\gamma - \alpha) \) implies \( \gamma \in \langle \gamma' \rangle. \)

Proof. First, suppose that there exists \( \nu \in \mathbb{Z}_l \) such that \( \gamma = (\gamma')^\nu \) and \( \alpha = (\alpha')^\nu. \) Then

\[
\gamma - \alpha = (\gamma')^\nu - (\alpha')^\nu = \sum_{i=0}^{\infty} \binom{\nu}{i} (\gamma' - 1)^i - \sum_{i=0}^{\infty} \binom{\nu}{i} (\alpha' - 1)^i
\]

is divisible by \( \gamma' - \alpha'. \)

Next, suppose \( (\gamma' - \alpha') \mid (\gamma - \alpha) \). By Lemma 4.2, there exists a set of free generators \( \{\gamma_1, \ldots, \gamma_r\} \) of \( \Gamma \) with \( \gamma_1 = \gamma' \). Let \( \Gamma^* \) be the closed subgroup of \( \Gamma \) generated by \( \{\gamma_2, \ldots, \gamma_r\} \), and set \( \Lambda_{\mathcal{O}}^{\text{def}} = \mathcal{O}[[\Gamma^*]]. \) Consider the surjective homomorphism \( \Lambda_{\mathcal{O}} \twoheadrightarrow \Lambda_{\mathcal{O}}^{\text{prim}} \) of \( \mathcal{O} \)-algebras, defined by \( \gamma_1 = \gamma' \mapsto \alpha' \in 1 + m \subset \mathcal{O}^\times \subset (\Lambda_{\mathcal{O}})^\times \) and \( \gamma_i \mapsto \gamma_i \) (\( i = 2, \ldots, r \)). Following the decomposition \( \Gamma \cong (\gamma')^{\mathbb{Z}_l} \times \Gamma^* \), write \( \gamma = (\gamma')^\nu \gamma^* \), where \( \nu \in \mathbb{Z}_l \), \( \gamma^* \in \Gamma^* \). Then the image of \( \gamma - \alpha \) in \( \Lambda_{\mathcal{O}}^{\text{prim}} \) is \( (\alpha')^\nu \gamma^* - \alpha \), which must be 0 by assumption. Thus, \( \gamma^* = (\alpha')^{-\nu} \alpha \in \mathcal{O} \) in \( \Lambda_{\mathcal{O}}^{\text{prim}} \), which first implies \( \gamma^* = 1 \) (i.e. \( \gamma = (\gamma')^\nu \)) and then \( (\alpha')^{-i} \alpha = 1 \) (i.e. \( \alpha = (\alpha')^\nu \)), as desired. \( \square \)

The cyclotomic character \( w : \Gamma \to 1 + \mathbb{Z}_l \) induces naturally a continuous surjective homomorphism of \( \mathbb{Z}_l \)-algebras \( \psi_w : \Lambda = \mathbb{Z}_l[[\Gamma]] \to \mathbb{Z}_l \) such that \( \psi_w(\gamma) = w(\gamma) \) if \( \gamma \in \Gamma \).

**Definition 4.4.** Let \( M \) be a \( \Lambda \)-module. We define

\[
I_M \overset{\text{def}}{=} \langle \gamma - \alpha \mid \gamma \in \Gamma, \alpha \in 1 + \mathbb{Z}_l, \text{ and } \text{Ann}_\Lambda(x) = \langle \gamma - \alpha \rangle_\Lambda \text{ for some } x \in M \setminus \{0\} \rangle
\]

which is an ideal of \( \Lambda. \) (For the map \( \mathbb{Z}_l^\times \to 1 + \mathbb{Z}_l, \alpha \mapsto \overline{\alpha}, \) see Notations.) Note that if \( M \subset M' \) then \( I_M \subset I_{M'} \). Further, \( I_M = I_{M_{\Lambda_{\text{tor}}}} \).

Of particular interest to us is the ideal \( I_J \) of \( \Lambda. \)

**Lemma 4.5.** Let \( \{\gamma_1, \ldots, \gamma_r\} \) be a set of free generators of \( \Gamma. \) Consider the following ideals of \( \Lambda: \bar{I} \overset{\text{def}}{=} \ker(\psi_w); I \overset{\text{def}}{=} \langle \gamma \mid \gamma \in \Gamma \rangle; \) and \( I' = I'_{\gamma_1, \ldots, \gamma_r} \overset{\text{def}}{=} \langle \gamma_1 - w(\gamma_1), \ldots, \gamma_r - w(\gamma_r) \rangle. \) Then the following equalities hold: \( I_J = \bar{I} = I = I'. \) (In particular, \( I' \) does not depend on the choice of \( \{\gamma_1, \ldots, \gamma_r\} \).

Proof. The inclusions \( I' \subset I \subset \bar{I} \) clearly hold for any \( \{\gamma_1, \ldots, \gamma_r\} \). We show \( \bar{I} \subset I' \) for any \( \{\gamma_1, \ldots, \gamma_r\}. \) As \( I' \subset \bar{I}, \) we have a surjective homomorphism \( \Lambda/I' \to \Lambda/\bar{I} = \mathbb{Z}_l. \) Using the isomorphism \( \Lambda \cong \mathbb{Z}_l[[T_1, \ldots, T_r]]; \gamma_i \mapsto 1 + T_i \) (\( 1 \leq i \leq r), \) one sees easily that \( \Lambda/I' \cong \mathbb{Z}_l, \) hence the homomorphism \( \Lambda/I' \to \Lambda/\bar{I} \) is an isomorphism. Thus, the equalities \( I' = I = \bar{I} \) follow. (In particular, \( I' \) does not depend on the choice of \( \{\gamma_1, \ldots, \gamma_r\} \).)
Next, we prove $I_J \subset I$. For each $p \in \mathfrak{P} \setminus \Sigma$ with $\mu_l \subset \kappa(p)$, take an element $\tilde{\gamma}_p \in \Gamma$ as in the paragraph preceding Lemma 4.2. Then, by Proposition 4.1,

$$J_p = \text{Ind}_{\Gamma}^{\mathfrak{F}_l} Z_l(1) \simeq \Lambda/\langle \gamma_p - \epsilon_p w(\gamma_p) \rangle = \Lambda/\langle \tilde{\gamma}_p^{l_{mp}} - \epsilon_p w(\tilde{\gamma}_p)^{l_{mp}} \rangle.$$

Further, write $E_p$ for the set of $l_{mp}$-th roots of $\epsilon_p$ in $\mathfrak{F}_l$, and set $\mathcal{O}_{E_p} \overset{\text{def}}{=} Z_l[E_p] \subset \mathfrak{F}_l$ and $\Lambda_{E_p} \overset{\text{def}}{=} \Lambda_{\mathcal{O}_{E_p}}$. More concretely, if $\epsilon_p = 1$ (resp. $\epsilon_p = -1$), $E_p = \mu_{l_{mp}}$ (resp. $E_p = \mu_{l_{mp}+1 \setminus l_{2mp}}$) and $\mathcal{O}_{E_p} = Z_l[\zeta]$, where $\zeta$ is a primitive $l_{mp}$-th (resp. $2m_{mp}+1$-th) root of unity in $\mathfrak{F}_l$. Now, one has

$$J_p \hookrightarrow J_p \otimes \mathbb{Z}_l \mathcal{O}_{E_p} \simeq \Lambda_{E_p}/\langle \tilde{\gamma}_p^{l_{mp}} - \epsilon_p w(\tilde{\gamma}_p)^{l_{mp}} \rangle \Lambda_{E_p} \hookrightarrow \prod_{\eta \in E_p} \Lambda_{E_p}/\langle \tilde{\gamma}_p - \eta w(\tilde{\gamma}_p) \rangle \Lambda_{E_p}.$$

Here, the first injection comes from the fact that $\mathcal{O}_{E_p}$ is free (of rank $> 0$) as a $\mathbb{Z}_l$-module, while the second injection comes from the fact that $\Lambda_{E_p} (\simeq \mathcal{O}_{E_p}[[T_1, \ldots, T_r]])$ is a unique factorisation domain. Set $J_{p, \eta} \overset{\text{def}}{=} \Lambda_{E_p}/\langle \tilde{\gamma}_p - \eta w(\tilde{\gamma}_p) \rangle \Lambda_{E_p}$. Thus, one has $J = \prod_p J_p \hookrightarrow \prod_p \prod_{\eta \in E_p} J_{p, \eta}$ as $\Lambda$-modules. Let $x \in J \setminus \{0\}$ such that $\text{Ann}_{\Lambda}(x) = \langle \gamma - \alpha \rangle \Lambda \subset I_J$ with $x \in \Gamma$ and $\alpha \in 1 + l\mathbb{Z}_l$. Via the above injections, $x$ is identified with $(x_{p, \eta})_{p, \eta}$, where $x_{p, \eta} \in J_{p, \eta}$ for $p \in \mathfrak{P} \setminus \Sigma$ with $\mu_l \subset \kappa(p)$ and $\eta \in E_p$. As $x \neq 0$, there exists $(p, \eta)$ such that $x_{p, \eta} \neq 0$, and

$$\gamma - \alpha \in \text{Ann}_{\Lambda}(x) \subset \text{Ann}_{\Lambda_{E_p}}(x_{p, \eta}) = \langle \tilde{\gamma}_p - \eta w(\tilde{\gamma}_p) \rangle \Lambda_{E_p},$$

where the equality follows from the fact that $\Lambda_{E_p}/\langle \tilde{\gamma}_p - \eta w(\tilde{\gamma}_p) \rangle \Lambda_{E_p}$ is an integral domain (cf. Lemma 4.2, (v) $\implies$ (vii)). Now, by Lemma 4.3, there exists $\nu \in \mathbb{Z}_l$ such that $\gamma = \tilde{\gamma}_p^\nu$ and $\alpha = (w(\tilde{\gamma}_p))^\nu = \eta^\nu w(\tilde{\gamma}_p)^\nu$. Further, one has $\alpha w(\tilde{\gamma}_p)^{-\nu} = \eta^\nu \in \mathbb{Z}_l^\times \cap (O_{E_p}^\times)_{\text{tor}} = (\mathbb{Z}_l^\times)_{\text{tor}}$. As $\alpha w(\tilde{\gamma}_p)^{-\nu} \in (\mathbb{Z}_l^\times)_{\text{tor}}$, one has $\alpha = w(\tilde{\gamma}_p)^\nu = w(\tilde{\gamma}_p)^\nu$. Thus, $\gamma - \alpha = \tilde{\gamma}_p^\nu - w(\tilde{\gamma}_p)^\nu = \tilde{\gamma}_p^{\nu} - w(\tilde{\gamma}_p)^\nu \in I_J$ as desired.

Finally, we prove $I' = I_{\gamma_1, \ldots, \gamma_r} \subset I_J$ (for some choice of a set of free generators $\{\gamma_1, \ldots, \gamma_r\}$ of $\Gamma$). Let $\{v_1, \ldots, v_r\}$ be a basis of $\Gamma \otimes \mathbb{F}_l$, then by Chebotarev’s density theorem there exists $\{p_1, \ldots, p_r\} \subset \mathfrak{P} \setminus \Sigma$ such that $\mu_l \subset \kappa(p_i)$, $1 \leq i \leq r$, and that the Frobenius element $\gamma_{p_i}$ at $p_i$ maps to $v_i \in \Gamma \otimes \mathbb{F}_l$, $1 \leq i \leq r$. By Nakayama’s lemma $\{\gamma_{p_1}, \ldots, \gamma_{p_r}\}$ is a set of free generators of $\Gamma$. Fix any $i \in \{1, \ldots, r\}$. By Proposition 4.1, we have $J_{p_i} \simeq \Lambda/\langle \gamma_{p_i} - \epsilon_{p_i} w(\gamma_{p_i}) \rangle$ as $\Lambda$-modules. Let $t_{p_i}$ be the element of $J_{p_i} \subset J$ corresponding to $1 \in \Lambda/\langle \gamma_{p_i} - \epsilon_{p_i} w(\gamma_{p_i}) \rangle$ under this isomorphism. Then $\text{Ann}_{\Lambda}(t_{p_i}) = \langle \gamma_{p_i} - \epsilon_{p_i} w(\gamma_{p_i}) \rangle \Lambda$, hence by definition, $\gamma_{p_i} - w(\gamma_{p_i}) = \gamma_{p_i} - \epsilon_{p_i} w(\gamma_{p_i}) \in I_J$. As $i \in \{1, \ldots, r\}$ is arbitrary, this shows $I' = I_{\gamma_{p_1}, \ldots, \gamma_{p_r}} \subset I_J$, as desired. This finishes the proof of Lemma 4.5.

**Lemma 4.6.** Assume $r \geq 2$ and let $f \in \Lambda \setminus \{0\}$. Then $I_{f_J} = I_J$.

**Proof.** As $fJ \subset J$, one has $I_{fJ} \subset I_J$. So, we show the converse $I_J \subset I_{fJ}$. As in the last part of the proof of Lemma 4.5, let $\{v_1, \ldots, v_r\}$ be a basis of $\Gamma \otimes \mathbb{F}_l$, and take $\{p_1, \ldots, p_r\} \subset \mathfrak{P} \setminus \Sigma$ such that $\mu_l \subset \kappa(p_i)$, $1 \leq i \leq r$, and that $\gamma_{p_i} \in \Gamma$ maps to $v_i \in \Gamma \otimes \mathbb{F}_l$, $1 \leq i \leq r$. Thus, $\{\gamma_{p_1}, \ldots, \gamma_{p_r}\}$ is a set of free generators of $\Gamma$, and $I_J = I' = \langle \gamma_{p_i} - \epsilon_{p_i} w(\gamma_{p_i}) \rangle$, $1 \leq i \leq r \rangle \Lambda$ (cf. loc. cit.). Further, $J_{p_i} \simeq \Lambda/\langle \gamma_{p_i} - \epsilon_{p_i} w(\gamma_{p_i}) \rangle$ as $\Lambda$-modules; $\Lambda/\langle \gamma_{p_i} - \epsilon_{p_i} w(\gamma_{p_i}) \rangle$ is an integral domain
(cf. Lemma 4.2, (ii) $\implies$ (vi)); and $fJ_{p_i} \subset fJ$. Accordingly, if $f \not\in \langle \gamma_{p_i} - \epsilon_{p_i}w(\gamma_{p_i}) \rangle$ and $t_{p_i}$ is the element of $J_{p_i}(\subset J)$ corresponding to $1 \in \Lambda/\langle \gamma_{p_i} - \epsilon_{p_i}w(\gamma_{p_i}) \rangle$, then $\text{Ann}_\Lambda(f t_{p_i}) = \langle \gamma_{p_i} - \epsilon_{p_i}w(\gamma_{p_i}) \rangle$, hence $\gamma_{p_i} - w(\gamma_{p_i}) = \gamma_{p_i} - \epsilon_{p_i}w(\gamma_{p_i}) \in I_{fJ}$. Thus, to prove $I_J = I'_{\gamma_{p_1}, \ldots, \gamma_{p_r}} \subset I_{fJ}$, it suffices to show that given $f \in \Lambda \setminus \{0\}$, we can choose $\{p_1, \ldots, p_r\}$ as above such that $\langle \gamma_{p_i} - \epsilon_{p_i}w(\gamma_{p_i}) \rangle \nmid f$ for $1 \leq i \leq r$. Further, (as $\Lambda \cong \mathbb{Z}[T_1, \ldots, T_r]$ is a unique factorisation domain) this would follow if one shows that for each $i \in \{1, \ldots, r\}$, there exists an infinite subset $\{p_{i,j}\}_{j \geq 1} \subset \mathfrak{Primes}_K \setminus \Sigma$ such that $\mu_i \subset \kappa(p_{i,j})$ for every $j \geq 1$; $\gamma_{p_{i,j}} \in \Gamma$ maps to $v_i \in \Gamma \otimes \mathbb{Z}_l \mathbb{F}_l$; and $\gamma_{p_{i,j}} - \epsilon_{p_{i,j}}w(\gamma_{p_{i,j}})$ is coprime to $\gamma_{p_{i,j}'} - \epsilon_{p_{i,j}'}w(\gamma_{p_{i,j}'})$ for every $j \neq j'$. (Note that this is not possible if $r = 1$.)

We choose such an infinite set $\{p_{i,j}\}_{j \geq 1}$ inductively. Assume that $p_{i,1}, \ldots, p_{i,t}$ have been chosen as above. Since $\Gamma \setminus (\Gamma_{p_{i,1}} \cup \ldots \cup \Gamma_{p_{i,t}})$ is open and nonempty (as $r \geq 2$), Chebotarëv’s density theorem ensures that there exists $p_{i,t+1} \in \mathfrak{Primes}_K \setminus \Sigma$ such that $\mu_i \subset \kappa(p_{i,t+1})$; $\gamma_{p_{i,t+1}} \in \Gamma$ maps to $v_i \in \Gamma \otimes \mathbb{Z}_l \mathbb{F}_l$; and $\gamma_{p_{i,t+1}} \in \Gamma \setminus (\Gamma_{p_{i,1}} \cup \ldots \cup \Gamma_{p_{i,t}})$. The $\{p_{i,j}\}_{j \geq 1}$ being chosen in this way, we claim that $\gamma_{p_{i,j}} - \epsilon_{p_{i,j}}w(\gamma_{p_{i,j}})$ is coprime to $\gamma_{p_{i,j}'} - \epsilon_{p_{i,j}'}w(\gamma_{p_{i,j}'})$ for every $j \neq j'$. Indeed, suppose otherwise and assume $j < j'$ without loss of generality. As $\gamma_{p_{i,j}} \in \Gamma_{\text{prim}}$ and $\epsilon_{p_{i,j}}w(\gamma_{p_{i,j}}) \in 1 + l\mathbb{Z}_l$, $\gamma_{p_{i,j}} - \epsilon_{p_{i,j}}w(\gamma_{p_{i,j}})$ is a prime element of $\Gamma$, hence one must have $\langle \gamma_{p_{i,j}} - \epsilon_{p_{i,j}}w(\gamma_{p_{i,j}}) \rangle \mid \langle \gamma_{p_{i,j}'} - \epsilon_{p_{i,j}'}w(\gamma_{p_{i,j}'}) \rangle$. Then, by Lemma 4.3, there exists $\nu \in \mathbb{Z}_l$ such that $\gamma_{p_{i,j}'} = \gamma_{p_{i,j}} + \nu \epsilon_{p_{i,j}}w(\gamma_{p_{i,j}})$, which contradicts our choice of $p_{i,j}$. This finishes the proof of Lemma 4.6. □

**Lemma 4.7.** Let $M \subset J$ be a $\Lambda$-submodule which is $\Lambda$-cofinite (cf. Notations). Then $I_M = I_J$.

**Proof.** As $M \subset J$, one has $I_M \subset I_J$. So, we show the converse $I_J \subset I_M$. The exact sequence $0 \to M \to J \to J/M \to 0$ induces an exact sequence $0 \to M_{\Lambda, \text{tor}} \to J_{\Lambda, \text{tor}} \to (J/M)_{\Lambda, \text{tor}}$, and since $J/M$ is a finitely generated $\Lambda$-module, then its finite quotient $(J/M)_{\Lambda, \text{tor}}$ is a finitely generated torsion $\Lambda$-module, hence there exists $f \in \Lambda \setminus \{0\}$ such that $f(J/M)_{\Lambda, \text{tor}} = 0$. In particular, $fJ_{\Lambda, \text{tor}} \subset M_{\Lambda, \text{tor}} \subset M$.

First, assume $r \geq 2$. Then one has

$I_J = I_{fJ} = I_{fJ_{\Lambda, \text{tor}}} \subset I_M, \tag{1}$

as desired, where the first equality follows from Lemma 4.6. Next, assume $r = 1$ and fix a generator $\gamma$ of $\Gamma$. Set $S \overset{\text{def}}{=} \{p \in \mathfrak{Primes}_K \setminus \Sigma \mid \mu_1 \subset \kappa(p), \gamma_{p} \in \Gamma_{\text{prim}}\}$, which is an infinite set by Chebotarëv’s density theorem. For each $p \in S$, write $\gamma_{p} = \gamma^\alpha_p$ with $\alpha_p \in \mathbb{Z}_l^\times$. Then

$J_p \simeq \Lambda/\langle \gamma_p - \epsilon_{p}w(\gamma_{p}) \rangle = \Lambda/\langle \gamma^\alpha_p - \epsilon_{p}w(\gamma^\alpha_{p}) \rangle = \Lambda/\langle \gamma - \epsilon_{p}w(\gamma) \rangle \simeq \mathbb{Z}_l$.

by Lemma 4.3. (Observe $\epsilon^\alpha_{p} = \epsilon_{p}$. In particular, for every $x_p \in J_p \setminus \{0\}$, one has $\text{Ann}_\Lambda(x_p) = \langle \gamma - \epsilon_{p}w(\gamma) \rangle$. For each $\epsilon \in \{\pm 1\}$, set $S_{\epsilon} \overset{\text{def}}{=} \{p \in S \mid \epsilon_{p} = \epsilon\}$, or, equivalently, $S_{+1} = \{p \in S \mid \mu_{j} \subset \kappa(p)\}$ and $S_{-1} = \{p \in S \mid \mu_{j} \not\subset \kappa(p)\}$. One has $S = S_{+1} \coprod S_{-1}$, hence at least one of $S_{\epsilon}$ ($\epsilon \in \{\pm 1\}$) is an infinite set. Fix such an $\epsilon$. Set $J_{S_{\epsilon}} \overset{\text{def}}{=} \prod_{p \in S_{\epsilon}} J_p$. Then, for every $x \in J_{S_{\epsilon}} \setminus \{0\}$, one has $\text{Ann}_\Lambda(x) = \langle \gamma - \epsilon_{p}w(\gamma) \rangle$. Now, as $J/M$ (resp. $J_{S_{\epsilon}}$) is finitely generated (resp. not finitely generated) as a $\Lambda$-module, $J_{S_{\epsilon}} \cap M = \ker(J_{S_{\epsilon}}(\subset J) \to J/M) \neq \{0\}$. So, take $x \in J_{S_{\epsilon}} \cap M \setminus \{0\}$. Then $\text{Ann}_\Lambda(x) = \langle \gamma - \epsilon_{p}w(\gamma) \rangle$. Thus, by definition, $I_M \supset \langle \gamma - \epsilon_{p}w(\gamma) \rangle = \langle \gamma - \epsilon_{p}w(\gamma) \rangle = I_J$, as desired, where the last equality follows from Lemma 4.5. This finishes the proof of Lemma 4.7. □
Proposition 4.8. \( \cap_{M \subset P} \text{\( \Lambda \)-cofinite} I_M = I_J. \)

Proof. Recall the exact sequence (4.1): \( 0 \to J \to P \to P/\Sigma \to 0. \) As \( P/\Sigma \) is a finitely generated \( \Lambda \)-module (cf. [Nguyen-Quang-Do], Proposition 1.1 and Théorème 1.4), \( J \) is \( \Lambda \)-cofinite in \( P. \) Thus, \( \cap_{M \subset P} \text{\( \Lambda \)-cofinite} I_M \subset I_J. \) To show the converse, let \( M \) be any \( \Lambda \)-submodule of \( P \) which is \( \Lambda \)-cofinite in \( P. \) Then the exact sequence \( 0 \to M \cap J \to J \to P/M \) shows that \( M \cap J \) is \( \Lambda \)-cofinite in \( J. \) Thus, one has \( I_M \supset I_{M \cap J} = I_J, \) where the equality follows from Lemma 4.7. As \( M \) is arbitrary, this shows \( \cap_{M \subset P} \text{\( \Lambda \)-cofinite} I_M \supset I_J. \) This finishes the proof of Proposition 4.8. \( \square \)

We are now ready to bear the fruits of characterising the cyclotomic character \( w: \Gamma \to 1 + \mathbb{Z}_l \subset \mathbb{Z}_l \times. \) Let \( \phi_\Lambda: \Lambda \twoheadrightarrow \Lambda/I_J \) be the natural projection which maps \( \mathbb{Z}_l \subset \Lambda \) isomorphically onto \( \Lambda/I_J \) (cf. Lemma 4.5 and its proof). In what follows we will identify \( \Lambda/I_J \) with \( \mathbb{Z}_l \) via this isomorphism.

Proposition 4.9. Let \( \phi_\Gamma: \Gamma \subset \Lambda \times \to \mathbb{Z}_l \times \) be the restriction of \( \phi_\Lambda \) to \( \Gamma. \) Then \( \phi_\Gamma = w \) holds. In particular, the cyclotomic character \( w \) of \( \Gamma \) can be recovered group-theoretically from \( G_2^K. \)

Proof. The first assertion follows from the equality \( I_J = I \) (hence \( \phi_\Lambda = \psi_w \)) in Lemma 4.5. The second assertion follows from this and Proposition 4.8, since \( \Lambda \) and \( P, \) hence also \( \cap_{M \subset P} \text{\( \Lambda \)-cofinite} I_M = I_J, \) can be recovered group-theoretically from \( G_2^K, \) as follows from the various definitions. \( \square \)

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