HARDY UNIQUENESS PRINCIPLE FOR THE LINEAR
SCHRÖDINGER EQUATION ON QUANTUM REGULAR TREES

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Abstract. In this paper we consider the linear Schrödinger equation (LSE) on a regular
tree with the last generation of edges of infinite length and analyze some unique contin-
uation properties. The first part of the paper deals with the LSE on the real line with a
piece-wise constant coefficient and uses this result in the context of regular trees. The sec-
ond part treats the case of a LSE with a real potential in the framework of a star-shaped
graph.

1. Introduction

For any function \( f \in L^2(\mathbb{R}) \) we consider its Fourier transform

\[
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx, \quad \xi \in \mathbb{R}.
\]

With the above definition in mind, the well known Hardy’s uniqueness principle (HUP) [8,
Theorem 2], asserts that if \( f \) and \( \hat{f} \) are both \( O(e^{-\frac{1}{2}x^2}) \), then \( f = g = Ae^{-\frac{1}{2}x^2} \), with \( A \) a
constant, and if one is \( o(e^{-\frac{1}{2}x^2}) \), then both are identically zero. As a consequence, if

\[
f(x) = O(e^{-\alpha x^2}) \quad \text{and} \quad \hat{f}(\xi) = O(e^{-\beta \xi^2})
\]

with \( \alpha, \beta > 0 \) such that \( \alpha \beta > 1/4 \), then \( f \equiv 0 \). This result is sharp, in the sense that if
\( \alpha \beta = 1/4 \) then \( f \) is a multiple of \( e^{-\alpha x^2} \). Morgan [13] extends this result to any conjugate
exponents \( p \) and \( p' = \frac{p}{p-1} \) with \( p > 2 \). More precisely, if

\[
f(x) = O(e^{-\alpha x^p}) \quad \text{and} \quad \hat{f}(\xi) = O(e^{-\beta \xi^{p'}})
\]

with \( \alpha, \beta > 0 \) such that \( \alpha^{1/p} \beta^{1/p'} > \frac{1}{p^{1/p} p'^{1/p'}} \cos(\frac{\pi p'}{2})^{1/p'} \), then \( f \equiv 0 \). This result is also
sharp. One-sided versions of these results are obtained by Nazarov [14, Theorem 2.3]: for
\( p \in [2, \infty] \) if

\[
f(x) = O(e^{-\alpha x^p}) \quad \text{and} \quad \hat{f}(\xi) = O(e^{-\beta \xi^{p'}})
\]

with \( \alpha, \beta > 0 \) such that \( \alpha^{1/p} \beta^{1/p'} > \frac{1}{p^{1/p} p'^{1/p'}} \sin(\frac{\pi p'}{p}) \), then \( f \equiv 0 \). The exponents in this
case are also the best possible.

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Cowling and Price [4] extend to $L^p, L^q$ versions: if $1 \leq p, q \leq \infty$ with at least one of them finite and

$$\|e^{\alpha x^2} f\|_{L^p(\mathbb{R})} + \|e^{\beta x^2} \hat{f}\|_{L^q(\mathbb{R})} < \infty,$$

with $\alpha, \beta > 0$, such that $\alpha \beta > 1/4$, then $f \equiv 0$. The proofs of the above results use complex analysis techniques, and similar results in terms of the unique solution in $C(\mathbb{R}, L^2(\mathbb{R}))$ of the linear Schrödinger equation

$$\begin{cases}
i u_t(t, x) + \Delta u(t, x) = 0, & x \in \mathbb{R}, t \neq 0, \\
u(0) = u_0, & x \in \mathbb{R}
\end{cases},$$

(1.1)

can be obtained, see for e.g. [3]. Using the Fourier transform the solution $u$ of the above system can be written as

$$u(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x|^2}{4t}} \left( \frac{e^{-\frac{1}{4t} |x|^2} u_0}{(2t)^{\frac{1}{2}}}. \right).$$

This representation and the above property of the Fourier transform show that the unique solution of system (1.1) satisfying $u(0, x) = O(e^{-\alpha x^2})$, $u(T, x) = O(e^{-\beta x^2})$ as $|x| \to \infty$, with

$$\alpha \beta > \frac{1}{16T^2},$$

vanishes identically. $L^p$-versions of these results hold also under the same assumption. For convenience, in the following we will consider the case $T = 1$.

In this paper we obtain similar results for the Schrödinger equation on trees. Let us consider the Schrödinger equation on a tree $\Gamma$:

$$\begin{cases}
i u_t(t, x) + \Delta_\Gamma u(t, x) = 0, & x \in \Gamma, t \neq 0, \\
u(0) = u_0, & x \in \Gamma,
\end{cases},$$

(1.2)

where with $\Delta_\Gamma$ is the Laplace operator on $\Gamma$ with the Kirchhoff coupling condition at the vertices (see section 2 for a precise definition).

Our main result concerning the HUP for the above system is obtained in the context of regular trees. These are particular cases of trees having the property that all the edges of the same generation have the same length and all the vertices of the same generation have equal number of children. In the following we write $f \lesssim g$ if there exists a positive constant $C$, depending on $f$ and $g$, such that $f \leq C g$.

**Theorem 1.1.** Let $\Gamma$ be a regular tree and $\alpha$ and $\beta$ such that $\alpha \beta > 1/16$. Any solution $u \in C(\mathbb{R}, L^2(\Gamma))$ of problem (1.2) that satisfies

$$|u(0, x)| \lesssim e^{-\alpha x^2}, |u(1, x)| \lesssim e^{-\beta x^2}, \forall x \in \Gamma$$

vanishes identically.

Using the arguments in [9] one can reduce the properties of the solutions of the LSE on a regular tree to the analysis of the LSE involving a piecewise constant coefficient $\sigma$. 


Theorem 1.1 is a consequence of the following result for the linear Schrödinger equation with a piecewise constant coefficient $\sigma : \mathbb{R} \to \mathbb{R}$ taking a finite number of positive values:

$$\begin{cases} 
  iu_t(t, x) + \partial_x (\sigma \partial_x u)(t, x) = 0, & x \in \mathbb{R}, t \neq 0, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}.
\end{cases} \tag{1.3}$$

For a precise statement we introduce the two values of $\sigma$ at $\pm \infty$:

$$\sigma_- = \lim_{x \to -\infty} \sigma(x), \quad \sigma_+ = \lim_{x \to +\infty} \sigma(x).$$

**Theorem 1.2.** Any solution $u \in C(\mathbb{R}, L^2(\mathbb{R}))$ of system (1.3) satisfying for some positive $\alpha, \beta$ one of the following assumptions

(i) $|u(0, x)| \lesssim e^{-\alpha x^2}$, $|u(1, x)| \lesssim e^{-\beta x^2}$, as $x \to -\infty$, $\alpha\beta > \frac{1}{16\sigma_-^2}$;

(ii) $|u(0, x)| \lesssim e^{-\alpha x^2}$, $|u(1, x)| \lesssim e^{-\beta x^2}$, as $x \to +\infty$, $\alpha\beta > \frac{1}{16\sigma_+^2}$;

(iii) $|u(0, x)| \lesssim e^{-\alpha x^2}$, $|u(1, x)| \lesssim e^{-\beta x^2}$, as $|x| \to \infty$, $\alpha\beta > \frac{1}{16\max\{\sigma_-^2, \sigma_+^2\}}$,

vanishes identically. Moreover, in the case when $\sigma$ is a two-step piecewise constant function (i.e., it takes only two values), these exponents are sharp.

In spirit of [4], $L^2$-versions may be obtained, but it is beyond the scope of this paper.

In the case of Schrödinger equation with a potential $V = (V_1, \ldots, V_N) : [0, 1] \times \Gamma \to \mathbb{C}$ we can prove a similar result in the case of a star-shaped tree. Here $\Gamma$ is viewed as a collection of $N$ infinite intervals $(0, \infty)$ coupled at the origin. We consider the following critical exponent

$$\gamma_\Gamma = \frac{1}{2} \begin{cases} 
  1, & N \text{ even}, \\
  m + \frac{1}{m}, & N = 2m + 1.
\end{cases} \tag{1.4}$$

**Theorem 1.3.** Let $\alpha, \beta$ such that $\alpha\beta > 4\gamma_\Gamma^4$. Assume the solution $u \in C([0, 1], L^2(\Gamma))$ of equation

$$u_t = i(\Delta_\Gamma + V(t, x))u \quad \text{in } [0, 1] \times \Gamma$$

satisfies

$$\|e^{\alpha x^2} u(0)\|_{L^2(\Gamma)} + \|e^{\beta x^2} u(1)\|_{L^2(\Gamma)} < \infty,$$

where $V(t, x) = V_1(x) + V_2(t, x)$ with $V_1$ real-valued, $\|V_1\|_{L^\infty(\Gamma)} \leq M_1$ and

$$\sup_{t \in [0, 1]} \|e^{\alpha|\beta| x^2 \sqrt{m(1-t)+1}} V_2(t)\|_{L^\infty(\Gamma)} < +\infty.$$ 

Then $u$ vanishes identically.
The above result is not sharp. In fact when all the components of \( V \) are equal, i.e. \( V_1 \equiv V_2 \equiv \cdots \equiv V_N \), the result can be improved by using the same strategy as in the proof of Theorem 1.1 of making the sum of the components and using the real line result. In this case \( \gamma_\Gamma \) corresponds to the one in [6], \( \gamma_\Gamma = 1/\sqrt{8} \).

The paper is organised as follows. In Section 2 we present the notations and preliminaries about metric graphs and the Schrödinger equation on a metric graph. In Section 3 we consider the simple case of a star-shaped tree and give a sketch of how Theorem 1.1 can be proven in this particular case. Also we show how Theorem 1.2 implies Theorem 1.1. Theorem 1.2 is proved in Section 4. Sections 5 and 6 are devoted to the case of the LSE with a potential on a star-shaped tree.

2. Notations and Preliminaries

In this section we present some generalities about metric graphs and introduce the Laplace operator on such structure. Let \( \Gamma = (V, E) \) be a graph where \( V \) is a set of vertices and \( E \) the set of edges. For each \( v \in V \) we denote \( E_v = \{ e \in E : v \in e \} \). We assume that \( \Gamma \) is a finite connected graph. The edges could be of finite length and then their ends are vertices of \( V \) or they have infinite length and then we assume that each infinite edge is a ray with a single vertex belonging to \( V \) (see [12] for more details on graphs with infinite edges).

We fix an orientation of \( \Gamma \) and for each finite oriented edge \( e \), we have an initial vertex \( I(e) \) and a terminal one \( T(e) \). In the case of infinite edges we have only initial vertices. We identify every edge \( e \) of \( \Gamma \) with an interval \( I_e \), where \( I_e = [0, l_e] \) if the edge is finite and \( I_e = [0, \infty) \) if the edge is infinite. This identification introduces a coordinate \( x_e \) along the edge \( e \). In this way \( \Gamma \) becomes a metric space, called metric graph [12].

We identify any function \( u \) on \( \Gamma \) with a collection \( \{u^e\}_{e \in E} \) of functions \( u^e \) defined on the edges \( e \) of \( \Gamma \). Each \( u^e \) can be considered as a function on the interval \( I_e \). In fact, we use the same notation \( u^e \) for both the function on the edge \( e \) and the function on the interval \( I_e \) identified with \( e \). For a function \( u : \Gamma \to \mathbb{C}, u = \{u^e\}_{e \in E} \), we denote by \( f(u) : \Gamma \to \mathbb{C} \) the family \( (f(u^e))_{e \in E} \), where \( f(u^e) : I_e \to \mathbb{C} \).

The space \( L^p(\Gamma) \), \( 1 \leq p < \infty \) consists of all functions \( u = \{u_e\}_{e \in E} \) on \( \Gamma \) that belong to \( L^p(I_e) \) for each edge \( e \in E \) and

\[
\|u\|_{L^p(\Gamma)}^p = \sum_{e \in E} \|u^e\|_{L^p(I_e)}^p < \infty.
\]

Similarly, the space \( L^\infty(\Gamma) \) consists of all functions that belong to \( L^\infty(I_e) \) for each edge \( e \in E \) and

\[
\|u\|_{L^\infty(\Gamma)} = \max_{e \in E} \|u^e\|_{L^\infty(I_e)} < \infty.
\]

The Sobolev space \( H^m(\Gamma) \), with \( m \geq 1 \) an integer, consists of all functions with components that belong to \( H^m(I_e) \) for each \( e \in E \) and

\[
\|u\|_{H^m(\Gamma)}^2 = \sum_{e \in E} \|u^e\|_{H^m(I_e)}^2 < \infty.
\]
These are Hilbert spaces with the inner products

\[(u, v)_{L^2(\Gamma)} = \sum_{e \in E} (u^e, v^e)_{L^2(I_e)} = \sum_{e \in E} \int_{I_e} u^e(x)\overline{v^e}(x) \, dx\]

and

\[(u, v)_{H^m(\Gamma)} = \sum_{e \in E} (u^e, v^e)_{H^m(I_e)} = \sum_{e \in E} \sum_{k=0}^m \int_{I_e} \frac{d^k u^e}{dx^k} \frac{d^k v^e}{dx^k} \, dx.\]

Notice that a function from \(H^m(\Gamma)\) has continuous components on the interior of edges, but there is no information about the continuity at the coupling at the vertices. A function \(u = \{u^e\}_{e \in E}\) is continuous if and only if \(u^e\) is continuous on \(I_e\) for every \(e \in E\), and moreover, it is continuous at the vertices of \(\Gamma\):

\[u^e(v) = u^{e'}(v), \quad \forall \, e, e' \in E_v.\]

We introduce the Laplace operator \(\Delta_\Gamma\) on the graph \(\Gamma\), with Kirchhoff coupling condition. This is a standard procedure and we refer to [2] for a complete description. The domain of \(\Delta_\Gamma\) (see [2]) is the space of all continuous functions on \(\Gamma\), \(u = \{u^e\}_{e \in E}\), such that for every edge \(e \in E\), \(u^e \in H^2(I_e)\), and satisfying the following Kirchhoff-type condition:

\[\sum_{e \in \mathcal{T}(v)} u^e_x(I_e -) = \sum_{e \in \mathcal{I}(v)} u^e_x(0+) \quad \text{for all } v \in V.\]

It acts as the second derivative along the edges

\[(\Delta_\Gamma u)^e = (u^e_{xx}) \quad \text{for all } e \in E, u \in D(\Delta_\Gamma).\]

It is easy to verify that \((\Delta_\Gamma, D(\Delta_\Gamma))\) is a linear, unbounded, self-adjoint, dissipative operator on \(L^2(\Gamma)\), i.e. \((\Delta_\Gamma u, u)_{L^2(\Gamma)} \leq 0\) for all \(u \in D(\Delta_\Gamma)\). Since \(C^\infty_C(\Gamma)\), the space of functions which are \(C^\infty\) on each edge and vanish outside some bounded set of \(\Gamma\), is included in \(D(\Delta_\Gamma)\) we obtain that \(D(\Delta_\Gamma)\) is dense in any \(L^p(\Gamma), 1 \leq p < \infty\). All self-adjoint extensions of the Laplacian on such quantum graphs have been described in [11] in terms of coupling conditions. Using the properties of the operator \(\Delta_\Gamma\) we obtain as a consequence of the Hille-Yosida theorem the following well-posedness result.

**Theorem 2.1.** For any \(u_0 \in D(\Delta_\Gamma)\) there exists a unique solution \(u(t)\) of system (1.2) that satisfies \(u \in C(\mathbb{R}, D(\Delta_\Gamma)) \cap C^1(\mathbb{R}, L^2(\Gamma))\). Moreover, for any \(u_0 \in L^2(\Gamma)\), there exists a unique solution \(u \in C(\mathbb{R}, L^2(\Gamma))\) that satisfies

\[\|u(t)\|_{L^2(\Gamma)} = \|u_0\|_{L^2(\Gamma)} \quad \text{for all } t \in \mathbb{R}.\]

3. Schrödinger equation with Kirchhoff coupling conditions

3.1. **The star-shaped tree.** Let us give first a proof of Theorem 1.1 in the particular case of a star-shaped tree with \(N\) edges, in anticipation of the strategy that one can develop in the case of general regular trees. For any \(u_0 = (u_{0k})_{k=0}^N \in D(\Delta_\Gamma)\) system...
(1.2) can be written in an explicit way as follows: \( u_k \in C(\mathbb{R}, H^2(0, \infty)) \cap C^1(\mathbb{R}, L^2(0, \infty)) \), \( k \in \{1, \ldots, N\} \),

\[
\begin{aligned}
&i \partial_t u_k + \partial_{xx} u_k = 0, \quad t \neq 0, x > 0, k \in \{1, \ldots, N\}, \\
&u_k(t, 0) = u_j(t, 0), \quad k, j \in \{1, \ldots, N\}, \\
&\sum_{k=1}^N \partial_x u_k(t, 0) = 0, \quad t \neq 0.
\end{aligned}
\]

(3.1)

We can consider the case \( \alpha = \beta \), the other case can be reduced to this one by using the so-called Appell transformation (see Section 7 for a precise definition). Denote by \( S \) the sum of all the components of \( u \):

\[
S(t, x) = \sum_{k=1}^N u_k(t, x).
\]

It follows that \( S \) satisfies the Schrödinger equation on the half-line with Neumann boundary condition at \( x = 0 \), \( S_x(t, 0) = 0 \). Moreover, \( S \) satisfies \( |S(0, x)| + |S(1, x)| \lesssim e^{-\alpha x^2} \).

Denoting by \( \tilde{S} \) the even extension of \( S \) we obtain that it satisfies the Schrödinger equation on the whole line

\[
i \tilde{S}_t + \tilde{S}_{xx} = 0, \quad x \in \mathbb{R}, \quad t \neq 0.
\]

Using the classical result on the real line we conclude that \( \tilde{S} \equiv 0 \) so \( S \equiv 0 \). Going back to \( u_k \), \( k = 1, \ldots, N \) we obtain that each component satisfies the Schrödinger equation on the half line with Dirichlet boundary condition at \( x = 0 \), \( u_k(t, 0) = 0 \). Making an odd extension \( \tilde{u}_k \), one obtains a solution of the linear Schrödinger equation on the whole line that decays as follows \( |\tilde{u}_k(1, x)| + |\tilde{u}_k(0, x)| \lesssim e^{-\alpha x^2} \). Then \( \tilde{u}_k \equiv 0 \), so \( u_k \equiv 0 \).

**Remark 3.1.** This assumption \( \alpha \beta > 1/16 \) is sharp. In the case of the star shaped tree the solution \( (u_k)_{k=1}^N \) of system (1.2) can be computed explicitly (see [1] for \( N = 3 \))

\[
u_k(t, x) = \int_0^\infty k_t(x - y)u_{0k}(y) \, dy + \int_0^\infty k_t(x + y)\left(\frac{2}{N} \sum_{j=1}^N u_{0j} - u_{0k}\right)(y) \, dy,
\]

where \( k_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} \). When \( \alpha \beta = 1/16 \), we consider as initial data

\[
u_{0,j}(x) = e^{-\alpha x^2} e^{-\frac{|x|^2}{4t}}, \quad k = 1, \ldots, N.
\]

Using the fact that all \( u_{0,k} \) are equal and the invariance of \( e^{-x^2} \) w.r.t. the Fourier transform, we obtain that for all \( 1 \leq k \leq N \)

\[
u_k(1, x) = (k_1 * u_{0,k})(x) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{|x|^2}{4}} e^{-\frac{|x|^2}{16\alpha}}.
\]

**3.2. Piecewise constant coefficients and LSE on regular trees.** We will show how one can apply Theorem 1.2 in order to obtain the same principle in the case of regular trees with Kirchhoff coupling condition. In order to give a clear and detailed proof, we borrow the notations from [9] and recall some of the needed key results. Also, for simplicity, we restrict ourselves to a particular regular tree and we explain the changes that appear in the case of a general regular tree after the proof.
Proof of Theorem 1.1. Following [9], we consider the regular tree as in Figure 1, with each internal vertex having other two children nodes, the edges of the same generation have the same length, with the last generation being edges of infinite length.

Figure 1. Regular tree with $n + 1 = 3$ generations of edges, 2 descendants from each vertex.

Let us assume we have $n$ generations of vertices and, correspondingly, $n + 1$ generations of edges, and present their indexing. Consider indices of type $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \{1, 2\}^k$ and $|\bar{\alpha}| = k$ the number of its components. Denote by $O$ the root of the tree, by $O_{\bar{\alpha}}$ and $e_{\bar{\alpha}}$ the remaining vertices and edges, respectively. From each vertex $O_{\bar{\alpha}}$ with $|\bar{\alpha}| \leq n$ there are two edges that branch out: $e_{\bar{\alpha}\beta}$, with $\bar{\alpha}\beta = (\alpha_1, \alpha_2, \ldots, \alpha_k, \beta), \beta \in \{1, 2\}$. In the case when $|\bar{\alpha}| \leq n - 1$, the endpoint of $e_{\bar{\alpha}\beta}$ is $O_{\bar{\alpha}\beta}$, otherwise, i.e. if $|\bar{\alpha}| = n$, the two edges that branch out are infinite strips.

Having these new notations in mind, a function $u : \Gamma \to \mathbb{C}$ is a collection of functions $\{u_{\bar{\alpha}}\}_{\bar{\alpha}}$ defined on each edge, $u_{\bar{\alpha}} : e_{\bar{\alpha}} \to \mathbb{C}$, each edge being identified with the real sub-interval $[0, l_{|\bar{\alpha}|})$, with $l_{|\bar{\alpha}|}$ the length of $e_{\bar{\alpha}}$ if $|\bar{\alpha}| \leq n - 1$, and $[0, \infty)$ if $|\bar{\alpha}| = n$. Denoting by

$$I_k = \begin{cases} (a_{k-1}, a_k), & \text{if } 1 \leq k \leq n, \\ (a_n, \infty), & \text{if } k = n + 1, \end{cases}$$

with $a_0 = 0, a_{k+1} = a_k + l_{k+1}, k = 0, n - 1, a_{n+1} = \infty$, system (3.1) is equivalent (after a space translation) with

$$\begin{cases} iu_t^\alpha(t, x) + u_{xx}^\alpha(t, x) = 0, & t \neq 0, x \in I_{|\bar{\alpha}|}, 1 \leq |\bar{\alpha}| \leq n + 1, \\ u^\alpha(t, a_{|\bar{\alpha}|}) = u_{\alpha\beta}^\beta(t, a_{|\bar{\alpha}|}), & \beta \in \{1, 2\}, 1 \leq |\bar{\alpha}| \leq n, \\ u^1(t, 0) = u^2(t, 0), \\ u_x^\alpha(t, a_{|\bar{\alpha}|}) = \sum_{\beta=1}^{2} u_{x\beta}^\beta(t, a_{|\bar{\alpha}|}), & 1 \leq |\bar{\alpha}| \leq n, \\ u^1_x(t, 0) + u^2_x(t, 0) = 0, \\ u^\alpha(0, x) = u_0^\alpha(x). \end{cases}$$

(3.2)
For every $\bar{\alpha}$ with $1 \leq |\bar{\alpha}| \leq n + 1$, consider the averaged sum functions

$$Z^{\bar{\alpha}} : J_{\bar{\alpha}} := \bigcup_{j=0}^{n+1-|\bar{\alpha}|} I_{|\bar{\alpha}|+j} \rightarrow \mathbb{C}$$

as

$$Z^{\bar{\alpha}}(t, x) = \frac{\sum_{|\gamma|=j} u_{\bar{\alpha} \gamma}(t, x)}{2^j}, \quad x \in I_{|\bar{\alpha}|+j}, \quad j = 0, n + 1 - |\bar{\alpha}|.$$

Note that

$$Z^{\bar{\alpha}}(\cdot, x) = u^{\bar{\alpha}}(\cdot, x), \quad x \in I_{|\bar{\alpha}|}. \quad (3.3)$$

Consider now

$$Z(t, x) = \frac{Z^1(t, x) + Z^2(t, x)}{2}, \quad t \in \mathbb{R}, x \in (0, \infty),$$

which satisfies $Z_x(t, 0) = 0$, $t \neq 0$, $Z(t, a_k^-) = Z(t, a_k^+)$ and $Z_x(t, a_k^-) = 2Z_x(t, a_k^+)$ for all $1 \leq k \leq n$. Let us introduce the sequence $(\tilde{a}_k)_{k=0}^{2n+2}$ defined by

$$\tilde{a}_k = \begin{cases} 
-a_{n+1-k}, & \text{if } 0 \leq k \leq n, \\
 a_{k-(n+1)}, & \text{if } n+1 \leq k \leq 2n+2.
\end{cases}$$

It follows that $v(t, x)$, the even extension of the function $Z$, satisfies

$$\begin{cases} 
iv_t(t, x) + v_{xx}(t, x) = 0, & t \neq 0, x \in (\tilde{a}_k, \tilde{a}_{k+1})1 \leq k \leq 2n, \\
v(t, \tilde{a}_k^-) = v(t, \tilde{a}_k^+), & 1 \leq k \leq 2n + 1, \\
v_x(t, \tilde{a}_k^-) = \frac{1}{2}v_x(t, \tilde{a}_k^+), & 1 \leq k \leq n, \\
v_x(t, \tilde{a}_{n+1}^-) = 0 = v_x(t, \tilde{a}_{n+1}^+), \\
v_x(t, \tilde{a}_k^-) = 2v_x(t, \tilde{a}_k^+), & n + 2 \leq k \leq 2n + 1, \\
v(0, x) = v_0(x), & x \in x \in (\tilde{a}_k, \tilde{a}_{k+1})1 \leq k \leq 2n.
\end{cases} \quad (3.4)$$

We consider

$$w(t, T_k(x)) = v(t, x), \quad t \in \mathbb{R}, x \in (\tilde{a}_k, \tilde{a}_{k+1}), 0 \leq k \leq 2n + 1,$$

where each $T_k : (\tilde{a}_k, \tilde{a}_{k+1}) \rightarrow (b_k, b_{k+1}), 0 \leq k \leq 2n + 1$ is a one-to-one linear map that satisfies $(T_k)_x = \mu_k$. The idea behind this linear transformation is that as long as $v_x(\tilde{a}_k^-) = \eta_k v_x(\tilde{a}_k^+)$ we can construct a piecewise constant coefficient $\sigma$ such that $\sigma(x) = \mu_k^2$ on $(b_k, b_{k+1})$ with $\mu_{k-1} = \mu_k/\eta_k$ and $w$ to satisfy

$$\begin{cases} 
iv_t(t, x) + \partial_x(\sigma \partial_x w)(t, x) = 0, & x \in \mathbb{R}, t \neq 0, \\
w(0, x) = w_0(x), & x \in \mathbb{R}.
\end{cases}$$

The particular structure of $(3.4)$ where the first half of $\eta$'s are equal with $1/2$ and the second half are equal $2$ allow us to consider $T_k$'s such that $(T_k)_x = 2^{(n+1/2-k)-(n+1/2)}$ and

$$\sigma(x) = 2^{2n+1-2k-(2n+1)}, x \in (b_k, b_{k+1}), 0 \leq k \leq 2n + 1.$$
Recall that $u(0, x) = O(e^{-\alpha x^2})$ and $u(1, x) = O(e^{-\beta x^2})$ which implies that
\[ u^\alpha(0, x) = O(e^{-\alpha x^2}) \quad \text{and} \quad u^\alpha(1, x) = O(e^{-\beta x^2}), \quad \forall \, |\bar{\alpha}| \leq n + 1, \]
which in view of the previous arguments, one gets that
\[ w(0, x) = O(e^{-\alpha x^2}) \quad \text{and} \quad w(1, x) = O(e^{-\beta x^2}). \]
Thus, since from the definition of $\sigma$ we get that $\sigma_\pm = 1$, by Theorem 1.2, $w(t, x) = 0$, for $x \in \mathbb{R}$ and $t \in [0, 1]$. This implies that
\[ Z(t, x) = \frac{Z_1(t, x) + Z_2(t, x)}{2} = 0, \quad x \in [0, \infty), \quad t \in [0, 1]. \quad (3.5) \]

We would like to conclude that $Z_1$ and $Z_2$ vanish for $x \in [0, \infty)$ and $t \in [0, 1]$, and thus, by (3.3), $u^1$ and $u^2$ vanish for $x \in I_1$ and $t \in [0, 1]$. Consider, as in [9], the difference functions
\[ \tilde{Z}^1 = Z - Z_1 \quad \text{and} \quad \tilde{Z}^2 = Z - Z_2. \quad (3.6) \]

Since $\tilde{Z}^1(t, 0) = 0$, making an odd extension $\tilde{Z}^{1,\text{odd}}$ to the whole real line, it satisfies an equation similar to (3.4) except the fact that $v_\alpha(t, \bar{a}_{n+1}) = v_\alpha(t, \bar{a}_{n+1})$ not necessarily vanishes and the initial data is $\tilde{Z}^{1,\text{odd}}(0, x)$. Since $\tilde{Z}^{1,\text{odd}}(0, x)$ and $\tilde{Z}^{1,\text{odd}}(1, x)$ are of order $O(e^{-\alpha x^2})$ and $O(e^{-\beta x^2})$, respectively, repeating the previous steps, one arrives finally to the conclusion that $\tilde{Z}^1$ vanishes for all $x \in [0, \infty)$ and $t \in [0, 1]$. Together with (3.5) and (3.6), we get that $Z^1(t, x) = 0$, for $x \in [0, \infty)$ and $t \in [0, 1]$. Similarly, one gets that $Z^2(t, x) = 0$, for $x \in [0, \infty)$ and $t \in [0, 1]$. Thus,
\[ u^1(t, x) = 0 \quad \text{and} \quad u^2(t, x) = 0, \quad x \in I_1, \quad t \in [0, 1]. \]

The vanishing property for the other components $u^{\bar{\alpha}}, 1 < |\bar{\alpha}| \leq n + 1$, follows by induction. More precisely, assume that $Z^{\alpha}$ vanishes for $x \in J_\alpha$ and $t \in [0, 1]$, for some $|\bar{\alpha}| = k$, and consider the difference functions
\[ \tilde{Z}^{\alpha\bar{\beta}}(t, x) = Z^{\alpha\bar{\beta}}(t, x) - Z^\alpha(t, x), \quad x \in J_{k+1}. \]

It follows (see for example [9]) that for $k \leq n - 1$, $\tilde{Z}^{\alpha\bar{\beta}}$ satisfies
\[
\begin{align*}
\left\{ 
\begin{array}{ll}
  & t \neq 0, x \in \bigcup_{m=k+1}^{n+1} I_m, \\
  i\tilde{Z}_{t}^{\alpha\bar{\beta}}(t, x) + \tilde{Z}_{xx}^{\alpha\bar{\beta}}(t, x) = 0, \\
  & t \neq 0, \\
  \tilde{Z}^{\alpha\bar{\beta}}(t, a_k) = 0, \\
  \tilde{Z}^{\alpha\bar{\beta}}(t, a_{m-}) = \tilde{Z}^{\alpha\bar{\beta}}(t, a_{m+}), \quad k + 1 \leq m \leq n, \\
  \tilde{Z}_{x}^{\alpha\bar{\beta}}(t, a_{m-}) = 2\tilde{Z}_{xx}^{\alpha\bar{\beta}}(t, a_{m+}), \quad k + 1 \leq m \leq n, \\
  \tilde{Z}^{\alpha\bar{\beta}}(0, x) = Z_0^{\alpha\bar{\beta}}(x), \quad x \in \bigcup_{m=k+1}^{n+1} I_m, 
\end{array}
\right.
\end{align*}
\]
In particular, one finally arrives to the step function
\[ d \]
and its even extension satisfies system (3.4) with initial data modified accordingly and with \( \tilde{k} \)
and thus, similarly as in the case of \( \tilde{k} \)
If \( k \leq n - 1 \), after a translation to move the point \( x = a_k \) to the origin \( x = 0 \), proceeding similarly as in the case of \( \tilde{Z}^1 \), one finally gets that
\[ Z^{\tilde{\alpha}}(t, x) = 0, \quad x \in J_{k+1}, \quad t \in [0, 1], \]
and thus,
\[ u^{\tilde{\alpha}}(t, x) = 0, \quad x \in I_{k+1}, \quad t \in [0, 1]. \]
If \( k = n \), making an odd extension of \( \tilde{Z}^{\tilde{\alpha}} \) to the whole real line, since \( \tilde{Z}^{\tilde{\alpha}}(0, x) = O(e^{-\alpha x^2}) \) and \( \tilde{Z}^{\tilde{\alpha}}(1, x) = O(e^{-\beta x^2}) \), by the classical Hardy uncertainty principle for the LSE on \( \mathbb{R} \) follows the desired result. □

**Extension to general regular trees.** In the proof of Theorem 1.1, the regular tree was assumed such that all vertices have two descendants. Let us review the modifications that appear in the case of a regular tree with all the vertices from the \( 0 \leq k \leq n \) generation having \( d_{k+1} \) descendants (edges). In this case, the indexing is of type \( \bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \{1, 2, \ldots, d_1\} \times \{1, 2, \ldots, d_2\} \times \cdots \times \{1, 2, \ldots, d_k\} \). From each vertex \( O_{\bar{\alpha}} \) with \( |\bar{\alpha}| = k \leq n \) there are \( d_{k+1} \) edges that branch out, \( e^{\alpha_{\beta_{k+1}}} \), with \( \bar{\alpha} \bar{\beta}_{k+1} = (\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_{k+1}) \), with \( \beta_{k+1} \in \{1, 2, \ldots, d_{k+1}\} \), having endpoints \( O_{\alpha\beta_{k+1}} \), and if \( |\bar{\alpha}| = n \), the \( d_{n+1} \) that branch out from the vertices of the last generation, are infinite strips. In this view, the function \( \{ u_{\bar{\alpha}} \}_{\bar{\alpha}} \) modifies accordingly. Furthermore, in equation (3.2), for each \( |\bar{\alpha}| = k \), \( \beta \) replaces with \( \beta_{k+1} \) and the sums are indexed by \( \beta_{k+1} = 1, d_{k+1} \).

The averaged sum functions become, for each \( |\bar{\alpha}| = k \),
\[ Z^{\bar{\alpha}}(t, x) = \frac{\sum_{|\gamma| = j} u^{\bar{\alpha} \gamma}(t, x)}{d_{|\bar{\alpha}|+1} \cdots d_{|\bar{\alpha}|+k}}, \quad x \in I_{|\bar{\alpha}|+j}, j = 0, n + 1 - |\bar{\alpha}|. \]
The new function \( Z \) is
\[ Z(t, x) = \frac{Z^1(t, x) + Z^2(t, x) + \cdots + Z^{d_1}(t, x)}{d_1}, \quad t \in \mathbb{R}, x \in (0, \infty), \]
and its even extension satisfies system (3.4) with initial data modified accordingly and with \( d_{k+1} \) instead of 2. This latter modification is carried out then throughout the entire proof. In particular, one finally arrives to the step function
\[ \sigma(x) = \begin{cases} 
\left( \frac{d_2 \cdots d_{n+1-k}}{d_2 \cdots d_{n+1}} \right)^2, & x \in (b_k, b_{k+1}), 0 \leq k \leq n - 1, \\
\left( \frac{d_2 \cdots d_{n+1}}{d_2 \cdots d_{n+1}} \right)^2, & x \in (b_k, b_{k+1}), n \leq k \leq n + 1, \\
\left( \frac{d_2 \cdots d_{n+1}}{d_2 \cdots d_{n+1}} \right)^2, & x \in (b_k, b_{k+1}), n + 2 \leq k \leq 2n + 1, 
\end{cases} \]
Taking then $\tilde{Z}^j = Z - Z^j$, $j = 1, d_1$, the proof follows similarly, keeping these changes accordingly. Again, in this case $\sigma_\pm = 1$.

4. PROOF OF THEOREM 1.2

Let $N \geq 2$, and consider a partition of the real line $l_0 = -\infty < l_1 < l_2 < \cdots < l_{N-1} < l_N = \infty$, and on each interval $I_i = (l_{i-1}, l_i)$ assume that $\sigma(x) = a_i^2$ is constant, with $a_i > 0$, for all $i = 1, N$. Since one can always reduce to the case $l_1 = 0$ and assume that $l_j = (j - 1)l$ by refining the partition of the $l_j$’s, from now on we will consider this special partition of $R$. This special choice will be used in this section.

Let $u_0$ be as in Theorem 1.2. Then it belongs to $L^2(R)$. Since the operator $L : \mathcal{D}(L) \subseteq L^2(R) \to L^2(R)$, acting as $Lu = \partial_x(\sigma \partial_x u)$ is self-adjoint, it generates a unitary group, i.e. there exists a unique mild solution in $L^2(R)$ for (1.2).

In order to prove Theorem 1.2, we would like first to obtain an explicit representation of the solution $u$ of equation (1.3) with $\sigma$ as explained at the beginning of this section. For reasons which will be clearer in what follows, it is sufficient to compute the solution only for $x \leq 0$. Following [7], $u$ can be written for all $t \neq 0$ and $x \leq 0$ as

\[
u(t, x) = \int_{-\infty}^{0} p^{1,1}_t(x, y)u_0(y) \, dy + \sum_{j=2}^{N-1} \int_{-\infty}^{(j-1)l} p^{1,j}_t(x, y)u_0(y) \, dy + \int_{(N-2)l}^{\infty} p^{1,N}_t(x, y)u_0(y) \, dy,\]

where

\[
\begin{align*}
p^{1,j}_t(x, y) &= \int_{\mathbb{R}} e^{-i\xi^2t} \left[ C_{1j}^{-}(\xi) e^{i\xi(a_1x-a_jy)} + C_{1j}^{+}(\xi) e^{i\xi(a_1x+a_jy)} \right] d\xi, \quad y \in I_j, \quad j = 1, N, \\
\end{align*}
\]

with

\[
C_{11}^{-}(\xi) = \frac{a_1}{2\pi}, \quad C_{11}^{+}(\xi) = 0
\]

and

\[
\begin{bmatrix}
C_{1j}^{-}(\xi) \\
C_{1j}^{+}(\xi)
\end{bmatrix} = \tilde{T}_{j-1}(\xi) \cdot \ldots \cdot \tilde{T}_1(\xi) \begin{bmatrix}
C_{11}^{-}(\xi) \\
C_{11}^{+}(\xi)
\end{bmatrix}.
\]

The complexly conjugated matrices $\tilde{T}_j(\xi)$, $1 \leq j \leq N - 1$, are of the type

\[
\tilde{T}_j(\xi) = \frac{\varepsilon_j}{2a_j} \begin{bmatrix} e^{-i\xi\delta_j(j-1)l} & \gamma_j e^{i\xi\varepsilon_j(j-1)l} \\ \gamma_j e^{-i\xi\varepsilon_j(j-1)l} & e^{i\xi\delta_j(j-1)l} \end{bmatrix} = \frac{\varepsilon_j}{2a_j} \begin{bmatrix} \lambda_j(\xi) & \mu_j(\xi) \\ \mu_j(\xi) & \lambda_j(\xi) \end{bmatrix}
\]

with

\[
\delta_j := a_j - a_{j+1}, \quad \varepsilon_j := a_j + a_{j+1}, \quad \gamma_j := \frac{\delta_j}{\varepsilon_j}, \\
\lambda_j(\xi) = e^{-i\xi\delta_j(j-1)l}, \quad \mu_j(\xi) = e^{i\xi\varepsilon_j(j-1)l}.
\]

In particular, taking $j = N$ in (4.4) and using (4.3)

\[
\begin{bmatrix}
C_{1N}^{-}(\xi) \\
0
\end{bmatrix} = \tilde{T}_{N-1}(\xi) \cdot \ldots \cdot \tilde{T}_1(\xi) \begin{bmatrix}
a_1/(2\pi) \\
C_{11}^{+}(\xi)
\end{bmatrix}.
\]
This allows us to obtain $C_{11}^{-1}$ and $C_{1N}^{-1}$ in terms of the matrices $\bar{T}_k$, $k = 1, N$. Thus, any other $C_{1k}^\pm$ can be written as

$$
\begin{bmatrix}
C_{1k}^- (\xi) \\
C_{1k}^+ (\xi)
\end{bmatrix} = (\bar{T}_{N-1} (\xi) \cdot \ldots \cdot \bar{T}_k (\xi))^{-1}
\begin{bmatrix}
C_{1N}^- (\xi) \\
0
\end{bmatrix}, \quad k = 1, N - 1.
$$

(4.9)

Our aim is to deduce first an easier way to handle the expression for the coefficients $C_{1k}^\pm (\xi)$, for $k = 1, \ldots, N$. While one can compute them inspired by [7], for clarity of the exposition we prefer to make it explicit. Their representation will be given in terms of two $C$ sequences of functions $ar{E}_{j,k} (\xi), F_{j,k} (\xi), 1 \leq k \leq j \leq N - 1$ defined as follows:

$$
\begin{cases}
E_{j,k} (\xi) = E_{j-1,k} (\xi) + \gamma_j e^{2i\xi l (a_{k+1} + \ldots + a_j)} \bar{F}_{j-1,k} (\xi), j > k, \\
\bar{F}_{j,k} (\xi) = \bar{F}_{j-1,k} (\xi) + \gamma_j e^{-2i\xi l (a_{k+1} + \ldots + a_j)} E_{j-1,k} (\xi),
\end{cases}
$$

(4.10)

and $E_{k,k} (\xi) = 1$, $\bar{F}_{k,k} (\xi) = \gamma_k$. Recursively, we get for all $i_{k+1}, \ldots, i_j \in \{0, 1\}$ the existence of some constants $c_{i_{k+1}, \ldots, i_j}, \bar{c}_{i_{k+1}, \ldots, i_j}$, such that for all $1 \leq k < j \leq N - 1$

$$
\begin{cases}
E_{j,k} (\xi) = \sum_{i_0 \in \{0, 1\}} c_{i_{k+1}, \ldots, i_j} e^{2i\xi l (i_{k+1} a_{k+1} + \ldots + i_j a_j)}, \\
\bar{F}_{j,k} (\xi) = \sum_{i_0 \in \{0, 1\}} \bar{c}_{i_{k+1}, \ldots, i_j} e^{-2i\xi l (i_{k+1} a_{k+1} + \ldots + i_j a_j)},
\end{cases}
$$

(4.11)

where $\sum_{i_0 \in \{0, 1\}} = \sum_{i_{k+1} \in \{0, 1\}} \ldots \sum_{i_j \in \{0, 1\}}$, depending on the indexes $i_n$ appearing in the complex exponentials.

In this view, let us first prove the following lemma.

**Lemma 4.1.** Let $\lambda_1, \mu_1, \ldots, \lambda_{N-1}, \mu_{N-1}$ as in (4.5). For any $1 \leq k < j \leq N - 1$,

$$
[T_j (\xi) \cdot \ldots \cdot T_k (\xi)]_{21} = \frac{\varepsilon_k \cdot \ldots \cdot \varepsilon_j}{2j-k+1} \bar{\lambda}_k (\xi) \cdot \ldots \cdot \bar{\lambda}_j (\xi) e^{-2i\xi l (\varepsilon_k a_k + \ldots + \varepsilon_j a_j)} \bar{F}_{j,k},
$$

$$
[T_j (\xi) \cdot \ldots \cdot T_k (\xi)]_{22} = \frac{\varepsilon_k \cdot \ldots \cdot \varepsilon_j}{2j-k+1} \bar{\lambda}_k (\xi) \cdot \ldots \cdot \bar{\lambda}_j (\xi) E_{j,k}.
$$

**Proof of Lemma 4.1.** Taking into account the form of the matrices $\bar{T}_j (\xi)$ in (4.5), we observe that their product is of type

$$
\bar{T}_j (\xi) \cdot \ldots \cdot \bar{T}_k (\xi) = \begin{bmatrix}
A_{j,k} (\xi) & \bar{B}_{j,k} (\xi) \\
B_{j,k} (\xi) & \bar{A}_{j,k} (\xi)
\end{bmatrix}.
$$

(4.12)

In the following, we obtain an explicit representation of the matrices $A_{j,k}$ and $B_{j,k}$. For any $j > k$ we have

$$
\begin{bmatrix}
A_{j,k} (\xi) \\
B_{j,k} (\xi)
\end{bmatrix} = \frac{\varepsilon_j}{2a_j} \begin{bmatrix}
\lambda_j (\xi) & \bar{\mu}_j (\xi) \\
\mu_j (\xi) & \bar{\lambda}_j (\xi)
\end{bmatrix} \begin{bmatrix}
A_{j-1,k} (\xi) \\
B_{j-1,k} (\xi)
\end{bmatrix},
$$

(4.13)

with

$$
\begin{bmatrix}
A_{k,k} (\xi) \\
B_{k,k} (\xi)
\end{bmatrix} = \frac{\varepsilon_k}{2a_k} \begin{bmatrix}
\lambda_k (\xi) \\
\mu_k (\xi)
\end{bmatrix}.
$$

(4.14)
Denoting
\[
\begin{align*}
A_{j,k}(\xi) &= \frac{\varepsilon_k \cdots \varepsilon_j}{2^{j-k+1}a_k \cdots a_j} \lambda_k(\xi) \cdots \lambda_j(\xi) E_{j,k}(\xi) \\
B_{j,k}(\xi) &= \frac{\varepsilon_k \cdots \varepsilon_j}{2^{j-k+1}a_k \cdots a_j} \bar{\lambda}_k(\xi) \cdots \bar{\lambda}_j(\xi) F_{j,k}(\xi)
\end{align*}
\] (4.15)
and using that \( \lambda_j(\xi) = e^{-i\xi\delta_j(j-1)^t} \) and \( \mu_j(\xi) = \gamma_j e^{-i\xi\varepsilon_j(j-1)^t} \), with \( \delta_j, \varepsilon_j \) and \( \gamma_j \) as in (4.6), we obtain that, in order to verify (4.13), \( E_{j,k} \) and \( F_{j,k} \) satisfy
\[
\begin{align*}
E_{j,k}(\xi) &= E_{j-1,k}(\xi) + \gamma_j e^{2i\xi((a_{k+1}+\cdots+a_j))} \cdot e^{2i\xi((k-1)a_k)F_{j-1,k}(\xi)} \\
F_{j,k}(\xi) &= F_{j-1,k}(\xi) + \gamma_j e^{-2i\xi((a_{k+1}+\cdots+a_j))} \cdot e^{-2i\xi((k-1)a_k)E_{j-1,k}(\xi)}
\end{align*}
\]
with \( E_{k,k}(\xi) = 1, \) \( F_{k,k}(\xi) = \gamma_k e^{-2i\xi((k-1)a_k)} \). Introducing \( \tilde{F}_{j,k}(\xi) = e^{2i\xi((k-1)a_k)F_{j,k}(\xi)} \) we obtain (4.10).

In view of the notations in (4.12)
\[
[T_j(\xi) \cdots T_k(\xi)]_{21} = B_{j,k}(\xi), \quad [T_j(\xi) \cdots T_k(\xi)]_{22} = \tilde{A}_{j,k}(\xi)
\]
the desired result follows by using (4.15). \( \square \)

We now give the precise expansions of the coefficients \( C_{1k}^+(\xi), k = 1, N \), besides \( C_{11}^- (\xi) \) and \( C_{1N}^- (\xi) \) in (4.3). In order to enlighten the expressions, set
\[
\alpha_k := \frac{\varepsilon_1 \cdots \varepsilon_{k-1}}{2^{k-1}a_1 \cdots a_{k-1}} (1 - \gamma_1^2) \cdots (1 - \gamma_{k-1}), \quad 2 \leq k \leq N. \tag{4.16}
\]

**Lemma 4.2.** The coefficients \( C_{1k}^+(\xi) \), can be written as follows
\[
C_{11}^-(\xi) = \frac{a_1}{2\pi}, \quad C_{11}^+(\xi) = -C_{11}^-(\xi) \frac{\tilde{F}_{N-1,1}(\xi)}{E_{N-1,1}(\xi)},
\]
\[
C_{1N}^-(\xi) = \frac{\alpha_N}{\bar{\lambda}_1(\xi) \cdots \bar{\lambda}_{N-1}(\xi)} \frac{C_{11}^-(\xi)}{E_{N-1,1}(\xi)}, \quad C_{1N}^+(\xi) = 0,
\]
and for \( 2 \leq k \leq N - 1, \)
\[
C_{1k}^-(\xi) = \frac{\alpha_k}{\bar{\lambda}_1(\xi) \cdots \bar{\lambda}_{k-1}(\xi)} C_{11}^-(\xi) \frac{\bar{E}_{N-1,k}(\xi)}{E_{N-1,1}(\xi)},
\]
\[
C_{1k}^+(\xi) = -\frac{\alpha_k}{\bar{\lambda}_1(\xi) \cdots \bar{\lambda}_{k-1}(\xi)} C_{11}^-(\xi) e^{-2i\xi((k-1)a_k)} \frac{\tilde{F}_{N-1,k}(\xi)}{E_{N-1,1}(\xi)}.
\]

**Proof.** We first emphasize that \([\tilde{T}_{N-1}(\xi) \cdots \tilde{T}_1(\xi)]_{22} \neq 0 \) for all \( \xi \in \mathbb{R} \). This follows from (4.8) since \( a_1 \neq 0 \) (otherwise all coefficients in the matrix would be 0 and the matrix would not be invertible). This implies that \( A_{N-1,1}(\xi) \) and \( E_{N-1,1}(\xi) \) do not vanish on the real line. From (4.8), we get immediately the expression for \( C_{11}^+(\xi) \). We deduce, substituting \( C_{11}^-(\xi), \) that
\[
C_{1N}^-(\xi) = C_{11}^-(\xi) \cdot \frac{|A_{N-1,1}(\xi)|^2 - |B_{N-1,1}(\xi)|^2}{A_{N-1,1}(\xi)}. \tag{4.17}
\]
Notice that due to (4.13), we have for all $1 \leq k < j \leq N - 1$

$$|A_{j,k}(\xi)|^2 - |B_{j,k}(\xi)|^2 = \left(\frac{\varepsilon_j}{2\alpha_j}\right)^2 \cdot (1 - \gamma_j^2) \left[|A_{j-1,k}(\xi)|^2 - |B_{j-1,k}(\xi)|^2\right] = \ldots = $$

$$= \frac{(\varepsilon_{k+1} \cdots \varepsilon_j)^2}{(2^{-k+1}a_{k+1} \cdots a_j)^2} (1 - \gamma_{k+1}^2) \cdots (1 - \gamma_j^2) \left[|A_{k,k}(\xi)|^2 - |B_{k,k}(\xi)|^2\right]$$

$$= \frac{(\varepsilon_k \cdots \varepsilon_j)^2}{(2^{-k+1}a_k \cdots a_j)^2} (1 - \gamma_k^2) \cdots (1 - \gamma_j^2), \tag{4.18}$$

where the last inequality follows from (4.14). Thus, (4.17) rewrites as

$$C_{1N}^-(\xi) = C_{11}^-(\xi) \cdot \frac{(\varepsilon_1 \cdots \varepsilon_{N-1})^2}{(2^{-N+1}a_1 \cdots a_{N-1})^2} (1 - \gamma_1^2) \cdots (1 - \gamma_{N-1}^2) \cdot \frac{1}{[T_{N-1}(\xi) \cdots \xi + 1}_{22}.\]$$

Let us now handle the coefficients $C_{1k}^\pm(\xi)$, $2 \leq k \leq N - 1$. By (4.12), one can check that

$$(\bar{T}_1(\xi) \cdots \bar{T}_k(\xi))^{-1} = \frac{1}{|A_{j,k}(\xi)|^2 - |B_{j,k}(\xi)|^2} \left[ \frac{\bar{A}_{j,k}(\xi)}{-\bar{B}_{j,k}(\xi)} \right] \cdot \frac{1}{\varepsilon_k \cdots \varepsilon_j} (1 - \gamma_k^2) \cdots (1 - \gamma_j^2) \left[ \frac{\bar{A}_{j,k}(\xi)}{-\bar{B}_{j,k}(\xi)} \right]$$

where the last identity follows by (4.18). Taking now $j = N - 1$ in the above relation, by (4.9) we obtain

$$\begin{bmatrix} C_{1k}^- \varepsilon \varepsilon_{N-1})^2 & (2^{-N+1}a_1 \cdots a_{N-1})^2 \\ C_{1k}^+ \varepsilon_1 \varepsilon_{N-1})^2 & (2^{-N+1}a_1 \cdots a_{N-1})^2 \end{bmatrix}(1 - \gamma_k^2) \cdots (1 - \gamma_{N-1}^2) \frac{1}{[T_{N-1}(\xi) \cdots \xi + 1}_{22} \cdot C_{1N}^-(\xi),$$

which implies together with (4.12) that for $2 \leq k \leq N - 1$

$$\begin{bmatrix} C_{1k}^- \varepsilon \varepsilon_{N-1})^2 & (2^{-N+1}a_1 \cdots a_{N-1})^2 \\ C_{1k}^+ \varepsilon_1 \varepsilon_{N-1})^2 & (2^{-N+1}a_1 \cdots a_{N-1})^2 \end{bmatrix}(1 - \gamma_k^2) \cdots (1 - \gamma_{N-1}^2) \frac{1}{[T_{N-1}(\xi) \cdots \xi + 1}_{22} \cdot C_{1N}^-(\xi),$$

Substituting now the previously obtained expression for $C_{1N}^-(\xi)$, we get for $2 \leq k \leq N - 1$

$$\begin{bmatrix} C_{1k}^- \varepsilon \varepsilon_{N-1})^2 & (2^{-N+1}a_1 \cdots a_{N-1})^2 \\ C_{1k}^+ \varepsilon_1 \varepsilon_{N-1})^2 & (2^{-N+1}a_1 \cdots a_{N-1})^2 \end{bmatrix}(1 - \gamma_k^2) \cdots (1 - \gamma_{N-1}^2) \frac{1}{[T_{N-1}(\xi) \cdots \xi + 1}_{22} \cdot C_{1N}^-(\xi),$$

The explicit values given in Lemma 4.1 complete the proof. □

Notice that in Lemma 4.2, $\bar{E}_{N-1,1}(\xi)$ appears as the denominator of the coefficients. In view (4.11) and using that $\bar{E}_{N-1,1}(\xi)$ is non-vanishing on the real line, by Wiener’s
Theorem in [15, Theorem 18.21], we immediately get that
\[
\frac{1}{E_{N-1,1}(\xi)} = \sum_{n_0 \geq 0} c_{n_2 \ldots n_{N-1}} e^{-2i\xi(l(n_2a_2 + \ldots + n_{N-1}a_{N-1})}. 
\]
In what follows we prove that the infinite sum is indexed only by nonnegative integers. This will be crucial in the proof of the Theorem 1.2.

**Lemma 4.3.** There exist constants \((c_{n_0})_{n_0 \geq 0}\) such that
\[
\frac{1}{E_{N-1,1}(\xi)} = \sum_{n_0 \geq 0} c_{n_2 \ldots n_{N-1}} e^{-2i\xi(l(n_2a_2 + \ldots + n_{N-1}a_{N-1}).
\]

**Proof.** We will prove that all the functions \(E_{j,1}(\xi), j = 1, \ldots, N-1\) can be represented as follows:
\[
\frac{1}{E_{j,1}(\xi)} = \sum_{n_0 \geq 0} c_{n_2 \ldots n_{j-1}} e^{2i\xi(l(n_2a_2 + \ldots + n_{j-1}a_{j-1})}. \tag{4.19}
\]

We use (4.10), for \(k = 1 \) and \(j = 1, N-1\) to obtain the following recurrences
\[
\begin{cases}
E_{j,1}(\xi) = E_{j-1,1}(\xi) + \gamma_j e^{2i\xi(l(a_2 + \ldots + a_j)}) \tilde{F}_{j-1,1}(\xi) \\
\tilde{F}_{j,1}(\xi) = \tilde{F}_{j-1,1}(\xi) + \gamma_j e^{-2i\xi(l(a_2 + \ldots + a_j))} E_{j-1,1}(\xi)
\end{cases} \tag{4.20}
\]
with \(E_{1,1}(\xi) = 1, \tilde{F}_{1,1}(\xi) = \gamma_1\), where \(\gamma_1\) is given in (4.6).

We divide the proof in two steps.

**Step 1:** We prove that for any \(j = 1, \ldots, N-1\), \(|\tilde{F}_{j,1}(\xi)/E_{j,1}(\xi)| < 1\). Defining \(w_j = \tilde{F}_{j,1}(\xi)/E_{j,1}(\xi)\) we obtain that it verifies
\[
w_j = \frac{w_{j-1} + b_j}{1 + b_j w_{j-1}}, j \geq 2,
\]
where \(b_j = \gamma_j e^{-2i\xi(l(a_2 + \ldots + a_j)})\) and \(w_1 = \gamma_1\). Since \(|b_j| < 1\), the map \(z \rightarrow \frac{z+b_j}{1+b_jz}\) maps the complex unit disk \(|z| < 1\) to itself. Using that \(|w_1| < 1\) and an inductive argument we obtain that \(|w_j| < 1\) for all \(j \geq 2\) so \(|\tilde{F}_{j,1}(\xi)/E_{j,1}(\xi)| < 1\) for \(j = 1, \ldots, N-1\).

**Step 2:** We prove identity (4.19). We first recall representation (4.11) of \(\tilde{F}_{j,1}\)
\[
\tilde{F}_{j,1}(\xi) = \sum_{i_2, \ldots, i_j \in \{0,1\}} \tilde{c}_{i_2, \ldots, i_j} e^{-2i\xi(l(i_2a_2 + \ldots + i_ja_j)}.
\]
It follows that the following product
\[
e^{2i\xi(l(a_2 + \ldots + a_j))} \tilde{F}_{j,1}(\xi) = \sum_{i_2, \ldots, i_j \in \{0,1\}} \tilde{c}_{i_2, \ldots, i_j} e^{2i\xi(l((1-i_2)a_2 + \ldots + (1-i_j)a_j)}
\]
has a representation of the form (4.19).
Let us now prove \((4.19)\) by an inductive argument. For \(j = 1\) it is obvious. Let us assume that the representation is true for \(j - 1\) and prove it for \(j \geq 2\). The recurrences in \((4.20)\) give us that
\[
\frac{1}{E_{j,1}(\xi)} = \frac{1}{E_{j-1,1}(\xi)} \frac{1}{1 + \gamma_j e^{2i\xi(a_2 + \cdots + a_j)} \tilde{F}_{j-1,1}(\xi)/E_{j-1,1}(\xi)}.
\]
Since \(1/E_{j-1,1}(\xi)\) admits representation \((4.19)\) it is sufficient to analyse the second factor in the above identity. Using Step 1 we can write
\[
\frac{1}{1 + \gamma_j e^{2i\xi(a_2 + \cdots + a_j)} \tilde{F}_{j-1,1}(\xi)/E_{j-1,1}(\xi)} = \sum_{n \geq 0} (-\gamma_j e^{2i\xi a_j})^n \left(\frac{e^{2i\xi(a_2 + \cdots + a_j)} \tilde{F}_{j-1,1}(\xi)}{E_{j-1,1}(\xi)}\right)^n.
\]
Since both factors \(e^{2i\xi(a_2 + \cdots + a_j)} \tilde{F}_{j-1,1}(\xi)\) and \(1/E_{j-1,1}(\xi)\) admit a representation as the one in \((4.19)\) it follows that their \(n\)-th power also has a such representation. Thus the term in the left hand side has the desired representation which finishes the proof. \(\square\)

For any \(t, x \in \mathbb{R}\), we set
\[
h_t(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi^2 t} e^{i\xi x} \frac{1}{E_{N-1,1}(\xi)} \, d\xi.
\]
(4.21)

In particular, when \(N = 2\), \(h_t = k_t\), \(k_t\) being the classical Schrödinger kernel.

**Lemma 4.4.** Let \(N \geq 2\) and \(\alpha_k\), for \(k = \frac{2}{\pi} N\) as in \((4.16)\). The kernels \(p_t^{1,k}\) given in \((4.2)\) can be expressed as
\[
p_t^{1,1}(x, y) = a_1 k_t(a_1 x - a_1 y) - a_1 \sum_{i \in \{0,1\}} \tilde{c}_{i_2,\ldots,i_{N-1}} h_t[a_1 x + a_1 y - 2l(i_2 a_2 + \ldots + i_{N-1} a_{N-1})],
\]
\[
p_t^{1,N}(x, y) = a_1 \alpha_N h_t[a_1 x - a_N(y - (N - 2)l) - l(a_2 + a_3 + \ldots + a_{N-1})]
\]
and for \(2 \leq k \leq N - 1\), \(N \geq 3\),
\[
p_t^{1,k}(x, y) = a_1 \alpha_k \sum_{i \in \{0,1\}} c_{i_{k+1},\ldots,i_{N-1}} h_t[a_1 x - a_k(y - (k - 1)l) - l \sum_{j=2}^{k} a_j - 2l \sum_{j=k+1}^{N-1} i_j a_j]
\]
\[
- a_1 \alpha_k \sum_{i \in \{0,1\}} \tilde{c}_{i_{k+1},\ldots,i_{N-1}} h_t[a_1 x - a_k((k - 1)l - y) - l \sum_{j=2}^{k} a_j - 2l \sum_{j=k+1}^{N-1} i_j a_j].
\]

**Proof.** In view of \((4.2)\), \((4.11)\) and Lemma 4.2, we can rewrite the kernel \(p_t^{1,1}\) in terms of the functions \(k_t\) and \(h_t\) as
\[
p_t^{1,1}(x, y) = \int_{\mathbb{R}} e^{-i\xi^2 t} \left[C^{(-)}_{11}(\xi) e^{i\xi(a_1 x - a_1 y)} + C^{(+)}_{11}(\xi) e^{i\xi(a_1 x + a_1 y)}\right] \, d\xi
\]
\[
= a_1 \int_{\mathbb{R}} e^{-i\xi^2 t} e^{i\xi(a_1 x - a_1 y)} \, d\xi - a_1 \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi^2 t} \frac{1}{E_{N-1,1}(\xi)} \, d\xi
\]
\[
= a_1 \int_{\mathbb{R}} e^{-i\xi^2 t} e^{i\xi(a_1 x - a_1 y)} \, d\xi - a_1 \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi^2 t} \frac{1}{E_{N-1,1}(\xi)} \, d\xi
\]
In the case of $2 \leq k \leq N-1$, by (4.2),
\[ p_i^{1,k}(x, y) = \int_{\mathbb{R}} e^{-i\xi^2 t} \left[ C_{1k}^-(\xi)e^{i\xi(\alpha_1 x - a_k y)} + C_{1k}^+(\xi)e^{i\xi(\alpha_1 x + a_k y)} \right] d\xi =: I_{1,k}^-(t; x, y) + I_{1,k}^+(t; x, y). \]

Let us first remark that
\[ \lambda_1(\xi) \ldots \lambda_{k-1}(\xi) = \exp \left( i\xi (-a_2 \ldots - a_{k-1} + (k-2)a_k) \right), \quad k \geq 2. \]

Let us first treat the integral $I_{1,k}^-(t; x, y)$. By Lemma 4.2 and (4.7), we can rewrite it as
\[ I_{1,k}^-(t; x, y) = a_1 \alpha_k \sum_{i_0 \in \{0,1\}} c_{i_0, i_{k+1}, \ldots, i_{N-1}} h_t \left[ a_1 x - a_k y - l \sum_{j=2}^{k} a_j + (k-1)a_k l - 2l \sum_{j=k+1}^{N-1} i_j a_j \right]. \]

In the case of $I_{1,k}^+(t; x, y)$, we similarly arrive to
\[ I_{1,k}^+(t; x, y) = -a_1 \alpha_k \int_{\mathbb{R}} e^{-i\xi^2 t} e^{i\xi(\alpha_1 x + a_k y)} \lambda_1(\xi) \ldots \lambda_{k-1}(\xi) \frac{E_{N-1,k}(\xi)}{E_{N-1,1}(\xi)} d\xi \]
\[ = -a_1 \alpha_k \sum_{i_0 \in \{0,1\}} \tilde{c}_{i_0, i_{k+1}, \ldots, i_{N-1}} h_t \left[ a_1 x + a_k y - l \sum_{j=2}^{k} a_j - (k-1)a_k l - 2l \sum_{j=k+1}^{N-1} i_j a_j \right]. \]

Thus, for all $2 \leq k \leq N-1$ we can write $p_i^{1,k}$ as
\[ p_i^{1,k}(x, y) = a_1 \alpha_k \sum_{i_0 \in \{0,1\}} c_{i_0, i_{k+1}, \ldots, i_{N-1}} h_t \left[ a_1 x - a_k y - l \sum_{j=2}^{k} a_j + (k-1)a_k l - 2l \sum_{j=k+1}^{N-1} i_j a_j \right] \]
\[ - a_1 \alpha_k \sum_{i_0 \in \{0,1\}} \tilde{c}_{i_0, i_{k+1}, \ldots, i_{N-1}} h_t \left[ a_1 x + a_k y - l \sum_{j=2}^{k} a_j - (k-1)a_k l - 2l \sum_{j=k+1}^{N-1} i_j a_j \right]. \]

In the case of $p_i^{1,N}$, we have again by (4.2) and Lemma 4.2
\[ p_i^{1,N}(x, y) = \int_{\mathbb{R}} e^{-i\xi^2 t} \left[ C_{1N}^-(\xi)e^{i\xi(\alpha_1 x - a_N y)} + C_{1N}^+(\xi)e^{i\xi(\alpha_1 x + a_N y)} \right] d\xi \]
\[ = a_1 \alpha_N \int_{\mathbb{R}} e^{-i\xi^2 t} e^{i\xi(\alpha_1 x - a_N y)} \lambda_1(\xi) \ldots \lambda_{N-1}(\xi) \frac{d\xi}{E_{N-1,1}(\xi)} \]
\[ = a_1 \alpha_N h_t \left[ (a_1 x - a_N y) - l(a_2 + a_3 + \ldots + a_{N-1}) + (N-2)a_N l \right]. \]

This finishes the proof.
Based on these new representations in Lemma 4.4 of the kernels in (4.2), we can rewrite the solution $u$ expressed in (4.1) in a more useful way.

**Lemma 4.5.** Let $u_0 \in L^2(\mathbb{R})$. There exists a function $\psi$, depending on $u_0$ and $(a_i)^N_{i=1}$, supported in $(0, \infty)$ such that for $t \neq 0$ and $x \leq 0$, the solution of (1.2) can be written as

$$u(t, x) = \int_{\mathbb{R}} k_t(a_1x - y)u_0\left(\frac{y}{a_1}\right)1_{(\infty, 0)}(y) dy + \int_{\mathbb{R}} h_t(a_1x - y)\psi(y) dy.$$  

(4.22)

**Proof.** We use (4.1) and Lemma 4.4. Using the first term in the representation of $p_t^{1,1}$ in Lemma 4.4 and a change of variables $y \rightarrow y/a_1$ we obtain the first term in the right hand side of (4.22):

$$a_1 \int_{-\infty}^0 k_t(a_1x-a_1y)u_0(y) dy = \int_{-\infty}^0 k_t(a_1x-y)u_0\left(\frac{y}{a_1}\right) dy = \int_{\mathbb{R}} k_t(a_1x-y)u_0\left(\frac{y}{a_1}\right)1_{(-\infty, 0)}(y) dy.$$

We will prove now that all the other terms in the representation of $u$ are of the form $(h_t * \psi)(a_1x)$ for some function $\psi$ having the support in $(0, \infty)$.

We remark that for any $1 \leq k \leq N-1$ and $y \in I_k$ the new variable

$$z = a_k((k-1)l - y) + l \sum_{j=2}^k a_j + 2l \sum_{j=k+1}^{N-1} i_j a_j$$

runs over the positive real numbers. We use this change of variables to obtain the existence of a function $\psi$ depending on $u_0$ and all the parameters involved in the definition of variable $z$ to obtain that

$$\int_{I_k} h_t[a_1x - a_k((k-1)l - y) - l \sum_{j=2}^k a_j - 2l \sum_{j=k+1}^{N-1} i_j a_j] u_0(y) dy = \int_0^\infty h_t(a_1x - z)\psi(z) dz.$$

We proceed in the same way with the terms containing $a_1x - aky$. For any $2 \leq k \leq N$ and $y \in I_k$ the variable

$$z = a_k(y - (k-1)l) + l \sum_{j=2}^k a_j + 2l \sum_{j=k+1}^{N-1} i_j a_j$$

runs over positive real numbers. Thus there exists a function $\psi$ such that

$$\int_{I_k} h_t[a_1x - a_k(y - (k-1)l) - l \sum_{j=2}^k a_j - 2l \sum_{j=k+1}^{N-1} i_j a_j] u_0(y) dy = \int_0^\infty h_t(a_1x - z)\psi(z) dz.$$

In all the cases we obtain that the integrals are of the form $(h_t * \psi)(a_1x)$ for some function $\psi$ supported on the positive axis. Summing all these functions we obtain the desired representation for the solution $u$. \qed
Lemma 4.6. Let $u$ be a solution of (1.3), such that

$$|u(0, x)| = O(e^{-\alpha x^2}), \quad |u(1, x)| = O(e^{-\beta x^2}), \quad \text{as} \ x \to -\infty,$$

for some $\alpha, \beta > 0$ with $\sqrt{\alpha \beta} > a_1^2/4$. Then, $u(t, x) = 0$ for all $t \in \mathbb{R}$ and $x \leq 0$.

**Proof.** We use the representation in Lemma 4.3, to write $h_t(x)$ as

$$h_t(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it^2 t^2} e^{ix\xi} \sum_{n_0 \geq 0} c_{n_2, \ldots, n_{N-1}} e^{-2i\xi(n_2a_2 + \ldots + n_{N-1}a_{N-1})} d\xi$$

$$= \sum_{n_0 \geq 0} c_{n_2, \ldots, n_{N-1}} k_t[x - 2l(n_2a_2 + \ldots + n_{N-1}a_{N-1})]$$

Using (4.22), we have for $t \neq 0$ and $x \leq 0$, that $u(t, x) = (k_t * \eta)(a_1 x)$ with

$$\eta(y) = u_0 \left( \frac{y}{a_1} \right) \mathbb{1}_{(-\infty, 0)}(y) + \sum_{n_0 \geq 0} c_{n_2, \ldots, n_{N-1}} \psi(y - 2l(n_2a_2 + \ldots + n_{N-1}a_{N-1})), \quad y \in \mathbb{R}.$$  

Using the explicit representation of $k_t$ we have

$$u(t, x) = \frac{1}{\sqrt{4\pi i t}} e^{\frac{|x|^2}{4 t}} \int_{\mathbb{R}} e^{-\frac{a_1 y^2}{2} t} e^{i\pi y \eta(y)} dy = \frac{1}{\sqrt{2i t}} e^{\frac{|x|^2}{4 t}} e^{i\pi \eta(y)} \left( \frac{a_1 x}{2t} \right).$$

Since $|u(1, x)| = O(e^{-\beta x^2})$ as $x \to -\infty$ we obtain

$$|e^{\frac{|x|^2}{4} \eta(x)}| = O(e^{-\frac{\beta x^2}{a_1^2}}), \quad \text{as} \ x \to -\infty.$$  

Since $\text{supp}(\psi) \subseteq (0, \infty)$ and $a_2, \ldots, a_{N-1} > 0$, we have for any $y \leq 0$ that $\eta(y) = u_0(y/a_1)$.

The property $|u(0, x)| = O(e^{-\alpha x^2})$ as $x \to -\infty$ gives us that

$$|e^{\frac{|x|^2}{4} \eta(x)}| = O(e^{-\frac{\alpha x^2}{a_1^2}}), \quad \text{as} \ x \to -\infty.$$  

Thus, by [14, Theorem 2.3 (B)], it follows that as long as $\sqrt{\alpha \beta} > a_1^2/4$ with $\alpha, \beta > 0$, we must have $\eta \equiv 0$ on $\mathbb{R}$, which implies $u(t, x) = 0$, for $t \neq 0$ and $x \leq 0$, which completes the proof. \hfill \Box

We are ready to prove the main result of this section, Theorem 1.2.

**Proof of Theorem 1.2.** We prove the first part since the other two follow from the first one. We will proceed by induction. For $N \geq 1$ let $P(N)$ be the statement: For any $a_1, \ldots, a_N > 0$, if the solution $u_{\sigma_N}$ of the equation

$$\begin{cases}
  iu_t(t, x) + \partial_x(\sigma_N \partial_x u)(t, x) = 0, & x \in \mathbb{R}, t \neq 0, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}
\end{cases}$$

with the piecewise constant function $\sigma_N$ given by $\sigma_1(x) = a_1^{-2}$ if $N = 1$ and for $N \geq 2$

$$\sigma_N(x) = a_k^{-2}, \quad x \in I_k, \quad k = 1, \ldots, N,$$  

satisfies for some positive numbers $\alpha$ and $\beta$ with $\alpha \beta > a_1^2/16$,

$$u_{\sigma_N}(0, x) = O(e^{-\alpha x^2}), \quad u_{\sigma_N}(1, x) = O(e^{-\beta x^2}), \quad \text{as} \ x \to -\infty,$$  

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then \( u_{\sigma_N} \equiv 0 \).  

When \( N = 1 \) let us consider \( u_{\sigma_1} \), solution of  

\[
iu(t, x) + \frac{1}{a_1^2} \partial_{xx}^2 u(t, x) = 0, \quad x \in \mathbb{R}, t \neq 0
\]

that satisfies \( u(0, x) = O(e^{-\alpha x^2}) \) and \( u(1, x) = O(e^{-\beta x^2}) \) as \( x \to -\infty \), for some positive numbers \( \alpha \) and \( \beta \) with \( \alpha \beta > a_1^4/16 \). We consider \( v(t, x) = u_{\sigma_1}(t, x/a_1) \) and apply the results for the one dimensional LSE [14, Theorem 2.3 (B)] to conclude that \( u_{\sigma_1} \equiv 0 \).

Assume now that \( P(N) \), holds true, and we want to prove that \( P(N + 1) \) also holds true, i.e. we want to show that for any \( a_1, \ldots, a_{N+1} > 0 \) and \( \sigma_{N+1}(x) = a_k^{-2}, \quad x \in I_k \), \( k = 1, \ldots, N + 1 \), the solution \( u_{\sigma_{N+1}} \) of the equation

\[
iu(t, x) + \partial_x (\sigma_{N+1} \partial_x u)(t, x) = 0, \quad x \in \mathbb{R}, t \neq 0
\]

satisfying for some positive numbers \( \alpha \) and \( \beta \) with \( \alpha \beta > a_1^4/16 \)

\[
u_{\sigma_{N+1}}(0, x) = O(e^{-\alpha x^2}), \quad u_{\sigma_{N+1}}(1, x) = O(e^{-\beta x^2}), \quad x \to -\infty,
\]

vanishes identically, \( u_{\sigma_{N+1}} \equiv 0 \).

Fix \( a_1, \ldots, a_N, a_{N+1} > 0 \) and consider the corresponding piecewise constant function \( \sigma_{N+1} \). Then, by Lemma 4.6 applied for \( \sigma_{N+1} \), since \( \alpha \beta > a_1^4/16 \), the solution \( u_{\sigma_{N+1}} \) vanishes for all \( t \in \mathbb{R} \) and \( x \leq 0 \). Then, one can check that \( u_{\sigma_{N+1}} \) is solution to

\[
\begin{cases}
iu(t, x) + \partial_x (\sigma_{N+1} \partial_x u)(t, x) = 0, & x \in \mathbb{R}, t \neq 0, \\
u(0) = u_0, & x \in \mathbb{R}
\end{cases}
\]

with \( \sigma_N = a_2^{-2} \) when \( N = 1 \) and for \( N \geq 2 \)

\[
\sigma_N(x) = \begin{cases}
a_2^{-2}, & x \in I_1 := (-\infty, l), \\
a_3^{-2}, & x \in I_2 := (l, 2l), \\
\vdots \\
a_{N+1}^{-2}, & x \in I_N := ((N-1)l, +\infty).
\end{cases}
\]

The translated function \( v_{\sigma_N}(t, x) := u_{\sigma_{N+1}}(t, x + l) \) is solution of

\[
\begin{cases}
v_t(t, x) + \partial_x (\sigma_N \partial_x v)(t, x) = 0, & x \in \mathbb{R}, t \neq 0, \\
v(0, x) = u_0(x + l), & x \in \mathbb{R}
\end{cases}
\]

with \( \sigma_N(x) = a_{k+1}^{-2}, \quad x \in I_k \), \( k = 1, \ldots, N \). Since \( u_{\sigma_{N+1}} \) vanishes for all \( t \in \mathbb{R} \) and \( x \leq 0 \) function \( v \) satisfies

\[
v_{\sigma_N}(0, x) = 0, \quad v_{\sigma_N}(1, x) = 0, \quad \text{for } x < -l.
\]

Thus

\[
v_{\sigma_N}(0, x) = O(e^{-\alpha x^2}), \quad v_{\sigma_N}(1, x) = O(e^{-\beta x^2}), \quad x \to -\infty,
\]

for any \( \alpha \) and \( \beta \) satisfying \( \alpha \beta > 0 \), in particular, for some positive numbers \( \alpha \) and \( \beta \) satisfying \( \alpha \beta > a_2^4/16 \) and thus, by the induction hypothesis, \( v_{\sigma_N} \equiv 0 \). Finally, we get that \( u_{\sigma_{N+1}} \equiv 0 \) and, therefore, \( P(N + 1) \) also holds true.
In order to complete the proof of Theorem 1.2, let us show that in the case of two steps piecewise-constant function $\sigma$, i.e. $N = 2$, the exponents are sharp. More precisely, let

$$\sigma(x) = \begin{cases} a_1^{-2}, & x \in I_1 := (-\infty, 0), \\ a_2^{-2}, & x \in I_2 := (0, \infty), \end{cases}$$

with $a_1, a_2 > 0$. We note that when $N = 2$ in view of (4.21) $h_t = k_t$. Using the representation of $u$ above (see also [7]), the solution $u$ of system (1.3) with $\sigma$ as above, can be written as

$$u(t, x) = \begin{cases} (k_t \ast \psi)(a_1 x), & x < 0, \\ (k_t \ast \psi)(a_2 x), & x > 0, \end{cases}$$

with

$$\left\{ \begin{array}{l} \psi(y) = u_0 \left( \frac{y}{a_1} \right) \mathbb{1}_{(-\infty, 0)}(y) + \frac{a_2 - a_1}{a_1 + a_2} u_0 \left( -\frac{y}{a_1} \right) \mathbb{1}_{(0, \infty)}(y) + \frac{2a_1}{a_1 + a_2} u_0 \left( \frac{y}{a_2} \right) \mathbb{1}_{(0, \infty)}(y), \\ \tilde{\psi}(y) = \frac{2a_2}{a_1 + a_2} u_0 \left( \frac{y}{a_1} \right) \mathbb{1}_{(-\infty, 0)}(y) + u_0 \left( \frac{y}{a_2} \right) \mathbb{1}_{(0, \infty)}(y) + \frac{a_1 - a_2}{a_1 + a_2} u_0 \left( -\frac{y}{a_2} \right) \mathbb{1}_{(-\infty, 0)}(y). \end{array} \right.$$ 

Taking as initial data

$$u_0(x) = \begin{cases} e^{-a_1^2 x^2 - ia_1^2 x^2}, & x \leq 0, \\ e^{-a_2^2 x^2 + ia_2^2 x^2}, & x > 0, \end{cases}$$

the solution at $t = 1$ can be written as

$$u(1, x) = \begin{cases} \sqrt{\frac{1}{1}} e^{ia_1^2 x^2 + e^{-|x|^2} \left( \frac{2a_1 x}{4} \right)}, & x \leq 0, \\ \sqrt{\frac{1}{1}} e^{ia_2^2 x^2 + e^{-|x|^2} \left( \frac{2a_2 x}{4} \right)}, & x > 0, \end{cases}$$

and letting $\alpha = \min\{a_1^2, a_2^2\}$, one gets a nonzero solution satisfying

$$|u(0, x)| \lesssim e^{-\alpha x^2}, \quad |u(1, x)| \lesssim e^{-\alpha x^2}, \quad \text{as } |x| \to \infty.$$

The proof is now complete. \hfill \Box

5. A Carleman Inequality

Let $\Gamma$ be a star-shaped graph, $N$ the number of its edges and the critical exponent $\gamma_\Gamma$ as in (1.4). In the following, we will obtain a Carleman inequality on $\Gamma$ on which we rely the proof of Theorem 1.3. Let consider the set $Z_{\text{comp}}$ defined by

$$Z_{\text{comp}} = \left\{ q = (q_j)_{j = 1}^N \in C([0, T] \times \Gamma), \quad q_j \in C^{1,2}([0, T] \times [0, \infty)) \quad \forall j = 1, N \text{ s.t.} \quad q_j(t, 0) = q_l(t, 0) \quad \forall 1 \leq j, l \leq N, \quad \sum_{j=1}^N q_{j,x}(t, 0) = 0, \quad t \in [0, T] \right\}$$
It is clear that $Z_{\text{comp}}$ is densely embedded in $C([0,T], D(\Delta \Gamma))$. Consider also the weight function $\varphi = (\varphi_j)_{j=1,N}$ given by

$$\varphi_j(t, x) = \mu |\alpha_j x + R t(1-t)|^2 - \frac{(1+\epsilon)R^2 t(1-t)}{16\mu} \quad \forall j = 1,N,$$

for some $\mu > 0$, $\epsilon > 0$, $R > 0$ and $\alpha_j \in \mathbb{R}$ for all $j \in 1,N$, such that

$$\sum_{j=1}^N \varphi_j(x,t,0) = 0, \quad (5.1)$$

i.e., in terms of the vector $\alpha = (\alpha_j)_{j=1,N}$, $\sum_{j=1}^N \alpha_j = 0$. Moreover, one can observe that $\varphi_j(0,t) = \varphi_l(0,t)$, for any $1 \leq j,l \leq N$ and $t \in [0,T]$, so the weight function $\varphi$ belongs to $Z_{\text{comp}}$.

In the proof of the Carleman inequality, we will make use of $N$ weights $(\varphi^k)_{k=1,N}$, with coefficients $(\alpha^k)_{k=1,N}$ such that

(i) If $N$ is even, $\alpha^1 = (1,-1,\ldots,1,-1)$ and $\alpha^k$ is a cyclic permutation of $\alpha^{k-1}$, for all $k = 2,\ldots,N$.

(ii) If $N = 2m+1$, $\alpha^1 = (-1,\ldots,-1,\frac{m+1}{m},\ldots,\frac{m+1}{m})$ and $\alpha^k$ is a cyclic permutation of $\alpha^{k-1}$, for all $k = 2,\ldots,N$.

The particular form of the vectors $\alpha^k$ satisfies the following properties that will be used in the proof of a Carleman inequality:

$$\sum_k \alpha^k_j = 0, \quad \forall j = 1,\ldots,N,$$

and the sum $\sum_k (\alpha^k_j)^2$ is independent on $j$.

**Lemma 5.1.** (Carleman Inequality) Let us consider the weights introduced above. The following inequality

$$\frac{R^2 \epsilon}{8\mu} \sum_{k=1}^N \|e^{\varphi^k}q\|_{L^2([0,1] \times \Gamma)}^2 \leq \sum_{k=1}^N \|e^{\varphi^k}(\partial_t + i\Delta \Gamma)q\|_{L^2([0,1] \times \Gamma)}^2$$

holds for all $\epsilon > 0$, $\mu > 0$, $R > 0$ and $q \in Z_{\text{comp}}$.

**Proof.** Writing explicitly the above norms we will prove that

$$\frac{R^2 \epsilon}{8\mu} \sum_k \sum_{j=1}^N \int_0^1 \int_0^\infty |e^{\varphi^k_j(x,t)}q_j(t,x)|^2 \, dx \, dt \leq \sum_k \sum_{j=1}^N \int_0^1 \int_0^\infty |e^{\varphi^k_j(x,t)}(\partial_t + i\partial_{xx})q_j(t,x)|^2 \, dx \, dt.$$

In the following, for the sake of reading we will not make precise the time dependence unless it is necessary. Following [10], for each $k \in \{1,\ldots,N\}$, we denote $u^k := e^{\varphi^k}q$, and

$$w^k := e^{\varphi^k}(\partial_t + i\Delta \Gamma)q = e^{\varphi^k}(\partial_t + i\Delta \Gamma)e^{-\varphi^k}u^k.$$
Then,
\[
\sum_k \|\omega^k\|_{L^2([0,1] \times \Gamma)} \geq \sum_k \sum_{j=1}^N 4 \int_0^1 \int_0^\infty \varphi_{j,x}^k |u^k_{j,x}|^2 \, dx \, dt + 43 \left( \int_0^1 \int_0^\infty \varphi_{j,xt}^k u^k_{j,x} \, dx \, dt \right)
\]
\[
+ \int_0^1 \int_0^\infty |u_j^k|^2 \left[ -(\varphi_{j,xx}^k - \varphi_{j,x}^k) + 4(\varphi_{j,x}^k)^2 \varphi_{j,xx}^k \right] \, dx \, dt + BT(0),
\]
where the boundary term at \( x = 0 \) is given by
\[
BT(0) = 2 \sum_k \sum_{j=1}^N \int_0^1 \varphi_{j,x}^k(0) |u^k_{j,x}(0)|^2 \, dt + \sum_k \sum_{j=1}^N \int_0^1 (-\varphi_{j,xx}^k(0) + 2(\varphi_{j,x}^k(0)^3)) |u(0)|^2 \, dt
\]
\[
+ 2 \sum_k \sum_{j=1}^N \int_0^1 \varphi_{j,xx}^k(0) \Re(u(0)\bar{u}_{j,x}^k(0)) \, dt + 2 \sum_k \sum_{j=1}^N \int_0^1 \varphi_{j,t}^k(0) \Re(-iu(0)\bar{u}_{j,x}^k(0)) \, dt
\]
\[
+ \sum_k \sum_{j=1}^N \int_0^1 i\varphi_{j,x}^k(0) [u(0)\bar{u}_t(0) - u_t(0)\bar{u}(0)] \, dt =: J_1 + J_2 + J_3 + J_4 + J_5,
\]
with \( u(0) = u_j^k(0, t) \) and \( \varphi(0) = \varphi_j^k(0, t) \), and similarly for the times derivatives.

In view of property (5.1) we immediately obtain that \( J_5 = 0 \). We now proceed with the other terms. For the first one we have
\[
J_1 = 2 \sum_k \sum_{j=1}^N \int_0^1 \varphi_{j,x}^k(0) |\varphi_{j,x}^k(0)q_j(0) + q_{j,x}(0)|^2 e^{2\varphi_j^k(0)} \, dt
\]
\[
= 2 \sum_k \sum_{j=1}^N \int_0^1 (\varphi_{j,x}^k(0))^3 |u(0)|^2 \, dt + 2 \sum_k \sum_{j=1}^N \int_0^1 \varphi_{j,x}^k(0) e^{2\varphi_j^k(0)} |q_{j,x}(0)|^2 \, dt
\]
\[
+ 4 \sum_k \sum_{j=1}^N \int_0^1 \varphi_{j,x}^k(0) \Re(\varphi_{j,x}^k(0)q_j(0)\bar{q}_{j,x}(0)) e^{2\varphi_j^k(0)} \, dt
\]
\[
= 2 \int_0^1 |u(0)|^2 \sum_k \sum_{j=1}^N (\varphi_{j,x}^k(0))^3 \, dt + 2 \int_0^1 e^{2\varphi(0)} \sum_k \sum_{j=1}^N \varphi_{j,x}^k(0) |q_{j,x}(0)|^2 \, dt
\]
\[
+ 4 \Re \int_0^1 u(0) e^{\varphi(0)} \sum_{j=1}^N q_{j,x}(0) \sum_k (\varphi_{j,x}^k(0))^2 \, dt.
\]

We prove that the last two term vanish. Using that \( \sum_k \varphi_{j,x}^k(0) = 0 \) for all \( j = 1, \ldots, N \), we obtain that the second one vanishes. Since \( \sum_k (\varphi_{j,x}^k(0))^2 \) is independent of \( j \) and \( \sum_{j=1}^N q_{j,x}(0) = 0 \), the last term in the above right hand side is zero. For any \( j = 1, \ldots, N \), \( \sum_k (\varphi_{j,x}^k(t,0))^3 = A(t) \) where \( A(t) \geq 0 \). In particular, when \( N \) is even \( A(t) = 0 \). This gives us that \( J_1 \geq 0 \). In the case of \( J_2 \) we use that all the third order derivatives of \( \varphi^k \) vanish
and we have $J_2 = J_1 \geq 0$. In the case of $J_3$ we use that

$$J_3 = \sum_k \sum_{j=1}^N \int_0^1 \varphi_{j,xx}^k(0) \Re[\mathbf{u}(0)e^{\varphi(0)}(\varphi_{j,x}^k(0)\bar{\varphi}_{j}(0) + \bar{\varphi}_{j,x}(0)))] \, dt$$

$$= \sum_k \sum_{j=1}^N \int_0^1 \varphi_{j,xx}^k(0) \Re[\mathbf{u}(0)|^2 \varphi_{j,x}^k(0) + \mathbf{u}(0)e^{\varphi(0)}\bar{\varphi}_{j,x}(0)))] \, dt.$$ 

Since $\varphi_j^k(0) = \varphi_j^k(0)$, for any $k, j, \tilde{k}, \tilde{j} \in \{1, \ldots, N\}$ and $\sum_k \varphi_{j,xx}(0)$ does not depend on $j$, we have that the sum of the last term vanishes and therefore,

$$J_3 = \int_0^1 |\mathbf{u}(0)|^2 \sum_k \sum_{j=1}^N \varphi_{j,xx}^k(0) \varphi_{j,x}^k(0) \, dt = 4R\mu^2 \int_0^1 |\mathbf{u}(0)|^2 t(1 - t) \, dt \sum_k \sum_{j=1}^N (\alpha_j^k)^3 \geq 0.$$ 

Denoting by $\varphi(t, 0)$ the common value of $\varphi_{j,t}^k(t, 0), 1 \leq j, k \leq N$ we get

$$J_4 = 2 \sum_k \sum_{j=1}^N \int_0^1 \varphi(t, 0) \Re[-i\mathbf{u}(0)(\varphi_{j,x}^k(0)\bar{\varphi}_{j}(0) + \bar{\varphi}_{j,x}(0))e^{\varphi(0)}] \, dt$$

$$= 2 \sum_k \sum_{j=1}^N \int_0^1 \varphi(t, 0) \Re[-i\mathbf{u}(0)\varphi_{j,x}^k(0)\bar{\mathbf{u}}(0) - i\mathbf{u}(0)\bar{\varphi}_{j,x}(0)e^{\varphi(0)}] \, dt$$

$$= 2N \int_0^1 \varphi(t, 0)e^{\varphi(0)}\Re[-i\mathbf{u}(0)\sum_{j=1}^N \bar{\varphi}_{j,x}(0)] \, dt = 0,$$

where we used the fact that $\varphi_{j,x}^k$ are real valued functions. The above estimates show that $BT(0) \geq 0$ and therefore

$$\sum_k \|\mathbf{u}^k\|_{L^2((0,1] \times \Gamma)}^2 \geq \sum_k \sum_{j=1}^N \int_0^1 \int_0^\infty 4\varphi_{j,xx}^k |u_{j,x}^k|^2 \, dx \, dt + 4\Re \int_0^1 \int_0^\infty \varphi_{j,xt}^k u_{j,x}^k \bar{u}_{j,x}^k \, dx \, dt$$

$$+ \sum_k \sum_{j=1}^N \int_0^1 \int_0^\infty |u_{j,x}^k|^2 [-\varphi_{j,xx}^k + \varphi_{j,tt}^k + 4(\varphi_{j,x}^k)^2 \varphi_{j,xx}^k] \, dx \, dt.$$ 

Notice that for all $k$ and $j$ in $\{1, \ldots, N\}$, $\varphi_{j,xx}(x, t) = 0$ and $\varphi_{j,xx}(x, t) = 2\mu(\alpha_j^k)^2$. Then, we make squares and we can write

$$\sum_k \|\mathbf{u}^k\|_{L^2((0,1] \times \Gamma)}^2 \geq \sum_k \sum_{j=1}^N \int_0^1 \int_0^\infty \left|2\sqrt{\varphi_{j,xx}^k} u_{j,x}^k - \frac{i\varphi_{j,xt}^k}{\sqrt{\varphi_{j,xx}^k}} u_j^k \right|^2 \, dx \, dt$$

$$+ \sum_k \sum_{j=1}^N \int_0^1 \int_0^\infty \left[\varphi_{j,tt}^k + 4\varphi_{j,xx}(\varphi_{j,x}^k)^2 - \frac{\left(\varphi_{j,xt}^k\right)^2}{\varphi_{j,xx}^k}\right] |u_{j,x}^k|^2 \, dx \, dt.$$
From the definition of $u_j^k$ we get that
\[
\varphi_{j,t}^k + 4 \varphi_{j,xx}^k (\varphi_{j,xx}^k)^2 - \frac{(\varphi_{j,xx}^k)^2}{\varphi_{j,xx}^k} = 32(\alpha_j^k)^4 \mu^3 (\alpha_j^k x + R t(1-t))^2 - 4 R \mu (\alpha_j^k x + R t(1-t)) + \frac{R^2(1+\epsilon)}{8 \mu} = 32(\alpha_j^k)^4 \mu^3 (\alpha_j^k x + R t(1-t) - \frac{R}{16 \mu^2 (\alpha_j^k)^4})^2 - \frac{R^2}{8 \mu (\alpha_j^k)^4} + \frac{R^2(1+\epsilon)}{8 \mu}
\]
\[
\geq \frac{R^2}{8 \mu} \left( 1 - \frac{1}{(\alpha_j^k)^4} + \epsilon \right) \geq \frac{R^2 \epsilon}{8 \mu},
\]
where the last inequality holds due to the fact that $|\alpha_j^k| \geq 1$, for all $k$ and $j$ in $\{1, \ldots, N\}$. Finally this implies that
\[
\sum_k \|u_j^k\|^2_{L^2([0,1] \times \Gamma)} \geq \frac{R^2 \epsilon}{8 \mu} \sum_j \sum_{k=1}^N \int_0^1 \int_0^\infty |u_j^k|^2 \, dx \, dt,
\]
which concludes the result. \hfill \Box

6. **Proof of Theorem 1.3**

Before proving Theorem 1.3, we need to study the behavior of a solution of the Schrödinger equation

\[
\begin{align*}
    \begin{cases}
        u_t = i(\Delta + V(t, x)) u & \text{in } [0,1] \times \Gamma, \\
        u(0) = u_0, x \in \Gamma. 
    \end{cases}
\end{align*}
\]
(6.1)
in the star-shaped graph $\Gamma$ with Gaussian decay at $t = 0$ and $t = 1$. More precisely, we need to show that such a solution has Gaussian decay at any time in between.

Through this section, we will denote by $\| \cdot \|_2$ and $\| \cdot \|_\infty$ the $L^2(\Gamma)$ and $L^\infty(\Gamma)$ norms.

**Theorem 6.1.** Assume that $u$ in $C([0,1], L^2(\Gamma))$ verifies (6.1), $V(t, x) = V_1(x) + V_2(t, x)$ where $V_1$ is real-valued, $\|V_1\|_\infty \leq M_1$ and that there are two positive numbers $\alpha$ and $\beta$ such that

\[
\|e^{\alpha|x|^2} u(0)\|_2, \|e^{\beta|x|^2} u(1)\|_2, \text{ and sup}_{[0,1]} \|e^{\alpha |x|} V_2(t)\|_\infty < +\infty.
\]

Then, there is a constant $N = N(\alpha, \beta)$ such that

\[
\|e^{\alpha |x|} u(t)\|_2 \leq e^{N(M_1 + M_2 + M_3^2 + M_4^2)} \|e^{\alpha |x|} u(0)\|_2 \|e^{\beta |x|^2} u(1)\|_2 \sup_{\sqrt{\alpha}} e^{\alpha |x|} V_2(t)\|_\infty < +\infty.
\]

when $0 \leq t \leq 1$, $M_2 = \sup_{[0,1]} \|e^{\alpha |x|} V_2(t)\|_\infty < +\infty$. Moreover

\[
\|\sqrt{t(1-t)} e^{\alpha |x|} \nabla u\|_{L^2([0,1] \times \Gamma)} \leq N e^{N(M_1 + M_2 + M_3^2 + M_4^2)} \left[ \|e^{\alpha |x|^2} u(0)\|_2 + \|e^{\beta |x|^2} u(1)\|_2 \right].
\]
The proof of this Theorem follows very closely the proof of the result in the real line, given in [5], and therefore we skip the details.

**Proof of Theorem 1.3.** Using the Appell transform (see Section 7) we can consider the case \( \alpha = \beta = \gamma > 2\gamma^2 \). The subsequent formal computations are justified by Theorem 6.1. Since \( \gamma > 2\gamma^2 \), we can choose \( \mu > 1/2 \) and \( \epsilon > 0 \) such that

\[
\frac{(2\gamma^2)2(1+\epsilon)^{3/2}}{2(1-\epsilon)^3} < \frac{(2\gamma^2)2}{1+\epsilon}, \tag{6.2}
\]

and the smooth functions \( \theta_M \) and \( \eta_R \), for \( M \gg R > 2 \), verifying

\[
\theta_M(x) = \begin{cases} 1, & x \in [0, M] \\ 0, & x \in (2M, \infty) \end{cases}, \quad 0 \leq \theta_M \leq 1,
\]

\[
\eta_R(t) = \begin{cases} 1, & t \in \left[ \frac{1}{R}, 1 - \frac{1}{R} \right] \\ 0, & t \in \left[ 0, \frac{1}{2R} \right] \cup \left[ 1 - \frac{1}{2R}, 1 \right] \end{cases}, \quad 0 \leq \eta_R \leq 1.
\]

We define the space-time truncation of \( u \)

\[
q_j(t, x) = \theta_M(x)\eta_R(t)u_j(t, x),
\]

and since \( q = (q_j)_{j=1}^N \in \mathcal{Z}_{\text{comp}} \), we can use the previous Carleman estimate. Note also that

\[
(\partial_t + i\partial_{xx})q_j = \theta_M\eta_R(\partial_t + i\partial_{xx})u_j + \eta_R^\prime \theta_M u_j + (\theta_M'' \eta_R u_j + 2\theta_M' \eta_R u_{j,x}).
\]

Therefore, in view of the Carleman estimates (5.2)

\[
\frac{R^2\epsilon}{8\mu} \sum_{k=1}^N \|e^{\phi^k}q_j\|_{L^2([0,1] \times \Gamma)}^2 \leq \sum_{k=1}^N \sum_{j=1}^N \int_0^1 \int_0^\infty |e^{\phi^k}V_j q_j|^2 + |e^{\phi^k}\eta_R^\prime \theta_M u_j|^2 \, dx \, dt \\
+ \sum_{k=1}^N \sum_{j=1}^N \int_0^1 \int_0^\infty |e^{\phi^k}\eta_R(\theta_M'' u_j + 2\theta_M' u_{j,x})|^2 \, dx \, dt. \tag{6.3}
\]

Since

\[
\sum_{k=1}^N \sum_{j=1}^N \int_0^1 \int_0^\infty |e^{\phi^k}V_j q_j|^2 \, dx \, dt \leq \|V\|_{\infty}^2 \sum_{k=1}^N \sum_{j=1}^N \int_0^1 \int_0^\infty |e^{\phi^k}q_j|^2 \, dx \, dt,
\]

taking \( R \geq 2\sqrt{\frac{8\mu\epsilon}{\epsilon}} \|V\|_{\infty} \), this term can be absorbed in the left-hand side. In the following computations, all the constants involved may depend on the behavior of the solution at times \( t = 0 \) and \( t = 1 \), the potential, and also on the parameters \( \gamma, \mu \) or \( \epsilon \), but they will not depend on \( R \) and \( M \).

Since the integrand in the second term is in fact supported in \( x \in [0, 2M] \) and \( t \in \left[ \frac{1}{2R}, \frac{1}{R} \right] \cup \left[ 1 - \frac{1}{R}, 1 - \frac{1}{2R} \right] \) by (6.2) we have that for such \( x \) and \( t \)

\[
\varphi_j^k(t, x) \leq \mu[(\alpha_j^k)^2 x^2 + R^2 t^2 (1-t)^2 + 2\alpha_j^k x R t (1-t)]
\]
\[ \leq (\alpha_j^k)^2 \mu (1 + \epsilon) x^2 + R^2 t^2 (1 - t)^2 \mu \left( 1 + \frac{1}{\epsilon} \right) \leq \max_{k,j} |\alpha_j^k|^2 \mu (1 + \epsilon) x^2 + \frac{\gamma}{(2 \gamma R)^2 \epsilon}. \]

Since \(|\alpha_j^k| \leq 2 \gamma R\) the second inequality in (6.2) gives us that
\[ \varphi_j^k (t, x) \leq \gamma x^2 + \frac{\gamma}{(2 \gamma R)^2 \epsilon}. \]

Hence, we can estimate all the terms uniformly in \(k\) and obtain that
\[
\begin{align*}
\sum_{k=1}^{N} \sum_{j=1}^{N} \int_0^1 \int_0^\infty |e^{\gamma k \eta R (\theta'^{\gamma}_M u_j + 2 \theta^\prime_M u_{j,x})}|^2 \, dx \, dt &\leq NR^2 e^{\frac{2 \gamma R^2}{2 \gamma R^2}} \sum_{j=1}^{N} \int_0^1 \int_0^\infty |e^{\gamma x^2} u_j|^2 \, dx \, dt \\
&\lesssim R^2 \sup_{t \in [0, 1]} \|e^{\gamma x^2} u(t)\|_2^2.
\end{align*}
\]

Taking into account that the integrand in the last term of (6.3) is supported now in \(x \in [M, 2M] \) and \(t \in \left[ \frac{1}{2R}, 1 - \frac{1}{2R} \right] \), similarly as before we get
\[ \varphi_j^k (x, t) \leq (\alpha_j^k)^2 \mu (1 + \epsilon) x^2 + R^2 t^2 (1 - t)^2 \mu \left( 1 + \frac{1}{\epsilon} \right) \]
\[ \leq \gamma x^2 + \frac{R^2}{16} \mu \left( 1 + \frac{1}{\epsilon} \right) \leq \gamma x^2 + \frac{R^2 \gamma}{16 (2 \gamma R)^2 \epsilon}. \]

Therefore,
\[
\sum_{k=1}^{N} \sum_{j=1}^{N} \int_0^1 \int_0^\infty |e^{\gamma k \eta R (\theta'^{\gamma}_M u_j + 2 \theta^\prime_M u_{j,x})}|^2 \, dx \, dt \\
\leq \frac{N}{M^2} e^{C_2 R^2} \sum_{j=1}^{N} \int_0^1 \int_0^\infty |e^{\gamma x^2} (u_j + u_{j,x})|^2 \, dx \, dt \\
\leq \frac{C_1}{M^2} e^{C_2 R^2} \left[ \sum_{j=1}^{N} \int_0^1 \int_0^\infty |e^{\gamma x^2} u_j|^2 \, dx \, dt + \sum_{j=1}^{N} \int_0^1 \int_0^\infty |e^{\gamma x^2} u_{j,x}|^2 \, dx \, dt \right] \\
\leq \frac{C_1}{M^2} e^{C_2 R^2} \left[ \sup_{t \in [0, 1]} \|e^{\gamma x^2} u(t)\|_2^2 + R^2 \int_0^1 \int_0^\infty t(1-t)|e^{\gamma x^2} u_{j,x}|^2 \, dx \, dt \right].
\]

Hence, thanks to Theorem 6.1 the last term on (6.3) is bounded by
\[
\sum_{k=1}^{N} \sum_{j=1}^{N} \int_0^1 \int_0^\infty |e^{\gamma k \eta R (\theta'^{\gamma}_M u_j + 2 \theta^\prime_M u_{j,x})}|^2 \, dx \, dt \leq \frac{C_1}{M^2} R^2 e^{C_2 R^2} \mathcal{K}.
\]

Gathering now all these estimates, we have that
\[
\frac{R^2 \epsilon}{8 \mu} \sum_{k=1}^{N} \sum_{j=1}^{N} \|e^{\varphi_j^k} q_j\|_{L^2([0, 1] \times \Gamma)}^2 \leq CR^2 + \frac{C_1 R^2}{M^2} e^{C_2 R^2} \mathcal{K}.
\]
On the other hand for each $j$ we can find a $k$ such that $\alpha_j^k = -1$. By discarding all the other values of $k$,

$$\sum_{j=1}^N \sum_{k=1}^N \|e^{\gamma_j^k}q_j\|_{L^2([0,1] \times \Gamma)}^2 \geq \sum_{j=1}^N \|e^{\gamma_j^k-x+Rt(1-t)^2-(1+\epsilon)\frac{R^2t(1-t)}{16\mu}}q_j\|_{L^2([0,1] \times \Gamma)}^2.$$ 

If $x \in \left[0, \frac{\epsilon(1-\epsilon)^2R}{4}\right]$ and $t \in \left[\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}\right]$, then $\theta_M = 1$ for $M \gg R$ and $\eta_K = 1$ for $1/R < (1-\epsilon)/2$. Moreover, in this region, $| -x + Rt(1-t) | > Rt(1-t) - x > R(1-\epsilon)^2/4$, so

$$\mu(-x + Rt(1-t))^2 - \frac{(1+\epsilon)R^2t(1-t)}{16\mu} \geq \frac{1}{4} \frac{R^2}{16\mu} (4\mu^2(1-\epsilon)^6 - (1+\epsilon)^3) > 0,$$

since $\mu > \frac{(1+\epsilon)^{3/2}}{2(1-\epsilon)^3}$. Hence, there exists a constant $C_{\gamma,\epsilon}$ such that

$$\sum_{j=1}^N \|e^{\gamma_j^k-x+Rt(1-t)^2-(1+\epsilon)\frac{R^2t(1-t)}{16\mu}}q_j\|_{L^2([0,1] \times \Gamma)}^2 \geq e^{C_{\gamma,\epsilon}R^2} \sum_{j=1}^N \|u_j\|_{L^2\left([\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}] \times [0, c(1-\epsilon)^2R]\right)}^2.$$ 

Thus we show that there exists a positive constant $C_{\gamma,\epsilon,V}$ such that

$$e^{C_{\gamma,\epsilon}R^2} \sum_{j=1}^N \|u_j\|_{L^2\left([\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}] \times [0, c(1-\epsilon)^2R]\right)} \leq C_{\gamma,\epsilon,V} + C_{\gamma,\epsilon,V} e^{C_2R^2} \frac{e^{M^2\mathcal{K}}}{M^2}.$$ 

By letting $M$ tend to infinity, we have

$$e^{C_{\gamma,\epsilon}R^2} \sum_{j=1}^N \|u_j\|_{L^2\left([\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}] \times [0, c(1-\epsilon)^2R]\right)} \leq C_{\gamma,\epsilon,V}.$$ 

Next, using the error estimate

$$\mathcal{N}^{-1} \|u_j(0)\|_2 \leq \|u_j(t)\|_2 \leq \mathcal{N} \|u_j(0)\|_2, \quad \mathcal{N} = e^{\sup_{[0,1]} \|3V(t)\|_\infty},$$

which can be proved in the same way as the analogous estimate in the real line, and

$$\|u_j(t)\|_2 \leq \|u_j(t)\|_{L^2\left([0, c(1-\epsilon)^2R]\right)} + e^{-\gamma R^2 \frac{t(1-\epsilon)^4}{16}} C_{\gamma,\epsilon,V}, \quad 0 \leq t \leq 1,$$

we show that there exists a positive constant

$$\tilde{C}_{\gamma,\epsilon} = \min \left\{ C_{\gamma,\epsilon}, \frac{\gamma\epsilon^2(1-\epsilon)^4}{16} \right\},$$

such that $e^{\tilde{C}_{\gamma,\epsilon}R^2} \|u(0)\|_2 \leq C_{\gamma,\epsilon,V}$. Indeed,

$$e^{\tilde{C}_{\gamma,\epsilon}R^2} \|u(0)\|_2 \leq \mathcal{N} e^{\tilde{C}_{\gamma,\epsilon}R^2} \sum_{j=1}^N \|u_j(t)\|_2 \leq \mathcal{N} e^{\tilde{C}_{\gamma,\epsilon}R^2} \sum_{j=1}^N \|u_j(0)\|_2 \leq \mathcal{N} e^{\tilde{C}_{\gamma,\epsilon}R^2} \sum_{j=1}^N \|u_j(t)\|_2 \leq \mathcal{N} e^{\tilde{C}_{\gamma,\epsilon}R^2} \sum_{j=1}^N \|u_j(0)\|_2.$$
\[
\lesssim N e^{C \gamma, \varepsilon R^2} \sum_{j=1}^{N} \left\| u_j \right\|_{L^2\left(\left[0, \frac{\alpha (1-\varepsilon)^2 R^4}{4} \right]\right)} + e^{C \gamma, \varepsilon R^2} \frac{C_\gamma, \varepsilon R^2}{t_0}.
\]

and hence, integrating the last inequality in \( t \in \left[\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right], \)

\[
C_\varepsilon e^{C \gamma, \varepsilon R^2} \left\| u(0) \right\|_2 \leq N e^{C \gamma, \varepsilon R^2} \sum_{j=1}^{N} \left\| u_j \right\|_{L^2\left(\left[\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right] \times \left[0, \frac{\alpha (1-\varepsilon)^2 R^4}{4}\right]\right)} + C_\varepsilon e^{C \gamma, \varepsilon R^2} \frac{(1-\varepsilon)^4}{t_0} \leq C.
\]

Letting \( R \) go to infinity, we conclude \( u \equiv 0. \)

\[\square\]

7. Appendix. Appell Transform

In order to study the behavior of solutions of the Schrödinger equation with a potential, we will restrict ourselves to the case where the rates of decay at times \( t = 0 \) and \( t = 1 \) are the same. We will reduce from the general case to this case by means of the so-called Appell transformation (see [5] for the proof).

**Lemma 7.1.** Assume that \( u(s, y) \) verifies

\[
\partial_s u = (A + iB)(\Delta u + V(s, y)u + F(s, y)), \quad \text{in } [0, 1] \times \Gamma,
\]

where \( A + iB \neq 0, \alpha \) and \( \beta \) are positive, \( \gamma \in \mathbb{R} \), and set

\[
\tilde{u}(t, x) = \left(\frac{(\alpha \beta)^{1/4}}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}\right)^{1/2} \mathbf{u} \left(\frac{\sqrt{\beta t}}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}, \frac{(\alpha \beta)^{1/4} x}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}\right) e^{\frac{(\sqrt{\alpha - \sqrt{\beta}})(s, y)^2}{2(\sqrt{\alpha(1-t)} + \sqrt{\beta t})}}.
\]

Then \( \tilde{u} \) verifies

\[
\partial_t \tilde{u} = (A + iB)(\Delta \tilde{u} + \tilde{V}(t, x)\tilde{u} + \tilde{F}(t, x)), \quad \text{in } [0, 1] \times \Gamma,
\]

with

\[
\tilde{V}(t, x) = \left(\frac{(\alpha \beta)^{1/4}}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}\right)^5 V \left(\frac{\sqrt{\beta t}}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}, \frac{(\alpha \beta)^{1/4} x}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}\right),
\]

\[
\tilde{F}(t, x) = \left(\frac{(\alpha \beta)^{1/4}}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}\right)^{5/2} F \left(\frac{\sqrt{\beta t}}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}, \frac{(\alpha \beta)^{1/4} x}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}\right) e^{\frac{(\sqrt{\alpha - \sqrt{\beta}})(s, y)^2}{2(\sqrt{\alpha(1-t)} + \sqrt{\beta t})}}.
\]

Moreover

\[
\left\| e^{\gamma |s|^2} \tilde{F}(t) \right\| = \left\| e^{\gamma |s|^2} \mathbf{F}(s) \right\| e^{\frac{2 \sqrt{\alpha - \sqrt{\beta}}}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}} + \frac{(\sqrt{\alpha - \sqrt{\beta}} A)}{4(A^2 + B^2)(\sqrt{\alpha(1-t)} + \sqrt{\beta t})}}|y|^2 \left\| \mathbf{F}(s) \right\|
\]

and

\[
\left\| e^{\gamma |s|^2} \tilde{u}(t) \right\| = \left\| e^{\gamma |s|^2} \mathbf{u}(s) \right\| e^{\frac{2 \sqrt{\alpha - \sqrt{\beta}}}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}} + \frac{(\sqrt{\alpha - \sqrt{\beta}} A)}{4(A^2 + B^2)(\sqrt{\alpha(1-t)} + \sqrt{\beta t})}}|y|^2 \left\| \mathbf{u}(s) \right\|
\]

when \( s = \frac{\sqrt{\beta t}}{\sqrt{\alpha(1-t)} + \sqrt{\beta t}}. \)

The proof is based on explicit computations and the fact that the first derivative of function \( x \to \exp(-x^2) \) vanishes at \( x = 0. \)
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