Inelastic interaction of nearly equal solitons for the quartic gKdV equation

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Abstract

This paper presents a complete description of the interaction of two solitons with nearly equal speeds for the quartic (gKdV) equation

$$\partial_t u + \partial_x (\partial_x^2 u + u^4) = 0, \quad t, x \in \mathbb{R}.$$ \hfill (0.1)

For \(c > 0, y_0 \in \mathbb{R}\), we call soliton a solution of (0.1) of the form \(R_{c,y_0}(t,x) = Q_c(x - ct - y_0)\), where \(Q''_c + Q^4_c = cQ_c\). Since (0.1) is not an integrable model, the general question of the collision of two given solitons \(R_{c_1,y_1}, R_{c_2,y_2}\) with \(c_1 \neq c_2\) is an open problem.

We focus on the special case where the two solitons have nearly equal speeds: let \(U(t)\) be the solution of (0.1) satisfying

$$\lim_{t \to -\infty} \|U(t) - Q_{c_1^-}(\cdot - c_1^- t) - Q_{c_2^-}(\cdot - c_2^- t)\|_{H^1} = 0,$$

for \(\mu_0 = (c_2^- - c_1^-)/(c_1^- + c_2^-) > 0\) small. By constructing an approximate solution of (0.1), we prove in particular that, for all time \(t \in \mathbb{R}\),

$$U(t) = Q_{c_1(t)}(x - y_1(t)) + Q_{c_2(t)}(x - y_2(t)) + w(t) \quad \text{ where } \|w(t)\|_{H^1} \leq |\ln \mu_0| \mu_0^2,$$

with \(y_1(t) - y_2(t) > 2|\ln \mu_0| + O(1)\). These estimates mean that the two solitons are preserved by the interaction and that for all time they are separated by a large distance, as in the case of the integrable KdV equation in this regime.

However, unlike in the integrable case, we prove that the collision is not perfectly elastic, in the following sense:

$$\lim_{t \to +\infty} c_1(t) > c_2^- \quad \text{ and } \quad \lim_{t \to +\infty} c_2(t) < c_1^- \quad \text{ and } \quad w(t) \not\to 0 \text{ in } H^1 \text{ as } t \to +\infty.$$

1 Introduction

We consider the generalized KdV equation

$$\partial_t u + \partial_x (\partial_x^2 u + u^p) = 0, \quad t, x \in \mathbb{R}.$$ \hfill (1.1)

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Recall that \( p = 2 \) and \( 3 \) correspond respectively to the (KdV) and (mKdV) equations, which are completely integrable models. In this paper, we focus on the nonintegrable case \( p = 4 \).

As usual, we call solitons solutions of \((1.1)\) of the form \( R_{c,y_0}(t,x) = Q_c(x - ct - y_0) \), for \( c > 0 \), \( y_0 \in \mathbb{R} \), where \( Q_c(x) = c^{\frac{1}{p-1}}Q(\sqrt{c}x) \) and \( Q \) satisfies
\[
Q'' + Q^p = Q, \quad Q(x) = \left( \frac{p + 1}{2\cosh^2(c^{-\frac{1}{2}}x)} \right)^{\frac{1}{p-1}}.
\]

1.1 Review of the collision problem for the (gKdV) model

We start with the classical integrable (KdV) equation
\[
\partial_t u + \partial_x (\partial_x^2 u + u^2) = 0. \tag{1.2}
\]

First, it is very well-known that the (KdV) equation has explicit pure \( N \)-soliton solutions (see [11], [36], [30]). Namely, for any given \( c_1 > \ldots > c_N > 0 \), \( y_1^-, \ldots, y_N^- \in \mathbb{R} \), there exists an explicit multi-soliton solution \( u(t,x) \) of \((1.2)\) which satisfies
\[
\lim_{t \to \pm \infty} \left\| u(t) - \sum_{j=1}^N Q_{c_j}(-c_j t - y_j^+) \right\|_{H^1(\mathbb{R})} = 0,
\]
for some \( y_j^+ \) such that the shifts \( \Delta_j = y_j^+ - y_j^- \) depends on the \((c_k)\). Recall that explicit formulas for such solutions were derived using the inverse scattering transform.

Recall also that before the discovery of these explicit solutions, Fermi, Pasta and Ulam [7] and Zabusky and Kruskal [40] discovered numerically remarkable phenomena related to solitons collision. Lax ([17]) has developed a mathematical framework to study these problems, known now as complete integrability and then other decisive developments appeared, such as the inverse scattering transform (for a review on this theory, we refer for example to Miura [30]).

The \( N \)-solitons are fundamental in studying the properties of general solutions of equation \((1.2)\) in particular because of the so-called decomposition property (Kruskal [15], Eckhaus and Schuur [3], [34], Cohen [3]), which states that the asymptotic behavior in large time of any sufficiently regular and decaying solution is governed by a finite number of solitons.

Stability and asymptotic stability of \( N \)-solitons were studied by Maddocks and Sachs [19] in \( H^N \) by variational techniques and in the energy space \( H^1 \) by Martel, Merle and Tsai [29].

Second, recall that LeVeque [18] investigated the behavior of the explicit 2-soliton \( U_{c_1,c_2} \) satisfying
\[
\lim_{t \to \pm \infty} \left\| U_{c_1,c_2}(t) - Q_{c_1}(-c_1 t - y_1^+) - Q_{c_2}(-c_2 t - y_2^+) \right\|_{H^1(\mathbb{R})} = 0, \tag{1.3}
\]
in the asymptotic \( \mu_0 = \frac{c_2 - c_1}{c_1 + c_2} > 0 \) small i.e. for nearly equal solitons. In [18], the following estimate
\[
\sup_{t,x \in \mathbb{R}} \left| U_{c_1,c_2}(t,x) - Q_{c_1(t)}(x-y_1(t)) - Q_{c_2(t)}(x-y_2(t)) \right| \leq C \mu_0^2, \tag{1.4}
\]
is proved for some explicit functions $c_j(t), y_j(t)$. Moreover, it is proved that
\[
\min_{t \in \mathbb{R}} (y_1(t) - y_2(t)) = 2|\ln \mu_0| + O(1),
\] (1.5)
which means that the minimum separation between the two solitons goes to $\infty$ as $\mu_0 \to 0$. See also [6], where Ei and Ohta investigated formally the dynamics of interacting pulses for several models.

Apart from integrability theory, questions on interaction of solitons have been studied since the 60’s from both numerical and experimental points of view. Fermi, Pasta and Ulam [7], Zabusky and Kruskal [40] and Zabusky [39] have introduced nonlinear systems and computed interaction of nonlinear objects by numerics. Since then, many other systems have been studied numerically. Bona et al. [1], and Kalisch and Bona [12], focused on the problem of collision of two solitary waves for the Benjamin and the BBM equations. Shih [35] studied the case of the generalized KdV equations with nonlinearities $|u|^p$ for some non integer values of $p$. Recently, Craig et al. [4] presented new numerical and experimental works for the water wave problem. These works give evidence that in general, unlike for the pure $N$-solitons of the integrable case, the collision of two solitary waves fails to be elastic by a small dispersion. For experimental literature, see e.g. Weidman and Maxworthy [37], Hammack et al. [8], Craig et al. [4].

We now recall two recent mathematical works related to the interaction of solitons for the (gKdV) equations, which is otherwise a widely open question.

First, Mizumachi [31] studied rigorously the interaction of two solitons of nearly equal speeds for (1.1) both for integrable ($p = 3$) and nonintegrable ($p = 4$) cases. Consider $u_0$ close to the sum of two solitons $Q(x) + Q_c(x + Y_0)$, where $Y_0 > 0$ is large and $c$, close to 1, satisfies $c - 1 \leq e^{-\frac{1}{2}Y_0}$. Let $u(t)$ be the corresponding solution of (1.1). If $c - 1 = e^{-\frac{1}{2}Y_0} > 0$, the quicker soliton is initially on the left of the other soliton: one could think that the two solitons have to cross at some positive finite time. However, Mizumachi proved (see Theorem 1.1 in [31]) that the interaction of the two solitons being repulsive, for $c - 1$ small enough, the two solitons remain separated for all positive time and eventually $u(t)$ behaves as
\[
u(t) = Q_{c_1^+}(. - c_1^+ t - y_1^+) + Q_{c_2^+}(. - c_2^+ t - y_2^+) + w(t, x),
\] (1.6)
for large time, for some $c_1^+ > c_2^+$ close to 1 and $w$ small in some space. The analysis part in [31] relies on scattering techniques due to Hayashi and Naumkin [9, 10] and on the use of spaces of exponentially decaying functions (introduced in this context by Pego and Weinstein [32]).

Interestingly, note that using Mizumachi’s result for $t < 0$, backwards in time (i.e. using the symmetry $x \rightarrow -x, t \rightarrow -t$ of (1.1)), one can construct a class of global solutions $u(t)$ of (1.1) such that for all $t \in \mathbb{R}, u(t)$ is close to the sum of two separated solitons with nearly equal speeds, where the minimal separation between the two solitons is large, and satisfying (1.6) both at $t \sim -\infty$ and $t \sim +\infty$.

The situation is thus at the main order similar to the one described in the integrable case by LeVeque [18]. However, after Mizumachi’s work, two important questions remain open in this regime for the nonintegrable case: the global stability in the energy space $H^1$ of the 2-soliton structure and the existence or nonexistence of pure 2-soliton solutions. By
analogy with the integrable case, the expression pure 2-soliton denotes a solution of (1.1) which satisfies
\[ \lim_{t \to \pm \infty} \| u(t) - Q_{c_1^{-}}(\cdot, -c_1^{-}t - y_{1}^{-}) - Q_{c_2^{-}}(\cdot, -c_2^{-}t - y_{2}^{-}) \|_{H^1} = 0. \] (1.7)

Note that if (1.7) holds both at \(-\infty\) and \(+\infty\), then necessarily \(c_j^- = c_j^+\) for \(j = 1, 2\) (see Lemma 4.1 and [25], pp. 68, 69).

Second, in a series of recent works ([26], [25]), the authors of the present paper have addressed the problem of collision of two solitons of (1.1) for a general nonlinearity in the case where one soliton is supposed to be large with respect to the other one, i.e. assuming that the ratio of the speeds \(c_1, c_2\) of the solitons satisfies \(c = c_1/c_2 \ll 1\).

In this regime, we were able to compute an approximate solution as a series of powers of \(c\) (in some sense) describing the collision up to any prescribed order of \(c\), which allowed us to understand the collision phenomenon.

In [25], under general assumptions on the nonlinearity, it is proved that the two solitons are preserved at the main order by the collision. While it is natural for the large soliton to be preserved by perturbation using standard \(H^1\) stability results, it is surprising that in a general nonintegrable situation, the small soliton is also preserved.

In [26], concerning the special case of quartic nonlinearity \(p = 4\) in (1.1), still for \(c \ll 1\), the perturbation due to the collision could be computed in more details. In particular, an explicit lower bound on the defect due to the collision was obtained. As a consequence, there exists no pure 2-soliton solution in this context.

These results have been later extended to the case of the BBM equation in [28].

1.2 Main results

In this paper, we focus on the quartic (gKdV) equation
\[ \partial_t u + \partial_x (\partial_x^2 u + u^4) = 0, \quad t, x \in \mathbb{R}. \] (1.8)

Recall that the Cauchy problem for (1.8) is globally well-posed in \(H^1\) (see Kenig, Ponce and Vega [14]), and that any \(H^1\) solution \(u(t, x)\) of (1.8) satisfies for all \(t \in \mathbb{R},\)

\[ \int u^2(t) = M(u(t)) = M(u(0)) \quad \text{(mass)} \] (1.9)

\[ \int (\partial_x u)^2(t) - \frac{2}{5} u^5(t) = \mathcal{E}(u(t)) = \mathcal{E}(u(0)) \quad \text{(energy)} \] (1.10)

Our objective is to describe the interaction of two solitons with almost equal speeds for (1.8) and in particular to answer in this specific regime the two main questions raised above: Is the 2-soliton structure stable globally in time in \(H^1\)? Does there exist a pure 2-soliton solution?

For our first result, Theorem 1, we focus on special solutions of (1.8) which behave as a 2-soliton asymptotically as \(t \to -\infty\). Recall that the existence (as well as uniqueness properties) of such solutions was proved in [20] (see also [29]). For \(c_2^- - c_1^- > 0\) small, and any \(x_1^-, x_2^-\), let \(u(t)\) be the unique solution of (1.8) such that
\[ \lim_{t \to -\infty} \| u(t) - Q_{c_1^-}(\cdot, -c_1^-t - x_1^-) - Q_{c_2^-}(\cdot, -c_2^-t - x_2^-) \|_{H^1} = 0. \] (1.11)
Let
\[ c_0 = \frac{c_1^- + c_2^-}{2}, \quad \mu_0 = \frac{c_2^- - c_1^-}{c_1^- + c_2^-}, \quad y_1^- = x_1^- \sqrt{c_0^-}, \quad y_2^- = x_2^- \sqrt{c_0^-}. \] (1.12)

Then
\[ U(t, x) = c_0^{-1/3} u \left( c_0^{-3/2} t, c_0^{-1/2} (x + t) \right) \] (1.13)
solves
\[ \partial_t U + \partial_x (\partial_x^2 U - U + U^4) = 0, \quad t, x \in \mathbb{R}, \] (1.14)
and it is the unique solution of (1.14) satisfying
\[ \lim_{t \to -\infty} \| U(t) - Q_{1-\mu_0}(\cdot + \mu_0 t - y_1^-) - Q_{1+\mu_0}(\cdot - \mu_0 t - y_2^-) \|_{H^1} = 0. \] (1.15)

This means that from the general case (1.11), we can reduce ourselves to a symmetric situation for the asymptotic speeds at \(-\infty\).

In this context, we now state our main results.

**Theorem 1** (Inelastic interaction of two solitons with nearly equal speeds). There exist \( C, c, \sigma, \mu_* > 0 \) such that the following holds. For \( 0 < \mu_0 < \mu_* \), let \( U(t) \) be the unique solution of (1.14) such that
\[ \lim_{t \to -\infty} \| U(t) - Q_{1-\mu_0}(\cdot + \mu_0 t + \frac{1}{2} Y_0 + \ln 2) - Q_{1+\mu_0}(\cdot - \mu_0 t - \frac{1}{2} Y_0 - \ln 2) \|_{H^1(\mathbb{R})} = 0, \] (1.16)
where \( Y_0 = |\ln(\mu_0^2/\alpha)| \) and \( \alpha = 12(10)^{2/3}(\int Q^2)^{-1} \). Then

(i) **Global behavior of 2-solitons.** There exist \( \mu_1(t), \mu_2(t), y_1(t), y_2(t) \) of class \( C^1 \) such that
\[ w(t, x) = U(t) - Q_{1+\mu_1(t)}(\cdot - y_1(t)) - Q_{1+\mu_2(t)}(\cdot - y_2(t)) \]
satisfies, for all \( t \in \mathbb{R} \),
\[ \| w(t) \|_{H^1(\mathbb{R})} \leq C |\ln \mu_0|^{1/2} \mu_0^2, \quad \min_{t \in \mathbb{R}} (y_1(t) - y_2(t)) - Y_0 \leq C |\ln \mu_0 \|^{3/2}, \] (1.17)
\[ \sum_{j=1, 2} |\mu_j(t) + (-1)^j \mu_0 \tanh(\mu_0 t)| + \sum_{j=1, 2} |y_j(t) - \mu_j(t)| \leq C |\ln \mu_0|^2 \mu_0^2. \] (1.18)

(ii) **Asymptotics and defect.** The limits \( \mu_1^+ = \lim_{t \to +\infty} \mu_1, \mu_2^+ = \lim_{t \to +\infty} \mu_2 \) exist and
\[ \lim_{t \to +\infty} \| w(t) \|_{H^1(\mathbb{R})} = 0, \quad \liminf_{t \to +\infty} \| w(t) \|_{H^1(\mathbb{R})} \geq c\mu_0^3, \] (1.19)
\[ c\mu_0^5 \leq \mu_1^+ - \mu_0 \leq C |\ln \mu_0|^{2\sigma} \mu_0^4, \quad c\mu_0^5 \leq -\mu_2^+ - \mu_0 \leq C |\ln \mu_0|^{2\sigma} \mu_0^4. \] (1.20)

It follows immediately from the lower bound (1.19) that **no pure 2-soliton exists**, which is a new result in this regime.

As a consequence of the proof of Theorem 1** the 2-soliton structure is globally stable in the energy space \( H^1 \).
Theorem 2 (Stability result in the energy space). There exist $C, \sigma, \mu_0 > 0$, such that the following holds. Let $\tilde{u}_0 \in \mathbb{R}$ and $\tilde{Y}_0 > 0$ be such that

$$\mu_0 = \left( \frac{\tilde{\mu}_0^2}{2} + 4\alpha e^{-\tilde{Y}_0} \right)^{1/2} < \mu_0,$$

(1.21)

where $\alpha$ is defined in Theorem 1. Let $u_0 \in H^1$ be such that

$$\|u_0 - Q_{1-\tilde{\mu}_0}(\cdot - \frac{1}{2}\tilde{Y}_0) - Q_{1+\tilde{\mu}_0}(\cdot + \frac{1}{2}\tilde{Y}_0)\|_{H^1(\mathbb{R})} \leq \omega \mu_0,$$

(1.22)

where $0 < \omega < |\ln \mu_0|^{-2}$, and let $u(t)$ be the solution of (1.14) such that $u(0) = u_0$. Then, there exist $T(t), X(t)$ of class $C^1$ such that, for all $t \in \mathbb{R}$,

$$\|u(t + T(t), \cdot + X(t)) - U(t)\|_{H^1(\mathbb{R})} + |X(t)| + \mu_0|\dot{T}(t)| \leq C\omega \mu_0 + C|\ln \mu_0|^{\sigma} \mu_0^{3/2},$$

(1.23)

where $U(t)$ is the solution defined in Theorem 1.

Comments on the results:

1. For the specific solution $U(t)$ considered in Theorem 1, the dynamics of the parameters $\mu_j(t), y_j(t)$ are closely related to the function

$$Y(t) = Y_0 + 2 \ln(\cosh(\mu_0 t))$$

which solves $\ddot{Y} = 2\alpha e^{-Y}, \lim_{t \to \pm\infty} \dot{Y} = \pm 2\mu_0, \dot{Y}(0) = 0.$

This choice is motivated by the symmetry in time of $Y(t)$. But by time and space translations and by the scaling argument (1.12), (1.13), we can extend Theorem 1 to any solution of (1.8) satisfying (1.11).

More detailed information on the behavior of $U(t)$ and the parameters $\mu_j(t), y_j(t)$ is available in Proposition 4.1. Using refined asymptotic techniques (see [24] and [26]), one can prove in the context of Theorem 1 that $\lim_{t \to +\infty} (y_1(t) - c_1^1 t)$ and $\lim_{t \to -\infty} (y_2(t) - c_2^1 t)$ exist.

Finally, from the critical Cauchy theory developed for (1.8) by Tao [33], one expects that in the context of Theorem 1 $w(t)$ scatters as $t \to +\infty$. In particular, apart from the two main solitons, the solution $U(t)$ should not contain any other soliton but only dispersion.

2. Theorems 1 and 2 completely answer the two questions raised before concerning the interaction of two solitons of almost equal speeds.

Note in particular that the lower bounds in estimates (1.19) and (1.20) measure the defect of $U(t)$ at $+\infty$; in other words, they quantify in the energy space $H^1$ the inelastic character of the collision of 2 solitons of (1.14) in the regime where $\mu_0$ is small. Comparing (1.17) and (1.19), and lower and upper bounds in (1.20), we see that there is a gap between the lower and the upper bounds for the size of the defect. It is an open problem. A similar open question related to the size of the defect appears in [26].

The information on the limiting values of the scaling parameters $\mu_1^+ \mu_2^+$ is more precise that the information one could deduce from (1.18). To prove (1.20), we use energy and mass conservations (see (1.9)–(1.10)) and the fact that $w(t)$ goes to zero locally around each soliton. In this way, we obtain sharp information on the scaling parameters, and in particular the monotonicity formulas: $\mu_1^+ > \mu_0, \mu_2^+ < -\mu_0$ (see Lemma 4.1).

Theorem 2 is a global stability result concerning solutions which are close to the sum of two solitons of nearly equal speeds. A sharper result is presented in Proposition 4.2.
3. The proof of Theorem 1 relies on new computations, in particular for the construction of a relevant approximate solution. The strategy developed in this paper is expected to be quite general: it will be extended to the BBM equation in [27] and can also be extended to the (gKdV) equation with general nonlinearity. The proof of the lower bounds on $\|w(t)\|_{H^1}$ as $t \to +\infty$ in (1.19) is in the spirit of Liouville theorem for the (gKdV) equation – see e.g. [22].

1.3 Strategy of the proofs

Recall that for the integrable KdV equation, the interaction of two solitons of almost equal speeds can be completely described using the explicit formulas for 2-solitons. From [18], considering $u(t)$ a typical 2-soliton solution of (1.2) with almost equal speeds, the two solitons remain well separated for all time and eventually exchange their speeds:

$$u(t) \sim Q_{c_1}(\cdot - y_1(t)) + Q_{c_2}(\cdot - y_2(t)),
\quad y_1(t) - y_2(t) = y(t) \sim Y(t),
\quad \lim_{-\infty} c_1 = \lim_{+\infty} c_2 = c_1^-,
\quad \lim_{-\infty} c_2 = \lim_{+\infty} c_1 = c_2^-;$$

where $Y(t)$ is solution of $\ddot{Y} = 2\bar{\alpha}e^{-Y}$ for some $\bar{\alpha}$. Their interaction is repulsive and since solitons have exponentially decay in space, they interact weakly. Moreover, the solution has a symmetry with respect to the transformation $x \to -x$, $t \to -t$.

In this paper, we focus on the quartic (gKdV) equation, which is not integrable and not close to any integrable model. Recall that for this equation, we have described in [26] the collision of two solitons with very different speeds. Indeed, considering two solitons $Q_{c_1}, Q_{c_2}$ such that $c_2 \gg c_1$, it follows from [26] that the two soliton structure is stable and that the collision is almost elastic but not exactly elastic. Let us sketch the main steps of the proofs in [26]:

1. First, we construct an approximate solution to the problem in the collision region. The approximate solution has the form of a series in terms of $c = c_1/c_2$ and involves a delicate algebra.

2. Second, using asymptotic arguments, we justify that the solution is close to the approximate solution (so that the description of the collision given by the approximate solution is relevant) and we control the solution in large time, i.e. for $|t| > T$.

3. Finally, we prove the inelastic character of the collision by a further analysis of the approximate solution. The defect is due to a nonzero extra term in the approximate solution after recomposition of the series. Thus, the defect is a direct consequence of the algebra underlying the construction of the approximate solution.

Turning back to our problem, keeping in mind the intuition of the integrable case, we prove that in the case of two solitons with almost same speeds, the description given in (1.25) persists at the main order for the quartic (gKdV) equation and we describe precisely the interaction. Let us present the strategy of the proofs of the main results.

Theorem 2 is a direct consequence of the proof of Theorem 1. The proof of Theorem 1, as in [26], follows from a combination of three different types of arguments.

1. We construct an approximate solution to the problem in terms of a series in $e^{-y(t)}$ where $y(t) = y_1(t) - y_2(t)$ is the distance between the two solitons. Whereas the first order of the interaction of the two solitons is $e^{-y(t)}$, we are able to compute the solution up to order $e^{-\frac{3}{2}y(t)}$ – see Proposition 2.1 (this improves the ansatz of [31] limited to (1.25)).
construction implies that the soliton parameters \(c_1(t), c_2(t), y_1(t), y_2(t)\) have to satisfy an approximate differential system.

Note from Proposition 2.1 that the approximate solution contains a tail of order \(e^{-\frac{3}{2}y(t)}\) between the two solitons (as in the (KdV) case, see [27]), which is relevant in the description of the exact solution, see Remark 2. We will see that the inelasticity is not related to this tail of order \(e^{-\frac{3}{2}y(t)}\).

It is only at order \(e^{-\frac{3}{2}y(t)}\) in the construction of the approximate solution that our analysis points out a deep difference between (KdV) and nonintegrable (gKdV). For the quartic (gKdV) case, one cannot build an approximate solution at order \(e^{-\frac{3}{2}y(t)}\) in the energy space. Indeed, at this order, a tail necessarily appears in the approximate solution at \(\infty\) in space. We then have to cut off this tail to obtain a rougher approximate solution in the energy space.

Note that the construction of the approximate solution in the present paper is completely new. Since for all time the distance \(y(t)\) between the two solitons is very large, and since the interactions are exponential in \(y(t)\), the approximate solution is found by separation of the three variables: \(e^{-\frac{3}{2}y(t)}\) and the coordinates of the two solitons. The approximate solution is thus of different nature compared to the one in [26]. We believe that the techniques of the present paper should have wide applicability to other models (see the case of the (BBM) equation in [27]).

(2) After the approximate solution is constructed, we introduce the following decomposition of the solution \(U(t)\) defined in Theorem 1:

\[
U(t, x) = Q_{c_1}(t)(x - y_1(t)) + Q_{c_2}(t)(x - y_2(t)) + W(t, x) + \varepsilon(t, x),
\]

where \(Q_{c_1}(t)(x - y_1(t)) + Q_{c_2}(t)(x - y_2(t)) + W(t, x)\) is the modulated approximate solution and \(\varepsilon(t)\) is a rest term. To prove stability of the two soliton structure, we have to control both the parameters \(c_j(t)\) and \(y_j(t)\) and the rest term \(\varepsilon(t)\).

The control of the rest term \(\varepsilon(t)\) uses variants of techniques developed earlier for large time stability and asymptotic stability of solitons and multi-solitons for the (gKdV) equations in the energy space ([35], [22], [29], [20]). These techniques involve: Liapunov functionals related to the stability of solitons, Virial identity and the introduction of almost monotone variants of the conservation laws (1.9)–(1.10) around each soliton (consequences of the Kato identity [13]). Recall that these techniques apply only in situations where the solitons are decoupled and were key arguments in several recent developments on global behavior of solutions of the (gKdV) equation: blow up in the \(L^2\) critical case ([23]), asymptotic stability, stability of multi-solitons and rigidity properties of the flow of the (gKdV) equations around solitons.

As a conclusion of this analysis on the rest term \(\varepsilon(t)\) and of the control of the dynamical system satisfied by the parameters, we obtain the stability results of Theorem 1. In particular, the dynamics of the parameters can be approximated by the simple ODE:

\[
\ddot{Y} = 2\alpha e^{-Y}, \quad Y(0) = Y_0, \quad \dot{Y}(0) = 0,
\]

and for all time \(t\), \(\|\varepsilon(t)\|_{H^1} \leq e^{-\frac{5}{2} Y_0}\). Observe that this ODE is the same as in the (KdV) case and seems to be universal in this type of problems.

(3) Finally, we prove by contradiction that in the quartic case the interaction of two solitons always produces a nonzero residual. The proof of the lower bounds on the defect involves some more refined arguments but is based on the same idea. Assume that \(U(t)\) is a pure 2-soliton solution, the contradiction then follows from the following two facts:
On the one hand, by uniqueness properties \((20)\), \(U(t)\) satisfies \(U(t, x) = U(-t, -x)\) up to translation in space and time. As a consequence, the parameters \(c_j(t), y_j(t)\) also have a symmetry property.

On the other hand, the dynamical system satisfied by \(c_j(t), y_j(t)\) is not symmetric by the transformation \(x \rightarrow -x, t \rightarrow -t\) at order \(e^{-\frac{3}{2}y(t)}\). Indeed, the approximate solution is not symmetric since it has a tail of this order on the left of the two solitons, to match the behavior of \(U(t)\) at \(t \rightarrow -\infty\).

Since \(U(t)\) is assumed to be pure at \(\pm\infty\), it has special space decay properties at the left of the soliton, so that one can use a special functional related to \(L^1\) to refine the dynamical system. Now, solving the refined non symmetric dynamical system, we find a contradiction with the symmetry properties of \(y_1(t)\) and \(y_2(t)\).

The above strategy to prove inelasticity is similar to the proof of a Liouville property, in the spirit of e.g. \([22]\) and \([23]\). Moreover, the \(L^1\) functional mentioned above was introduced to prove instability results for \((gKdV)\) equations, see \([2]\) and \([23]\). Note that the arguments in this step are different from the ones in \([24]\), where the defect is a direct consequence of a defect in the approximate solution.

We summarize the organization of the paper. In Section 2, we construct an approximate solution. Section 3 is devoted to preliminary decomposition and stability results. We then prove the stability part of Theorems 1 and 2 in Section 4. Finally, in Section 5, we prove the inelasticity of the interaction.

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2 Construction of an approximate solution

We denote by \(\mathcal{Y}\) the set of functions \(f \in C^\infty(\mathbb{R}, \mathbb{R})\) such that

\[
\forall j \in \mathbb{N}, \exists C_j, r_j > 0, \forall x \in \mathbb{R}, \quad |f^{(j)}(x)| \leq C_j(1 + |x|)^r_j e^{-|x|}.
\]

Proposition 2.1. There exist unique \(A_j(x), B_j(x), D_j(x), \alpha, \beta, \delta, a, b_j, d_j (j = 1, 2)\), \(\sigma \geq 3\) and \(0 < \mu_0 < 1/10\) such that for any \(0 < \mu_0 < \mu_\ast\), the following hold.

(i) Properties of \(A_j, B_j, D_j\) and \(b_j\).

\[
A_j, B_j, D_j \in L^\infty(\mathbb{R}), \quad A_j', B_j', D_j' \in \mathcal{Y},
\]

\[
- \lim_{\pm\infty} A_1 = \lim_{\pm\infty} A_2 = \pm \theta_A, \quad \theta_A = (10)^{2/3} \int_{Q} \frac{Q}{Q^2}, \quad \lim_{+\infty} D_1 = \lim_{+\infty} D_2 = 0, \quad (2.1)
\]

\[
- \lim_{+\infty} B_1 = \lim_{+\infty} B_2 = 0, \quad \lim_{-\infty} B_1 = - \lim_{-\infty} B_2 < 0,
\]

and \(A_j, B_j, D_j\) satisfy the orthogonality conditions of Lemmas \([2.3] [2.5] and [2.6]\).

Moreover,

\[
b_1 \neq b_2. \quad (2.2)
\]
(ii) **Definition of the approximate solution** \( V_0(x; \Gamma) \). For \( \Gamma = (\mu_1, \mu_2, y_1, y_2) \), define

\[
V_0(x; \Gamma) = Q_{1+\mu_1}(x - y_1) + Q_{1+\mu_2}(x - y_2) \\
+ e^{-(y_1 - y_2)}(A_1(x - y_1) + A_2(x - y_2)) \\
- 2(10)^{-2/3} \theta_A(\mu_1 - \mu_2)xQ(x - y_1)Q(x - y_2) \\
+ (y_1 - y_2)e^{-(y_1 - y_2)}(\mu_1 B_1(x - y_1) + \mu_2 B_2(x - y_2)) \\
+ e^{-(y_1 - y_2)}(\mu_1 D_1(x - y_1) + \mu_2 D_2(x - y_2)).
\]

(2.3)

(iii) **Equation of** \( V_0(x; \Gamma(t)) \). Let \( I \) be some time interval and \( \Gamma(t) = (\mu_1(t), \mu_2(t), y_1(t), y_2(t)) \) be a \( C^1 \) function defined on \( I \) such that, for some constant \( K > 1 \),

\[
\forall t \in I, \quad Y_0 - 1 \leq y_1(t) - y_2(t) \leq K Y_0, \quad |\mu_1(t)| \leq 2\mu_0, \quad |\mu_2(t)| \leq 2\mu_0, \quad \mu_1(t) + \mu_2(t) \leq Y_0^2 e^{-Y_0}, \quad |y_1(t) + y_2(t)| \leq Y_0^4 e^{-Y_0/4},
\]

where

\[
Y_0 = |\ln(\mu_0^2/\alpha)| \quad \text{and} \quad \alpha = \frac{12(10)^{2/3}}{\int Q^2}.
\]

(2.6)

Let

\[
V_0(t, x) = V_0(x; \Gamma(t)), \quad y(t) = y_1(t) - y_2(t).
\]

(2.7)

Then, on \( I \), \( V_0(t, x) \) solves

\[
\partial_t V_0 + \partial_x (\partial_x^2 V_0 - V_0 + V_0^4) = \bar{E}(V_0) + E_0(t, x)
\]

(2.8)

where

\[
\bar{E}(V_0) = \sum_{j=1,2} (\mu_j - \mathcal{M}_j) \frac{\partial V_0}{\partial \mu_j} - \sum_{j=1,2} (\mu_j - \dot{\mu}_j - \mathcal{N}_j) \frac{\partial V_0}{\partial y_j},
\]

(2.9)

\[
\mathcal{M}_1(t) = \alpha e^{-y(t)} + \beta \mu_1(t)y(t)e^{-y(t)} + \delta \mu_1(t)e^{-y(t)},
\]

\[
\mathcal{M}_2(t) = -\alpha e^{-y(t)} - \beta \mu_2(t)y(t)e^{-y(t)} - \delta \mu_2(t)e^{-y(t)},
\]

\[
\mathcal{N}_1(t) = a e^{-y(t)} + b_1 \mu_1(t)y(t)e^{-y(t)} + d_1 \mu_1(t)e^{-y(t)},
\]

\[
\mathcal{N}_2(t) = a e^{-y(t)} + b_2 \mu_2(t)y(t)e^{-y(t)} + d_2 \mu_2(t)e^{-y(t)}.
\]

(2.10)

and for some \( C = C(K) > 0 \),

\[
\forall t \in I, \quad \sup_{x \in \mathbb{R}} \left\{ \left( 1 + e^{\frac{1}{2}(x - y_1(t))} \right) |E_0(t, x)| \right\} \leq C (1 + Y_0^\sigma) e^{-Y_0 e^{-y(t)}}.
\]

(2.11)

Sections 2.1–2.5 are devoted to the proof of Proposition 2.1.

Since the function \( V_0 \) above which solves an approximate equation is not in \( H^1 \) (it has nonzero limits at infinity in \( x \)), we have to localize the result, by introducing an \( L^2 \) approximation of \( V_0 \), using a suitable cut-off function. Note that in the integrable case \( p = 2 \), one would have obtained \( V_0 \) in \( L^2 \) using the same scheme - it is thus related to nonintegrability (see [27]).
Let $\psi : \mathbb{R} \to [0, 1]$ be a $C^\infty$ function such that
\[
\psi' \geq 0, \psi \equiv 0 \text{ on } \mathbb{R}^-, \psi \equiv 1 \text{ on } \left[\frac{1}{2}, +\infty\right),
\] (2.12)

As a consequence of Proposition 2.1 and direct computations and estimates, we obtain the following result.

**Proposition 2.2** ($L^2$ approximate solution). Under the assumptions of Proposition 2.1 (i)–(iii), let
\[
V(x; \Gamma) = V_0(x; \Gamma)\psi\left(e^{-\frac{1}{2}Y_0 x} + 1\right), \quad V(t, x) = V(x; \Gamma(t)).
\] (2.13)

Then,
(i) Closeness to the sum of two solitons.
\[
\|V - \{Q_{1+\mu_1}(-, -y_1) + Q_{1+\mu_2}(-, -y_2)\}\|_{L^\infty} \leq C e^{-y},
\] (2.14)
\[
\|V - \{Q_{1+\mu_1}(-, -y_1) + Q_{1+\mu_2}(-, -y_2)\}\|_{H^1} \leq C \sqrt{y} e^{-y}.
\] (2.15)

(ii) Equation of $V(t, x)$.
\[
\partial_t V + \partial_x (\partial_x^2 V - V + V^4) = \tilde{E}(V) + E(t, x)
\] (2.16)

where
\[
\tilde{E}(V) = \sum_{j=1,2} (\tilde{\mu}_j - M_j) \frac{\partial V}{\partial \mu_j} - \sum_{j=1,2} (\mu_j - \tilde{\mu}_j - N_j) \frac{\partial V}{\partial y_j},
\] (2.17)

and for some $C = C(K) > 0$,
\[
\forall t \in I, \quad \sup_{x \in \mathbb{R}} \left\{\left(1 + e^{\frac{1}{2}\left(x - y(t)\right)}\right)|E(t, x)|\right\} \leq CY_0^\sigma e^{-Y_0} e^{-y(t)},
\] (2.18)
\[
\|E(t)\|_{H^1} \leq CY_0^\sigma e^{-\frac{3}{2}Y_0 - y(t)}.
\]

Proposition 2.2 is proved in Section 2.6.

### 2.1 Preliminary expansion

We set $(j = 1, 2)$
\[
\tilde{R}_j(t, x) = Q_{1+\mu_j(t)}(x - y_j(t)), \quad R_j(t, x) = Q(x - y_j(t)),
\]
\[
\Lambda \tilde{R}_j(t, x) = \Lambda Q_{1+\mu_j(t)}(x - y_j(t)), \quad \Lambda R_j(t, x) = \Lambda Q(x - y_j(t)),
\]
and similarly for $\Lambda^2 R_j$, where $\Lambda Q_c$ and $\Lambda^2 Q_c$ are defined in Claim A.2.

We introduce the notation
\[
r(t) = O_k, \text{ for } k \geq 1, \text{ if } \exists \sigma \geq 0 \text{ s.t. } \sup_{t \in I} \{|e^{\sigma(t)}|r(t)|\} \leq C(1 + Y_0^\sigma)e^{-(k-1)Y_0},
\]
\[
f(t, x) = O_k, \text{ for } k \geq 1, \text{ if } \sup_{x \in \mathbb{R}} \left\{(1 + e^{\frac{1}{2}(x - y(t))})|f(t, x)|\right\} = O_k.
\] (2.19)

Define
\[
S(v) = \partial_t v + \partial_x \left(\partial_x^2 v - v + v^4\right),
\]
and $M_j, N_j$ as in (2.10) for $\alpha, \beta, \delta$ and $a, b_j, d_j$ to be determined. We look for an approximate solution of $S(v) = 0$ under the form $v(t, x) = v(x; \Gamma(t))$,

$$v = \tilde{R}_1 + \tilde{R}_2 + w,$$

where $w(t, x) = w(x; \Gamma(t))$, so that using the equation of $Q_c$ (see (A.1)), and $\frac{\partial}{\partial y_1} \tilde{R}_1 = \Lambda \tilde{R}_1$,

$$S(v) = \tilde{E}(v) + F + \tilde{F} + G(w) + H(w)$$

(2.20)

where

$$\tilde{E}(v) = \sum_{j=1,2} (\mu_j - M_j) \frac{\partial v}{\partial \mu_j} - \sum_{j=1,2} (\mu_j - \tilde{y}_j - N_j) \frac{\partial v}{\partial y_j}$$

$$F = \partial_x \left( (\tilde{R}_1 + \tilde{R}_2)^4 - \tilde{R}_1^4 - \tilde{R}_2^4 \right)$$

$$\tilde{F} = M_1 \Lambda \tilde{R}_1 + M_2 \Lambda \tilde{R}_2 + N_1 \partial_x \tilde{R}_1 + N_2 \partial_x \tilde{R}_2,$$

and

$$G(w) = \partial_x \left[ \frac{\partial^2 w - w}{w} + 4 \left( \tilde{R}_1^3 + \tilde{R}_2^3 \right) w \right] + \sum_{j=1,2} \mu_j \frac{\partial w}{\partial y_j}$$

$$H(w) = \partial_x \left[ \left( \tilde{R}_1 + \tilde{R}_2 + w \right)^4 - \left( \tilde{R}_1 + \tilde{R}_2 \right)^4 + 4 \left( \tilde{R}_1^3 + \tilde{R}_2^3 \right) w \right]$$

$$+ \sum_{j=1,2} M_j \frac{\partial w}{\partial \mu_j} - \sum_{j=1,2} N_j \frac{\partial w}{\partial y_j}.$$

In the rest of Section 2.1, we perform preliminary expansions of $F$ and $\tilde{F}$.

**Lemma 2.1 (Expansion of $F$).** Under the assumptions of Proposition 2.1.

$$F = \partial_x \left( (\tilde{R}_1 + \tilde{R}_2)^4 - \tilde{R}_1^4 - \tilde{R}_2^4 \right) = F_A + F_B + F_D + O_2$$

where

$$F_A = 4(10)^{1/3} e^{-y} \partial_x \left[ e^{-x-y_1} R_1^3 + e^{x-y_2} R_2^3 \right],$$

$$F_B = 2(10)^{1/3} y e^{-y} \partial_x \left[ \mu_1 e^{-x-y_1} R_1^3 + \mu_2 e^{x-y_2} R_2^3 \right],$$

$$F_D = 4(10)^{1/3} \mu_1 e^{-y} \partial_x \left[ e^{-x-y_1} R_1^3 \left( \frac{2}{3} R_1 + \frac{1}{2} (x - y_1) R_1 + \frac{3}{2} (x - y_1) \partial_x R_1 \right) \right]$$

$$+ 4(10)^{1/3} \mu_2 e^{-y} \partial_x \left[ e^{x-y_2} R_2^3 \left( \frac{2}{3} R_2 - \frac{1}{2} (x - y_2) R_2 + \frac{3}{2} (x - y_2) \partial_x R_2 \right) \right].$$

**Proof.** Before starting the proof of Lemma 2.1, we claim the following estimates.

**Claim 2.1.** Assuming (2.4) - (2.6), the following hold

$$\bar{R}_1(t, x) + |\partial_x \bar{R}_1(t, x)| \leq C e^{-(1-2\mu_0) |x-y_1(t)|},$$

(2.21)

$$\left| \tilde{R}_1(t, x) - \{ R_1(t, x) + \mu_1(t) \Lambda R_1(t, x) \} \right| \leq C \mu_0^2 (1 + |x - y_1(t)|^2) e^{-(1-2\mu_0) |x-y_1(t)|},$$

(2.22)
together with similar estimates for $\tilde{R}_2$.

Moreover, for $\omega \geq 0$, 

$$
\left(1 + \sum_{j=1,2} |x - y_j|^\omega \right) e^{-(1-2\mu_0)|x-y_1|} e^{-(1-2\mu_0)|x-y_2|} = O_1,
$$

$$
\int \left(1 + \sum_{j=1,2} |x - y_j|^\omega \right) e^{-(1-2\mu_0)|x-y_1|} e^{-(1-2\mu_0)|x-y_2|} dx \leq C \left(1 + |y|^{1+\omega} \right) e^{-y}.
$$

**Proof of Claim 2.7** Since $Q(x) + |Q'(x)| \leq Ce^{-|x|}$ and $|\mu_1(t)| \leq 2\mu_0$, we have

$$
|\tilde{R}_1(t, x) + |\partial_x \tilde{R}_1(t, x)| \leq Ce^{-\sqrt{1-2\mu_0}|x-y_1(t)|} \leq Ce^{-(1-2\mu_0)|x-y_1(t)|}.
$$

To prove (2.22), we use the Taylor formula in the $\mu$ variable (recall the notation $\Lambda Q$ from Claim 2.2)

$$
\tilde{R}_1(t, x) = Q_1 + \mu_1(x - y_1) = Q(x - y_1) + \mu_1 \Lambda Q(x - y_1)
$$

$$
+ \mu_1^2 \int_0^1 (1-s) \Lambda^2 Q_1 + \mu_1 (x - y_1) ds.
$$

From (A.7), $|\Lambda^2 Q_1 + \mu_1(x)| \leq C(1 + |x|^2) e^{-(1-2\mu_0)|x|}$ and (2.22) follows.

To prove (2.23) and (2.24), we argue as follows. For $y_2 < x < y_1$, we have

$$
e^{-(1-2\mu_0)|x-y_1|} e^{-(1-2\mu_0)|x-y_2|} = e^{-(1-2\mu_0)y} \leq Ce^{-y},$$

since (2.4) implies $2\mu_0 y \leq C\mu_0 Y_0 \leq 1$ for $Y_0$ large enough by (2.6).

For $x > y_1 > y_2$, we have

$$
e^{-(1-2\mu_0)|x-y_1|} e^{-(1-2\mu_0)|x-y_2|} = e^{-(1-2\mu_0)(2x-y_1-y_2)} = e^{-2(1-2\mu_0)/(x-y_1)} e^{-(1-2\mu_0)y} \leq Ce^{-\frac{3}{2}|x-y_1|} e^{-y}.
$$

Arguing similarly for the case $x < y_2 < y_1$, we prove (2.23) and (2.24). \qed

We expand $F$,

$$
F = \partial_x \left( 4\tilde{R}_1^2 \tilde{R}_2 + 4\tilde{R}_1 \tilde{R}_2^3 + 6\tilde{R}_1^2 \tilde{R}_2^2 \right).
$$

We have immediately $\partial_x \left( \tilde{R}_1^2 \tilde{R}_2^2 \right) = O_2$ (see (2.21) and (2.23)).

Now, we focus on the term $\partial_x \left( \tilde{R}_1^3 \tilde{R}_2 \right)$. Using (2.21), (2.22), (2.23), $|\mu_j(t)| \leq 2\mu_0 \leq Ce^{-\frac{1}{2}y_0}$, and the expression of $\Lambda Q$ in (A.8), we obtain

$$
\partial_x \left( \tilde{R}_1^3 \tilde{R}_2 \right) = \partial_x \left( \left( R_1^3 + 3\mu_1 R_1^2 \Lambda R_1 \right) \left( R_2 + \mu_2 \Lambda R_2 \right) \right) + O_2
$$

$$
= \partial_x \left( R_1^3 R_2 + 3\mu_1 R_1^2 \Lambda R_1 R_2 + \mu_2 R_1^3 \Lambda R_2 \right) + O_2.
$$

Now, using the asymptotic behavior of $Q$, $Q'$ and $\Lambda Q$ at $+\infty$ (see (A.17) and (A.18)), we find

$$
\partial_x \left( \tilde{R}_1^3 \tilde{R}_2 \right) = \left(10\right)^{1/3} e^{-y} \partial_x \left[ e^{-(x-y_1)} R_1^3 \right]
$$

$$
+ \left(10\right)^{1/3} \mu_1 e^{-y} \partial_x \left[ e^{-(x-y_1)} R_1^2 \Lambda R_1 \right]
$$

$$
+ \left(10\right)^{1/3} \mu_2 e^{-y} \partial_x \left[ e^{-(x-y_1)} R_1^3 \right] + O_2.
$$
Therefore, using (2.5) and then the expression of $\Lambda Q$, we obtain
\[
\partial_x \left( \tilde{R}_1^3 \tilde{R}_2 \right) = (10)^{1/3} e^{-y} \partial_x \left[ e^{-(x-y_1)} R_1^3 \right] \\
+ \frac{1}{2} (10)^{1/3} \mu_1 e^{-y} \partial_x \left[ e^{-(x-y_1)} R_1^3 \right] \\
+ (10)^{1/3} \mu_1 e^{-y} \partial_x \left[ 3e^{-(x-y_1)} R_1^2 \Lambda R_1 - \left( \frac{1}{2} - \frac{1}{2} (x - y_1) \right) e^{-(x-y_1)} R_1^3 \right] + O_2 \\
= (10)^{1/3} e^{-y} \partial_x \left[ e^{-(x-y_1)} R_1^3 \right] \\
+ \frac{1}{2} (10)^{1/3} \mu_1 e^{-y} \partial_x \left[ e^{-(x-y_1)} R_1^3 \right] \\
+ (10)^{1/3} \mu_1 e^{-y} \partial_x \left[ e^{-(x-y_1)} R_1^2 \left( \frac{2}{3} R_1 + \frac{1}{2} (x - y_1) R_1 + \frac{2}{2} (x - y_1) \partial_x R_1 \right) \right] + O_2.
\]

Similar computations give
\[
\partial_x \left( \tilde{R}_1 \tilde{R}_2^3 \right) = (10)^{1/3} e^{-y} \partial_x \left[ e^{(x-y_2)} R_2^3 \right] \\
+ \frac{1}{2} (10)^{1/3} \mu_2 e^{-y} \partial_x \left[ e^{(x-y_2)} R_2^3 \right] \\
+ (10)^{1/3} \mu_2 e^{-y} \partial_x \left[ e^{(x-y_2)} R_2^2 \left( \frac{3}{2} R_2 - \frac{1}{2} (x - y_2) R_2 + \frac{2}{2} (x - y_2) \partial_x R_2 \right) \right] + O_2.
\]

Lemma 2.1 is proved by combining these computations. 

Lemma 2.2 (Expansion of $\tilde{F}$).

\[
\tilde{F} = \mathcal{M}_1 \Lambda \tilde{R}_1 + \mathcal{M}_2 \Lambda \tilde{R}_2 + N_1 \partial_x \tilde{R}_1 + N_2 \partial_x \tilde{R}_2 = \tilde{F}_A + \tilde{F}_B + \tilde{F}_D + O_2,
\]

where
\[
\tilde{F}_A = \alpha e^{-y} \Lambda R_1 + a e^{-y} \partial_x R_1 - \alpha e^{-y} \Lambda R_2 + a e^{-y} \partial_x R_2, \\
\tilde{F}_B = \beta \mu_1 e^{-y} \Lambda R_1 + b_1 \mu_1 e^{-y} \partial_x R_1 - \beta \mu_2 e^{-y} \Lambda R_2 + b_2 \mu_2 e^{-y} \partial_x R_2 \\
\tilde{F}_D = \delta \mu_2 e^{-y} \Lambda R_2 - \alpha \mu_1 e^{-y} \Lambda^2 R_1 + \alpha \mu_1 e^{-y} \partial_x \Lambda R_1 \\
- \delta \mu_2 e^{-y} \Lambda R_2 - \alpha \mu_2 e^{-y} \Lambda^2 R_2 + \alpha \mu_2 e^{-y} \partial_x \Lambda R_2.
\]

Proof. Recall the expressions of $\mathcal{M}_j$ and $N_j$ in (2.10). We expand
\[
e^{-y} \Lambda \tilde{R}_1 = e^{-y} \Lambda R_1 + \mu_1 e^{-y} \Lambda^2 R_1 + O_2 \\
e^{-y} \partial_x \tilde{R}_1 = e^{-y} \partial_x R_1 + \mu_1 e^{-y} \partial_x \Lambda R_1 + O_2
\]
and similarly for $\tilde{R}_2$, using $|\mu_1| \leq 2 \mu_0 \leq C e^{-\frac{1}{2}x_0}$. Thus,
\[
\tilde{F} = \mathcal{M}_1 \Lambda R_1 + \mathcal{M}_2 \Lambda R_2 + N_1 \partial_x R_1 + N_2 \partial_x R_2 \\
+ \mathcal{M}_1 \mu_1 \Lambda^2 R_1 + \mathcal{M}_2 \mu_2 \Lambda^2 R_1 + N_1 \mu_1 \partial_x \Lambda R_1 + N_2 \partial_x \Lambda R_2 + O_2.
\]

Some terms in the last line are $O_2$, using again $|\mu_1| \leq 2 \mu_0 \leq C e^{-\frac{1}{2}x_0}$, so that
\[
\tilde{F} = \mathcal{M}_1 \Lambda R_1 + \mathcal{M}_2 \Lambda R_2 + N_1 \partial_x R_1 + N_2 \partial_x R_2 \\
+ \alpha \mu_1 e^{-y} \Lambda^2 R_1 - \alpha \mu_2 e^{-y} \Lambda^2 R_2 + \alpha \mu_1 e^{-y} \partial_x \Lambda R_1 + \alpha \mu_2 e^{-y} \partial_x \Lambda R_2 + O_2,
\]
and the expressions of $\tilde{F}_A$, $\tilde{F}_B$ and $\tilde{F}_D$ follow from expanding $\mathcal{M}_j$ and $N_j$ in the first line. 

2.2 Determination of $A_1, A_2$

**Lemma 2.3** (Definition and equation of $w_A$). Let

\[
\alpha = 12 (10)^{2/3} \frac{1}{\int Q^2}, \quad \theta_A = (10)^{2/3} \frac{\int Q}{\int Q^2}.
\]

(i) There exist $a$ and $\hat{A}_1 \in \mathcal{Y}$ such that $A_1 = \hat{A}_1 + \theta_A Q'$ solves

\[
(-LA_1)' + 4\theta_A (Q^3)' + \alpha \Lambda Q + aQ' = -4(10)^{1/3} (e^{-x}Q^3)',
\]

\[
\int A_1 Q' = \int (A_1 + \theta_A) Q = 0.
\]

(ii) Set $A_2(x) = A_1(-x)$ and

\[
w_A(t, x) = e^{-y(t)} (A_1(x - y_1(t)) + A_2(x - y_2(t)).
\]

Then,

\[
F_A + \tilde{F}_A + G(w_A) = -2(10)^{-2/3}\theta_A (\mu_1 - \mu_2) R_1 R_2 + e^{-y} [\mu_1 S_1 (x - y_1) + \mu_2 S_2 (x - y_2)] + \mathcal{O}_2,
\]

where $S_1 \in \mathcal{Y}, S_2(x) = -S_1(-x)$.

Moreover,

\[
\left| \int w_A R_1 \right| + \left| \int w_A \partial_x R_1 \right| + \left| \int w_A R_2 \right| + \left| \int w_A \partial_x R_2 \right| = \mathcal{O}_2.
\]

**Proof.** Proof of (i). First, we determine the unique possible value of $\alpha$. Indeed, assume that $A_1$ satisfies the above equation, multiplying by $Q$, integrating and using $LQ' = 0$ and parity properties, we find (using (A.13) and (A.14))

\[
\alpha \int Q \Lambda Q = \alpha \frac{1}{6} \int Q^2 = -4(10)^{1/3} \int (e^{-x}Q^3)' Q = (10)^{1/3} \int e^{-x}Q^4 = 2(10)^{2/3}.
\]

Second, we determine the unique possible value of $\theta_A$. Let $A_1 = \hat{A}_1 + \theta_A \frac{Q'}{Q}$, using (A.11), we find for $\hat{A}_1$:

\[
(-L\hat{A}_1)' - \theta_A \left(-\frac{36}{5} Q^3 + \frac{99}{25} Q^6\right) + 4\theta_A (Q^3)' + \alpha \Lambda Q + aQ' = -4(10)^{1/3} (e^{-x}Q^3)'.
\]

To find $\hat{A}_1$ in $\mathcal{Y}$, which implies $L\hat{A}_1 \in \mathcal{Y}$, we need, using (A.12), (A.13) and (A.14),

\[
\theta_A \int \left(\frac{36}{5} Q^3 - \frac{99}{25} Q^6\right) + \alpha \int \Lambda Q = 2\theta_A - \alpha \frac{1}{6} \int Q = 0.
\]

Finally, we prove the existence of $\hat{A}_1$ and $a$. Let $Z \in \mathcal{Y}, \int ZQ' = 0$ be such that

\[
Z' = \theta_A \left(-\frac{36}{5} Q^3 + \frac{99}{25} Q^6\right) - 4\theta_A (Q^3)' - \alpha \Lambda Q - 4(10)^{1/3} (e^{-x}Q^3)'.
\]
Now, we compute \( F \) or this term, we use the following claim.

Thus, using the expressions of \( F \) and Claim 2.1, we have

\[
\int \theta - \int \phi = 0. \tag{2.5}
\]

Then, it suffices to solve \(-LA_1 + aQ = Z\). By Claim A.1, there exists a unique \( A \in Y \), \( \int \theta A = 0 \) such that \(-LA = Z\). Thus, we set \( \hat{A}_1 = A - aQ \), which from \( \{A.3\} \) solves the equation for all \( a \). Finally, since \( \int QAQ \neq 0 \) (see \( \{A.13\} \)), we uniquely fix \( a \) so that \( \int (\hat{A}_1 + \theta A)Q = 0 \). It is now straightforward to check that \( A_1 = \hat{A}_1 + \theta A \) satisfies (i).

Proof of (ii). First, by the parity properties of \( Q \), \( A_2(x) = A_1(-x) \) satisfies

\[
(-LA_2)' + 4\theta_A(Q^3)' - \alpha LA + aQ' = -4(10)^{1/3} (e^x Q^3)'.
\]

Now, we compute \( F_A + \hat{F}_A + G(w_A) \). Using Claim 2.1, we have

\[
G(w_A) = \partial_x (\partial_x^2 w_A - w_A + 4 (R_1^3 + R_2^3) w_A)
+ 12\partial_x ((\mu_1 R_1^2 \Lambda R_1 + \mu_2 R_2^2 \Lambda R_2) w_A) + \mu_1 \frac{\partial w_A}{\partial y_1} + \mu_2 \frac{\partial w_A}{\partial y_2} + O_2.
\]

First,

\[
\partial_x (\partial_x^2 w_A - w_A + 4 (R_1^3 + R_2^3) w_A)
= e^{-y} (-LA_1 + 4\theta_A Q^3)'(x - y) + e^{-y} \partial_x (4R_1^2(A_2(x - y) - \theta A))
+ e^{-y} (-LA_2 + 4\theta_A Q^3)'(x - y) + e^{-y} \partial_x (4R_2^3(A_1(x - y) - \theta A)).
\]

Using the estimate

\[
|A_2(x - y) - \theta A| \leq C(1 + |x - y|)^{\omega} e^{-y(x - y)} \text{ for } x > y \tag{2.25}
\]

and Claim 2.1, we have

\[
e^{-y} R_1^3(A_2(x - y) - \theta A) = O_2 \text{ and similarly } e^{-y} R_2^3(A_1(x - y) - \theta A) = O_2.
\]

Thus, using the expressions of \( F_A \) and \( \hat{F}_A \) in Lemmas 2.1 and 2.2 and the equations of \( A_1 \) and \( A_2 \), we find

\[
F_A + \hat{F}_A + \partial_x (\partial_x^2 w_A - w_A + 4 (R_1^3 + R_2^3) w_A) = O_2.
\]

Second, by similar arguments,

\[
12\partial_x ((\mu_1 R_1^2 \Lambda R_1 + \mu_2 R_2^2 \Lambda R_2) w_A) = 12\mu_1 e^{-y} \partial_x (R_1^2 \Lambda R_1(A_1(x - y) + \theta A))
+ 12\mu_2 e^{-y} \partial_x (R_2^2 \Lambda R_2(A_2(x - y) + \theta A)) + O_2.
\]

Finally, we compute \( \mu_1 \frac{\partial w_A}{\partial y_1} + \mu_2 \frac{\partial w_A}{\partial y_2} \). We have

\[
\frac{\partial w_A}{\partial y_1} = -w_A - e^{-y} A_1'(x - y_1), \quad \frac{\partial w_A}{\partial y_2} = w_A - e^{-y} A_2'(x - y_2).
\]

Thus, using (2.25),

\[
\mu_1 \frac{\partial w_A}{\partial y_1} + \mu_2 \frac{\partial w_A}{\partial y_2} = -(\mu_1 - \mu_2)w_A - \mu_1 e^{-y} A_1'(x - y_1) - \mu_2 e^{-y} A_2'(x - y_2)
= -\theta A(\mu_1 - \mu_2)e^{-y}(\frac{\partial_x R_1}{R_1} - \frac{\partial_x R_2}{R_2})
- \mu_1 e^{-y}(2\hat{A}_1 + A_1'(x - y_1)) - \mu_2 e^{-y}(2\hat{A}_2 + A_2'(x - y_2) + O_2.
\]

For this term, we use the following claim.
Claim 2.2. Let \( P_1(x) = P(-x) \), \( P_2(x) = P(x) \). Then,

\[
e^{-y} \left( \frac{\partial_x R_1}{R_1} - \frac{\partial_x R_2}{R_2} \right) = 2(10)^{-2/3} R_1 R_2 - e^{-y} P_1(x - y_1) - e^{-y} P_2(x - y_2) + O_2.
\]

Indeed, in the region \( x - y_2 > \frac{y}{2} \), from (A.10) and the definition of \( P \) (see (A.19)), we have

\[
e^{-y} \left( \frac{\partial_x R_1}{R_1} + P_1(x - y_1) - \frac{\partial_x R_2}{R_2} + P_2(x - y_2) \right) = 2(10)^{-1/3} e^{-y} e^{-(x-y_1)} R_1 + O_2,
\]

and from (A.17), in the same region,

\[
2(10)^{-2/3} R_1 R_2 = 2(10)^{-1/3} e^{-y} e^{-(x-y_1)} R_1 + O_2.
\]

In the complementary region \( x - y_1 < -\frac{y}{2} \), we argue similarly.

Using Claim 2.2, we obtain

\[
\mu_1 \frac{\partial w_A}{\partial y_1} + \mu_2 \frac{\partial w_A}{\partial y_2} = -\theta_A (\mu_1 - \mu_2) R_1 R_2 - \mu_1 e^{-y} (-2\theta_A P_1 + 2\dot{A}_1 + A'_1)(x - y_1) - \mu_2 e^{-y} (2\theta_A P_2 - 2\dot{A}_2 + A'_2)(x - y_2) + O_2.
\]

Combining these computations, we obtain

\[
F_A + \tilde{F}_A + G(w_A) = -\theta_A (\mu_1 - \mu_2) R_1 R_2 + 12 \partial_x \left( Q^2 \Lambda Q (A_1 + \theta_A) \right) + 2\theta_A P_1 - (2\dot{A}_1 + \partial_x A_1) (x - y_1) + \mu_2 e^{-y} \left( 12 \partial_x \left( Q^2 \Lambda Q (A_2 + \theta_A) \right) - 2\theta_A P_2 + (2\dot{A}_2 - \partial_x A_2) \right) (x - y_2),
\]

so that

\[
S_1 = 12 (Q^2 \Lambda Q (A_1 + \theta_A)) + 2\theta_A P(-x) - 2\dot{A}_1 - A'_1.
\]

Using (2.25), we have

\[
\int w_A R_1 = e^{-y} \int [A_1 Q + \theta_A Q] + O_2 = O_2,
\]

and similarly for the other scalar products. \( \square \)

2.3 Nonlocalized \( O_{3/2} \) term

Lemma 2.4. Let

\[
w_Q = -2(10)^{-2/3} \theta_A (\mu_1 - \mu_2) x R_1 R_2.
\]

Then

\[
G(w_Q) = 2(10)^{-2/3} \theta_A (\mu_1 - \mu_2) R_1 R_2 + 2(10)^{-1/3} \theta A \mu_1 ye^{-y} e^{-(x-y_1)} \left( 3(R_1 - \partial_x R_1 - \partial_x (R^4_1)) + R_1^4 \right) - 2(10)^{-1/3} \theta A \mu_2 ye^{-y} e^{-(x-y_2)} \left( 3(R_2 + \partial_x R_2 + \partial_x (R^4_2)) + R_2^4 \right) + e^{-y} \left( \mu_1 \tilde{S}_1 (x - y_1) + \mu_2 \tilde{S}_2 (x - y_2) \right) + O_2,
\]

where \( \tilde{S}_1 \in \mathcal{Y} \) and \( \tilde{S}_1(x) = -\tilde{S}_2(-x) \).
Proof. The proof is based on Claim A.3 in Appendix A. First, arguing as in the proof of Lemma 2.3, we have
\[ G(w_Q) = \partial_x (\partial_x^2 w_Q - w_Q + 4(R_1 + R_2 w_Q) + O_2. \]
Moreover, since \( x = \frac{1}{2}(x - y_1 + x - y_2) + \frac{1}{2}(y_1 + y_2) \), using (2.5), we have
\[ w_Q = -\frac{1}{2} \theta_A (\mu_1 - \mu_2)(x - y_1 + x - y_2) R_1 R_2 + O_2. \]
Therefore, using Claim A.3 and the asymptotics of Q from A.17, we get
\[ G(w_Q) = (10)^{-2/3} \theta_A (\mu_1 - \mu_2) \partial_x \{ - \partial_x^2 ((x - y_1 + x - y_2 ) R_1 R_2) + (x - y_1 + x - y_2) R_1 R_2 \}
- 4(R_1 + R_2)((x - y_1 + x - y_2) R_1 R_2) + O_2 \]
\[ = 2(10)^{-2/3} \theta_A (\mu_1 - \mu_2) R_1 R_2 \]
\[ + 2(10)^{-1/3} \theta_A \mu_1 e^{-y} e^{-(x-y_1)} (3(R_1 - \partial_x R_1 - \partial_x(R_1^4) + R_1^2)) \]
\[ - 2(10)^{-1/3} \theta_A \mu_2 e^{-y} e^{-(x-y_2)} (3(R_2 + \partial_x R_2 + \partial_x(R_2^4) + R_2^2)) \]
\[ + e^{-y} (\mu_1 \tilde{S}_1(x - y_1) + \mu_2 \tilde{S}_2(x - y_2)) + O_2, \]
where \( \tilde{S}_1 \) and \( \tilde{S}_2 \) satisfy the desired conditions. \( \square \)

2.4 Determination of \( B_1, B_2 \) and \( D_1, D_2 \)

Lemma 2.5 (Definition and equation of \( w_B \)). Let
\[ Z(x) = -2(10)^{1/3} \left( e^{-x} Q^3 \right)' - 2(10)^{-1/3} \theta_A e^{-x} (3(Q - Q' - (Q^4)) + Q^4), \]
\[ \beta = \frac{6}{Q^2} (10)^{2/3}, \quad \theta_B = \frac{3}{2} \left( 10 \right)^{-2/3} \frac{\int Q}{\int Q^2} > 0. \]

(i) There exist unique \( b_1 \) and \( \tilde{B}_1 \in \mathcal{U} \) such that \( B_1 = \tilde{B}_1 + \theta_B \left( 1 + Q' \right) \) satisfies
\[ (-LB_1)' + \beta \Lambda Q + b_1 Q' = Z, \quad \int B_1 Q' = \int B_1 Q = 0. \]

(ii) There exist unique \( b_2 \) and \( \tilde{B}_2 \in \mathcal{U} \) such that \( B_2 = \tilde{B}_2 - \theta_B \left( 1 + Q' \right) \) satisfies
\[ (-LB_2)' + \beta \Lambda Q + b_2 Q' = -Z(-x), \quad \int B_2 Q' = \int (B_2 - 2\theta_B) Q = 0. \]
Moreover,
\[ b_1 \neq b_2. \quad (2.26) \]

(iii) Set
\[ w_B(t, x) = ye^{-y(t)} (\mu_1 B_1(x - y_1(t)) + \mu_2 B_2(x - y_2(t))). \]
Then,
\[ F_A + \tilde{F}_A + G(w_A) + G(w_Q) + F_B + \tilde{F}_B + G(w_B) \]
\[ = e^{-y} \left[ \mu_1(S_1 + \tilde{S}_1)(x - y_1) + \mu_2(S_2 + \tilde{S}_2)(x - y_2) \right] + O_2, \quad (2.27) \]
\[ \left| \int w_B R_1 \right| + \left| \int w_B \partial_x R_1 \right| + \left| \int w_B R_2 \right| + \left| \int w_B \partial_x R_2 \right| = O_{5/2}. \quad (2.28) \]
Proof. We follow the strategy of the proof of Lemma 2.3. The only difference is that we now look for solutions $B_1, B_2$ both with limit 0 at $+\infty$.

Proof of (i). We find the value of $\beta$ from the equation of $B_1$ multiplied by $Q$, using (A.14),

$$\beta \frac{1}{6} \int Q^2 = \int ZQ = \frac{1}{2} (10)^{1/3} \int e^{-x} Q^4 - 2 (10)^{-1/3} \theta_A \left( \frac{3}{2} \int e^{-x} Q^2 - \frac{7}{5} \int e^{-x} Q^5 \right) = (10)^{2/3}. $$

Next, since

$$Z = -2 (10)^{-1/3} \theta_A \left( 3 \int e^{-x} (Q - Q') - 2 \int e^{-x} Q^4 \right) = 2 \theta_A,$$

from (A.17) and (A.14), we find $\theta_B$ by integrating the equation of $B_1$ ($2 \theta_B = \int (-LB_1)'$)

$$2 \theta_B = \beta \frac{1}{6} \int Q + \int Z = 3 (10)^{2/3} \frac{\int Q}{\int Q^2}.$$  

We now obtain the existence of $\hat{B}_1 \in \mathcal{Y}$ as in the proof of Lemma 2.3 with $b_1$ uniquely chosen so that $\int B_1 Q = 0$ and $\int B_1 Q' = 0$.

Proof of (ii). We solve the equation of $B_2$ exactly in the same way. We check that the values of $\beta$ and $\theta_B$ are suitable to solve the problem, and we obtain unique $\hat{B}_2 \in \mathcal{Y}$ and $b_2$ so that $\int B_2 Q' = \int (B_2 - 2 \theta_B) Q = 0$.

We now check that $b_1 \neq b_2$. Let $B(x) = \hat{B}_1(x) - \hat{B}_2(-x) = B_1(x) - B_2(-x) - 2 \theta_B$. Then $B \in \mathcal{Y}$ and

$$(-LB)' + 16 \theta_B (Q^3)' + (b_1 - b_2) Q' = 0, \quad \int BQ = -4 \theta_B \int Q.$$

By integration

$$-LB + 16 \theta_B Q^3 + (b_1 - b_2) Q = 0.$$

Multiplying the equation of $B$ by $\Lambda Q$ and using $L(\Lambda Q) = -Q$ (see (A.9)), we find

$$-4 \theta_B \int Q + 16 \theta_B \int Q^3 \Lambda Q + (b_1 - b_2) \int Q \Lambda Q = 0.$$

Since $\int Q^3 \Lambda Q = \frac{5}{24} \int Q$ and $\int Q \Lambda Q = \frac{1}{6} \int Q^2$ (see (A.13)) we obtain finally

$$b_1 - b_2 = 4 \theta_B \int \frac{Q}{Q^2} \neq 0.$$

Proof of (iii). We finish the proof of Lemma 2.5 as the one of Lemma 2.3. In particular, using the limits of $B_1$ and $B_2$ at $\pm \infty$, and (2.5),

$$G(w_B) = \mu_1 y e^{-y} (-LB_1)'(x - y_1) + \mu_2 y e^{-y} (-LB_2 - 8 \theta_B Q^3)'(x - y_2) + O_2.$$

This, combined with the equations of $B_1$ and $B_2$ and Lemmas 2.1, 2.2, 2.3 and 2.4 proves (2.27). Note that $w_B$ is not in $L^2$ since it has a nonzero limit at $-\infty$. However, it has exponential decay as $x \to +\infty$. This allows us to prove that all rest terms are indeed of the form $O_2$ (see notation $O_2$ in (2.19)).

The control of the various scalar products is easily obtained as in Lemma 2.3 from the properties of $B_1, B_2$. 

\[\square\]
We claim without proof the following existence result.

Lemma 2.6 (Definition and equation of \(w_D\)). Let

\[
S = -4(10)^{1/3} \left( e^{-x}Q^2 \left( \frac{2}{3}Q + \frac{1}{2}xQ + \frac{2}{3}xQ' \right) \right)' - \alpha \Lambda^2 Q + a(\Lambda Q)' - S_1 - \tilde{S}_1.
\]

(i) There exist unique \(\delta, \theta_D, d_1\) and \(\hat{D}_1 \in \mathcal{Y}\) such that \(D_1 = \hat{D}_1 + \theta_D \left( 1 + \frac{Q}{Q'} \right)\) satisfies

\[
(-LD_1)' + \delta \Lambda Q + d_1 Q' = S(x), \quad \int D_1 Q' = \int D_1 Q = 0.
\]

(ii) There exist unique \(d_2\) and \(\hat{D}_2 \in \mathcal{Y}\) such that \(D_2 = \hat{D}_2 - \theta_D \left( 1 + \frac{Q}{Q'} \right)\) satisfies

\[
(-LD_2)' - 8\theta_D(Q^3)' - \delta \Lambda Q + d_2 Q' = -S(-x), \quad \int D_2 Q' = \int (D_2 - 2\theta_D)Q = 0.
\]

(iii) Set

\[
w_D(t, x) = e^{-v(t)} \left( \mu_1 D_1(x - y_1(t)) + \mu_2 D_2(x - y_2(t)) \right).
\]

Then,

\[
F_A + F_A + G(w_A) + G(w_Q) + F_B + F_B + \tilde{F}_B + G(w_B) + F_D + \tilde{F}_D + G(w_D) = O_2,
\]

\[
\left| \int w_D R_1 \right| + \left| \int w_D \partial_x R_1 \right| + \left| \int w_D R_2 \right| + \left| \int w_D \partial_x R_2 \right| = O_{5/2}.
\]

The proof is exactly the same as the one of Lemma 2.5 except that we do not need the values of \(\delta, \theta_D\) and \(d_1 - d_2\). The exact expressions of \(S_1\) and \(\tilde{S}_1\) are thus not needed.

2.5 End of the proof of Proposition 2.1

Set

\[
V_0 = \tilde{R}_1 + \tilde{R}_2 + W_0, \quad \tilde{R}_1 = w_A + w_Q + w_B + w_D.
\]

From the preliminary expansion (2.20), we have

\[
S(V_0) = \tilde{E}(V_0) + E_0, \quad E_0 = F + \tilde{F} + G(W_0) + H(W_0).
\]

In view of notation (2.19), estimate (2.11) holds true for some \(\sigma > 0\) provided that \(E_0 = O_2\). From Lemmas 2.1, 2.2 and 2.6 we have \(F + \tilde{F} + G(W_0) = O_2\). Thus, we only have to check that \(H(W_0) = O_2\).

First,

\[
\partial_x \left[ \left( \tilde{R}_1 + \tilde{R}_2 + W_0 \right)^4 - \left( \tilde{R}_1 + \tilde{R}_2 \right)^4 + 4 \left( \tilde{R}_1^3 + \tilde{R}_2^3 \right) W_0 \right] = O_2
\]

since this term is quadratic in \(W_0\).

Second, since \(|M_j| + |N_j| \leq Ce^{-y}\), we also obtain

\[
\sum_{j=1,2} M_j \frac{\partial W_0}{\partial \mu_j} - \sum_{j=1,2} N_j \frac{\partial W_0}{\partial y_j} = O_2.
\]

Thus, Proposition 2.1 is proved.
2.6 Proof of Proposition 2.2

Denote \( \tilde{x} = e^{-\frac{i}{2}Y_0}x + 1 \). Multiplying (2.8) by \( \psi(\tilde{x}) \) and using \( \tilde{E}(V_0)\psi(\tilde{x}) = \tilde{E}(V) \), we observe that \( V(t, x) \) solves
\[
\partial_t V + \partial_x (\partial_x^2 V - V + V^4) = \tilde{E}(V) + E(t, x),
\]
where
\[
E(t, x) = E_0(t, x)\psi(\tilde{x}) + e^{-\frac{i}{2}Y_0}\psi''(\tilde{x})V_0 + 3e^{-Y_0}\psi''(\tilde{x})\partial_x V_0 + 3e^{-\frac{i}{2}Y_0}\psi'(\tilde{x})\partial_x^2 V_0
- e^{-\frac{i}{2}Y_0}\psi'(\tilde{x})V_0 + \psi(\tilde{x})(\psi^3(\tilde{x}) - 1)\partial_x(V_0^4) + 4\psi^3(\tilde{x})\psi'(\tilde{x})V_0^4.
\]

Note that
\[
\|\psi(\tilde{x})/(1 + e^{\frac{i}{2}(x-y_1(t))})\|_{L^2} + \|\psi'(\tilde{x})\|_{L^2} + \|\psi''(\tilde{x})\|_{L^2} + \|\psi'''(\tilde{x})\|_{L^2} \leq C e^{\frac{i}{2}Y_0}.
\]
Moreover, by the properties of \( \psi \), and \( |y_j(t)| \leq KY_0 \) (combine (2.4)–(2.5)), we have
\[
(\|\psi(\tilde{x})\| + \|\psi'(\tilde{x})\| + \|\psi''(\tilde{x})\| + \|\psi'''(\tilde{x})\|) \left( e^{-\frac{1}{2}|x-y_1|} + e^{-\frac{1}{2}|x-y_2|} \right) \leq C \exp(-\frac{1}{4}e^{\frac{i}{2}Y_0}).
\]

First, using (2.11) and (2.30),
\[
\|E_0(t, x)\psi(\tilde{x})\|_{L^2} \leq CY_0^\sigma e^{-Y_0}e^{-y(t)}\|\psi(\tilde{x})/(1 + e^{\frac{i}{2}(x-y_1(t))})\|_{L^2} \leq CY_0^\sigma e^{-\frac{i}{4}Y_0}e^{-y(t)}.
\]

Second, by the structure of \( V_0 \), we easily check that
\[
|V_0| + |\partial_x V_0| + |\partial_x^2 V_0| \leq C \left( e^{-\frac{1}{2}|x-y_1|} + e^{-\frac{1}{2}|x-y_2|} \right) + CY_0 e^{-\frac{i}{4}Y_0}e^{-y(t)},
\]
and all the other terms in \( E \) are controled in \( L^2 \) as desired using (2.32) combined with (2.31). The estimate in \( H^1 \) is obtained similarly.

3 Preliminary stability arguments

3.1 Stability of the 2-soliton structure in the interaction region

We start by decomposing any solution of (1.14) close the approximate solution \( V \) (introduced in Proposition 2.2). See Appendix B for the proof.

**Lemma 3.1** (Decomposition around the approximate solution). There exists \( \omega_0 > 0, C > 0, \bar{y}_0 > 0 \) such that if \( u(t) \) is a solution of (1.14) on some time interval \( I \) satisfying for \( 0 < \omega < \omega_0, y_0 > \bar{y}_0 \)
\[
\forall t \in I, \inf_{y_1, y_2 > y_0} \|u(t) - V(\cdot; (0, 0, y_1, y_2))\|_{H^1} \leq \omega,
\]
then there exists a unique decomposition \( (\Gamma(t), \varepsilon(t)) \) of \( u(t) \) on \( I \),
\[
u(t, x) = V(x; \Gamma(t)) + \varepsilon(t, x), \quad (\Gamma(t) = (\mu_1(t), \mu_2(t), y_1(t), y_2(t)) \ \text{of class } C^1,
\]
such that \( \forall t \in I, \)
\[
\int \varepsilon(t)\tilde{R}_1(t) = \int \varepsilon(t)\partial_x \tilde{R}_1(t) = \int \varepsilon(t)\tilde{R}_2(t) = \int \varepsilon(t)\partial_x \tilde{R}_2(t) = 0,
\]
\[
y(t) = y_1(t) - y_2(t) > y_0 - C\omega, \quad \|\varepsilon(t)\|_{H^1} + |\mu_1(t)| + |\mu_2(t)| \leq C\omega,
\]
both integrable and nonintegrable cases. Approximate solutions at long time stability results in the interaction region. They will allow us to compare the mass is related to Weinstand's approach for stability of one soliton [38] and to Kato identity (1.14) equation (see [13]). These techniques have been developed in [22], [29], [20] and extended in [24] and have had decisive applications to long time issues for (1.1) : blow up in the critical case, stability and asymptotic stability of multi-solitons in the energy space in both integrable and nonintegrable cases.

In the next proposition, we present almost monotonicity laws which are essential in proving long time stability results in the interaction region. They will allow us to compare the approximate solution $V(t, x)$ with exact solutions. The functional is different depending on whether $\mu_1(t) > \mu_2(t)$ or $\mu_1(t) < \mu_2(t)$. The introduction of such variants of the energy and mass is related to Weinstein’s approach for stability of one soliton [38] and to Kato identity for the (1.14) equation (see [13]). These techniques have been developed in [22], [29], [20] and extended in [24] and have had decisive applications to long time issues for (1.1) : blow up in the $L^2$ critical case, stability and asymptotic stability of multi-solitons in the energy space in both integrable and nonintegrable cases.

Fix a constant $0 < \rho < 1/32$ and set

$$\varphi(x) = \frac{2}{\pi} \arctan(\exp(8\rho x)), \quad \text{so that } \lim_{-\infty} \varphi = 0, \lim_{\infty} \varphi = 1,$$

$$\forall x \in \mathbb{R}, \varphi(-x) = 1 - \varphi(x), \quad \varphi'(x) = \frac{8\rho}{\pi \cosh(8\rho x)},$$

$$|\varphi''(x)| \leq 8\rho|\varphi'(x)|, \quad |\varphi'''(x)| \leq (8\rho)^2|\varphi'(x)|.$$

**Proposition 3.1 (Almost monotonicity laws).** Under the assumptions of Lemma 3.1, let

$$\mathcal{F}_+(t) = \int \left[ \left((\partial_x \epsilon)^2 + \epsilon^2 - \frac{2}{5} \left((\epsilon + V)^5 - V^5 - 5V^4 \epsilon \right) \right) + \varepsilon^2 \Phi(t, x) \right] \, dx,$$

where $\Phi(t, x) = \mu_1(t) \varphi(x) + \mu_2(t) (1 - \varphi(x))$;

$$\mathcal{F}_-(t) = \int \left[ \left((\partial_x \epsilon)^2 + \epsilon^2 - \frac{2}{5} \left((\epsilon + V)^5 - V^5 - 5V^4 \epsilon \right) \right) \Phi_1(t, x) + \varepsilon^2 \Phi_2(t, x) \right] \, dx,$$

where

$$\Phi_1(t, x) = \frac{\varphi(x)}{(1 + \mu_1(t))^2} + \frac{1 - \varphi(x)}{(1 + \mu_2(t))^2}, \quad \Phi_2(t, x) = \frac{\mu_1(t) \varphi(x)}{(1 + \mu_1(t))^2} + \frac{\mu_2(t) (1 - \varphi(x))}{(1 + \mu_2(t))^2}.$$

There exists $C > 0$ such that

$$\|\varepsilon(t)\|_{H^1}^2 \leq C \mathcal{F}_+(t), \quad \|\varepsilon(t)\|_{H^1}^2 \leq C \mathcal{F}_-(t).$$
Moreover,
(i) If \( t \in I \) is such that \( \mu_1(t) \geq \mu_2(t) \) and \( y_2(t) \leq -\frac{1}{4} y(t), \ y_1(t) \geq \frac{1}{4} y(t), \) then
\[
\frac{d}{dt} F_+(t) \leq C \|\varepsilon\|_{L^2}^2 \left[ e^{-\frac{3}{2} \varepsilon} + (|\mu_1| + |\mu_2| + \|\varepsilon\|_{L^2})(e^{-2\rho y} + \|\varepsilon\|_{L^2}) \right] + C \|\varepsilon\|_{H^1} \|E\|_{H^1}. \tag{3.11}
\]
(ii) If \( t \in I \) is such that \( \mu_2(t) \geq \mu_1(t) \) and \( y_2(t) \leq -\frac{1}{4} y(t), \ y_1(t) \geq \frac{1}{4} y(t), \) then
\[
\frac{d}{dt} F_-(t) \leq C \|\varepsilon\|_{L^2}^2 \left[ e^{-\frac{3}{2} \varepsilon} + (|\mu_1| + |\mu_2| + \|\varepsilon\|_{L^2})(e^{-2\rho y} + \|\varepsilon\|_{L^2}) \right] + C \|\varepsilon\|_{H^1} \|E\|_{H^1}. \tag{3.12}
\]

See proof of Proposition 3.1 in Appendix B.

3.2 Stability of the two soliton structure for large time

In this section, we claim a stability result for the two soliton structure for large time, i.e. far away from the interaction time. The argument is similar to the one of Proposition 3.1. See a sketch of proof in Appendix B.

**Proposition 3.2** (Stability for large time). There exists \( C > 0 \) such that for \( \mu_0 > 0 \) and \( \omega > 0 \) small enough, if \( u(t) \) is an \( H^1 \) solution of (1.14) satisfying
\[
\|u(t_0) - Q_{1-\mu_0}(\cdot + \mu_0 t_0) - Q_{1+\mu_0}(\cdot - \mu_0 t_0)\|_{H^1(\mathbb{R})} \leq \omega \mu_0, \tag{3.13}
\]
for some \( t_0 < - (\rho \mu_0)^{-1} |\log \mu_0| \), then there exist \( y_1(t), y_2(t) \) and \( \mu_1^+, \mu_2^+ \) such that

(i) For all \( t_0 \leq t \leq - (\rho \mu_0)^{-1} |\log \mu_0| \),
\[
\|u(t) - Q_{1-\mu_0}(\cdot - y_1(t)) - Q_{1+\mu_0}(\cdot - y_2(t))\|_{H^1(\mathbb{R})} \leq C \omega \mu_0 + C \exp (-4\rho \mu_0 |t|),
\]
\[
y_1(t) - y_2(t) \geq \frac{3}{2} \mu_0 |t|,
\]
\[
| - \mu_0 - \dot{y}_1(t) | + | \mu_0 - \dot{y}_2(t) | \leq C \omega \mu_0 + C \exp (-4\rho \mu_0 |t|).
\]

(ii) For all \( t \leq t_0 \),
\[
\|u(t) - Q_{1-\mu_0}(\cdot - y_1(t)) - Q_{1+\mu_0}(\cdot - y_2(t))\|_{H^1(\mathbb{R})} \leq C \omega \mu_0 + C \exp (-4\rho \mu_0 |t_0|),
\]
\[
y_1(t) - y_2(t) \geq \frac{3}{2} \mu_0 |t|,
\]
\[
| - \mu_0 - \dot{y}_1(t) | + | \mu_0 - \dot{y}_2(t) | \leq C \omega \mu_0 + C \exp (-4\rho \mu_0 |t_0|).
\]

(iii) Asymptotic stability.
\[
\lim_{t \to -\infty} \|u(t) - Q_{1+\mu_1^+}(\cdot - y_1(t)) - Q_{1+\mu_2^+}(\cdot - y_2(t))\|_{H^1(\mathbb{R})} = 0,
\]
\[
\lim_{t \to -\infty} \dot{y}_1(t) = \mu_1^+, \quad \lim_{t \to -\infty} \dot{y}_2(t) = \mu_2^+,
\]
\[
| \mu_1^+ + \mu_0 | + | \mu_2^+ - \mu_0 | \leq C \omega \mu_0 + C \exp (-4\rho \mu_0 |t_0|).
\]

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Remark 1. Using the invariance of the $gKdV$ equation by the transformation  

$$x \rightarrow -x, \quad t \rightarrow -t,$$  

(3.17)
a statement similar to Proposition 3.2 holds for $t_0 > (\rho\mu_0)^{-1}|\log \mu_0|$.

Corollary 3. Let $u(t)$ be the unique solution of (1.14) satisfying  

$$\lim_{t \rightarrow -\infty} \|u(t) - Q_{1-\mu_0}(\cdot + \mu_0 t) - Q_{1+\mu_0}(\cdot - \mu_0 t)\|_{H^1} = 0.$$  

Then, for all $t \leq - (\rho\mu_0)^{-1}|\log \mu_0|$,  

$$\|u(t) - Q_{1-\mu_0}(\cdot + \mu_0 t) - Q_{1+\mu_0}(\cdot - \mu_0 t)\|_{H^1} \leq \exp(-4\rho \mu_0 |t|).$$  

(3.18)

We refer to Theorem 1 in [20] for the existence and uniqueness of the solution $U(t)$.

Proof of Corollary 3 assuming Proposition 3.2 For fixed $t$, we can pass to the limit $\omega \rightarrow 0$, $t_0 \rightarrow -\infty$ in (3.14). Then, we integrate the estimates on $\dot{y}_1(t)$ and $\dot{y}_2(t)$ (see (3.14)) from $-\infty$ to $t$.  

4 Stability of the 2-soliton structure

In this section, using the approximate solution constructed in Propositions 2.1 and 2.2 and the asymptotic arguments of Section 3, we prove the stability part of Theorem 1 and Theorem 2. These properties are much more refined than the asymptotic results obtained in [29]; they rely on the approximate solution constructed in Section 2 and on a reduction to a finite dimensional dynamical system. In particular, they cannot be derived from [29].

4.1 Description of the global behavior of the asymptotic 2-soliton solution

Recall that $0 < \rho < 1/16$, $\alpha > 0$ and $\sigma \geq 3$ are defined respectively in Propositions 3.1 and 2.1. We also recall the following notation from the Introduction  

$$Y_0 = |\ln(\mu_0^2/\alpha)| \quad \text{or equivalently} \quad \mu_0 = \sqrt{\alpha e^{-\frac{1}{2}Y_0}},$$  

(4.1)

$$Y(t) = Y_0 + 2 \ln(\cosh(\mu_0 t)) \text{ solution of } \dot{Y} = 2\alpha e^{-Y}, \quad \lim_{t \rightarrow -\infty} \dot{Y}(t) = -2\mu_0, \dot{Y}(0) = 0.$$  

(4.2)

Note that $\dot{Y}(t) = 2\mu_0 \tanh(\mu_0 t)$ and, for all $t \in \mathbb{R}$,  

$$0 \leq Y(t) - (Y_0 + 2\mu_0 |t| - 2 \ln 2) \leq 2\exp(-2\mu_0 |t|).$$  

(4.3)

Proposition 4.1 (Description of the 2-soliton solution in the interaction region).

Let $U(t)$ be the unique solution of (1.14) such that  

$$\lim_{t \rightarrow -\infty} \|U(t) - Q_{1-\mu_0}(\cdot + \frac{1}{2}Y(t)) - Q_{1+\mu_0}(\cdot - \frac{1}{2}Y(t))\|_{H^1(\mathbb{R})} = 0.$$  

(4.4)

Let $T > 0$ be such that $Y(T) = 400\rho^{-2}Y_0$. Then, for $\mu_0 > 0$ small enough, there exists $(\Gamma(t), \varepsilon(t)) \in C^1$ such that for all $t \in [-T, T]$,  

$$U(t, x) = V(t; \Gamma(t)) + \varepsilon(t, x), \quad \Gamma(t) = (\mu_1(t), \mu_2(t), y_1(t), y_2(t)).$$  

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\[ |\tilde{\mu}(t)| \leq Y_0^2 e^{-Y_0}, \quad |\tilde{y}(t)| \leq Y_0^4 e^{-\frac{1}{2}Y_0}, \quad (4.5) \]
\[ |\mu(t) - \dot{Y}(t)| \leq CY_0^{\sigma+1} e^{-\frac{1}{2}Y_0}, \quad |y(t) - Y(t)| \leq CY_0^{\sigma+2} e^{-\frac{3}{2}Y_0}, \quad (4.6) \]
\[ \|\varepsilon(t)\|_{H^1} \leq CY_0^{\sigma} e^{\frac{3}{2}Y_0}, \quad (4.7) \]

where
\[ \mu(t) = \mu_1(t) - \mu_2(t), \quad y(t) = y_1(t) - y_2(t), \]
\[ \tilde{\mu}(t) = \mu_1(t) + \mu_2(t), \quad \tilde{y}(t) = y_1(t) + y_2(t). \quad (4.8) \]

Moreover, there exists \( t_0 \) such that
\[ |t_0| \leq CY_0^{\sigma} e^{-\frac{4}{3}Y_0}, \quad \mu(t_0) = 0; \forall t \in [-T, t_0), \mu(t) < 0; \forall t \in (t_0, T], \mu(t) > 0. \quad (4.9) \]

**Proof.** It follows from Corollary 3 and (4.3) that for all \( t \leq - (\rho \mu_0)^{-1} \log \mu_0 \),
\[ \|U(t) - Q_{1-\mu_0} (x + \frac{1}{2} Y(t)) - Q_{1+\mu_0} (x - \frac{1}{2} Y(t))\|_{H^1} \leq \exp \left(-2\rho Y(t) \right). \quad (4.10) \]
This estimate is the starting point of our analysis.

Let \( 0 < \epsilon < \frac{1}{100} \) to be chosen later, and let \( T > T' > T'' > 0 \) be defined as follows
\[ Y(T) = 400 \rho^{-2} Y_0, \quad Y(T') = 40 \rho^{-1} Y_0, \quad Y(T'') = Y_0 + \epsilon^2. \quad (4.11) \]

Note that by the explicit expression of \( Y(t) \) in (4.2), for \( 0 < C < C' \) independent of \( \epsilon \), we have
\[ CY_0^2 e^{\frac{1}{2}Y_0} \leq T' < T < C' Y_0 e^{\frac{1}{2}Y_0}, \]
\[ C e^{-\frac{1}{2}Y_0} \leq T'' < C' \epsilon e^{-\frac{1}{2}Y_0}, \quad \dot{Y}(T'') \geq C \epsilon e^{-\frac{1}{2}Y_0}. \quad (4.12) \]

Applying estimate (4.10) at \( t = -T \), and using (4.11), we find
\[ \|U(-T) - Q_{1-\mu_0} (x + \frac{1}{2} Y(T)) - Q_{1+\mu_0} (x - \frac{1}{2} Y(T))\|_{H^1} \leq C e^{-\rho Y(T)} e^{-10 Y(T')} \quad (4.13) \]

For \( t \geq -T \), as long as \( u(t) \) stays close to the sum of two solitons, we introduce the decomposition \((\Gamma(t), \varepsilon(t))\) of \( U(t) \) as constructed in Lemma 3.1. At \( t = -T \), from (4.13), we obtain
\[ \|\varepsilon(-T)\|_{H^1} + |\mu_1(-T) + \mu_0| + |\mu_2(-T) - \mu_0| \leq C e^{-\rho Y(T)} e^{-10 Y(T')}, \quad (4.14) \]
\[ |y_1(-T) - \frac{1}{2} Y(T)| + |y_2(-T) + \frac{1}{2} Y(T)| \leq C e^{-\rho Y(T)} e^{-10 Y(T')}. \quad (4.15) \]

Let \( C_3 > C_2 > C_1 > 1 \) to be chosen later, and consider the following estimates:

For \( t \in [-T, T'] \),
\[ e^{\frac{2}{3}Y_0} \|\varepsilon(t)\|_{H^1} + e^{-\frac{1}{2}Y_0} |y(t) - Y(t)| + e^{\frac{1}{2}Y_0} |\mu(t) - \dot{Y}(t)| \leq C_1 Y_0^\sigma e^{-(1-\rho)Y(T')} e^{-\rho Y(t)}. \quad (4.16) \]

For \( t \in [-T', T''] \),
\[ \|\varepsilon(t)\|_{H^1} + |\mu(t) - \dot{Y}(t)| \leq C_2 Y_0^\sigma e^{-\frac{1}{2}Y_0} e^{-Y(t)}, \quad |y(t) - Y(t)| \leq C_2 Y_0^\sigma e^{-\frac{1}{2}Y_0}. \quad (4.17) \]

For \( t \in [T'', T] \),
\[ Y_0^2 e^{-\frac{3}{2}Y_0} \|\varepsilon(t)\|_{H^1} + Y_0 e^{\frac{1}{2}Y_0} |\mu(t) - \dot{Y}(t)| + |y(t) - Y(t)| \leq C_3 Y_0^\sigma e^{\frac{1}{2}Y_0}. \quad (4.18) \]
From (4.14)-(4.15) and the continuity of $t \mapsto u(t)$ in $H^1$, it follows that we can define

$$T^* = \sup \{ t \in [-T, T] \text{ such that } (4.5)-(4.16)-(4.17)-(4.18) \text{ hold on } [-T, t] \}.$$ 

Now, we aim at proving that $T^* = T$ by strictly improving estimates (4.5)-(4.16)-(4.17)-(4.18) on $[-T, T^*]$ for $C_1, C_2, C_3$ large enough (independent of $Y_0$) and $Y_0$ large enough (possibly depending on $C_1, C_2, C_3$).

**Step 1.** Preliminary simplification of the dynamical system.

**Claim 4.1** (Simplified dynamical system). For all $t \in [-T, T^*],

\[
|\dot{\mu} - 2ae^{-y}| \leq CY_0^\sigma e^{-Y_0} e^{-Y(t)} + C\|\varepsilon(t)\|^2_{L^2}, \\
|\dot{y} - \mu| \leq C|\mu|Y_0e^{-Y(t)} + CY_0^\sigma e^{-Y_0} e^{-Y(t)} + C\|\varepsilon(t)\|_{L^2}, \\
|\ddot{\mu}| \leq C|\mu|Y_0e^{-Y(t)} + CY_0^\sigma e^{-Y_0} e^{-Y(t)} + C\|\varepsilon(t)\|^2_{L^2}, \\
|\ddot{y} - \ddot{\mu}| \leq C|\varepsilon(t)|_{L^2}. \tag{4.19}
\]

**Proof of Claim 4.1**. First, on $[-T, T^*]$, (2.4)-(2.5) hold and so by (2.13),

\[
\int E(t) \dot{Q}_j(t) \leq CY_0^\sigma e^{-Y_0} e^{-Y(t)}, \quad \|E(t)\|_{H^1} \leq CY_0^\sigma e^{-\frac{3}{2}Y_0} e^{-Y(t)} \tag{4.20}
\]

Thus, by (3.6), for some constant $C = C(K) > 0$,

\[
|\dot{\mu}_j - \mathcal{M}_j| \leq CY_0^\sigma e^{-Y_0} e^{-Y(t)} + CY_0\|\varepsilon(t)\|_{L^2} e^{-Y(t)} + C\|\varepsilon(t)\|^2_{L^2}, \tag{4.21}
\]

\[
|\mu_j - \dot{\mu}_j - \mathcal{N}_j| \leq CY_0^\sigma e^{-Y_0} e^{-Y(t)} + C\|\varepsilon(t)\|_{L^2}. \tag{4.22}
\]

Second, the following estimates are obtained by combining the definitions of $\mathcal{M}_j$ and $\mathcal{N}_j$ in (2.10):

\[
|\dot{\mu} - \{2ae^{-y} + \beta \mu \dot{y}e^{-y} + \delta \mu e^{-y}\}| \leq \sum_{j=1,2} |\dot{\mu}_j - \mathcal{M}_j|,
\]

\[
|\ddot{\mu} - \{\beta \mu \dot{y}e^{-y} + \delta \mu e^{-y}\}| \leq \sum_{j=1,2} |\ddot{\mu}_j - \mathcal{M}_j|,
\]

\[
|\dot{y} - \{\mu + b_+ \mu \dot{y}e^{-y} + b_- \mu \dot{y}e^{-y} + d_+ \mu e^{-y} + d_- \mu e^{-y}\}| \leq \sum_{j=1,2} |\mu_j - \dot{\mu}_j - \mathcal{N}_j|, \tag{4.23}
\]

\[
|\ddot{y} - \{\dot{\mu} - 2ae^{-y} + b_- \mu \dot{y}e^{-y} + b_+ \mu \dot{y}e^{-y} + d_- \mu e^{-y} + d_+ \mu e^{-y}\}| \leq \sum_{j=1,2} |\mu_j - \dot{\mu}_j - \mathcal{N}_j|,
\]

where

\[
b_+ = -\frac{1}{2}(b_1 + b_2), \quad b_- = \frac{1}{2}(b_1 - b_2), \quad d_+ = -\frac{1}{2}(d_1 + d_2), \quad d_- = -\frac{1}{2}(d_1 - d_2). \tag{4.24}
\]

Using the previous estimates, (4.5), and $\sigma \geq 3$, this implies (4.19).

**Step 2.** Bootstrap estimates.

**Bootstrap of (4.5).** From the definition of $T^*$ and (4.19), we improve (4.5) on $[-T, T^*]$. Note that using (4.16)-(4.17)-(4.18), we have for all $t \in [-T, T^*],

\[
\|\varepsilon(t)\|_{H^1} \leq C_3Y_0^\sigma e^{-\frac{3}{2}Y_0}, \quad |\mu(t) - \dot{Y}(t)| \leq C_3Y_0^{\sigma+1} e^{-\frac{3}{2}Y_0}, \quad |y(t) - Y(t)| \leq C_3Y_0^{\sigma+2} e^{-\frac{3}{2}Y_0}. \tag{4.25}
\]
Concerning $\bar{\mu}$, by (4.19) and (4.25), for $Y_0$ large enough,
\[ |\bar{\mu}| \leq CY_0 e^{-\frac{1}{2}Y_0} e^{-Y(t)} + CCY_0^2 e^{-\frac{3}{2}Y_0}. \]

Thus, by direct integration, using the expression of $Y(t)$, $T < CY_0 e^{\frac{1}{2}Y_0}$ (see (4.12)) and (4.14), we find for all $t \in [-T, T^*], |\bar{\mu}(t)| \leq CY_0 e^{-Y_0}$, for $Y_0$ large enough, and thus by possibly taking a larger $Y_0$,
\[ \forall t \in [-T, T^*], \ |\bar{\mu}(t)| \leq \frac{1}{2}Y_0^2 e^{-Y_0}. \quad (4.26) \]

Concerning $\bar{y}$, we proceed similarly. By (4.19) and (4.25), for $Y_0$ large enough, $|\bar{y}| \leq Y_0^2 e^{-Y_0} + Ce^{-Y_0}$, and by direct integration, $T < CY_0 e^{\frac{1}{2}Y_0}$ and (4.14), we find for all $t \in [-T, T^*], |\bar{y}(t)| \leq CY_0^3 e^{-\frac{1}{2}Y_0}$. Finally, taking $Y_0$ large enough, we find
\[ \forall t \in [-T, T^*], \ |\bar{y}(t)| \leq \frac{1}{2}Y_0^3 e^{-\frac{1}{2}Y_0}. \quad (4.27) \]

It follows that
\[ T^* = \sup \{ t \in [-T, T] \text{ such that } (4.16)-(4.17)-(4.18) \text{ hold on } [-T, t] \}. \]

Bootstrap of (4.16) on $[-T, -T']$. Now, we prove that $T^* > -T'$ for $C_1 > 1$ large enough. First, we estimate $\varepsilon(t)$ on $[-T, T^*]$. We use the functional $F_-(t)$ defined in (3.9) since $\mu_2 \geq \mu_1$ on $[-T, T']$. Using (3.11), (4.20) and (4.16), we have
\[ \frac{d}{dt} F_-(t) \leq CC_1 Y_0^{2\sigma} e^{-Y_0} e^{-(1-\rho)Y(T')} e^{-(1+\rho)Y(t)} + CC_1^2 Y_0^{4\sigma} e^{-Y_0} e^{-2(1-\rho)Y(T')} e^{-3\rho Y(t)} + CC_1 Y_0^{4\sigma} e^{-Y_0} e^{-4(1-\rho)Y(T')} e^{-4\rho Y(t)} \leq 2CC_1 Y_0^{2\sigma} e^{-Y_0} e^{-2(1-\rho)Y(T')} e^{-2\rho Y(t)}, \]
for $Y_0$ large enough (depending on $C_1$). By integrating over $[-T, T^*]$, using $\dot{Y}(t) \geq Ce^{-\frac{1}{2}Y_0}$, we obtain, for $C_1$ large,
\[ F_-(t) - F_-(T^*) \leq CC_1 Y_0^{2\sigma} e^{-\frac{1}{2}Y_0} e^{-2(1-\rho)Y(T')} e^{-2\rho Y(t)}. \]

Using (3.10) and (4.14), we obtain
\[ \forall t \in [-T, T^*], \ \|\varepsilon(t)\|_{H^1} \leq C \sqrt{C_1 Y_0^{\sigma} e^{-\frac{1}{2}Y_0} e^{-(1-\rho)Y(T')} e^{-\rho Y(t)}}. \quad (4.28) \]

Second, we control $\mu(t)$ and $y(t)$ on $[-T, T^*]$. By (4.14)-(4.15), we have
\[ |\mu(\bar{\mu}(T)) + |\bar{y}(T) - Y(T)| + |\bar{y}(t)| \leq Ce^{-\rho Y(T)} e^{-3Y(T')}, \quad (4.29) \]
and using (4.19) and (4.16) on $[-T, T^*]$, we have
\[ |\bar{\mu} - 2Ce^{-Y}| \leq CY_0^{\sigma} e^{-Y_0} e^{-(1-\rho)Y(T')} e^{-\rho Y(t)}, \quad (4.30) \]
\[ |\bar{y} - \bar{\mu}| \leq C \sqrt{C_1 Y_0^{\sigma} e^{-\frac{1}{2}Y_0} e^{-(1-\rho)Y(T')} e^{-\rho Y(t)}}, \quad (4.31) \]
by choosing $Y_0$ large enough (depending on $C_1$). Moreover, $\dot{Y} = 2Ce^{-Y}$, so that by (4.30) and (4.13),
\[ |\bar{\mu} - \bar{\mu}| \leq CY_0^{\sigma} e^{-Y_0} e^{-(1-\rho)Y(T')} e^{-\rho Y(t)}. \]
Integrating over \([-T, t]\) using \(\hat{Y}(t) \geq Ce^{-\frac{1}{2}Y_0}\) and (4.29), we find \(\forall t \in [-T, T^*]\),

\[
|\mu(t) - \hat{Y}(t)| \leq CY_0^\sigma e^{-\frac{1}{2}Y_0}e^{-(1-\rho)Y(T')}e^{-\rho Y(t)}. \tag{4.32}
\]

Inserting this into (4.31), and integrating using (4.29), we find \(\forall t \in [-T, T^*]\),

\[
|y(t) - Y(t)| \leq C\sqrt{C_1Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-(1-\rho)Y(T')}e^{-\rho Y(t)}}. \tag{4.33}
\]

Now, we choose \(C_1\) large enough so that by (4.28), (4.32), (4.33), we have

\[
e^{\frac{1}{2}Y_0}\|\varepsilon(t)\|_{H^1} + e^{-\frac{1}{2}Y_0}|y(t) - Y(t)| + e^{\frac{1}{2}Y_0}|\mu(t) - \hat{Y}(t)| \leq (C_1/2)Y_0^\sigma e^{-(1-\rho)Y(T')}e^{-\rho Y(t)}. \tag{4.34}
\]

By (4.26)-(4.27) and continuity arguments, it follows that for such \(C_1, T^* > -T'\). The constant \(C_1\) is now fixed.

**Bootstrap of (4.17).** Let \(C_2 > 2C_1\) to be chosen later. From the previous step, we have

\[
||\varepsilon(-T')||_{H^1} \leq C_1Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-Y(T')}, \quad |y(-T') - Y(-T')| \leq C_1Y_0^\sigma e^{\frac{1}{2}Y_0}e^{-Y(T')},
\]

\[
|\mu(-T') - \hat{Y}(-T')| \leq C_1Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-Y(T')}, \tag{4.35}
\]

As before, the objective is to prove \(T^* > T''\) for \(C_2 > C_1\) large enough by strictly improving (4.17) on \([-T', T^*]\).

First, we prove that \(T^* > -T''\). We work on the time interval \([-T', t^*]\), where \(t^* = \min(-T'', T^*)\). To estimate \(\varepsilon(t)\), we argue as before; using (3.11) and (4.17), we have

\[
\frac{d}{dt}F_\varepsilon(t) \leq CC_2Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-2Y(t)} + CC_2Y_0^\sigma e^{-(1+\rho)Y_0}e^{-2Y(t)} + CC_2Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-4Y(t)}
\]

\[
\leq 2CC_2Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-2Y(t)},
\]

for \(Y_0\) large (depending on \(C_2\)). Using \(\hat{Y}(t) \geq \hat{Y}(T'') \geq Ce^{-\frac{1}{2}Y_0}\) (see (4.12)), (3.10) and (4.35), we obtain

\[
\forall t \in [-T', t^*], \quad \|\varepsilon(t)\|_{H^1} \leq C\left(\sqrt{\frac{C_2}{\varepsilon} + C_1Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-Y(t)}} \right) \leq 2C\sqrt{\frac{C_2}{\varepsilon}Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-Y(t)}}, \tag{4.36}
\]

by choosing \(C_2\) large enough (depending on \(\varepsilon\)).

Now, we turn to \(\mu(t)\) and \(y(t)\). We obtain from (4.19)

\[
|\dot{\mu} - 2\alpha e^{-y}| \leq CY_0^\sigma e^{-\frac{1}{2}Y_0}e^{-Y(t)},
\]

\[
|\dot{y} - \mu| \leq C\sqrt{\frac{C_2}{\varepsilon}Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-Y(t)}}. \tag{4.37}
\]

We need an approximate conserved quantity for the dynamical system; let

\[
H(t) = \mu^2(t) + 4\alpha e^{-y(t)}. \tag{4.38}
\]

Then, \(\hat{H}(t) = 2\mu\dot{\mu} - 4\alpha \dot{y}e^{-y}\), so that by (4.37) and \(|\mu(t)| \leq Ce^{-\frac{1}{2}Y_0}\),

\[
|\hat{H}| \leq CY_0^\sigma e^{-\frac{1}{2}Y_0}e^{-Y(t)} + C\sqrt{\frac{C_2}{\varepsilon}Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-2Y(t)}} \leq C\sqrt{\frac{C_2}{\varepsilon}Y_0^\sigma e^{-\frac{1}{2}Y_0}e^{-Y(t)}}.
\]
By (4.35) and \((\dot{Y})^2 + 4ae^{-Y} = 4ae^{-Y_0}\), we have
\[ |H(-T') - 4ae^{-Y_0}| \leq CY_0^2 e^{-\frac{3}{2}Y_0} e^{-Y(T')} . \]

Thus, integrating on \([-T', t]\), for all \(t \in [-T', t^*]\), using \(|\dot{Y}(t)| \geq Cee^{-\frac{3}{2}Y_0}\), we get
\[ |H(t) - 4ae^{-Y_0}| \leq C e^{-3/2} \sqrt{C_2Y_0^2 e^{-\frac{3}{2}Y_0} e^{-Y(t)}} . \]

Using (4.37), we have \(|\dot{y} - \mu| \leq C \sqrt{C_2} \sqrt{Y_0^2 e^{-\frac{3}{2}Y_0} e^{-Y(t)}}\), and thus
\[ |(\dot{y}(t))^2 + 4ae^{-y(t)} - 4ae^{-Y_0}| \leq C e^{-3/2} \sqrt{C_2Y_0^2 e^{-\frac{3}{2}Y_0} e^{-Y(t)}} \leq 2C e^{-3/2} \sqrt{C_2Y_0^2 e^{-\frac{3}{2}Y_0} e^{-Y(t)}} . \]

Set \(a_0 = C/(2\alpha e^{3/2})\), so that
\[ 4\alpha(1 - a_0)e^{-y(t)} \leq 4\alpha e^{-Y_0} - (\dot{y}(t))^2 \leq 4\alpha(1 + a_0)e^{-y(t)} . \] (4.39)

For \(s \in [-S_0, s^*]\), where \(S_0 = \sqrt{\alpha}e^{-\frac{1}{2}Y_0}T\), \(s^* = \sqrt{\alpha}e^{-\frac{1}{2}Y_0}t^* \leq -s_1 = -\sqrt{\alpha}e^{-\frac{1}{2}Y_0}T^p \leq -C\), we define
\[ f(s) = \exp \left( \frac{1}{2} \left( y \left( \frac{1}{\sqrt{\alpha}} e^{\frac{3}{2}Y_0} \right) - Y_0 \right) \right) . \] (4.40)

For \(s < s^*\), we have by (4.2) and (4.17)
\[ Y \left( \frac{1}{\sqrt{\alpha}} e^{\frac{1}{2}Y_0} \right) - Y_0 \geq 2 \ln(\cosh(s^*)) \geq C\epsilon, \quad \text{and so} \quad f(s) \geq 1 + C\epsilon . \] (4.41)

Next, we have \(\dot{f}(s) = \frac{1}{2\sqrt{\alpha}} e^{\frac{3}{2}Y_0} \dot{y} \left( \frac{1}{\sqrt{\alpha}} e^{\frac{3}{2}Y_0} \right) f(s)\), and by (4.39), we find
\[ 1 - a_0 \leq f^2 - (\dot{f})^2 \leq 1 + a_0 . \]

We define
\[ f_+(s) = \frac{f(s)}{\sqrt{1 - a_0}}, \quad f_-(s) = \frac{f(s)}{\sqrt{1 + a_0}}, \]
so that by (4.41), for \(Y_0\) large enough (depending on \(\epsilon\) and \(C_2\)), we have \(f_+(s) > f_-(s) > 1\) on \([-S_0, s^*]\), and \(f_{\pm}\) satisfy
\[ f_+^2 - (\dot{f}_+)^2 \geq 1, \quad f_-^2 - (\dot{f}_-)^2 \leq 1 . \]

Let now \(g_\pm > 0\) be such that \(\cosh(g_\pm) = f_\pm\). Then \((\dot{g}_+)^2 \leq 1\) and \((\dot{g}_-)^2 \geq 1\). Since \(\dot{g}_- < 0\) (it has the sign of \(\dot{y}\) and \(\mu\)), we find:
\[ \forall s \in [-S_0, s^*], \quad \dot{g}_+(s) \geq -1 \quad \text{and} \quad \dot{g}_-(s) \leq 1 . \]

At \(s = -S_0\), by (4.35), we have
\[ |f(-S_0) - \cosh S_0| \leq C e^{\frac{1}{2}(Y(T') - Y_0)} (y(-T') - Y(T')) \leq CY_0^2 e^{-\frac{3}{2}Y_0} e^{-\frac{1}{2}Y(T')} \leq C a_0 , \]
and so
\[ |\cosh(g_\pm(-S_0)) - \cosh(S_0 + \frac{1}{2}a_0)| \leq C a_0 \quad \Rightarrow \quad |g_\pm(-S_0) - S_0| \leq CY_0^2 e^{-\frac{3}{2}Y_0} \leq C a_0 . \]
Integrating on $[-S_0, s]$, for all $s \in [-S_0, s^*]$, we find

$$g_+(s) \geq -s - C a_0, \quad g_-(s) \leq -s + C a_0.$$  

Thus, $\forall s \in [-S_0, s^*]$,

$$\cosh(s + C' a_0) \leq \sqrt{1 - a_0} \cosh(s + C a_0) \leq f(s) \leq \sqrt{1 + a_0} \cosh(s - C a_0) \leq \cosh(s - C' a_0),$$

and since $\cosh(s) = \exp(\frac{1}{2}(Y(t) - Y_0))$, $t = \frac{1}{\sqrt{\alpha}} e^{\frac{1}{2} Y_0} s$ (see (4.2)),

$$Y(t + \frac{C}{\sqrt{\alpha}} e^{\frac{1}{2} Y_0} a_0) \leq y(t) \leq Y(t - \frac{C}{\sqrt{\alpha}} e^{\frac{1}{2} Y_0} a_0)$$

so that

$$\forall t \in [-T', t^*], \quad |y(t) - Y(t)| \leq C a_0 = C e^{\frac{3}{2}} \sqrt{C_2 Y_0^\sigma} e^{-\frac{3}{2} Y_0}.$$  

(4.42)

By (4.37) and (4.42), we deduce

$$|\dot{\mu} - 2a e^{-Y}| \leq C Y_0^\sigma e^{-Y_0} e^{-Y(t)} + C e^{-Y(t)} a_0.$$ 

Integrating on $[-T', t^*]$, using $|\dot{Y}(t)| \geq C e^{-\frac{1}{2} Y_0}$ and (4.35), we find

$$|\mu(t) - \dot{Y}(t)| \leq C e^{-5/2} \sqrt{C_2 Y_0^\sigma} e^{-\frac{5}{2} Y_0} e^{-Y(t)}.$$  

(4.43)

By (4.36), (4.42), (4.43), assuming $C_2 > C_2^e = C/\epsilon^5$, for $C > 0$ large enough, we strictly improve (4.17) and thus we obtain $t^* = -T''$ and by continuity $-T'' < T^*$ for such $C_2$.

Now, assuming $T^* < T''$, we argue on $[-T'', T^*]$ in order to prove by contradiction that $T^* > T''$. First, by (4.17), we have

$$|e^{-y(t)} - e^{-Y(t)}| \leq 2 e^{-Y(t)} |y(t) - Y(t)| \leq C C_2 Y_0^\sigma e^{-\frac{3}{2} Y_0},$$

and thus, by (4.19), (4.17) and $\ddot{Y} = 2 a e^{-Y}$, we obtain

$$|\dot{\mu} - \ddot{Y}| \leq C C_2 Y_0^\sigma e^{-\frac{3}{2} Y_0}.$$ 

By integration on $[-T'', t]$, for all $t \in [-T'', T^*]$, and using (4.43) at $t = -T''$, we find

$$|\mu(t) - \dot{Y}(t)| \leq C e^{-5/2} \sqrt{C_2 Y_0^\sigma} e^{-\frac{5}{2} Y_0} + C C_2 Y_0^\sigma e^{-\frac{3}{2} Y_0}.$$  

(4.44)

From (4.44), (4.19) and (4.17), we have ($Y_0$ large enough)

$$|\dot{y} - \ddot{Y}| \leq C e^{-5/2} \sqrt{C_2 Y_0^\sigma} e^{-\frac{5}{2} Y_0} + C C_2 Y_0^\sigma e^{-\frac{3}{2} Y_0},$$

and thus, integrating on $[-T'', t]$, for all $t \in [-T'', T^*]$, and using (4.42) at $t = -T''$, we find

$$|y(t) - Y(t)| \leq C e^{3/2} \sqrt{C_2 Y_0^\sigma} e^{-\frac{3}{2} Y_0} + C C_2 Y_0^\sigma e^{-\frac{3}{2} Y_0}.$$  

(4.45)

From (4.44) and (4.45), we fix $\epsilon > 0$ (independent of $C_2$) small enough so that

$$|\mu(t) - \dot{Y}(t)| \leq C' \sqrt{C_2 Y_0^\sigma} e^{-\frac{5}{2} Y_0} + \frac{1}{16} C C_2 Y_0^\sigma e^{-\frac{3}{2} Y_0},$$

$$|\dot{y}(t) - \ddot{Y}(t)| \leq C' \sqrt{C_2 Y_0^\sigma} e^{-\frac{3}{2} Y_0} + \frac{1}{16} C C_2 Y_0^\sigma e^{-\frac{3}{2} Y_0}.$$
Thus, $\epsilon > 0$ being fixed, for $C_2$ large enough ($C_2 \geq 16(C')^2$), we obtain

$$|\mu(t) - \dot{Y}(t)| \leq \frac{1}{8} C_2 Y_0^\sigma e^{-\frac{2}{5} Y_0}, \quad |\dot{y}(t) - \dot{Y}(t)| \leq \frac{1}{8} C_2 Y_0^\sigma e^{-\frac{2}{5} Y_0}. \quad (4.46)$$

On $[-T'', T']$, since $|\dot{\mu}(t) - \dot{Y}(t)| \leq C Y_0^\sigma e^{-\frac{2}{5} Y_0}$ and $\ddot{Y} \geq \frac{3}{4} \alpha e^{-Y_0}$, we have $\ddot{\mu} \geq \alpha e^{-Y_0}$. The fact that $\dot{Y}(0) = 0$ and this lower bound on $\ddot{\mu}$ implies, if $T'$ is large enough, the existence of a unique $t_0 \in [-T'', T']$ such that $\mu(t) < 0$ if $t \in [-T'', t_0]$, $\mu(t_0) = 0$ and $\mu(t) > 0$ for $t \in [-T', T']$, moreover, $|t_0| \leq C Y_0 e^{-\frac{2}{5} Y_0}$.

Finally, to control $\|\varepsilon(t)\|_{H^1}$ on $[-T'', T']$, we use the function $F_+(t)$ introduced in Proposition 3.1 on $[-T'', t_0]$ and the function $F_-(t)$ introduced in Proposition 3.1 on $[t_0, T']$. As before, we get the following bound on $\varepsilon(t)$ using (3.11), (3.12), (4.17) and (4.36) (recall that $\epsilon > 0$ has been fixed)

$$\forall t \in [-T'', T'], \quad \|\varepsilon(t)\|_{H^1} \leq C \sqrt{C_2 Y_0^\sigma} e^{-\frac{2}{5} Y_0} \leq \frac{C_2}{8} Y_0^\sigma e^{-\frac{2}{5} Y_0}, \quad (4.47)$$

for $C_2$ large enough. Thus, combining (4.46) and (4.47), we have proved for $t \in [-T', T']$, (note that for $t \in [-T'', T']$, $|Y(t) - Y_0| \leq C \epsilon^2 < \frac{1}{2}$ so that $e^{-Y_0} \leq 2 e^{-Y(t)}$)

$$\|\varepsilon(t)\|_{H^1} + |\mu(t) - \dot{Y}(t)| \leq (C_2/2) Y_0^\sigma e^{-\frac{2}{5} Y_0} e^{-Y(t)}, \quad |\dot{y}(t) - \dot{Y}(t)| \leq (C_2/2) Y_0^\sigma e^{-\frac{2}{5} Y_0}, \quad (4.48)$$

which improves strictly (4.17) on $[-T'', T']$ and thus, $T' > T''$.

**Bootstrap of (4.18).** Now, we prove $T' = T$. Let us first estimate $\|\varepsilon(t)\|_{H^1}$ on $[T'', T']$, using the function $F_+(t)$ defined in Proposition 3.1 (3.11). We get

$$\frac{d}{dt} F_+(t) \leq C C_3 Y_0^{2\sigma} e^{-2 Y_0} e^{-Y(t)} + C C_3 Y_0^{2\sigma} e^{-3 Y_0} e^{-\rho Y_0} + C C_3 Y_0^{4\sigma} e^{5 Y_0}. \quad (4.49)$$

Integrating on $[T'', t]$, for all $t \in [T'', T']$, using $\dot{Y}(t) \geq C e^{-\frac{2}{5} Y_0}$ (if $\epsilon > 0$ is fixed), $|T'| \leq C Y_0 e^{-\frac{2}{5} Y_0}$ and $|F_+(T'')| \leq C Y_0^{2\sigma} e^{-\frac{2}{5} Y_0}$ (from (4.17)), we obtain, for $Y_0$ large enough (depending on $C_3$),

$$\forall t \in [T'', T'], \quad \|\varepsilon(t)\|_{H^1} \leq C \sqrt{C_3 Y_0^\sigma} e^{-\frac{2}{5} Y_0}. \quad (4.49)$$

Next, we control the dynamics of $\mu$ and $y$. We need to argue differently on two regions in time: $[T'', T'']$ and $[T'', T']$, where

$$T'' \in (T'', T)$$

is such that $M Y_0 e^{-Y(T'')} = e^{-Y_0}$, for $M > 10$ large enough to be chosen later.

First, we prove that $T' > T''$, setting $t' = \min(T', T'')$ and strictly improving estimates (4.18) on $[T'', t']$. By (4.19), (4.18) and (4.49), we have

$$|\dot{\mu} - 2 \alpha e^{-y}| \leq C Y_0^\sigma e^{-Y_0} e^{-Y(t)} + CC_3 Y_0^{2\sigma} e^{-\frac{2}{5} Y_0}, \quad |\dot{y} - \mu| \leq C \sqrt{C_3 Y_0^\sigma} e^{-\frac{2}{5} Y_0}, \quad (4.50)$$

As in the proof of (4.17), we set $H(t) = (\dot{y})^2 + 4 \alpha e^{-y}$, so that by (4.50),

$$|\dot{H}| \leq CC_3^2 Y_0^{2\sigma} e^{-3 Y_0} + C \sqrt{C_3 Y_0^\sigma} e^{-\frac{2}{5} Y_0} e^{-Y(t)}. \quad (4.50)$$
By \([4.17]\), taken at \(t = T''\), we have
\[
|\mu(T'') - \hat{Y}(T'')| \leq CY_0^0 e^{-\frac{5}{4}Y_0}, \quad |y(T'') - Y(T'')| \leq CY_0^0 e^{-\frac{3}{4}Y_0}.
\]
Hence, by \((\hat{Y})^2 + 4\alpha e^{-Y} = 4\alpha e^{-Y_0}\), we obtain \(|H(T'') - 4\alpha e^{-Y_0}| \leq CY_0^0 e^{-\frac{3}{4}Y_0}\). Thus, integrating in \([T'', t]\), for all \(t \in [T'', t^*]\), we get
\[
|H(t) - 4\alpha e^{-Y_0}| \leq C \sqrt{C_3} Y_0^\sigma e^{-\frac{3}{4}Y_0}.
\]
Since \(T'' \leq t \leq t^* \leq T'''\), we have \(e^{-Y_0} = MY_0 e^{-Y(T''')} \leq M Y_0 e^{-Y(t)}\), and so
\[
|H(t) - 4\alpha e^{-Y_0}| \leq CM \sqrt{C_3} Y_0^\sigma e^{-\frac{3}{4}Y_0} e^{-Y(t)}.
\]
Using \(|\mu - \hat{y}| \leq CM \sqrt{C_3} Y_0^\sigma e^{-\frac{3}{4}Y_0} e^{-Y(t)}\), we obtain
\[
|y^2 + 4\alpha e^{-y} - 4\alpha e^{-Y_0}| \leq CM \sqrt{C_3} Y_0^\sigma e^{-\frac{3}{4}Y_0} e^{-Y(t)} \leq 2CM \sqrt{C_3} Y_0^\sigma e^{-\frac{3}{4}Y_0} e^{-y(t)}.
\]
Let \(b_0 = 2CM \sqrt{C_3} Y_0^\sigma e^{-\frac{3}{4}Y_0}\). Applying the same strategy as before, we set \(f\) as in \([4.40]\),
\[
f_+(s) = \frac{f(s)}{\sqrt{1 - b_0}}, \quad f_-(s) = \frac{f(s)}{\sqrt{1 + b_0}},
\]
and \(\cosh(g_+) = f_+\). Arguing as in the proof of \([4.17]\), using \([4.51]\) and \([4.17]\) at \(t = T''\), we obtain
\[
\dot{g}_+ \leq 1, \quad |g_+(s_1) - s_1| \leq Cb_0, \quad \text{and so} \quad g_+(s) \leq s + Cb_0,
\]
\[
\dot{g}_- \geq 1, \quad |g_-(s_1) - s_1| \leq Cb_0, \quad \text{and so} \quad g_-(s) \geq s - Cb_0.
\]
We deduce the following
\[
\forall t \in [T'', t^*], \quad |y(t) - Y(t)| \leq Cb_0 = CM \sqrt{C_3} Y_0^\sigma e^{-\frac{3}{4}Y_0}.
\]
Thus,
\[
|\mu - 2\alpha e^{-Y}| \leq Cb_0 e^{-Y(t)},
\]
and by integration, using \([4.17]\),
\[
\forall t \in [T'', t^*], \quad |\mu(t) - \hat{Y}(t)| \leq CM \sqrt{C_3} Y_0^\sigma e^{-\frac{3}{4}Y_0}.
\]
Therefore, from \([4.49]\), \([4.52]\) and \([4.53]\), we see that for \(C_3 > C_3^M = CM^2\), for \(C > 0\) large enough, we strictly improve estimate \([4.18]\) and thus prove \(t^* = T'''\), and by continuity \(T^* \in (T'''', T]\).

Second, we prove that \(T^* = T\), arguing on \([T'''', T]^*\). This is where we will need to fix \(M\) large enough. Using \([4.18]\), we have
\[
|e^{-y} - e^{-Y}| \leq CC_3 Y_0^\sigma e^{-\frac{3}{4}Y_0} e^{-Y(t)}.
\]
Thus, by \(\hat{Y} = 2\alpha e^{-Y}\) and \([4.50]\), we obtain
\[
|\hat{\mu} - \hat{Y}| \leq CC_3 Y_0^\sigma e^{-\frac{3}{4}Y_0} e^{-Y(t)} + CC_3^2 Y_0^2 \sigma e^{-\frac{3}{4}Y_0}.
\]
Integrating on $[T'', t]$, for all $t \in [T'', T^*)$, using $\dot{Y} \geq Ce^{-\frac{1}{2}Y_0}$, and $T'' \leq CY_0e^{\frac{1}{2}Y_0}$, and \((4.53)\) at $t^* = T''$, we obtain
\[
|\mu(t) - \dot{Y}(t)| \leq CC_3 Y_0^{\sigma+2} e^{-\frac{4}{5}Y_0} e^{-Y(T'')} + CM \sqrt{C_3} Y_0^{\sigma+1} e^{-\frac{2}{3}Y_0}
\leq C \left( \frac{C_3}{M} + M \sqrt{C_3} \right) Y_0^{\sigma+1} e^{-\frac{2}{3}Y_0}.
\]
(4.54)

Since $|\dot{y} - \mu| \leq C \sqrt{C_3} Y_0^\sigma e^{-\frac{2}{5}Y_0}$, we get
\[
|\dot{y} - \dot{Y}| \leq C \left( \frac{C_3}{M} + M \sqrt{C_3} \right) Y_0^{\sigma+1} e^{-\frac{2}{5}Y_0}.
\]

By integration, using $T \leq CY_0 e^{\frac{1}{2}Y_0}$, and \((4.52)\) we obtain
\[
|y(t) - Y(t)| \leq C \left( \frac{C_3}{M} + M \sqrt{C_3} \right) Y_0^{\sigma+2} e^{-\frac{3}{5}Y_0}.
\]
(4.55)

Fix $M > 1$ large enough so that \((4.54)\) and \((4.55)\) imply
\[
|\mu(t) - \dot{Y}(t)| \leq \left( \frac{1}{4} C_3 + MC \sqrt{C_3} \right) Y_0^{\sigma+1} e^{-\frac{3}{5}Y_0},
\]
\[
|y(t) - Y(t)| \leq \left( \frac{1}{4} C_3 + MC \sqrt{C_3} \right) Y_0^{\sigma+2} e^{-\frac{2}{3}Y_0}.
\]

Then, $M$ being fixed to such value, we can choose $C_3$ large enough so that for all $t \in [T'', T^*)$
\[
Y_0^2 e^{Y_0} \|e(t)\|_{H^1} + Y_0 e^{Y_0} |\mu(t) - \dot{Y}(t)| + |y(t) - Y(t)| \leq (C_3/2) Y_0^{\sigma+2} e^{-\frac{3}{5}Y_0}.
\]
(4.56)

This proves $T^* = T$ and Proposition 4.1 is proved.

For future reference, we observe that from \((4.6)\), \((4.7)\), \((2.18)\), \((4.21)\), \((4.22)\), for $t \in [-T, T]$,
\[
|\dot{\mu}_j - \mathcal{M}_j| \leq CY_0^\sigma e^{-2Y_0}, \quad |\mu_j - \dot{y}_j - N_j| \leq CY_0^\sigma e^{-\frac{2}{3}Y_0}.
\]
(4.57)

4.2 Conclusion of the proof of the stability of the 2-soliton structure

In this section, we finish the proof of the stability part of Theorem 1.

Proof of \((1.17)\)–\((1.18)\) and partial proof of \((1.19)\) and \((1.20)\). Let $T > 0$ be defined as in Proposition 4.1. We prove the existence of $\mu_j(t)$ and $y_j(t)$ and estimates \((1.17)\)–\((1.18)\) separately on $(-\infty, -T]$, $[-T, T]$ and $[T, +\infty)$. It is straightforward that the functions $\mu_j(t)$ and $y_j(t)$ can be adjusted to have $C^1$ regularity on $\mathbb{R}$.

For $t < -T$, estimate \((4.10)\) clearly implies \((1.17)\)–\((1.18)\).

On $[-T, T]$, \((1.17)\)–\((1.18)\) are direct consequences of \((4.5)\)–\((4.7)\) and \((2.15)\) (comparing in $H^1$ the approximate solution with the sum of two solitons).

Remark 2. By \((4.7)\) and the definition of $V$ (see \((2.3)\) and \((2.13)\)), for $t \in [-T, T]$,
\[
\|U(t) - \tilde{R}_1(t) - \tilde{R}_2(t) - e^{-y(t)}(A_1(\cdot, -y_1(t)) + A_2(\cdot, -y_2(t)))\|_{H^1} \leq CY_0^\sigma e^{-\frac{2}{3}Y_0},
\]
(4.58)

where for $t$ close to 0, the term $e^{-y}(A_1(x - y_1) + A_2(x - y_2))$ is indeed relevant as a correction term in the computation of $U(t)$, From the behavior at $\pm \infty$ of the functions $A_1, A_2$ (see Lemma 4.2), this term decays exponentially for $x > y_1(t)$ and $x < y_2(t)$ but it contains a tail for $y_2(t) < x < y_1(t)$. Note that a similar tail appears in the integrable case $p = 2$, see \[27\].
Now, we consider the region $t \geq T$. By (4.13) and (4.11), we have $T > \frac{10}{\sqrt{\alpha}} \rho^{-1} Y_0 e^{\frac{Y_0}{2}} > 10(\mu_0)^{-1} \ln \mu_0$. From (4.5), (4.6), (4.7) and (2.15) written at $t = T$, we have
\[
\|U(T) - Q_{1+\mu_1}(T). - y_1(T)) - Q_{1+\mu_2}(T). - y_2(T))\| \leq CY_0^2 e^{-\frac{Y_0}{2}} \leq C' Y_0^2 e^{-\frac{Y_0}{2}} \mu_0,
\]
where $|\mu_1(T) - \mu_0| + |\mu_2(T) + \mu_0| \leq CY_0^2 e^{-\frac{Y_0}{2}}$.

Therefore, we can apply Proposition 3.2 backwards (i.e. for $t \geq T$ – see Remark 1), with $\omega = C' Y_0^2 e^{-\frac{Y_0}{2}}$. There exist $y_1(t), y_2(t)$ and $\mu_1^+ = \lim_{+\infty} \mu_1, \mu_2^+ = \lim_{+\infty} \mu_2$, such that
\[
w(t) = U(t) - Q_{1+\mu_1}(T). - y_1(t)) - Q_{1+\mu_2}(T). - y_2(t))
\]
satisfies
\[
\sup_{t \in [T, +\infty]} \|w(t)\|_{H^1} \leq CY_0^2 e^{-\frac{Y_0}{2}}, \quad \lim_{t \to +\infty} \|w(t)\|_{H^1(x \geq -100/99 t)} = 0, \\
|\mu_1^+ - \mu_0| + |\mu_2^+ + \mu_0| \leq CY_0^2 e^{-\frac{Y_0}{2}}, \quad \lim_{+\infty} \overset{j}{\hat{y}} = \mu_j^+ \quad (j = 1, 2).
\]
(4.59)

Finally, using the conservation laws and the above asymptotics for $w(t)$, we claim the following refined estimates on the limiting scaling parameters:
\[
0 \leq \mu_1^+ - \mu_0 \leq CY_0^2 e^{-2Y_0}, \quad 0 \leq -\mu_2^+ - \mu_0 \leq CY_0^2 e^{-2Y_0},
\]
which is a consequence of (4.59) and the following lemma.

**Lemma 4.1** (Monotonicity of the speeds by conservation laws). There exists $C > 0$ such that
\[
\frac{1}{C} e^{Y_0} \lim_{t \to +\infty} \sup_{t \in [T, +\infty]} \|w(t)\|_{H^1}^2 \leq \frac{\mu_1^+}{\mu_0} - 1 \leq C e^{Y_0} \lim_{t \to +\infty} \inf_{t \in [T, +\infty]} \|w(t)\|_{H^1}^2 \leq CY_0^2 e^{-\frac{Y_0}{2}}, \\
\frac{1}{C} e^{Y_0} \lim_{t \to +\infty} \sup_{t \in [T, +\infty]} \|w(t)\|_{H^1}^2 \leq -\frac{\mu_2^+}{\mu_0} - 1 \leq C e^{Y_0} \lim_{t \to +\infty} \inf_{t \in [T, +\infty]} \|w(t)\|_{H^1}^2 \leq CY_0^2 e^{-\frac{Y_0}{2}}.
\]

*Proof.* We write the mass and energy conservation for $U(t)$ (see (1.9) and (1.10)) and then pass to the limit $t \to -\infty, t \to +\infty$, using (4.59). We deduce the existence of the limits $\lim_{+\infty} M(w)$ and $\lim_{+\infty} E(w)$ and the following identities
\[
M(Q_{1+\mu_0}) + M(Q_{1-\mu_0}) = M(Q_{1+\mu_1^+}) + M(Q_{1+\mu_2^+}) + \lim_{+\infty} M(w), \quad (4.61) \\
E(Q_{1+\mu_0}) + E(Q_{1-\mu_0}) = E(Q_{1+\mu_1^+}) + E(Q_{1+\mu_2^+}) + \lim_{+\infty} E(w). \quad (4.62)
\]

Let
\[
\nu_1 = \frac{E(Q_{1+\mu_0}) - E(Q_{1+\mu_1^+})}{M(Q_{1+\mu_1^+}) - M(Q_{1+\mu_0})}, \quad \nu_2 = \frac{E(Q_{1-\mu_0}) - E(Q_{1+\mu_2^+})}{M(Q_{1+\mu_2^+}) - M(Q_{1+\mu_0})},
\]
so that by (A.10) and (4.59),
\[
\left| \frac{\nu_1 - 1}{\mu_0} - 1 \right| \leq \frac{1}{4}, \quad \left| \frac{\nu_2 - 1}{-\mu_0} - 1 \right| \leq \frac{1}{4},
\]
We combine (4.61) and (4.62) to get
\[
\lim_{+\infty} E(w) = \nu_1 (M(Q_{1+\mu_1^2}) - M(Q_{1+\mu_0})) + \nu_2 (M(Q_{1+\mu_2^2}) - M(Q_{1-\mu_0})),
\]
\[
= (\nu_1 - \nu_2)(M(Q_{1+\mu_1^2}) - M(Q_{1+\mu_0})) - \nu_2 \lim_{+\infty} M(w),
\]
\[
= (\nu_1 - \nu_2)(M(Q_{1-\mu_0}) - M(Q_{1+\mu_2^2})) - \nu_1 \lim_{+\infty} M(w).
\]
Since \(\|w\|_{L^\infty} \leq C \|w\|_{H^1} \leq CY_0^2 e^{-\frac{\beta}{2}Y_0^2}\), we have
\[
\frac{1}{2} \lim_{+\infty} \sup \|w\|_{H^1}^2 < \lim E(w) + \nu_2 \lim_{+\infty} M(w) < 2 \lim \inf \|w\|_{H^1}^2,
\]
\[
\frac{1}{2} \lim_{+\infty} \sup \|w\|_{H^1}^2 < \lim E(w) + \nu_1 \lim_{+\infty} M(w) < 2 \lim \inf \|w\|_{H^1}^2.
\]
Using \(\frac{d}{dc}Q_c|_{c=1} > 0\), we finish the proof of Lemma 4.1. \(\square\)

### 4.3 Proof of Theorem 2

First, we claim the following sharp stability result to be proved in Appendix B.

**Proposition 4.2.** Let \(U\) be defined as in Theorem 1. For \(\mu_0 > 0\) small enough, if \(u(t)\) is a solution of (1.14) such that
\[
\|u(T_1) - U(T_1)\|_{H^1} = \omega \mu_0 \tag{4.63}
\]
for some \(T_1\), where \(0 < \omega < |\ln \mu_0|^{-2}\), then there exist \(t \in \mathbb{R} \mapsto (T(t), X(t)) \in \mathbb{R}^2\) such that
\[
\forall t \in \mathbb{R}, \quad \|u(t + T(t), . + X(t)) - U(t)\|_{H^1} + |X(t)| + e^{-\frac{1}{2}Y_0|T(t)|} \leq C \omega \mu_0. \tag{4.64}
\]

We continue the proof of Theorem 2. Let \(\tilde{\mu}_0 \in \mathbb{R}\) and \(\tilde{Y}_0 > 0\) be such that
\[
\mu_0 = \left(\tilde{\mu}_0^2 + 4\alpha e^{-\tilde{Y}_0}\right)^{1/2}
\]
is small enough, let \(u_0 \in H^1\) be as in (4.22) and let \(u(t)\) be the corresponding solution of (1.14). We assume that \(\tilde{\mu}_0 \leq 0\), the proof being the same in the case \(\tilde{\mu}_0 > 0\) by using the transformation \(x \to -x, \ t \to -t\) and translation in space invariance.

For this value of \(\mu_0 > 0\), let \(U(t, x)\) and \(Y(t)\) be defined as in Theorem 1 and Sections 4.1 and 4.2. Recall that for all \(t\), \(\dot{Y}^2(t) + 4\alpha e^{-Y(t)} = 4\mu_0^2\). Since \(\tilde{Y}_0 > Y_0\), there exists \(\tilde{T}_0 < 0\) such that \(Y(\tilde{T}_0) = \tilde{Y}_0\), so that \(\dot{Y}(\tilde{T}_0) = 2\tilde{\mu}_0\). We claim that for some \(X_1 \in \mathbb{R}\),
\[
\|U(\tilde{T}_0, \cdot + X_1) - Q_{1-\tilde{\mu}_0}(\cdot - \frac{1}{2}Y_0) - Q_{1+\tilde{\mu}_0}(\cdot + \frac{1}{2}Y_0)\|_{H^1} \leq C|\ln \mu_0|^{\sigma + 2} \mu_0^{3/2}. \tag{4.65}
\]
Indeed, if \(\tilde{T}_0 < -T\) then we simply use (4.10). Otherwise, by Proposition 4.1 applied at \(t = \tilde{T}_0\):
\[
\|y(\tilde{T}_0) - Y(\tilde{T}_0)\| \leq C|\ln \mu_0|^{\sigma + 2} \mu_0^{3/2}, \quad \||\varepsilon(\tilde{T}_0)\|_{H^1} \leq C|\ln \mu_0|^{\sigma + 5/2},
\]
\[
|\mu_1(\tilde{T}_0) + \frac{1}{2}\dot{Y}(\tilde{T}_0)| \leq C|\ln \mu_0|^2 \mu_0^2, \quad |\mu_2(\tilde{T}_0) - \frac{1}{2}\dot{Y}(\tilde{T}_0)| \leq C|\ln \mu_0|^2 \mu_0^2,
\]
and (4.65) then follows from (2.15).

Combining (4.65) and (4.22), we get
\[
\|u_0 - U(\tilde{T}_0, \cdot + X_1)\|_{H^1} \leq C\omega \mu_0 + C|\ln \mu_0|^{\sigma + 2} \mu_0^{3/2},
\]
and by Proposition 4.2 this implies (1.23) for some \(\sigma\).
5 Nonexistence of a pure 2-soliton and interaction defect

In this section, we complete the proof of Theorem 1 by proving the lower bounds in (1.19) and (1.20).

5.1 Refined control of the translation parameters

From the analysis of Section 4, the error term in the dynamical system for \( \mu_j(t), y_j(t) \) is not sharp enough to justify rigorously the defect in the interaction. Now we introduce specific functionals \( J_j(t) \) related to the translation parameters \( y_j(t) \) to obtain a sharper version of the dynamical system. Recall that \( \tilde{R}_j \) and \( \Lambda \tilde{R}_j \) are defined at the beginning of Section 2.1.

Lemma 5.1. Under the assumptions of Proposition 4.1 for \( j = 1, 2 \), let

\[
J_j(t) = \frac{1}{\int Q \Lambda Q} \int \varepsilon(t,x) J_j(t,x) dx \quad \text{where} \quad J_j(t,x) = \int_{-\infty}^{x} \Lambda \tilde{R}_j(t,y) dy.
\]

Then \( J_j(t) \) is well-defined and the following hold

(i) Estimates on \( J_j \).

\[
\forall t \in [-T,T], \quad |J_1(t)| + |J_2(t)| \leq CY_0^{\sigma + 1} e^{-\frac{5}{4} Y_0}.
\]

(ii) Equation of \( J_j \). For \( j = 1, 2 \),

\[
\forall t \in [-T,T], \quad \left| \frac{d}{dt} J_j(t) - (\mu_j - \dot{y}_j - N_j) \right| \leq CY_0^{\sigma + 2} e^{-\frac{7}{4} Y_0}.
\]

Remark 3. The constant \( \int Q \Lambda Q \) is not zero (see (A.13)).

Note also from (A.13) that the \( \int \Lambda Q \neq 0 \), and so the function \( J_j(x) \) is bounded but has a nonzero limit as \( x \to +\infty \). Therefore, \( J_j(t) \) is not well-defined for a general \( \varepsilon \in H^1 \). Part of the proof of Lemma 5.1 consists on obtaining decay in space for \( \varepsilon(t) \), in order to give sense to \( J_1 \) and \( J_2 \).

Remark 4. Estimate (5.3) says formally that \( \mu_j - \dot{y}_j - N_j \) is of order \( O(T/4) \), which is a decisive improvement with respect to (4.57) (gain of a factor \( e^{-\frac{1}{2} Y_0} \)).

Proof. Preliminary estimates. We work under the assumptions of Proposition 4.1 and on the interval \([-T,T]\). First, we claim without proof exponential uniform decay properties of \( U(t) \) on the right \((x > y_1(t))\).

Claim 5.1 (Decay estimate on \( u(t) \)). There exist \( C > 1 \) and \( \rho_0 < 1 \) such that for all \( t \in [-T,T], \) for all \( X_0 > 1 \),

\[
\int_{x > X_0 + y_1(t)} (U^2 + (\partial_x U)^2)(t,x) dx \leq C e^{-\rho_0 x_0}.
\]

Recall that the proof of Claim 5.1 is a consequence of

\[
\lim_{t \to -\infty} \|U(t)\|_{H^1(x > \frac{1}{2}|t|)} = 0
\]

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combined with known monotonicity arguments, see e.g. [16].

Estimate of $J$. Note that $J$ does not belong to $L^2$ (see Remark 3) but satisfies

$$\sup_{x \in \mathbb{R}} \left\{ \left(1 + e^{-\frac{1}{2}(x-y_j(t))}\right) |J_j(t,x)| \right\} \leq C.$$  \hfill (5.5)

It follows from (5.4), (5.5), (4.7) and the decomposition of $U(t)$ in Lemma 3.1 that

$$|J_1(t)| \leq C \int_{x < y_1(t)} |\varepsilon(t,x)||J_j(t,x)| dx + C \int_{y_1(t) \leq x < y_1(t)+10\rho_0^{-1}Y_0} |\varepsilon(t,x)| dx$$

$$+ C \int_{x > y_1(t)+10\rho_0^{-1}Y_0} |\varepsilon(t,x)||J_j(t,x)| dx$$

$$\leq C(1 + Y_0) \|\varepsilon(t)\|_{L^\infty} + C \int_{x > y_1(t)+10\rho_0 Y_0} |U(t,x)| + CE^{-5Y_0}$$

$$\leq CY_0^{\sigma+1}e^{-\frac{\sigma}{2}Y_0}.$$  

Moreover, using $y_1(t) - y_2(t) = y(t) \leq Y(T) \leq CY_0$, one gets by similar arguments $|J_2(t)| \leq CY_0^{\sigma+1}e^{-\frac{\sigma}{2}Y_0}$.

Equation of $J_1$. To prove (5.3), we make use of the equation of $\varepsilon$ (see (3.4)), and of the special algebraic structure of the approximate solution $V(t,x)$ introduced in Propositions 2.1 and 2.2. We have

$$\left( \int Q\Lambda Q \right) \frac{d}{dt} J_1(t) = \int (\partial_t \varepsilon) J_1 + \int \varepsilon \partial_t J_1.$$  

First observe that

$$\partial_t J_1(x) = \int_{-\infty}^x \partial_t (\Lambda \tilde{R}_1)(y) dy = \int_{-\infty}^x \left\{ \mu_1 \frac{\partial \tilde{R}_1}{\partial \mu_1} + \tilde{y}_1 \frac{\partial \tilde{R}_2}{\partial \tilde{y}_1} \right\}(y) dy.$$  

Thus, by $|M_j| + |N_j| \leq Ce^{-\frac{\sigma}{2}Y_0}$, $|\mu_j(t)| \leq Ce^{-\frac{\sigma}{2}Y_0}$, (4.57) and (5.4), arguing as in the proof of (5.2),

$$\left| \int \varepsilon \partial_t J_1 \right| \leq CY_0^{\sigma+1}e^{-\frac{\sigma}{2}Y_0}. $$  \hfill (5.6)

Next, using (3.4) and $\partial_x J_1 = \Lambda \tilde{R}_1$, we have

$$\int (\partial_t \varepsilon) J_1 = \int (\partial_x^2 \varepsilon - \varepsilon + (V + \varepsilon)^4 - V^4) \Lambda \tilde{R}_1 - \int E J_1 + \int \tilde{E}(V) J_1.$$  

For the first term, i.e. $\int (\partial_x^2 \varepsilon - \varepsilon + (V + \varepsilon)^4 - V^4) \Lambda \tilde{R}_1$, we argue as the proof of Lemma 3.1. Using $L\Lambda Q = -Q$ and $\int \varepsilon \tilde{R}_1 = 0$, (1.7) and the definition of $V$ (see Proposition 2.2), we obtain

$$\left| \int (\partial_x^2 \varepsilon - \varepsilon + (V + \varepsilon)^4 - V^4) \Lambda \tilde{R}_1 \right| \leq CY_0^2 e^{-\frac{\sigma}{2}Y_0} ||\varepsilon||_{L^2} + C ||\varepsilon||_{L^2}^2 \leq CY_0^{\sigma+2}e^{-\frac{\sigma}{2}Y_0}.$$  

By (4.20) and (5.5), we have

$$\left| \int E J_1 \right| \leq CY_0^{\sigma}e^{-2Y_0}.$$

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Next, we consider the term \( \int \tilde{E}(V) J_1 \). From the definition of \( \tilde{E}(V) \) in (2.17), the structure of \( V_0 \) and \( V \), see (2.3) and (2.13) (see also (2.2)), and (4.57), we have

\[
\sup_{x \in \mathbb{R}} \left\{ (1 + e^{\frac{1}{2} (x - y_1(t))}) \left| \tilde{E}(V) - \sum_{j=1,2} (\mu_j - \tilde{y}_j - N_j) \partial_x \tilde{R}_j \right| \right\} \leq CY_0^\sigma e^{-2Y_0}. \tag{5.7}
\]

Thus, by (5.5), we obtain

\[
\left| \int \tilde{E}(V) J_1 - (\mu_1 - \tilde{y}_1 - N_1) \int (\partial_x \tilde{R}_1) J_1 \right| \leq CY_0^\sigma e^{-2Y_0}. \tag{5.8}
\]

Finally, using \( \int (\partial_x \tilde{R}_1) J_1 + \int QS \leq C|\mu_1(t)| \leq Ce^{-\frac{1}{2}Y_0} \) and (4.57), we obtain (5.3).

The proof for \( \frac{d}{dt} J_2 \) is exactly the same.

\( \square \)

5.2 Preliminary symmetry arguments

First, we claim some additional information on the parameters of the solution \( U(t) \), under the assumptions of Proposition 4.1

**Claim 5.2.** For all \( t \in [-T, T] \),

\[
|\mu_1(t) - \mu_2(-t)| \leq CY_0^{\sigma+3} e^{-\frac{3}{2}Y_0}. \tag{5.9}
\]

**Proof of (5.9).** From (4.16) and \( \tilde{Y}(0) = 0 \), we have \( |\mu_1(0) - \mu_2(0)| \leq CY_0^{\sigma+1} e^{-\frac{1}{2}Y_0} \). From (4.5), \( |y(t) - y(-t)| \leq |y(t) - Y(t)| + |Y(t) - y(-t)| \leq CY_0^{\sigma+2} e^{-\frac{2}{5}Y_0} \). Thus, by (4.57) and the expression of \( M_j \) in (2.10), we obtain

\[
\hat{\mu}_1(t) - \left\{ \alpha e^{-y(t)} + \beta \mu_1(t) y(t) e^{-y(t)} + \delta \mu_1(t) e^{-y(t)} \right\} \leq CY_0^\sigma e^{-2Y_0},
\]

\[
-\hat{\mu}_2(-t) - \left\{ \alpha e^{-y(-t)} + \beta \mu_2(-t) y(-t) e^{-y(-t)} + \delta \mu_2(-t) e^{-y(-t)} \right\} \leq CY_0^\sigma e^{-2Y_0},
\]

and so

\[
-\hat{\mu}_2(-t) - \left\{ \alpha e^{-y(-t)} + \beta \mu_2(-t) y(t) e^{-y(t)} + \delta \mu_2(-t) e^{-y(t)} \right\} \leq CY_0^{\sigma+2} e^{-\frac{3}{2}Y_0}.
\]

By \( T = CY_0^{\frac{3}{2}Y_0} \), it follows that for all \( t \in [-T, T] \), \( |\mu_1(t) - \mu_2(-t)| \leq CY_0^{\sigma+3} e^{-\frac{5}{2}Y_0}. \)

The next lemma claims that if the asymptotic 2-soliton solution \( U(t) \) considered in Proposition 4.1 has an approximate symmetry property (i.e. \( U(t, x) - U(-t + t_0, -x + x_0) \) is small for some \( t_0, x_0 \)) then the corresponding parameters (\( \Gamma(t) \) in Proposition 4.1) also have some symmetry properties, despite the fact that the decomposition itself is not symmetric (see the definition of \( V(t, x) \) in Propositions 2.1 and 2.2). This result relies in particular on parity properties of \( A_j \), \( B_j \) and \( D_j \) and on the choice of orthogonality conditions for \( B_j \), \( D_j \) in Section 2.

**Lemma 5.2.** Let \( t_0, x_0 \) be such that \( |t_0| \leq 1, |x_0| \leq 1 \). Under the assumptions of Proposition 4.1 for all \( t \in [-T, T] \),

\[
|\mu_1(t) - \mu_2(-t + t_0)| + |y_1(t) + y_2(-t + t_0) - x_0| 
\leq C\|U(t, x) - U(-t + t_0, -x + x_0)\|_{H^1} + CY_0^5 e^{-2Y_0}. \tag{5.10}
\]

In particular, assume that \( U(t, x) = U(-t + t_0, -x + x_0) \) for some \( t_0, x_0 \), then

\[
|\mu_1(t) - \mu_2(-t + t_0)| + |y_1(t) + y_2(-t + t_0) - x_0| \leq CY_0^5 e^{-2Y_0}. \tag{5.11}
\]
Remark 5. Assuming \( \| U(t, x) - U(-t + t_0, -x + x_0) \|_{H^1} \leq C Y_0^{\sigma + 3} e^{-\frac{3}{2} Y_0} \), it follows from (5.10) that the following hold

\[
|x_0| \leq C Y_0^2 e^{-\frac{3}{2} Y_0}, \quad |t_0| \leq C Y_0^{\sigma + 3} e^{-\frac{3}{2} Y_0}. \tag{5.12}
\]

Indeed, on the one hand, estimate \( |x_0| \leq C Y_0^2 e^{-\frac{3}{2} Y_0} \) follows from (4.5) taken at time \( t = t_0/2 \) and (5.10) taken at \( t = t_0/2 \).

On the other hand, from (5.9) and (5.10), we have \( |\mu_1(t) - \mu_1(t - t_0)| \leq C Y_0^{\sigma + 3} e^{-\frac{3}{2} Y_0} \). Since \( \mu_1(t) \geq C e^{-Y_0} \) for \( |t| \) close to 0, we obtain \( |t_0| \leq C Y_0^{\sigma + 3} e^{-\frac{3}{2} Y_0} \).

Proof. Set

\[
[U](t, x) = U(t, x) - U(-t + t_0, -x + x_0),
\]

\[
[\tilde{R}_j](t, x) = \tilde{R}_j(t, x) \psi \left( e^{-\frac{3}{2} Y_0} x + 1 \right) - \tilde{R}_j(-t + t_0, -x + x_0) \psi \left( e^{-\frac{3}{2} Y_0} (-x + x_0) + 1 \right),
\]

\[
[w_A](t, x) = w_A(t, x) \psi \left( e^{-\frac{3}{2} Y_0} x + 1 \right) - w_A(-t + t_0, -x + x_0) \psi \left( e^{-\frac{3}{2} Y_0} (-x + x_0) + 1 \right),
\]

\[
[t](t, x) = \varepsilon(t, x) - \varepsilon(-t + t_0, -x + x_0),
\]

\[
[y_j](t)(j \neq k),
\]

\[
|\mu_1(t)| = \mu_k(-t + t_0), \quad |y_j(t)| = y_j(t) + y_k(-t + t_0) - x_0 \quad (j \neq k),
\]

and similar definitions for \([R_j], [w[A], [w_B], [w_D], \) where \( w_A, w_Q, w_B \) and \( w_D \) are defined in the proof of Proposition 2.1 so that by the expression of \( V \) in Proposition 2.2 and the decomposition \( U(t, x) = V(x; \Gamma(t)) + \varepsilon(t, x) \), the following holds

\[
[U] = [\tilde{R}_1] + [\tilde{R}_2] + [w_A] + [w_Q] + [w_B] + [w_D] + [\varepsilon].
\]

We now compute and estimate the various terms above. First, by the following elementary claim, for \( \mu \) and \( y \) small (see e.g. proof of Claim 2.1)

\[
\{|Q_1 + \mu (x - y) - Q(x)| - \{\mu \Lambda Q(x) - y Q'(x)\}| \leq C (|\mu|^2 + |y|^2) e^{-\frac{3}{2} |x|},
\]

the parity of \( Q \) and the properties of the cut-off function \( \psi \) (see (2.12)) we observe that (for \( Y_0 \) large)

\[
|\tilde{R}_j| - |\mu_j| \Lambda \tilde{R}_j - [y_j] \partial_x \tilde{R}_j| \leq C e^{-\frac{3}{2} |x-y|} (|\mu_j|^2 + |y_j|^2 + e^{-100 Y_0}). \tag{5.13}
\]

Note also

\[
|\tilde{R}_j| - |y_j| \partial_x \tilde{R}_j| \leq C |y_j|^2 e^{-|x-y|}, \quad ||R_j| R_k||_{H^1} \leq C |y_j| Y_0 e^{-Y_0} \quad (j \neq k). \tag{5.14}
\]

Then, we note that

\[
y(t) - y(-t + t_0) = [y_1] - [y_2] \quad \text{and so} \quad |y(\varepsilon(t)) - e^{-y(-t + t_0)}| \leq C e^{-y(t)} (||y_1|| + ||y_2||). \tag{5.15}
\]

From Section 2, we know that \( A_1(x) = A_2(-x) \) and so \( A_j(-x + x_0 - y_j(-t + t_0)) = A_k(x - x_0 + y_j(-t + t_0)) \) \((k \neq j)\). In particular, using also (5.15), we deduce

\[
||w_A||_{H^1} \leq C \sqrt{y} e^{-y(t)} (||y_1|| + ||y_2||) + e^{-100 Y_0}. \tag{5.16}
\]
Next,\[
[w_Q] = \theta_A \{(\mu(t)+\mu(-t-t_0))xR_1R_2-\mu(-t-t_0)\{xR_1R_2+(x-x_0)(R_1-R_1)(R_2-R_2)\}\} + O(e^{-100\gamma_0})
= \theta_A \{(\mu(t)+\mu(-t-t_0))xR_1R_2
\quad -\mu(-t-t_0)(xR_1R_2+(x-x_0)(R_1-R_2)[R_2]+[R_1][R_2])\} + O(e^{-100\gamma_0}).
\]
Using $|\mu(t)+\mu(-t-t_0)| \leq ||\mu_1|| + ||\mu_2||$ and
\[
-x_0 = \frac{1}{2}[y_1] + \frac{1}{2}[y_2] - \frac{1}{2}(y_1+y_2)(-t+t_0) - \frac{1}{2}(y_1(t)+y_2(t)),
\]
we obtain by (1.5)
\[
|x_0| \leq C(||y_1|| + ||y_2|| + Y_0^4 e^{-\frac{1}{2}Y_0}).
\]
Thus, using \(\|R_1R_2\|_{H^1} \leq C\sqrt{\gamma} e^{-y} \) (see (2.21)), \(|\mu(t)| \leq C e^{-\frac{1}{2}Y_0} \) (see (1.4)) and (5.14), we obtain
\[
\|\left\{\begin{array}{c}
[w_Q]\n\end{array}\right\}\|_{H^1} \leq C e^{-\frac{1}{4}Y_0} (||\mu_1|| + ||\mu_2|| + ||y_1|| + ||y_2||) + CY_0^5 e^{-2Y_0}.
\]
(5.17)

Summarizing, so far, we have obtained
\[
\left\{\begin{array}{c}
\|\tilde{R}_1 + [R_2] + [w_A] + [w_Q] - \left\{\begin{array}{c}
[\mu_1]\Lambda \tilde{R}_1 + [\mu_2]\Lambda \tilde{R}_1 - [y_1]\partial_x \tilde{R}_1 - [y_2]\partial_x \tilde{R}_2\end{array}\right\}\right\}_{H^1}
\leq C e^{-\frac{1}{4}Y_0} (||\mu_1|| + ||\mu_2|| + ||y_1|| + ||y_2||) + CY_0^5 e^{-2Y_0}.
\]
(5.18)

Now, using orthogonality properties of $B_j$, $D_j$ and $\varepsilon$, we claim
\[
\left|\int [w_B]\tilde{R}_j \right| + \left|\int [w_B]\partial_x \tilde{R}_j \right| + \left|\int [w_D]\tilde{R}_j \right| + \left|\int [w_D]\partial_x \tilde{R}_j \right| + \left|\int \varepsilon \tilde{R}_j \right| + \left|\int \varepsilon \partial_x \tilde{R}_j \right| \leq C e^{-Y_0} (||y_1|| + ||y_2|| + ||\mu_1|| + ||\mu_2||) + CY_0 e^{-2Y_0}.
\]
(5.19)

Let us assume (5.19) and finish the proof of (5.10). Since $\int [U]\tilde{R}_j + \int [U]\partial_x \tilde{R}_j \leq C \|U\|$, and $\int \tilde{R}_j \Lambda \tilde{R}_j \geq c_0$, $\int QQ' = \int Q'\Lambda Q = 0$, by combining (5.18) and (5.19), we obtain
\[
||y_1|| + ||y_2|| + ||\mu_1|| + ||\mu_2|| \leq C \|U\|_{H^1} + C e^{-\frac{1}{4}Y_0} (||y_1|| + ||y_2|| + ||\mu_1|| + ||\mu_2||) + CY_0^5 e^{-2Y_0},
\]
and (5.10) follows for $Y_0$ large enough.

Proof of (5.19). Recall that the orthogonality conditions chosen on $B_1$, $B_2$ implies approximate orthogonality conditions on $w_B$, see (2.28). We deduce
\[
\left|\int [w_B]\tilde{R}_j \right| + \left|\int [w_B]\partial_x \tilde{R}_j \right| 
\leq \|w_B\|_{L^\infty} (\|\tilde{R}_1 - R_1\|_{H^1} + \|\tilde{R}_2 - R_2\|_{H^1}) + \|w_B\|_{L^\infty} (||R_1||_{H^1} + ||R_2||_{H^1}) + CY_0^5 e^{-\frac{1}{2}Y_0}
\leq C \|w_B\|_{L^\infty} (e^{-\frac{1}{2}Y_0} + ||y_j||) + C e^{-2Y_0} \leq CY_0 e^{-\frac{1}{2}Y_0} (e^{-\frac{1}{2}Y_0} + ||y_j||) + C e^{-2Y_0},
\]
and similary for $w_D$.

Finally, by the orthogonality conditions $\int \varepsilon \tilde{R}_j = \int \varepsilon \partial_x \tilde{R}_j = 0$ and (5.13), we obtain
\[
\left|\int \varepsilon \tilde{R}_j \right| + \left|\int \varepsilon \partial_x \tilde{R}_j \right| \leq C \|\varepsilon\|_{H^1} (||y_j|| + ||\mu_j||) \leq CY_0^5 e^{-\frac{1}{2}Y_0} (||y_j|| + ||\mu_j||),
\]
which finishes the proof of (5.19).
5.3 Nonexistence of a pure 2-soliton solution

Proposition 5.1. Under the assumptions of Theorem 1, the solution $U(t)$ of (1.14) satisfying (1.4) is not an asymptotic 2-soliton solution at $+\infty$. Equivalently,

$$
\lim_{t \to +\infty} \|w(t)\|_{H^1(\mathbb{R})} \neq 0.
$$

(5.20)

In Section 5.4 we shall prove a statement stronger than Proposition 5.1. However, we give a direct proof for Proposition 5.1 at this point because of its own interest and simplicity. Indeed, the nonexistence of a global 2-soliton solution can be seen as a rigidity property, and its proof can now be obtained by a simple symmetry argument. The proof of the lower bound stated in Section 5.4 is based on a similar symmetry argument but also requires the sharper stability result proved in Proposition 4.2.

Proof of Proposition 5.1. The proof is by contradiction. We assume that $U(t)$ is a pure 2-soliton solution at $+\infty$.

Step 1. By Lemma 4.1 and the uniqueness of asymptotic pure 2-soliton solutions (see [20], Theorem 1), $\mu_1^+ = \mu_0$, $\mu_2^+ = -\mu_0$ and there exist $t_0$, $x_0 \in \mathbb{R}$ such that, for all $t$, $x \in \mathbb{R}$,

$$
U(t, x) = U(-t + t_0, -x + x_0).
$$

(5.21)

Moreover, by Proposition 4.1, $x_0$ and $t_0$ are small.

We denote

$$
\nu(t) = \mu_1(t) - \mu_2(-t + t_0), \quad z(t) = y_1(t) + y_2(-t + t_0) - x_0.
$$

(5.22)

By (5.21) and Lemma 5.2, we have

$$
\forall t \in [-T,T], \quad |\nu(t)| + |z(t)| \leq CY_0^5 e^{-2Y_0}.
$$

(5.23)

In Step 2, we see that (5.23) combined with Lemma 5.1 provides a contradiction.

Step 2. We claim

$$
|J_1(t)| + |J_2(t)| \leq CY_0^{\sigma+1} e^{-\frac{5}{4}Y_0},
$$

(5.24)

$$
\left| \frac{d}{dt} \left( z(t) - ((b_- y(t) + (b_- + d_-)) e^{-y(t)}) + (J_1(t) - \hat{J}_2(-t + t_0)) \right) \right| \leq CY_0^{\sigma+2} e^{-\frac{5}{4}Y_0}.
$$

(5.25)

We postpone the proof of (5.24)–(5.25) and we obtain a contradiction. By integration of (5.25), (5.24) and (1.12), we obtain, for any $t_1, t_2 \in [-T,T],$

$$
\left| (z(t_1) - ((b_- y(t_1) + (b_- + d_-)) e^{-y(t_1)}) - (z(t_2) - ((b_- y(t_2) + (b_- + d_-)) e^{-y(t_2)}) \right| \leq CY_0^{\sigma+3} e^{-\frac{5}{4}Y_0}.
$$

(5.26)

Thus, by (5.26), for any $t_1, t_2 \in [-T,T]$, for $k = 1 + \frac{d_-}{b_-}$, $(b_- \neq 0),$

$$
|((y(t_1) + k) e^{-y(t_1)} - (y(t_2) + k) e^{-y(t_2)})| \leq CY_0^{\sigma+3} e^{-\frac{5}{4}Y_0}.
$$

(5.27)
But taking $0 < t_1 < t_2 < T$ such that $y(t_1) = Y_0 + 1$, $y(t_2) = Y_0 + 2$ and then $Y_0$ large enough, \eqref{5.27} implies
\begin{equation}
|Y_0e^{-(Y_0+1)} - Y_0e^{-(Y_0+2)}| \leq Ce^{-Y_0},
\end{equation}
which is a contradiction for $Y_0$ large enough.

Now, we prove \eqref{5.24}–\eqref{5.25}. Estimate \eqref{5.24} is exactly \eqref{5.2}. Next, by \eqref{5.3} and the expression of $N_j$ in \eqref{2.10}, we have
\begin{equation}
\begin{aligned}
\dot{y}_1 &= \mu_1 - ae^{-y} - b_1\mu_1e^{-y} - d_1\mu_1e^{-y} - \hat{J}_1 + O(Y_0^{\sigma+2}e^{-\frac{2}{4}Y_0}), \\
\dot{y}_2 &= \mu_2 - ae^{-y} - b_2\mu_2e^{-y} - d_2\mu_2e^{-y} - \hat{J}_2 + O(Y_0^{\sigma+2}e^{-\frac{2}{4}Y_0}).
\end{aligned}
\end{equation}

Moreover,
\begin{equation}
y(t) - y(-t + t_0) = (y_1(t) - y_2(t)) - (y_1(-t + t_0) - y_2(-t + t_0)) \\
= (y_1(t) + y_2(-t + t_0) - x_0) - (y_2(t) + y_1(-t + t_0) - x_0),
\end{equation}
and so by \eqref{5.23}, $|y(t) - y(-t + t_0)| \leq CY_0^5e^{-2Y_0}$, so that
\begin{equation}
|e^{-y(t)} - e^{-y(-t+t_0)}| \leq CY_0^5e^{-3Y_0}.
\end{equation}

By \eqref{5.28} and \eqref{5.29}, we have
\begin{equation}
\begin{aligned}
\dot{z}(t) &= \dot{y}_1(t) - \dot{y}_2(-t + t_0) \\
&= \nu(t) - (b_1 - b_2)\mu_1e^{-y} - (d_1 - d_2)\mu_1e^{-y} - (\hat{J}_1(t) - \hat{J}_2(-t + t_0)) + O(Y_0^{\sigma+2}e^{-\frac{2}{4}Y_0}),
\end{aligned}
\end{equation}

Let (see \eqref{4.24})
\begin{equation}
b_- = -\frac{1}{2}(b_1 - b_2) \neq 0, \quad d_- = -\frac{1}{2}(d_1 - d_2).
\end{equation}

Since $|\mu_1 + \mu_2| \leq CY_0e^{-Y_0}$ (see \eqref{4.10}), we have $|\mu_1 - \frac{1}{2}\mu| \leq CY_0e^{-Y_0}$ and thus, by $|\mu - \hat{y}| \leq C e^{-Y_0}$ (see \eqref{4.11}), we obtain $|\mu_1e^{-y} - \frac{1}{2}\hat{y}e^{-y}| \leq CY_0e^{-2Y_0}$.

We obtain
\begin{equation}
\dot{z} = \nu + b_-\hat{y}e^{-y} + d_-\hat{y}e^{-y} - (\hat{J}_1(t) - \hat{J}_2(-t + t_-)) + O(Y_0^{\sigma+2}e^{-\frac{2}{4}Y_0}),
\end{equation}
where $|\nu(t)| \leq CY_0^5e^{-2Y_0}$ from \eqref{5.23}. Thus, by elementary computations, we now obtain \eqref{5.25}. \tag*{\Box}

5.4 Lower bound on the defect

**Proposition 5.2.** Under the assumptions of Theorem 7, there exists $c > 0$ such that,
\begin{equation}
\liminf_{t \to +\infty} \|w(t)\|_{H^1} \geq cY_0e^{-\frac{3}{2}Y_0},
\end{equation}
\begin{equation}
\mu_1^+ - \mu_0 \geq cY_0^2e^{-\frac{5}{2}Y_0}, \quad -\mu_2^+ - \mu_0 \geq cY_0^2e^{-\frac{5}{2}Y_0}.
\end{equation}
Proof. It suffices to prove (5.30), since estimate (5.31) then follows from Lemma 4.1. Let \( \epsilon > 0 \) arbitrary, and suppose for the sake of contradiction that

\[
\liminf_{t \to +\infty} \|w(t)\|_{H^1} \leq \epsilon Y_0 e^{-\frac{3}{2}Y_0}. \tag{5.32}
\]

Step 1. First, we claim that there exist \( \tilde{T}(t) \), \( \tilde{X}(t) \) such that, for all \( t \in \mathbb{R} \),

\[
\|U(t, x) - U(-t + \tilde{T}(t), -x + \tilde{X}(t))\|_{H^1} + |\tilde{X}(t)| + e^{-\frac{3}{2}Y_0}|\tilde{T}(t)| \leq C_0 Y_0 e^{-\frac{3}{2}Y_0} \tag{5.33}
\]

Proof of (5.33). From (5.32), there exists \( T_1 \) arbitrarily large with \( \|w(T_1)\|_{H^1} \leq 2\epsilon Y_0 e^{-\frac{3}{2}Y_0} \). By Lemma 4.1 it also follows from (5.32) that

\[
0 \leq \mu_1^+ - \mu_0 \leq C e^2 Y_0^2 e^{-\frac{3}{2}Y_0}, \quad 0 \leq -(\mu_2^+ + \mu_0) \leq C e^2 Y_0^2 e^{-\frac{3}{2}Y_0}.
\]

In particular, for all \( t \)

\[
\left\| \sum_{j=1,2} Q_{1+\mu_j^+}(\cdot, -y_j(t)) - (Q_{1+\mu_0}(\cdot, -y_1(t)) + Q_{1-\mu_0}(\cdot, -y_2(t))) \right\|_{H^1} \leq C e^2 Y_0^2 e^{-\frac{3}{2}Y_0}.
\]

From the behavior of \( U(t) \) as \( t \to -\infty \), and the information above, there exists \( T_2 > T \) and \( X \in \mathbb{R} \) such that

\[
\|U(T_1, x) - U(-T_2, -x + X)\|_{H^1} \leq 2\epsilon Y_0 e^{-\frac{3}{2}Y_0} + C e^2 Y_0^2 e^{-\frac{3}{2}Y_0} \leq 3\epsilon Y_0 e^{-\frac{3}{2}Y_0}, \tag{5.34}
\]

taking \( Y_0 \) large enough. From Proposition 4.2 (sharp global stability of the 2-soliton structure), it follows that there exist \( \tilde{T}(t) \) and \( \tilde{X}(t) \) such that (5.33) follows.

Step 2. Conclusion of the proof of Proposition 5.2. As in the proof of Proposition 5.1 take \( 0 < t_1 < t_2 \) such that \( Y(t_1) = Y_0 + 1 \) and \( Y(t_2) = Y_0 + 2 \). Note that \( t_2 - t_1 < C e^2 Y_0 \) by \( \dot{Y}(t) > c_0 e^{-\frac{3}{2}Y_0} \) for \( c_0 > 0 \) on \([t_1, t_2]\).

Note that for \( t \in [-T, T] \), \( \tilde{T}(t) \) and \( \tilde{X}(t) \) are small. Applying Lemma 5.2 for all \( t \in [t_1, t_2] \), we obtain (for \( Y_0 \) large enough depending on \( \epsilon \))

\[
|\mu_1(t) - \mu_2(-t + \tilde{T}(t))| + |y_1(t) + y_2(-t + \tilde{T}(t)) - \tilde{X}(t)| \leq C_0 e^{-\frac{3}{2}Y_0}.
\]

By (5.33), for all \( t \in [t_1, t_2] \), we have \( |\tilde{T}(t) - \tilde{T}(t_1)| \leq C_0 Y_0 e^{-\frac{3}{2}Y_0} \), \( |\tilde{X}(t) - \tilde{X}(t_1)| \leq C_0 Y_0 e^{-Y_0} \)
and thus,

\[
\forall t \in [t_1, t_2], \quad |\mu_2(-t + \tilde{T}(t_1)) - \mu_2(-t + \tilde{T}(t))| \leq C_0 Y_0 e^{-\frac{3}{2}Y_0}, \quad |y_2(-t + \tilde{T}(t_1)) - \tilde{X}(t_1) - (y_2(-t + \tilde{T}(t)) - \tilde{X}(t))| \leq C_0 Y_0 e^{-Y_0}.
\]

Therefore, setting

\[
\nu(t) = \mu_1(t) - \mu_2(-t + \tilde{T}(t_1)), \quad z(t) = y_1(t) + y_2(-t + \tilde{T}(t_1)) - \tilde{X}(t_1),
\]

we obtain

\[
|\nu(t)| \leq C_0 Y_0 e^{-\frac{3}{2}Y_0}, \quad |z(t)| \leq C_0 e^{-Y_0}. \tag{5.35}
\]
Now, we obtain a contradiction following the strategy of the proof of Proposition 5.1. Arguing as in the proof of (5.24) and (5.25), using (5.35) we get
\[ |J_1(t)| + |J_2(t)| \leq CY_0^\sigma + e^{-\frac{5}{2}Y_0}, \] (5.36)
\[ \left| \frac{d}{dt} \left( z\{b_- y(t) + (b_- + d_-)\}e^{-y(t)} \right) + (\tilde{J}_1(t) - \tilde{J}_2(-t + \tilde{T}(t_1))) \right| \leq C\epsilon Y_0 e^{-\frac{3}{2}Y_0}. \] (5.37)

Integrating (5.37) on \([t_1, t_2]\) using \(t_2 - t_1 < Ce^{Y_0}\), and then using (5.36) and we obtain
\[ \left| (z(t_1) - \{b_- y(t_1) + (b_- + d_-)\}e^{-y(t_1)}) - (z(t_2) - \{b_- y(t_2) + (b_- + d_-)\}e^{-y(t_2)}) \right| \leq C\epsilon Y_0 e^{-Y_0}. \] (5.38)

Thus, by (5.35), for \(k = 1 + \frac{d}{b_-} (b_- \neq 0)\),
\[ |(y(t_1) + k)e^{-y(t_1)} - (y(t_2) + k)e^{-y(t_2)}| \leq C\epsilon Y_0 e^{-Y_0}. \] (5.39)

But using \(Y(t_1) = Y_0 + 1\) and \(Y(t_2) = Y_0 + 2\), (5.39) is a contradiction for \(\epsilon\) small enough and \(Y_0\) large enough. \(\Box\)

**A Preliminary results on solitons**

### A.1 Linearized operator, identities and asymptotics

Recall
\[ Q_c(x) = c^{1/3} Q \left( \sqrt{c} x \right), \quad Q''_c + Q_c^4 = c Q_c, \quad \text{for } c > 0, \] (A.1)
where
\[ Q(x) = \left( \frac{5}{2} \right)^{1/3} \cosh^{-2/3} \left( \frac{3}{2} x \right) \] (A.2)
solves
\[ Q'' + Q^4 = Q \quad \text{and} \quad (Q')^2 + \frac{2}{5} Q^5 = Q^2 \quad \text{on } \mathbb{R}. \] (A.3)

**Claim A.1** (Properties of the linearized operator \(L\)). The operator \(L\) defined in \(L^2(\mathbb{R})\) by
\[ Lf = -f'' + f - 4Q^3 f \] (A.4)
is self-adjoint and satisfies the following properties:

(i) **First eigenfunction**: \(LQ \frac{1}{2} = -\frac{21}{4} Q \frac{1}{2};\)

(ii) **Second eigenfunction**: \(LQ' = 0;\) the kernel of \(L\) is \(\{\lambda Q', \lambda \in \mathbb{R}\};\)

(iii) **For any function** \(h \in L^2(\mathbb{R})\) orthogonal to \(Q'\) for the \(L^2\) scalar product, there exists a unique function \(f \in H^2(\mathbb{R})\) orthogonal to \(Q'\) such that \(Lf = h;\) moreover, if \(h\) is even (respectively, odd), then \(f\) is even (respectively, odd).

(iv) **Suppose that** \(f \in H^2(\mathbb{R})\) **is such that** \(Lf \in \mathcal{Y},\) **then** \(f \in \mathcal{Y}.\)
(v) There exists $\lambda > 0$ such that for all $f \in H^1(\mathbb{R})$,
\[
\int Q f = \int Q' f = 0 \Rightarrow (L f, f) \leq \lambda \|f\|_{H^1}^2.
\] (A.5)

These properties of the linearized operator are standard (see e.g. [26], Lemma 2.2). See 38 for property (v).

Claim A.2. (i) Scaling. Let
\[
\Lambda Q_c = \left( \frac{d}{dc} Q\right)_{|c'=c}, \quad \Lambda^2 Q_c = \left( \frac{d^2}{dc^2} Q\right)_{|c'=c}.
\] (A.6)

Then,
\[
\Lambda Q_c = \frac{1}{c} \left( \frac{1}{3} Q_c + \frac{1}{2} x Q'_c \right), \quad \Lambda^2 Q_c = \frac{1}{c^2} \left( -\frac{2}{9} Q_c + \frac{1}{12} x Q'_c + \frac{1}{4} x^2 Q''_c \right),
\] (A.7)
\[
\Lambda Q = \Lambda Q_1 = \frac{1}{3} Q + \frac{1}{2} x Q', \quad \Lambda^2 Q = \Lambda^2 Q_1 = -\frac{2}{9} Q + \frac{1}{12} x Q' + \frac{1}{4} x^2 Q''.
\] (A.8)

(ii) Some explicit antecedents for $L$.
\[
LQ = -3Q^4, \quad L(\Lambda Q) = -Q, \quad L(Q') = 0,
\] (A.9)
\[
\left( \frac{Q'}{Q} \right)' = -\frac{3}{5} Q^3, \quad \lim_{\pm \infty} \frac{Q'}{Q} = \pm 1,
\] (A.10)
\[
L \left( \frac{Q'}{Q} \right) = \frac{Q'}{Q} - \frac{11}{5} Q^2 Q', \quad \left( L \left( \frac{Q'}{Q} \right) \right)' = -\frac{36}{5} Q^3 + \frac{99}{25} Q^6.
\] (A.11)

(iii) Integral identities. For $r \geq 1, c > 0$,
\[
\int Q^{r+3} = \frac{5r}{2r+3} \int Q^r, \quad \int (Q')^2 = \frac{3}{7} \int Q^2,
\] (A.12)
\[
\int \Lambda Q = -\frac{1}{6} \int Q, \quad \int Q^3 \Lambda Q = \frac{5}{24} \int Q, \quad \int Q(\Lambda Q) = \frac{1}{6} \int Q^2,
\] (A.13)
\[
\frac{7}{5} \int e^{-x} Q^5 = \frac{3}{2} \int e^{-x} Q^2, \quad \int e^{-x} Q^4 = 2(10)^{1/3}, \quad \int Q^3 = \frac{10}{3},
\] (A.14)
\[
\int Q_c^2 = c^{1/6} \int Q_c^2, \quad \mathcal{E}(Q_c) = c^{7/6} \mathcal{E}(Q), \quad \mathcal{E}(Q) = -\frac{1}{7} \int Q^2, \quad -\frac{d}{dc} \mathcal{E}(Q_c) = c \frac{d}{dc} \int Q_c^2 > 0.
\] (A.15)

(iv) Asymptotics as $x \to +\infty$
\[
Q(x) = (10)^{1/3} e^{-x} + O(e^{-4x}), \quad Q'(x) = -(10)^{1/3} e^{-x} + O(e^{-4x}),
\] (A.17)
\[
\Lambda Q(x) = (10)^{1/3} \left( \frac{1}{3} - \frac{x}{2} \right) e^{-x} + O(x e^{-4x}).
\] (A.18)

Let
\[
P = \frac{Q'}{Q} - 1 + 2(10)^{-1/3} e^x Q.
\] (A.19)

Then,
\[
\forall x \in \mathbb{R}, \quad |P(x)| \leq C e^{-2|x|}.
\] (A.20)
Proof. These results are easily obtained for the equation of $Q$. See e.g. [26], Appendix C.1 and Claim 2.1.

We only prove (A.20) concerning the function $P$. First, since \(\lim_{\pm \infty} \frac{Q'}{Q} = \mp 1\) and \(\lim_{+\infty} 2(10)^{-1/3}e^x Q = 2\), \(\lim_{-\infty} 2(10)^{-1/3}e^x Q = 0\), we have \(\lim_{\pm \infty} P = 0\). Moreover, we have from (A.10),

\[
P' = -\frac{3}{5} Q^3 + 2(10)^{-1/3}(e^x Q)'
\]

and using the explicit expression of $Q$, we find \(|(e^x Q)'| \leq CE^{-3x}\) for $x > 0$ and \(|(e^x Q)'| \leq CE^{2x}\) for $x < 0$. This proves (A.20) by integration. \(
\)

A.2 Approximate antecedent of $R_1R_2$

Claim A.3. Let $x_1 = x - y_1$ and $x_2 = x - y_2$. Then

\[
\partial_x \left( -\partial_x^2 ((x_1 + x_2)R_1R_2) + (x_1 + x_2)R_1R_2 - 4(R_1^3 + R_2^3)(x_1 + x_2)R_1R_2 \right)
\]

\[
= 2R_1R_2 + 3y(R_1 - \partial_x R_1 - \partial_x (R_1^3))R_2 - yR_1^4 \partial_x R_2
\]

\[
+ 3y(R_2 + \partial_x R_2 + \partial_x (R_2^2))R_1 + yR_2^2 \partial_x R_1
\]

\[
+ (-R_1^4 - 6x_1 \partial_x (R_1^3))R_2 - 2x_1 R_1^4 \partial_x R_2 - R_1^4 R_2
\]

\[
+ 6x_1 (R_1 - \partial_x R_1)R_2 + 6(R_1 - \partial_x R_1) \partial_x R_2 - 6(R_1 - \partial_x R_1)R_2
\]

\[
+ (-R_2^4 - 6x_2 \partial_x (R_2^3))R_1 - 2x_2 R_2^4 \partial_x R_1 - R_2^4 R_1
\]

\[
- 6x_2 (R_2 + \partial_x R_2)R_1 - 6(\partial_x R_2 + R_2)(\partial_x R_1) - 6(\partial_x R_2 + R_2)R_1.
\]

Proof. We start with a formula for two general functions $f_1$, $f_2$,

\[
\partial_x \left( -\partial_x^2 (f_1(x_1)f_2(x_2)) + f_1(x_1)f_2(x_2) - 4(R_1^3 + R_2^3)(f_1(x_1)f_2(x_2)) \right)
\]

\[
= \partial_x \left[ f_2(x_2)(Lf_1)(x_1) + f_1(x_1)(Lf_2)(x_2) - 2f_1'(x_1)f_2'(x_2) - f_1(x_1)f_2(x_2) \right]
\]

\[
= f_2'(x_2)(Lf_1)'(x_1) + f_1(x_1)(Lf_2)'(x_2)
\]

\[
+ (Lf_1 - f_1 - 2f_1''(x_1)) f_2'(x_2) + (Lf_2 - f_2 - 2f_2''(x_2)) f_1'(x_1).
\]

We apply this formula with $f_1 = xQ$ and $f_2 = Q$. Note that using (A.9), we have

\[
LQ = -3Q^4, \quad LQ - Q - 2Q'' = -3Q - Q^4,
\]

\[
L(xQ) = xLQ - 2Q' = -3xQ^4 - 2Q',
\]

\[
L(xQ) - xQ - 2(xQ)'' = -x(3Q + Q^4) - 6Q'.
\]

Therefore, one gets

\[
\partial_x \left( -\partial_x^2 (x_1R_1R_2) + x_1R_1R_2 - 4(R_1^3 + R_2^3)x_1R_1R_2 \right)
\]

\[
= (-R_1^4 - 3x_1 \partial_x (R_1^3))R_2 - x_1 R_1^4 \partial_x R_2 - 3x_1 R_1 R_2^3 (R_1^4) - (R_1 + x_1 \partial_x (R_1^3))R_2^3
\]

\[
- 3x_1(\partial_x R_1)R_2 - 3x_1 R_1(\partial_x R_2) - 6(\partial_x R_1)(\partial_x R_2) - 5R_1 R_2.
\]

Similarly,

\[
\partial_x \left( -\partial_x^2 (x_2R_2R_1) + x_2R_2R_1 - 4(R_1^3 + R_2^3)x_2R_2R_1 \right)
\]

\[
= (-R_2^4 - 3x_2 \partial_x (R_2^3))R_1 - x_2 R_2^4 \partial_x R_1 - 3x_2 R_2 \partial_x (R_1^4) - (R_2 + x_2 \partial_x (R_2^3))R_1^4
\]

\[
- 3x_2(\partial_x R_2)R_1 - 3x_2 R_2(\partial_x R_1) - 6(\partial_x R_2)(\partial_x R_1) - 5R_2 R_1.
\]
Thus,
\[
\partial_x \left( -\partial_x^2 ((x_1 + x_2) R_1 R_2) + (x_1 + x_2) R_1 R_2 - 4(R_1^3 + R_2^3)(x_1 + x_2) R_1 R_2 \right)
= (-R_1^4 - 3x_1 R_1^3 R_2) R_2 - x_1 R_1^3 \partial_x R_2 - 3x_1 R_1 R_2^2 (R_1^3) - (R_1 + x_1 \partial_x (R_1))(R_1^3) R_2
+ (-R_2^4 - 3x_2 \partial_x (R_2^3)) R_1 - x_2 R_2^3 \partial_x R_1 - 3x_2 R_2 \partial_x (R_1^3) - (R_2 + x_2 \partial_x (R_2))(R_1^3)
- 12(\partial_x R_1)(\partial_x R_2) - 10R_1 R_2
- 3x_1 (\partial_x R_1) R_2 - 3x_1 R_1 (\partial_x R_2) - 3x_2 (\partial_x R_2) R_1 - 3x_2 R_2 (\partial_x R_1).
\]

First, the terms in the last line are handled as follows (recall that \(x_2 - x_1 = y\))
\[
- 3x_1 (\partial_x R_1) R_2 - 3x_1 R_1 (\partial_x R_2) - 3x_2 (\partial_x R_2) R_1 - 3x_2 R_2 (\partial_x R_1)
= 3x_1 (R_1 - \partial_x R_1) R_2 - 3x_1 R_1 (R_2 + \partial_x R_2) - 3x_2 (R_2 + \partial_x R_2) R_1 + 3x_2 R_2 (R_1 - \partial_x R_1)
= 6x_1 (R_1 - \partial_x R_1) R_2 + 3y (R_1 - \partial_x R_1) R_2 - 6x_2 (R_2 + \partial_x R_2) R_1 + 3y (R_2 + \partial_x R_2) R_1.
\]

Second,
\[
- 12(\partial_x R_1)(\partial_x R_2)
= 6(R_1 - \partial_x R_1)(\partial_x R_2) - 6(\partial_x R_2 + R_2)(\partial_x R_1) - 6R_1 (\partial_x R_2) + 6R_2 (\partial_x R_1)
= 6(R_1 - \partial_x R_1)(\partial_x R_2) - 6(\partial_x R_2 + R_2)(\partial_x R_1)
+ 6R_1 (R_2 + \partial_x R_2) - 6R_2 (R_1 - \partial_x R_1) + 12R_1 R_2.
\]

Gathering these computations, we obtain
\[
\partial_x \left( -\partial_x^2 ((x_1 + x_2) R_1 R_2) + (x_1 + x_2) R_1 R_2 - 4(R_1^3 + R_2^3)(x_1 + x_2) R_1 R_2 \right)
= 2R_1 R_2 + 3y (R_1 - \partial_x R_1) R_2 + 3y (R_2 + \partial_x R_2) R_1
+ (-R_1^4 - 3x_1 R_1^3 R_2) R_2 - x_1 R_1^3 \partial_x R_2 - 3x_1 R_1 R_2^2 (R_1^3) - (R_1 + x_1 \partial_x (R_1))(R_1^3) R_2
+ 6x_1 (R_1 - \partial_x R_1) R_2 + 6(R_1 - \partial_x R_1)(\partial_x R_2) - 6(R_1 - \partial_x R_1) R_2
+ (-R_2^4 - 3x_2 \partial_x (R_2^3)) R_1 - x_2 R_2^3 \partial_x R_1 - 3x_2 R_2 \partial_x (R_1^3) - (R_2 + x_2 \partial_x (R_2))(R_1^3)
- 6x_2 (R_2 + \partial_x R_2) R_1 - 6(\partial_x R_2 + R_2)(\partial_x R_1) - 6(R_2 + \partial_x R_2) R_1
+ 2R_1 R_2 + 3y (R_1 - \partial_x R_1 - \partial_x (R_1^3)) R_2 - y R_1^4 \partial_x R_2
+ 3y (R_2 + \partial_x R_2 + \partial_x (R_2^3)) R_1 + y R_2^3 \partial_x R_1
+ (-R_1^4 - 6x_1 \partial_x (R_1^3)) R_2 - 2x_1 R_1^3 \partial_x R_2 - R_1^4 R_2
+ 6x_1 (R_1 - \partial_x R_1) R_2 + 6(R_1 - \partial_x R_1)(\partial_x R_2) - 6(R_1 - \partial_x R_1) R_2
+ (-R_2^4 - 6x_2 \partial_x (R_2^3)) R_1 - 2x_2 R_2^3 \partial_x R_1 - R_2^4 R_1
- 6x_2 (R_2 + \partial_x R_2) R_1 - 6(\partial_x R_2 + R_2)(\partial_x R_1) - 6(R_2 + \partial_x R_2) R_1,
\]
which finishes the proof of Claim A.3. \(\square\)

**B Modulation and monotonicity arguments**

**B.1 Proof of Lemma 3.1**

Let
\[
\mathcal{N}(\omega_0, y_0) = \{ u \in H^1(\mathbb{R}); \inf_{y_1 \neq y_2 \neq y_0} \| u - V(\cdot; (0, 0, y_1, y_2)) \|_{H^1} \leq \omega_0 \},
\]
where \(V(x; \Gamma)\) is defined in Proposition 2.2.

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Claim B.1 (Time independent modulation). There exist \( \omega_0, \tilde{\gamma}_0 > 0 \) and a unique \( C^1 \) map \( \Gamma = (\mu_1, \mu_2, y_1, y_2) : V(\omega_0, \tilde{\gamma}_0) \to (0, \infty)^2 \times \mathbb{R}^2 \) such that if \( u \in V(\omega, y_0) \) for \( 0 < \omega \leq \omega_0, y_0 \geq \tilde{\gamma}_0 \)

\[
\varepsilon(x) = u(x) - V(x; \Gamma),
\]

then, for \( j = 1, 2, \)

\[
\int \varepsilon Q_{1+\mu_j}(\cdot - y_j) = \int \varepsilon Q'_{1+\mu_j}(\cdot - y_j) = 0
\]

and

\[
y_1 - y_2 > y_0 - C\omega, \quad \|\varepsilon\|_{H^1} + |\mu_1| + |\mu_2| \leq C\omega.
\]

Proof. The proof, based on the implicit function theorem, is similar to the one of Lemma 8 in [29], the only difference being that we perform modulation around the map \((\mu_1, \mu_2, y_1, y_2) \mapsto V(x; (\mu_1, \mu_2, y_1, y_2))\) instead of the family of sums of two solitons. By the properties of \( V \) (see (2.15) and (1.2) below) and (A.13), the nondegeneracy conditions are the same as in [29]. \( \square \)

The existence, uniqueness and continuity of \( \Gamma(t) \) is a consequence of Claim B.1 applied to \( u(t) \) for all \( t \in I \).

The equation of \( \varepsilon(t) \) is easily deduced from (1.11) and (2.16). Next, we prove the estimates on \( \hat{\Gamma}(t) \), i.e. (3.6), omitting standard regularization arguments to justify the computations. First, we expand \( \frac{d}{dt} \int \varepsilon \tilde{R}_1 \). Using (3.4), we obtain

\[
0 = \frac{d}{dt} \int \varepsilon \tilde{R}_1 = \int \varepsilon \partial_t \tilde{R}_1 + \int (\partial^2 \varepsilon - \varepsilon + 4V^3\varepsilon) \partial_x \tilde{R}_1 + \int ((V + \varepsilon)^4 - V^4 - 4V^3\varepsilon) \partial_x \tilde{R}_1
\]

\[+ E\tilde{R}_1 - (\mu_1 - M_1) \int \frac{\partial V}{\partial \mu_1} \tilde{R}_1 - (\mu_2 - M_2) \int \frac{\partial V}{\partial \mu_2} \tilde{R}_1
\]

\[+ (\mu_1 - \tilde{\gamma}_1 - N_1) \int \frac{\partial V}{\partial y_1} \tilde{R}_1 + (\mu_2 - \tilde{\gamma}_2 - N_2) \int \frac{\partial V}{\partial y_2} \tilde{R}_1.
\]

We claim the following estimates.

Claim B.2. Assuming (3.5),

\[
\left| \int \tilde{R}_1 \tilde{R}_2 \right| \leq C(y + 1)e^{-y}, \quad (B.1)
\]

\[
j = 1, 2, \quad \left\| \frac{\partial V}{\partial \mu_j} - \Lambda \tilde{R}_j \right\|_{H^1} + \left\| \frac{\partial V}{\partial y_j} + \partial_x \tilde{R}_j \right\|_{L^\infty} + \frac{1}{\sqrt{y}} \left\| \frac{\partial V}{\partial y_j} + \partial_x \tilde{R}_j \right\|_{H^1} \leq Ce^{-y}. \quad (B.2)
\]

Indeed, under assumption (3.5), (B.1) is a consequence of (2.24) – see also proof of Claim 2.1. Moreover, (B.2) is a consequence of the explicit expression of \( V \) (see (2.3) and (2.13)) and the properties of \( A_j, B_j \) and \( D_j \) (see proof of Proposition 2.1).

By (B.1), (B.2), (2.14), \( LQ' = 0 \) ((A.9)) and \( \int Q'Q = 0 \), we get

\[
0 = \mu_1 \int \varepsilon \Lambda \tilde{R}_1 + (\mu_1 - \tilde{\gamma}_1) \int \varepsilon \partial_x \tilde{R}_1 + \|\varepsilon(t)\|_{L^2} O(ye^{-y}) + O (\|\varepsilon\|_{L^2}^2)
\]

\[- \int E\tilde{R}_1 - (\mu_1 - M_1) \int \tilde{R}_1 \Lambda \tilde{R}_1 + O(e^{-y}) + (\mu_2 - M_2)O(y^2e^{-y})
\]

\[+ (\mu_1 - \tilde{\gamma}_1 - N_1)O(e^{-y}) + (\mu_2 - \tilde{\gamma}_2 - N_2)O(ye^{-y}).
\]
Hence, by \( \left| \int \tilde{R}_1 \Lambda \tilde{R}_1 \right| \geq c_0 > 0 \) (see (A.13)), for \( y \) large and \( \varepsilon \) small, we get

\[
|\dot{\mu}_1 - \mathcal{M}_1| \leq C \left( \|\varepsilon\|^2 \mathcal{L}_2 + ye^{-y}\|\varepsilon\|_{L^2} + \int |E\tilde{R}_1| \right)
+ Cy^2e^{-y}|\dot{\mu}_2 - \mathcal{M}_2| + Ce^{-y}|\mu_1 - \dot{\nu}_1 - \mathcal{N}_1| + Cy e^{-y}|\mu_2 - \dot{\nu}_2 - \mathcal{N}_2|.
\]

Similarly, expanding \( 0 = \frac{d}{dt} \int \varepsilon \partial_x \tilde{R}_1 \), we obtain

\[
|\mu_1 - \dot{\nu}_1 - \mathcal{N}_1| \leq C \left( \|\varepsilon\|^2 \mathcal{L}_2 + \int |E\partial_x \tilde{R}_1| \right)
+ C(\|\varepsilon\|_{L^2} + e^{-\frac{1}{2}y})|\dot{\nu}_1 - \mathcal{M}_1| + Ce^{-\frac{1}{2}y}|\dot{\mu}_2 - \mathcal{M}_2| + Cy e^{-y}|\mu_2 - \dot{\nu}_2 - \mathcal{N}_2|.
\]

Combining these two estimates, together with similar estimates for \(|\dot{\mu}_2 - \mathcal{M}_2|\) and \(|\mu_2 - \dot{\nu}_2 - \mathcal{N}_2|\) for \( y_0 \) large and \( \omega_0 \) small, (3.5) is proved.

**B.2 Proof of Proposition 3.1**

**Proof of (3.10).** The proof of (3.10) is standard under the orthogonality conditions (3.3), see for example Lemma 4 in [29]. Recall that the proof is mainly based on coercivity property of the operator \( L \) under orthogonality conditions, i.e. Lemma A.1(v) and on localization arguments. Indeed, we observe in particular that locally around each soliton \( \tilde{R}_j \), both functionals behave essentially as

\[
\int (\partial_x \varepsilon)^2 + (1 + \mu_j)\varepsilon^2 - 4\tilde{R}_j^2 \varepsilon,
\]

which is a rescaled version of \( (L\varepsilon, \varepsilon) \).

**Proof of (3.11).** We start with the following preliminary estimates.

**Claim B.3.**

\[
\left\| V - \tilde{R}_1 - \tilde{R}_2 \right\|_{L^\infty} \leq Ce^{-y}, \tag{B.3}
\]

\[
\left\| \partial_x V - \tilde{E}(V) + \sum_{j=1,2} \mu_j \partial_x \tilde{R}_j \right\|_{L^\infty} \leq Ce^{-y}, \tag{B.4}
\]

\[
\left\| (\Phi - \mu_j)e^{-\frac{1}{2}|x-\nu_j|} \right\|_{L^\infty} + \left\| V \partial_x \Phi \right\|_{L^\infty} \leq C(|\mu_1| + |\mu_2|)e^{-2\rho y}, \tag{B.5}
\]

\[
\left\| \Phi \partial_x V - \sum_{j=1,2} \mu_j \partial_x \tilde{R}_j \right\|_{L^\infty} \leq C(|\mu_1| + |\mu_2|)e^{-2\rho y} + Ce^{-y}, \tag{B.6}
\]

\[
\left\| \tilde{E}(V) - \sum_{j=1,2} (\dot{\mu}_j - \mathcal{M}_j) \Lambda \tilde{R}_j + \sum_{j=1,2} (\mu_j - \dot{\nu}_j - \mathcal{N}_j) \partial_x \tilde{R}_j \right\|_{H^1}
\leq C \left( \|\varepsilon\|_{L^2} + \int |E|(\tilde{R}_1 + \tilde{R}_2) \right) \sqrt{ye^{-y}}. \tag{B.7}
\]
These estimates follow from (2.14), (2.3), (2.13), (2.17), (3.7), (3.6), and (B.2).

Let

\[ \Theta = \|\varepsilon\|_{L^2}^2 \left[ e^{-\frac{3}{2}y} + (|\mu_1| + |\mu_2| + \|\varepsilon\|_{L^2}) (e^{-2\rho y} + \|\varepsilon\|_{L^2}) \right] + \|\varepsilon\|_{H^1} \|E\|_{H^1}. \]

Now, we compute \( \frac{d}{dt} \mathcal{F}_+(t) \):

\[
\frac{1}{2} \frac{d}{dt} \mathcal{F}_+(t) = \int \partial_t \varepsilon \left( -\partial_x^2 \varepsilon + \varepsilon - ((\varepsilon + V)^4 - V^4) + \varepsilon \Phi \right) - \int \partial_t V \left( (V + \varepsilon)^4 - V^4 - 4V^3 \varepsilon \right) + \frac{1}{2} \int \varepsilon^2 \partial_x \Phi = F_1 + F_2 + F_3.
\]

Observe that \( \partial_x \Phi = (\mu_1 - \mu_2) \varphi' \geq 0 \) in the present situation. We claim

\[ F_1 + F_2 \leq C\Theta \quad \text{and} \quad F_3 \leq C\Theta. \]

First, using the equation of \( \varepsilon \) (i.e. (3.4)),

\[
F_1 = -\int (\partial_x^2 \varepsilon + \varepsilon - ((\varepsilon + V)^4 - V^4)) \partial_x (\Phi \varepsilon) - \int \bar{E}(V) (\partial_x^2 \varepsilon + \varepsilon \Phi - ((\varepsilon + V)^4 - V^4)) - \int E (\partial_x^2 \varepsilon + \varepsilon \Phi - ((\varepsilon + V)^4 - V^4)) = F_{1,1} + F_{1,2} + F_{1,3}.
\]

Integrating by parts and using \( |\varphi'''| \leq (8\rho)^2 |\varphi'| \leq (1/16) |\varphi'| \),

\[
F_{1,1} = -\frac{3}{2} \int (\partial_x \varepsilon)^2 \partial_x \Phi - \frac{1}{2} \int \varepsilon^2 \partial_x \Phi + \frac{1}{2} \int \varepsilon^2 \partial_x^3 \Phi + \int ((\varepsilon + V)^4 - V^4) \partial_x (\Phi \varepsilon)
\leq -\frac{3}{2} \int (\partial_x \varepsilon)^2 \partial_x \Phi - \frac{3}{8} \int \varepsilon^2 \partial_x \Phi + \int ((\varepsilon + V)^4 - V^4) \partial_x (\Phi \varepsilon).
\]

Then, we decompose \( F_{1,2} \) as follows

\[
F_{1,2} = -\int \bar{E}(V) (\partial_x^2 \varepsilon + \varepsilon \Phi - 4V^3 \varepsilon) + \int \bar{E}(V) ((\varepsilon + V)^4 - V^4 - 4V^3 \varepsilon).
\]

First, by (B.7) and (3.6), integrating by parts and using Cauchy Schwarz inequality,

\[
\left| \int (\bar{E}(V) - \sum_{j=1,2} (\mu_j - M_j) \Lambda \tilde{R}_j + \sum_{j=1,2} (\mu_j - \bar{y}_j - N_j) \partial_x \tilde{R}_j) (\partial_x^2 \varepsilon + \varepsilon \Phi - 4V^3 \varepsilon) \right|
\leq C\|\varepsilon\|_{H^1} \left( \|\varepsilon\|_{L^2} + \int |E|(\tilde{R}_1 + \tilde{R}_2) \right) \sqrt{y}e^{-y} \leq C\Theta.
\]

Second, by (B.3), (B.5) and (A.9), (3.3), (3.6),

\[
\left| \int (\sum_{j=1,2} (\mu_j - M_j) \Lambda \tilde{R}_j + \sum_{j=1,2} (\mu_j - \bar{y}_j - N_j) \partial_x \tilde{R}_j) (\partial_x^2 \varepsilon + \varepsilon \Phi - 4V^3 \varepsilon) \right|
\leq C\|\varepsilon\|_{H^1} \left( \|\varepsilon\|_{L^2} + \int |E|(\tilde{R}_1 + \tilde{R}_2) \right) (C(|\mu_1| + |\mu_2|)e^{-2\rho y} + \sqrt{y}e^{-y}) \leq C\Theta.
\]
Thus,
\[ \left| \int \tilde{E}(V) \left( -\partial_x^2 \varepsilon + \varepsilon + \varepsilon \Phi - 4V^3 \varepsilon \right) \right| \leq C \Theta. \]

Moreover, integrating by parts,
\[ |F_{1,3}| \leq C \|E\|_{H^1} \|\varepsilon\|_{H^1} \leq C \Theta. \]

Next, we combine \( F_2 = -\int \partial_x V ((V + \varepsilon)^4 - V^4 - 4V^3 \varepsilon) \) with the remaining terms from \( F_{1,1} \) and \( F_{1,2} \). Using (B.6) and (B.4), we have
\[
\int ((\varepsilon + V)^4 - V^4) \partial_x \Phi \varepsilon + \int \tilde{E}(V)((\varepsilon + V)^4 - V^4 - 4V^3 \varepsilon) + F_2 \\
= O \left( \|\varepsilon\|^2_{L^2}(e^{-\varepsilon} + e^{-2\rho y(|\mu_1| + |\mu_2|)}) \right) \\
+ \int ((\varepsilon + V)^4 - V^4) \partial_x \Phi \varepsilon + \int ((\varepsilon + V)^4 - V^4 - 4V^3 \varepsilon) \Phi \partial_x V.
\]

Moreover, integrating by parts,
\[
\int \Phi ((\varepsilon + V)^4 - V^4) \partial_x \varepsilon + \int \Phi((\varepsilon + V)^4 - V^4 - 4V^3 \varepsilon) \partial_x V \\
= \int \Phi \partial_x \left( \frac{1}{6} (\varepsilon + V)^5 - \frac{1}{5} V^5 - V^4 \varepsilon \right) \right) = - \int (\partial_x \Phi) \left( \frac{1}{5} (\varepsilon + V)^5 - \frac{1}{3} V^5 - V^4 \varepsilon \right),
\]

and so by (B.5),
\[
\left| \int ((\varepsilon + V)^4 - V^4) \partial_x \Phi \varepsilon + \int ((\varepsilon + V)^4 - V^4 - 4V^3 \varepsilon) \Phi \partial_x V \right| \\
= \left| \int \partial_x \Phi \left( \varepsilon (V + \varepsilon)^4 - \frac{1}{6} (\varepsilon + V)^5 + \frac{1}{3} V^5 \right) \right| \leq C \|V(\partial_x \Phi)\|_{L^\infty} \|\varepsilon\|^2_{L^2} + C \|\varepsilon\|_{H^1} \int \varepsilon^2 \partial_x \Phi \\
\leq C \Theta + C \|\varepsilon\|_{H^1} \int \varepsilon^2 \partial_x \Phi.
\]

We deduce \( F_1 + F_2 \leq C \Theta. \)

Finally
\[ F_3 = \frac{1}{2} \int (\dot{\mu}_1 \varphi + \dot{\mu}_2 (1 - \varphi)) \varepsilon^2, \]

so that by \( |M_j| \leq Ce^{-\varepsilon} \) and (3.6),
\[ |F_3| \leq C(e^{-\varepsilon} + \sum_{j=1,2} |\mu_j - M_j|) \|\varepsilon\|^2_{H^1} \leq C \Theta. \]

**Proof of (3.12).** Since \( \mu_2(t) \geq \mu_1(t) \) we have\[ \frac{1}{(1 + \mu_1(t))^\gamma} \geq \frac{1}{(1 + \mu_2(t))^\gamma}, \] \( \partial_x \Phi_1 \geq 0 \) and \( \partial_x \Phi_2 \leq 0. \) Note also that by explicit computations, for \( \mu_j \) small enough:
\[ |\partial_x \Phi_1 + 2\partial_x \Phi_2| \leq C(|\mu_1| + |\mu_2|) \partial_x \Phi_1; \]

and similarly to Claim (B.3) (B.4)-(B.6), we have
\[
\left\| \Phi_1 (\partial_t V - \tilde{E}(V)) + \Phi_2 \partial_x V \right\|_{L^\infty} + \|V(\partial_x \Phi_1)\|_{L^\infty} + \|V(\partial_x \Phi_2)\|_{L^\infty} \\
\leq C(|\mu_1| + |\mu_2|) e^{-2\rho y} + Ce^{-\varphi},
\]

(3.12)
We compute \( \frac{1}{2} \frac{d}{dt} \mathcal{F}^{-}(t) \):

\[
\frac{1}{2} \frac{d}{dt} \mathcal{F}^{-}(t) = \frac{1}{2} \int \left\{ \left[ (\partial_x \epsilon)^2 + \epsilon^2 - \frac{3}{8} ((\epsilon + V)^5 - V^5 - 5V^4 \epsilon) \right] \partial_t \Phi_1 + \epsilon^2 \partial_t \Phi_2 \right\} \\
+ \int \partial_t \epsilon \left( -\partial_x^2 \epsilon + \epsilon - ((\epsilon + V)^4 - V^4) \right) \Phi_1 - \int \partial_t \epsilon \partial_x \epsilon \partial_x \Phi_1 + \int \partial_t \epsilon \epsilon \Phi_2 \\
- \int \partial_t V \left( (\epsilon + V)^4 - V^4 - 4V^3 \epsilon \right) \Phi_1.
\]

The first term is treated as the term \( F_3 \) above. By (B.11), we thus obtain

\[
\frac{1}{2} \frac{d}{dt} \mathcal{F}^{-}(t) = \int \partial_t \epsilon \left( -\partial_x^2 \epsilon + \epsilon - ((\epsilon + V)^4 - V^4) \right) \Phi_1 - \int \partial_t \epsilon \partial_x \epsilon \partial_x \Phi_1 + \int \partial_t \epsilon \epsilon \Phi_2 \\
- \int \tilde{E}(V) \left( (\epsilon + V)^4 - V^4 - 4V^3 \epsilon \right) \Phi_1 + \int \Phi_2 \partial_x V \left( (\epsilon + V)^4 - V^4 - 4V^3 \epsilon \right) + O(\Theta).
\]

Using the equation of \( \epsilon \) and integrating by parts, arguing as in the proof of (3.11), we get

\[
\int \partial_t \epsilon \left( -\partial_x^2 \epsilon + \epsilon - ((\epsilon + V)^4 - V^4) \right) \Phi_1 - \int \partial_t \epsilon \partial_x \epsilon \partial_x \Phi_1 + \int \partial_t \epsilon \epsilon \Phi_2 \\
- \int \tilde{E}(V) \left( (\epsilon + V)^4 - V^4 - 4V^3 \epsilon \right) \Phi_1 \\
\leq -\frac{1}{2} \int \langle \partial_x^2 \epsilon \rangle^2 \partial_x \Phi_1 - \frac{7}{8} \int \langle \partial_x \epsilon \rangle^2 \partial_x \Phi_1 - \frac{3}{8} \int \epsilon^2 \partial_x \Phi_1 + \int \tilde{E}(V) \epsilon \Phi_2 + C \Theta. \tag{B.12}
\]

Next, by integration by parts, (3.7) and (B.11), and arguing as in the proof of (B.8),

\[
- \int \partial_t \epsilon \partial_x \epsilon \partial_x \Phi_1 = \int \left( -\partial_x^2 \epsilon + \epsilon - ((\epsilon + V)^4 - V^4) \right) \partial_x (\partial_x \epsilon \partial_x \Phi_1) + (\tilde{E}(V) + E) \partial_x \epsilon \partial_x \Phi_1 \\
\leq - \int \langle \partial_x^2 \epsilon \rangle^2 \partial_x \Phi_1 - \frac{7}{8} \int \langle \partial_x \epsilon \rangle^2 \partial_x \Phi_1 + C \| \epsilon \|_{H^3} \int \epsilon^2 \partial_x \Phi_1 + C \Theta.
\]

Finally, again by integration by parts, (3.7) and (B.8),

\[
\int \partial_t \epsilon \epsilon \Phi_2 + \int \tilde{E}(V) \epsilon \Phi_2 + \int \Phi_2 \partial_x V \left( (\epsilon + V)^4 - V^4 - 4V^3 \epsilon \right) \\
= - \int \left( -\partial_x^2 \epsilon + \epsilon - ((\epsilon + V)^4 - V^4) \right) \partial_x (\epsilon \Phi_2) - \int E \epsilon \Phi_2 \\
+ \int \Phi_2 \partial_x V \left( (\epsilon + V)^4 - V^4 - 4V^3 \epsilon \right) \leq \frac{3}{2} \int \langle \partial_x \epsilon \rangle^2 |\partial_x \Phi_2| + \frac{3}{4} \int \epsilon^2 |\partial_x \Phi_2| + C \Theta.
\]

Using (B.10), we check that the two positive terms above are compensated by the term (B.12) up to terms of order \( \Theta \), and the proof of (3.12) is now complete.

\[\textbf{B.3 Proof of Proposition 3.2}\]

By classical arguments (based on the implicit function theorem – see e.g. Lemma 3.1, Lemma B.1 and [29]), there exists \( \omega_1 > 0, \bar{y}_0 > 1 \) such that if

\[
\inf_{y_1 - y_2 > \bar{y}_0} \| u(t) - Q_{1 - \mu_0} (\cdot - y_1) - Q_{1 + \mu_0} (\cdot - y_2) \|_{H^3} \leq \omega_1 \tag{B.13}
\]
then $u(t)$ can be decomposed as follows

$$u(t, x) = \mathcal{R}_1(t, x) + \mathcal{R}_2(t, x) + \tilde{\varepsilon}(t, x), \quad (B.14)$$

where

$$\mathcal{R}_1(t, x) = Q_{1-\mu_0}(x - y_1(t)), \quad \mathcal{R}_2(t, x) = Q_{1+\mu_0}(x - y_2(t)) \quad (B.15)$$

and $y_j(t)$ are $C^1$ functions uniquely chosen so that

$$\int \tilde{\varepsilon}(t, x) \partial_x \mathcal{R}_j(t, x) dx = 0. \quad (B.16)$$

Moreover, $\|\tilde{\varepsilon}\|_{H^1} \leq C\omega_1$. Note that this decomposition is similar to the one of Lemma 3.1 except that for simplicity, we do not adjust the scaling parameter by modulation (it is not required here since the variation of the scalings of the solitons is quadratic in $\tilde{\varepsilon}$ in this regime – see Claim B.4 where we control $\int \tilde{\varepsilon}\mathcal{R}_j$ using the conservation laws). Moreover we use modulation around the family of sums of two solitons, not around a more sophisticated approximate solution such as $V$.

Let $y(t) = y_1(t) - y_2(t)$. For future reference, note that

$$\int \mathcal{R}_2(t) \mathcal{R}_1(t) \leq Ce^{-\frac{2}{3}y(t)} \quad (B.17)$$

(see the proof of a similar estimate in the proof of Claim 2.1). Note that one cannot obtain an estimate of the form $Ce^{-y}$ in this situation.

By (A.1) and the equation of $u(t)$, the functions $\tilde{\varepsilon}(t, x)$ and $y_j(t)$ satisfy the following equation

$$\partial_t \tilde{\varepsilon} + \partial_x \left( \partial_x^2 \tilde{\varepsilon} - \tilde{\varepsilon} + (\mathcal{R}_1 + \mathcal{R}_2 + \tilde{\varepsilon})^4 - (\mathcal{R}_1 + \mathcal{R}_2)^4 \right)$$

$$= -\partial_x \left( (\mathcal{R}_1 + \mathcal{R}_2)^4 - \mathcal{R}_1^4 \right) + (\dot{y}_1 + \mu_0)\partial_x \mathcal{R}_1 + (\dot{y}_2 - \mu_0)\partial_x \mathcal{R}_2. \quad (B.18)$$

By (B.17), as in the proof of Lemma 3.1 we obtain

$$|\mu_0 + \dot{y}_1| + |\mu_0 - \dot{y}_2| \leq C \left( \|\tilde{\varepsilon}\|_{H^1} + e^{-\frac{2}{3}y} \right). \quad (B.19)$$

**Proof of (3.14).** For $C_2 > 2$ to be chosen later, assume (3.13) and define

$$T^* = \sup \left\{ t_0 < T < -(\rho\mu_0)^{-1}\log \mu_0 \mid \text{such that, for all } t_0 < t < T, \ u(t) \text{ satisfies (B.13)}, \right\}$$

$$T^* \leq C_2 \omega_1 \mu_0 + C_2 e^{-4\rho\mu_0|t|} \text{ and } y(t) > \frac{3}{2} \rho \mu_0 |t|. \}$$

Note that for $C^*$ large enough, $T^*$ is well-defined by (3.13) and by continuity of $u(t)$ in $H^1$.

We prove that $T^* < -(\rho\mu_0)^{-1}\log \mu_0$, for $C^*$ large enough, assuming by contradiction that $T^* < -(\rho\mu_0)^{-1}\log \mu_0$ and working on the time interval $[t_0, T^*]$.

First, we claim the following control of the scaling directions of $\tilde{\varepsilon}(t)$.

**Claim B.4.** For all $t \in [t_0, T^*]$,

$$\int \tilde{\varepsilon}(t, x) \partial_x \mathcal{R}_j(t, x) dx \leq C \left( \mu_0^{-1} \sup_{[t_0, t]} \|\tilde{\varepsilon}\|_{H^1}^2 + \sup_{[t_0, t]} e^{-\frac{2}{3}y} + \mu_0 \omega \right). \quad (B.20)$$
Proof of Claim [B.4]. We obtain (B.4) by expanding \( u(t) = \overline{R}_1(t) + \overline{R}_2(t) + \overline{\varepsilon}(t) \) in the conservation laws (1.9) and (1.10) using (A.1) and (B.17).

\[
M(u(t)) = M(\overline{R}_1(t)) + M(\overline{R}_2(t)) + 2 \int \overline{\varepsilon}(t_0) \overline{R}_1(t_0) \\
+ 2 \int \overline{\varepsilon}(t_0) \overline{R}_2(t_0) + O(e^{-\frac{3}{4}y(t_0)}) + O(\|\overline{\varepsilon}(t_0)\|_{H^1}^2) \\
= M(u(t)) = M(\overline{R}_1(t)) + M(\overline{R}_2(t)) + 2 \int \overline{\varepsilon}(t) \overline{R}_1(t) \\
+ 2 \int \overline{\varepsilon}(t) \overline{R}_2(t) + O(e^{-\frac{3}{4}y(t)}) + O(\|\overline{\varepsilon}(t)\|_{H^1}^2);
\]

\[
\mathcal{E}(u(t)) = \mathcal{E}(\overline{R}_1(t)) + \mathcal{E}(\overline{R}_2(t)) - 2(1 - \mu_0) \int \overline{\varepsilon}(t_0) \overline{R}_1(t_0) \\
- 2(1 + \mu_0) \int \overline{\varepsilon}(t_0) \overline{R}_2(t_0) + O(e^{-\frac{3}{4}y(t_0)}) + O(\|\overline{\varepsilon}(t_0)\|_{H^1}^2) \\
= \mathcal{E}(u(t)) = \mathcal{E}(\overline{R}_1(t)) + \mathcal{E}(\overline{R}_2(t)) - 2(1 - \mu_0) \int \overline{\varepsilon}(t) \overline{R}_1(t) \\
- 2(1 + \mu_0) \int \overline{\varepsilon}(t) \overline{R}_2(t) + O(e^{-\frac{3}{4}y(t)}) + O(\|\overline{\varepsilon}(t)\|_{H^1}^2);
\]

Using \( M(\overline{R}_j(t_0)) = M(\overline{R}_j(t)), \mathcal{E}(\overline{R}_j(t_0)) = \mathcal{E}(\overline{R}_j(t)), \|\overline{\varepsilon}(t_0)\|_{H^1} \leq C\mu_0\bar{\omega} \), and combining the above estimates, we find (B.20).

Now, we use a functional \( \tilde{\mathcal{F}} \) similar to \( \mathcal{F}_- \) introduced in Proposition 3.1. Let

\[
\tilde{\mathcal{F}}(t) = \int \left[ (\partial_x \varepsilon)^2 + \varepsilon^2 - \frac{2}{5} ((\varepsilon + \overline{R}_1 + \overline{R}_2)^5 - (\overline{R}_1 + \overline{R}_2)^5 - 2(\varepsilon - (\overline{R}_1 + \overline{R}_2)^4) \right] \tilde{\Phi}_1 + \int \varepsilon^2 \tilde{\Phi}_2,
\]

where, \( \varphi \) being defined in (3.7),

\[
\tilde{\Phi}_1(x) = \frac{\varphi(x)}{1 - \mu_0} + \frac{1}{(1 + \mu_0)^2}, \quad \tilde{\Phi}_2(x) = \frac{-\mu_0\varphi(x)}{1 - \mu_0} + \frac{\mu_0(1 - \varphi(x))}{(1 + \mu_0)^2}.
\]

We perform similar (and simpler) computations as the ones of Propositions 3.1 and 3.1 (scaling parameters and \( \Phi_j \) are time independent). We obtain

\[
\frac{d}{dt} \tilde{\mathcal{F}}(t) \leq C\|\varepsilon\|_{L^2} (e^{-2\rho_0 \|\varepsilon\|_{L^2}} + \omega^{-\frac{3}{4}}).
\]

(B.21)

In particular, we point out that the term \( \|\varepsilon\|_{H^1}^4 \) does not appear in this estimate, since it was only due to the scaling modulation in the proof of Proposition 3.1.

Note also that from (3.13), we have \( \|\overline{\varepsilon}(t_0)\|_{H^1} \leq C\omega\mu_0 \) and thus \( \tilde{\mathcal{F}}(t_0) \leq C\omega^2\mu_0^2 \). Integrating (B.21) from \( t_0 \) to \( t \) \((t_0 < t < T^*)\), using the definition of \( T^* \), we obtain, for \( t_0 \leq t \leq T^* \), for \( \mu_0 \) small enough (possibly depending on \( C^* \))

\[
\tilde{\mathcal{F}}(t) \leq C\frac{C^2}{\mu_0} e^{-3\rho_0 |t|} (e^{-8\rho_0 |t|} + \omega^2\mu_0^2) + C\omega^2\mu_0^2 \\
\leq C\frac{C^2}{\mu_0^2} e^{-8\rho_0 |t|} + C\omega^2\mu_0^2.
\]

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Moreover, from (B.10), (B.20) and standard arguments,

$$\sup_{[t_0,T^*]} \|\bar{\varepsilon}\|^2_{H^1} \leq C \sup_{[t_0,T^*]} \mathcal{F} + C^2 \sup_{[t_0,T^*]} \sum_{j=1,2} \left( \int \bar{\varepsilon}(t) \mathcal{R}_j(t) \right)^2 \leq C_1 C_2^2 \mu_0^2 e^{-8\rho \mu_0 |t|} + C_1 e^{-y} + C_1 \omega^2 \mu_0^2,$$

(B.22)

where $C_1 > 0$ is independent of $C_s$. Choosing $C_s^2 > 4C_1$ and then $\mu_0$ small enough, from (B.22), we get for all $t \in [t_0, T^*],$

$$\|\bar{\varepsilon}(t)\|^2_{H^1} \leq \frac{1}{2} C_2^2 e^{-8\rho \mu_0 |t|} + \frac{1}{2} C_2^2 \omega^2 \mu_0^2.$$

Thus, by a standard continuity argument, we have just contradicted the definition of $T^*$.

Finally, from (B.19), we have $|\dot{y} + 2\mu_0| \leq C_1 \sup_{[t_0,T^*]} \|\bar{\varepsilon}\|_{H^1} + C_1 \bar{e}^{-\frac{7}{4} y}$, and thus $y(t) > \frac{7}{4} \mu_0 |t|$.

Proof of (3.15). The proof of (3.15) is completely similar, replacing the functional $\mathcal{F}$ by a similar functional inspired by the functional $\mathcal{G}_+$ in (3.3).

Proof of (3.10). The stability result being established for all $t < t_0$, the asymptotic stability is a consequence of [22] and [21].

**B.4 Proof of Proposition 4.2**

The proof is similar to the one of Propositions 5.1 and 5.2. We obtain a better result (no exponential error term), since Propositions 5.1 and 5.2 compare an exact solution with an approximate solution, whereas Proposition 4.2 compares two exact solutions.

We assume $T_1 \in [-T, T]$ (the case $|T_1| > T$ is similar) and we prove the stability result on $(-\infty, T_1]$, the stability proof for $[T_1, +\infty)$ following from similar arguments.

For $X_1, X_2 \in \mathbb{R}$, let $U_{X_1,X_2}$ be the unique solution of (1.14) such that

$$\lim_{t \to -\infty} \|U_{X_1,X_2}(t) - Q_{1-\mu_0}(x + \mu_0 t - X_1) - Q_{1+\mu_0}(x - \mu_0 t - X_2)\|_{H^1} = 0. \quad (B.23)$$

Then, as a direct consequence of the uniqueness of the asymptotic 2-soliton solution for given parameters (see [20], Theorem 1), for any $Y_1, Y_2 \in \mathbb{R}$, one has

$$U_{Y_1,Y_2}(t, x) = U_{X_1,X_2}(t - T_0, x - X_0)$$

where $X_0 = \frac{1}{2}(Y_1 - X_1) + \frac{1}{2}(Y_2 - X_2), \ T_0 = \frac{1}{2\mu_0} \left((Y_1 - X_1) - (Y_2 - X_2)\right). \quad (B.24)$

In particular, the map $(X_1, X_2) \mapsto U_{X_1,X_2}$ is smooth and

$$\frac{\partial U_{X_1,X_2}}{\partial X_j} = (-1)^j \frac{1}{2\mu_0} \frac{\partial U_{X_1,X_2}}{\partial t} - \frac{1}{2} \frac{\partial U_{X_1,X_2}}{\partial x}. \quad (B.25)$$

For $C^* > 2$ to be chosen, we define

$$T^* = \inf \{t \leq T_1 \ ; \text{ s.t. for all } t \leq t' \leq T_1, \ \inf_{X_1,X_2} \|u(t') - U_{X_1,X_2}\|_{H^1} \leq C^* \omega \mu_0\}. \quad (B.26)$$
By the assumption on \( u(T_1) \) and continuity of \( u(t) \) in \( H^1 \), \( T^* < T_1 \) is well-defined. We prove that \( T^* = \infty \) by contradiction: we assume \( T^* > \infty \) and we strictly improve the estimate of \( \inf_{X_1, X_2} \| u(t) - U_{X_1, X_2} \|_{H^1} \) on \([T^*, T_1]\), which contradicts the definition of \( T^* \).

By Proposition 4.1, \( U_{X_1, X_2} \) is close to the sum of two separated solitons. Thus, on \([T^*, T_1]\), for \( \omega \) small enough, we use modulation theory (as in the proof of Lemma 3.1) to prove the existence of \( X_1(t), X_2(t) \) such that for \( \bar{U}(t, x) = U_{X_1(t), X_2(t)}(t, x) \),

\[
u(t, x) = \bar{U}(t, x) + \bar{\varepsilon}(t, x), \quad \int \bar{\varepsilon} \frac{\partial \bar{U}}{\partial x} = 0. \tag{B.27}\]

(Note that this orthogonality condition is similar to \( \int \varepsilon \partial_x \tilde{R}_j = 0 \) in Lemma 3.1.) Moreover, there exist \( \tilde{\mu}_j(t), \tilde{y}_j(t) \) such that

\[
\left\| \bar{U}(t) - Q_1 + \tilde{\mu}_1(t)(\cdot - \tilde{y}_1(t)) - Q_1 + \tilde{\mu}_2(t)(\cdot - \tilde{y}_2(t)) \right\|_{H^1} \leq C|\ln \mu_0|^{1/2} \mu_0^2. \tag{B.28}\]

Next, \( \|\bar{\varepsilon}(t)\|_{H^1} \leq C\omega \mu_0 \),

\[
\partial_t \bar{\varepsilon} + \partial_x (\partial_x^2 \bar{\varepsilon} - \bar{\varepsilon}) + (\bar{U} + \bar{\varepsilon})^4 + \sum_{j=1,2} \tilde{x}_j \frac{\partial \bar{U}}{\partial x_j} = 0, \tag{B.29}\]

and \( |\tilde{x}_1| + |\tilde{x}_2| \leq C\|\bar{\varepsilon}\|_{H^1}. \tag{B.30}\)

From Proposition 4.1 there exists \( t_0 \) such that \( \tilde{\mu}_1(t) > \tilde{\mu}_2(t) \) if \( t > t_0 \) and \( \tilde{\mu}_1(t) < \tilde{\mu}_2(t) \) if \( t < t_0 \). Assume that \( t_0 < T^* \). In this case, to control \( \bar{\varepsilon}(t) \) on \([T^*, T_1]\) we use the functional

\[
\tilde{F}(t) = \int \left[ (\partial \bar{\varepsilon})^2 + \bar{\varepsilon}^2 - \frac{2}{5}((\bar{\varepsilon} + \bar{U})^5 - \bar{U}^5 - 5\bar{U}^4 \bar{\varepsilon}) \right] \tilde{\Phi} + \int \bar{\varepsilon}^2 \tilde{\Phi}_2,
\]

similar to \( F_+(t) \), for \( \tilde{\Phi}_j \) defined from \( \tilde{\mu}_j(t) \) as in Proposition 3.1. To treat the case \( T^* < t_0 \), one uses a functional similar to \( F_+(t) \).

**Claim B.5.** For all \( t \in [T^*, T_1] \),

\[
\frac{d}{dt} \tilde{F}(t) \geq -C\|\bar{\varepsilon}(t)\|_{L^2}^2 e^{-\frac{(\bar{\varepsilon} + \bar{\mu})Y_0}{\bar{\varepsilon}}} (C > 0), \tag{B.31}\]

\[
\tilde{F}(t) \geq \lambda\|\bar{\varepsilon}(t)\|_{L^2}^2 - C \sum_{j=1,2} \left( \int \bar{\varepsilon}(t) Q_1 + \tilde{\mu}_j(t)(x - \tilde{y}_j(t)) \right)^2. \tag{B.32}\]

\[
\left| \int \bar{\varepsilon}(t) Q_1 + \tilde{\mu}_j(t)(x - \tilde{y}_j(t)) \right| \leq C \omega \mu_0, \tag{B.33}\]

where \( \lambda > 0 \) and \( C > 0 \) are independent of \( C^* \).

Assuming this claim, we integrate (B.31) on \([T^*, T_1]\), and then use a combination of (B.32) and (B.33) to contradict the definition of \( T^* \), for \( C^* \) large enough and \( \mu_0 \) small enough. Note that the estimates on \( |\tilde{X}| \) and \( |\bar{T}| \) follow from (B.30) and (B.24).
Proof of Claim B.5. To prove (B.31), we use the same argument as in the proof of Propositions 3.1 and 3.2 except that in the present situation there is no error term $E(t,x)$, and no scaling parameter. Moreover, to replace (B.11), we use the following estimate

$$
\| \Phi_2 \partial_x \tilde{U} - \Phi_1 \partial_x (\partial_x^2 \tilde{U} - \tilde{U} + \hat{U}^4) \|_{L^2} \leq Ce^{-\left(\frac{1}{2} + \rho\right)\gamma_0}.
$$

Note that (B.32) is a standard coercivity property, see Claim A.1.

Finally, we prove (B.33). On $[T^*, T_1]$, we have from (B.29) (see also proof of Lemma 3.1)

$$
\frac{d}{dt} \int \tilde{e}(t) Q_{1+\tilde{\mu}_j(t)}(x - \tilde{y}_j(t)) \leq C|\ln \mu_0|\mu_0^2 \|\tilde{e}(t)\|_{H^1} + C\|\tilde{e}(t)\|_{H^1}^2
$$

$$
\leq CC^*\omega \mu_0^2 (|\ln \mu_0|\mu_0 + C^*|\ln \mu_0|^{-2}),
$$

since $0 < \omega < |\ln \mu_0|^{-2}$. Integrating on $(t, T_1)$, for $t \in [T^*, T_1]$, using $0 < T < C|\ln(\mu_0)\mu_0^{-1}$, we find

$$
\int \tilde{e}(t) Q_{1+\tilde{\mu}_j(t)}(x - \tilde{y}_j(t)) \leq C\omega \mu_0 + CC^*\omega \mu_0 (|\ln \mu_0|\mu_0^2 + C^*|\ln \mu_0|^{-1}) \leq 2C\omega \mu_0,
$$

for $\mu_0$ small enough depending on $C^*$.

References

[1] J.L. Bona, W.G. Pritchard and L.R. Scott, Solitary-wave interaction, Phys. Fluids 23, 438, (1980).

[2] J. L. Bona, P. E. Souganidis and W. A. Strauss, Stability and instability of solitary waves of Korteweg-de Vries type, Proc. Roy. Soc. London Ser. A 411 (1987), 395–412.

[3] A. Cohen, Existence and regularity for solutions of the Korteweg–de Vries equation, Arch. Rat. Mech. Anal. 71 (1979), 143–175.

[4] W. Craig, P. Guyenne, J. Hammack, D. Henderson and C. Sulem, Solitary wave interactions. Phys. Fluids 18, (2006), 057106.

[5] W. Eckhaus and P. Schuur, The emergence of solutions of the Korteweg–de Vries equation from arbitrary initial conditions, Math. Meth. Appl. Sci., 5, (1983) 97–116.

[6] S.-I. Ei and T. Ohta, Equation of motion for interacting pulses, Physical Review E, 50 (1994), 4672–4678.

[7] E. Fermi, J. Pasta and S. Ulam, Studies of nonlinear problems, I, Los Alamos Report LA1940 (1955); reproduced in Nonlinear Wave Motion, A.C. Newell, ed., American Mathematical Society, Providence, R. I., 1974, pp. 143–156.

[8] J. Hammack, D. Henderson, P. Guyenne and Ming Yi, Solitary-wave collisions, in Proceedings of the 23rd ASME Offshore Mechanics and Artic Engineering (A symposium to honor Theodore Yao-Tsu Wu), Vancouver, Canada, June 2004 (World Scientific, Singapore, 2004).
[9] N. Hayaski and P. Naumkin, Large time asymptotics of solutions to the generalized Korteweg-de Vries equation, J. Funct. Anal. 159 (1998), 110–136.

[10] N. Hayaski and P. Naumkin, On the modified Korteweg-de Vries equation, Math. Phys. Anal. Geom. 4 (2001), 197–201.

[11] R. Hirota, Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, Phys. Rev. Lett., 27 (1971), 1192–1194.

[12] H. Kalisch and J.L. Bona, Models for internal waves in deep water, Discrete and Continuous Dynamical Systems, 6 (2000), 1–20.

[13] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation. Studies in applied mathematics, 93–128, Adv. Math. Suppl. Stud., 8, Academic Press, New York, 1983.

[14] C.E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46, (1993) 527–620.

[15] M. D. Kruskal, The Korteweg-de Vries equation and related evolution equations, in Nonlinear Wave Motion, A.C. Newell, ed., American Mathematical Society, Providence, R. I., 1974, pp. 61–83.

[16] C. Laurent and Y. Martel, Smoothness and exponential decay of $L^2$-compact solutions of the generalized KdV equations. Comm. Partial Differential Equations 28 (2003), 2093–2107.

[17] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. 21, (1968) 467–490.

[18] R. LeVeque, On the interaction of nearly equal solitons in the KdV equations, SIAM J. Appl. Math. 47 (1987), 254–262.

[19] J. H. Maddocks and R. L. Sachs, On the stability of KdV multi-solitons, Comm. Pure Appl. Math. 46 (1993), 867–901.

[20] Y. Martel, Asymptotic $N$–soliton–like solutions of the subcritical and critical generalized Korteweg–de Vries equations, Amer. J. Math. 127 (2005), 1103-1140.

[21] Y. Martel, Linear problems related to asymptotic stability of solitons of the generalized KdV equations, SIAM J. Math. Anal. 38 (2006), 759–781.

[22] Y. Martel and F. Merle, Asymptotic stability of solitons for subcritical generalized KdV equations, Arch. Ration. Mech. Anal. 157, (2001) 219–254.

[23] Y. Martel and F. Merle, Stability of blow-up profile and lower bounds for blow-up rate for the critical generalized KdV equation. Ann. of Math. 155 (2002), 235–280.

[24] Y. Martel and F. Merle, Refined asymptotics around solitons for the gKdV equations with a general nonlinearity, Discrete Contin. Dyn. Syst. 20 (2008), 177–218.
[25] Y. Martel and F. Merle, Stability of two soliton collision for nonintegrable gKdV equations, Comm. Math. Phys. 286 (2009), 39–79.

[26] Y. Martel and F. Merle, Description of two soliton collision for the quartic gKdV equation, submitted preprint. [http://arxiv.org/abs/0709.2672](http://arxiv.org/abs/0709.2672)

[27] Y. Martel and F. Merle, Inelastic interaction of nearly equal solitons for the BBM equation, preprint.

[28] Y. Martel, F. Merle and T. Mizumachi, Description of the inelastic collision of two solitary waves for the BBM equation. To appear in Arch. Rat. Mech. Anal.

[29] Y. Martel, F. Merle and Tai-Peng Tsai, Stability and asymptotic stability in the energy space of the sum of $N$ solitons for the subcritical gKdV equations, Commun. Math. Phys. 231, (2002) 347–373.

[30] R.M. Miura, The Korteweg–de Vries equation: a survey of results, SIAM Review 18, (1976) 412–459.

[31] T. Mizumachi, Weak interaction between solitary waves of the generalized KdV equations, SIAM J. Math. Anal. 35 (2003), 1042–1080.

[32] R. L. Pego and M. I. Weinstein, Asymptotic stability of solitary waves, Commun. Math. Phys., 164 (1994) 305–349.

[33] T. Tao, Scattering for the quartic generalised Korteweg-de Vries equation, J. Diff. Eqs. 232 (2007), 623–651.

[34] P. C. Schuur, Asymptotic analysis of solitons problems, Lecture Notes in Math. 1232 (1986), Springer-Verlag, Berlin.

[35] L.Y. Shih, Soliton–like interaction governed by the generalized Korteweg-de Vries equation, Wave motion 2 (1980), 197–206.

[36] M. Wadati and M. Toda, The exact $N$–soliton solution of the Korteweg–de Vries equation, J. Phys. Soc. Japan 32, (1972) 1403–1411.

[37] P.D. Weidman and T. Maxworthy, Experiments on strong interactions between solitary waves, J. Fluids Mech. 85, (1978) 417–431.

[38] M.I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Comm. Pure Appl. Math. 39, (1986) 51–68.

[39] N.J. Zabusky, Solitons and energy transport in nonlinear lattices, Computer Physics Communications, 5 (1973), 1–10.

[40] N.J. Zabusky and M.D. Kruskal, Interaction of “solitons” in a collisionless plasma and recurrence of initial states, Phys. Rev. Lett. 15 (1965), 240–243.