Portfolio Optimization managing Value at Risk under heavy tail distribution

Subhojit Biswas, Mrinal K.Ghosh and Diganta Mukherjee

August 13, 2019

Abstract

We consider an investor, whose portfolio consists of a single risky asset and a risk free asset, who wants to maximize his expected utility of the portfolio subject to managing the Value at Risk (VaR) assuming a heavy tailed distribution of the stock prices return. We use a stochastic maximum principle to formulate the dynamic optimisation problem. The equations which we obtain does not have any explicit analytical solution, so we look for accurate approximations to estimate the value function and optimal strategy. As our calibration strategy is non-parametric in nature, no prior knowledge on the form of the distribution function is needed. We also provide detailed empirical illustration using real life data. Our results show close concordance with financial intuition. We expect that our results will add to the arsenal of the high frequency traders.

Keywords: Portfolio Optimization, Hamiltonian system, Heavy tailed distribution, Stochastic maximum principle

AMS Classification: 91G10, 91G80

1 Introduction

1.1 Background and Motivation

Risk management occurs everywhere in the financial world. There are lot of places where risk managements are done such as it occurs when an investor buys low-risk government bonds over riskier corporate bonds, bank performing a credit check on an individual before issuing a personal line of credit, stockbrokers buying assets like options & futures in their portfolio and money managers using strategies like portfolio and investment diversification to mitigate or effectively manage risk. Inadequate risk management can result in severe consequences such as the sub prime mortgage meltdown in 2007 that helped trigger the Great Recession that stemmed from poor risk-management decisions. In the financial world the performance of the portfolio is associated with the risk and portfolio management is primarily risk management. A common definition of investment risk is a deviation from an expected outcome, which we can benchmark with the market parameters. The deviation can be positive or...
negative. How Do Investors Measure Risk? Investors use a variety of tactics to ascertain risk. One of the most
commonly used risk metrics is Value at Risk (VaR), a statistical measure of the riskiness of financial entities or
portfolios of assets. It is defined as the maximum dollar amount expected to be lost over a given time horizon, at a
pre-defined confidence level. There are also other risk measure metric used in the market such as Sharpe Ratio or
Expected Shortfall (ES). Our main focus in this paper will be Value at Risk (VaR).

Our aim is to find a strategy for the investor such that the VaR at a certain quantile level is manages, i.e. kept
above a critical level with a high probability. There are lot of constraints which are needed to be addressed in a
portfolio. Some of them are the target return from the portfolio and the transaction cost. Here both the factors
are addressed in the portfolio optimization problem and how the investor’s are going to follow a recursive optimal
policy so that the VaR is managed at the same time at a desired level along with optimizing the expected return.
The return of the assets are considered to follow a heavy tail distribution whose higher order moments do not exist
and in this paper we have proposed a continuous time dynamic framework for the investor on how to handle the
heavy tail distribution for a known or an unknown distribution function of return. Our proposal does not require
knowledge about the exact form of the distribution. We take recourse to non-parametric calibration techniques to
handle general unknown distribution functions.

1.2 Literature Review

Interest rate risk immunization is one of the key concerns for fixed income portfolio management. In recent years,
risk measures (e.g. value-at-risk and conditional value-at-risk) as tools for the formation of an optimum investment
portfolio has gained traction. The article by Mato [5] aims to discuss this issue. The work by Agarwal & Sircar [3]
which gives an idea of portfolio optimization under drawdown constraint and stochastic Sharpe ratio tells us how
the stochastic differential equation of the asset return can be converted in to the stochastic differential equation
of the quantile. Fotios [6] aims to test empirically the performance of different models in measuring VaR and
ES in the presence of heavy tails in returns using historical data. Daily returns are modelled with empirical (or
historical), Gaussian, Generalized Pareto (peak over threshold (POT) technique of extreme value theory (EVT)).
Assessing financial risk and portfolio optimization using a multivariate market model with returns assumed to fol-
low a multivariate normal tempered stable distribution (i.e. this distribution is a mixture of the multivariate normal
distribution and the tempered stable subordinator) can be seen in the paper of Young Shin Kim [7]. Several authors
have considered the optimal portfolio problem under drawdown constraint. The first to comprehensively study this
problem over infinite time horizon in a market setting with single risky asset modelled as a geometric Brownian
motion with constant volatility (log normal model) was [8]. Dynamic programming was used to solve the maxi-
mization problem of the long term growth rate of the expected utility of the wealth. [9] streamlined the analysis of
[8] and extended the results to the case when there are multiple risky assets. The paper by Samuelson [10] gives an
idea of portfolio selection by stochastic dynamic programming. Finally, very relevant for our empirical analysis,
we mention the paper by Sahalia [12] who gives an idea of using a non parametric estimator for the State Price
Densities implicit in option prices.

1.3 Our Contribution

In this article the investor is worried about when to build up on stocks or liquidate the stock when dealing with
heavy tail distribution of the return of the stock prices and tries to optimize the portfolio based on Value at Risk
(VaR). The investor’s portfolio has one risky asset and a risk free asset. We consider the quantiles of the heavy
tailed distribution which are asymptotically jointly multivariate normal; allowing us to formulate the stochastic
differential equation for the quantiles. Then we use a stochastic maximum principle to formulate the dynamic optimisation problem (Yong & Zhou [13]). The equations which we obtain does not have any explicit analytical solution, so we look for accurate approximations to estimate the value function and optimal strategy. As our calibration strategy is non-parametric in nature, no prior knowledge on the form of the distribution function is needed.

1.4 Organization of the paper

In section 2 we considered the quantiles of the heavy tailed distribution which are asymptotically multivariate normal with a very simple co-variance structure and a mean. The stochastic differential equation was derived for the quantiles and Hamilton-Jacobi-Bellman equation for the optimal portfolio problem under certain assumptions and present the analytical formula for the optimal portfolio strategy in terms of the value function. Examples and numerical results are included in the subsections of section 3. Finally section 4 concludes.

2 Formulation and Analysis

We assume the existence of a friction-less financial market. In our portfolio we consider a risky asset denoted by S and a risk-free asset, such as bank account, providing a risk-free rate of interest given by a scalar constant $r > 0$. Let the return of the risky asset at a time instant be given by $dX_t = \frac{dS_t}{S_t}$ where $X_t$ follows a heavy tail distribution with population c.d.f. $F(x)$, which is assumed continuous and differentiable to at least second order. Heavy tailness of the distribution does not allow us to formulate the linear stochastic differential equation for the return of the stock. Instead, we will focus on two quantiles $X_{(p_1)}$ and $X_{(p_2)}$, for $0 < p_1 < p_2 < 1$, and take recourse to some usual asymptotics using the following proposition from Beach [1].

Let a random sample of size N be given from this population and let the observations be ordered by size from the smallest $(X_{(1)})$ to the largest $(X_{(N)})$ so that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(N)}$. Then let the sample quantile $\xi_{(p)}$ be defined as the $r$-th order statistic, $X_{(r)}$, where $r = \lfloor Np \rfloor$ denotes the greatest integer less than or equal to $Np$. If $F$ is strictly monotonic, $\xi_{(p)}$ has the property of strong or almost sure consistency [2]. Note that this result does not require the existence of moments for $F(x)$, which is often the case for heavy tailed distributions.

**Proposition:** If $F$ is differentiable at $\xi_{p}$ for $p \in \{p_1, p_2\}$ with density $f(\xi_{(p_1)}) = f_1$ and $f(\xi_{(p_2)}) = f_2$, then $\xi_{(p)}$’s are asymptotically multivariate normal with a simple co-variance structure. From Lemma 1 of [1] we can write this for two quantiles as

$$\Lambda = \begin{bmatrix} p_1(1-p_1) & p_2(1-p_2) \\ p_1(1-p_1) & p_2(1-p_2) \end{bmatrix} \begin{bmatrix} f_1 f_2 \\ f_1 f_2 \end{bmatrix},$$

On the basis of asymptotic normality, the equation of motion for the quantiles can be written as

$$\begin{bmatrix} dX_{(p_1)} \\ dX_{(p_2)} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \begin{bmatrix} p_1(1-p_1) & p_2(1-p_2) \\ p_1(1-p_1) & p_2(1-p_2) \end{bmatrix} \begin{bmatrix} dB_{t}^{(1)} \\ dB_{t}^{(2)} \end{bmatrix},$$

where $dB_{t}^{(1)}$ and $dB_{t}^{(2)}$ are two Brownian motions related as,

$$dB_{t}^{(2)} = \rho dB_{t}^{(1)} + \sqrt{1-\rho^2} dB_{t}^{(3)},$$

with

$$dY_{t} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \begin{bmatrix} p_1(1-p_1) & p_2(1-p_2) \\ p_1(1-p_1) & p_2(1-p_2) \end{bmatrix} \begin{bmatrix} dB_{t}^{(1)} \\ dB_{t}^{(2)} \end{bmatrix}.$$
where \( \rho = \frac{\frac{p_1(1-p_2)}{f_1 f_2}}{\frac{p_1(1-p_1)}{f_1} \frac{p_2(1-p_2)}{f_2}} = \frac{\sqrt{p_1} \sqrt{1-p_2}}{\sqrt{p_2} \sqrt{1-p_1}} \)

For reducing the number of parameters, we subtract the expected value of one of the quantiles from the data. Thus, the expected value of one of the quantile becomes zero and the values of others are actually relative to this one. Applying this methodology the equation of motion now becomes

\[
\begin{bmatrix}
    dX_{(p_1)} \\
    dX_{(p_2)}
\end{bmatrix} =
\begin{bmatrix}
    0 & \frac{p_1(1-p_1)}{f_1 f_2} & \frac{p_1(1-p_2)}{f_1 f_2} \\
    \mu_2 & \frac{p_1(1-p_1)}{f_1 f_2} & \frac{p_1(1-p_2)}{f_1 f_2}
\end{bmatrix}
\begin{bmatrix}
    dB_i^{(1)} \\
    dB_i^{(2)}
\end{bmatrix}.
\]

Simplifying notation, we denote \( X_{(p_1)} \) by \( X_1 \) and \( X_{(p_2)} \) by \( X_2 \) to write

\[
\begin{bmatrix}
    dX_1 \\
    dX_2
\end{bmatrix} =
\begin{bmatrix}
    b_1 & \frac{p_1(1-p_1)}{f_1 f_2} & \frac{p_1(1-p_2)}{f_1 f_2} \\
    b_2 & \frac{p_1(1-p_1)}{f_1 f_2} & \frac{p_1(1-p_2)}{f_1 f_2}
\end{bmatrix}
\begin{bmatrix}
    dB_i^{(1)} \\
    dB_i^{(2)}
\end{bmatrix}.
\]

where \( b_1 = 0 \) and \( b_2 = \mu_2 \). So finally

\[
dX_1 = \frac{p_1(1-p_1)}{f_1^2} dB_i^{(1)} + \frac{p_1(1-p_2)}{f_1 f_2} dB_i^{(2)}
\]
\[
dX_2 = b_2 dt + \frac{p_1(1-p_2)}{f_1 f_2} dB_i^{(1)} + \frac{p_2(1-p_2)}{f_2^2} dB_i^{(2)}.
\]

We denote the wealth process of an investor by \( L \) who invests \( \pi_t \) portion of it in risky asset and the remaining in the bank which is a risk-free asset.

\[
\begin{align*}
    dL_t &= L_t (1 - \pi_t) dt + L_t \pi_t dX_1 \\
    \text{or, } dL_t &= L_t (1 - \pi_t) dt + L_t \pi_t \left( \frac{p_1(1-p_1)}{f_1^2} dB_i^{(1)} + \frac{p_1(1-p_2)}{f_1 f_2} dB_i^{(2)} \right) \\
    \text{or, } dL_t &= L_t (1 - \pi_t) dt + L_t \pi_t \frac{p_1(1-p_1)}{f_1^2} dB_i^{(1)} + L_t \pi_t \frac{p_1(1-p_2)}{f_1 f_2} dB_i^{(2)}.
\end{align*}
\]

In this work, we propose an investment framework that encourages managing the Value at Risk, while maximizing the median value of the utility function \( U \) satisfying:

**Assumption 1.** The terminal utility function \( U : (0, 1) \rightarrow R \) is smooth, strictly increasing and strictly concave.

In particular, we specialise to the constant relative risk aversion utility function \( U(x) = \frac{x}{\gamma} \) for explicit exposition of the derivation of our results. For the objective function, we take the usual time discounted aggregate utility. Then the objective function which needs to be maximized may be written as

\[
\begin{align*}
    \text{maximize } & \mathbb{E} \left( \int_0^T e^{-\beta t} \frac{L_t^Y}{Y} dt + L_T \right) \\
    \text{subject to } & \Pr \left( L_t (r - r \pi_t + b_2 \pi_t) dt + L_t \pi_t \frac{p_1(1-p_2)}{f_1 f_2} dB_i^{(1)} + L_t \pi_t \frac{p_2(1-p_2)}{f_2^2} dB_i^{(2)} \geq Q_{0.05} \right) \geq 0.95
\end{align*}
\]

where \( \beta > 0 \) is the rate of discount over time, \( \gamma \in (0, 1) \) is the risk aversion parameter, \( p_1 = 0.05 \), \( p_2 = 0.5 \) and
since it is continuous we have considered the equality constraint.

The above optimization problem with state constraint can be solved using Maximum principle and Stochastic Hamiltonian system. Using the notation from Yong & Zhou [13]. Equation (1) represents the state equation and equation (2) represents the objective utility function and the state constraint. If we use the notation then we can define the Hamiltonian as,

\[ H(t, L_t, \pi_t, s, q, w, \psi^0, \psi) := -\psi^0 f(t, L_t, \pi_t) - <\psi, f_1(t, L_t, \pi_t)> + <s, b(t, L_t, \pi_t)> + q\sigma_1(t, L_t, \pi_t) + w\sigma_2(t, L_t, \pi_t) \]

The constraint can be written as

\[ \mathbb{E} \int_0^T \frac{1}{T} \mathbb{I} \left( L_t(r-r\pi_t + b_2\pi_t)dt + L_t\pi_t p_1(1-p_2) dB_t^{(1)} + L_t\pi_t p_2(1-p_2) dB_t^{(2)} \geq Q_{0.05} \right) dt \]

Since it is not continuous, we approximate it to a continuous and differentiable function with the help of the Sigmoid function where the \( \alpha \) needs to be selected in such a way that it is very very close to the indicator function, Let us consider,

\[ d_t = \left( L_t(r-r\pi_t + b_2\pi_t)dt + L_t\pi_t p_1(1-p_2) dB_t^{(1)} + L_t\pi_t p_2(1-p_2) dB_t^{(2)} \right) \]

\[ f_1(t, L_t, \pi_t) = \begin{cases} 
0 & d_t \leq Q_{0.05} - \epsilon \\
\frac{1}{1+e^{-\alpha(d_t-Q_{0.05})}} & Q_{0.05} - \epsilon \leq d_t \leq Q_{0.05} + \epsilon \\
1 & d_t \geq Q_{0.05} - \epsilon 
\end{cases} \]

We can show that \( f_1(t, L_t, \pi_t) \) have the same probability as the actual function,

\[ \int_{-\epsilon}^{\epsilon} \frac{1}{1+e^{-\alpha(d_t-Q_{0.05})}} dd_t \leq \int_{-\epsilon}^{\epsilon} dd_t \]

As,

\[ \frac{1}{1+e^{-\alpha(d_t-Q_{0.05})}} \leq 1 \]

Also,

\[ \int_{-\epsilon}^{\epsilon} dd_t = 2\epsilon \]

\[ \int_{-\epsilon}^{\epsilon} \frac{1}{1+e^{-\alpha(d_t-Q_{0.05})}} dd_t \leq 2\epsilon \]

As \( \epsilon \) may be chosen to be arbitrarily small, we can say that

\[ \int_{-\epsilon}^{\epsilon} \frac{1}{1+e^{-\alpha(d_t-Q_{0.05})}} dd_t \approx 0. \]

Now defining the expressions used in equation (3) with our state parameters, we see that

\[ b(t, L_t, \pi_t) = L_t(1-\pi_t)r, \]

\[ \sigma_1(t, L_t, \pi_t) = L_t\pi_t \frac{p_1(1-p_1)}{f_1^2}, \]
We can represent this in terms of an indicator function as,

$$\sigma_2(t, L_t, \pi_t) = L_t \pi_t \frac{p_1(1 - p_2)}{f_1 f_2},$$

and $f(t, L_t, \pi_t) = e^{-\beta_1 \frac{L_t^T}{Y}}$.

Thus, collecting all terms, we get

$$H(t, L_t, \pi_t, s, q, w, \psi^0, \psi) = -\psi_0 e^{-\beta_1 \frac{L_t^T}{Y}} - \psi f_1(t, L_t, \pi_t) + s L_t (1 - \pi_t) r + q L_t \pi_t \frac{p_1(1 - p_1)}{f_1^2} + w L_t \pi_t \frac{p_1(1 - p_2)}{f_1 f_2}$$

First order derivative $f'_1(t, L_t, \pi_t)$ is given by

$$f'_1(t, L_t, \pi_t) = \frac{1}{T} \begin{cases} 
0 & d_t \leq Q_{0.05} - \varepsilon \\
\alpha \left( \frac{d_t}{L_t} \right) e^{-\alpha (d_t - Q_{0.05})} & Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \\
0 & d_t \geq Q_{0.05} - \varepsilon 
\end{cases}$$

We can represent this in terms of an indicator function as,

$$f'_1(t, L_t, \pi_t) = \frac{1}{T} \alpha \left( \frac{d_t}{L_t} \right) e^{-\alpha (d_t - Q_{0.05})} \left( 1 + e^{-\alpha (d_t - Q_{0.05})} \right)^{-2} \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right)$$

or,

$$f'_1(t, L_t, \pi_t) = \frac{1}{T} \alpha \left( \frac{d_t}{L_t} \right) e^{-\alpha (d_t - Q_{0.05})} \left( 1 + 2e^{-\alpha (d_t - Q_{0.05})} \right)^{-2} \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right)$$

Using binomial approximation, we rewrite this as

$$f'_1(t, L_t, \pi_t) = \frac{1}{T} \alpha \left( \frac{d_t}{L_t} \right) e^{-\alpha (d_t - Q_{0.05})} \left( 1 - 2e^{-\alpha (d_t - Q_{0.05})} \right) \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right)$$

Thus, $(L_t, \pi_t)$ are the optimal points for the constrained problem.

From the Theorem 6.1 of Yong & Zhou [13] we can state that for an optimal solution

$$\exists (\psi^0, \psi) \text{ s.t. } \psi^0 \geq 0, |\psi^0|^2 + |\psi|^2 = 1$$

$$< \psi, Z > + \int_0^T \mathbb{E} f_1(t, L_t, \pi_t) \geq 0, \forall Z \in [0.95, 1] \text{ and } (s(.), q(.), w(.)), (S(.), Q(.), W(.)) \text{ satisfying the adjoint equations},$$

$$ds(t) = -H_{L_t}(t, L_t, \pi_t, s(t), q(t), w(t), \psi^0, \psi) dt + q(t)d_{B^1_t} + w(t)d_{B^2_t}$$

Let us consider,

$$k_1 = \left\{ \frac{1}{T} \alpha \left( \frac{d_t}{L_t} \right) e^{-\alpha (d_t - Q_{0.05})} \left( 1 - 2e^{-\alpha (d_t - Q_{0.05})} \right) \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right) \right\}.$$
Again using approximations,
\[
k_1 = \left\{ \frac{1}{T} \alpha \left( \frac{d_t}{L_t} \right) \left( 1 - \alpha (d_t - Q_{0.05}) \right) \left( 2 \alpha (d_t - Q_{0.05}) - 1 \right) \| \left( Q_{0.05} - \epsilon \leq d_t \leq Q_{0.05} + \epsilon \right) \right\},
\]

or \[
k_1 = \left\{ \frac{1}{T} \alpha \left( \frac{d_t}{L_t} \right) \left( -1 + \alpha (d_t - Q_{0.05}) - 2 \alpha^2 (d_t - Q_{0.05})^2 \right) \| \left( Q_{0.05} - \epsilon \leq d_t \leq Q_{0.05} + \epsilon \right) \right\},
\]

or \[
k_1 = \left\{ \frac{\alpha}{T L_t} \left( -d_t + \alpha (d_t^2 - Q_{0.05} d_t) - 2 \alpha^2 (d_t^3 - 2 d_t^2 Q_{0.05} + d_t Q_{0.05}^2) \right) \| \left( Q_{0.05} - \epsilon \leq d_t \leq Q_{0.05} + \epsilon \right) \right\}
\]

and finally \[
k_1 = \left\{ \frac{\alpha}{T L_t} \left( -(1 + \alpha Q_{0.05} + 2 \alpha^2 Q_{0.05}^2) \left( d_t + d_t^2 (\alpha + 4 \alpha^2 Q_{0.05}) - 2 \alpha^2 d_t^3 \right) \right) \| \left( Q_{0.05} - \epsilon \leq d_t \leq Q_{0.05} + \epsilon \right) \right\},
\]

where \[
ds(t) = \left\{ \psi^0 e^{-\beta t} L_t \gamma - 1 + \psi k_1 - s(t)(1 - \pi_t) r - q(t) \pi_t \frac{p_1(1 - p_1)}{f_1^2} - w(t) \pi_t \frac{p_1(1 - p_2)}{f_1 f_2} \right\} dt + q(t) dB_t^{(1)} + w(t) dB_t^{(2)} \right\}
\]

and \(s(T) = 0\).

Taking \(q(t) = 0\), \(w(t) = 0\) we have
\[
ds(t) = \left\{ \psi^0 e^{-\beta t} L_t \gamma - 1 + \psi k_1 - s(t)(1 - \pi_t) r \right\} dt
\]

or \[
ds(t) + s(t)(1 - \pi_t) r = \psi k_1 + \psi^0 e^{-\beta t} L_t \gamma - 1
\]

Integrating we have,
\[
s(t) = \frac{\psi k_1 + \psi^0 e^{-\beta t} L_t \gamma - 1}{(1 - \pi_t) r} + c_1 e^{-(1 - \pi_t) r t}.
\]

Putting the terminal constraint we get,
\[
s(t) = \frac{\psi k_1 + \psi^0 e^{-\beta t} L_t \gamma - 1}{(1 - \pi_t) r} (1 - e^{-(1 - \pi_t) r t}.
\]

Again, second order derivative \(f''_1(t, L_t, \pi_t)\) is given by
We can also represent this in terms of indicator function as
\[
f_f^n(t, L_t, \pi_t) = \begin{cases} 
0 & d_t \leq Q_{0.05} - \varepsilon \\
\frac{\left(\frac{\alpha_d}{L_t}\right)^2 e^{-\alpha(d_t - Q_{0.05})}}{1 + e^{-\alpha(d_t - Q_{0.05})}} - 2 \left(\frac{\alpha_{d_t}}{T}ight) e^{-\alpha(d_t - Q_{0.05})} & Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \\
0 & d_t \geq Q_{0.05} - \varepsilon 
\end{cases}
\]

Doing the usual approximations,
\[
k_2 = \frac{1}{T} \left[ \left(\frac{\alpha_d}{L_t}\right)^2 e^{-\alpha(d_t - Q_{0.05})} - 2 \left(\frac{\alpha_d}{L_t}\right) e^{-\alpha(d_t - Q_{0.05})} \right] \mathbb{I} \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right)
\]

Or, \( k_2 \approx \frac{1}{T} \left[ \left(\frac{\alpha_d}{L_t}\right) k_1 \left(1 - e^{-\alpha(d_t - Q_{0.05})}\right) \left(1 - e^{-\alpha(d_t - Q_{0.05})}\right) \right] \mathbb{I} \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right)

\[
k_2 = \frac{1}{T} \left[ \left(\frac{\alpha_d}{L_t}\right) k_1 \left(1 - e^{-\alpha(d_t - Q_{0.05})}\right)^2 \right] \mathbb{I} \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right)
\]

Doing the usual approximations,
\[
k_2 = \frac{1}{T} \left[ \left(\frac{\alpha_d}{L_t}\right) k_1 \left(\alpha(d_t - Q_{0.05})\right)^2 \right] \mathbb{I} \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right)
\]

\[
k_2 = \frac{\alpha}{T L_t} \left[ d_t k_1 \left(\alpha^2(d_t^2 - 2d_tQ_{0.05} + Q_{0.05}^2)\right) \right] \mathbb{I} \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right)
\]

\[
k_2 = \frac{\alpha}{T L_t} \left[ k_1 \left(\alpha^2(d_t^2 - 2d_t^2Q_{0.05} + d_t^2Q_{0.05}^2)\right) \right] \mathbb{I} \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right)
\]
Clearly the adapted solution to equations (4) and (5) is given by the following pairs:

\[ S(t) = \frac{\psi k_2 + \psi^0 e^{-\beta t} (\gamma - 1) L^2}{2(1 - \bar{\pi}_t) r + (\bar{\pi}_t \frac{p_1(1-p_1)}{f_1^2})^2 + (\bar{\pi}_t \frac{p_1(1-p_2)}{f_1 f_2})^2} \left( 1 - e^{-\left( \frac{2(1-\bar{\pi}_t)r + (\bar{\pi}_t \frac{p_1(1-p_1)}{f_1^2})^2 + (\bar{\pi}_t \frac{p_1(1-p_2)}{f_1 f_2})^2}{f_1^2} \right)} \right) \]

Clearly the adapted solution to equations (4) and (5) is given by the following pairs:
\[(s(t), q(t), w(t)) = \left(\frac{\psi k_1 + \psi_0 e^{-\beta t} L_r^{\gamma-1}}{(1 - \pi_r) r} (1 - e^{-(1 - \pi_r) r}), 0, 0\right)\]

and

\[(S(t), Q(t), W(t)) = \left(\frac{\psi k_2 + \psi_0 e^{-\beta t} (\gamma - 1) L_r^{\gamma-2}}{(2(1 - \pi_r) r + (\pi_r F_1^{(1-p_1)})^2 + (\pi_r F_1^{(1-p_2)})^2)} \left(1 - e^{-\left(2(1 - \pi_r) r + (\pi_r F_1^{(1-p_1)})^2 + (\pi_r F_1^{(1-p_2)})^2\right)}\right), 0, 0\right).\]

Define,

\[H(t, L_r, \pi_r) = G(t, L_r, \pi_r, s(t), s(t)) + \sigma_1(t, L_r, \pi_r)[q(t) - S(t)\sigma_1(t, L_r, \pi_r)] + \sigma_2(t, L_r, \pi_r)[w(t) - S(t)\sigma_2(t, L_r, \pi_r)]\]

where,

\[G(t, L_r, \pi_r, s(t), S(t)) = \frac{1}{2} \sigma_1^2(t, L_r, \pi_r)S(t) + \frac{1}{2} \sigma_2^2(t, L_r, \pi_r)S(t) - \psi_0 f(t, L_r, \pi_r) - \psi f_1(t, L_r, \pi_r) > < s, b(t, L_r, \pi_r) >\]

or

\[G(t, L_r, \pi_r, s(t), S(t)) = -\psi_0 e^{-\beta t} \frac{\pi_r^\gamma}{\gamma} - \psi f_1(t, L_r, \pi_r) + s(t)L_r(1 - \pi_r)r + \frac{1}{2} S(t) \left(\frac{L_r \pi_r P_1(1 - p_1)}{f_1^2} + \frac{L_r \pi_r P_1(1 - p_2)}{f_1 f_2}\right)\]

Thus,

\[H(t, L_r, \pi_r) = \psi_0 e^{-\beta t} \frac{L_r^\gamma}{\gamma} - \psi f_1(t, L_r, \pi_r) + \psi k_1 + \psi_0 e^{-\beta t} L_r^{\gamma-1} \frac{1}{(1 - \pi_r) r} \left(1 - e^{-\left(2(1 - \pi_r) r + (\pi_r F_1^{(1-p_1)})^2 + (\pi_r F_1^{(1-p_2)})^2\right)}\right) \times \]

\[\left(\frac{L_r \pi_r P_1(1 - p_1)}{f_1^2} + \frac{L_r \pi_r P_1(1 - p_2)}{f_1 f_2}\right) - \frac{L_r \pi_r P_1(1 - p_1)}{f_1^2} + \frac{\psi k_2 + \psi_0 e^{-\beta t} \frac{\pi_r^\gamma}{\gamma}}{(2(1 - \pi_r) r + (\pi_r F_1^{(1-p_1)})^2 + (\pi_r F_1^{(1-p_2)})^2)} \left(1 - e^{-\left(2(1 - \pi_r) r + (\pi_r F_1^{(1-p_1)})^2 + (\pi_r F_1^{(1-p_2)})^2\right)}\right) \times \]

\[\frac{L_r \pi_r P_1(1 - p_1)}{f_1^2} - \frac{L_r \pi_r P_1(1 - p_2)}{f_1 f_2} \left(1 + \frac{\psi k_2 + \psi_0 e^{-\beta t} \frac{\pi_r^\gamma}{\gamma}}{(2(1 - \pi_r) r + (\pi_r F_1^{(1-p_1)})^2 + (\pi_r F_1^{(1-p_2)})^2)} \left(1 - e^{-\left(2(1 - \pi_r) r + (\pi_r F_1^{(1-p_1)})^2 + (\pi_r F_1^{(1-p_2)})^2\right)}\right) \frac{L_r \pi_r P_1(1 - p_2)}{f_1 f_2}\right)\]
or

\[ H(t, \vec{L}_t, \pi_t) = -\psi^0 e^{-\beta t} \tilde{L}_t^T - \psi f_1(t, L_t, \pi_t) + \frac{\psi k_1 + \psi^0 e^{-\beta t} \tilde{L}_t^{T - 1}}{(1 - \pi_t)} (1 - e^{-(1 - \pi_t) r}) L_t (1 - \pi_t) r + \]

\[ \frac{\psi k_2 + \psi^0 e^{-\beta t} (\gamma - 1) \tilde{L}_t^{T - 2}}{(2 (1 - \pi_t) r + (\pi_t P_1 L_t^{(1 - p_1)})^2 + (\pi_t P_1 L_t^{(1 - p_2)})^2)} \left( 1 - e^{-2 (1 - \pi_t) r + (\pi_t P_1 L_t^{(1 - p_1)})^2 + (\pi_t P_1 L_t^{(1 - p_2)})^2} \right)^t \times \]

\[ \left( \frac{1}{2} \left( L_t \pi_t P_1 (1 - p_1) \right) \right) + \frac{1}{2} \left( L_t \pi_t P_1 (1 - p_2) \right)^2 - L_t \pi_t p_1 (1 - p_1) L_t \pi_t p_1 (1 - p_2) - \]

\[ L_t \pi_t p_1 (1 - p_2) L_t \pi_t p_1 (1 - p_2) \]

For analytical convenience, we assume that the function \( H(t, L_t, \pi_t) \) is integrable for all \( \pi_t \) and is differentiable w.r.t. \( \pi_t \). Assume that there is a random variable \( Z \) such that \( \frac{dH(t, L_t, \pi_t)}{d\pi_t} \leq Z \) a.s for all \( \pi_t \) and \( \mathbb{E}(Z) < \infty \). Then using the exchange property for the expectation and the derivative, we can write,

\[ \mathbb{E} \left( \frac{dH(t, L_t, \pi_t)}{d\pi_t} \right) = \frac{d}{d\pi_t} \left( \mathbb{E}(H(t, L_t, \pi_t)) \right) \]

So,

\[ \mathbb{E}(H(t, L_t, \pi_t)) = -\psi^0 e^{-\beta t} \tilde{L}_t^{T - 1} \psi \times \operatorname{Pr} \left( L_t (r - r \pi_t + b_2 \pi_t) dt + L_t \pi_t p_1 (1 - p_2) f_1 f_2 \right) \mathbb{E}(L_t (1 - \pi_t) r = \)

\[ \frac{\mathbb{E}(\psi k_1 + \psi^0 e^{-\beta t} \tilde{L}_t^{T - 1})}{(1 - \pi_t) r} \left( (1 - e^{-(1 - \pi_t) r}) L_t (1 - \pi_t) r - \right) \]

\[ \frac{\mathbb{E}(\psi k_2 + \psi^0 e^{-\beta t} (\gamma - 1) \tilde{L}_t^{T - 2})}{(2 (1 - \pi_t) r + (\pi_t P_1 L_t^{(1 - p_1)})^2 + (\pi_t P_1 L_t^{(1 - p_2)})^2)} \left( 1 - e^{-2 (1 - \pi_t) r + (\pi_t P_1 L_t^{(1 - p_1)})^2 + (\pi_t P_1 L_t^{(1 - p_2)})^2} \right)^t \times \]

\[ \left( \frac{1}{2} \left( L_t \pi_t P_1 (1 - p_1) \right) \right) + \frac{1}{2} \left( L_t \pi_t P_1 (1 - p_2) \right)^2 - L_t \pi_t p_1 (1 - p_1) L_t \pi_t p_1 (1 - p_2) - \]

\[ L_t \pi_t p_1 (1 - p_2) L_t \pi_t p_1 (1 - p_2) \]

Now,

\[ \mathbb{E}(\psi k_1 + \psi^0 e^{-\beta t} \tilde{L}_t^{T - 1}) = \psi \mathbb{E}(k_1) + \psi^0 e^{-\beta t} \tilde{L}_t^{T - 1} \]

\[ k_1 = \left\{ \frac{\alpha}{T L_t} \left( - (1 + \alpha Q_{0.05} + 2 \alpha^2 Q_{0.05}) d_t + d_t^2 (\alpha + 4 \alpha^2 Q_{0.05}) - 2 \alpha^2 d_t^3 \right) \Pi \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right) \right\} \]

\[ \mathbb{E}(k_1) = \left\{ \frac{\alpha}{T L_t} \left( - (1 + \alpha Q_{0.05} + 2 \alpha^2 Q_{0.05}) \mathbb{E}(d_t) + \mathbb{E}(d_t^2) (\alpha + 4 \alpha^2 Q_{0.05}) - 2 \alpha^2 \mathbb{E}(d_t^3) \right) \right\} \]

\[ \mathbb{E}(d_t) = L_t (r - r \pi_t + b_2 \pi_t) dt \]

\[ \mathbb{E}(d_t^2) = L_t^2 \pi_t \left\{ \frac{p_1^2 (1 - p_2)^2}{f_1^2 f_2^2} + \frac{p_2^2 (1 - p_2)^2}{f_2^4} + \frac{p_1 p_2 (1 - p_2)^2}{f_1 f_2^3} \right\} \]
\[ \mathbb{E}(d_t^2) = 0 \]

\[
\mathbb{E}(k_1) = \left\{ \frac{\alpha}{T L_t} \left( - (1 + \alpha Q_{0.05} + 2\alpha^2 Q_{0.05}^2) (L_t - r \pi_t + b_2 \pi_t) dt + \left( \frac{p_1^2(1-p_2)^2}{f_1^2 f_2^2} + \frac{p_2^2(1-p_2)^2}{f_2^4} + \frac{p_1 p_2 (1-p_2)^2}{f_1 f_2^2} \rho \right) dt \right) (\alpha + 4\alpha^2 Q_{0.05}) \right\}
\]

\[
\mathbb{E}(\psi_k + \psi^0 e^{-\beta_t L_t \gamma^{-1}}) = \psi \left\{ \frac{\alpha}{T} \left( - (1 + \alpha Q_{0.05} + 2\alpha^2 Q_{0.05}^2) ((r - r \pi_t + b_2 \pi_t) dt + \left( \frac{p_1^2(1-p_2)^2}{f_1^2 f_2^2} + \frac{p_2^2(1-p_2)^2}{f_2^4} + \frac{p_1 p_2 (1-p_2)^2}{f_1 f_2^2} \rho \right) dt \right) (\alpha + 4\alpha^2 Q_{0.05}) \right\} + \psi^0 e^{-\beta_t L_t \gamma^{-1}}
\]

\[
\mathbb{E}(\psi_k + \psi^0 e^{-\beta_t (\gamma - 1) L_t \gamma^{-2}}) = \psi \mathbb{E}(k_2) + \psi^0 e^{-\beta_t (\gamma - 1) L_t \gamma^{-2}}
\]

\[
\mathbb{E}(k_2) = \frac{\alpha}{T L_t} \left[ \frac{\alpha^3}{T L_t^2} \left( - (1 + \alpha Q_{0.05} + 2\alpha^2 Q_{0.05}^2) (Q_{0.05}^2 \mathbb{E}(d_t^2)) \right) \right]
\]

\[
\mathbb{E}(\psi_k + \psi^0 e^{-\beta_t (\gamma - 1) L_t \gamma^{-2}}) = \psi \frac{\alpha^4}{T^2} \left[ - (1 + \alpha Q_{0.05} + 2\alpha^2 Q_{0.05}^2) (Q_{0.05}^2) \left( \frac{\pi_t^2}{f_1^2 f_2^2} \left( \frac{p_1^2(1-p_2)^2}{f_1^2 f_2^2} + \frac{p_2^2(1-p_2)^2}{f_2^4} + \frac{p_1 p_2 (1-p_2)^2}{f_1 f_2^2} \rho \right) dt \right) \right] + \psi^0 e^{-\beta_t (\gamma - 1) L_t \gamma^{-2}}
\]

Let us use the following short-hands for ease of exposition,

\[ g_t = (r - r \pi_t + b_2 \pi_t), \]

\[ g_1 = \left\{ \frac{p_1^2(1-p_2)^2}{f_1^2 f_2^2} + \frac{p_2^2(1-p_2)^2}{f_2^4} + \frac{p_1 p_2 (1-p_2)^2}{f_1 f_2^2} \rho \right\} \]

and \[ g_2 = -(1 + \alpha Q_{0.05} + 2\alpha^2 Q_{0.05}^2). \]

Now using the above definition, we can rewrite
\[\mathbb{E}(\psi k_1 + \psi^0 e^{-\beta t} L_t \gamma^{-1}) = \psi \left\{ \frac{\alpha}{T} \left( g_2 g_1 dt + \left( L_t \pi_t^2 g_1 dt \right) (\alpha + 4 \alpha^2 Q_{0.05}) \right) \right\} + \psi^0 e^{-\beta t} L_t \gamma^{-1}\]

And, \[\mathbb{E}(\psi k_2 + \psi^0 e^{-\beta t} (\gamma - 1)L_t \gamma^{-2}) = \psi \frac{\alpha^4}{T^2} \left[ g_2 Q_{0.05}^2 \left( \pi_t^2 g_1 dt \right) \right] + \psi^0 e^{-\beta t} (\gamma - 1)L_t \gamma^{-2}\]

\[\mathbb{E}(H(t, L_t, \pi_t)) = -\psi^0 e^{-\beta t} \frac{L_t \gamma}{\gamma} - \psi \times \text{Pr} \left( L_t g_1 dt + L_t \pi_t^2 \frac{p_1(1 - p_2)}{f_1 f_2} dB_1^{(1)} + L_t \pi_t^2 \frac{p_2(1 - p_2)}{f_2^2} dB_1^{(2)} \geq Q_{0.05} \right) + \left\{ \frac{\alpha}{T} \left( g_2 g_1 dt + \left( L_t \pi_t^2 g_1 dt \right) (\alpha + 4 \alpha^2 Q_{0.05}) \right) \right\} + \psi^0 e^{-\beta t} L_t \gamma^{-1}\]

\[\left( \frac{\psi \alpha^4}{T^2} g_2 Q_{0.05}^2 \left( \pi_t^2 g_1 dt \right) + \psi^0 e^{-\beta t} (\gamma - 1)L_t \gamma^{-2} \right) \left( 1 - e^{-\left( 2(1 - \pi_t) r + (\pi_t \frac{p_1(1 - p_2)}{f_1 f_2})^2 + (\pi_t \frac{p_1(1 - p_2)}{f_1 f_2})^2 \right)} \right) \times \left( \frac{1}{2} L_t \pi_t^2 \frac{p_1(1 - p_1)}{f_1^2} + \frac{1}{2} \left( L_t \pi_t^2 \frac{p_1(1 - p_2)}{f_1 f_2} \right)^2 - L_t \pi_t^2 \frac{p_1(1 - p_1)}{f_1^2} L_t \pi_t^2 \frac{p_1(1 - p_2)}{f_1 f_2} \right) \]

Differentiating it and equating it to zero we will get the optimal control \( \pi_t \),

\[\frac{d}{d\pi_t}(\mathbb{E}(H(t, L_t, \pi_t))) = -\psi \frac{d f_1(t, L_t, \pi_t)}{d\pi_t} + \left\{ \frac{\alpha}{T} \left( g_2 g_1 dt + \left( L_t \pi_t^2 g_1 dt \right) (\alpha + 4 \alpha^2 Q_{0.05}) \right) \right\} + \psi^0 e^{-\beta t} L_t \gamma^{-1}\]

\[\left( \frac{\psi \alpha^4}{T^2} g_2 Q_{0.05}^2 \left( \pi_t^2 g_1 dt \right) + \psi^0 e^{-\beta t} (\gamma - 1)L_t \gamma^{-2} \right) \left( 1 - e^{-\left( 2(1 - \pi_t) r + (\pi_t \frac{p_1(1 - p_2)}{f_1 f_2})^2 + (\pi_t \frac{p_1(1 - p_2)}{f_1 f_2})^2 \right)} \right) \times \left( \left( L_t \pi_t^2 \frac{p_1(1 - p_1)}{f_1^2} \right) \left( L_t - L_t \right) + \left( L_t \pi_t^2 \frac{p_1(1 - p_2)}{f_1 f_2} \right) \left( L_t - L_t \right) \right) = 0\]
Linearizing the first order condition for an explicit solution, we get

\[
\frac{d}{d\pi_t}(\mathbb{E}(H_t, \bar{L}_t, \pi_t))) = -\alpha \left( \bar{L}_t (b_2 - r) + \bar{L}_t \frac{p_1 (1 - p_2)}{f_1 f_2} dB_t^{(1)} + \bar{L}_t \frac{p_2 (1 - p_2)}{f_2^2} dB_t^{(2)} \right) \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right) + \\
\psi \left\{ \frac{1}{2} \left( g_2 \bar{g}_t dt + \left( \bar{L}_t \pi_t^2 g_1 dt \right) (\alpha + 4\alpha^2 Q_{0.05}) \right) \right\} + \psi^0 e^{-\beta L_t \gamma^{-1}} \\
\psi \left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(1 - \frac{\beta \gamma}{\alpha} \right) \beta L_t \gamma^{-1}} \right) \\
\left( \frac{a^2 g_2 \bar{g}_t \pi_t^2}{1 - \pi_t + (\pi_t^2 g_1 + \bar{L}_t \pi_t^2 \gamma^{-2})} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right) \\
\left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right) \\
\left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right) \\
\left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right) \\
\left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right)
\]

Linearizing the first order condition for an explicit solution, we get

\[
0 = -\alpha \left( \bar{L}_t (b_2 - r) + \bar{L}_t \frac{p_1 (1 - p_2)}{f_1 f_2} dB_t^{(1)} + \bar{L}_t \frac{p_2 (1 - p_2)}{f_2^2} dB_t^{(2)} \right) \left( Q_{0.05} - \varepsilon \leq d_t \leq Q_{0.05} + \varepsilon \right) + \\
\psi \left\{ \frac{1}{2} \left( g_2 \bar{g}_t dt + \left( \bar{L}_t \pi_t^2 g_1 dt \right) (\alpha + 4\alpha^2 Q_{0.05}) \right) \right\} + \psi^0 e^{-\beta L_t \gamma^{-1}} \\
\psi \left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(1 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right) \\
\left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right) \\
\left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right) \\
\left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right) \\
\left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right) \\
\left( \frac{2(1 - \pi_t) r}{(1 - \bar{L}_t) \pi_t} \right) \left( 1 - e^{-\left(2 - \frac{\beta \gamma}{\alpha} \beta L_t \gamma^{-1} \right)} \right)
\]

3 Numerical Example

For the numerical illustration, data used are daily closing price of the stock “Entergy Corporation” in the time range 31st August, 2009 till 30th August, 2013 [Quantopian][2018]. The return is calculated for this data.

Assuming that the distribution function is not known for the return of the stock prices we try to fit a distribution using kernel density estimator (KDE). It is a non parametric way to estimate the probability density function of a random variable. KDE is a fundamental data smoothing problem based on the finite data sample we choose. If we
have \((x_1, x_2, ..., x_n)\) as the independent univariate samples coming from an unknown distribution \(f\) then we can write

\[
f_h(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right)
\]

where \(K\) is the kernel which is a non-negative function and \(h > 0\) is a smoothing parameter called the bandwidth [Markovich 2007]. Using the KDE function we get the following estimate of the parameters:

- kernel density estimate = log-quadratic fitting
- bandwidth (bw) = 0.00271447

Now calculating the \(Q_{0.05}\) and the probability value at \(p_1 = 0.05\) and \(p_2 = 0.5\) quantiles, we estimate the values of the other parameters. The value of the sigmoid parameter \(\alpha\) is set at 10. The parameter values are shown in the following table. We are going to vary \((\psi, \psi^0), \gamma, \beta\) and \(r\) and see the effects on the strategy and portfolio wealth. The choices for these are also shown in the table below.

| \(Q_{0.05}\)   | -0.0171 |
| \(f_{0.05}\)   | 8.2581  |
| \(f_{0.5}\)    | 40.8509 |
| \(\alpha\)     | 10      |
| \(b_2\)        | 0.0167  |
| \(\rho\)       | 0.23    |
| \(\varepsilon\) | 0.001   |
| \((\psi, \psi^0)\) | (0.8, 0.6), (0.6, 0.8), (0.95, 0.312), (0.312, 0.95) |
| \(\gamma\)     | 0.3, 0.5, 0.7 |
| \(\beta\)      | 0.01, 0.02, 0.03 |
| \(r\)          | 0.0001, 0.00014, 0.0004 |

Taking a short interval of time, \(t = 45\), \(dt = 1\), \(L_0 = 40\) and \(\pi_0 = 0.2\), solving the above problem piece wise in time for random \(dB^{(1)}\) and \(dB^{(2)}\), we can make the approximation accurate while linearizing. Also we use the simulation from the Bi-variate Normal Distribution in R using Gibbs sampler to get the two Brownian motion. To illustrate our calculation steps, we take the following values \(\psi = 0.8\), \(\psi^0 = 0.6\), \(\gamma = 0.5\) and \(\beta = 0.2\). Putting this values in (6) for \(r = 0.00014\) we get,

\[
0 = \left( (-0.13248) + (-0.0006)dB^{(1)} + (-0.0012)dB^{(2)} \right) \left(Q_{0.05} - \varepsilon \leq dt \leq Q_{0.05} + \varepsilon \right) \\
\left( L_t(0.0028 + 0.3312\bar{\pi})dt + L_t\bar{\pi}(0.0015)dB^{(1)} + L_t\bar{\pi}(0.003)dB^{(2)} - 0.658621 \right) \\
+ \left( -0.00000123 - 0.000146\bar{\pi} + L_t\bar{\pi}^2(0.00001)dt \right) + 0.6e^{-0.02t}\bar{L_t}^{-0.5} \approx 0.0014t 
\]

Putting the value \(r = 0.0004\) in (6) we get,

\[
0 = \left( (-0.1304) + (-0.0006)dB^{(1)} + (-0.0012)dB^{(2)} \right) \left(Q_{0.05} - \varepsilon \leq dt \leq Q_{0.05} + \varepsilon \right) \\
\left( L_t(0.008 + 0.326\bar{\pi})dt + L_t\bar{\pi}(0.0015)dB^{(1)} + L_t\bar{\pi}(0.003)dB^{(2)} - 0.658621 \right) \\
+ \left( -0.00000353 - 0.000143\bar{\pi} + L_t\bar{\pi}^2(0.00001)dt \right) + 0.6e^{-0.02t}\bar{L_t}^{-0.5} \approx 0.0004t 
\]

15
Putting the values \( r = 0.0001 \) in (6) we get,

\[
0 = \left( -0.1328 + (-0.0006)dB_t^{(1)} + (-0.0012)dB_t^{(2)} \right) e^{-0.0005t} + e^{-0.0005t} + e^{-0.0005t}
\]

\[
\left( L_t(0.002 + 0.332\pi t)dt + L_t\pi_t(0.0015)dB_t^{(1)} + L_t\pi_t(0.003)dB_t^{(2)} - 0.658621 \right) +
\left( -0.00000088 - 0.000146\pi t + L_t\pi_t^2(0.00001)dt + 0.6e^{-0.002t}L_t^{-0.5} \right) 0.0001t \quad (9)
\]

The other expressions for alternative parametric configurations are similar.

We generate 100 sample paths each for all the configurations and find 95% confidence interval for optimal strategy from all these random samples of the Brownian motion. First, keeping other things fixed, we vary the rate of interest \( r = 0.0001, 0.00014 \) and 0.0004. First, when \( r = 0.0004 \) as used in equation (8), drawing the plot we see that it lies between the range (0.0175, 0.0275) as shown in the figure (1b). Similarly we can show that the portfolio wealth lies in the range (37.5, 44) as shown in the figure (2b).

The 95% confidence interval for optimal strategy for all the random samples of the Brownian motion for \( r = 0.0001 \) as used in equation (9) shows that it lies in the range (0.0325, 0.0425) as shown in the figure (1c). Similarly we can show that the portfolio wealth lies in the range (37.5, 44) as shown in the figure (2c).

Finally, the 95% confidence interval for optimal strategy for all the random samples of the Brownian motion for \( r = 0.00014 \) as used in equation (7) shows that it lies in the range (0.03, 0.041) as shown in the figure (7a). Similarly we can show that the portfolio wealth lies in the range (38, 46) as shown in the figure (8a). From the above discussion and plots we can conclude that as the rate of interest increases the amount of wealth to be invested on stock decreases. This is intuitive as when the riskless asset becomes more attractive, it makes sense to invest more in it so that risk is minimised without compromising too much on return.

Next, we are going to consider the scenario where weights \((\psi, \psi^0)\) are modified keeping other parameters constant. As we change the weights of the optimizing function from 0.6 to 0.8 and the constraint weights from 0.8 to 0.6 and plot the optimal strategy for \( r = 0.00014 \), refer figure (7a) and figure (5b). When we are using the weights for objective function as 0.312 and the constraint as 0.95, refer figure (5c) and when we interchange the weights accordingly refer figure (5d). From these plots we can see that as we increase the objective weights and give less weightage to subject constraint then the amount to be invested in stocks should be more. Again, this result is in line with financial intuition.

Looking in to the portfolio wealth as we change the weights of the optimizing function from 0.6 to 0.8 and the constraint weights from 0.8 to 0.6 and plot the wealth for \( r = 0.00014 \), refer figure (8a) and figure (4b). When we are using the weights for objective function as 0.312 and the constraint as 0.95, refer figure (4c) and when we interchange the weights accordingly, refer figure (4d). From these graphs we can see that as we increase the objective weights and give less weightage to the constraint, then portfolio wealth increases. Again, giving less weightage to the constraint means taking more risk (compromising on the VaR) in order to increase the expected (median) gain. So, naturally, in such a case the portfolio wealth increases.

Next, we would like to see the effect of the risk aversion parameter \( \gamma \) keeping all other factors constant. When we are going to decrease \( \gamma \) from 0.5 to 0.3 and plot the optimal strategy and compare, refer figure (7a) and (5b). Again plotting the optimal while taking \( \gamma = 0.7 \) refer figure (5c). We notice that all the three graphs are similar so there is no specific change in the optimal strategy if we change \( \gamma \).

If we plot the wealth for different levels of risk aversion, as illustrated by the three values of \( \gamma \) we have chosen, we can see that for all the cases the wealth accumulates in the same way, refer figure (8a, 6b and 6c).

16
Figure 1: Average optimal strategy and its corresponding 95% interval for first sub-figure when $r = 0.0004$, second sub-figure when $r = 0.0001$ and third sub-figure when $r = 0.0014$
Figure 2: Portfolio wealth and its corresponding 95% interval for first sub-figure when $r = 0.0004$, second sub-figure when $r = 0.0001$ and third sub-figure when $r = 0.00014$
Figure 3: Average optimal strategy and its corresponding 95% interval $r = 0.00014$, $\gamma = 0.5$, $\beta = 0.02$. 

(a) $\psi = 0.8$, $\psi^0 = 0.6$ 

(b) $\psi = 0.6$, $\psi^0 = 0.8$ 

(c) $\psi = 0.95$, $\psi^0 = 0.312$ 

(d) $\psi = 0.312$, $\psi^0 = 0.95$
(a) $\psi = 0.8, \psi_0 = 0.6$

(b) $\psi = 0.6, \psi_0 = 0.8$

(c) $\psi = 0.95, \psi_0 = 0.312$

(d) $\psi = 0.312, \psi_0 = 0.95$

Figure 4: Average portfolio wealth and its corresponding 95% interval $r = 0.00014, \gamma = 0.5, \beta = 0.02$
Figure 5: Average optimal strategy and its corresponding 95\% interval $r = 0.00014$, $\psi = 0.8$, $\psi^0 = 0.6$, $\beta = 0.02$
Figure 6: Average wealth and its corresponding 95% interval $r = 0.00014$, $\psi = 0.8$, $\psi^0 = 0.6$, $\beta = 0.02$
Finally, we see the effect of a change in the time discounting, \( \beta \), keeping all other factors constant. When we decrease \( \beta \) from 0.02 to 0.01 and plot the optimal strategy and compare, refer figure (7a) and (7b). Again plotting the optimal path while taking \( \beta = 0.03 \) refer figure (7c). We notice that all the three graphs are similar so there is no specific change in the optimal strategy if we change \( \beta \). But the variability of the wealth process, as illustrated by the width of the confidence interval, increases as we increase \( \beta \).

If we plot the wealth for all the three values of \( \beta \) we can see that for all the cases the wealth accumulates in the same way, refer figures 8a, 8b and 8c.

4 Concluding Remarks

In this paper we have considered the optimisation problem for an investor whose portfolio consists of a single risky asset and a risk free asset. She wants to maximize her expected utility of the portfolio subject to managing the Value at Risk (VaR) assuming a heavy tailed distribution of the stock prices return. We assumed that the quantiles of the heavy tail distribution asymptotically follows normal distribution, allowing us to formulate the stochastic differential equation for the quantiles.

We assume that the investor tries to optimize the portfolio based on Value at Risk (Var). We use a stochastic maximum principle to formulate the dynamic optimisation problem (Yong & Zhou [13]). The equations which we obtain does not have any explicit analytical solution, so we look for accurate approximations to estimate the value function and optimal strategy. As our calibration strategy is non-parametric in nature, no prior knowledge on the form of the distribution function is needed.

We finally provide detailed empirical illustration based on data centric parameter values calibrated from a real life data and a range of choices for the subjective parameters. Our results show close concordance with financial intuition. As this kind of a risk tolerance based portfolio optimization exercise has not been attempted in continuous time before, our results are expected to add to the arsenal of the portfolio managers who deals in high frequency trading.

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Figure 7: Average optimal strategy and its corresponding 95% interval $r = 0.00014$, $\psi = 0.8$, $\psi^0 = 0.6$, $\gamma = 0.5$
Figure 8: Average wealth and its corresponding 95% interval $r = 0.00014$, $\psi = 0.8$, $\psi^0 = 0.6$, $\gamma = 0.5$
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