A New Analysis of an Adaptive Convex Mixture: A Deterministic Approach

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Abstract—We introduce a new analysis of an adaptive mixture method that combines outputs of two constituent filters running in parallel to model an unknown desired signal. This adaptive mixture is shown to achieve the mean square error (MSE) performance of the best constituent filter, and in some cases outperforms both, in the steady-state. However, the MSE analysis of this mixture in the steady-state and during the transient regions uses approximations and relies on statistical models on the underlying signals and systems. Hence, such an analysis may not be useful or valid for signals generated by various real life systems that show high degrees of nonstationarity, limit cycles and, in many cases, that are even chaotic. To this end, we perform the transient and the steady-state analysis of this adaptive mixture in a “strong” deterministic sense without any approximations in the derivations or statistical assumptions on the underlying signals such that our results are guaranteed to hold. In particular, we relate the time-accumulated squared estimation error of this adaptive mixture at any time to the time-accumulated squared estimation error of the optimal convex mixture of the constituent filters directly tuned to the underlying signal in an individual sequence manner.

Index Terms—Deterministic, adaptive mixture, convexly constrained, steady-state, transient.

I. INTRODUCTION

The problem of estimating an unknown desired signal is heavily investigated in the adaptive signal processing literature. However, in various applications, certain difficulties arise in the estimation process due to the lack of structural and statistical information about the data model that relates the observation process to the desired signal. To resolve this lack of information, mixture approaches are proposed that adaptively combine outputs of multiple constituent algorithms performing the same task [1–3]. These parallel running algorithms can be seen as alternative hypotheses for modeling, which can be exploited for both performance improvement and robustness. Along these lines, a convexly constrained mixture method that combines outputs of two adaptive filters is introduced in [2]. In this approach, the outputs of the constituent algorithms are adaptively combined under a convex constraint to minimize the final MSE. This adaptive mixture is shown to be universal with respect to the input filters in a certain stochastic sense such that it achieves (and in some cases outperforms) the MSE performance of the best constituent filter in the mixture in the steady-state. However, the MSE analysis of this adaptive mixture for the steady-state and during the transient regions uses approximations, e.g., separation assumptions, and relies on statistical models on the signals and systems, e.g., nonstationarity data models [2–4].

Nevertheless, signals produced by various real life systems, such as in underwater acoustic communication applications, show high degrees of nonstationarity, limit cycles and, in many cases, are even chaotic so that they hardly fit to assumed statistical models. Hence an analysis based on certain statistical assumptions or approximations may not useful or adequate under these conditions. To this end, we refrain from making any statistical assumptions on the underlying signals and present an analysis that is guaranteed to hold for any bounded arbitrary signal without any approximations. In particular, we relate the performance of this adaptive mixture to the performance of the optimal convex combination that is directly tuned to the underlying signal and outputs of the constituent filters in a deterministic sense. Naturally, this optimal convex combination can only be chosen in hindsight after observing the whole signal and outputs a priori (before we even start processing the data). In this sense, we provide both the transient and steady-state analysis of the adaptive mixture in a deterministic sense without any assumptions on the underlying signals or any approximations in the derivations. Our results are guaranteed to hold in an individual sequence manner.

After we provide a brief system description in Section II, we present a deterministic analysis of the convexly constrained adaptive mixture method in Section III, where the performance bounds are given as a theorem and a lemma. The letter concludes with certain remarks.

II. PROBLEM DESCRIPTION

In this framework, we have a desired signal \( \{y(t)\}_{t \geq 1} \), where \(|y(t)| \leq Y < \infty \), and two constituent filters running in parallel producing \( \{\hat{y}_1(t)\}_{t \geq 1} \) and \( \{\hat{y}_2(t)\}_{t \geq 1} \), respectively, as the estimations (or predictions) of the desired signal \( \{y(t)\}_{t \geq 1} \). We assume that \( Y \) is known. Here, we have no restrictions on \( \hat{y}_1(t) \) or \( \hat{y}_2(t) \), e.g., these outputs are not required to be causal, however, without loss of generality, we assume \( |\hat{y}_1(t)| \leq Y \) and \( |\hat{y}_2(t)| \leq Y \), i.e., these outputs can be clipped to the range \([-Y, Y]\) without sacrificing performance under the squared error. As an example, the desired signal and outputs of the first stage filters can be single realizations generated under the framework of [2]. At each time \( t \), the convexly constrained algorithm receives an input vector \( x(t) \equiv [\hat{y}_1(t) \, \hat{y}_2(t)]^T \) and outputs

\[
\hat{y}(t) = \lambda(t)\hat{y}_1(t) + (1 - \lambda(t))\hat{y}_2(t) = [\lambda(t) \, (1 - \lambda(t))].x(t),
\]
where $0 \leq \lambda(t) \leq 1$, as the final estimate. The final estimation error is given by $e(t) = y(t) - \hat{y}(t)$. The combination weight $\lambda(t)$ is trained through an auxiliary variable using a stochastic gradient update to minimize the squared final estimation error as

$$\lambda(t) = \frac{1}{1 + e^{-\rho(t)}},$$

$$\rho(t+1) = \rho(t) - \mu \nabla_{\rho} e^2(t)|_{\rho=\rho(t)}$$

$$= \rho(t) + \mu e(t) \lambda(t) (1 - \lambda(t))(\hat{y}_1(t) - \hat{y}_2(t)), \quad (2)$$

where $\mu > 0$ is the learning rate. The combination parameter $\lambda(t)$ in (1) is constrained to lie in $[\lambda^+, (1 - \lambda^+)]$, $0 < \lambda^+ < 1/2$ in (2), since the update in (2) may slow down when $\lambda(t)$ is too close to the boundaries. We follow the same restriction and analyze (2) under this constraint.

When applied to any sequence $\{y(t)\}_{t \geq 1}$, the algorithm in (1) yields the total accumulated loss

$$L_n(\hat{y}, y) \triangleq \sum_{t=1}^{n} (y(t) - \hat{y}(t))^2$$

for any $n$. Although, we use the time-accumulated squared error as the performance measure, our results can be readily extended to the exponentially weighted accumulated squared error. We next provide deterministic bounds on $L_n(\hat{y}, y)$ with respect to the best convex combination $\min_{\beta \in [0, 1]} L_n(\hat{y}_\beta, y)$, where

$$L_n(\hat{y}_\beta, y) = \sum_{t=1}^{n} (y(t) - \hat{y}_\beta(t))^2$$

and $\hat{y}_\beta(t) \triangleq \beta \hat{y}_1(t) + (1 - \beta) \hat{y}_2(t)$, that holds uniformly in an individual sequence manner without any stochastic assumptions on $y(t)$, $\hat{y}_1(t)$, $\hat{y}_2(t)$ or $n$. Note that the best convex combination $\min_{\beta \in [0, 1]} L_n(\hat{y}_\beta, y)$, which we compare the performance against, can only be determined after observing the entire sequences, i.e., $\{y(t)\}, \{\hat{y}_1(t)\}$ and $\{\hat{y}_2(t)\}$, in advance for all $n$.

### III. A Deterministic Analysis

In this section, we first relate the accumulated loss of the adaptive mixture to the accumulated loss of the best convex combination that minimizes the accumulated loss in the following theorem. Then, we demonstrate that one cannot improve the convergence rate of this upper bound using our methodology directly and the Kullback-Leibler (KL) divergence as the distance measure by providing counter examples as a lemma. We emphasize that although the steady-state and transient MSE performances of the convexly constrained mixture algorithms are analyzed with respect to the constituent filters, we perform the steady-state and transient analysis without any stochastic assumptions or use any approximations in the following theorem.

**Theorem:** The algorithm given in (2), when applied to any sequence $\{y(t)\}_{t \geq 1}$, with $|y(t)| \leq Y < \infty$, yields, for any $n$ and any $\epsilon > 0$

$$\frac{L_n(\hat{y}, y)}{n} - \left(\frac{2\epsilon + 1}{1 - \epsilon^2}\right) \min_{\beta \in [0, 1]} \left\{ \frac{L_n(\hat{y}_\beta, y)}{n} \right\} \leq O\left(\frac{1}{n\epsilon}\right), \quad (3)$$

where $\hat{y}_\beta(t) = \beta \hat{y}_1(t) + (1 - \beta) \hat{y}_2(t)$, $z \triangleq \frac{1 - 4\lambda^+ (1 - \lambda^+)}{1 + 4\lambda^+ (1 - \lambda^+)} < 1$ and step size $\mu = \frac{4\epsilon + 2 + 2\epsilon^2}{2\epsilon + 1} < \frac{1}{\lambda^+}$, provided that $\lambda(t) \in [\lambda^+, 1 - \lambda^+]$, $0 < \lambda^+ < 1/2$, for all $t$ during the adaptation.

Equation (3) provides the exact trade-off between the transient and steady-state performances of the adaptive mixture in a deterministic sense without any assumptions or approximations. From (3) we observe that the convergence rate of the right hand side is $O\left(\frac{1}{n\epsilon}\right)$ and, as in the stochastic case, to get a tighter asymptotic bound with respect to the optimal convex combination of the filters, we require a smaller $\epsilon$, i.e., smaller learning rate $\mu$, which increases the right hand side of (3). Although this result is well-known in the adaptive filtering literature and appears widely in stochastic contexts, however, this trade-off is guaranteed to hold in here without any statistical assumptions or approximations. Note that the optimal convex combination in (3), i.e., minimizing $\beta$, depends on the entire signal and outputs of the constituent filters for all $n$.

**Proof:** To prove the theorem, we use the approach introduced in (3) (and later used in [5]) based on measuring progress of an adaptive algorithm using certain distance measures. We first convert (2) to a direct update on $\lambda(t)$ and use this direct update in the proof. Using $e^{-\rho(t)} = \frac{1 - \lambda(t)}{\lambda(t)}$ from (1), the update in (2) can be written as

$$\lambda(t + 1) = \frac{1}{1 + e^{-\rho(t + 1)}} = \frac{1}{1 + \frac{1 - \lambda(t)}{\lambda(t)} e^{-\mu e(t)\lambda(t)(1 - \lambda(t))} \hat{y}_1(t) - \hat{y}_2(t)}$$

$$= \frac{\lambda(t) e^{\mu e(t)\lambda(t)(1 - \lambda(t))} \hat{y}_1(t) + (1 - \lambda(t)) e^{\mu e(t)\lambda(t)(1 - \lambda(t))} \hat{y}_2(t)}{\lambda(t) e^{\mu e(t)\lambda(t)(1 - \lambda(t))} \hat{y}_1(t) + (1 - \lambda(t)) e^{\mu e(t)\lambda(t)(1 - \lambda(t))} \hat{y}_2(t)}, \quad (4)$$

Unlike [5] (Lemma 5.8), our update in (4) has, in a certain sense, an adaptive learning rate $\mu \lambda(t) (1 - \lambda(t))$ which requires different formulation, however, follows similar lines of [5] in certain parts.

Here, we first define $\hat{y}_\beta(t) \triangleq \beta \hat{y}_1(t) + (1 - \beta) \hat{y}_2(t) = u^T \omega(t)$, where $\beta \in [0, 1]$ and $u \triangleq [\beta \ 1 - \beta]^T$. At each adaptation, the progress made by the algorithm towards $u$ at time $t$ is measured as $d(u, u(t)) = d(u, u(t + 1))$, where $\omega(t) \triangleq [\lambda(t) (1 - \lambda(t))]^T$ and $d(u, u) \triangleq \sum_{i=1}^{2} u_i \ln(u_i/w_i)$ is the Kullback-Leibler divergence [6], $u \in [0, 1]^2$, $w \in [0, 1]^2$. We require that this progress is at least $a(y(t) - \hat{y}(t))^2 - b(y(t) - \hat{y}(t))^2$ for certain $a, b, \mu [5, 6]$, i.e.,

$$a(y(t) - \hat{y}(t))^2 - b(y(t) - \hat{y}(t))^2$$

$$\leq |d(u, u(t)) - d(u, u(t + 1))|$$

$$= \beta \ln \left(\frac{\lambda(t + 1)}{\lambda(t)}\right) + (1 - \beta) \ln \left(\frac{1 - \lambda(t + 1)}{1 - \lambda(t)}\right), \quad (5)$$

which yields the desired deterministic bound in (3) after telescoping.
Defining $\zeta(t) = e^{\mu e(t)\lambda(t)(1-\lambda(t))}$, we have from (4)

$$\beta \ln \left( \frac{\lambda(t+1)}{\lambda(t)} \right) + (1-\beta) \ln \left( \frac{1-\lambda(t+1)}{1-\lambda(t)} \right) = \hat{y}_\beta(t) \ln \zeta(t) - \ln(\lambda(t)\zeta(t)\hat{y}_1(t)) + (1-\lambda(t))\zeta(t)\hat{y}_2(t).$$

(6)

Using the inequality $x^2 \leq 1-x(1-\alpha)$ for $\alpha \geq 0$ and $x \in [0, 1]$ from (5), we have

$$\zeta(t)^{\hat{y}_1(t)} = (\zeta(t)^{2Y})^{\frac{\hat{y}_1(t)+Y}{2Y}} \zeta(t)^{-Y} \leq (1-\zeta(t)^{2Y}),$$

which implies in (6)

$$\ln \left( \lambda \zeta \hat{y}_1(t) + (1-\lambda) \zeta \hat{y}_2(t) \right) \leq \ln \left( \zeta \hat{y}_1 \hat{y}_2 \right) - Y \ln \zeta(t) + \ln \left( 1 - \frac{\hat{y}_1(t) + Y}{2Y} (1 - \zeta(t)^{2Y}) \right),$$

(7)

where $\hat{y}_1(t) = \lambda(t)\hat{y}_1(t) + (1-\lambda(t))\hat{y}_2(t)$. As in (5), one can bound (7) using $\ln(1-q(1-e^p)) \leq pq + \frac{p^2}{8}$ for $0 \leq q < 1$ (originally from (6))

$$\ln \left( \lambda \zeta \hat{y}_1(t) + (1-\lambda) \zeta \hat{y}_2(t) \right) \leq -Y \ln \zeta(t) + (\hat{y}_1(t) + Y) \ln \zeta(t) + \frac{Y^2(\ln \zeta(t))^2}{2} - 2 \ln(\zeta(t)).$$

(8)

Using (9) in (6) yields

$$\beta \ln \left( \frac{\lambda(t+1)}{\lambda(t)} \right) + (1-\beta) \ln \left( \frac{1-\lambda(t+1)}{1-\lambda(t)} \right) \geq \hat{y}_\beta(t) + (1-\hat{y}_\beta(t)), \quad \ln(\zeta(t) - (\hat{y}_1(t) - (\hat{y}_1(t) - (\hat{y}_2(t) + (1 - \hat{y}_2(t)))) \ln(\zeta(t) - \frac{Y^2(\ln \zeta(t))^2}{2}.$$ 

(9)

From now on, we omit $\beta$ of $\hat{y}_\beta(t)$. We observe from (5) and (9) that to prove the theorem, it is sufficient to show that $G(y(t), \hat{y}(t), \hat{y}_\beta(t), \zeta(t)) \leq 0$, where

$$G(y(t), \hat{y}(t), \hat{y}_\beta(t), \zeta(t)) \triangleq -\hat{y}_\beta(t) + (y(t) + \hat{y}(t)) \ln \zeta(t) + \frac{Y^2(\ln \zeta(t))^2}{2} + a(y(t) - \hat{y}(t))^2 - b(y(t) - \hat{y}_\beta(t))^2.$$ 

(10)

For fixed $y(t), \hat{y}(t), \zeta(t)$, $G(y(t), \hat{y}(t), \hat{y}_\beta(t), \zeta(t))$ is maximized when $\frac{\partial G}{\partial \hat{y}_\beta(t)} = 0$, i.e., $\hat{y}_\beta(t) - y(t) + \ln(\frac{\zeta(t)}{\hat{y}_\beta(t)}) = 0$ since $\frac{\partial^2 G}{\partial \hat{y}_\beta(t)^2} = -2b < 0$, yielding $\hat{y}_\beta(t)^* = y(t) - \ln(\frac{\zeta(t)}{\hat{y}_\beta(t)})$ Note that while taking the partial derivative of $G(\cdot)$ with respect to $\hat{y}_\beta(t)$ and finding $\hat{y}_\beta(t)^*$, we assume that all $y(t), \hat{y}(t), \zeta(t)$ are fixed, i.e., their partial derivatives with respect to $\hat{y}_\beta(t)$ is zero. This yields an upper bound on $G(\cdot)$ in terms of $\hat{y}_\beta(t)$. Hence, it is sufficient to show that $G(y(t), \hat{y}(t), \hat{y}_\beta(t)^*, \zeta(t)) \leq 0$ such that (5)

$$G(y(t), \hat{y}, \hat{y}_\beta(t)^*, \zeta(t)) = -\left( y(t) + \frac{\ln \zeta(t)}{2b} \right) \ln \zeta(t) + (\hat{y}(t) + Y) \ln \zeta(t) + \frac{Y^2(\ln \zeta(t))^2}{2} + a(y(t) - \hat{y}(t))^2 - \frac{(\ln \zeta(t))^2}{2}.$$ 

(11)

$$= a(y(t) - \hat{y}(t))^2 - (y(t) - \hat{y}(t)) \ln \zeta(t) + \frac{(\ln \zeta(t))^2}{4b} + \frac{Y^2(\ln \zeta(t))^2}{2}.$$ 

(12)

For (12) to be negative, defining $k = \frac{\lambda(t)(1-\lambda(t))}$ and

$$H(k) = k^2 \mu^2 (\frac{Y^2}{2} + \frac{1}{4b}) - \mu k + a,$$

it is sufficient to show that $H(k) \leq 0$ for $k \in [\lambda^+(1-\lambda^+), \frac{1}{2}]$, i.e., $k \in [\lambda^+(1-\lambda^+), \frac{1}{2}]$ when $\lambda(t) \in [\lambda^+, (1-\lambda^+)]$, since $H(k)$ is a convex quadratic function of $k$, i.e., $\frac{\partial^2 H}{\partial k^2} > 0$. Hence, we require the interval where the function $H(\cdot)$ is negative should include $[\lambda^+(1-\lambda^+), \frac{1}{2})$, i.e., the roots $k_1$ and $k_2$ (where $k_2 \leq k_1$) of $H(\cdot)$ should satisfy $k_1 \geq \frac{1}{4}$ and $k_2 \leq \lambda^+(1-\lambda^+)$, where

$$k_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4\mu a (\frac{Y^2}{2} + \frac{1}{4b})}}{2\mu} = \frac{1 \pm \sqrt{1 - 4a\mu}}{2\mu}.$$ 

(13)

and

$$s \triangleq \left( \frac{Y^2}{2} + \frac{1}{4b} \right).$$

To satisfy $k_1 \geq 1/4$, we straightforwardly require from (13)

$$\frac{2 + 2\sqrt{1 - 4a\mu}}{s} \geq \mu.$$ 

To get the tightest upper bound for (13), we set

$$\mu = \frac{2 + 2\sqrt{1 - 4a\mu}}{s},$$

i.e., the largest allowable learning rate.

To have $k_2 \leq \lambda^+(1-\lambda^+)$ with $\mu = \frac{2 + 2\sqrt{1 - 4a\mu}}{s}$, from (13) we require

$$\frac{1 - \sqrt{1 - 4a\mu}}{4(1 + \sqrt{1 - 4a\mu})} \leq \lambda^+(1-\lambda^+).$$

(14)

Equation (14) yields

$$as = a \left( \frac{Y^2}{2} + \frac{1}{4b} \right) \leq 1 - \frac{z^2}{4},$$

(15)

where

$$z = 1 - 4\lambda^+(1-\lambda^+) \frac{1 + 4\lambda^+(1-\lambda^+)}{1 + 4\lambda^+(1-\lambda^+)}.$$
and $z < 1$ after some algebra.

To satisfy (15), we set $b = \frac{Y^2}{Y^2(2\epsilon + 1)}$ for any (or arbitrarily small) $\epsilon > 0$ that results

$$a \leq \frac{(1 - z^2)\epsilon}{Y^2(2\epsilon + 1)}. \quad (16)$$

To get the tightest bound in (5), we select $a = \frac{(1 - z^2)\epsilon}{Y^2(2\epsilon + 1)}$ in (16). Such selection of $a$, $b$ and $\mu$ results in (5)

$$\left(\frac{(1 - z^2)\epsilon}{Y^2(2\epsilon + 1)}\right) (y(t) - \hat{y}(t))^2 - \left(\frac{\epsilon}{Y^2}\right) (y(t) - \hat{y}_\beta(t))^2 \leq \beta \ln \left(\frac{\lambda(t + 1) + (1 - \beta) \ln \left(1 - \frac{1 - \lambda(t + 1)}{1 - \lambda(t)}\right)}{\lambda(t)}\right). \quad (17)$$

After telescoping, i.e., summation over $t$, $\sum_{t=1}^n (17)$ yields

$$\alpha L_n(y, \hat{y}) - b \min_{\beta \in [0, 1]} \left\{ \frac{L_n(\hat{y}_\beta, y)}{n} \right\} \leq \beta \ln \left(\frac{\lambda(t + 1) + (1 - \beta) \ln \left(1 - \frac{1 - \lambda(t + 1)}{1 - \lambda(t)}\right)}{\lambda(t)}\right) \leq O(1), \quad (18)$$

which is the desired bound.

Note that using $b = \frac{Y}{2\epsilon^2}$, $a = \frac{(1 - z^2)\epsilon}{Y^2(2\epsilon + 1)}$ and $s = \left(\frac{Y^2}{2\epsilon} + \frac{1}{2}\right)$, we get

$$\mu = \frac{2 \epsilon + 1}{\epsilon} = \frac{2 + 2z}{Y^2},$$

after some algebra, as in the statement of the theorem. This concludes the proof of the theorem. \hfill \Box

In the following lemma, we show that the order of the upper bound using the KL divergence as the distance measure under the same methodology cannot be improved by presenting an example in which the bound on $b$ is of the same order as that given in the theorem.

**Lemma:** For positive real constants $a$, $b$ and $\mu$ which satisfies (5) for all $|y(t)| \leq Y$, $|\hat{y}_1(t)| \leq Y$ and $|\hat{y}_2(t)| \leq Y$ and $\lambda(t) \in [\lambda^+, (1 - \lambda^+)]$, we require

$$b \geq \frac{a}{4} + \frac{1}{16\lambda^+ (1 - \lambda^+)}.$$  \hfill (19)

**Proof:** Since the inequality in (5) should be satisfied for all possible $y(t)$, $\hat{y}_1(t)$, $\hat{y}_2(t)$, $\beta$ and $\lambda(t)$, the proper values of $a$, $b$ and $\mu$ should satisfy (5) for any particular selection of $y(t)$, $\hat{y}_1(t)$, $\hat{y}_2(t)$, $\beta$ and $\lambda(t)$. First we consider $y(t) = \hat{y}_1(t) = Y$, $\hat{y}_2(t) = 0$, $\beta = 1$ and $\lambda(t) = \lambda^+$ (or, similarly, $y(t) = \hat{y}_1(t) = Y$, $\hat{y}_2(t) = -Y$ and $\lambda(t) = \lambda^+$). In this case, we have

$$a(Y - \lambda^+ Y)^2 \leq -\ln(\lambda^+ + (1 - \lambda^+)e^{\mu(Y - \lambda^+ Y)\lambda^+(1 - \lambda^+)(-Y)} \leq -\lambda^+ \ln(1 - \mu(1 - \lambda^+)^2\lambda^+ Y(1 - \lambda^+)(-Y)) \leq \mu(1 - \lambda^+)^3\lambda^+ Y^2, \quad (19)$$

where (19) follows from the Jensen’s Inequality for concave function $\ln(\cdot)$. By (20), we have

$$\mu \geq \frac{a}{\lambda^+ (1 - \lambda^+)}. \quad (20)$$

For another particular case where $\hat{y}_1(t) = Y$, $y(t) = \hat{y}_2(t) = 0$, $\beta = 1$ and $\lambda(t) = 1/2$, we have

$$\frac{Y}{2} \leq a \left(\frac{Y}{2}\right)^2 - b(-Y)^2 \leq -\ln\left(\frac{1}{2} + \frac{1}{2} e^{\mu(-\frac{1}{2})\frac{1}{2}(-Y)}\right) \leq -\frac{1}{2} Y^2.$$  \hfill (21)

where (22) also follows from the Jensen’s Inequality. By (22), we have

$$b \geq \frac{a}{4} + \frac{\mu}{16} \geq \frac{a}{4} + \frac{1}{16\lambda^+ (1 - \lambda^+)}, \quad (23)$$

where (23) follows from (21), which finalizes the proof. \hfill \Box

**IV. CONCLUSION**

In this paper, we introduce a new and deterministic analysis of the convexly constrained adaptive mixture of [2] without any statistical assumptions on the underlying signals or any approximations in the derivations. We relate the time-accumulated squared estimation error of this adaptive mixture at any time to the time-accumulated squared estimation error of the optimal convex combination of the constituent filters that can only be chosen in hindsight. We refrain from making statistical assumptions on the underlying signals and our results are guaranteed to hold in an individual sequence manner. We also demonstrate that the proof methodology cannot be changed directly to obtain a better bound, in the convergence rate, on the performance by providing counter examples. To this end, we provide both the transient and steady state analysis of this adaptive mixture in a deterministic sense without any assumptions on the underlying signals or without any approximations in the derivations.

**REFERENCES**

[1] A. C. Singer and M. Feder, “Universal linear prediction by model order weighting,” *IEEE Transactions on Signal Processing*, vol. 47, no. 10, pp. 2685–2699, 1999.

[2] J. Arenas-Garcia, A. R. Figueiras-Vidal, and A. H. Sayed, “Mean-square performance of a convex combination of two adaptive filters,” *IEEE Transactions on Signal Processing*, vol. 54, no. 3, pp. 1078–1090, 2006.

[3] V. H. Nascimento, M. T. M. Silva, R. Candido, and J. Arena-Garcia, “A transient analysis for the convex combination of adaptive filters,” *IEEE/SP 15th Workshop on Statistical Signal Processing*, pp. 53–56, 2009.

[4] S. S. Kozat, A. T. Erdogan, A. C. Singer, and A. H. Sayed, “Steady state MSE performance analysis of mixture approaches to adaptive filtering,” *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4050–4063, August 2010.

[5] J. Kivinen and M. K. Warmuth, “Exponentiated gradient versus gradient descent for linear predictors,” *Journal of Information and Computation*, vol. 132, pp. 1–64, 1997.

[6] N. Cesa-Bianchi, Y. Freund, D. Haussler, D. P. Helmbold, R. E. Schapire, and M. K. Warmuth, “How to use expert advice,” *Journal of the ACM*, vol. 44, no. 3, pp. 427–485, 1997.