Exact periodic solutions of shell models of turbulence

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Abstract

We derive exact analytical solutions of the GOY shell model of turbulence. In the absence of forcing and viscosity we obtain closed form solutions in terms of Jacobi elliptic functions. With three shells the model is integrable. In the case of many shells, we derive exact recursion relations for the amplitudes of the Jacobi functions relating the different shells and we obtain a Kolmogorov solution in the limit of infinitely many shells. For the special case of six and nine shells, these recursion relations are solved giving specific analytic solutions. Some of these solutions are stable whereas others are unstable. All our predictions are substantiated by numerical simulations of the GOY shell model. From these simulations we also identify cases where the models exhibit transitions to chaotic states lying on strange attractors or ergodic energy surfaces.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The behaviour of turbulent systems is fairly well understood from numerical simulations of the Navier–Stokes equations but a deep analytical understanding of fully developed turbulence based on these equations is still lacking (see for instance [7] and references therein). A fully developed turbulent state is obtained when the Reynolds number of the flow grows towards infinity which, for instance can be realized in the limit of vanishing viscosity. This limit of the Navier–Stokes equations is known to be singular and when the viscosity is identically zero the system is usually termed the Euler equations (without external forcing) which are believed to exhibit a singularity in a finite time.

Direct numerical simulations of the Navier–Stokes equations at high Reynolds numbers are generally very cumbersome and in recent decades shell models of turbulence have become strong alternatives to direct simulations [1, 9, 10, 23]. Indeed, using shell model approaches
one easily simulates Reynolds numbers of the order from $10^{10}$ up to $10^{14}$. Shell models provide excellent agreement with statistical experimental data, such as structure functions and probability distribution functions [1]. In contrast, shell models generally fail to provide knowledge about geometrical structures as these models are too purely resolved in Fourier space.

In this paper we take our starting point in shell models with relatively few shells in the Euler case of vanishing viscosity and no external forcing. This might not be a very realistic case for a real fluid flow but we are nevertheless able to solve the shell model exactly in this limit. In the case of three shells we find the full dynamical solutions in terms of Jacobi elliptic functions. We furthermore show that the solutions are stable as is confirmed by numerical simulations. In addition, we find that no other solutions are possible meaning that the GOY shell model with three shells is here shown to be fully integrable. If we consider this system as a low-dimensional version of the Euler equations (possessing the same basic structure and symmetries as the real Euler equations), our results lead to the interesting conclusion that despite the driving there are no finite time singularities for such a system.

It is well known that the GOY shell model possesses a period three symmetry [1,9,10,23]. We therefore attack the model with six and nine shells and again find analytical solutions in terms of Jacobi elliptic functions. We obtain the interesting results that some of these periodic solutions are unstable whereas others are stable. This observation is very much related to previous studies on periodic solutions in the GOY model [13], in other turbulent systems [6,14,15,19–21] and in the Kuramoto–Shivashinsky (KS) equation [4,8]. Kawahara and Kida identified periodic orbits in plane Couette flow [14] and van Veen and Kida have looked for periodic orbits in symmetry reduced models of fully developed turbulence [15,20]. Toh and Itano [19] and Walaffe [21] have identified periodic structures in channel flows and Eckhardt et al have studied pipe flows [6]. The Kuramoto–Shivashinsky system is a field equation that does not model a turbulent fluid but nevertheless possesses chaotic states developed from nonlinear mode interactions. Reference [4] identified a multitude of unstable periodic solutions in the KS equation. However, Frisch and co-workers found that the phase space of the KS equations contains small isolated areas which include stable periodic solutions [8]. It is nevertheless believed that a chaotic attractor of a large dimension in a high-dimensional space can be thought of as composed of an ordered infinity of unstable periodic orbits (‘Hopf’s last hope’, see [4,5]). Indeed long unstable periodic orbits have been obtained for the GOY model subjected to forcing and viscosity [13] thus supporting this picture of a strange attractor. Our results therefore bridge these observations by finding exact analytical periodic solutions to the unforced GOY model which appear to be both stable and unstable. It is important to note that our analytical solutions possess a phase freedom; so our results define a continuous infinity of exact solutions for the GOY model. Since we shall show (see appendix A) that for infinitely many shells a Kolmogorov solution is obtained, this means that the periodic solutions are physically important and constitute a nontrivial part of the phase space.

We also note that one can obtain similar Jacobi type solutions for the Sabra shell model [16] although the specific recursion relations for the amplitudes of the Jacobi elliptic functions possess different conjugations (see appendix B). For the Sabra model self-similar quasi-soliton solutions have been identified in [17].

When a small forcing term is added we find that some but not all of the stable solutions become unstable. Numerically we observe that the dynamics becomes either chaotic on a strange attractor in the 12-dimensional space or quasiperiodic with a regular time evolution. The transition can be driven by varying the initial condition in the sixth shell and we show numerically examples of such transitions.
2. Solution of a three-shell model

In this section we shall find solutions to the GOY model with three shells for the cases with no forcing where the phases are constant and where the phases are variable. This model is integrable. Further, we also consider the model with (an imaginary) force and find a solution. The main point in this investigation is that it will serve as a warm-up exercise for the solution with an arbitrary number of shells given in the next section with further details in appendix A.

2.1. Constant phases

The GOY model is a rough approximation to the Navier–Stokes equations and is formulated on a discrete set of $k$-values, $k_n = r^n$. In terms of a complex Fourier mode, $u_n$, of the velocity field the model reads

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = \delta n \sum_{k} \left(\frac{u_n^{*+1} u_n^{-1}}{r} - \frac{1 - \delta}{r^2} u_n^{*+1} u_n^{-2}\right) + \delta n,$$

with suitable boundary conditions [1,9,10,23]. $f$ is an external, constant forcing (here on the first mode) and $\nu$ is the kinematic viscosity. From these equations we have energy $\sum |u_n|^2$ as well as helicity $\sum (-1)^n |u_n|^2$ (for 3D where $\delta = 1 - 1/r$) and enstrophy $\sum k_n^2 |u_n|^2$ (for 2D where $\delta = 1 + 1/r^2$) conservations [2, 11, 12].

The mathematics of the three-shell case has been studied before [3] in a reduced version of 2D MHD with interactions between only three wave vectors. The structure of these equations is the same as the three-shell GOY model discussed below. However, the physics of the model in [3] is in 2D and is not the same as the GOY model, where we can study different dimensions via different conservation laws.

Initially, we shall for simplicity solve the three-shell model without any force or viscosity (i.e. $f = \nu = 0$) and with constant phases of the velocity functions $u_n$. It is then easy to see that the sum of these constant phases must be $\pi/2 \mod \pi$. Therefore we replace $u_1$ by $iu_1$ and take the new $u_1$ as well as $u_2$ and $u_3$ to be real. The three-shell equations then become

$$\frac{du_1}{dt} = ru_2(t) u_3(t),$$

$$\frac{du_2}{dt} = -r \delta u_1(t) u_3(t)$$

and

$$\frac{du_3}{dt} = -r (1 - \delta) u_1(t) u_2(t).$$

From these equations we have energy conservation as well as helicity (3D) or enstrophy (2D) conservations. However, the three-shell model has two other conservation laws. Combining (3) and (4) we easily find

$$u_3(t)^2 = L + \frac{1 - \delta}{\delta} u_2(t)^2.$$

Here $L$ is a constant which is determined by the initial conditions. In a similar manner we obtain from equations (2) and (3)

$$u_1(t)^2 = M - \frac{1}{\delta} u_2(t)^2,$$

where $M$ is again a constant determined by the initial conditions. Thus,

$$L = u_3(0)^2 - \frac{1 - \delta}{\delta} u_2(0)^2, \quad M = u_1(0)^2 + \frac{1}{\delta} u_2(0)^2.$$

We note that $L + M$ is the total energy.
For the moment we disregard the possibility of forcing. Later we shall show that even if a constant imaginary forcing is included, the model is fully integrable.

From (3) we have
\[
\frac{du_2}{u_1 u_3} = -r \delta \, dt.
\] (8)

By means of equations (5) and (6) we can express \( u_1 \) and \( u_3 \) in terms of \( u_2 \) and equation (8) then gives
\[
\int_{u_2(0)}^{u_2(t)} \frac{du_2}{\sqrt{L + \frac{1 - \delta}{\delta} u_2^2 \left( M - \frac{1}{\delta} u_2^2 \right)}} = -r \delta t.
\] (9)

This solution is thus in general an elliptic integral. If we substitute \( y = u_2/\sqrt{\delta M} = cn_x \), where \( cn \) is the Jacobi elliptic function, we obtain
\[
u_2(t) = \sqrt{\delta} |u_1(0)|^2 + |u_2(0)|^2 \times cn \left( \sqrt{\delta r} |u_3(0)|^2 + (1 - \delta)|u_1(0)|^2 t + cn^{-1} \left( \frac{u_2(0)}{\sqrt{\delta |u_1(0)|^2 + |u_2(0)|^2}} \right) \right).
\] (10)

where the modulus of the Jacobi elliptic function is given by
\[
k^2 = \frac{(1 - \delta)M}{L + (1 - \delta)M} = (1 - \delta) \frac{|u_1(0)|^2 + 1/\delta |u_2(0)|^2}{|u_3(0)|^2 + (1 - \delta)|u_1(0)|^2}.
\] (11)

Also, \( cn^{-1} \) is the inverse Jacobi function. The Jacobi functions depend on the elliptic integrals \((k^2 = 1 - k^2)\)
\[
K \equiv K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \quad K' \equiv K(k').
\] (12)

By use of the conservation laws equations (5) and (6) we can easily find the other \( u_3 \):\n\[
u_1(t) = i \sqrt{|u_1(0)|^2 + 1/\delta |u_2(0)|^2} \times sn \left( \sqrt{\delta r} |u_3(0)|^2 + (1 - \delta)|u_1(0)|^2 t + cn^{-1} \left( \frac{u_2(0)}{\sqrt{\delta |u_1(0)|^2 + |u_2(0)|^2}} \right) \right).
\] (13)

where we reintroduced the phase in \( u_1 \) and
\[
u_3(t) = \sqrt{|u_3(0)|^2 + (1 - \delta)|u_1(0)|^2} \times dn \left( \sqrt{\delta r} |u_3(0)|^2 + (1 - \delta)|u_1(0)|^2 t + cn^{-1} \left( \frac{u_2(0)}{\sqrt{\delta |u_1(0)|^2 + |u_2(0)|^2}} \right) \right).
\] (14)

Here we used that the Jacobi elliptic functions satisfy
\[
\text{sn}^2 x + \text{cn}^2 x = 1, \quad \text{dn}^2 x + k^2 \text{sn}^2 x = 1,
\] (15)

and their derivatives satisfy
\[
\frac{d \text{sn} x}{dx} = \text{cn} x \text{dn} x, \quad \frac{d \text{cn} x}{dx} = -\text{sn} x \text{dn} x, \quad \frac{d \text{dn} x}{dx} = -k^2 \text{sn} x \text{dn} x.
\] (16)

\(^1\) For the moment we disregard the possibility \( L = 0 \).
These relations clearly show the intimate connection between the Jacobi elliptic functions and the three-shell model. In the next sections we shall see that this remark can be generalized to an arbitrary number of shells.

As to the integrability of the three-shell model, one can argue geometrically as follows: since the total energy is conserved, the time evolution is confined to the surface of a sphere in $R^3$. Now take any one of the other two conserved quantities, equations (5) and (6). This corresponds to a cylinder along the $u_3$-axis with elliptical cross section. The intersection between this cylinder and the sphere then gives the line along which the time evolution can proceed. The second conserved quantity could limit everything to points, but it does not do so, because it is not linearly independent of the other two. So this is then the geometrical origin of the solutions in terms of elliptic functions.

We mention that the elliptic functions can be defined by (see [22])

$$u = \int_0^{\text{sn}(u,k)} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} \quad \text{and} \quad u = \int_{\text{cn}(u,k)}^1 \frac{dt}{\sqrt{(1-t^2)(k^2 + k^2 t^2)}}$$

They have product representations and Fourier expansions

$$\text{sn}(u,k) = 2q^{1/4}k^{-1/2} \sin x \prod_{n=1}^{\infty} \frac{1 - 2q^{2n} \cos 2x + q^{4n}}{1 - 2q^{2n} \cos 2x + q^{4n-2}}$$

$$= \frac{2\pi}{Kk} \sum_{n=0}^{\infty} q^{n+1/2} \frac{\sin(2n+1)x}{1 - q^{2n+1}},$$

(18)

$$\text{cn}(u,k) = 2q^{1/4}k^{-1/2}k^{1/2} \cos x \prod_{n=1}^{\infty} \frac{1 + 2q^{2n} \cos 2x + q^{4n}}{1 - 2q^{2n} \cos 2x + q^{4n-2}}$$

$$= \frac{2\pi}{Kk} \sum_{n=0}^{\infty} q^{n+1/2} \frac{\cos(2n+1)x}{1 + q^{2n-1}},$$

(19)

and

$$\text{dn}(u,k) = k^{1/2} \prod_{n=1}^{\infty} \frac{1 + 2q^{2n-1} \cos 2x + q^{4n-2}}{1 - 2q^{2n-1} \cos 2x + q^{4n-2}} = \frac{\pi}{2k} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n \cos 2nx}{1 + q^{2n}}. \quad (20)$$

Here $x = \pi u/2K$ and $q = \exp(-\pi K'/K)$. The constants $K$ and $K'$ can be obtained from the hypergeometric function

$$K = \frac{\pi}{2} \, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; k^2 \right) \quad \text{and} \quad K' = \frac{\pi}{2} \, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1 - k^2 \right). \quad (21)$$

The hypergeometric function $\, _2F_1(a, b; c; z)$ is analytic in the plane cut from $z = 1$ to $\infty$. Since $k^2$ depends on $\delta$ through equation (11) all results can be analytically continued in the cut plane from 2D to 3D.

It should be noted that for small $k$ the Fourier expansions in practice only contain very few terms, since $q$ is small. This is also applied when $k$ is not so small. For example, if $k^2 = \frac{1}{2}$, then $K = K'$ and hence $q = e^{-\gamma}$, so $q^n$ is fast decreasing. However, when $k$ approaches 1, a large number of terms are needed in the Fourier transforms because in this case $q$ is not small. This follows because $q$ can be expressed in terms of $k$ by the formula

$$q = \lambda + 2k^5 + 15k^9 + 150k^{13} + \cdots \quad \lambda = \frac{1}{2} \, \frac{1 - (1 - k^2)^{1/4}}{1 + (1 - k^2)^{1/4}}. \quad (22)$$
In the case $L = 0$, i.e. $u_3(0)^2 = (1 - \delta)/\delta$ $u_2(0)^2$, solution (9) does not give an oscillating solution any longer. Instead one obtains

$$u_2(t) = u_2(0)$$

$$\times \frac{2\sqrt{u_1(0)^2 + u_2(0)^2}/\delta e^{-\sqrt{M\delta(1 - \delta)t}}}{\sqrt{u_1(0)^2 + u_2(0)^2}/\delta + u_1(0) + (\sqrt{u_1(0)^2 + u_2(0)^2}/\delta - u_1(0))e^{-2\sqrt{M\delta(1 - \delta)t}}}.$$  \hspace{1cm} (23)

In this case $k = 1$, and obviously an infinite number of Fourier modes are needed to produce the exponentially decreasing behaviour (23). From equation (23) one can easily obtain $u_1$ by the use of the conservation law and (6):

$$u_1(t) = \sqrt{u_1(0)^2 + u_2(0)^2}/\delta$$

$$\times \frac{\sqrt{u_1(0)^2 + u_2(0)^2}/\delta + u_1(0) - (\sqrt{u_1(0)^2 + u_2(0)^2}/\delta - u_1(0))e^{-2\sqrt{M\delta(1 - \delta)t}}}{\sqrt{u_1(0)^2 + u_2(0)^2}/\delta + u_1(0) + (\sqrt{u_1(0)^2 + u_2(0)^2}/\delta - u_1(0))e^{-2\sqrt{M\delta(1 - \delta)t}}}.$$ \hspace{1cm} (24)

Thus, at $t \to \infty$ all the energy has gone to the $u_1$-mode.

In three dimensions $1 - \delta$ is positive, whereas in two dimensions this quantity is negative. From (11) it then follows that $k^2 > 0$ in 3D, whereas $k^2 < 0$ in 2D if $|u_3(0)|^2 > (\delta - 1)|u_1(0)|^2$.

The Jacobi elliptic functions can be continued to negative $k^2$, so this does not pose any problem. The continued functions are given by

$$\text{sn}(x, ik) = \frac{1}{\sqrt{1 + k^2}} \frac{\sqrt{x + k^2}}{\sqrt{1 + k^2}}, \quad \text{cn}(x, ik) = \frac{\sqrt{x + k^2}}{\sqrt{1 + k^2}},$$

$$\text{dn}(x, ik) = \frac{1}{\sqrt{1 + k^2}} \frac{\sqrt{x + k^2}}{\sqrt{1 + k^2}}.$$ \hspace{1cm} (25)

For $k \to 0 \pm$ these functions are continuous.

Solutions (10)–(14) simplify if $u_1(0) = 0$ and $u_2(0)^2/\delta = u_3(0)^2$. From (11) we see that in this case $k^2 = 1 - \delta$, and

$$u_1(t) = i \frac{a}{\sqrt{\delta}} \text{sn}(art, \sqrt{1 - \delta}), \quad u_2(t) = a \text{cn}(art, \sqrt{1 - \delta}), \quad u_3(t) = \frac{a}{\sqrt{\delta}} \text{dn}(art, \sqrt{1 - \delta}).$$ \hspace{1cm} (26)

Here $a = u_2(0)$. In figure 1 we show these exact solutions by direct numerical integration of the GOY model with three shells (with specific initial conditions). Note the familiar behaviour of the Jacobi elliptic functions. Two different values of the parameter $\delta$ are shown in order to vary the elliptic functions parameter $k^2 = 1 - \delta$. The solutions are stable and the numerical integrations can proceed indefinitely.

We also mention that if $k = 0$, corresponding to $\delta = 1$, $\text{sn}$ and $\text{cn}$ become $\text{sin}$ and $\text{cos}$, respectively, and $\text{dn}$ approaches the constant one. From equation (11) we see that $k^2 = 0$ if $\delta = 1$. This case has been studied (including viscosity) in [18].

2.2. Solution with variable phases

We shall now discuss the case of variable phases. In forming the energy or other conserved quantities they do not enter directly. However, in this section we shall see that the phases do play a dynamical role. On letting the phases vary a new solution emerges, which can still be
expressed in terms of a Jacobian elliptic function, but which has one more free parameter than solutions (10)–(14).

The three-shell model with variable phases defined by $u_n = |u_n|e^{i\phi_n}$ can be rewritten as six equations:

$$\frac{d|u_1|}{r \, dr} = |u_2||u_3| \sin(\phi_1 + \phi_2 + \phi_3), \quad \frac{d|u_2|}{r \, dr} = -\delta |u_1||u_3| \sin(\phi_1 + \phi_2 + \phi_3)$$

and

$$\frac{d|u_3|}{r \, dr} = -(1 - \delta)|u_2||u_1| \sin(\phi_1 + \phi_2 + \phi_3),$$

(27)

as well as

$$|u_1| \frac{d\phi_1}{r \, dr} = |u_2||u_3| \cos(\phi_1 + \phi_2 + \phi_3), \quad |u_2| \frac{d\phi_2}{r \, dr} = -\delta |u_1||u_3| \cos(\phi_1 + \phi_2 + \phi_3)$$

and

$$|u_3| \frac{d\phi_3}{r \, dr} = -(1 - \delta)|u_1||u_2| \cos(\phi_1 + \phi_2 + \phi_3),$$

(28)

From the last equations we easily derive

$$\frac{d(\phi_1 + \phi_2 + \phi_3)}{r \, dr} = -\cos(\phi_1 + \phi_2 + \phi_3) \left[ -\frac{|u_2||u_3|}{|u_1|} + \delta \frac{|u_1||u_3|}{|u_2|} + (1 - \delta) \frac{|u_1||u_2|}{|u_3|} \right].$$

(29)

On the left we can use equations (27) in the square brackets to get

$$\frac{d(\phi_1 + \phi_2 + \phi_3)}{dr} = \cot(\phi_1 + \phi_2 + \phi_3) \frac{d\ln(|u_1||u_2||u_3|)}{dr},$$

(30)

from which we get

$$\cos(\phi_1 + \phi_2 + \phi_3) = \frac{N}{|u_1||u_2||u_3|},$$

(31)

where $N$ is a constant. We see that if the phases are constants, i.e. $N = 0$, their sum must be $\pi/2$ mod $\pi$ (i.e. $\pi/2, 3\pi/2, \ldots$), as already used in the previous subsection.

From the cosine in (31) we can compute the sine and insert it in equations (27). By this procedure we arrive at

$$\frac{d|u_1|}{r \, dr} = \frac{1}{|u_1|} \sqrt{|u_1|^2|u_2|^2|u_3|^2 - N^2}, \quad \frac{d|u_2|}{r \, dr} = -\delta \frac{1}{|u_2|} \sqrt{|u_1|^2|u_2|^2|u_3|^2 - N^2}$$

and

$$\frac{d|u_3|}{r \, dr} = -(1 - \delta) \frac{1}{|u_3|} \sqrt{|u_1|^2|u_2|^2|u_3|^2 - N^2}.$$

(32)
In this equation we have inserted a positive square root, corresponding to a positive sine. Since the left-hand sides of equations (32) can also be negative in some range of \( t \) values, it follows that for such values of \( t \) a negative sign should be chosen in front of the square roots in equations (32), corresponding to negative values of the sine. In order not to make the notation too clumsy we do not indicate this explicitly. Further, from equation (27) we can easily derive like equations (5) and (6)

\[
|u_3|^2 = L + \frac{1 - \delta}{\delta} |u_2|^2 \quad \text{and} \quad |u_1|^2 = M - \frac{1}{\delta} |u_2|^2. \tag{33}
\]

This allows us to express any \( |u_n| \) in terms of the other two \( |u| \). Using this in equations (32) we obtain for example for \( |u_2| \) as the equation of motion,

\[
\frac{d|u_2|^2}{2 \delta \sqrt{|u_2|^2(M - |u_2|^2/\delta)(L + |u_2|^2(1 - \delta)/\delta) - N^2}} = -d(rt). \tag{34}
\]

This is again an elliptic type of integral. However, it is slightly more complicated than the solutions considered in the previous subsection.

Inside the square root we have a polynomial of third degree in \( |u_2|^2 \). If we introduce the zeros of this polynomial, this is the prototype of an integral which can be inverted by the use of theta functions [22]. However, instead of seeking the most general solution, in the following we shall study only a limited range of parameters corresponding to a limited range of the initial values of the \( u \).

To obtain a relatively simple solution we must remember that the Jacobi elliptic functions are not really independent, as is seen from equation (15). Therefore one could attempt to make an ansatz in such a manner that inside the square root we have three squares of elliptic functions, which requires

\[
|u_1|^2 = A_1 + B_1 \text{sn}^2(art), \quad |u_2|^2 = A_2 + B_2 \text{cn}^2(art) \quad \text{and} \quad |u_3|^2 = A_3 + B_3 \text{dn}^2(art). \tag{35}
\]

With a suitable choice of the \( A \) and \( B \) constants the square root in equation (34) could then produce the product \( \text{sn}(art) \text{cn}(art) \text{dn}(art) \) to be matched by the derivative in the numerator, \( d \text{cn}^2(art) = -2 \text{cn}(art) \text{sn}(art) \text{dn}(art) d(art) \) (see equation (16)).

Instead of proceeding from integral (34) it is actually easier to insert ansatz (35) directly in the equations of motion (32). This gives a number of algebraic equations. When we differentiate the functions (35) in order to insert them on the left-hand sides in (32) we get a result proportional to \( B_n \text{sn}(art) \text{cn}(art) \text{dn}(art)/|u_n| \) with \( n = 1, 2, 3 \) by use of equations (16). To match the right-hand sides of equations (32) we therefore need

\[
\sqrt{|u_1|^2|u_2|^2|u_3|^2 - N^2} \propto \text{sn}(art) \text{cn}(art) \text{dn}(art). \tag{36}
\]

This means that the following conditions must be satisfied:

\[
aB_1 = \sqrt{B_1B_2B_3}, \quad aB_2 = \delta \sqrt{B_1B_2B_3}, \quad k^2 aB_3 = (1 - \delta) \sqrt{B_1B_2B_3}. \tag{37}
\]

As before in equation (26), we take \( k^2 = 1 - \delta \). Then the restrictions (37) are easily solved

\[
B_1 = a^2/\delta, \quad B_2 = a^2, \quad B_3 = a^2/\delta. \tag{38}
\]

The remaining conditions then become

\[
0 = -A_3 a^4/\delta - A_2 (1 - \delta) a^4/\delta^2 + A_1 (1 - \delta) a^4/\delta, \tag{39}
\]

\[2 \text{ In arriving at these results we have first considered the term in the products of the functions (35) which obviously produces the wanted result. Then we have combined the other terms to be coefficients of } \text{sn}^2(art), \text{ sn}^2(art) \text{ and } \text{sn}^2(art) \text{ by use of the relations (15). These coefficients must vanish, which is the requirement in (39)-(41).} \]
which is the requirement that the coefficient of $sn^4(art)$ vanish, and
\[ N^2 = A_1A_2A_3 + A_1A_3a^2 + A_1A_2a^2/\delta + A_1a^4/\delta, \] (40)
which expresses the condition that the coefficient of $sn^0(art)$ vanishes, and
\[ 0 = A_2A_3a^2/\delta - A_1A_3a^2 + A_3a^2/\delta - A_1A_2(1 - \delta)a^2/\delta \\
+ A_2a^4/\delta^2 - A_1a^4/\delta - A_1(1 - \delta)a^4/\delta, \] (41)
which expresses the condition that the coefficient of $sn^2(art)$ vanishes.

The solutions of (39)–(41) should be such that the square root on the right-hand sides of equation (32) should be real. This is not a trivial requirement. One can study this question in general by first solving for $A_3$ in terms of $A_1$ and $A_2$ by use of equation (39), and (41) then becomes a second order equation which gives $A_2$ in terms of $A_1$. In this way we obtain
\[ A_3 = (1 - \delta)(A_1 - A_2/\delta), \] (42)
\[ A_2 = \frac{\delta}{2(1 - \delta)} \left( a^2 + (1 - \delta)A_1 \pm \sqrt{a^4 + (1 - \delta)A_1 \left( 2\frac{a^2\delta^2 - 2}{\delta^2} - 3(1 - \delta)A_1 \right)} \right), \] (43)
as well as
\[ N^2 = A_1^2 \delta(1 - \delta) + A_1^2a^2(2 - \delta) + A_1a^4/\delta. \] (44)
The last restriction can be compared with equation (31), which expresses the constant $N$ in terms of the initial values of the phases and the $|u|$s. Using equation (38) to find the initial $|u|$s and comparing this with equation (44) we see that ansatz (35) is valid only when the cosine of the sum of the initial phases is equal to plus/minus one, i.e.
\[ \phi_1(0) + \phi_2(0) + \phi_3(0) = 0(\text{mod } \pi). \] (45)
This does not mean that the phases at later times are restricted in this way.

The requirements (42), (43) and (44) do not give acceptable (real) results for all values of $A_1$ and $a$, since we need to ensure that in (44) $N^2 > 0$ and that the square root in (43) should be real. Let us therefore give a numerical example where the results are good. We have found that if $\delta = 1 - 1/r = 1/2$ (3D, with the often used value $r = 2$) and $a = 1$ the following values are completely consistent (the square root is real!):
\[ A_1 = 0.309, \quad A_2 = 0.577, \quad A_3 = -0.423, \quad N^2 = 0.7685. \] (46)
For initial values of the phases and the $|u|$s which are not covered by the ansatz (35) one must go back to the standard elliptic integral (34) and treat it in other ways than done here.

Solution (35) can be used to obtain an integral representation for the phases. We can insert the cosine from equation (31) in the equations of motion for the phases (28), and we get
\[ \phi(t) = \phi(0) + N \int_0^t \frac{r \, dt}{A_1 + (a^2/\delta)sn^2(art)}, \] (47)
with similar expressions for the other phases.

It should also be mentioned that it is possible to take into account the viscosity terms in the equations for the phases. The result is
\[ \frac{d\phi_1}{dr} = \frac{N}{|u_1|^2} e^{-v(k_1^2 + k_2^2 + k_3^2)} |u_1|^2, \] (48)
with similar expressions for the other phases. Unfortunately we have not been able to solve for the modulus $|u_1|$ when viscosity is included.

In conclusion we have seen that if the phases are allowed to vary, this situation gives rise to new solutions (35). So, although the phases do not appear in the energy and other conserved quantities and one might therefore consider them unphysical, this is not true because of their dynamical importance. As the phases are allowed to vary continuously this indeed outlines an infinity of exact periodic solutions to the GOY models with three shells.
2.3. Inclusion of a constant force

We shall now consider the case where a constant force \( f \) is taken to be imaginary, \( f \rightarrow if \), is included in the first shell, so that (2) reads

\[
\frac{du_1}{dt} = ru_2(t)u_3(t) + f, \tag{49}
\]

whereas the other two equations (3) and (4) are unchanged. For this reason we derive as in equation (5)

\[
u_3(t)^2 = L + \frac{1 - \delta}{\delta}u_2(t)^2. \tag{50}\]

However, equation (6) is now replaced by

\[
\frac{1}{2}u_1(t)^2 - f \int_0^t dt' u_1(t') = \frac{1}{2}M - \frac{1}{2\delta}u_2(t)^2. \tag{51}\]

Here the initial conditions again determine the constants \( M \) and \( L \). The integral on the left-hand side can be determined from equations (3) and (50),

\[
-r\delta \int_0^t dt' u_1(t') = \sqrt{\frac{\delta}{1-\delta}} \ln \left( \frac{u_2(t) + \sqrt{u_2(t)^2 + \frac{L\delta}{1-\delta}}}{u_2(0) + \sqrt{u_2(0)^2 + \frac{L\delta}{1-\delta}}} \right). \tag{53}\]

Inserting this in equation (3) we obtain by use of equation (50)

\[
-r\delta t = \int \frac{du_2}{u_1u_3} = \int_{u_2(0)}^{u_2(t)} \frac{du_2}{\sqrt{L + \frac{1 - \delta}{\delta}u_2^2}} \times \left( M - \frac{1}{\delta}u_2^2 - 2f \frac{1}{r\sqrt{\delta(1-\delta)}} \ln \left( \frac{u_2 + \sqrt{u_2^2 + \frac{L\delta}{1-\delta}}}{u_2(0) + \sqrt{u_2(0)^2 + \frac{L\delta}{1-\delta}}} \right) \right). \tag{54}\]

This in principle expresses \( u_2 \) as a function of \( t \), and \( u_1 \) can then be obtained from (51), whereas \( u_3 \) can be obtained from (50). However, due to the logarithm in the denominator we cannot use elliptic functions to invert equation (54).

If \( L = 0 \) and \( f/r \) is large and positive, it is possible to obtain an asymptotic expression for integral (54) by introducing the variable

\[
z = M - \frac{1}{\delta}u_2^2 - \frac{2f}{r\sqrt{\delta(1-\delta)}} \ln \frac{u_2}{u_2(0)}. \tag{55}\]

By a partial integration one then obtains

\[
-r\sqrt{\delta(1-\delta)}rt = -\int_0^{u_2(0)} \frac{dz}{\sqrt{z} \left( 2u_2^2/\delta + 2f/r\sqrt{\delta(1-\delta)} \right)} = -\left[ \frac{2\sqrt{z}}{2u_2^2/\delta + 2f/r\sqrt{\delta(1-\delta)}} \right]_0^{z_0} \left( 1 + O \left( \frac{1}{f} \right) \right). \tag{56}\]
Here $z_0$ is obtained from substitution (55) just by inserting the upper limit $u_2(t)$ instead of the integration variable $u_2$. With the assumption $u_2 \ll f|\ln u_2|$ we can invert this expression and obtain

$$u_2(t) \approx u_2(0) e^{-\sqrt{\delta(1-\delta)} f r^2/2}.$$  

This expression works quite well for large $f$s, as one can see by solving equation (49) numerically. Thus the assumption $u_2 \ll f|\ln u_2|$ is self-consistent.

For $f < 0$ we have not been able to find asymptotic results. Numerically one can easily see that in this case $u_2(t)$ oscillates. In figure 2 we show numerical solutions of the GOY model with three shells with a small forcing added. The solutions are surprisingly still stable even though energy is now added into the system. We observe the interesting phenomenon that the dn-function undergoes a period doubling when the value of the forcing is changed. We were however not able to identify a series of period doubling eventually leading into a chaotic state.

3. Many shells

In this section we shall extend one of the results obtained in the last section. We start by defining the functions

$$J_{a1} = \text{sn}(art + b), \quad J_{a2} = \text{cn}(art + b), \quad J_{a3} = \text{dn}(art + b),$$

$$J_{a4} = \text{sn}(art + b), \quad J_{a5} = \text{cn}(art + b), \quad J_{a6} = \text{dn}(art + b),$$

$$J_{a7} = \text{sn}(art + b), \ldots,$$

where the symbol $J_a$ stands for Jacobi. In general $J_{an} = J_{a(n-3)}$. The $J_a$s satisfy the differential equations

$$dJ_a/dt = \epsilon_a J_{a+1}J_{a+2}.$$  

Here

$$\epsilon_n = +1 \text{ if } n = 1 (\text{mod } 3), \quad \epsilon_n = -1 \text{ if } n = 2 (\text{mod } 3), \quad \epsilon_n = -k^2 \text{ if } n = 3 (\text{mod } 3).$$

The modulus $k^2$ is to be determined by the requirement that the functions

$$u_n = A_n J_{an}(art + b)$$

are periodic solutions of shell models of turbulence.
should satisfy the shell equations up to some order. Similarly, the constants \( a, b \) and \( A_n \) should be restricted from this requirement. Equation (61) is of course a generalization of (10), (13) and (14). As an example we show in figure 3 two solutions to the unforced GOY model with six shells. By just changing the initial conditions in the sixth shell, the solutions change from being quasiperiodic to a chaotic signal. Thus in the 12-dimensional phase space stable periodic and quasiperiodic islands are situated between chaotic areas as is a well-known scenario for conservative systems of low dimension.

The shell model without viscosity and forcing

\[
\frac{du_n^*}{dr} = -ik_n \left( u_{n+1}u_{n+2} - \frac{\delta}{r} u_{n-1}u_{n+1} - \frac{1 - \delta}{r^2} u_{n-1}u_{n-2} \right)
\]  

(62)
is satisfied exactly by ansatz (61), because by direct insertion we get

\[
\text{LHS} = A_n^*a_\epsilon \epsilon_n J_{a_{n+1}}(at) J_{a_{n+2}}(at)
\]

\[
= -ik_n \left( A_{n+1}A_{n+2}J_{a_{n+1}}J_{a_{n+2}} - \frac{\delta}{r} A_{n-1}A_{n+1}J_{a_{n-1}}J_{a_{n+1}} - \frac{1 - \delta}{r^2} A_{n-1}A_{n-2}J_{a_{n-1}}J_{a_{n-2}} \right),
\]

(63)

and using that the \( J \)s are defined mod 3 in the index we get

\[ J_{a_{n-1}} = J_{a_{n+2}} \quad \text{(in second and third terms)}, \quad J_{a_{n-2}} = J_{a_{n+1}} \quad \text{(in third term)}, \]

(64)

which shows that the \( J \)s cancel out on both sides of equation (63). The \( A \)s should therefore satisfy

\[
ra A_n^* \epsilon_n = -ik_n \left( A_{n+1}A_{n+2} - \frac{\delta}{r} A_{n-1}A_{n+1} - \frac{1 - \delta}{r^2} A_{n-1}A_{n-2} \right).
\]

(65)

This can be rewritten as a recursion relation

\[
A_{n+2} = \frac{ir a A_n^* \epsilon_n}{k_n A_{n+1}} + \frac{\delta}{r} A_{n-1} + \frac{1 - \delta}{r^2} \frac{A_{n-1}A_{n-2}}{A_{n+1}}.
\]

(66)

We have not succeeded in showing that this relation always has a solution. However, it is instructive to compare with the results in the last section. Let us start to solve recursion

\[
A_{n+2} = \frac{ir a A_n^* \epsilon_n}{k_n A_{n+1}} + \frac{\delta}{r} A_{n-1} + \frac{1 - \delta}{r^2} \frac{A_{n-1}A_{n-2}}{A_{n+1}}.
\]

(66)

We have not succeeded in showing that this relation always has a solution. However, it is instructive to compare with the results in the last section. Let us start to solve recursion
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relation (66),

\[ A_3 = \frac{iaA_1^*}{A_2}, \]  

\[ A_4 = \frac{1}{rA_1^*} (\delta |A_1|^2 - |A_2|^2) \]  

and

\[ A_5 = \frac{1}{rA_2^* (\delta |A_1|^2 - |A_2|^2)} \left( -k^2 a^2 |A_1|^2 + (\delta^2 - \delta + 1) |A_1|^2 |A_2|^2 - \delta |A_2|^4 \right). \]  

Subsequent coefficients get more complex. The next one is

\[ A_6 = \frac{ia(A_4^* + r(1 - \delta)A_1^*)}{r^3 A_5} + \frac{ia\delta A_1^*}{r A_2}, \]  

Here equations (68) and (69) should be inserted.

To gain some insight into the recursion scheme let us start by assuming that there are only three shells. Then we should require that

\[ A_4 = 0, \]  

i.e. from (68)

\[ |A_2| = \sqrt{\delta} |A_1|. \]  

Further, from the requirement \( A_5 A_4 = 0 \) one obtains

\[ 0 = -k^2 a^2 |A_1|^2 + (\delta^2 - \delta + 1) |A_1|^2 |A_2|^2 - \delta |A_2|^4. \]  

Inserting (71) this gives

\[ k^2 = (1 - \delta) \frac{|A_2|^2}{a^2}, \]  

which is exactly the same as the previously obtained result (11).

We mention that usually the GOY model is just truncated at some shell number, in this case \( n = 3 \). However, in dealing with the analytic solutions this is absolutely necessary since we need the information contained when the two subsequent shell amplitudes (in this case \( n = 4 \) and 5) are put equal to zero, leading to a value for \( k^2 \), which shows that only the first amplitude is a free parameter. Therefore it is necessary to proceed in the somewhat unconventional way followed above, also when we consider an arbitrary but finite number of shells, as we shall see in the following.

With six shells\(^3\) we need

\[ A_7 = 0 = -\frac{iaA_5^*}{r^2 A_6} + \frac{\delta}{r} A_4 + \frac{1 - \delta}{r^2} \frac{iaA_1^* A_4}{A_2 A_6} \]  

and

\[ A_7 A_8 = 0 = -\frac{iaA_6^*}{r^3} + \frac{(1 - \delta)}{r^2} A_5 A_4. \]  

Equations (74) and (75) can in principle be used to determine \( k^2 \). The algebra is, however, quite involved.

It is possible to simplify the rather tedious algebra by introducing the following ansatz,

\[ A_4 = \alpha_4 A_1, \quad A_5 = \alpha_5 A_2, \quad \text{and} \quad A_6 = \alpha_6 A_3. \]  

Here we have assigned \( A_1 \) and \( A_4 \) the phase \( \pi/2 \), whereas the other \( A_\text{s} \) are real. In the following, we take all the \( A_\text{s} \) to be real. From equation (68) we see that

\[ A_2^2 = (\delta - r\alpha_4) A_1^2. \]  

\(^3\) Because of the special role played by modulus 3 in the GOY model it is safer to end \( n \) at 3, 6, 9, 12, ...
From equation (69) we similarly get

\[ k^2 = (-r^2 \alpha_4 \alpha_5 + 1 - \delta + r \delta \alpha_4) A_2^2 / a^2, \tag{78} \]

Next we consider recursion relation (70) for \( A_6 \) which gives

\[ \alpha_6 = \frac{\alpha_4 + r (1 - \delta)}{r^3 \alpha_5} + \frac{\delta}{r}. \tag{79} \]

The conditions for the shell model to stop at six shells are represented by equations (74) and (75), which lead to

\[ \frac{\delta}{r} \alpha_4 \alpha_6 = \frac{\alpha_5}{r^2} (\delta - r \alpha_4) - \frac{1 - \delta}{r^2} \alpha_4 \tag{80} \]

and

\[ (1 - \delta) \alpha_4 \alpha_5 = \frac{\alpha_6}{r^3} (-r^2 \alpha_4 \alpha_5 + 1 - \delta + r \delta \alpha_4), \tag{81} \]

respectively. In the last equation we used \( k^2 \) from equation 78).

From these three equations we can obtain the three \( \alpha \)s in terms of the parameters \( r, \delta \). This elimination procedure leads to a sixth order equation for \( \alpha_4 \), and from the solutions of this equation the other \( \alpha \)s can then be obtained.

This procedure leads to extremely messy expressions which are not useful, since we do not have analytic solutions of sixth order equations. Instead, it is much simpler to assign numerical values for the parameters. Taking \( r = 2 \) and \( \delta = 1/2 \) (3D) we obtain from equations (79), (80) and (81) by use of NSolve in Mathematica (applied to these three simultaneous equations) the following real solutions:

\[ \alpha_4 = 0.065 277 183 514 53, \quad \alpha_5 = 0.670 441 370 5072, \quad \alpha_6 = 0.448 614 903 251 81 \]
\[ \alpha_4 = -0.678 864 569 1146, \quad \alpha_5 = -0.056 426 934 955 29, \quad \alpha_6 = -0.461 396 585 557 51, \]
\[ \alpha_4 = -17.345 971 520 666, \quad \alpha_5 = 1.399 595 443 2919, \quad \alpha_6 = -1.209 883 604 1347, \]

not acceptable (see remark below equation (83)): \[ \alpha_4 = 0.625 261, \quad \alpha_5 = -0.309 106, \quad \alpha_6 = -0.407 242. \tag{82} \]

From equation (77) there are relations between \( A_1 \) and \( A_2 \), and using the numbers in (82) we get

\[ A_2^2 / A_1^2 = \delta - r \alpha_4 = 0.369 446, 1.857 73, 35.1919, \tag{83} \]

respectively. Here the fourth line in equation (82) has been ignored, since it does not lead to a real ratio \( A_2 / A_1 \), as assumed in the above derivation. Thus there are only three acceptable solutions.

For \( k^2 \) we obtain

\[ \frac{k^2 a^2}{A_2^2} = 0.390 22, -0.332 09, 80.263 366, \tag{84} \]

respectively.

In figures 4 and 5 we show two of the exact solutions to the model GOY with six shells as given in (82) by the first two sets of \( \alpha_n \)-values. In the first case, shown in figure 4, we obtain the familiar Jacobi elliptic function with the exact relationships between the amplitude as predicted by theory. Notice that the amplitude of \( \text{Im} u_1 \) comes to exactly 1 even though the
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Figure 4. The solutions to the GOY model with six shells without forcing and viscosity for $\delta = 0.5$ with the following scaling coefficients: $\alpha_4 = 0.065277 183514.53$, $\alpha_5 = 0.670441 370507.2$, $\alpha_6 = 0.448614 90325181$ and parameter values $a = 1$, $b = 0$, $A_1 = 1$. This leads to the following initial values: $\text{Re } u_1(0) = \text{Im } u_1(0) = \text{Im } u_2(0) = \text{Im } u_3(0) = \text{Re } u_4(0) = \text{Im } u_4(0) = \text{Im } u_5(0) = \text{Im } u_6(0) = 0.0$, $\text{Re } u_2(0) = 0.607820697$, $\text{Re } u_3(0) = 1.645222028$, $\text{Re } u_9(0) = 0.407507916$, $\text{Re } u_6(0) = 0.73807128$. The following shell model variables are plotted: (a) $\text{Im } u_1(t)$, $\text{Im } u_4(t)$; (b) $\text{Re } u_2(t)$, $\text{Re } u_5(t)$; (c) $\text{Re } u_3(t)$, $\text{Re } u_6(t)$.

Figure 5. The solutions to the GOY model with six shells without forcing and viscosity for $\delta = 0.5$. We apply the following scaling coefficients: $\alpha_4 = -0.678864 5691146$, $\alpha_5 = -0.056426 93495529$, $\alpha_6 = -0.461396 58555751$. Setting $a = 10$, $b = 0$, $A_1 = 1$ give the following initial conditions: $\text{Re } u_1(0) = \text{Im } u_1(0) = \text{Im } u_2(0) = \text{Im } u_3(0) = \text{Re } u_4(0) = \text{Im } u_4(0) = \text{Im } u_5(0) = \text{Im } u_6(0) = 0.0$, $\text{Re } u_2(0) = 0.607820697$, $\text{Re } u_3(0) = 1.362985377$, $\text{Re } u_9(0) = 7.336835887$, $\text{Re } u_5 = -0.079609087$, $\text{Re } u_6 = -3.385191019$.

The GOY model has been modified by changing the order of the complex conjugates of the velocity amplitudes resulting in the Sabra model [16]. We have checked that the Jacobi elliptic functions are also exact solutions to this model. One can derive recursion relations such as equations (79), (80) and (81) which have a similar structure but the conjugations and a sign appear differently. This is discussed in some detail in appendix B.

In order to study the stability of the solutions further, we have initiated the integration with the exact solutions shown in figure 4 but with the addition of a small force. As is seen from the simulation (shown in figure 6), the solution starts out to be stable, looking very similar to the exact solution; however in the long run it also becomes unstable and chaotic. Nevertheless, we have observed numerically that the existence of this transition depends very much on the value of the added forcing $f$. For some values of $f$, the solutions remain stable although...
Figure 6. Solutions to the GOY model with six shells and with the same parameter values as in figure 4 but with the addition of a small forcing Re $f = \text{Im} f = 0.001$. Note that the solutions appear to be stable for a long time but turn into chaotic trajectories eventually. Plotted are (a) Re $u_1(t)$; (b) Re $u_4(t)$.

slightly changed away from the exact solutions. For other values of $f$ we observe the scenario depicted in figure 6. We shall not elaborate further on these transitions in this paper.

It is well known for the GOY shell model that at the parameter value $\delta = 1.25$ the enstrophy $H = \sum_{n=1}^{N} k_{n}^{2} |u_{n}|^{2}$ is a conserved quantity. This point therefore corresponds to a two-dimensional turbulence with a forward cascade of enstrophy and an inverse cascade of energy [1]. We have therefore also looked for exact solutions to the GOY model with six shells at this point and by using $r = 2$ and $\delta = 1.25$ we obtain as before from equations (79), (80) and (81) by the use of NSolve in Mathematica the following real solutions:

$$
\alpha_4 = 0.289785, \quad \alpha_5 = 2.21803, \quad \alpha_6 = 0.613153,
$$

$$
\alpha_4 = -0.108771, \quad \alpha_5 = 0.112449, \quad \alpha_6 = -0.0517172,
$$

$$
\alpha_4 = 0.0856125, \quad \alpha_5 = 0.160463, \quad \alpha_6 = 0.302194,
$$

$$
\alpha_4 = 0.0982932, \quad \alpha_5 = 0.359568, \quad \alpha_6 = 0.485351
$$

(85)

with the following relations between the amplitudes $A_2$ and $A_1$

$$
A_2^{2}/A_1^{2} = 0.67043, 1.46754, 1.07878, 1.05341
$$

(86)

and the elliptic parameter $k$ determined by:

$$
k^2 a_2^2/A_2^2 = -2.09654, -0.473003, -0.0909193, -0.145639.
$$

(87)

Of these four solutions to the 2D analogy for turbulence we have found numerically that number one, three and four on the list are unstable whereas number two is stable. As examples of this we show in figure 7 some of the solutions, the ones corresponding to the second and the fourth in the list. As compared with the 3D case, we observe that the solutions in figure 7(b) become unstable even faster and that they clearly behave very chaotically.

In appendix A, we present solutions for the case of nine shells. Furthermore, we derive a general recursion relation which allows us to solve the $\alpha_n$ coefficients for any number of shells. If the number of shells goes to infinity one can see from these recursion relations that the only scaling law solution behaves like $\alpha_n \sim k_n^{-4}$ as expected from Kolmogorov theory [1].

We end this section by pointing out that the solutions based on ansatz (76) are special. Presumably the general recursion relations (66) for the amplitudes of the Jacobi functions can have more general solutions, where the constant phases could also enter the recursions. In any case, the solutions that we found represent an infinite set of solutions, since the continuous parameters $A_1, A_2, a, b$ can vary freely. Alternatively, an analysis of equations (61) and (66)
Figure 7. Plot of the Jacobi elliptic functions for six shells for the case of $\delta = 1.25$ for enstrophy conservation corresponding to 2D turbulence for two different sets of $\alpha_n$ values from (85) (the initial conditions are adjusted accordingly). (a) $\alpha_4 = -0.108771, \alpha_5 = 0.112449, \alpha_6 = -0.0517172$. Here the solutions in terms of elliptic functions are stable. (b) $\alpha_4 = 0.0982932, \alpha_5 = 0.359568, \alpha_6 = 0.485351$. The solutions ‘follow’ the elliptic solutions for a few periods and then clearly become unstable.

shows that the free parameters are $u_1(0), u_2(0), u_3(0)$ and $A_1$. Since these quantities in general are complex, this represents an $\infty^8$-continuous set of parameters.

## 4. Conclusions

We have derived a series of exact analytical solutions to the GOY shell model in the absence of forcing and viscosity. In this model a shell couples to nearest and next-nearest neighbour shells and the very construction of these couplings directly suggests that the solutions should be formulated in terms of Jacobi elliptic functions. We have presented closed form Jacobi solutions in the case of three shells at which point the GOY model is completely integrable and possesses a continuous infinity of stable solutions. We have further derived exact recursion relations for the amplitudes of the elliptic functions in the case of 6, 9, 12, etc shells which is related to the period three symmetry of the GOY model. Some of these solutions are stable and others are unstable. In the last cases the trajectories lie on chaotic energy surfaces. This picture of an infinity of solutions, some either periodic or quasiperiodic and others chaotic, is common for conservative systems where the trajectories will move on surfaces of constant energy. Our results therefore complement the picture presented for the GOY model with forcing and viscosity obtained in [13] where long periodic unstable solutions are found embedded in the large dimensional phase space of the dynamics of a GOY model with many shells. In this limit of many shells we have found solutions that behave according to Kolmogorov scaling theory for energy spectra, which shows the physical relevance of our periodic solutions. Indeed, we have also observed that adding a small force, thus destroying the conservative property of the system, changes the solutions. Some of them are destabilized but some might remain periodic even when a force is added. The case of a forced GOY model is closer to a realistic system for turbulence and it is for future research to investigate how our solutions might be relevant for a turbulent state when a true energy cascade over many scales is present.

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Appendix A. Solutions with nine shells

In this appendix we shall discuss the case of nine shells. Following the same procedure as in the main text we obtain (the equations for \( n < 7 \) are the same as before, but for completeness we include the \( n = 6 \) equation (79) below)

\[
\alpha_6 = \frac{\alpha_4 + r(1 - \delta) + \delta}{r^3 \alpha_5} + \frac{\delta}{r},
\]

(88)

\[
\alpha_7 = -\frac{\alpha_5}{r^4 \alpha_6} + \frac{\delta}{r} + \frac{(1 - \delta) \alpha_4}{r^2 \alpha_6} + \frac{\delta}{r},
\]

(89)

and

\[
\alpha_8 = -r^2 \alpha_4 \alpha_5 + 1 - \delta + r \delta \alpha_4 + \frac{\alpha_6}{r^3 \alpha_7} + \frac{\delta}{r} + \frac{1 - \delta}{r^2} \alpha_5 \alpha_6 + \frac{\delta}{r},
\]

(90)

and

\[
\alpha_9 = \frac{\alpha_7}{r^6 \alpha_8} + \frac{\delta}{r} + \frac{1 - \delta}{r^2} \alpha_5 \alpha_6
\]

(91)

and

\[
\frac{\delta}{r} \alpha_7 \alpha_9 = \frac{\alpha_8}{r^7} (\delta - r \alpha_4) - \frac{1 - \delta}{r^2} \alpha_7 \alpha_6,
\]

(92)

as well as

\[
(1 - \delta)(\delta - \alpha_4 \alpha_5 + 1 - \delta + r \delta \alpha_4).
\]

(93)

These coupled equations can be solved using the simple Mathematica program (\( \delta = .5, r = 2 \))

\[
\text{NSolve}\left[\{t == (x + 1)/(2 y) + 1/4, r == -y(1/2 - 2 x)/(2^4 z) + 1/4 x + 1/2^3 x / z, s == -(4 xy + 1/2 + x) z / (2^5 r) + 1/4 y + 1/2^3 x y / r, t == r / (2^6 s) + 1/4 z + 1/2^3 y z / s, 1/4 r t == s^2 (1/2 - 2 x) - 1/2^3 z r, 1/2 r s == t / 2^6 (-4 x y + 1/2 + x)\}, \{x, y, z, r, s, t\}\right],
\]

(94)

where the notation is

\[
a_4 = x, a_5 = y, a_6 = z, a_7 = r, a_8 = s, a_9 = t.
\]

(95)

One obtains 32 solutions, some of which are complex, and some which lead to \( A^2 < 0 \). Only seven of these solutions are acceptable. The reader who is interested in the actual numbers should just run the above program. Increased accuracy can be obtained by putting \( d \) at the end of the square bracket in \( (94) \), where \( d \) is the number of wanted decimals.

For the \( a \)s we can also obtain in general recursion relations by the use of equation (66).

Defining \( A_n = a_n A_1 \) if \( n = 4 \mod(3) \), \( A_n = a_n A_2 \) if \( n = 5 \mod(3) \) and \( A_n = a_n A_3 \) if \( n = 6 \mod(3) \), we obtain for \( n \geq 4 \)

\[
a_{n+2} = \frac{\tilde{e}_n a_n}{r^{n-1} \alpha_{n+1}} + \frac{\delta}{r} \alpha_{n-1} + \frac{1 - \delta}{r^2} \frac{a_{n-1}}{\alpha_{n+1}},
\]

(96)

where we defined \( a_1 = a_2 = a_3 = 1 \). Here \( (n = 0, 1, 2, 3, \ldots) \)

\[
\tilde{e}_n a_n = -\frac{\delta}{r} \alpha_4, \tilde{e}_0 a_n = -(1 - \delta + r \delta \alpha_4 - r^2 \alpha_4 \alpha_5).
\]

(97)

In addition there are two ‘stop-equations’: suppose we stop at the number of shells \( 3 N \), then we must satisfy that \( A_{3 N + 1} = 0 \) and \( A_{3 N + 1} A_{3 N + 2} = 0 \), which lead to the two equations

\[
\delta \alpha_{3 N - 2} a_{3 N} = \frac{\alpha_{3 N - 1}}{r^{3 N - 1}} (\delta - r \alpha_4) - \frac{1 - \delta}{r} \alpha_{3 N - 3} \alpha_{3 N - 2}
\]

(98)
and
\[
(1 - \delta)\alpha_{3N-2}\alpha_{1N-1} = \frac{\alpha_{3N}}{r^{3N-3}} (1 - \delta + r\delta\alpha_4 - r^2\alpha_2). \tag{99}
\]
By means of equations (96), (98) and (99) it is easy to write down the relevant equations for any number of shells. The remaining problem is the numerical solution of these equations. We are not able to show the existence of acceptable solutions in general, but have checked that they exist for six and nine shells.

In the case with an infinite number of shells, \( N \to \infty \), the ‘stop-equations’ (98) and (99) should of course not be applied so we are left only with equation (96). If we make the ansatz \( \alpha_n \sim r^{\beta_n} \), then the only solution is
\[
\beta_n = -\frac{n}{3}, \tag{100}
\]
which is fully consistent with the Kolmogorov theory for turbulence. The first term in equation (96) is totally subdominant with this ansatz.

Appendix B. Solutions to the Sabra model

In this appendix we shall derive results similar to those in section 3 for the Sabra model [16]. This model is given by
\[
\dot{u}_n = i r^n \left[ u_{n+1} u_{n+2} - \frac{\delta}{r} u_{n-1} u_{n+1} + \frac{1 - \delta}{r^2} u_{n-1} u_{n-2} \right]. \tag{101}
\]
We can now easily check that if ansatz (61) is inserted in this equation, as in the GOY model, the Jacobi functions cancel on both sides and we are left with the following recursion relation:
\[
A_{n+2} = - \frac{i \alpha A_n}{A^n} + \frac{\delta}{r} \frac{A_{n+1}}{A_{n+1}} - \frac{1 - \delta}{r^2} \frac{A_{n-1} A_{n-2}}{A_{n+1}}, \tag{102}
\]
This is the Sabra version of analogous GOY model equation (66). We can then proceed to solve for the first few \( A_s \),
\[
A_3 = - \frac{i \alpha A_1}{A^*_1}, \tag{103}
\]
which is similar to equation (67),
\[
A_4 = - \frac{A_2}{r A^*_1 A^*_2} (\delta |A_1|^2 - |A_2|^2), \tag{104}
\]
similarly to equation (68), and
\[
A_5 = \frac{A_1^2 A_2}{r (A_2^*)^2 |A_1|^2 (\delta |A_1|^2 - |A_2|^2)} \left[ -a^2 k^2 |A_1|^2 + (\delta^2 - \delta + 1) |A_1|^2 |A_2|^2 + \delta |A_2|^4 \right]. \tag{105}
\]
which is similar to equation (69). One can of course continue as in section 3 to obtain further equations from general recursion relation (102).

For the case of three shells we should again require that \( A_4 = 0 \), which gives
\[
|A_2|^2 = \delta |A_1|^2. \tag{106}
\]
Requiring further that \( A_5 A_4 = 0 \), we obtain
\[
a^2 k^2 = (1 - \delta) |A_2|^2. \tag{107}
\]
Equations (106) and (107) are completely the same as the analogous equations obtained in the GOY model in section 3. However, in the case of more shells this is certainly not expected to continue.
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