A Democratic Seesaw Quark Mass Matrix
Related to the Charged Lepton Masses

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Abstract

We investigate a seesaw type mass matrix \( M_f \approx m_L M_F^{-1} m_R \) for quarks and leptons, \( f \), under the assumptions that the matrices \( m_L \) and \( m_R \) have common structures for the quarks and leptons, and that the matrix \( M_F \) characterizing the heavy fermion sector has the form \( [(\text{unit matrix})+ (\text{democratic-type matrix})] \). We obtain well-satisfied relations for quark masses and mixings related to the charged lepton masses.
I. INTRODUCTION

Why is the top quark mass $m_t$ so enhanced compared with the bottom quark mass $m_b$? Why is the $u$-quark mass $m_u$ of the order of the $d$-quark mass $m_d$? In most models, in order to understand $m_t \gg m_b$, it is inevitable to bring in a parameter which takes hierarchically different values between up- and down-quark sectors. However, from the point of view of the “democracy of families”, such a hierarchical difference seems to be unnatural. What is of great interest to us is whether we can find a model in which $M_u$ and $M_d$ are almost symmetric in their matrix structures and in their parameter values.

Recently, by applying the so-called “seesaw” mechanism [1] to quark mass matrix [2], the authors [3] have proposed a model which provides explanations of both $m_t \gg m_b$ and $m_u \sim m_d$, while keeping the model “almost” up-down symmetric. The essential idea is as follows: the mass matrices $M_f$ of quarks and leptons $f_i$ ($i = 1, 2, 3$: family index) are given by

$$M_f \simeq -m_L M_F^{-1} m_R,$$

where $F_i$ denote heavy fermions $U_i$, $D_i$, $N_i$ and $E_i$, corresponding to $f_i = u_i, d_i, \nu_i$ and $e_i$, respectively. They have assumed that the mass matrix $m_L$ ($m_R$) between $f_L$ ($f_R$) and $F_R$ ($F_L$) is common to all $f = u, d, \nu, e$ (i.e., independently of up-/down- and quark-/lepton- sectors) and $m_R$ is proportional to $m_L$, i.e., $m_R = \kappa m_L$. The variety of $M_f$ ($f = u, d, \nu, e$) comes only from the variety of the heavy fermion matrix $M_F$ ($F = U, D, N, E$). If we take a parametrization such that the parameter value in the up-quark sector gives $\det M_U \simeq 0$, while, in down-quark sector, a value slightly deviated from that in $M_U$ does not yield $\det M_D \simeq 0$ any longer, the model can provide $m_t \gg m_b$, keeping the model “almost” up-down symmetric, because of the factor $M_F^{-1}$ in the seesaw expression (1.1). On the other hand, they have taken $M_F = m_0 \lambda O_f$ as the form of the heavy fermion mass matrix $M_F$, where

$$O_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + b_f e^{i \beta_f} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \equiv 1 + 3 b_f e^{i \beta_f} X,$$

and $\lambda$ is an enhancement factor with $\lambda \gg \kappa \gg 1$. Note that the inverse of the matrix $O_f$ is again given by the form [(unit matrix) + (democratic matrix)], i.e.,

$$O_f^{-1} = 1 + 3 a_f e^{i \alpha_f} X,$$
with
\[ a_f e^{i\alpha_f} = \frac{b_f e^{i\beta_f}}{1 + 3b_f e^{i\beta_f}}. \] (1.4)

Thus, we can provide top-quark mass enhancement \( m_t \gg m_b \) in the limit of \( b_u e^{i\beta_u} \to -1/3 \), because it leads to \( |a_u| \to \infty \). On the other hand, since a democratic mass matrix \([4]\) makes only one family heavy, we can keep \( m_u \sim m_d \).

They have taken
\[ m_L = \frac{1}{\kappa} m_R = m_0 Z \equiv m_0 \left( \begin{array}{ccc} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{array} \right), \] (1.5)

where \( z_i \) are normalized as \( z_1^2 + z_2^2 + z_3^2 = 1 \) and given by
\[ \frac{z_1}{\sqrt{m_e}} = \frac{z_2}{\sqrt{m_\mu}} = \frac{z_3}{\sqrt{m_\tau}} = \frac{1}{\sqrt{m_e + m_\mu + m_\tau}}, \] (1.6)
in order to give the charged lepton mass matrix \( M_e \) for the case \( b_e = 0 \), i.e., \( M_e = m_0(\kappa/\lambda)Z^2 \). They have obtained \([3]\) reasonable quark mass ratios and Kobayashi-Maskawa (KM) \([5]\) matrix parameters by taking \( \kappa/\lambda = 0.02, b_u = -1/3, \beta_u = 0, b_d \simeq -1 \) and \( \beta \simeq -18^\circ \).

However, in their study \([3]\), the KM matrix parameters have been evaluated only numerically, and have not been expressed analytically in terms of charged lepton masses and mass matrix parameters \( \kappa/\lambda, b_f \) and \( \beta_f \). Therefore, we cannot see the dependencies of these parameters. For example, they predicted a value \( |V_{cb}| = 0.0598 \), which is somewhat large compared with the recent experimental value \([6]\) \( |V_{cb}| = 0.041 \pm 0.003 \). However, we cannot see whether the discrepancy is a fatal defect in this model or not. One of the purposes of the present paper is to express our predictions of \( |V_{ij}| \) in terms of charged lepton mass ratios and the mass matrix parameters \( \kappa/\lambda, b_f \) and \( \beta_f \). We will obtain sum rules on quark mass ratios and \( |V_{ij}| \), which are satisfied independently of those adjustable parameters.

Since they put stress on the “economy of adjustable parameters” of the model, their predictions were done by adjusting only three parameters \( \kappa/\lambda, b_d \) and \( \beta_d \). As a result, some of the predictions were in poor agreement with experiment.
Another purpose of the present paper is to improve such disagreements by changing the model slightly. We will be able to fit all the numerical predictions within experimental error. Also, a possible shape of the unitary triangle $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$ in our model will be discussed.

II. A UNIVERSAL SEESAW MASS MATRIX WITH A DEMOCRATIC $M_F$

In the present model, quarks and leptons $f_i$ belong to $f_L = (2, 1)$ and $f_R = (1, 2)$ of $SU(2)_L \times SU(2)_R$ and heavy fermions $F_i$ are vector-like, i.e., $F_L = (1, 2)$ and $F_R = (1, 1)$. The $SU(2)_L$ and $SU(2)_R$ symmetries are broken by Higgs bosons $\phi_L = (\phi^+_L, \phi^0_L)$ and $\phi_R = (\phi^+_R, \phi^0_R)$, respectively. We assume that these Higgs bosons couple to the fermions universally, but with the degree of freedom of their phases, as follows:

$$H_{Yukawa} = \sum_{i=1}^{3} (\bar{u} d)_{L_i} \left( y_{Li} \exp(i \delta_{Li}) \right) \left( \begin{array}{c} \phi^+_L \\ \phi^0_L \end{array} \right) D_{Ri}$$

$$+ \sum_{i=1}^{3} (\bar{u} d)_{L_i} \left( y_{Li} \exp(i \delta_{Li}) \right) \left( \begin{array}{c} -\phi^0_L \\ -\phi^-_L \end{array} \right) U_{Ri}$$

$$+ h.c. + (L \leftrightarrow R) + [(u, d, U, D) \to (\nu, e, N, E)] , \hspace{1cm} (2.1)$$

where $y_{Li}$ and $y_{Ri}$ are real parameters, and they are universal for the quark and lepton sectors. Therefore, the mass matrix which is sandwiched by $(\bar{f}_L, f_L)$ and $(\bar{f}_R, f_R)^T$ is given by a $6 \times 6$ matrix

$$M = \left( \begin{array}{cc} 0 & m^f_L \\ m^f_R & M_F \end{array} \right) = m_0 \left( \begin{array}{cc} 0 & P^f_L Z \\ \kappa P^f_R Z & \lambda O_f \end{array} \right) , \hspace{1cm} (2.2)$$

where $m_L = y_{L_i} \langle \phi^0_L \rangle$, $P^f_L$ and $P^f_R$ are phase matrices given by

$$P^f_L = \text{diag}(\exp(i \delta^f_{L1}), \exp(i \delta^f_{L2}), \exp(i \delta^f_{L3})) , \hspace{1cm} (2.3)$$

and (2.3) with $(L \to R)$, and the matrices $Z$ and $O_f$ are defined by (1.5) and (1.2), respectively. The previous model [3] corresponds to the model (2.2) with $P^f_R = 1$.

The KM matrix parameters are dependent only on

$$P^f_L P^d_L \equiv P = \text{diag}(e^{i \delta_1}, e^{i \delta_2}, e^{i \delta_3}) . \hspace{1cm} (2.4)$$
Of the three parameters $\delta_i$ ($i = 1, 2, 3$), only two are observable. Without loosing generality, we can put $\delta_1 = 0$. In the present model, the nine observable quantities (five quark mass ratios and four KM matrix parameters) are described by the seven parameters ($\kappa/\lambda, b_u, b_d, \beta_u, \beta_d, \delta_2, \delta_3$). Since we put the ansatz “maximal top-quark-mass enhancement” according to the Ref. [3], we fix $b_u$ and $\beta_u$ at $b_u = -1/3$ and $\beta_u = 0$. However, we still possess five free parameters. In order to economize on the number of the free parameters, we will give some speculation on these parameters in the final section. On the other hand, since the phases $\delta_{Li}^e$ and $\delta_{Ri}^e$ are not observable, we can put $P_L^e = P_R^e = 1$.

### III. QUARK MASS RATIOS IN TERMS OF CHARGED LEPTON MASSES

First, let us discuss the fermion mass spectra. Note that for the case $b_f = -1/3$ the seesaw expression (1.1) is not valid any longer because of $\text{det} M_F = 0$. In Fig. 1, we illustrate the numerical behavior of fermion masses $m_f$ versus the parameter $b_f$ which has been evaluated from the $6 \times 6$ matrix (2.2) without approximation. As seen in Fig. 1, the third fermion is sharply enhanced at $b_f = -1/3$ for $\beta_f = 0$. The calculation for the case $b_f \simeq -1/3$ must be done carefully.

For the case of $\lambda \gg \kappa \gg 1$, by expanding the eigenvalues $m_f$ ($i = 1, 2, 3$) of the mass matrix (2.2) in $\kappa/\lambda$, we obtain the following expressions of $m_f$:

\[
\left( \frac{m_f}{m_0} \right)^2 = \frac{2\sigma^2}{\rho^2 f(b, \beta)} \left( 1 + \sqrt{1 - \frac{4\sigma^2 g(b, \beta)}{\rho^4 f^2(b, \beta)}} \right)^{-1} \left( \frac{\kappa}{\lambda} \right)^2 + O \left( \frac{\kappa^4}{\lambda^4} \right), \quad (3.1)
\]

\[
\left( \frac{m_2}{m_0} \right)^2 = \frac{\rho^2 f(b, \beta)}{g(b, \beta)} \left( 1 + \sqrt{1 - \frac{4\sigma^2 g(b, \beta)}{\rho^4 f^2(b, \beta)}} \right) \left( 1 + \sqrt{1 + 4\rho^2 \frac{f(b, \beta) h(b, \beta)}{g^2(b, \beta)}} \right)^{-1} \left( \frac{\kappa}{\lambda} \right)^2
+ O \left( \frac{\kappa^4}{\lambda^4} \right), \quad (3.2)
\]

\[
\left( \frac{m_3}{m_0} \right)^2 = 3g(b, \beta) \left[ \left( \frac{\kappa}{\lambda} \right)^2 + 6h(b, \beta) \left( 1 + \sqrt{1 + 4\rho^2 \frac{f(b, \beta) h(b, \beta)}{g^2(b, \beta)}} \right) \right]^{-1} \left( \frac{\kappa}{\lambda} \right)^2
+ O \left( \frac{\kappa^4}{\lambda^4} \right), \quad (3.3)
\]
where

\[ f(b, \beta) = (1 + b)^2 - 2(1 + 2b) \frac{\sigma}{\rho^2} - 4b \left( 1 - 2 \frac{\sigma}{\rho^2} \right) \sin^2 \frac{\beta}{2}, \]

\[ g(b, \beta) = (1 + 2b)^2 - 2(1 + b)(1 + 3b)\rho - 8b(1 - 2\rho) \sin^2 \frac{\beta}{2}, \]

\[ h(b, \beta) = (1 + 3b)^2 - 12b \sin^2 \frac{\beta}{2}, \]

\[ \rho = z_1^2 z_2^2 + z_2^2 z_3^2 + z_3^2 z_1^2, \]

\[ \sigma = z_1^2 z_2^2 z_3^2, \]

and for simplicity we have denoted \( b_f \) and \( \beta_f \) as \( b \) and \( \beta \). The explicit expressions of the up-quark masses at \( b_u \simeq -\frac{1}{3} \) and the down-quark masses at \( b_d \simeq -1 \) are as follows:

\[ m_u \simeq \frac{3\sigma}{2\rho} \left( 1 + \frac{3\sigma}{4\rho^2} - \frac{3}{2} \varepsilon_u \right) \frac{\kappa}{\lambda} m_0 \simeq \frac{3m_e \kappa}{2m_\tau \lambda} m_0, \]

\[ m_e \simeq 2\rho \left[ 1 - \frac{3\sigma}{4\rho^2} - \frac{9}{2} \left( 1 - \frac{8}{3} \rho \right) \varepsilon_u \right] \frac{\kappa}{\lambda} m_0 \simeq \frac{2m_\mu \kappa}{m_\tau \lambda} m_0, \]

\[ m_t \simeq \frac{1}{\sqrt{3}} \frac{1}{\sqrt{1 + 27\varepsilon_u^2 \lambda^2 / \kappa^2}} m_0 \simeq \frac{1}{\sqrt{3}} m_0, \]

\[ m_d \simeq \frac{\sigma}{2|\sin(\beta_d/2)|\rho} \left( 1 + \frac{1}{2} \varepsilon_d \right) \frac{\kappa}{\lambda} m_0 \simeq \frac{1}{2|\sin(\beta_d/2)|} \frac{m_e \kappa}{m_\tau \lambda} m_0, \]

\[ m_s \simeq 2 \left( 1 + \frac{3}{2} \varepsilon_d - 2 \sin^2 \frac{\beta_d}{2} \right) \left| \sin \frac{\beta_d}{2} \right| \frac{\kappa}{\lambda} m_0 \simeq 2 \left| \sin \frac{\beta_d}{2} \right| \frac{m_\mu \kappa}{m_\tau \lambda} m_0, \]

\[ m_b \simeq \frac{1}{2} \left( 1 - \frac{1}{2} \varepsilon_d + \frac{5}{2} \sin^2 \frac{\beta_d}{2} \right) \frac{\kappa}{\lambda} m_0 \simeq \frac{1}{2 \lambda} m_0, \]

where small parameters \( \varepsilon_u \) and \( \varepsilon_d \) are defined by

\[ b_u = -\frac{1}{3} + \varepsilon_u, \]

\[ b_d = -1 + \varepsilon_d. \]
Here, we have taken $\beta_u = 0$, because top-quark enhancement is caused only for the case of $\beta_u = 0$ (see Fig. 1). For down-quark masses, we have shown only the expressions for $b_d \simeq -1$ and $1 \gg \sin \beta_d \neq 0$, because from the numerical study in Ref. [3], we have seen that the observed down-quark mass spectrum is in favor of $b_d \simeq -1$ and $\beta_d \simeq -20^\circ$.

The expressions (3.9) – (3.14) lead to the following sum rules which are almost independent of the parameters $\kappa/\lambda$, $\varepsilon_u$, $\varepsilon_d$ and $\beta_d$:

$$
\frac{m_u}{m_c} \simeq \frac{3}{4} \frac{m_e}{m_\mu}, \quad (3.16)
$$

$$
\frac{m_c}{m_b} \simeq \frac{4}{m_\tau}, \quad (3.17)
$$

$$
\frac{m_d m_s}{m_b^2} \simeq \frac{4}{m_\tau^2} \frac{m_e m_\mu}{m_\mu}, \quad (3.18)
$$

$$
\frac{m_u}{m_d} \simeq \frac{3 m_s}{m_c} \simeq \frac{3 m_d}{4 m_b} \frac{m_\tau}{m_\mu} \simeq 3 \left| \sin \frac{\beta_d}{2} \right| . \quad (3.19)
$$

The expression (3.16) has been obtained by the model $M_u \propto Z O_f^{-1} Z$ in Ref.[7].

In the limit of unbroken SU(2)$_L \times$SU(2)$_R$, i.e., $m_L = m_R = 0$, heavy fermion masses $m_{F_i'}$ are given by

$$
m_{F_1'} = m_{F_2'} = \lambda m_0 ,
$$

$$
m_{F_3'} = \sqrt{1 + 6 b_f \cos \beta_f + 9 b_f^2} \lambda m_0 , \quad (3.20)
$$

where $F_i'$ are mass-eigenstates for the mass matrix $M_F = m_0 \lambda O_f$ given by (1.2). As seen from (3.20), the minimum condition of the sum of the up-heavy-quark masses leads to $\beta_u = 0$ and $b_u = -1/3$. Therefore, the ansatz “maximal top-quark-mass enhancement” can be replaced by another expression that the parameters $(b_u, \beta_u)$ are fixed such that the sum of the up-heavy-quark masses becomes a minimum.

For the case of $Z \neq 0$, the heavy fermion masses are given by

$$
m_4^e \simeq m_5^e \simeq m_6^e \simeq \lambda m_0 , \quad (3.21)
$$
\[ m_4^u \simeq \frac{1}{\sqrt{3}} \kappa m_0, \quad m_5^u \simeq m_6^u \simeq \lambda m_0, \quad (3.22) \]

\[ m_4^d \simeq m_5^d \simeq \lambda m_0, \quad m_6^d \simeq 2\sqrt{1 + 3 \sin^2(\beta_d/2)} \lambda m_0, \quad (3.23) \]

where the numbering of \( m_i^f \) has been defined as \( m_4^f \leq m_5^f \leq m_6^f \) in the mass eigenstates \( F_i' \) \((i = 1, 2, 3)\). Note that only the fourth up-quark \( u_4 \) \((\equiv U_3')\) is remarkably light compared with other heavy fermions. The enhancement of the top-quark \( u_3 \) \((\equiv t)\) is caused at the cost of the lightening of \( U_3' \). Since the mass ratio \( m_4^u/m_3^u \) is given by

\[ m_4^u/m_t \simeq \kappa \quad (3.24) \]

and \( \kappa \) is of the order of \( m(W_R)/m(W_L) \), we can expect the observation of the fourth up-quark \( u_4 \) at an energy scale at which we can observe the right-handed weak bosons \( W_R \).

**IV. KM MATRIX PARAMETERS IN TERMS OF CHARGED LEPTON MASSES**

We diagonalize the \( 6 \times 6 \) mass matrix (2.2) by the following two steps. As the first step, we transform the mass matrix \( M \) (2.2) into

\[
M' = \begin{pmatrix}
M_{11}' & 0 \\
0 & M_{22}'
\end{pmatrix} \equiv \begin{pmatrix}
M_f & 0 \\
0 & M_F'
\end{pmatrix}.
\quad (4.1)
\]

At the second step, the \( 3 \times 3 \) matrix \( M_f \equiv M_{11}' \) with \( P_L^f P_R^f = 1 \), i.e., \( \tilde{M}_f \equiv P_L^{f\dagger} M_f P_R^{f\dagger} \), is diagonalized by two unitary matrices \( U_L^f \) and \( U_R^f \) as

\[
U_L^f \tilde{M}_f U_R^{f\dagger} = D_f,
\quad (4.2)
\]

where \( D_f = \text{diag}(m_1^f, m_2^f, m_3^f) \). The KM matrix \( V \) is given by

\[
V = U_L^u P U_L^{d\dagger},
\quad (4.3)
\]

where the phase matrix \( P \) is defined by (2.4).

For the up-quark sector, we put an ansatz “maximal top quark mass enhancement”, i.e., we assume that \( b_u = -1/3 \) and \( \beta_u = 0 \). Then, the unitary matrix \( U_L^u \) is given by
For the down-quark sector, we use the following approximate expression for $b_d = -1$,

\[
 U^d_L \simeq \begin{pmatrix}
 1 & -\frac{z_1}{2z_2} & -\frac{z_1}{2z_3} \\
 \frac{z_1}{2z_2} & 1 & -\frac{z_2}{z_3} \\
 \frac{z_1}{z_3} & \frac{z_2}{z_3} & 1
\end{pmatrix},
\]  

(4.4)

Here, the expression (4.5) is valid only for a sizable value of $\beta_d$, i.e., for $(z_1/z_3)^2 < \beta_d^2 \ll 1$.

Without losing generality, we can take

\[
 P = \text{diag}(1, e^{i\delta_2}, e^{i\delta_3}),
\]  

(4.6)

so that we obtain the following KM matrix elements:

\[
 V_{12} \simeq -\frac{z_1}{2z_2} \left( \frac{i e^{i\beta_d/2}}{\sin \frac{\beta_d}{2}} - e^{i\delta_2} \right),
\]  

(4.7)

\[
 V_{23} \simeq -\frac{z_2}{z_3} \left( \frac{2 - e^{i\beta_d}}{5 - 4 \cos \beta_d} e^{i\delta_2} + e^{i\delta_3} \right),
\]  

(4.8)

\[
 V_{13} \simeq -\frac{z_1}{2z_3} \left[ \frac{2 - e^{i\beta_d}}{5 - 4 \cos \beta_d} \left( 2 - e^{i\delta_2} \right) + e^{i\delta_3} \right],
\]  

(4.9)
\[ V_{31} \simeq \frac{z_1}{z_3} \left[ 1 + \frac{i e^{-i\beta_d/2}}{2 \sin \frac{\beta_d}{2}} \left( e^{i\delta_2} + e^{i\delta_3} \right) \right]. \] (4.10)

Eq. (4.7) leads to

\[ |V_{us}| \simeq \frac{z_1}{2z_2} \frac{1}{|\sin \frac{\beta_d}{2}|} \sqrt{1 + 2 \sin \frac{\beta_d}{2} \sin \left( \frac{\beta_d}{2} - \delta_2 \right) + \sin^2 \frac{\beta_d}{2}}. \] (4.11)

If we assume \(|\delta_2| \ll |\beta_d| \ll 1\), we obtain the well-known formula \([8]\)

\[ |V_{us}| \simeq \sqrt{m_d/m_s}, \] (4.12)

from (3.12) and (3.13).

Since we have already known that \(\sin^2(\beta_d/2) \simeq 0.025\) from the observed value of \(m_s/m_c\) and \(z_2/z_3 \simeq 0.24\) from the observed value of \(m_\mu/m_\tau\), we must take \(\delta_3 - \delta_2 \simeq \pi\) in order to understand the observed value \(|V_{cb}| \simeq 0.041\) \([6]\). When we put \(\delta = \delta_3 - \delta_2 - \pi\ (|\delta| \ll 1)\), we obtain

\[ |V_{cb}| \simeq 2 \frac{z_2}{z_3} \frac{\left| \sin \frac{\beta_d}{2} + \sin \frac{\delta}{2} \right|}{\sqrt{5 - 4 \cos \beta_d}} \simeq m_s \sqrt{m_c m_b} \frac{1 + \sin \frac{\delta}{2}/\sin \frac{\beta_d}{2}}{\sqrt{1 + 8 \sin^2 \frac{\beta_d}{2}}}, \] (4.13)

where we have used (3.10), (3.13) and (3.14).

Similarly, we obtain

\[ |V_{ub}| \simeq \frac{z_1}{z_3} \frac{\left| \sin \frac{\beta_d}{2} + \sin \frac{\delta}{2} + 2 \sin \frac{\delta_2}{2} \right|}{\sqrt{5 - 4 \cos \beta_d} \sin \beta_d}, \] (4.14)

so that

\[ \left| \frac{V_{ub}}{V_{cb}} \right| \simeq \frac{z_1}{2z_2} \frac{\left| 1 + \frac{2 \sin \frac{\delta_2}{2}}{\sin \frac{\beta_d}{2} + \sin \frac{\delta}{2}} \right|}{\sqrt{\frac{m_u}{2m_c}} \frac{1 + \frac{2 \sin \frac{\delta_2}{2}}{\sin \frac{\beta_d}{2} + \sin \frac{\delta}{2}}}}. \] (4.15)
or

\[
\frac{|V_{ub}|}{|V_{cb}|} \simeq |V_{us}| \sin \frac{\beta_d}{2} \left(1 + \frac{2 \sin \frac{\delta}{2}}{\sin \frac{\beta_d}{2} + \sin \frac{\delta}{2}}\right). \tag{4.16}
\]

For \(|V_{td}|\), we obtain

\[
|V_{td}| \simeq \frac{z_1}{z_3} \left|1 + \frac{\sin \frac{\beta}{2}}{\sin \frac{\beta_d}{2}}\right|, \tag{4.17}
\]

or

\[
\left|\frac{V_{td}}{V_{cb}}\right| \simeq \left|\frac{V_{us}}{V_{cb}}\right| \frac{1 + 8 \sin^2 \frac{\beta_d}{2}}{1 + \sin^2 \frac{\beta_d}{2} + 2 \sin \frac{\beta_d}{2} \sin \left(\frac{\beta_d}{2} - \delta\right)} \tag{4.18}
\]

The rephasing invariant \(J\) [9] is expressed in terms of \(|V_{ij}|\) as follows [10]:

\[
J^2 = |V_{us}|^2 |V_{cb}|^2 |V_{ub}|^2 \left(1 + |V_{us}|^2 - |V_{cb}|^2 - \omega\right)
- \frac{1}{4} \left[|V_{us}|^2 |V_{cb}|^2 - \left(|V_{us}|^2 + |V_{cb}|^2\right) |V_{ub}|^2 + \left(1 - |V_{ub}|^2\right) \omega\right]^2, \tag{4.19}
\]

where

\[
\omega = |V_{cd}|^2 - |V_{us}|^2 = |V_{td}|^2 - |V_{cb}|^2 = |V_{ub}|^2 - |V_{td}|^2. \tag{4.20}
\]

By using (4.18), i.e., \(|V_{td}|^2 \simeq |V_{us}|^2 |V_{cb}|^2\), and the observed relation \(|V_{us}|^2 \gg |V_{cb}|^2 \gg |V_{ub}|^2\), we obtain

\[
|J| \simeq \sqrt{1 - \frac{1}{4} \frac{|V_{ub}/V_{cb}|^2}{|V_{us}|^2} |V_{us}| |V_{cb}| |V_{ub}|}. \tag{4.21}
\]

V. NUMERICAL RESULTS OF THE KM MATRIX PARAMETERS

In the previous section, we have obtained approximate expressions for the KM matrix elements \(|V_{ij}|\). The results for \(|V_{us}|\), \(|V_{cb}|\), and \(|V_{td}|\), i.e., (4.11), (4.13) and (4.17) are in excellent agreement with the results from numerical evaluation of
the $6 \times 6$ mass matrix (2.2). However, for the matrix element $|V_{ub}|$, the numerical value of (4.14) is somewhat in disagreement with that which is directly evaluated from the diagonalization of the $6 \times 6$ mass matrix (2.2). This means that the approximate expressions (4.4) and (4.5) are not sufficient to evaluate a small mixing element such as $|V_{ub}|$. However, the expression (4.14) is still useful for describing the dominant behavior of $|V_{ub}|$.

In order to complement the study of the previous section, in the present section, we shall present a numerical study of the $6 \times 6$ mass matrix (1.4). The results for the mass eigenstates given in Sec. III are valid with good accuracy, so we confine our numerical study to that of the KM matrix parameters (for the numerical study of the mass eigenvalues, see Ref. [3]).

As the numerical inputs, according to Ref. [3], we use $\kappa/\lambda = 0.02$, $b_u = -1/3$, $\beta_u = 0$, $b_d = -1$ and $\beta_d = -18^\circ$, which are required for a reasonable fit with the observed quark masses. Our interest is in the behavior of $|V_{ij}|$ versus the phase parameters $\delta_2$ and $\delta_3$ defined by (2.4), because in the previous study [3], the degree of freedom of the phases ($\delta_2, \delta_3$) was not taken into consideration. In Fig. 2, we illustrate the allowed regions of ($\delta_2, \delta_3$) which give the observed values of $|V_{us}|$, $|V_{cb}|$ and $|V_{ub}|$ [6]:

$$|V_{us}| = 0.2205 \pm 0.0018,$$
$$|V_{cb}| = 0.041 \pm 0.003,$$
$$|V_{ub}|/|V_{cb}| = 0.08 \pm 0.02.$$ (5.1)

We have two allowed regions of ($\delta_2, \delta_3$): we obtain the predictions

$$|V_{us}| = 0.2195, \quad |V_{cb}| = 0.0388, \quad |V_{ub}| = 0.0028,$$

$$|V_{ub}|/|V_{cb}| = 0.072, \quad |V_{td}| = 0.0105, \quad J = -1.7 \times 10^{-5},$$ (5.2)

for ($\delta_2, \delta_3$) = ($0^\circ, 186^\circ$) and

$$|V_{us}| = 0.2211, \quad |V_{cb}| = 0.0411, \quad |V_{ub}| = 0.0027,$$

$$|V_{ub}|/|V_{cb}| = 0.065, \quad |V_{td}| = 0.0092, \quad J = -2.3 \times 10^{-5},$$ (5.3)

for ($\delta_2, \delta_3$) = ($4^\circ, 208^\circ$). In the latter case ($\delta_2 = 4^\circ$), we have taken such a value of $\delta_3$ at which the Wolfenstein parameter [11] $\rho$ [which is defined by $V_{ub} \equiv |V_{us}|/|V_{cb}|(\rho - i\eta)$] takes $\partial \rho/\partial \delta_3 = 0$ by way of trial.
In Fig. 3, we show the possible unitary-triangle shape of the present model on the \((\rho, \eta)\) plane. The vertex \((\rho, \eta)\) moves on the circle which is denoted by the solid line in Fig. 3 according as the parameter \(\delta_3\) varies from 0° to 360°. For reference, we have shown the constraints [12] from the observed values \(|V_{ub}/V_{cb}|, \Delta m_{Bd}\) and \(\varepsilon_K\). Both triangles which correspond to the cases \((\delta_2, \delta_3) = (0°, 186°)\) and \((4°, 208°)\) satisfy these constraints safely.

VI. DISCUSSIONS

In conclusion, we have obtained relations among quark mass ratios and KM matrix parameters on the basis of the democratic seesaw mass matrix (2.2). The sum rules given in III and IV are well satisfied by the observed values.

In the present model, there are seven parameters. Two of these seven, \((b_u, \beta_u)\), have been fixed as \((b_u, \beta_u) = (-1/3, 0)\) by putting the ansatz “maximal top-quark-mass enhancement” (or “minimal up-heavy-quark masses”). The values \((b_d, \beta_d)\) have been fixed as \((b_d, \beta_d) \simeq (-1, -18°)\) from the phenomenological study [3]. Why does the parameter \(b_f\) take \(b_e = 0, b_u = -1/3\) and \(b_d = -1\)?

If we consider an SU(3)-family symmetry, the parameter \(b_f e^{i\beta_f}\) gives a measure of its symmetry breaking. The symmetry is exactly unbroken for the charged heavy leptons \(E_i\), i.e., \(m_{E_1} = m_{E_2} = m_{E_3} = \lambda m_0\), in the limit of \(m_L = m_R = 0\). For up-heavy-quarks, the symmetry is badly broken, i.e., \(m_{U_1} = m_{U_2} = \lambda m_0\) and \(m_{U_3} = 0\). If the values \((b_f, \beta_f)\) are governed by a rule, the rule should be independent of low energy phenomena, i.e., the values \((b_f, \beta_f)\) should be determined only by the dynamics of heavy fermions \(F_i\), independently of that of quarks and leptons \(f_i\). For example, let us direct our attention to the deviation

\[
\Delta m_F = m_{F_3} - m_{F_2} = m_{F_3} - m_{F_1} = \left(\sqrt{1 + 6b_f \cos \beta_f + 9b_f^2} - 1\right) \lambda m_0 , \tag{6.1}
\]

which is derived from (3.20). If we assume that \(\Delta m_U + \Delta m_D = 0\), then we obtain

\[
b_d = -\frac{1}{3} \left(1 - 2 \sin^2 \frac{\beta_d}{2} + 2 \sqrt{1 - \frac{1}{2} \sin^2 \frac{\beta_d}{2}}\right) , \tag{6.2}
\]

which gives \(b_d \simeq -1\) for \(|\beta_d| \ll 1\). The similar ansatz \(\Delta m_E + \Delta m_N = 0\), i.e., \(\Delta m_N = 0\), predicts \(b_\nu = 0\) or

\[
b_\nu = -\frac{2}{3} \cos \beta_\nu . \tag{6.3}
\]
The latter solution predicts $b_\nu \simeq -2/3$ for $|\beta_\nu| \ll 1$.

Another interesting speculation on $b_f$ is as follows: If we plot the values of $b_f$ and the electric charges $Q_f$ on the $(b_f, Q_f)$ plane, the points $(b_e, Q_e) = (0, -1)$, $(b_d, Q_d) = (-1, -1/3)$, $(b_u, Q_u) = (-1/3, 2/3)$ take three corners of a square on $(b_f, Q_f)$. The remaining corner is assigned to $(b_\nu, Q_\nu) = (2/3, 0)$. In other words, the parameter $b_f$ is given by an empirical relation

$$\frac{3}{2} b_f = Q_f - \frac{1}{2} N_B + (N_L - 3N_B) ,$$

where $N_L$ and $N_B$ are lepton- and baryon-numbers, respectively.

Whether these speculations are justified or not will be checked by seeing whether neutrino masses and mixings can be described by a similar model with $b_\nu \simeq \pm 2/3$. A study of neutrino mixings based on the democratic seesaw mass matrix model will be given elsewhere [13].

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Figure Captions

Fig. 1. Masses $m_i^f$ ($i = 1, \cdots, 6$) versus $b_f$ for the case of $\kappa = 10$ and $\kappa/\lambda = 0.02$. The solid and broken lines denote for the cases of $\beta_f = 0$ and $\beta_f = -20^\circ$, respectively. The parameters $\kappa$ and $\lambda$ are defined by (2.2). At $b_f = 0$, the charged lepton masses $m_e$, $m_\mu$ and $m_\tau$ have been used as input values for the parameters $z_i$. For up- and down-quark sectors, the values $b_u = -1/3$ and $b_d = -1$ are chosen from the phenomenological study [3] of the observed quark masses.

Fig. 2. Constraints on the phase parameters ($\delta_2, \delta_3$) from the experimental values $|V_{us}| = 0.2205 \pm 0.0018$ (dotted lines), $|V_{cb}| = 0.041 \pm 0.003$ (solid lines) and $|V_{ub}/V_{cb}| = 0.08 \pm 0.02$ (dashed lines). The hatched areas denote the allowed regions.

Fig. 3. Trajectories of the vertex ($\rho, \eta$) of the unitary triangle for the cases $\delta_2 = 0^\circ$ and $\delta_2 = 4^\circ$. The points $\circ$, $\Box$, $\Diamond$ and $\triangle$ denote the vertex ($\rho, \eta$) for $\delta_3 = 180^\circ$, $190^\circ$, $200^\circ$ and $210^\circ$, respectively. The other parameters are fixed to $\kappa = 10$, $\kappa/\lambda = 0.02$, $b_u = -1/3$, $\beta_u = 0$, $b_d = -1$, and $\beta_d = -18^\circ$ from the observed quark mass ratios. The solid, broken and dot-dashed lines denote constraints from $|V_{ub}/V_{cb}|$, $|\Delta m_{B_d}|$ and $\varepsilon_K$. The two triangles correspond to the cases $(\delta_2, \delta_3) = (0^\circ, 186^\circ)$ and $(4^\circ, 208^\circ)$, respectively.
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Fig. 3
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