Local existence and uniqueness for the frictional Newton-Schrödinger equation in three dimensions

Ali BenAmor
I.P.E.I. Tunis
2, Rue Jawaher Lel Nehru
1008, Tunis
Tunisia
ali.benamor@ipeit.rnu.tn

Philippe Blanchard
Fakultät für Physik
Uni. Bielefeld
D-33501 Bielefeld
Germany
blanchard@physik.uni-bielefeld.de

Abstract

We prove, in this paper, local existence and uniqueness of solution for the frictional Newton-Schrödinger equation in three dimensions. Further we show that the blow-up alternative holds true as well as the continuous dependence of the solution w.r.t. the initial data. Our method is rather direct and based essentially on a fixed point-type theorem due to Weissinger and an approximation process.

Key words: Nonlocal nonlinearity, Duhamel’s formula, local existence.

1 Introduction

We consider the frictional Newton-Schrödinger equation (frNSE for short) in three dimensions:

\begin{equation}
\begin{cases}
  i \frac{\partial \psi}{\partial t} = -\hbar \Delta \psi + i4\pi \frac{G m^2}{\hbar} \psi \int_{\mathbb{R}^3} \frac{|\psi(y)|^2}{|\cdot-y|} dy - i4\pi \frac{G m^2}{\hbar} \psi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\psi(y)|^2|\psi(z)|^2}{|z-y|} dy dz \\
  \psi(0) = \varphi
\end{cases}
\end{equation}

where $\varphi$ is a given element from the Sobolev space $W^{1,2} := W^{1,2}(\mathbb{R}^3)$.

This equation is the limit case as $R \to 0$ of the alternative (frNSE) considered by Diosi [Dio07]. Indeed he considered a (frNSE) with kernel $U(x, x')$ in the second and third term of (1.1) that behaves like

\begin{equation}
U(x, x') \sim \begin{cases}
  c + \frac{1}{2}m\omega_G^2|x - x'|^2, & \text{if } |x - x'| << R \\
  -Gm^2|x - x'|^{-1}, & \text{if } |x - x'| >> R
\end{cases},
\end{equation}

where $R > 0$.

As observed by Diosi, equation (1.1) differs from the usual Newton-Schrödinger equation
(NSE for short) \cite{Dio07, Dio84, PM98, TM01, TM99, Adl07}:

\[
\begin{cases}
i \frac{\partial \psi}{\partial t} = -\frac{\hbar}{2m} \Delta \psi - 4\pi \frac{Gm^2}{\hbar} \int_{\mathbb{R}^3} \frac{|\psi(y)|^2}{|y-y'|} dy, \\
\psi(0) = \varphi,
\end{cases}
\tag{1.3}
\]

essentially by the term \(-i\), which is responsible for the friction, and also the last term (for normalization).

By the way, we would like to mention that the (NSE) in 1, 2 and 3 dimensions has a long standing history. It was already investigated in the 50’s by Pekar \cite[pp.29-34]{Pek54}. Since that time, there is a huge literature dealing with the equation, we cite, as examples, the papers of Lieb \cite{Lie77}, Penrose \cite{Pen96}, Nawa-Ozawa \cite{NO92}, the very instructive book of Cazenave \cite{Caz03} (and references therein), \cite{Kat89} and recently \cite{MXZ07}. The problem of existence of bound states for the 1-dimensional (NSE) was already investigated in a recent paper of Choquard and Stubbe \cite{CS07}. Also the pseudo-relativistic (NSE) was the subject of papers by Fröhlich et al. \cite{FLL07, FL07}.

However, to our best knowledge, the frictional Newton-Schrödinger equation was not treated in the literature. Unlike the (NSE), the (frNSE) cannot have a stationary solution and has no energy. Further, due the occurrence of complex coefficients, one cannot expect the conservation of charge for equation (1.1). Therefore all proof-methods based on ‘energy functional’ and ‘conservation laws’ arguments do not work any more to prove existence of solutions in this situation.

Also, since the nonlinearity is nonlocal, Kato’s method \cite{Kat87, Kat94} (based essentially on the use of a fixed point theorem and on Strichartz estimates) does not work; but maybe its generalization given in \cite[p.98]{Caz03}.

Here we shall propose an alternative method which, for proving local existence and continuous dependence does not rely on Strichartz estimates, and hence applicable also for the equation with space variable lying in a subset of \(\mathbb{R}^3\).

Our main goal, in this paper, is to prove local existence and uniqueness of (weak)-solution for the (frNSE) within the space \(X := L^\infty(I,W^{1,2})\), where \(I\) is an interval of the real line containing 0. We will also show that the blow-up alternative holds true as well as the continuous dependence w.r.t. the initial data of the solution.

To reach our purposes we will use the following strategy: Truncate the Newton kernel by eliminating the singularity lying on the diagonal. This consists to introduce the sequence of integral operators

\[
K_n \phi := \int_{\{|y| > 1/n\}} | \cdot - y |^{-1} \phi(y) \, dy.
\]

We obtain in this manner a new function representing the interaction given by (up to a complex factor) \(f_n(\phi) = \phi K_n(|\phi|^2)\) that preserves the space \(W^{1,2}\). In this stage, using Duhamel’s formula together with a fixed point argument (Satz of Weissinger) we prove existence and uniqueness of solution for the approximate problem (replace \(\phi K(|\phi|^2)\) by \(f_n(\phi)\)). The main ingredient of the proof, in this step, is the local uniform Lipschitz property of the \(f_n\)’s.
After this step we prove that the approximate solutions tend to the solution of (1.1). This will rely upon the crucial property that functions $g_1, g_2$ (defined below) are weakly continuous. Finally continuous dependence w.r.t. initial data will be established.

2 Preparatory results

The aim of this section is the proof of some preparatory results for the local existence and uniqueness theorem.

We set $W^{-1,2}$ the dual space of $W^{1,2}$ and introduce the functions

$$g_1 : W^{1,2} \to W^{-1,2}, \quad \psi \mapsto \psi \int_{\mathbb{R}^3} |\psi(y)|^2 \, dy,$$

and

$$g_2 : W^{1,2} \to W^{-1,2}, \quad \psi \mapsto \psi \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\psi(x)|^2 |\psi(y)|^2 \, dy \, dx,$$  

Later we shall prove that $g_1, g_2$ are well-defined, continuous and are bounded on bounded sets.

For a fixed $\varphi \in W^{1,2}$, by a weak solution of (1.1) we mean a function

$$\psi \in L^\infty (I, W^{1,2}) \cap W^{1,\infty} (I, W^{-1,2}),$$

that satisfies satisfies

$$i \frac{\partial \psi}{\partial t} = -\frac{\hbar}{2m} \Delta \psi + i4\pi \frac{Gm^2}{\hbar} g_1(\psi) - i4\pi \frac{Gm^2}{\hbar} g_2(\psi) \text{ in } W^{-1,2} \text{ for a.e. } t \in I, \psi(0) = \varphi. \quad (2.3)$$

For the convenience of the reader we reproduce the comments made by Cazenave (see [Caz83, pp.56-57]) to visualize that the position of the problem, in this manner is coherent. From the already indicated properties of both functions $g_1, g_2$, we observe that if function $\psi \in X := L^\infty (I, W^{1,2})$ then both $g_1(\psi), g_2(\psi)$ are in the space $L^\infty (I, W^{-1,2})$, yielding

$$-\frac{\hbar}{2m} \Delta \psi + i4\pi \frac{Gm^2}{\hbar} g_1(\psi) - i4\pi \frac{Gm^2}{\hbar} g_2(\psi) \in L^\infty (I, W^{-1,2}).$$

Thus if $\psi$ satisfies the first part of (2.3) then it is in $W^{1,\infty} (I, W^{-1,2})$.

Furthermore, the fact that $\psi \in X \cap W^{1,\infty} (I, W^{-1,2})$ yields that $\psi \in C(I, L^2(\mathbb{R}^3))$, which implies that the second identity in (2.3) is meaningful.

From now on we set

$$\alpha_1 := \frac{\hbar}{2m}, \quad \alpha_2 := 4\pi G \frac{m^2}{\hbar},$$

and for every $\psi \in W^{1,2}$

$$K \psi := \int_{\mathbb{R}^3} \frac{\psi(y)}{|\cdot - y|} \, dy, \quad G_1(\psi) := \int_{\mathbb{R}^3} |\psi(x)|^2 K(|\psi(x)|^2) \, dx$$
Let us recall some classical inequalities that we shall use many times. First, the Sobolev inequality: For every $p \in [2,6]$ there is a constant $C_{\text{sob}}(p)$ such that

$$\left( \int_{\mathbb{R}^3} |\psi|^p \, dx \right)^{2/p} \leq C_{\text{sob}}(p) \left( \int_{\mathbb{R}^3} |\nabla \psi|^2 \, dx + \int_{\mathbb{R}^3} |\psi|^2 \, dx \right), \quad \forall \psi \in W^{1,2}. \quad (2.4)$$

Second the inequality satisfied by Riesz potentials (see [SC 01]): For every $1 < p < 3/2$, set $q := \frac{3p}{3-2p}$. Then there is a constant $C_{\text{Riesz}}(p)$ such that

$$\|K\psi\|_{L^q} \leq C_{\text{Riesz}}(p) \|\psi\|_{L^p}, \quad \forall \psi \in L^p. \quad (2.5)$$

As a notation we shall designate by $\int \cdots \, dx$ the integral on $\mathbb{R}^3$ w.r.t. Lebesgue measure of a given function.

**Lemma 2.1.** The functions $g_1, g_2$ are well defined. Moreover they are bounded on bounded sets.

**Proof.** For every $\varphi, \psi \in W^{1,2}$. Making use of inequality (2.4) together with (2.5) we get

$$\int |\varphi g_1(\psi)| \, dx \leq \left( \int |\varphi\psi|^{\frac{2}{3}} \right)^{2/3} \left( \int (K(|\psi|^2))^{\frac{2}{3}} \right)^{5/3} \leq C_{\text{Riesz}} C_{\text{sob}} C_{\text{sob}}^\prime \|\varphi\|_{W^{1,2}} \|\psi\|_{W^{1,2}}^2. \quad (2.6)$$

Hence for every fixed $\psi \in W^{1,2}$, the function

$$T := W^{1,2} \to \mathbb{R}, \quad \varphi \mapsto \text{Re} \left( \int \varphi g_1(\psi) \, dx \right)$$

is linear and continuous. Thus for every $\psi \in W^{1,2}$, $g_1(\psi) \in W^{-1,2}$ and

$$\|T \circ g_1(\psi)\|_{W^{-1,2}} \leq C_{\text{Riesz}} C_{\text{sob}} C_{\text{sob}}^\prime \|\psi\|_{W^{1,2}}^3$$

yielding that $g_1$ is bounded on bounded sets. For $g_2$, we have $g_2(\psi) = \psi G_1(\psi)$ and by the same inequalities

$$G_1(\psi) \leq \left( \int |\psi|^{\frac{2}{3}} \right)^{2/3} \cdot C_{\text{Riesz}} \left( \int |\psi|^{\frac{30}{17}} \right)^{\frac{4}{30}} \leq C_{\text{Riesz}} C_{\text{sob}} \|\psi\|_{W^{1,2}}^3. \quad (2.7)$$

Thus $g_2$ is well defined, bounded on bounded sets and is even in $W^{1,2}$. □

The functions $g_1, g_2$ enjoy further interesting properties especially the local Lipschitz property:

**Lemma 2.2.** The function $g_1$ satisfies the following properties: There is $r_1, r_2 \in [2,6)$ such that

i) $g_1 \in C(W^{1,2}, L^{r_1})$.

ii) For every $0 < M < \infty$, there is a constant $C(M)$ such that

$$\|g_1(\varphi) - g_1(\psi)\|_{L^{r_2}} \leq C(M) \|\varphi - \psi\|_{L^{r_1}},$$

for every $\varphi, \psi \in W^{1,2}$ such that $\|\varphi\|_{W^{1,2}} \leq M$, $\|\psi\|_{W^{1,2}} \leq M.$
Proof. We have for every $\psi \in W^{1,2}$, $g_1(\psi) = \psi K(\vert \psi \vert^2)$ and $\psi \in L^p$ for every $2 \leq p \leq 6$.

i): We must first show that there is $\rho \in [2,6)$ such that

$$\forall \psi \in W^{1,2}, \; g(\psi) \in L^{\rho'}.$$ 

Let $\rho \in [2,6)$ be fixed and $r$ a real number such that

$$ (1) : 1 < r < \frac{3}{3-\rho'}, \; (2) : 2 \leq r \rho' \leq 6 \text{ and } (3) : 1 < r < \frac{12}{6+\rho'}$$

(2.8)

Note that if $\psi \in L^{r\rho'}$ and $K(\vert \psi \vert^2) \in L^{r\rho'}$, then by Hölder inequality $\psi K(\vert \psi \vert^2) \in L^{\rho'}$. Therefore we will prove that if $\psi \in W^{1,2}$ and if $r, \rho$ satisfy the above conditions then (i) is fulfilled as well as (ii) with $r_1 = r \rho'$.

Now if $\psi \in W^{1,2}$ and if $r, \rho$ satisfy (1-2) of (2.8) then $r \rho' > 3$. Thus setting $p = \frac{3r \rho'}{3+2r \rho'}$ then $1 < p < 3/2$. Activating inequality (2.5) together with inequality (2.4) gives

$$\left( \int (K(\vert \psi \vert^2))^{r \rho'} \, dx \right)^{\frac{1}{r \rho'}} \leq C_{\text{Riesz}} \left( \int \vert \psi \vert^{2p} \, dx \right)^{\frac{1}{p}} \leq C_{\text{Riesz}} C_{\text{sob}}^2 \Vert \psi \Vert^2_{W^{1,2}}.$$ 

Next from Hölder inequality, we achieve

$$\left( \int \vert g_1(\psi) \vert^{\rho'} \, dx \right)^{\frac{1}{\rho'}} \leq \left( \int \vert \psi \vert^{r \rho'} \right)^{\frac{1}{r \rho'}} \left( \int (K(\vert \psi \vert^2))^{r \rho'} \, dx \right)^{\frac{1}{r \rho'}} \leq C_{\text{sob}}'' C_{\text{Riesz}} C_{\text{sob}}^2 \Vert \psi \Vert^3_{W^{1,2}},$$

(2.9)

yielding that $g_1 : W^{1,2} \rightarrow L^{\rho'}$ is well defined.

Continuity: Let $(\psi_k) \subset W^{1,2}$ converging in $W^{1,2}$ to $\psi$. Then

$$\int \vert g_1(\psi_k) - g_1(\psi) \vert^{\rho'} \, dx \leq 2^{\rho'-1} \int \vert (\psi_k - \psi) K(\vert \psi_k \vert^2) \vert^{\rho'} \, dx + 2^{\rho'-1} \int \vert \psi K(\vert \psi_k \vert^2 - \vert \psi \vert^2) \vert^{\rho'} \, dx.$$ 

The preceding calculus shows that

$$\int \vert g_1(\psi_k) - g_1(\psi) \vert^{\rho'} \, dx \leq C \Vert \psi_k - \psi \Vert_{W^{1,2}} \Vert \psi_k \Vert^2_{W^{1,2}} + C \Vert \psi \Vert_{W^{1,2}} \Vert \psi_k \Vert^2 - \Vert \psi \Vert^2 \Vert_{L^{\rho'}} \rightarrow 0 \text{ as } k \rightarrow \infty,$$

(2.10)

and $g_1$ is continuous.

ii) Fix $M > 0$ and $\varphi, \psi \in W^{1,2}$ such that $\Vert \varphi \Vert_{W^{1,2}}, \Vert \psi \Vert_{W^{1,2}} \leq M$; Let $r, \rho$ be as in (2.8). Then

$$g_1(\varphi) - g_1(\psi) = \varphi K(\vert \varphi \vert^2) - \psi K(\vert \psi \vert^2) = (\varphi - \psi) K(\vert \varphi \vert^2) + \psi K(\vert \varphi \vert^2 - \vert \psi \vert^2).$$

Thus

$$\Vert g(\varphi) - g(\psi) \Vert_{L^{\rho'}} \leq \Vert (\varphi - \psi) K(\vert \varphi \vert^2) \Vert_{L^{\rho'}} + \Vert \psi K(\vert \varphi \vert^2 - \vert \psi \vert^2) \Vert_{L^{\rho'}}.$$ 

Set $p = \frac{3 \rho'}{3+2r \rho'}$. As before from Hölder, Riesz and Sobolev inequalities we get

$$\Vert (\varphi - \psi) K(\vert \varphi \vert^2) \Vert_{L^{\rho'}} \leq C \Vert \varphi - \psi \Vert_{L^{r \rho'}} \Vert \varphi \Vert^2_{L^{1,2}},$$

(2.11)
\[ \|\psi K(\varphi^2 - \psi^2)\|_{L^{r'}} \leq C\|\psi\|_{L^{r'}}\|K(\varphi^2 - \psi^2)\|_{L^{r'}} \leq C\|\varphi\|_{L^{r'}}\|(\varphi - \psi)(\varphi + \psi)\|_{L^{r'}}. \] (2.12)

From the conditions imposed on \( r, \rho \) we conclude that for \( \beta = \frac{r\rho'}{p} \) we have \( \beta > 1 \) and \( \beta'p \in [2, 6] \). Thus by the same arguments and observing that \( \beta p = r\rho' \), we get

\[ \|(\varphi - \psi)(\varphi + \psi)\|_{L^p} \leq \|\varphi - \psi\|_{L^{\beta p}}\|(\varphi + \psi)\|_{L^{\beta'p}} \leq 2CM_{\text{sob}}\|\varphi - \psi\|_{L^{r'}p}. \] (2.13)

Finally putting all together, we get

\[ \|g_1(\varphi) - g_1(\psi)\|_{L^{r'}} \leq (CM^2 + 2C_{\text{sob}}M^2)\|\varphi - \psi\|_{L^{r'}p}. \] (2.14)

and (ii) is proved.

In order to show that the function \( g_2 \) satisfies similar conditions as \( g_1 \), we shall need further auxiliary results concerning the function \( g_1 \).

**Lemma 2.3.** For every \( \psi \in W^{1,2} \), we have

\[ \|g_1(\psi)\|_{L^2}^2 \leq C_{\text{Riesz}}C_{\text{sob}}\|\psi\|_{W^{1,2}}^4. \] (2.15)

**Proof.** It suffices to prove the inequality for positive functions from \( W^{1,2} \). Let \( \psi \) be such a function and \( \alpha \geq 3 \). Then \( 2 < 2\alpha' \leq 6, 1 < \gamma := \frac{6\alpha}{3 + 4\alpha} < 3/2 \) and \( 2\gamma \in [2, 6] \). Thus by Hölder inequality together with Riesz and Sobolev inequalities we obtain

\[ \|g_1(\psi)\|_{L^2}^2 \leq \left(\int \psi^{2\alpha'}\right)^{1/\alpha'}\left(\int (K(\psi^2))^{2\alpha}\right)^{1/\alpha} \leq C_{\text{sob}}C_{\text{Riesz}}\psi_{W^{1,2}}^2\left(\int \psi^{2\gamma} \, dx\right)^{1/\gamma} \]
\[ \leq C_{\text{sob}}C'_{\text{sob}}C_{\text{Riesz}}\|\psi\|_{W^{1,2}}^4. \] (2.16)

Here are the main properties of the function \( g_2 \)

**Lemma 2.4.** The function, \( g_2 \) is well defined and satisfies:

i) \( g_2 \in C(W^{1,2}, W^{-1,2}) \).

ii) \( g_2 \in C(W^{1,2}, L^2) \).

iii) For every \( M > 0 \) and every \( \varphi, \psi \in W^{1,2} \) such that \( \|\varphi\|_{W^{1,2}}, \|\psi\|_{W^{1,2}} \leq M \), we have

\[ \|g_2(\varphi) - g_2(\psi)\|_{L^2} \leq M^3 A\|\varphi - \psi\|_{L^2}, \]

where \( A \) is a constant depending only on Riesz and Sobolev constants.
Proof. Clearly $g_2$ is well defined. The property (i) follows from (ii), and the latter one follows from (iii) if we just show that $g_2$ from $W^{1,2}$ into $L^2$ is well defined. On the other we observe that $g_2(\psi) = \psi G_1(\psi)$ and the function $G_1 : W^{1,2} \to \mathbb{R}$ is well defined. Thus $g_2$ is well defined as well from $W^{1,2}$ into $L^2$.

(iii) Let $M > 0$ and $\varphi, \psi \in W^{1,2}$ such that $\|\varphi\|_{W^{1,2}}, \|\psi\|_{W^{1,2}} \leq M$. Then

$$
\|g_2(\varphi) - g_2(\psi)\|_{L^2} \leq G_1(\varphi)\|\varphi - \psi\|_{L^2} + |G_1(\varphi) - G_1(\psi)|\|\psi\|_{L^2}.
$$

(2.17)

We now proceed to establish the sought estimate for each term of the latter inequality separately. For the first term of RHS, making use of Lemma 2.3, we get

$$
G_1(\varphi) = \int |\varphi|^2 K(|\varphi|^2) \, dx \leq \|\varphi\|_{L^2}^2 \|g_1(\varphi)\|_{L^2} \leq M^3 (C_{Riesz} C_{\text{sob}} C_{\text{sob}}')^{1/2}.
$$

(2.18)

yielding

$$
G_1(\varphi)\|\varphi - \psi\|_{L^2} \leq M^3 (C_{Riesz} C_{\text{sob}} C_{\text{sob}}')^{1/2}\|\varphi - \psi\|_{L^2}.
$$

(2.19)

To estimate the second term, we introduce the function

$$
\Gamma := [0, 1] \to \mathbb{R}, \ t \mapsto G_1(t\varphi + (1 - t)\psi).
$$

(2.20)

Obviously $G_1$ is of class $C^1$ (Fréchet) and $G'_1 = 4g_1$. Thus $\Gamma$ is differentiable as well and

$$
\Gamma'(t) = 4 \text{Re} \int g_1(t\varphi + (1 - t)\psi)(\varphi - \psi) \, dx.
$$

(2.21)

Hence

$$
|\Gamma(1) - \Gamma(0)| = |G_1(\varphi) - G_1(\psi)| \leq 4 \sup_{t \in [0, 1]} \int g_1(t\varphi + (1 - t)\psi)(\varphi - \psi) \, dx
\leq 4\|\varphi - \psi\|_{L^2} \sup_{t \in [0, 1]} \|g_1(t\varphi + (1 - t)\psi)\|_{L^2}
\leq 4(C_{Riesz} C_{\text{sob}} C_{\text{sob}}')^{1/2}\|\varphi - \psi\|_{L^2}(\|\varphi\|_{L^2} + \|\psi\|_{L^2})^2
\leq 16M^2(C_{Riesz} C_{\text{sob}} C_{\text{sob}}')^{1/2}\|\varphi - \psi\|_{L^2}.
$$

(2.22)

Finally putting equations (2.19) and (2.22) together yields the result, which completes the proof.

\[
\square
\]

Lastly we establish the most important feature of functions $g_1, g_2$, namely their weak continuity.

In the sequel we denote by

$$
\tilde{K}_n \phi := \int_{\{|1 - y| \leq 1/n\}} \frac{|\phi(y)|^2}{|y|} \, dy.
$$

(2.23)

We shall also designate by $B$ various open balls in $\mathbb{R}^3$. 

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Lemma 2.5.  

i) The function $g_1 : W^{1,2} \to L^\rho'$ is continuous w.r.t. the weak topologies of $W^{1,2}$ and $L^\rho'$.  

ii) The function $g_2 : W^{1,2} \to L^2$ is continuous w.r.t. the weak topologies of $W^{1,2}$ and $L^2$.

Proof. We will repeatedly use the known fact that for every open ball $B \subset \mathbb{R}^3$ and every $2 \leq s < 6$ the space $W^{1,2}$ embeds compactly into $L^s(B)$.  

(i): Let $(\psi_n) \subset W^{1,2}$, $\psi \subset W^{1,2}$ such that $\psi_n \rightharpoonup \psi$ in $W^{1,2}$, $B$ be an open ball in $\mathbb{R}^3$ and $w \in L^\rho$. Then

$$
\int w(g_1(\psi_n) - g_1(\psi)) \, dx = \int (\psi_n - \psi)wK(|\psi_n|^2) \, dx + \int \psi wK(|\psi_n|^2 - |\psi|^2) \, dx \tag{2.24}
$$

We decompose the first integral into

$I_1(n) := \int (\psi_n - \psi)wK(|\psi_n|^2) \, dx = \int_B (\psi_n - \psi)wK(|\psi_n|^2) \, dx + \int_{B^c} (\psi_n - \psi)wK(|\psi_n|^2) \, dx$

Let $\rho$ and $r$ be the exponents given by Lemma 2.2. By Hölder’s inequality and the computations made in the proof of Lemma 2.2 we get

$$
\int_B (\psi_n - \psi)wK(|\psi_n|^2) \, dx \leq \|w\|_{L^\rho(B)}\left(\int_B |\psi_n - \psi|^{r \rho'} \, dx\right)^{1/r \rho'} \left(\int_B (|K(\psi_n)|^2)^{r' \rho'} \, dx\right)^{1/r' \rho'} \\
\leq C\|w\|_{L^\rho(B)}\|\psi_n\|_{W^{1,2}}^2 \left(\int_B |\psi_n - \psi|^{r \rho'} \, dx\right)^{1/r \rho'} \rightarrow 0, \quad (2.25)
$$

by the fact that $2 \leq r \rho' < 6$ and the compactness of the embedding of $W^{1,2}$ into $L^{r \rho'}$. By the same arguments we get

$$
\int_{B^c} (\psi_n - \psi)wK(|\psi_n|^2) \, dx \leq \|w\|_{L^\rho(B^c)}\left(\int_{B^c} |\psi_n - \psi|^{r \rho'} \, dx\right)^{1/r \rho'} \left(\int_{B^c} (|K(\psi_n)|^2)^{r' \rho'} \, dx\right)^{1/r' \rho'} \\
\leq C\|w\|_{L^\rho(B^c)}\|\psi_n\|_{W^{1,2}}^2 \left(\int_{B^c} |\psi_n - \psi|^{r \rho'} \, dx\right)^{1/r \rho'} \leq C\|w\|_{L^\rho(B^c)}. \quad (2.26)
$$

Now given $\epsilon > 0$, we choose $B$ so that $\|w\|_{L^\rho(B^c)} < \epsilon$ and get

$$
\int_{B^c} (\psi_n - \psi)wK(|\psi_n|^2) \, dx \leq \epsilon, \quad \forall \epsilon > 0, \quad (2.27)
$$

yielding the convergence toward zero of $I_1(n)$. We also decompose the second integral

$I_2(n) := \int \psi wK(|\psi_n|^2 - |\psi|^2) \, dx = \int_B \psi wK(|\psi_n|^2 - |\psi|^2) \, dx + \int_{B^c} \psi wK(|\psi_n|^2 - |\psi|^2) \, dx

= \int (|\psi_n|^2 - |\psi|^2)K(1_B \psi) \, dx + \int_{B^c} \psi wK(|\psi_n|^2 - |\psi|^2) \, dx,$
and rewrite the integral

\[ I'_2(n) := \int (|\psi_n|^2 - |\psi|^2) K(1_B w \psi) \, dx = \int_B (|\psi_n|^2 - |\psi|^2) K(1_B w \psi) \, dx + \int_{B^c} (|\psi_n|^2 - |\psi|^2) K(1_B w \psi) \, dx. \]

Choose \( p \) so that \( 1 \leq p < \frac{6\rho}{7\rho - 6} \). Then \( 1 < p < \frac{3}{2} \). Setting \( \alpha = \frac{p}{p-1} \) and \( \beta = \frac{3\alpha}{2\alpha + 3} \), yields

\[ 1 < p < \frac{3}{2} \quad \text{and} \quad 2 \leq \frac{\rho \beta}{\rho - \beta} < 6. \]  

(2.28)

Using Hölder’s inequality together with Riesz’s potential estimate (2.5) we obtain

\[ \int_B \left| |\psi_n|^2 - |\psi|^2 \right| K(1_B w \psi) \, dx \leq \| |\psi_n|^2 - |\psi|^2 \|_{L^p(B)} \| K(w \psi) \|_{L^{\rho}(B)} \leq C(M) \| \psi_n - \psi \|_{L^{2p}(B)} \| w \|_{L^p(B)} \| \psi \|_{L^{\rho}(B^c)}. \]

(2.29)

Taking conditions (2.28) into account yields

\[ \| \psi \|_{L^{\rho \frac{\alpha}{\rho - \beta}}(B)} < \infty \quad \text{and} \quad \lim_{n \to \infty} \| \psi_n - \psi \|_{L^{2p}(B)} = 0. \]

We conclude thereby that the latter integral tends to zero as \( n \to \infty \).

By the same way we get the estimate

\[ \int_{B^c} \left| |\psi_n|^2 - |\psi|^2 \right| K(1_B w \psi) \, dx \leq \| |\psi_n|^2 - |\psi|^2 \|_{L^p(B^c)} \| K(1_B w \psi) \|_{L^{\rho}(B^c)} \leq C(M) \| K(1_B w \psi) \|_{L^{\rho}(B^c)}. \]

(2.30)

The already made calculus shows that

\[ \int |K(1_B w \psi)|^\alpha \, dx < \infty. \]

(2.31)

Whence choosing, for every \( \epsilon > 0 \), a \( B \) such that \( \| K(1_B w \psi) \|_{L^{\rho}(B^c)} < \epsilon \), gives

\[ \int_{B^c} \left| |\psi_n|^2 - |\psi|^2 \right| K(1_B w \psi) \, dx < \epsilon, \quad \forall \ \epsilon > 0, \]

(2.32)

and thereby \( \lim_{n \to \infty} I'_2(n) = 0 \). Finally writing

\[ \int_{B^c} w \psi K(|\psi_n|^2 - |\psi|^2) \, dx = \int (|\psi_n|^2 - |\psi|^2) K(1_{B^c} w \psi) \, dx \]

and using the same techniques as before we obtain

\[ \lim_{n \to \infty} \int_{B^c} w \psi K(|\psi_n|^2 - |\psi|^2) \, dx < \epsilon, \quad \forall \ \epsilon > 0. \]

(2.33)
Thus \( \lim_{n \to \infty} (I_1(n) + I_2(n)) = 0 \), which was to be proved.

(ii): Let \((\psi_n), \psi\) be as before. We rewrite

\[
g_2(\psi_n) = \psi_n(\psi_n, g_1(\psi_n))_{L^p, L^{p'}} = \psi_n(\psi_n, g_1(\psi_n) - g_1(\psi))_{L^p, L^{p'}}
+ \psi_n(\psi_n - \psi, g_1(\psi))_{L^p, L^{p'}} + \psi_n(\psi, g_1(\psi))_{L^p, L^{p'}}
- \psi(\psi, g_1(\psi))_{L^p, L^{p'}} = g_2(\psi), \text{ in } L^2 \text{ as } n \to \infty. \tag{2.34}
\]

which finishes the proof. \(\square\)

3 Existence and uniqueness

Thanks to the properties of \(g_1, g_2\), we conclude that a function \(\psi \in L^\infty(I, W^{1,2})\) solves the (frNSE) if and only if it satisfies Duhamel’s formula:

\[
\psi(t) = e^{i\alpha t H} \varphi + i\alpha_2 \int_0^t e^{i\alpha_1 (t-s) H} g_1(\psi(s)) \, ds - i\alpha_2 \int_0^t e^{i\alpha_1 (t-s) H} g_2(\psi(s)) \, ds, \forall t \in I, \tag{3.1}
\]

where \(H\) stands for Laplace operator on the Euclidean space \(\mathbb{R}^3\).

Observe that since \(g_1\) does not preserve the space \(W^{1,2}\) it is not possible to use a fixed point argument to solve the problem directly. Also the occurrence of complex coefficients in the (frNSE) does not permit use of classical results, especially those based on 'conservation laws' (see [Caz(3)]) to solve the problem. Instead we shall truncate the Newton kernel, construct a sequence of approximate solutions then pass to the limit.

To that end we introduce the sequence of functions \((f_n), n \in \mathbb{N}^*:\)

\[
f_n(\psi) := \psi \int_{\{|y| > 1/n\}} \frac{\psi(y)^2}{|\cdot - y|} \, dy := \psi K_n(|\psi|^2).
\]

The function \(f_n\) enjoys the following properties

**Lemma 3.1.**

i) For every \(\psi \in W^{1,2}, |f_n(\psi)| \leq |g_1(\psi)|.\)

ii) For every \(\psi \in W^{1,2}, f_n(\psi) \in W^{1,2}.\)

iii) Let \(\rho, r_1\) be the exponents given by Lemma 2.2. Then for every \(0 < M < \infty,\) there is a constant \(C(M)\) such that for each \(n \in \mathbb{N}^*,\)

\[
\|f_n(\varphi) - f_n(\psi)\|_{L^{p'}} \leq C(M)\|\varphi - \psi\|_{L^{r_1}}.
\]

for every \(\varphi, \psi \in W^{1,2}\) such that \(\|\varphi\|_{W^{1,2}} \leq M, \|\psi\|_{W^{1,2}} \leq M.\)

**Proof.** The proof of property (i) is obvious. To prove (ii) observe that by (i) since for every \(\psi \in L^2, g_1(\psi) \in L^2\) then \(f_n(\psi) \in L^2\) as well. Now a direct computation yields that for every \(\psi \in W^{1,2}, \nabla f_n(\psi) = K_n(|\psi|^2)\nabla \psi + \psi \nabla K_n(|\psi|^2)\) and

\[
|K_n(|\psi|^2)\nabla \psi| \leq n\|\psi\|^2_{L^2}\|\nabla \psi\| \in L^2, \tag{3.2}
\]
The proof of (iii) follows by using (i), the fact that \( |K_n \phi| \leq K(|\phi|) \) and Lemma 2.2.

The most important statement of the latter lemma is property (iii), which indicates that the Lipschitz constant, as well as the exponents \( \rho \) and \( r_1 \), are independent of the integer \( n \).

Consider now the approximate problem

\[
\begin{align*}
\left\{ \begin{array}{l}
i \frac{\partial \psi}{\partial t} = -\alpha_1 \Delta \psi + i \alpha_2 f_n(\psi) - i \alpha_2 g_2(\psi) \\
\psi(0) = \varphi
\end{array} \right. \quad (3.4)
\end{align*}
\]

As before, we assert that \( \psi_n \in L^\infty(I, W^{1,2}) \) solves (3.4) if and only if it satisfies the related Duhamel’s formula

\[
\psi_n(t) = e^{i \alpha_1 t H} \varphi + i \alpha_2 \int_0^t e^{i \alpha_1 (t-s) H} f_n(\psi_n(s)) \, ds - i \alpha_2 \int_0^t e^{i \alpha_1 (t-s) H} g_2(\psi_n(s)) \, ds. \quad (3.5)
\]

Thanks to the already observed fact that both \( f_n \) and \( g_2 \) preserve the space \( W^{1,2} \), it is possible to solve the latter equation via a fixed point-argument. However, we shall use that argument not for the operator defined by the RHS of (3.5), but for some power of it. To this end we use a theorem due to Weissinger (see [Heu06]):

**Theorem 3.1. (Weissinger)** Let \( Y \) be a Banach space and \( (\alpha_k) \) a sequence of positive numbers such that \( \sum \alpha_k < \infty \). Let \( F \subset Y \) be closed and \( A : F \to F \) be an operator such that

\[
\| A^k \varphi - A^k \psi \| \leq \alpha_k \| \varphi - \psi \|, \quad \forall k \in \mathbb{N} \text{ and } \forall \varphi, \psi \in Y.
\]

Then \( A \) possesses a unique fixed point. Furthermore the fixed point can be obtained as the limit of the sequence defined by \( \psi_0 = \Phi \in F \), and \( \psi_{k+1} = A \psi_k \).

We shall also make use of the known fact that the operator \( e^{itH} \) is unitary on each of the spaces \( L^2, W^{1,2} \) and \( W^{-1,2} \).

Now let \( M > 0, T > 0 \) and \( \varphi \in W^{1,2} \) be given. Set \( I = [-T, T], \ X := L^\infty(I, W^{1,2}) \) and

\[
F_M := \{ \psi \in X : \| \psi - e^{i H} \varphi \|_X \leq M \}.
\]

We will first determine \( T \) so that for every integer \( n \) the operators

\[
A_n : F_M \to X, \ \psi \mapsto e^{i \alpha_1 t H} \varphi + i \alpha_2 \int_0^t e^{i \alpha_1 (t-s) H} f_n(\psi(s)) \, ds - i \alpha_2 \int_0^t e^{i \alpha_1 (t-s) H} g_2(\psi(s)) \, ds
\]

maps \( F_M \) into itself.

For \( \psi \in X \) set

\[
S_n \psi := \int_0^t e^{i \alpha_1 (t-s) H} f_n(\psi(s)) \, ds \quad \text{and} \quad U \psi := \int_0^t e^{i \alpha_1 (t-s) H} g_2(\psi(s)) \, ds. \quad (3.6)
\]
Lemma 3.2. Let $M > 0$ and $\varphi \in W^{1,2}$ such that $\|\varphi\|_{W^{1,2}} \leq M$. There is $0 < T = T(M)$ such that for every $n \in \mathbb{N}$ the operator $A_n$ maps $F$ into itself.

Proof. Let $M > 0$ be fixed and $\psi \in F_M$. Then

$$\|S_n\psi\|_{W^{1,2}} + \|U\psi\|_{W^{1,2}} = \|S_n\psi\|_{W^{-1,2}} + \|U\psi\|_{W^{1,2}}$$

$$\leq \int_0^{|t|} \|e^{i\alpha_1(t-s)H}\|_{W^{-1,2},W^{-1,2}} \|f_n(\psi(s))\|_{W^{-1,2}} ds$$

$$+ \int_0^{|t|} \|e^{i\alpha_1(t-s)H}\|_{W^{1,2},W^{1,2}} \|g_2(\psi(s))\|_{W^{1,2}} ds$$

$$\leq C_1 \int_0^{|t|} \|f_n(\psi(s))\|_{L^p} ds + C_2 \int_0^{|t|} \|\psi(s)\|^4_{W^{1,2}} ds$$

$$\leq C_1(M) \int_0^{|t|} \|g_1(\psi(s))\|_{L^p} ds + C_2(M) \int_0^{|t|} \|\psi(s)\|^4_{W^{1,2}} ds$$

$$\leq C_1(M) |t| \left(\|\psi\|_X + \|\psi\|^4_X\right),$$

yielding, for $0 < |t| \leq T$

$$\|A_n\psi - e^{iH}\varphi\|_X \leq \alpha_2 \|S_n\psi\|_X + \alpha_2 \|U\psi\|_X$$

$$\leq C(M)T(M + \|\varphi\|_{W^{1,2}})(1 + (M + \|\varphi\|_{W^{1,2}})^3).$$

Finally we choose $T$ small so that

$$2MC(M)(1 + 8M^3)T \leq M,$$

which completes the proof.

Now we proceed to show that for each integer $n$, operators $A_n$ satisfy the conditions demanded by Weissinger’s theorem.

Lemma 3.3. Let $M > 0$, $T > 0$ and $\varphi \in W^{1,2}$, $\|\varphi\|_{W^{1,2}} \leq M$ be fixed. Then there is a constant $C$ depending only on $M$, Riesz’s and Sobolev constants such that for every $t \in [-T, T]$, every integer $k, n$ and every $\psi_1, \psi_2 \in X$ such that $\|\psi_1\|_X, \|\psi_2\|_X \leq M$ we have

$$\|A_n^k\psi_1(t) - A_n^k\psi_2(t)\|_{W^{1,2}} \leq \frac{(Ct)^k}{k!}\|\psi_1 - \psi_2\|_X.$$
Proof. The proof runs by induction, with the help of the local Lipschitz property of both functions $g_1, g_2$.

We will only give the idea how to get the estimate for the $k = 1$. For general $k$ the estimate follows by direct induction.

Without loss of generality we assume that $t \geq 0$. Let $\psi_1, \psi_2 \in X$ be such that $\|\psi_1\|_X \leq M, \|\psi_2\|_X \leq M$. Let $r_1$ be as given by Lemma 2.2. Then

\[
\|S_n \psi_1(t) - S_n \psi_2(t)\|_{W^{1,2}} = \|S_n \psi_1(t) - S_n \psi_2(t)\|_{W^{-1,2}} \\
\leq \int_0^t \|e^{i\alpha_1(t-s)H}||_{W^{-1,2}, W^{-1,2}} f_n(\psi_1(s)) - f_n(\psi_2(s))|_{W^{-1,2}} ds \\
\leq \int_0^t \|f_n(\psi_1(s)) - f_n(\psi_2(s))|_{L^{\rho'}} ds \\
\leq C \int_0^t \|g_1(\psi_1(s)) - g_1(\psi_2(s))|_{L^{\rho'}} ds \\
\leq C(M) \int_0^t \|g_1(\psi_1(s)) - g_1(\psi_2(s))|_{L^1} ds \\
\leq C(M) \int_0^t \|\psi_1(s) - \psi_2(s)|_{W^{1,2}} ds \\
\leq tC(M) \|\psi_1 - \psi_2\|_X. 
\]

Here we used the fact that $L^{\rho'}$ embeds continuously into $W^{-1,2}$. By the same ideas we achieve

\[
\|U \psi_1(t) - U \psi_2(t)\|_{W^{1,2}} \leq \int_0^t \|e^{i\alpha_1(t-s)H}||_{W^{-1,2}, W^{-1,2}} g_2(\psi_1(s)) - g_2(\psi_2)|_{W^{-1,2}} ds \\
\leq C(M) \int_0^t \|\psi_1(s) - \psi_2(s)|_{W^{1,2}} ds \leq C(M) \|\psi_1 - \psi_2\|_X. 
\]

Thus

\[
\|A_n \psi_1(t) - A_n \psi_2(t)\|_{W^{1,2}} \leq C(M) t \|\psi_1 - \psi_2\|_X. \quad (3.10) 
\]

We are now in position to affirm the local solvability of the approximate problem (3.1).

**Theorem 3.2.** Let $M > 0$ and $\varphi \in W^{1,2}$ such that $\|\varphi\|_{W^{1,2}} \leq M$ be fixed. Then there is $T_M > 0$ such that for every $n \in \mathbb{N}^*$ problem (3.4) has a unique solution, $\psi_n$, in the space $L^\infty([-T_M, T_M], W^{1,2})$. Further the solution may be gained as the limit of the sequence $\psi_0 = \Phi, \psi_{k+1} = A_n \psi_k$, where $\Phi$ is any element from $L^\infty([-T_M, T_M], W^{1,2})$.

Proof. Making use of Duhamel’s formula, we have simply to check that assumptions of Weissinger’s theorem are fulfilled.
Let $M > 0$ and $\varphi \in W^{1,2}$, $\|\varphi\|_{W^{1,2}} \leq M$ be fixed. By Lemma 3.2, for every $n \in \mathbb{N}^*$, there is $T := T_M > 0$ such that operators $A_n$ map the closed ball of $X := L^\infty([-T, T], W^{1,2})$ of radius $M$ and centered on $e^{itH} \varphi$, $F$ into itself.

Setting $\beta_k := \frac{(C(M)T)^k}{k!}, \forall k \in \mathbb{N}$, we obtain by Lemma 3.3

$$\|A_n^k \psi_1 - A_n^k \psi_2\|_X \leq \beta_k \|\psi_1 - \psi_2\|_X, \forall k \in \mathbb{N}, n \in \mathbb{N}^*$$

(3.11)

with $\sum_{k=0}^\infty \beta_k = \exp(C(M)T)$, which completes the proof.

For our later purposes we establish continuous dependence of the approximate solution (solution of the approximate problem) w.r.t. the initial data.

**Proposition 3.1.** Let $0 < M' < M$ and $\varphi, \tilde{\varphi} \in W^{1,2}$ be such that $\|\varphi - \tilde{\varphi}\|_{W^{1,2}} \leq M - M'$. Set $T := \min(T_M(\varphi), T_M(\tilde{\varphi}))$ and $\psi_n$, resp. $\tilde{\psi}_n$, the local solution of (3.4) with $\psi_n(0) = \varphi$, resp. $\tilde{\psi}_n(0) = \tilde{\varphi}$. Then there is a constant $C$ such that

$$\sup_{|t| \leq T} \|\psi_n(t) - \tilde{\psi}_n(t)\|_{W^{1,2}} \leq C\|\varphi - \tilde{\varphi}\|_{W^{1,2}}, \forall n \in \mathbb{N}^*.$$  

(3.12)

**Proof.** As observed in Theorem 3.3 the solution $\psi_n$ on $[-T, T]$ is given as the limit of the sequence

$$\Phi_0 \in \{u \in L^\infty([-T, T], W^{1,2}) : \|u - e^{itH} \varphi\|_{L^\infty([-T, T], W^{1,2})} \leq M\}, \Phi_{k+1} = A_n\Phi_k.$$  

(3.13)

By the conditions imposed on $M, M', \varphi$ and $\tilde{\varphi}$ we have $\|\tilde{\psi}_n - e^{iH} \tilde{\varphi}\|_{L^\infty([-T, T], W^{1,2})} \leq M$. Thus we can choose $\Phi_0 = \tilde{\psi}_n$ on $[-T, T]$.

Setting $h_n := i\alpha_2(f_n - g_n)$ we get: $\forall t \in [-T, T],

$$\tilde{\psi}_n(t) = e^{i\alpha_1 t H} \tilde{\varphi} + \int_0^t e^{i\alpha_1 (t-s) H} h_n(\tilde{\psi}_n(s)) \, ds = e^{i\alpha_1 t H} \tilde{\varphi} + \int_0^t e^{i\alpha_1 (t-s) H} h_n(\Phi_0(s)) \, ds.$$  

Thus, for every $t \in [-T, T]$, we have

$$\Phi_1(t) - \Phi_0(t) = \Phi_1(t) - \tilde{\psi}_n(t) = e^{i\alpha_1 t H} (\varphi - \tilde{\varphi}),$$  

(3.14)

and

$$\sup_{|t| \leq T} \|\Phi_1(t) - \Phi_0(t)\|_{W^{1,2}} \leq \|\varphi - \tilde{\varphi}\|_{W^{1,2}}.$$  

(3.15)

On the other hand we have, for every $n, k \in \mathbb{N}$

$$\psi_n - \Phi_k = \sum_{j=k+1}^\infty (\Phi_{j+1} - \Phi_k),$$  

(3.16)

yielding for $n = 1$ and for $\beta_k, C$ as given in the proof of Theorem 3.3

$$\|\psi_n - \Phi_1\|_{L^\infty([-T, T], W^{1,2})} \leq \left(\sum_{k=2}^\infty \beta_k\right) \|\Phi_1 - \Phi_0\|_{L^\infty([-T, T], W^{1,2})} \leq \exp(C T)\|\varphi - \tilde{\varphi}\|_{W^{1,2}}.$$  

(3.17)
Putting all together gives
\[ \|\psi_n - \tilde{\psi}_n\|_{L^\infty([-T,T],W^{1,2})} \leq \exp(CT)\|\varphi - \tilde{\varphi}\|_{W^{1,2}}, \ \forall, \ n \in \mathbb{N} \] (3.18)
and the proof is finished.

We stress that the constant occurring in the estimate given by Proposition 3.3 does not depend on \( n \) but only on \( M \) and \( M' \).

Next we shall rely on the Theorem 3.2 result to prove local existence of solutions for (1.1).

**Theorem 3.3.** Let \( M > 0 \) and \( \varphi \in W^{1,2} \), \( \|\varphi\|_{W^{1,2}} \leq M \) be fixed. Then there is \( T_M > 0 \) such that problem (1.1) has a solution, \( \psi \in L^\infty([-T_M,T_M],W^{1,2}) \).

**Proof.** On the light of Theorem 3.2, there is \( T := T_M \) and a sequence of approximate solutions \( (\psi_n) \subset X := L^\infty([-T,T],W^{1,2}) \) of problem (3.4). Thus, for every \( n \), \( \psi_n \) satisfies
\[ i\frac{\partial \psi_n}{\partial t} = -\alpha_1 \Delta \psi_n + i\alpha_2 f_n(\psi_n) - i\alpha_2 g_2(\psi_n), \ \text{in} \ W^{-1,2}. \] (3.19)

Making use of the uniform boundedness of \( (\psi_n) \) in \( X \), we achieve
\[ \|\frac{\partial \psi_n}{\partial t}\|_{W^{-1,2}} \leq C(M) \left( \|\nabla \psi_n\|_{W^{1,2}} + \|f_n(\psi_n)\|_{W^{-1,2}} + \|g_2(\psi_n)\|_{W^{1,2}} \right) \]
\[ \leq C(M) \left( 1 + \|f_n(\psi_n)\|_{L^\rho'} + \|g_2(\psi_n)\|_{W^{1,2}} \right) \]
\[ \leq C(M) \left( 1 + \|f_n(\psi_n)\|_{L^\rho'} + 1 \right) \leq C(M) \left( 2 + \|g_1(\psi_n)\|_{L^\rho'} \right) \]
\[ \leq C(M). \] (3.20)

Therefore the sequence \( (\psi_n) \) is uniformly bounded in
\[ Y := X \cap W^{1,\infty}([-T,T],W^{-1,2}). \]

Thus (see [Caz03, Proposition 1.3.14]) there is \( \psi \in Y \) and a subsequence which we denote also by \( (\psi_n) \) such that
\[ \psi_n(t) \rightharpoonup \psi(t) \ \text{in} \ W^{1,2}, \ \forall t \in [-T,T]. \] (3.21)

Thus \( \psi(0) = \varphi \).

Let \( \tilde{K}_n \) be the operators defined by
\[ \tilde{K}_n \phi := \int_{|y| < 1/n} \frac{\phi(y)}{|y|} \, dy, \ n \in \mathbb{N}^+. \] (3.22)

Having Duhamel’s formula (for \( \psi_n \)'s) in hand and rewriting
\[ f_n(\psi_n) = \psi_n K_n(|\psi_n|^2) = \psi_n (K - \tilde{K}_n)(|\psi_n|^2) = g_1(\psi_n) - \psi_n \tilde{K}_n(|\psi_n|^2), \] (3.23)
we get that for every $\phi \in L^2 \cap L^\rho$,

$$
< \psi_n(t), \phi >_{L^\rho, L^\rho'} =< e^{i\alpha_1 t H} \varphi, \phi >_{L^\rho, L^\rho'} + i\alpha_2 \int_0^t < e^{i\alpha_1 (t-s) H} g_1(\psi_n(s)), \phi >_{L^\rho, L^\rho'} ds

- i\alpha_2 \int_0^t < e^{i\alpha_1 (t-s) H} (\psi_n(s) \tilde{K}_n(|\psi_n|^2)(s)), \phi >_{L^\rho, L^\rho'} ds

- i\alpha_2 \int_0^t < e^{i\alpha_1 (t-s) H} g_2(\psi_n(s)), \phi >_{L^\rho, L^\rho'} ds.
$$

(3.24)

We claim that $\| \tilde{K}_n \|_{L^p, L^p} \to 0$ for every $1 < p < \infty$. Indeed: For every $\phi \in L^p$, setting $q$ the conjugate exponent of $p$ and

$$
G_n(x, y) := 1_{|x - y| < 1/n} |x - y|^{-1},
$$

we get

$$
|\tilde{K}_n \phi(x)| \leq (C_n := \sup_x \int G_n(x, y) dy)^{1/q} \int G_n(x, y) |\phi(y)|^p dy
$$

(3.25)

and thereby

$$
\| \tilde{K}_n \|_{L^p, L^p} \leq C_n = c/n^2 \to 0 \text{ as } n \to \infty.
$$

(3.26)

Thus we get by Proposition 2.5 and use of the fact that $e^{itH}$ maps continuously $L^\rho'$ into $L^\rho$ for every $t \neq 0$, together with dominated convergence theorem, that for every $\phi \in L^2 \cap L^\rho'$,

$$
< \psi_n(t), \phi >_{L^\rho, L^\rho'} \to < e^{i\alpha_1 t H} \varphi, \phi >_{L^\rho, L^\rho'} + i\alpha_2 \int_0^t < e^{i\alpha_1 (t-s) H} g_1(\psi(s)), \phi >_{L^\rho, L^\rho'} ds

- i\alpha_2 \int_0^t < e^{i\alpha_1 (t-s) H} g_2(\psi(s)), \phi >_{L^\rho, L^\rho'} ds =< \psi(t), \phi >_{L^\rho, L^\rho'},
$$

(3.27)

yielding therefore

$$
\psi(t) = e^{i\alpha_1 t H} \phi + i\alpha_2 \int_0^t e^{i\alpha_1 (t-s) H} g_1(\psi(s)) ds - i\alpha_2 \int_0^t e^{i\alpha_1 (t-s) H} g_2(\psi(s)) ds.
$$

(3.28)

Whence $\psi$ satisfies

$$
i \frac{\partial \psi}{\partial t} = -\alpha_1 \Delta \psi + i\alpha_2 f - i\alpha_2 g, \text{ in } W^{-1,2},
$$

(3.29)

and $\psi$ is a solution of equation (1.1).

Uniqueness: Follows from [Caz03, Proposition 4.2.3, p.85].

\[ \square \]

**Proposition 3.2.** (Blow-up alternative) The blow-up alternative holds true for the solution of the (frNSE).
The proof is quite standard so we omit it.
Yet we will describe how does the local solution behaves w.r.t. the initial data.

**Proposition 3.3.** (continuous dependence) Let \((\varphi_k)_k \subset W^{1,2}\) and \(\varphi \subset W^{1,2}\) be such that \(\|\varphi_k - \varphi\|_{W^{1,2}} \to 0\). Set \(\tilde{\psi}_k\) resp. \(\psi\) the local solution of the frictional Newton-Schrödinger equation with initial data \(\varphi_k\), resp. \(\varphi\). Then there is \(T > 0\) such that \(\lim_{k \to \infty} \|\psi - \psi_k\|_{L^\infty((-T,T),W^{1,2})} = 0\).

**Proof.** Set \(\psi_{n,k}\), resp. \(\psi_n\) the solution of the approximate Newton-Schrödinger equation with initial data \(\varphi_k\), resp. \(\varphi\). Making use of Proposition 3.1, there are constants \(C, T > 0\) depending only on \(\|\varphi\|_{W^{1,2}}\) such that for large \(k\)

\[
\sup_{t \in [-T,T]} \|\psi_{n,k}(t) - \psi_{n,k}(t)\|_{W^{1,2}} \leq C \|\varphi_k - \varphi\|_{W^{1,2}}, \quad \forall n \in \mathbb{N}. \tag{3.30}
\]

By the proof of the existence of Theorem 3.3 together with the uniqueness we conclude that for large \(k\)

\[
\psi_n \rightharpoonup \psi, \quad \psi_{n,k} \rightharpoonup \tilde{\psi}_k, \quad \in W^{1,2}, \quad \forall t \in [-T,T]. \tag{3.31}
\]

Whence by the weak lower semi-continuity of the norm we get for large \(k\)

\[
\|\psi_n(t) - \tilde{\psi}_k(t)\|_{W^{1,2}} \leq \liminf_{n \to \infty} \|\psi_n(t) - \psi_{n,k}(t)\|_{W^{1,2}} \leq C \|\varphi_k - \varphi\|_{W^{1,2}}, \quad \forall t \in [-T,T], \tag{3.32}
\]

yielding the result.

\(\square\)

### 4 Concluding remarks

We would like to stress that our method (except maybe for the proof of uniqueness) still works in a general domain of \(\mathbb{R}^3\). However, if \(\Omega \subset \mathbb{R}^3\) is bounded then, thanks to the properties of the Newton kernel on bounded subsets, it is possible to use an \(L^2\)-Gronwall-type inequality to get the uniqueness.

At this stage, we mention that our method suggests an abstract framework for solving evolution equations related to some classes of positive operators.

Finally, we indicate some open problems related to the (frNSE). The first one is, of course, that dealing with the global existence of the solution. Here we expect that a global solution would exists provided the energy of the initial data is small enough. We are yet working in this direction. Furthermore if a global solution exits it is interesting to ask about its large time behavior. For the (NSE), this question was already investigated by Wada [Wad01].

The second one is much more complicated: Having the frictional Newton-Schrödinger equation proposed by Diosi [Dio07] (which is still unsolved to our best knowledge!) in mind, one is tempted to replace Diosi’s kernel by an other one, say

\[
N(x,y) = \int \int G(x,z)G(y,z') \, d\mu(z) \, d\mu(z'), \tag{4.1}
\]
where $G$ is positive, symmetric and $\mu$ is a positive Radon measure. The immediate question that arises is under which conditions on the measure $\mu$ and on $G$ has the related (frNSE) a solution(s)? Is it local or global and is the related (frNSE) well-posed? In this stage, to illuminate the way, one has first to look for the problem with the kernel proposed by Diosi.

The last problem is the obvious generalization of the above questions in higher dimensions.

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