THE Λ-ADIC EICHLER-SHIMURA ISOMORPHISM AND p-ADIC ÉTALE COHOMOLOGY

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Abstract. We give a new proof of Ohta’s Λ-adic Eichler-Shimura isomorphism using p-adic Hodge theory and the results of Bloch-Kato and Hyodo on p-adic étale cohomology.

1. Introduction

We study the relationship between the ordinary p-adic étale cohomology of the modular curve and the group of ordinary cusp forms with coefficients in \( \mathbb{Z}_p \). The p-adic étale cohomology of the special fiber is central to our approach. We show that the group of ordinary cusp forms is dual to the ordinary cohomology of the special fiber (Theorem 1.1). We also show that the cohomology of the special fiber satisfies a control theorem – that is, the tower of special fibers as the power of \( p \) in the level increases is controlled by the Iwasawa algebra of diamond operators (Theorem 1.2). Finally, we show that the ordinary cohomology of the modular curve has a two step filtration coming from the cohomology of the special fiber (Theorem 1.3).

As applications of our results, we give new proofs of Hida’s control theorem (Corollary 1.4) and of Ohta’s Λ-adic Eichler-Shimura isomorphism (Corollary 1.6). Our proofs are simpler and have the advantage of removing some technical hypotheses from the statements. Moreover, formerly delicate questions about the comparison between Hecke- and Galois-actions are now clear (Corollary 1.8).

The utility of Ohta’s Λ-adic Eichler-Shimura isomorphism in Iwasawa theory is already well-known. It was used by Ohta in [O2] to give a simplified version of Mazur and Wiles’ proof of the Iwasawa Main Conjecture. It was also used by him in [O4] to show a finer version of the Main Conjecture when a certain Hecke algebra is Gorenstein. It was further used by Sharifi in [S] in the formulation of his conjectures, and by Fukaya and Kato in [K-F] in their partial solution of Sharifi’s conjectures.

Ohta’s proof of his Eichler-Shimura isomorphism relies on the study of “good quotients” of \( p \)-divisible groups of modular Jacobians, as introduced by Mazur and Wiles ([M-W]). This proof is very special to the case of modular curves and seems impossible to generalize to the case of higher dimensional Shimura varieties.

In contrast, our proof uses only techniques of \( p \)-adic Hodge theory. In some sense, this resolves Ohta’s question ([O1, pg. 52]) of whether his theory can be reconciled with Faltings’s. The main result in integral \( p \)-adic Hodge theory we use is from Bloch and Kato’s seminal work [B-K]. This theory works for ordinary varieties of any dimension. Therefore, our methods should generalize to other Shimura varieties if it can be shown that “the ordinary part is ordinary” (cf. Lemma 1.1).
1.1. Main Results. We first fix some notation in order to state our main results.

1.1.1. Notation. Let $p$ be a prime number and $N$ be a number prime to $p$, and assume $Np > 4$. Let $X_r = X_1(Np^r)/\mathbb{Z}_p$ be the modular curve. Let $X_r$ (resp. $Y_r$) be its generic (resp. special) fiber. Let $S_2(Np^r)_{\mathbb{Z}_p}$ be the group of weight 2 cusp forms of level $Np^r$ with coefficients in $\mathbb{Z}_p$.

For a $\mathbb{Z}_p$-module $M$, let $M^\vee = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$. If $M$ has an unramified $G_{\mathbb{Q}_p}$-action, let $D(M) = (M \otimes W(\mathbb{F}_p))^G_{r=1}$. Note that $D(M)$ is non-canonically isomorphic to $M$ ([K-P] Proposition 1.7.6]).

Let $\Delta_r$ denote the kernel of the natural map $\mathbb{Z}_p^\times \to (\mathbb{Z}/p^r\mathbb{Z})^\times$. Let $\Lambda = \mathbb{Z}_p[[\Delta_1]]$, and let $\Lambda_r = \mathbb{Z}_p[[\Delta_1/\Delta_r]]$.

A superscript $\text{ord}$ will indicate the $T(p)$-ordinary part (in the sense of Hida) (cf. Section 2.2.4).

1.1.2. New results. Our first main result can be thought of as a version of Serre duality for integral cusp forms.

Theorem 1.1. There is a natural isomorphism

$$S_2(Np^r)_{\mathbb{Z}_p}^{\text{ord}} \cong D(H^1(Y_r, \mathbb{Z}_p)^{\text{ord}})^{\vee}$$

for all $r$.

Our second main result is a version of Hida’s control theorem for the cohomology of the special fiber. Let $H(Y_r) = \lim_{r} H^1(Y_r, \mathbb{Z}_p)^{\text{ord}}$ where the transition maps are the trace maps. It is naturally a $\Lambda$-module via the action of diamond operators (Section 2.2.3).

Theorem 1.2. The $\Lambda$-module $H(Y)$ is free of finite rank. The natural maps

$$H(Y) \otimes_{\Lambda} \Lambda_r \to H(Y_r)$$

are isomorphisms for all $r > 0$.

Our third result gives a filtration of $H^1(X_r, \mathbb{Z}_p)^{\text{ord}}$ in terms of the cohomology of the special fiber.

Theorem 1.3. There is a natural exact sequence

$$0 \to H^1(Y_r, \mathbb{Z}_p)^{\text{ord}} \to H^1(X_r, \mathbb{Z}_p)^{\text{ord}} \to (H^1(Y_r, \mathbb{Z}_p)^{\text{ord}})^{\vee}(\kappa^{-1}(-)^{-1}) \to 0$$

for all $r > 0$. The first map is the natural one, and the second map comes from Poincaré duality (cf. Section 2.2.2).

The $(\kappa^{-1}(-)^{-1})$ on the right side indicates that the action of $\sigma \in G_{\mathbb{Q}_p}$ is twisted by $\kappa(\sigma)^{-1}(\sigma)^{-1}$, where $\kappa$ is the cyclotomic character, and $\langle \sigma \rangle = \langle a \rangle$ is the diamond operator for $a$ such that $\sigma(\zeta_{Np^r}) = \zeta_{Np^r}^a$.

1.1.3. Applications. As an application of our results, we give new proofs and mild extensions of results of Hida and Ohta.

Let $H(X_r) = H^1(X_r, \mathbb{Z}_p)^{\text{ord}}$ and let $H(X) = \lim_{r} H(X_r)$.

Corollary 1.4 (Hida’s Control Theorem). The $\Lambda$-module $H(X)$ is free of finite rank. The natural maps

$$H(X) \otimes_{\Lambda} \Lambda_r \to H(X_r)$$

are isomorphisms for all $r > 0$. 

Remark 1.5. This result is due to Hida ([H1]) in the case \( p \geq 5 \). Ours result requires only that \( Np > 4 \) (this assumption is made so that \( \mathfrak{X}_r \) is a nice moduli space).

Proof. By Theorem 1.2 and Theorem 1.3, we have that \( H(X_r) \) is a free \( \Lambda_r \)-module of finite rank that is independent of \( r \). This implies the first statement. The second statement follows from Theorem 1.2 and Theorem 1.3 by the five lemma. \( \square \)

Let \( S_\Lambda = \lim_{\leftarrow} S_2(Np^r)_{\mathbb{Z}_p}^{\text{ord}} \).

Corollary 1.6 (Ohta’s \( \Lambda \)-adic Eichler-Shimura isomorphism). There is a canonical exact sequence

\[
0 \to H(X)_{\text{sub}} \to H(X) \to H(X)_{\text{quo}} \to 0
\]

classified by the following two facts

- The action of the inertia subgroup \( I \) of \( G_{\mathbb{Q}_p} \) on \( H(X)_{\text{sub}} \) is trivial
- There is an isomorphism of Hecke modules \( H(X)_{\text{quo}} \cong S_\Lambda \).

Moreover, \( H(X)_{\text{sub}} \) is a free Hecke module of rank 1.

Proof. It follows from Theorem 1.3 that \( H(Y_r) \) is isomorphic to \( H(X_r)^I \), the inertia fixed part. It follows from Theorem 1.1 that there is an isomorphism of Hecke modules \( H(Y_r)^\vee \cong S_2(Np^r)_{\mathbb{Z}_p}^{\text{ord}} \), and therefore an isomorphism of Hecke modules \( H(Y_r) \cong (S_2(Np^r)_{\mathbb{Z}_p}^{\text{ord}})^\vee \). The first part of the theorem follows from these and Corollary 1.4. The second part follows from the fact that \( S_\Lambda \) is a dualizing module over the Hecke algebra. \( \square \)

Remark 1.7. Our Corollary 1.6 is a refinement of Ohta’s result [O1] in a few ways. One is that our isomorphism is defined over \( \mathbb{Z}_p \), where Ohta’s is over \( \mathcal{O}_L \), where \( L/\mathbb{Q}_p \) contains all roots of unity. Another is that Ohta’s result requires \( p > 3 \), where ours requires only that \( Np > 4 \).

One advantage of our method of proof is that comparison of the Hecke- and Galois-actions is easy to see. Indeed, we have a new proof of the following result of Fukaya-Kato that was proven with great difficulty by a different method in [K-F, Proposition 1.8.1]. Earlier partial results were obtained by Mazur-Wiles and by Ohta, by still different methods.

Corollary 1.8. In the notation of Corollary 1.6, the action of \( T(p) \) on \( H(X)_{\text{sub}} \) coincides with the action of the geometric Frobenius \( \varphi \in G_{\mathbb{Q}_p} \).

Proof. By the proof of Corollary 1.6, this follows from the corresponding statement for \( H(Y_r) \), which is proved in Lemma 3.3. \( \square \)

Remark 1.9. After submitting an earlier version of this paper, we learned about the preprints [C]. In that paper, Cais gives new results in this area and some new proofs of results of Ohta and Hida. In particular, he gives proofs of Hida’s control theorem and Ohta’s Eichler-Shimura isomorphism. His results assume that \( p > 2 \) and \( Np > 4 \). His proofs are different from ours and do not use the \( p \)-adic cohomology of the special fiber.

1.2. Method of proof. We outline the idea of the proofs of our results.
1.2.1. *The proof of Theorem 1.1.* The idea is that cusp forms are given by differential forms, and so, by Serre duality, should be dual to coherent cohomology. In characteristic $p$, coherent cohomology is almost the same as $p$-adic cohomology (Section 5). The difficulty is that $X_r$ is not smooth, and so the dualizing sheaf is not the sheaf of 1-forms. Still, we show that ordinary cusp forms are given by ordinary sections of the dualizing sheaf (Section 4). This uses the geometry of the $T(p)$-correspondence in characteristic $p$.

1.2.2. *The proof of Theorem 1.2.* We study the geometry of the special fiber. The special fiber is known to be a union of Igusa curves crossing at supersingular points (this is reviewed in Section 2). Using a result of Mazur and Wiles on the geometry of the $T(p)$ correspondence, we show that the ordinary part of the cohomology of the special fiber is equal to the ordinary cohomology of a certain one of the Igusa curves (Section 3). But the Igusa curves form a tower of finite Galois covers, so the result follows from a standard result in étale cohomology.

1.2.3. *First proof of Theorem 1.3.* Using Theorem 1.1 to show that the ranks are correct, we notice that the sequence is exact if and only if the morphism $H(Y_r) \to H(X_r)$ is a split injection of $\mathbb{Z}_p$-modules. This is clear from the sequence of low degree terms in the Leray spectral sequence (Section 6). However, we find this proof unsatisfying because it is very special to the case of $H^1$ of curves.

1.2.4. *Second proof of Theorem 1.3.* We first prove Theorem 1.3 in the case $r = 1$ as follows: we extend the base ring in order to obtain a semi-stable model, and then use the method of [B-K] (Section 7). The result for general $r$ is then obtained from this using Theorem 1.2 (Section 8).

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1.4. **Notation.** A bar will indicate algebraic or integral closure (i.e. $\overline{\mathbb{Q}_p}$, $\mathbb{Z}_p$), or base change to such (i.e. $\mathbb{X}_r = X_r \times_{\text{Spec} \mathbb{Q}_p} \text{Spec} \overline{\mathbb{Q}_p}$).

For a morphism of rings $R \to S$ and an $R$-module $M$, we use the notation $M_S := M \otimes_R S$. Similarly, for a morphisms of schemes $X \to S$ and $T \to S$ we use the notation $X_T = X \times_S T$. If $T = \text{Spec} R$ is affine, we write $X_R$ instead of $X_T$.

For a field $F$, we let $G_F = \text{Gal}(F/F)$.

2. **Modular Curves and Hecke Operators**

We review some of the theory of modular curves and Hecke operators. For the most part we follow [K-F] Chapter 1]. This theory is well-known, but there are many non-canonical choices to be made. The main purpose of this section is to carefully state which choices we make. Different choices are made in each of the
papers [K-F], [G], [M-W], [O1]. We point out when a choice is made: see Remarks 2.1 and 2.2. To see the effect of the different choices, see [K-F, esp. Sections 1.4, 1.7.15, and 1.7.16].

2.1. Modular Curves. For more details on modular curves, see [K-M] and [D-R].

2.1.1. Definition of $X(M)$. We define $Y(Np^r)$ as a functor from $\mathbb{Z}_p$-schemes to sets. It takes $S$ to the set of isomorphism classes of triples $(E, \alpha, \beta)$ where $E$ is an elliptic curve over $S$, $\beta : (\mathbb{Z}/N\mathbb{Z})^2 \sim \rightarrow E[N]$ is an isomorphism and $\alpha : (\mathbb{Z}/p^r\mathbb{Z})^2 \rightarrow E(S)$ is a Drinfeld level structure. This means that there is an equality of effective Cartier divisors $E[p^r] = \sum_{x \in (\mathbb{Z}/p^r\mathbb{Z})^2} [\alpha(x)]$ on $E$.

Then $Y(Np^r)$ is represented by a regular 2-dimensional scheme which we also denote by $Y(Np^r)$. It has a compactification by adding cusps, which we denote by $X(Np^r)$. It is a proper, regular scheme, flat and of relative dimension 1 over $\mathbb{Z}_p$. Since $X(N)$ is smooth over $\mathbb{Z}_p$, the isomorphism $\beta$ will not play a role in our analysis of the geometry of $X(Np^r)$, and we will often refer to a point $(E, \alpha) \in X(Np^r)$, with the additional data $\beta$ dropped from the notation.

2.1.2. Action of $GL_2$. There is a right-action of the group $GL_2(\mathbb{Z}/Np^r\mathbb{Z})$ on $X(Np^r)$. An element $g = (g_N, g_p) \in GL_2(\mathbb{Z}/N\mathbb{Z}) \times GL_2(\mathbb{Z}/p^r\mathbb{Z})$ acts by $g(E, \alpha, \beta) = (E, \alpha \circ g_p, \beta \circ g_N)$, where the action of $GL_2(\mathbb{Z}/Np^r\mathbb{Z})$ on $(\mathbb{Z}/Np^r\mathbb{Z})^2$ is the left action on column vectors.

2.1.3. Definition of $X_r$. Let $X_r = X_1(Np^r)/\mathbb{Z}_p$ be the quotient of $X(Np^r)$ by the subgroup $\Gamma_1(Np^r) = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2(\mathbb{Z}/Np^r\mathbb{Z})$. It is the moduli space of triples $(E, P, b)$, where $P \in E$ is a point of exact order $p^r$ ([K-M] pg. 99) and $b$ is a point of order $N$. Again, the point $b$ plays little role and we will often drop it from the notation.

The scheme $X_r$ is again regular, proper and flat over $\mathbb{Z}_p$. We write $X_r$ (resp. $Y_r$) for the generic (resp. special) fiber. It is known that $X_r$ is smooth, and that $Y_r$ is a local complete intersection. In particular, the morphism $X_r \rightarrow \mathbb{Z}_p$ is Gorenstein.

Remark 2.1. There is a lack of symmetry in this definition. We could instead consider the quotient of $X(Np^r)$ by the subgroup $\left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\}$. This is the approach taken in, for example, [G] and [O1].
2.1.4. Geometry of $Y_r$. The geometry of $Y_r$ is well-understood: it is a union of $r + 1$ irreducible components, all intersecting transversely at each supersingular point. We label the irreducible components $Y^{(a)}_r$ where $0 \leq a \leq r$. Note that a point $(E, P) \in Y$ with $E$ an ordinary elliptic curve gives a degree $p^r$ isogeny $\pi : E \to E'$ with $(P) = \ker(\pi)$. In characteristic $p$, all such isogenies are isomorphic to $V^a F^b$, where $F$ (resp. $V$) is the Frobenius (resp. Verschiebung) isogeny and $a + b = r$. The component $Y^{(a)}_r$ corresponds to the points where $\pi \sim V^a F^b$.

2.1.5. Ordinary locus. We let $S_r \subset Y_r$ be the closed subscheme consisting of supersingular points, and we let $Y^{(a)}_{r, \text{ord}}$ denote $Y^{(a)}_r - S_r$ (and generally, a subscript $\text{ord}$ will indicate removing the supersingular points).

2.1.6. Igusa curves. Let $Ig(p^r)$ be the étale cover of $Y_{0, \text{ord}}$ defined as follows. For a point $E \in Y_{0, \text{ord}}$, to give a point of $Ig(p^r)$ lying over it is to give a generator $P \in \ker(V^r : E(p^r) \to E)$. This is an étale cover with Galois group $(\mathbb{Z}/p^r \mathbb{Z}) \times$. For $a = 0, \ldots, r$, there are diagrams

\[
\begin{array}{ccc}
Ig(p^a) & \longrightarrow & Y^{(a)}_{r, \text{ord}} \\
\downarrow & & \downarrow \\
Y_{0, \text{ord}} & \overset{F^r}{\longrightarrow} & Y_{0, \text{ord}}
\end{array}
\]

Moreover, the induced map $Ig(p^a) \to (Y^{(a)}_{r, \text{ord}})_{\text{red}}$ is an isomorphism. Counting degrees, we see that $Y^{(a)}_{r, \text{ord}}$ has multiplicity $\phi(p^r)/\phi(p^a)$, and, in particular, that $Y^{(r)}_{r, \text{ord}}$ is reduced.

2.2. Hecke Operators. We now recall the definition and some properties of the Hecke operators $T(n)$ and the diamond operators $\langle a \rangle$. We follow [K-F, Section 1.2].

2.2.1. Diamond operators. For $a \in (\mathbb{Z}/p^r \mathbb{Z})^\times \subset (\mathbb{Z}/Np^r \mathbb{Z})^\times$, we define the diamond operator $\langle a \rangle$ to be the automorphism of $X(Np^r)$ induced by the matrix

\[
\begin{pmatrix}
1/a & 0 \\
0 & a
\end{pmatrix} \in GL_2(\mathbb{Z}/Np^r \mathbb{Z}).
\]

It descends to an automorphism of $\mathcal{X}_r$; in terms of the moduli problem, it is given by $\langle a \rangle(E, P) = (E, aP)$.

2.2.2. $T(n)$ operators. Let $l$ be a prime number. We will soon define correspondences

\[
\begin{array}{ccc}
\mathcal{X}(r; l) & \xrightarrow{\pi} & \mathcal{X}_r \\
\psi_l & \longrightarrow & \mathcal{X}(r; l)
\end{array}
\]

where $\psi_l$ and $\pi$ are finite maps. We then define the Hecke operators $T(l)$ as $(\psi_l)_* \pi^*$. We then define $T(n)$ for any integer $n$ as the usual linear combinations of $T(l)$ for $l|n$ (cf. eg. [K-F, Section 1.2.4]).

The scheme $\mathcal{X}(r; l)$ is defined to be the quotient of $\mathcal{X}(Nl p^r)$ by the subgroup

\[
\left\{ \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \mid c = 0, \ d \equiv 1 \pmod{Np^r} \right\} \subseteq GL_2(\mathbb{Z}/Nl p^r \mathbb{Z}).
\]
Then π is the natural projection and ψ_l is the map induced by \( \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \).

In terms of the moduli problem, a point of \( X(r; l) \) is a triple \((E, P, C)\) where \((E, P) \in \mathcal{X}_r\) and \(C \subset E\) is a subgroup scheme such that there is étale locally an isomorphism \( \mathbb{Z}/Np^r \mathbb{Z} \to C \) given by \( 1 \mapsto Q \) such that \( lQ = P \).

The map π is then given by \((E, P, C) \mapsto (E, P)\) and the map ψ_l is given by \((E, P, C) \mapsto (E/Np^r C, Q \mod Np^r C)\).

**Remark 2.2.** There is also the correspondence \( T^*(l) = π_*(\psi_l)^*\), but we do not consider it here. Many authors use the notation \( U_l \) for \( T(l) \) if \( l \nmid Np \) and reserve \( T(l) \) for \( l \mid Np \). There is also some disagreement about which correspondence should be called \( T(l) \) and which \( T^*(l) \): our correspondence \( T(p) \) is called \( U_p^* \) in [M-W] pg. 237.

### 2.2.3. Iwasawa and Hecke algebras

Let \( \Delta_r \) denote the kernel of the natural map \( \mathbb{Z}_p^r \to (\mathbb{Z}/p^r \mathbb{Z})^\times \). We identify \( \Delta_1/\Delta_r \) with the Sylow-p subgroup of \((\mathbb{Z}/p^r \mathbb{Z})^\times\).

Let \( \Lambda = \mathbb{Z}_p[[\Delta_1]] \), the completed group ring, and let \( \Lambda_r = \mathbb{Z}_p[\Delta_1/\Delta_r] \). Then \( \Lambda \) is a local ring, and \( \Lambda = \lim \Lambda_r \) where the transition maps are the natural surjections.

We let \( \mathfrak{h}_r \) denote the \( \mathbb{Z}_p\)-subalgebra of \( \text{End}_{\mathbb{Z}_p}(H^1(\mathcal{X}_r(\mathbb{C}), \mathbb{Z}_p)) \) generated by all \( \langle a \rangle \) for \( a \in (\mathbb{Z}/p^r \mathbb{Z})^\times \) and all \( T(n) \) for \( n \geq 1 \). We consider \( \mathfrak{h}_r \) as a \( \Lambda_r \)-algebra via the homomorphism induced by \( \Delta_1/\Delta_r \subset (\mathbb{Z}/p^r \mathbb{Z})^\times \to \mathfrak{h}_r \) given by \( a \mapsto \langle a \rangle \).

### 2.2.4. Ordinary component

For an \( \mathfrak{h}_r \)-module \( M \) that is of finite type over a \( \mathbb{Z}_p \)-algebra we let the **ordinary component**, denoted \( M^{\text{ord}} \), be the largest direct summand on which \( T(p) \) acts invertibly. The theory of the ordinary component was developed by Hida.

If \( M \) is finite type over \( \mathbb{Z}_p \) then \( \lim_{n \to \infty} T(p)^{n!} \) exists and is an idempotent of \( \text{End}(M) \), and \( M^{\text{ord}} \) is the corresponding summand (cf. [H2] pg. 86). Then it is clear that the functor \( M \mapsto M^{\text{ord}} \) is exact on this category.

If \( M \) is a finite dimensional \( \mathbb{F}_p \)-vector space, then \( M^{\text{ord}} \) is the direct sum of all the \( T(p) \)-eigenspaces with non-zero eigenvalues. This follows from the definition by linear algebra.

### 2.3. Poincaré duality

The usual Poincaré pairing

\[ H^1(\mathcal{X}_r, \mathbb{Z}_p) \times H^1(\mathcal{X}_r, \mathbb{Z}_p)(1) \to \mathbb{Z}_p \]

exchanges \( T(p) \) and \( T^*(p) \), and so is unsuitable for discussing ordinary parts.

### 2.3.1. Twisted pairing

Following Ohta, we use a twisted Poincaré pairing which is equivariant for the the Hecke action (cf. [K-T] Section 1.6] for more details). In particular, we have a perfect pairing of free \( \mathbb{Z}_p \)-modules

\( (\ , \ )_r : H^1(\mathcal{X}_r, \mathbb{Z}_p)^{\text{ord}} \times H^1(\mathcal{X}_r, \mathbb{Z}_p)^{\text{ord}}(\kappa(-)) \to \mathbb{Z}_p \)

where the \( \kappa(-) \) on the right side indicates that the action of \( \sigma \in G_{\mathbb{Q}_p} \) is twisted by \( \kappa(\sigma)(\sigma) \), where \( \kappa \) is the cyclotomic character, and \( (\sigma) = \langle a \rangle \) for the \( a \) such that \( \sigma(\zeta_{p^r}) = \zeta_{p^r}^a \). It satisfies:

1. For any \( a \in \mathfrak{h}_r \), \( (ax, y)_r = (x, ay)_r \).
2. For any \( \sigma \in G_{\mathbb{Q}_p} \), \( (\sigma x, \sigma y)_r = (x, y)_r \).
2.3.2. Isotropy. We now define the sequence of Theorem \[\text{1.3}\] and show that it is a complex. The map

$$H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord} \to H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord}$$

is the composition

$$H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord} \cong H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord} \to H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord}$$

where the first map is given by proper base change. Note that the image of $H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord}$ under this map has trivial action by the inertia subgroup of $G_{\mathbb{Q}_p}$.

The map

$$H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord} \to (H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord})^{\vee}(\kappa^{-1}(-)^{-1})$$

is the dual of the above map under the Poincaré pairing $(\cdot, \cdot)_r$.

To show that the resulting sequence

$$H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord} \to H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord} \to (H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord})^{\vee}(\kappa^{-1}(-)^{-1})$$

is a complex, it suffices to show that $H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord}$ is isotropic with respect to $(\cdot, \cdot)_r$. But this is clear from the invariance of $(\cdot, \cdot)_r$ under the Galois action, using the fact that the inertia group acts trivially.

2.4. Semi-stable model. We note that $\mathcal{X}_1$ is not semi-stable because $Y_1$ is not reduced. However, it is well-known that there is a semi-stable model $\mathcal{X}$ for $X_1$ over $\mathbb{Z}_p[\zeta_p]$, which we describe below. We will let $X'$ and $Y'$ denote the generic and special fibers of $\mathcal{X}'$.

2.4.1. Definition of $\mathcal{X}'$. Let $\mathcal{X}'$ be the quotient of $\mathcal{X}(Np)$ by the subgroup

$$\left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subseteq GL_n(\mathbb{Z}/Np\mathbb{Z}).$$

It is the moduli space of triples $(E, P, Q)$, where $E \in \mathcal{X}_0$, $\pi : E \to E'$ is an isogeny of degree $p$, and $P \in \ker(\pi)$, $Q \in \ker(\pi^t)$ are generators (\cite{KM} (pg. 100)).

2.4.2. Semistability. Let $\zeta_p \in \overline{\mathbb{Q}}_p$ be a primitive $p$-th root of unity. There is a natural map $\mathcal{X}' \to \text{Spec}(\mathbb{Z}_p[\zeta_p])$ given by $(P, Q)_\pi$, where $(\cdot, \cdot)_\pi$ is the Weil pairing associated to $\pi$ (\cite{KM} 1.2.8 (pg. 87-91)). Then $\mathcal{X}'$ is semi-stable over $\mathbb{Z}_p[\zeta_p]$ (\cite{DR} Théorème 2.12 (pg. 108)).

2.4.3. $\mathcal{X}'$ is a model for $X_1$. There is a natural map $\mathcal{X}' \to \mathcal{X}_1$ given by $(E, P, Q) \mapsto (E, P)$. It induces an isomorphism

$$X' \cong X_1 \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_p).$$

Indeed, over $\mathbb{Q}_p(\zeta)$ we can recover $Q$ as the unique point such that $(P, Q)_\pi = \zeta_p$.

2.4.4. Galois action. The above isomorphism gives an action of $\Gamma = \text{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$ on $X'$. This action extends to $\mathcal{X}'$ as follows. Let $\bar{d} \in (\mathbb{Z}/p\mathbb{Z})^\times$, and let $\sigma_d \in \Gamma$ be the element that satisfies $\sigma_d(\zeta_p) = \zeta_p^{d^{-1}}$. Then $\sigma_d(E, P, Q) = (E^{\sigma_d}, P^{\sigma_d}, d^{-1}Q^{\sigma_d})$ where the superscript $\sigma_d$ indicates the action on the base scheme.
2.4.5. Stratification of \( Y' \). For \( i = 0, 1 \), let \( Y'^{(i)} \) be the preimage of \( Y_1^{(i)} \).

The natural map \( Y'^{(1)} \to Y_1^{(1)} \) is an isomorphism: the inverse is given by \((E, P) \mapsto (E, P, 0)\). Indeed, if \((E, P, Q) \in Y'\) maps to a point of \( Y_1^{(1)} \), then \( \pi \) is isomorphic to \( V \). Then \( \pi^t \) is isomorphic to \( F \) and so \( Q \) generates \( \ker(V : E^{(p)} \to E) \).

There is an isomorphism \( Y'^{(0)} \cong Ig(p) \) given by \((E, P, Q) \mapsto (E, Q)\). Indeed, if \((E, P, Q) \in Y'\) maps to a point of \( Y'^{(0)} \), then we must have \( \pi = F \) and \( \pi^t = V \) and so \( Q \) generates \( \ker(V : E^{(p)} \to E) \).

For \( d \in (\mathbb{Z}/p\mathbb{Z})^\times \), the action of the element \( \sigma_d \in \Gamma \) on \( Y' \) induces the automorphism \( 1 \times \langle d - 1 \rangle \) of \( Y_1^{(1)} \times Ig(p) \) under the above isomorphisms.

3. The ordinary component of the special fiber

In this section, we compute the ordinary component of the cohomology of the special fiber using the geometry of the \( T(p) \) correspondence.

3.1. The geometry of \( T(p) \). We consider the action of \( T(p) \) on \( Y_r \), as in, for example, [M-W] or [U]. In this subsection, we consider \( T(p) \) as a homomorphism

\[
T(p) : \text{Div}(Y_r) \to \text{Div}(Y_r)
\]

given by \( T(p) = (\psi_p)_* \pi^* \), where \( \psi_p \) and \( \pi \) are as in Section 2.2.2 and where the upper and lower star indicate the usual pullback and pushforward of divisors.

3.1.1. Formula. Writing down the definition of \( T(p) \) from Section 2.2.2 in terms of its action on points, we obtain the following formula. If \((E, P) \in Y_r\) with \( E \) an elliptic curve, then

\[
T(p)(E, P) = \sum_{Q \in \overline{E}/PQ = P} (E/C_P, \overline{Q}),
\]

where \( C_P \) is the finite subgroup scheme whose associated Cartier divisor is

\[
\sum_{i=0}^{p-1} [ip^{r-1} P].
\]

Note that \( C_P = \ker(F) \) if \((E, P) \in Y_r^{(a)} \) for \( 0 \leq a < r \).

3.1.2. Action on the strata. We can deduce the action of \( T(p) \) on the stratification \( Y_r = \cup_a Y_r^{(a)} \).

Lemma 3.2 (Mazur-Wiles). Let \( x_a \in \text{Div}(Y_r^{(a)}) \) for \( a = 0, \ldots, r \). Then we have

\[
T(p)x_0 = Fr_*x_0 + y_1 \text{ for some } y_1 \in \text{Div}(Y_r^{(1)})
\]

\[
T(p)x_a \in \text{Div}(Y_r^{(a+1)}) \text{, for } a = 1, \ldots, r - 1
\]

\[
T(p)x_r = Fr_*x_r
\]

where \( Fr \) denotes the relative Frobenius, and \( Fr_* \) and \( Fr^* \) are the usual pushforward and pullback of divisors.

For \( s \in S_r \), we have \( T(p)s = pFr_*s \).
Remark 3.3. If we let \( x = \sum_{i=0}^{r} x_r \in \text{Div}(Y_r) \), then we can rewrite the first part of the lemma as

\[
T(p)x \in \begin{pmatrix}
Fr_* & * & 0 & 0 & \cdots & 0 \\
0 & 0 & * & 0 & \cdots & 0 \\
0 & 0 & 0 & * & \cdots & 0 \\
\vdots \\
0 & \cdots & 0 & 0 & 0 & * \\
0 & \cdots & 0 & 0 & 0 & Fr^* \\
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
0 \\
\vdots \\
0 \\
x_r \\
\end{pmatrix}
\]

in matrix notation.

Proof. The ordinary part can be deduced as in [M-W] Section 2.9, Proposition 1 (pg. 253) and [M-W] Section 3.3, Proposition 2 (pg. 271). The last sentence can be deduced from this, as in \([G]\) pg. 472. However, since our notation differs from both [M-W] and \([G]\), we illustrate the idea of the proof.

Let \( x_0 = (E, P) \in Y_r^{(0)} \). Since the isogeny associated to \( P \) is purely inseparable, we must have \( P = 0 \) and \( C_P = \ker(F) \). Then the sum (3.1) is over \( Q \) such that \( pQ = 0 \). Either \( Q = 0 \), in which case

\[ (E/C_P, Q) = (E/\ker(F), 0) \cong (E^{(p)}, 0) = Fr_*(E, P), \]

or \( Q \neq 0 \), in which case \( (E/C_P, Q) \in Y_r^{(1)} \). The other ordinary cases can be handled similarly – for more details see [M-W].

We now explain the last sentence. By the above computation, we have that \( T(p) \) and \( Fr^* \), as functions \( Y_r^{(r)} \to \text{Pic}(Y_r) \), agree on the dense subset \( Y_r^{(r) ord} \). Since \( Y_r^{(r) ord} \) is separated, they must agree on \( S_r \) also. To see that \( Fr^* \) is supersingular, we note that if \( (E', P') \in Fr^{-1}(E, P) \) for \( (E, P) \in S_r \), then \( E'^{(p)} = E \) and so \( E'^{(p)} = E'^{(p')} \). Thus, the last isomorphism uses the fact that \( E' \) is supersingular. \( \square \)

3.2. Action on cohomology. Using Lemma 3.2 we can better understand the ordinary component of the cohomology of the special fiber.

3.2.1. Geometric and algebraic ordinariness. The next lemma shows the relationship between the (geometric) ordinary locus of Section 2.1.5 and the (algebraic) ordinary component of Section 2.2.4.

Lemma 3.4. Let \( Y \) be one of the schemes \( \overline{Y}_r \) or \( \overline{Y}_r^{(a)} \). For each \( i \geq 0 \), the natural maps

\[ H^i_c(Y_{ord}, \mathbb{Z}_p) \to H^i(Y, \mathbb{Z}_p) \]

induce isomorphisms

\[ H^i_c(Y_{ord}, \mathbb{Z}_p)^{ord} \cong H^i(Y, \mathbb{Z}_p)^{ord}. \]

In particular, \( H^i(Y, \mathbb{Z}_p)^{ord} = 0 \) for \( i \neq 1 \).

Proof. Using the long exact sequence associated to \( Y_{ord} \subset Y \) (\([D]\) Section 5.1.16 (pg. 350)):

\[ \cdots \to H^{i-1}(\overline{S}_r, \mathbb{Z}_p) \to H^i_c(Y_{ord}, \mathbb{Z}_p) \to H^i(Y, \mathbb{Z}_p) \to H^i(\overline{S}_r, \mathbb{Z}_p) \to \cdots \]

we see that it suffices to show

\[ H^i(\overline{S}_r, \mathbb{Z}_p)^{ord} = 0. \]
We see from the previous lemma that it suffices to show that $pFr_\ast$ acts topologically nilpotently on $H^1(\mathcal{S}_r, \mathbb{Z}_p)$. But, indeed, the action of $Fr_\ast$ on $H^1(\mathcal{S}_r, \mathbb{Z}_p)$ is through the trace map on the sheaf $\mathbb{Z}_p$, which is multiplication by $p$ on stalks.

The last statement follows from the fact that $H^0_c(Y_{\text{ord}}, \mathbb{Z}_p) = 0 = H^2(Y, \mathbb{Z}_p)$ for dimension reasons.

3.2.2. **Description of the ordinary part.** We can now describe the ordinary part and compare the Hecke and Galois actions.

**Lemma 3.5.** There is a natural isomorphism

$$H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{\text{ord}} \cong H^1(\mathcal{Y}_{r, r}^{(r)}, \mathbb{Z}_p)^{\text{ord}}$$

On $H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{\text{ord}}$ we have $T(p) = \varphi$, where $\varphi \in G_{\mathbb{Z}_p}$ is the geometric Frobenius.

*Proof.* By the previous lemma, we have

$$H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{\text{ord}} \cong \left( \bigoplus_{a=0}^r H^1_c(\mathcal{Y}_{r, r,\text{ord}}^{(a)}, \mathbb{Z}_p) \right)^{\text{ord}}.$$  

The action of $Fr_\ast$ on $H^1(\mathcal{Y}_r^{(0)}, \mathbb{Z}_p)$ is through the trace map on the sheaf $\mathbb{Z}_p$, which is multiplication by $p$ on stalks, and so it acts as 0 on $H^1(\mathcal{Y}_r^{(0)}, \mathbb{Z}/p\mathbb{Z})$. Using Lemma 3.2 we see that the only summand of $H^1(\mathcal{Y}_r, \mathbb{Z}/p\mathbb{Z})^{\text{ord}}$ on which $T(p)$ can have a non-zero eigenvector is $H^1(\mathcal{Y}_{r, r,\text{ord}}^{(r)}, \mathbb{Z}/p\mathbb{Z})$. We see that

$$H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{\text{ord}} / pH^1(\mathcal{Y}_r, \mathbb{Z}_p)^{\text{ord}} \cong H^1(\mathcal{Y}_r, \mathbb{Z}/p\mathbb{Z})^{\text{ord}}$$

$$\cong H^1(\mathcal{Y}_{r, r,\text{ord}}^{(r)}, \mathbb{Z}/p\mathbb{Z})^{\text{ord}}$$

$$\cong H^1(\mathcal{Y}_r^{(r)}, \mathbb{Z}_p)^{\text{ord}} / pH^1(\mathcal{Y}_r^{(r)}, \mathbb{Z}_p)^{\text{ord}}.$$  

This implies that

$$H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{\text{ord}} = H^1(\mathcal{Y}_r^{(r)}, \mathbb{Z}_p)^{\text{ord}}$$

on which $T(p)$ acts as $Fr_\ast^\ast$, which acts on étale cohomology by the geometric Frobenius. \hfill \square

3.2.3. **Control for $Y_r$.** Let $H(Y_r) = H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{\text{ord}}$. We have the following corollary, which implies Theorem 1.2.

**Corollary 3.6.** Let $r > s > 0$ be integers. Consider the natural map $Y_r \to Y_s$. The trace map on cohomology induces an isomorphism

$$H(Y_r) \otimes_{\Lambda_r} \Lambda_s \cong H(Y_s)$$  

of $\Lambda_s$-modules. Moreover, $H(Y_r)$ is a free $\Lambda_r$-module of finite rank.

*Proof.* For any integer $t > 0$ we have by Lemmas 3.4 and 3.5

$$H^1(\mathcal{Y}_t, \mathbb{Z}_p)^{\text{ord}} = H^1_c(\mathcal{Y}_t^{(t)}, \mathbb{Z}_p)^{\text{ord}}.$$  

The natural map $\mathcal{Y}_t^{(t)}_{t,\text{ord}} \to \mathcal{Y}_0$ is an étale cover with Galois group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. We see that $\mathcal{Y}_r^{(r)}_{r,\text{ord}} \to \mathcal{Y}_s^{(s)}_{s,\text{ord}}$ is an étale cover with Galois group $\Delta_s / \Delta_r$.

We use the following fact from étale cohomology, due to Deligne.
Fact 3.7. Let $\pi : X \to Y$ be a finite étale Galois cover with Galois group $G$. Let $A$ be a commutative ring, and let $\mathcal{F}$ be a constructible sheaf of $A$-modules on $Y$ whose stalks are perfect as $A$-modules. Then the trace map induces an isomorphism

$$R\Gamma_c(X, \mathcal{F}) \, \overset{\frac{c}{d}}{\otimes}_{A[G]} \, A \to R\Gamma_c(Y, \mathcal{F}).$$

Proof. We have $R\Gamma_c(X, \mathcal{F}) \cong R\Gamma_c(Y, \pi_*\mathcal{F})$. Then

$$R\Gamma_c(Y, \pi_*\mathcal{F}) \, \overset{\frac{c}{d}}{\otimes}_{A[G]} \, A \cong R\Gamma_c(Y, \pi_*\mathcal{F} \overset{1}{\otimes}_{A[G]} A)$$

by [D] Proposition 5.2.9 (pg. 358) and Théorème 5.3.6 (pg. 364)]. The trace map induces an isomorphism

$$R\Gamma_c(Y, \pi_*\mathcal{F} \overset{\frac{c}{d}}{\otimes}_{A[G]} A) \to R\Gamma_c(Y, \mathcal{F}).$$

We apply this to the case where $\pi$ is $\varphi^{(r)} \to \varphi^{(s)}$. We have

$$R\Gamma_c(\varphi^{(r)} \otimes_{\Lambda_s} \Lambda_s) \to R\Gamma_c(\varphi^{(s)} \otimes_{\Lambda_s} \Lambda_s).$$

By the vanishing of $H^1_c(\varphi^{(r)} \otimes_{\Lambda_s} \Lambda_s)$ for $i \neq 1$ (Lemma 3.4), this induces

$$H^1_c(\varphi^{(r)} \otimes_{\Lambda_s} \Lambda_s) \to H^1_c(\varphi^{(s)} \otimes_{\Lambda_s} \Lambda_s).$$

This completes the proof of the first statement.

For the second statement, notice that the argument above also implies that $\text{Tor}^1_c(H(Y_r), \Lambda_s) = 0$ for any $s < r$. We know that $H(Y_1)$ is a free $\Lambda_1$-module of finite rank, say $d$. Lift any isomorphism $\Lambda_1^{\otimes d} \to H(Y_1)$ to a surjection $\Lambda_1^{\otimes d} \to H(Y_r)$ (by Nakayama’s lemma). Then $\text{Tor}^1_c(H(Y_r), \Lambda_1) \to \ker(\phi) \otimes_{\Lambda_s} \Lambda_1$. We have $\ker(\phi) \otimes_{\Lambda_s} \Lambda_1 = 0$ and so, by Nakayama’s lemma, $\ker(\phi) = 0$.

3.2.4. Relation to the semistable model. We can also apply the ideas of this section to $Y'$, the special fiber of the semistable model of $X_1$.

Lemma 3.8. The natural map $Y' \to Y_1$ induces an isomorphism

$$H^1(Y_1, \mathbb{Z}_p)^{\text{ord}} \to H^1(Y', \mathbb{Z}_p)^{\text{ord}}$$

In particular, the geometric action of $\Gamma = \text{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$ on $Y'$ induces the trivial action on $H^1(Y', \mathbb{Z}_p)^{\text{ord}}$.

Proof. Similarly to Lemma 3.4, we have

$$H^1(Y', \mathbb{Z}_p)^{\text{ord}} \to H^1(Y'^{(1)}, \mathbb{Z}_p)^{\text{ord}}.$$
4. Cusp forms and the dualizing sheaf

4.1. Duality. We recall some general theory of duality. For more details, see [D-R II.2].

**Theorem 4.1** (Grothendieck). Let $f : X \to Y$ be a proper, finite type morphism of noetherian schemes. Then there is a right adjoint $f^!$ to $Rf_*$. It has the following properties

1) If $f$ is Cohen-Macaulay of pure relative dimension $d$, then $f^! \mathcal{O}_Y$ has only non-zero cohomology in degree $-d$, and we call $H^{-d}(f^! \mathcal{O}_Y) = \Omega_{X/Y}$ the dualizing sheaf.

2) If, in addition, $f$ is Gorenstein, then $\Omega_{X/Y}$ is invertible.

3) If, in addition, $f$ is smooth, then $\Omega_{X/Y} = \Omega^d_{X/Y}$, the sheaf of top forms.

If $Y = \text{Spec} A$ is affine, we denote $\Omega_{X/Y}$ by $\Omega_{X/A}$, or, when no confusion can arise, simply by $\Omega_X$.

4.2. Cusp forms. We recall some notions about cusp forms of weight 2.

4.2.1. **Definition of $S_2(Np^r)_{\mathbb{Z}_p}$.** We define $S_2(Np^r)_{\mathbb{Q}_p}$, the space of cusp forms of weight 2 as

$$S_2(Np^r)_{\mathbb{Q}_p} = H^0(X_r, \Omega^1_{X_r}).$$

We would also like to discuss $S_2(Np^r)_{\mathbb{Z}_p}$, the cusp forms with integral coefficients. In the case $r = 0$, we can define $S_2(N)_{\mathbb{Z}_p} = H^0(X_0, \Omega^1_{X_0}).$ However, when $r > 0$, $X_r/\mathbb{Z}_p$ is not smooth, and so the sheaf $\Omega^1_{X_r}$ is not as nice. Instead, we define

$$S_2(Np^r)_{\mathbb{Z}_p} = \{ f \in S_2(Np^r)_{\mathbb{Q}_p} \mid a_n(f) \in \mathbb{Z}_p \text{ for all } n > 0 \},$$

where $a_n(f)$ are the coefficients in the $q$-expansion. The next proposition shows why this definition agrees with the above definition in the case $r = 0$.

**Proposition 4.2.** (cf. [G] Proposition 8.4] There is a natural injective map

$$H^0(X_r, \Omega_{X_r}) \to S_2(Np^r)_{\mathbb{Z}_p}.$$

It is an isomorphism after tensoring $\mathbb{Q}_p$. If $r = 0$, then it is an isomorphism.

**Proof.** There is clearly an injective map

$$H^0(X_r, \Omega_{X_r}) \to H^0(X_r, \Omega_{X_r})$$

by restriction. Since $X_r/\mathbb{Q}_p$ is smooth, $\Omega_{X_r} = \Omega^1_{X_r}$, and so the codomain is $S_2(Np^r)_{\mathbb{Q}_p}$. We claim that the image is contained in $S_2(Np^r)_{\mathbb{Z}_p}$. We just need to check that the $q$-expansion is integral. However, the $q$-expansion is just the pull-back by the cusp $\infty : S = \text{Spec} \mathbb{Z}_p((q)) \to X_r$, so if $f \in H^0(X_r, \Omega_{X_r})$, then $\infty^* f \in H^0(S, \Omega^1_{S/\mathbb{Z}_p})$, and the result is clear.

To see the last statement, notice that, for any $r$, an element of $S_2(Np^r)_{\mathbb{Z}_p}$ gives a meromorphic section of the line bundle $\Omega_{X_r}$. The poles form a divisor supported on the special fibre. Since the $q$-expansion is integral, the divisor doesn’t intersect the component containing the image of $\infty$. In the case $r = 0$, the special fibre is irreducible, and so we see that the divisor must be empty.
4.3. Ordinary cusp forms. For $r > 0$, the map in Proposition 4.2 is not an isomorphism. However, it does induce an isomorphism on ordinary parts. The idea of the proof comes from [O1] Proposition 3.4.8.

**Theorem 4.3.** The map in Proposition 4.2 induces an isomorphism  
\[ H^0(\mathcal{X}_r, \Omega_{\mathcal{X}})^{ord} \cong S_2(Np)^{ord}. \]

**Proof.** We give the proof in the case $r = 1$, for simplicity. The proof in the general case is the same but more cumbersome.

We first note that, since it is an isomorphism after tensoring $\mathbb{Q}_p$, it is enough to show that it is an isomorphism after tensoring any integral extension of $\mathbb{Z}_p$; taking intersections then gives the result. We will prove it over $\mathcal{O}_K$. Let $\pi = 1 - \zeta_p \in \mathcal{O}_K$, a uniformizer.

Let $\infty'$ be a cusp of $\mathcal{X}_1$ whose image is in $Y_1^{(0)}$ and that is defined over $\mathcal{O}_K$. Note that, by Lemma 3.2, $T(p) = Fr^* = 0$ on $H^0(Y_1^{(0)}, \Omega_{Y_1}^{1, (0)})$. If $g \in H^0(\mathcal{X}_1, \Omega_{\mathcal{X}_1})\mathcal{O}_K$, then

\[ \infty'^*(T(p)g) \equiv \infty'^*(Fr^*g) \equiv 0 \mod \pi. \]

Let $f \in S_2(Np)^{ord}_{\mathcal{O}_K}$. Since the cokernel is $p$-torsion, there exists a least non-negative integer $n$ such that $g = \pi^n f \in H^0(\mathcal{X}_1, \Omega_{\mathcal{X}_1})^{ord}_{\mathcal{O}_K}$. Then, by (*), we have $\infty'^*(T(p)g) \in \pi\mathcal{O}_K[[q]]$, and so $\infty'^*(g) \in \pi\mathcal{O}_K[[q]]$, since $g$ is ordinary. Thus $\infty'^*(\pi^{n-1} f) \in \mathcal{O}_K[[q]]$.

Since $\infty'^* f \in \mathcal{O}_K[[q]]$ already, we have that $\infty^* g \in \pi^n \mathcal{O}_K[[q]]$. Thus, if $n > 0$, $\infty^*(\pi^{n-1} f) \in \mathcal{O}_K[[q]]$ and $\infty'^*(\pi^{n-1} f) \in \mathcal{O}_K[[q]]$, and so $\pi^{n-1} f \in H^0(\mathcal{X}_r, \Omega_{\mathcal{X}_r})^{ord}$, contradicting the minimality of $n$. Therefore $n = 0$ and $f$ is in the image, completing the proof. \qed

5. The dualizing sheaf and the special fiber

5.1. Coherent and $p$-adic cohomology. For a variety in characteristic $p$, we can compare the coherent and $p$-adic cohomologies using the Artin-Schreier sequence. The result we need is the following.

**Lemma 5.1.** There is a natural isomorphism  
\[ H^1(\nabla_r, \mathbb{Z}_p)^{ord} \otimes W(\mathbb{F}_p) \to H^1(\mathcal{X}_{r,W}(\mathbb{F}_p), \mathcal{O}_{\mathcal{X}_{r,W}(\mathbb{F}_p)})^{ord} \]

for any $r > 0$.

**Proof.** On $\mathcal{X}_{r,W}(\mathbb{F}_p)$, there is a natural map of sheaves $\mathbb{Z}_p \to \mathcal{O}_{\mathcal{X}_{r,W}(\mathbb{F}_p)}$. This induces a map on cohomology  
\[ H^1(\mathcal{X}_{r,W}(\mathbb{F}_p), \mathbb{Z}_p) \to H^1(\mathcal{X}_{r,W}(\mathbb{F}_p), \mathcal{O}_{\mathcal{X}_{r,W}(\mathbb{F}_p)}). \]

Proper base change, extension of scalars, and ordinary parts give the map in question.

To show that it is an isomorphism, it is enough to show that it is an isomorphism modulo $p$. The map modulo $p$ comes from extending scalars in the first map of the Artin-Schreier sequence  
\[ 0 \to H^1(\nabla_r, \mathbb{Z}/p\mathbb{Z})^{ord} \to H^1(\nabla_r, \mathcal{O}_r)^{ord} \xrightarrow{1-F_r} H^1(\nabla_r, \mathcal{O}_r)^{ord} \to 0. \]
Since $F_r$ acts as an isomorphism on $H^1(Y_r, \mathcal{O}_{Y_r})^{\text{ord}}$ (by Lemma 3.2), $p$-linear algebra gives us that the map
\[ H^1(Y_r, \mathbb{Z}/p\mathbb{Z})^{\text{ord}} \otimes \mathbb{F}_p \to H^1(Y_r, \mathcal{O}_{Y_r})^{\text{ord}}. \]
is an isomorphism. □

5.1.1. The functor $D$. For a finitely generated $\mathbb{Z}_p$-module $M$ with an unramified action of $G_{\mathbb{Q}_p}$, let
\[ D(M) = (M \otimes \mathbb{Z}_p W(\overline{\mathbb{F}_p}))^{F_r=1} = \{ x \in M \otimes \mathbb{Z}_p W(\overline{\mathbb{F}_p}) \mid (F_r \otimes F_r)(x) = x \} \]
where $F_r \in G_{\mathbb{Q}_p}$ is an arithmetic Frobenius. This functor was introduced by Fukaya-Kato and they prove that there is (non-canonical) isomorphism of $\mathbb{Z}_p$-modules $D(M) \cong M$ ([K-F, Proposition 1.7.6]).

Corollary 5.2. We have a natural isomorphism
\[ H^0(X_r, \Omega^n_{X_r})^{\text{ord}} \to D(H^1(Y_r, \mathbb{Z}_p)^{\text{ord}}). \]
Proof. By duality, we have a natural isomorphism
\[ H^0(X_r, \Omega^n_{X_r})^{\text{ord}} \cong H^1(X_r, \mathcal{O}_{X_r})^{\text{ord}}. \]
Noting the isomorphism, $H^1(X_r, \mathcal{O}_{X_r}) \cong H^1(X_r, W(\overline{\mathbb{F}_p}), \mathcal{O}_{X_r, W(\overline{\mathbb{F}_p})})^{F_r=1}$, the result follows from the previous lemma. □

5.1.2. Combining results. Combining Corollary 5.2 with Theorem 4.3, we obtain the proof of Theorem 1.1.

Theorem 1.1. There is a natural isomorphism
\[ S_2(Np^r)^{\text{ord}} \cong D(H^1(Y_r, \mathbb{Z}_p)^{\text{ord}}). \]
for all $r$.

6. First proof of Theorem 1.3

In this section, we complete the proof of Theorem 1.3.

6.1. An observation from commutative algebra. We recall the following simple fact from commutative algebra.

Lemma 6.1. Let $M$ be a free $\mathbb{Z}_p$ module of rank $2n$, and let
\[ (\cdot, \cdot) : M \times M \to \mathbb{Z}_p \]
be a perfect pairing. Let $N$ be a free submodule of $M$ of rank $n$ and let $i : N \to M$ denote the inclusion. Suppose $N$ is isotropic with respect to $(\cdot, \cdot)$. Then the following are equivalent:
1) $N$ is a direct summand
2) $N/pN \xrightarrow{i} M/pM$ is injective
3) $M \xrightarrow{i^\vee} N^\vee$ is surjective
4) The sequence $0 \to N \xrightarrow{i} M \xrightarrow{i^\vee} N^\vee \to 0$ is exact.
Proof: The equivalence of the first 3 items is standard, and clearly 4) implies 3). For 1) implies 4), note that, by the perfectness of the pairing, $M \rightarrow N^\vee$ is surjective. Moreover, since $N$ is isotropic, $M \rightarrow N^\vee$ factors through $M/N$. Then $M/N \rightarrow N^\vee$ is a surjection of free $\mathbb{Z}_p$-modules of rank $n$, so it is an isomorphism by Nakayama’s lemma.

6.2. Application to the complex from Section 2.3.2

Proposition 6.2. The sequence
\[ 0 \rightarrow H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord} \rightarrow H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord} \rightarrow (H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord})^\vee(1-1^-1) \rightarrow 0 \]
of Section 2.3.2 is exact if and only if $H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord} \rightarrow H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord}$ is a split injection of $\mathbb{Z}_p$-modules.

Proof. Since $H^1(\mathcal{Y}_r, \mathbb{Z}_p)$ and $H^1(\mathcal{X}_r, \mathbb{Z}_p)$ are $p$-torsion-free, it suffices, by Lemma 6.1, to show that
\[ \text{rank}_{\mathbb{Z}_p} H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord} = 2 \cdot \text{rank}_{\mathbb{Z}_p} H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord}. \]
However, it is well-known that $H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^1(\mathcal{X}_r, \mathbb{Q}_p)^{ord}$ a free $\mathfrak{h}_r^{ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ module of rank 2 and that $\mathfrak{h}_r^{ord} \otimes \mathbb{Q}_p = \text{Hom}_{\mathbb{Z}_p}(S_2(\mathbb{N}p^r)^{ord}, \mathbb{Q}_p)$. We see that $\text{rank}_{\mathbb{Z}_p} H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord} = 2 \cdot \text{rank}_{\mathbb{Z}_p} S_2(\mathbb{N}p^r)^{ord}_{\mathbb{Q}_p}$ and, by Theorem 1.1, we have
\[ \text{rank}_{\mathbb{Z}_p} H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord} = \text{rank}_{\mathbb{Z}_p} S_2(\mathbb{N}p^r)^{ord}_{\mathbb{Q}_p}. \]
This completes the proof. \[\square\]

Now now have a simple proof that $H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord} \rightarrow H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord}$ is a split injection of $\mathbb{Z}_p$-modules.

Lemma 6.3. For any $n \geq 1$ the composite map
\[ H^1(\mathcal{Y}_r, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow H^1(\mathcal{X}_r, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow H^1(\mathcal{X}_r, \mathbb{Z}/p^n\mathbb{Z}) \]
is injective. The inverse limit map
\[ H^1(\mathcal{Y}_r, \mathbb{Z}_p) \rightarrow H^1(\mathcal{X}_r, \mathbb{Z}_p) \]
is a split injection.

Proof. For the first statement, the first map is an isomorphism by the proper base change theorem, and the second map is injective by the sequence of low-degree terms for the Leray spectral sequence.

For the second statement, the injectivity is clear by the left-exactness of inverse limit. Moreover, since $H^2(\mathcal{Y}_r, \mathbb{Z}_p) = 0$ and $H^2(\mathcal{X}_r, \mathbb{Z}_p)$ is torsion-free, we have a commutative diagram
\[
\begin{array}{ccc}
H^1(\mathcal{Y}_r, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{i} & H^1(\mathcal{X}_r, \mathbb{Z}/p\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^1(\mathcal{Y}_r, \mathbb{Z}_p)/pH^1(\mathcal{Y}_r, \mathbb{Z}_p) & \rightarrow & H^1(\mathcal{X}_r, \mathbb{Z}_p)/pH^1(\mathcal{X}_r, \mathbb{Z}_p)
\end{array}
\]
and so the splitting follows from Lemma 6.1. \[\square\]

This lemma implies that $H^1(\mathcal{Y}_r, \mathbb{Z}_p)^{ord} \rightarrow H^1(\mathcal{X}_r, \mathbb{Z}_p)^{ord}$ is a split injection of $\mathbb{Z}_p$-modules. By Proposition 6.2 this completes the proof of Theorem 1.3.
6.3. The remaining sections. This completes the proofs of all the results stated in the introduction. In the remaining sections, we give a second proof of Theorem 1.3. The reason for doing this is that we see that the proof of Lemma 6.3 cannot generalize to any higher dimensional case. It may be possible to generalize the method of our second proof.

7. Second proof I: the method of Bloch and Kato

We use the results of Bloch and Kato [B-K] (as extended by Hyodo [H] to the semi-stable case) to compute the ordinary cohomology of $X_1$.

7.1. The method of Bloch-Kato. Let $K = \mathbb{Q}_p(\zeta_p)$. We consider the cohomology of $X_1$ by using the semi-stable model $X'/\mathcal{O}_K$ (2.4). We have the pair of Cartesian squares

$$
\begin{array}{ccc}
X' & \xrightarrow{j} & X' \\
\downarrow & & \downarrow \\
\text{Spec}K & \xrightarrow{i} & \text{Spec}O_K
\end{array}
\quad
\begin{array}{ccc}
Y' & \xleftarrow{i} & Y'
\downarrow & & \downarrow \\
\text{Spec}O_K & \xleftarrow{j} & \text{Spec}O_K
\end{array}
\quad
\begin{array}{ccc}
\text{Spec}F_p & \xleftarrow{i} & \text{Spec}F_p
\downarrow & & \downarrow \\
\text{Spec}O_K & \xleftarrow{j} & \text{Spec}O_K
\end{array}
\quad
\begin{array}{ccc}
\text{Spec}O_K & \xrightarrow{i} & \text{Spec}O_K
\downarrow & & \downarrow \\
\text{Spec}F_p & \xrightarrow{j} & \text{Spec}F_p
\end{array}
$$

The goal of $[B-K]$ is to understand the $p$-adic nearby cycles $M_n^q = i^* R^q j_* \mathbb{Z}/p^n\mathbb{Z}(q)$.

7.1.1. Log Hodge-Witt sheaves. The method of Bloch-Kato-Hyodo is to compare the sheaves $M_n^q$ to the logarithmic Hodge-Witt sheaves $W_n^{1,\omega_{Y'},\log}$, whose definitions and properties we now briefly recall from [H]. Let $\omega_{X'}^1$ be the sheaf of one-forms with (at worst) log poles at the supersingular points, and $\omega_{Y'}^1 = \omega_{X'}^1 \otimes \mathcal{O}_{Y'}$. It is the dualizing sheaf on $Y'$.

There is a (Frobenious anti-linear) Cartier operator $C$ on $\omega_{Y'}^1$, and we define $W_1\omega_{Y',\log} = \omega_{Y',\log} = \ker(1 - C^{-1} : \omega_{Y'}^1)$. The sheaves $W_n\omega_{Y',\log}$ can be defined and they fit in exact sequences

$$0 \rightarrow \omega_{Y',\log}^1 \rightarrow W_n\omega_{Y',\log}^1 \rightarrow W_{n-1}\omega_{Y',\log}^1 \rightarrow 0.$$ 

We can define $W\omega_{Y',\log}^1$ as the inverse limit of the $W_n\omega_{Y',\log}^1$.

7.1.2. The ordinary part is ordinary. The results of Bloch-Kato and Hyodo apply in the case of ordinary varieties – varieties for which $H^i(Y, d\omega_Y^1) = 0$ for all $i, j \geq 0$. Our $X'$ is not an ordinary variety. However, the important observation is the coincidence of two uses of the adjective "ordinary":

**Lemma 7.1.** The ordinary part of $X'$ is ordinary. That is, $H^i(Y', d\mathcal{O}_{Y'})^{\text{ord}} = 0$ for $i = 0, 1$.

**Proof.** This follows from the shape of the correspondence

$$T(p) = \begin{pmatrix} Fr_* & * \\ 0 & Fr^* \end{pmatrix}$$

in a similar manner to Lemma 3.3.

Since $d\mathcal{O}_{Y'} \cong \mathcal{O}_{Y'}/\mathcal{O}_{Y'}^p$ we see that both $Fr^*$ and $Fr_*$ act as 0, and so $T(p)$ acts nilpotently. \qed
The results of Bloch-Kato. We see that the arguments of Bloch-Kato and Hyodo apply for the ordinary parts. In particular, we have the following analog of [B-K, Theorem 9.2, Corollary 9.4].

**Theorem 7.2.** The spectral sequence

\[ E_{s,t}^2 = H^s(Y', \mathcal{M}_n^{(-t)})^{\text{ord}} \implies H^{s+t}(X, \mathbb{Z}/p\mathbb{Z})^{\text{ord}} \]

degenerates at the \( E_2 \) page. Moreover, we have isomorphisms

\[ H^q(Y', \mathcal{M}_n^{\text{ord}}) \cong H^q(Y, W_n\omega_1^{1/\nu}Y', \log)^{\text{ord}}. \]

**Proof.** Indeed, examining the proofs of [B-K, Theorem 9.2, Corollary 9.4], we see that the ordinary assumption is only used in the proof of the vanishing theorem [B-K, Corollary 8.2, pg. 142]. However, we see that if we take the ordinary component of each exact sequence appearing in that proof, then Lemma 7.1 is enough to show the vanishing theorem for the ordinary component.

**Corollary 7.3.** We have a natural exact sequence

\[ 0 \to H^1(Y, \mathbb{Z}/p\mathbb{Z})^{\text{ord}} \to H^1(X, \mathbb{Z}/p\mathbb{Z})^{\text{ord}} \to H^0(Y, W_1\omega_1^{1/\nu}Y', \log)^{\text{ord}}(-1) \to 0. \]

**Proof.** Follows from the degeneration of the spectral sequence.

7.2. **Duality on \( Y' \).** We describe Serre duality for the logarithmic Hodge-Witt sheaves.

**Proposition 7.4.** The natural pairing

\[ H^1(Y', \mathbb{Z}/p\mathbb{Z}) \times H^0(Y', W_1\omega_1^{1/\nu}Y', \log) \to H^1(Y', W_1\omega_1^{1/\nu}Y', \log) \to \mathbb{Z}/p \]

is perfect.

**Proof.** It is enough to show that the pairing modulo \( p \)

\[ H^1(Y, \mathbb{Z}/p\mathbb{Z}) \times H^0(Y, W_1\omega_1^{1/\nu}Y', \log) \to \mathbb{Z}/p \]

is perfect.

We have the perfect pairing

\[ H^1(Y', \mathcal{O}_{Y'}) \times H^0(Y', \omega_1^{1/\nu}) \to \mathbb{F}_p. \]

of Serre duality. Under this pairing, the Frobenius on \( H^1(Y', \mathcal{O}_{Y'}) \) is dual to the the inverse Cartier operator on \( H^0(Y, \omega_1^{1/\nu}) \). The result now follows from the following lemma in \( p \)-linear algebra.

**Lemma 7.5.** Let \( V \) be a finite-dimensional \( \mathbb{F}_p \)-vector spaces and \( V^* \) its dual space. Let \( \phi \) be a Frobenius-linear endomorphism of \( V \). Then the natural pairing on \( V \times V^* \) induces a perfect pairing

\[ \ker(1 - \phi : V) \times \ker(1 - \phi^* : V^*) \to \mathbb{F}_p. \]
7.3. The result for $X_1$. The following result is the case $r = 1$ of Theorem 1.3

Proposition 7.6. The complex

$$0 \rightarrow H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}} \rightarrow H^1(X_1, \mathbb{Z}_p)^{\mathrm{ord}} \rightarrow (H^1(Y_1, \mathbb{Z}_p)^{\mathrm{ord}})^{\vee} (\kappa^{-1}(-)^{-1}) \rightarrow 0$$

constructed in Section 2.3.2 is acyclic.

Proof. To show that it is acyclic, it suffices to show it is so as a sequence of $\mathbb{Z}_p$-modules (ignoring the Galois action). We compare it to the sequence of Corollary 7.3.

Recall the notation $H(Y_r) = H^1(\mathbb{Y}_r, \mathbb{Z}_p)^{\mathrm{ord}}$ and $H(X_r) = H^1(\mathbb{X}_r, \mathbb{Z}_p)^{\mathrm{ord}}$. The purpose of this section is to prove the following proposition.

Proposition 8.1. For any $r \geq 1$, the map $H(Y_r) \rightarrow H(X_r)$ is a split injection of $\Lambda_r$-modules

Note that, by Proposition 6.2, this completes the second proof of Theorem 1.3

Proof. In the case $r = 1$, this follows from Proposition 7.6 and Lemma 6.1. We now turn to the case of general $r$. By Lemma 6.1 it suffices to show that $H(X_r) \rightarrow H(Y_r)^{\vee}$ is surjective.

Consider the commutative diagram of $\Lambda_1$-modules

$$
\begin{array}{ccc}
H(Y_1) & \longrightarrow & H(X_1) \\
\downarrow & & \downarrow \\
H(Y_r)_{\Lambda_1} & \longrightarrow & H(X_r)_{\Lambda_1}
\end{array}
$$

where the vertical arrows are the trace maps. The right vertical arrow is an isomorphism by Theorem 1.2. The top horizontal arrow is a split injection of $\Lambda_1$-modules, by the $r = 1$ case already proven. Therefore the bottom horizontal map is a split injection of $\Lambda_1$-modules.

Using Lemma 6.1 we see that the map $H(X_r)_{\Lambda_1} \rightarrow \text{Hom}_{\Lambda_1}(H(Y_r)_{\Lambda_1}, \Lambda_1)$ given by the Poincaré pairing is surjective. Since, by Theorem 1.2 $H(Y_r)$ is $\Lambda_r$-free, it is easy to see that the natural map

$$\text{Hom}_{\Lambda_1}(H(Y_r)_{\Lambda_1}, \Lambda_1) \rightarrow (H(Y_r)^{\vee})_{\Lambda_1}$$

is an isomorphism. The composite $H(X_r)_{\Lambda_1} \rightarrow \text{Hom}_{\Lambda_1}(H(Y_r)_{\Lambda_1}, \Lambda_1) \rightarrow (H(Y_r)^{\vee})_{\Lambda_1}$ is the same as the natural map $H(X_r) \rightarrow H(Y_r)^{\vee}$ tensored by $\Lambda_1$. By Nakayama’s lemma, we have that $H(X_r) \rightarrow H(Y_r)^{\vee}$ is surjective.
References

[B-K] Bloch, S.; Kato, K. p-adic tale cohomology. Inst. Hautes tudes Sci. Publ. Math. No. 63 (1986), 107-152.
[C] Cais, B. The Geometry of Hida Families and Λ-adic Hodge theory. Preprint.
[D] Deligne, P. Cohomologie à supports propres. Exposé XVII in “Théorie des topos et cohomologie étale des schémas (SGA 4, Tome 3)”. Lecture Notes in Math., Vol 305, Springer-Verlag, Berlin-New York, 1973. vi+640 pp. 14-06.
[D-R] Deligne, P.; Rapoport, M. Les schmas de modules de courbes elliptiques. (French) Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 143-316. Lecture Notes in Math., Vol. 349, Springer, Berlin, 1973.
[H] Hyodo, Osamu. A note on p-adic tale cohomology in the semistable reduction case. Invent. Math. 91 (1988), no. 3, 543-557.
[K-F] Fukaya, T; Kato, K. On conjectures of Sharifi. Preprint.
[K-M] Katz, N.; Mazur, B. Arithmetic moduli of elliptic curves. Annals of Mathematics Studies, 108. Princeton University Press, Princeton, NJ, 1985. xiv+514 pp. ISBN: 0-691-08349-5; 0-691-08352-5.
[G] Gross, B. A tameness criterion for Galois representations associated to modular forms (mod S). Duke Math. J. 61 (1990), no. 2, 445-517.
[H1] Hida, H. Galois representations into $GL_2(\mathbb{Z}_p[[X]])$ attached to ordinary cusp forms. Invent. Math. 85 (1986), no. 3, 545-613.
[H2] Hida, H. p-adic automorphic forms on Shimura varieties. Springer Monographs in Mathematics. Springer-Verlag, New York, 2004. xii+390 pp. ISBN: 0-387-20711-2
[M-W] Mazur, B; Wiles, A. Class fields of abelian extensions of $\mathbb{Q}$. Invent. Math. 76 (1984), no. 2, 179-330.
[O1] Ohta, M. On the p-adic Eichler-Shimura isomorphism for Λ-adic cusp forms. J. Reine Angew. Math. 463 (1995), 49-98.
[O2] Ohta, M. Ordinary p-adic tale cohomology groups attached to towers of elliptic modular curves. Compositio Math. 115 (1999), no. 3, 241-301.
[O3] Ohta, M. Ordinary p-adic tale cohomology groups attached to towers of elliptic modular curves. II. Math. Ann. 318 (2000), no. 3, 557-583.
[O4] Ohta, M. Companion forms and the structure of p-adic Hecke algebras. II. J. Math. Soc. Japan 59 (2007), no. 4, 913-951.
[S] Sharifi, R. A reciprocity map and the two-variable p-adic L-function. Ann. of Math. (2) 173 (2011), no. 1, 251-300.
[U] Ulmer, D. On universal elliptic curves over Igusa curves. Invent. Math. 99 (1990), no. 2, 377-391.

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