Two types of generalized integrable decompositions and new solitary-wave solutions for the modified Kadomtsev-Petviashvili equation with symbolic computation

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Abstract

The modified Kadomtsev-Petviashvili (mKP) equation is shown in this paper to be decomposable into the first two soliton equations of the 2N-coupled Chen-Lee-Liu and Kaup-Newell hierarchies by respectively nonlinearizing two sets of symmetry Lax pairs. In these two cases, the decomposed (1+1)-dimensional nonlinear systems both have a couple of different Lax representations, which means that there are two linear systems associated with the mKP equation under the same constraint between the potential and eigenfunctions. For each Lax representation of the decomposed (1+1)-dimensional nonlinear systems, the corresponding Darboux transformation is further constructed such that a series of explicit solutions of the mKP equation can be recursively generated with the assistance of symbolic computation. In illustration, four new families of solitary-wave solutions are presented and the relevant stability is analyzed.

Keywords: Modified Kadomtsev-Petviashvili equation; Integrable decompositions; Darboux transformations; Solitary-wave solutions; Symbolic computation

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1. Introduction

According to Lax’s theory \cite{1}, a given NLEE is said to be integrable if it arises as the compatibility condition of two linear eigenvalue equations which are usually called a Lax pair and comprised of the spatial part and the temporal part. Although it is not an easy work to find the Lax pair associated with an integrable NLEE, one can relate a properly-chosen spectral problem to a hierarchy of soliton equations whose Lax pairs have the same spatial part but different temporal parts \cite{2}. In the past several decades, some important and typical (1+1)-dimensional integrable hierarchies have been established and well understood, including the Ablowitz-Kaup-Newell-Segur (AKNS) \cite{3}, Wadati-Konno-Ichikawa (WKI) \cite{4}, Kaup-Newell (KN) \cite{5} and Levi \cite{6} hierarchies. Today, the Lax pair has been playing a considerable role in studying the integrable properties of NLEEs such as the Hamiltonian structures, conservation laws and symmetry classes \cite{7}.

For describing various complex nonlinear phenomena of our realistic world, the higher-dimensional NLEEs appear very attractive in many fields of physical and engineering sciences \cite{8}. However, due to the higher space dimensions, those higher-dimensional nonlinear systems often exhibit more intricate properties (e.g., the integrability of NLEEs in 3+1 dimensions is still not a well-solved problem \cite{9}) and admit more abundant soliton structures \cite{10}. It is mentioned that by dimensional splitting the higher-dimensional problem can be reduced to several lower-dimensional ones which are easier to treat with the available tools \cite{11, 12}. In recent studies, many (2+1)-dimensional integrable NLEEs have shown to be relevant with some known (1+1)-dimensional soliton equations by the nonlinearization of their Lax pairs and adjoint Lax pairs \cite{12, 13, 14, 15}. Through such a decomposition, a submanifold of solutions for the given (2+1)-dimensional integrable NLEE is obtainable by solving the resulting (1+1)-dimensional integrable systems.

The modified Kadomtsev-Petviashvili (mKP) equation \cite{16}, as one of the most important integrable NLEEs in 2+1 dimensions, has been derived in many physical applications such as the propagation of ion-acoustic waves in a plasma with non-isothermal electrons \cite{17} and the electromagnetic wave description in an isotropic charge-free infinite ferromagnetic thin film \cite{18}. It has been found that the mKP equation is able to be decomposed into the first two nontrivial nonlinear systems in the Burgers hierarchy \cite{15}, two-coupled Korteweg-de Vries (KdV) hierarchy \cite{19}, two-coupled Chen-Lee-Liu (CLL) hierarchy \cite{15, 20}, two-coupled KN hierarchy \cite{21, 22} and some other soliton hierarchies \cite{23}. But to our knowledge none has given a systematic way of determining its all decompositions to two (1+1)-dimensional integrable NLEEs in the same hierarchy, which means that some new or more generalized integrable decompositions for the mKP equation have not been uncovered as yet.

In Ref. \cite{15}, the authors have pointed out that the following mKP equation

$$ q_t = \frac{1}{4} \left( q_{xxx} - 6 q^2 q_x - 6 q_x \partial_x^{-1} q_y + 3 \partial_x^{-1} q_{yy} \right), \tag{1.1} $$

can be constrained into the two-coupled CLL and high-order CLL systems by imposing
the nonlinearization on both the associated Lax pair and another auxiliary Lax pair. The present paper is intended to make a further investigation on Eqn. (1.1) by proposing two types of generalized integrable decompositions which respectively reduce Eqn. (1.1) to the first two nontrivial members in the $2N$-coupled CLL and KN hierarchies. On this basis, our next concern is to derive the Lax representations of the decomposed (1+1)-dimensional nonlinear systems and construct their respective Darboux transformations by which some new solitary-wave solutions are expected to be revealed for Eqn. (1.1).

2. Proposal of generalized integrable decompositions

As indicated in Refs. [15, 16], Eqn. (1.1) is associated with the following linear system

\begin{align}
    u_y &= L_1 u, \quad L_1 = \partial_x^2 - 2q \partial_x, \quad (2.1a) \\
    u_t &= M_1 u, \quad M_1 = \partial_x^3 - 3q \partial_x^2 + \frac{3}{2} (q^2 - q_x - \partial_x^{-1}q_y) \partial_x, \quad (2.1b)
\end{align}

from which we follow the definition of the adjoint of a differential operator (for a differential operator in the form $\Omega = \Sigma a_k \partial^k_x$, its adjoint form is taken as $\Omega^* = \Sigma (-\partial_x)^k a_k$, where the asterisk denotes the adjoint operator [13]) and obtain other three linear systems respectively with respect to $v = (u_x)^*$, $m = u^*$ and $p = v^*$, as follows:

\begin{align}
    v_y &= L_2 v, \quad L_2 = -\partial_x^2 - 2q \partial_x, \quad (2.2a) \\
    v_t &= M_2 v, \quad M_2 = \partial_x^3 + 3q \partial_x^2 + \frac{3}{2} (q^2 + q_x - \partial_x^{-1}q_y) \partial_x, \quad (2.2b) \\
    m_y &= -L_1^* m, \quad L_1^* = \partial_x^2 + 2q \partial_x + 2q_x, \quad (2.3a) \\
    m_t &= -M_1^* m, \quad M_1^* = -\partial_x^3 - 3q \partial_x^2 - \frac{3}{2} (q^2 + 3q_x - \partial_x^{-1}q_y) \partial_x \\
    &\quad + \frac{3}{2} (q_y - 2q q_x - q_{xx}), \quad (2.3b) \\
    p_y &= -L_2^* p, \quad L_2^* = -\partial_x^2 + 2q \partial_x + 2q_x, \quad (2.4a) \\
    p_t &= -M_2^* p, \quad M_2^* = -\partial_x^3 + 3q \partial_x^2 - \frac{3}{2} (q^2 - 3q_x - \partial_x^{-1}q_y) \partial_x \\
    &\quad + \frac{3}{2} (q_y - 2q q_x + q_{xx}), \quad (2.4b)
\end{align}

where System (2.2) has also been given in Ref. [15], while Systems (2.3) and (2.4) are exhibited here for the first time. By direct calculations, we find that the compatibility conditions $v_{yt} = v_{ty}$, $m_{yt} = m_{ty}$ and $p_{yt} = p_{ty}$ all give rise to Eqn. (1.1), which suggests that Systems (2.2)–(2.4) are other three different Lax pairs of Eqn. (1.1).

Assuming $u_j$ and $v_j$ ($j = 1, 2, \ldots, N$) respectively satisfy Systems (2.1) and (2.2), we introduce the following potential constraint [13]

\[ q_I = -\frac{1}{2} \sum_{j=1}^{N} u_j v_j, \quad (2.5) \]
into Systems (2.1) and (2.2), and obtain the following 2N-coupled CLL system [24],

\[ u_{j,y} - u_{j,xx} - \sum_{k=1}^{N} u_k v_k u_{j,x} = 0, \quad v_{j,y} + v_{j,xx} - \sum_{k=1}^{N} v_k u_k v_{j,x} = 0, \quad (j = 1, 2, \ldots, N), \quad (2.6) \]

and its high-order generalization

\[
\begin{align*}
&u_{j,t} - u_{j,xxx} - \frac{3}{2} \sum_{k=1}^{N} u_k v_k u_{j,xx} - \frac{3}{4} \left[ \left( \sum_{k=1}^{N} u_k v_k \right)^2 + 2 \sum_{k=1}^{N} u_{k,x} v_k \right] u_{j,x} = 0, \\
&(j = 1, 2, \ldots, N), \quad (2.7a) \\
&v_{j,t} - v_{j,xxx} + \frac{3}{2} \sum_{k=1}^{N} v_k u_k v_{j,xx} - \frac{3}{4} \left[ \left( \sum_{k=1}^{N} v_k u_k \right)^2 - 2 \sum_{k=1}^{N} v_{k,x} u_k \right] v_{j,x} = 0, \\
&(j = 1, 2, \ldots, N). \quad (2.7b)
\end{align*}
\]

Similarly, if we constrain the potential as

\[ q_{II} = -\frac{1}{2} \sum_{j=1}^{N} m_j p_j, \quad (2.8) \]

where \( m_j \) and \( p_j \) (\( j = 1, 2, \ldots, N \)) satisfy Systems (2.3) and (2.4), respectively, then Eqns. (2.3a) and (2.4a) are nonlinearized into the 2N-coupled KN system [24],

\[
\begin{align*}
m_{j,y} + m_{j,xx} - \left( \sum_{k=1}^{N} k p_k m_j \right)_{x} &= 0, \\
p_{j,y} - p_{j,xx} - \left( \sum_{k=1}^{N} p_k m_k p_j \right)_{x} &= 0, \\
&\quad (j = 1, 2, \ldots, N), \quad (2.9)
\end{align*}
\]

and Eqns. (2.3b) and (2.4b) become

\[
\begin{align*}
m_{j,t} - m_{j,xxx} - \frac{3}{2} \left[ \left( \sum_{k=1}^{N} m_k p_k \right)^2 m_j - \sum_{k=1}^{N} m_k p_k m_{j,x} - \sum_{k=1}^{N} m_{k,x} p_k m_j \right]_{x} &= 0, \\
&(j = 1, 2, \ldots, N), \quad (2.10a) \\
p_{j,t} - p_{j,xxx} - \frac{3}{2} \left[ \left( \sum_{k=1}^{N} p_k m_k \right)^2 p_j + \sum_{k=1}^{N} p_k m_k p_{j,x} + \sum_{k=1}^{N} p_{k,x} m_k p_j \right]_{x} &= 0, \\
&(j = 1, 2, \ldots, N). \quad (2.10b)
\end{align*}
\]

which is a generalized high-order version of System (2.9). Without any difficulty, one can check that the above two constrained potentials \( q_I \) and \( q_{II} \) both satisfy Eqn. (1.1) exactly. Thus, we have got two generalized integrable decompositions for Eqn. (1.1), as follows:
Decomposition I: If \((u_j, v_j) (j = 1, 2, \ldots, N)\) is a compatible solution of Systems \((2.6)\) and \((2.7)\), then the function \(q_I\) determined by Expression \((2.5)\) solves the mKP equation \((1.1)\).

Decomposition II: If \((m_j, p_j) (j = 1, 2, \ldots, N)\) is a compatible solution of Systems \((2.9)\) and \((2.10)\), then the function \(q_{II}\) determined by Expression \((2.8)\) solves the mKP equation \((1.1)\).

Note that the two proposed decompositions in Refs. [15, 20] and Refs. [21, 22], respectively, correspond to the special cases of Decompositions I and II when \(N = 1\). Moreover, the authors in Refs. [21, 22] have also not shown the association of the decomposition there with the Lax pairs for the mKP equation.

Under the potential constraints \((2.5)\) and \((2.8)\), we can gain much information about the mKP equation \((1.1)\) from Systems \((2.6)–(2.7)\) and \((2.9)–(2.10)\) by means of various known effective approaches. In the following, to explore more unrevealed solutions (especially the solitary-wave solutions) of Eqn. \((1.1)\), we will deal with the decomposed \((1+1)\)-dimensional nonlinear systems by employing the Darboux transformation method which has been proved to be an excellent technique for analytically studying integrable NLEEs and soliton problems [25] in that it gives the general procedure to recursively generate a series of explicit solutions including the multi-soliton solutions from an initial solution [26]. Once the Darboux transformation for a given NLEE is constructed, one only needs to solve a linear differential system (i.e., the Lax pair with an initial potential) and perform tedious but not complicated algebraic operations [27]. Hereby, an obvious advantage of the Darboux transformation lies in that the iterative algorithm is purely algebraic and very computerizable by virtue of symbolic computation [28, 31, 32].

3. Lax representations and Darboux transformations of Systems \((2.6)\) and \((2.7)\)

It is possible that an integrable nonlinear system could be associated with several linear spectral problems, which might lead to different Darboux transformations. We consider the \((N+1) \times (N+1)\) linear eigenvalue problem and interestingly find that Systems \((2.6)\) and \((2.7)\) admit two different kinds of Lax representations, in which the first one is of the form

\[
\begin{align*}
\psi_x &= U^{(1)} \psi = \left[ \lambda U_0^{(i)} + U_1^{(i)} \right] \psi, \\
\psi_y &= V^{(1)} \psi = \left[ \lambda^2 V_0^{(i)} + \lambda V_1^{(i)} + V_2^{(i)} \right] \psi, \\
\psi_t &= W^{(1)} \psi = \left[ \lambda^3 W_0^{(i)} + \lambda^2 W_1^{(i)} + \lambda W_2^{(i)} + W_3^{(i)} \right] \psi,
\end{align*}
\]

where \(\lambda\) is the eigenvalue parameter, \(\psi = (\psi_1, \psi_2, \ldots, \psi_{N+1})^T\) (the superscript \(T\) denotes the vector transpose) is the vector eigenfunction, the matrices \(U_i^{(i)}, V_i^{(i)}\) and \(W_i^{(i)} (i = 0, 1; k = 0, 1, 2; l = 0, 1, 2, 3)\) are expressible in the form

\[
\begin{align*}
V_0^{(i)} &= 2U_0^{(i)}, & W_0^{(i)} &= 4U_0^{(i)}, & W_1^{(i)} &= 2V_1^{(i)}, \\
U_0^{(i)} &= \begin{pmatrix} 1 & 0 \\ V & -I \end{pmatrix}, & U_1^{(i)} &= \begin{pmatrix} 0 & -U \\ O & -\frac{1}{2}VU \end{pmatrix},
\end{align*}
\]
with $I$ as the $N \times N$ identity matrix, $A = UVUV$, $U = (u_1, u_2, \ldots, u_N)$, $V = (v_1, v_2, \ldots, v_N)^T$ and $O_1 = (0, 0, \ldots, 0)^T$. Based on the matrix-form inverse scattering formulation in Ref. [20], the second Lax representation of Systems (2.6) and (2.7) is presented as follows:

\[
\begin{align*}
\Psi_x &= U^{(2)} \Psi = \left[ \lambda^2 V_0^{(2)} + \lambda U_1^{(2)} + U_2^{(2)} \right] \Psi, \\
\Psi_y &= V^{(2)} \Psi = \left[ \lambda^3 V_0^{(2)} + \lambda^2 V_2^{(2)} + \lambda^2 V_3^{(2)} + V_4^{(2)} \right] \Psi, \\
\Psi_t &= W^{(2)} \Psi = \left[ \lambda^6 W_0^{(2)} + \lambda^5 W_1^{(2)} + \lambda^4 W_2^{(2)} + \lambda^3 W_3^{(2)} + \lambda^2 W_4^{(2)} + \lambda W_5^{(2)} + W_6^{(2)} \right] \Psi,
\end{align*}
\]

with

\[
\begin{align*}
V_0^{(2)} &= -2 U_0^{(2)}, & W_0^{(2)} &= 4 U_0^{(2)}, & V_1^{(2)} &= -2 U_1^{(2)}, \\
W_1^{(2)} &= 4 U_1^{(2)}, & W_2^{(2)} &= -2 V_2^{(2)}, & W_3^{(2)} &= -2 V_3^{(2)}, \\
U_0^{(2)} &= \begin{pmatrix} -1 & O_1^T \\ O_1 & I \end{pmatrix}, & U_1^{(2)} &= \begin{pmatrix} 0 & U \\ V & O_2 \end{pmatrix}, & U_2^{(2)} &= \begin{pmatrix} 0 & O_1^T \\ O_1 & -\frac{1}{2} VU \end{pmatrix}, \\
V_2^{(2)} &= \begin{pmatrix} UV & O_1^T \\ O_1 & VU \end{pmatrix}, & V_3^{(2)} &= \begin{pmatrix} 0 & \frac{1}{2} UV - V_x \\ \frac{1}{2} UV - V_x & O_2 \end{pmatrix}, \\
V_4^{(2)} &= \begin{pmatrix} 0 & O_1^T \\ O_1 & -\frac{1}{4} VUVU + \frac{1}{2} (V_x U - VU_x) \end{pmatrix}, \\
W_4^{(2)} &= \begin{pmatrix} -\frac{1}{2} A + UV_x - U_x V & O_1^T \\ O_1 & \frac{1}{2} VUVU + VU_x - V_x U \end{pmatrix}, \\
W_5^{(2)} &= \begin{pmatrix} 0 & \frac{1}{2} AU + \frac{1}{2} (U_x V - UV_x) U + UVU_x + U_x U \frac{1}{2} V(UV_x - U_x V) U + V_x UU + \frac{1}{2} V(UV_x - U_x V) U \frac{1}{2} V(UV_x - U_x V) U \end{pmatrix}, \\
W_6^{(2)} &= \begin{pmatrix} \frac{1}{2} & O_1^T \\ O_1 & -\frac{1}{4} VAU - VU_{xx} - V_x U - VVU_x + V_x UVU + \frac{1}{2} V(UV_x - U_x V) U \end{pmatrix}
\end{align*}
\]

where $O_2$ is the $N \times N$ zero matrix, $A$, $I$, $U$, $V$ and $O_1$ have been defined as above. Here, it is easy to verify that Systems (2.6) and (2.7) can be respectively derived from the zero-
curvature conditions \(U_y^{(i)} - V_x^{(i)} + [U^{(i)}, V^{(i)}] = 0\) and \(U_t^{(i)} - W_x^{(i)} + [U^{(i)}, W^{(i)}] = 0\) \((i = 1, 2)\), where the brackets represent the usual matrix commutator.

It is known that the Darboux transformation is actually a gauge transformation which relates two different solutions of the same linear system \([25]\). Starting from System \((3.1)\), we construct the first Darboux transformation for Systems \((2.6)\) and \((2.7)\) in the form

\[
\hat{\psi} = (\lambda \Delta^{(1)} - \Delta^{(1)}S^{(1)}) \Psi, \quad \Delta^{(1)} = \begin{pmatrix} \Delta_{11}^{(1)} & \Delta_{12}^{(1)} \\ \Delta_{21}^{(1)} & \Delta_{22}^{(1)} \end{pmatrix}, \quad S^{(1)} = \begin{pmatrix} S_1^{(1)} & S_2^{(1)} \\ S_3^{(1)} & S_4^{(1)} \end{pmatrix}, \quad (3.16)
\]

where \(\Delta_{11}^{(1)} = \delta_{11}^{(1)}, \Delta_{21}^{(1)} = (\delta_{12}^{(1)}, \ldots, \delta_{1,N+1}^{(1)}), \Delta_{31}^{(1)} = (\delta_{21}^{(1)}, \ldots, \delta_{N+1,1}^{(1)})^T, \Delta_{41}^{(1)} = (\delta_{ik}^{(1)})_{2 \leq i, k \leq N+1}, \Delta_{12}^{(1)} = s_1^{(1)}, \Delta_{22}^{(1)} = (s_{12}^{(1)}, \ldots, s_{1,N+1}^{(1)}), \Delta_{32}^{(1)} = (s_{21}^{(1)}, \ldots, s_{N+1,1}^{(1)})^T, \Delta_{42}^{(1)} = (s_{ik}^{(1)})_{2 \leq i, k \leq N+1}, \delta_{ik}^{(1)}\) and \(s_{ik}^{(1)} (1 \leq i, k \leq N+1)\) are all the functions of \(x, y\) and \(t\) to be determined, \(\hat{\psi}\) is required to satisfy System \((3.1)\) with \(U^{(1)}, V^{(1)}\) and \(W^{(1)}\) replaced respectively by \(\hat{U}^{(1)}, \hat{V}^{(1)}\) and \(\hat{W}^{(1)}\) in which the old potentials \((u_j, v_j)\) are transformed into new ones \((\hat{u}_j, \hat{v}_j)\) \((j = 1, 2, \ldots, N)\).

From the knowledge of the Darboux transformation, we know that the matrices \(\Delta^{(1)}, S^{(1)}\), \(\hat{U}_i^{(1)}, \hat{V}_k^{(1)}\) and \(\hat{W}_l^{(1)}\) \((i = 0, 1; k = 0, 1, 2; l = 0, 1, 2, 3)\) must satisfy the following equations:

\[
\begin{align*}
\Delta^{(1)} & \Delta_0^{(1)} - \hat{U}_0^{(1)} \Delta^{(1)} = 0, \\
\Delta^{(1)} & \Delta_1^{(1)} U_1^{(1)} - \Delta^{(1)} S^{(1)} U_0^{(1)} + \hat{U}_0^{(1)} \Delta^{(1)} S^{(1)} - \hat{U}_1^{(1)} \Delta^{(1)} = 0, \\
\Delta^{(1)} & \Delta_2^{(1)} S^{(1)} + \Delta^{(1)} S^{(1)} U_1^{(1)} - \hat{U}_1^{(1)} \Delta^{(1)} S^{(1)} = 0, \\
\Delta^{(1)} & \Delta_3^{(1)} V_1^{(1)} - \Delta^{(1)} S^{(1)} V_0^{(1)} + \hat{V}_0^{(1)} \Delta^{(1)} S^{(1)} - \hat{V}_1^{(1)} \Delta^{(1)} = 0, \\
\Delta^{(1)} & \Delta_4^{(1)} V_2^{(1)} - \Delta^{(1)} S^{(1)} V_1^{(1)} + \hat{V}_1^{(1)} \Delta^{(1)} S^{(1)} - \hat{V}_2^{(1)} \Delta^{(1)} = 0, \\
\Delta^{(1)} & \Delta_5^{(1)} S^{(1)} + \Delta^{(1)} S^{(1)} V_2^{(1)} - \hat{V}_2^{(1)} \Delta^{(1)} S^{(1)} = 0, \\
\Delta^{(1)} & \Delta_6^{(1)} W_2^{(1)} - \Delta^{(1)} S^{(1)} W_1^{(1)} + \hat{W}_1^{(1)} \Delta^{(1)} S^{(1)} - \hat{W}_2^{(1)} \Delta^{(1)} = 0, \\
\Delta^{(1)} & \Delta_7^{(1)} W_3^{(1)} - \Delta^{(1)} S^{(1)} W_2^{(1)} + \hat{W}_2^{(1)} \Delta^{(1)} S^{(1)} - \hat{W}_3^{(1)} \Delta^{(1)} = 0, \\
\Delta^{(1)} & \Delta_8^{(1)} S^{(1)} + \Delta^{(1)} S^{(1)} W_3^{(1)} - \hat{W}_3^{(1)} \Delta^{(1)} S^{(1)} = 0.
\end{align*}
\]

By Eqns. \((3.17a)\) and \((3.17b)\), we can directly compute out:

\[
\begin{align*}
\Delta_2^{(1)} & = O_1^{(1)}, \quad \Delta_4^{(1)} = O_2, \\
\delta_{11,x}^{(1)} & = \delta_{11}^{(1)} S_2^{(1)} V - \hat{U} \Delta_3^{(1)}, \\
\Delta_3^{(1)} & = 2 \Delta_4^{(1)} S_3^{(1)} + \Delta_4^{(1)} S_2^{(1)} V + \Delta_4^{(1)} S_4^{(1)} V - \Delta_4^{(1)} V S_4^{(1)} - \frac{1}{2} \hat{V} \hat{U} \Delta_3^{(1)},
\end{align*}
\]

with

\[
\hat{U} = (\delta_{11}^{(1)} U - 2 \delta_{11}^{(1)} S_2^{(1)}) (\Delta_4^{(1)})^{-1}, \quad \hat{V} = (2 \Delta_4^{(1)} + \Delta_4^{(1)} V) / \delta_{11}^{(1)}.
\]

Then, using the above results, it can be found that Eqn. \((3.17d)\) is lead to be satisfied automatically, while Eqns. \((3.17c), (3.17g)\) and \((3.17h)\) yield the following constraint conditions...
on $\delta_{11}^{(1)}$, $\Delta_3^{(1)}$ and $\Delta_4^{(1)}$ as

\begin{align}
\Delta_{4,y} &= O_2, \quad \Delta_{4,t} = O_2, \\
\delta_{11,y}^{(1)} &= 2\delta_{11}^{(1)}S_2^{(1)}V S_1^{(1)} - \frac{1}{2}\hat{U}\hat{V}\hat{U}\Delta_3^{(1)} - \hat{U}_x\Delta_3^{(1)} + \frac{1}{2}\delta_{11}^{(1)}S_2^{(1)}VUV - \delta_{11}^{(1)}S_2^{(1)}V_x \\
&\quad + 2\delta_{11}^{(1)}US_3^{(1)} - 4\delta_{11}^{(1)}S_2^{(1)}S_3^{(1)},
\end{align}

\begin{align}
\delta_{11,t}^{(1)} &= \frac{1}{2}\delta_{11}^{(1)}S_2^{(1)}UV_x V - \frac{1}{2}\delta_{11}^{(1)}S_2^{(1)}VUV_x + \frac{1}{8}\delta_{11}^{(1)}S_2^{(1)}VA + \hat{U}\hat{V}\hat{U}\Delta_4^{(1)}S_3^{(1)} - \hat{U}_{xx}\Delta_3^{(1)} \\
&\quad + \delta_{11}^{(1)}\hat{U}\hat{V}_x S_1^{(1)} - \delta_{11}^{(1)}\hat{U}_x V S_1^{(1)} - \frac{1}{2}\delta_{11}^{(1)}\hat{A}S_1^{(1)} + \hat{U}\hat{V}\hat{U}\Delta_3^{(1)}S_1^{(1)} + 2\hat{U}_x\Delta_3^{(1)}S_1^{(1)} \\
&\quad + \delta_{11}^{(1)}S_1^{(1)}U_x V - \delta_{11}^{(1)}S_1^{(1)}UV_x + \frac{1}{2}\delta_{11}^{(1)}S_1^{(1)}A - \delta_{11}^{(1)}S_2^{(1)}V_x UV + \delta_{11}^{(1)}S_2^{(1)}V_{xx} \\
&\quad + \frac{1}{2}\hat{U}\hat{V}_x\hat{U}\Delta_3^{(1)} - \hat{U}\hat{V}_x\Delta_3^{(1)} - \frac{1}{2}\hat{U}_x\hat{V}\hat{U}\Delta_3^{(1)} - \frac{1}{4}\hat{A}\hat{U}\Delta_3^{(1)} + 2\hat{U}_x\Delta_4^{(1)}S_3^{(1)},
\end{align}

\begin{align}
\Delta_{3,yy}^{(1)} &= \frac{1}{2}\Delta_3^{(1)}S_2^{(1)}VUV + \frac{1}{2}\Delta_4^{(1)}S_2^{(1)}VUV - 4\Delta_3^{(1)}S_2^{(1)}S_3^{(1)} - \Delta_4^{(1)}S_4^{(1)}V_x \\
&\quad + 2\Delta_3^{(1)}US_3^{(1)} + \Delta_4^{(1)}UVS_3^{(1)} + 4\Delta_4^{(1)}S_3^{(1)}S_1^{(1)} + 2\Delta_3^{(1)}S_2^{(1)}V S_1^{(1)} \\
&\quad + 2\Delta_4^{(1)}S_4^{(1)}V S_1^{(1)} - \frac{1}{2}\hat{V}\hat{U}\Delta_3^{(1)} + \frac{1}{2}\hat{V}\hat{U}\Delta_3^{(1)} - \frac{1}{4}\hat{V}\hat{U}\Delta_3^{(1)} \\
&\quad + \Delta_4^{(1)}V_x S_1^{(1)} - \frac{1}{2}\Delta_4^{(1)}UVV S_1^{(1)} - \Delta_3^{(1)}UVS_1^{(1)} - 2\Delta_4^{(1)}V(S_1^{(1)})^2 \\
&\quad + \Delta_3^{(1)}S_1^{(1)}UV - 2\Delta_4^{(1)}V S_2^{(1)}S_3^{(1)} + 4\Delta_3^{(1)}S_3^{(1)}UV - \Delta_3^{(1)}S_1^{(1)}V_x,
\end{align}

\begin{align}
\Delta_{3,tt}^{(1)} &= \Delta_3^{(1)}S_1^{(1)}U_x V - \Delta_3^{(1)}S_1^{(1)}UV_x + \frac{1}{2}\Delta_3^{(1)}S_1^{(1)}A - \Delta_4^{(1)}S_3^{(1)}UV_x + \Delta_4^{(1)}S_4^{(1)}U_x V \\
&\quad + \frac{1}{2}\hat{V}\hat{U}\Delta_4^{(1)}S_3^{(1)} - \Delta_3^{(1)}S_2^{(1)}V_x UV - \frac{1}{2}\Delta_3^{(1)}S_2^{(1)}VUV_x + \frac{1}{2}\Delta_3^{(1)}S_2^{(1)}V_x V \\
&\quad + \Delta_3^{(1)}S_2^{(1)}V_{xx} + \frac{1}{4}\Delta_4^{(1)}S_4^{(1)}VA + \Delta_4^{(1)}S_4^{(1)}V_{xx} + \hat{V}\hat{U}\Delta_3^{(1)}S_1^{(1)} - \hat{V}_x\Delta_3^{(1)}S_1^{(1)} \\
&\quad + \frac{1}{2}\hat{V}\hat{U}\Delta_3^{(1)}S_1^{(1)} + \hat{V}_x\hat{U}\Delta_4^{(1)}S_1^{(1)} + \frac{1}{2}\hat{V}\hat{U}\Delta_3^{(1)}S_1^{(1)} - \frac{1}{2}\hat{V}_x\hat{V}\Delta_3^{(1)}S_1^{(1)} \\
&\quad + \frac{1}{2}\hat{V}_x\hat{V}\Delta_3^{(1)}S_1^{(1)} - \frac{1}{4}\hat{V}\hat{A}\Delta_3^{(1)}S_1^{(1)} - \hat{V}_{xx}\Delta_3^{(1)}S_1^{(1)} - \frac{1}{2}\hat{V}_x\triangle_3^{(1)}S_1^{(1)} \\
&\quad + \frac{1}{4}\Delta_3^{(1)}S_2^{(1)}VA - \Delta_4^{(1)}S_4^{(1)}V_x UV - \frac{1}{2}\Delta_4^{(1)}S_4^{(1)}VUV_x + \frac{1}{2}\Delta_4^{(1)}S_4^{(1)}V_x V \\
&\quad + \frac{1}{4}\hat{V}\hat{U}\hat{V}_x\hat{U}\Delta_3^{(1)} + \frac{1}{2}\hat{V}_x\hat{V}\hat{U}\Delta_3^{(1)} - \frac{1}{4}\hat{V}_x\hat{V}\Delta_3^{(1)} \\
&\quad + \hat{V}\hat{U}_x\Delta_4^{(1)}S_3^{(1)} - \frac{1}{8}\hat{V}\hat{A}\Delta_3^{(1)} - \hat{V}_x\hat{U}\Delta_4^{(1)}S_3^{(1)} + \frac{1}{2}\Delta_4^{(1)}S_4^{(1)}A,
\end{align}

and three redundant equations:

\begin{align}
\delta_{11}^{(1)}UVS_2^{(1)} &= -\frac{1}{2}\delta_{11}^{(1)}UVU - 2\delta_{11}^{(1)}S_2^{(1)}V S_2^{(1)} + \delta_{11}^{(1)}S_2^{(1)}VU + 4\delta_{11}^{(1)}S_2^{(1)}S_4^{(1)} \\
&\quad + \frac{1}{2}\hat{U}\hat{V}\hat{U}\Delta_4^{(1)} - 2\delta_{11}^{(1)}US_4^{(1)} + \hat{U}_x\Delta_4^{(1)} + 2\delta_{11}^{(1)}S_1^{(1)}U - \delta_{11}^{(1)}U_x = 0,
\end{align}
Here, after substitution of Eqns. (3.19)–(3.21), Eqns. (3.27)–(3.29) are proved to be completely covered by Eqn. (3.30a). In other words, Eqns. (3.27)–(3.29) are all satisfied identically
\[
\Delta_3^{(1)} U_3 V - \Delta_3^{(1)} V U_3 + \frac{1}{2} \Delta_3^{(1)} A - \Delta_4^{(1)} V_2 U V - \frac{1}{2} \Delta_4^{(1)} V U V + \frac{1}{2} \Delta_4^{(1)} V U_3 V
\]
\[
+ \frac{1}{4} \Delta_4^{(1)} V A - 2 \Delta_3^{(1)} S_1^{(1)} U V - 2 \Delta_4^{(1)} S_3^{(1)} U V - \Delta_3^{(1)} S_2^{(1)} V U V - \frac{1}{4} \delta_1^{(1)} \hat{V} A
\]
\[
+ 2 \Delta_4^{(1)} S_4^{(1)} V_x + 2 \Delta_3^{(1)} S_2^{(1)} V_x - 2 \hat{V} \hat{U} \Delta_3^{(1)} S_1^{(1)} + \delta_1^{(1)} \hat{V} \hat{U} \hat{S}_1^{(1)} - 2 \delta_1^{(1)} \hat{V}_x S_1^{(1)}
\]
\[
+ \Delta_4^{(1)} V_{xx} + \hat{V} \hat{U}_x \Delta_3^{(1)} - \hat{V}_x \Delta_3^{(1)} + \frac{1}{2} \hat{V} \hat{U} \hat{U} \Delta_3^{(1)} + \hat{V}_x \hat{U} \Delta_1^{(1)} - \delta_1^{(1)} \hat{V}_{xx}
\]
\[
+ \frac{1}{2} \delta_1^{(1)} \hat{V} \hat{U}_x \hat{V} - \frac{1}{2} \delta_1^{(1)} \hat{V} \hat{U}_x V - \Delta_4^{(1)} S_4^{(1)} V U V - 2 \hat{V} \hat{U} \Delta_1^{(1)} S_3^{(1)} = 0,
\]

(3.28)
\[
\frac{1}{2} \delta_1^{(1)} U V_x U - \delta_1^{(1)} U V U_x - \frac{1}{2} \delta_1^{(1)} U_x V U - \hat{U} \hat{V} \Delta_4^{(1)} S_4^{(1)} - 2 \hat{U}_x \Delta_4^{(1)} S_4^{(1)} - \delta_1^{(1)} \hat{U} \hat{V}_x S_2^{(1)}
\]
\[
+ \delta_1^{(1)} \hat{U}_x \hat{V} S_2^{(1)} - \delta_1^{(1)} S_1^{(1)} V_x U - \hat{U} \hat{V} \Delta_3^{(1)} S_2^{(1)} - 2 \hat{U}_x \Delta_3^{(1)} S_2^{(1)} - \frac{1}{2} \hat{U} \hat{V}_x \Delta_4^{(1)}
\]
\[
+ \hat{U} \hat{V} \hat{U}_x \Delta_4^{(1)} + \frac{1}{2} \hat{U}_x \hat{V} \Delta_4^{(1)} + \delta_1^{(1)} S_1^{(1)} V U V + \frac{1}{2} \delta_1^{(1)} S_2^{(1)} V U V + 2 \delta_1^{(1)} S_1^{(1)} U_x
\]
\[
+ \delta_1^{(1)} S_2^{(1)} V U_x + \frac{1}{2} \delta_1^{(1)} \hat{A} S_2^{(1)} - \delta_1^{(1)} U_{xx} + \hat{U}_{xx} \Delta_4^{(1)} - \frac{1}{4} \delta_1^{(1)} \hat{A} U + \frac{1}{4} \hat{A} \hat{U} \Delta_4^{(1)} = 0,
\]

(3.29)

where \( \hat{A} = \hat{U} \hat{V} \hat{U} \hat{V} \).

The removal of \( \hat{U}_1^{(1)}, \hat{V}_2^{(1)} \) and \( \hat{W}_3^{(1)} \) respectively from Eqns. (3.17c), (3.17d) and (3.17e) by virtue of other six equations in System (3.17) gives
\[
S_2^{(1)} = \left[ U_0^{(1)} (S_1^{(1)}) S_1^{(1)} \right] + \left[ U_1^{(1)} S_1^{(1)} \right],
\]

(3.30a)
\[
S_2^{(1)} = \left[ V_0^{(1)} (S_1^{(1)}) S_1^{(1)} \right] + \left[ V_1^{(1)} S_1^{(1)} S_1^{(1)} \right] + \left[ V_2^{(1)} S_1^{(1)} \right],
\]

(3.30b)
\[
S_2^{(1)} = \left[ W_0^{(1)} (S_1^{(1)})^2 S_1^{(1)} \right] + \left[ W_1^{(1)} S_1^{(1)} S_1^{(1)} \right] + \left[ W_2^{(1)} S_1^{(1)} S_1^{(1)} \right] + \left[ W_3^{(1)} S_1^{(1)} \right].
\]

(3.30c)

Here, after substitution of Eqns. (3.19)–(3.21), Eqns. (3.27)–(3.29) are proved to be completely covered by Eqn. (3.30a). In other words, Eqns. (3.27)–(3.29) are all satisfied identically with the identity of Eqn. (3.30a). In a similar way like in Ref. [25], we take \( S^{(1)} \) as
\[
S^{(1)} = H^{(1)} \Lambda^{(1)} (H^{(1)})^{-1},
\]

(3.31)

with
\[
H^{(1)} = (h_1^{(1)}, h_2^{(1)}, \ldots, h_{N+1}^{(1)}), \quad \Lambda^{(1)} = \text{diag}(\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_{N+1}^{(1)}),
\]

(3.32)

where \( h_k^{(1)} = (h_{1k}^{(1)}, h_{2k}^{(1)}, \ldots, h_{N+1,k}^{(1)})^T \) corresponds to the column solution of System (3.1) with \( \lambda = \lambda_k^{(1)} (\lambda_i^{(1)} \neq \lambda_k^{(1)} \text{ when } i \neq k; 1 \leq i, k \leq N + 1) \), namely,
\[
H_x^{(1)} = U_0^{(1)} H^{(1)} \Lambda^{(1)} + U_1^{(1)} H^{(1)},
\]

(3.33a)
\[
H_y^{(1)} = V_0^{(1)} H^{(1)} (\Lambda^{(1)})^2 + V_1^{(1)} H^{(1)} \Lambda^{(1)} + V_2^{(1)} H^{(1)},
\]

(3.33b)
\[
H_t^{(1)} = W_0^{(1)} H^{(1)} (\Lambda^{(1)})^3 + W_1^{(1)} H^{(1)} (\Lambda^{(1)})^2 + W_2^{(1)} H^{(1)} \Lambda^{(1)} + W_3^{(1)} H^{(1)}.
\]

(3.33c)

By straightforward substitution of Expression (3.31) with Eqns. (3.32) and (3.33), one can easily verify the identity of Eqns. (3.17c), (3.17d) and (3.17e) (details ignored). Thus, collecting what have been obtained above, we get the following:
Darboux transformation A: The matrices \( \hat{U}^{(1)} \), \( \hat{V}^{(1)} \) and \( \hat{W}^{(1)} \) have the same forms as \( U^{(1)} \), \( V^{(1)} \) and \( W^{(1)} \) under the linear transformation \( (3.16) \) with Eqns. \( (3.31)-(3.33) \), where \( \Delta_2^{(1)} = O_1^T \), \( \Delta_4^{(1)} \) is an arbitrary invertible constant matrix, \( \delta_{11}^{(1)} \) and \( \Delta_3^{(1)} \) are determined by Eqns. \( (3.19), (3.20), (3.23)-(3.26) \), and the relationship between the old and new potentials is constructed by Transformation \( (3.21) \).

The second Darboux transformation of Systems \( (2.6) \) and \( (2.7) \) based on System \( (3.7) \) is assumed as the following form \( (3.30) \)

\[
\tilde{\psi} = (\lambda^2 \Delta^{(2)} - \lambda \Delta^{(2)} S_{1}^{(2)} - \Delta^{(2)} S_{T}^{(2)}) \psi,
\]

with

\[
\Delta^{(2)} = \left( \begin{array}{cc} \Delta_1^{(2)} & \Delta_2^{(2)} \\ \Delta_3^{(2)} & \Delta_4^{(2)} \end{array} \right), \quad S_{1}^{(2)} = \left( \begin{array}{c} 0 \\ S_{3}^{(2)} \\ O_{2} \end{array} \right), \quad S_{T}^{(2)} = \left( \begin{array}{c} S_{1}^{(2)} \\ O_{1} \\ S_{4}^{(2)} \end{array} \right),
\]

where \( \Delta_1^{(2)} = \delta_{11}^{(2)}, \Delta_2^{(2)} = (\delta_{12}, \ldots, \delta_{1N+1}), \Delta_3^{(2)} = (\delta_{21}, \ldots, \delta_{N+1,1})^T, \Delta_4^{(2)} = (\delta_{ik})_{2 \leq i, k \leq N+1}, S_{1}^{(2)} = (s_{11}^{(2)}, \ldots, s_{1N+1}^{(2)}), S_{2}^{(2)} = (s_{21}^{(2)}, \ldots, s_{N+1,1}^{(2)})^T, S_{3}^{(2)} = (s_{3k}^{(2)})_{1 \leq i, k \leq N+1}, S_{4}^{(2)} \)

and \( s_{ik}^{(2)} \) \( (1 \leq i, k \leq N + 1) \) are all the functions of \( x, y \) and \( t \) to be determined by a set of equations, as below:

\[
\Delta^{(2)} U_0^{(2)} - \tilde{U}_0^{(2)} \Delta^{(2)} = 0,
\]

\[
\Delta^{(2)} U_1^{(2)} - \Delta^{(2)} S_{1}\cdot U_0^{(2)} + \tilde{U}_0^{(2)} \Delta^{(2)} S_{1} - \tilde{U}_1^{(2)} \Delta^{(2)} = 0,
\]

\[
\Delta^{(2)} U_2^{(2)} - \Delta^{(2)} S_{1}\cdot U_1^{(2)} + \tilde{U}_1^{(2)} \Delta^{(2)} S_{2} - \tilde{U}_2^{(2)} \Delta^{(2)} = 0,
\]

\[
\Delta^{(2)} V_2^{(2)} - \Delta^{(2)} S_{1}\cdot V_1^{(2)} + \tilde{V}_1^{(2)} \Delta^{(2)} S_{2} - \tilde{V}_2^{(2)} \Delta^{(2)} = 0,
\]

\[
\Delta^{(2)} V_3^{(2)} - \Delta^{(2)} S_{1}\cdot V_2^{(2)} + \tilde{V}_2^{(2)} \Delta^{(2)} S_{3} - \tilde{V}_3^{(2)} \Delta^{(2)} = 0,
\]

\[
\Delta^{(2)} V_4^{(2)} - \Delta^{(2)} S_{1}\cdot V_3^{(2)} + \tilde{V}_3^{(2)} \Delta^{(2)} S_{4} - \tilde{V}_4^{(2)} \Delta^{(2)} = 0,
\]

\[
\Delta^{(2)} W_4^{(2)} - \Delta^{(2)} S_{1}\cdot W_3^{(2)} + \tilde{W}_3^{(2)} \Delta^{(2)} S_{2} - \tilde{W}_4^{(2)} \Delta^{(2)} S_{T}^{(2)} = 0
\]

and  
\[
\Delta^{(2)} W_3^{(2)} - \Delta^{(2)} S_{1}\cdot W_2^{(2)} + \tilde{W}_2^{(2)} \Delta^{(2)} S_{3} - \tilde{W}_3^{(2)} \Delta^{(2)} S_{T}^{(2)} = 0
\]
\[ + \tilde{W}_4^{(2)} \Delta^{(2)} S_4^{(2)} - \tilde{W}_4^{(2)} \Delta^{(2)} = 0, \]  
\[ \Delta^{(2)} W_6^{(2)} - \Delta^{(2)} S_6^{(2)} W_4^{(2)} - \Delta^{(2)} S_6^{(2)} W_3^{(2)} + \tilde{W}_3^{(2)} \Delta^{(2)} S_T^{(2)} \]  
\[ + \tilde{W}_3^{(2)} \Delta^{(2)} S_3^{(2)} - \tilde{W}_3^{(2)} \Delta^{(2)} = 0, \]  
\[ \Delta^{(2)} W_6^{(2)} - \Delta^{(2)} S_6^{(2)} W_4^{(2)} - \Delta^{(2)} S_6^{(2)} W_3^{(2)} + \tilde{W}_3^{(2)} \Delta^{(2)} S_T^{(2)} \]  
\[ + \tilde{W}_3^{(2)} \Delta^{(2)} S_3^{(2)} - \tilde{W}_3^{(2)} \Delta^{(2)} = 0, \]  
\[ \Delta^{(2)} S_{1,T}^{(2)} + \Delta^{(2)} S_{1,t}^{(2)} + \Delta^{(2)} S_T^{(2)} W_6^{(2)} + \Delta^{(2)} S_T^{(2)} \]  
\[ - \tilde{W}_5^{(2)} \Delta^{(2)} S_T^{(2)} - \tilde{W}_5^{(2)} \Delta^{(2)} S_T^{(2)} = 0, \]  
\[ \Delta^{(2)} S_{1,T}^{(2)} + \Delta^{(2)} S_{1,t}^{(2)} + \Delta^{(2)} S_T^{(2)} W_6^{(2)} - \tilde{W}_6^{(2)} \Delta^{(2)} S_T^{(2)} = 0, \]

in which \( \tilde{U}_i^{(2)}, \tilde{V}_k^{(2)} \) and \( \tilde{W}_l^{(2)} \) are required to have the same forms as \( U_i^{(2)}, V_k^{(2)} \) and \( W_l^{(2)} \) (0 \( \leq i \leq 2; 0 \leq k \leq 4; 0 \leq l \leq 6 \) except that \((u_j, v_j)\) are replaced respectively with \((\tilde{u}_j, \tilde{v}_j)\) \((j = 1, 2, \ldots, N)\).

From Eqns. (3.36a) and (3.36b), we have the following results:
\[ \Delta_4^{(2)} = O_7, \quad \Delta_3^{(2)} = O_1, \]  
\[ \tilde{U} = (\delta_{11}^{(2)} U - 2 \delta_{21}^{(2)} S_2^{(2)}) (\Delta_{11}^{(2)})^{-1}, \quad \tilde{V} = (\Delta_{11}^{(2)} V + 2 \Delta_{11}^{(2)} S_3^{(2)}) / \delta_{11}^{(2)}, \]

which are further substituted into Eqns. (3.36c), (3.36d)–(3.36h) and (3.36k)–(3.36m), determining that \( \delta_{11}^{(2)} \) and \( \Delta_4^{(2)} \) should satisfy the conditions as
\[ \Delta_{4,x}^{(2)} = O_2, \quad \Delta_{4,y}^{(2)} = O_2, \quad \Delta_{4,t}^{(2)} = O_2, \]  
\[ \delta_{11,x}^{(2)} = \delta_{11}^{(2)} S_2^{(2)} V - \delta_{11}^{(2)} U S_3^{(2)} + 2 \delta_{11}^{(2)} S_2^{(2)} S_3^{(2)}, \]  
\[ \delta_{11,y}^{(2)} = \tilde{U} \tilde{V} \delta_{11}^{(2)} S_1^{(2)} + \frac{1}{2} \delta_{11}^{(2)} S_2^{(2)} V U V - \delta_{11}^{(2)} S_2^{(2)} V x - \delta_{11}^{(2)} S_1^{(2)} U V \]  
\[ - \frac{1}{2} \tilde{U} \tilde{V} \tilde{U} \Delta_{11}^{(2)} S_3^{(2)} - \tilde{U} x \Delta_{11}^{(2)} S_3^{(2)}, \]  
\[ \delta_{11,t}^{(2)} = \frac{1}{4} \delta_{11}^{(2)} S_2^{(2)} V A - \delta_{11}^{(2)} S_2^{(2)} V U V - \frac{1}{2} \delta_{11}^{(2)} S_2^{(2)} V U V x + \delta_{11}^{(2)} S_2^{(2)} V x x \]  
\[ + \frac{1}{2} \delta_{11}^{(2)} S_2^{(2)} V U x V - \frac{1}{2} \delta_{11}^{(2)} S_1^{(2)} A + \delta_{11}^{(2)} S_1^{(2)} U V x - \delta_{11}^{(2)} S_1^{(2)} U x V \]  
\[ + \frac{1}{2} \tilde{U} \tilde{V} \tilde{U} \Delta_{11}^{(2)} S_3^{(2)} - \frac{1}{4} \tilde{A} \tilde{U} \Delta_{11}^{(2)} S_3^{(2)} - \tilde{U} \tilde{V} \tilde{U} \Delta_{11}^{(2)} S_3^{(2)} + \frac{1}{2} \delta_{11}^{(2)} \tilde{A} S_1^{(2)} \]  
\[ + \delta_{11}^{(2)} \tilde{U} x \tilde{V} S_1^{(2)} - \frac{1}{2} \tilde{U} x \tilde{V} \tilde{U} \Delta_{11}^{(2)} S_3^{(2)} - \delta_{11}^{(2)} \tilde{U} \tilde{V} x S_1^{(2)} - \tilde{U} x x \Delta_{11}^{(2)} S_3^{(2)}, \]

with \( \tilde{A} = \tilde{U} \tilde{V} \tilde{U} \tilde{V} \), and yielding the following redundant equations:
\[ \frac{1}{2} \delta_{11}^{(2)} U V U - \frac{1}{2} \tilde{U} \tilde{V} \tilde{U} \Delta_{11}^{(2)} - \tilde{U} x \Delta_{11}^{(2)} - \tilde{U} \tilde{V} \delta_{11}^{(2)} S_2^{(2)} + 2 \delta_{11}^{(2)} S_1^{(2)} U \]  
\[ - 2 \tilde{U} \Delta_{11}^{(2)} S_3^{(2)} - \delta_{11}^{(2)} S_2^{(2)} V U = 0, \]  
\[ \frac{1}{2} \Delta_{11}^{(2)} V U V - \Delta_{11}^{(2)} V x - \frac{1}{2} \delta_{11}^{(2)} \tilde{V} \tilde{U} V + \delta_{11}^{(2)} \tilde{V} x + \Delta_{11}^{(2)} S_3^{(2)} U V + 2 \Delta_{11}^{(2)} S_4^{(2)} V \]  
\[ - 2 \delta_{11}^{(2)} \tilde{V} S_1^{(2)} + \tilde{V} U \Delta_{11}^{(2)} S_3^{(2)} = 0, \]
\[
\frac{1}{4} \delta^{(2)}_{i1} A U - \frac{1}{2} \delta^{(2)}_{i1} U V U_x U + \delta^{(2)}_{i1} U V U_x + \frac{1}{2} \delta^{(2)}_{i1} U_x V U + \delta^{(2)}_{i1} U_{xx} - \frac{1}{4} \tilde{A} \tilde{U} \Delta^{(2)}_4 \\
+ \frac{1}{2} \tilde{U} \tilde{V} \tilde{U} \Delta^{(2)}_4 + \delta^{(2)}_{i1} S^{(2)}_S U V U - \frac{1}{2} \tilde{U} \tilde{V} \tilde{U} \Delta^{(2)}_4 - \tilde{U} \Delta^{(2)}_3 - \delta^{(2)}_{i1} S^{(2)}_S U V_U \\
+ \delta^{(2)}_{i1} S^{(2)}_S U x U - \tilde{U} \tilde{V} \Delta^{(2)}_4 + 2 \delta^{(2)}_{i1} S^{(2)}_S x U - \delta^{(2)}_{i1} \tilde{U} \tilde{S}^{(2)}_S - \frac{1}{2} \delta^{(2)}_{i1} \tilde{A} S^{(2)}_S \\
+ \delta^{(2)}_{i1} \tilde{U} \tilde{V} \tilde{S}^{(2)}_S - \tilde{U} \tilde{V} \Delta^{(2)}_4 S^{(2)}_S - 2 \tilde{U} \Delta^{(2)}_3 S^{(2)}_S - \frac{1}{2} \delta^{(2)}_{i1} S^{(2)}_S \tilde{V} U V U \\
- \tilde{U} \tilde{V} \Delta^{(2)}_4 - 2 \tilde{U} \Delta^{(2)}_3 = 0, \\
(3.45)
\]

where the above four equations, by substitution of Eqns. (3.38)–(3.41), can be completely covered by Eqns. (3.47) and (3.48) as below.

From Eqns. (3.36d)–(3.36e), (3.36f)–(3.36g) and (3.36h)–(3.36i), we remove \( \tilde{U}^{(2)}_1, \tilde{U}^{(2)}_2, \tilde{V}^{(2)}_3, \tilde{V}^{(2)}_4, \tilde{W}^{(2)}_6 \) and \( \tilde{W}^{(2)}_6 \) by use of other equations in System (3.36), and obtain

\[
S^{(2)}_{\perp x} = [U^{(2)}_0 S^{(2)}_{\perp} + U^{(2)}_1 S^{(2)}_{\perp} + U^{(2)}_x S^{(2)}_{\perp}] + [U^{(2)}_0 S^{(2)}_{\perp} + U^{(2)}_1 S^{(2)}_{\perp}]
\]

\[
S^{(2)}_{\perp y} = [U^{(2)}_0 S^{(2)}_{\perp} + U^{(2)}_1 S^{(2)}_{\perp} + [U^{(2)}_0 S^{(2)}_{\perp} + U^{(2)}_1 S^{(2)}_{\perp}]
\]

\[
S^{(2)}_{\perp y} = [V^{(2)}_0 (S^{(2)}_{\perp} + S^{(2)}_{\perp}) + [V^{(2)}_1 (S^{(2)}_{\perp} + S^{(2)}_{\perp})]
\]

\[
S^{(2)}_{\perp y} = [V^{(2)}_0 (S^{(2)}_{\perp} + S^{(2)}_{\perp}) + [V^{(2)}_1 (S^{(2)}_{\perp} + S^{(2)}_{\perp})]
\]

\[
S^{(2)}_{\perp y} = [V^{(2)}_0 (S^{(2)}_{\perp} + S^{(2)}_{\perp}) + [V^{(2)}_1 (S^{(2)}_{\perp} + S^{(2)}_{\perp})]
\]

\[
S^{(2)}_{\perp y} = [V^{(2)}_0 (S^{(2)}_{\perp} + S^{(2)}_{\perp}) + [V^{(2)}_1 (S^{(2)}_{\perp} + S^{(2)}_{\perp})]
\]

\[
S^{(2)}_{\perp y} = [V^{(2)}_0 (S^{(2)}_{\perp} + S^{(2)}_{\perp}) + [V^{(2)}_1 (S^{(2)}_{\perp} + S^{(2)}_{\perp})]
\]

\[
S^{(2)}_{\perp y} = [V^{(2)}_0 (S^{(2)}_{\perp} + S^{(2)}_{\perp}) + [V^{(2)}_1 (S^{(2)}_{\perp} + S^{(2)}_{\perp})]
\]

\[
S^{(2)}_{\perp t} = [W^{(2)}_6 + W^{(2)}_5 S^{(2)}_S + W^{(2)}_4 S^{(2)}_S + W^{(2)}_3 S^{(2)}_S + X_1 S^{(2)}_S
\]

\[
S^{(2)}_{\perp t} = [W^{(2)}_6 + W^{(2)}_5 S^{(2)}_S + W^{(2)}_4 S^{(2)}_S + W^{(2)}_3 S^{(2)}_S + X_1 S^{(2)}_S
\]

\[
S^{(2)}_{\perp t} = [W^{(2)}_6 + W^{(2)}_5 S^{(2)}_S + W^{(2)}_4 S^{(2)}_S + W^{(2)}_3 S^{(2)}_S + X_1 S^{(2)}_S
\]

\[
S^{(2)}_{\perp t} = [W^{(2)}_6 + W^{(2)}_5 S^{(2)}_S + W^{(2)}_4 S^{(2)}_S + W^{(2)}_3 S^{(2)}_S + X_1 S^{(2)}_S
\]

\[
S^{(2)}_{\perp t} = [W^{(2)}_6 + W^{(2)}_5 S^{(2)}_S + W^{(2)}_4 S^{(2)}_S + W^{(2)}_3 S^{(2)}_S + X_1 S^{(2)}_S
\]

\[
S^{(2)}_{\perp t} = [W^{(2)}_6 + W^{(2)}_5 S^{(2)}_S + W^{(2)}_4 S^{(2)}_S + W^{(2)}_3 S^{(2)}_S + X_1 S^{(2)}_S
\]

\[
S^{(2)}_{\perp t} = [W^{(2)}_6 + W^{(2)}_5 S^{(2)}_S + W^{(2)}_4 S^{(2)}_S + W^{(2)}_3 S^{(2)}_S + X_1 S^{(2)}_S
\]

\[
S^{(2)}_{\perp t} = [W^{(2)}_6 + W^{(2)}_5 S^{(2)}_S + W^{(2)}_4 S^{(2)}_S + W^{(2)}_3 S^{(2)}_S + X_1 S^{(2)}_S
\]
where $X_1$ and $X_2$ are defined as

$$X_1 = [W_0^{(2)} S_+^{(2)} S_\perp^{(2)}, S_\perp^{(2)}] + [W_0^{(2)} (S_+^{(2)})^2 S_\perp^{(2)}, S_\perp^{(2)}] + [W_0^{(2)} S_+^{(2)} (S_\perp^{(2)})^2, S_\perp^{(2)}] + [W_0^{(2)} (S_+^{(2)})^3 S_\perp^{(2)}, S_\perp^{(2)}] + [W_0^{(2)} (S_+^{(2)})^4 S_\perp^{(2)}, S_\perp^{(2)}] + [W_1^{(2)} S_\perp^{(2)} S_\perp^{(2)}] + [W_1^{(2)} (S_\perp^{(2)})^2, S_\perp^{(2)}] + [W_1^{(2)} S_\perp^{(2)} S_\perp^{(2)}] + [W_2^{(2)} (S_\perp^{(2)})^2, S_\perp^{(2)}] + [W_2^{(2)} S_\perp^{(2)} S_\perp^{(2)}] + [W_3^{(2)} S_\perp^{(2)} S_\perp^{(2)}] + [W_4^{(2)} S_\perp^{(2)} S_\perp^{(2)}]$$

$$X_2 = [W_0^{(2)} S_+^{(2)} S_\perp^{(2)}, S_\perp^{(2)}] + [W_0^{(2)} (S_+^{(2)})^3, S_\perp^{(2)}] + [W_0^{(2)} S_\perp^{(2)} S_\perp^{(2)}] + [W_0^{(2)} S_\perp^{(2)} S_\perp^{(2)}] + [W_1^{(2)} (S_\perp^{(2)})^2, S_\perp^{(2)}] + [W_1^{(2)} S_\perp^{(2)} S_\perp^{(2)}] + [W_2^{(2)} (S_\perp^{(2)})^2, S_\perp^{(2)}] + [W_2^{(2)} S_\perp^{(2)} S_\perp^{(2)}] + [W_3^{(2)}, S_\perp^{(2)}] + [W_4^{(2)}, S_\perp^{(2)}].$$

(3.53)

(3.54)

In order to make Eqns. (3.47)–(3.52) satisfied, it is sufficient to require the following relation (proof omitted for brevity):

$$S_\perp^{(2)} H^{(2)} \Lambda^{(2)} (H^{(2)})^{-1} + S_\perp^{(2)} = H^{(2)} (\Lambda^{(2)})^2 (H^{(2)})^{-1},$$

(3.55)

with

$$H^{(2)} = (h_1^{(2)}, h_2^{(2)}, \ldots, h_N^{(2)}), \quad \Lambda^{(2)} = \text{diag}(\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_N^{(2)}),$$

(3.56)

where $h_i^{(2)} = (h_{i1}^{(2)}, h_{i2}^{(2)}, \ldots, h_{iN+1}^{(2)})^T$ is the column solution of System (3.7) with $\lambda = \lambda_i^{(2)}$ ($\lambda_i^{(2)} \neq \lambda_k^{(2)}$ when $i \neq k$; $1 \leq i, k \leq N + 1$), namely,

$$H_x^{(2)} = U_0^{(2)} H^{(2)} (\Lambda^{(2)})^2 + U_1^{(2)} H^{(2)} \Lambda^{(2)} + U_2^{(2)} H^{(2)},$$

(3.57a)

$$H_y^{(2)} = V_0^{(2)} H^{(2)} (\Lambda^{(2)})^4 + V_1^{(2)} H^{(2)} (\Lambda^{(2)})^3 + V_2^{(2)} H^{(2)} (\Lambda^{(2)})^2$$

$$+ V_3^{(2)} H^{(2)} \Lambda^{(2)} + V_4^{(2)} H^{(2)},$$

(3.57b)

$$H_i^{(2)} = W_0^{(2)} H^{(2)} (\Lambda^{(2)})^6 + W_1^{(2)} H^{(2)} (\Lambda^{(2)})^5 + W_2^{(2)} H^{(2)} (\Lambda^{(2)})^4 + W_3^{(2)} H^{(2)} (\Lambda^{(2)})^3$$

$$+ W_4^{(2)} H^{(2)} (\Lambda^{(2)})^2 + W_5^{(2)} H^{(2)} \Lambda^{(2)} + W_6^{(2)} H^{(2)}.$$ 

(3.57c)

**Darboux transformation B:** The matrices $U_i^{(2)}$, $V_i^{(2)}$ and $W_i^{(2)}$ have the same forms as $U_i^{(2)}$, $V_i^{(2)}$ and $W_i^{(2)}$ ($0 \leq i \leq 2$; $0 \leq k \leq 4$; $0 \leq l \leq 6$) under the linear transformation (3.34) with Eqns. (3.55)–(3.57), where $\Delta_2 = O_1$, $\Delta_4 = O_1$, $\Delta_6 = \text{an arbitrary invertible constant matrix}$, $\delta^{(2)}_{11}$ is determined by Eqns. (3.40)–(3.42), and the relationship between the old and new potentials is constructed by Transformation (3.38).

4. Lax representations and Darboux transformations of Systems (2.9) and (2.10)

Likewise, we find that Systems (2.9) and (2.10) are also related to two different types of Lax representations, which are respectively written as
\[
\Phi_\gamma = U^{(3)} \Phi = [\lambda U_0^{(3)} + U_1^{(3)}] \Phi, \\
\Phi_\delta = V^{(3)} \Phi = [\lambda^2 V_0^{(3)} + \lambda V_1^{(3)} + V_2^{(3)}] \Phi, \\
\Phi_\epsilon = W^{(3)} \Phi = [\lambda^3 W_0^{(3)} + \lambda^2 W_1^{(3)} + \lambda W_2^{(3)} + W_3^{(3)}] \Phi,
\]

with

\[
V_0^{(3)} = -2 U_0^{(3)}, \quad W_0^{(3)} = 4 U_0^{(3)}, \quad W_1^{(3)} = -2 V_1^{(3)}, \\
U_0^{(3)} = \begin{pmatrix} 1 & O_1^T \\ O_2 & -I \end{pmatrix}, \quad U_1^{(3)} = \begin{pmatrix} 0 & M \\ O_1 & O_2 \end{pmatrix}, \quad V_1^{(3)} = \begin{pmatrix} MP & -2 M \\ PMP + P_x & -PM \end{pmatrix}, \\
V_2^{(3)} = \begin{pmatrix} 0 & MPM - M_x \\ O_1 & O_2 \end{pmatrix}, \quad W_3^{(3)} = \begin{pmatrix} 0 & \frac{3}{2} BM - \frac{3}{2} M_x PM - \frac{3}{2} MPM + M_{xx} \\ O_1 & O_2 \end{pmatrix}, \\
W_2^{(3)} = \begin{pmatrix} \frac{3}{2} B + MP_x - M_x P & -2 MPM + 2 M_x \\
\frac{3}{2} PB + \frac{3}{2} P_x MP + \frac{3}{2} PMP + P_{xx} - \frac{3}{2} PMP + PM_x - P_x M \end{pmatrix},
\]

and

\[
\Phi_\zeta = U^{(4)} \Phi = [\lambda^2 U_0^{(4)} + \lambda U_1^{(4)}] \Phi, \\
\Phi_\eta = V^{(4)} \Phi = [\lambda^4 V_0^{(4)} + \lambda^3 V_1^{(4)} + \lambda^2 V_2^{(4)} + \lambda V_3^{(4)}] \Phi, \\
\Phi_\kappa = W^{(4)} \Phi = [\lambda^6 W_0^{(4)} + \lambda^5 W_1^{(4)} + \lambda^4 W_2^{(4)} + \lambda^3 W_3^{(4)} + \lambda^2 W_4^{(4)} + \lambda W_5^{(4)}] \Phi,
\]

with

\[
V_0^{(4)} = 2 U_0^{(4)}, \quad W_0^{(4)} = 4 U_0^{(4)}, \quad V_1^{(4)} = 2 U_1^{(4)}, \\
W_1^{(4)} = 4 U_1^{(4)}, \quad W_2^{(4)} = 2 V_2^{(4)}, \quad W_3^{(4)} = 2 V_3^{(4)}, \\
U_0^{(4)} = \begin{pmatrix} -1 & O_1^T \\ O_2 & I \end{pmatrix}, \quad U_1^{(4)} = \begin{pmatrix} 0 & M \\ -P & O_2 \end{pmatrix}, \quad V_2^{(4)} = \begin{pmatrix} -MP & O_1^T \\ O_1 & PM \end{pmatrix}, \\
V_3^{(4)} = \begin{pmatrix} 0 & MPM - M_x \\ -PMP - P_x & O_2 \end{pmatrix}, \\
W_4^{(4)} = \begin{pmatrix} -\frac{3}{2} B + M_x P - MP_x & O_1^T \\ O_1 & \frac{3}{2} PMP - P_x M - PM_x \end{pmatrix}, \\
W_5^{(4)} = \begin{pmatrix} 0 & \frac{3}{2} BM - \frac{3}{2} MPM + \frac{3}{2} M_x PM + M_{xx} \\ -\frac{3}{2} PB - \frac{3}{2} P_x MP + \frac{3}{2} PMP + P_{xx} \end{pmatrix},
\]

where \( \Phi = (\phi_1, \phi_2, \ldots, \phi_{N+1})^T \), \( M = (m_1, m_2, \ldots, m_N) \), \( P = (p_1, p_2, \ldots, p_N)^T \), \( B = MPMP \), \( O_1, O_2 \) and \( I \) have the same definitions as in Section 3. The zero-curvature conditions \( U_\gamma^{(i)} - V_\gamma^{(i)} + [U^{(i)}, V^{(i)}] = 0 \) and \( U_\iota^{(i)} - W_\iota^{(i)} + [U^{(i)}, W^{(i)}] = 0 \) \((i = 3, 4)\) give rise to Systems (2.9) and (2.10), respectively.
Following the procedure in Section 3, we can also arrive at two Darboux transformations for Systems \((2.9)\) and \((2.10)\), as follows:

**Darboux transformation C:** The linear system \((4.1)\) is kept invariant by the gauge transformation

\[
\hat{\phi} = (\lambda \Delta^{(3)} - \Delta^{(3)} S^{(3)}) \phi, \quad \Delta^{(3)} = \begin{pmatrix} \Delta^{(3)}_x & \Delta^{(3)}_y \\ \Delta^{(3)}_y & \Delta^{(3)}_x \end{pmatrix}, \quad S^{(3)} = \begin{pmatrix} S^{(3)}_1 & S^{(3)}_2 \\ S^{(3)}_3 & S^{(3)}_4 \end{pmatrix},
\]

(4.13)

where \(S^{(3)} = H^{(3)} \Lambda^{(3)} (H^{(3)})^{-1}, \Lambda^{(3)} = \text{diag}(\lambda_1^{(3)}, \lambda_2^{(3)}, \ldots, \lambda_{N+1}^{(3)}), H^{(3)} = (h_1^{(3)}, h_2^{(3)}, \ldots, h_{N+1}^{(3)})\) with \(h_k^{(3)}\) \((1 \leq k \leq N+1)\) as the column solutions of System \((4.1)\) for different eigenvalues \(\lambda = \lambda_k^{(3)}, \Delta^{(3)}_2 = O^T_1, \Delta^{(3)}_1 = \delta^{(3)}_1, \Delta^{(3)}_3 = \delta^{(3)}_1 \left( \delta^{(3)}_2, \ldots, \delta^{(3)}_{N+1} \right)^T, \Delta^{(3)}_4 = \left( \delta^{(3)}_{ik} \right)_{1 \leq i, k \leq N+1}\) obey the following conditions:

\[
\begin{align*}
\delta^{(3)}_{11,x} &= \hat{M}\Delta^{(3)}_3 + \delta^{(3)}_{11} S^{(3)}_2 P, \\
\delta^{(3)}_{11,y} &= 2 \hat{M}\Delta^{(3)}_3 S^{(3)}_1 - \delta^{(3)}_{11} \hat{M}\hat{P}S^{(3)}_1 + \hat{M}\hat{P}\hat{M}\Delta^{(3)}_3 - \hat{M}\Delta^{(3)}_3 + \delta^{(3)}_{11} S^{(3)}_2 \hat{P}MP + \\
&+ \delta^{(3)}_{11} S^{(3)}_2 P_x + 2 \hat{M}\Delta^{(3)}_4 S^{(3)}_3 + \delta^{(3)}_{11} S^{(3)}_4 MP, \\
\delta^{(3)}_{11,t} &= \delta^{(3)}_{11} \hat{M}\hat{P}S^{(3)}_1 - \delta^{(3)}_{11} \hat{M}\hat{P}S^{(3)}_1 - \delta^{(3)}_{11} \hat{M}\hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_1 + \frac{3}{2} \hat{B}\hat{M}\Delta^{(3)}_3 \\
&+ 2 \hat{M}\hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3 - 2 \hat{M}\Delta^{(3)}_3 S^{(3)}_3 - \frac{3}{2} \hat{M}\hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3 - \frac{3}{2} \hat{M}\hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3 \\
&+ \hat{M}\Delta^{(3)}_3 + \frac{3}{2} \delta^{(3)}_{11} S^{(3)}_1 B + \delta^{(3)}_{11} S^{(3)}_1 MP_x - \delta^{(3)}_{11} S^{(3)}_1 M_x P - 2 \hat{M}\Delta^{(3)}_4 S^{(3)}_3 \\
&+ \frac{3}{2} \delta^{(3)}_{11} S^{(3)}_2 \hat{P}MP_x + \frac{3}{2} \delta^{(3)}_{11} S^{(3)}_2 P_x MP + \delta^{(3)}_{11} S^{(3)}_2 P_{xx} + \frac{3}{2} \delta^{(3)}_{11} S^{(3)}_2 \hat{P}MP, \\
\Delta^{(3)}_{3,x} &= 2 \Delta^{(3)}_4 S^{(3)}_3 - \Delta^{(3)}_3 P S^{(3)}_1 + \Delta^{(3)}_3 S^{(3)}_2 P + \Delta^{(3)}_4 S^{(3)}_4 P, \\
\Delta^{(3)}_{3,y} &= \Delta^{(3)}_3 S^{(3)}_2 P M P + \Delta^{(3)}_3 S^{(3)}_2 P_x + \Delta^{(3)}_4 S^{(3)}_4 P M P + \Delta^{(3)}_4 S^{(3)}_4 P_x + \hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_1 \\
&+ \delta^{(3)}_{11} \hat{P}MPS^{(3)}_1 + \Delta^{(3)}_4 S^{(3)}_3 MP + \hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3 - \delta^{(3)}_{11} \hat{P}S^{(3)}_1, \\
\Delta^{(3)}_{3,t} &= \frac{3}{2} \Delta^{(3)}_3 S^{(3)}_1 B + \frac{3}{2} \Delta^{(3)}_3 S^{(3)}_1 B + \Delta^{(3)}_3 S^{(3)}_1 M P_x - \Delta^{(3)}_3 S^{(3)}_1 M_x P - \delta^{(3)}_{11} \hat{P}S^{(3)}_1 \\
&+ \Delta^{(3)}_3 S^{(3)}_3 P M P_x + \frac{3}{2} \Delta^{(3)}_3 S^{(3)}_2 \hat{P}MP + \frac{3}{2} \Delta^{(3)}_3 S^{(3)}_2 \hat{P}MP_x + \frac{3}{2} \Delta^{(3)}_3 S^{(3)}_2 P_x MP + \\
&+ \Delta^{(3)}_3 S^{(3)}_2 P_{xx} + \frac{3}{2} \Delta^{(3)}_4 S^{(3)}_4 \hat{P}MP + \frac{3}{2} \Delta^{(3)}_4 S^{(3)}_4 P_x MP + \frac{3}{2} \Delta^{(3)}_4 S^{(3)}_4 P_x MP \\
&+ \hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3 - \hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3 + \frac{3}{2} \hat{P}\hat{M}\hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3 \\
&+ \frac{3}{2} \hat{P}\hat{M}\hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3 - \frac{3}{2} \delta^{(3)}_{11} \hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3 M_x P - \hat{P}\hat{M}\Delta^{(3)}_4 S^{(3)}_3 \\
&+ \Delta^{(3)}_4 S^{(3)}_4 P_{xx} + \frac{3}{2} \delta^{(3)}_{11} \hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_3, \\
\Delta^{(3)}_{4,x} &= -2 \Delta^{(3)}_3 S^{(3)}_2 - \Delta^{(3)}_3 P S^{(3)}_2 - \Delta^{(3)}_3 M, \\
\Delta^{(3)}_{4,y} &= -\Delta^{(3)}_3 P M P - 2 \Delta^{(3)}_3 S^{(3)}_1 M - 2 \Delta^{(3)}_4 S^{(3)}_2 M - \Delta^{(3)}_3 S^{(3)}_2 P M - \delta^{(3)}_{11} \hat{P}x S^{(3)}_1 \\
&+ \hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_2 - \Delta^{(3)}_4 S^{(3)}_4 P M - \delta^{(3)}_{11} \hat{P}\hat{M}\Delta^{(3)}_3 S^{(3)}_2 + \Delta^{(3)}_3 M_x + \hat{P}\hat{M}\Delta^{(3)}_4 S^{(3)}_4, \\
\end{align*}
\]

(4.14-4.21)
\[ \Delta_{4,t}^{(3)} = \frac{3}{2} \Delta_{3}^{(3)} MPMx - \frac{3}{2} \Delta_{3}^{(3)} BM + \frac{3}{2} \Delta_{3}^{(3)} MxPM - \Delta_{3}^{(3)} M_{xx} + 2 \Delta_{3}^{(3)} S_{1}^{(3)} Mx \]
\[ + 2 \Delta_{4}^{(3)} S_{3}^{(3)} Mx - 2 \Delta_{4}^{(3)} S_{1}^{(3)} MPM - 2 \Delta_{4}^{(3)} S_{3}^{(3)} MPM + \Delta_{4}^{(3)} S_{2}^{(3)} PMx \]
\[ + \Delta_{4}^{(3)} S_{4}^{(3)} PMx - \Delta_{4}^{(3)} S_{2}^{(3)} PMx - \frac{3}{2} \Delta_{3}^{(3)} S_{2}^{(3)} PMPM - \Delta_{4}^{(3)} S_{4}^{(3)} P_{x} M \]
\[ + \tilde{P}_{x} \hat{M} \Delta_{3}^{(3)} S_{2}^{(3)} - \frac{3}{2} \Delta_{3}^{(3)} S_{4}^{(3)} PMPM - \hat{P} \hat{M}_{x} \Delta_{3}^{(3)} S_{2}^{(3)} - \frac{3}{2} \delta_{11}^{(3)} \hat{P} \hat{B} S_{2}^{(3)} \]
\[ + \frac{3}{2} \hat{P} \hat{M} \hat{P} \hat{M}_{x} \Delta_{3}^{(3)} S_{2}^{(3)} - \frac{3}{2} \delta_{11}^{(3)} \hat{P} \hat{M}_{x} \Delta_{4}^{(3)} S_{4}^{(3)} - \frac{3}{2} \delta_{11}^{(3)} \hat{P} \hat{M}_{x} \hat{P} S_{2}^{(3)} \],
\[ (4.22) \]

with
\[ \hat{M} = (\delta_{11}^{(3)} M + 2 \delta_{11}^{(3)} S_{2}^{(3)}) (\Delta_{4}^{(3)})^{-1}, \quad \hat{P} = (\Delta_{4}^{(3)} P + 2 \Delta_{3}^{(3)}) / \delta_{11}^{(3)}. \]  
\[ (4.23) \]

**Darboux transformation D:** The linear system (4.6) is kept invariant by the gauge transformation
\[ \tilde{\phi} = (\lambda^{2} \Delta^{(4)} - \lambda \Delta^{(4)} S_{1}^{(4)} - \Delta^{(4)} S_{4}^{(4)}) \phi, \]
\[ (4.24) \]

with
\[ \Delta^{(4)} = \begin{pmatrix} \Delta_{1}^{(4)} & \Delta_{2}^{(4)} \\ \Delta_{3}^{(4)} & \Delta_{4}^{(4)} \end{pmatrix}, \quad S_{1}^{(4)} = \begin{pmatrix} 0 & S_{2}^{(4)} \\ S_{3}^{(4)} & O_{2} \end{pmatrix}, \quad S_{4}^{(4)} = \begin{pmatrix} S_{1}^{(4)} & O_{1}^{T} \\ O_{1} & S_{4}^{(4)} \end{pmatrix}, \]
\[ (4.25) \]
\[ S_{1}^{(4)} H^{(4)} \Lambda^{(4)} (H^{(4)})^{-1} + S_{4}^{(4)} = H^{(4)} (\Lambda^{(4)})^{2} (H^{(4)})^{-1}, \]
\[ (4.26) \]

where \( \Lambda^{(4)} = \text{diag}(\lambda_{1}^{(4)}, \lambda_{2}^{(4)}, \ldots, \lambda_{N+1}^{(4)}) \), \( H^{(4)} = (h_{1}^{(4)}, h_{2}^{(4)}, \ldots, h_{N+1}^{(4)}) \) with \( h_{k}^{(4)} \) (1 \( \leq k \leq N + 1 \) as the column solutions of System (4.6) for different eigenvalues \( \lambda = \lambda_{k}^{(4)}, \Delta_{2}^{(4)} = O_{1}^{T}, \Delta_{3}^{(4)} = O_{1} \), \( \Delta_{4}^{(4)} = \delta_{11}^{(4)}, \Delta_{4}^{(4)} = (\delta_{ik}^{(4)})_{2 \leq i, k \leq N + 1} \) obey the following conditions:

\[ \delta_{11,x}^{(4)} = 2 \delta_{11}^{(4)} S_{2}^{(4)} S_{3}^{(4)} - \delta_{11}^{(4)} S_{2}^{(4)} P - \delta_{11}^{(4)} M S_{3}^{(4)}, \]
\[ \delta_{11,y}^{(4)} = \delta_{11}^{(4)} \tilde{M} \tilde{P} S_{1}^{(4)} - \delta_{11}^{(4)} S_{2}^{(4)} MPM - \delta_{11}^{(4)} S_{2}^{(4)} P_{x} - \delta_{11}^{(4)} S_{1}^{(4)} MP \]
\[ + \tilde{M}_{x} \Delta_{4}^{(4)} S_{3}^{(4)} - \tilde{M} \tilde{P} \tilde{M} \Delta_{4}^{(4)} S_{3}^{(4)}, \]
\[ \delta_{11,z}^{(4)} = -3 \delta_{11}^{(4)} S_{2}^{(4)} PB - \frac{3}{2} \delta_{11}^{(4)} S_{2}^{(4)} PMP - \frac{3}{2} \delta_{11}^{(4)} S_{2}^{(4)} P_{xx} + \delta_{11}^{(4)} \tilde{M}_{x} \tilde{P} S_{2}^{(4)} - \frac{3}{2} \delta_{11}^{(4)} S_{1}^{(4)} MP_{x} + \tilde{M}_{x} \tilde{P} \tilde{M} \Delta_{4}^{(4)} S_{3}^{(4)} \]
\[ + \frac{3}{2} \tilde{M} \tilde{P} \tilde{M}_{x} \Delta_{4}^{(4)} S_{3}^{(4)} - \delta_{11}^{(4)} \tilde{M}_{x} \tilde{P} S_{1}^{(4)} - \frac{3}{2} \tilde{B} \tilde{M} \Delta_{4}^{(4)} S_{3}^{(4)} + \frac{3}{2} \delta_{11}^{(4)} \tilde{B} S_{1}^{(4)} \]
\[ + \delta_{11}^{(4)} S_{3}^{(4)} M_{x} P - \tilde{M}_{xx} \Delta_{4}^{(4)} S_{3}^{(4)}, \]
\[ \Delta_{4,x}^{(4)} = \Delta_{4}^{(4)} S_{3}^{(4)} M + \Delta_{4}^{(4)} P S_{2}^{(4)} - 2 \Delta_{4}^{(4)} S_{3}^{(4)} S_{2}^{(4)}, \]
\[ \Delta_{4,y}^{(4)} = \Delta_{4}^{(4)} S_{3}^{(4)} MPM - \Delta_{4}^{(4)} S_{3}^{(4)} M_{x} + \Delta_{4}^{(4)} S_{4}^{(4)} PM - \tilde{P} \tilde{M} \Delta_{4}^{(4)} S_{4}^{(4)} \]
\[ + \delta_{11}^{(4)} \tilde{P}_{x} S_{2}^{(4)} + \delta_{11}^{(4)} \tilde{P} \tilde{M} \tilde{P} S_{2}^{(4)}, \]
\[ (4.27) \]
Δ_t^{(4)} = \frac{3}{2} \Delta_4^{(4)} S_3^{(4)} BM - \frac{3}{2} \Delta_4^{(4)} S_3^{(4)} MPM - \frac{3}{2} \Delta_4^{(4)} S_3^{(4)} M_{xx} + \Delta_4^{(4)} S_3^{(4)} M_{x} M + \Delta_4^{(4)} S_3^{(4)} M_{xx} + \Delta_4^{(4)} S_3^{(4)} P_x M - \Delta_4^{(4)} S_3^{(4)} P M_x + \frac{3}{2} \delta^{(4)}_1 \tilde{P} M \tilde{P} S_2^{(4)} + \frac{3}{2} \delta^{(4)}_1 \tilde{P} M \tilde{P} S_2^{(4)} - \frac{3}{2} \tilde{P} \tilde{M} \tilde{P} M \Delta_4^{(4)} S_3^{(4)} - \tilde{P} \tilde{M} \Delta_4^{(4)} S_3^{(4)} + \tilde{P} \tilde{M} \Delta_4^{(4)} S_3^{(4)} + \delta^{(4)}_1 \tilde{P} x S_2^{(4)}, \quad (4.32)

with

\tilde{M} = (\delta^{(4)}_1 M - 2 \delta^{(4)}_1 S_2^{(4)}) (\Delta_4^{(4)})^{-1}, \quad \tilde{P} = (\Delta_4^{(4)} P - 2 \Delta_4^{(4)} S_3^{(4)}) / \delta^{(4)}_1. \quad (4.33)

5. New solitary-wave solutions with symbolic computation

From the previous results, we can gain a series of explicit solutions for the mKP equation (1.1) by the following iterative procedure:

1) For the initial potentials (U, V) and (M, P), solve the linear systems (3.1), (3.14), and (4.6) with different eigenvalues \( \lambda^{(k)}_i \) for column solutions \( k^{(i)} \) (1 \( \leq k \) \( \leq \ N + 1; 1 \leq i \leq 4 \)).

2) Work out the matrices \( S^{(1)}, S^{(2)}, S^{(3)}, S^{(4)} \) and \( \Delta^{(i)} \) (1 \( \leq i \leq 4 \)), and yield the new potentials \( (\hat{U}, \hat{V}), (\hat{U}, \hat{V}), (\hat{M}, \hat{P}) \) and \( (\hat{M}, \hat{P}) \) via Transformations (3.21), (3.38), (4.23) and (4.33).

3) Substitute \( (\hat{U}, \hat{V}), (\hat{U}, \hat{V}) \) into Expression (2.5) and \( (\hat{M}, \hat{P}), (\hat{M}, \hat{P}) \) into Expression (2.8), and obtain four families of solutions for the mKP equation (1.1) of the form

\[
q^{(1)}_N = -\frac{1}{2} \hat{U} \hat{V} = -\frac{1}{2} (U - 2 S_2^{(1)}) [2 (\Delta_4^{(1)})^{-1} \Delta_3^{(1)} + V], \quad (5.1)
\]

\[
q^{(2)}_N = -\frac{1}{2} \tilde{U} \tilde{V} = -\frac{1}{2} (U - 2 S_2^{(2)}) (V + 2 S_3^{(2)}), \quad (5.2)
\]

\[
q^{(3)}_N = -\frac{1}{2} \hat{M} \hat{P} = -\frac{1}{2} (M + 2 S_2^{(3)}) [2 (\Delta_4^{(3)})^{-1} \Delta_3^{(3)} + P], \quad (5.3)
\]

\[
q^{(4)}_N = -\frac{1}{2} \tilde{M} \tilde{P} = -\frac{1}{2} (M - 2 S_2^{(4)}) (P - 2 S_3^{(4)}). \quad (5.4)
\]

Based on the above explained procedure, in what follows we shall combine Darboux transformations A–D with Decompositions I and II to construct explicit solutions of Eqn. (1.1) by starting with the trivial seed solutions of Systems (2.6)–(2.7) and (2.9)–(2.10). For instance, we solve the linear system (3.1) with \( N = 1 \), \( u_1 = v_1 = 0 \), \( \lambda = \lambda^{(k)}_1 \) (\( k = 1, 2; \lambda^{(1)}_1 \neq \lambda^{(2)}_1 \)) and get the following:

\[
h^{(1)}_{11} = \alpha^{(1)}_1 \exp [\lambda^{(1)}_1 x + 2 (\lambda^{(1)}_1)^2 y + 4 (\lambda^{(1)}_1)^3 t], \quad (5.5a)
\]

\[
h^{(1)}_{21} = \alpha^{(2)}_1 \exp [-\lambda^{(1)}_1 x - 2 (\lambda^{(1)}_1)^2 y - 4 (\lambda^{(1)}_1)^3 t], \quad (5.5b)
\]

\[
h^{(1)}_{12} = \alpha^{(1)}_2 \exp [\lambda^{(2)}_1 x + 2 (\lambda^{(2)}_1)^2 y + 4 (\lambda^{(2)}_1)^3 t], \quad (5.5c)
\]

\[
h^{(1)}_{22} = \alpha^{(2)}_2 \exp [-\lambda^{(2)}_1 x - 2 (\lambda^{(2)}_1)^2 y - 4 (\lambda^{(2)}_1)^3 t], \quad (5.5d)
\]

17
where $\lambda_1^{(1)}, \lambda_2^{(1)}$ and $\alpha_1^{(1)}$ ($l = 1, 2, 3, 4$) are all nonzero constants. From Eqn. (3.31), the element $s_{12}^{(1)}$ of matrix $S^{(1)}$ is expressed as

$$s_{12}^{(1)} = \frac{(\lambda_1^{(1)} - \lambda_2^{(1)}) h_{11}^{(1)} h_{12}^{(1)}}{h_{12}^{(1)} h_{21}^{(1)} - h_{11}^{(1)} h_{22}^{(1)}}. \quad (5.6)$$

Proceedingly, by performing symbolic manipulations on Eqns. (3.20), (3.25) and (3.26) with substitution of Expressions (5.5a)-(5.5d), the function $\delta_{21}^{(1)}$ is figured out as follows:

$$\delta_{21}^{(1)} = \frac{\delta_{22}^{(1)} \alpha_1^{(1)} \alpha_2^{(1)} \alpha_3^{(1)} \alpha_4^{(1)} (\lambda_1^{(1)} - \lambda_2^{(1)})}{\alpha_1^{(1)} \alpha_2^{(1)} \lambda_2^{(1)} (h_{11}^{(1)})^2 - \alpha_3^{(1)} \alpha_4^{(1)} \lambda_1^{(1)} (h_{11}^{(1)})^2}, \quad (5.7)$$

with $\delta_{22}^{(1)}$ as an arbitrary nonzero constant. To this point, the first family of solitary-wave solutions for Eqn. (1.1) is obtained as

$$q_1^{(1)} = \frac{2 \alpha_1^{(1)} \alpha_2^{(1)} \alpha_3^{(1)} \alpha_4^{(1)} (\lambda_1^{(1)} - \lambda_2^{(1)})^2 \text{sech}^2 \xi^{(1)}}{(\gamma_1^{(1)} + \gamma_3^{(1)} \tan \xi^{(1)}) (\gamma_2^{(1)} + \gamma_4^{(1)} \tanh \xi^{(1)})}, \quad (5.8)$$

with

$$\xi^{(1)} = (\lambda_1^{(1)} - \lambda_2^{(1)}) x + 2 [(\lambda_1^{(1)})^2 - (\lambda_2^{(1)})^2] y + 4 [(\lambda_1^{(1)})^3 - (\lambda_2^{(1)})^3] t,$$

$$\gamma_1^{(1)} = \alpha_1^{(1)} \alpha_4^{(1)} - \alpha_2^{(1)} \alpha_3^{(1)}, \quad \gamma_2^{(1)} = \alpha_1^{(1)} \alpha_4^{(1)} \lambda_1^{(1)} - \alpha_2^{(1)} \alpha_3^{(1)} \lambda_2^{(1)},$$

$$\gamma_3^{(1)} = \alpha_2^{(1)} \alpha_3^{(1)} + \alpha_1^{(1)} \alpha_4^{(1)}, \quad \gamma_4^{(1)} = \alpha_1^{(1)} \alpha_4^{(1)} \lambda_1^{(1)} + \alpha_2^{(1)} \alpha_3^{(1)} \lambda_2^{(1)}.$$

In a like manner, we can present other three families of solitary-wave solutions for Eqn. (1.1):

$$q_1^{(2)} = \frac{2 (\alpha_2^{(2)} [(\lambda_1^{(2)})^2 - (\lambda_2^{(2)})^2] \text{sech} 2 \xi^{(2)})}{\gamma_2^{(2)} \text{sech} 2 \xi^{(2)} + \gamma_3^{(2)} \tanh 2 \xi^{(2)} - \gamma_4^{(2)}}, \quad (5.10)$$

with

$$\xi^{(2)} = -[(\lambda_1^{(2)})^2 - (\lambda_2^{(2)})^2] x + 2 [(\lambda_1^{(2)})^4 - (\lambda_2^{(2)})^4] y - 4 [(\lambda_1^{(2)})^6 - (\lambda_2^{(2)})^6] t,$$

$$\gamma_1^{(2)} = \alpha_1^{(2)} \alpha_3^{(2)} \alpha_4^{(2)} [(\lambda_1^{(2)})^2 - (\lambda_2^{(2)})^2], \quad \gamma_2^{(2)} = \alpha_1^{(2)} \alpha_2^{(2)} \alpha_3^{(2)} \alpha_4^{(2)} [(\lambda_1^{(2)})^2 + (\lambda_2^{(2)})^2],$$

$$\gamma_3^{(2)} = [(\alpha_2^{(2)} \alpha_3^{(2)})^2 - (\alpha_1^{(2)} \alpha_4^{(2)})^2] \lambda_1^{(2)} \lambda_2^{(2)}, \quad \gamma_4^{(2)} = [(\alpha_2^{(2)} \alpha_3^{(2)})^2 + (\alpha_1^{(2)} \alpha_4^{(2)})^2] \lambda_1^{(2)} \lambda_2^{(2)}.$$
with
\[
\xi^{(4)} = \left[ (\lambda_1^{(4)})^2 - (\lambda_2^{(4)})^2 \right] x + 2 \left[ (\lambda_1^{(4)})^4 - (\lambda_2^{(4)})^4 \right] y + 4 \left[ (\lambda_1^{(4)})^6 - (\lambda_2^{(4)})^6 \right] t,
\]
\[
\gamma_1^{(4)} = \alpha_1^{(4)} \alpha_2^{(4)} \alpha_3^{(4)} \alpha_4^{(4)} \left( [\lambda_1^{(4)})^2 - (\lambda_2^{(4)})^2 \right],
\gamma_2^{(4)} = \alpha_1^{(4)} \alpha_2^{(4)} \alpha_3^{(4)} \alpha_4^{(4)} \left( (\lambda_1^{(4)})^2 + (\lambda_2^{(4)})^2 \right),
\gamma_3^{(4)} = \left[ (\alpha_1^{(4)} \alpha_2^{(4)})^2 - (\alpha_2^{(4)} \alpha_3^{(4)})^2 \right] \lambda_1^{(4)} \lambda_2^{(4)},
\gamma_4^{(4)} = \left[ (\alpha_2^{(4)} \alpha_4^{(4)})^2 + (\alpha_1^{(4)} \alpha_4^{(4)})^2 \right] \lambda_1^{(4)} \lambda_2^{(4)}.
\]  

We recall that \( q_i^{(4)} \ (i = 1, 2, 3, 4) \) are four new families of solitary-wave solutions for Eqn. (1.1) since they are different from the conventional solitary-wave solutions in terms of a finite series of tanh and/or sech functions. Through the qualitative analysis, it is easy to see that the functions \( q_i^{(4)} \ (i = 1, 2, 3, 4) \) have no singularity and exhibit stable bell profiles for \((\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)}, \lambda_1^{(i)}, \lambda_2^{(i)}) \in \mathcal{A}^{(i)} \cup \mathcal{B}^{(i)} \), where \( \mathcal{A}^{(i)} \) and \( \mathcal{B}^{(i)} \) are defined by
\[
\mathcal{A}^{(i)} = \{ (\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)}, \lambda_1^{(i)}, \lambda_2^{(i)}) \mid \alpha_1^{(i)} \alpha_2^{(i)} < 0, \alpha_1^{(i)} \alpha_4^{(i)} > 0, \lambda_1^{(i)} \lambda_2^{(i)} > 0, \lambda_1^{(i)} \neq \lambda_2^{(i)} \},
\mathcal{B}^{(i)} = \{ (\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)}, \lambda_1^{(i)}, \lambda_2^{(i)}) \mid \alpha_1^{(i)} \alpha_2^{(i)} > 0, \alpha_1^{(i)} \alpha_4^{(i)} < 0, \lambda_1^{(i)} \lambda_2^{(i)} > 0, \lambda_1^{(i)} \neq \lambda_2^{(i)} \},
\mathcal{A}^{(2)} = \{ (\alpha_2^{(2)}, \alpha_3^{(2)}, \alpha_4^{(2)}, \lambda_2^{(2)}) \mid \alpha_2^{(2)} \alpha_3^{(2)} \alpha_4^{(2)} < 0, \lambda_2^{(2)} > 0, \lambda_1^{(2)} \neq \lambda_2^{(2)} \},
\mathcal{B}^{(2)} = \{ (\alpha_2^{(2)}, \alpha_3^{(2)}, \alpha_4^{(2)}, \lambda_2^{(2)}) \mid \alpha_2^{(2)} \alpha_3^{(2)} \alpha_4^{(2)} > 0, \lambda_2^{(2)} < 0, \lambda_1^{(2)} \neq \lambda_2^{(2)} \},
\mathcal{A}^{(3)} = \{ (\alpha_3^{(3)}, \alpha_4^{(3)}, \lambda_3^{(3)}, \lambda_2^{(3)}) \mid \alpha_3^{(3)} \alpha_4^{(3)} < 0, \alpha_4^{(3)} \alpha_3^{(3)} > 0, \lambda_3^{(3)} \lambda_2^{(3)} > 0, \lambda_1^{(3)} \neq \lambda_2^{(3)} \},
\mathcal{B}^{(3)} = \{ (\alpha_3^{(3)}, \alpha_4^{(3)}, \lambda_3^{(3)}, \lambda_2^{(3)}) \mid \alpha_3^{(3)} \alpha_4^{(3)} > 0, \alpha_4^{(3)} \alpha_3^{(3)} < 0, \lambda_3^{(3)} \lambda_2^{(3)} > 0, \lambda_1^{(3)} \neq \lambda_2^{(3)} \},
\mathcal{A}^{(4)} = \{ (\alpha_4^{(4)}, \lambda_4^{(4)} \lambda_2^{(4)}) \mid \alpha_4^{(4)} \lambda_4^{(4)} \lambda_2^{(4)} < 0, \lambda_4^{(4)} \lambda_2^{(4)} > 0, \lambda_1^{(4)} \neq \lambda_2^{(4)} \},
\mathcal{B}^{(4)} = \{ (\alpha_4^{(4)}, \lambda_4^{(4)} \lambda_2^{(4)}) \mid \alpha_4^{(4)} \lambda_4^{(4)} \lambda_2^{(4)} > 0, \lambda_4^{(4)} \lambda_2^{(4)} < 0, \lambda_1^{(4)} \neq \lambda_2^{(4)} \}.
\]

6. Conclusions and discussions

In this paper, we have shown that the mKP equation (1.1) can be reduced to the first two nontrivial nonlinear systems in the 2N-coupled CLL and KN hierarchies, i.e., Systems (2.6)–(2.7) and (2.9)–(2.10), by imposing the potential constraints (2.5) and (2.8) on Systems (2.1)–(2.2) and (2.3)–(2.4), respectively. Furthermore, it has been found that the 2N-coupled CLL and high-order CLL systems possess two different Lax representations (3.1) and (3.7), and the 2N-coupled KN and high-order KN systems also admit two different Lax representations (4.1) and (4.6). For these four Lax representations, we have constructed the corresponding Darboux transformations by which abundant explicit solutions of Eqn. (1.1) can be obtained in a recursive manner. Through one-time iteration of those Darboux transformations, four new families of solitary-wave solutions have been presented and the relevant stability has been analyzed. Finally, we would like to discuss the following issues:

1. As far as we know, there are usually two ways of finding the integrable decompositions for a (2+1)-dimensional integrable NLEE: the first is to choose a proper (1+1)-
dimensional soliton hierarchy and relate its first two nontrivial members to the desired \((2+1)\)-dimensional equation \([19, 20, 23]\); the second is to directly nonlinearize one or two Lax pairs of the \((2+1)\)-dimensional integrable equation into two \((1+1)\)-dimensional nonlinear systems \([12, 13, 14, 15, 20]\). Obviously, the proposal of Decompositions I and II in Section 2 is based on the second decomposition method. In addition, it is noted that the potential constraint \((2.5)\) or \((2.8)\), which originates from the Bargmann symmetry constraint \([34]\), might also be applicable to some other \((2+1)\)-dimensional integrable NLEEs with the availability of two symmetry Lax pairs like Systems \((2.1)-(2.2)\) and \((2.3)-(2.4)\). Special attention should be paid to that Systems \((2.1)\) and \((2.2)\) can be transformed to each other with 
\[
[q(x, y, t), u(x, y, t)] \leftrightarrow [q(-x, -y, -t), v(-x, -y, -t)],
\]
so do Systems \((2.3)\) and \((2.4)\) with 
\[
[q(x, y, t), m(x, y, t)] \leftrightarrow [q(-x, -y, -t), p(-x, -y, -t)].
\]

2. From the derivation of Darboux transformations A–D, we infer that for the following two general linear spectral problems
\[
\begin{align*}
\Psi_x &= [\lambda Q_0^{(1)} + Q_1^{(1)}] \Psi, \\
\Phi_y &= [\lambda^2 Q_0^{(2)} + \lambda Q_1^{(2)} + Q_2^{(2)}] \Phi,
\end{align*}
\]
where \(\lambda\) is the eigenvalue parameter, \(\Psi = (\psi_1, \psi_2, \ldots, \psi_N)^T\), \(\Phi = (\phi_1, \phi_2, \ldots, \phi_N)^T\) are the vector eigenfunctions, \(Q_i^{(1)}\) and \(Q_k^{(2)}\) \((i = 0, 1; k = 0, 1, 2)\) are all the \(N \times N\) matrices, the corresponding Darboux transformations can be respectively taken as
\[
\begin{align*}
\hat{\Psi} &= (\lambda \Delta_1 - \Delta_1 S) \Psi, \\
\hat{\Phi} &= (\lambda^2 \Delta_2 - \lambda \Delta_2 S \perp - \Delta_2 S \top) \Phi,
\end{align*}
\]
where \(\Delta_1, \Delta_2, S, S \perp\) and \(S \top\) are five \(N \times N\) undetermined matrices, \(\hat{\Psi}\) and \(\hat{\Phi}\) respectively satisfy Eqns. \((6.1)\) and \((6.2)\) with \(Q_i^{(1)}\) and \(Q_k^{(2)}\) replaced by \(\hat{Q}_i^{(1)}\) and \(\hat{Q}_k^{(2)}\) \((i = 0, 1; k = 0, 1, 2)\). By utilizing Ansätze \((6.3)\) and \((6.4)\), one can attempt to construct the Darboux transformations for many hierarchies of soliton equations.

3. It is emphasized that the iterative algorithm of the Darboux transformation can be easily achieved on the computerized symbolic computation systems such as Mathematica and Maple. Accordingly, if we choose \(N > 1\) and make the successive iteration of Darboux transformations A–D, many more complicated explicit solutions of Eqn. \((1.1)\) will be unearthed, which might be different from those previously obtained.

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