MIRROR SYMMETRY FOR NON-ABELIAN LANDAU-GINZBURG MODELS

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Abstract

We consider Landau-Ginzburg models stemming from non-Abelian groups comprised of non-diagonal symmetries and we describe a rule for the mirror LG model. In particular, we present the non-Abelian dual group $G^\star$, which serves as the appropriate choice of group for the mirror LG model. We also describe an explicit mirror map between the A-model and the B-model state spaces for two examples.

1 Introduction

Mirror symmetry is most easily explained for Calabi-Yau manifolds which come in dual pairs. The physics of string theory produces an A-model and a B-model for each Calabi-Yau manifold. Mirror symmetry essentially says that the A-model for a Calabi-Yau manifold is “the same” as the B-model on its mirror dual—meaning they produce the same physics.

Landau-Ginzburg models are built from an invertible polynomial $W$ and a group $G \subset G_{\text{max}}^W$. The Landau-Ginzburg (LG) Mirror Symmetry Conjecture predicts that for a large class of invertible polynomials $W$ with a group $G$ of admissible symmetries of $W$, there is a dual polynomial $W^T$ and dual group $G^T$ of symmetries of $W^T$ such that the Landau-Ginzburg A-model for the pair $(W, G)$ is isomorphic to the B-model construction for the pair $(W^T, G^T)$.

In the past, the Landau-Ginzburg models studied have stemmed from Abelian groups comprised of the so-called diagonal symmetries. There has been much interest in understanding the mirror symmetry for non-Abelian cases, but until now we have been missing a definition of the mirror model. One of the main issues for the definition of the dual group construction for LG models was that the dual group was only defined for groups of diagonal symmetries. In this article, we describe the non-Abelian dual group $G^\star$, which extends the Landau-Ginzburg Mirror Symmetry Conjecture to LG models built from non-Abelian groups. We describe the construction of the A- and B-models as graded vector spaces, and provide an explicit isomorphism between A- and B-model state spaces. This has several similarities with the mirror map defined by Krawitz [7] for Abelian LG models. There are still many hurdles for considering structures beyond vector spaces, the foremost being the lack of definition of even a Frobenius product on the B-side.

This construction of $G^\star$ was independently discovered by Ebeling and Gusein-Zade ([4], [5]). Here we focus on finding a mirror map between the...
A- and B-models, while their work pertains more to the Euler characteristic. We will begin by describing the polynomial $W$, and then the group $G$.

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### 2 Preliminary Definitions

**Definition 2.1.** A polynomial $W : \mathbb{C}^n \to \mathbb{C}$ is **nondegenerate** if it has an isolated critical point at the origin of $\mathbb{C}^n$.

**Definition 2.2.** A polynomial is **quasihomogeneous** if there exist positive rational numbers $(q_1, \ldots, q_n)$ so that for every $c \in \mathbb{C}^*$, we have

$$W(c^{q_1} x_1, \ldots, c^{q_n} x_n) = cW(x_1, \ldots, x_n).$$

The numbers $(q_1, \ldots, q_n)$ are called the **weights** of the polynomial $W$. These will be used later to construct an important symmetry of $W$ called the exponential grading operator.

**Definition 2.3.** A nondegenerate, quasihomogeneous polynomial is **invertible** if the weights are unique and the polynomial has the same number of monomials as variables.

**Example 2.4.** Consider the polynomial $W : \mathbb{C}^4 \to \mathbb{C}$ defined by

$$W = x_1^4 + x_2^4 + x_3^4 + x_4^4.$$ 

First, $W$ is nondegenerate since it has a unique critical point at $(0, 0, 0, 0)$. Next, note that $W$ is quasihomogeneous with weights $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ since for every $c \in \mathbb{C}^*$, we have

$$W(c^{\frac{1}{4}} x_1, c^{\frac{1}{4}} x_2, c^{\frac{1}{4}} x_3, c^{\frac{1}{4}} x_4) = (c^{\frac{1}{4}} x_1)^4 + (c^{\frac{1}{4}} x_2)^4 + (c^{\frac{1}{4}} x_3)^4 + (c^{\frac{1}{4}} x_4)^4 = c(x_1^4 + x_2^4 + x_3^4 + x_4^4) = cW(x_1, x_2, x_3, x_4).$$

Clearly this choice of weights is unique. Also, we can see that $W$ has four monomials and four variables, hence $W$ is invertible.

**Definition 2.5 (Mukai [8]).** Let $W : \mathbb{C}^n \to \mathbb{C}$ be an invertible polynomial with weights $(q_1, \ldots, q_n)$. Then the **maximal symmetry group** of $W$, denoted $G_W^{\text{max}}$, is defined as follows:

$$G_W^{\text{max}} := \{ g \in GL_n(\mathbb{C}) | (g \cdot W)(x_1, \ldots, x_n) = W(x_1, \ldots, x_n) \quad \text{and} \quad g_{ij} = 0 \text{ if } q_i \neq q_j \}$$
Definition 2.6. The diagonal symmetry group of $W$ is the group of diagonal linear transformations, defined

$$G_W^{\text{diag}} := \{(g_1, \ldots, g_n) \in (C^*)^n | W(g_1 x_1, \ldots, g_n x_n) = W(x_1, \ldots, x_n)\}.$$ 

The second definition is the standard definition of diagonal symmetries, as in [6]. Note that $G_W^{\text{diag}}$ is a subgroup of $G_W^{\text{max}}$. We can view the elements of $G_W^{\text{diag}}$ as diagonal matrices and it is a standard fact that the entries $g_i$ as above are roots of unity. For simplicity, we will typically represent these symmetries additively as $n$-tuples of rational numbers as follows:

$$(e^{2\pi i a_1}, \ldots, e^{2\pi i a_n}) \leftrightarrow (a_1, \ldots, a_n) \in (Q/Z)^n.$$ 

In fact, it is a fact that $G_W^{\text{diag}}$ is generated by the entries of the inverse of the exponent matrix $A_W^{-1}$, which we define below.

Two other important subgroups of $G_W^{\text{max}}$ are $J_W$, and $SL_W^{\text{diag}}$, the group of matrices in $G_W^{\text{diag}}$ whose determinant is 1. The group $J_W$ is generated by the group generated by the exponential grading operator $j_W$, which we write additively as $(q_1, \ldots, q_n)$, where $q_1, \ldots, q_n$ are the weights of $W$. Explicitly, we write these groups as follows:

$$J_W = \langle j_W \rangle = \langle (q_1, \ldots, q_n) \rangle$$

$$SL_W^{\text{diag}} = \text{SL}(n, C) \cap G_W^{\text{diag}}.$$ 

Example 2.7. For $W = x_1^4 + x_2^4 + x_3^4 + x_4^4$, we have

$$J_W = \langle (1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \rangle \text{ and } SL_W^{\text{diag}} = \langle (1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (\frac{2}{4}, \frac{1}{4}, 1, 0), (\frac{1}{4}, \frac{2}{4}, 1, 0) \rangle.$$ 

These two groups are the dual of each other, under BHK mirror symmetry, discovered by Berglund– Hübsch [3] and Krawitz [7]. BHK mirror symmetry associates to an LG model $(W, G)$ another pair LG model $(W^T, G^T)$ which we work towards next.

Definition 2.8. Let $W : C^n \to C$ be an invertible polynomial. If we write $W = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}}$, then the associated exponent matrix is defined to be $A_W = (a_{ij})$. Then the dual polynomial $W^T$ is the invertible polynomial defined by the matrix $A_W^T$.

Example 2.9. For $W = x_1^4 + x_2^4 + x_3^4 + x_4^4$, we have

$$A_W = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = A_W^T.$$ 

Hence in this case, $W^T = W$. 

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Definition 2.10. The dual group of a subgroup $G \subseteq G_{W}^{\text{diag}}$ is the set

$$G^{T} = \{ g \in G_{W}^{\text{diag}} | gA_{W}h^{T} \in \mathbb{Z} \text{ for all } h \in G \}.$$ 

Example 2.11. Earlier we claimed that the dual group of $J_{W}$ is $SL_{W}^{\text{diag}}$ for our choice of $W$. Observe

$$(J_{W})^{T} = \{ g \in G_{W}^{\text{diag}} | gA_{W}h^{T} \in \mathbb{Z} \text{ for all } h \in J_{W} \}.$$ 

Since $g \in G_{W}^{\text{diag}}$ and $h \in J_{W}$, then $g = (a_{1}, a_{2}, a_{3}, a_{4})$ and $h = (b_{1}, b_{2}, b_{3}, b_{4})$ where $a_{1}, a_{2}, a_{3}, a_{4}, b \in \{0, 1, 2, 3\}$. Then

$$gA_{W}h^{T} = b(a_{1} + a_{2} + a_{3} + a_{4})$$

This is in an integer for all $b \in \mathbb{Z}$ if and only if $(a_{1} + a_{2} + a_{3} + a_{4}) \in \mathbb{Z}$, implying $g \in SL_{W}^{\text{diag}}$. Hence $(J_{W})^{T} = SL_{W}^{\text{diag}}$.

In fact, it is true that $J_{W}^{T} = SL_{W}^{\text{diag}}$ for any choice of $W$ (Artebani [1]).

As mentioned previously, most of the work done with Landau-Ginzburg models has been with groups of diagonal matrices. Next we consider a group with a permutation as one of the generators, which is a non-diagonal matrix.

Example 2.12. Consider the subgroup

$$G = \langle j_{W}, (123) \rangle \subset G_{W}^{\text{max}}, \text{ where } (123) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Even though $G$ contains non-diagonal matrices, it is actually still Abelian since the generators commute, seen below:

$$(123)j_{W} = \begin{pmatrix} 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} = j_{W}(123).$$ 

Although $G$ is Abelian in the above example, its dual group is non-Abelian. However, we cannot use the previously mentioned definition for $G^{T}$ since $G$ is not a subset of $G_{W}^{\text{diag}}$, as required by Definition 2.10.

Definition 2.13. An element of $G_{W}^{\text{max}}$ is called a pure permutation if it acts on $\mathbb{C}[x_{1}, \ldots, x_{n}]$ by simply permuting the variables.

Notice that a pure permutation can only permute variables that have the same weight.
Definition 2.14. Let $G \subset G_W^\text{max}$ be a group of the form

$$G = H \cdot K,$$

where $H \subset G_W^{\text{diag}}$ and $K \subseteq G$ is a group of pure even permutations in $G$. We define the non-Abelian dual group of $G$ to be

$$G^* = K \cdot H^T \subseteq \text{GL}_n(\mathbb{C}).$$

Example 2.15. For $G = \langle j_W, (123) \rangle \subset G_W^\text{max}$, we have

$$G^* = \langle (123) \rangle \cdot (j_W)^T = \langle (123) \rangle \cdot \text{SL}_W^{\text{diag}}.$$

Explicitly, the elements of $G^*$ are of the form $(123)^k(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}, \frac{a}{2})$, where $a_1 + a_2 + a_3 + a_4 \in 4\mathbb{Z}$ and $0 \leq k \leq 2$. As stated earlier, in this example $G^*$ is non-Abelian. For instance, consider $(123)(\frac{1}{2}, \frac{1}{2}, 0), (132)(\frac{1}{2}, \frac{1}{2}, 0) \in G^*$. Observe

$$[(123)(\frac{1}{2}, \frac{1}{2}, 0)](123)(\frac{1}{2}, \frac{1}{2}, 0) = (\frac{3}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}),$$

whereas

$$[(132)(\frac{1}{2}, \frac{1}{2}, 0)](123)(\frac{1}{2}, \frac{1}{2}, 0) = (\frac{3}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}).$$

Now we have defined a rule relating two LG models $(W, G)$ and $(W^T, G^*)$. In the next sections we will construct the A- and B-model state spaces.

3 The A-model State Space

The A-model vector space is referred to as the state space. The construction for the state space, as found e.g. in [8] and [2], requires an invertible polynomial $W$ and an admissible subgroup of $G_W^\text{max}$. One of the important pieces is the Milnor ring, which we now define.

Definition 3.1. The Milnor ring of a polynomial $W$ is defined to be

$$\mathcal{Q}_W = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle.$$ 

Definition 3.2. Let $W : \mathbb{C}^n \to \mathbb{C}$ be a nondegenerate quasihomogeneous polynomial with unique weights $(q_1, \ldots, q_n)$, and let $G$ be a subgroup of $G_W^\text{max}$. Then $G$ is admissible if $G$ contains $j_W = (q_1, \ldots, q_n)$.

Given $g \in G$, we denote by $\text{fix}(g)$ the locus of points in $\mathbb{C}^n$ fixed by $g$. The group $G$ acts on $\mathbb{C}[x_1, \ldots, x_n]$ in a natural way. In the definition below, $G$ also acts on the volume form in the same way.

Definition 3.3. Let $W$ be an invertible polynomial and $G$ be an admissible subgroup of $G_W^\text{max}$. The state space for the A-model is defined as

$$\mathcal{A}_{W,G} = \left( \bigoplus_{g \in G} \mathcal{Q}_W|_{\text{fix}(g)} \cdot \omega_g \right)^G,$$

where $\omega_g$ is a volume form on the fixed locus of $g$. 

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We will use the notation $[P, g]$ to denote an element of $Q_{W|_{\text{fix}(g)}} \cdot \omega_g$, often suppressing the volume form where convenient. The volume form can be easily determined by $g$. We can form a basis of $A_{W,G} \cdot \omega_g$ using sums of the form

$$\sum_{g_i \in [g]} [P, g_i],$$

where $g_i$ are the group elements in the same conjugacy class $[g]$ of $G$, and $P \in Q_{W|_{\text{fix}(g)}}$.

For Abelian A-models, we can rewrite the state space definition as

$$A_{W,G} = \bigoplus_{g \in G} (Q_{W|_{\text{fix}(g)}} \cdot \omega_g)^G$$

as the action of $G$ preserves each summand. But if $G$ is non-Abelian, then for $h \in G$,

$$h \cdot (Q_{W|_{\text{fix}(g)}} \cdot \omega_g) \subseteq Q_{W|_{\text{fix}(gh^{-1}g)}} \cdot \omega_{gh^{-1}g}.$$

\textbf{Example 3.4.} Let $W = x_1^4 + x_2^3 + x_3^2 + x_4^4$ and $G = \langle j_W, (123) \rangle$. Since in this case, $G$ is an Abelian group, the conjugacy class for each $g \in G$ contains only $g$. Hence we can choose a basis of $A_{W,G}$ consisting of elements of the form $[P, g]$ (i.e. single terms, instead of sums). The elements of $G$ can be expressed as $(123)^a j_W^b$ with $0 \leq a \leq 2$ and $0 \leq b \leq 3$. For each $g \in G$, we will need to find the basis elements of $(Q_{W|_{\text{fix}(g)}})^G$. The choices of $g$ can be broken down into three different cases.

\textbf{Case 1:} $g = (0,0,0,0)$.

When $g = (0,0,0,0)$, then $W|_{\text{fix}(g)} = W$, so the Milnor ring is of $W$ is

$$Q_W = \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{\langle 4x_1^4, 4x_2^3, 4x_3^2, 4x_4^4 \rangle} = \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{\langle x_1, x_2, x_3, x_4 \rangle}.$$}

The elements of $Q_W$ are sums of elements in the set $\{x_1^a x_2^b x_3^c x_4^d | 0 \leq a, b, c, d \leq 2\}$. The volume form $\omega_g$ in this case is $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. To find the elements of $(Q_W \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4))^G$ we look for $p(x) \in Q_W$ such that $p(x) \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$ is invariant under $j_W$ and $(123)$, the generators of $G$. The volume form is invariant under $j_W$ since

$$(e^{2\pi i} dx_1) \wedge (e^{2\pi i} dx_2) \wedge (e^{2\pi i} dx_3) \wedge (e^{2\pi i} dx_4) = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$}

It is also invariant under $(123)$ since

$$dx_2 \wedge dx_3 \wedge dx_1 \wedge dx_4 = -dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_4$$
$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

Thus, in this sector we only need to be concerned that the actual polynomial $p(x)$ is invariant under $j_W$ and $(123)$. 6
In order to be invariant under (123), the polynomial must be symmetric with respect to $x_1$, $x_2$, and $x_3$ and polynomials invariant under $j_W$ must have exponents in each term sum to a multiple of 4; for example, the polynomial $x_1x_2x_3x_4$ is invariant under both $j_W$ and (123). Elements of $(\mathbb{Q}_W : \omega G)$ can also be sums of elements in $\mathbb{Q}_W$. Consider $x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2 \in \mathbb{Q}_W$. Applying $j_W$ to $x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2$ gives

$$\left(\frac{2\pi i}{4}x_1\right)^2\left(\frac{2\pi i}{4}x_4\right)^2 + \left(\frac{2\pi i}{4}x_2\right)^2\left(\frac{2\pi i}{4}x_4\right)^2 + \left(\frac{2\pi i}{4}x_3\right)^2\left(\frac{2\pi i}{4}x_4\right)^2$$

$$= \left(\frac{4\pi i}{4}x_1\right)^2\left(\frac{4\pi i}{4}x_4\right)^2 + \left(\frac{4\pi i}{4}x_2\right)^2\left(\frac{4\pi i}{4}x_4\right)^2 + \left(\frac{4\pi i}{4}x_3\right)^2\left(\frac{4\pi i}{4}x_4\right)^2$$

$$= x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2.$$

Applying (123) gives

$$x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2 = x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2,$$

and thus this element of the Milnor ring (including its volume form) is invariant under all the generators of $G$, so it is invariant under $G$.

In the same way, we find that the invariant elements of the Milnor ring in the identity sector are of the form $P = p(x) \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$, where $p(x)$ is one of the following polynomials:

1
$x_1x_2x_3x_4$
$x_1^2x_2x_3x_4$
$x_1^2x_2^2 + x_1^2x_3^2$x_2^2x_3^2 + x_2^2x_4^2
$x_1x_2x_3^2 + x_1^2x_2x_3 + x_1x_3^2x_3$
$x_1x_2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2$
$x_1x_2^2x_4 + x_2x_3^2x_4 + x_3^2x_4^2$
$x_2^2x_3x_4 + x_2^2x_3x_4 + x_3^2x_1x_4$
$x_2^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2$

The 9 dimensional vector space generated by these elements is called the *untwisted broad sector* of $A_{W,G}$.

**Case 2:** $g = (123)$ or $g = (132)$  

Let $g = (123)$. To find the fixed locus of (123), we look for eigenvectors with an eigenvalue of 1. Diagonalizing gives (123) = $QDQ^{-1}$, where $D = \text{diag}(1, 1, e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{4}})$ and

$$Q = \begin{bmatrix}
1 & 1 & e^{\frac{2\pi i}{4}} & e^{\frac{2\pi i}{4}} \\
0 & 1 & e^{\frac{2\pi i}{4}} & e^{\frac{4\pi i}{4}} \\
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}.$$
Thus the eigenvectors with eigenvalue 1 are \((1, 1, 1, 0)\) and \((0, 0, 0, 1)\), and the span of these two vectors is the fixed locus of \(g = (123)\). If we call the coordinates of these two vectors \(y_1\) and \(y_4\), then we have \(W|_{\text{fix}(g)} = c_1y_1^4 + y_4^4\) for some constant \(c_1\). The value of \(c_1\) does not matter for our purposes, since the Milnor ring is simply \(\mathbb{C}[y_1, y_4]/(y_1^3, y_4^3)\). The volume form here is \(dy_1 \wedge dy_4 = (dx_1 + dx_2 + dx_3) \wedge dx_4\). This is invariant under \((123)\), which acts trivially when considering only \(y_1\) and \(y_4\). However, this volume form is not invariant under \(jW\) since
\[
(e^{2\pi i} y_1)(e^{2\pi i} y_4) = -y_1 y_4 \neq y_1 y_4.
\]
To balance this, in order for an element of the Milnor ring to be invariant under \(jW\), it must be a polynomial where each term has degree equal to 2 mod 4. This means the degree must be 2 since the elements of \(\mathbb{C}[y_1, y_4]/(y_1^3, y_4^3)\) have the exponents on \(y_1\) and \(y_4\) capped at 2. This gives us three invariant polynomials:
\[
y_1^2 = (x_1 + x_2 + x_3)^2
\]
\[
y_1 y_4 = (x_1 + x_2 + x_3)x_4
\]
\[
y_4^2 = x_4^2
\]
Each one of these, together with the volume form, is another element in a basis.

The case of \(g = (132)\) is almost identical. The matrix \(P\) is different, but the vectors \((0,0,1,1)\) and \((1,1,0,0)\) still correspond to the eigenvalues of 1, and they produce the same Milnor ring, volume form, and invariant entries.

**Case 3: Other Values of \(g\)**

The fixed locus of \(g \in G\) consists of all eigenvectors with an eigenvalue of 1. The eigenvalues of \(jW\) are all \(e^{2\pi i}\), so \(g = jW\) has no fixed locus. Thus \(W|_{\text{fix}(jW)} = 0\), which is invariant under \(jW\) and \((123)\). This implies that for \(g = jW\), we get \(\mathcal{Q}_{W_{\text{fix}(g)}} \cdot \omega_g \cong \mathbb{C}\). There is a natural basis for \(\mathbb{C}\), namely 1, so this sector only produces a single basis element of \(\mathcal{A}_{W,G}\), being \([1, jW]\). Sectors with \(\text{fix}(g) = 0\) are called narrow sectors. Similarly, \((jW)^2\) and \((jW)^3\) produce narrow sectors as well.

Next we look at \(g = (123)jW\). As seen in case 2, the eigenvalues of \((123)\) are 1, 1, \(e^{2\pi i}\), and \(e^{2\pi i}\), so the eigenvalues of \((123)jW\) are \(e^{2\pi i}, e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}}\), and \(e^{\frac{2\pi i}{3}}\). None of these are 1, so \((123)jW\) produces another narrow sector. Similarly, we can see that \((123)jW, (123)(jW)^2, (123)(jW)^3, (132)(jW), (132)(jW)^2, \) and \((132)(jW)^3\) have no eigenvalues equal to 1, so they are also narrow sectors. In total, there are 9 narrow sectors in \(\mathcal{A}_{W,G}\).

To conclude this example, we have found 9 narrow sectors, the untwisted locus has dimension 9, and two more broad sectors each contributes dimension 3 to the state space. Hence \(\mathcal{A}_{W,G}\) has dimension 24.

### 3.1 A-model grading

The A-model can also be given a bigrading structure as well which will be preserved under mirror symmetry. This bigrading is similar to the Hodge grading...
for Calabi-Yau manifolds. Since mirror symmetry for Calabi-Yau manifolds rotates the Hodge diamond, we expect some similar phenomenon for LG models.

**Definition 3.5 (Mukai [8])**. Let $G$ be a finite subgroup of the symmetry group of some non-degenerate quasihomogeneous polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. We define the *age* of $g \in G$ as

$$\text{age } g = \frac{1}{2\pi i} \sum_{j=1}^{n} \log(\lambda_j),$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $g$ and the branch of the logarithmic function is chosen to satisfy $0 \leq \log(z) < 2\pi i$ for $z \in \mathbb{C}^*$ such that $|z| = 1$.

**Example 3.6**. When $g$ is diagonal, then $\lambda_j$ will take on the value of the sole entry in the $j^{th}$ column. Since the entries are all of the form $e^{2\pi i a_j}$, the age is just $\sum_{j=1}^{n} a_j$. Hence for diagonal symmetries, we can just take the sum of the entries when they are written in additive form. For $j_W = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ from the A-model vector space in the previous section, we just have

$$\text{age}(j_W) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1.$$

**Definition 3.7**. The A-model has a **bigrading**, defined as

$$(\deg P + \text{age } g - \text{age } j_W, N_g - \deg P + \text{age } g - \text{age } j_W),$$

where $N_g$ is the dimension of the fixed locus of $g$. In this notation, note that the volume form $\omega_g$ contributes to $\deg P$.

**Example 3.8**. Let’s continue with the A-model vector space from the previous section to turn it into a graded vector space. Since $\text{age}(j_W) = 1$, the bigrading for each element reduces to

$$(\deg P + \text{age } g - 1, N_g - \deg P + \text{age } g - 1),$$

and we can break up the process into the same three cases as before depending on $g$.

**Case 1: $g = (0, 0, 0, 0)$**

When $g$ is the identity, we get $\text{age } g = 0$ and $N_g = 4$, so the bigrading simplifies to

$$(\deg P - 1, 4 - \deg P - 1) = (\deg P - 1, 3 - \deg P),$$

and the bigrading is dependent just on $\deg P$. Since $P = p(x) \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$, then $\deg P$ will be $1 + \deg p(x)$. Recall that there were 9 polynomials in our basis for this choice of $g$. A few examples of $\deg P$ can be seen...
Case 2: \( g = (123) \) or \( g = (132) \)

Recall that in this case, the fixed locus was spanned by the two vectors \( y_1 = x_1 + x_2 + x_3 \) and \( y_4 = x_4 \), so \( N_8 = 2 \). To find age((123)) and age((132)), recall that the eigenvalues of (123) and (132) are 1, 1, \( e^{\frac{2\pi i}{3}} \), and \( e^{\frac{4\pi i}{3}} \). Then

\[
\frac{1}{2\pi i} \sum_{j=1}^{n} \log(\lambda_j) = \frac{1}{2\pi i} (\log(1) + \log(1) + \log(e^{\frac{2\pi i}{3}}) + \log(e^{\frac{4\pi i}{3}}))
\]

\[
= \frac{1}{2\pi i} (0 + 0 + \frac{2\pi i}{3} + \frac{4\pi i}{3})
\]

\[
= \frac{1}{2\pi i} (2\pi i) = 1,
\]

so age((123)) and age((132)) are both 1. Next we need to find \( \deg P \) for the three polynomials we found earlier. It is easiest to think about the polynomials as elements of the Milnor ring \( C[y_1, y_4]/(y_1^3, y_4^3) \). There were 3 elements in

\[
\deg(1 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = 1
\]

\[
\deg(x_1 x_2 x_3 x_4 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = 2
\]

\[
\deg(x_1^2 x_2^2 x_3^2 x_4^2 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = 3
\]

\[
\deg(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = 2
\]
the G-invariant subspace of the Milnor ring, and their degrees are computed below:

\[
\begin{align*}
\deg(y_1^2 \cdot (dy_1 \wedge dy_4)) &= 1 \\
\deg(y_1y_4 \cdot (dy_1 \wedge dy_4)) &= 1 \\
\deg(y_2^2 \cdot (dy_1 \wedge dy_4)) &= 1
\end{align*}
\]

Thus the bigrading of each of these elements is the same, which is

\[
(\deg P + \text{age } g - 1, N_g - \deg P + \text{age } g - 1) = (1, 1).
\]

**Case 3: Other Values of** \(g\)

For all other choices of \(g\), we know that \(g\) creates a narrow sector. So the fixed locus has dimension 0, that is, \(N_g = 0\). The formula for bigrading thus reduces to

\[
(\text{age } g - \text{age } jW, \text{age } g - \text{age } jW) = (\text{age } g - 1, \text{age } g - 1)
\]

Hence in this case, the only thing we need to actually compute is \(\text{age } g\). When \(g\) is a multiple of \((jW)^n\), we can simply add up the components to get

\[
\text{age}(jW) = 1, \text{age}((jW)^2) = 2 \text{ and } \text{age}((jW)^3) = 3,
\]

and their bigrading is shown below.

| basis element  | bigrading |
|----------------|-----------|
| \([1, jW]\)    | (0, 0)    |
| \([1, (jW)^2]\) | (1, 1)    |
| \([1, (jW)^3]\) | (2, 2)    |

The rest of the elements are non-diagonal, so we must find the eigenvalues as in the last case. The resulting age is the same for all of them, which is 2. Thus the bigrading for the rest of the narrow sectors is \((1, 1)\). We have now covered all cases, and below we can see all basis elements and their bigrading:
If we arrange these as a Hodge diamond, we have

1 1

20

1 1

1

4 The B-model State Space

Having constructed the A-model as a graded vector space, we can begin our construction of the B-model. We expect the B-model to be isomorphic to the
A-model; in our example, this means that the B-model should also have dimension 24 with the same bigrading as the A-side.

**Definition 4.1.** Let $W$ be an invertible polynomial and $H \subset SL^\text{max}_W$. The state space for the B-model is defined as

$$B_{W,H} = \left( \bigoplus_{h \in H} \mathcal{Q}_{W|_{\text{fix}(h)}} \cdot \omega_h \right)^H,$$

where $\omega_h$ is a volume form on the fixed locus of $h$. This is exactly analogous to Definition 3.3 except that the associated group $H$ has different requirements than the group $G$ used for the A-model. If we used $(W, G)$ to construct the A-model, then Karwitz showed that the B-model state space associated to $\left( W^T, G^T \right)$ will be isomorphic to the A-model state space for $(W, G)$ (7).

For groups of non-diagonal matrices, in order for mirror symmetry to hold, we replace $G^T$ by the non-Abelian dual group $G^*$, defined in Definition 2.14.

**Example 4.2.** Let $W = x_1^4 + x_2^4 + x_3^4 + x_4^4$, $G = \langle j_W, (123) \rangle$, and $H = G^*$. Recall from Example 2.15 that $G^* = \langle (123) \rangle \cdot SL^\text{diag}_W$. The elements of $SL^\text{diag}_W$ are of the form $(123)^k \left( \frac{a_1}{a_2}, \frac{a_3}{a_4}, \frac{a_3}{a_4}, \frac{a_4}{a_4} \right)$; again, this notation refers to a 4x4 diagonal matrix with diagonal entries on the complex unit circle. The entries also satisfy $4(a_1 + a_2 + a_3 + a_4)$; this is required to be in $SL_4$. Alternately, the elements are generated by $(123)$, $j_W$, $K$, and $L$, where $j_W = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \right)$, $K = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0 \right)$, and $L = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0 \right)$.

As we begin to construct $B_{W,G^*}$, we need to pay attention to centralizers and conjugacy classes. On the A-side, $j_W$ commuted with $(123)$, so the centralizer of every element was $G$ and the conjugacy class of every element was itself. That is not the case for $G^*$. We need to consider only one representative from each conjugacy class, and we only need to check invariance over the centralizer, not the whole group.

**Case 1: $g = (0, 0, 0, 0)$**

Given that we are using the same polynomial as for the A-model, the Milnor ring here will be exactly the same. However, the list of polynomials invariant under $G^*$ will not be the same as that for $G$, since $G^*$ has different generators. Since $(123), j_W \in G^*$, this list of polynomials will be a subset of the 9 from earlier, but we also need to check if those 9 polynomials are invariant under $K = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0 \right)$, and $L = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0 \right)$ as well. The only polynomials that will work are those where each term have the same exponent for $x_1, x_2$, and $x_3$. An example of a polynomial that isn’t invariant under $K$ is $x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2$ since

$$\left( e^{\frac{4\pi}{3}} x_1 \right)^2 \left( e^{\frac{4\pi}{3}} x_2 \right)^2 + \left( e^{\frac{4\pi}{3}} x_1 \right)^2 \left( e^{\frac{4\pi}{3}} x_3 \right)^2 + \left( e^{\frac{4\pi}{3}} x_2 \right)^2 \left( e^{\frac{4\pi}{3}} x_3 \right)^2 \neq x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2.$$

This $G^*$-invariant subspace has dimension 3 spanned by $1, x_1 x_2 x_3 x_4$, and $x_1^2 x_2^2 x_3^2 x_4^2$ (again suppressing the volume form).
**Case 2: \( g = (123) \) or \( g = (132) \)**

Much like in the previous case, we know that the polynomials in either of these sectors will be a subset of those found in the A-model. Recall there were the same three polynomials for both choices of \( g \). One can check that all three are invariant under \( K \) and \( L \) too, meaning that this case yields the same polynomials as on the A-side.

**Case 3: \( g \) is a narrow sector without permutations**

This case means \( g = (a_1^4, a_2^4, a_3^4, a_4^4) \), for \( 1 \leq a_i \leq 3 \). Any sector where \( a_1, a_2, a_3, \) and \( a_4 \) are all nonzero will fix nothing, so it will be narrow. Since the sum of \( a_1, a_2, a_3, \) and \( a_4 \) must be a multiple of four, then \( (a_1, a_2, a_3, a_4) \) will need to be an ordering of one the following:

\[
(3, 3, 3, 3) \\
(3, 3, 1, 1) \\
(3, 2, 2, 1) \\
(2, 2, 2, 2) \\
(1, 1, 1, 1)
\]

The 3 choices from the above where the components are all equal are powers of \( jW \). In any of those 3 cases, the conjugacy class is trivial since they will commute with \( (123), jW, K, \) and \( L \).

There are 12 different orderings of \( (1, 2, 2, 3) \). Conjugation by \( jW, K, \) or \( L \) does nothing, but conjugation by \( (123) \) creates a conjugacy class of size 3, implying there will be \( 12/3 = 4 \) conjugacy classes of this type. There are 6 orderings of \( (1, 1, 3, 3) \), so this choice gives \( 6/3 = 2 \) additional conjugacy classes. The powers of \( jW \) give three more classes. Thus in this case we found a total of 9 conjugacy classes. The sums of the elements in each conjugacy class form a basis vector for a narrow sector. A few examples of these are the following:

\[
[1, jW] \\
[1, (\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4})] + [1, (\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})] + [1, (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4})] \\
[1, (\frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4})] + [1, (\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})] + [1, (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4})] \\
\]

Later we will show that these 9 sums of narrow sectors correspond to the 9 untwisted broad sectors from the A-model.

**Case 4: \( g \) is diagonal with at least one component equal to 0**

Again, we have \( g = (a_1^4, a_2^4, a_3^4, a_4^4) \), but with \( 0 \leq a_i \leq 3 \) and at least one of the \( a_i \)'s is 0.

The sectors where exactly one of the \( a_i \)'s is 0 fix those coordinates. We need to check for invariance under the centralizer, which includes powers of \( jW, K, \) and \( L \) but not \( (123) \). If exactly one \( a_i \) is 0, then our resulting polynomial has
only one variable (with degree less than 3) and a \( dx_i \) volume form, so it cannot be invariant under \( j_W \).

If exactly two of the \( a_i \) values are zero, two standard basis vectors are included in the fixed locus. There are six ways to choose these two, and there are three in each conjugacy class, so we only need to consider two classes. Without loss of generality assume these are \( x_1 \) and \( x_2 \). Then the Milnor ring of \( W|_{\text{fix}(g)} \) is \( \mathbb{C}[x_1, x_2]/\langle x_1^3, x_2^3 \rangle \), with the volume form being \( (dx_1 \wedge dx_2) \). Invariance under \( J \) tells us that each of these must be of the form \( x_2^r x_1^s \), \( x_1 x_2^r \), or \( x_2^{3-r} \). However, these polynomials are not invariant under both \( K \) and \( L \), so we get no additional basis elements from this subcase.

If three \( a_i \) values are 0, the fourth must be as well or else \( g \) would not be in \( SL_{W}^{\text{diag}} \). This implies that \( g = (0, 0, 0, 0) \), which would be a repeat of case 1. Thus case 4 yields no contribution to the state space.

Case 5: \( g \) is a narrow sector with permutations

Finally, we move on to sectors with factors of \((123)\) and \((132)\). In particular, \((123)j_W, (123)(j_W)^2, (123)(j_W)^3, (123)j_W, (123)(j_W)^2, \) and \((132)(j_W)^2\) all have trivial fixed locus as we have seen in Case 3 of Example 3.4, and they still don’t on this side, so they are narrow. However, on the B-side, these elements have nontrivial conjugacy classes. Let’s consider a specific example, say \((123)j_W \).

Conjugating \((123)j_W\) by \( K \) yields the following:

\[
K[(123)j_W]K^{-1} = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 \\
\frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

In this computation, the nonzero entries of the matrix are to be understood as the exponent of \( e^{2\pi i} \).

This conjugation is equivalent to multiplying on the left of \((123)j_W\) by \( K^2L \). Similarly, conjugating \((123)j_W\) by \( L \) is equivalent to multiplying on the left by \( K^3L \). Together the conjugacy class of \((123)\) reaches \((123)K^iL^j\) for any \( i, j \in \{0, 1, 2, 3\} \). The same is true for the other five classes, producing 6 more narrow sectors. One can easily check that \( x \in \text{fix}(g) \) if and only if \( h^{-1} \cdot x \in \text{fix}(h^{-1}gh) \), so the conjugates of narrow group elements remain narrow.

In conclusion, the B-model state space contains 3 basis elements from the unwisted broad sector \((0, 0, 0, 0)\), 6 basis elements from the two twisted broad sectors \((123)\) and \((132)\), 9 narrow sectors from case 3, and 6 more narrow sectors from case 5, for a total of 24 basis elements. Recall that there were 24 basis elements in the A-model as well, which is sufficient for showing that the A-
and B-models are isomorphic as vector spaces.

### 4.1 B-model grading

Just like with the A-model, there is also a bigrading on the B-model state space.

**Definition 4.3.** The B-model bigrading is defined for an element $[P, g]$ to be

$$(\deg P + \text{age } g - \text{age } j_W, \deg P + \text{age } g^{-1} - \text{age } j_W).$$

**Example 4.4.** The A-model had 24 basis elements, with 20 of them having a bigrading of $(1, 1)$ and 1 of each of the following: $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(2, 2)$. For the A- and B-models to be isomorphic as graded vector spaces, we should see the same breakdown of elements for the B-model too.

As with the A-model, we know that $\text{age}(j_W) = 1$. Hence the bigrading for all of the elements in the B-model can be reduced to

$$(\deg P - 1, \deg P + \text{age } g^{-1} - 1).$$

As with the A-model, we will handle this by cases. Many of the basis elements on this side will be sums of elements, but we only need to look at one term in the sum since they will all have the same degree.

**Case 1: $g = (0, 0, 0, 0)$**

As with the A-model, we will have $\text{age } g = 0$ and thus $\text{age } g^{-1} = 0$ as well since $(0, 0, 0, 0)$ is its own inverse. There are only three polynomials to check in this sector, shown below:

\[
\begin{align*}
\deg(1 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) &= 1 \\
\deg(x_1x_2x_3x_4 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) &= 2 \\
\deg(x_1^2x_2^2x_3^2 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) &= 3
\end{align*}
\]

Hence the bigrading for the elements in this sector is

$$(\deg P - 1, \deg P - 1),$$

where $\deg P$ is found above.

**Case 2: $g = (123)$ or $g = (132)$**

Recall from the A-model that $\text{age}((123)) = 1$ and $\text{age}((132)) = 1$. Also, the polynomials in this case are exactly the same as those from the A-model, where we found $\deg P = 1$ for all such polynomials. Thus the bigrading for the basis elements in these sectors is $(1, 1)$.

**Case 3: $g$ is a narrow sector without permutations**

There are a total of 9 sectors in this case, with 3 having a basis elements with trivial conjugacy class and the other 6 having a conjugacy class of size 3. All of
the polynomials in these sectors will have degree 0, so the bigrading depends just on age $g$ and age $g^{-1}$. When $g = (jW)^i$ where $1 \leq i \leq 3$, then age $g = i$ and age $g^{-1} = 4 - i$. Hence the bigrading for these 3 elements is

$$(\text{age } g - 1, \text{age } g^{-1} - 1),$$

where age $g$ and age $g^{-1}$ are given above.

An example of one of the 6 basis elements with non trivial conjugacy class is

$$[1, (\frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4})] + [1, (\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})] + [1, (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4})].$$

To find the degree of this element, we need to only look at one term in the summation. Since all of the group elements are diagonal, we can simply sum up the components of $g$ to find age $g$, which is 2 in this case. Looking at $g^{-1}$ will also yield an age of 2. In fact, the associated $g$ and $g^{-1}$ for all 6 of these basis elements have an age of 2. Thus the bigrading for all of them will be $(2 - 1, 2 - 1) = (1, 1)$.

Case 4: $g$ is a narrow sector with permutations

This case yields 6 such basis elements, each with a conjugacy class of size 16. These conjugacy classes are found through conjugating $(123)jW, (123)(jW)^2, (123)(jW)^3, (132)jW, (132)(jW)^2, (132)(jW)^3$ by $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, 0)$ and $(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, 0)$. Since all of these sectors are narrow, we have deg $P = 0$, so once again the bigrading depends solely on age $g$ and age $g^{-1}$. All of these elements appeared in the A-model as well, where we found that they all have an age of 2. The inverse of $(123)^i(jW)^j$ is $(123)^{3-i}(jW)^{4-j}$, which also has an age of 2. One can check that each conjugate will also have age 2. Thus the bigrading for all of the elements in this case is $(2 - 1, 2 - 1) = (1, 1)$. 

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As with the A-model, we now present of the basis elements in the B-model with their bigrading:

| B-model basis element | bigrading |
|------------------------|-----------|
| [1, (0, 0, 0, 0)]      | (0, 0)    |
| [x_1 x_2 x_3 x_4, (0, 0, 0, 0)] | (1, 1) |
| [x_1^2 x_2 x_3 x_4, (0, 0, 0, 0)] | (2, 2) |
| [(x_1 + x_2 + x_3)^2, (123)] | (1, 1) |
| [(x_1 + x_2 + x_3)x_4, (123)] | (1, 1) |
| [x_2^3, (123)]         | (1, 1)    |
| [(x_1 + x_2 + x_3)^2, (132)] | (1, 1) |
| [(x_1 + x_2 + x_3)x_4, (132)] | (1, 1) |
| [x_1 x_2, (132)]       | (1, 1)    |
| [1, jw]                | (0, 2)    |
| [1, (jw)^2]            | (1, 1)    |
| [1, (jw)^3]            | (2, 0)    |
| [1, (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})] | (1, 1) |
| + [1, (\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})] + [1, (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4})]| |
| [1, (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})] | (1, 1) |
| + [1, (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4})] | | |
| + [1, (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})] | | |
| + [1, (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4})] | | |
| + [1, (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})] | (1, 1) |
| + [1, (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})] + [1, (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4})] | | |
| + [1, (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})] | | |
| + [1, (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4})] | | |
| + [1, (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})] | (1, 1) |
| + [1, (123)jw] + (15 elements in conj. class) | (1, 1) |
| [1, (123)(jw)^2] + (15 elements in conj. class) | (1, 1) |
| [1, (123)(jw)^3] + (15 elements in conj. class) | (1, 1) |
| [1, (132)jw] + (15 elements in conj. class) | (1, 1) |
| [1, (132)(jw)^2] + (15 elements in conj. class) | (1, 1) |
| [1, (132)(jw)^3] + (15 elements in conj. class) | (1, 1) |

If we arrange these as a hodge diamond, we have

```
  1
  1  20  1
  1
```
5 The Mirror Map

The mirror map is an isomorphism between the A- and B-model. Thus far we have shown that both the A- and B-models have 24 basis elements with the same amount of elements for each bigrading. While this in itself would be sufficient for claiming they are isomorphic, we aim to create a canonical map which will better demonstrate which elements on one side correspond to elements on the other. In particular the mirror map should exchange narrow and broad sectors.

We start with the part of the map is already laid out for us by matching the 4 elements on either side with unique bigrading.

| bigrading | A-model | B-model |
|-----------|---------|---------|
| (0,0)     | $[1, jW]$ | $[1, (0,0,0,0)]$ |
| (2,2)     | $[1, (jW)^3]$ | $[x_1^2 x_2^2 x_3^2 x_4^2, (0,0,0,0)]$ |
| (0,2)     | $[1, (0,0,0,0)]$ | $[1, jW]$ |
| (2,0)     | $[x_1^2 x_2^2 x_3^2 x_4^2, (0,0,0,0)]$ | $[1, (jW)^3]$ |

This illuminates 2 more corresponding elements:

| bigrading | A-model | B-model |
|-----------|---------|---------|
| (1,1)     | $[1, (jW)^2]$ | $[x_1 x_2 x_3 x_4, (0,0,0,0)]$ |
| (1,1)     | $[x_1 x_2 x_3 x_4, (0,0,0,0)]$ | $[1, (jW)^2]$ |

A nice generalization of the six element maps above can be seen by

$$[1, (jW)^i] \leftrightarrow [x_1^{i-1} x_2^{i-1} x_3^{i-1} x_4^{i-1}, (0,0,0,0)].$$

Recall the dimension of the untwisted broad sector in the A-model was 9, meaning there are still 6 remaining in a basis. These 6 elements map to the 6 narrow sectors in the B-model which have a conjugacy class of size 3. Specifically, we map the element on the A-side whose polynomial has the same permutation structure as the group elements on the B-side. One explicit example is given by mapping the A-model element

$$[x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2, (0,0,0,0)]$$

to the B-model element

$$[1, (\frac{3}{4}, \frac{3}{4}, \frac{1}{4})] + [1, (\frac{3}{4}, \frac{1}{4}, \frac{3}{4})] + [1, (\frac{1}{4}, \frac{3}{4}, \frac{3}{4})].$$

All 6 are given below, although the entire conjugacy class is not written out for the B-model elements. The bigrading is also left out, but all of the following
elements have a bigrading of $(1,1)$.

| A-model | B-model |
|---------|---------|
| $x_1^2 x_2^2 + x_1 x_3^2 + x_2^2 x_3^2, (0,0,0,0)$ | $[1, (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})] + (2 \text{ others})$ |
| $x_1 x_2 x_3 + x_1^2 x_2 x_3 + x_1 x_2 x_3^2, (0,0,0,0)$ | $[1, (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})] + (2 \text{ others})$ |
| $x_1 x_2 x_3^2 + x_1 x_2 x_3 + x_2 x_3 x_4, (0,0,0,0)$ | $[1, (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})] + (2 \text{ others})$ |
| $x_1 x_2^2 x_4 + x_2 x_3 x_4 + x_1^2 x_3 x_4, (0,0,0,0)$ | $[1, (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})] + (2 \text{ others})$ |
| $x_1^2 x_4 + x_2^2 x_4 + x_3^2 x_4, (0,0,0,0)$ | $[1, (\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})] + (2 \text{ others})$ |

There are now 12 basis elements left to be mapped in both models, with 6 being twisted broad sectors from $g = (123)$ or $g = (132)$ and 6 being narrow sectors, where $g$ is a product of a permutation and a power of $j_W$. While they all have the same grading, the mirror map should map broad sectors to narrow sectors and narrow sectors to broad sectors, so we will do the same here.

| A-model | B-model |
|---------|---------|
| $[(x_1 + x_2 + x_3)^2, (123)]$ | $[1, (123)j_W] + (15 \text{ others})$ |
| $[(x_1 + x_2 + x_3)x_4, (123)]$ | $[1, (123)(j_W)^2] + (15 \text{ others})$ |
| $[(x_4)^2, (123)]$ | $[1, (123)(j_W)^3] + (15 \text{ others})$ |
| $[(x_1 + x_2 + x_3)^2, (132)]$ | $[1, (132)j_W] + (15 \text{ others})$ |
| $[(x_1 + x_2 + x_3)x_4, (132)]$ | $[1, (132)(j_W)^2] + (15 \text{ others})$ |
| $[(x_4)^2, (132)]$ | $[1, (132)(j_W)^3] + (15 \text{ others})$ |

| $[1, (123)j_W]$ | $[(x_1 + x_2 + x_3)^2, (123)]$ |
| $[1, (123)(j_W)^2]$ | $[(x_1 + x_2 + x_3)x_4, (123)]$ |
| $[1, (123)(j_W)^3]$ | $[(x_4)^2, (123)]$ |
| $[1, (132)j_W]$ | $[(x_1 + x_2 + x_3)^2, (132)]$ |
| $[1, (132)(j_W)^2]$ | $[(x_1 + x_2 + x_3)x_4, (132)]$ |
| $[1, (132)(j_W)^3]$ | $[(x_4)^2, (132)]$ |

This completes the mirror map, and we have explicitly shown that as graded vector spaces we have

$$A_{W,G} \cong B_{Wj_G},$$

as bigraded vector spaces.

6 Another Example

While the previous example was a great starting place, the mirror map left a bit to be desired given that 20 of the 24 basis elements had the same bigrading of
(1,1). Moving up to a higher degree polynomial will create A- and B-models with larger bases and more variety in their bigrading, illuminating a clearer picture of the mirror map.

Let \( W = x_1^5 + x_2^3 + x_3^3 + x_4^2 + x_5^2 \) and \( G = \langle j_W, (12)(34), (13)(24) \rangle \), where

\[
(12)(34) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
(13)(24) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Then \( W^T = W \) and the non-Abelian dual group of \( G \) is

\[ G^* = \langle (12)(34), (13)(24) \rangle \cdot SL_W^{\text{diag}}, \]

where \( SL_W^{\text{diag}} = \langle j_W, (0, \frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{1}{5}), (0, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}), (0, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}) \rangle. \)

As with the previous example, the goal is to show that \( A_{W,G} \cong B_{W^T,G^*} \) as graded vector spaces. While the work for this example is a bit more tedious, it follows the same recipe as the previous example, so we leave the details to the reader and simply provide the mirror map.

The first eight elements listed below follow the mirror map described by Krawitz [7].

| Bigrading | A-model     | B-model     |
|-----------|-------------|-------------|
| (0,0)     | \([1, j_W]\) | \([1, (0,0,0,0,0)]\) |
| (1,1)     | \([1, (j_W)^2]\) | \([x_1 x_2 x_3 x_4, (0,0,0,0,0)]\) |
| (2,2)     | \([1, (j_W)^3]\) | \([x_1^2 x_2^2 x_3^2 x_4^2, (0,0,0,0,0)]\) |
| (3,3)     | \([1, (j_W)^4]\) | \([x_1^3 x_2^3 x_3^3 x_4^3, (0,0,0,0,0)]\) |
| (0,3)     | \([1, (0,0,0,0,0)]\) | \([1, j_W]\) |
| (1,2)     | \([x_1 x_2 x_3 x_4, (0,0,0,0,0)]\) | \([1, (j_W)^2]\) |
| (2,1)     | \([x_1^2 x_2^2 x_3 x_4^2, (0,0,0,0,0)]\) | \([1, (j_W)^3]\) |
| (3,0)     | \([x_1^3 x_2^3 x_3^3 x_4^3, (0,0,0,0,0)]\) | \([1, (j_W)^4]\) |
The following 28 corresponding elements have a bigrading of $(1, 2)$. On the A side these come from untwisted sectors, and on the B side these are from the narrow sectors. This again follows the patter described in Section 3.

| A-model                                      | B-model                                      |
|----------------------------------------------|----------------------------------------------|
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_1^3x_4^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
| $[x_1^2x_2^2 + x_1^3x_3^2 + x_2^3x_1^2, (0,0,0,0)]$ | $[1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3$ $\text{others})$ |
The following 28 corresponding elements have a bigrading of \((2, 1)\). Again the basis elements on the A side come from the untwisted sector and the elements on the B side are narrow.

| A-model | B-model |
|---------|---------|
| \(x_1^2 x_2^2 x_4^2 + x_1^2 x_2^2 x_3^2 + x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_1^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_5^2 + x_1^2 x_2^2 x_4^2 + x_1^2 x_2^2 x_5^3 + x_1^2 x_2^2 x_5 x_2^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_3^3 + x_1^2 x_2^2 x_3^2 + x_1^2 x_2^2 x_3 x_2^2 + x_1^2 x_2^2 x_3 x_1^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_2^3 + x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2 x_2^2 + x_1^2 x_2^2 x_2 x_1^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_1^2 x_2^2 + x_1^2 x_2^2 x_1^2 x_1^2 + x_1^2 x_2^2 x_2^2 x_2^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_1^2 x_2^2 + x_1^2 x_2^2 x_1^2 x_1^2 + x_1^2 x_2^2 x_2^2 x_1^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_1^2 x_2^2 + x_1^2 x_2^2 x_2^2 x_2^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2 x_2^2 + x_1^2 x_2^2 x_2 x_1^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_1^2 x_2^2 + x_1^2 x_2^2 x_1^2 x_1^2 + x_1^2 x_2^2 x_2^2 x_2^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2 x_2^2 + x_1^2 x_2^2 x_2 x_1^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_1^2 x_2^2 + x_1^2 x_2^2 x_1^2 x_1^2 + x_1^2 x_2^2 x_2^2 x_2^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2 x_2^2 + x_1^2 x_2^2 x_2 x_1^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_1^2 x_2^2 + x_1^2 x_2^2 x_1^2 x_1^2 + x_1^2 x_2^2 x_2^2 x_2^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2^2 + x_1^2 x_2^2 x_2 x_2^2 + x_1^2 x_2^2 x_2 x_1^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
| \(x_1^2 x_2^2 x_1^2 x_2^2 + x_1^2 x_2^2 x_1^2 x_1^2 + x_1^2 x_2^2 x_2^2 x_2^2, (0, 0, 0, 0)\) | \([1, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})] + (3 \text{ others})\) |
In the previous example we stated that there was a correspondence between broad sectors and narrow sectors. We can see it more clearly in this example below since each A-model element needs to match with a B-model element with the same bigrading and there is more variety in this example.

| Bigrading | A-model        | B-model                                                      |
|-----------|----------------|--------------------------------------------------------------|
| (1, 1)    | $[1,((12)(34))]_{jw}$ | $[(x_1 + x_2)(x_3 + x_4), (12)(34)]$                          |
| (2, 2)    | $[1,((12)(34))_{jw}(jw)^2]$ | $[(x_1 + x_2)^2(x_3 + x_4)^2, (12)(34)]$                      |
| (1, 1)    | $[1,((12)(34))]_{jw}$ | $[x_2^3, (12)(34)]$                                           |
| (2, 2)    | $[1,((12)(34))_{jw}(jw)^4]$ | $[(x_1 + x_2)^3(x_3 + x_4)^3, (12)(34)]$                      |
| (1, 1)    | $[1,((13)(24))]_{jw}$ | $[(x_1 + x_3)(x_2 + x_4), (13)(24)]$                          |
| (2, 2)    | $[1,((13)(24))_{jw}(jw)^2]$ | $[(x_1 + x_3)^2(x_2 + x_4)^2, (13)(24)]$                      |
| (1, 1)    | $[1,((13)(24))]_{jw}$ | $[x_3^2, (13)(24)]$                                           |
| (2, 2)    | $[1,((13)(24))_{jw}(jw)^4]$ | $[(x_1 + x_3)^3(x_2 + x_4)^3, (13)(24)]$                      |
| (1, 1)    | $[1,((14)(23))]_{jw}$ | $[(x_1 + x_4)(x_2 + x_3), (14)(23)]$                          |
| (2, 2)    | $[1,((14)(23))_{jw}(jw)^2]$ | $[(x_1 + x_4)^2(x_2 + x_3)^2, (14)(23)]$                      |
| (1, 1)    | $[1,((14)(23))]_{jw}$ | $[x_4^2, (14)(23)]$                                           |
| (2, 2)    | $[1,((14)(23))_{jw}(jw)^4]$ | $[(x_1 + x_4)^3(x_2 + x_3)^3, (14)(23)]$                      |
| (1, 2)    | $[(x_1 + x_2)(x_3 + x_4), (12)(34)]$ | $[1,((12)(34))_{jw}], + (24 \text{ others})$                  |
| (2, 1)    | $[(x_1 + x_2)^2(x_3 + x_4)^2x_5^3, (12)(34)]$ | $[1,((12)(34))_{jw}(jw)^2], + (24 \text{ others})$           |
| (1, 2)    | $[x_2^3, (12)(34)]$ | $[1,((12)(34))_{jw}(jw)^3], + (24 \text{ others})$           |
| (2, 1)    | $[(x_1 + x_2)^3(x_3 + x_4)^3, (12)(34)]$ | $[1,((12)(34))_{jw}(jw)^4], + (24 \text{ others})$           |
| (1, 2)    | $[(x_1 + x_3)(x_2 + x_4), (13)(24)]$ | $[1,((13)(24))_{jw}], + (24 \text{ others})$                  |
| (2, 1)    | $[(x_1 + x_3)^2(x_2 + x_4)^2x_5^3, (13)(24)]$ | $[1,((13)(24))_{jw}(jw)^2], + (24 \text{ others})$           |
| (1, 2)    | $[x_3^2, (13)(24)]$ | $[1,((13)(24))_{jw}(jw)^3], + (24 \text{ others})$           |
| (2, 1)    | $[(x_1 + x_3)^3(x_2 + x_4)^3, (13)(24)]$ | $[1,((13)(24))_{jw}(jw)^4], + (24 \text{ others})$           |
| (1, 2)    | $[(x_1 + x_4)(x_2 + x_3), (14)(23)]$ | $[1,((14)(23))_{jw}], + (24 \text{ others})$                  |
| (2, 1)    | $[(x_1 + x_4)^2(x_2 + x_3)^2x_5^3, (14)(23)]$ | $[1,((14)(23))_{jw}(jw)^2], + (24 \text{ others})$           |
| (1, 2)    | $[x_4^2, (14)(23)]$ | $[1,((14)(23))_{jw}(jw)^3], + (24 \text{ others})$           |
| (2, 1)    | $[(x_1 + x_4)^3(x_2 + x_3)^3, (14)(23)]$ | $[1,((14)(23))_{jw}(jw)^4], + (24 \text{ others})$           |
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