Isoperimetric bounds for Wentzel-Laplace eigenvalues on Riemannian manifolds

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Abstract. In this paper, we investigate eigenvalues of the Wentzel-Laplace operator on a bounded domain in some Riemannian manifold. We prove asymptotically optimal estimates, according to the Weyl’s law through bounds that are given in terms of the isoperimetric ratio of the domain. Our results show that the isoperimetric ratio allows to control the entire spectrum of the Wentzel-Laplace operator in various ambient spaces.

1 Introduction

Let \( n \geq 2 \) and \( (M, g) \) be an \( n \)-dimensional Riemannian manifold. Let \( \Omega \subset M \) be a bounded domain with smooth boundary \( \Gamma \). We denote by \( \Delta \) and \( \Delta_\Gamma \) the Laplace-Beltrami operators acting on functions on \( M \) and \( \Gamma \), respectively. Notice that, in conformance with conventions in computational geometry, we define the Laplacian with negative sign, that is the negative divergence of the gradient operator. The gradient operators on \( M \) and \( \Gamma \) will be denoted by \( \nabla \) and \( \nabla_\Gamma \) respectively, the outer normal derivative on \( \Gamma \) by \( \partial_n \). Throughout the paper we denote by \( d_M \) and \( d_\Gamma \) the Riemannian volume elements of \( M \) and \( \Gamma \).

Given an arbitrary constant \( \beta \in \mathbb{R} \), consider the following eigenvalue problem on \( \Omega \):

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
\beta \Delta_\Gamma u + \partial_n u &= \lambda u \quad \text{on } \Gamma.
\end{align*}
\]  

(Wentzel Problem) (1.1)

In what follows, we will assume that \( \beta \), which we refer to as the boundary parameter, is non-negative. In this case, the Wentzel eigenvalues form a
discrete sequence that can be arranged as
\[ 0 = \lambda_{W,0}^\beta < \lambda_{W,1}^\beta \leq \lambda_{W,2}^\beta \leq \cdots \leq \lambda_{W,k}^\beta \leq \cdots \to \infty. \tag{1.2} \]
We adopt the convention that each eigenvalue is repeated according to its multiplicity.

The boundary condition in (1.2), which we call Wentzel boundary condition, was initially introduced in [15], in order to find the most general boundary conditions for which the associated operator generates a Markovian semigroup. It is often considered in a more general form cf.[6, (1.2)],[7, (2.32)]. Or sometimes, it subordinates the heat equation as in [11, (1.3)] see also [6]. A good discussion on motivations and the physical interpretation of Wentzel boundary conditions can be found in [10].

The present paper we use valuable tools to find bounds in terms of geometric quantities in order to estimate Wentzel eigenvalues. These bounds are optimal according to the asymptotic behaviour of the eigenvalues given by the Weyl law (2.8).

The eigenvalue problem of the Laplacian with Wentzel boundary condition has only recently been significantly investigated. When \( \beta = 0 \), the eigenvalue problem (1.1) reduced to the so called Steklov eigenvalue problem. An advanced reference providing an overview on the Steklov problem, is [9]. As in [7], the problem (1.1) can be viewed as a perturbed (as opposite to unperturbed when \( \beta = 0 \)) Steklov problem.

The most relevant works on bounds for eigenvalues of the Wentzel-Laplace operator have been done in [4, 17, 5]. Dambrine, Kateb and Lamboley [4] obtained a first upper bound for the first non-trivial eigenvalue \( \lambda_{W,1}^\beta \) in terms of purely geometric quantities if \( \Omega \) is a bounded domain in \( \mathbb{R}^n \):

Let \( \wedge(\Omega) \) denote the spectral radius of the matrix
\[
P(\Omega) \overset{\text{def}}{=} \left( \int_{\Gamma} \delta_{ij} - n_i n_j d\Gamma \right)_{i,j=1,\ldots,n}.
\]

The following inequality holds:
\[
\lambda_{W,1}^\beta \leq \frac{\text{Vol}(\Omega) + \beta \wedge(\Omega)}{\frac{1}{n} \text{Vol}(\Omega) \left[ 1 + c_n \left( \frac{\text{Vol}(\Omega) \Delta B}{\text{Vol}(B)} \right)^2 \right]} \quad \text{with} \quad c_n := \frac{(\sqrt{2} - 1)(n + 1)}{4n}.
\tag{1.3}
\]

Here, \( B \) is the ball having the same volume as \( M \) and with the same center of mass than \( \Gamma \) and \( \omega_n \) denotes the volume of the \( n \)-dimensional Euclidean unit ball. Equality holds in (1.3) if \( M \) is a ball. In [17], Wang and Xia proved the following bound for the same eigenvalue:
\[
\lambda_{W,1}^\beta \leq \frac{n \text{Vol}(\Omega) + \beta(n - 1) \text{Vol}(\Gamma)}{n \text{Vol}(\Omega) (\text{Vol}(\Omega) \omega_n^{-1})^{\frac{1}{n}}}.
\tag{1.4}
\]

They also present a bound for \( \lambda_{W,1}^\beta \) in non-Euclidean case, when the Ricci curvature of \( M \) and the principle curvatures of \( \Gamma \) are bounded. Going further, Du-Wang-Xia provided the following isoperimetric bound for the first \( n(n}$
being the dimension) eigenvalues, when $M$ is immersed in an Euclidean space $\mathbb{R}^N$ equipped with the canonical Euclidean metric. If $H$ is the mean curvature vector field of $\Gamma$ in $\mathbb{R}^N$ then one has

$$\frac{1}{n-1} \sum_{j=1}^{n-1} \lambda_{W,j}^\beta \leq \sqrt{\left[n \text{Vol}(M) + (n-1)\beta \text{Vol}(\Gamma)\right]} \int_{\Gamma} |H|^2 d\Gamma \over \text{Vol}(\Gamma).$$

(1.5)

When $N = n$, that is, $M$ is a bounded domain of $\mathbb{R}^N$, then equality holds in (1.5) if and only if $M$ is a ball.

The aim of this work is to go even further and provide uniform isoperimetric bounds for all the eigenvalues of (1.1). If $\Omega$ is a domain of an $n$-dimensional complete Riemannian manifold $(M, g)$, with boundary $\Gamma$, the isoperimetric ratio of $\Omega$ is defined by $I(\Omega) := \frac{\text{Vol}(\Gamma)}{\text{Vol}(\Omega)}$. In the numerator $\text{Vol}$ stands for the $(n-1)$-Riemannian volume and for the $n$-Riemannian volume from $g$ in the denominator.

Our first result provides an upper bound in the case of Euclidean domains. We respectively denote $\omega_n$ and $\rho_{n-1} = n \omega_n$ the volumes of the unit ball and the unit sphere in the $n$-dimensional Euclidean space.

**Theorem 1.1.** Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded euclidean domain with smooth boundary $\Gamma$. Then, for every $k \geq 1$, one has

$$\lambda_{W,k}^\beta(\Omega) \leq \zeta_1(n) \left(\frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)}\right)^{1-\frac{2}{n}} \left(\frac{k}{\text{Vol}(\Gamma)}\right)^{\frac{2}{n}} + \zeta_2(n) \left[I(\Omega)^{1+\frac{2}{n-1}} \left[\frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta\right]\right] \left(\frac{k}{\text{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} , (1.6)$$

where $\zeta_1(n) := 2^{10(n+1)} \omega_n^{\frac{2}{n}}$ and $\zeta_2(n) := 2^{10(n+3)} \omega_n^{\frac{1}{n}}$.

**Corollary 1.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded euclidean domain of dimension $n \geq 3$ with smooth boundary $\Gamma$. Then, for every $k \geq 1$, we have

$$\lambda_{W,k}^\beta(\Omega) \leq C_1(\Omega, \beta) + C_2(\Omega, \beta) \left(\frac{k}{\text{Vol}(\Gamma)}\right)^{\frac{2}{n-1}} . (1.7)$$

Here $C_1(\Omega, \beta)$ and $C_2(\Omega, \beta)$ are geometric constants given by:

$$C_2(\Omega, \beta) = \zeta_2(n) I(\Omega)^{1+\frac{2}{n-1}} \left[\frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta\right] + 1$$

$$C_1(\Omega, \beta) = \zeta_1(n) \left(\frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)}\right)^{n-2} C_2(\Omega, \beta).$$

The constants $\zeta_1(n)$ and $\zeta_2(n)$ are the same as in Theorem 1.1.

For bounded domains in Riemannian manifold with Ricci curvature bounded from below, we have an isoperimetric upper bound, which also depends on the infimum isoperimetric ratio that we define as follows:
Definition 1.3. Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \leq 2\) and \(\Omega\) a bounded domain in \(M\). The infimum isoperimetric ratio of \(\Omega\) is the quantity \(I_0(\Omega) := \inf\{I(U) : U \text{ open set in } \Omega\}\). In particular, if \(\Omega\) is an Euclidean domain, one has \(I_0(\Omega) = I_0(\mathbb{R}^n) = n\omega_1^n\).

Theorem 1.4. Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 3\) with non-negative Ricci curvature. Let \(\Omega \subset M\) be a bounded domain with smooth boundary \(\Gamma\). Then for every \(k \geq 1\), we have
\[
\lambda^\beta_{W,k}(\Omega) \leq c_1(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n}} + c_2(n) \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}},
\] where \(c_1(n) := 2^{10(n+1)}\omega_n^\frac{2}{n}\) and \(c_2(n) := 2^{5(n+5)}\rho_n^\frac{2}{n-1}\).

Remark 1.5. It is not usually simple to gauge this quantity \(I_0(\Omega)\). It is not easy to determine the best constant in the isoperimetric inequality for domains in many complete Riemannian manifolds. For example, as we see in Corollary 1.7, the longstanding conjecture known as the Cartan-Hadamard conjecture, is about sharp isoperimetric inequalities in complete Riemannian manifolds with negative sectional curvature.

Theorem 1.6. Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 3\) with Ricci curvature bounded from below by \(-(n-1)\kappa^2, \kappa \in \mathbb{R}_{>0}\). Let \(\Omega \subset M\) be a bounded domain with smooth boundary \(\Gamma\). Then for every \(k \geq 1\), we have
\[
\lambda^\beta_{W,k}(\Omega) \leq A(\Omega, \beta) + B(\Omega, \beta) \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}},
\] where
\[
A(\Omega, \beta) = \kappa^2 \zeta(n) \left\{ 1 + \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} + \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \right\},
\]
\[
B(\Omega, \beta) = \zeta(n) \left\{ 1 + \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \right\},
\]
\(\zeta(n)\) being a constant depending only on the dimension \(n\).

Theorems 1.4 and 1.6 emanate from a generic result (Theorem 3.6) that we prove in Section 3.

Besides the Euclidean case, we have at least one other situation where we know something about the quantity \(I_0(\Omega)\). The so called Cartan-Hadamard conjecture, proved in dimensions \(n = 2\) by Weil [16], \(n = 3\) by Kleiner [12] and \(n = 4\) by Croke [3], states that any bounded domain in a smooth Cartan-Hadamard manifold of dimension \(n\) satisfies
\[
I(\Omega) \geq C(n)
\]
where $C(n)$ is a dimensional constant. Very recently, Ghomi and Spruck (2019) in [8] proposed a solution in all dimensions. This leads to the following corollary.

**Corollary 1.7.** Let $(M, g)$ be a smooth Cartan-Hadamard manifold of dimension $n \geq 3$ with Ricci curvature bounded from below by $-(n-1)\kappa^2$, $\kappa \in \mathbb{R}_{>0}$ and $\Omega \subset M$ a bounded domain with smooth boundary $\Gamma$. Then for every $k \geq 1$, we have

$$\lambda_{W,k}^\beta(\Omega) \leq A(\Omega, \beta) + B(\Omega, \beta) \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}},$$

(1.10)

where

$$A(\Omega, \beta) = \kappa^2 \zeta(n) \left\{ 1 + \left( \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{\frac{1}{n}} + I(\Omega)^{1+\frac{2}{n-1}} \left[ \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \right\},$$

$$B(\Omega, \beta) = \zeta(n) \left\{ 1 + I(\Omega)^{1+\frac{2}{n-1}} \left[ \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \right\},$$

$\zeta(n)$ being a constant depending on the dimension $n$.

**Plan of the paper.** The proofs of Theorems 1.1 1.4 and 1.6 are presented in Section 4. We present the proof of Theorem 3.6 in Section 3, following some technical results. We devote Section 2 to briefly summarize properties of the Wentzel-Laplace eigenvalues.

## 2 Wentzel-Laplace operator and functional framework

Consider the map $\wedge : L^2(\Gamma) \rightarrow L^2(\Omega)$ related to the Dirichlet problem

$$\begin{cases}
\Delta u = 0 \quad \text{in } \Omega, \\
u|_{\Gamma} = f \quad \text{on } \Gamma,
\end{cases}$$

(2.1)

which associate to any $f \in L^2(\Gamma)$ its harmonic extension i.e. the unique function $u$ in $L^2(\Omega)$ satisfying (2.1). This map is well defined from $L^2(\Gamma)$ (respectively, $H^{\frac{s}{2}}(\Gamma)$) to $L^2(\Omega)$ (respectively, $H^{1}(\Omega)$). See [14, p. 320, Prop 1.7], for more details. By $H^{s}(\Omega)$ and $H^{s}(\Gamma)$, we denote the Sobolev spaces of order $s$ on $\Omega$ and $\Gamma$, respectively, and $u|_{\Gamma} \in H^{\frac{s}{2}}(\Gamma)$ stands for the trace of $u \in H^{1}(\Omega)$ at the boundary $\Gamma$. This will also be denoted simply by $u$, if no ambiguity can result.

Then the Dirichlet-to-Neumann operator is defined by

$$N_{D}: H^{\frac{s}{2}}(\Gamma) \rightarrow H^{-\frac{s}{2}}(\Gamma)$$

$$f \mapsto \partial_{n}(\wedge f).$$

(2.2)

Again $\partial_{n}$ stands for the normal derivative at the boundary $\Gamma$ of $\Omega$ with unit normal vector $n$ pointing outwards.
For all $u \in H^{1/2}(\Gamma)$, we define the operators $B_0 = N_D$ (in the operator sense). For $\beta > 0$, we define for all $u \in H^1(\Gamma)$ $C_\beta u \overset{\text{def}}{=} \beta \Delta u$ and $B_\beta \overset{\text{def}}{=} B_0 + C_\beta$. The eigenvalue sequence $\{\lambda_{W,k}^\beta\}_{k=0}^\infty$ given in (1.2) can be interpreted as the spectrum associated with the operator $B_\beta$ and is subject to the following min-max characterization (see e.g., [13, Thm1.2] and [7, (2.33)]):

**Min-max principle.** Let $\mathfrak{V}(k)$ denotes the set of all $k$-dimensional subspaces of $\mathfrak{V}_\beta$ which is defined by

\[
\mathfrak{V}_0 \overset{\text{def}}{=} \{ (u, u_\Gamma) \in H^1(\Omega) \times H^{1/2}(\Gamma) : u_\Gamma = u|_\Gamma \}, \\
\mathfrak{V}_\beta \overset{\text{def}}{=} \{ (u, u_\Gamma) \in H^1(\Omega) \times H^1(\Gamma) : u_\Gamma = u|_\Gamma \}, \quad \beta > 0.
\]

Of course, for all $\beta > 0$, we have $\mathfrak{V}_\beta \subset \mathfrak{V}_0$. For every $k \in \mathbb{N}$, the $k$Th eigenvalue of the Wentzel-Laplace operator $B_\beta$ satisfies

\[
\lambda_{W,k}^\beta = \min_{V \in \mathfrak{V}(k) \setminus \{0\}} \max_{u \in V} R_\beta(u), \quad k \geq 0,
\]

where $R_\beta(u)$, the Rayleigh quotient for $B_\beta$, is given by

\[
R_\beta(u) \overset{\text{def}}{=} \frac{\int_\Omega |\nabla u|^2 dM + \beta \int_\Gamma |\nabla u_\Gamma|^2 d\Gamma}{\int_\Gamma u_\Gamma^2 d\Gamma}, \quad \text{for all } u \in \mathfrak{V}_\beta \setminus \{0\}.
\]

**Asymptotic behaviour.** The eigenvalues for the Dirichlet-to-Neumann map $B_0 = N_D$ are those of the well known Steklov problem.

\[
\begin{cases}
\Delta u = 0, & \text{dans } \Omega, \\
\partial_n u = \lambda^S u, & \text{sur } \Gamma.
\end{cases}
\]

A good discussion of this problem can be found in [9]. The Steklov eigenvalues are then $\{\lambda_{W,k}^0\}_{k=0}^\infty$ which we shall denote equivalently as $\{\lambda_{S,k}\}_{k=0}^\infty$. They behave according to the following asymptotic formula:

\[
\lambda_{S,k} = C_n k^{\frac{1}{n-1}} + O(k^{\frac{1}{n-1}}), \quad k \to \infty.
\]

where $C_n = \frac{2\pi}{(\omega_{n-1} \text{Vol}(\Gamma))^{\frac{1}{n-1}}}$, The reader can refer to [13, section 4]. For $\beta > 0$, the Weyl asymptotic for $\lambda_{W,k}^\beta$ can be deduced directly from properties of perturbed forms using the asymptotic behaviour of the spectrum of $C_\beta$ by Hörmander:

\[
\lambda_{C_\beta,k} = \beta C_n^2 k^{\frac{2}{n-1}} + O(k^{\frac{2}{n-1}}), \quad k \to \infty.
\]

The Weyl law for eigenvalues on problem (1.1) reads

\[
\lambda_{W,k}^\beta = \beta C_n^2 k^{\frac{2}{n-1}} + O(k^{\frac{2}{n-1}}), \quad k \to \infty.
\]

A complete and detailed discussion about the spectral properties of the Wentzel Laplacian can be found in [7, Section 2] and for the asymptotic in (2.8), the reader can refer to [7, Prop 2.7 and (2.37)].
3 General inequality

In this section, we establish some needed technical results and the major result in this paper used to prove our main theorems. Let \( n \geq 2 \) and \((M, g)\) be an \( n\)-dimensional Riemannian manifold. Let \( \Omega \subset M \) a bounded domain with smooth boundary \( \Gamma \). Let \( r \in \mathbb{R}_{>0} \), we denote by \( B(x, r) = \{ p \in M, d(x, p) < r \} \) the metric ball of radius \( r \) centered at \( x \in M \), where \( d \) is the Riemannian distance associated to the metric \( g \). We assume \( \Gamma \) satisfies the following hypothesis:

\((H_0):\) There exists a radius \( r_-(\Gamma) > 0 \) and a constant \( C \in \mathbb{N}_{>1} \) such that for all \( x \in \Gamma \) and \( r < r_-(\Gamma) \), one has

\[
\Vol(B(x, r)) < C\omega_n r^n \quad \text{and} \quad \Vol(\partial B(x, r)) < C\rho_{n-1} r^{n-1}.
\]

(3.1)

Here \( \partial B(x, r) \) denotes the geodesic sphere of radius \( r \) centered at \( x \).

Lemma 3.1. Let \((M, g)\), \( \Omega \) and \( \Gamma \) be as above. For every \( K \in \mathbb{N} \), let \( r_K \) be an associated “maximal” radius defined by

\[
r_K := \left( \frac{\Vol(\Omega)}{2} \right)^{\frac{1}{n}} \left( \frac{I_0(\Omega)}{KC\rho_{n-1}} \right)^{\frac{1}{n-1}}.
\]

(3.2)

Let \( \{x_j\}_{j=1}^K \) be an arbitrary set of points in \( \Gamma \). Then for every \( r > 0 \) satisfying both \( r < \frac{1}{2}r_-(\Omega) \) and \( r \leq \frac{1}{2}r_K \), one has

\[
\Vol\left( \Gamma \setminus \bigcup_{j=1}^K B(x_j, 2r) \right) > 0.
\]

(3.3)

Proof. We denote by \( \Omega_0 \) (respectively \( \Gamma_0 \)) the subset \( \Omega \setminus \bigcup_{j=1}^K B(x_j, 2r) \) (respectively \( \Gamma \setminus \bigcup_{j=1}^K B(x_j, 2r) \)). One can think of \( \overline{\Omega}_0 \) as a holed cheese. Since the boundary of \( \Omega_0 \), that we denote by \( \partial \Omega_0 \), is contained in the union \( \Gamma_0 \setminus \left( \bigcup_{j=1}^K \Vol(\partial B(x_j, 2r)) \right) \), one has

\[
\Vol(\Gamma_0) \geq \Vol(\partial \Omega_0) - \sum_{j=1}^K \Vol(\partial B(x_j, 2r))
\]

\[
= I(\Omega_0) \Vol(\Omega_0) \omega_n^{n-1} - \sum_{j=1}^K \Vol(\partial B(x_j, 2r)),
\]

where \( \partial B(x_j, 2r) = \{ p \in M \mid d(x_j, p) = 2r \} \).

Then, since \( 2r < r_-(\Gamma) \), one has

\[
\Vol(\Gamma_0) + KC\rho_{n-1}(2r)^{n-1} > I(\Omega_0) \Vol(\Omega_0) \omega_n^{n-1}.
\]

(3.4)

Now, we assume that

\[
I_0(\Omega) \omega_n^{n-1} \Vol(\Omega) - KC\rho_{n-1}^{n-1}(2r)^n \geq 0.
\]

(3.5)
Noticing that $I_0(\Omega) \leq I_0(\mathbb{R}^n)$, we have then
\[
I(\Omega_0) \text{Vol}(\Omega_0)^{\frac{n-1}{n}} > I(\Omega_0)[\text{Vol}(\Omega) - KC\omega_n(2r)^n]^{\frac{n-1}{n}}
\]
\[
\geq [I_0(\Omega)^{\frac{n}{n-1}} \text{Vol}(\Omega) - KC\rho_{n-1}^{\frac{n}{n-1}}(2r)^n]^{\frac{n-1}{n}}.
\]
Replacing in (3.4), this leads to the following inequality:
\[
\text{Vol}(\Gamma_0) > [I_0(\Omega)^{\frac{n}{n-1}} \text{Vol}(\Omega) - KC\rho_{n-1}^{\frac{n}{n-1}}(2r)^n]^{\frac{n-1}{n}} - KC\rho_{n-1}(2r)^{n-1}. \quad (3.6)
\]
The right hand side is non-negative if
\[
\left(\frac{I_0(\Omega)}{KC\rho_{n-1}}\right)^{\frac{n}{n-1}} \text{Vol}(\Omega) \geq (2r)^n. \quad (3.7)
\]
We notice that
\[
\frac{1}{(KC)^{\frac{1}{n-1}} + 1} \geq \frac{1}{2}. \quad (3.7)
\]
Inequality (3.7) is then satisfied whenever
\[
r \leq \frac{1}{2} \left(\frac{I_0(\Omega)}{KC\rho_{n-1}}\right)^{\frac{1}{n-1}} \left(\frac{\text{Vol}(\Omega)}{2}\right)^{\frac{1}{n}} = \frac{2}{2} (KC)^{\frac{1}{n(n-1)}} r_K. \quad (3.8)
\]
This implies that $\text{Vol}(\Gamma_0) > 0$, under the assumption in (3.5). However, (3.5) can be written as
\[
r \leq \frac{1}{2} \left(\frac{\text{Vol}(\Omega)}{KC}\right)^{\frac{1}{n}} \left(\frac{I_0(\Omega)}{\rho_{n-1}}\right)^{\frac{1}{n-1}} = \frac{2}{2} (KC)^{\frac{1}{n(n-1)}} r_K. \quad (3.8)
\]
Since $1 \leq 2^\frac{1}{n} (KC)^{\frac{1}{n(n-1)}}$, (3.8) is satisfied by assumption. This ends the proof.

**Lemma 3.2.** Let the assumptions of Lemma 3.1 be fulfilled. We define
\[
K_0 := \left[\frac{I_0(\Omega)}{C\rho_{n-1} - \rho(\Gamma)}^{\frac{n}{n-1}} \left(\frac{\text{Vol}(\Omega)}{2}\right)^{\frac{1}{n-1}}\right] + 1, \quad (3.9)
\]
where \(\left\lfloor \cdot \right\rfloor\) denotes the floor function, so that $r_K < r(\Gamma)$ if $K \geq K_0$. Let \(\{x_j\}_{j=1}^K\) be an arbitrary set of points in $\Gamma$. Then, for every $K \geq K_0$ and $0 < r \leq \frac{1}{16} r_k$, we have
\[
\text{Vol} \left(\Gamma \setminus \bigcup_{j=1}^K B(x_j, 2r)\right) > \left(\frac{r}{r_K}\right)^{\frac{n-1}{n}} I_0(\Omega) \text{Vol}(\Omega)^{\frac{n-1}{n}}. \quad (3.10)
\]
**Proof.** From (3.6) in the proof of Lemma 3.1, one has
\[
\text{Vol}(\Gamma_0) > [I_0(\Omega)^{\frac{n}{n-1}} \text{Vol}(\Omega) - KC\rho_{n-1}^{\frac{n}{n-1}}(2r)^n]^{\frac{n-1}{n}} - KC\rho_{n-1}(2r)^{n-1}.
\]
Setting $\alpha := \frac{r_K}{r}$ (we notice that $\alpha \geq 2^4$ since $r \leq \frac{1}{16} r_k$), we have
\[
(2r)^n = \left(\frac{2}{\alpha} r_K\right)^n = \left(\frac{2}{\alpha}\right)^n \text{Vol}(\Omega)\left(\frac{I_0(\Omega)}{KC\rho_{n-1}}\right)^{\frac{n}{n-1}}
\]
\[
\leq \frac{1}{KC\rho_{n-1}} \left(\frac{2\rho_{n-1}}{\alpha}\right)^n I_0(\Omega) \text{Vol}(\Omega), \quad (3.11)
\]
where we have used that $KC \geq 1$. On the other hand,
\[(2r)^{n-1} = \left(\frac{2}{\alpha}r_K\right)^{n-1} = \left(\frac{2}{\alpha}\right)^{n-1} \left(\frac{\text{Vol}(\Omega)}{2}\right)^{\frac{n-1}{n}} \frac{I_0(\Omega)}{KC\rho_{n-1}} \leq \frac{1}{KC\rho_{n-1}} \left(\frac{2}{\alpha}\right)^{n-1} I_0(\Omega)^{\frac{n}{n-1}} \text{Vol}(\Omega). \quad (3.12)\]

From inequalities (3.11) and (3.12), we get
\[
\text{Vol}\left(\Gamma \setminus \bigcup_{j=1}^{K} B(x_j, 2r)\right) > \left[\left(1 - \left(\frac{2}{\alpha}\right)^{\frac{n-1}{n}}\right)^{n-1} - \left(\frac{2}{\alpha}\right)^{\frac{n-1}{n}}\right] I_0(\Omega)\text{Vol}(\Omega)^{\frac{n-1}{n}}. \quad (3.13)
\]

We notice that, since $\alpha > 2$,
\[
\left(1 - \left(\frac{2}{\alpha}\right)^{\frac{n-1}{n}}\right)^{n-1} - \left(\frac{2}{\alpha}\right)^{\frac{n-1}{n}} \geq (1 - \alpha^{-1})^{\frac{n-1}{n}} - \alpha^{-\frac{n-1}{n}} = \alpha^{-\frac{n-1}{n}} \left[(\alpha - 1)^{\frac{n-1}{n}} - 1\right] \geq \alpha^{-\frac{n-1}{n}} [15^{\frac{n-1}{n}} - 1].
\]

It follows that
\[
\text{Vol}\left(\Gamma \setminus \bigcup_{j=1}^{K} B(x_j, 2r)\right) > \alpha^{-\frac{n-1}{n}} I_0(\Omega)\text{Vol}(\Omega)^{\frac{n-1}{n}},
\]
since $15^{\frac{n-1}{n}} \geq 2$ for every $n \geq 2$. □

Let $(M, g)$, $\Omega$ and $\Gamma$ be as described above and $r \in \mathbb{R}_{>0}$. The external covering number $N_{r}^{\text{ext}}(\Gamma)$ of $\Gamma$ in $M$ with respect to $r$ is defined as the fewest number of points $x_1, \ldots, x_N \in M$ such that the balls $B(x_1, r), \ldots, B(x_N, r)$ cover $\Gamma$. Lemmas 3.1 and 3.2 imply the following principal lemma.

**Lemma 3.3.** Let $n \geq 2$ and $(M, g)$ be an $n$-dimensional Riemannian manifold. Let $\Omega \subset M$ a bounded domain with smooth boundary $\Gamma$. Then for every $K \geq K_0$ and $0 < r \leq \frac{1}{2}r_K$,

i. $K < N_{r}^{\text{ext}}(\Gamma)$.

ii. If in addition $r \leq \frac{1}{16}r_K$ then for every arbitrary set of points $\{x_j\}_{j=1}^{K}$ in $M$, one has
\[
\text{Vol}(\Gamma \setminus \bigcup_{j=1}^{K} B(x_j, r)) > \left(\frac{r}{r_K}\right)^{\frac{n-1}{n}} I_0(\Omega)\text{Vol}(\Omega)^{\frac{n-1}{n}}. \quad (3.14)
\]
Proof. Suppose $N_r^{ext}(\Gamma) \leq K$ and let $\{B(x_j, r)\}_{j=1}^{N_r^{ext}(\Gamma)}$ be a minimal covering of $\Gamma$. By the minimality assumption, every $B(x_j, r)$ intersects $\Gamma$. For $j \in \{1, \ldots, N_r^{ext}(\Gamma)\}$, let $x_j' \in B(x_j, r) \cap \Gamma$, one has

$$B(x_j, r) \subset B(x_j', 2r), \quad \text{for every } i \in \{1, \ldots, N_r^{ext}(\Gamma)\}.$$ 

This implies

$$\text{Vol} \left( \Gamma \setminus \bigcup_{j=1}^{N_r^{ext}(\Gamma)} B(x_j', 2r) \right) \leq \text{Vol} \left( \Gamma \setminus \bigcup_{j=1}^{N_r^{ext}(\Gamma)} B(x_j, r) \right).$$

We complete the family $\{B(x_j', 2r)\}_{j=1}^{N_r^{ext}(\Gamma)}$ to $\{B(x_j', 2r)\}_{j=1}^{K}$ by setting $x_j := x_1'$ for $N_r^{ext}(\Gamma) < j \leq K$. Then, applying Lemma 3.1, we have

$$\text{Vol}(\Gamma \setminus \bigcup_{j=1}^{N_r^{ext}(\Gamma)} B(x_j, r)) \geq \text{Vol}(\Gamma \setminus \bigcup_{j=1}^{N_r^{ext}(\Gamma)} B(x_j', 2r))$$

$$= \text{Vol}(\Gamma \setminus \bigcup_{j=1}^{K} B(x_j', 2r)) > 0.$$ 

Hence, it is contradictory to $\Gamma \subset \bigcup_{j=1}^{N_r^{ext}(\Gamma)} B(x_j, r)$. To prove (ii.), we notice that if $B(x_j, r) \cap \Gamma \neq \emptyset$ then $B(x_j, r) \subset B(x_j', 2r)$ with $x' \in \Gamma$. The inequality follows the applying Lemma 3.2. $\square$

The next lemma of Colbois and Maerten [2] provides the final ingredient to prove the most technical results in this paper presented in Theorems 3.5 and 3.6.

Let $(X, d)$ be a complete, locally compact metric space. Let $\varepsilon \in \mathbb{N}$ and $N : (0, \rho] \rightarrow \mathbb{N}_{\geq 2}$ an increasing function. We say that $(X, d)$ satisfies the $(N, \varepsilon)$-covering property if each ball of radius $r$ can be covered by $N(r)$ balls of radius $\frac{\rho}{\varepsilon}$. In order to simplify notation, we will write $N_r$ instead of $N(r)$.

We denominate capacitor any couple $(A, B)$ of subsets such that $\emptyset \neq A \subset B \subset X$. Two capacitors $(A_1, B_1)$ and $(A_2, B_2)$ are disjoint if $B_1 \cap B_2 = \emptyset$. A family of capacitors is a finite set of capacitors in $X$ that are pairwise disjoint.

**Lemma 3.4 (Colbois-Maerten, 2008).** Let $(X, d, \mu)$ be a complete, locally compact metric measure space satisfying the $(N, 4)$-covering property with $N : (0, \text{diam}(X)] \rightarrow \mathbb{N}_{\geq 2}$ a discrete positive function. Let $r > 0$ and $K \in \mathbb{N}$ such that for every $x \in X$, $\mu(B(x, r)) \leq \frac{\mu(X)}{2KN^2r}$. Then there exists a family of $K$ capacitors $\{(A_i, B_i)\}_{1 \leq i \leq K}$ with the following properties for $1 \leq i, j \leq K$

1. $\mu(A_i) \geq \frac{\mu(X)}{2KN^2}$,
2. $B_i = A_i^r := \{x \in X, d(x, A_i) < r\}$ is the $r$-neighbourhood of $A_i$ and $d(B_i, B_j) > 2r$ whenever $i \neq j$. 

Theorem 3.5. Let \((M, g)\) be a complete Riemannian manifold of dimension \(n \geq 2\). Let \(\Omega \subset M\) be a bounded domain whose boundary \(\Gamma\) is a smooth hypersurface satisfying \((H_0)\). We assume that \(M\), with respect to the distance associated to the metric \(g\) satisfies the \((N, 4)\)-covering property for some discrete positive function \(N\).

Then, for every integer \(k \geq \frac{1}{4}K_0\) (\(K_0\) is the same as in \((3.9)\)), one has

\[
\lambda_{W,k}^\beta(\Omega) \leq C_1 \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1 - \frac{2}{n}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n}} + C_2 \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1 + \frac{2}{n - 1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n - 1}},
\]

(3.15)

where \(C_1 = 2^8(C\omega_n)^{\frac{n}{2}} N_r^2\), \(C_2 = 2^{21}(C\rho_{n-1})^{\frac{n}{2-n}} N_r\) and \(r := \frac{1}{64}(4k)\).

Proof. The methods we use in this proof are inspired by [1]. We consider the metric measure space \((M, d, \mu)\), where \(d\) is the distance from the metric \(g\) and \(\mu\) is the Borel measure with support \(\Gamma\) defined for each Borelian \(A\) in \(M\) by

\[
\mu(A) := \int_{A \cap \Gamma} d\Gamma.
\]

Fix \(K = 4k\) and choose in \(M\) a family of points \(\{x_j\}_{j=1}^K\) satisfying

\[
\begin{cases}
B(x_j, 2r) \cap B(x_i, 2r) = \emptyset & \text{for all } 1 \leq i \neq j \leq K, \\
\mu(B(x_1, r)) \geq \mu(B(x_2, r)) \geq \ldots \geq \mu(B(x_K, r)) \geq \mu(B(x, r)),
\end{cases}
\]

(3.16)

for all \(x \in M_0 := M \setminus \bigcup_{j=1}^K B(x_j, 4r)\). This can be done inductively, selecting the point \(x_1\) such that

\[
\mu(B(x_1, r)) = \sup \{\mu(B(x, r)), x \in M\},
\]

and the points \(x_j\), for \(j = 2, \ldots, K\), such that

\[
\mu(B(x_j, r)) = \sup \{\mu(B(x, r)), x \in M \setminus \bigcup_{i=1}^{j-1} B(x_i, 4r)\}.
\]

There are two possible cases:

Case \(\mu(B(x_K, r)) \leq \frac{\mu(M)}{4K N_r^2}\). We consider the metric measure space \((M_0, d, \mu_0)\) where \(\mu_0\) is defined by

\[
\mu_0(A) := \int_{A \cap \Gamma_0} d\Gamma, \quad \Gamma_0 := \Gamma \setminus \bigcup_{i=1}^K B(x_i, 4r).
\]

for every Borelian \(A\) in \(M\). Since \(4r = \frac{1}{16}rK\), it follows from Lemma 3.3, that

\[
\mu_0(M_0) = \text{Vol}(\Gamma_0) > \frac{1}{16^{n-1}} I_0(\Omega) \text{Vol}(\Omega)^{\frac{n-1}{n}}.
\]
From (3.16) one has $\mu_0(B(x, r)) \leq \mu(B(x, r)) \leq \frac{\mu(M)}{4K\sqrt{r}}$ for every $x \in M_0$. Applying Lemma 3.4, we have a family of $K$ capacitors $\{(A_i, B_i)\}_{1 \leq i \leq K}$ with the following properties for $1 \leq i, j \leq K$:

1. $\mu_0(A_i) \geq \frac{\mu_0(M_0)}{2N_rK}$,
2. $B_i = A_i^r = \{x \in X, d(x, A_i) < r\}$ is the $r$-neighborhood of $A_i$ and $d(B_i, B_j) > 2r$ whenever $i \neq j$.

We notice that $\mu_0(M_0) = \text{Vol}(\Gamma_0)$. For each $1 \leq j \leq K$, we consider the function $\varphi_j$ supported in $A_j^r$ defined by

$$\varphi_j(x) := \begin{cases} 1 - \frac{d(A_j, x)}{r} & \forall x \in A_j^r, \\ 0 & \forall x \in M \setminus A_j^r. \end{cases}$$

It follows that $R_\beta(\varphi_j) \leq \frac{\int_{\Omega \cap A_j^r} |\nabla \varphi_j|^2 d_M + \beta \int_{\Gamma \cap A_j^r} |\nabla \varphi_j|^2 d_S}{\int_{\Gamma \cap A_j^r} \varphi_j^2 d_S}.$

i) One has

$$\int_{\Omega \cap A_j^r} |\nabla \varphi_j|^2 d_M \leq \frac{1}{r^2 \text{Vol}(\Omega \cap A_j^r)}.$$ 

The $A_j^r$’s are pairwise disjoint then $\sum_{j=1}^{4k} \text{Vol}(\Omega \cap A_j^r) \leq \text{Vol}(\Omega)$. We deduce that at least $2k$ of the $A_j^r$’s satisfy

$$\text{Vol}(\Omega \cap A_j^r) \leq \frac{\text{Vol}(\Omega)}{k}. \tag{3.17}$$

Up to re-ordering, we assume that for the first $2k$ of the $A_j^r$’s we have (3.17). Hence,

$$\int_{\Omega \cap A_j^r} |\nabla \varphi_j|^2 d_M \leq \frac{1}{r^2} \frac{\text{Vol}(\Omega)}{k}, \quad \forall 1 \leq j \leq 2k.$$ 

ii) By the same arguments, at least $k$ of the $A_j^r$’s satisfy

$$\text{Vol}(\Gamma \cap A_j^r) \leq \frac{\text{Vol}(\Gamma)}{k}. \tag{3.18}$$

Up to re-ordering, we assume that for the first $k$ of the $A_j^r$’s (3.18) holds. Hence,

$$\int_{\Gamma \cap A_j^r} |\nabla f_i|^2 d_M \leq \frac{1}{r^2} \frac{\text{Vol}(\Omega)}{k}, \quad \forall 1 \leq j \leq k.$$ 

Since $\int_{\Gamma \cap A_j} \varphi_j^2 d_S \geq \int_{\Gamma_0 \cap A_j} d_S = \mu_0(A_j) \geq \frac{\text{Vol}(\Gamma_0)}{8N_rk}$, we have

$$R_\beta(\varphi_j) \leq \frac{8N_rk}{\text{Vol}(\Gamma_0)} \left[ \frac{1}{r^2} \frac{\text{Vol}(\Omega)}{k} + \beta \frac{1}{r^2} \frac{\text{Vol}(\Gamma)}{k} \right]$$

$$= \frac{8N_r}{r^2 \text{Vol}(\Gamma_0)} [\text{Vol}(\Omega) + \beta \text{Vol}(\Gamma)].$$

However,

$$\frac{1}{r^2} = \left( \frac{2^6}{r(4k)} \right)^2 = 2^{12} \left( \frac{2}{\text{Vol}(\Omega)} \right)^{\frac{2}{n}} \left( \frac{4kC\rho_{n-1}}{I_0(\Omega)} \right)^{\frac{2}{n-1}}$$
and
\[
\text{Vol}(\Gamma_0) > \frac{1}{16^{\frac{n-1}{n}}} I_0(\Omega) \text{Vol}(\Omega)^{\frac{n-1}{n}}.
\]
Thus,
\[
\frac{1}{r^2 \text{Vol}(\Gamma_0)} \leq 2^{19} \frac{(kC\rho_{n-1})^{\frac{2}{n-1}}}{\text{Vol}(\Omega)^{1+\frac{2}{n-1}} I_0(\Omega)^{1+\frac{2}{n-1}}}.
\]
We get
\[
R_\beta(\varphi_j) \leq 2^{21} N_r \frac{(kC\rho_{n-1})^{\frac{2}{n-1}}}{\text{Vol}(\Omega)^{1+\frac{2}{n-1}}} \left[ \text{Vol}(\Omega) + \beta \text{Vol}(\Gamma) \right]
\]
\[
\leq 2^{21} N_r (C\rho_{n-1})^{\frac{2}{n-1}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \frac{\text{Vol}(\Gamma)^{1+\frac{2}{n-1}}}{\text{Vol}(\Omega)^{1+\frac{2}{n-1}}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right]
\]
\[
= 2^{21} N_r (C\rho_{n-1})^{\frac{2}{n-1}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Gamma)^{\frac{n-1}{n}}}{I(\Omega)^{1+\frac{n-1}{n}}} + \beta \right].
\]
Hence,
\[
\lambda_{W,k}^{\beta}(\Omega) \leq \max_{1 \leq j \leq k} R_\beta(\varphi_j)
\]
\[
\leq 2^{21} N_r (C\rho_{n-1})^{\frac{2}{n-1}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Gamma)^{\frac{n-1}{n}}}{I(\Omega)^{1+\frac{n-1}{n}}} + \beta \right].
\]
Case \( \mu(B(x_K, r)) > \frac{\mu(X)}{4KN_r^2} \). From (3.16) one has \( \mu(B(x_j, r)) \geq \frac{\mu(X)}{4KN_r^2} \) for every \( 1 \leq j \leq K \). We consider, for \( 1 \leq j \leq 4k \), the function \( f_j \) supported in \( B_j := B(x_j, 2r) \) and defined by
\[
f_j(x) := \begin{cases} 
\min\{1, 2 - \frac{d(x_j, x)}{r}\} & \forall x \in B_j, \\
0 & \forall x \in M \setminus B_j.
\end{cases}
\]
Set \( A_j := B(x_j, r) \), then the Rayleigh quotient of \( f_j \) satisfies
\[
R_\beta(f_j) \leq \frac{\int_{\Omega \cap B_j} |\nabla f_j|^2 \text{d}M + \beta \int_{\Gamma \cap B_j} |\nabla f_j|^2 \text{d}r}{\int_{\Gamma \cap A_j} f_j^2 \text{d}\Gamma}.
\]
i) Since for every \( x \in A_j \), \( f_j(x) = 1 \), one has
\[
\int_{\Gamma \cap A_j} f_j^2 \text{d}\Gamma \geq \int_{\Gamma \cap A_j} \text{d}\Gamma > \mu(A_j) > \frac{\text{Vol}(\Gamma)}{16N_r^2 k}.
\]
ii) Set for \( x \in M \), \( d_j(x) := \text{dist}(x, x_j) \), then
\[
|\nabla f_j| \leq |\nabla(2 - \frac{d_j(x)}{r})| = \left| \frac{1}{r} \nabla(d_j(x)) \right| \leq \frac{1}{r}.
\]
By Hölder’s inequality, we have
\[\int_{\Omega \cap B_j} |\nabla f_j|^2 d_M \leq \left( \int_{\Omega \cap B_j} |\nabla f_j|^n d_M \right)^{\frac{2}{n}} \left( \int_{\Omega \cap B_j} d_M \right)^{1-\frac{2}{n}} \leq \left( \frac{1}{r^n \Vol(B_j)} \right)^{\frac{2}{n}} (\Vol(\Omega \cap B_j))^{1-\frac{2}{n}}.\]

Notice that $B_j \cap \Gamma \supset A_j \cap \Gamma \neq \emptyset$. Let $x'_j \in B_j \cap \Gamma$, one has $B_j \subset B(x'_j, 4r)$. Since $4r \leq r_K < r_-(\Gamma)$,
\[\Vol(B_j) \leq \Vol(B(x'_j, 4r)) < C\omega_n (4r)^n.\]

In addition, since the $B_j$’s are pairwise disjoint, we have
\[\sum_{j=1}^{4k} \Vol(\Omega \cap B_j) \leq \Vol(\Omega).\]

We deduce that at least $2k$ of $B_j$’s satisfy
\[\Vol(\Omega \cap B_j) \leq \frac{\Vol(\Omega)}{k}. \quad (3.20)\]

Up to re-ordering, we assume that for the first $2k$ of the $B_j$’s (3.20) holds. Hence,
\[\int_{\Omega \cap B_j} |\nabla f_j|^2 \leq (C\omega_n 4^n)^{\frac{2}{n}} \left( \frac{\Vol(\Omega)}{k} \right)^{1-\frac{2}{n}}, \quad \forall \ 1 \leq j \leq 2k.\]

iii) Again the $B_j$’s are pairwise disjoint then $\sum_{j=1}^{4k} \Vol(\Gamma \cap B_j) \leq \Vol(\Gamma)$. Hence at least $k$ of the $B_j$’s satisfy
\[\Vol(\Gamma \cap B_j) \leq \frac{\Vol(\Gamma)}{k}. \quad (3.21)\]

Up to re-ordering, we assume that for the first $k$ of the $B_j$’s, inequality holds (3.21). Thus,
\[\int_{\Gamma \cap B_j} |\nabla f_i|^2 \leq \frac{1}{r^2} \left( \frac{\Vol(\Gamma)}{k} \right)^{1-\frac{n-2}{n}}, \quad \forall \ 1 \leq j \leq k.\]

Hence, one has
\[
R_\beta(f_j) \leq \frac{16N_r^2 k}{\Vol(\Gamma)} \left[ (C\omega_n 4^n)^{\frac{2}{n}} \left( \frac{\Vol(\Omega)}{k} \right)^{1-\frac{2}{n}} + \beta \frac{1}{r^2} \frac{\Vol(\Gamma)}{k} \right] \\
\leq 2^8 N_r^2 (C\omega_n)^{\frac{2}{n}} \left( \frac{k}{\Vol(\Gamma)} \right)^{\frac{2}{n-1}} \left( \frac{\Vol(\Omega)}{\Vol(\Gamma)} \right)^{1-\frac{2}{n}} \\
+ \beta 2^{10} N_r^2 (C\rho_{n-1})^{\frac{2}{n-1}} \left( \frac{k}{\Vol(\Gamma)} \right)^{\frac{2}{n-1}} \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{\frac{2}{n-1}}.
\]
Since \( \frac{I(\Omega)}{I_0(\Omega)} \geq 1 \), regarding the right hand side of (3.19), we have
\[
R_\beta(f_j) \leq 2^8 N_r^2(C\omega_n)^\frac{2}{n} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^\frac{2}{n} \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} + \beta 2^1 n_\Gamma^2(C\rho_{n-1})^{\frac{n}{n-1}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}}.
\]
Then, in this case
\[
\lambda_{W,k}^\beta(\Omega) \leq \max_{1 \leq j \leq k} R_\beta(\varphi_j)
\leq 2^8 N_r^2(C\omega_n)^\frac{2}{n} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^\frac{2}{n} \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} + \beta 2^1 n_\Gamma^2(C\rho_{n-1})^{\frac{n}{n-1}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}}.
\]
From (3.19) and (3.22), in both possible cases we have
\[
\lambda_{W,k}^\beta(\Omega) \leq C_1 \left( \frac{k}{\text{Vol}(\Gamma)} \right)^\frac{2}{n} \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} + C_2 \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}},
\] where \( C_1 := 2^8 N_r^2(C\omega_n)^\frac{2}{n} \) and \( C_2 := 2^1 N_r(C\rho_{n-1})^{\frac{n}{n-1}} \). This ends the proof.

When \( n \geq 3 \), Theorem 3.5 can be extended to cover all eigenvalues as follows:

**Theorem 3.6.** Let \((M, g)\) be a complete Riemannian manifold of dimension \( n \geq 3 \) and let \( \Omega \subset M \) be a bounded domain whose boundary \( \Gamma \) is a smooth hypersurface satisfying the hypothesis \((H_0)\). We assume that \( M \), with respect to the distance associated to the metric \( g \) satisfies the \((N, 4)\)-covering property for some discrete positive function \( N \).

Then, for every \( k \geq 1 \), one has
\[
\lambda_{W,k}^\beta(\Omega) \leq C(\Omega, \beta) + C_1 \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n}} + C_2 \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}},
\]
where the constants \( C_1 \) and \( C_2 \) are the same as in Theorem 3.5 and
\[
C(\Omega, \beta) := \frac{C_1}{(C\rho_{n-1} r_{\Gamma}^n)^{\frac{2}{n}}} \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} + \frac{C_2}{(C\rho_{n-1} r_{\Gamma}^n)^{\frac{2}{n-1}}} \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right].
\]
Proof. For $1 \leq k < K_0$, one has
\begin{equation}
\lambda_{W,k}^{\beta}(\Omega) \leq \lambda_{W,K_0}^{\beta}(\Omega)
\leq C_1 \left( \frac{K_0}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}}
+ C_2 \left( \frac{K_0}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)^{\frac{1}{n-1}}}{I(\Omega)^{1+\frac{1}{n-1}}} + \beta \right].
\end{equation}

However, $K_0 \leq \frac{I_0(\Omega)}{C_{\rho_{n-1}r_-(\Gamma)n-1}} \left( \frac{\text{Vol}(\Omega)}{2} \right)^{\frac{n-1}{n}} + 1$, using the triangle inequality, we obviously have for every $p \in \mathbb{N} \geq 2$
\begin{align*}
K_0^{\frac{2}{p}} &\leq \left( \frac{I_0(\Omega)}{C_{\rho_{n-1}r_-(\Gamma)n-1}} \left( \frac{\text{Vol}(\Omega)}{2} \right)^{\frac{n-1}{n}} \right)^{\frac{2}{p}} + 1 \\
&\leq \left( \frac{\text{Vol}(\Gamma)}{2^{\frac{n-1}{p}} C_{\rho_{n-1}r_-(\Gamma)n-1} I(\Omega)} \right)^{\frac{2}{p}} + 1 \\
&\leq \left( \frac{\text{Vol}(\Gamma)}{C_{\rho_{n-1}r_-(\Gamma)n-1}} \right)^{\frac{2}{p}} + k^{\frac{2}{p}}.
\end{align*}

We set $C_3 := \left( \frac{1}{C_{\rho_{n-1}r_-(\Gamma)n-1}} \right)^{\frac{2}{p}}$, replacing in (3.26), we get
\begin{align*}
\lambda_{W,k}^{\beta}(\Omega) &\leq C_1 \left\{ C_3 + \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{p}} \right\} \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}}
+ C_2 \left\{ C_3 + \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{p-1}} \right\} \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{p-1}} \left[ \frac{\text{Vol}(\Omega)^{\frac{1}{p-1}}}{\text{Vol}(\Gamma)^{1+\frac{1}{p-1}}} + \beta \right].
\end{align*}

Rearranging terms in above inequality, we have
\begin{align*}
\lambda_{W,k}^{\beta}(\Omega) &\leq C(\Omega, \beta) + C_1 \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n}}
+ C_2 \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{p-1}} \left[ \frac{\text{Vol}(\Omega)^{\frac{1}{p-1}}}{\text{Vol}(\Gamma)^{1+\frac{1}{p-1}}} + \beta \right] \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{p}},
\end{align*}
where
\begin{align*}
C(\Omega, \beta) := &\frac{C_1}{\left( C_{\rho_{n-1}r_-(\Gamma)n-1} \right)^{\frac{2}{p}} \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}}} \\
&+ \frac{C_2}{\left( C_{\rho_{n-1}r_-(\Gamma)n-1} \right)^{\frac{2}{p-1}} \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{p-1}} \left[ \frac{\text{Vol}(\Omega)^{\frac{1}{p-1}}}{\text{Vol}(\Gamma)^{1+\frac{1}{p-1}}} + \beta \right]}.
\end{align*}

The result follows applying Theorem 3.5 when $k \geq K_0$. □
4 Proof of main theorems

Proof of Theorem 1.1. We have in the Euclidean case:

\[ r_{-}(\Gamma) = +\infty, \quad C = 2, \quad I_{0}(\Omega) = I_{0}(\mathbb{R}^{n}) = n\omega_{n}^{\frac{1}{n}}, \quad N_{r} \leq 32^{n}, \forall \ r > 0. \]

Applying Theorem 3.6, we get for every \( k \geq 1 \)

\[
\lambda_{W,k}^{\beta}(\Omega) \leq \zeta_{1}(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n}} \\
+ \zeta_{2}(n) \left( \frac{I(\Omega)}{I_{0}(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}}, \tag{4.1}
\]

where \( \zeta_{1}(n) := 2^{10(n+1)}\omega_{n}^{\frac{2}{n}} \) and \( \zeta_{2}(n) := 2^{10(n+3)}\rho_{n-1}^{\frac{2}{n-1}} \). The result follows replacing \( I_{0}(\Omega) \) by \( n\omega_{n}^{\frac{1}{n}} \). \( \square \)

Proof of Corollary 1.2. From Theorem 1.1, one has

\[
\lambda_{W,k}^{\beta}(\Omega) \leq \zeta_{1}(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n}} \\
+ \zeta_{2}(n)I(\Omega)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}}, \tag{4.2}
\]

1. If \( \zeta_{1}(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} \left( \frac{\text{Vol}(\Gamma)}{k} \right)^{\frac{2}{n(n-1)}} < 1 \), then

\[
\lambda_{W,k}^{\beta}(\Omega) < C_{2}(\Omega, \beta) \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}}.
\]

2. Otherwise, \( \zeta_{1}(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} \left( \frac{\text{Vol}(\Gamma)}{k} \right)^{\frac{2}{n(n-1)}} \geq 1 \). That is,

\[
\frac{k}{\text{Vol}(\Gamma)} \leq \left[ \zeta_{1}(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{\frac{n-2}{n}} \right]^{\frac{n(n-1)}{2}},
\]

\[
\left\{ \begin{array}{l}
\left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \leq \zeta_{1}^{n-1}(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{(n-2)(n-1)} \\
\left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{n-2}{n-1}} \leq \zeta_{1}^{n}(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{n-2} \end{array} \right. \tag{4.3}
\]
Replacing in (4.2), we get
\[
\lambda_{W,k}^\beta (\Omega) \leq \zeta_1^n (n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{n-2}
+ \zeta_1^n (n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{n-2} \zeta_2 (n) I(\Omega)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] 
\leq \zeta_1^n (n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{n-2} \left\{ 1 + \zeta_2 (n) I(\Omega)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \right\} 
= C_1 (\Omega, \beta)
\]

In both cases, one has \( \lambda_{W,k}^\beta (\Omega) \leq C_1 (\Omega, \beta) + C_2 (\Omega, \beta) \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \). \qed

**Proof of Theorem 1.6.** Let \( r > 0 \), we denote by \( \nu(n, -\kappa^2, r) \) (respectively \( \nu_\partial(n, -\kappa^2, r) \)) the volume of a ball (respectively a sphere) of radius \( r \) in the constant curvature model space \( M_{-\kappa^2}^n \). As a consequence of the relative Bishop-Gromov volume comparison theorem, we have the following volume and area comparisons, for every \( r > 0 \) and \( x \in M \):
\[
\text{Vol}(B(x, r)) \leq \nu(n, -\kappa^2, r) \quad \text{and} \quad \text{Vol}(\partial B(x, r)) \leq \nu_\partial(n, -\kappa^2, r).
\]
The sphere of radius \( r \) in the model space \( M_{-\kappa^2}^n \) has area
\[
\nu_\partial(n, -\kappa^2, r) = \rho_{n-1} s_{n-\kappa^2}(r)^{n-1}
\]
and the ball of radius \( r \) has volume
\[
\nu(n, -\kappa^2, r) = \rho_{n-1} \int_0^r s_{n-\kappa}(t)^{n-1} dt,
\]
where \( s_{n\kappa} : \mathbb{R} \rightarrow \mathbb{R} \) is defined by
\[
s_{n\kappa}(t) = \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}t) & \text{if } \kappa > 0 \\
t & \text{if } \kappa = 0 \\
\frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}t) & \text{if } \kappa < 0.
\end{cases}
\]
\[
\rho_{n-1} \int_0^r s_{n-\kappa^2}(r)^{n-1} dt = \rho_{n-1} \int_0^r \left( \frac{1}{\kappa} \sinh(\kappa t) \right)^{n-1} dt 
\leq \rho_{n-1} \int_0^r [te^{\kappa t}]^{n-1} dt 
\leq \rho_{n-1} e^{r(n-1)\kappa} \int_0^r t^{n-1} dt 
\leq \omega_n r^ne^{r(n-1)\kappa}
\]
and
\[
\rho_{n-1}s_{n-\kappa^2}(r)^{n-1} \leq e^{r(n-1)\kappa} \rho_{n-1} r^{n-1}.
\]
Hence, for every \( 0 < r < 1 \) and \( x \in M \), we have
\[
\text{Vol}(B(x, r)) < C\omega_n r^n \quad \text{and} \quad \text{Vol}(\partial B(x, r)) < C\rho_{n-1} r^{n-1}, \tag{4.4}
\]
with \( C := e^p\kappa \).
On the other hand, for every $0 < r < 1$ and $x \in M$, $B(x, r)$ can be covered by $N := 2^{5n}e^{4r(n-1)\kappa} < 5^{5n}e^{4(n-1)\kappa}$ balls of radius $\frac{r}{8}$. Indeed, take \( \{B(x_i, \frac{r}{8})\}_{i=1}^N \) a maximal family of disjoint balls with center $x_i \in B(x, r)$. By the maximality assumption, the family $\{B(x_i, \frac{r}{8})\}_{i=1}^N$ covers $B(x, r)$. Let $i_0 \in \{1, \ldots, N\}$ such that

$$\text{Vol} \left( B(x_{i_0}, \frac{r}{8}) \right) = \min_{1 \leq i \leq N} \text{Vol} \left( B(x_i, \frac{r}{8}) \right).$$

Then one has

$$N \text{Vol}(B(x_{i_0}, \frac{r}{8})) \leq \sum_{1 \leq i \leq N} \text{Vol}(B(x_i, \frac{r}{8}))$$

since the balls $B(x_i, \frac{r}{8})$ are pairwise disjoint. In addition, $B(x_i, \frac{r}{8}) \subset B(x_i, r + \frac{r}{8})$ for every $x_i \in B(x, r)$. Hence $N \text{Vol}(B(x_{i_0}, \frac{r}{8})) \leq \text{Vol}(B(x, \frac{r}{8}))$,

$$N \leq \frac{\text{Vol}(B(x, \frac{2r}{8}))}{\text{Vol}(B(x_{i_0}, \frac{r}{8}))} \leq \frac{\text{Vol}(B(x, 2r))}{\text{Vol}(B(x_{i_0}, \frac{r}{8}))} \leq \frac{\text{Vol}(B(x_{i_0}, 4r))}{\text{Vol}(B(x_{i_0}, \frac{r}{8}))}. $$

Using the Relative volume comparison theorem (Bishop 1964, Gromov 1980), one has

$$\frac{\text{Vol}(B(x_{i_0}, 4r))}{\text{Vol}(B(x_{i_0}, \frac{r}{8}))} \leq \frac{\nu(n, -\kappa, 4r)}{\nu(n, -\kappa, \frac{r}{8})}, $$

where $\nu(n, \kappa, r)$ denotes the volume of a ball of radius $r$ in the constant curvature model space $M_n^\kappa$. Then

$$N \leq \frac{\int_0^{4r} \sinh^{n-1}((\kappa t)) dt}{\int_0^{\frac{r}{8}} \sinh^{n-1}((\kappa t)) dt} \leq \frac{\int_0^{4r} [((\kappa t)e^{\kappa t})]^{n-1} dt}{\int_0^{\frac{r}{8}} ((\kappa t))^{n-1} dt} \leq \frac{e^{4r(n-1)\kappa} \int_0^{4r} t^{n-1} dt}{\int_0^{\frac{r}{8}} t^{n-1} dt} = 2^{5n}e^{4r(n-1)\kappa} < 2^{5n}e^{4(n-1)\kappa}. $$

Then applying Theorem 3.6 with

$$r_-(\Gamma) = 1, \quad C = e^{n\kappa}, \quad N_r = 2^{5n}e^{4(n-1)\kappa},$$

we get, for every $k \geq 1$,

$$\lambda_{W,k}^\beta(\Omega) \leq e^{c_0(n)\kappa} \left\{ C(\Omega, \beta) + c_1(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \right\} + c_2(n) \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}}, $$

where

$$C(\Omega, \beta) := c_1'(n) \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} + c_2'(n) \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right]$$

and the constants $c_0(n), c_1(n), c_2(n), c_1'(n)$ and $c_2'(n)$ depend only on $n$. 
Following the same arguments as the proof of Corollary 1.2, we have for every $k \geq 1$, one has
\[
\lambda_{W,k}^\beta(\Omega) \leq e^{c_0(n)\kappa} \left\{ \overline{C}_1(\Omega, \beta) + \overline{C}_2(\Omega, \beta) \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \right\},
\]
where $\overline{C}_2(\Omega, \beta) = 1 + c_2(n) \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right]$ and $\overline{C}_1(\Omega, \beta) = C(\Omega, \beta) + c_1^0(n) \overline{C}_2(\Omega, \beta)$.

- If $\kappa \leq 1$, then
\[
\lambda_{W,k}^\beta(\Omega) \leq e^{c_0(n)} \left\{ \overline{C}_1(\Omega, \beta) + \overline{C}_2(\Omega, \beta) \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n-1}} \right\},
\]
which implies (1.9).

- Otherwise, we assume that $\text{Ric}(M, g) \geq -(n - 1)\kappa^2 g$ with $\kappa > 1$. Then the Ricci curvature $\text{Ric}(M, \tilde{g})$ of the rescaled metric $\tilde{g} := \kappa^2 g$ is bounded by $-(n - 1)\tilde{g}$. We mark with a tilde quantities associated with the metric $\tilde{g}$, while those unmarked with such will be still associated with the metric $g$. Then we have
\[
\lambda_{W,k}^\beta(\tilde{\Omega}) \leq e^{c_0(n)} \left\{ \overline{C}_1(\tilde{\Omega}, \beta) + \overline{C}_2(\tilde{\Omega}, \beta) \left( \frac{k}{\text{Vol}(\tilde{\Gamma})} \right)^{\frac{2}{n-1}} \right\}. \tag{4.5}
\]
However $\text{Vol}(\tilde{\Omega}) = \text{Vol}_\beta(\Omega) = \kappa^n \text{Vol}(\Omega)$ and $\text{Vol}(\tilde{\Gamma}) = \kappa^{n-1} \text{Vol}(\Gamma)$. Thus,
\[
\overline{C}_2(\tilde{\Omega}, \beta) = 1 + c_2(n) \left( \frac{I(\tilde{\Omega})}{I_0(\Omega)} \right)^{1+\frac{2}{n-1}} \left[ \frac{\text{Vol}(\tilde{\Omega})}{\text{Vol}(\tilde{\Gamma})} + \beta \right] = 1 + c_2(n) \left( \frac{\text{Vol}(\tilde{\Omega})}{\text{Vol}(\tilde{\Gamma})} \right)^{1+\frac{2}{n-1}} \left[ \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right]. \tag{4.6}
\]
Likewise, since
\[
C(\tilde{\Omega}, \beta) = c_1'(n) \left( \frac{\text{Vol}(\tilde{\Omega})}{\text{Vol}(\tilde{\Gamma})} \right)^{1-\frac{2}{n}} + c_2'(n) \left( \frac{I(\tilde{\Omega})}{I_0(\tilde{\Omega})} \right)^{1+\frac{2}{n-1}} \left[ \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right],
\]
we have
\[
\overline{C}_1(\tilde{\Omega}, \beta) = C(\tilde{\Omega}, \beta) + c_1^0(n) \overline{C}_2(\tilde{\Omega}, \beta)
= c_1^0(n) + c_1'(n) \left( \frac{\text{Vol}(\tilde{\Omega})}{\text{Vol}(\tilde{\Gamma})} \right)^{1-\frac{2}{n}}
+ \left( c_1^0(n)c_2(n) + c_2'(n) \right) \left( \frac{I(\tilde{\Omega})}{I_0(\tilde{\Omega})} \right)^{1+\frac{2}{n-1}} \left[ \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right]. \tag{4.7}
\]
We set $\zeta(n) := \max\{1, c_2(n), c^n_3(n), c^n_3(n)c_2(n) + c^n_2(n)\}$ so that

$$\overline{C}_2(\Omega, \beta) \leq \zeta(n) \left\{ 1 + \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n}} \left[ \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \right\}$$

and

$$\overline{C}_1(\Omega, \beta) \leq \zeta(n) \left\{ 1 + \left( \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} + \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n}} \left[ \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \right\}.$$

In addition, since $\kappa > 1$, for all $u \in \mathcal{V}_\beta \setminus \{0\}$ we have

$$\bar{R}_\beta(u) = \frac{\kappa \int_\Omega |\nabla u|^2 dM + \beta \int_\Gamma |\nabla u|^2 d\Gamma}{\kappa^2 \int_\Gamma u^2 d\Gamma} \geq \frac{1}{\kappa^2} R_\beta(u).$$

Every orthonormal basis of a $k$-dimensional subspaces $V \in \mathcal{V}(k)$ of $\mathcal{V}_\beta$ remains orthogonal with the metric $\tilde{g}$, then using the variation characterisation with (4.5), (4.6) and (4.7), we have

$$\lambda^2_{W,k}(\Omega) \leq \kappa^2 \lambda^2_{W,k}(\Omega) \leq \kappa^2 e^{c_0(n)} \left\{ \overline{C}_1(\Omega, \beta) + \overline{C}_2(\Omega, \beta) \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n}} \right\}
\leq \kappa^2 \zeta(n) \left\{ 1 + \left( \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} \right)^{1-\frac{2}{n}} + \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n}} \left[ \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \right\}
+ \zeta(n) \left\{ 1 + \left( \frac{I(\Omega)}{I_0(\Omega)} \right)^{1+\frac{2}{n}} \left[ \kappa \frac{\text{Vol}(\Omega)}{\text{Vol}(\Gamma)} + \beta \right] \right\} \left( \frac{k}{\text{Vol}(\Gamma)} \right)^{\frac{2}{n}},$$

where $\zeta(n) = e^{c_0(n)}\zeta(n)$ is a dimensional constant.

\(\square\)

**Proof of Theorem 1.4.** We have for every $r > 0$, $s_{n_0}(r) = r$, then for every $r > 0$ and $x \in M$, one has

$$\text{Vol}(B(x, r)) \leq v(n, 0, r) = \rho_{n-1} \int_0^r t^{n-1} dt = \omega_n r^n$$

and

$$\text{Vol}(\partial B(x, r)) \leq \nu_0(n, 0, r) = \rho_{n-1}(r)^{n-1}, \quad \forall x \in M.$$

On the other hand, for every $r > 0$ and $x \in M$, $B(x, r)$ can be covered by $N := 32^n$ balls of radius $\frac{r}{8}$. As above, we take $\{B(x_i, \frac{r}{8})\}_{i=1}^N$ a maximal family of disjoint balls with center $x_i \in B(x, r)$. By the maximality assumption, the family $\{B(x_i, \frac{r}{8})\}_{i=1}^N$ covers $B(x, r)$. Let $i_0 \in \{1, \ldots, N\}$ such that

$$\text{Vol}(B(x_{i_0}, \frac{r}{8})) = \min_{1 \leq i \leq N} \text{Vol}(B(x_i, \frac{r}{8})).$$

Then, since the balls $B(x_i, \frac{r}{8})$ are pairwise disjoint, one has

$$N \text{Vol}(B(x_{i_0}, \frac{r}{8})) \leq \sum_{1 \leq i \leq N} \text{Vol}(B(x_i, \frac{r}{8})).$$
In addition, \( B(x_i, \frac{r}{8}) \subset B(x, r + \frac{r}{8}) \) for every \( x_i \in B(x, r) \), so
\[
N \text{Vol}(B(x_{i_0}, \frac{r}{8})) \leq \text{Vol}(B(x, \frac{9r}{8})).
\]
Hence,
\[
N \leq \frac{\text{Vol}(B(x, \frac{9r}{8}))}{\text{Vol}(B(x_{i_0}, \frac{r}{8}))} \leq \frac{\text{Vol}(B(x, 2r))}{\text{Vol}(B(x_{i_0}, \frac{r}{8}))} \leq \frac{\text{Vol}(B(x_{i_0}, 4r))}{\text{Vol}(B(x_{i_0}, \frac{r}{8}))}.
\]
Using the volume comparison theorem (Bishop 1964, Gromov 1980), one has
\[
\frac{\text{Vol}(B(x_{i_0}, 4r))}{\text{Vol}(B(x_{i_0}, \frac{r}{8}))} \leq \frac{\omega_n(4r)^n}{\omega_n(\frac{r}{8})^n} \leq 32^n.
\]
Then follows from Theorem 3.6 with \( r_-(\Gamma) = +\infty \), \( C = 2 \) and \( N_r = 32^n \).

□

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