Applying Residue Theorem to Compute Real Definite Integral

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Abstract: The Residue Theorem is one of the fundamental theorems in complex analysis. It can be used to integrate some types of real functions that are difficult, or even unable, to be expressed in terms of fundamental functions. This paper reevaluates the application of the Residue Theorem in the real integration of one type of function that decay fast.

1. Introduction

The Residue Theorem, also known as the Cauchy’s residue theorem, is a useful tool when computing integrals of holomorphic functions defined on a subset of complex plane $U\setminus\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, where $U$ is a connected open subset of a complex plane and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are a finite collection of points on the complex plane. The Residue Theorem states that if a function $f$ is complex-analytic on a closed, clockwise contour $C$, then the value of the integral is $2\pi i$ times the sum of the residues of $f$ at the poles inside $C$.

To apply the Residue Theorem when integrating a real, definite function $f(x)$, first we need to find a function, $g(z)$, that equals $f(x)$ or closely related to it [1-3]. We will then choose a closed contour $C$ that contains a part or the whole of the real axis, which depends on the domain of the real integral. The integration along the chosen contour, except the part on the real axis, must be zero. Next, the Residue Theorem is applied to compute the integral of $g(z)$ over contour $C$ by computing the residue of the poles of $g(z)$. Since the real part or imaginary part of the original function $f(x)$ will be left, its integration can be obtained [4, 5].

This paper mainly summarizes how to compute real integrals using the residue theorem. First, zeros and poles of holomorphic function are discussed. Then, a brief review of the residue theorem is given. And last, we detail the procedure of computing real integrals of decaying functions.

2. Main work

Definition 1 If $f$ is a holomorphic function on the region $\Omega$, if $z_0 \in \Omega$, we say $z_0$ is a zero of the function $f$.

Remark: the holomorphic function only has isolated zeros. If $z_0$ is a zero of function $f$, then there exists $r > 0$, such that $|f(z_0)| > 0$ for all $z \in B_r(z_0) \setminus \{z_0\}$ [2].

Theorem 2. Suppose $f$ is holomorphic on $\Omega$ with isolated zero $z_0 \in \Omega$, then there exist

A neighborhood, $U$, of $z_0$, an integer $n$ and a non-vanishing holomorphic function $g$, such that

$$f(z) = (z - z_0)^n g(z)$$
let \( r \) be such that \( \overline{D_r(z_0)} \subseteq \Omega \). Since \( f \) is holomorphic on \( \Omega \), the Taylor expansion of \( f \) can be written as:

\[
f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k
\]

for all \( z \in D_r(z_0) \).

Since \( z_0 \) is an isolated zero, we can not have \( a_k = 0 \) for all \( k \in \mathbb{N} \).

Let \( n \in \mathbb{N} \) be the smallest integer such that \( a_n \neq 0 \).

Then

\[
f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k = (z-z_0)^n \sum_{k=n}^{\infty} a_k(z-z_0)^{k-n}
\]

Let \( g(z) = \sum_{k=n}^{\infty} a_k(z-z_0)^{k-n} \).

**Definition 3.** In the context of theorem 2, we say \( f \) has a zero of order \( n \).

When \( n = 1 \), we say \( z_0 \) a simple zero.

**Definition 4.** Suppose \( f \) is holomorphic on \( D_r(z_0) - z_0 \) for some \( r > 0 \) and \( z_0 \in C \). We say \( z_0 \) is a pole of \( f \) if, on a neighborhood of \( z_0 \), the function \( \frac{1}{f(z)} \) is holomorphic.

**Theorem 5.** If \( f \) is holomorphic on \( \Omega \) with a pole at \( z_0 \). Then there exists a neighborhood U of \( z_0 \) in \( \Omega \cup \overline{z_0} \), a unique \( n \in \mathbb{N} \) and non-vanishing holomorphic function such that

\[
f(z) = \frac{1}{(z-z_0)^n} h(z)
\]

**Definition 6.** In the context of the previous theorem, we say \( f \) has a pole of order \( n \) at \( z_0 \).

When \( n = 1 \), we say \( z_0 \) a simple pole.

Remark: When \( z_0 \) is a pole, holomorphic function \( f \) does not have Taylor expansion but something similar called Laurent series expansion.

**Theorem 7.** When \( f \) has a pole of order \( n \) at \( z_0 \), then for a neighborhood U of \( z_0 \),

\[
f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \ldots + \frac{a_{-1}}{(z-z_0)} + G(z)
\]

where \( a_{-n}, a_{-n+1}, \ldots, a_{-1} \in \mathbb{C} \) and \( G(z) \) is a holomorphic function.

**Definition 8.** In the context of theorem 7, \( \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \ldots + \frac{a_{-1}}{(z-z_0)} \) is called the principal part of \( f \) and \( a_1 \) is called residue theorem of \( f \) at \( z_0 \) and is denoted as \( \text{res}_{z_0}(f) \).

Note that \( k \neq 1 \), the term \( a_{-k}(z-z_0)^{-k} \) in the principal part has \( \frac{a_{-k}}{(z-z_0)^{-k+1}} \) as a primitive. Thus if \( P(z) \) is the principal part of \( f \) at the point \( z_0 \) and \( z_0 \) is inside \( C \), then

\[
\frac{1}{2\pi i} \int_C P(z)dz = a_{-1}
\]

for any circle \( C \). If \( P(z) \) is the principle part of \( f \) at the point \( z_0 \) and \( z_0 \) is outside \( C \), then

\[
\frac{1}{2\pi i} \int_C P(z)dz = 0
\]

for any circle \( C \). Therefore, in order to compute the integral of \( f \), it suffices to compute the residue of \( f \) at \( z_0 \).

**Theorem 9.** If \( f \) has a pole of order \( n \) at \( z_0 \), then

\[
\text{res}_{z_0}(f) = \frac{1}{(n-1)!} g^{(n-1)}(z_0)
\]

where \( g(z) = (z-z_0)^n f(z) \).

By the expansion of \( g(z) \) near \( z_0 \) we have:

\[
g(z) = (z-z_0)^n f(z) = a_{-n} + a_{-n+1}(z-z_0) + \ldots + a_{-1}(z-z_0)^{n-1} + (z-z_0)^n G(z).
\]

Thus

\[
g^{(n-1)}(z_0) = (n-1)! a_{-1} = (n-1)! \text{res}_{z_0}(f)
\]

**Theorem 10 (The residue theorem).** Suppose \( f \) is holomorphic an a neighborhood of a circle \( C \) and its interior \( D \), except for a pole \( z_0 \in D \) then
\[ \int_C f(z)dz = 2\pi i \times res_z(f) \]

Proof: Let \( n \in N \) be the order of the pole and expansion of \( f \)
\[
f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z)
\]
holds for all \( z \in D \). Notice that integration for the integrand \( \frac{a_{-k}}{z-z_0} \) is \( 2\pi i \cdot a_{-1} \). The integration for the integrand \( \frac{a_{-k}}{(z-z_0)^k} \) is \( 2\pi i \cdot a_{-1} \) when \( k \neq 1 \).

(Corollary 11): Suppose \( f \) is holomorphic a neighborhood of a circle \( C \) and its interior disc \( D \), except for a pole \( z_1, \ldots, z_k \in D \) then
\[
\int_C f(z)dz = 2\pi i \times \sum_{i=1}^k res_z(f)
\]

Definition 12: If \( f \) is a holomorphic function defined on the upper or lower half complex plane satisfying for some constant bigger than one when \( |z| \) is large enough. We say \( f \) decays faster.

Example. \( \frac{1}{z^{a+1}} \) decays faster than \( \frac{1}{z} \) since the limit of \( \frac{1}{z^{a+1}} \) when \( z \) goes to infinity is zero.

Theorem 13. If \( f \) decays faster than \( \frac{1}{z} \), then \( \lim_{R \to \infty} \int_{C_R} f(z)dz = 0 \), where \( C_R \) is the semicircle \( Re^{i\theta}, 0 < \theta < \pi \).

Proof: \( \left| \int_{C_R} f(z)dz \right| \leq \int_{C_R} |f(z)||dz| \leq \int_{C_R} M \cdot R \cdot d\theta = \frac{M\pi}{R^{a-1}}. \)

Notice that \( a \) is bigger than one. Thus when \( R \) goes to the positive infinity, the integrals along the \( C_R \) equals to zero.

Theorem 13 guarantee the possibility of computing real integrations via the tool of the complex analysis. In details, if the holomorphic function \( f \) is real-valued along the real axis, then the integration of \( f(z) \) along the real axis equals the integration
\[
\int_{-\infty}^{+\infty} f(x)dx
\]

If the holomorphic function \( f \) both is real-valued along the real axis and decays faster than \( \frac{1}{z} \) and \( z_1, \ldots, z_k \) are all poles on the half upper (lower) complex plane, then it holds that
\[
\int_{-\infty}^{+\infty} f(x)dx \lim_{R \to \infty} \int_{C_R \cup [-R,R]} f(z)dz = 2\pi i \times \sum_{i=1}^k res_z(f)
\]

Next, we will give one to show the powerfulness of this complex analysis technique in computing some type of real integrals.

For real integration such as \( \int_{-\infty}^{+\infty} \frac{1}{(1+z^2)^2}dz \), the common method is to utilize the fundamental theorem of Calculus, but it is cumbersome to compute a primitive of the function \( \frac{1}{(1+z^2)^2} \).

Let \( \frac{f(z) be \frac{1}{(1+z^2)^2} \). It is clear that \( f(z) \) decays faster than \( \frac{1}{z} \).

Considering path composed of the semicircle \( Re^{i\theta}(0 < \theta < \pi) \) with the interval \([-R,R]\), it is clear that \( C_R \cup [-R,R] \) is a closed contour. By theorem 13, it is obvious that
\[
\lim_{R \to \infty} \int_{C_R} f(z)dz = 0.
\]

Then \( \int_{C_R \cup [-R,R]} f(z)dz = \int_{[-R,R]} f(z)dz \). Notice that
\[ \int_{C_R \cup [-R,R]} f(z) \, dz = 2\pi i \text{ times the sum of residues inside of the closed contour} \]

\[ C_R \cup [-R,R]. \text{ Let } R \text{ goes to infinity, the integration of } f \text{ along with the interval from } -R \text{ to } R \]

\[ \text{becomes the integration of } f \text{ along the real axis. And} \]

\[ \lim_{R \to \infty} \int_{C_R \cup [-R,R]} f(z) \, dz = 2\pi i \times \sum_{i=1}^{k} \text{res}_{z_i}(f) \]

where \( z_1, \ldots, z_k \) are all poles on the half upper complex plane. Finally, we compute the needed residues. Notice that \( f(z) \) has two poles of order two.

One pole is \( -i \) and another pole is \( +i \). However, only \( z = i \) is inside the contour. Thus we compute the residue at the point \( z = i \).

Let \( g(z) \) be the function \( (z - i)^2 f(z) \). It follows that \( g(z) = (z+i)^2 \).

It is easy to get that \( r \text{es}_{z_i}(f) = g'(i) = \frac{1}{4i} \).

Thus, \[ \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} \, dx = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}. \]

Let’s considering another example \( f(z) \).

Take \( f(z) \) as \( \frac{1}{1+z^2} \). Notice that the limit of \( \frac{z^2}{1+z^2} \) is zero when \( z \) goes to infinity. It is obvious that \( f(z) \) decays faster than \( \frac{1}{z} \). Considering semicircle \( Re^{i\theta} \) \((0 < \theta < \pi)\) combined with interval \([-R, R]\), it is easy to know that \( C_R \cup [-R,R] \) is a closed contour. Based on theorem 13, it holds that \[ \lim_{R \to \infty} \int_{C_R} f(z) \, dz = 0. \]

Then \[ \int_{C_R} f(z) \, dz = \int_{[-R,R]} f(z) \, dz. \]

According to the residue theorem, \[ \int_{C_R \cup [-R,R]} f(z) \, dz = 2\pi i \times \sum_{i=1}^{k} r \text{es}_{z_i}(f) \]

where \( z_1, \ldots, z_k \) are all poles on the half upper complex plane. Hence, in order to evaluate \[ \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx, \] it suffices to compute all the residues of \( \frac{1}{1+z^2} \) on the half upper complex plane.

Notice that \( f(z) \) has four simple poles. However, only \( z_1 = e^{i\frac{\pi}{4}} \) and \( z_2 = \) \( \) are inside the contour.

Let \( g(z) \) be the function \( (z - z_1) f(z) \). It follows that the limit of \( g(z) \) when \( z \) approaches \( z_1 \) is \( e^{i\frac{\pi}{4}} \).

Let \( h(z) \) be the function \( (z - z_2) f(z) \). It follows that the limit of \( g(z) \) when \( z \) approaches \( z_2 \) is \( e^{i\frac{\pi}{4}} \).

Thus, \[ \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = 2\pi i \times \left( e^{i\frac{\pi}{4}} + e^{i\frac{\pi}{4}} \right) = \pi \sqrt{2}. \]
3. Conclusion

For holomorphic function that decays faster than $\frac{1}{z}$ and is real-valued along the real axis, the real integration of such function along the real axis equals to $2\pi i$ multiple of the sum of all residues of such function on the half upper (lower) complex plane.

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