ABSTRACT. We study the solid-on-solid interface model above a horizontal wall in three dimensional space, with an attractive interaction when the interface is in contact with the wall, at low temperatures. There is no bulk external field. The system presents a sequence of layering transitions, whose levels increase with the temperature, before reaching the wetting transition.
1. Introduction and results

We consider the square lattice $\mathbb{Z}^2$, and to each site $x = (x_1, x_2) \in \mathbb{Z}^2$ we associate an integer variable $\phi_x \geq 0$ which represents the height of the interface at this site. The system is first considered in a finite box $\Lambda \subset \mathbb{Z}^2$ with fixed values of the heights outside. Each interface configuration on $\Lambda$: $\{\phi_x\}, \ x \in \Lambda$, denoted $\phi_{\Lambda}$, has an energy defined by the Hamiltonian

$$H_{\Lambda}^{WAB}(\phi_{\Lambda} | \bar{\phi}) = 2J_{AB} \sum_{\langle x,x' \rangle \cap \Lambda \neq \emptyset} |\phi_x - \phi_{x'}| + 2(J_{W,A} + J_{AB}) \sum_{x \in \Lambda} (1 - \delta(\phi_x)) + 2J_{WB} \sum_{x \in \Lambda} \delta(\phi_x)$$

(1.1)

or equivalently, simplifying the notation to $J = J_{AB}$ and $K = J_{WB} - J_{WA}$, by the Hamiltonian

$$H_{\Lambda}(\phi_{\Lambda} | \bar{\phi}) = 2J \sum_{\langle x,x' \rangle \cap \Lambda \neq \emptyset} |\phi_x - \phi_{x'}| + 2(K - J) \sum_{x \in \Lambda} \delta(\phi_x)$$

(1.2)

where $J > 0$ and $K \in \mathbb{R}$, the function $\delta$ equals 1 when $\phi_x = 0$, and 0 otherwise, and $|\Lambda|$ is the number of sites in $\Lambda$. The first sum is taken over all nearest neighbors pairs $\langle x, x' \rangle \subset \mathbb{Z}^2$ such that at least one of the sites belongs to $\Lambda$, and one takes $\phi_x = \bar{\phi}_x$ when $x \notin \Lambda$, the configuration $\bar{\phi}$ being the boundary condition, assumed to be uniformly bounded.

In the space $\mathbb{R}^3$, the region obtained as the union of all unit cubes centered at the sites of the three dimensional lattice $\Lambda \times \left(\frac{1}{2} + \mathbb{Z}\right)$, which satisfy $0 < x_3 < \phi(x_1, x_2)$, is supposed to be occupied by fluid $A$, the union of all unit cubes centered at the sites which satisfy $x_3 > \phi(x_1, x_2)$ is supposed to be occupied by fluid $B$. The common boundary between these regions is a surface in $\mathbb{R}^3$, the microscopic interface $\mathcal{I}$. The union of all unit cubes centered at the sites which satisfy $x_3 < 0$ is considered as the substrate, also called the wall $W$.

The considered system differs from the usual SOS model by the restriction to non-negative height variables and the introduction of the second and third sums in the Hamiltonian. This term describes the interaction with the substrate.

The probability of the configuration $\phi_{\Lambda}$ at the inverse temperature $\beta = 1/kT$ is given by the finite volume Gibbs measure

$$\mu_{\Lambda}(\phi_{\Lambda} | \bar{\phi}) = \Xi(\Lambda, \bar{\phi})^{-1} \exp \left( - \beta H_{\Lambda}(\phi_{\Lambda} | \bar{\phi}) \right),$$

(1.3)

where $\Xi(\Lambda, \bar{\phi})$ is the partition function

$$\Xi(\Lambda, \bar{\phi}) = \sum_{\phi_{\Lambda}} \exp \left( - \beta H_{\Lambda}(\phi_{\Lambda} | \bar{\phi}) \right).$$

(1.4)
Local properties at equilibrium can be described by correlation functions between the heights on finite sets of sites, such as,
\[
\langle f(\phi_{x_1}, \ldots, \phi_{x_k}) \rangle_{\Lambda} = \sum_{\phi_{\Lambda}} \mu_{\Lambda}(\phi_{\Lambda} | \bar{\phi}) f(\phi_{x_1}, \ldots, \phi_{x_k}).
\]

We next briefly recall the setting of the wetting transition, as developed by Fröhlich and Pfister (refs. [6], [7], [8]) for the semi-infinite Ising model. We do not formally translate these results to the SOS context, so they should be taken as descriptive background.

Let \( \Lambda \subset \mathbb{Z}^2 \) be a rectangular box of sides parallel to the axes. Consider the boundary condition \( \bar{\phi}_x = 0 \), for all \( x \notin \Lambda \), and write \( \Xi(\Lambda, 0) \) for the corresponding partition function. The associated free energy per site,
\[
\tau_{WB} = 2(J_{WA} + J_{AB}) - \lim_{\Lambda \to \infty} \frac{1}{\beta |\Lambda|} \ln \Xi(\Lambda, 0),
\]
represents the surface tension between the medium \( B \) and the substrate \( W \).

This limit (1.6) exists and \( \tau_{WB} \leq 2 \min\{J_{WB}, J_{WA} + J_{AB}\} \). One can introduce the densities
\[
\rho_z = \lim_{\Lambda \to \infty} \sum_{z'=0} \langle \delta(\phi_x - z') \rangle_{\Lambda}^{(0)}, \quad \rho_0 = \lim_{\Lambda \to \infty} \langle \delta(\phi_x) \rangle_{\Lambda}^{(0)},
\]
where \( (0) \) denotes the height-0 boundary condition. Their connection with the surface free energy is given by the formula
\[
\tau_{WB}(\beta, K) = \tau_{WB}(\beta, 0) + 2 \int_0^K \rho_0(\beta, K') dK'.
\]

The surface tension \( \tau_{WA} \) between the fluid \( A \) and the substrate is \( \tau_{WA} = J_{WA} \). In order to define the surface tension \( \tau^{AB} \) associated to a horizontal interface between the fluids \( A \) and \( B \) we consider the ordinary SOS model, with height-0 boundary condition, and Hamiltonian
\[
H^{SOS}_\Lambda(\phi_{\Lambda} | \bar{\phi}) = 2J_{AB} \sum_{\langle x, x' \rangle \cap \Lambda \neq \emptyset} (1 + \left| \phi_x - \phi_{x'} \right|)
\]
The corresponding free energy gives \( \tau^{AB} \), obeying \( \tau^{AB} \leq 2J_{AB} \). With the above definitions, we have
\[
\tau^{WA}(\beta) + \tau^{AB}(\beta) \geq \tau_{WB}(\beta, K).
\]
and the right hand side in (1.10) is a monotone increasing and concave (and hence continuous) function of the parameter \( K \). This follows from relation (1.8) where the integrand is a positive decreasing function of \( K \). Moreover, when \( K \geq J \) equality is satisfied in (1.10).

In the thermodynamic description of wetting, the partial wetting situation is characterized by the strict inequality in equation (1.10), which can occur
only if $K < J$, as assumed henceforth. We must have then $\rho_0 > 0$. The complete wetting situation is characterized by the equality in (1.10). If this occurs for some $K$, say $K' < J$, then equation (1.8) tells us that this condition is equivalent to $\rho_0 = 0$. Then both conditions, the equality and $\rho_0 = 0$, hold for any value of $K$ in the interval $(K', J)$.

On the other hand, we expect that $\rho_0 = 0$ implies also that $\rho_z = 0$, for any positive integer $z$. This indicates that, in the limit $\Lambda \to \infty$, we are in the $A$ phase of the system, despite the height-0 boundary condition, so that the medium $B$ cannot reach anymore the wall. This means also that the Gibbs state of the SOS model does not exist in this case.

That such a situation of complete wetting is present for some $K < J$ does not follow, however, from the above results. This statement, as far as we know, remains an open problem for the semi-infinite Ising model in 3 dimensions. For the model (1.2) an answer to this problem has been given by Chalker [4]:

Chalker’s theorem: The following propositions hold

\begin{align*}
\text{(1.11)} & \quad \text{if } 2\beta(J-K) > -\ln \frac{1-e^{-2\beta J}}{10(1-e^{-2\beta J})}, \text{ then } \rho_0 > 0, \\
\text{(1.12)} & \quad \text{if } 2\beta(J-K) < -\ln(1-e^{-8\beta J}), \text{ then } \rho_0 = 0.
\end{align*}

Thus, for any given values of $J$ and $K$, there is a temperature below which the interface is almost surely bound and another higher one above which it is almost surely unbound and complete wetting occurs. An illustration of these results, in the plane of the parameters $(K, \beta^{-1})$, is given in Figure 1.

Here we investigate the intermediate region not covered by this theorem, when the temperature is low enough. We will prove that a sequence of layering transitions occurs before the system attains complete wetting. We shall use the following notation:

\begin{align*}
\text{(1.13)} & \quad u = 2\beta(J-K), \quad t = e^{-4\beta J}.
\end{align*}

The variable $t$ may be viewed as the cost of each pair of plaquettes comprising the interface, and $u$ is the gain per site of contact of the interface with the substrate.

**Theorem 1.1.** Let the integer $n \geq 0$ be given. For each $\epsilon > 0$ there exists a value $t_0(n, \epsilon) > 0$ such that, if the parameters $t, u, s$ satisfy $0 < t < t_0(n, \epsilon)$ and

\begin{align*}
-\ln(1-t^2) + (2+\epsilon)t^{n+3} < u < -\ln(1-t^2) + (2-\epsilon)t^{n+2} & \quad \text{if } n \geq 1, \\
-\ln(1-t^2) + (2+\epsilon)t^3 < u < \sqrt{t} & \quad \text{if } n = 0,
\end{align*}

then the following statements hold: (1) The free energy $\tau_{WB}$ is an analytic function of the parameters $t, u$. (2) There is a unique translation invariant Gibbs state $\mu_n$ satisfying $\mu_n(\{\phi_x \neq n\}) = O(t^2)$ for all $x$. (3) The density is $\rho_0 > 0$.  

An illustration for this theorem, in the plane \((K, \beta^{-1})\), is given in Figure 2. It reflects the principle that if the parameter \(K\) is kept fixed, which seems natural since it depends on the properties of the substrate, then the level \(n\) of the unique translation invariant Gibbs state increases when the temperature is increased. Rigorously, this principle would be in part a consequence of (1.14) if we could take \(\epsilon = 0\).

Figure 1. Illustration of Chalker’s results. The slope of the “partial wetting curve” at \((K = J, \beta^{-1} = 0)\) is \(-2 \ln 2\). The approximate value of the roughening temperature for the SOS model with values in \(\mathbb{Z}\), expected near \(\beta R J \simeq 0.4\), indicates the scale on the vertical axis.

Figure 2. Qualitative picture of the unicity regions established in Theorem 1.1. There are small gaps between the regions, due to \(\epsilon > 0\). Actual unicity regions should be larger. Magnified view around the point \((K = J, \beta^{-1} = 0)\).
Remarks:

(1) The analyticity of the free energy comes from the existence of a convergent cluster expansion for this system. This implies the analyticity, in a direct way, of some correlation functions and, in particular, of the density $\rho_0$.

(2) By unicity of the translation invariant Gibbs state we mean that the average over translations of local observables in finite volume Gibbs states with any uniformly bounded boundary conditions lead to the same infinite volume measure, independent of the boundary conditions.

(3) The condition $\rho_0 > 0$ means that the interface remains at a finite distance from the wall and hence, we have partial wetting. We can see from Theorem 1.1 that the region where this condition holds is much larger, at low temperatures, than the region initially proved by Chalker. It comes very close to the line above which it is known that complete wetting occurs.

(4) The values $t_0(n, \epsilon)$ for which we are able to prove Theorem 1.1 satisfy $t_0(n, \epsilon) \to 0$ when $n \to \infty$ or $\epsilon \to 0$.

The dependence of $t_0(n, \epsilon)$ on $\epsilon$, satisfying Remark 4, may be understood as follows. One believes that the regions of unicity of the state extend in such a way that two neighboring regions, say those corresponding to the levels $n$ and $n + 1$, will have a common boundary where the two states $\mu_n$ and $\mu_{n+1}$ coexist. At this boundary there will be a first order phase transition, since the two Gibbs states are different. The curve of coexistence does not necessarily exactly coincide with the curve $u = -\ln(1 - t^2) + 2t^{n+3}$. Theorem 1.1 says that it is however very near to it, if the temperature is sufficiently low.

Let us formulate in the following statement the kind of theorem that we expect, though we are not able to prove it. We think that such a statement could be proved using an extension of the Pirogov-Sinai theory [13]. This would certainly require some additional work and, in particular, a refinement of the notion of contours.

Statement: For each integer $n \geq 0$, there exists $t_0(n) > 0$ and a continuous function $u = \psi_{n+1}(t)$ on the interval $0 < t < t_0(n)$ such that the statements of Theorem 1.1 hold for $t$ in this interval, in the region where $\psi_{n+1}(t) < u < \psi_n(t)$ and for $n = 0$, in the region $\psi_1(t) < u$. When $u = \psi_{n+1}(t)$ the two Gibbs states, $\mu_n$ and $\mu_{n+1}$, coexist.

The existence of such a sequence of layering transitions has been proved for the SOS model with a bulk external field. See the works by Dinaburg, Mazel [5], Cesi, Martinelli [3] and Lebowitz, Mazel [11]. This model has the same set of configurations as the model considered here, but a different energy: The second term in (1.2) has to be replaced by the term $+h \sum_{x \in \Lambda} \phi_x$ to obtain
the Hamiltonian of the model with an external magnetic field. Effectively, each model incorporates a potential $V(\phi_x)$, with $V(n) = hn$ in the earlier work and $V(n) = 2(J - K)\delta(n)$ here. The purely local nature of the latter potential makes the layering transitions perhaps a priori less natural, and more difficult to prove. Nonetheless, the method that we follow for the proof of Theorem 1.1 is inspired by the method developed for the study of the model with external field and we shall have occasion to refer at various points to the works mentioned above. The most important difference between the two systems concerns the restricted ensembles and the computation of the associated free energies, a point that will be discussed in Sections 3 and 4. In ref. [5] an analogous result to that of Theorem 1.1 of the present work has been proved for the model with external field. In ref. [3] these results were extended to a proof of a theorem analogous to the Statement above. Ref. [11] contains new inequalities for the model, allows $t_0 > 0$ to be independent of $n$ (so that, for small $t$, there is an infinite sequence of layering transitions) and strengthens the form of uniqueness established for the Gibbs state.

References [3] and [11] make use of an “infinite-height” boundary condition which dominates all other boundary conditions, in the FKG sense. The fact that the measure given by this boundary condition is well-defined depends on the fact that the interaction $h\phi_x \to \infty$ as the height $\phi_x \to \infty$. The analogous statement fails in our context here, and the finite-volume measure with infinite-height boundary condition does not exist. Lacking this tool, we are restricted to results analogous to [5].

Without bulk external field, the problem of layering transitions has been considered before for the Ising model, heuristically and numerically by several authors, and mathematically mostly by Basuev ([11] and references therein). Ref. [11] deals mostly with the case of a positive bulk field, coexisting with a surface field, but the case of zero bulk field is considered in section 9 of [11].

Theorem 1.1 indicates that infinitely many phase transition lines start from the point $K = J, \beta^{-1} = 0$ in the plane $(K, \beta^{-1})$. Where and how do these lines end? Five possible scenarios are shown in Fig. 3 (a-e). Cases (a-b-c) are inspired by the analog diagrams of Binder and Landau for the Ising model (Fig. 1 p. 2 in [2]).
Fig. 3. Tentative phase diagrams (a)-(e)
2. Cylinder models

We consider the model in a box $\Lambda$ under the constant boundary condition $\phi_x = n$, for any given integer $n \geq 0$. As we have seen, every configuration $\phi_\Lambda$ can naturally be considered as a surface $I$ imbeded into $\mathbb{R}^3$. Under the given boundary condition all points of the boundary $\partial I$ are at height $n$, and we can write

$$\beta H_\Lambda(\phi_\Lambda| n) = 2\beta J(|I| - |\Lambda|) - u|I \cap W|,$$

where $W$ is the horizontal plane at height 0. The possible interfaces are connected sets of unit squares, also called plaquettes, and $| \cdot |$ denotes the number of these plaquettes.

Dinaburg and Mazel [5] have shown that such an interface can also be obtained in a unique way from the horizontal plane $P_n$, at height $n$, by adding “positive cylinders” and digging “negative cylinders”. The order of operations is not specified, but one may decide that larger ones are placed first.

Formally, a cylinder, $\gamma = (\tilde{\gamma}, E, I)$, is defined by its base perimeter $\tilde{\gamma}$, a closed path of bonds in the dual lattice $\mathbb{Z}^2 + (1/2, 1/2)$, and two different non-negative integers: $E$, the exterior level, and $I$, the interior level. The closed path $\tilde{\gamma}$ must remain a single closed path when self-intersections are removed by connecting south to west and north to east as in Figure 4. Then $\tilde{\gamma}$ can be slightly deformed into a simple path by rounding corners.

![Figure 4. Rounding corners at self-intersections to make a simple path](image)

The sign of $\gamma$ is defined as $S(\gamma) = \text{sign}(I(\gamma) - E(\gamma))$. The length of $\gamma$ is defined as $L(\gamma) = |I(\gamma) - E(\gamma)|$. The interior, $\bar{\gamma}$, is defined to be the set of sites $x \in \mathbb{Z}^2$ enclosed by $\tilde{\gamma}$.

Next, one defines the notion of compatibility of two cylinders in such a way as to have a one–to–one correspondence between the set of configurations $\phi_\Lambda$ and the set of all compatible sets of cylinders. We shall not use here the notion of weak compatibility considered in ref. [5].

Two cylinders $\gamma, \gamma'$ are compatible, written $\gamma \sim \gamma'$, if either condition (1) or condition (2), together with condition (3), hold:

1. $S(\gamma) = S(\gamma')$, $\bar{\gamma} \neq \bar{\gamma'}$ and either $\bar{\gamma} \cap \bar{\gamma'} = \emptyset$ and $\bar{\gamma} \cap \tilde{\gamma'} = \emptyset$, or $\bar{\gamma} \subset \gamma'$, or $\gamma' \subset \bar{\gamma}$,
\[ S(\gamma) = -S(\gamma'), \bar{\gamma} \neq \bar{\gamma}' \] and either
\[ \bar{\gamma} \cap \bar{\gamma}' = \emptyset, \quad \text{or} \quad \bar{\gamma} \subset \bar{\gamma}' \quad \text{and} \quad \bar{\gamma} \cap \bar{\gamma}' = \emptyset. \]

(3) \[ E(\gamma) = E(\gamma') \quad \text{if} \quad \bar{\gamma} \cap \bar{\gamma}' = \emptyset, \]
\[ I(\gamma) = E(\gamma') \quad \text{if} \quad \bar{\gamma}' \subset \bar{\gamma}, \]
where \( \bar{\gamma} \cap \bar{\gamma}' = \emptyset \) is decided after rounding SW and NE corners as in Figure 4.

Examples of compatible cylinders are shown on Figure 5.

Two cylinders \( \gamma', \gamma'' \) are \textit{separated} by a cylinder \( \gamma \), if \( \bar{\gamma}' \neq \bar{\gamma} \neq \bar{\gamma}'' \) and either
\[ \bar{\gamma}' \subset \bar{\gamma} \subset \bar{\gamma}'', \quad \text{or} \quad \bar{\gamma}'' \subset \bar{\gamma} \subset \bar{\gamma}'', \]

\[ \text{or} \quad \bar{\gamma}' \subset \bar{\gamma} \quad \text{and} \quad \bar{\gamma}'' \subset \bar{\gamma} \quad \text{and} \quad \bar{\gamma}' \subset \bar{\gamma}''. \]

Let \( \Gamma = \{ \gamma_i \} \) be a set of cylinders. We say that this set \( \Gamma \) is a \textit{compatible} set of cylinders if any two of its cylinders not separated by a third one are compatible. We denote by \( \Gamma_{ext} \) the set of all \textit{external} cylinders in \( \Gamma \), i.e., the set of all \( \gamma \) such that \( \bar{\gamma} \) is not contained in the interior of any other cylinder in \( \Gamma \). We write \( E(\Gamma) = n \) if \( E(\gamma) = n \) for all \( \gamma \in \Gamma_{ext} \). The partition function with constant boundary conditions can now be expressed as a sum over compatible sets of cylinders with suitable weights.

To any cylinder \( \gamma = (\bar{\gamma}, E, I) \) we assign the statistical weight
\[ \varphi(\gamma) = \varphi_{t,u}(\gamma) = \exp \left( -2\beta J L(\gamma) |\bar{\gamma}| + u |\bar{\gamma}| (\delta(I) - \delta(E)) \right) \]
(2.2)
\[ = t \frac{1}{2} L(\gamma) |\bar{\gamma}| \exp \left( u |\bar{\gamma}| (\delta(I) - \delta(E)) \right) \]

Then, taking formula (2.1) into account, we obtain
(2.3) \[ \Xi(\Lambda, n) = e^{u\delta(n)|\Lambda|} \sum_{\Gamma \in C(\Lambda, n)} \prod_{\gamma \in \Gamma} \varphi(\gamma), \]
where the sum runs over the set \( C(\Lambda, n) \) of all compatible sets of cylinders, \( \Gamma \), on \( \Lambda \), such that \( E(\Gamma) = n \).
3. Restricted ensembles

We come back to the problem of phase transitions in the SOS model with a wall. It has been recognized that for an interesting class of systems, including our model, one needs some extension of the Pirogov-Sinai theory of phase transitions, though a general theory of the concerned systems does not exist. In such an extension certain states, called the restricted ensembles, play the role of the ground states in the usual theory. They can be defined as a Gibbs probability measure on certain subsets of configurations.

In the present case we shall consider, for each \( n = 0, 1, 2, \ldots \), subsets of configurations which are in some sense near to the constant configurations \( \phi_x \equiv n \). The precise definition is as follows. A cylinder \( \gamma \) is called elementary if

\[
\text{diam} \tilde{\gamma} \leq 3k + 3,
\]

where \( k \) is a given positive integer. Since we are dealing with integer numbers we shall use \( \ell^1 \) distance for this diameter, and the \( \ell^1 \) norm \( |x| = |x_1| + |x_2| \) on \( \mathbb{Z}^2 \). We use the notation \( C^\text{el}_k(\Lambda, n) \) for the set of finite compatible sets of elementary cylinders, that is, the set of all \( \Gamma \in C(\Lambda, n) \) that contain only elementary cylinders.

The Gibbs measure defined on the subset \( C^\text{el}_k(\Lambda, n) \) is the restricted ensemble corresponding to level \( n \). The associated free energy (times \( \beta \)) per unit area is

\[
f_k(n) = -\lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \ln Z_k(\Lambda, n),
\]

where

\[
Z_k(\Lambda, n) = \sum_{\phi_\Lambda \in C^\text{el}_k(\Lambda, n)} \exp(-\beta H_\Lambda(\phi_\Lambda|n))
= e^{u\delta(n)|\Lambda|} \sum_{\Gamma \in C^\text{el}_k(\Lambda, n)} \prod_{\gamma \in \Gamma} \varphi(\gamma).
\]

Using the cluster expansion technique we shall prove, in Section 4, that one is able to compute the free energies \( f_k(n) \) as convergent power series in the variable \( t \). The radius of convergence depends on the choice of the restricted set configurations \( C^\text{el}_k(\Lambda, n) \), that is on the value of \( k \) used in its definition. It happens, as will be seen in the proof, that \( f_k(n) \) differs from the other free energies \( f_k(n') \) at least by an order \( t^{3n+3} \). Therefore, when we want to study the level \( n \), we have to consider in the definition of \( C^\text{el}_k(\Lambda, n) \) all the cylinders that can contribute with a weight at least of order \( t^{3n+3} \). A consistent choice for this purpose will be to take \( k = \max(2n, 8) \).

At this point one is able to study the phase diagram of the restricted ensembles. The restricted ensemble at level \( n \) is said to be dominant for some given
values of the parameters $u, t, k$, if $f_k(n) = \min_h f_k(h)$. In the next proposition we summarize the results that will be proved concerning the regions in the plane $t, u$, where, for each $n$, the associated restricted ensemble at level $n$ is the dominant restricted ensemble.

**Proposition 3.1.** Let the integer $n \geq 0$ be given and choose $k \geq 8$. Let $a, b \geq 0$ and $0 < t \leq t_1(k) = (3k + 3)^{-1}$. If $n \geq 1$ and

$$- \ln(1 - t^2) + (2 + a)t^{n+3} \leq u \leq - \ln(1 - t^2) + (2 - b)t^{n+2},$$

or if $n = 0$ and

$$- \ln(1 - t^2) + (2 + a)t^3 \leq u \leq t^{1/2},$$

then we have

$$f_k(n) \leq f_k(h) - at^{3n+3} + O(t^{3n+4}),$$

uniformly in $n \geq 0, k \geq \max(8, n), h \geq n + 1$, (3.6)

$$f_k(n) \leq f_k(n - 1) - bt^{3n} + O(t^{3n+1}) \text{ if } n \geq 1, \text{ and}$$

$$f_k(n) \leq f_k(h) - 2t^{3h+3} + O(t^{3h+4}) \text{ for } h \leq n - 2,$$

both uniformly in $n \geq 1, k \geq \max(8, n), 0 \leq h \leq n - 1$. (3.7)

"Uniformly" in (3.6) means that $|O(t^{3n+4})|$ is bounded from above by $t^{3n+4}$ times a constant independent of $k, n, h$ in the given ranges, and similarly for (3.7).

The bound of $t^{1/2}$ in (3.5) is somewhat arbitrary, mainly present to ensure convergence of the cluster expansion in the form that we use.

Proposition 3.1 is proved in Section 4. Then the proof of Theorem 1.1 will consist of showing that the phase diagram of the pure phases at low temperature is close to the phase diagram of the dominant restricted ensembles.

4. Proof of Proposition 3.1

Throughout the paper, $K_i$ are constants which do not depend on any of the parameters in the problem; in particular they are independent of heights $h, n$.

In order to discuss the cluster expansion for the free energies $f_k(n)$ associated to the restricted ensembles, some definitions are needed. We first introduce the notion of elementary perturbation, defined as a compatible set of elementary cylinders $\omega \in C_k^e(\Lambda, n)$ that contains a unique cylinder which is external in the set.

Thus, if $\omega$ is an elementary perturbation we may write $\omega = (\gamma^{ext}, \{\gamma_i\})$ as a compatible set of elementary cylinders such that $\gamma_i \subset \gamma^{ext}$ for all $i$ and $E(\omega) = E(\gamma^{ext}) = n$. The set $\gamma^{ext}$ is called the support of the elementary
perturbation and is denoted by $\text{Supp} \omega$. To each elementary perturbation the following statistical weight is assigned:

$$\varphi(\omega) = \varphi(\gamma^{\text{ext}}) \prod_i \varphi(\gamma_i).$$

(4.1)

When computing these weights the following notation will be useful:

$$2\|\omega\| = L(\gamma^{\text{ext}})|\tilde{\gamma}^{\text{ext}}| + \sum_i L(\gamma_i)|\tilde{\gamma}_i|,$$

(4.2)

so that $\|\omega\|$ represents half the number of vertical plaquettes.

Let now $\Gamma \in C^k(\Lambda, n)$ be any compatible set of elementary cylinders. It is then possible to write $\Gamma$ as the disjoint union of elementary perturbations

$$\Gamma = \omega_1 \cup \ldots \cup \omega_r$$

(4.3)

in a unique way. This shows that there is a one-to-one correspondence between the restricted set of configurations $C^k(\Lambda, n)$ and the set of all compatible sets of elementary perturbations, with the following definitions.

Two elementary perturbations $\omega$ and $\omega'$ are compatible if their supports do not intersect and, moreover, $\omega \cup \omega' \in C(\Lambda, n)$. This last condition enters only when the boundaries of the supports of $\omega$ and $\omega'$ have a common part and is satisfied as soon as $S(\gamma^{\text{ext}}) = -S(\gamma'^{\text{ext}})$. A set of elementary perturbations is a compatible set if any two perturbations in the set are compatible. The following is clear.

**Lemma 4.1.** The partition function (3.3) may be written as

$$Z_k(\Lambda, n) = e^{u\delta(n)|\Lambda|} \sum_{\{\omega_i\} \in C^k(\Lambda, n)} \prod_i \varphi(\omega_i),$$

(4.4)

where the sum runs over all compatible sets of elementary perturbations contained in $\Lambda$ such that $E(\omega) = n$.

This lemma and the compatibility definitions tell us that the restricted ensemble can be considered as a gas of elementary perturbations, in technical terms as a polymer system, for which a cluster expansion theory can be applied. See, for instance, refs. [9], [10], [12].

A cluster $X$ from $C^k(\Lambda, n)$ can be defined as a finite sequence of elementary perturbations $X = (\omega_1, \ldots, \omega_r)$, where some perturbations may be repeated, such that the set $\{\omega_1, \ldots, \omega_r\}$ is connected by incompatibility relations $\omega \not\sim \omega'$. (This notion of connection is formalized below.) $\text{Supp} X$ will denote the union of the supports of the elementary perturbations in $X$.

Then there is a function $\varphi^T_n(X)$, the truncated function, defined on the set of clusters (independent of $\Lambda$ and of the order of perturbations in $X$), such
that one can write

$$- \ln Z_k(\Lambda, n) = -u\delta(n)|\Lambda| - \sum_{X: \text{Supp } X \subseteq \Lambda} \varphi^T_u(X)$$

whenever the series converges, where

$$\varphi^T_u(X) = a^T(X) \prod_{i=1}^r \varphi(\omega_i).$$

and $a^T(X)$ is a signed combinatoric factor, defined in (4.7) below.

More formally, one considers a countable set $P$; the elements of this set are the abstract polymers, the notation $\alpha \not\sim \alpha'$ means that the two polymers $\{\alpha, \alpha'\} \subset P$ are not compatible, and the weight $\varphi(\alpha)$ is a complex valued function on $P$. To any finite sequence of polymers $(\alpha_1, \ldots, \alpha_r)$ there corresponds a function $X = X(\alpha)$ on $P$, with non-negative integer values, such that $\sum_{\alpha} X(\alpha) = r \geq 1$. $X(\alpha)$ is the multiplicity of $\alpha$ in the sequence. Associated to the same $X$ we have $r!/X!$ ordered sequences, where $X! = \prod_{\alpha} X(\alpha)!$. Given $X$ we construct the graph $g(X)$ with vertices $1, \ldots, r$ and edges $\{i, j\}$ corresponding to pairs such that $\alpha_i \not\sim \alpha_j$ or $\alpha_i = \alpha_j$. If $g(X)$ is a connected graph we say that $X$ is a cluster, and then

$$a^T(X) = (X!)^{-1} \sum_g (-1)^{|g|},$$

where the sum runs over all spanning connected subgraphs of $g(X)$, and $|g|$ is the number of edges of $g$. If $r = 1$, then $a^T = 1$.

Let us recall the following theorem on the convergence of cluster expansions (see [12].)

**Convergence theorem:** For a polymer system $P$, assume that there is a positive function $\mu(\alpha)$, $\alpha \in P$, such that for all $\alpha \in P$,

$$|\varphi(\alpha)| \leq \mu(\alpha) \exp \left( - \sum_{\alpha': \alpha' \not\sim \alpha} \mu(\alpha') \right).$$

Then for all $\alpha \in P$ we have

$$\sum_{X: \alpha \in X} |\varphi^T_u(X)| \leq \mu(\alpha)$$

and

$$\sum_X X(\alpha)|\varphi^T_u(X)| \leq \varphi(\omega)e^{\sum_{\omega': \omega' \not\sim \omega} \mu(\omega')} \leq e^{\mu(\alpha)} - 1.$$
We now apply this theorem to the free energies of the restricted ensembles. As a consequence of (3.2) and (4.5) we have the formal expansion

\begin{equation}
\tag{4.11}
\varphi_k(h) = -u\delta(h) - \sum_{X: \text{Supp} X \ni 0} \frac{1}{|\text{Supp} X|} \varphi^T_u(X).
\end{equation}

**Lemma 4.2.** Let \( k \geq 8, t \leq t_1(k) = (3k + 3)^{-4}, u \leq t^{1/2} \). Then the expansion (4.11) of the free energy is an absolutely convergent power series in \( t \). Moreover, with \( s = te^{t/4} \) and \( \mu(\omega) = \varphi_{s,0}(\omega) \), the following bounds are satisfied for all elementary perturbations \( \omega \):

\begin{equation}
\tag{4.12}
\sum_{\omega': \omega \not\sim \omega} \mu(\omega') < 3000s^2 \cdot 9|\gamma_{\omega}^{\text{ext}}| < s^{1/2}|\gamma_{\omega}^{\text{ext}}|,
\end{equation}

\begin{equation}
\tag{4.13}
|\varphi(\omega)| \leq \mu(\omega) \exp \left( -\sum_{\omega': \omega \not\sim \omega} \mu(\omega') \right),
\end{equation}

\begin{equation}
\tag{4.14}
\sum_{X: \omega \in X} |\varphi^T_u(X)| \leq \mu(\omega),
\end{equation}

\begin{equation}
\tag{4.15}
\sum_{X} X(\omega)|\varphi^T_u(X)| \leq \varphi(\omega)e^{\sum_{\omega': \omega \not\sim \omega} \mu(\omega')} \leq e^{\mu(\omega)} - 1.
\end{equation}

Here the sums are over elementary perturbations \( \omega' \) or clusters \( X \), of an arbitrary fixed external height.

**Proof.** In our case, in the Convergence Theorem we can take \( \mu(\omega) = \varphi_{s,0}(\omega) \), the same as the weight \( \varphi(\omega) = \varphi_{t,u}(\omega) \) on the set of elementary perturbations, but for some \( s > t \) in place of \( t \), and \( u \) replaced by 0. Then if \( \gamma_{\omega}^{\text{ext}} \) is the exterior cylinder of \( \omega \), we have

\begin{equation}
\tag{4.16}
\sum_{\omega: \gamma_{\omega}^{\text{ext}} = \gamma} \mu(\omega) < \mu(\gamma) \left( 1 + 2 \sum_{h=1}^{\infty} s^{h/2} \right) 4|\gamma| = \mu(\gamma)(1 - 2s^{1/2})^{-4|\gamma|} < \mu(\gamma) \exp(9s^{1/2}|\gamma|),
\end{equation}

and then

\begin{equation}
\tag{4.17}
\sum_{\omega: \gamma_{\omega}^{\text{ext}} = \gamma} \mu(\omega) < \frac{2s^{||\gamma||/2}}{1 - s^{||\gamma||/2}} \exp(9s^{1/2}|\gamma|) < 3s^{||\gamma||/2} \exp(9s^{1/2}|\gamma|).
\end{equation}

We have first bounded the contributions of all other cylinders of \( \omega \) by the contributions of all possible sets of vertical plaquettes that project on bonds of the dual lattice inside \( \hat{\gamma} \) or on \( \hat{\gamma}_i \); we take \( 4|\gamma| \) as a bound for the number of such bonds. We have used \( (1 - 2s^{1/2})^{-1} < e^{9s^{1/2}/4}, \) assuming \( 2s^{1/2} < 0.21 \).
A useful inequality is

(4.18) \[ \sum_{l=m}^{\infty} l^2 x^l \leq \frac{5}{4} m^2 x^m \text{ for all } m \geq 4, x \leq \frac{1}{2}. \]

Now \( 2 \cdot 3^{\ell-1} \) is a bound on the number of \( \tilde{\gamma} \) of length \( \ell \) passing through a given dual lattice site. Further, \( \gamma_{\text{ext}} \), being an elementary cylinder, has interior containing fewer than \( (3k + 3)^2/4 \) lattice sites, or also fewer than \( (\ell/4)^2 \) lattice sites, so using (4.17) and (4.18) we obtain for all \( k \geq 1 \):

\[
\sum \mu(\omega) < 2 \expleft(9s^{1/2}(3k + 3)^2/4\right) \sum_{\ell=4}^{\infty} \left(\frac{\ell}{4}\right)^2 (3s^{1/2})^\ell
\]

\[
< \left(\frac{3}{2}\right) s^2 \exp\left(9s^{1/2}(3k + 3)^2/4\right)
\]

(4.19)

\[
< 3000s^2 \text{ if } s \leq 2(3k + 3)^{-4},
\]

where we used (4.18) and \( \exp(9s^{1/2}(3k + 3)^2/4) \leq \exp(9e^{1/54}/4) < 10. \)

For \( s \) as above with \( k \geq 2 \), the argument of the exponential function in condition (4.8) is therefore bounded as

\[
\sum_{\omega': \omega' \neq \omega} \mu(\omega') \leq (|\tilde{\gamma}_{\omega}'| + |\gamma_{\omega}'| + 4) \sum_{\omega': \text{Supp } \omega' \neq 0} \mu(\omega')
\]

\[
< 3000s^2 \cdot 9|\tilde{\gamma}_{\omega}'|
\]

(4.20)

\[
< \frac{1}{20} s^{1/2} |\tilde{\gamma}_{\omega}'|,
\]

where the factor in front in the first inequality reflects the fact that given \( \omega \) there is a set of at most \( |\tilde{\gamma}_{\omega}'| + |\gamma_{\omega}'| + 4 \) sites such that every \( \omega' \) incompatible with \( \omega \) must contain one of these sites in its support. This proves (4.12).

By (4.20), in order to obtain (4.13), which corresponds to (4.8), it suffices that

(4.21) \[ \varphi_{t,0}(\omega) \leq \varphi_{s,0}(\omega) e^{-s^{1/2} |\tilde{\gamma}_{\omega}'|}. \]

Now

(4.22) \[ \frac{\varphi_{t,0}(\omega)}{\varphi_{s,0}(\omega)} \leq \frac{\varphi_{t,0}(\gamma_{\omega}'\text{ext})}{\varphi_{s,0}(\gamma_{\omega}'\text{ext})} e^{u |\tilde{\gamma}_{\omega}'|} \leq \left(\frac{t}{s}\right) |\tilde{\gamma}_{\omega}'|/2 e^{s^{1/2} |\tilde{\gamma}_{\omega}'|}, \]

so (4.21) reduces to

(4.23) \[ t^{1/2} |\tilde{\gamma}_{\omega}'| \leq s^{1/2} |\tilde{\gamma}_{\omega}'| e^{-2s^{1/2} |\tilde{\gamma}_{\omega}'|}. \]

Using \( s \leq 2(3k + 3)^{-4}, |\tilde{\gamma}_{\omega}'| \leq (3k + 3)^2/4 \) and the isoperimetric inequality \( |\tilde{\gamma}_{\omega}'| \geq 4|\gamma_{\omega}'|^{1/2} \), we see that

(4.24) \[ 2s^{1/4} |\tilde{\gamma}_{\omega}'| \leq (3k + 3) s^{1/4} |\gamma_{\omega}'|^{1/2} < 2^{-7/4} |\gamma_{\omega}'|, \]
so it suffices for (4.23) that
\[ t \leq s e^{-(s/8)^{1/4}}. \]
For (4.25), in turn, it suffices that
\[ t e^{t^{1/4}} \leq s \leq 2t. \]
Thus for \( k \geq 2, \) \( t \leq (3k + 3)^{-4} \) and \( s(t) = te^{t^{1/4}}, \) (4.13) is satisfied, and (4.14), (4.15) follow by the Convergence Theorem.

We can use (4.14) to establish convergence of (4.11), as follows. Let \( n(X) = |\{ \omega : X(\omega) \geq 1\}| \) be the number of distinct elementary perturbations in the cluster \( X. \) Then by (4.14) and (4.19) we have
\[
\sum_{X: \text{Supp} X \ni 0} \frac{1}{|\text{Supp} X|} |\varphi^T_u(X)| = \sum_{\omega, 0 \in \gamma_{ext}} \frac{1}{|\text{Supp} \omega|} \sum_{X \ni \omega} \frac{1}{n(X)} |\varphi^T_u(X)| \\
\leq \sum_{\omega, 0 \in \gamma_{ext}} \frac{1}{|\text{Supp} \omega|} \mu(\omega) \leq \infty.
\]
(4.27)\]

Henceforth, dealing with \( t \leq (3k + 3)^{-4}, \) we use the function \( \mu(\omega) = \varphi_{s,0}(\omega) \) defined in the last proof, with \( s = s(t) \equiv te^{t^{1/4}}, \) so
\[ s \leq s_k \equiv 2(3k + 3)^{-4} \leq 2/3^{12} \text{ for } k \geq 8. \]
From (4.13) we have \( \varphi = \varphi_{t,u} \leq \mu. \) In view of (3.6) we observe that
\[ s^{3h+4} = O(t^{3h+4}) \text{ uniformly in } h \leq k, t \leq (3k + 3)^{-4} \text{ and } k \geq 8. \]
One may replace \( 3h + 4 \) here with \( 3h + r \) for any fixed \( r. \)

Next, we are going to compute the relevant terms in the expansion (4.5).

In Figure 6 are represented the clusters that contribute to the difference \( f_k(h + 1) - f_k(h), \) classified according to the number of contacts with the wall, with leading terms singled out, when \( h \geq 2. \) The picture is a schematic representation as the elementary perturbations are actually three-dimensional objects. Either they touch the wall when placed on the interface at level \( h, \) cases (a) to (d), or when placed on the interface at level \( h + 1, \) cases (e) to i).

The elementary perturbations in (a) and (e) are cylinders with one plaquette as base, with \( \| \omega \| = 2h \) in case (a) and \( \| \omega \| = 2h + 2 \) in case (e). Those in (c) and i) are cylinders with two plaquettes as base and we have \( \| \omega \| = 3h \) and \( \| \omega \| = 3h + 3, \) respectively. The perturbations in (b) touch the wall only with one plaquette and satisfy \( \| \omega \| \geq 2h + 1. \) In case (d) they touch with two plaquettes and \( \| \omega \| \geq 3h + 1. \) Finally, cases (f), (g) and (h) correspond to
perturbations that touch the wall with one plaquette, obtained as a continuation of those in cases (b), (c), and (d), and satisfy $\|\omega\| \geq 2h + 3$ in case (f), $\|\omega\| = 3h + 2$ in case (g) and $\|\omega\| \geq 3h + 3$ in case (h).

From these considerations and (4.11) we obtain, for $k \geq 1, h \geq 4$,

$$f_k(h + 1) - f_k(h) = At^{2h}e^u + P_h(t)e^u + Ct^{3h}e^{2u} + Q_h(t)e^{2u}$$

$$- At^{2h} - P_h(t) - Ct^{3h} - Q_h(t)$$

$$- Et^{2h+2}e^u - P_h(t)t^2e^u - Gt^{3h+2}e^u - 2t^2Q_h(t)e^u - It^{3h+3}e^{2u}$$

$$+ V_h(t, u),$$

where $V_h(t, u)$ contains only terms of order $t^{3h+4}$, or smaller, in $t$. In the right side of (4.30), the contributions to $f_k(h)$ from the relevant clusters are in the first line, and the contributions to $f_k(h + 1)$ are in the second and third lines. The coefficients $A, C, E$, etc. are the number of elementary perturbations per

Figure 6. Clusters as in (4.30)
The terms containing $P_h(t)$ correspond to the contributions coming from clusters of types (b) and (f). $P_h(t)$ is a convergent series

$$P_h(t) = B_1 t^{2h+1} + B_2 t^{2h+2} + \cdots + B_{h+3} t^{3h+3} + \cdots$$

where $B_j, j = 1, \ldots, \infty$, is the number, at each site, of the perturbations in case (b) such that $||\omega|| = 2h + j$, plus the number (times their coefficient $a^T(\cdot)$) of clusters of height $h$ of order $t^{2h+j}$. In particular, the leading coefficient $B_1 = 4$ for $h = 0$, and $B_1 = 0$ for $h = 1$. The terms containing $Q_h(t)$ correspond to the contributions coming from cases (d) and (h), and analogously

$$Q_h(t) = D_1 t^{3h+1} + D_2 t^{3h+2} + D_3 t^{3h+3} + \cdots,$$

with leading coefficient $D_1 = 16$ for $h \geq 2$ and $D_1 = 1$ for $h = 1$. Here $D_1 = 16$ comes from perturbations with 2 cubes in the top and bottom layer, with each of the cubes attached to any of the 4 sides of a 1 × 1 column. The dependence of $P_h, Q_h, V_h$ on $k$ is suppressed in the notation. The error term $V_h(t, u)$ excludes those terms of order $t^{3h+4}$ or smaller which are already accounted for in $P_h(t)$ or $Q_h(t)$. Regrouping terms in (4.30) we obtain for $h \geq 4$:

$$f_k(h + 1) - f_k(h) = (t^{2h} + P_h(t))(e^u - 1 - t^2 e^u) + (2t^{3h} + Q_h(t))(e^{2u} - 1 - 2t^2 e^u)$$

$$- 2t^{3h+3} e^{2u} + V_h(t, u). \quad (4.34)$$

We also need the analog of (4.34) for $1 \leq h \leq 3$, which is similar but includes additional terms. For $j = 3h + 2, 3h + 3$ and $m \geq 2$ let $L_{jm}(h)$ be the number (per site, and incorporating the combinatorial factors $a^T(X)$) of clusters of external level $h$ incorporating $m$ horizontal plaquettes touching the wall and $2j$ vertical plaquettes, which are not counted in the top row of Figure 6. From the isoperimetric inequality, $L_{jm}(h) = 0$ for $m > j^2/4h^2$. Define

$$R_h(t, u) = \sum_{j=3h+2}^{3h+3} \sum_{2 \leq m \leq j^2/4h^2} L_{jm}(h)t^i(e^{mu} - 1).$$

Then for $h = 2, 3$,

$$f_k(h + 1) - f_k(h) = (t^{2h} + P_h(t))(e^u - 1 - t^2 e^u) + (2t^{3h} + Q_h(t))(e^{2u} - 1 - 2t^2 e^u)$$

$$- 2t^{3h+3} e^{2u} + R_h(t, u) + V_h(t, u). \quad (4.35)$$
For $h = 1$ we have additional terms with $j = 3h + 1 = 4$:

$$f_k(2) - f_k(1) = (t^2 + P_1(t))(e^u - 1 - t^2e^u) + (2t^3 + Q_1(t))(e^{2u} - 1 - 2t^2e^u)$$

$$+ L_{43}(1)t^4(e^{3u} - 1 - 3t^2e^u) + L_{44}(1)t^4(e^{4u} - 1 - 4t^2e^u)$$

$$+ L_{42}(1)t^4(e^{2u} - 1 - 2t^2e^u) - 2t^6e^{2u} + R_1(t,u) + V_1(t,u).$$

Note that $L_{43}(1) = 6$ is the number (per site) of downward perturbations of height 1 consisting of 3 cubes, adjacent via common faces, and $L_{44}(1) = 1$ is the number (per site) consisting of a $2 \times 2$ block of cubes. $L_{42}(1) = -5/2$ incorporates contributions from (i) single perturbations which consist of a pair of downward cubes with bases touching by a northeast-southwest corner, and (ii) clusters consisting of two single-downward-cube perturbations which may coincide, or may be adjacent with bases touching by either an edge or a northeast-southwest corner.

Finally, for $h = 0$ we have

$$f_k(1) - f_k(0) = u - t^2(1 + e^u - e^{-u}) - 2t^3(1 + e^{2u} - e^{-2u}) + V_0(t,u).$$

The elementary perturbations that contribute to this difference outside of $V_0(t,u)$ are only the cylinders of type (a) and (c) in Figure 6 with height equal to 1, i.e., having as base one or two plaquettes, and $\|\omega\| = 2$ or 3, respectively. However, one has now to consider the upward as well as the downward cylinders.

In order that a given $n \geq 4$ be the optimum interface height, $u$ should be chosen so that (4.34) is negative for $h < n$ and positive for $h \geq n$. At the crossover point at which the right side of (4.34) is 0, at least to order $t^{3h+3}$, it is easily seen that $e^u - 1 - t^2e^u$ is approximately $2t^{h+3}$, so that the term $t^h(e^u - 1 - t^2e^u)$ from perturbations of types (a) and (e) in Figure 6 cancels the term $2t^{3h+3}e^{2u}$ from perturbations of type i); all other terms are then of smaller order. To make the given $n$ be optimal, therefore, ignoring error terms one should have

$$e^u - 1 - t^2e^u \begin{cases} < 2t^{h+3} & \text{for all } h < n, \\ \geq 2t^{h+3} & \text{for all } h \geq n, \end{cases}$$

which (after allowing room for error terms) yields the interval of $u$ values given by (1.14). The essential cancellation of type (a), (e) and i) terms here contrasts with the external-field case in [1], where the balancing is between type (e) and the external field, with all other perturbation types contributing only to smaller order.

The major difficulty in making this idea rigorous is that to establish an optimal $n$ we need control of error terms uniformly in $h$. Thus we need bounds on $P_h$, $Q_h$ and $V_h$ which are uniform in $h$, $k$. This requires a preliminary lemma. Given a cluster $X$, an incompatibility path in $X$ is a finite sequence $(\alpha_0, \ldots, \alpha_m)$
of elementary perturbations in \( X \) satisfying \( \alpha_{i-1} \neq \alpha_i \) or \( \alpha_{i-1} = \alpha_i \) for all \( 1 \leq i \leq m \); such a sequence may be viewed as a path in the incompatibility graph \( g(X) \). The \textit{length} of the path is \( m \); we say the path is \textit{minimal} if there is no strictly shorter path from \( \alpha_0 \) to \( \alpha_m \) in \( g(X) \). Given a cluster \( X \) containing a designated \( \alpha_0 \) which we call the root, we say a perturbation \( \omega \) in \( X \) is \textit{beyond} another perturbation \( \alpha \) in \( X \) (relative to the root \( \alpha_0 \)) if \( \omega \notin \{ \alpha_0, \alpha \} \) and every path from \( \alpha_0 \) to \( \omega \) in \( g(X) \) contains \( \alpha \). The \textit{outer leaf} in \( X \) of a perturbation \( \alpha \in X \) is

\[
L_{out}(\alpha) = \{ \omega \in X : \omega \text{ is beyond } \alpha \},
\]

the \textit{leaf} is

\[
L(\alpha) = L_{out}(\alpha) \cup \{ \alpha \},
\]

and the \textit{stem} of \( \alpha \) in \( X \) is

\[
S(\alpha) = X \setminus L(\alpha).
\]

Note \( \{ \alpha \} = L(\alpha) \cap S(\alpha) \), and the leaf and stem are both clusters. The multiplicity of \( \alpha \) in \( S(\alpha) \) is then \( X(\alpha) \), and the multiplicity of \( \alpha \) in \( L(\alpha) \) is \( 1 \). The leaf \( L(\alpha) \) may be just \( \{ \alpha \} \), meaning the outer leaf is empty; this is always the case if \( X(\alpha) \geq 2 \). For \( X \) containing a minimal incompatibility path \(( \alpha_0, \ldots, \alpha_m \)\( )\), we can then describe the structure of \( g(X) \) as follows. For \( 1 \leq i \leq m - 1 \) we say \( \alpha_i \) is \textit{critical} (for \( \alpha_0 \to \alpha_m \) if \( \alpha_m \) is beyond \( \alpha_i \). Let \( i_1 < \cdots < i_l \) be the indices of the critical perturbations \( \alpha_i \). The 0th bead consists of those \( \omega \in X \) which are not beyond \( \alpha_{i_1} \), the \( l \)th bead consists of \( \alpha_{i_l} \) and those \( \omega \in X \) which are beyond \( \alpha_{i_l} \), and for \( 1 \leq j \leq l - 1 \), the \( j \)th bead consists of \( \alpha_{i_j} \) and those \( \omega \in X \) which are beyond \( \alpha_{i_j} \) but not beyond \( \alpha_{i_{j+1}} \). Note that the intersection of the \(( j - 1) \)st and \( j \)th beads is \( \{ \alpha_{i_j} \} \).

Let \( G(\alpha_0, \ldots, \alpha_m) \) denote the set of clusters \( X \) for which \( (\alpha_0, \ldots, \alpha_m) \) is a minimal incompatibility path in \( g(X) \).

**Lemma 4.3.** (i) Let \( m \geq 0 \) and let \( \alpha_0, \ldots, \alpha_m \) be perturbations with the same \textit{external height} \( h \geq 0 \). Then for all \( u, t \) as in Lemma 4.2 and \( \mu = \varphi_{s,0} \) with \( s = s(t) = te^{t/4} \),

\[
(4.39) \quad \sum_{X \in G(\alpha_0, \ldots, \alpha_m)} |\varphi_u^T(X)| \leq 3^{3m+1} \mu(\alpha_0) \prod_{i=1}^m \varphi(\alpha_i)
\]

and

\[
(4.40) \quad \sum_{X \in G(\alpha_0, \ldots, \alpha_m)} X(\alpha_0) |\varphi_u^T(X)| \leq 3^{3m+2} \mu(\alpha_0) \prod_{i=1}^m \varphi(\alpha_i).
\]

(ii) For \( m = 1 \) the path assumption \( X \in G(\alpha_0, \alpha_1) \) in \((4.39)\) can be removed: for any two distinct elementary perturbations \( \alpha_0, \alpha_1 \) with the same external
height \( h \geq 0, \)
\[
\sum_{X: \alpha_0, \alpha_1 \in X} |\varphi_u^T(X)| \leq 3^4 \mu(\alpha_0) \varphi(\alpha_1)
\]
and
\[
\sum_{X: \alpha_0, \alpha_1 \in X} X(\alpha) |\varphi_u^T(X)| \leq 3^5 \mu(\alpha_0) \varphi(\alpha_1).
\]

(iii) For each elementary perturbation \( \alpha, \)
\[
\sum_{X: X(\alpha) \geq 2} |\varphi_u^T(X)| \leq 3^4 \mu(\alpha) \varphi(\alpha)
\]
and
\[
\sum_{X: X(\alpha) \geq 2} X(\alpha) |\varphi_u^T(X)| \leq 2 \cdot 3^5 \mu(\alpha) \varphi(\alpha)
\]

Proof. (i) For \( m = 0 \) this is a consequence of the Convergence Theorem, so consider \( m \geq 1. \) Let \( X \) be a cluster in which \( (\alpha_0, \ldots, \alpha_m) \) is a minimal incompatibility path. Inductively we first define \( X_m, T_m \) to be the leaf and stem, respectively, of \( \alpha_m \) in \( X, \) with \( \alpha_0 \) as root, then define \( X_{m-1}, T_{m-1} \) to be the leaf and stem of \( \alpha_{m-1} \) in \( T_m, \) continuing this way until \( X_1, T_1 \) are the leaf and stem of \( \alpha_1 \) in \( T_2. \) Then we define \( X_0 = T_1. \)

An alternative description is as follows. If \( \alpha_i \) is non-critical for \( \alpha_0 \rightarrow \alpha_m \) and is part of the \( j \)th bead, then the leaf \( X_i \) is part of the \( j \)th bead as well, and the stem \( T_i \) consists of beads 0 through \( j \), with the outer leaves of \( \alpha_i, \alpha_{i+1}, \ldots \) removed from the \( j \)th bead. If \( \alpha_i \) is critical, then (i) \( X(\alpha_i) = 1, \) (ii) \( \alpha_i \) is the intersection of the \((j-1)st\) and \( j \)th beads for some \( 1 \leq j \leq m, \) (iii) \( X_i \) consists of the \( j \)th bead with the outer leaves of \( \alpha_{i+1}, \alpha_{i+2}, \ldots \) removed, and (iv) the stem \( T_i \) consists of beads 0 through \( j - 1. \)

Thus each bond in \( g(X) \) is in exactly one graph \( g(X_i), \) and each spanning graph of \( X \) is uniquely obtained as a union of spanning graphs of each cluster \( X_i. \) Further, from the bead description, each \( \alpha_i, i \geq 1, \) appears with multiplicity 1 in \( X_i, \) and with multiplicity \( X(\alpha_i) \) in one \( X_j \) with \( j < i; \) in other \( X_l \) it appears with multiplicity 0. It follows that
\[
a^T(X) = \prod_{i=0}^{m} a^T(X_i),
\]
\[
\sum_{j=0}^{m} X_j(\alpha_i) = X(\alpha_i) + 1, \quad i = 0, \ldots, k,
\]
and

\[
|\varphi^T_u(X)| = \frac{\prod_{i=0}^{m} |\varphi^T(X_i)|}{\prod_{i=1}^{m} \varphi(\alpha_i)}.
\]

We now define modifications of the weights \(\varphi\) and \(\mu\) as follows. Let \(I_4(\alpha_0, \ldots, \alpha_m)\) denote the set of perturbations which are incompatible with at least 4 of the perturbations \(\alpha_i\), let \(\hat{\varphi}(\omega) = \begin{cases} 0, & \text{if } \omega \in I_4(\alpha_1, \ldots, \alpha_m), \\ \frac{1}{3}, & \text{if } \omega \in \{\alpha_1, \ldots, \alpha_m\}, \\ \varphi(\omega), & \text{otherwise}, \end{cases} \)

and let \(\hat{\mu}(\omega) = \begin{cases} 0, & \text{if } \omega \in I_4(\alpha_1, \ldots, \alpha_m), \\ \frac{1}{3}, & \text{if } \omega \in \{\alpha_1, \ldots, \alpha_m\}, \\ 3\mu(\omega), & \text{otherwise}, \end{cases} \)

and let \(\hat{\varphi}^T_u(\cdot)\) be the corresponding weight for clusters, as in (4.6). From the definition of minimal path, for each \(X \in G(\alpha_0, \ldots, \alpha_m)\) and \(\omega \in X\) there are at most 3 values \(i\) for which \(\omega \not\sim \alpha_i\), so \(X \cap I_4(\alpha_0, \ldots, \alpha_m) = \emptyset\). Hence for every perturbation \(\omega\), by (4.20),

\[
\sum_{\omega' \not\sim \omega} \hat{\mu}(\omega') \leq 3 \cdot \frac{1}{3} + 3 \sum_{\omega' \not\sim \omega} \mu(\omega') \leq 1 + \frac{1}{4} s^{1/2} |\bar{\gamma}_{\omega}^{ext}|.
\]

Since \(s \leq 2(3k + 3)^{-4}, |\bar{\gamma}_{\omega}^{ext}| \leq (3k + 3)^2/4 \) and \(k \geq 8\), we see that the right side of (4.47) is bounded by \(\log 3\). It is then easily checked that the condition (4.18) is satisfied in the Convergence Theorem for \(\hat{\varphi}\) and \(\hat{\mu}\). Therefore using
and the Convergence Theorem we have

\[ \sum_{X \in \mathcal{G}(\alpha_0, \ldots, \alpha_m)} X(\alpha_0) |\varphi^T_u(X)| \]

\[ = \sum_{X \in \mathcal{G}(\alpha_0, \ldots, \alpha_m)} X(\alpha_0) \frac{\prod_{i=0}^{m} |\varphi^T_u(X_i)|}{\prod_{i=1}^{m} \varphi(\alpha_i)} \]

\[ = \sum_{X \in \mathcal{G}(\alpha_0, \ldots, \alpha_m)} X(\alpha_0) |\hat{\varphi}^T_u(X_0)| \frac{\prod_{i=1}^{m} (\varphi(\alpha_i) X(\alpha_i) + 1 |\hat{\varphi}^T_u(X_i)|)}{\prod_{i=1}^{m} \varphi(\alpha_i)} \]

\[ \leq \left( \prod_{i=1}^{m} 81 \varphi(\alpha_i) \right) \sum_{X \in \mathcal{G}(\alpha_0, \ldots, \alpha_m)} X(\alpha_0) \prod_{i=0}^{m} |\hat{\varphi}^T_u(X_i)| \]

\[ \leq \left( \prod_{i=1}^{m} 81 \varphi(\alpha_i) \right) \left( \sum_{X: X \ni \alpha_0} X(\alpha_0) |\hat{\varphi}^T_u(X)| \right) \prod_{i=1}^{m} \left( \sum_{X: X \ni \alpha_i} |\hat{\varphi}^T_u(X)| \right) \]

\[ \leq 81^m (e^{\hat{\mu}(\alpha_0)} - 1) \prod_{i=1}^{m} \varphi(\alpha_i) \mu(\alpha_i) \]

\[ \leq 3^{3m+2} \mu(\alpha_0) \prod_{i=1}^{m} \varphi(\alpha_i). \]

This proves (4.40). The proof of (4.39) is essentially the same, with the \( X(\alpha_0) \) factors removed.

(ii) In part (i), the path assumption was used only to create the bead description, and to ensure that for relevant clusters \( X \) and \( \omega \in X \), there are at most 3 values \( i \) for which \( \omega \not\sim \alpha_i \). Neither of these considerations is needed for \( m = 1 \) so the same proof applies.

(iii) We wish to use (ii). We define a new polymer system for which the set \( \mathcal{P}^* \) of polymers consists of the set \( \mathcal{P}_{el} \) of elementary perturbations and one additional polymer \( \alpha^* \). This \( \alpha^* \) is a "copy of \( \alpha \)" in the sense that we define the weight \( \varphi^* \) on \( \mathcal{P} \) by \( \varphi^* = \varphi \) on \( \mathcal{P}_{el} \) and \( \varphi^*(\alpha^*) = \varphi(\alpha) \), and define \( \alpha^* \) to be compatible with the same elementary perturbations as \( \alpha \). \( \mu^* \) is defined analogously, and the corresponding truncated function \( (\varphi^*)^T_u \) given by the analog of (4.6). There is a natural projection \( Q \) from clusters in \( \mathcal{P}^* \) to clusters in \( \mathcal{P}_{el} \) defined by replacing each copy of \( \alpha^* \) with a copy of \( \alpha \), and for \( Y \) a cluster in \( \mathcal{P}^* \) we have

\[ (\varphi^*)^T_u(Y) = \left( Y(\alpha) + Y(\alpha^*) \right) \frac{Y(\alpha)}{Y(\alpha)} \varphi^T_u(Q(Y)), \]
because the sum over graphs in (4.17) is the same in $Y$ as in $Q(Y)$. Therefore for each cluster $X$ in $\mathcal{P}_{el}$ with $X(\alpha) \geq 2$,
\[ \sum_{Y:Q(Y) = X \atop \alpha, \alpha^* \in Y} |(\varphi^*)^T_u(Y)| = (2^{X(\alpha)} - 2)|\varphi^T_u(X)| \geq |\varphi^T_u(X)|. \]

We would like to conclude from (ii) (with $\alpha_0 = \alpha^*, \alpha_1 = \alpha$) that
\[ \sum_{X: X(\alpha) \geq 2} |\varphi^T_u(X)| \leq \sum_{Y: \alpha, \alpha^* \in Y} |(\varphi^*)^T_u(Y)| \leq 3^4 \mu^*(\alpha^*) |\varphi^*(\alpha)|. \]

Since the proof of (ii) uses the Convergence Theorem, we need to know that (4.19) remains valid for $\hat{\varphi}$ and $\hat{\mu}$ when one more term $\hat{\mu}(\alpha^*) = 3\mu(\alpha)$ is added to the sum there. But this follows from the fact that (4.41) remains valid with the extra term. Thus we can indeed apply (ii), proving (4.43). The proof of (4.44) is similar, using the fact that $Y(\alpha) + Y(\alpha^*) = X(\alpha)$ when $Q(Y) = X$. \qed

Lemma 4.3(i) can be used to help control the contribution to $V_h(t, u)$ from clusters of large diameter (at least $16h$). We say that an elementary perturbation $\omega$ is \textit{simple} if $\omega$ consists of a single downward cylinder $\gamma$ with $L(\gamma) = 1$. We say a cluster $X$ is \textit{simple} if every elementary perturbation in $X$ is simple. For $A \subset \mathbb{Z}^2$ we define the site boundary $\partial_s A = \{y \in A : d(y, A^c) = 1\}$.

**Lemma 4.4.** Let $u, t$ be as in Lemma 4.3, let $s = s(t) = te^{\alpha/4}$ and let $h \geq 0$.

(i) For all $0 \neq x \in \mathbb{Z}^2$,
\[ (4.49) \sum_{X:x \in \text{Supp}(X)} |\varphi^T_u(X)| \leq 10(180s)^{|x|^2}, \]
where the sum is over clusters from $\mathcal{C}^H(\mathbb{Z}^2, h)$.

(ii) For $\zeta$ an elementary perturbation and $x \in \mathbb{Z}^2$ with $d(\text{Supp} \zeta, x) = r \geq 1$,
\[ (4.50) \sum_{X: \zeta \subseteq X \atop x \in \text{Supp}(X)} |\varphi^T_u(X)| \leq 3^9 r \mu(\zeta)(180s)^{r+1}, \]

and
\[ (4.51) \sum_{X: \zeta \subseteq X \atop x \in \text{Supp}(X)} X(\zeta) |\varphi^T_u(X)| \leq 3^{10} r \mu(\zeta)(180s)^{r+1}. \]

**Proof.** We consider (i) first. We have $0, x \in \text{Supp}(X)$ if and only if there exists a minimal incompatibility path $(\alpha_0, \ldots, \alpha_m)$ in $g(X)$ with $0 \in \tilde{\gamma}_{\alpha_0}^\text{ext}, x \in \tilde{\gamma}_{\alpha_m}^\text{ext}$. Given $X$, let $U_m(X)$ denote the set of all $x$ for which such a minimal path exists, with length $m \geq 0$. Then
\[ (4.52) \sum_{X:0,x \in \text{Supp}(X)} |\varphi^T_u(X)| = \sum_{m=0}^{\infty} \sum_{X:x \in U_m(X)} |\varphi^T_u(X)|, \]
and from Lemma 4.3(i) and a slight modification of (4.17) we have for \( m \geq 0 \),
\[
\sum_{X : x \in U_m(X)} |\varphi_u^T(X)| \leq \sum_{(a_0, \ldots, a_m) \in \bar{\gamma}_{ext}} \sum_{X \in \mathcal{G}(a_0, \ldots, a_m)} |\varphi_u^T(X)|
\leq \sum_{(a_0, \ldots, a_m) \in \bar{\gamma}_{ext}} 3^{2m+1} \prod_{i=1}^m \varphi(\alpha_i)
\leq 3^{3m+1} \sum_{(a_0, \ldots, a_m) \in \bar{\gamma}_{ext}} \left( \sum_{\omega \in \bar{\gamma}_{ext}} \mu(\omega) \right)
\leq 3^{3m+2} \sum_{(a_0, \ldots, a_m) \in \bar{\gamma}_{ext}} \prod_{i=0}^m \mu(\alpha_i).
\]

(4.53)

In these sums, \((\alpha_0, \ldots, \alpha_m)\) represents a sequence of perturbations which form a minimal path in some cluster \( X \); when such an \( X \) exists, one such \( X \) consists of the perturbations \( \alpha_0, \ldots, \alpha_m \) with multiplicity 1 each. In the third line of (4.53) we identify a simple \( \alpha_i \) and its unique cylinder, in a mild abuse of notation. We will show by induction on \( m \) that

(4.54)
\[
\sum_{(a_0, \ldots, a_m) \in \bar{\gamma}_{ext}} \prod_{i=0}^m \mu(\alpha_i) \leq (180s)^{|x|+m+2}.
\]

For \( m = 0 \) this is a simple Peierls-type bound: a cylinder \( \alpha \) with 0, \( x \in \bar{\alpha} \) must have
\[
|\bar{\alpha}| \geq (2|x| + 4) \lor 4|\bar{\alpha}|^{1/2},
\]
and there are at most \( 3^l \) base perimeters \( \bar{\gamma} \) of length \( l \) through a given site, so using (4.18),

(4.55)
\[
\sum_{\alpha \text{ simple}} \mu(\alpha) \leq \sum_{l \geq 2|x|+4} \left( \frac{l}{4} \right)^2 3^l s^{l/2} \leq \frac{5}{64} (2|x| + 4)^2 (3s^{1/2})^2 s^{1/2} \leq (18s)^{|x|+2}.
\]

Now suppose (4.54) holds for \( m = 0, \ldots, j - 1 \) for some \( j \geq 1 \). For \( x \in \mathbb{Z}^2 \) let \( R(x) \) be the (possibly degenerate) rectangle with opposite corners 0 and \( x \); then for all \( y \),
\[
|x - y| = |x| - |y| + 2 \text{dist}(y, R(x)),
\]
where dist denotes $\ell^1$ distance. Given $(\alpha_0, \ldots, \alpha_j)$ with $0 \in \mathring{\gamma}^\text{ext}_{\alpha_0}, x \in \mathring{\gamma}^\text{ext}_{\alpha_j}$, there must exist $y, y' \in \mathbb{Z}^2$, either equal or adjacent, with $y \in \mathring{\gamma}^\text{ext}_{\alpha_{j-1}}, y' \in \mathring{\gamma}^\text{ext}_{\alpha_j}$. Therefore using (4.55),

$$\sum_{(\alpha_0, \ldots, \alpha_j) \in \mathring{\gamma}^\text{ext}_{\alpha_0}, x \in \mathring{\gamma}^\text{ext}_{\alpha_j}} \prod_{i=0}^j \mu(\alpha_i) \leq \sum_{y \in \mathbb{Z}^2} \sum_{y' \in \mathbb{Z}^2 \atop |y' - y| \leq 1} (180s)^{|y|+j+1}(18s)^{|x-y'|+2} \leq 18s(180s)^{j+1} \sum_{y \in \mathbb{Z}^2} 5(180s)^{|y|}(18s)^{|x-y|} \leq 5(18s)^{|x|+1}(180s)^{j+1} \sum_{y \in \mathbb{Z}^2} 10^{|y|}(18s)^{2 \text{dist}(y,R(x))}. \tag{4.56}$$

Now

$$\sum_{y \in \mathbb{Z}^2} 10^{|y|}(18s)^{2 \text{dist}(y,R(x))} \leq \sum_{j=1}^{\infty} j 10^{|x|-j+1} \cdot \sum_{y \in \mathbb{Z}^2 \setminus R(x)} 10^{|y|}(18s)^{2 \text{dist}(y,R(x))} \leq \left(\frac{10}{9}\right)^2 10^{|x|} + 10^{|x|} \cdot 8000s^2 \leq 2 \cdot 10^{|x|}. \tag{4.57}$$

This and (4.56) show that (4.54) holds for $j$, and the induction is complete. Since $s \leq 2 \cdot 3^{-12}$, with (4.52) and (4.53) this shows that

$$\sum_{X:0, x \in \text{Supp}(X)} |\varphi_v^T(X)| \leq \sum_{m=0}^{\infty} 3^{4m+2}(180s)^{|x|+m+2} \leq 10(180s)^{|x|+2}. \tag{4.58}$$

This completes the proof of (i).

For (ii) we may assume $x = 0$. If $0 \in \text{Supp} \zeta$, (4.50) follows from the Convergence Theorem, so we assume $r \geq 1$. Given $X$ let $U_m(X, \zeta)$ denote the set of all $y \in \mathbb{Z}^2$ with $d(y, \text{Supp} \zeta) \leq 1$ for which there exists a minimal incompatibility path $(\alpha_0, \ldots, \alpha_m)$ in $g(X)$, with $\alpha_0 = \zeta$, $y \in \text{Supp} \alpha_1$ and
0 ∈ \text{Supp} \alpha_m. Then using Lemma 4.3(i) as in (4.53), and using (4.54),

\[
\sum_{X: \zeta \in X, \theta \in \text{Supp}(X)} |\varphi^T_u(X)| \leq \sum_{y: d(y, \text{Supp} \zeta) \leq 1} \sum_{m=1}^{\infty} \sum_{X: y \in U_m(X, \zeta)} |\varphi^T_u(X)|
\]

\[
\leq \sum_{y: d(y, \text{Supp} \zeta) \leq 1} \sum_{m=1}^{\infty} \sum_{(\alpha_0, \ldots, \alpha_m) \in \text{Supp} \alpha_m} \sum_{X: y \in \mathcal{I}(\alpha_0, \ldots, \alpha_m)} |\varphi^T_u(X)|
\]

\[
\leq \mu(\zeta) \sum_{y: |y| \geq r-1} \sum_{m=1}^{\infty} 3^{4m+2} \sum_{(\alpha_1, \ldots, \alpha_m) \in \mathcal{I}_m} \prod_{i=1}^{m} \mu(\alpha_i)
\]

\[
\leq 3^{3}\mu(\zeta) \sum_{y: |y| \geq r-1} (180s)|y| + 2
\]

(4.59)

\[
\leq 3^{3}r \mu(\zeta)(180s)^{r+1}.
\]

This proves (4.50). The proof of (4.51) is similar, using (4.40) in place of (4.39) in Lemma 4.3(i). □

We say a cylinder \(\gamma\) in an elementary perturbation is **touching** if \(I(\gamma) = 0\). A touching cylinder is **small** if \(|\gamma| \leq 6\) and **big** if \(|\gamma| \geq 8\). We write \(\omega: h \to 0\), and say \(\omega\) is **touching**, to designate that \(\omega\) is an elementary perturbation from height \(h\) containing a touching cylinder, and we write \(X: h \to 0\) to designate that the cluster \(X\) contains such an elementary perturbation. We say that a touching elementary perturbation \(\omega: h \to 0\) is **multi-touching** if \(\omega\) includes two or more touching cylinders (possibly with nested interiors); otherwise \(\omega\) is **single-touching**. We write \(\mathcal{C}^{si}_k(\Lambda, h)\) for the set of all single-touching elementary perturbations \(\omega: h \to 0\). For cylinders \(\gamma_1, \gamma_2\) we write \(\gamma_1 \prec \gamma_2\) to mean that \(\gamma_1\) is a proper subset of \(\gamma_2\), \(E(\gamma_1) = I(\gamma_2)\) and \(\gamma_1, \gamma_2\) are not separated by any cylinder.

For each \(x \in \mathbb{Z}^2\) and each \(\omega: h \to 0\) with \(\bar{\omega} \ni x\) and \(x\) at height 0 in \(\omega\), there exists a unique maximal compatible family \(\mathcal{T}_x(\omega) = \{\gamma_1 \prec \cdots \prec \gamma_r\}\) with

(4.60)

\[
x \in \gamma_1, \quad I(\gamma_1) = 0 \quad \text{and} \quad \gamma_r = \gamma^\text{ext}_\omega.
\]

This family also makes up an elementary perturbation; we call this special nested type of perturbation a **tornado** above \(x\). We write \(\gamma^\omega_1\) for the innermost cylinder in \(\mathcal{T}_x(\omega)\), and denote by \(T^h_\omega\) the set of all tornadoes (for a given \(h\)) above \(x\).
We say a tornado $\omega : h \to 0, \omega = \{\gamma_1 \prec \cdots \prec \gamma_r\}$, is *semi-monotone* if $I(\gamma_r) > \cdots > I(\gamma_1)$. The tornado is *fully monotone* if also $h > I(\gamma_r)$. Given a tornado $\alpha : h \to 0, \alpha = \{\gamma_1 \prec \cdots \prec \gamma_r = \gamma^\text{ext}\}$, we can construct a semi-monotone tornado $\mathcal{M}(\alpha) : h \to 0$ from $\alpha$ as follows. Let $i_1 < \cdots < i_m = r$ be the indices $i$ for which either $i = r$ or $I(\gamma_i) < \min_{j>i} I(\gamma_j)$. For $l \leq m - 1$ define the cylinder $\zeta_l = (\tilde{\gamma}_{i_l}, I(\gamma_{i_{l+1}}), I(\gamma_{i_l}))$, and let $\zeta_m = \gamma_{i_m}$. Note that $\zeta_l$ is obtained by truncating the top of the cylinder $\gamma_{i_l}$ from its height $E(\gamma_{i_l})$ to the possibly lower height $I(\gamma_{i_{l+1}})$, that is, we retain only the portion of $\gamma_{i_l}$ which projects below all cylinders which are larger, in the ordering of $\prec$. Finally let $\mathcal{M}(\alpha) = \{\zeta_1 \prec \cdots \prec \zeta_m\}$ denote the resulting monotone tornado, and for an elementary perturbation $\omega : h \to 0$ and a site $x$ at height 0 in $\omega$ let $\mathcal{M}_x(\omega) = \mathcal{M}(T_x(\omega))$. We denote by $M^x_h$ and $F^x_h$ the sets of all semi-monotone and fully monotone tornadoes, respectively, above $x$, for a given $h$.

Let $\Omega_a$ and $\Omega_c$ denote the sets of elementary perturbations of types (a), (c), respectively, in Figure 6, at arbitrary location. (The fixed height $h$ is suppressed in the notation but should be clear from the context.)

**Proposition 4.5.** There exists $C$ as follows. Let $k \geq 8$, $t < t_1(k)$ and $s = s(t) = te^{1/4}$. Then for all $h \geq 1$,

\begin{equation}
|P_h(t)| \leq 3^6 s^{2h+1}, \quad |Q_h(t)| \leq 3^{16} s^{3h+1},
\end{equation}

and for all $h \geq 0$ and $u < t^{1/2}$,

\begin{equation}
|V_h(t, u)| \leq Cs^{3h+4}.
\end{equation}

**Proof.** As in (4.16) we have for all $h \geq 1$,

\begin{equation}
|P_h(t)| \leq \sum_{\omega \in M^0_h} \sum_{|\gamma^T_u| = 1, \omega \in \Omega_a} \sum_{\alpha : h \to 0} \sum_{X : h \to 0, \alpha \in X} |\varphi^{T}_u(X)|
\leq \sum_{\omega \in M^0_h} \sum_{|\gamma^T_u| = 1, \omega \in \Omega_a} \mu(\alpha)
\leq \sum_{\omega \in M^0_h} \sum_{|\gamma^T_u| = 1, \omega \in \Omega_a} \mu(\omega) \exp(9s^{1/2} |\gamma^\text{ext}_\omega|)
\leq 25 \sum_{\omega \in M^0_h} \mu(\omega),
\end{equation}

(4.63)
where the third inequality follows from the Convergence Theorem. Similarly for all $h \geq 1$ we have

\begin{equation}
|Q_h(t)| \leq 25 \sum_{\omega \in M^0_h \atop \|\omega\| = 6, \omega \not\in \Omega(c)} \mu(\omega).
\end{equation}

Before bounding the right sides of (4.63) and (4.64), we will obtain bounds for (parts of) $|V_h(t, u)|$ also in terms of sums of weights $\mu(\cdot)$. We will then bound these sums of weights in a somewhat unified way.

For $V_h$ with $h \geq 1$, we split the corresponding sum over clusters into several parts, according to the external height ($h$ or $h + 1$) and according to five types. For height $h$:

(i) Type 1 consists of those clusters $X$ in which some perturbation contains a big touching cylinder.

(ii) Type 2 consists of those clusters, not of type 1, in which some perturbation is multi-touching.

(iii) Type 3 consists of those clusters, not of types 1 and 2, of diameter at least $16h$, in which some perturbation is touching.

(iv) Type 4 consists of those clusters, not of types 1, 2, 3, in which there are two or more (possibly equal) touching perturbations.

Note that outside of these four types, the other clusters $X : h \to 0$ are types (a)–(d) in Figure 6, and for types 3 and 4, all touching perturbations must be of these types (a)–(d). For height $h + 1$, there is a fifth type: we can take a cluster $X : h \to 0$ of Types 1–4, “lift” it everywhere by one height unit to create a cluster $X^+ : h + 1 \to 1$, and then extend it downward to height 0 by either (A) lowering the interior height of a single small cylinder in $X^+$ from 1 to 0, or (B) adding a new compatible small cylinder $\gamma$ to $X^+$ with $E(\gamma) = 1, I(\gamma) = 0$. Thus for height $h + 1$:

(v) Type 5 consists of clusters created by extension, as in (A) or (B) above, of a cluster $X : h \to 0$ of types 1–4.

For $h \geq 1$ and $i = 1, 2, 3, 4$ we write $W_{h,i}(t, u)$ for the contribution to the sum $V_h(t, u)$ from clusters $X : h \to 0$ of type $i$, and $W_{h+1,5}(t, u)$ for the contribution to $V_h(t, u)$ from clusters $X : h + 1 \to 0$ of type 5. Then

\begin{equation}
V_h(t, u) = \sum_{i=1}^{4} \left( W_{h,i}(t, u) - W_{h,i}(t, 0) - W_{h+1,i}(t, u) \right) - W_{h+1,5}(t, u).
\end{equation}
We begin with Type 1. As in (4.63), using the argument of (4.16) to bound the effect of summing over $\alpha$, we have for all $h \geq 1$:

$$ |W_{h,1}(t,u)| \leq \sum_{\omega \in M_h^0} \sum_{\omega : h=0} \sum_{\alpha \in X} \frac{1}{|\gamma_1^\beta|} |\varphi_u(X)| $$

$$ \leq 25 \sum_{\omega \in M_h^0} \frac{1}{|\gamma_1^\beta|} \mu(\omega). \tag{4.66} $$

Turning to Type 2, let $\omega$ be a multi-touching perturbation with no big touching cylinders, so all touching cylinders in $\omega$ have disjoint interiors, and suppose $0$ and some site $x \neq 0$ are in the interiors of some such disjoint touching cylinders. Then $|x| \geq 2$. In $\omega$ there must exist an innermost cylinder $\gamma$ with both $0, x \in \gamma$, and we have $|\gamma| \geq 2(|x| + 2) \geq 8$ and $I(\gamma) = j$ for some $j \geq 1$. The tornadoes $T_0(\omega)$ and $T_x(\omega)$ must both include $\gamma$, and the cylinders larger than $\gamma$ in the ordering of the tornado are the same in both tornadoes. Thus there is a unique triple $T_{0,x}(\omega) = (\eta, \zeta, \xi)$ associated to $\omega$ and $x$, in which $\zeta, \xi$ are tornadoes:

$$ \zeta = \{\zeta_1 \prec \cdots \prec \zeta_r\} \text{ with } 0 \in \tilde{\zeta}_1, |\tilde{\zeta}_1| \leq 6, I(\zeta_i) \geq 1 \text{ for all } i \geq 2, \text{ and } \zeta : j \to 0, $$

$$ \xi = \{\xi_1 \prec \cdots \prec \xi_q\} \text{ with } x \in \tilde{\xi}_1, |\tilde{\xi}_1| \leq 6, I(\xi_i) \geq 1 \text{ for all } i \geq 2, \text{ and } \xi : j \to 0, $$

and

$$ \eta = \{\eta_1 \prec \cdots \prec \eta_m\}, \text{ with } |\tilde{\eta}_1| \geq 2(|x| + 2), I(\eta_i) \geq 1 \text{ for all } i \geq 1, \text{ and } \eta : h \to j, \tag{4.67} $$

with $\gamma = \eta_1, \tilde{\zeta}_r \cap \tilde{\xi}_q = \phi, \zeta_r \prec \eta_1$ and $\xi_q \prec \eta_1$. Here $\eta : h \to j$ means the innermost cylinder $\eta_1$ has interior height $j$. We write $T_{0,x}^0$ for the set of all such triples $(\eta, \zeta, \xi)$ (with $\gamma, j$ arbitrary.) We call a perturbation $\eta$ as in (4.67) a cloud of height $j$, and write $C_k^j(\Lambda, h, j)$ for the set of all such clouds. Note that the cylinders in $\eta, \zeta, \xi$ together make up an elementary perturbation, so our earlier definition gives the weight $\mu(\eta, \zeta, \xi) = \mu(\eta)\mu(\zeta)\mu(\xi)$. The total depth of a tornado or cloud $\eta = \{\eta_1 \prec \cdots \prec \eta_m\}$ is

$$ D(\eta) = \sum_{i=1}^m L(\eta_i). $$
Again analogously to (4.63) we have for all \( h \geq 1 \):

\[
|W_{h,2}(t, u)| \leq \sum_{x: 2 \leq |x| \leq 3k+3} \sum_{(\eta, \zeta, \xi) \in T^{(h)}_{h,x}} \sum_{\alpha \to x \in X} \sum_{\alpha \to x \in X} |\varphi_{\alpha}(X)|
\]

\[
\leq \sum_{x: 2 \leq |x| \leq 3k+3} \sum_{(\eta, \zeta, \xi) \in T^{(h)}_{h,x}} \sum_{\alpha \to x \in X} \mu(\alpha)
\]

\[
\leq 25 \sum_{x: 2 \leq |x| \leq 3k+3} \sum_{(\eta, \zeta, \xi) \in T^{(h)}_{h,x}} \mu(\eta)\mu(\zeta)\mu(\xi)
\]

\[
\leq 25^3 \sum_{x: 2 \leq |x| \leq 3k+3} \sum_{j \geq 1} \left( \sum_{\eta \in C_{\Lambda,\eta}^{(j)}(\Lambda, \eta, j)} \mu(\eta) \right) \left( \sum_{\zeta \in T^{(h)}_{j}} \mu(\zeta) \right)^2
\]

(4.68)

Here in the last inequality, we have bounded the sum over \( T^{(h)}_{j} \) in terms of the sum over \( M^{(j)} \), using the argument of (4.16).

Next consider Type 3. We conclude from Lemma 4.4(i) that for all \( h \geq 1 \),

\[
|W_{h,3}(t, u)| \leq \sum_{x: |x|=8h-2} \sum_{X: x \in \text{Supp}(X)} |\varphi_{u}(X)|
\]

\[
\leq 10 \sum_{x: |x|=8h-2} (180s)^{|x|+2}
\]

\[
\leq 40(8h-2)(180s)^{8h}
\]

\[
\leq s^{3h+4},
\]

(4.69)

where in the last inequality we used \( s \leq s_8 = 2 \cdot 3^{-12} \). Note that (4.69) (excluding the first inequality) is also valid for \( h = 0 \), if we replace \( 8h-2 \) with 8.
Now we consider Type 4. We use Lemma 4.3(ii,iii) and once more reason analogously to (4.63) to obtain that, for all $h \geq 1$,

$$|W_{h,4}(t, u)|$$

$$\leq \sum_{x \in \mathbb{Z}^2} \sum_{\omega \in M_0^h \ |T^2| \leq 16} \sum_{\zeta, \xi, \eta : |T^2| \leq 6} \sum_{\alpha : M_0^h(\zeta) = \omega} \sum_{\xi : M_0^h(\xi) = \omega} \sum_{\zeta : M_0^h(\zeta) = \alpha} \sum_{\xi : M_0^h(\xi) = \alpha} |\phi^T_u(X)|$$

$$\leq 3^6 \left( \sum_{\omega \in M_0^h \ |T^2| \leq 6} \sum_{\zeta, \xi, \eta : |T^2| \leq 6} \mu(\zeta) \right) \left( \sum_{\alpha : M_0^h(\alpha) = \omega} \sum_{\xi : M_0^h(\xi) = \alpha} \varphi(\xi) \right)$$

$$\leq 3^6 \cdot 2(16h + 1)^2 \left( \sum_{\omega \in M_0^h \ |T^2| \leq 2} \mu(\omega) \right)^2$$

(4.70)

$$\leq 3^{17} h^2 \left( \sum_{\omega \in M_0^h \ |T^2| \leq 2} \mu(\omega) \right)^2.$$

Here on the right side of the first inequality, for terms with $\zeta = \xi$ we interpret $\zeta, \xi \in X$ as meaning $X(\zeta) \geq 2$ and use Lemma 4.3(iii).

Last, for Type 5 we observe that when we create a cluster by extension, lifting a cluster $X$ and adding a small cylinder to some elementary perturbation $\alpha \in X$, in the factor $a^T$ from (4.7), $X!$ is increased by a factor of $X(\alpha)$ and (as in the proof of Lemma 4.3(iii)) the sum over subgraphs is unchanged. With this observation, we can decompose the sum $W_{h+1,5}(t, u)$ into sums $W_{h+1,5,i}(t, u)$, $i = 1, 2, 3, 4$, according to the type $i$ of the elementary perturbation that is extended, and then apply a modified version of the first inequality in each of (4.66), (4.68), (4.69) and (4.70) in which each term $|\phi^T_u(X)|$ is multiplied by $X(\alpha)|\gamma^T_1|(t^2 + 4t^3)$ (or similar with $\omega$ in place of $\alpha$), as follows.
First, similarly to (4.66), for \( h \geq 1 \),

\[
|W_{h+1,1}(t, u)| \leq (t^2 + 4t^3) \sum_{\omega \in M_h^0} \sum_{\alpha : h \to 0} \sum_{\alpha \in X} X(\alpha) |\phi_u^T(X)|
\]

\[
\leq s^2 \sum_{\omega \in M_h^0} \sum_{|\tilde{x}| \geq 8} (e^{\mu(\alpha)} - 1)
\]

(4.71)

\[
\leq 3^3 s^2 \sum_{\omega \in M_h^0} \mu(\omega).
\]

Second, similarly to (4.68), for \( h \geq 1 \),

\[
|W_{h+1,2}(t, u)|
\]

\[
\leq (t^2 + 4t^3) \sum_{x: 2 \leq |x| \leq 3k+3} \sum_{(\eta, \zeta, \xi) \in T_h^{0,x}} \sum_{\alpha : h \to 0} \sum_{\alpha \in X} |\eta|_1 X(\alpha) |\phi_u^T(X)|
\]

\[
\leq 2s^2 \sum_{x: 2 \leq |x| \leq 3k+3} \sum_{(\eta, \zeta, \xi) \in T_h^{0,x}} \sum_{\alpha : h \to 0} (e^{\mu(\alpha)} - 1)
\]

\[
\leq 3^4 s^2 \sum_{x: 2 \leq |x| \leq 3k+3} \sum_{(\eta, \zeta, \xi) \in T_h^{0,x}} \mu(\eta) \mu(\zeta) \mu(\xi)
\]

(4.72)

\[
\leq 3^{10} s^2 \sum_{x: 2 \leq |x| \leq 3k+3} \sum_{j \geq 1} \left( \sum_{\eta \in C_j^0(A, h, j)} \mu(\eta) \right) \left( \sum_{\zeta \in M_0} \mu(\zeta) \right)^2.
\]

Third, for an elementary perturbation \( \omega \), let \( r(\omega) = \max\{d(x, 0) : x \in \text{Supp}(\omega)\} \). We claim that for all \( h, l \geq 1 \),

(4.73)

\[
\sum_{\omega \in M_h^0} \sum_{|\tilde{x}| \leq 5} \frac{\mu(\omega)}{r(\omega) = l} \leq 3^{10} l^2 (9s)^l s^{2h}.
\]
LAYERING AND WETTING TRANSITIONS

Assuming this claim, similarly to (4.63), (4.64) and (4.69), using Lemma 4.4(ii) and \( s \leq s_8 \) we obtain that for \( h \geq 1 \),

\[
|W_{h+1,5,3}(t, u)| \\
\leq (t^2 + 4t^3) \left( \sum_{l=0}^{8h-3} \sum_{\omega \in \mathcal{M}_0^b} \sum_{\alpha, h \to 0} \sum_{x : |x| = 8h - 2} \sum_{X : h \to 0} \sum_{x \in \text{Supp}(X)} |\bar{\gamma}_{\omega}^0| \cdot X(\alpha) \cdot |\varphi_u^T(X)| \right) \\
\leq s^2 \left( \sum_{l=0}^{8h-3} \sum_{\omega \in \mathcal{M}_0^b} \sum_{\alpha, h \to 0} \sum_{x \in \text{supp}(X)} 4(8h - 2)3^{10}(8h - 2 - l)\mu(\alpha)(180s)^{8h-1-l} \\
+ 3 \sum_{\omega \in \mathcal{M}_0^b} \sum_{\alpha, h \to 0} \sum_{x \in \text{supp}(X)} \mu(\alpha) \right) \\
\leq 25s^2 \left[ 256 \cdot 3^{10} h^2 \sum_{l=0}^{8h-3} \left( \sum_{\omega \in \mathcal{M}_0^b} \mu(\omega) \right)(180s)^{8h-1-l} + 3 \sum_{\omega \in \mathcal{M}_0^b} \mu(\omega) \right] \\
\leq 25s^2 \left[ 256 \cdot 3^{10} h^2(180s)^{8h-1} + s^2 \sum_{l=0}^{8h-3} l^2 20^{-l} + 4 \cdot 3^{10}(8h - 2)^2 (9s)^{8h-2}s^2 h \right] \\
\leq 160000h^2 180^{-2h}(180s)^{10h} \\
\leq (180s)^{10h}.
\]
Fourth, similarly to (4.70), using Lemma 4.3(ii,iii), for $h \geq 1$,

$$|W_{h+1,5,4}(t, u)|$$

$$\leq (t^2 + 4t^3) \sum_{x \in \mathbb{Z}^2} \sum_{|x| \leq 16h} \sum_{\omega \in M_h^0} \sum_{|\chi| \leq 6} \sum_{\zeta : \mu h = 0} \sum_{\xi, h = 0} \sum_{M_0(\zeta) = \omega \ M_x(\xi) = \alpha} |\eta| X(\zeta) |\varphi_T^T(X)|$$

$$\leq 2s^2 \sum_{x \in \mathbb{Z}^2} \sum_{|x| \leq 16h} \sum_{\omega \in M_h^0} \sum_{|\chi| \leq 6} \sum_{\zeta : \mu h = 0} \sum_{\xi, h = 0} \sum_{M_0(\zeta) = \omega \ M_x(\xi) = \alpha} 2 \cdot 3^5 \mu(\zeta) \varphi(\xi)$$

$$\leq 4 \cdot 3^5 s^2 \sum_{x \in \mathbb{Z}^2} \sum_{|x| \leq 16h} \sum_{\omega \in M_h^0} \sum_{|\chi| \leq 6} \mu(\zeta) \left( \sum_{\omega \in M_h^0} \sum_{|\chi| \leq 6} \varphi(\xi) \right)^2$$

$$\leq 8 \cdot 3^5 (16h + 1)^2 s^2 \left( 25 \sum_{\omega \in M_h^0} \mu(\omega) \right)^2$$

$$\leq 3^{18} h^2 s^2 \left( \sum_{\omega \in M_h^0} \mu(\omega) \right)^2.$$

We now establish bounds for the right sides of (4.63), (4.64), (4.66), (4.68), (4.70), (4.71), (4.72), and (4.75), and establish the claim (4.73). We first do this with $M_h^0$ replaced by $F_h^0$. In fact we claim that for all $h \geq 1, q \geq 3$,

$$S_1(h, 2q) \equiv \sum_{\omega \in F_h^0 \atop |\zeta| = 1, \omega \in \Omega(\alpha) \atop |\zeta| \geq 2q} \mu(\omega) \leq 6q^2 (9s)^q s^{2h-2} \prod_{j=1}^{h-1} (1 + (3^{11}s)^{j-1})$$

and when $q \geq 4$,

$$S_2(h, 2q) \equiv \sum_{\omega \in F_h^0 \atop |\zeta| = 6, \omega \in \Omega(\xi) \atop |\zeta| \geq 2q} \mu(\omega) \leq 4q^2 (9s)^q s^{3h-3} \prod_{j=1}^{h-1} (1 + (3^{11}s)^{j-1})$$

and

$$S_3(h, 2q) \equiv \sum_{\omega \in F_h^0 \atop |\zeta| \geq 8 \atop |\zeta| \geq 2q} \mu(\omega) \leq q^2 (9s)^q (3^{11}s)^{h-1}.$$
Of course the products in (4.76) and (4.77) are bounded in $h$ (bounded by 12, in fact, since $s \le s_8$), so they can be replaced by constants, but their presence simplifies the induction. For $h = 1$ we have

$$S_1(1, 2q) = S_2(1, 2q) = 0 \quad \text{for all } q \ge 3,$$

and we have the Peierls bound from (4.18):

$$S_3(1, 2q) \le \sum_{l \ge 2q} \left( \frac{l}{4} \right)^2 3^l s^{l/2} \le q^2 (9s)^q \quad \text{for all } q \ge 4.$$  

Now suppose that (4.76)—(4.78) are valid for $h = 1, \ldots, m$, and all given values of $q$, for some $m \ge 1$, and consider $h = m + 1$. Considering the effect of removing the lowest layer of cubes from a fully monotone tornado in $F^h_{m+1}$ we see that

$$S_1(m + 1, 2q) \le s^2 \left( S_1(m, 2q) + 4s^{3m} \delta_{(q=3)} + S_2(m, 2q) + S_3(m, 2q) \right) \le 6q^2 (9s)^q s^{2m} \left( 1 + \frac{2}{3^9} s^{m-1} + \frac{2}{3} s^{m-1} + \frac{1}{6} (3^{11} s^2)^{m-1} \right) \prod_{j=1}^{m-1} (1 + (3^{11} s)^{j-1})$$

$$\le 6q^2 (9s)^q s^{2m} \prod_{j=1}^{m} (1 + (3^{11} s)^{j-1}),$$

where we used $s \le s_8$. Similarly,

$$S_2(m + 1, 2q) \le s^3 \left( S_2(m) + 4S_3(m) \right) \le 4q^2 (9s)^q s^{3m-3} \left( 1 + (3^{11} s)^{m-1} \prod_{j=1}^{m-1} (1 + (3^{11} s)^{j-1}) \right)$$

$$= 4q^2 (9s)^q s^{3m-3} \prod_{j=1}^{m} (1 + (3^{11} s)^{j-1}),$$

and

$$S_3(m + 1, 2q) \le S_3(1, 8) S_3(m, 2q) \le 16(9s)^4 \cdot q^2 (9s)^q (3^{11} s^4)^{m-1} \le q^2 (9s)^q (3^{11} s^4)^{m},$$

so (4.76)—(4.78) are valid for $h = m + 1$ as well, establishing the claim.

We would now like to replace $F^0_h$ with $M^0_h$ in (4.76)—(4.78). If $\omega : h \to 0, \omega = \{ \gamma_1 \prec \cdots \prec \gamma_k = \gamma^\text{ext}_h \}$, is a semi-monotone tornado for some $h \ge 1$, and $I(\gamma^\text{ext}_h) = h + j$ for some $j \ge 1$, then $\{ \gamma_1 \prec \cdots \prec \gamma_{k-1} \}$ is a fully monotone
tornado from \( h + j \) to 0. Therefore for \( h \geq 1 \) and \( q \geq 3 \), using (4.18) and (4.76), and using \( S_1(h + j, 6) \leq 2s^{2(h+j)} \) for \( s \leq s_8 \),

\[
\sum_{\omega \in M^0_h \atop |\omega_i^0| = 1, \omega \not\in \Omega(a) \atop |\omega_i^0| \geq 2q} \mu(\omega) \leq S_1(h, 2q) + \sum_{j \geq 1} \left( 2s^{2(h+j)} + S_1(h + j, 6) \right) \sum_{l \geq 2q} \left( \frac{l}{4} \right)^2 3^l s^{j(l)/2} \\
\leq S_1(h, 2q) + 3s^{2h} \sum_{l \geq 2q} \left( \frac{l}{4} \right)^2 3^l s^{j(l+1)/2} \\
\leq 72q^2(9s)^q s^{2h-2} + q^2(9s)^q s^{2h+2} \\
\leq 3^4 q^2(9s)^q s^{2h-2}.
\]

(4.83)

Similarly, for \( h \geq 1 \) and \( q \geq 4 \), using \( S_2(h + j, 8) \leq 19s^{3(h+j)} \) for \( s \leq s_8 \),

\[
\sum_{\omega \in M^0_h \atop |\omega_i^0| = 6, \omega \not\in \Omega(c) \atop |\omega_i^0| \geq 2q} \mu(\omega) \leq S_2(h, 2q) + \sum_{j \geq 1} \left( 3s^{3(h+j)} + S_2(h + j, 8) \right) \sum_{l \geq 2q} \left( \frac{l}{4} \right)^2 3^l s^{j(l)/2} \\
\leq S_2(h, 2q) + 20s^{3h} \sum_{l \geq 2q} \left( \frac{l}{4} \right)^2 3^l s^{j(l+1)/2} \\
\leq 48q^2(9s)^q s^{3h-2} + 7q^2(9s)^q s^{3h+3} \\
\leq 3^4 q^2(9s)^q s^{3h-3}.
\]

(4.84)

and using \( S_3(h + j, 8) \leq (3^{11} s^4)^{h+j} \),

\[
\sum_{\omega \in M^0_h \atop |\omega_i^0| \geq 8 \atop |\omega_i^0| \geq 2q} \mu(\omega) \leq S_3(h, 2q) + \sum_{j \geq 1} S_3(h + j, 8) \sum_{l \geq 2q} \left( \frac{l}{4} \right)^2 3^l s^{j(l)/2} \\
\leq S_3(h, 2q) + \left( 3^{11} s^4 \right)^{h} \sum_{l \geq 2q} \left( \frac{l}{4} \right)^2 3^l \left( 3^{11} s^{(l+8)/2} \right)^j \\
\leq q^2(9s)^q(3^{11} s^4)^{h-1} + q^2(9s)^q(3^{11} s^4)^{h+1} \\
\leq 2q^2(9s)^q(3^{11} s^4)^{h-1}.
\]

(4.85)

From (4.63), (4.64), (4.83) with \( q = 3 \) and (4.84) with \( q = 4 \), we obtain that for \( h \geq 1 \),

\[
|P_h(t)| \leq 3^{12} s^{2h+1}, \quad |Q_h(t)| \leq 3^{15} s^{3h+1}.
\]

(4.86)

From (4.66) and (4.85) with \( q = 4 \), we obtain that for \( h \geq 4 \),

\[
|W_{h,1}(t, u)| \leq 30(3^{11} s^4)^{h} \leq 3^{48} s^{3h+4}.
\]

(4.87)
Note that the first inequality in (4.87), but not the second, is valid for \( h = 1, 2, 3 \). From (4.70), (4.83) with \( q = 3 \) and (4.84) with \( q = 4 \), we obtain that for \( h \geq 4 \),

\[
(4.88) \quad |W_{h,4}(t, u)| \leq 3^{17} h^2 \left( s^{2h} + 3^{12} s^{2h+1} + s^{3h} + 16 \cdot 3^{12} s^{3h+1} \right)^2 \leq 3^{22} s^{3h+4}.
\]

To control \( |W_{h,2}(t, u)| \) we need a bound for the sum over clouds \( \eta \) on the right side of (4.68). We proceed by induction on the total depth \( D(\eta) \). We claim that for all \( d \geq 1 \), \( |x| \geq 2 \) and all \( h, j \geq 1 \) with \( d \geq |h - j| \lor 1 \),

\[
(4.89) \quad \sum_{\eta \in C^\ast_{\text{cl}}(\Lambda, h, j)} \mu(\eta) \leq (18s)^d(|x|+2).
\]

Write \( V(d, h, j, |x|) \) for the sum in (4.89). For \( d = 1 \), (4.89) is a consequence of a Peierls argument and (4.18): for \( h \geq 1, j = h \pm 1 \) and \( |x| \geq 2 \), similarly to (4.55) we have

\[
(4.90) \quad V(1, h, j, |x|) \leq \sum_{l \geq 2(|x|+2)} 2 \left( \frac{l}{4} \right)^2 3^l s^{l/2} \leq \frac{5}{8} (|x| + 2)^2 (9s)^{|x|+2} \leq (18s)^{|x|+2}.
\]

Now let \( m \geq 1 \) and suppose (4.89) is valid whenever \( d \leq m \). A cloud \( \eta \) of depth \( m + 1 \) can be obtained by adding a cylinder of length 1 to a cloud of depth \( m \), or by extending the length of the innermost cylinder \( \eta_1 \) by 1. Therefore using (4.90), for \( h, j \geq 1 \),

\[
(4.91) \quad V(m + 1, h, j, |x|) \leq \left( V(m, h, j - 1, |x|) + V(m, h, j + 1, |x|) \right) \sum_{l \geq 2(|x|+2)} \left( \frac{l}{4} \right)^2 3^l s^{l/2} \leq (18s)^m(|x|+2) (18s)^{|x|+2} \leq (18s)^{(m+1)(|x|+2)},
\]

so (4.89) is valid for \( d = m + 1 \) as well, establishing the claim.

Consider now \( W_{h,2}(t, u) \). Observe that the quantity in parentheses in (4.88) is actually a bound for the quantity in parentheses on the right side of (4.70), and hence it is also a bound for the quantity which is squared on the right side of (4.68). With this fact, summing (4.89) over \( d \) with \( d \geq |h - j| \lor 1 \), and
plugging into (4.68), we obtain analogously to (4.69) that for \( h \geq 4 \),
\[
|W_{h,2}(t, u)| \leq 25^3 \sum_{x: 2 \leq |x| \leq 3k+3} \sum_{j \geq 1} 2(18s)^{|h-j||V_1|(|x|+2)} \cdot \left( s^{2j} + 3^{12} s^{2j+1} + s^{3j} + 16 \cdot 3^{12} s^{3j+1} \right)^2 \\
\leq 10 \cdot 25^3 \sum_{j \geq 1} s^{4j} \sum_{x: |x| \geq 2} (18s)^{|h-j||V_1|(|x|+2)} \\
\leq 3^4 \cdot 25^3 \sum_{j \geq 1} (18s)^{4(|h-j||V_1|)} \\
\leq \frac{18}{17} \cdot 3^4 \cdot 25^3 (18s)^{4(h-1)} s^4 \\
\leq 3^{42} s^{3h+4}. 
\]  
(4.92)

Turning to the sums \( W_{h+1,5,i}(t, u) \), similarly to the bound (4.87) on (4.66), for \( h \geq 4 \) (4.71) leads to
\[
|W_{h+1,5,1}(t, u)| \leq 3^{48} s^{3h+6}, 
\]  
(4.93)
and similarly to the bound (4.92) on (4.68), (4.72) leads to
\[
|W_{h+1,5,2}(t, u)| \leq 3^{43} s^{3h+6}. 
\]  
(4.94)

For \( W_{h+1,5,3}(t, u) \) with \( h \geq 1 \), let us establish the claim (4.73) to complete the proof of (4.74). For \( l = 0 \) the sum on the left side of (4.73) has only one term, \( s^{2h} \), so (4.73) holds. For \( l = 1 \), by (4.76) the left side of (4.73) is bounded by
\[
S_1(h, 6) + 4s^{3h} \leq 3^{12} s^{2h+1}. 
\]

For \( l \geq 2 \), by (4.76) and (4.77) the left side of (4.73) is bounded by
\[
S_1(h, 2l + 4) + S_2(h, 2l + 4) \leq 2 \cdot 3^9 l^2 (9s)^l s^{2h} + 3^9 l^2 (9s)^l s^{3h-1} \\
\leq 3^{10} l^2 (9s)^l s^{2h}, 
\]  
(4.95)
which establishes the claim. Finally, using the bound (4.88) for the right side of (4.70), we see from (4.75) that
\[
|W_{h+1,5,4}(t, u)| \leq 3^{23} s^{3h+6}. 
\]  
(4.96)
Combining (4.74), (4.93), (4.94), and (4.96) and using \( s \leq s_8 \) yields that for \( h \geq 4 \),
\[
|W_{h+1,5}(t, u)| \leq 3^{26} s^{3h+4}. 
\]  
(4.97)
Then from (4.65), (4.69), (4.84), (4.88), (4.92) and (4.97), again for \( h \geq 4 \),
\[
|V_h(t, u)| \leq 3^{49} s^{3h+4}, 
\]  
(4.98)
which with (4.86) proves the proposition for these \(h\).

To deal with \(h = 1, 2, 3\) we first observe that the bounds (4.69), (4.73), (4.74) and (4.86) remain valid for these \(h\), so we may restrict attention to elementary perturbations and clusters whose support is contained in \(\{x : |x| < 8h\}\), and we need only establish (4.98). Similarly, for \(h = 0\) we may assume all supports are in \(\{x : |x| < 8\}\). Let \(C_h\) be the set of all clusters consisting only of perturbations with such supports. For a cluster \(X\) let \(m(X)\) denote the number of horizontal plaquettes in \(X\) in contact with the wall in the case \(h \geq 1\), and the negative of the number not in contact with the wall, in the case \(h = 0\). Let \(\|X\|\) denote half the number of vertical plaquettes in \(X\), so that \(\varphi^T_u(X) = a^T(X)t\|X\|e^{um(X)}\). It is easily seen that \(|m(X)| \leq 4\|X\|\) for all \(0 \leq h \leq 3\), and \(X \in C_h\). Note now that the absolute convergence of the cluster expansion, established in Lemma 4.2, means that provided \(k \geq 8\), for all \(0 \leq h \leq 3\) we have, using (4.9) and (4.19),

\[
(4.99) \quad t \leq t_1(8), \quad u \leq t^{1/2} \implies \sum_{X \in C_h} |a^T(X)|t\|X\|e^{um(X)} < 3000s^2 \cdot 2(8(h+1))^2
\]

and hence

\[
\sum_{X \in C_h, \|X\| \geq 3h+4} |a^T(X)|t\|X\|e^{um(X)} \leq \left(\frac{t}{t_1(8)}\right)^{3h+4} \sum_{X \in C_h, \|X\| \geq 3h+4} |a^T(X)|t_1(8)\|X\|e^{um(X)}
\]

\[
< \left(\frac{t}{t_1(8)}\right)^{3h+4} 6000 s(t_1(8))^2 \cdot (8(h+1))^2
\]

\[
< Ct^{3h+4},
\]

with \(C = t_1(8)^{-11} \times 7000 \times 24^2\). This shows that (4.98) holds, with a different constant, for \(0 \leq h \leq 3\), and completes the proof. \(\square\)

For later use, we note that by combining (4.63), (4.83), (4.64), (4.84), (4.98) and (4.100), we see that there exist numbers \(C, K_1\) such that for \(h \geq 1, k \geq 8, t < t_1(k)\) and \(u \leq \sqrt{t}\),

\[
\sum_{X : h \to 0 \atop \text{Supp } X \geq 0} \frac{1}{|\text{Supp } X|} |\varphi^T_u(X)|
\]

\[
\leq (t^{2h} + 25 \cdot 3^{12} s^{2h+1})e^u + (t^{3h} + 25 \cdot 3^{12} s^{3h+1})e^{2u} + Cs^{3h+4}
\]

\[
(4.101) \quad \leq 2t^{2h} + K_1 s^{2h+1}.
\]

Here we use the fact that the bound (4.98) is obtained by adding a bound for the sum of the absolute values of all terms corresponding to clusters \(X : h \to 0\) to a similar bound for clusters \(X : h+1 \to 0\). In the case \(h = 0\), in place of (4.101) we have using (4.69) (modified for \(h = 0\) in the manner noted there)
and (4.100) that for some $K_2$,

\[ (4.102) \quad \sum_{X \in \text{Supp} X \geq 0} \frac{1}{|\text{Supp} X|} \left| \varphi_u^T(X) \right| \leq t^2 e^{-u} + 2t^3 e^{-2u} + Ct^4 + s^4 \leq t^2 + K_2 t^3. \]

We continue with the proof of Proposition 3.1. We first assume $u = t^2 + O(t^4)$, in agreement with (3.4) for $n \geq 2$. Then $e^{mu} - 1 - mt^2 e^u = O(t^4)$ for $m = 2, 3, 4$, and $R_h(t, u) = O(t^{3h+4})$ for $h = 1, 2, 3$. Taking (4.34)–(4.36) into account, it follows using Proposition 4.5 that for $h \geq 1$,

\[ f_k(h + 1) - f_k(h) \]

\[ (4.103) \quad = (t^{2h} + P_h(t)) \left( \exp(u + \ln(1 - t^2)) - 1 \right) - 2t^{3h+3} + O(s^{3h+4}), \]

with the $O(s^{3h+4})$ uniform in $h \geq 1$ and $k \geq 8$, provided we restrict to $t < t_1(k)$.

If we assume that $n \geq 2$ and the first inequality in equation (3.3) holds, then as a consequence of equation (4.103) we have $\forall h \geq 1$,

\[ f_k(h + 1) - f_k(h) \geq (2 + a)t^{2h+n+3} - 2t^{3h+3} + O(s^{3h+4}) + O(s^{2h+n+4}) \]

and in particular,

\[ f_k(n + 1) - f_k(n) \geq at^{3n+3} + O(s^{3n+4}) \]

and

\[ f_k(h + 1) - f_k(h) \geq (2 + a)t^{2h+n+3} + O(s^{2h+n+4}), \quad \forall h \geq n + 1. \]

Here all $O(\cdot)$ terms are uniform in $h \geq 1, n \geq 1, k \geq 8$ provided we restrict to $t < t_1(k)/2$. Summing these increments we obtain

\[ f_k(n) \leq f_k(h) - at^{3n+3} + O(s^{3n+4}), \quad \text{uniformly in } k \geq 8, n \geq 1, h \geq n + 1. \]

Restricting to $n \leq k$ allows us to replace $O(s^{3n+4})$ with $O(t^{3n+4})$, by (1.29), so (3.6) is proved.

If instead we assume that the second inequality in equation (3.4) holds with $n \geq 2$, then again from (4.103), $\forall h \geq 1$,

\[ f_k(h + 1) - f_k(h) \leq (2 - b)t^{2h+n+2} - 2t^{3h+3} + O(s^{3h+4}) + O(s^{2h+n+3}), \]

and in particular, for $n \geq 2$,

\[ f_k(n) - f_k(n - 1) \leq -bt^{3n} + O(s^{3n+1}) \]

and

\[ f_k(h) - f_k(h - 1) \leq -2t^{3h} + O(s^{3h+1}), \quad \forall 2 \leq h \leq n - 1, \]

again with uniformity in $h \geq 1, n \geq 2, k \geq 8$ for $O(\cdot)$ terms provided we restrict to $t < t_1(k)$. From (4.37) we also have

\[ f_k(1) - f_k(0) \leq -2t^3 + O(t^4), \]
uniformly in \( k \geq 8 \) for these same \( t \) values. Summing the increments we obtain

\( f_k(n) \leq f_k(h) - 2t^{3h+3} + O(s^{3h+4}) \), uniformly in \( k \geq 8, n \geq 2, 0 \leq h \leq n - 2 \)

and

\( f_k(n) \leq f_k(n - 1) - bt^{3n} + O(s^{3n+1}) \), uniformly in \( k \geq 8, n \geq 2 \).

As with (4.107), restricting to \( n \leq k \) makes (3.7) a consequence of (4.112) and (4.113).

In the \( n = 1 \) case, we have

\( u = t^2 + O(t^3), e^{mu} - 1 = mt^2e'^{u} = m(u + \log(1 - t^2)) + O(t^4) \) for \( m = 2, 3, 4 \) and \( R_h(t, u) = O(t^{3h+4}) \) for \( h = 1, 2, 3 \). Therefore by (4.34)–(4.36), for \( h \geq 1 \), in place of (4.103) we have

\[
 f_k(h + 1) - f_k(h) = (t^{2h} + O(t^{2h+1})) (u + \ln(1 - t^2)) - 2t^{3h+3} + O(s^{3h+4}),
\]

with the \( O(\cdot) \) uniform in \( h \geq 2 \) and \( k \geq 8 \). Assuming the first inequality in (3.4) it follows that

\[
 f_k(h + 1) - f_k(h) \geq (2 + a)t^{2h+4} + O(t^{2h+5}), \quad \forall \ h \geq 2,
\]

and

\[
 f_k(2) - f_k(1) \geq at^6 + O(t^7).
\]

Now (3.6) follows from (4.115) and (4.116). From (4.37) we obtain

\[
 f_k(1) - f_k(0) \leq -bt^3 + O(t^4),
\]

which proves (3.7).

It remains only to consider the case \( n = 0 \). Here the assumption is \(-\ln(1 - t^2) + (2 + a)t^3 \leq u < \sqrt{t} \). Let us first assume that \( u \) is at most of the order \( t^2 \). Then

\[
 f_k(1) - f_k(0) = u - t^2 - 2t^3 + O(t^4),
\]

which implies that

\[
 f_k(1) - f_k(0) \geq at^3 + O(t^4).
\]

But in fact, \( u = O(t^2) \) is stronger than necessary. We notice that, from (4.37) and Proposition 4.5 uniformly in \( u < t^{1/2} \),

\[
 f_k(1) - f_k(0) = u(1 - 2t^2) - t^2 + O(t^3) + O(t^2u^3),
\]

which is greater than \( u/2 \) provided \( 3t^2 < u < t^{1/2} \) and \( t \) is sufficiently small. This ends the proof of Proposition 3.1.
5. Proof of Theorem 1.1

We consider the model in a rectangular box $\Lambda$, with sides parallel to the axes, under the constant boundary condition $\bar{\phi}_x = n$, for any given integer $n \geq 0$. First, we are going to rewrite the partition function $\Xi(\Lambda, n)$, expression (2.3), using some grouping of the nonelementary cylinders. This will allow us to describe the model as a polymer system. The next task will be to study the cluster expansion that can be obtained by this method.

For this purpose, given $\Gamma \in \mathcal{C}(\Lambda, n)$, we define $\Gamma_k^L \subset \Gamma$ (the subset of large cylinders), as the set of cylinders obtained by removing from $\Gamma$ all those cylinders which are elementary. In all this section, we take $k \geq \max(2n, 8)$. $\Gamma_k^L$ is still a (possibly empty) compatible set of cylinders, because the operation of removing from $\Gamma$ a cylinder together with all the other cylinders contained in it does not spoil the compatibility. We set

$$\mathcal{C}_k^L(\Lambda, n) = \{ \Gamma \in \mathcal{C}(\Lambda, n) : \Gamma = \Gamma_k^L \}.$$  

Then, according to (2.3), we can write

$$\Xi(\Lambda, n) = e^{u\delta(n)\vert\Lambda\vert} \sum_{\Gamma \in \mathcal{C}(\Lambda, n)} \prod_{\gamma \in \Gamma} \varphi(\gamma) \tag{5.1}$$

We define a contour as a compatible set $\Gamma \in \mathcal{C}_k^L(\Lambda, n)$ of nonelementary cylinders, such that:

1. there exists a unique cylinder which is external in $\Gamma$,
2. if $\gamma \in \Gamma$ and $I(\gamma) = n$, then there is no other $\gamma' \in \Gamma$ such that $\bar{\gamma}' \subset \bar{\gamma}$.
3. Condition (2) is equivalent to
4. if $\gamma \in \Gamma$ is not external, then $E(\gamma) \neq E(\Gamma) = n$.

Any given contour $\Gamma$ can be written as a set $\Gamma = \{ \gamma^{ext}, \gamma_i, \gamma^{int}_j \}$ where $\gamma^{ext}$ is the unique external cylinder in $\Gamma$, $\gamma^{int}_j$ are the cylinders that satisfy $I(\gamma^{int}_j) = n$, and $\gamma_i$ are the remaining cylinders in $\Gamma$.

With these notations, we define

$$\text{Supp}(\Gamma) = \bar{\gamma}^{ext} \setminus (\bigcup_j \bar{\gamma}_j^{int}), \tag{5.2}$$

$$\text{Supp}^i(\Gamma) = \bar{\gamma}_i \setminus (\bigcup_{\bar{\gamma} \subset \bar{\gamma}_i} \bar{\gamma}), \tag{5.3}$$

$$\text{Supp}^{ext}(\Gamma) = \bar{\gamma}^{ext} \setminus (\bigcup_{\gamma \neq \bar{\gamma}^{ext}} \bar{\gamma}). \tag{5.4}$$

The set $\text{Supp}(\Gamma)$ is called the support of $\Gamma$. The sets (5.3), associated to the cylinders $\gamma_i$, together with the set (5.4), associated to $\gamma^{ext}$, are mutually disjoint subsets of $\text{Supp}(\Gamma)$ and their union coincides with this support.
As an explanation for these definitions let us notice that \( E(\Gamma) = E(\gamma^{ext}) = I(\gamma_j^{int}) = n \). On the other hand, if we consider the interface that contains only the contour \( \Gamma \), then the associated configuration \( \phi_\Lambda \) has \( \phi_x = n \) if \( x \not\in \text{Supp}(\Gamma) \) and \( \phi_x \neq n \) if \( x \in \text{Supp}(\Gamma) \). It takes constant values in \( \text{Supp}^{ext}(\Gamma) \), namely \( \phi_x = I(\gamma^{ext}) \), and in each \( \text{Supp}^i(\Gamma) \), where we have \( \phi_x = I(\gamma_i) \).

We say that two contours \( \Gamma \) and \( \Gamma' \) are compatible if their supports do not intersect and \( \Gamma \cup \Gamma' \in C_L^k(\Lambda, n) \). This last condition enters only when the boundaries of the supports of \( \Gamma \) and \( \Gamma' \) have a non-empty intersection, otherwise it is already satisfied.

Notice that, from definition (5.2), \( \partial \text{Supp}(\Gamma) = \tilde{\gamma}^{ext} \cup \left( \cup_j \tilde{\gamma}_j^{int} \right) \), so the compatibility condition \( \Gamma \cup \Gamma' \in C_L^k(\Lambda, n) \) concerns only these particular cylinders. Since, among the compatibility conditions (see Section 2), condition (3) is already satisfied, they have only to agree with the signs according to conditions (1) and (2). More precisely, any two cylinders \( \gamma \) and \( \gamma' \), not necessarily compatible, such that \( \tilde{\gamma} \cap \tilde{\gamma}' \neq \emptyset \), satisfy the sign condition if,

\[
S(\gamma) = S(\gamma'), \text{ if } \tilde{\gamma} \subset \tilde{\gamma}', \text{ and } S(\gamma) = -S(\gamma'), \text{ if } \tilde{\gamma} \subset \Lambda \setminus \tilde{\gamma}'.
\]

This leads to the following equivalent definition.

Two contours \( \Gamma \) and \( \Gamma' \), such that \( E(\Gamma) = E(\Gamma') \), are compatible if their supports do not intersect and if, in addition, we have: \( S(\gamma^{ext}) = -S(\gamma'^{ext}) \), if \( \tilde{\gamma}^{ext} \cap \tilde{\gamma}'^{ext} \neq \emptyset \), \( S(\gamma^{ext}) = S(\gamma_j^{int}) \), if \( \tilde{\gamma}^{ext} \cap \tilde{\gamma}_j^{int} \neq \emptyset \) and \( S(\gamma'^{ext}) = S(\gamma_j^{int}) \), if \( \tilde{\gamma}'^{ext} \cap \tilde{\gamma}_j^{int} \neq \emptyset \).

The set of contours \( \{\Gamma_i\} \) is a compatible set, if any two contours in it are compatible. Let \( \Gamma \in C_L^k(\Lambda, n) \). Then it is possible to write \( \Gamma \) as the disjoint union

\[
\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_r
\]

in such a way that, for each \( i = 1, \ldots, r \), \( \Gamma_i \) is a contour. The decomposition (5.6) is unique.

At this point we can write the first sum in expression (5.1) as a sum over compatible sets of contours \( \{\Gamma_i\} \). The sum over the elementary contours in (5.1) decomposes then into a product of sums associated to the different regions of \( \Lambda \) determined by the contours, and gives rise to some partition functions that we are going to discuss.

Some remarks on the geometry of contours will be helpful. We notice that the interiors of the cylinders in \( \Gamma \) are partially ordered by inclusion. The maximal one in the sense of this partial order is \( \gamma^{ext} \). The \( \gamma_j^{int} \) are minimal elements, but in general not the only ones. Given a cylinder in \( \Gamma \), denoted \( \gamma_0 \), that is not a minimal element, there are other cylinders \( \gamma_r \in \Gamma \), different from
γ₀, that are maximal elements in the set of cylinders contained in γ₀. This is the situation specified by our notation γᵣ ≺ γ₀.

The cylinder γ₀ can be either γₑᵣ or one of the γᵢ. Let Λ′ be one of the sets in definitions (5.3) or (5.4), accordingly. Then we may write Λ′ ⊂ γ₀ \ (∪ᵣ γᵣ), and we have ∂Λ′ ⊂ γ₀ ∪ (∪ᵣ γᵣ), where the γᵣ are, as before, the cylinders such that γᵣ ≺ γ₀. We denote by (Λ′, n′) the set of elementary perturbations ω belonging to C_k(Λ′, n′), hence with support contained in Λ′ and satisfying E(ω) = n′. We denote by (Λ′, n′)* the subset of (Λ′, n′) consisting of the elementary perturbations ω for which in addition γₑᵣ satisfies the sign condition (5.5) with respect to all the cylinders γ₀ and γᵣ for which the base perimeters have a non-empty intersection with γₑᵣ.

We introduce the partition function

\[ Z_k(Λ′, n′) = \sum_{\{ω_j\} ∈ (Λ′, n′)*} \prod_j ϕ(ω_j), \]

where the sum runs over all compatible sets of elementary perturbations \{ω_j\}, whose elements belong to (Λ′, n′)*. Here k is the value used in the definition (3.1) of the elementary cylinders. We notice that if n′ = I(γ₀), then Γ ∪ {ω_j} is a compatible set of cylinders, i.e., Γ ∪ {ω_j} ∈ C(Λ, n). But the partition functions (5.7) are well defined, and will be used, also when n′ ≠ I(γ₀).

In the case Λ′ = Supp(Γ), definition (5.2), we also introduce the partition function Z_k(Λ′, n′) by the same formula (5.7). Since ∂ Supp(Γ) = γₑᵣ ∪ (∪₀ γ_jᵣ), now the cylinders γₑᵣ and γ_jᵣ play the role of γ₀ and γᵣ, in the sign condition, when γₑᵣ has a non-empty intersection with ∂ Supp(Γ).

Given a compatible set \{Γ_i\} of contours, we introduce also the partition function Z_k(Λ′, n) associated to the complementary region Λ′ = Λ \ (∪ᵢ Supp(Γ_i)), again by equation (5.7). In this case the sign condition for an elementary perturbation ω has to be verified with respect to all the boundaries ∂ Supp(Γ_i). Since E(ω) = n, the set ω ∪ (∪ᵢ Γ_i) is a compatible set of cylinders, i.e., ω ∪ (∪ᵢ Γ_i) ∈ C(Λ, n).

Now, for each contour Γ = {γₑᵣ, γᵢ, γ_jᵣ} in C_k(Λ, n), we define the statistical weight

\[ ϕ(Γ) = ϕ(γₑᵣ) \left( \prod_i ϕ(γᵢ) \right) \left( \prod_j ϕ(γ_jᵣ) \right) \]

\[ \times \frac{Z_k^*(\text{Supp}^{ext}(Γ), I(γₑᵣ)) \prod_i Z_k^*(\text{Supp}^{int}(Γ), I(γᵢ))}{Z_k^*(\text{Supp}(Γ), n)}. \]

For the convergence of (5.43) in Lemma 5.6 below, it is essential to have the denominator present in (5.8), so that the energy-entropy benefit of choosing any
Lemma 5.1. We have
\[ \Xi(\Lambda, n) = e^{u\delta(n)|\Lambda|} \times \sum_{\{\Gamma_i\}} Z_k^\ast(\Lambda \setminus (\cup_i \text{Supp}(\Gamma_i)), n) \prod_i \left( \varphi(\Gamma_i) Z_k^\ast(\text{Supp}(\Gamma_i), n) \right). \] (5.9)

The sum runs over all compatible sets \( \{\Gamma_i\} \) of contours contained in \( \Lambda \) such that \( E(\Gamma_i) = n \), for all \( i \).

Proof. The proof follows from formula (5.1) and the above definitions and remarks. This lemma corresponds to lemma 2.4 of ref. [5].

Note that the height \( n \) appears in the partition function \( Z_k^\ast(\text{Supp}(\Gamma_i), n) \) on the right side of (5.9), rather than the actual height \( I(\gamma_i) \) of the base of \( \Gamma_i \), because the factor \( \varphi(\Gamma_i) \) contains the ratio of partition functions that appears in (5.8).

We still need the following definition of compatibility. We say that an elementary perturbation \( \omega \), with external cylinder denoted \( \gamma^\text{ext}_\omega \) and external level \( E(\omega) = n \), is compatible with the contour \( \Gamma = (\gamma^\text{ext}, \gamma_i, \gamma^\text{int}_j) \), of external level \( E(\Gamma) = n \), if either

(i) \( \tilde{\gamma}^\text{ext}_\omega \subset \text{Supp}(\Gamma) \), and then: \( S(\gamma^\text{ext}_\omega) = S(\gamma^\text{ext}) \) if \( \tilde{\gamma}^\text{ext}_\omega \cap \tilde{\gamma}^\text{ext} \neq \emptyset \), \( S(\gamma^\text{ext}_\omega) = -S(\gamma^\text{int}_j) \) if \( \tilde{\gamma}^\text{ext}_\omega \cap \tilde{\gamma}^\text{int}_j \neq \emptyset \), or,

(ii) \( \tilde{\gamma}^\text{ext}_\omega \subset \Lambda \setminus \text{Supp}(\Gamma) \), and then: \( S(\gamma^\text{ext}_\omega) = -S(\gamma^\text{ext}) \) if \( \tilde{\gamma}^\text{ext}_\omega \cap \tilde{\gamma}^\text{ext} \neq \emptyset \), and \( S(\gamma^\text{ext}_\omega) = S(\gamma^\text{int}_j) \) if \( \tilde{\gamma}^\text{ext}_\omega \cap \tilde{\gamma}^\text{int}_j \neq \emptyset \).

We say that a set \( \{\Gamma_i, \omega_j\} \) is a compatible set of contours and elementary perturbations if any two contours in the set are compatible, as well as any two elementary perturbations in it and, also, any elementary perturbation in the set is compatible with any contour in the set.

Lemma 5.2. We have
\[ \Xi(\Lambda, n) = e^{u\delta(n)|\Lambda|} \sum_{\{\Gamma_i, \omega_j\}} \prod_i \varphi(\Gamma_i) \prod_j \varphi(\omega_j). \] (5.10)

The sum runs over all compatible sets \( \{\Gamma_i, \omega_j\} \) of contours and elementary perturbations contained in \( \Lambda \) such that \( E(\Gamma_i) = E(\omega_j) = n \), for all \( i, j \).

Proof. The lemma follows from the above definition of compatibility between contours and elementary perturbations. The set of contours \( \{\Gamma_i\} \) being fixed, the sum over all compatible sets \( \{\omega_j\} \) of elementary perturbations, such that also \( \{\Gamma_i, \omega_j\} \) is a compatible set, gives the product of the partition functions in equation (5.9). \( \square \)
As a consequence of the lemma we see that the system of contours and elementary perturbations has become a polymer model.

Our next task is to find a suitable estimate for the statistical weight of a contour. To this end we first derive, from equation (5.8), a more convenient expression for this weight. We introduce the partition functions

\[ (5.11) \quad \tilde{Z}_k(\Lambda', n') = e^{u \delta(n')|\Lambda'|} Z_k(\Lambda', n') \]

with the “correct” factor \( e^{u \delta(n')|\Lambda'|} \), where \( Z^*_k(\Lambda', n') \) is from (5.7). They differ from the \( Z_k(\Lambda', n') \) considered before, definition (3.3) and (4.4), by the compatibility condition on the elementary perturbations that touch the boundary \( \partial \Lambda' \), assumed in the definition of \( Z^*_k \). Only boundary terms are therefore affected by this condition.

For a contour \( \Gamma = \{ \gamma^\text{ext}, \gamma^\text{int}_i, \gamma^\text{int}_j \} \) we define \( \|\Gamma\| \) to be half the number of vertical plaquettes in \( \Gamma \), that is,

\[ 2\|\Gamma\| = |\gamma^\text{ext}| L(\gamma^\text{ext}) + \sum_i |\tilde{\gamma}_i| L(\gamma_i) + \sum_j |\tilde{\gamma}^\text{int}_j| L(\gamma^\text{int}_j). \]

**Lemma 5.3.** The statistical weight of a contour can also be written as

\[ (5.12) \quad \varphi(\Gamma) = t^\|\Gamma\| \frac{\tilde{Z}_k(\text{Supp}^\text{ext}(\Gamma), I(\gamma^\text{ext})) \prod_i \tilde{Z}_k(\text{Supp}(\Gamma), I(\gamma_i))}{\tilde{Z}_k(\text{Supp}(\Gamma), n)}. \]

**Proof.** In expression (5.8) replace each \( \varphi(\gamma) \) by its value (2.2). Then the claim in the Lemma follows, except for a factor \( e^{uG} \) with

\[
G = (\delta(I(\gamma^\text{ext})) - \delta(n))|\gamma^\text{ext}| + \sum_i (\delta(I(\gamma_i)) - \delta(E(\gamma_i)))|\tilde{\gamma}_i| \\
+ \sum_j (\delta(n) - \delta(E(\gamma^\text{int}_j)))|\tilde{\gamma}^\text{int}_j| - \delta(I(\gamma^\text{ext}))|\text{Supp}^\text{ext}(\Gamma)| \\
- \sum_i \delta(I(\gamma_i))|\text{Supp}(\Gamma)| + \delta(n)|\text{Supp}(\Gamma)|.
\]

Then the identity \( G = 0 \) will complete the proof. First, we see that the coefficient of \( \delta(n) \) is zero because \( |\text{Supp}(\Gamma)| = |\gamma^\text{ext}| - \sum_j |\gamma^\text{int}_j| \). Next, consider the terms where the cylinders \( \gamma^\text{ext}, \tilde{\gamma}_i \prec \gamma^\text{ext} \) and \( \gamma^\text{int}_j \prec \gamma^\text{ext} \), appear. We have then \( I(\gamma^\text{ext}) = E(\gamma_i) = E(\gamma^\text{int}_j) \) and

\[ |\gamma^\text{ext}| - \sum_i |\tilde{\gamma}_i| - \sum_j |\gamma^\text{int}_j| = |\text{Supp}^\text{ext}(\Gamma)|. \]

This implies that, in \( G \), the terms concerning \( \gamma^\text{ext} \) and the considered \( \gamma^\text{int}_j \) cancel, and, for each of the considered \( \tilde{\gamma}_i \), only the term \( \delta(I(\gamma_i))|\tilde{\gamma}_i| \) remains. In the next step, we apply the same arguments to each of these \( \tilde{\gamma}_i \). After a certain number of similar steps we arrive at the minimal \( \gamma_i \) that do not contain
any other cylinder of the contour, and thus we obtain $G = 0$, completing the proof of Lemma 5.3.

In order to estimate the statistical weight of a contour, given by equation (5.12), we first observe that

$$
\tilde{Z}_k(\text{Supp}^\text{ext}(\Gamma), I(\gamma^\text{ext})) \prod_i \tilde{Z}_k(\text{Supp}_i(\Gamma), I(\gamma_i)) \leq \tilde{Z}_k(\text{Supp}^\text{ext}(\Gamma), I(\gamma^\text{ext})) \prod_i \tilde{Z}_k(\text{Supp}_i(\Gamma), I(\gamma_i)).
$$

We have then, for this purpose, to estimate the quotient

$$
\frac{\tilde{Z}_k(\Lambda', h)}{\tilde{Z}_k(\Lambda', n)} \text{ when } h \neq n.
$$

From the definition of a contour we always have the level $n = E(\Gamma)$ in the denominator of (5.14), while only levels $h \neq n$ appear in the numerator. By the definition of $\tilde{Z}_k$ we may write

$$
\ln \frac{\tilde{Z}_k(\Lambda', h)}{\tilde{Z}_k(\Lambda', n)} = -|\Lambda'| (f_k(h) - f_k(n)) + \text{“boundary terms”}.
$$

Now, assume that inequalities (3.4) in Proposition 3.1 are satisfied, with $a = b = \epsilon$ in the hypotheses, so that the restricted ensemble at level $n$ is a dominant state.

Concerning the “boundary terms” in the right hand side of (5.15), let $\mathcal{D}(\Lambda', h)$ denote the set of all clusters $X$ satisfying $(\text{Supp} X) \cap \Lambda' \neq \emptyset$ for which either (i) $(\text{Supp} X) \cap (\Lambda')^c \neq \emptyset$, or (ii) $\text{Supp} X \subset \Lambda'$ and the sign condition is violated by some $\omega \in X$. From (4.5) and (4.11), for $h \geq 0$,

$$
\ln \tilde{Z}_k(\Lambda', h) + |\Lambda'| f_k(h) = -\sum_{x \in \Lambda'} \sum_{X \in \mathcal{D}(\Lambda', h): \text{Supp} X \ni x} \frac{1}{|\Lambda' \cap \text{Supp} X|} \varphi_u^T(X).
$$

Let $h \geq 1$ and recall that $s = te^{t/4}$. For $x \in \Lambda'$ with $d(x, (\Lambda')^c) \geq 8h$ we use (4.69) (excluding the first inequality there) and discard the factor $1/|\Lambda' \cap \text{Supp} X|$; assuming $t < t_1(k)$,

$$
\sum_{X \in \mathcal{D}(\Lambda', h): \text{Supp} X \ni x} \frac{1}{|\Lambda' \cap \text{Supp} X|} \varphi_u^T(X) \leq \sum_{y \in \partial \Lambda'} \sum_{x \in \Lambda': h \rightarrow 0} \varphi_u^T(Y) \leq s^{3h+4}.
$$
For \( x \in \Lambda' \) with \( d(x, (\Lambda')^c) < 8h \) we use (4.101): assuming \( s < 1/K_1 \),

\[
(5.18) \quad \sum_{X \in D(\Lambda', h) : \text{Supp } X \ni x} \frac{1}{|\Lambda' \cap \text{Supp } X|} |\varphi_u^T(X)| \leq 2t^{2h} + K_1s^{2h+1} \leq 3s^{2h}.
\]

Combining these we obtain that for \( h \geq 1 \),

\[
(5.19) \quad \log \frac{Z_k(\Lambda', h)}{Z_k(\Lambda', n)} \leq -|\Lambda'|(f_k(h) - f_k(n)) + K_2 |\varphi_u^T| \partial \Lambda'|s^{2h_1} + 2|\Lambda'|s^{3h_4 + 4}.
\]

For \( h = 0 \), (4.69) is valid with \( 8h - 2 \) replaced by \( 8 \), as noted there, so (5.17) is valid for \( x \) with \( d(x, (\Lambda')^c) \geq 9 \). For \( x \) with \( d(x, (\Lambda')^c) \leq 8 \) we use (4.102): assuming \( t < 1/K_2 \),

\[
(5.19) \quad \log \frac{Z_k(\Lambda', h)}{Z_k(\Lambda', n)} \leq -|\Lambda'|(f_k(h) - f_k(n)) + K_2 |\varphi_u^T| \partial \Lambda'|s^{2h_1} + 2|\Lambda'|s^{3h_4 + 4}.
\]

Hence in place of (5.19) we have for \( h = 0 \):

\[
(5.21) \quad \log \frac{Z_k(\Lambda', h)}{Z_k(\Lambda', n)} \leq -|\Lambda'|(f_k(h) - f_k(n)) + K_2 |\varphi_u^T| \partial \Lambda'|s^{2h_1} + 2|\Lambda'|s^{3h_4 + 4}.
\]

Write \( h_1 \) for \( \max(h, 1) \) and \( n_1 \) for \( \max(n, 1) \). By (3.6) and (3.7) in Proposition 3.1 for some \( K_3 \) we have

\[
(5.22) \quad f_k(h) - f_k(n) \geq \begin{cases} 2t^{3h+3} - K_3 t^{3h+4} & \text{for } h \leq n - 2, \\ \varepsilon t^{3h+3} - K_3 t^{3h+4} & \text{for } h = n - 1, \\ \varepsilon t^{3h+3} - K_3 t^{3h+4} & \text{for } h \geq n + 1. \end{cases}
\]

Provided \( t \leq \min(t_1(k), (\varepsilon \wedge 1)/2(K_3 + 2)) \), this and (5.19), (5.21) show that for some \( K_4 \), for \( h \leq n - 2 \),

\[
\log \frac{Z_k(\Lambda', h)}{Z_k(\Lambda', n)} \leq -|\Lambda'|(f_k(h) - f_k(n)) + K_4 h_1^2 |\partial \Lambda'|s^{2h_1} + 2|\Lambda'|s^{3h_4 + 4}
\]

(5.23) and similarly, considering \( h = n - 1 \),

\[
\log \frac{Z_k(\Lambda', h - 1)}{Z_k(\Lambda', n)} \leq -|\Lambda'|(f_k(h) - f_k(n)) + K_4 h_1^2 |\partial \Lambda'|s^{2h_1} + 2|\Lambda'|s^{3h_4 + 4}
\]

(5.24) while for \( h \geq n + 1 \),

\[
\log \frac{Z_k(\Lambda', h)}{Z_k(\Lambda', n)} \leq -|\Lambda'|(f_k(h) - f_k(n)) + K_4 n_1^3 |\partial \Lambda'|s^{2n_1} + 2|\Lambda'|s^{3n_4 + 4}
\]

(5.25)
Here we have used the fact that $s^{3h+4} \leq 2t^{3h+4}$ for $t \leq t_1(k)$ and $h \leq n \leq k/2$.

Thus we obtain the following lemma.

**Lemma 5.4.** Assume that inequalities (1.14) in Theorem 1.1 are satisfied for some $\varepsilon > 0$. Then there exist $K_5, K_6$ such that for $n, h \geq 0$, $h \neq n$, $k \geq \max(8, 2n)$ and $t \leq t_2(k, \varepsilon) \equiv \min(t_1(k), K_5(\varepsilon \wedge 1))$, we have

\[
\frac{\tilde{Z}_k(\Lambda', h)}{\tilde{Z}_k(\Lambda', n)} \leq \exp \left( - |\Lambda'| \frac{\varepsilon}{2} t^{3n+3} + |\partial \Lambda'| K_6 s^2 \right),
\]

where $s = t e^{t_1/4}$.

This lemma is the analog of Lemma 2.5 in ref. [5], though the definitions are not exactly the same. The factor $\frac{\varepsilon}{2} t^{3n+3}$ represents the cost per unit area for the region $\Lambda'$ to be at a suboptimal height $h \neq n$.

As a consequence of Lemma 5.3, (5.14) and Lemma 5.4 we obtain the following estimate on the statistical weight of a contour:

\[
\varphi(\Gamma) \leq t^{||\Gamma||} \exp \left( |\partial \text{Supp}^\text{ext}(\Gamma)| K_6 s^2 + \sum_i |\partial \text{Supp}^i(\Gamma)| K_6 s^2 \right.
\]

\[
- \left( |\text{Supp}^\text{ext}(\Gamma)| \frac{\varepsilon}{2} t^{3n+3} + \sum_i |\text{Supp}^i(\Gamma)| \frac{\varepsilon}{2} t^{3n+3} \right)
\]

\[
\leq t^{||\Gamma||} \exp \left( \left( |\tilde{\gamma}^\text{ext}| + 2 \sum_i |\tilde{\gamma}_i| + \sum_j |\tilde{\gamma}_j^{\text{int}}| \right) K_6 s^2 - |\text{Supp}(\Gamma)| \frac{\varepsilon}{2} t^{3n+3} \right).
\]

We now consider the convergence of the cluster expansion for the logarithm of $\Xi(\Lambda, n)$, as written in expression (5.10). That is, we consider the convergence of the following expansion of the surface tension:

\[
\tau^\text{WB} - 2(J_{WA} + J_{AB}) = - \lim_{\Lambda \to \mathbb{Z}^2} \frac{1}{\beta |\Lambda|} \ln \Xi(\Lambda, n) = - \frac{u}{\beta} \delta(n) - \frac{1}{\beta} \sum_{X \supseteq 0} \frac{1}{|\text{Supp} X|} \varphi_\beta^T(X).
\]

The sum runs over the clusters of contours $\Gamma$ and elementary perturbations $\omega$ such that $E(\Gamma) = E(\omega) = n$ and we use the notation

\[
\text{Supp} X = (\cup_{\Gamma:X(\Gamma) \geq 1} \text{Supp}(\Gamma)) \cup (\cup_{\omega:X(\omega) \geq 1} \text{Supp} \omega).
\]

We will apply the Convergence Theorem in Section 4 with

\[
\mu(\Gamma) = s^{||\Gamma||} e^{\frac{1}{6} \text{Supp} \Gamma((16t)^{3k+4})} \tilde{Z}_k \left( \frac{\text{Supp}^\text{ext}(\Gamma), I(\gamma^\text{ext})}{\text{Supp}(\Gamma), n} \right) \prod_i \tilde{Z}_k \left( \frac{\text{Supp}^i(\Gamma), I(\gamma_i)}{\text{Supp}(\Gamma), n} \right).
\]
\[ \mu(\Gamma) \leq s^{\|\Gamma\|} \exp \left( \left( |\tilde{\gamma}^{\text{ext}}| + 2 \sum_i |\tilde{\gamma}_i| + \sum_j |\tilde{\gamma}^{\text{int}}_j| \right) K_0 s^2 - |\text{Supp}(\Gamma)| \frac{\varepsilon}{4} t^{3n+3} \right), \]

provided
\[ (16t)^{3k+4} < \frac{\varepsilon}{4} t^{3n+3}, \]
which is satisfied as soon as \( k \geq \max(8, 2n) \) and \( t \leq \varepsilon^{1/12}/2000 \).

The statistical weight (5.27), (5.30) is given in terms of factors \( t^{(1/2)|\tilde{\gamma}| L(\gamma)} \) associated to the cylinders of the contour, but in contrast to the situation for clusters, the set of perimeters of these cylinders is not connected. The following two lemmas show how one can nevertheless include the set of perimeters in a connected structure, in such a way that the weights become summable, as needed in condition (4.8).

**Lemma 5.5.** Let \( k \) and \( n \) be integers with \( n \geq 0 \) and \( k \geq \max(2n, 8) \). Consider rooted trees \( T \), oriented from the root \( r(T) \) outwards. Let each vertex \( v \) carry an integer label \( n_v \), and each edge \( g \) an integer label \( p_g \). Denote by \( d(v) = d(v, T) \) the number of edges starting from vertex \( v \), and let these edges be ordered, 1 through \( d(v) \). Let \( T \) be the set of all such labeled trees in which the labels satisfy
\[ 2n_v \geq d(v), \quad n_v \geq 3k + 4, \quad p_g \geq 0. \]
Let \( c > 0, \epsilon_1 > 0, t > 0 \) and \( s = te^{t/4} \) with
\[ cs < \frac{1}{4}, \quad t \leq (3k + 3)^{-4}, \quad t^{3(k-n)-1} \leq \epsilon_1. \]
Assign to every tree \( T \) a weight
\[ \mu(T) = \prod_{v \in T} \left( \frac{2n_v}{d(v)} \right)^{n_v} \prod_{g \in T} \exp \left( - \epsilon_1 p_g t^{3n+3} \right), \]
Then for \( m \geq 3k + 4, \)
\[ \sum_{T \in T: n_r(T) = m} \mu(T) \leq \left( cs \epsilon_1^{-1} t^{3(k-n)+1} \right)^m. \]

**Proof.** This lemma corresponds to Lemma 2.13 of ref. [5]. The proof proceeds by induction on the number \( |T| \) of vertices of \( T \), starting with
\[ \sum_{T: n_r(T) = m} \mu(T) = (cs)^m \]
and then assuming

\[(5.37) \sum_{T: n_r(T) = m \atop |T| \leq q} \mu(T) \leq \left( c s e^{3 \epsilon_1^{-1} t^3(k-n)+1} \right)^m \]

for all \( m \), for some \( q \geq 1 \), so that also

\[(5.38) \sum_{T: |T| \leq q} \mu(T) \leq \frac{\left( c s e^{3 \epsilon_1^{-1} t^3(k-n)+1} \right)^{3k+4}}{1 - c s e^{3 \epsilon_1^{-1} t^3(k-n)+1}} \leq e^{2cs} \left( c s e^{3 \epsilon_1^{-1} t^3(k-n)+1} \right)^{3k+4}.\]

Any tree with \( |T| = q + 1 \) can be decomposed into its root \( v_0 \), edges \( g_1, \ldots, g_f \) which lead to the vertices \( v_1, \ldots, v_f \), and subtrees \( T_1, \ldots, T_f \), such that \( |T_i| \leq q \) and \( r(T_i) = v_i \). We have the formal identity

\[(5.39) \sum_{T: n_r(T) = m} \mu(T) = (cs)^m \sum_{f=0}^{2m} \binom{2m}{f} \prod_{i=1}^{f} \left( \sum_{p_i=0}^{\infty} \exp \left( - \epsilon_1 p_i t^{3n+3} \right) \sum_{T_i \in T} \mu(T_i) \right) \]

and the related induction bound

\[(5.40) \sum_{T: n_r(T) = m \atop |T| \leq q+1} \mu(T) \leq (cs)^m \sum_{f=0}^{2m} \binom{2m}{f} \left( \frac{1}{1 - \exp(-\epsilon_1 t^{3n+3})} e^{2cs} \left( c s e^{3 \epsilon_1^{-1} t^3(k-n)+1} \right)^{3k+4} \right)^f \leq (cs)^m \left( 1 + \epsilon_1 t^{3n+3} \right)^{-1} e^{2cs} \left( c s e^{3 \epsilon_1^{-1} t^3(k-n)+1} \right)^{3k+4} 2m \]

\[(5.40) \leq \left( c s e^{3 \epsilon_1^{-1} t^3(k-n)+1} \right)^m,\]

where \( (5.33) \) was used in the last inequality, concluding the proof of the Lemma. \( \square \)
Lemma 5.6. Assume that for all contours $\Gamma$,

$$\mu(\Gamma) \leq s^{\|\Gamma\|} \exp \left( 2K_6 \left( |\bar{\gamma}^{\text{ext}}| + \sum_i |\bar{\gamma}_i| + \sum_j |\bar{\gamma}_j^{\text{int}}| \right) s^2 - |\text{Supp}(\Gamma)| \epsilon_1 t^{3n+3} \right),$$

with $\epsilon_1 > 0$, $k \geq \max(2n, 8)$, $K_6$ from Lemma 5.4, $s = t^{\epsilon_1/4}$ and

$$t < (4K_6)^{-1} \wedge (3k + 3)^{-4}, \quad t^{3(k-n)-1} \leq \epsilon_1/4.$$

Then

$$\sum_{\Gamma: \text{Supp}(\Gamma) \ni 0} \mu(\Gamma) \leq (16t)^{3k+4}.$$

Proof. For each base perimeter $\bar{\gamma}$ of a cylinder $\gamma \in \Gamma$, $\gamma \neq \gamma^{\text{ext}}$, we select as a designated point $z(\bar{\gamma}) \in \bar{\gamma}$ the lowest among the leftmost points in $\bar{\gamma}$. From each $z(\bar{\gamma})$, draw a horizontal line segment (necessarily a lattice path in $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$) leftward from $z(\bar{\gamma})$ until it reaches the base perimeter of some other cylinder in $\Gamma$. The line segment may have length 0. Reverse the orientation of this segment so it points rightward. Note these segments are necessarily disjoint for distinct $\bar{\gamma}$ from $\Gamma$, and contained in $\text{Supp}(\Gamma)$. We obtain a rooted labeled tree as in Lemma 5.5, connecting all the $\bar{\gamma} \in \Gamma$, by taking each $\bar{\gamma}$ from $\Gamma$ to be a vertex $v$, with label $2n_v = |\bar{\gamma}|$ and with $\bar{\gamma}^{\text{ext}}$ as root, and taking each line segment to be an oriented edge $g$, with label $p_g$ equal to its length; the ordering of the edges emanating from each $v$ is given by counterclockwise ordering of the segments emanating from $\bar{\gamma}$, starting from $z(\bar{\gamma})$.

We may then identify all those rooted labeled trees with vertices $\bar{\gamma}$ which are isomorphic as rooted trees and which have the same values $n_v, p_g$, taking into account the ordering of edges; the resulting structure defines a rooted labeled tree $T = T(\Gamma)$ of the type in Lemma 5.5. In order to apply Lemma 5.5, we need

$$\sum_{\Gamma: T(\Gamma) = T} \mu(\Gamma) \leq \mu(T).$$

Given $T$, to specify a contour $\Gamma$ with $T(\Gamma) = T$, we need first to choose for the root $r(T)$ a cylinder $\gamma$ with $|\bar{\gamma}| = 2n_{r(T)}$ and $0 \in \bar{\gamma}$, then choose $d(r(T))$ sites along $\bar{\gamma}$ to be the starting points of the line segments, and then choose a length $p_g$ for each such line segment $g$ emanating from $\bar{\gamma}$. Second we must choose for each nonroot vertex $v$ a cylinder $\gamma$ with $|\bar{\gamma}| = 2n_v$ for which $\bar{\gamma}$ passes through a fixed site (say $(\frac{1}{2}, \frac{1}{2})$), then (as with the root) choose $d(v)$ sites along $\bar{\gamma}$ and a length $p_g$ for each segment $g$ emanating from $\bar{\gamma}$. Thus the factor $\left( \binom{2n_v}{d(v)} \right) \left( \binom{2n_{r(T)}}{d(r(T))} \right)$
in (5.34) is the number of choices of starting points of $d(v)$ segments starting from $\tilde{\gamma}$ with $|\tilde{\gamma}| = 2n_v$. The factor with $p_g$ is obtained from (5.41) using
\begin{equation}
\text{Supp}(\Gamma) \geq \sum_{g \in T} p_g
\end{equation}
and the power of $cs$ is obtained using (5.41) and the bound
\begin{equation}
\sum_{\gamma, |\tilde{\gamma}| = 2n_v} s^{4|\tilde{\gamma}|L(\gamma)} e^{2|\tilde{\gamma}|K_6t^2} < 2n_v^2 (1 - s^{n_v})^{-1} (3^2 s e^{4K_6t^2})^{n_v}
\end{equation}
\begin{equation}
< 3n_v^2 (10s)^{n_v}
\end{equation}
We see that we fit into the conditions of Lemma 5.5, with $c = 14$. Then the right-hand side of (5.35) is bounded by $(15 t)^m$. Summing over $m \geq 3k + 4$ proves Lemma 5.6. \hfill \square

Applying Lemma 5.6 with $\varepsilon_1 = \varepsilon/4$, we obtain
\begin{equation}
\sum_{\Gamma^\prime: \Gamma^\prime \not\sim \Gamma} \mu(\Gamma^\prime) \leq \sum_{x \in \text{Supp}(\Gamma)} \sum_{\Gamma^\prime: \text{Supp}(\Gamma^\prime) \ni x} \mu(\Gamma^\prime) \leq |\text{Supp}(\Gamma)|(16 t)^{3k+4}.
\end{equation}
Similarly,
\begin{equation}
\sum_{\Gamma^\prime: \Gamma^\prime \not\sim \omega} \mu(\Gamma^\prime) \leq \sum_{x \in \tilde{\gamma}_\omega^{\text{ext}}} \sum_{\Gamma^\prime: \text{Supp}(\Gamma^\prime) \ni x} \mu(\Gamma^\prime) \leq |\tilde{\gamma}_\omega^{\text{ext}}|(16 t)^{3k+4}.
\end{equation}
Recall $t_2(k, \varepsilon)$ from Lemma 5.4, and recall the sufficient condition $t \leq \varepsilon^{1/12}/2000$ for (5.31). Note that since $k - n \geq k/2$, to satisfy the second inequality in (5.42) with $\varepsilon_1 = \varepsilon/4$, it suffices that $t \leq (\varepsilon/16)^{2/3k}/12$, and this in turn follows from the condition $t \leq \varepsilon^{1/12}/2000$. After reducing $K_5$ if necessary, this last condition follows from the condition $t \leq K_5(\varepsilon \vee 1)$ in Lemma 5.4.

**Lemma 5.7.** Let the level $n$ be given and let $k \geq \max(8, 2n)$. Assume that the inequalities (1.14) in Theorem 1.1 are satisfied. Then for $t \leq t_5(k, \varepsilon) \equiv \min(t_1(k)/12, K_5(\varepsilon \vee 1))$, the cluster expansion (5.28) of the surface tension converges. Here $K_5$ is from Lemma 5.4.

**Proof.** We must check (4.8) with $\varphi(\omega) = \varphi_t, u(\omega), \mu(\omega) = \varphi_s, 0(\omega), s = te^{1/4}$, with $\varphi(\Gamma)$ given by (5.8) and (5.12), bounded as in (5.27), and with $\mu(\Gamma)$ given by (5.29), bounded as in (5.30).

In the case associated to an elementary perturbation $\omega$, (4.8) takes the form
\begin{equation}
|\varphi(\omega)| \leq \mu(\omega) e^{-\sum_{\omega' \neq \omega} \mu(\omega') - \sum_{\Gamma^\prime \neq \omega} \mu(\Gamma^\prime)}.
\end{equation}
Since \( \omega \) is elementary we have both \( |\bar{\gamma}_{\omega}^{ext}| \leq (3k + 3)^2/4 \) and \( |ar{\gamma}_{\omega}^{int}| \leq |\bar{\gamma}_{\omega}^{ext}|^2/4 \), so also \( |\bar{\gamma}_{\omega}^{ext}| \leq (3k + 3)|\bar{\gamma}_{\omega}^{ext}|/4 \). Hence from (4.12) and (5.48) we have

\[
\begin{align*}
\frac{s^{1/2}|\bar{\gamma}_{\omega}^{ext}|}{\sum_{\omega': \omega' \neq \omega} \mu(\omega')} + \sum_{\Gamma': \Gamma' \neq \omega} \mu(\Gamma') \\
\leq 2s^{1/2}|\bar{\gamma}_{\omega}^{ext}| + (16t)^{3k4} |\bar{\gamma}_{\omega}^{ext}|
\end{align*}
\]

(5.50)

Then as in (4.22) and (4.23), (5.49) will be satisfied if for every elementary cylinder \( \gamma \) of length one,

\[
t^{1/2} < s^{1/2}e^{-16t^{3k4} + (16t)^{3k4}} |\bar{\gamma}|
\]

(5.51)

or, since \( s = te^{1/4} \),

\[
\frac{1}{2}t^{1/4} > 2^{1/4}s^{1/4} + (16t)^{3k4},
\]

(5.52)

which is easily verified for \( t < 1/32 \) with \( k \geq 8 \).

In the case associated to a contour \( \Gamma = (\gamma, \gamma, \gamma_{int}) \), (4.8) takes the form

\[
|\varphi(\Gamma)| \leq \mu(\Gamma)e^{-\sum_{\omega \neq \Gamma} \mu(\omega') - \sum_{\Gamma' \neq \Gamma} \mu(\Gamma')}
\]

(5.53)

From (4.19), a slight variant of the first inequality in (4.20), and (5.47), taking into account the compatibility condition between elementary perturbations and contours given before Lemma 5.2, we have

\[
\sum_{\omega': \omega' \neq \Gamma} \mu(\omega') + \sum_{\Gamma': \Gamma' \neq \Gamma} \mu(\Gamma')
\]

\[
\leq 3000s^2 \left( 2|\bar{\gamma}^{ext}| + \sum_j (|\bar{\gamma}^{int}_j| + 4) \right) + |\text{Supp}(\Gamma)|(16t)^{3k4}
\]

(5.54)

\[
\leq 6000s^2 \left( |\bar{\gamma}^{ext}| + \sum_j |\bar{\gamma}^{int}_j| \right) + |\text{Supp}(\Gamma)|(16t)^{3k4}.
\]

From (5.12) and (5.29) we have

\[
\frac{\varphi(\Gamma)}{\mu(\Gamma)} = \left( \frac{t}{s} \right) \frac{1}{2} \frac{\|\Gamma\|}{\|\Gamma\|} e^{-|\text{Supp} \Gamma|(16t)^{3k4}},
\]

(5.55)

and hence a sufficient condition for (5.53) is

\[
t^{1/2}\|\Gamma\| \leq s^{1/2}\|\Gamma\| e^{-10000s^2 \left( |\bar{\gamma}^{ext}| + \sum_j |\bar{\gamma}^{int}_j| \right)},
\]

(5.56)

which is satisfied for \( s = te^{1/4} \), \( t < t_1(k) \) and \( k \geq \max(8, 2n) \). Then the Convergence Theorem gives (4.9), that is,

\[
\sum_{X \ni \omega} |\varphi^T_u(X)| \leq \mu(\omega), \quad \sum_{X \ni \Gamma} |\varphi^T_u(X)| \leq \mu(\Gamma).
\]

(5.57)
As in (4.27), these together with (4.19) and Lemma 5.6 yield convergence of the cluster expansion (5.28) for the surface tension. We also obtain convergence of cluster expansions for the correlation functions, see e.g. (14) in [12]. □

For $t_5(k, \varepsilon)$ from Lemma 5.7, after a further reduction of $K_5$ if necessary, we have

$$t_5(\max(2n, 8), \varepsilon) \geq \min\left(\frac{1}{(6n + 3)^4}, K_5(\varepsilon \vee 1)\right).$$

Therefore (1) in Theorem 1.1 with (5.58)

$$t_0(n, \varepsilon) = \min\left(\frac{1}{(6n + 3)^4}, K_5(\varepsilon \vee 1)\right)$$

follows as a corollary, in the same way as done in [11] for the model with an external field.

We turn to the proof of (2) in Theorem 1.1, which we obtain from convergence of the cluster expansion in a similar way as was done in [11]. As in the case of the SOS model in an external field, FKG inequalities can be used to reduce the problem to constant boundary conditions. We restrict attention to rectangular $\Lambda$. Let $h \neq n$ be taken as a constant boundary condition. This can be obtained from the $n$-boundary condition by (i) requiring the presence of a contour $\Gamma_0 = \{\gamma^{ext}, \gamma_i, \gamma_j^{int}\}$ such that

- $\gamma^{ext} = \Lambda$
- $I(\gamma^{ext}) = h$
- $\text{Supp}^{ext}(\Gamma_0) \supset \{x \in \Lambda : d(x, \Lambda^c) = 1\}$,

where $d(\cdot)$ is the $\ell^\infty$-distance (or the euclidean distance), and (ii) replacing the weight $\varphi(\Gamma_0)$ from (5.8) with a weight $\varphi^*(\Gamma_0)$ similar to (5.8) except that the partition function $Z_k^* (\text{Supp}^{ext}(\Gamma), I(\gamma^{ext}))$ excludes elementary perturbations with support intersecting $\{x \in \Lambda : d(x, \Lambda^c) = 1\}$. Then Lemma 5.1 and Lemma 5.2 will hold for the corresponding partition function, denoted $\Xi(\Lambda, n, h)$, with summations including a $\Gamma_0$ as above:

$$\Xi(\Lambda, n, h) = e^{u_d(n)|\Lambda|} \sum_{\Gamma_0} \varphi^*(\Gamma_0) \sum_{\{\Gamma_i, \omega_j\}} \prod_i \varphi(\Gamma_i) \prod_j \varphi(\omega_j).$$

The sum runs over all compatible sets $\{\Gamma_0, \{\Gamma_i, \omega_j\}\}$ of contours and elementary perturbations contained in $\Lambda$ such that $E(\Gamma_0) = E(\Gamma_i) = E(\omega_j) = n$, for all $i, j$. The probability that the configuration includes a given $\Gamma_0$ is

$$\mu_\Lambda(\Gamma_0|n, h) = \Xi(\Lambda, n, h)^{-1} e^{u_d(n)|\Lambda|} \varphi^*(\Gamma_0) \sum_{\{\Gamma_i, \omega_j\} \sim \Gamma_0} \prod_i \varphi(\Gamma_i) \prod_j \varphi(\omega_j)$$

where the sum is over compatible families $\{\Gamma_i, \omega_j\}$ compatible with $\Gamma_0$. 


Lemma 5.8. Given $\varepsilon > 0$ and $n \geq 0$, there exists $t_0(\varepsilon, n)$ as follows. Let $t < t_0(\varepsilon, n)$ and let $c_{n,h,\varepsilon,t} = 9|n-h|\varepsilon^{-1}t^{-3n-3}\ln t^{-1}$. Then for contours $\Gamma_0$ as in (i) above,

\begin{equation}
\sum_{\Gamma_0: \text{Supp } \Gamma_0 \ni \partial \Lambda} \mu_A(\Gamma_0|n,h) < t^{n-h}\|\partial \Lambda\|.
\end{equation}

Proof. The sum over $\{\Gamma_i, \omega_j\}$ in (5.60) is a partition function which can be exponentiated, with a cluster expansion obeying the same bounds as the full partition function, only with fewer terms:

\begin{equation}
\sum_{\{\Gamma_i, \omega_j\} \sim \Gamma_0} \prod_i \varphi(\Gamma_i) \prod_j \varphi(\omega_j) = \exp\left(\sum_{X \sim \Gamma_0} \varphi^T_u(X)\right)
\end{equation}

where the clusters $X$ are clusters of polymers contained in $\Lambda$, compatible with $\Gamma_0$. A particular $\Gamma_0$ consisting of two cylinders is $\Gamma_00 = \{\gamma^{\text{ext}}, \gamma^{\text{int}}\}$ with $\gamma^{\text{ext}}$ as in (i) and $\gamma^{\text{int}} = \{x \in \Lambda : d(x, \Lambda^c) > 1\}$, $I(\gamma^{\text{int}}) = n$. Then for each $\Gamma_0$,

\begin{equation}
\mu_A(\Gamma_0|n,h) = \frac{\varphi^*(\Gamma_0)}{\varphi^*(\Gamma_00)} \exp\left(-\sum_{X \sim \Gamma_00} \varphi^T_u(X)\right)
\end{equation}

Indeed the sum in (5.63) may be written as

\begin{equation}
\sum_{X \sim \Gamma_00} \sum_{x \in \Lambda} \sum_{X \sim \Gamma_00, \text{Supp } X \ni x} \varphi^T_u(X) = \frac{\varphi^T(X)}{|\text{Supp } X|}.
\end{equation}

For each $x$, analogously to Lemma 5.7 we have a convergent cluster expansion, uniformly in $n$ and $h$, with leading term $t^2$ corresponding to a unit cube excitation up or down. The conditions in the sum over $X$ does not allow both up and down excitations. Given the hypotheses over $t$, the remainder $O(t^3)$ can be uniformly bounded by $t^2/2$. Therefore the sum in (5.63) is positive.

The same argument, starting from (5.12) shows that

\begin{equation}
\varphi^*(\Gamma_00) > t^{n-h}\|\partial \Lambda\| e^{-32|\partial \Lambda|} \quad \text{and} \quad \varphi^*(\Gamma_0) < e^{32|\partial \Lambda|} \varphi(\Gamma_0).
\end{equation}

This and $\mu_A(\Gamma_00|n,h) < 1$ in (5.63) give

\begin{equation}
\mu_A(\Gamma_0|n,h) < t^{-n-h}\|\partial \Lambda\| e^{-6t^2|\partial \Lambda|} \varphi(\Gamma_0).
\end{equation}

From (5.27) and $t < s$, we see that $e^{t^{3n+3}|\text{Supp } \Gamma|} \varphi(\Gamma)$ obeys the hypotheses of Lemma 5.6 with $\varepsilon_1 = \varepsilon/4$, so that

\begin{equation}
\sum_{\Gamma: \text{Supp } (\Gamma) \ni 0} e^{t^{3n+3}|\text{Supp } \Gamma|} \varphi(\Gamma) \leq (16 t)^{3\kappa+4}
\end{equation}
and therefore
\[(5.68) \quad \sum_{\Gamma: \text{Supp}(\Gamma) \ni 0 | \text{Supp}(\Gamma) | > c_{n,h,\varepsilon,t} | \partial \Lambda |} \varphi(\Gamma) \leq (16t)^{3k+4} \exp \left( -\frac{\varepsilon}{4} t^{3n+3} c_{n,h,\varepsilon,t} | \partial \Lambda | \right).
\]

Hence, with (5.66),
\[(5.69) \quad \sum_{| \text{Supp} \Gamma_0 | > c_{n,h,\varepsilon,t} | \partial \Lambda |} \mu_\Lambda(\Gamma_0 | n,h) < | \Lambda | (16t)^{3k+4} \exp \left( 6t^2 | \partial \Lambda | - \frac{\varepsilon}{4} t^{3n+3} c_{n,h,\varepsilon,t} | \partial \Lambda | \right),
\]
which implies (5.61).

Lemma 5.8 shows that a translation invariant Gibbs state obtained from the \(h\) boundary condition is the same as that obtained from the \(n\)-boundary condition. We continue the proof of (2) in Theorem 1.1, going back to the \(n\)-boundary condition and estimating:
\[
\rho_0 = \mu_n(\{ \phi_0 = 0 \}) > \Xi(\Lambda, n) e^{u\delta(n) | \Lambda |} \sum_{\text{Supp} \omega \ni 0} \varphi(\omega) \sum_{\{ \Gamma_i, \omega_j \} \sim \omega} \prod_i \varphi(\Gamma_i) \prod_j \varphi(\omega_j)
= \Xi(\Lambda, n) e^{u\delta(n) | \Lambda |} \sum_{\text{Supp} \Gamma \ni 0} \varphi(\Gamma) \sum_{\{ \Gamma_i, \omega_j \} \sim \Gamma} \prod_i \varphi(\Gamma_i) \prod_j \varphi(\omega_j)
= \Xi(\Lambda, n) e^{u\delta(n) | \Lambda |} \sum_{\text{Supp} \omega \ni 0} \varphi(\omega) e^{\sum_{X \ni \omega} \varphi_T^\omega(X)} + \sum_{\text{Supp} \Gamma \ni 0} \varphi(\Gamma) e^{\sum_{X \ni \Gamma} \varphi_T^\Gamma(X)}
< 3000 t^2 e^{2t^{1/4}} + (16t)^{3k+4},
\]
using (4.19) and Lemma 5.6.

Similarly, for the proof of (3) in Theorem 1.1, using a cylinder \(\omega\) with \(|\tilde{\omega}| = 4\) and \(L(\omega) = n, \ n \neq 0\) for definiteness,
\[
\rho_0 = \mu_n(\{ \phi_0 = 0 \}) > \Xi(\Lambda, n) e^{u\delta(n) | \Lambda |} \varphi(\omega) \sum_{\{ \Gamma_i, \omega_j \} \sim \omega} \prod_i \varphi(\Gamma_i) \prod_j \varphi(\omega_j)
= \Xi(\Lambda, n) e^{u\delta(n) | \Lambda |} \varphi(\omega) e^{\sum_{X \ni \omega} \varphi_T^\omega(X)}
= \varphi(\omega) e^{\sum_{x \ni \omega} \varphi_T^\omega(x)}
= t^{2n} e^{O(t^2)};
\]
completing the proof of Theorem 1.1.
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Department of Mathematics KAP 108, University of Southern California, Los Angeles, CA 90089-2532 USA
E-mail address: alexandr@usc.edu

Laboratoire de Physique Théorique et Modélisation (CNRS, UMR 8089), Université de Cergy-Pontoise, 95302 Cergy-Pontoise, France
E-mail address: Francois.Dunlop@u-cergy.fr

Centre de Physique Théorique, CNRS, Case 907, 13288 Marseille cedex 9, France
E-mail address: miracle@cpt.univ-mrs.fr