MAXIMAL $L_q$-REGULARITY OF NONLOCAL PARABOLIC EQUATIONS IN HIGHER ORDER BESSEL POTENTIAL SPACES

NIKOLAOS ROIDOS AND YUANZHEN SHAO

ABSTRACT. We consider fractional parabolic equations with variable coefficients and establish maximal $L_q$-regularity in Bessel potential spaces of arbitrary nonnegative order. As an application, we show higher order regularity and instantaneous smoothing for the fractional porous medium equation and for a nonlocal Kirchhoff equation.

1. INTRODUCTION

The mathematical study of diffusion has been very successful in modelling and analyzing a variety of phenomena in physics, chemistry, biology, material sciences, population dynamics and finance. However, standard diffusion models are incapable of describing long-memory or long-range interactions in real processes and substances and thus fail to explain a number of phenomena occurring in recent experiments, cf. [14, 15, 16, 21, 23]. This observation explains the recent surge of the study of nonlocal differential equations. In this article, we will focus on a class of nonlocal parabolic equations.

It is well known that maximal $L_q$-regularity theory has been playing an important role in the study of nonlinear parabolic equations. In particular, maximal regularity theory is capable of handling systems (and thus tensor-valued equations) of parabolic equations and higher order equations, which are generally not accessible in many other traditional approaches, like monotone operator techniques, a priori estimates and Leray-Schauder continuation techniques.

The main theme of the paper is to investigate the maximal $L_q$-regularity of the following nonlocal parabolic equation

\[
\begin{cases}
\partial_t u + w\mathcal{L}^\sigma u = f & \text{on } M \times (0, \infty); \\
u(0) = u_0 & \text{on } M.
\end{cases}
\]

Here $(M, g)$ is either an $n$-dimensional smooth closed Riemannian manifold or the Euclidean space $(\mathbb{R}^n, g_0)$ with $g_0$ being the standard Euclidean metric.

2020 Mathematics Subject Classification. 35B65, 35K59, 35K65, 35R01, 35R11, 76S05.

Key words and phrases. maximal $L_q$-regularity, nonlinear nonlocal diffusion, fractional porous medium equation, nonlocal Kirchhoff diffusion, smoothing for solutions.
in $\mathbb{R}^n$. In this article, a closed manifold always refers to one that is compact and without boundary. The coefficient $w$ belongs to $BC^r(M)$ with $r \in (0, \infty)$. Briefly speaking, a function belongs to $BC^k(M)$ if its derivatives up to order $k$ are continuous and bounded. See Section 2.1 for the precise definitions of function spaces. For any $u \in C^\infty(M,TM^\otimes \otimes T^*M^\otimes)$, i.e. smooth tensor-valued functions, the linear differential operator $\mathcal{L}$ is defined by

$$\mathcal{L} u = \nabla^*(a \nabla u) \quad \text{with} \quad a \in BC^\infty(M),$$

where $\nabla^*$ is the formal adjoint of

$$\nabla : C^\infty(M,TM^\otimes \otimes T^*M^\otimes) \to C^\infty(M,TM^\otimes \otimes T^*M^\otimes).$$

In addition, we assume that there exists a constant $c > 0$ such that $c^{-1} \leq a \leq c$. Lower regularity of $a$ or tensor-valued $a$ can be imposed. For the sake of simplicity, we will confine our discussion to the case $a \in BC^\infty(M)$ in this manuscript. Further, $\sigma \in (0,1)$ and $\mathcal{L}^\sigma$ is understood as the fractional power of $\mathcal{L}$.

Given a continuously and densely embedded Banach couple $X_1 \overset{\ldots}{\hookrightarrow} X_0$, assume that the linear operator $-A$, with $\text{dom}(A) = X_1$, generates a strongly continuous analytic semigroup on $X_0$. For any $q \in (1, \infty)$, the following abstract Cauchy problem

$$\begin{cases}
\partial_t u(t) + Au(t) = f(t), \quad t \geq 0 \\
\quad \quad u(0) = u_0
\end{cases}$$

is said to have maximal $L_q$-regularity if for any $f \in L_q(\mathbb{R}^+,X_0)$ and $u_0 \in X_{1/q,q} := (X_0,X_1)_{1-1/q,q}$, (1.3) has a unique solution

$$u \in L_q(\mathbb{R}^+,X_1) \cap H^1_q(\mathbb{R}^+,X_0).$$

Here $(\cdot,\cdot)_{\theta,q}$ with $\theta \in (0,1)$ is the real interpolation method, cf. [1] Example I.2.4.1, and $H^1_q$ is the usual Bessel potential space. Symbolically, we denote the maximal $L_q$-regularity property by

$$A \in MR_q(X_1,X_0).$$

In combination with an abstract Theorem by P. Clément and S. Li, cf. Theorem 6.1 maximal $L_q$-regularity of (1.1) can be used to establish the local well-posedness of a large class of quasilinear parabolic equations including

$$\begin{cases}
\partial_t u(t) + w(u(t)) \mathcal{L}^\sigma u(t) = F(t,u(t)) \quad \text{on} \quad M \times (0,\infty); \\
u(0) = u_0, \quad \text{on} \quad M,
\end{cases}$$

where $w \in C^1((U,BC^r(M))$ and $F \in C^{1-1-}([0,T_0] \times U, H^s_p(M,TM^\otimes \otimes T^*M^\otimes))$.
for some \( U \subseteq X_{1/q, q} \) open and \( T_0 > 0 \), and
\[
\begin{align*}
    u_0 \in B_{p,q}^{r+2\sigma-2\sigma/q}(M, TM^{\otimes \sigma} \otimes T^*M^{\otimes \tau}).
\end{align*}
\]
Here \( r > s \geq 0 \) and \( H^s_p(M, TM^{\otimes \sigma} \otimes T^*M^{\otimes \tau}) \), \( B_{p,q}^{r+2\sigma-2\sigma/q}(M, TM^{\otimes \sigma} \otimes T^*M^{\otimes \tau}) \) are tensor-valued Bessel potential and Besov spaces, respectively. See Section 2.1 for details.

Particularly, (1.4) includes the fractional porous medium equation of the form
\[
\begin{align*}
    \partial_t u + (\Delta_g)^\sigma(|u|^{m-1}u) &= 0 \quad \text{on } M \times (0, \infty); \\
    u(0) &= u_0 \quad \text{on } M,
\end{align*}
\]
which has been extensively investigated in the last decade. See [8, 9, 10, 11, 12, 36, 37, 49, 50, 51] for instance. For an approach to the above problem in the framework of pseudodifferential operators, we also refer to the theory developed in [24, 25, 26].

In the realm of nonlocal parabolic problems, maximal regularity theory has been successfully applied to several models, cf. [2, 3, 27, 28, 29, 30]. However, when the leading term is nonlocal in space and has a variable multiplier, e.g. \( wL^{\sigma} \) in (1.1), an essential difficulty arises. Indeed, in the conventional approaches, the first step of obtaining the maximal regularity property of a parabolic problem is to study a constant-coefficient problem by means of proper harmonic analysis techniques. Then the perturbation theory of \( R \)-sectorial operators, cf. Definition 2.7, and a freezing-of-coefficient method can be used to extend the maximal regularity property to equations/systems with variable coefficients. When the leading nonlocal operator is accompanied by a variable coefficient, due to the spatial nonlocality, the standard freezing-of-coefficients method no longer applies.

In recent work [42], we have overcome the aforementioned difficulty and established the \( L_q \)-maximal regularity of (1.1) for the scalar case and \( X_0 = L_q(M) \). The main contribution of this article is to extend our previous results to tensor-valued equations and \( X_0 = H^s_p(M) \) for arbitrary \( s \geq 0 \), i.e. Bessel potential spaces of arbitrary non-negative order. As a direct consequence, immediate regularization of the corresponding nonlocal diffusion can be obtained. We expect that the methods in this and our previous work [42] will serve as the stepping stone to the study of more general nonlocal parabolic equations and systems.

This manuscript is organized as follows. In Section 2.1, we give the precise definitions of tensor-valued function spaces and present various properties of those spaces. In Section 2.2, we introduce some crucial functional analytic concepts in the study of maximal \( L_q \)-regularity like sectorial operators, bounded imaginary powers and \( R \)-sectorial operators. In Section 3, we prove that the \( H^s_p \)-realization of the operator \( c + \mathcal{L} \) has bounded imaginary powers for sufficiently large constant \( c > 0 \). In Section 4, we obtain
the $R$-sectoriality of the fractional operator $\mathcal{L}^\sigma$. Section 4 is devoted to the proof of the maximal $L_q$-regularity of (1.1). The key component of the proof consists of several novel commutator estimates. In Section 5, we apply the maximal $L_q$-regularity result to two nonlocal quasilinear parabolic equations. Lastly, the arguments for closed manifolds and Euclidean spaces are essentially the same. The only differences appear in Step 2 of the proof of Theorem 3.1 and the proof of Lemma 5.1. Therefore, in this article, we will mainly focus on the case of closed manifolds and point out changes for $(\mathbb{R}^n, g_n)$ where necessary.

**Notations:** For any two Banach spaces $X$ and $Y$, 

$$X \cong Y$$

means that they are equal in the sense of equivalent norms. The notations 

$$X \hookrightarrow Y, \quad X \overset{d}{\hookrightarrow} Y$$

mean that $X$ is continuously embedded and further densely embedded into $Y$, respectively. $\mathcal{L}(X, Y)$ denotes the set of all bounded linear maps from $X$ to $Y$, and 

$$\mathcal{L}(X) := \mathcal{L}(X, X).$$

Furthermore, $\mathcal{L}_{is}(X, Y)$ stands for the subset of $\mathcal{L}(X, Y)$ consisting of all bounded linear isomorphisms from $X$ to $Y$. Given a densely-defined operator $\mathcal{A}$ in $X$, $\text{dom}(\mathcal{A})$ stands for the domain of $\mathcal{A}$. In addition, $\mathbb{N}$ denotes the set of nonzero natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. 

2. Preliminaries

2.1. **Function spaces.** Given $\eta, \tau \in \mathbb{N}_0$, we define the $(\eta, \tau)$–tensor bundle of $M$ as 

$$T^\eta_\tau M := TM^{\otimes \eta} \otimes T^*M^{\otimes \tau},$$

where $TM$ and $T^*M$ are the tangent and the cotangent bundle of $M$, respectively. Let $T^\eta_\tau M$ denote the $C^\infty(M)$–module of all smooth sections of $T^\eta_\tau M$.

Throughout the rest of this paper, we will adopt the following conventions.

- $(M, g)$ is either an $n$-dimensional closed manifold or $(\mathbb{R}^n, g_n)$.
- $p$ always denotes a generic point on $M$.
- $1 < p, q < \infty, k \in \mathbb{N}_0$ and $r, s \geq 0$.
- $\eta, \tau \in \mathbb{N}_0$, $V = V^\eta_\tau := \{T^\eta_\tau M, (\cdot | \cdot)_g\}$.
- $\nabla$ is the extension of the Levi-Civita connection over $T^\eta_\tau M$.
- $|a|_g := \sqrt{(a | a)_g}$ for all $a \in V$ is the (vector bundle) norm induced by $g$. 

The Sobolev space $W^k_p(M,V)$ is defined as the completion of $C^\infty_0(M,V)$, the space of smooth and compactly supported tensor-valued functions, in $L_{1,loc}(M,V)$ with respect to the norm

$$
\| \cdot \|_{k,p} : u \mapsto \left( \sum_{i=0}^{k} \| \nabla^i u \|_p \right)^{\frac{1}{p}}.
$$

It is clear that $W^0_p(M,V) = L_p(M,V)$. The Bessel potential spaces are defined by means of interpolation

$$
H^s_p(M,V) := \begin{cases} 
[W^k_p(M,V), W^{k+1}_p(M,V)]_{s-k} & \text{for } k < s < k+1, \\
[W^{k-1}_p(M,V), W^{k+1}_p(M,V)]_{1/2} & \text{for } s = k \in \mathbb{N}, \\
L_p(M,V) & \text{for } s = 0.
\end{cases}
$$

Here $[\cdot, \cdot]_\theta$ is the complex interpolation method [1, Example I.2.4.2]. In particular, by [4] Corollary 7.2 (i), $H^k_p(M,V) = W^k_p(M,V)$. We denote the norm of $H^s_p(M,V)$ by $\| \cdot \|_{s,p}$.

The following interpolation theory for Bessel potential spaces is proved in [4] Corollary 7.2 (ii), [47] Theorem 2.4.2 and [48] Theorem 7.4.4].

**Proposition 2.1.** Suppose that $0 \leq s_0 < s_1 < \infty$ and $\theta \in (0,1)$. Then

$$
H^s_p(M,V) = [H^{s_0}_p(M,V), H^{s_1}_p(M,V)]_{\theta}
$$

and

$$
B^{s_0}_{p,q}(M,V) = (H^{s_0}_p(M,V), H^{s_1}_p(M,V))_{\theta,q},
$$

where $s_\theta = (1 - \theta)s_0 + \theta s_1$.

Here, $B^{s}_{p,q}(M,V)$ is a Besov space, cf. [47] [48]. We define

$$
BC^k(M,V) := \{ u \in C^k(M,V) : \| u \|_{k,\infty} < \infty, \| \cdot \|_{k,\infty} \},
$$

where $\| u \|_{k,\infty} := \max_{0 \leq i \leq k} \| \nabla^i u \|_\infty$. Set

$$
BC^\infty(M,V) := \bigcap_k BC^k(M,V)
$$

endowed with the conventional projective topology. Then

$$
bC^k(M,V) := \text{the closure of } BC^\infty(M,V) \text{ in } BC^k(M,V).
$$

Letting $k < s < k+1$, the Hölder space $BC^s(M,V)$ is defined by

$$
BC^s(M,V) := (bC^k(M,V), bC^{k+1}(M,V))_{s-k,\infty}.
$$

Here $(\cdot, \cdot)_{s,\infty}$ is again the real interpolation method.

When $s \in (0,1)$, a function $u \in BC^s(M,V)$ if $u \in BC(M,V) = BC^0(M,V)$ and

$$
\| u \|_{s,\infty} = \| u \|_{\infty} + \sup_{p,q \in M} \frac{|u(p) - u(q)|_g}{d(p,q)^s} < \infty,
$$

where $d(p,q)$ is the distance between $p$ and $q$.
where \( d = d(p, q) \) the geodesic distance between two points \( p, q \in M \) with respect to the metric \( g \). This alternative characterization is well-known for Euclidean spaces. When \((M, g)\) is a closed manifold, it can be proved via localization.

**Proposition 2.2.** (i) For any \( s_2 > s_1 > s_0 \geq 0 \), we have
\[
H^s_p(M, V) \hookrightarrow B^s_{p,q}(M, V) \hookrightarrow H^s_p(M, V).
\]
Suppose that \( s > r + n/p \) and \( r \geq 0 \). Then
\[
H^s_p(M, V) \hookrightarrow BC^r(M, V) \quad \text{and} \quad B^s_{p,q}(M, V) \hookrightarrow BC^r(M, V).
\]
(ii) Let \( v \in H^s_p(M, V) \). When \( s \geq \nu \), given any \( w \in BC^\nu(M) \) we have
\[
\|vw\|_{H^s_p(M, V)} \leq C\|w\|_{BC^\nu(M)}\|v\|_{H^s_p(M, V)},
\]
for certain \( C > 0 \). Further, if \( w \in H^\xi_{q+\frac{n}{q}}(M) \) for some \( \xi > \nu \geq 0 \) and \( q \in (1, \infty) \), then
\[
\|vw\|_{H^s_p(M, V)} \leq C_0\|w\|_{H^\xi_{q+\frac{n}{q}}(M)}\|v\|_{H^s_p(M, V)},
\]
for certain \( C_0 > 0 \).
(iii) Let \((M, g)\) be an \( n\)-dimensional smooth closed Riemannian manifold. When \( s > n/p \), \( \mathfrak{H}^s(M) \) is a Banach algebra (up to an equivalent norm) for \( \mathfrak{H} \in \{H_p, B_{p,q}\} \), i.e. there exists \( C_1 > 0 \) depending only on \( s, p, q, \) and \( n \), such that
\[
\|uv\|_{\mathfrak{H}^s(M)} \leq C_1\|u\|_{\mathfrak{H}^s(M)}\|v\|_{\mathfrak{H}^s(M)}, \quad \text{for all } u, v \in \mathfrak{H}^s(M).
\]
In addition, \( \mathfrak{H}^s(M) \) is closed under holomorphic functional calculus, that is if \( v \in \mathfrak{H}^s(M) \) and \( f \) is an analytic function in some neighborhood of \( \text{Ran}(v) = \{v(p) \in \mathbb{C} \mid p \in M\} \), then \( f(v) \in \mathfrak{H}^s(M) \). Furthermore, if \( U \) is a bounded set in \( \mathfrak{H}^s(M) \) consisting of functions \( u \) satisfying \( \text{Re}(u) \geq c \), for certain \( c > 0 \) depending on \( U \), then the set \( \{u^{-1} : u \in U\} \) is also bounded in \( \mathfrak{H}^s(M) \).

**Proof.** (i) The embeddings \((2.2)\) are shown in [48, Theorem 7.4.2 (2) and (5)] for scalar functions and the proof for tensor-valued case is similar. The embeddings \((2.3)\) follow from [4, Theorem 14.2] and \((2.2)\).

(ii) \((2.4)\) follows from [4] Theorem 9.2 by choosing the weight function \( \rho \equiv 1 \). Then \((2.5)\) is a direct consequence of \((2.3)\) and \((2.4)\).

(iii) The fact that \( \mathfrak{H}^s(M) \) is a Banach algebra is a direct consequence of [4, Theorem 9.3] by choosing the weight function equal to one. For the closedness under the holomorphic functional calculus, by following the proof of [10, Lemma 6.2], let \( B_j, R = B_M(p_j, R), j \in \{1, \ldots, N\} \) with \( N \in \mathbb{N} \), be an open cover of \( M \) consisting of geodesic balls of radius \( R > 0 \), centred at \( p_j \in M \). Moreover, let \( \phi_j \), \( j \in \{1, \ldots, N\} \), be a subordinated partition of unity and assume that the closure of each \( B_j, R/2, j \in \{1, \ldots, N\} \), is
contained in a single coordinate chart. Take \( u \in \mathfrak{F}^s(M) \) and assume that \( u \) is pointwise invertible. Let \( \omega : \mathbb{R} :\rightarrow [0,1] \) be a smooth nonincreasing function that equals 1 on \([0,1/2]\) and 0 on \([3/4, +\infty)\), and define
\[
 u_j(p) = \omega \left( \frac{d(p,p_j)}{2R} \right) u(p) + \left( 1 - \omega \left( \frac{d(p,p_j)}{2R} \right) \right) u(p_j), \quad p \in M, \; j = 1, \ldots, N.
\]
Choose \( R \) sufficiently small such that
\[
 \| u(\cdot) - u(p_j) \|_\infty \leq \frac{1}{2} |u(p_j)|, \quad \text{for each} \quad j \in \{1, \ldots, N\}.
\]
For each \( j \in \{1, \ldots, N\} \) the push-forward of \( \eta_j = u_j - u(p_j) \) belongs to \( \mathfrak{F}^s(\mathbb{R}^n) \). By [13, Theorems 6 and 10], so does the push-forward of \( \eta_j(u(p_j) + \eta_j)^{-1} \). Therefore, \( \eta_j(u(p_j) + \eta_j)^{-1} \) belongs to \( \mathfrak{F}^s(M) \). We have
\[
 u_j^{-1} = (u(p_j) + \eta_j)^{-1} = \frac{1}{u(p_j)}(1 - \eta_j(u(p_j) + \eta_j)^{-1}), \quad j \in \{1, \ldots, N\},
\]
so that \( u_j^{-1} \in \mathfrak{F}^s(M) \) for each \( j \). Then, the identity
\[
 1 = \sum_{j=1}^N \phi_j = \left( \sum_{j=1}^N \phi_j u_j^{-1} \right) u
\]
shows that \( u^{-1} \in \mathfrak{F}^s(M) \) as well; recall here that \( \phi_j u = \phi_j u_j \) for each \( j \). The closedness under holomorphic functional calculus follows immediately by the expression
\[
 f(u) = \frac{1}{2\pi i} \int_{\Gamma} f(-\lambda)(u + \lambda)^{-1} d\lambda,
\]
where \( \Gamma \) is a finite simple path around \( \text{Ran}(-u) \), in the area of holomorphicity of \( f \). The boundedness of the set \( \{u^{-1} \mid u \in U\} \) follows by the above construction. \( \square \)

2.2. Functional analytic tools. We introduce some tools from function analysis. The reader may refer to the treatises [1], [20] and [39] for more details of these concepts.

Let \( X_1 \overset{d}{\hookrightarrow} X_0 \) be a continuously and densely injected complex Banach couple.

**Definition 2.3** (Dissipativity). A linear operators \( \mathcal{A} \) in \( X_0 \) with \( \text{dom}(\mathcal{A}) = X_1 \) is called dissipative if for all \( \lambda > 0 \) and \( x \in X_1 \)
\[
 \| (\lambda - \mathcal{A}) x \| \geq \lambda \| x \|.
\]

**Definition 2.4** (Sectoriality). Let \( \mathcal{P}(K, \theta), \; K \geq 1, \; \theta \in [0,\pi) \), be the class of all closed densely defined linear operators \( \mathcal{A} \) in \( X_0 \) such that
\[
 \Sigma_\theta := \{ \lambda \in \mathbb{C} : |\text{arg}(\lambda)| \leq \theta \} \cup \{0\} \subset \rho(-\mathcal{A})
\]
The holomorphic functional calculus for sectorial operators in the class $z$ typical example are the complex powers; for $\Re(z) > 0$ they are defined by the Dunford integral formula, see, e.g. [20, Theorem 1.7]. A
\begin{equation}
\rho \left( (2.6) \right)
\end{equation}
in particular, the sectorial bound of $S$ 
Recall that $\mathcal{P}(K, \theta) \subset \mathcal{P}(2K + 1, \phi)$ for some $\phi \in (0, \pi)$, see, e.g. [1, (III.4.6.4)-(III.4.6.5)], and similarly for the class $\mathcal{S}(\theta)$. Hence, whenever $A \in \mathcal{P}(\theta)$ or $A \in \mathcal{S}(\theta)$ we can always assume that $\theta > 0$. Moreover, for any $\rho > 0$ and $\theta \in (0, \pi)$, let the counterclockwise oriented path
\begin{equation}
\Gamma_{\rho, \theta} = \{ re^{i\theta} \in \mathbb{C} : r \geq \rho \} \cup \{ re^{i\phi} \in \mathbb{C} : \theta \leq \phi \leq 2\pi - \theta \}.
\end{equation}

The holomorphic functional calculus for sectorial operators in the class $\mathcal{P}(\theta)$ is defined by the Dunford integral formula, see, e.g. [20, Theorem 1.7]. A typical example are the complex powers; for $\Re(z) < 0$ they are defined by
\begin{equation}
\mathcal{A}^z = \frac{1}{2\pi i} \int_{\Gamma_{\rho, \theta}} (-\lambda)^z (A + \lambda)^{-1} d\lambda,
\end{equation}
where $\rho > 0$ is sufficiently small. The family $\{ \mathcal{A}^z \}_{\Re(z) < 0}$ together with $\mathcal{A}^0 = I$ is a strongly continuous analytic semigroup on $X_0$, see, e.g. [1, Theorem III.4.6.2 and Theorem III.4.6.5]. Moreover, each $\mathcal{A}^z$, $\Re(z) < 0$, is an injection and the complex powers for positive real part $\mathcal{A}^{-z}$ are defined by $\mathcal{A}^{-z} = (\mathcal{A}^z)^{-1}$, see, e.g. [1, (III.4.6.12)]. By Cauchy’s theorem we can deform the path in (2.6) and define the imaginary powers $\mathcal{A}^t$, $t \in \mathbb{R}\setminus\{0\}$, as the closure of the operator
\begin{equation}
\mathcal{A}^t = \frac{\sin(it)}{it} \int_{\theta}^{+\infty} s^t (A + s)^{-2} ds \quad \text{in } \text{dom}(A),
\end{equation}
see, e.g. [1, (III.4.6.21)]. For the properties of the complex powers of sectorial operators, we refer to [1, Theorem III.4.6.5]. Concerning the imaginary powers, the following property can be satisfied.

**Definition 2.5** (Bounded imaginary powers). Let $A \in \mathcal{P}(0)$ in $X_0$ and assume that there exist some $\delta, M > 0$ such that $\mathcal{A}^t \in \mathcal{L}(X_0)$ and $\| \mathcal{A}^t \|_{\mathcal{L}(X_0)} \leq M$ when $t \in (-\delta, \delta)$. Then, $\mathcal{A}^t \in \mathcal{L}(X_0)$ for each $t \in \mathbb{R}$ and there exist some $\phi, M > 0$ such that $\| \mathcal{A}^t \|_{\mathcal{L}(X_0)} \leq M e^{\phi|t|}$, $t \in \mathbb{R}$; in this case we say that $A$ has bounded imaginary powers and denote it by $A \in BIP(\phi)$. 
The following property, stronger than the boundedness of the imaginary powers, can also be satisfied by operators in the class \( \mathcal{P}(\theta) \).

**Definition 2.6** (Bounded \( H^\infty \)-calculus). Let \( \theta \in (0, \pi) \), \( \phi \in [0, \theta) \), \( A \in \mathcal{P}(\theta) \) and let \( H^\infty_0(\phi) \) be the space of all bounded holomorphic functions \( f : \mathbb{C} \setminus \Sigma_\phi \to \mathbb{C} \) satisfying

\[
|f(\lambda)| \leq c\left(\frac{|\lambda|}{1 + |\lambda|^2}\right)^\eta \quad \text{for any} \quad \lambda \in \mathbb{C} \setminus \Sigma_\phi
\]

and some \( c, \eta > 0 \) depending on \( f \). Any \( f \in H^\infty_0(\phi) \) defines an element \( f(-A) \in \mathcal{L}(X_0) \) by

\[
f(-A) = \frac{1}{2\pi i} \int_{\Gamma_\phi} f(\lambda)(A + \lambda)^{-1}d\lambda.
\]

We say that the operator \( A \) has bounded \( H^\infty \)-calculus of angle \( \phi \), and we denote by \( A \in \mathcal{H}_{}^\infty(\phi) \), if there exists some \( C > 0 \) such that

\[
\|f(-A)\|_{\mathcal{L}(X_0)} \leq C \sup_{\lambda \in \mathbb{C} \setminus \Sigma_\phi} |f(\lambda)| \quad \text{for any} \quad f \in H^\infty_0(\phi).
\]

**Definition 2.7** (R-boundedness). A set \( E \subset \mathcal{L}(X_0) \) is called R-bounded if for every \( T_1, \ldots, T_N \in E \) and \( x_1, \ldots, x_N \in X_0 \), \( N \in \mathbb{N} \), we have

\[
\left\| \sum_{k=1}^N \epsilon_k T_k x_k \right\|_{L_2((0,1),X_0)} \leq C \left\| \sum_{k=1}^N \epsilon_k x_k \right\|_{L_2((0,1),X_0)}
\]

for certain \( C > 0 \), where \( \{\epsilon_k\}_{k=1}^\infty \) is the sequence of Rademacher functions. The infimum of all such constants \( C > 0 \) is called the R-bound of \( E \).

Denote by \( \mathcal{R}(\theta) \), \( \theta \in [0, \pi) \), the class of all operators \( A \in \mathcal{S}(\theta) \) in \( X_0 \) such that the set \( E = \{\lambda(A + \lambda)^{-1} : \lambda \in \Sigma_\theta \setminus \{0\}\} \) is R-bounded. If \( A \in \mathcal{R}(\theta) \) then \( A \) is called R-sectorial of angle \( \theta \) and the R-bound of \( E \) is called the R-sectorial bound of \( A \).

Given any \( T > 0 \), recall the embedding

\[
L_q((0, T), X_1) \cap H^1_q((0, T), X_0) \hookrightarrow C([0, T], X_1) \quad (q, q) \quad \text{cf. [II Theorem III.4.10.2]}, \quad \text{and} \quad X_{1/q,q} := (X_0, X_1)_{1/q,q}.
\]

Let \( J = (0, T) \) and

\[
\mathbb{E}_0(J) := L_q(J, X_0), \quad \mathbb{E}_1(J) := L_q(J, X_1) \cap H^1_q(J, X_0).
\]

For any \( A \in \mathcal{S}(\theta) \), \( \theta \in (\pi/2, \pi) \), with \( \text{dom}(A) = X_1 \),

\[
A \in \mathcal{M}\mathcal{R}_q(X_1, X_0)
\]

holds iff

\[
(\partial_t + A, \gamma_0) \in \mathcal{L}(\mathbb{E}_1(J), \mathbb{E}_0(J) \times X_{1/q,q}),
\]

where \( \gamma_0 \) is the trace map at 0, i.e. \( \gamma_0(u) = u(0) \).
If we restrict to the class of UMD (unconditionality of martingale differences property, see, e.g. [11 Section III.4.4]) Banach spaces, then we have the following.

**Theorem 2.8** (Kalton and Weis, [31 Theorem 6.5] or [52 Theorem 4.2]). If \( X_0 \) is UMD and \( A \in \mathcal{R}(\theta) \) in \( X_0 \) with \( \text{dom}(A) = X_1 \) and \( \theta \in (\pi/2, \pi) \), then \( A \in \mathcal{MR}_q(X_1, X_0) \) for all \( q \in (1, \infty) \).

### 3. Imaginary Powers of Elliptic Operators

Let \( \mathcal{L}_s^{\eta, \tau} \) denote the \( H^s_P(M,V) \)-realization of \( \mathcal{L} \), where \( \mathcal{L} \) is defined in \([17,2]\). Note that when \( \eta + \tau > 0 \) and \( a \equiv 1 \), \( \mathcal{L}_s^{\eta, \tau} \) is the Bochner Laplacian, cf. [5 Example 1.6]. In the sequel, we will omit the indices \( \eta, \tau \) whenever the choice of \( V \) is clear from the context. If, further, the choice of \( s \) is immaterial, we will simply use \( \mathcal{L} \).

The sectoriality of \( \mathcal{L} \) can be shown by modifying an argument in E. Davies [19]. This idea has been adopted in [41] to show the sectoriality of a class of singular operators acting on scalar function.

**Theorem 3.1.** \( \mathcal{L}_s^{\eta, \tau} \in \mathcal{S}(\theta) \) for some \( \theta > \pi/2 \).

**Proof.** Choosing a local orthonormal frame of vector fields \( \{e_j\}_{j=1}^n \) satisfying \( \nabla e_k e_j = 0 \) and with dual covector fields \( \{e^j\}_{j=1}^n \). We set

\[
e_{(i)} = e_{i_1} \otimes \cdots \otimes e_{i_r}, \quad e^{(i)} = e^{i_1} \otimes \cdots \otimes e^{i_r},
\]

where \( (i) = (i_1, \ldots, i_r) \in \mathbb{J}^r := \{1, \ldots, n\}^r \) for any \( r \in \mathbb{N}_0 \). We express any \( a \in V_{s+\eta}^{\eta, \tau} \) by

\[
a = a_{(j)(r)}^{(i)(s)} e_{(i)(s)} \otimes e^{(j)(r)}
\]

where \( (i)(r) \in \mathbb{J}^r, (j)(s) \in \mathbb{J}^s \) and \( (i)(s) = (i_1, \ldots, i_{\eta}, s_1, \ldots, s_{\tau}) \) and \( (j)(r) = (j_1, \ldots, j_{\tau}, r_1, \ldots, r_{\eta}) \).

It follows from [46 Appendix C, Equation (1.35)] that

\[
\mathcal{L}u = - \sum_{j=1}^n \left[ a \nabla e_j \nabla e_j u + \text{div}(ae_j) \nabla e_j u \right] = - ag^* \otimes \mathcal{J} \cdot \nabla^2 u - \sum_{j=1}^n \nabla e_j a \nabla e_j u
\]

(3.1)

where \( \text{div} \) is the divergence operator defined in [46 Chapter 10, Equation (1.39)] and \( \mathcal{J} \in T_{\tau+\eta}^{\eta, \tau} M \) is defined by \( \mathcal{J}(i)(s) = \delta(i)(s) \) with

\[
\delta(i)(s) = \begin{cases} 1 & \text{if } (i) = (r), (j) = (s) \\ 0 & \text{otherwise.} \end{cases}
\]

Further, the notation \( \cdot \) denotes the complete contraction, cf. [5 p. 4] and \( g^* \) is the covariant metric induced by \( g \) on \( T^*M \). Then, it follows
from [5] Theorem 1.30 (i) that for \( k \in \mathbb{N} \), \( \mathcal{L}^{2k,\tau}_0 \) is a closed operator with \( \operatorname{dom}(\mathcal{L}^{2k,\tau}_0) = H^{2k+2}_p(M, V) \). By the interpolation theory and (2.1), we conclude that \( \mathcal{L}^{p,\tau}_s \) is a closed operator with \( \operatorname{dom}(\mathcal{L}^{p,\tau}_s) = H^{s+2}_p(M, V) \) for all \( s \geq 0 \).

Step 1: \( s = 0 \) and \( p = 2 \)

Since the \( L^2(M, V) \)-realization of \( \mathcal{L} \) is self-adjoint and positive semi-definite, we infer that \( \mathcal{L}_0 \) is dissipative. In particular, \(-\mathcal{L}_0|_{H^2_0(M, V)} \in \mathcal{S}(\theta)\) for any \( \theta \in (0, \pi) \), see, e.g. [1] Theorem III.4.6.7], which implies that it generates an analytic semigroup on \( L^2(M, V) \). Moreover, the Lumer-Phillips’ theorem implies that this semigroup is an \( L^2 \)-contraction.

\( \mathcal{L}_0 \) is associated with a quadratic form \( b : H^1_2(M, V) \times H^1_2(M, V) \to \mathbb{C} \) defined by

\[
b(u, v) = (a \nabla u | \nabla v)|_g, \quad u, v \in H^1_2(M, V).
\]

In the rest of Step 1, we will follow the idea in the proof of [35, Theorem 2.5]. First, direct computations show that \( (|u|_g - 1)^+ \text{sign} u \in H^1_2(M, V) \) whenever \( u \in H^1_2(M, V) \), where

\[
\text{sign} u := \begin{cases} u/|u|_g, & u \neq 0; \\ 0, & u = 0. \end{cases}
\]

Choosing a local orthonormal frame of vector fields \( \{e_j\}_{j=1}^n \) with \( \nabla e_i e_j = 0 \), then locally it holds that

\[
\nabla e_i |u|_g = \frac{\nabla e_i (u|\nabla|_g)}{2|u|_g} = \frac{(\nabla e_i u|\nabla|_g)}{2|u|_g} + \frac{(u|\nabla|_g)}{2|u|_g} = \text{Re} \left( \frac{\nabla e_i u|\nabla|_g}{|u|_g} \right).
\]

Therefore, \( \nabla |u|_g = \text{Re} \frac{\nabla u \cdot \nabla}{|u|_g} \). When \( |u|_g \geq 1 \), it follows from (3.2) that

\[
\nabla e_i [(|u|_g - 1)\text{sign} u] = \nabla e_i u - \nabla e_i \left( \frac{u}{|u|_g} \right) = \frac{|u|_g - 1}{|u|_g} \nabla e_i u + \frac{1}{|u|_g^2} u \nabla e_i |u|_g
\]

\[= \frac{|u|_g - 1}{|u|_g} \nabla e_i u + u \text{Re} \left( \frac{\nabla e_i u|\nabla|_g}{|u|_g^3} \right).
\]

We thus infer that

\[
\nabla [(|u|_g - 1)\text{sign} u] = \frac{|u|_g - 1}{|u|_g} \nabla u + \frac{\text{Re}(\nabla u \cdot \nabla) \otimes u}{|u|_g^3}.
\]

This implies that when \( |u|_g \geq 1 \)

\[
\text{Re} \left( a \nabla u |\nabla[(|u|_g - 1)\text{sign} u]|_g \right) = a |\nabla u|_g^2 \left( \frac{|u|_g - 1}{|u|_g} \right) + a \frac{|\text{Re}(\nabla u \cdot \nabla)|_g^2}{|u|_g^3} \geq 0.
\]

Therefore

\[
\text{Re} b(u, (|u|_g - 1)^+ \text{sign} u) \geq 0.
\]
Denote by $\langle \cdot, \cdot \rangle$ the inner product of $L_2(M, V)$. Given any $\lambda > 0$, let $u = \lambda(\lambda + \mathcal{L})^{-1}f$ in the above inequality for some $f \in L_2(M, V) \cap L_\infty(M, V)$ with $\|f\|_\infty \leq 1$. Then
\begin{align*}
0 \leq \text{Re} \langle \mathcal{L} \lambda(\lambda + \mathcal{L})^{-1}f\rangle, (|\lambda(\lambda + \mathcal{L})^{-1}f|_g - 1)^+ \text{sign}(\lambda(\lambda + \mathcal{L})^{-1}f) \\
= \lambda \text{Re} \int_M (|f| \text{sign}(\lambda(\lambda + \mathcal{L})^{-1}f)_g - |\lambda(\lambda + \mathcal{L})^{-1}f|_g)(|\lambda(\lambda + \mathcal{L})^{-1}f|_g - 1)^+ d\mu_g.
\end{align*}
But $\|f\|_\infty \leq 1$ implies that $|(f| \text{sign}(\lambda(\lambda + \mathcal{L})^{-1}f)_g| \leq 1$ a.e. and thus (3.3) holds true. We thus conclude that $|\lambda(\lambda + \mathcal{L})^{-1}f|_g \leq 1$ a.e.

From the standard semigroup theory, it follows that
$$e^{-t\mathcal{L}} u = \lim_{n \to \infty} \left[ \frac{1}{t} \left(1 + \frac{n}{t} \mathcal{L} \right)^{-n} u \right], \quad u \in L_\infty(M, V) \cap L_2(M, V).$$

We thus infer the $L_\infty$-contraction of the semigroup $\{e^{-t\mathcal{L}}\}_{t \geq 0}$, i.e.
$$\|e^{-t\mathcal{L}} u\|_\infty \leq \|u\|_\infty, \quad u \in L_\infty(M, V) \cap L_2(M, V).$$

Step 2: $s = 0$ and $p \in (1, \infty)$

The proof follows a classic idea in [19] Chapter 4, which was originally presented for scalar functions. By a duality argument, we can prove
$$\|e^{-t\mathcal{L}} u\|_1 \leq \|u\|_1, \quad u \in L_1(M, V).$$

Then, the Riesz-Thorin interpolation theorem implies that
$$\|e^{-t\mathcal{L}} u\|_p \leq \|u\|_p, \quad u \in L_p(M, V).$$

When $(M, g)$ is a closed manifold, the Hölder’s inequality and the strong continuity of $\{e^{-t\mathcal{L}}\}_{t \geq 0}$ in $L_2(M, V)$ show that for all $u \in L_2(M, V)$
$$\lim_{t \to 0^+} \|e^{-t\mathcal{L}} u - u\|_1 \leq \lim_{t \to 0^+} \|e^{-t\mathcal{L}} u - u\|_2(\text{vol}(M))^{1/2} = 0.$$

Since $L_2(M, V)$ is dense in $L_1(M, V)$, we thus obtain the strong continuity of $\{e^{-t\mathcal{L}}\}_{t \geq 0}$ in $L_1(M, V)$.

When $(M, g) = (\mathbb{R}^n, g_n)$, following the proof of [19] Theorem 1.4.1, we can show that $\{e^{-t\mathcal{L}}\}_{t \geq 0}$ is strongly continuous in $L_1(M)$. Note that for every $u = (u_1, \ldots, u_n) \in L_1(M, V)$ with $u_j \in L_1(M)$
$$e^{-t\mathcal{L}} u = (e^{-t\mathcal{L}} u_1, \ldots, e^{-t\mathcal{L}} u_n).$$

This implies the strong continuity of $\{e^{-t\mathcal{L}}\}_{t \geq 0}$ in $L_1(M, V)$.

By the interpolation theory, we get the strong continuity in $L_p(M, V)$ for $1 < p < 2$ and a standard duality argument yields the same for $2 < p < \infty$.
Then we can follow the Stein interpolation argument in [19, Theorem 1.4.2] and prove that \( \{e^{-tL}\}_{t \geq 0} \) can be extended to an analytic semigroup on \( L_p(M, V) \) in a sector \( \Sigma_\phi \) with

\[
\phi \geq \frac{\pi}{2} \left( 1 - \left| \frac{2}{p} - 1 \right| \right), \quad p \in (1, \infty).
\]

By the standard semigroup theory, this implies \( \mathcal{L}_0 \in S(\theta) \) with \( \theta > \frac{\pi}{2} \).

Step 3: \( s > 0 \)

The proof for this case follows by an analogous argument as for the case \( s > 0 \) and \( 1 < p < \infty \) in the proof of [42, Theorem 5.1].

First we will show that

\[
(3.4) \quad \Sigma_\theta \setminus \{0\} \subset \rho(-\mathcal{L}_s) \quad \text{and} \quad (\lambda + \mathcal{L}_0)^{-1}|_{H^s_p(M,V)} = (\lambda + \mathcal{L}_s)^{-1}, \quad \lambda \in \Sigma_\theta \setminus \{0\},
\]

where \( \theta \) is the sectorial angle of \( \mathcal{L}_0 \) asserted in Step 2. It is sufficient to verify the identities

\[
(\lambda + \mathcal{L}_0)^{-1}(\lambda + \mathcal{L}_s) = I \quad \text{and} \quad (\lambda + \mathcal{L}_s)(\lambda + \mathcal{L}_0)^{-1} = I, \quad \lambda \in \Sigma_\theta \setminus \{0\},
\]
on \( H^{s+2}_p(M,V) \) and \( H^s_p(M,V) \), respectively. The first one is trivial. For the second one, let \( u \in H^0_p(M,V) \) such that \( (\lambda + \mathcal{L})u \in H^s_p(M,V) \). If \( s \in (0, 2] \), then we have that \( u, \mathcal{L}u \in H^s_p(M,V) \), i.e. \( u \) belongs to the domain of \( \mathcal{L} \) in \( H^s_p(M,V) \), which implies that \( u \in H^{s+2}_p(M,V) \). The higher values of \( s \) can be treated by iteration.

Step 3a: \( s \in 2\mathbb{N} \)

We proceed by induction. Assume that the result holds for some \( s \in \mathbb{N} \). For each \( v \in H^{s+2}_p(M,V) \), we have

\[
\|\lambda(\lambda + \mathcal{L}_s + 2)^{-1}v\|_{H^{s+2}_p(M,V)} = \|\lambda(\lambda + \mathcal{L}_s)^{-1}v\|_{H^{s+2}_p(M,V)} \leq C_1 \left( \|\lambda(\lambda + \mathcal{L}_s)^{-1}v\|_{H^s_p(M,V)} + \|\mathcal{L}_s(\lambda(\lambda + \mathcal{L}_s)^{-1}v)\|_{H^s_p(M,V)} \right) = C_1 \left( \|\lambda(\lambda + \mathcal{L}_s)^{-1}v\|_{H^s_p(M,V)} + \|\lambda(\lambda + \mathcal{L}_s)^{-1}\mathcal{L}_sv\|_{H^s_p(M,V)} \right) \leq C_2 \left( \|v\|_{H^s_p(M,V)} + \|\mathcal{L}_sv\|_{H^s_p(M,V)} \right) \leq C_3 \|v\|_{H^{s+2}_p(M,V)},
\]

for certain \( C_1, C_2, C_3 > 0 \) independent of \( \lambda \in \Sigma_\theta \setminus \{0\} \).

Step 3b: \( s \in \mathbb{R} \)

The result follows by Proposition 2.1 and the interpolation theory. More precisely, for each \( s \in (k, k+2) \), \( k \in \mathbb{N}_0 \), and each \( \lambda \in \Sigma_\theta \), we have

\[
\lambda(\lambda + \mathcal{L})^{-1} \in \mathcal{L}(H^k_p(M,V)) \quad \text{and} \quad \lambda(\lambda + \mathcal{L})^{-1} \in \mathcal{L}(H^{k+2}_p(M,V)),
\]

with norm independent of \( \lambda \in \Sigma_\theta \setminus \{0\} \). So the required estimate is obtained by [34, Theorem 2.6] and Proposition 2.1. \( \square \)
Remark 3.2. For any $p \in (1, \infty)$, $\theta \in [0, \pi)$ and $\phi > 0$ there exists a $c > 0$ such that $c + \mathcal{L}_0 \in \mathcal{R}(\theta) \cap \mathcal{BIP}(\phi)$. This follows by [6] Corollary 10.4 in combination with [18] Theorem 4. Note that, by (3.1), the symbol of $\mathcal{L}_0$ is defined by

$$[b \mapsto a g^* \otimes \mathcal{J} \cdot (\xi \circ \otimes b)] = [b \mapsto a|\xi|^2 b], \quad b \in V, \xi \in T^*M.$$ 

Therefore, $\mathcal{L}_0$ is $\phi$-elliptic in the sense of [6] Theorem 10.3], for arbitrary small $\phi > 0$.

**Proposition 3.3 (\textit{BIP} for higher $s$).** There exists a $c > 0$ with the following property: for any $s \geq 0$ and any $\phi > 0$, we have $c + \mathcal{L}_s \in \mathcal{BIP}(\phi)$.

**Proof.** We proceed by induction and interpolation. For $s = 0$, the result holds true due to Remark 3.2. Assume that the statement holds for certain $s \in \mathbb{N}_0$. Let $c > 0$ be as in Remark 3.2. By Remark 3.2 and the argument leading to (3.4), for any $\varepsilon > 0$ and any $t \in \mathbb{R}$, we have that

$$\nu \geq \nu, \quad \text{let } u \in H^{s+2}_p(M, V).$$

By the boundedness of the imaginary powers of $c + \mathcal{L}_s$, in particular by [1] Lemma III.4.7.4 (ii)], we have that

$$\| (c + \mathcal{L}_{s+2})^{-\varepsilon + it} u \|_{H^{s+2}_p(M, V)} = \| (c + \mathcal{L}_s)^{-\varepsilon + it} u \|_{H^{s+2}_p(M, V)}$$

for certain $C_1, C_2 > 0$ independent of $\varepsilon$ and $t$. Hence, from [1] Lemma III.4.7.4 (i), we deduce that $c + \mathcal{L}_{s+2} \in \mathcal{BIP}(\phi)$.

By [18] Theorem 2.6, Proposition [2.1 and (3.5)], for each $\rho \in (0, 1)$ we have

$$\| (c + \mathcal{L}_{s+\rho})^{-\varepsilon + it} \|_{L(H^{s+\rho}_p(M, V))} \leq C_3 \left( \| (c + \mathcal{L}_s)^{-\varepsilon + it} \|_{L(H^{s+2}_p(M, V))} \right)^{1-\rho} \left( \| (c + \mathcal{L}_{s+2})^{-\varepsilon + it} \|_{L(H^{s+2}_p(M, V))} \right)^{\rho},$$

for certain $C_3 > 0$ independent of $\varepsilon$ and $t$. Hence, again by [1] Lemma III.4.7.4 (ii), we obtain that

$$\| (c + \mathcal{L}_{s+\rho})^{-\varepsilon + it} \|_{L(H^{s+\rho}_p(M, V))} \leq C_4 (e^{\phi|t|})^{1-\rho}(e^{\phi|t|})^\rho,$$

for some $C_4 > 0$ independent of $\varepsilon$ and $t$. The result then follows by [1] Lemma III.4.7.4 (i)]. \qed
Following the discussion in [43, Section 4], we can show for any \( c \geq 0 \)

\[
J_c^\sigma u := \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty x^{\sigma - 1} (c + \mathcal{L}^\eta_{s,\tau}) (x + c + \mathcal{L}^\eta_{s,\tau})^{-1} u \, dx
\]

is well-defined for all \( u \in H^{s+2}_p(M,V) = \text{dom}(\mathcal{L}^\eta_{s,\tau}). \) Indeed, we have proved that a formula similar to (4.1), [43, (4.1)], holds true for \( c = 0 \) and an operator \( -\Delta_{F,p}. \) See [43, pp. 15-17]. The proof only relies on the fact that \( -\Delta_{F,p} \in S(\theta) \) for some \( \theta > 0. \)

Note that (4.1) is exactly Balakrishnan’s formula for fractional powers of dissipative operators. By [7, (2.7)], \((c + \mathcal{L}^\eta_{s,\tau})^\sigma \) is the smallest closed extension of \( J_c^\sigma. \)

Therefore, (4.1) converges for all \( u \in \text{dom}((c + \mathcal{L}^\eta_{s,\tau})^\sigma) \) in \( H^{s+2}_p(M,V). \) The domain \( \text{dom}((c + \mathcal{L}^\eta_{s,\tau})^\sigma) \) is independent of \( c \geq 0, \) cf. [45, Lemma 2.3.5].

Due to Proposition 3.3 for certain \( c > 0, \) the operator \( c + \mathcal{L}^\eta_{s,\tau} \) has bounded imaginary powers. By Proposition 2.1 [1, (I.2.9.8)] and [45, Lemma 2.3.5], we infer that

\[
\text{dom}((\mathcal{L}^\eta_{s,\tau})^\sigma) = H^{s+2}_p(M,V).
\]

**Proposition 4.1.** For any \( s \geq 0 \) and any \( \theta > 0, \) there exists a \( c > 0 \) such that \( c + (\mathcal{L}_s)^\sigma \in \mathcal{R}(\theta). \)

**Proof.** We follow the ideas in Step 2 of the proof of [32, Theorem 6.2]. Let \( c_1 > 0 \) be fixed and sufficiently large. By [32, Theorem 1.1], \((c + \mathcal{L}_s)^\sigma \in \mathcal{R}(\theta)\) for each \( c \geq c_1. \) Moreover, by [40, Lemma 2.6] and the estimate in Part (i) in the proof of [32, Theorem 1.1], the \( R \)-sectorial bound of \((c + \mathcal{L}_s)^\sigma\) is uniformly bounded in \( c \geq c_1. \) Let \( \xi > 1 \) be fixed. Again by [40, Lemma 2.6], the operator \((c + \mathcal{L}_s)^\sigma + c^{\sigma + \xi}\) is \( R \)-sectorial and its \( R \)-sectorial bound can be chosen uniformly bounded in \( c \geq c_1. \) By [32, (2.18)] we have

\[
\|(c + \mathcal{L}_s)^\sigma - (\mathcal{L}_s)^\sigma)((c + \mathcal{L}_s)^\sigma + c^{\sigma + \xi})^{-1}\|_{L(H^p_p(M,V))} \leq \frac{C c^{\sigma + \xi}}{c^{\sigma + \xi}}
\]

for certain \( C > 0 \) only depending on the sectorial bound of \((c + \mathcal{L}_s)^\sigma \in \mathcal{S}(0)\) and \( \sigma. \) By noting that

\[
(\mathcal{L}_s)^\sigma + c^{\sigma + \xi} = (c + \mathcal{L}_s)^\sigma + c^{\sigma + \xi} + (\mathcal{L}_s)^\sigma - (c + \mathcal{L}_s)^\sigma,
\]

after taking \( c \geq c_1 \) sufficiently large, we obtain the result by perturbation, see, e.g. [32, Theorem 1].
5. \(L_q\)-Maximal regularity

Suppose that
\[
  w \in BC^r(M) \quad \text{for some} \quad r \in (0, \infty),
\]
together with the following convention: if \(M = (\mathbb{R}^n, g_n)\), then we assume that there exists a constant \(w_{\infty} > 0\) such that
\[
  \|w - w_{\infty}\|_{L_\infty(\mathbb{R}^n \setminus B_M(0, \tilde{R}))} \to 0 \quad \text{as} \quad \tilde{R} \to \infty.
\]
Assume, in addition, that there exists a constant \(c_0 > 0\) such that
\[
  w > c_0.
\]

Let \(s \in [0, r)\) and \(1 < p, q < \infty\). In this section, we will show that
\[
  w \mathcal{L}^\sigma \in \mathcal{MR}_q(H^{s+2\sigma}_p(M, V), H^s_p(M, V)).
\]

Let \(f \in L_q(J, H^s_p(M))\), where \(J = (0, T)\) with \(T > 0\). Consider the Cauchy problem:
\[
  \begin{aligned}
  \partial_t u + w \mathcal{L}^\sigma u &= f; \\
  u(0) &= 0.
  \end{aligned}
\]

Our goal is to prove that \((5.4)\) admits a unique solution
\[
  u \in L_q(J, H^{s+2\sigma}_p(M, V)) \cap H^1_q(J, H^s_p(M, V)).
\]

Let \(R, \tilde{R} > 0\) and let \(\tilde{\omega} : \mathbb{R} \to [0, 1]\) being a smooth non-increasing function that equals 1 on \([0, 1/2]\) and 0 on \([3/4, \infty)\). Choose a finite open cover \(U_j\) of \((M, g)\), where \(j \in \{1, \ldots, N\}\) when \((M, g)\) is a closed manifold and \(j \in \{0, 1, \ldots, N\}\) when \((M, g) = (\mathbb{R}^n, g_n)\), such that the following properties are fulfilled.

We let \(U_j = B_{j,R} = B_M(p_j, R)\) being geodesic balls with radius \(R\) on \(M\) centered at \(p_j \in M\), \(j = 1, \ldots, N\). Moreover, we define
\[
  (5.5) \quad w_{j,R}(p) = \tilde{\omega}
  \left(\frac{d(p, p_j)}{2R}\right)w(p) + \left(1 - \tilde{\omega}
  \left(\frac{d(p, p_j)}{2R}\right)\right)w(p_j), \quad p \in M, \quad j = 1, \ldots, N.
\]

In the case of \(M = (\mathbb{R}^n, g_n)\), we assume that \(U_j = B_{j,R}, j = 1, \ldots, N,\) cover the closure of \(B_M(0, \tilde{R})\) and we further choose \(U_0 = \mathbb{R}^n \setminus B_M(0, \tilde{R})\). In this case, we also define
\[
  (5.6) \quad w_{0,\tilde{R}}(p) = \left(1 - \tilde{\omega}
  \left(\frac{d(p, 0)}{2\tilde{R}}\right)\right)w(p) + \tilde{\omega}
  \left(\frac{d(p, 0)}{2\tilde{R}}\right)w_{\infty}, \quad p \in M,
\]
where \(w_{\infty}\) is defined in \((5.2)\).

Lemma 5.1. Assume that \(w\) satisfies \((5.1)-(5.2)\) and \(r \in (0, 1]\). For any \(\alpha \in [0, r)\) and \(\varepsilon > 0\), there exists an \(R_0 > 0\) such that
\[
  \|w_{j,R} - w(p_j)\|_{\alpha, \infty} < \varepsilon \quad \text{for each} \quad j \in \{1, \ldots, N\}, \quad \text{whenever} \quad R \in (0, R_0).
\]
In addition, if \( M = (\mathbb{R}^n, g_n) \), then there exists \( \tilde{R} > 0 \) such that
\[
\| w_{0, \tilde{R}} - w_\infty \|_{a, \infty} < \varepsilon, \text{ whenever } \tilde{R} > \tilde{R}.
\]

**Proof.** By the given condition, we immediately have
\[
\| w_{j, \tilde{R}} - w(p_j) \|_{\infty} \to 0 \text{ for each } j \in \{1, \ldots, N\}
\]
as \( R \to 0^+ \). Let
\[
f_{j, R}(p) = w_{j, R}(p) - w(p_j) = \tilde{\omega} \left( \frac{d(p, p_j)}{2R} \right) \left( w(p) - w(p_j) \right).
\]
Then for \( p, q \in B_M(p_j, 2R) \) with \( p \neq q \), it follows from (5.1) that
\[
\left| f_{j, R}(p) - f_{j, R}(q) \right| \frac{d(p, q)^\alpha}{d(p, q)^\alpha} \leq \frac{\tilde{\omega} \left( \frac{d(p, p_j)}{2R} \right) - \tilde{\omega} \left( \frac{d(q, p_j)}{2R} \right)}{d(p, q)^\alpha} |w(p) - w(p_j)| + \frac{\tilde{\omega} \left( \frac{d(q, p_j)}{2R} \right)}{d(p, q)^\alpha} |w(p) - w(q)|
\]
\[
\leq \frac{\tilde{\omega} \left( \frac{d(p, p_j)}{2R} \right) - \tilde{\omega} \left( \frac{d(q, p_j)}{2R} \right)}{d(p, q)^\alpha} \left| \frac{d(p, p_j)}{2R} - \frac{d(q, p_j)}{2R} \right|^\alpha |w(p) - w(p_j)| + CR^{r-\alpha}
\]
for certain \( C > 0 \). If \( M = (\mathbb{R}^n, g_n) \), then let
\[
g_{0, \tilde{R}}(p) = w_{0, \tilde{R}}(p) - w_\infty = \left( 1 - \tilde{\omega} \left( \frac{d(p, 0)}{2R} \right) \right) (w(p) - w_\infty).
\]
For any \( p, q \in \mathbb{R}^n \setminus B_M(0, \tilde{R}) \) with \( p \neq q \), we have
\[
\left| g_{0, \tilde{R}}(p) - g_{0, \tilde{R}}(q) \right| \frac{d(p, q)^\alpha}{d(p, q)^\alpha} \leq |w(q) - w_\infty| \left| \tilde{\omega} \left( \frac{d(p, 0)}{2R} \right) - \tilde{\omega} \left( \frac{d(q, 0)}{2R} \right) \right| + \left| 1 - \tilde{\omega} \left( \frac{d(p, 0)}{2R} \right) \right| \left| \frac{d(p, 0)}{2R} - \frac{d(q, 0)}{2R} \right|^\alpha \frac{d(p, q)^\alpha}{d(p, q)^\alpha}
\]
\[
(5.7)
\]
Due to (5.2), the first term on the right hand side of (5.7) is \( \leq C \tilde{R}^{r-\alpha} \). For the second term on the right hand side of (5.7), if \( d(p, q) \leq \varepsilon_0 \), for some \( \varepsilon_0 > 0 \), then it is \( \leq C \varepsilon_0^{r-\alpha} \varepsilon / 2 \), by taking \( \varepsilon_0 \) small enough. If \( d(p, q) \geq \varepsilon_0 \), then
\[
\left| g_{0, \tilde{R}}(p) - g_{0, \tilde{R}}(q) \right| \frac{d(p, q)^\alpha}{d(p, q)^\alpha} \leq C \varepsilon_0^{-\alpha} \left| w(p) - w_\infty \right| + \left| w_\infty - w(q) \right| \leq \varepsilon / 2,
\]
by choosing \( \tilde{R} \) sufficiently large due to (5.2). The result follows by writing \( M \times M = \{(p, q) \in \mathbb{R}^n \times \mathbb{R}^n \mid d(p, q) \leq \varepsilon_0\} \cup \{(p, q) \in \mathbb{R}^n \times \mathbb{R}^n \mid d(p, q) \geq \varepsilon_0\} \).

\( \square \)
Lemma 5.2. If $\phi \in BC^\infty(M)$, then for any $s \geq 0$ and $c > 0$
\[ [\phi, (c + \mathcal{L})^\sigma] \in \mathcal{L}(H^{s+2\sigma}_p(M, V), H^{s+1-\varepsilon}_p(M, V)) \]
for any $\varepsilon > 0$. Moreover
\[ \| [\phi, (c + \mathcal{L})^\sigma]\|_{\mathcal{L}(H^{s+2\sigma}_p(M, V), H^{s+1-\varepsilon}_p(M, V))} \leq M = M(c_0), \quad c > c_0, \]
for any fixed $c_0 > 0$.

Proof. Let $A = c + \mathcal{L}$. Then, for any $u \in H^{s+2\sigma}_p(M, V)$, (5.1) implies
\[ (\phi A^\sigma - A^\sigma \phi)u = \frac{\sin(\pi \sigma)}{\pi} \int_0^{+\infty} x^{\sigma-1} \left\{ [\phi, A](A + x)^{-1} + A[\phi, (A + x)^{-1}] \right\} u \, dx \]
\[ = -\frac{\sin(\pi \sigma)}{\pi} \int_0^{+\infty} x^{\sigma}(A + x)^{-1}[\phi, (A + x)^{-1}]u \, dx. \]

Note that $[A, \phi]$ is a first order differential operator. When $x > 1$, Lemma 2.3.3 implies
\[ \| x^{\sigma}(A + x)^{-1}[\phi, (A + x)^{-1}]u \|_{s+1-\varepsilon, p} \leq \| [A, \phi]x^{\sigma-1}A^{-\frac{1}{2}-\frac{\varepsilon}{2}}A^{1-\sigma-\frac{\varepsilon}{2}}(A + x)^{-1}A^{\sigma+\frac{\varepsilon}{2}-\frac{1}{2}}u \|_{s+1-\varepsilon, p} \leq Cx^{-1-\frac{\varepsilon}{4}}u \|_{s+2\sigma, p} \]
for $\varepsilon > 0$ sufficiently small; and when $x \leq 1$, letting $B = \frac{\varepsilon}{2} + \mathcal{L}$, we have
\[ \| x^{\sigma}(A + x)^{-1}[\phi, (A + x)^{-1}]u \|_{s+1-\varepsilon, p} \leq \| [A, \phi]x^{\sigma-1}B^{\frac{1}{2}+\frac{\varepsilon}{2}}B^{1-\sigma-\frac{\varepsilon}{2}}(A + x)^{-1}B^{\sigma+\frac{\varepsilon}{2}-\frac{1}{2}}u \|_{s+1-\varepsilon, p} \leq C \left( \frac{c}{2} + x \right)^{-1-\frac{\varepsilon}{4}}u \|_{s+2\sigma, p}. \]

These two estimates establish the assertion. \hfill \Box

Proposition 5.3. Suppose that $w$ satisfies (5.1)-(5.3) with $r \in (0, 1]$ and let $f \in L_q(J, H^s_p(M, V))$ for some $s \in [0, r)$. Then there exists a unique
\[ u \in L_q(J, H^{s+2\sigma}_p(M, V)) \cap H^1_q(J, H^s_p(M, V)) \]
solving (5.4).

Proof. The result follows by similar steps as in the proof of [42 Theorem 6.2], where we have to take Lemma 5.2 into account. More precisely, if $w_{j, R}$, $j \in \{1, \ldots, N\}$, $R > 0$, are as in (5.5)-(5.6), let
\[ w_{j, R}A^\sigma = w_j A^\sigma + (w_{j, R} - w_j) A^\sigma : H^{s+2\sigma}_p(M, V) \to H^s_p(M, V), \]
where $A = c_0 + \mathcal{L}$, $c_0 > 0$; and in the case of $M = (\mathbb{R}^n, g_n)$, we define
\[ w_{0, R}A^\sigma = w_\infty A^\sigma + (w_{0, R} - w_\infty) A^\sigma : H^{s+2\sigma}_p(M, V) \to H^s_p(M, V). \]
Note that, by Proposition 2.2 (ii), elements in $BC^r(M)$ act by multiplication as bounded maps on $H^s_p(M, V)$. Therefore, for every $\theta \in (\pi/2, \pi)$ and every $c > 0$, by Lemma 5.1 [5.8] and [32, Theorem 1], after choosing $R$ sufficiently small and $\tilde{R}, N$ large enough, both operators (5.8)-(5.9) belong to $R(\theta)$. As a consequence, due to standard sectoriality of $w_{j,R}A^\sigma, j \in \{1, \ldots, N\}, w_0, \tilde{R}A^\sigma$ and $A^\sigma$, [42] (6.45) holds true.

Set $J = \{1, \ldots, N\}$ when $(M, g)$ is a closed manifold or $J = \{0, 1, \ldots, N\}$ when $(M, g) = (\mathbb{R}^n, g_n)$. Moreover, we put $R_j = R$ when $j \in \{1, \ldots, N\}$ and $R_0 = \tilde{R}$. Choose $\phi_j \in BC^\infty(M), j \in J$, to be a partition of unity subordinated to the cover $\{U_j\}_{j \in J}$. Moreover, let $\psi_j \in BC^\infty(M), j \in J$, supported on $U_j$, taking values on $[0, 1]$ and satisfying $\psi_j \equiv 1$ on the support of $\phi_j$. Then by Lemma 5.2, similarly to [42, (6.47)], for sufficiently large $c$, we can construct a left inverse $L(\lambda)$ of $wA^\sigma + c + \lambda, \lambda \in \Sigma_\theta$, that belongs to the space $\mathcal{L}(H^s_p(M, V), H^{s+2\sigma}_p(M, V))$. More precisely, we have that

$$L(\lambda) = \sum_{k=0}^{\infty} Q^k(\lambda) R(\lambda), \quad \lambda \in \Sigma_\theta,$$

where

$$Q(\lambda) = \sum_{j \in J} \psi_j (w_{j,R_j}A^\sigma + c + \lambda)^{-1} w_{j,R_j} [A^\sigma, \phi_j]$$

and

$$R(\lambda) = \sum_{j \in J} \psi_j (w_{j,R_j}A^\sigma + c + \lambda)^{-1} \phi_j.$$

Furthermore, similarly to [42] (6.48) we can show that $L(\lambda)$ is also a right inverse of $wA^\sigma + c + \lambda, \lambda \in \Sigma_\theta$. After having the above expression of the resolvent of $wA^\sigma + c$, we can show $R$-sectoriality of angle $\theta$ for this operator as in the proof of [42, Theorem 6.2], i.e. similarly to [42, (6.49)] and the estimates below. Next, $R$-sectoriality for $wL^\sigma + c$ for large $c$ is obtained by the Step 2 of the proof of [42, Theorem 6.2]. Then the result follows by Theorem 2.8. \[\square\]

Remark 5.4. The proof of Proposition 5.3 is based on the generalization of freezing-of-coefficients method to the case of non-local operator of certain type. Such an extension was first demonstrated in the proof of [42, Theorem 6.2]. One of the main ingredients of the proof is the observation that the commutator of the fractional powers of the Laplacian and a function in the class $BC^\infty(M)$ is indeed of lower order in a sectoriality sense, see Lemma 5.2. Moreover, instead of using an $\varepsilon - C_\varepsilon$ argument as in the classical case (i.e. the case of differential operators in $L^p$-spaces, see, e.g. the proof of [20, Theorem 5.7]), similarly to the proof of [42, Theorem 6.2], we proceed by using the decay properties of the resolvent of a sectorial operator, i.e. [45, Lemma 2.3.3], in order to construct a left and right inverse for $wA^\sigma + c + \lambda, \lambda \in \Sigma_\theta$. 
With a little abuse of notation, we denote
\[ F := [\nabla, \mathcal{L}] =: \nabla \mathcal{L} - \mathcal{L} \nabla, \]
which is a second order differential operator. Then, we have
\[ [\nabla, (\lambda + \mathcal{L})^{-1}] = - (\lambda + \mathcal{L} inverse)^{-1} F (\lambda + \mathcal{L} inverse)^{-1} \]
for all \( \lambda \in \Sigma_\theta \), where \( \theta > \pi/2 \) is the sectorial angle of \( \mathcal{L} \) asserted in Theorem 3.1. Given any \( \delta > 0 \) and \( u \in H^{\pi+2\pi+\delta}(M, V) \), since
\[
\nabla \mathcal{L}(\lambda + \mathcal{L})^{-1} u = \nabla u - \lambda \nabla (\lambda + \mathcal{L})^{-1} u = \mathcal{L}(\lambda + \mathcal{L})^{-1} \nabla u - \lambda (\nabla, (\lambda + \mathcal{L})^{-1} u) \]
we have
\[
\nabla \mathcal{L}^\sigma u = \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty x^{\sigma-1} \nabla \mathcal{L}(x + \mathcal{L})^{-1} u \, dx \\
= \mathcal{L}^\sigma \nabla u - \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty x^{\sigma}(x + \mathcal{L})^{-1} F (x + \mathcal{L})^{-1} u \, dx.
\]

To estimate (\(*\)), we first note that for \( x > 1 \)
\[
\| (x + \mathcal{L} inverse)^{-1} F (x + \mathcal{L} inverse)^{-1} u \|_{s,p} \leq \frac{M}{x} \| F (x + \mathcal{L})^{-1} u \|_{s,p} \\
\leq \frac{M}{x} \| (x + \mathcal{L})^{-1} u \|_{s+2,p} \\
\leq \frac{M}{x} \left[ \| \mathcal{L} (x + \mathcal{L})^{-1} u \|_{s,p} + \| (x + \mathcal{L})^{-1} u \|_{s,p} \right] \\
\leq \frac{M}{x} \frac{C}{x^{\sigma+\pi}} \| u \|_{s+2\sigma+2\pi,p} + \frac{C}{x} \| u \|_{s,p}
\]
for \( \varepsilon > 0 \) sufficiently small. The last step follows from [45] Lemma 2.3.3]. When \( x \leq 1 \), we will use the following lemma.

Lemma 5.5. Let \( X_1 \overset{d}{\to} X_0 \) be a pair of Banach spaces, where \( X_j \) is equipped with norm \( \| \cdot \|_j \). Suppose that \( \theta \in (\pi/2, \pi) \), and \( A \in \mathcal{S}(\theta) \) with domain \( \text{dom}(A) = X_1 \). Let \( (\alpha, p), (\beta, p) \in \{(0,1) \times [1, \infty) \} \cup \{(1, \infty)\} \) with \( \beta \geq \alpha \). There exists \( C = C(p, \alpha, \beta) > 0 \) such that for all \( t \in (0, 1] \)
\[
\| (t - A)^{-1} \|_{\mathcal{L}(X_0, X_1), (\alpha, p), (X_0, X_1), \beta, p} \leq C t^{\beta-1-\alpha}.
\]

Proof. It follows from [33, Propositions 2.2.2 and 2.2.9] that
\[
\| t^{\beta-\alpha} e^{tA} \|_{\mathcal{L}(X_0, X_1), (\alpha, p), (X_0, X_1), \beta, p} \leq C = C(p, \alpha, \beta).
\]
Since
\[(t + A)^{-1} = \int_0^\infty e^{-ts}e^{-sA}ds,\]
we can compute for all \(u \in (X_0, X_1)_{\alpha,p}\)
\[\| (t + A)^{-1}u \|_{(X_0, X_1)_{\beta,p}} \leq \int_0^\infty e^{-ts}\|e^{-sA}u\|_{(X_0, X_1)_{\beta,p}}ds \leq C\|u\|_{(X_0, X_1)_{\alpha,p}} \int_0^\infty e^{-ts}s^{\alpha-\beta}ds \leq C\|u\|_{(X_0, X_1)_{\alpha,p}} t^{\beta-1-\alpha} \int_0^\infty e^{-s}s^{\alpha-\beta}ds.\]

By Lemma 5.5, when \(x \leq 1\), we have an even better estimate
\[\| (x + \mathcal{L}^\eta)_{s,\tau+1}^{-1}F(x + \mathcal{L}^\eta)_{s,\tau}^{-1}u \|_{s,p} \leq \frac{M}{x}\| (x + \mathcal{L})^{-1}u \|_{s+2,p} \leq \frac{M}{x^{1+\sigma-\varepsilon}}\|u\|_{s+2\sigma,p}.\]

Therefore,
\[(5.11)\]
\[\|\nabla, \mathcal{L}^\sigma\|_{s,p} =: \|\nabla(\mathcal{L}^\eta)_{s,\tau}^\sigma u - (\mathcal{L}^\eta)_{s-1,\tau+1}^\sigma (\nabla u)\|_{s,p} \leq M\|u\|_{s+2\sigma+2\varepsilon,p}.\]

**Theorem 5.6.** Assume that \(w \in BC^\varepsilon(M)\) satisfies (5.1)–(5.3) and let
\[f \in L_q(J, H^s_p(M))\]
for some \(p, q \in (1, \infty)\) and \(s \in [0, r]\).

Then the solution to (5.4) satisfies
\[(5.12)\]
\[u \in H^1_q(J, H^s_p(M, V)) \cap L_q(J, H^{s+2\sigma}_p(M, V)),\]
i.e.
\[w\mathcal{L}^\sigma \in \mathcal{M}R_q(H^{s+2\sigma}_p(M, V), H^s_p(M, V)).\]

**Proof.** When \(s \in [0, 1]\), the assertion is already proved. Consider the case \(s \in [1, 2)\) and \(r > 1\). Choose \(\varepsilon > 0\) so small that \(s - 1 + 2\varepsilon < 1\). Taking \(\nabla\) on both sides of (5.4) yields
\[\partial_t v + w(\mathcal{L}^\eta)_{s-1,\tau+1}^\sigma v = \nabla f - \nabla w \otimes (\mathcal{L}^\eta)_{s,\tau}^\sigma u - w[\nabla, (\mathcal{L})^\sigma]u,\]
where \(v = \nabla u\). By Proposition 5.3 we already know that
\[u \in L_q(J, H^{s-1+2\sigma+2\varepsilon}_p(M, V)) \cap H^1_q(J, H^{s-1+2\varepsilon}_p(M, V)).\]
The standard pointwise multiplication theory, cf. [4, Theorem 9.2], implies
\[\nabla w \otimes (\mathcal{L}^\eta)_{s,\tau}^\sigma u \in L_q(J, H^{s-1}_p(M, V^\eta_{\tau+1}));\]
(5.11) gives
\[\nabla, (\mathcal{L}^\sigma)u \in L_q(J, H^{s-1}_p(M, V^\eta_{\tau+1})).\]
Note that in the above step, we need \( u \in L_q(J, H_p^{s-1+2\sigma+2\epsilon}(M, V)) \) in view of (5.11). It follows from Proposition 5.3 that \( v \in L_q(J, H_p^{s-1+2\sigma}(M, V_{r+1})) \cap H^1_q(J, H_p^{s-1}(M, V_{r+1})). \) This proves (5.12) for \( s \in [1, 2). \) The general case follows by induction. □

6. Applications

In this section, we will apply Theorem 5.6 and the following theorem by P. Clément and S. Li to study two quasilinear parabolic equations.

**Theorem 6.1** (Clément and Li, [17, Theorem 2.1]). Suppose that \( X_1 \hookrightarrow X_0 \) is a continuously and densely injected complex Banach couple. Let \( U \) be an open subset of \( (X_0, X_1) \) with \( \frac{s}{q} \leq 1 \), where \( q \in (1, \infty) \). Consider the problem

\[
\begin{cases}
  u'(t) + A(u(t))u(t) = F(t, u(t)), & t \geq 0 \\
  u(0) = u_0,
\end{cases}
\]

where \( u_0 \in U \). Assume that:

(H1) \( A \in C^1(U, \mathcal{L}(X_1, X_0)). \)

(H2) \( F \in C^{1,1}([0, T_0] \times U, X_0) \) for some \( T_0 > 0. \)

(H3) \( A(u_0) \in M^{R_q}(X_1, X_0). \)

Then, there exists a \( T \in (0, T_0] \) and a unique \( u \in H^1_q((0, T), X_0) \cap L_q((0, T), X_1) \) solving (6.1).

**6.1. Fractional porous medium equation.** As an application, we consider first the following fractional porous medium equation (FPME)

\[
\begin{cases}
  \partial_t u + (-\Delta)^\sigma |u|^{m-1}u = f & \text{on } M \times (0, \infty); \\
  u(0) = u_0 & \text{on } M,
\end{cases}
\]

where \((M, g)\) is an \( n\)-dimensional closed manifold, \( \sigma \in (0, 1) \) and \( f \in C([0, T_0], H^s_p(M)) \) for some \( T_0 > 0, s \geq 0 \) and \( p \in (1, \infty) \). Further, \( \Delta = -\nabla^* \circ \nabla \) is the Laplace-Beltrami operator, cf. (3.1).

**Theorem 6.2** (Smoothing for the FPME). Let \( u_0 \in B^{s+2\sigma-2\sigma/q}_p(M) \) for some \( q \in (1, \infty) \), where \( p \) and \( s \) are as in (6.3). Assume that \( 2\sigma > 2\sigma/q + n/p \) and \( u_0 > c \) on \( M \), for certain \( c > 0 \). Then, there exists a \( T \in (0, T_0] \) and a unique

\[
\begin{align*}
  u & \in L_q((0, T), H_p^{s+2\sigma}(M)) \cap H^1_q((0, T), H_p^s(M)) \\
  & \hookrightarrow C([0, T], B^{s+2\sigma-2\sigma/q}_p(M))
\end{align*}
\]
Let $\Gamma$ be a finite positively oriented simple path in $\alpha$ for any $\nu > 0$.

By Proposition 2.2, we have

$$\int f \in \bigcap_{\nu > 0} L_{\nu}((0, T_0), H^\nu_p(M)) \cap C^\nu((0, T_0), H^\nu_p(M)),$$

then $u$ satisfies the regularity

$$u \in \bigcap_{\nu > 0} C^\nu((0, T), H^\nu_p(M)).$$

**Proof.** We consider first the problem

$$\begin{cases}
\partial_t w + mw^{m-1}(\Delta)^{\sigma} w = mw^{m-1}f & \text{on } M \times (0, \infty); \\
w(0) = u_0^m & \text{on } M.
\end{cases}$$

Concerning (6.4)-(6.5), we will apply the theorem of P. Clément and S. Li, i.e. Theorem 6.1, to the above equation and then we will recover the required existence and regularity result for the original problem. Define the Banach spaces $X_0 = H^s_p(M), X_1 = H^{s+2\sigma}_p(M)$, the operator family $A(\cdot) = A_s(\cdot) = m(\cdot)^{\frac{m-1}{m}}(-\Delta_s)^{\sigma}$, where $\Delta_s$ denotes the map $\Delta : H^{s+2}_p(M) \to H^s_p(M)$, and let the potential term $F(\cdot) = m(\cdot)^{\frac{m-1}{m}} f$. Note that Proposition 2.1 implies

$$B^{s+2\sigma-2\sigma/q}_p(M) \doteq (X_0, X_1)_{1-1/q, q}.$$

By Proposition 2.2, we have

$$u_0^\alpha \in B^{s+2\sigma-2\sigma/q}_p(M) \hookrightarrow H^s_p(M) \hookrightarrow BC^\sigma(M),$$

for any $\alpha \in \mathbb{R}$ and $s + n/p < r + n/p < \xi < s + 2\sigma - 2\sigma/q$. By the relation

$$|v - mu_0^{m-1}| \leq \|v - mu_0^{m-1}\|_\infty \leq C_1\|v - mu_0^{m-1}\|_{B^{s+2\sigma-2\sigma/q}_p},$$

valid for certain $C_1 > 0$, we choose an open ball $U$ in $B^{s+2\sigma-2\sigma/q}_p(M)$ around $mu_0^{m-1}$ of sufficiently small radius, such that

$$\text{Re}(v) \geq c/2 \quad \text{for each} \quad v \in U.$$

Let $\Gamma$ be a finite positively oriented simple path in $\{z \in \mathbb{C} | \text{Re}(z) > 0\}$ that surrounds $\{\text{Ran}(v) | v \in U\}$. For each $v_1, v_2 \in U$ we have

$$v_1^\alpha - v_2^\alpha = \frac{1}{2\pi i} \int_\Gamma \left( \frac{\lambda^\alpha}{\lambda - v_1} - \frac{\lambda^\alpha}{\lambda - v_2} \right) d\lambda = \frac{v_1 - v_2}{2\pi i} \int_\Gamma \frac{\lambda^\alpha}{(\lambda - v_1)(\lambda - v_2)} d\lambda.$$

Hence, by Proposition 2.2 (ii)

$$\|A_s(v_1) - A_s(v_2)\|_{\mathcal{L}(H^{s+2\sigma}_p(M), H^s_p(M))}$$

$$= \|(v_1^{m-1} - v_2^{m-1})(-\Delta_s)^{\sigma}\|_{\mathcal{L}(H^{s+2\sigma}_p(M), H^s_p(M))}$$

$$\leq C_2\|v_1^{m-1} - v_2^{m-1}\|_{\mathcal{L}(H^s_p(M))}$$

$$\leq C_3\|v_1^{m-1} - v_2^{m-1}\|_{H^s_p(M)} \leq C_4\|v_1^{m-1} - v_2^{m-1}\|_{B^{s+2\sigma-2\sigma/q}_p}.$$
for certain $C_2, C_3, C_4 > 0$, so that

\[
\|A_s(v_1) - A_s(v_2)\|_{L(H^{s+2\sigma}_p(M), H^s_p(M))} \leq C_5 \|v_1 - v_2\|_{B^{s+2\sigma-2\sigma/q}_{p,q}}
\]

for some $C_5 > 0$ due to (6.8) and (6.10).

Furthermore, for each $t_1, t_2 \in [0, T_0]$, by Proposition 2.2 (ii), we have

\[
\|F(v_1, t_1) - F(v_2, t_2)\|_{s, p}
= m\|((v_1 - v_2) + m^{-1} f(t_1)) + v_2/m (f(t_1) - f(t_2))\|_{s, p}
\leq C_6 \|v_1 - v_2\|_{H^s_p(M)} \|f\|_{C([0, T_0], H^s_p(M))}
+ \|v_2/m \|_{H^s_p(M)} \|f(t_1) - f(t_2)\|_{s, p}
\leq C_7 \|v_1 - v_2\|_{B^{s+2\sigma-2\sigma/q}_{p,q}} \|f\|_{C([0, T_0], H^s_p(M))}
+ \|v_2/m \|_{B^{s+2\sigma-2\sigma/q}_{p,q}} \|f(t_1) - f(t_2)\|_{s, p}
\leq C_8 \|v_1 - v_2\|_{B^{s+2\sigma-2\sigma/q}_{p,q}} + |t_1 - t_2|
\]

(6.12)

for some $C_6, C_7, C_8 > 0$, where we have used (6.8) and (6.10) once more.

Clearly, $A_s(u_0)$ has maximal $L_q$-regularity due to Theorem 5.6 and (6.8). By Theorem 6.1, there exists a $T \in (0, T_0]$ and a unique

\[
w \in H^s_p((0, T), H^s_p(M)) \cap L_q((0, T), H^{s+2\sigma}_p(M))
\]

solving (6.7). In addition, due to (2.7), we also have

(6.13)

\[
w \in C([0, T], B^{s+2\sigma-2\sigma/q}_{p,q}(M)).
\]

Hence, by choosing $T > 0$ small enough we can make $w(t) \in U$ for each $t \in [0, T)$. Then, due to Proposition 2.2 (iii) and (6.8), for any $\alpha \in \mathbb{R}$

(6.14)

\[
w^\alpha \in C([0, T], B^{s+2\sigma-2\sigma/q}_{p,q}(M)).
\]

By the relation

(6.13), (6.15) and Proposition 2.2 we deduce that $w^{1/m} \in H^{1/2}_p((0, T), H^s_p(M))$.

Furthermore, due to the Banach algebra property of $H^{s+2\sigma}_p(M)$ and (6.13), we see that $w^{1/m}(t) \in H^{s+2\sigma}_p(M)$ for almost all $t \in [0, T]$, so that the function $u = w^{1/m}$ satisfies $\partial_t u + (-\Delta)^\sigma(u^{m}) = f$ for almost all $t \in [0, T]$. We estimate

\[
\|u(t)\|_{s+2\sigma} \leq C_9 (\|u(t)\|_{s, p} + \|(-\Delta)^\sigma u(t)\|_{s, p})
\leq C_9 (\|u(t)\|_{s, p} + \|\partial_t u(t)\|_{s, p} + \|f(t)\|_{s, p}),
\]

for certain $C_9 > 0$ and almost all $t \in [0, T]$. By integrating the above inequality over $t \in [0, T]$, we obtain (6.4). Then, (6.5) follows by (2.7).
Concerning (6.6), we will apply the smoothing result \[11\] Theorem 3.1 to (6.7) and then we will recover again the required regularity for \(u\). Hence, we examine the conditions (i), (ii) and (iii) of \[11\] Theorem 3.1. We choose the Banach scales 
\[
Y_j^0 = H_p^{s+jb}(M), \quad Y_j^1 = H_p^{s+2\sigma+jb}(M), \quad j \in \mathbb{N}_0,
\]
where \(b \in (0, 2\sigma - \frac{2\sigma}{q} - \frac{2}{p})\) is fixed. Moreover, choose \(A(\cdot), F\) as before and let \(Z = \{v \in U \mid \text{Im}(v) = 0\}\).

Condition (i). By the previous step, we have the existence of \(w\) as in (6.13) satisfying \(w(t) \in Z\) for all \(t \in [0, T]\); here we have taken the complex conjugate to (6.7) and then used the above uniqueness result, i.e. we have obtained in addition that \(\text{Im}(w(t)) = 0, \quad t \in [0, T]\). By this observation, (6.8), (6.9), (6.14) and Theorem 5.6, we also have that, for each \(t \in [0, T]\), the operator \(A(w(t)) \in \mathcal{MR}_q(Y_1^0, Y_0^0)\). Finally, due to (6.11) and (6.14), we deduce that \(A(w(\cdot)) \in C([0, T], L(Y_1^0, Y_0^0))\).

Condition (ii). Let \(h \in Z \cap (Y_j^0, Y_j^1)_{1-\frac{2}{q}, q}, \quad j \in \mathbb{N}\). By Proposition 2.2 (i) and (6.8), we have
\[
h \in B_p^s + jb + 2\sigma - \frac{2\sigma}{q} (M) \hookrightarrow H_p^s(M) \hookrightarrow BC^{\sigma_j}(M),
\]
where \(s + (j + 1)b + n/p < \xi_j < \sigma_j < s + jb + 2\sigma - \frac{2\sigma}{q}\). Thus, due to (6.9) and Theorem 5.6 we obtain that \(A(h) \in \mathcal{MR}_q(Y_j^{j+1}, Y_0^{j+1})\).

Now let \(\eta \in C([0, T], Z \cap (Y_j^0, Y_j^1))_{1-\frac{2}{q}, q}\). Similarly to (6.11), by (6.9) and Proposition 2.2 we get
\[
\|A_{s+(j+1)b}(\eta(t_1)) - A_{s+(j+1)b}(\eta(t_2))\|_{L(H_p^{s+(j+1)b+2\sigma}(M), H_p^{s+(j+1)b}(M))} \leq C_{10}\|\eta^m(t_1) - \eta^m(t_2)\|_{L(H_p^s(M))} \leq C_{11}\|\eta^m(t_1) - \eta^m(t_2)\|_{H_p^s(M)} \leq C_{12}\|\eta^m(t_1) - \eta^m(t_2)\|_{B_p^s, Y_j^{j+2\sigma - \frac{2\sigma}{q}}} \leq C_{13}\|\eta(t_1) - \eta(t_2)\|_{B_p^s, Y_j^{j+2\sigma - \frac{2\sigma}{q}}}
\]
for some \(C_{10}, C_{11}, C_{12}, C_{13} > 0\), where \(t_1, t_2 \in [0, T]\). This implies that 
\[ A_{s+(j+1)b}(\eta(\cdot)) \in C([0, T], L(Y_j^{j+1}, Y_0^{j+1}))\].

Condition (iii). Similarly to (6.12) we have
\[
\|F(\eta(\cdot), \cdot)\|_{s+(j+1)b, p} \leq C_{14}\|\eta^m(\cdot)\|_{B_p^{s+jb+2\sigma - \frac{2\sigma}{q}}} \|F\|_{s+(j+1)b, p}
\]
for certain \(C_{14} > 0\). By Proposition 2.2 (iii), the set \(\eta^m(t), \quad t \in [0, T]\), is bounded in \(B_p^{s+jb+2\sigma - \frac{2\sigma}{q}}(M)\), so that \(F(\eta(\cdot), \cdot) \in L_q((0, T), Y_0^{j+1})\).
We conclude that for each $\delta \in (0, T)$
\[ w \in \bigcap_{\nu > 0} L_q((\delta, T), H^{\nu+2\sigma}_p(M)) \cap H^1_q((\delta, T), H_\nu(M)). \]

By the same argument as before, we can pass the above regularity to $u$ so that by [1, (I.2.5.2)], Proposition 2.1 and (2.7) we also have
\[ u \in \bigcap_{\nu > 0} H^1_q((\delta, T), H^\nu_p(M)) \cap C([\delta, T), H^\nu_p(M)). \]

Thus, by Proposition 2.2
\[ u^{m-1} \partial_t u \in \bigcap_{\nu > 0} L_q((\delta, T), H^\nu_p(M)), \]
so that, by differentiating (6.2) over time we find that $u \in H^2_q((\delta, T), H^\nu_p(M))$ for all $\nu > 0$. Then (6.6) follows by iteration.

Higher regularity for solutions of (6.2) in $\mathbb{R}^n$ was recently proved in [49] by differentiating in time or by applying $(-\Delta)^\sigma$ to the equation and then using successively a Hölder continuity result, see [49, Section 6]. In the proof of Theorem 6.2 above, we follow a totally different method, based on abstract maximal regularity theory, which is of particular interest by itself.

6.2. Nonlocal Kirchhoff diffusion problem. The following parabolic Kirchhoff type problem
\[ \begin{cases} \partial_t u(t) - M(\|\nabla u\|_2^2)\Delta u(t) = 0, & t \geq 0, \\ u(0) = u_0, \end{cases} \]
has been studied by many authors; see [22] and the references therein. Here $M \in C^1_{-1}([0, \infty), [0, \infty))$.

We will consider a tensor-valued non-local version of (6.18), namely,
\[ \begin{cases} \partial_t u(t) + M(\|(-\Delta)^{\sigma/2} u\|_2^2)(-\Delta)^{\sigma/2} u(t) = F(u), & t \geq 0, \\ u(0) = u_0. \end{cases} \]

This equation has been explored in [22]. Here $(\mathcal{M}, g)$ is either an $n$-dimensional closed manifold or $\mathbb{R}^n$. Let $X^s = H^{s+2\sigma}_p(M, V)$ and $X^s_0 = H^s_p(M, V)$. Put
\[ U^s_p = \{ v \in (X^s_0, X^s_1)_{1-1/p, p} : M(\|(-\Delta)^{\sigma/2} v\|_2^2) \neq 0 \}, \quad s + \sigma > 2\sigma/p, \]
and assume that
\[ F \in C^1_{-1}(U^s_p, X^s_0). \]

Similar problems have been investigated in [38, 53] for the scalar case with $F$ independent of $u$ and $F(u) = |u|^r - 2u$, where $1 < r < \infty$. The theorem below generalizes the previous results [38, 53] on nonlocal Kirchhoff equations to the tensor-valued case and extends the admissible class of nonlinearities. In particular, our result applies to systems of nonlocal Kirchhoff equations.
By Theorem 5.6, Theorem 6.1 and [41, Theorem 3.1], we immediately have the following result.

**Theorem 6.3.** Assume that (6.19) and (6.21) are satisfied. For any $p \in (2, \infty)$ and $u_0 \in U^0_p$, the equation (6.20) has a unique solution $u \in L_p((0, T), X^0_1) \cap H^1_p((0, T), X^0_0)$ such that $M(\|(-\Delta)^{\sigma/2} u(t)\|_2^2) \neq 0$ for all $t \in (0, T)$. Moreover, $u \in L_p((\varepsilon, T), X^s_1) \cap H^1_p((\varepsilon, T), X^s_0)$ for all $s \geq 0$ and $\varepsilon \in (0, T)$.

**References**

[1] H. Amann, *Linear and quasilinear parabolic problems: Volume I. Abstract linear theory*, Monographs in Mathematics 89, Birkhäuser Verlag 1995.

[2] H. Amann, *Time-delayed Perona-Malik type problems*, Acta Math. Univ. Comenian. (N.S.) 76 (2007), no. 1, 15–38.

[3] H. Amann, *Maximal regularity and quasilinear parabolic boundary value problems*, Recent advances in elliptic and parabolic problems, 1-17, World Sci. Publ., Hackensack, NJ, 2005.

[4] H. Amann, *Function spaces on singular manifolds*, Math. Nachr. 286 (2013), no. 5-6, 436–475.

[5] H. Amann, *Cauchy problems for parabolic equations in Sobolev-Slobodeckii and Hölder spaces on uniformly regular Riemannian manifolds*, J. Evol. Equ. 17 (2017), no. 1, 51–100.

[6] H. Amann, M. Hieber, G. Simonett, *Bounded $H_\infty$-calculus for elliptic operators*, Differential Integral Equations 7 (1994), no. 3-4, 613–653.

[7] A. V. Balakrishnan, *Fractional powers of closed operators and the semi-groups generated by them*, Pacific J. Math. 10 (1960), no. 2, 419–437.

[8] M. Bonforte, A. Figalli, X. Ros-Oton, *Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains*, Comm. Pure Appl. Math. 70 (2017), no. 8, 1472–1508.

[9] M. Bonforte, A. Figalli, J. L. Vázquez, *Sharp global estimates for local and nonlocal porous medium-type equations in bounded domains*, Anal. PDE 11 (2018), no. 4, 945–982.

[10] M. Bonforte, Y. Sire, J. L. Vázquez, *Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains*, Discrete Contin. Dyn. Syst. 35 (2015), no. 12, 5725–5767.

[11] M. Bonforte, J. L. Vázquez, *A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains*, Arch. Ration. Mech. Anal. 218 (2015), no. 1, 317–362.

[12] M. Bonforte, J. L. Vázquez, *Fractional nonlinear degenerate diffusion equations on bounded domains part I. Existence, uniqueness and upper bounds*, Nonlinear Anal. 131 (2016), 363–398.

[13] G. Bourdaud, W. Sickel, *Composition operators on function spaces with fractional order of smoothness*, RIMS Kokyuroku Bessatsu B 26 (2011), 93–132.

[14] N. Lopez Cardozo, *Perturbative transport studies in fusion plasmas*, Plasma Phys. and Controlled Fusion 37 (1995), 799–852.

[15] B. A. Carreras, *Progress in anomalous transport research in toroidal magnetic confinement devices*, IEEE Transactions of Plasma Science, 25 (1997), no. 6, 1281–1321.
B. A. Carreras, B.V. Milligen, M.A. Pedroza et al., Long-range time correlations in plasma edge turbulence, Phys. Rev. Lett. 80 (1998), 4438–4441.

P. Clément, S. Li, Abstract parabolic quasilinear equations and application to a groundwater flow problem, Adv. Math. Sci. Appl. 3 (1993/94), Special Issue, 17–32.

P. Clément, J. Prüss, An operator-valued transference principle and maximal regularity on vector-valued $L_p$-spaces, In: G. Lumer, L. Weis (Eds.), Proc. of the 6th International Conference on Evolution Equations, Marcel Dekker 2001.

E. B. Davies, Heat kernels and spectral theory, Cambridge Tracts in Mathematics 92, Cambridge University Press 1989.

R. Denk, M. Hieber, J. Prüss, $B$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166, no. 788, 2003.

K. Gentle, G. Cima, H. Gasquet, G.A. Hallock et al., An experimental counter-example to the local transport paradigm, Phys. Plasmas 2 (1995), no. 6, 2292–2298.

M. Gobbino, Quasilinear degenerate parabolic equations of Kirchhoff-type, Math. Methods Appl. Sci. 22 (1999), no. 5, 375–388.

R. J. Goldstone, Energy confinement scaling in tokamaks: some implications of recent experiments with ohmic and strong auxiliary heating, Plasma Phys. and Controlled Fusion 26 (1984), 87–103.

G. Grubb, Fractional-order operators: boundary problems, heat equations, In: L. G. Rodino and J. Toft (eds.), Mathematical Analysis and Applications–Plenary Lectures, Springer Proceedings in Mathematics & Statistics 262, Springer Verlag 2018.

G. Grubb, Limited regularity of solutions to fractional heat and Schrödinger equations, Discrete Contin. Dyn. Syst. 39 (2019), no. 6, 3609–3634.

G. Grubb, Regularity in $L_p$ Sobolev spaces of solutions to fractional heat equations, J. Funct. Anal. 274 (2018), no. 9, 2634–2660.

P. Guidotti, A new nonlocal nonlinear diffusion of image processing, J. Differential Equations 246 (2009), no. 12, 4731–4742.

P. Guidotti, J. Lambers, Two new nonlinear nonlocal diffusions for noise reduction, Journal of Mathematical Analysis and Vision 33 (2009), no. 1, 25–37.

P. Guidotti, Anisotropic diffusions of image processing from Perona-Malik, In Variational Methods for Evolving Objects, 131–156, Adv. Stud. Pure Math. 67, 2015.

P. Guidotti, Y. Shao, Wellposedness of a nonlocal nonlinear diffusion equation of image processing, Nonlinear Anal. 150 (2017), 114–137.

N. J. Kalton, L. Weis, The $H^\infty$-calculus and sums of closed operators, Math. Ann. 321 (2001), no. 2, 319–345.

P. C. Kunstmann, L. Weis, Perturbation theorems for maximal $L_p$-regularity, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 30 (2001), no. 2, 415–435.

A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Birkhäuser Verlag 1995.

A. Lunardi, Interpolation theory, Lecture Notes Scuola Normale Superiore 16, Edizioni della Normale 2018.

E. M. Ouhabaz, $L^\infty$–contractivity of semigroups generated by sectorial forms, J. London Math. Soc. (2) 46 (1992), no. 3, 529–542.

A. Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez, A fractional porous medium equation, Adv. Math. 226 (2011), no. 2, 1378–1409.

A. Pablo, F. Quirós, A. Rodríguez, J. L. Vázquez, A general fractional porous medium equation, Comm. Pure Applied Math. 65 (2012), no. 9, 1242–1284.

P. Pucci, M. Xiang, B. Zhang, A diffusion problem of Kirchhoff type involving the nonlocal fractional $p$-Laplacian, Discrete Contin. Dyn. Syst. 37 (2017), no. 7, 4035–4051.

J. Prüss, G. Simonett, Moving interfaces and quasilinear parabolic evolution equations, Monographs in Mathematics 105, Birkhäuser Verlag 2016.
[40] N. Roidos, E. Schrohe, Existence and maximal $L^p$-regularity of solutions for the porous medium equation on manifolds with conical singularities, Comm. Partial Differential Equations 41 (2016), no. 9, 1441–1471.
[41] N. Roidos, E. Schrohe, Smoothness and long time existence for solutions of the porous medium equation on manifolds with conical singularities, Comm. Partial Differential Equations 43 (2018), no. 10, 1456–1484.
[42] N. Roidos, Y. Shao, The fractional porous medium equation on manifolds with conical singularities I, J. Evol. Equ. 22 (2022), no. 1, article no. 8.
[43] N. Roidos, Y. Shao, The fractional porous medium equation on manifolds with conical singularities II, Math. Nachr., to appear. arXiv:1908.07138v3.
[44] Y. Shao, Singular parabolic equations of second order on manifolds with singularities, J. Differential Equations 260 (2016), no. 2, 1747–1800.
[45] H. Tanabe, Equations of evolution, Monographs and Studies in Mathematics 6, Pitman Publishing 1979.
[46] M.E. Taylor, Partial differential equations II. Qualitative studies of linear equations. Second edition. Applied Mathematical Sciences, 116. Springer, New York, 2011.
[47] H. Triebel, Theory of function spaces, Monographs in Mathematics 78, Birkhäuser Verlag 1983.
[48] H. Triebel, Theory of function spaces II, Monographs in Mathematics 84, Birkhäuser Verlag 1992.
[49] J. L. Vázquez, A. de Pablo, F. Quirós, A. Rodríguez, Classical solutions and higher regularity for nonlinear fractional diffusion equations, J. Eur. Math. Soc. 19 (2017), no. 7, 1949–1975.
[50] J. L. Vázquez, Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type, J. Eur. Math. Soc. 16 (2014), no. 4, 769–803.
[51] J. L. Vázquez, The mesa problem for the fractional porous medium equation, Interfaces Free Bound. 17 (2015), no. 2, 261–286.
[52] L. Weis, Operator–valued Fourier multiplier theorems and maximal $L_p$-regularity, Math. Ann. 319 (2001), no. 4, 735–758.
[53] M. Xiang, V.D. Rădulescu, B. Zhang, Nonlocal Kirchhoff diffusion problems: local existence and blow-up of solutions, Nonlinearity 31 (2018), no. 7, 3228–3250.

Department of Mathematics, University of Patras, 26504 Rio Patras, Greece

Email address: roidos@math.upatras.gr

Department of Mathematics, The University of Alabama, Box 870350, Tuscaloosa, AL 35487-0350, USA

Email address: yshao8@ua.edu