A CRITIQUE ON THE SUPPLEMENTARY CONDITIONS
OF RARITA-SCHWINGER FRAMEWORK

M. KIRCHBACH\textsuperscript{a}, D. V. AHLUWALIA\textsuperscript{a,b}

\textsuperscript{a} Theoretical Physics Group, Facul. de Fisica
Univ. Aut. de Zacatecas, Zacatecas, ZAC 98062, Mexico
\textsuperscript{b} Mail stop H-846, Los Alamos National Laboratory
Los Alamos, NM 87545, USA

Abstract

After a brief review of the celebrated 1941 paper of Rarita and Schwinger on the
theory of particles with half-integral spins, we present an \textit{ab initio} construct of the
representation space relevant for the description of spin-$\frac{3}{2}$ particles. The chosen
example case of spin-$\frac{3}{2}$ shows that covariance of a wave equation, and that of the
imposed supplementary conditions, alone is not a sufficient criterion to guarantee the
compatibility of a framework with relativity -- a lesson already arrived by Velo and
Zwanziger. Here this same lesson is shown to be true at the level of the representation
space without invoking any interactions. The presented detailed analysis forces us
to abandon the single-spin interpretation of the Rarita and Schwinger framework,
and suggests a new interpretation that fully respects the relativity theory.

1 Introduction

Higher spin fields have attracted attention since the very early days of particle
physics. After the introduction of the neutrino into the theory of $\beta$-decay,
Oppenheimer on the basis of then-existing data put forward the suggestion
that electron neutrino may have a spin of three half, and carry mass $\textsuperscript{[1]}$. The
problem of neutrino mass is being currently addressed experimentally\textsuperscript{[\dag]} while
the Oppenheimer’s suggestion of a spin-$\frac{3}{2}$ neutrino was immediately ruled out
by a set of two papers $\textsuperscript{[7,8]}$.

Though this month (July 2001) falls on the sixtieth anniversary of Ref. $\textsuperscript{[7]}$, the
interest in Rarita-Schwinger formalism remains unabated as more and more

$\textsuperscript{1}$ More precisely, what the current experiments are probing is the phenomena of
flavor oscillations. In such a scenario a flavor eigenstate is a linear superposition of
different mass eigenstates$\textsuperscript{[2-6]}$. 

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baryonic resonances of higher spins are found in particle detectors on the one hand, and as theorists realize that for one reason, or another, higher spins may play a pivotal role in the unification of gravity with other interactions. Yet, this sixty-year old formalism remains vexing to theorists to some extent. This circumstance arises due to difficulties with the quantization of Rarita-Schwinger fields on the one hand, and their tachyonic propagation on the other. We conjecture that the known difficulties associated with the Rarita-Schwinger formalism may take their origin from the improper treatment and interpretation of the underlying representation space. In conjunction with Ref. [9], this paper is a preliminary step towards exploring this idea.

Here we first retrace the arguments of Rarita and Schwinger, and then immediately proceed to construct the representation space defined by Eq. (1), below. This would allow us to present an essentially self-contained completion of the Rarita-Schwinger framework that is consistent with the relativity theory at the kinematic level. At the same time it will allow us to point out where and how the inconsistency in the canonical wisdom on the Rarita-Schwinger framework enters.

Our considerations will be confined to spin-$\frac{3}{2}$. No new conceptual difficulties are expected to occur for spins $s > \frac{3}{2}$.

2 Rarita-Schwinger framework for spin-$\frac{3}{2}$

The Rarita and Schwinger spinor-vector, $\psi_\mu$, transforms as a finite dimensional non-unitary representation of the Lorentz group,

$$\psi_\mu : \left( \left( \frac{1}{2}, 0 \right) \oplus \left( 0, \frac{1}{2} \right) \right) \otimes \left( \frac{1}{2}, \frac{1}{2} \right)$$ (1)

In configuration space, it satisfies the Dirac equation

$$\left( i\gamma^\lambda \partial_\lambda - m \right) \psi_\mu(x) = 0,$$ (2)

for each of the Lorentz indices associated with the $\left( \frac{1}{2}, \frac{1}{2} \right)$ representation space. As is evident from Eq. (1), $\psi_\mu(x)$ contains 16 degrees of freedom. In the original interpretation, those were (correctly)interpreted to be distributed over two Dirac spinors, $\partial_\mu \psi_\mu(x)$ and $\gamma^\mu \psi_\mu(x)$, and the eight degrees of freedom required for the description of a spin-$\frac{3}{2}$ particle and its antiparticle. The idea was put forward to nullify the indicated Dirac spinors in the hope that in this way they will be removed from the representation space. In doing so one eventually
would end up with eight degrees of freedom as required for the relativistic description of a spin-$\frac{3}{2}$ field.

That, $\psi_I(x) \equiv \partial^\mu \psi_\mu(x)$, and, $\psi_{II}(x) \equiv \gamma^\mu \psi_\mu(x)$, indeed satisfy the Dirac equation is immediately seen from the following two simple exercises:

**I.** Taking the divergence of Eq. (2) leads to

$$
\partial^\mu (i\gamma^\lambda \partial_\lambda - m) \psi_\mu(x) = \left( i\gamma^\lambda \partial_\lambda - m \right) \partial^\mu \psi_\mu(x) = 0 ,
$$

(3)

and therefore to

$$
(i\gamma^\lambda \partial_\lambda - m) \psi_I(x) = 0
$$

(4)

The situation with $\psi_{II}(x)$ is slightly trickier, though not fatally.

**II.** In nullifying $\psi_I(x)$, i.e., in setting $\psi_I(x) = 0$, allows for $\psi_{II}(x)$ to satisfy a Dirac equation (with the wrong sign for the mass term):

$$
(i\gamma^\lambda \partial_\lambda - m) \psi_{II}(x) = 2i \psi_I(x) - \gamma^\mu (i\partial_\lambda \gamma^\lambda + m) \psi_\mu(x)
$$

(5)

The second term on the right-hand side of the above equation carries a wrong sign for the mass term (if $\psi_{II}(x)$ is to satisfy the Dirac equation). This, however, can be corrected by replacing the 1941 Rarita-Schwinger suggestion of $\psi_{II}$ by $\psi'_{II}(x) \equiv \gamma^5 \psi_{II}(x)$. Then, it is clear that $\psi'_{II}(x)$ satisfies the Dirac equation. This is an important point as regards the relative intrinsic parities of the two spin-$\frac{1}{2}$ particles contained in the representation space (1). However, for the rest of this paper we shall ignore this “minor” matter of inconsistency in the Rarita-Schwinger framework without affecting our essential conclusions in any way. However, the reader should keep the presence of $\gamma^5$ in $\psi'_{II}(x)$ in mind while applying the framework to physical problems.

In summary, the Rarita-Schwinger framework for spin-$\frac{3}{2}$ consists of Eq. (2), supplemented by the conditions:

$$
\gamma^\mu \psi_\mu(x) = 0 ,
$$

(6)

$$
\partial^\mu \psi_\mu(x) = 0 .
$$

(7)

This framework is then claimed to describe a pure spin-$\frac{3}{2}$ system, despite a parenthetical remark in the original paper of Rarita and Schwinger which read, “it [the square of the intrinsic angular momentum] will not have this value $\left[\frac{3}{2}\right]$ in an arbitrary reference frame.” While our analysis will explicitly support this remark, we will show that despite covariance of the system of Eqs. (2), (6), and (7), the Rarita-Schwinger framework is incompatible with the theory of relativity.
3 Kinematic structure of the Rarita-Schwinger framework

The most noted problems with the above summarized framework have been
given by Johnson and Sudarshan [10], on one hand, and by Velo and Zwanziger
[11] on the other. These authors studied propagation of Rarita-Schwinger field
in an external electromagnetic potential. In particular, Velo and Zwanziger
came to the conclusion that “the main lesson to be drawn ... is that special
relativity is not automatically satisfied by writing equations that transform
covariantly. In addition, the solutions must not propagate faster than light.”

Here, essentially the same is shown to be true purely at the level of the represen-
tation space without invoking any interactions — provided that one takes
due care, beyond the work of Refs. [10,11], of the \((\tfrac{1}{2}, \tfrac{1}{2})\) sector of the the-
ory. The detailed analysis presented in here forces us to abandon a single-spin
interpretation of the Rarita and Schwinger framework, and suggests a new
interpretation that fully respects the relativity theory — at least at the kine-
matic level. The new interpretation of the representation space defined by
Eq. (1) will require us to abandon the supplementary conditions, (7) and (6),
and force us to interpret this space as a multi-spin object containing two spin
half objects of opposite relative intrinsic parities, and a spin three-half object.

Our entire analysis, unless otherwise made apparent, will be done in the mo-
mentum space.

3.1 Incompatibility of the Rarita-Schwinger framework with theory of rela-
tivity

The un-truncated Rarita-Schwinger representation space is a direct product of
a spinor and a Lorentz vector. The objects which span the spinor and vector
sectors of the theory are obtained by applying the \((\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2})\) boost to the
following rest, i.e., \(\vec{p} = 0\), spinors:

\footnote{We do not hasten to study as to what happens when interactions are introduced. The reason is simple: if the kinematic structure itself is acausal, or pathological in any manner, then these same elements would come to plague us later when interactions are introduced. In particular, we draw attention to Eq. (16) of Ref. [9] which indicates as to what could have gone wrong even with the completeness relation for the \((\tfrac{1}{2}, \tfrac{1}{2})\) sector of the theory. For any theory that does not satisfy the correct completeness relation, quantization is bound to be problematic.}
The laboratory frame is populated with spinors and vectors for all values of $s$ also exist in the original frame. It is by this “Wigner argument” that the boosted objects should be treated.

A careful reader has perhaps already noted that the application of the boost operators takes one from the original laboratory frame to a boosted frame. However, as no inertial frame is a preferred frame, the boosted objects should also exist in the original frame. It is by this “Wigner argument” that the laboratory frame is populated with spinors and vectors for all values of $\vec{p}$.

While all $\psi_i(\vec{p})$, $i = 1, 2, 3, 4$, carry well-defined spin, i.e. $s = \frac{1}{2}$, same is not true for $w_\zeta(\vec{p})$, $\zeta = 1, 2, 3, 4$. However, for $\vec{p} = \vec{0}$, the latter, for $\zeta = 1, 2, 3$ are eigenstates of spin one, while the $\zeta = 4$ case yields spin zero. The interested reader will find that this result is in accord with observation of Rarita and Schwinger in the context of spin three half — see, the parenthetic remark after Eq. (2) of Ref. [7]. Complementary details on the $(\frac{1}{2}, \frac{1}{2})$ representation space can be found in Ref. [9].

\[
\psi_1(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi_2(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \psi_3(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \psi_4(\vec{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix},
\]

and by boosting the following $(\frac{1}{2}, \frac{1}{2})$ Lorentz rest-frame vectors:

\[
w_1(\vec{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w_2(\vec{0}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad w_3(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad w_4(\vec{0}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.
\]

The $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ and the $(\frac{1}{2}, \frac{1}{2})$ boosts are in turn given by,

\[
\kappa^{(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})} = \kappa^{(\frac{1}{2}, 0)} \oplus \kappa^{(0, \frac{1}{2})}, \quad \kappa^{(\frac{1}{2}, \frac{1}{2})} = \kappa^{(\frac{1}{2}, 0)} \otimes \kappa^{(0, \frac{1}{2})},
\]

with

\[
\kappa^{(\frac{1}{2}, 0)} = \frac{1}{\sqrt{2m(E+m)}} \left[ (E+m)I_2 + \vec{\sigma} \cdot \vec{p} \right], \quad \kappa^{(0, \frac{1}{2})} = \frac{1}{\sqrt{2m(E+m)}} \left[ (E+m)I_2 - \vec{\sigma} \cdot \vec{p} \right].
\]

All notational details are those of Ref. [9].

While all $\psi_i(\vec{p})$, $i = 1, 2, 3, 4$, carry well-defined spin, i.e. $s = \frac{1}{2}$, same is not true for $w_\zeta(\vec{p})$, $\zeta = 1, 2, 3, 4$. However, for $\vec{p} = \vec{0}$, the latter, for $\zeta = 1, 2, 3$ are eigenstates of spin one, while the $\zeta = 4$ case yields spin zero. The interested reader will find that this result is in accord with observation of Rarita and Schwinger in the context of spin three half — see, the parenthetic remark after Eq. (2) of Ref. [7]. Complementary details on the $(\frac{1}{2}, \frac{1}{2})$ representation space can be found in Ref. [9].
After rotation by the matrix,

\[
S = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & i & -i & 0 \\
-i & 0 & 0 & i \\
1 & 0 & 0 & 1 \\
0 & i & i & 0
\end{pmatrix},
\]

(13)

obtained in Ref. [9], the \( w_\zeta(\vec{p}) \), carry the usual (contravariant)Lorentz index. We denote this \( S \)-rotated object by \([W_\zeta(\vec{p})]^{\mu} \), or simply as \( W_\zeta^\mu(\vec{p}) \) In this language (and in momentum space), the 16 objects that span the representation space defined in Eq. (1) are obtained as:

\[
\psi^\mu_{i\zeta}(\vec{p}) \equiv \psi_i(\vec{p}) \otimes W_\zeta^\mu(\vec{p}).
\]

(14)

In order for \( \gamma^\mu \psi^\mu_{i\zeta}(\vec{p}) \) to identically vanish for all values of \( i \) and \( \zeta \),

\[
(E + m)^2 - \vec{p}^2 = 0.
\]

(15)

Solving for \( E \),

\[
E = -m \pm \sqrt{\vec{p}^2}.
\]

(16)

As such, the group velocity associated with the Rarita-Schwinger field turns out to be:

\[
\vec{v}_g \equiv \frac{\partial E}{\partial \vec{p}} = 1 \hat{\vec{p}}
\]

(17)

That is, implementing the supplementary condition (6) requires the group velocity associated with the Rarita Schwinger field to be unity (i.e. equals the velocity of light). This value is independent of mass of the Rarita-Schwinger field under consideration. Consequently, we conclude that the covariance of a set of equations alone is not sufficient to warrant consistency with the theory of relativity. One must further demand obtaining the correct dispersion relation.

3.2 The truncation of the \( \left( \frac{1}{2}, \frac{1}{2} \right) \) sector as the origin of difficulties of the Rarita-Schwinger framework

The supplementary condition (6) involves not only a summation over the Lorentz indices, but also involves a transformation on the relevant spinorial
elements. In contrast, the supplementary condition (7) sums out the Lorentz index, and without any further transformation on the spinorial element sets it equal to zero. It is therefore instructive to look at the Lorentz-index defining \((\frac{1}{2}, \frac{1}{2})\) representation space, to gain further insight in the representation space (1).

A direct calculation of the divergency of each one of the four Lorentz vectors \(W_\zeta^\mu(\vec{p})\), \(\zeta = 1, 2, 3, 4\), leads to

\[
p_\mu W_\zeta^\mu(\vec{p}) = c_\zeta (m^2 - p^2) = 0 \quad \text{for} \quad \zeta = 1, 2, 3,
\]

\[
p_\mu W_4^\mu(\vec{p}) = \frac{i}{m} p^2.
\]

Here, \(c_1 = i(p_x + ip_y)\), \(c_2 = -ip_z\), and \(c_3 = -i(p_x - ip_y)\). As long as the first supplementary condition on \(p_\mu \psi_\mu(\vec{p})\) operates onto the Lorentz index only, the latter equations show that it checks consistency with the mass-shell relation \(E^2 - \vec{p}^2 = m^2\). For massive particles this condition is fulfilled only for vectors \(W_1^\mu(\vec{p}), W_2^\mu(\vec{p})\), and \(W_3^\mu(\vec{p})\), and is not satisfied at all for the vector \(W_4^\mu(\vec{p})\). This calculation shows that imposing the supplementary condition (7) onto the Rarita-Schwinger field restricts the underlying four vectorial degrees of freedom to only three. Yet, as shown in Ref. [9], the three vectors \(W_1^\mu(\vec{p}), W_2^\mu(\vec{p})\) and \(W_3^\mu(\vec{p})\) are not eigenstates of the squared angular momentum \(\vec{J}^2\) and do not lend themselves to pure spin-1 states. Rather they are eigenstates of the parity operator, which in the considered representation space is nothing but the matrix of the metric tensor \(\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). Consequently, though this supplementary condition restricts the four degrees of freedom of the \((\frac{1}{2}, \frac{1}{2})\) representation space to only three, it does not restrict the spin degrees of freedom to spin-1 only.\(^2\)

The spin-0 piece is still there and mixes up with spin-1 within \(W_\zeta(\vec{p})\) (for \(\zeta = 1, 2, 3\)). The immediate consequence of the covariant inseparability of the \((\frac{1}{2}, \frac{1}{2})\) space into spin-0 and spin-1 is the covariant inseparability of the

\(^2\) Moreover, these three Lorentz vectors cannot span the \((\frac{1}{2}, \frac{1}{2})\) space in the same mathematical sense as do the four Dirac spinors in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation space. It was shown in Ref. [9] that this serious drawback is immediately rectified by incorporating, \(W_4^\mu(\vec{p})\), the fourth natural companion of the three \(W_1^\mu(\vec{p}), W_2^\mu(\vec{p})\) and \(W_3^\mu(\vec{p})\). By doing so, the \((\frac{1}{2}, \frac{1}{2})\) representation space, in exact parallel of the Dirac’s \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation space, carries objects that have positive as well as negative norm before quantization. In addition, both of these spaces now become endowed with covariantly separated sectors of definite parities. We expect this new parallelism to circumvent the difficulties of quantization pointed out by Weinberg [12].
Rarita-Schwinger field into a spin-$\frac{3}{2}$ and two spin-$\frac{1}{2}$ components.

Only within the rest frame, or in the helicity basis, does the separation between spin-0 and spin-1 take place [9].

The essential additional physics lies in the fact that the Proca equation

$$\partial_{\mu}F^{\mu\nu} + m^2 A^{\nu} = 0$$

by construction satisfies, $\partial_{\nu}A^{\nu} = 0$ (for $m \neq 0$). However, as shown in Ref. [9], “$\partial_{\nu}A^{\nu} = 0$” cannot be satisfied for all relevant degrees of freedom in the massive ($\frac{1}{2}, \frac{1}{2}$) representation space without violating the completeness relation. While the wave equation satisfied by the ($\frac{1}{2}, \frac{1}{2}$) spanning $W_{\mu}^{\nu}(\vec{p})$, contains all solutions of the Proca equation the converse is not true. The wave equation for $W_{\mu}^{\nu}(\vec{p})$, which carry with them a completeness relation exactly paralleling the Dirac construct for spin-$\frac{1}{2}$, is [9]:

$$\left(\Lambda_{\mu \nu}p^\mu p^\nu - \epsilon m^2 I_4\right)_{\alpha \beta} W_{\zeta}^{\beta}(\vec{p}) = 0,$$

where $\epsilon$ equals +1 for $\zeta = 4$ and is −1 for $\zeta = 1, 2, 3$. The $\Lambda_{\mu \nu}$ matrices are:

$$\Lambda_{00} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Lambda_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Lambda_{33} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Lambda_{01} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_{03} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
\( \Lambda_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \) \( (21) \)

The remaining \( \Lambda_{\mu\nu} \) are obtained from the above expressions by noting: \( \Lambda_{\mu\nu} = \Lambda_{\nu\mu} \).

For this reason, and several others given in Ref. [9], the Proca equation is not endowed with the complete physical content of the massive \( \left( \frac{1}{2}, \frac{1}{2} \right) \) representation space. In correcting the Rarita-Schwinger framework for the above incompleteness one is forced to drop the supplementary conditions and required to reinterpret the \( \psi_\mu \) as a multi-spin(multi-parity) object.

3.3 Multi-spin character of \( \psi_\mu \) from the perspective of the Pauli-Lubanski vector

Here we confirm the multi-spin valued nature of \( \psi_\mu \) from a slightly different perspective but the equation of motion. We use instead the properties of the squared-length of the Pauli-Lubanski vector. We shall make our analysis in four steps. In the first step we simply recall the definition of the Pauli-Lubanski vector, and that of the associated Casimir invariant, \( C_2 \). In the second step we consider \( C_2 \) for the \( (j,0) \oplus (0,j) \) representation space. The third step does the same for the \( (j,j) \) representation spaces by considering the special case of \( j = \frac{1}{2} \). Finally, the fourth step is devoted to \( \psi_\mu(\vec{p}) \).

I. The Pauli-Lubanski vector is defined as

\[ \mathcal{W}^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} I_{\nu\rho} P_\sigma. \] \( (22) \)

Here, \( \epsilon_{\mu\nu\rho\sigma} \) is the standard Levi-Cevita symbol in four dimensions, while \( I_{\nu\rho} \) denote the generators of the Lorentz group

\[ I_{0i} = \kappa_i, \quad I_{ij} = \epsilon_{ijk}J^k, \] \( (23) \)

with \( \kappa_i \) and \( J_k \) being in turn the \( i \)th and \( k \)th components of boost and rotation generators, respectively. The final expression for the Pauli-Lubanski vector in terms of boost and rotation generators reads

\[ \mathcal{W}^\mu = \left( -\vec{J} \cdot \vec{P}, \quad -\vec{J}P_0 + \vec{\kappa} \times \vec{P} \right). \] \( (24) \)
Correspondingly, its squared length is obtained as

\[ C_2 \equiv \mathcal{W}^\mu \mathcal{W}_\mu = (\vec{J} \cdot \vec{P})^2 - \left( -\vec{J}P_0 + \vec{\kappa} \times \vec{P} \right)^2. \]  

(25)

So far this result is entirely general, i.e., it is true for an arbitrary representation space of the Lorentz group.

II. The detailed physical content of \( C_2 \) is determined by the representation space under consideration. The most straightforward interpretation of \( C_2 \) occurs for the \((j,0)\) and \((0,j)\) representation spaces. For these spaces the boost generators are entirely determined by the \((2j + 1) \times (2j + 1)\) matrices associated with the generators of rotations,

\[ \vec{\kappa}^{(j,0)} = +i\vec{J}, \quad \vec{\kappa}^{(0,j)} = -i\vec{J}. \]  

(26)

Consequently, the \( C_2 \)'s take the simple form

\[ C_2^{(j,0)} = C_2^{(0,j)} = (\vec{J} \cdot \vec{P})^2 - (\vec{J}P_0)^2 + (\vec{J} \times \vec{P})^2 \]  

(27)

In order to obtain a parity covariant framework one constructs \((j,0) \oplus (0,j)\) representation spaces. It was shown in Refs. [13,14] that each of these representation spaces is \(2(2j + 1)\) dimensional, is endowed with a covariant parity bifurcation into \((2j + 1)\) subspaces of opposite relative intrinsic parities, and supports particles and antiparticles of spin-\(j\). For \( j = \frac{1}{2} \), one obtains the standard Dirac representation space – provided one seeks objects that are eigenstates of the charge operator. Under similar assumption, for \( j = 1 \) one obtains the Bargmann-Wightman-Wigner (BWW) representation space. Since

\[ \vec{J}^{(\frac{1}{2},0) \oplus (0,\frac{1}{2})} = \begin{pmatrix} \vec{\sigma}/2 & 0 \\ 0 & -\vec{\sigma}/2 \end{pmatrix}, \quad \vec{J}^{(1,0) \oplus (0,1)} = \begin{pmatrix} \vec{S} & 0 \\ 0 & -\vec{S} \end{pmatrix}. \]  

(28)

where \( \vec{\sigma} \) are the usual Pauli matrices, and \( \vec{S} \) are \(3 \times 3\) spin-1 matrices, it readily verifies that

\[ C_2^{(\frac{1}{2},0) \oplus (0,\frac{1}{2})} \psi_i^{(\frac{1}{2},0) \oplus (0,\frac{1}{2})}(\vec{p}) = -\frac{3}{4}m^2 \psi_i^{(\frac{1}{2},0) \oplus (0,\frac{1}{2})}(\vec{p}), \]  

(29)

\[ C_2^{(1,0) \oplus (0,1)} \psi_i^{(1,0) \oplus (0,1)}(\vec{p}) = -2m^2 \psi_i^{(1,0) \oplus (0,1)}(\vec{p}), \]  

(30)

where \( i = 1, 2, 3, 4 \) for the Dirac spinors, and \( i = 1, 2, 3, 4, 5, 6 \) for the BWW objects. Thus, we explicitly verify that these representation spaces are correctly associated with pure spin one half, and one, respectively:
It is useful to summarize the following observations for the \((j, 0) \oplus (0, j)\) representation space:

1. The eigenstates of the \(C_2^{(j,0)\oplus(0,j)}\)’s are also eigenstates of \((\vec{J}^{(j,0)\oplus(0,j)})^2\). Mathematically, this happens because commutator of \(C_2\) and \(\vec{J}^2\) vanishes,

\[
\left[ C_2, \vec{J}^2 \right]_{(j,0)\oplus(0,j)} = 0. 
\] (31)

2. The \(2(2j + 1)\) dimensional space bifurcates into two sectors of opposite relative intrinsic parties. Each of these sectors carries dimensionality \(2j + 1\), and are related to each other by the action of a charge conjugation operator.

III. For other representation spaces no simple relationship, such as given by Eqs. (26), exists between the generators of boosts and the generators of rotations. For this reason the result just summarized for the \((j, 0) \oplus (0, j)\) space no longer remains true.

Specifically, consider the \(\left(\frac{1}{2}, \frac{1}{2}\right)\) representation space. It is spanned by the four \(W_\zeta^\mu(\vec{p})\), \(\zeta = 1, 2, 3, 4\). The \(C_2^{\left(\frac{1}{2}, \frac{1}{2}\right)}\) is obtained by substituting the \(\left(\frac{1}{2}, \frac{1}{2}\right)\) generators of rotations and boosts into Eq. (25). These, for the specific realization of the \(W_\zeta^\mu(\vec{p})\), are the \(S\)-transformed generators given in Eqs. (4), (5), (6), and (7) of Ref. [9]. A detailed calculation then shows that,

\[
C_2^{\left(\frac{1}{2}, \frac{1}{2}\right)} W_\zeta^\mu(\vec{p}) = -m^2 \lambda_\zeta W_\zeta^\mu(\vec{p}), \quad \text{(no sum on } \zeta) \right). 
\] (32)

where \(\lambda_\zeta = 2\), for \(\zeta = 1, 2, 3\); and \(\lambda_\zeta = 0\) for \(\zeta = 4\).

To avoid confusion, we note that \(\lambda_\zeta\) can also be read off from Eq. (25) by going to the rest frame. The \(C_2\), in general, while acting upon a mass eigenstate yields, \(-m^2 \vec{J}^2\). Because of that the eigenvalues of the squared-length of the Pauli-Lubanski vector, i.e. \(C_2\), at rest can be given the interpretation of, \(-m^2 \lambda_\zeta = -m^2 j_\zeta(\bar{j}_\zeta + 1)\). Based upon this finding, valid solely at rest, the impression arises that the eigenstates of the second Casimir of the Poincaré group carry definite mass and a pure spin. This impression, as regards the spin, is in general not correct.\(^3\) Indeed, while the eigenvalues of \(C_2\), in being a Casimir operator, are frame independent, their association with the eigenvalues of \(\vec{J}^2\) is frame dependent. In the most general case the eigenvalues of \(C_2\) arise as a consequence of a delicate cancellation between the actions of all the

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\(^3\) Further, in reference to footnote [1], quantum framework allows for linear superposition of mass eigenstates. This has important implications not only for neutrino oscillation phenomenology but it also has important significance for the meaning of gravitational potential in the quantum framework[15].
terms on the right hand side of Eq. (24) upon the state vectors. As a result of these cancellations, even though the eigenvalues of $C_2$ numerically coincide with the eigenvalues of $-m^2\vec{J}^2$, at rest, it has not to be confused with the latter. Only for the $(j,0) \oplus (0,j)$ representation space, the $\vec{J}^2$ and $C_2$ have simultaneous eigen-basis (in all inertial frames).

To summarize,

1. The eigenstates of the $C_2^{(\frac{1}{2} \cdot \frac{1}{2})}$'s are, in general, not eigenstates of $-m^2 \left(\vec{J}(\frac{1}{2},\frac{1}{2})\right)^2$. Mathematically, it derives it origin from the fact that the following commutator does not vanish,

$$[C_2, \vec{J}^2]^{(\frac{1}{2},\frac{1}{2})} \neq 0.$$  \hspace{1cm} (33)

2. The 4 dimensional space bifurcates into two sectors of opposite relative intrinsic parties. The one sector has dimensionality of 3, while the second sector has dimensionality of unity. The charge conjugate sector is obtained by complex conjugating the $W^\mu_\nu(\vec{p})$.

### IV.

For representation spaces of the Rarita-Schwinger type even though the action of $C_2$, associated with the representation space defined by Eq. (1), on mass eigenstates $\psi_\mu(\vec{p})$ formally yields, $-m^2\lambda$, with $\lambda = \frac{3}{4}$, twice, and $\lambda = \frac{15}{4}$, once, such states, in general, are not eigenstates of $\vec{J}^2$ — where $\vec{J}$ is the appropriate generator of rotation. For the representation space defined by Eq. (1), it is possible, in a rest frame, say, to identify the eigenstates of $\vec{J}^2$ as two objects of spin one half, and an object of spin three half. However, this separation into spin-states is not covariant in general — even though the associated $C_2$ divides the space (1) into sub-spaces that carry eigenvalues, $-m^2j(j+1)$, with $j = 1/2, 1/2, 3/2$. The latter eigenvalues of $C_2$ no longer carry meaning of “spin” in the sense of being eigenstates of $\vec{J}^2$ (which they are not). This interpretation is in accord with the conjecture that one of us advanced some years ago while studying the baryonic spectra [16,17]. It arises as a consequence of Eqs. (31) and (33), which imply,

$$[C_2, \vec{J}^2]^{[(\frac{1}{2},0)\oplus(0,\frac{1}{2})]\otimes(\frac{1}{2},\frac{1}{2})} \neq 0.$$  \hspace{1cm} (34)

on the one hand, and on the other hand due to Eqs. (7) and (6), which despite their covariance, are inadmissible on grounds of being inconsistent with the theory of relativity.
Covariance of a set of equations alone does not guarantee their compatibility with the theory of relativity. Specifically, the supplementary condition (6) forces the massive Rarita-Schwinger field to have a group velocity of unity (clearly in violation of the theory of relativity). The supplementary condition (7) cannot be implemented for all four degrees of freedom that span the massive \( \left( \frac{1}{2}, \frac{1}{2} \right) \) representation space — the failure arises with \( W_4^\mu(\vec{p}) \).

Once a charge conjugate part of a representation is incorporated in the framework, the eigenvalues of the Casimir invariant \( C_2 \) split the representation space (1) into \( 2(2j + 1) \) dimensional subspaces, with \( j = \frac{1}{2} \) twice, and \( j = \frac{3}{2} \) once.

These subspaces further subdivide into sectors of definite relative intrinsic parities. In general, however, these subspaces, do not carry a definite spin. That is, they are not eigenstates of the square of the relevant generators of the rotations, \( \vec{J}^2 \). The result readily extends to Rarita-Schwinger fields of spin greater than three half.

Only in the \((j, 0) \oplus (0, j)\) representation spaces do these subspaces carry definite values of \( \vec{J}^2 \), \( C_2 \), and the parity operator as well.

We are thus left with no choice but to abandon a single-spin interpretation of the representation space defined in Eq. (1). Once that is done \( \psi_\mu \) contains two spin one half particles of opposite relative parities, and a spin three half particle. In the absence of any interaction these particles are mass degenerate. Furthermore, the bifurcation into spin one half and spin three half occurs only in the rest and helicity frames. In general, the particles in this representation do not carry a well-defined spin. In this representation space the Dirac operator \( (\gamma^\lambda p_\lambda \pm m) \) annihilates the spinorial sector of the \( \psi_\mu(\vec{p}) \), while a new operator, \( (\Lambda_\lambda p^\lambda p^\nu \pm m^2 I_4) \), annihilates the vector sector of \( \psi_\mu(\vec{p}) \).

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\footnote{Such a charge conjugate sector, e.g., is already present in the \( \left( \frac{1}{2}, 0 \right) \oplus \left( 0, \frac{1}{2} \right) \) part of the Rarita-Schwinger field. For the \( \left( \frac{1}{2}, \frac{1}{2} \right) \) representation space it can be shown to be brought in by the operation of complex conjugation.}
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