Stabilization of the Homotopy Groups of the Moduli Spaces of \(k\)-Higgs Bundles

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Ronald A. Zúñiga-Rojas

Centro de Investigaciones Matemáticas y Metamatemáticas CIMM
Universidad de Costa Rica UCR
San José 11501, Costa Rica
e-mail: ronald.zunigarojas@ucr.ac.cr

Abstract. The work of Hausel proves that the Białynicki-Birula stratification of the moduli space of rank two Higgs bundles coincides with its Shatz stratification. He uses that to estimate some homotopy groups of the moduli spaces of \(k\)-Higgs bundles of rank two. Unfortunately, those two stratifications do not coincide in general. Here, the objective is to present a different proof of the stabilization of the homotopy groups of \(\mathcal{M}^k(2, d)\), and generalize it to \(\mathcal{M}^k(3, d)\), the moduli spaces of \(k\)-Higgs bundles of degree \(d\), and ranks two and three respectively, over a compact Riemann surface \(X\), using the results from the works of Hausel and Thaddeus, among other tools.

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Introduction

In this work, we estimate some homotopy groups of the moduli spaces of \(k\)-Higgs bundles \(\mathcal{M}^k(r, d)\) over a compact Riemann surface \(X\) of genus \(g > 2\). This space was first introduced by Hitchin [20]; and then, it was worked by Hausel [16], where he estimated some of the homotopy groups working the particular case of rank two, and denoting \(\mathcal{M}^\infty = \lim_{k \to \infty} \mathcal{M}^k\) as the direct limit of the sequence.

The co-prime condition \(\gcd(r, d) = 1\) implies that \(\mathcal{M}^k(r, d)\) is smooth. We shall do the estimate with Higgs bundles of fixed determinant \(\det(E) = \Lambda \in \mathcal{J}^d\), where \(\mathcal{J}^d\) is the Jacobian of degree \(d\) line bundles on \(X\), to ensure that \(\mathcal{N}(r, d)\) and \(\mathcal{M}(r, d)\) are simply connected.

Denote \(\mathcal{M}_\Lambda^k\) as the moduli space of \(k\)-Higgs bundles with determinant \(\Lambda\), and \(\mathcal{M}_\Lambda^\infty = \lim_{k \to \infty} \mathcal{M}_\Lambda^k\) as the direct limit of these moduli spaces, as before. Hence, the group action \(\pi_1(\mathcal{M}_\Lambda^k) \times \mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k\) will be trivial.

Hausel [16] estimates the homotopy groups \(\pi_n(\mathcal{M}^k(2, 1))\) using two main tools: first the coincidence mentioned before between the Białynicki-Birula stratification and the Shatz stratification; and second, the well-behaved embeddings \(\mathcal{M}^k(2, 1) \hookrightarrow \mathcal{M}^{k+1}(2, 1)\). These inclusions are also well-behaved in general for \(\gcd(r, d) = 1\); nevertheless, those two stratifications above mentioned do not coincide in general (see for instance [11]).

In this paper, our estimate is based on the embeddings

\[\mathcal{M}^k(r, d) \hookrightarrow \mathcal{M}^{k+1}(r, d)\]
and their good behavior, notwithstanding the non-coincidence between stratifications when the rank is \( r = 3 \). The paper is organized as follows: in section 1 we recall some facts about vector bundles and Higgs bundles; in section 2, we present the cohomology ring \( H^n(\mathcal{M}^k) \); in section 3, we discuss the most relevant results about the cohomology and the moduli of the moduli spaces \( \mathcal{M}^k \); finally, in section 4, subsection 4.1 we estimate the homotopy groups of \( \mathcal{M}^k \) under the assumption that \( \pi_1(\mathcal{M}^k) \) acts trivially on \( \pi_n(\mathcal{M}^{\infty}, \mathcal{M}^k) \), and hence, in subsection 4.2 we present and prove the main result:

**Theorem.** (Corollary 4.14) Suppose the rank is either \( r = 2 \) or \( r = 3 \), and \( \text{GCD}(r, d) = 1 \). Then, for all \( n \) exists \( k_0 \), depending on \( n \), such that

\[
\pi_j \left( \mathcal{M}_n^k(r, d) \right) \xrightarrow{\pi} \pi_j \left( \mathcal{M}^\infty_n(r, d) \right)
\]

for all \( k \geq k_0 \) and for all \( j \leq n - 1 \).

1 Preliminary definitions

Let \( X \) be a compact Riemann surface of genus \( g > 2 \) and let \( K = T^*X \) be the canonical line bundle of \( X \). Note that, algebraically, \( X \) is also a nonsingular complex projective algebraic curve.

**Definition 1.1.** A *Higgs bundle* over \( X \) is a pair \((E, \Phi)\) where \( E \to X \) is a holomorphic vector bundle and \( \Phi : E \to E \otimes K \) is an endomorphism of \( E \) twisted by \( K \), which is called a *Higgs field*. Note that \( \Phi \in H^0(X; \text{End}(E) \otimes K) \).

**Definition 1.2.** For a vector bundle \( E \to X \), we denote the *rank* of \( E \) by \( \text{rk}(E) = r \) and the *degree* of \( E \) by \( \text{deg}(E) = d \). Then, for any smooth bundle \( E \to X \) the *slope* is defined to be

\[
\mu(E) := \frac{\text{deg}(E)}{\text{rk}(E)} = \frac{d}{r}.
\]

A vector bundle \( E \to X \) is called *semistable* if \( \mu(F) \leq \mu(E) \) for any \( F \) such that \( 0 \subseteq F \subseteq E \). Similarly, a vector bundle \( E \to X \) is called *stable* if \( \mu(F) < \mu(E) \) for any nonzero proper subbundle \( 0 \subsetneq F \subsetneq E \). Finally, \( E \) is called *polystable* if it is the direct sum of stable subbundles, all of the same slope.

**Definition 1.3.** A subbundle \( F \subset E \) is said to be \( \Phi \)-*invariant* if \( \Phi(F) \subset F \otimes K \). A Higgs bundle is said to be *semistable* [respectively, *stable*] if \( \mu(F) \leq \mu(E) \) [resp., \( \mu(F) < \mu(E) \)] for any nonzero \( \Phi \)-invariant subbundle \( F \subset E \) [resp., \( F \subsetneq E \)]. Finally, \((E, \Phi)\) is called *polystable* if it is the direct sum of stable \( \Phi \)-invariant subbundles, all of the same slope.

Fixing the rank \( \text{rk}(E) = r \) and the degree \( \text{deg}(E) = d \) of a Higgs bundle \((E, \Phi)\), the isomorphism classes of polystable bundles are parametrized by a quasi-projective variety: the moduli space \( \mathcal{M}(r, d) \). Constructions of this space can be found in the work of Hitchin [20], using gauge theory, or in the work of Nitsure [27], using algebraic geometry methods.

An important feature of \( \mathcal{M}(r, d) \) is that it carries an action of \( \mathbb{C}^* \): \( z \cdot (E, \Phi) = (E, z \cdot \Phi) \). According to Hitchin [20], \((\mathcal{M}, I, \Omega)\) is a Kähler manifold, where \( I \) is its complex structure and \( \Omega \) its corresponding Kähler form. Furthermore, \( \mathbb{C}^* \) acts on \( \mathcal{M} \) biholomorphically with respect to the complex structure \( I \) by the action mentioned above, where the Kähler form \( \Omega \) is invariant under the induced action \( e^{i\theta} \cdot (E, \Phi) = (E, e^{i\theta} \cdot \Phi) \) of the circle \( \mathbb{S}^1 \subset \mathbb{C}^* \).
Besides, this circle action is Hamiltonian, with proper momentum map $f : \mathcal{M} \to \mathbb{R}$ defined by:

$$f(E, \Phi) = \frac{1}{2\pi} \|\Phi\|_{L^2}^2 = \frac{i}{2\pi} \int_X \text{tr}(\Phi\Phi^*),$$

(1.2)

where $\Phi^*$ is the adjoint of $\Phi$ with respect to the hermitian metric on $E$ which provides the Hitchin-Kobayashi correspondence (see Hitchin [20]), and $f$ has finitely many critical values.

There is another important fact mentioned by Hitchin (see the original version in Frankel [8], and its application to Higgs bundles in Hitchin [20]): the critical points of $f$ are exactly the fixed points of the circle action on $\mathcal{M}$.

If $(E, \Phi) = (E, e^{i\theta}\Phi)$ then $\Phi = 0$ with critical value $c_0 = 0$. The corresponding critical submanifold is $F_0 = f^{-1}(c_0) = f^{-1}(0) = \mathcal{N}$, the moduli space of semistable bundles. On the other hand, when $\Phi \neq 0$, there is a type of algebraic structure for Higgs bundles introduced by Simpson [29]: a variation of Hodge structure, or simply a VHS, for a Higgs bundle $(E, \Phi)$ is a decomposition:

$$E = \bigoplus_{j=1}^n E_j$$

such that $\Phi : E_j \to E_{j+1} \otimes K$ for $j \leq n - 1$ and $\Phi(E_n) = 0$. (1.3)

It has been proved by Simpson [30] that the fixed points of the circle action on $\mathcal{M}(r, d)$, and so, the critical points of $f$, are these VHS, where the critical values $c_\lambda = f(E, \Phi)$ will depend on the degrees $d_j$ of the components $E_j \subset E$. By Morse theory, we can stratify $\mathcal{M}$ in such a way that there is a nonzero critical submanifold $F_\lambda := f^{-1}(c_\lambda)$ for each nonzero critical value $0 \neq c_\lambda = f(E, \Phi)$ where $(E, \Phi)$ represents a fixed point of the circle action, or equivalently, a VHS. We then say that $(E, \Phi)$ is an $(r_1, \ldots, r_n)$-VHS, where $r_j = \text{rk}(E_j) \ \forall j$.

**Definition 1.4.** A holomorphic triple on $X$ is a triple $T = (E_1, E_2, \phi)$ consisting of two holomorphic vector bundles $E_1 \to X$ and $E_2 \to X$ and a homomorphism $\phi : E_2 \to E_1$, i.e., an element $\phi \in H^0(\text{Hom}(E_2, E_1))$.

There are certain notions of $\sigma$-degree:

$$\text{deg}_\sigma(T) := \text{deg}(E_1) + \text{deg}(E_2) + \sigma \cdot \text{rk}(E_2),$$

and $\sigma$-slope:

$$\mu_\sigma(T) := \frac{\text{deg}_\sigma(T)}{\text{rk}(E_1) + \text{rk}(E_2)}$$

which give rise to notions of $\sigma$-stability of triples. The reader may consult the works of Bradlow and García-Prada [5]; Bradlow, García-Prada and Gothen [6]; and Muñoz, Ortega and Vázquez-Gallo [26] for the details.

With this notions, one can construct:

$$\mathcal{N}_\sigma = \mathcal{N}_\sigma(r, d) = \mathcal{N}_\sigma(r_1, r_2, d_1, d_2)$$

the moduli space of $\sigma$-polystable triples $T = (E_1, E_2, \phi)$ such that $\text{rk}(E_j) = r_j$ and $\text{deg}(E_j) = d_j$, and

$$\mathcal{N}^\sigma_\sigma = \mathcal{N}^\sigma_\sigma(r, d)$$
the moduli space of \( \sigma \)-stable triples, where \((r, d) = (r_1, r_2, d_1, d_2)\) is the type of the triple \(T = (E_1, E_2, \phi)\).

We mention the moduli space \( N_\sigma(r_1, r_2, d_1, d_2) \) of \( \sigma \)-stable triples because they are closely related to some of the critical submanifolds \( F_\lambda \).

**Definition 1.5.** Fix a point \( p \in X \), and let \( O_X(p) \) be the associated line bundle to the divisor \( p \in \text{Sym}^1(X) = X \). A \( k \)-Higgs bundle (or Higgs bundle with poles of order \( k \)) is a pair \((E, \Phi^k)\) where:

\[
E \xrightarrow{\Phi^k} E \otimes K \otimes O_X(kp) = E \otimes K(kp)
\]

and where the morphism \( \Phi^k \in H^0(X, \text{End}(E) \otimes K(kp)) \) is what we call a Higgs field with poles of order \( k \). The moduli space of \( k \)-Higgs bundles of rank \( r \) and degree \( d \) is denoted by \( \mathcal{M}^k(r, d) \). For simplicity, we will suppose that \( \text{GCD}(r, d) = 1 \), and so, \( \mathcal{M}^k(r, d) \) will be smooth.

There is an embedding

\[
i_k: \mathcal{M}^k(r, d) \to \mathcal{M}^{k+1}(r, d)
\]

\[
[(E, \Phi^k)] \mapsto [(E, \Phi^k \otimes s_p)]
\]

where \( 0 \neq s_p \in H^0(X, O_X(p)) \) is a nonzero fixed section of \( O_X(p) \).

All the results mentioned for \( \mathcal{M}(r, d) \), hold also for \( \mathcal{M}^k(r, d) \).

### 2 Generators for the Cohomology Ring

According to Hausel and Thaddeus [17, (4.4)], there is a universal family \((E^k, \Phi^k)\) over \( X \times \mathcal{M}^k \) where

\[
\begin{align*}
E^k &\to X \times \mathcal{M}^k(r, d) \\
\Phi^k &\in H^0(\text{End}(E^k) \otimes \pi^*_X(K(kp)))
\end{align*}
\]

and from now on, we will refer \((E^k, \Phi^k)\) as a universal \( k \)-Higgs bundle. Note that \((E^k, \Phi^k)\) satisfies the Universal Property: in general, for any family \((F^k, \Psi^k)\) over \( X \times M \), there is a morphism \( \eta: M \to \mathcal{M}^k \) such that \((\text{Id}_X \times \eta)^*(E^k, \Phi^k) = (F^k, \Psi^k)\). It means that, for \( M = \mathcal{M}^k \) whenever exists \((F^k, \Psi^k)\) such that

\[
(E^k, \Phi^k)_P \cong (F^k, \Psi^k)_P \quad \forall P = (E, \Phi^k) \in \mathcal{M}^k(r, d),
\]

then, there exists a unique bundle morphism \( \xi: F^k \to E^k \) such that

\[
\begin{array}{ccc}
F^k & \xrightarrow{\exists \xi} & E^k \\
\downarrow p_2 & & \downarrow p_1 \\
X \times \mathcal{M}^k(r, d) & \xrightarrow{\xi} & \end{array}
\]

commutes: \( p_2 = p_1 \circ \xi \).

The universal bundle extends then to the following: if \((E^k, \Phi^k)\) and \((F^k, \Psi^k)\) are families of stable \( k \)-Higgs bundles parametrized by \( \mathcal{M}^k(r, d) \), such that \((E^k, \Phi^k)_P \cong (F^k, \Psi^k)_P \)
\((\mathbb{P}^k, \Psi^k)\) for all \(P = (E, \Phi^k) \in \mathcal{M}^k(r, d)\), then there is a line bundle \(\mathcal{L} \to \mathcal{M}^k(r, d)\) such that
\[
(\mathbb{P}^k, \Phi^k) \cong (\mathbb{P}^k \otimes \pi_2^*(\mathcal{L}), \Psi^k \otimes \pi_2),
\]
where \(\pi_2: X \times \mathcal{M}^k(r, d) \to \mathcal{M}^k(r, d)\) is the natural projection and the endomorphisms satisfy \(\Phi^k \cong \Psi^k \otimes \pi_2(\sigma_P) \cong \Psi^k\), where \(\sigma_P\) is a section of \(X \times \mathcal{M}^k \to \mathcal{M}^k\). For more details, see Hausel and Thaddeus \[17, (4.2)\].

**Remark 2.1.** Do not confuse \(\pi_2\) with \(p_2\) (neither \(\pi_1\) with \(p_1\)); \(\pi_j\) are the natural projections of the cartesian product, while \(p_j\) are the bundle surjective maps:

\[
\begin{array}{c}
\mathbb{P}^k \\
\downarrow p_1 \\
X \times \mathcal{M}^k(r, d) \\
\downarrow \pi_1 \\
X \\
\end{array}
\quad
\begin{array}{c}
\mathbb{P}^k \\
\downarrow p_2 \\
\mathcal{M}^k(r, d) \\
\downarrow \pi_2 \\
\mathcal{M}^k(r, d) \\
\end{array}
\]

If we consider the embedding \(i_k: \mathcal{M}^k(r, d) \to \mathcal{M}^{k+1}(r, d)\) for general rank, we get that:

**Proposition 2.2.** Let \((\mathbb{P}^k, \Phi^k)\) be a universal Higgs bundle. Then:
\[
(\text{Id}_X \times i_k)^* (\mathbb{P}^{k+1}) \cong \mathbb{P}^k.
\]

**Proof.** Note that
\[
(\mathbb{P}^k, \Phi^k \otimes \pi_1^*(s_P)) \to X \times \mathcal{M}^k
\]
is a family of \((k + 1)\)-Higgs bundles on \(X\), where \(\pi_1: X \times \mathcal{M}^k \to X\) is the natural projection. So, by the universal property:
\[
(\mathbb{P}^k, \Phi^k \otimes \pi_1^*(s_P)) = (\text{Id}_X \times i_k)^* (\mathbb{P}^{k+1}, \Phi^{k+1}).
\]

Consider
\[
\text{Vect}(X) := \left\{ V \to X : V \text{ is a top. vector bundle} \right\} / \cong
\]
the set of equivalence classes of topological vector bundles taken by isomorphism between them. Define the operation
\[
[V] \oplus [W] := [V \oplus W]
\]
and consider the abelian semi-group \((\text{Vect}(X), \oplus)\). Denote by
\[
K^0(X) = K\left(\text{Vect}(X)\right) := \left\{ [V] - [W] \right\} / \sim
\]
the abelian \(K\)-group of topological vector bundles on \(X\), where
\[
[V] - [W] \sim [V \oplus U] - [W \oplus U]
\]

for every topological vector bundle $U \to X$.

Let $K^1(X)$ be the odd $K$-group of $X$ and let

$$K^*(X) = K^0(X) \oplus K^1(X)$$

be the $K$-ring described by Atiyah [1, Chapter II].

In this case, $K^*(X)$ is torsion free since the Riemann surface $X$ is also a projective algebraic variety. Then, as a consequence of the K"unneth Theorem (see for instance Atiyah [1, Corollary 2.7.15.] or [2, Main Theorem]), there is an isomorphism:

$$\left( K^0(X) \otimes K^0(\mathcal{M}^k) \right) \oplus \left( K^1(X) \otimes K^1(\mathcal{M}^k) \right) \cong K^0(X \times \mathcal{M}^k)$$

(2.3)

The reader may see Markman [25] for the details. Furthermore, Markman [25] chooses bases $\{x_1, \ldots, x_{2g}\} \subset K^1(X)$, and $\{x_{2g+1}, \ldots, x_{2g+2}\} \subset K^0(X)$ to get a total basis

$$\{x_1, \ldots, x_{2g}, x_{2g+1}, x_{2g+2}\} \subset K^*(X) = K^0(X) \oplus K^1(X),$$

and, since there is a universal bundle $\mathcal{E}^k \to X \times \mathcal{M}^k$, we get the K"unneth decomposition:

$$[\mathcal{E}^k] = \sum_{j=0}^{2g} x_j \otimes e_j^k$$

where $x_0 \in K^0(X) = \text{span}\{x_{2g+1}, x_{2g+2}\}$, $e_0^k \in K^0(\mathcal{M}^k)$, $x_j \in K^1(X)$, and $e_j^k \in K^1(\mathcal{M}^k)$ for $j = 1, \ldots, 2g$. Finally, Markman [25] considers the Chern classes $c_j(e_i^k) \in H^{2j}(\mathcal{M}^k, \mathbb{Z})$ for $e_i^k \in K^*(\mathcal{M}^k)$ and proves that:

**Theorem 2.3** (Markman [25, Th. 3]). *The cohomology ring $H^*(\mathcal{M}^k(r, d), \mathbb{Z})$ is generated by the Chern classes of the K"unneth factors of the universal vector bundle.*

\[ \square \]

3 Preliminary Results

Let $i_k: \mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ be the embedding given by the tensorization map of the $k$-Higgs field $\Phi^k \mapsto (E, \Phi^k \otimes s_p)$, where $s_p$ is a fixed nonzero section of $L_p$. We want to prove that the map

$$\pi_j(i_k): \pi_j(\mathcal{M}^k(r, d)) \to \pi_j(\mathcal{M}^{k+1}(r, d))$$

stabilizes as $k \to \infty$. But first, we need to present some preliminary results to conclude that.

**Proposition 3.1.** Let $i_k: \mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ be the embedding above mentioned. Consider the $K$-classes $e_i^k \in K(\mathcal{M}^k)$. Then

$$i_k^*(c_j(e_i^{k+1})) = c_j(e_i^k).$$

**Proof.** By Proposition 2.2, and by the naturality of the Chern classes:

$$\sum_{j=0}^{2g} x_j \otimes e_j^k = [\mathcal{E}^k] = [(\text{Id}_X \times i_k)^*(\mathcal{E}^{k+1})] = \sum_{j=0}^{2g} x_j \otimes i_k^*(e_j^{k+1})$$

we have that $i_k^*(e_i^{k+1}) = e_i^k$ and hence $i_k^*(c_j(e_i^{k+1})) = c_j(e_i^k)$. \[ \square \]
Corollary 3.2. Let $i_k: \mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ be the embedding above mentioned. Then, the induced cohomology homomorphism $i_k^*: H^*(\mathcal{M}^{k+1}, \mathbb{Z}) \to H^*(\mathcal{M}^k, \mathbb{Z})$ is surjective.

Proof. The result is an immediate consequence of Theorem 2.3 and Proposition 3.1. □

Definition 3.3. A gauge transformation is an automorphism of $E$. Locally, a gauge transformation $g \in \text{Aut}(E)$ is a $C^\infty(E)$-function with values in $GL_r(\mathbb{C})$. A gauge transformation $g$ is called unitary if $g$ preserves a hermitian inner product on $E$. We will denote $\mathcal{G}$ as the group of unitary gauge transformations. Atiyah and Bott [3] denote $\overline{\mathcal{G}}$ as the quotient of $\mathcal{G}$ by its constant central $U(1)$-subgroup. We will follow this notation too. Moreover, denote $B\mathcal{G}$ and $B\overline{\mathcal{G}}$ as the classifying spaces of $\mathcal{G}$ and $\overline{\mathcal{G}}$, respectively.

We get the fibration

$$BU(1) \to B\mathcal{G} \to B\overline{\mathcal{G}}$$

of classifying spaces, which splits actually as the product

$$B\overline{\mathcal{G}} \cong BU(1) \times B\overline{\mathcal{G}}.$$ 

Then, the generators of $H^*(B\mathcal{G})$ give generators for $H^*(B\overline{\mathcal{G}})$ and so, $B\overline{\mathcal{G}}$ is a free graded commutative algebra on those generators, since $B\mathcal{G}$ is, and consequently, $B\overline{\mathcal{G}}$ is free of torsion. The reader may see Atiyah and Bott [3, Sec. 9.] and Hausel [16, Chap. 3] for the details.

Let $\mathcal{M}^\infty := \lim_{k \to \infty} \mathcal{M}^k = \bigcup_{k=0}^\infty \mathcal{M}^k$ be the direct limit of the spaces $\{\mathcal{M}^k(r,d)\}_{k=0}^\infty$. Hausel and Thaddeus [17] prove that:

Theorem 3.4 (Hausel and Thaddeus [17, (9.7)]). The classifying space of $\overline{\mathcal{G}}$ is homotopically equivalent to the direct limit of the spaces $\mathcal{M}^k(r,d)$:

$$B\overline{\mathcal{G}} \simeq \mathcal{M}^\infty = \lim_{k \to \infty} \mathcal{M}^k.$$ □

Assumption 3.5. Unless otherwise stated, from now on, we will assume that the rank is either $r = 2$ or $r = 3$.

Theorem 3.6. $H^*(\mathcal{M}^k(r,d))$ is torsion free for all $k$.

Proof. The proof uses the following result of Frankel [8, Corollary 1]:

$$F^k_\lambda \text{ is torsion free} \forall \lambda \Leftrightarrow \mathcal{M}^k \text{ is torsion free.}$$

In fact, the result of Frankel is more general. The specific case of moduli spaces of Higgs bundles holds because the proper momentum Hitchin map $f(E, \Phi)$ described in (1.2) is a perfect Morse-Bott function, since we are taking $\text{GCD}(r,d) = 1$.

In both cases, $r = 2$ and $r = 3$, the moduli space of stable vector bundles corresponds to the first critical submanifold: $F_0 = f^{-1}(c_0) = f^{-1}(0) = \mathcal{N}$, which is indeed torsion free (see Atiyah and Bott [3, Theorem 9.9]).
1. When $\text{rk}(E) = 2$, Hitchin notes that the nontrivial critical submanifolds, or $(1, 1)$-VHS, are of the form

$$F_{d_1}^k = \left\{ (E, \Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \phi^k_{21} & 0 \end{pmatrix}) \middle| \begin{array}{l} \deg(E_1) = d_1, \quad \deg(E_2) = d_2, \\
\text{rk}(E_1) = 1, \quad \text{rk}(E_2) = 1, \\
\phi^k_{21}: E_1 \to E_2 \otimes K(kp) \end{array} \right\}$$

and $F_{d_1}^k$ is isomorphic to the moduli space of $\sigma_H$-stable triples $\mathcal{N}_{\sigma_H}(1, 1, \tilde{d}, d_1)$, where $\sigma_H$ is giving by $\sigma_H = \deg \left( K(kp) \right) = 2g - 2 + k$ and $\tilde{d} = d_2 + 2g - 2 + k$, by the map:

$$(E_1 \otimes E_2, \Phi^k) \mapsto (E_2 \otimes K(kp), E_1, \phi^k_{21}).$$

Furthermore, by Hitchin [20], $\mathcal{N}_{\sigma_H}(1, 1, \tilde{d}, d_1)$ is isomorphic to the cartesian product $\mathcal{J}^{d_1}(X) \times \text{Sym}^{d-d_1}(X)$. Hence:

$$F_{d_1}^k \cong \mathcal{J}^{d_1}(X) \times \text{Sym}^{d-d_1}(X)$$

which, by Macdonald [23, (12.3)], is indeed torsion free.

2. When $\text{rk}(E) = 3$, there are three kinds of nontrivial critical submanifolds:

2.1. (1, 2)-VHS of the form

$$F_{d_1}^k = \left\{ (E, \Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \phi^k_{21} & 0 \end{pmatrix}) \middle| \begin{array}{l} \deg(E_1) = d_1, \quad \deg(E_2) = d_2, \\
\text{rk}(E_1) = 1, \quad \text{rk}(E_2) = 2, \\
\phi^k_{21}: E_1 \to E_2 \otimes K(kp) \end{array} \right\}.$$

In this case, there are isomorphisms between the $(1, 2)$-VHS and the moduli spaces of triples $F_{d_1}^k \cong \mathcal{N}_{\sigma_H(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$, where $\tilde{d}_1 = d_2 + 2(2g - 2 + k)$ and $\tilde{d}_2 = d_1$, and where the isomorphism is giving by a map similar to the above mentioned.

By Muñoz, Ortega, Vázquez-Gallo [26, Theorem 4.8. and Lemma 6.1.], when working with $\mathcal{N}_{\sigma}(2, 1, \tilde{d}_1, \tilde{d}_2)$, they find that either the flip loci $S_{\sigma_c}^+$ is the projectivization of a bundle of rank $r^+ = \tilde{d}_1 - d_M - \tilde{d}_2$ over

$$\mathcal{J}^{d_M}(X) \times \mathcal{J}^{d_2}(X) \times \text{Sym}^{r^+}(X)$$

where $d_M = \frac{\sigma_c + \tilde{d}_1 + \tilde{d}_2}{3} \in \mathbb{Z}$, or the flip loci $S_{\sigma_c}^-$ is the projectivization of a bundle of rank $r^- = 2d_M - \tilde{d}_1 + g - 1$ over

$$\mathcal{J}^{d_M}(X) \times \mathcal{J}^{d_2}(X) \times \text{Sym}^{r^-}(X)$$

with $d_M \in \mathbb{Z}$ as above. Hence, by Macdonald [23, (12.3)], the flip loci $S_{\sigma_c}^+$ and $S_{\sigma_c}^-$ are free of torsion for $\sigma_c \in I$. Therefore, $\mathcal{N}_{\sigma_H(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ is also torsion free, and so is $F_{d_1}^k$.

The fact that $\mathcal{N}_{\sigma_H(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ is torsion free since the flip loci are, follows from the next lemma:

**Lemma 3.7.** Let $M$ be a complex manifold, and let $\Sigma \subset M$ be a complex submanifold. Let $\tilde{M}$ be the blow-up of $M$ along $\Sigma$. Then

$$H^*(\tilde{M}, \mathbb{Z}) \cong H^*(M, \mathbb{Z}) \oplus H^{*+2}(\Sigma, \mathbb{Z}) \oplus \cdots \oplus H^{*+2n-2}(\Sigma, \mathbb{Z})$$

where $n$ is the rank of $\mathcal{N}_{\Sigma/M}$, the normal bundle of $\Sigma$ in $M$. 

Proof. (Lemma 3.7)
Let \( E = \mathbb{P}(\mathcal{N}_\Sigma/M) \) be the projectivized normal bundle of \( \Sigma \) in \( M \), sometimes called \textit{exceptional divisor}. The result follows from the fact that the additive cohomology of the blow-up \( H^*(\tilde{M}, \mathbb{Z}) \), can be expressed as:

\[
H^*(\tilde{M}) \cong \pi^*H^*(M) \oplus H^*(E)/\pi^*H^*(\Sigma)
\]

(see for instance Griffiths and Harris [12, Chapter 4.,Section 6.]), and the fact that \( H^*(E) \) is a free module over \( H^*(\Sigma) \) via the injective map \( \pi^*: H^*(\Sigma) \to H^*(E) \) with basis

\[ 1, c, \ldots, c^{n-1}, \]

where \( c \in H^2(E) \) is the first Chern class of the tautological line bundle along the fibres of the projective bundle \( E \to \Sigma \) (see the general version at Husemoller [21, Chapter 17.,Theorem 2.5.]).

2.2. \((2, 1)\)-VHS of the form

\[
F^k_{d_2} = \left\{ (E, \Phi^k) = (E_2 \oplus E_1, \begin{pmatrix} 0 & 0 \\ \varphi^k_{21} & 0 \end{pmatrix}) \right\} \text{ where } \deg(E_2) = d_2, \deg(E_1) = d_1, \text{ and } \varphi^k_{21}: E_2 \to E_1 \otimes K(kp)
\]

By symmetry, similar results can be obtained using the isomorphisms between the \((2, 1)\)-VHS and the moduli spaces of triples:

\[ F^k_{d_2} \cong \mathcal{N}_{\sigma_H(k)}(1, 2, \tilde{d}_1, \tilde{d}_2), \]

and the dual isomorphisms

\[ \mathcal{N}_{\sigma_H(k)}(2, 1, \tilde{d}_1, \tilde{d}_2) \cong \mathcal{N}_{\sigma_H(k)}(1, 2, -\tilde{d}_2, -\tilde{d}_1) \]

between moduli spaces of triples.

2.3. \((1, 1, 1)\)-VHS of the form

\[
F^k_{d_1, d_2, d_3} = \left\{ (E, \Phi^k) = (E_1 \oplus E_2 \oplus E_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi^k_{21} & 0 & 0 \\ 0 & \varphi^k_{32} & 0 \end{pmatrix}) \right\} \text{ where } \deg(E_j) = d_j, \text{ and } \varphi^k_{ij}: E_j \to E_i \otimes K
\]

Finally, we know that

\[
F^k_{d_1, d_2, d_3} \xrightarrow{\cong} \text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X) \times \mathcal{J}^{d_3}(X)
\]

\[
(E, \Phi^k) \mapsto (\text{div}(\varphi^k_{21}), \text{div}(\varphi^k_{32}), E_3),
\]

where \( m_i = d_{i+1} - d_i + \sigma_H \), and so, by Macdonald [23, (12.3)] there is nothing to worry about torsion.

\[ \square \]

Corollary 3.8.

\[ \lim_{\leftarrow} H^*(\mathcal{M}^k, \mathbb{Z}) \cong H^*(\mathcal{M}^\infty, \mathbb{Z}) \cong H^*(\mathcal{B}\mathcal{G}, \mathbb{Z}) \]

\[ \square \]
Corollary 3.9. For each \( n \geq 0 \) there is a \( k_0 \), depending on \( n \), such that
\[
i_k^*: H^j(M^{k+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(M^k, \mathbb{Z})
\]
is an isomorphism for all \( k \geq k_0 \) and for all \( j \leq n \).

By the Universal Coefficient Theorem for Cohomology (see for instance Hatcher [14, Theorem 3.2. and Corollary 3.3.]), we get

Lemma 3.10. For each \( n \geq 0 \) there is a \( k_0 \), depending on \( n \), such that
\[
H_j(M^\infty, M^k; \mathbb{Z}) = 0
\]
for all \( k \geq k_0 \) and for all \( j \leq n \).

Proof. The embedding \( i_k: M^k(r, d) \to M^{k+1}(r, d) \) is injective, and by Corollary 3.2 we know that \( i_k^*: H^j(M^k, \mathbb{Z}) \leftarrow H^j(M^{k+1}, \mathbb{Z}) \) is surjective for all \( k \). Hence, by the Universal Coefficient Theorem, we get that the following diagram
\[
\begin{array}{cccccc}
0 & \xrightarrow{0} & \Ext(H_{j-1}(M^k), \mathbb{Z}) & \xrightarrow{i_k^*} & H^j(M^k, \mathbb{Z}) & \xrightarrow{(i_k)_*} & \Hom(H_j(M^k), \mathbb{Z}) & \xrightarrow{0} \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \xrightarrow{0} & \Ext(H_{j-1}(M^{k+1}), \mathbb{Z}) & \xrightarrow{i_k^*} & H^j(M^{k+1}, \mathbb{Z}) & \xrightarrow{(i_k)_*} & \Hom(H_j(M^{k+1}), \mathbb{Z}) & \xrightarrow{0}
\end{array}
\]
commutes. By Theorem 3.6 \( H^*(M^k, \mathbb{Z}) \) is torsion free, and so, by Corollary 3.9, for all \( n \geq 0 \), there is \( k_0 \), depending on \( n \), such that
\[
H_j(M^k(r, d), \mathbb{Z}) \xrightarrow{\cong} H_j(M^{k+1}(r, d), \mathbb{Z}) \xrightarrow{\cong} H_j(M^\infty(r, d), \mathbb{Z})
\]
for all \( k \geq k_0 \) and for all \( j \leq n \). Hence
\[
H_j(M^\infty, M^k; \mathbb{Z}) = 0
\]
for all \( k \geq k_0 \) and for all \( j \leq n \).

Proposition 3.11. For general rank \( r \), denoting \( M^k = M^k(r, d) \) for simplicity, and \( N = N(r, d) \) as the moduli of stable bundles, the following diagram commutes
\[
\begin{array}{ccc}
\pi_1(M^k) & \xrightarrow{\cong} & \pi_1(M^{k+1}) \\
\downarrow & & \downarrow \\
\pi_1(N) & \xrightarrow{\cong} & \pi_1(N)
\end{array}
\]

Proof. It is an immediate consequence of the result proved by Bradlow, García-Prada and Gothen [7, Proposition 3.2.] using Morse theory.

Proposition 3.12. For all \( k \in \mathbb{N} \), there is an isomorphism between the fundamental group of \( M^k \) and the fundamental group of the direct limit of the spaces \( \{M^k(r, d)\}_{k=0}^\infty \):
\[
\pi_1(M^k) \xrightarrow{\cong} \pi_1(M^\infty).
\]
Proof. Using the generalization of Van Kampen’s Theorem presented by Fulton [9], and using the fact that \( M^k \hookrightarrow M^{k+1} \) are embeddings of Deformation Neighborhood Retracts (DNR), i.e. every \( M^k(r, d) \) is the image of a map defined on some open neighborhood of itself and homotopic to the identity (see for instance Hausel and Thaddeus [17, (9.1)]), we can conclude that \( \pi_1 \left( \lim_{k \to \infty} M^k \right) = \lim_{k \to \infty} \pi_1 (M^k) \).

Remark 3.13. By Atiyah and Bott [3] we have:

\[
\pi_1(N) \cong H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g},
\]

and hence, by Proposition 3.11 and Proposition 3.12:

\[
\pi_1(M^k) \cong \pi_1(M^\infty) \cong \mathbb{Z}^{2g}.
\]

We will need the following version of Hurewicz Theorem, presented by Hatcher [14, Theorem 4.37.] (see also James [22]). Hatcher first mentions that, in the relative case when \((X, A)\) is an \((n-1)\)-connected pair of path-connected spaces, the kernel of the Hurewicz map

\[
h : \pi_n(X, A) \to H_n(X, A; \mathbb{Z})
\]

contains the elements of the form \([\gamma][f] - [f]\) for \([\gamma] \in \pi_1(A)\). Hatcher defines \(\pi'_n(X, A)\) to be the quotient group of \(\pi_n(X, A)\) obtained by factoring out the subgroup generated by the elements of the form \([\gamma][f] - [f]\), or the normal subgroup generated by such elements in the case \(n = 2\) when \(\pi_2(X, A)\) may not be abelian, then \(h\) induces a homomorphism \(h' : \pi'_n(X, A) \to H_n(X, A; \mathbb{Z})\). The general form of Hurewicz Theorem presented by Hatcher deals with this homomorphism:

**Theorem 3.14. (Hurewicz Theorem)**

If \((X, A)\) is an \((n-1)\)-connected pair of path-connected spaces, with \(n \geq 2\) and \(A \neq \emptyset\), then \(h' : \pi'_n(X, A) \to H_n(X, A; \mathbb{Z})\) is an isomorphism and \(H_j(X, A; \mathbb{Z}) = 0\) for \(j \leq n-1\).

**Definition 3.15.** i. The determinant of a vector bundle \(E \to X\) of rank \(r\) is a line bundle giving by the exterior power of the vector bundle. It gives a natural map of the form:

\[
\det : \mathcal{N} \longrightarrow \mathcal{J}^d
\]

\[
E \longmapsto \det(E) = \bigwedge^r E
\]

where \(\mathcal{N} = \mathcal{N}(r, d)\) is the moduli space of stable bundles \(E \to X\) of rank \(r\) and degree \(d\), and \(\mathcal{J}^d\) is the Jacobian of \(X\). Fixing a line bundle \(\Lambda \to X\), \(\Lambda \in \mathcal{J}^d\), the fibre \(\mathcal{N}_\Lambda = \mathcal{N}_\Lambda(r, d) := \det^{-1}(\Lambda)\) is the moduli space of stable bundles with fixed determinant.

ii. Together with the trace, the determinant allows us to define the map

\[
\zeta : \mathcal{M}^k(r, d) \longrightarrow \mathcal{J}^d \times H^0(X, K(kp))
\]

\[
(E, \Phi) \longmapsto \left(\det(E), \text{tr}(\Phi)\right)
\]

and to consider the fibre \(\mathcal{M}^k_\Lambda(r, d) := \zeta^{-1}(\Lambda, 0)\) which is the moduli space of \(k\)-Higgs bundles with fixed determinant and trace zero.
There is an important result of Atiyah and Bott [3] that is relevant to mention here:

**Theorem 3.16** (Atiyah and Bott [3, (9.12.)]). The moduli space $N_\Lambda(r, d)$ of stable bundles of fixed determinant $\Lambda$, with $\text{GCD}(r, d) = 1$, is simply connected.

By Proposition 3.11 holds also for fixed determinant:

$$\mathcal{M}_\Lambda^\infty(r, d) \simeq BG \tilde{G}$$

(see Hausel and Thaddeus [17]). Nevertheless, Corollary 3.2 does not adapt in a straightforward way, as we shall see in subsection 4.2.

ii. The moduli space $\mathcal{M}_\Lambda^k(r, d)$ is simply connected because Proposition 3.11 holds also for fixed determinant $k$-Higgs bundles. So, $\pi_1(\mathcal{M}_\Lambda^k)$ acts trivially on $\pi_n(\mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k)$.

## 4 Main Results

### 4.1 General Results

Here, we will concern the moduli spaces $\mathcal{M}^k(r, d)$ of $k$-Higgs bundles, where the results are true under the condition that $\pi_1(\mathcal{M}^k)$ acts trivially on all the higher relative homotopy groups of the pair $(\mathcal{M}^\infty, \mathcal{M}^k)$. However, we do not know if this condition is true or not.

**Lemma 4.1.** If $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$, then for all $n \geq 0$ exists $k_0$, depending on $n$, such that $\pi_j(\mathcal{M}^\infty, \mathcal{M}^k) = 0$ for all $k \geq k_0$ and for all $j \leq n$.

**Proof.** The proof proceeds by induction on $m \in \mathbb{N}$ for $2 \leq m \leq n$. The first induction step is trivial because

$$\pi_1(\mathcal{N}) = \pi_1(\mathcal{M}) = \pi_1(\mathcal{M}^k) = \pi_1(\mathcal{M}^\infty)$$

by Proposition 3.11. For $m = 2$ we need $\pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ to be abelian. Consider the sequence

$$\pi_2(\mathcal{M}^\infty) \to \pi_2(\mathcal{M}^\infty, \mathcal{M}^k) \to \pi_1(\mathcal{M}^k) \to \pi_1(\mathcal{M}^\infty) \to \pi_1(\mathcal{M}^\infty, \mathcal{M}^k) \to 0$$

where $\pi_2(\mathcal{M}^\infty) \to \pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ is surjective, $\pi_1(\mathcal{M}^k) \xrightarrow{\cong} \pi_1(\mathcal{M}^\infty)$ are isomorphic, and hence $\pi_1(\mathcal{M}^\infty, \mathcal{M}^k) = 0$. So, $\pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ is a quotient of the abelian group $\pi_2(\mathcal{M}^\infty)$, and so it is also abelian.

Finally, suppose that the statement is true for all $j \leq m - 1$ for $2 \leq m \leq n$. So, $(\mathcal{M}^\infty, \mathcal{M}^k)$ is $(m - 1)$-connected, i.e.

$$\pi_j(\mathcal{M}^\infty, \mathcal{M}^k) = 0 \quad \forall j \leq m - 1.$$ 

For $m \geq 2$, by Hurewicz Theorem 3.14,

$$h' : \pi_m(\mathcal{M}^\infty, \mathcal{M}^k) \xrightarrow{\cong} H_m(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z})$$

is an isomorphism. By Lemma 3.10, there is an integer $k_0$, depending on $m$, such that $H_m(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0$ for all $k \geq k_0$. Hence, if $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, then

$$\pi_m(\mathcal{M}^\infty, \mathcal{M}^k) = \pi_m(\mathcal{M}^\infty, \mathcal{M}^k) = 0$$

finishing the induction process.

□
Corollary 4.2. If $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$, then for all $n \geq 0$ exists $k_0$, depending on $n$, such that

$$\pi_j(\mathcal{M}^k) \xrightarrow{\cong} \pi_j(\mathcal{M}^\infty)$$

for all $k \geq k_0$ and for all $j \leq n - 1$. □

4.2 Fixed determinant case

The main goal here, is to avoid the hypothesis of the trivial action of the fundamental group on the relative homotopy group: $\pi_1(\mathcal{M}^k) \circ \pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$. So, we want to get the analogue of Lemma 4.1 for $\mathcal{M}_k^\Lambda$, the moduli space of $k$-Higgs bundles with fixed determinant, since $\mathcal{M}_k^\Lambda$ is simply connected. To do that, we will need the analogue of Corollary 3.2, and then the analogue of Lemma 3.10 also for $\mathcal{M}_k^\Lambda$.

The analogue of Corollary 3.2 for $\mathcal{M}_k^\Lambda$ is not immediate. Note that the group of $r$-torsion points in the Jacobian:

$$\Gamma = \text{Jac}(r) := \{ L \to X \text{ line bundle} : L^r \cong \mathcal{O}_X \}$$

acts on $\mathcal{M}_k^\Lambda(r, d)$ by tensorization:

$$(E, \Phi^k) \mapsto (E \otimes L, \Phi^k \otimes \text{id}_L).$$

Hence, $\Gamma$ acts on $H^*(\mathcal{M}_k^\Lambda, \mathbb{Z})$ for all $k$. This cohomology splits in a $\Gamma$-invariant part and in a complement which is called by Hausel and Thaddeus [19] as the “variant part”:

$$H^*(\mathcal{M}_k^\Lambda, \mathbb{Z}) = H^*(\mathcal{M}_k^\Lambda, \mathbb{Z})^\Gamma \oplus H^*(\mathcal{M}_k^\Lambda, \mathbb{Z})^{\text{var}}. \quad (4.1)$$

This decomposition appears in various cohomology calculations, see e.g., Hitchin [20] for rank two, Gothen [10] for rank three, Hausel [16] also for rank two, Bento [4] for the explicit calculations for rank two and rank three, and Hausel and Thaddeus [19] for general rank.

The analogue of Corollary 3.2 for $\mathcal{M}_k^\Lambda$ will be obtained for each of the pieces in the last direct sum $(4.1)$ separately:

- For $H^*(\mathcal{M}_k^\Lambda, \mathbb{Z})^\Gamma$:

It follows from the corresponding result for $H^*(\mathcal{M}^k, \mathbb{Z})$ because there is a surjection $H^*(\mathcal{M}^k, \mathbb{Z}) \to H^*(\mathcal{M}_k^\Lambda, \mathbb{Z})^\Gamma$.

Recall that, for general rank $r$, the moduli space of stable vector bundles corresponds to the first critical submanifold: $F_0 = f^{-1}(c_0) = f^{-1}(0) = \mathcal{N}(r, d)$. The group $\Gamma$ acts trivially on $H^*(\mathcal{N}, \mathbb{Z})$, and there is a surjection

$$H^*(\mathcal{N}, \mathbb{Z}) \to H^*(\mathcal{N}_\Lambda, \mathbb{Z}).$$

The reader may see Atiyah and Bott [3, Prop. 9.7.] for the details.

For the rank $r = 2$ case, a nontrivial critical submanifold of $\mathcal{M}_k^\Lambda(2, 1)$, is a so-called $(1, 1)$-VHS:

$$F_{d_1}^k(\Lambda) = \left\{ (E, \Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \bigg| \begin{array}{l} \deg(E_j) = d_j, \ \text{rk}(E_j) = 1, \\ \varphi_{21}^k : E_1 \to E_2 \otimes K(kp), \\ E_1 E_2 = \Lambda \end{array} \right\},$$

which is a $2^{2g}$-covering with covering group the $2$-torsion points in the Jacobian $\Gamma \cong (\mathbb{Z}_2)^{2g}$. Hence, the results of Betti numbers presented by Bento [4, Prop. 2.2.3.] let us conclude the following:
Proposition 4.3. The cohomology map

$$H^*(\text{Sym}^m(X), \mathbb{Z}) \to H^*(F^k_{d_1}(\Lambda), \mathbb{Z})$$

induced by the $\Gamma$-covering $F^k_{d_1}(\Lambda) \to \text{Sym}^m(X)$ where $m = d_2 - d_1 + 2g - 2 + k$, is injective, and its image is the $\Gamma$-invariant subgroup $H^*(F^k_{d_1}(\Lambda), \mathbb{Z})^\Gamma$. □

Corollary 4.4. There exists a surjection

$$H^*(\mathcal{M}^k(2, 1), \mathbb{Z}) \to H^*(\mathcal{M}^k_\Lambda(2, 1), \mathbb{Z})^\Gamma.$$

□

When $r = 3$, the group of 3-torsion points in the Jacobian looks like $\Gamma \cong (\mathbb{Z}_3)^{2g}$, and the nontrivial critical submanifolds of $\mathcal{M}^k(3, d)$ are VHS either of type $(1, 2), (2, 1)$ or $(1, 1, 1)$, where the cohomology of the $(1, 2)$ and $(2, 1)$ VHS is invariant under the action of $\Gamma$, and the $(1, 1, 1)$-VHS is a $3^{2g}$-covering of $\text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X)$ with covering group $\Gamma \cong (\mathbb{Z}_3)^{2g}$. Hence:

Proposition 4.5.

$$H^*(F^k_{d_1}(\Lambda), \mathbb{Z}) = H^*(F^k_{d_1}(\Lambda), \mathbb{Z})^\Gamma \text{ and } H^*(F^k_{d_2}(\Lambda), \mathbb{Z}) = H^*(F^k_{d_2}(\Lambda), \mathbb{Z})^\Gamma$$

where

$$F^k_{d_1}(\Lambda) = \left\{(E, \Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \mid \deg(E_1) = d_1, \ \deg(E_2) = d_2, \ \rk(E_1) = 1, \ \rk(E_2) = 2, \ \varphi_{21}^k : E_1 \to E_2 \otimes K(kp) \right\}$$

and

$$F^k_{d_2}(\Lambda) = \left\{(E, \Phi^k) = (E_2 \oplus E_1, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \mid \deg(E_2) = d_2, \ \deg(E_1) = d_1, \ \rk(E_2) = 2, \ \rk(E_1) = 1, \ \varphi_{21}^k : E_2 \to E_1 \otimes K(kp) \right\}$$

are the $(1, 2)$ and $(2, 1)$-VHS of $\mathcal{M}^k(3, d)$ respectively, with

$$\frac{d}{3} \leq d_1 \leq \frac{d}{3} + \frac{2g - 2 + k}{2} \text{ and } \frac{2d}{3} \leq d_2 \leq \frac{2d}{3} + \frac{2g - 2 + k}{2}.$$

Furthermore:

$$H^*(F^k_{m_1m_2}(\Lambda), \mathbb{Z}) = H^*(F^k_{m_1m_2}(\Lambda), \mathbb{Z})^\Gamma \oplus H^*(F^k_{m_1m_2}(\Lambda), \mathbb{Z})^\text{var}$$

and the cohomology map

$$H^*(\text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X), \mathbb{Z}) \to H^*(F^k_{m_1m_2}(\Lambda), \mathbb{Z})$$

induced by the $\Gamma$-covering $F^k_{m_1m_2}(\Lambda) \to \text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X)$ where $F^k_{m_1m_2}(\Lambda) = \left\{(E, \Phi^k) = (E_1 \oplus E_2 \oplus E_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21}^k & 0 & 0 \\ 0 & \varphi_{32}^k & 0 \end{pmatrix}) \mid \deg(E_i) = d_j, \ \rk(E_j) = 1, \ \varphi_{ij} : E_j \to E_i \otimes K(kp) \right\}$, is the $(1, 1, 1)$-VHS of $\mathcal{M}^k_\Lambda(3, d)$ with $m_j = d_{j+1} - d_j + 2g - 2 + k$, is injective, and its image is the $\Gamma$-invariant subgroup $H^*(F^k_{m_1m_2}(\Lambda))^\Gamma$. □
Corollary 4.6. There exists a surjection
\[ H^* \left( \mathcal{M}^k(3, d), \mathbb{Z} \right) \twoheadrightarrow H^* \left( \mathcal{M}^k_\Lambda(3, d), \mathbb{Z} \right)^\Gamma. \]

The reader may see Bento [4], Gothen [10] and also Hausel and Thaddeus [19], for details. Using the results above, we get:

Lemma 4.7. The induced cohomology homomorphism restricted to the $\Gamma$-invariant cohomology of the moduli spaces of $k$-Higgs bundles with fixed determinant $\Lambda$
\[ i_k^*: H^* (\mathcal{M}^{k+1}_\Lambda (r, d), \mathbb{Z})^\Gamma \twoheadrightarrow H^* (\mathcal{M}^k (r, d), \mathbb{Z})^\Gamma \]
is surjective.

Proof. It is enough to note that the following diagram
\[
\begin{array}{c}
H^* (\mathcal{M}^{k+1}, \mathbb{Z})
\downarrow
\downarrow
\downarrow
\downarrow
i_k^*
\end{array}
\begin{array}{c}
H^* (\mathcal{M}^k, \mathbb{Z})
\downarrow
\downarrow
\downarrow
\downarrow
i_k^*
\end{array}
\begin{array}{c}
H^* (\mathcal{M}^{k+1}_\Lambda, \mathbb{Z})^\Gamma
\downarrow
\downarrow
\downarrow
\downarrow
\end{array}
\begin{array}{c}
H^* (\mathcal{M}^k_\Lambda, \mathbb{Z})^\Gamma
\end{array}
\]
commutes, where the top arrow is surjective by Corollary 3.2, and the descending arrows are surjective because of Corollary 4.4 and Corollary 4.6.

- For $H^* (\mathcal{M}_\Lambda, \mathbb{Z})^{\text{var}}$:
First, note that with fixed determinant $\Lambda$ the critical submanifolds of type $(1,1)$ and $(1,1,1)$ are $r^{2g}$-coverings with covering group $\Gamma \cong (\mathbb{Z}_r)^{2g}$, with $r = 2$ or $r = 3$ (see Bento [4] Prop. 2.2.1. and Lemma 2.4.4.). Furthermore, when $r = 3$ the cohomology of $(1,2)$ and $(2,1)$ critical submanifolds is $\Gamma$-invariant. Then, only the cohomology of $(1,1)$-VHS and $(1,1,1)$-VHS split in the $\Gamma$-invariant part and the variant complement, for rank $r = 2$ and $r = 3$, respectively. Hence:
\[
H^* \left( \mathcal{M}^k_\Lambda (2, 1), \mathbb{Z} \right)^{\text{var}} = \bigoplus_{d_1 > \frac{1}{2}} H^* \left( F^k_{d_1} (\Lambda) \right)^{\text{var}} \quad \text{and}
\]
\[
H^* \left( \mathcal{M}^k_\Lambda (3, d), \mathbb{Z} \right)^{\text{var}} = \bigoplus_{(m_1, m_2) \in \Omega_{d_k}} H^* \left( F^k_{m_1 m_2} (\Lambda), \mathbb{Z} \right)^{\text{var}}
\]
where $d_k = \deg \left( K \otimes \mathcal{O}_X (kp) \right) = \deg \left( K(kp) \right) = 2g - 2 + k$, $\frac{1}{2} < d_1 < \frac{1+d_k}{2}$ according to Hitchin [20] for $(1,1)$-VHS in rank two, and $(m_1, m_2) \in \Omega_{d_k}$ where $M_j := E_j^* E_{j+1} K(kp)$, $m_j := \deg (M_j) = d_{j+1} - d_j + d_k$, and the set of indexes
\[
\Omega_{d_k} = \left\{ (m_1, m_2) \in \mathbb{N} \times \mathbb{N} : \begin{pmatrix} 2m_1 + m_2 < 3d_k \\ m_1 + 2m_2 < 3d_k \\ m_1 + 2m_2 \equiv d(3) \end{pmatrix} \right\}
\]
Lemma 4.8. Let $F^k_{d_1}(\Lambda)$ be a $(1, 1)$-VHS of $\mathcal{M}^k_2(2, 1)$ and let $m = d_2 - d_1 + 2g - 2 + k$. Then

$$H^j(F^k_{d_1}(\Lambda), \mathbb{Z})_{\text{var}} \neq 0 \iff j = m.$$  

Proof. See Bento [4, Prop. 2.2.4].

Lemma 4.9. Let $F^k_{m_1m_2}(\Lambda)$ be a $(1, 1, 1)$-VHS of $\mathcal{M}^k_A(3, d)$. Then

$$H^i(F^k_{m_1m_2}(\Lambda), \mathbb{Z})_{\text{var}} \neq 0 \iff i = m_1 + m_2,$$

where $m_j = d_{j+1} - d_j + d_k$.

Proof. See Bento [4, Prop. 2.4.4].

Then, in both cases, when $r = 2$ and when $r = 3$, the cohomology groups with integer coefficients are torsion free:

- If $r = 2$, we have just one nonzero component, $H^m(F^k_{d_1}(\Lambda), \mathbb{Z})_{\text{var}}$ which is the sum of $2^{2g}$ copies of $H^m(\text{Sym}^m(X), \mathbb{Z})$, since $F^k_{d_1}(\Lambda) \to \text{Sym}^m(X)$ is a $(\mathbb{Z}_2)^{2g}$-covering.

- Similarly, if $r = 3$, the nonzero component is $H^{m_1+m_2}(F^k_{m_1m_2}(\Lambda), \mathbb{Z})_{\text{var}}$ which is the sum of $3^{2g}$ copies of $H^{m_1+m_2}(\text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X), \mathbb{Z})$, since

$$F^k_{m_1m_2}(\Lambda) \to \text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X)$$

is a $(\mathbb{Z}_3)^{2g}$-covering.

They are torsion free by Macdonald [23, (12.3)]. The reader may consult Bento [4, Chap. 2] for the details. Hence, we get

Lemma 4.10. Let $i_k : \mathcal{M}^k_A \hookrightarrow \mathcal{M}^{k+1}_A$ be the embedding given by the tensorization map $(E, \Phi^k) \mapsto (E, \Phi^k \otimes s_p)$ as above mentioned. Then, the induced cohomology homomorphism $i^*_k : H^*(\mathcal{M}^{k+1}_A, \mathbb{Z})_{\text{var}} \to H^*(\mathcal{M}^k_A, \mathbb{Z})_{\text{var}}$ is surjective, restricted to the variant complement.

This latter method only works with rank $r = 2$ or $r = 3$, but not in general. The difficulty in calculating $H^*(\mathcal{M}_A, \mathbb{Z})_{\text{var}}$ for general rank is explained also on Hausel and Thaddeus [19].
Finally, we may conclude the following:

**Corollary 4.11.** Let \( i_k: \mathcal{M}_k^k \hookrightarrow \mathcal{M}_k^{k+1} \) be the embedding above mentioned. Then, the induced cohomology homomorphism

\[
i_k^*: H^*(\mathcal{M}_k^{k+1}, \mathbb{Z}) \to H^*(\mathcal{M}_k^k, \mathbb{Z})
\]

is surjective.

**Proof.** It is enough to see that the cohomology of \( \mathcal{M}_k^k \) splits in the \( \Gamma \)-invariant part and the variant complement:

\[
H^*(\mathcal{M}_k^k, \mathbb{Z}) = H^*(\mathcal{M}_k^k, \mathbb{Z})^\Gamma \oplus H^*(\mathcal{M}_k^k, \mathbb{Z})^{\text{var}}
\]

and so, the result follows from Lemma 4.7 and Lemma 4.10.

**Lemma 4.12.** For all \( n \) exists \( k_0 \), depending on \( n \), such that

\[
H_j(\mathcal{M}_k^\infty, \mathcal{M}_k^k; \mathbb{Z}) = 0
\]

for all \( k \geq k_0 \) and for all \( j \leq n \).

**Theorem 4.13.** For all \( n \) exists \( k_0 \), depending on \( n \), such that

\[
\pi_j(\mathcal{M}_k^\infty, \mathcal{M}_k^k) = 0
\]

for all \( k \geq k_0 \) and for all \( j \leq n \).

**Proof.** The proof is quite similar to the proof of Lemma 4.1, using now Corollary 4.11 and Lemma 4.12, and so, we have a new advantage: \( \mathcal{M}_k^k \) is simply connected, hence the action \( \pi_1(\mathcal{M}_k^k) \cap \pi_n(\mathcal{M}_k^\infty, \mathcal{M}_k^k) \) is trivial.

**Corollary 4.14.** For all \( n \) exists \( k_0 \), depending on \( n \), such that

\[
\pi_j(\mathcal{M}_k^k) \xrightarrow{\cong} \pi_j(\mathcal{M}_k^\infty)
\]

for all \( k \geq k_0 \) and for all \( j \leq n - 1 \).

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