A weighted limit theorem for periodic Hurwitz zeta-function

Oleg Lukasčonok

Department of Mathematics and Informatics, Vilnius University
Naugarduko 24, LT-03225 Vilnius
E-mail: lukasonokoleg@gmail.com

Abstract. In the paper, a weighted limit theorem for weakly convergent probability measures on the complex plane for the periodic Hurwitz zeta function is obtained.

Keywords: periodic Hurwitz zeta function, probability measure, weak convergence.

Introduction

Let $\{a_m: m \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be a periodic with the least period $k \in \mathbb{N}$ sequence of complex numbers, and $\alpha \in (0, 1]$ be a fixed parameter. The periodic Hurwitz zeta-function $\zeta(s, \alpha; a)$, $s = \sigma + it$, is defined, for $\sigma > 1$, by Dirichlet series

$$\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s}.$$ 

For $\sigma > 1$, the periodicity of $a$ implies the equality

$$\zeta(s, \alpha; a) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\left(s, \frac{\alpha + l}{k}\right),$$

where

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \quad \sigma > 1,$$

is the classical Hurwitz zeta-function. Since the function $\zeta(s, \alpha)$ has a simple pole at $s = 1$ with residue $1$, equality (1) gives analytic continuation for $\zeta(s, \alpha; a)$ to the whole complex plane, except maybe, for a simple pole at $s = 1$. If

$$a := \frac{1}{k} \sum_{l=0}^{k-1} a_l = 0,$$

then the function $\zeta(s, \alpha; a)$ is entire, while, in the case $a \neq 0$, the point $s = 1$ is a simple pole with residue $a$.

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space $S$, and by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Suppose that $\alpha$ is transcendental. Then in [2], by the way, it was obtained that, for $\sigma > \frac{1}{2}$, probability measure

$$\frac{1}{T} \text{meas}\{t \in [0, T]: \zeta(\sigma + it, \alpha; a) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the explicitly given probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$. 
A weighted limit theorem for periodic Hurwitz zeta-function

The aim of this note is to prove a weighted limit theorem on the complex plane for the function \( \zeta(s, \alpha; a) \). Let \( w(t) \) be a positive function of bounded variation on \([T_0, \infty), T_0 > 0\), such that

\[
\lim_{T \to \infty} U(T, w) = \lim_{T \to \infty} \int_{T_0}^{T} w(t) \, dt = +\infty.
\]

Also, we require that, for \( \sigma > \frac{1}{2}, \sigma \neq 1 \), and all \( v \in \mathbb{R} \), the estimate

\[
\int_{T_0}^{T_0+v} w(u-v)\left|\zeta(\sigma+it, \alpha, a)\right|^2 \, dt \ll U\left(1+|v|\right)
\]

should be satisfied. Denote by \( I_A(t) \) the indicator function of a set \( A \), and define the probability measure

\[
P_{T,\sigma,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{t: \zeta(\sigma+it, \alpha, a) \in A\}} \, dt, \quad A \in \mathcal{B}(\mathbb{C}).
\]

**Theorem 1.** Suppose that \( \alpha \) is transcendental, \( \sigma > \frac{1}{2}, \) and that the weight function satisfies the condition (2). Then, on \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\), there exists a probability measure \( P_\sigma \) such that the measure \( P_{T,\sigma,w} \) converges weakly to \( P_\sigma \) as \( T \to \infty \).

1 Auxiliary results

We start with a weighted limit theorem on the infinite-dimensional torus. Let

\[
\Omega = \prod_{m=0}^{\infty} \gamma_m,
\]

where \( \gamma_m = \{s \in \mathbb{C}: |s| = 1\} \) for all \( m \in \mathbb{N}_0 \). With the product topology and pointwise multiplication, the torus \( \Omega \) is a compact topological Abelian group. Therefore, on \((\Omega, \mathcal{B}(\Omega))\), the probability Haar measure \( m_H \) can be defined. Define the probability measure

\[
Q_{T,w}(A) = \frac{1}{U} \int_{T_0}^{T} w(t) I_{\{(m+\alpha)^{-1}: m \in \mathbb{N}_0\} \in A\}} \, dt, \quad A \in \mathcal{B}(\Omega).
\]

**Lemma 1.** Suppose that \( \alpha \) is transcendental. Then the probability measure \( Q_{T,w} \) converges weakly to \( m_H \) as \( T \to \infty \).

**Proof.** Denote by \( \mathbb{Z} \) the set of all integers. Then the dual group (the character group) of \( \Omega \) is isomorphic to \( \mathbb{D} = \bigoplus_{m=0}^{\infty} \mathbb{Z}_m \), where \( \mathbb{Z}_m = \mathbb{Z} \) for all \( m \in \mathbb{N}_0 \). Let \( \omega(m) \) be the projection of \( \omega \in \Omega \) to the coordinate space \( \gamma_m, m \in \mathbb{N}_0 \).

An element \( \bar{k} = (k_0, k_1, \ldots) \in \mathbb{D} \), where only a finite number of integers \( k_m, m \in \mathbb{N}_0 \), are distinct from zero, acts on \( \Omega \) by

\[
\omega \to \omega^{\bar{k}} = \prod_{m=0}^{\infty} \omega^{k_m}(m), \quad \omega \in \Omega.
\]

Liet. matem. rink. Proc. LMS, Ser. A, 53, 2012, 60–65.
Therefore, the Fourier transform $g_{T,w}(k)$ of the measure $Q_{T,w}$ is

$$g_{T,w}(k) = \int_{\Omega} \left( \prod_{m=0}^{\infty} \omega^{k_m}(m) \right) dQ_{T,w} = \frac{1}{U} \int_{T_0}^{T} w(t) \prod_{m=0}^{\infty} (m+\alpha)^{-atk_m} dt.$$  \hspace{1cm} (3)

Since $\alpha$ is transcendental, the set \{log$(m+\alpha)$: $m \in \mathbb{N}_0$\} is linearly independent over the field of rational numbers. Therefore, $\sum_{m=0}^{\infty} k_m \log(m+\alpha) = 0$ if and only if $k = 0$.

If $k \neq 0$, then we have

$$\int_{T_0}^{T} w(t) \prod_{m=0}^{\infty} (m+\alpha)^{-atk_m} dt = \int_{T_0}^{T} w(t) e^{-it\sum_{m=0}^{\infty} k_m \log(m+\alpha)} dt = \left( -i \sum_{m=0}^{\infty} k_m \log(m+\alpha) \right)^{-1} \int_{T_0}^{T} w(t) e^{-it\sum_{m=0}^{\infty} k_m \log(m+\alpha)} dt = O\left( \sum_{m=0}^{\infty} k_m \ln(m+\alpha) \right)^{-1},$$

where, as above, only a finite number of integers $k_m$ are distinct from zero. This, together with (3), shows that

$$\lim_{T \to \infty} g_{T,w}(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

and the lemma follows from a continuity theorem on compact groups.

Now let $\sigma_1 > \frac{1}{2}$ be fixed, and, for $m, n \in \mathbb{N}_0$,

$$v_n(m, \alpha) = e^{-\left(\frac{m+\alpha}{\sigma_1}\right)^{\sigma_1}}.$$

Then it is easy to show that the series

$$\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m v_n(m, \alpha)}{(m+\alpha)^s}$$

converges absolutely for $\sigma > \frac{1}{2}$. Consider the probability measure

$$P_{T,n,\sigma,w}(A) = \frac{1}{u} \int_{T_0}^{T} w(t) | \{ t : \zeta_n(t+\sigma+i, t, \alpha, a) \in A \} | dt, \quad A \in B(\mathbb{C}).$$

**Lemma 2.** Suppose that $\alpha$ is transcendental and $\sigma > \frac{1}{2}$. Then, on $(\mathbb{C}, B(\mathbb{C}))$, there exists a probability measure $P_{n,\sigma}$ such that the measure $P_{T,n,\sigma,w}$ converges weakly to $P_{n,\sigma}$ as $T \to \infty$.

**Proof.** Define the function $h_{n,\sigma} : \Omega \to \mathbb{C}$ by the formula

$$h_{n,\sigma}(\omega) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) v_n(m, \alpha)}{(m+\alpha)^{\sigma}}, \quad \omega \in \Omega.$$
The absolute convergence of the series \(4\) implies the continuity of the function \(h_{n,\sigma}\). Since
\[
h_{n,\sigma}((m+\alpha)^{-it}; m \in \mathbb{N}_0) = \zeta_n(\sigma + it, \alpha; a),
\]
hence, using Theorem 5.1 from [Liet. matem. rink. Proc. LMS, Ser. A, 53] and Lemma 2 we obtain that the measure \(P_{T,n,\sigma,w}\) converges weakly to \(mHh_{n,\sigma}^{-1}\) as \(T \to \infty\).

For the proof of Theorem 1, it remains to pass from the function \(\zeta_n(s, \alpha; a)\) to \(\zeta(s, \alpha; a)\). For this, the following statement will be applied.

**Lemma 3.** Suppose that \(\sigma > \frac{1}{2}\), and the condition (1) holds. Then
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t) |\zeta(\sigma + it, \alpha; a) - \zeta_n(\sigma + it, \alpha; a)| \, dt = 0.
\]

**Proof.** The function \(\zeta_n(s, \alpha; a)\) can be written in the form
\[
\zeta_n(s, \alpha; a) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \zeta(s + z, \alpha; a) l_n(z, \alpha) \frac{dz}{z},
\]
where
\[
l_n(z, \alpha) = \frac{z}{\sigma_1} \Gamma\left(\frac{z}{\sigma_1}\right)(n + \alpha)^z,
\]
and \(\Gamma(s)\) denotes the Euler gamma function. From this, using the residue theorem, we derive that
\[
\zeta_n(s, \alpha; a) = \frac{1}{2\pi i} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \zeta(s, \alpha; a) l_n(z, \alpha) \frac{dz}{z} + \zeta(s, \alpha; a) + \frac{l_n(1-s, \alpha)}{1-s},
\]
where \(\sigma_2 > \sigma_1\), and \(\sigma_2 < \sigma\). Therefore, as \(T \to \infty\),
\[
\frac{1}{U} \int_{T_0}^{T} w(t) |\zeta(\sigma + it, \alpha; a) - \zeta_n(\sigma + it, \alpha; a)| \, dt
\ll \int_{-\infty}^{\infty} |l_n(\sigma_2 - \sigma + iv, \alpha)| \left( \int_{T_0+\nu}^{T+\nu} w(t-v) |\zeta(\sigma, \alpha; a)| \, dt \right) \, dv + O(e^{-cT}).
\]

In view of (2), we find that
\[
\frac{1}{U} \int_{T_0+\nu}^{T+\nu} w(t-v) |\zeta(\sigma + it, \alpha; a)| \, dt
\ll \frac{1}{U} \left( \int_{T_0+\nu}^{T+\nu} w(t-v) \, dt \right)^{\frac{1}{2}} \left( \int_{T_0+\nu}^{T+\nu} w(t-v) |\zeta(\sigma + i \cdot t, \alpha; a)|^2 \, dt \right)^{\frac{1}{2}}
\ll (1 + |v|).
\]

Thus, by (5),
\[
\limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t) |\zeta(\sigma + it, \alpha; a) - \zeta_n(\sigma + it, \alpha; a)| \, dt
\ll \int_{-\infty}^{\infty} |l_n(\sigma_2 - \sigma + iv, \alpha)| (1 + |v|) \, dv.
\]

Liet. matem. rink. Proc. LMS, Ser. A, 53, 2012, 60–65.
Since $\sigma_2 - \sigma < 0$, the definition of $l_n(z, \alpha)$ shows that
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} |l_n(\sigma_2 - \sigma + iv, \alpha)| (1 + |v|) \, dv = 0,
\]
and the lemma follows from (6).

2 Proof Theorem 1

Now we a ready to prove Theorem 1. First we observe that family of probability measures $\{P_{n, \sigma}; n \in \mathbb{N}\}$, where $P_{n, \sigma}$ is the limit measure in Lemma 2, is tight. Really, for arbitrary $M > 0$,
\[
\frac{1}{U} \int_{T_0}^{T} w(t)^I(t; |\zeta_n(\sigma + it, \alpha; a)| > M) \, dt \ll \frac{1}{MU} \int_{T_0}^{T} w(t)|\zeta_n(\sigma + it, \alpha; a)| \, dt. \tag{7}
\]
Moreover, by (2) and Lemma 3,
\[
\sup_{n \in \mathbb{N}_0} \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t)|\zeta_n(\sigma + it, \alpha; a)| \, dt \\
\ll 1 + \limsup_{T \to \infty} \frac{1}{U} \left( \int_{T_0}^{T} w(t) \, dt \int_{T_0}^{T} w(t)|\zeta(\sigma + it, \alpha; a)| \, dt \right)^\frac{1}{2} \ll R < \infty. \tag{8}
\]

Now let $M = M_\epsilon = Re^{-1}$, where $\epsilon > 0$ is arbitrary number. Then (7), (8) and Theorem 2.1 of [1] give, for all $n \in \mathbb{N}_0$,
\[
P_{n, \sigma}(\{s \in \mathbb{C}; |s| > M_\epsilon\}) \leq \liminf_{T \to \infty} P_{T, n, \sigma, w}(\{s \in \mathbb{C}; |s| > M_\epsilon\}) \\
= \liminf_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t)^I(t; |\zeta_n(\sigma + it, \alpha; a)| > M_\epsilon) \, dt \\
\leq \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t)^I(t; |\zeta_n(\sigma + it, \alpha; a)| > M_\epsilon) \, dt \leq \epsilon. \tag{9}
\]

Define $K_\epsilon = \{s \in \mathbb{C}; |s| \leq M_\epsilon\}$. Then the set $K_\epsilon$ is compact, and, in view of (9), for all $n \in \mathbb{N}_0$,
\[
P_{n, \sigma}(K_\epsilon) \geq 1 - \epsilon.
\]
The tightness of $\{P_{n, \sigma}; n \in \mathbb{N}\}$ implies its relative compactness. Therefore, there exists a sequence $\{P_{nk, \sigma}\} \subset \{P_{n, \sigma}\}$ such that $P_{nk, \sigma}$ converges weakly to a certain probability measure $P_\sigma$ on $(\mathbb{C}, B(\mathbb{C}))$ as $k \to \infty$. Let $\theta = \theta_T$ be a random variable on a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ having the distribution
\[
\mathbb{P}(\theta \in A) = \frac{1}{U} \int_{T_0}^{T} w(t)^A(t) \, dt, \quad A \in B(\mathbb{C}).
\]
Suppose that $X_n(\sigma)$ is a complex-valued random variable with the distribution $P_{n, \sigma}$. Then the above remark implies the relation
\[
X_{nk}(\sigma) \overset{D}{\longrightarrow} P_\sigma, \tag{10}
\]
A weighted limit theorem for periodic Hurwitz zeta-function

where \( \overset{D}{\rightarrow} \) denotes the convergence in distribution. Define

\[ X_{T,n}(\sigma) = \zeta_n(\sigma + i\theta_T, \alpha; a). \]

Then, by Lemma 2,

\[ X_{T,n}(\sigma) \overset{D}{\rightarrow} X_n(\sigma). \quad (11) \]

Putting \( X_T(\sigma) = \zeta(\sigma + i\theta_T, \alpha; a) \), we have from Lemma 3 that, for every \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} P\left( \left| X_T(\sigma) - X_{T,n}(\sigma) \right| \geq \varepsilon \right) \\
\leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(t) \left| \zeta(\sigma + it, \alpha; a) - \zeta_n(\sigma + it, \alpha; a) \right| dt = 0.
\]

This, (10), (11), and Theorem 4.2 of [1] now show that

\[ X_T(\sigma) \overset{D}{\rightarrow} P_\sigma. \]

References

[1] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York, 1968.

[2] A. Rimkevičienė. Limit theorems for periodic Hurwitz zeta-function. *Šiauliai Math. Semin.*, 5(13):55–69, 2010.