CONVEX DOMAINS OF FINSLER AND RIEMANNIAN MANIFOLDS

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Abstract. A detailed study of the notions of convexity for a hypersurface in a Finsler manifold is carried out. In particular, the infinitesimal and local notions of convexity are shown to be equivalent. Our approach differs from Bishop’s one in his classical result [8] for the Riemannian case. Ours not only can be extended to the Finsler setting but it also reduces the typical requirements of differentiability for the metric and it yields consequences on the multiplicity of connecting geodesics in the convex domain defined by the hypersurface.

1. Introduction

Convexity is a central concept in different branches of Mathematics and, thus, it admits different definitions depending on the used viewpoint. In Riemannian Geometry, there are two natural definitions for the convexity of a smooth hypersurface \( \partial D \) which bounds a domain \( D \), i.e. a connected open subset: \( \partial D \) is infinitesimally convex if its second fundamental form, with respect to the inner normal, is positive semi-definite at any \( p \in \partial D \) and locally convex if the exponential of the tangent space \( T_p \partial D \), restricted to some neighborhood of 0, does not intersect \( D \). Infinitesimally convex hypersurfaces naturally arise from regular values of smooth convex functions. On the other hand, the domain \( D \) is called convex when each two \( x, y \in D \) can be joined by a non-necessarily unique geodesic which minimizes the distance in \( D \). When the closure \( \overline{D} \) is complete, the convexity of \( D \) must be equivalent to the convexity of its boundary; in order to prove this claim, an intermediate notion such as geometric convexity for \( \partial D \) becomes useful (see the next section for exhaustive definitions and details).

The consistency of the approach relies then on the equivalence between the infinitesimal and local notions of convexity. The fact that the former implies the latter is not as trivial as it sounds: do Carmo and Warner [14] prove it when the ambient Riemannian manifold \((M, g_R)\) has constant curvature and Bishop [8] in the general situation. Bishop’s proof reduces the problem to dimension 2 and, to this end, a family of surfaces which sweep out a neighborhood of \( p \) is constructed. As a consequence, a uniform bound for some focal distances in the family is required, and

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smoothability $C^4$ is imposed on the metric $g_R$. This requirement on smoothability seems strong and Bishop himself suggests that it may be non-optimal. Notice also that the interplay between bounds on curvature, convexity of domains or functions and smoothability, becomes a classical topic in Differential Geometry [1, 2, 13, 20, 30].

Most elements of this Riemannian setting can be transplanted to the Finslerian one. But, as pointed out by Borisenko and Olin [9, Remark 1], an important difficulty now appears: Bishop’s technique only works for Berwald spaces, where the Chern connection becomes a linear connection on the tangent bundle (cf. [3, §10]). In the general case, the relation between the convexity of the domain and its boundary is not clear and one is lead to some more strict notions of convexity as a technical assumption (cf. for example [1, 30]). The situation is even worse for non-reversible Finsler metrics, as there is no a priori a clear equivalent hypothesis to the completeness of $\mathcal{D}$.

The aim of the present article is to give a definitive answer to these questions, showing the natural equivalence of the different convexities.

As a preliminary step, in Section 2 the different notions of convexity are reviewed, explaining their extensions to the Finslerian case and checking that, in the non-reversible case, it is equivalent to assume any notion of convexity for the original Finsler metric $F$ and its reversed metric $\tilde{F}$.

In Section 3, the equivalence between infinitesimal and local convexities is proved. Indeed the following result holds:

**Theorem 1.1.** [Finslerian Bishop’s Theorem] Let $M$ be a smooth manifold, endowed with a Finsler metric whose fundamental tensor $(4)$ is $C^1_{1,1}$ (i.e. its components are $C^1$ in $TM \setminus 0$ with locally Lipschitz derivatives) and let $N \subset (M, F)$ be a $C^2_{1,1}$ embedded hypersurface (i.e., $N$ is locally regarded as the inverse image of some $C^2_{1,1}$ regular function).

Let $p \in N$ and choose a transverse direction as inner pointing in some neighborhood $U$ of $p$. If $N$ is infinitesimally convex in $U \cap N$, then $N$ is locally convex at $p$ (and, thus, on all $U \cap N$).

Notice that this extension of Bishop’s theorem to the Finsler case is also useful in the Riemannian setting, as in this case only $C^{1,1}$ differentiability is required for the metric. In regard to the hypersurface $N$, it is not clear if more specific techniques on regularity may reduce $C^2_{1,1}$ in $C^2$. However, as pointed out by Li and Nirenberg [22], the hypothesis $C^2_{1,1}$ is the natural regularity assumption when the distance function to a boundary is considered (see also [12, Sect. 4, 5]).

Theorem 1.1, combined with the straightforward implications discussed in Section 2, yields the full equivalences among the notions of convexity for the boundary of a domain, namely:

**Corollary 1.2.** Let $D$ be a $C^2_{1,1}$ domain (i.e. an open connected subset of $M$ whose boundary is locally defined as a level set of a $C^2_{1,1}$ function) in a manifold endowed with a Finsler metric whose fundamental tensor is $C^1_{1,1}$ on $TM \setminus 0$. It is equivalent for $\partial D$ to be: (a) infinitesimally convex, (b) geometrically convex and (c) locally convex.

For a domain $D$ the correspondence of the equivalent notions of convexity for $\partial D$ and the convexity of $D$ is summarized in the following result proved in Section
4, which involves the balls for the symmetrized distance $d_s$ of the pseudo-distance associated to $F$:

**Theorem 1.3.** Let $D$ be a $C^{2,1}_{\text{loc}}$ domain of a smooth manifold $M$ endowed with a Finsler metric $F$ having $C^{1,1}_{\text{loc}}$ fundamental tensor and such that the intersection of the closed symmetrized balls $\overline{B}_s(p, r)$ with $\overline{D}$ is compact.

Then, $D$ is convex if and only if $\partial D$ is convex (in any one of the equivalent sense in Corollary 1.2). Moreover, in this case, any pair of points in $D$ can be joined by infinitely many connecting geodesics contained in $D$ and having diverging lengths if $D$ is not contractible.

Some remarks are in order. First, the compactness of the intersections $\overline{D} \cap \overline{B}_s(p, r)$, plays a role analogous to that of the completeness of $\overline{D}$ in the Riemannian setting. This becomes natural after [12], where a correspondence between some elements in Lorentzian and Finslerian geometries is exploited. For example, the compactness of the symmetrized balls $\overline{B}_s(p, r)$ (which is a condition weaker than forward or backward completeness of $F$) yields the existence of a minimizing geodesic between any two points in a Finsler manifold. Our approach to the problem of the convexity of a domain uses variational methods which directly yield multiplicity results by standard arguments. The proof is based on a penalization technique which goes back to Gordon [19]. Roughly speaking, the lack of completeness impedes the energy functional (see (5) below) to satisfy the Palais-Smale condition, so that many classical results in critical point theory are not applicable. Thus, the functional is modified by adding a penalizing term which becomes infinite close to the boundary and a family of penalized functionals $(J_\varepsilon)_{\varepsilon > 0}$ is considered.

However, there are interesting differences with respect to the Riemannian setting studied in [19]. In fact, the critical points of such functionals (which are approximating solutions) are $C^1$ curves having supports in $D$. They are continuously twice differentiable only on the open subset of the domain of parametrization where their velocity vector field is not zero. Anyway, due to the particular conservation law they satisfy (25), the set where their derivative vanishes is not negligible and the passage to the limit ($\varepsilon \to 0$) in the penalization technique is much more delicate than in the Riemannian case. Another difficulty is connected with the fact that a result analogous to the Nash isometric embedding theorem (that in several papers about this topic, allows one to avoid many technicalities, see for example [19, 7, 17, 16]) does not hold in general for a Finsler manifold, [28, Theorem 1.1].

Theorem 1.3 extends to domains of Finsler manifolds a result about Finsler metrics in $\mathbb{R}^N$ (see [18, Theorem 1, p. 250]), stating the existence of a geodesic of length equal to the Finslerian distance between any two points in $\mathbb{R}^N$. Such a result is an emblematic example of the application of the direct method in the calculus of variations (cf. [18, Preface]). Namely it comes from a minimization argument, based on the lower-semicontinuity with respect to the $C^0$-topology of the energy integral of a Finsler metric (see [18, Lemma 5, p. 259]). The role of the assumption about the compactness of the sets $\overline{B}_s(p, r) \cap \overline{D}$ in Theorem 1.3 is taken in [18] by the existence of a constant $a > 1$ such that

$$\frac{1}{a} |y| \leq F(x, y) \leq a |y|$$

for any $x \in \mathbb{R}^N, y \in \mathbb{R}^N$.

The result in [18] can be extended to a domain $D$ in $\mathbb{R}^N$ but, as the authors themselves observe [18, Remark 3, p. 254], it is not easy to give and to check convexity
conditions on $\partial D$ ensuring that a minimizing geodesic in $D$ exists. Our result on the equivalence between different notions of convexity for $\partial D$ and Theorem 1.3 aim also to fill that gap between Riemannian and Finsler geometry.

2. Preliminaries

Hopf-Rinow theorem states that metric (or equivalently geodesic) completeness of a Riemannian manifold $(M, g_R)$ is a sufficient condition for convexity.

When a domain $D$ of a Riemannian manifold $M$ is considered, suitable convexity assumptions are needed in order to control the lack of completeness. As pointed out by Gordon [19], on Riemannian manifolds this problem is interesting also because of its relation, via the Jacobi metric, to the problem of connecting two points by means of a trajectory of fixed energy for a Lagrangian system. In the case of a Finsler manifold the study of convexity of a domain is interesting also in connection with the existence of lightlike and timelike geodesics connecting a point with a line in an open region of a stationary spacetime (see [11]).

At first we review the different notions of convexity for the boundary of a smooth domain $D$ of a Riemannian manifold (see [4] and [27] where also non differentiable boundaries are considered).

We say that the boundary $\partial D$ of $D$ at a point $x \in \partial D$ is

- **infinitesimally convex** if the second fundamental form $\sigma_x$, with respect to the interior normal, is positive semi-definite;
- **locally convex** if there exists a neighborhood $U \subset M$ of $x$ such that

  $\exp_x (T_x \partial D) \cap (U \cap D) = \emptyset$.  \hspace{1cm} (1)

In order to apply variational methods to the study of geodesic connectedness, a characterization of the infinitesimal convexity is useful. Indeed, note that for each $x \in \partial D$ a neighborhood $U \subset M$ of $x$ and a differentiable function (with the same degree of differentiability of $\partial D$) $\phi : U \to \mathbb{R}$ exist such that

$$
\begin{cases}
\phi^{-1}(0) = U \cap \partial D \\
\phi > 0 & \text{on } U \cap D \\
d\phi(x) \neq 0 & \text{for every } x \in U \cap \partial D
\end{cases}
$$

and then the following holds:

- $\partial D$ is infinitesimally convex at $x \in \partial D$ if and only if for one (and then for all) function $\phi$ satisfying (2):

  $$H_\phi(x)[y, y] \leq 0 \quad \text{for every } y \in T_x \partial D.$$ 

Easily, the local convexity at $x$ implies the infinitesimal one. For the converse, one has to assume that the infinitesimal convexity holds on a neighborhood of $x$; in this case, Bishop [8] proved that the converse holds if the metric is $C^4$. Notice that $\partial D$ is assumed to be an embedded manifold in $M$, so that the function $\phi$ in (2) can be found as defined on all $M$. Thus, the following global definitions (equivalent at least in the $C^4$ case) can be given:

- $\partial D$ is **infinitesimally convex** if so it is at any point;
• \(\partial D\) is variationally convex if for one, and then for all, function \(\phi\) on \(M\) such that

\[
\begin{align*}
\phi^{-1}(0) & = \partial D \\
\phi & > 0 \quad \text{on } D \\
d\phi(x) & \neq 0 \quad \text{for every } x \in \partial D
\end{align*}
\]

it holds

\[
H_{\phi}(x)[y,y] \leq 0 \quad \text{for every } x \in \partial D, \ y \in T_x\partial D.
\]

• \(\partial D\) is locally convex if so it is at any point.

There is also another definition of convexity which comes out to be equivalent to the previous ones.

• \(\partial D\) is geometrically convex if for any \(p,q \in D\) the range of any geodesic \(\gamma: [a,b] \to \overline{D}\) such that \(\gamma(a) = p\) and \(\gamma(b) = q\) satisfies

\[
\gamma([a,b]) \subset D.
\]

This definition is intermediate between infinitesimal/variational convexity and local convexity. In fact, under geometric convexity, any geodesic \(\rho: [-\epsilon,\epsilon] \to M\) with initial velocity in \(T_x\partial D\) will remain in \(M \setminus D\) reducing eventually \(\epsilon > 0\). This implies infinitesimal convexity at \(x\) but, in order to obtain local convexity, one must ensure that the same \(\epsilon\) can be chosen in all the directions. Bishop’s result ensures the equivalence in the \(C^4\) case (see also [16] for a different technique in one of the implications). As we will see in Proposition 3.2 and Corollary 3.4, the equivalences hold even for \(C^{1,1}\) Riemannian metrics, as these equivalences hold in the general Finslerian case whenever the fundamental tensor has such a level of regularity.

There are also different ways to prove that, for a complete \(M\) (or equivalently \(\overline{D}\)), the boundary \(\partial D\) is convex if and only if the domain \(D\) is convex (see the review [27]).

We deal here with convexity of a domain \(D\) of a Finsler manifold \(M\), so let us recall some basic notions in Finsler Geometry.

A Finsler structure on a smooth finite dimensional manifold \(M\) is a function \(F: TM \to [0, +\infty)\) which is continuous on \(TM\), smooth on \(TM \setminus 0\), vanishing only on the zero section, fiberwise positively homogeneous of degree one, i.e. \(F(x,\lambda y) = \lambda F(x,y)\), for all \(x \in M\), \(y \in T_xM\) and \(\lambda > 0\), and which has fiberwise strictly convex square i.e. the matrix

\[
g(x,y) = \begin{bmatrix} 1 \frac{\partial^2(F^2)}{2 \partial y^i \partial y^j}(x,y) \end{bmatrix}
\]

is positively defined for any \((x,y) \in TM \setminus 0\). Typically, the word “smooth” means \(C^\infty\) and one can maintain this here for the manifold \(M\). Nevertheless, in order to obtain a sharp result on differentiability, \(F\) smooth will mean that the fundamental tensor \(g\) in (4) is \(C^{1,1}_{\text{loc}}\). Obviously, this will hold for the Finsler metric associated to a \(C^{1,1}\) Riemannian metric as well as when \(F\) is a \(C^{3,1}_{\text{loc}}\) function on \(TM \setminus 0\).

The length of a piecewise smooth curve \(\gamma: [a,b] \to M\) with respect to the Finsler structure \(F\) is defined by

\[
\ell_F(\gamma) = \int_a^b F(\gamma, \dot{\gamma}) \, ds
\]
hence the distance between two arbitrary points \( p, q \in M \) is given by
\[
d(p, q) = \inf_{\gamma \in \mathcal{P}(p, q; M)} \ell_F(\gamma),
\]
where \( \mathcal{P}(p, q; M) \) is the set of all piecewise smooth curves \( \gamma: [a, b] \to M \) with \( \gamma(a) = p \) and \( \gamma(b) = q \). The distance function is non-negative and satisfies the triangle inequality, but it is not symmetric since \( F \) is only positively homogeneous of degree one in \( y \). So for any point \( p \in M \) and for all \( r > 0 \) we can define two different balls centered at \( p \) and having radius \( r \): the forward ball \( B^+(p, r) = \{ q \in M \mid d(p, q) < r \} \) and the backward one \( B^-(p, r) = \{ q \in M \mid d(q, p) < r \} \). Analogously, it makes sense to give two different notions of Cauchy sequences and completeness: a sequence \( (x_n)_n \subset M \) is a forward (resp. backward) Cauchy sequence if for all \( \varepsilon > 0 \) there exists an index \( \nu \in \mathbb{N} \) such that, for all \( m \geq n \geq \nu \), it is \( d(x_n, x_m) < \varepsilon \) (resp. \( d(x_m, x_n) < \varepsilon \)); consistently a Finsler manifold is forward complete (resp. backward complete) if every forward (resp. backward) Cauchy sequence converges. It is well known that both the topology induced by the forward balls and that induced by the backward ones agree with the underlying manifold topology. Moreover suitable versions of the Hopf-Rinow theorem hold (cf. [3, Theorem 6.6.1]) stating, in particular, the equivalence of forward (resp. backward) completeness and the compactness of closed and forward (resp. backward) bounded subsets of \( M \). The validity of one of these properties implies the existence of a geodesic connecting any pair of points in \( M \) and minimizing the Finslerian distance, i.e. the convexity of \( M \). Geodesics can be defined in different ways using different connections defined on the pulled-back bundle \( \pi^*TM, \pi: TM \to M \), (cf. [3, Chapter 2]) or as critical points of the length functional (cf. [3, Proposition 5.1.1] for details); furthermore (cf. for example [11, Proposition 2.3]) a smooth curve \( \gamma \) on \( [a, b] \) is a geodesic parameterized with constant speed (i.e \( s \mapsto F(\gamma(s), \dot{\gamma}(s)) = \text{const.} \)) if and only if it is a critical point of the energy functional
\[
J(\gamma) = \frac{1}{2} \int_a^b F^2(\gamma, \dot{\gamma}) \, ds
\]
defined on the manifold of \( H^1 \) curves having fixed endpoints. Thus in local coordinates \( \gamma \) satisfies the equations
\[
\ddot{\gamma}^i(s) + g^{ij}(\gamma, \dot{\gamma}) \left( \frac{1}{2} \partial_{y^j x^k} F^2(\gamma, \dot{\gamma}) \dot{\gamma}^k - \frac{1}{2} \partial_{x^j} F^2(\gamma, \dot{\gamma}) \right) = 0.
\]
Here \( g^{ij} \) are the components of the inverse matrix of fundamental tensor \( g \), \( \partial_{x^j}, \partial_{y^j x^k} \) are the symbols of the partial derivatives with respect to the variables \( x^i, y^j \) and we adopt the usual Einstein’s summation convention. Using the structural equations defining the Chern connection (see [29, Theorem 5.2.2]) it can be proved that the functions \( G^i(x, y) = g^{ij}(x, y) \left( \frac{1}{2} \partial_{y^j x^k} F^2(x, y) y^k - \frac{1}{2} \partial_{x^j} F^2(x, y) \right), (x, y) \in TM \setminus 0 \) are equal to \( \Gamma_{jk}^i(x, y)y^jy^k \), where \( \Gamma_{jk}^i \) are the components of the Chern connection.\(^1\) Therefore geodesics equations become
\[
\ddot{\gamma}^i(s) + \Gamma_{jk}^i(\gamma(s), \dot{\gamma}(s)) \dot{\gamma}^j(s) \dot{\gamma}^k(s) = 0.
\]
Remark 2.1. Consider the symmetrized distance on \( M \)
\[
d_s(p, q) = \frac{1}{2} \left( d(p, q) + d(q, p) \right).
\]
\(^1\)To see that use Eqs. (5.2), (5.7) and the formula after equation (5.31) in [29], besides the fact that the functions \( G^i \) are positively homogeneous of degree 2 in the \( y \) variable.
and denote by $B_s$ the balls associated to $d_s$. It results that if the Heine-Borel property holds, i.e. for all $x \in M$, $r > 0$, the closed balls $\mathcal{B}_s(x, r)$ are compact (or equivalently the subsets $\mathcal{B}_s^+(x, r_1) \cap \mathcal{B}_s^-(y, r_2)$ are compact for any $x, y \in M, r_1, r_2 > 0$), then the metric space $(M, d_s)$ is complete (cf. [12, Proposition 2.2]).

This condition implies convexity (cf. [12, Theorem 5.2]). It is worth to stress that the Hopf-Rinow theorem in general does not hold for the metric $d_s$. For instance, Example 2.3 in [12] exhibits a non compact, $d_s$-bounded Randers space whose symmetrized distance $d_s$ is complete.

Now, let $N$ be a hypersurface of $M$ and choose a (unit) normal vector $n$ at some $x \in N$ (namely, the hyperplane parallel to $T_x N$ through $n$ is tangent to the $F$-unit ball at $T_x M$). The normal curvature $\Lambda_n$ at a point $x \in N$ in a direction $y \in T_x N$ is defined by

$$\Lambda_n(y) = g(x, n)[\nabla_{\dot{\gamma}(s)} \dot{\gamma}(s) |_{s=0}, n]$$

where $\gamma: [\epsilon, \epsilon] \to N$ is a geodesic for the Finsler metric induced by $F$ on $N$ such that $\gamma(0) = x$, $\dot{\gamma}(0) = y$, $\nabla_{\dot{\gamma}} \dot{\gamma}$ is the covariant derivative of $\dot{\gamma}$ along $\gamma$ in $(M, F)$ (see [29, §5.3]). Observe that the definition of the normal curvature $\Lambda_n(y)$ in (8) differs from the one in [29, §14.2] for a minus sign.

When we deal with domains $D$ of Finsler manifolds, we say (cf. [29, Proposition 14.2.1]) that the boundary $\partial D$ of $D$ at a point $x \in \partial D$ is

- *infinitesimally convex* if the normal curvature with respect to the normal vector pointing into $D$ is non-negative or equivalently if for a function $\phi$ as in (2):

$$H_\phi(x, y)[y, y] \leq 0 \quad \text{for every } y \in T_x \partial D,$$

where $H_\phi$ is the Finslerian Hessian of $\phi$ (see [29, §14.1]) defined, for each $(x, y) \in TM \setminus 0$, as $H_\phi(x, y)[y, y] = \frac{\partial^2 \phi}{\partial y^i \partial y^j}(x)y^iy^j$, being $\gamma$ the geodesic of $(M, F)$ (parameterized with constant speed) such that $\gamma(0) = x$ and $\dot{\gamma}(0) = y$. Taking into account the equation (7) satisfied by constant speed geodesics, we get in local coordinates

$$H_\phi(x, y)[y, y] = \frac{\partial^2 \phi}{\partial x^i \partial x^j}(x)y^iy^j - \frac{\partial \phi}{\partial x^k}(x)\Gamma^k_{ij}(x, y)y^iy^j,$$

being $\Gamma^k_{ij}(x, y)$ the components of the Chern connection of $(M, F)$.

In general, Finsler metrics are *non-reversible*, which means $F(x, -y) \neq F(x, y)$ on $TM$, thus we can define the *reversed* Finsler metric $\bar{F}$ as $\bar{F}(x, y) = F(x, -y)$ for each $(x, y) \in TM$.

If $F$ is non-reversible and $\gamma$ is a geodesic on $[0, 1]$, the reversed curve $\dot{\gamma}(s) = \gamma(1 - s)$ in general is not a geodesic of $F$, but it is a geodesic for $\bar{F}$ (this can be easily seen by using the fact that geodesics are the curves that locally minimize the length functional and that $\ell_F(\dot{\gamma}) = \ell_{\bar{F}}(\dot{\gamma})$).

**Remark 2.2.** The notions of infinitesimal convexity for $F$ and $\bar{F}$ are equivalent: indeed, if $x \in D, y \in T_x \partial D$ and $\gamma$ is the geodesic for $(M, \bar{F})$ such that $\dot{\gamma}(0) = x, \dot{\gamma}(0) = -y$, then $\gamma(s) = \gamma(1 - s)$ is a geodesic in $(M, F)$ (and $\gamma(0) = x, \dot{\gamma}(0) = y$); since the components $\Gamma^i_{jk}$ of the Chern connection of $\bar{F}$ satisfy $\Gamma^i_{jk}(x, y) = -\Gamma^i_{jk}(x, y)$, we have $H_{\bar{F}}(x, -y)[-y, -y] = H_\phi(x, y)[y, y] \leq 0$. Moreover, associated to a non-reversible metric, there are two exponential maps: one, denoted by exp, associated to the geodesics of $F$, the other one, denoted by $\exp$ associated to $\bar{F}$, cf. [3, Chapter 6]). The definition of local convexity in [29, p. 216] is indeed
equivalent to require that (1) holds for both the exponential maps. Recall also that the definition of geometric convexity for the Riemannian setting can be extended trivially to the Finslerian one, and becomes equivalent for geodesics of $F$ and $\tilde{F}$.

3. Convexity of the Boundary

As in the Riemannian case, local convexity at one point of an hypersurface in a Finsler manifold implies infinitesimal convexity (cf. [29, Theorem 14.2.3]).

We pointed out that, except for Berwald spaces, Bishop’s theorem (which ensures that the converse is true in a Riemannian manifold, when infinitesimal convexity holds in a neighborhood of the point in the hypersurface) seems to be an open issue in the Finslerian setting. We see here that in fact Bishop’s theorem is true also for any Finsler manifold.

Throughout this section $D$ denotes a $C^{2,1}$ domain, that is, there exists a $C^{2,1}$ function $\phi$ which satisfies (3). Notice that then $\partial D$ is endowed with an intrinsic $C^{1,1}$ structure.

We start by giving two results which generalize the analogous ones in [16] where geometric convexity of a $C^3$ domain of a complete Riemannian manifold is studied. The differential inequality in the next lemma is less restrictive than the one in [16, Lemma 9].

**Lemma 3.1.** If $\psi \in C^2([0,b], \mathbb{R})$ is a non-negative function verifying

$$\begin{cases}
\ddot{\psi} \leq A(\psi + |\dot{\psi}|) \\
\psi(0) = 0, \dot{\psi}(0) = 0
\end{cases}$$

for some $A > 0$, then $\psi \equiv 0$ on $[0,b]$.

**Proof.** By contradiction, assume that a non-trivial solution of (11) exists. If $\dot{\psi} \geq 0$ on the interval $[0,b]$, then integrating on $[0,t]$, $0 < t \leq b$, both hand sides in (11) we get

$$\dot{\psi}(t) \leq A \left( \int_0^t \psi(s) \, ds + \psi(t) \right).$$

Integrating again we obtain

$$\psi(t) \leq (Ab + 1) \int_0^t \psi(s) \, ds$$

and from Gronwall’s inequality we get that $\psi \equiv 0$ in $[0,b]$. Hence we can assume that a point $\bar{t} \in [0,b]$, say $\bar{t} = 0$, exists such that $\dot{\psi}$ has indefinite sign in a right neighborhood of 0. So a sequence $(t_m)_m$ converging to zero exists such that each $t_m$ is a maximum point of $\psi$:

$$\psi(t_m) \to 0, \quad \dot{\psi}(t_m) = 0, \quad \ddot{\psi}(t_m) \leq 0.$$

Now, let $\psi_m : [0,b] \to \mathbb{R}$ be the unique solution of

$$\begin{cases}
\ddot{\phi} = A(\phi + \dot{\phi}) \\
\phi(t_m) = \psi(t_m), \dot{\phi}(t_m) = 0.
\end{cases}$$

We are going to prove that

$$\psi(t) < \psi_m(t), \quad \text{for all } t \in [t_m, b].$$

(13)
Notice that easy computations give
\[ \dot{\psi}_m(t) = C_m e^{\lambda_- t} + C_+ e^{\lambda_+ t} \]
where \( \lambda_- < 0 < \lambda_+ \) are the two roots of \( \lambda^2 - A \lambda - A = 0 \) and \( C_m, C_+ \) are strictly positive constants obtained imposing the initial conditions in (12). Thus \( \dot{\psi}_m(t) > 0 \) for any \( t \) and, as \( \dot{\psi}_m(t_m) = 0 \), \( \dot{\psi}_m \) is strictly increasing and positive on \( [t_m, b] \).

Since \( \psi_m(t_m) = \dot{\psi}(t_m) \), \( \dot{\psi}(t_m) = 0 = \dot{\psi}(t_m) \), \( \dot{\psi}_m(t_m) > 0 \geq \dot{\psi}(t_m) \), inequality (13) is true in a right neighborhood of \( t_m \). To prove that it holds on the whole interval \( [t_m, b] \), assume by contradiction that there exists a point \( \tilde{t} \in [t_m, b] \) such that \( \dot{\psi}_m(\tilde{t}) = \dot{\psi}(\tilde{t}) \). Let \( c > t_m \) be the minimum of the set
\[ A = \{ t \in [t_m, b] \mid \dot{\psi}(t) = \dot{\psi}_m(t) \}. \]
Hence \( \dot{\psi}(t) < \dot{\psi}_m(t) \) for \( t \in ]t_m, c[ \) and \( \dot{\psi}_m(c) \leq \dot{\psi}(c) \). As \( 0 = \dot{\psi}(t_m) < \dot{\psi}_m(c) \leq \dot{\psi}(c) \), we can consider \( c_0 = \max \{ t \in [t_m, c[ \mid \dot{\psi}(t) = 0 \} \). It is \( \dot{\psi} > 0 \) in \( ]c_0, c[ \) and \( c_1 \in ]c_0, c[ \) exists such that \( \dot{\psi}(c_1) = \dot{\psi}_m(c_1) \) and \( \dot{\psi}(\tilde{t}) < \dot{\psi}_m(t) \) if \( t \in ]c_0, c_1[ \). Thus, for any \( t \in ]c_0, c_1[ \), the following inequalities holds
\[ \dot{\psi}(\tilde{t}) \leq A(\psi(t) + \dot{\psi}(t)) < A(\dot{\psi}_m(t) + \dot{\psi}_m(t)) = \dot{\psi}_m(t) \]
and we get a contradiction observing that
\[ \dot{\psi}(c_1) = \int_{c_0}^{c_1} \dot{\psi}(t) \, dt < \int_{c_0}^{c_1} \dot{\psi}_m(t) \, dt < \dot{\psi}_m(c_1) = \dot{\psi}(c_1). \]
Inequality (13) allows us to complete the proof. Indeed, as \( \psi(t_m) \rightarrow 0 \), by smooth dependence of the solutions of (12) by initial conditions, the sequence \( (\psi_m(t_m))_m \) goes to 0, for each \( t \in ]0, b[ \).

The following crucial proposition holds. This will turn out to be a strengthening of geometric convexity, as it forbids the possibility of tangency to the boundary for geodesics in \( \overline{D} \) but not lying in \( \partial D \).

**Proposition 3.2.** Assume that \( \partial D \) is infinitesimally convex in a neighborhood \( U \) of \( p \in \partial D \). Let \( \gamma : [0, b] \rightarrow U \) be a geodesic which satisfies \( \gamma(0) = p, \gamma([0, b[) \subset U \cap D \). Then, \( \gamma'(0) \notin T_p \partial D \).

**Proof.** Assume by contradiction that \( \gamma'(0) \in T_p \partial D \). We are going to prove that \( \sigma > 0 \) exists such that \( \gamma([0, \sigma[) \subset \partial D \), getting a contradiction.

Without loss of generality, we also assume that \( \gamma \) is parameterized with unit Finslerian speed, i.e. \( F(\gamma(s), \dot{\gamma}(s)) = 1 \) for all \( s \in [0, b] \). Take a chart \( (V, (x^i)_{i=1,\ldots,n}) \) of \( M \) centered at \( p \), with \( V \subset U \) and adapted to \( D \) (i.e. function \( \phi \) defining the boundary of \( D \) is given as a coordinate \( x^i \), say \( \phi = x^n \)). In what follows, \( | \cdot | \) denotes the Euclidean norm on \( \varphi(V) \subset \mathbb{R}^n \), \( \nabla^0 \) its associated gradient and, with an abuse of notation, the symbols which denote elements in \( M \) or \( TM \) remain unchanged for the induced ones by means of \( \varphi \) in \( \mathbb{R}^n \) or \( \mathbb{R}^{2n} \). In particular, \( \nabla^0 \varphi \equiv \partial_{x^n} \). Assuming also that the closure \( \overline{V} \) is compact, \( a > 0 \) exists such that:
\[ \frac{1}{a} |y| \leq F(x, y) \leq a|y|, \text{ for every } (x, y) \in TV. \]  
(14)

Now, define the natural projection map on \( x^n = 0 \). More precisely, let \( \eta \) be a local flow around \( p \), i.e. for some \( \epsilon > 0 \), \( \tilde{W} = \varphi^{-1}([0, \epsilon[) \subset V \):
\[ \eta : [-\epsilon, \epsilon]\times W \rightarrow V \subset M, \quad \eta(t, (x^1, \ldots, x^n)) = (x^1, \ldots, x^{n-1}, x^n - t). \]
Moreover, \( \phi(\eta(\phi(w), w)) = 0 \), and the projection \( \Pi : W \rightarrow \partial D \) is defined as:

\[
\Pi(w) = \eta(\phi(w), w).
\]

As \( \gamma(0) \in \Pi(W) \), \( \sigma > 0 \) exists such that \( \gamma(s) \in W \) for all \( s \in [-\sigma, \sigma] \). Consider the projected curve \( \gamma_\Pi : [-\sigma, \sigma] \rightarrow \partial D \) of \( \gamma \) on \( \partial D \) given by \( \gamma_\Pi(s) = \Pi(\gamma(s)) \). Since \( \dot{\gamma}_\Pi(s) \in T_{\gamma_\Pi(s)}\partial D \), by (9) we have

\[
H_\phi(\gamma_\Pi(s), \dot{\gamma}_\Pi(s))[\dot{\gamma}_\Pi(s)] \leq 0, \quad \text{for every } s \in [-\sigma, \sigma].
\]

Set \( \rho(s) = \phi(\gamma(s)) \), it follows \( \ddot{\rho}(s) = H_\phi(\gamma(s), \dot{\gamma}(s))[\dot{\gamma}(s), \dot{\gamma}(s)] \). Moreover

\[
\dot{\gamma}_\Pi(s) = d\Pi(\gamma(s))[\dot{\gamma}(s)] = \partial_D \eta(\rho(s), \gamma(s))\rho(s) + \partial_D \eta(\rho(s), \gamma(s))[\dot{\gamma}(s)].
\]

Using the local expression of the Hessian of \( \phi \) (see (10)), from (15), we get on \([0, \sigma]\):

\[
\ddot{\rho}(s) = (H_\phi)_{ij}(\gamma(s), \dot{\gamma}(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s)
\]

\[
\leq (H_\phi)_{ij}(\gamma(s), \dot{\gamma}(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s) - (H_\phi)_{ij}(\gamma_\Pi(s), \dot{\gamma}_\Pi(s))\dot{\gamma}^i_\Pi(s)\dot{\gamma}^j_\Pi(s)
\]

\[
= (H_\phi)_{ij}(\gamma(s), \dot{\gamma}(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s) - (H_\phi)_{ij}(\gamma_\Pi(s), \dot{\gamma}_\Pi(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s)
\]

\[
+ (H_\phi)_{ij}(\gamma(s), \dot{\gamma}_\Pi(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s) \leq \dot{\gamma}^i(s)\dot{\gamma}^j(s) - \dot{\gamma}^i_\Pi(s)\dot{\gamma}^j_\Pi(s).
\]

From (14), recalling that \( \gamma \) is parameterized with Finslerian unit speed, using the fact that the second derivatives of \( \phi \) are Lipschitz functions and the \( \Gamma^k_{ij}(x, y) \) are smooth on \( TW \setminus 0 \), we get

\[
\left[(H_\phi)_{ij}(\gamma(s), \dot{\gamma}(s)) - (H_\phi)_{ij}(\gamma_\Pi(s), \dot{\gamma}_\Pi(s))\right]\dot{\gamma}^i(s)\dot{\gamma}^j(s)
\]

\[
\leq \|H_\phi(\gamma(s), \dot{\gamma}(s)) - H_\phi(\gamma_\Pi(s), \dot{\gamma}_\Pi(s))\|a^2
\]

\[
\leq a_1(|\gamma(s) - \gamma_\Pi(s)| + |\dot{\gamma}(s) - \dot{\gamma}_\Pi(s)|),
\]

(18)

where \( \|\cdot\| \) denotes the norm on the space of bounded bilinear operator on \( \mathbb{R}^n \times \mathbb{R}^n \). As \( \eta \) is a \( C^{2,1} \) map, we obtain

\[
|\gamma(s) - \gamma_\Pi(s)| = |\eta(0, \gamma(s)) - \eta(\rho(s), \gamma(s))| \leq a_2\rho(s).
\]

(19)

Moreover, from (14)

\[
|(H_\phi)_{ij}(\gamma_\Pi(s), \dot{\gamma}_\Pi(s))\dot{\gamma}^i(s) + \dot{\gamma}^i_\Pi(s))\dot{\gamma}^j(s) - \dot{\gamma}^i_\Pi(s))\dot{\gamma}^j_\Pi(s)| \leq a_3|\dot{\gamma}(s) - \dot{\gamma}_\Pi(s)|
\]

(20)

and from (16)

\[
|\dot{\gamma}(s) - \dot{\gamma}_\Pi(s)|
\]

\[
= |\partial_\eta(0, \gamma(s))[\dot{\gamma}(s)] - \partial_\eta(\rho(s), \gamma(s))[\dot{\gamma}(s)] - \partial_\eta(\rho(s), \gamma(s))[\dot{\gamma}(s)]| \leq a_4\rho(s) + a_5|\dot{\rho}(s)|.
\]

(21)

Thus from (17), (18), (19), (20), (21), we easily get

\[
\ddot{\rho}(s) \leq a_6(\rho(s) + |\dot{\rho}(s)|)
\]

and, since \( \rho \geq 0 \), \( \rho(0) = \dot{\rho}(0) = 0 \), Lemma 3.1 implies that \( \rho = 0 \) on \([0, \sigma]\). \( \square \)

**Remark 3.3.** Since infinitesimal convexity with respect to the metric \( F \) is equivalent to that for the reversed metric \( \tilde{F} \) (see Remark 2.2), an analogous statement holds also for geodesics of \( \tilde{F} \) (or in other words for the reversed curves obtained from the geodesics of \((M, F)\)) assuming local infinitesimal convexity of \( \partial D \) with respect to \( F \).

**Corollary 3.4.** If \( \partial D \) is infinitesimally convex then \( \partial D \) is geometrically convex.
Proof. Otherwise, a geodesic \( \gamma : [0, 1] \to \overline{D} \) with \( \gamma(0), \gamma(1) \in D \) and \( c \in [0, 1] \) exist such that \( \gamma(c) \in \partial D \) and \( \gamma([c, 1]) \subset D \). Necessarily, \( \dot{\gamma}(c) \in T_p \partial D \), in contradiction with Proposition 3.2.

Using Proposition 3.2, the following lemma can be proved in a standard way (cf. [31, §4]). We observe that differently from [31], where strict convexity of the boundary is imposed, we cannot state that the closure of \( D \cap B^+(p, \delta) \) is completely convex.\(^2\) In fact infinitesimal convexity does not exclude the case in which the points \( p_1, p_2 \in \partial D \cap B^+(p, \delta) \) are connected by a geodesic included in \( \partial D \).

**Lemma 3.5.** Assume that \( \partial D \) is infinitesimally convex in a neighborhood \( U \) of \( p \in \partial D \). Then a small enough convex ball \( B^+(p, \delta) \) exists such that \( \partial D \cap B^+(p, \delta) \subset U \) and for each \( p_1, p_2 \in D \cap B^+(p, \delta) \) the (unique) geodesic in \( B^+(p, \delta) \) which connects \( p_1 \) with \( p_2 \) is included in \( D \).

**Proof.** Set \( C = D \cap B^+(p, \delta) \) and let \( A \) be the set of points \( (p_1, p_2) \in C \times C \) that can be connected by a geodesic having support in \( D \). We are going to prove that \( A = C \times C \). Since \( C \) is connected, it is enough to prove that \( A \) is non-empty, open and closed in \( C \times C \). Clearly each couple \( (p_1, p_2) \in C \times C \) can be connected by a constant geodesic, hence \( A \neq \emptyset \). If \( (p_1, p_2) \in A \) and \( \gamma \) is the unique geodesic connecting them and whose inner points are in \( D \cap B^+(p, \delta) \), by smooth dependence of geodesics by initial conditions, we can consider two small enough neighborhoods \( U_1 \) and \( U_2 \) of \( p_1 \) and, respectively, \( p_2 \) in \( C \), such that the geodesic in \( B^+(p, \delta) \) connecting \( p_1 \in U_1 \) to \( p_2 \in U_2 \) lies in a small neighborhood of \( \gamma \), hence its points are contained in \( D \). Now let \( (p_1, p_2) \in A \subset C \times C \) and consider a sequence \( (p_n^1, p_n^2) \subset A \) converging to \( (p_1, p_2) \). Up to reparametrizations, the geodesics \( \gamma_n \) connecting \( p_n^1 \) to \( p_n^2 \) converge uniformly to the geodesic \( \gamma \) contained in \( B^+(p, \delta) \) and connecting \( p_1 \) to \( p_2 \). Thus \( \gamma \) lies in \( \overline{B} \). As the points \( p_1 \) and \( p_2 \) are in \( D \), from Proposition 3.2, \( \gamma \) cannot be tangent to \( \partial D \) at any of its inner points.

**Proof of Theorem 1.1.** Assume by contradiction that \( N \) is not locally convex at \( p \in N \). Denoting by \( D \) the inner domain of \( N \), then a sequence of tangent vectors \( v_n \in T_p N \) exists such that \( v_n \to 0 \) and each \( p_n = \exp(v_n) \) or \( q_n = \text{exp}(v_n) \) belongs to \( D \cap U \). To fix ideas, let us assume that \( (p_n = \exp(v_n)) \) exists such that \( p_n \in D \cap U \). From Remark 3.3 the other case can be proved in the same manner.

Let \( B^+(p, \delta) \) be as in Lemma 3.5. With no loss of generality, we can assume that all \( p_n \) belong to \( B^+(p, \delta) \cap D \) hence, from Lemma 3.5, each unit speed geodesic \( \alpha_n : [0, b_n] \to U \) which connects \( p_n \) with the first point \( p_1 \) of the sequence is included in \( D \cap B^+(p, \delta) \). As \( B^+(p, \delta) \) is relatively compact the sequence of curves \( (\alpha_n) \) uniformly converges to a curve \( \alpha : [0, b] \to \overline{D} \cap \overline{B}^+(p, \delta) \). In fact, \( \alpha \) is also a geodesic which connects \( p \) with \( p_1 \) and, thus, it coincides with the original geodesic \( \gamma_1 \), up to a reparameterization. Since \( p_1 \in D \cap U \), \( \gamma_1 \) must definitively leave the boundary \( N \) of \( D \). Thus, denoting by \( s_M \in [0, b] \) the maximum value of the parameter such that \( \gamma_1(s) \in N \), it must be \( \gamma_1(s_M) \in T_{\gamma_1(s_M)} N \) and \( \gamma_1([s_M, b]) \subset U \cap D \). On the other hand, from Proposition 3.2, \( \gamma(s_M) \notin T_{\gamma_1(s_M)} N \). This contradiction concludes the proof. \(\square\)

\(^2\)We recall that the closure of a subset \( X \) of a Riemannian or Finsler manifold is said completely convex if any two points in \( \overline{X} \) can be connected by a geodesic lying in \( X \), with the possible exception of either one endpoint or both.
4. Convexity of a Domain

In the following, we denote by $D$ a $C^{2,1}_{\text{loc}}$ domain of a Finsler manifold $(M,F)$. Let us set, for any $p,q \in D$,

$$\Omega(p,q;D) = \{ \gamma : [0,1] \to D \mid \gamma \text{ absolutely continuous,} \int_0^1 h(\gamma)[\dot{\gamma},\dot{\gamma}] \, ds < +\infty, \gamma(0) = p, \gamma(1) = q \},$$

where $h$ is any complete auxiliary Riemannian metric on $M$. It is well known that $\Omega(p,q;D)$ is an infinite dimensional manifold (cf. e.g. [21]). Let us denote the function $F^2$ by $G$; then consider the functional

$$J : \Omega(p,q;D) \to \mathbb{R}, \quad J(\gamma) = \frac{1}{2} \int_0^1 G(\gamma,\dot{\gamma}) \, ds \tag{22}$$

which is a $C^1$ functional with locally Lipschitz differential (see [24, Theorem 4.1]). A critical point $\gamma$ of $J$ is a curve $\gamma \in \Omega(p,q;D)$ such that $dJ(\gamma) = 0$. As recalled in Section 1, the critical points of $J$ on $\Omega(p,q;M)$ are smooth curves and are all and only the geodesics of the Finsler manifold $(M,F)$ connecting the point $p$ to the point $q$. Thus, looking for geodesics joining two arbitrary points $p,q \in D$ and having support in $D$, is equivalent to seek for the critical points of $J$. From the viewpoint of critical point theory, a natural condition to assume on $J$ would be to satisfy the so-called Palais-Smale condition. In fact, it is proved in [12] that this condition is fulfilled when the symmetrized balls of $(M,F)$ are compact. But in our setting, as $D$ is an open subset of $M$, the manifold $\Omega(p,q;D)$ is not complete in any reasonable sense. As a consequence, Palais-Smale sequences may converge to curves touching the boundary $\partial D$. In order to overcome this difficulty, we use a penalization method.

For any $\varepsilon \in [0,1]$, we consider on $\Omega(p,q;D)$ the functional

$$J_\varepsilon(\gamma) = J(\gamma) + \int_0^1 \frac{\varepsilon}{\phi^2(\gamma)} \, ds \tag{23}$$

where $J$ and $\phi$ were respectively introduced in (22), (3).

The presence of the penalizing term

$$\gamma \in \Omega(p,q;D) \mapsto \int_0^1 \frac{\varepsilon}{\phi^2(\gamma)} \, ds,$$

combined with the lack of regularity of $G$ on the zero section, makes the study of regularity and existence of critical points of $J_\varepsilon$ more subtle than the one for the unpenalized functional $J$ and that of the analogous Riemannian problem.

Lemma 4.1. For any $\varepsilon \in [0,1]$, let $\gamma_\varepsilon \in \Omega(p,q;D)$ be a critical point of $J_\varepsilon$ in (23). Then $\gamma_\varepsilon$ is $C^1$ and, for any $\bar{s} \in [0,1]$ such that $\gamma_\varepsilon(\bar{s}) \neq 0$, said $(U,\Phi)$ a chart of $M$ such that $\gamma_\varepsilon(\bar{s}) \in U$, $\gamma_\varepsilon$ is twice differentiable on the open subset $\gamma_\varepsilon^{-1}(U)$ and there it satisfies

$$\gamma_\varepsilon''(s) + \Gamma^i_{jk}(\gamma_\varepsilon(s),\dot{\gamma}_\varepsilon(s),\ddot{\gamma}_\varepsilon(s)) \gamma_\varepsilon^i(s) \gamma_\varepsilon^j(s) \gamma_\varepsilon^k(s) = -\frac{2\varepsilon}{\phi^3(\gamma_\varepsilon(s))} \partial_{2,k} \phi(\gamma_\varepsilon(s)) g^{ki}(\gamma_\varepsilon(s),\dot{\gamma}_\varepsilon(s)). \tag{24}$$

Moreover a constant $E_\varepsilon(\gamma_\varepsilon) \in \mathbb{R}$ exists such that

$$E_\varepsilon(\gamma_\varepsilon) = \frac{1}{2} G(\gamma_\varepsilon,\dot{\gamma}_\varepsilon) - \frac{\varepsilon}{\phi^2(\gamma_\varepsilon)} \text{ on } [0,1]. \tag{25}$$
Proof: Let us consider a finite covering of the range of $\gamma_\varepsilon$ made by local charts $(U_k, \Phi_k)$ of $M$ and let $(TU_k, T\Phi_k)$ be the corresponding charts of $TM$. Let us define the intervals $I_k = \gamma_\varepsilon^{-1}(U_k) = \{ s_k, s_{k+1} \mid [0, 1]$. As $\gamma_\varepsilon \in \Omega(p, q; D)$ is a critical point of $J_\varepsilon$, if $Z \in T\gamma_\varepsilon \Omega(p, q; D)$ has compact support in the interval $I_k$, we have

$$dJ_\varepsilon(\gamma_\varepsilon)[Z] = \frac{1}{2} \int_{I_k} (\partial_x G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon)[Z] + \partial_y G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon)[\dot{Z}]) \, ds - \int_0^1 \frac{2\varepsilon}{\phi^3(\gamma_\varepsilon)} \partial_x \phi(\dot{\gamma}_\varepsilon)[Z] \, ds = 0. \quad (26)$$

With an abuse of notation, in the integral above we denoted by $L$ the function $G \circ (T\Phi_k)^{-1}$ and by $\gamma_\varepsilon$ the curve $\Phi_k \circ \gamma_\varepsilon$ (so that the derivatives $\partial_x G$ and $\partial_y G$ have to be intended as the corresponding derivatives in $\Phi_k(U_k) \times \mathbb{R}^n$ of the function $G \circ (T\Phi_k)^{-1}$).

Evaluating (26) on any smooth vector field $Z$ along $\gamma_\varepsilon$ with compact support on the interval $I_k$, we get

$$\int_{I_k} \left( H + \frac{1}{2} \partial_y G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon) \right) [\dot{Z}] \, ds = 0, \quad (27)$$

where $H = H(s)$ is the covector field along $\gamma_\varepsilon$ defined as

$$H(s) = - \int_{s_k}^s \frac{1}{2} \partial_x G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon) - \frac{2\varepsilon}{\phi^3(\gamma_\varepsilon)} \partial_x \phi(\dot{\gamma}_\varepsilon) \, dt.$$

Equation (27) implies that a constant covector $W \in \mathbb{R}^n$, with $n = \dim M$, exists such that

$$H(s) + \frac{1}{2} \partial_y G(\gamma_\varepsilon(s), \dot{\gamma}_\varepsilon(s)) = W \quad \text{a.e. on } I_k; \quad (28)$$

since $H$ is continuous, the function $s \in I_k \mapsto \partial_y G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon)$ is also continuous. Now fix $x \in D$ and consider the map $L_x: y \in \mathbb{R}^n \mapsto \partial_y G(x, y) \in \mathbb{R}^n$. It can be proved that $L_x$ is a homeomorphism of $\mathbb{R}^n$, hence the function $s \in I_k \mapsto L_{\gamma_\varepsilon(s)} \circ L_{\gamma_\varepsilon(s)}(\dot{\gamma}_\varepsilon(s)) = L_{\gamma_\varepsilon(s)}(\partial_y G(\gamma_\varepsilon(s), \dot{\gamma}_\varepsilon(s)))$ is continuous, thus $\gamma_\varepsilon$ is a $C^1$ curve (cf. [11, Proposition 2.3] for details). From (28) and the fact that $G$ is fiberwise strictly convex, as in [10, Proposition 4.2] the implicit function theorem implies that $\gamma_\varepsilon$ is actually twice differentiable at each $s$ where $\dot{\gamma}_\varepsilon(s) \neq 0$. Thus we infer (24) (recall Eqs. (6)–(7)). Let us consider now the open subsets $A_{k, \varepsilon} = \{ s \in I_k \mid \dot{\gamma}_\varepsilon(s) \neq 0 \}$.

From (28), we get

$$\frac{d}{ds} \frac{1}{2} \partial_y G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon) = \frac{1}{2} \partial_x G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon) - \frac{2\varepsilon}{\phi^3(\gamma_\varepsilon)} \partial_x \phi(\dot{\gamma}_\varepsilon),$$

and then the energy $E_{k, \varepsilon}(\gamma_\varepsilon) = \frac{1}{2} \partial_y G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon)[\dot{\gamma}_\varepsilon] - \frac{1}{4} G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon) - \frac{\varepsilon}{\phi^3(\gamma_\varepsilon)}$ is constant on every connected component of $A_{k, \varepsilon}$. As $G$ is positively homogeneous of degree 2, from Euler’s theorem:

$$E_{k, \varepsilon}(\gamma_\varepsilon) = \frac{1}{2} G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon) - \frac{\varepsilon}{\phi^3(\gamma_\varepsilon)} \quad \text{on each connected component of } A_{k, \varepsilon}.$$

Since the function $s \in I_k \mapsto \frac{1}{2} G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon) - \frac{\varepsilon}{\phi^3(\gamma_\varepsilon)}$ is continuous, it has to be $E_{k, \varepsilon}(\gamma_\varepsilon) = -\frac{\varepsilon}{\phi^3(\gamma_\varepsilon)}$ on $I_k \setminus A_{k, \varepsilon}$. By standard arguments, constants $E_{k, \varepsilon}(\gamma_\varepsilon)$ must agree, hence a real constant $E_\varepsilon(\gamma_\varepsilon)$ exists such that (25) holds. \qed
To prove that functionals $J_\varepsilon$ satisfy the Palais-Smale condition, we adapt Gordon’s lemma, (cf. e.g. [5]), which essentially is the core of the penalization technique. We report here the proof for the reader’s convenience.

**Lemma 4.2.** Let $D$ be a domain of a Finsler manifold $(M, F)$ and assume that $\overline{B_s(p, r)} \cap \overline{D}$ is compact, for any closed ball $\overline{B_s(p, r)} \subset M$. Let $(\gamma_m)_m$ be a sequence in $\Omega(p, q; D)$ such that

$$\sup_{m \in \mathbb{N}} \int_0^1 G(\gamma_m, \dot{\gamma}_m) \, ds = k^2 < +\infty$$

for some $k > 0$, and assume that $(s_m)_m$ is a sequence in $[0, 1]$ such that

$$\lim_{m \to +\infty} \phi(\gamma_m(s_m)) = 0.$$

Then

$$\lim_{m \to +\infty} \int_0^1 \frac{1}{\phi^2(\gamma_m)} \, ds = +\infty.$$

**Proof.** Observe that the supports of the curves $\gamma_m$ are contained in the intersection $\overline{B_s(p, k)} \cap \overline{B_s(q, k)} \cap \overline{D}$, which is a compact subset of $\overline{D}$ in our assumptions (see Remark 2.1). Indeed by (29), for each $s \in [0, 1]$, we have

$$d(p, \gamma_m(s)) \leq \int_0^s F(\gamma_m, \dot{\gamma}_m) \, dt \leq \int_0^1 F(\gamma_m, \dot{\gamma}_m) \, dt \leq \left( \int_0^1 G(\gamma_m, \dot{\gamma}_m) \, dt \right)^{1/2} \leq k$$

and likewise

$$d(\gamma_m(s), q) \leq \int_s^1 F(\gamma_m, \dot{\gamma}_m) \, dt \leq \int_0^1 F(\gamma_m, \dot{\gamma}_m) \, dt \leq \left( \int_0^1 G(\gamma_m, \dot{\gamma}_m) \, dt \right)^{1/2} \leq k.$$

Hence a positive constant $C_1$ exists such that

$$\frac{1}{C_1} |y|_{h(x)}^2 \leq G(x, y) \leq C_1 |y|_{h(x)}^2,$$

for every $x \in K$ and for every $y \in T_x M$. From Hölder inequality and (30) we have that for each $s \in [s_m, 1]$

$$\phi(\gamma_m(s)) - \phi(\gamma_m(s_m)) = \int_{s_m}^s h(\gamma_m)[\nabla^0 \phi(\gamma_m), \dot{\gamma}_m] \, dt \leq C_2 (s - s_m)^{1/2}$$

(31)

where $C_2$ is a positive constant depending on $K$, but independent from $m$. From (31) we get

$$\frac{1}{\phi^2(\gamma_m(s))} \geq \frac{1}{2(C_2^2(s - s_m) + \phi^2(\gamma_m(s_m)))}.$$  

(32)

Moreover from (31), recalling that for each $m \in \mathbb{N}$, $\gamma_m(1) = q$, we deduce that a positive constant $C_3$ exists such that $1 - s_m \geq C_3$ for all $m \in \mathbb{N}$. Therefore integrating both hand sides of (32) on the interval $[s_m, 1]$ we get the thesis. □

Let $\mathcal{M}$ be a Banach manifold. We recall that a $C^1$ functional $f: \mathcal{M} \to \mathbb{R}$ satisfies the Palais-Smale condition if every sequence $(x_m)_m$ such that $(f(x_m))_m$ is bounded and $df(x_m) \to 0$, as $m \to +\infty$, admits a converging subsequence.

**Proposition 4.3.** Under the assumptions of Theorem 1.3, then

(i) for any $\varepsilon \in [0, 1]$ and for any $c \in \mathbb{R}$, the sublevels $J_\varepsilon^c = \{ x \in \Omega(p, q; D) \mid J_\varepsilon(x) \leq c \}$ are complete metric subspaces of $\Omega(p, q; D)$.
(ii) for any \( \varepsilon \in [0, 1] \), \( J_\varepsilon \) satisfies the Palais-Smale condition.

**Proof.** (i) Fix \( \varepsilon \in [0, 1] \) and \( c \in \mathbb{R} \). Let \( (\gamma_m)_m \) be a Cauchy sequence in \( J_\varepsilon \); then it is a Cauchy sequence also in \( \Omega(p, q; M) \) and it uniformly converges to a curve \( \gamma \) with support in \( \overline{D} \). By Lemma 4.2 it follows

\[
d = \inf \{ \phi(\gamma_m(s)) | s \in [0, 1], m \in \mathbb{N} \} > 0.
\]

Thus \( \phi(\gamma(s)) \geq d \) for any \( s \in [0, 1] \), hence \( \gamma \in \Omega(p, q; D) \) and, by the continuity of \( J_\varepsilon \), \( J_\varepsilon(\gamma) \leq c \), as required.

(ii) We can adapt the proof of the Palais-Smale condition for the unperturbed functional \( J \) on \( \Omega(p, q; M) \) in [11, Theorem 3.1]. Indeed, reasoning as in the first part of the proof of Lemma 4.2 we have that, if \( (\gamma_m)_m \) is a Palais-Smale sequence, then a compact subset \( K \subset \overline{D} \) containing the supports of the curves \( \gamma_m \) exists and (30) holds. Therefore for all \( s_1, s_2 \in [0, 1] \) a positive constant \( C \) exists such that

\[
d_h(\gamma_m(s_1), \gamma_m(s_2)) \leq \int_{s_1}^{s_2} |\gamma'_m|_h ds \leq C|s_1 - s_2|^{1/2}
\]

for all \( m \in \mathbb{N} \), where \( d_h \) is the distance associated to the auxiliary Riemannian metric \( h \). Hence, from the Ascoli-Arzelà theorem, there exists a subsequence, denoted again by \( (\gamma_m)_m \), which uniformly converges to a curve \( \gamma \) whose support is in \( \overline{D} \). From Lemma 4.2, \( \gamma([0, 1]) \subset D \), otherwise \( J_\varepsilon(\gamma_m) \to +\infty \), in contradiction with the assumption that \( (\gamma_m)_m \) is a Palais-Smale sequence. Now consider a smooth curve \( w \in \Omega(p, q; D) \) which approximates \( \gamma \) and a natural chart \((\Omega(p, q; D), \exp_w^{-1})\) centered at \( w \), of the manifold \( \Omega(p, q; D) \) (cf. [21, Corollary 2.3.15]). Set \( \xi_m = \exp_w^{-1}(\gamma_m) \). We have

\[
d_J(\gamma_m)[d\exp_w(\xi_m)[\xi_m - \xi_n]] =
\]

\[
d_J(\gamma_m)[d\exp_w(\xi_m)[\xi_m - \xi_n]] - 2\varepsilon \int_0^1 h(\gamma_m)[\nabla h(\gamma_m), d\exp_w(\xi_m)[\xi_m - \xi_n]]
\]

\[
\phi(\gamma_m)
\]

(33)

As \( (\gamma_m)_m \) uniformly converges to \( \gamma \), \( \xi_m - \xi_n \to 0 \) uniformly as \( m, n \to +\infty \), hence the second term in the right-hand side of (33) goes to zero as \( m, n \to +\infty \). Since \( (\gamma_m)_m \) is a Palais-Smale sequence, also the left-hand side goes to zero and then

\[
d_J(\gamma_m)[d\exp_w(\xi_m)[\xi_m - \xi_n]] \to 0,
\]

(34)

as \( m, n \to +\infty \). From (34), the rest of the proof follows step by step that in [11, Theorem 3.1].

**Remark 4.4.** By Proposition 4.3, for any \( \varepsilon \in [0, 1] \), \( J_\varepsilon \) has a minimum point \( \gamma_\varepsilon \in \Omega(p, q; D) \); then \( k > 0 \) exists such that

\[
J_\varepsilon(\gamma_\varepsilon) \leq k \text{ for all } \varepsilon \in [0, 1],
\]

(35)

since \( J_\varepsilon(\gamma_\varepsilon) \leq J_\varepsilon(\gamma_1) \leq J_1(\gamma_1) \). Moreover, from (25) we get

\[
E_\varepsilon(\gamma_\varepsilon) = J_\varepsilon(\gamma_\varepsilon) - 2\int_0^1 \frac{\varepsilon}{\phi^2(\gamma_\varepsilon)} ds \leq k \text{ for all } \varepsilon \in [0, 1]
\]

thus

\[
\frac{1}{2}G(\gamma_\varepsilon(s), \dot{\gamma}_\varepsilon(s)) \leq k + \frac{\varepsilon}{\phi^2(\gamma_\varepsilon(s))} \text{ for all } \varepsilon \in [0, 1] \text{ and } s \in [0, 1].
\]

(36)

Next we give an a priori estimate about the critical points of the functionals \( J_\varepsilon \).
**Lemma 4.5.** Under the assumptions of Theorem 1.3, let \((\gamma_\varepsilon)_{\varepsilon \in [0,1]}\) be a family in \(\Omega(p,q;D)\) such that, for any \(\varepsilon \in [0,1]\), \(\gamma_\varepsilon\) is a critical point of \(J_\varepsilon\) and let \(k \in \mathbb{R}\) be such that (35) holds. Then, set

\[
\lambda_\varepsilon(s) = \frac{2\varepsilon}{\phi^2(\gamma_\varepsilon(s))} \quad \text{for all } \varepsilon \in [0,1] \text{ and } s \in [0,1],
\]

\(\varepsilon_0 \in [0,1]\) exists such that \(||\lambda_\varepsilon\|_\varepsilon \in [0,\varepsilon_0]||\) is bounded, where

\[
||\lambda_\varepsilon||_\infty = \max_{s \in [0,1]} \lambda_\varepsilon(s).
\]

**Proof.** Let \((\gamma_\varepsilon)_{\varepsilon \in [0,1]}\) be a family of critical points of \(J_\varepsilon\) satisfying (35) and let us set for any \(\varepsilon \in [0,1]\), \(s \in [0,1]\), \(\rho_\varepsilon(s) = \phi(\gamma_\varepsilon(s))\) and \(\rho_\varepsilon(s) = \min_{s \in [0,1]} \rho_\varepsilon(s)\). It is enough to prove the thesis when

\[
\lim_{m \to +\infty} \rho_{\varepsilon_m}(s_{\varepsilon_m}) = 0,
\]

where \((\varepsilon_m)_m\) is any infinitesimal and decreasing sequence in \([0,1]\). Note also that, reasoning as in the first part of the proof of Lemma 4.2, the supports of the curves \((\gamma_\varepsilon)_{\varepsilon \in [0,1]}\) are contained in a compact subset.

We distinguish the following two cases.

**First case:** Let \(s_\varepsilon \in A_\varepsilon = \{s \in [0,1] \mid \gamma_\varepsilon(s) \neq 0\}\); then from Lemma 4.1, \(\gamma_\varepsilon\) is \(C^2\) in a neighborhood of \(s_\varepsilon\) and \(\dot{\rho}_\varepsilon(s_\varepsilon) = 0\), \(\dot{\rho}_\varepsilon(s_\varepsilon) \geq 0\), where

\[
\dot{\rho}_\varepsilon(s) = \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\gamma_\varepsilon(s))\gamma_j^i(\gamma_\varepsilon(s))\dot{\gamma}_j^i(\gamma_\varepsilon(s)) + \frac{\partial \phi}{\partial x_i}(\gamma_\varepsilon(s))\dot{\gamma}_i(\gamma_\varepsilon(s))\gamma_j^i(\gamma_\varepsilon(s))
\]

(37)

In this case the proof is essentially the same as for domains in a Riemannian manifold (cf. [17]). Indeed, taking into account that (24) holds on a neighborhood of \(s_\varepsilon\), we get:

\[
0 \leq \dot{\rho}_\varepsilon(s_\varepsilon) = \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\gamma_\varepsilon(s_\varepsilon))\gamma_j^i(\gamma_\varepsilon(s_\varepsilon))\dot{\gamma}_j^i(\gamma_\varepsilon(s_\varepsilon))
\]

As the components of the Chern connection are positively homogeneous of degree 0 with respect to \(y\), \(\Gamma^i_{jk}(\gamma_\varepsilon(s_\varepsilon), \dot{\gamma}_\varepsilon(s_\varepsilon)) = \Gamma^i_{jk}(\gamma_\varepsilon(s_\varepsilon), \dot{\gamma}_\varepsilon(s_\varepsilon))\), by the fact that the supports of the curves \(\gamma_\varepsilon\) are contained in a compact subset of \(M\), the first two terms in the right-hand side of (37) can be bounded by \(k_1 G(\gamma_\varepsilon(s_\varepsilon), \dot{\gamma}_\varepsilon(s_\varepsilon))\), for a positive constant \(k_1\). Analogously, since 0 is a regular value for \(\phi\) and the matrix \([g^{ki}(x,y)]\) is positive definite, for all \((x,y) \in TM \setminus 0\), and positively homogeneous of degree 0, a positive constant \(k_2\) exists such that

\[
\frac{\partial \phi}{\partial x^k}(\gamma_\varepsilon(s_\varepsilon))\frac{\partial \phi}{\partial x^l}(\gamma_\varepsilon(s_\varepsilon))g^{kl}(\gamma_\varepsilon(s_\varepsilon), \dot{\gamma}_\varepsilon(s_\varepsilon)) > k_2.
\]

Hence, from (37) and (36) we get

\[
0 \leq k_1 G(\gamma_\varepsilon(s_\varepsilon), \dot{\gamma}_\varepsilon(s_\varepsilon)) - k_2 \frac{2\varepsilon}{\phi^2(\gamma_\varepsilon(s_\varepsilon))}
\]

\[
\leq k_1 \left(2k + \frac{2\varepsilon}{\phi^2(\gamma_\varepsilon(s_\varepsilon))}\right) - k_2 \frac{2\varepsilon}{\phi^3(\gamma_\varepsilon(s_\varepsilon))}
\]
and then
\[
\frac{\varepsilon}{\phi^4(\gamma_\varepsilon(s_\varepsilon))} \leq c \left( k + \frac{\varepsilon}{\phi^2(\gamma_\varepsilon(s_\varepsilon))} \right)
\] (38)
and the thesis follows.

Second case: Let \( s_\varepsilon \in B_\varepsilon = [0, 1] \setminus A_\varepsilon \); we can further distinguish the following possibilities:

(a) \( s_\varepsilon \) is an isolated point of \( B_\varepsilon \);

(b) \( s_\varepsilon \) is an accumulation point not in the interior of \( B_\varepsilon \);

(c) \( s_\varepsilon \) is in the interior of \( B_\varepsilon \).

In case (a), recalling that \( \rho_\varepsilon \) is \( C^1 \), we have \( \tilde{\rho}_\varepsilon(s_\varepsilon) = 0 \) and a neighborhood \( U(s_\varepsilon) \) of \( s_\varepsilon \), exists such that on \( U(s_\varepsilon) \), \( \tilde{\rho}_\varepsilon(s) \geq 0 \) for \( s > s_\varepsilon \), \( \tilde{\rho}_\varepsilon(s) \leq 0 \) for \( s < s_\varepsilon \); hence, for each \( s \in U(s_\varepsilon) \setminus \{s_\varepsilon\} \), \( \tilde{\rho}_\varepsilon(s) \) exists and, according to the mean value theorem, \( \tilde{\rho}_\varepsilon(s) \geq 0 \). Let us then consider a sequence \( (s_{\varepsilon,m})_m \) in \( U(s_\varepsilon) \setminus \{s_\varepsilon\} \) converging to \( s_\varepsilon \); reasoning as in the first step of the proof we get for any \( m \in \mathbb{N} \):

\[
\frac{\varepsilon}{\phi^4(\gamma_\varepsilon(s_{\varepsilon,m}))} \leq c \left( k + \frac{\varepsilon}{\phi^2(\gamma_\varepsilon(s_{\varepsilon,m}))} \right)
\]

and, passing to the limit, we get again formula (38) for \( s_\varepsilon \).

In case (b) there exists in \( B_\varepsilon \), a strictly monotone sequence, say strictly increasing, \((s_{\varepsilon,m})_m\) converging to \( s_\varepsilon \). We know that \( \tilde{\rho}_\varepsilon(s_{\varepsilon,m}) = 0 \) for any \( m \in \mathbb{N} \). Applying Rolle’s theorem to function \( \tilde{\rho}_\varepsilon \) on each \([s_{\varepsilon,m-1}, s_{\varepsilon,m}]\), we get another sequence, say \((s_{\varepsilon,m}^2)\), in \( A_\varepsilon \) which tends to \( s_\varepsilon \) and such that \( \tilde{\rho}_\varepsilon(s_{\varepsilon,m}^2) = 0 \) for any \( m \in \mathbb{N} \). Then as in the first step we get:

\[
\frac{\varepsilon}{\phi^4(\gamma_\varepsilon(s_{\varepsilon,m}^2))} \leq c \left( k + \frac{\varepsilon}{\phi^2(\gamma_\varepsilon(s_{\varepsilon,m}^2))} \right)
\]

and, passing to the limit, we still get formula (38).

Case (c) has to be ruled out: indeed the interior of \( B_\varepsilon \) is empty. For if, assume that a neighborhood \( U(s_\varepsilon) \) in \( B_\varepsilon \) exists on which \( \gamma_\varepsilon \) has zero derivative. Take then a vector \( z \in T_{\gamma_\varepsilon(s_\varepsilon)}D \) such that \( \partial_y \phi(\gamma_\varepsilon(s_\varepsilon))[z] > 0 \) and a variation vector field \( Z \) such that \( Z(s_\varepsilon) = z \), with support in \( U(s_\varepsilon) \), in such a way that \( \partial_x \phi(\gamma_\varepsilon)[Z] \geq 0 \) on \( U(s_\varepsilon) \).

As \( \gamma_\varepsilon \) is a critical point of \( J_\varepsilon \) we get a contradiction. Indeed, as \( \partial_x G(\gamma_\varepsilon, \gamma_\varepsilon) = 0 \) and \( \partial_y G(\gamma_\varepsilon, \gamma_\varepsilon) = 0 \), it follows

\[
0 = dJ_\varepsilon(\gamma_\varepsilon)[Z] = \frac{1}{2} \int_{U(s_\varepsilon)} (\partial_x G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon)[Z] + \partial_y G(\gamma_\varepsilon, \dot{\gamma}_\varepsilon)[\dot{Z}]) \, ds - 2\varepsilon \int_{U(s_\varepsilon)} \frac{\partial_x \phi(\gamma_\varepsilon)[Z]}{\phi^4(\gamma_\varepsilon)} \, ds
\]
\[
= -2\varepsilon \int_{U(s_\varepsilon)} \frac{\partial_x \phi(\gamma_\varepsilon)[Z]}{\phi^4(\gamma_\varepsilon)} \, ds < 0.
\]

The lemma above is fundamental in order to conclude the limit process. Indeed we can state the following proposition.

**Proposition 4.6.** Let \( (\gamma_\varepsilon)_{\varepsilon \in [0, 1]} \) be a family in \( \Omega(p, q; D) \) such that for any \( \varepsilon \in [0, 1] \) \( \gamma_\varepsilon \) is a critical point of \( J_\varepsilon \) and let \( k > 0 \) be such that (35) holds. Then, a subsequence \((\varepsilon_m)_m\) in \([0, 1]\) exists such that

1. \( (\gamma_{\varepsilon_m})_m \) strongly converges to a curve \( \gamma \in \Omega(p, q; M) \) whose support is contained in \( D \).
(2) \((λ_\varepsilon)_m\) weakly converges to \(λ \in L^2([0,1],\mathbb{R})\):

(3) the limit curve \(γ\) is \(C^1\) and, for any \(\bar{s} \in [0,1]\) such that \(γ(\bar{s}) \neq 0\), said \((U, Φ)\) a chart of \(M\) such that \(γ(\bar{s}) \in U\), \(γ\) has \(H^2,2\)-regularity on an open subset of \(γ^{-1}(U)\) containing the point \(\bar{s}\) and there it satisfies a.e. the equations

\[ 2\dot{γ}^i + Γ^i_{jk}(γ, \dot{γ})\dot{γ}^j\dot{γ}^k = -λ\partial_{x_k}\phi(γ)g^k(γ, \dot{γ}), \]  

(39)

Proof. Statements (1) and (2) respectively follow by an argument analogous to that used in Proposition 4.3 and by Lemma 4.5. Let us prove (39). As observed in Section 1, we cannot proceed as in previous references on the topic, because therein, by the Nash embedding theorem, the Riemannian manifold \(M\) is treated as a closed submanifold of an Euclidean space \(\mathbb{R}^N\) (see [25] for the existence of a closed isometric embedding) and some arguments based on the vector space structure of the Hilbert space \(H^{1,2}([0,1],\mathbb{R}^N)\) are used (cf. [17, Lemma 4.6, Lemma 4.7]). Instead, our proof relies on a local representation of the Lagrangian \(G\) as in [6].

Let \(\mathcal{A} = \{(V_i, Φ_i)\}\) be a smooth atlas of \(M\) and \(\mathcal{T}\mathcal{A} = \{(TV_i, TΦ_i)\}\) the corresponding atlas of \(TM\). Let us consider, for any \(V_i \in \mathcal{A}\), the Lagrangian \(G_{V_i} : Φ_i(V_i) \times \mathbb{R}^n \rightarrow \mathbb{R}\), \(G_{V_i}(q, v) = G \circ (TΦ_i)^{-1}(q, v)\), where \(n = \dim M\). Note that the Lagrangians \(G_{V_i}\) are positively homogeneous of degree 2 with respect to the variable \(v \in \mathbb{R}^n\). Let us consider the functionals

\[ j_{V_i}(q) = \frac{1}{2} \int_{I_q} G_{V_i}(q(s), \dot{q}(s)) \, ds, \]

where \(q\) is any curve in \(Φ_i(V_i)\), namely \(q: I_q \subset \mathbb{R} \rightarrow Φ_i(V_i)\). Then we set

\[ j_{ε_m, V_i}(q) = j_{V_i}(q) + \int_{I_q} \frac{ε_m}{ϕ_{V_i}(q(s))} \, ds, \]

where \(ϕ_{V_i} = ϕ \circ (Φ_i)^{-1} : Φ_i(V_i) \rightarrow \mathbb{R}\). Now let \(\{(U_k, Φ_k)\}\) be a finite covering of \(γ\) such that, for each \(k\), \((U_k, Φ_k) \in \mathcal{A}\). Let \(I_k = γ^{-1}(U_k) \subset [0,1]\); clearly for \(m\) large enough it is also \(γ_{ε_m}(I_k) \subset U_k\). We have

\[ J_{ε_m}(γ_{ε_m}) = \sum_k J_{ε_m, U_k}(q_{mk}) \]  

(40)

where \((q_{mk}, \dot{q}_{mk}) = TΦ_k(γ_{ε_m}|_{I_k})\). Let \((q_k, Z_k) = TΦ_k(Z|_{I_k})\), where \(Z \in T_y(p, q; M)\). Clearly we can view the vector field \(s \mapsto Z_k(s) \in \mathbb{R}^n\) along the curve \(q_{mk}\). Then we have

\[ dJ_{ε_m, U_k}(q_{mk})[Z_k] = \]

\[ = \frac{1}{2} \int_{I_k} (\partial_q G_{U_k}(q_{mk}, \dot{q}_{mk})[Z_k] + \partial_q G_{U_k}(q_{mk}, \dot{q}_{mk})[\dot{Z}_k]) \, ds \]

\[ - 2ε_m \int_{I_k} \frac{∂q G_{U_k}(q_{mk})[Z_k]}{ϕ_{U_k}(q_{mk})} \, ds \]

\[ = \frac{1}{2} \int_{I_k} (|q_{mk}|^2 ψ_{mk} + |q_{mk}| χ_{mk}) \, ds - \int_{I_k} 2ε_m \frac{∂q G_{U_k}(q_{mk})[Z_k]}{ϕ_{U_k}(γ_{ε_m}(s))} \, ds, \]  

(41)

where for a.e. \(s \in I_k\), \(ψ_{mk} = ∂_q G_{U_k}(q_{mk}, \dot{q}_{mk})[Z_k]\), \(χ_{mk} = ∂_s G_{U_k}(q_{mk}, \dot{q}_{mk})[\dot{Z}_k]\). Observe that, for each \(k \in \mathbb{N}\), the sequences of functions \((ψ_{mk})_m\) and \((χ_{mk})_m\) are bounded in \(L^∞(I_k, \mathbb{R})\) and since \(q_{km} \rightarrow q_k\) in \(H^1(I_k, Φ_k(U_k))\), by the Lebesgue
dominated convergence theorem we deduce that the first integral in (41) converges to
\[
\frac{1}{2} \int_{I_k} \left( \partial_q G_{U_k}(q_k, \dot{q}_k)[Z_k] + \partial_s G_{U_k}(q_k, \dot{q}_k)[\dot{Z}_k] \right) \, ds,
\]
as \(m \to +\infty\). On the other hand from the weak convergence of \((\lambda_{\varepsilon_m})_m\) to \(\lambda\) and the uniform convergence of \((q_{mk})_m\) to \(q_k\), we get
\[
\int_{I_k} \lambda_{\varepsilon_m}(s) \partial_q \phi_{U_k}(q_{mk})[Z_k] \, ds 
\rightarrow \int_{I_k} \lambda(s) \partial_q \phi_{U_k}(q_k)[Z_k] \, ds, \quad \text{as } m \to +\infty.
\]
Summing over \(k\), from (40), we get
\[
\sum_k d_j_{\varepsilon_m,U_k}(q_{mk})[Z_k] = dJ_{\varepsilon_m}(\gamma_m)[Z_m] = 0,
\]
where \(Z_m\) is the vector field along \(\gamma_m\) obtained patching together the vector fields \((T\Phi_k)^{-1}(Z_k)\). Thus, from (41)–(42) we deduce that \(\gamma\) satisfies the following equation
\[
\frac{1}{2} \int_0^1 (\partial_x G(\gamma, \dot{\gamma})[Z] + \partial_q G(\gamma, \dot{\gamma})[\dot{Z}]) \, ds - \int_0^1 \lambda(s) \partial_x \phi(\gamma)[Z] \, ds = 0
\]
(here we use the same abuse of notation as in the proof of Lemma 4.1). Consider now any smooth vector field \(Z\) along \(\gamma\) with compact support in the interval \(I_k = \gamma^{-1}(U_k) = [s_k, s_{k+1}] \subset [0, 1]\); then
\[
\int_{I_k} \left( H + \frac{1}{2} \partial_x G(\gamma, \dot{\gamma}) \right)[\dot{Z}] \, ds = 0,
\]
where \(H = H(s)\) is the covector field along \(\gamma\) defined as
\[
H(s) = -\int_{s_k}^s \left( \frac{1}{2} \partial_x G(\gamma, \dot{\gamma}) - \lambda \partial_x \phi(\gamma) \right) \, dt.
\]
Reasoning as in Lemma 4.1, we get that \(\gamma\) is a \(C^1\) curve. Since \(\lambda \in L^2([0, 1], \mathbb{R})\), as in the proof of Lemma 4.1, now we get that \(\gamma\) has \(H^{2,2}\)-regularity on an open neighborhood of any point \(\bar{s} \in [0, 1]\) where \(\dot{\gamma}(\bar{s}) \neq 0\) and thus (39) holds. \(\square\)

**Proof of Theorem 1.3.** The implication “\(D\) convex \(\Rightarrow \partial D\) convex” is trivial by using infinitesimal convexity. In fact, if \(\partial D\) is not infinitesimally convex then the normal curvature at some point \(x \in \partial D\) is negative and the corresponding geodesic yields an immediate contradiction.

For the converse, reasoning as in Remark 4.4, we can consider a family \((\gamma_\varepsilon)_{\varepsilon \in [0,1]}\) in \(\Omega(p, q; D)\) such that, for any \(\varepsilon \in [0,1]\), \(\gamma_\varepsilon\) is a minimum point of \(J_\varepsilon\) and a constant \(k > 0\) such that, (35) holds. Then, by Proposition 4.6 a subsequence \((\gamma_{\varepsilon_m})\) exists converging to a curve \(\gamma \in \Omega(p, q; M)\) which satisfies (39) for a.e. \(s \in V\), where \(V\) is an open neighborhood of any point \(s_0 \in A_\gamma = \{s \in [0,1] \mid \dot{\gamma}(s) \neq 0\}\). Let \(s_0 \in A_\gamma\) be such that \(\gamma(s_0) \in D\); then, as \((\gamma_{\varepsilon_m})_m\) uniformly converges to \(\gamma\), \(\nu \in \mathbb{N}\) and \(\delta > 0\) exist such that
\[
d = \inf \{\phi(\gamma_{\varepsilon_m}(s)) \mid s \in [s_0 - \delta, s_0 + \delta], m \geq \nu\} > 0.
\]
Then \((\lambda_{\varepsilon_m})_m\) uniformly converges to 0 on \([s_0 - \delta, s_0 + \delta]\), where it must be \(\lambda(s) = 0\) for a.e. \(s\) (recall (2) of Proposition 4.6). Let now \(I\) be a measurable subset in \([0,1]\) with strictly positive measure and assume that \(\gamma(s) \in \partial D\) for any \(s \in I\). Set
$I^* = I \cap A_\gamma$. Each $s \in I^*$ is a minimum point of $\rho(s) = \phi(\gamma(s))$ hence, from (39), for a.e. $s \in I^*$ we have

$$0 \leq \ddot{\rho}(s) = \frac{\partial^2 \phi}{\partial x^i \partial x^j}(\gamma(s)) \dot{\gamma}^i(s) \dot{\gamma}^j(s) - \frac{\partial \phi}{\partial x^i}(\gamma(s)) \Gamma^i_{jk}(\gamma(s)) \dot{\gamma}^j(s) \dot{\gamma}^k(s) - \lambda(s) \frac{\partial \phi}{\partial x^i}(\gamma(s)) \frac{\partial \phi}{\partial x^k}(\gamma(s)) g^{ik}(\gamma(s), \dot{\gamma}(s)).$$

Since $\dot{\gamma}(s) \in T_{\gamma(s)} \partial D$ for all $s \in I$, from (9), the fact that 0 is a regular value for $\phi$ and that the matrix $[g^{ik}(x, y)]$ is positive definite, we have $\lambda(s) = 0$, for a.e. $s \in I^*$. Summing up, we have proved that $\lambda(s) = 0$, for a.e. $s \in A_\gamma$. By (39), this means that $\gamma$ is a geodesic on each connected component of $A_\gamma$ and the function $E(\gamma) = G(\gamma(s), \dot{\gamma}(s))$ is constant on such connected components. Since $\gamma$ is a $C^1$ curve on $[0, 1]$, such constants must agree on the whole interval $[0, 1]$, therefore $A_\gamma = [0, 1]$ and $\gamma$ is a geodesic joining $p, q \in D$. Finally, as the boundary is convex, the range of $\gamma$ is contained in $D$.

Moreover $D$ is convex. Indeed, since $J$ is a continuous functional, recalling that $\gamma_{c,m}$ is a minimum for $J_{c,m}$ and $(\gamma_{c,m})_m$ converges to $\gamma$ in $\Omega(p, q; D)$ (and therefore $\inf \{\phi(\gamma_{c,m}(s)) \mid s \in [0, 1], m \in \mathbb{N}\} > 0$), we get

$$J(\gamma) = \lim_{m} J(\gamma_{c,m}) \leq \lim_{m} J_{c,m}(\gamma_{c,m}) \leq \lim_{m} J_{c,m}(\gamma) = J(\gamma),$$

for any other curve $\tilde{\gamma} \in \Omega(p, q; D)$. Hence $\gamma$ is a minimum for $J$ and therefore also for the length functional $\ell_F$.

Now we pass to prove multiplicity of geodesics connecting the points $p$ and $q$ and having support contained in $D$, under the assumption that $D$ is not contractible. This is a quite standard application of Lusternik-Schnirelman theory and its proof is the same as in the case of a domain in a Riemannian manifold. We observe also that such geodesics necessarily have different supports, except if a closed geodesic crosses the given points. We sketch the proof for the reader convenience. We recall that given a topological space $X$ the Lusternik-Schnirelman category of $A \subset X$, denoted by $\text{cat}_X(A)$, is defined as the minimum number of closed contractible subsets of $X$ needed to cover $A$. By definition $\text{cat}_X(A) = +\infty$ if the covering cannot be realized by a finite number of subsets. We introduce an auxiliary complete Riemannian metric $h$ on $M$ and, using a suitable deformation of the flow of the vector field $\frac{\nabla h \phi}{1 + |\nabla h \phi|^2}$, we can construct as in [23, Proposition 4.4.8] a $C^{1,1}$ diffeomorphism $\psi$ of $D$ onto the subset $D \setminus D_\delta$, with $D_\delta = \{x \in D \mid \phi(x) < \delta\}$. We can use $\psi$ to define, by composition, a locally Lipschitz map on $\Omega(p, q; D)$ that maps any sublevel $J_c = \{\gamma \in \Omega(p, q; D) \mid J(\gamma) \leq c\}$, $c > 0$, into the intersection of another sublevel $J_{c'}$, $c' > 0$, with the set of the curves in $\Omega(p, q; D)$ having support in $D \setminus D_\delta$. This is enough to get $\text{cat}_{\Omega(p, q; D)}(J^c) < +\infty$. By a result of E. Fadell and A. Husseini [15], if $D$ is not contractible, a sequence $(K_m)_m$ of compact subsets of $\Omega(p, q; D)$ exists such that, for each $m \in \mathbb{N}$, $\text{cat}_{\Omega(p, q; D)}(K_m) \geq m$; hence for such $m$ and for every $\varepsilon > 0$

$$c_{\varepsilon,m} = \inf_{A \in \Gamma_m} \sup_{\gamma \in A} J_\varepsilon(\gamma),$$

where $\Gamma_m = \{A \subset \Omega(p, q; D) \mid \text{cat}_{\Omega(p, q; D)}(A) \geq m\}$, is a real number. Thus $c_{\varepsilon,m}$ is a critical value of functional $J_\varepsilon$ (see for example [26]). Observe that for a fixed $c > 0$, there must exist $m(c) \in \mathbb{N}$ such that, for any $A \in \Gamma_m(c)$, $A \cap (\Omega(p, q; D) \setminus J^c) \neq \emptyset$, otherwise $\text{cat}_{\Omega(p, q; D)}(J^c) = +\infty$ (we recall that if $A \subset B$, then $\text{cat}_X(A) \leq
Therefore, for each $\varepsilon > 0$, we have $c \leq c_{\varepsilon,m(\varepsilon)} \leq \sup_{\gamma \in K_{m}(\varepsilon)} J(\gamma)$ and passing to the limit on $\varepsilon \to 0$, by Proposition 4.6 and the first part of this proof, we get a critical value $c_{m(\varepsilon)} \geq c > 0$ of $J$ and therefore a geodesic $\gamma_{m}$ in $D$ connecting $p$ to $q$. Since $c$ was arbitrarily chosen, we obtain in this way a sequence $(\gamma_{m})_{m} \subset \Omega(p,q;D)$ of geodesics such that, as $m \to +\infty$, $J(\gamma_{m}) \to +\infty$ and hence $\ell_{F}(\gamma_{m}) \to +\infty$ as well.

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