Magnetic phenomena at and near \( \nu = \frac{1}{2} \) and \( \frac{1}{2} \): theory, experiment and interpretation.

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I show that the Hamiltonian theory of Composite Fermions (CF) is capable of yielding a unified description in agreement with recent experiments on polarization \( P \) and relaxation rate \( 1/T_1 \) in quantum Hall states at filling \( \nu = p/(2ps + 1) \), at and near \( \nu = \frac{1}{2} \) and \( \frac{1}{2} \) (\( s = 1, 2 \)) at zero and nonzero temperatures. I show how rotational invariance and two dimensionality can make the underlying interacting theory behave like a free one in a limited context.

Recent experiments on quantum Hall systems have focused on spin degrees of freedom. At \( T = 0 \), critical fields \( B^c \) at which gapped states jump from one quantized value of polarization \( P \) to another \( 1 \) and at which \( P \) saturates for gapless states have been measured \( \nu = 1 \). The temperature dependence of \( P \) \( 3 \), and \( 1/T_1 \) \( 1 \) were measured for gapless states. These data pose new constraints and challenges for theorists. Why do CF appear to be strongly interacting, in that it takes two substantially different masses \( m_a \) and \( m_p \) to describe activation and polarization phenomena; and yet noninteracting, in that certain polarization phenomena can be fit by free fermions? Which mass should we use at \( T > 0 \)?

The Hamiltonian theory of CF, developed for small \( q \) \( 1 \) and extended for all \( q \) \( 1 \), gives a unified account of these experiments and answers the question raised above. Once a single parameter characterizing each sample is obtained by fitting the theory to one data point, the values at other fields, tilts and temperatures are predicted.

This fermionic theory is defined in first quantization by the following equations. Readers not interested in the details may safely go on Eqn. \( 7 \):

\[
H = \int \frac{d^2q}{(2\pi)^2} \frac{v(q)}{2} e^{-\frac{q^2}{2}} \langle \tilde{p}^0(q) \tilde{p}^0(-q) \rangle \tag{1}
\]

\[
\tilde{p}^0(q) = \tilde{p}(q) - c^2 \tilde{\chi}(q) \exp(-q^2/8ps) \tag{2}
\]

\[
\tilde{p}(q) = \sum_j \exp(-i\mathbf{q} \cdot \mathbf{r}_j - \frac{l^2}{2c} \bar{\mathbf{z}} \cdot \mathbf{r}_j) \tag{3}
\]

\[
\tilde{\chi}(q) = \sum_j \exp(-i\mathbf{q} \cdot \mathbf{r}_j + \frac{l^2}{2c} \mathbf{r}_j) \tag{4}
\]

\[
0 = \tilde{\chi}(q)|\text{Physical State}| \tag{5}
\]

\[
c^2 = \frac{2ps}{2ps + 1} \quad l = \frac{1}{\sqrt{cB}} \tag{6}
\]

where \( \Pi \) is the kinetic momentum of the CF in the field \( B^c = \frac{B}{2ps + 1} \), \( h = c = 1 \), and the incoherent spin index has been suppressed. Referring to Ref. \( 1 \) for details, I merely state that these equations provide an operator realization of the idea that in the lowest Landau Level (LL) electrons bind to 2s-fold vortices of charge \( -c^2 = -2ps/(2ps + 1) \) (in electronic units) to form composite fermions of charge \( e^* = e/(2ps + 1) \). Since the CF charge and dipole moment are built in at tree level, one expects a Hartree-Fock approximation to be reliable. \( 1 \)

I will use the Zhang-Das Sarma \( 1 \) potential

\[
v(q) = \frac{2\pi c^2 e^{-q\Lambda}}{\varepsilon q} \equiv \frac{2\pi c^2 e^{-q\Lambda}}{\varepsilon q} \tag{7}
\]

with \( e^2/(c\k_B) \simeq 50\sqrt{B(T)^6 K} \). The value of \( \Lambda \) used to describe experiment will be discussed later. The Gaussian in Eqn. \( 6 \) (absent in the small \( q \) theory) ensure that even for \( \lambda = 0 \), (coulomb case) \( q \) integrals converge.

Note that \( H = H_0 + H_1 \) is of a very unusual form. Consider \( H_0 \), the single particle part, given by the diagonal terms in the double sum \( \tilde{p}^0(q) \cdot \tilde{p}^0(-q) \). In the simplest case \( \nu = \frac{1}{2} \), \( s = 1, p = \infty, c = 1 \):

\[
H_0 = \sum_i 2 \int \frac{d^2q}{(2\pi)^2} \sin^2 \left[ \frac{q \times k(l^2/2)}{2} \right] v(q) e^{-(q^2)/2} \tag{8}
\]

where \( k \) is the particle momentum. The CF kinetic energy originates from the interaction \( v(q) \) as it should, and is quadratic only at small \( k \). \( H_1 \) also has nonstandard terms that appear with definite relative strengths. At \( \nu \neq \frac{1}{2}, \frac{1}{4} \), \( H \) is even more complicated. Fortunately, free-fermion states with filled LL and single particle-hole excitations thereof, are its Hartree-Fock (HF) eigenstates, and form the basis for the following computations.

Let us begin with gapped states at \( T = 0 \), and assume unit area. Let \( \{p - r, r\} \) denote a state with \( p - r \) spin-up and \( r \) spin-down LL of CF's and hence a polarization \( P = (p - 2r)/p \). Its energy is given in HF by

\[
E(p - r, r) = \langle p - r, r|H|p - r, r \rangle. \tag{9}
\]

The Zeeman energy, ignored here, can be trivially included. The critical field \( B^c \) for the transition from \( r \) to \( r + 1 \) is found by equating \( E(p - r - 1, r + 1) - E(p, r + 1) \) to the corresponding Zeeman energy difference. All expressions can all be evaluated in closed form. \( 12 \)

In the gapless case, \( P \) is continuous and found by minimizing the Zeeman energy plus the HF energy in a state.
with Fermi momenta $k_{\pm F}$ for the spin up/down species. Note that $k_{+ F}^2 + k_{- F}^2 = k_F^2 = 4\pi n$, where $k_F$ is the momentum of the fully polarized sea, and $n$ the total density. At $T > 0$, a HF calculation using these states gives $P$ and $1/T_1$ as a function of $T$.

Before turning to numbers, I address and interpret a strange regularity in the $T > 0$ results, already encountered by Park and Jain [13], namely that they can be fit by a theory of free fermions with Fermi momenta $k_{\pm F}$ that occupy LL with a gap $\Delta_p = eB^*/m_p$. In this case would have

$$E(p - r, r) - E(p - r - 1, r + 1) = \frac{n(p - 2r - 1)}{n} \Delta_p$$

(10)

since $(n/p)$ spin-up fermions of energy $(p - r - 1 + \frac{1}{2})\Delta_p$ drop to the spin-down level with energy $(r + \frac{1}{2})\Delta_p$.

Suppose we evaluate the left-hand-side of Eqn. (10) in the HF approximation to $H$ and define

$$\Delta_p(r) = \frac{e}{\xi}(0.00660, 0.00649, 0.00641)$$

(11)

which describe $(6, 0) \rightarrow (5, 1), (5, 1) \rightarrow (4, 2),$ and $(4, 2) \rightarrow (3, 3).$ This was true for every fraction and every value of $\lambda I$ looked at. Yet I knew that $H$ was definitely not free since the activation gap $\Delta_a$ to make a widely separated particle hole pair differs from $\Delta_p$ by factors like 2 or 4 (depending on $\lambda$) and turning off $H_I$ makes a substantial difference.

Likewise at $v = \frac{1}{2}$ and $\frac{1}{2}$ the energy cost of flipping one spin could be fit to $(k_{+ F}^2 - k_{- F}^2)/2m_p$ where $m_p$ is independent of $P$ and matches the value found at nearby gapped states. This too is surprising given that $\mathcal{E}$, the HF energies of particles on top either sea, does not even vary as $k_{\pm F}^2$! For example at $v = \frac{1}{2}$ and $\lambda = 1$

$$\frac{\mathcal{E}(k_{\pm F})}{(e/\xi)} = a \left(\frac{k_{\pm F}}{k_F}\right)^2 + b \left(\frac{k_{\pm F}}{k_F}\right)^4$$

(13)

where $a = .075, b = -.030.$

I will now explain these results for both gapped and gapless states. Consider $E(S)$, the ground state energy density as a function of $S = n P$, with up/down particles contributing $\pm 1$ to $S$. By rotational invariance

$$E(S) = E(0) + \frac{\alpha}{2} S^2 + O(S^4)$$

(14)

Assume $O(S^4)$ terms are negligible. (We will see that this does not correspond to a free-fermion assumption, or in the gapless case, even a $k^2$ kinetic energy.)

Consider now the gapless case. When $dn$ particles go from from spin-down to spin-up,

$$dE = \alpha S dS = \alpha S (2dn)$$

(15)

$$= \alpha \frac{k_{+ F}^2 - k_{- F}^2}{4\pi} (2dn)$$

(16)

using Luttinger’s theorem on the volumes of the Fermi seas. We see that $dE$ has precisely the form of the kinetic energy difference of particles of mass $m_p$ given by

$$\frac{1}{m_p} = \frac{\alpha}{\pi}$$

(17)

Thus $m_p$ is really a susceptibility. Note that the free-field form of $dE$ comes from $E \approx S^2 \alpha$ and $d = 2$: in $d = 3$, we would have $dE/dn \approx S \approx (k_{+ F}^2 - k_{- F}^2)$ which no one would interpret as a difference of kinetic energies.

For gapped states, if $E \approx S^2$ is again assumed, the energies $E(p, r)$ will obey Eqn. (10) with an $r$-independent $\Delta_p$. (Here one must replace $2S$ $dS$ by the discrete expression $S(p - r, r)^2 - S(p - r - 1, r + 1)^2$.)

To understand the smallness of the $O(S^4)$ term, consider Eqn. (13). The cost of transferring a particle from the top of one sea to the top of the other is

$$\frac{dE}{(e/\xi)} = \frac{(a + b)}{(k_{+ F}^2 - k_{- F}^2)}$$

(18)

Thus $k^4$ terms in $\mathcal{E}(k_{\pm F})$ are not the cause of the $S^4$ term. However, a small $k^6$ term in $\mathcal{E}(k_{\pm F})$, will generate small quartic terms in $dE$ and $E(S)$.

To understand why the $k^6$ term is so small, we turn to Eqn. (8) for $H_0$. Expanding the $\sin^2$ in a series, we find the $k^6$ term is down by a factor of at least 15 (50) relative to the $k^2$ term, at $\lambda = 0$ ($\lambda = 1$), all the way up to $k = k_F$. Presumably this feature (and its counterpart in the gapped case) persists in the HF approximation to $H$ and keeps $E(S)$ essentially quadratic, which in turn mimics free-field behavior.

Now I present the HF results for $(r$-independent) $m_p$ and $\Delta_p = eB^*/m_p$. At and near $\nu = \frac{1}{2}$ and $\nu = \frac{1}{2}$, for .75 $< \lambda < 2$, $m_p$ may be approximated by

$$\frac{1}{m_p(2)} = \frac{e^2}{\xi} C_p^{(2)}(\lambda)$$

(19)

$$\frac{1}{m_p(4)} = \frac{e^2}{\xi} C_p^{(4)}(\lambda)$$

(20)

For fractions like 2/5, not too close to 1/2, I will use the actual $m_p$ in comparing to experiment.

The transition $|p - r, r| \rightarrow |p - r - 1, r + 1|$ occurs when

$$g \frac{eB^*}{2m_e \cos \theta} = (p - 2r - 1)\Delta_p$$

(21)

where $g = .44, m_e$ is the electron mass, and $\theta$ is the tilt.

In the gapless case the total energy density $E_T(S)$,

$$E_T(S) = \frac{\alpha}{2} S^2 - g \frac{eB^*_S}{2m_e \cos \theta}$$

(22)
where $\alpha = \pi/m_p$, is minimized (for $P \leq 1$) to give $P$:

\begin{align*}
P &= \frac{1}{\cos \theta} \left( \frac{B}{B_c} \right)^{\lambda/4} \quad \nu = \frac{1}{2} \quad \text{B in Tesla} \quad (23)

P &= \frac{1}{\cos \theta} \left( \frac{B}{B_c} \right)^{\lambda/4} \quad \nu = \frac{1}{4} \quad \text{B in Tesla} \quad (24)
\end{align*}

At $T \neq 0$, a HF calculation, in which the nonquadratic dispersion in Eqn. (1) and the corresponding density of states play a central role, yields $P$ and $1/T_1$.

We now compare to some experiments, starting with $\nu = 1/2$ and $T > 0$. Consider first Dementyev et al. who find $P = .75$ for $B = B_\perp = 5.52T$ at 300 mK. An LDA calculation for the potential would give $\lambda \simeq 1$ for this sample density. However, since this number does not include the effects of disorder, I have chosen instead to match my HF results with the above data point, (which gives $\lambda = 1.75$) and see to what extent a sole parameter $\lambda$, can describe $P$ and $1/T_1$ for the given sample at a given $B_\perp$, but various temperatures and tilts. Since there does not exist a model, including disorder, for how $\lambda$ should vary with tilt I include no such variation.

Figure 1 compares the HF calculation to the data. Dementyev et al. had pointed out that a two parameter fit (using a mass $m$ and interaction $J$), led to four disjoint pairs of values for these four curves. Given that $H$ is neither free nor of the standard form ($p^2/2m + V(x)$) this is to be expected. By contrast, a single $\lambda$ is able to describe the data here since $H$ has the right functional form. Given how the theory fits the data (up to the Fermi energy of $\simeq 1^\circ K$, beyond which point the size of the CF exceeds the spacing between them and the CF concept breaks down) it is clear that changing the data point used to fix $\lambda$ will be inconsequential.

If $P$ were computed from the LDA value $\lambda \simeq 1$, it would be down by 15-50% as $T$ drops from $1^\circ K$ to $0^\circ K$. Thus disorder and LL mixing are quite important, and at present theory cannot describe experiments ab initio. Instead the present work establishes a phenomenological, nontrivial and nonobvious fact that a single $\lambda$ parameter, (like $g$ or $\varepsilon$) determined from one data point, can describe both $P$ and $1/T_1$ for the given sample under a variety of conditions. That the fitted $\lambda$ is larger than the LDA value, makes sense, as both disorder and LL mixing will lower the gap and raise $\lambda$.

Consider next sample M280 of Melinte et al. which had $P = .76$ at $0.06^\circ K$ and $B = B_\perp = 7.1T$, from which I deduced $\lambda = 1.6$. Figure 2 compares my $T$-dependence with data. The initial rise of $P$ with temperature was also seen by Chakroborthy and Pietlianen [12]. As for the tilted sample, I do not understand the disagreement (not found in the Dementyev case).

Now consider Kukushkin et al. who measure $P(B)$ at $0^\circ K$ by extrapolation. They vary both $B = B_\perp$ and the density $n$ at each $\nu$ and observe, in addition to the spatially uniform states of polarization labeled by $r$, narrow steps of intermediate polarization, probably the inhomogeneous states proposed by Murthy [16]. In comparing to theory, I will take the midpoints of these steps as the transition points between spatially uniform states.

I consider $B^2$'s at which the systems at $1/4, 1/2, 3/7, 4/9$, and $1/2$ lose full polarization ($r = 0$ for gapped cases, saturation for the gapless cases) and, for $4/9$, also the $r = 1$ transition, $(3, 1) \rightarrow (2, 2)$. I fit $\lambda$ to the $\nu = 3/7$ transition $(3, 0) \rightarrow (2, 1)$ at $B^2 = 4.5T$. I obtain $\lambda_{3/7} = 1.42$ on solving Eqn. (21):

\begin{equation}
\frac{e^2}{2m_e} \frac{B^2}{(e^2/\varepsilon l)} = \frac{2\Delta_p [(3, 0) \rightarrow (2, 1)]}{(e^2/\varepsilon l)} = \frac{2(0.0117)}{\lambda_{3/7}^{7/4}} \quad (25)
\end{equation}
For transitions at other $B_\perp$ and $n$, I need the corresponding $\lambda$'s. One can argue [13] that $\Lambda \simeq n^{-1/3}$ so that $\lambda \simeq n^{-1/3} B_\perp^{1/2} \simeq B_\perp^{1/6} \nu^{-1/3}$, and

$$\lambda_\nu = \lambda_3 / \left( 4.5 B_\perp^{1/6} / \nu^{-1/3} \right)^{1/3} = .83 B_\perp^{1/6} / \nu^{1/3}. \quad (26)$$

Given $\lambda$ one finds $B^e$ using Eqn. (24) for gapped cases and Eqns. (24, 25) for saturation in the gapless cases. The results are summarized in Table I.

| $\nu$ | comment          | $B^e$ (exp.) | $B^e$ (theo) | $\nu B^e$ (exp.) |
|-------|------------------|--------------|--------------|------------------|
| 4/9   | (3, 1) $\rightarrow$ (2, 2) | 2.7 T        | 1.6 T        | 1.2              |
| 2/5   | (2, 0) $\rightarrow$ (1, 1) | 3 T          | 2.65 T       | 1.2              |
| 1/4   | saturation       | 5.2 T        | 4.4 T        | 1.3              |
| 3/7   | (3, 0) $\rightarrow$ (2, 1) | 4.5 T        | 4.5 T        | 1.93             |
| 4/5   | (4, 0) $\rightarrow$ (3, 1) | 5.9 T        | 5.9 T        | 2.62             |
| 1/2   | saturation       | 9.3 T        | 11.8 T       | 4.65             |

Table I: Critical fields based on a fit at 3/7. The rows are ordered by the last column which measures density.

Note that in rows above (below) 3/7, where I fit $\lambda$, the predicted $B^e$'s are lower (higher) than the observed values, i.e., the actual $\lambda$'s are less (more) than what Eqn. (24) gives. This is consistent with the expectation that interactions (neglected in Eqn. (26)) will increase the effective thickness with increased density. If I fit to the 2/5 point, I obtain similar numbers, with the agreement worsening as we move off in density from 2/5.

The work of Yeh et al. [2] which involves an interplay of $m_a$ and $m_p$, and the work of Du et al. [4] which deals with $\nu > 1/2$, fall outside the present approach designed for $\nu < 1/2$, unless further assumptions about the role of filled (electronic) LL and particle-hole symmetry are made and justified. A sequel dealing with $m_a$ [8] will address these issues and provide details of the results merely sketched here. While the level crossings at the Fermi surface of Ref. [2] can be understood in the interacting theory in the sense described here, crossings below are probably related to inhomogeneous states [10].

In summary, the Hamiltonian theory of CF [6,7] provides a quantitative description of several magnetic phenomena in FQH states at zero and nonzero temperatures. A single parameter $\lambda$ extracted from one data point, provides a reasonable characterization of each sample at other temperatures, tilts, fillings and $B_\perp$. This theory accommodates enough interactions between CF to produce two distinct masses $m_a$ and $m_p$, and yet a mechanism for simulating free-fermion behavior in calculations of $B^e$ at $T = 0$. At $T \neq 0$, the interacting theory, with its (nonquadratic) dispersion relation and density of states, is needed to correctly predict $P$ and $1/T_1$.

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