Approximation of functions belonging to $L[0, \infty)$ by product summability means of its Fourier-Laguerre series

Kejal Khatri* and Vishnu Narayan Mishra

Abstract: In this paper, we have proved the degree of approximation of functions belonging to $L[0, \infty)$ by Harmonic-Euler means of its Fourier-Laguerre series at $x = 0$. The aim of this paper is to concentrate on the approximation properties of the functions in $L[0, \infty)$ by Harmonic-Euler means of its Fourier-Laguerre series associated with the function $f$.

Keywords: degree of approximation; Harmonic-Euler means; Fourier-Laguerre series; orthogonal polynomials and special functions

Mathematics subject classifications: 40C05; 40D25; 40G05; 41A25; 42A10

1. Introduction

Various researchers such as Gupta (1971), Singh (1977), Beohar and Jadia (1980), Lal and Nigam (2001), Nigam and Sharma (2010), Krasniqi (2013) and Sonker (2014) obtained the degree of approximation of $L[0, \infty)$ by Cesàro, Harmonic, Nörlund, Euler, $(C, 1)$ $(E, q)$, $(C, 2)(E, q)$ and Cesàro means, respectively. The degree of approximation of functions belonging to various classes through trigonometric Fourier approximation using different summability methods.
methods with monotone rows has been proved by many investigators like Khan (1974, 1973–1974, 1982), Mishra (2007), Mishra, Khatri, and Mishra (2012a, 2012b, 2013), Mishra and Khatri (2014), Mishra, Khatri, Mishra, and Deepmala (2014). A number of researchers Liu and Srivastava (2006), Alzer, Karayannakis, and Srivastava (2006), Bor, Srivastava, and Sulaiman (2012), Choi and Srivastava (1991) have proved interesting results in sequences and series using different type of linear summability operators. In Alghamdi and Mursaleen (2013) discussed Hankel matrix transformation of the Walsh-Fourier series and Alotaibi, and Mursaleen (2013) studied on applications of Hankel and regular matrices in Fourier series. In 2014, Mursaleen, and Mohiuddine (2014) discussed convergence methods for double sequences. In this paper, We have extended the previous known results which have already discussed above. The product summability methods are more powerful than the individual summability methods and thus give an approximation for wider class of functions than the individual methods.

Analysis of signals or time functions are of great importance, because it convey information or attributes of some phenomenon. The engineers and scientists use properties of Fourier approximation for designing digital filters. Especially, Psarakis, and Moustakides (1997) presented a new $L_2$ based method for designing the Finite Impulse Response (FIR) digital filters and get corresponding optimum approximations having improved performance.

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of $n^{th}$ partial sums $\{s_n\}$. Let $\{p_n\}$ be a non-negative sequence of constants, real or complex, and let us write

$$P_n = \sum_{k=0}^{n} p_k \neq 0 \ \forall \ n \geq 0, \ p_{-1} = 0 = P_{-1} \ \text{and} \ P_n \to \infty \ \text{as} \ n \to \infty.$$  

The series $\sum_{n=0}^{\infty} a_n$ is said to be Harmonic ($H_1$) - summable to $s$, if

$$H_1^n = \frac{1}{\log n} \sum_{k=0}^{n} \frac{s_k}{(n-k+1)} \to s, \ \text{as} \ n \to \infty.$$  

This method was introduced by Riesz (1924).

The $(E, 1)$ means is defined as the $n^{th}$ partial sum of $(E, 1)$ summability and we denote it by $E_1^n$. If

$$E_1^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k \to s, \ \text{as} \ n \to \infty,$$

the series $\sum_{n=0}^{\infty} a_n$ is said to be $(E, 1)$ - summable to sum $s$ Hardy (1949).

The product of $H_1$ summability with a $E^1$ summability defines $H_1 \cdot E^1$ summability. Thus the $H_1 \cdot E^1$ mean is given by

$$t_{n}^{\text{HE}} = \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(n-k+1)} E_1^k = \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \binom{k}{i} s_i.$$  

If $t_{n}^{\text{HE}} \to s$ as $n \to \infty$, then the infinite series $\sum_{n=0}^{\infty} a_n$ is said to be $H_1 \cdot E^1$ summable to the sum $s$.

The Fourier-Laguerre expansion of a function $f(x) \in L[0, \infty)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} b_n (f) L_n^{(\nu)}(x),$$  

(2)
where

\[ L_n^{(\beta)}(x) = \frac{1}{n!} x^\beta e^{-x} \frac{d^n}{dx^n} [x^{n+\beta} e^{-x}], \]

and \( L_n^{(\beta)}(x) \) stands the \( n^{th} \) degree Laguerre polynomial of order \( \beta > -1 \), defined by the generating function

\[ \sum_{n=0}^{\infty} L_n^{(\beta)}(x) y^n = (1 - y)^{-\beta-1} e^{-\frac{y}{1-y}}, \]

provided the integral in (3) exists. The elementary properties of Laguerre polynomials can be seen in Rainville (1960) and Szegö (1975). Let \( S_n(f;x) = \sum_{k=0}^{n} b_k(f) L_k^{(\beta)}(x) \), denote the partial sums, called Fourier-Laguerre polynomials of degree \( n \), of the first \( (n+1) \) terms of the Fourier-Laguerre series of \( f \) in (4). At the point \( x = 0 \),

\[ S_n(f;0) = \sum_{k=0}^{n} b_k(f) L_k^{(\beta)}(0) = \frac{1}{\Gamma(\beta+1)} \int_{0}^{\infty} z^{\beta} e^{-z} f(z) \sum_{k=0}^{n} L_k^{(\beta)}(z) \, dz, \]

since \( L_n^{(\beta)}(0) = \left( \begin{array}{c} k + \beta \\ \beta \end{array} \right) \) and \( \sum_{k=0}^{n} L_k^{(\beta)}(z) = L_n^{(\beta+1)}(z) \). Thus using \( S_n(f;0) \) and (1), we get

\[ t_n^{HE}(f;0) = \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(n-k+1) \Gamma(\beta+1) 2^k} \sum_{i=0}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) \int_{0}^{\infty} z^{\beta} e^{-z} f(z) L_n^{(\beta+1)}(z) \, dz. \]

We write

\[ \psi(z) = \frac{z^{\beta} e^{-z} [f(z) - f(0)]}{\Gamma(\beta+1)}. \]

2. Main result

The degree of approximation of functions belonging to \( L[0, \infty) \) by different matrix summability methods using Fourier-Laguerre expansion (2) at the point \( x = 0 \) has been determined by various investigators such as Gupta (1971), Singh (1977), Beohar and Jadia (1980), Lal and Nigam (2001), Nigam and Sharma (2010), Krasniqi (2013) and Sonker (2014). But till now, nothing seems to have been done so far to obtain the degree in approximation of functions \( f \in L[0, \infty) \) by its using Fourier-Laguerre expansion (2) at the point \( x = 0 \) using Harmonic-Euler summability methods with a suitable set of conditions and prove the following theorem:

**Theorem 2.1** If \( \{p_n\} \) is a positive non-increasing sequence of real number and the degree of approximation of Fourier-Laguerre expansion (2) at the point \( x = 0 \) using Harmonic-Euler summability means is given by

\[ t_n^{HE}(f;0) - f(0) = o(\eta(P_n)), \]

provided that

\[ \Psi(q) = \int_{0}^{1} |\psi(z)| \, dz = o(q^{\beta+1} \eta(1/q)), \quad q \to 0, \]
\[ \int_{a}^{\infty} e^{iz} \mathbf{A}(z) \, dz = \alpha(P_{n}^{2\beta+1/4} \eta(P_{n})), \]  

(9)

\[ \int_{P_{n}}^{\infty} e^{iz} \mathbf{A}(z) \, dz = \alpha(\eta(P_{n})), \quad P_{n} \to \infty \text{ as } n \to \infty, \]  

(10)

\{ \eta(q)/q \} \text{ is non-increasing in } q. \quad (11)

where \( \alpha \) is a fixed positive constant, \( \beta \in (-1, -1/2) \) and \( \eta(q) \) is a positive monotonic increasing function of \( q \) such that \( \eta(P_{n}) \to \infty \) as \( P_{n} \to \infty \) (as \( n \to \infty \)).

**Note 1.** Using condition (11), we get the inequality: \( \eta \left( \frac{e}{P_{n}} \right) \leq \pi \eta \left( \frac{1}{P_{n}} \right) \) for \( \left( \frac{e}{P_{n}} \right) \geq \left( \frac{1}{P_{n}} \right) \).

### 3. Lemmas

We use the following lemmas in the proof of Theorem 2.1.

**Lemma 3.1** Let \( \beta \) be an arbitrary real number, \( a \) and \( \alpha \) be fixed positive constants. Then

\[ L_{n}^{\beta}(X) = \begin{cases} O(P_{n}^{\beta}), & \text{if } 0 \leq X \leq a/P_{n}, \\ O\left(X^{-2(-1/4)\alpha}P_{n}^{2\beta-1/4}\right), & \text{if } a/P_{n} \leq X \leq \alpha, \end{cases} \]  

(12)

as \( P_{n} \to \infty \) as \( n \to \infty \).

**Proof** The proof is similar as in Szegö (1975, p. 177).

**Lemma 3.2** Let \( \beta \) be an arbitrary real number, \( \alpha > 0 \) and \( 0 < \xi < 4 \). Then

\[ \max e^{-x} X^{\beta/2+1/4} |L_{n}^{\beta}(X)| = \begin{cases} O\left(P_{n}^{\beta/2-1/4}\right), & \text{if } a \leq x \leq (4 - \xi)P_{n}, \\ O\left(P_{n}^{\beta/2-1/12}\right), & \text{if } x \geq \alpha, \end{cases} \]  

(13)

as \( P_{n} \to \infty \) as \( n \to \infty \).

**Proof** The proof is similar as in Szegö (1975, p. 177).

### Proof of Theorem 2.1

\[ t_{n}^{\text{HE}}(f;0) - f(0) = \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(n-k+1)2^{k}} \sum_{i=0}^{k} \binom{k}{i} \left( \frac{1}{\Gamma(\beta + 1)} \right) \int_{0}^{\infty} e^{-z} \psi(z) L_{n}^{\beta+1}(z) \, dz - f(0) \]  

(14)

\[ = \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(n-k+1)2^{k}} \sum_{i=0}^{k} \binom{k}{i} \left( \frac{1}{\Gamma(\beta + 1)} \right) \int_{0}^{\infty} e^{-z} (f(z) - f(0)) L_{n}^{\beta+1}(z) \, dz \]  

\[ = \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(n-k+1)2^{k}} \sum_{i=0}^{k} \binom{k}{i} \psi(z) L_{n}^{\beta+1}(z) \, dz \]  

\[ = \frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(n-k+1)2^{k}} \sum_{i=0}^{k} \binom{k}{i} \left( \frac{1}{\alpha/P_{n}} \right) \psi(z) L_{n}^{\beta+1}(z) \, dz \]  

\[ = \sum_{i=0}^{4} I_{i}, \quad \text{say}, \]
where

\[ |I_1| \leq \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) \left| \psi(z) \right| |L_n^{(\beta+1)}(z)| \, dz \]

\[ = \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) O(P_n^{\beta+1}) O\left( \frac{\alpha^{\beta+1} \eta(P_n/\alpha)}{P_n^{\beta+1}} \right) \]

\[ = O(P_n^{\beta+1}) O\left( \frac{\alpha^{\beta+1} \eta(P_n/\alpha)}{P_n^{\beta+1}} \right) \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) \]  

(15)

\[ \text{using Lemma 3.1 (first part) and condition (8).} \]

\[ |I_2| \leq \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) \left| \psi(z) \right| |L_n^{(\beta+1)}(z)| \, dz \]

\[ = \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) O(P_n^{2\beta+1/4}) \int_{a/P_n}^{a/P_n} |\psi(z)| z^{-(2\beta+3)/4} \, dz \]

\[ = O(P_n^{2\beta+1/4}) \int_{a/P_n}^{a/P_n} |\psi(z)| z^{-(2\beta+3)/4} \, dz \]

\[ = o(\eta(P_n)), \]

(16)

\[ \text{using Lemma 3.1 (second part) and condition (8), integrating by parts and using the argument as in Krasniqi (2013) and Nigam and Sharma (2010).} \]

\[ |I_3| \leq \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) \left| e^{z/2} z^{-(2\beta+3)/4} |\psi(z)| e^{-z/2} z^{(2\beta+5)/4} |L_n^{(\beta+1)}(z)| \right| \, dz \]

\[ = \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) O(P_n^{2\beta+1/4}) \int_{a/P_n}^{a/P_n} |e^{z/2} z^{-(2\beta+3)/4} |\psi(z)| \, dz \]

\[ = O(P_n^{2\beta+1/4}) O\left( P_n^{-2\beta+1/4} \right) O(\eta(P_n)) \]

\[ = o(\eta(P_n)), \]

(17)

\[ \text{using Lemma 3.2 and condition (9).} \]

\[ |I_4| \leq \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) \left| e^{z/2} z^{-(3\beta+5)/6} |\psi(z)| e^{-z/2} z^{(3\beta+5)/6} |L_n^{(\beta+1)}(z)| \right| \, dz \]

\[ = \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) O(P_n^{\beta+1/2}) \int_{a/P_n}^{a/P_n} |e^{z/2} z^{-(3\beta+5)/6} |\psi(z)| \, dz \]

\[ = \frac{1}{\log n} \sum_{k=0}^{n-1} \frac{1}{(n-k+1)^2} \sum_{i=0}^{k} \left( \frac{k}{(n-k+1)^2} \right) O(P_n^{\beta+1/2}) \int_{a/P_n}^{a/P_n} |e^{z/2} z^{1/3} |\psi(z)| \, dz \]

\[ = O(P_n^{\beta+1/2}) O\left( P_n^{-\beta+1/2} \right) o(\eta(P_n)) \]

\[ = o(\eta(P_n)), \]

(18)

\[ \text{using Lemma 3.2 and condition (10), combining (15)–(18) and putting into (14). The proof of the theorem is completed.} \]
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Author details
Kejal Khatri
E-mail: kejal090@gmail.com
ORCID ID: http://orcid.org/0000-0002-3425-1727

Vishnu Narayan Mishra
E-mails: vishnunaraynmishra@gmail.com, vishnunaraynmishra@yahoo.co.in
ORCID ID: http://orcid.org/0000-0002-2159-7710

1 Department of Applied Mathematics & Humanities, S. V. National Institute of Technology, Surat 395 007, Gujarat, India.

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