On the Topology of the Symmetry Group of the Standard Model

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We study the topological structure of the symmetry group of the standard model, $G_{SM} = U(1) \times SU(2) \times SU(3)$. Locally, $G_{SM} \cong S^1 \times (S^3)^2 \times S^5$. For $SU(3)$, which is an $S^3$-bundle over $S^5$ (and therefore a local product of these spheres) we give a canonical gauge i.e. a canonical set of local trivializations. These formulae give the matrices of $SU(3)$ in terms of points of spheres. Globally, we prove that the characteristic function of $SU(3)$ is the suspension of the Hopf map $S^3 \xrightarrow{h} S^2$. We also study the case of $SU(n)$ for arbitrary $n$, in particular the cases of $SU(4)$, a flavour group, and of $SU(5)$, a candidate group for grand unification. We show that the 2-sphere is also related to the fundamental symmetries of nature due to its relation to $SO^0(3,1)$, the identity component of the Lorentz group, a subgroup of the symmetry group of several gauge theories of gravity.

1. INTRODUCTION

As is well known, the symmetry group of the electroweak and strong forces (standard model) before spontaneous symmetry breaking is given by (Taylor, 1976)

$$G_{SM} = U(1) \times SU(2) \times SU(3)$$

After symmetry breaking, however, $G_{SM}$ breaks down to $G_{SM}' = U(1) \times SU(3)$, the remaining exact symmetry of the electromagnetic and color forces. In any case,

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$U(1)$ is the circle or 1-sphere $S^1$, the unit complex numbers; while $SU(2)$, given by all complex matrices $A = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ with $\det A = 1$ is the 3-sphere $S^3$ or unit quaternions, since if $z = \alpha + i\beta$ and $w = \gamma + i\delta$ then the condition of unit determinant is $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

In the mathematical literature it is well known that for all $n = 2, 3, \ldots$ the groups $SU(n)$ are principal $SU(n-1)$-bundles over the $(2n-1)$-spheres i.e. that one has pairs of maps (Steenrod, 1951)

$$SU(n-1) \xrightarrow{\iota} SU(n) \xrightarrow{\pi_n} S^{2n-1}$$

where $\iota$ is the canonical inclusion and $\pi_n$ maps a matrix $A$ to $Ae_0$, where $e_0$ is the vector whose entries are all 0 except the last one which is 1. In particular for $n = 3$ one has the $SU(2)$-bundle

$$SU(2) \rightarrow SU(3) \xrightarrow{\pi_3} S^5$$

which in particular means that locally $SU(3) \cong S^5 \times S^3$ since $SU(2) \cong S^3$; moreover, according to the theory of bundles, the isomorphism classes of $SU(2)$-bundles over $S^5$ are in one-to-one correspondence with the 4-th homotopy group of $SU(2)$ i.e. $k_{SU(2)}(S^5) \leftrightarrow \Pi_4(SU(2)) \cong \Pi_4(S^3) \cong \mathbb{Z}_2 = \{0, 1\}$: 0 corresponds to the trivial bundle, $S^5 \times S^3$ while 1 corresponds to $SU(3)$ (see Section 3.2). In other words, $SU(3)$, the symmetry group of the strong interactions, is the unique (up to isomorphism) non-trivial $SU(2)$-bundle over the 5-sphere, and as this result shows, it is also constructed from spheres, though not globally. This means that

$$G_{SM} = S^1 \times (S^3)^2 \times S^5$$

and, after symmetry breaking, $G'_{SM} = S^1 \times S^3 \times S^5$.

For higher $n$, however, uniqueness is lost since, for example, for $n = 4$ and $n = 5$ one has the bundles

$$SU(3) \rightarrow SU(4) \xrightarrow{\pi_4} S^7$$

and

$$SU(4) \rightarrow SU(5) \xrightarrow{\pi_5} S^9$$

respectively, and $k_{SU(3)}(S^7) \leftrightarrow \Pi_6(SU(3)) \cong \mathbb{Z}_6$ and $k_{SU(4)}(S^9) \leftrightarrow \Pi_8(SU(4)) \cong \mathbb{Z}_{24}$ (EDM, 1993). Notice however that locally any $SU(n)$ is a topological product of odd-dimensional spheres: $SU(4) = S^7 \times SU(3) = S^7 \times S^5 \times S^3$, $SU(5) = S^9 \times SU(4) = S^9 \times S^7 \times S^5 \times S^3$, $SU(n) = S^{2n-1} \times S^{2n-3} \times \ldots \times S^5 \times S^3$. This
expression allows us to define a formula which gives, in a canonical way, any element of $SU(n)$ in terms of points of spheres.

It is interesting to remark here that a typical gauge theory, say on Minkowski space-time $M^4$, is a theory of connections on the trivial bundle $M^4 \times G$, where $G$ is the symmetry group, coupled to sections of associated bundles; in the case of the strong and weak interactions the groups themselves are principal bundles: for the weak case, $SU(2)$ is the total space of the Hopf bundle $S^1 \to S^3 \stackrel{\kappa_2}{\to} S^2$.

In Section 2 we briefly review the global construction of the bundle $SU(2) \to SU(3) \stackrel{\pi_3}{\to} S^5$ and construct a canonical set of local trivializations of $SU(3)$, starting from the (canonical) homogeneous coordinates on $\mathbb{C}P^2$, the complex projective plane. These formulae exhibit $SU(3)$ as a local product of spheres and moreover, give explicit expressions for all the matrices of $SU(3)$ in terms of the spheres $S^5$ and $S^3$.

The above choice of coordinates is natural since for all $n \geq 2$, $S^{2n-1}$ is a principal bundle over $\mathbb{C}P^{n-1}$ with fiber $S^1$ (complex Hopf bundles):

$$
\begin{array}{ccc}
S^1 & \downarrow & \\
SU(n-1) & \to & SU(n) \stackrel{\pi_n}{\to} S^{2n-1} \\
& \downarrow^{\kappa_n} & \\
& \mathbb{C}P^{n-1} & \\
\end{array}
$$

and $\mathbb{C}P^{n-1}$ has $n$ canonical charts defining its homogeneous coordinates. Then the bundle $\pi_n$, for all $n$, can be locally trivialized in a canonical way, $n$ being the number of local trivializations. In the following the Hopf map $\kappa_2$ will be denoted by $h$. If $\left(\begin{array}{c} z \\ w \end{array}\right) \in S^3 \subset \mathbb{C}^2$ ($|z|^2 + |w|^2 = 1$) and $\mathbb{C}P^1$ is identified with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ then $h$ is given by

$$
h\left(\begin{array}{c} z \\ w \end{array}\right) = \begin{cases} z/w, w \neq 0 \\ \infty, w = 0. \end{cases}
$$

It can be proved that $h$ is essential i.e. it is not homotopic to a constant map (Spanier, 1966).

In Section 3 we prove that the characteristic (or clutching) map of $SU(3)$ is the suspension of the Hopf map $S^3 \stackrel{h}{\to} S^2$. Since the clutching map allows to construct the bundle, then $SU(3)$ is built from information contained in the Hopf map. This map, besides having great importance in homotopy theory, plays a relevant rôle
in physics e.g. in the geometrical description of the spin $\frac{1}{2}$ system (Ashtekar and Schilling, 1994; Corichi and Ryan, 1997), and the Dirac monopole of unit magnetic charge (Wu and Yang, 1975).

In Section 4 we investigate the general case of $SU(n)$ and, using a result of Steenrod for $U(n)$, we prove that for even $n$, $n \geq 2$, the characteristic map $g_{n+1} : S^{2n} \to SU(n)$ of $SU(n+1)$ is a homotopy lifting of the $(2n - 3)$-th suspension of $h$. (If $Z \xrightarrow{p} Y$ is a projection and $X \xrightarrow{f} Y$ is a continuous function, then $p$ lifts $f$ if there is a continuous function $X \xrightarrow{g} Z$ such that $p \circ g = f$. The lifting is up to homotopy if $p \circ g \sim f$.) The case of $SU(5)$ is interesting since it is a candidate group for grand unification (Mohapatra, 1986). On the other hand, for odd $n$, $n \geq 3$, $\pi_n \circ g_{n+1}$ is inessential. A particular case is $SU(4)$, which is a flavour group.

In Section 5 we briefly discuss how the 2-sphere $S^2$ appears in the context of the symmetry group of the fundamental interactions, due to the canonical isomorphism between its conformal group and $SO^0(3,1)$, the proper orthochronous Lorentz group.

2. THE BUNDLE $S^3 \to SU(3) \xrightarrow{\pi_3} S^5$

2.1. The groups $U(3)$ and $SU(3)$

The $n$-dimensional complex vector space $\mathbb{C}^n$ equipped with the Hermitian scalar product $\langle \bar{z}, \bar{w} \rangle = \sum_{i=1}^{n} \bar{z}_i w_i$ is a Hilbert space. The $n \times n$ complex matrices which leave $\langle , \rangle$ invariant form the group $U(n)$ i.e. $U(n) = \text{Aut}(\mathbb{C}^n, \langle , \rangle)$: the group of automorphisms of $\mathbb{C}^n$ as a Hilbert space. If $A \in U(n)$ and $A^*$ is the transpose conjugate matrix, then $A^* A = I$ i.e. $A^* = A^{-1}$, so $| \det A | = 1$ and $\dim_{\mathbb{R}} U(n) = n^2$.

The topology of $U(n)$ is inherited from the vector space of $n \times n$ complex matrices, which is isomorphic to Euclidean space $E^{2n^2}$. $U(n)$ is a Lie group and $SU(n)$ is the closed Lie subgroup consisting of matrices whose determinant is 1. Since $U(n)$ is compact, $SU(n)$ is also compact.

For $n = 3$, $SU(3)$ is 2-connected i.e. $\Pi_k(SU(3)) = 0$ for $k = 1, 2$, and $\Pi_3(SU(3)) \cong \mathbb{Z})$. Topologically, $U(3) \cong SU(3) \times U(1)$, so $U(3)$ is connected but not 1-connected since $\Pi_1(U(3)) \cong \mathbb{Z}$.

2.2. The inclusion and action $SU(2) \to SU(3)$

Let $\iota : SU(2) \to SU(3), \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} z & w & 0 \\ -\bar{w} & \bar{z} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be the inclusion of $SU(2)$
into $SU(3)$, call $SU(2)' = \iota(SU(2))$. Clearly $SU(2)' \cong SU(2)$, both topologically and as a group. The right action $SU(3) \times SU(2)' \to SU(3)$ is given by matrix multiplication $(B,A) \mapsto BA$ i.e.

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \sigma & \varepsilon & \varphi \\ \kappa & \lambda & \mu \end{pmatrix} \begin{pmatrix} z & w & 0 \\ -\bar{w} & \bar{z} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha z - \beta \bar{w} & \alpha w + \beta \bar{z} & \gamma \\ \delta z - \varepsilon \bar{w} & \delta w + \varepsilon \bar{z} & \varphi \\ \kappa z - \lambda \bar{w} & \kappa w + \lambda \bar{z} & \mu \end{pmatrix}$$

Let $q: SU(3) \to SU(3)/SU(2)'$ be the quotient map, i.e. $q(B) = [B]$, where $SU(3)/SU(2)'$ is the orbit space $\{[B]\}_{B \in SU(3)'}$ with the quotient topology, and $[B] = BSU(2)'$, in particular $[I] = SU(2)'$. Notice that $|\gamma|^2 + |\varphi|^2 + |\mu|^2 = 1$ i.e. $\begin{pmatrix} \gamma \\ \varphi \\ \mu \end{pmatrix} \in S^5 \subset \mathbb{C}^3$. It is easy to verify that the following diagram commutes:

$$\xymatrix{SU(3) \ar[r]^q & SU(3)/SU(2)' \ar[r]^\kappa & S^5 \ar[r]^{\pi_3} & S^5}$$

where $\pi_3(B) = \begin{pmatrix} \gamma \\ \varphi \\ \mu \end{pmatrix}$ and $\kappa([B]) = B \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, in particular $\kappa([I]) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $\kappa$ turns out to be a homeomorphism with inverse $\kappa^{-1} \begin{pmatrix} \gamma \\ \varphi \\ \mu \end{pmatrix} = B'SU(2)'$ for any $B' = \begin{pmatrix} \gamma \\ \varphi \\ \mu \end{pmatrix} \in SU(3)$ (an explicit formula for $\kappa^{-1}$ will be given in Section 2.3).

Clearly $\pi_3^{-1} \left( \begin{pmatrix} \gamma \\ \varphi \end{pmatrix} \right) = B'SU(2)' \cong SU(2)' \cong SU(2) \cong S^3$, so $S^3$ is the fiber of $\pi_3$.

2.3. Local trivializations

Consider the $S^1$-bundle $S^5 \xrightarrow{\kappa_3} \mathbb{C}P^2$, where the complex projective plane is the space of complex lines through the origin in $\mathbb{C}^3$; $\mathbb{C}P^2$ has three canonical charts given by the open sets $V_k = \{z(\mathbb{C} \setminus \{0\}) \mid \bar{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \text{ and } z_k \neq 0\} \subset \mathbb{C}P^2$, for $k = 1,2,3$, and the homeomorphisms $V_k \to \mathbb{C}^2$ map $z(\mathbb{C} \setminus \{0\})$ to $(\xi_i, \xi_j) = (z_i/z_k, z_j/z_k)$ with $i,j,k$ in cyclic order, the $\xi_i$ are called homogeneous coordinates.
Then the pre-images of $V_k$ by the projection $\kappa_3$ define three open sets in $S^5$ given by $U_k = \kappa_3^{-1}(V_k) = \left\{ \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} \right) \mid z_k \neq 0 \right\} \equiv S^5_k \subset S^5$; $\bigcup_{i=1}^3 U_i = S^5$ with $U_i \cap U_j \neq \emptyset$ for all $i, j$ and $(0, 0, 1) \in U_3 \subset S^5$ but $(0, 0, 1) \notin U_1, U_2$. Notice that the complements of $S^5_k$ with respect to $S^5$ are homeomorphic to $S^3$: $(S^5_3)^c = S^5 \setminus S^5_3 = \left\{ \left( \begin{array}{c} z_1 \\ z_2 \\ 0 \end{array} \right) \mid |z_1|^2 + |z_2|^2 = 1 \right\} \cong S^3$, and analogous formulae for $S^5_1$ and $S^5_2$: $(S^5_k)^c$ are closed sets in $S^5$. We shall trivialize the bundle $SU(3) \to S^5$ over the $U_k$’s.

In order to construct local sections of the bundle $\pi_3$, consider the following complex matrices:

$$
\tilde{C} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}, \quad \text{and } \tilde{A} = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}
$$

It is easy to verify that the three column vectors in each of them are linearly independent if, respectively, $c \neq 0$, $b \neq 0$ and $a \neq 0$. In the three cases we take $|a|^2 + |b|^2 + |c|^2 = 1$. By the Gram-Schmidt procedure we can construct unitary matrices $\tilde{C}$, $\tilde{B}$, and $\tilde{A}$ and then, multiplying each of them from the right by the matrix $\left( \begin{array}{cc} z & 0 \\ 0 & I \end{array} \right)$, where $z^{-1} = \text{det}\tilde{C}$, $\text{det}\tilde{B}$, or $\text{det}\tilde{A}$, we obtain the following elements of $SU(3)$:

$$
C = \begin{pmatrix} |a|^2 & -ab & a \\ \frac{c\sqrt{1-|b|^2}}{1-|b|^2} & \frac{\sqrt{1-|b|^2}}{1-|b|^2} & 0 \\ -\bar{a}b & \bar{b}c & \frac{b}{\sqrt{1-|b|^2}} \end{pmatrix}, \quad B = \begin{pmatrix} |a|^2 & -ac & a \\ \frac{-b\sqrt{|a|^2 + |b|^2}}{\sqrt{|a|^2 + |b|^2}} & \frac{\sqrt{|a|^2 + |b|^2}}{\sqrt{|a|^2 + |b|^2}} & 0 \\ \frac{-\bar{a}c}{\sqrt{|a|^2 + |b|^2}} & \frac{-\bar{b}c}{\sqrt{|a|^2 + |b|^2}} & \frac{b}{\sqrt{|a|^2 + |b|^2}} \end{pmatrix}, \quad A = \begin{pmatrix} |a|^2 & -ac & a \\ \frac{\sqrt{1-|c|^2}}{\sqrt{1-|c|^2}} & \frac{\sqrt{1-|c|^2}}{\sqrt{1-|c|^2}} & 0 \\ \frac{-b\sqrt{1-|c|^2}}{a\sqrt{1-|c|^2}} & \frac{-b\sqrt{1-|c|^2}}{a\sqrt{1-|c|^2}} & \frac{b}{\sqrt{1-|c|^2}} \end{pmatrix}
$$

with $\text{det}\tilde{C} = c/|c|$, $\text{det}\tilde{B} = -b/|b|$ and $\text{det}\tilde{A} = a/|a|$.

(These formulae give an explicit expression for the map $\kappa^{-1}$ of Section 2.2: given $\left( \begin{array}{c} \gamma \\ \varphi \\ \mu \end{array} \right) \in S^5$, we choose $B'$ equal to $A$, $B$ or $C$ if, respectively, $\gamma$, $\varphi$ or $\mu$ is
\( \neq 0 \). We define local sections \( \sigma_k: S^5_k \rightarrow SU(3) \) as follows:

\[
\begin{align*}
\sigma_1: S^5_1 & \rightarrow SU(3), \\
\begin{pmatrix} a \\ b \\ c \end{pmatrix} & \mapsto \sigma_1 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A, \\
\sigma_2: S^5_2 & \rightarrow SU(3), \\
\begin{pmatrix} a \\ b \\ c \end{pmatrix} & \mapsto \sigma_2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = B, \\
\sigma_3: S^5_3 & \rightarrow SU(3), \\
\begin{pmatrix} a \\ b \\ c \end{pmatrix} & \mapsto \sigma_3 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = C.
\end{align*}
\]

If \( \pi: P \rightarrow X \) is a principal \( G \)-bundle over \( X \), and \( \sigma_\beta: U_\beta \rightarrow P \) are local sections, then \( \varphi_\beta: \pi^{-1}(U_\beta) \equiv P_\beta \rightarrow U_\beta \times G \), where \( \varphi_\beta(p) = (x, \gamma_\beta(p)) \) with \( x = \pi(p) \) and \( p = \sigma_\beta(\pi(p)) \cdot \gamma_\beta(p) \), are local trivializations. If \( \varphi_\alpha \) and \( \varphi_\beta \) are local trivializations and \( U_\alpha \cap U_\beta \neq \emptyset \), then \( \varphi_\beta \circ \varphi_\alpha^{-1}: (U_\alpha \cap U_\beta) \times G \rightarrow (U_\alpha \cap U_\beta) \times G \) satisfies \( \varphi_\beta \circ \varphi_\alpha^{-1}(x,g) = (x,g_{\beta\alpha}(x)) \), where \( g_{\beta\alpha}: U_\alpha \cap U_\beta \rightarrow G \) are the transition functions and \( \sigma_\beta(x) \cdot g_{\beta\alpha}(x) = \sigma_\alpha(x) \). In our case, with \( G = SU(2), P = SU(3), X = S^5 \), \( \beta = k = 1, 2, 3, U_k = S^5_k \), and \( P_k = SU(3)_k = \left\{ \begin{pmatrix} \cdot \cdot \cdot z_1 \\ \cdot \cdot \cdot z_2 \\ \cdot \cdot \cdot z_3 \end{pmatrix} \in SU(3) \mid z_k \neq 0 \right\} \), the local trivializations of the \( SU(2) \)-bundle \( p \equiv \pi_3: SU(3) \rightarrow S^5 \) are:

\[
\begin{align*}
\varphi_1: SU(3)_1 & \rightarrow S^5_1 \times SU(2)', \\
\varphi_1(R) & = (p(R), (\sigma_1(p(R)))^{-1}R) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, A^*R, \\
\varphi_2: SU(3)_2 & \rightarrow S^5_2 \times SU(2)', \\
\varphi_2(S) & = (p(S), (\sigma_2(p(S)))^{-1}S) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, B^*S, \\
\varphi_3: SU(3)_3 & \rightarrow S^5_3 \times SU(2)', \\
\varphi_3(T) & = (p(T), (\sigma_3(p(T)))^{-1}T) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, C^*T.
\end{align*}
\]

The matrices \( A^*R, B^*S \) and \( C^*T \) are of the form \( \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \) with \( D \in SU(2) \). \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) exhibit the local structure of \( SU(3) \). The transition functions are:

\[
g_{12}: S^5_1 \cap S^5_2 \rightarrow SU(2)', \\
g_{12} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^*B,
\]

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$g_{23} : S_2^5 \cap S_3^5 \rightarrow SU(2)'$, \( g_{23} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = B^* C, \)

and

$g_{31} : S_3^5 \cap S_1^5 \rightarrow SU(2)'$, \( g_{31} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = C^* A. \)

Notice that \( I \in SU(3)_3 \) but \( I \notin SU(3)_j \) for \( j = 1, 2 \), so \( SU(3)_3 \) is an open neighbourhood of the identity. The inverses of the local trivializations are given by

\[
\psi_3 : S_3^5 \times SU(2) \rightarrow SU(3)_3, \quad \left( \begin{array}{c} a \\ b \\ c \end{array} \right), R_3 \mapsto CR_3',
\]

\[
\psi_2 : S_2^5 \times SU(2) \rightarrow SU(3)_2, \quad \left( \begin{array}{c} a \\ b \\ c \end{array} \right), R_2 \mapsto BR_2',
\]

and

\[
\psi_1 : S_1^5 \times SU(2) \rightarrow SU(3)_1, \quad \left( \begin{array}{c} a \\ b \\ c \end{array} \right), R_1 \mapsto AR_1'
\]

with \( \psi_i = \varphi_i^{-1} \) after identifying \( SU(2) \cong SU(2)' \), and \( R_k' = \left( \begin{array}{cc} R_k & 0 \\ 0 & 0 \end{array} \right) \), \( k = 1, 2, 3 \). These formulae give all elements of \( SU(3) \) in terms of points of the 3- and 5-spheres.

With the help of the above formulae the set of matrices of \( SU(3) \) can be divided into seven disjoint subsets: \( SU(3)_{123}, SU(3)_{i,j,k} \) and \( SU(3)_{i,j,k} \), respectively the pieces of \( SU(3) \) lying over \( S_{123}^5 = S_1^5 \cap S_2^5 \cap S_3^5 \), \( S_{i,j,k}^5 = S_5^5 \setminus (S_j^5 \cup S_k^5) \) and \( S_{i,j,k}^5 = S_i^5 \cap S_j^5 \setminus S_{123}^5 \), with \( i, j, k \in \{1, 2, 3\} \) in cyclic order:

\[
\left( \begin{array}{ccc}
\frac{|c|^2 z/c + ab \bar{w}}{\sqrt{1-|b|^2}} & \frac{|c|^2 w/c - ab \bar{z}}{\sqrt{1-|b|^2}} & a \\
-\bar{w} \sqrt{1-|b|^2} & \bar{z} \sqrt{1-|b|^2} & b \\
-\bar{a} z + bc \bar{w} & -\bar{a} w + bc \bar{z} & c
\end{array} \right) \in SU(3)_{123},
\]

\[
|a|^2 + |b|^2 + |c|^2 = 1.0 < |a|, |b|, |c| < 1;
\]

\[
\left( \begin{array}{ccc}
0 & 0 & e^{i \varphi} \\
0 & we^{-i \varphi} & 0 \\
\bar{w} & \bar{z} & 0
\end{array} \right) \in SU(3)_{1,23};
\]
\[
\left(\begin{array}{ccc}
-ze^{-i\varphi} & -we^{-i\varphi} & 0 \\
0 & 0 & e^{i\varphi} \\
-w & \bar{z} & 0
\end{array}\right) \in SU(3)_{2,31};
\]
\[
\left(\begin{array}{ccc}
z e^{-i\varphi} & w e^{-i\varphi} & 0 \\
-w & \bar{z} & 0 \\
0 & 0 & e^{i\varphi}
\end{array}\right) \in SU(3)_{3,12};
\]
\[
\left(\begin{array}{ccc}
-z\bar{b} & -w\bar{b} & a \\
\frac{a}{b} & w|a|^2 & b \\
-w & \frac{a}{\bar{z}} & 0
\end{array}\right) \in SU(3)_{12,3},
\]
\[
|a|^2 + |b|^2 = 1, \ a, b \neq 0;
\]
\[
\left(\begin{array}{ccc}
-z|b| & -w|b| & 0 \\
\frac{\bar{b}w\bar{c}}{|b|} & \frac{\bar{b}w\bar{c}}{|b|} & b \\
-|b|\bar{w} & \frac{b}{|b|\bar{z}} & c
\end{array}\right) \in SU(3)_{23,1},
\]
\[
|b|^2 + |c|^2 = 1, \ b, c \neq 0;
\]
\[
\left(\begin{array}{ccc}
\frac{|c|^2 z}{c} & \frac{|c|^2 w}{c} & a \\
-w & \frac{c}{\bar{z}} & 0 \\
-\bar{a} & -\bar{a}w & c
\end{array}\right) \in SU(3)_{31,2},
\]
\[
|a|^2 + |c|^2 = 1, \ a, c \neq 0; \text{with } |z|^2 + |w|^2 = 1 \text{ and } \varphi \in [0, 2\pi).
\]

**Remark.** The above results can be extended to the bundles
\[
U(n - 1) \to U(n) \xrightarrow{p_n} S^{2n - 1}
\]
i.e. to the unitary groups. In particular for \(n = 3\) we have the pair of maps
\[
U(2) \xrightarrow{\iota} U(3) \xrightarrow{p_3} S^5
\]
with \(U(2) = \left\{ \left(\begin{array}{cc}
z & w \\
\bar{w}e^{i\lambda} & -\bar{z}e^{i\lambda}
\end{array}\right) \mid |z|^2 + |w|^2 = 1, \lambda \in [0, 2\pi) \right\} \). The local trivializations of \(U(3), \phi_k : U(3)_k \to S^5_k \times U(2)'_k, \ k = 1, 2, 3 \) and where \(U(2)' = \iota(U(2))\) are given by the same formulae as those for \(SU(3)\), with the matrices \(A, B\) and \(C\) respectively replaced by the matrices \(\hat{A}, \hat{B}\) and \(\hat{C}\).

### 3. \(SU(3)\) FROM THE \(N = 2\) HOPF BUNDLE

#### 3.1. Suspension
The suspension of a topological space \( X \) is the quotient space given by

\[
SX = \frac{X \times I}{X \times \{0\}, X \times \{1\}} = \{(x,t) | (x,t) \in X \times I\}
\]

with

\[
[x,t] = \begin{cases} 
(x,t), & t \in (0,1) 
X \times \{0\}, & t = 0 
X \times \{1\}, & t = 1
\end{cases}
\]

This means that in the product \( X \times I \), \( X \times \{0\} \) has been identified to one point, and \( X \times \{1\} \) has been identified to another point. Intuitively, it is clear that \( SS^0 \cong S^1 \), \( SS^1 \cong S^2 \), ..., \( SS^{n-1} \cong S^n \). The suspension of a continuous function is defined by \( Sf([x,t]) = [f(x), t] \), which satisfies the functorial properties \( Sid_X = id_{SX} \) and \( Sg \circ f = Sg \circ Sf \) if \( f : X \to Y \) and \( g : Y \to Z \). If \( p_Z : Z \times I \to SZ \) is the projection \( p(z,t) = [z,t] \) then the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \iota_0 & & \downarrow \iota_0 \\
X \times I & \xrightarrow{f \times id} & Y \times I \\
p_X & & \downarrow p_Y \\
SX & \xrightarrow{Sf} & SY
\end{array}
\]

If \( H : X \times I \to Y \) is a homotopy between \( h_0 \) and \( h_1 \) then \( SH : SX \times I \to SY \) given by \( SH([x,t], t') = [H(x,t'), t] \) i.e. \( (SH)_t = SH_t \) is a homotopy between \( Sh_0 \) and \( Sh_1 \). \( SH \) is called the suspension of the homotopy. Then there is a well defined function between homotopy classes of maps \( S : [X, Y] \to [SX, SY] \), \([f] \to S([f]) := [Sf] \).

If \( X \) is a pointed space with base point \( x_0 \), then the reduced suspension of \( X \), \( S_r X \) is defined by

\[
S_r X = \frac{X \times I}{X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I}
\]

i.e. all the points in \( X \times \{0\} \), \( X \times \{1\} \) and \( \{x_0\} \times I \) are identified to one point. In this case its elements are given by

\[
[x,t] = \begin{cases} 
X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I, & x = x_0, \text{ all } t \in I 
\{(x,t)\}, & t = 0 \text{ or } 1, \text{ all } x \in X 
\end{cases}
\]

\( \tilde{x}_0 = [x_0, t] \) is the base point of \( S_r X \). If \( f : X \to Y \) preserves the base points i.e. if \( f(x_0) = y_0 \), then \( S_r f(\tilde{x}_0) = y_0 \), and if \( h_0 \sim h_1 \) (rel \( x_0 \)) then \( Sh_0 \sim \sim Sh_1 \) (rel \( \tilde{x}_0 \)).
(rel $x_0$ means that the homotopy $H$ preserves the base point.) Also, there is a homeomorphism $\varphi^{-1}_{n+1} : S_t S^n \to S^{n+1}$ given by

$$\varphi^{-1}_{n+1}([x, t]) = \begin{cases} p^{-1}_-(2t\vec{x} + (1 - 2t)x_0), & t \in [0, 1/2] \\ p^+_{-(2 - 2t)\vec{x} + (2t - 1)x_0), & t \in [1/2, 1] \end{cases}$$

where: $\vec{x}_0 = (1, 0, \ldots, 0) \in S^{n+1} = \{(x_1, \ldots, x_{n+2}) \mid \sum_{i=1}^{n+2} x_i^2 = 1\} \subset \mathbb{R}^{n+2}$ is the base point, $S^n = \{\vec{x} \in S^{n+1} \mid x_{n+2} = 0\}$, and if $H_+ = \{\vec{x} \in S^{n+1} \mid x_{n+2} \geq 0\}$, $H_- = \{\vec{x} \in S^{n+1} \mid x_{n+2} \leq 0\}$ and $D^{n+1} = \{(x_1, \ldots, x_{n+1}, 0) \mid \sum_{i=1}^{n+1} x_i^2 \leq 1\}$ then $p_{(+)} : H_+ \to D^{n+1}$ are the homeomorphisms given by $(x_1, \ldots, x_{n+2}, 0) \mapsto (x_1, \ldots, x_{n+1}, 0)$; inverses $(x_1, \ldots, x_{n+2}, 0) \mapsto (x_1, \ldots, x_{n+1}, 0)$ with inverses $(x_1, \ldots, x_{n+1}, 0) \mapsto (x_1, \ldots, x_{n+1}, +(-)\sqrt{1 - \sum_{i=1}^{n+1} x_i^2})$ respectively (Spanier, 1966). In particular,

$$\vec{x}_0 = S^n \times \{0\} = S^n \times \{1\} = \{\vec{x}_0\} \times I \varphi^{-1}_{n+1} \vec{x}_0$$

and if $\vec{x} \neq \vec{x}_0$ then $[\vec{x}, 1/2] = \{(\vec{x}, 1/2)\} \varphi^{-1}_{n+1} \vec{x}$. The inverse homeomorphism is given by the following formulae: $\vec{x}_0 \mapsto \vec{x}_0$, if $\vec{x} \in S^n$ and $\vec{x} \neq \vec{x}_0$ then $\vec{x} \mapsto [\vec{x}, 1/2] = \{(\vec{x}, 1/2)\}$, $(0, \ldots, 0, 1) = N$ (north pole) $\mapsto [-\vec{x}_0, 3/4] = \{(-\vec{x}, 3/4)\}$, $(0, \ldots, -1) = S$ (south pole) $\mapsto [-\vec{x}_0, 1/4] = \{(-\vec{x}, 1/4)\}$, and if $\vec{x} \in H_+ \setminus S^n$, $\vec{x} \neq S, N$, then

$$\varphi_{n+1}(\vec{x}) = [\vec{z}(\vec{x}), t_{+}(\vec{x})] = \{(\vec{z}(\vec{x}), t_{+}(\vec{x})\{\vec{x})\}$$

with

$$\vec{z}(\vec{x}) = \frac{2(1 - x_1)x_1 - x_2^2, 2(1 - x_1)x_2, \ldots, 2(1 - x_1)x_{n+1}}{2(1 - x_1) - x_{n+2}}$$

and $t_{+}(-)(\vec{x}) = 1/2 + (-)\frac{x_{n+2}^2}{4(1 - x_1)}$.

3.2. $SU(3)$ from the Hopf map $h$

Let $G$ be a path connected topological group. Then the set of isomorphism classes of principal $G$-bundles over the $n$-sphere $k_G(S^n)$ is in one-to-one correspondence with $\Pi_{n-1}(G)$ (Steenrod, 1951). This can be understood from the fact that the $n$-sphere can be covered by two open sets $U_1, U_2$, which are homeomorphic to $n$-balls and contain $S^{n-1}$, and the fact that any bundle over an $n$-ball is trivial. Using these trivializations there is only one transition function $g_{12} : U_1 \cap U_2 \to G$,
for a bundle $\xi$. Then we associate to $\xi$ the map $g_{12} \mid_{S^{n-1}} : S^{n-1} \to G$, called the characteristic map of $\xi$. Therefore, if the characteristic maps of two bundles are in the same homotopy class, then the corresponding bundles are isomorphic, and a bundle is trivial if and only if its characteristic map is null-homotopic. Notice that in our construction of the local charts for $SU(3)$ we have used a different trivialization. In the following we shall consider the bundles $SU(n-1) \to SU(n) \overset{\pi_{n}}{\to} S^{2n-1}$ and call $g_n : S^{2n-2} \to SU(n-1)$ the corresponding characteristic maps.

To study the case $n=3$ we need the following

**Proposition.** The successive suspensions of the Hopf map, $S_r h : S^4 \to S^3$, $S^{2r}_r h : S^5 \to S^4$, ... are essential (Steenrod and Epstein, 1962).

As a consequence, we have the

**Proposition.** $SU(3)$ is determined by the suspension of the Hopf map.

**Proof.** For $n = 3$, $k_{SU(2)}(S^5) \cong [S^4, S^3] \cong \Pi_3(S^3) \cong \mathbb{Z}_2 = \{0, 1\}$ and $g_3 : S^4 \to S^3$. By the proposition above $S_r h$ is essential. To see that $g_3$ is also essential we will show that the bundle $SU(3) \overset{\pi_{3}}{\to} S^5$ is not trivial. By (EDM, 1993), $\Pi_4(SU(3)) \cong 0$, on the other hand $\Pi_4(S^3 \times SU(2)) \cong \Pi_4(S^5) \times \Pi_4(S^3) \cong 0 \times \mathbb{Z}_2 \cong \mathbb{Z}_2$. Hence $SU(3)$ is not isomorphic to the trivial bundle. Since $\Pi_4(S^3) \cong \mathbb{Z}_2$ we have that $[g_3] = [S_r h]$. QED

4. **THE CASE OF $SU(N)$**

4.1. **$SU(4)$**

For $n = 4$, $k_{SU(3)}(S^7) \cong [S^6, SU(3)] \cong \Pi_6(SU(3)) \cong \mathbb{Z}_6$ which has two generators. This means that up to isomorphism there are five nontrivial $SU(3)$- bundles over $S^7$, one of them being $SU(4)$ since $\Pi_6(SU(4)) \cong 0$ and $\Pi_6(S^7 \times SU(3)) \cong \Pi_6(S^7) \times \Pi_6(SU(3)) \cong \mathbb{Z}_6$. Let $g_4 : S^6 \to SU(3)$ be its characteristic map. If $g_4$ were a homotopy lifting of $S^3_r h$, then there should exist a lifting $g'_4$ by $\pi_3 : SU(3) \to S^5$ of $S^3_r h : S^6 \to S^5$ i.e. a commuting diagram

$$
\begin{array}{ccc}
S^6 & \overset{g'_4}{\rightarrow} & SU(3) \\
\uparrow S^3_r h & \downarrow \pi_3 & \downarrow S^5 \\
\end{array}
$$

with $g'_4 \sim g_4$. We have the

**Proposition.** $\pi_3$ does not lift $S^3_r h$. 


Proof. We will show that the homomorphism $\pi_{3*} : \Pi_6(SU(3)) \to \Pi_6(S^5)$ is zero. This implies that any map $S^6 \xrightarrow{f} S^5$ which factorizes through $\pi_3$ i.e. a map for which there exists a map $S^6 \xrightarrow{g} SU(3)$ such that $\pi_3 \circ g \sim f$, is null-homotopic. The result now follows from this since $S^3_\pi h$ is essential.

Consider the long exact homotopy sequence (Steenrod, 1951) of the principal bundle $SU(2) \to SU(3) \xrightarrow{\pi_3} S^5$:

$$
\ldots \to \Pi_6(SU(3)) \xrightarrow{\pi_{3*}} \Pi_6(S^5) \xrightarrow{\delta} \Pi_5(S^3) \xrightarrow{\iota_*} \Pi_5(SU(3)) \to \ldots
$$

This gives an exact sequence:

$$
\ldots \to \mathbb{Z}_6 \xrightarrow{\beta} \mathbb{Z}_2 \xrightarrow{\gamma} \mathbb{Z}_2 \xrightarrow{\alpha} \mathbb{Z} \to \ldots
$$

where we called $\beta$, $\gamma$ and $\alpha$ the homomorphisms corresponding to $\pi_{3*}$, $\delta$ and $\iota_*$, respectively. Since $\mathbb{Z}_2$ is a torsion group and $\mathbb{Z}$ is torsion free, then the homomorphism $\alpha$ is zero. Therefore $\gamma$ is an isomorphism i.e. $\ker(\gamma) = \ker(\delta) = \{0\} = \text{Im}(\pi_{3*})$ i.e. $\pi_{3*} = 0$. QED

Remark. This result can also be obtained from the general theorem proved in Section 4.4. However, the proof given above is simpler.

4.2. $(H, f)$-structures

Let $H$ and $G$ be topological groups (e.g. Lie groups), $\xi_H : H \to PH \xrightarrow{\pi_H} BH$ and $\xi_G : G \to PG \xrightarrow{\pi_G} BG$ their universal bundles, and $f : H \to G$ a topological group homomorphism. Then the action $\bar{f} : H \times G \to G$, $\bar{f}(h, g) = f(h)g$ induces the associated principal $G$-bundle $((\xi_H)_G : G \to PH \times_H G \to BH$ with total space $PH \times_H G = \{(a, g) \mid (p, g) \in PH \times G, [a, g] = \{(ah, f(h^{-1})g)\}_{h \in H}$, action $(PH \times_H G) \times G \to PH \times_H G$ given by $[a, g].g' = [a, gg']$, and projection $(\pi_H)_G([a, g]) = \pi_H(a)$. $PH \times_H G$ is isomorphic to the pull-back bundle $(Bf)^*(PG)$, where the induced function $Bf : BH \to BG$ is uniquely defined up to homotopy.

If $HTop$ is the category of paracompact topological spaces and homotopy classes of maps, and $Set$ is the category of sets and functions, then for each topological group $K$ there are two cofunctors $k_K$ and $[ , BK]$ from $HTop$ to $Set$ such that, for each topological group homomorphism $f : H \to G$ there are natural transformations $f_* : k_H \to k_G$ and $Bf_* : [ , BH] \to [ , BG]$, and natural equivalences $\psi_H$ and $\psi_G$ which make the following functorial diagram commutative:

$$
\begin{array}{ccc}
k_H & \xrightarrow{f_*} & k_G \\
\psi_H & \uparrow & \psi_G \\
[ , BH] & \xrightarrow{Bf_*} & [ , BG]
\end{array}
$$
So, for each paracompact topological space $X$ the following set theoretic diagram commutes:

$$
\begin{array}{ccc}
  k_H(X) & \xrightarrow{f_*} & k_G(X) \\
  \psi_H \uparrow \cong & & \psi_G \uparrow \cong \\
  [X, BH] & \xrightarrow{Bf_*} & [X, BG]
\end{array}
$$

where: $k_K(X) = \{ \text{isomorphism classes of principal } K\text{-bundles over } X \}$, $[X, BK] = \{ \text{unbased homotopy classes of maps from } X \text{ to } BK \}$, $\psi_K([\alpha]) = [\alpha^*(PK)]$, $f_*([\eta]) = [\xi]$ with $\eta : H \to E \xrightarrow{q} X$ and $\xi : G \to E \times_H G \xrightarrow{\Phi} X$, and $Bf_*([\alpha]) = [Bf \circ \alpha]$. If $[\xi] \in k_G(X)$, then $f_*^{-1}([\xi])$ is the set of $(H, f)$-structures on $\xi$; this set can be empty. So, $\xi$ has a $(H, f)$-structure if and only if there exists a map $\alpha : X \to BH$ such that $Bf \circ \alpha \sim F$, where $F$ is the classifying map of $\xi$. One can show that this definition is equivalent to the existence of a $G$-bundle isomorphism

$$
E \times_H G \xrightarrow{\Phi} P \xrightarrow{\pi} X
$$

where $H \to E \xrightarrow{q} X$ is a principal $H$-bundle or, equivalently, to the bundle map

$$
\begin{array}{ccc}
  E \times H & \xrightarrow{\varphi \times f} & P \times G \\
  \kappa \downarrow & & \psi \downarrow \\
  E & \xrightarrow{\varphi} & P \\
  q \downarrow & & \pi \downarrow \\
  X & & X
\end{array}
$$

where $\varphi = \tilde{\varphi} \circ \varphi_f$ with $\varphi_f : E \to E \times_H G$ given by $\varphi_f(a) = [a, e]$ ($e$ is the unit of $G$) (Aguilar and Socolovsky, 1997a). One says that $(E, \varphi)$ is an $(H, f)$-structure on $G \to P \xrightarrow{\pi} X$.

In the case of smooth bundles, if the Lie group homomorphism $H \xrightarrow{f} G$ is an embedding i.e. an injective immersion, then $E$ is called a reduction of $P$ to $H$. In this setting, one has the following

**Proposition.** If $f$ is an embedding, then $\varphi$ is also an embedding.

**Proof.** Since $\varphi = \tilde{\varphi} \circ \varphi_f$ and $\tilde{\varphi}$ is a diffeomorphism, then $\varphi$ is an embedding if and only if $\varphi_f$ is an embedding; we shall show that $\varphi_f$ is an embedding. i) $\varphi_f$ is injective: Let $\varphi_f(a_1) = \varphi_f(a_2)$ i.e. $[a_1, e] = [a_2, e]$, since $[a, e] = \{ (ah, f(h^{-1})) \}_{h \in H}$ then there must exist $h \in H$ such that $(a_1, e) = (a_2h, f(h^{-1}))$ i.e. $a_1 = a_2h$ and $f(h^{-1}) = e$, but $f$ is injective, so $h^{-1} = h = e'$, the identity in $H$, and then $a_1 = a_2$. 

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ii) $d\varphi_f$ is injective at each $a_0 \in E$: Consider the commutative diagram

$$
\begin{array}{ccc}
E \times G & \xrightarrow{i} & E \\
\downarrow{\varphi_f} & & \downarrow{p} \\
E \times_H G & \xrightarrow{\alpha_{(a_0,e)}} & T_{[a_0,e]}(E \times H \; G) \rightarrow 0
\end{array}
$$

where $i(a) = (a, e)$ and $p(a, e) = [a, e]$. By (Greub et al., 1973) $H \rightarrow E \times G \rightarrow E \times_H G$ is a principal $H$-bundle, so fixing $(a_0, e) \in E \times G$ there is a map $\alpha_{(a_0,e)} : H \rightarrow E \times G$, given by $\alpha_{(a_0,e)}(h) = (a_0, e) \cdot h = (a_0 h, f(h^{-1}))$, in particular $\alpha_{(a_0,e)}(e') = (a_0, e)$.

One then has the following diagram of vector spaces:

$$
0 \rightarrow T_{e'}H \xrightarrow{(d\alpha_{(a_0,e)})e'} T_{(a_0,e)}(E \times G) \xrightarrow{(dp)(a_0,e)} T_{[a_0,e]}(E \times H \; G) \rightarrow 0
$$

where the horizontal sequence is exact and the triangle commutes. If $p_1$ and $p_2$ are respectively the projections of $E \times G$ onto $E$ and $G$, then $p_1 \circ i(a) = a$ i.e. $p_1 \circ i = id_E$ and $p_2 \circ i(a) = p_2(a, e) = e$ i.e. $p_2 \circ i = const.$, hence $(d(p_1 \circ i))a = (d(id_E))a = id_{T_{a_0}E}$ and $(d(p_2 \circ i))a = (d(const.))a = 0$. Therefore $(di)_{a_0}(v) = ((d(p_1 \circ i))a_0(v), (d(p_2 \circ i))a_0(v)) = (v, 0)$. On the other hand, $p_1 \circ \alpha_{(a_0,e)}(h) = p_1(a_0 h, f(h^{-1})) = a_0 h := \alpha_{a_0}(h)$, $p_2 \circ \alpha_{(a_0,e)}(h) = p_2(a_0 h, f(h^{-1})) = f \circ \gamma(h)$, where $\gamma : H \rightarrow H$ is given by $\gamma(h) = h^{-1}$. Therefore $(d\alpha_{(a_0,e)})e'(w) = ((d(p_1 \circ \alpha_{(a_0,e)}))e'(w), (d(p_2 \circ \alpha_{(a_0,e)}))e'(w)) = ((d\alpha_{a_0})e'(w), (df)e' \circ (d\gamma)e'(w))$. Let $(r, s) \in Im((di)_{a_0}) \cap Im((d\alpha_{(a_0,e)})e')$, then $s = 0$ and hence $0 = (df)e'((d\gamma)e'(w))$, therefore $(d\gamma)e'(w) = 0$ because $f$ is an immersion and, since $\gamma$ is a diffeomorphism, $w = 0$, so $r = (d\alpha_{a_0})e'(0) = 0$ i.e. $Im((di)_{a_0}) \cap Im((d\alpha_{(a_0,e)})e') = \{0\}$. Finally, let $v \in ker((d\varphi_f)_{a_0})$, then $0 = (dp)(a_0,e)((di)_{a_0}(v))$ i.e. $(di)_{a_0}(v) \in ker((dp)(a_0,e)) = Im((d\alpha_{(a_0,e)})e')$ i.e. $(di)_{a_0}(v) = 0$. Since $i$ is an embedding, then $v = 0$ i.e. $(d\varphi_f)_{a_0}$ is one-to-one. QED

Remark. One often finds in the literature (Kobayashi and Nomizu, 1963; Trautman, 1984) that to define a reduction to a Lie subgroup $H \subset G$, $\varphi$ is required to be an embedding. The proposition above shows that this is a consequence of the fact that $H \rightarrow G$ is an embedding.

4.3. $SU(n) \rightarrow SU(n + 1)^\pi_{n+1}S^{2n+1}$ as an $(SU(n), i)$-structure on $U(n) \rightarrow U(n + 1)^{\pi_{n+1}}S^{2n+1}$

Proposition. For $n = 1, 2, 3, \ldots$, the bundle $\pi_{n+1}$ is a reduction of the bundle
\[ p_{n+1} \text{ i.e. one has the } U(n)\text{-bundle isomorphism given by the commutative diagram} \]

\[
\begin{array}{ccc}
(SU(n+1) \times SU(n) U(n)) \times U(n) & \xrightarrow{\tilde{\varphi} \times \text{id}} & U(n+1) \times U(n) \\
\downarrow \lambda & & \downarrow \psi \\
SU(n+1) \times SU(n) U(n) & \xrightarrow{\varphi} & U(n+1) \\
q_{n+1} \searrow & & \searrow p_{n+1} \\
\end{array}
\]

where \( \lambda([D, A], B) = [D, AB], \) \( q_{n+1}[D, A] = \pi_{n+1}(D) = De_0, \) \( \psi(C, B) = Cj(B), \) \( p_{n+1}(C) = Ce_0, \) and \( \tilde{\varphi}, \) and \( j \) are given below.

**Proof.** Consider the inclusion \( SU(n+1) \xrightarrow{\varphi} U(n+1); \) one can easily show that this is a smooth bundle map between the principal \( SU(n)\)-bundle \( \pi_{n+1} \) and the principal \( U(n)\)-bundle \( p_{n+1}. \) Therefore, by (Greub et al, 1973) the map \( \tilde{\varphi} \) given by \( \tilde{\varphi}(\{D, A\}) = Dj(A) \) where \( j \) is the inclusion \( U(n) \to U(n+1) \) with \( j(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \) is a smooth bundle isomorphism. The inverse of \( \tilde{\varphi} \) is given as follows: if \( C \in U(n+1) \) then \( C = Dl(l(detC))^{-1} \in SU(n+1) \) and \( l : U(1) \to U(n+1) \) is the inclusion \( l(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \) then \( [D, l(detC)] = \tilde{\varphi}^{-1}(C). \) QED

**Remark.** Notice that if \( \iota : SU(n) \to U(n) \) is the inclusion, then \( p_n \circ \iota = \pi_n. \)

### 4.4. Proof of the main result

**Proposition.** Let \( H \) and \( G \) be path connected topological groups such that each one has the homotopy type of a CW-complex and let \( f : H \to G \) be a continuous homomorphism. Then, in the following diagram each horizontal function is a bijection and each square commutes, for \( n = 1, 2, 3, \ldots \)

\[
\begin{array}{cccc}
[S^{2n}, G] \xrightarrow{\mu_{G#}} [S^{2n}, \Omega BG] & \xrightarrow{\text{adj}G*} & [S_rS^{2n}, BG] & \xrightarrow{\psi_G} & k_G(S^{2n+1}) \\
\uparrow f# & \uparrow \Omega B f# & \uparrow Bf* & \uparrow \psi* & \uparrow f* \\
[S^{2n}, H] \xrightarrow{\mu_{H#}} [S^{2n}, \Omega BH] & \xrightarrow{\text{adj}H*} & [S_rS^{2n}, BH] & \xrightarrow{\psi_H} & k_H(S^{2n+1}) \\
\end{array}
\]

where \( k_K, f, Bf \) and \( \psi_K \) have been defined before, \( \Omega BK \) is the loop space of \( BK, \) and \( f#, \mu_{K#}, \Omega B f# \) and \( \text{adj}K* \) are given by \( f_#([\delta]) = [f \circ \delta], \mu_{K#}([\sigma]) = [\mu_K \circ \sigma] \) (\( \mu_K \) is defined below), \( \Omega B f#([\kappa]) = [\Omega B f \circ \kappa] \) with \( \Omega B f : \Omega BH \to \Omega BG \) given by \( \Omega B f(\gamma) = Bf \circ \gamma, \) and \( \text{adj}K*([\alpha]) = [\text{adj}K(\alpha)] \) with \( \text{adj}K(\alpha)([z, t]) = \alpha(z)(t), \) \( t \in [0, 1]. \) The set \( [S^{2n}, K] = \Pi_{2n}(K) \) corresponds to the characteristic maps for the \( K\)-principal bundles over \( S^{2n+1}. \)

**Proof.** The commutativity of the third square has been proved in Section 4.2, with \( S^{2n+1} = X. \) The natural equivalence \( \text{adj}K \) is given by the exponential law in
function spaces (Spanier, 1966). By (Switzer, 1975) there exist homotopy equivalences $\mu_K$ such that the diagram

$$
\begin{array}{ccc}
H & \xrightarrow{f} & G \\
\mu_H \uparrow & & \uparrow \mu_G \\
\Omega BH & \xrightarrow{\Omega Bf} & \Omega BG
\end{array}
$$

commutes up to homotopy, therefore $\mu_K#$ is a bijection and the first square commutes. Finally, notice that in the diagram of the proposition we are dealing with based homotopy classes of maps. This corresponds to based principal bundles. However, since we are taking path connected topological groups, the function that forgets the basepoints is a bijection between based bundles and the usual unbased bundles of section 4.2. QED

**Proposition.** For even $n$, $n \geq 2$, the clutching map $g_{n+1}$ of the principal bundle $SU(n) \to SU(n+1) \overset{\pi_{n+1}}{\longrightarrow} S^{2n+1}$ is a homotopy lifting of the $(2n-3)-th$ reduced suspension of the Hopf map $h$. For odd $n$, $n \geq 3$, $\pi_n \circ g_{n+1}$ is inessential.

**Proof.** We apply the previous proposition to the case $H = SU(n), G = U(n)$, and $f = \iota$ (the inclusion), for $n = 2, 3, \ldots$. By the proposition in Section 4.3, $[\xi] = \iota_*([\eta])$ with $\xi : U(n) \to U(n + 1) \overset{\pi_{n+1}}{\longrightarrow} S^{2n+1}$ and $\eta : SU(n) \to SU(n + 1) \overset{\pi_{n+1}}{\longrightarrow} S^{2n+1}$. Then one has the commutative diagram

$$
\begin{array}{ccc}
[T'_{n+1}] & \in & [S^{2n}, U(n)] \\
\downarrow & \mu \downarrow & \downarrow k_{U(n)}(S^{2n+1}) \\
[g_{n+1}] & \in & [S^{2n}, SU(n)] \\
\end{array}
$$

where $\mu = \psi U(n) \circ \text{adj}_{U(n)} \circ \mu U(n)#$ and $\nu = \psi SU(n) \circ \text{adj}_{SU(n)} \circ \mu SU(n)#$ are bijections, and $T'_{n+1}$ is the clutching map for the principal bundle $\xi$ (Steenrod, 1951). Then $[T'_{n+1}] = \mu^{-1}([\xi]) = \mu^{-1}(\iota_*([\eta])) = \mu^{-1} \circ \iota_* \circ \nu([g_{n+1}]) = \mu^{-1} \circ \iota_* \circ \nu([g_{n+1}]) = \iota_#([g_{n+1}]) = [\iota \circ g_{n+1}]$ and therefore $T'_{n+1} \sim \iota \circ g_{n+1}$.

Consider the following diagram:

$$
\begin{array}{ccc}
U(n-1) & \downarrow & U(n) \\
\downarrow & & \downarrow \\
S^{2n} & T'_{n+1} & \overset{\pi_n}{\longrightarrow} S^{2n-3} \text{h} \quad S^{2n-1}
\end{array}
$$

Steenrod (Steenrod, 1951) proved that for $n$ even, $n \geq 2$, $p_n \circ T'_{n+1} \sim S^{2n-3} \text{h} i.e.$ the diagram commutes up to homotopy, while for $n$ odd, $n \geq 3$, $p_n \circ T'_{n+1} \sim \text{const.}
Then, \( p_n \circ T'_{n+1} \sim p_n \circ (\iota \circ g_{n+1}) = (p_n \circ \iota) \circ g_{n+1} = \pi_n \circ g_{n+1} \)

\[ \sim \begin{cases} \mathbb{S}^{2n-3n}, & n \text{ even} \\ \text{const.}, & n \text{ odd} \end{cases} \]

QED

5. \( S^2 \) AND RELATIVITY

As is well known, the Lorentz group, the group of linear transformations of Minkowski space-time which preserves the scalar product \( \langle x, y \rangle = x^T \eta y \) where

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

is the Minkowskian metric, is a subgroup of the symmetry group of several gauge theories of gravity (Hehl et al, 1976; Basombrío, 1980). This means that \( O(3,1) \) is a subgroup of the structure group of the corresponding principal bundles. The relationship between these theories and the 2-sphere (the Riemann sphere \( \mathbb{C} \cup \{\infty\} \)) comes from the fact that there is a canonical isomorphism between the connected component of \( O(3,1) \), the proper orthochronous Lorentz group \( SO^0(3,1) \) and the group of conformal (Moebius) transformations of \( S^2 \), \( Conf(S^2) \). We recall that \( Conf(S^2) \) is the set of all invertible transformations of the Riemann sphere which preserves the angles between curves and that at each point multiply all the tangent vectors by a fixed positive number.

Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be an element of \( GL_2(\mathbb{C}) \), we define a Moebius transformation \( m : S^2 \to S^2 \) as follows: if \( c \neq 0 \), then

\[
z \mapsto \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq -d/c \\
\infty & \text{if } z = -d/c
\end{cases}
\]

and

\[
\infty \mapsto a/c;
\]

and, if \( c = 0 \), then

\[
\begin{cases} z \mapsto \frac{a}{d}z + \frac{b}{d} \\
\infty \mapsto \infty
\end{cases}
\]

It is then easy to verify that the following diagram commutes:

\[
\begin{array}{cccc}
\mathbb{Z}_2 & \xrightarrow{\psi} & SO^0(3,1) & \\ \downarrow & & \xrightarrow{\tau} & \xrightarrow{\lambda} \\ Conf(S^2) \\
\xrightarrow{\psi} & & \xrightarrow{\tau} & \xrightarrow{\lambda}
\end{array}
\]
where: i) the projections $\psi$ and $\lambda$ are two-to-one group homomorphisms, respectively given by

$$\psi \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cccc} |a|^2 + |b|^2 + |c|^2 + |d|^2 & Re(ab + cd) & Im(ab + cd) & |a|^2 - |b|^2 + |c|^2 - |d|^2 \\ Re(ac + bd) & Re(ad + bc) & Im(ad - bc) & Re(ac - bd) \\ -Im(ac + bd) & Im(ad - bc) & -Im(ad + bc) & -Im(ac - bd) \\ |a|^2 + |b|^2 - |c|^2 - |d|^2 & Re(ab - cd) & Im(ab - cd) & |a|^2 - |b|^2 - |c|^2 + |d|^2 \end{array} \right)$$

with $\psi \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \psi \left( \begin{array}{cc} -a & -b \\ -c & -d \end{array} \right) = l$ (Penrose and Rindler, 1984), and $\lambda(g/\sqrt{\det g}) = m$ with $\lambda(g/\sqrt{\det g}) = \lambda(-g/\sqrt{\det g})$; and ii) $\tau(l) = m$ is the desired isomorphism. $SL_2(\mathbb{C}) \rightarrow SO^0(3, 1)$ and $SL_2(\mathbb{C}) \rightarrow Conf(S^2)$ are $\mathbb{Z}_2$- principal bundles.

Thus we conclude that the symmetry group of the standard model $G'_{SM}$, when gravitation is included, locally contains, as a space, $S^1 \times (S^3)^2 \times S^5 \times Conf(S^2)$.

**Remark.** In the framework of the theory of categories, functors, and natural transformations, some of the geometrical objects of the previous sections, e.g. spheres and the Hopf map, have a natural origin. This suggests a possible relation between symmetries in nature, and therefore conservation laws, and some of the most general mathematical concepts. The basic idea is that of a representable functor (Aguilar and Socolovsky, 1997b).

**REFERENCES**

Aguilar, M. A., and Socolovsky, M. (1997a). Reductions and extensions in bundles and homotopy, *Advances in Applied Clifford Algebras*, 7 (S), 487-494.

Aguilar, M. A., and Socolovsky, M. (1997b). Naturalness of the space of States in Quantum Mechanics, *International Journal of Theoretical Physics*, 36, 883-921.

Ashtekar, A., and Schilling, T.A. (1995). Geometry of Quantum Mechanics, *AIP Conference Proceedings*, 342, 471-478.

Basombrio, F. G. (1980). A Comparative Review of Certain Gauge Theories of the Gravitational Field, *General Relativity and Gravitation*, 12, 109-136.

Corichi, A., and Ryan, Jr., M. P. (1997). Quantization of nonstandard Hamiltonian systems, *Journal of Physics A: Mathematical and General*, 30, 3553-3572.
Greub, W., Halperin, S. and Vanstone, R. (1973). *Connections, Curvature and Cohomology*, Academic Press, New York.

Hehl, F. W., von der Heyde, P., and Kerlick, G. D. (1976). General relativity with spin and torsion: Foundations and prospects, *Reviews of Modern Physics*, 48, 393-416.

Itô, K. ed. (1993). *Encyclopedic Dictionary of Mathematics*, The Mit Press, Cambridge, Mass.

Kobayashi, S., and Nomizu, K. (1963). *Foundations of Differential Geometry*, Vol. I, Wiley, New York.

Mohapatra, R. N. (1986). *Unification and Supersymmetry*, Springer-Verlag, New York.

Penrose, R., and Rindler, W. (1984). *Spinors and Space-time*, Vol. 1, Cambridge University Press, Cambridge.

Spanier, E. H. (1966). *Algebraic Topology*, Springer-Verlag, New York.

Steenrod, N. (1951). *The Topology of Fibre Bundles*, Princeton University Press, Princeton, New Jersey.

Steenrod, N., and Epstein, D. B. A. (1962). *Cohomology operations*, Annals of Mathematical Studies, 50, Princeton University Press, Princeton, New Jersey.

Switzer, R. (1975). *Algebraic Topology- Homotopy and Homology*, Springer-Verlag, New York.

Taylor, J. C. (1976). *Gauge Theories of Weak Interactions*, Cambridge University Press, Cambridge.

Trautman, A. (1984). *Differential Geometry for Physicists*, Bibliopolis, Napoli.

Wu, T. T., and Yang, C. N. (1975). Concept of nonintegrable phase factors and global formulation of gauge fields, *Physical Review D*, 12, 3845-3857.