ANALYZING RELEVANCE VECTOR MACHINES USING A SINGLE PENALTY APPROACH
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Abstract

Relevance vector machine (RVM) is a popular sparse Bayesian learning model typically used for prediction. Recently it has been shown that improper priors assumed on multiple penalty parameters in RVM may lead to an improper posterior. Currently in the literature, the sufficient conditions for posterior propriety of RVM do not allow improper priors over the multiple penalty parameters. In this article, we propose a single penalty relevance vector machine (SPRVM) model in which multiple penalty parameters are replaced by a single penalty and we consider a semi Bayesian approach for fitting the SPRVM. The necessary and sufficient conditions for posterior propriety of SPRVM are more liberal than those of RVM and allow for several improper priors over the penalty parameter. Additionally, we also prove the geometric ergodicity of the Gibbs sampler used to analyze the SPRVM model and hence can estimate the asymptotic standard errors associated with the Monte Carlo estimate of the means of the posterior predictive distribution. Such a Monte Carlo standard error cannot be computed in the case of RVM, since the rate of convergence of the Gibbs sampler used to analyze RVM is not known. The predictive performance of RVM and SPRVM is compared by analyzing three real life datasets.

Keywords: cross validation, geometric ergodicity, improper prior, Monte Carlo standard errors, posterior propriety, reproducing kernel Hilbert spaces.

1 Introduction

Let \( \{(y_i, x_i) : i = 1, 2, \cdots, n\} \) denote the training dataset where \( y_i \in \mathcal{R} \) is the \( i^{th} \) observation of the response variable and \( x_i \in \mathcal{R}^p \) is the \( p \) dimensional covariate vector associated with \( y_i \). For such a dataset, often the objective is to come up with a function \( h \), such that the

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response variable \( y_i \) can be expressed as \( y_i = h(x_i) + \epsilon_i \ \forall \ i = 1, 2, \cdots, n \) where \( h : \mathcal{R}^p \to \mathcal{R} \) and \( \{\epsilon_i\}_{i=1}^n \) are the errors. Many times, for a previously unobserved \( p \) dimensional covariate vector, the function \( h \) is utilized to predict its associated response variable. If \( p \) is small, then the function \( h \) can be estimated using the nonparametric approach of a Nadaraya-Watson type estimator. In this approach, the errors are assumed to be uncorrelated, have a zero mean and a constant variance. For higher dimensions, kernel density estimation might not work well, and hence Nadaraya Watson type estimators are not recommended when \( p \) is large. Thus, in cases where \( p \) is large but smaller than \( n \), one can use the ordinary least squares (OLS) method to estimate the function \( h \). In OLS, \( h \) is estimated from a class of linear models by minimizing the quadratic loss function.

In recent years, there is a plethora of datasets wherein \( p \) is far greater than \( n \). Such datasets are often referred to as high dimensional datasets. Examples of these can be found in the field of genetics, nutrition, chemical engineering etc. In such cases, the methods described before are no longer applicable. A possible solution in such cases is to use the least absolute shrinkage and selection operator (LASSO) proposed by Tibshirani (1996) that estimates the function \( h \) from a class of linear models by minimizing the quadratic loss function with respect to an \( L_1 \) penalty. Another option is to utilize the ridge estimator proposed by Hoerl and Kennard (1970) that is similar to LASSO, but uses an \( L_2 \) penalty. There are other penalized regression variants of LASSO and ridge proposed in the literature (see eg. Zou and Hastie (2005)). Bayes and empirical Bayes versions have also been developed using the connection between the penalized estimates and the posterior mode corresponding to appropriately chosen prior densities on the regression coefficients (see Park and Casella (2008), Kyung et al. (2010) and Roy and Chakraborty (2017)). Parameter estimation in the Bayesian models is generally carried out using Markov chain Monte Carlo (MCMC) samplers. In traditional as well as Bayesian versions, a drawback of these penalized regression methods is that the function \( h \) is restricted among the class of linear models.

If one wishes to explore a more general class of models, a common strategy is to take a reproducing kernel Hilbert space (RKHS) approach to estimate the function \( h \). Such an estimate of the function \( h \) was found by Wahba (1990) by solving the Tikhonov regularization over RKHS. This RKHS based solution allows us to reduce the complexity of the model.
matrix from $p$ to $n$ dimensions. This pleasing property of the RKHS based solution was utilized by Tipping (2001) to propose the relevance vector machine (RVM) (see also Tipping (2000) and Bishop and Tipping (2000)).

RVM is a hierarchical Bayesian model in which the finite dimensional solution found by Wahba (1990) was utilized as the mean structure of the data model. It can be analyzed using either proper or improper priors over the hyperparameters and Tipping (2001) presents both cases. Assuming improper priors is fine as long as the posterior propriety has been established. Recently, Dixit and Roy (2021) provide necessary and sufficient conditions for posterior propriety of RVM and prove that improper priors assumed by Tipping (2001) lead to improper posteriors. Thus, in order to conduct valid Bayesian analysis, one needs to either use proper priors or other improper priors that satisfy the sufficient conditions. For additional details about RVM and some other kernel methods see Clarke and Clarke (2018).

In the past, Fokoué et al. (2011) have attempted to implement RVM using conjugate proper priors over its hyperparameters. In that case, the full conditional distributions of the parameters involved in RVM are well known distributions which are easy to simulate from and hence can be utilized to construct an RVM Gibbs sampler. Further, for a previously unobserved $p$ dimensional covariate vector, the response variable can be predicted by utilizing the RVM Gibbs sampler iterations to produce a Monte Carlo estimate of the mean of the posterior predictive distribution. A Monte Carlo estimate should ideally be accompanied by a valid standard error estimate, so that the user is aware about the uncertainty associated with the estimate. In order to compute Monte Carlo standard errors for Markov chain samples, one needs to establish a Markov chain central limit theorem (CLT), which in turn depends on the rate of convergence of the Markov chain (see Jones and Hobert (2001)). Currently in the literature, the rate of convergence of the Gibbs sampler implemented by Fokoué et al. (2011) is not known and hence the Markov chain CLT is not guaranteed. Thus, in the case of RVM, one cannot compute the standard errors associated with the Monte Carlo estimate of the mean of the posterior predictive distribution.

Mallick et al. (2005) proposed RKHS based hierarchical Bayesian classification models using both single and multiple shrinkage parameters which are also known as penalty parameters. RVM proposed by Tipping (2001) is a RKHS based hierarchical Bayesian regression
model based on multiple penalty parameters. In this article we propose to replace these multiple penalty parameters by a single penalty parameter. We propose to name this new model as single penalty relevance vector machine (SPRVM) and analyze it using a semi Bayesian approach. In SPRVM, conjugate priors are assumed on a few parameters and since SPRVM is primarily used for prediction, other parameters are estimated using cross validation. Further, in the case of SPRVM, the posterior predictive distribution is not known in closed form, and a Gibbs sampler is implemented to produce a Monte Carlo estimate of the mean of the posterior predictive distribution. Additionally, we also prove that the Gibbs sampler implemented in the case of SPRVM converges at a geometric rate, and hence the Markov chain CLT is guaranteed. Thus, in the case of SPRVM, asymptotically valid standard error estimates can be attached to a Monte Carlo estimate of the mean of the posterior predictive distribution. This is an advantage of SPRVM over RVM. Furthermore, we show that unlike RVM, there is significant overlap in the necessary and sufficient conditions for posterior propriety of the SPRVM allowing improper priors on the penalty parameter. Finally, in the context of three real life datasets, we observe that the predictive performance of SPRVM is as good as the RVM.

The article is structured as follows. In Section 2, we provide details about RVM and its associated Gibbs sampler. In Section 3, we introduce and provide details about SPRVM. In Section 4, we analyze some real life datasets obtained from the field of genetics, nutrition and chemical engineering to compare the predictive performance of RVM and SPRVM. In this section, we also discuss the marginal likelihood approach of estimating a few SPRVM model parameters and some concluding remarks are provided in Section 5.

2 Relevance Vector Machine

Let \( y = (y_1, y_2, \cdots, y_n) \) be the vector of standardized responses where \( y_i \in \mathcal{R} \). Recall that \( x_i \in \mathcal{R}^p \) denote the covariate vector associated with the \( i^{th} \) observation. Let \( K_\theta \) be the \( n \times (n + 1) \) kernel matrix whose \( i^{th} \) row is given by \( K_\theta^T = (1, k_{\theta i1}, k_{\theta i2}, \cdots, k_{\theta in}) \) where \( \{k_{\theta ij} = k_\theta(x_i, x_j) : i, j = 1, 2, \cdots, n\} \) are the values of the reproducing kernel and \( \theta \) is a kernel parameter that is typically tuned using cross validation. Also, let \( \beta = (\beta_0, \beta_1, \cdots, \beta_n) \). Then,
the RVM proposed by Tipping (2001) is as follows,

\[
y | \beta, \sigma^2, \theta \sim N_n(K_{\theta} \beta, \sigma^2 I),
\]

\[
\beta | \lambda_0, \lambda_1, \ldots, \lambda_n \sim N_{n+1}(0, D^{-1}) \quad \text{with} \quad D = diag(\lambda_0, \lambda_1, \ldots, \lambda_n),
\]

\[
\pi(\lambda_i) \propto \lambda_i^{a-1} \exp\{-b\lambda_i\} \quad \forall i = 0, 1, 2, \ldots, n,
\]

\[
\pi\left(\frac{1}{\sigma^2}\right) \propto \left(\frac{1}{\sigma^2}\right)^{c-1} \exp\left\{-\frac{d}{\sigma^2}\right\},
\]

where \((a, b, c, d)\) are hyperparameters that are specified by the user. In the above model, Tipping (2001) assumed that \(1/\sigma^2\) and \(\{\lambda_i\}_{i=0}^n\) are apriori independent. Further, \(\beta\) and \(1/\sigma^2\) are also assumed to be apriori independent. The posterior density of the parameters in RVM, indexed by \(\theta\), is as follows,

\[
\pi(\beta, 1/\sigma^2, \lambda_0, \lambda_1, \ldots, \lambda_n | y, \theta) = \frac{f(y | \beta, \sigma^2, \theta) \pi(\beta, 1/\sigma^2, \lambda_0, \lambda_1, \ldots, \lambda_n)}{m_\theta(y)},
\]

where \(f(y | \beta, \sigma^2, \theta)\) is the data model given in (2.1a), \(\pi(\beta, 1/\sigma^2, \lambda_0, \lambda_1, \ldots, \lambda_n)\) is the joint prior density obtained from (2.1b) - (2.1d) and \(m_\theta(y)\) is the marginal likelihood which is also known as the normalizing constant. When the posterior density given in (2.2) is integrated over the entire parametric space, the integral is equal to 1, provided the normalizing constant exists. Therefore, the posterior distribution is proper if and only if \(m_\theta(y) < \infty\).

In Bayesian analysis, prior information available with the researcher is generally incorporated by choosing the user defined hyperparameters accordingly. In the case of RVM, the prior assumed can be either proper or improper depending upon the choice of hyperparameters and Tipping (2001) explored both cases. The improper prior assumed by Tipping (2001) can be obtained by choosing \((a, b, c, d)\) to be \((0, 0, 0, 0)\). Dixit and Roy (2021) proved that for this choice of the user specified hyperparameters, \(m_\theta(y)\) is infinity, and hence the RVM implemented by Tipping (2001) using improper priors is based on an improper posterior distribution. Given the posterior impropriety of RVM for the hyperparameters used by Tipping (2001), we choose to implement RVM using priors that satisfy the sufficient conditions for posterior propriety derived by Dixit and Roy (2021). RVM is typically used for predicting the response variable say \(y_{\text{new}}\) for a previously unobserved \(p\) dimensional covariate vector say \(x_{\text{new}}\). Such a prediction is often based on the posterior predictive distribution of the model,
which is given by,
\[
f(y_{\text{new}}|y, \theta) = \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+2}} f(y_{\text{new}}|\beta, \sigma^2, \theta) \pi(\beta, 1/\sigma^2, \lambda_0, \lambda_1, \ldots, \lambda_n|y, \theta) d\beta d \frac{1}{\sigma^2} d\lambda_0 d\lambda_1 \cdots d\lambda_n.
\]
Further, the mean of the above posterior predictive density can be reported as the predicted value associated with \(x_{\text{new}}\) and is given by,
\[
E(y_{\text{new}}|y, \theta) = K^T_{\theta,\text{new}} \hat{\beta}_R
\]
where \(K^T_{\theta,\text{new}} = (1, k_\theta(x_{\text{new}}, x_1), k_\theta(x_{\text{new}}, x_2), \ldots, k_\theta(x_{\text{new}}, x_n))\) and \(\hat{\beta}_R\) is the posterior mean of the parameter \(\beta\) in the case of RVM model.

Although posterior propriety is guaranteed for priors that satisfy the sufficient conditions for posterior propriety derived by Dixit and Roy (2021), the marginal likelihood is still analytically intractable, and hence the posterior density given in (2.2) and \(\hat{\beta}_R\) in (2.3) are not available in closed form. A Monte Carlo estimate for \(\hat{\beta}_R\) can be obtained by implementing a Gibbs sampler with invariant density (2.2). The full conditional distributions of \((\beta, 1/\sigma^2, \lambda_0, \lambda_1, \cdots, \lambda_n)\), required to implement the Gibbs sampler are as follows:
\[
\beta|\cdot \sim N_{n+1}\left((K^T_\theta K_\theta + D\sigma^2)^{-1}K^T_\theta y, (K^T_\theta K_\theta \frac{1}{\sigma^2} + D)^{-1}\right)
\]
\[
\frac{1}{\sigma^2}|\cdot \sim \text{Gamma}\left(\frac{n}{2} + c, \frac{1}{2}||y - K_\theta \beta||^2 + d\right)
\]
\[
\lambda_i|\cdot \sim \text{Gamma}\left(a + \frac{1}{2}, \frac{\beta^2}{2} + b\right) \quad \forall \ i = 0, 1, 2, \cdots, n.
\]

Thus, RVM Gibbs sampler given above is a two component fixed scan sampler in which, for every iteration, \((1/\sigma^2, \{\lambda_i\}_{i=0}^n)\) is drawn given \(\beta\) and then \(\beta\) is drawn given the other variables. Thus, an estimate of the mean of the posterior predictive distribution is given by
\[
\hat{E}(y_{\text{new}}|y, \tilde{\theta}) = K^T_{\tilde{\theta},\text{new}} \hat{\beta}_{R,M},
\]
where \(\tilde{\theta}\) is the estimate of \(\theta\) found using cross validation, \(K^T_{\tilde{\theta},\text{new}}\) is as defined previously and \(\hat{\beta}_{R,M}\) is the Monte Carlo estimate of the posterior mean of \(\beta\), based on \(M\) iterations of the RVM Gibbs sampler.

The choice of \(M\) depends on the Monte Carlo standard error (MCSE) associated with the estimate given in (2.4). If the MCSE associated with (2.4) is deemed large, then it
can be reduced by choosing a larger $M$. On the other hand, if the error is small, computing resources can be conserved by choosing a smaller $M$. But, since the rate of convergence of the above RVM Gibbs sampler is not known, we cannot compute the Monte Carlo standard error associated with the Monte Carlo estimate given in (2.4). Thus, in the case of RVM, there are no guidelines for choosing a suitable $M$. Additionally, if proper priors are assumed in RVM, it requires the specification of user defined hyperparameters $(a, b, c, d)$. Specifying these hyperparameters to assume a non-informative proper prior can be challenging. Therefore, Fokuné et al. (2011) proposed to reduce the number of hyperparameters by assuming an extended hierarchical prior structure. The sufficient conditions for posterior propriety of RVM derived by Dixit and Roy (2021) allow for impropriety over $1/\sigma^2$ but not over $\{\lambda_i\}_{i=0}^n$. Hence in the data analysis section of this article, for $1/\sigma^2$ we assume an improper prior, $\pi(1/\sigma^2) \propto \sigma^2$ which can be obtained by choosing $c = d = 0$, and in the case of $\{\lambda_i\}_{i=0}^n$, for the sake of implementation, we choose $a = 0.001$ and $b = 0.01$ which yields a proper Gamma prior with a mean of 0.1 and a variance of 10.

3 Single Penalty Relevance Vector Machine

The improper prior assumed by Tipping (2001) was looked upon to be non-informative and hyperparameter free. But since it leads to an improper posterior distribution, one cannot implement RVM using that improper prior. In this section we will replace multiple penalty parameters with a single penalty parameter and simplify the prior structure to propose single penalty relevance vector machine (SPRVM).

Let $\{(y_i, x_i) : i = 1, 2, \cdots, n\}$ be the training data containing standardized responses and their corresponding covariate vectors, $\beta$ be the vector of coefficient parameters and $K_\theta$ be the $n \times (n+1)$ kernel matrix, where $y_i, x_i, \beta$ and $K_\theta$ are as defined previously in Section 2. Then we propose SPRVM as follows,

$$y|\beta, \xi, \theta \sim N(K_\theta \beta, \xi^{-1} I), \quad (3.1a)$$

$$\beta|\lambda \sim N(0, \lambda^{-1} I), \quad (3.1b)$$

$$\pi(\lambda) \propto \lambda^{a-1} \exp\{-b\lambda\} \quad (3.1c)$$
where \((a, b)\) are user specified hyperparameters. If a Gamma prior is assumed on \(\xi\), then a Gibbs sampler can be implemented. Such an MCMC sampler does not work well in practice since the traceplot for the \(\xi\) parameter reveals mixing issues. Therefore, for SPRVM, we do not assume any prior over \(\xi\). For SPRVM, the posterior density of parameters \((\beta, \lambda)\), indexed by \(\theta\) and \(\xi\), is as follows,

\[
\pi(\beta, \lambda|y, \xi, \theta) = \frac{f(y|\beta, \xi, \theta)\pi(\beta, \lambda)}{m_{\theta, \xi}(y)}, \tag{3.2}
\]

where \(f(y|\beta, \xi, \theta)\) is the data model given in (3.1a), \(\pi(\beta, \lambda)\) is the joint prior density following from (3.1b)-(3.1c) and \(m_{\theta, \xi}(y)\) is the marginal likelihood which is given by,

\[
m_{\theta, \xi}(y) = \int_{\mathbb{R}^{n+1} \times \mathbb{R}^+} f(y|\beta, \xi, \theta)\pi(\beta, \lambda) \, d\beta \, d\lambda. \tag{3.3}
\]

As mentioned previously in Section 2, the posterior density given in (3.2) is proper if and only if the marginal likelihood exists, i.e., if \(m_{\theta, \xi}(y) < \infty\). For SPRVM, the necessary conditions for the posterior propriety are as follows.

**Theorem 3.1** Consider the SPRVM given in (3.1), then, for \(b = 0\), which leads to the prior \(\pi(\lambda) \propto \lambda^{a-1}\), a necessary condition for the propriety of the posterior density (3.2) is \(a \in \left(-(n + 1)/2, 0\right)\).

A proof of Theorem 3.1 is given in the Appendix B.

The improper priors that do not satisfy the above necessary conditions will lead to an improper posterior. To identify improper priors that will lead to a proper posterior, we need to derive sufficient conditions. Since the full conditional distributions of \((\beta, \lambda)\) are known, we can construct a Gibbs sampler to explore the analytically intractable posterior density (3.2). The conditionals required for the implementation of the SPRVM Gibbs sampler are as follows,

\[
\beta|\cdot \sim N_{n+1}\left((K_\theta^T K_\theta + \lambda \xi^{-1} I)^{-1} K_\theta^T y, (K_\theta^T K_\theta \xi + \lambda I)^{-1}\right) \tag{3.4a}
\]

\[
\lambda|\cdot \sim Gamma\left(\frac{n + 1}{2} + a, \frac{\beta^T \beta}{2} + b\right). \tag{3.4b}
\]
Let $\{(\beta^{(j)}, \lambda^{(j)})\}_{j=0}^{\infty}$ be the fixed scan two component Markov chain associated with the SPRVM Gibbs sampler. Such a Gibbs sampler is geometrically ergodic if there exists a positive real valued function $G$ and a constant $\rho \in [0,1)$ such that,

$$||P^t((\beta_0, \lambda_0), \cdot) - \Pi(\cdot|y)||_{TV} \leq G(\beta_0, \lambda_0)\rho^t \quad \forall \ t = 1, 2, \cdots$$

(3.5)

where $|| \cdot ||_{TV}$ denotes the total variation norm, $P^t((\beta_0, \lambda_0), \cdot)$ denotes the probability distribution of the SPRVM Markov chain started at $(\beta_0, \lambda_0)$ after $t$ steps and $\Pi(\cdot|y)$ is the probability measure corresponding to the posterior density given in (3.2). If the geometric ergodicity of the SPRVM Gibbs sampler is established, then under finite moments, a CLT is guaranteed for the posterior mean estimates of $(\beta, \lambda)$ computed using the SPRVM Gibbs sampler draws (see Roberts and Rosenthal (1997)). The geometric ergodicity of SPRVM Gibbs sampler defined in (3.5) is proved in the following theorem.

**Theorem 3.2** The SPRVM Gibbs sampler $\{(\beta^{(j)}, \lambda^{(j)})\}_{j=0}^{\infty}$ is geometrically ergodic if conditions (i), (ii) and (iii) given below are satisfied.

(i) Either $b > 0$ or $a < b = 0$.

(ii) There exists $s \in (0, 1]$ such that,

$$\frac{\Gamma\left(\frac{n+1}{2} + a - s\right)}{\Gamma\left(\frac{n+1}{2} + a\right)} < 2^s.$$

(iii) The kernel matrix $K_\theta$ defined earlier in Section 2 is such that,

$$\frac{k_{\theta ij}}{k_{\theta jj}} \neq 1 \text{ and } k_{\theta jj} \neq 0 \ \forall i, j = 1, 2, \cdots, n \text{ and } i \neq j.$$

A proof of Theorem 3.2 is given in the Appendix B.

**Remark 1** Taking $s = 1$, condition (ii) of Theorem 3.2 holds for $a > -(n-2)/2$.

**Remark 2** The following are some examples of reproducing kernels typically used in sparse Bayesian learning models.
• **Gaussian kernel:**

\[ k_{\theta ij} = k_{\theta}(x_i, x_j) = \exp \left\{ -\frac{||x_i - x_j||^2}{\theta^2} \right\} \quad \forall \ i, j = 1, 2, \ldots, n, \]

where \( \theta \in \mathcal{R}_+ \) and \( || \cdot || \) denotes the Euclidean norm.

• **Laplace kernel:**

\[ k_{\theta ij} = k_{\theta}(x_i, x_j) = \exp \left\{ -\frac{||x_i - x_j||}{\theta} \right\} \quad \forall \ i, j = 1, 2, \ldots, n, \]

where \( \theta \in \mathcal{R}_+ \).

• **Polynomial kernel:**

\[ k_{\theta ij} = k_{\theta}(x_i, x_j) = (1 + x_i^T x_j)^\theta \quad \forall \ i, j = 1, 2, \ldots, n, \]

where \( \theta \in \mathcal{N} \).

Note that for each of the above three kernels, the condition (iii) of Theorem 3.2 will be satisfied if \( x_i \neq x_j \ \forall \ i, j = 1, 2, \ldots, n \) and \( i \neq j \).

Since the conditions for geometric ergodicity are sufficient for posterior propriety, a large class of improper priors guarantee posterior propriety for SPRVM. There is also a significant overlap in the necessary and sufficient conditions for posterior propriety. The necessary and sufficient conditions for posterior propriety of RVM derived by Dixit and Roy (2021) do not have any overlap in them. In fact, the sufficient conditions in Dixit and Roy (2021) do not allow for any prior impropriety in multiple penalty parameters of RVM. Given the sufficient conditions for posterior propriety of SPRVM, we propose to assume the following improper prior on the penalty parameter \( \lambda \),

\[ \pi(\lambda) \propto \frac{1}{\lambda^2}. \quad (3.6) \]

From Remark \[ \square \] for \( n \geq 5 \), the above improper prior satisfies the sufficient condition for posterior propriety of SPRVM. Thus, the above improper prior allows SPRVM to have a non-informative prior structure without the difficulty of specifying any hyperparameters and also leads to a proper posterior as long as \( n \geq 5 \). Thus, SPRVM is able to achieve the objective of specifying a non-informative improper prior which leads to a proper posterior.
In SPRVM, majority of parameters are estimated by the Gibbs sampler given in (3.4) and since SPRVM is primarily used for prediction, the remaining parameters i.e. the precision parameter, $\xi$, and the kernel parameter, $\theta$, are estimated using cross validation. We also tried the approach of estimating $\xi$ and $\theta$ by optimizing the marginal likelihood, however, the prediction performance of this approach was found to be poor. Additional details and illustrations about the marginal likelihood approach can be found in section 3.

In the case of SPRVM, prediction for the response variable say $y_{new}$ for a previously unobserved $p$ dimensional covariate vector say $x_{new}$ is based on posterior predictive distribution, which is given by

$$f(y_{new}|y, \tilde{\xi}, \tilde{\theta}) = \int_{\mathbb{R}^{n+1} \times \mathbb{R}_+} f(y_{new}|\beta, \tilde{\xi}, \tilde{\theta}) \pi(\beta, \lambda|y, \tilde{\xi}, \tilde{\theta}) \, d\beta \, d\lambda,$$

where $\tilde{\xi}$ and $\tilde{\theta}$ are the estimates of $\xi$ and $\theta$ found using cross validation.

As observed in the case of RVM, the estimate of the mean of the above posterior predictive distribution which is reported as the predicted response corresponding to $x_{new}$ is given by

$$\hat{E}(y_{new}|y, \tilde{\xi}, \tilde{\theta}) = K_{\tilde{\theta}, new}^T \hat{\beta}_{S,M},$$

(3.7)

where $K_{\tilde{\theta}, new}$ is as defined previously in (2.3) and $\hat{\beta}_{S,M}$ is the estimate of the posterior mean of $\beta$ found by $\sum_{j=1}^{M} \beta^{(j)}/M$ where $\beta^{(j)}$'s are samples from the SPRVM Gibbs sampler given in (3.4).

From Theorem 3.2 we know that SPRVM Gibbs sampler converges at a geometric rate. Therefore, using Theorem 3.2 and assuming $E[\beta^T \beta|y] < \infty$, the following central limit theorem holds,

$$\sqrt{M} \left( \hat{\beta}_{S,M} - \bar{\beta}_S \right) \to N(0, \Sigma) \quad \text{as} \quad M \to \infty,$$

where $\bar{\beta}_S$ is the posterior mean of $\beta$ in the case of SPRVM model and $\Sigma$ is the asymptotic covariance matrix. If the posterior mean estimate i.e. $\hat{\beta}_{S,M}$ could be based on $M$ iid observations, then $\Sigma$ can be easily estimated using sample covariance matrix. But since $\hat{\beta}_{S,M}$ is based on $M$ draws from the SPRVM Gibbs sampler, the draws are correlated and hence estimating $\Sigma$ is challenging. In the case of geometrically ergodic Markov chains, consistent batch means and spectral variance estimators for $\Sigma$ can be derived (see eg. Vats et al. (2019)) and these estimators are available in the mcmcse R package contributed by Flegal et al. (2017).
In the case of SPRVM, the estimate of the standard error associated with the Monte Carlo estimate in (3.7) is given by,

\[ \hat{SE}(K_{new}^T \hat{\beta}_{S,M}) = \sqrt{K_{new}^T (\hat{\Sigma} / M) K_{new}}, \]

(3.8)

where \( \hat{\Sigma} \) is a consistent estimator of \( \Sigma \). Thus, in SPRVM, we can provide a Monte Carlo estimate of the mean of the posterior predictive distribution along with a valid estimate of its standard error.

### 4 Data Analysis

In order to compare the predictive performance of RVM and SPRVM, we implement these two methods on high dimensional datasets in the field of genetics, nutrition and chemical engineering. For each dataset, we split the dataset into training and testing sets. The model is fitted on the training set, and the testing set is utilized to compute the root mean squared prediction error. For both the methods we use the Gaussian kernel. For RVM, the kernel parameter \( \theta \) and for SPRVM, the precision parameter, \( \xi \), and the kernel parameter, \( \theta \), are tuned by conducting a 10 fold cross validation. The average root mean squared prediction error (RMSPE) is computed based on 20 random splitting of the datasets into training and testing sets of size \( n^* \) and \( (n - n^*) = 10 \), respectively. The details of the three datasets are as follows:

**Gene dataset:** In order to study the genetics of mice population, an experiment was conducted by Lan et al. (2006). For the experiment, a total of \( n = 60 \) mice were available. Among those 60 mice, 31 were females and 29 were males. From each mouse, genetic information corresponding to 22575 genes was collected. Several physiological phenotypes were also collected. We will attempt to predict the physiological phenotype named stearoyl-CoA desaturase (SCD1) using the genetic and gender \( (p = 22576) \) information available. This dataset was analyzed in the past by Zhang et al. (2009) and Bondell and Reich (2012). It can be accessed at <http://www.ncbi.nlm.nih.gov/geo>; accession number GSE3330.

**Gas dataset:** In recent years, chemical engineers have attempted to obtain the octane number of gasoline samples using near infrared (NIR) spectrum measurements. We will
work with the gasoline dataset available in the pls R package and will attempt to predict
the octane number of the gasoline sample using NIR spectrum measurements. The data
was collected by Kalivas (1997), and the pls R package was contributed by Mevik et al.
(2016). The dataset consists of 60 gasoline samples. For each sample, octane number and
NIR spectra measurements from 900 nm to 1700 nm in 2nm intervals are provided in the
dataset. Thus, the dataset consists of \( n = 60 \) observations and \( p = 401 \) variables.

**Cookie dataset:** In the field of nutrition, researchers are often interested in finding out the
fat content of food items. The ppls R package provides a cookie dataset which consists of data
on 72 cookie dough samples. For each sample, fat content and NIR spectra measurements
from 1100 nm to 2498 nm at 2 nm intervals are provided in the dataset. In this exercise,
our objective will be to predict the fat content using NIR spectrum measurements. The
R package ppls was provided by Kraemer et al. (2008), and the dataset was collected by
Osborne et al. (1984). This dataset was analyzed in the past by Brown et al. (2001) among
others. Among the 72 observations, 2 are outliers which are often excluded from analysis.
Thus, the dataset consists of \( n = 70 \) observations and \( p = 700 \) variables.

For RVM and SPRVM Gibbs samplers, we run four independent chains using over dis-
persed starting values for 5000 iterations and assess convergence using potential scale re-
duction factor (PSRF) proposed by Gelman and Rubin (1992). The PSRF values for all
the variables in RVM and SPRVM were close to 1. We also investigated the corresponding
traceplots and observed that the MCMC sampler was fairly stable and there were no signs
of non convergence. Thus, for both RVM and SPRVM, in order to draw observations from
the posterior predictive distribution, the corresponding Gibbs samplers were run for 10000
iterations out of which first 5000 were treated as burn-in.

Table 1: Comparing the predictive performance of RVM and SPRVM using RMSPE

| Method   | Cookie dataset | Gas dataset | Gene dataset |
|----------|----------------|-------------|--------------|
| RVM      | 0.2445         | 0.1816      | 0.6446       |
| SPRVM    | 0.2379         | 0.1725      | 0.5852       |
| SPRVM-ML | 0.3675         | 0.1668      | 0.6137       |
In Table 1 we observe that the predictive performance of SPRVM is either similar or slightly better than that of RVM. The advantage of SPRVM over RVM is that, we can provide an asymptotically valid standard error estimate along with the Monte Carlo estimate of the mean of the posterior predictive distribution. To provide an illustration, for the gas dataset, consider an out of sample observation in which \( y_{new} = 1.0237 \). The Monte Carlo estimate of mean of the posterior predictive distribution for that observation was found to be 0.9589 in the case of RVM and 0.9468 in the case of SPRVM. Further, in the case of SPRVM, using (3.8), the associated Monte Carlo standard error was found to be 0.0022. Thus, in the case of SPRVM, we are able to quantify the uncertainty associated with our Monte Carlo estimate.

**Estimating (\( \theta, \xi \)):** For SPRVM, so far we have discussed estimating the precision parameter, \( \xi \), and the kernel parameter, \( \theta \), using cross validation. Another approach to estimating these parameters is by optimizing the marginal likelihood. The marginal likelihood for SPRVM is given in (3.3). Since the data model and the prior on \( \beta \) are both normal, we can integrate it out and a simplified version of marginal likelihood is then given by,

\[
m_{\theta, \xi}(y) = \int_{\mathbb{R}^+} f(y|\lambda, \xi, \theta) \pi(\lambda) \, d\lambda,
\]  

where,

\[
f(y|\lambda, \xi, \theta) = \frac{\xi^{-1/2}}{(2\pi)^{n/2}} \lambda^{(n+1)/2} |K^T K + \lambda \xi^{-1} I|^{-1/2} \exp \left\{ -\frac{1}{2} y^T (\xi^{-1} I + \lambda^{-1} K K^T)^{-1} y \right\}
\]  

and \( \pi(\lambda) \) is as given in (3.6). The estimate of \( \xi \) and \( \theta \) found by optimizing (4.1) is then given by, \((\hat{\xi}, \hat{\theta}) = \arg\max m_{\theta, \xi}(y)\).

To assess the predictive performance of the above approach, we implement it on the datasets mentioned earlier. In Table 1, for the Cookie dataset, the predictive performance of the SPRVM marginal likelihood (SPRVM-ML) approach is significantly worse than that of the SPRVM cross validation approach (SPRVM). This indicates that optimizing the marginal likelihood need not be optimal from a prediction standpoint. Hence, for SPRVM, we recommend taking a cross validation approach to estimating the precision parameter, \( \xi \), and the kernel parameter, \( \theta \).
5 Conclusion

In this article we have proposed to analyze RVM using a single penalty parameter instead of multiple penalty parameters. The single penalty relevance vector machine (SPRVM) model was analyzed using a semi Bayesian approach. In the case of SPRVM, the sufficient conditions for posterior propriety allow for several improper priors over the penalty parameter. Currently in the literature, improper prior is not allowed over any of the penalty parameters in RVM. Additionally, we also prove the geometric ergodicity of the Gibbs sampler used to analyze the SPRVM model, and hence using the Markov chain CLT, we can calculate standard errors associated with the Monte Carlo estimate of the mean of the posterior predictive distribution. Such a measure of uncertainty cannot be computed in the case of RVM since the rate of convergence of the RVM Gibbs sampler is currently not known in the literature. Thus, the SPRVM model proposed in this article has advantages over the RVM.

6 Appendix A: Some Useful Lemmas

Notation: From here on, to simplify notations we will drop the subscript $\theta$ and write $K_\theta$ as $K$ and $k_{\theta ij}$ as $k_{ij} \forall i, j = 1, 2, \cdots, n$.

Lemma 1 Let $y$ be an $n$ dimensional vector, $K$ be an $n \times (n + 1)$ matrix and $s > 0$. There exists a finite constant $Q$ depending on $y$ and $K$ such that

$$
\left(y^T K (K^T K + \lambda \xi^{-1} I)^{-2} K^T y\right)^s \leq Q.
$$

Proof: By definition, $K^T K = \sum_{i=1}^n t_i t_i^T$ where $t_i^T$ is the $i^{th}$ row of the matrix $K$ for all $i = 1, 2, \cdots, n$. The vector $y$ can be expressed as, $y = \sum_{j=1}^n b_j e_j$ where for each $j$, $b_j \in \mathcal{R}$ and $e_j$ is the $j^{th}$ unit vector with 1 in the $j^{th}$ place and 0 everywhere else, $j = 1, 2, \cdots, n$. Therefore,

$$
y^T K (K^T K + \lambda \xi^{-1} I)^{-2} K^T y = \left( \sum_{i=1}^n b_i e_i^T K \right) (K^T K + \lambda \xi^{-1} I)^{-2} \left( \sum_{j=1}^n b_j K^T e_j \right) = \sum_{i=1}^n \sum_{j=1}^n b_i b_j t_i^T \left( \sum_{k=1}^n t_k t_k^T + \lambda \xi^{-1} I \right)^{-2} t_j. \quad (6.1)
$$
Using Lemma 3 of Khare and Hobert (2011),
\[
\left( \sum_{k=1}^{n} t_k t_k^T + \lambda \xi^{-1} I \right)^{-2} t_i \leq Q_i \quad \forall \ i = 1, 2, \ldots, n,
\]
where \( \{Q_i : i = 1, 2, \ldots, n\} \) are constants that depends on \( n, t_1, t_2, \ldots, t_n \).

By Cauchy-Schwartz inequality,
\[
\left[ t_i^T \left( \sum_{k=1}^{n} t_k t_k^T + \lambda \xi^{-1} I \right)^{-2} t_j \right]^2 \leq \left[ t_i^T \left( \sum_{k=1}^{n} t_k t_k^T + \lambda \xi^{-1} I \right)^{-2} t_i \right] \left[ t_j^T \left( \sum_{k=1}^{n} t_k t_k^T + \lambda \xi^{-1} I \right)^{-2} t_j \right] \leq Q_i Q_j \quad \forall \ i, j = 1, 2, \ldots, n.
\]

The proof follows from (6.1) and (6.2) with \( Q = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |b_i b_j| \sqrt{Q_i Q_j} \right)^s \).

**Lemma 2** Suppose \( K \) is a \( n \times (n + 1) \) kernel matrix defined previously in Section 2 that satisfies condition (iii) of Theorem 3.2. Then, \( K \) is a full row rank matrix.

**Proof:** Let \( \alpha_i \in \mathbb{R} \) for all \( i = 1, 2, \ldots, n \). We need to show that \( \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0 \) is the only solution that satisfies the following equations,

\[
\sum_{i=1}^{n} \alpha_i = 0 \quad (6.3)
\]

\[
\sum_{i=1}^{n} \alpha_i k_{ij} = 0 \quad \forall \ j = 1, 2, \ldots, n. \quad (6.4)
\]

Using (6.3) we get,
\[
\alpha_j = -\sum_{i=1}^{n} \alpha_i \quad \forall \ j = 1, 2, \ldots, n. \quad (6.5)
\]

Using (6.4) we get,
\[
\alpha_j = \frac{-1}{k_{jj}} \sum_{i=1}^{n} \alpha_i k_{ij} \quad \forall \ j = 1, 2, \ldots, n. \quad (6.6)
\]

Further, using (6.5) and (6.6), we get,
\[
\sum_{i=1}^{n} \alpha_i \left( 1 - \frac{k_{ij}}{k_{jj}} \right) = 0 \quad \forall \ j = 1, 2, \ldots, n. \quad (6.7)
\]

Using condition (iii) of Theorem 3.2 \( \exists \gamma_1 \in \mathbb{R} - \{0\} \) and \( \gamma_2 \in \mathbb{R} - \{0\} \) such that,
\[
\gamma_1 \sum_{i=1}^{n} \alpha_i \leq \sum_{i=1}^{n} \alpha_i \left( 1 - \frac{k_{ij}}{k_{jj}} \right) \leq \gamma_2 \sum_{i=1}^{n} \alpha_i \quad \forall \ j = 1, 2, \ldots, n. \quad (6.8)
\]
• Case 1: If $\gamma_1, \gamma_2 > 0$ or $\gamma_1, \gamma_2 < 0$, then from (6.7) and (6.8) we have,

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i = 0 \quad \forall \ j = 1, 2, \cdots, n. \quad (6.9)$$

Using (6.3) and (6.9), $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ is the only possible solution.

• Case 2: If $\gamma_1 > 0$ and $\gamma_2 < 0$, then from (6.7) and (6.8) we have,

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \leq 0 \quad \forall \ j = 1, 2, \cdots, n. \quad (6.10)$$

Using (6.3) and (6.10), $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ is the only possible solution.

• Case 3: If $\gamma_1 < 0$ and $\gamma_2 > 0$, then from (6.7) and (6.8) we have,

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i \geq 0 \quad \forall \ j = 1, 2, \cdots, n. \quad (6.11)$$

Using (6.3) and (6.11), $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ is the only possible solution.

Thus, combining Case 1, Case 2 and Case 3, $K$ is a full row rank matrix, i.e. $\text{rank}(K) = n$.

Lemma 3 Suppose $K$ is a kernel matrix as defined in Section 2 that satisfies condition (iii) of Theorem 3.2 and $s \in (0, 1]$, then,

$$\left[ \text{tr}\left((K^TK\xi + \lambda I)^{-1}\right) \right]^s \leq \xi^{-s}\left[ \text{tr}\left((K^TK)^+\right) \right]^s + \lambda^{-s},$$

where $(K^TK)^+$ denotes the Moore Penrose inverse of $K^TK$.

Proof: Let $O\Psi O^T$ be the spectral decomposition of $K^TK$ where $O$ is an orthogonal matrix such that its columns $\{o_i\}_{i=1}^{n+1}$ are eigenvectors of $K^TK$ and $\Psi = \text{diag}(\psi_1, \psi_2, \cdots, \psi_{n+1})$ is a diagonal matrix whose diagonal elements are eigenvalues of $K^TK$. Then,

$$\left(K^TK\xi + \lambda I\right)^{-1} = O\left(\Psi\xi + \lambda I\right)^{-1}O^T. \quad (6.12)$$

As in Abrahamsen and Hobert (2017), let $\Psi^+$ be a $(n + 1) \times (n + 1)$ diagonal matrix whose $i^{th}$ diagonal element is given by,

$$\psi_i^+ = \psi_i^{-1}(1 - I_{\{0\}}(\psi_i)) \quad \forall \ i = 1, 2, \cdots, n + 1.$$

Thus, combining Case 1, Case 2 and Case 3, $K$ is a full row rank matrix, i.e. $\text{rank}(K) = n$. 

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$$\psi_i^+ = \psi_i^{-1}(1 - I_{\{0\}}(\psi_i)) \quad \forall \ i = 1, 2, \cdots, n + 1.$$
Further,
\[
\left( \psi_i \xi + \lambda \right)^{-1} \leq \xi^{-1} \psi_i^+ + \lambda^{-1} I_{\{0\}}(\psi_i^+) \quad \forall \quad i = 1, 2, \ldots, n + 1.
\]
So,
\[
\left( \Psi \xi + \lambda I \right)^{-1} \leq \xi^{-1} \Psi^+ + \lambda^{-1} (I - P_\Psi)
\]  
(6.13)
where \( P_\Psi \) is a \((n + 1) \times (n + 1)\) diagonal matrix whose \( i^{th} \) diagonal element is \( 1 - I_{\{0\}}(\psi_i) \).

Using (6.12) and (6.13), we get,
\[
\left( K^T K \xi + \lambda I \right)^{-1} \leq \xi^{-1} O \Psi^+ O^T + \lambda^{-1} O (I - P_\Psi) O^T
\]  
\[= \xi^{-1} (K^T K)^+ + \lambda^{-1} O (I - P_\Psi) O^T. \]  
(6.14)

Let \( \tilde{O} \) be submatrix of \( O \) consisting of columns \( \{o_i\}_{i \in A} \) where \( A = \{i \in \{1, 2, \ldots, n + 1\} : \psi_i > 0\} \) then,
\[
OP_\Psi O^T = \sum_{i \in A} o_i o_i^T = \tilde{O} \tilde{O}^T.
\]

Further, \( \tilde{O} \tilde{O}^T \) is an orthogonal projection onto \( K^T K \) since \( \{o_i\}_{i \in A} \) forms an orthogonal basis for the column space of \( K^T K \). Therefore,
\[
O(I - P_\Psi) O^T = I - P_{K^T K},
\]  
(6.15)
where \( P_{K^T K} \) denotes orthogonal projection onto column space of \( K^T K \).

Using (6.14) and (6.15) and since \( s \in (0, 1] \),
\[
\left( K^T K \xi + \lambda I \right)^{-1} \leq \xi^{-1} (K^T K)^+ + \lambda^{-1} (I - P_{K^T K})
\]
\[
\therefore \left[ tr \left( (K^T K \xi + \lambda I)^{-1} \right) \right]^s \leq \xi^{-s} \left[ tr \left( (K^T K)^+ \right) \right]^s + \lambda^{-s} \left[ tr (I - P_{K^T K}) \right]^s. \]  
(6.16)

Further, using Lemma 2,
\[
tr(I - P_{K^T K}) = tr(I) - tr(P_{K^T K}) = (n + 1) - \text{rank}(K) = 1.
\]  
(6.17)

Using (6.16) and (6.17), we get,
\[
\left[ tr \left( (K^T K \xi + \lambda I)^{-1} \right) \right]^s \leq \xi^{-s} \left[ tr \left( (K^T K)^+ \right) \right]^s + \lambda^{-s}.
\]
Hence proved.
Lemma 4  Consider the following integral,
\[
\int_{\mathcal{R}+} \frac{t^{-(a+1)}}{(g + t)^{(n+1)/2}} dt,
\]
where \(g\) and \(a\) are constants. The above integral is finite iff \(a \in (-\frac{n+1}{2}, 0)\).

Proof: Suppose \(t = g \tan^2 \omega\), then the above integral becomes,
\[
2g^{-(a+(n+1)/2)} \int_0^{\pi/2} \frac{(\tan^2 \omega)^{-(a+1)}}{(\sec^2 \omega)^{(n+1)/2}} \tan \omega \sec^2 \omega d\omega.
\]
Let \(z = \sec^2 \omega\), then the above integral becomes,
\[
g^{-(a+(n+1)/2)} \int_1^{\infty} (z-1)^{-(a+1)} z^{-(n+1)/2} dz.
\]
The above integral is finite iff \(a \in (-\frac{n+1}{2}, 0)\). Hence proved.

7 Appendix B: Proof of Theorems

Proof of Theorem 3.1

From (4.1),
\[
m_{\theta,\xi}(y) = \int_{\mathcal{R}+} f(y|\lambda, \xi, \theta) \pi(\lambda) d\lambda.
\]
Using (4.2),
\[
m_{\theta,\xi}(y) = \int_{\mathcal{R}+} \frac{\xi^{-1/2}}{(2\pi)^{n/2}} \lambda^{(n+1)/2} |K^TK + \lambda \xi^{-1}I|^{-1/2} \exp \left\{ -\frac{1}{2} y^T (\xi^{-1} I + \lambda^{-1} K^T K)^{-1} y \right\} \lambda^{a-1} d\lambda
\]
\[
\geq \frac{\xi^{-1/2}}{(2\pi)^{n/2}} \exp \left\{ -\frac{\xi}{2} y^T y \right\} \int_{\mathcal{R}+} \frac{\lambda^{a-1}}{\left( \xi^{-1} + \frac{\psi_{max}}{\lambda} \right)^{\frac{n+1}{2}}} d\lambda,
\]
where \(\psi_{max}\) is the maximum eigenvalue of \(K^TK\). Using the transformation \(t = 1/\lambda\), the above integral becomes,
\[
m_{\theta,\xi}(y) \geq \frac{\xi^{-1/2}}{(2\pi)^{n/2}} \exp \left\{ -\frac{\xi}{2} y^T y \right\} \left[ \frac{1}{\psi_{max}} \right]^{(n+1)/2} \int_{\mathcal{R}+} \frac{t^{-(a+1)}}{\left( \frac{\xi^{-1}}{\psi_{max} + t} \right)^{\frac{n+1}{2}}} dt.
\]
Using Lemma 4, the above integral is finite iff \( a \in (-(n + 1)/2, 0) \). Hence proved.

**Proof of Theorem 3.2**

Since, SPRVM Gibbs sampler is a two block Gibbs sampler, the two sub-chains \( \{ \beta^{(j)} \}_{j=0}^{\infty} \) and \( \{ \lambda^{(j)} \}_{j=0}^{\infty} \) are themselves Markov chains. Further, the rate of convergence of the three chains \( \{ \beta^{(j)}, \lambda^{(j)} \}_{j=0}^{\infty}, \{ \beta^{(j)} \}_{j=0}^{\infty} \) and \( \{ \lambda^{(j)} \}_{j=0}^{\infty} \) is the same (see Roberts and Rosenthal (2001)). Therefore, if we prove the geometric ergodicity of one of the chains, it holds for all the three chains. We will work with the \( \{ \lambda^{(j)} \}_{j=0}^{\infty} \) chain. The Markov transition density associated with the \( \{ \lambda^{(j)} \}_{j=0}^{\infty} \) chain is given by,

\[
p_t(\tilde{\lambda}|\lambda) = \int_{\mathbb{R}^{n+1}} \pi(\tilde{\lambda}|\beta, y) \pi(\beta|\lambda, y) \, d\beta,
\]

where \( \pi(\tilde{\lambda}|\beta, y) \) is the density corresponding to the full conditional distribution given in (3.4b) and \( \pi(\beta|\lambda, y) \) is the density of the full conditional distribution given in (3.4a).

We define the drift function as follows,

\[
v(\tilde{\lambda}) = \tilde{\lambda}^m + \tilde{\lambda}^{-s}, \tag{7.1}
\]

where \( m \in (0, 1) \) is a positive constant that is determined in the proof and \( s \in (0, 1] \) is a constant that satisfies condition (ii) in Theorem 3.2.

Since the above drift function is unbounded off compact sets and \( \{ \lambda^{(j)} \}_{j=0}^{\infty} \) is a Feller chain, geometric ergodicity of the \( \{ \lambda^{(j)} \}_{j=0}^{\infty} \) chain is established by proving the following drift condition (see Meyn and Tweedie (1993))

\[
E[v(\tilde{\lambda})|\lambda] = \int_{\mathbb{R}^+} v(\tilde{\lambda}) \, p_t(\tilde{\lambda}|\lambda) \, d\tilde{\lambda} \leq L + \rho v(\lambda)
\]

where \( L > 0 \) and \( \rho \in (0, 1) \) are finite constants.

Note that,

\[
E[v(\tilde{\lambda})|\lambda] = E[E[v(\tilde{\lambda})|\beta]|\lambda]. \tag{7.2}
\]

We start with the inner expectation in (7.2). Also, first consider \( b > 0 \),

\[
E[\tilde{\lambda}^m|\beta] = \Gamma(a + m + \frac{n+1}{2}) \frac{\beta^T \beta}{2}^{-m} \leq \frac{\Gamma(a + m + \frac{n+1}{2})}{\Gamma(a + \frac{n+1}{2})} b^{-m}. \tag{7.3}
\]
Now we consider the outer expectation in (7.2). From (7.3) we get,

\[ E[\tilde{\lambda}^m|\lambda] \leq \frac{\Gamma(a+m+n_{\perp}+1)}{\Gamma(a+n_{\perp}+1/2)} b^{-m}. \]  

(7.4)

Next,

\[ E[\tilde{\lambda}^{-s}|\beta] = \frac{\Gamma(a-s+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \left( \frac{\beta^T \beta}{2} + b \right)^s \leq \frac{\Gamma(a-s+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \left( \frac{(\beta^T \beta)^s}{2^s} + b^s \right). \]  

(7.5)

From (7.3), using Lemma 1 and Lemma 3 we have,

\[ E[\tilde{\lambda}^{-s}|\lambda] \leq \frac{\Gamma(a-s+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \left( \frac{1}{2^s} E[(\beta^T \beta)^s|\lambda] + b^s \right) \]

\[ \leq \frac{\Gamma(a-s+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \left( \frac{1}{2^s} \left( E[(\beta^T \beta)|\lambda]\right)^s + b^s \right) \]

\[ \leq \frac{\Gamma(a-s+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \left\{ \frac{1}{2^s} \left[ y^T K(K^T K + \lambda \xi^{-1} I)^{-2} K^T y + tr \left( (K^T K \xi + \lambda I)^{-1} \right) \right]^s + b^s \right\} \]

\[ \leq \frac{\Gamma(a-s+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \left[ Q + (2b)^s + \xi^{-s} \left( tr \left( (K^T K)^+ \right) \right)^s + \lambda^{-s} \right] \]

\[ \leq L_0 + \frac{\Gamma(a-s+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \frac{1}{2^s} \lambda^{-s} \]  

(7.6)

where \( L_0 = \frac{\Gamma(a-s+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \frac{1}{2^s} \left[ Q + (2b)^s + \xi^{-s} \left( tr \left( (K^T K)^+ \right) \right)^s \right]. \)

Using (7.1), (7.3) and (7.6), we get,

\[ E[v(\tilde{\lambda})|\lambda] \leq L_1 + \rho_0 \ v(\lambda) \]

where \( L_1 = L_0 + \frac{\Gamma(a+m+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \ b^{-m} \) and \( \rho_0 = \frac{\Gamma(a-s+n_{\perp}+1/2)}{\Gamma(a+n_{\perp}+1/2)} \frac{1}{2^s} \) are finite constants. Further, using condition (ii) of Theorem 3.2, \( \rho_0 \in (0, 1) \). Thus proving geometric ergodicity of the SPRVM Gibbs sampler for \( b > 0 \).

Now consider \( b = 0 \) and \( a < 0 \). Let \( \Sigma_\beta \) denote the covariance matrix of \( \beta|\lambda, y \) i.e.

\[ \Sigma_\beta = (K^T K \xi + \lambda I)^{-1} \implies \Sigma_\beta^{-1} = K^T K \xi + \lambda I. \]

As defined in Lemma 3 let \( O \Psi O^T \) be spectral decomposition of \( K^T K \) where \( \Psi = diag(\psi_1, \psi_2, \cdots, \psi_{n+1}) \). Also, let \( \psi_{\text{max}} = \max\{\psi_1, \psi_2, \cdots, \psi_{n+1}\} \). Therefore,

\[ \Sigma_\beta^{-1} \leq (\psi_{\text{max}} \xi + \lambda) I \]

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\[ \Rightarrow \beta^T \Sigma^{-1}_\beta \beta \leq \beta^T (\psi_{\max} \xi + \lambda) I \beta \]

\[ \Rightarrow (\beta^T \Sigma^{-1}_\beta \beta)^m \geq \left[ \beta^T (\psi_{\max} \xi + \lambda) I \beta \right]^{-m}. \] (7.7)

Now, \( \beta^T \Sigma^{-1}_\beta \beta \lambda, y \) has a non central \( \chi^2 \) distribution with \( n + 1 \) degrees of freedom. Using Lemma 4 of Román and Hobert (2012), for \( m \in (0, 1) \), we get,

\[ E[(\beta^T \Sigma^{-1}_\beta \beta)^{-m} | \lambda] \leq 2^{-m} \frac{\Gamma(\frac{n+1}{2} - m)}{\Gamma(\frac{n+1}{2})}. \] (7.8)

Now, using (7.7) and (7.8),

\[ E[(\beta^T \beta)^{-m} | \lambda] = (\psi_{\max} \xi + \lambda)^m \left[ \beta^T (\psi_{\max} \xi + \lambda) I \beta \right]^{-m} \]

\[ \leq (\psi_{\max} \xi + \lambda)^m E[(\beta^T \Sigma^{-1}_\beta \beta)^{-m} | \lambda] \]

\[ \leq ((\psi_{\max} \xi)^m + \lambda^m) 2^{-m} \frac{\Gamma(\frac{n+1}{2} - m)}{\Gamma(\frac{n+1}{2})}. \] (7.9)

Since,

\[ E[\tilde{\lambda}^m | \beta] = \frac{\Gamma(a + m + \frac{n+1}{2})}{\Gamma(a + \frac{n+1}{2})} \left[ \psi_{\max} \xi \right]^m, \]

using (7.9) we have,

\[ E[\tilde{\lambda}^m | \lambda] \leq \frac{\Gamma(a + m + \frac{n+1}{2}) \Gamma(\frac{n+1}{2} - m)}{\Gamma(a + \frac{n+1}{2})} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+1}{2})} ((\psi_{\max} \xi)^m + \lambda^m). \] (7.10)

Using (7.1), (7.6) and (7.10),

\[ E[v(\tilde{\lambda}) | \lambda] \leq \tilde{L}_0 + L_1 + \rho_0 \lambda^{-s} + \rho_1 \lambda^m \]

where \( \rho_0 \) is as defined before and

\[ \tilde{L}_0 = \frac{\Gamma(a - s + \frac{n+1}{2})}{\Gamma(a + \frac{n+1}{2})} \left[ Q + \xi^{-s} (tr \left( (K^T K)^+ \right)) \right]^s, \]

\[ L_1 = \frac{\Gamma(a + m + \frac{n+1}{2}) \Gamma(\frac{n+1}{2} - m)}{\Gamma(a + \frac{n+1}{2}) \Gamma(\frac{n+1}{2})} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+1}{2})} \psi_{\max} \xi^m, \]

\[ \rho_1 = \frac{\Gamma(a + m + \frac{n+1}{2}) \Gamma(\frac{n+1}{2} - m)}{\Gamma(a + \frac{n+1}{2}) \Gamma(\frac{n+1}{2})}. \]

For \( m \in (0, 1) \cap (0, -a) \), Román and Hobert (2012) have shown that \( \rho_1 < 1 \). Let \( L^* = \tilde{L}_0 + L_1 \) and \( \rho^* = \max\{\rho_0, \rho_1\} \). Then for \( b = 0 \) and \( a < 0 \),

\[ E[v(\tilde{\lambda}) | \lambda] \leq L^* + \rho^* v(\lambda) \]

where \( L^* \) and \( \rho^* \) are finite constants. Further, \( \rho^* \in (0, 1) \) since \( \rho_0 \in (0, 1) \) and \( \rho_1 \in (0, 1) \).

Thus, we have proved geometric ergodicity of \( \{\lambda^{(j)}\}_{j=0}^{\infty} \) for \( b = 0 \) and \( a < 0 \). Hence proved.
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