CHOW GROUPS AND EQUIVARIANT GEOMETRY

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Introduction. Throughout, ‘variety’ will mean ‘separated scheme of finite type over Spec(\(\mathbb{C}\))’. A G-variety will mean a variety endowed with the action of a linear algebraic group G. The existence of functorial mixed Hodge structures on the rational cohomology, Borel-Moore homology, equivariant cohomology, etc., of a variety will be freely used (see [D]). For homology (Borel-Moore, equivariant, etc.), \(H_*\) will be called pure if each \(H_i\) is a pure Hodge structure of weight \(-i\).

Theorem 1. Let X be a variety on which a linear algebraic group acts with finitely many orbits. If the (Borel-Moore) homology \(H_*(X; \mathbb{Q})\) is pure (for instance, if X is rationally smooth and complete), then the cycle class map

\[ CH_*(X)_{\mathbb{Q}} \xrightarrow{\sim} H_*(X; \mathbb{Q}), \]

from rational Chow groups to homology, is a degree doubling isomorphism.

This extends a result of Fulton-MacPherson-Sottile-Sturmfels [FMSS] from solvable groups to arbitrary linear algebraic groups. The price paid is that while most of the results of [FMSS] hold integrally, we deal exclusively with rational coefficients. Our arguments are quite different from those in [FMSS]. In particular, Theorem 1 is deduced from statements in the equivariant context.

For a G-variety X, write \(A^G_*(X)_{\mathbb{Q}}\) for its rational equivariant Chow groups. Let \(H^G_*(X; \mathbb{Q})\) denote the G-equivariant (Borel-Moore) homology of X, and let \(W_*\) be the weight filtration on \(H^G_*(X; \mathbb{Q})\).

Theorem 2. Let G be a linear algebraic group acting on a variety X. Assume X admits finitely many orbits. Then the cycle class map

\[ A^G_i(X)_{\mathbb{Q}} \xrightarrow{\sim} W_{-2i}H^G_{2i}(X; \mathbb{Q}) \]

is an isomorphism for each \(i \in \mathbb{Z}\).

This is established by mimicking B. Totaro’s arguments from [T1]. Combined with Lemma 7, it yields the equivariant analogue of Theorem 1.

Corollary 3. Let G be a linear algebraic group acting on a variety X. Assume X admits finitely many orbits, and that \(H^G_*(X; \mathbb{Q})\) is pure. Then the cycle class map

\[ A^G_*(X)_{\mathbb{Q}} \xrightarrow{\sim} H^G_*(X; \mathbb{Q}) \]

is a degree doubling isomorphism.

Now Theorem 1 follows via restriction from the equivariant to the non-equivariant context, using a result of M. Brion (Lemma 4).

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Proofs

Preliminaries. Let $X$ be a $G$-variety. Write $\bar{H}^G_*(X; \mathbb{Q}(j))$ for the equivariant motivic cohomology of $X$ (with ‘coefficients’ in $\mathbb{Q}(j)$). Write $\bar{H}^G_*(X; \mathbb{Q}(j))$ for the equivariant motivic (Borel-Moore) homology of $X$. In terms of the higher equivariant Chow groups $A^G_p(X, k)$ of [EG]:

$$\bar{H}^G_i(X; \mathbb{Q}(j)) = A^G_{i-j}(X, i-2j) \otimes \mathbb{Q}.$$ 

In particular, $\bar{H}^G_{2i}(X; \mathbb{Q}(i)) = A^G_i(X) \otimes \mathbb{Q}$. It will be notationally convenient to set $A^G_{-2i} = \bar{H}^G_{2i}(\text{Spec}(\mathbb{C}); \mathbb{Q}(i))$.

Given a group morphism $H \to G$, there is a restriction map $A^G_* \to A^H_*$. There are analogous restriction maps for motivic homology.

Lemma 4. If $G$ is connected, then restriction induces an isomorphism:

$$\mathbb{Q} \otimes_{A^G_*} A^G_i(X) \otimes \mathbb{Q} \cong CH_*(X) \otimes \mathbb{Q}.$$ 

Proof. If $G$ is reductive, then this is [B, Corollary 6.7(i)]. In general, if $U \subseteq G$ is the unipotent radical, then $G/U$ is reductive, and restriction yields an isomorphism $A^G_{G/U} \cong A^G_G$. Similarly for motivic homology (see [T2, Lemma 2.18]).

There is a natural map $\bar{H}^G_{2i}(X; \mathbb{Q}(j)) \to W_{-2j}H^G_{2i}(X; \mathbb{Q})$. See [T1, §4] for a cogent explanation of this. The map

$$\bar{H}^G_{2i}(X; \mathbb{Q}(i)) = A^G_i(X) \otimes \mathbb{Q} \to \bar{H}^G_{2i}(X; \mathbb{Q})$$

is the cycle class map.

Weak property. A $G$-variety $X$ satisfies the weak property if the cycle class map

$$\bar{H}^G_{2i}(X; \mathbb{Q}(i)) = A^G_i(X) \otimes \mathbb{Q} \to W_{-2i}H^G_{2i}(X; \mathbb{Q})$$

is an isomorphism for each $i \in \mathbb{Z}$.

Strong property. A $G$-variety $X$ satisfies the strong property if it satisfies the weak property and the map

$$\bar{H}^G_{2i+1}(X; \mathbb{Q}(i)) \to \text{gr}^W_{-2i}H^G_{2i+1}(X; \mathbb{Q})$$

is surjective for each $i \in \mathbb{Z}$. Here $\text{gr}^W_*$ denotes the associated graded with respect to the weight filtration $W_*$.

Lemma 5. Let $G$ be a linear algebraic group, and let $K \subseteq G$ be a closed subgroup. Then $G/K$ satisfies the strong property (as a $G$-variety).

Proof. The map $A^G_{G/K} \otimes \mathbb{Q} \cong H^G_*(G/K; \mathbb{Q})$ is a degree doubling isomorphism (see [T2, Theorem 2.14]).

Lemma 6. Let $X$ be a $G$-variety, $Z \subseteq X$ a $G$-stable closed subvariety, and $U = X - Z$ the open complement. If $U$ satisfies the strong property and $Z$ the weak, then $X$ satisfies the weak property.
**Proof.** We have a morphism of long exact sequences:

\[
\begin{align*}
\tilde{H}^G_{2i+1}(U; \mathbb{Q}(i)) \rightarrow \tilde{H}^G_{2i}(Z; \mathbb{Q}(i)) \rightarrow \tilde{H}^G_{2i}(X; \mathbb{Q}(i)) \rightarrow \tilde{H}^G_{2i}(U; \mathbb{Q}(i)) \rightarrow 0 \\
\text{gr}_2^W H^G_{2i+1}(U) \rightarrow W_{-2i} \tilde{H}^G_{2i}(Z) \rightarrow W_{-2i} \tilde{H}^G_{2i}(X) \rightarrow W_{-2i} \tilde{H}^G_{2i}(U) \rightarrow 0
\end{align*}
\]

where ‘$\mathbb{Q}$’ has been omitted from the notation in the bottom row due to typesetting considerations. The first vertical map is surjective (strong property for $U$). The second and fourth vertical maps are isomorphisms by the weak property for $Z$ and $U$ respectively. So the third vertical map must also be an isomorphism. \hfill $\square$

**Proof of Theorem 2.** Combine Lemma 5 and Lemma 6.

**Proof of Corollary 3.** Combine Theorem 2 with the following observation.

**Lemma 7.** Let $X$ be a variety on which a linear algebraic group $G$ acts with finitely many orbits. Then the (Borel-Moore) homology $H_*(X; \mathbb{Q})$ is a successive extension of Hodge structures of type $(n, n)$.

**Proof.** We may assume $X = G/K$, where $K \subset G$ is a closed subgroup. Now $H^*(G/K; \mathbb{Q})$ is the $K$-equivariant cohomology of $G$. Consider the usual simplicial variety computing this (see [D, §6]). Filtering by skeleta yields a spectral sequence whose $E_1$ entries are of the form $H^q(K^{	imes p} \times G; \mathbb{Q})$ [D, Proposition 8.3.5]. Now recall that the cohomology of a linear algebraic group is of type $(n, n)$ [D, §9.1]. \hfill $\square$

**Proof of Theorem 1.** We may assume that $G$ is connected. Let $H^G_*$ denote the equivariant cohomology ring of a point. Purity and Lemma 7 imply that $H_*(X; \mathbb{Q})$ is concentrated in even degrees. Purity also implies that the natural map

\[
\mathbb{Q} \otimes_{H^G_*} H^G_*(X; \mathbb{Q}) \xrightarrow{\sim} H_*(X; \mathbb{Q})
\]

is an isomorphism. Further, the cycle class map $A^G_*(X) \mathbb{Q} \xrightarrow{\sim} H^G_*(X; \mathbb{Q})$ is an isomorphism by Corollary 3, since purity of $H_*(X; \mathbb{Q})$ implies purity of $H^G_*(X; \mathbb{Q})$. Thus, combined with Lemma 4, we obtain a commutative diagram:

\[
\begin{array}{c}
A^G_*(X) \mathbb{Q} \rightarrow \mathbb{Q} \otimes_{A^G_*} A^G_*(X) \mathbb{Q} \xrightarrow{\sim} CH_*(X) \mathbb{Q} \\
\quad \downarrow \quad \downarrow \\
H^G_*(X; \mathbb{Q}) \rightarrow \mathbb{Q} \otimes_{H^G_*} H^G_*(X; \mathbb{Q}) \xrightarrow{\sim} H_*(X; \mathbb{Q})
\end{array}
\]

Consequently, $CH_*(X) \mathbb{Q} \xrightarrow{\sim} H_*(X; \mathbb{Q})$ is an isomorphism.

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The Appalachians