The commutant of simple modules over almost commutative algebras

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Abstract

Let $B$ be a finitely generated algebra over a field $k$. Then $B$ is called a Jacobson algebra if every semiprime ideal of $B$ is semiprimitive. We will discuss several conditions, all involving the commutant of simple $B$-modules, which imply that $B$ is Jacobson. In particular, we will recover the well-known result that every finitely generated almost commutative algebra is Jacobson. The same holds true for $\mathbb{N}$-filtered $k$-algebras $B$ with a locally finite filtration such that the associated graded $k$-algebra is left-noetherian.

Introduction

Following Dixmier [3] we shall recall the proofs of the following three well-known results which are fundamental for the representation theory of universal enveloping algebras:

Let $k$ be a field and $\mathfrak{g}$ a finite-dimensional Lie algebra over $k$. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. Then we have:

1. If $M$ is a simple $U(\mathfrak{g})$-module, then its commutant $\text{End}_{U(\mathfrak{g})}(M)$ is algebraic over $k$. (This assertion is not obvious if $M$ is infinite-dimensional as a $k$-vector space.)

2. The associative $k$-algebra $U(\mathfrak{g})$ is a Jacobson algebra which means that every prime ideal is an intersection of primitive ideals.

3. In particular $U(\mathfrak{g})$ is Jacobson-semisimple.

These assertions can be proved using variants of Grothendieck’s generic flatness lemma. In algebraic geometry generic flatness means the following: If $A$ is a noetherian integral domain, $B$ a finitely generated commutative $A$–algebra, and $M$ a finitely generated $B$-module, then there exists a non-zero $f \in A$ such that the localization $M_f = A_f \otimes_A M$
of $M$ is free and hence flat. Duflo [7] and Quillen [12] showed that generic freeness over $k[X]$ is satisfied for simple $k[X] \otimes_k B$-modules where $B$ is a finitely generated almost commutative $k$-algebra, compare Theorem 14 and Theorem 15. This includes all quotients $B$ of universal enveloping algebras of finite-dimensional Lie algebras. Artin, Small, and Zhang [2] proved that generic freeness holds for simple $k[X] \otimes_k B$-modules where $B$ is a $k$-algebra endowed with a locally finite $\mathbb{N}$-filtration such that the associated graded algebra is noetherian.

These results about generic freeness can be used to prove that algebras over commutative Jacobson rings are also Jacobson. This was done by Duflo [7] in the almost commutative case and by Szczepanski [14] in the $\mathbb{N}$-filtered case.

As a motivation we give an application of the above results. Let $k$ be an algebraically closed field. Let $P$ be a primitive ideal of $U(g)$ and $M$ a simple $U(g)$-module such that $P = \text{Ann}_U(g)(M) = k1$ because $\text{End}_U(g)(M)$ is algebraic over $k$ and $k$ is algebraically closed. Let $Z(g)$ denote the center of $U(g)$. Since $Z(g)/(Z(g) \cap P)$ can be regarded as a subalgebra of $\text{End}_U(g)(M)$, we conclude $Z(g)/(Z(g) \cap P) \cong k$. Thus there exists a unique homomorphism $\chi_P : Z(g) \to k$ such that $\ker \chi_P = Z(g) \cap P$, which is called the central character of $P$. Since $U(g)$ is semiprimitive, we know that $W \in Z(g)$ and $\chi_P(W) = 0$ for all $P \in \text{Prim}(U(g))$ implies $W = 0$.

Let $S(g)$ be the symmetric algebra of $g$ and $\bar{\gamma} : S(g) \to U(g)$ the Duflo map which is, in a certain sense not to be explained here, a modification of the symmetrization map

$$\beta(X_1 \cdots X_n) = \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}.$$

The Poincaré-Birkhoff-Witt theorem implies that $\beta$ and $\bar{\gamma}$ are $\text{ad}(g)$-invariant linear isomorphisms. Let $I(g)$ denote the subalgebra of all $\text{ad}(g)$-invariants of $S(g)$. In 1969 Duflo proved that the restriction $\gamma : I(g) \to Z(g)$ of $\bar{\gamma}$ is an isomorphism of algebras, the so-called Duflo isomorphism. See [5] for the case of solvable and semisimple Lie algebras, and [6] for the general case. Duflo’s idea can be described as follows: In order to prove $\gamma(X_1X_2) = \gamma(X_1)\gamma(X_2)$ for all $X_1, X_2 \in I(g)$, it suffices, in view of the above considerations, to verify

$$\chi_P(\gamma(X_1X_2)) = \chi_P(\gamma(X_1)) \chi_P(\gamma(X_2))$$

for all $P \in \text{Prim}(U(g))$. The latter identity is an immediate consequence of Duflo’s character formula which states: If $f \in g^*$, $P$ is the primitive ideal of $U(g)$ associated to $f$ via the Kirillov map, and $\chi_f = \chi_P$ is its central character, then

$$\chi_f(\gamma(X)) = X(f)$$

for all $X \in I(g)$. But the proof of Duflo’s character formula is demanding and requires a detailed knowledge of the representation theory and the primitive ideal theory of $U(g)$. 
The importance of the results $U(g)$ semiprimitive and $\text{End}_{U(g)}(M)$ algebraic over $k$ in the context of Duflo isomorphism motivated us to elaborate on their proofs.

**Jacobson algebras**

Let $B$ be a unital ring. An ideal $P$ of $B$ is called primitive if there exists a (non-zero) simple $B$-module $M$ with annihilator

$$P = \text{Ann}_B(M) = \{ a \in B : b \cdot m = 0 \text{ for all } m \in M \}.$$ 

An ideal $I$ of $B$ is said to be semiprimitive if it is an intersection of primitive ones. The **Jacobson radical** $J(B)$ is defined as the intersection of all primitive ideals of $B$.

We shall recall some facts about simple modules. If $L$ is a maximal left ideal of $B$, then $M = B/L$ is a simple $B$-module whose annihilator is given by

$$\text{Ann}_B(M) = \{ a \in B : a \cdot B \subset L \}.$$ 

Conversely, if $M$ is a simple $B$-module and $\xi \in M$ is non-zero, then $L = \{ a \in B : a \cdot \xi = 0 \}$ is a maximal left ideal and $M \cong B/L$ in a canonical way. Since maximal left ideals exist by Zorn’s Lemma, the existence of simple modules and primitive ideals is guaranteed.

It is known that maximal ideals are primitive: Let $B$ be a ring and $I$ a maximal ideal of $B$. By Zorn’s Lemma there exists a maximal left ideal $L$ of $B$ with $I \subset L$. Now $M = B/L$ is a simple $B$-module with $I \subset \text{Ann}_B(M)$. Since $I$ is maximal, we see that $I = \text{Ann}_B(M)$ is primitive.

**Lemma 1.** If $B$ is commutative, then every primitive ideal of $B$ is maximal.

**Proof.** Let $I$ be a primitive ideal of $B$, $I = \text{Ann}_B(M)$ for some simple $B$-module $M$. Let $\xi \in M$ be non-zero. Then $L = \{ a \in B : a \cdot \xi = 0 \}$ is a maximal (left) ideal of $B$. Clearly $I \subset L$. If $a \in I$, then it follows $a \cdot M = a \cdot (B \cdot \xi) = B \cdot (a \cdot \xi) = 0$. Thus $I = L$ is maximal.

We refrain from giving the various characterizations of the Jacobson radical and content ourselves with the following observation.

**Lemma 2.** If $a \in J(B)$, then $1 + xa$ is left invertible for all $x \in B$.

**Proof.** Suppose that $1 + xa$ is not left invertible. Then $B(1 + xa) \neq B$. By Zorn’s Lemma there exists a maximal left ideal $L$ with $1 + xa \in L$ and $1 \notin L$. Since $B/L$ is a simple $B$-module and $a \in J(B)$, it follows $a \in aB \subset L$. This implies $1 = (1 + xa) - xa \in L$, a contradiction.
Next we shall introduce the Baer radical of $B$. Let $I$ be an ideal of $B$ with $I \neq B$. We say that $I$ is prime if the following condition is satisfied: If $J_1$ and $J_2$ are ideals of $B$ with $J_1 J_2 \subseteq I$, then $J_1 \subseteq I$ or $J_2 \subseteq I$. The radical $\sqrt{I}$ of $I$ is defined as the intersection of all prime ideals containing $I$. Further we say that $I$ is semiprime if $I = \sqrt{I}$. By Zorn’s Lemma there exist maximal ideals containing $I$. And since maximal ideals are prime, this implies that $\sqrt{I} \neq B$ is well-defined. Clearly $I \subseteq \sqrt{I}$ and $\sqrt{I} = \sqrt{\sqrt{I}}$. This shows that $\sqrt{I}$ is the smallest semiprime ideal of $B$ containing $I$. Finally the **Baer radical** $\sqrt{0}$ is defined to be the intersection of all prime ideals of $B$. In the literature $\sqrt{0}$ is also known as the prime radical, the lower nilradical or the Baer-McCoy radical of $B$.

It is easy to see that primitive ideals are prime: Let $M$ be a simple module such that $I = \text{Ann}_B(M)$. Let $J_1, J_2$ be ideals of $B$ such that $J_1 J_2 \subseteq I$. If $J_2 \not\subseteq I$, then $J_2 \cdot M$ is a non-zero submodule and hence $J_2 \cdot M = M$ because $M$ is simple. This yields $J_1 \cdot M = J_1 J_2 \cdot M = I \cdot M = 0$ and hence $J_1 \subseteq I$. Thus $I$ is prime. In particular we see that the Jacobson radical contains the Baer radical: $\sqrt{0} \subseteq J(B)$.

In the proof of the next lemma we will need the following fact: If $I$ is a semiprime ideal of $B$ and $J$ is an ideal of $B$ with $J^2 \subseteq I$, then it follows $J \subseteq I$. This can be seen as follows: If $P$ is prime ideal with $J^2 \subseteq I \subseteq P$, then it follows $J \subseteq P$. Intersecting all these $P$ gives $J \subseteq \sqrt{J} = I$.

**Lemma 3.** If $I$ is a nilpotent ideal of $B$, then $I \subseteq \sqrt{0}$.

**Proof.** Suppose that $I$ is nilpotent and $I \not\subseteq \sqrt{0}$. Then there is some $n \geq 1$ such that $I^n \not\subseteq \sqrt{0}$ and $I^{n+1} \subseteq \sqrt{0}$. Let $k$ be the smallest integer $\geq (n+1)/2$. Clearly $(I^k)^2 \subseteq I^{n+1} \subseteq \sqrt{0}$. Since $\sqrt{0}$ is semiprime, this implies $I^k \subseteq \sqrt{0}$, a contradiction. \hfill $\square$

A ring $B$ is called (semi-)prime or (semi-)primitive if its zero ideal is (semi-)prime or (semi-)primitive respectively. In the literature semiprimitive rings are also known as Jacobson-semisimple rings. Clearly $I$ is a semiprime or semiprimitive ideal of $B$ if and only if $B/I$ is a semiprime or semiprimitive ring respectively.

For noetherian rings we obtain additional results.

**Proposition 4.** Let $B$ be a left noetherian ring. Then the Baer radical $\sqrt{0}$ is a nilpotent ideal of $B$. Thus $B$ has a maximal nilpotent ideal.

**Proof.** The first step is to prove that $B$ contains a maximal nilpotent ideal. Suppose that this is not the case. Let $J_0$ be an arbitrary nilpotent ideal of $B$. Since $J_0$ is not maximal, there exists a nilpotent ideal $J_1$ of $B$ such that $J_1 \not\subseteq J_0$. Clearly $J_0 + J_1$ is a nilpotent ideal which properly contains $J_0$. By induction we obtain a strictly increasing sequence of nilpotent (two-sided) ideals, in contradiction to the assumption that $B$ left noetherian.

Let $J$ denote the maximal nilpotent ideal of $B$. Clearly $J \subseteq \sqrt{0}$ by Lemma 3. Suppose
that \( J \) is not semiprime. Then Proposition 3.3 (ii) of the Appendix shows that there exists an ideal \( J_1 \) of \( B \) with \( J_1^2 \subset J \) and \( J_1 \not\subset J \). Since \( J \) is nilpotent, it follows that \( J + J_1 \) is also nilpotent, in contradiction to the maximality of \( J \). Thus \( J \) is semiprime. This shows that \( \sqrt{0} \subset \sqrt{J} = J \) so that \( J = \sqrt{0} \) is nilpotent.

The next proposition is crucial for the proof of Proposition 7.

**Proposition 5.** Let \( B \) be a left noetherian ring. Then every nil right ideal of \( B \) is nilpotent.

**Proof.** Let \( R \) be a nil right ideal of \( B \). Let \( J = \sqrt{0} \) denote the Baer radical of \( B \) which is known to be the maximal nilpotent ideal of \( B \) by Proposition 4. Consequently we must prove \( R \subset J \). Suppose that \( R \not\subset J \). Since \( B \) is right noetherian, there exists some \( a \in R \setminus J \) such that the left ideal \( l(a) = \{ x \in B : xa \in J \} \) is maximal among all left ideals of this form. Let \( y \in B \) be arbitrary. Our aim is to prove \( ay \in J \). If \( ay \in J \), then we are done. So let us assume \( ay \not\in J \). For \( ay \in L \) and \( L \) is nil, there exists some \( k > 1 \) such that \( (ay)^{k-1} \not\in J \) and \( (ay)^k \in J \). Since \( l(a) \subset l((ay)^{k-1}) \), it follows \( l((ay)^{k-1}) = l(a) \) by the maximality of \( l(a) \). This implies \( ay \in l(a) \) and hence \( aya \in J \). Since \( aBa \subset J \) and \( J \) is semiprime, it follows \( a \in J \). This contradiction proves \( R \subset J \).

The preceding two propositions as well as the subsequent theorem remain valid after exchanging the words left and right.

The next result is fundamental.

**Lemma 6 (Schur).** If \( M \) is a simple left \( B \)-module, then the commutant

\[
D = \text{End}_B(M) = \{ \varphi : M \to M : \varphi \text{ is } B\text{-linear} \}
\]

is a division ring. In particular the center \( Z(D) \) of \( D \) is a field.

**Proof.** First of all, \( D = \text{End}_B(M) \) is a (not necessarily commutative) unital ring. If \( \varphi \in D, \varphi \neq 0 \), then it follows \( \text{im} \varphi = M \) and \( \ker \varphi = 0 \) because \( \text{im} \varphi \neq 0 \) and \( \ker \varphi \neq M \) are submodules of the simple \( B \)-module \( M \). This means that \( \varphi \) is bijective so that \( \varphi^{-1} \) exists. Clearly \( \varphi^{-1} \in D \). This proves \( D \) to be a division ring. Obviously \( Z(D) \) is a field.

In order to obtain information about the given ring \( B \) we will introduce an additional variable \( X \) and consider (simple) modules \( M \) over the ring \( B[X] \) of all polynomials in one indeterminate with coefficients in \( B \). This striking idea can be traced back to the famous one-sided article [13] of Rabinovich from 1929. Note that any \( B[X] \)-module structure boils down to a \( B \)-module \( M \) together with a map \( \varphi(m) = X \cdot m \) in its commutant. Clearly \( M \) is \( B[X] \)-simple if and only if \( M \) does not admit proper \( \varphi \)-invariant \( B \)-submodules.

Now let \( A \) be a commutative unital ring and \( B \) a unital \( A \)-algebra. One might think of
Thus we know that there exists a polynomial \( a(X) \) in \( A \) as a subring of the center \( Z(B) \) of the ring \( B \). Note that the commutant \( \text{End}_B(M) \) of any \( B \)-module \( M \) becomes an \( A \)-algebra via \( (z \cdot \varphi)(m) = z \cdot \varphi(m) \). We say that \( \varphi \in \text{End}_B(M) \) is integral over \( A \) if there exist elements \( z_0, \ldots, z_{n-1} \in A \) such that
\[
\varphi^n + \sum_{k=0}^{n-1} z_k \varphi^k = 0.
\]
In the sequel we shall investigate whether the following condition is satisfied:

\[\text{(E)} \text{ If } M \text{ is a simple } B[X]\text{-module, then } \varphi(m) = X \cdot m \text{ is integral over } A.\]

Let \( P = \text{Ann}_{B[X]}(M) \) denote the primitive ideal of \( B[X] \) associated to \( M \). Since \( A \cap P = \text{Ann}_A(M) \) is a prime ideal of \( A \), it follows that \( A/A \cap P \) is an integral domain. Note that \( M \) can be regarded as an \( A/A \cap P \)-module. We point out that (E) is satisfied if and only if \( \varphi(m) = X \cdot m \) is integral over \( A/A \cap P \). This is the case if and only if \( X \) is integral as an element of the integral domain and \( A/A \cap P \)-algebra \( A[X]/A[X] \cap P \).

The next proposition is due to Duflo, see Théorème 1 of [7]. This result seems to be inspired by an earlier work of Amitsur on the radical of polynomial rings, see [1].

**Proposition 7.** Let \( A \) be a commutative ring and \( B \) be a left noetherian \( A \)-algebra. If \( B \) satisfies (E), then the Jacobson radical and the Baer radical of \( B \) coincide.

**Proof.** It suffices to show that \( J(B) \subset \sqrt{0} \). Let \( a \in J(B) \) be arbitrary. First we shall prove that \( B[X]/(1-aX) = B[X] \). Suppose that this is not the case so that there exists a maximal left ideal \( L \) of \( B[X] \) such that \( 1-aX \in L \) and \( 1 \notin L \). Then \( M = B[X]/L \) is a simple \( B[X]\)-module and \( 1+L \in M \) is non-zero. Since \( X \) is in the center of \( B[X] \), the map
\[
\varphi : M \to M, \quad \varphi(p + L) = X \cdot (p + L) = Xp + L,
\]
is \( B[X] \)-linear. Clearly \( \varphi(1+L) = X+L \neq 0 \). As \( M \) is simple, Lemma 6 implies that \( \varphi \) is invertible. By assumption \( \varphi \in \text{End}_{B[X]}(M) \) is integral over \( A \). Thus there exist \( z_0, \ldots, z_{n-1} \in A \) such that
\[
\varphi^n + \sum_{k=0}^{n-1} z_k \varphi^k = 0.
\]
Dividing by \( \varphi^n \) we obtain \( 1 + f(\varphi^{-1}) = 0 \) where \( f = \sum_{k=0}^{n-1} z_k Y^{n-k} \) is a polynomial with coefficients in \( A \). From \( aX + L = 1 + L \) we deduce
\[
\varphi^k(a^k \cdot (1+L)) = a^k X^k + L = (aX)^k + L = (aX + L)^k = 1 + L
\]
for all \( k \geq 0 \) which shows that
\[
(1 + f(a)) \cdot (1+L) = (1 + f(\varphi^{-1}))(1+L) = 0.
\]
On the other hand, \( f(a) \in J(B) \) because \( f \) has constant term zero. Hence \( 1 + f(a) \) is left invertible in \( B \) by Lemma 2. This contradiction proves \( B[X]/(1-aX) = B[X] \).

Thus we know that there exists a polynomial \( p = \sum_{k=0}^{m} a_k X^k \) in \( B[X] \) such that
\[
1 = p(1-aX) = a_0 + \sum_{k=1}^{m} (a_k - a_{k-1} a) X^k + a_m a X^{m+1}.
\]
Comparing the coefficients we find $a_k = a^k$ for $0 \leq k \leq m$ and $a^{m+1} = a^m a = 0$, which proves $a$ to be nilpotent. Thus $J(B)$ is a nil ideal. As $B$ is left-noetherian, Proposition [8] shows that $J(B)$ is a nilpotent ideal. Finally Lemma [8] implies that $J(B)$ is contained in $\sqrt{0}$.

Next we recall the definition of a Jacobson algebra which goes back to Duflo [7]. The motivation is to generalize the assertion of Hilbert’s Nullstellensatz to the non-commutative setting. To this end we recall the following version of the Nullstellensatz:

Let $k$ be an algebraically closed field and $I$ a proper ideal of the commutative polynomial algebra $B = k[X_1, \ldots, X_n]$. Let $\mathcal{V}(I) = \{ \lambda \in k^n : p(\lambda) = 0 \text{ for all } p \in I \}$ be the set of common zeros of $I$ and $r(I)$ the ideal of all $p \in B$ such that $p^n \in I$ for some $n \geq 1$. The Nullstellensatz asserts

$$r(I) = \{ p \in B : p(\lambda) = 0 \text{ for all } \lambda \in \mathcal{V}(I) \}.$$ 

This means that $r(I)$ is equal to the intersection of the maximal ideals $\{ I_\lambda : \lambda \in \mathcal{V}(I) \}$ of $B$ where $I_\lambda = \{ p \in B : p(\lambda) = 0 \}$. In particular it follows $r(I) = \sqrt{I}$, in accordance with the usual notation. If $k$ is an arbitrary field, then one might expect that $\sqrt{I}$ equals the intersection of all maximal ideals containing $I$, whereas these maximal ideals need not be of the form $I_\lambda$ for some $\lambda \in k$. Finally one might ask for a generalization of the assertion of the Nullstellensatz to non-commutative rings $B$. The decisive idea is to replace maximal ideals by primitive ones. This step is encouraged by the fact that primitive ideals play a very prominent role in representation theory. Taking into account that $I = \sqrt{I}$ semiprime is necessary for $I$ to be an intersection of primitive ideals we end up with

**Definition 8.** A ring $B$ is called a Jacobson ring if every semiprime ideal of $B$ is semiprimitive.

The next theorem is an immediate consequence of Proposition [7]

**Theorem 9.** Let $A$ be a commutative ring and $B$ a left noetherian $A$-algebra satisfying condition (E). Then $B$ is a Jacobson ring.

**Proof.** Let $I$ be a semiprime ideal of $B$. Then $\bar{B} = B/I$ is a left noetherian semiprime ring. Note that $\bar{B}[X]$ is a quotient of $B[X]$ so that every $\bar{B}[X]$-module can be regarded as a $B[X]$-module. Thus condition (E) implies that $\varphi = X \cdot m$ is integral over $A$ for every simple $\bar{B}[X]$-module $M$. Now Proposition [7] yields $J(\bar{B}) = \sqrt{\bar{B}} = 0$ which shows that $I$ is semiprimitive. □

Duflo proved that finitely generated almost commutative algebras over commutative Jacobson rings are Jacobson, see Théorème 3 of [7]. As a first step in this direction he considers almost commutative algebras over arbitrary fields. In this context the following condition plays a crucial role: Let $k$ be a field and $B$ an arbitrary $k$-algebra.

**(D)** If $M$ is a simple $B[X]$-module, then $\varphi(m) = X \cdot m$ is algebraic over $k$. 
According to Theorem 9 this condition implies the Jacobson property. By contrast Irving preferred the following stronger requirement.

(I) If $M$ is a simple $B[X]$-module, then $\text{End}_{B[X]}(M)$ is algebraic over $k$.

However, we do not follow Irving [10] in saying that $B[X]$ satisfies the Nullstellensatz if condition (I) holds true.

In [12] Quillen discovered that the generic flatness lemma can be used to prove that the commutant of a simple module is algebraic. We shall explain this important fact in detail.

First we recall some definitions. Let $A$ be a commutative ring and $f \in A$ not nilpotent defining the multiplicative subset $S_f = \{f^k : k \geq 0\}$ of $A$. Then $A_f = S_f^{-1}A$ is called a simple localization. Roughly speaking, $A_f$ is generated by $A$ and $f^{-1}$. Note that $A_f \neq 0$ because $f$ is not nilpotent. We say that an $A$-module $M$ is \textbf{generically free} if there exists a non-nilpotent $f \in A$ such that $M_f = A_f \otimes_A M$ is a free $A_f$-module. In the sequel we will use the fact that the localization functor is exact which means that $A_f$ is a flat $A$-module.

Returning to the original situation we suppose that $B$ is a $k$-algebra and that $M$ is a simple $B[X]$-module. Since $k[X]$ is contained in the center of $B[X]$, $M$ becomes a module over the principal ideal domain $A = k[X]$ such that the actions of $A$ and $B[X]$ commute. Hence the set

$$T_A(M) = \{m \in M : a \cdot m = 0 \text{ for some } a \in A\}$$

of $A$-torsion elements is a $B[X]$-submodule of $M$. If $T_A(M) \neq 0$, then there exist non-zero elements $m \in M$ and $f \in A$ such that $f \cdot m = 0$. Then

$$f \cdot M = f \cdot (B[X] \cdot m) = B[X] \cdot (f \cdot m) = 0$$

which means $f(\varphi) = 0$. Thus $\varphi(m) = X \cdot m$ is algebraic over $k$. In order to establish property (D) we must exclude the case $T_A(M) = 0$. (In this case $M$ is flat, but not necessarily free over $A$.)

According to Quillen we introduce the following condition.

(Q0) Every simple $B[X]$-module is generically free over $k[X]$.

In the proof of Theorem 11 we will need the following auxiliary result.

\textbf{Lemma 10.} \textit{Let $k$ be a field. Then $k[X]$ contains infinitely many monic irreducibles.}

\textit{Proof.} If $k$ is infinite, then $J = \{X - \lambda : \lambda \in k\}$ is an infinite set of irreducibles with leading coefficient 1. Next we assume that $k$ is finite. Suppose that the set $J$ of all
irreducible and monic polynomials in \( k[X] \) is finite. Set \( n = \max\{\deg(q) : q \in J\} \) and define \( Q \) as the product of all \( q \in J \). Since \( \deg(Q) > n \), it follows that \( 1 + Q \) is reducible. Hence there exist \( p, q \in k[X] \) with \( 1 + Q = pq \) and \( p \) irreducible. Clearly \( p \mid Q \) so that there is a \( q' \in k[X] \) with \( Q = pq' \). This implies \( pq = 1 + Q = 1 + pq' \) and \( p(q - q') = 1 \), a contradiction. Thus \( J \) is infinite.

The next result is due to Quillen, see [12]. Since the proof in [12] is quite succinct, we reproduce the elaborate argument given by Dixmier in Lemme 2.6.4 of [3].

**Theorem 11.** Let \( B \) be a unital \( k \)-algebra. Then \((Q0)\) implies \((D)\).

**Proof.** Let \( M \) be a simple \( B[X] \)-module. As above we regard \( M \) as a module over \( A = k[X] \). Suppose that \( \varphi(m) = X \cdot m \) is transcendental over \( k \). By \((Q0)\) there exists a non-zero \( f \in A \) such that \( M_f = A_f \otimes_A M \) is a free \( A_f \)-module.

First of all we observe that the natural map \( \eta : M \rightarrow M_f, \eta(m) = 1 \otimes m \), is an injection: Its kernel \( \ker \eta = T_{S_f}(M) = \{m \in M : f^n \cdot m = 0 \text{ for some } n \geq 0\} \) is trivial because \( \varphi \) is transcendental. Thus \( M_f \neq 0 \).

By Lemma [10] we can choose an irreducible polynomial \( g \in A \) not dividing \( f \). Then \( A_f \rightarrow A_f, p \mapsto gp \) is not surjective because \( f^n \not\in gA \) for all \( n \geq 0 \).

Next we define \( \Psi : M_f \rightarrow M_f, \Psi(p \otimes m) = (gp) \otimes m = p \otimes (g \cdot m) \). On the one hand, \( M_f \cong A_f^{(1)} \neq 0 \) is free and \( \Psi : A_f^{(1)} \rightarrow A_f^{(1)} \) has the form \( \Psi(p)_i = gp_i \). This shows that \( \Psi \) is not surjective. On the other hand, \( \im g(\varphi) \) is a non-zero \( B[X] \)-submodule of \( M \) because \( \varphi \in \End_{B[X]}(M) \) is transcendental over \( k \). As \( M \) is simple, it follows \( \im g(\varphi) = M \). This shows \( \im \Psi = M_f \) because \( \Psi(p \otimes m) = p \otimes g(\varphi)(m) \) and \( M_f = A_f \otimes_A M \) is generated by elementary tensors. This contradiction proves \((D)\). \( \square \)

Next we will prove that \((Q0)\) holds for all finitely generated almost commutative algebras. Let \( A \) be a commutative ring and \( B \) a unital \( A \)-algebra. Suppose that there exists an \( A \)-submodule \( \mathfrak{g} \) of \( B \) such that \( xy - yx \in \mathfrak{g} \) for all \( x, y \in B \). Then \( B = \bigoplus_{n=0}^{\infty} \mathfrak{g}^n \). In particular this means that \( \mathfrak{g} \) is a Lie subalgebra of \( B \) with respect to the canonical Lie algebra structure \([x, y] = xy - yx\) of \( B \). Further we assume that \( \mathfrak{g} \) is a finitely generated \( A \)-module. If \( x_1, \ldots, x_d \) are generators of \( \mathfrak{g} \), then \( \mathfrak{g}^n \) is, modulo \( \mathfrak{g}^{n-1} \) and as an \( A \)-module, generated by the set of all products of the form \( x^{\nu} = x_1^{\nu_1} \cdots x_d^{\nu_d} \) with \( \nu \in \mathbb{N}^d \) and \( |\nu| = \sum_{j=0}^{d} \nu_j = n \). We say that \((B, \mathfrak{g})\) is a finitely generated almost commutative \( A \)-algebra if the above conditions are satisfied.

By means of the universal property of universal enveloping algebras we obtain the following characterization.

**Lemma 12.** Let \( A \) be a commutative ring. If \( \mathfrak{g} \) is a Lie algebra over \( A \) which is finitely generated as an \( A \)-module, then every quotient of its universal enveloping algebra \( U(\mathfrak{g}) \) is a finitely generated almost commutative \( A \)-algebra. Conversely, if \((B, \mathfrak{g})\) is finitely generated and almost commutative, then \( B \) is isomorphic to a quotient of \( U(\mathfrak{g}) \).
Theorem 14 generalizes the generic flatness lemma of algebraic geometry, compare Lemma 6.7 in [9] Exposé IV of SGA 1960-61. Grothendieck’s proof of generic freeness for modules over finitely generated commutative algebras uses the Noether normalization Lemma and Krull dimension theory. Here we restate a more elementary proof due to Dixmier [4] and Duflo [7] only relying on the fact that the localization functor is exact.

A filtration of an $A$-module $M$ is a sequence $M_n$ of $A$-submodules of $M$ such that $M_{n-1} \subseteq M_n$ for all $n \geq 0$, and $M = \sum_{n=0}^{\infty} M_n$.

**Lemma 13.** Let $A$ be a commutative ring and $M$ a left $A$-module endowed with a filtration $\{M_n : n \geq -1\}$. Suppose that $f \in A$ is non-nilpotent such that $f \cdot M_n \subset M_{n-1}$ whenever $M_n/M_{n-1}$ is not free. Then $A_f \otimes_A M$ is a free $A_f$-module.

**Proof.** The images $L_n$ of the canonical injections $i_n : A_f \otimes_A M_n \rightarrow A_f \otimes_A M$ form a filtration of the $A_f$-module $A_f \otimes_A M$. We claim that $L_n/L_{n-1}$ is free over $A_f$ for all $n \geq 0$: First we treat the case $M_n/M_{n-1}$ not free.

For all $r \in A_f$ and $m \in M_n$. This proves $L_n = L_{n-1}$ so that $L_n/L_{n-1} = 0$. Next we assume that $M_n/M_{n-1} \cong A^{(f)}$ is free. As $A_f \otimes_A -$ commutes with direct sums, it follows that $A_f \otimes_A (M_n/M_{n-1}) \cong A_f \otimes_A A^{(f)} \cong A_f^{(f)}$ is a free $A_f$-module. Since $A_f$ is flat, it follows that $L_n/L_{n-1} \cong A_f \otimes_A (M_n/M_{n-1})$ is free over $A_f$ for all $n \geq 0$. This shows that $A_f \otimes_A M$ is free. \hfill \square

The next theorem is due to Duflo, see Théorème 2 of [7]. The idea of the proof goes back to Dixmier [4]. Compare also Lemme 2.6.3 of Dixmier’s book [3]. The proof is a bit technical and needs some preparation. For $d \geq 1$ we interpret $\mathbb{N}^d$ as the set of all multi-indices. The order of $\nu \in \mathbb{N}^d$ is given by $|\nu| = \sum_{j=1}^{d} \nu_j$. Let $\preceq$ denote the unique total order on $\mathbb{N}^d$ satisfying the following two properties:

1. If $\mu, \nu \in \mathbb{N}^d$ and $|\mu| < |\nu|$, then $\mu \preceq \nu$.

2. If $\mu, \nu \in \mathbb{N}^d$ such that $|\mu| = |\nu|$ and if there exists $1 \leq k \leq n$ such that $\mu_j = \nu_j$ for $1 \leq j \leq k-1$ and $\mu_k < \nu_k$, then $\mu \preceq \nu$.

As a poset $(\mathbb{N}^d, \preceq)$ is isomorphic to $(\mathbb{N}, \leq)$ in a canonical way. By abuse of notation we write $\nu - 1$ for the maximum of the set of all $\mu \in \mathbb{N}^d$ such that $\mu \preceq \nu$ and $\mu \neq \nu$, provided that $\nu \neq 0$.

**Theorem 14.** Let $A$ be an integral domain and $(B, g)$ a finitely generated almost commutative $A$-algebra. Then the following enhancement of (Q0) holds true: If $M$ a cyclic $B$-module, then there is a non-zero $f \in A$ such that $A_f \otimes M$ is a free $A_f$-module.

**Proof.** Let $M$ be a cyclic $B$-module and $\xi \in M$ with $M = B \cdot \xi$. Let $x_1, \ldots, x_d$ be generators of the $A$-submodule $g$ of $B$. As usual we write $x^\nu = x_1^{\nu_1} \cdots x_d^{\nu_d}$ for $\nu \in \mathbb{N}^d$. Since $B = \sum_{n=0}^{\infty} g^n = \sum_{\nu \in \mathbb{N}^d} A \cdot x^\nu$, it is evident that $M_\nu = \sum_{\mu \preceq \nu} A \cdot x^\mu \cdot \xi$ defines a
Proof. Let $B$ be an almost commutative $k$-algebra. Then $B$ satisfies condition (D). In particular $B$ is a Jacobson algebra and the commutant $\text{End}_B(M)$ of any simple $B$-module is algebraic over $k$.

**Theorem 15.** Let $k$ be a field and $(B, g)$ a finitely generated almost commutative $k$-algebra. Then $B$ satisfies condition (D). In particular $B$ is a Jacobson algebra and the commutant $\text{End}_B(M)$ of any simple $B$-module is algebraic over $k$.

**Proof.** By Theorem [14] we know that $B$ has property (Q0). Theorem [11] implies (D). Thus $B$ is Jacobson by Theorem [9]. Since any $\varphi \in \text{End}_B(M)$ gives rise to a simple $B[X]$-module, it follows by (D) that $\varphi$ is algebraic over $k$.

**Lemma 16.** Let $k$ be an algebraically closed field and $B$ be a finitely generated commutative $k$-algebra. If $I$ is a primitive (i.e., maximal) ideal of $B$, then there exists a homomorphism $\chi : B \rightarrow k$ such that $I = \ker \chi$.

**Proof.** Let $M$ be a simple $B$-module with $I = \text{Ann}_B(M)$. Observe that $B/I$ can be regarded as a subring of $\text{End}_B(M)$ because $B$ is commutative. Since $k$ is algebraically closed and $\text{End}_B(M)$ is algebraic over $k$ by Theorem [15], it follows $B/I \cong k$. Thus the canonical projection $\chi : B \rightarrow B/I \cong k$ has the desired properties.

Finally, applying these results to the commutative algebra $B = k[X_1, \ldots, X_n]$, where $k$ algebraically closed, we rediscover Hilbert’s Nullstellensatz: Any semiprime ideal $I = \sqrt{I}$ of $B$ is equal to the intersection of the maximal ideals containing it. As the homomorphisms $\chi : B \rightarrow k$ are known in this case, every maximal ideal is of the form $I_\lambda = \{ p \in B : p(\lambda) = 0 \}$ for some $\lambda \in k^n$. 

filtration of the $A$-module $M$ such that $M_\nu/M_{\nu-1}$ is cyclic for all $\nu$. Let us consider the ideals

$$I_\nu = \{ a \in A : a \cdot M_\nu \subset M_{\nu-1} \} = \{ a \in A : a \cdot x^{\nu} \cdot \xi \in M_{\nu-1} \}$$

of $A$. Clearly $M_\nu/M_{\nu-1} \cong A$ is free provided that $I_\nu = 0$. Let $\Lambda = \{ \nu \in \mathbb{N}^d : I_\nu \neq 0 \}$.

In view of Lemma [13] it now suffices to prove that the ideal $J = \bigcap \{ I_\nu : \nu \in \Lambda \}$ is non-zero. The problem is to treat the case where $\Lambda$ happens to be infinite.

If $\nu \in \mathbb{N}^d$ is arbitrary, $e_j \in \mathbb{N}^d$ is the $j^{th}$ canonical basis vector, and $a \in I_{\nu}$, then

$$a \cdot x^{\nu+e_j} \cdot \xi = x_j \cdot a \cdot x^{\nu} \cdot \xi + a \cdot [x_1^{\nu_1} \cdots x_{j-1}^{\nu_{j-1}} x_j x_{j+1}^{\nu_{j+1}} \cdots x_{d}^{\nu_d} \cdot \xi \in M_{\nu+e_j-1}$$

which shows $a \in I_{\nu+e_j}$. Thus $I_\nu \subset I_{\nu+\alpha}$ for all $\alpha \in \mathbb{N}^d$ and hence $\Lambda + \mathbb{N}^d = \Lambda$. Let $\Lambda_0$ denote the set of all $\gamma \in \Lambda$ such that $\gamma \neq \nu + e_j$ for all $\nu \in \Lambda$ and $1 \leq j \leq n$. It is easy to see that $\Lambda_0$ is a finite subset of $\Lambda$ such that $\Lambda = \Lambda_0 + \mathbb{N}^d$. Since $A$ is an integral domain and $I_\nu \subset I_{\nu+\alpha}$, it follows that $J = \bigcap \{ I_\gamma : \gamma \in \Lambda_0 \}$ is non-zero. For every $0 \neq f \in J$ we have $f \cdot M_\nu \subset M_{\nu-1}$ for all $\nu \in \Lambda$. This completes the proof. \( \square \)

The preceding theorem can be generalized to the case of finitely generated $B$-modules. We omit the details.

The next result appears as a corollary of the preceding achievements. It is a generalization of a result of Krull in the commutative case, see Satz 1 and Satz 2 of [11]. Compare also Theorem 3 of Goldmann [8].
Theorem 17. Let $A$ be a commutative noetherian Jacobson ring and $B$ a finitely generated almost commutative $A$-algebra. Then $B$ is a Jacobson algebra and the commutant $\text{End}_B(M)$ of any simple $B$-module is integral over $A$.

Proof. First of all, $B$ is known to be left noetherian. According to Theorem 9 we must verify (E). Let $M$ be a simple $B[X]$-module with annihilator $P = \text{Ann}_B(M)$ and $\varphi(m) = X \cdot m$. As we have already seen, it suffices to prove that $\varphi$ is integral over the integral domain $\bar{A} = A/A \cap P$. For the convenience of the reader we reproduce the proof of Proposition 1 of [7] which states that $\bar{A}$ is a field.

By Theorem 13 there exists a non-zero $f \in \bar{A}$ such that $M_f = \bar{A}_f \otimes_{\bar{A}} M$ is a free $\bar{A}_f$-module. Suppose that the natural map $\eta : M \rightarrow M_f$, $\eta(m) = 1 \otimes m$, has a non-trivial kernel $\ker \eta = TS_f(M) \neq 0$. Then there exists a $k \geq 1$ and a non-zero $m \in M$ such that $f^k \cdot m = 0$. Since $M$ is simple, it follows $f^k \cdot M = 0$. Hence $f^k = 0$. For $\bar{A}$ has no zero divisors, we get $f = 0$, a contradiction. Thus $\eta$ is injective. In particular $M_f$ is non-zero.

First we prove that $\bar{A}_f$ is a field. Let $a \in \bar{A}_f$ be non-zero. Clearly $\mu_a : M_f \rightarrow M_f$, $\mu_a(m) = a \cdot m$, is a non-zero element of $\text{End}_B(M)$. By Schur’s Lemma $\mu_a$ is invertible. Let $\{m_i : i \in I\}$ be a basis of $M_f$ and choose $i \in I$. One can find $b, b_j \in \bar{A}_f$ such that

$$\mu_a^{-1}(m_i) = b \cdot m_i + \sum_{j \neq i} b_j \cdot m_j.$$ 

Applying $\mu_a$ to both sides and comparing coefficients, we obtain $ab = 1$ so that $a$ is invertible in $\bar{A}_f$. Consequently $\bar{A}_f$ is a field.

Since $\bar{A}$ is a Jacobson ring, there exists a maximal ideal $I$ of $\bar{A}$ such that $f \notin I$. As $f + I$ is invertible in $\bar{A}/I$, there exists a unique ring homomorphism $\psi : \bar{A}_f \rightarrow \bar{A}/I$ such that $\psi(x) = x + I$ for all $x \in \bar{A}$. It follows that $\psi$ is injective because $\bar{A}_f$ is a field. This proves $I = 0$. Thus $\bar{A}$ itself is a field.

Since $\bar{B} = B/P$ is an almost commutative $\bar{A}$-algebra, Theorem 14 implies that $\varphi$ is algebraic over $\bar{A}$. This proves (E).

In respect of Irving’s condition (I) we state the following generic freeness property.

(Q1) If $M$ is a simple $B[X]$-module and $\varphi \in \text{End}_{B[X]}(M)$, then $Y \cdot m = \varphi(m)$ defines a generically free $k[Y]$-module.

In the situation of (Q1) we can regard $M$ as a simple module over the $k$-algebra $B[X,Y] = B[X][Y] = B[X] \otimes_k k[Y]$. Applying Theorem 14 to $B[X][Y]$, we obtain

Lemma 18. Condition (Q1) implies (I).

Finally we add the conditions (A) the commutant $\text{End}_B(M)$ of any simple $B$-module $M$ is algebraic over $k$ and (J) $B$ is a Jacobson algebra. Altogether these conditions are
related as follows:

\[ \begin{array}{ccc}
  \text{(Q1)} & \rightarrow & \text{(Q0)} \\
  \downarrow & & \downarrow \\
  \text{(I)} & \rightarrow & \text{(D)} \\
  \downarrow & & \downarrow \\
  \text{(A)} & & \text{(J)} \\
\end{array} \]

Here the implication \((D) \Rightarrow (J)\) is valid only if \(B\) is left (or right) noetherian. The others hold true in general.

**Modules over filtered algebras**

Some results of the preceding section can be generalized to filtered \(k\)-algebras such that the associated graded algebra is noetherian. To begin with, let \(A\) be a commutative ring and \(B\) an \(A\)-algebra. An increasing sequence \(F = \{B_n : n \geq -1\}\) of \(A\)-submodules of \(B\) is called a filtration of \(B\) if the following conditions are satisfied: \(B_{-1} = 0\), \(A \cdot 1 \subseteq B_0\), \(\sum_{n=0}^{\infty} B_n = B\), and \(B_m B_n \subseteq B_{m+n}\). We say that \((B, F)\) is an \(\mathbb{N}\)-filtered \(A\)-algebra. The filtration is locally finite if \(B_n\) is a finitely generated \(A\)-module for all \(n \geq 0\).

Subject to a given filtration \(F\) of \(B\) we define the \(A\)-modules \(\bar{B}_n = B_n / B_{n-1}\) and \(\bar{B} = \bigoplus_{n=0}^{\infty} \bar{B}_n\). It is easy to see that

\[ \bar{B}_m \times \bar{B}_n \rightarrow \bar{B}_{m+n}, \quad (x + B_{m-1})(y + B_{n-1}) = xy + B_{m+n-1} \]

is well-defined and turns \(\bar{B}\) into an \(\mathbb{N}\)-graded \(A\)-algebra. We call \(\bar{B} = \text{gr}(B, F)\) the associated graded algebra of \(B\).

Let \(B\) be an \(A\)-algebra. If \(g\) is an \(A\)-submodule of \(B\) such that \([g, g] \subseteq g\) and \(B = \sum_{n=0}^{\infty} g^n\), then \(B_n = g^n + B_{n-1}\) yields a filtration \(F\) of \(B\) such that the associated graded algebra \(\text{gr}(B, F)\) is commutative, i.e., such that \([B_n, B_m] \subseteq B_{n+m-1}\) for all \(m, n \geq 0\). Generalizing the notion of finitely generated almost commutative algebras given in the previous section we state

**Definition 19.** A filtered \(A\)-algebra \((B, F)\) is called almost commutative if \(\text{gr}(B, F)\) is commutative. If in addition \(\text{gr}(B, F)\) is finitely generated, then \((B, F)\) is said to be of finite type.

Conversely, if \((B, F)\) is almost commutative, then \(B_1\) is a Lie subalgebra of \(B\), but \(B = \sum_{n=0}^{\infty} B_1^n\) is not apparent. If in addition \((B, F)\) is of finite type, then \(B_1\) is finitely generated.

Suppose that \(A\) is a noetherian commutative ring and \(B\) is an almost commutative \(A\)-algebra of finite type. Then it follows by the Hilbert basis theorem that \(\text{gr}(B, F)\) is
a noetherian $A$-algebra.

Let $(B, F)$ be a filtered $A$-algebra. For every finitely generated $B$-module $M$ with generators $\xi_1, \ldots, \xi_r$ we consider the filtration

$$M_n = \sum_{j=1}^r B_n \cdot \xi_j$$

of $M$. This means that $E = \{ M_n : n \geq -1 \}$ is an increasing sequence of $A$-submodules of $M$ such that $M_{-1} = 0$, $\sum_{n=1}^\infty M_n = M$, and $B_m \cdot M_n \subset M_{m+n}$. Here the action of $A$ on $M$ is given by $a \cdot \xi = (a \cdot 1) \cdot \xi$. Note that the definition of $E$ depends on the choice of the generators $\xi_1, \ldots, \xi_r$ of $M$ and the filtration of $F$ of $B$. Now we set $
abla M = M_n/M_{n-1}$ and $\nabla = \bigoplus_{n=0}^\infty \nabla M$. One verifies easily that

$$\nabla B_m \times \nabla M_n \longrightarrow \nabla M_{m+n}, (x + B_{m-1}) \cdot (m + L_{n-1}) = x \cdot m + L_{m+n-1}$$

turns $\nabla = \text{gr}(M, E)$ into a finitely generated graded $\nabla B$-module.

As from now let $k$ be a field and $B$ a $k$-algebra with filtration $\mathcal{F} = \{ B_n : n \geq -1 \}$. The $k[X]$-algebra $B' = B[X]$ carries a natural filtration $\mathcal{F}'$ given by

$$B'_n = B_n[X] = \{ p \in B[X] : p(k) \in B_n \text{ for all } k \geq 0 \}$$

and the associated graded $k[X]$-algebra $\text{gr}(B', \mathcal{F}')$ is isomorphic to $\nabla B[X] = \text{gr}(B, \mathcal{F})[X]$.

We remind the reader that we are interested in finding convenient conditions which are sufficient for $B$ to be a Jacobson algebra. To this end we introduce

(G0) Every cyclic graded $B[X]$-module is generically free over $k[X]$.

The next proposition can be found (more or less explicitly) in the work of Quillen [12], Dixmier [3], and Artin-Small-Zhang [2].

**Proposition 20.** Let $(B, \mathcal{F})$ be an $\mathbb{N}$-filtered $k$-algebra. Then (G0) implies (Q0).

**Proof.** Let $M$ be a simple $B[X]$-module and $\xi \in M$ non-zero. As above we define a filtration $M_n = B_n[X] \cdot \xi$ of $M$ and consider the associated graded $B[X]$-module $\nabla M$. Note that $\nabla M$ is cyclic over $B[X]$. Put $A = k[X]$. By (G0) there exists a non-zero $f \in A$ such that

$$\nabla M_f = A_f \otimes_A \nabla M \cong \bigoplus_{n=0}^\infty A_f \otimes_A (M_n/M_{n-1})$$

is free. (Here we used the fact that $A_f \otimes_A -$ commutes with direct sums.) Since $A_f$ is a principal ideal domain, it follows that the $A_f$-submodules $A_f \otimes_A (M_n/M_{n-1})$ of $\nabla M_f$ are free for all $n$. We know that

$$0 \longrightarrow A_f \otimes_A M_{n-1} \overset{\eta_{n-1}}{\longrightarrow} A_f \otimes_A M_n \longrightarrow A_f \otimes_A (M_n/M_{n-1}) \longrightarrow 0$$
is exact because $A_f$ is flat over $A$. Consequently the canonical images $L_n$ of $A_f \otimes_A M_n$ in $M_f = A_f \otimes_A M$ form a filtration of $M_f$ such that

$$L_n/L_{n-1} \cong (A_f \otimes_A M_n) / \eta_{n-1}(A_f \otimes_A M_{n-1}) \cong A_f \otimes_A (M_n/M_{n-1})$$

is free over $A_f$ for all $n \geq 0$. Hence it follows that $A_f \otimes_A M_f$ is a free $A_f$-module. This proves (Q0).

In the sequel we shall restrict ourselves to filtered algebras $(B, F)$ such that $\text{gr}(B, F)$ is left noetherian. As we will see next, a necessary condition for the latter is that $B$ itself is left noetherian: Let $A$ be a ring and $(B, F)$ an $A$-algebra with filtration $F = \{B_n : n \geq -1\}$. Let $\text{gr}_n : B_n \rightarrow B_n/B_{n-1}$ denote the canonical maps. If $L$ is a left ideal of $B$, then

$$\text{gr}(L) = \bigoplus_{n=0}^{\infty} \text{gr}_n(L \cap B_n)$$

is a left ideal of $\text{gr}(B)$. Clearly $L_1 \subseteq L_2$ implies $\text{gr}(L_1) \subseteq \text{gr}(L_2)$.

**Lemma 21.** If $L_1$ and $L_2$ are left ideals of $B$ such that $L_1 \subseteq L_2$ and $L_1 \neq L_2$, then $\text{gr}(L_1) \neq \text{gr}(L_2)$.

**Proof.** Suppose that $\text{gr}(L_1) = \text{gr}(L_2)$. Let $n \geq 0$ be minimal with $L_1 \cap B_n \neq L_2 \cap B_n$. Choose $b \in L_2 \cap B_n$ such that $b \notin L_1$. Since $\text{gr}(L_1) = \text{gr}(L_2)$, there is a $c \in L_1 \cap B_n$ such that $\text{gr}_n(c) = \text{gr}_n(b)$. By the minimality of $n$ we conclude that $\gamma - c \in L_1 \cap B_{n-1}$. But this implies $b = (b - c) + c \in L_1$, a contradiction.

Suppose $\{L_n : n \geq 0\}$ is a chain of left ideals of $B$. Then $\{\text{gr}(L_n) : n \geq 0\}$ is a chain of left ideals of $\text{gr}(B, F)$ which becomes stationary provided that $\text{gr}(B, F)$ is left noetherian. By Lemma 21 it follows that $\{L_n : n \geq 0\}$ is stationary. This proves $B$ to be left noetherian.

**Theorem 22.** Let $B$ be a $k$-algebra endowed with a locally finite filtration $F$ such that the associated graded $k$-algebra $\bar{B} = \text{gr}(B, F)$ is left noetherian. Then $B$ satisfies condition (G0). In particular $B$ is a Jacobson algebra and $\text{End}_B(M)$ is algebraic over $k$ for every simple $B$-module $M$.

**Proof.** As in (G0) let $\bar{M}$ be a cyclic graded $\bar{B}[X]$-module. If $\xi \in \bar{M}_0$ is a cyclic vector, then $L = \{p \in \bar{B}[X] : p \cdot \xi = 0\}$ is a left ideal of $\bar{B}[X]$ such that $\bar{M} \cong \bar{B}[X]/L$. In particular it follows that $\bar{M}$ is noetherian. Furthermore the summands of the grading

$$\bar{M} = \bigoplus_{k=0}^{\infty} \bar{M}_k = \bigoplus_{k=0}^{\infty} \bar{B}_k[X] \cdot \xi$$

are finitely generated $k[X]$-modules. Set $A = k[X]$. Obviously the set

$$\bar{T} = T_A(\bar{M}) = \{ \bar{m} \in \bar{M} : \text{ there exists a non-zero } g \in A \text{ such that } g \cdot \bar{m} = 0 \}$$
of $A$-torsion elements of $\bar{M}$ is a $\bar{B}[X]$-submodule. Since $\bar{M}$ is noetherian, we know that $\bar{T} = \sum_{j=0}^{N} \bar{B}[X] \cdot \eta_j$ is finitely generated. As $\eta_j \in \bar{T}$, there exist $0 \neq f_j \in A$ such that $f_j \cdot \eta_j = 0$. Setting $f = \prod_{j=0}^{N} f_j \neq 0$ we find that $f \cdot \bar{T} = 0$. We consider the simple localization $\bar{M}_f = S^{-1} A$ of $A$ by $S = \{ f^n : n \geq 0 \}$. Next we observe that the $A_f$-torsion of the localization $\bar{M}_f = A_f \otimes_A \bar{M}$ of $M$ is zero:

\[ T_{A_f}(\bar{M}_f) = \{ \frac{\bar{m}}{s} : \bar{m} \in T_A(\bar{M}) \text{ and } s \in S \} = 0. \]

For $A_f$ is a Prüfer Domain, it follows that $\bar{M}_f$ is a flat $A_f$-module. Consequently all summands of the decomposition $\bar{M}_f = \bigoplus_{k=0}^{\infty} A_f \otimes_A \bar{M}_n$ are flat. Further the $A_f \otimes_A \bar{M}_n$ are projective because they are finitely generated over $A_f$. Moreover, they are even free because $A_f$ is a principal ideal domain. Altogether we see that $\bar{M}_f$ is a free $A_f$-module. This proves (G0). Since gr$(B, F)$ and hence $B$ are noetherian, Proposition 20 and Theorem 11 imply that $B$ is a Jacobson algebra satisfying (A).

We sustain our search for sufficient conditions for $B$ to be Jacobson. The aim is to introduce a condition (G1) on the level of associated graded algebras and modules which is stronger than (Q1). To this end we suppose that the $k$-algebra $B' = B[X]$ carries a filtration $\mathcal{F}' = \{ B'_n : n \geq -1 \}$ which need not be induced by a filtration $\mathcal{F}$ of $B$ as above. However, the $k[Y]$-algebra $B'[Y]$ is endowed with the natural filtration $\mathcal{F}''$ induced by $\mathcal{F}'$. In particular gr$(B'[Y], \mathcal{F}'') \cong$ gr$(B', \mathcal{F})|Y]$. Let $M$ be a simple $B'$-module and $\varphi \in \text{End}_{B'}(M)$. As usual we regard $M$ as a $B'[Y]$-module via $Y \cdot m = \varphi(m)$ and form the associated graded gr$(B'[Y], \mathcal{F}'')$-module $\bar{M}$. Note that $\bar{M}$ is cyclic over gr$(B', \mathcal{F})$. As a variant of (G0) we implement

\begin{itemize}
  \item[(G1)] Every graded gr$(B', \mathcal{F})|Y]$-module $\bar{M}$ which is cyclic over gr$(B', \mathcal{F})$ is generically free over $k[Y]$.
\end{itemize}

Applying Proposition 20 to $(B'[Y], \mathcal{F}'')$ we obtain

**Corollary 23.** Condition (G1) implies (Q1).

**Remark 24.** It is an interesting question whether the preceding results can be generalized to filtered algebras over arbitrary Jacobson rings. In [14] it is proven that if $A$ is a noetherian Jacobson ring and $(B, \mathcal{F})$ is an $A$-algebra with a locally finite filtration such that gr$(B, \mathcal{F})$ is noetherian, then $B$ is a Jacobson algebra. The crucial step in the proof is to show that $A/A \cap P$ is a field for every primitive ideal $P$ of $B$. 

Appendix A: More on filtered algebras

Let $I$ be an ideal of an algebra $B$ with filtration $\mathcal{F} = \{B_n : n \geq -1\}$. Let $\pi : B \rightarrow B/I$ denote the canonical map. Then $\tilde{\mathcal{F}} = \{\pi(B_n) : n \geq -1\}$ is a filtration of $B/I$. Further the maps $\pi_n : B_n/B_{n-1} \rightarrow \pi(B_n)/\pi(B_{n-1})$ define a homomorphism of $\text{gr}(B, \mathcal{F})$ onto $\text{gr}(B/I, \tilde{\mathcal{F}})$ with kernel $\text{gr}(I)$. From this we deduce that $B/I$ is finitely generated and almost commutative whenever $B$ is. Further if $B$ is an almost commutative algebra of finite type, so is $B/I$.

The following observation seems to be appropriate: If $(B, \mathcal{F})$ is a finitely generated, almost commutative $A$-algebra, then $\text{gr}(B, \mathcal{F})$ is a finitely generated commutative $A$-algebra, but the converse fails. The notion of an almost commutative algebra $(B, \mathcal{F})$ of finite type is more general.

Under the additional assumption that $A$ is a principal ideal domain, the preceding result can be generalized to the case of almost commutative $A$-algebras $B$ of finite type. The next proposition is contained implicitly in Quillen [12]. See also Lemme 2.6.4 in [3].

Let $k$ be a field, $A$ a commutative $k$-algebra, and $(B, \mathcal{F})$ an $A$-algebra with filtration $\mathcal{F} = \{B_n : n \geq -1\}$. Clearly $B' = A \otimes_k B$ becomes an $A$-algebra via

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2) \quad \text{and} \quad a \cdot (a_1 \otimes b_1) = (aa_1) \otimes b_1 .$$

Further $B'_n = A \otimes_k B_n$ defines a filtration $\mathcal{F}'$ of the $A$-algebra $B'$. Note that $B'$ can also be regarded as a filtered $k$-algebra. Since $A \otimes_k -$ is exact and commutes with direct sums, it follows that

$$B'_n / B'_{n-1} = (A \otimes_k B_n) / (A \otimes_k B_{n-1}) \cong A \otimes_k (B_n / B_{n-1})$$

are isomorphic as $A$-modules, and

$$\text{gr}(B', \mathcal{F}') = \bigoplus_{n=0}^{\infty} B'_n / B'_{n-1} = A \otimes_k \left( \bigoplus_{n=0}^{\infty} B_n / B_{n-1} \right) \cong A \otimes_k \text{gr}(B, \mathcal{F})$$

as $A$-algebras. From the last equation we deduce

1. If $(B, \mathcal{F})$ is a (finitely generated) almost commutative $k$-algebra, then $(B', \mathcal{F}')$ is also a (finitely generated) almost commutative $A$-algebra.

2. If $\text{gr}(B, \mathcal{F})$ is a finitely generated $k$-algebra, then $\text{gr}(B', \mathcal{F}')$ is a finitely generated $A$-algebra. If in addition $A$ is finitely generated as a $k$-algebra, then $\text{gr}(B', \mathcal{F}')$ is also a finitely generated $k$-algebra.

Appendix B: Localizations

In this section we collect some results about localizations. In particular we prove that every localization $S^{-1}A$ of a commutative ring $A$ is a flat (right) $A$-module.
Let $A$ be a commutative ring and $S$ a multiplicative subset of $A$ which means $1 \in S$, $0 \notin S$, and $s,t \in S$ implies $st \in S$. In this section we shall discuss the notion of a localization of $A$ with respect to $S$. Generalizing the definition of the field of fractions of an integral domain we consider the following equivalence relation on $S \times A$: $(s,a) \sim (t,b)$ if and only if there exists $v \in S$ such that $vta = vsb$. Let $\frac{a}{s}$ denote the equivalence class of $(s,a)$, and $S^{-1}A$ the set of all equivalence classes. One verifies easily that

$$
\frac{a}{s} + \frac{b}{t} = \frac{ta + sb}{st} \quad \text{and} \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}
$$

give a well-defined addition and multiplication on $S^{-1}A$. Further we define a map $\iota : A \rightarrow S^{-1}A$, $\iota(a) = \frac{a}{1}$. Then the localization $(S^{-1}A, \iota)$ of $A$ with respect to $S$ has the following properties:

- $S^{-1}A$ is a ring and $\iota : A \rightarrow S^{-1}A$ is a ring homomorphism.
- $\iota(s)$ is invertible for all $s \in S$.
- If $B$ is a ring and $\varphi : A \rightarrow B$ a ring homomorphism such that $\varphi(s)$ is invertible for all $s \in S$, then there exists a unique homomorphism $\tilde{\varphi} : S^{-1}A \rightarrow B$ such that $\varphi = \tilde{\varphi} \circ \iota$.
- $S^{-1}A$ is generated by $\iota(A)$ and $\iota(S)^{-1}$.
- $\ker \iota = \{ a \in A : as = 0 \text{ for some } s \in S \}$.

Clearly the first three properties determine $(S^{-1}A, \iota)$ up to isomorphisms. Next we shall discuss the ideal theory of $A$ and $S^{-1}A$. If $J$ is an ideal of $A$, then

$$
\text{ext}(J) = (S^{-1}A) \cdot \iota(J) = \left\{ \frac{a}{s} : a \in J \text{ and } s \in S \right\}
$$

is an ideal of $S^{-1}A$ called the extension of $J$ in $S^{-1}A$. If $I$ is an ideal of $S^{-1}A$, then

$$
\text{res}(I) = \iota^{-1}(I) = \left\{ a \in A : \frac{a}{1} \in I \right\}
$$

is an ideal of $A$, the restriction of $I$ to $A$. Obviously

$$
I = \text{ext}(\text{res}(I)) \quad \text{and} \quad J \subset \text{res}(\text{ext}(J)).
$$

Note that $I \neq S^{-1}A$ is proper if and only if $\text{res}(I) \cap S = \emptyset$. If $I$ is prime, so is $\text{res}(I)$. If $J$ is prime and $J \cap S = \emptyset$, then $\text{ext}(J)$ is also prime. In this case $J = \text{res}(\text{ext}(J))$. Furthermore, if $J_1 \subset J_2$, then $\text{ext}(J_1) \subset \text{ext}(J_2)$, and if $I_1 \subset I_2$, then $\text{res}(I_1) \subset \text{res}(I_2)$.

**Definition 25.** Let $A$ be a commutative ring. Let $\text{Spec}(A)$ denote the set of all prime ideals of $A$. If $J$ is an ideal of $A$, then $h(J) = \{ P \in \text{Spec}(A) : J \subset P \}$ is called the hull of $J$. Conversely, if $X \subset \text{Spec}(A)$, then the ideal $k(X) = \bigcap \{ P : P \in X \}$ is called the kernel of $X$. We say that a subset $X$ of $\text{Spec}(A)$ is closed if and only if $X = h(J)$ for some ideal $J$ of $A$. The space $\text{Spec}(A)$ endowed with this so-called hull-kernel topology is the spectrum of $A$. 
The preceding observations show that the spectrum \( \text{Spec}(S^{-1}A) \) of the localization of \( A \) with respect to \( S \) can be identified with the subset \( \{ P \in \text{Spec}(A) : P \cap S = \emptyset \} \) of the spectrum of \( A \) by means of the homeomorphisms ext and res.

**Lemma 26.** If \( A \) is a principal ideal domain, so is \( S^{-1}A \).

**Proof.** Obviously \( S^{-1}A \) is an integral domain. Let \( I \) be an ideal of \( S^{-1}A \). Since \( A \) is a principal ideal ring, there exists some \( b \in A \) such that \( \text{res}(I) = A \cdot b \). This implies

\[
I = \text{ext}(\text{res}(I)) = (S^{-1}A) \cdot \frac{b}{1}.
\]

Now let \( P \) be a left \( A \)-module. Then the functors \( \text{Hom}_A(-, P) \) and \( \text{Hom}_A(P, -) \) are left exact. One can prove that \( P \otimes_A - \) is also right exact.

**Definition 27.** Let \( A \) be a ring and \( P \) a right \( A \)-module. We say that \( P \) is a flat \( A \)-module if the functor \( P \otimes_A - \) is exact.

Since \( P \otimes_A - \) is already known to be right exact, it follows that \( P \) is \( A \)-flat if and only if the following condition is satisfied: If \( \varphi : L \to M \) is an injective homomorphism of left \( A \)-modules, then \( 1 \otimes \varphi : P \otimes_A L \to P \otimes_A M \) is injective.

To prove that localizations \( S^{-1}A \) are \( A \)-flat, i.e., that the functor \( (S^{-1}A) \otimes_A - \) is exact, we introduce localizations of \( A \)-modules. Let \( A \) be a commutative ring, \( S \) a multiplicative subset of \( A \), and \( M \) a left \( A \)-module. In analogy to the definition of \( S^{-1}A \) we consider the following equivalence relation on \( S \times M \): \( (s, m) \sim (t, n) \) if and only if there exists a \( v \in S \) such that \( vt \cdot m = vs \cdot n \). Let \( \frac{m}{s} \) denote the equivalence class of \( (s, m) \), and \( S^{-1}M \) the set of all such equivalence classes. It is easy to see that

\[
\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st} \quad \text{and} \quad \frac{a}{r} \cdot \frac{m}{s} = \frac{a \cdot m}{rs}
\]
gives a well-defined \( S^{-1}A \)-module structure on \( S^{-1}M \).

**Lemma 28.** Let \( A \) be a commutative ring and \( S \) a multiplicative subset of \( A \). If \( M \) is a left \( A \)-module, then \( (S^{-1}A) \otimes_A M \) and \( S^{-1}M \) are isomorphic as left \( S^{-1}A \)-modules.

**Proof.** By the universal property of the balanced tensor product we obtain a map

\[
\Phi : (S^{-1}A) \otimes_A M \to S^{-1}M , \quad \Phi(\frac{a}{r} \otimes m) = \frac{a \cdot m}{r}.
\]

One checks that \( \Phi \) is a homomorphism of left \( S^{-1}A \)-modules. On the other hand we have a map

\[
\Psi : S^{-1}M \to (S^{-1}A) \otimes_A M , \quad \Psi(\frac{m}{s}) = \frac{1}{s} \otimes m.
\]
We check that $\Psi$ is well-defined: Let $m,n \in M$ and $s,t \in S$ such that $\frac{m}{s} = \frac{n}{t}$. Hence there exists a $v \in S$ such that $vt \cdot m = vs \cdot n$. This implies

$$\Psi\left(\frac{m}{s}\right) = \frac{1}{s} \otimes m = \frac{1}{vts} \otimes (vt \cdot m) = \frac{1}{vts} \otimes (vs \cdot n) = \frac{1}{t} \otimes n = \Psi\left(\frac{n}{t}\right).$$

Obviously $\Psi$ is also a homomorphism of $S^{-1}A$-modules. Further $\Psi \circ \Phi = \text{Id}$ and $\Phi \circ \Psi = \text{Id}$. \hfill $\square$

**Proposition 29.** Let $A$ be a commutative ring and $S$ a multiplicative subset of $A$. Then $S^{-1}A$ is a flat $A$-module.

**Proof.** Let $\varphi : L \rightarrow M$ be an injective homomorphism of left $A$-modules. By the preceding considerations it suffices to prove that $1 \otimes \varphi : (S^{-1}A) \otimes_A L \rightarrow (S^{-1}A) \otimes_A M$ is injective. By means of the isomorphisms $\Phi$ and $\Psi$ given in Lemma 28 we see that $1 \otimes \varphi$ corresponds to the homomorphism

$$\bar{\varphi} : S^{-1}L \rightarrow S^{-1}M, \quad \bar{\varphi}\left(\frac{l}{s}\right) = \frac{\varphi(l)}{s}.$$

It suffices to check that $\bar{\varphi}$ is injective: Let $l \in L$ and $s \in S$ such that $0 = \bar{\varphi}\left(\frac{l}{s}\right) = \frac{\varphi(l)}{s}$. Then there exists $v \in S$ such that $0 = v \cdot \varphi(l) = \varphi(v \cdot l)$. Hence $0 = v \cdot l$ because $\varphi$ is injective. This proves $\frac{l}{s} = 0$. \hfill $\square$

**Appendix C: Semiprimitive and semiprime ideals**

Here we collect some properties of (semi-)primitive and (semi-)prime ideals in non-commutative rings.

**Definition 30.** Let $B$ be a ring and $\Lambda$ a subset of $B$. We say that $\Lambda$ is $m$-closed if, for any $a,b \in \Lambda$, there exists some $x \in B$ such that $axb \in \Lambda$. Further $\Lambda$ is called $p$-closed if, for any $a \in \Lambda$, there exists some $x \in B$ such that $axa \in \Lambda$.

Prime ideals are characterized easily as follows.

**Proposition 31.** Let $B$ be a ring and $I$ an ideal of $B$ with $I \neq B$. Then there are equivalent:

(i) $I$ is prime.

(ii) $a_1Ba_2 \subseteq I$ implies $a_1 \in I$ or $a_2 \in I$.

(iii) $B \setminus I$ is an $m$-closed subset of $B$.

**Proof.** First we prove (i) $\Rightarrow$ (ii). Suppose that $I$ is prime. Let $a_1,a_2$ be in $B$ such that $a_1Ba_2 \subseteq I$. Then $(Ba_1B)(Ba_2B) \subseteq I$. Since $I$ is prime, it follows $Ba_1B \subseteq I$ or $Ba_2B \subseteq I$, and hence $a_1 \in I$ or $a_2 \in I$ because $B$ is unital. This proves (ii). Obviously (iii) is the contrapositive of (ii). Thus (ii) $\Leftrightarrow$ (iii). It remains to prove (ii) $\Rightarrow$ (i). Let $J_1$ and $J_2$ be ideals of $B$ such that $J_1J_2 \subseteq I$. Suppose that $J_2 \not\subseteq I$ and choose $b \in J_2 \setminus I$. For every $a \in J_1$ we have $aBb \subseteq J_1J_2 \subseteq I$ and thus $a \in I$ by (ii). This proves $J_1 \subseteq I$. Hence $I$ is prime. \hfill $\square$
Before we state a similar characterization for semiprime ideals, we prove the following auxiliary result.

**Lemma 32.** Let $B$ be a ring. If $\Lambda$ is a $p$-closed subset of $B$ and $x \in \Lambda$, then there exists a countable $m$-closed subset $\Lambda_0$ of $B$ with $\Lambda_0 \subset \Lambda$ and $x \in \Lambda_0$.

**Proof.** By induction we define a sequence $\{x_n : n \geq 0\}$ of elements of $\Lambda$ as follows: Set $x_0 = x$. If $x_0, \ldots, x_n$ are defined, then there is a $y \in B$ such that $x_nyx_n \in \Lambda$ because $\Lambda$ is $p$-closed. We set $x_{n+1} = x_nyx_n$. Finally we define $\Lambda_0 = \{x_n : n \geq 0\}$. Now we must prove that $x_mBx_n \cap \Lambda_0 \neq \emptyset$ for all $m, n \geq 0$: Suppose that $m \leq n$. Then $x_{n+1} \in x_nBx_n \subset x_mBx_n$ and $x_{n+1} \in \Lambda_0$. The case $m \geq n$ is similar. □

**Proposition 33.** Let $B$ be a ring and $I \neq B$ an ideal of $B$. Then there are equivalent:

(i) $I$ is semiprime.

(ii) If $J$ is an ideal of $B$ such that $J^2 \subset I$, then $J \subset I$.

(iii) $aBa \subset I$ implies $a \in I$.

(iv) $B \setminus I$ is a $p$-closed subset of $B$.

**Proof.** First we prove (i)⇒(ii). Suppose that $I$ is semiprime and let $J$ be an ideal of $B$ such that $J^2 \subset I$. Let $P$ be an arbitrary prime ideal of $B$ such that $I \subset P$. From $J^2 \subset P$ it follows $J \subset P$. Intersecting all these $P$ we conclude $J \subset \sqrt{I} = I$ which proves (ii). Next we verify (ii)⇒(iii). Let $a \in B$ such that $aBa \subset I$. Then it follows $(BaB)(BaB) \subset I$ and hence $BaB \subset I$ by (ii). Thus $a \in I$ because $B$ is unital. This proves (iii). Obviously (iv) is the contraposition of (iii) so that (iii)⇔(iv). Finally we establish (iv)⇒(i). Suppose that $B \setminus I$ is $p$-closed. We must show that $\sqrt{I} \subset I$. Let $a \in B \setminus I$. From Lemma 32 we deduce that there exists an $m$-closed subset $\Lambda_0$ of $B \setminus I$ such that $a \in \Lambda_0$. By Proposition 31 we know that $P = B \setminus \Lambda_0$ is prime. Since $a \notin P$, we conclude $a \notin \sqrt{I}$. □

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