CLASSIFICATION OF $K$-TYPE FORMULAS FOR THE HEISENBERG ULTRAHYPERBOLIC OPERATOR $\Box_s$ FOR $\tilde{SL}(3, \mathbb{R})$ AND TRIDIAGONAL DETERMINANTS FOR LOCAL HEUN FUNCTIONS

TOSHIHISA KUBO AND BENT ØRSTED

Abstract. The $K$-type formulas of the space of $K$-finite solutions to the Heisenberg ultrahyperbolic equation $\Box_s f = 0$ for the non-linear group $\tilde{SL}(3, \mathbb{R})$ are classified. This completes a previous study of Kable for the linear group $SL(m, \mathbb{R})$ in the case of $m = 3$, as well as generalizes our earlier results on a certain second order differential operator. As a by-product we also show several properties of certain sequences $\{P_j(x; y)\}_{j=0}^\infty$ and $\{Q_j(x; y)\}_{j=0}^\infty$ of tridiagonal determinants, whose generating functions are given by local Heun functions. In particular, it is shown that these sequences satisfy a certain arithmetic-combinatorial property, which we refer to as a palindromic property. We further show that classical sequences of Cayley continuants $\{Cay_j(x; y)\}_{j=0}^\infty$ and Krawtchouk polynomials $\{K_j(x; y)\}_{j=0}^\infty$ also admit this property. In the end a new proof of Sylvester’s formula for certain tridiagonal determinant $Sylv(x; n)$ is provided from a representation theory point of view.

Contents

1. Introduction
2. Peter–Weyl theorem for the space of $K$-finite solutions
3. Specialization to $(\tilde{SL}(3, \mathbb{R}), B)$
4. Heisenberg ultrahyperbolic operator for $\tilde{SL}(3, \mathbb{R})$
5. Hypergeometric model $d\pi^{II}_n(D^s_k)f(t) = 0$
6. Heun model $d\pi^{I}_n(D^s_k)f(t) = 0$
7. Sequences $\{P_k(x; y)\}_{k=0}^\infty$ and $\{Q_k(x; y)\}_{k=0}^\infty$ of tridiagonal determinants
8. Cayley continuants $\{Cay_k(x; y)\}_{k=0}^\infty$ and Krawtchouk polynomials $\{K_k(x; y)\}_{k=0}^\infty$
9. Appendix A: local Heun functions
10. Appendix B: A proof of Sylvester’s formula
References

1. Introduction

The representation theory of a reductive Lie group $G$ is intimately tied to the analysis on corresponding flag manifolds $G/P$, where $P$ is a parabolic subgroup. For a representation of $P$, one considers the corresponding homogeneous vector bundle over the flag manifold $G/P$ and the space of
sections either in the smooth sense or in some square integrable sense. Understanding the structure of this space of sections is crucial, for example, whether there are interesting subspaces invariant under the left regular action of $G$. This might for example happen if there is given a $G$-invariant differential equation on the space of sections, so that its solutions form an invariant subspace. In order to analyze such a space of solutions, it is convenient to use the analogue of Fourier series, namely, via the maximal compact subgroup $K$ in $G$; here $G = KP$, so the flag manifold $G/P$ is also a homogeneous space for $K$, and sections can be described according to how they transform under the action of $K$.

In the case that we consider in this paper, there will be a one-parameter family $\Box_s$ of natural invariant differential operators and explicit spaces of solutions. It turns out that these will be related to classical function theory, namely, hypergeometric functions and local Heun functions, and some of the relevant identities analogous to classical tridiagonal determinants of Sylvester type and Cayley type. Further, the representations obtained from the solution space to the differential equation $\Box_s f = 0$ include ones, which are to be thought of as minimal representations (in some sense). We provide a new aspect on a connection between the representation theory of reductive groups, ordinary differential equations in the complex domains, and sequences of polynomials.

The aim of this paper is threefold. The first is the classification of $K$-type formulas of the space of $K$-finite solutions to the Heisenberg ultrahyperbolic equation $\Box_s f = 0$ for $\widetilde{SL}(3, \mathbb{R})$. The second is a study of certain sequences $\{ P_k(x; y) \}_{k=0}^\infty$ and $\{ Q_k(x; y) \}_{k=0}^\infty$ of tridiagonal determinants arising from the study of the $K$-finite solutions to $\Box_s f = 0$. The third is an application of the arguments for $\{ P_k(x; y) \}_{k=0}^\infty$ and $\{ Q_k(x; y) \}_{k=0}^\infty$ to classical sequences of Cayley continuants $\{ \text{Cay}_k(x; y) \}_{k=0}^\infty$ and Krowtchouk polynomials $\{ K_k(x; y) \}_{k=0}^\infty$. We describe these three topics in detail now.

1.1. Heisenberg ultrahyperbolic operator $\Box_s$. For a moment, let $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{C})$ with $m \geq 3$ and take a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that there exists a parabolic subalgebra $\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ with Heisenberg nilpotent radical $\mathfrak{n}_0$, namely, $\dim_{\mathbb{R}}[\mathfrak{n}_0, \mathfrak{n}_0] = 1$. The Heisenberg condition on $\mathfrak{n}_0$ forces $\mathfrak{g}_0$ to be either $\mathfrak{sl}(m, \mathbb{R})$ or $\mathfrak{su}(p, q)$ with $p + q = m$. We write $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ for the complexification of $\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$. We write $\mathcal{U}(\mathfrak{g})$ for the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

In [18], under the framework of $\mathfrak{g}$ and $\mathfrak{p}$ as above, Kable introduced a one-parameter family $\Box_s$ of differential operators with $s \in \mathbb{C}$ as an example of conformally invariant systems ([2, 3]). The operator $\Box_s$ is referred to as the Heisenberg ultrahyperbolic operator ([18]). For instance, for $m = 3$, it is defined as

$$\Box_s = R((XY + YX) + s[X, Y]),$$

(1.1)

where $R$ denotes the infinitesimal right translation and $X, Y$ are certain nilpotent elements in $\mathfrak{sl}(3, \mathbb{R})$ (see [3, 1]). The differential operator $\Box_s$ for $m = 3$ in [11] in particular recovers the second order differential operator studied in [27] as the case of $s = 0$. Some algebraic and analytic properties of the Heisenberg ultrahyperbolic operator $\Box_s$ as well as its generalizations are investigated in [17, 18, 19, 20] for a linear group $SL(m, \mathbb{R})$.

From a viewpoint of intertwining operators, the Heisenberg ultrahyperbolic operator $\Box_s$ is an intertwining differential operator between parabolically induced representations for $G \supset P$, where
\(G\) and \(P\) are Lie groups with Lie algebras \(\mathfrak{g}_0\) and \(\mathfrak{p}_0\), respectively. Thus the space \(\text{Sol}(\square_s)\) of smooth solutions to the equation \(\square_s f = 0\) in the induced representation is a subrepresentation of \(G\), and also the space \(\text{Sol}(\square_s)_K\) of \(K\)-finite solutions is a \((\mathfrak{g}, K)\)-module. We then consider the following problem.

**Problem 1.2.** Classify the \(K\)-type formulas for \(\text{Sol}(\square_s)_K\).

In [18], an attempt for this direction was made for \(G = SL(m, \mathbb{R})\). In the paper, Kable introduced a notion of \(\mathcal{H}\)-modules and developed an algebraic theory for them. Although the theory is powerful to compute the dimension of the \(\mathfrak{t} \cap \mathfrak{m}\)-invariant subspace of the space of \(K\)-type solutions for each \(K\)-type in \(\text{Sol}(\square_s)_K\), the determination of the explicit \(K\)-type formulas was not achieved.

In this paper, we consider \(\widetilde{SL}(3, \mathbb{R})\) for \(G\). As \(M\) is isomorphic to the quaternion group \(Q_8\) of order 8. Let \(K\) be a maximal compact subgroup of \(G\) so that \(G = KAN\) is an Iwasawa decomposition of \(G\). We have \(K \simeq SU(2) \simeq Spin(3)\).

Let \(\text{Irr}(M)\) and \(\text{Irr}(K)\) denote the set of equivalence classes of irreducible representations of \(M\) and \(K\), respectively. As \(M \simeq Q_8\), the set \(\text{Irr}(M)\) may be given as

\[
\text{Irr}(M) = \{ (\pm, \pm), (\pm, -\pm), (-\pm, \pm), (-\pm, -\pm), \mathbb{H} \},
\]

where \((\pm, \pm)\) are some characters (see Section 3.4 for the definition) and \(\mathbb{H}\) stands for the unique two-dimensional genuine representation of \(M\). The character \((\pm, \pm)\) is, for instance, the trivial character. Let \((n/2)\) denote the irreducible finite-dimensional representation of \(K \simeq Spin(3)\) with dimension \(n + 1\). Then we have

\[
\text{Irr}(K) = \{ (n/2) : n \in \mathbb{Z}_{\geq 0} \}.
\]

For \(\sigma \in \text{Irr}(M)\) and a character \(\lambda\) of \(A\), we write

\[
I(\sigma, \lambda) = \text{Ind}_{B}^{G}(\sigma \otimes (\lambda + \rho) \otimes 1)
\]

for the representation of \(G\) induced from the representation \(\sigma \otimes (\lambda + \rho) \otimes 1\) of \(B = MAN\), where \(\rho\) is half the sum of the positive roots corresponding to \(B\). We realize the induced representation \(I(\sigma, \lambda)\) on the space of smooth sections for a \(G\)-equivariant homogeneous vector bundle over \(G/B\). Let \(\text{Diff}_G(I(\sigma_1, \lambda_1), I(\sigma_2, \lambda_2))\) denote the space of intertwining differential operators from \(I(\sigma_1, \lambda_1)\) to \(I(\sigma_2, \lambda_2)\).

Let \(\mathfrak{g}\) denote the complexified Lie algebra of \(\mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{R})\). Since \(\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})\), the Heisenberg ultrahyperbolic operator \(\square_s\) is given as in [11]. We then set

\[
D_s := (XY + YX) + s[X, Y] \in \mathcal{U}(\mathfrak{g}),
\]

(1.3)
so that \( \Box_s = R(D_s) \). It follows from Proposition 4.3 that we have

\[
R(D_s) \in \text{Diff}_G(I((+,+), -\bar{\rho}(s)), I((-,-), \bar{\rho}(-s))),
\]

where \( \bar{\rho}(s) \) is a certain weight determined by \( \bar{\rho} := \rho/2 \) (see (4.1)). It is further shown that, for \( \sigma \in \text{Irr}(M) \),

\[
R(D_s) \otimes \text{id}_\sigma \in \text{Diff}_G(I(\sigma, -\bar{\rho}(s)), I((-,-) \otimes \sigma, \bar{\rho}(-s))),
\]

where \( \text{id}_\sigma \) denotes the identity map on \( \sigma \).

For notational convenience we consider \( R(D_s) \) in place of \( R(D_s) \), where \( s \) denotes the complex conjugate of \( s \in \mathbb{C} \) (see Section 4.1 for the details). We define

\[
\text{Sol}(s; \sigma) := \text{the space of smooth solutions to } (R(D_s) \otimes \text{id}_\sigma)f = 0,
\]

\[
\text{Sol}(s; \sigma)_K := \text{the space of } K\text{-finite solutions to } (R(D_s) \otimes \text{id}_\sigma)f = 0. \tag{1.4}
\]

It follows from a Peter–Weyl theorem for the solution space (Theorem 2.11) that the \((\mathfrak{g}, K)\)-module \( \text{Sol}(s; \sigma)_K \) decomposes as

\[
\text{Sol}(s; \sigma)_K \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (n/2) \otimes \text{Hom}_M (\text{Sol}(s; n), \sigma), \tag{1.5}
\]

where \( \text{Sol}(s; n) \) is the space of \( K \)-type solutions to \( D_s \) (without complex conjugation on \( s \)) on the \( K \)-type \( (n/2) = (\delta, V_{(n/2)}) \in \text{Irr}(K) \), that is,

\[
\text{Sol}(s; n) = \{ v \in V_{(n/2)} : d\delta(D_s^b)v = 0 \}. \tag{1.6}
\]

(see (4.7) and (4.6)). Here \( d\delta \) is the differential of \( \delta \) and \( D_s^b \) denotes the compact model of \( D_s \) (see Definition 2.5). Then the \( K \)-type decomposition \( \text{Sol}(s; \sigma)_K \) for \((s, \sigma) \in \mathbb{C} \times \text{Irr}(M) \) is explicitly given as follows.

**Theorem 1.7.** The following conditions on \((s, \sigma) \in \text{Irr}(M) \times \mathbb{C} \) are equivalent.

(i) \( \text{Sol}(s; \sigma) \neq \{0\} \).

(ii) One of the following conditions holds.

- \( \sigma = (+,+) : s \in \mathbb{C} \).
- \( \sigma = (-,-) : s \in \mathbb{C} \).
- \( \sigma = (-,+), s \in 1 + 4\mathbb{Z} \).
- \( \sigma = (+,-), s \in 3 + 4\mathbb{Z} \).
- \( \sigma = \mathbb{H} : s \in 2\mathbb{Z} \).

Further, the \( K \)-type formulas for \( \text{Sol}(s; \sigma)_K \) are given as follows.

(1) \( \sigma = (+,+): \)

\[
\text{Sol}(s; (+,+))_K \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (2n) \quad \text{for all } s \in \mathbb{C}.
\]

(2) \( \sigma = (-,-): \)

\[
\text{Sol}(s; (-,-))_K \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (1 + 2n) \quad \text{for all } s \in \mathbb{C}.
\]
(3) $\sigma = (-,+)$:

$$\text{Sol}(s; (-,+))_K \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} ((|s| + 1)/2 + 2n) \quad \text{for} \quad s \in 1 + 4\mathbb{Z}.$$ 

(4) $\sigma = (+,-)$:

$$\text{Sol}(s; (+,-))_K \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} ((|s| + 1)/2 + 2n) \quad \text{for} \quad s \in 3 + 4\mathbb{Z}.$$ 

(5) $\sigma = \mathbb{H}$:

$$\text{Sol}(s; \mathbb{H})_K \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} ((|s| + 1)/2 + 2n) \quad \text{for} \quad s \in 2\mathbb{Z}.$$ 

We shall deduce Theorem [1.7] from Theorem [5.36] at the end of Section 5. The proof is not case-by-case analysis on $\sigma \in \text{Irr}(M)$; each $M$-representation $\sigma$ is treated uniformly via a recipe for the $K$-type decomposition of $\text{Sol}(s; \sigma)_K$. See Section 4.3 for the details of the recipe. In regard to Theorem [1.7] we shall also classify the space $\text{Sol}(s; n)$ of $K$-type solutions for all $n \in \mathbb{Z}_{\geq 0}$. This is done in Theorems [5.16] and 6.7. Recall from (1.4) that $\text{Sol}(s; \sigma)_K$ concerns $R(D_s)$. Theorem [1.7] shows that the $K$-type decompositions are in fact independent of taking the complex conjugate on the parameter $s \in \mathbb{C}$.

There are several remarks on the space $\text{Sol}(s; n)$ of $K$-type solutions and the $K$-type decompositions of $\text{Sol}(s; \sigma)_K$. First, as mentioned above, the dimensions of $\mathfrak{k} \cap \mathfrak{m}$-invariant subspaces of the spaces of $K$-type solutions are determined in [18] for $SL(m, \mathbb{R})$ with arbitrary rank $m \geq 3$. In particular, as $\mathfrak{k} \cap \mathfrak{m} = \{0\}$ for $m = 3$, the dimensions $\dim_{\mathbb{C}} \text{Sol}(s; n)$ for $n \in 2\mathbb{Z}_{\geq 0}$ are obtained in the cited paper ([18 Thm. 5.13]). We note that our normalization on $s \in \mathbb{C}$ is different from one for $z \in \mathbb{C}$ in [18] by $s = -2z$. In the paper, factorization formulas of certain tridiagonal determinants are essential to compute $\dim_{\mathbb{C}} \text{Sol}(s; n)$. For further details on the factorization formulas, see the remark after (1.22) below. By making use of differential equations, we determine the dimensions $\dim_{\mathbb{C}} \text{Sol}(s; n)$ for all $n \in \mathbb{Z}_{\geq 0}$ independently to the results of [18].

Table 1 summarizes the results of [18] and this paper, concerning the $K$-type decompositions of $\text{Sol}(s; \sigma)_K$ for $\mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{R})$.

| $\mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{R})$ | $\dim_{\mathbb{C}} \text{Sol}(s; n)$ | $K$-type decomposition of $\text{Sol}(s; \sigma)_K$ |
|--------------------------------|--------------------------------|----------------------------------|
| [18] | Done for $SL(3, \mathbb{R})$ | Not obtained |
| this paper | Done for $\tilde{SL}(3, \mathbb{R})$ | Done for $\tilde{SL}(3, \mathbb{R})$ |

Secondly, Tamori recently investigates in [36] the representations on the space of smooth solutions to the same differential equation as the Heisenberg ultrahyperbolic equation $R(D_s)f = 0$, as part of his thorough study on minimal representations. In particular, the $K$-type formula $\text{Sol}(s; \mathbb{H})_K$ for the case $\sigma = \mathbb{H}$ is determined in [36 Prop. 5.4.6]. The representations on $\text{Sol}(s; \mathbb{H})$ for $s \in 2\mathbb{Z}$ are genuine representations and not unitarizable unless $s = 0$ ([36 Rem. 5.4.5]). We remark that, both in [36] and in this paper, the $K$-type formula for $\text{Sol}(s; \mathbb{H})_K$ is obtained by realizing the $K$-type
solutions in $\text{Sol}(s; n)$ as hypergeometric polynomials; nevertheless, the methods are rather different. For instance, some combinatorial computations are carried out in [39], whereas we simply solve the hypergeometric equation. Further, we also give the $K$-type solutions by Heun polynomials. We illustrate our methods in detail in Section 1.3.

Lastly, the $K$-type formulas $\text{Sol}(0; \sigma)_K$ for the case of $s = 0$ are previously classified by ourselves in [27, Thm. 1.6]. In this case only three $M$-representations $\sigma = (+, +), (-, -), \mathbb{H}$ contribute to $\text{Sol}(0; \sigma) \neq \{0\}$ with explicit $K$-type formulas

\[
\text{Sol}(0; (+, +))_K \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (2n),
\]
\[
\text{Sol}(0; (-, -))_K \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (1 + 2n),
\]
\[
\text{Sol}(0; \mathbb{H})_K \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} ((1/2) + 2n).
\]

It was quite mysterious that only the series of $(3/2) + 2\mathbb{Z}_{\geq 0}$ does not appear in the $K$-type formulas. Theorem 1.7 now gives an answer for this question.

The representations realized on $\text{Sol}(0; \sigma)$ for $\sigma = (+, +), (-, -), \mathbb{H}$ are known to be unitarizable and the resulting representations are the ones attached to the minimal nilpotent orbit ([33]). For instance, the representation realized on $\text{Sol}(0; \mathbb{H})$ is the genuine representation so-called Torasso’s representation ([39]). Here are two remarks on a recent progress on the unitarity of the representations on $\text{Sol}(0; \sigma)$. First, Dahl recently constructed in [9] the unitary structures for the three unitarizable representations on $\text{Sol}(0; \sigma)$ by using the Knapp–Stein intertwining operator and the Fourier transform on the Heisenberg group. Furthermore, Frahm also recently gives the $L^2$-models of these unitary representations in [11], as part of his intensive study on the $L^2$-realizations of the minimal representations realized on the solution space of conformally invariant systems.

1.3. Hypergeometric and Heun’s differential equations. The main idea for accomplishing Theorem 1.7 is use of the decomposition formula (1.5) and a refinement of the method applied for the case $s = 0$ in [27]. The central problem is classifying the space $\text{Sol}(s; n)$ of $K$-type solutions. In order to do so, one needs to proceed the following two steps.

Step 1: Classify $(s, n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}$ so that $\text{Sol}(s; n) \neq \{0\}$.

Step 2: Classify the $M$-representations on $\text{Sol}(s; n) \neq \{0\}$.

In [27] (the case $s = 0$), via the polynomial realization

\[
\text{Irr}(K) \simeq \{(\pi_n, \text{Pol}_n[t]) : n \in \mathbb{Z}_{\geq 0}\}
\]

of $\text{Irr}(K)$ (see [3,14]), we carried out the two steps by realizing $\text{Sol}(0; n)$ in (1.6) as

\[
\text{Sol}(0; n) = \{p(t) \in \text{Pol}_n[t] : d\pi_n^1(D_0^\flat)p(t) = 0\},
\]

the space of polynomial solutions to an ordinary differential equation $d\pi_n^1(D_0^\flat)p(t) = 0$ via a certain identification $\Omega^1 : \mathfrak{t} \cong \mathfrak{sl}(2, \mathbb{C})$ (see Sections 3.2 and 4.2). It turned out that the differential equation
that the differential equation \( d\pi_n^1(D_0^t)p(t) = 0 \) is a hypergeometric differential equation. Steps 1 and 2 were then easily carried out by observing the Gauss hypergeometric functions arising from \( d\pi_n^1(D_0^t)p(t) = 0 \).

In this paper we apply the same idea for the general case \( s \in \mathbb{C} \). Nonetheless, in the general case, Steps 1 and 2 do not become as simple as the case of \( s = 0 \), as the differential equation \( d\pi_n^1(D_s^t)p(t) = 0 \) is turned out to be Heun’s differential equation

\[
D_H(-1, -\frac{ns}{4}; -\frac{n}{2}, -\frac{n-1}{2}, \frac{1}{2}, \frac{1-n-s}{2}; t^2)p(t) = 0
\]  

(1.9)

with the \( P \)-symbol

\[
P \left\{ \begin{array}{cccc}
0 & 1 & -1 & \infty \\
0 & 0 & -\frac{n}{4} & t^2 \\
\frac{1}{2} & \frac{1+n+s}{2} & -\frac{n-1}{2} & -\frac{ns}{4} \\
\end{array} \right\}
\]

(See Section 9.1 for the definition of \( D_H(a, q; \alpha, \beta, \gamma, \delta; z) \) and the notation of the \( P \)-symbol.) As such, one needs to deal with local Heun functions at \( 0 \) ([38]); these are not as easy to handle as hypergeometric functions, classifying such parameters is not an easy problem at all, since only known are necessary conditions in which local Heun functions are reduced to polynomials. To resolve this problem, inspired by a work [36] of Tamori, we use a different identification \( \Omega^H: \mathfrak{t} \cong \mathfrak{sl}(2, \mathbb{C}) \) so that the differential equation \( d\pi_n^{H}(D_s^t)p(t) = 0 \) becomes again a hypergeometric equation, namely,

\[
D_F(-\frac{n}{2}, -\frac{n+s-1}{4}, \frac{3-n+s}{4}; t^2)p(t) = 0
\]

(1.10)

(See [4,39] for the definition of \( D_F(a, b, c; z) \).) For the detail, see Sections 3.2 and 4.2. We accomplished the \( K \)-type formulas of \( \text{Sol}(s; \sigma)_K \) by using the hypergeometric model \( d\pi_n^{H}(D_s^t)p(t) = 0 \) (see Theorem 5.36).

The Heun model \( d\pi_n^H(D_s^t)p(t) = 0 \) and the hypergeometric model \( d\pi_n^{H}(D_s^t)p(t) = 0 \) are related by a Cayley transform \( \pi_n(k_0) \) given by an element \( k_0 \) of \( K \) ([4,18]). Namely, we have

\[
d\pi_n^H(D_s^t)p(t) = 0 \iff d\pi_n^1(D_s^t)\pi_n(k_0)p(t) = 0
\]

(see Proposition 4.19). For instance, put

\[
u_{[s,n]}(t) := Hl(-1, -\frac{ns}{4}; -\frac{n}{2}, -\frac{n-1}{2}, \frac{1}{2}, \frac{1-n-s}{2}; t^2)
\]

(1.11)

and

\[
a_{[s,n]}(t) := F(-\frac{n}{2}, -\frac{n+s-1}{4}, \frac{3-n+s}{4}; t^2),
\]

where \( Hl(a, q; \alpha, \beta, \gamma, \delta; z) \) denotes the local Heun function at \( z = 0 \) and \( F(a, b, c; z) \equiv zF_1(a, b, c; z) \) is the Gauss hypergeometric function. Then, for

\[
D_H^{[s,n]} := D_H(-1, -\frac{ns}{4}; -\frac{n}{2}, -\frac{n-1}{2}, \frac{1}{2}, \frac{1-n-s}{2}; t^2)
\]

in (1.9) and

\[
D_F^{[s,n]} := D_F(-\frac{n}{2}, -\frac{n+s-1}{4}, \frac{3-n+s}{4}; t^2)
\]
1.4. Palindromic property of a sequence \( \{p_k(x; y)\}_{k=0}^{\infty} \) of polynomials. In the last half of this paper, we shall discuss certain arithmetic-combinatorial properties of four sequences of polynomials of two variables, namely,

\[
\{P_k(x; y)\}_{k=0}^{\infty}, \quad \{Q_k(x; y)\}_{k=0}^{\infty}, \quad \{\text{Cay}_k(x; y)\}_{k=0}^{\infty}, \quad \text{and} \quad \{K_k(x; y)\}_{k=0}^{\infty},
\]
where \( \{P_k(x; y)\}_{k=0}^{\infty} \) and \( \{Q_k(x; y)\}_{k=0}^{\infty} \) are obtained from the study of the Heun model \( d\pi_{(n)}^+(D_0^{p})p(t) = 0 \). Since we could not find the property in consideration, we refer to it as a palindromic property.

As such, we shall digress for a moment from the main results of this paper.

To state the general definition, let \( \{p_k(x; y)\}_{k=0}^{\infty} \) be a sequence of polynomials of two variables with \( p_0(x; y) = 1 \), and \( \{a_k\}_{k=0}^{\infty} \) a sequence of non-zero numbers. Let \( d: \mathbb{Z}_{\geq 0} \to \mathbb{R} \) be a given map.

For \( n \in \mathbb{Z}_{\geq 0} \), we put

\[
\text{Sol}_k(p; n) := \{ s \in \mathbb{C} : p_k(s; n) = 0 \}.
\]

**Definition 1.15** (palindromic property). A pair \( \{(p_k(x; y))_{k=0}^{\infty}, \{a_k\}_{k=0}^{\infty}\} \) is said to be a palindromic pair with degree \( d(n) \) if, for each \( n \in d^{-1}(\mathbb{Z}_{\geq 0}) \), there exists a map

\[
\theta_{(p; n)}: \text{Sol}_{d(n)+1}(p; n) \to \{\pm 1\}
\]

such that, for all \( s \in \text{Sol}_{d(n)+1}(p; n) \), we have \( p_k(s; n) = 0 \) for \( k \geq d(n) + 1 \) and

\[
p_k(s; n) = \frac{\theta_{(p; n)}(s) p_{d(n)-k}(s; n)}{a_{d(n)-k}} \quad \text{for} \quad k \leq d(n).
\]

(1.16)

We call the identity (1.16) the palindromic identity and \( \theta_{(p; n)}(s) \) its sign factor. Further, for a given palindromic pair \( \{(p_k(x; y))_{k=0}^{\infty}, \{a_k\}_{k=0}^{\infty}\} \), the sequence \( \{a_k\}_{k=0}^{\infty} \) is said to be an associated sequence of \( \{(p_k(x; y))_{k=0}^{\infty}\} \). If there exits a sequence \( \{a_k\}_{k=0}^{\infty} \) such that \( \{(p_k(x; y))_{k=0}^{\infty}, \{a_k\}_{k=0}^{\infty}\} \) is a palindromic pair, then we say that \( \{(p_k(x; y))_{k=0}^{\infty}\} \) admits a palindromic property.

We note that our notion of palindromic property does not concern polynomials themselves but sequences of polynomials; for instance, each \( p_k(x; n) \) is not necessarily a palindromic polynomial.

**Example 1.17.** Here are some simple examples.

(a) If \( p_0(x; y) = 1 \) and \( p_k(x; y) = 0 \) for all \( k \geq 1 \), then, for any sequence \( \{a_k\}_{k=0}^{\infty} \) of non-zero numbers, the pair \( \{(p_k(x; y))_{k=0}^{\infty}, \{a_k\}_{k=0}^{\infty}\} \) is a palindromic pair with degree 0.

(b) If \( p_0(x; y) = p_1(x; y) = 1 \) and \( p_k(x; y) = 0 \) for all \( k \geq 2 \), then, for any sequence \( \{a_k\}_{k=0}^{\infty} \) of non-zero numbers with \( a_0 = \pm a_1 \), the pair \( \{(p_k(x; y))_{k=0}^{\infty}, \{a_k\}_{k=0}^{\infty}\} \) is a palindromic pair with degree 1.

(c) If \( p_k(x; y) = x^k \), then, for any sequence \( \{a_k\}_{k=0}^{\infty} \) of non-zero numbers, the pair \( \{(x^k)_{k=0}^{\infty}, \{a_k\}_{k=0}^{\infty}\} \) is a palindromic pair with degree 0.

We shall provide four examples for which the degree \( d(n) \) is not constant. (See Sections 1.7, 1.8 and 1.9)

Now suppose that \( \{(p_k(x; y))_{k=0}^{\infty}, \{a_k\}_{k=0}^{\infty}\} \) is a palindromic pair with degree \( d(n) \), where the associated sequence \( \{a_k\}_{k=0}^{\infty} \) is of the form \( a_k = (c_k)! \) for some sequence \( \{c_k\}_{k=1}^{\infty} \) of non-negative integers with \( c_0 = 1 \). Then, as \( p_0(x; y) = 1 \) by definition, the palindromicity of \( \{(p_k(x; y))_{k=0}^{\infty}, \{a_k\}_{k=0}^{\infty}\} \) implies that, for \( n \in d^{-1}(\mathbb{Z}_{\geq 0}) \), the \( d(n) \)th term \( p_{d(n)}(x; y) \) satisfies the identity

\[
p_{d(n)}(s; n) = \theta_{(p; n)}(s) (c_{d(n)})! \quad \text{for} \quad s \in \text{Sol}_{d(n)+1}(p; n).
\]

(1.18)
We refer to the identity (1.18) as the *factorial identity* of $p_{d(n)}(x; n)$ on $\text{Sol}_{d(n)+1}(p; n)$. Further, we mean by the *refinement* of a factorial (resp. palindromic) identity a factorial (resp. palindromic) identity for each explicit value of $s \in \text{Sol}_{d(n)+1}(p; n)$.

In order to show the refinements of a palindromic property and factorial identity for $\{p_k(x; y)\}_{k=0}^\infty$, it is important to understand the zero set $\text{Sol}_{d(n)+1}(p; n)$ of $p_{d(n)+1}(x; n)$. For the purpose we also consider the *factorization formula* of the polynomial $p_{d(n)+1}(x; n)$.

In summary, we shall investigate the following properties for the sequences of polynomials $\{p_k(x; y)\}_{k=0}^\infty$ in (1.14):

- Factorization formula of $p_{d(n)+1}(x; n)$;
- Palindromic property of $\{p_k(x; y)\}_{k=0}^\infty$;
- Factorial identity of $p_{d(n)}(x; n)$;
- Refinement of a factorial identity of $p_{d(n)}(x; n)$.

It is remarked that factorization formulas for $\{P_k(x; y)\}_{k=0}^\infty$ and $\{Q_k(x; y)\}_{k=0}^\infty$ will be given in a more general situation. We are going to describe these topics in detail now.

1.5. **Sequences $\{P_k(x; y)\}_{k=0}^\infty$ and $\{Q_k(x; y)\}_{k=0}^\infty$ of tridiagonal determinants.** Recall from (1.12) that the local Heun function $u_{[s;n]}(t)$ in (1.11) is a local solution at $t = 0$ to the Heun differential equation (1.9). Let $v_{[s;n]}(t)$ be the second solution (see (1.35)). It will be shown in Proposition (1.10) that $u_{[s;n]}(t)$ and $v_{[s;n]}(t)$ may be given in power series representations as

$$u_{[s;n]}(t) = \sum_{k=0}^\infty P_k(s; n) \frac{t^{2k}}{(2k)!} \quad \text{and} \quad v_{[s;n]}(t) = \sum_{k=0}^\infty Q_k(s; n) \frac{t^{2k+1}}{(2k+1)!},$$

where $P_k(x; y)$ and $Q_k(x; y)$ are certain $k \times k$ tridiagonal determinants (see Section 7.1). For instance, the $\frac{n}{2} \times \frac{n}{2}$ tridiagonal determinant $P_{\frac{n}{2}}(x; n)$ for $y = n \in 2\mathbb{Z}_\geq 0$ is given as $P_0(x; 0) = 1$, $P_1(x; 2) = 2x$, and for $n \in 4 + 2\mathbb{Z}_\geq 0$,

$$P_{\frac{n}{2}}(x; n) = \begin{vmatrix} nx & 1 \cdot 2 & 3 \cdot 4 & \cdots & 5 \cdot 6 & \cdots & -(n-8)x & -(n-3)(n-2) & -3 \cdot 4 & -(n-4)x \\ -(n-1)n & (n-4)x & (n-8)x & \cdots & -5 \cdot 6 & \cdots & -(n-8)x & (n-3)(n-2) & -3 \cdot 4 & -(n-4)x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}. \quad (1.19)$$

Similarly, the $\frac{n^2}{2} \times \frac{n^2}{2}$ tridiagonal determinant $Q_{\frac{n^2}{2}}(x; n)$ for $y = n \in 2(1 + \mathbb{Z}_\geq 0)$ is given as $Q_0(x; 2) = 1$, $Q_1(x; 4) = 2x$, and for $n \in 6 + 2\mathbb{Z}_\geq 0$,

$$Q_{\frac{n^2}{2}}(x; n) = \begin{vmatrix} (n-2)x & 2 \cdot 3 & 4 \cdot 5 & \cdots & 6 \cdot 7 & \cdots & -(n-10)x & -(n-4)(n-3) & -4 \cdot 5 & -(n-6)x \\ -(n-2)(n-1) & (n-6)x & (n-10)x & \cdots & -6 \cdot 7 & \cdots & -(n-10)x & (n-4)(n-3) & -4 \cdot 5 & -(n-6)x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}. \quad (1.20)$$
1.6. Factorization formulas of $P_{(n+2)/2}(x;n)$ and $Q_{(n+1)/2}(x;n)$. In 1854, Sylvester observed that an $(n + 1) \times (n + 1)$ centrosymmetric tridiagonal determinant

$$\text{Sylv}(x; n) := \begin{vmatrix} x & 1 & & & \\ n & x & 2 & & \\ & n-1 & x & 3 & \\ & & \cdots & \cdots & \cdots \\ & & & 3 & x & n-1 \\ & & & & 2 & x \\ & & & & & 1 \end{vmatrix} \quad (1.21)$$

satisfies the following formula (1.35).

$$\text{Sylv}(x; n) = \begin{cases} (x^2 - 1^2)(x^2 - 3^2) \cdots (x^2 - (n - 2)^2)(x^2 - n^2) & \text{if } n \text{ is odd,} \\
(x^2 - 2^2)(x^2 - 4^2) \cdots (x^2 - (n - 2)^2)(x^2 - n^2) & \text{if } n \text{ is even.} \end{cases} \quad (1.22)$$

By utilizing some results for the Heun model $d\pi_1(D_0^\pm)p(t) = 0$, we show that $P_{(n+2)/2}(x;n)$ and $Q_{(n+1)/2}(x;n)$ enjoy similar but more involved factorization formulas. See Theorems 7.18 and 7.22 for the details. Based on the factorization formula, we also express $P_{(n+1)/2}(x;n)$ for $n$ odd in terms of the Sylvester determinant $\text{Sylv}(x; n)$ (Corollary 7.27).

Here are some remarks on the factorization formulas in order. First, the factorization formulas $P_{(n+2)/2}(x;n)$ and $Q_{n/2}(x;n)$ for $n \in 2\mathbb{Z}$ also follow from a more general formula in [18] Prop. 5.11, which is obtained via some clever trick based on a change of variables and some general formula on tridiagonal determinants (see, for instance, [51] p. 52 and [18] Lem. 5.10). As mentioned in the remark after Theorem 1.7 above, the factorization formulas for $P_{(n+2)/2}(x;n)$ and $Q_{n/2}(x;n)$ play an essential role in [18] to determine $\dim \text{Sol}(s;n)$. In this paper we somewhat do this process backwards including the case of $n$ odd. Schematically, the difference is described as follows.

- [18]: “factorization $\rightarrow \dim \text{Sol}(s;n)$” \hspace{1cm} ($n \in 2\mathbb{Z}_{\geq 0}$)
- this paper: “$\text{Sol}(s;n) \rightarrow \text{Heun} \rightarrow \text{factorization}”$ \hspace{1cm} ($n \in \mathbb{Z}_{\geq 0}$)

(It is recalled that, as the linear group $SL(m, \mathbb{R})$ is considered in [18], only even $n \in 2\mathbb{Z}_{\geq 0}$ appear for the space $\text{Sol}(s;n)$ of $K$-type solutions for $SO(3) \subset SL(3, \mathbb{R})$ in the case of $m = 3$, whereas we handle all $n \in \mathbb{Z}_{\geq 0}$ in this paper, as $\text{Spin}(3) \subset \tilde{SL}(3, \mathbb{R})$ is in consideration.)

The techniques used in [18] can also be applied for the case of $n$ odd. Nevertheless, it requires some involved computations; for instance, the trick used in the cited paper does not make the situation as simple as for the case of $n$ even, and further, the general formula ([5] p. 52] and [18] Lem. 5.10)] cannot be applied either. By simply applying the results for the Heun model $d\pi_1(D_0^\pm)p(t) = 0$, we successfully avoid such computations.

1.7. Palindromic properties for $\{P_k(x;y)\}_{k=0}^{\infty}$ and $\{Q_k(x;y)\}_{k=0}^{\infty}$. We next describe the palindromic properties and factorial identities for $\{P_k(x;y)\}_{k=0}^{\infty}$ and $\{Q_k(x;y)\}_{k=0}^{\infty}$. The palindromic properties are given as follows.

**Theorem 1.23** (Theorems 7.29 and 7.41). The pair $\{(P_k(x;y))_{k=0}^{\infty}, \{(2k)!\}_{k=0}^{\infty}\}$ is a palindromic pair with degree $\frac{\alpha}{2}$. Similarly, $\{(Q_k(x;y))_{k=0}^{\infty}, \{(2k+1)!\}_{k=0}^{\infty}\}$ is a palindromic pair with degree $\frac{\alpha-2}{2}$.
For the case for \( n \) odd, see Propositions 4.39 and 4.41. The key idea of proofs of Theorem 1.23 is to use a symmetry of the generating functions \( u_{[k,n]}(t) \) and \( v_{[k,n]}(t) \) with respect to the non-trivial Weyl group element \( m_{\frac{1}{2}}^{\frac{1}{3}} \{3,1,1\} \) of SU(2) in the form of the inversion. See Section 7.4 for the details.

It follows from Theorem 1.23 that \( P_{2z}(x;y) \) (1.19) and \( Q_{2z-2}(x;y) \) (1.20) satisfy factorial identities (see Corollaries 7.37 and 7.45). Corollaries 1.24 and 1.26 below are the refinements of the identities via the factorization formulas (Theorem 7.18) of \( P_{2z}(x;n) \) and \( Q_{2z}(x;n) \).

**Corollary 1.24** (Refinement of the factorial identity of \( P_{2z}(x;n) \)). Let \( n \in 2Z_{\geq 0} \). Then the following hold.

1. \( n \equiv 0 \pmod{4} \): The values of \( P_{2z}(s;n) \) are
   \[
P_{2z}(s;n) = n! \text{ for all } s \in \mathbb{C}.
   \]
2. \( n \equiv 2 \pmod{4} \): The values of \( P_{2z}(s;n) \) for \( s = \pm 1, \pm 5, \ldots, \pm (n - 2) \) are given as
   \[
P_{2z}(s;n) = \begin{cases} 
n! & \text{if } s = 1, 5, 9, \ldots, n - 2, \\
-n! & \text{if } s = -1, -5, -9, \ldots, -(n - 2). \end{cases}
   \]

**Corollary 1.26** (Refinement of the factorial identity of \( Q_{2z-2}(x;n) \)). Let \( n \in 2(1 + Z_{\geq 0}) \). Then the following hold.

1. \( n \equiv 0 \pmod{4} \): The values of \( Q_{2z-2}(s;n) \) for \( s = \pm 3, \pm 7, \ldots, \pm (n - 1) \) are given as
   \[
   Q_{2z-2}(s;n) = \begin{cases} 
   (n - 1)! & \text{if } s = 3, 7, 11, \ldots, n - 1, \\
   -(n - 1)! & \text{if } s = -3, -7, -11, \ldots, -(n - 1). \end{cases}
   \]
2. \( n \equiv 2 \pmod{4} \): The values of \( Q_{2z-2}(s;n) \) are
   \[
   Q_{2z-2}(s;n) = (n - 1)! \text{ for all } s \in \mathbb{C}.
   \]

We shall give proofs of Corollaries 1.24 and 1.26 in Sections 7.4.1 and 7.4.2.

### 1.8. Sequence \( \{C_{k}(x;y)\}_{k=0}^{\infty} \) of Cayley continuants

In 1858, four years after the observation of Sylvester on (1.22), Cayley considered in [4] a sequence \( \{C_{k}(x;y)\}_{k=0}^{\infty} \) of \( k \times k \) tridiagonal determinants \( C_{k}(x;y) \) and expressed each \( C_{k}(x;n) \) in terms of the Sylvester determinant \( \text{Sylv}(x;n) \) (see, for instance, [4, 31] and [30, p. 429])). The first few terms are given as

\[
C_{0}(x;y) = 1, \quad C_{1}(x;y) = x, \quad C_{2}(x;y) = \begin{vmatrix} x & 1 \\ y & x \end{vmatrix}, \quad C_{3}(x;n) = \begin{vmatrix} x & 1 & 2 \\ y & x & 1 \\ y - 1 & x \end{vmatrix}, \ldots.
\]

The \((n+1)\)th term \( C_{n+1}(x;n) \) with \( y = n \) is nothing but the Sylvester determinant \( C_{n+1}(x;n) = \text{Sylv}(x;n) \). Further, the \( n \)th term \( C_{n}(x;n) \) may be thought of as the “almost” Sylvester determinant, as it is given as

\[
C_{n}(x;n) = \begin{vmatrix} x & 1 & 2 & 3 & \ldots & \ldots & n-1 \\ n & x & 3 & \ldots & \ldots & 2 \\ n - 1 & x & & \ldots & \ldots & 3 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 3 & x & n - 1 & \ldots & \ldots & 2 \\ 2 & x & & \ldots & \ldots & 3 \end{vmatrix}
\]
(compare (1.28) with (1.21)). Following [31], we refer to each \( \text{Cay}_k(x; y) \) as a Cayley continuant.

As part of the third aim of this paper, we shall show the palindromic property of the Cayley continuants \( \{ \text{Cay}_k(x; y) \}_{k=0}^\infty \) as follows.

**Theorem 1.29 (Theorem 8.8).** The pair \( (\{ \text{Cay}_k(x; y) \}_{k=0}^\infty, \{ k! \}_{k=0}^\infty) \) is a palindromic pair with degree \( n \).

For the palindromic identity, see Theorem 8.8. As for the cases of \( \{ P_k(x; y) \}_{k=0}^\infty \) and \( \{ Q_k(x; y) \}_{k=0}^\infty \), we prove Theorem 1.29 by observing a symmetry of the generating function of the Cayley continuants \( \{ \text{Cay}_k(x; y) \}_{k=0}^\infty \) with respect to the inversion (see Section 8.2).

Theorem 1.29 implies that the almost Sylvester determinant \( \text{Cay}_n(x; y) \) (1.28) also satisfies a factorial identity. Corollary 1.30 below is the refinement of the factorial identity (see Theorem 8.8) of \( \text{Cay}_n(x; n) \) on \( \text{Sol}_{n+1}(\text{Cay}; n) \) via Sylvester’s factorization formula (1.22) of \( \text{Sylv}(x; n) = \text{Cay}_{n+1}(x; n) \).

**Corollary 1.30 (Refinement of the factorial identity of \( \text{Cay}_n(x; n) \)).** For \( n \) even, the values of the almost Sylvester determinant \( \text{Cay}_n(s; n) \) for \( s = 0, \pm 2, \ldots, \pm (n - 2), \pm n \) are given as follows.

- \( n \equiv 0 \pmod{4} \):
  \[
  \text{Cay}_n(s; n) = \begin{cases} 
  n! & \text{if } s = 0, \pm 4, \ldots, \pm (n - 4), \pm n, \\
  -n! & \text{if } s = \pm 2, \pm 6, \ldots, \pm (n - 2).
  \end{cases}
  \]

- \( n \equiv 2 \pmod{4} \):
  \[
  \text{Cay}_n(s; n) = \begin{cases} 
  -n! & \text{if } s = 0, \pm 4, \ldots, \pm (n - 4), \pm n, \\
  n! & \text{if } s = \pm 2, \pm 6, \ldots, \pm (n - 2).
  \end{cases}
  \]

Similarly, for \( n \) odd, the values of \( \text{Cay}_n(s; n) \) for \( s = \pm 1, \pm 3, \ldots, \pm (n - 2), \pm n \) are given as follows.

- \( n \equiv 1 \pmod{4} \):
  \[
  \text{Cay}_n(s; n) = \begin{cases} 
  n! & \text{if } s = 1, -3, 5, \ldots, -(n - 2), n, \\
  -n! & \text{if } s = -1, 3, -5, \ldots, n - 2, -n.
  \end{cases}
  \]

- \( n \equiv 3 \pmod{4} \):
  \[
  \text{Cay}_n(s; n) = \begin{cases} 
  -n! & \text{if } s = 1, -3, 5, \ldots, n - 2, -n, \\
  n! & \text{if } s = -1, 3, -5, \ldots, -(n - 2), n.
  \end{cases}
  \]

We shall give a proof of Corollary 1.30 at the end of Section 8.2.

It has been more than 160 years since Cayley continuants \( \{ \text{Cay}_k(x; y) \}_{k=0}^\infty \) were introduced in [4]. We therefore suppose that Theorem 1.29 and Corollary 1.30 are already in the literature. Yet, in contrast to a large variety of a proof for Sylvester’s formula (1.22) (see the remark after Theorem 7.22), we could not find them at all. It will be quite surprising if the classical identities in Corollary 1.30 have not been found over a century.

It is also remarked that Sylvester’s formula (1.22) readily follows from the representation theory of \( \mathfrak{sI}(2, \mathbb{C}) \) without any computation. Nonetheless, it seems not in the literature either. We shall
then provide a proof of (1.22) from a representation theory point of view (see Section 10). The idea of the proof is in principle the same as the one discussed in Section 1.3 for the classification of K-type formulas. Furthermore, by applying the idea for determining the generating functions of \(\{P(x; y)\}_{k=0}^\infty\) and \(\{Q(x; y)\}_{k=0}^\infty\), we also give in Proposition 8.3 another proof for Sylvester’s formula (1.22) as well as the generating function of the Cayley continuants \(\{\text{Cay}_k(x; y)\}_{k=0}^\infty\). (Thus we give two different proofs of Sylvester’s formula (1.22) in this paper.)

1.9. **Sequence \(\{K_k(x; y)\}_{k=0}^\infty\) of Krawtchouk polynomials.** For \(k \in \mathbb{Z}_{\geq 0}\), let \(K_k(x; y)\) be a polynomial of homogeneous degree \(k\) such that \(K_k(x; n)\) for \(y = n \in \mathbb{Z}_{\geq 0}\) is a Krawtchouk polynomial in the sense of [28, p. 137], namely,

\[
K_k(x; y) = \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{y-x}{k-j}
\]

(see Definition 8.11). In this paper we also call the polynomials \(K_k(x; y)\) of two variables Krawtchouk polynomials. From the observation of the generating functions of \(\text{Cay}_k(x; y)\) and \(K_k(x; y)\), it is immediate that

\[
K_k(x; y) = \frac{\text{Cay}_k(y - 2x; y)}{k!}
\]

(see (8.3) and (8.13)). In the last part of this paper, via the identity (1.32), we deduce the palindromic property of the Krawtchouk polynomials \(\{K_k(x; y)\}_{k=0}^\infty\) from that of the Cayley continuants \(\{\text{Cay}_k(x; y)\}_{k=0}^\infty\) as follows.

**Theorem 1.33 (Theorem 8.19).** The pair \(\{\{K_k(x; y)\}_{k=0}^\infty, \{1\}_{k=0}^\infty\}\) is a palindromic pair with degree \(n\).

See Theorem 8.19 for the palindromic identity. As the associated sequence \(\{1\}_{k=0}^\infty\) for \(\{K_k(x; y)\}_{k=0}^\infty\) is of the form \(1 = 1!\) for all \(k\), the \(n\)th term \(K_n(x; y)\) satisfies a factorial identity. Corollary 1.34 below is the refinement of the factorial identity of \(K_n(x; n)\) on \(S\mathcal{O}_{n+1}(K, n)\) (see Theorem 8.19).

**Corollary 1.34 (Refinement of the factorial identity of \(K_n(x; n)\)).** Given \(n \in \mathbb{Z}_{\geq 0}\), the values of \(K_n(s; n)\) for \(s = 0, 1, 2, \ldots, n\) are given as follows.

\[
K_n(s; n) = \begin{cases} 
1 & \text{if } s = 0, 2, 4, \ldots, n_{\text{even}}, \\
-1 & \text{if } s = 1, 3, 5, \ldots, n_{\text{odd}},
\end{cases}
\]

where \(n_{\text{odd}}\) and \(n_{\text{even}}\) are defined as

\[
n_{\text{odd}} := \begin{cases} 
n & \text{if } n \text{ is odd}, \\
n - 1 & \text{if } n \text{ is even}
\end{cases} \quad \text{and} \quad n_{\text{even}} := \begin{cases} 
n - 1 & \text{if } n \text{ is odd}, \\
n & \text{if } n \text{ is even}.
\end{cases}
\]

We shall give a proof of Corollary 1.34 at the end of Section 8.3 as a corollary of Theorem 8.19. We remark that the identities (1.35) can also be easily shown directly from the definition (1.31) by an elementary observation.

The Krawtchouk polynomials \(\{K_k(x; y)\}_{k=0}^\infty\) are also a classical object at least for the case of \(y = n \in \mathbb{Z}_{\geq 0}\). Therefore Theorem 1.33 may be a known as, for instance, an exercise problem; nonetheless, we could not find it in the literature.
1.10. Summary of the data for palindromic properties. To close this introduction, we summarize some data on the palindromic property of the sequence \( \{ p_k(x; y) \}_{k=0}^{\infty} \) of polynomials in \( (1.14) \). For the associated sequence \( \{ a_k \}_{k=0}^{\infty} \) of \( \{ p_k(x; y) \}_{k=0}^{\infty} \), we write

\[
g_{[x;y]}(t) := \sum_{k=0}^{\infty} p_k(x; y) \frac{a_k}{t^k}
\]

where \( \{ b_k \}_{k=0}^{\infty} \) is some sequence of non-negative integers. Table 2 exhibits the generating function \( g_{[x;y]}(t) \), the associated sequence \( \{ a_k \}_{k=0}^{\infty} \), the sequence \( \{ b_k \}_{k=0}^{\infty} \) of exponents, the degree \( d(n) \), and the sign factor \( \theta_{(p;n)}(s) \) for \( \{ p_k(x; y) \}_{k=0}^{\infty} \) in \( (1.14) \). (For the sign factors \( \theta_{(p;n)}(s) \) and \( \theta_{(Q;n)}(s) \), see \( (7.28) \) and \( (7.40) \), respectively.)

| \( p_k(x; y) \) | \( g_{[x;y]}(t) \) | \( a_k \) | \( b_k \) | \( d(n) \) | \( \theta_{(p;n)}(s) \) |
|-----------------|-----------------|----------|----------|-------------|----------------|
| \( K_k(x; y) \) | \( (1 + t)^{y-x}(1 - t)^x \) | 1 | \( k \) | \( n \) | \( (-1)^s \) |
| \( \text{Cay}_k(x; y) \) | \( (1 + t)^{\frac{x+y}{2}}(1 - t)^{\frac{x-y}{2}} \) | \( k! \) | \( k \) | \( n \) | \( (-1)^{\frac{n-s}{2}} \) |
| \( P_k(x; y) \) | \( u_{[x;y]}(t) \) | \( (2k)! \) | \( 2k \) | \( \frac{n}{2} \) | \( (7.28) \) |
| \( Q_k(x; y) \) | \( v_{[x;y]}(t) \) | \( (2k+1)! \) | \( 2k+1 \) | \( \frac{n-2}{2} \) | \( (7.40) \) |

1.11. Organization. We now outline the rest of this paper. This paper consists of ten sections including this introduction. In Section 2, we overview a general framework established in [27] for the Peter–Weyl theorem \( (1.5) \) for the space of \( \mathcal{K} \)-finite solutions to intertwining differential operators. In Section 3, we collect necessary notation and normalizations for \( \tilde{SL}(3, \mathbb{R}) \). Two identifications \( \Omega^J: \mathfrak{t} \rightarrow \mathfrak{sl}(2, \mathbb{C}) \) for \( J = I, II \) are discussed in this section.

The purpose of Section 4 is to recall from [18] the Heisenberg ultrahyperbolic operator \( \Box_s = R(D_s) \) for \( \tilde{SL}(3, \mathbb{R}) \) and to study the associated differential equation \( d\pi^J_n(D^s_k)p(t) = 0 \). In this section we identify the equation \( d\pi^J_n(D^s_k)p(t) = 0 \) with Heun’s differential equation \( (J = I) \) and the hypergeometric differential equation \( (J = II) \) via the identifications \( \Omega^J: \mathfrak{t} \rightarrow \mathfrak{sl}(2, \mathbb{C}) \). At the end of this section we give a recipe to classify the space \( \text{Sol}(s; n) \) of \( K \)-type solutions to \( \Box_s = R(D_s) \).

In accordance with the recipe given in Section 4, we show the \( K \)-type decompositions of \( \mathcal{Sol}(s; \sigma)_K \) in the hypergeometric model \( d\pi^H_n(D^s_k)p(t) = 0 \) in Section 5. This is accomplished in Theorem 5.36. We then convert the results in the hypergeometric model \( d\pi^H_n(D^s_k)p(t) = 0 \) to the Heun model \( d\pi^H_n(D^s_k)p(t) = 0 \) by a Cayley transform \( \pi_n(k_0) \) in Section 6.

Sections 7 and 8 are devoted to applications to sequences of polynomials. In Section 7, by utilizing the results in the Heun model \( d\pi^H_n(D^s_k)p(t) = 0 \), we investigate two sequences \( \{ P_k(x; n) \}_{k=0}^{\infty} \) and \( \{ Q_k(x; n) \}_{k=0}^{\infty} \) of tridiagonal determinants. The main results of this sections are factorization formulas (Theorems 7.18 and 7.22) and palindromic properties (Theorems 7.29 and 7.41). We also show an expression of \( P_{\frac{n+1}{2}}(x; n) \) for \( n \) odd in terms of the Sylvester determinant \( \text{Sylv}(x; n) \) (Corollary 7.27). In Section 8, we show the palindromic properties for Cayley continuants \( \{ \text{Cay}_k(x; y) \}_{k=0}^{\infty} \) and Krawtchouk polynomials \( \{ K_k(x; y) \}_{k=0}^{\infty} \). These are achieved in Theorems 8.8 and 8.19, respectively.
The last two sections are appendices. In order to study \( \{P_k(x;n)\}_{k=0}^\infty \) and \( \{Q_k(x;n)\}_{k=0}^\infty \), we use some general facts on the coefficients of a power series expression of local Heun functions. We collect those facts in Section 9. In Section 10, for future possible convenience, we give a proof of Sylvester's formula (1.22) from an \( \mathfrak{sl}(2,\mathbb{C}) \) point of view.

2. Peter–Weyl theorem for the space of \( K \)-finite solutions

The aim of this section is to recall from [27, Sect. 2] a general framework established in [27]. In particular, we give a Peter–Weyl theorem for the space of \( K \)-finite solutions to intertwining differential operators between parabolically induced representations. This is done in Theorem 2.11.

2.1. General framework. Let \( G \) be a reductive Lie group with Lie algebra \( \mathfrak{g}_0 \). Choose a Cartan involution \( \theta : \mathfrak{g}_0 \to \mathfrak{g}_0 \) and write \( \mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{s}_0 \) for the Cartan decomposition of \( \mathfrak{g}_0 \) with respect to \( \theta \). Here \( \mathfrak{t}_0 \) and \( \mathfrak{s}_0 \) stand for the +1 and -1 eigenspaces of \( \theta \), respectively. We take maximal abelian subspaces \( \mathfrak{a}_0^{\min} \subset \mathfrak{s}_0 \) and \( \mathfrak{t}_0^{\min} \subset \mathfrak{m}_0^{\min} := Z_{\mathfrak{h}_0}(\mathfrak{a}_0^{\min}) \) such that \( \mathfrak{h}_0 := \mathfrak{a}_0^{\min} \oplus \mathfrak{t}_0^{\min} \) is a Cartan subalgebra of \( \mathfrak{g}_0 \).

For a real Lie algebra \( \mathfrak{n}_0 \), we denote by \( \mathfrak{n} \) the complexification of \( \mathfrak{n}_0 \). For instance, the complexifications of \( \mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{a}_0^{\min}, \) and \( \mathfrak{m}_0^{\min} \) are denoted by \( \mathfrak{g}, \mathfrak{h}, \mathfrak{a}, \) and \( \mathfrak{m}^{\min} \), respectively. We write \( \mathcal{U}(\mathfrak{n}) \) for the universal enveloping algebra of a Lie algebra \( \mathfrak{n} \).

Let \( \Delta \equiv \Delta(\mathfrak{g}, \mathfrak{h}) \) be the set of roots with respect to the Cartan subalgebra \( \mathfrak{h} \) and \( \Sigma \equiv \Sigma(\mathfrak{g}_0, \mathfrak{a}_0) \) denote the set of restricted roots with respect to \( \mathfrak{a}_0 \). We choose a positive system \( \Delta^+ \) and \( \Sigma^+ \) in such a way that \( \Delta^+ \) and \( \Sigma^+ \) are compatible. We denote by \( \rho \) for half the sum of the positive roots.

Let \( \mathfrak{n}_0^{\min} \) be the nilpotent subalgebra of \( \mathfrak{g}_0 \) corresponding to \( \Sigma^+ \), so that \( \mathfrak{p}_0^{\min} := \mathfrak{m}_0^{\min} \oplus \mathfrak{a}_0^{\min} \oplus \mathfrak{n}_0^{\min} \) is a Langlands decomposition of a minimal parabolic subalgebra \( \mathfrak{p}_0^{\min} \) of \( \mathfrak{g}_0 \). Fix a standard parabolic subalgebra \( \mathfrak{p}_0 \supset \mathfrak{p}_0^{\min} \) with Langlands decomposition \( \mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0 \). Let \( P \) be a parabolic subalgebra of \( G \) with Lie algebra \( \mathfrak{p}_0 \). We write \( P = MAN \) for the Langlands decomposition of \( P \) corresponding to \( \mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0 \).

For \( \mu \in \mathfrak{a}^* \cong \text{Hom}_\mathbb{C}(\mathfrak{a}_0, \mathbb{C}) \), we define a one-dimensional representation \( C_\mu \) of \( A \) as \( a \mapsto e^{\mu(a)} \) for \( a \in A \). Then, for a finite-dimensional representation \( W_\sigma = (\sigma, W) \) of \( M \) and weight \( \lambda \in \mathfrak{a}^* \), we define an \( MA \)-representation \( W_{\sigma,\lambda} \) as

\[
W_{\sigma,\lambda} := W_\sigma \otimes C_\lambda.
\]

(We note that the definition of \( W_{\sigma,\lambda} \) was slightly different from the one in [27].) As usual, by letting \( N \) act on \( W_{\sigma,\lambda} \) trivially, we regard \( W_{\sigma,\lambda} \) as a representation of \( P \). We identify the Fréchet space \( C^\infty(G/P, W_{\sigma,\lambda}) \) of smooth sections for the \( G \)-equivariant homogeneous vector bundle \( W_{\sigma,\lambda} := G \times_P W_{\sigma,\lambda} \to G/P \) as \( C^\infty(G/P, W_{\sigma,\lambda}) \cong (C^\infty(G) \otimes W_{\sigma,\lambda})^P \), that is,

\[
C^\infty(G/P, W_{\sigma,\lambda}) \cong \left\{ f \in C^\infty(G) \otimes W_{\sigma,\lambda} : f(\text{gman}) = \sigma(m)^{-1}e^{-\lambda}(a)f(g) \text{ for all } \text{man} \in MAN \right\},
\]

where \( G \) acts by left translation. Then we realize a parabolically induced representation

\[
I_P(\sigma, \lambda) := \text{Ind}_P^G(\sigma \otimes (\lambda + \rho) \otimes 1)
\]  
(2.1)

of \( G \) on \( C^\infty(G/P, W_{\sigma,\lambda+\rho}) \).
For pairs \((\sigma, \lambda), (\eta, \nu)\) of finite-dimensional representations \(\sigma, \eta\) of \(M\) and weights \(\lambda, \nu \in \mathfrak{a}^*\), we write \(\text{Hom}_G(I_P(\sigma, \lambda), I_P(\eta, \nu))\) for the space of intertwining operators from \(I_P(\sigma, \lambda)\) to \(I_P(\eta, \nu)\). Then we set

\[
\text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu)) := \text{Diff}(I_P(\sigma, \lambda), I_P(\eta, \nu)) \cap \text{Hom}_G(I_P(\sigma, \lambda), I_P(\eta, \nu)),
\]

where \(\text{Diff}(I_P(\sigma, \lambda), I_P(\eta, \nu))\) is the space of differential operators from \(I_P(\sigma, \lambda)\) to \(I_P(\eta, \nu)\).

Let \(\text{Irr}(M)_{\text{fin}}\) be the set of equivalence classes of irreducible finite-dimensional representations of \(M\). As \(M\) is not connected in general, we write \(M_0\) for the identity component of \(M\). Let \(\text{Irr}(M/M_0)\) denote the set of irreducible representations of the component group \(M/M_0\). Via the surjection \(M \to M/M_0\), we regard \(\text{Irr}(M/M_0)\) as a subset of \(\text{Irr}(M)_{\text{fin}}\). Let \(\text{id}_\xi\) denote the identity map on \(\xi \in \text{Irr}(M/M_0)\). Lemma 2.2 below shows that the tensored operator \(\mathcal{D} \otimes \text{id}_\xi\) to \(\mathcal{D} \in \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu))\) is also an intertwining differential operator.

**Lemma 2.2** ([27, Lems. 2.17, 2.21]). Let \(\mathcal{D} \in \text{Diff}_G(I_P(\sigma, \lambda), I_P(\eta, \nu))\). For any \(\xi \in \text{Irr}(M/M_0)\), we have

\[
\mathcal{D} \otimes \text{id}_\xi \in \text{Diff}_G(I_P(\sigma \otimes \xi, \lambda), I_P(\eta \otimes \xi, \nu)).
\]

If the \(M\)-representation \(\sigma\) is the trivial character \(\sigma = \chi_{\text{triv}}\), then Lemma 2.2 in particular shows that differential operator \(\mathcal{D} \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\eta, \nu))\) yields

\[
\mathcal{D} \otimes \text{id}_\xi \in \text{Diff}_G(I_P(\xi, \lambda), I_P(\eta \otimes \xi, \nu)) \quad \text{for} \ \xi \in \text{Irr}(M/M_0). \tag{2.3}
\]

### 2.2. Peter–Weyl theorem for \(\text{Sol}_{\text{gr}(\chi, \lambda)}(\xi)_K\)

For a later purpose, we specialize the representations \(\sigma, \eta\) of \(M\) to be \(\sigma = \chi_{\text{triv}}\) and \(\eta = \chi\), where \(\chi\) is some character of \(M\). In this situation, it follows from [23] that, for \(\mathcal{D} \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))\) and \(\xi \in \text{Irr}(M/M_0)\), we have

\[
\mathcal{D} \otimes \text{id}_\xi \in \text{Diff}_G(I_P(\xi, \lambda), I_P(\chi \otimes \xi, \nu)).
\]

By the duality theorem between intertwining differential operators and homomorphisms between generalized Verma modules (see, for instance, [6, 23, 25]), any intertwining differential operator \(\mathcal{D} \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))\) is of the form \(\mathcal{D} = R(u)\) for some \(u \in \mathcal{U}(\mathfrak{n})\), where \(R\) denotes the infinitesimal right translation of \(\mathcal{U}(\mathfrak{g})\) and \(\mathfrak{n}\) is the opposite nilpotent radical to \(\mathfrak{n}\), namely,

\[
\text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu)) = \{R(u) \text{ for some } u \in \mathcal{U}(\mathfrak{n})\}.
\]

In particular, intertwining differential operators \(\mathcal{D}\) are determined by the complex Lie algebra \(\mathfrak{g}\) and independent of its real forms.

It follows from [15, Lem. 2.1] and [27, Lem. 2.24] that there exists a \((\mathcal{U}(\mathfrak{t} \cap \mathfrak{m}), K \cap M)\)-isomorphism

\[
\mathcal{U}(\mathfrak{t}) \otimes_{\mathcal{U}(\mathfrak{t} \cap \mathfrak{m})} C_{\chi_{\text{triv}}} \xrightarrow{\sim} \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} C_{\chi_{\text{triv}}, - (\lambda + \rho)}, \quad u \otimes 1_{\chi_{\text{triv}}} \mapsto u \otimes (1_{\chi_{\text{triv}}} \otimes 1_{- (\lambda + \rho)}), \tag{2.4}
\]

where \(K \cap M\) acts on \(\mathcal{U}(\mathfrak{t}) \otimes_{\mathcal{U}(\mathfrak{t} \cap \mathfrak{m})} C_{\chi_{\text{triv}}}\) and \(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} C_{\chi_{\text{triv}}, - (\lambda + \rho)}\) diagonally. Via the isomorphism (2.4), we define a compact model of \(u^\flat \in \mathcal{U}(\mathfrak{t})\) of \(u \in \mathcal{U}(\mathfrak{n})\) as follows.

**Definition 2.5.** For \(R(u) \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))\) with \(u \in \mathcal{U}(\mathfrak{n})\), we denote by \(u^\flat \in \mathcal{U}(\mathfrak{t})\) an element of \(\mathcal{U}(\mathfrak{t})\) such that the identity

\[
u^\flat \otimes (1_{\chi_{\text{triv}}} \otimes 1_{- (\lambda + \rho)}) = u \otimes (1_{\chi_{\text{triv}}} \otimes 1_{- (\lambda + \rho)}). \tag{2.6}
\]
holds in $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} C_{\chi_{\text{triv}}, -(\lambda + \rho)}$.

**Remark 2.7.** We remark that compact model $u^\flat$ is not unique; any choice of $u^\flat$ satisfying the identity (2.6) is acceptable.

Let $\text{Irr}(K)$ be the set of equivalence classes of irreducible representations of the maximal compact subgroup $K$ with Lie algebra $\mathfrak{k}_0$. For $V_\delta := (\delta, V) \in \text{Irr}(K)$ and $u \in \mathcal{U}(\tilde{\mathfrak{n}})$, we define a subspace $\text{Sol}^{(u)}(\delta)$ of $V_\delta$ by

$$\text{Sol}^{(u)}(\delta) := \{ v \in V_\delta : d\delta(\tau(u^\flat))v = 0 \}.$$  \hfill (2.8)

Here $d\delta$ denotes the differential of $\delta$ and $\tau$ denotes the conjugation $\tau : \mathfrak{g} \to \mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$, that is, $\tau(X_1 + \sqrt{-1}X_2) = X_1 - \sqrt{-1}X_2$ for $X_1, X_2 \in \mathfrak{g}_0$.

**Lemma 2.9** ([27, Lem. 2.41]). The space $\text{Sol}^{(u)}(\delta)$ is a $K \cap M$-representation.

We set

$$\text{Sol}^{\text{fin}}(\delta) := \text{Sol}^{(u)}(\delta) \cap V_\delta^{\text{fin}},$$

where $V_\delta^{\text{fin}}$ is the subspace of $\mathfrak{k} \cap \mathfrak{m}$-invariant vectors of $V_\delta$. Clearly $\text{Sol}^{\text{fin}}(\delta)$ is a $K \cap M$-subrepresentation of $\text{Sol}^{(u)}(\delta)$. Further, since $\mathfrak{k} \cap \mathfrak{m}$ acts on $\xi \in \text{Irr}(M/M_0)$ trivially, we have

$$\text{Hom}_{K \cap M} \left( \text{Sol}^{(u)}(\delta), \xi \right) \neq \{0\} \quad \text{if and only if} \quad \text{Hom}_{K \cap M} \left( \text{Sol}^{\text{fin}}(\delta), \xi \right) \neq \{0\}$$ \hfill (2.10)

via the composition of maps $K \cap M \hookrightarrow M \to M/M_0$.

Let $I_P(\chi_{\text{triv}}, \lambda)_K$ denote the $(\mathfrak{g}, K)$-module consisting of the $K$-finite vectors of $I_P(\chi_{\text{triv}}, \lambda)$. Then, given $R(u) \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))$ and $\xi \in \text{Irr}(M/M_0)$, we set

$$\text{Sol}^{(u, \lambda)}(\xi)_K := \{ f \in I_P(\xi, \lambda)_K : (R(u) \otimes \text{id}_\xi)f = 0 \}.$$  \hfill (2.11)

For the sake of future convenience, we now give a slight modification of a Peter–Weyl theorem [27, Thm. 1.2] for $\text{Sol}^{(u, \lambda)}(\xi)_K$, although such a modification is not necessary for the main objective of this paper. We remark that this modified version is a more direct generalization of the argument given after the proof of [15, Thm. 2.6] than [27, Thm. 1.2].

**Theorem 2.11** (Peter–Weyl theorem for $\text{Sol}^{(u, \lambda)}(\xi)_K$). Let $R(u) \in \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, \nu))$ and $\xi \in \text{Irr}(M/M_0)$. Then the $(\mathfrak{g}, K)$-module $\text{Sol}^{(u, \lambda)}(\xi)_K$ can be decomposed as a $K$-representation as

$$\text{Sol}^{(u, \lambda)}(\xi)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Hom}_{K \cap M} \left( \text{Sol}^{\text{fin}}(\delta), \xi \right).$$ \hfill (2.12)

**Proof.** It follows from [27, Thm. 1.2] that the space $\text{Sol}^{(u, \lambda)}(\xi)_K$ can be decomposed as

$$\text{Sol}^{(u, \lambda)}(\xi)_K \simeq \bigoplus_{\delta \in \text{Irr}(K)} V_\delta \otimes \text{Hom}_{K \cap M} \left( \text{Sol}^{(u)}(\delta), \xi \right).$$

Now the equivalence (2.10) concludes the theorem. \hfill $\square$

When $G$ is a split real group and $P = MAN$ is a minimal parabolic subgroup of $G$, we have $K \cap M = M$, $M/M_0 = M$, and $\mathfrak{k} \cap \mathfrak{m} = \{0\}$. Thus in this case Theorem 2.11 is simplified as follows.
Corollary 2.13. Suppose that $G$ is split real and $P = MAN$ is minimal parabolic subgroup of $G$. Let $R(u) ∈ \text{Diff}_G(I_P(\chi_{\text{triv}}, \lambda), I_P(\chi, ν))$ and $σ ∈ \text{Irr}(M)$. Then the $(g, K)$-module $\text{Sol}_{(u, \lambda)}(σ)_K$ can be decomposed as

$$\text{Sol}_{(u, \lambda)}(σ)_K ≅ \bigoplus_{δ ∈ \text{Irr}(K)} V_δ ⊗ \text{Hom}_M (\text{Sol}_u(δ), σ) \quad (2.14)$$

3. Specialization to $(\widetilde{SL}(3, \mathbb{R}), B)$

The purpose of this section is to specialize the general theory discussed in Section 2 to a pair $(\widetilde{SL}(3, \mathbb{R}), B)$, where $B$ is a minimal parabolic subgroup of $\widetilde{SL}(3, \mathbb{R})$. In this section we in particular give a recipe for computing $K$-type formulas of the space $\text{Sol}_{(u, \lambda)}(σ)_K$ of $K$-finite solutions for $R(u) ∈ \text{Diff}_G(I_B(\chi_{\text{triv}}, \lambda), I_B(\chi, ν))$. It is described in Section 3.7.

3.1. Notation and normalizations. We begin by recalling from [27, Sect. 4.1] the notation and normalizations for $\widetilde{SL}(3, \mathbb{R})$. Let $G = \widetilde{SL}(3, \mathbb{R})$ with Lie algebra $g_0 = \mathfrak{sl}(3, \mathbb{R})$. Fix a Cartan involution $θ : g_0 → g_0$ such that $θ(U) = −U^t$. We write $\mathfrak{t}_0$ and $\mathfrak{s}_0$ for the +1 and −1 eigenspaces of $θ$, respectively, so that $g_0 = \mathfrak{t}_0 ⊕ \mathfrak{s}_0$ is a Cartan decomposition of $g_0$. We put $a_0 := \text{span}_R \{E_{ii} − E_{i+1,i+1} : i = 1, 2\}$ and $n_0 := \text{span}_R \{E_{12}, E_{23}, E_{13}\}$, where $E_{ij}$ denote matrix units. Then $b_0 := a_0 ⊕ n_0$ is a minimal parabolic subalgebra of $g_0$.

Let $K, A,$ and $N$ be the analytic subgroups of $G$ with Lie algebras $\mathfrak{k}_0, \mathfrak{a}_0,$ and $\mathfrak{n}_0$, respectively. Then $G = KAN$ is an Iwasawa decomposition of $G$. We write $M = Z_K(\mathfrak{a}_0)$, so that $B := MAN$ is a minimal parabolic subgroup of $G$ with Lie algebra $\mathfrak{b}_0$.

We denote by $\mathfrak{g}$ the complexification of the Lie algebra $g_0$ of $G$. A similar convention is employed also for subgroups of $G$; for instance, $\mathfrak{b} = \mathfrak{a} ⊕ \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. We write $\tilde{\mathfrak{n}}$ for the nilpotent radical opposite to $\mathfrak{n}$.

Let $Δ ≡ Δ(\mathfrak{g}, \mathfrak{a})$ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We denote by $Δ^+$ and $Π$ the positive system corresponding to $\mathfrak{b}$ and the set of simple roots of $Δ^+$, respectively. We have $Π = \{ε_1 − ε_2, \varepsilon_2 − \varepsilon_3\}$, where $ε_j$ are the dual basis of $E_{jj}$ for $j = 1, 2, 3$. The root spaces $g_{ε_1 − ε_2}$ and $g_{ε_2 − ε_3}$ are then given as $g_{ε_1 − ε_2} = \mathbb{C}E_{12}$ and $g_{ε_2 − ε_3} = \mathbb{C}E_{23}$. We write $ρ$ for half the sum of the positive roots, namely, $ρ = \varepsilon_1 − \varepsilon_3$.

We define $X, Y ∈ \mathfrak{g}$ as

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.1)$$

Then $X$ and $Y$ are root vectors for $−(ε_1 − ε_2)$ and $−(ε_2 − ε_3)$, respectively. The opposite nilpotent radical $\tilde{\mathfrak{n}}$ is thus given as $\tilde{\mathfrak{n}} = \text{span}\{X, Y, [X, Y]\}$.

3.2. Two identifications for $\mathfrak{so}(3, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C})$. As $\mathfrak{t}_0 ≅ \mathfrak{so}(3) ≅ \mathfrak{su}(2)$, we have $\mathfrak{k} = \mathfrak{so}(3, \mathbb{C}) ≅ \mathfrak{sl}(2, \mathbb{C})$. For later applications we consider two identifications of $\mathfrak{so}(3, \mathbb{C})$ with $\mathfrak{sl}(2, \mathbb{C})$:

$$Ω^J : \mathfrak{so}(3, \mathbb{C}) \xrightarrow{\cong} \mathfrak{sl}(2, \mathbb{C}) \quad \text{for} \ J = I, II.$$

Loosely speaking, these identifications are described as follows.
(1) Identification $\Omega^I$: $\mathfrak{so}(3, \mathbb{C}) \xrightarrow{\Omega^I} \mathfrak{sl}(2, \mathbb{C})$: This is an identification via a Lie algebra isomorphism $\Omega^I_0: \mathfrak{so}(3) \xrightarrow{\sim} \mathfrak{u}(2)$. Schematically, we have

\[
\begin{array}{ccc}
\mathfrak{so}(3, \mathbb{C}) & \xrightarrow{\Omega^I} & \mathfrak{sl}(2, \mathbb{C}) \\
\otimes_{\mathbb{C}} & \text{f.f.} & \otimes_{\mathbb{C}} \\
\mathfrak{so}(3) & \xrightarrow{\sim} & \mathfrak{su}(2)
\end{array}
\]

(2) Identification $\Omega^II$: $\mathfrak{so}(3, \mathbb{C}) \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C})$: This is an identification independent of $\mathfrak{su}(2)$. Schematically, we have

\[
\begin{array}{ccc}
\mathfrak{so}(3, \mathbb{C}) & \xrightarrow{\Omega^II} & \mathfrak{sl}(2, \mathbb{C}) \\
\otimes_{\mathbb{C}} & \text{f.f.} & \otimes_{\mathbb{C}} \\
\mathfrak{so}(3) & \xrightarrow{\sim} & \mathfrak{su}(2)
\end{array}
\]

Let $E_+, E_-$, and $E_0$ be the elements of $\mathfrak{sl}(2, \mathbb{C})$ defined as

\[
E_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(3.2)

We now describe the identifications $\Omega^I$ and $\Omega^II$ in detail separately.

3.2.1. Identification $\mathfrak{k} \simeq \mathfrak{sl}(2, \mathbb{C})$ via $\Omega^I$. We start with the identification $\Omega^I: \mathfrak{k} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C})$. First observe that $\mathfrak{k}_0 = \mathfrak{so}(3)$ is spanned by the three matrices

\[
B_1 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

with commutation relations

\[
[B_1, B_2] = B_3, \quad [B_1, B_3] = -B_2, \quad \text{and} \quad [B_2, B_3] = B_1.
\]

(3.3)

On the other hand, the Lie algebra $\mathfrak{su}(2)$ is spanned by

\[
A_1 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 := \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}
\]

with commutation relations

\[
[A_1, A_2] = 2A_3, \quad [A_1, A_3] = -2A_2, \quad \text{and} \quad [A_2, A_3] = 2A_1.
\]

Then one may identify $\mathfrak{k}_0$ with $\mathfrak{su}(2)$ via the linear map

\[
\Omega^I_0: \mathfrak{k}_0 \xrightarrow{\sim} \mathfrak{su}(2), \quad B_j \mapsto \frac{1}{2} A_j \quad \text{for} \ j = 1, 2, 3.
\]

(3.4)

Let $Z_+, Z_-, Z_0$ be the elements of $\mathfrak{k} = \mathfrak{so}(3, \mathbb{C})$ defined as

\[
Z_+ := B_2 - \sqrt{-1}B_3, \quad Z_- := -(B_2 + \sqrt{-1}B_3), \quad Z_0 := [Z_+, Z_-] = -2\sqrt{-1}B_1.
\]

(3.5)

Then we have

\[
\mathfrak{k} = \text{span}\{Z_+, Z_-, Z_0\}.
\]

(3.6)
As \( A_2 - \sqrt{-1}A_3 = 2E_+ \) and \(- (A_2 + \sqrt{-1}A_3) = 2E_- \), the isomorphism (3.4) yields a Lie algebra isomorphism
\[
\Omega^I: \mathfrak{t} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C}), \quad Z_j \mapsto E_j \quad \text{for } j = +, -, 0. \tag{3.7}
\]

3.2.2. Identification \( \mathfrak{t} \simeq \mathfrak{sl}(2, \mathbb{C}) \) via \( \Omega^I \). We next discuss the identification \( \Omega^I: \mathfrak{t} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C}) \). Let \( W_+, W_-, \) and \( W_0 \) be the elements of \( \mathfrak{so}(3, \mathbb{C}) \) defined by
\[
W_+ := B_1 + \sqrt{-1}B_3, \quad W_- := -B_1 + \sqrt{-1}B_3, \quad W_0 := [W_+, W_-] = -2\sqrt{-1}B_2. \tag{3.8}
\]
Then we have
\[
\mathfrak{t} = \text{span}\{W_+, W_-, W_0\}. \tag{3.9}
\]
It follows from the commutation relations (3.3) that the triple \( \{W_+, W_-, W_0\} \) forms an \( \mathfrak{sl}(2) \)-triple, namely, \([W_+, W_-] = W_0, [W_0, W_+] = 2W_+, \) and \([W_0, W_-] = -2W_- \). Therefore, \( \mathfrak{t} \) may be identified with \( \mathfrak{sl}(2, \mathbb{C}) \) via the Lie algebra isomorphism
\[
\Omega^I: \mathfrak{t} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C}), \quad W_j \mapsto E_j \quad \text{for } j = +, -, 0. \tag{3.10}
\]

3.3. A realization of the subgroup \( M = Z_K(a_0) \). As \( K \) is isomorphic to \( SU(2) \), we realize \( M = Z_K(a_0) \) as a subgroup of \( SU(2) \) via the isomorphism (3.7). To do so, first observe that the adjoint action \( \text{Ad of } SU(2) \) on \( \mathfrak{su}(2) \) yields a two-to-one covering map \( SU(2) \rightarrow \text{Ad}(SU(2)) \simeq SO(3) \) on \( SO(3) \). We realize \( \text{Ad}(SU(2)) \) as a matrix group with respect to the ordered basis \( \{A_2, A_1, A_3\} \) of \( \mathfrak{su}(2) \) in such a way that the elements \( \pm m_j^1 \in SU(2) \) with
\[
m_0^1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_1^1 := \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad m_2^1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad m_3^1 := \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \tag{3.11}
\]
are mapped to \( \pm m_j^1 \mapsto m_j \in SO(3) \) for \( j = 0, 1, 2, 3 \), where
\[
m_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad m_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

One can easily check that the map \( \pm m_j^1 \mapsto m_j \) respects the Lie algebra isomorphism \( \Omega^I: \mathfrak{t} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C}) \) in (3.7) ([27 Lem. 4.6]). Namely, for \( Z \in \mathfrak{t} \), we have
\[
\Omega^I(\text{Ad}(m_j)Z) = \text{Ad}(m_j^1)\Omega^I(Z) \quad \text{for } j = 0, 1, 2, 3.
\]
As \( Z_{SO(3)}(a_0) = \{m_0, m_1, m_2, m_3\} \), we then realize \( M \) as a subgroup of \( SU(2) \) as
\[
M = \{ \pm m_0^1, \pm m_1^1, \pm m_2^1, \pm m_3^1 \}.
\]
The subgroup \( M \) is isomorphic to the quaternion group \( Q_8 \), a non-commutative group of order 8.

3.4. Irreducible representations \( \text{Irr}(M) \) of \( M \). In order to compute a \( K \)-type formula via (2.14), we next discuss the sets \( \text{Irr}(M) \) and \( \text{Irr}(K) \) of equivalence classes of irreducible representations of \( M \) and \( K \), respectively. We first consider \( \text{Irr}(M) \) via the isomorphism (3.7). As \( M \) is isomorphic to the quaternion group \( Q_8 \), the set \( \text{Irr}(M) \) consists of four characters and one 2-dimensional irreducible
Table 3. Character table for $(\varepsilon, \varepsilon')$ for $m_j^1$

|       | $\pm m_j^0$ | $\pm m_j^1$ | $\pm m_j^2$ | $\pm m_j^3$ |
|-------|-----------|-----------|-----------|-----------|
| $(+,+)$ | 1     | 1     | 1     | 1     |
| $(+,-)$ | 1     | -1    | -1    | 1     |
| $(-,+)$ | 1     | -1    | 1     | -1    |
| $(-,-)$ | 1     | 1     | -1    | -1    |

representation. For $\varepsilon, \varepsilon' \in \{\pm\}$, we define a character $\chi^{SO(3)}_{\varepsilon, \varepsilon'} : Z_{SO(3)}(a_0) \to \{\pm 1\}$ of $Z_{SO(3)}(a_0)$ as

$$
\chi^{SO(3)}_{\varepsilon, \varepsilon'}(\text{diag}(a_1, a_2, a_3)) := |a_1|_\varepsilon |a_3|_{\varepsilon'},
$$

where $|a|_+ := |a|$ and $|a|_- := a$. Via the character $\chi^{SO(3)}$ of $Z_{SO(3)}(a_0)$, we define a character $\chi(\varepsilon, \varepsilon') : M \to \{\pm 1\}$ of $M$ as

$$
\chi(\varepsilon, \varepsilon')(\pm m_j^1) := \chi(\varepsilon, \varepsilon')(m_j) \quad \text{for} \quad j = 0, 1, 2, 3. \tag{3.12}
$$

We often abbreviate $\chi(\varepsilon, \varepsilon')$ as $(\varepsilon, \varepsilon')$. The character $(+,+)$, for instance, is the trivial character of $M$. Table 3 illustrates the character table for $(\varepsilon, \varepsilon') = \chi(\varepsilon, \varepsilon')$. The set $\text{Irr}(M)$ may then be described as follows:

$$
\text{Irr}(M) = \{(+,+), (+,-), (-,+), (-,-), \mathbb{H}\}, \tag{3.13}
$$

where $\mathbb{H}$ is the unique genuine 2-dimensional representation of $M \simeq Q_8$.

3.5. **Irreducible representations** $\text{Irr}(K)$ of $K$. We next consider a polynomial realization of $\text{Irr}(K)$. Let $\text{Pol}[t]$ be the space of polynomials of one variable $t$ with complex coefficients and set

$$
\text{Pol}_n[t] := \{p(t) \in \text{Pol}[t] : \text{deg} p(t) \leq n\}.
$$

Then we realize the set $\text{Irr}(K)$ of equivalence classes of irreducible representations of $K \simeq SU(2)$ as

$$
\text{Irr}(K) \simeq \{(\pi_n, \text{Pol}_n[t]) : n \in \mathbb{Z}_{\geq 0}\}, \tag{3.14}
$$

where the representation $\pi_n$ of $SU(2)$ on $\text{Pol}_n[t]$ is defined as

$$
(\pi_n(g)p)(t) := (ct + d)^n p \left( \frac{at + b}{ct + d} \right) \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}. \tag{3.15}
$$

It follows from (3.15) that the elements $m_j^1$ defined in (3.11) act on $\text{Pol}_n[t]$ via $\pi_n$ as

$$
m_1^1 : p(t) \mapsto (\sqrt{-1})^n p(-t); \quad m_2^1 : p(t) \mapsto t^n p \left( \frac{-1}{t} \right); \quad m_3^1 : p(t) \mapsto (-\sqrt{-1})^n p \left( \frac{1}{t} \right). \tag{3.16}
$$

Let $d\pi_n$ be the differential of the representation $\pi_n$. As usual we extend $d\pi_n$ complex-linearly to $\mathfrak{sl}(2, \mathbb{C})$ and also naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$. It follows from (3.2) and (3.15) that $E_+, E_-$, and $E_0$ act on $\text{Pol}_n[t]$ via $d\pi_n$ as

$$
d\pi_n(E_+) = -\frac{d}{dt}, \quad d\pi_n(E_-) = -nt + t^2 \frac{d}{dt}, \quad \text{and} \quad d\pi_n(E_0) = -2t \frac{d}{dt} + n. \tag{3.17}
$$
3.6. Peter–Weyl theorem for $\mathcal{S}ol_{(u;\lambda)}(\sigma)_K$ with the polynomial realization of $\text{Irr}(K)$. Let $\Omega: \mathfrak{k} \cong \mathfrak{sl}(2, \mathbb{C})$ be an identification of $\mathfrak{k}$ with $\mathfrak{sl}(2, \mathbb{C})$ such as (3.7) and (3.10). Then elements $F \in \mathcal{U}(\mathfrak{k})$ can act on $\text{Pol}_n[t]$ via $d\pi_n$ as $d\pi_n(\Omega(F))$. To simplify the notation we write

$$d\pi_n^\Omega(F) = d\pi_n(\Omega(F)) \quad \text{for } F \in \mathcal{U}(\mathfrak{k}).$$

(3.18)

Then, for $R(u) \in \text{Diff}_G(I((+,+), \lambda), I(\chi, \nu))$ with $(+, +)$ the trivial character of $M$, we set

$$\text{Sol}_{(u)}(n) := \{p(t) \in \text{Pol}_n[t] : d\pi_n^\Omega(\tau(u^h))p(t) = 0\},$$

(3.19)

where $u^h \in \mathcal{U}(\mathfrak{h})$ is a compact model of $u \in \mathcal{U}(\mathfrak{n})$ (see Definition 2.5). Then the $K$-type decomposition of $\text{Sol}_{(u;\lambda)}(\sigma)_K$ in (2.14) with the polynomial realization (3.14) of $\text{Irr}(K)$ becomes

$$\text{Sol}_{(u;\lambda)}(\sigma)_K \simeq \bigoplus_{n \geq 0} \text{Pol}_n[t] \otimes \text{Hom}_M(\text{Sol}_{(u)}(n), \sigma).$$

(3.20)

We remark that since $K \cap M = M$ and $\mathfrak{m} = \{0\}$, the $(\mathcal{U}(\mathfrak{k} \cap \mathfrak{m}), K \cap M)$-isomorphism (2.4) reduces to an $M$-isomorphism $\mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{h})} \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathcal{U}(\mathfrak{g})$. In particular a compact model $u^h \in \mathcal{U}(\mathfrak{k})$ of $u \in \mathcal{U}(\mathfrak{n})$ is indeed unique in the present situation (see Remark 2.7).

3.7. Recipe for determining the $K$-type formula for $\text{Sol}_{(u;\lambda)}(\sigma)_K$. For later convenience we summarize a recipe for computing the $K$-type formula for $\text{Sol}_{(u;\lambda)}(\sigma)_K$ via (3.20). Let $R(u) \in \text{Diff}_G(I((+,+), \lambda), I(\chi, \nu))$.

Step 0: Choose an identification $\Omega: \mathfrak{k} \cong \mathfrak{sl}(2, \mathbb{C})$.

Step 1: Find the compact model $u^h$ of $u$ (see Definition 2.5).

Step 2: Find the explicit formula of the differential operator $d\pi_n^\Omega(\tau(u^h))$.

Step 3: Solve the differential equation $d\pi_n^\Omega(\tau(u^h))p(t) = 0$ and classify $n \in \mathbb{Z}_{\geq 0}$ such that $\text{Sol}_{(u)}(n) \neq \{0\}$.

Step 4: For $n \in \mathbb{Z}_{\geq 0}$ with $\text{Sol}_{(u)}(n) \neq \{0\}$, classify the $M$-representations on $\text{Sol}_{(u)}(n)$.

Step 5: Given $\sigma \in \text{Irr}(M)$, classify $n \in \mathbb{Z}_{\geq 0}$ with $\text{Sol}_{(u)}(n) \neq \{0\}$ such that

$$\text{Hom}_M(\text{Sol}_{(u)}(n), \sigma) \neq \{0\}.$$
Let $X$ and $Y$ be the elements of $\mathfrak{n}$ defined in (3.1). For $s \in \mathbb{C}$, we define $D_s$ as
$$D_s := (XY + YX) + s[X,Y] \in U(\mathfrak{g}).$$

Then the second order differential operator
$$R(D_s) = R((XY + YX) + s[X,Y])$$
is called the Heisenberg ultrahyperbolic operator for $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. For the general definition for $\mathfrak{sl}(m, \mathbb{C})$ with arbitrary rank $m \geq 3$, see [18, Sect. 3].

Recall from Section 3.1 that the simple roots $\alpha, \beta \in \Pi$ are realized as $\alpha = \varepsilon_1 - \varepsilon_2$ and $\beta = \varepsilon_2 - \varepsilon_3$. Then, for $s \in \mathbb{C}$, we set
$$\tilde{\rho}(s) := \tilde{\rho} - s\tilde{\rho}^\perp$$
with
$$\tilde{\rho} := \frac{1}{2}(\varepsilon_1 - \varepsilon_3) \quad \text{and} \quad \tilde{\rho}^\perp := \frac{1}{2}(\varepsilon_1 + \varepsilon_3).$$

Remark that as $\tilde{\rho}^\perp \notin \mathfrak{a}^*$, we have $\rho(s) \notin \mathfrak{a}^*$ unless $s = 0$. For $\sigma \in \text{Irr}(M)$, we write
$$I(\sigma, \tilde{\rho}(s)) = I_B(\sigma, \tilde{\rho}(s)|\mathfrak{a}^*)$$
for the parabolically induced representation $I_B(\sigma, \tilde{\rho}(s)|\mathfrak{a}^*)$ defined as in (2.1). Proposition 4.3 below shows that the operator $R(D_s)$ is indeed an intertwining differential operator between parabolically induced representations.

**Proposition 4.3.** We have
$$R(D_s) \in \text{Diff}_G(I((+,+), -\tilde{\rho}(s)), I((+-), \tilde{\rho}(-s))).$$
Consequently, for $\sigma \in \text{Irr}(M)$, we have
$$R(D_s) \otimes \text{id}_\sigma \in \text{Diff}_G(I(\sigma, -\tilde{\rho}(s)), I((-\sigma), \tilde{\rho}(-s))).$$

**Proof.** The first assertion readily follows from [18, Lem. 3.2] and [27, Lem. 6.4]. Lemma 2.2 then concludes the second. □

Our aim is to compute the branching law of the space $\text{Sol}_{(D_s; -\tilde{\rho}(s))}(\sigma)_K$ of $K$-finite solutions to $(R(D_s) \otimes \text{id}_\sigma)f = 0$. The $K$-type formula in (4.4) with $(w; \lambda) = (D_s; -\tilde{\rho}(s))$ gives
$$\text{Sol}_{(D_s; -\tilde{\rho}(s))}(\sigma)_K \simeq \bigoplus_{n \geq 0} \text{Pol}_n[t] \otimes \text{Hom}_M(\text{Sol}_{(D_s)}(n), \sigma)$$
with
$$\text{Sol}_{(D_s)}(n) = \{p(t) \in \text{Pol}_n[t] : d\pi_n^M(\tau(D_s^\flat))p(t) = 0\}.$$ 

Observe that, for $X$ and $Y$ in (3.1), we have $X, Y \in \mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{R})$. Thus $\tau(D_s^\flat)$ for $D_s = (XY + YX) + s[X,Y]$ is simply given as
$$\tau(D_s^\flat) = \tau(D_s)^\flat = D_s^\flat,$$
where $\bar{s}$ denotes the complex conjugate of $s \in \mathbb{C}$. Then we write
$$\text{Sol}(s; n) := \text{Sol}_{(D_s)}(n)$$
Similarly, we put

\[ K = \{ p(t) \in \text{Pol}_n[t] : d^n_{\pi_n}(\tau(D^h_s)) p(t) = 0 \} \]

where

\[ d^n_{\pi_n}(D^h_s) p(t) = 0. \]

Similarly, we put

\[ \text{Sol}(s; \sigma)_K := \text{Sol}(D_{z; \bar{p}(s)}^{(\sigma)}). \]

Then the \( K \)-type formula (4.4) yields

\[ \text{Sol}(s; \sigma)_K \cong \bigoplus_{n \geq 0} \text{Pol}_n[t] \otimes \text{Hom}_M (\text{Sol}(s; n), \sigma). \]

Hereafter we consider the branching law of \( \text{Sol}(s; \sigma)_K \) instead of \( \text{Sol}(D_{z; \bar{p}(s)}^{(\sigma)}). \)

In order to solve the equation \( d^n_{\pi_n}(D^h_s) p(t) = 0 \) in (4.6), we next find the compact model \( D^h_s \) of \( D_s \), which is an element of \( U(\mathfrak{f}) \) such that

\[ D^h_s \otimes (\mathbb{1}_{(+,+)} \otimes \mathbb{1}_{\bar{p}(s) - \rho}) = D_s \otimes (\mathbb{1}_{(+,+)} \otimes \mathbb{1}_{\bar{p}(s) - \rho}) \]

in \( U(\mathfrak{g}) \otimes U(\mathfrak{b}) \mathbb{C}_{(+,+), \bar{p}(s) - \rho}. \) For \( X, Y \in \mathfrak{f} \) in (3.1), define \( X^\flat, Y^\flat \in \mathfrak{f} \) as

\[ X^\flat := X + \theta(X) \quad \text{and} \quad Y^\flat := Y + \theta(Y), \]

where \( \theta \) is the Cartan involution defined as \( \theta(U) = -U^t \).

**Lemma 4.8.** We have

\[ D^h_s = X^\flat Y^\flat + Y^\flat X^\flat + s[X^\flat, Y^\flat] \in U(\mathfrak{f}). \]

**Proof.** One can easily verify that

\[ (X^\flat Y^\flat + Y^\flat X^\flat + s[X^\flat, Y^\flat]) \otimes (\mathbb{1}_{(+,+)} \otimes \mathbb{1}_{\bar{p}(s) - \rho}) = D_s \otimes (\mathbb{1}_{(+,+)} \otimes \mathbb{1}_{\bar{p}(s) - \rho}) \]

in \( U(\mathfrak{g}) \otimes U(\mathfrak{b}) \mathbb{C}_{(+,+), \bar{p}(s) - \rho}. \)

\[ \square \]

**4.2.** Relationship between \( d\pi^I_n(D^h_s) \) and \( d\pi^I_n(D^h_s) \). In Section 3.2, we discussed two identifications of \( \mathfrak{f} \) with \( \text{sl}(2, \mathbb{C}) \), namely,

\[ \Omega^I : \mathfrak{f} \rightleftharpoons \text{sl}(2, \mathbb{C}) \quad \text{and} \quad \Omega^I : \mathfrak{f} \rightleftharpoons \text{sl}(2, \mathbb{C}). \]

Thus compact model \( D^h_s \) may act on \( \text{Pol}_n[t] \) via \( d\pi_n \) as

\[ d\pi_n(\Omega^I( (D^h_s))) \quad \text{and} \quad d\pi_n(\Omega^I( (D^h_s))). \]

As for the notation \( d\pi^I_n(F) = d\pi_n(\Omega(F)) \) in (3.15), we abbreviate (4.11) as

\[ d\pi^I_n(D^h_s) = d\pi_n(\Omega^I(D^h_s)) \quad \text{for} \ J \in \{ I, \ II \}. \]

We next discuss a relationship between \( d\pi^I_n(D^h_s) \) and \( d\pi^I_n(D^h_s) \).

We begin with the expression of \( \Omega^I(D^h_s) \) in terms of the \( \text{sl}(2) \)-triple \( \{ E_+, E_-, E_0 \} \) in (3.2). First, recall from (3.6) and (3.9) that \( \mathfrak{f} \) can be given as

\[ \mathfrak{f} = \text{span}\{ Z_+, Z_-, Z_0 \} = \text{span}\{ W_+, W_-, W_0 \}. \]


where \( \{Z_+, Z_-, Z_0\} \) and \( \{W_+, W_-, W_0\} \) are the \( \mathfrak{sl}(2) \)-triples of \( \mathfrak{t} \) defined in (3.5) and (3.8), respectively.

**Lemma 4.12.** The compact model \( D^b_s = X^b Y^b + Y^b X^b + s[X^b, Y^b] \) is expressed as

\[
D^b_s = \frac{1}{2} ((W_+ + W_-)W_0 - (s - 1)(W_+ - W_-)).
\]  

**(Proof.** A direct computation shows that \( X^b \) and \( Y^b \) are expressed in terms of \( \{Z_+, Z_-, Z_0\} \) and \( \{W_+, W_-, W_0\} \) as

\[
X^b = \frac{1}{2} (Z_+ + Z_-) = -\sqrt{-1} \frac{1}{2}(W_+ + W_-),
\]
\[
Y^b = \frac{1}{2} (Z_+ - Z_-) = \sqrt{-1} \frac{1}{2} W_0.
\]

By substituting these expressions into \( D^b_s \), one obtains the lemma. \( \square \)

**Lemma 4.15.** The elements \( \Omega^I(D^b_s) \) and \( \Omega^{II}(D^b_s) \) in \( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C})) \) are given as

\[
\Omega^I(D^b_s) = \frac{1}{2} ((E_+ + E_-)(E_+ - E_-) - (s - 1)E_0),
\]
\[
\Omega^{II}(D^b_s) = \frac{1}{2} ((E_+ + E_-)E_0 - (s - 1)(E_+ - E_-)).
\]

**(Proof.** This follows from (3.7), (3.10), and Lemma 4.12. \( \square \)

We set

\[
k_0 := \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{-1} & 1 \\ -1 & -\sqrt{-1} \end{pmatrix} \in SU(2).
\]  

**Proposition 4.19.** We have

\[
\Omega^I(D^b_s) = \text{Ad}(k_0)\Omega^{II}(D^b_s).
\]  

Consequently, for \( p(t) \in \text{Pol}_n[t] \), the following are equivalent:

(i) \( d\pi^J_n(D^b_s)p(t) = 0 \);

(ii) \( d\pi^J_n(D^b_s)\pi_n(k_0)p(t) = 0 \).

**(Proof.** One can easily check that

\[
E_+ + E_- = -\text{Ad}(k_0)(E_+ + E_-),
\]
\[
E_+ - E_- = \sqrt{-1}\text{Ad}(k_0)E_0.
\]

Now the identity (4.20) follows from Lemma 4.15. Since \( d\pi^J_n(D^b_s) = d\pi_n(\Omega^J(D^b_s)) \) for \( J \in \{I, II\} \), the equivalence between (i) and (ii) readily follows from (4.20). \( \square \)

For \( J \in \{I, II\} \), we write

\[
\text{Sol}_J(s; n) = \{ p(t) \in \text{Pol}_n[t] : d\pi^J_n(D^b_s)p(t) = 0 \}.
\]  

(4.21)
In place of $\text{Sol}(s; n)$ with $\text{Sol}_J(s; n)$, the $K$-type decomposition (4.7) becomes

$$\text{Sol}(s; \sigma)_K \simeq \bigoplus_{n \geq 0} \text{Pol}_n[t] \otimes \text{Hom}_M(\text{Sol}_J(s; n), \sigma).$$

(4.22)

It follows from Proposition 4.19 that $\pi_n(k_0)$ yields a linear isomorphism

$$\pi_n(k_0): \text{Sol}(s; n) \overset{\sim}{\longrightarrow} \text{Sol}_I(s; n).$$

(4.23)

For $m_j^I \in M$ for $j = 0, 1, 2, 3$ defined in (3.11), we define $m_j^II \in M$ as

$$m_j^II := k_{-1}^0 m_j^I k_0$$

so that the following diagram commutes.

A direct computation shows that $m_j^II$ are given as follows.

$$m_0^II = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_1^II = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad m_2^II = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad m_3^II = -\begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.$$

(4.25)

By (3.11) and (4.25), we have

$$m_0^II = m_0^I, \quad m_1^II = m_2^I, \quad m_2^II = m_1^I, \quad m_3^II = -m_3^I.$$

(4.26)

It then follows from (3.16) that $m_j^II$ act on $\text{Pol}_n[t]$ via $d\pi_n$ as

$$m_1^II: p(t) \mapsto t^n p \left(-\frac{1}{t}\right); \quad m_2^II: p(t) \mapsto (\sqrt{-1})^n p(-t); \quad m_3^II: p(t) \mapsto (\sqrt{-1}t)^n p \left(\frac{1}{t}\right).$$

(4.27)

Table 4 is the character table for $(\varepsilon, \varepsilon')$ with $m_j^II$.

### Table 4. Character table for $(\varepsilon, \varepsilon')$ for $m_j^II$

|   | $\pm m_0^II$ | $\pm m_1^II$ | $\pm m_2^II$ | $\pm m_3^II$ |
|---|--------------|--------------|--------------|--------------|
| 1 | 1            | 1            | 1            | 1            |
| 2 | 1            | -1           | -1           | 1            |
| 3 | 1            | -1           | 1            | -1           |
| 4 | 1            | 1            | -1           | -1           |

4.3. **Differential equation** $d\pi_n^J(D_s^\emptyset)f(t) = 0$ for $J \in \{I, II\}$. As indicated in the recipe in Section 3.7 to determine the $K$-type decomposition of $\text{Sol}(s; n)_K$, it is crucial to determine the space $\text{Sol}_J(s; n)$ of $K$-type solutions, for which one needs to find polynomial solutions $p(t) \in \text{Pol}_n[t]$ to differential equations $d\pi_n^J(D_s^\emptyset)f(t) = 0$ for $J \in \{I, II\}$. For this purpose we next investigate these differential equations.
4.3.1. **Differential equation** $d\pi_n^I(D_s^h)f(t) = 0$. We start with the differential equation $d\pi_n^I(D_s^h)f(t) = 0$. The explicit formula of $d\pi_n^I(D_s^h)$ is given as follows.

**Lemma 4.28.** We have

$$-2\sqrt{-1} d\pi_n^I(D_s^h) = (1 - t^4)\frac{d^2}{dt^2} + 2 ((n - 1)t^2 + s) t \frac{d}{dt} - n ((n - 1)t^2 + s).$$

(4.29)

**Proof.** Recall from (4.16) that we have

$$-2\sqrt{-1} \Omega^I(D_s^h) = (E_+ + E_-)(E_+ - E_-) - (s - 1)E_0.$$

Now the proposed identity follows from a direct computation with (3.17).

In order to study the differential equation $d\pi_n^I(D_s^h)f(t) = 0$, we set

$$\mathcal{D}_H(a, q; \alpha, \beta, \gamma, \delta; z) := \frac{d^2}{dz^2} + \left(\gamma \frac{\delta}{z - 1} + \frac{\varepsilon}{z - a}\right) \frac{d}{dz} + \frac{\alpha\beta z - q}{z(z - 1)(z - a)},$$

(4.30)

where $a, q, \alpha, \beta, \gamma, \delta, \varepsilon$ are complex parameters with $a \neq 0, 1$ and $\gamma + \delta + \varepsilon = \alpha + \beta + 1$. Then the differential equation

$$\mathcal{D}_H(a, q; \alpha, \beta, \gamma, \delta; z)f(z) = 0$$

(4.31)

is called **Heun’s differential equation**. For a brief account of the equation (4.31), see Section 9.1. The differential equation $d\pi_n^I(D_s^h)f(t) = 0$ can be identified with Heun’s differential equation as in Lemma 4.32 below. (We are grateful to Hiroyuki Ochiai for pointing it out.)

**Lemma 4.32.** We have

$$-\frac{d\pi_n^I(D_s^h)}{4} = \mathcal{D}_H(-1, -\frac{ns}{4}; -\frac{n}{2}, -\frac{n - 1}{2}, 1 - \frac{n - s}{2}; t^2).$$

(4.33)

**Proof.** This follows from a direct computation with a change of variables $z = t^2$ on (4.29). □

By Lemma 4.32 to determine the space $\text{Sol}_I(s; n)$ of $K$-type solutions to $d\pi_n^I(D_s^h)f(t) = 0$, it suffices to find polynomial solutions to the Heun equation

$$\mathcal{D}_H(-1, -\frac{ns}{4}; -\frac{n}{2}, -\frac{n - 1}{2}, 1 - \frac{n - s}{2}; t^2)f(t) = 0.$$

(4.34)

Let $Hl(a, q; \alpha, \beta, \gamma, \delta; z)$ denote the power series solution $Hl(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{r=0}^{\infty} c_r z^r$ with $c_0 = 1$ to (4.31) at $z = 0$ (38). We set

$$u_{[s; n]}(t) := Hl(-1, -\frac{ns}{4}; -\frac{n}{2}, -\frac{n - 1}{2}, 1 - \frac{n - s}{2}; t^2),$$

(4.35)

$$v_{[s; n]}(t) := tHl(-1, -\frac{(n - 2)s}{4}; -\frac{n - 2}{2}, -\frac{n - 2}{2}, 1 - \frac{n - s}{2}; t^2).$$

(4.36)

It follows from (9.1) in Section 9 that $u_{[s; n]}(t)$ and $v_{[s; n]}(t)$ are linearly independent solutions at $t = 0$ to the Heun equation (4.33). Therefore we have

$$\text{Sol}_I(s; n) \subset \mathbb{C}u_{[s; n]}(t) \oplus \mathbb{C}v_{[s; n]}(t).$$

We shall classify the parameters $(s, n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}$ such that $u_{[s; n]}(t), v_{[s; n]}(t) \in \text{Sol}_I(s; n)$ in Section 6 (see Propositions 6.5 and 6.6).
Remark 4.37. Let $F(a, b, c; z) \equiv {}_2F_1(a, b, c; z)$ denote the Gauss hypergeometric function. In [29 (3.8)], Maier found the following Heun-to-Gauss reduction formula:

$$HI(-1, 0; \alpha, \beta, \gamma, \frac{\alpha + \beta - \gamma + 1}{2}; t) = F\left( \frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma + 1}{2}; t^2 \right).$$

We remark that the case of $s = 0$ for $d\pi_n^I(D_0^s) f(t) = 0$ gives special cases of this identity. Indeed, it follows from [27, Lem. 7.3] that a change of variables $z = t^4$ yields the identity

$$\frac{\sqrt{-1}}{32t^2} d\pi_n(D_0^s) = D_F\left(-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}; t^4 \right),$$

where $D_F(a, b, c; z)$ denotes the differential operator

$$D_F(a, b, c; z) = z(1 - z) \frac{d^2}{dz^2} + (c - (a + b + 1)z) \frac{d}{dz} - ab$$

such that

$$D_F(a, b, c; z)f(z) = 0$$

is the hypergeometric differential equation. Thus the equation $d\pi_n^I(D_0^s) f(t) = 0$ is also equivalent to the hypergeometric equation

$$D_F\left(-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}; t^4 \right) f(t) = 0.$$  \hspace{1cm} (4.40)

Then (4.33) and (4.40) yield the following Heun-to-Gauss reductions:

$$HI(-1, 0; -\frac{n}{2}, -\frac{n-1}{2}, \frac{1}{2}, -\frac{n-1}{2}; t) = F\left(-\frac{n}{4}, -\frac{n-1}{4}, \frac{3}{4}; t^2 \right);$$

$$HI(-1, 0; -\frac{n-1}{2}, -\frac{n-2}{2}, \frac{3}{2}, -\frac{n-1}{2}; t) = F\left(-\frac{n-1}{4}, -\frac{n-2}{4}, \frac{5}{4}; t^2 \right).$$

4.3.2. Differential equation $d\pi_n^{II}(D_s^s) f(t) = 0$. We next consider $d\pi_n^{II}(D_s^s) f(t) = 0$. The equation $d\pi_n^{II}(D_s^s) f(t) = 0$ can be identified with the hypergeometric equation $D_F(a, b, c; z)f(z) = 0$ as follows.

Lemma 4.43. We have

$$\frac{d\pi_n^{II}(D_s^s)}{16t} = D_F\left(-\frac{n}{2}, -\frac{n+s-1}{4}, \frac{3-n+s}{4}; t^2 \right).$$

Proof. First, it follows from (4.17) and (3.17) that $d\pi_n^{II}(D_s^s)$ is given as

$$2d\pi_n^{II}(D_s^s) = 2(1-t^2) t \frac{d^2}{dt^2} + ((s+3n-3)t^2 + (s-n+1)) \frac{d}{dt} - n(s+n-1)t.$$  \hspace{1cm} (4.44)

Then a direct computation with a change of variable $z = t^2$ concludes the lemma. \hfill $\Box$

Similar to the equation $d\pi_1^I(D_s^s) f(t) = 0$, Lemma 4.43 shows that, to determine the space $\text{Sol}_{II}(s; n)$ of $K$-type solutions to $d\pi_n^{II}(D_s^s) f(t) = 0$, it suffices to find polynomial solutions to the hypergeometric equation

$$D_F\left(-\frac{n}{2}, -\frac{n+s-1}{4}, \frac{3-n+s}{4}; t^2 \right) f(t) = 0.$$  \hspace{1cm} (4.45)
We set
\[ a_{[s;n]}(t) := F\left(-\frac{n}{2}, -\frac{n+s-1}{4}, \frac{3-n+s}{4}; t^2\right), \]
\[ b_{[s;n]}(t) := t^{\frac{1+n-s}{2}} F\left(-\frac{n+s-1}{4}, -\frac{s-1}{2}, \frac{5+n-s}{4}; t^2\right). \]

Then \( a_{[s;n]}(t) \) and \( b_{[s;n]}(t) \) form fundamental set of solutions to (4.45) for suitable parameters \((s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}\) for which \( a_{[s;n]}(t) \) and \( b_{[s;n]}(t) \) are well-defined. Therefore we have
\[ \text{Sol}_{\Pi}(s;n) \subset \mathbb{C}a_{[s;n]}(t) \oplus \mathbb{C}b_{[s;n]}(t). \]

The precise conditions of \((s,n)\) such that \( a_{[s;n]}(t), b_{[s;n]}(t) \in \text{Sol}_{\Pi}(s;n) \) will be investigated in Section 5.1.

4.4. Recipe for the \( K \)-type decomposition of \( \text{Sol}(s;\sigma)_K \). In Section 3.7 we gave a general recipe to determine the \( K \)-type formula \((3.20)\) of \( \text{Sol}_{(u;\lambda)}(\sigma)_K \). We modify the recipe in such a way that it will fit well for \( \text{Sol}(s;\sigma)_K \) with
\[ \text{Sol}(s;\sigma)_K \simeq \bigoplus_{n \geq 0} \text{Pol}_n[t] \otimes \text{Hom}_M(\text{Sol}_J(s;n), \sigma). \]

We first fix an identification
\[ \Omega^J: \mathfrak{k} \xrightarrow{\sim} \mathfrak{sl}(2, \mathbb{C}) \quad \text{for} \ J \in \{I, \Pi\}. \]

Then the \( K \)-type formula of \( \text{Sol}(s;\sigma)_K \) may be determined in the following three steps.

**Step A:** Classify \((s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}\) such that \( \text{Sol}_J(s;n) \neq \{0\} \). It follows from (4.36) and (4.46) that this is indeed equivalent to classifying \((s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}\) such that
- \( u_{[s;n]}(t) \in \text{Pol}_n[t] \) or \( v_{[s;n]}(t) \in \text{Pol}_n[t] \) for \( J = I \);
- \( a_{[s;n]}(t) \in \text{Pol}_n[t] \) or \( b_{[s;n]}(t) \in \text{Pol}_n[t] \) for \( J = \Pi \).

**Step B:** For \((s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}\) with \( \text{Sol}_J(s;n) \neq \{0\} \), classify the \( M \)-representations on \( \text{Sol}_J(s;n) \).

**Step C:** Given \( \sigma \in \text{Irr}(M) \), classify \((s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}\) with \( \text{Sol}_J(s;n) \neq \{0\} \) such that
\[ \text{Hom}_M(\text{Sol}_J(s;n), \sigma) \neq \{0\}. \]

In Sections 5 and 6, we shall proceed Steps A, B, and C for \( J = \Pi \) and \( J = I \), respectively.

5. Hypergeometric model \( d\pi^\Pi_n(D^\Pi_s)f(t) = 0 \)

The aim of this section is to classify the \( K \)-type formulas for \( \text{Sol}(s;\sigma)_K \) by using the hypergeometric model \( d\pi^\Pi_n(D^\Pi_s)f(t) = 0 \). The decomposition formulas are achieved in Theorem 5.36.

5.1. The classification of \( \text{Sol}_{\Pi}(s;n) \). As Step A of the recipe in Section 4.4 we first wish to classify \((s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}\) such that \( \text{Sol}_{\Pi}(s;n) \neq \{0\} \), where
\[ \text{Sol}_{\Pi}(s;n) = \{ p(t) \in \text{Pol}_n[t] : d\pi^\Pi_n(D^\Pi_s)p(t) = 0 \}. \]
Recall from [44] that we have
\[
\text{Sol}_\Pi(s; n) \subset \mathbb{C}a_{[s;n]}(t) \oplus \mathbb{C}b_{[s;n]}(t)
\]
with
\[
a_{[s;n]}(t) = F\left(-\frac{n}{2}, -\frac{n + s - 1}{4}, \frac{3 - n + s}{4}; t^2\right),
\]
\[
b_{[s;n]}(t) = t^{\frac{n + s - 1}{2}} F\left(-\frac{n + s - 1}{4}, -\frac{s - 1}{2}, \frac{5 + n - s}{4}; t^2\right).
\]

It thus suffices to classify \((s, n)\) such that \(a_{[s;n]}(t) \in \text{Pol}_n[t]\) or \(b_{[s;n]}(t) \in \text{Pol}_n[t]\).

Given \(n \in \mathbb{Z}_{\geq 0}\), we define \(I_{0}^{\pm}, J_{0}, I_{1}, I_{2}^{\pm}, J_{2}, I_{3} \subset \mathbb{Z}\) as follows:
\[
I_{0}^{\pm} := \{ \pm(3 + 4j) : j = 0, 1, \ldots, \left[\frac{n}{4}\right] - 1 \} \quad (n \in 4\mathbb{Z}_{\geq 0});
\]
\[
J_{0} := \{ 1 + 4j : j = 0, 1, \ldots, \left[\frac{n}{4}\right] - 1 \} \quad (n \in 4\mathbb{Z}_{\geq 0});
\]
\[
I_{1} := \{ \pm 4j : j = 0, 1, \ldots, \left[\frac{n}{4}\right] \};
\]
\[
I_{2}^{\pm} := \{ \pm(1 + 4j) : j = 0, 1, \ldots, \left[\frac{n}{4}\right] \};
\]
\[
J_{2} := \{ 3 + 4j : j = 0, 1, \ldots, \left[\frac{n}{4}\right] - 1 \};
\]
\[
I_{3} := \{ \pm(2 + 4j) : j = 0, 1, \ldots, \left[\frac{n}{4}\right] \}.
\]

We start by observing when \(a_{[s;n]}(t) \in \text{Pol}_n[t]\). A key observation is that, for \((a, b, c) \in \mathbb{C}\) for which the hypergeometric function \(F(a, b, c; z)\) is well-defined, we have
\[
F(a, b, c; z) \in \text{Pol}_n[z] \iff a \in \{0, -1, \ldots, -n\} \text{ or } b \in \{0, -1, \ldots, -n\}.
\]

**Proposition 5.4.** The following conditions on \((s, n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}\) are equivalent.

(i) \(a_{[s;n]}(t) \in \text{Pol}_n[t]\).

(ii) One of the following conditions hold:
   (a) \(n \equiv 0 \pmod{4} : s \in \mathbb{C} \setminus J_{0}\);
   (b) \(n \equiv 1 \pmod{4} : s \in I_{1}\);
   (c) \(n \equiv 2 \pmod{4} : s \in \mathbb{C} \setminus J_{2}\);
   (d) \(n \equiv 3 \pmod{4} : s \in I_{3}\).

**Proof.** The proposition follows from a careful observation for the parameters of \(a_{[s;n]}(t)\) in (5.1). Indeed, suppose that \(n\) is even. Then the parameters of \(a_{[s;n]}(t)\) imply that \(a_{[s;n]}(t) \notin \text{Pol}_n[t]\) if and only if the following conditions are satisfied
\[
\frac{3 - n + s}{4} \in -\mathbb{Z}_{\geq 0}, \quad -\frac{n}{2} < \frac{3 - n + s}{4}, \quad \text{and} \quad -\frac{n + s - 1}{4} < \frac{3 - n + s}{4},
\]
which is equivalent to \(s \in J_{0}\) for \(n \equiv 0 \pmod{4}\) and \(s \in J_{2}\) for \(n \equiv 2 \pmod{4}\). Now the assertions for (a) and (c) follow from the contrapositive of the arguments.
Next suppose that $n$ is odd. In this case it follows from (5.3) that $a_{s,n}(t) \in \mathcal{P}_n[t]$ if and only if
\[
0 \leq \frac{n + s - 1}{4} \leq \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad \frac{n + s - 1}{4} \in \mathbb{Z}_{\geq 0},
\]
which is equivalent to $s \in I_1$ for $n \equiv 1 \pmod{4}$ and $s \in I_3$ for $n \equiv 3 \pmod{4}$. Here we remark that, for $s \in I_1 \cup I_3$, we have $\frac{3-n-s}{4} \notin \mathbb{Z}$. This concludes the proposition.

Suppose that $a_{s,n}(t) \in \mathcal{P}_n[t]$. Then generically we have $\deg a_{s,n}(t) = n$. Lemma 5.5 below classifies the singular parameters of $s \in \mathbb{C}$ for $a_{s,n}(t)$ in a sense that $\deg a_{s,n}(t) < n$ (see Section 5.2.4).

**Lemma 5.5.** Suppose that $n \equiv k \pmod{4}$ and $s \in \mathbb{C} \setminus J_k$ for $k = 0, 2$. Then the following conditions on $(s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}$ are equivalent.

(i) $\deg a_{s,n}(t) < n$.
(ii) $\deg a_{s,n}(t) = \frac{n + s - 1}{2}$.
(iii) One of the following conditions holds:
- (a) $n \equiv 0 \pmod{4}$ : $s \in I_0^-$;
- (b) $n \equiv 2 \pmod{4}$ : $s \in I_2^-$.

**Proof.** It follows from the parameters of $a_{s,n}(t)$ that $\deg a_{s,n}(t) < n$ if and only if
\[
2 \cdot \frac{n + s - 1}{4} < n \quad \text{and} \quad \frac{n + s - 1}{4} \in \mathbb{Z}_{\geq 0},
\]
which shows the equivalence between (i) and (ii). Moreover, (5.6) is equivalent to
\[
s \in \{ -n + 1 + 4j : j = 0, 1, 2, \ldots, n \frac{1}{2} - 1 \}.
\]
One can readily verify that, under the condition $s \notin J_k$, (5.7) is indeed equivalent to $s \in I_k^-$ for $k = 0, 2$.

We next consider $b_{s,n}(t)$ in (5.2).

**Proposition 5.8.** The following conditions on $(s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}$ are equivalent.

(i) $b_{s,n}(t) \in \mathcal{P}_n[t]$ with $b_{s,n}(t) \neq a_{s,n}(t)$.
(ii) One of the following conditions holds.
- (a) $n \equiv 0 \pmod{4}$ : $s \in I_0^+ \cup I_0^- \cup J_0$;
- (b) $n \equiv 1 \pmod{4}$ : $s \in I_1$;
- (c) $n \equiv 2 \pmod{4}$ : $s \in I_2^+ \cup I_2^- \cup J_2$;
- (d) $n \equiv 3 \pmod{4}$ : $s \in I_3$.

**Proof.** Observe that if $b_{s,n}(t) \in \mathcal{P}_n[t]$, then the exponent $\frac{1+n-s}{2}$ for $t^{\frac{1+n-s}{2}}$ in (5.2) must satisfy $\frac{1+n-s}{2} \in \mathbb{Z}_{\geq 0}$, which in particular forces $\frac{5+n-s}{4} \notin -\mathbb{Z}_{\geq 0}$. Moreover, if $\frac{1+n-s}{2} = 0$, then
\[
b_{n+1,n}(t) = F\left(-\frac{n}{2}, -\frac{n}{2}; 1; t^2\right) = a_{n+1,n}(t).
\]
Consequently, we have \( b_{[s;n]}(t) \in \text{Pol}_n[t] \) with \( b_{[s;n]}(t) \neq a_{[s;n]}(t) \) if and only if either
\[
\frac{1 + n - s}{2} \in 1 + \mathbb{Z}_{\geq 0}, \quad \frac{n + s - 1}{4} \in \mathbb{Z}_{\geq 0}, \quad \text{and} \quad \frac{1 + n - 2}{2} + 2 \cdot \frac{n + s - 1}{4} \leq n, \tag{5.9}
\]
or
\[
\frac{1 + n - s}{2} \in 1 + \mathbb{Z}_{\geq 0}, \quad \frac{s - 1}{2} \in \mathbb{Z}_{\geq 0}, \quad \text{and} \quad \frac{1 + n - 2}{2} + 2 \cdot \frac{s - 1}{4} \leq n. \tag{5.10}
\]
A direct observation shows that the conditions (5.9) and (5.10) are equivalent to
\[
s \in (-n + 1 + 4\mathbb{Z}_{\geq 0}) \cap (-\infty, n + 1) \tag{5.11}
\]
and
\[
n \text{ is even and } s \in (1 + 2\mathbb{Z}_{\geq 0}) \cap (-\infty, n + 1), \tag{5.12}
\]
respectively. One can directly verify that (5.11) and (5.12) are equivalent to the conditions on \( s \in \mathbb{C} \) stated in Proposition 5.8 for \( n \equiv k \pmod{4} \) for \( k = 0, 1, 2, 3 \). Indeed, if \( n \equiv 0 \pmod{4} \), then (5.11) and (5.12) are equivalent to \( s \in I_0^+ \cup J_0 \) and \( s \in I_0^- \cup J_0 \), respectively. Since the other three cases can be shown similarly, we omit the proof. □

It follows from Propositions 5.4 and 5.8 that if \( n \equiv k \pmod{4} \) for \( k = 0, 2 \), then
\[
\text{Sol}_I(s; n) = \mathbb{C}a_{[s;n]}(t) \oplus \mathbb{C}b_{[s;n]}(t) \quad \text{for } s \in I_k^+ \cup I_k^-.
\tag{5.13}
\]

For a later purpose for determining the \( M \)-representations on \( \text{Sol}_I(s; n) \), for such \( n \equiv k \pmod{4} \), we define
\[
c_{[s;n]}^\pm(t) := a_{[s;n]}(t) \pm C(s; n)b_{[s;n]}(t) \quad \text{for } s \in I_k^-,
\]
where
\[
C(s; n) := \begin{cases} \left( \frac{-n}{2} \right)_{\left[ \frac{n}{4} \right]} & \text{if } n \equiv 2 \pmod{4} \text{ and } s = -1, \\ \left( \frac{-n + s + 1}{2} \right)_{\left( -\frac{n + s + 1}{4} \right)_{\left( -\frac{n + s + 1}{4} \right)}} \cdot \frac{1}{4} & \text{otherwise}. \end{cases}
\tag{5.14}
\]

Here \((\ell, m)\) stands for the shifted factorial, namely, \((\ell, m) = \frac{\Gamma(\ell + m)}{\Gamma(\ell)}\). Then, for \( n \equiv k \pmod{4} \) for \( k = 0, 2 \), the space \( \text{Sol}_I(s; n) \) may be described as
\[
\text{Sol}_I(s; n) = \begin{cases} \mathbb{C}a_{[s;n]}(t) \oplus \mathbb{C}b_{[s;n]}(t) & \text{if } s \in I_k^+, \\ \mathbb{C}c_{[s;n]}^+(t) \oplus \mathbb{C}c_{[s;n]}^-(t) & \text{if } s \in I_k^- \end{cases}
\tag{5.15}
\]

We then summarize the parameters \((s, n)\) such that \( \text{Sol}_I(s; n) \neq \{0\} \) as follows.

**Theorem 5.16.** The following conditions on \((s, n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}\) are equivalent.

(i) \( \text{Sol}_I(s; n) \neq \{0\} \).

(ii) One of the following conditions is satisfied.

- \( n \equiv 0 \pmod{4} : s \in \mathbb{C} \).
- \( n \equiv 1 \pmod{4} : s \in I_1 \).
- \( n \equiv 2 \pmod{4} : s \in \mathbb{C} \).
- \( n \equiv 3 \pmod{4} : s \in I_3 \).

Further, for such \((s, n)\), the space \( \text{Sol}_I(s; n) \) may be given as follows.
(1) \( n \equiv 0 \pmod{4} \):

\[
\text{Sol}_\Pi(s; n) = \begin{cases}
    C_{a_{[s,n]}}(t) & \text{if } s \in \mathbb{C}\backslash(I_0^+ \cup I_0^- \cup J_0), \\
    C_{a_{[s,n]}}(t) \oplus C_{b_{[s,n]}}(t) & \text{if } s \in I_0^+,
    \\
    C_{b_{[s,n]}}(t) & \text{if } s \in J_0,
    \\
    C_{c_{[s,n]}^+}(t) \oplus C_{c_{[s,n]}^-}(t) & \text{if } s \in I_0^-.
\end{cases}
\]

(2) \( n \equiv 1 \pmod{4} \):

\[
\text{Sol}_\Pi(s; n) = C_{a_{[s,n]}}(t) \oplus C_{b_{[s,n]}}(t) \quad \text{for } s \in I_1.
\]

(3) \( n \equiv 2 \pmod{4} \):

\[
\text{Sol}_\Pi(s; n) = \begin{cases}
    C_{a_{[s,n]}}(t) & \text{if } s \in \mathbb{C}\backslash(I_2^+ \cup I_2^- \cup J_2), \\
    C_{a_{[s,n]}}(t) \oplus C_{b_{[s,n]}}(t) & \text{if } s \in I_2^+,
    \\
    C_{b_{[s,n]}}(t) & \text{if } s \in J_2,
    \\
    C_{c_{[s,n]}^+}(t) \oplus C_{c_{[s,n]}^-}(t) & \text{if } s \in I_2^-.
\end{cases}
\]

(4) \( n \equiv 3 \pmod{4} \):

\[
\text{Sol}_\Pi(s; n) = C_{a_{[s,n]}}(t) \oplus C_{b_{[s,n]}}(t) \quad \text{for } s \in I_3.
\]

Proof. This is a summary of the results in Propositions 5.3 and 5.8 and (5.15).

Remark 5.17. Theorem 5.16 shows that the structure of \( \text{Sol}_\Pi(s; n) \) (hypergeometric model) is somewhat complicated. It will be shown in Theorem 6.7 that that of \( \text{Sol}_I(s; n) \) (Heun model) is more straightforward.

Remark 5.18. Theorem 5.16 also completely classifies the dimension \( \dim_{\mathbb{C}} \text{Sol}_I(s; n) \). When \( n \) is even, the dimension \( \dim_{\mathbb{C}} \text{Sol}_I(s; n) \) was determined in [16, Thm. 5.13] by factorization formulas of certain tridiagonal determinants (see Theorem 7.18 and Remark 7.21). We shall study such determinants in Section 7 from a different perspective from [16].

5.2. The \textbf{M-representations on} \( \text{Sol}_\Pi(s; n) \). As Step B of the recipe in Section 4.4 we next classify the \( M \)-representations on \( \text{Sol}_\Pi(s; n) \). Here is the classification.

Theorem 5.19. For each \((s;n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0} \) determined in Theorem 5.16, the \( M \)-representations on \( \text{Sol}_\Pi(s; n) \) are classified as follows.

(1) \( n \equiv 0 \pmod{4} \):

\[
\text{Sol}_\Pi(s; n) \simeq \begin{cases}
    (+,+), & \text{if } s \in \mathbb{C}\backslash(I_0^+ \cup I_0^- \cup J_0), \\
    (+,+) \oplus (+,-), & \text{if } s \in I_0^+,
    \\
    (+,+), & \text{if } s \in J_0,
    \\
    (+,+) \oplus (-,+), & \text{if } s \in I_0^-.
\end{cases}
\]

(2) \( n \equiv 1 \pmod{4} \):

\[
\text{Sol}_\Pi(s; n) \simeq \mathbb{H} \quad \text{for } s \in I_1.
\]
(3) \( n \equiv 2 \pmod{4} \):

\[
\text{Sol}_{\Pi}(s; n) \simeq \begin{cases} 
(\neg, \neg) & \text{if } s \in \mathbb{C}\setminus(I_2^+ \cup I_2^- \cup J_2), \\
(\neg, \neg) \oplus (\neg, +) & \text{if } s \in I_2^+, \\
(\neg, \neg) & \text{if } s \in J_2, \\
(\neg, \neg) \oplus (+, \neg) & \text{if } s \in I_2^-.
\end{cases}
\]

(4) \( n \equiv 3 \pmod{4} \):

\[
\text{Sol}_{\Pi}(s; n) \simeq \mathbb{H} \quad \text{for } s \in I_3.
\]

Here the characters \((\varepsilon, \varepsilon')\) stand for the ones on \(\mathbb{C}a_{[s; n]}(t)\), \(\mathbb{C}b_{[s; n]}(t)\), and \(C_{[s; n]}^\pm(t)\) at the same places in Theorem \[5.16\].

We prove Theorem \[5.19\] by considering the following cases separately.

- Case 1: \( n \) is odd.
- Case 2: \( n \) is even. Let \( k \in \{0, 2\} \).
  - Case 2a: \( n \equiv k \pmod{4} \) and \( s \in \mathbb{C}\setminus(I_k^- \cup J_k) \)
  - Case 2b: \( n \equiv k \pmod{4} \) and \( s \in I_k^+ \cup J_k \)
  - Case 2c: \( n \equiv k \pmod{4} \) and \( s \in I_k^- \)

5.2.1. Case 1. We start with the case that \(\text{Sol}_{\Pi}(s; n)\) is a two-dimensional representation of \(M\).

**Lemma 5.20.** Suppose that \( n \equiv k \pmod{4} \) and \( s \in I_k \) for \( k = 1, 3 \). Then, as an \(M\)-representation, we have

\[
\mathbb{C}a_{[s; n]}(t) \oplus \mathbb{C}b_{[s; n]}(t) \simeq \mathbb{H}.
\]

**Proof.** In this case \(a_{[s; n]}(t)\) and \(b_{[s; n]}(t)\) are even and odd functions, respectively. Then the assertion can be shown by essentially the same argument as the one for \[27\] Prop. 6.11 by replacing \(u_n(t)\) and \(v_n(t)\) with \(a_{[s; n]}(t)\) and \(b_{[s; n]}(t)\), respectively. Hence we omit the proof. \(\square\)

5.2.2. Cases 2a. We next consider the characters on \(\mathbb{C}a_{[s; n]}(t)\).

**Lemma 5.21.** Suppose that \( n \equiv k \pmod{4} \) and \( s \in \mathbb{C}\setminus(I_k^- \cup J_k) \) for \( k = 0, 2 \). Then \(M\) acts on \(\mathbb{C}a_{[s; n]}(t)\) as a character.

**Proof.** We only give a proof for the case \( k = 0 \), namely, \( n \equiv 0 \pmod{4} \) and \( s \in \mathbb{C}\setminus(I_0^- \cup J_0) \); the other case can be shown similarly. As

\[
\mathbb{C}\setminus(I_0^- \cup J_0) = (\mathbb{C}\setminus(I_0^+ \cup I_0^- \cup J_0)) \cup I_0^+,
\]

we consider the cases \( s \in \mathbb{C}\setminus(I_0^+ \cup I_0^- \cup J_0) \) and \( s \in I_0^+ \), separately.

First suppose that \( s \in \mathbb{C}\setminus(I_0^+ \cup I_0^- \cup J_0) \). By Theorem \[5.16\] we have \(\text{Sol}_{\Pi}(s; n) = \mathbb{C}a_{[s; n]}(t)\). Since \(\text{Sol}_{\Pi}(s; n)\) is an \(M\)-representation by Lemma \[27\] the assertion clearly holds for this case. We next suppose that \( s \in I_0^+ \). In this case we have \(\text{Sol}_{\Pi}(s; n) = \mathbb{C}a_{[s; n]}(t) \oplus \mathbb{C}b_{[s; n]}(t)\) by Theorem \[5.16\] As the exponent \(\frac{1+n-s}{2}\) for \(t^{\frac{1+n-s}{2}}\) of \(b_{[s; n]}(t)\) is odd, \(a_{[s; n]}(t)\) and \(b_{[s; n]}(t)\) are an even and odd function, respectively. The transformation laws \[4.27\] imply that the action of \(M\) on \(\text{Pol}_n[t]\) preserves the parities of the polynomials for \(n\) even. Hence \(M\) acts both on \(\mathbb{C}a_{[s; n]}(t)\) and \(\mathbb{C}b_{[s; n]}(t)\) as a character. \(\square\)
We next determine the characters on \(a_{[s;n]}(t)\) explicitly. It follows from Lemma 5.21 that \(a_{[s;n]}(t)\) has degree \(\deg a_{[s;n]}(t) = n\) for \((s, n)\) with \(n \equiv k \pmod{4}\) and \(s \in \mathbb{C} \setminus (I_k^+ \cup J_k^+)\) for \(k = 0, 2\). Thus, in this case, the hypergeometric polynomial \(a_{[s;n]}(t)\) is given as

\[
a_{[s;n]}(t) = \sum_{j=0}^{n/2} A_j(s; n) t^{2j}
\]  

(5.22)

with

\[
A_j(s; n) = \left(\frac{-n/2, j}{\sqrt{-1}}\right) \left(\frac{-n+s-1/4, j}{\sqrt{-1}}\right) \cdot \frac{1}{j!},
\]

where \((\ell, m)\) stands for the shifted factorial. Recall from (4.25) that \(M = \{\pm m_j^H : j = 0, 1, 2, 3\}\) with

\[
m_0^H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_1^H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad m_2^H = \begin{pmatrix} \sqrt{-1} & \sqrt{-1} \\ 0 & \sqrt{-1} \end{pmatrix}, \quad m_3^H = -\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.
\]

As \(m_3^H = m_1^H m_2^H\), it suffices to check the actions of \(m_1^H\) and \(m_2^H\) on \(a_{[s;n]}(t) = \sum_{j=0}^{n/2} A_j(s; n) t^{2j}\).

**Proposition 5.23.** Under the same hypothesis in Lemma 5.21, the character \((\varepsilon, \varepsilon')\) on \(Ca_{[s;n]}(t)\) is given as follows.

1. \(n \equiv 0 \pmod{4}\) and \(s \in \mathbb{C} \setminus (I_0^- \cup J_0)\):

\[Ca_{[s;n]}(t) \simeq (+, +).\]

2. \(n \equiv 2 \pmod{4}\) and \(s \in \mathbb{C} \setminus (I_2^- \cup J_2)\):

\[Ca_{[s;n]}(t) \simeq (-, -).\]

**Proof.** Since the second assertion can be shown similarly, we only give a proof for the first assertion. Let \(n \equiv 0 \pmod{4}\) and \(s \in \mathbb{C} \setminus (I_0^- \cup J_0)\). We wish to show that both \(m_1^H\) and \(m_2^H\) act trivially. First, it is easy to see that the action of \(m_2^H\) is trivial. Indeed, since \(n \equiv 0 \pmod{4}\) and \(a_{[s;n]}(t)\) is an even function, the transformation law of (4.27) shows that

\[
m_2^H : a_{[s;n]}(t) \mapsto (\sqrt{-1})^n a_{[s;n]}(-t) = a_{[s;n]}(t).
\]

In order to show that \(m_1^H\) also acts trivially, observe that, by (4.27) and (5.22), we have

\[
m_1^H : a_{[s;n]}(t) \mapsto t^n a_{[s;n]}(-1) = \sum_{j=0}^{n/2} A_{n/2-j}(s; n) t^{2j} = \sum_{j=0}^{n/2} A_j(s; n) t^{2j},
\]

On the other hand, by Lemma 5.21 and Table 4, \(m_1^H\) acts on \(a_{[s;n]}(t)\) by \(\pm 1\). Therefore,

\[
\sum_{j=0}^{n/2} A_{n/2-j}(s; n) t^{2j} = \pm \sum_{j=0}^{n/2} A_j(s; n) t^{2j}.
\]

An easy computation shows that we have \(A_{n/2}(s; n) = 1 = A_0(s; n)\), which forces \(t^n a_{[s;n]}(-1/2) = a_{[s;n]}(t)\). Hence \(m_1^H\) also acts on \(Ca_{[s;n]}(t)\) trivially.

5.2.3. Cases 2b. Next we consider the characters on \(Cb_{[s;n]}(t)\).
Proposition 5.24. Suppose that $n \equiv k \pmod{4}$ and $s \in I_k^{+} \cup J_k$ for $k = 0, 2$. Then $M$ acts on $C_{b_{[s;n]}}(t)$ as a character as follows.

(1) $n \equiv 0 \pmod{4}$ and $s \in I_0^{+} \cup J_0$:

\[
C_{b_{[s;n]}}(t) \simeq \begin{cases} 
(+,-) & \text{if } s \in I_0^+, \\
(+,+). & \text{if } s \in J_0.
\end{cases}
\]

(2) $n \equiv 2 \pmod{4}$ and $s \in I_2^{+} \cup J_2$:

\[
C_{b_{[s;n]}}(t) \simeq \begin{cases} 
(-,+). & \text{if } s \in I_2^+, \\
(-,-). & \text{if } s \in J_2.
\end{cases}
\]

Proof. Since the assertions can be shown similarly to Lemma 5.21 and Proposition 5.23, we omit the proof. We remark that it is already shown in the proof of Lemma 5.21 that $M$ acts on $C_{b_{[s;n]}}(t)$ as a character for $(s, n)$ with $n \equiv 0 \pmod{4}$ and $s \in I_0^+$.

\[\square\]

5.2.4. Case 2c. Now we consider $C_{c_{[s;n]}}(t)$ for $n \equiv k \pmod{4}$ and $s \in I_k^{-}$ for $k = 0, 2$, where $C_{c_{[s;n]}}(t)$ is given as

\[
c_{[s;n]}^{+}(t) = a_{[s;n]}(t) \pm C(s; n)b_{[s;n]}(t)
\]

with $C(s; n)$ in (5.14).

Observe that, in this case, $a_{[s;n]}(t)$ has degree $\deg a_{[s;n]}(t) = \frac{n+s-1}{2} < n$ by Lemma 5.5. Therefore, $a_{[s;n]}(t)$ is of the form

\[
a_{[s;n]}(t) = \sum_{j=0}^{(n+s-1)/4} \tilde{A}_j(s; n)t^{2j}.
\]

(5.25)

To give an explicit formula of $\tilde{A}_j(s; n)$, first remark that when $n \equiv 2 \pmod{4}$ and $s = -1$, we have $\frac{n+s-1}{4} = \frac{3-n+s}{4} \in \mathbb{Z}_{\geq 0}$. Namely, for $n = 4\ell + 2$ and $s = -1$, the hypergeometric polynomial $a_{[-1; 4\ell+2]}(t)$ is

\[
a_{[-1; 4\ell+2]}(t) = F(-2\ell - 1, -\ell, -\ell; t^2).
\]

(5.26)

In this paper we regard (5.26) as

\[
a_{[-1; 4\ell+2]}(t) = \sum_{j=0}^{k} \frac{(-2\ell - 1, j)}{j!} t^{2j}
\]

so that $a_{[-1; 4\ell+2]}(t)$ and $b_{[-1; 4\ell+2]}(t)$ still form a fundamental solutions to the hypergeometric equation (4.15). Therefore $\tilde{A}_j(s; n)$ in (5.25) is given as

\[
\tilde{A}_j(s; n) = \begin{cases} 
\frac{(-\frac{n+d}{2})}{j!} & \text{if } n \equiv 2 \pmod{4} \text{ and } s = -1 \\
\frac{(-\frac{n+d}{2})}{j!} \left( \frac{\Delta + 1}{\Delta + 2j} \right) & \text{otherwise}.
\end{cases}
\]

In particular, by (5.14), we have

\[
C(s; n) = \tilde{A}_{n+s-1}(s; n).
\]

(5.27)
It follows from (5.2) that $b_{[s;n]}(t)$ is given as

$$b_{[s;n]}(t) = t^{\frac{n+s-1}{2}} \sum_{j=0}^{(n+s-1)/4} B_j(s;n) t^{2j}$$

with

$$B_j(s;n) = \frac{(-\frac{n+s-1}{4},j)(-\frac{s-1}{4})}{(\frac{n+s-1}{4} - j)} \frac{1}{j!}.$$

Lemma 5.29 below plays a key role to determine the $M$-representation on $\mathbb{C}c_{[s;n]}^\pm(t)$.

**Lemma 5.29.** Suppose that $n \equiv k \pmod{4}$ and $s \in I_k^-$ for $k = 0, 2$. Then,

$$\pi_n(m_{1}^\Pi)a_{[s;n]}(t) = C(s;n)b_{[s;n]}(t).$$

**Proof.** We only show the case of $k = 0$; the other case can be shown similarly, including the exceptional case for $n \equiv 2 \pmod{4}$ and $s = -1$. Let $n \equiv 0 \pmod{4}$ and $s \in I_0^-$. It follows from (5.13) that we have

$$\text{Sol}_{\Pi}(s;n) = \mathbb{C}a_{[s;n]}(t) \oplus \mathbb{C}b_{[s;n]}(t).$$

Therefore there exist some constants $c_1, c_2 \in \mathbb{C}$ such that

$$\pi_n(m_1^\Pi)a_{[s;n]}(t) = c_1a_{[s;n]}(t) + c_2b_{[s;n]}(t).$$

(5.30)

On the other hand, by (4.27) and (5.25), we have

$$\pi_n(m_1^\Pi)a_{[s;n]}(t) = t^n a_{[s;n]}(t) = \sum_{j=0}^{(n+s-1)/4} A_j(s;n) t^{n-2j}.$$

(5.31)

Since

$$n - \frac{n+s-1}{2} > \frac{n+s-1}{2} = \text{deg} a_{[s;n]}(t)$$

for $n \equiv 0 \pmod{4}$ and $s \in I_0^-$, all exponents $n-2j$ for $t^{n-2j}$ in (5.31) is $n-2j > \text{deg} a_{[s;n]}(t)$. Therefore,

$$\pi_n(m_1^\Pi)a_{[s;n]}(t) = c_2b_{[s;n]}(t).$$

Further, by (5.31) and (5.28), we then have

$$\widetilde{A}_{n-2j}(s;n) = c_2 B_0(s;n) = c_2.$$

Now (5.27) concludes the assertion.

**Proposition 5.32.** Suppose that $n \equiv k \pmod{4}$ and $s \in I_k^-$ for $k = 0, 2$. Then $M$ acts on $\mathbb{C}c_{[s;n]}^\pm(t)$ as the following characters.

1. $n \equiv 0 \pmod{4}$ and $s \in I_0^-$. 
   $$\mathbb{C}c_{[s;n]}^+(t) \simeq (+,+) \quad \text{and} \quad \mathbb{C}c_{[s;n]}^-(t) \simeq (-,+).$$

2. $n \equiv 2 \pmod{4}$ and $s \in I_2^-$. 
   $$\mathbb{C}c_{[s;n]}^+(t) \simeq (-,-) \quad \text{and} \quad \mathbb{C}c_{[s;n]}^-(t) \simeq (+,-).$$
Proof. As in Lemma 5.29, we only show the case of \( k = 0 \); the other case can be treated similarly (including the exceptional case for \( n \equiv 2 \) (mod 4) and \( s = -1 \)). Let \( n \equiv 0 \) (mod 4) and \( s \in I_0^- \). To show the assertion it suffices to consider only \( m_1^H \). Indeed, since the exponent \( \frac{1+n-s}{2} \) for \( t^{\frac{1+n-s}{2}} \) in (5.28) is even, \( a_{[s,n]}(t) \) and \( b_{[s,n]}(t) \) are both even functions; thus, \( m_1^H \) acts on \( Cc_{[s,n]}^\pm(t) \) trivially by (4.27). To consider the action of \( m_1^H \) on \( Cc_{[s,n]}^\pm(t) \), observe that, by Lemma 5.29, we have

\[
\pi_n(m_1^H)b_{[s,n]}(t) = C(s; n)^{-1}a_{[s,n]}(t).
\]

Therefore, for \( \varepsilon \in \{+,-\} \), \( m_1^H \) transforms \( c_{[s,n]}^\varepsilon(t) \) as

\[
m_1^H: c_{[s,n]}^\varepsilon(t) \mapsto \varepsilon c_{[s,n]}^\varepsilon(t).
\]

Now Table 4 concludes the assertion. \( \square \)

5.3. The classification of \( \text{Hom}_M(\text{Sol}_H(s; n), \sigma) \). As Step C in the recipe in Section 4.4. we now classify \( (\sigma, s, n) \in \text{Irr}(M) \times C \times Z_{\geq 0} \) such that \( \text{Hom}_M(\text{Sol}_H(s; n), \sigma) \neq \{0\} \). Given \( \sigma \in \text{Irr}(M) \), we define \( I^\pm((+,-)), I^\pm((-,-)), I(\Pi) \subset C \times Z_{\geq 0} \) as follows.

\[
\begin{align*}
I^+(+,-) & := \{(s, n) \in (3 + 4Z_{\geq 0}) \times (4Z_{\geq 0}) : n > s\}, \\
I^-(+,-) & := \{(s, n) \in -(1 + 4Z_{\geq 0}) \times (2 + 4Z_{\geq 0}) : n > |s|\}, \\
I^+(+-) & := \{(s, n) \in (1 + 4Z_{\geq 0}) \times (2 + 4Z_{\geq 0}) : n > s\}, \\
I^-(+-) & := \{(s, n) \in -(3 + 4Z_{\geq 0}) \times (4Z_{\geq 0}) : n > |s|\}, \\
I(\Pi) & := \{(s, n) \in (2Z) \times (1 + 2Z_{\geq 0}) : n > |s|\}.
\end{align*}
\]

Theorem 5.34. The following conditions on \( (\sigma, s, n) \in \text{Irr}(M) \times C \times Z_{\geq 0} \) are equivalent.

(i) \( \text{Hom}_M(\text{Sol}_H(s; n), \sigma) \neq \{0\} \).

(ii) \( \dim_C \text{Hom}_M(\text{Sol}_H(s; n), \sigma) = 1 \).

(iii) One of the following conditions holds.

- \( \sigma = (+,+): (s, n) \in C \times 4Z_{\geq 0} \).
- \( \sigma = (-,-): (s, n) \in C \times (2 + 4Z_{\geq 0}) \).
- \( \sigma = (+,-): (s, n) \in I^+((+,-)) \cup I^-((+,-)) \).
- \( \sigma = (-,+): (s, n) \in I^+((-,-)) \cup I^-((-,-)) \).
- \( \sigma = \Pi: (s, n) \in I(\Pi) \).

Further, for such \((\sigma, s, n)\), the space \( \text{Hom}_M(\text{Sol}_H(s; n), \sigma) \) is given as follows.

1. \( \sigma = (+,+): \) For \( n \in 4Z_{\geq 0} \), we have

\[
\text{Hom}_M(\text{Sol}_H(s; n), (+,+)) = \begin{cases} 
C a_{[s,n]}(t) & \text{if } s \in C \setminus (J_0^- \cup J_0), \\
C b_{[s,n]}(t) & \text{if } s \in J_0, \\
C c_{[s,n]}^+(t) & \text{if } s \in I_0^-.
\end{cases}
\]

2. \( \sigma = (-,-): \) For \( n \in 2 + 4Z_{\geq 0} \), we have

\[
\text{Hom}_M(\text{Sol}_H(s; n), (-,-)) = \begin{cases} 
C a_{[s,n]}(t) & \text{if } s \in C \setminus (I_2^- \cup J_2), \\
C b_{[s,n]}(t) & \text{if } s \in J_2, \\
C c_{[s,n]}^+(t) & \text{if } s \in I_2^-.
\end{cases}
\]
(3) \( \sigma = (+,-) \): We have
\[
\text{Hom}_M(\text{Sol}_I(s; n), (+,-)) = \begin{cases} 
\mathbb{C} b_{[s;n]}(t) & \text{if } (s, n) \in I^+((+,-)), \\
\mathbb{C} c_{[s;n]}(t) & \text{if } (s, n) \in I^-((+,-)). 
\end{cases}
\]

(4) \( \sigma = (-,+) \): We have
\[
\text{Hom}_M(\text{Sol}_I(s; n), (-,+)) = \begin{cases} 
\mathbb{C} b_{[s;n]}(t) & \text{if } (s, n) \in I^+((-,+)), \\
\mathbb{C} c_{[s;n]}(t) & \text{if } (s, n) \in I^-((-,+)). 
\end{cases}
\]

(5) \( \sigma = \mathbb{H} \): We have
\[
\text{Hom}_M(\text{Sol}_I(s; n), \mathbb{H}) = \mathbb{C} \varphi_{II}^{(s;n)} \quad \text{for } (s, n) \in I(\mathbb{H}),
\]
where \( \varphi_{II}^{(s;n)} \) is a non-zero \( M \)-isomorphism
\[
\varphi_{II}^{(s;n)} : \text{Sol}_I(s; n) \xrightarrow{\sim} \mathbb{H}.
\]

Proof. The theorem simply follows from Theorems 5.16 and 5.19. Indeed, for instance, suppose that \( \sigma = (+,+) \). It then follows from Theorem 5.19 that \( \text{Hom}_M(\text{Sol}_I(s; n), (+,+)) \neq \{0\} \) if and only if \( n \equiv 0 \pmod{4} \). Moreover, Theorems 5.16 and 5.19 show that
\[
\text{Hom}_M(\text{Sol}_I(s; n), (+,+)) = \begin{cases} 
\mathbb{C} a_{[s;n]}(t) & \text{if } s \in \mathbb{C} \setminus (I_0^- \cup J_0), \\
\mathbb{C} b_{[s;n]}(t) & \text{if } s \in J_0, \\
\mathbb{C} c_{[s;n]}^+(t) & \text{if } s \in I_0^-.
\end{cases}
\]
This concludes the assertion for \( \sigma = (+,+). \) Similarly, suppose that \( \sigma = \mathbb{H} \). Then, by Theorem 5.19, we have \( \text{Hom}_M(\text{Sol}_I(s; n), \mathbb{H}) \neq \{0\} \) if and only if \( (s, n) \in I(\mathbb{H}) \). Furthermore, in this case, \( \text{Sol}_I(s; n) \simeq \mathbb{H} \). Therefore, \( \text{Hom}_M(\text{Sol}_I(s; n), \mathbb{H}) \) is spanned by a non-zero \( M \)-isomorphism \( \varphi_{II}^{(s;n)} : \text{Sol}_I(s; n) \xrightarrow{\sim} \mathbb{H} \). Since the other cases can be handled similarly, we omit the proof.

Remark 5.35. As for Theorem 5.16 it will be shown in Theorem 6.9 that \( \text{Hom}_M(\text{Sol}_I(s; n), \sigma) \) is simpler than \( \text{Hom}_M(\text{Sol}_I(s; n), \sigma) \).

Now we give the \( K \)-type formulas for \( \text{Sol}(s; n)_K \) for each \( \sigma \in \text{Irr}(M) \) in the polynomial realization (3.14) of \( \text{Irr}(K) \).

**Theorem 5.36.** The following conditions on \((\sigma, s) \in \text{Irr}(M) \times \mathbb{C}\) are equivalent.

(i) \( \text{Sol}(s; \sigma)_K \neq \{0\} \).

(ii) One of the following conditions holds.

- \( \sigma = (+,+): s \in \mathbb{C} \).
- \( \sigma = (-,-): s \in \mathbb{C} \).
- \( \sigma = (+,-): s \in 3 + 4\mathbb{Z} \).
- \( \sigma = (-,+): s \in 1 + 4\mathbb{Z} \).
- \( \sigma = \mathbb{H}: s \in 2\mathbb{Z} \).

Further, the \( K \)-type formulas for \( \text{Sol}(s; \sigma)_K \) may be given as follows.

1. \( \sigma = (+,+): \)
\[
\text{Sol}(s; (+,+))_K \simeq \bigoplus_{n \geq 0} \text{Pol}_{4n}[t] \quad \text{for all } s \in \mathbb{C}.
\]
(2) \(\sigma = (-,-)\):

\[\text{Sol}(s;(-,-))_K \simeq \bigoplus_{n \geq 0} \text{Pol}_{2+4n}[t] \quad \text{for all } s \in \mathbb{C}.\]

(3) \(\sigma = (+,-)\):

\[\text{Sol}(s;(+,-))_K \simeq \bigoplus_{n \geq 0} \text{Pol}_{|s|+1+4n}[t] \quad \text{for } s \in 3 + 4\mathbb{Z}.\]

(4) \(\sigma = (-,+)\):

\[\text{Sol}(s;(-,))_K \simeq \bigoplus_{n \geq 0} \text{Pol}_{|s|+1+4n}[t] \quad \text{for } s \in 1 + 4\mathbb{Z}.\]

(5) \(\sigma = \mathbb{H}\):

\[\text{Sol}(s;\mathbb{H})_K \simeq \bigoplus_{n \geq 0} \text{Pol}_{|s|+1+4n}[t] \quad \text{for } s \in 2\mathbb{Z}.\]

Proof. We only give a proof for \(\sigma = (+,-)\); the other cases may be handled similarly. Suppose that \(\sigma = (+,-)\). By (4.22) with \(J = \mathbb{II}\), we have

\[\text{Sol}(s; (+,-))_K \simeq \bigoplus_{n \geq 0} \text{Pol}_n[t] \otimes \text{Hom}_M(\text{Sol}_\mathbb{II}(s;n),(+,-)).\]

Thus, \(\text{Sol}(s; (+,-))_K \neq \{0\}\) if and only if \(\text{Hom}_M(\text{Sol}_\mathbb{II}(s;n),(+,-)) \neq \{0\}\) for some \(n \in \mathbb{Z}_{\geq 0}\), which is further equivalent to

\[s \in (3 + 4\mathbb{Z}_{\geq 0}) \cup (-1 + 4\mathbb{Z}_{\geq 0})) = 3 + 4\mathbb{Z}\]

by Theorem 5.34. This shows the assertion between (i) and (ii) for \(\sigma = (+,-)\).

To determine the explicit branching law, observe that, by the Frobenius reciprocity, we have

\[\text{Hom}_K(\text{Pol}_n[t],\text{Sol}(s; (+,-))_K) \neq \{0\} \quad \text{if and only if} \quad \text{Hom}_M(\text{Sol}_\mathbb{II}(s;n),(+,-)) \neq \{0\},\]

which is equivalent to

\[n \in 4\mathbb{Z}_{\geq 0} \text{ with } n > s \text{ for } s \in 3 + 4\mathbb{Z}_{\geq 0}\]

or

\[n \in 2 + 4\mathbb{Z}_{\geq 0} \text{ with } n > |s| \text{ for } s \in -(1 + 4\mathbb{Z}_{\geq 0})\]

by Theorem 5.34. It also follows from Theorem 5.34 that \(\text{Sol}(s; (+,-))_K\) is multiplicity-free. Therefore, we have

\[\text{Sol}(s; (+,-))_K \simeq \begin{cases} \bigoplus_{n \equiv 0 \pmod{4}} \text{Pol}_n[t] & \text{if } s \in 3 + 4\mathbb{Z}_{\geq 0}, \\ \bigoplus_{n \equiv 2 \pmod{4}} \text{Pol}_n[t] & \text{if } s \in -(1 + 4\mathbb{Z}_{\geq 0}), \end{cases}\]

which is equivalent to the proposed formula. \(\square\)

Remark 5.37. The \(K\)-type formula for the case of \(\sigma = \mathbb{H}\) is obtained also in [36, Prop. 5.4.6].

We close this section by giving a proof of Theorem 1.7 in the introduction.
6. Heun model $dπ_n^1(D^s_0)f(t) = 0$

The purpose of this short section is to interpret the results in Section 5 to the Heun model $dπ_n^1(D^s_0)f(t) = 0$. In particular we give a variant of Theorem 5.34 for $\text{Hom}_M(\text{Sol}(s;n), \sigma)$ in Theorem 6.9. We remark that the results in this section will play a key role to compute certain tridiagonal determinants in Section 7.

6.1. Relationships between $a_{[s;n]}(t), b_{[s;n]}(t), c_{[s;n]}(t)$ and $u_{[s;n]}(t), v_{[s;n]}(t)$. As for the hypergeometric model $dπ_n^1(D^s_0)f(t) = 0$, we start with Step A of the recipe in Section 4.4, that is, the classification of $(s, n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}$ such that $\text{Sol}(s; n) \neq \{0\}$. Recall from (4.36) that we have

$$\text{Sol}_I(s; n) \subset \mathbb{C}u_{[s;n]}(t) \oplus \mathbb{C}v_{[s;n]}(t)$$

with

$$u_{[s;n]}(t) = \text{H}(1, -\frac{ns}{4}; \frac{n-1}{2}, \frac{n-1}{2}, \frac{1-n-s}{2}; t^2),$$

$$v_{[s;n]}(t) = t\text{H}(1, -\frac{n-2}{4}; \frac{n-1}{2}, \frac{n-2}{2}, \frac{1-n-s}{2}; t^2).$$

Thus, equivalently, we wish to classify $(s, n)$ such that $u_{[s;n]}(t) \in \text{Pol}_n[t]$ or $v_{[s;n]}(t) \in \text{Pol}_n[t]$.

To make full use of the results for $\text{Sol}_I(s; n)$, we first consider a transfer of elements in $\text{Sol}_{II}(s; n)$ to $\text{Sol}_I(s; n)$. Recall from (4.18) and (4.23) that

$$k_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{-1} & 1 \\ -1 & \sqrt{-1} \end{pmatrix} \in SU(2)$$

gives an $M$-isomorphism

$$\pi_n(k_0): \text{Sol}_{II}(s; n) \sim \text{Sol}_I(s; n).$$

For $p(t), q(t) \in \text{Pol}_n[t]$ with $\pi_n(k_0)p(t) \in \mathbb{C}q(t)$, we write

$$p(t) \sim_{k_0} q(t).$$

Then, for $n \in 2\mathbb{Z}_{\geq 0}$, the polynomials $a_{[s;n]}(t), b_{[s;n]}(t), c_{[s;n]}(t) \in \text{Sol}_{II}(s; n)$ are transferred to $u_{[s;n]}(t), v_{[s;n]}(t) \in \text{Sol}_I(s; n)$ via $\pi_n(k_0)$ as follows.

Lemma 6.3. Let $n \in 2\mathbb{Z}_{\geq 0}$. We have the following.

1. $n \equiv 0 \pmod{4}$:
   a. $a_{[s;n]}(t) \sim_{k_0} u_{[s;n]}(t)$ for $s \in \mathbb{C}\setminus (I^-_0 \cup J_0)$.
   b. $b_{[s;n]}(t) \sim_{k_0} u_{[s;n]}(t)$ for $s \in J_0$.
   c. $c_{[s;n]}(t) \sim_{k_0} u_{[s;n]}(t)$ for $s \in I^-_0$.
   d. $b_{[s;n]}(t) \sim_{k_0} v_{[s;n]}(t)$ for $s \in I^+_0$.
is an even function, as and some constants We only give a proof of (1)(a); the other cases can be shown similarly. Let

\text{Proof.} We only give a proof of (1)(a); the other cases can be shown similarly. Let \( n \equiv 0 \mod 4 \) and \( s \in \mathbb{C}\setminus(I_0 \cap J_0) \). It follows from Theorem \[5.16\] that in this case \( a_{[s;n]}(t) \in \text{Sol}_{II}(s;n) \) and thus \( \pi_n(k_0)a_{[s;n]}(t) \in \text{Sol}_I(s;n) \). By [6.11], this implies that \( \pi_n(k_0)a_{[s;n]}(t) = c_1u_{[s;n]}(t) + c_2v_{[s;n]}(t) \) for some constants \( c_1, c_2 \in \mathbb{C} \). We wish to show \( c_2 = 0 \). To do so, it suffices to show that \( \pi_n(k_0)a_{[s;n]}(t) \) is an even function, as \( u_{[s;n]}(t) \) and \( v_{[s;n]}(t) \) are even and odd functions, respectively. It follows from Theorem [5.19] that \( m_1^I \) acts on \( a_{[s;n]}(t) \) trivially. Since the linear map \( \pi_n(k_0) : \text{Sol}_{II}(s;n) \to \text{Sol}_I(s;n) \) is an \( M \)-isomorphism, by [4.24], this implies that \( m_1^I \) acts on \( \pi_n(k_0)a_{[s;n]}(t) \) trivially. Then, by (3.16) and the assumption \( n \equiv 0 \mod 4 \), we have

\[
\pi_n(k_0)a_{[s;n]}(-t) = (\sqrt{-1})^n\pi_n(k_0)a_{[s;n]}(-t) = \pi_n(m_1^I)\pi_n(k_0)a_{[s;n]}(t) = \pi_n(k_0)a_{[s;n]}(t).
\]

Now the assertion follows. \( \square \)

\textbf{Remark 6.4.} Lemma [6.3] in particular gives Gauss-to-Heun transformations. Moreover, by Remark [4.37] the transformations (1)(a) and (2)(a) reduce to Gauss-to-Gauss transformations for \( s = 0 \).

\textbf{Proposition 6.5.} Given \( n \in \mathbb{Z}_{\geq 0} \), the following conditions on \( s \in \mathbb{C} \) are equivalent.

(i) \( u_{[s;n]}(t) \in \text{Pol}_n[t] \).

(ii) One of the following holds.

(a) \( n \equiv 0 \mod 4 \) : \( s \in \mathbb{C} \).

(b) \( n \equiv 1 \mod 4 \) : \( s \in I_1 \).

(c) \( n \equiv 2 \mod 4 \) : \( s \in I_2^+ \cup I_2^- \).

(d) \( n \equiv 3 \mod 4 \) : \( s \in I_3 \).

\text{Proof.} For the case of \( n \) even, the assertions simply follow from Theorem [5.16] and Lemma [6.3]. For the case of \( n \) odd, suppose that \( n \equiv 1 \mod 4 \). By Theorem [5.16] we have \( \text{Sol}_{II}(s;n) \neq \{0\} \) if and only if \( s \in I_1 \). Moreover, for such \( s \in I_1 \), the dimension \( \dim_{\mathbb{C}} \text{Sol}_{II}(s;n) \) is \( \dim_{\mathbb{C}} \text{Sol}_{II}(s;n) = 2 \). Thus \( \dim_{\mathbb{C}} \text{Sol}_I(s;n) = 2 \) for \( s \in I_1 \). Now the assertion follows from (6.11). Since the case of \( n \equiv 3 \mod 4 \) can be handled similarly, we omit the proof. \( \square \)
Proposition 6.6. Given $n \in \mathbb{Z}_{\geq 0}$, the following conditions on $s \in \mathbb{C}$ are equivalent.

(i) $v_{[s,n]}(t) \in \text{Pol}_n[t]$.

(ii) One of the following holds.

(a) $n \equiv 0 \pmod{4}$ : $s \in I_0^+ \cup I_0^-$.
(b) $n \equiv 1 \pmod{4}$ : $s \in I_1$.
(c) $n \equiv 2 \pmod{4}$ : $s \in \mathbb{C}$.
(d) $n \equiv 3 \pmod{4}$ : $s \in I_3$.

Proof. Since the proof is similar to the one for Proposition 6.5, we omit the proof. \qed

Theorem 6.7. The following conditions on $(s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}$ are equivalent.

(i) $\text{Sol}_1(s;n) \not= \{0\}$.

(ii) One of the following conditions is satisfied.

- $n \equiv 0 \pmod{4}$ : $s \in \mathbb{C}$.
- $n \equiv 1 \pmod{4}$ : $s \in I_1$.
- $n \equiv 2 \pmod{4}$ : $s \in \mathbb{C}$.
- $n \equiv 3 \pmod{4}$ : $s \in I_3$.

Moreover, for such $(s,n)$, the space $\text{Sol}_1(s;n)$ may be described as follows.

(1) $n \equiv 0 \pmod{4}$ :

$$\text{Sol}_1(s;n) = \begin{cases} \mathbb{C}u_{[s,n]}(t) & \text{if } s \in \mathbb{C} \setminus (I_0^+ \cup I_0^-), \\ \mathbb{C}u_{[s,n]}(t) \oplus \mathbb{C}v_{[s,n]}(t) & \text{if } s \in I_0^+, \\ \mathbb{C}u_{[s,n]}(t) \oplus \mathbb{C}v_{[s,n]}(t) & \text{if } s \in I_0^-. \end{cases}$$

(2) $n \equiv 1 \pmod{4}$ :

$$\text{Sol}_1(s;n) = \mathbb{C}u_{[s,n]}(t) \oplus \mathbb{C}v_{[s,n]}(t) \quad \text{for } s \in I_1.$$  

(3) $n \equiv 2 \pmod{4}$ :

$$\text{Sol}_1(s;n) = \begin{cases} \mathbb{C}v_{[s,n]}(t) & \text{if } s \in \mathbb{C} \setminus (I_2^+ \cup I_2^-), \\ \mathbb{C}u_{[s,n]}(t) \oplus \mathbb{C}v_{[s,n]}(t) & \text{if } s \in I_2^+, \\ \mathbb{C}u_{[s,n]}(t) \oplus \mathbb{C}v_{[s,n]}(t) & \text{if } s \in I_2^- \end{cases}.$$  

(4) $n \equiv 3 \pmod{4}$ :

$$\text{Sol}_1(s;n) = \mathbb{C}u_{[s,n]}(t) \oplus \mathbb{C}v_{[s,n]}(t), \quad \text{for } s \in I_3.$$  

Proof. This is a summary of the results in Propositions 6.5 and 6.6. \qed

Theorem 6.8. For each $(s,n) \in \mathbb{C} \times \mathbb{Z}_{\geq 0}$ determined in Theorem 6.7, the $M$-representations on $\text{Sol}_{\Pi}(s;n)$ are classified as follows.

(a) $n \equiv 0 \pmod{4}$ :

$$\text{Sol}_{\Pi}(s;n) \simeq \begin{cases} (+,+), & \text{if } s \in \mathbb{C} \setminus (I_0^+ \cup I_0^-), \\ (+,+) \oplus (+,-) & \text{if } s \in I_0^+ \cup I_0^-, \\ (+,+) \oplus (+,-) & \text{if } s \in I_0^- \cup I_0^+ \cup I_0^- \cup I_0^+. \end{cases}$$
Here the characters \((\varepsilon, \varepsilon')\) stand for the ones on \(C_{u([s;n])}(t)\) and \(C_{v([s;n])}(t)\) at the same places in Theorem 6.7.

**Proof.** The case of \(n\) odd, the assertions simply follow from Theorem 5.19 via the \(M\)-isomorphism \(\pi_n(k_0)\): \(\text{Sol}_1(s; n) \sim \text{Sol}_1(s; n)\). Similarly, the assertions for \(n\) even are drawn by Lemma 6.3, Theorem 6.7, and Propositions 5.23, 5.24, and 5.32. \(\square\)

Recall from \([5.33]\) the subsets \(I^\pm((+,+)), I^\pm((-,-)), I(\mathbb{H}) \subset \mathbb{Z} \times \mathbb{Z}_{\geq 0}\). We now give the explicit description of \(\text{Hom}_M(\text{Sol}_1(s; n), \sigma) \neq \{0\}\).

**Theorem 6.9.** The following conditions on \((\sigma, s, n) \in \text{Irr}(M) \times \mathbb{C} \times \mathbb{Z}_{\geq 0}\) are equivalent.

(i) \(\text{dim}_C \text{Hom}_M(\text{Sol}_1(s; n), \sigma) \neq \{0\}\).

(ii) \(\dim_C \text{Hom}_M(\text{Sol}_1(s; n), \sigma) = 1\).

(iii) One of the following conditions holds.

- \(\sigma = (+,+): (s, n) \in \mathbb{C} \times 4\mathbb{Z}_{\geq 0}\).
- \(\sigma = (-,-): (s, n) \in \mathbb{C} \times (2 + 4\mathbb{Z}_{\geq 0})\).
- \(\sigma = (+,-): (s, n) \in I^\pm((+,+)) \cup I^\pm((-,-))\).
- \(\sigma = (-,+): (s, n) \in I^\pm((-,+)) \cup I^\pm((+,+))\).
- \(\sigma = \mathbb{H}: (s, n) \in I(\mathbb{H})\).

Moreover, for such \((\sigma, s, n)\), the space \(\text{Hom}_M(\text{Sol}_1(s; n), \sigma)\) is given as follows.

1. \(\sigma = (+,+): \) For \(n \in 4\mathbb{Z}_{\geq 0}\), we have

\[
\text{Hom}_M(\text{Sol}_1(s; n), (+,+)) = C_{u([s;n])}(t) \quad \text{for all } s \in \mathbb{C}.
\]

2. \(\sigma = (-,-): \) For \(n \in 2 + 4\mathbb{Z}_{\geq 0}\), we have

\[
\text{Hom}_M(\text{Sol}_1(s; n), (-,-)) = C_{v([s;n])}(t) \quad \text{for all } s \in \mathbb{C}.
\]

3. \(\sigma = (+,-): \) We have

\[
\text{Hom}_M(\text{Sol}_1(s; n), (+,-)) = \begin{cases} 
C_{u([s;n])}(t) & \text{if } (s, n) \in I^\pm((+,+)), \\
C_{v([s;n])}(t) & \text{if } (s, n) \in I^\pm((-,-)).
\end{cases}
\]

4. \(\sigma = (-,+): \) We have

\[
\text{Hom}_M(\text{Sol}_1(s; n), (-,+)) = \begin{cases} 
C_{u([s;n])}(t) & \text{if } (s, n) \in I^\pm((-,-)), \\
C_{v([s;n])}(t) & \text{if } (s, n) \in I^\pm((+,+)).
\end{cases}
\]
We give factorization formulas for $P_{\text{tridiagonal}}$ determinants associated to polynomial solutions to the Heun model. Since the theorem can be shown similarly to Theorem 5.34, we omit the proof.

**Proof.** Since the theorem can be shown similarly to Theorem 5.34, we omit the proof. □

**7. Sequences** $\{P_k(x; y)\}_{k=0}^\infty$ and $\{Q_k(x; y)\}_{k=0}^\infty$ of tridiagonal determinants

The aim of this section is to discuss about two sequences $\{P_k(x; y)\}_{k=0}^\infty$ and $\{Q_k(x; y)\}_{k=0}^\infty$ of $k \times k$ tridiagonal determinants associated to polynomial solutions to the Heun model $d\pi_n(D^s_5)f(t) = 0$. We give factorization formulas for $P_{\left[\frac{n+2}{2}\right]}(x; n)$ and $Q_{\left[\frac{n+1}{2}\right]}(x; n)$ as well as palindromic property of $\{P_k(x; y)\}_{k=0}^\infty$ and $\{Q_k(x; y)\}_{k=0}^\infty$ for $n$ even. (See Definition 1.15 for the definition of palindromic property.) These are achieved in Theorems 7.18 and 7.22 (factorization formulas), and 7.29 and 7.44 (palindromic property).

**7.1. Sequences** $\{P_k(x; y)\}_{k=0}^\infty$ and $\{Q_k(x; y)\}_{k=0}^\infty$ of tridiagonal determinants. We start with the definitions of $\{P_k(x; y)\}_{k=0}^\infty$ and $\{Q_k(x; y)\}_{k=0}^\infty$. Let $a(x)$ and $b(x)$ be two polynomials such that

$$a(x) = 2x(2x - 1) \quad \text{and} \quad b(x) = 2x(2x + 1).$$

(7.1)

For instance, for $x = 1, 2, 3, \ldots$, we have

$$a(1) = 1 \cdot 2, \quad a(2) = 3 \cdot 4, \quad a(3) = 5 \cdot 6, \ldots, \quad b(1) = 2 \cdot 3, \quad b(2) = 4 \cdot 5, \quad b(3) = 6 \cdot 7, \ldots.$$  

Similarly, for $x = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, we have

$$a\left(\frac{1}{2}\right) = 0 \cdot 1, \quad a\left(\frac{3}{2}\right) = 2 \cdot 3, \quad a\left(\frac{5}{2}\right) = 4 \cdot 5, \ldots, \quad b\left(\frac{1}{2}\right) = 1 \cdot 2, \quad b\left(\frac{3}{2}\right) = 3 \cdot 4, \quad b\left(\frac{5}{2}\right) = 5 \cdot 6, \ldots.$$  

Clearly, the polynomials $a(x)$ and $b(x)$ satisfy

$$a(x) = b\left(\frac{2x - 1}{2}\right) \quad \text{and} \quad b(x) = a\left(\frac{2x + 1}{2}\right).$$

(7.2)

We define $k \times k$ tridiagonal determinants $P_k(x; y)$ and $Q_k(x; y)$ in terms of $a(x)$ and $b(x)$ as follows.

(1) $P_k(x; y)$:

- $k = 0 : P_0(x; y) = 1$,
- $k = 1 : P_1(x; y) = yx$. 

Remark 7.4. Moreover, for instance, for $Q_k(x,y)$:

- $k \geq 2 : P_k(x;y) = \begin{vmatrix} yx & a(1) & \cdots & \cdots & a(2k) \\ -a \left( \frac{y}{2} \right) & (y-4)x & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ -a \left( \frac{y-2k+2}{2} \right) & \cdots & \cdots & (-4k+8)x & a(k-1) \\ -a \left( \frac{y-2k+4}{2} \right) & \cdots & \cdots & (y-4k+4)x & \end{vmatrix}$

(2) $Q_k(x;y)$:

- $k = 0 : Q_0(x;y) = 1$, 
- $k = 1 : Q_1(x;y) = (y-2)x$, 
- $k \geq 2 : Q_k(x;y) = \begin{vmatrix} (y-2)x & b(1) & \cdots & \cdots & b(2k) \\ -b \left( \frac{y}{2} \right) & (y-6)x & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ -b \left( \frac{y-2k+2}{2} \right) & \cdots & \cdots & (y-4k+6)x & b(k-1) \\ -b \left( \frac{y-2k+4}{2} \right) & \cdots & \cdots & (y-4k+2)x & \end{vmatrix}$

Example 7.3. Here are a few examples of $P_k(x;4)$ and $Q_k(x;6)$ for $k = 2, 3, 4$.

(1) $P_k(x;4)$:

$P_2(x;4) = \begin{vmatrix} 4x & 1 \cdot 2 \\ -3 \cdot 4 & 0 \end{vmatrix}$, \hspace{1cm} $P_3(x;4) = \begin{vmatrix} 4x & 1 \cdot 2 & 3 \cdot 4 \\ -3 \cdot 4 & 0 & 3 \cdot 4 \\ -1 \cdot 2 & -4x \end{vmatrix}$, \hspace{1cm} $P_4(x;4) = \begin{vmatrix} 4x & 1 \cdot 2 & 3 \cdot 4 & 5 \cdot 6 \\ -3 \cdot 4 & 0 & 3 \cdot 4 & 0 \\ -1 \cdot 2 & -4x & 0 & -12x \end{vmatrix}$

(2) $Q_k(x;6)$:

$Q_2(x;6) = \begin{vmatrix} 4x & 2 \cdot 3 \\ -4 \cdot 5 & 0 \end{vmatrix}$, \hspace{1cm} $Q_3(x;6) = \begin{vmatrix} 4x & 2 \cdot 3 & 4 \cdot 5 \\ -4 \cdot 5 & 0 & 4 \cdot 5 \\ -2 \cdot 3 & -4x \end{vmatrix}$, \hspace{1cm} $Q_4(x;6) = \begin{vmatrix} 4x & 2 \cdot 3 & 4 \cdot 5 & 6 \cdot 7 \\ -4 \cdot 5 & 0 & 4 \cdot 5 & 0 \\ -2 \cdot 3 & -4x & 0 & -12x \end{vmatrix}$

Moreover, for instance, for $y = 5$ and $k = 3$, we have

$P_3(x;5) = \begin{vmatrix} 5x & 1 \cdot 2 \\ -4 \cdot 5 & 0 \\ -2 \cdot 3 & -3x \end{vmatrix}$ and \hspace{1cm} $Q_3(x;5) = \begin{vmatrix} 3x & 2 \cdot 3 \\ -3 \cdot 4 & 0 \\ -1 \cdot 2 & -5x \end{vmatrix}$.

Remark 7.4. In general $P_k(x;y)$ and $Q_k(x;y)$ satisfy the following properties for specific $y$ and $k$.

(1) If $y = n \in 2 + 2\mathbb{Z}_{\geq 0}$, then $P_{\frac{n+2}{2}}(x;n)$ is anti-centrosymmetric:

$$P_{\frac{n+2}{2}}(x;n) = \begin{vmatrix} nx & a(1) \\ -a \left( \frac{n}{2} \right) & (n-4)x & a(2) \\ \vdots & \vdots & \ddots & \cdots \\ -a \left( \frac{n-2}{2} \right) & (n-8)x & a(3) \\ \vdots & \vdots & \cdots & \cdots \\ -a(2) & -(n-4)x & a \left( \frac{n}{2} \right) \\ -a(1) & -nx \end{vmatrix}.$$ (7.5)
(2) If \( y = n \in 4 + 2\mathbb{Z}_{\geq 0} \), then \( Q_{\frac{n}{2}}(x; n) \) is also anti-centrosymmetric:

\[
Q_{\frac{n}{2}}(x; n) = \begin{vmatrix}
(n-2)x & b(1) & b(2) & b(3) \\
-b\left(\frac{n-2}{2}\right) & (n-6)x & b(2) & b(3) \\
& -b\left(\frac{n-4}{2}\right) & (n-10)x & b(3) \\
& & \ldots & \ldots & \ldots \\
& & & -b(2) & -(n-6)x & b\left(\frac{n-2}{2}\right) \\
& & & & -b(1) & -(n-2)x
\end{vmatrix}.
\] (7.6)

(3) It follows from (7.2) that for \( y = n \in 3 + 2\mathbb{Z}_{\geq 0} \), we have

\[
P_{\frac{n+1}{2}}(x; n) = \begin{vmatrix}
(n-2)x & a(1) & a(2) & a(3) \\
-b\left(\frac{n-1}{2}\right) & (n-4)x & a(2) & a(3) \\
& -b\left(\frac{n-3}{2}\right) & (n-8)x & a(3) \\
& & \ldots & \ldots & \ldots \\
& & & -b(2) & -(n-6)x & a\left(\frac{n-1}{2}\right) \\
& & & & -b(1) & -(n-2)x
\end{vmatrix}.
\] (7.7)

\[
Q_{\frac{n+1}{2}}(x; n) = \begin{vmatrix}
(n-2)x & b(1) & b(2) & b(3) \\
-a\left(\frac{n-1}{2}\right) & (n-6)x & b(2) & b(3) \\
& -a\left(\frac{n-3}{2}\right) & (n-10)x & b(3) \\
& & \ldots & \ldots & \ldots \\
& & & -a(2) & -(n-4)x & b\left(\frac{n-1}{2}\right) \\
& & & & -a(1) & -nx
\end{vmatrix}.
\] (7.8)

We also have \( P_1(x; 1) = x \) and \( Q_1(x; 1) = -x \). Therefore,

\[
Q_{\frac{n+1}{2}}(x; n) = (-1)^{\frac{n+1}{2}} P_{\frac{n+1}{2}}(x; n) \quad \text{for } n \in 1 + 2\mathbb{Z}_{\geq 0}.
\] (7.9)

The tridiagonal determinants \( P_k(x; y) \) and \( Q_k(x; y) \) enjoy the following property.

**Lemma 7.10.** For \( k \in \mathbb{Z}_{\geq 0} \), we have

\[
P_k(-x; y) = (-1)^k P_k(x; y) \quad \text{and} \quad Q_k(-x; y) = (-1)^k Q_k(x; y).
\]

**Proof.** The identities for \( k = 0, 1 \) clearly hold by definition. The assertions for \( k \geq 2 \) follow from the following general property of \( k \times k \) tridiagonal determinants:

\[
\begin{vmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
& \ldots & \ldots \\
c_{k-2} & a_{k-2} & \ldots \\
& c_{k-1} & a_{k-1} \\
& \llap{-}a_k & \llap{-}a_k
\end{vmatrix} = (-1)^k \begin{vmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
& \ldots & \ldots \\
c_{k-2} & a_{k-2} & \ldots \\
& c_{k-1} & a_{k-1} \\
& \llap{-}a_k & \llap{-}a_k
\end{vmatrix}.
\] (7.11)

From the next subsections \( y \) is taken to be \( y = n \in \mathbb{Z}_{\geq 0} \) and we shall discuss several properties of \( \{P_k(x; n)\}_{k=0}^{\infty} \) and \( \{Q_k(x; n)\}_{k=0}^{\infty} \).

### 7.2. Generating functions of \( \{P_k(x; n)\}_{k=0}^{\infty} \) and \( \{Q_k(x; n)\}_{k=0}^{\infty} \)

We first give the generating functions of \( \{P_k(x; n)\}_{k=0}^{\infty} \) and \( \{Q_k(x; n)\}_{k=0}^{\infty} \). Let \( u[s;n](t) \) be the local Heun function defined in (4.34). Write \( u[s;n](t) = \sum_{k=0}^{\infty} U_k(s; n)t^{2k} \) for the power series expansion at \( t = 0 \). It follows from
Proposition 7.16. We have

\[ u_{[s,n]}(t) = \sum_{k=0}^{\infty} P_k(s;n) \frac{t^{2k}}{(2k)!} \quad \text{and} \quad v_{[s,n]}(t) = \sum_{k=0}^{\infty} Q_k(s;n) \frac{t^{2k+1}}{(2k+1)!}. \]

Proof. We only show the identity for \( u_{[s,n]}(t) \); the assertion for \( v_{[s,n]}(t) \) can be shown similarly. We wish to show that \( U_k(s;n) = P_k(s;n)/(2k)! \) for all \( k \in \mathbb{Z}_{\geq 0} \). By definition, it is clear that \( U_0(s;n) = 1 = P_0(s;n) \) and \( U_1(s;n) = ns/2 = P_1(s;n)/2! \). We then assume that \( k \geq 2 \). It follows from (7.13) that \( U_k(s;n) \) can be given as

\[ U_k(s;n) = \frac{(-1)^k}{\prod_{j=1}^{k} a(j)} \frac{P_k(s;n)}{P_k(s;-n)} = \frac{P_k(s;n)}{\prod_{j=1}^{k} a(j)}. \]

Here, Lemma 7.10 is applied from the second identity to the third. Now the assertion follows from the identity \( \prod_{j=1}^{k} a(j) = (2k)! \).

For \( R_k(s;n) \in \{ P_k(s;n), Q_k(s;n) \} \), we define

\[ \text{Sol}_k(R;n) := \{ s \in \mathbb{C} : R_k(s;n) = 0 \}. \]

We recall from (5.3) the subsets \( I_0^+, I_1, I_2^+, I_3 \subset \mathbb{Z} \).
Corollary 7.17. Let \( n \in 2\mathbb{Z}_{\geq 0} \). Then \( \text{Sol}_{\frac{n+2}{2}}(P; n) \) and \( \text{Sol}_{\frac{n}{2}}(Q; n) \) are given as

\[
\text{Sol}_{\frac{n+2}{2}}(P; n) = \begin{cases} 
\mathbb{C} & \text{if } n \equiv 0 \pmod{4}, \\
I_2^+ \cup I_2^- & \text{if } n \equiv 2 \pmod{4} 
\end{cases}
\]

and

\[
\text{Sol}_{\frac{n}{2}}(Q; n) = \begin{cases} 
I_0^+ \cup I_0^- & \text{if } n \equiv 0 \pmod{4}, \\
\mathbb{C} & \text{if } n \equiv 2 \pmod{4} 
\end{cases}
\]

Further, for \( n \in 1 + 2\mathbb{Z}_{\geq 0} \), the sets \( \text{Sol}_{\frac{n+1}{2}}(P; n) \) and \( \text{Sol}_{\frac{n+1}{2}}(Q; n) \) are given as

\[
\text{Sol}_{\frac{n+1}{2}}(P; n) = \text{Sol}_{\frac{n+1}{2}}(Q; n) = \begin{cases} 
I_1 & \text{if } n \equiv 1 \pmod{4}, \\
I_3 & \text{if } n \equiv 3 \pmod{4}. 
\end{cases}
\]

Proof. It follows from Propositions 6.5 and 7.16 that the following conditions on \( s \in \mathbb{C} \) are equivalent for \( n \in 2\mathbb{Z}_{\geq 0} \):

(i) \( s \in \text{Sol}_{\frac{n+2}{2}}(P; n) \);

(ii) \( u_{[s:n]}(t) \in \text{Pol}_n[t] \);

(iii) \( s \in \begin{cases} 
\mathbb{C} & \text{if } n \equiv 0 \pmod{4}, \\
I_2^+ \cup I_2^- & \text{if } n \equiv 2 \pmod{4} 
\end{cases} \)

This concludes the assertion for \( \text{Sol}_{\frac{n+2}{2}}(P; n) \). The other cases can be shown similarly. \( \square \)

7.3. Factorization formulas of \( P_{\frac{n+2}{2}}(x; n) \) and \( Q_{\frac{n+1}{2}}(x; n) \). We next show the factorization formulas for \( P_{\frac{n+2}{2}}(x; n) \) and \( Q_{\frac{n+1}{2}}(x; n) \) for \( n \) even, and for \( P_{\frac{n+1}{2}}(x; n) \) and \( Q_{\frac{n+1}{2}}(x; n) \) for \( n \) odd. We remark that any product of the form \( \prod_{\ell=0}^{j-1} c_{\ell} \) with \( c_{\ell} \in \text{Pol}_n[t] \) (in particular, \( c_{\ell} \in \mathbb{C} \)) is regarded as \( \prod_{\ell=0}^{j-1} c_{\ell} = 1 \).

7.3.1. Factorization formulas of \( P_{\frac{n+2}{2}}(x; n) \) and \( Q_{\frac{n}{2}}(x; n) \). We start with \( P_{\frac{n+2}{2}}(x; n) \) and \( Q_{\frac{n}{2}}(x; n) \n\) for \( n \) even (see (7.5) and (7.6)). For \( n \in 2\mathbb{Z}_{\geq 0} \), let \( \alpha_n \) and \( \beta_n \) be the products of the coefficients of \( x \) on the main diagonal of \( P_{\frac{n+2}{2}}(x; n) \) and \( Q_{\frac{n}{2}}(x; n) \), respectively. Namely, we have

\[
\alpha_n = \prod_{\ell=0}^{\frac{n}{4}} (n - 4\ell) \quad \text{and} \quad \beta_n = \prod_{\ell=0}^{\frac{n-2}{4}} (n - 2 - 4\ell).
\]

It is remarked that \( \alpha_n \) and \( \beta_n \) may be given as follows.

\[
\alpha_n = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{4}, \\
(-4)^{\frac{n+2}{2}} \prod_{\ell=0}^{\frac{n-2}{2}} (1 + 2\ell)^2 & \text{if } n \equiv 2 \pmod{4} 
\end{cases}
\]

and

\[
\beta_n = \begin{cases} 
(-4)^{\frac{n}{4}} \prod_{\ell=0}^{\frac{n-4}{2}} (1 + 2\ell)^2 & \text{if } n \equiv 0 \pmod{4}, \\
0 & \text{if } n \equiv 2 \pmod{4} 
\end{cases}
\]
Theorem 7.18 (Factorization formulas of $P_{\frac{n+2}{2}}(x; n)$ and $Q_{\frac{n}{2}}(x; n)$). For $n \in 2\mathbb{Z}_{\geq 0}$, the polynomials $P_{\frac{n+2}{2}}(x; n)$ and $Q_{\frac{n}{2}}(x; n)$ are either 0 or factored as follows.

$$P_{\frac{n+2}{2}}(x; n) = \begin{cases} 0 & \text{ if } n \equiv 0 \pmod{4}, \\ \alpha_n \prod_{\ell=0}^{\frac{n-2}{2}} (x^2 - (4\ell + 1)^2) & \text{ if } n \equiv 2 \pmod{4} \end{cases}$$  \hspace{1cm} (7.19)

and

$$Q_{\frac{n}{2}}(x; n) = \begin{cases} 0 & \text{ if } n \equiv 0 \pmod{4}, \\ \beta_n \prod_{\ell=0}^{\frac{n-2}{2}} (x^2 - (4\ell + 3)^2) & \text{ if } n \equiv 2 \pmod{4}. \end{cases}$$  \hspace{1cm} (7.20)

Proof. We only demonstrate the proof for $P_{\frac{n+2}{2}}(x; n)$; the assertion for $Q_{\frac{n}{2}}(x; n)$ can be shown similarly. It follows from Corollary [7.17] that

$$\text{Sol}_{\frac{n+2}{2}}(P; n) = \begin{cases} C & \text{ if } n \equiv 0 \pmod{4}, \\ I_2^+ \cup I_2^- & \text{ if } n \equiv 2 \pmod{4}. \end{cases}$$

In particular, as $\alpha_n$ is the product of the coefficients of $x$ in $P_{\frac{n+2}{2}}(x; n)$, we have

$$P_{\frac{n+2}{2}}(x; n) = \begin{cases} 0 & \text{ if } n \equiv 0 \pmod{4}, \\ \alpha_n \prod_{s \in I_2^+ \cup I_2^-} (x - s) & \text{ if } n \equiv 2 \pmod{4}. \end{cases}$$

Now the assertion follows from the identity $\prod_{s \in I_2^+ \cup I_2^-} (x - s) = \prod_{\ell=0}^{\frac{n-2}{2}} (x^2 - (4\ell + 1)^2)$.

Remark 7.21. The factorization formulas (7.19) and (7.20) can also be obtained from [18, Prop. 5.11]. In fact, in [18], the dimension $\text{dim}_C \text{Sol}(s; n)(= \text{dim}_C \text{Sol}(s; n))$ was determined by using (7.19) and (7.20) (see Remark 5.18).

7.3.2. Factorization formula of $P_{\frac{n+1}{2}}(x; n)$. We next consider $P_{\frac{n+1}{2}}(x; n)$ and $Q_{\frac{n+1}{2}}(x; n)$ for $n$ odd (see (7.7) and (7.8)). As shown in (7.9), the determinant $Q_{\frac{n+1}{2}}(x; n)$ is given as

$$Q_{\frac{n+1}{2}}(x; n) = (-1)^{\frac{n+1}{2}} P_{\frac{n+1}{2}}(x; n) \quad \text{ for } n \in 1 + 2\mathbb{Z}_{\geq 0}. $$

It thus suffices to only consider $P_{\frac{n+1}{2}}(x; n)$. For $n$ odd, let $\gamma_n$ be the product of the coefficients of $x$ on the main diagonal of $P_{\frac{n+1}{2}}(x; n)$, namely,

$$\gamma_n = \prod_{\ell=0}^{\frac{n-1}{2}} (n - 4\ell).$$

We remark that $\gamma_n$ may be given as

$$\gamma_n = \begin{cases} (-1)^{\frac{n+1}{2}} \prod_{\ell=0}^{\frac{n-3}{2}} (3 + 2\ell) & \text{ if } n \equiv 1 \pmod{4}, \\ (-1)^{\frac{n+1}{2}} \prod_{\ell=0}^{\frac{n-3}{2}} (3 + 2\ell) & \text{ if } n \equiv 3 \pmod{4}. \end{cases}$$
Theorem 7.22 (Factorization formula of $P_{n+1}(x;n)$). For $n \in 1+2\mathbb{Z}_{\geq 0}$, the polynomial $P_{n+1}(x;n)$ is factored as

$$P_{n+1}(x;n) = \gamma_n \prod_{\ell=0}^{n-1} (x - (n - 1) + 4\ell).$$

(7.23)

Equivalently, we have

$$P_{n+1}(x;n) = \begin{cases} 
\gamma_n x \prod_{\ell=1}^{n-1} (x^2 - (4\ell)^2) & \text{if } n \equiv 1 \pmod{4}, \\
\gamma_n \prod_{\ell=0}^{n-1} (x^2 - (4\ell + 2)^2) & \text{if } n \equiv 3 \pmod{4}.
\end{cases}$$

Proof. Since the argument is similar to that for Theorem 7.18, we omit the proof. \qed

The factorization formula of $P_{n+1}(x;n)$ in (7.23) suggests that one may express $P_{n+1}(x;n)$ in terms of the so-called Sylvester determinant $\text{Sylv}(x;n)$ (1.21). The factorization formula (7.25) below for $\text{Sylv}(x;n)$ was first observed by Sylvester in 1854 ([35]). Later, a number of people such as Askey ([1]), Edelman–Kostlan ([10]), Holtz ([13]), Kac ([21]), Mazza ([30, p. 442]), Taussky-Todd ([37]), among others, gave several proofs. For more details about the Sylvester determinant $\text{Sylv}(x;n)$ as well as some related topics, see, for instance, [7, 8, 31, 32] and references therein. We remark that the factorization formula (7.25) also readily follows from a general theory of $\mathfrak{sl}(2, \mathbb{C})$ representations (see Proposition 8.3 and Section 10).

Fact 7.24 (Sylvester’s factorization formula). Let $n \in 1+\mathbb{Z}_{\geq 0}$. The Sylvester determinant $\text{Sylv}(x;n)$ in (1.21) is factored as

$$\text{Sylv}(x;n) = \prod_{\ell=0}^{n-1} (x - n + 2\ell).$$

(7.25)

Equivalently, we have

$$\text{Sylv}(x;n) = \begin{cases} 
\prod_{\ell=0}^{n-1} (x^2 - (2\ell + 1)^2) & \text{if } n \text{ is odd}, \\
x \prod_{\ell=1}^{n-1} (x^2 - (2\ell)^2) & \text{if } n \text{ is even}.
\end{cases}$$

(7.26)

Corollary 7.27. For $n$ odd, we have

$$P_{n+1}(x;n) = 2^{n+1} \text{Sylv}(\frac{1}{2}; \frac{n-1}{2}) \text{Sylv}(\frac{x}{2}; \frac{n-1}{2}).$$

Proof. By a direct computation one finds

$$\prod_{\ell=0}^{n-1} (x - n + 4\ell) = 2^{n+1} \text{Sylv}(\frac{1}{2}; \frac{n-1}{2})$$

and

$$\prod_{\ell=0}^{n-1} (x - (n - 1) + 4\ell) = 2^{n+1} \text{Sylv}(\frac{x}{2}; \frac{n-1}{2}).$$

Now the proposed identity follows from Theorem 7.22 as $\gamma_n = \prod_{\ell=0}^{n-1} (n - 4\ell)$. \qed
7.4. Palindromic properties of \( \{P_k(x; y)\}_{k=0}^{\infty} \) and \( \{Q_k(x; y)\}_{k=0}^{\infty} \). We finish this section by showing the palindromic properties of \( \{P_k(x; y)\}_{k=0}^{\infty} \) and \( \{Q_k(x; y)\}_{k=0}^{\infty} \) (see Definition 1.15). For \( s \in \mathbb{R}\setminus\{0\} \), we set

\[
\text{sgn}(s) := \begin{cases} +1 & \text{if } s > 0, \\ -1 & \text{if } s < 0. 
\end{cases}
\]

### 7.4.1. Palindromic property of \( \{P_k(x; y)\}_{k=0}^{\infty} \)

We start with \( \{P_k(x; y)\}_{k=0}^{\infty} \). Recall from Corollary 7.17 that \( \text{Sol}_{\frac{n+2}{2}}(P; n) \) for \( n \in 2\mathbb{Z}_{\geq 0} \) is given as

\[
\text{Sol}_{\frac{n+2}{2}}(P; n) = \begin{cases} \mathbb{C} & \text{if } n \equiv 0 \pmod{4}, \\ I_2^+ \cup I_2^- & \text{if } n \equiv 2 \pmod{4}. 
\end{cases}
\]

We then define a map \( \theta_{(P,n)} : \text{Sol}_{\frac{n+2}{2}}(P; n) \rightarrow \{\pm1\} \) as

\[
\theta_{(P,n)}(s) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{4}, \\ \text{sgn}(s) & \text{if } n \equiv 2 \pmod{2}. 
\end{cases}
\] (7.28)

**Theorem 7.29** (Palindromic property of \( \{P_k(x; y)\}_{k=0}^{\infty} \)). The pair \( \{P_k(x; y)\}_{k=0}^{\infty}, \{(2k)\}_{k=0}^{\infty} \) is a palindromic pair with degree \( \frac{n}{2} \). Namely, let \( n \in 2\mathbb{Z}_{\geq 0} \) and \( s \in \text{Sol}_{\frac{n+2}{2}}(P; n) \). Then we have \( P_k(s; n) = 0 \) for \( k \geq \frac{n+2}{2} \) and

\[
\frac{P_k(s; n)}{(2k)!} = \theta_{(P,n)}(s) \frac{P_{\frac{n}{2}-k}(s; n)}{(n-2k)!} \quad \text{for } k \leq \frac{n}{2}.
\] (7.30)

Equivalently, for \( k \leq \frac{n}{2} \), the palindromic identity (7.30) is given as follows.

1. \( n \equiv 0 \pmod{4} \): We have

\[
\frac{P_k(s; n)}{(2k)!} = \frac{P_{\frac{n}{2}-k}(s; n)}{(n-2k)!} \quad \text{for all } s \in \mathbb{C}.
\] (7.31)

2. \( n \equiv 2 \pmod{4} \): We have

\[
\frac{P_k(s; n)}{(2k)!} = \text{sgn}(s) \frac{P_{\frac{n}{2}-k}(s; n)}{(n-2k)!} \quad \text{for } s \in I_2^+ \cup I_2^-.
\] (7.32)

**Proof.** Take \( s \in \text{Sol}_{\frac{n+2}{2}}(P; n) \). It follows from the equivalence given in the proof of Corollary 7.17 that \( P_k(s; n) = 0 \) for \( k \geq \frac{n+2}{2} \). Thus, to prove the theorem, it suffices to show (7.31) and (7.32) for \( k \leq \frac{n}{2} \). We first show that

\[
\frac{P_k(s; n)}{(2k)!} = \pm \frac{P_{\frac{n}{2}-k}(s; n)}{(n-2k)!}.
\] (7.33)

It follows from Theorem 6.8 and Corollary 7.17 that \( M \) acts on \( u_{[s;n]}(t) \) as a character \( \chi_{(\varepsilon,\varepsilon')} \) in (3.12); in particular, we have

\[
u_{[s;n]}(t) = \chi_{(\varepsilon,\varepsilon')} (m_2^1)_{\pi_n} (m_2^1) u_{[s;n]}(t). 
\] (7.34)

By (3.16), the identity (7.33) is equivalent to

\[
u_{[s;n]}(t) = \chi_{(\varepsilon,\varepsilon')} (m_2^1) t^n u_{[s;n]} \left( -\frac{1}{t} \right).
\] (7.35)
As \( s \in \text{Sol}_{n+2}(P;n) \), by Proposition 7.16 we have

\[
u_{[s;n]}(t) = \sum_{k=0}^{\frac{n}{2}} P_k(s;n) \frac{t^{2k}}{(2k)!}.
\]

The identity (7.35) thus yields the identity

\[
\sum_{k=0}^{\frac{n}{2}} P_k(s;n) \frac{t^{2k}}{(2k)!} = \chi_{(\varepsilon,\varepsilon')}(m_2^1) \sum_{k=0}^{\frac{n}{2}} P_{\frac{n}{2}-k}(s;n) \frac{t^{2k}}{(n-2k)!}.
\]

(7.36)

Since \( \chi_{(\varepsilon,\varepsilon')}(m_2^1) \in \{ \pm 1 \} \), the identity (7.33) follows from (7.36).

In order to show (7.31) and (7.32), observe that Theorem 6.8 shows that the character \( \chi_{(\varepsilon,\varepsilon')} \) is given as

- \( \chi_{(\varepsilon,\varepsilon')} = \chi_{(+,+)} \) for \( n \equiv 0 \pmod{4} \) and \( s \in \mathbb{C} \),
- \( \chi_{(\varepsilon,\varepsilon')} = \chi_{(-,+)} \) for \( n \equiv 2 \pmod{4} \) and \( s \in I_2^+ \),
- \( \chi_{(\varepsilon,\varepsilon')} = \chi_{(+,-)} \) for \( n \equiv 2 \pmod{4} \) and \( s \in I_2^- \).

By Table 3, we have \( \chi_{(+,+)}(m_2^1) = \chi_{(-,+)}(m_2^1) = 1 \) and \( \chi_{(+,-)}(m_2^1) = -1 \). Now (7.31) and (7.32) follow from (7.36). \( \square \)

Theorem 7.29 in particular implies the factorial identity of \( P_{\frac{n}{2}}(s;n) \) (see (1.19)) as follows.

**Corollary 7.37** (Factorial identity of \( P_{\frac{n}{2}}(x;n) \)). Let \( n \in 2\mathbb{Z}_{\geq 0} \). Then the following hold.

1. \( n \equiv 0 \pmod{4} \) : We have

\[
P_{\frac{n}{2}}(s;n) = n! \quad \text{for all } s \in \mathbb{C}.
\]

2. \( n \equiv 2 \pmod{4} \) : We have

\[
P_{\frac{n}{2}}(s;n) = \text{sgn}(s)n! \quad \text{for } s \in I_2^+ \cup I_2^-.
\]

(7.38)

**Proof.** This is simply the case of \( k = \frac{n}{2} \) in (7.31) and (7.32). \( \square \)

We now give a proof for Corollary 1.24 in the introduction.

**Proof of Corollary 1.24.** It follows from Corollary 7.37 that it suffices to show (1.25). Suppose \( n \equiv 2 \pmod{4} \). Since

\[
I_2^+ \cup I_2^- = \left\{ \pm(4j+1) : j = 0, 1, 2, \ldots, \frac{n-2}{4} \right\},
\]

we have

\[
I_2^+ \cup I_2^- = \{ \pm 1, \pm 5, \pm 9 \ldots, \pm(n-2) \}.
\]

The identity (7.38) then concludes the assertion. \( \square \)

Theorem 7.29 shows that \( \{ P_k(x;y) \}_{k=0}^\infty, \{ (2k)! \}_{k=0}^\infty \) is a palindromic pair for \( n \) even. Proposition 7.39 below shows that it is not the case for \( n \) odd.

**Proposition 7.39.** The pair \( \{ P_k(x;y) \}_{k=0}^\infty, \{ (2k)! \}_{k=0}^\infty \) is not a palindromic pair for \( n \) odd.
Proof. Proof by contradiction. Suppose the contrary, that is, there exist \( n_0 \in 1 + 2\mathbb{Z}_{\geq 0} \) and a map \( d: \mathbb{Z}_{\geq 0} \to \mathbb{R} \) with \( d(n_0) \in \mathbb{Z}_{\geq 0} \) such that, for all \( s \in \text{Sol}_{d(n_0)+1}(P; n_0) \), we have \( P_k(s; n_0) = 0 \) for \( k \geq d(n_0) + 1 \) and

\[
\frac{P_k(s; n_0)}{(2k)!} = \pm \frac{P_{d(n_0)}(s; n_0)}{(2d(n_0) - 2k)!} \quad \text{for} \quad k \leq d(n_0).
\]

Take \( s \in \text{Sol}_{d(n_0)+1}(P; n_0) \). It follows from Proposition 7.16 that the palindromic property of \( \{P_k(x; y)\}_{k=0}^\infty, \{(2k)!\}_{k=0}^\infty \) implies

\[
u(s; n_0)(t) = \sum_{k=0}^{d(n_0)} P_k(s; n_0) \frac{t^{2k}}{(2k)!} \in \text{Pol}_{d(n_0)}[t]
\]

and

\[
u^{d(n_0)}\nu(s; n_0) \left( \frac{1}{t} \right) = \pm \nu(s; n_0)(t),
\]

which is, by (3.16), equivalent to

\[
\pi_{d(n_0)}(m^1_{2n})\nu(s; n_0)(t) = \pm \nu(s; n_0)(t).
\]

Since \( \nu(s; n_0)(-t) = \nu(s; n_0)(t) \), this further implies that \( C\nu(s; n_0)(t) \) is a one-dimensional representation of \( M \) for such \( (s, n_0) \). On the other hand, as \( n_0 \in 1 + 2\mathbb{Z}_{\geq 0} \), it follows from Proposition 6.5 that, for \( n_0 \equiv \ell \ (\text{mod} \ 4) \) for \( \ell = 1, 3 \), we have

\[
s \in I_\ell \cap \text{Sol}_{d(n_0)+1}(P; n_0).
\]

In particular, by Theorem 6.8, the space \( C\nu(s; n_0)(t) \oplus C\nu(s; n_0)(t) \) forms the unique two-dimensional irreducible representation of \( M \), which is a contradiction. Now the proposed assertion follows. \( \square \)

7.4.2. Palindromic property of \( \{Q_k(x; y)\}_{k=0}^\infty \). We next consider \( \{Q_k(x; y)\}_{k=0}^\infty \). Corollary 7.17 shows that

\[
\text{Sol}_\mathfrak{H}(Q; n) = \begin{cases} I_0^+ \cup I_0^- & \text{if} \ n \equiv 0 \ (\text{mod} \ 4), \\ \mathbb{C} & \text{if} \ n \equiv 2 \ (\text{mod} \ 4). \end{cases}
\]

We then define a map \( \theta_{(Q,n)}: \text{Sol}_\mathfrak{H}(Q; n) \to \{\pm 1\} \) as

\[
\theta_{(Q,n)}(s) = \begin{cases} \text{sgn}(s) & \text{if} \ n \equiv 0 \ (\text{mod} \ 4), \\ 1 & \text{if} \ n \equiv 2 \ (\text{mod} \ 4). \end{cases} \tag{7.40}
\]

Theorem 7.41 (Palindromic property of \( \{Q_k(x; y)\}_{k=0}^\infty \)). The pair \( \{\{Q_k(x; y)\}_{k=0}^\infty, \{(2k)!\}_{k=0}^\infty \} \) is a palindromic pair with degree \( \frac{n-2}{2} \). Namely, let \( n \in 2(1 + \mathbb{Z}_{\geq 0}) \) and \( s \in \text{Sol}_\mathfrak{H}(Q; n) \). Then we have \( Q_k(s; n) = 0 \) for \( k \geq \frac{n}{2} \) and

\[
\frac{Q_k(s; n)}{(2k+1)!} = \theta_{(Q,n)}(s) \frac{Q_{\frac{n-2}{2}-k}(s; n)}{(n-2k-1)!} \quad \text{for} \quad k \leq \frac{n-2}{2}. \tag{7.42}
\]

Equivalently, for \( k \leq \frac{n}{2} - 1 \), the palindromic identity (7.42) is given as follows.

(1) \( n \equiv 0 \ (\text{mod} \ 4) \): We have

\[
\frac{Q_k(s; n)}{(2k+1)!} = \text{sgn}(s) \frac{Q_{\frac{n-2}{2}-k}(s; n)}{(n-2k-1)!} \quad \text{for} \ s \in I_0^+ \cup I_0^- \tag{7.43}
\]
We have
\[
\frac{Q_k(s; n)}{(2k + 1)!} = \frac{Q_{\frac{n-2}{2} - k}(s; n)}{(n - 2k - 1)!}
\]
for all \(s \in \mathbb{C}\).

(7.44)

Proof. Since the argument goes similarly to the one for Theorem 7.29, we omit the proof. \(\square\)

Corollary 7.45 (Factorial identity of \(Q_{\frac{n-2}{2}}(s; n)\)). Let \(n \in 2(1 + \mathbb{Z}_{\geq 0})\). Then the following hold for \(Q_{\frac{n-2}{2}}(s; n)\) (see (1.20)).

(1) \(n \equiv 0 \pmod{4}\): We have
\[
Q_{\frac{n-2}{2}}(s; n) = \text{sgn}(s)(n-1)!
\]
for \(s \in I^+_0 \cup I^-_0\).

(7.46)

(2) \(n \equiv 2 \pmod{4}\): We have
\[
Q_{\frac{n-2}{2}}(s; n) = (n-1)!
\]
for all \(s \in \mathbb{C}\).

Proof. This is the case of \(k = \frac{n-2}{2}\) in (7.43) and (7.44). \(\square\)

Here is a proof of Corollary 1.26 in the introduction.

Proof of Corollary 1.26. By Corollary 7.45, it suffices to show (1.27). Suppose \(n \equiv 0 \pmod{4}\). Since
\[
I^+_0 \cup I^-_0 = \left\{ \pm (4j + 3) : j = 0, 1, 2, \ldots, \frac{n - 4}{4} \right\},
\]
we have
\[
I^+_0 \cup I^-_0 = \left\{ \pm 3, \pm 7, \pm 11, \ldots, \pm (n - 1) \right\}.
\]
Then (7.46) concludes the proposed identity. \(\square\)

We close this section by stating about the palindromity of the pair \(\{Q_k(x; y)\}_{k=0}^\infty, \{(2k+1)!\}_{k=0}^\infty\) for \(n\) odd.

Proposition 7.47. The pair \(\{Q_k(x; y)\}_{k=0}^\infty, \{(2k+1)!\}_{k=0}^\infty\) is not a palindromic pair for \(n\) odd.

Proof. As the proof is similar to the one for Proposition 7.39, we omit the proof. \(\square\)

8. Cayley continuants \(\{\text{Cay}_k(x; y)\}_{k=0}^\infty\) and Krawtchouk polynomials \(\{K_k(x; y)\}_{k=0}^\infty\)

In Section 7, it was shown that \(\{P_k(x; y)\}_{k=0}^\infty\) and \(\{Q_k(x; y)\}_{k=0}^\infty\) admit palindromic properties. In this short section we show that the sequences of Cayley continuants \(\{\text{Cay}_k(x; y)\}_{k=0}^\infty\) and Krawtchouk polynomials \(\{K_k(x; y)\}_{k=0}^\infty\) also admit palindromic properties. These are achieved in Theorem 8.8 for Cayley continuants and in Theorem 8.19 for Krawtchouk polynomials.

8.1. Cayley continuants \(\{\text{Cay}_k(x; y)\}_{k=0}^\infty\). We start with the definition of \(\{\text{Cay}_k(x; y)\}_{k=0}^\infty\). For each \(k \in \mathbb{Z}_{\geq 0}\), the \(k \times k\) tridiagonal determinants Cay\(_k(x; y)\) are defined as follows (\(\text{Cay}_0(x : y) = 1\), \(\text{Cay}_1(x; y) = x\), and \(p. 429\)).
Following [31], we refer to each $\text{Cay}_k(x; y)$ as a Cayley continuant. There are some combinatorial interpretations of them. For instance, they can be thought of as a generalization of the raising factorial (shifted factorial) $x^k := x(x+1) \cdots (x+k-1)$, the falling factorial $x^k := x(x-1) \cdots (x-k+1)$, and the factorial $k!$. Indeed, we have

\[ \text{Cay}_k(x; y) = x^k, \quad \text{Cay}_k(x; -x) = x^k, \quad \text{Cay}_k(1; -1) = k!. \]

For more details on a relationship with combinatorics, see [31].

Remark 8.1. When $y = n \in 1 + \mathbb{Z}_{\geq 0}$, the Cayley continuant $\text{Cay}_{n+1}(x; n)$ is the Sylvester determinant $\text{Sylv}(x; n)$ (see (1.21)). Thus, by (7.25), we have

\[ \text{Cay}_{n+1}(x; n) = \text{Sylv}(x; n) = \prod_{\ell=0}^{n}(x-n+2\ell). \quad (8.2) \]

The generating function of the Cayley continuants $\{\text{Cay}_k(x; y)\}_{k=0}^{\infty}$ is known and several proofs are in the literature (see, for instance, [21, 31, 37]). In Proposition 8.3 below, by utilizing the idea for the proof of Proposition 7.16, we shall provide another proof for the generating function as well as Sylvester’s formula (8.2). We remark that our argument for the formula (8.2) can be more abstract. (See Section 10.)

**Proposition 8.3.** We have

\[ (1 + t)^{\frac{x+y}{2}}(1 - t)^{\frac{y-x}{2}} = \sum_{k=0}^{\infty} \text{Cay}_k(x; y) \frac{t^k}{k!}. \quad (8.4) \]

In particular,

\[ \text{Cay}_{n+1}(x; n) = \prod_{\ell=0}^{n}(x-n+2\ell). \]

**Proof.** Let $E_+$ and $E_-$ be the elements of $\mathfrak{sl}(2, \mathbb{C})$ in (3.2). It then follows from (3.17) that the matrix $d\pi_n(E_+ + E_-)_B$ with respect to the ordered basis $B := \{1, t, t^2, \ldots, t^n\}$ is given as

\[
\begin{pmatrix}
0 & -1 & -2 & -3 & \cdots & -(n-1) & -n \\
-n & 0 & -2 & 0 & \cdots & -(n-1) & -n \\
-(n-1) & 0 & -3 & 0 & \cdots & -(n-1) & -n \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-3 & 0 & \cdots & \cdots & \cdots & 0 & \cdots \\
0 & -2 & \cdots & \cdots & \cdots & 0 & \cdots \\
-2 & \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\
1 & -1 & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}
\]
Therefore the Sylvester determinant $\text{Cay}_{n+1}(x; n) (= \text{Sylv}(x; n))$ is the characteristic polynomial of $d\pi_n(E_+ + E_-)$; thus, to show \([8.2]\), it suffices to find the eigenvalues of $d\pi_n(E_+ + E_-)$. We set

$$\mathcal{D}_{[x; n]} := (1 - t^2) \frac{d}{dt} + nt - x.$$ 

By \((3.17)\), for $p(t) \in \text{Pol}_n[t]$, we have

$$(d\pi_n(E_+) + d\pi_n(E_-) + x)p(t) = 0 \iff \mathcal{D}_{[x; n]}p(t) = 0.$$ 

Thus it further suffices to determine $s \in \mathbb{C}$ for which $\mathcal{D}_{[x; n]}p(t) = 0$ has a non-zero solution.

By separation of variables, any solution to $\mathcal{D}_{[x; n]}f(t) = 0$, where $f(t)$ is not necessarily a polynomial, is of the form

$$f(t) = c \cdot (1 + t) \frac{n+x}{2}(1 - t) \frac{n-x}{2}$$

for some constant $c$. It is clear that $(1 + t) \frac{n+x}{2}(1 - t) \frac{n-x}{2}$ becomes a polynomial if and only if $x = n - 2\ell$ for $\ell = 0, 1, \ldots, n$. Now Sylvester’s formula \((8.2)\) follows.

In order to show \((8.4)\), suppose that $\sum_{k=0}^\infty h_k(x; n)t^k$ is the power series solution of $\mathcal{D}_{[x; n]}f(t) = 0$ with $h_0(x; n) = 1$. Then, by a direct observation, the coefficients $h_k(x; n)$ for $k \geq 1$ satisfy the following recurrence relations:

$$h_1(x; n) = x,$$

$$h_{k+1}(x; n) = \frac{x}{k+1}h_k(x; n) + \frac{k-n-1}{k+1}h_{k-1}(x; n).$$

It then follows from Lemma\([9.5]\) in Section\([9]\) with \((7.11)\) that each $h_k(x; n)$ is given as

$$h_k(x; n) = \text{Cay}_k(x; n) = \frac{\text{Cay}_k(x; n)}{k!}.$$ 

Thus $\sum_{k=0}^\infty \text{Cay}_k(x; n)t^k$ is a solution of $\mathcal{D}_{[x; n]}f(t) = 0$; consequently, there exists a constant $c$ such that

$$c \cdot (1 + t) \frac{n+x}{2}(1 - t) \frac{n-x}{2} = \sum_{k=0}^\infty \frac{\text{Cay}_k(x; n)}{k!}t^k.$$ 

As both sides are equal to 1 at $t = 0$, we have $c = 1$; thus, the identity \((8.4)\) holds for $y = n \in \mathbb{Z}_{\geq 0}$. Since both sides of \((8.4)\) are well-defined for any $y \in \mathbb{C}$ for suitable values of $t$, the desired identity now holds.

**Remark 8.5.** The differential operator $d\pi_n(E_+ + E_-)$ is related to the study of $K$-type decomposition of the space of $K$-finite solutions to intertwining differential operators. In fact, let $X$ be an nilpotent element of $\mathfrak{sl}(2, \mathbb{C})$ defined in \((3.1)\). Then $d\pi_n(E_+ + E_-)$ is the realization of the differential operator $R(X)$ on the $K$-type $\text{Pol}_n[t]$ via the identification $\Omega^1 : \mathfrak{k} \rightarrow \mathfrak{sl}(2, \mathbb{C})$ in \((3.7)\) (see [27, Sect. 5]).

We write

$$g^{C}_{[x; n]}(t) := (1 + t) \frac{n+x}{2}(1 - t) \frac{n-x}{2}.$$
and set
\[ \text{Sol}_k(\text{Cay}; n) := \{ s \in \mathbb{C} : \text{Cay}_k(s; n) = 0 \}. \]

It follows from (8.2) (or, equivalently, from (8.4)) that
\[ \text{Sol}_{n+1}(\text{Cay}; n) = \{ n - 2\ell : \ell = 0, \ldots, n \}. \] (8.6)

**Corollary 8.7.** The following conditions of \( s \in \mathbb{C} \) are equivalent.

(i) \( g^C_{[s;n]}(t) \in \text{Pol}_n[t] \).

(ii) \( s \in \text{Sol}_{n+1}(\text{Cay}; n) \).

**Proof.** A direct consequence of Proposition 8.3. \( \square \)

8.2. **Palindromic property for \( \{\text{Cay}_k(x; y)\}_{k=0}^\infty \).** Now we show the palindromic property and factorial identity for the Cayley continuants \( \{\text{Cay}_k(x; y)\}_{k=0}^\infty \).

**Theorem 8.8** (Palindromic property of \( \{\text{Cay}_k(x; y)\}_{k=0}^\infty \) and factorial identity of \( \text{Cay}_n(x; n) \)). The pair \( \{\text{Cay}_k(x; y)\}_{k=0}^\infty, \{k!\}_{k=0}^\infty \) is a palindromic pair with degree \( n \). Namely, let \( n \in \mathbb{Z}_{\geq 0} \) and \( s \in \text{Sol}_{n+1}(\text{Cay}; n) \). Then we have \( \text{Cay}_k(s; n) = 0 \) for \( k \geq n + 1 \) and
\[
\frac{\text{Cay}_k(s; n)}{k!} = (-1)^{\frac{n-k}{2}} \frac{\text{Cay}_{n-k}(s; n)}{(n-k)!} \quad \text{for } k \leq n.
\]

In particular we have
\[
\text{Cay}_n(s; n) = (-1)^{\frac{n}{2}} \frac{n!}{2^n} \quad \text{for } s \in \text{Sol}_{n+1}(\text{Cay}; n).
\] (8.9)

**Proof.** Take \( s \in \text{Sol}_{n+1}(\text{Cay}; n) \). It follows from Corollary 8.7 that, for such \( s \), we have
\[
g^C_{[s;n]}(t) = \sum_{k=0}^{n} \frac{\text{Cay}_k(s; n)}{k!} t^k,
\] (8.10)

namely, \( \text{Cay}_k(s; n) = 0 \) for \( k \geq n + 1 \). Further, (8.10) shows that \( t^n g^C_{[s;n]} \left( \frac{1}{t} \right) \) is given as
\[
t^n g^C_{[s;n]} \left( \frac{1}{t} \right) = \sum_{k=0}^{n} \frac{\text{Cay}_{n-k}(s; n)}{(n-k)!} t^k.
\]

On the other hand, as \( g^C_{[x;n]}(t) = (1 + t)^{\frac{n}{2}} (1 - t)^{\frac{n}{2}} \), we also have
\[
t^n g^C_{[x;n]} \left( \frac{1}{t} \right) = (-1)^{\frac{n}{2}} g^C_{[s;n]}(t).
\]

Therefore,
\[
\sum_{k=0}^{n} \frac{\text{Cay}_{n-k}(s; n)}{(n-k)!} t^k = (-1)^{\frac{n-s}{2}} \sum_{k=0}^{n} \frac{\text{Cay}_k(s; n)}{k!} t^k,
\]

which yields
\[
\frac{\text{Cay}_k(x; n)}{k!} = (-1)^{\frac{n-s}{2}} \frac{\text{Cay}_{n-k}(x; n)}{(n-k)!} \quad \text{for all } k \leq n.
\]

The second identity is the case \( k = n \). This concludes the theorem. \( \square \)

We close this section by showing a proof of Corollary 1.30 in the introduction.
Proof of Corollary 1.30. As Cay\(_n(x; n) = \text{Sylv}(x; n)\), it follows from Sylvester’s factorization formula (1.22) or (7.26) that
\[
\text{Sol}_{n+1}(\text{Cay}; n) = \{ s \in \mathbb{C} : \text{Sylv}(s; n) = 0 \} = \begin{cases} 
\{0, \pm 2, \pm 4, \ldots , \pm n\} & \text{if } n \text{ is even}, \\
\{\pm 1, \pm 3, \pm 5, \ldots , \pm n\} & \text{if } n \text{ is odd}.
\end{cases}
\]
Now the proposed identities are direct consequences of (8.9).

8.3. Krawtchouk polynomials \(\{K_k(x; y)\}_{k=0}^{\infty}\). From the palindromic property of the Cayley continuants \(\{\text{Cay}_k(x; y)\}_{k=0}^{\infty}\), one may deduce that of a certain sequence \(\{K_k(x; y)\}_{k=0}^{\infty}\) of polynomials, where the polynomials \(K_k(x; n)\) for \(y = n \in \mathbb{Z}_{\geq 0}\) are Krawtchouk polynomials in the sense of [28, p. 130] (equivalently, the case of \(p = 2\) for [28, p. 151]). We close this section by discussing the palindromic property of \(\{K_k(x; y)\}_{k=0}^{\infty}\).

We start with the definition of \(K_k(x; y)\).

Definition 8.11. For \(k \in \mathbb{Z}_{\geq 0}\), we define a polynomial \(K_k(x; y)\) of \(x\) and \(y\) as
\[
K_k(x; y) := \sum_{j=0}^{k} (-1)^j \binom{x}{j} (y-x-k-j),
\]
where the binomial coefficient \(\binom{a}{m}\) is defined as
\[
\binom{a}{m} = \begin{cases} \frac{a(a-1)\cdots(a-m+1)}{m!} & \text{if } m \in 1 + \mathbb{Z}_{\geq 0}, \\
1 & \text{if } m = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

For \(y = n \in \mathbb{Z}_{\geq 0}\), the polynomials \(K_k(x; n)\) are called Krawtchouk polynomials ([28, p. 130]). In this paper, we also call the polynomials \(K_k(x; y)\) of two variables Krawtchouk polynomials.

Remark 8.12. Symmetric Krawtchouk polynomials (see, for instance, [24, p. 237]) are used to show Sylvester’s factorization formula (8.2) in [1]. We remark that the definition of the Krawtchouk polynomial \(K_k(x; n)\) in this paper is different from one in the cited paper; in particular, \(K_k(x; n)\) is non-symmetric.

It readily follows from Definition 8.11 that the Krawtchouk polynomials \(\{K_k(x; y)\}_{k=0}^{\infty}\) have the following generating function:
\[
(1 + t)^{y-x} (1-t)^x = \sum_{k=0}^{\infty} K_k(x; y) t^k. \tag{8.13}
\]

By comparing (8.13) with the generating function (8.4) of the Cayley continuants \(\{\text{Cay}_k(x; y)\}_{k=0}^{\infty}\), we have
\[
K_k(x; y) = \frac{\text{Cay}_k(y-2x; y)}{k!}. \tag{8.14}
\]
Then \(\text{Cay}_k(x; y)\) may be expressed explicitly as follows. Here we remark that in Proposition 8.15 below we use the notation \((m)^{j} := m(m+1)\cdots(m+j-1)\) for the raising factorial (shifted
factorial) instead of \((m, j)\) in (5.14) to make the comparison with the falling factorial \((m)_{j} := m(m - 1) \cdots (m - j + 1)\) clearer.

**Proposition 8.15.** We have

\[
\text{Cay}_k(x; y) = \sum_{j=0}^{k} \binom{k}{j} \left( \frac{x + y}{2} \right)^j \left( \frac{x - y}{2} \right)^{k-j}.
\]  

(8.16)

**Proof.** By (8.4) and (8.13), we have

\[
\text{Cay}_k(x; y) = (k!) \cdot K_k(y - x^2; y) = (k!) \cdot \sum_{j=0}^{k} (-1)^j \binom{y-x}{j} \binom{y+x}{k-j}.
\]

As \((-1)^j(m)_{j} = (-m)^{j}\), this concludes the proposed identity. \(\square\)

**Remark 8.17.** The identity (8.16) is also shown in [31, p. 356] slightly differently.

Via the identity (8.13), we are now going to give the factorization formula, the palindromic property, and the factorial identity for the Krawtchouk polynomials \(\{K_k(x; y)\}_{k=0}^{\infty}\).

**Proposition 8.18 (Factorization formula of \(K_{n+1}(x; n)\)).** For given \(n \in \mathbb{Z}_{\geq 0}\), we have

\[
K_{n+1}(x; n) = \frac{(-2)^{n+1}}{(n+1)!} \prod_{\ell=0}^{n} (x - \ell) = (-2)^{n+1} \binom{x}{n+1}.
\]

**Proof.** The proposed identity simply follows from (8.14) and Sylvester’s factorization formula (8.2). \(\square\)

We next show the palindromic property and factorial identity for Krawtchouk polynomials \(\{K_k(x; y)\}\). We denote by \(\{1\}_{k=0}^{\infty}\) the sequence \(\{a_k\}_{k=0}^{\infty}\) such that \(a_k = 1\) for all \(k\).

**Theorem 8.19 (Palindromic property of \(\{K_k(x; y)\}_{k=0}^{\infty}\) and factorial identity of \(K_n(x; n)\)).** The pair \((\{K_k(x; y)\}_{k=0}^{\infty}, \{1\}_{k=0}^{\infty})\) is a palindromic pair with degree \(n\). Namely, let \(n \in \mathbb{Z}_{\geq 0}\) and \(s \in \text{Sol}_{n+1}(K; n)\). Then we have \(K_k(s; n) = 0\) for \(k \geq n + 1\) and

\[
K_k(s; n) = (-1)^s K_{n-k}(s; n) \quad \text{for } k \leq n.
\]

In particular, we have

\[
K_n(s; n) = (-1)^s \quad \text{for } s \in \text{Sol}_{n+1}(K; n).
\]  

(8.20)

**Proof.** The proof is the same as that of Theorem 8.8 we omit the proof. \(\square\)

We close this section by giving a proof of Corollary 1.34 in the introduction.

**Proof of Corollary 1.34** It follows from Proposition 8.18 that

\[
\text{Sol}_{n+1}(K; n) = \{s \in \mathbb{C} : K_{n+1}(s; n) = 0\} = \{0, 1, 2, \ldots, n\}.
\]

Then (8.20) concludes the proposed identity. \(\square\)
9. Appendix A: Local Heun functions

In this appendix we collect several facts and lemmas for the local Heun function $H_l(a, q; \alpha, \beta, \gamma, \delta; z)$ at $z = 0$ that are used in the main part of this paper.

9.1. General facts. As in (4.30), we set

$$D_H(a, q; \alpha, \beta, \gamma, \delta; z) := \frac{d^2}{dz^2} \left( \gamma z + \frac{\delta}{z - 1} + \frac{\varepsilon}{z - a} \right) \frac{d}{dz} + \frac{\alpha \beta z - q}{z(z - 1)(z - a)}$$

with $\gamma + \delta + \varepsilon = \alpha + \beta + 1$. Then the equation

$$D_H(a, q; \alpha, \beta, \gamma, \delta; z) f(z) = 0$$

is called Heun’s differential equation. The $P$-symbol is

$$P \begin{cases} 0 & 1 & a & \infty \\ 0 & 0 & 0 & \alpha & z \\ 1 - \gamma & 1 - \delta & 1 - \varepsilon & \beta \end{cases}.$$ 

Let $H_l(a, q; \alpha, \beta, \gamma, \delta; z)$ stands for the local Heun function at $z = 0$ ([38]). As in [38], we normalize $H_l(a, q; \alpha, \beta, \gamma, \delta; z)$ so that $H_l(a, q; \alpha, \beta, \gamma, \delta; 0) = 1$. It is known that, for $\gamma \notin \mathbb{Z}$, the functions

$$H_l(a, q; \alpha, \beta, \gamma, \delta; z) \quad \text{and} \quad z^{1-\gamma} H_l(a, q'; \alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \delta; z)$$

with $q' := q + (1 - \gamma)(\varepsilon + a \delta)$ are two linearly independent solutions at $z = 0$ to the Heun equation $D_H(a, q; \alpha, \beta, \gamma, \delta; z) f(z) = 0$. (See, for instance, [29] and [34, p. 99].)

Remark 9.2. In [34, p. 99], there seems to be a typographical error on the formula

$$\lambda' = \lambda + (a + b + 1 - c)(1 - c).$$

This should read as

$$\lambda' = \lambda + (a + b + 1 - c + (t - 1)d)(1 - c).$$

Let $H_l(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{k=0}^{\infty} c_k z^k$ be the power series expansion at $z = 0$. In our normalization we have $c_0 = 1$. Then $c_k$ for $k \geq 1$ satisfy the following recurrence relations (see, for instance, [38, p. 34]).

$$-q + a \gamma c_1 = 0,$$

$$P_k c_{k-1} - (Q_k + q) c_k + R_k c_{k+1} = 0,$$

where

$$P_k = (k - 1 + \alpha)(k - 1 + \beta),$$

$$Q_k = k[(k - 1 + \gamma)(1 + a) + a \delta + \varepsilon],$$

$$R_k = (k + 1)(k + \gamma)a.$$

Lemma [36] below shows that each coefficient $c_k$ for $H_l(a, q; \alpha, \beta, \gamma, \delta; z) = \sum_{k=0}^{\infty} c_k z^k$ has a determinant representation.
Lemma 9.5 (see, for instance, [26, Lem. B.1]). Let \( \{a_k\}_{k \in \mathbb{Z}_{\geq 0}} \) be a sequence with \( a_0 = 1 \) generated by the following relations:

- \( a_1 = A_0 \);
- \( a_{k+1} = A_k a_k + B_k a_{k-1} \) (\( k \geq 1 \))

for some \( A_0, A_k, B_k \in \mathbb{C} \). Then \( a_k \) can be expressed as

\[
a_k = \begin{bmatrix}
  A_0 & -1 \\
  B_1 & A_1 & -1 \\
  & B_2 & A_2 & -1 \\
  & & \ddots & \ddots & \ddots \\
  & & & B_{k-2} & A_{k-2} & -1 \\
  & & & & B_{k-1} & A_{k-1}
\end{bmatrix}
\]

9.2. The local solutions \( u_{[s;n]}(t) \) and \( v_{[s;n]}(t) \). Now we consider the following local solutions to (4.33).

\[
\begin{align*}
  u_{[s;n]}(t) &= Hl(-1, -\frac{ns}{4}; -\frac{n}{2}, -\frac{n-1}{2}, \frac{1-n-s}{2}; t^2), \\
  v_{[s;n]}(t) &= tHl(-1, -\frac{n-2}{4}s; -\frac{n-1}{2}, -\frac{n-2}{2}, \frac{3-n-s}{2}, \frac{1-n-s}{2}; t^2).
\end{align*}
\]  

(9.6)  

(9.7)

Lemma 9.8. Let \( u_{[s;n]}(t) = \sum_{k=0}^{\infty} U_k(s;n)t^{2k} \) be the power series expansion of \( u_{[s;n]}(t) \) at \( t = 0 \). Then the coefficients \( U_k(s;n) \) for \( k \geq 1 \) satisfy the following recurrence relations.

\[
\begin{align*}
  U_1(s;n) &= E_0^u, \\
  U_{k+1}(s;n) &= E_k^u U_k(s;n) + F_k^u U_{k-1}(s;n),
\end{align*}
\]

where

\[
E_0^u = \frac{ns}{2}, \quad E_k^u = \frac{(n-4k)s}{(2k+1)(2k+2)} \quad \text{and} \quad F_k^u = \frac{(n-2k+1)(n-2k+2)}{(2k+1)(2k+2)}.
\]  

(9.9)

Proof. This follows from (9.3) and (9.4) for the specific parameters in (9.6). \( \square \)

Lemma 9.10. Let \( v_{[s;n]}(t) = \sum_{k=0}^{\infty} V_k(s;n)t^{2k+1} \) be the power series expansion of \( v_{[s;n]}(t) \) at \( t = 0 \). Then the coefficients \( V_k(s;n) \) (\( k \geq 1 \)) satisfy the following recurrence relations.

\[
\begin{align*}
  V_1(s;n) &= E_0^v, \\
  V_{k+1}(s;n) &= E_k^v V_k(s;n) + F_k^v V_{k-1}(s;n),
\end{align*}
\]

where

\[
E_0^v = \frac{(n-2)s}{6}, \quad E_k^v = \frac{(n-4k-2)s}{(2k+2)(2k+3)} \quad \text{and} \quad F_k^v = \frac{(n-2k)(n-2k+1)}{(2k+2)(2k+3)}.
\]  

(9.11)

Proof. This follows from (9.3) and (9.4) for the specific parameters in (9.7). \( \square \)

It follows from Lemmas 9.8, 9.10 and 9.5 that \( U_k(s;n) \) and \( V_k(s;n) \) have determinant representations

\[
U_k(s;n) = \begin{vmatrix}
  E_0^u & -1 \\
  F_1^u & E_1^u & -1 \\
  & F_2^u & E_2^u & -1 \\
  & & \ddots & \ddots & \ddots \\
  & & & F_{k-2}^u & E_{k-2}^u & -1 \\
  & & & & F_{k-1}^u & E_{k-1}^u
\end{vmatrix}
\]  

(9.12)
10. Appendix B: A proof of Sylvester’s formula

In this short appendix we provide a proof of Sylvester’s factorization formula

\[ \text{Sylv}(x; n) = \prod_{\ell=0}^{n} (x - n + 2\ell), \quad (10.1) \]

based on a general theory of \( \mathfrak{sl}(2, \mathbb{C}) \) representations. For other proofs and related topics, see the remark after Theorem 7.22 and Proposition 8.3. For possible wide audience of this appendix, we shall discuss a proof in two ways; one of which is abstract in nature (Section 10.1) and the other uses a concrete realization of irreducible representations (Section 10.2). We remark that the arguments in this appendix can be thought of as a baby case of the one discussed in Section 7.2.

10.1. Sylvester determinant \( \text{Sylv}(x; n) \) and \( \mathfrak{sl}(2, \mathbb{C}) \) representations I. Let \( E_+, E_- \), and \( E_0 \) be the elements of \( \mathfrak{sl}(2, \mathbb{C}) \) defined in (3.2), and let \( (d\rho_n, V_n) \) be an irreducible representation of \( \mathfrak{sl}(2, \mathbb{C}) \) with \( \dim \mathbb{C} V_n = n + 1 \). Then there exists an ordered basis \( \mathcal{B} \) of \( V_n \) such that the matrix \( d\rho_n(E_+ + E_-)_{\mathcal{B}} \) with respect to \( \mathcal{B} \) is given as

\[
d\rho_n(E_+ + E_-)_{\mathcal{B}} = \begin{pmatrix}
0 & -1 & & & \\
-1 & 0 & -2 & & \\
& -n & 0 & -3 & \\
& & -(n-1) & 0 & -n \\
& & & -(n-1) & 0 \\
& & & & -1 \\
\end{pmatrix} \quad (10.2)
\]

(cf. [14] Sect. 7.2]). Then the Sylvester determinant \( \text{Sylv}(x; n) \) (17.25)) is the characteristic polynomial of \( d\rho_n(E_+ + E_-)_{\mathcal{B}} \). As the leading coefficient of \( \text{Sylv}(x; n) \) is one, this implies that

\[ \text{Sylv}(x; n) = \prod_{\lambda \in \text{Spec}(d\rho_n(E_+ + E_-))} (x - \lambda), \quad (10.3) \]

where \( \text{Spec}(T) \) denotes the set of eigenvalues of a linear map \( T \). Thus, to show (10.1), it suffices to find the eigenvalues of \( d\rho_n(E_+ + E_-) \). Since \( E_+ + E_- \) is conjugate \( E_0 \) via an element of \( SU(2) \) (cf. [22] Thm. 4.34]), it is further sufficient to determine \( \text{Spec}(d\rho_n(E_0)) \). It is well-known that the eigenvalues of \( d\rho_n(E_0) \) are \( n - 2j \) for \( j = 0, 1, 2, \ldots, n-1, n \) (cf. [14] Sect. 7.2]). Therefore we have

\[ \text{Spec}(d\rho_n(E_+ + E_-)) = \{ n - 2j : j = 0, 1, 2, \ldots, n-1, n \}. \]

Now the desired factorization formula (10.1) follows from (10.3).
10.2. **Sylvester determinant** \( \text{Sylv}(x; n) \) and \( \mathfrak{sl}(2, \mathbb{C}) \) representations II. We now give some details of the arguments given in Section 10.1 with a concrete realization of irreducible \( \mathfrak{sl}(2, \mathbb{C}) \) representations.

Let \((\pi_n, \text{Pol}_n[t])\) be the irreducible representation of \( SU(2) \) defined in (3.15) and \((d\pi_n, \text{Pol}_n[t])\) the corresponding irreducible representation of \( \mathfrak{sl}(2, \mathbb{C}) \), so that \( d\pi_n(E_j) \) for \( j = +, -, 0 \) are the differential operators given in (3.17). It is easily checked that the matrix \( d\pi_n(E_+ + E_-) \) with respect to the ordered basis \( \mathcal{B} := \{1, t, t^2, \ldots, t^n\} \) is given as (10.2); therefore, we have

\[
\text{Sylv}(x; n) = \prod_{\lambda \in \text{Spec}(d\pi_n(E_+ + E_-))} (x - \lambda).
\] (10.4)

On the other hand, the matrix \( d\pi_n(E_0) \) is given as

\[
d\pi_n(E_0) = \text{diag}(n, n - 2, n - 4, \ldots, -n + 2, -n),
\]

where \( \text{diag}(m_0, m_1, \ldots, m_n) \) denotes a diagonal matrix. Thus the eigenvalues of \( d\pi_n(E_0) \) on \( \text{Pol}_n[t] \) are \( n - 2j \) for \( j = 0, 1, 2, \ldots, n - 1, n \).

For

\[
g_0 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in SU(2),
\]

we have

\[
\text{Ad}(g_0)(E_+ + E_-) = E_0.
\]

Thus,

\[
d\pi_n(E_0) = \pi_n(g_0)d\pi_n(E_+ + E_-)\pi_n(g_0)^{-1}.
\]

Therefore,

\[
\text{Spec}(d\pi_n(E_+ + E_-)) = \{n - 2j : j = 0, 1, 2, \ldots, n - 1, n\}.
\] (10.5)

Now (10.4) and (10.5) conclude the desired formula.

**Acknowledgements.** Part of this research was conducted during a visit of the first author at the Department of Mathematics of Aarhus University. He is appreciative of their support and warm hospitality during his stay.

The authors are grateful to Anthony Kable for his suggestion to study the \( K \)-type formulas for the Heisenberg ultrahyperbolic operator. They would also like to express their gratitude to Hiroyuki Ochiai for the helpful discussions on the hypergeometric equation and Heun equation, especially for Lemma 4.32. Their deep appreciation also goes to Hiroyoshi Tamori for sending his Ph.D. dissertation, from which they thought of an idea for the hypergeometric model.

The first author was partially supported by JSPS Grant-in-Aid for Young Scientists (JP18K13432).

**References**

1. R. Askey, *Evaluation of Sylvester type determinants using orthogonal polynomials*, Advances in Analysis, World Sci. Publ., 2005, pp. 1–16.
2. L. Barchini, A.C. Kable, and R. Zierau, *Conformally invariant systems of differential equations and prehomogeneous vector spaces of Heisenberg parabolic type*, Publ. Res. Inst. Math. Sci. 44 (2008), no. 3, 749–835.
3. *Conformally invariant systems of differential operators*, Adv. Math. 221 (2009), no. 3, 788–811.
4. A. Cayley, *On the determination of the value of a certain determinant*, Quart. J. Math. II (1858), 163–166, The collected mathematical papers of Arthur Cayley, VOL. III, pp. 120–123.
5. P.A. Clement, *A class of triple-diagonal matrices for test purposes*, SIAM Rev. 1 (1959), 50–52.
6. D.H. Collingwood and B. Shelton, A duality theorem for extensions of induced highest weight modules, Pacific J. Math. 146 (1990), no. 2, 227–237.
7. C.M. da Fonseca and E. Kilic, An observation on the determinant of a Sylvester-Kac type matrix, An. Ştiinţ. Univ. “Ovidius” Constanţa Ser. Mat. 28 (2020), 111–115.
8. C.M. da Fonseca, D.A. Mazilu, I. Mazilu, and H.T. Williams, The eigenpairs of a Sylvester-Kac type matrix associated with a simple model for one-dimensional deposition and evaporation, Appl. Math. Lett. 26 (2013), 1206–1211.
9. T.T. Dahl, Knapp-Stein Operators and Fourier Transformations, Ph.D. thesis, Arhus University, 2019.
10. A. Edelman and E. Kostlan, The road from Kac’s matrix to Kac’s random polynomials, in: J. Lewis (Ed.), Proc. of the Fifth SIAM Conf. on Applied Linear Algebra (Philadelphia), SIAM, 1994, pp. 503–507.
11. J. Frahm, Conformally invariant differential operators on Heisenberg groups and minimal representations, preprint (2020), arXiv:2012.05952.
12. N. Gogin and M. Hirvensalo, On the generating function of discrete Chebyshev polynomials, Zap. Nauchn. Sem. POMI 448 (2016), 124–134.
13. O. Holtz, Evaluation of Sylvester type determinants using block-triangularization, Advances in Analysis, World Sci. Publ., 2005, pp. 395–405.
14. J.E. Humphreys, Introduction to Lie algebras and representation theory, Grad. Texts in Math., vol. 9, Springer-Verlag, New York, 1972.
15. A.C. Kable, K-finite solutions to conformally invariant systems of differential equations, Tohoku Math. J. 63 (2011), no. 4, 539–559, Centennial Issue.
16. , Conformally invariant systems of differential equations on flag manifolds for $G_2$ and their K-finite solutions, J. Lie Theory 22 (2012), no. 1, 93–136.
17. , The Heisenberg ultrahyperbolic equation: The basic solutions as distributions, Pacific J. Math 258 (2012), no. 4, 839–894.
18. , The Heisenberg ultrahyperbolic equation: K-finite and polynomial solutions, Kyoto J. Math 52 (2012), no. 4, 839–894.
19. , On certain conformally invariant systems of differential equations II: Further study of type A systems, Tsukuba J. Math. 39 (2015), 39–81.
20. , The structure of the space of polynomial solutions to the canonical central systems of differential equations on the block Heisenberg groups: A generalization of a theorem of Karánya, Tohoku Math. J. 70 (2018), 523–545.
21. M. Kac, Random walk and the theory of Brownian motion, Amer. Math. Monthly 54 (1947), 369–390.
22. A.W. Knapp, Lie groups beyond an introduction, second edition, Progr. Math., vol. 140, Birkhäuser, Boston Press, New York, 2002.
23. T. Kobayashi and M. Pevzner, Differential symmetry breaking operators. I. General theory and F-method, Selecta Math. (N.S.) 22 (2016), 801–845.
24. R. Koekoek, P.A. Lesky, and R.F. Swarttouw, Hypergeometric Orthogonal Polynomials and Their q-Analogues, Springer Monographs in Mathematics, Springer-Verlag, Berlin Heidelberg, 2010.
25. B. Kostant, Verma modules and the existence of quasi-invariant differential operators, Noncommutative harmonic analysis, Lecture Notes in Math., vol. 466, Springer, 1975, pp. 101–128.
26. G. Kristensson, Second order differential equations: Special functions and their classification, Springer, New York, 2010.
27. T. Kubo and B. Ørsted, On the space of K-finite solutions to intertwining differential operators, Represent. Theory 23 (2019), 213–248.
28. F.J. MacWilliams and N.J.A. Sloane, The theory of error-correcting codes, North-Holland Mathematical Library, vol. 16, North-Holland Publishing Co., Amsterdam, 1977.
29. R.S. Maier, On reducing the Heun equation to the hypergeometric equation, J. Differential Equations 213 (2005), 171–203.
30. T. Muir, The Theory of Determinants in the Historical Order of Development, vol. II, 1911, University of Michigan Library, 2006, Reprinted.
31. E. Munarini and D. Torri, Cayley continuants, Theoret. Comput. Sci. 347 (2005), 353–369.
32. K. Nomura and P. Terrerlager, Krawtchouk polynomials, the Lie algebra sl$_2$, and Leonard pairs, Linear Algebra Appl. 437 (2012), 345–375.
33. J. Rawnsley and S. Sternberg, On representations associated to the minimal nilpotent coadjoint orbit of $SL(3,\mathbb{R})$, Amer. J. Math. 104 (1982), no. 6, 1153–1180.
34. S.Y. Slavyanov and W. Lay, Special functions. A unified theory based on singularities, Oxford Mathematical Monographs, Oxford Science Publications, Oxford University Press, Oxford, 2000, With a foreword by Alfred Seeger.
35. J.J. Sylvester, Théorème sur les déterminants de M. Sylvester, Nouvelles annales de mathématiques: journal des candidats aux écoles polytechnique et normale, le série 13 (1854), 305.
36. H. Tamori, *Classification and Construction of minimal representations*, Ph.D. thesis, The University of Tokyo, 2020.

37. O. Taussky and J. Todd, *Another look at a matrix of Mark Kac*, Linear Algebra Appl. **150** (1991), 341–360.

38. A. Ronveaux (ed.), *Heun’s differential equations*, Oxford Science Publications, Oxford University Press, Oxford, 1995, With contributions by F.M. Arscott, S. Yu. Slavyanov, D. Schmidt, G. Wolf, P. Maroni, and A. Duval.

39. P. Torasso, *Quantification géométrique, opérateurs d’entrelacement et représentations unitaires de (SL)3(R)*, Acta Math. **150** (1983), no. 3-4, 153–242.

40. R. Vidunas and G. Filipuk, *Parametric transformations between the Heun and Gauss hypergeometric functions*, Funkcial. Ekvac. **56** (2013), 271–321.

41. _____, *A classification of coverings yielding Heun-to-hypergeometric reductions*, Osaka J. Math. **51** (2014), 867–903.

Faculty of Economics, Ryukoku University, 67 Tsukamoto-cho, Fukakusa, Fushimi-ku, Kyoto 612-8577, Japan

Email address: toskubo@econ.ryukoku.ac.jp

Department of Mathematics, Aarhus University, Ny Munkegade 118 DK-8000 Aarhus C Denmark

Email address: orsted@imf.au.dk