On the Dynamics of Phase Transitions in Relativistic Scalar Field Theory¹

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Received February 12, 2021; revised May 22, 2021; accepted May 22, 2021

Abstract—We study the Kibble–Zurek scaling dynamics and universal second order phase transitions in a relativistic scalar field theory in 1 + 1 dimensions. Using tensor networks techniques as a non-perturbative non-equilibrium numerical tool for quantum field theory, we perform an analysis of the formation of topological defects in a non-equilibrium quantum system, as a realistic analogue toy model of the formation of cosmological defects in the large-scale structure of the Early Universe.

DOI: 10.1134/S1063772921100152

1. INTRODUCTION

Dynamics of topological defects in course of non-equilibrium symmetry breaking phase transitions is a topic of wide interest in various areas of physics, from cosmology to condensed matter [1–7] and low temperature systems. A current challenge in modern cosmology is whether the vacuum contains topological defects generated during the initial phase transitions in the Early Universe, such as domain walls, cosmic strings and monopoles.

We employ tensor networks (TN) and the time dependent variational principle to predict topological defect formation and the Kibble–Zurek scenario [8, 9] for a system undergoing a symmetry breaking phase transition in the context of a scalar field theory. We are interested in the non-equilibrium time evolution of a quantum system in 1 + 1 space-time dimensions which is driven through a phase transition, and the associated two point function responsible for the defect formation.

In this paper we use uniform matrix product states (MPS), in particular uniform MPS [10] to simulate the formation of topological defects within a relativistic φ⁴ scalar field theory [11, 12] in d = (1 + 1) dimensions [13]. The MPS is considered as an approximation of a defect state. We calculate the two point function in momentum space Gₖ(k) during a quantum phase transition [14] using a finite size lattice MPS approximation and show that Gₖ(k) is a measure of the defect formation and the defect density with two scales, the defect width dₖ and the defect density n, respectively.

The study of topological defects in the early Universe has a long interdisciplinary history, at the interface between particle physics, field theory, condensed matter, and cosmology. Current theoretical models predict a variety of topological defects that would have formed during the early evolution of the universe. The understanding of the large-scale formation of such structures within the inflationary dynamics of the early Universe would provide valuable information on the first moments around 10–35 s after the Big Bang.

Topological defects are relevant to cosmology, providing mechanisms for generating the initial density fluctuations and anisotropies in the cosmic microwave background, explaining the cosmological phase transitions generated by the primordial expansion and cooling of the Universe, predicting observational signatures concerning the non-random distribution of matter on various cosmological scales and leading to several constraints which are significant to very early universe cosmology, such as the symmetry-breaking phase transition or the non-vanishing net baryon to entropy ratio.

During a cosmological transition at around at around 10²⁷ K, the symmetry of vacuum was most likely broken and spacelike separated vacuum regions were formed. If the phase of the vacuum is disordered, a number of topological defects may occur. In a condensed matter system that is driven through a second-order phase transition [15], various topological defects, such as vortices or domain walls are formed at

¹ Paper presented at the Fourth Zeldovich meeting, an international conference in honor of Ya.B. Zeldovich held in Minsk, Belarus on September 7–11, 2020. Published by the recommendation of the special editors: S.Ya. Kilin, R. Ruffini, and G.V. Vereshchagin.
a density dependent on the speed of the underlying transition.

The universal mechanism of defect formation is known as the Kibble–Zurek Mechanism (KZM) [16, 17] and has been confirmed by various experiments. The KZM mechanism predicts the universal power-law scaling of the defect density dynamics across a continuous (second-order) quantum phase transition and describes the scaling laws when the system is driven through the transition. We revisit the KZM mechanism by studying the out of equilibrium dynamics across a second order phase transition and the phase ordering induced by a quench in the system.

2. DEFECT FORMATION IN $\phi^4$ SCALAR FIELD THEORY

In a $\phi^4$ scalar field theory, the two point function in $d = (1 + 1)$ momentum space for defect dynamics contains a contribution given by the density of defects $n$ and secondary contribution given by the defect width $d$ corresponding to separate scales in the system. The defect density scales universally with the equilibrium critical exponents at a scale characterizing the rate of the phase transition. The dynamical behavior of the system contains only two relevant scales, given by the defect density $n$ and the defect width $d$. If the two scales are sufficiently far from each other, the corresponding two point function in momentum space will be approximated only by the contribution of the defect distribution $G_{\text{def}}(k/n)$ and the contribution given by the defect profile $G_{\text{def}}(kd_{\text{def}})$.

In a lattice theory, the defects would be approximated with field configurations interpolating between different sign vacua, therefore the defect density would be given by the counting of the field configuration zeroes. In a quantum field theory, the explicit counting of the defects in the system becomes ultraviolet (UV) sensitive, therefore it can’t be used to approximate the defect density in the system. As the solution has to be UV independent [18], the defect dynamics is given by the long distance dynamics, where the UV behavior can be ignored. Different quantum phases in the ground state are described by the long range order (LRO) critical regime. Therefore, the two point function characterizes the formation of defects via the defect profile term and provides a full quantum description for the defect density in the system. Near the critical point, the analytic solution is obtained by universality means [19]. The critical behavior is described by the Ising universality class via Kramers–Wannier duality that establishes a relation between the disorder operators for the two-point correlation functions at different couplings.

The $\phi^4$ scalar field undergoes a second-order quantum phase transition in its ground state $\Omega(g)$ and can be described by the action

$$S[\phi] = \int dx dt \left[ \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\partial_x \phi)^2 - \frac{\mu^2}{4!} \phi^4 - \frac{\lambda_2}{2!} \phi^4 \right],$$

where the bare coupling is $g_0 = \lambda_0 / \mu_0$ and the reduced coupling $\epsilon = (g_0 - g_c) / |g_c|$. If such system undergoes a symmetry breaking phase transition, a number of defects will be present and the two point function $G_2(k) = \langle \phi(-k)\phi(k) \rangle$ can be written as

$$G_2(k) = \frac{v^2}{n} G_{\text{cor}}(k/n) G_{\text{def}}(kd_{\text{def}})$$

with $v$ as the vacuum expectation value and $d_{\text{def}}$ as the defect width.

The defect profile contribution takes the form

$$G_{\text{def}}(kd_{\text{def}}) = \frac{k^2}{4v^2} |\phi_{\text{def}}(k)|^2 = \left( \frac{1}{2} \frac{\pi kd_{\text{def}}}{\sinh \left( \frac{1}{2} \pi kd_{\text{def}} \right)} \right)^2.$$  

The theory has a global $\mathbb{Z}_2$ symmetry with $\phi \rightarrow -\phi$. Here, the defect spatial profile is given by $G_{\text{def}}(kd_{\text{def}})$, while defect spatial distribution will be described by $G_{\text{cor}}(k/n)$. At the same time, $G_{\text{cor}}(k/n)$ takes the form

$$G_{\text{cor}}(k/n) = \alpha_i e^{-\delta_{\text{cor}}(k/n)^2} + \frac{\beta_i}{1 + \beta_2(k/n)^2}.$$  

The defect density can be approximated as

$$n \approx \xi^{-d_{\text{co}}},$$

where $d_{\text{co}}$ is the defect co-dimension, given by the difference between the dimension of space and the dimension of the defect.

Let us assume that the system driven through a second order phase transition is described by the complete set of couplings $g = \{g_1, g_2, \ldots, g_N\}$ with one coupling $g_i$ modified by a process outside the system.

The transition has a universal character as in the symmetric phase, the length scale $\xi$ has also universal properties inherited from the dynamical scale, together with the critical exponents of the transition. The density of topological defects obeys a power law in the quench rate, with an exponent given by a combination between the critical exponents of the second order phase transition.
In this case, the temperature difference between
the current state of the system and the critical
temperature is approximated by the reduced coupling $\epsilon$

$$\epsilon = \frac{(g - g_c)}{g_c},$$  \hspace{1cm} (6)

where $\epsilon = 0$ is the critical point at $g = g_c$.

In this scenario, the second order phase transition
will be associated to a global symmetry breaking of a
symmetry group $G$ generating what is known as
dependent reduced coupling $\epsilon(t)$, a
symmetry broken phase will occur beyond the critical
point. The system in the symmetry broken phase can
continue to evolve with a correlation length $\xi$ that
changes very fast during the materialization process on
a timescale $t_{\text{therm}}$. Near the critical point [20], an
equilibrium state may be described by a time-scale $\tau$ 
given by the divergent relaxation time

$$\tau \approx t_0 |\epsilon|^{-\mu}, \hspace{1cm} \epsilon > 0.$$  \hspace{1cm} (8)

The quench time-scale $\tau_Q$ will lead to a finite trans-
transition rate corresponding to $\epsilon = -t/t_Q$, while the sys-

tem dynamics will be described by both the contribu-
tions from states breaking the $Z_2$ symmetry.

When the second-order phase transition is crossed,
the system no longer evolves adiabatically due to the
critical slowing down near the critical point. If the
ground state of the system is characterized by a topoi-

gically nontrivial vacuum, disparate local ground
state regions will lead to the topological defect forma-
tion. The dynamics is described by the order param-
ter $\phi$, with $\phi = 0$ for the symmetric case $\epsilon > 0$ and
$\phi \neq 0$ in the symmetry broken case $\epsilon < 0$. The order
parameter function $\varphi$ is a thermodynamic function that is dif-
ferent in each phase, and therefore can be used to dis-

After the initial relaxation, the system is charac-
terized by a distribution of defects with the defect den-
sity $n$. The correlation length $\xi$ associated with the
order parameter two point function takes the divergent form

$$\xi = \xi_0 |\epsilon|^{-\nu}, \hspace{1cm} \epsilon > 0.$$  \hspace{1cm} (7)

Moreover, the materialization scale of the defects is
much larger than any other scale. The phase transition
will occur at the universal length scale $\xi$. Any physical
state of a system that is initially in equilibrium, will be
excited in the neighbourhood of a second order phase
transition, with a finite correlation length during the
broken symmetry phase. In the symmetry broken
phase, the quantity $\xi$ sets the correlation scale of the
domains present in the system as a consequence of the
defect formation.

However, as the order parameter two point function
diverges, near the critical point in the broken sym-
metry phase, the correlation length becomes infinite.
The system that is in its ground state is driven through
a phase transition by the time dependent bare mass
$\mu^2(t)$. After the symmetry is broken, the two point
function $G_x(k) = \langle \psi(t, -k) \psi(t, k) \rangle$ is known.

Using the time dependent variational principle, the
time evolution of the state $|\psi\rangle$ can be analyzed. The
quantity $\bar{G}_x(k)$ can be associated to the quantity
$G_{\text{def}}(k)$ determined via the KZM mechanism by the
distribution of defects. In general, if the system is con-
sidered in equilibrium in the symmetric phase and

driven via a time dependent reduced coupling $\epsilon(t)$, a
symmetry broken phase will occur beyond the critical
point. The system in the symmetry broken phase can
continue to evolve with a correlation length $\xi$ that
changes very fast during the materialization process on
a timescale $t_{\text{therm}}$. Near the critical point [20], an
equilibrium state may be described by a time-scale $\tau$
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tions from states breaking the $Z_2$ symmetry.

Secondly, the system remains in equilibrium if the
relaxation time-scale $\tau$ is shorter than the rate that $\epsilon(t)$
changes, or $|\dot{\epsilon}|/|\epsilon|$. The system will be put out of equili-

In particular, if this change is slow, $\tau$ has the uni-

versal behavior from (8) and the adiabaticity break-
down occurs at the time $\dot{t}$ such as

$$\dot{t} = \tau(\dot{t}).$$  \hspace{1cm} (9)

The adiabatic impulse approximation associated
with the KZM mechanism requires that if the change
is sufficiently slow, the correlation lengths in the initial
state and during the symmetry breaking are exactly
equal $\xi = \xi$ and will control the defect density $n$ in
the system. Additionally, the correlation length $\xi$ is
preserved for long time after the second order phase
transition has ended and therefore has a universal char-

$$\xi = \xi_0 (\tau_0^{-1} \tau_Q)^{1/\nu}. \hspace{1cm} (11)$$

However, the relaxation timescale is given by the
inverse of the gap $\Delta$, associated with the scalar mass
$m_\phi$ with $\Delta = m_\phi = \xi^{-1}$ (Lorentz invariance) and equal
critical exponents $\mu$ and $\nu$. Therefore the correlation length $\xi$ (11) will become

$$\xi = \xi_0 (\Delta_0 \tau_Q)^{1/\nu}. \hspace{1cm} (12)$$
with $\Delta_0$ the vanishing gap term near the critical point
\[ \Delta = m_c = \Delta_0 |t|^\alpha. \] (13)

The $\phi^4$ scalar field critical exponents can be described by the universality class in the classical $d = 1 + 1$ Ising model with $\nu = 1$, $\xi \sim \xi_0$ (freeze-out condition) and the scaling laws
\[ \hat{t} = -\Delta_0^{-1/2} t_0^{-1/2}, \] (14)
\[ \hat{\xi} = -\Delta_0^{-1/2} \xi_0^{-1/2}, \] (15)
\[ \hat{\xi} \approx \xi_0 \Delta_0^{-1/2} t_0^{-1/2}. \] (16)

The simplest type of soliton which arises in $\phi^4$ theory in $(1+1)$ dimensions in the symmetry broken phase is a topological defect of co-dimension $d_{\text{co}} = 1$ known as a kink. The equations of motion have non-trivial solutions that carry a topological charge $Q = \pm 1$
\[ Q = \frac{1}{2\nu}(\phi(\infty) - \phi(-\infty)) \] (17)
with $d_k = \sqrt{-2/m_u^2}$ the defect width and two sign vacuum solutions $+\nu$ corresponding to different field configurations
\[ \phi_{\text{def}}(x) = \nu \tanh \left( \frac{x}{d_{\text{def}}} \right), \] (18)
where $\nu$ is the vacuum expectation value [21].

The defect density becomes
\[ n \sim \tau_0^{-1/2}. \] (19)
For $\tau_0$ very small, the equilibrium is lost and the defect density is
\[ n \sim \tau_0^{-1/3}. \] (20)

The defect density is approximated from the $k = 0$ condition: $G_2 = G_q$ and depends on the vacuum expectation values $\nu$ and $G_{q}^{(2)}(k)$
\[ n_{k=0} = |G_2 - G_{2}^{(2)}(0)|/\nu^2. \] (21)

The $G_2(k)$ dynamics scales are $n$ and $d_{\text{def}}$ corresponding to a classical scalar field and the vacua contributions from the quantum excitations [22] during the unitary time evolution of the system with the vacuum two point function $G_{2}^{(2)}(k)$ and a two point function of the matter particles $G_{\text{mat}}(k)$. The defect production containing the full quantum corrections $G_q(k)$ scales as
\[ G_q(k) = \frac{\nu^2}{n} G_{\text{mat}}(k/n) G_{\text{def}}(kd_{\text{def}}) + G_{2}^{(2)}(k) + G_{\text{mat}}(k) \] (22)
The Hamiltonian (25) has a bare mass
\[ \mu_0^2(t) = -\frac{\dot{\phi}^2}{\tau_0} + \mu_0^2(t = 0), \quad t < t_f, \]
\[ \mu_0^2(t) = \mu_0^2(t_f), \quad t \geq t_f. \] (26)

Using Källén–Lehmann spectral representation of the two point function for a time dependent ground state with Lorentz invariance, we relate the theory to the non-interacting Feynman propagator \( D_F(x - y; M^2) \) acting as
\[ \langle \Omega | F(x) F(y) | \Omega \rangle = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) D_F(x - y; M^2) \] (27)
with \( x \) and \( y \) the space-time coordinates, \( \rho(M^2) \) the spectral density
\[ \rho(M^2) = \sum_k (2\pi) \delta(M^2 - m_k^2) \left| \langle \Omega | \phi(0) | \lambda_k \rangle \right|^2 \] (28)
and \( | \lambda_k \rangle \) the zero-momentum energy eigenstate. The Feynman propagator in \( d = (1 + 1) \) is
\[ D_F(r; M^2) = \frac{1}{2\pi} K_0(Mr) \] (30)
with \( K_0(z) \) the modified Bessel function of the second kind. At strong couplings, a Bessel-squared function corresponding to two non-interacting excitations can be used for the two point function which will be approximated as
\[ G_2(r) = \int_0^\infty \frac{dM^2}{4\pi} \rho(M^2) K_0(Mr). \] (31)

The two point function \( G_2(k = 0) \) in the broken symmetry phase scales as predicted and is a correct picture of the defect density \( n_{k=0} \) (21) if \( G_2 = G_4 \) for \( k = 0 \). The system is initially in equilibrium for \( G_2(k = 0) = G_4^2(k = 0) \) and becomes excited close to the critical point, allowing an approximation of \( \hat{e} \) with the quench rate \( \tau_0 \) such that \( \hat{e} \sim \tau_0^{1/2} \) when \( \tau_0 \) is sufficiently large to probe the critical region. In the broken symmetry phase, \( G_2(k = 0) \) will grow, however the scaling from \( \hat{e} \) still remains in place.

The evolution of \( G_2 \) for \( k = 0 \) in the symmetric phase is a measure of the correlation length \( \xi_0 \), to the equilibrium state \( G_4^2 \). The two-point function \( G_2(k = 0) = \langle \psi(t) | \phi(-k) \phi(k) | \psi(t) \rangle \) is approximated via a discrete cosine transform of \( G_4(r) \), matching the non-equilibrium two point function from KZM predictions, such that such that \( G_2(k) = G_4(k) \).

3. CONCLUSIONS

In this paper, we use tensor network techniques, specifically matrix product states to study the non-equilibrium dynamics of defect formation via KZM mechanism in the special case of relativistic \( \phi^4 \) scalar field theory and \( d = (1 + 1) \) dimensions for a quantum system undergoing a symmetry breaking phase transition via the KZM mechanism.

Furthermore, we find that the two point function in a free scalar field theory is a good measure tool of the defect density and the contributions of the defect profile are a powerful signature of the defect formation in the system. We find that MPS is an efficient tool to analyze the contributions of defect excitations to the ground state observables close to a critical point, useful for studying defect formation in field cosmology.

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