A Novel Spectral Modified Pell Polynomials for Solving Singular Differential Equations

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ABSTRACT
This paper studies the modified Pell polynomials. Some important properties of modified Pell polynomials are presented. An exact formula of modified Pell polynomials derivative in terms of modified Pell themselves is first derived with the proof and then a new relationship is constructed which relates the modified Pell polynomials expansion coefficients of a derivative in terms of their original expansion coefficients. An interesting new formula for product operational matrix of modified Pell polynomials is also derived in this work. With modified Pell polynomials expansion scheme, the powers \( \{1, x, \ldots, x^n\} \) are expressed in terms of such polynomials. The main goal of all presented formulas is to simplify the original equations and the determination of the coefficients of expansion based on modified Pell polynomials will be easy. Spectral techniques together with all the derived formulas of modified Pell polynomials are utilized to solve some singular initial value problems. Three test examples are solved in this work to illustrate the validity of the proposed method. The computational method is replaced by exact and explicit formulas. More accurate results are obtained than those presented by other existing methods in the literature.

KEYWORDS: Modified Pell polynomials; singular initial value problem; operation matrix; spectral method; product matrix of two modified Pell polynomials.

INTRODUCTION
The mathematical formulations of many physical problems occurs in the various disciplines such as quantum mechanics, fluid dynamic and elasticity [1-5], often lead to either nonlinear ordinary differential equations or singular ordinary differential equations [1-3]. Accurate and efficient approximate methods are often required for the solution of such types of ordinary differential equations. Several techniques are available in the literature. A basis function is needed with interesting properties. Polynomials and wavelets are basis functions, which obtain a suitable place for many problems [4-10]. One of the important families of polynomials is modified Pell polynomials. Due to it's implicitly, modified Pell polynomials has become an effective tool for solving differential equations. In this study, an approximate solution based on such polynomials is developed to solve special singular initial value problems. Two different approaches are suitable for solving differential equations. The first approach based on reducing the differential equations into integral equations through
introduction. Approximating different singles contained in the equation by truncated orthogonal series, and using their operational matrix of integration to estimate the integral operations [11-13]. The second approach is based on applying the operation matrix of derivative to reduce the original problem into system of equations [14-18]. The purpose of this paper is to introduce modified Pell spectral method for solving singular initial value problems (I.V.P.) with this method, the given D.E. and its related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of Pell series solution. This method is useful to obtain the approximate solution of I.V.P., no need to linearization or discretization and large computational work. It has been used to solve effectively, easily and accurately a large class of problems with approximations. The goal of this work is to derive new properties concerning modified Pell polynomials and then utilized them to evaluate the unknown coefficients and find an approximate solution. The article is organized as follows: in section 2, the definition of modified Pell polynomials is given. The main results of this paper are described in sections 3-6. Lemma 1 it greatly simplifies the calculation. Lemma 2 gives the relationship between the power and modified Pell polynomials. Lemma 3 reveals the product of two modified Pell polynomials in an explicit form. The derivative of modified Pell polynomials and the relationship between the coefficients is included in section 4. This is of great significance in numerical analysis and it has made new contributions to the study of spectral method. Section 7 concerns with the implementation of modified Pell polynomials spectral method for solving some singular initial value problems. The conclusion is listed in section 8.

Definitions of Modified Pell Polynomials

**First-level sub-title**

For \( n \geq 1 \), modified Pell polynomials \( q_n(x) \), is defined by the following recurrence relations

\[
q_{n+1}(x) = 2xq_n(x) + q_{n-1}(x)
\]

(1)

with the initial conditions

\[
q_0(x) = 1, \quad q_1(x) = x.
\]

(2)

The characteristic equation of recurrence relation Eq. 1 is

\[ t^2 - xt - 1 = 0, \]

this equation has two real roots

\[
a_1 = \frac{x + \sqrt{x^2 + 4}}{2}, \quad a_2 = \frac{x - \sqrt{x^2 + 4}}{2}.
\]

From Eqns. 1 and 2, the following modified Pell polynomials are obtained

\[
q_0(x) = 1
\]

\[
q_1(x) = x
\]

\[
q_2(x) = 2x^2 + 1
\]

\[
q_3(x) = 4x^3 + 3x
\]

\[
q_4(x) = 8x^4 + 8x^2 + 1
\]

\[
q_5(x) = 16x^5 + 20x^3 + 5x
\]

\[
q_6(x) = 32x^6 + 48x^4 + 18x^2 + 1
\]

\[
q_7(x) = 64x^7 + 112x^5 + 56x^3 + 7x
\]

The first eight coefficients of \( q_n(x) \) are arranged in Table 1. Let \( q_{nj} \) denote the element in row \( n \) and column \( j \), where \( j \geq 1, n \geq 1 \), it is obvious form Table 1 that

\[
q_{n1}(x) = \begin{cases} 
1 & \text{if } n \text{ even} \\
0 & \text{otherwise} 
\end{cases}
\]

In general,

\[
q_{nm} = 2q_{n-1,m-1} + 2q_{n-2,m}, \quad q_{nn} = 2^{n-1}, \quad n > m
\]

**Table 1.** The first eight coefficients of \( q_n(x) \)

| \( n \) | \( x^0 \) | \( x^1 \) | \( x^2 \) | \( x^3 \) | \( x^4 \) | \( x^5 \) | \( x^6 \) | \( x^7 \) |
|------|------|------|------|------|------|------|------|------|
| 0    | 1    |      |      |      |      |      |      |      |
| 1    | 0    | 1    |      |      |      |      |      |      |
| 2    | 1    | 0    | 2    |      |      |      |      |      |
| 3    | 0    | 0    | 0    | 4    |      |      |      |      |
| 4    | 1    | 0    | 8    | 0    | 8    |      |      |      |
| 5    | 0    | 5    | 0    | 20   | 0    | 16   |      |      |
| 6    | 1    | 0    | 18   | 0    | 48   | 0    | 32   |      |
| 7    | 0    | 7    | 0    | 56   | 0    | 112  | 0    | 64   |

Representing Derivatives of \( q_n(x) \) by Modified Pell Polynomials

The objective of this subsection is to state and prove a new analytical formula representing explicitly the first derivative of modified Pell polynomials in terms of modified Pell polynomials themselves. As a direct consequence, a novel spectral algorithm based on the derivative for modified Pell polynomials is proposed.

**Lemma (1):**

The explicit expression for modified Pell polynomials derivative is given by

\[
\dot{q}_n(x) = 2n \sum_{i=1}^{n} \delta_{(n+i) \text{ odd}}(-1)^{\frac{[i]}{2}} q_{n-i}(x)
\]

for \( n \) even

(3)

\[
\dot{q}_n(x) = 2n \sum_{i=0}^{n-1} \delta_{(n+i) \text{ even}}(-1)^{\frac{[i]}{2}} q_{n-i}(x) + nq_0
\]

for \( n \) odd

(4)
\[\hat{f}(x) = \sum_{i=0}^{n-1} c_i q_i(x).\]

The relationship between the coefficients \(a_i\) and \(c_i\) is given by
\[c_n = 2n a_n,\]
\[c_{n-1} = 2(n-1) a_{n-1},\]
\[c_r = -c_{r+1} + 2r a_r, \quad r = n - 2, n - 3, \ldots, 2\]
with \(c_1 = -c_3 + 2a_1.\)

The Derivative of Modified Pell Polynomials and The Relationship Between the Coefficients

Lemma (2):
The product of two modified Pell Polynomials is given by the following formula
\[q_n(x) q_m(x) = \begin{cases} q_{n+m}(x) - q_{n-m}(x), & n \text{ odd} \\ 2q_{n+m}(x) + q_{n-m}(x), & n \text{ even} \end{cases}\]
when \(m \geq n\) (7)

Proof:
Eq. 7 is an identity for \(n = 0\), since \(q_0 = 1\), it then follows that
\[q_0 q_m(x) = \frac{1}{2} (q_m(x) + q_m(x)) = \frac{1}{2} (2q_m(x)) \quad (8)\]
Multiply both sides of Eq. 8 by \(2x\) yield
\[q_m(x)(2xq_0) = \frac{1}{2} (2xq_m(x)) \quad (9)\]
By applying the basic recurrence relationship for modified Pell polynomials
\[q_m(x) = 2xq_{m-1}(x) + q_{m-2}(x)\]
or \(2xq_m(x) = q_m(x) - q_{m-2}(x)\) (10)
With the aid of Eq. 10, one can get
\[q_m(x) q_1(x) = \frac{1}{2} (q_{m+1}(x) - q_{m-1}(x)).\]
That is Eq. 7 is true for \(n = 1\), therefore; it is true for \(n\) since it is true for \(n = 0\) and \(n = 1\).
Now, Eq. 7 must be valid for \(n + 1\)
\[q_n(x) q_m(x) = \frac{1}{2} (q_{n+m}(x) - q_{n-m}(x)).\]
Multiply both sides by \(2x\)
\[q_m(x)(2xq_n(x)) = \frac{1}{2} (2xq_{n+m}(x) - 2xq_{n-m}(x))\]
\[q_m(x)(q_{n+1}(x) - q_{n-1}(x)) = \frac{1}{2} (q_{n+m+1}(x) - q_{n-m-1}(x)) - q_{n-m+1}(x) + q_{n-m-1}(x) + q_{m+1}(x) + q_{m+1}(x) - q_{n-m+1}(x)\]
\[q_n(x) q_{m+1}(x) = \frac{1}{2} [q_{n+m+1}(x) - q_{n-m-1}(x)]\]
The relationship Between The Power \( \{1, x, x^2, \ldots, x^n\} \) and Modified Pell Polynomials

The relationship between the power \( \{1, x, x^2, \ldots, x^n\} \) and modified Pell polynomials is illustrated in this subsection.

\[ x^n = 2^{1-n} \sum_{k=0}^{[n/2]} (-1)^{k-1} \binom{n}{k} q_{n-2k}(x), \text{ n odd} \quad (11) \]

\[ x^n = 2^{1-n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} q_{n-2k}(x) + \]

\[ (-1)^n 2^{-n} \binom{n}{2} q_0(x) \quad (12) \]

for even \( n \), where \( 1 = q_0 \)

**Proof:**

The mathematical induction is used to prove the Lemma. First, Eq. 11 is true for \( n = 1 \), by direct calculation, since \( x^n = x = \frac{1}{2^n} (q_1) = q_1 \)

Assume that Eq. 11 is true for a particular positive integer \( n = 2m + 1 \), that is

\[ x^{2m+1} = 2^{-2m} \sum_{k=0}^{\lfloor 2m+1/2 \rfloor} (-1)^k \binom{2m+1}{k} q_{2m-2k+1}(x) \]

It must be valid for \( n = 2m + 3 \)

\[ x^{2m+3} = \frac{1}{2^{2m+2}} \left[ q_{2m+1}(x) \right. \]

\[ - \left( \binom{2m+1}{1} q_{2m-1}(x) \right. \]

\[ \left. + \left( \binom{2m+1}{2} q_{2m-3}(x) + \ldots \right) \right] q_2(x) \]

\[ = \frac{1}{2^{2m+2}} \left[ q_{2m+1}(x) q_2(x) \right. \]

\[ - \left( \binom{2m+1}{1} q_{2m-1}(x) q_2(x) \right. \]

\[ \left. + \left( \binom{2m+1}{2} q_{2m-3}(x) q_2(x) + \ldots \right) \right] \]

\[ = \frac{1}{2^{2m+2}} \left[ q_{2m+3}(x) - q_{|2m-1|}(x) \right. \]

\[ - \left( \binom{2m+1}{1} q_{2m+1}(x) \right. \]

\[ - q_{|2m-3|}(x) \right) \]

\[ + \left( \binom{2m+3}{1} q_{|2m-1|}(x) \right. \]

\[ - q_{2m-5}(x) \left. \right] \]

\[ = \frac{1}{2^{2m+2}} \left[ q_{2m+3}(x) - \left( \binom{2m+1}{1} \right) q_{2m+1}(x) \right. \]

\[ + \left( \binom{2m+1}{2} q_{2m-1}(x) \right. \]

\[ \left. + \left( \binom{2m+1}{3} q_{2m-3}(x) \right) \right] \]

\[ \left. - \binom{2m+1}{1} q_{2m+1}(x) \right) \]

\[ + \binom{2m+3}{1} q_{|2m-1|}(x) \]

\[ - q_{2m-5}(x) \left. \right] \]

which is the required result. By the same way, one can prove when \( n \) is even.

**Implementation of Modified Pell Spectral Method on Some Problems**

A special class of singular initial value problem is considered in this section.

\[ f(x) + \frac{m}{x} f(x) + g(x)f(x) = h(x) \quad (13) \]

subject to \( f(0) + \bar{f}(0) = \alpha \) \quad (14)

which arising in physiology: The unknown function \( f(x) \) is approximated by

\[ f(x) = c^T q(x) \quad (15) \]

where: \( c \) and \( q(x) \) are defined by \( c^T = [c_0 \ c_1 \ \ldots \ c_n]^T \) and

\[ q(x) = [q_0(x) \ q_1(x) \ \ldots \ q_n(x)] \]

one can obtain

\[ f(x) = c^T q(x) = b^T q(x) \quad (16) \]

\[ f(x) = c^T q(x) = d^T q(x) \quad (17) \]

rewrite \( g(x)f(x) \) and \( h(x) \) as

\[ g(x)f(x) = h^T q(x) \quad (18) \]

and \( h(x) = h^T q(x) \quad (19) \]

by substituting Eqns. 15-19 into Eq. 13, yields

\[ b^T q(x) + \frac{m}{x} d^T q(x) + g^T q(x) = h^T q(x) \quad (20) \]

Rewrite Eq. 20 as follows

\[ x b^T q(x) + mb^T q(x) + xg^T q(x) = x h^T q(x) \quad (21) \]

In addition, using Eqns. 14-15 to get

\[ c^T q(0) + b^T q(0) = \alpha \quad (22) \]

Now, the resulting Eq. 21 generates \( n \) linear equations together with Eq. 22, which can be solved to obtain the vector, \( c^T \) and hence the approximate solution \( f_n(x) \) can be obtained.

**Test Example 1**

Consider the following singular initial value problem
subject to the condition \( f(0) + \frac{\dot{f}(0)}{x} = 0 \) \hspace{1cm} (24)

This example is solved by modified Pell polynomials spectral method with \( n = 4 \)

\[
\begin{align*}
f(x) &= c q^T(x) \\
\dot{f}(x) &= b q^T(x) \\
\ddot{f}(x) &= d q^T(x) \\
x f(x) &= d_1 q^T(x), \quad x f(x) = c_1 q^T(x) \\
12x - 10x^5 &= h_1 q^T(x)
\end{align*}
\]

where

\[
\begin{align*}
c &= [a_0 \ a_1 \ a_2 \ a_3 \ a_4] \\
b &= [(a_1 - 3a_3) \ (4a_2 - 8a_4) \ 6a_3 \ 8a_4 \ 0] \\
d_1 &= [-12a_3 \ (4a_2 - 56a_4) \ 12a_3 \ 24a_4 \ 0] \\
d &= [(4a_2 - 32a_4) \ 24a_3 \ 48a_4 \ 0 \ 0] \\
h_1 &= \begin{bmatrix} 23 & 0 & 50 & 0 & -5 \\ 0 & 4 & 0 & 16 & 0 \\ \end{bmatrix} \\
q_4^T &= [q_0 \ q_1 \ q_2 \ q_3 \ q_4]^T \\
q_5^T &= [q_0 \ q_1 \ q_2 \ q_3 \ q_4 \ q_5]^T
\end{align*}
\]

and Eq. 24 can be approximated to obtain the following equation

\[
[a_0 \ a_1 \ a_2 \ a_3 \ a_4] q_4^T(x) = 0 
\]

Eq. 23 leads to the equation:

\[
d_1 d_1^T(x) + 2bd_1^T(x) - 10c_1 d_1^T(x) = h_1 d_5^T(x) 
\]

From Eqns. 25-26, the following five linear equations are obtained:

\[
\begin{align*}
a_0 + a_1 + a_2 + 3a_3 + a_4 &= 0 \\
-5a_4 &= \frac{5}{8}, & 5a_3 &= 0, & 45a_4 &= -5a_2 = \frac{5}{16} \\
29a_3 - 5a_1 &= 0
\end{align*}
\]

The modified Pell polynomials coefficients are given by the vector \( c = \begin{bmatrix} \frac{-5}{8} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \end{bmatrix} \). The approximate solution of Eq. 23 in comparison with the exact solution is listed in Figure 1.

The results section is where you tell the reader the basic descriptive information about the scales you used (report the mean and standard deviation for each scale). If you have more than 3 or 4 variables in your paper, you might want to put this descriptive information in a table to keep the text from being too choppy and bogged down (see the APA manual for ideas on creating good tables). In the results section, you also tell the reader what statistics you conducted to test your hypothesis (theses) and what the results indicated. In this paper, you conducted bivariate correlation (s) to test your hypothesis.
The obtained approximate solution of Eq. 27 in comparison with the exact solution is listed in Figure 2.

**Test Example 3**  
Consider the nonhomogeneous singular initial value problem [18]  
\[ \ddot{f}(x) + \frac{8}{x} \dot{f}(x) + xf(x) = x^5 - x^4 + 44x^2 - 30x, \]  
(28)  

Together with \( f(0) = 0, \dot{f}(0) = 0 \) applying the modified Pell method presented in section 4 for this problem. The modified Pell coefficients \( c_i \)'s are obtained,  
\[ c = \begin{bmatrix} \frac{3}{8} & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{bmatrix} \]  
The obtained approximate solution is listed in comparison with other methods and the exact solution \( f(x) = x^4 - x^3 \) in Table 2. A comparison of ADM and HWCM (\( N = 16 \)) solutions listed in [18] with our method (\( N = 4 \)) is also included in Table 2 and Figure 3 against the exact solution.

| Table 2. Comparison of ADM and HWCM (\( N = 16 \)) solutions with our method (\( N = 4 \)) against the exact solution. |
|---|---|---|---|---|
| \( x=1/32 \) | ADM | HWCM | Exact | Proposed Method |
| 1 | 0 | -0.000048 | -0.000029 | -0.000049 |
| 3 | 0 | -0.0000640 | -0.0000746 | -0.0000746 |
| 5 | 0 | -0.003104 | -0.003218 | -0.003218 |
| 7 | 0 | -0.0080343 | -0.008177 | -0.008177 |
| 9 | 0 | -0.015832 | -0.015990 | -0.015990 |
| 11 | 0 | -0.026497 | -0.026656 | -0.026656 |
| 13 | 0 | -0.039660 | -0.039809 | -0.039809 |
| 15 | 0 | -0.054590 | -0.054717 | -0.054717 |
| 17 | 0 | -0.070188 | -0.070281 | -0.070281 |
| 19 | 0 | -0.084990 | -0.085036 | -0.085036 |
| 21 | 0 | -0.097163 | -0.097151 | -0.097151 |
| 23 | 0 | -0.1045115 | -0.104430 | -0.104430 |
| 25 | 0 | -0.104470 | -0.104308 | -0.104308 |
| 27 | 0 | -0.094110 | -0.093855 | -0.093855 |
| 29 | 0 | -0.070136 | -0.069777 | -0.069777 |
| 31 | -0.028411 | -0.028885 | -0.028410 | -0.028410 |

**Figure 3.** Comparison of approximate solutions in [18] with our method (\( N = 4 \)) against the exact solution.

The error analysis for different values of \( N \) is given in Table 3.

**Table 3.** The error analysis for different values of \( N \)  

| \( N \) | ADM \( L_{\infty} \) | HWCM \( L_{\infty} \) | HWAGM \( L_{\infty} \) | Proposed Method |
|---|---|---|---|---|
| 8 | 1.0043e-01 | 1.6292e-05 | 4.0839e-04 | 0 |
| 16 | 1.0443e-01 | 4.7476e-05 | 1.0293e-04 | - |
| 32 | 1.0520e-01 | 1.2686e-04 | 3.0278e-05 | - |
| 64 | 1.054e-01 | 3.2718e-05 | 8.0095e-06 | - |
| 128 | 1.045e-01 | 8.3037e-06 | 2.0090e-06 | - |
| 256 | 1.0546e-01 | 2.0904e-06 | 9.0998e-07 | - |

**CONCLUSIONS**  
This work concerns with modified Pell polynomials. The analytical formula associated with the first derivative of modified Pell polynomials is constructed in terms of modified Pell polynomials themselves. Then, the relationship between the coefficients in the differentiated expansions of modified Pell polynomials and those of the original expansion is formulated. Other important properties of modified Pell polynomials are derived in this article. As an application and with the help of the obtained formulas, spectral solutions of some singular initial value problems are proposed.

**REFERENCES**

[1] Dkhilali F., Borchani S. M., Rasheed M., Barella R., Shihab S., Guidara K., & Megdiche M. (2018). Characterizations and morphology of sodium tungstate particles. Royal Society open science, (8) 1-16.  
[2] TSAY, S. C., & LEE, T. T. (1986). Solutions of integral equations via Taylor series. International Journal of Control, 44(3), 701-709.  
[3] Al-Rawy S. N. (2006). On the Solution of Certain Fractional Integral Equations, kirkuk university journal for scientific studies, (1) 125-136.  
[4] Kadkhoda, N. (2020). A numerical approach for solving variable order differential equations using Bernstein polynomials. Alexandria Engineering Journal, 59(5), 3041-3047.  
[5] Shihab S. N. & Abdalrehman A. A. (2012). Numerical solution of calculus of variations by using the second Chebyshev wavelets, Engineering and Technology Journal, 30(18):3219-3229.  
[6] Yuanlu, L. I. (2010). Solving a nonlinear fractional differential equation using Chebyshev wavelets. Communications in Nonlinear Science and Numerical Simulation, 15(9), 2284-2292.  
[7] Dehghan, M., & Saadatmandi, A. (2008). Chebyshev finite difference method for Fredholm integro-differential equation. International Journal of Computer Mathematics, 85(1), 123-130.
[8] Kojabad, E. A., & Rezapour, S. (2017). Approximate solutions of a sum-type fractional integro-differential equation by using Chebyshev and Legendre polynomials. Advances in Difference Equations, 2017(1), 1-18. https://doi.org/10.1186/s13662-017-1404-y

[9] El-Gendi, S. E. (1969). Chebyshev solution of differential, integral and integro-differential equations. The Computer Journal, 12(3), 282-287. https://doi.org/10.1093/comjnl/12.3.282

[10] Clenshaw, C. W., & Norton, H. J. (1963). The solution of nonlinear ordinary differential equations in Chebyshev series. The Computer Journal, 6(1), 88-92. https://doi.org/10.1093/comjnl/6.1.88

[11] Sezer, M., & Kaynak, M. (1996). Chebyshev polynomial solutions of linear differential equations. International Journal of Mathematical Education in Science and Technology, 27(4), 607-618. https://doi.org/10.1080/0020739960270414

[12] Zhu, L., & Fan, Q. (2012). Solving fractional nonlinear Fredholm integro-differential equations by the second kind Chebyshev wavelet. Communications in nonlinear science and numerical simulation, 17(6), 2333-2341. https://doi.org/10.1016/j.cnsns.2011.10.014

[13] Henderson, M., & Matthews, R. (1995). Permutation properties of Chebyshev polynomials of the second kind over a finite field. Finite Fields and Their Applications, 1(1), 115-125. https://doi.org/10.1016/filtfa.1995.1008

[14] Mohsen, R., & Mohammad H. (1996). Application of Legendre series to the control problems governed by linear parabolic equations. Mathematics and computers in simulation, 42(1), 77-84. https://doi.org/10.1016/0378-4754(95)00133-6

[15] Sarhan M. A., SHIHAB S., & RASHEED M. (2021). Some Results on a Two Variables Pell Polynomials. Al-Qadisiyah Journal of Pure Science. 26(1) 55-70. https://doi.org/10.29350/jops.2021.26.11246

[16] Aziz S. H., SHIHAB S., & RASHEED M. (2021). On Some Properties of Pell Polynomials. Al-Qadisiyah Journal of Pure Science. 26(1) 39-54. https://doi.org/10.29350/jops.2020.26.11237

[17] Maleknejad, K., & Mirzaee, F. (2005). Using rationalized Haar wavelet for solving linear integral equations. Applied Mathematics and Computation, 160(2), 579-587. https://doi.org/10.1016/j.amc.2003.11.036

[18] Shiralashetti, S. C., & Kumbinarasaiah, S. (2018). Hermite wavelets operational matrix of integration for the numerical solution of nonlinear singular initial value problems. Alexandria engineering journal, 57(4), 2591-2600. https://doi.org/10.1016/j.aej.2017.07.014