FIRST COHOMOLOGY GROUPS OF THE AUTOMORPHISM GROUP OF A FREE GROUP WITH COEFFICIENTS IN THE ABELIANIZATION OF THE IA-AUTOMORPHISM GROUP

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Abstract. We compute a twisted first cohomology group of the automorphism group of a free group with coefficients in the abelianization of the IA-automorphism group of a free group. In particular, we show that it is generated by two crossed homomorphisms constructed with the Magnus representation and the Magnus expansion due to Morita and Kawazumi respectively. As a corollary, we see that the first Johnson homomorphism does not extend to the automorphism group of a free group as a crossed homomorphism for the rank of the free group is greater than 4.

1. Introduction

Let \(F_n\) be a free group of rank \(n \geq 2\) with basis \(x_1, \ldots, x_n\), and \(\text{Aut} F_n\) the automorphism group of \(F_n\). The study of the (co)homology groups of \(\text{Aut} F_n\) with trivial coefficients has been developed for these twenty years by many authors. There are several remarkable computations. Gersten [7] showed \(H_2(\text{Aut} F_n, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\) for \(n \geq 5\). Hatcher and Vogtmann [8] showed \(H_q(\text{Aut} F_n, \mathbb{Q}) = 0\) for \(n \geq 1\) and \(1 \leq q \leq 6\), except for \(H_4(\text{Aut} F_4, \mathbb{Q}) = \mathbb{Q}\). Furthermore, recently Galatius [6] showed that the stable integral homology groups of \(\text{Aut} F_n\) are isomorphic to those of the symmetric group \(S_n\) of degree \(n\). In particular, from his results, we see that the stable rational homology groups \(H_q(\text{Aut} F_n, \mathbb{Q})\) of \(\text{Aut} F_n\) are trivial for \(n \geq 2q + 1\).

In this paper, we are interested in twisted (co)homology groups of \(\text{Aut} F_n\) from a viewpoint of the study of the Johnson homomorphism of \(\text{Aut} F_n\). Let \(H\) be the abelianization of \(F_n\). The group \(\text{Aut} F_n\) naturally acts on \(H\) and its dual group \(H^* := \text{Hom}_{\mathbb{Z}}(H, \mathbb{Z})\). There are a few computation for the (co)homology groups of \(\text{Aut} F_n\) with coefficients in \(H\) and \(H^*\). Hatcher and Wahl [9] showed that the stable homology groups of \(\text{Aut} F_n\) with coefficients in \(H\) are trivial using the stability of the homology groups of the mapping class groups of certain 3-manifolds. In our previous papers [23] and [29], we studied the stable twisted first and second (co)homology groups of \(\text{Aut} F_n\) with coefficients in \(H\) and \(H^*\), using the presentation for \(\text{Aut} F_n\) due to Gersten [7]. In particular, we obtained \(H^1(\text{Aut} F_n, H) = \mathbb{Z}\) for \(n \geq 4\), and \(H_2(\text{Aut} F_n, H^*) = 0\) for \(n \geq 6\).

Our research mentioned above is inspired by Morita’s work for the mapping class group of a surface. For \(g \geq 1\), let \(\Sigma_{g,1}\) be a compact oriented surface of genus \(g\) with one boundary component, and \(\mathcal{M}_{g,1}\) the mapping class group of \(\Sigma_{g,1}\). Namely, \(\mathcal{M}_{g,1}\) is

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the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma_{g,1}$ which fix the boundary pointwise. The action of $\mathcal{M}_{g,1}$ on the fundamental group of $\Sigma_{g,1}$ induces a natural homomorphism $\mathcal{M}_{g,1} \to \text{Aut} F_{2g}$. It is known that this homomorphism is injective for any $g \geq 1$ due to classical works by Dehn and Nielsen. Then we can consider $H$ as $\mathcal{M}_{g,1}$-modules for $n = 2g$. We remark that $H^*$ is canonically isomorphic to $H$ by the Poincaré duality. In [14], Morita computed $H^1(\mathcal{M}_{g,1}, H) = \mathbb{Z}$ for $g \geq 2$, and $H_2(\mathcal{M}_{g,1}, H) = 0$ for $g \geq 12$. (See also [15].) In particular, he showed that a crossed homomorphism induced from the Magnus representation of $\mathcal{M}_{g,1}$ generates $H^1(\mathcal{M}_{g,1}, H)$.

In general, the groups $\text{Aut} F_n$ and $\mathcal{M}_{g,1}$ share many similar algebraic properties. If a certain result for either $\text{Aut} F_n$ or $\mathcal{M}_{g,1}$ is obtained, it would be natural to ask whether the corresponding result holds or not for the other. As far as we compare the Morita’s works with ours, it seems that $\text{Aut} F_n$ and $\mathcal{M}_{g,1}$ behave similarly with respect to the low dimensional twisted (co)homology groups.

In this paper, we consider another $\text{Aut} F_n$-module other than $H$ and $H^*$. Let $\rho : \text{Aut} F_n \to \text{Aut} H$ be the natural homomorphism induced from the abelianization of $F_n$. We identify $\text{Aut} H$ with $\text{GL}(n, \mathbb{Z})$ by fixing a basis of $H$ induced from that of $F_n$. The kernel of $\rho$ is called the IA-automorphism group of $F_n$, denoted by $\text{IA}_n$. The IA-automorphism group is a free group analogue of the Torelli subgroup of the mapping class group. Although the study of the IA-automorphism group has a long history since its finite many generators were obtained by Magnus [12] in 1935, the combinatorial group structure of $\text{IA}_n$ is still quite complicated. For instance, any presentation for $\text{IA}_n$ is not known in general. Nielsen [21] showed that $\text{IA}_2$ coincides with the inner automorphism group, hence, is a free group of rank 2. For $n \geq 3$, however, $\text{IA}_n$ is much larger than the inner automorphism group $\text{Inn} F_n$. Krstić and McCool [11] showed that $\text{IA}_3$ is not finitely presentable. For $n \geq 4$, it is not known whether $\text{IA}_n$ is finitely presentable or not. On the other hand, the abelianization $V$ of $\text{IA}_n$ is completely determined by recent independent works of Cohen-Pakianathan [3, 4], Farb [5] and Kawazumi [10]. From their results, we have $V \cong H^* \otimes \mathbb{Z} \Lambda^2 H$ as a $\text{GL}(n, \mathbb{Z})$-module.

Let $L$ be a commutative ring which does not contain any 2-torsions. In this paper, we determine the stable first cohomology group of $\text{Aut} F_n$ with coefficients in $V_L := V \otimes \mathbb{Z} L$. Here the ring $L$ is regarded as a trivial $\text{Aut} F_n$-module. Our main theorem is

**Theorem 1.** (= Theorem 4.7) For $n \geq 5$, if $L$ does not contain any 2-torsions,

$$H^1(\text{Aut} F_n, V_L) = L^\otimes 2.$$  

We also show that the generators of $H^1(\text{Aut} F_n, V_L)$ are constructed by the Magnus representation and the Magnus expansion due to Morita [17] and Kawazumi [10] respectively. These are denoted by $f_M$ and $f_K$. (For details, see Section 4.)

The computation of Theorem 1 is motivated by a result for the mapping class group $\mathcal{M}_{g,1}$ due to Morita. In [16], he computed the first cohomology group of $\mathcal{M}_{g,1}$ with coefficients in $\Lambda^3 H$, the free part of the abelianization of the Torelli subgroup $I_{g,1}$ of $\mathcal{M}_{g,1}$. In particular, he showed $H^1(\mathcal{M}_{g,1}, \Lambda^3 H) = \mathbb{Z}^\otimes 2$ for $g \geq 3$. Hence, we also see that the corresponding result of $\mathcal{M}_{g,1}$ holds for $\text{Aut} F_n$ in this case.
In order to show the theorem above, we use the Nielsen’s presentation for Aut $F_n$. (See Subsection 2.2.) One of advantages of the generators-and-relations calculation is that by this method, we can determine $H^1(\text{Aut} F_n, V_L)$ for many $L$ at the same time. For example, $L = \mathbb{Z}, \mathbb{Z}/p\mathbb{Z}$ for any integer $p \in \mathbb{Z}$ such that $(p, 2) = 1$.

Now, as an application of Theorem 1, we can see that the first Johnson homomorphism $\tau_1 : \text{IA}_n \to V$ does not extend to Aut $F_n$ as a crossed homomorphism directly. (See [17] or [25] for the definition of the Johnson homomorphism, for example.) More precisely, for any commutative ring $L$, let denote $\tau_{1,L}$ the composition of the first Johnson homomorphism $\tau_1$ and the natural projection $V \to V_L$. Then we have

**Proposition 1.** (= Proposition 5.1) Let $L$ be a commutative ring which does not contain both any 2-torsions and 1/2. Then for $n \geq 5$, there is no crossed homomorphism from Aut $F_n$ to $V_L$ which restriction to $\text{IA}_n$ coincides with $\tau_{1,L}$.

We should remark that if a commutative ring $L$ contains 1/2, then the first Johnson homomorphism $\tau_1 : \text{IA}_n \to V_L$ extends to a crossed homomorphism $\text{Aut} F_n \to V_L$ due to Kawazumi [10]. He explicitly construct a crossed homomorphism, denoted by $f_K$ in this paper, which restriction to $\text{IA}_n$ coincides with $\tau_{1,L}$ using the theory of Magnus expansions. On the other hand, as to the mapping class group, it has already known by Morita [18] that if $L$ contains 1/2 then the first Johnson homomorphism

$$
\tau_1 : \mathcal{I}_{g,1} \to \Lambda^3 H \otimes \mathbb{Z} L
$$

of the mapping class group is uniquely extends to $\mathcal{M}_{g,1}$ as a crossed homomorphism where $\mathcal{I}_{g,1}$ denotes the Torelli subgroup of $\mathcal{M}_{g,1}$. Hence, we see that the groups Aut $F_n$ and $\mathcal{M}_{g,1}$ also share a common property with respect to the extension of the first Johnson homomorphism.

At the end of the paper, we consider the outer automorphism group Out $F_n$. In particular, we show

**Proposition 2.** (= Proposition 5.2) Let $L$ be a commutative ring which does not contain any 2-torsions. Then for $n \geq 5$,

$$
H^1(\text{Out} F_n, V_L) = L.
$$

This paper consists of six sections. In Section 2, we fix some notation and conventions. Then we recall the Nielsen’s finite presentation for Aut $F_n$. In Section 3, we construct two crossed homomorphisms $f_M$ and $f_K$ from Aut $F_n$ into $V_L$ for any commutative ring $L$. In Section 4, we compute the twisted first cohomology groups of Aut $F_n$ using the Nielsen’s presentation. In Section 5, we consider two applications. One is non-extendability of the Johnson homomorphism. The other is a computation of the twisted first cohomology group of the outer automorphism group of a free group.

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2. Preliminaries

In this section, after fixing some notation and conventions, we recall the Nielsen’s finite presentation for Aut $F_n$, which is used to compute the first cohomology groups in Section 4. Then we also recall the IA-automorphism group of a free group and its abelianization.

2.1. Notation and conventions.

Throughout the paper, we use the following notation and conventions. Let $G$ be a group and $N$ a normal subgroup of $G$.

- The abelianization of $G$ is denoted by $G^{ab}$.
- The automorphism group Aut $F_n$ of $F_n$ acts on $F_n$ from the right. For any $\sigma \in$ Aut $F_n$ and $x \in G$, the action of $\sigma$ on $x$ is denoted by $x^\sigma$.
- For an element $g \in G$, we also denote the coset class of $g$ by $g \in G/N$ if there is no confusion.
- Let $L$ be an arbitrary commutative ring. For any $\mathbb{Z}$-module $M$, we denote $M \otimes \mathbb{Z} L$ by the symbol obtained by attaching a subscript $L$ to $M$, like $M_L$ or $M^L$. Similarly, for any $\mathbb{Z}$-linear map $f : A \to B$, the induced $L$-linear map $A_L \to B_L$ is denoted by $f_L$ or $f^L$.
- For elements $x$ and $y$ of $G$, the commutator bracket $[x, y]$ of $x$ and $y$ is defined to be $[x, y] := xyx^{-1}y^{-1}$.
- For a group $G$ and a $G$-module $M$, we set

$$\text{Cros}(G, M) := \{ f : G \to M \mid f : \text{crossed homomorphism} \},$$

$$\text{Prin}(G, M) := \{ g : G \to M \mid g : \text{principal homomorphism} \}.$$  

2.2. Nielsen’s Presentation.

For $n \geq 2$, let $F_n$ be a free group of rank $n$ with basis $x_1, \ldots, x_n$. Let $P, Q, S$ and $U$ be automorphisms of $F_n$ given by specifying its images of the basis $x_1, \ldots, x_n$ as follows:

|   | $x_1$ | $x_2$ | $x_3$ | $\cdots$ | $x_{n-1}$ | $x_n$ |
|---|-------|-------|-------|-----------|-----------|-------|
| $P$ | $x_2$ | $x_1$ | $x_3$ | $\cdots$ | $x_{n-1}$ | $x_n$ |
| $Q$ | $x_2$ | $x_3$ | $x_4$ | $\cdots$ | $x_n$ | $x_1$ |
| $S$ | $x_1^{-1}$ | $x_2$ | $x_3$ | $\cdots$ | $x_{n-1}$ | $x_n$ |
| $U$ | $x_1 x_2$ | $x_2$ | $x_3$ | $\cdots$ | $x_{n-1}$ | $x_n$ |
In 1924, Nielsen [22] showed that the four elements above generate Aut $F_n$. Furthermore, he obtained a first finite presentation for Aut $F_n$.

**Theorem 2.1 (Nielsen [22]).** For $n \geq 2$, Aut $F_n$ is generated by $P$, $Q$, $S$ and $U$ subject to relations:

(N1): $P^2, Q^n, S^2$,
(N2): $(QP)^{n-1}$,
(N3): $(SPU)^2$,
(N4): $[P, Q^{-1}PQ], \quad 2 \leq l \leq n/2$,
(N5): $[S, Q^{-1}PQ], [S, PQ]$,
(N6): $(PS)^4$,
(N7): $[U, Q^{-2}PQ^2], [U, Q^{-2}UQ^2], \quad n \geq 3$,
(N8): $[U, Q^{-2}S^2Q], [U, SUS]$,
(N9): $[U, PQ^{-1}PQ^{-1}PQ]$, $[U, PQ^{-1}SU]$,
(N10): $[U, PQ^{-1}PQUPQ^{-1}PQ], [U, PQ^{-1}SU]$,
(N11): $U^{-1}PUPSUSPS$,
(N12): $(PQ^{-1}UQ)^2UQ^{-1}U^{-1}QU^{-1}$.

Let $H$ be the abelianization of $F_n$, and $H^* := \text{Hom}_\mathbb{Z}(H, \mathbb{Z})$ the dual group of $H$. Let $e_1, \ldots, e_n$ be the basis of $H$ induced from $x_1, \ldots, x_n$, and $e_1^*, \ldots, e_n^*$ its dual basis of $H^*$. Here we remark the actions of the generators $P$, $Q$, $S$ and $U$ on $e_i$'s and $e_i^*$'s. In this paper, according to the usual custom, any Aut $F_n$-module is considered as a left Aut $F_n$-module. Namely, for any element $\sigma \in \text{Aut} F_n$, the action of $\sigma$ on $e_i$ is given by $\sigma \cdot e_i := x_i^\sigma^{-1} \in H$. In particular, the actions of $P$, $Q$, $S$ and $U$ on $e_i$ and $e_i^*$ are given by

\[
P \cdot e_k = \begin{cases} e_2, & k = 1, \\
e_1, & k = 2, \\
e_k, & k \neq 1, 2, \end{cases} \quad P \cdot e_k^* = \begin{cases} e_2^*, & k = 1, \\
e_1^*, & k = 2, \\
e_k^*, & k \neq 1, 2, \end{cases}
\]

\[
Q \cdot e_k = \begin{cases} e_n, & k = 1, \\
e_k, & k \neq 1, \end{cases} \quad Q \cdot e_k^* = \begin{cases} e_n^*, & k = 1, \\
e_k^*, & k \neq 1, \end{cases}
\]

\[
S \cdot e_k = \begin{cases} -e_1, & k = 1, \\
e_k, & k \neq 1, \end{cases} \quad S \cdot e_k^* = \begin{cases} -e_1^*, & k = 1, \\
e_k^*, & k \neq 1, \end{cases}
\]

\[
U \cdot e_k = \begin{cases} e_1 - e_2, & k = 1, \\
e_k, & k \neq 1, \end{cases} \quad U \cdot e_k^* = \begin{cases} e_1^* + e_1, & k = 2, \\
e_k^*, & k \neq 2. \end{cases}
\]

2.3. IA-automorphism group.

Here we recall the IA-automorphism group of a free group. Fixing the basis $e_1, \ldots, e_n$ of $H$, we identify Aut $H$ with $\text{GL}(n, \mathbb{Z})$. The kernel of the natural homomorphism $\rho : \text{Aut} F_n \to \text{GL}(n, \mathbb{Z})$ induced from the abelianization of $F_n$ is called the IA-automorphism group of $F_n$, denoted by $\text{IA}_n$. Magnus [22] showed that for any $n \geq 3$, the group $\text{IA}_n$ is finitely generated by automorphisms

\[
K_{ij} : \begin{cases} x_i \mapsto x_j^{-1}x_ix_j, \\
x_t \mapsto x_t, \quad (t \neq i) \end{cases}
\]
for distinct \( i, j \in \{1, 2, \ldots, n\} \) and
\[
K_{ijk} : \begin{cases} 
    x_i & \mapsto x_ix_jx_kx_j^{-1}x_k^{-1}, \\
    x_t & \mapsto x_t, 
\end{cases} \quad (t \neq i)
\]
for distinct \( i, j, k \in \{1, 2, \ldots, n\} \) such that \( j > k \).

Recently, Cohen-Pakianathan [3, 4], Farb [5] and Kawazumi [10] independently determined the abelianization of \( \text{IA}_n \). More precisely, they showed
\[
\text{IA}_n^{ab} \cong H^* \otimes_\mathbb{Z} \Lambda^2 H
\]
as a \( \text{GL}(n, \mathbb{Z}) \)-module. This abelianization is induced from the first Johnson homomorphism
\[
\tau_1 : \text{IA}_n \to \text{Hom}_\mathbb{Z}(H, \Lambda^2 H) = H^* \otimes_\mathbb{Z} \Lambda^2 H
\]
defined by \( \sigma \mapsto (x \mapsto x^{-1}x^\sigma) \). (For a basic material concerning the Johnson homomorphism, see [17] and [25] for example.) In this paper, we identify \( \text{IA}_n^{ab} \) with \( H^* \otimes_\mathbb{Z} \Lambda^2 H \) through \( \tau_1 \). Then, (the coset classes of) the Magnus generators
\[
K_{ij} = e_i^* \otimes e_i \wedge e_j, \quad K_{ijk} = e_i^* \otimes e_j \wedge e_k
\]
form a basis of \( \text{IA}_n^{ab} \) as a free abelian group. In the following, for simplicity, we write \( V \) for \( \text{IA}_n^{ab} \), and set
\[
e_{i,j,k}^i := e_i^* \otimes e_j \wedge e_k
\]
for any \( i, j \) and \( k \). Moreover, we consider the set
\[
I := \{(i, j, k) \mid 1 \leq i \leq n, \ 1 \leq j < k \leq n\}
\]
of the indices of the basis of \( V \).

Finally, we recall the inner automorphism group. For each \( 1 \leq i \leq n \), set
\[
\iota_i := K_{1i}K_{2i} \cdots K_{ni} \in \text{IA}_n,
\]
and let \( \text{Inn} F_n \) be a subgroup of \( \text{IA}_n \) generated by \( \iota_i \) for \( 1 \leq i \leq n \). The group \( \text{Inn} F_n \) is called the inner automorphism group of \( F_n \), and is a free group with basis \( \iota_1, \ldots, \iota_n \). We remark that the abelianization \( (\text{Inn} F_n)^{ab} \) is naturally isomorphic to \( H \) as a \( \text{GL}(n, \mathbb{Z}) \)-module. Furthermore, the inclusion \( \text{Inn} F_n \hookrightarrow \text{IA}_n \) induces a \( \text{GL}(n, \mathbb{Z}) \)-equivariant injective homomorphism
\[
H = (\text{Inn} F_n)^{ab} \to \text{IA}_n^{ab} = H^* \otimes_\mathbb{Z} \Lambda^2 H
\]
between their abelianizations.

3. Construction of crossed homomorphisms

In this section, for any commutative ring \( L \), we introduce two crossed homomorphisms \( f_M \) and \( f_K \) from \( \text{Aut} F_n \) into \( V_L = V \otimes_\mathbb{Z} L \), due to Morita [17] and Kawazumi [10] respectively. We remark that in their papers, the action of \( \text{Aut} F_n \) on \( F_n \) is considered as the left one. Hence, in this paper, whenever we use their notation and have to consider the left action of \( \text{Aut} F_n \) on \( F_n \), we use \( \sigma(x) := x^\sigma \) for any \( \sigma \in \text{Aut} F_n \) and \( x \in F_n \).
3.1. Morita’s construction.

First we construct a crossed homomorphism $f_M$ from $\text{Aut} F_n$ into $V$ using the Magnus representation of $\text{Aut} F_n$ due to Morita [17]. Let

$$\frac{\partial}{\partial x_j} : \mathbb{Z}[F_n] \rightarrow \mathbb{Z}[F_n]$$

be the Fox’s free derivations for $1 \leq j \leq n$. (For a basic material concerning with the Fox’s derivation, see [1] for example.) Let $\partial_j : \mathbb{Z}[F_n] \rightarrow \mathbb{Z}[F_n]$ be the Fox’s free derivations for $1 \leq j \leq n$. Let $\overline{\cdot} : \mathbb{Z}[F_n] \rightarrow \mathbb{Z}[F_n]$ be the antiautomorphism induced from the map $y \mapsto y^{-1} \in F_n$, and $\alpha : \mathbb{Z}[F_n] \rightarrow \mathbb{Z}[H]$ the ring homomorphism induced from the abelianization $F_n \rightarrow H$. For any matrix $A = (a_{ij}) \in \text{GL}(n, \mathbb{Z}[F_n])$, set $A^\alpha = (a_{ij}^\alpha) \in \text{GL}(n, \mathbb{Z}[H])$. Then a map $r_M : \text{Aut} F_n \rightarrow \text{GL}(n, \mathbb{Z}[H])$ defined by

$$\sigma \mapsto \left( \frac{\partial \sigma(x_j)}{\partial x_i} \right)^\alpha$$

is called the Magnus representation of $\text{Aut} F_n$. We remark that $r_M$ is not a homomorphism but a crossed homomorphism. Namely, $r_M$ satisfies

$$r_M(\sigma \tau) = r_M(\sigma) \cdot r_M(\tau)^{\sigma^*}$$

for any $\sigma, \tau \in \text{Aut} F_n$ where $r_M(\tau)^{\sigma^*}$ denotes the matrix obtained from $r_M(\tau)$ by applying a ring homomorphism $\sigma^* : \mathbb{Z}[H] \rightarrow \mathbb{Z}[H]$ induced from $\sigma$ on each entry. (For detail for the Magnus representation, see [17].)

Now, observing the images of the Nielsen’s generators by $\text{det} \circ r_M$, we verify that $\text{Im}(\text{det} \circ r_M)$ is contained in a multiplicative abelian subgroup $\pm H$ of $\mathbb{Z}[H]$. In order to modify the image of $\text{det} \circ r_M$, we consider the signature of $\text{Aut} F_n$. For any $\sigma \in \text{Aut} F_n$, set $\text{sgn}(\sigma) := \text{det}(\rho(\sigma)) \in \{\pm 1\}$, and define a map $f_M : \text{Aut} F_n \rightarrow \mathbb{Z}[H]$ by

$$\sigma \mapsto \text{sgn}(\sigma) \text{ det}(r_M(\sigma))$$

Then the map $f_M$ is also crossed homomorphism which image of is contained in a multiplicative abelian subgroup $H$ in $\mathbb{Z}[H]$. In the following, we identify the multiplicative abelian group structure of $H$ with the additive one.

Finally, for any commutative ring $L$, by composing $f_M$ with a natural homomorphism $H \rightarrow V \rightarrow V_L$ induced from the inclusion $\text{Im} F_n \hookrightarrow \text{IA}_n$ and the projection $V \rightarrow V_L$, we obtain an element in $\text{Cros}(\text{Aut} F_n, V_L)$, also denoted by $f_M$.

3.2. Kawazumi’s construction.

Next, we construct another crossed homomorphism from $\text{Aut} F_n$ into $V_L$ using the Magnus expansion of $F_n$ due to Kawazumi [10]. (For a basic material for the Magnus expansion, see Chapter 2 in [2].)

Let $\hat{T}$ be the complete tensor algebra generated by $H$. For any Magnus expansion $\theta : F_n \rightarrow \hat{T}$, Kawazumi define a map

$$\tau_1^\theta : \text{Aut} F_n \rightarrow H^* \otimes_{\mathbb{Z}} H^\otimes 2$$
called the first Johnson map induced by the Magnus expansion $\theta$. The map $\tau_1^\theta$ satisfies
$$
\tau_1^\theta(\sigma)([x]) = \theta_2(x) - |\sigma|^{\otimes 2}\theta_2(\sigma^{-1}(x))
$$
for any $x \in F_n$, where $[x]$ denotes the coset class of $x$ in $H$, $\theta_2(x)$ is the projection of $\theta(x)$ in $H^{\otimes 2}$, and $|\sigma|^{\otimes 2}$ denotes the automorphism of $H^{\otimes 2}$ induced by $\sigma \in \text{Aut} F_n$. This shows that $\tau_1^\theta$ is a crossed homomorphism from $\text{Aut} F_n$ to $H^* \otimes \mathbb{Z} H^{\otimes 2}$. In [10], he also showed that $\tau_1^\theta$ does not depend on the choice of the Magnus expansion $\theta$, and that the restriction of $\tau_1^\theta$ to $\text{IA}_n$ is a homomorphism satisfying
$$
\tau_1^\theta(K_{ij}) = e_i^* \otimes e_i \otimes e_j - e_i^* \otimes e_j \otimes e_i, \quad \tau_1^\theta(K_{ijk}) = e_i^* \otimes e_j \otimes e_k - e_i^* \otimes e_k \otimes e_j.
$$

Now, for any commutative ring $L$, compose $\tau_1^\theta$ and a natural projection $H^* \otimes \mathbb{Z} \Lambda^2 H \to V_L$. Then we obtain an element in $\text{Cros}(\text{Aut} F_n, V_L)$. In this paper, we denote it by $f_K$. For $L = \mathbb{Z}$, from the result of Kawazumi as mentioned above, we see that the restriction of $f_K$ to $\text{IA}_n$ coincides with the double of the first Johnson homomorphism $\tau_1$. Namely, we have
$$
f_K(K_{ij}) = 2e_i^* \otimes e_j, \quad f_K(K_{ijk}) = 2e_k^* \otimes e_j.
$$
(See [17] or [25] for the definition of the Johnson homomorphism, for example.) Conversely, if $L$ contains $1/2$, the composition of the first Johnson homomorphism $\tau_1$ and the natural projection $V \to V_L$ extends to $\text{Aut} F_n$ as a crossed homomorphism.

### 3.3. Some observations.

In this subsection, we consider another crossed homomorphism $f_N$ in $\text{Cros}(\text{Aut} F_n, V_L)$ constructed from $f_M$ and $f_K$. It is used to determine the first cohomology group $H^1(\text{Aut} F_n, V_L)$ in Section 4.

To begin with, we see the images of the crossed homomorphisms $f_M$ and $f_K$. From the definition, we have
$$
f_M(\sigma) := \begin{cases}
-(e_{1,2}^2 + e_{1,3}^3 + \cdots + e_{1,n}^n), & \sigma = S, \\
0, & \sigma = P, Q, U
\end{cases}
$$
and
$$
f_K(\sigma) := \begin{cases}
-e_{1,2}^1, & \sigma = U, \\
0, & \sigma = P, Q, S.
\end{cases}
$$
These are obtained by straightforward calculation. We leave it to the reader as exercises.

Next, for elements
$$
a_{j,k}^i := \begin{cases}
0, & i \neq j, k, \\
1, & i = k, \\
-1, & i = j,
\end{cases}
$$
in $L$ for $(i, j, k) \in I$, set
$$
a := \sum_{(i,j,k) \in I} a_{j,k}^i e_{j,k}^i \in V_L,
$$
and let \( f_a \in \text{Prin}(\text{Aut} F_n, V_L) \) be a principal homomorphism associated to \( a \in V_L \). Namely, for any \( \sigma \in \text{Aut} F_n \), it holds

\[
f_a(\sigma) = \sigma \cdot a - a,
\]

\[
= \begin{cases} 
0, & \sigma = P, Q, \\
-2(e_{1,2}^2 + e_{1,3}^3 + \cdots + e_{1,n}^n), & \sigma = S, \\
e_{1,2}^1 - (e_{2,3}^3 + e_{2,4}^4 + \cdots + e_{2,n}^n), & \sigma = U.
\end{cases}
\]

In fact, we have

\[
f_a(P) = -(a_{1,2}^2 + a_{1,2}^1)e_{1,2}^1 + \sum_{k=3}^{n}(a_{2,k}^2 - a_{1,k}^1)e_{1,k}^1 \\
- (a_{1,2}^1 + a_{1,2}^2)e_{1,2}^2 + \sum_{k=3}^{n}(a_{1,k}^1 - a_{2,k}^2)e_{2,k}^2 \\
+ \sum_{k=3}^{n}\{(a_{2,k}^k - a_{1,k}^k)e_{1,k}^k + (a_{1,k}^k - a_{2,k}^k)e_{2,k}^k\}, \\
= 0,
\]

\[
f_a(Q) = \sum_{1\leq j<i\leq n-1}(a_{j+1,i+1}^{i+1} - a_{j,i}^{i})e_{j,i}^i + \sum_{j=1}^{n-1}(-a_{1,j+1}^{1} - a_{j,n}^{n})e_{j,n}^n \\
+ \sum_{1\leq i<j\leq n-1}(a_{i+1,j+1}^{i+1} - a_{i,j}^{i})e_{i,j}^i, \\
= 0,
\]

\[
f_a(S) = \sum_{2\leq i\leq n} -2a_{i,i}^i e_{1,i}^i = -2(e_{1,2}^2 + e_{1,3}^3 + \cdots + e_{1,n}^n)
\]

and

\[
f_a(U) = a_{1,2}^2 e_{1,2}^1 + \sum_{3\leq i\leq n} -a_{i,i}^i e_{2,i}^i + \sum_{3\leq i\leq n}(a_{2,i}^2 - a_{1,i}^1)e_{2,i}^1, \\
= e_{1,2}^1 - (e_{2,3}^3 + e_{2,4}^4 + \cdots + e_{2,n}^n).
\]

Now, we define \( f_N := 2f_M - f_K - f_a \in \text{Cros}(\text{Aut} F_n, V_L) \). From the arguments above, we have

\[
f_N(\sigma) = \begin{cases} 
\begin{align*}
& e_{2,3}^3 + e_{2,4}^4 + \cdots + e_{2,n}^n, & \sigma = U, \\
& 0, & \sigma = P, Q, S.
\end{align*}
\end{cases}
\]

We use \( f_N \) in Section 4.

### 4. The First Cohomology Group

In the following, we always assume that \( L \) is a commutative ring which does not contain any 2-torsions. Set \( V_L := V \otimes_{\mathbb{Z}} L \) as above. In this section, by using the Nielsen’s presentation for \( \text{Aut} F_n \), we show

**Theorem 4.1.** For \( n \geq 5 \),

\[
H^1(\text{Aut} F_n, V_L) = L^\otimes 2.
\]
Here we give the outline of the computation. Let $F$ be a free group with basis $P$, $Q$, $S$ and $U$, and $\varphi : F \to \text{Aut} F_n$ the natural projection. Then the kernel $R$ of $\varphi$ is a normal closure of the relators $(N1)$, $\ldots$, $(N12)$. Considering the five-term exact sequence of the Lyndon-Hochshild-Serre spectral sequence of the group extension

$$1 \to R \to F \to \text{Aut} F_n \to 1,$$

we obtain an exact sequence

$$0 \to H^1(\text{Aut} F_n, V_L) \to H^1(F, V_L) \to H^1(R, V_L)^F.$$

Observing this sequence at the cocycle level, we also obtain an exact sequence

$$0 \to \text{Cros}(\text{Aut} F_n, V_L) \to \text{Cros}(F, V_L) \xrightarrow{\iota^*} \text{Cros}(R, V_L)$$

where $\iota^*$ is a map induced from the inclusion $\iota : R \hookrightarrow F$. Hence we can consider $\text{Cros}(\text{Aut} F_n, V_L)$ as a subgroup consisting of elements of $\text{Cros}(F, V_L)$ which are killed by $\iota^*$. Hence, we can determine $\text{Cros}(\text{Aut} F_n, V_L)$ by using the relators of the Nielsen’s presentation, and hence $H^1(\text{Aut} F_n, V_L)$.

**Proof of Theorem 4.1.** First, we consider the abelian group structure of $\text{Cros}(F, V_L)$. For any $\sigma \in F$ and a crossed homomorphism $f \in \text{Cros}(F, V_L)$, set

$$f(\sigma) := \sum_{(i,j,k) \in I} a^i_{j,k}(\sigma)e^i_{j,k} \in V_L$$

for $a^i_{j,k}(\sigma) \in L$. Since $F$ is a free group generated by $P$, $Q$, $S$ and $U$, by the universality of a free group, the crossed homomorphism $f$ is completely determined by $a^i_{j,k}(\sigma)$ for $\sigma = P$, $Q$, $S$ and $U$. More precisely, a map

$$\text{Cros}(F, V_L) \to L^{\oplus 2n^2(n-1)}$$

defined by

$$f \mapsto \left( a^i_{j,k}(P), a^i_{j,k}(Q), a^i_{j,k}(S), a^i_{j,k}(U) \right)_{(i,j,k) \in I}$$

is an isomorphism as an abelian group. In the following, through this map, we identify $\text{Cros}(F, V_L)$ with $L^{\oplus 2n^2(n-1)}$.

In the following, we show that each $f \in \text{Cros}(\text{Aut} F_n, V_L) \subset \text{Cros}(F, V_L)$ is determined by at most

$$a^i_{j,k}(Q), \quad 1 \leq i \leq n - 1, \quad 1 \leq j < k \leq n,$$

$$a^1_{j,k}(U), \quad 1 \leq j < k \leq n,$$

$$a^2_{1,2}(S), \quad a^3_{2,3}(U).$$

(2)

Namely, we show that a map

$$\Phi : \text{Cros}(\text{Aut} F_n, V_L) \to L^{\oplus (n^3 - n^2 + 4)/2}$$

defined by

$$f \mapsto \left( (a^i_{j,k}(Q))_{i \neq n, 1 \leq j < k \leq n}, (a^1_{j,k}(U))_{1 \leq j < k \leq n}, a^2_{1,2}(S), a^3_{2,3}(U) \right)$$

is injective. (Later, we see that these elements uniquely determine a crossed homomorphism from $\text{Aut} F_n$ into $V_L$.) To see this, it suffices to show that each of $a^i_{j,k}(\sigma)$ for
\[ \sigma = P, Q, S \text{ and } U, \text{ and } (i, j, k) \in I \text{ other than } 2 \text{ is written as a linear combination of } 2. \text{ In this process, we use the relators of the Nielsen’s presentation.} \\

To do this, we prepare some notation. Let \( f \in \text{Cros}(\text{Aut } F_n, V_L) \). We denote by \( W \) the quotient \( L \)-module of a free \( L \)-module spanned by \( a_{j,k}^i(\sigma) \) for \( \sigma = P, Q, S \text{ and } U, \text{ and } (i, j, k) \in I \) by a submodule generated by all linear relations obtained from \( \nu^i(f) = 0 \). Then each of the coefficients of \( e_{j,k}^i \) in \( f(\sigma) \) is considered as an element in \( W \). Furthermore, we denote by \( \overline{W} \) the quotient module of \( W \) by the submodule generated by all elements in \( 2 \). We use \( \doteq \) for the equality in \( \overline{W} \). In the following, from Step I to Step V, we show 

\[ a_{j,k}^i(\sigma) \doteq 0 \]

for \( \sigma = P, Q, S \text{ and } U, \text{ and any } (i, j, k) \in I. \)

For the convenience, we also write \( a \doteq a' \) for \( a, a' \in V_L \) if each of the coefficients of \( e_{j,k}^i \) in \( a \) is equal to that in \( a' \) in \( \overline{W} \).

**Step I.** (Proof for \( a_{j,k}^n(Q) \doteq 0 \).) 
From the relation (N1): \( Q^n = 1 \), we obtain

\[ f(Q^n) = (1 + Q + Q^2 + \cdots + Q^{n-1})f(Q) = 0. \]

For any \( 1 \leq j < k \leq n \), observing the coefficient of \( e_{j,k}^n \) in the equation above, we see

\[
\begin{align*}
a_{j,k}^n(Q) &= a_{j+1,k+1}^1(Q) + a_{j+2,k+2}^2(Q) + \cdots + a_{j+n-k}^{n-k}(Q) \\
&- a_{1,j+n-k+1}^{n-k+1}(Q) - \cdots - a_{k-1,j+n-k}^{n-j}(Q) + a_{1,k-j+1}^{n-j+1}(Q) + \cdots + a_{j-1,k-1}^{n-1}(Q) \doteq 0,
\end{align*}
\]

and hence \( a_{j,k}^n(Q) \doteq 0 \). Therefore we see \( f(Q) \doteq 0 \).

**Step II.** (Some relations among \( a_{j,k}^1(P) \) and \( a_{j,k}^1(S) \).) 
Here we consider some linear relations among \( a_{j,k}^1(P) \) and \( a_{j,k}^1(S) \).

From the relation (N1): \( P^2 = 1 \), we see

\[ f(P^2) = (1 + P)f(P) \doteq 0. \]

Observing the coefficients of \( e_{j,k}^1, e_{1,k}^1, e_{1,2}^1, e_{2,k}^1 \) and \( e_{1,k}^i \) in the equation above, we obtain

\[
\begin{align*}
(3) \quad &a_{j,k}^1(P) + a_{j,k}^2(P) \doteq 0, \quad 3 \leq j < k \leq n, \\
(4) \quad &a_{1,k}^1(P) + a_{2,k}^2(P) \doteq 0, \quad 3 \leq k \leq n, \\
(5) \quad &a_{1,2}^1(P) - a_{1,2}^2(P) \doteq 0, \\
(6) \quad &a_{2,k}^1(P) + a_{1,k}^2(P) \doteq 0, \quad 3 \leq k \leq n, \\
(7) \quad &a_{i,k}^1(P) + a_{2,k}^i(P) \doteq 0, \quad 3 \leq i, k \leq n
\end{align*}
\]

respectively.

On the other hand, from the relation (N1): \( S^2 = 1 \), we see

\[ f(S^2) = (1 + S)f(S) \doteq 0. \]
Observing the coefficients of $e_{i,k}^j$ for $2 \leq i \leq n$ and $2 \leq j < k \leq n$ in the equation above, we see $2a_{j,k}^i(S) = 0$. Since $L$ does not contain any 2-torsions, we obtain $a_{j,k}^i(S) = 0$. Similarly, from the coefficients of $e_{1,k}^1$ for $2 \leq k \leq n$, we see $a_{1,k}^1(S) = 0$.

These relations are often used later.

**Step III.** (Proof for $a_{j,k}^i(U) = 0$) This step consists of six parts. 

(i) (Proof for $a_{1,2}^2(U) = a_{1,2}^2(U) = 0$) From the relation (N3): $(PSU)^2 = 1$, we have

$$
(PSU + 1)f(PSU) = (PSU + 1)(f(P) + Pf(S) + Psf(P) + PSPf(U)) = 0
$$

(8)
The actions of $PS$, $PSP$ and $PSPU$ on $e_k$ and $e_k^*$ are given by

$$
PS \cdot e_k = \begin{cases} 
-e_2, & k = 1, \\
 e_1, & k = 2, \\
e_k, & k \neq 1,2, 
\end{cases}

PS \cdot e_k^* = \begin{cases} 
-e_2^*, & k = 1, \\
e_1^*, & k = 2, \\
e_k^*, & k \neq 1,2, 
\end{cases}

PSP \cdot e_k = \begin{cases} 
-e_2, & k = 2, \\
e_k, & k \neq 2, 
\end{cases}

PSP \cdot e_k^* = \begin{cases} 
-e_2^*, & k = 2, \\
e_k^*, & k \neq 2, 
\end{cases}

PSPU \cdot e_k = \begin{cases} 
e_1 + e_2, & k = 1, \\
e_2, & k = 2, \\
e_k, & k \neq 1,2, 
\end{cases}

PSPU \cdot e_k^* = \begin{cases} 
e_2^* + e_1^*, & k = 2, \\
e_k^*, & k \neq 2, 
\end{cases}

$$

Using this, we see that the coefficient of $e_{2,k}^i$ for $3 \leq i, k \leq n$ in $(PSU + 1)f(PSPU)$ is equal to the coefficient of $e_{1,k}^i$ in $f(PSPU)$, and to

$$
a_{1,k}^i(P) + a_{1,2}^i(S) + a_{1,2}^i(P) + a_{1,2}^i(U) = a_{1,k}^i(U)
$$

by the argument above in Step III. Hence we obtain $a_{1,k}^i(U) = 0$. Similarly, from the coefficient of $e_{1,2}^2$ in $(PSU + 1)f(PSPU)$, which is equal to the coefficient of $e_{1,2}^2$ in $f(PSPU)$ times $-1$, we see

$$
-(a_{1,2}^2(P) - a_{1,2}^1(S) - a_{1,2}^1(P) + a_{1,2}^2(U)) = -a_{1,2}^2(U) = 0.
$$

(ii) (Proof for $a_{1,k}^2(U) = 0$ for $3 \leq k \leq n$. In general, if elements $\sigma, \tau \in \text{Aut} F_n$ are commute, we have $f(\sigma) + \sigma f(\tau) = f(\tau) + \tau f(\sigma)$ from a relation $\sigma \tau = \tau \sigma$. Hence, we see

$$
(\tau - 1)f(\sigma) = (\sigma - 1)f(\tau).
$$

Applying (9) for $\sigma = U$ and $\tau = Q^{-(l-1)}UQ^{l-1}$ for $3 \leq l \leq n - 1$, we have

$$
(Q^{-(l-1)}UQ^{l-1} - 1)f(U) = (U - 1)f(Q^{-(l-1)}UQ^{l-1})
$$

$$
= (U - 1)(f(Q^{-(l-1)} + Q^{-(l-1)}UQ^{l-1} + Q^{-(l-1)}Uf(Q^{l-1})).
$$

Since $f(Q) = 0$ by Step I, we see

$$
(Q^{-(l-1)}UQ^{l-1} - 1)f(U) \doteq (U - 1)Q^{-(l-1)}f(U).
$$

(10)
Here the actions of $Q^{-(l-1)}UQ^{l-1}$ on $e_k$ and $e_k^*$ are given by

$$Q^{-(l-1)}UQ^{l-1} \cdot e_k = \begin{cases} e_{l+1}, & k = l, \\ e_l, & k = l+1, \\ e_k, & k \neq l, l+1. \end{cases}$$

$$Q^{-(l-1)}UQ^{l-1} \cdot e_k^* = \begin{cases} e_{l+1}^* + e_l^*, & k = l, \\ e_l^*, & k = l+1, \\ e_k^*, & k \neq l, l+1. \end{cases}$$

Observing this, we see the coefficient of $e_{2,l}^{l+1}$ of $(Q^{-(l-1)}UQ^{l-1} - 1)f(U)$ is equal to 0. On the other hand, that of $(U - 1)Q^{-(l-1)}f(U)$ is equal to

the coefficient of $e_{1,l}^{l+1}$ of $Q^{-(l-1)}f(U)$ times $-1$,

and to

the coefficient of $e_{1,n+2-l}^2$ of $f(U)$.

Hence, we see

$$a_{1,n+2-l}^2(U) = 0$$

for $3 \leq l \leq n - 1$. Namely, $a_{1,k}^2(U) = 0$ for $3 \leq k \leq n - 1$.

Similarly, applying (9) for $\sigma = U$ and $\tau = Q^{-(l-1)}PQ^{l-1}$ for $3 \leq l \leq n - 1$, we have

$$(Q^{-(l-1)}PQ^{l-1} - 1)f(U) \equiv (U - 1)Q^{-(l-1)}f(P).$$

Here the actions of $Q^{-(l-1)}PQ^{l-1}$ on $e_k$ and $e_k^*$ are given by

$$Q^{-(l-1)}PQ^{l-1} \cdot e_k = \begin{cases} e_{l+1}, & k = l, \\ e_l, & k = l+1, \\ e_k, & k \neq l, l+1. \end{cases}$$

$$Q^{-(l-1)}PQ^{l-1} \cdot e_k^* = \begin{cases} e_{l+1}^* + e_l^*, & k = l, \\ e_l^*, & k = l+1, \\ e_k^*, & k \neq l, l+1. \end{cases}$$

Using this, from the coefficient of $e_{1,l}^2$ in (11), we see

$$a_{1,l+1}^2(U) - a_{1,l}^2(U) \equiv 0$$

for $3 \leq l \leq n - 1$. In particular, we have $a_{1,n}^2(U) = a_{1,n-1}^2(U) \equiv 0$.

(iii) (Proof for $a_{j,k}^2(U) \equiv 0$ for $2 \leq j < k \leq n$.) First, we show $a_{2,k}^2(U) \equiv 0$. Now, observing the coefficients of $e_{1,k}^1$ and $e_{2,k}^1$ for $3 \leq k \leq n$ in $(PSPU + 1)f(PSPU)$, which are equal to those of $2e_{1,k}^1 + e_{1,k}^2$ and $-e_{2,k}^2 + e_{2,k}^1$ in $f(PSPU)$ respectively, we obtain

$$2a_{1,k}^1(P) + a_{1,k}^2(P) + 2a_{2,k}^2(S) + a_{2,k}^1(S) + 2a_{2,k}^2(P) - a_{2,k}^1(P) + 2a_{1,k}^1(U) - a_{1,k}^2(U) \equiv 0,$$

$$-a_{2,k}^2(P) + a_{1,k}^2(P) - a_{1,k}^1(S) + a_{2,k}^1(S) - a_{1,k}^1(P) - a_{2,k}^1(P) - a_{2,k}^2(U) - a_{1,k}^2(U) \equiv 0.$$
Using (11), and \( a_{1,k}^1(S) = a_{2,k}^2(S) = 0 \), we have

12. \[
a_{1,k}^2(P) + a_{2,k}^1(P) - a_{2,k}^1(S) - a_{1,k}^2(U) + 2a_{1,k}^1(U) = 0,
\]

13. \[
a_{1,k}^2(P) + a_{2,k}^1(S) - a_{2,k}^1(P) - a_{2,k}^2(U) - a_{1,k}^2(U) = 0.
\]

Then, considering (12) \(-\) (13), we obtain \( a_{2,k}^2(U) = -2a_{1,k}^1(U) = 0 \) for \( 3 \leq k \leq n \).

Next, we show \( a_{j,k}^2(U) = 0 \) for \( 3 \leq j < k \leq n \). For \( 3 \leq j < l \leq n - 1 \), from the coefficient of \( e_{j,l+1}^2 \) in (10), we see \( a_{j,k}^2(U) = 0 \). Similarly, for \( 3 \leq l \leq n - 2 \), from the coefficient of \( e_{l+1,n}^2 \) in (10), we see \( a_{l,n}^2(U) = 0 \). To see \( a_{n-1,n}^2(U) = 0 \), we use (14). Observing the coefficient of \( e_{n-2,n}^2 \) in (11) for \( l = n - 2 \), we see \( a_{n-1,n}^2(U) = a_{n-2,n}^2(U) = 0 \).

(iv) (Proof for \( a_{1,2}^1(U) = 0 \) for \( 3 \leq i \leq n \).) Observing the coefficient of \( e_{1,2}^1 \) in (10), we see \( a_{1,2}^1(U) = 0 \) for \( 3 \leq l \leq n - 1 \). To show \( a_{1,2}^3(U) = 0 \), considering the coefficient of \( e_{1,2}^3 \) in (11), we see \( a_{1,2}^3(U) = a_{1,2}^4(U) = 0 \).

(v) (Proof for \( a_{j,k}^i(U) = 0 \) for \( 3 \leq i \leq n \) and \( 3 \leq j < k \leq n \).) First, we consider the case where \( i \neq j, k \) and \( j \neq k - 1 \), and show that \( a_{j,k}^i(U) = a_{n,n}^i(U) \). For \( 3 \leq l \leq n - 1 \) and \( j, k \neq l, l + 1 \), from the coefficient of \( e_{j,k}^l \) in (11), we see

\[ a_{j,k}^l(U) = a_{j,k}^{l+1}(U). \]

Similarly, observing the coefficients of \( e_{j,l+1}^l \) and \( e_{l+1,k}^l \) in (11), we obtain

14. \[ a_{j,k}^l(U) = a_{j,k}^{l+1}(U), \quad 3 \leq j < l \leq n - 1, \]

15. \[ a_{l+1,k}^l(U) = a_{l+1,k}^{l+1}(U), \quad 3 \leq l < k - 1 \leq n - 1 \]
respectively. Using these equations, we obtain,

16. \[ a_{j,k}^i(U) = a_{j,k}^{i+1}(U) = \cdots = a_{j,k}^n(U), \quad k + 1 \leq i \]

17. \[ a_{j,k}^i(U) = a_{j,k}^{i+1}(U) = \cdots = a_{j,k}^{k-1}(U) = a_{j,k}^k(U) = a_{j,k-1}(U) = a_{j,k-1}(U), \quad j + 1 \leq i \leq k - 1, \]

18. \[ a_{j,k}^i(U) = a_{j,k}^{i+1}(U) = \cdots = a_{j,k}^{j-1}(U) = a_{j,k}^j(U) = a_{j-1,k}(U) = a_{j-1,k}(U), \quad i \leq j - 1. \]

Hence it suffices to show that \( a_{n,n}^j(U) = 0 \) for \( j < k - 1 \) and \( k \leq n - 1 \). The reason why we consider only \( k \leq n - 1 \) is that an element type of \( a_{n,n}^j(U) \) never appear in the above. Then, observing the coefficient of \( e_{j+1,k}^n \) in (11) for \( l = j \), we obtain the required result.

Next, we consider the other cases. If \( j = k - 1 \), by the same argument as (18) and (16), we have

\[
a_{k-1,k}^i(U) = \begin{cases} a_{k-1,k}^{k-2}(U) = a_{k-2,k-1}^0(U) = 0, & i \leq k - 2, \\ a_{k-1,k}^{k+1}(U) = a_{k-1,k}^0(U) = 0, & i \geq k + 1. \end{cases}
\]
For the case where \( i = j, k \), we prepare some relations as follows. By the coefficients of \( e_{j,i}^l \) and \( e_{i,k}^l \) in (11), we see
\[
(19) \quad a_{j,l}^i(U) = a_{j,l+1}^{i+1}(U), \quad 3 \leq j < l \leq n - 1,
\]
\[
(20) \quad a_{i,k}^l(U) = a_{l+1,k}^{l+1}(U), \quad 3 \leq l < k - 1 \leq n - 1
\]
respectively. Then by (19) and (20),
\[
a_{j,i}^l(U) = a_{j,i}^{j+1}(U), \quad a_{i,k}^l(U) = a_{i,k}^{k-1}(U).
\]
Hence it suffices to show that \( a_{i,l+1}^l(U) = a_{i,l+1}^{l+1}(U) = 0 \) for \( 3 \leq l \leq n - 1 \). From the coefficients of \( e_{i,l+1}^l \) in (11) and (10), we have
\[
a_{i,l+1}^l(U) = -a_{i,l+1}^{l+1}(U), \quad a_{i,l+1}^{l+1}(U) = 0
\]
for \( 3 \leq l \leq n - 1 \) respectively. This shows the required result.

(vi) (Proof for \( a_{i,k}^1(U) = 0 \) for \( 3 \leq i, k \leq n \).) First, we consider the case where \( i \neq k \). For \( k \neq l + 1 \), in the equation (10), the coefficient of \( e_{2,k}^l \) of \((Q^{-(l-1)}UQ^{l-1} - 1)f(U)\) is \( a_{2,k}^{l+1}(U) \). On the other hand, that of \((U - 1)Q^{-(l-1)}f(U)\) is equal to the coefficient of \( Q_{1,k}^l \) of \( Q^{-(l-1)}f(U) \) times \(-1\), and hence \( a_{k-l+1,n-2-i}(U) \). Therefore, we have
\[
(21) \quad a_{2,k}^{l+1}(U) = a_{k-l+1,n-2-i}(U) = 0
\]
for \( 3 \leq l \leq n - 1 \) and \( k \neq l + 1 \).

Similarly, from the coefficient of \( e_{2,l+1}^3 \) in (10), we see
\[
a_{2,l+1}^3(U) = -a_{2,n+2-l}^{n+4-l}(U) = 0
\]
from (21) for \( 4 \leq l \leq n - 1 \). To show \( a_{2,n}^3(U) = 0 \), we consider a relation
\[
[U, Q^{-(n-2)}PUP^{-1}Q^{(n-2)}] = 1.
\]
Using (9), we have
\[
(Q^{-(n-2)}PUP^{-1}Q^{(n-2)} - 1)f(U) = (U - 1)f(Q^{-(n-2)}PUP^{-1}Q^{(n-2)})
= (U - 1)Q^{-(n-2)}(f(P) + Pf(U) - PUP^{-1}f(P)).
\]
Here the actions of \( Q^{-(l-2)}PUP^{-1}Q^{(l-2)} \) for \( 2 \leq l \leq n \) on \( e_k \) and \( e_k^* \) are given by
\[
Q^{-(l-2)}PUP^{-1}Q^{(l-2)} \cdot e_k = \begin{cases} e_{l-1} - e_l, & k = l, \\ e_k, & k \neq l, \end{cases}
\]
\[
Q^{-(l-2)}PUP^{-1}Q^{(l-2)} \cdot e_k^* = \begin{cases} e_{l-1}^* + e_l^*, & k = l - 1, \\ e_k^*, & k \neq l - 1. \end{cases}
\]
Then the coefficient of \( e_{2,n-1}^3 \) of \((Q^{-(n-2)}PUP^{-1}Q^{(n-2)} - 1)f(U)\) is given by \(-a_{2,n}^3(U)\). On the other hand, that of \((U - 1)Q^{-(n-2)}(f(P) + Pf(U) - PUP^{-1}f(P))\) is equal to the coefficient of \( e_{1,n-1}^3 \) of \( Q^{-(n-2)}(f(P) + Pf(U) - PUP^{-1}f(P)) \) times \(-1\), and to
\[
\text{the coefficient of } e_{2,3}^3 \text{ of } f(P) + Pf(U) - PUP^{-1}f(P),
\]
and to
\[ a_{2,3}^5(P) + a_{1,3}^5(U) - a_{2,3}^5(P) = a_{1,3}^5(U). \]
Hence we obtain \( a_{2,n}^3(U) \equiv -a_{1,3}^5(U) \equiv 0. \)

Finally, we consider the case where \( i = k. \) In (10), the coefficients of \( e_{2,l+1}^i \) of \( (Q^{-l-1})UQ^{-l-1} - 1) f(U) \) is equal to
\[
(a_{2,l+1}^i(U) - a_{2,l}^i(U) - a_{2,l}^{i+1}(U) \equiv a_{2,l+1}^i(U) - a_{2,l}^i(U).
\]
On the other hand, that of \((U - 1)Q^{-l-1} f(U)\) is equal to
the coefficient of \( e_{1,l+1}^i \) of \( Q^{-l-1} f(U) \) times \(-1,\)
and to \( a_{2,2-l+n}^1(U) \equiv 0. \) Then we obtain \( a_{2,l+1}^1(U) \equiv a_{2,l}^1(U) \) for \( 3 \leq l \leq n - 1, \) and hence
\[
a_{2,n}^n(U) \equiv a_{2,n-1}^{n-1}(U) \equiv \cdots \equiv a_{2,3}^3(U) \equiv 0.
\]
Therefore we see \( a_{i,j,k}^i(U) \equiv 0 \) for any \((i, j, k) \in I. \) This shows that \( f(U) \equiv 0. \) This completes the proof of Step III.

**Step IV. (Proof for \( a_{i,j,k}^i(P) \equiv 0. \))**

By a result obtained in Step II, it suffices to show \( a_{i,j,k}^i(P) \equiv 0 \) for \( i \neq 2. \)

(i) (Proof for \( a_{1,j,k}^1(P) \equiv 0 \) for \( 1 \leq j < k \leq n. \)) From a relation \( (PQ^{-1}UQ)^2 = UQ^{-1}UQU^{-1} \) and a result \( f(Q) \equiv f(U) \equiv 0 \) as above, we see
\[
(1 + PQ^{-1}UQ)f(PQ^{-1}UQ) \equiv f(UQ^{-1}UQU^{-1}) \equiv 0,
\]
and hence
\[
(1 + PQ^{-1}UQ)f(P) \equiv 0.
\]
The actions of \( PQ^{-1}UQ \) on \( e_k \) and \( e_k^* \) are given by
\[
PQ^{-1}UQ \cdot e_k = \begin{cases} 
eq 2, & k = 1, \\ e_1 - e_3, & k = 2, \\ e_k, & k \neq 1, 2, \end{cases}
\]
and
\[
PQ^{-1}UQ \cdot e_k^* = \begin{cases} e_2^*, & k = 1, \\ e_1^*, & k = 2, \\ e_1^* + e_3^*, & k = 3, \\ e_k^*, & k \neq 1, 2, 3. \end{cases}
\]
Using this, we see that the coefficients of \( e_{2,3}^2 \) and \( e_{3,k}^2 \) in (22) are calculated as
\[
a_{2,3}^2(P) + a_{1,3}^1(P) - a_{1,2}^1(P) \equiv 0,
\]
\[
a_{3,k}^2(P) + a_{3,k}^1(P) - a_{2,k}^1(P) \equiv 0, \quad 4 \leq k \leq n
\]
respectively. Then from (1) and (3), we obtain \( a_{1,2}^1(P) \equiv 0 \) and \( a_{2,k}^1(P) \equiv 0 \) for \( 4 \leq k \leq n. \)

Next, applying (9) for \( \sigma = P \) and \( \tau = Q^{-l-1}UQ^{-l-1} \) for \( 3 \leq l \leq n - 1, \) we have
\[
(Q^{-l-1}UQ^{-l-1} - 1) f(P) \equiv (U - 1)Q^{-l-1}UQ^{-l-1} \equiv 0.
\]
By the coefficient of \( e_{2,l+1}^1 \) in the equation above, we see \( -a_{2,l}^1(P) \equiv 0. \) In particular, \( a_{2,3}^1(P) \equiv 0. \) Furthermore, from the coefficient of \( e_{1,l+1}^1, \) we see
\[
a_{1,l}(P) \equiv 0, \quad 3 \leq l \leq n - 1.
Now, from a relation \((QP)^{n-1} = 1\),
\[(1 + QP + \cdots + (QP)^{n-2})(f(Q) + Qf(P)) = 0,\]
and hence
\[(24) \quad (1 + QP + \cdots + (QP)^{n-2})Qf(P) = 0.\]
By the coefficient of \(e_{1,2}^1\) in (24), we see
\[a_{2,3}^2(P) + a_{2,4}^2(P) + \cdots + a_{2,n}^2(P) - a_{1,2}^2(P) = 0.\]
Then using (4) and (5), we obtain \(a_{1,n}^1(P) = 0\).

Subsequently, we consider \(a_{j,k}^1(P)\) for \(3 \leq j < k \leq n\). From the coefficient of \(e_{j,l+1}^1\) in (23) for \(3 \leq j < l \leq n - 1\), we obtain
\[(25) \quad a_{j,l}^1(P) = 0.\]
On the other hand, from the coefficient of \(e_{i,n}^1\) in (23), we see \(a_{i,n}^1(P) = 0\) for \(3 \leq j \leq n - 2\). Furthermore, observing the coefficient of \(e_{n-1,n}^1\) in (24), we see
\[-a_{1,2}^2(P) + a_{1,3}^2(P) + a_{3,4}^2(P) + \cdots + a_{n-2,n-1}^2(P) + a_{n-1,n}^2(P) = 0,\]
and hence \(a_{1,n-1,n}^1(P) = -a_{n-1,n}^2(P) = 0\). Therefore we have \(a_{j,k}^1(P) = a_{j,k}^2(P) = 0\) for any \(1 \leq j < k \leq n\).

(ii) (Proof for \(a_{i,k}^1(P) = 0\) for \(3 \leq i \leq n\) and \(2 \leq k \leq n\).) First, we consider the case where \(i \neq k\). Observing the coefficients of \(e_{1,k}^i\) and \(e_{1,l+1}^i\) in (23), we see
\[(26) \quad a_{1,k}^{i+1}(P) = 0, \quad 3 \leq l \leq n - 1, \quad 2 \leq k \neq l + 1,\]
\[(27) \quad a_{1,l}^{i+1}(P) = 0, \quad 4 \leq l \leq n - 1\]
respectively. Hence, it suffices to show that \(a_{1,n}^{i+1}(P) = a_{1,n}^2(P) = 0\).

By the coefficient of \(e_{1,2}^3\) in (24), we see
\[a_{2,3}^4(P) + a_{2,4}^5(P) + \cdots + a_{2,n-1}^n(P) + a_{2,n}^1(P) - a_{1,2}^3(P) = 0.\]
From (7),
\[-a_{1,3}^4(P) - a_{1,4}^5(P) - \cdots - a_{1,n-1}^n(P) + a_{1,n}^1(P) = a_{1,2}^3(P),\]
and hence \(a_{1,2}^3(P) = 0\). Similarly, the coefficient of \(e_{1,n}^3\) in (24), we see
\[-a_{1,2}^4(P) + a_{2,3}^5(P) + \cdots + a_{2,n-2}^n(P) + a_{2,n-1}^1(P) + a_{1,n}^3(P) = 0,\]
and
\[-a_{1,2}^4(P) - a_{1,3}^5(P) - \cdots - a_{1,n-2}^n(P) + a_{1,n-1}^1(P) = a_{1,n}^3(P),\]
and hence \(a_{1,n}^3(P) = 0\).

Next, we consider the case where \(i = k\). By the coefficient of \(e_{1,l+1}^i\) in (23),
\[(28) \quad a_{1,l+1}^{i+1}(P) = a_{1,l}^i(P) - a_{1,l}^{i+1}(P) = a_{1,l}^i(P), \quad 3 \leq l \leq n - 1,\]
Hence it suffices to show that \(a_{1,i}^3(P) = 0\). On the other hand, by the coefficient of \(e_{2,3}^1\) in (22), we see
\[a_{2,3}^1(P) + a_{1,3}^2(P) - a_{1,2}^3(P) - a_{1,2}^3(P) + a_{1,3}^3(P) = 0,\]
and hence $a_{1,3}^3(P) \doteq 0$.

From the argument above, we obtain $a_{i,k}^i(P) \doteq 0$ for $3 \leq i \leq n$ and $2 \leq k \leq n$. We remark that this also shows that $a_{2,k}^i(P) \doteq 0$ for $3 \leq i \leq n$ and $3 \leq k \leq n$ by (7).

(iii) (Proof for $a_{i,k}^i(P) \doteq 0$ for $3 \leq i \leq n$ and $3 \leq j < k \leq n$.) First, we consider the case where $i \geq 4$. By the coefficient of $e_{j,k}^i$ in (23), we see

$$a_{i,k}^{i+1}(P) \doteq 0$$

for $3 \leq l \leq n - 1$ and $j, k \neq l + 1$. Hence $a_{j,k}^i(P) \doteq 0$ for $4 \leq i \leq n$ and $i \neq j, k$.

If $i = j$ or $i = k$, observe the coefficients of $e_{i,l+1}^i, e_{j,l+1}^j$ and $e_{l+1,k}^l$ in (23). Then we see

(iii) $a_{i,k}^{i+1}(P) \doteq 0$, 3 \leq l \leq n \label{eq9}$

(iii) $a_{i,k}^{i+1}(P) - a_{j,l}^{i+1}(P) - a_{j,l}^i(P) \doteq 0$, 3 \leq l \leq n - 1, j \neq l \label{eq10}$

(iii) $a_{i,k}^{i+1}(P) - a_{i,l}^i(P) - a_{i,l}^{i+1}(P) \doteq 0$, 3 \leq l \leq n - 1, l + 1 < k \label{eq11}$

respectively. By (29), the equations (30) and (31) are equivalent to

(iii) $a_{j,l+1}^{i+1}(P) \doteq a_{j,l}^i(P)$, $a_{i,k}^{i+1}(P) \doteq a_{i,l}^i(P)$

respectively. Using this and (30), we see

(iii) $a_{j,n}^n(P) \doteq a_{j,n-1}^{n-1}(P) \doteq \cdots \doteq a_{j,j+1}^{j+1}(P) \doteq 0$

(iii) $a_{j,j+1}^j(P) \doteq a_{j-1,j+1}^{j-1}(P) \doteq \cdots \doteq a_{j,j+1}^3(P)$

for $3 \leq j \leq n - 1$. Hence the proof of Step IV is finished if we show $a_{j,k}^i(P) \doteq 0$ for $3 \leq j < k \leq n$.

From the coefficients of $e_{j,l+1}^3$ and $e_{l+1,n}^3$ in (23), we see

(iii) $a_{j,l}^3(P) \doteq 0$, 3 \leq j \leq l \leq n - 1, \label{eq12}$

(iii) $a_{l,n}^3(P) \doteq 0$, 4 \leq l \leq n - 2 \label{eq13}$

respectively. Hence it suffices to show $a_{3,n}^3(P) \doteq a_{n-1,n}^3(P) \doteq 0$. Then observing the coefficients of $e_{3,n}^3$ and $e_{n-1,n}^3$ in (24), we obtain

(iii) $a_{1,4}^4(P) - a_{3,5}^5(P) - \cdots - a_{n-2,n}^n(P) + a_{1,n-1}^1(P) + a_{3,n}^3(P) \doteq 0$,

(iii) $- a_{1,n}^4(P) + a_{1,3}^5(P) + a_{3,4}^6(P) + \cdots + a_{n-3,n-2}^n(P) + a_{1,n-2,n-1}^1(P) + a_{3,n-1,n}^3(P) \doteq 0$.

These equations induce the required results. Therefore we obtain $a_{j,k}^i(P) \doteq 0$ for any $(i,j,k) \in I$. This shows that $f(P) \doteq 0$. This completes the proof of Step IV.

Step V. (The rest of the proof for $a_{j,k}^i(S) \doteq 0$.)

Here we show that $a_{i,k}^i(S) \doteq 0$ for $i, k \geq 2$, and $a_{j,k}^j(S) \doteq 0$ for $2 \leq j < k \leq n$.

By the relation (N11): $SUSPS = PU^{-1}PU$, we have

(iii) $(1 + SU + SUSP)f(S) = f(PU^{-1}PU) \doteq 0$. 

The actions of $SU$ and $SUSP$ on $e_k$ and $e_k^*$ are given by
\[
SU \cdot e_k = \begin{cases} 
-e_1 - e_2, & k = 1, \\
e_k, & k \neq 1,
\end{cases}
SU \cdot e_k^* = \begin{cases} 
-e_1^*, & k = 1, \\
e_1^* + e_2^*, & k = 2, \\
e_k^*, & k \neq 1, 2,
\end{cases}
\]
\[
SUSP \cdot e_k = \begin{cases} 
e_2, & k = 1, \\
e_1 + e_2, & k = 2, \\
e_k, & k \neq 1, 2
\end{cases}
SUSP \cdot e_k^* = \begin{cases} 
-e_1^* + e_2^*, & k = 1, \\
e_1^*, & k = 2, \\
e_k^*, & k \neq 1, 2,
\end{cases}
\]
Using this, for $3 \leq k \leq n$, from the coefficients $e_{1,k}^1$ and $e_{1,k}^2$ in (33), we obtain
\[
2a_{1,k}^1(S) + a_{1,k}^2(S) + a_{2,k}^2(S) - a_{2,k}^1(S) = 0,
\]
\[
a_{2,k}^1(S) = 0
\]
respectively. Hence $a_{1,k}^1(S) = a_{2,k}^1(S) = 0$.

Next, we show $a_{1,k}^i(S) = 0$ for $i \geq 3$ and $k \geq 2$. Applying (9) for $\sigma = S$ and $\tau = QP$, we have
\[
(34) \quad (QP - 1)f(S) = (S - 1)f(QP) = 0.
\]
The actions of $QP$ on $e_k$ and $e_k^*$ are given by
\[
QP \cdot e_k = \begin{cases} 
e_1, & k = 1, \\
e_n, & k = 2, \\
e_{k-1}, & k \neq 1, 2
\end{cases}
QP \cdot e_k^* = \begin{cases} 
e_1^*, & k = 1, \\
e_n^*, & k = 2, \\
e_{k-1}^*, & k \neq 1, 2
\end{cases}
\]
Using this, from the coefficients $e_{1,k}^i$ in (34), we see
\[
35) \quad a_{1,k+1}^{i+1}(S) = a_{1,k}^i(S), \quad 2 \leq i, k \leq n - 1,
\]
\[
36) \quad a_{1,2}^{i+1}(S) = a_{1,n}^i(S), \quad 2 \leq i \leq n - 1, \quad k = n.
\]
From (35), if $i \leq k$,
\[
a_{1,k}^i(S) = a_{1,k-1}^{i-1}(S) \cdots \cdots a_{1,k+2-i}^2(S) = 0,
\]
and hence from (35) if $i > k$,
\[
a_{1,k}^i(S) = a_{1,k-1}^{i-1}(S) \cdots \cdots a_{1,2}^{i+2-k}(S) = 0.
\]
Finally, we show $a_{1,k}^1(S) = 0$ for $3 \leq j < k \leq n$. From the coefficient $e_{1,k}^1$ in (34), we see
\[
a_{1,j+1,k+1}^1(S) = a_{j,k}^1(S)
\]
for $2 \leq j < k \leq n - 1$. This shows that for $3 \leq j < k \leq n$,
\[
a_{1,k}^1(S) = a_{j-1,k-1}^{j-1}(S) \cdots \cdots a_{2,k+2-j}^1(S) = 0.
\]
Therefore we obtain $a_{1,k}^i(S) = 0$ for any $(i, j, k) \in I$. This completes the proof of Step V.

From the argument above, we verify that any $a_{j,k}^i(\sigma) \in W$ belongs to the submodule generated by (2). Namely, a crossed homomorphism $f \in \text{Cros}(\text{Aut} F_n, V_L)$ is determined by (2). In other words, the map $\Phi : \text{Cros}(\text{Aut} F_n, V_L) \to L^{\oplus(n^3-n^2+4)/2}$
is injective. Let $W' = L^\oplus(n^3 - n^2 + 4)/2$ be the target of the map above. We consider $\text{Cros}(\text{Aut } F_n, V_L)$ as a submodule of $W'$. In the next Step, we study the quotient $L$-module $W'/\text{Prin}(\text{Aut } F_n, V_L)$.

**Step VI.** (The structure of $W'/\text{Prin}(\text{Aut } F_n, V_L)$.)

Here, we show that $W'/\text{Prin}(\text{Aut } F_n, V_L)$ is a free $L$-module of rank 2. For any element

$$a := \sum_{(i,j,k) \in I} a_{j,k}^i e_{j,k}^i \in V_L,$$

let $f_a : \text{Aut } F_n \to V$ be the principal homomorphism associated to $a$. For example,

$$f_a(Q) = \sum_{i \neq n, 1 \leq j < k \leq n-1} (a_{j+1,k+1}^i - a_{j,k}^i) e_{j,k}^i + \sum_{1 \leq i,j \leq n-1} (-a_{1,j+1}^i - a_{j,n}^i) e_{j,n}^i$$

$$+ \sum_{1 \leq j < k \leq n} (a_{j+1,k+1}^1 - a_{j,k}^n) e_{j,k}^n + \sum_{1 \leq j < k \leq n} (-a_{1,j+1}^1 - a_{j,n}^n) e_{j,n}^n$$

and

$$f_a(U) = \sum_{2 \leq k \leq n} a_{1,k}^2 e_{1,k}^2 + \sum_{3 \leq k \leq n} (a_{2,k}^2 - a_{1,k}^2) e_{2,k}^1 + \sum_{3 \leq j \leq n} a_{1,k}^j e_{j,k}^j$$

$$+ \sum_{3 \leq j \leq n} -a_{1,k}^2 e_{2,j}^2 + \sum_{3 \leq i,k \leq n} -a_{1,k}^i e_{i,k}^i.$$  

In order to determine the $L$-module structure of $W'/\text{Prin}(\text{Aut } F_n, V_L)$, it suffices to find the elementary divisors of an $n^2(n - 1)/2 \times (n^3 - n^2 + 4)/2$ matrix:

$$A := \begin{pmatrix} a_{1,1}^1(Q) & a_{2,1}^2(Q) & \cdots & a_{n,1}^{n-1}(Q) & a_{1,1}^1(U) & a_{1,2}^2(S) & a_{1,3}^3(U) \\ a_{1,1}^2 & A_{1,1} & \cdots & A_{1,n-1} & A_{1,n} & A_{1,n+1} & A_{1,n+2} \\ a_{2,1}^1 & A_{2,1} & \cdots & A_{2,n-1} & A_{2,n} & A_{2,n+1} & A_{2,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1}^1 & A_{n,1} & \cdots & A_{n,n-1} & A_{n,n} & A_{n,n+1} & A_{n,n+2} \end{pmatrix}$$

which row is indexed by $a_{j,k}^i$s, and which column is indexed by $i$. Here each $A^{p,q}$ is a block matrix defined as follows. First, we consider the case where $1 \leq q \leq n - 1$. For $\sigma = P, Q, S$ and $U$, set

$$f_a(\sigma) := \sum_{(i,j,k) \in I} a_{j,k}^i(\sigma) e_{j,k}^i \in V_L.$$  

Then, for any $1 \leq j_2 < k_2 \leq n$, we have

$$a_{j_2,k_2}^q(Q) = \sum_{(p,j_1,k_1) \in I} C_{(j_1,k_1), (j_2,k_2)}^{p,q} a_{j_1,k_1}^p.$$
for some $C_{(j_1, k_1), (j_2, k_2)} \in L$. Then the matrix $A^{p,q}$ is defined by

$$A^{p,q} := \begin{pmatrix}
  a^p_{1,2} & a^p_{1,3} & \cdots & a^p_{n-1,n} \\
  C^{p,q}_{(1,2), (1,2)} & C^{p,q}_{(1,2), (1,3)} & \cdots & C^{p,q}_{(1,2), (n-1,n)} \\
  C^{p,q}_{(1,3), (1,2)} & C^{p,q}_{(1,3), (1,3)} & \cdots & C^{p,q}_{(1,3), (n-1,n)} \\
  \vdots & \vdots & \ddots & \vdots \\
  C^{p,q}_{(n-1,n), (1,2)} & C^{p,q}_{(n-1,n), (1,3)} & \cdots & C^{p,q}_{(n-1,n), (n-1,n)}
\end{pmatrix}$$

where the rows are indexed by $a^p_{j,k}$s according to the usual lexicographic order on the set $\{(j, k) | 1 \leq j < k \leq n\}$. Similarly, the columns are indexed by $a^q_{j,k}$s.

By an argument similar to the above, the block matrices $A^{p,n}$, $A^{p,n+1}$ and $A^{p,n+2}$ for $1 \leq p \leq n$ are defined from $a^1_{j,k}(U)s$, $a^2_{1,2}(S)$ and $a^3_{2,3}(U)$ respectively.

Set

$$A' := \begin{pmatrix}
  a^1_{j,k} & a^2_{j,k} & \cdots & a^{n-1}_{j,k} & a^1_{j,k}(U) \\
  A^{1,1} & A^{1,2} & \cdots & A^{1,n-1} & A^{1,n} \\
  A^{2,1} & A^{2,2} & \cdots & A^{2,n-1} & A^{2,n} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  A^{n,1} & A^{n,2} & \cdots & A^{n,n-1} & A^{n,n}
\end{pmatrix}.$$

In the following, we prove that all elementary divisors of $A$ are equal to $1 \in L$ by showing that $A'$ can be transformed into the identity matrix with only the elementary column operations. Then we conclude $W'/\text{Prin}(\text{Aut } F_n, V_L) \cong L^{n^2}$.

First, we transform the $a^1_{j,k}(U)$ columns of $A'$. To do this, we use the followings. From $f_\alpha(Q)$ and $f_\alpha(U)$ as above, the $a^1_{j,k}(Q)$ columns, the $a^1_{j,k}(U)$ columns for $j \neq 2$ and the $a^1_{2,k}(U)$ columns of $A'$ are given by

$$a^1_{1,k} \begin{pmatrix} -E & O \\ O & -1 \end{pmatrix}, \quad a^1_{j,k}(U) \begin{pmatrix} O \\ a^1_{j,k} \end{pmatrix}, \quad a^1_{1,2} \begin{pmatrix} O \\ -E \end{pmatrix}, \quad a^1_{1,k} \begin{pmatrix} O \\ -E \end{pmatrix}$$

respectively. Here $E$ denotes the identity matrix.

Let us consider the $a^1_{2,k}(U)$ columns of $A'$. Add the $a^1_{1,k}(U)$ columns to $a^1_{2,k}(U)$ columns for $3 \leq k \leq n$, and minus $a^1_{1,k}(Q)$ columns from the $a^1_{2,k}(U)$ columns for $3 \leq k \leq n$. Subsequently, by subtracting the $a^1_{1,2}(U)$ column from the $a^1_{2,n}(U)$ column,
we see that the $a_{2,k}(U)$ columns of $A'$ are transformed into

\[
\begin{pmatrix}
a_{2,1}(U) \\
\vdots \\
a_{2,k} \\
X \\
\vdots \\
a_{2,n}(U)
\end{pmatrix}
\]

where

\[
X = \begin{pmatrix}
a_{2,1}(U) & a_{2,4}(U) & \cdots & a_{2,n}(U) \\
1 & 0 & \cdots & 0 & 0 \\
-1 & 1 & & & 0 \\
0 & & & & \\
& & & & \\
& & \ddots & & \\
& & & & 1 & 0
\end{pmatrix}
\]

It is easily seen that $X$ can be transformed into the identity matrix with the elementary column operations. Hence the $a_{j,k}(U)$ columns of $A'$ are transformed into

\[
\begin{pmatrix}
a_{j,1}(U) \\
\vdots \\
a_{j,k} \\
E \\
\vdots \\
a_{j,n}(U)
\end{pmatrix}
\]

(37)

Next, for any $1 \leq j \leq n - 1$, we consider the $a_{j,k}(Q)$ columns given by

\[
\begin{pmatrix}
a_{j,1}(Q) & a_{j,n}(Q) \\
\vdots & \\
a_{j,k} & O \\
\vdots & \\
a_{j,n} & O \\
\vdots & \\
a_{j+1,k+1} & E \\
\vdots & \\
a_{j+1,1} & O
\end{pmatrix}
\]

Multiplying each column by $-1$ and using (37), we can transform the $a_{j,k}(Q)$ columns into

\[
\begin{pmatrix}
a_{j,1}(Q) \\
\vdots \\
a_{j,k} \\
E \\
\vdots \\
a_{j,n}(Q)
\end{pmatrix}
\]
Now, we consider the $a_{j,k}^2(Q)$ columns given by

$$
\begin{pmatrix}
\vdots & O & O & O & \cdots & O & O \\
\vdots & -E & O & O & \cdots & O & O \\
a_{1,k}^2 & O & -1 & O & \cdots & 0 & 0 \\
a_{1,n}^2 & O & 0 & O & \cdots & 0 & 0 \\
a_{2,l}^2 & O & 0 & -E & \cdots & O & O \\
a_{2,n}^2 & O & 0 & O & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n-1,n}^2 & O & 0 & O & \cdots & 0 & -1 \\
a_{1,2}^3 & O & -1 & O & \cdots & 0 & 0 \\
a_{1,3}^3 & O & 0 & O & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{1,n}^3 & O & O & O & \cdots & 0 & -1 \\
a_{2,k+1}^3 & E & O & O & \cdots & O & O \\
a_{3,l+1}^3 & O & O & E & \cdots & O & O \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n-1,n}^3 & O & O & O & \cdots & 1 & 0 \\
\vdots & O & O & O & \cdots & 0 & O \\
\end{pmatrix}
$$

respectively. Similarly, using (37) and the multiplication of $-1$, we can transform the $a_{j,k}^2(Q)$ columns of $A'$ into

$$
\begin{pmatrix}
\vdots \\
\vdots \\
a_{j,k}^3 \\
\vdots \\
\vdots \\
a_{j,k}^2 \\
\end{pmatrix}
\begin{pmatrix}
O \\
E \\
O \\
\end{pmatrix}
$$

respectively.

By the same argument as above, for any $3 \leq p \leq n - 1$, we can transform the $a_{j,k}^p(Q)$ columns of $A'$ into

$$
\begin{pmatrix}
\vdots \\
\vdots \\
a_{j,k}^{p+1} \\
\vdots \\
\vdots \\
a_{j,k}^p \\
\end{pmatrix}
\begin{pmatrix}
O \\
E \\
O \\
\end{pmatrix}
$$
recursive from \( p = 3 \) to \( n - 1 \). From the argument above, we can transform \( A' \) into

\[
A' := \begin{pmatrix}
\begin{array}{cccc}
a^1_{j,k}(Q) & a^2_{j,k}(Q) & \cdots & a^{n-1}_{j,k}(Q) \\
E & O & \cdots & O \\
a^2_{j,k} & O & \cdots & O \\
a^3_{j,k} & O & \cdots & E \\
\vdots & \vdots & \ddots & \vdots \\
\end{array}
\end{pmatrix}
\]

and into the identity matrix with only the elementary column operations. Therefore we conclude that all elementary divisors of \( A \) are equal to 1 ∈ \( L \). In particular, observing the process of the transformation of \( A \) as mentioned above, we see that a map \( \Phi' : W' \to \mathbb{L}^{\oplus 2} \) defined by

\[
\left((a^i_{j,k}(Q))_{i \neq n, 1 \leq j < k \leq n}, (a^1_{j,k}(U))_{1 \leq j < k \leq n}, a^2_{1,2}(S), a^3_{2,3}(U)\right) \mapsto (a^2_{1,2}(S), a^3_{2,3}(U))
\]

induces an isomorphism

\[
W' / \text{Prin}(\text{Aut } F_n, V_L) \cong \mathbb{L}^{\oplus 2}.
\]

Finally, for the crossed homomorphisms \( f_M, f_N \in \text{Cros}(F, V_L) \) defined in Section 3 we see

\[
\Phi'(f_M) = (-1, 0), \quad \Phi(f_N) = (0, 1).
\]

Hence

\[
H^1(\text{Aut } F_n, V_L) \cong W' / \text{Prin}(\text{Aut } F_n, V_L) \cong \mathbb{L}^{\oplus 2}.
\]

This completes the proof of Theorem 4.1. \( \square \)

From Theorem 4.1, we see that the crossed homomorphisms \( f_M \) and \( f_N \), and hence \( f_M \) and \( f_K \), generate \( H^1(\text{Aut } F_n, V_L) \) for \( n \geq 5 \).

5. Some Applications

In this section, we consider the first Johnson homomorphism and the first cohomology group of the outer automorphism group.

5.1. The first Johnson homomorphism.

Here we show

**Proposition 5.1.** Let \( L \) be a commutative ring which does not contain both any 2-torsions and \( 1/2 \). Then for \( n \geq 5 \), there is no crossed homomorphism from \( \text{Aut } F_n \) to \( V_L \) which restriction to \( \text{IA}_n \) coincides with \( \tau_{1,L} \).

**Proof of Proposition 5.1.** Assume that the first Johnson homomorphism \( \tau_{1,L} \) extends to \( \text{Aut } F_n \) as a crossed homomorphism. By Theorem 4.1, \( \tau_{1,L} \) is cohomologous to \( a f_M + b f_K \) for some \( a, b \in L \). Then, for distinct \( i, j, k \) and \( j < k \), observing \( f_K(K_{ijk}) = 2e^i_{j,k} \), we see

\[
e^i_{j,k} = \tau_{1,L}(K_{ijk}) = 2be^i_{j,k}.
\]

This shows \( 2b = 1 \). It is contradiction to the hypothesis of \( L \). This completes the proof of Proposition 5.1. \( \square \)
5.2. Outer automorphism group.

Here, we compute the first cohomology group of the outer automorphism group $\text{Out} F_n := \text{Aut} F_n / \text{Inn} F_n$ of $F_n$ with coefficients in $V_L$ for any commutative ring $L$ which does not have any 2-torsions.

**Proposition 5.2.** Let $L$ be as above. Then, for $n \geq 5$, \[ H^1(\text{Out} F_n, V_L) = L. \]

**Proof of Proposition 5.2.** Considering the five-term exact sequence of \[ 1 \to \text{Inn} F_n \to \text{Aut} F_n \to \text{Out} F_n \to 1, \] we have \[ 0 \to H^1(\text{Out} F_n, V_L) \to H^1(\text{Aut} F_n, V_L) \xrightarrow{\alpha} H^1(\text{Inn} F_n, V_L) \to H^2(\text{Out} F_n, V_L). \]

From Theorem 4.1, $H^1(\text{Aut} F_n, V_L)$ can be identified with a free $L$-module $L^{\oplus 2}$ generated by $f_M$ and $f_K$. Since

\[ \alpha(f_M) = (n - 1) \sum_{i=1}^{n} \iota_i^* \otimes (e_{1,i} + e_{2,i} + \cdots + e_{n,i}), \]

\[ \alpha(f_K) = 2 \sum_{i=1}^{n} \iota_i^* \otimes (e_{1,i} + e_{2,i} + \cdots + e_{n,i}), \]

the image of $\alpha$ is contained in a free $L$-module $L$ generated by

\[ \sum_{i=1}^{n} \iota_i^* \otimes (e_{1,i} + e_{2,i} + \cdots + e_{n,i}). \]

Then $\alpha$ is considered as an $L$-linear homomorphism $L^{\oplus 2} \to L$ which matrix representation is $(n - 1 \ 2)$. Using the elementary operations, we can transform it into

\[ \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \ n : \text{even}, \]

\[ \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \ n : \text{odd}. \]

In both cases, the kernel of this homomorphism is isomorphic to $L$. This completes the proof of Proposition 5.2.

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