Pascal’s triangle, Hoggatt matrices, and analogous constructions
Johann Cigler

Abstract
We give an overview about some elementary properties of Hoggatt matrices, which are generalizations of Pascal’s triangle, and study $q$–analogs and Fibonacci analogs and derive a common generalization.

1. Introduction
In [4] Daniel C. Fielder and Cecil O. Alford defined generalizations of Pascal’s triangle which they called Hoggatt triangles. We give an overview about some elementary properties of these triangles and their $q$–analogs and give a common generalization with Fibonacci polynomials. I want to thank Christian Krattenthaler for help with some determinants and hypergeometric identities.

Let us first introduce some notations which emphasize the analogy with Pascal’s triangle. Let $d$ be a positive integer. We write

$$\langle n \rangle_d = \binom{n + d - 1}{d} = \frac{(d + 1)(d + 2) \cdots (d + n - 1)}{(n - 1)!}$$

and

$$\langle n \rangle_d! = \prod_{j=1}^{n} \langle j \rangle_d$$

and define

$$\langle n \rangle_d \langle k \rangle_d = \frac{\langle n \rangle_d \langle n-1 \rangle_d}{\langle k \rangle_d \langle k-1 \rangle_d} = \prod_{j=0}^{k-1} \frac{\langle n-j \rangle_d}{\langle k-j \rangle_d} = \frac{\langle n \rangle_d!}{\langle k \rangle_d! \langle n-k \rangle_d!}$$

for $0 \leq k \leq n$ and $\langle n \rangle_d \langle k \rangle_d = 0$ for $k > n$.

Following [4] we call the matrix

$$H_d = \left( \begin{array}{c} \langle n \rangle_d \\ \langle k \rangle_d \end{array} \right)_{n,k \geq 0}$$

the Hoggatt matrix or Hoggatt triangle of order $d$.

Email: johann.cigler@univie.ac.at

Key words and phrases: Pascal’s triangle, $q$–analog, Narayana numbers, semistandard Young tableaux, Fibonomial coefficients, Fibonacci polynomials
All entries of these matrices are nonnegative integers. This is of course true for Pascal’s triangle $H_1 = \binom{n}{k}_{n,k \geq 0}$ because of the recursion $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

For $d = 2$ the numbers $\binom{n+1}{2} = T_n$ are the triangle numbers $1, 3, 6, 10, \ldots$ and
\[
\binom{n}{2} = \frac{n!(n+1)!}{2^n}, \text{ because } \binom{1}{2} = 1 \text{ and by induction } \\
\binom{n}{2} = \binom{n-1}{2} \binom{n}{2} = \frac{(n-1)! n(n+1)}{2^{n-1}} = \frac{n!(n+1)!}{2^n}.
\]

The entries
\[
\binom{n}{k} = \frac{n!(n+1)!}{2^n} \cdot \frac{2^{n-k}}{k!(k+1)! (n-k)!(n-k+1)!} = \frac{1}{k+1} \binom{n}{k} \binom{n+1}{k}
\]
are Narayana numbers and $H_2$ is the well-known Narayana triangle (cf. e.g. [9] and OEIS [8], A001263)

\[
\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 3 & 1 & & & & \\
1 & 6 & 6 & 1 & & & \\
1 & 10 & 20 & 10 & 1 & & \\
1 & 15 & 50 & 50 & 15 & 1 \\
\end{array}
\]

For $3 \leq d \leq 9$ the corresponding triangles appear in OEIS [8], A056939, A056940, A056941, A142465, A142467, A142468, A174109.

From $\binom{n}{k} = \frac{(d+n-k) \cdots (d+n-1)}{(n-k) \cdots (n-1)} \binom{n-1}{k}$ we see that the first terms of $H_d$ are

\[
\begin{pmatrix}
1 \\
1 & 1 \\
1 & d+1 \\
\frac{1}{2} & \frac{(d+1)(d+2)}{2} \\
\frac{1}{6} & \frac{(d+1)(d+2)(d+3)}{6} \\
\frac{1}{24} & \frac{(d+1)(d+2)(d+3)(d+4)}{24} \\
\end{pmatrix}
\]
It is also well-known and easy to verify that
\[
\left\langle \frac{n}{k} \right\rangle_2 = \det \begin{pmatrix} \binom{n}{k} & \binom{n+1}{k+1} \\ \binom{n+1}{k} & \binom{n+2}{k+1} \end{pmatrix} = \det \begin{pmatrix} \binom{n}{k} & \binom{n}{k-1} \\ \binom{n}{k+1} & \binom{n}{k} \end{pmatrix}
\] (4)

These determinants show that all elements \( \left\langle \frac{n}{k} \right\rangle_2 \) are integers.

2. Main properties

For each \( n \) the entries \( \left\langle \frac{n}{k} \right\rangle_d \) are palindromic with center of symmetry at \( \frac{n}{2} \) since
\[
\left\langle \frac{n}{k} \right\rangle_d = \left\langle \frac{n}{n-k} \right\rangle_d.
\] (5)

They are also unimodal with center of symmetry at \( \frac{n}{2} \), which means that
\[
\left\langle \frac{n}{0} \right\rangle_d \leq \left\langle \frac{n}{1} \right\rangle_d \leq \cdots \leq \left\langle \frac{n}{\frac{n}{2}} \right\rangle_d \leq \left\langle \frac{n}{n-1} \right\rangle_d \geq \left\langle \frac{n}{n} \right\rangle_d.
\]

Due to symmetry it suffices to show that we have \( \left\langle \frac{n}{k} \right\rangle_d \leq \left\langle \frac{n}{k+1} \right\rangle_d \) or equivalently
\[
\left\langle \frac{k+1}{d} \right\rangle_d \leq \left\langle \frac{n-k}{d} \right\rangle_d \text{ for } k < \left\lfloor \frac{n}{2} \right\rfloor.
\]

This is true because for each \( j \) we have \( k+1+j \leq n-k+j \).

Let us also mention some alternative formulas.

Proposition 1

\[
\left\langle n \right\rangle_d! = \prod_{j=0}^{d-1} \frac{(n+j)!}{(d-j)^{n+j}}.
\] (6)

and
\[
\left\langle \frac{n}{k} \right\rangle_d = \prod_{j=0}^{k-1} \left\langle \frac{n-j}{k-j} \right\rangle_d = \prod_{j=0}^{d-1} \binom{k}{k+j} = \prod_{j=0}^{d-1} \binom{n+d-1}{n+d-1-j}.
\] (7)
Proof

Let \( f(n) \) denote the right-hand side of (6). Then

\[
 f(1) = \prod_{j=0}^{d-1} (1 + j)! = 1 = \binom{1}{d}!
\]

and by induction

\[
 f(n) = \prod_{j=0}^{d-1} \frac{(n + j)!}{(d - j)^{\sum_{j}^d} (n + d - 1)!} = \prod_{j=0}^{d-1} \frac{(n + j - 1)!}{(d - j)^{\sum_{j}^d} (n + d - 1)!} = \left( \frac{n + d - 1}{d} \right) f(n - 1) = \binom{n}{d} \binom{n - 1}{d}!.
\]

Identities (7) follow from

\[
 \prod_{j=0}^{d-1} \binom{n + j}{k} = \prod_{j=0}^{d-1} \frac{n + j - 1}{k - j} = \binom{n - 1}{d} \prod_{j=0}^{d-1} \frac{n + j}{k + j - 1} = \binom{n - 1}{d} \prod_{j=0}^{d-1} \frac{n + j - 1}{k + j - 1} = \binom{n}{d} \prod_{j=0}^{d-1} \frac{n + j}{k + j}.
\]

An analog of (4) is

**Theorem 2**

\[
 \binom{n}{k}_d = \det \left( \begin{array}{c} n + j + i \\ k + j \end{array} \right)_{i,j=0}^{d-1} = \det \left( \begin{array}{c} n - j \\ k - j \end{array} \right)_{i,j=0}^{d-1}.
\]

These determinants show that all \( \binom{n}{k}_d \) are integers.

Proof

Let us first prove the left-hand side.

\[
 \binom{n + i + j}{k + j} = \frac{(n + i + j)!}{(k + j)!(n + i - k)!} = \frac{j!(n + i)!}{(k + j)!(n + i - k)!} = \frac{j!(n + i)!}{(k + j)!(n + i - k)!} = \binom{n + i + j}{k + j}
\]

implies

\[
 \det \left( \begin{array}{c} n + i + j \\ k + j \end{array} \right)_{i,j=0}^{d-1} = \prod_{j=0}^{d-1} \frac{j!(n + j)!}{(k + j)!(n + j - k)!} \det \left( \begin{array}{c} n + j \\ j \end{array} \right)_{i,j=0}^{d-1}
\]

with

\[
 \prod_{j=0}^{d-1} \frac{j!(n + j)!}{(k + j)!(n + j - k)!} = \prod_{j=0}^{d-1} \frac{n + j}{k + j} = \binom{n}{k}_d.
\]
It remains to prove that
\[
\det\left(\begin{pmatrix} n + i + j \end{pmatrix}_{i,j=0}^{d-1}\right) = 1. \tag{9}
\]

To this end let \( \Delta \) be the difference operator on the polynomials defined by
\[
\Delta f(x) = f(x + 1) - f(x). \]

It satisfies \( \Delta \left(\begin{pmatrix} x \\ n \end{pmatrix}\right) = \left(\begin{pmatrix} x+1 \\ n \end{pmatrix} - \begin{pmatrix} x \\ n \end{pmatrix}\right) = \begin{pmatrix} x \\ n-1 \end{pmatrix} \) and therefore
\[
\Delta^k \left(\begin{pmatrix} x \\ n \end{pmatrix}\right) = \begin{pmatrix} x \\ n-k \end{pmatrix}. \]

Writing \( \Delta = E - 1 \) with \( Ef(x) = f(x+1) \) we get
\[
\Delta^k = (E - 1)^k = \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} E^{k-j} \]
and thus \( \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \begin{pmatrix} x + k - j \\ n \end{pmatrix} = \begin{pmatrix} x \\ n-k \end{pmatrix} \).

Since \( \begin{pmatrix} x \\ n-k \end{pmatrix} = 0 \) for \( k > n \) and \( \begin{pmatrix} x \\ n-n \end{pmatrix} = 1 \) the matrix \( \left(\begin{pmatrix} x \end{pmatrix}_{i,j=0}^{d-1} \right) \) is upper triangular with all entries 1 in the main diagonal. This implies
\[
\det\left(\begin{pmatrix} n + i + j \end{pmatrix}_{i,j=0}^{d-1}\right) = \det\left(\begin{pmatrix} x \end{pmatrix}_{i,j=0}^{d-1} \right) = 1.
\]

To compute the determinant \( \det\left(\begin{pmatrix} n \\ k + j - i \end{pmatrix}_{i,j=0}^{d-1}\right) = \det\left(\begin{pmatrix} n \\ k + j - i \end{pmatrix}_{i,j=0}^{d}\right) \)
we use formula (3.12) in [6] for \( q = 1 \):
\[
\det\left(\begin{pmatrix} A \\ L + j \end{pmatrix}_{i,j=1}^{d}\right) = \prod_{1 \leq i < j \leq n} \left( L_i - L_j \right) \prod_{i=1}^{n} (A + i - 1)! \over \prod_{i=1}^{n} (L_i + n)! \prod_{i=1}^{n} (A - L_i - 1)!
\]
Choosing \( A = n, \ L_i = k - i \) and \( n = d \) this gives
\[
\det\left(\begin{pmatrix} n \\ k + j - i \end{pmatrix}_{i,j=1}^{d}\right) = \prod_{j=0}^{d-1} \frac{\prod_{j=0}^{n} (n + j)!}{\prod_{j=0}^{n} (k + j)! \prod_{j=0}^{n} (n - k + j)!} = \prod_{j=0}^{d-1} \binom{n + j}{k} = \binom{n}{k}_{d} \tag{10}
\]
Another determinant representation has been given in [7]:

**Corollary 3**

\[
\binom{n}{k}_{d} = \det\left(\begin{pmatrix} n + i \\ k + j \end{pmatrix}_{i,j=0}^{d-1}\right). \tag{11}
\]
Proof

If we subtract row $i-1$ from row $i$ in
\[
\binom{n+i}{k+j}_{i,j=0}^{d-1}
\]
the new row $i$ has the entries
\[
\binom{n+i-1}{k+j-1}.
\]
If we do this for $i=d-1,d-2,\ldots,1$ the new matrix has the first row unchanged
and the rest is the matrix
\[
\binom{n+i-1}{k+j-1}_{i=1}^{d-1}.
\]
If we iterate this we arrive at
\[
\binom{n}{k+i}_{i,j=0}^{d-1}.
\]

Remark

In [14] and [7] these “MacMahon determinants” have been proved with the condensation
method (cf. [6], Proposition 10). We will use this method in Theorem 8 for the proof of a $q$–
analog.

There is a nice generalization of the formula
\[
\sum_{n \geq 0} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}.
\]

In [12] Robert A. Sulanke introduced Narayana numbers $N(d,n,k)$ of dimension $d$. His
results imply Theorem 4 which we state without proof.

Theorem 4

\[
(1-x)^{dk+1} \sum_{n \geq 0} \binom{n+k}{k} x^n = \sum_{j=0}^{(d-1)(k-1)} N(d,k,j)x^j.
\]

For $d=3$ the polynomials
\[
\sum_{j=0}^{2(k-1)} N(3,k,j)x^j
\]
are
\[
1, 1+3x+x^2, 1+10x+20x^2+10x^3+x^4,
\]
\[
1+22x+113x^2+119x^3+113x^4+22x^5+x^6, \ldots.
\]

For $d=2$ we get the Narayana numbers $N(2,n,k) = \frac{1}{k+1} \binom{n}{k} \binom{n-1}{k} = N_{n,k}$ in the usual
notation. In our notation $N_{n,k} = \binom{n-1}{k-2}$.

Let us give a direct proof for this case.

Theorem 5

\[
\sum_{j=0}^{k-1} j \binom{k-1}{j} x^j = \sum_{n \geq 0} \binom{n+k}{k} x^n.
\]
Proof

Since \( \binom{k+1}{j+1} = \frac{k+1}{j+1} \binom{k}{j} \) (14) is equivalent with

\[
(1-x)^{2k+1} \sum_{n=0}^{k} \binom{n+k}{k} \left( \binom{n+k+1}{k} x^n \right) = \sum_{j=0}^{k-1} \binom{k-1}{j} \left( \binom{k+1}{j+1} x^j \right).
\] (15)

If \( D = \frac{d}{dx} \) denotes the differentiation operator we get

\[
\sum_{n=0}^{k} \binom{n+k}{k} \left( \binom{n+k+1}{k} x^n \right) = \frac{D^k}{k!} \sum_{n=0}^{k} \binom{n+k}{k} x^{n+k+1} = \frac{D^k}{k! \ (1-x)^{k+1}} \frac{x^{k+1}}{k!} = \frac{D^k \ (1-(1-x))^{k+1}}{k! \ (1-x)^{k+1}}
\]

\[
= \frac{D^k}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k+1}{j} (1-x)^{j-k} = \sum_{j=0}^{k-1} (-1)^j \binom{k+1}{j} \left( \binom{j-k-1}{k} \right) (1-x)^{j-k}.
\]

It remains to show that

\[
\sum_{j=0}^{k-1} (-1)^j \binom{k+1}{j} \left( \binom{2k-j}{k} \right) (1-x)^j = \sum_{j=0}^{k-1} \binom{k-1}{j} \left( \binom{k+1}{j+1} \right) x^j.
\] (16)

Comparing the coefficient of \( z^k \) in

\[
\sum_{j=\ell}^{k-1} \binom{k-1}{j} \binom{k+1}{\ell} x^j z^{\ell+k+1} = (1+z)^{k-\ell} (x+z)^{k+1} = (1+z)^{k-\ell} (x-1+z)^{k+1} = \sum_{j=\ell}^{k+1} \binom{k+1}{j} (x-1)^j (1+z)^{2k-j}
\]

\[
= \sum_{j=\ell}^{k+1} \binom{k+1}{j} (x-1)^j \left( \binom{2k-j}{\ell} \right) z^j
\]
gives (16) and thus (15).

3. A combinatorial interpretation

There is an interesting combinatorial interpretation which I owe to Qiaochu Yuan [13]:

\[
\binom{n}{k} \text{ is the number of semistandard Young tableaux with shape } d^k \ (a \ box \ with \ d \ columns \ and \ k \ rows) \ and \ entries \ in \ \{1, \cdots, n\}. \ This \ is \ equivalent \ with \ all \ k \times d - matrices } (a_{i,j}) \ with \ entries \ in \ \{1, \cdots, n\}, \ such \ that \ a_{i,j} \leq a_{i,j+1} \ and \ a_{i,j} < a_{i+1,j} \ for \ all \ i, j.
\]

For \( d = 1 \) this is equivalent with choosing \( k \) different numbers from \( \{1, \cdots, n\} \).

In general the number of such matrices is given by the semistandard hook length formula (cf. [11]) which gives
$$\prod_{i=1}^{k} \prod_{j=1}^{d} \frac{n-i+j}{(k-i)+(d-j)+1} = \prod_{i=1}^{k} \frac{(n-i+1)(n-i+2)\cdots(n-i+d)}{(k-i+1)(k-i+2)\cdots(k-i+d)}$$

$$= \prod_{i=0}^{k-1} \frac{(n-i)(n-i+1)\cdots(n-i+d-1)}{(k-i)(k-i+1)\cdots(k-i+d-1)} = \prod_{i=0}^{k-1} \frac{d}{k-i+d-1} = \prod_{i=0}^{k-1} \frac{k-i}{d} = \frac{n}{k}. $$

The Jacobi-Trudi identities (cf. [11]) give

**Theorem 6**

$$\langle n \rangle = \det\left(\begin{array}{c} n+d+j-1 \\ \vdots \\ n \end{array}\right)_{i,j=0}^{k-1}. \quad (17)$$

We now give an elementary

**Proof**

We consider more generally $\det\left(\begin{array}{c} x_i+j \\ \vdots \\ n-1 \end{array}\right)_{i,j=0}^{k-1}$. This is a polynomial in the indeterminates $x_0, \ldots, x_{k-1}$ of degree $\leq n-1$ in each variable. It vanishes for $x_i = x_j$ which gives the factor $\prod_{0 \leq i < j \leq k-1} (x_j-x_i)$. Since all entries of row $i$ have the factor $x_i(x_i-1)\cdots(x_i-n+k+1)$

the determinant has the factor $\prod_{0 \leq i < j \leq k-1} (x_j-x_i)\prod_{i=0}^{k-1} x_i(x_i-1)\cdots(x_i-n+k+1)$.

This also is a polynomial of degree $n$ in each variable. Therefore, there exists a constant $c$ such that

$$\det\left(\begin{array}{c} x_i+j \\ \vdots \\ n-1 \end{array}\right)_{i,j=0}^{k-1} = c \prod_{0 \leq i < j \leq k-1} (x_j-x_i)\prod_{i=0}^{k-1} x_i(n-k). \quad (18)$$

To compute $c$ we choose $x_i = n-1-i$. Then $\left(\begin{array}{c} x_i+j \\ \vdots \\ n-1 \end{array}\right)_{i,j=0}^{k-1}$ is a right triangle matrix with

$$\left(\begin{array}{c} x_i+i \\ \vdots \\ n-1 \end{array}\right)_{i,j=0}^{k-1} = 1 \text{ und therefore } \det\left(\begin{array}{c} x_i+j \\ \vdots \\ n-1 \end{array}\right)_{i,j=0}^{k-1} = 1.$$

On the right-hand side of (18) we get

$$c \prod_{0 \leq i < j \leq k-1} (x_j-x_i)\prod_{i=0}^{k-1} x_i(n-k) = c \prod_{0 \leq i < j \leq k-1} (i-j)\prod_{i=0}^{k-1} x_i(n-k)$$

$$= c(-1)^{\binom{k}{2}} \prod_{i=0}^{k-1} \frac{n-1-i}{k-1-i}$$
Setting $f(k) = \frac{\prod_{i=0}^{k-1} i!(n-1-i)(k-1-i)}{k!}$ we get

$$
\frac{f(k)}{f(k-1)} = \frac{\prod_{i=0}^{k-1} i!(n-1-i)(k-1-i)}{\prod_{i=0}^{k-2} i!(n-1-i)(k-2-i)} = (k-1)! \prod_{i=0}^{k-2} \frac{(k-2-i)!}{(k-1-i)!} = (n-k+1)^k
$$

and therefore $f(k) = \prod_{j=0}^{k-1} (n-j)^i$.

Thus

$$
\det\left(\begin{pmatrix} x_i + j \\ n-1 \end{pmatrix}\right)_{i,j=0}^{k-1} = (-1)^\binom{k}{2} \frac{1}{\prod_{j=0}^{k-1} (n-j)^j} \prod_{0 \leq i \leq j \leq k-1} (x_j - x_i) \prod_{i=0}^{k-1} \left(x_i \right)_{n-k} \right). \quad (19)
$$

To compute $\det\left(\begin{pmatrix} d - i + n - 1 + j \\ n-1 \end{pmatrix}\right)_{i,j=1}$ we choose $x_i = d - i + n - 1$ and get

$$
(-1)^\binom{k}{2} \frac{1}{\prod_{j=0}^{k-1} (n-j)^j} \prod_{0 \leq i \leq j \leq k-1} \left(\begin{pmatrix} d - i + n - 1 \\ n-1 \end{pmatrix}\right)^j = \prod_{i=0}^{k-1} \left(\begin{pmatrix} d - i + n - 1 \\ n-1 \end{pmatrix}\right)^j (n-j)(n-k)! = \frac{(d - i + n - 1)!}{(n-i-1)!} (n-k)! (n-k+1)!...
$$

because

$$
\prod_{i=0}^{k-1} \frac{(n-i-1)!}{(n-i)!(n-k)!} = \prod_{i=0}^{k-1} \frac{(n-i-1)!}{(n-i)!(n-k)!} = (n-1)!(n-2)!\cdots(n-k)! (n-k+1)!^{k-1} (n-k)!^{k-1} = 1.
$$

4. q-analogs

The above constructions have straightforward $q$-analogs. For a real number $q$ with $|q| < 1$

$$
[n]_q = 1 + q + \cdots + q^{n-1}, \quad [n]_q! = \prod_{j=1}^{n} [j]_q \quad \text{and} \quad \left[\begin{pmatrix} n \\ k \end{pmatrix}\right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \prod_{j=0}^{k-1} \frac{1 - q^{n-j}}{1 - q^{k-j}}.
$$
As is well known the $q$–binomial coefficients $\binom{n}{k}_q$ satisfy
\begin{align}
\binom{n}{k}_q &= q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q + q^{n-k} \binom{n-1}{k-1}_q,
\end{align}
and are therefore polynomials in $q$ with nonnegative integer coefficients.

For later use let us mention the following $q$–analog of the binomial theorem (cf. e.g. [1])
\begin{align}
\prod_{j=0}^{n-1} (1 - q^j x) &= \sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \binom{n}{j}_q x^j.
\end{align}

We define $\binom{n}{d}_q = \binom{n+d-1}{d}_q$, $\binom{n}{d}_q = \binom{n}{d}_q$ and get
\begin{align}
\binom{n}{k}_d &= \frac{\binom{n}{d}_q}{\binom{k}{d}_q \binom{n-k}{d}_q} = \prod_{j=0}^{k-1} \binom{n-j}{d}_q = \prod_{j=0}^{d-1} \binom{n+j}{k}_q = \prod_{j=0}^{d-1} \binom{n+d-1}{k+j}_q.
\end{align}

From Theorem 7 we see that these are also polynomials in $q$ with integer coefficients.

For $d = 2$ we get
\begin{align}
\binom{n}{k}_{2,q} &= \frac{1}{[k+1]_q} \binom{n}{k}_q \binom{n+1}{k}_q.
\end{align}

This gives the triangle
\begin{align}
\begin{pmatrix}
1 \\
1 \\
1 + q + q^2 \\
(1+q^2)(1+q+q^2) \\
(1+q^2)(1+q+q^2)(1+q^2+q^4)
\end{pmatrix}
\end{align}

As analog of (8) we get
\begin{align}
\binom{n}{k}_{d,q} &= \det\left(\binom{n+i+j}{k+j}_q\right)_{i,j=0}^{d-1} = \det\left(q^{\binom{i+j}{2}} \binom{n}_{k-i+j}_q\right)_{i,j=0}^{d-1}.
\end{align}
**Proof**

From
\[ \begin{bmatrix} n+i+j \\ k+j \end{bmatrix}_q = \frac{[n+i+j]_q!}{[k+j]_q ![n-k+i]_q!} = \frac{[j]_q ![n+i]_q!}{[k+j]_q ![n-k+i]_q!} \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q \] we get

\[ \det \left( \begin{bmatrix} n+i+j \\ k+j \end{bmatrix}_q \right)_{i,j=0}^{d-1} = \prod_{i=0}^{d-1} \frac{[j]_q ![n+i]_q!}{[k+j]_q ![n-k+i]_q!} \det \left( \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q \right)_{i,j=0}^{d-1} \]

with
\[ \prod_{j=0}^{d-1} \frac{[j]_q ![n+j]_q!}{[k+j]_q ![n-k+j]_q!} = \prod_{j=0}^{d-1} \begin{bmatrix} n+j \\ k+j \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_{d,q}. \]

It remains to compute \( \det \left( \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q \right)_{i,j=0}^{d-1}. \)

Using the identity
\[ \sum_{i=0}^{j} (-1)^{i-j} \begin{bmatrix} i \\ \ell \end{bmatrix}_q q^{(i-j)/2} \begin{bmatrix} n+\ell+j \\ n-j \end{bmatrix}_q = q^{j(n+i)} \begin{bmatrix} n+j \\ j-i \end{bmatrix}_q \] (24)

we see that \( \left( (-1)^{i-j} q^{-j/2} \begin{bmatrix} i \\ \ell \end{bmatrix}_q \right)_{i,j=0}^{d-1} \left( \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q \right)_{i,j=0}^{d-1} \) is an upper triangular matrix with entries \( q^{i+j^2} \) in the main diagonal. This implies

\[ \det \left( \begin{bmatrix} n+i+j \\ j \end{bmatrix}_q \right)_{i,j=0}^{d-1} = q^{d^2/2} \sum_{j=0}^{d} j^2. \] (25)

To prove (24) we consider
\[ \begin{bmatrix} x \\ k \end{bmatrix}_q = \prod_{j=0}^{k-1} \frac{[x-j]_q}{[k-j]_q} = \prod_{j=0}^{k-1} \frac{q^x-q^j}{q^k-q^j} \] as a polynomial in \( q^x \) with coefficients in \( Q(q) \). If we define the operator \( E \) on these polynomials by

\[ Ef(q^x) = f \left( q^{x+1} \right) \]

then we get \( E \begin{bmatrix} x \\ n \end{bmatrix}_q = \begin{bmatrix} x+1 \\ n \end{bmatrix}_q \). Let now \( \Delta = E - 1 \) be the difference operator.

We have
\[ \Delta \begin{bmatrix} x \\ n \end{bmatrix}_q = \begin{bmatrix} x+1 \\ n \end{bmatrix}_q - \begin{bmatrix} x \\ n \end{bmatrix}_q = q^{x+1} \begin{bmatrix} x \\ n-1 \end{bmatrix}_q. \]

More generally we get by induction
\[ (E-1)(E-q) \cdots (E-q^{k-1}) \begin{bmatrix} x \\ n \end{bmatrix}_q = q^{k(n+k-\ell)} \begin{bmatrix} x \\ n-k \end{bmatrix}_q, \]
because

\[
\begin{align*}
\left( E - q^{k-1} \right) q^{(k-1)(x+k-1-n)} & \left[ \begin{array}{c} x \\ n - k + 1 \end{array} \right]_q = q^{(k-1)(x+k-n)} \left[ \begin{array}{c} x + 1 \\ n - k + 1 \end{array} \right]_q - q^{(k-1)(x+k-n)} \left[ \begin{array}{c} x \\ n - k + 1 \end{array} \right]_q \\
& = q^{(k-1)(x+k-n)} q^{x+k-n} \left[ \begin{array}{c} x \\ n - k \end{array} \right]_q = q^{x+k-n} \left[ \begin{array}{c} x \\ n - k \end{array} \right]_q.
\end{align*}
\]

Finally by (21) we have \( (E - 1)(E - q)\cdots (E - q^{k-1}) = \sum_{j=0}^{k} (-1)^j q^{(j+1)\left[ k \right]_q} E^{k-j}. \)

For the computation of \( \det \left( q^{j\left( \begin{array}{c} 0 \\ k-j \end{array} \right)_q} \left[ \begin{array}{c} n \\ k-j+i \end{array} \right]_q \right)_{i,j=0}^{d-1} = \det \left( q^{j\left( \begin{array}{c} 0 \\ k-j \end{array} \right)_q} \left[ \begin{array}{c} n \\ k-j+i \end{array} \right]_q \right)_{i,j=1}^{d} \) we use [6], formula (3.12):

\[
\det \left( q^{j\left[ n \right]_q} \left[ \begin{array}{c} A \\ L_i + j \end{array} \right]_q \right)_{i,j=1}^{n} = q^{j\left[ n \right]_q} \prod_{1 \leq i < j \leq n} \left[ L_i - L_j \right]_q \prod_{i=1}^{n} \left[ A + i - 1 \right]_q \prod_{i=1}^{n} \left[ A - L_i - 1 \right]_q.
\]

First we write

\[
\det \left( q^{j\left( \begin{array}{c} 0 \\ k-j \end{array} \right)_q} \left[ \begin{array}{c} n \\ k-j+i \end{array} \right]_q \right)_{i,j=1}^{d-1} = \det \left( q^{j\left( \begin{array}{c} 0 \\ k-j \end{array} \right)_q} \left[ \begin{array}{c} n \\ k-j+i \end{array} \right]_q \right)_{i,j=1}^{d} = \sum_{j=1}^{d} \det \left( q^{j\left[ n \right]_q} \left[ \begin{array}{c} n \\ k-j \end{array} \right]_q \right)_{i,j=1}^{d}.
\]

Then we choose \( A = n, \ L_i = k-i, \ n = d \) and get

\[
\sum_{j=1}^{d} -k\left( \begin{array}{c} d+1 \\ 2 \end{array} \right)_q \det \left( q^{(k-1)\left[ \begin{array}{c} n \\ k-i \end{array} \right]_q} \right)_{i,j=1}^{d} = \sum_{j=1}^{d} -k\left( \begin{array}{c} d+1 \\ 2 \end{array} \right)_q \sum_{i=0}^{d-k} \left[ \prod_{j=0}^{d-1} \left[ j \right]_q \prod_{j=0}^{d-1} \left[ n+j \right]_q \right] \prod_{j=0}^{d-1} \left[ j \right]_q \prod_{j=0}^{d-1} \left[ n+k+j \right]_q \\
= \prod_{j=0}^{d-1} \left[ k+j \right]_q \prod_{j=0}^{d-1} \left[ n+k+j \right]_q = \prod_{j=0}^{d-1} \left[ k+j \right]_q \prod_{j=0}^{d-1} \left[ n+k+j \right]_q = \binom{n}{k}_d.
Theorem 8
\[
\langle n \rangle_{d,q} = q^{-\binom{d}{2}} \left[ \begin{array}{c} n+i \\ k+j \end{array} \right]_{i,j=0}^{d-1} \det \left( \begin{array}{c} n+i \\ k+j \end{array} \right)_{i,j=0}^{d-1}.
\] (26)

Proof
We use Dodgson’s condensation method (cf. [6], Proposition 10, and [14]). Let
\[
X(d,n,k) = \det \left( \begin{array}{c} n+i \\ k+j \end{array} \right)_{i,j=0}^{d-1}.
\]
By condensation we get
\[
X(d,n,k) = X(d-1,n,k)X(d-1,n+1,k+1) - X(d-1,n+1,k)X(d-1,n,k+1) \quad \frac{X(d-2,n+1,k+1)}{X(d-2,n+1,k+1)}.
\]
The same identity holds for \( X(d,n,k) = q^{-\binom{d}{2}} \left[ \begin{array}{c} n \langle n \rangle_{d,q} \\ k \langle k \rangle_{d,q} \end{array} \right] \). For we have
\[
\frac{q^{-\binom{d-1}{2}} \langle n \rangle_{d-1,q}^{n} \langle n+1 \rangle_{d-1,q}^{n+1}}{q^{-\binom{d}{2}} \langle n \rangle_{d,q}^{n} \langle n+1 \rangle_{d,q}^{n+1}} = 1 \quad \frac{[n-k+d-1]_q}{[d-1]_q}
\]
and
\[
\frac{q^{-\binom{d-2}{2}} \langle n+1 \rangle_{d-2,q}^{n+1} \langle n \rangle_{d-1,q}^{n}}{q^{-\binom{d}{2}} \langle n \rangle_{d,q}^{n} \langle n+1 \rangle_{d-2,q}^{n+1}} = 1 \quad \frac{[n-k]_q}{q^{-\binom{d}{2}} [d-1]_q},
\]
which implies
\[
\frac{X(d-1,n,k)X(d-1,n+1,k+1) - X(d-1,n+1,k)X(d-1,n,k+1)}{X(d,n,k)X(d-2,n+1,k+1)} = \frac{[n-k+d-1]_q - [n-k]_q}{q^{-\binom{d}{2}} [d-1]_q} = 1.
\]
Identity (26) holds for \( d = 0 \) and \( d = 1 \) and therefore (26) holds for all \( d \) by induction.

A \( q \)-analog of (17) is

Theorem 9
\[
\det \left( \begin{array}{c} n+d+j-i-1 \\ n-1 \end{array} \right)_{i,j=0}^{k-1} = q^{-\binom{k}{2}} \langle n \rangle_{d,q}. \] (27)
Proof

Here we use [6], formula (3.11):

\[
\det\left( \begin{array}{c} L_i + A + j \\ L_i + j \end{array} \right)_{i,j=1}^n = \sum_{i=1}^n \prod_{i<j<\infty} \frac{\prod_{i}^{n} [L_i - L_j]_q}{\prod_{i}^{n} [L_i + n]_q} \prod_{i}^{n} [L_i + A + 1]_q \prod_{i}^{n} [A + 1 - i]_q.
\]

We choose \( L_i = d - i \), \( A = n - 1 \) and \( n = k \) and get

\[
\det\left( \begin{array}{c} n + d + j - i - 1 \\ n - 1 \end{array} \right)_{j,i=0}^{k-1} = q^{\binom{k}{2}} \prod_{j=0}^{k-1} \frac{n - j + d - 1}{d - k + j - 1} = q^{\binom{k}{2}} \prod_{j=0}^{k-1} \frac{n + j}{k + j} = q^{\binom{k}{2}} \binom{n}{k}.
\]

Remark

Since the determinant (27) is closely related to semistandard Young Tableaux it would make sense from this point of view to define \( q \)–Hoggatt matrices with entries \( q^{\binom{k}{2}} \binom{n}{k} \) instead of \( \binom{n}{k} \). For \( d = 1 \) this means to replace the \( q \)–binomial coefficients \( \binom{n}{k} \) by their companion form \( q^{\binom{k}{2}} \binom{n}{k} \). This would give the nice generating function

\[
\sum_{k=0}^n q^{\binom{k}{2}} \binom{n}{k} x^k = (1 + x)(1 + qx)\cdots(1 + q^{n-1}x) \text{ for row } n \text{ of the matrix.}
\]

For \( d = 2 \) we would get

\[
\sum_{k=0}^n q^{k(n+1)} \binom{n}{k} = C_{n+1}(q) = \frac{1}{[n+2]} \binom{2n+2}{n+1},
\]

which is a nice \( q \)–analogue of the fact that the sum of the Narayana numbers \( \binom{n}{k} \) are the Catalan numbers \( C_{n+1} \).

As \( q \)–analogue of (13) we state
Conjecture 10

For positive integers $d, k$ we have

\[
(1 - x)(1 - qx) \cdots (1 - q^{dk} x) \sum_{n \geq 0} \binom{n+k}{k} x^n = \sum_{j=0}^{(d-1)(k-1)} N(d, k, j, q)x^j. \tag{28}
\]

where the coefficients $N(d, k, j, q)$ are palindromic polynomials in $q$ with nonnegative coefficients.

For $d = 2$ this reduces to

\[
(1 - x)(1 - qx) \cdots (1 - q^{2k} x) \sum_{n \geq 0} \binom{n+k}{k} x^n = \sum_{j=0}^{k-1} q^{(j+1)} \binom{k-1}{j} x^j. \tag{29}
\]

The sums $\sum_{j=0}^{(d-1)(k-1)} N(d, k, j, q) = C^{(d)}_n (q) = \left[ dn \right]_q ! \prod_{j=0}^{d-1} \left[ n+j \right]_q !$ are the $d$–dimensional $q$–Catalan numbers.

5. Fibonacci-Hoggatt triangles

Let $F_n = \sum_{j=0}^{\lfloor n/2 \rfloor} \left( \binom{n-1-j}{j} \right)$ denote the Fibonacci numbers which satisfy $F_n = F_{n-1} + F_{n-2}$ with initial values $F_0 = 0$ and $F_1 = 1$ and $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Let us write $(n)_F = F_n, (n)_F! = F_n F_{n-1} \cdots F_1$ and define the Fibonomial coefficients by

\[
\binom{n}{k}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_k F_{k-1} \cdots F_1} \frac{(n)_F !}{(k)_F ! (n-k)_F !}.
\]

The first terms are (cf. OEIS [8], A010048)

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{pmatrix},
\]

The Fibonacci numbers satisfy

\[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}^n = \begin{pmatrix}
F_{n+1} & F_n \\
F_n & F_{n+1}
\end{pmatrix}.
From \[ \left( \begin{array}{cc} F_{n-1} & F_n \\ F_n & F_{n+1} \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right)^{n-k} \left( \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right)^k = \left( \begin{array}{cc} F_{n-k-1} & F_{n-k} \\ F_{n-k} & F_{n-k+1} \end{array} \right) \left( \begin{array}{cc} F_{k-1} & F_k \\ F_k & F_{k+1} \end{array} \right) \]
we get by comparing the top right elements
\[ F_n = F_{k+1}F_{n-k} + F_{n-k-1}F_k. \]
This is equivalent with
\[
\binom{n}{k} = F_{k+1}\binom{n-1}{k} + F_{n-k-1}\binom{n-1}{k-1},
\]
which shows that the Fibonomials \( \binom{n}{k} \) are nonnegative integers.

L. Carlitz [3] found the analog of (21)
\[
h_n(x) = \sum_{j=0}^{n} (-1)^{j+1/2} \binom{n}{j} x^j = \prod_{j=0}^{n} \left(1 - \alpha^{n-j-1} \beta^j x\right).
\]
Let us reproduce his proof. In formula (21) we set \( q = \beta/\alpha \) and get
\[
\prod_{j=0}^{n-1} \left(1 - \frac{\beta^j}{\alpha^j} x\right) = \sum_{j=0}^{n} (-1)^j \left(\frac{\beta}{\alpha}\right)^{j/2} \binom{n}{j} \alpha^{j-n} x^j
\]
with
\[
\binom{n}{j} = \frac{(1-q^n)\cdots(1-q^{n-j+1})}{(1-q^j)\cdots(1-q)} = \frac{1}{\alpha^{(n-j)j}} \left(\alpha^n - \beta^n\right)\cdots\left(\alpha^{n-j+1} - \beta^{n-j+1}\right) = \alpha^{j-n} \binom{n}{j}.
\]
This gives
\[
\prod_{j=0}^{n-1} \left(1 - \alpha^{-j} \beta^j x\right) = \sum_{j=0}^{n} (-1)^j \alpha^{j/2} \beta^{j-n} \binom{n}{j} \alpha^{j-n} x^j.
\]
Replacing \( x \to \alpha^{-1} x \) we get
\[
\prod_{j=0}^{n-1} \left(1 - \alpha^{-n-j} \beta^j x\right) = \prod_{j=0}^{n} (-1)^j \alpha^{j/2} \beta^{j-n} \binom{n}{j} x^j = \prod_{j=0}^{n} (-1)^j \binom{n}{j} x^j.
\]
Since \( \alpha^n + \beta^n = L_n \) are the Lucas numbers \( 2, 1, 3, 4, 7, 11, 18, \cdots \), we see that
\[
h_n(x) = \prod_{j=0}^{n-1} \left(1 - \alpha^{-n-j} \beta^j x\right) = (1-x^n) \left(1 - \beta^{-1} x\right) \prod_{j=0}^{n-2} \left(1 + \alpha^{n-2-j} \beta^{-1} x\right)
\]
\[
= (1-x^n) \left(1 - \beta^{-1} x\right) \prod_{j=0}^{n-3} \left(1 + \alpha^{n-3-j} \beta^j x\right) = (1-L_{n-1} x + (-1)^{n-1} x^2) p_{n-2}(-x).
\]
This gives
\[ h_k(x) = \sum_{j=0}^{k} (-1)^j \binom{j+1}{2} \binom{k}{j} x^j = \prod_{j=0}^{k} u_{k-j} \binom{(-1)^j x}{j} \] (32)

with
\[ u_k(x) = 1 - L_{k-1} x + (-1)^{k-1} x^2, \]
\[ u_0(x) = 1 - x, \quad u_1(x) = 1. \] (33)

As analog of (12) we get
\[ \frac{1}{h_{k+1}(x)} = \frac{1}{\sum_{j=0}^{k+1} (-1)^j \binom{j+1}{2} \binom{k+1}{j} x^j} = \sum_{n \geq 0} \binom{n+k}{k} x^n. \] (34)

**Proof**

Since \( h_1(x) = 1 - x \) and \( h_2(x) = 1 - x - x^2 \) identity (34) is true for \( k = 0 \) and \( k = 1 \).

By (32) identity (34) is equivalent with \( u_{k+1}(x) \sum_{n \geq 0} \binom{n+k}{k} x^n = \sum_{n \geq 0} \binom{n+k-2}{k-2} (-x)^n \), i.e.
\[ (1 - L_k x + (-1)^k x^2) \sum_{n \geq 0} \frac{F_{n+1} \cdots F_{n+k}}{F_1 \cdots F_k} x^n = \sum_{n \geq 0} \frac{F_{n+1} \cdots F_{n+k-2}}{F_1 \cdots F_{k-2}} (-x)^n. \]

This is equivalent with
\[ F_{n+k-1} F_{n+k} - L_k F_n F_{n+k-1} + (-1)^k F_{n+1} F_n = (-1)^n F_{k-1} F_k, \]

which is easily verified.

In [5] the authors studied Fibo-Narayana numbers defined by
\[ \left\langle \frac{n}{k} \right\rangle = \frac{1}{F_{k+1}} \binom{n}{k} \binom{n+1}{k} = \frac{1}{F_{n+1}} \binom{n+1}{k} \binom{n+1}{k+1}. \]

Let us more generally define Fibo-Hoggatt numbers
\[ \left\langle \frac{n}{k} \right\rangle = \prod_{j=0}^{k-1} \frac{n-j}{k-j}. \] (35)
with \( \langle n \rangle_{d,F} = \binom{n+d-1}{d} \) and consider the corresponding Fibonacci-Hoggatt matrices 

\[
H_{d,F} = \begin{pmatrix}
\langle n \rangle \\
\langle k \rangle_{d,F}
\end{pmatrix}_{n,k \geq 0}.
\]

As in (7) we get

\[
\langle n \rangle_{d,F} = \prod_{j=0}^{d-1} \binom{n+j}{k+j}_F = \prod_{j=0}^{d-1} \binom{n+d-1}{k+j}_F.
\] (36)

For example for \( d = 3 \) we get

\[
\begin{pmatrix}
\langle n \rangle \\
\langle k \rangle_{3,F}
\end{pmatrix}_{i,j=0} = \begin{pmatrix}
1 & 1 & 1 \\
1 & 3 & 1 \\
1 & 15 & 1 \\
1 & 60 & 60 & 1 \\
1 & 260 & 5200 & 260 & 1
\end{pmatrix}.
\]

As an analog of the first identity (8) we get

**Theorem 11**

\[
\langle n \rangle_{d,F} = \frac{\det\left(\binom{n+i+j}{k+j}_F\right)_{i,j=0}^{d-1}}{\det\left(\binom{n+i+j}{j}_F\right)_{i,j=0}^{d-1}}.
\] (37)

**Proof**

This follows from

\[
\det\left(\binom{n+i+j}{k+j}_F\right)_{i,j=0}^{d-1} = \det\left(\binom{n+i+j}{j}_F\right)_{i,j=0}^{d-1}
\]

\[
= \prod_{j=0}^{d-1} \frac{(j)_F!(n+j)_F!}{(k+j)_F!(n-k+j)_F!}
\]

\[
= \prod_{j=0}^{d-1} \frac{(j)_F!(n+j)_F!}{(k+j)_F!(n-k+j)_F!} \det\left(\binom{n+i+j}{j}_F\right)_{i,j=0}^{d-1}
\]

if we observe that
\[ \prod_{j=0}^{d-1} \frac{(j)_F! (n+j)_F!}{(k+j)_F! (n-k+j)_F!} = \prod_{j=0}^{d-1} \frac{n+j}{k+j} \left( \begin{array}{c} n \\ k \end{array} \right)_F = \left( \begin{array}{c} n \\ k \end{array} \right)_{d,F}. \]

If we set \( a(d,n) = \det \left( \begin{array}{c} n+i+j \\ j \end{array} \right)_F \right)_{i,j=0}^{d-1} \) then we get \( a(2,n) = F_n \), but for \( n > 3 \) no other interpretation seems to be known. For example for \( d = 3 \) we get \( 1,5,7,53,187,853, \ldots \).

As analog of (10) we get as special case of Theorem 15

**Theorem 12**

\[ \det \left( -1 \right)^{\frac{d-1}{2}} \left( \begin{array}{c} n \\ k-i+j \end{array} \right)_F \right)_{i,j=0}^{d-1} = \left( \begin{array}{c} n \\ k \end{array} \right)_{d,F}. \]  

(38)

This shows that all \( \left( \begin{array}{c} n \\ k \end{array} \right)_{d,F} \) are positive integers.

There is also a nice analog of (17):

**Theorem 13**

\[ \left( \begin{array}{c} n \\ k \end{array} \right)_{d,F} = \left( -1 \right)^{d(\frac{k}{2})} \det \left( \begin{array}{c} n+d+j-i-1 \\ n-1 \end{array} \right)_F \right)_{i,j=0}^{k-1}. \]  

(39)

**Proof**

By Binet’s formula we have \( F_n = \alpha^{n-1} \) with \( \alpha = \frac{3+\sqrt{5}}{2} = \frac{1}{\alpha^2} \). If we set \( \alpha = q \), then we get

\[ F_n = (-q)^{\frac{1}{2} \frac{n}{2}} \frac{1-q^n}{1-q}. \]  

(40)

This implies

\[ \left( \begin{array}{c} n \\ k \end{array} \right)_F = (-q)^{\frac{k^2-nk}{2}} \left[ \begin{array}{c} n \\ k \end{array} \right]_q. \]  

(41)
We can now formulate $\det\left(\begin{array}{ccc} n+d+j-i-1 \\ n-1 \end{array}\right)_{F, j=0}^{k-1}$ in terms of $q -$ binomial coefficients as

$$\det\left(\begin{array}{ccc} n+d+j-i-1 \\ n-1 \end{array}\right)_{F, j=0}^{k-1} = \det\left((-q)\frac{(1-q)(d+j-i)}{2}\right)^{k-1} \det\left(\begin{array}{ccc} n+d+j-i-1 \\ d+j-i \end{array}\right)_{i,j=0}^{k-1}.$$

Using the above result we get by (41)

$$\det\left(\begin{array}{ccc} n+d+j-i-1 \\ n-1 \end{array}\right)_{F, j=0}^{k-1} = (-q)\frac{(1-q)d}{2} \prod_{j=0}^{k-1} \left(\begin{array}{ccc} n+j+d-1 \\ d \end{array}\right)_{F}^{k-j} \prod_{j=0}^{k-1} \left(\begin{array}{ccc} n+j+d-1 \\ d \end{array}\right)_{F}^{k-j} = (-1)^{\binom{k}{2}} \prod_{j=0}^{k-1} \left(\begin{array}{ccc} n+j+d-1 \\ d \end{array}\right)_{F}^{k-j} \prod_{j=0}^{k-1} \left(\begin{array}{ccc} n+j+d-1 \\ d \end{array}\right)_{F}^{k-j} = (-1)^{\binom{k}{2}} \binom{n}{k}_{d,F}^{k,j}.$$
6. A common generalization

A generalization which contains all above cases is given by the Fibonacci polynomials

\[ F_n(s, t) = \sum_{j=0}^{n-1} \binom{n-1-j}{j} s^{n-2j}. \tag{42} \]

They satisfy \( F_n(s, t) = s F_{n-1}(s, t) + t F_{n-2}(s, t) \) with initial values \( F_0(s, t) = 0 \) and \( F_1(s, t) = 1 \).

For \( s = 2 \) and \( t = -1 \) we have \( F_n(2,-1) = n \) which gives the original Hoggatt matrices, for \( s = 1 + q \) and \( t = -q \) we get \( F_n(1+q,-q) = [n]_q \) which gives the \( q \)– analogs and for \( s = t = 1 \) we get the Fibonacci analogs.

Binet’s formulae give

\[ F_n(s, t) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \]

with \( \alpha = \frac{s + \sqrt{s^2 + 4t}}{2} \) and \( \beta = \frac{s - \sqrt{s^2 + 4t}}{2} \).

We also have

\[
\begin{pmatrix}
0 & 1 \\
t & s
\end{pmatrix}^n = \begin{pmatrix}
t F_{n-1}(s, t) & F_n(s, t) \\
t F_n(s, t) & F_{n+1}(s, t)
\end{pmatrix}. 
\]

From

\[
\begin{pmatrix}
t F_{n-1}(s, t) & F_n(s, t) \\
t F_n(s, t) & F_{n+1}(s, t)
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
t & s
\end{pmatrix}^{-k} \begin{pmatrix}
0 & 1 \\
t & s
\end{pmatrix}^k = \begin{pmatrix}
t F_{n-k-1}(s, t) & F_{n-k}(s, t) \\
t F_{n-k}(s, t) & F_{n-k+1}(s, t)
\end{pmatrix} \begin{pmatrix}
t F_{k-1}(s, t) & F_k(s, t) \\
t F_k(s, t) & F_{k+1}(s, t)
\end{pmatrix}
\]

we get by comparing the top right elements

\[ F_n(s, t) = F_{k+1}(s, t) F_{n-k}(s, t) + t F_{n-k-1}(s, t) F_k(s, t). \]

This is equivalent with

\[
\binom{n}{k}_{F(s,t)} = F_{k+1}(s, t) \binom{n-1}{k}_{F(s,t)} + t F_{n-k-1}(s, t) \binom{n-1}{k-1}_{F(s,t)}, \tag{43}
\]

which shows that the Fibonomials

\[
\binom{n}{k}_{F(s,t)} = \prod_{j=0}^{k-1} \frac{F_{n-j}(s, t)}{F_{k-j}(s, t)} \tag{44}
\]

are polynomials in \( s,t \) with nonnegative integer coefficients.

A combinatorial proof of this fact has been given in [2] and an arithmetic one in [10].
The first terms of the Fibonomial triangle are

\[
\begin{pmatrix}
1 \\
1 & 1 \\
1 & s & 1 \\
1 & s^2 + t & s^2 + t & 1 \\
1 & s(s^2 + 2t) & (s^2 + t)(s^2 + 2t) & s(s^2 + 2t) & 1 \\
1 & s^2 + 3s^2t + t^2 & (s^2 + 2t)(s^4 + 3s^2t + t^2) & (s^2 + 2t)(s^4 + 3s^2t + t^2) & s^4 + 3s^2t + t^2 & 1
\end{pmatrix}
\]

As above we get

\[h_k(x,s,t) = \sum_{j=0}^{k} (-1)^{j+1} \binom{j+1}{2} \binom{j}{k} x^j F(x,t) = \prod_{j=0}^{k} u_{k-j}(x,s,t)\]

with \( u_0(x,s,t) = 1 - L_{n-1}(s,t) x + (-t)^{n-1} x^2 \) for \( n > 1 \) and \( u_0(x,s,t) = 1 \) and \( u_1(x,s,t) = 1 - x \), where the Lucas polynomials \( L_n(s,t) \) satisfy \( L_n(s,t) = sL_{n-1}(s,t) + tL_{n-2}(s,t) \) with initial values \( L_0(s,t) = 2 \) and \( L_1(s,t) = s \).

This implies as before that

\[
\frac{1}{h_{k+1}(x,s,t)} = \frac{1}{\sum_{j=0}^{k+1} (-1)^{j+1} \binom{j+1}{2} \binom{j}{k+1} x^j F(x,t)} = \sum_{n=0}^{k} \binom{n+k}{k} x^n. \tag{46}
\]

The Hoggatt coefficients can be defined by

\[
\binom{n}{k}_{d,F(x,t)} = \prod_{j=0}^{k-1} \binom{n-j+d-1}{d}_{F(x,t)} \binom{k-j+d-1}{d}_{F(x,t)}. \tag{47}
\]

An extension of a result which in [5] has been obtained for \( d = 2 \) is

**Theorem 15**

\[
\det \left\{ (-t)^{i+j} \binom{n}{k-j+i}_{F(x,t)} \right\}_{i,j=0}^{d-1} = \binom{n}{k}_{d,F(x,t)}. \tag{48}
\]

This implies that all Hoggatt coefficients are polynomials in \( s,t \) with integer coefficients.
Proof

By Binet’s formula we have \( F_n(s,t) = \alpha^{n-1} \left( \frac{\beta}{\alpha} \right)^n \) with \( \frac{\beta}{\alpha} = -\frac{t}{\alpha^2} \). If we set \( \frac{\beta}{\alpha} = q \), then we get

\[
F_n = \left( -\frac{q}{t} \right)^{1-n} \frac{1-q^n}{1-q}.
\] (49)

This implies

\[
\binom{n}{k}_{F(s,t)} = \left( -\frac{q}{t} \right)^{\frac{k^2-nk}{2}} \binom{n}{k}.
\] (50)

We can now formulate \( \det \left( (-t)^{\binom{i-j}{2}} \binom{n}{k-j+i}_{F(s,t)} \right)_{i,j=0}^{d-1} \) in terms of \( q \)-binomial coefficients as

\[
\det \left( (-t)^{\binom{i-j}{2}} \binom{n}{k-j+i}_{F(s,t)} \right)_{i,j=0}^{d-1} = \det \left( (-t)^{\binom{i-j}{2}} \left( -\frac{q}{t} \right)^{\frac{(k-j+i)(k-j+i-n)}{2}} \binom{n}{k-j+i} \right)_{i,j=0}^{d-1} = (-1)^{\frac{k(k-n)d}{2}} t^2 q^{\frac{k(k-n)d}{4}} \det \left( q^{\binom{i-j}{2}} \binom{n}{k-j+i} \right)_{i,j=0}^{d-1}.
\]

The last determinant has been computed above as

\[
\left\langle \frac{n}{k} \right\rangle_{d,q} = \prod_{j=0}^{d-1} \frac{\binom{n+j}{k}}{\binom{k+j}{k}\binom{k+n+j}{k}} = \prod_{j=0}^{d-1} \frac{-\frac{t}{q}}{\frac{k^2-(n+j)k}{2}} \prod_{j=0}^{d-1} \frac{\binom{n+j}{k}_{F(s,t)}}{\binom{k+n+j}{k}_{F(s,t)}} = \left( -\frac{t}{q} \right)^{\frac{k(n-k)d}{2}} \left\langle \frac{n}{k} \right\rangle_{d,F(s,t)}.
\]

This gives

\[
\det \left( (-t)^{\binom{i-j}{2}} \binom{n}{k-j+i}_{F(s,t)} \right)_{i,j=0}^{d-1} = (-1)^{\frac{k(k-n)d}{2}} t^2 q^{\frac{k(k-n)d}{4}} \left( -\frac{t}{q} \right)^{\frac{k(n-k)d}{2}} = \left\langle \frac{n}{k} \right\rangle_{d,F(s,t)}.
\]

The same proof as above gives
Theorem 16

\[
\det \left( \begin{array}{c}
 n + d + j - i - 1 \\
 n - 1 \\
\end{array} \right)_{F(s,t)}^{k-1}
= (-1)^{\frac{k}{2}} \binom{n}{k}_{d,F(s,t)}. 
\]  
(51)

Let us also mention

Theorem 17

\[
\frac{\det \left( \begin{array}{c}
 n + i + j \\
 k + j \\
\end{array} \right)_{F(s,t)}^{d-1}}{\det \left( \begin{array}{c}
 n + i + j \\
 j \\
\end{array} \right)_{F(s,t)}^{d-1}} = \binom{n}{k}_{d,F(s,t)}. 
\]  
(52)

and

\[
\frac{\det \left( \begin{array}{c}
 n + i + k \\
 k + j \\
\end{array} \right)_{F(s,t)}^{d-1}}{\det \left( \begin{array}{c}
 n + i \\
 j \\
\end{array} \right)_{F(s,t)}^{d-1}} = \binom{n+k}{k}_{d,F(s,t)}. 
\]  
(53)

The proof follows in the same way as in Theorem 11.

Let us consider two extreme special cases.

Taking limits for \( t \to 0 \) we get \( F_n(s,0) = s^{n-1} \) for \( n \geq 1 \) and \( F_0(s,0) = 0 \).

For the Fibonomials we get \( \binom{n}{k}_{F(s,0)} = s^{k(n-k)} \) for \( 0 \leq k \leq n \).

This follows by induction from (43). From (47) we get that the entries of the Hoggatt matrices are

\[
\binom{n}{k}_{F(s,0)} = s^{dk(n-k)}. 
\]

Taking limits for \( s \to 0 \) gives more interesting results.

\( F_{2n}(0,t) = 0 \) and \( F_{2n+1}(0,t) = t^n \) by the definition of the Fibonacci polynomials.

The Lucas polynomials reduce to \( L_{2n}(0,t) = 2t^n \) and \( L_{2n+1}(0,t) = 0 \).
Therefore we get

\[ h_{2n}(x,0,t) = \left( t^{2n-1} x^2 - 1 \right)^n \]  

and

\[ h_{2n+1}(x,0,t) = (1 - t^n x) \left( 1 + t^n x \right)^n. \]  

Comparing with (45) we get

\[
\binom{2n}{2j}_{F(0,j)} = \binom{n}{j} t^{2j(n-j)}, \quad \binom{2n}{2j+1}_{F(0,j)} = 0, \\
\binom{2n+1}{2j}_{F(0,j)} = \binom{n}{j} t^{2j(n+1-2j)}, \quad \binom{2n+1}{2j+1}_{F(0,j)} = \binom{n}{j} t^{2j(2n+1-2j)/n}. 
\]  

For example

\[
\left( \frac{n}{k} \right)_{F(0,j)} \bigg|_{n,k=0}^7 = \begin{pmatrix} 1 & 1 \\ 1 & 0 & 1 \\ 1 & t & t & 1 \\ 1 & 0 & 2t^2 & 0 & 1 \\ 1 & t^2 & 2t^3 & 2t^3 & t^2 & 1 \\ 1 & 0 & 3t^4 & 0 & 3t^4 & 0 & 1 \end{pmatrix}. 
\]  

Let us also mention the Hoggatt triangle \( H_{2,F(0,j)}. \)

Here we get by (56)

\[
\left\langle \frac{n}{k} \right\rangle_{2,F(0,j)} = \prod_{j=0}^{k-1} \frac{\left( \frac{n+1-j}{2} \right)_{F(0,j)}}{\left( \frac{k+1-j}{2} \right)_{F(0,j)}} = \prod_{j=0}^{k-1} \frac{n+1-j}{2} t^{n-j} \left( \frac{n}{2} \right) \left( \frac{n+1}{2} \right) \left( \frac{k}{2} \right) \left( \frac{k+1}{2} \right). 
\]  

for \( 0 \leq k \leq n \) and \( = 0 \) else.

The first terms are

\[
\left( \frac{n}{k} \right)_{2,F(0,j)} \bigg|_{n,k=0}^7 = \begin{pmatrix} 1 & 1 \\ 1 & t & 1 \\ 1 & 2t^2 & 2t^2 & 1 \\ 1 & 2t^3 & 4t^3 & 2t^3 & 1 \\ 1 & 3t^4 & 6t^6 & 6t^6 & 3t^4 & 1 \\ 1 & 3t^5 & 9t^8 & 9t^8 & 9t^8 & 3t^5 & 1 \end{pmatrix}. 
\]
For $t = 1$ this is OEIS [8], A088855. For $t = 1$ the row sums are \( \binom{n+1}{\frac{n+1}{2}} \) and for $t = -1$ the sum of row $2n$ is the Catalan number $C_n$ and the sum of row $2n - 1$ is the central binomial coefficient $\binom{2n}{n}$.

A companion to Conjecture 13 is

**Conjecture 18**

\[
\left( \sum_{j=0}^{dk+1} (-1)^{\frac{j+1}{2}} \binom{dk+1}{j} t^j \right) \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = N(d,k,s,t,x) \tag{60}
\]

is a polynomial of degree $(d - 1)(k - 1)$.

For $d = 2$ we get more precisely

\[
\sum_{j=0}^{2k+1} (-1)^{\frac{j+1}{2}} \binom{2k+1}{j} t^j \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \sum_{j=0}^{k-1} \binom{k-1}{j} t^{j+1} x^j \tag{61}
\]

For $s \to 0$ identity (61) reduces to

\[
(1-t^k x)^{k+1} (1+t^k x)^k \sum_{n=0}^{\infty} \binom{n+k}{\frac{k}{2}} \binom{n+k+1}{\frac{k+1}{2}} x^n = \sum_{j=0}^{k} \binom{k-1}{\frac{j}{2}} \binom{k}{\frac{j}{2}} t^j x^j \tag{62}
\]

Let us prove this identity. It suffices to consider $t = 1$.

Let $k = 2\ell$. Then

\[
\sum_{n=0}^{\infty} \binom{2n+k}{\frac{k}{2}} \binom{2n+k+1}{\frac{k+1}{2}} x^{2n} = \sum_{n=0}^{\infty} \binom{n+\ell}{\ell}^2 x^{2n},
\]

\[
\sum_{n=0}^{\infty} \binom{2n+1+k}{\frac{k}{2}} \binom{2n+1+k+1}{\frac{k+1}{2}} x^{2n+1} = \sum_{n=0}^{\infty} \binom{n+\ell}{\ell} \binom{n+1+\ell}{\ell} x^{2n+1}.
\]

Thus the left-hand side is
\[(1-x)(1-x^2)^{2^t}\left(\sum_{n=0}^{\ell} \binom{n+\ell}{\ell} x^{2n} + \sum_{n=0}^{\ell} \binom{n+1+\ell}{\ell} x^{2n+1}\right)\]

\[= (1-x)(1-x^2)^{2^t}\left(\sum_{n=0}^{\ell+1} \frac{\ell+1}{1 \cdot 1+1} x^n + (\ell+1)x \sum_{n=0}^{\ell+1} \frac{\ell+1+2}{2} x^n\right).\]

The right-hand side reduces in an analogous way to \( \sum_{n=0}^{\ell+1} \frac{-\ell, -\ell+1}{1} x^n + \ell x \sum_{n=0}^{\ell+1} \frac{-\ell+1, -\ell+1}{2} x^n.\)

Euler’s transformation formula (cf. [1], (2.2.7))

\[\sum_{n=0}^{\ell} \frac{a_n}{c} ; z = (1-z)^{-a-b} \sum_{n=0}^{\ell} \frac{a-a, c-b}{c} ; z\]

gives

\[\sum_{n=0}^{\ell} \frac{\ell+1, \ell}{1 \cdot 1; x^2} (1-x^2)^{2^t} = \sum_{n=0}^{\ell} \frac{-\ell, 1-\ell}{1 \cdot 1; x^2},\]

\[\sum_{n=0}^{\ell} \frac{\ell+1, \ell+1}{2 \cdot 1; x^2} (1-x^2)^{2^t} = \sum_{n=0}^{\ell} \frac{1-\ell, 1-\ell}{2 \cdot 1; x^2}.\]

By comparing coefficients we get

\[\sum_{n=0}^{\ell} \frac{\ell+1, \ell}{1 \cdot 1; x^2} + \ell x \sum_{n=0}^{\ell} \frac{\ell+1, \ell+1}{2 \cdot 1; x^2} = (1-x) \sum_{n=0}^{\ell} \frac{\ell+1, \ell+1}{1 \cdot 1; x^2} + (\ell+1)x(1-x) \sum_{n=0}^{\ell} \frac{\ell+1, \ell+2}{2 \cdot 1; x^2}\]

which proves (62) for even \( k.\) In a similar way the formula is proved for odd \( k.\)

References

[1] George E. Andrews, Richard Askey and Ranjan Roy, Special Functions, Encyclopedia of Mathematics and its Applications 71
[2] Curtis Bennett, Juan Carrillo, John Machacek and Bruce E. Sagan, Combinatorial Interpretations of Lucas Analogues of Binomial Coefficients and Catalan Numbers, Ann.Comb. 24 (2020), 503-530
[3] Leonard Carlitz, The characteristic polynomial of a certain matrix of binomial coefficients, Fibonacci Quarterly 3(2) (1965), 81-89
[4] Daniel C. Fielder and Cecil O. Alford, On a conjecture by Hoggatt with extensions to Hoggatt sums and Hoggatt triangles, The Fibonacci Quarterly, 27(2):160–168, May 1989.
[5] Kristina Garrett and Kendra Killpatrick, A recursion for the Fibonarayana and the generalized Naryana numbers, arXiv:1910.08855
[6] Christian Krattenthaler, Advanced Determinant Calculus, Séminaire Lotharingien Combin. 42 (1999), Article B42q
[7] Ana Luzón, Manuel A. Morón and José L. Ramírez, On Ward’s differential calculus, Riordan matrices and Sheffer polynomials, Linear Algebra Appl. 610 (2021), 440-473
[8] OEIS, The Online Encyclopedia of Integer Sequences, http://oeis.org/
[9] T. Kyle Petersen, Eulerian numbers, Birkhäuser 2015
[10] Bruce E. Sagan and Jordan Tirrell, Lucas atoms, Advances in Mathematics 374 (2020), 107387
[11] Richard P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Studies in Advanced Mathematics 62, 1999
[12] Robert A. Sulanke, Generalizing Narayana and Schröder numbers to higher dimensions, Electr. J. Comb. 11 (2004), R 54
[13] Qiaochu Yuan, Answer to https://math.stackexchange.com/questions/3990327/
[14] Doron Zeilberger, Reverend Charles to the aid of Major Percy and Fields Medalist Enrico, Amer. Math. Monthly 103 (6) (1996), 501-502