Transitions in a probabilistic interface growth model

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Abstract. We study a generalization of the Wolf–Villain (WV) interface growth model based on a probabilistic growth rule. In the WV model, particles are randomly deposited onto a substrate and subsequently move to a position nearby where the binding is strongest. We introduce a growth probability which is proportional to a power of the number \( n_i \) of bindings of the site \( i \): \( p_i \propto n_i^\nu \).

Through extensive simulations, in \((1+1)\) dimensions, we find three behaviors depending on the \( \nu \) value: (i) if \( \nu \) is small, a crossover from the Mullins–Herring to the Edwards–Wilkinson (EW) universality class; (ii) for intermediate values of \( \nu \), a crossover from the EW to the Kardar–Parisi–Zhang (KPZ) universality class; and, finally, (iii) for large \( \nu \) values, the system is always in the KPZ class.

In \((2+1)\) dimensions, we obtain three different behaviors: (i) a crossover from the Villain–Lai–Das Sarma to the EW universality class for small \( \nu \) values; (ii) the EW class is always present for intermediate \( \nu \) values; and (iii) a deviation from the EW class is observed for large \( \nu \) values.

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1. Introduction

As interface growth is ubiquitous in nature, it has constituted an important subject of research in the past few decades [1, 2]. In general, the thickness of these interfaces, also named the roughness $w$, is a signature of the nonequilibrium growth conditions under which the interface has formed and their evolution can be characterized by the Family–Vicsek scaling relation [3]

$$w(L, t) = L^\alpha f\left(\frac{t}{L^z}\right),$$  \hspace{1cm} (1)

where $L$ is the linear size of the system and $t$ is the evolution time. The scaling function $f(x)$ behaves as

$$f(x) \sim \begin{cases} x^\beta, & \text{for } x \ll 1 \\ \text{const,} & \text{for } x \gg 1 \end{cases}$$  \hspace{1cm} (2)

where $\beta = \alpha/z$. Then, the roughness $w$ grows with $t^\beta$ until saturation occurs at a value $w_{\text{sat}}$, named the saturation roughness, which behaves with the system size $L$ as a power law, according to $L^\alpha$. The saturation time $t_{\text{sat}}$ also grows with the system size as $L^z$. The exponents $\alpha$, $\beta$ and $z$ are known as the roughness, growth and dynamic exponents, respectively.

This scaling analysis led to a considerable advance in the understanding of the roughening of interface growth and allows us to define some universality classes which are related to a stochastic differential equation for the height $h(\vec{x}, t)$ of the system at position $\vec{x}$ and time $t$. Lai and Das Sarma [4] propose a general form for these equations:

$$\frac{\partial h}{\partial t} = \nu_0 \nabla^2 h + \nu_1 \nabla^4 h + \cdots + \lambda_0 (\nabla h)^2 + \lambda_1 (\nabla^2 h)^2 + \lambda_2 \nabla (\nabla h)^3 + \cdots + \eta,$$  \hspace{1cm} (3)

where the parameters $\nu_i$ are related to the linear terms, the parameters $\lambda_i$ are related to the nonlinear terms and $\eta$ is a white noise.
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Table 1. The growth exponent $\beta$ and the roughness exponent $\alpha$ associated with the main terms of equation (2) (left column) for $d = 1$ (center) and $d = 2$ (right) [1].

| Term                | $\beta$ | $\alpha$ | $\beta$ | $\alpha$ |
|---------------------|---------|----------|---------|----------|
| $\nabla^2 h$        | 1/4     | 1/2      | 0 (log) | 0 (log)  |
| $(\nabla h)^2$     | 1/3     | 1/2      | 0.24    | 0.4      |
| $\nabla^4 h$       | 3/8     | 3/2      | 1/4     | 1        |
| $\nabla^2(\nabla h)^2$ | 1/3  | 1       | 1/5     | 2/3      |
| $\nabla(\nabla h)^3$ | 3/10 | 3/4      | 1/6     | 1/2      |

In the hydrodynamic limit ($L \to \infty$ and $t \to \infty$), the $\lambda_0$ term dominates and the asymptotic behavior is governed by the KPZ universality class [5]. This equation is the simplest nonlinear differential equation for describing a kinetic growth process and it has an exact solution only in $(1 + 1)$ dimensions where the scaling exponents are $\alpha = 1/2$ and $z = 3/2$.

For models where the up–down symmetry is present, $\lambda_0 = 0$ [1] and the $\nu_0$ term of equation (3) dominates. In the asymptotic limit, we recover the linear equation proposed by Edwards and Wilkinson [6] in studying a sedimentation process. This equation has an exact solution and the scaling exponents are given by $\alpha = (2 - d)/2$, where $d$ is the dimension of the substrate, and $z = 2$.

However, for short and intermediate times, the process can present a behavior which is a representation of the dominant term before the hydrodynamic limit. An example is provided by the linear equation proposed by Wolf and Villain [7] and by Das Sarma and Tamborenea [8] for introducing surface diffusion, where the term $\nu_1$ dominates for short times. Equation (3) just with this term is linear and the exact solution gives $\alpha = (4 - d)/2$ and $z = 4$. In Table 1 we show the growth and the roughness exponents associated with the main terms of equation (3), for $d = 1$ and 2.

The determination of the asymptotic universality class of a growth model may be a difficult task due to the extremely long time for which the initial behavior is seen, which can mask the crossover to another behavior [9]. To investigate this problem, various techniques have been developed; as examples we can cite the investigation of the universality class using the surface diffusion currents [10,11], the application of the noise reduction technique to models with limited mobility [12], the investigation of finite size effects in the DT and WV models [13], and the application of the scaling transformation to the stochastic equation of the WV model [14,15].

In this paper, we propose a probabilistic interface growth model that is a generalization of the WV model where an adatom diffuses to maximize its coordination number. In our model, the incoming particle searches for a site where the number of bonds is higher and it has a probability of being incorporated at this site. The probability depends on the number of bonds that the particle would have at this site and it has a parameter $\nu$ such that, for $\nu = 0$, the WV model is recovered. For low values of $\nu$, the crossover to the asymptotic universality class occurs early, and for high values, the nonlinear KPZ universality class appears. A similar strategy was used in the investigation of the on-lattice Eden model [16], where the authors show that the introduction of such probability growth leads to the growth of on-lattice Eden clusters without the undesirable
Figure 1. Interface morphology obtained after $10^5$ time steps in a $(1 + 1)$-dimensional lattice of size $L = 400$. We used in these simulations $\nu = 0.0, 0.1, 0.50$ and $2.0$ for (a), (b), (c) and (d), respectively.

anisotropy effects. This strategy was initially proposed to generate an isotropic cluster of the on-lattice diffusion limited aggregation model [17]. However, a later work [18] shows that in the noiseless limit, instead of isotropic patterns, a $45^\circ$ ($30^\circ$) rotation in the anisotropy directions of the clusters grown on square (triangular) lattices was observed.

The model proposed is described in section 2 and the results in section 3. Section 4 shows some conclusions.

2. The probabilistic growth rule

We study a model where the particles are randomly deposited on a $d$-dimensional lattice of linear size $L$. The height of the interface at time $t$ is represented by $h_i(t)$ and the number of bonds of each site by $n_i(t)$, with $i = 1, 2, \ldots, L^d$. The initial condition is given by $h_i(0) = 0$ and $n_i(0) = 1$, $\forall i$, i.e., at the beginning of the simulation the interface is flat and a particle deposited at the site $i$ will have $n_i = 1$ bonds, $\forall i$. A particle is deposited at random and searches in its neighborhood for the site with the largest number of bonds, and this site is chosen as the growing site. If the number of bonds of the particle cannot be increased in the neighborhood, the particle will choose the deposition site as the growing site and, if it has more than one site with the same number of bonds, one of them is chosen at random. One particle is incorporated to the growing site with a probability given by

$$p_i = \left(\frac{n_i}{n_{\text{max}}}\right)^\nu$$

where $n_{\text{max}} = 2d + 1$ is the maximum number of bonds of one site, which is the case for a particle deposited at a site with its neighborhood fully occupied in a $d$-dimensional
Figures 1(b) and 1(c), and in figures 2(b) and (c), an additional increase in the $\nu$ value leads to an increase in this attenuation until...
Figure 3. Temporal roughness evolution for a (1 + 1)-dimensional system of size $L = 10^5$ for various values of the parameter $\nu$ as indicated in the legend. Notice that, to improve the visualization, the results for $\nu \leq 2$ are shown in (a), while those for $\nu \geq 2$ are shown in (b).

The ridgeline landscape disappears completely, even for larger systems sizes and times that are asymptotically large. Therefore, in both (1 + 1) and (2 + 1) dimensions, the effect of the $\nu$ parameter is to smooth the interface and this effect can be easily understood through an analysis of the growth probability. As observed previously, when we increase the $\nu$ parameter, the growth rate of sites with $n < n_{\text{max}}$ decreases, and hence the rates of appearance of peaks are reduced.

In order to quantitatively characterize the dependence of the roughening process using the parameter $\nu$, we consider the roughness of the interface; this involves the determination of the root mean square fluctuation in its height, i.e.,

$$w(t) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (h_i(t) - \bar{h}(t))^2}$$

(5)

where $N = L^d$ is the number of sites in the substrate, $h_i(t)$ is the height of the site $i$ at time $t$ and $\bar{h}(t)$ is the mean height of the interface [1]. We carried out simulations for systems of size ranging from $L = 10$ up to $10^5$ for $d = 1$, while for $d = 2$, we used systems from $L = 10$ up to 500. The $\nu$ parameter was varied from 0 to 8 and from 0 to 4, for (1 + 1) and (2 + 1) dimensions, respectively.

As we can observe in figure 3, for the system in (1 + 1) dimensions, an important consequence of the introduction of the growth probability is the arising of different behaviors. If the $\nu$ value is set to zero (the upper curve in figure 3(a)), we find $\beta \approx 0.375$ throughout all simulations, which means that the $\nabla^4 h$ term of equation (3) dominates. However, for sufficiently small $\nu$ values ($\nu = 0.1$), we observe a crossover in the $\beta$ exponent from a value of $\approx 0.375$ to a value of $\approx 0.28$, very close to $\beta = 1/4$, which means that the $\nabla^2$ term of equation (3) begins to prevail. This crossover was previously observed for the WV original model [10,20]. For a small increase in the $\nu$ parameter, e.g. for $\nu = 0.5$ in

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Figure 4. Saturation roughness as a function of the system size obtained for a (1 + 1)-dimensional system with distinct $\nu$ values indicated in the legend of the figure.

Figure 3(a), the growth exponent is $\approx 0.28$ in all simulations. However, for further increase in the $\nu$ parameter ($\nu = 2.0$), we observe the exponent $\beta \approx 0.33$ after a crossover time. Finally, for the largest $\nu$ values (the bottom three curves in figure 3(b)), we can observe $\beta \approx 0.33$ in all simulations. This new asymptotic growth exponent close to 1/3 is directly related to the rejection of the particle which chose a site with a very low growth probability. Therefore, this asymptotic value can be associated with the KPZ universality class, which is similar to the restricted solid-on-solid model proposed by Kim and Kosterlitz [19].

Corroborating the scaling of the roughness evolution, the saturation roughness also exhibits a crossover between different roughness exponents, as we can see in figure 4. When low values of the $\nu$ parameter are used, for small values of $L$, the roughness exponent exhibits a value close to 3/2, as observed for the $\nabla^4 h$ term, and a value close to 1/2 for the largest values of the system size $L$. However, when large $\nu$ values are used, this crossover disappears and the observed roughness exponent is always close to 1/2, a value obtained for both the EW ($\nabla^2 h$ term) and KPZ ($\nabla h$) terms) universality classes.

For (2 + 1) dimensions we obtained a scenario similar to that observed for (1 + 1) dimensions. Figures 5 and 6 show the log-log and semi-log plots for the time evolution of the roughness and the system size dependence of the saturation roughness, respectively. As can be seen in these figures, for values of $\nu \leq 0.5$ a crossover from a power law to a logarithmic growth was observed for both the roughness and the saturation roughness scaling. The initial values obtained for the growth ($\beta \approx 0.22$) and roughness ($\alpha \approx 0.78$) exponents are very close to those found to the universality class of the $\nabla^4 h$ term, for which $\beta = 1/5$ and $\alpha = 2/3$, and the asymptotic behavior is logarithmic (the growth and roughness exponents in the EW universality class are null, which indicates a logarithmic behavior). For larger values of $\nu$, the scaling is always logarithmic.
Figure 5. Roughness time evolution for distinct $\nu$ values (0.0, 0.1, 0.5, 1.0 and 2.0) for a (2 + 1)-dimensional system of linear size $L = 500$. In (a), a log–log plot is shown, and in (b), a semi-log plot is shown.

Figure 6. Saturation roughness $w_{\text{sat}}$ as a function of the system size $L$ for distinct $\nu$ values in a (2 + 1)-dimensional system. In (a), a log–log plot is shown, and in (b), a semi-log plot is shown.

The results observed using intermediate $\nu$ values for both 1+1 and 2+1 dimensions, indicating the EW asymptotic universality class, are in good agreement with results found in the literature [10, 11, 14, 15, 21]. Besides, the time to the asymptotic behavior appearing decreases as $\nu$ is increased, so intermediate values of $\nu$ can be used to anticipate the asymptotic universality class of the WV model.
4. Conclusions

In this work we studied a modified Wolf–Villain growth model with a probabilistic growth rule proportional to the power of the coordination number of a growing site, i.e., $p \propto n^\nu$. Simulations in $(1 + 1)$ dimensions indicate that, for low $\nu$ values, the model presents a crossover from the $\nabla^4 h$ to the $\nabla^2 h$ behavior, in agreement with the original model. We observe that the crossover time separating these two behaviors decreases as $\nu$ increases. This fact indicates that intermediate values of $\nu$ can be used to anticipate the asymptotic universality class of the WV model. However, for intermediate $\nu$ values, the model presents a crossover from the linear $\nabla^2 h$ to the $(\nabla h)^2$ nonlinear behavior and, for larger $\nu$ values, the model falls into the KPZ universality class. The presence of the KPZ asymptotic universality class in this modified version can be easily explained. The probability of growth in this version introduces a refusal of a particle. As known in the literature [19], the refusal of a particle is one of the basic conditions for the appearance of the KPZ universality class. In $(2 + 1)$ dimensions, the model presents the same type of crossover and, as in the $(1+1)$-dimensional case, the crossover time is dependent on the $\nu$ value. For larger values, we found that the model belongs to the EW universality class. In further work, we intend to apply the probabilistic rule to models, e.g. the DT model, that take into account the relaxation to the neighbor, considering its coordination number.

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