In this study, we implemented a new numerical method known as the Chebyshev Pseudospectral method for solving nonlinear delay differential equations having fractional order. The fractional derivative is defined in Caputo manner. The proposed method is simple, effective, and straightforward as compared to other numerical techniques. To check the validity and accuracy of the proposed method, some illustrative examples are solved by using the present scenario. The obtained results have confirmed the greater accuracy than the modified Laguerre wavelet method, the Chebyshev wavelet method, and the modified wavelet-based algorithm. Moreover, based on the novelty and scientific importance, the present method can be extended to solve other nonlinear fractional-order delay differential equations.

1. Introduction

Fractional calculus is used in various branches of mathematics due to its numerous applications in modeling different physical phenomena in engineering and science. The concept of fractional calculus has been derived from the fact \( D^\alpha (f(x)) \), where alpha is noninteger. Later on, different scientists such as Riemann–Liouville, Euler, Leibniz, L’Hospital, Bernoulli, and Wallis have devoted their work to this research area. Fractional calculus has numerous applications in different field of sciences. For example dynamic of viscoelastic materials [1], electromagnetism [2], fluid mechanics [3], propagation of spherical flames [4], and viscoelastic materials [5].

In our real life, DEs are used to develop a different number of physical problems. Some are more complex and cannot be modeled with the help of simple differential equations. For these complex problems, a new technique has been used by the researchers known as fractional differential equations (FDEs). In the mathematical modeling of real-world physical problems, FDEs have been widespread due to their numerous applications in engineering and real-life sciences problems [6–9], such as economics [10], solid mechanics [11], continuum and statistical mechanics [12], oscillation of earthquakes [13], dynamics of interfaces between soft-nanoparticles and rough substrates [14], fluid-dynamic traffic model [15], colored noise [16], solid mechanics [11], anomalous transport [17], and bioengineering [18–20].

Delay differential equations (DDEs) have a wide range of applications in engineering and science. Delay differential equation simplifies the ordinary differential equation, depends on the past data, and is suitable for physical systems. Nowadays, researchers pay more attention to FDDEs as compared to DEs because a slight delay has a large effect. In this regard, numerous papers have been dedicated to the study of the numerical solution of FDDEs. FDDs have been widespread in mathematical modelings, such as population
dynamics, epidemiology, immunology, physiology, and neural networks [21–25].

In literature, there is no precise technique for finding an exact or analytical solution for every FDEs; the researcher’s effort is to find the numerical solution of FDEs. Various methods have been implemented for solving these problems numerically. The well-known among these methods are new predictor corrector method (NPCM) [26], adomain decomposition method (ADM) [27], Legendre pseudospectral method (LSM) [28], kernel method (KM) [29], LMS method (LMSM) [30], Adams–Bashforth–Moulton algorithm (ABMA) [31], extend predictor corrector method (EPCM) [32], simplified reproducing kernel method (SRKM) [29], variation iteration method (VIM) [33], homotopy perturbation method (HPM) [34], Galerkin method (GM) [35], Runge–Kutta-type methods (RKMM) [36], Bernoulli wavelet method (BWM) [37], and modified Laguerre wavelet method [38] have been used for the analytical and numerical solution of FDEs.

In the present work, CPM is extended for the solutions of FDEs. The results we obtained are compared with other methods, which show that CPM has good convergence rate than other methods. We focus on FDE of the form
\[
D^\gamma u (t) = g(u, f (u), f (h(u))),
\]
with the following boundary conditions:
\[
f (c) = \alpha_0, \quad f (d) = \alpha_1, \quad f (u) = \zeta (u), \quad u \in [c_0, c],
\]
where \( h \) is the delay function which is to be assumed continues in the interval \([c, d]\) and satisfies the inequality \( c_0 \leq h(u) \leq d \) for some fix real constant \( c_0 \) for \( u \in [c, d] \) and \( \zeta \in C[c_0, c] \).

The following is a summary of the paper’s structure. In Section 2, we introduce some fundamental fractional calculus definitions and mathematical techniques that will be useful in our later study. The approximation of the fractional derivative \( D^\gamma u (t) \) is obtained in Section 3. Section 4 describes the Chebyshev collocation method’s application to the solution of eq. (1). As a result, a set of algebraic equations is created, and the solution to the problem in question is presented. Section 5 provides some numerical results to help clarify the method.

2. Basic Definitions of Fractional Derivatives

**Definition 1.** A real function, \( g(u), u > 0 \), is said to be in the space \( \mathbb{C}_p^m, \mu \in \mathbb{R} \), if there exists a real number \( p > \mu \), such that \( g(u) = u^\mu \gamma_1 (u) \), where \( \gamma_1 (u) \in [0, \infty) \), and it is said to be in the space \( \mathbb{C}_p^m \) if and only if \( g^{(m)} \in \mathbb{C}_p^m, m \in \mathbb{N} \).

**Definition 2.** In Caputo manner, the derivative having fractional-order \( D^\gamma g(u) \) is given as below:
\[
D^\gamma g(u) = \frac{1}{\Gamma (j - \gamma)} \int_0^u (u - t)^{j - \gamma - 1} g^{(m)} (t)dt, \quad u > 0, \quad j - 1 < \gamma < j.
\]

The order of the derivative is \( \gamma > 0 \), and the lowest integer greater than \( \gamma \) is \( j \in \mathbb{N} \) and \( g \in C_0^m \).

We have the Caputo derivative [39]:
\[
D^\gamma C = 0, \quad C \text{ is a constant,}
\]
\[
D^\gamma u^\alpha = \begin{cases} 0 \text{ for } \alpha \in \mathbb{N}_0 \text{ and } \alpha < [\gamma] & \text{if } \alpha \in \mathbb{N}_0 \text{ and } \alpha \geq [\gamma], \end{cases}
\]
where the lowest integer larger than or equal to \( \gamma \) is denoted by the ceiling function \([\gamma]\) and \( \mathbb{N}_0 = 1, 2, \ldots \). Remember that the Caputo differential operator is the same as the normal differential operator of the integer order for \( \gamma \in \mathbb{N} \). Fractional differentiation is a linear operation, just like integer-order differentiation:
\[
D^\gamma (\phi g (u) + \mu h (u)) = \phi D^\gamma g (u) + \mu D^\gamma h (u),
\]
where \( \phi \) and \( \mu \) are constants.

3. Chebyshev Series Expansion is Used to Approximate a Caputo Derivative

On the interval \([-1, 1]\), Chebyshev polynomials are defined and, with the help of recurrence formulae, explained as [40, 41]
\[
T_{j+1} (u) = 2uT_j (u) - T_{j-1} (u), \quad j = 1, 2, \ldots,
\]
where \( T_0 (u) = 1 \) and \( T_1 (u) = u. \) The Chebyshev polynomial analytical form for degree \( j \) is defined as [41]
\[
T_j (u) = \frac{1}{2} \sum_{r=0}^{[j/2]} (-1)^r \frac{j-r-1)!}{r!(j-2r)!} (2u)^{j-2r}.
\]

If we apply the Chebyshev polynomials over the \([0, 1]\) interval, we explain the Chebyshev shifted polynomials \( \tilde{T}_j (u) \). These are described in the sense of Chebyshev polynomials \( T_j (u) \) as [41]
\[
\tilde{T}_j (u) = T_j (2u - 1).
\]
And recurrence formula is as follows:
\[
\tilde{T}_{j+1} (u) = \frac{2}{2} \tilde{T}_j (u) - \tilde{T}_{j-1} (u), \quad j = 1, 2, \ldots,
\]
where \( \tilde{T}_0 (u) = 1 \) and \( \tilde{T}_1 (u) = 2u - 1 \). The orthogonality condition is [42]
\[
\int_0^1 \frac{\tilde{T}_j (u) \tilde{T}_m (u)}{\sqrt{1 - u^2}} du = \begin{cases} 0 \text{ if } m \neq j, & m = j \neq 0, \\
\pi/2 & m = j = 0.
\end{cases}
\]

Now, we can use the well-known relation,
\[
\tilde{T}_j (u) = T_{2j} (\sqrt{u}),
\]

Complexity
and equation (8) to get shifted Chebyshev polynomials analytical form having order $j$ as

$$
\tilde{T}_j(u) = \sum_{r=0}^{j} (-1)^r 2^{2j-2r-j/2} j!(2j-r-1)!/r!(2j-2r)! (\chi)^{j-r}. \quad (13)
$$

A function $f(u) \in L_2[0,1]$ may be described in terms of Chebyshev shifted polynomials as

$$
f(u) = \sum_{j=1}^{\infty} c_j \tilde{T}_j(u), \quad (14)
$$

where the coefficients $c_j, j=1,2,\ldots$, are given by

$$
c_0 = \frac{1}{\pi} \int_{0}^{1} g(u)\tilde{T}_0(u) \frac{du}{\sqrt{u-u^2}} \quad \text{and} \quad c_n = \frac{2}{\pi} \int_{0}^{1} g(u)\tilde{T}_n(u) \frac{du}{\sqrt{u-u^2}}. \quad (15)
$$

Only Chebyshev shifted polynomials first $(m+1)$-terms are considered in practice. Thus,

$$
f_m(u) = \sum_{j=0}^{m} c_j \tilde{T}_j(u). \quad (16)
$$

### 3.1. Chebyshev Truncation Theorem [43].

The sum of the absolute values of all the disregarded coefficients limits the inaccuracy in approximating $f(u)$ by the sum of its first $m$ terms. That is, assuming

$$
f_m(u) = \sum_{k=0}^{m} c_k T_k(u), \quad (17)
$$

then, for all $f(u)$, all $m$, and all $u \in [-1,1]$, we obtain

$$
D^\alpha f_m(u) = \sum_{j=0}^{m} c_j \tilde{T}_j(u) \quad j = [\alpha], [\alpha] + 1, \ldots, m. \quad (18)
$$

The following is the result of combining (21)–(23):

$$
E_T(m) = |f(u) - f_m(u)| \leq \sum_{k=m+1}^{\infty} |c_k|. \quad (19)
$$

**Proof.** For any $u \in [-1,1]$ and all $k$, the Chebyshev polynomials are bounded by 1, $|T_k(u)| \leq 1$. As a result, the $k$th term is restricted by $|c_k|$. By subtracting the reduced series from the infinite series, bounding each term in the difference, and then summing the bounds, the theorem can be derived.

The following theorem contains the main approximate formula for the fractional derivative of $f(u)$.

### 3.2. Theorem [44].

Assume $\alpha > 0$ and that $f(u)$ is estimated by the shifted Chebyshev polynomials as in (16). Then,

$$
D^\alpha f_m(u) = \sum_{j=0}^{m} c_j b_j^{\alpha} u^{j-r}, \quad (19)
$$

where $b_j^{\alpha}$ is given by

$$
b_j^{\alpha} = (-1)^r 2^{2j-2r-r} j!(2j-r-1)!/(r!(2j-2r)\Gamma(j-r+1-\alpha)). \quad (20)
$$

**Proof.** Since Caputo fractional differentiation is a linear operation, we have

$$
D^\alpha f_m(u) = \sum_{j=0}^{m} c_j D^\alpha(\tilde{T}_j(u)). \quad (21)
$$

Now, to evaluate $D^\alpha(\tilde{T}_j(u))$, applying to equations (4) and (5)–(13),

$$
D^\alpha(\tilde{T}_j(u)) = \sum_{r=0}^{j} (-1)^r 2^{2j-2r} j!(2j-r-1)!/(r!(2j-2r)\Gamma(j-r+1-\alpha)) u^{j-r}, \quad j = [\alpha], [\alpha] + 1, \ldots, m. \quad (22)
$$

The following is the result of combining (21)–(23):

$$
D^\alpha(f_m(u)) = \sum_{j=0}^{m} c_j b_j^{\alpha} u^{j-r}, \quad (23)
$$

Test example: consider formula (19) with $f(u) = u^2, m = 2$. The shifted series of $u^2$ is

$$
u^2 = c_0 \tilde{T}_0(u) + c_1 \tilde{T}_1(u) + c_2 \tilde{T}_2(u) = 3/8 \tilde{T}_0(u) + 1/2 \tilde{T}_1(u) + 1/8 \tilde{T}_2(u) \quad (24)
$$

and
\[
\frac{1}{D^2(u^2)} = \sum_{j=1}^{n-1} c_j \frac{1}{2} u^{j-1} = c_1 \frac{1}{2} u^2 \\
+ c_2 \frac{1}{2} u^3 + c_3 \frac{1}{2} u^2
\]
which yields the same result as evaluating \( D^{1/2}(u^2) \) by relation (5).

\[
\sum_{j=1}^{n-1} \sum_{i=0}^{m} c_{j,i} u^{j-2r-a} = g \left( u, \sum_{j=1}^{m} c_j \tilde{T}_j(u), \sum_{j=1}^{m} c_j \tilde{T}_j(h(u)) \right), \quad 0 < u < 1, m + 1 < \alpha < m.
\]

Now, we collocate (23) at points \( u_p, p = 0, 1, 2, \ldots, m - \lceil \alpha \rceil \):

\[
\sum_{j=1}^{n-1} \sum_{i=0}^{m} c_{j,i} u_p^{j-2r-a} = g \left( u_p, \sum_{j=1}^{m} c_j \tilde{T}_j(u_p), \sum_{j=1}^{m} c_j \tilde{T}_j(h(u_p)) \right), \quad u_p, p = 0, 1, \ldots m - \lceil \alpha \rceil, m + 1 < \alpha < m.
\]

Using (22) in the boundary conditions (2), we may construct the following \( \lceil \alpha \rceil \) algebraic equations from (24) and \( \lceil \alpha \rceil \) algebraic equations from (26). As a result, we have total \( m + 1 \) linear or nonlinear algebraic equations that can be easily solved using matrices for unknowns \( c_{j,i}, j = 0, 1, 2, \ldots, m \), to find out an estimated solution \( \mu_m(\psi) \).

5. Numerical Representation

In this section, we solve some delay problems. The results we obtained are compared with other methods. All the numerical results are obtained using MAPLE.

### Problem 1

Consider the FDDE:

\[
\frac{d^\alpha f(u)}{du} = \frac{1}{2} \exp \left( \frac{u}{2} \right) + \frac{1}{2} f(u), \quad 0 < \alpha \leq 1,
\]
subject to the initial conditions \( f(0) = 1 \), having accurate solution \( f(u) = \exp u \) at \( \alpha = 1 \).

The exact solution and CPM solution are given in Table 1. Table 2 shows CPM and CWM error comparison at \( m = 4 \) which confirm that CPM converges quickly as compared to CWM. We illustrate the accurate and estimated solutions for \( m = 4 \) in Figure 1, while Figure 2 shows the error comparison of both methods. Also, Figure 3 provides the graphical layout of the solution of example 1 at various fractional orders. It can be seen that the solutions of CPM are in good agreement to the actual solution than that of CWM.

### Problem 2

Consider the nonlinear DDE:

\[
\frac{d^\alpha f(u)}{du} = 1 - 2 f^2 \left( \frac{u}{2} \right), \quad 0 \leq u \leq 1, 1 < \alpha \leq 2,
\]
subjects to the initial condition \( f(0) = 1, f'(0) = 0 \).

The accurate solution of this equation for \( \alpha = 2 \) is \( f(u) = \cos(u) \). The exact solution and CPM solution are shown in Table 3. Table 4 shows the error comparison of CPM at \( m = 3 \) and MWBA at \( m = 20 \) which confirm that CPM converges quickly as compare to MLWM. The estimated and accurate solutions are illustrated in Figure 4, whereas Figure 5 shows the error comparison of both methods. In addition, the convergence phenomena of the solutions at different fractional orders can be seen in Figure 6. The results of the presented method are better than those of the MWBA method for example 2.

### Problem 3

Consider the fractional DDE of the form

\[
\frac{d^\alpha f(u)}{du} = f^3 \left( \frac{u}{2} \right) + \frac{3}{4} f(u) - u^2 + 2, \quad 0 \leq u \leq 1, 1 < \alpha \leq 2,
\]
with initial conditions \( f(0) = f'(0) = 0 \).
The exact solution of this equation for $\alpha = 2$ is $f(u) = u^2$. The exact solution and CPM solution are shown in Table 5. Table 6 shows the error comparison of CPM at $m = 3$ and MWLM at $m = 5$ which confirm that CPM converges quickly as compare to MLWM. We illustrate the accurate and estimated solutions for $m = 3$ in Figure 7, while Figure 8 shows the error comparison of both methods. The results of the presented method are better than those of the MWBA method for example 3.

Problem 4. Consider the following nonlinear delay differential equation with boundary conditions $f(0) = 1$ and $f(1) = 1$:
\[
\frac{d^\alpha}{du^\alpha} f(u) = \frac{8}{3} \frac{d}{du} \left( f\left(\frac{u}{2}\right)\right) f(u) + 8u^2 f\left(\frac{u}{2}\right) - \frac{4}{3} - \frac{22}{3} u - 7u^2 - \frac{5}{3} u^3, \quad 1 < \alpha \leq 2.
\]

The accurate solution of this equation for \( \alpha = 2 \) is \( f(u) = 1 + u - u^3 \). The exact and CPM solution are shown in Table 7. Table 8 shows the error comparison of CPM at \( m = 4 \) and MWBA at \( m = 8 \) which confirm that CPM converges quickly as compare to MWBA. The estimated and accurate solutions are illustrated in Figure 9, while Figure 10 shows the error comparison of both methods. It can be seen that our method is more accurate.
Problem 5. Consider the FDDE

\[
\frac{d^\alpha f(u)}{du} + f(u) + f(u - 0.3) = \exp^{-u^{1.3}}, \quad 1 < \alpha < 2, \quad 0 < \alpha < 1,
\]

having initial conditions \( f(0) = 1, f'(0) = -1 \), and \( f''(0) = 1 \).

The accurate solution of this problem for \( \alpha = 3 \) is \( f(u) = \exp^{-u} \). The exact and CPM solutions are shown in
Table 9. Table 10 shows the error comparison of CPM and CWM at \( m = 6 \) which confirm that CPM converges quickly as compared to CWM. We illustrate the accurate and estimated solutions for \( m = 6 \) in Figure 11, while Figure 12 shows the error comparison of both methods. In Figure 13, the solution for example 4.5 at different fractional orders is calculated. It is confirmed that the solution at various fractional order approaches towards the integer-order solution. The results of the presented method are better than those of the CWM method for this problem.
Table 5: Exact, CPM solution, and CPM A.E of problem 3 at \( m = 3 \).

| \( u \) | Exact | CPM | CPM error |
| --- | --- | --- | --- |
| 0  | 0.000 000 000 000 00 | 0.000 000 000 000 00 | 4.000 000 000 000 00 \( E^{+00} \) |
| 0.10 | 0.010 000 000 000 00 | 0.010 000 000 000 00 | 1.200 000 000 000 00 \( E^{−29} \) |
| 0.20 | 0.040 000 000 000 00 | 0.040 000 000 000 00 | 2.400 000 000 000 00 \( E^{−29} \) |
| 0.30 | 0.090 000 000 000 00 | 0.090 000 000 000 00 | 3.700 000 000 000 00 \( E^{−29} \) |
| 0.40 | 0.160 000 000 000 00 | 0.160 000 000 000 00 | 5.000 000 000 000 00 \( E^{−29} \) |
| 0.50 | 0.250 000 000 000 00 | 0.250 000 000 000 00 | 6.400 000 000 000 00 \( E^{−29} \) |
| 0.60 | 0.360 000 000 000 00 | 0.360 000 000 000 00 | 7.800 000 000 000 00 \( E^{−29} \) |
| 0.70 | 0.490 000 000 000 00 | 0.490 000 000 000 00 | 9.400 000 000 000 00 \( E^{−29} \) |
| 0.80 | 0.640 000 000 000 00 | 0.640 000 000 000 00 | 1.100 000 000 000 00 \( E^{−28} \) |
| 0.90 | 0.810 000 000 000 00 | 0.810 000 000 000 00 | 1.280 000 000 000 00 \( E^{−28} \) |
| 1.0  | 1.000 000 000 000 00 | 1.000 000 000 000 00 | 1.400 000 000 000 00 \( E^{−28} \) |

Table 6: Absolute error (A.E) comparison of CPM and other different methods of problem 3.

| \( u \) | CPM A.E at \( m = 3 \) | MLWM A.E at \( m = 5 \) |
| --- | --- | --- |
| 0  | 4.000 000 000 000 00 \( E^{+00} \) | 1.414 213 56 \( E^{−09} \) |
| 0.10 | 1.210 000 000 000 00 \( E^{−29} \) | 4.758 097 40 \( E^{−08} \) |
| 0.20 | 2.480 000 000 000 00 \( E^{−29} \) | 9.693 080 00 \( E^{−08} \) |
| 0.30 | 3.740 000 000 000 00 \( E^{−29} \) | 1.470 106 25 \( E^{−07} \) |
| 0.40 | 5.000 000 000 000 00 \( E^{−29} \) | 1.980 200 00 \( E^{−07} \) |
| 0.50 | 6.400 000 000 000 00 \( E^{−29} \) | 2.509 000 00 \( E^{−07} \) |
| 0.60 | 7.800 000 000 000 00 \( E^{−29} \) | 3.055 000 00 \( E^{−07} \) |
| 0.70 | 9.400 000 000 000 00 \( E^{−29} \) | 3.624 000 00 \( E^{−07} \) |
| 0.80 | 1.100 000 000 000 00 \( E^{−28} \) | 4.220 000 00 \( E^{−07} \) |
| 0.90 | 1.280 000 000 000 00 \( E^{−28} \) | 4.840 000 00 \( E^{−07} \) |
| 1.0  | 1.400 000 000 000 00 \( E^{−28} \) | 5.510 000 00 \( E^{−07} \) |

Figure 7: The exact and CPM solution graph for problem 3.
Figure 8: Error graph of MLWM and CPM for problem 3.

Table 7: Exact, CPM solution, and CPM A.E at $m = 4$ of problem 4.

| $u$ | Exact | CPM | CPM error |
|-----|-------|-----|-----------|
| 0   | 1.00000000000000 | 1.00000000000000 | 2.0000000000E-40 |
| 0.10| 1.09900000000000 | 1.09900000000000 | 3.0000000000E-39 |
| 0.20| 1.19200000000000 | 1.19200000000000 | 6.0000000000E-39 |
| 0.30| 1.27300000000000 | 1.27300000000000 | 7.0000000000E-39 |
| 0.40| 1.33600000000000 | 1.33600000000000 | 1.0000000000E-38 |
| 0.50| 1.37500000000000 | 1.37500000000000 | 1.3000000000E-38 |
| 0.60| 1.38400000000000 | 1.38400000000000 | 1.6000000000E-38 |
| 0.70| 1.35700000000000 | 1.35700000000000 | 1.9000000000E-38 |
| 0.80| 1.28800000000000 | 1.28800000000000 | 2.3000000000E-38 |
| 0.90| 1.17100000000000 | 1.17100000000000 | 2.5000000000E-38 |
| 1.0 | 1.00000000000000 | 1.00000000000000 | 2.9000000000E-38 |

Table 8: Absolute error (A.E) comparison of CPM and other different methods for problem 4.

| $u$ | CPM error at $m = 4$ | MWBA error at $m = 8$ |
|-----|---------------------|----------------------|
| 0   | 2.0000000000E-40    | 1.2000000000E-29     |
| 0.10| 3.0000000000E-39    | 1.0000000000E-29     |
| 0.20| 6.0000000000E-39    | 1.0000000000E-29     |
| 0.30| 7.0000000000E-39    | 1.0000000000E-29     |
| 0.40| 1.0000000000E-38    | 1.0000000000E-29     |
| 0.50| 1.3000000000E-38    | 1.0000000000E-29     |
| 0.60| 1.6000000000E-38    | 1.0000000000E-29     |
| 0.70| 1.9000000000E-38    | 1.0000000000E-29     |
| 0.80| 2.3000000000E-38    | 2.0000000000E-29     |
| 0.90| 2.5000000000E-38    | 2.0000000000E-29     |
| 1.0 | 2.9000000000E-38    | 1.5000000000E-29     |
Figure 9: The exact and CPM solution graph for problem 4.

Figure 10: Error graph of MWBA and CPM for problem 4.

Table 9: Exact, CPM solution, and CPM error of problem 5 for $m = 6$.

| $u$  | Exact             | CPM               | CPM (A.E)          |
|------|-------------------|-------------------|--------------------|
| 0    | 1.000 000 000 000 000 | 1.000 000 000 000 000 | 0.000 000 000 $E + 00$ |
| 0.01 | 0.990 049 833 749 168 | 0.990 049 833 749 168 | 2.6 506 231 572 $E − 16$ |
| 0.02 | 0.980 198 673 306 755 | 0.980 198 673 306 738 | 1.6 837 479 038 $E − 14$ |
| 0.03 | 0.970 445 533 548 508 | 0.970 445 533 548 318 | 1.9 035 198 323 $E − 13$ |
| 0.04 | 0.960 789 439 152 323 | 0.960 789 439 151 262 | 1.0 614 659 655 $E − 12$ |
| 0.05 | 0.951 229 424 500 714 | 0.951 229 424 496 695 | 4.0 185 154 936 $E − 12$ |
| 0.06 | 0.941 764 533 584 249 | 0.941 764 533 572 341 | 1.1 907 912 782 $E − 11$ |
| 0.07 | 0.932 393 819 905 948 | 0.932 393 819 876 151 | 2.9 797 628 776 $E − 11$ |
| 0.08 | 0.923 116 346 386 636 | 0.923 116 346 320 752 | 6.5 884 200 594 $E − 11$ |
| 0.09 | 0.913 931 185 271 228 | 0.913 931 185 138 694 | 1.3 253 379 871 $E − 10$ |
| 0.10 | 0.904 837 418 035 960 | 0.904 837 417 788 512 | 2.4 744 798 291 $E − 10$ |
Table 10: Absolute error (A.E) comparison of CPM and other different methods of problem 5 at $m = 6$.

| $u$  | CPM A.E                | CWM A.E                |
|------|------------------------|------------------------|
| 0    | $0.0000000000E + 00$   | $0.0000000000E + 00$   |
| 0.01 | $2.6506231572E + 16$   | $8.20000E - 09$        |
| 0.02 | $1.6837479038E + 14$   | $6.68000E - 08$        |
| 0.03 | $1.9035198323E + 13$   | $2.28800E - 07$        |
| 0.04 | $1.0614659655E + 12$   | $5.50500E - 07$        |
| 0.05 | $4.0185154936E + 11$   | $1.09130E - 06$        |
| 0.06 | $1.1907912872E + 11$   | $1.91420E - 06$        |
| 0.07 | $2.9797628776E + 11$   | $3.08520E - 06$        |
| 0.08 | $6.5854682284E + 11$   | $4.67410E - 06$        |
| 0.09 | $1.3253379871E + 10$   | $6.75420E - 06$        |
| 0.10 | $2.4744798291E + 10$   | $9.40260E - 06$        |

Figure 11: The exact and CPM solution graph for problem 5.

Figure 12: Error graph of CWM and CPM for problem 5.
6. Conclusion

In this study, we applied the Chebyshev pseudospectral method for solving fractional delay differential equations. The technique is easy to implement and show good convergence rate than other methods. Some examples are solved which shows the effectiveness of the present method. The results we obtained are compared with other methods such as modified wavelet-based algorithm (MWBA), modified Laguerre wavelet method (MLWM), Chebyshev wavelet method (CWM). It is clear from comparison that CPM has higher accuracy than all these methods. Although, CPM can easily be extended to other fractional delay or non-delay models of physics and real-life sciences.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant CODE: 22UQU4310396DSR02.

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