On a logarithmic sum related to the Selberg sieve

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Abstract

We study the sum \( \Sigma_q(U) = \sum_{d,e \leq U} \mu(d)\mu(e) \left[ \frac{U}{d} \right] \log \left( \frac{U}{d} \right) \left[ \frac{U}{e} \right] \log \left( \frac{U}{e} \right) \), \( U > 1 \), so that a continuous, monotonic and explicit version of Selberg’s sieve can be stated.

Thanks to Barban–Vehov (1968), Motohashi (1974) and Graham (1978), it has been long known, but never explicitly, that \( \Sigma_1(U) \) is asymptotic to \( \log(U) \). In this article, we discover not only that \( \Sigma_q(U) \sim q \phi(q) \log(U) \) for all \( q \in \mathbb{Z}^+ \), but also we find a closed-form expression for its secondary order term of \( \Sigma_q(U) \), a constant \( s_q \), which we are able to estimate explicitly when \( q = v \in \{1, 2\} \). We thus have \( \Sigma_v(U) = v \phi(v) \log(U) - s_v + O_v \left( \frac{1}{U^{\frac{1}{3}}} \right) \), for some explicit constant \( K_v > 0 \)

and \( s_1 = 0 \).

As an application, we show how our result gives an explicit version of the Brun–Titchmarsh theorem within a range.

1 Notation and basic definitions

Throughout the present work the variable \( p \) denotes a prime number, \( q \) denotes an arbitrary positive integer and the function \( X > 0 \mapsto \log^+(X) \) corresponds to \( \max \{ \log(X), 0 \} \). We also use the \( O^* \) notation: we write \( f(X) = O^*(h(X)) \), as \( X \to a \) to indicate that \( |f(X)| \leq h(X) \) in a neighborhood of \( a \), where, in absence of precision, \( a \) corresponds to \( \infty \). Finally, we consider the Euler \( \varphi \) and Kappa \( \kappa \) functions: let \( s \) be any complex number, we define \( \varphi_s : \mathbb{Z}^+ \to \mathbb{C} \) as \( q \mapsto q^s \prod_{p \mid q} \left( 1 - \frac{1}{p^s} \right) \) and \( \kappa_s : \mathbb{Z}^+ \to \mathbb{C} \) as \( q \mapsto q^s \prod_{p \mid q} \left( 1 + \frac{1}{p^s} \right) \).

2 Introduction

Let \( U_1 > 1 \) and \( U_0 > 0 \) such that \( U_1 > U_0 \). Consider the Barban–Vehov weights \( d \in \mathbb{Z}^+ \mapsto L_d = \log^+ \left( \frac{U_1}{d} \right) - \log^+ \left( \frac{U_0}{d} \right) \); \( d \mapsto L_d \) is continuous on \( (0, \infty) \) and satisfies \( \frac{L_1}{\log(U_1)} = 1 \) and \( \frac{L_d}{\log(U_1)} = 0 \) for all \( d \geq U_1 \). Therefore \( \left\{ \frac{L_d}{\log(U_1)} \right\}_{d=1}^\infty \) is a sequence of

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parameters as in Selberg’s sieve that are also continuous and monotonic. In particular, if \( U_1 > 2 \), then the sum

\[
\sum_{d \leq U} \left( \sum_{d \mid n} \frac{\mu(d) L_d}{\log(U)} \right)^2
\]

sifts the prime numbers in the interval \([U_1, X]\), where \( v \in \{1, 2\} \).

Our main motivation is to give a continuous and monotonic version of the Selberg sieve, which, by stating it explicitly, has important consequences, as Theorem 2 below. Here, the condition \( U_1^2 \leq X \) is mandatory. However, it is significant that Graham, in [8], carries out a non-explicit asymptotic analysis for the sum (2.1) that considers not only the case \( U_1^2 \leq X \) but also \( X \leq U_1^2 \) (see [8] §4).

Sums like the one in (2.1) have been studied non-explicitly, with \( \frac{1}{\log(U)} \) replaced by \( d \mapsto \mathbb{1}_{(d \leq U)}(d) \), by Dress, Iwaniec and Tenenbaum in [5] and recently by de la Bretèche, Dress and Tenenbaum in [3]. In this case, the analogous main term coefficient \( \sum_{d \leq U} \frac{\mu(d) \mu \ast (e)}{|[d, e]|} \) converges to a positive constant whose rigorous estimation has been given by Helfgott in [9, Prop. 6.30], showing that \( \sum_{d \leq U} \frac{\mu(d) \mu \ast (e)}{|[d, e]|} = 0.440729 + O^*(0.0000213) \).

In this article, we study the asymptotic expression of (2.1) with a particular choice of Barban–Vehov weights, namely, \( U_1 = U > 1 \) and \( U_0 = 1 \), so that \( L_d \) becomes the one parameter logarithmic weight \( d \mapsto \log^+(\frac{U}{d}) \). With this choice, in our main result, we conclude in an explicit manner not only that the main term coefficient \( \sum_{d \leq U} \frac{\mu(d) \mu \ast (e)}{|[d, e]|} L_d \) of (2.1) is asymptotic to \( \frac{q}{\varphi(q)} \log(U) \) for all \( q \in \mathbb{Z}_{>0} \), as shown in [8] without coprimality conditions, but we are also able to obtain its second order term, which is a constant value for all \( q \in \mathbb{Z}_{>0} \). It is stated in [5] and reads as follows.

**Theorem 1.** Let \( U > 1 \). Then for all \( q \in \mathbb{Z}_{>0} \), one can determine explicit constants \( s_q \) and \( K_q > 0 \) such that

\[
\sum_{d \leq U} \frac{\mu(d) \mu \ast (e)}{|[d, e]|} L_d L_e = \frac{q}{\varphi(q)} \log(U) - s_q + O_q^* \left( \frac{K_q}{\log(U)} \right).
\]

In particular,

\[
s_v = \begin{cases} 
0.60731 \ldots & \text{if } v = 1, \\
1.4728 \ldots & \text{if } v = 2,
\end{cases}
\]

and, if \( U \geq 10^{\frac{21}{13}} \), we can select \( K_1 = 12.2109 \), \( K_2 = 3.7903 \).

The error term magnitude in the theorem above has been obtained by resting on modern estimations given by Balazard [1], Bordelès [3], El-Marraki [7], Helfgott [9] §6 and Ramaré [17], [18], among others; nonetheless, when using non-explicit tools, one should expect an error of magnitude \( e^{-w \sqrt{\log(U)}} \) for some constant \( w > 0 \). Furthermore, in order to derive Theorem 1 one must differ from a classical prime-number-theorem-like approach (as in [8], for example), as it gives ineffective or well explicit estimations that involve huge numbers, thus too inconvenient or impractical to be used. For this reason, in order to succeed, we introduce in [5] some averaging functions involving the Möbius function, from which we can derive explicit estimations that are crucial for the results given in sections §1 and §3.

Our work is general enough to consider any coprimality condition, and not the specific ones \( q = v \in \{1, 2\} \) that we have chosen to work with to derive explicit constants. Hence, we are able to derive Theorem 5.1 for any \( q \in \mathbb{Z}_{>0} \), provided that the specific constant \( K_q \) is found. In that case, given its closed-form and with the help of analogous results to propositions 8.3 and 8.8 any constant \( s_q \) with \( q > 2 \) may be rigorously estimated.
Subsequently, closely following [11 §3.2], Theorem 1 can be applied to derive an explicit version of the Brun–Titchmarsh theorem within a range, by providing a uniform upper bound for the number of primes in arithmetic progressions. That is, if we consider the quantity \( \pi(X; q, a) = \{ p \leq X, p \equiv a \pmod{q} \} \), we have the following.

**Theorem 2 (Brun–Titchmarsh inequality).** Let \( a, q \in \mathbb{Z}_{>0} \) such that \( (a, q) = 1 \). Let \( Y \) be a real number such that \( Y \geq 10^{29}q \). Then for all \( X \geq 0 \),

\[
\pi(X + Y; q, a) - \pi(X; q, a) \leq \frac{2Y}{\varphi(q) \log \left( \frac{Y}{q} \right)} \left( 1 - \frac{0.0271}{\log \left( \frac{Y}{q} \right)} \right).
\]

### 3 Bounds on functions involving the Möbius function

In §3 we will need to reduce our estimations to some well-known and simpler functions. For any \( X > 0 \), define \( m_q(X) = \sum_{n \leq X} \frac{\mu(n)}{n} \) and consider

\[
\hat{m}_q(X) = \sum_{n \leq X, (n, q) = 1} \frac{\mu(n)}{n} \log \left( \frac{X}{n} \right), \quad \hat{m}_q(X) = \sum_{n \leq X, \frac{n}{q} < X} \frac{\mu(n)}{n} \log^2 \left( \frac{X}{n} \right),
\]

\[
m_q(X) = \sum_{n \leq X, (n, q) = 1} \frac{\mu(n)}{n} \log \left( \frac{X}{n} \right), \quad \check{m}_q(X) = \sum_{n \leq X, \frac{n}{q} < X} \frac{\mu(n)}{n} \log^2 \left( \frac{X}{n} \right).
\]

It is straightforward to see by summation by parts that \( \check{m}_q \) and \( \hat{m}_q \), as well as \( m_q \) and \( \check{m}_q \), are related by the following identity.

**Lemma 3.1.** Let \( X \geq 1 \). We have

\[
\int_1^X \hat{m}_q(s) \frac{ds}{s} = \frac{1}{2} \hat{m}_q(X), \quad \int_1^X \check{m}_q(s) \frac{ds}{s} = \frac{1}{2} \check{m}_q(X).
\]

Moreover, we have the following explicit estimations

\[
|\hat{m}_q(X) - 1| \leq \frac{1}{\sqrt{X}}, \quad \text{if } 0 < X \leq 10^{12} \quad \square \text{ Lemma 5.9},
\]

\[
|\hat{m}_q(X) - 2 \log(X) + 2\gamma| \leq \frac{4e^{-\gamma}}{\sqrt{X}}, \quad \text{if } 0 < X \leq 10^{12} \quad \square \text{ Lemma 5.9},
\]

\[
|\check{m}_q(X) - 2 \log(X) + 2\gamma| \leq \frac{1}{103 \log(X)}, \quad \text{if } X \geq 9 \quad \square \text{ Thm. 1.8},
\]

where the first bound in each case has been obtained with the help of computer calculations using an implementation of interval arithmetic.

Consider now \( q, d \in \mathbb{Z}_{>0} \). We write \( d \mid q^\infty \), meaning that \( d \) is in the set \( \{ d', p \mid d' \Rightarrow p \mid q \} \). If \( n \in \mathbb{Z}_{>0} \), then \( (q^\infty, n) \) is the greatest divisor \( d' \) of \( n \) such that \( d' \mid q^\infty \). Therefore \( (q^\infty, n) = 1 \) if and only if \( (n, q) = 1 \); otherwise, \( (q^\infty, n) \) and \( (n, q) \) may differ. With this definition, and by using the following identity, established for example throughout [8, Lemma 2] and in [9, Eq. (5.72)], one can study a sum with coprimality conditions from the same sum without such conditions.

**Lemma 3.2.** We have the identity \( \sum_{d \mid q^\infty, \frac{n}{d} \mid n} \mu \left( \frac{n}{d} \right) = \mu(n) \mathds{1}_{\{(n, q) = 1\}}(n) \). Hence, for any function \( h : \mathbb{Z}_{>0} \to \mathbb{C} \), we have the formal identity

\[
\sum_{(n, q) = 1} \frac{\mu(n)}{n} h(n) = \sum_{d \mid q^\infty} \frac{1}{d} \sum_{n} \frac{\mu(n)}{n} h(dn).
\]

Lemma 3.3 is meaningful since, as Ramaré points out in [17, §1], on using merely a Möbius inversion \(\sum_{d\mid q, d\mid n} \mu(d) = 1 \times I_{\{(n, q)=1\}}(n)\) in (3.3), one would have been taken back to a sum having, again, coprimality conditions. For example, with the help of Lemma 3.2, we have

**Lemma 3.3.** Let \(X > 0\) and \(\theta = 1 - \frac{1}{\log(10^{12})}\). Then

\[
\left| \hat{m}_q(X) - \frac{q}{\varphi(q)} \right| \leq \frac{\sqrt{q}}{\varphi\left(\frac{q}{2}\right)} \frac{1}{\sqrt{X}} + \frac{q^\theta}{\varphi(q)} \frac{I_{\{X \geq 10^{12}\}}(X)}{389 \log(X)}.
\]

The proof of the Lemma 3.3 is given by Helfgott in [9, Prop 5.15]; although it is given in the range \(X \geq 1\), it is not difficult to derive it for any \(X > 0\) as long as the term of order \(\frac{1}{\log(X)}\) is considered for sufficiently large values of \(X\). This proof consists on finding a way to put together the bounds (3.3) so that resulting estimation comes from a direct application of identity (3.3); it is indeed very convenient to have general inequalities that prevent us from splitting a summation into many ranges that are not in general simple to handle on their own. Inspired by this remark, we derive our first result, that will help us to further understand the estimations given in lemmas 3.1 and 3.2.

**Lemma 3.4.** Let \(X > 0\) and \(\theta = 1 - \frac{1}{\log(10^{12})}\). Define \(f_q : X \geq 1 \mapsto \log(X) - \gamma - \sum_{p\mid q} \frac{\log(p)}{p-1}\). Then

\[
\left| \hat{m}_q(X) - \frac{2q}{\varphi(q)} f_q(X) \right| \leq \frac{\sqrt{q}}{\varphi\left(\frac{q}{2}\right)} \frac{4e^{\frac{s}{2}} - 1}{\sqrt{X}} + \frac{q^\theta}{\varphi(q)} \frac{I_{\{X \geq 10^{12}\}}(X)}{103 \log(X)}.
\]

**Proof.** By taking \(h(n) = (\max \{\log\left(\frac{n}{q}\right), 0\})^2\) in Lemma 3.2, we derive

\[
\hat{m}_q(X) = \sum_{d \mid q^\infty} \frac{1}{d} \sum_{n \leq X} \frac{\mu(n)}{n} \log^2 \left(\frac{X}{dn}\right) = \sum_{d \mid q^\infty} \frac{1}{d} \hat{m}\left(\frac{X}{d}\right).
\]

Consider the Dirichlet series \(\sum_{d \mid q^\infty} \frac{1}{d^s}\); it converges to \(\frac{q^s}{\varphi_s(q)}\) for all \(s \in \mathbb{C}\) such that \(\Re(s) > 0\). Subsequently, we can differentiate it to obtain

\[
- \sum_{d \mid q^\infty} \frac{\log(d)}{d} = \left(\sum_{d \mid q^\infty} \frac{1}{d^s}\right)' = \left(\frac{q^s}{\varphi_s(q)}\right)' = \left(\frac{q^s}{\varphi_s(q)} \sum_{p \mid q} \frac{-\log(p)}{p^s - 1}\right)_s = \left(\frac{q^s}{\varphi_s(q)} \sum_{p \mid q} \frac{-\log(p)}{p^s - 1}\right)_{s=1}.
\]

On the other hand, by combining the bounds given in (3.3), we have that

\[
\left| \sum_{d \mid q^\infty} \frac{1}{d} \left( \hat{m}\left(\frac{X}{d}\right) - 2 \log \left(\frac{X}{d}\right) + 2\gamma \right) \right| \leq \frac{1}{\sqrt{X}} \sum_{d \mid q^\infty} \frac{4e^{\frac{s}{2}} - 1}{\sqrt{d}} + \sum_{d \mid q^\infty} \frac{I_{\{X \geq 10^{12}\}}(d)}{103 d \log\left(\frac{d}{X}\right)}
\]

\[
\leq \frac{\sqrt{q}}{\varphi\left(\frac{q}{2}\right)} \frac{4e^{\frac{s}{2}} - 1}{\sqrt{X}} + \frac{I_{\{X \geq 10^{12}\}}(X)}{103 \log(X)} \sum_{d \mid q^\infty} \frac{1}{d^\theta} = \frac{\sqrt{q}}{\varphi\left(\frac{q}{2}\right)} \frac{4e^{\frac{s}{2}} - 1}{\sqrt{X}} + \frac{q^\theta}{\varphi(q)} \frac{I_{\{X \geq 10^{12}\}}(X)}{103 \log(X)}
\]

where, recalling the definition of \(\theta\), we have used that the function \(d \mapsto \frac{1}{d^{-\theta} \log\left(\frac{d}{X}\right)}\) is decreasing for \(1 \leq d \leq \frac{X}{10^{12}}\). We conclude the result from (3.7), by identifying the following identity

\[
\sum_{d \mid q^\infty} \frac{1}{d} \left( \log \left(\frac{X}{d}\right) - \gamma \right) = \frac{q}{\varphi(q)} \left( \log(X) - \gamma - \sum_{p \mid q} \frac{\log(p)}{p - 1} \right),
\]

that can be replaced in the leftmost expression of inequality (3.8).  \(\square\)
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Notice that (3.5) is valid since the bounds (3.1) hold regardless of whether or not \( \frac{X}{\theta} \geq 1 \); this is the reason why we do not incorporate a range condition on the variable \( d \) in the outer sum of the expression (3.9) and also why the values of \( X \) and \( d \) remain independent.

We provide now the main term of the function \( X \mapsto \tilde{m}_q(X) \). It might come as a surprise that \( a_q \), defined below, is closely related to the function that optimizes the Selberg sieve (refer to \([11, \S 3.2]\)) introduced in Lemma 4.7.

**Lemma 3.5.** Let \( X > 0 \) and \( \theta = 1 - \frac{1}{\log(10 + g)} \). Then

\[
\left| \tilde{m}_q(X) - \frac{2\zeta(2)\kappa(q)}{q} (\log(X) - a_q) \right| \leq p_{\frac{q}{2}}(q) \frac{4e^{\gamma} - 1}{\sqrt{X}} P_{\frac{q}{2}} \frac{P_0 \mathbb{1}_{X \geq 10^{12}}(X)}{103 \log(X)},
\]

where

\[ a_q = \sum_{p} \frac{\log(p)}{p(p-1)} + \gamma + \frac{\log(p)}{p}, \quad \sum_{p} \frac{\log(p)}{p(p-1)} + \gamma = 1.332 \ldots, \]

\[ p_{\alpha}(q) = \prod_{p \mid q} \left( 1 + \frac{p^{1-\alpha}}{p+1-p^{1-\alpha}} \right), \quad P_{\frac{q}{2}} = 3.575, \quad P_0 = 1.693. \]

**Proof.** Observe that for any square-free \( n, \alpha(n) = \sum_{d \mid n} \mu(d) \). Therefore

\[ \tilde{m}_q(X) = \sum_{1 \leq n \leq X} \mu(n) \log^2 \left( \frac{X}{n} \right) \sum_{d \mid n} \frac{\mu(d)}{\kappa(d)} = \sum_{(d,q) = 1} \frac{\mu^2(d)}{d \kappa(d)} \tilde{m}_{dq} \left( \frac{X}{d} \right). \]  

(3.10)

By (3.10), we are now able to derive the main term of \( \tilde{m}_q \left( \frac{X}{d} \right) \) in a similar manner to the obtainment of expression (3.5). Indeed, by using that \( X \) and \( d \) are independent variables, and with the help of Lemma 3.4, we have

\[
\sum_{(d,q) = 1} \frac{\mu^2(d)}{d \kappa(d)} \left[ \tilde{m}_{dq} \left( \frac{X}{d} \right) - 2dq \frac{\varphi(d) \log \left( \frac{X}{d} \right)}{\varphi(d)} \right] \leq \frac{4e^{\gamma} - 1}{\sqrt{X}} \sum_{(d,q) = 1} \frac{\mu^2(d)}{\varphi(d)} \varphi(\varphi(q)) \frac{1_{X \geq 10^{12}}(X)}{103} \sum_{(d,q) = 1} \frac{\mu^2(d)}{d^{1-\theta} \varphi(\varphi(q)) \log \left( \frac{X}{d} \right)} \leq p_{\frac{q}{2}}(q) \frac{4e^{\gamma} - 1}{\sqrt{X}} P_{\frac{q}{2}} \frac{P_0 \mathbb{1}_{X \geq 10^{12}}(X)}{103 \log(X)},
\]

where we have used that \( d \mapsto \frac{1}{d^{1-\theta} \log \left( \frac{X}{d} \right)} \) is decreasing for \( 1 \leq d \leq \frac{X}{10^6} \); thereafter, we have completed the above summations on the variable \( d \) to derive a convergent sum, obtaining, for \( \alpha \in \left\{ \frac{1}{2}, \theta \right\} \),

\[ p_{\alpha}(q) = \frac{q^\alpha}{\varphi(q)} \prod_{p \mid q} \left( 1 + \frac{1}{(p^\alpha - 1)(p+1)} \right)^{-1} = \prod_{p \mid q} \frac{p+1}{p+1-p^{1-\alpha}}, \]

\[ \prod_{p} \left( 1 + \frac{1}{(p^\alpha - 1)(p+1)} \right) \in \left\{ 3.574, P_{\alpha} \right\}, \quad \text{if } \alpha = \frac{1}{2}, \quad \left\{ 1.692, P_{\alpha} \right\}, \quad \text{if } \alpha = \theta. \]
Lemma 3.6. We do not bother to perform the involved calculations, since it is a result that has already been proved, by different means, in [9, Prop. 6.8]; it reads as follows.

\[
\sum_{d \mid (d,q)=1} \frac{\mu^2(d)}{\varphi(d)} \frac{\log(p)}{p-1} = \sum_{p \mid q} \frac{\log(p)}{(p-1)(p^2-1)} + \sum_{(e,pq)=1} \frac{\mu^2(e)}{\varphi(e)} \kappa(e) = \sum_{p \mid q} \frac{\log(p)}{p^2(p-1)}, \tag{3.11}
\]

where \(F_0(0) = \zeta(2) \frac{\kappa(q)}{q}\). Therefore, from (3.11), we derive

\[
\frac{2q}{\varphi(q)} \sum_{d \mid (d,q)=1} \frac{\mu^2(d)}{\varphi(d)} \kappa(d) \left( \log \left( \frac{X}{d} \right) - \gamma - \sum_{p \mid dq} \frac{\log(p)}{p-1} \right) = \frac{2q}{\varphi(q)} F_0(0) \left( \log(X) - \sum_{p \mid (p-1)} \frac{\log(p)}{p^2(p-1)} \right)
\]

Therefore, from (3.11), we derive

\[
\sum_{p} \frac{\log(p)}{p(p-1)} + \gamma \in [1.33258227221663, 1.3325822916778]. \tag{3.12}
\]

A similar treatment to (3.11) allows us to derive the main term of \(\tilde{m}_q(X)\) and thus Lemma 3.6. We do not bother to perform the involved calculations, since it is a result that has already been proved, by different means, in [9, Prop. 6.8]; it reads as follows.

**Lemma 3.6.** Let \(X > 0\) and \(\theta = 1 - \frac{1}{\log(10^2)}\). Then

\[
\left| \tilde{m}_q(X) - \frac{\zeta(2) \kappa(q)}{q} \right| \leq \frac{P_\pm}{\sqrt{X}} + \frac{P_\theta}{\log(X)},
\]

where \(P_\pm, P_\theta, P_\pm(\theta)\) and \(P_\theta(\theta)\) are defined in Lemma 3.5.

Consider now \(h_q : s \geq 1 \mapsto \sum_{d \mid (d,q)=1} \frac{\mu(d)}{\varphi(d)} \kappa(d) \left( \tilde{m}_q \left( \frac{s}{d} \right) - \frac{s^2}{6} \frac{\kappa(dq)}{dq} \right)^2 \). By expanding the square, we readily have (see [9, Lemma 6.10])

\[
\sum_{d \mid (d,q)=1} \frac{\mu(d)}{\kappa(d)} \frac{\mu(d)^2}{\varphi(d)} \frac{s^2}{d} \tilde{m}_q^2 \left( \frac{s}{d} \right) = h_q(s) + \frac{\pi^2 \kappa(q)}{3q} \tilde{m}_q(s) + \frac{\pi^2 \kappa(q)}{6 \varphi(q)}, \tag{3.13}
\]

so that, as the above left-hand side is a finite sum, the function \(h_q\) is well-defined. In order to estimate \(h_q\), we will need the following result. It is a general tool that helps in deriving the correct order of arithmetic averages that are weighted by suitable negative powers of logarithms. It will also be useful to analyze propositions 4.17 and 4.18.
Proposition 3.7. Let $Z, X, m, n$ be real numbers such that $m \geq 1$ and $1 \leq Z < X$. Then, a) if $m = 1$,

$$
\int_{1}^{Z} \frac{du}{u^{m} \log^{n} \left(\frac{X}{u} \right)} = \begin{cases} 
\log \left( \frac{\log(X)}{\log(\frac{X}{Z})} \right) & \text{if } n = 1, \\
\frac{1}{n-1} \left( \frac{1}{\log^{n-1}(\frac{X}{Z})} - \frac{1}{\log^{n-1}(\frac{X}{u})} \right) & \text{if } n \neq 1, 
\end{cases}
$$

b) if $m > 1$ and $n > 0$, then

$$
\int_{1}^{Z} \frac{du}{u^{m} \log^{n} \left(\frac{X}{u} \right)} \leq \frac{1}{m-1} \left( \frac{1}{\log^{n}(\frac{X}{Z})} + \frac{1}{\log^{n}(\frac{X}{u})} Z^{m-1} \right).
$$

Proof. If $m = n = 1$, we have $\int_{1}^{Z} \frac{du}{u^{m} \log^{n} (\frac{X}{u})} = \left[ - \log \left( \log \left( \frac{X}{u} \right) \right) \right]_{1}^{Z}$; if $m = 1$, and $n \neq 1$, we have $\int_{1}^{Z} \frac{du}{u^{m} \log^{n} (\frac{X}{u})} = \left[ \frac{1}{n-1} \log^{-n} \left(\frac{X}{u} \right) \right]_{1}^{Z}$, whence a).

With respect to b), if $n > 0$, the function $u \mapsto \log^{-n} \left(\frac{X}{u} \right)$ is increasing for $1 \leq u < X$ and since $m > 1$, for any $0 < k < 1$, in particular for $k = \frac{1}{2}$, we conclude that

$$
\int_{1}^{Z} \frac{du}{u^{m} \log^{n} \left(\frac{X}{u} \right)} \leq \frac{1}{m-1} \left( \frac{1}{\log^{n}(\frac{X}{Z})} + \frac{1}{\log^{n}(\frac{X}{u})} Z^{m-1} \right).
$$

Proposition 3.8. Let $q \in \mathbb{Z}_{>0}$. The integral $\int_{X}^{\infty} \frac{h_{q}(s)}{s} ds$ converges and defines a constant depending on $q$. Moreover, for any $X > 0$, we have the following tail order estimation

$$
\int_{X}^{\infty} \frac{h_{q}(s)}{s} ds = O_{q} \left( \frac{1}{\log(X)} \right).
$$

Proof. Given [20] Thm. 3.3], we can derive a more theoretical proof. By Lemma 3.6, for any $d \in \mathbb{Z}_{>0}$ with $(d, q) = 1$, the main term of $\tilde{h}_{dq}$ is $\frac{\mu^{2}(d)}{s \log^{2} \left( \frac{X}{q} \right)}$ and we have the bound

$$
|h_{q}(s)| \leq \sum_{d \leq s \atop (d, q) = 1} \frac{\mu^{2}(d)}{s \kappa(d)^{2}} \left( A_{q}^{d}(s)^{2} + 2A_{q}^{d}(s)B_{q}^{d}(s) + B_{q}^{d}(s)^{2} \right),
$$

where

$$
A_{q}^{d}(s) = P_{q}^{d}(dq) \frac{\sqrt{d}}{\sqrt{s}}, \quad B_{q}^{d}(s) = P_{q}^{d}(dq) \frac{1_{(s \geq 10^{12})}(d)}{389 \log \left( \frac{X}{q} \right)}.
$$

Observe that

$$
\sum_{d \leq s \atop (d, q) = 1} \frac{\mu^{2}(d)}{s \kappa(d)^{2}} A_{q}^{d}(s)^{2} = \frac{P_{q}^{2}(q)^{2}}{s} \sum_{d \leq s \atop (d, q) = 1} \frac{\mu^{2}(d)P_{q}(dq)^{2}d}{\kappa(d)^{2}} \leq \frac{a_{1}(q) \log(s) + a_{2}(q)}{s}, \quad (3.14)
$$

and, by using Proposition 3.7

$$
\sum_{d \leq s \atop (d, q) = 1} \frac{\mu^{2}(d)}{s \kappa(d)^{2}} B_{q}^{d}(s)^{2} = \frac{P_{q}^{2}(q)^{2}}{389} \sum_{d \leq s \atop (d, q) = 1} \frac{\mu^{2}(d)P_{q}(dq)^{2}d}{\kappa(d)^{2} \log^{2} \left( \frac{X}{q} \right)} \leq \frac{b_{q}}{\log^{2}(s)}, \quad (3.15)
$$
for some positive values $a_q^{(1)}$, $a_q^{(2)}$ and $b_q$ depending solely on $q$. From both estimations above, the sum $\sum_{(d,q)=1} d^{-\gamma} \sum_{s=1}^{\infty} A_q^d(s) B_q^d(s)$ can be bounded by Cauchy–Schwarz inequality, giving a term of order $O_q \left( \frac{1}{\sqrt{\log(s)}} \right)$. 

Finally, as $\int \frac{\log(s)+1}{s^2} ds = \frac{\log(s)+2}{s^2}$ and $\int \frac{ds}{s \log^2(s)} = \frac{1}{\log(s)}$, we derive from (3.14), (3.15) and Cauchy–Schwarz inequality for integrals that the integral $\int_{\infty}^{\infty} \frac{h_q(s)}{s^2} ds$ converges, and further that $\int_{\infty}^{\infty} \frac{h_q(s)}{s^2} ds = O_q \left( \frac{1}{\sqrt{\log(s)}} \right)$. 

In [6,7] we will need explicit estimations for $h_v$, $v \in \{1,2\}$. As predicted by the estimations [3.12, 3.14] and thanks to [6] Prop. 6.14 and [9] Prop. 6.17, we have 

**Proposition 3.9.** For any $s \geq 1$, $|h_v(s)|$, $v \in \{1,2\}$, is at most 

\[ T_v^{(s)} = \begin{cases} 
T_v^{(2)} = 3.83717, & \text{if } 1 \leq s \leq 10^{12}, \\
T_v^{(3)} = 4.89606, & \text{if } s \geq 10^{12}, \\
T_v^{(4)} = 0.000033536, & \text{if } s \geq 10^{12}, \\
T_v^{(5)} = 0.0000615022, & \text{if } s \geq 10^{12},
\end{cases} \]

where

\[ T_v^{(2)} = 3.83717, \quad T_v^{(3)} = 4.89606, \quad T_v^{(4)} = 0.000033536, \quad T_v^{(5)} = 0.0000615022. \]

## 4 A logarithmic sum involving the Möbius function

In order to start our analysis, let $U \geq 10^7$. Consider a parameter $1 < Z < U$ such that $\frac{Z}{U} \geq 20$ and $Z \geq 4 \times 10^9$, and write

\[
\sum_{d,e \ (d,e,v)=1} \mu(d)\mu(e) L_d L_e = \sum_{\ell \leq U} \frac{\mu^2(\ell)}{\ell} \sum_{r_1 \neq r_2} \sum_{\ell_1 \neq \ell_2} \sum_{(r_1, r_2) = 1} \mu(r_1) \mu(r_2) L_{\ell r_1} L_{\ell r_2} = S_1 + S_{II}.
\]

where

\[
S_1 = S_1(U, Z, v) = \sum_{\ell \leq Z} \frac{\mu^2(\ell)}{\ell} \sum_{(r_1, r_2) = 1} \mu(r_1) \mu(r_2) L_{\ell r_1} L_{\ell r_2},
\]

\[
S_{II} = S_{II}(U, Z, v) = \sum_{\ell \leq Z} \frac{\mu^2(\ell)}{\ell} \sum_{(r_1, r_2) = 1} \mu(r_1) \mu(r_2) L_{\ell r_1} L_{\ell r_2}.
\]

The reason why the above sums have been introduced is related to the obtainment of actual error terms, which otherwise fail to arise. Indeed, in order to deal with lower order terms, two different approaches are required; one for $S_1^{(v)}$ and another for $S_{II}^{(v)}$, neither of them being satisfactory when applied to both sums at once.

We first show an estimation for $S_1$, where the convergence of the integral below is assured by Proposition 3.3. Its proof is given in [6,7].

**Lemma 4.1.** Let $U \geq 10^7$ and $v \in \{1,2\}$. If $Z$ is a real number such that $\frac{Z}{U} \geq 20$, then

\[
S_1 = \frac{6v}{\pi^2 K(v)} \int_1^{\infty} \frac{h_v(s)}{s} ds + \tilde{m}_v \left( \frac{U}{Z} \right) - \frac{v}{\varphi(v)} \log \left( \frac{U}{Z} \right) + O^*(\left( \frac{T_v^{(4)}}{\sqrt{Z}} \right) + \frac{6v}{\pi^2 K(v)} \left( \Psi_v \left( \frac{Z \log \left( \frac{Z}{U} \right)}{U} + \frac{Z}{U} + \frac{2T_v^{(4)}}{\log \left( \frac{Z}{U} \right)} \right) \right),
\]

where $\Psi_v(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{\Gamma(s+1)} x^s ds$ and $\Gamma(s)$ is the Gamma function.
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where \( T_v^{(1)} = \begin{cases} 11.188 & \text{if } v = 1, \\ 0.802 & \text{if } v = 2, \end{cases} \quad \Psi_v = \left( T_v^{(2)} + \frac{T_v^{(3)}}{\log(20)} \right), \) and

\( T_v^{(2)}, T_v^{(3)} \) and \( T_v^{(4)} \) are defined in Proposition 3.3.

On the other hand, by M"{o}bius inversion, and recalling 3.11, we can write

\[ S_H = \sum_{\ell \leq Z} \frac{\mu^2(\ell)}{\ell} \sum_{d \leq Z} \frac{\mu(d)}{d^2} \hat{m}_{\ell dv} \left( \frac{U}{\ell d} \right)^2. \tag{4.4} \]

As the main term of \( X \mapsto \hat{m}_{\ell dv}(X) \) is \( \frac{\ell dv}{\varphi(\ell dv)} \), we use Lemma 3.3 to derive three more summations. Namely, we write \( S_H = 2\psi^{(1)}_H - S^{(2)}_H + \psi^{(3)}_H \), where

\[ S^{(1)}_H = \frac{v}{\varphi(v)} \log(U) - \frac{1}{2} \hat{m}_v \left( \frac{U}{Z} \right) + O^* \left( \frac{\Psi_v^1 \log^2 \left( \frac{U}{Z} \right)}{\sqrt{Z}} \right), \]

where

\[ \Psi_v^1 = \begin{cases} 9.416 & \text{if } v = 1, \\ 3.109 & \text{if } v = 2. \end{cases} \]

Additionally, by Lemma 3.3 we can replace \( \hat{m}_v \left( \frac{U}{Z} \right) \) in lemmas 3.1 and 1.2 by \( \frac{2}{\varphi(v)} \left( \log \left( \frac{U}{Z} \right) - \gamma - \sum_{p|v} \frac{\log(p)}{p-1} \right) \)

\[ + O^* \left( \frac{\sqrt{Z}}{\varphi^{(1)}(q)} \frac{1}{\sqrt{1 + \log(Z)}} + \frac{\varphi^0}{\varphi(v)} \frac{1}{103 \log \left( \frac{U}{Z} \right)} \right). \]

Lemma 4.2. Let \( U \geq 10^7, 0 < Z < U \) such that \( \frac{U}{Z} \geq 20 \) and \( v \in \{1, 2\} \). We have that

\[ S^{(1)}_H = \frac{v}{\varphi(v)} \log(U) - \frac{1}{2} \hat{m}_v \left( \frac{U}{Z} \right) + O^* \left( \frac{\Psi_v^1 \log^2 \left( \frac{U}{Z} \right)}{\sqrt{Z}} \right), \]

where

\[ \Psi_v^1 = \begin{cases} 9.416 & \text{if } v = 1, \\ 3.109 & \text{if } v = 2. \end{cases} \]

Lemma 4.3. Let \( 1 < Z < U \) such that \( \frac{U}{Z} \geq 20 \) and \( Z \geq 4 \times 10^5 \). Then

\[ S^{(2)}_H = \frac{v}{\varphi(v)} \left( \log(Z) + \gamma + \sum_{p|v} \frac{\log(p)}{p-1} \right) + O^* \left( \frac{\Psi_v^2 \sqrt{Z}}{U} + \frac{\Psi_v^3 Z}{U} \right), \]

where

\[ \Psi_v^2 = \begin{cases} 2.394 & \text{if } v = 1, \\ 6.771 & \text{if } v = 2, \end{cases} \quad \Psi_v^3 = \begin{cases} 4.634 & \text{if } v = 1, \\ 1.888 & \text{if } v = 2. \end{cases} \]

With respect to \( Z \), it will be clear in 3.4 and 47 why we will end up selecting \( Z = cU^{\frac{2}{3}} \), with \( c \in \{10, 16, 70\} \). With this choice, we will deduce the following.
Lemma 4.4. Let \( v \in \{1, 2\} \). We have the following estimation
\[
|S_3^{(3)}| \leq \frac{\Upsilon_v^{(4)} \log(U)}{3U^{\frac{1}{8}}} + \frac{\Upsilon_v^{(4)}}{U^{\frac{1}{8}}} + \frac{1_{\{U \geq 10^{12}\}}(U)\Upsilon_v^{(5)}}{\log(U)},
\]
where
\[
\Upsilon_v^{(4)} = \begin{cases} 
305.594 & \text{if } v = 1, \\
74.712 & \text{if } v = 2,
\end{cases}
\]
if \( U \geq 10^7 \) and \( Z = 10U^{\frac{1}{2}} \),
\[
\Upsilon_v^{(4)} = \begin{cases} 
0.0402 & \text{if } v = 1, \\
0.012 & \text{if } v = 2,
\end{cases}
\]
if \( U \geq 10^{12} \) and \( Z = 16U^{\frac{1}{2}} \),
\[
\Upsilon_v^{(5)} = \begin{cases} 
0.0434 & \text{if } v = 1, \\
0.0128 & \text{if } v = 2,
\end{cases}
\]
if \( U \geq 10^{12} \) and \( Z = 70U^{\frac{1}{2}} \),
\[
\Upsilon_v^{(5)} = \begin{cases} 
0.0612 & \text{if } v = 1, \\
0.0177 & \text{if } v = 2.
\end{cases}
\]

Lemma 4.5. Let \( X > 0 \). The following estimation holds.
\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} = \frac{q}{\pi(q) \pi^2} \left( \log(X) + b_q \right) + O^* \left( \sqrt{\frac{X}{\psi}} \frac{1.044 \prod_{p \mid q} \Phi(p) 0.223}{\sqrt{X}} \right),
\]
where
\[
b_q = \sum_p \frac{2 \log(p)}{p^2 - 1} + \gamma + \sum_{p \mid q} \frac{\log(p)}{p + 1}, \quad \sum_p \frac{2 \log(p)}{p^2 - 1} + \gamma = 1.717\ldots
\]

Proposition 4.6. Let \( X \geq 10 \) and \( v \in \{1, 2\} \). Then
\[
\frac{1}{\log^2(X)} \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\sqrt{\ell} \varphi^\frac{3}{2}(\ell)} \log \left( \frac{X}{\ell} \right) \leq \psi_v = \begin{cases} 
1.426, & \text{if } v = 1, \\
0.712, & \text{if } v = 2.
\end{cases}
\]

Proof. Let \( q \in \mathbb{Z}_{>0} \). By applying \([20]\) Thm. 3.3] with \( f(p) = \frac{1}{\sqrt{\varphi(p)}} = \frac{1}{\sqrt{\psi(p)}} \), \( \alpha = 1, \beta = \frac{3}{2} \) and \( 0 \leq \delta = \frac{1}{3} < \frac{1}{2} \), we obtain that
\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\sqrt{\ell} \varphi^\frac{3}{2}(\ell)} = k_q F \left( \log(X) + f_q \right) + O^* \left( \frac{1}{X^{\frac{1}{8}}} \right),
\]
where

\[ k_q = \prod_{p \neq q} \left( 1 - \frac{p+1}{p^2 - p^2 + \sqrt{p} + 1} \right), \quad l_q = \prod_{p \neq q} \left( 1 + \frac{p^2 - 2p + 1}{p^2 - p^2 + p + 1} \right), \]

\[ f_q = -\sum_p \left( \frac{\sqrt{p} - 2}{p - \sqrt{p} + 1} \right) \log(p) + \gamma + \sum_p \log(p), \]

\[ F = \zeta(\frac{4}{3}) \in [2.173, 2.174], \quad -\sum_p \left( \frac{\sqrt{p} - 2}{p - \sqrt{p} + 1} \right) \log(p) + \gamma \in [0.367, 0.37], \]

\[ f = \Delta_L \prod_p \left( 1 + \frac{p^2 + 1}{p^2 (\sqrt{p} - 1)} \right) \in [46.722, 53.877]. \]

Therefore, when \( q = v \in \{1, 2\} \) and \( X \geq C = 10^7 \), we derive from (4.9) that

\[ \sum_{\ell \leq X} \mu^2(\ell) \log \left( \frac{X}{\ell} \right) \text{ can be expressed as} \]

\[ \int_1^X \left( k_q F (\log(t) + f_v) + O^* \left( \frac{1}{\log(C)} \right) \right) \frac{dt}{t \log^2(X)} - 2.174 k_v \left( \frac{1}{2} + \frac{1}{\log(C)} \right) + 161.63 \left( \frac{1}{\log(C)} \right) = \begin{cases} 1.344 & \text{if } v = 1, \\ 0.218 & \text{if } v = 2. \end{cases} \]

where we have used that \( f_q > 0 \) for all \( q \in \mathbb{Z}_{>0} \). On the other hand, for all \( 10 \leq X \leq 10^7 \),

\[ \frac{1}{\log^2(X)} \times \sum_{\ell \leq X} \mu^2(\ell) \log \left( \frac{X}{\ell} \right) \leq \begin{cases} 1.426 & \text{if } v = 1, \\ 0.712 & \text{if } v = 2. \end{cases} \]

The result is concluded by defining \( \psi_v \) as the maximum between the bounds given in (4.10) and (4.11).

**Proof of Lemma 4.1.** Conditions \( \ell r_i \leq U \) and \( Z < \ell \) imply that \( r_i \leq \frac{U}{\ell} \) for \( i = 1, 2 \).

Therefore, from definition (4.2), we derive

\[ S_l = \sum_{r_1, r_2 \leq \frac{U}{\ell}} \mu(r_1) \mu(r_2) \sum_{Z < \ell \leq U} \mathcal{L}_{Z_{\ell v}} \mathcal{L}_{r_1 r_2}. \]

On the other hand, with the help of Lemma 4.1, for any \( t > Z \), we have that

\[ A_q(t) = \sum_{Z < \ell \leq t} \mu^2(\ell) \frac{q}{\ell} \kappa(q) \log \left( \frac{t}{Z} \right) + O^* \left( \left( \frac{1}{\sqrt{q}} \frac{2.088}{\sqrt{Z}} \right) \right). \]

Moreover, by considering a monotone continuous function \( L^* \) on \([1, U]\), such that \( L^* (U) = 0 \), as \( L^* \) is of bounded variation, we can apply summation by parts and derive

\[ \sum_{Z < \ell \leq U} \frac{\mu^2(\ell)}{\ell} L^*(\ell) = - \int_Z^U A_q(t) dL^*(t) \]

\[ = \frac{6}{\pi^2} \frac{q}{\kappa(q)} \int_Z^U \frac{L^*(t)}{t} dt + O^* \left( \frac{2.088}{\sqrt{Z}} \frac{0.223}{\sqrt{q}} \right), \]

(4.13)
since \([A_q(t)L^*(t)]_Z^U = \left[ \frac{\log \left( \frac{U}{Z} \right)}{U} \right] L^*(t)_Z^U = 0\) and \(f_U^U dL^*(t) = -L^*(Z)\).

In particular, by taking \(L^*(t) = I_{r_1} L_{r_2} = \log^2 \left( \frac{U}{r_1} \right) \log^2 \left( \frac{U}{r_2} \right)\), with \(r_1, r_2 \leq \frac{U}{Z}\), we have a monotone decreasing function on \((0, \infty)\), thus of bounded variation, such that \(L_{U/r_1} L_{U/r_2} = 0\) and \(L^*(Z) = L_{Z/r_1} L_{Z/r_2} = \log \left( \frac{U}{Z/r_1} \right) \log \left( \frac{U}{Z/r_2} \right)\). Further, with this choice of \(L\), by taking \(q = r_1 r_2 v\) and replacing \((4.13)\) into the innermost summation of \((4.12)\), \(S_1\) equals

\[
\sum_{r_1, r_2 \leq \frac{U}{Z}} \frac{\mu(r_1) \mu(r_2)}{r_1 r_2} \left( \frac{6}{\pi^2} \kappa(r_1 r_2) \int_{Z}^{U} L_{r_1} L_{r_2} \frac{dt}{t} \right) + O^* \left( \frac{2.088 \prod_{r \mid r_1 r_2} 0.223 \sqrt{r_1 r_2 v} \log \left( \frac{U}{Z/r_1} \right) \log \left( \frac{U}{Z/r_2} \right)}{\varphi \left( \frac{U}{Z/r_1} \right) \sqrt{Z}} \right) = \frac{6}{\pi^2} \kappa(v) \int_{Z}^{U} \sum_{r_1, r_2 \leq \frac{U}{Z}} \frac{\mu(r_1) \mu(r_2)}{r_1 r_2} \kappa(r_1 \kappa(r_2)}{t} dt + r_v(U),
\]

where condition \(r_1 \leq \frac{U}{Z}\) above is encoded by the definition of \(L_{r_i}, i \in \{1, 2\}\), and by the range of \(t\). The remainder term \(r_v\) can be estimated by defining \(Q_v : X \mapsto \sum_{r \leq X} \frac{\mu(r)}{r^2} \log \left( \frac{X}{r} \right)\) and using Proposition 4.6 as follows

\[
|r_1(U)| \leq \frac{2.088}{\sqrt{Z}} \left( \frac{2 \times 0.222 \sqrt{2 \sqrt{2} - 1} Q_1 \left( \frac{U}{Z} \right) Q_1 \left( \frac{U}{Z} \right) + Q_1^4 \left( \frac{U}{Z} \right) \right) \leq \frac{11.188 \log^4 \left( \frac{U}{Z} \right)}{\sqrt{Z}} = T_1^4 \left( \frac{U}{Z} \right),
\]

\[
|r_2(U)| \leq \frac{0.464 \sqrt{Z}}{\varphi \left( \frac{U}{Z} \right)} Q_2 \left( \frac{U}{Z} \right) \leq \frac{0.802 \log^4 \left( \frac{U}{Z} \right)}{\sqrt{Z}} = T_2^4 \left( \frac{U}{Z} \right),
\]

where we have used that \(\frac{U}{Z} \geq 10\).

With respect to the main term of \(S_1\) given in \((4.13)\), recall the function \(\tilde{m}\) defined in \((3.16)\) and observe that

\[
\int_{Z}^{U} \sum_{r_1, r_2} \frac{\mu(r_1) \mu(r_2)}{r_1 r_2} \frac{L_{r_1} L_{r_2}}{t} dt = \int_{1}^{U} \sum_{d, v \mid r_1, r_2} \frac{\mu(d)}{\kappa(d)^2} \tilde{m}_d^2 \left( \frac{U}{d} \right) \frac{ds}{s},
\]

where the change of variables \(s = \frac{U}{d}\) has been performed, which is valid by Lemma 3.1

Hence, by combining Lemma 3.1 and equations \((3.13)\), \((4.17)\), \(\int_{Z}^{U} \sum_{r_1, r_2} \frac{\mu(r_1) \mu(r_2)}{r_1 r_2} L_{r_1} L_{r_2} dt\) equals

\[
\int_{1}^{U} \frac{h_v(s)}{s} ds + \frac{\pi^2 \kappa(v)}{6 v} \tilde{h}_v \left( \frac{U}{Z} \right) - \frac{\pi^2 \kappa(v)}{6 \varphi(v)} \log \left( \frac{U}{Z} \right).
\]

Thus, by recalling \((4.13)\), \(S_1\) may be expressed as

\[
\frac{6}{\pi^2} \frac{v}{\kappa(v)} \int_{1}^{U} \frac{h_v(s)}{s} ds + \tilde{h}_v \left( \frac{U}{Z} \right) - \frac{v}{\varphi(v)} \log \left( \frac{U}{Z} \right) + O^* \left( \frac{T_1^4 \log^4 \left( \frac{U}{Z} \right)}{\sqrt{Z}} \right).
\]
4.2 The sum $S^n_1$

By Proposition 3.8, $\int_1^\infty \frac{\mu(s)}{s} ds$ converges. Therefore, from Equation (4.18), we may write

$$S^n_1 = \frac{6\nu}{\pi^2 \kappa(v)} \int_1^\infty \frac{\mu(s)}{s} ds + \frac{\sum_{\ell \leq X}}{\phi(v)} \left( \frac{U}{Z} \right) - \frac{v}{\phi(v)} \log \left( \frac{U}{Z} \right) + O^* \left( \frac{T_v^{(1)} \log^4 \left( \frac{U}{Z} \right)}{\sqrt{Z}} + \frac{6\nu}{\pi^2 \kappa(v)} \int_1^\infty \frac{|h_v(s)|}{s} ds \right).$$

(4.19)

Now, by Proposition 3.9, whenever $10^{12} \leq \frac{U}{Z}$, we have

$$\left| \int_1^\infty \frac{h_v(s)}{s} ds \right| \leq \int_1^\infty \frac{T_v^{(4)}}{s \log^4(s)} ds = \frac{T_v^{(4)}}{\log \left( \frac{U}{Z} \right)},$$

(4.20)

whereas, if $20 \leq \frac{U}{Z} \leq 10^{12}$, we have

$$\left| \int_1^\infty \frac{h_v(s)}{s} ds \right| \leq \Psi_v \left( \frac{Z \log \left( \frac{U}{Z} \right)}{U} - \frac{Z}{U} \right) + \frac{T_v^{(4)}}{\log \left( \frac{U}{Z} \right)},$$

(4.21)

where $\Psi_v = \left( T_v^{(2)} + \frac{T_v^{(3)}}{\log \left( 2^{247} \right)} \right)$, by using that $\int_1^\infty \frac{\log(s)}{s} ds = -\frac{\log(s)+1}{s}$.

Finally, from (4.20), (4.21) and the definitions of $T_v^{(1)}, T_v^{(2)}, T_v^{(3)}, T_v^{(4)}$ and $\Psi_v$, we derive the result. \qed

4.2 The sum $S^n_1$

By \cite{20} Lemma 4.7, we have that

**Lemma 4.7.** Let $X > 0$. The following estimation holds

$$\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\phi(\ell)} = \frac{\varphi(q)}{q} \left( \log(X) + a_q \right) + O^* \left( \frac{A_q \prod_{p \nmid q} q^{0.493}}{\sqrt{X}} \right),$$

(4.22)

where $a_q$ is defined as in Lemma 3.5 and $A_q = \prod_{p \nmid q} \left( 1 + \frac{p-2}{p^{\frac{p-1}{2}} \sqrt{p+2}} \right)$.

**Proposition 4.8.** Let $X \geq 10$ and $v \in \{1, 2\}$. Then

$$\frac{1}{\log^2(X)} \sum_{\ell \leq X} \frac{\mu^2(\ell) A_\ell}{\phi(\ell)} \log \left( \frac{X}{\ell} \right) \leq \eta_v = \begin{cases} 1.256 & \text{if } v = 1, \\ 0.717 & \text{if } v = 2. \end{cases}$$

Proof. As $\sum_{\ell \leq X} \frac{\mu^2(\ell) A_\ell}{\phi(\ell)} \log \left( \frac{X}{\ell} \right) = \int_1^X \left( \sum_{\ell \leq t} \frac{\mu^2(\ell) A_\ell}{\phi(\ell)} \right) dt$, it suffices to analyze the sum inside the integral. By \cite{20} Thm. 3.3], with $f(p) = \frac{A_p}{p^{\alpha}}$, $\alpha = 1$, $\beta = \frac{3}{2}$ and $\delta = \frac{1}{4}$, we derive that

$$\sum_{\ell \leq X} \frac{\mu^2(\ell) A_\ell}{\phi(\ell)} = j_q G \left( \log(X) + g_q \right) + O^* \left( \frac{k_q G}{X^2} \right).$$
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where

\[ j_q = \prod_{p \mid q} \left( 1 - \frac{A_p}{p - 1 + A_p} \right), \quad G = \prod_p \left( 1 + \frac{A_p - 1}{p} \right) \in [1.603, 1.604], \]

\[ g_q = \sum_p \frac{\log(p)(p - 1 - (p - 2)A_p)}{(A_p + p - 1)(p - 1)} + \gamma + \sum_{p \mid q} \log(p)A_p, \]

\[ k_q = \prod_{\ell \mid q} \left( 1 + \frac{2(p - 1) - A_p p + p^2}{(p - 1)p^2 + A_p (p + p^2) - p + 1} \right), \]

\[ \sum_p \frac{\log(p)(p - 1 - (p - 2)A_p)}{(A_p + p - 1)(p - 1)} + \gamma \in [0.381, 0.383], \]

\[ g = \Delta \prod_p \left( 1 + \frac{p(A_p - 1) + A_p p^2 + 1}{(p - 1)p^2} \right) \in [30.767, 35.984]. \]

Therefore, as \( \int_1^X \frac{\log(t)}{t} dt = \frac{\log^2(X)}{2} \) and \( g_q > 0 \) for all \( q \in \mathbb{Z}_{>0} \), we derive for all \( X \geq C = 10^7 \) and \( q = v \in \{1, 2\} \) that \( \frac{\log^2(X)}{X} \times \sum_{\ell \leq X} \frac{\mu^2(\ell)A_\ell}{\ell} \log \left( \frac{X}{\ell} \right) \) may be estimated as

\[ \int_1^X \left( j_v G (\log(t) + g_v) + O^* \left( \frac{k_v g_v}{t^2} \right) \right) \frac{dt}{t \log^2(X)} \leq 1.604 j_v \left( \frac{1}{2} + \frac{g_v}{\log(C)} \right) + \frac{107.952 k_v}{\log^2(C)} = \begin{cases} 1.256 & \text{if } v = 1, \\ 0.717 & \text{if } v = 2. \end{cases} \]  

(4.23)

On the other hand, for all \( 10 \leq X \leq 10^7 \),

\[ \frac{1}{\log^2(X)} \times \sum_{\ell \leq X} \frac{\mu^2(\ell)A_\ell}{\ell} \log \left( \frac{X}{\ell} \right) \leq \begin{cases} 1.1 & \text{if } v = 1, \\ 0.695 & \text{if } v = 2. \end{cases} \]  

(4.24)

The result is concluded by defining \( \eta_0 \) as the maximum between the bounds given in (4.23) and (4.24).

**Proof of Lemma 4.2**  From (4.3) and the definition of \( \tilde{m}_q \) given in (3.1), observe that

\[ S_{(1)}^{(1)} = \frac{v}{\varphi(v)} \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{n \leq U} \frac{\mu(n)}{\varphi(n)} \log \left( \frac{U}{\ell n} \right), \]

(4.25)

where we have used that for any square-free \( n \), \( \sum_{d \mid n} \frac{\mu(d)}{\varphi(d)} = \frac{n}{\varphi(n)} \). Moreover, from (4.25), we obtain that \( \frac{\mu(n)}{\varphi(n)} S_{(1)}^{(1)} \) equals

\[ \sum_{\ell \leq U} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{n \leq U} \frac{\mu(n)}{\varphi(n)} \log \left( \frac{U}{\ell n} \right) - \sum_{\ell \leq U} \frac{\mu^2(\ell)}{\varphi(\ell)} \sum_{n \leq U} \frac{\mu(n)}{\varphi(n)} \log \left( \frac{U}{\ell n} \right) \]

\[ = \log(U) - \sum_{n \leq U} \frac{\mu(n)}{\varphi(n)} \sum_{\ell \leq U} \frac{\mu^2(\ell)}{\varphi(\ell)} \log \left( \frac{U}{\ell n} \right), \]

(4.26)

where in the above first summation, we have used Möbius inversion.
Now, on using Lemma 4.2 and summation by parts, we deduce that \( \sum_{\ell < \ell \leq U} \frac{\mu^2(\ell)}{\varphi(\ell)} \log \left( \frac{U}{\ell n} \right) \) may be estimated as

\[
\int_{Z}^{U} \left( \frac{\varphi(nv)}{nv} \log \left( \frac{t}{Z} \right) + O^* \left( \frac{A_{nv} 8.801 \prod_{Z|n,v} 0.493}{\sqrt{Z}} \right) \right) \frac{dt}{t} = \frac{\varphi(nv)}{2nv} \log^2 \left( \frac{U}{Z} \right) + O^* \left( \frac{A_{nv} 8.801 \prod_{Z|n,v} 0.493 \log \left( \frac{Z}{n} \right)}{\sqrt{Z}} \right). \tag{4.27}
\]

Replacing \( \text{(4.27)} \) into the second term of \( \text{(4.26)} \) gives further

\[
\sum_{n \leq U \atop \ell < \ell \leq U} \frac{\mu(n)}{\varphi(n)} \sum_{n \leq U \atop \ell < \ell \leq U} \frac{\mu^2(\ell)}{\varphi(\ell)} \log \left( \frac{U}{\ell n} \right) = \frac{\varphi(v)}{2v} \sum_{n \leq U} \frac{\mu(n)}{n} \log^2 \left( \frac{U}{n} \right) + O^* \left( \frac{8.801 A_v}{\sqrt{Z}} \sum_{n \leq U} \frac{\prod_{Z|n,v} 0.493 \mu^2(n)A_n}{\varphi(n)} \log \left( \frac{U}{n} \right) \right). \]

The above main term corresponds to \( \frac{\varphi(v)}{2v} \tilde{n}_v \left( \frac{U}{Z} \right) \). As for the error term, it can be estimated by means of Proposition 4.8 if \( v = 2 \), the factor \( \prod_{Z|n,v} 0.493 \) is always present, whereas, if \( v = 1 \), we have

\[
\sum_{n \leq U} \frac{\prod_{Z|n,v} 0.493 \mu^2(n)A_n}{\varphi(n)} \log \left( \frac{U}{n} \right) = \sum_{n \leq U} \frac{\prod_{Z|n,v} 0.493 \mu^2(n)A_n}{\varphi(n)} \log \left( \frac{U}{n} \right)
\]

\[
= 0.493 A_2 \sum_{n \leq U} \frac{\mu^2(n)A_n}{\varphi(n)} \log \left( \frac{U}{n} \right) + \sum_{n \leq U} \frac{\mu^2(n)A_n}{\varphi(n)} \log \left( \frac{U}{n} \right)
\]

\[
\leq 0.493 \eta_2 \log^2 \left( \frac{U}{2Z} \right) + \eta_2 \log^2 \left( \frac{U}{Z} \right) \leq (0.493 \eta_2 + \eta_2) \log^2 \left( \frac{U}{Z} \right), \tag{4.28}
\]

where we have used that \( A_2 = 1 \) and, since \( \frac{U}{2Z} \geq 10 \), Proposition 4.8. We conclude the result by defining

\[
\gamma_v^{(1)} = \begin{cases} 
4.339 \eta_2 + 8.801 \eta_2 & \text{if } v = 1, \\
4.339 \eta_2 & \text{if } v = 2.
\end{cases}
\]

Remark 4.9. The error term given in \( \text{(4.27)} \) has arisen due to the fact that we are studying a sum whose range starts sufficiently away from 1; if the summation \( \sum_{Z < \ell \leq U} \frac{\mu^2(\ell)}{\varphi(\ell)} \log \left( \frac{U}{\ell n} \right) \) would have started from 1 (or any admissible constant value) rather than \( Z \), a remainder term of order \( \log^2(U) \) would have appeared in Lemma 4.2, thus not even providing an asymptotic estimation for \( S_{II}^{(1)} \). This fact justifies why we have split the expression \( \text{(4.1)} \) into two summations.

4.3 The sum \( S_{II}^{(2)} \)

We need a series of lemmas that rely on an interval arithmetic computations within a range, using specifically that \( v \in \{1, 2\} \). As those calculations may be performed for any \( q \in \mathbb{Z}_{>0} \), Theorem 5.1 holds true.
Proposition 4.10. Let $X \geq 20$ and $v \in \{1, 2\}$. Then

$$X \times \sum_{\ell \geq X \atop \ell \equiv v \pmod{q}} \frac{\mu^2(\ell)}{\varphi(\ell)^2} \leq \varphi_v^{(1)} = \begin{cases} 2.373 & \text{if } v = 1, \\ 0.726 & \text{if } v = 2. \end{cases}$$

Proof. By applying [20, Thm. 4.3.1] with $f(p) = \frac{1}{\varphi(p)^2} = \frac{1}{(p-1)^2}$, $\alpha = 2$ and $\beta = 3$, we have

$$\sum_{\ell \leq X \atop \ell \equiv q \pmod{q}} \frac{\mu^2(\ell)}{\varphi(\ell)^2} = \sum_{\ell \leq X \atop \ell \equiv q \pmod{q}} \frac{\mu^2(\ell)}{\varphi(\ell)^2} - \frac{u_q}{X} I + O^* \left( \frac{v_q I(q)}{X^2} \right),$$

where

$$u_q = \prod_{p \not| q} \left( 1 - \frac{p}{p^2 - p + 1} \right), \quad I = \prod_p \left( 1 + \frac{1}{p(p-1)} \right) \in [1.943, 1.944],$$

$$v_q = \prod_{p \not| q} \left( 1 + \frac{p^2 - 4p + 2}{(\sqrt{p} - 1)(p-1)^2 + 2p - 1} \right),$$

$$i(q) = \begin{cases} 0.32 \prod_{p \not| q} \left( 1 + \frac{2p-1}{\sqrt{p} - 1} \right) & \in [19.326, 19.328], \text{ if } 2 \not| q, \\ 0.238 \prod_{p \not| q} \left( 1 + \frac{2p-1}{\sqrt{p} - 1} \right) & \in [14.373, 14.374], \text{ if } 2 \mid q. \end{cases}$$

Therefore, for all $X \geq C = 10^6$ and $q = v \in \{1, 2\}$, we deduce from (4.29) that

$$X \times \sum_{\ell \equiv X \atop \ell \equiv v \pmod{q}} \frac{\mu^2(\ell)}{\varphi(\ell)^2} \leq \frac{u_v I}{\sqrt{C}} \leq \begin{cases} 1.963 & \text{if } v = 1, \\ 0.654 & \text{if } v = 2. \end{cases}$$

On the other hand, we have that for all $20 \leq X \leq 10^6$,

$$X \times \sum_{\ell \geq X \atop \ell \equiv v \pmod{q}} \frac{\mu^2(\ell)}{\varphi(\ell)^2} \leq \begin{cases} 2.373 & \text{if } v = 1, \\ 0.726 & \text{if } v = 2. \end{cases}$$

Finally, we define $\varphi_v^{(1)}$ by taking the maximum between the bounds (4.31) and (4.32), respectively.

Proposition 4.11. Let $X \geq 4 \times 10^5$ and $v \in 1, 2$. Then

$$\frac{1}{X} \times \sum_{\ell \leq X \atop \ell \equiv v \pmod{q}} \frac{\mu^2(\ell)^2}{\varphi(\ell)^2} \leq \varphi_v^{(2)} = \begin{cases} 1.954 & \text{if } v = 1, \\ 0.651 & \text{if } v = 2. \end{cases}$$

Proof. By using summation by parts in Equation (4.29), we obtain

$$\sum_{\ell \leq X \atop \ell \equiv q \pmod{q}} \frac{\mu^2(\ell)^2}{\varphi(\ell)^2} = u_q I X + O^* \left( 5v_q i(q) \sqrt{X} \right),$$

where $u_q$, $I$, $v_q$ and $i(q)$ are defined in Proposition 4.10. Note that it was not necessary to calculate the main term of (4.29). Hence, when $X \geq C = 10^6$ and $q = v \in \{1, 2\}$, we have

$$\frac{1}{X} \times \sum_{\ell \leq X \atop \ell \equiv v \pmod{q}} \frac{\mu^2(\ell)^2}{\varphi(\ell)^2} \leq \frac{1}{X} \times \sum_{\ell \leq X \atop \ell \equiv v \pmod{q}} \frac{\mu^2(\ell)^2}{\varphi(\ell)^2} \leq \begin{cases} 1.954 & \text{if } v = 1, \\ 0.651 & \text{if } v = 2. \end{cases}$$

(4.33)
On the other hand, for all $X$ such that $4 \times 10^5 \leq X \leq 10^8$,

$$\frac{1}{X} \times \sum_{\ell \leq X} \mu^2(\ell) \ell^2 \phi(\ell)^2 \leq \begin{cases} 1.944 & \text{if } v = 1, \\ 0.648 & \text{if } v = 2. \end{cases}$$  \hspace{1cm} (4.34)

The result is concluded by taking the maximum between the bounds (4.33) and (4.34), which we define as $\phi(2) = 1$, $\nu(m) = \frac{p}{p-2}$, if $p > 3$. The following result is interesting since it describes the function whose average has an asymptotic expression with constant term equal to $\gamma + \sum_{p \mid q} \frac{\log(p)}{p-1}$, so that the infinite summation $T^2_f$ considered in [20 Thms. 3.3, 4.6] vanishes.

**Lemma 4.12.** Let $X > 0$. Then

$$\sum_{\ell \leq X} \frac{\mu^2(\ell) \nu(\ell)}{\ell} = H \left( \log(X) + \gamma + \frac{\log(2)}{2} \right) + O^* \left( \frac{5.193}{X} \right),$$  \hspace{1cm} (4.35)

$$\sum_{\ell \leq X} \frac{\mu^2(\ell) \nu(\ell)}{\ell} = \frac{H}{2} \left( \log(X) + \gamma + \log(2) \right) + O^* \left( \frac{2.565}{X} \right),$$

where $H = 1.514 \ldots$

**Proof.** Let $q \in \mathbb{Z}_{>0}$. By [20 Thm. 4.6] with $f(\ell) = \frac{\nu(\ell)}{\ell}$, $\alpha = 1$ and $\beta = 2$, we derive that

$$\sum_{\ell \leq X} \frac{\mu^2(\ell) \nu(\ell)}{\ell} = m_q H \left( \log(X) + h_q \right) + O^* \left( \frac{n_q h(q)}{\sqrt{X}} \right),$$

where

$$m_q = \prod_{2 \mid q} \frac{1}{2} \prod_{p \mid q} \left( 1 - \frac{1}{p-1} \right), \quad n_q = \prod_{p \mid q} \left( 1 - \frac{p-4}{(\sqrt{p-1})(p-2)+2} \right),$$

$$H = \prod_{p \geq 3} \left( 1 + \frac{1}{p(p-2)} \right) \in [1.514, 1.515],$$

$$h(q) = \begin{cases} 0.47 \prod_{p \geq 3} \left( 1 + \frac{2}{(\sqrt{p-1})(p-2)} \right) & \text{if } 2 \nmid q, \\ 0.233 \prod_{p \geq 3} \left( 1 + \frac{2}{(\sqrt{p-1})(p-2)} \right) & \text{if } 2 \mid q, \end{cases} \in [5.192, 5.193],$$

and, as $1 - f(p)p + 2f(p) = 1 - \frac{p}{p-2} + \frac{2}{p-2} = 0$ for $p \geq 3$, $h_q = \sum_{2 \nmid q} \frac{\log(2)}{2} + \gamma + \sum_{p \mid q} \frac{\log(p)}{p-1}$. The result is concluded by considering $q = v \in \{1, 2\}$. \hfill \square

**Proof of Lemma 4.3.** By recalling (4.10), we have that $S^{(2)}(U)$ is equal to

$$\frac{v^2}{\phi(v)^2} \sum_{\ell \leq Z} \frac{\mu^2(\ell) \phi(\ell)^2}{(d, \ell) = 1} \sum_{d} \phi(\ell) - \frac{v^2}{\phi(v)^2} \sum_{\ell \leq Z} \frac{\mu^2(\ell) \phi(\ell)^2}{(d, \ell) = 1} \sum_{d} \phi(\ell) = (4.36)$$

$$\frac{v^2}{\phi(v)^2} \sum_{\ell \leq Z} \frac{\mu^2(\ell) \phi(\ell)^2}{(d, \ell) = 1} \sum_{d} \phi(\ell) - \frac{v^2}{\phi(v)^2} \sum_{\ell \leq Z} \frac{\mu^2(\ell) \phi(\ell)^2}{(d, \ell) = 1} \sum_{d} \phi(\ell),$$
The inner sum of the second right hand term of (4.30) above can be estimated with the help of Proposition [4.10] and indeed, for any \( \ell \leq Z \), we have \( \frac{U}{\ell} \geq \frac{U}{Z} \geq 20 \), so that, by recalling the definition of \( \varphi_2^{(1)} \),

\[
\left| \sum_{d > \frac{U}{1}} \mu(d) \varphi(d)^2 \right| \leq \sum_{d > \frac{U}{1}} \mu^2(d) \varphi(d)^2 \frac{\ell}{\varphi(\ell)^2} \leq \sum_{d > \frac{U}{1}} \mu^2(d) \varphi(d)^2 \frac{\varphi(v)^{9}}{U}.
\]

Hence,

\[
\frac{v^2}{\varphi(v)^2} \sum_{\ell \leq Z} \frac{\mu^2(\ell)\ell}{\varphi(\ell)^2} \sum_{d > \frac{U}{\ell}} \frac{\mu(d)}{\varphi(d)^2} \leq \frac{\varphi_1^{(1)} v^2}{\varphi(v)^2} \frac{1}{U} \sum_{\ell \leq Z} \frac{\mu^2(\ell)\ell}{\varphi(\ell)^2} \leq \frac{\varphi_3^{(3)} Z}{U},
\]

where \( \varphi_3^{(3)} = \frac{\varphi_1^{(1)} v^2}{\varphi(v)^2} \), by using that \( Z \geq 4 \times 10^5 \) and the definition of \( \varphi_2^{(2)} \).

On the other hand, as \( \prod_{p \mid \ell v} \left( 1 - \frac{1}{p-1} \right) = 0 \) if \( 2 \nmid \ell \) and \( \frac{\mu(\ell)}{\varphi(\ell)} = \frac{\mu(\ell)}{\varphi(\ell)} \prod_{p \mid \ell} \left( 1 - \frac{1}{p-1} \right) \),

for all square-free numbers \( \ell \) such that \( (\ell, 2) = 1 \), we have that \( \sum_{\ell \leq Z} \frac{\mu^2(\ell)\ell}{\varphi(\ell)^2} \prod_{p \mid \ell} \left( 1 - \frac{1}{p-1} \right) \)

is equal to

\[
\frac{2}{v} \prod_{p \geq 3} \left( 1 - \frac{1}{(p-1)^2} \right) \sum_{\ell \leq \frac{Z}{2}} \frac{\mu^2(\ell)\ell}{\ell \varphi(\ell)^2}.
\]

Recall now Lemma [4.12] by using the definition of \( H \) given in [4.35], we have

\[
\prod_{p \geq 3} \left( 1 - \frac{1}{(p-1)^2} \right) \sum_{\ell \leq \frac{Z}{2}} \frac{\mu^2(\ell)\ell}{\ell \varphi(\ell)^2} = \frac{1}{2} \left( \log(Z) + \gamma + \sum_{p \mid v} \log(p) \right) + O^{*}\left( \frac{1}{p-1} \right) \frac{\sqrt{v} 2.565}{\sqrt{Z}}.
\]

Therefore, by putting everything together, we derive from estimations (4.39) and (4.40) that

\[
\frac{v^2}{\varphi(v)^2} \sum_{\ell \leq Z} \frac{\mu^2(\ell)\ell}{\varphi(\ell)^2} \prod_{p \mid \ell v} \left( 1 - \frac{1}{(p-1)^2} \right) = \frac{v}{\varphi(v)} \left( \log(Z) + \gamma + \sum_{p \mid v} \log(p) \right)
\]

\[
+ O^{*}\left( \frac{v^2}{\varphi(v)} \prod_{p \geq 3} \left( 1 - \frac{1}{(p-1)^2} \right) \frac{\sqrt{v} 2.565}{\sqrt{Z}} \right).
\]

The result is concluded by defining \( \varphi_2^{(3)} \) as the resulting bound on the error term given in (4.41), upon replacing either \( v = 1 \) or \( v = 2 \) and observing that

\[
\prod_{p \geq 3} \left( 1 - \frac{1}{(p-1)^2} \right) = \frac{1}{H} \in [0.66, 0.661].
\]
4.4 The sum $s_{II}^{(3)}$ and choice of parameter

As for section 4.3.3 we need a series of results that rely on an interval arithmetic computations that may be carried out for any $q \in \mathbb{Z}_{>0}$.

Proposition 4.13. Let $X > 0$. Then

\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{\ell}^2(\ell)} = f_q \mathbf{D} (\log(X) + \delta_q) + O^* \left( \frac{16.12.218 h_q}{X^{\frac{1}{2}}} \right),
\]

\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{\ell}^2(\ell)} = f_q DX + O^* \left( \frac{4.04.545 h_q X^{\frac{1}{2}}}{} \right),
\]

where

\[
f_q = \prod_{p \nmid q} \left( 1 - \frac{1}{p - 2\sqrt{p} + 2} \right),
\]

\[
\delta_q = -\sum_p \frac{(2\sqrt{p} - 3) \log(p)}{(p - 2\sqrt{p} + 2)(p - 1)} + \gamma + \sum_{p \nmid q} \frac{\log(p)}{p - 2\sqrt{p} + 2},
\]

\[
h_q = \prod_{p \nmid q} \left( 1 + \frac{p - 4\sqrt{p} - p^{\frac{1}{2}} + 2}{(\sqrt{p} - 1)^2 p^{\frac{3}{2}} + 2\sqrt{p} + p^{\frac{1}{2}} - 1} \right), \quad \mathbf{D} = 15.033 \ldots
\]

Proof. By applying [20, Thm. 3.3] with $f(p) = \frac{1}{\varphi_{\ell}^2(p)} = \frac{1}{p^{1-1/2}}$, $\alpha = 1$, $\beta = \frac{3}{2}$ and $0 \leq \delta = \frac{1}{2} < \frac{3}{4}$, we obtain

\[
\frac{\varphi(q) H_q^\delta(0)}{q} = D \prod_{p \nmid q} \left( 1 - \frac{1}{p - 2\sqrt{p} + 2} \right),
\]

\[
\frac{\kappa_{1-\delta}(q) H_q^{1-\delta}(-\delta)}{q^{1-\delta}} = d \prod_{p \nmid q} \left( 1 + \frac{p - 4\sqrt{p} - p^{\frac{1}{2}} + 2}{(\sqrt{p} - 1)^2 p^{\frac{3}{2}} + 2\sqrt{p} + p^{\frac{1}{2}} - 1} \right).
\]

where

\[
D = \prod_p \left( 1 + \frac{2}{p(p - \sqrt{p} - 1)} \right) \in [15.033, 15.034],
\]

\[
d = \Delta^\delta \prod_p \left( 1 + \frac{2p/p^{\frac{3}{2}} - 1}{p^{\frac{3}{2}}(\sqrt{p} - 1)^2} \right) \in [1470.434, 1616.218].
\]

On the other hand, but always according to [20, Thm. 3.2.1], we have that

\[
T_q^\alpha = -\sum_{p \nmid q} \frac{(2\sqrt{p} - 3) \log(p)}{(p - 2\sqrt{p} + 2)(p - 1)},
\]

so that, by defining $\gamma_q = \sum_{p \nmid q} \frac{\log(p)}{p - 2\sqrt{p} + 2}$, we have

\[
\delta_q = -\sum_p \frac{(2\sqrt{p} - 3) \log(p)}{(p - 2\sqrt{p} + 2)(p - 1)} + \gamma + \gamma_q,
\]

where

\[
-\sum_p \frac{(2\sqrt{p} - 3) \log(p)}{(p - 2\sqrt{p} + 2)(p - 1)} + \gamma \in [-1.16, -1.158],
\]

whence Equation (4.42). Finally, a summation by parts allows us to derive expression (4.43) from (4.42).
The shape of the above error term becomes impractical when one wants to provide an overall estimation. It is there when one can take advantage of computer calculations under interval arithmetic.

**Proposition 4.14.** Let $X \geq 20$ and $v \in \{1, 2\}$. Then

$$\frac{1}{\log(X)} \times \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)} \leq \chi_{v}^{(1)} = \begin{cases} 15.136 & \text{if } v = 1, \\ 2.216 & \text{if } v = 2. \end{cases}$$

**Proof.** Observe that, for all $X$ such that $20 \leq X \leq 5 \times 10^8$,

$$\frac{1}{\log(X)} \times \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)} \leq 14.088, \quad \frac{1}{\log(X)} \times \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)} \leq 2.137. \tag{4.44}$$

On the other hand, by Proposition 4.13, When $q = v \in \{1, 2\}$ and $X \geq 5 \times 10^8$, we conclude from (4.42) that

$$\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)} \leq 15.034 \left(\log(X) - 1.158\right) + \frac{1616.218}{X^+} \leq \left(15.034 + \frac{1616.218}{C_+ \log(C)}\right) \log(X) \leq 15.136 \log(X),$$

$$\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)} \leq 2.202 \left(\log(X) - 0.566\right) + \frac{213.493}{X^+} \leq \left(2.202 + \frac{213.493}{C_+ \log(C)}\right) \log(X) \leq 2.216 \log(X). \tag{4.45}$$

The result is concluded by defining $\chi_{v}^{(1)}$ as the maximum between the bounds (4.44) and (4.45).

**Proposition 4.15.** Let $X \geq 1$ and $v \in \{1, 2\}$. Then

$$\frac{1}{X} \times \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)^2} \leq \chi_{v}^{(2)} = \begin{cases} 20.125 & \text{if } v = 1, \\ 2.875 & \text{if } v = 2. \end{cases}$$

**Proof.** Observe that for all $X$ such that $1 \leq X \leq 5 \times 10^8$,

$$\frac{1}{X} \times \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)^2} \leq 16.769, \quad \frac{1}{X} \times \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)^2} \leq 2.507. \tag{4.46}$$

On the other hand, by Proposition 4.13 when $q = v \in \{1, 2\}$ and $X \geq 5 \times 10^8$, (4.43) tells us that

$$\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)^2} \leq \left(15.034 + \frac{4040.545}{C_+}\right) X \leq 20.125 \times X,$$

$$\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi_{2}(\ell)^2} \leq \left(15.034 f_{2} + \frac{4040.545 \times h_{2}}{C_+}\right) X \leq 2.875 \times X. \tag{4.47}$$

The result is thus obtained by defining $\chi_{v}^{(2)}$ as the maximum between the bounds (4.46) and (4.47).
Proposition 4.16. Let $X$, $Z$ such that $1 \leq Z < X$ and $v \in \{1, 2\}$. Then

$$\sum_{\ell \leq Z} \frac{\mu^2(\ell)\ell}{\varphi(Z)} \log \left( \frac{X}{\ell} \right) \leq \chi^{(2)}_v Z \left( \log \left( \frac{X}{Z} \right) + 1 \right),$$

where $\chi^{(2)}_v$ is defined as in Proposition 4.15.

Proof. By summation by parts and Proposition 4.15, we derive

$$\sum_{\ell \leq Z} \frac{\mu^2(\ell)\ell}{\varphi(Z)} \log \left( \frac{X}{\ell} \right) \leq \chi^{(2)}_v Z \log \left( \frac{X}{Z} \right) + \chi^{(2)}_v (Z - 1).$$

By [20] Thm. 3.3 and summation by parts, we know how to detect the order of the summation given in Lemma 4.17 by applying Proposition 3.7. It is of order $\log^{-2}(X)$; by using [20] Thm. 4.6, summation by parts and Proposition 3.7, we could derive an estimation for it. Nonetheless, given that this sum is involved in a small value, presented in the second term of the bounds (5.3), we have chosen to proceed faster by observing that the non-weighted sum in the statement below is convergent.

Proposition 4.17. Let $X \geq 10^{12}$, $\theta = 1 - \frac{1}{\log(10^{12})}$ and $v \in \{1, 2\}$. Then

$$\log^2(X) \times \sum_{d \leq X} \frac{\mu^2(d)}{d^{2-2\theta} \varphi(d)^2 \log^2 \left( \frac{X}{d} \right)} \leq 1.549 \quad \text{if } v = 1,$$

$$3.232 \quad \text{if } v = 2.$$

Proof. Define $f(p) = \frac{\mu^2(p)}{d_2^{2-2\theta} \varphi(p)^2 \log^2 \left( \frac{X}{p} \right)}$ on prime numbers and extend it to a multiplicative function. Consider $W$ such that $1 < W < X$; by using the bound $\log^{-1} \left( \frac{X}{d} \right) \leq \log^{-1}(W)$, for $1 \leq d \leq \frac{X}{W}$, and writing $\log(X) = \log \left( \frac{X}{d} \right) + \log(d)$, we derive that $\sum_{d \leq \frac{X}{W}} \frac{\mu^2(d)f(d)\log^2(X)}{d^2 \log^2 \left( \frac{X}{d} \right)}$ is bounded from above by

$$\sum_{d \leq \frac{X}{W}} \frac{\mu^2(d)f(d)}{d^2} + \sum_{d \leq \frac{X}{W}} \frac{\mu^2(d)f(d)\log(d)}{d^2} \left( \frac{2}{\log(W)} + \frac{\log(d)}{\log^2(W)} \right).$$

Observe now that the functions $t \mapsto \frac{\log(t)}{t^2}$ and $t \mapsto \frac{\log^2(t)}{t^2}$ have a global maximum at $t = e^2$, with value $\frac{\log(e^2)}{e^2} = \frac{2}{e^2}$, and at $t = e^4$, with value $\frac{16}{e^4}$, respectively. Hence, from [138], $\sum_{d \leq \frac{X}{W}} \frac{\mu^2(d)f(d)\log^2(X)}{d^2 \log^2 \left( \frac{X}{d} \right)}$ is at most

$$\prod_{p \mid q} \left( 1 + \frac{f(p)}{p^2} \right) \left( 1 + \frac{1}{e \log(W)} + \frac{16}{e^2 \log^2(W)} \right) \prod_{p \mid q} \left( 1 + \frac{f(p)}{p^2} \right).$$

On the other hand, on using the definition of $f$, we obtain

$$\prod_{p \mid q} \left( 1 + \frac{f(p)}{p^2} \right) = \prod_{p \mid q} \left( 1 - \frac{1}{p^{2-2\theta}(p^\theta - 1)^2 + 1} \right) \prod_p \left( 1 + \frac{1}{p^{2-2\theta}(p^\theta - 1)^2} \right),$$

$$\prod_{p \mid q} \left( 1 + \frac{f(p)}{p^2} \right) = \prod_{p \mid q} \left( 1 - \frac{1}{p^{2-2\theta}(p^\theta - 1)^2 + 1} \right) \prod_p \left( 1 + \frac{1}{p^{2-2\theta}(p^\theta - 1)^2} \right),$$
Proposition 4.18. Let $X \geq 10^{12}$, $\theta = 1 - \frac{1}{\log(10^{12})}$ and $q \in \mathbb{Z}_{>0}$. Let $c$ and $\varepsilon$ be two real numbers such that $1 < Z = cX^\varepsilon < X$ and $0 < \varepsilon < 1 - \frac{\log(c)}{\log(10^{12})}$. Then

$$\log(X) \times \sum_{\ell \leq Z} \frac{\mu^2(\ell)}{\ell \log^2 \left( \frac{\ell}{\varphi(\ell)} \right)} \left( \frac{\ell^\theta}{\varphi(\ell)} \right)^2 \leq \xi_{\{c,\varepsilon, q\}},$$

for some explicit constant $\xi_{\{c,\varepsilon, q\}} > 0$. In particular, we may define $\xi_1^{(10)} = \xi_{\{10, \frac{4}{5}, 1\}} = 6.151$, $\xi_2^{(10)} = \xi_{\{10, \frac{6}{5}, 2\}} = 1.849$, $\xi^{(16)}_2 = \xi_{\{16, \frac{4}{5}, 2\}} = 1.982$ and $\xi_1^{(70)} = \xi_{\{70, \frac{4}{5}, 1\}} = 6.638$.

Proof. Define $f(p) = \frac{1}{p^{\varepsilon - 1}(p-1)^2}$. As $\theta > \frac{1}{2}$, we can use [20, Thm. 4.6] with $\alpha = 1$ and $\beta = 1 + \theta$ to estimate the above sum without the weight $\log(\frac{X}{\ell})$. We derive that

$$\sum_{\ell \leq Z} \frac{\mu^2(\ell)}{\ell \log^2 \left( \frac{\ell}{\varphi(\ell)} \right)} \left( \frac{\ell^\theta}{\varphi(\ell)} \right)^2$$

may be estimated as

$$M_q(Z) + O^* \left( \frac{y_q j(q)}{\sqrt{Z}} \right) = x_q J (\log(Z) + j_q) + O^* \left( \frac{y_q j(q)}{\sqrt{Z}} \right),$$

where

$$x_q = \prod_{p \nmid q} \left( 1 - \frac{p - 1}{p^{\varepsilon - 2\theta}(p^{\theta} - 1)^2 + 2p^{1-\theta} - p^{\varepsilon - 2\theta} - 1} \right),$$

$$j_q = -\sum_{p} \log(p) \left( \frac{2p^{1-\theta} - p^{1-2\theta} + 2}{p^{\varepsilon - 2\theta}(p^{\theta} - 1)^2 + 2p^{1-\theta} - 1} \right),$$

$$y_q = \prod_{p \nmid q} \left( 1 - \frac{2p^{2\theta} - 4p^{\theta} + 2}{p^{\varepsilon - 1}(p^{\theta} - 1)^2 + 2p^{1-\theta} - 1} \right),$$

$$J = \prod_{p} \left( 1 - \frac{2p^{1-\theta} - p^{1-2\theta} - 1}{p^{\varepsilon - 2\theta}(p^{\theta} - 1)^2 + 1} \right) \in [2.088, 2.089],$$

$$= -\sum_{p} \log(p) \left( \frac{2p^{1-\theta} - p^{1-2\theta} + 2}{p^{\varepsilon - 2\theta}(p^{\theta} - 1)^2 + 1} \right) + \gamma \in [0.859, 0.86],$$

$$j^{(q)} = \begin{cases} 0.325 \prod_{p} \left( 1 - \frac{2p^{1-1}}{p^{\varepsilon - 1}(p^{\theta} - 1)^2} \right) \in [24.267, 24.269], & \text{if } 2 \nmid q, \\
0.233 \prod_{p} \left( 1 - \frac{2p^{1-1}}{p^{\varepsilon - 1}(p^{\theta} - 1)^2} \right) \in [17.335, 17.337], & \text{if } 2 | q. \end{cases}$$

Now, by integration by parts, we derive from (4.50) that the summation $\sum_{\ell \leq Z} \frac{\mu^2(\ell)}{\ell \log^2 \left( \frac{\ell}{\varphi(\ell)} \right)} \left( \frac{\ell^\theta}{\varphi(\ell)} \right)^2$ equals

$$\int_{1}^{Z} \frac{M_q^{(1)}(t)}{\log^2 \left( \frac{X}{t} \right)} dt + \frac{M_q^{(1)}(1)}{\log^2(X)} + y_q j^{(q)} O^* \left( \frac{1}{\sqrt{Z} \log^2 \left( \frac{X}{\ell} \right)} + \int_{1}^{Z} \frac{2dt}{t^\theta \log^2 \left( \frac{X}{t} \right)} \right),$$

where

$$M_q(1) = \prod_{p \nmid q} \left( 1 - \frac{p - 1}{p^{\varepsilon - 2\theta}(p^{\theta} - 1)^2 + 2p^{1-\theta} - p^{\varepsilon - 2\theta} - 1} \right),$$

$$M_q(t) = \prod_{p \nmid q} \left( 1 - \frac{p - 1}{p^{\varepsilon - 2\theta}(p^{\theta} - 1)^2 + 2p^{1-\theta} - p^{\varepsilon - 2\theta} - 1} \right).$$
Notice that the integral in right hand side of (4.51) can be bounded with the help of Proposition 3.7, giving an upper bound for the above error term of the form

\[
y_q \sum_{\ell \leq \omega} \frac{1}{\sqrt{\log^2 \left( \frac{y}{x} \right)}} + \frac{4}{\log^3 \left( \frac{y}{x} \right)} + \frac{4}{Z \log^3 \left( \frac{y}{x} \right)}
\]

\[
\leq \frac{y_q \sum_{\ell \leq \omega} \frac{1}{\sqrt{\log^2 \left( \frac{y}{x} \right)}} W_{(c,\varepsilon)} \log(X)}{\log \left( \frac{10^{12(1-\varepsilon)}}{c} \right) (1 - \varepsilon - \frac{\log(c)}{\log(10^{12-\varepsilon})}) \log(X)} = \frac{Y_{(c,\varepsilon,\varepsilon)}}{\log(X)},
\]

where we have used that \(0 < \varepsilon < 1\) and \(1 < \frac{10^{12(1-\varepsilon)}}{c} \leq \frac{X}{y}\) and where \(W_{(c,\varepsilon)}\) is defined as

\[
\frac{1}{\sqrt{c 10^{6\varepsilon}}} + \frac{4}{\log \left( \frac{10^{12(1-\varepsilon)}}{c} \right) c^{10^{3\varepsilon}} \log \left( \frac{10^{12(1-\varepsilon)}}{c} \right)}.
\]

On the other hand, the main term of Equation (4.51) can be bounded with the help of Proposition 3.7, giving that \(\int_1^y \frac{M_q(t)}{\log^2(t)} \, dt + \frac{M_q(1)}{\log^2(X)}\) is at most

\[
x_q J \left( \frac{1}{\log \left( \frac{y}{x} \right)} - \frac{1}{\log(X)} + \frac{i_q}{\log^2(X)} \right) \leq \frac{X_{(c,\varepsilon,\varepsilon)}}{\log(X)},
\]

where we have used that \(i_q\) is positive for all \(q \in \mathbb{Z}_{>0}\) and

\[
X_{(c,\varepsilon,\varepsilon)} = x_q J \left( \frac{1}{1 - \varepsilon - \frac{\log(v)}{\log(10^{12-\varepsilon})}} + \frac{i_q}{\log(10^{12})} \right).
\]

The result is achieved by considering \(v = \ell \in \{1, 2\}\), setting \(\xi_{(c,\varepsilon,\varepsilon)}\) as any upper bound of \(X_{(c,\varepsilon,\varepsilon)} + Y_{(c,\varepsilon,\varepsilon)}\) and defining \(\xi_{(c,\varepsilon,\varepsilon)}\) as a reasonable value of \(\xi_{(c,\varepsilon,\varepsilon)}\), for \(c \in \{10, 16, 70\}\). □

**Remark 4.19.** As long as \(1 < Z < X\), and regardless of the choice of \(Z\), the expressions (4.52) and (4.53) tell us that \(\sum_{\ell \leq Z \,(\ell,q)=1} \frac{\mu^2(\ell)}{\log^2(\ell)} + \sum_{d \leq Z \,(d,v)=1} \frac{\mu^2(d)}{\varphi_d(d)} \left( \frac{ld}{\varphi_d(ld)} \right)^2 \ll q \frac{1}{\log(\frac{y}{x})}\).

As \(S^{(3)}\) involves a summation of small terms, given by the extraction from \(S_{11}\) of the main terms of the functions \((\ell, d) \mapsto m_{\ell d v}\), we expect it to be small. As it turns out, this is so provided that \(U\) is sufficiently large.

By using the inequality \((A_1 + B_1)^2 \leq (1 + \omega) A_2^2 + (1 + \frac{1}{A}) B_2^2\), valid for any \(0 < A_1 \leq A_2\), \(0 < B_1 \leq B_2\) and \(\omega > 0\), we obtain that \(|S^{(3)}_{11}|\) is at most

\[
\frac{1 + \omega v}{U} \sum_{\ell \leq Z \,(\ell,q)=1} \frac{\mu^2(\ell)}{\log^2(\ell)} + \sum_{d \leq Z \,(d,v)=1} \frac{\mu^2(d)}{\varphi_d(d)} \left( \frac{\ell d}{\varphi_d(\ell d)} \right)^2
\]

\[
= \frac{1}{389^2} \left( 1 + \frac{1}{\omega v} \right) \varphi_2(v) \sum_{\ell \leq Z \,(\ell,q)=1} \frac{\mu^2(\ell)}{\ell} + \sum_{d \leq Z \,(d,v)=1} \frac{\mu^2(d)}{d} \left( \frac{\ell d}{\varphi_d(\ell d)} \right)^2.
\]

On the other hand, as \(20 \leq \frac{X}{Z} \leq \frac{X}{y}\) we can apply Proposition 3.14 and then Proposition 4.16 to bound the double sum given in (4.54) as

\[
\sum_{\ell \leq Z \,(\ell,q)=1} \frac{\mu^2(\ell)}{\varphi_2(\ell)^2} \sum_{d \leq Z \,(d,v)=1} \frac{\mu^2(d)}{\varphi_2(\ell)^2} \leq \frac{1}{X_{(c,\varepsilon,\varepsilon)}} \left( \log \left( \frac{U}{Z} \right) + 1 \right),
\]

\[
\sum_{\ell \leq Z \,(\ell,q)=1} \frac{\mu^2(\ell)}{\varphi_2(\ell)^2} \sum_{d \leq Z \,(d,v)=1} \frac{\mu^2(d)}{\varphi_2(d)^2} \leq \frac{1}{X_{(c,\varepsilon,\varepsilon)}} \left( \log \left( \frac{U}{Z} \right) + 1 \right),
\]

\[
\sum_{\ell \leq Z \,(\ell,q)=1} \frac{\mu^2(\ell)}{\varphi_2(\ell)^2} \sum_{d \leq Z \,(d,v)=1} \frac{\mu^2(d)}{\varphi_2(d)^2} \leq \frac{1}{X_{(c,\varepsilon,\varepsilon)}} \left( \log \left( \frac{U}{Z} \right) + 1 \right).
\]
where \( \chi_v^{(3)} = \chi_v^{(1)} \chi_v^{(2)} \). Furthermore, Proposition 4.17 with \( X = U \geq 10^{12} \) gives the following estimation

\[
\sum_{d \leq U, (\ell, v) = 1} \frac{\mu^2(d)}{d^2 \varphi(d)^2} \leq \sum_{d \leq U, (\ell, v) = 1} \frac{\mu^2(d)}{d^2 \varphi(d)^2} \leq \frac{\tau_v}{\log^2 \left( \frac{U}{Z} \right)},
\]

so that, when \( U \geq 10^{12} \),

\[
\sum_{\ell \leq Z} \sum_{(\ell, v) = 1} \frac{\mu^2(\ell)}{\ell \log^2 \left( \frac{\ell}{\varphi(\ell)} \right)} \leq \frac{\tau_v}{\log^2 \left( \frac{U}{Z} \right)}.
\]

**Choice of parameter.** We conclude from estimations (4.56), (4.57) and Remark 4.19 that, as long as \( \frac{Z}{U} \geq 20 \) (or rather, \( \frac{Z}{U} > 1 \)), we have

\[
S_{\text{II}}^{(3)} \ll \frac{Z}{U} \log \left( \frac{U}{Z} \right) + \frac{Z}{U} + \frac{1}{\log \left( \frac{Z}{U} \right)}.
\]

Therefore, by recalling (4.14), we derive from lemmas 4.12, 4.13 and Equation (4.58) that the error term of \( \sum_{d, e, (\ell, v) = 1} \frac{\mu^2(d)}{\varphi(d) \varphi(e)} \log \log(N) = S_{\text{I}} + S_{\text{II}} \) has magnitude at most

\[
\frac{\log^4 \left( \frac{U}{Z} \right)}{\sqrt{Z}} + \frac{\log^2 \left( \frac{U}{Z} \right)}{\sqrt{Z}} + \frac{Z}{U} \log \left( \frac{U}{Z} \right) + \frac{1}{\log \left( \frac{Z}{U} \right)}.
\]

From the bound \( \frac{\log^4 \left( \frac{U}{Z} \right)}{\sqrt{Z}} \leq \frac{\log^4(U)}{\sqrt{Z}} \), as we are aiming for an error term as small as possible, we may suppose that \( \log^4(U) \ll Z \) for all \( a > 0 \). On the other hand, in order to minimize the contribution of \( -\log^{-1} \left( \frac{U}{Z} \right) \) to the order of \( \log(U) \), given that \( \log^{-1}(U) \ll \log^{-1} \left( \frac{U}{Z} \right) \) and that \( \log^{-1} \left( \frac{U}{\log(U)} \right) \) is of strictly higher order than \( \log^{-1}(U) \) for any \( a > 0 \) and \( 0 < \varepsilon < 1 \), it is plausible to suppose that \( Z = O(U^c) \) for some \( c > 1 \). Hence the overall magnitude of the expression given in (4.59), regardless of coprimality conditions, is \( \log^{-1}(U) \), as predicted in Theorem 1. Furthermore, the magnitude of the secondary terms therein is at most \( \log^4 \left( \frac{U}{Z} \right) \left( \frac{1}{U^{1/2}} + \frac{Z}{U} \right) = O(\log^4(U) (U^{-\frac{1}{2}} + U^{-(1-c)}) \right) \), and it achieves its minimal order when \( \varepsilon = \frac{5}{3} \). We can set then \( Z = cU^{\frac{5}{3}} \) for some constant \( c > 0 \).

**Proof of Lemma 4.4.** Let \( c \in \{10, 16, 70\} \). As \( 0 < \varepsilon = \frac{5}{3} < 1 - \frac{\log(70)}{\log(10^{12})} \), we can apply the particular case of Proposition 4.15 to estimate (4.57), considered only when \( U \geq 10^{12} \). Moreover, by equations (4.54), (4.55) and estimation (4.56), we have that \( |S_{\text{II}}^{(3)}| \) is bounded from above by

\[
(1 + \omega_v) \frac{\chi_v^{(1)} \chi_v^{(2)} v}{\varphi(\varphi(v)^{2})} \left( \frac{\log(U)}{3U^{\frac{1}{2}}} + \frac{1}{U^{\frac{1}{2}}} \right) + \left( 1 + \frac{1}{\omega_v} \right) \frac{\tau_v \xi_v^{(c)} v^{20}}{\varphi(\varphi(v)^{2})} \frac{\log(U)}{389^2 \log(U)}.
\]

On the other hand, observe that

\[
\frac{1}{389} \frac{\tau_v \xi_v^{(c)} v^{20}}{\varphi(\varphi(v)^{2})} \leq \max_{c \in \{10, 16, 70\}} \left\{ \frac{1}{389} \frac{\tau_v \xi_v^{(c)} v^{20}}{\varphi(\varphi(v)^{2})} \right\} \leq \begin{cases} 304.593 & \text{if } v = 1, \\ 74.211 & \text{if } v = 2, \\ 0.078 & \text{if } v = 1, \\ 0.046 & \text{if } v = 2. \\ \end{cases}
\]
Hence, the estimation for the first term of (5.2) is numerically very big and we seek for a value \( \omega_v \) that will not make it much more bigger upon multiplying it by \((1 + \omega_v)\). On the other hand, as long as \( \frac{1}{389} (1 + \frac{1}{\omega_v}) \) remains small, the second term estimation therein will be numerically small. Thus, we may naturally suppose that \((1 + \omega_v)\) is independent of the value of \( c \), and define

\[
\Upsilon_v^{(4)} = \frac{(1 + \omega_v)^{\chi_v^{(1)}(x_v)^2} v}{\varphi_g(v)^2}, \quad \Upsilon_v^{(5)} = \frac{1}{389^2} \left(1 + \frac{1}{\omega_v}\right) \tau_v \varphi_g(v)^2. 
\]

□

5 Main term and conclusion

We choose three values of \( c \): if \( U \geq 10^7 \), set \( Z = 10U^{\frac{3}{5}} \); if \( U \geq 10^{12} \) and \( v = 2 \), set \( Z = 16U^{\frac{3}{5}} \) and, if \( U \geq 10^{12} \) and \( v = 1 \), set \( Z = 70U^{\frac{3}{5}} \). With these choices, we have that \( U^2 \geq 10 \) and \( Z \geq 4 \times 10^5 \). Thus, conditions on propositions 4.3, 4.8, 4.10 are satisfied with \( X = \frac{Z}{U} \) and estimations (4.135) and (4.137) and, respectively, correct. Furthermore, as \( Z \geq 4 \times 10^5 \), Proposition 4.11 can be applied to derive inequality (4.138). Finally, Proposition 4.13 holds with \( X = \frac{Z}{U} \geq 20 \), so that we can derive (4.139).

By recalling identity (4.14), we can combine lemmas 4.14, 4.2, 4.3 and 4.4 to derive the following estimation,

\[
\sum_{d,v (d,v) = 1} \frac{\mu(d) \mu(v)}{d,v} \log^+ \left( \frac{U}{d} \right) \log^+ \left( \frac{U}{v} \right) = \frac{\gamma}{\varphi(v)} \log(U) - s_v + \Xi_v(U), \tag{5.1}
\]

where \( s_v = \frac{\gamma}{\varphi(v)} \left( \gamma + \sum_{p|v} \frac{\log(p)}{p-1} + \frac{\varphi(v)}{\pi^2 \kappa(v)} \int_1^{\infty} \frac{\text{Li}(s)}{s} ds \right) \), \( v \in \{1, 2\} \) and

\[
|\Xi_v(U)| \leq \Upsilon_v^{(1)} \log^4 \left( \frac{U}{\kappa(v)} \right) + \frac{2\Upsilon_v^{(1)} \log^2 \frac{U}{\kappa(v)}}{2\sqrt{\kappa(v)}} \frac{20\Psi_v \log \left( \frac{U}{\kappa(v)} \right)}{\pi^2 \kappa(v)} \frac{U}{U^{\frac{3}{5}}} + \\
\frac{\Upsilon_v^{(4)} \log(U)}{3U^{\frac{3}{5}}} + \frac{60\Psi_v \log(U)}{\pi^2 \kappa(v)} \frac{U}{U^{\frac{3}{5}}} + \Upsilon_v^{(2)} \frac{c_v \Upsilon_v^{(3)}}{U^{\frac{3}{5}}} + \Upsilon_v^{(4)} \frac{c_v \Upsilon_v^{(3)}}{U^{\frac{3}{5}}} + \\
\frac{36\Upsilon_v^{(4)} \log^5 \left( \frac{U}{\kappa(v)} \right)}{\tau_v \pi^2 \kappa(v)} \frac{1}{1 - \frac{\log(c_v)}{\log(U)}} \log(U) + \frac{1}{1 + \frac{\log(U)}{\log(U)}} \log(U) \tag{5.2}. 
\]

If \( U \geq 10^7 \), we use that \( \left( 1 - \frac{\log(c_v)}{\log(U)} \right)^{-1} \leq \left( 1 - \frac{\log(U)}{\log(U)} \right)^{-1} \) to merge the last two terms of estimation (5.2) to the order \( \frac{1}{\log(U)} \). Moreover, by using that \( \log \left( \frac{U}{2} \right) \leq \frac{1}{8} \log(U) \), we can merge the remaining lower order terms to the second order \( \frac{\log^3(U)}{U^{\frac{3}{5}}} \).

If \( U \geq 10^{12} \), we can obtain a numerical bound for \( \Xi_v(U), v \in \{1, 2\} \), by observing that the last term in (5.2) appears only if \( U \geq 10^{12} \), that \( \left( 1 - \frac{\log(c_v)}{\log(U)} \right)^{-1} \left( 1 - \frac{\log(U)}{\log(U)} \right)^{-1} \), with \( c_1 = 70, c_2 = 16 \), and that the functions \( t \geq c_v \mapsto \log \left( \frac{1}{t} \right) t^{-\frac{3}{5}}, t \geq c_v \mapsto \log^2 \left( \frac{1}{t^2} \right) t^{-\frac{3}{5}} \) and \( t > 0 \mapsto \frac{\log(t)}{t^{\frac{3}{5}}} \) are all decreasing for \( t \geq 10^{12} \).

Subsequently, by observing that \( t \geq c_v \mapsto \log^5 \left( \frac{1}{t^2} \right) t^{-\frac{3}{5}} \) is also decreasing for \( t \geq 10^{12} \), we can merge the whole expression (5.2) to the order \( \frac{1}{\log(U)} \).
Furthermore, by observing that the results given in [11, 12, 13] and [14] can be worked out for any fixed $q \in \mathbb{Z}_{>0}$, we can derive a general equation, similar to (5.1), in which $v$ is replaced by $q$, and whose remainder term $\Xi_q(U)$ is of order $\frac{1}{\log(U)}$. Thus, by putting everything together, we deduce our main result.

**Theorem 5.1.** Let $U > 1$. Then for all $q \in \mathbb{Z}_{>0}$, one can determine an explicit constant $K_q > 0$ such that

$$
\sum_{(d,e)=1}^{\mu(d)\mu(e)} \frac{d}{[d,e]} \log^+ \left( \frac{U}{d} \right) \log^+ \left( \frac{U}{e} \right) = \frac{q}{\varphi(q)} \log(U) - s_q + \Xi_q(U),
$$

and $\Xi_q(U) = O_q^* \left( \frac{K_q}{\min(d,e)} \right)$, where, by recalling the definition of $h_q$ given in (3.8), we have

$$s_q = \frac{q}{\varphi(q)} \left( \gamma + \sum_{p^s q^r} \frac{\log(p)}{p-1} - \frac{6}{\pi^2} \frac{\varphi(q)}{\kappa(q)} \int_{1}^{\infty} \frac{h_q(s)}{s} ds \right).$$

In particular, if $U \geq 10^7$, we have

$$|\Xi_1(U)| \leq \frac{0.079 \log^4(U)}{U^2} + \frac{0.0404}{\log(U)}, \quad |\Xi_2(U)| \leq \frac{0.0145 \log^4(U)}{U^2} + \frac{0.0122}{\log(U)},$$

(5.3)

If $U \geq 10^{\frac{2}{97}}$, we have

$$|\Xi_1(U)| \leq 0.42425, \quad |\Xi_2(U)| \leq 0.13169,$$

(5.4)

$$|\Xi_1(U)| \leq \frac{12.2109}{\log(U)}, \quad |\Xi_2(U)| \leq \frac{3.7903}{\log(U)}.$$

(5.5)

**The specific constant in the case** $v \in \{1, 2\}$. An estimation of $\int_1^{10^6} \frac{h_v(s)}{s} ds$ is provided in [9, Proposition 6.26]. Inspired by this calculation, we obtain

$$\int_1^{10^6} \frac{h_v(s)}{s} ds \in \left\{ \begin{array}{ll}
[-0.0495100113498 - 0.0495100106266] & \text{if } v = 1,
[2.63481269161, 2.63481271383] & \text{if } v = 2.
\end{array} \right.$$  

Moreover, similar to the estimation given in [12], we have that $\int_1^{\infty} \frac{h_v(s)}{s} ds$ can be expressed as

$$\int_1^{10^6} \frac{h_v(s)}{s} ds + O^* \left( \Psi'_v \Omega + \frac{2T_v^{(4)}}{\log(10^{1/2})} \right),$$

where $\Psi'_v = T_v^{(2)} + \frac{T_v^{(3)}}{\log(10^6)}$ and $\Omega = \frac{\log(10^6)}{10^6} + \frac{\log(10^{1/2})}{10^{1/2}} + \frac{1}{10^{1/2}}$. Thereupon, the constant $s_v$ coming from Theorem 5.1 can be estimated as follows

$$s_1 \in [\gamma + \frac{6}{\pi^2} \times [0.049510010626, 0.0495100113498] + O^*(0.00000322413) \in [0.60731091805, 0.60731736674],$$

$$s_2 \in [2(\gamma + \log(2)) + \frac{4}{\pi^2} \times [-2.63481271383, -2.63481269161] + O^*(0.00000552290) \in [1.47287079666, 1.47288185146],$$

(5.6)

so that (2.2) is confirmed.

Finally, since we have carefully estimated the value of $s_v$, $v \in \{1, 2\}$, by assuming that $U \geq 10^7$, and since $10^7$ is a moderate value, we can ask about the behavior of (5.1) when $1 \leq U \leq 10^7$. Indeed, this study was carried out by a program, inspired by [9, §6.6.1], that run for about 6 weeks combined with a recurrence algorithm, whose output we write below.
Proposition 5.2. Let \( 1 \leq U \leq 10^7 \) and \( v \in \{1, 2\} \). Then the expression
\[
\left| \sum_{(d,e) \neq (1,1)} \mu(d)\mu(e) \log^+ \left( \frac{\ell}{d} \right) \log^+ \left( \frac{\ell}{e} \right) \right| \leq \frac{2}{\log(Y)} \left( -0.73643 + \frac{\Xi_2(\sqrt{Y})}{2} + 4 \frac{1}{\pi^2} + \frac{\ell}{Y^2} \right)^2.
\]

By supposing that \( Y \) is minimized when \( U \) is very close to \( 3^+ \) and \( 1^+ \), respectively, the error constants being much smaller for larger \( U \).

6 On the Brun–Tichmarsh theorem

By proceeding as in [11, §3.2], we finally see how we can apply Theorem [11] to derive an explicit result about the distribution of the prime numbers.

Let \( X, Y \in \mathbb{R}_{\geq 1} \) and define \( S(X, Y; P) = \# \{ n \in (X, X+Y] \cap \mathbb{Z}_{\geq 0}, (n, P) = 1 \} \).

As per the discussion in [11], we know that \( \left\{ \frac{1}{\log(U)} \right\}_{U \geq 1} \), \( U > 2 \), is a sequence of parameters as in the Selberg sieve. Hence, by taking in particular \( P' = \prod_{2 < p \leq U} p \), by [11, Eq. (3.10)], we readily have that \( S(X, Y; 2P') \) is at most
\[
\frac{Y}{4074} \sum_{(d,e) = 1} \frac{\mu(d)\mu(e)}{d,e} \log(Y) + \frac{4}{\log(Y)} \left( \sum_{d \neq 1} \frac{\mu^2(d)}{d} \right)^2.
\]

Observe that the above right hand term can be estimated as
\[
\sum_{(d,e) = 1} \frac{\mu^2(d)}{d} \log(Y) = \int_0^Y \left( \sum_{d \leq t} \mu^2(d) \right) \frac{dt}{t} = \frac{4}{\pi^2} U + O^* \left( \ell U \right);
\]

where, by using [9, Lemma 5.2], the constant \( \ell \) is defined as \( 2 \left( 1 - \frac{1}{25} \right) \).

Therefore, by using Theorem [11] and estimation [5.6], we have that \( S(X, Y; 2P') \) is bounded from above by
\[
\frac{Y}{4074} \left( 2 \log(Y) - 1.47287 + \Xi_2(Y) \right) + \frac{4}{\log(Y)} \left( \frac{4}{\pi^2} U + \ell U \right)^2.
\]

From [6.1], we see immediately that in order to derive a non-trivial estimation for \( S(X, Y; 2P') \), and in general, to derive a continuous and monotonic version of Selberg sieve, we must have \( U^2 \ll Y \). Moreover, the magnitude of the bound [6.1] is \( \frac{Y}{\log(Y)} \), which is minimized when \( U \) is as large as possible. For numerical simplicity, we take \( U = \sqrt{Y} \).

Therefore [6.1] can be written as
\[
\frac{2Y}{\log(Y)} + \frac{4Y}{\log(Y)} \left( -0.73643 + \frac{\Xi_2(\sqrt{Y})}{2} + 4 \frac{1}{\pi^2} + \ell Y^{-\frac{1}{2}} \right)^2.
\]

By supposing that \( Y \geq 10^{25} \), we may use [6.4] to bound \( \Xi_2(\sqrt{Y}) \). Moreover, by using the bound \( \frac{1}{\pi^2} + \ell Y^{-\frac{1}{2}} \leq \frac{1}{\pi^2} + \ell 10^{-\frac{1}{2}} \), we derive from [6.2] that
\[
S(X, Y; 2P') \leq \frac{2Y}{\log(Y)} \frac{0.05426 Y}{\log^2(Y)}.
\]

In particular, if \( Y \geq 10^{25} \), by [11, Eq. (3.3)] and [6.3], we derive that for all \( X \geq 0 \),
\[
\pi(X + Y) - \pi(X) \leq \frac{2Y}{\log(Y)} - \frac{0.05426 Y}{\log^2(Y)} + \sqrt{Y} \leq \frac{2Y}{\log(Y)} \left( 1 - \frac{0.0271}{\log(Y)} \right).
\]
where we have used that \( \frac{\log^2(Y)}{\sqrt{Y}} \leq \frac{\log^2(10^{25})}{\sqrt{10^{25}}} \).

Now, if \( P \) is any positive integer, not necessarily of the form \( 2P' \) considered in \([5,3]\), we can derive from \([11, \text{Thm. 3.6}] \) and \((6.3)\) that for any \( Y \geq 10^{25} \),

\[
S(X, Y, P) \leq \prod_{p \leq \sqrt{Y}} \left( 1 - \frac{1}{p} \right)^{-1} \times \left( \frac{2Y}{\log(Y)} - 0.05426Y \right). \tag{6.5}
\]

On the other hand, suppose that \( Y \geq 10^{25}q \), where \( a, q, P \in \mathbb{Z}_{>0} \) are such that \( (a, q) = (P, q) = 1 \). We derive from \([11, \text{Thm. 3.8}] \) that the number of integers \( n \) such that \( X < n \leq X + Y \), \( n \equiv a \mod q \) and \( (n, P) = 1 \) can be bounded from above by

\[
\prod_{p \leq \sqrt{Y} \atop p \nmid q} \left( 1 - \frac{1}{p} \right)^{-1} \times \left( \frac{2Y}{q \log \left( \frac{Y}{q} \right)} - 0.05426Y \right). \tag{6.6}
\]

Therefore, we can generalize \((6.4)\) as follows.

**Theorem 6.1 (Brun–Titchmarsh inequality).** Let \( a, q \in \mathbb{Z}_{>0} \) such that \( (a, q) = 1 \) and \( Y \) be a real number such that \( Y \geq 10^{25}q \). Then for all \( X \geq 0 \),

\[
\pi(X + Y; q, a) - \pi(X; q, a) \leq \frac{2Y}{\varphi(q) \log \left( \frac{Y}{q} \right)} \left( 1 - \frac{0.0271}{\log \left( \frac{Y}{q} \right)} \right).
\]

**Proof.** As \( (a, q) = 1 \), if \( p \equiv a \mod q \), then \( p \nmid q \). Proceed as in \([11, \text{Thm. 3.9}] \). By selecting \( P = \prod_{p \nmid q, p \leq \sqrt{Y}} p \), the number of primes \( p \) such that \( X < p \leq X + Y \), \( p \equiv a \mod q \) and \( (p, P) = 1 \) can be bounded with the help of \((6.6)\), giving the upper bound

\[
\prod_{p \leq \sqrt{Y} \atop p \nmid q} \left( 1 - \frac{1}{p} \right)^{-1} \times \left( \frac{2Y}{q \log \left( \frac{Y}{q} \right)} - 0.05426Y \right) \leq \frac{Y}{\varphi(q) \log \left( \frac{Y}{q} \right)} \left( 2 - \frac{0.05426}{\log \left( \frac{Y}{q} \right)} \right). \tag{6.7}
\]

It remains to consider the primes \( X < p \leq X + Y \) such that \( p \equiv a \mod q \) and \( p \nmid P \); by definition of \( P \), its cardinality is at most \( \sqrt{Y} \). Hence, by using that \( \frac{\varphi(q) \log^2 \left( \frac{Y}{q} \right)}{q \sqrt{Y}} \leq \frac{\log^2(10^{25})}{\sqrt{10^{25}}} \), we conclude from \((6.7)\) that \( \pi(X + Y; q, a) - \pi(X; q, a) \) is at most

\[
\frac{Y}{\varphi(q) \log \left( \frac{Y}{q} \right)} \left( 2 - \frac{0.05426}{\log \left( \frac{Y}{q} \right)} \right) + \frac{\varphi(q) \log^2 \left( \frac{Y}{q} \right)}{q \sqrt{Y}} \log \left( \frac{Y}{q} \right) \leq \frac{Y}{\varphi(q) \log \left( \frac{Y}{q} \right)} \left( 2 - \frac{0.05426}{\log \left( \frac{Y}{q} \right)} \right). \tag{6.8}
\]

\( \square \)

The reader may refer to \([10, 12, 19]\) for further insights about the Brun–Titchmarsh theorem.
7 Computational matters

The procedure for calculating converging expressions or constants in this article is standard: we set a precision value, calculate them by a recurrence algorithm until its precision value, and then, if needed, we numerically bound the remaining term, which will be small. These calculations were carried out under the interval arithmetic header intdouble14.2.h, implemented by Platt in C++ (used for example in [14]), providing results with double precision, higher performance and faster speed, when compared to the ARB package of Sage.

The constant c, introduced in §6.2, plays an important role in deriving Theorem 6.1. As discussed in §5, supposing that $U \geq 10^E$, for $E \geq 7$, the parameter $Z = c_E U^3$ must satisfy $U \geq 20Z$ and $Z \geq 4 \times 10^3$. Also, by Proposition 4.18, we must have $c_E < 10^4$. Therefore, we take $c_E \in \left[\frac{4 \times 10^3}{10^3}, \min\left\{\frac{10^4}{20}, 10^4\right\}\right]$. In particular, if $E = 7$, $c_E \in (8.617, 10.773)$, whereas, if $E = 25$, $c_E \in (0.001, 733.9)$. With the latter choice, as we want to obtain a numerical bound in (5.2), we additionally suppose that the three functions $t \geq 10^{\frac{32}{5}} \Rightarrow \log^3 \left(\frac{1}{t^c}\right) \leq A$, $A \in \{1, 2, 4\}$ are decreasing, so that $c_E \leq \frac{10^4}{e}$ and $c_E \in (0.001, 268.837)$. Further, in order to merge (5.2) to the order $\frac{1}{\log(U)}$, we consider $A \in \{1, 2, 3, 5\}$, so that $c_E \in (0.001, 98.9)$.

For the sake of simplicity, we choose $c$ to be integer. The value $c = 10$ considered in the range $U \geq 10^7$, has been chosen to diminish the numerical contribution of the term of magnitude $\frac{\log^3(U)}{U^3}$ to the error (5.3). The value $c = 16$, considered in the range $U \geq 10^{\frac{32}{5}}$, has been chosen to provide the optimal value $B$ in Theorem 6.1, given our exposition. Moreover, in this range, the value $c = 16$, with $v = 2$, is also optimal for merging the bound (5.2) to the expression $\frac{B}{\log(U)}$, as $c$ contributes to $B$ only through $\Sigma_2(U)$, introduced in (5.3); by the same reasoning, when $v = 1$, we choose $c = 70$ to merge the bound (5.2) to the expression $\frac{B}{\log(U)}$, described in (5.3). See the table below.

| $c$ | $B$  | $\gamma_0$ | $c$ | $B$  | $\gamma_0$ | $c$ | $B$  | $\gamma_0$ |
|-----|------|------------|-----|------|------------|-----|------|------------|
| 0.5 | -0.8264 | 12.4874 | 16 | 0.0271 | 0.683 | 60 | -0.0372 | 0.4249 |
| 6   | -0.0069 | 1.4208 | 17 | 0.0269 | 0.6578 | 70 | -0.0199 | 0.4242 |
| 7   | 0.0036 | 1.2519 | 38 | 0.0004 | 0.4573 | 98 | -0.10184 | 0.439 |
| 15  | 0.027 | 0.7118 | 39 | -0.0013 | 0.4541 | 268 | -0.3566 | - |

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References

[1] M. Balazard, Elementary remarks on Möbius’ function. Proc. Steklov Inst. Math. 276 (2012), 33–39.

[2] M.B. Barban, P.P. Vehov, An extremal problem. (Russian) Trudy Moskov. Mat. Obsč. 18 1968 83–90.

[3] O. Bordelès, Some explicit estimates for the Möbius function. Journal of Integer Sequences, Vol. 18 (2015), Article 15.11.1.
[4] A.C. Cojocaru, M.R. Murty, *An introduction to sieve methods and their applications*. Cambridge University Press (2005).

[5] R. De La Bretèche, F. Dress, G. Tenenbaum, *Remarques sur une somme liée à la fonction de Möbius*. (French) Mathematika 166 (2020), 416-421.

[6] F. Dress, H. Iwaniec, G. Tenenbaum, *Sur une somme liée à la fonction de Möbius*. Journal für die reine und angewandte Mathematik (1983), Volume: 340, page 53-58.

[7] M. El Marraki, *Fonction sommatoire de la fonction μ de Möbius, majorations effectives fortes*. J. Théorie Nombres Bordeaux 7 (1995), 407–433.

[8] S. Graham, *An asymptotic estimate related to Selberg’s sieve*. Journal of Number Theory 10, 83-94 (1978).

[9] H. Helfgott, *The ternary Goldbach conjecture*. Book accepted for publication in *Ann. of Math. Studies*. https://webusers.imj-prg.fr/harald.helfgott/anglais/book.html (version 09/2019).

[10] J. Maynard, *On the Brun-Titchmarsh theorem*. Acta Arithmetica (2013), Volume: 157, Issue: 3, page 249-296.

[11] H.L. Montgomery, R.C. Vaughan, *Multiplicative number theory: I. Classical theory*. Cambridge University Press (2007).

[12] H.L. Montgomery, R.C. Vaughan, *On the large sieve*. Mathematika 20 (1973), 119-134.

[13] Y. Motohashi (apud [8]), *On a problem in the theory of sieve methods*. (Japanese) Res. Inst. Math. Sci. Kyoto Univ. Kokyuroko 222 (1974), 9-50.

[14] Y. Motohashi, *A multiple sum involving the Möbius function*. Publ. Inst. Math. (Beograd) (N.S.) 76(90) (2004), 31–39.

[15] D.J. Platt, *Numerical computations concerning the GRH*. Math. Comp. 85 (2016), 3009-3027.

[16] O. Ramaré, *On Šnirel’man’s constant*. Annali della Scuola Normale Superiore di Pisa, Classi di Scienze 4° serie, tome 22, n°4 (1995), pp. 645-706.

[17] O. Ramaré, *Explicit estimates on the summatory functions of the Moebius function with coprimality restrictions*. Acta Arith. 165 (2014), no. 1, 1–10.

[18] O. Ramaré, *Explicit estimates on several summatory functions involving the Moebius function*. Mathematics of Computation Volume 84, Number 293, May 2015, pp. 1359–1387.

[19] A. Selberg, *Collected Papers II: Lectures on Sieves*. Springer Collected Works in Mathematics (1991)

[20] S. Zuniga Alterman, *Explicit averages of square-free supported functions: beyond the convolution method*. Accepted. To appear in Colloquium Mathematicum (2021).