A NOTE ON THE PICARD NUMBER OF SINGULAR FANO 3-FOLDS

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ABSTRACT. Using a construction due to C. Casagrande and further developed by the author in [DN12], we prove that the Picard number of a non-smooth Fano 3-fold with isolated factorial canonical singularities, is at most 6.

INTRODUCTION

Let $X$ be a Fano 3-fold. If $X$ is smooth, we know from the classification results in [MM81], that its Picard number $\rho_X$ is at most 10. Moreover, if $\rho_X \geq 6$, then $X$ is isomorphic to a product $S \times \mathbb{P}^1$, where $S$ is a smooth Del Pezzo surface.

If $X$ is singular, bounds for $\rho_X$ are known only in particular cases. If $X$ is toric and has canonical singularities, then $\rho_X \leq 5$ ([Bat82] and [WW82]). If $X$ has Gorenstein terminal singularities, then $\rho_X \leq 10$, because $X$ has a smoothing which preserves $\rho_X$ (see [Nam97, Theorem 11] and [JR11, Theorem 1.4]). If, instead, $X$ has Gorenstein canonical singularities, it does not admit, in general, a smooth deformation (see [Pro05, Example 1.4] for an example). In this setting, the following holds.

Theorem 0.1. [DN12, Theorem 1.3] Let $X$ be a 3-dimensional $\mathbb{Q}$-factorial Gorenstein Fano variety with isolated canonical singularities. Then $\rho_X \leq 10$.

The proof of this theorem uses a construction introduced by C. Casagrande in [Cas12], and relies on the result of [BCHM10] that Fano varieties are Mori dream spaces (see [HK00] for the definition).

In this paper, using the same construction, we show that the bound given by Theorem 0.1 can be improved if $X$ is actually singular and its singularities are also factorial. Our result is the following.

Theorem 0.2. Let $X$ be a non-smooth factorial Fano 3-fold with isolated canonical singularities. Then $\rho_X \leq 6$.

In the first section of this paper, we recall some preliminary results from [DN12]; the second section contains the proof of Theorem 0.2 and an observation concerning the case $\rho_X = 6$.

Notation and terminology

We work over the field of complex number.

Let $X$ be a normal variety. We call $X$ Fano if $-K_X$ has a multiple which is an ample Cartier divisor. We denote by $X_{reg}$ the non-singular locus of $X$. We say that $X$ is $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier, i.e. it admits a multiple which is Cartier. We call $X$ factorial if all its local rings are UFD; by [Har77, II, Proposition 6.11], this implies that every Weil divisor of

This work has been partially supported by PRIN 2009 “Moduli, strutture geometriche e loro applicazioni”.

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X is Cartier. We refer the reader to [KM98] for the definition and properties of terminal and canonical singularities. If X has canonical singularities, it is called Gorenstein if its canonical divisor $K_X$ is a Cartier divisor.

We denote with $\mathcal{N}_1(X)$ (resp. $\mathcal{N}^1(X)$) the vector space of one-cycles (resp. $\mathbb{Q}$-Cartier divisors) with real coefficients, modulo the relation of numerical equivalence. The dimension of these two real vector spaces is, by definition, the Picard number of $X$, and is denoted by $\rho_X$. We denote by $[C]$ (resp. $[D]$) the numerical equivalence class of a one-cycle (resp. a $\mathbb{Q}$-Cartier divisor).

Given $[D] \in \mathcal{N}^1(X)$, we set $D^\perp := \{ \gamma \in \mathcal{N}_1(X) | D \cdot \gamma = 0 \}$, where $\cdot$ denotes the intersection product. We define $\text{NE}(X) \subset \mathcal{N}_1(X)$ as the convex cone generated by classes of effective curves and $\text{NE}(X)$ is its closure. An extremal ray $R$ of $X$ is a one-dimensional face of $\text{NE}(X)$. We denote by Locus($R$) the subset of $X$ given by the union of curves whose class belongs to $R$.

A contraction of $X$ is a projective surjective morphism with connected fibers $\varphi : X \to Y$ onto a projective normal variety $Y$. It induces a linear map $\varphi_* : \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$ given by the push-forward of one-cycles. We set $\text{NE}(\varphi) := \text{NE}(X) \cap \text{ker}(\varphi_*)$. We say that $\varphi$ is $K_X$-negative if $K_X \cdot \gamma < 0$ for every $\gamma \in \text{NE}(\varphi)$.

The exceptional locus of $\varphi$ is the locus where $\varphi$ is not an isomorphism; we denote it by $\text{Exc}(\varphi)$. We say that $\varphi$ is of fiber type if $\dim(X) > \dim(Y)$, otherwise $\varphi$ is birational. We say that $\varphi$ is elementary if $\dim(\ker(\varphi_*)) = 1$. In this case $\text{NE}(\varphi)$ is an extremal ray of $\text{NE}(X)$; we say that $\varphi$ (or $\text{NE}(\varphi)$) is divisorial if $\text{Exc}(\varphi)$ is a prime divisor of $X$ and it is small if its codimension is greater than 1.

An elementary contraction from a 3-fold $X$ is called of type $(2, 1)$ if $\varphi$ is $K_X$-negative, birational, $\dim(\text{Exc}(\varphi)) = 2$ and $\dim(\varphi(\text{Exc}(\varphi))) = 1$.

If $D \subset X$ is a Weil divisor and $i : D \to X$ is the inclusion map, we set $\mathcal{N}_1(D, X) := i_* \mathcal{N}_1(D) \subseteq \mathcal{N}_1(X)$.

1. Preliminaries

In the following statement, we collect some results from [DN12]. For the reader’s convenience, we recall here the main steps of their proof. We refer the reader to [DN12] Theorem 2.2] for the properties of contractions of type $(2, 1)$ defined on mildly singular 3-folds.

**Lemma 1.1.** [DN12 Theorem 1.2 and its proof - Remark 5.2] Let $X$ be a $\mathbb{Q}$-factorial Gorenstein Fano 3-fold with isolated canonical singularities. Suppose $\rho_X \geq 6$. Then there exist morphisms

$$
\psi : X \to \mathbb{P}^1 \text{ and } \xi : X \to S,
$$

where $S$ is a normal surface with $\rho_S = \rho_X - 1$, and the morphism

$$
\pi := (\xi, \psi) : X \to S \times \mathbb{P}^1
$$

is finite.

Moreover there exist extremal rays $R_0, \ldots, R_m$ ($m \geq 3$) in $\text{NE}(X)$ such that:

- each $R_i$ is of type $(2, 1)$;
- $\text{NE}(\psi) = R_0 + \cdots + R_m$;
- for $i = 0, \ldots, m$, set $E_i = \text{Locus } R_i$ and $Q = \text{NE}(\xi)$. Then

$$
\psi(E_i) = \mathbb{P}^1, \quad \mathcal{N}_1(E_i, X) = \mathbb{R} R_i \oplus \mathbb{R} Q \quad \text{and} \quad Q \subseteq \bigcap_{i=0}^m E_i^\perp;
$$
• $\psi$ factors as $X \xrightarrow{\sigma} \tilde{X} \rightarrow \mathbb{P}^1$, where $\sigma$ is birational, $\tilde{X}$ is a Fano 3-fold with canonical isolated singularities, $\text{NE}(\sigma) = R_1 + \cdots + R_s$, with $m \geq s \in \{\rho_X - 2, \rho_X - 3\}$ and $\sigma(E_1), \ldots, \sigma(E_s) \subset \tilde{X}$ are pairwise disjoint.

**Proof.** By [DN12] Remark 5.2, the assumption $\rho_X \geq 6$ implies that all the assumptions of [DN12] Theorem 1.2 are satisfied, from which the existence of the finite morphism $\pi$. The properties of its projections $\psi$ and $\xi$ follow by their construction, that we briefly recall. All the details can be found in the proof of [DN12] Theorem 1.2.

By [DN12] Proposition 3.5, there exists an extremal ray $R_0 \subset \text{NE}(X)$ of type $(2,1)$. Set $E_0 = \text{Locus}(R_0)$; we have $\dim N_1(E_0, X) = 2$. As in [DN12] Lemma 3.1, we may find a Mori program

\[ X = X_0 \xrightarrow{\sigma_0} X_1 \longrightarrow \cdots \longrightarrow X_{k-1} \xrightarrow{\sigma_{k-1}} X_k \xrightarrow{\varphi} Y \]

where $X_1, \ldots, X_k$ are $\mathbb{Q}$-factorial 3-folds with canonical singularities and, for each $i = 0, \ldots, k-1$, there exists a $K_{X_i}$-negative extremal ray $Q_i \subset \text{NE}(X_i)$ such that $\sigma_i$ is either its contraction, if $Q_i$ is divisorial, or its flip, if it is small. Moreover, if $(E_0)_i \subset X_i$ is the transform of $E_0$ and $(E_0)_{i_0} := E_0$, then $(E_0)_i : Q_i > 0$. Finally, $\varphi$ is a fiber type contraction to a $\mathbb{Q}$-factorial normal variety $Y$.

Let us set

\[ \{i_1, \ldots, i_s\} := \{i \in \{0, \ldots, k-1\} | \text{codim} N_1(D_{i+1}, X_{i+1}) = \text{codim} N_1(D_i, X_i) - 1\}. \]

Then, by [DN12] Lemma 3.3, $s \in \{\rho_X - 2, \rho_X - 3\}$ (in particular $s \geq 3$); moreover, for every $j \in \{1, \ldots, s\}$, $Q_{i_j}$ is a divisorial ray, $\sigma_{i_j}$ is a birational contraction of type $(2,1)$ and, if $E_j \subset X$ is the transform of the exceptional divisors of the contraction $\sigma_{i_j}$ as above, then $E_1, \ldots, E_s$ are pairwise disjoint.

Since $s \geq 3$, [DN12] Proposition 3.5 assures that, for each $j = 1, \ldots, s$, there exists an extremal ray $R_j \subset \text{NE}(X)$ of type $(2,1)$ such that $E_j = \text{Locus}(R_j)$. The divisor $-K_X + E_1 + \cdots + E_s$ comes out to be nef, and its associated contraction $\sigma : X \to \tilde{X}$ verifies

\[ \ker(\sigma_s) = \mathbb{R}R_1 + \cdots + \mathbb{R}R_s \quad \text{and} \quad \text{Exc}(\sigma) = E_1 \cup \cdots \cup E_s. \]

It is thus possible to look at $\sigma$ a part of a Mori program as in (1.1), and to find a fiber type contraction $\varphi : \tilde{X} \to Y$ giving rise to a morphism $\psi := \varphi \circ \sigma : X \to Y$ as in the statement. In particular, we have $\text{NE}(\psi) = R_0 + \cdots + R_m$, where $m \geq s$ and $R_{s+1}, \ldots, R_m$ are extremal rays of type $(2,1)$. We notice that, since $\dim(X) = 3$, we have $Y \cong \mathbb{P}^1$ by [DN12] Remark 4.2.

The second projection $\xi$ arises as the contraction associated to a certain nef divisor defined as a combination of the prime divisors $E_0, \ldots, E_m$ constructed above (recall that $E_i = \text{Locus}(R_i)$ for $i = 0, \ldots, m$). It is an elementary contraction and the one-dimensional subspace generated by $\text{NE}(\xi)$ belongs to $N_1(E_i, X)$ for every $i = 0, \ldots, m$. \( \square \)

2. **Theorem 0.2**

**Proof of Theorem 0.2.** Let us prove that, if $\rho_X \geq 7$, then the morphism $\pi : X \to S \times \mathbb{P}^1$ given by Lemma 1.1 is an isomorphism. This will give a contradiction with our assumptions on the singularities of $X$, since $S \times \mathbb{P}^1$ is smooth or has one-dimensional singular locus.

We are in the setting of Lemma 1.1 let us keep its notations. By [AW97] Corollary 1.9 and Theorem 4.1(2)], the general fiber of $\xi$ is a smooth rational curve, and the other fibers have
at most two irreducible components (that might coincide) whose whose reduced structures are isomorphic to \(\mathbb{P}^1\).

Our assumptions imply that \(S\) is factorial: if \(C \subset S\) is a Weil divisor, its counterimage \(D := \xi^{-1}(C) \subset X\) is a Cartier divisor, because \(X\) is factorial. Moreover \(D \cdot Q = 0\) (where \(Q = \text{NE}(\xi)\)), because \(D\) is disjoint from the general fiber of \(\xi\). Then \(D = \xi^*(C')\) for a certain Cartier divisor \(C'\) on \(S\). But then \(C = C'\) is Cartier.

Fix \(i = 0, \ldots, m\); let \(\varphi_i : X \to Y_i\) be the contraction of \(R_i\) and set \(G_i := \varphi_i(E_i) \subset Y_i\), \(T_i := \xi(E_i) \subset S\):

\[
\begin{array}{ccc}
\varphi_i|E_i & \longrightarrow & E_i \\
\downarrow & & \downarrow \\
G_i & \longrightarrow & T_i.
\end{array}
\]

Notice that \(T_i \subset S\) is a curve. Indeed, by Lemma 1.1 \(E_i \cdot Q = 0\), which implies that \(T_i \subset S\) is a (Cartier) divisor and \(E_i = \xi^*(T_i)\).

Let \(f_i\) be the general fiber of \(\varphi_i\). Since \(f_i\) is a smooth rational curve which dominates \(T_i\), \(T_i\) is a (possibly singular) rational curve. The same conclusion holds for \(G_i\), which is dominated by any smooth curve contained in a fiber of \(\xi\) over \(T_i\).

We have

\[-1 = E_i \cdot f_i = \xi^*(T_i) \cdot f_i = T_i^2 \cdot \deg(\xi|f_i),\]

from which \(-T_i^2 = \deg(\xi|f_i) = 1\). Then the general fiber \(g\) of \(\xi\) over \(T_i\) is a smooth rational curve. Indeed, \(g\) has no embedded points, and if, by contradiction, the 1-cycle associated to \(g\) is of the type \(C_1 + C_2\), then \(g\) would intersect \(f_i\) in at least two (distinct or coincident) points.

This is impossible because \(g\) is general and \(\deg(\xi|f_i) = 1\).

Then \(E_i\) is smooth along the general fibers of both \(\varphi_i\) and \(\xi\); we deduce that \(E_i\) is smooth in codimension one. Moreover \(E_i\) is a Cohen-Macaulay variety, because \(X\) is factorial. Then, by Serre’s criterion, \(E_i\) is normal. Then the finite morphism \((\xi|E_i, \varphi_i|E_i) : E_i \to T_i \times G_i\), which has degree one, factors through the normalization of the target: there is a commutative diagram

\[
\begin{array}{ccc}
E_i & \longrightarrow & \mathbb{P}^1 \\
\downarrow & & \downarrow \\
T_i \times G_i.
\end{array}
\]

Since \(\tau\) is finite of degree one, by Zariski Main Theorem, it is an isomorphism. Thus \(E_i \cong \mathbb{P}^1 \times \mathbb{P}^1\), and \(\xi|E_i : E_i \to T_i \cong \mathbb{P}^1\) and \(\varphi_i|E_i : E_i \to G_i \cong \mathbb{P}^1\) are the projections. In particular, since both \(E_i\) and \(T_i\) are Cartier divisors, they are contained in the smooth loci of, respectively, \(X\) and \(S\).

We have

\[
(K_X - \xi^*(K_S)) \cdot f_i = (K_{E_i} - \xi_{E_i}^*(K_{T_i})) \cdot f_i = (\varphi_{i|E_i}^*(K_{G_i})) \cdot f_i = 0.
\]

Let \(F\) be a general fiber of \(\psi : X \to \mathbb{P}^1\). Then \(F\) is a smooth Del Pezzo surface and, by Lemma 1.1 \(\mathcal{N}_i(F) \subset \sum \mathbb{R}[f_i]\); thus \(K_X - \xi^*(K_S)\) is numerically trivial in \(F\). Moreover \(\zeta := \xi|F : F \to S\) is a finite morphism of degree \(d := \deg(\pi)\) and

\[
K_F = (K_X)|_F = (\xi^*(K_S))|_F = \zeta^* K_S;
\]
in particular \( \zeta \) is unramified in the open subset \( \xi^{-1}(S_{\text{reg}}) \), which contains \( E_i \cap F \) for every \( i = 0, \ldots, m \).

Set \( \tilde{F} := \sigma(F) \subset \tilde{X} \), where \( \sigma : X \to \tilde{X} \) is the birational contraction given by Lemma 1.1: then \( \tilde{F} \) is again a smooth Del Pezzo surface and \( \sigma|_F : F \to \tilde{F} \) is a contraction. For every \( i = 1, \ldots, s \), the intersection \( E_i \cap F \) is the union of \( d \) disjoint curves numerically equivalent to \( f_i \); in particular \( \sigma|_F \) realizes \( F \) as the blow-up of \( \tilde{F} \) along \( s \cdot d \) distinct points (where \( s = \rho_X - \rho_{\tilde{X}} \)). Then, recalling that \( s \geq \rho_X - 3 \) and \( \rho_X \geq 7 \), we get

\[
9 \geq \rho_F = \rho_{\tilde{F}} + s \cdot d \geq 1 + 4d,
\]

and then \( d \leq 2 \). Moreover, if \( d = 2 \), then \( \rho_F = 9 \) and, by 2.1,

\[
1 = K_{\tilde{F}}^2 = \zeta^*(K_S) \cdot K_F = 2(K_S)^2,
\]

which is impossible because \( S \) is factorial and thus \( K_S^2 \) is integral. Hence \( d = \deg(\zeta) = \deg(\pi) = 1 \) and the statement is proved.

The case \( \rho_X = 6 \) is more complicated to analyze. Indeed, though Lemma 1.1 still holds in that case, we are not able to conclude that \( \pi \) is an isomorphism and that, as a consequence, \( X \) is smooth.

**Proposition 2.1.** Let \( X \) be a factorial Fano 3-fold with isolated canonical singularities and with \( \rho_X = 6 \). If \( X \) is not smooth, there exists a finite morphism of degree 2

\[
\pi : X \to S \times \mathbb{P}^1,
\]

where \( S \) is a singular Del Pezzo surface with factorial canonical singularities, \( \rho_S = 5 \), \( (K_S)^2 = 1 \). Moreover the ramification locus of \( \pi \) contains a surface \( R \) which dominates \( S \).

**Proof.** We argue as in the proof of Theorem 0.2 and we use the same notations. Since \( X \) is not smooth, the degree of \( \pi \) must be 2. Exactly as in the above case, we have

\[
(2.2) \quad K_F = (K_X)|_F = (\xi^*(K_S))|_F = (\xi^*K_S)|_F = \zeta^*K_S,
\]

and

\[
(2.3) \quad \rho_F = 10 - (K_F)^2 = 10 - 2(K_S)^2,
\]

so that \( \rho_F \) needs to be even. Since \( \rho_X = 6 \), we have \( s \in \{3, 4\} \), and then

\[
9 \geq \rho_F = \rho_{\tilde{F}} + 2s.
\]

Thus the only possibility is that \( \rho_{\tilde{F}} = 2 \) and \( \rho_F = 8 \). By (2.3), we get \( (K_S)^2 = 1 \).

Let us call \( R \) the ramification divisor (possibly trivial) of \( \pi \). Let \( C \) be the general fiber of \( \xi \). Then \( C \cong \mathbb{P}^1 \) and \( \psi_C : \mathbb{P}^1 \to \mathbb{P}^1 \) is finite of degree 2. By Hurwitz’s formula we have \( R \cdot C = 2 \), and hence \( R \) is not trivial and it dominates \( S \).

**Acknowledgments.** This paper is part of my PhD thesis; I am deeply grateful to my advisor Cinzia Casagrande for her constant guidance.
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