Fatigue life prediction method of mechanical parts based on Weibull distribution

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Abstract. Aiming at the randomness of fatigue life of mechanical parts, the reliability analysis and fatigue life prediction of mechanical parts are carried out through the Weibull distribution of three parameters. The process of solving the complex transcendental equation is analyzed, and the method of solving the likelihood function by unconstrained optimization method is proposed to estimate the values of the three parameters of Weibull distribution. A complete and accurate Weibull distribution model was established to analyze and calculate the data of gear tooth fracture, and the accuracy of the model was verified. The fatigue life of mechanical parts could be predicted by this model.

Keywords: Fatigue life; Weibull distribution; Failure probability; Unconstrained optimization.

1. Introduction
As an important part of mechanical structure, the reliability of mechanical parts directly affects whether the whole mechanical system can work normally. Fatigue life of mechanical parts is an important index to ensure the normal operation of the whole mechanical system. Rayleigh distribution, normal distribution, lognormal distribution, exponential distribution and weibull distribution can be used to simulation analysis the fatigue life of mechanical parts [1]. However the reliability data of mechanical parts always get through a large number of experiences. Then, similar distribution functions can be found after analysis and suitables means can be finded. Finally, we can get precise data. Through many experiments, analyses and calculations, it is considered that three-parameters weibull is the most suitable probability distribution to describe the reliability of mechanical products, mechanical parts, engineering materials and structural components. In the three-parameter weibull distribution, the accuracy of the parameters directly affects the analysis result of the final reliability. Mapping method, regression analysis and method of moment estimation method can solve the necessary parameters, but these methods have their own disadvantages: drawing method is simple but its accuracy is not high, and the maximum likelihood method with high precision, evaluation method is more difficult [2]. In this paper, the unconstrained optimization method is mainly used to solve the likelihood function to solve the Weibull probability distribution function parameters.
2. Weibull Distribution

2.1. Three Parameters Of The Weibull Distribution.

A large number of experiments have proved that Weibull distribution has a very wide applicability. Because it is derived from the weakest link model or series model, it is especially applicable to the reliability of mechanical parts. By adjusting each parameter of the distribution, different distribution forms can be formed, so the life characteristic model of different types of products can be established [3]. The fatigue life of many mechanical parts and mechanical products is consistent with weibull distribution.

The three parameters of the three-parameter Weibull distribution are the shape parameter, the dimension parameter, and the position parameter, which greatly improve the accuracy of the distribution. The distribution function is as follows:

\[ F(t) = 1 - \exp\left[-\left(\frac{t - \gamma}{\lambda}\right)^k\right], t > \gamma \]  

(1)

In this formula, \( k \geq 0, \lambda > 0, \gamma \geq 0 \), as \( X \sim \text{Wei}(\lambda, k; \gamma) \). The probability density distribution function is:

\[ f(t; \lambda, k, \gamma) = \frac{k}{\lambda} \left(\frac{t - \gamma}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{t - \gamma}{\lambda}\right)^k\right] \]

(2)

In this formula, \( t \)- Life random variable
\( k \)- Shape parameters, the shape of Weibull curve can be changed by changing K value, and the magnitude of K value is related to the failure mechanism of the product;
\( \lambda \)- Dimension parameter, also known as characteristic life. That is, the life when the reliability is 63.2%. The stretching of the Weibull curve in the coordinate system can be controlled by changing the value of \( \lambda \);
\( \gamma \)- Position parameter, also known as minimum life, By changing the value of \( \gamma \), the initial position of the Weibull curve in the coordinate system can be controlled, that is, there is a latency for a period of time when \( \gamma > 0 \).

Figure 1. (a) Shape parameter variation diagram of probability density distribution (b) Variation diagram of size parameters of probability density distribution

Figure 1(a) displays the parameter variation images of Probability density distribution shape, because \( \gamma \) is location parameters. Just decided to weibull curve at the x axis starting position, so temporarily by \( \gamma = 0 \), makes the position of the image from \( x = 0 \), set for size parameter \( \lambda = 1 \), only change the value of the shape parameter, \( k \), got the figure 1(a). Caused by the change of the \( k \) function image change, can easily analyze the failure mechanism of different products. When \( k < 1 \), the image is similar with the
gamma distribution image; When k=1, the image is similar with the exponential distribution image. When k>1, the image is close to the normal distribution image. Therefore, we can adapt the failure mechanism of various products by changing the shape parameter k.

Figure 1(b) displays the parameter variation images of Probability density distribution size, because \( \gamma \) is position parameters, just decided to weibull curve at the x axis starting position. So temporarily by \( \gamma = 0 \), makes the position of the image from x = 0, the shape parameter k =1, only change size parameter \( \lambda \) values, got the figure 1(b). The change in the slope of a function graph caused by the change in \( \lambda \), It is also call characteristic life time difference. When \( \lambda <1 \), the slope of the curve changes a lot; When \( \lambda =1 \), the slope of the curve is relatively moderate; When \( \lambda >1 \), as \( \lambda \) goes up, the slope of the curve gets flatter and flatter.

Through the above analysis, it is known that by changing the values of the three parameters, different distribution models can be obtained to adapt to different situations. Therefore, the reliability of products and parts can be analyzed by solving the three parameters of weibull distribution, the failure and life of products and parts can be predicted.

2.2. Weibull Distribution Function Form.
If the unreliability function and reliability function of the three-parameter Weibull distribution are F (t) and R (t), respectively, then the unreliability function F (t) is:

\[
F(t) = 1 - \exp\left(-\left(\frac{t-\gamma}{\lambda}\right)^k\right)
\]  \hspace{1cm} (3)

The reliability function R (t) obtained from (3) is:

\[
R(t) = \exp\left(-\left(\frac{t-\gamma}{\lambda}\right)^k\right)
\]  \hspace{1cm} (4)

Calculate the relatively accurate values of the three parameters of the Weibull distribution, and bring the values into equation (3) (4), that is, the relatively accurate images of product reliability and cumulative failure probability can be obtained. According to the images, the accuracy of the model can be compared with the test data, and the product life can be predicted by the image curve.

3. Reliability Analysis Of Gear Teeth

3.1. Model establishment.
From the above analysis, it can be seen that the accuracy of the three parameters directly affects the accuracy of Weibull distribution for product reliability analysis and life prediction. The commonly used methods for solving parameters are mapping method, regression analysis, moment estimation method, Bayes estimation method and maximum likelihood estimation method [4]. This paper uses the method of solving likelihood function to solve the parameters of Weibull probability distribution function.

Firstly, the maximum likelihood function of the probability density function of Weibull distribution is estimated, and the maximum likelihood function is established:

\[
Max : L(k, \lambda, \gamma) = \prod_{i=1}^{n} f(t_i, k, \lambda, \gamma) = \prod_{i=1}^{n} \left(\frac{k}{\lambda}\right)^{k-1} \left(\frac{t_i - \gamma}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{t_i - \gamma}{\lambda}\right)^k\right]
\]  \hspace{1cm} (5)

If you take a logarithmic logarithm to (5), you can get:

\[
Min : L_i = -\ln L(k, \lambda, \gamma) = -\sum_{i=1}^{n} \ln \left[\left(\frac{k}{\lambda}\right)^{k-1} \left(\frac{t_i - \gamma}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{t_i - \gamma}{\lambda}\right)^k\right]\right]
\]  \hspace{1cm} (6)

The maximum likelihood estimation can be obtained by finding the partial derivative of the seemingly function:

\[
\frac{\partial \ln[L(k, \lambda, \gamma)]}{\partial k} = 0
\]  \hspace{1cm} (7)
Expand (7) (8) (9) to get:

\[
\sum_{i=1}^{n} \left[ \frac{1}{k} \ln(t_i - \gamma) - \ln \lambda - \frac{(t_i - \gamma)^k}{\lambda} \ln(t_i - \gamma) \right] + \sum_{i=1}^{n} \left[ -\frac{(t_i - \gamma)^k}{\lambda} \ln(t_i - \gamma) \right] = 0
\]  

(10)

\[
\sum_{i=1}^{n} \left[ -\frac{k-1}{t_i - \gamma} + \frac{k}{\lambda} (t_i - \gamma)^{k-1} \right] + \sum_{i=1}^{n} \left[ \frac{k}{\lambda} (t_i - \gamma)^{k-1} \right] = 0
\]  

(11)

\[
\sum_{i=1}^{n} \left[ -\frac{k}{\lambda} + \frac{k}{\lambda} (t_i - \gamma)^k \right] + \sum_{i=1}^{n} \left[ \frac{k}{\lambda} (t_i - \gamma)^k \right] = 0
\]  

(12)

Due to the above (10) (11) (12) three formula is too complex transcendental equation, It can be used iterative method to solve, such as Newton-Raphson method [5]. but with access to a lot of data and found that this method solving process is extremely complex, and convergence because influenced by the initial value is bigger, lead to poor convergence, sometimes even can't get a relatively accurate parameter values. Therefore, the method of unconstrained optimization is used to solve the likelihood function.

There are two main methods to solve unconstrained optimization problems, inculding direct search method and gradient method. The direct search method is applicable to the situation where the objective function is highly nonlinear and there is no derivative or derivative or derivative is difficult to solve. The commonly used direct search methods are simplex method, hooke-jeeves search method and Pavell conjugate direction method, etc. [6, 7].In the case that the derivative of the function can be found, the gradient method is a better method, which USES the function gradient and the Hessian matrix construction algorithm, can get faster convergence speed. Common gradient methods include Newton method, Marquart method, conjugate gradient method and quasi-newton method [8]. Establish the unconstrained optimization model:

\[
\begin{align*}
\min F(t) \\
t = [t_1, t_2, \ldots, t_n] \in \mathbb{R}^n
\end{align*}
\]  

(13)

The method to solve the optimal solution \((t^*, F^*)\) of the above problems is called unconstrained optimization method. General idea of the algorithm is selecting an initial point as the starting point of the search, generating the search direction one by one according to the prescribed path, taking the appropriate step size, finding the new iteration point of the function according to the search direction, until find the best advantage. The optimal design is to minimize the value of the objective function. so the search direction is the negative direction of the point, so that the value of the function drops the fastest near the point, this method is the fastest drop method.

According to the context, the mathematical model of the solution can be established, and the steepest descent method can be used to find the optimal solution \((t^*, F^*)\) \(x_1 = \gamma, x_2 = k, x_3 = \lambda\), due to formula (6) and formula (13) could get:

\[
\begin{align*}
\text{Min} : L_4 = -\ln L(x_2, x_3, x_1) &= -\sum_{i=1}^{n} \ln \left\{ \frac{x_2}{x_3} \left( \frac{L_i - x_1}{x_3} \right)^{k-1} \exp\left[ -\left( \frac{L_i - x_1}{x_3} \right)^{k} \right] \right\}
\end{align*}
\]  

(14)

The solution of the equation (14) is obtained by the steepest descent method, that is, the three parameters of the Weibull distribution function.
3.2. Reliability Analysis Of Gear Teeth.

According to the formula derived in 3.1, the reliability analysis and life prediction of gear teeth are carried out. Taking the test of distribution of fatigue fracture stress cycles of gear teeth as an example, the specific test data are based on the test data of gear tooth fracture test in reference [11]. According to the maximum likelihood estimation and unconstrained optimization algorithm introduced above, the parameters of the three-parameter Weibull distribution are estimated by finding the optimal solution of the formula (14), and the estimated values of the parameters are as follows: \( k = 1.129 \), \( \lambda = 4642.000 \), \( \gamma = 3941.700 \), \( L_1 = 666.769 \). Then it is known that the gear life distribution function is:

\[
F(t) = 1 - e^{\left(\frac{t - 3941.7}{4642}\right)^{1.129}}
\]

The reliability distribution function of the gear is:

\[
R(t) = e^{\left(\frac{t - 3941.7}{4642}\right)^{1.129}}
\]

The life distribution function curve is shown in figure 2(a), the reliability distribution function curve is shown in figure 2(b):

![Figure 2](image)

**Figure 2.** (a) Life distribution function of gear (b) Reliability distribution curve of gear

According to the relationship among reliability function, failure density function and failure rate function, \( \eta(t) = f(t)/R(t) \), the expression of failure rate density function of gear can be obtained as follows:

\[
\eta(t) = k \left(\frac{t - \gamma}{\lambda}\right)^{k-1} \exp\left[-\left(\frac{t - \gamma}{\lambda}\right)^k\right] \exp\left[-\left(\frac{t - \gamma}{\lambda}\right)^k\right]
\]

By inserting the resulting three-parameter estimates (17), the failure rate function of the gear can be obtained, as shown in figure 3:

![Figure 3](image)

**Figure 3.** Failure rate function of gear

As can be seen from figures 2 and 3, the problem of tooth breaking will hardly occur before the number of stress cycles reaches $3 \times 10^3$ times, and when the number of stress cycles reaches $1.5 \times 10^4$ times, the reliability tends to be stable, but the reliability is very low. At this time, it is very easy to break teeth, so the gear should be replaced in time to prevent accidents.

4. Summary
Through the unconstrained optimization method to solve the likelihood function to solve the parameter process of the Weibull probability distribution function, we can see that, compared with the iterative method to solve the likelihood function, the method mentioned in this paper has more simple formula and avoids solving the transcendental equation and the problem of non-convergence. Through the unconstrained optimization method to solve the likelihood function to solve the parameter process of the Weibull probability distribution function, we can see that, compared with the iterative method to solve the likelihood function, the method mentioned in this paper has more simple formula and avoids solving the transcendental equation and the problem of non-convergence. At the same time, because the loss degree and service life of mechanical parts have a great influence on the stability of the whole mechanical structure, the life and remaining life of mechanical parts can be analyzed and evaluated based on Weibull distribution. It can better help some mechanical parts of mechanical structures in engineering to be maintained and updated.

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