We investigate numerically the model proposed in Sahoo \textit{et al.} (2017 \textit{Phys. Rev. Lett.} 118, 164501) where a parameter $\lambda$ is introduced in the Navier–Stokes equations such that the weight of homochiral to heterochiral interactions is varied while preserving all original scaling symmetries and inviscid invariants. Decreasing the value of $\lambda$ leads to a change in the direction of the energy cascade at a critical value $\lambda_c \sim 0.3$. In this work, we perform numerical simulations at varying $\lambda$ in the forward energy cascade range and at changing the Reynolds number $Re$. We show that for a fixed injection rate, as $\lambda \to \lambda_c$, the kinetic energy diverges with a scaling law $E \propto (\lambda - \lambda_c)^{-2/3}$. The energy spectrum is shown to display a larger bottleneck as $\lambda$ is decreased. The forward heterochiral flux and the inverse homochiral flux both increase in amplitude as $\lambda_c$ is approached while keeping their difference fixed and equal to the injection rate. As a result, very close to $\lambda_c$ a stationary state is reached where the two opposite fluxes are of much higher amplitude than the mean flux and large fluctuations are observed. Furthermore, we show that intermittency as $\lambda_c$ is approached is reduced. The possibility of obtaining a statistical description of regular Navier–Stokes turbulence as an expansion around this newly found critical point is discussed.

This article is part of the theme issue ‘Scaling the turbulence edifice (part 2)’.

1. Introduction

In a turbulent flow, energy is injected at large scales and dissipated at much smaller scales by viscosity [1]. A transfer of energy is thus required from one scale to the other that is achieved by the energy cascade caused by the nonlinearity of the Navier–Stokes equations (NSEs).
The fundamental idea of the energy cascade across scales was first introduced by Richardson [2] and later quantified by Kolmogorov [3]. Under the assumption of scale-similarity, Kolmogorov predicted a power-law behaviour for the energy spectrum $E(k) \propto k^{-5/3}$ and for the scaling of the moments of velocity’s differences across a distance $r$: $\langle |\delta u|^p \rangle \propto (er)^{p/3}$. However, overwhelming experimental and numerical evidence have shown that the process of transferring energy from one scale to the other is not self-similar and that there exist anomalous exponents. 

There have been various attempts to predict and explain the observed exponents, see e.g. [5–8]. However, all these attempts are based on simplified phenomenological assumptions and no exact or systematic derivation of the anomalous corrections directly from the NSEs has been proposed so far. As a result, the existence of exactly solvable limits from where to develop perturbative or asymptotic expansions has been long sought. The main theoretical obstacle to attack three-dimensional turbulent flows comes from being out-of-equilibrium, with anomalous scaling laws and stronger and stronger non-Gaussian small-scale statistics at increasing Reynolds numbers. In general, no universal recipes exist for the treatment of out-of-equilibrium problems. Equilibrium Gaussian or quasi-Gaussian systems have predictable statistics but are only met in fluid dynamics for the truncated Euler equations where only a finite number of Fourier modes are kept. In this case, energy is conserved exactly and no finite energy flux through scales exists [9–11]. Although an expansion from such a state to a weakly cascading case can be performed [12] it seems unlikely to serve as a starting point to recover regular Navier–Stokes turbulence. Solvable out-of-equilibrium states over which an expansion could be carried out have been sought with the use of re-normalization group theory. Some of such studies consider deviations from a power-law force spectrum $F \propto k^{-(d+\epsilon)}$, where $\epsilon = d - 2$ (with $d=3$ the dimension) corresponds to a molecular background noise, representing fluctuations in an equilibrium fluid at absolute temperature and $\epsilon = 4$ corresponds to real turbulence [13–16]. Other studies consider expansions from critical dimensions $d \gg 3$ where turbulence is conjectured to follow mean field dynamics, or from $d = 4/3$ where the finite flux spectrum coincides with thermal spectrum, or at changing the couplings among triads in Fourier space [17–24]. The effect of helicity has also been investigated with these techniques and found to play a minor role [25–27]. Higher or non-integer dimensions however are not physically realizable and can be studied and tested with the use of numerical simulations only [28–32].

More recently, systems where one dimension is compactified were demonstrated to result in a transition from three-dimensional behaviour with a forward energy cascade, towards a two-dimensional behaviour where energy cascades backward [33–35]. In this kind of transition, the systems dimension $d$ does not vary continuously from $d=3$ to $d=2$ as in the previous considerations but have the advantage of being physically realizable. Similar transitions have been observed in a variety of physical systems at varying control parameters, including the rotation intensity, magnetic fields, stratification and forcing properties. For a recent review see [36]. In most of these cases, the transition from forward to inverse cascade occurred through a split state where both fluxes exist simultaneously. Recently, a variant of the NSEs was introduced, with a dimensionless control parameter, $\lambda$, weighing the relative importance of homochiral and heterochiral triads and developing an abrupt transition from forward to inverse cascade at a critical value $\lambda_c$ [37]. Right at the critical value, a new singular state exists where energy does not cascade, either forward or inverse without the flow being necessarily at equilibrium.

Here, we study in greater detail the behaviour of the system when $\lambda$ is close but above this critical value. We argue that as the critical point is approached the flow is closer and closer to a flux-loop state where the mean energy flux towards the small scales is subdominant and large turbulent fluctuations that transfer energy both to large and small scales develop. We also demonstrate that intermittency in our model is reduced as $\lambda \to \lambda_c$. This could indicate that the
fluctuation at $\lambda = \lambda_c$ has a more tractable statistics and could serve as starting point for perturbative expansion towards real Navier–Stokes turbulence at $\lambda = 1$.

2. Formulation

(a) Helical decomposition

Let $\mathbf{u}(t, x)$ be a zero-mean divergence-free vector field defined in a cubic triple periodic domain of side $L$. Its Fourier transform $\hat{\mathbf{u}}_k(t)$ is given by

$$\hat{\mathbf{u}}_k(t) = \frac{1}{L^3} \int \mathbf{u}(t, x) e^{-i\mathbf{k} \cdot \mathbf{x}} \, dx^3$$

and

$$\mathbf{u}(t, x) = \sum_k \hat{\mathbf{u}}_k(t) e^{i\mathbf{k} \cdot \mathbf{x}},$$

(2.1)

where the three component complex vector $\hat{\mathbf{u}}_k$ satisfies $\mathbf{k} \cdot \hat{\mathbf{u}}_k = 0$ due to the divergence-free condition. Thus, each $\hat{\mathbf{u}}_k$ has two independent degrees of freedom. A convenient way to express these degrees of freedom is using the helical mode decomposition [38–40], where $\hat{\mathbf{u}}_k(t)$ is decomposed in two helical modes

$$\hat{\mathbf{u}}_k(t) = \hat{\mathbf{u}}_k^+(t) \mathbf{h}_k^+ + \hat{\mathbf{u}}_k^-(t) \mathbf{h}_k^-.$$  

(2.2)

Here, $\hat{\mathbf{u}}_k^\pm(t)$ are two independent complex scalar amplitudes and the orthogonal unit vectors $\mathbf{h}_k^\pm$ are given by

$$\mathbf{h}_k^\pm = \frac{\mathbf{k} \times \hat{\mathbf{e}}}{\sqrt{2|\mathbf{k} \times \hat{\mathbf{e}}|}} \pm i \frac{\mathbf{k} \times \hat{\mathbf{e}}}{\sqrt{2|\mathbf{k} \times \hat{\mathbf{e}}|}},$$

(2.3)

where $\hat{\mathbf{e}}$ is an arbitrary vector non-parallel to $\mathbf{k}$. The vectors $\mathbf{h}_k^\pm$ are eigen-functions of the curl satisfying $i \mathbf{k} \times \mathbf{h}_k^\pm = \pm \mathbf{k} \times \mathbf{h}_k^\pm$ with $k = |\mathbf{k}|$ and $\mathbf{h}_k^0 \cdot (\mathbf{h}_k^\pm)^* = \mathbf{h}_k^0 \cdot \mathbf{h}_k^\pm = \mathbf{h}_k^\pm \cdot \mathbf{h}_k^\pm = \delta_{s_1, s_2}$ (where $s_1 = \pm 1$ and $\delta_{s_1, s_2}$ are the Kronecker delta). Using this decomposition, we can split the real vector field $\mathbf{u}(t, x)$ in two helical fields as $\mathbf{u}(t, x) = \mathbf{u}^+(t, x) + \mathbf{u}^-(t, x)$ with $\mathbf{u}^\pm(t, x)$ given by

$$\mathbf{u}^\pm(t, x) = \sum_{k \neq 0} \hat{\mathbf{u}}_k^\pm(t) \mathbf{h}_k^\pm e^{i\mathbf{k} \cdot \mathbf{x}}.$$  

(2.4)

Note that in (2.4), we assume that $\mathbf{u}$ has a zero-mean value $\langle \mathbf{u} \rangle = 0$. Otherwise, the $\mathbf{k} = 0$ mode (that cannot be written as a helical mode and remains constant in time) should be kept in the expansion. We avoid such complication by removing the $\mathbf{k} = 0$ mode by an appropriate Galilean transformation.

The velocity field $\mathbf{u}(t, x)$ is evolved in time based on the NSEs

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f},$$  

(2.5)

where $P$ is the pressure enforcing incompressibility $\nabla \cdot \mathbf{u} = 0$, $\nu$ is the viscosity and $\mathbf{f}$ is an external body force. In particular, using the helical decomposition the Navier–Stokes can be written as

$$\partial_t \mathbf{u}^\pm = \mathbb{P}^\pm \left[ \sum_{s_2, s_3} (\mathbf{u}^{s_2} \times \mathbf{w}^{s_3}) \right] + \nu \nabla^2 \mathbf{u}^\pm + \mathbf{f}^\pm,$$  

(2.6)

where $s_i = \pm 1$, $\mathbf{w}^\pm = \nabla \times \mathbf{u}^\pm$, $\mathbb{P}^\pm$ is a projector to the helical base

$$\mathbb{P}^\pm [\mathbf{u}(t, x)] = \sum_k \mathbf{h}_k^\pm \cdot \hat{\mathbf{u}}_k(t) e^{i\mathbf{k} \cdot \mathbf{x}}$$  

(2.7)

and $\mathbf{f}^\pm = \mathbb{P}^\pm [\mathbf{f}]$. Note that the helical base is an incompressible base and projecting in to it eliminates the pressure. From the eight nonlinear terms that appear in equation (2.6), one for each sign combination $s_1, s_2, s_3$, the six that involve different signs are responsible for transferring energy to the small scales while the two homochiral terms $s_1 = s_2 = s_3 = \pm 1$ transfer energy in the large scales [41,42]. In the absence of forcing, viscosity (and in the absence of singularities) the
evolution of \( \mathbf{u}(t, x) \) conserves two ideal invariants the energy \( \mathcal{E} \) and the helicity \( \mathcal{H} \)
\[
\mathcal{E} = \frac{1}{2} \int |\mathbf{u}|^2 \, dx^3 = \frac{1}{2} \sum s \sum_k |\tilde{u}_k^s|^2 \quad \text{and} \quad \mathcal{H} = \frac{1}{2} \int \mathbf{u} \cdot \mathbf{w} \, dx^3 = \frac{1}{2} \sum s \sum_k sk|\tilde{u}_k^s|^2,
\]
(2.8)

It is worth noting that while \( \mathcal{E} \) is a positive quantity, the helicity can take either sign.

(b) Homochiral Navier–Stokes

When only the homochiral terms are kept in the NSEs the system reduces to
\[
\partial_t \mathbf{u}^\pm = \mathbb{P}^s[(\mathbf{u}^\pm \times \mathbf{w}^\pm)] + \nu \nabla^2 \mathbf{u}^\pm + \Phi^\pm.
\]
(2.9)

In this case, the two helical fields \( \mathbf{u}^\pm \) evolve independently, with the nonlinearity conserving their energy and helicity
\[
\mathcal{E}^\pm = \frac{1}{2} \sum k |\tilde{u}_k^\pm|^2 \quad \text{and} \quad \mathcal{H}^\pm = \frac{1}{2} \sum k |\tilde{u}_k^\pm|^2.
\]
(2.10)

However, unlike the NSE, in the homochiral version the two helicities are sign definite quantities with \( \mathcal{H}^+ > 0 \) and \( \mathcal{H}^- < 0 \). The sign definiteness of the helicity has a profound impact on the cascade. As shown in [43,44], it leads to an inverse cascade. In fact, when \( \mathcal{H}^\pm \) are sign definite one can show that a simultaneous forward cascade of \( \mathcal{E}^\pm \) and \( \mathcal{H}^\pm \) is incompatible (using similar arguments to Fjørtoft [45] for the dual cascade of energy and enstrophy in two dimensions, see [36] §3.5). On the contrary, for the original Navier–Stokes case (\( \mathcal{H} = 1 \)), and in the presence of a large-scale helical forcing, the energy and helicity cascades are observed to be forward as originally proposed in [46] and later verified numerically [42,47].

(c) The \( \lambda \)-Navier–Stokes model

The different roles played by the homochiral and heterochiral interactions led Sahoo et al. [37] to propose a model that transitions from the forward cascading Navier–Stokes (2.6) to the inverse cascading homochiral Navier–Stokes (2.9) varying continuously a parameter \( \lambda \). In detail, the model reads
\[
\partial_t \mathbf{u}^s = \mathbb{P}^s[(\mathbf{u}^s \times \mathbf{w}^s)] + \lambda \mathbb{P}^s[(\mathbf{u}^{-s} \times \mathbf{w}^s) + (\mathbf{u}^s \times \mathbf{w}^{-s}) + (\mathbf{u}^{-s} \times \mathbf{w}^{-s})] + \nu \nabla^2 \mathbf{u}^s + \Phi^s,
\]
(2.11)

For \( \lambda = 1 \), homochiral and heterochiral terms are balanced so that one recovers the NSE (2.6) where energy cascades forward. For \( \lambda = 0 \), the heterochiral terms are eliminated and the system reduces to the homochiral NSE (2.9). For any finite value of \( \lambda \), the system (2.11) has exactly the same ideal invariants as the NSE \( \mathcal{E} = \mathcal{E}^+ + \mathcal{E}^- \) and helicity \( \mathcal{H} = \mathcal{H}^+ + \mathcal{H}^- \). As for the original NSE, \( \mathcal{H} \) is not a sign definite quantity and thus it poses no restriction in the direction of the energy cascade. One cannot thus trivially predict the direction of the energy transfer when \( \lambda \neq 0 \).

In [37], it was shown that as the parameter \( \lambda \) was varied from 1 to 0 a change of the cascade direction was observed from a forward to an inverse cascade. In the limit of infinite Reynolds number, this transition was shown to converge to a critical discontinuous transition at a critical value \( \lambda_c \simeq 0.3 \) such that for \( \lambda < \lambda_c \) all injected energy cascades to large scales while for \( \lambda > \lambda_c \) all energy cascades to the small scales.

3. Numerical set-up

In the present work, we are going to investigate the limit \( \lambda \to \lambda_c \) from above \( \lambda > \lambda_c \). To do that, we perform numerical simulations of the \( \lambda \)-Navier–Stokes system (2.11) in a triple periodic cubic domain of size \( 2\pi L \). The velocity field \( \mathbf{u} \) is evolved using a pseudo-spectral code with \( 2/3 \) dialiasing and a second-order Runge–Kutta method for the time advancement. We use a uniform grid with \( N \) grid points in each direction. The values of \( N \) used varied from \( N = 128 \) to \( N = 1024 \) depending on the Reynolds number used.
Table 1. Resolutions used for all the simulations performed. It is worth noting that values of $\lambda$ closer to the critical value $\lambda_c \simeq 0.3$ require less resolution for the same value of $Re_\epsilon$ but require longer computational time to converge to a statistically steady state.

| $\lambda$ | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 | 0.60 | 0.80 | 1.00 |
|-----------|------|------|------|------|------|------|------|------|
| $Re_\epsilon = 500$ | 128  | 128  | 256  | 256  | 256  | 256  | 256  | 256  |
| $Re_\epsilon = 840$ | 128  | 256  | 256  | 256  | 256  | 256  | 512  | 1024 |
| $Re_\epsilon = 2500$ | —    | 512  | 512  | —    | —    | —    | 512  | 1024 |
| $Re_\epsilon = 6300$ | —    | 1024 | 1024 | —    | —    | —    | 1024 | —    |

In the examined range of $\lambda > \lambda_c$, the cascade is forward so we pick the forcing to act only on Fourier modes that lie inside a sphere of radius $k_f = 2/L$. The phases of the forced Fourier modes $\tilde{f}_k$ are changed randomly at every time step so that the forcing is delta correlated in time and injects energy on average at a fixed rate denoted here by $\epsilon$, and with zero helicity injection. Given the input parameters of our system, the only other non-dimensional number, besides $\lambda$, is the $\epsilon$-based Reynolds number

$$Re_\epsilon \equiv \frac{\epsilon^{1/3} L^{4/3}}{\nu}. \quad (3.1)$$

Small resolution runs $N \leq 256$ started from random initial data and were evolved until a statistically steady state is reached where all quantities fluctuate around a mean value and the energy injection is balanced by the energy dissipation. Larger resolution runs $N \geq 512$ started with initial conditions obtained from smaller resolution runs extrapolated to the new grid. They were then evolved until a statistically steady state is reached.

A table of the parameters of our runs is given in table 1. Let us also stress that simulations with values of $\lambda$ on the other side of the transition $\lambda < \lambda_c$ are hindered from the fact that in such a range one would need to fully resolve also the inverse energy cascade range and this is by far too demanding for the scopes of this work.

4. Results

(a) Energy balance relations

Figure 1 shows the time evolution of the total energy, $E$, for the smallest value of $Re_\epsilon = 500$ examined and for different values of $\lambda$. As the value of $\lambda$ approaches its critical value $\lambda_c \simeq 0.3$ the mean (time averaged) energy is increased, as also are the fluctuations around the mean value. This behaviour could in part be anticipated since by taking the limit $\lambda \to \lambda_c$ we reduce the efficiency of the flow to transport energy to the small scales so the amplitude of the turbulent fluctuations has to increase to compensate this lack of efficiency and maintain the flux of energy to the small scales fixed and equal to the injection rate $\epsilon$. In figure 2b, we show the time-averaged energy as a function of $\lambda$ for the different values of $Re_\epsilon$. Different symbols are used for the different values of $Re_\epsilon$ as indicated in the legend. For large values of $\lambda$, the amplitude of the mean energy is practically independent on the value of $Re_\epsilon$ (in the range examined) and is weakly dependent on $\lambda$. However, as $\lambda$ is decreased close to $\lambda_c$ the mean energy increases displaying a divergence at $\lambda_c$. Close to $\lambda_c$ the mean energy strongly depends on the value of $Re_\epsilon$, increasing as $Re_\epsilon$ is increased. This implies that for values of $\lambda$ close to $\lambda_c$ we have not yet reached the asymptotic state $Re_\epsilon \to \infty$ where energy saturation is independent on the value of viscosity.

An alternative way to plot the same data is to study the ratio

$$C_D = \frac{2\pi \epsilon L e}{U^3}. \quad (4.1)$$
Figure 1. Evolution of energy, $\mathcal{E}$, as a function of time for $Re_\epsilon = 500$ and different values of $\lambda$. (Online version in colour.)

Figure 2. (a) Energy of the flow at steady state as a function of the parameter $\lambda$. Different symbols correspond to different Reynolds numbers. The dashed lines gives the prediction (4.2). The inset shows the same data as a function of $\lambda - \lambda_c$ in a log-log scale. (b) Normalized energy dissipation rate $C_D$ as a function of $\lambda$. (Online version in colour.)

where $U$ is the root mean square value of the velocity $U = \sqrt{2\mathcal{E}}$. This ratio expresses the efficiency of turbulent fluctuations of a given amplitude to cascade energy to the small scales and sometimes it is called the normalized dissipation rate. It is a fundamental property of turbulence that $C_D$ remains finite in the $Re_\epsilon \to \infty$ limit, resulting in finite dissipation of energy in the zero viscosity limit. This quantity is plotted in the right panel of figure 2. For $\lambda = 1$, $C_D \simeq 0.4$ that is close to reported values ([48]) but is decreasing as $\lambda$ is decreased. The data indicate that it linearly approaches zero as $\lambda \to \lambda_c$, so that $C_D \propto (\lambda - \lambda_c)$. Right at the critical point $\lambda = \lambda_c$, the flow is inefficient to cascade the energy to the small scales. At this critical point, the flow evolution is limited only by viscous effects at the forcing scale to saturate the energy injection and the amplitude of the fluctuations would diverge in the $\nu \to 0$ limit.

The linear approach to zero can be re-interpreted to find the divergence observed in the energy in figure 2a as

$$\mathcal{E} \propto \frac{\mathcal{E}^{2/3}}{(\lambda - \lambda_c)^{2/3}}. \quad (4.2)$$

The dashed line in this figures shows that indeed this scaling is compatible with the data.

There are a few comments that should follow the result in equation (4.2). First of all, we should stress again that the increase in the energy as $\lambda \to \lambda_c$ is approached is due a reduced efficiency of the flow to cascade energy to smaller scales. This has some direct consequences. If we define the
Reynolds number based on the r.m.s. velocity of the flow
\[ Re_u \equiv \frac{UL}{v}, \quad (4.3) \]
the two definitions \( Re_u, Re_v \) are not equivalent but \( Re_u \propto (\lambda - \lambda_c)^{-1/3} Re_v \).

We also need to comment on the two limits \( \lambda \to \lambda_c \) and \( Re_v \to \infty \). For any value of \( \lambda > \lambda_c \), the normalized dissipation rate \( C_D \) will remain strictly positive in the \( Re_v \to \infty \) limit. On the other hand for \( \lambda = \lambda_c \), where there is no cascade to the small scales, velocity fluctuations saturate with amplitudes such that \( \epsilon \propto \nu L^2 / L^2 \) leading to the estimate \( C_D \propto Re_v^{-3/2} \) which becomes zero at infinite \( Re_v \).

(b) Spectral properties

Modifying the efficiency of the flow to cascade the energy to small scales will affect the turbulent scale-by-scale energy budget. Further understanding can be obtained by looking at the spectral distribution of energy. The spherically averaged energy spectrum \( E(k) \) is defined as
\[ E(k) = \sum_{k-|q|<k+1} (|\tilde{u}_q^+|^2 + |\tilde{u}_q^-|^2), \quad (4.4) \]
and expresses the amount of energy in a spherical shell in Fourier space with unit width. We note that the \( \lambda \) model used here has exactly the same scaling symmetries as the NSEs. Therefore, dimensional analysis will imply again a Kolmogorov energy spectrum \( E(k) \propto k^{-5/3} \).

Figure 3a–d displays \( E(k) \) for different values of \( Re_v \) and \( \lambda \). The spectra are compensated by \( k^{5/3} \) so that a Kolmogorov spectrum would appear as flat. The x-axis has been re-scaled by the Kolmogorov dissipation wavenumber
\[ k_v = \left( \frac{\epsilon}{\nu} \right)^{1/4}, \quad (4.5) \]
so that the spectra of different \( Re_v \) collapse together at large wavenumbers.

The \( \lambda = 1 \) case shows the typical large \( Re_v \) behaviour of a turbulent flow for which the power-law behaviour \( k^{-\alpha} \) (with \( \alpha \sim -5/3 \)) is followed by a bottleneck increase of the compensated spectrum. The bottleneck behaviour is well documented in the literature [49,50] and is roughly explained as a pile-up of the cascading energy when the viscous cut-off is reached. In [51], it was argued that when a very high order hyperviscosity is used the bottleneck is increased approaching a thermalized state. Thermalized states manifest themselves in conservative systems like the spectrally truncated Euler equations in which energy is equally distributed among all modes leading to the energy spectrum \( E(k) \propto k^2 / 11 \). The transition of hyperviscous runs to a thermalized state has been recently demonstrated in [52]. For regular viscosity, the bottleneck has thus been interpreted as a partial thermalization.

As \( \lambda \) is decreased this behaviour starts to change. The bottleneck appears to increase in amplitude and covers a wider range of wavenumbers. This tendency is demonstrated in figure 4 where the spectra are plotted from different values of \( \lambda \). For \( \lambda = 0.8 \), one can still observe a \( k^{-5/3} \) range but for smaller values of \( \lambda \) it is hard to observe a power-law range with the present resolution. At the smallest values of \( \lambda \) examined \( \lambda = 0.35 \) and \( \lambda = 0.40 \), the bottleneck covers the whole range of wavenumbers. Therefore, as \( \lambda \to \lambda_c \), the range of wavenumbers which follow partial thermalization increases. It is impossible, within the given resolution limitations, to precisely estimate the scaling of the extension of the bottleneck effects as a function of \( \lambda \) and Reynolds number. An understanding of why this excess of thermalization occurs as the critical point is approached is obtained by looking at the spectral energy fluxes.

The energy flux gives the rate that energy flows across a particular scale. It is defined as
\[ \Pi(k) = -\sum_s \langle u_k^s \cdot u \cdot \nabla u \rangle, \quad (4.6) \]
where \( u_k^s \) stands for the velocity field filtered so that only wavenumbers with norm \(|k| < k\) are kept. It has been shown [42] that can be decomposed to a homochiral part stemming from
same chirality interactions and a heterochiral part stemming from cross-chirality interactions. The homochiral flux is defined as

\[
\Pi^\text{homo}(k) = - \sum_s \langle \mathbf{u}_s \cdot \mathbb{P}^s ([\mathbf{u}^s \times \mathbf{w}^s]) \rangle,
\]

(4.7)
while the heterochiral flux is defined as

\[ \Pi_{\text{hete}}(k) = -\lambda \sum_s \langle u^<_k \cdot P^s [(u^s \times w)^s] + (u^s \times w^s) \rangle \]. \tag{4.8} \]

The total flux is equal to the sum of the two

\[ \Pi_{\text{hete}}(k) = \Pi_{\text{homo}}(k) + \Pi_{\text{hete}}(k). \tag{4.9} \]

In figure 5a–d, we show the fluxes for four different values of \( \lambda = 0.35, 0.4, 0.6, 1.0 \) for the largest resolutions attained. The time-averaged total energy flux \( \Pi(k) \) is shown with a dark blue solid line. It has been decomposed to its homochiral (red dashed line) and a heterochiral component (purple dashed-dot line). With a light blue lines the instantaneous total energy flux is shown for several different times. In all cases, the total energy flux is positive and equal to the energy injection/dissipation rate, while the homochiral flux is negative and the heterochiral flux is positive. For the Navier–Stokes case \( \lambda = 1 \), the negative homochiral flux constitutes a small fraction of the total flux (about 10%) so that \( -\Pi_{\text{homo}} \ll \Pi_{\text{hete}} \). Small fluctuations around the time-averaged value are observed in the instantaneous fluxes. As \( \lambda \) approaches the critical value \( \lambda_c \) the amplitude of negative homochiral flux and the positive heterochiral flux both increase, keeping of course their sum fixed to the injection rate. As a result, the two competing processes for the transfer of energy to smaller and larger scales come closer together in amplitude making their relative difference smaller and smaller. This leads to also an increase in the amplitude of the fluctuations around the mean value observed. On the one hand, the increase of fluctuations with respect to the mean could be a potential indication that the system is driven toward a sort of quasi-thermal state, that could interpret the large bottleneck as partial thermalization as it [51]. On the other hand, the existence of two counter-directional fluxes one driving energy toward small scales and the other oppositely with clear non-zero average and amplitudes that becomes larger and larger by approaching \( \lambda_c \) is an indication that the fluid remains out of equilibrium but in a flux-loop state where finite fluxes exist that however cancel each other [36].

(c) Intermittency

A key property of Navier–Stokes turbulent cascade is the presence of intermittency manifesting itself as a breaking of scale-similarity, with stronger events appearing as smaller scales are examined. Such deviations from scale similarity are measured by examining the scaling behaviour of structure functions of different order \( S_n(r) = \langle (\delta u_r)^n \rangle \), where \( \delta u_r \) stands for either the longitudinal increment, i.e. when \( r \) is parallel to \( u(x,t) - u(x+r,t) \), or for the transverse, when \( r \) it is perpendicular to the velocity increment, and the brackets stand for a space and time average. It is well known that in homogeneous and isotropic Navier–Stokes three-dimensional turbulence, structure functions enjoy anomalous scaling, \( S_n(r) \sim r^{\zeta n} \), with power-law behaviours and exponents that depart from the Kolmogorov mean-field prediction, \( \zeta_n \neq n/3 \) [30]. This is a signature of intermittency, i.e. that normalized and standardized probability distribution functions (PDFs) of velocity increments cannot be superposed at changing the distance \( r \) and develop a stronger and stronger departure from Gaussian statistics. As a result, by simultaneously decreasing the scale and increasing Reynolds one can obtain turbulent states that are further and further away from a quasi-equilibrium distribution [30]. In our simulations, the limitation on the numerical resolution, imposed by the need to perform many different investigations for different \( \lambda \) values, and the appearance of a strong bottleneck by approaching \( \lambda_c \) result in the absence of a well-developed scaling range. As a result, we refrain from giving any quantitative measurements on the scaling exponents \( \zeta_n \). On the other hand, in figure 6, we show
Figure 5. Fluxes: total time averaged (blue solid line), homochiral time averaged (red dashed line) and heterochiral time averaged (purple line), instantaneous total fluxes (light blue). (Online version in colour.)

Figure 6. (a) The kurtosis $K_{\delta u}$ of the parallel velocity difference as a function of $r$ for different values of $\lambda$. (b) $K_{\delta u} - 3$ as a function of $\lambda - \lambda_c$ for different values of $r$ in a log log scale. Data are obtained from the highest $Re_\epsilon$ runs. (Online version in colour.)

A dimensionless measurement of the departure from Gaussianity by plotting the kurtosis of the velocity increments PDF at changing $r$ and for different $\lambda$ for the highest Reynolds number

$$K_{\delta u}(r) \equiv \frac{\langle (\delta u_r)^4 \rangle}{\langle (\delta u_r)^2 \rangle^2}. \quad (4.10)$$

As one can see in figure 6a, the NSE case for $\lambda = 1$ shows the classical behaviour of $K_{\delta u}$ increasing by decreasing $r$, going from the Gaussian value approximately 3 for $r \sim L$ to the highly non-Gaussian and intermittent plateau at approximately 7 inside the viscous range, $r \to 0$. On the other range by decreasing $\lambda$ and approaching $\lambda_c$, we have a strong reduction in the inertial
range values and also a corresponding reduction for the viscous plateau, indicating that the flow is approaching a closer and closer Gaussian distribution at all scales when $\lambda \to \lambda_c$. Similarly, on figure 6b, we re-plot the data by showing the values of the flatness at changing the distance $r/L$ and for different $\lambda$. From the latter plot, there is a clearer tendency towards the mean-field Gaussian value, approximately 3, by approaching $\lambda_c$ even though the behaviour is not extremely well developed. Higher Reynolds numbers are probably needed in order to enhance the critical behaviour.

Finally, because intermittency depends also on $Re_\epsilon$ we verify that the decrease of intermittency observed in the previously examined figures is a trend that persists for all $Re_\epsilon$ examined. In figure 7, we plot the kurtosis of the $z$ vorticity component

$$K_{w_z} \equiv \frac{\langle w_z^4 \rangle}{\langle w_z^2 \rangle^2},$$

as a function of $\lambda$ for all examined runs. The symbols for the different $Re_\epsilon$ used are the same as in figure 2. All data indicate that as $\lambda_c$ is approached intermittency tends to decrease. In particular, for large values of $\lambda$ the kurtosis is increased as $Re_\epsilon$ is increased, something well known in turbulence theory due to intermittency [53,54] while as $\lambda_c$ is approached, the values appear to weakly depend on $Re_\epsilon$ and tend to the Gaussian value, $\to 3$ for all $Re_\epsilon$.

5. Conclusion

Numerical results for a variant of the NSEs have been presented. The model is characterized by a dimensionless control parameter $\lambda$ that weights homochiral and heterochiral interactions such as to link in a continuous way the forward cascading Navier–Stokes case for $\lambda = 1$ and the inverse cascading homochiral Navier–Stokes for $\lambda = 0$. In the investigation both $\lambda$ and the Reynolds number $Re_\epsilon$ were varied in order to investigate the behaviour of the system for the different values of $\lambda$ as $Re_\epsilon \to \infty$.

We have focused on the transition from the $\lambda = 1$ case to case observed at $\lambda_c \sim 0.3$ where the direct energy transfer stops and a highly complex statistical state develops. We have shown that by approaching $\lambda_c$ from above we have a tendency to develop a more and more intense spectral viscous bottleneck. We should also note here that the limits $\lambda \to \lambda_c$ and $Re_\epsilon \to \infty$ do not necessarily commute. That is, when the $\lambda$ limit is taken first the bottleneck would occupy all scales, while when $Re_\epsilon \to \infty$ at fixed $\lambda$ a power-law inertial range is present. Furthermore, in
the $\lambda \rightarrow \lambda_c$ limit larger and large fluctuations around the mean energy flux exist whose energy is diverging like $(\lambda - \lambda_c)^{2/3}$, accompanied by larger and larger heterochiral and homochiral opposite contributions to the energy cascade. As a result, we are closer and closer to a turbulent state where the finite counter-directing fluxes cancel each other leading to a subdominant mean energy flux. We refer to this state as flux-loop. Such flux-loop states have been met before in the literature in different contexts [55–59] but are inadequately explored, and it is not known if methods from equilibrium dynamics could be applied to them.

Another direction that could be followed would be the use of renormalization group techniques for the $\lambda$-Navier–Stokes system in $d$ dimensions. This could lead to a $\lambda$-dependence of the renormalized transport coefficients that could change sign or the appearance of new $\lambda$-dependent terms. Ideally, an optimal path in the $(\lambda, d)$ could exist where the $d=3, \lambda=1$ case could be solved for.

In the present investigation, the statistics is observed to have a decreasing kurtosis for both velocity increments in the inertial range and vorticity components as the flux-loop state is approached, $\lambda \rightarrow \lambda_c$. The results thus indicate the possibility that the flux-loop state could have Gaussian or quasi-Gaussian statistics. The entangled presence of non-zero fluxes and reduction of non-Gaussian contributions opens the way to perturbative approaches of intermittency, offering also a unique testing bed for any new theory of turbulence because of the possibility to change the Navier–Stokes statistics as a function of a free control parameter.

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**Authors’ contributions.** All authors participated in the analytical computations. A.v.K. drafted the manuscript and performed the numerical simulations. All authors read, edited and approved the manuscript.

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