OPTIMAL INDIRECT STABILITY OF A WEAKLY DAMPED
ELASTIC ABSTRACT SYSTEM OF SECOND ORDER
EQUATIONS COUPLED BY VELOCITIES

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Abstract. In this paper, by means of the Riesz basis approach, we study
the stability of a weakly damped system of two second order evolution equa-
tions coupled through the velocities (see (1.1)). If the fractional order damping
becomes viscous and the waves propagate with equal speeds, we prove exponen-
tial stability of the system and, otherwise, we establish an optimal polynomial
decay rate. Finally, we provide some illustrative examples.

1. Introduction. In this paper, we investigate the energy decay rate of the fol-
lowing abstract system of second order evolution equations
\[
\begin{aligned}
&u_{tt} + aAu + A^\gamma u_t + \alpha y_t = 0, \\
y_{tt} + Ay - \alpha u_t = 0,
\end{aligned}
\]  
(1.1)
where $a > 0$, $\gamma \leq 0$, $\alpha \in \mathbb{R}^*$ is the coupling parameter and $A$ is a self-adjoint,
coercive operator with a compact resolvent in a separable Hilbert space $H$
and with simple spectrum. The fractional damping term $A^\gamma u_t$ is only applied at the
first equation and the second equation is indirectly damped through the coupling
between the two equations. The fractional order damping of the type $A^\gamma$, arising
from the material property, has been introduced in [19] and, in the cases $\gamma \in \{0, \frac{1}{2}, 1\}$, is referred to as the so-called viscous damping, square-root (or structural)
damping, and Kelvin-Voigt damping respectively. If $\gamma = \frac{1}{2}$, it was shown in [19]
that the semigroup corresponding to the damped elastic model
\[ u_{tt} + Au + A^\gamma u_t = 0, \]
is analytical, while the subsequent works in [20] and [21] showed that the semigroup
is still analytical for $\frac{1}{2} \leq \gamma \leq 1$ but is only of Gevrey class for $0 < \gamma < \frac{1}{2}$. 

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exponential stability, polynomial stability, optimal stability, spectrum method.
In [32], Liu and Zhang studied the energy decay rate of the weakly damped elastic abstract system described by
\[
\begin{cases}
    u_{tt} + Au + Bu_t = 0, \\
    u(0) = u_0, \quad u_t(0) = u_1,
\end{cases}
\tag{1.2}
\]
where $A$ is a self-adjoint, positive definite operator on a Hilbert space $H$. The dissipation operator $B$ is another positive operator satisfying $cA^\gamma u \leq Bu \leq CA^\gamma u$ for some constants $0 < c < C$. When $\gamma < 0$, they proved that the energy of System (1.2) has a polynomial decay rate of type $t^{\frac{1}{\gamma}}$ and that this decay rate is in some sense optimal. Regarding System (1.1) when $\alpha = 0$, it reduces to System (1.2) with $B = A^\gamma$ and $a = 1$. In this case, we recover the results of [32]. When the coupling acts through displacements, Alabau, Cannarsa, and Komornik studied in [5] the stability of the following abstract system of coupled equations
\[
\begin{cases}
    u_{tt} + A_1 u + Bu_t + \alpha y = 0, \\
    y_{tt} + A_2 y + \alpha u = 0,
\end{cases}
\tag{1.3}
\]
where $A_1$ and $A_2$ are self-adjoint positive linear operators in a Hilbert space $H$ and $B$ is self-adjoint positive bounded operator in $H$. They proved that the energy of System (1.3) has a polynomial decay rate. When $A_1 = A_2$ and $B = Id$, System (1.3) is the closest to our system (1.1) in the case $\gamma = 0$. Nonetheless, in [5], the coupling acts through displacements while in the present work the coupling acts through the velocities.

Loreti and Rao studied in [34] the stability of the following abstract system of coupled equations
\[
\begin{cases}
    u_{tt} + Au + A^\gamma u_t + \alpha y = 0, \\
    y_{tt} + Ay + \alpha u = 0,
\end{cases}
\tag{1.4}
\]
where $\gamma < 0$, $\alpha \in \mathbb{R}^*$ and $A$ is a self-adjoint, coercive operator with a compact resolvent in a separable Hilbert space $H$. They proved that System (1.4) is not exponentially stable and an optimal polynomial energy decay rate of type $t^{-\tau(\gamma)}$ is obtained where
\[
\tau(\gamma) = \begin{cases}
    \frac{1}{\gamma + 1}, & -\frac{1}{2} \leq \gamma \leq 0, \\
    -\frac{1}{\gamma}, & -\frac{1}{2} \geq \gamma.
\end{cases}
\]
Consequently, the energy achieves its maximum optimal decay rate $t^{-\frac{1}{\gamma}}$ when $\gamma = -\frac{1}{2}$. System (1.4) is the closest to our system (1.1) in the case $\gamma < 0$. However, the coupling in [34] acts through displacements while in this paper the coupling acts through the velocities. Indeed, the transmission between the two equations depends on the nature of the coupling; for instance, see [4, 13, 5, 14], and [27].

The fact that only one equation of the coupled system is damped refers to the so-called class of indirect stabilization problems. The concept of indirect damping mechanisms has been introduced by Russell in [40]. That paper is one of the firsts to give an algebraic characterization of coupled indirectly damped vibration models. Before we start our study, we recall some results concerning the stability of systems of two equations coupled by velocities. In [27], Kapitonov considers a pair of coupled hyperbolic systems in some open subset of a domain. One of these systems contains locally distributed damping. Under certain conditions imposed on the subset where the damping terms is effective, a uniform decay of the energy is established. The results are proved by using multiplier techniques. Khodja and
Bader in [14] study the stability of a system of coupled one-dimensional wave equations posed on a finite interval \((0, 1)\) with only one internal or boundary control. They show that the internal damping applied to only one of the equations never gives exponential stability if the wave speeds are different. If the wave speeds are the same, they present necessary and sufficient conditions for stability. In addition, the simultaneous boundary stabilization of the same system is also studied in [14]. Let us mention the additional references [3, 16, 2, 18, 7, 15], and [13] for indirect stabilization of coupled equations via one order terms. Next, we recall some of the results related to the stability of two equations coupled through displacements. In [4], Alabau considers coupled equations with only one boundary control where it is shown for different examples such as the wave equations or the Kirchhoff plates that the full system can be strongly stabilized provided that the coupling parameter is sufficiently small. In such a case, the author proves that the energy decays polynomially with explicit polynomial decay rate for sufficiently smooth solutions and these results are extended to the case of two coupled wave equations with different speeds of propagation under a condition on the ratio of the two speeds and for \(n\)-dimensional intervals. In [5], Alabau et al., study the indirect internal stabilization of weakly coupled equations where the damping is effective in the whole domain. The authors prove that the behaviour of the first equation is sufficient to stabilize the total system and to get a polynomial decay for sufficiently smooth solutions. In [11], Alabau and Léautaud extend the result of [5] to a system of two coupled equations with a coupling operator. Under certain assumptions on the coupling operator, they prove the polynomial stability the system. Furthermore, we mention [6, 30, 24, 41] and [9] for indirect stabilization of coupled equations via zero order terms.

Last but not least, in addition to the previously cited papers, the stability of abstract system with memory term has been studied in [10] and [8]. Notwithstanding to the previously cited papers, we recall some results concerning the exponential or polynomial indirect stability of systems which arise from physical problems. For example, we quote [1, 12, 23, 25, 31, 37, 43, 44] for the Bresse system and [2, 7, 15, 17, 28, 36, 39] for the Timoshenko system. The Bresse system is usually considered in the study of elastic structures of the arcs type (see [29]) while the Timoshenko system is usually considered in describing the transverse vibration of a beam and it ignores damping effects of any nature (see [42]).

The aim of the present paper consists in studying the stability of the indirectly damped System (1.1). For this purpose, we write System (1.1) as the differential system

\[ U_t = AU, \]

where

\[ U = \begin{pmatrix} u \\ v \\ y \\ z \end{pmatrix} \quad \text{and} \quad AU = \begin{pmatrix} v \\ -Au - A^2v + \alpha z \\ z \\ -Ay + \alpha v \end{pmatrix}, \]

where \( U \in H \) and \( A \) is an unbounded operator on \( H \). We use \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) to denote the inner product and the induced norm respectively on \( H \). Since the resolvent of \( A \) turns out to be compact in \( H \), there exists a non decreasing sequence \( (\mu_n)_{n \geq 1} \) tending to infinity and an orthonormal basis \( (e_n)_{n \geq 1} \) of \( H \) such that, for \( n \geq 1 \), \( Ae_n = \mu_n^2 e_n \). We assume the spectrum of \( A \) is simple, i.e., the sequence \( (\mu_n)_{n \geq 1} \) is increasing. Our goal in this paper is to establish the optimal stability of System
(1.1) using the spectral method for the operator $A$. For this aim, we study the effect of both the fractional order damping of type $A^\gamma$ and the speeds of the two wave equations on this spectrum and we prove that the latter is made of two branches $(\lambda_{1,n})_{n \in \mathbb{N}}$ and $(\lambda_{2,n})_{n \in \mathbb{N}}$, whose asymptotics, as $n$ tends to infinity, are given next.

**Case 1.** Assume that $\gamma = 0$. If $a = 1$, i.e., when the two waves propagate with equal speed, we prove that the spectrum of $A$ has an asymptotic expansion, as $n$ tends to infinity, given by

$$
\lambda_{1,n}^\pm = \pm i \mu_n - \frac{1}{4} \frac{1}{\sqrt{1 - 4a^2}} + o(1) \quad \text{and} \quad \lambda_{2,n}^\pm = \pm i \mu_n - \frac{1}{4} \frac{1}{\sqrt{1 - 4a^2}} + o(1),
$$

see Lemma 3.2). Note here that if $4a^2 > 1$, $\sqrt{1 - 4a^2}$ actually denotes the imaginary number $i\sqrt{4a^2 - 1}$. We then prove that the energy of the system is (uniformly) exponentially stable. If $a \neq 1$, i.e., when the waves propagate with different speeds, we show that the spectrum of $A$ has asymptotic expansion, as $n$ tends to infinity, given by

$$
\lambda_{1,n}^\pm = \pm i \sqrt{a} \mu_n - \frac{1}{2} + o(1) \quad \text{and} \quad \lambda_{2,n}^\pm = \pm i \left( \mu_n - \frac{\alpha^2}{(a - 1) \mu_n} \right) - \frac{\alpha^2}{2 (a - 1)^2 \mu_n^2} + o(1),
$$

see Lemma 4.3). Thus, the real part corresponding to the first branch of eigenvalues is uniformly bounded and the real part corresponding to the second branch of eigenvalues is of magnitude $\mu_n^{-2}$. Therefore, we prove that the total energy decays at the optimal rate $1/t$.

**Case 2.** Assume that $\gamma < 0$. If $a = 1$, then the real parts of $\lambda_{1,n}^\pm$ and $\lambda_{2,n}^\pm$ are of magnitude $\mu_n^{2\gamma}$ (see Lemma 4.2) and the total energy decays at the optimal rate $t^{-\frac{\gamma}{2}}$. If $a \neq 1$, then the real part corresponding to the first branch of eigenvalues is of magnitude $\mu_n^{\gamma - 2}$ and the real part corresponding to the second branch of eigenvalues is of magnitude $\mu_n^{2\gamma - 2}$ (see Lemma 4.3), yielding a decay rate of the optimal total energy equal to $t^{-\frac{\gamma}{2}}$. From the above results, we deduce that the maximum decay rate is achieved when $\gamma$ tends to zero. Therefore, a stronger damping term $A^\gamma u_t$ does not necessarily give a better decay rate of the total energy, as it is expected. A good damping term should transmit the damping from one wave to another before the directly damped wave dies out or loses its energy. This effect of a good damping term is interpreted by the real parts of the eigenvalues. Consequently, the results of this paper show that a suitable weaker damping term can compensate the lack of feedback on the second equation of System (1.1). It seems interesting to consider coupled systems of the type (2.1) with different operators $A_1, A_2$. Indeed the same results could be obtained without essential difficulty in the case $A_2 = A_1^2$. But in general we can no longer calculate explicitly the eigenvalues as in Lemmas 3.2, 4.2, 4.3.

This paper is organized as follows. In Section 2, we set the framework of System (1.1) and we establish the characteristic equation satisfied by the eigenvalues of the operator $A$. Next, in Section 3, relying on the spectral method, we prove the exponential stability of System (2.1) when $a = 1$ and $\gamma = 0$. In Section 4, we consider the other cases of $a$ and $\gamma$. We prove the optimal polynomial energy decay rate of type $t^{-\delta(\gamma)}$ of System (2.1), where

$$
\delta(\gamma) = \begin{cases} 
-\gamma, & \text{if } a = 1 \text{ and } \gamma < 0, \\
\frac{1}{1 - \gamma}, & \text{if } a \neq 1 \text{ and } \gamma \leq 0.
\end{cases}
$$
Finally, in Section 5, we examine some applications for our study.

2. Characteristic equation and Riesz basis method. In this paper, we consider the following abstract system of second order evolution equations given by

\[
\begin{cases}
    u_{tt} + aAu + A^\gamma u_t + \alpha y_t = 0, \\
y_{tt} + Ay - \alpha u_t = 0,
\end{cases}
\]

(2.1)

where \(a > 0\), \(\gamma \leq 0\), \(\alpha \in \mathbb{R}^*\), the operator \(A\) is a self-adjoint coercive operator with compact resolvent in a separable Hilbert space \(H\). Let us define the energy space \(H = D(A^{1/2}) \times H \times D(A^{1/2}) \times H\) equipped with the following norm

\[
\| (u, v, y, z) \|_H^2 = a \left\| A^{1/2}u \right\|^2 + \left\| A^{1/2}y \right\|^2 + \|v\|^2 + \|z\|^2,
\]

where \(\| \cdot \|\) denotes the norm in \(H\). We define the linear unbounded operator \(A\) in \(H\) by

\[
\begin{pmatrix}
u \\
v \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
v \\
-aAu - A^\gamma v - \alpha z \\
z \\
-Ay + \alpha v
\end{pmatrix}.
\]

Therefore, we can write System (2.1) as an evolution equation

\[
\begin{cases}
    U_t(x, t) = AU(x, t), \\
    U(x, 0) = u_0(x),
\end{cases}
\]

(2.2)

where \(u_0 = (u_0, v_0, y_0, z_0)^T \in H\).

One clearly has that \(A\) is a maximal dissipative operator on \(H\) and, thanks to the Lumer-Phillips theorem (see [33, 38]), we deduce that \(A\) generates a \(C_0\)-semigroup of contractions \(e^{tA}\) in \(H\) and therefore (2.1) is well-posed. Moreover, the energy of System (2.1) is given by

\[
E(t) = \frac{1}{2} a \left( \left\| A^{1/2}u \right\|^2 + \left\| A^{1/2}y \right\|^2 + \|v\|^2 + \|z\|^2 \right),
\]

where

\[
E'(t) = -\left\| A^{1/2}u_t \right\|^2 \leq 0.
\]

Hence, the energy of System (2.1) is decaying. Before starting the main results of this work, we introduce here the notions of stability that we encounter in this work.

Definition 2.1. Assume that \(A\) is the generator of a \(C_0\)-semigroup of contractions \(e^{tA}\) on a Hilbert space \(H\). The \(C_0\)-semigroup \(e^{tA}\) is said to be

1. Exponentially (or uniformly) stable if there exist two positive constants \(M\) and \(\epsilon\) such that

\[
\| e^{tA}x_0 \|_H \leq Me^{-\epsilon t} \| x_0 \|_H, \quad \forall \ t > 0, \ \forall \ x_0 \in H.
\]

(2.3)
2. Polynomially stable if there exists two positive constants $C$ and $\alpha$ such that
\[ \|e^{tA}x_0\|_H \leq Ct^{-\alpha}\|Ax_0\|_H, \quad \forall t > 0, \forall x_0 \in D(A). \] (2.4)

In that case, one says that solutions of (2.2) decay at a rate $t^{-\alpha}$. The $C_0$-semigroup $e^{tA}$ is said to be polynomially stable with optimal decay rate $t^{-\alpha}$ (with $\alpha > 0$) if it is polynomially stable with decay rate $t^{-\alpha}$ and, for any $\varepsilon > 0$ small enough, solutions of (2.2) do not decay at a rate $t^{-(\alpha-\varepsilon)}$.

Note that, in the definition of polynomially stable, one can replace $A$ by $A^\theta$ for some positive real number $\theta$ and the constants $C, \alpha$ hence depend on $\theta$.

**Definition 2.2.** Let $(e^{tA})_{t \geq 0}$ be a $C_0$-semigroup of contractions generated by the operator $A$ on a Hilbert space $H$. Let $(\lambda_{k,n})_{1 \leq k \leq K, n \geq 1}$ denotes the $k$th branch of eigenvalues of $A$ and $\{e_{k,n}\}_{1 \leq k \leq K, n \geq 1}$ the system of eigenvectors which forms a Riesz basis in $H$. Then the fractional power $A^\theta$ of $A$ with $\theta \in \mathbb{R}$ is defined by
\[ D(A^\theta) = \left\{ u \in H : \sum_{k=1}^{K} \sum_{n \geq 1} |\lambda_{k,n}^\theta|^2 \|\langle u, e_{k,n}\rangle_H\|^2 < \infty \right\} \]
and for all $u \in D(A^\theta)$, we have
\[ A^\theta u = \sum_{k=1}^{K} \sum_{n \geq 1} \lambda_{k,n}^\theta \langle u, e_{k,n}\rangle_H e_{k,n}. \]

Our subsequent findings on exponential stability will rely on the following result from [22, 35], which gives necessary and sufficient conditions for a semigroup to be exponentially stable.

**Proposition 2.3** (cf. [22, 35]). Let $(A, D(A))$ be an unbounded linear operator on $H$ with compact resolvent. Assume that $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions $(e^{tA})_{t \geq 0}$. Moreover, suppose that the eigenvectors and the root vectors of $A$ form a Riesz basis in $H$ and that the multiplicity of the eigenvalues of $A$ are uniformly bounded. Then, $(e^{tA})_{t \geq 0}$ is exponentially stable if and only if its spectral bound $s(A)$, defined as
\[ s(A) = \sup \{ \Re(\lambda) : \lambda \in \sigma(A) \}, \]
is negative.

If the semigroup fails to be exponentially stable, we search for another type of decay rate such polynomial stability. In that case, the following proposition from [34] provides a useful way to even characterize optimal polynomial stability.

**Proposition 2.4** (cf. Theorem 2.1 in [34]). Let $(e^{tA})_{t \geq 0}$ be a $C_0$-semigroup of contractions generated by the operator $A$ on a Hilbert space $H$. Let $(\lambda_{k,n})_{1 \leq k \leq K, n \geq 1}$ denotes the $k$-th branch of eigenvalues of $A$ and $\{e_{k,n}\}_{1 \leq k \leq K, n \geq 1}$ the system of eigenvectors which forms a Riesz basis in $H$. Assume that for each $1 \leq k \leq K$ there exist a positive sequence $(\mu_{k,n})_{n \geq 1}$ tending to infinity and two positive constants $\alpha_k \geq 0$, $\beta_k > 0$ such that
\[ \Re(\lambda_{k,n}) \leq -\frac{\beta_k}{\mu_{k,n}} \quad \text{and} \quad |\Im(\lambda_{k,n})| \geq \mu_{k,n} \quad \forall n \geq 1. \]
Then, for every $\theta > 0$, there exists a constant $M > 0$ such that, for every $u_0 \in D(A^\theta)$, one has
\[
\|e^{tA}u_0\|_H \leq \|A^\theta u_0\|_H \frac{M}{\theta^\delta} \quad \forall t > 0,
\]
where $\delta$ is given by
\[
\delta := \min_{1 \leq k \leq K} \frac{1}{\alpha_k} = \frac{1}{\alpha_l}.
\tag{2.5}
\]
Moreover, if there exists two constants $c_1 > 0$, $c_2 > 0$ such that
\[
\Re(\lambda_{k,n}) \geq -c_1 \mu_k^{2} \quad \text{and} \quad |\Im(\lambda_{k,n})| \leq c_2 \mu_k^{1} \quad 1 \leq k \leq K, \quad \forall n \geq 1,
\]
then $(e^{tA})_{t \geq 0}$ is polynomially stable with optimal decay rate $t^{-\delta}$, where $\delta$ is given in (2.5).

In this work, to check the decay rate, we rely on the Riesz basis method in which we first determine the characteristic equation satisfied by the spectrum. Since the resolvent of $A$ is compact in $H$, there exists an increasing sequence $(\mu_n)_{n \geq 1}$ tending to infinity and an orthonormal basis $(e_n)_{n \geq 1}$ of $H$ such that
\[
Ae_n = \mu_n e_n \quad \forall n \geq 1.
\tag{2.6}
\]
In turn, to study the spectrum of System (2.1), let $\lambda$ be an eigenvalue of the operator $A$ and $U = (u, v, y, z)^T$ a corresponding eigenvector. Therefore, we have $AU = \lambda U$.

Equivalently, we have the following system
\[
\begin{cases}
  v = \lambda u, \\
  -aAu - A^\gamma v - \alpha z = \lambda v, \\
  z = \lambda y, \\
  -Ay + \alpha v = \lambda z.
\end{cases}
\tag{2.7}
\]
Similar to the analysis done in [34], we will see in Proposition 3.6 and in Proposition 4.8 that, for every $n \geq 1$, there exists $(B_n, C_n) \neq (0, 0)$ such that the eigenvector $U$ of $A$ is of the form
\[
u = B_n e_n, \quad v = \lambda B_n e_n, \quad y = C_n e_n, \quad z = \lambda C_n e_n.
\tag{2.8}
\]
Inserting (2.8) into (2.7) and using (2.6), we obtain
\[
\begin{cases}
  (a\mu_n^{2} + \lambda^2 + \lambda \mu_n^{2\gamma}) B_n e_n + \alpha \lambda C_n e_n = 0, \\
  -\alpha \lambda B_n e_n + (\mu_n^{2} + \lambda^2) C_n e_n = 0,
\end{cases}
\tag{2.9}
\]
which has a non-trivial solution $(B_n, C_n) \neq (0, 0)$ if and only if $\lambda$ is a solution of the equation
\[
a\mu_n^{4} + \lambda \left((a + 1) \lambda + \mu_n^{2\gamma}\right) \mu_n^{2} + \lambda^2 \left(\lambda^2 + \lambda \mu_n^{2\gamma} + \alpha^2\right) = 0,
\tag{2.10}
\]
that we refer to as the characteristic equation associated with the eigenvalue $\mu_n^2$ of $A$. The four roots of this equation are eigenvalues of $A$ and called the eigenvalues of $A$ corresponding to $\mu_n$. We also have the following result.

**Lemma 2.5.** Let $\lambda_n = \lambda_{j,n}^\pm$, $j = 1, 2$ be one of the fourth eigenvalues of $A$ corresponding to $\mu_n$. Then, there exists two positive constants $m$, $M$, such that, for $n$ large enough,
\[
m \leq \left|\frac{\lambda_n}{\mu_n}\right| \leq M.
\tag{2.11}
Proof. Set \( Z_n = \frac{\lambda_n}{\mu_n} \). Then, from (2.10), one has that \( Z_n \) is one of the four roots of the polynomial \( f_n \) of degree four given by
\[
f_n(Z) = Z^4 + \mu_n^{2\gamma-1}Z^3 + (a + 1 + \frac{\alpha^2}{\mu_n^2})Z^2 + \mu_n^{2\gamma-1}Z + a.
\]
Let \( g \) be the polynomial of degree four given by \( g(Z) = Z^4 + (a + 1)Z^2 + a \), which has exactly four non-zero roots. Since the coefficients of \( f_n \) converge to those of \( g \) as \( n \) tends to infinity, one gets the result.

3. Exponential stability. In this Section, we consider the case where \( a = 1 \) and \( \gamma = 0 \). Our main result is the following theorem.

**Theorem 3.1.** If \( a = 1 \) and \( \gamma = 0 \), then System (2.1) is exponentially stable.

For the proof of Theorem 3.1, we first need to study the asymptotic behaviour of the spectrum of \( \mathcal{A} \) and, in that direction, we have the following Lemma.

**Lemma 3.2.** Assume that \( a = 1 \) and \( \gamma = 0 \). Then, for \( n \geq 1 \) large enough, the four eigenvalues of \( \mathcal{A} \) corresponding to the eigenvalue \( \mu_n^2 \) of \( \mathcal{A} \) and denoted \( \lambda_{1,n}^\pm, \lambda_{2,n}^\pm \), satisfy the following asymptotic expansions

**Case 1.** If \( 0 < \alpha^2 \leq \frac{1}{4} \), then
\[
\begin{align*}
\lambda_{1,n}^\pm &= \pm \frac{i}{4} \sqrt{1 - 4\alpha^2} + O\left(\frac{1}{\mu_n}\right), \\
\lambda_{2,n}^\pm &= \pm \frac{i}{4} \sqrt{1 - 4\alpha^2} + O\left(\frac{1}{\mu_n}\right).
\end{align*}
\]  

**Case 2.** If \( \alpha^2 > \frac{1}{4} \), then
\[
\begin{align*}
\lambda_{1,n}^\pm &= \pm \frac{i}{4} \sqrt{4\alpha^2 - 1} + O\left(\frac{1}{\mu_n}\right), \\
\lambda_{2,n}^\pm &= \pm \frac{i}{4} \sqrt{4\alpha^2 - 1} + O\left(\frac{1}{\mu_n}\right).
\end{align*}
\]

Proof. We divide the proof into two cases.

**Case 1.** If \( 0 < \alpha^2 \leq \frac{1}{4} \), from (2.10), we get that
\[
\begin{align*}
\lambda_{1,n}^2 + \frac{1 - \sqrt{1 - 4\alpha^2}}{2}\lambda_{1,n} + \mu_n^2 &= 0 \quad \text{and} \quad \lambda_{2,n}^2 + \frac{1 + \sqrt{1 - 4\alpha^2}}{2}\lambda_{2,n} + \mu_n^2 = 0.
\end{align*}
\]  

For \( n \) large, solving equation (3.3), we obtain
\[
\begin{align*}
\lambda_{1,n}^\pm &= \frac{1}{4}\left(-1 + \sqrt{1 - 4\alpha^2}\right) \pm \frac{i}{4} \sqrt{16\mu_n^2 - \left(-1 + \sqrt{1 - 4\alpha^2}\right)^2}, \\
\lambda_{2,n}^\pm &= \frac{1}{4}\left(-1 - \sqrt{1 - 4\alpha^2}\right) \pm \frac{i}{4} \sqrt{16\mu_n^2 - \left(-1 - \sqrt{1 - 4\alpha^2}\right)^2}.
\end{align*}
\]  

Moreover, we have
\[
\begin{align*}
\frac{i}{4} \left[16\mu_n^2 - \left(-1 \pm \sqrt{1 - 4\alpha^2}\right)^2\right]^\frac{1}{2} &= \pm i\mu_n \left[1 + O\left(\frac{1}{\mu_n}\right)\right]^\frac{1}{2} \\
&= \pm i\mu_n + O\left(\frac{1}{\mu_n}\right).
\end{align*}
\]  

Substituting (3.5) into (3.4), we obtain (3.1).
Case 2. If $\alpha^2 > \frac{1}{4}$, from (2.10), we obtain
\[
\lambda_{1,n}^2 + \frac{1 - i \sqrt{4 \alpha^2 - 1}}{2} \lambda_{1,n} + \mu_n^2 = 0, \quad \lambda_{2,n}^2 + \frac{1 + i \sqrt{4 \alpha^2 - 1}}{2} \lambda_{2,n} + \mu_n^2 = 0. \tag{3.6}
\]
then for $n$ large, solving equation (3.6), we obtain
\[
\lambda_{1,n}^\pm = \frac{1}{4} \left( -1 \pm \sqrt{2 \sqrt{16 \mu_n^4 + 4(2 \alpha^2 - 1)\mu_n^2 + \alpha^4 - 8 \mu_n^2 - 2 \alpha^2 + 1}} \right)
+ \frac{i}{4} \left( \sqrt{4 \alpha^2 - 1} \pm \sqrt{2 \sqrt{16 \mu_n^4 + 4(2 \alpha^2 - 1)\mu_n^2 + \alpha^4 + 8 \mu_n^2 + 2 \alpha^2 - 1}} \right), \tag{3.7}
\]
and
\[
\lambda_{2,n}^\pm = \frac{1}{4} \left( -1 \pm \sqrt{2 \sqrt{16 \mu_n^4 + 4(2 \alpha^2 - 1)\mu_n^2 + \alpha^4 - 8 \mu_n^2 - 2 \alpha^2 + 1}} \right)
+ \frac{i}{4} \left( -\sqrt{4 \alpha^2 - 1} \pm \sqrt{2 \sqrt{16 \mu_n^4 + 4(2 \alpha^2 - 1)\mu_n^2 + \alpha^4 + 8 \mu_n^2 + 2 \alpha^2 - 1}} \right), \tag{3.8}
\]
since
\[
\begin{cases}
\sqrt{2 \sqrt{16 \mu_n^4 + 4(2 \alpha^2 - 1)\mu_n^2 + \alpha^4 - 8 \mu_n^2 - 2 \alpha^2 + 1}} = O \left( \frac{1}{\mu_n} \right), \\
\sqrt{2 \sqrt{16 \mu_n^4 + 4(2 \alpha^2 - 1)\mu_n^2 + \alpha^4 + 8 \mu_n^2 + 2 \alpha^2 - 1}} = 4 \mu_n + O \left( \frac{1}{\mu_n} \right),
\end{cases} \tag{3.9}
\]
then inserting (3.9) into (3.7) and (3.8), we get (3.2). Thus, the proof is complete.

We next provide the form of the eigenvectors and root vectors of $A$. We start with the following Lemma.

Lemma 3.3. If $a = 1$ and $\gamma = 0$, then the eigenvectors of $A$ take the following form.

Case 1. If $0 < \alpha^2 \leq \frac{1}{4}$, then we have
\[
e^{\pm}_{1,n} = B^{\pm}_{1,n} \begin{pmatrix} \frac{e_n}{\lambda_{1,n}^\pm}, e_n, \beta_1 e_n, \beta_1 e_n \end{pmatrix}^T,
\]
\[
e^{\pm}_{2,n} = C^{\pm}_{2,n} \begin{pmatrix} \frac{e_n}{\lambda_{2,n}^\pm}, \delta_1 e_n, \beta_1 e_n, \beta_1 e_n \end{pmatrix}^T, \tag{3.10}
\]
where $B^{\pm}_{1,n}, C^{\pm}_{2,n} \in \mathbb{C}$ and $\beta_1 = \frac{2 \alpha}{-1 + \sqrt{1 - 4 \alpha^2}}, \delta_1 = -1 + \sqrt{1 - 4 \alpha^2}$.  

Case 2. If $\alpha^2 > \frac{1}{4}$, then we have
\[
e^{\pm}_{1,n} = B^{\pm}_{1,n} \begin{pmatrix} \frac{e_n}{\lambda_{1,n}^\pm}, e_n, \beta_3 e_n, \beta_3 e_n \end{pmatrix}^T,
\]
\[
e^{\pm}_{2,n} = C^{\pm}_{2,n} \begin{pmatrix} \frac{e_n}{\lambda_{2,n}^\pm}, \delta_3 e_n, \beta_3 e_n, \beta_3 e_n \end{pmatrix}^T, \tag{3.11}
\]
where $B_{1,n}^\pm, C_{2,n}^\pm \in \mathbb{C}$ and $\beta_3 = \frac{2\alpha}{-1 + i\sqrt{4\alpha^2 - 1}}$, $\delta_3 = \frac{-1 + i\sqrt{4\alpha^2 - 1}}{2\alpha}$.

**Proof.** Let $\lambda_{1,n}^\pm, \lambda_{2,n}^\pm$ be the solutions of (3.3). Setting

$$B_{1,n} = \frac{B_{1,n}^\pm}{\lambda_{1,n}^\pm} \quad \text{and} \quad C_{2,n} = \frac{C_{2,n}^\pm}{\lambda_{2,n}^\pm}$$

in (2.9), we get

$$C_{1,n} = \frac{\alpha}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} B_{1,n}^\pm \quad \text{and} \quad B_{2,n} = \frac{(\lambda_{2,n}^\pm)^2 + \mu_n^2}{\alpha (\lambda_{2,n}^\pm)^2} C_{2,n}^\pm.$$ 

Therefore, from (2.8), we obtain

$$e_{1,n}^\pm = B_{1,n}^\pm \left( \frac{e_n}{\lambda_{1,n}^\pm}, e_n, \frac{\alpha e_n}{(\lambda_{1,n}^\pm)^2 + \mu_n^2}, \frac{\alpha \lambda_{1,n}^\pm e_n}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} \right)^\top,$$

$$e_{2,n}^\pm = C_{2,n}^\pm \left( \frac{(\lambda_{2,n}^\pm)^2 + \mu_n^2}{\alpha (\lambda_{2,n}^\pm)^2} e_n, e_n, \frac{\lambda_{2,n}^\pm e_n}{\alpha (\lambda_{2,n}^\pm)^2}, \frac{e_n}{\alpha (\lambda_{2,n}^\pm)^2} \right)^\top \quad (3.12)$$

are the eigenvectors corresponding to the four eigenvalues $\lambda_{1,n}^\pm, \lambda_{2,n}^\pm$, where $B_{1,n}^\pm, C_{2,n}^\pm \in \mathbb{C}$. We next divide the argument into two cases.

**Case 1.** If $0 < \alpha^2 \leq \frac{1}{4}$, then from (3.3), we get

$$\frac{1}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} = \frac{2}{(-1 + \sqrt{1 - 4\alpha^2}) \lambda_{1,n}^\pm},$$

$$\frac{1}{(\lambda_{2,n}^\pm)^2 + \mu_n^2} = \frac{2}{1 + \sqrt{1 - 4\alpha^2} \lambda_{2,n}^\pm}. \quad (3.13)$$

Inserting (3.13) into (3.12), we get (3.10).

**Case 2.** If $\alpha^2 > \frac{1}{4}$, then from (3.6), we get

$$\frac{1}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} = \frac{2}{(-1 + i\sqrt{4\alpha^2 - 1}) \lambda_{1,n}^\pm},$$

$$\frac{1}{(\lambda_{2,n}^\pm)^2 + \mu_n^2} = \frac{2}{-1 + i\sqrt{4\alpha^2 - 1} \lambda_{2,n}^\pm}. \quad (3.14)$$

Inserting (3.14) into (3.12), we get (3.11). Thus, the proof is complete.

We now search for the asymptotic behaviour of the eigenvectors of $A$. From Lemma 3.2, we remark that if $\alpha^2 = \frac{1}{4}$, we have double eigenvalues. In this case, we look for the corresponding root vectors.

**Lemma 3.4.** If $a = 1$ and $\gamma = 0$, then the eigenvectors $e_{1,n}^\pm, e_{2,n}^\pm$ of $A$ satisfy the following asymptotic expansion.

**Case 1.** If $0 < \alpha^2 < \frac{1}{4}$, then we have

$$e_{1,n}^\pm = \frac{1}{\beta_1^\pm} \left( \frac{e_n}{\pm i\mu_n}, \frac{\beta_1 e_n}{\pm i\mu_n}, \frac{\beta_1 e_n}{\pm i\mu_n} \right)^\top + O \left( \frac{1}{\mu_n^2} \right),$$

$$e_{2,n}^\pm = \frac{1}{\delta_1^\pm} \left( \frac{\delta_1 e_n}{\pm i\mu_n}, \frac{\delta_1 e_n}{\pm i\mu_n}, \frac{\delta_1 e_n}{\pm i\mu_n} \right)^\top + O \left( \frac{1}{\mu_n^2} \right), \quad (3.15)$$
where \( \beta_1 = \frac{2\alpha}{-1 + \sqrt{1 - 4\alpha^2}} \), \( \delta_1 = -\frac{1 + \sqrt{1 - 4\alpha^2}}{2\alpha} \), \( \beta_1^+ = \sqrt{2 + 2|\beta_1|^2} \) and \( \delta_1^+ = \sqrt{2 + 2|\delta_1|^2} \).

**Case 2.** If \( \alpha^2 = \frac{1}{4} \), we suppose that \( \alpha = \frac{1}{2} \) since the analysis follows similarly if \( \alpha = -\frac{1}{2} \), then the eigenvectors \( e_n^\pm \) of \( A \) satisfy the following asymptotic expansion

\[
e_n^\pm = \frac{1}{2} \left( e_n^{\pm i\mu_n}, e_n, e_n^{\mp i\mu_n}, -e_n \right)^\top + O \left( \frac{1}{\mu_n^2} \right),
\]

and the root vectors of \( A \) satisfy the following asymptotic expansion

\[
e_n^\pm = \frac{1}{\sqrt{2}} \left( e_n^{\pm i\mu_n}, e_n, 0, 0 \right)^\top + \left( O \left( \frac{1}{\mu_n^2} \right), 0, 0, O \left( \frac{1}{\mu_n} \right) \right)^\top.
\]

**Case 3.** If \( \alpha^2 > \frac{1}{4} \), then we have

\[
e_{1,n}^\pm = \frac{1}{\beta_3} \left( e_n, e_n, e_n^{\pm i\mu_n}, e_n \right)^\top + O \left( \frac{1}{\mu_n^2} \right),
\]

\[
e_{2,n}^\pm = \frac{1}{\delta_3} \left( e_n, e_n, e_n^{\pm i\mu_n}, e_n \right)^\top + O \left( \frac{1}{\mu_n^2} \right),
\]

where \( \beta_3 = \frac{2\alpha}{-1 + i\sqrt{4\alpha^2 - 1}} \), \( \delta_3 = -\frac{1 + i\sqrt{4\alpha^2 - 1}}{2\alpha} \), \( \beta_3^+ = \sqrt{2 + 2|\beta_3|^2} \) and \( \delta_3^+ = \sqrt{2 + 2|\delta_3|^2} \).

**Proof.** When \( a = 1 \) and \( \gamma = 0 \), we must now subdivide the proof into three cases.

**Case 1.** If \( 0 < \alpha^2 < \frac{1}{4} \), then from (3.1), we obtain

\[
\frac{1}{\lambda_{1,n}^\pm} = \frac{1}{\pm i\mu_n} + O \left( \frac{1}{\mu_n^2} \right) \quad \text{and} \quad \frac{1}{\lambda_{2,n}^\pm} = \frac{1}{\pm i\mu_n} + O \left( \frac{1}{\mu_n^2} \right).
\]

Setting \( B_{1,n}^\pm = \frac{1}{\beta_1} \) and \( C_{2,n}^\pm = \frac{1}{\delta_1} \) in (3.10), then using (3.19), we get (3.15).

**Case 2.** If \( \alpha^2 = \frac{1}{4} \), we suppose that \( \alpha = \frac{1}{2} \) since the analysis follows similarly if \( \alpha = -\frac{1}{2} \). In this case, since (2.10) admits two double solutions \( \lambda_n^\pm \), the associated eigenvectors are given by

\[
e_n^\pm = B_n^\pm \left( e_n, e_n, e_n, e_n \right)^\top,
\]

where \( B_n^\pm \in \mathbb{C} \). Furthermore, from (3.4), we have

\[
\frac{1}{\lambda_n^\pm} = \frac{1}{\pm i\mu_n} \left[ 1 + O \left( \frac{1}{\mu_n} \right) \right] = \frac{1}{\pm i\mu_n} + O \left( \frac{1}{\mu_n^2} \right).
\]

Setting \( B_n^\pm = \frac{1}{\beta_1} \) in (3.20) and using (3.21), we get (3.16). We next look for corresponding root vectors

\[
e_n^\pm = \left( \tilde{u}_n^\pm, \tilde{v}_n^\pm, \tilde{y}_n^\pm, \tilde{z}_n^\pm \right)^\top,
\]

such that

\[
(\lambda_n^\pm I - A) e_n^\pm = e_n^\pm.
\]
Equivalently, we have
\[
\begin{align*}
\lambda^\pm_n \tilde{u}^\pm_n - \tilde{v}^\pm_n &= \frac{e_n}{\lambda^\mp_n}, \\
\lambda^\pm_n \tilde{v}^\pm_n + A \tilde{u}^\pm_n + \tilde{v}^\pm_n + \alpha \tilde{z}^\pm_n &= e_n,
\end{align*}
\] (3.22)

where
\[
\tilde{B}^\pm_n = \begin{pmatrix}
\lambda^\mp_n & 1 \\
-\alpha & \lambda^\pm_n
\end{pmatrix}.
\]

Setting \( \tilde{u}^\pm_n = c^\pm e_n \) and \( \tilde{v}^\pm_n = d^\pm e_n \) in (3.22), we get
\[
\tilde{v}^\pm_n = \left( c^\pm \lambda^\pm_n - \frac{1}{\lambda^\mp_n} \right) e_n \quad \text{and} \quad \tilde{z}^\pm_n = \left( d^\pm \lambda^\pm_n + \frac{1}{\lambda^\mp_n} \right) e_n,
\] (3.23)

where the constants \( c^\pm \) and \( d^\pm \) satisfy
\[
\begin{align*}
\left( \mu^2 + (\lambda^\pm_n)^2 + \lambda^\pm_n \right) c^\pm + \alpha \lambda^\pm_n d^\pm &= 2 + \frac{1}{2\lambda^\mp_n}, \\
-\alpha \lambda^\pm_n c^\pm + \left( \mu^2 + (\lambda^\pm_n)^2 \right) d^\pm &= -2 - \frac{1}{2\lambda^\mp_n}.
\end{align*}
\] (3.24)

Since \( \lambda^\pm_n \) satisfy (2.10), then the first equation of (3.24) can be reduced to the second one. Therefore, taking
\[
d^\pm = 0 \quad \text{and} \quad c^\pm = \frac{4\lambda^\pm_n + 1}{(\lambda^\pm_n)^2}
\]
in (3.23), we get
\[
\tilde{v}^\pm_n = \tilde{B}^\pm_n \begin{pmatrix} \frac{4\lambda^\pm_n + 1}{(\lambda^\pm_n)^2} e_n, 4e_n, 0, \frac{e_n}{\lambda^\pm_n} \end{pmatrix}^\top,
\] (3.25)

where \( \tilde{B}^\pm_n \in \mathbb{C} \). From (3.4) and (3.21), we obtain
\[
\frac{4\lambda^\pm_n + 1}{(\lambda^\pm_n)^2} = \frac{4}{\pm i\mu_n} + O \left( \frac{1}{\mu_n^2} \right) \quad \text{and} \quad \frac{1}{\lambda^\pm_n} = O \left( \frac{1}{\mu_n} \right).
\] (3.26)

Finally, setting \( B^\pm_n = \frac{1}{4\sqrt{2}} \) in (3.25), then using (3.26), we get (3.17).

**Case 3.** If \( \alpha^2 > \frac{1}{4} \), then from (3.2), we obtain
\[
\frac{1}{\lambda^\pm_{1,n}} = \frac{1}{\pm i\mu_n} + O \left( \frac{1}{\mu_n^2} \right) \quad \text{and} \quad \frac{1}{\lambda^\pm_{2,n}} = \frac{1}{\pm i\mu_n} + O \left( \frac{1}{\mu_n^2} \right).
\] (3.27)

Setting \( B^\pm_{1,n} = \frac{1}{\beta_1^\pm} \) and \( C^\pm_{2,n} = \frac{1}{\delta_1^\pm} \) in (3.10), then using (3.27), we get (3.18). Thus, the proof is complete.

Let now \( E^\pm_{1,n}, E^\pm_{2,n} \) be linearly independent eigenvectors of the decoupled system (corresponding to \( \alpha = 0 \)). Then one has
\[
E^\pm_{1,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_n, e_n, 0, 0 \end{pmatrix}^\top \quad \text{and} \quad E^\pm_{2,n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0, 0, e_n, e_n \end{pmatrix}^\top.
\] (3.28)

Moreover, for every \( n \geq 1 \), define
\[
W_n = \text{span} \{ E^+_{1,n}, E^-_{1,n}, E^+_{2,n}, E^-_{2,n} \},
\] (3.29)
and
\[
V_n = \text{span} \{ e^+_n, e^-_n, e^+_2, e^-_2 \}, \quad \tilde{V}_n = \text{span} \{ e^+_n, e^-_n, \tilde{e}^+_n, \tilde{e}^-_n \}.
\] (3.30)

From what precedes, one gets the following corollary.
Corollary 3.5. If \( a = 1 \) and \( \gamma = 0 \), then from Lemma 3.4, the following relationship holds.

**Case 1.** If \( 0 < \alpha^2 < \frac{1}{4} \), then

\[
(e_{1,n}^+, e_{1,n}^-, e_{2,n}^+, e_{2,n}^-) = (E_{1,n}^+, E_{1,n}^-, E_{2,n}^+, E_{2,n}^-) L_n, \tag{3.31}
\]

where

\[
L_n = \sqrt{2} \begin{pmatrix}
\frac{1}{\beta_1^2} & 0 & \frac{\delta_1}{\beta_1^2} & 0 \\
0 & \frac{1}{\beta_1^2} & 0 & \frac{\delta_1}{\beta_1^2} \\
\frac{\delta_1}{\beta_1^2} & 0 & \frac{1}{\beta_1^2} & 0 \\
0 & \frac{\delta_1}{\beta_1^2} & 0 & \frac{1}{\beta_1^2}
\end{pmatrix} + O \left( \frac{1}{\mu_n^2} \right). \tag{3.32}
\]

**Case 2.** If \( \alpha^2 = \frac{1}{4} \), then

\[
(e_n^+, e_n^-, \tilde{e}_n^+, \tilde{e}_n^-) = (E_{1,n}^+, E_{1,n}^-, E_{2,n}^+, E_{2,n}^-) \tilde{L}_n, \tag{3.33}
\]

where

\[
\tilde{L}_n = \frac{1}{2} \begin{pmatrix}
\sqrt{2} & 0 & 2 & 0 \\
0 & \sqrt{2} & 0 & 2 \\
-\sqrt{2} & 0 & 0 & 0 \\
0 & -\sqrt{2} & 0 & 0
\end{pmatrix} + O \left( \frac{1}{\mu_n} \right). \tag{3.34}
\]

**Case 3.** If \( \alpha^2 > \frac{1}{4} \), then

\[
(e_{1,n}^+, e_{1,n}^-, e_{2,n}^+, e_{2,n}^-) = (E_{1,n}^+, E_{1,n}^-, E_{2,n}^+, E_{2,n}^-) \tilde{L}_n, \tag{3.35}
\]

where

\[
\tilde{L}_n = \sqrt{2} \begin{pmatrix}
\frac{1}{\beta_3^2} & 0 & \frac{\delta_3}{\beta_3^2} & 0 \\
0 & \frac{1}{\beta_3^2} & 0 & \frac{\delta_3}{\beta_3^2} \\
\frac{\delta_3}{\beta_3^2} & 0 & \frac{1}{\beta_3^2} & 0 \\
0 & \frac{\delta_3}{\beta_3^2} & 0 & \frac{1}{\beta_3^2}
\end{pmatrix} + O \left( \frac{1}{\mu_n^2} \right). \tag{3.36}
\]

In the sequel, our aim is to prove the following proposition.

**Proposition 3.6.** Suppose that \( a = 1 \) and \( \gamma = 0 \). Then the following holds true.

(i): If \( \alpha^2 \neq \frac{1}{4} \), then the set \( \{e_{1,n}^+, e_{1,n}^-, e_{2,n}^+, e_{2,n}^-\}_{n \geq 1} \) of eigenvectors of \( A \) forms a Riesz basis in \( \mathcal{H} \). In particular, all eigenvectors of \( A \) are of the form given in (2.8).

(ii): If \( \alpha^2 = \frac{1}{4} \), then the set \( \{e_{n}^+, e_{n}^-, \tilde{e}_{n}^+, \tilde{e}_{n}^-\}_{n \geq 1} \) of eigenvectors and root vectors of \( A \) forms a Riesz basis in \( \mathcal{H} \). In particular, all eigenvectors of \( A \) are of the form given in (2.8).

To prove Proposition 3.6, we first recall Lemma 3.1 in [34].

**Proposition 3.7** (Lemma 3.1 in [34]). Let \( \{X_n\}_{n \geq 1} \) be a Riesz basis of subspaces in a Hilbert space \( \mathcal{H} \) and \( \{Y_n\}_{n \geq 1} \) a Riesz sequence of subspaces in \( \mathcal{H} \). Assume that there exist a sequence of isomorphisms \( \{L_n\}_{n \geq 1} \) from \( X_n \) onto \( Y_n \) and positive constants \( m, M \) independent of \( n \) such that

\[
\forall \ x_n \in X_n, \forall \ n \geq 1, \quad m \|x_n\| \leq \|L_n x_n\| \leq M \|x_n\|. \tag{3.37}
\]
Assume furthermore that there exist a Riesz basis \( \{ f_{n,i}\}_{1 \leq i \leq I_n} \) in each \( X_n \) and positive constants \( c, C \) independent of \( n \) such that, for every \( x_n = \sum_{i=1}^{I_n} \alpha_{n,i} f_{n,i} \),

one has

\[
c \sum_{i=1}^{I_n} |\alpha_{n,i}|^2 \leq \|x_n\|^2 \leq C \sum_{i=1}^{I_n} |\alpha_{n,i}|^2.
\] (3.38)

Then the sequence

\[
g_{n,i} = L_n f_{n,i} \quad \forall n \geq 1, \quad 1 \leq i \leq I_n.
\] (3.39)

forms a Riesz basis in \( H \).

Proof of Proposition 3.6. First, we prove that \( \{ W_n \}_{n \geq 1} \), defined in (3.29), is a Riesz basis of subspaces of \( H \). Let

\[
U = (u, v, y, z)^T \in H.
\]

Since \( (e_n)_{n \geq 1} \) is a Hilbert basis of \( H \), then

\[
u = \sum_{n \geq 1} u_n e_n \quad v = \sum_{n \geq 1} v_n e_n \quad y = \sum_{n \geq 1} y_n e_n \quad z = \sum_{n \geq 1} z_n e_n.
\]

Hence,

\[
U = \sum_{n \geq 1} (u_n e_n, v_n e_n y_n e_n, z_n e_n)^T
\]

\[
= \frac{1}{\sqrt{2}} \sum_{n \geq 1} \left[ (v_n + i\mu_n u_n) E_{1,n}^+ + (v_n - i\mu_n u_n) E_{1,n}^- \\
+ (z_n + i\mu_n y_n) E_{2,n}^+ + (z_n - i\mu_n y_n) E_{2,n}^-. \right]
\]

Therefore, for every \( U \in H \), \( U \) can be written as \( \sum_{n \geq 1} U_n \) with \( U_n \in W_n \). Moreover, for every pair of different positive integers \( (n, m) \), we have that \( U_n \) and \( U_m \) are perpendicular since \( W_n \) and \( W_m \) are. Therefore \( \|U\|_H^2 = \sum_{n \geq 1} \|U_n\|^2 \) and \( U \) can be uniquely written as \( \sum_{n \geq 1} U_n \). This yields that \( \{ W_n \}_{n \geq 1} \) is a Riesz basis of subspaces in \( H \). We prove similarly that \( \{ V_n \}_{n \geq 1} \) and \( \{ \tilde{V}_n \}_{n \geq 1} \) form a Riesz sequence of subspaces in \( H \). Next, we divide the proof into three cases: \( 0 < \alpha^2 < \frac{1}{4}, \alpha^2 = \frac{1}{4} \) and \( \alpha^2 > \frac{1}{4} \). Since the argument is entirely similar for the three cases, we only provide one of them.

If \( \alpha^2 \neq \frac{1}{4} \), then from Corollary 3.5, we remark that \( L_n \) has a constant leading term which is invertible. This, together with the fact that \( L_n \) is invertible for every \( n \geq 1 \), implies condition (3.37). Moreover, the condition (3.38) is satisfied since \( \{ E_{1,n}^+, E_{1,n}^-, E_{2,n}^+, E_{2,n}^- \} \) forms a Hilbert basis in the subspace \( W_n \). Then applying Proposition 3.7, we obtain that the system of eigenvectors \( \{ e_{1,n}^+, e_{1,n}^-, e_{2,n}^+, e_{2,n}^- \}_{n \geq 1} \) forms a Riesz basis in \( H \). Thus, the proof is complete.
Proof of Theorem 3.1. From Lemma 3.2, the large eigenvalues \( \lambda_{k,n}^{\pm} \) of \( A \) satisfy the following estimation
\[
\Re \{ \lambda_{k,n}^{\pm} \} = \begin{cases} 
-\frac{1}{4} \pm \frac{1}{4} \sqrt{1-4\alpha^2} + O \left( \frac{1}{\mu_n} \right), & \text{if } 0 < \alpha^2 < \frac{1}{4}, \\
-\frac{1}{4} + O \left( \frac{1}{\mu_n} \right), & \text{if } \alpha^2 \geq \frac{1}{4},
\end{cases}
\]
and
\[
|\Im \{ \lambda_{k,n}^{\pm} \}| \geq \mu_n.
\]
In addition to that, from Proposition 3.6, the system of eigenvectors and root vectors of \( A \) forms a Riesz basis of \( \mathcal{H} \). Then, applying Proposition 2.3, we get that System (2.1) is exponentially stable. Thus, the proof is complete.

4. Polynomial stability. In this Section, we consider the remaining cases when \( a = 1 \) and \( \gamma < 0 \) or when \( a \neq 1 \) and \( \gamma \leq 0 \). In these cases, we prove that System (2.1) is polynomially stable. More precisely, we find the optimal polynomial decay rate. Our main result in this Section is the following theorem.

**Theorem 4.1.** There exists a positive constant \( C > 0 \) such that for every \( u_0 \in D(A) \), the energy of System (2.1) has the polynomial decay rate
\[
E(t) \leq \frac{C}{\delta(\gamma)} \|Au_0\|_{\mathcal{H}}^2, \quad \forall \ t > 0,
\]
where
\[
\delta(\gamma) = \begin{cases} 
-\frac{1}{\gamma}, & \text{if } a = 1 \text{ and } \gamma < 0, \\
\frac{1}{1-\gamma}, & \text{if } a \neq 1 \text{ and } \gamma \leq 0.
\end{cases}
\]
In addition, the energy decay rate in (4.1) is optimal according to Definition 2.1.

For the proof of Theorem 4.1, we need, as in the previous case, to study the asymptotic behavior of the eigenvalues \( \lambda_{1,n}^{\pm} \), \( \lambda_{2,n}^{\pm} \) and the corresponding eigenvectors \( e_{1,n}^{\pm} \), \( e_{2,n}^{\pm} \). We start with the case when \( a = 1 \) and \( \gamma < 0 \).

**Lemma 4.2.** Suppose that \( a = 1 \) and \( \gamma < 0 \). Let \( N \in \mathbb{N} \) be the integer part of \( \frac{1}{2} - \gamma \), i.e., the unique integer such that
\[
2N \leq 1 - 2\gamma < 2N + 2.
\]
Then the eigenvalues \( \lambda_{1,n}^{\pm} \), \( \lambda_{2,n}^{\pm} \) of System (2.1) satisfy the following asymptotic expansions
\[
\lambda_{1,n}^{\pm} = \begin{cases} 
\pm i\mu_n + \frac{i\alpha}{2} - \frac{1}{4\mu_n^{2\gamma}} + O \left( \frac{1}{\mu_n^{\min(1,-4\gamma)}} \right), & \text{if } 0 < -2\gamma < 1, \\
\pm i\mu_n + \frac{i\alpha}{2} \pm i \sum_{k=1}^{N} \frac{\alpha_k}{\mu_n^{k-1}} - \frac{1}{4\mu_n^{2\gamma}} + O \left( \frac{1}{\mu_n^{\min(2N+1,1-2\gamma)}} \right), & \text{if } -2\gamma \geq 1,
\end{cases}
\]
\[
\lambda_{2,n}^{\pm} = \begin{cases} 
\pm i\mu_n - \frac{i\alpha}{2} - \frac{1}{4\mu_n^{2\gamma}} + O \left( \frac{1}{\mu_n^{\min(1,-4\gamma)}} \right), & \text{if } 0 < -2\gamma < 1, \\
\pm i\mu_n - \frac{i\alpha}{2} \pm i \sum_{k=1}^{N} \frac{\alpha_k}{\mu_n^{k-1}} - \frac{1}{4\mu_n^{2\gamma}} + O \left( \frac{1}{\mu_n^{\min(2N+1,1-2\gamma)}} \right), & \text{if } -2\gamma \geq 1,
\end{cases}
\]
where \( \alpha_1 = \frac{\alpha^2}{8} \) and \( \alpha_k = \frac{(2k-3)! \alpha^{2k} (-1)^{k-1}}{2^{4k-k(k-2)!}}, \ \forall \ k \geq 2. \)

Proof. Assume that \( \alpha = 1 \) and \( \gamma < 0. \) For \( n \) large enough, from (2.10), we get that

\[
\left\{ \begin{align*}
\lambda_{1,n}^2 + \frac{\mu_n^{2\gamma} - i \sqrt{4\alpha^2 - \mu_n^{4\gamma}}}{2} \lambda_{1,n} + \mu_n^2 &= 0, \\
\lambda_{2,n}^2 + \frac{\mu_n^{2\gamma} + i \sqrt{4\alpha^2 - \mu_n^{4\gamma}}}{2} \lambda_{2,n} + \mu_n^2 &= 0.
\end{align*} \right. \tag{4.4}
\]

Solving Equation (4.4), we obtain

\[
\lambda_{1,n}^\pm = \frac{1}{4} \left( -\mu_n^{2\gamma} \pm \sqrt{2\sqrt{16\mu_n^4 + (8\alpha^2 - 4\mu_n^{4\gamma})\mu_n^2 + \alpha^4 - 8\mu_n^2 - 2\alpha^2 + \mu_n^{4\gamma}}} \right)
+ \frac{i}{4} \left( \sqrt{4\alpha^2 - \mu_n^{4\gamma}} \pm \sqrt{2\sqrt{16\mu_n^4 + (8\alpha^2 - 4\mu_n^{4\gamma})\mu_n^2 + \alpha^4 + 8\mu_n^2 + 2\alpha^2 - \mu_n^{4\gamma}}} \right). \tag{4.5}
\]

and

\[
\lambda_{2,n}^\pm = \frac{1}{4} \left( -\mu_n^{2\gamma} \pm \sqrt{2\sqrt{16\mu_n^4 + (8\alpha^2 - 4\mu_n^{4\gamma})\mu_n^2 + \alpha^4 - 8\mu_n^2 - 2\alpha^2 + \mu_n^{4\gamma}}} \right)
- \frac{i}{4} \left( \sqrt{4\alpha^2 - \mu_n^{4\gamma}} \pm \sqrt{2\sqrt{16\mu_n^4 + (8\alpha^2 - 4\mu_n^{4\gamma})\mu_n^2 + \alpha^4 + 8\mu_n^2 + 2\alpha^2 - \mu_n^{4\gamma}}} \right). \tag{4.6}
\]

We have

\[
\sqrt{2\sqrt{16\mu_n^4 + 4(2\alpha^2 - \mu_n^{4\gamma})\mu_n^2 + \alpha^4 - 8\mu_n^2 - 2\alpha^2 + \mu_n^{4\gamma}}} = O \left( \frac{1}{\mu_n^{2-2\gamma}} \right), \tag{4.7}
\]

and

\[
\sqrt{4\alpha^2 - \mu_n^{4\gamma}} = 2\alpha + O \left( \frac{1}{\mu_n^{2-4\gamma}} \right). \tag{4.8}
\]

and

\[
\sqrt{16\mu_n^4 + 4(2\alpha^2 - \mu_n^{4\gamma})\mu_n^2 + \alpha^4 + 8\mu_n^2 + 2\alpha^2 - \mu_n^{4\gamma}}
= 2\sqrt{2\mu_n} \sqrt{1 + \frac{\alpha^2}{4\mu_n^2}} + O \left( \frac{1}{\mu_n^{2-4\gamma}} \right) + 1 + \frac{\alpha^2}{4\mu_n^2} + O \left( \frac{1}{\mu_n^{2-4\gamma}} \right). \tag{4.9}
\]

We next proceed by studying two cases.

**Case 1.** If \( 0 < -2\gamma < 1, \) then \( N = 0 \) and, from (4.9), we obtain

\[
4\mu_n \sqrt{1 + \frac{\alpha^2}{4\mu_n^2}} + O \left( \frac{1}{\mu_n^{2-4\gamma}} \right) = 4\mu_n \sqrt{1 + O \left( \frac{1}{\mu_n^2} \right)} = 4\mu_n + O \left( \frac{1}{\mu_n} \right). \tag{4.10}
\]

Inserting (4.7), (4.8), (4.9) and (4.10) into (4.5) and (4.6), we get

\[
\lambda_{1,n}^\pm = \pm i\mu_n + \frac{i\alpha}{2} - \frac{1}{4\mu_n^{2-2\gamma}} + O \left( \frac{1}{\mu_n^{\min(1, -4\gamma)}} \right), \tag{4.11}
\]
and
\[
\lambda_{2,n}^\pm = \pm i\mu_n - \frac{i\alpha}{2} - \frac{1}{4\mu_n - 2\gamma} + O \left( \frac{1}{\min(1, -4\gamma)} \right). \tag{4.12}
\]

**Case 2.** If \(-2\gamma \geq 1\), then from (4.11) and (4.12), we need to increase the order of the finite expansion. Consider the integer part \(N \in \mathbb{N}\) of \(\frac{1}{2} - \gamma\), which is positive. Setting \(x = \frac{\alpha^2}{4\mu_n^2} + O \left( \frac{1}{\mu_n^{2\gamma}} \right)\), we have
\[
(1 + x)^{\frac{1}{2}} = 1 + \sum_{k=1}^{N} \beta_k x^k + O \left( x^{N+1} \right), \tag{4.13}
\]
where
\[
\beta_1 = \frac{1}{2} \quad \text{and} \quad \beta_k = \frac{(2k - 3)!(-1)^{k-1}}{2^{2k-2}(k-2)!k!}, \quad \forall \ k \geq 2.
\]
Therefore,
\[
\left( 1 + \frac{\alpha^2}{4\mu_n^2} + O \left( \mu_n^{4\gamma-2} \right) \right)^{\frac{1}{2}} = 1 + \sum_{k=1}^{N} \beta_k \left( \frac{\alpha^2}{4\mu_n^2} \right)^k + O \left( \frac{1}{\mu_n^{2\gamma}} \right) + O \left( \frac{1}{\mu_n^{2N+2}} \right). \tag{4.14}
\]
Taking \(\alpha_k = \frac{\alpha^2 \beta_k}{2^{2k}}\) for every positive integer \(k\) and noticing that \(2N + 2 \leq 2 - 4\gamma\), we obtain
\[
\left( 1 + \frac{\alpha^2}{4\mu_n^2} + O \left( \mu_n^{4\gamma-2} \right) \right)^{\frac{1}{2}} = 1 + \sum_{k=1}^{N} \frac{\alpha_k}{\mu_n^{2k}} + O \left( \frac{1}{\mu_n^{2N+2}} \right). \tag{4.15}
\]
Inserting (4.14) into (4.9), we get
\[
\frac{1}{4} \sqrt{16\mu_n^4 + 4(2\alpha^2 - \mu_n^{2\gamma})\mu_n^2 + \alpha^4 + 8\mu_n^2 + 2\alpha^2 - \mu_n^{4\gamma}}
= \mu_n + \sum_{k=1}^{N} \frac{\alpha_k}{\mu_n^{2k}} + O \left( \frac{1}{\mu_n^{2N+1}} \right). \tag{4.15}
\]
Substituting (4.7), (4.8) and (4.15) into (4.5), then using the condition \(1 - 2\gamma \leq 2N + 2\), we obtain
\[
\lambda_{1,n}^\pm = -\frac{1}{4\mu_n - 2\gamma} + i \left( \pm \mu_n + \frac{\alpha}{2} \pm \sum_{k=1}^{N} \frac{\alpha_k}{\mu_n^{2k-1}} \right) + O \left( \frac{1}{\min(2N+1, 1-2\gamma)} \right). \tag{4.16}
\]
Then, from (4.11) and (4.16), we get (4.2). On the other hand, inserting (4.7), (4.8) and (4.15) into (4.4), then using the condition \(1 - 2\gamma \leq 2N + 2\), we get
\[
\lambda_{2,n}^\pm = -\frac{1}{4\mu_n - 2\gamma} + i \left( \pm \mu_n - \frac{\alpha}{2} \pm \sum_{k=1}^{N} \frac{\alpha_k}{\mu_n^{2k-1}} \right) + O \left( \frac{1}{\min(2N+1, 1-2\gamma)} \right). \tag{4.17}
\]
Finally, from (4.12) and (4.17), we obtain (4.3). Thus, the proof is complete.

Before we study the asymptotic behavior of the eigenvalues in case \(a \neq 1\) and \(\gamma \leq 0\), we prove the following lemma. We now study the asymptotic behavior of the eigenvalues in the case when \(a \neq 1\) and \(\gamma \leq 0\). We prove the following lemma.
Suppose that $a \neq 1$ and $\gamma \leq 0$. Let $N \in \mathbb{N}$ be the integer part of $\frac{1}{2} - \gamma$. Then, we have

$$\lambda_{1,n}^{\pm} = \begin{cases} 
\pm i \sqrt{a} \mu_n - \frac{1}{2 \mu_n^{2 \gamma}} + O \left( \frac{1}{\mu_n} \right) & \text{if } 0 \leq -2\gamma < 1, \\
\pm i \sqrt{a} \mu_n + i \sqrt{a} \sum_{\ell=1}^{N} C_1(\ell) \frac{1}{\mu_n^{2 \ell-1}} + O \left( \frac{1}{\mu_n^{2N+1}} \right) & \text{if } -2\gamma \geq 1 
\end{cases}$$

(4.18)

and

$$\lambda_{2,n}^{\pm} = \begin{cases} 
\pm i \mu_n + \frac{ia^2}{(a-1) \mu_n} - \frac{\alpha^2}{2(a-1) \mu_n^{2-2\gamma}} + O \left( \frac{1}{\mu_n^3} \right) & \text{if } 0 \leq -2\gamma < 1, \\
\pm i \mu_n + i \sum_{\ell=1}^{N+1} C_2(\ell) \frac{1}{\mu_n^{2 \ell-1}} - \frac{\alpha^2}{2(a-1) \mu_n^{2-2\gamma}} + O \left( \frac{1}{\mu_n^{2N+3}} \right) & \text{if } -2\gamma \geq 1, 
\end{cases}$$

(4.19)

where, for $1 \leq \ell \leq N + 1$, $C_1(\ell)$ and $C_2(\ell)$ are real numbers depending only on $a$ and $\alpha$.

For the proof of Lemma 4.3, we need the following lemmas.

**Lemma 4.4.** Suppose that $a \neq 1$ and $\gamma \leq 0$. Let $N \in \mathbb{N}$ be the integer part of $\frac{1}{2} - \gamma$. Then the eigenvalues $\lambda_{1,n}^{\pm}$, $\lambda_{2,n}^{\pm}$ satisfy the following asymptotic expansion

$$\left\{ \begin{array}{l}
\lambda_{1,n}^{\pm} = \pm i \sqrt{a} \mu_n - \frac{1}{2 \mu_n^{2 \gamma}} + O \left( \frac{1}{\mu_n} \right), \\
\frac{(\lambda_{1,n}^{\pm})^2}{a} + \mu_n^2 + \sum_{k=1}^{N} \frac{D(k)}{(\lambda_{1,n}^{\pm})^{2k-2} + \frac{i D_1^+}{\mu_n^{1-2\gamma}}} + O \left( \frac{1}{\mu_n^{2N+1}} \right) = 0, \text{ if } -2\gamma \geq 1
\end{array} \right.$$ 

(4.20)

and

$$\left\{ \begin{array}{l}
\lambda_{2,n}^{\pm} = \pm i \mu_n + \frac{ia^2}{(a-1) \mu_n} - \frac{\alpha^2}{2(a-1) \mu_n^{2-2\gamma}} + O \left( \frac{1}{\mu_n^3} \right), \\
(\lambda_{2,n}^{\pm})^2 + \mu_n^2 - \sum_{k=1}^{N+1} \frac{D(k)}{(\lambda_{2,n}^{\pm})^{2k-2} + \frac{i D_2^+}{\mu_n^{1-2\gamma}}} + O \left( \frac{1}{\mu_n^{2N+2}} \right) = 0, \text{ if } -2\gamma \geq 1, 
\end{array} \right.$$ 

(4.21)

where

$$D_1^\pm = \pm \frac{1}{\sqrt{a}}, \quad D_2^\pm = \pm \frac{\alpha^2}{(a-1)^2}, \quad D(1) = \frac{\alpha^2}{a-1}$$

and for every $k \geq 2$, we have

$$D(k) = \frac{2a^{k-1} \alpha^2 (2k-3)!}{(a-1)^{2k-1} (k-2)! k!}.$$ 

**Proof.** First, from (2.10), we have

$$\frac{(a + 1) (\lambda_{1,n}^{\pm})^2 + \lambda_{1,n}^{\pm} \mu_n^{2 \gamma} - (a - 1) (\lambda_{1,n}^{\pm})^2 P(\lambda_{1,n}^{\pm}, \mu_n)}{2a} + \mu_n^2 = 0$$

(4.22)

and

$$\frac{(a + 1) (\lambda_{2,n}^{\pm})^2 + \lambda_{2,n}^{\pm} \mu_n^{2 \gamma} + (a - 1) (\lambda_{2,n}^{\pm})^2 P(\lambda_{2,n}^{\pm}, \mu_n)}{2a} + \mu_n^2 = 0.$$ 

(4.23)
where

\[ P(\lambda, \mu_n) = \left( 1 - \frac{4a\alpha^2}{(a-1)^2\lambda^2} - \frac{2}{(a-1)\lambda\mu_n^{-2\gamma}} + \frac{1}{(a-1)^2\left(\lambda\mu_n^{-2\gamma}\right)^2} \right)^{\frac{1}{2}}. \]

The proof is based on finding the asymptotic expansion of (4.22) and (4.23). We note that the order of expansion is chosen so that the term equivalent to \( \Re(\lambda_n) \) appears. First, we show (4.20). Consequently, let \( \lambda_n = \lambda_{1.n}^\pm \). We consider two cases.

**Case 1.** If \( 0 \leq -2\gamma < 1 \) (in this case \( N = 0 \)), then from Lemma 2.5, we get

\[ P(\lambda_n, \mu_n) = \left( 1 - \frac{2}{(a-1)\lambda_n\mu_n^{-2\gamma}} + O\left( \frac{1}{\mu_n^2} \right) \right)^{\frac{1}{2}} = 1 - \frac{1}{(a-1)\lambda_n\mu_n^{-2\gamma}} + O\left( \frac{1}{\mu_n^2} \right). \]

Inserting (4.24) into (4.22), then using Lemma 2.5, we obtain

\[ \left( \frac{\lambda_{1.n}^\pm}{a} \right)^2 + \frac{\lambda_{1.n}^\pm}{a\mu_n^{-2\gamma}} + \mu_n^2 + O(1) = 0. \]

Solving (4.25), we obtain

\[ \lambda_{1.n}^\pm = \pm i\sqrt{a\mu_n - \frac{1}{2\mu_n^{-2\gamma}}} + O\left( \mu_n \right). \]

**Case 2.** If \( -2\gamma \geq 1 \), then from (4.26), the order of expansion increases and \( N \geq 1 \).
We have

\[ P(\lambda_n, \mu_n) = 1 + \sum_{k=1}^{N} \beta_k x^k + O(x^{N+1}), \]

where we have set

\[ x = -\frac{4a\alpha^2}{(a-1)^2\lambda_n^2} - \frac{2}{(a-1)\lambda_n\mu_n^{-2\gamma}} + \frac{1}{(a-1)^2\left(\lambda_n\mu_n^{-2\gamma}\right)^2}. \]

From Lemma 2.5, we have

\[ x = -\frac{4a\alpha^2}{(a-1)^2\lambda_n^2} - \frac{2}{(a-1)\lambda_n\mu_n^{-2\gamma}} + O\left( \frac{1}{\mu_n^{2-4\gamma}} \right) \]

and for every \( k \geq 2 \) (in case \( N \geq 2 \), we have

\[ x^k = \left( -\frac{4a\alpha^2}{(a-1)^2\lambda_n^2} \right)^k + O\left( \frac{1}{\mu_n^{2k-2\gamma-1}} \right). \]

Inserting (4.29) and (4.30) into (4.27) and using the condition \( 1 - 2\gamma < 2N + 2 \), we get

\[ P(\lambda_n, \mu_n) = 1 + \sum_{k=1}^{N} \beta_k \left( -\frac{4a\alpha^2}{(a-1)^2\lambda_n^2} \right)^k \]

\[ -\frac{1}{(a-1)\lambda_n^{-2\gamma}} + O\left( \frac{1}{\mu_n^{\min(2-4\gamma,2-2\gamma,2N+2)}} \right). \]
Since $N \geq 1$ and $2N \leq 1 - 2\gamma$, then
\[
\min (2 - 4\gamma, 3 - 2\gamma, 2N + 2) = 2N + 2.
\] (4.32)

Therefore,
\[
P (\lambda_n, \mu_n) = 1 + \frac{N}{k} \beta_k \left( \frac{-4a\alpha^2}{(a - 1)^2 \lambda_n^2} \right)^k - \frac{1}{(a - 1) \lambda_n \mu_n^{2\gamma}} + O \left( \frac{1}{\mu_n^{2N + 2}} \right). \tag{4.33}
\]

Substituting (4.33) into (4.22), then using Lemma 2.5, we obtain
\[
\frac{\left(\lambda_{1,n}^\pm\right)^2}{a} + \frac{\lambda_{1,n}^\pm}{a \mu_n^{2\gamma}} + \mu_n^2 + \sum_{k=1}^{N} \frac{D(k)}{(\lambda_{1,n}^\pm)^{2k-2}} + O \left( \frac{1}{\mu_n^{2N}} \right) = 0. \tag{4.34}
\]

From (4.34) and Lemma 2.5, we obtain
\[
\frac{(\lambda_{1,n}^\pm)^2}{a} + \mu_n^2 + O (1) = 0. \tag{4.35}
\]

Consequently, we have
\[
\frac{\lambda_{1,n}^\pm}{a \mu_n^{2\gamma}} = \pm \frac{i}{a \sqrt{\mu_n^{1-2\gamma}}} + O \left( \frac{1}{\mu_n^{1-2\gamma}} \right). \tag{4.36}
\]

Inserting (4.36) into (4.34), then using the condition $1 - 2\gamma \geq 2N$ and Lemma 2.5, we get
\[
\frac{(\lambda_{1,n}^\pm)^2}{a} + \frac{iD_{1,n}^\pm}{\mu_n^{1-2\gamma}} + \mu_n^2 + \sum_{k=1}^{N} \frac{D(k)}{(\lambda_{1,n}^\pm)^{2k-2}} + O \left( \frac{1}{\mu_n^{2N}} \right) = 0. \tag{4.37}
\]

Finally, from (4.26) and (4.37), we get (4.20).

Our next aim is to prove (4.21) by similar computations. Let $\lambda_n = \lambda_{2,n}^\pm$. We have two cases.

**Case 1.** If $0 \leq -2\gamma < 1$, then
\[
P (\lambda_n, \mu_n) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + O (x^4),
\]
where $x$ is defined in (4.28). Using Lemma 2.5, we get
\[
P (\lambda_n, \mu_n) = 1 - \frac{2a\alpha^2}{(a - 1)^2 \lambda_n^2} - \frac{1}{(a - 1) \lambda_n \mu_n^{2\gamma}} - \frac{2a\alpha^2}{(a - 1)^3 \lambda_n^2 \mu_n^{2\gamma}} + O \left( \frac{1}{\mu_n^4} \right). \tag{4.38}
\]

Inserting (4.38) into (4.23) and using Lemma 2.5, we obtain
\[
\frac{(\lambda_{2,n}^\pm)^2}{a} + \frac{\alpha^2}{\mu_n^{1-2\gamma} \lambda_{2,n}^\pm} - \frac{\alpha^2}{a - 1} + O \left( \frac{1}{\mu_n^2} \right) = 0. \tag{4.39}
\]

From (4.39) and Lemma 2.5, we have
\[
(\lambda_{2,n}^\pm)^2 = -\mu_n^2 + O (1). \tag{4.40}
\]

Consequently, we have
\[
- \frac{\alpha^2}{(a - 1)^2 \mu_n^{2\gamma} \lambda_{2,n}^\pm} = \pm \frac{i\alpha^2}{(a - 1)^2 \mu_n^{1-2\gamma}} + O \left( \frac{1}{\mu_n^{3-2\gamma}} \right). \tag{4.41}
\]

Substituting (4.41) into (4.39), we get
\[
(\lambda_{2,n}^\pm)^2 + \mu_n^2 - \frac{\alpha^2}{a - 1} \pm \frac{i\alpha^2}{(a - 1)^2 \mu_n^{1-2\gamma}} + O \left( \frac{1}{\mu_n^2} \right) = 0. \tag{4.42}
\]
Solving (4.42), we obtain
\[
\lambda_{2,n}^\pm = \pm i\mu_n \mp \frac{i\alpha^2}{2(a - 1) \mu_n} - \frac{\alpha^2}{2(a - 1)^2 \mu_n^{2-2\gamma}} + O\left(\frac{1}{\mu_n^3}\right). 
\] (4.43)

**Case 2.** If \(-2\gamma \geq 1\), then the order of expansion equal to \(N\) is not sufficient. Consequently, we let
\[
P(\lambda_n, \mu_n) = 1 + \sum_{k=1}^{N+1} \beta_k x_k + O\left(x^{N+2}\right),
\] (4.44)
where \(x\) is given in (4.28). Using Lemma 2.5, we obtain
\[
\beta_1 x + \beta_2 x^2 = \sum_{k=1}^{2} \beta_k \left( - \frac{4a\alpha^2}{(a - 1)^2 \lambda_n^2} \right)^k - \frac{1}{(a - 1) \lambda_n \mu_n^{2\gamma}} - \frac{2a\alpha^2}{(a - 1)^2 \lambda_n \mu_n \gamma} + O\left(\frac{1}{\mu_n^{4-4\gamma}}\right).
\] (4.45)

If \(N \geq 2\), then for every \(k \geq 3\), we have
\[
x^k = \left( - \frac{4a\alpha^2}{(a - 1)^2 \lambda_n^2} \right)^k + O\left(\frac{1}{\mu_n^{2k-2\gamma-1}}\right).
\] (4.46)

Inserting (4.45) and (4.46) into (4.44), we get
\[
P(\lambda_n, \mu_n) = 1 + \sum_{k=1}^{N+1} \beta_k \left( - \frac{4a\alpha^2}{(a - 1)^2 \lambda_n^2} \right)^k - \frac{1}{(a - 1) \lambda_n \mu_n^{2\gamma}} - \frac{2a\alpha^2}{(a - 1)^2 \lambda_n \mu_n \gamma} + O\left(\frac{1}{\mu_n^{S(N, \gamma)}}\right),
\] (4.47)
where
\[
S(N, \gamma) = \min(4 - 4\gamma, 5 - 2\gamma, 2N + 4).
\]

Since \(N \geq 1\) and \(2N \leq 1 - 2\gamma\), we can easily check that
\[
S(N, \gamma) = 2N + 4.
\] (4.48)

From (4.47) and (4.48), we get
\[
P(\lambda_n, \mu_n) = 1 + \sum_{k=1}^{N+1} \beta_k \left( - \frac{4a\alpha^2}{(a - 1)^2 \lambda_n^2} \right)^k - \frac{1}{(a - 1) \lambda_n \mu_n^{2\gamma}} - \frac{2a\alpha^2}{(a - 1)^2 \lambda_n \mu_n \gamma} + O\left(\frac{1}{\mu_n^{2N+4}}\right).
\] (4.49)

Inserting (4.49) into (4.23), then using Lemma 2.5, we obtain
\[
(\lambda_{2,n}^\pm)^2 + \mu_n^2 - \frac{\alpha^2}{(a - 1)^2 \mu_n^{2-2\gamma}} \lambda_{2,n}^\pm - \sum_{k=1}^{N+1} \frac{D(k)}{(\lambda_{2,n}^\pm)^{2k-2}} + O\left(\frac{1}{\mu_n^{2N+2}}\right) = 0.
\] (4.50)

From (4.50) and Lemma 2.5, we remark that
\[
(\lambda_{2,n}^\pm)^2 = -\mu_n^2 + O\left(1\right).
\] (4.51)

Consequently, we obtain
\[
- \frac{\alpha^2}{(a - 1)^2 \mu_n^{2-2\gamma}} \lambda_{2,n}^\pm = \pm \frac{i\alpha^2}{(a - 1)^2 \mu_n^{1-2\gamma}} + O\left(\frac{1}{\mu_n^{3-2\gamma}}\right).
\] (4.52)
Substituting (4.52) into (4.50), we get
\[
(\lambda_{2,n}^\pm)^2 + \mu_n^2 - \sum_{k=1}^{N+1} \frac{D(k)}{\lambda_{2,n}^\pm} + \frac{iD^\pm_2}{\mu_n-2\gamma} + O\left(\frac{1}{\mu_n^{2N+2}}\right) = 0.
\] (4.53)

Finally, from (4.43) and (4.53), we get (4.21). Thus, the proof is complete. \(\square\)

Next, when \(-2\gamma \geq 1\), we try to replace the powers of \(\lambda_n\) in (4.20) and in (4.21) with powers of \(\mu_n\) as shown in the following lemma.

**Lemma 4.5.** Suppose that \(a \neq 1\) and \(\gamma \leq 0\). Let \(N \in \mathbb{N}\) be the integer part of \(\frac{1}{2} - \gamma\). If \(-2\gamma \geq 1\), then the eigenvalues \(\lambda_{1,n}^\pm, \lambda_{2,n}^\pm\) satisfy the following asymptotic expansion
\[
\frac{(\lambda_{1,n}^\pm)^2}{a} + \mu_n^2 + \sum_{k=1}^{N} \frac{F_1(k)}{\mu_n^{2k-2}} + \frac{iD^\pm_1}{\mu_n^{1-2\gamma}} + O\left(\frac{1}{\mu_n^{2N}}\right) = 0
\] (4.54)

and
\[
(\lambda_{2,n}^\pm)^2 + \mu_n^2 - \sum_{k=1}^{N+1} \frac{F_2(k)}{\mu_n^{2k-2}} + \frac{iD^\pm_2}{\mu_n^{1-2\gamma}} + O\left(\frac{1}{\mu_n^{2N+2}}\right) = 0,
\] (4.55)

where, for \(1 \leq k \leq N + 1\), the real numbers \(F_1(k)\) and \(F_2(k)\) depend only on \(a\) and \(\alpha\).

**Proof.** We first show (4.55) by proceeding in two steps. We start by proving the following asymptotic estimate
\[
(\lambda_{2,n}^\pm)^2 + \mu_n^2 - \sum_{k=1}^{N} \frac{F_2(k)}{\mu_n^{2k-2}} + O\left(\frac{1}{\mu_n^{2N}}\right) = 0,
\] (4.56)

where, for \(1 \leq k \leq N + 1\), the numbers \(F_2(k)\) are real numbers depending only on \(a\) and \(\alpha\). For this aim, using (4.21), Lemma 2.5, and the fact that \(2N - 2 < 2N \leq 1 - 2\gamma\), we get
\[
(\lambda_{2,n}^\pm)^2 + \mu_n^2 - \sum_{k=1}^{N} \frac{D(k)}{(\lambda_{2,n}^\pm)^{2k-2}} + O\left(\frac{1}{\mu_n^{2N}}\right) = 0.
\] (4.57)

If \(N \leq 3\), then from (4.57) and Lemma 2.5, we obtain
\[
(\lambda_{2,n}^\pm)^2 + \mu_n^2 - \frac{\alpha^2}{a - 1} + O\left(\frac{1}{\mu_n^2}\right) = 0.
\] (4.58)

From (4.58), we get
\[
\sum_{k=1}^{3} \frac{D(k)}{(\lambda_{2,n}^\pm)^{2k-2}} = \sum_{k=1}^{3} \frac{F_2(k)}{\mu_n^{2k-2}} + O\left(\frac{1}{\mu_n^6}\right),
\] (4.59)

where
\[
F_2(1) = \frac{\alpha^2}{a - 1}, \quad F_2(2) = -\frac{aa^4}{(a - 1)^3} \quad and \quad F_2(3) = \frac{a(a + 1)\alpha^5}{(a - 1)^5}.
\]

Inserting (4.59) into (4.57), we obtain
\[
(\lambda_{2,n}^\pm)^2 + \mu_n^2 - \sum_{k=1}^{3} \frac{F_2(k)}{\mu_n^{2k-2}} + O\left(\frac{1}{\mu_n^6}\right) = 0.
\] (4.60)
Therefore, if $N \leq 3$, then from (4.60), we get (4.56). Next, if $3 < N \leq 5$, then from (4.60), we obtain
\[
\sum_{k=1}^{5} \frac{D(k)}{(\lambda_{2,n}^{\pm})^{2k-2}} = \sum_{k=1}^{5} \frac{F_2(k)}{\mu_k^{2k-2}} + O \left( \frac{1}{\mu_n^{10}} \right),
\]
where
\[
F_2(4) = -\frac{a\alpha^8 (a^2 + 3\alpha + 1)}{(a-1)^7} \quad \text{and} \quad F_2(5) = \frac{a\alpha^{10} (a^2 + 5\alpha + 1)}{(a-1)^9}.
\]
Substituting (4.61) into (4.57), we get
\[
(\lambda_{2,n}^{\pm})^2 + \mu_n^2 - \sum_{k=1}^{5} \frac{F_2(k)}{\mu_k^{2k-2}} + O \left( \frac{1}{\mu_n^{10}} \right) = 0.
\]
Therefore, if $3 < N \leq 5$, then from (4.62), we get (4.56). Similarly, if $N > 5$, we iterate the above the process in order to get (4.56).

Our next goal is to prove (4.55). From (4.56), we get
\[
\frac{1}{\left(\lambda_{2,n}^{\pm}\right)^2} = -\frac{1}{\mu_n^2} - \frac{1}{\mu_n^2} \sum_{j=1}^{N-1} \left( \sum_{k=1}^{N} \frac{F_2(k)}{\mu_k^{2k}} \right)^j + O \left( \frac{1}{\mu_n^{2N+2}} \right).
\]
Therefore, we have
\[
\frac{1}{\left(\lambda_{2,n}^{\pm}\right)^2} = -\frac{1}{\mu_n^2} - \frac{1}{\mu_n^2} \sum_{j=1}^{N-1} \left( \sum_{k=1}^{N} \frac{F_2(k)}{\mu_k^{2k}} \right)^j + O \left( \frac{1}{\mu_n^{2N+2}} \right).
\]
From (4.63) we can find a real number $F_2(N+1)$ depending on $a$ and $\alpha$ such that
\[
\sum_{k=1}^{N+1} \frac{D(k)}{(\lambda_{2,n}^{\pm})^{2k-2}} = \sum_{k=1}^{N+1} \frac{F_2(k)}{\mu_k^{2k-2}} + O \left( \frac{1}{\mu_n^{2N+2}} \right).
\]
Inserting (4.64) into (4.21), we get (4.55).

Next, we show (4.54). If $N = 1$, then from (4.20), we obtain (4.54). Otherwise, if $N \geq 2$, then similar to (4.56), using (4.20), Lemma 2.5, and the fact that $2N - 4 < 2N - 2 \leq -1 - 2\gamma$, we get
\[
\frac{\left(\lambda_{1,n}^{\pm}\right)^2}{a} + \mu_n^2 + \sum_{k=1}^{N-1} \frac{F_1(k)}{\mu_k^{2k-2}} + O \left( \frac{1}{\mu_n^{2N-2}} \right) = 0,
\]
where for every $k = 1, \ldots, N - 1$ the numbers $F_1(k)$ are real numbers depending only on $a$ and $\alpha$. Moreover, similar to (4.64), using (4.65), it follows that
\[
\sum_{k=1}^{N} \frac{D(k)}{(\lambda_{1,n}^{\pm})^{2k-2}} = \sum_{k=1}^{N} \frac{F_1(k)}{\mu_k^{2k-2}} + O \left( \frac{1}{\mu_n^{2N}} \right),
\]
where $F_1(N)$ is a real number depending on $a$ and $\alpha$. Substituting (4.66) into the second estimation of (4.20), we get (4.54). Thus, the proof is complete.

**Proof of Lemma 4.3.** Suppose that $a \neq 1$ and $\gamma \leq 0$. First, if $0 \leq -2\gamma < 1$, then the estimations (4.18) and (4.19) are obtained from (4.20) and (4.21). Otherwise, if $-2\gamma \geq 1$, then from (4.54), we get
\[
\lambda_{1,n}^{\pm} = \pm i \alpha (1 + x)^{\frac{1}{2}}.
\]
where
\[ x = \sum_{k=1}^{N} F_1(k) \mu_n^{2k} + iD_1^\pm \mu_n^{3-2\gamma} + O \left( \frac{1}{\mu_n^{2N+2}} \right). \]  
(4.68)

Therefore, we have
\[ (1 + x)^{\frac{1}{2}} = 1 + \sum_{j=1}^{N} \beta_j x^j + O \left( x^{N+1} \right), \]  
(4.69)

where, for every \( j \geq 2 \), we have
\[ x^j = \left( \sum_{k=1}^{N} F_1(k) \mu_n^{2k} \right)^j + O \left( \frac{1}{\mu_n^{2N+2}} \right) + O \left( \frac{1}{\mu_n^{2N+4}} \right). \]  
(4.70)

Since \( 1 - 2\gamma \geq 2N \), then for every \( 2 \leq j \leq N \), we have
\[ x^j = \left( \sum_{k=1}^{N} F_1(k) \mu_n^{2k} \right)^j + O \left( \frac{1}{\mu_n^{2N+2}} \right) \]  
(4.71)

and
\[ x^{N+1} = \frac{1}{\mu_n^{2N+2}} \left( \sum_{k=1}^{N} F_1(k) \mu_n^{2k} \right)^{N+1} + O \left( \frac{1}{\mu_n^{2N+4}} \right) = O \left( \frac{1}{\mu_n^{2N+2}} \right). \]  
(4.72)

Inserting (4.71) and (4.72) into (4.69), we obtain
\[ (1 + x)^{\frac{1}{2}} = 1 + \sum_{j=1}^{N} \beta_j \left( \sum_{k=1}^{N} F_1(k) \mu_n^{2k} \right)^j + O \left( \frac{1}{\mu_n^{2N+2}} \right). \]  
(4.73)

On the other hand, for every \( 1 \leq j \leq N \), there exist real numbers \( C_1(\ell) \), \( 1 \leq \ell \leq N \), depending only on \( a \) and \( \alpha \), such that
\[ \sum_{j=1}^{N} \beta_j \left( \sum_{k=1}^{N} F_1(k) \mu_n^{2k} \right)^j = \sum_{\ell=1}^{N} C_1(\ell) \mu_n^{2\ell} + O \left( \frac{1}{\mu_n^{2N+2}} \right). \]  
(4.74)

Substituting (4.74) into (4.73), we obtain
\[ (1 + x)^{\frac{1}{2}} = 1 + \frac{iD_1^\pm}{2\mu_n^{3-2\gamma}} + \sum_{\ell=1}^{N} C_1(\ell) \mu_n^{2\ell} + O \left( \frac{1}{\mu_n^{2N+2}} \right). \]  
(4.75)

Inserting (4.75) into (4.67), we get second estimation of (4.18). Next, for \( \lambda_{2,n}^\pm \), from (4.55), we obtain
\[ \lambda_{2,n}^\pm = \pm i\mu_n \left( 1 - \sum_{k=1}^{N+1} F_2(k) \mu_n^{2k} + iD_2^\pm \mu_n^{3-2\gamma} + O \left( \frac{1}{\mu_n^{2N+4}} \right) \right)^\frac{1}{2}. \]  
(4.76)

Similar to (4.75), we can show that
\[ \left( 1 - \sum_{k=1}^{N+1} F_2(k) \mu_n^{2k} + iD_2^\pm \mu_n^{3-2\gamma} + O \left( \frac{1}{\mu_n^{2N+4}} \right) \right)^\frac{1}{2} \]  
\[ = 1 + \frac{iD_2^\pm}{2\mu_n^{3-2\gamma}} + \sum_{\ell=1}^{N+1} C_2(\ell) \mu_n^{2\ell} + O \left( \frac{1}{\mu_n^{2N+2}} \right), \]  
(4.77)
where for every $1 \leq \ell \leq N + 1$, the numbers $C_2(\ell)$ are real numbers depending only on $a$ and $\alpha$. Inserting (4.77) into (4.76), we obtain second estimation of (4.19).

Thus, the proof is complete.

We now study the asymptotic behavior of the eigenvectors in the different cases $a = 1$ and $\gamma < 0$ or $a \neq 1$ and $\gamma \leq 0$. We prove the following lemma.

**Lemma 4.6.** If $a = 1$ and $\gamma < 0$ or $a \neq 1$ and $\gamma \leq 0$, and $N$ equal to the integer part of $\frac{1}{2} - \gamma$, then the eigenvectors $e_{1,n}^\pm$ and $e_{2,n}^\pm$ of System (2.1) satisfy the following asymptotic expansion

If $a = 1$ and $\gamma < 0$, then

$$e_{1,n}^\pm = \frac{1}{2} \left( \begin{array}{c} e_n \\
\pm i\mu_n \
\end{array} \right) + O \left( \frac{1}{\mu_n^2}, O \left( \frac{1}{\mu_n^{\min(2,1-2\gamma)}} \right) \right)$$

and

$$e_{2,n}^\pm = \frac{1}{2} \left( \begin{array}{c} e_n \\
\pm i\mu_n \
\end{array} \right) + O \left( \frac{1}{\mu_n^{\min(2,1-2\gamma)}}, O \left( \frac{1}{\mu_n^{-2\gamma}} \right) \right).$$

If $a \neq 0$ and $\gamma \leq 0$, then

$$e_{1,n}^\pm = \frac{1}{\sqrt{2}} \left( \begin{array}{c} e_n \\
\pm i\sqrt{a\mu_n} \
0 \
\end{array} \right) + O \left( \frac{1}{\mu_n^2}, O \left( \frac{1}{\mu_n^2}, O \left( \frac{1}{\mu_n^2} \right) \right) \right)$$

and

$$e_{2,n}^\pm = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\
0 \\
e_n \
\end{array} \right) + O \left( \frac{1}{\mu_n^2}, O \left( \frac{1}{\mu_n^2}, O \left( \frac{1}{\mu_n^2} \right) \right) \right).$$

**Proof.** Let $\lambda_{1,n}^\pm$, $\lambda_{2,n}^\pm$ be the solutions of (3.3). Setting

$$B_{1,n} = \frac{B_{1,n}^\pm}{\lambda_{1,n}^\pm} \quad \text{and} \quad C_{2,n} = \frac{C_{2,n}^\pm}{\lambda_{2,n}^\pm},$$

in (2.9), we get

$$C_{1,n} = \frac{\alpha B_{1,n}^\pm}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} \quad \text{and} \quad B_{2,n} = \frac{(\lambda_{2,n}^\pm)^2 + \mu_n^2}{\alpha (\lambda_{2,n}^\pm)^2} C_{2,n}^\pm.$$

Therefore, from (2.8), we obtain

$$e_{1,n}^\pm = B_{1,n}^\pm \left( \begin{array}{c} e_n \\
\lambda_{1,n}^\pm \
\frac{\alpha e_n}{(\lambda_{1,n}^\pm)^2 + \mu_n^2}, \frac{\alpha \lambda_{1,n}^\pm e_n}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} \end{array} \right)^\top,$$

$$e_{2,n}^\pm = C_{2,n}^\pm \left( \begin{array}{c} 0 \\
0 \\
\lambda_{2,n}^\pm \
\frac{\alpha \lambda_{2,n}^\pm e_n}{(\lambda_{2,n}^\pm)^2 + \mu_n^2} \end{array} \right)^\top,$$

are the eigenvectors corresponding to the four eigenvalues $\lambda_{1,n}^\pm$, $\lambda_{2,n}^\pm$, where $B_{1,n}^\pm$, $C_{2,n}^\pm \in \mathbb{C}$. Now, we prove (4.78) and (4.79). From (4.2) and (4.3), we have

$$\lambda_{1,n}^\pm = \pm i\mu_n + O(1) \quad \text{and} \quad \lambda_{2,n}^\pm = \pm i\mu_n + O(1).$$
Therefore,
\[
\frac{1}{\lambda_{1,n}^\pm} = \frac{1}{\pm i\mu_n} + O\left(\frac{1}{\mu_n^2}\right) \quad \text{and} \quad \frac{1}{\lambda_{2,n}^\pm} = \frac{1}{\pm i\mu_n} + O\left(\frac{1}{\mu_n^2}\right). \tag{4.84}
\]
Next, from (4.4), we obtain
\[
\begin{align*}
\begin{cases}
\alpha & (\lambda_{1,n}^\pm)^2 + \mu_n^2 = 1 \mp \mu_n + O\left(\frac{1}{\mu_n^2}\right), \\
\alpha \lambda_{1,n}^\pm & \frac{\alpha}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} = -i + O\left(\frac{1}{\mu_n^2}\right),
\end{cases}
\end{align*}
\tag{4.85}
\]
From (4.84) and (4.85), we get
\[
\begin{align*}
\begin{cases}
\alpha & (\lambda_{1,n}^\pm)^2 + \mu_n^2 = 1 \mp \mu_n + O\left(\frac{1}{\mu_n^2}\right), \\
\alpha \lambda_{1,n}^\pm & \frac{\alpha}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} = -i + O\left(\frac{1}{\mu_n^2}\right),
\end{cases}
\end{align*}
\tag{4.86}
\]
and
\[
\begin{align*}
\begin{cases}
\alpha & (\lambda_{2,n}^\pm)^2 + \mu_n^2 = 1 \mp \mu_n + O\left(\frac{1}{\mu_n^2}\right), \\
\alpha \lambda_{2,n}^\pm & \frac{\alpha}{(\lambda_{2,n}^\pm)^2 + \mu_n^2} = -i + O\left(\frac{1}{\mu_n^2}\right),
\end{cases}
\end{align*}
\tag{4.87}
\]
Setting \(B_{1,n}^\pm = \frac{1}{2}\) in the first equation of (4.82), then using (4.84) and (4.86) we get (4.78). Finally, setting \(C_{2,n}^\pm = \frac{1}{2}\) in the second equation of (4.82), then using (4.84) and (4.87), we obtain (4.79). Our next aim is to prove (4.80) and (4.81). From (4.18), we have
\[
\lambda_{1,n}^\pm = \pm i\sqrt{a}\mu_n + O\left(\frac{1}{\min(1,1-2\gamma)}\right). \tag{4.88}
\]
Consequently, we obtain
\[
\begin{align*}
\begin{cases}
\frac{1}{\lambda_{1,n}^\pm} & = \frac{1}{\pm i\sqrt{a}\mu_n} + O\left(\frac{1}{\mu_n^2}\right), \\
\alpha & \frac{\alpha}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} = O\left(\frac{1}{\mu_n^2}\right), \\
\alpha \lambda_{1,n}^\pm & = O\left(\frac{1}{\mu_n^2}\right), \\
\frac{\alpha}{(\lambda_{1,n}^\pm)^2 + \mu_n^2} & = O\left(\frac{1}{\mu_n^2}\right). 
\end{cases}
\end{align*}
\tag{4.89}
\]
Setting \(B_{1,n}^\pm = \frac{1}{\sqrt{2}}\) in the first estimation of (4.82), then using (4.89), we get (4.80). On the other hand, from (4.19), we have
\[
\lambda_{2,n}^\pm = \pm i\mu_n + O\left(\frac{1}{\mu_n}\right). \tag{4.90}
\]
Consequently, we obtain
\[
\begin{align*}
\begin{cases}
\frac{1}{\lambda_{2,n}^\pm} & = \frac{1}{\pm i\mu_n} + O\left(\frac{1}{\mu_n^2}\right), \\
\lambda_{2,n}^\pm & = O\left(\frac{1}{\mu_n}\right), \\
\frac{(\lambda_{2,n}^\pm)^2 + \mu_n^2}{\alpha \lambda_{2,n}^\pm} & = O\left(\frac{1}{\mu_n}\right), \\
\alpha & = O\left(\frac{1}{\mu_n}\right). 
\end{cases}
\end{align*}
\tag{4.91}
\]
Finally, setting $C_{2,n}^\pm = \frac{1}{\sqrt{2}}$ in the second estimation of (4.82), then using (4.91), we obtain (4.81). Thus, the proof is complete.

Similar to Corollary 3.5, we have the following Corollary.

**Corollary 4.7.** From Lemma 4.6, we deduce that

$$（e_1^+, e_2^+, e_3^+, e_4^+, e_5^+）= (E_1^+, E_2^+, E_3^+, E_4^+, E_5^+)$$

where

$$L_n = \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix} + O \left( \frac{1}{\mu_n} \right), \quad \text{if } a = 1 \text{ and } \gamma < 0,$$

$$I + O \left( \frac{1}{\mu_n} \right), \quad \text{if } a \neq 1 \text{ and } \gamma \leq 0,$$

where $I$ denotes the identity matrix.

Similar to Proposition 3.6, we can prove the following proposition.

**Proposition 4.8.** Whether $a = 1$ and $\gamma < 0$ or $a \neq 1$ and $\gamma \leq 0$, the system of eigenvectors $\{e_1^+, e_2^+, e_3^+, e_4^+, e_5^+\}_{n \geq 1}$ of $A$ given in Lemma 4.6 forms a Riesz basis in $\mathcal{H}$. In particular, all eigenvectors of $A$ are of the form given in (2.8).

**Proof of Theorem 4.1.** First, if $a = 1$ and $\gamma < 0$, then from Lemma 4.2, we remark that $\Re(\lambda_n) + \frac{1}{4\mu_n} = o(1)$, as $n$ tends to infinity. Therefore, by Proposition 2.4, we get (4.1) where $\delta(\gamma) = -\frac{1}{\gamma}$. Next, if $a \neq 1$ and $\gamma \leq 0$, then from Lemma 4.3, we remark that

$$\Re(\lambda_{1,n}^+) \sim -\frac{1}{2\mu_n^2\gamma}, \quad \text{and} \quad \Re(\lambda_{2,n}^+) \sim -\frac{\alpha^2}{2(a-1)^2\mu_n^2\gamma}.$$ 

Therefore, by Proposition 2.4, we get (4.1) where $\delta(\gamma) = \frac{1}{\gamma}$. Furthermore, from Proposition 4.8, the system of eigenvectors of $A$ forms a Riesz basis in $\mathcal{H}$. Then, applying Proposition 2.4, we get the optimal polynomial energy decay rate given in (4.1). Thus, the proof is complete.

5. **Examples.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary $\Gamma$.

**Example 1.** Consider the system of weakly coupled wave equations

$$\begin{cases} u_{tt} - a\Delta u + (-\Delta)^\gamma u_{tt} + \alpha y_t = 0 \quad \text{in } \Omega, \\ y_{tt} - \Delta y - \alpha u_t = 0 \quad \text{in } \Omega, \\ u = y = 0 \quad \text{on } \Gamma, \end{cases}$$

with the following initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x),$$

where $\gamma \leq 0$, $a > 0$, and $\alpha$ is a real number. We define the operator $A$ in $L^2(\Omega)$ by

$$A = -\Delta \quad \text{with} \quad D(A) = H^2(\Omega) \cap H^1_0(\Omega).$$

We easily get that $A$ is a densely defined, closed, self-adjoint and coercive operator with compact resolvent in $L^2(\Omega)$. We also assume that the spectrum of $A$ is simple. Note that this assumption is generic (in the Baire sense) with respect to the domain $\Omega$, according to [20]. Therefore:
When $a = 1$ and $\gamma = 0$, applying Theorem 3.1, we obtain an exponential energy decay rate given by

$$\|e^{tA}u_0\|_H \leq M e^{-\epsilon t} \|u_0\|_H, \quad t > 0, \quad u_0 \in \mathcal{H}.$$  

When $a = 1$ and $\gamma < 0$ or when $a \neq 0$ and $\gamma \leq 0$, applying Theorem 4.1, we obtain an optimal polynomial energy decay rate of the form

$$E(t) \leq \frac{C}{t^{\delta(\gamma)}} \|u_0\|^2_{D(A)}, \quad t > 0, \quad u_0 \in D(A).$$  

**Example 2.** Consider the system of weakly coupled plate equations given by

$$\begin{cases} u_{tt} + a\Delta^2 u + (\Delta^2)\gamma u_t + \alpha y_t = 0 \quad \text{in } \Omega, \\ y_{tt} + \Delta^2 y - \alpha u_t = 0 \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma, \end{cases}$$

with the following initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x),$$

where $\gamma \leq 0$, $a > 0$ and $\alpha$ is a real number. We define the operator $A$ in $L^2(\Omega)$ by

$$A = \Delta^2 \quad \text{with} \quad D(A) = H^4(\Omega) \cap H^2(\Omega).$$

Here, $A$ is a densely defined, closed, self-adjoint and coercive operator with compact resolvent in $L^2(\Omega)$ and we furthermore assume that the spectrum of $A$ is simple, assumption which holds generically with respect to the domain $\Omega$, according to [26]. Then:

Applying Theorem 3.1 with $a = 1$ and $\gamma = 0$, we get

$$\|e^{tA}u_0\|_H \leq M e^{-\epsilon t} \|u_0\|_H, \quad t > 0, \quad u_0 \in \mathcal{H}.$$  

Applying Theorem 4.1, we obtain

$$E(t) \leq \frac{C}{t^{\delta(\gamma)}} \|u_0\|^2_{D(A)}, \quad t > 0, \quad u_0 \in D(A).$$

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