Real C*-Algebras, United K-Theory, and the K"unneth Formula

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March 29, 2022

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Abstract

We define united K-theory for real C*-algebras, generalizing Bousfield’s topological united K-theory. United K-theory incorporates three functors — real K-theory, complex K-theory, and self-conjugate K-theory — and the natural transformations among them. The advantage of united K-theory over ordinary K-theory lies in its homological algebraic properties, which allow us to construct a K"unneth-type, non-splitting, short exact sequence whose middle term is the united K-theory of the tensor product of two real C*-algebras $A$ and $B$ which holds as long as the complexification of $A$ is in the bootstrap category $N$. Since united K-theory contains ordinary K-theory, our sequence provides a way to compute the K-theory of the tensor product of two real C*-algebras.

As an application, we compute the united K-theory of the tensor product of two real Cuntz algebras. Unlike in the complex case, it turns out that the isomorphism class of the tensor product $O^R_{k+1} \otimes O^R_{l+1}$ is not determined solely by the greatest common divisor of $k$ and $l$. Hence, we have examples of non-isomorphic, simple, purely infinite, real C*-algebras whose complexifications are isomorphic.
Introduction

In 1982, Claude Schochet ([22]) proved the existence of an exact K"unneth-type sequence
\[ 0 \to K_*(A) \otimes K_*(B) \to K_*(A \otimes B) \to \text{Tor}(K_*(A), K_*(B)) \to 0 \]
which holds for complex C*-algebras A and B in which A is in the bootstrap category \( \mathcal{N} \).
Recall that \( \mathcal{N} \) is the smallest subcategory of complex, separable, nuclear C*-algebras which contains
the separable type I C*-algebras; which is closed under the operations of taking
inductive limits, stable isomorphisms, and crossed products by \( \mathbb{Z} \) and \( \mathbb{R} \); and which satisfies
the two out of three rule for short exact sequences (i.e. if \( 0 \to A \to B \to C \to 0 \) is exact
and two of A, B, C are in \( \mathcal{N} \), then the third is also in \( \mathcal{N} \)).

In this paper, we will develop an analogous K"unneth sequence for real C*-algebras. A real
C*-algebra \( A \) is a real Banach *-algebra which satisfies the C*-equation \( \|a^*a\| = \|a\|^2 \) as well
as the axiom that \( 1 + a^*a \) is invertible in the unitization \( A^+ \) for all \( a \in A \). This is equivalent
by [17] to saying that \( A \) is *-isometrically isomorphic to a norm-closed adjoint-closed algebra
of operators on a real Hilbert space.

If \( A \) is any real C*-algebra, it has a unique complexification \( C \otimes A \) (sometimes denoted
\( A_c \) in this paper) which is a complex C*-algebra (see [12] or [17]). It is often useful to
complexify a real C*-algebra since complex C*-algebras are better understood. However,
the complexification by itself loses. For example, the real C*-algebras \( \mathbb{R} \oplus \mathbb{R} \) and \( \mathbb{C} \) become
isomorphic upon complexification. Later in this paper we will describe examples of distinct
real simple purely infinite C*-algebras whose complexifications are isomorphic.

Given two real C*-algebras A and B, let \( A \otimes_{\mathbb{R}} B \) be the real spacial tensor product. The
problem set before us is to express the \( K \)-theory of \( A \otimes_{\mathbb{R}} B \) in terms of the \( K \)-theory of A
and of B in the same way that Schochet’s K"unneth-type sequence expresses the \( K \)-theory
of the tensor product of two complex C*-algebras in terms of the \( K \)-theory of the factors.

A key step in Schochet’s proof (following Atiyah in [2]) is to build a free geometric
resolution of \( K_*(B) \) for any complex C*-algebra B. This is a length one resolution
\[ 0 \to K_*(F_1) \xrightarrow{\mu_1} K_*(F_0) \xrightarrow{\mu_0} K_*(B) \to 0 \]
where \( K_*(F_i) \) are free (in the category of \( \mathbb{Z} \)-graded \( K_*(\mathbb{C}) \)-modules) and the homomorphisms
\( \mu_i \) are induced by homomorphisms on the level of C*-algebras.

Unfortunately, if B is a real C*-algebra, \( K_*(B) \) does not necessarily have projective
dimension one in the category of \( K_*(\mathbb{R}) \)-modules. Thus, it is impossible to build a length-one
free resolution of $K_*(B)$, so Schochet’s proof does not adapt to give a proof of a Künneth-type sequence for real C*-algebras. In fact, counter-examples showing that the analogous sequence is not exact can be found in [2] (for the topological case) and [3] (for a simpler example of C*-algebras).

United $K$-theory is the way around the problem of projective dimension. Following A.K. Bousfield’s work in the topological case (see [5]), we will define $K^{CRT}(A)$, the united $K$-theory of a real C*-algebra, as a triple of $\mathbb{Z}$-graded abelian groups — ordinary $K$-theory, complex $K$-theory, and self-conjugate $K$-theory — together with their respective module structures and interrelating natural transformations. This functor takes values in the category $CRT$ consisting of so-called CRT-modules.

As in the topological case (and drawing on a theorem of Bousfield), we find that $K^{CRT}(B)$ has projective dimension one in the category $CRT$ and so the homological algebraic obstruction to producing a Künneth Formula for real $K$-theory disappears. Using this result, we are able to produce free geometric resolutions and subsequently we prove the following product formula.

**Main Theorem** Let $A$ and $B$ be real C*-algebras with $C \otimes A \in N$. Then there is a short exact sequence

$$0 \rightarrow K^{CRT}(A) \otimes_{CRT} K^{CRT}(B) \rightarrow K^{CRT}(A \otimes B) \rightarrow \text{Tor}_{CRT}(K^{CRT}(A), K^{CRT}(B)) \rightarrow 0.$$ 

This Künneth Sequence partially generalizes Schochet’s result in [22] in the sense that by restricting attention to the complex part of each of the CRT-modules in the exact sequence, we recover Schochet’s Künneth sequence for the complex C*-algebras $C \otimes A$ and $C \otimes B$.

At the same time, it generalizes Bousfield’s Künneth sequence for topological united $K$-theory stated (but not proven) in [6]. Indeed, Bousfield’s result can be recovered by taking $A = C(X, \mathbb{R})$ and $B = C(Y, \mathbb{R})$.

It is also the case that $K^{CRT}(B)$ has injective dimension one in the category $CRT$, a property that we intend to exploit in a subsequent paper to prove a universal coefficient theorem for united $K$-theory, analogous to the universal coefficient theorem for complex C*-algebras of [20].

Since united $K$-theory incorporates ordinary $K$-theory, the main product theorem provides a way of computing the ordinary $K$-theory of the tensor product of two real C*-algebras. As an example, we compute the united $K$-theory of the tensor product of two real Cuntz algebras $O_{k+1}$ and $O_{k+1}$. In the case of complex Cuntz algebras, the $K$-theory of the tensor product — and therefore, the isomorphism class of the tensor product by the classification
Theorems of Phillips and Kirchberg ([18] and [15]) — only depends on the greatest common divisor of $k$ and $l$. In the real case however, we have found that the greatest common divisor is not enough to determine $K$-theory. The $K$-theory of $\mathcal{O}_{k+1}^\mathbb{R} \otimes \mathcal{O}_{k+1}^\mathbb{R}$ is a function of the greatest common divisor of $k$ and $l$ as well as the values of $k$ and $l$ modulo 4. As a particular example, it turns out that the real C*-algebras $\mathcal{O}_3^\mathbb{R} \otimes \mathcal{O}_5^\mathbb{R}$ and $\mathcal{O}_3^\mathbb{R} \otimes \mathcal{O}_5^\mathbb{R}$ are not isomorphic, even though their complexifications $\mathcal{O}_3 \otimes \mathcal{O}_3$ and $\mathcal{O}_3 \otimes \mathcal{O}_5$ are isomorphic.

The organization of this paper is as follows. In Section 1, we define and develop united $K$-theory. In Section 2, we compute the united $K$-theory for the real C*-algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{T}$. These are the prototypical free CRT-modules and form the building blocks for the free geometric resolution which we construct for an arbitrary real C*-algebra $B$. Section 3 is entirely algebraic, wherein we define and develop the theory of tensor products in the category CRT so that $K_{CRT}(A) \otimes_{CRT} K_{CRT}(B)$ and $\text{Tor}_{CRT}(K_{CRT}(A), K_{CRT}(B))$ make sense in the statement of the main theorem. Finally, in Section 4 we prove the main theorem and in Section 5 we apply the main theorem to compute the united $K$-theory of the tensor product of two arbitrary Cuntz algebras.

Acknowledgments: The content of this paper represents work done for my dissertation at the University of Oregon. I gratefully acknowledge my debt to my advisor, Chris Phillips, for his patience and helpfulness, both before and after graduation.

1 United $K$-theory

We begin by recording the definition of united $K$-theory, even though the terms contained in it are as yet undefined. It is the task of this chapter to make full sense of the definition.

Definition 1.1. Let $A$ be a real C*-algebra. The united $K$-theory of $A$ is the triple of $\mathbb{Z}$-graded abelian groups

$$K_{CRT}(A) := \{KO_*(A), KU_*(A), KT_*(A)\}$$

together with the eight natural transformations $\{r, c, \varepsilon, \zeta, \psi_U, \psi_T, \gamma, \tau\}$.

The $K$-groups comprising united $K$-theory will be defined in Section 1.2 and the natural transformations among the three graded groups will be described in Section 1.2. In Section 1.3 we will record some important properties of the category CRT which is the target category of United $K$-theory. Finally in Section 1.4 we will prove the existence of three long exact sequences involving the three graded groups comprising united $K$-theory.
1.1 Real, Complex, and Self-Conjugate $K$-theory

Recall from Section 1.4 of [24] that the $K$-theory of a real unital $C^*$-algebra $A$ is defined to be the Grothendieck group of equivalence classes of stable projections in $\bigcup_{k \in \mathbb{Z}^+} M_k(A)$. The equivalence relation can be taken in the sense of partial isometries, unitaries, or homotopies. If $A$ is not unital then $K$-theory is defined to be the kernel of the map $K(A^+) \to K(\mathbb{R})$. For $n \geq 0$, we define $K_n(A) = K(S^nA)$ where $S^nA$ denotes the $n$-fold suspension. Finally, for $n < 0$, $K_n(A)$ is defined using the 8-fold periodicity. Note that in this paper, all $K$-groups are considered to be graded over $\mathbb{Z}$.

This definition of $K$-theory is identical with the familiar definition of $K$-theory for complex $C^*$-algebra (see [3], [16], or [25]) in the sense that if $A$ is a complex $C^*$-algebra, the groups obtained would be the same whether $A$ were considered a complex $C^*$-algebra or just a real $C^*$-algebra with a forgotten complex structure.

In this paper, we will use the phrase “real $K$-theory” to refer to the $K$-theory of a real $C^*$-algebra as described above while the phrase “complex $K$-theory” will refer to the $K$-theory of the complexification of a real $C^*$-algebra, as in the following definitions.

**Definition 1.2 (Real $K$-theory).** $KO_n(A) := K_n(A)$ for all $n \in \mathbb{Z}$.

**Definition 1.3 (Complex $K$-theory).** $KU_n(A) := K_n(C \otimes A)$ for all $n \in \mathbb{Z}$.

The real $K$-theory $KO_*(A)$ is a $\mathbb{Z}$-graded module over the ring $KO_*(\mathbb{R})$, which according to [24] on p. 23, is given as below in gradings 0 through 8.

\[
KO_*(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus 0 \oplus \mathbb{Z} \oplus 0 \oplus 0 \oplus 0 \oplus \mathbb{Z}
\]

We recall that $KO_*(\mathbb{R})$ is a graded skew-commutative ring generated by the unit $1_o \in KO_0(\mathbb{R})$, $\eta_o \in KO_1(\mathbb{R})$, $\xi \in KO_4(\mathbb{R})$, and the Bott element $\beta_o \in KO_8(\mathbb{R})$. These generators are subject to the relations $2\eta_o = 0$, $(\eta_o)^3 = 0$, $\eta_o \cdot \xi = 0$, and $\xi^2 = 4\beta_o$. Since the Bott element $\beta_o \in KO_8(\mathbb{R})$ is invertible, multiplication by $\beta_o$ induces a period 8 isomorphism of $KO_*(A)$.

The complex $K$-theory $KU_*(A)$ is a $\mathbb{Z}$-graded module over the ring $KU_*(\mathbb{R})$.

\[
KU_*(\mathbb{R}) = \mathbb{Z} \oplus 0 \oplus \mathbb{Z} \oplus 0 \oplus \mathbb{Z} \oplus 0 \oplus 0 \oplus \mathbb{Z} \oplus \mathbb{Z}
\]

The ring $KU_*(\mathbb{R})$ is a free commutative polynomial ring over $\mathbb{Z}$ generated by the Bott element $\beta_u \in KU_2(\mathbb{R})$ and its inverse $\beta_u^{-1} \in KU_{-2}(\mathbb{R})$. In particular, the well-known Bott periodicity isomorphism for complex $K$-theory is induced by multiplication by $\beta_u$. 


Finally, we define self-conjugate $K$-theory. Consider the real C*-algebra

$$T = \{ f \in C([0,1], \mathbb{C}) \mid f(0) = \overline{f(1)} \}.$$ 

If $A$ is any other real C*-algebra, we have

$$T \otimes A \cong \{ f \in C([0,1], \mathbb{C} \otimes A) \mid f(0) = \overline{f(1)} \}$$

where the conjugation of $\mathbb{C} \otimes A$ is defined by $\overline{\lambda \otimes a} = \overline{\lambda} \otimes a$.

**Definition 1.4 (Self-conjugate $K$-theory).** $KT_n(A) := K_n(T \otimes A)$ for all $n \in \mathbb{Z}$.

The motivation for this definition comes from the topological case. Topological self-conjugate $K$-theory was first defined by D.W. Anderson in [1] in terms of self-conjugate vector bundles. A self-conjugate vector bundle is a complex vector bundle together with a given conjugate linear automorphism. In the case of real C*-algebras, such an object is replaced by a self-conjugate projection, which is a projection $p$ in a matrix algebra over $\mathbb{C} \otimes A$ together with a given equivalence to the conjugate projection $\overline{p}$. If by “equivalence” we mean homotopy equivalence, this is a continuous path of projections from $p$ to $\overline{p}$, which is the same thing as a projection in a matrix algebra over $T \otimes A$.

The rest of this section is devoted to developing the theory of self-conjugate $K$-theory for real C*-algebras. A priori, $KT_*(A)$ has a periodicity of period 8 since $T \otimes A$ is a real C*-algebra, but we will eventually show that it has period 4 (see the discussion following Corollary [4]). Since $T$ is commutative there is a pairing $T \otimes T \to T$ which makes $KT_*(\mathbb{R})$ into a ring and makes $KT_*(A)$ into a $KT_*(\mathbb{R})$-module. To compute the structure ring $KT_*(\mathbb{R})$, we must develop the relationship between self-conjugate and complex $K$-theory, particularly the long exact sequence relating the two.

Let $\zeta$ be the C*-algebra homomorphism from $T$ to $\mathbb{C}$ defined by $\zeta(f) = f(0)$ for any $f \in T$. If $A$ is any real C*-algebra, then the homomorphism

$$\zeta \otimes \text{id} : T \otimes A \to \mathbb{C} \otimes A$$

induces a natural transformation $KT_n(A) \to KU_n(A)$, which we also call $\zeta$ or $\zeta_n$. Similarly, the conjugation $\psi_U : \mathbb{C} \to \mathbb{C}$ defines an involutive natural transformation $(\psi_U)_n : KU_n(A) \to KU_n(A)$ for any C*-algebra $A$.

Finally, the inclusion $\gamma : SC \to T$ defines a natural transformation $\gamma_n : KU_n(A) \to KT_{n-1}(A)$ as follows. For $n \geq 1$, the transformation $\gamma_n$ is induced by the homomorphism

$$S^n(\mathbb{C} \otimes A) \to S^{n-1}(T \otimes A)$$
where by convention we say that it is the outermost (or leftmost) suspension that gets used by $\gamma_n$ and the innermost $n - 1$ suspensions go along for the ride. For $n < 0$, $\gamma_n$ is defined using the periodicity isomorphisms.

**Theorem 1.5.** There is a natural long exact sequence

$$\cdots \rightarrow KU_{n+1}(A) \xrightarrow{\gamma_n} KT_n(A) \xrightarrow{\zeta} KU_n(A) \xrightarrow{1-\psi_U} KU_n(A) \rightarrow \cdots.$$  

**Proof.** The short exact sequence

$$0 \rightarrow SC \otimes A \xrightarrow{\gamma_n} T \otimes A \xrightarrow{\zeta} C \otimes A \rightarrow 0$$

induces the long exact sequence

$$\cdots \rightarrow K_n(SC \otimes A) \xrightarrow{\gamma_n} K_n(T \otimes A) \xrightarrow{\zeta} K_n(C \otimes A) \xrightarrow{\delta} K_{n-1}(SC \otimes A) \rightarrow \cdots.$$  

Since $K_{n-1}(SC \otimes A) \cong KU_n(A)$, this is the long exact sequence we want. It only remains to identify the index $\delta$. We will prove that $\delta = -(1 - \psi_U)$. This is sufficient since the sign of the homomorphism does not affect the exactness of the sequence.

We first consider the case $n = 1$. Since the index map $\delta$ is most conveniently defined in terms of unitaries, we consider the unitary definition of $K$-theory, denoted by $KU^U_1(A)$, and show that the following diagram commutes. Here $\Theta$ is the usual isomorphism from $KU^U_1(A)$ to $KU_0(SA)$ as described in Section 7.2 of [25].

$$
\begin{array}{ccc}
KU^U_1(A) & \xrightarrow{\psi_U-1} & KU^U_1(A) \\
\downarrow{\delta} & & \downarrow{\Theta} \\
KU_0(SA) & &
\end{array}
$$

Let $u$ be a unitary element of $M_k(C \otimes A)$. Then $\delta([u]) = [V_i p_k V_i^*] - [p_k]$ where $V_i$ is a unitary lift of $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$ in $M_{2k}(T \otimes A)$. In other words, $V_i$ may be taken to be any path in $U_{2k}(A)$ from $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} \pi & 0 \\ 0 & \pi \end{smallmatrix}\right)$. The class $[V_i p_k V_i^*] - [p_k]$ is ostensibly an element of $KU_0(C(S^1, A))$; but since it is in the kernel of the point-evaluation map $KU_0(C(S^1, A)) \rightarrow KU_0(A)$, it determines a unique element of $KU_0(SA)$.

On the other hand, $\Theta(\psi_U - 1)[u] = \Theta[\overline{w}u^*] = [W_i p_k W_i^*] - [p_k]$ where $W_i$ is any path in $U_{2k}(A)$ from $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} \pi & 0 \\ 0 & \pi \end{smallmatrix}\right)$. Now, $\overline{w}u^*$ is stably path-connected to $u^*\overline{w}$ through unitaries so we can replace $W_i$ by any path $\widetilde{W}_i$ from $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} w\pi & 0 \\ 0 & w\pi \end{smallmatrix}\right)$. Furthermore, the class $[\widetilde{W}_i p_k \widetilde{W}_i^*]$ is equivalent to the class $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)\widetilde{W}_i p_k \widetilde{W}_i^* \left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right)^*$. Therefore, in the definition of
δ, we take \( V_t \) to be the path \( (u \ 0 \ 0) \) to \( (\bar{u} \ 0 \ 0) \) and it follows that \( \theta(\psi_U - 1)[u] = [V_t p_k V_t^*] - [p_k] = \delta([u]) \).

We have proven that \( \delta = -(1 - \psi_U) \) for \( n = 1 \) and for all \( A \). The general case follows using suspensions and periodicity. \( \square \)

**Corollary 1.6.** The self-conjugate \( K \)-theory of \( \mathbb{R} \) in degrees zero through eight is

\[
KT_*(\mathbb{R}) = \mathbb{Z} \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}
\]

**Proof.** Take \( A = \mathbb{R} \) in the sequence of Theorem 1.5 and use

\[
KU_n(\mathbb{R}) = \begin{cases} \mathbb{Z} & n \equiv 0 \pmod{2} \\ 0 & n \equiv 2 \pmod{2} \end{cases}
\]

and

\[
(\psi_U)_n = \begin{cases} 1 & n \equiv 0 \pmod{4} \\ -1 & n \equiv 2 \pmod{4} \end{cases}
\]

where \( (\psi_U)_* : KU_n(\mathbb{R}) \to KU_n(\mathbb{R}) \) to compute \( KT_*(\mathbb{R}) \). \( \square \)

Let \( 1_T \in KT_0(\mathbb{R}) \) be the class of the unit. Since \( \zeta \) is a unital homomorphism, it sends \( 1_T \) to \( 1_U \in KU_0(\mathbb{R}) \). The calculation above also reveals that \( \zeta : KT_4(\mathbb{R}) \to KU_4(\mathbb{R}) \) is an isomorphism. We let \( \beta_T \) denote the generator of \( KT_4(\mathbb{R}) \) satisfying \( \zeta(\beta_T) = \beta_T^2 \). We let \( \eta_T = \gamma(\beta_T) \) denote the non-zero element of \( KT_1(\mathbb{R}) \) and we let \( \omega \) denote \( \gamma(\beta_T^2) \), which is a generator of \( KT_3(\mathbb{R}) \).

Since \( \zeta \) is a ring homomorphism and an isomorphism in degree \( 4k \) for all integers \( k \), it follows that the element \( \beta_T \) is invertible, where the inverse is given by the element \( \zeta^{-1}(\beta_T^{-2}) \). This fact and the relations \( (\eta_T)^2 = 0, \omega^2 = 0, \) and \( \eta_T \cdot \omega = \omega \cdot \eta_T = 0 \) (which are forced by dimension considerations) completely determine the ring structure of \( KT_*(\mathbb{R}) \).

Since \( \mathbb{C} \) and \( T \) are commutative, there are external products in real, complex, and self-conjugate \( K \)-theory for any real \( \text{C}^* \)-algebras \( A \) and \( B \).

\[
\alpha_O : KO_m(A) \otimes KO_n(B) \to KO_{m+n}(A \otimes B)
\]
\[
\alpha_U : KU_m(A) \otimes KU_n(B) \to KU_{m+n}(A \otimes B)
\]
\[
\alpha_T : KT_m(A) \otimes KT_n(B) \to KT_{m+n}(A \otimes B)
\]

We will use the notation \( x \cdot_O y, x \cdot_U y, \) and \( x \cdot_T y \) to denote the real, complex, and self-conjugate products respectively.

8
1.2 Natural Transformations

In this subsection we define the rest of the eight natural transformations which interrelate the real, complex, and self-conjugate $K$-theories. These transformations are

$$
c_n : KO_n(A) \longrightarrow KU_n(A) \quad r_n : KU_n(A) \longrightarrow KO_n(A)
$$

$$
\varepsilon_n : KO_n(A) \longrightarrow KT_n(A) \quad \zeta_n : KT_n(A) \longrightarrow KU_n(A)
$$

$$
(\psi_U)_n : KU_n(A) \longrightarrow KU_n(A) \quad (\psi_T)_n : KT_n(A) \longrightarrow KT_n(A)
$$

$$
\gamma_n : KU_n(A) \longrightarrow KT_{n-1}(A) \quad \tau_n : KT_n(A) \longrightarrow KO_{n+1}(A)
$$

The transformations $\zeta$, $\psi_U$, and $\gamma$ have already been defined, induced by C*-algebra homomorphisms on the appropriate C*-algebras. In the same way, $c$, $r$, $\varepsilon$, and $\psi_T$ are induced by C*-algebra homomorphisms defined as follows. We take $c$ and $\varepsilon$ to be the unital inclusion maps. The realification $r$ is defined by $r(x + iy) = (x, -y)$ for all $x + iy \in \mathbb{C}$. Finally, let $f \in T$ be a path from $f(0)$ to $f(1) = \overline{f(0)}$. We define $\psi_T(f) \in T$ to be the reverse path from $f(1)$ to $f(0)$ — that is, $\psi_T(f)(s) = f(1 - s)$ for all $s \in [0, 1]$.

Defining $\tau$ will take a little more work. First, we will define two homomorphisms $\sigma_1$ and $\sigma_2$ from $T$ to $C(S^1, M_4(\mathbb{R}))$.

To define $\sigma_1$, let $U_s$ be any path in $U_4(\mathbb{R}) = O(4)$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

to

$$
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
$$

Let $f$ be an element of $T$ and write $f(0) = x + iy \in \mathbb{C}$. Then $f(1) = x - iy$. Now $U_s \cdot \left( r(f(1)) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \cdot U_s^*$ is a path in $M_4(\mathbb{R})$

$$
\begin{pmatrix}
x & y & 0 & 0 \\
-y & x & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

to

$$
\begin{pmatrix}
x & -y & 0 & 0 \\
y & x & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

On the other hand, $r(f)$ is a path from $(x, y, x, y)$ to $(x, y, y, x)$. We define $\sigma_1(f)$ to be the concatenation

$$
\left( r(f) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \# U_s \cdot \left( r(f(1)) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \cdot U_s^*
$$
which is a loop in $M_4(\mathbb{R})$ based at

$$\left( r(f(0)) \begin{array}{c} 0 \\ 0 \end{array} \right) = \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We define $\sigma_2(f)$ to be simply the constant path with value $r(f(0)) = (x, y, 0) \in M_2(\mathbb{R}) \subset M_4(\mathbb{R})$.

Now each of the homomorphisms $\sigma_1$ and $\sigma_2$ induces, for any real C*-algebra $A$, a natural homomorphism

$$(\sigma_i)_*: K_i_0(A) \longrightarrow KO_0(C(S^1, M_4(A))) \cong KO_0(C(S^1, A)).$$

We define $\tau = (\sigma_1)_* - (\sigma_2)_*$. This homomorphism takes values in $KO_0(C(S^1, A))$. But elements in the image of $\tau$ are in the kernel of the base-point evaluation map

$$KO_0(C(S^1, A)) \longrightarrow KO_0(A)$$

and hence determine elements of $KO_0(SA) = KO_1(A)$. Therefore $\tau$ is a homomorphism $KT_0(A) \rightarrow KO_1(A)$. By use of suspensions and periodicity, we then have a natural transformation

$$\tau_n: KT_n(A) \rightarrow KO_{n+1}(A)$$

for all $n \in \mathbb{Z}$.

**Proposition 1.7.** For any real C*-algebra $A$, the natural transformations among $KO_*(A)$, $KU_*(A)$, and $KT_*(A)$ satisfy the following relations

$$rc = 2, \quad \psi_u \beta_u = -\beta_u \psi_u, \quad \xi = r \beta^2 \psi$$

$$cr = 1 + \psi_u, \quad \psi_T \beta_T = \beta_T \psi_T, \quad \omega = \beta_T \gamma$$

$$r = r \gamma, \quad \varepsilon \beta_o = \beta^2 \varepsilon, \quad \beta_T \varepsilon \bar{\sigma} = \varepsilon \beta_T + \eta_T \beta_T$$

$$c = r \gamma, \quad \zeta \beta_T = \beta^2 \zeta, \quad \varepsilon \zeta = 1 + \psi_T$$

$$r^2 = 1, \quad \gamma \beta^2_u = \beta_T \gamma, \quad \gamma \sigma_T = 1 - \psi_T$$

$$r^2 = 1, \quad \tau \beta^2_T = \beta_o \tau, \quad \tau = -\tau \psi_T$$

$$r \varepsilon = \varepsilon, \quad \gamma = \gamma \psi_u, \quad \tau \beta_T \varepsilon = 0$$

$$\zeta \gamma = 0, \quad \eta_o = \tau \varepsilon, \quad \varepsilon \xi = 2 \beta_T \varepsilon$$

$$\zeta = \psi_u \zeta, \quad \eta_T = \gamma \beta_u \zeta, \quad \xi \tau = 2 \tau \beta_T.$$
In this section, we will prove the relations in the first two columns. The relations in the third column will be proven in subsequent sections concluding in Section 2.2. As we go, we will be careful to use only those relations which have been proven.

Lemma 1.8. Let $x_1 \in KO_i(A)$, $x_2 \in KO_j(B)$, $y_1 \in KU_k(C)$, $y_2 \in KU_l(D)$, $z_1 \in KT_m(E)$, and $z_2 \in KT_n(F)$. Then

1. $\psi_U(y_1 \cdot_U y_2) = \psi_U(y_1) \cdot_U \psi_U(y_2)$
2. $\zeta(z_1 \cdot_T z_2) = \zeta(z_1) \cdot_U \zeta(z_2)$
3. $\gamma(y_1 \cdot_U \zeta(z_1)) = \gamma(y_1) \cdot_T z_1$
4. $\gamma(\zeta(z_1) \cdot_U y_1) = (-1)^m z_1 \cdot_T \gamma(y_1)$
5. $c(x_1 \cdot_O x_2) = c(x_1) \cdot_U c(x_2)$
6. $r(c(x_1) \cdot_U y) = x_1 \cdot_O r(y)$
7. $r(y \cdot_U c(x_1)) = r(y) \cdot_U x_1$
8. $\varepsilon(x_1 \cdot_O x_2) = \varepsilon(x_1) \cdot_T \varepsilon(x_2)$
9. $\psi_T(z_1 \cdot_T z_2) = \psi_T(z_1) \cdot_T \psi_T(z_2)$
10. $\tau(z_1 \cdot_T \varepsilon(x_1)) = \tau(z_1) \cdot_O x_1$
11. $\tau(\varepsilon(x_1) \cdot_T z_1) = (-1)^i x_1 \cdot_O \tau(z_1)$

Proof. Parts (1), (2), (5), (8), and (9) follow from the facts that $\psi_U$, $\zeta$, $c$, $\varepsilon$, and $\psi_T$ are ring homomorphisms on the level of real C*-algebras.

To show part (3), we first show that the diagram

\[
\begin{array}{ccc}
SC \otimes T & \xrightarrow{\gamma \otimes 1} & SC \otimes C \\
\downarrow \gamma & & \downarrow \gamma \\
T \otimes T & \rightarrow & T
\end{array}
\]

commutes up to homotopy, where $\cdot$ represents multiplication in $C$ or $T$. Indeed, for $f \in SC$, $g \in T$, and $s \in [0, 1]$; let $F_s(f, g) \in T$ be defined by $F_s(f, g)(t) = f(t) \cdot g(st)$. Since $F_s(f, g)(0) = F_s(f, g)(1) = 0$, we see that $F_s(f, g)$ is in $T$ for each $s$. Furthermore, $F_0(f, g) = \gamma(\alpha_U(f \otimes \zeta(g)))$ and $F_1(f, g) = \alpha_T(\gamma(f) \otimes g)$. 

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This implies that, up to homotopy, the diagram
\[
\begin{array}{ccc}
S^k(C \otimes C) \otimes S^m(T \otimes A) & \xrightarrow{1 \otimes \zeta} & S^k(C \otimes C) \otimes S^m(C \otimes A) \\
\downarrow \gamma \otimes 1 & & \downarrow \gamma \\
S^{k-1}(T \otimes C) \otimes S^m(T \otimes A) & & S^{k+m-1}(T \otimes C \otimes A)
\end{array}
\]
commutes (It is at this point where we rely on our convention regarding suspensions in the definition of $\gamma_k$.) and hence in $K$-theory the formula $\gamma(z_1 \cdot_U \zeta(y_1)) = \gamma(z_1) \cdot_T y_1$ holds. Part (4) follows from part (3) using skew-commutativity.

Part (6) follows from the following commutative diagram which expresses that $r$ is an $\mathbb{R}$-module homomorphism:
\[
\begin{array}{ccc}
\mathbb{R} \otimes \mathbb{C} & \xrightarrow{id \otimes r} & \mathbb{R} \otimes M_2(\mathbb{R}) \\
\downarrow c \otimes id & & \downarrow \\
\mathbb{C} \otimes \mathbb{C} & \xrightarrow{r} & \mathbb{C} \otimes M_2(\mathbb{R})
\end{array}
\]

Now we move to part (10). For each $i$, the homomorphism $\sigma_i : T \to C(S^1, M_4(\mathbb{R}))$ is an $\mathbb{R}$-module homomorphism. Therefore, the following diagram commutes:
\[
\begin{array}{ccc}
T \otimes \mathbb{R} & \xrightarrow{\sigma_i \otimes id} & C(S^1, M_4(\mathbb{R})) \otimes \mathbb{R} \\
\downarrow id \otimes \varepsilon & & \downarrow \\
T \otimes T & \xrightarrow{\sigma_i} & C(S^1, M_4(\mathbb{R}))
\end{array}
\]
This establishes the formula $(\sigma_i)_*(z_1 \cdot_T \varepsilon(x_1)) = (\sigma_i)_*(z_1) \cdot_T x_1$ for each $i$. Therefore, since $\tau = (\sigma_1)_* - (\sigma_2)_*$, it follows that $\tau(z_1 \cdot_T \varepsilon(x_1)) = \tau(z_1) \cdot_T x_1$.

Finally, using skew-commutativity, parts (4), (7), and (11) follow from parts (3), (6), and (10).

\[\square\]

\textit{Beginning of Proof of \[.\]}. The composition $r \circ c$ from $\mathbb{R}$ to $M_2(\mathbb{R})$ is given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ which induces multiplication by 2 on real $K$-theory. The composition $c \circ r$ from $\mathbb{C}$ to $M_2(\mathbb{C})$ is given by $x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$. This map is unitarily equivalent (via the unitary $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$) to the map $x + iy \mapsto \begin{pmatrix} x + iy & 0 \\ 0 & x - iy \end{pmatrix}$, which induces $1 + \psi_U$ on complex $K$-theory. This proves the first two relations in the first column.

Next we prove the relation $r = \tau \gamma$. Let $p$ be a projection in $SC \otimes A$ and consider it (via the inclusion $\gamma$) as an element of $T \otimes A$. Then
\[
\tau \gamma[p] = \sigma_1[p] - \sigma_2[p] = [r(p) \# U_s \cdot r(p(1)) \cdot U_s^*] - [r(p(0))].
\]
But \( r(p(0)) \) and \( U_s \cdot r(p(1)) \cdot U_s^* \) are both identically zero since \( p(0) = p(1) = 0 \). Therefore, \( \tau \gamma[p] = [r(p)] \).

The next five relations in the first column are satisfied directly on the level of \( \text{C}^* \)-algebra homomorphisms. The relations \( \zeta = \psi_v \zeta \) and \( \gamma = \gamma \psi_v \) follow from the exact sequence of Theorem 1.8.

Now we prove the first six relations in the second column. In preparation, we recall the formulas \( \psi_v(\beta_u) = -\beta_u \) and \( \zeta(\beta_T) = \beta_T^2 \). Furthermore, we have the formulas \( \epsilon(\beta_o) = \beta_T^2 \) and \( \psi_T(\beta_T) = \beta_T \). Indeed, the first identity is established from \( \zeta \circ \epsilon = c \) and \( c(\beta_o) = \beta_U^4 \) using the fact that \( \zeta_8 \) is an isomorphism. The second identity is established from the identities \( \psi_v(\beta_U^2) = \beta_U^2 \) and \( \zeta \psi_T = \psi_v \zeta \) (the latter holds on the level of \( \text{C}^* \)-algebra homomorphisms) using the fact that \( \zeta_4 \) is an isomorphism taking \( \beta_T \) to \( \beta_U^2 \).

Let \( x \in KO_n(A) \), \( y \in KU_n(A) \), and \( z \in KT_n(A) \). Then using Lemma 1.8 we have

\[
\begin{align*}
\psi_v \beta_y(y) &= \psi_v(\beta_u \cdot y) = \psi_v(\beta_u) \cdot \psi_v(y) = -\beta_u \cdot \psi_v(y) = -\beta_u \psi_v(y), \\
\psi_T \beta_T(z) &= \psi_T(\beta_T \cdot_T z) = \psi_T(\beta_T) \cdot_T \psi_T(z) = \beta_T \cdot_T \psi_T(z) = \beta_T \psi_T(z), \\
\epsilon \beta_o(x) &= \epsilon(\beta_o \cdot_o x) = \epsilon(\beta_o) \cdot_T \epsilon(x) = \beta_T^2 \cdot_T \epsilon(x) = \beta_T^2 \epsilon(x), \\
\zeta \beta_T(z) &= \zeta(\beta_T \cdot_T z) = \zeta(\beta_T) \cdot_T \zeta(z) = \beta_T^2 \cdot_T \zeta(z) = \beta_T^2 \zeta(z), \\
\gamma \beta_U^2(y) &= \gamma(\beta_U^2 \cdot_U y) = \gamma(\beta_T) \cdot_T \gamma(y) = \beta_T \cdot_T \gamma(y), \\
\text{and} \quad \tau \beta_T^2(z) &= \tau(\beta_T \cdot_T z) = \tau(\epsilon(\beta_o) \cdot_T z) = \beta_o \cdot_T \tau(z) = \beta_o \tau(z).
\end{align*}
\]

We end by proving the relations \( \eta_O = \tau \epsilon \) and \( \eta_T = \gamma \beta_T \zeta \). First we prove that \( \tau \epsilon(1_o) = \eta_o = KO_1(\mathbb{R}) \). Now

\[
\tau(1_T) = \sigma_1(1_T) - \sigma_2(1_T) = [1_2 \# U_s \cdot 1_2 \cdot U_s^*] - [1_2] = [U_s \cdot (\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}) \cdot U_s^*] - [(0 \begin{smallmatrix} 0 \\ 1 \end{smallmatrix})].
\]

But since \( U_s \) is a path in \( M_4(\mathbb{R}) \) from \( \begin{smallmatrix} 1_2 & 0 \\ 0 & 1_2 \end{smallmatrix} \) to \( \begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix} \) where \( u = \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \), we have

\[
\tau \epsilon(1_o) = \tau(1_T) = \Theta[(0 \begin{smallmatrix} 0 \\ 1 \end{smallmatrix})] = \Theta[(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})] = \Theta[-1] = \eta_o.
\]

This proves that \( \tau \epsilon(1_o) = \tau(1_T) = \eta_o \). In general, we let \( x \in KO_n(A) \) and we use Lemma 1.8, parts (8) and (10):

\[
\tau \epsilon(x) = \tau \epsilon(1_o \cdot_o x) = \tau(\epsilon(1_o) \cdot_T \epsilon(x)) = \tau \epsilon(1_o) \cdot_T x = \eta_o \cdot_T x.
\]

Therefore \( \eta_o = \tau \epsilon \).
Finally, we compute
\[ \gamma_\beta \zeta(1_T) = \gamma(\beta_U \cdot 1_U) = \gamma(\beta_U) = \eta_T \in KO_1(\mathbb{R}). \]
In general, then, we have for any \( z \in KT_n(A) \),
\[ \gamma_\beta \zeta(z) = \gamma_\beta \zeta(1_T \cdot Tz) = \gamma_\beta (\zeta(1_T) \cdot \zeta(z)) = \gamma(\beta_U \zeta(1_T) \cdot \zeta(z)) = (\gamma \beta_\zeta(1_T) \cdot \zeta(z)) = \eta_T \cdot Tz \]
using Lemma 1.8, parts (2) and (3).

1.3 CRT-modules

This section is a summary of sections 1, 2, and 3 of [5] describing the category \( CRT \) which is the target category of united \( K \)-theory.

Definition 1.9. (Section 2.1 of [5]) A \( CRT \)-module is a triple \( M = \{M^O, M^U, M^T\} \) of \( \mathbb{Z} \)-graded abelian groups where \( M^O \) is a \( KO_*(\mathbb{R}) \)-module, \( M^U \) is a \( KU_*(\mathbb{R}) \)-module, and \( M^T \) is a \( KT_*(\mathbb{R}) \)-module. In addition to the operations implied by the module structures, each object \( M \) comes with \( CRT \)-operations
\[
\begin{align*}
  c_n &: M^O_n \rightarrow M^U_n \\
  \varepsilon_n &: M^O_n \rightarrow M^T_n \\
  (\psi_U)_n &: M^U_n \rightarrow M^U_n \\
  \gamma_n &: M^U_n \rightarrow M^T_{n-1} \\
  (\psi_T)_n &: M^T_n \rightarrow M^T_n \\
\end{align*}
\]
which are \( KO_*(\mathbb{R}) \)-module homomorphisms and satisfy all the relations of Proposition 1.7.

Definition 1.10. A \( CRT \)-morphism \( \phi: M \rightarrow N \) is a triple \( \phi = \{\phi^O, \phi^U, \phi^T\} \) where \( \phi^O: M^O \rightarrow N^O \) is a graded \( KO_*(\mathbb{R}) \)-module homomorphism, \( \phi^U: M^U \rightarrow N^U \) is a graded \( KU_*(\mathbb{R}) \)-module homomorphism, and \( \phi^T: M^T \rightarrow N^T \) is a graded \( KT_*(\mathbb{R}) \)-module homomorphism. Furthermore, \( \phi \) must commute with the eight transformations \( r, c, \varepsilon, \zeta, \psi_U, \psi_T, \gamma, \) and \( \tau \).

Definition 1.11. The abelian category \( CRT \) is the category whose objects are \( CRT \)-modules and whose morphisms are \( CRT \)-morphisms.

Theorem 1.12. United \( K \)-theory is a natural, exact, homotopy invariant, continuous, stable, covariant functor from the category of real \( C^* \)-algebras to the category \( CRT \).
Proof. These properties all hold for ordinary real $K$-theory. They hold then for for complex and self-conjugate $K$-theory because the operation of tensoring a C*-algebra by $\mathbb{C}$ or by $T$ commutes with stabilization and is natural, exact, homotopy invariant, continuous, and covariant. Finally, since the homomorphisms involved with these properties are natural, they commute with the internal CRT-module homomorphisms and thus are CRT-module homomorphisms.

We now record some general results about the category CRT from [5] which will be necessary in what follows.

**Definition 1.13.** (Section 2.3 of [5]) An object $M = \{M^O, M^U, M^T\}$ in the category CRT is said to be acyclic if the sequences

\begin{align*}
\cdots & \longrightarrow M^U_{n+1} \xrightarrow{\gamma} M^T_n \xrightarrow{\zeta} M^U_n \xrightarrow{1-\psi_U} M^U_n \longrightarrow \cdots \\
\cdots & \longrightarrow M^O_n \xrightarrow{\eta_O} M^O_{n+1} \xrightarrow{c} M^U_{n+1} \xrightarrow{r\beta_U^{-1}} M^O_{n-1} \longrightarrow \cdots \\
\cdots & \longrightarrow M^O_n \xrightarrow{\eta_T} M^O_{n+2} \xrightarrow{\epsilon} M^T_{n+2} \xrightarrow{\tau\beta_T^{-1}} M^O_{n-1} \longrightarrow \cdots
\end{align*}

are exact.

We will show in Section 1.4 that the united $K$-theory of any real C*-algebra is acyclic. The acyclicity condition imposes a high degree of rigidity on the CRT-module, as demonstrated by the following propositions found in Section 2.3 of [5].

**Proposition 1.14.** Suppose $\phi: M \rightarrow N$ is a homomorphism of acyclic objects of CRT. If one of $\phi^O$, $\phi^U$, and $\phi^T$ is an isomorphism, then the other two are also isomorphisms.

**Proposition 1.15.** Suppose $M = \{M^O, M^U, M^T\}$ is an acyclic object of CRT. If one of the groups $M^O$, $M^U$, and $M^T$ is trivial, then the other two are also trivial.

The following two theorems of Bousfield will be used in Section 2 to form geometric, free, length-one resolutions of $K_{\text{CRT}}(B)$ for any real C*-algebra $B$.

**Theorem 1.16.** (Theorem 3.4 of [5]) An object $M$ in CRT has projective dimension at most 1 if and only if it is acyclic.
**Theorem 1.17.** *(Theorem 3.2 of [2])* For an object \( M \in CRT \), the following are equivalent

1. \( M \) is projective
2. \( M \) is free
3. \( M \) is acyclic and \( M^\vee \) is a free abelian group

Furthermore, Bousfield explicitly describes the free \( CRT \)-modules which are generated by one element. Arbitrary free \( CRT \)-modules are direct sums of monogenic free \( CRT \)-modules. We will come across examples of these monogenic free \( CRT \)-modules in Section 2.1 as the united \( K \)-theory of \( \mathbb{R}, \mathbb{C}, \) and \( T \).

### 1.4 Long Exact Sequences

**Theorem 1.18.** For any real \( C^* \)-algebra \( A \), the following sequences are exact:

\[
\cdots \to KU_{n+1}(A) \xrightarrow{\gamma} KT_n(A) \xrightarrow{\zeta} KU_n(A) \xrightarrow{1-\psi_U} KU_n(A) \xrightarrow{\phi} \cdots \tag{4}
\]

\[
\cdots \to KO_n(A) \xrightarrow{\eta_2} KO_{n+1}(A) \xrightarrow{\epsilon} KU_{n+1}(A) \xrightarrow{\tau_U} KO_{n-1}(A) \xrightarrow{\phi} \cdots \tag{5}
\]

\[
\cdots \to KO_n(A) \xrightarrow{\eta_2} KO_{n+2}(A) \xrightarrow{\epsilon} KT_{n+2}(A) \xrightarrow{\tau_U} KO_{n-1}(A) \xrightarrow{\phi} \cdots \tag{6}
\]

In the language of [2], this theorem says that \( K_{CRT}(A) \) is acyclic. It is an immediate corollary of Theorems 1.16 and 1.18 that \( K_{CRT}(A) \) has projective dimension at most 1 in the category \( CRT \). However, this corollary will not be justified until we finish the proof of Proposition 1.7. In this section, we will not use any of the unproven relations of that proposition, nor will we use any consequences of them.

In Section 1.1, we proved the exactness of Sequence 4. Sequence 5 can be found in [24] or [13], but we will give a new proof which avoids Clifford algebras. Using similar techniques we will also prove the exactness of Sequence 6 which is new in the non-commutative context; in the topological case, it is due to Anderson [1].

We will use the following notation in this section.
**Definition 1.19.** Let $A$ be any real $C^\ast$-algebra, let $X$ be any locally compact topological space, and let $\tau$ be any involution of $X$. Then $C(X, A)$ denotes the algebra of continuous functions from $X$ to $A$ and $C_0(X, A)$ is the subalgebra of those functions which vanish at infinity. We further define

\[ C(X; \tau) = \{ f \in C(X, \mathbb{C}) \mid f(\tau(x)) = \overline{f(x)} \} \]
\[ C_0(X; \tau) = \{ f \in C_0(X, \mathbb{C}) \mid f(\tau(x)) = \overline{f(x)} \} \]
\[ IA = C([0, 1], A) \]
\[ SA = C_0(\mathbb{R}, A) \]
\[ S^{-1}A = \{ f \in C_0(\mathbb{R}, \mathbb{C} \otimes A) \mid f(-x) = \overline{f(x)} \} \]
\[ S^{m,n}A = S^m(S^{-1})^n A . \]

The operation $A \mapsto SA$ is the usual suspension operation. The operation $A \mapsto S^{-1}A$ is an inverse suspension operation in the sense of the following proposition.

**Proposition 1.20.** The real $C^\ast$-algebras $\mathbb{R}$ and $S^{1,1}$ are $KK$-equivalent. Hence, for any real $C^\ast$-algebra $A$, there is an isomorphism $K(S^{m,n}A) \cong K_{m-n}(A)$.

In Section 1.5 of [24], Schröder adapts the Toeplitz algebra proof of Bott periodicity to the case of real $C^\ast$-algebras to show that $K_n(A) \cong K_n(S^{1,1}A)$. Our proposition is somewhat stronger, but the proof takes the same approach using the real Toeplitz algebra. The proof will follow the next lemma. Recall that a short exact sequence is semisplit if it has a linear completely positive contractive section.

**Lemma 1.21.** A short exact sequence of real $C^\ast$-algebras is semisplit if the short exact sequence obtained by complexifying is semisplit.

**Proof.** Let $s$ be a section for the complexified short exact sequence

\[ 0 \to \mathbb{C} \otimes A \to \mathbb{C} \otimes B \overset{1 \otimes \pi}{\to} \mathbb{C} \otimes C \to 0 \]

so that $(1 \otimes \pi) \circ s = 1$. If we simply restrict $s$ to $C = 1 \otimes C \subset \mathbb{C} \otimes C$, we cannot be guaranteed that the image will lie in $B$. However, we claim that the linear projection $f: \mathbb{C} \otimes B \to B$ defined by $f(b_1 + ib_2) = b_1$ is contractive and completely positive. In that case the composition $f \circ s|_{1 \otimes C} : C \to B$ is the desired section for the short exact sequence

\[ 0 \to A \to B \overset{\pi}{\to} C \to 0 . \]
To prove the claim, assume that $B$ is an algebra of operators on a real Hilbert space $\mathcal{H}$. Then $\mathbb{C} \otimes B$ is an algebra of operators on the complex Hilbert space $\mathbb{C} \otimes \mathcal{H}$.

We show that $f$ is positive using the condition (Theorem VIII.3.8 in [3]) that an operator $T$ on a Hilbert space is positive if and only if $\langle Tx, x \rangle \geq 0$ for all vectors $x$ in the Hilbert space. Let $b_1 + ib_2$ be an arbitrary positive element in $\mathbb{C} \otimes B$. Then $b_1$ is positive since for all $x \in \mathcal{H}$,

$$0 \leq \langle (b_1 + ib_2)x, x \rangle = \langle b_1 x, x \rangle + i\langle b_2 x, x \rangle .$$

Since $\langle b_1 x, x \rangle \in \mathbb{R}$, it follows that $\langle b_1 x, x \rangle \geq 0$ and that $\langle b_2 x, x \rangle = 0$ for all $x$. This shows that $f$ is positive. To show $f$ is completely positive repeat the argument, replacing $B$ with $M_n(B)$.

To show that $f$ contractive, we compute

$$\|b_1 + ib_2\| = \sup_{\|x_1 + ix_2\| = 1} \|(b_1 + ib_2)(x_1 + ix_2)\| \geq \sup_{\|x_1\| = 1} \|(b_1 + ib_2)(x_1)\| \geq \sup_{\|x_1\| = 1} \|b_1(x_1)\| = \|b_1\| .$$

Proof of Proposition 1.20. Let $\mathcal{T}$ denote the complex Toeplitz algebra generated by a single isometry $S$. Then as in Section V.1 of [11] or Section 11.2 of [25], $\mathcal{T}$ has an ideal isomorphic to the compact operators $\mathcal{K}$ which is the kernel of the homomorphism $\sigma: \mathcal{T} \to C(S^1, \mathbb{C})$ which sends $S$ to the identity function $z$. Thus we have the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\sigma} C(S^1, \mathbb{C}) \to 0 .$$

(7)

If we let $\mathcal{T}_\mathbb{R}$ be the real Toeplitz algebra generated by $S$, then $\mathcal{T}_\mathbb{R}$ contains the ideal $\mathcal{T}_\mathbb{R} \cap \mathcal{K} = \mathcal{K}_\mathbb{R}$, the algebra of real compact operators. The quotient is isomorphic to the real subalgebra of $C(S^1, \mathbb{C})$ that is generated by $z$, namely

$$C(S^1; \overline{\tau}) = \{ f \in C(S^1, \mathbb{C}) \mid f(\overline{\tau}) = \overline{f(x)} \} .$$

Therefore we have the short exact sequence

$$0 \to \mathcal{K}_\mathbb{R} \to \mathcal{T}_\mathbb{R} \xrightarrow{\sigma} C(S^1; \overline{\tau}) \to 0$$

(8)

as found in Section 1.5 of [24].
Now let $\pi : \mathcal{T}_R \to \mathbb{R}$ be the composition of $\sigma$ with evaluation at $1 \in S^1$ and let $\mathcal{T}_{R,0}$ be the kernel of $\pi$. Then $\mathcal{K}_R$ is an ideal of $\mathcal{T}_{R,0}$ with quotient
\[ \{ f \in C(S^1, \mathbb{C}) \mid f(1) = 0 \} \cong S^{-1}. \]
Therefore we have our third short exact sequence
\[ 0 \to \mathcal{K}_R \to \mathcal{T}_{R,0} \xrightarrow{\pi} S^{-1} \to 0. \quad (9) \]

Since $C(S^1, \mathbb{C})$ is nuclear, the theorem of Choi and Effros recorded as Theorem 15.8.3 of [3] implies that Sequence 8 has a completely positive section. Since Sequence 7 is the complexification of Sequence 8, Lemma 1.21 implies that Sequence 8 also has a completely positive section. Finally, since $\mathcal{T}_{R,0}$ is precisely the set of operators $T$ in $\mathcal{T}_R$ such that $\sigma(T)$ is in $S^{-1}$, this section restricts to a completely positive section for Sequence 9.

Then by Proposition 2.5.6 of [24], for any real separable C*-algebra $A$ the semisplit exact Sequence 9 induces long exact sequences
\[ \cdots \to KK_{n+1}(A, \mathcal{T}_{R,0}) \to KK_n(A, S^{1,1}) \xrightarrow{\delta} KK_n(A, \mathbb{R}) \to KK_n(A, \mathcal{T}_{R,0}) \to \cdots \quad (10) \]
and
\[ \cdots \to KK_n(\mathcal{T}_{R,0}, A) \to KK_n(\mathbb{R}, A) \xrightarrow{\delta} KK_n(S^{1,1}, A) \to KK_{n-1}(\mathcal{T}_{R,0}, A) \to \cdots \quad (11) \]
where in both cases $\delta$ is given by the intersection product with an element $\delta \in KK_0(S^{1,1}, \mathbb{R})$. The element $\delta$ is one half of our $KK$-equivalence.

We claim that $KK_n(A, \mathcal{T}_{R,0}) = KK_n(\mathcal{T}_{R,0}, A) = 0$. Using the long exact sequence on $K$-theory induced by
\[ 0 \to \mathcal{T}_{R,0} \to \mathcal{T}_R \xrightarrow{\pi} \mathbb{R} \to 0, \]
Schröder shows (in Section 1.5 of [24]) that $K_*(\mathcal{T}_{R,0}) = 0$ by showing that the homomorphism $\pi_*$ induced on $K$-theory has an inverse. The same techniques immediately generalize to show that the $KK$-theory homomorphisms $\pi_*$ and $\pi^*$ have inverses in the sequences
\[ \cdots \to KK_n(A, \mathcal{T}_{R,0}) \to KK_n(A, \mathcal{T}_R) \xrightarrow{\pi_*} KK_n(A, \mathbb{R}) \to KK_{n-1}(A, \mathcal{T}_{R,0}) \to \cdots \]
and
\[ \cdots \to KK_{n+1}(\mathcal{T}_{R,0}, A) \to KK_n(\mathbb{R}, A) \xrightarrow{\pi^*} KK_n(\mathcal{T}_R, A) \to KK_n(\mathcal{T}_{R,0}, A) \to \cdots \]
Therefore $KK_n(A, \mathcal{T}_{R,0}) = KK_n(\mathcal{T}_{R,0}, A) = 0$. 

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It follows that $\delta$ is an isomorphism in the Sequences 10 and 11. In particular, setting $A = \mathbb{R}$ in Sequence 10, we have an isomorphism

$$KK_0(\mathbb{R}, S^{1,1}) \xrightarrow{\delta} KK_0(\mathbb{R}, \mathbb{R}) \cong \mathbb{Z}.$$  

To find the inverse $KK$-element to $\delta$, let $\varepsilon \in KK_0(\mathbb{R}, S^{1,1})$ be the element such that $\varepsilon \cap \delta = 1_{\mathbb{R}} \in KK_0(\mathbb{R}, \mathbb{R})$. It remains only to show that $\delta \cap \varepsilon = 1 \in KK_0(S^{1,1}, S^{1,1})$.

Applying Sequence 11 with $A = S^{1,1}$ gives an isomorphism

$$KK_0(\mathbb{R}, S^{1,1}) \to KK_0(S^{1,1}, S^{1,1})$$

and we conclude that $KK_0(S^{1,1}, S^{1,1}) \cong \mathbb{Z}$. Now, we note that $\delta \cap \varepsilon$ is an idempotent since

$$(\delta \cap \varepsilon)^2 = \delta \cap (\varepsilon \cap \delta) \cap \varepsilon = \delta \cap 1 \cap \varepsilon = \delta \cap \varepsilon.$$  

But $\delta \cap \varepsilon$ is non-zero since $\delta \cap \varepsilon \cap \delta = \delta \neq 0$. Therefore, $\delta \cap \varepsilon$ must be the only non-zero idempotent in the ring $KK_0(S^{1,1}, S^{1,1}) \cong \mathbb{Z}$, namely the identity. \hfill \Box

We are now ready to prove the exactness of the sequences of Theorem 1.18.

**Construction of Sequence 5.** We start with the inclusion homomorphism $c: \mathbb{R} \to \mathbb{C}$ and construct the mapping cylinder and mapping cone. See Section 2 of [23] as a basic reference for mapping cylinders and mapping cones. In our case, the mapping cylinder is

$$Zc = \{(x, f) \in \mathbb{R} \oplus I\mathbb{C} \mid f(0) = c(x)\}$$

$$\cong \{f \in I\mathbb{C} \mid f(0) \in \mathbb{R} \subset \mathbb{C}\}$$

and the mapping cone is

$$Cc = \{(x, f) \in \mathbb{R} \oplus I\mathbb{C} \mid f(0) = c(x), f(1) = 0\}$$

$$\cong \{f \in I\mathbb{C} \mid f(0) \in \mathbb{R}, f(1) = 0\}$$

$$\cong \{f \in C_0(\mathbb{R}, \mathbb{C}) \mid f(-z) = f(z)\}$$

$$\cong S^{-1}$$

Then there is a short exact sequence

$$0 \to Cc \to Zc \to \mathbb{C} \to 0.$$  \hfill (12)
Since $Zc$ is homotopy equivalent to $\mathbb{R}$ and $Cc$ is isomorphic to $S^{-1}$, we have up to homotopy equivalence a sequence

$$0 \to S^{-1} \to \mathbb{R} \xrightarrow{c} C \to 0 \quad (13)$$

which, upon tensoring on the right by $A$ and applying $K$-theory, gives rise to a long exact sequence

$$\cdots \to KO_n(A) \xrightarrow{\alpha} KO_{n+1}(A) \xrightarrow{\epsilon} KU_{n+1}(A) \xrightarrow{\beta} KO_{n-1}(A) \to \cdots. \quad (14)$$

Let $\alpha$ be the element of $KK_0(S^{-1}, \mathbb{R}) \cong KK_1(\mathbb{R}, \mathbb{R})$ determined by the homomorphism $S^{-1} \to \mathbb{R}$ so that the $K$-theory homomorphism $\alpha$ in the sequence above is realized by multiplication by the element $\alpha$. Now, the index map in the long exact sequence above is induced by a homomorphism $SC \to S^{-1}$ (see Proposition 2.3 of [23]). Let $\beta \in KK(SC, S^{-1}) \cong KK_{-2}(\mathbb{C}, \mathbb{R})$ be the $KK$-element corresponding to this homomorphism.

We will show that $\alpha = \eta_0 \in KK_1(\mathbb{R}, \mathbb{R})$ and $\beta = \pm r\beta_{U}^{-1} \in KK_{-2}(\mathbb{C}, \mathbb{R})$. The first is easy since there are only two elements in $KK_1(\mathbb{R}, \mathbb{R}) \cong \mathbb{Z}$. The element $\alpha$ cannot be zero since then Sequence (14) would not be exact for $A = \mathbb{R}$.

To show that $\beta = \pm r\beta_{U}^{-1}$, we must first compute $KK_{-2}(\mathbb{C}, \mathbb{R})$. For this, we use the long exact sequence induced by Sequence (13) on the functor $KK(-, \mathbb{R})$,

$$KK_{-1}(\mathbb{R}, \mathbb{R}) \leftarrow KK_{-2}(\mathbb{R}, \mathbb{R}) \leftarrow KK_{-2}(\mathbb{C}, \mathbb{R}) \leftarrow KK_{0}(\mathbb{R}, \mathbb{R}) \leftarrow KK_{-1}(\mathbb{R}, \mathbb{R})$$

according to Proposition 2.5.4 of [24]. Since $KK_{-2}(\mathbb{R}, \mathbb{R}) \cong KK_{-1}(\mathbb{R}, \mathbb{R}) = 0$ it follows that there is an isomorphism $KK_{0}(\mathbb{R}, \mathbb{R}) \to KK_{-2}(\mathbb{C}, \mathbb{R})$ given by the intersection product with $\beta$. Therefore, $\beta$ is a generator of $KK_{-2}(\mathbb{C}, \mathbb{R}) \cong \mathbb{Z}$.

Now, the inverse Bott element $\beta_{U}^{-1}$ is a generator of $KK_{-2}(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$, so it remains only to prove that $r$ is a generator of $KK_{0}(\mathbb{C}, \mathbb{R}) \cong \mathbb{Z}$. For this, it is enough to show that $r: K_4(\mathbb{C}) \to K_4(\mathbb{R})$ is an isomorphism $\mathbb{Z} \to \mathbb{Z}$. We use the short exact sequence

$$0 \to K_4(\mathbb{R}) \xrightarrow{c} K_4(\mathbb{C}) \to K_4(\mathbb{R}) \to 0$$

induced by Sequence (13) and the fact that $K_2(\mathbb{R}) = \mathbb{Z}_2$ to deduce that $c: K_4(\mathbb{R}) \to K_4(\mathbb{C})$ is multiplication by $\pm 2$ from $\mathbb{Z}$ to $\mathbb{Z}$. Then the relation $cr = 2$ implies that $r_4$ is an isomorphism.

**Construction of Sequence [13].** To develop the third exact sequence, we follow the same procedure as above, starting with the homomorphism $\varepsilon: \mathbb{R} \to T$. 

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The mapping cylinder of $\varepsilon$ is

$$Z\varepsilon = \{(x,f) \in \mathbb{R} \oplus IT \mid f(0) = \varepsilon(x)\}$$

$$\cong \{f \in \mathbb{I}^{2}\mathbb{C} \mid f(s,0) = \overline{f(s,1)} \text{ for all } s \in I \text{ and } f(0,t_1) = f(0,t_2) \in \mathbb{R} \text{ for all } t_1, t_2 \in I\}$$

and the mapping cone is

$$C\varepsilon = \{(x,f) \in \mathbb{R} \oplus IT \mid f(0) = \varepsilon(x) \text{ and } f(1) = 0\}$$

$$\cong \{f \in \mathbb{I}^{2}\mathbb{C} \mid f(s,0) = \overline{f(s,1)} \text{ for all } s \in I, \quad f(0,t_1) = f(0,t_2) \in \mathbb{R} \text{ for all } t_1, t_2 \in I, \text{ and} \quad f(1,t) = 0 \text{ for all } t \in I\}$$

$$\cong \{f \in \mathbb{I}^{2}\mathbb{C} \mid f(s,0) = f(s,1) = f(1,t) = 0 \text{ and } f(0,t) = \overline{f(0,1-t)}\}$$

Call this last algebra $B$. We will prove that $B$ is isomorphic to $S^{-2}$. Recall that in previous proof, we found that

$$S^{-1} \cong \{f \in \mathbb{I}\mathbb{C} \mid f(0) \in \mathbb{R}, f(1) = 0\};$$

thus

$$S^{-2} \cong \{f \in \mathbb{I}\mathbb{C} \mid f(0) \in \mathbb{R}, f(1) = 0\} \otimes^2$$

$$\cong \{f \in \mathbb{I}^2(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \mid f(1,t) = f(s,1) = 0 \text{ for all } s \text{ and } t \in I, \quad f(0,t) \in \mathbb{R} \otimes \mathbb{C} \text{ for all } t \in I, \quad f(s,0) \in \mathbb{C} \otimes \mathbb{R} \text{ for all } s \in I\}.$$

Now, $\mathbb{C} \oplus \mathbb{C}$ is isomorphic to $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ via the homomorphism

$$\alpha(x,y) = \frac{1}{2}(x+y) \otimes 1 + \frac{1}{2}(x-y)i \otimes i.$$  

(This isomorphism is an isomorphism of complex C*-algebras when $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ receives the complex structure based on its first factor.) Under this isomorphism $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ is identified with $\{(x,\overline{x}) \mid x \in \mathbb{C}\} \subset \mathbb{C} \oplus \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}$ is identified with $\{(x,x) \mid x \in \mathbb{C}\}$.

Hence $S^{-2}$ is isomorphic to the subalgebra $B'$ of $\mathbb{I}^2(\mathbb{C} \oplus \mathbb{C}) \cong \mathbb{I}^2\mathbb{C} \oplus \mathbb{I}^2\mathbb{C}$ consisting of pairs $(f,g)$ satisfying

$$f(s,1) = f(1,t) = g(s,1) = g(1,t) = 0$$

$$f(0,t) = \overline{g(0,t)}, \text{ and}$$

$$f(s,0) = g(s,0).$$
Finally, the pasting map \((f, g) \mapsto h\) defined by
\[
h(s, t) = \begin{cases} 
g(s, 1 - 2t), & \text{if } t \leq \frac{1}{2}, 
g(s, 2t - 1), & \text{if } t \geq \frac{1}{2}.
\end{cases}
\]
gives an isomorphism from \(B'\) to \(B\). We have shown that \(C\varepsilon\) is isomorphic to \(S^{-2}\).

Then the short exact sequence
\[
0 \rightarrow C\varepsilon \rightarrow Z\varepsilon \rightarrow T \rightarrow 0
\]
becomes, up to homotopy, the sequence
\[
0 \rightarrow S^{-2} \rightarrow \mathbb{R} \xrightarrow{\varepsilon} T \rightarrow 0
\]
which gives rise to a long exact sequence
\[
\cdots \rightarrow KO_n(A) \xrightarrow{\alpha} KO_{n+2}(A) \xrightarrow{\varepsilon} KT_{n+2}(A) \xrightarrow{\beta} KO_{n-1}(A) \rightarrow \cdots.
\]
Let \(\alpha \in KK_2(\mathbb{R}, \mathbb{R})\) and \(\beta \in KK_{-3}(T, \mathbb{R})\) be the \(KK\)-elements that implement the homomorphisms in the above sequence.

We will show that \(\alpha = \eta_0^2\) and \(\beta = \pm \tau \beta^{-1}_r\). The element \(\alpha\) cannot be zero, since otherwise Sequence 10 above would not be exact for \(A = \mathbb{R}\). Since \(\eta_0^2\) is the only non-zero element of \(KK_2(\mathbb{R}, \mathbb{R}) \cong \mathbb{Z}_2\), we have \(\alpha = \eta_0^2\).

Sequence 15 gives rise to a long exact sequence
\[
KK_{-1}(\mathbb{R}, \mathbb{R}) \xleftarrow{\gamma} KK_{-3}(\mathbb{R}, \mathbb{R}) \xleftarrow{\tau} KK_{-3}(T, \mathbb{R}) \xleftarrow{\gamma} KK_0(\mathbb{R}, \mathbb{R}) \xleftarrow{\beta} KK_{-2}(\mathbb{R}, \mathbb{R})
\]
Since \(KK_{-2}(\mathbb{R}, \mathbb{R}) = KK_{-3}(\mathbb{R}, \mathbb{R}) = 0\) and since \(KK_0(\mathbb{R}, \mathbb{R}) \cong \mathbb{Z}\), this implies that \(KK_{-3}(T, \mathbb{R}) \cong \mathbb{Z}\) and the element \(\beta\) is a generator.

In the proof of the exactness of Sequence 14, we discovered that \(r\) is an isomorphism from \(KU_4(\mathbb{R}) \cong \mathbb{Z}\) to \(KO_4(\mathbb{R}) \cong \mathbb{Z}\). This isomorphism passes through \(KT_3(\mathbb{R}) \cong \mathbb{Z}\) via the factorization \(r = \tau \gamma\). So \(\tau\) must be a generator of \(KK_1(T, \mathbb{R}) \cong \mathbb{Z}\). Since \(\beta_r\) is a \(KK\)-equivalence, this implies \(\tau \beta^{-1}_r\) is a generator of \(KK_{-3}(T, \mathbb{R})\).

\[\square\]

2 Free Geometric Resolutions

In this section we show that given an arbitrary unital \(C^*\)-algebra \(A\), we can form a free geometric resolution of \(K^{CRT}(A)\). In the first section, we compute the united \(K\)-theory of the real \(C^*\)-algebras \(\mathbb{R}, \mathbb{C},\) and \(T\). These \(CRT\)-modules give us geometric realizations of singly-generated free \(CRT\)-modules and form the basic building blocks for the free geometric resolutions which will be obtained in Section 2.2.
2.1 Geometrically Realized Free CRT-Modules

The united $K$-theory of the real C*-algebras $\mathbb{R}$, $\mathbb{C}$, and $T$ are given in Tables [1], [2], and [3]. These tables give the groups $KO_n(A)$, $KU_n(A)$, and $KT_n(A)$ up to isomorphism as well as the operations $c_n$, $r_n$, $\varepsilon_n$, $\zeta_n$, $(\psi_U)_n$, $(\psi_T)_n$, $\gamma_n$, and $\tau_n$ which are written as multiplication by a certain number or matrix in terms of fixed ordered generators for each of the groups. These preferred generators are shown in the table for $K\text{CRT}(\mathbb{R})$. For $K\text{CRT}(\mathbb{C})$ and $K\text{CRT}(T)$ the preferred generators are described in the course of the computation in the text. These preferred generators are chosen to be stable under the periodicity isomorphisms. The $KO_*(\mathbb{R})$-, $KU_*(\mathbb{R})$-, and $KT_*(\mathbb{R})$-module structures are given implicitly by the relations $\eta_0 = \tau \varepsilon$, $\eta_T = \gamma \beta_U \zeta$, $\xi = \tau \gamma \beta_U^2 c$, and $\omega = \beta_T \gamma \zeta$. These last two relations have not been proven in general yet. However, our calculation of $K\text{CRT}(\mathbb{R})$ will imply that they hold for $A = \mathbb{R}$ and, following the computation, we use this to prove the general case.

Table 1: $K\text{CRT}(\mathbb{R})$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $\mathbb{Z} \cdot 1_0$ | $\mathbb{Z} \cdot \varepsilon_0$ | $\mathbb{Z} \cdot \zeta_0$ | $0$ | $\mathbb{Z} \cdot \xi$ | $0$ | $0$ | $0$ | $\mathbb{Z} \cdot \beta_0$ |
| $KU_n$ | $\mathbb{Z} \cdot 1_U$ | $0$ | $\mathbb{Z} \cdot \beta_U$ | $0$ | $\mathbb{Z} \cdot \beta_U^2$ | $0$ | $\mathbb{Z} \cdot \beta_U^3$ | $0$ | $\mathbb{Z} \cdot \beta_U^4$ |
| $KT_n$ | $\mathbb{Z} \cdot 1_T$ | $\mathbb{Z} \cdot \eta_T$ | $0$ | $\mathbb{Z} \cdot \omega$ | $\mathbb{Z} \cdot \beta_T$ | $\mathbb{Z} \cdot \beta_T \eta_T$ | $0$ | $\mathbb{Z} \cdot \beta_T \omega$ | $\mathbb{Z} \cdot \beta_T^2$ |

$|c_n| |r_n| |\varepsilon_n| |\zeta_n| |(\psi_U)_n| |(\psi_T)_n| |\gamma_n| |\tau_n|

1 | 1 | 1 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 2 |
2 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
1 | 1 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 1 |
1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
1 | 1 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 2 | 1 |

Let $F(b,n,\mathbb{R})$ (respectively $F(b,n,\mathbb{C})$ and $F(b,n,T)$) denote the free CRT-module with a single generator $b$ in the real (respectively complex and self-conjugate) part in degree $n$ (see Section 2 of [3]). Any free CRT-module can be obtained as a direct sum of these three monogenic free CRT-modules.

By direct comparison with Paragraph 2.4 of [3], we observe from our computations that $K\text{CRT}(\mathbb{R})$ is a free CRT-module with single generator $1_0 \in KO_0(\mathbb{R})$; $K\text{CRT}(\mathbb{C})$ is a free CRT-module with generator $\kappa_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in KU_0(\mathbb{C})$; and $K\text{CRT}(T)$ is a free CRT-module with generator $\chi = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in KT_{-1}(T)$. Therefore, any free CRT-module can be realized as
### Table 2: $K^{CRT}(C)$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| $KU_n$ | $\mathbb{Z} \oplus \mathbb{Z}$ | 0 | $\mathbb{Z} \oplus \mathbb{Z}$ | 0 | $\mathbb{Z} \oplus \mathbb{Z}$ | 0 | $\mathbb{Z} \oplus \mathbb{Z}$ | 0 | $\mathbb{Z} \oplus \mathbb{Z}$ |
| $KT_n$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $c_n$ | $(\frac{1}{1})$ | 0 | $(\frac{1}{1})$ | 0 | $(\frac{1}{1})$ | 0 | $(\frac{1}{1})$ | 0 | $(\frac{1}{1})$ |
| $r_n$ | $(1, 1)$ | 0 | $(1, 1)$ | 0 | $(1, 1)$ | 0 | $(1, 1)$ | 0 | $(1, 1)$ |
| $\varepsilon_n$ | 1 | 0 | $-1$ | 0 | 1 | 0 | $-1$ | 0 | 1 |
| $\zeta_n$ | $(\frac{1}{1})$ | 0 | $(\frac{1}{1})$ | 0 | $(\frac{1}{1})$ | 0 | $(\frac{1}{1})$ | 0 | $(\frac{1}{1})$ |
| $(\psi_U)_n$ | $(1, 0)$ | 0 | $(0, -1)$ | 0 | $(0, 0)$ | 0 | $(0, -1)$ | 0 | $(0, 0)$ |
| $(\psi_T)_n$ | 1 | $-1$ | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | 1 |
| $\gamma_n$ | $(1, 1)$ | 0 | $(1, -1)$ | 0 | $(1, 1)$ | 0 | $(1, -1)$ | 0 | $(1, 1)$ |
| $\tau_n$ | 0 | $-1$ | 0 | 1 | 0 | $-1$ | 0 | 1 | 0 |

### Table 3: $K^{CRT}(T)$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $KU_n$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $KT_n$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| $c_n$ | 1 | 0 | 0 | 2 | 1 | 0 | 0 | 2 | 1 |
| $r_n$ | 2 | 1 | 0 | 1 | 2 | 1 | 0 | 1 | 2 |
| $\varepsilon_n$ | $(\frac{1}{0})$ | 1 | 0 | $(\frac{1}{1})$ | 1 | 0 | $(\frac{1}{1})$ | 0 | $(\frac{1}{1})$ |
| $\zeta_n$ | $(1, 0)$ | 0 | 0 | $(1, 0)$ | 0 | 0 | $(1, 0)$ | 0 | $(1, 0)$ |
| $(\psi_U)_n$ | 1 | $-1$ | $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | 1 |
| $(\psi_T)_n$ | $(1, 0)$ | 1 | $-1$ | $(1, 0)$ | 1 | $-1$ | $(1, 0)$ | $-1$ | $(1, 0)$ |
| $\gamma_n$ | $(0, 1)$ | $-1$ | $(0, 1)$ | $-1$ | $(0, 1)$ | $-1$ | $(0, 1)$ | $-1$ | $(0, 1)$ |
| $\tau_n$ | $(1, 1)$ | 0 | 1 | $(1, -1)$ | 0 | 1 | $(1, -1)$ | 0 | $(1, -1)$ |
$K_{\text{CRT}}(F)$, where $F$ is a direct sum of suspensions of the algebras $\mathbb{R}$, $\mathbb{C}$, and $T$.

**Computation of Table 4.** We have already established the group and module structures of $KO_\ast(\mathbb{R})$, $KU_\ast(\mathbb{R})$, and $KT_\ast(\mathbb{R})$. For the computation of $c$, $r$, and $\psi_U$, we refer to [13] (III.5.19) or [5]. Alternatively, the values of $c$ and $r$ can be computed directly using Sequence 5 and $\psi_U$ using the relation $cr = 1 + \psi_U$. We computed $\zeta$ and $\gamma$ in Section 2.1 using Sequence 4. The remaining homomorphisms $\varepsilon$, $\psi_T$, and $\tau$ can be easily determined by the constraints $c = \zeta \varepsilon$, $\eta_O = \tau \varepsilon$, $\varepsilon r \zeta = 1 + \psi_T$, and $r = \tau \gamma$.

At this point, we pause to prove two more of the relations stated in Proposition 1.7 using the results of the computation of $K_{\text{CRT}}(\mathbb{R})$. These relations will be used in computing $K_{\text{CRT}}(\mathbb{C})$.

**Continuation of proof of Proposition 1.7.** First we prove $\xi = r \beta_U^2 c$. Let $x \in KO_\ast(A)$. Then using the relations $r = \tau \gamma$, $c = \zeta \varepsilon$ (from Proposition 1.7); Lemma 1.8 parts (3), (5), and (10); and $r \beta_U^2 c(1_O) = \xi$ from Table 1 we have

$$r \beta_U^2 c(x) = r \beta_U^2 c(1_O \cdot_O x)$$
$$= r \beta_U^2 (1_O \cdot_U c(x))$$
$$= \tau \gamma (\beta_U^2 \cdot_U \zeta(x))$$
$$= \tau (\gamma \beta_U^2 \cdot_T \varepsilon(x))$$
$$= \tau \gamma \beta_U^2 \cdot_O (x)$$
$$= r \beta_U^2 c(1_O) \cdot_O x$$
$$= \xi \cdot_O x.$$

Second, we prove $\omega = \beta_T \gamma \zeta$. Let $z \in KT_\ast(A)$. Then using $\gamma(1_U) = \beta_T^{-1} \omega$ (from Table 1) and Lemma 1.8 part (3), we have

$$\beta_T \gamma \zeta(z) = \beta_T \gamma(1_U \cdot_U \zeta(z))$$
$$= \beta_T (\gamma(1_U) \cdot_T z)$$
$$= \beta_T (\beta_T^{-1} \omega \cdot_T z)$$
$$= \omega \cdot_T z.$$


Computation of Table 2. We designate the preferred generators of \( KO_n(C) \) to be those elements which correspond to the preferred generators of \( KU_n(R) \).

Now, as in the proof of Theorem 1.18, \( C \oplus C \) is isomorphic to \( C \otimes R C \) via the isomorphism

\[
\alpha(\lambda_1, \lambda_2) = \frac{1}{2}(\lambda_1 + \lambda_2) \otimes 1 + \frac{1}{2}(\lambda_1 - \lambda_2)i \otimes i.
\]

Thus, \( KU_n(C) = K_n(C \otimes R C) \cong K_n(C \oplus C) \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \) if \( n \) is even and is 0 otherwise. We define the classes

\[
\kappa_1 = \alpha_*(1, 0) = \left[ \frac{1}{2}(1 \otimes 1) + \frac{1}{2}(i \otimes i) \right]
\]

\[
\kappa_2 = \alpha_*(0, 1) = \left[ \frac{1}{2}(1 \otimes 1) - \frac{1}{2}(i \otimes i) \right]
\]

in \( KU_0(C) \). Then the preferred generators of \( KU_{2n}(C) \) are set to be \( \beta^n_U \cdot \kappa_1 \) and \( \beta^n_U \cdot \kappa_2 \) in that order.

Now the complexification operation \( c: KO_n(C) \to KU_n(C) \) is computed by studying the composition

\[
C \cong R \otimes R C \to C \otimes R C \cong C \oplus C
\]

which is given by \( \lambda \mapsto (\overline{\lambda}, \lambda) \). Similarly, the operation \( \psi_U: KU_n(C) \to KU_n(C) \) is computed by studying the composition

\[
C \oplus C \cong C \otimes R C \xrightarrow{\psi_U \otimes 1} C \otimes R C \cong C \oplus C
\]

which is given by \( (\lambda_1, \lambda_2) \mapsto (\overline{\lambda_2}, \overline{\lambda_1}) \).

To compute \( KT_n(C) \), we use the exact sequence

\[
0 \to KT_n(C) \xrightarrow{\zeta} KU_n(C) \xrightarrow{1 - \psi_U} KU_n(C) \xrightarrow{\gamma} KT_{n-1}(C) \to 0
\]

for \( n \) even.

Now \( 1 - (\psi_U)_n = \left( \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right) \) for \( n \equiv 0 \pmod{4} \) and \( 1 - (\psi_U)_n = \left( \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right) \) for \( n \equiv 2 \pmod{4} \). So for any even \( n \), the kernel and the cokernel of \( 1 - (\psi_U)_n \) are both isomorphic to \( \mathbb{Z} \). Thus \( KT_n(C) \cong \mathbb{Z} \) for all \( n \). For each \( n \), choose the preferred generators of \( KT_n(C) \) such that \( \gamma_n \) and \( \zeta_n \) are as described in the table. Since \( \zeta \beta = \beta^2 U \zeta \) and \( \gamma \beta^2 U = \beta \gamma \), the collection of these preferred generators is stable under the periodicity isomorphism \( \beta_U \).

Finally, the operations \( r, \varepsilon, \psi_T, \) and \( \tau \) are determined by the relations \( cr = 1 + \psi_U, \)

\[
c = \zeta \varepsilon, \varepsilon r \zeta = 1 + \psi_T, \text{ and } r = \tau \gamma.
\]
Computation of Table 3. The structure of real $K$-theory is immediate, since $KO_n(T) \cong KT_n(\mathbb{R})$. We specify generators of $KO_*(T)$ which correspond to the previously specified generators of $KT_*(\mathbb{R})$. The $KO_*(\mathbb{R})$-module structure of $KO_*(T)$ is inferred from the $KT_*(\mathbb{R})$-module structure on $KT_*(\mathbb{R})$ via the inclusion $\varepsilon: \mathbb{R} \hookrightarrow T$. For example, for the element $\xi \in KO_4(\mathbb{R})$ and any $z \in KT_k(\mathbb{R}) = KO_k(T)$ we have

$$\xi \cdot z = \varepsilon(\xi) \cdot z = 2\beta_T \cdot z,$$

a fact which we shall use shortly.

For all $n$, we have $KU_n(T) \cong K_n(\mathbb{C} \otimes T) \cong KT_n(\mathbb{C}) \cong \mathbb{Z}$. Then use the sequence

$$KO_{n-1}(T) \xrightarrow{\eta_0} KO_n(T) \xrightarrow{\varepsilon} KU_n(T) \xrightarrow{r \beta U^1} KO_{n-2}(T) \xrightarrow{\eta_0} KO_{n-1}(T)$$

(17)

from Theorem 1.18 to compute $c$ and $r$.

Let $y_0 = c(1_T)$ be the preferred generator of $KU_0(T)$ and let $y_1$ be the generator of $KU_1(T)$ such that $c(\omega) = 2\beta_U \cdot y_1$. Then we take $\beta_U^n \cdot y_0$ to be the preferred generator of $KU_{2n}(T)$ and we take $\beta_U^n \cdot y_1$ to be the preferred generator of $KU_{2n+1}(T)$.

From these choices of generators and from Sequence 17, along with the relations $\psi_U c = c$, $\psi_U \beta_U = \beta_U \psi_U$, $rc = 2$ and $cr = 1 + \psi_U$, we can mostly compute the operations of $r$, $c$, and $\psi_U$. However, the values of $r_4$, $c_4$, $r_7$, and $c_7$ are only determined up to sign. The following calculations show that $r_4 = 2$ and $r_7 = 1$ and it follows that $c_4 = 1$ and $r_7 = 2$:

$$r(\beta_U^2 \cdot y_0) = r(\beta_U^2 \cdot c(1_T))$$

$$= r(\beta_U^2) \cdot 1_T$$

$$= \xi \cdot 1_T$$

$$= 2\beta_T$$

and

$$2r(\beta_U^3 \cdot y_1) = r(\beta_U^3 \cdot 2\beta_U y)$$

$$= r(\beta_U^3 \cdot c(\omega))$$

$$= r(\beta_U^3) \cdot \omega$$

$$= \xi \cdot \omega$$

$$= 2\beta_T \omega$$

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The groups $KT_\ast(T)$ can be computed immediately from the sequence

$$KU_{n+1}(T) \xrightarrow{1-\psi_U} KU_n(T) \xrightarrow{\tau} KT_n(T) \xrightarrow{\zeta} KU_n(T) \xrightarrow{1-\psi_U} KU_n(T) \quad (18)$$

using what we know about $KU_\ast(T)$ and $\psi_U$. Choose generators for $KT_n(T)$ so that $\gamma$ and $\zeta$ are as specified in the table (this choice is not unique).

To compute $\varepsilon$ we use the sequence

$$KO_{n-2}(T) \xrightarrow{\eta_0} KO_n(T) \xrightarrow{\varepsilon} KT_n(T) \xrightarrow{\tau_3^{-1}} KO_{n-3}(T) \xrightarrow{\eta_0} KO_{n-1}(T). \quad (19)$$

Note that in this sequence, $\eta_0^2 = 0$ for all $n$. It is immediate that $\varepsilon_1 = \varepsilon_5 = 1$.

Looking at Sequence (19) with $n = 0$, we see that the image of $\varepsilon_0$ is a subgroup of index 2 in $KT_0(T) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Since $\zeta_0\varepsilon_0 = c_0 = 1$, we conclude that $\varepsilon_0$ is either $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. By changing the basis of $KT_0(T)$ we may assume that $\varepsilon_0 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. (If a change is necessary, the new generators will be $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ given in terms of the old generators. This basis change does not disturb our previously computed representations of $\gamma$ and $\zeta$.)

From $\zeta_3\varepsilon_3 = c_3 = 2$, we see that $\varepsilon_3$ is $\left(\begin{smallmatrix} a \\ 2 \end{smallmatrix}\right)$ where $a$ is some integer. But the exact sequence above tells us that the image of $\varepsilon_3$ is a free summand of $KT_3(T) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Thus $a$ must be odd. Again we may change the basis $KT_3(T)$ so that $\varepsilon_3 = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$. (The new generators will be $\left(\begin{smallmatrix} a-1 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Again this change does not disturb our representations of $\gamma$ and $\zeta$.)

Using the same reasoning that we used in the computation of $\varepsilon_0$ above, we know that $\varepsilon_4$ is either $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. However at this point we do not have the freedom to change generators. (The generators of $KT_4(T)$ are to be the suspensions of the generators of $KT_0(T)$.) Instead, we compute $\tau_0 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ using the relations $\eta_0 = \tau \varepsilon$ and $r = \tau \gamma$. Then we use the relation $\tau \beta_r^{-1} \varepsilon = 0$ from Sequence (3) to see that $\varepsilon_4 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$.

The computation of $\varepsilon_7$ is similar to that of $\varepsilon_4$. First we use the relation $\zeta_7\varepsilon_7 = c_7 = 2$ and the exact Sequence (19) to deduce that $\varepsilon_7 = \left(\begin{smallmatrix} 2 \\ a \end{smallmatrix}\right)$ for some odd $a$. Independently, we compute $\tau_3 = \left(\begin{smallmatrix} -1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ using the relations $\eta_0 = \tau \varepsilon$ and $r = \tau \gamma$. Finally, we use the relation $\tau \beta_r^{-1} \varepsilon = 0$ to see that $a = 1$.

It remains to compute $\psi_r$ and $\tau$, which can easily be done using the relations $\varepsilon r \zeta = 1 + \psi_r$, $r = \tau \gamma$, and $\eta_0 = \tau \varepsilon$.

2.2 Geometrically Realized Surjections and Resolutions

In [22], Schochet produces a geometric resolution by first producing for any unital complex C*-algebra $A$ a homomorphism $\mu: F \to K(\mathcal{H}) \otimes A$ that is surjective on both $K_0$ and $K_1$.
where $F$ is a $C^*$-algebra such that $K_*(F)$ is free (Lemma 3.1 of [22]). Roughly, he used homomorphisms from $\mathbb{R}$ in order to hit each projection (representing an element in $K_0(A)$) and a homomorphism from $S\mathbb{R}$ in order to hit each unitary (representing an element in $K_1(A)$). Then he embedded $A$ in $K \otimes A$ in order to have room to combine the homomorphisms. Once $\mu$ is constructed, the resolution is completed using a mapping cone construction.

In what follows we will also produce a geometric resolution by first producing a homomorphism which is surjective on united $K$-theory. Now there are eight distinct groups in real $K$-theory and only two of them have simple intrinsic descriptions (projections for $K_0$ and unitaries for $K_1$). Therefore, we have to resort to suspensions, representing elements of $K_i(A)$ by projections in $S_i A$. Therefore the target of our homomorphism will be the stabilization of a certain eightfold suspension of $A$ described below.

We will make essential use of Schochet’s stabilization trick in order to have room to paste together the many homomorphisms that are necessary to hit the many elements of all the different groups of united $K$-theory. Finally, we will use a mapping cone construction similar to Schochet’s and the fact that $K^{\text{CRT}}(A)$ has projective dimension 1 in $\text{CRT}$ to turn a geometric surjective homomorphism into a geometric resolution of length 1.

Let $A$ be a real unital $C^*$-algebra. Let

$$S^+ A = \{ f \in C([0,1], A) \mid f(0) = f(1) \in \mathbb{R} \cdot 1_A \}$$

be the unitized suspension of $A$. We let $\mathfrak{G}A$ denote the unital algebra obtained by repeating this process eight times:

$$\mathfrak{G} A := (S^+)^8 A .$$

There is a split exact sequence

$$0 \to S^8 A \to \mathfrak{G} A \to (S^+)^7 \mathbb{R} \to 0$$

yielding the direct sum decomposition

$$K^{\text{CRT}}(\mathfrak{G} A) \cong K^{\text{CRT}}(S^8 A) \oplus K^{\text{CRT}}((S^+)^7 \mathbb{R}) .$$

There is also an isomorphism $K^{\text{CRT}}(A) \cong K^{\text{CRT}}(S^8 A)$ so $K^{\text{CRT}}(\mathfrak{G} A)$ carries the same essential information as $K^{\text{CRT}}(A)$. In our geometric resolution, we will replace $K^{\text{CRT}}(A)$ by $K^{\text{CRT}}(K(H) \otimes \mathfrak{G} A)$. The main theorems of this section are the following.
Proposition 2.1. Let $A$ be a real unital $C^*$-algebra. Then there is a real $C^*$-algebra $F$, a real Hilbert space $H$, and a $C^*$-algebra homomorphism $\mu : F \to \mathcal{K}(H) \otimes \mathfrak{A}$ such that $K^{CRT}(F)$ is a free CRT-module and the induced homomorphism

$$\mu_* : K^{CRT}(F) \to K^{CRT}(\mathcal{K}(H) \otimes \mathfrak{A}) \cong K^{CRT}(\mathfrak{A})$$

is a surjection.

Theorem 2.2. Let $A$ be a real unital $C^*$-algebra. Then there are real $C^*$-algebras $F_1$ and $F_0$, $C^*$-algebra homomorphisms $\mu_1$ and $\mu_2$, and a Hilbert space $H$ making the following sequence exact

$$0 \to K^{CRT}(F_1) \xrightarrow{(\mu_1)_*} K^{CRT}(F_0) \xrightarrow{(\mu_0)_*} K^{CRT}(\mathcal{K}(H) \otimes A) \to 0.$$ 

Furthermore, $K^{CRT}(F_0)$ and $K^{CRT}(F_1)$ are free CRT-modules.

We tackle the proof of Proposition 2.1 by considering the real, complex, and self-conjugate cases independently in the next lemmas. Once it is proven, it will be used with Lemma 1.7 to prove Theorem 2.2.

Lemma 2.3. Let $A$ be a real unital $C^*$-algebra. Then there is a real $C^*$-algebra $F$, a real Hilbert space $H$, and a $C^*$-algebra homomorphism $\mu : F \to \mathcal{K}(H) \otimes A$ such that $K^{CRT}(F)$ is a free CRT-module and the induced homomorphism

$$\mu_* : KO_0(F) \to KO_0(\mathcal{K}(H) \otimes A) \cong KO_0(A)$$

is a surjection.

Proof. Let $\{p_s\}_{s \in S}$ be a collection of projections $p_s \in \mathcal{K}(H_s) \otimes A$ such that $\{[p_s]\}_{s \in S}$ generates $KO_0(A)$. Here $S$ is a possibly uncountable index set $S$, and for each $s \in S$, $H_s$ is a finite dimensional Hilbert space. Then construct homomorphisms

$$\mu_s : \mathbb{R} \to \mathcal{K}(H_s) \otimes A$$

defined by $\mu_s(\lambda) = \lambda p_s$ so that $[p_s]$ is in the image of $(\mu_s)_*$.

Now form the Hilbert space direct sum $H = \bigoplus_{s \in S} H_s$ and the $C^*$-algebra direct sum $F = \bigoplus_{s \in S} \mathbb{R}$. Then we patch together the homomorphisms $\mu_s$ to form a homomorphism

$$\mu : F \to \mathcal{K}(H) \otimes A$$
by defining $\mu$ to be the composition

$$\mathbb{R} \xrightarrow{\mu_s} K(\mathcal{H}_s) \otimes A \hookrightarrow K(\mathcal{H}) \otimes A$$
onumber

on each summand $\mathbb{R}$ of $F$.

Recall that $\kappa_1 \in KU_0(\mathbb{C})$ is the free generator of $K_{CRT}(\mathbb{C})$. From Table 2 we have $r(\kappa_1) = 1_U$.

**Lemma 2.4.** Let $A$ be a real unital C*-algebra. Let $p$ be a projection in $\mathbb{C} \otimes A$. Then there is a C*-algebra homomorphism $\mu: \mathbb{C} \to M_2(A)$ such that the induced homomorphism

$$\mu_*: KU_0(\mathbb{C}) \to KU_0(M_2(A)) \cong KU_0(A)$$

carries $\kappa_1$ to $[p]$ modulo the image of $\beta_U^{-1}c$.

**Proof.** Define $\mu$ by $\mu(\lambda) = r(\lambda p) \in M_2(A)$. Then

$$r(\mu_*(\kappa_1) - [p]) = \mu_*(r(\kappa_1)) - r[p] = \mu_*[1] - r[p] = 0.$$ 

Therefore $\mu_*(\kappa_1) - [p]$ is in the kernel of $r$ which is the image of $\beta_U^{-1}c$. \qed

**Lemma 2.5.** Let $A$ be a real unital C*-algebra. Then there is a real C*-algebra $F$, a real Hilbert space $\mathcal{H}$, and a C*-algebra homomorphism $\mu: F \to K(\mathcal{H}) \otimes A$ such that $K_{CRT}(F)$ is a free CRT-module and the induced homomorphism

$$\mu_*: KU_0(F) \to KU_0(K(\mathcal{H}) \otimes A) \cong KU_0(A)$$

is a surjection modulo the image of $\beta_U^{-1}c$.

**Proof.** Again, choose a family of projections $p_s \in K(\mathcal{H}_s) \otimes \mathbb{C} \otimes A$ such that the classes $[p_s]$ generate $KU_0(A)$. By Lemma 2.4, we have homomorphisms

$$\mu_s: \mathbb{C} \to M_2(K(\mathcal{H}_s) \otimes A) \cong K(\mathcal{H}_s') \otimes A$$

such that $\mu_s(\kappa_1) = [p_s]$ modulo $\beta_U^{-1}c$. Here $\mathcal{H}_s'$ is taken to be the Hilbert space tensor product $\mathbb{R}^2 \otimes \mathcal{H}_s$ so that $M_2(\mathbb{R}) \otimes K(\mathcal{H}_s) \cong K(\mathbb{R}^2) \otimes K(\mathcal{H}_s) \cong K(\mathcal{H}_s')$.

Then the family $\mu_s$ is patched together as in the proof of Lemma 2.3 to form

$$\mu_s: F \to K(\mathcal{H}) \otimes A$$

where $F = \bigoplus_{s \in S} \mathbb{C}$ and $\mathcal{H} = \bigoplus_{s \in S} \mathcal{H}_s'$. \qed
We now turn to the self-conjugate case. In this case the target of our homomorphism will not be a matrix algebra over $A$, but a matrix algebra over $S^+A$. With respect to the natural decomposition

$$K^{\text{CRT}}(S^+A) \cong K^{\text{CRT}}(SA) \oplus K^{\text{CRT}}(\mathbb{R}) ,$$

let $\pi: K^{\text{CRT}}(S^+A) \to K^{\text{CRT}}(SA)$ be the projection onto the first summand. For the source of our homomorphism, we will use the algebra $T$. Recall that $K^{\text{CRT}}(T)$ is a free CRT-module generated by $\chi = (\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}) \in KT^{-1}(T)$. The only other fact we need to remember about $\chi$ is that $\tau(\chi) = 1_T \in KO_0(T)$ (see Table 3).

**Lemma 2.6.** Let $A$ be unital and let $p$ be a projection in $T \otimes A$. Then there is a $C^*$-algebra homomorphism $\mu: T \to M_{16}(S^+A)$ such that the composition

$$\pi \circ \mu_*: KT^{-1}(T) \to KT^{-1}(M_{16}(S^+A)) \to KT^{-1}(SA) \cong KT_0(A)$$

carries $\chi$ to $[p]$ modulo the image of $\beta_T^{-1}\varepsilon$.

**Proof.** Fix a unitary $u$ in $M_8(A)$ such that

$$u \begin{pmatrix} r(p(0)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1_2 - r(p(0)) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} u^* = \begin{pmatrix} 1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Recall from Section 1.2 that for $i = 1, 2$, we have homomorphisms

$$\sigma_i: T \otimes A \to C(S^1, M_4(A))$$

such that $\tau = (\sigma_1)_* - (\sigma_2)_*$. Then let $P' \in C(S^1, M_8(A))$ be the projection defined by

$$P' = u \begin{pmatrix} \sigma_1(p) & 0 \\ 0 & \sigma_2(1-p) \end{pmatrix} u^* .$$

Note that $P'$ is in $M_8(S^+A)$ since

$$P'(0) = P'(1) = u \begin{pmatrix} \sigma_1(p(0)) & 0 \\ 0 & \sigma_2(1-p(0)) \end{pmatrix} u^* = u \begin{pmatrix} r(p(0)) & 0 \\ 0 & 1-r(p(0)) \end{pmatrix} u^* = \begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix} .$$

Finally, let $P = \begin{pmatrix} P' & 0 \\ 0 & 0 \end{pmatrix} \in M_{16}(S^+A)$. 

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We will construct a homomorphism \( \mu: T \rightarrow M_{16}(S^+A) \) that sends the unit to \( P \). For this we first define the C*-algebra homomorphism \( \mu': T \rightarrow M_{16}(\mathbb{R}) \) as follows. Given \( f \in T \), let \( \zeta(f) = f(0) = x + iy \). Then define

\[
\mu'(f) = \begin{pmatrix}
x & 0 & 0 & 0 & -y & 0 & 0 & 0 \\
0 & x & 0 & 0 & 0 & -y & 0 & 0 \\
0 & 0 & x & 0 & 0 & 0 & -y & 0 \\
0 & 0 & 0 & x & 0 & 0 & 0 & -y \\
y & 0 & 0 & 0 & x & 0 & 0 & 0 \\
0 & y & 0 & 0 & 0 & x & 0 & 0 \\
0 & 0 & y & 0 & 0 & 0 & x & 0 \\
0 & 0 & 0 & y & 0 & 0 & 0 & x
\end{pmatrix}
\]

Now \( \mu'(f) \) commutes with the projection \( P \) for all \( f \) so the rule

\[
\mu(f) = \mu'(f) \cdot P
\]

defines a C*-algebra homomorphism which sends the unit to \( P \).

We claim that \( \mu_* \) carries \( 1_T \in KO_0(T) \) to \( \tau[p] + 2 \cdot 1_o \in KO_0(M_8(\mathbb{R}) \otimes S^+A) \cong KO_1(A) \oplus KO_0(\mathbb{R}) \). Indeed,

\[
\mu_*(1_T) = [P] = [\sigma_1(p)] + [\sigma_2(1 - p)] = [\sigma_1(p)] + [\sigma_2(1)] - [\sigma_2(p)] = \tau[p] + [r(1)] = \tau[p] + 2 \cdot 1_o.
\]

Then

\[
\tau(\pi \circ \mu_*)(\chi) = (\pi \circ \mu_*)\tau(\chi) = (\pi \circ \mu_*)(1_T) = \pi(\tau[p] + 2o) = \tau[p] .
\]

Therefore, \( (\pi \circ \mu_*)(\chi - [p]) \) is in the kernel of \( \tau \) which is the image of \( \beta_T^{-1}\varepsilon \), according to Sequence 6.

\[\square\]

**Lemma 2.7.** Let \( A \) be a real unital C*-algebra. Then there is a C*-algebra \( F \), a Hilbert space \( \mathcal{H} \), and a C*-algebra homomorphism \( \mu: F \rightarrow K(\mathcal{H}) \otimes S^+A \) such that \( K_CRT(F) \) is a free CRT-module and the induced homomorphism

\[
\mu_*: KT_{-1}(F) \rightarrow KT_{-1}(K(\mathcal{H}) \otimes S^+A) \cong KT_{-1}(S^+A) \cong KT_0(A) \oplus KT_{-1}(\mathbb{R})
\]

is a surjection modulo the image of \( \beta_T^{-1}\varepsilon \).
Proof. Let $\mu_0: \mathbb{R} \to S^+A = K(\mathcal{H}_0) \otimes S^+A$ be the unital inclusion where $\mathcal{H}_0$ is a one-dimensional real Hilbert space. The homomorphism $\mu_0$ is designed to insure that the image of $\mu_*$ includes $KT_{-1}(\mathbb{R}) \subset KT_{-1}(S^+(A))$. Now let $p_s \in K(\mathcal{H}_s) \otimes T \otimes A$ be a family of projections indexed by $S$ so that $\{[p_s]\}_{s \in S}$ and the image of $\beta_T^{-1}\varepsilon$ generate $KT_0(A)$. For each $s \in S$, let

$$\mu_s: T \longrightarrow M_{16}(K(\mathcal{H}_s) \otimes S^+A) \cong K(\mathcal{H}_s') \otimes S^+A$$

be given according to Lemma 2.6 such that $(\pi \circ (\mu_s)_*)\chi = [p_s]$ modulo $\beta_T^{-1}\varepsilon$.

Then, taking $F = \mathbb{R} \oplus \bigoplus_{s \in S} T$ and $\mathcal{H} = \mathcal{H}_0 \oplus \bigoplus_{s \in S} \mathcal{H}_s'$, we form

$$\mu: F \longrightarrow K(\mathcal{H}) \otimes S^+A$$

by patching together the homomorphisms $\mu_0$ and $\{\mu_s\}_{s \in S}$.

Proof of Proposition 2.1. Let $A$ be a given real unital $C^*$-algebra. Using the now familiar patching technique, it suffices to produce enough maps of the form

$$\mu_s: F_s \to K(\mathcal{H}_s) \otimes S^+A$$

(where $K^{\text{CRT}}(F_s)$ is a free $\text{CRT}$-module for each $s$) so that the images of $(\mu_s)_*$ generate $K^{\text{CRT}}(\mathcal{G}A)$. There will be 25 maps in all.

Recall that we have the decomposition

$$K^{\text{CRT}}(\mathcal{G}A) \cong K^{\text{CRT}}(S^8A) \oplus K^{\text{CRT}}((S^+)^7\mathbb{R})$$

arising from the split exact sequence

$$0 \to S^8A \to \mathcal{G}A \to (S^+)^7\mathbb{R} \to 0.$$ 

The first map in our collection is taken to be the section

$$s: (S^+)^7\mathbb{R} \to \mathcal{G}A$$

whose image in united $K$-theory is the second summand in the decomposition above.

Now for each integer $i$ satisfying $0 \leq i \leq 7$, we let

$$\mu^0_i: F^0_i \to K(\mathcal{H}^0_i) \otimes S^iA$$

be given as in Lemma 2.3 so that the image of $(\mu^0_i)_*$ contains $KO_0(S^iA) \cong KO_i(A)$. Then we suspend to get a map with the correct codomain:

$$S^{8-i}\mu_i^0: S^{8-i}(F^0_i) \to S^8(K(\mathcal{H}^0_i) \otimes A) \hookrightarrow K(\mathcal{H}^0_i) \otimes \mathcal{G}A.$$
The image of the maps \( \{ S^{8-i} \mu_i^O \}_{i=0}^7 \) induced on real K-theory is all of \( KO_* (S^8 A) \subset KO_* (\mathcal{G} A) \). It also follows, therefore, that the image of the maps induced on complex and self-conjugate K-theory contains the image of \( c \) (and therefore of \( \beta^{-1} U c \)) in \( KU_* (S^8 A) \) and the image of \( \varepsilon \) (and therefore of \( \beta_T^{-1} \varepsilon \)) in \( KT_* (S^8 A) \).

Similarly, we use Lemma 2.5 to build maps

\[ S^{8-i} \mu_i^{U} : S^{8-i} (F_i^U) \to S^8 (K (H_i^U) \otimes A) \hookrightarrow K (H_i^U) \otimes \mathcal{G} A \]

so that the image of the corresponding maps induced on K-theory contains \( KU_* (S^8 A) \) modulo \( \beta^{-1} U c \).

Finally, for each integer \( i \), Lemma 2.7 guarantees the existence of a map

\[ \mu_i^T : F_i^T \to K (H_i^T) \otimes S^i (S^i A) \]

such that the image of the map induced on self-conjugate K-theory, in degree \(-1\), is surjective modulo \( \beta_T^{-1} \varepsilon \). Then we suspend to get maps

\[ S^{8-i} \mu_i^T : S^{8-i} F_i^T \to K (H_i^T) \otimes S^{8-i} (S^i S^i A) \hookrightarrow K (H_i^T) \otimes \mathcal{G} A . \]

The image of the corresponding maps induced on self-conjugate K-theory lies in \( KT_* (\mathcal{G} A) \). But modulo the image of \( s_* \) and the image of \( \beta_T^{-1} \varepsilon \) we know that the image of \( S^{8-i} \mu_i^T \) contains all of \( KT_* (S^8 A) \).

Therefore, the images of the maps on united K-theory induced by the homomorphisms \( s, \mu_i^O, \mu_i^U, \) and \( \mu_i^T \) contain all of \( K^{CRT} (\mathcal{G} A) \) and hence these maps can be used to patch together a map \( \mu \) that is surjective on united K-theory.

**Completion of Proof of Proposition 1.7.** Given a real unital C*-algebra \( A \), let \( F \) and \( \mu \) be given as in the Proposition 2.1. Since \( K^{CRT} (F) \) is a free CRT-module, it follows in particular that the CRT-relations are satisfied. Therefore, the CRT-relations hold for the image of \( \mu_* \), which contains \( K^{CRT} (S^8 A) \cong K^{CRT} (A) \). So \( K^{CRT} (A) \) is a CRT-module for all real unital C*-algebras and thus, using a unitization argument, for all real C*-algebras.

Now that Proposition 1.7 is finally proven, it is also established that \( K^{CRT} (A) \) has projective dimension at most 1 by Theorem 1.16. This fact and Proposition 2.1 will both be used in the proof of Theorem 2.2.

**Proof of Theorem 2.2.** Given \( A \), we take \( F, \mu, \) and \( \mathcal{H} \) as in Proposition 2.1 so that

\[ K^{CRT} (F) \xrightarrow{\mu_*} K^{CRT} (K (\mathcal{H}) \otimes \mathcal{G} A) \]

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is a surjection.

Now let $F_0$ be the mapping cylinder of $\mu$ and let $F_1$ be the mapping cone of $\mu$:

$$
F_0 = \{(f, g) \in F \oplus C([0, 1], K(\mathcal{H}) \otimes \mathcal{G}A) \mid g(0) = \mu(f)\}
$$

$$
F_1 = \{(f, g) \in F \oplus C([0, 1], K(\mathcal{H}) \otimes \mathcal{G}A) \mid g(0) = \mu(f), g(1) = 0\}.
$$

The homomorphism $\pi : F_0 \to F$ defined by $\pi(f, g) = f$ is a homotopy equivalence. Furthermore, the homomorphism $\mu_0 : F_0 \to K(\mathcal{H}) \otimes \mathcal{G}A$ defined by $\mu_0(f, g) = g(1)$ is equivalent to $\mu$ in the sense that the diagram

$$
\begin{array}{ccc}
F_0 & \xrightarrow{\mu_0} & K(H) \otimes \mathcal{G}A \\
\downarrow{\pi} & & \searrow{\mu} \\
F & & \\
\end{array}
$$

commutes up to homotopy. Indeed, there is a homotopy from $\mu_0$ to the homomorphism $(f, g) \mapsto g(0)$ which would make the diagram commute on the nose.

The homomorphism $\mu_0$ is surjective and its kernel is $F_1$. Let $\mu_1 : F_1 \to F_0$ be the inclusion. Then we have the short exact sequence of C*-algebras

$$
0 \to F_1 \xrightarrow{\mu_1} F_0 \xrightarrow{\mu_0} K(\mathcal{H}) \otimes \mathcal{G}A \to 0.
$$

Since $(\mu_0)_*$ is surjective on united $K$-theory, the long exact sequence on united $K$-theory breaks down into a resolution of $K^{\text{CRT}}(\mathcal{G}A)$. Furthermore, $K^{\text{CRT}}(F_0) \cong K^{\text{CRT}}(F)$ is a free CRT-module. Since $K^{\text{CRT}}(\mathcal{G}A)$ has projective dimension at most 1 in CRT, the CRT-module $K^{\text{CRT}}(F_1)$ must also be projective (see Theorem 9.5 of [21]), hence free (by Theorem 1.17).

3 Tensor Products

Before we can state and prove the Künneth Theorem for united $K$-theory, we must discuss the tensor product in the category $CRT$. I am grateful to A.K. Bousfield for sharing with me some valuable unpublished notes on this topic ([7]). The following sections are largely drawn from those notes.

In Section 3.1.1, we define the tensor product in the category $CRT$. Given two $CRT$-modules $M$ and $N$, their tensor product $M \otimes_{\text{CRT}} N$ will also be a $CRT$-module. In Section 3.2 we prove some results about the behavior of tensor products when one of the factors is acyclic or when one of the factors is free. These results are important both for the proof of our Künneth formula and for purposes of computation (see Section 3).
3.1 Tensor Products in CRT

Let $M = \{M^O, M^U, M^T\}$ be a CRT-module. In particular $M^O$ is a left module over $KO_*(\mathbb{R})$. We give $M^O$ the structure of a right $KO_*(\mathbb{R})$ module by the rule $m \cdot x = (-1)^{|m||x|} x \cdot m$ for all $m \in M^O$ and $x \in KO_*(\mathbb{R})$ where $|m|$ and $|x|$ denote the graded degree of those elements. Similarly, $M^U$ and $M^T$ are given right $KU_*(\mathbb{R})$- and $KT_*(\mathbb{R})$-module structures, respectively.

Now if $N = \{N^O, N^U, N^T\}$ is also a CRT-module we have the tensor products $M^O \otimes_{KO_*(\mathbb{R})} N^O$, $M^U \otimes_{KU_*(\mathbb{R})} N^U$, and $M^T \otimes_{KT_*(\mathbb{R})} N^T$ using the right module structures on $M$ described above and the natural left module structures on $N$. These tensor products are again $KO_*(\mathbb{R})$-, $KU_*(\mathbb{R})$-, and $KT_*(\mathbb{R})$-modules, respectively.

**Definition 3.1.** Let $M$, $N$, and $P$ be CRT-modules. A CRT-pairing $\alpha: (M, N) \rightarrow P$ consists of three homomorphisms:

1. a $KO_*(\mathbb{R})$-module homomorphism $\alpha^O: M^O \otimes_{KO_*(\mathbb{R})} N^O \rightarrow P^O$
2. a $KU_*(\mathbb{R})$-module homomorphism $\alpha^U: M^U \otimes_{KU_*(\mathbb{R})} N^U \rightarrow P^U$
3. a $KT_*(\mathbb{R})$-module homomorphism $\alpha^T: M^T \otimes_{KT_*(\mathbb{R})} N^T \rightarrow P^T$.

We use the notation $\alpha^O(m_o \otimes n_o) = m_o \cdot_o n_o$, $\alpha^U(m_u \otimes n_u) = m_u \cdot_u n_u$, and $\alpha^T(m_t \otimes n_t) = m_t \cdot_t n_t$ to express these products. Furthermore, in order to be a CRT-pairing, the homomorphisms $\alpha^O$, $\alpha^U$, and $\alpha^T$ must satisfy the following properties for all $m_o \in M^O$, $n_o \in N^O$, $m_u \in M^U$, $n_u \in N^U$, $m_t \in M^T$, and $n_t \in N^T$:

1. $c(m_o \cdot_o n_o) = c(m_o) \cdot_o c(n_o)$
2. $r(c(m_o \cdot_u n_u)) = m_o \cdot_o r(n_u)$
3. $r(m_u \cdot_u c(n_o)) = r(m_u) \cdot_o n_o$
4. $\varepsilon(m_o \cdot_o n_o) = \varepsilon(m_o) \cdot_T \varepsilon(n_o)$
5. $\zeta(m_t \cdot_T n_t) = \zeta(m_t) \cdot_u \zeta(n_t)$
6. $\psi_U(m_u \cdot_u n_u) = \psi_U(m_u) \cdot_u \psi_U(n_u)$
7. $\psi_T(m_t \cdot_T n_t) = \psi_T(m_t) \cdot_T \psi_T(n_t)$
8. $\gamma(m_u \cdot_u \zeta(n_t)) = \gamma(m_u) \cdot_T n_T$
(9) \( \gamma(\zeta(m_T) \cdot_U n_U) = (-1)^{|m_T|} m_T \cdot_T \gamma(n_U) \)

(10) \( \tau(m_T \cdot_U \varepsilon(n_O)) = \tau(m_T) \cdot_O n_O \)

(11) \( \tau(\varepsilon(m_O) \cdot_T n_T) = (-1)^{|m_O|} m_O \cdot_O \tau(n_T) \)

(12) \( \varepsilon \tau(m_T \cdot_T n_T) = \varepsilon \tau(m_T) \cdot_T n_T + (-1)^{|m_T|} m_T \cdot_T \varepsilon \tau(n_T) + \eta_T(m_T \cdot_T n_T) \).

In the above definition, the first three equations are redundant in the sense that they can be derived from the others using the relations \( r = \tau \gamma \) and \( c = \zeta \varepsilon \). We include them explicitly on our list simply out of convenience, remembering that it is enough to check properties 4 through 12 to have a CRT-pairing.

The tensor product in the category of CRT-modules is defined as the solution of a universal property described in the statement of the following proposition.

**Proposition 3.2.** Let \( M \) and \( N \) be CRT-modules. Then there exists a unique CRT-module, denoted \( M \otimes_{\text{CRT}} N \), and CRT-pairing \( \iota : (M, N) \rightarrow M \otimes_{\text{CRT}} N \) such that for every CRT-pairing \( \alpha : (M, N) \rightarrow P \) there exists a unique CRT-module homomorphism \( \alpha : M \otimes_{\text{CRT}} N \rightarrow P \) making the diagram below commute.

\[
\begin{array}{c}
(M, N) \\
\downarrow \alpha
\end{array} \quad \begin{array}{c}
\quad \rightarrow \quad \quad M \otimes_{\text{CRT}} N \\
\downarrow \alpha
\end{array} \quad \begin{array}{c}
\quad \rightarrow \quad \quad P
\end{array}
\]

**Proof.** We construct \( M \otimes_{\text{CRT}} N \) as follows. Let \( R \) be the free CRT-module generated by the elements of \( M^O \times N^O, M^U \times N^U, \) and \( M^T \times N^T \). These generators are denoted as pure tensors \( m_O \otimes n_O, m_U \otimes n_U, \) and \( m_T \otimes n_T \). Let \( Q \) be the smallest CRT-submodule of \( R \) such that the composition pairing \( (M, N) \rightarrow R \rightarrow R/Q \) is a CRT-pairing. In other words, \( Q \) is the CRT-submodule generated by elements of \( R \) corresponding to the CRT-pairing relations in Definition 3.1.

**Definition 3.3.** Given a CRT-module \( M \), we define the suspension of \( M \) by shifting the degrees such that

\[
(\Sigma M)_k = M_{k+1}
\]

and the desuspension by

\[
(\Sigma^{-1} M)_k = M_{k-1}
\]
These are defined such that $K_{\text{CRT}}(S^n A) = \Sigma^n K_{\text{CRT}}(A)$ for any integer $n$.

The tensor product in the category of CRT-modules enjoys the usual properties associated with tensor products as stated in the following proposition. The proof uses standard techniques of homological algebra, the details of which can be found in Section IV.1 of [4].

**Proposition 3.4.** The tensor product functor is a symmetric bifunctor which in each argument is covariant, continuous, right exact, and commutes with suspensions and direct sums.

Since the tensor product is right exact and since there are enough projective modules in CRT to form projective resolutions, we also have derived functors $\text{Tor}^i_{\text{CRT}}(M, N)$ for all CRT-modules $M$ and $N$ (see [21], page 121).

The next three propositions characterize tensor products with monogenic free objects of CRT. They are due to Bousfield ([7]) and proofs can be found in [4].

**Proposition 3.5.** Let $N$ be an arbitrary CRT-module. Then

$$F(b, 0, \mathbb{R}) \otimes_{\text{CRT}} N \cong \left\{ \{ b \otimes n_o \mid n_o \in N^o \}, \right.$$

$$\left. \{ cb \otimes n_v \mid n_v \in N^v \}, \right.$$

$$\left. \{ \varepsilon b \otimes n_T \mid n_T \in N^T \} \right\} \cong N.$$

**Proposition 3.6.** Let $N$ be an arbitrary CRT-module. Then

$$F(b, 0, \mathbb{C}) \otimes_{\text{CRT}} N \cong \left\{ \{ r(b \otimes n_v) \mid n_v \in N^v \}, \right.$$

$$\left. \{ b \otimes n_v \mid n_v \in N^v \} \oplus \{ \psi_v b \otimes n_v \mid n_v \in N^v \}, \right.$$

$$\left. \{ \gamma(b \otimes n_v) \mid n_v \in N^v \} \oplus \{ \varepsilon r(b \otimes n_v) \mid n_v \in N^v \} \right\} \cong \{ N^u, N^u \oplus N^u, \Sigma N^u \oplus N^u \}.$$

**Proposition 3.7.** Let $N$ be an arbitrary CRT-module. Then

$$F(b, 0, T) \otimes_{\text{CRT}} N \cong \left\{ \{ \tau(b \otimes n_T) \mid n_T \in N^T \}, \right.$$

$$\left. \{ \zeta b \otimes n_v \mid n_v \in N^v \} \oplus \{ c \tau b \otimes n_v \mid n_v \in N^v \}, \right.$$

$$\left. \{ b \otimes n_T \mid n_T \in N^T \} \oplus \{ \varepsilon \tau b \otimes n_T \mid n_T \in N^T \} \right\} \cong \{ \Sigma^{-1} N^T, N^u \oplus \Sigma^{-1} N^v, N^T \oplus \Sigma^{-1} N^T \}.$$
We say that a CRT-module $N$ is flat if tensoring on the right by $N$ preserves exact sequences of CRT-modules.

**Corollary 3.8.** Free objects in CRT are flat.

*Proof.* It follows immediately from Propositions 3.5, 3.6, and 3.7 that $F(b,0,\mathbb{R})$, $F(b,0,\mathbb{C})$, and $F(b,0,T)$ are flat. Since all free CRT-modules are directed sums of suspensions and desuspensions of these three monogenic free objects, the corollary follows using Lemma 3.4.

**Corollary 3.9.** Suppose $M$ and $N$ are CRT-modules such that $M$ is acyclic. Then the natural homomorphism

$$M^u \otimes_{KU^*(\mathbb{R})} N^u \to (M \otimes_{CRT} N)^u$$

is an isomorphism.

*Proof.* Again, this follows from Propositions 3.5, 3.6, and 3.7 in case $M$ is free. Otherwise, $M$ has a free resolution

$$0 \to F_1 \to F_0 \to M \to 0$$

of length one by Theorem 1.16. Then consider the commutative diagram

```
0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
On the other hand, in the real and self-conjugate cases, there are homomorphisms
\[ M^o \otimes_{KO^*(R)} N^o \to (M \otimes_{CRT} N)^o \]
and
\[ M^r \otimes_{KT^*(R)} N^r \to (M \otimes_{CRT} N)^r \]
which are not isomorphisms in general.

### 3.2 Acyclic Objects and Tensor Products

In this section we will prove the following proposition.

**Proposition 3.10.** Let \( M \) be free and \( N \) be acyclic in CRT. Then \( M \otimes_{CRT} N \) is acyclic.

On the other hand, we will see in Table 11 that the tensor product of two arbitrary acyclic CRT-modules is not necessarily acyclic.

To prove Proposition 3.10 it suffices to assume that \( M \) is a free monogenic CRT-module.

In fact, since tensor products commute with the operation of suspension, it suffices to assume that \( M \) is either \( F(b, 0, \mathbb{R}) \), \( F(b, 0, \mathbb{C}) \), or \( F(b, 0, T) \) (since suspensions and products preserve acyclicity). The case \( M = F(b, 0, \mathbb{R}) \) is immediate since \( F(b, 0, \mathbb{R}) \otimes_{CRT} N \cong N \). The other two cases are addressed in the following lemmas.

**Lemma 3.11.** \( F(b, 0, \mathbb{C}) \otimes_{CRT} N \) is acyclic for any CRT-module \( N \).

**Proof.** Recall that
\[
F(b, 0, \mathbb{C}) \otimes_{CRT} N \cong \left\{ r(b \otimes n_u) \mid n_u \in N^u \right\},
\{ b \otimes n_1 + \psi_U b \otimes n_2 \mid n_i \in N^i \},
\{ \gamma(b \otimes n_1) + \varepsilon r(b \otimes n_2) \mid n_i \in N^u \}\right\}
\]

To show \( F(b, 0, \mathbb{C}) \otimes_{CRT} N \) is acyclic, there are three sequences that must be shown to be exact. This can be done directly, using the representation of \( F(b, 0, \mathbb{C}) \otimes_{CRT} N \) above. We illustrate by showing that the image of \( \gamma \) is the kernel of \( \zeta \) (which shows exactness at one point of the exact sequence relating the complex and self-conjugate parts).

Let \( b \otimes n_1 + \psi_U (b) \otimes n_2 \) be an arbitrary element of \( (F(b, 0, \mathbb{C}) \otimes_{CRT} N)^U \). Then
\[
\gamma(b \otimes n_1 + \psi_U (b) \otimes n_2) = \gamma(b \otimes n_1) + \gamma \psi_U (b \otimes \psi_U (n_2)) = \gamma(b \otimes (n_1 + \psi_U (n_2))).
\]
On the other hand, if $\gamma(b \otimes n_1) + \varepsilon r(b \otimes n_2)$ is an arbitrary element of $(F(b, 0, \mathbb{C}) \otimes_{\text{CRT}} N)^T$ then

$$\zeta(\gamma(b \otimes n_1) + \varepsilon r(b \otimes n_2)) = cr(b \otimes n_2) = (b \otimes n_2) + (\psi_U(b) \otimes \psi_U(n_2)) \, .$$

Therefore,

$$\ker \zeta = \{ \gamma(b \otimes n_1) + \varepsilon r(b \otimes n_2) \mid n_1 \in N^U \text{ and } n_2 = 0 \}$$

$$= \{ \gamma(b \otimes n) \mid n \in N^U \}$$

$$= \text{image } \gamma \, .$$

Before proving that $F(b, 0, T) \otimes N$ is acyclic, we need the following lemma concerning acyclic objects (compare [5], Paragraph 2.3).

**Lemma 3.12.** If $M$ is acyclic, then the sequence

$$\cdots \longrightarrow M_n^T \overset{\eta_T}{\longrightarrow} M_{n+2}^T \overset{(\zeta \varepsilon r)}{\longrightarrow} M_{n+2}^U \oplus M_{n+3}^U \overset{(\varepsilon \eta_T \gamma \beta U)}{\longrightarrow} M_{n+2}^T \longrightarrow \cdots$$

is exact.

**Proof.** Consider the following diagram.

$$\cdots \longrightarrow M_n^U \overset{\gamma}{\longrightarrow} M_n^T \overset{\zeta}{\longrightarrow} M_n^U \overset{1-\psi U}{\longrightarrow} M_{n+1}^U \overset{\varepsilon r \beta U}{\longrightarrow} M_{n+1}^U \overset{\gamma}{\longrightarrow} M_n^T \longrightarrow \cdots$$

In this diagram, the rows are exact since $M$ is acyclic. We check that the squares commute:

$$\beta_U(1-\psi_U)\beta_U^{-1} = 1 + \psi_U = cr = \zeta \varepsilon r \gamma \, ,$$

$$\gamma \beta_U^{-1} \zeta \varepsilon r = \beta_U^{-1} \gamma \beta_U \zeta \varepsilon r = \beta_U^{-1} \eta_U \varepsilon r = \beta_T^{-1} \varepsilon r \eta_U \, ,$$

$$= \varepsilon r \beta_T^{-1} \eta_U + \eta_U \beta_T^{-1} \eta_U = \varepsilon r \beta_T^{-1} \gamma \beta_U^{-1} \zeta = \varepsilon r \beta_U^{-1} \zeta$$

$$\zeta \varepsilon r \beta_U^{-1} = cr \beta_U^{-1} = (1 + \psi_U) \beta_U^{-1} = \beta_U^{-1} (1 - \psi_U) \, .$$

In this computation the relation $\eta_U \varepsilon r = \varepsilon r \eta_U$ follows from the fact that $\varepsilon$ and $\tau$ are $KO_s(\mathbb{R})$-module homomorphisms and the relation $\varepsilon (\eta_U) = \eta_U$.

Now the exact sequence is easily derived from this commutative ladder above by a diagram chase, recalling that $\eta_U = \gamma \beta_U \zeta$.
Lemma 3.13. If $N$ is acyclic then $F(b, 0, T) \otimes_{\text{CRT}} N$ is acyclic.

Proof. Recall that

$$F(b, 0, T) \otimes_{\text{CRT}} N \cong \{ \{ \tau(b \otimes n) \mid n \in N^T \},$$

$$\{ \zeta b \otimes n_1 + c\tau b \otimes n_2 \mid n_i \in N^U \},$$

$$\{ b \otimes n_1 + \varepsilon \tau b \otimes n_2 \mid n_i \in N^T \} \}.$$

Again, it can easily be shown by hand that the three sequences that need to be exact are exact. For this, the two exact sequences

$$\cdots \longrightarrow N_{n+1}^U \xrightarrow{\gamma} N_n^T \xrightarrow{\zeta} N_n^U \xrightarrow{1-\psi} N_n^U \longrightarrow \cdots \quad (21)$$

and

$$\cdots \longrightarrow N_n^T \xrightarrow{\eta_T} N_{n+2}^T \xrightarrow{(\zeta, \varepsilon)} N_{n+2}^U \oplus N_{n+3}^U \xrightarrow{(-\varepsilon\beta_U, \gamma\beta_U)} N_{n+2}^T \longrightarrow \cdots \quad (22)$$

must be used.

For example, to show that the image of $\gamma$ is the kernel of $\zeta$ in $F(b, 0, T) \otimes_{\text{CRT}} N$, we compute

$$\gamma(\zeta b \otimes n_1 + c\tau b \otimes n_2) = b \otimes \gamma(n_1) + \varepsilon \tau b \otimes \gamma(n_2)$$

and

$$\zeta(b \otimes n_1 + \varepsilon \tau b \otimes n_2) = \zeta b \otimes \zeta(n_1) + c\tau b \otimes \zeta(n_2).$$

Therefore,

$$\text{image } \gamma = \{ b \otimes n_1 + \varepsilon \tau b \otimes n_2 \mid n_1, n_2 \in \text{image } \gamma \}$$

$$= \{ b \otimes n_1 + \varepsilon \tau b \otimes n_2 \mid n_1, n_2 \in \ker \zeta \}$$

$$= \ker \zeta.$$

We will also show that the image of $\eta_T$ is the kernel of $c$ to illustrate how Sequence (22) is used:

$$\eta_T(\tau(b \otimes n)) = \tau(b \otimes \eta_T n)$$

and

$$c(\tau(b \otimes n)) = \zeta \varepsilon \tau(b \otimes n)$$

$$= \zeta(\varepsilon \tau b \otimes n + b \otimes \varepsilon \tau(n) + \eta_T b \otimes n)$$

$$= c\tau b \otimes \zeta(n) + \zeta b \otimes \varepsilon \tau(n).$$
Hence

\[
\text{image } \eta_o = \{ \tau(b \otimes n) \mid n \in \text{image } \eta_r \} = \{ \tau(b \otimes n) \mid n \in \ker \zeta \cap \ker c_T \} = \ker c.
\]

This completes the proof of Proposition 3.10

4 The K"unneth Formula

4.1 The Statement of the K"unneth Formula

In this section we will prove that the usual pairings in real, complex, and self-conjugate $K$-theory form a CRT-pairing in united $K$-theory. This pairing forms one of the maps of our K"unneth exact sequence, which is stated as a theorem near the end of this section.

Proposition 4.1. Let $A$ and $B$ be C*-algebras. Then there is a CRT-module homomorphism

\[ \alpha: K_{CRT}(A) \otimes_{CRT} K_{CRT}(B) \longrightarrow K_{CRT}(A \otimes B). \]

Proof. By Lemma 1.8, the homomorphisms

\[ \alpha_O: KO_*(A) \otimes_{KO_*(\mathbb{R})} KO_*(B) \longrightarrow KO_*(A \otimes B), \]
\[ \alpha_U: KU_*(A) \otimes_{KU_*(\mathbb{R})} KU_*(B) \longrightarrow KU_*(A \otimes B), \]
\[ \alpha_T: KT_*(A) \otimes_{KT_*(\mathbb{R})} KT_*(B) \longrightarrow KT_*(A \otimes B) \]

are known to satisfy all of the CRT-pairing relations of Definition 3.1 except the last. We must show that

\[ \varepsilon_T(z_1 \cdot_T z_2) = \varepsilon_T(z_1) \cdot_T z_2 + (-1)^{|z_1|} z_1 \cdot_T \varepsilon_T(z_2) + \eta_T(z_1 \cdot_T z_2) \]  \hspace{1cm} (23)

for any $z_1 \in KT_*(A)$ and $z_2 \in KT_*(B)$.

First, we suppose $z_1 = \varepsilon(x_1)$ for some $x_1 \in KO_*(A)$. In this case,

\[ \varepsilon_T(z_1 \cdot_T z_2) = \varepsilon_T(\varepsilon(x_1) \cdot_T z_2) = (-1)^{|x_1|} \varepsilon(x_1 \cdot_o \tau(z_2)) = (-1)^{|x_1|} \varepsilon(x_1) \cdot_T \varepsilon_T(z_2) = (-1)^{|z_1|} z_1 \cdot_T \varepsilon_T(z_2). \]
We also have

\[ \epsilon \tau(z_1) \cdot_T z_2 = \epsilon \tau(\epsilon(x_1)) \cdot_T z_2 \]
\[ = \epsilon(\eta_o \cdot_o x_1) \cdot_T z_2 \]
\[ = \eta_t(\epsilon(x_1) \cdot_T z_2) \]
\[ = -\eta_t(z_1 \cdot_T z_2) \]

(since \( 2\eta_t = 0 \)) so Equation 23 holds in this case.

Similarly, if \( z_2 = \epsilon(x_2) \) for some \( x_2 \in KO_*(B) \) we have \( \epsilon \tau(z_1 \cdot_T z_2) = \epsilon \tau(z_1) \cdot_T z_2 \) and \( (-1)^{|z_1|}z_1 \cdot_T \epsilon \tau(z_2) = -\eta_t(z_1 \cdot_T z_2) \) and so Equation 23 holds in this case, too.

Next, we prove the claim that Equation 23 holds for a pair of elements \( z_1 \) and \( z_2 \) if and only if it holds for the pair \( \beta_T z_1 \) and \( z_2 \). Indeed, suppose that it holds for \( z_1 \) and \( z_2 \). Then

\[ \epsilon \tau(\beta_T z_1 \cdot z_2) = \epsilon \tau(\beta_T z_1 \cdot z_2) \]
\[ = \beta_T \epsilon \tau(z_1 \cdot z_2) + \eta_t \beta_T(z_1 \cdot z_2) \]
\[ = \beta_T(\epsilon \tau z_1 \cdot z_2 + (-1)^{|z_1|}z_1 \cdot \epsilon \tau z_2 + \eta_t z_1 \cdot z_2) + \beta_T \eta_t(z_1 \cdot z_2) \]
\[ = \epsilon \tau(\beta_T z_1) \cdot z_2 + \eta_t(\beta_T z_1 \cdot z_2) + (-1)^{|z_1|} \beta_T z_1 \cdot \epsilon \tau(z_2) \]

using the relations \( \beta_T \epsilon \tau = \epsilon \tau + \eta_t \beta_T \) and \( 2\eta_t = 0 \). Conversely, suppose that Equation 23 holds for \( \beta_T z_1 \) and \( z_2 \). Then repeating the above calculation, we find that it holds for \( \beta_T^2 z_1 \) and \( z_2 \). But \( \beta_T^2 = \beta_o \) and all operations commute with \( \beta_o \). Since \( \beta_o \) is invertible, Equation 23 holds for \( z_1 \) and \( z_2 \).

Furthermore, because of the symmetry, Equation 23 holds for a pair \( z_1 \) and \( z_2 \) if and only if it holds for the pair \( z_2 \) and \( z_1 \).

We are now ready to prove Equation 23 in general. It suffices to assume that \( z_1 \) and \( z_2 \) are both elements in degree zero and that \( A \) and \( B \) are unital. Furthermore, we may assume that \( z_1 \) is represented by a projection in \( T \otimes A \) and that \( z_2 \) is represented by a projection in \( T \otimes B \) (rather than differences of projections in matrix algebras over \( T \otimes A \) and \( T \otimes B \)). We will prove Equation 23 for the pair \( \beta_T z_1 \) and \( \beta_T z_2 \). Using Lemma 2.3, let \( \mu_1: T \to M_{16}(S^+ A) \) and \( \mu_2: T \to M_{16}(S^+ B) \) be homomorphisms such that the equations \( (\mu_1)_*(\chi) = z_1 \in KT_{-1}(S^+ A) \) and \( (\mu_2)_*(\chi) = z_2 \in KT_{-1}(S^+ B) \), hold modulo \( \beta_T^{-1} \epsilon \). (Recall that \( \chi \in KT_{-1}(T) \) is the free generator of \( KT^{\text{crt}}(T) \).) Then the equations \( (\mu_1)_*(\beta_T \chi) = \beta_T z_1 \) and \( (\mu_2)_*(\beta_T \chi) = \beta_T z_2 \) hold modulo \( \epsilon \).

Using these maps, it suffices to consider the case \( A = B = T \) with the elements \( z_1, z_2 \in KT_3(T) \). Using the periodicity isomorphism \( \beta_T \) and the claim above, it is enough to consider
$z_1, z_2 \in KT_{-1}(T)$. In fact, we will establish the formula for any elements $z_1$ and $z_2 \in KT_{-1}(T)$.

We first prove that $KT_{-1}(T)$ is generated by the image of $\varepsilon : KO_{-1}(T) \to KT_{-1}(T)$ and the image of $i_* : KT_{-1}(\mathbb{R}) \to KT_{-1}(T)$ where $i$ is the inclusion $\mathbb{R} \hookrightarrow T$. From Table 3, we know that the image of $\varepsilon$ is the +1 eigenspace of $\psi_T$, forming a free summand of $KT_{-1}(T) \cong \mathbb{Z} \oplus \mathbb{Z}$. On the other hand, $\psi_T$ is multiplication by $-1$ on $KT_{-1}(\mathbb{R}) \cong \mathbb{Z}$ by Table 1; and therefore, the image of $i_*$ is contained in the $-1$ eigenspace of $\psi_T$.

Now consider the involution $\iota : T \otimes T \to T \otimes T$ which interchanges the two factors of the tensor product. The map induced by $\iota$ on $KT_{-1}(T) \cong K_{-1}(T \otimes T)$ interchanges the image of $\varepsilon$ and the image of $i_*$. Therefore $KT_{-1}(T)$ can be written as the direct sum of the image of $\varepsilon$ and the image of $i_*$, each of which is isomorphic to $\mathbb{Z}$.

Since Equation 23 holds if either $z_1$ or $z_2$ is in the image of $\varepsilon$, it suffices to suppose that both $z_1$ and $z_2$ are in the image of $i_*$. Therefore, it suffices to demonstrate that Equation 23 holds for $KT_{-1}(\mathbb{R}) \cong \mathbb{Z}$.

Now $KT_{-1}(\mathbb{R})$ is generated by $\beta_{-1}^{-1}\omega$. Referring to Table 4 we have $(\beta_{-1}^{-1}\omega)^2 = 0$, $\eta_T \beta_T = 0$, and $\varepsilon \tau (\beta_{-1}^{-1}\omega) = 2 \cdot 1_T$. Thus

\[
\varepsilon \tau (\beta_{-1}^{-1}\omega) \cdot_T \beta_{-1}^{-1}\omega + \beta_{-1}^{-1}\omega \cdot_T \varepsilon \tau (\beta_{-1}^{-1}\omega) + \eta_T (\beta_{-1}^{-1}\omega \cdot_T \beta_{-1}^{-1}\omega) \\
= 2 \cdot 1_T \cdot_T \beta_{-1}^{-1}\omega - \beta_{-1}^{-1}\omega \cdot_T 2 \cdot 1_T + 0 \\
= \varepsilon \tau (\beta_{-1}^{-1}\omega \cdot_T \beta_{-1}^{-1}\omega)
\]

proving that Equation 23 holds for $z_1 = z_2 = \beta_{-1}^{-1}\omega$. \hfill \qed

Now that the homomorphism $\alpha$ is defined, we are finally prepared to understand the statement of the main theorem. Recall that $\mathcal{N}$ is the bootstrap category of complex $C^*$-algebras described in the introduction.

**Theorem 4.2 (Künneth Theorem).** Let $A$ and $B$ be real $C^*$-algebras such that $\mathbb{C} \otimes A$ is in $\mathcal{N}$. Then there is a natural short exact sequence

\[
0 \longrightarrow K^{CRT}(A) \otimes_{CRT} K^{CRT}(B) \xrightarrow{\alpha} K^{CRT}(A \otimes B) \xrightarrow{\beta} \text{Tor}^{CRT}(K^{CRT}(A), K^{CRT}(B)) \longrightarrow 0 \quad (24)
\]

which does not necessarily split. Here $\alpha$ is the pairing described above and $\beta$ is a CRT homomorphism of degree $-1$. 

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We will prove this theorem in the next section. In Section 5.2 we will come across examples which show that the sequence does not split.

The following corollary states that the subcategory of C*-algebras $A$ for which the Künneth sequence holds for all $B$ is at least as large as we would expect. Let $\mathcal{N}_R$ be the bootstrap category of real C*-algebras: the smallest subcategory of real separable C*-algebras whose complexification is nuclear which contains real separable type I C*-algebras; which is closed under the operations of taking inductive limits, stable isomorphisms, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$; and which satisfies the two out of three rule for short exact sequences.

**Corollary 4.3.** Let $A$ and $B$ be real C*-algebras with $A \in \mathcal{N}_R$. Then the Künneth sequence is exact for $A$ and $B$.

**Proof.** Let $A_c$ denote the complexification $\mathbb{C} \otimes A$ of $A$. It suffices to show that the collection of real C*-algebras $A$ such that $A_c \in \mathcal{N}$ contains real separable type I C*-algebras; is closed under the operations of taking inductive limits, stable isomorphisms, and crossed products by $\mathbb{Z}$ and $\mathbb{R}$; and satisfies the two out of three rule for short exact sequences.

Suppose that $A$ is a real, separable, type I C*-algebra. Then $A_c$ is certainly separable. We show that $A_c$ is type I. Let $\pi$ be a non-zero, irreducible representation of $A_c$ on a complex Hilbert space $\mathcal{H}$. Then the restriction $\pi|_A$ of $\pi$ to $A$ is a non-zero representation of $A$ on the Hilbert space $\mathcal{H}$ (thought of as a real Hilbert space). If $\pi|_A$ is irreducible, then $\pi|_A(A_c)$ would contain a compact operator (since $A$ is type I) and then so must $\pi(A_c)$.

On the other hand, if $\pi|_A$ is not irreducible, then there is a non-zero $\xi \in H$ such that $E_1 = \pi(A)\xi \subseteq H$. Let $E_2 = \pi(iA)\xi = iE_1$. Then $E_1 + E_2 = \pi(A_c)\xi = H$. Now, $E_1 \cap E_2$ is a complex $A_c$-invariant subspace of $H$. Since $\pi$ is irreducible, either $E_1 \cap E_2 = 0$ or $E_1 \cap E_2 = H$. However, $E_1 \cap E_2 = H$ implies $E_1 = H$, contrary to hypothesis. Hence $E_1 \cap E_2 = 0$, so $H = E_1 \oplus E_2$. In this case, the representation $\pi$ of $A_c$ on $H$ is isomorphic to the complexification of the representation $\pi$ of $A$ on $E_1$. Since the former is irreducible, so is the latter. Therefore the latter contains a compact operator, and so does the former.

Either way, $\pi(A_c)$ contains a compact operator and therefore $A_c$ is a type I C*-algebra.

The operation of complexification commutes with the operations of taking inductive limits, of stabilizing, and of taking crossed products by by $\mathbb{Z}$ and by $\mathbb{R}$. Therefore the category in question is closed under these four operations.

Finally, since complexification preserves exact sequences, the collection of C*-algebras whose complexification is in $\mathcal{N}$ satisfies the two out of three rule. \qed
4.2 The Proof of the Küneth Formula

We first consider the case that $K^{CRT}(B)$ is a free CRT-module. In this case, the CRT-module $\text{Tor}_{CRT}(K^{CRT}(A), K^{CRT}(B))$ vanishes and so the claim that Sequence 24 is exact collapses to the statement that $\alpha$ is an isomorphism. This is proven below in Proposition 4.4. Once that is accomplished, we will prove Theorem 4.2 in the general case by using a geometric resolution of $B$.

Proposition 4.4. Let $A$ and $B$ be real C*-algebras such that $K^{CRT}(B)$ is free and $\mathbb{C} \otimes A \in \mathcal{N}$. Then $\alpha(A, B)$ is an isomorphism.

Proof. By Theorem 1.18 and Proposition 3.10, $K^{CRT}(A) \otimes_{CRT} K^{CRT}(B)$ and $K^{CRT}(A \otimes B)$ are both acyclic objects. By Proposition 1.14, it suffices to show that $\alpha_U : (K^{CRT}(A) \otimes_{CRT} K^{CRT}(B))^U \rightarrow (K^{CRT}(A \otimes B))^U$ is an isomorphism. Using Lemma 3.9, this homomorphism can be rewritten as

$$
\alpha_U : KU_*(A) \otimes_{KU_*}(\mathbb{R}) KU_*(B) \rightarrow KU_*(A \otimes B)
$$

Now since $K^{CRT}(B)$ is a free CRT-module, $KU_*(B)$ is a free abelian group by Theorem 3.2 of [3]. Furthermore $\mathbb{C} \otimes A$ is in the complex bootstrap category $\mathcal{N}$, so the Küneth formula for complex C*-algebras (Theorem 2.14 of [22]) shows that $\alpha_U$ is an isomorphism. 

The next proposition and the lemmas which follow will complete the proof of the Küneth Theorem.

Proposition 4.5. Suppose $A$ is a real C*-algebra such that $\mathbb{C} \otimes A$ is nuclear and $\alpha(A, B)$ is an isomorphism whenever $K^{CRT}(B)$ is a free CRT-module. Then for any C*-algebra $B$, the Küneth formula holds for the pair $(A, B)$.

Proof. Let $A$ be as specified in the hypothesis. Let $B$ be an arbitrary unital C*-algebra. Then, using Theorem 2.2, let

$$
0 \rightarrow F_1 \xrightarrow{\mu_1} F_0 \xrightarrow{\mu_0} K(H) \otimes \mathcal{G}B \rightarrow 0
$$

be a sequence of C*-algebras such that

$$
0 \rightarrow K^{CRT}(F_1) \xrightarrow{(\mu_1)_*} K^{CRT}(F_0) \xrightarrow{(\mu_0)_*} K^{CRT}(\mathcal{G}B) \rightarrow 0
$$

be a sequence of C*-algebras such that

$$
0 \rightarrow K^{CRT}(F_1) \xrightarrow{(\mu_1)_*} K^{CRT}(F_0) \xrightarrow{(\mu_0)_*} K^{CRT}(\mathcal{G}B) \rightarrow 0
$$
is a free resolution of $K^{\text{CRT}}(\mathcal{S}B)$.

Now, we take Sequence 25 and tensor everything on the left by the C*-algebra $A$ obtaining

$$0 \longrightarrow A \otimes F_1 \xrightarrow{1 \otimes \mu_1} A \otimes F_0 \xrightarrow{1 \otimes \mu_0} A \otimes K(\mathcal{H}) \otimes \mathcal{S}B \longrightarrow 0.$$ (27)

The condition that $C \otimes A$ is nuclear insures that tensoring by $A$ preserves exact sequences. We unsplice the induced long exact sequence on united $K$-theory, forming

$$0 \longrightarrow \text{coker}((1 \otimes \mu_1)_*) \xrightarrow{(1 \otimes \mu_0)*} K^{\text{CRT}}(A \otimes \mathcal{S}B) \xrightarrow{\delta} \ker((1 \otimes \mu_1)_*) \longrightarrow 0$$

where the connecting homomorphism $\delta$ is a map of degree $-1$.

It remains to make the identifications

$$\ker((1 \otimes \mu_1)_*) = \text{Tor}^{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(\mathcal{S}B))$$

$$\text{coker}((1 \otimes \mu_1)_*) = K^{\text{CRT}}(A) \otimes_{\text{CRT}} K^{\text{CRT}}(\mathcal{S}B)$$

which is done using the following commutative diagram.

$$\begin{array}{ccc}
0 & \longrightarrow & \text{Tor}^{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(\mathcal{S}B)) \\
\downarrow & & \downarrow \\
K^{\text{CRT}}(A) \otimes_{\text{CRT}} K^{\text{CRT}}(F_1) & \xrightarrow{\alpha(A,F_1)} & K^{\text{CRT}}(A \otimes F_1) \\
\downarrow 1 \otimes (\mu_1)_* & & \downarrow (1 \otimes \mu_1)_* \\
K^{\text{CRT}}(A) \otimes_{\text{CRT}} K^{\text{CRT}}(F_0) & \xrightarrow{\alpha(A,F_0)} & K^{\text{CRT}}(A \otimes F_0) \\
\downarrow 1 \otimes (\mu_0)_* & & \downarrow (1 \otimes \mu_0)_* \\
K^{\text{CRT}}(A) \otimes_{\text{CRT}} K^{\text{CRT}}(\mathcal{S}B) & \xrightarrow{\alpha(A,\mathcal{S}B)} & K^{\text{CRT}}(A \otimes \mathcal{S}B) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$ (28)

The vertical sequence on the left is exact, derived from the resolution given by Sequence 26. The vertical sequence on the right is also exact, induced by Sequence 27. The horizontal maps $\alpha(A, F_1)$ and $\alpha(A, F_0)$ are isomorphisms since $K^{\text{CRT}}(F_0)$ and $K^{\text{CRT}}(F_1)$ are free. The diagram then shows that the kernel of $(1 \otimes \mu_1)_*$ is isomorphic to the kernel of $1 \otimes (\mu_1)_*$ which, in turn, is isomorphic to $\text{Tor}^{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(\mathcal{S}B))$. Composing $\delta$ with this isomorphism, we obtain the homomorphism $\beta : K^{\text{CRT}}(A \otimes \mathcal{S}B) \rightarrow \text{Tor}^{\text{CRT}}(K^{\text{CRT}}(A), K^{\text{CRT}}(\mathcal{S}B))$ of the K"unneth formula.
Similarly, Diagram 28 shows that the cokernel of \((1 \otimes \mu)_*\) is isomorphic to the cokernel of \(1 \otimes (\mu)_*\), which, in turn, is isomorphic to \(K^{CRT}(A) \otimes_{CRT} K^{CRT}(\mathcal{G}B)\). Furthermore, in this last identification, the composition
\[
K^{CRT}(A) \otimes K^{CRT}(\mathcal{G}B) \to \text{coker}(1 \otimes (\mu)_*) \to K^{CRT}(A \otimes \mathcal{G}B)
\]
is the homomorphism \(\alpha(A, \mathcal{G}B)\).

This proves that the Künneth formula holds for the pair \((A, \mathcal{G}B)\). Recall that \(\mathcal{G}B\) is the eight-fold unital suspension of \(B\). By repeatedly applying Lemmas 4.6 and 4.7 below, it follows that the Künneth formula holds for the pair \((A, B)\).

A further application of Lemma 4.7 shows that the Künneth formula holds if \(B\) is not unital. \(\square\)

**Lemma 4.6.** The Künneth formula is satisfied for the pair \((A, B)\) if and only if it is satisfied for the pair \((A, SB)\).

**Proof.** Consider the following diagram.

\[
\begin{array}{ccccccc}
  0 & \rightarrow & K^{CRT}(A) \otimes_{CRT} K^{CRT}(B) & \rightarrow & K^{CRT}(A) \otimes_{CRT} K^{CRT}(SB) & \rightarrow & 0 \\
  & & \downarrow \alpha(A, B) & & \downarrow \alpha(A, SB) & & \\
  & & K^{CRT}(A \otimes B) & \rightarrow & K^{CRT}(A \otimes SB) & \rightarrow & 0 \\
  & & \downarrow \beta(A, B) & & \downarrow \beta(A, SB) & & \\
  & & \text{Tor}_{CRT}(K^{CRT}(A), K^{CRT}(B)) & \rightarrow & \text{Tor}_{CRT}(K^{CRT}(A), K^{CRT}(SB)) & \rightarrow & 0 \\
  & & 0 & & 0 & & \\
\end{array}
\]

The horizontal maps in this diagram are all CRT-module isomorphisms with degree \(-1\). Therefore, there exists a homomorphism \(\beta(A, B)\) making the left vertical sequence exact if and only if there exists a homomorphism \(\beta(A, SB)\) making the right vertical sequence exact. \(\square\)

**Lemma 4.7.** Let \(A\) and \(B\) be real C*-algebras with \(\mathbb{C} \otimes A\) nuclear. Suppose that the Künneth Formula holds for the pair \((A, B^+)\). Then it holds for the pair \((A, B)\).
Proof. We consider the short exact sequence

$$0 \rightarrow B \xrightarrow{i} B^+ \xrightarrow{\pi} \mathbb{R} \rightarrow 0.$$  

Because of the presence of a section $s: \mathbb{R} \rightarrow B^+$ such that $\pi \circ s = 1$ (and because $\mathbb{C} \otimes A$ is nuclear) the vertical sequences in the diagram below are split exact.

\[
\begin{array}{ccc}
0 & \quad & 0 \\
\downarrow & & \downarrow \\
K_{CRT}(A) \otimes_{CRT} K_{CRT}(B) & \xrightarrow{\alpha(A,B)} & K_{CRT}(A \otimes B) \\
\downarrow_{1 \otimes i_*} & & \downarrow_{(1 \otimes i)_*} \\
K_{CRT}(A) \otimes_{CRT} K_{CRT}(B^+) & \xrightarrow{\alpha(A,B^+)} & K_{CRT}(A \otimes B^+) \\
\downarrow_{1 \otimes \pi_*} & & \downarrow_{(1 \otimes \pi)_*} \\
K_{CRT}(A) \otimes_{CRT} K_{CRT}(\mathbb{R}) & \xrightarrow{\alpha(A,R)} & K_{CRT}(A \otimes \mathbb{R}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

Furthermore, the second and third rows are exact by hypothesis. In the first row, we define $\beta(A, B)$ to be the composition $(1 \otimes i)^{-1} \circ \beta(A, B^+) \circ (1 \otimes i)_*$. This makes the first row a complex and by the snake lemma (Exercise 6.16 of [21]) it is exact. □

The proof of Theorem 4.2 is completed by the following Proposition.

**Proposition 4.8.** The Künneth Sequence is natural with respect to homomorphisms of real $C^*$-algebras in either variable.

**Proof.** It is clear that $\alpha$ is natural. So it suffices to show that $\beta$ is natural with respect to either argument. First, we show that the homomorphism $\beta$ is independent of the choice of geometric resolution. Indeed, suppose that

$$0 \rightarrow E_1 \xrightarrow{\lambda_1} E_0 \xrightarrow{\lambda_0} \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B \rightarrow 0$$

and

$$0 \rightarrow F_1 \xrightarrow{\mu_1} F_0 \xrightarrow{\mu_0} \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B \rightarrow 0$$

are two geometric resolutions as in Theorem 2.2. Then we can form a third geometric resolution

$$0 \rightarrow D_1 \xrightarrow{\kappa_1} D_0 \xrightarrow{\kappa_0} \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B \rightarrow 0$$

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by defining $D_0 = E_0 \oplus F_0$ and $D_1$ to be the kernel of
\[
\kappa_0 = (\mu_0 \lambda_0): D_0 \to \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B.
\]

Then we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \equiv & & \\
0 & \rightarrow & D_1 & \rightarrow & D_0 & \rightarrow & \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B & \rightarrow & 0
\end{array}
\]

where $F_0 \rightarrow D_0$ is the inclusion onto the second summand of $D_0 = E_0 \oplus F_0$. This homomorphism of resolutions induces a homomorphism from each group of Diagram 28 to the corresponding group of a similar diagram based on the resolution with $D_1$ and $D_0$. The resulting commutative diagram shows that the homomorphism $\beta$ based on the resolution $D_1$ and $D_0$ is the same as the homomorphism $\beta$ based on the resolution with $F_1$ and $F_0$. Similarly, a diagram based on the inclusion onto the first summand shows that this homomorphism $\beta$ is the same as the one based on the resolution with $E_1$ and $E_0$. Therefore $\beta$ does not depend on the particular geometric resolution.

Now, if $\phi: A \rightarrow A'$ is a homomorphism of real $C^*$-algebras, then we choose a fixed resolution of $\mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B$ to compute $\beta(A, B)$ and $\beta(A', B)$. Because $\alpha$ is natural, there is a homomorphism induced by $\phi$ from each group of Diagram 28 to the corresponding groups of the same diagram with $A$ replaced by $A'$. Since $\beta$ is defined in terms of $\alpha$ by way of this commutative diagram, the diagram shows that $\beta$ is natural with respect to homomorphisms in the first argument.

Finally, suppose that we have a real $C^*$-algebra homomorphism $\phi: B \rightarrow B'$. Let $\mu: F \rightarrow \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B$ and $\mu': F' \rightarrow \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B'$ both be given by Proposition 2.1. Let $\phi$ also denote the induced homomorphism from $\mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B$ to $\mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B'$. Then there is a commutative diagram

\[
\begin{array}{ccc}
F & \xrightarrow{\mu} & \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B \\
\downarrow (1_0) & & \downarrow \phi \\
F \oplus F' & \xrightarrow{\mu''} & \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B'
\end{array}
\]

where $\mu'' = (\phi \circ \mu \circ \mu')$. The naturality of the mapping cone construction produces the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \phi & & \downarrow & & \\
0 & \rightarrow & F_1' & \rightarrow & F_0' & \rightarrow & \mathcal{K}(\mathcal{H}) \otimes \mathfrak{S}B' & \rightarrow & 0
\end{array}
\]
where $F_1$ and $F_0$ are the mapping cone and mapping cylinder of $\mu$ while $F'_1$ and $F'_0$ are the mapping cone and mapping cylinder of $\mu'$. This is a homomorphism from one geometric resolution to the other and it induces a family of homomorphisms from the groups of Diagram $\mathbb{D}$ to the groups of the same diagram with $B$ replaced by $B'$. Since $\beta$ does not depend on the particular geometric resolution, it follows that $\beta$ commutes with $\phi$.

5 Application: Real Cuntz Algebras

5.1 The United $K$-Theory of the Real Cuntz Algebras

For $k \geq 1$, the real Cuntz algebra $\mathcal{O}_{k+1}^R$ is the universal real C*-algebra generated by $k + 1$ isometries $S_1, S_2, \ldots, S_{k+1}$ subject to the relation $\sum_{i=1}^{k+1} S_i S_i^* = 1$ (see [24], page 4). The complexification of $\mathcal{O}_{k+1}^R$ is the complex Cuntz algebra $\mathcal{O}_{k+1}$ introduced in [9].

The $K$-theory of the complex Cuntz algebras was computed by Cuntz in [10] while the ordinary $K$-theory of the real Cuntz algebras was computed by Schröder (Theorem 1.6.8 in [24]). However, this last computation is not entirely correct, as it involves a mistaken solution to a certain extension problem. In this section we compute the united $K$-theory of the real Cuntz algebras. In particular, we find the correct real $K$-groups. In fact, it is the structure of united $K$-theory — namely, the relationship between real and complex $K$-theory — that allowed us to detect Schröder’s error and correctly solve the extension problem.

The form that $K_{\text{CRT}}(\mathcal{O}_{k+1}^R)$ takes depends on the congruence class of $k$ modulo 4. Tables 4, 5, and 6 below show the united $K$-theory in the case that $k$ is odd, the case that $k$ is congruent to 2 modulo 4, and the case that $k$ is congruent to 0 modulo 4, respectively.

Computation of Tables 4, 5, and 6. The complex $K$-theory of the real Cuntz algebras is the same as the ordinary $K$-theory of the complex Cuntz algebras. Hence, we refer to [10] for the values of $KU_*(\mathcal{O}_{k+1}^R)$. Furthermore, since $KU_0(\mathcal{O}_{k+1}^R) \cong \mathbb{Z}_k$ is generated by the class of the identity in $\mathcal{O}_{k+1}^R \subseteq \mathbb{C} \otimes \mathcal{O}_{k+1}^R$, we know that $KU_0(\mathcal{O}_{k+1}^R)$ is in the image of $c_0$. It follows that $(\psi_u)_n$ is the trivial involution for $n \equiv 0$ (mod 4) and multiplication by $-1$ for $n \equiv 2$ (mod 4).

Let $\phi_i : M_{(k+1)i} \rightarrow M_{(k+1)i+1}$ be the matrix algebra embedding of multiplicity $k + 1$ and let $A_{(k+1)\infty}$ be the real UHF algebra obtained by taking the direct limit of the system $\{M_{(k+1)i}, \phi_i\}$. We find the real $K$-theory of $A_{(k+1)\infty}$ in degrees 0 through 8 using the fact
Table 4: $K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R})$ for $k$ odd

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $\mathbb{Z}_k$ | 0 | 0 | 0 | $\mathbb{Z}_k$ | 0 | 0 | 0 | $\mathbb{Z}_k$ |
| $KU_n$ | $\mathbb{Z}_k$ | 0 | $\mathbb{Z}_k$ | 0 | $\mathbb{Z}_k$ | 0 | $\mathbb{Z}_k$ | 0 | $\mathbb{Z}_k$ |
| $KT_n$ | $\mathbb{Z}_k$ | 0 | 0 | $\mathbb{Z}_k$ | $\mathbb{Z}_k$ | 0 | 0 | $\mathbb{Z}_k$ | $\mathbb{Z}_k$ |
| $c_n$ | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 |
| $r_n$ | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| $\varepsilon_n$ | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 |
| $\zeta_n$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $(\psi_U)_n$ | 1 | 0 | $-1$ | 0 | 1 | 0 | $-1$ | 0 | 1 |
| $(\psi_T)_n$ | 1 | 0 | 0 | $-1$ | 1 | 0 | 0 | $-1$ | 1 |
| $\gamma_n$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $\tau_n$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 |

that $K$-theory passes through direct limits:

\[
KO_*(A_{(k+1)\infty}) = \begin{cases} 
\mathbb{Z}[\frac{1}{k+1}] & \text{for } k + 1 \text{ odd} \\
\mathbb{Z}_2 & \text{for } k + 1 \text{ even}
\end{cases}
\]

There is an automorphism $\alpha$ of $\mathcal{K} \otimes A_{(k+1)\infty}$ which induces multiplication by $\frac{1}{k+1}$ on real $K$-theory such that there is an isomorphism

$\mathcal{K} \otimes \mathcal{O}_{k+1}^\mathbb{R} \cong (\mathcal{K} \otimes A_{(k+1)\infty}) \times_{\alpha} \mathbb{Z}$.

(See page 51 of [24], following [1] in the complex case.)

We compute the $K$-theory of this crossed product, and hence of $\mathcal{O}_{k+1}^\mathbb{R}$, using the real $K$-theory version of the Pimsner-Voiculescu sequence (found as Theorem 1.5.5 in [24], following [19] in the complex case):

$KO_{n+1}(\mathcal{O}_{k+1}^\mathbb{R}) \to KO_{n}(A_{(k+1)\infty}) \xrightarrow{1-\alpha_*} KO_{n}(A_{(k+1)\infty}) \to KO_{n}(\mathcal{O}_{k+1}^\mathbb{R}) \to$

In the cases where $KO_{n}(A_{(k+1)\infty}) \cong \mathbb{Z}[\frac{1}{k+1}]$, the homomorphism $1 - \alpha_* = 1 - \frac{1}{k+1} = \frac{k}{k+1}$ is injective with cokernel isomorphic to $\mathbb{Z}_k$. In the cases where $k + 1$ is odd and $n$ is congruent to 1 or 2 (mod 8), the homomorphism $1 - \alpha_*$ is zero.

Using this, the Pimsner-Voiculescu sequence almost completely determines $KO_*(\mathcal{O}_{k+1}^\mathbb{R})$; the exception is $KO_2(\mathcal{O}_{k+1}^\mathbb{R})$ which is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$ in case $k + 1$ is odd. We show that it is $\mathbb{Z}_4$ for $k \equiv 2 \pmod{4}$ and it is $\mathbb{Z}_2^2$ for $k \equiv 0 \pmod{4}$.

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Table 5: $K^{CRT}(O_{k+1}^Z)$ for $k \equiv 2 \pmod{4}$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $Z_k$ | $Z_2$ | $Z_4$ | $Z_k$ | 0 | 0 | 0 | $Z_k$ | $Z_k$ |
| $KU_n$ | $Z_k$ | 0 | $Z_k$ | 0 | $Z_k$ | 0 | $Z_k$ | 0 | $Z_k$ |
| $KT_n$ | $Z_k$ | $Z_2$ | $Z_2$ | $Z_k$ | $Z_2$ | $Z_2$ | $Z_k$ | $Z_k$ | $Z_k$ |

- $e_n$: 1 0 $\frac{k}{2}$ 0 2 0 0 0 1
- $r_n$: 2 0 2 0 1 0 0 0 2
- $\varepsilon_n$: 1 1 1 $\frac{k}{2}$ 2 0 0 0 1
- $\zeta_n$: 1 0 $\frac{k}{2}$ 0 1 0 $\frac{k}{2}$ 0 1
- $(\psi_U)_n$: 1 0 $-1$ 0 1 0 $-1$ 0 1
- $(\psi_T)_n$: 1 1 1 $-1$ 1 1 1 $-1$ 1
- $\gamma_n$: 1 0 1 0 1 0 1 0 1
- $\tau_n$: 1 2 1 1 0 0 0 2 1

Table 6: $K^{CRT}(O_{k+1}^Z)$ for $k \equiv 0 \pmod{4}$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $Z_k$ | $Z_2$ | $Z_4$ | $Z_2$ | $Z_k$ | 0 | 0 | 0 | $Z_k$ |
| $KU_n$ | $Z_k$ | 0 | $Z_k$ | 0 | $Z_k$ | 0 | $Z_k$ | 0 | $Z_k$ |
| $KT_n$ | $Z_k$ | $Z_2$ | $Z_2$ | $Z_k$ | $Z_2$ | $Z_2$ | $Z_k$ | $Z_k$ | $Z_k$ |

- $e_n$: 1 0 $\left(\begin{array}{c}0 \\ \frac{k}{2}\end{array}\right)$ 0 2 0 0 0 1
- $r_n$: 2 0 $\left(\begin{array}{c}1 \\ 0\end{array}\right)$ 0 1 0 0 0 2
- $\varepsilon_n$: 1 1 $\left(\begin{array}{c}0 \\ 1\end{array}\right)$ $\frac{k}{2}$ 2 0 0 0 1
- $\zeta_n$: 1 0 $\frac{k}{2}$ 0 1 0 $\frac{k}{2}$ 0 1
- $(\psi_U)_n$: 1 0 $-1$ 0 1 0 $-1$ 0 1
- $(\psi_T)_n$: 1 1 1 $-1$ 1 1 1 $-1$ 1
- $\gamma_n$: 1 0 1 0 1 0 1 0 1
- $\tau_n$: 1 $\left(\begin{array}{c}1 \\ 0\end{array}\right)$ 1 1 0 0 0 2 1

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We will use the exact sequence

\[ \cdots \rightarrow KO_n(\mathcal{O}_{k+1}^\mathbb{R}) \xrightarrow{\eta_0} KO_{n+1}(\mathcal{O}_{k+1}^\mathbb{R}) \xrightarrow{c} KU_{n+1}(\mathcal{O}_{k+1}^\mathbb{R}) \xrightarrow{r\psi^{-1}} KO_{n-1}(\mathcal{O}_{k+1}^\mathbb{R}) \rightarrow \cdots. \] (29)

Since \( KU_3(\mathcal{O}_{k+1}^\mathbb{R}) \cong 0 \), we deduce that \( (\eta_0)_1 \) must be injective and \( (\eta_0)_2 \) must be surjective.

Suppose first that \( KO_2(\mathcal{O}_{k+1}^\mathbb{R}) \cong \mathbb{Z}_4 \). Then \( (\eta_0)_1 = 2 \) and \( (\eta_0)_2 = 1 \). Now the image of \( (\eta_0)_1 \) is the kernel of \( c_2 \), so we conclude that \( c_2 = \frac{k}{2} \). Also, since the kernel of \( (\eta_0)_2 \) is the image of \( r_2 \), we conclude that \( r_2 = 2 \). Therefore, the relation \( rc = 2 \) forces \( k \equiv 2 \) (mod 4).

On the other hand, suppose that \( KO_2(\mathcal{O}_{k+1}^\mathbb{R}) \cong \mathbb{Z}_2^2 \). Since \( (\eta_0)_1 \) is injective we can write \( (\eta_0)_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^2 \) (written in terms of appropriately chosen generators of \( KO_2(\mathcal{O}_{k+1}^\mathbb{R}) \)). Now \( KO_1(\mathcal{O}_{k+1}^\mathbb{R}) \) is in the image of \( \eta_0 \). Since \( \eta_0^3 = 0 \) we know that \( (\eta_0)_2 \) vanishes on the image of \( (\eta_0)_1 \). But \( (\eta_0)_2 \) is non-zero, so we conclude that \( (\eta_0)_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

Using Sequence 29 again, we determine \( r_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( c_2 = \begin{pmatrix} 0 \\ \frac{k}{2} \end{pmatrix} \). Then \( rc = 0 \) implies \( k \equiv 0 \) (mod 2), which implies \( k \equiv 0 \) (mod 4).

This completes the computation of \( KO_*(\mathcal{O}_{k+1}^\mathbb{R}) \). Now the computation of \( KT_*(\mathcal{O}_{k+1}^\mathbb{R}) \) and the remaining operations proceeds without difficulty using the exact sequence

\[ \cdots \rightarrow KU_{n+1}(\mathcal{O}_{k+1}^\mathbb{R}) \xrightarrow{\gamma} KT_n(\mathcal{O}_{k+1}^\mathbb{R}) \xrightarrow{\zeta} KU_n(\mathcal{O}_{k+1}^\mathbb{R}) \xrightarrow{1-\psi_U} KU_n(\mathcal{O}_{k+1}^\mathbb{R}) \rightarrow \cdots, \]

and the known values of \( \psi_U \), as well as the relations \( rc = 2 \), \( c = \zeta \varepsilon \), \( r = \tau \gamma \), and \( \varepsilon r \zeta = 1 + \psi_T \).

5.2 The United \( K \)-Theory of Tensor Products of Real Cuntz Algebras

In this section, we compute the united \( K \)-theory of \( \mathcal{O}_{k+1}^\mathbb{R} \otimes \mathcal{O}_{l+1}^\mathbb{R} \) where \( k \) and \( l \) are positive integers. Throughout, let \( g = \text{gcd}(k, l) \). Also if \( k \) and \( l \) are both even, let \( k', l' \in \mathbb{Z}_2 \) be defined by

\[
k' = \begin{cases} 
0 & \text{if } \frac{k}{2} \equiv 0 \pmod{g} \\
1 & \text{if } \frac{k}{2} \equiv \frac{g}{2} \pmod{g}
\end{cases}
\]

and

\[
l' = \begin{cases} 
0 & \text{if } \frac{l}{2} \equiv 0 \pmod{g} \\
1 & \text{if } \frac{l}{2} \equiv \frac{g}{2} \pmod{g}
\end{cases}
\]
The simplest case — to describe and to compute — is when one of the indices is odd. Table 7 shows the united K-theory of $\mathcal{O}_k^{R} \otimes \mathcal{O}_l^{R}$ if $k$ is odd.

If $k$ and $l$ are both even, the united K-theory of $\mathcal{O}_k^{R} \otimes \mathcal{O}_l^{R}$ falls into three cases: both $k$ and $l$ are congruent to 2 modulo 4; both $k$ and $l$ are congruent to 0 modulo 4; and $k$ is congruent to 2 and $l$ is congruent to 0 modulo 4. The results of these computations are shown in Tables 8, 9, and 10 respectively.

Table 7: $K^{\text{CRT}}(\mathcal{O}_k^{R} \otimes \mathcal{O}_l^{R})$ for $k$ odd

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | 0 | 0 | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | 0 | 0 | $\mathbb{Z}_g$ |
| $KU_n$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ |
| $KT_n$ | $\mathbb{Z}_g^2$ | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ | $\mathbb{Z}_g^2$ | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ | $\mathbb{Z}_g^2$ |
| $c_n$ | 1 | 1 | 0 | 0 | 2 | 2 | 0 | 0 | 1 |
| $r_n$ | 2 | 2 | 0 | 0 | 1 | 1 | 0 | 0 | 2 |
| $\epsilon_n$ | $(1 \ 0)$ | 1 | 0 | 0 | $(2 \ 0)$ | 2 | 0 | 0 | $(1 \ 0)$ |
| $\zeta_n$ | $(1 \ 0)$ | 1 | 0 | 0 | $(1 \ 0)$ | 1 | 0 | 0 | $(1 \ 0)$ |
| $\psi_U(n)$ | 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 |
| $\psi_T(n)$ | $(1 \ 0 \ -1)$ | 1 | 0 | -1 | $(1 \ 0 \ -1)$ | 1 | 0 | -1 | $(1 \ 0 \ -1)$ |
| $\gamma_n$ | 1 | $(0 \ 1)$ | 0 | 0 | 1 | $(0 \ 1)$ | 0 | 0 | 1 |
| $\tau_n$ | $(0 \ 2)$ | 0 | 0 | 1 | $(0 \ 1)$ | 0 | 0 | 2 | $(0 \ 2)$ |

Notice that, unlike in the complex case, it is not enough to know the greatest common divisor of $k$ and $l$. Suppose that $k$ and $l$ are both congruent to 2 modulo 4. Then we find the united K-theory in Table 8. If we replace $l$ by $2l$, then we find the united K-theory in Table 10. This change does not change the greatest common divisor, but it does change the K-theory and therefore the isomorphism class of the tensor product.

To be concrete, let $A = \mathcal{O}_3^{R} \otimes \mathcal{O}_3^{R}$ and $B = \mathcal{O}_5^{R} \otimes \mathcal{O}_5^{R}$. Then $A$ and $B$ are simple, real C*-algebras that are not isomorphic, distinguished by K-theory. However, $A_c$ and $B_c$ are simple complex C*-algebra that have the same K-theory and hence are isomorphic by the classification theorems of Phillips and Kirchberg ([18] and [15]).

Computation of Table 7. The complex Cuntz algebras are stably isomorphic to a crossed product of an AF-algebra by $\mathbb{Z}$ (see [8]). Hence all of complex Cuntz algebras are in Schochet’s bootstrap category $\mathcal{N}$. We are therefore entitled to use our Künneth sequence for the united K-theory of the product.
### Table 8: $K^{CRT}(O_{k+1}^R \otimes O_{l+1}^R)$ for $k \equiv l \equiv 2 \pmod{4}$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $KU_n$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $KT_n$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $c_n$ | 1 | 1 | $\left(\frac{1}{2} \ 0 \ \frac{1}{2}\right)$ | 0 | 1 | 2 | 0 | 0 | 1 |
| $r_n$ | 2 | 2 | 0 | $\left(\frac{1}{2} \ 0\right)$ | 2 | 1 | 0 | 0 | 2 |
| $\varepsilon_n$ | $\left(\frac{1}{2} \ 0\right)$ | 1 | $\left(\frac{1}{2} \ 0\right)$ | 0 | $\left(\frac{1}{2} \ 0\right)$ | 0 | $\left(\frac{1}{2} \ 0\right)$ | 0 | $\left(\frac{1}{2} \ 0\right)$ |
| $\zeta_n$ | $\left(\frac{1}{2} \ 0\right)$ | 1 | $\left(\frac{1}{2} \ 0\right)$ | 0 | $\left(\frac{1}{2} \ 0\right)$ | 0 | $\left(\frac{1}{2} \ 0\right)$ | 0 | $\left(\frac{1}{2} \ 0\right)$ |
| $(\psi_u)_n$ | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 |
| $(\psi_T)_n$ | $\left(-\frac{1}{2} \ 0\right)$ | 1 | $\left(-\frac{1}{2} \ 0\right)$ | −1 | $\left(-\frac{1}{2} \ 0\right)$ | 1 | $\left(-\frac{1}{2} \ 0\right)$ | −1 | $\left(-\frac{1}{2} \ 0\right)$ |
| $\gamma_n$ | 2 | $\left(\frac{1}{2} \ 0\right)$ | $\left(\frac{1}{2} \ 0\right)$ | 2 | $\left(\frac{1}{2} \ 0\right)$ | $\left(\frac{1}{2} \ 0\right)$ | 2 | $\left(\frac{1}{2} \ 0\right)$ | $\left(\frac{1}{2} \ 0\right)$ |
| $\tau_n$ | $\left(g+2 \ g\right)$ | $\left(g+2 \ g\right)$ | $\left(g+2 \ g\right)$ | $\left(g+2 \ g\right)$ | $\left(g+2 \ g\right)$ | $\left(g+2 \ g\right)$ | $\left(g+2 \ g\right)$ | $\left(g+2 \ g\right)$ | $\left(g+2 \ g\right)$ |

### Table 9: $K^{CRT}(O_{k+1}^R \otimes O_{l+1}^R)$ for $k \equiv l \equiv 0 \pmod{4}$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $KU_n$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $KT_n$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $c_n$ | 1 | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ |
| $r_n$ | 2 | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ |
| $\varepsilon_n$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ |
| $\zeta_n$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ | $\left(0 \ 0\right)$ |
| $(\psi_u)_n$ | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 |
| $(\psi_T)_n$ | $\left(-\frac{1}{2} \ 0\right)$ | 1 | $\left(-\frac{1}{2} \ 0\right)$ | −1 | $\left(-\frac{1}{2} \ 0\right)$ | 1 | $\left(-\frac{1}{2} \ 0\right)$ | −1 | $\left(-\frac{1}{2} \ 0\right)$ |
| $\gamma_n$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ |
| $\tau_n$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ | $\left(0 \ 1\right)$ |
Table 10: $K_{CRT}^{R}(O_{k+1}^{R} \otimes CRT_{R}^{R})$ for $k \equiv 2$ and $l \equiv 0 \pmod{4}$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $KO_n$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g \oplus \mathbb{Z}_g$ | $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_g$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_g$ | $\mathbb{Z}_g$ | $0$ | $0$ | $\mathbb{Z}_g$ |
| $KU_n$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ |
| $KT_n$ | $\mathbb{Z}_g^2$ | $\mathbb{Z}_g \oplus \mathbb{Z}_g$ | $\mathbb{Z}_g^2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_g$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_g$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_g$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_g$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_g$ | $\mathbb{Z}_g^2$ |

Let $k$ and $l$ be integers with $k$ odd. We must first compute $K_{CRT}^{R}(O_{k+1}^{R}) \otimes CRT_{R}^{R} K_{CRT}^{R}(O_{l+1}^{R})$ and $\text{Tor}_{CRT}(K_{CRT}^{R}(O_{k+1}^{R}), K_{CRT}^{R}(O_{l+1}^{R}))$. For this we must build a free resolution of $K_{CRT}^{R}(O_{k+1}^{R})$. In the case that $k$ is odd, this is easy since $K_{CRT}^{R}(O_{k+1}^{R})$ is generated by the class of the unit in $KO_0(O_{k+1}^{R})$. Let $F(b, 0, \mathbb{R})$ be the free $CRT$-module with a single generator in the real part in degree 0. Our free resolution for $k$ odd is:

$$0 \rightarrow F(b, 0, \mathbb{R}) \xrightarrow{\mu_1} F(b, 0, \mathbb{R}) \xrightarrow{\mu_0} K_{CRT}^{R}(O_{k+1}^{R}) \rightarrow 0$$

where $\mu_1$ is multiplication by $k$ and $\mu_0$ is the homomorphism which sends the generator $b$ in $F(b, 0, \mathbb{R})$ to the class of the unit $[1]$ in $KO_0(O_{k+1}^{R})$. When we tensor this resolution on the right by $K_{CRT}^{R}(O_{l+1}^{R})$, we obtain the homomorphism

$$K_{CRT}^{R}(O_{l+1}^{R}) \xrightarrow{\mu_1 \otimes 1} K_{CRT}^{R}(O_{l+1}^{R})$$

referring to Proposition 3.3. Now $K_{CRT}^{R}(O_{k+1}^{R}) \otimes CRT_{R}^{R} K_{CRT}^{R}(O_{l+1}^{R})$ is isomorphic to the cokernel of $\mu_1 \otimes 1$ and $\text{Tor}_{CRT}(K_{CRT}^{R}(O_{k+1}^{R}), K_{CRT}^{R}(O_{l+1}^{R}))$ is isomorphic to the kernel of $\mu_1 \otimes 1$. But $\mu_1 \otimes 1$ is multiplication by $k$. Therefore, the $CRT$-module $K_{CRT}^{R}(O_{k+1}^{R}) \otimes CRT_{R}^{R} K_{CRT}^{R}(O_{l+1}^{R})$ is easily computed and is shown in Table 10. The Tor group turns out to be the same in this case so we will not reproduce it in a separate table.
Table 11: $M = K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R}) \otimes_{\text{CRT}} K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R})$ for $k$ odd

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $M_O$ | $\mathbb{Z}_g$ | 0 | 0 | 0 | $\mathbb{Z}_g$ | 0 | 0 | 0 | $\mathbb{Z}_g$ |
| $M_U$ | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ |
| $M_T$ | $\mathbb{Z}_g$ | 0 | 0 | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ | 0 | 0 | $\mathbb{Z}_g$ | $\mathbb{Z}_g$ |
| $c_n$ | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 |
| $r_n$ | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| $\varepsilon_n$ | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 |
| $\zeta_n$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $(\psi_U)_n$ | 1 | 0 | $-1$ | 0 | 1 | 0 | $-1$ | 0 | 1 |
| $(\psi_T)_n$ | 1 | 0 | 0 | $-1$ | 1 | 0 | 0 | $-1$ | 1 |
| $\gamma_n$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $\tau_n$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 |

It remains to solve the extension problem given by the Künneth Sequence

$$0 \longrightarrow K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R}) \otimes_{\text{CRT}} K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R}) \xrightarrow{\alpha} K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R} \otimes \mathcal{O}_{l+1}^\mathbb{R}) \xrightarrow{\beta} \text{Tor}_{\text{CRT}}(K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R}), K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R})) \longrightarrow 0 .$$

We show that it splits in this case. Indeed, the only graded degree in which there is a possibility of a non-trivial extension is that for $KT_0(\mathcal{O}_{k+1}^\mathbb{R} \otimes \mathcal{O}_{l+1}^\mathbb{R})$ which is an extension of $\mathbb{Z}_g$ by $\mathbb{Z}_g$. However, since $\gamma_0$ is an isomorphism in the CRT-module $\text{Tor}_{\text{CRT}}(K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R}), K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R}))$ and since $\beta$ is an isomorphism in degree 1, we have a splitting $s = \gamma_1 \circ \beta^{-1} \circ \gamma_0^{-1}$ for $\beta$ in the self-conjugate part in degree 0.

The other three cases are somewhat more complicated than the case when $k$ is odd. We will sketch the computation of Table 9 as a representative example.

Sketch of Computation of Table 9. Let $k$ and $l$ be integers which are both multiples of 4. The united $K$-theory of $\mathcal{O}_{k+1}^\mathbb{R}$ is generated by two elements: one in $KO_0(\mathcal{O}_{k+1}^\mathbb{R})$ and one in $KO_2(\mathcal{O}_{k+1}^\mathbb{R})$. Thus we must start our resolution with a surjective homomorphism $\mu_0$ from $F(b, 0, \mathbb{R}) \oplus F(b, 2, \mathbb{R})$ to $K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R})$. The kernel of $\mu_0$ turns out to be isomorphic to $F(b, 0, \mathbb{C})$. Hence our resolution is

$$0 \rightarrow F(b, 0, \mathbb{C}) \xrightarrow{\mu_1} F(b, 0, \mathbb{R}) \oplus F(b, 2, \mathbb{R}) \xrightarrow{\mu_0} K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R}) \rightarrow 0$$

where $\mu_1$ sends the generator $b \in F(b, 0, \mathbb{C})_0^U$ to $b \oplus \beta^{-1} cb \in (F(b, 0, \mathbb{R}) \oplus F(b, 2, \mathbb{R}))_0^U$.  

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Table 12: \( M = K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R}) \otimes_{\text{CRT}} K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R}) \) for \( k \equiv l \equiv 0 \pmod{4} \)

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---------|---|---|---|---|---|---|---|---|---|
| \( M_0 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^3 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_g \) | \( \mathbb{Z}_2 \) | 0 | 0 | \( \mathbb{Z}_g \) |
| \( M_U \) | \( \mathbb{Z}_g \) | 0 | \( \mathbb{Z}_g \) | 0 | \( \mathbb{Z}_g \) | 0 | \( \mathbb{Z}_g \) | 0 | \( \mathbb{Z}_g \) |
| \( M_T \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_g \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_g \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_g \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2^2 \) | \( \mathbb{Z}_g \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_g \) |
| \( c_n \) | 1 | 0 | \((\frac{4}{5}, 0 \ 0)\) | 0 | \((0 \ 2)\) | 0 | 0 | 0 | 1 |
| \( r_n \) | 2 | 0 | \((\frac{4}{5}, 0)\) | 0 | \((0 \ 1)\) | 0 | 0 | 0 | 2 |
| \( \varepsilon_n \) | \((0 \ 0)\) | 1 | \((\frac{4}{5}, 0 \ 0)\) | \((0 \ 1)\) | \((0 \ 1)\) | 0 | 0 | 0 | \((0 \ 1)\) |
| \( \zeta_n \) | \((0 \ 1)\) | 0 | \((\frac{4}{5}, 0)\) | \((0 \ 1)\) | \((0 \ 1)\) | 0 | 0 | 0 | \((0 \ 1)\) |
| \((\psi_U)_n\) | \((0 \ 0)\) | 1 | \((\frac{4}{5}, 0 \ 0)\) | \((0 \ 1)\) | \((0 \ 1)\) | 0 | 0 | 0 | 2 |
| \((\psi_T)_n\) | \((1 \ 0)\) | 1 | \((\frac{4}{5}, 0 \ 0)\) | \((0 \ 1)\) | \((0 \ 1)\) | 0 | 0 | 0 | \((0 \ 1)\) |
| \( (\psi U) \) | \( (0 \ 1) \) | \( (0 \ 1) \) | \( (0 \ 1) \) | \( (0 \ 1) \) | \( (0 \ 1) \) | \( (0 \ 1) \) | \( (0 \ 1) \) | \( (0 \ 1) \) | \( (0 \ 1) \) |

Using Propositions 3.3 and 3.6, we study the tensor product of this resolution with \( K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R}) \). Then we obtain Tables 12 and 13 by finding the cokernel and kernel (respectively) of

\[
F(b, 0, \mathbb{C}) \otimes_{\text{CRT}} K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R}) \xrightarrow{\mu_1 \otimes 1} (F(b, 0, \mathbb{R}) \oplus F(b, 2, \mathbb{R})) \otimes_{\text{CRT}} K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R}).
\]

It remains to determine the extension

\[
0 \longrightarrow K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R}) \otimes_{\text{CRT}} K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R}) \xrightarrow{\alpha} K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R} \otimes \mathcal{O}_{l+1}^\mathbb{R}) \xrightarrow{\beta} \text{Tor}_{\text{CRT}}(K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R}), K^{\text{CRT}}(\mathcal{O}_{l+1}^\mathbb{R})) \longrightarrow 0.
\]

Fortunately, the structure imposed by the CRT-relations and the acyclicity of united K-theory will be enough to determine \( K^{\text{CRT}}(\mathcal{O}_{k+1}^\mathbb{R} \otimes \mathcal{O}_{l+1}^\mathbb{R}) \).

Let \( A \) denote \( \mathcal{O}_{k+1}^\mathbb{R} \otimes \mathcal{O}_{l+1}^\mathbb{R} \). We can immediately establish \( KO_*(A) \) in degrees 0, 2, 4, 6, and 7; \( KU_*(A) \) in all degrees; and \( KT_*(A) \) in degrees 2 and 6. The realification and complexification operations are also established in the even degrees.

In the following, we will make frequent use of the long exact sequence

\[
\cdots \longrightarrow KO_n(A) \xrightarrow{\eta_0} KO_{n+1}(A) \xrightarrow{c} KU_{n+1}(A) \xrightarrow{r \partial_U^{-1}} KO_{n-1}(A) \longrightarrow \cdots.
\]
Table 13: $M = \text{Tor}_{\text{CRT}}(K_{\text{CRT}}^{\mathcal{O}_{k+1}^{R}}, K_{\text{CRT}}^{\mathcal{O}_{l+1}^{R}})$ for $k \equiv l \equiv 0 \pmod{4}$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $M_{O}$ | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_g \frac{1}{2}$ | 0 | 0 | 0 | $\mathbb{Z}_g$ |
| $M_{L}$ | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_g$ |
| $M_{T}$ | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_g \frac{1}{2}$ | $\mathbb{Z}_g$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_g \frac{1}{2}$ | $\mathbb{Z}_g$ |
| $c_n$ | 1 | 0 | $\frac{1}{2}$ | 0 | 2 | 0 | 0 | 0 | 1 |
| $r_n$ | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| $\varepsilon_n$ | 1 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 1 |
| $\zeta_n$ | 1 | 0 | $\frac{1}{2}$ | 0 | 1 | 0 | $\frac{1}{2}$ | 0 | 1 |
| $(\psi_n)_n$ | 1 | 0 | $-1$ | 0 | 1 | 0 | $-1$ | 0 | 1 |
| $(\psi_T)_n$ | 1 | 0 | 1 | $-1$ | 1 | 0 | 1 | $-1$ | 1 |
| $\gamma_n$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $\tau_n$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 |

First, we show that $KO_1(A)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_g$, rather than $\mathbb{Z}_2 \mathbb{Z}_g$. Indeed, the kernel of $c_2: \mathbb{Z}_3 \rightarrow \mathbb{Z}_g$ must have at least two generators. Since the kernel of $c_2$ is the image of $(\eta_0)_1$, it follows that $KO_1(A)$ must have two generators.

Second, we show that $KO_5(A)$ is isomorphic to $\mathbb{Z}_g$. Indeed, $KO_6(A) = KO_7(A) = 0$, so the long exact sequence above shows that $r_5$ is an isomorphism. It also follows from this and the relation $rc = 2$ that the image of $c_5$ is $2\mathbb{Z}_g$.

Third, we show that $KO_3(A)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ rather than $\mathbb{Z}_2 \mathbb{Z}_4$. Assume that $KO_3(A) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Because the image of $c_5$ is $2\mathbb{Z}_g$, the realification map $r_3$ must take the generator of $KU_3(A)$ to an element of degree 2. On the other hand, the image of $(\eta_0)_3$ (which is the kernel of $c_4$) has two generators. Therefore, the image of $r_3$ cannot be the $\mathbb{Z}_2$ summand of $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. By re-choosing the basis, we may assume $r_3 = \left( \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right)$.

The image of $(\eta_0)_2$ is a copy of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. As there is only one such subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_4$, this determines $c_3 = \left( \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right)$. However, then we have $rc = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ which is not multiplication by 2. By this contradiction, we conclude that $KO_3(A) \cong \mathbb{Z}_2^2$. This completes the computation of the groups $KO_*(A)$.

Because $KO_6(A) \cong KO_7(A) \cong 0$, the long exact sequence

$$\cdots \rightarrow KO_n(A) \xrightarrow{\eta_0} KO_{n+2}(A) \xrightarrow{\varepsilon} KT_{n+2}(A) \xrightarrow{\tau_3^{-1}} KO_{n-1}(A) \rightarrow \cdots$$

shows that $\varepsilon_1$ and $\tau_3$ are isomorphisms. Thus we know $KT_*(A)$ in degrees 1 and 3, and hence
in degrees 5 and 7. It remains to find $KT_0(A) \cong KT_4(A)$. The sequence

$$KU_1(A) \xrightarrow{1-\psi_0} KU_1(A) \xrightarrow{\gamma} KT_0(A) \xrightarrow{\zeta} KU_0(A) \xrightarrow{1-\psi_0} KU_0(A)$$

becomes

$$\mathbb{Z}_g \xrightarrow{0} \mathbb{Z}_g \xrightarrow{\gamma} KT_0(A) \xrightarrow{\zeta} \mathbb{Z}_g \xrightarrow{0} \mathbb{Z}_g$$

so $KT_0(A)$ is an extension of $\mathbb{Z}_g$ by $\mathbb{Z}_g$. We claim that it is isomorphic to $\mathbb{Z}_g \oplus \mathbb{Z}_g$. For this we also need the sequence

$$KO_{-2}(A) \xrightarrow{\nu_0^2} KO_0(A) \xrightarrow{\varepsilon} KT_0(A) \xrightarrow{\tau_0^\beta^{-1}} KO_{-3}(A) \xrightarrow{\nu_0^2} KO_{-1}(A)$$

which becomes

$$0 \to \mathbb{Z}_g \xrightarrow{\varepsilon} KT_0(A) \to \mathbb{Z}_g \to 0.$$ 

Since $c_0 = \zeta_0 \varepsilon_0: \mathbb{Z}_g \to \mathbb{Z}_g$ is an isomorphism, the image of $\varepsilon_0$ has a trivial intersection with $\ker \zeta_0 = \text{image} \gamma_1$. Therefore, $KT_0(A) = \text{image} \varepsilon \oplus \text{image} \gamma \cong \mathbb{Z}_g \oplus \mathbb{Z}_g$.

Now that all of the groups of $K^{\text{CRT}}(A)$ have been established, the behavior of the operations can be computed using the exact sequences and the CRT-relations.

This example shows that the Künneth sequence for united $K$-theory does not split in general, even on the level of abelian groups.

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