ON A MIXED PROBLEM
FOR THE PARABOLIC LAMÉ TYPE OPERATOR

R. PUZYREV AND A. SHLAPUNOV

Abstract. We consider a boundary value problem for the parabolic Lamé type operator being a linearization of the Navier-Stokes’ equations for compressible flow of Newtonian fluids. It consists of recovering a vector-function, satisfying the parabolic Lamé type system in a cylindrical domain, via its values and the values of the boundary stress tensor on a given part of the lateral surface of the cylinder. We prove that the problem is ill-posed in the natural spaces of smooth functions and in the corresponding Hölder spaces; besides, additional initial data do not turn the problem to a well-posed one. Using the Integral Representation’s Method we obtain the Uniqueness Theorem and solvability conditions for the problem.

Introduction

Let, as usual, $\Delta_n$ be the Laplace operator, $\nabla_n$ be the gradient operator and $\text{div}_n$ be the divergence operator in $\mathbb{R}^n$, $n \geq 2$. The Navier-Stokes’ equations for compressible flow of Newtonian fluids over the four-dimensional domain $D \subset \mathbb{R}_x^3 \times \mathbb{R}_t$ under the action of a body force $F(x, t) = (F_1(x, t), F_2(x, t), F_3(x, t))$ can be written in the following form (see [1, §15, formulas (15.5), (15.6))):

$$
\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla_3 v \right) + \nabla_3 p - \text{div}_3 (\mu_1 \nabla_3 v) - \nabla_3 \left( \frac{\mu_1}{3} + \mu_2 \right) \text{div}_3 v - av = F, \tag{1}
$$

where $v(x, t) = (v_1(x, t), v_2(x, t), v_3(x, t))$ is the flow velocity, $\rho(x, t)$ is the fluid density, $p(x, t)$ is the pressure, $\mu_j(x, t)$ are (positive) viscosity coefficients and

$$
av = [(\nabla_3 \mu_1)^* \otimes \nabla_3 - (\nabla_3 \mu_1) \text{div}_3]v
$$

is the linear first order summand with $M_1^*$ being the adjoint matrix for a matrix $M_1$ and $M_1^* \otimes M_2$ being the Kronecker product of matrices $M_1^*$ and $M_2$. If the boundary $\partial D$ of $D$ is piece-wise smooth then the boundary conditions for this system often involve the force $\nu p - \sigma' v$ acting on the unit surface area where the force friction (or the boundary viscosity tensor) $\sigma'$ has the components

$$
\sigma'_{i,j} = \delta_{i,j} \mu_1 \sum_{k=1}^n \nu_k \frac{\partial}{\partial k} + \mu_1 \nu_j \frac{\partial}{\partial x_i} + (\mu_2 - 2\mu_1/3) \nu_i \frac{\partial}{\partial x_j}, \quad 1 \leq i, j \leq 3,
$$

with $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$ being the unit normal vector to the surface $\partial D$ and $\delta_{i,j}$ being the Kronecker symbol (see [1, §15, formula (15.12)])

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Since the density $\rho$ is positive, a proper linearization of the substantial derivative term $v \cdot \nabla_3 v$ turns (1) into a parabolic Lamé type system related to an unknown vector $u$:

$$L_4 u = \frac{\partial u}{\partial t} - L_3 u - \sum_{j=1}^{3} a_j(x,t) \frac{\partial u}{\partial x_j} - a_0(x,t) u = f$$

where $a_j(x,t)$, $0 \leq j \leq 3$, are $(3 \times 3)$ matrices with functional entries and

$$L_n = \text{div}_n \left( \mu \nabla_n \right) + \nabla_n \left( (\mu + \lambda) \text{div}_n \right), \quad n \geq 2$$

is the strongly elliptic (with respect to the space variables) formally self-adjoint Lamé type operator with the Lamé coefficients satisfying

$$\mu(x,t) > 0, \quad (\mu(x,t) + \lambda(x,t)) \geq 0.$$  

The regularity (smoothness) of the Lamé coefficients and the matrices $a_j(x,t)$ depends upon the regularity of the density $\rho$ and the viscosity coefficients $\mu_j$.

Note that if $\mu$ is constant, $\lambda + \mu = 0$ and $a_j = 0$, $0 \leq j \leq 3$, then $L_4$ reduces to the heat operator, though, of course, it is known that the heat equation is not ideal to model the process of the heat conduction.

Let $\Omega$ be a bounded domain (i.e. bounded open connected set) in $n$-dimensional real space $\mathbb{R}^n$ with the coordinates $x = (x_1, \ldots, x_n)$. As usual we denote by $\overline{\Omega}$ the closure of $\Omega$, and we denote by $\partial \Omega$ its boundary. In the sequel we assume that $\partial \Omega$ is piece-wise smooth. As $\partial \Omega$ is piece-wise smooth, the normal vector $\nu = (\nu_1, \ldots, \nu_n)$ is defined almost everywhere on $\partial \Omega$ and satisfies $\sum_{j=1}^{n} \nu_j^2 \neq 0$.

Let $\Omega_T = \{x \in \Omega, \ 0 < t < T\}$ an open cylinder, having the altitude $0 < T \leq +\infty$ and the base $\Omega$, in $(n + 1)$-dimensional real space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \{-\infty < t < +\infty\}$. Let also $\Gamma \subset \partial \Omega$ be a non empty connected open (in the topology of $\partial \Omega$) subset of $\partial \Omega$ and $\Gamma_T = \Gamma \times (0, T)$.

In the present paper we consider a mixed boundary problem for the parabolic system in the cylindrical domain $\Omega_T$

$$L_{n+1} = \frac{\partial}{\partial t} - L_n - Au,$$

where

$$Au = \sum_{j=1}^{n} a_j(x,t) \frac{\partial}{\partial x_j} + a_0(x,t),$$

the Lamé coefficients and the entries of the $(n \times n)$-matrices $a_j(x,t)$, $0 \leq j \leq n$, are $C^\infty$-smooth in a neighborhood of $\overline{\Omega}_T$ and real analytic with respect to the space variables in a neighborhood of $\overline{\Omega}$.

Instead of classical boundary value problems for parabolic equations (see, for instance, [2], [3], [4], [5]) we consider the ill-posed problem, consisting in finding a vector-function satisfying the corresponding parabolic equation in the cylinder via its values and the values of the boundary stress tensor with the components

$$\sigma_{i,j} = \mu \delta_{i,j} \sum_{k=1}^{n} \nu_k \frac{\partial}{\partial x_k} + \mu \nu_j \frac{\partial}{\partial x_i} + \lambda \nu_i \frac{\partial}{\partial x_j}, \quad 1 \leq i, j \leq n.$$  

(2)

on the given part $\Gamma_T$ of the lateral surface of the cylinder $\Omega_T$ (cf. [6]).

Using parabolic potentials we prove Uniqueness Theorem and obtain solvability conditions for the problem (cf. [7] related to similar results for the heat equation).

Actually, the approach was invented for the investigation of the famous ill-posed
Cauchy problem for elliptic equations (see, for instance, [8] for the Cauchy-Riemann operator, [9] for the elliptic Lamé operator and [10], [11], [12], for general systems with injective principal symbols).

1. Preliminaries

As usual, for $s \in \mathbb{Z}_+$ (here $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$) and an open subset $D \subset \mathbb{R}^m$ we denote $C^s(D)$ the set of all $s$ times continuously differentiable functions in $D$. The standard topology of this metrisable space induces uniform convergence on compact subsets in $D$ together with all the partial derivatives up to order $s$.

For $S \subset \partial D$ we denote $C^s(D \cup S)$ the set of such functions from the space $C^s(D)$ that all their derivatives up to order $s$ can be extended continuously onto $D \cup S$. The standard topology of this metrisable space induces uniform convergence on compact subsets in $D \cup S$ together with all the partial derivatives up to order $s$. In particular, for bounded domains, $C^s(D \cup \partial D) = C^s(\overline{D})$ is a Banach space. If $D$ is an unbounded then the Banach space $C^s_b(\mathbb{R}^+)$ consists of $s$-times differentiable functions in $\mathbb{R}^+$ and it is endowed with the standard $\sup$-norm. Then $C^s_b(\overline{D}) = C^s(\overline{D})$ for a bounded domain $D$.

Apart from the standard functional spaces, we need also spaces taking into account the specific properties of parabolic equations in $\mathbb{R}^{n+1} = \mathbb{R}^n \times (-\infty < t < +\infty)$. Namely, let $C^{1,0}(\Omega_T)$ be the set of continuous functions $u$ in $\Omega_T$, having in $\Omega_T$ continuous partial derivatives $u_{x_i}$, and let $C^{2,1}(\Omega_T)$ denote the set of continuous functions in $\Omega_T$, having in $\Omega_T$ continuous partial derivatives $u_{x_i}$, $u_{x_ix_j}$, $u_{tt}$. The standard topology of this metrisable space induces uniform convergence on compact subsets in $D$ together with all the partial derivatives used in its definition.

As before, for $S \subset \partial \Omega_T$ we denote by $C^{1,0}(\Omega_T \cup S)$ the set of such functions $u$ from the space $C^{1,0}(\Omega_T)$ that their derivatives $u_{x_i}$ can be extended continuously onto $\Omega_T \cup S$. The standard topology of this metrisable space induces uniform convergence on compact subsets of $\Omega_T \cup S$ of both the functional sequences and the corresponding sequences of first partial derivatives $x_i$. Clearly, $C^{1,0}(\Omega_T \cup \partial \Omega_T) = C^{1,0}(\overline{\Omega_T})$ is a Banach space. Similarly to the standard spaces, if $D \subset \mathbb{R}^{n+1}$ the Banach space $C^{1,0}_b(\mathbb{R}^+)$ consisting of bounded $C^{0,1}(\mathbb{R}^+)\lambda$-functions with bounded derivatives $u_{x_i}$ in $\mathbb{R}^+$ and it is endowed with the norm

$$
\|u\|_{C^{1,0}_b(\mathbb{R}^+)} = \sup_{(x,t) \in \mathbb{R}^+} |u(x,t)| + \sup_{(x,t) \in \mathbb{R}^+} \sum_{j=1}^n \left| \frac{\partial u(x,t)}{\partial x_j} \right|.
$$

Then $C^{0,1}_b(\overline{D}) = C^{0,1}(\overline{D})$ for a bounded domain $D$.

The space of $n$-vector-functions $u = (u_1, \ldots, u_n)$ of a class $\mathcal{C}$ will be denoted by $[\mathcal{C}]^n$.

Let now $\theta$ be such positive constant that for all $(x,t) \in \overline{\Omega_T}$ we have

$$
\mu(x,t) \geq \theta, (\lambda(x,t) + 2\mu(x,t)) \geq \theta.
$$

Then a direct calculation shows that for all $\zeta \in \mathbb{R}^n$ we have

$$
\det \left( \mu(x,t) |\zeta|^2 (\sqrt{-1})^2 I_n + (\lambda(x,t) + \mu(x,t)) \zeta^T (\sqrt{-1})^2 - \kappa I_n \right) =
$$

$$
(-1)^n (\mu(x,t)|\zeta|^2 + \kappa)^{n-1}((2\mu(x,t) + \lambda(x,t))|\zeta|^2 + \kappa)
$$
where $I_n$ is the unit $(n \times n)$ matrix and $\zeta^T$ is the transposed vector for $\zeta$. Hence the roots of this polynomial (with respect to $\kappa$) are
\[
\kappa_1(x, t, \zeta) = -(2\mu(x, t) + \lambda(x, t))|\zeta|^2, \quad \kappa_2(x, t, \zeta) = -\mu(x, t)|\zeta|^2
\]
and, for all $(x, t) \in \overline{\Omega_T}$, we have
\[
\max \left( \sup_{|\zeta|=1} \kappa_1(x, t, \zeta), \sup_{|\zeta|=1} \kappa_2(x, t, \zeta) \right) \leq -\theta,
\]
i.e. the operator $L_n$ is uniformly parabolic (according to Petrovskii) on $\overline{\Omega_T}$.

Now we assume that there is a $n$-dimensional domain $U \supset \Omega$ such that the Lamé coefficients $\mu(x, t)$, $\lambda(x, t)$ and the entries of the $(n \times n)$-matrices $a_j(x, t)$, $0 \leq j \leq n$, are $C^\infty$-smooth in $\overline{U_T}$ and real analytic with respect to the space variables in $U$.

Under the assumptions, the following properties hold true for parabolic operator $L_{n+1}$, which will be crucial for the approach below.

**Theorem 1.** Each weak solution $u$ to $L_{n+1}u = 0$ in the domain $\Omega_T \subset U_T$ belongs to $C^\infty(\Omega_T)$ and it is actually real analytic with respect to variables $x$ in $\Omega$.

**Theorem 2.** The operator $L_{n+1}$ has a fundamental solution in $U_T$, i.e. a $(n \times n)$-matrix $\Phi(x, t, y, \tau)$ satisfying
\[
(L_{n+1})_{x,t}\Phi(x, t, y, \tau) = 0, \quad (L_{n+1})_{y,\tau}\Phi(x, t, y, \tau) = 0, \quad \text{if} \ (x, t) \neq (y, \tau), \quad (3)
\]

with the formal adjoint operators
\[
(L_{n+1})^*_{y,\tau} = -\frac{\partial}{\partial \tau} - (L_n)y - A^*, \quad A^* = -\sum_{k=1}^{n} \frac{\partial}{\partial y_k}(a_k^*(y, \tau)\cdot) + a_0^*(y, \tau).
\]

**Proof** See, for instance, [4, Ch. 2].

We need a sort of an integral representation, similar to the famous Green Formula for the Laplace Operator, constructed with the use the fundamental solutions. More precisely, consider the cylinder type domain $\Omega_{T_1, T_2} = \Omega_{T_2} \setminus \Omega_{T_1}$ and a closed measurable set $S \subset \partial \Omega$.

Let $\sigma$ be the tensor with the components given by (2) and
\[
\tilde{\sigma} = \sigma - \sum_{k=1}^{n} a_k^*(x, t)\mu_k(x) + [(\nabla_n \mu(x, t))\nu^*(x) - \nu(x)(\nabla_n \mu(x, t))^*].
\]

For functions $f \in C(\overline{\Omega_{T_1, T_2}}), \ v \in C(S_T), \ w \in C(S_T), \ h \in C(\overline{\Omega})$ we set
\[
I_{\Omega, T_1}h(x, t) = \int_{\Omega} \Phi^*(x, t, y, T_1)h(y)dy, \quad (4)
\]
\[
G_{\Omega, T_1}f(x, t) = \int_{T_1} \int_{\Omega} \Phi^*(x, t, y, \tau)f(y, \tau)dyd\tau, \quad (5)
\]
\[
V_{S, T_1}v(x, t) = \int_{T_1} \int_{S} \Phi^*(x, t, y, \tau)v(y, \tau)ds(y)d\tau, \quad (6)
\]
Lemma 1. For all \( \text{Parabolic Potentials} \) with densities \( f, v, w \) and \( h \) respectively. In our situation these are convergent improper integrals depending on vector parameter \((x, t)\) in the neighborhood \( U \) of the cylinder \( \Omega_{T_1, T_2} \) in \( \mathbb{R}^{n+1} \) (see, for instance, \([2, \text{Ch. 4, §1}], [15, \text{Ch. 3, §10}], [3, \text{Ch. 1, §3 and Ch. 5, §2}]\)). The potential \( I_{\Omega, T_1}(h) \) is sometimes called \textit{Poisson type integral} for the Lam\'e type Operator, the functions \( G_{\Omega, T_1}(f), V_{S, T_1}(v), W_{S, T_1}(w) \) are often referred to as \textit{Parabolic Volume Potential}, \textit{Parabolic Single Layer Potential} and \textit{Parabolic Double Layer Potential} respectively.

**Proof.** Indeed, it follows from Gauß-Ostrogradskii Formula that

\[
\int_{\partial \Omega} v^* \sigma u = \int_{\partial \Omega} v^* (\mathcal{L}_n u + au) dy + \mathcal{D}_\Omega(u, v)
\]

for all \( u, v \in C^{1,0}(\Omega_{T_1, T_2}) \) with \( L_{n+1}u \in C(\Omega_{T_1, T_2}) \), where

\[
a u = [(\nabla_n \mu)^* \otimes \nabla_n - (\nabla_n \mu) \text{div}_n] u,
\]

\[
\mathcal{D}_\Omega(u, v) = \int_{\Omega} \left( \mu (\nabla_n v)^* \nabla_n u + \nabla_n u \right) dy.
\]

On the other hand, by Gauß-Ostrogradskii Formula,

\[
\int_{\partial \Omega} [(\nabla_n \mu(x, t)) v^* (x) - \nu(x)((\nabla_n \mu(x, t))^*)] ds(y) = \int_{\Omega} v^*(au - (a^* v)^* u) dy.
\]

Therefore

\[
\int_{\partial \Omega} (v^* \sigma u - (\tilde{\sigma} v)^* u) ds(y) = \int_{\Omega} (v^*(\mathcal{L}_n u + Au) - (\mathcal{L}_n v + A^* v)^* u) dy
\]

for all \( u, v \in C^{1,0}(\Omega) \) with \( \mathcal{L}_n u, \mathcal{L}_n v \in C(\Omega) \). Hence, again by Gauß-Ostrogradskii Formula, we obtain the (first) Green formula for the Lam\'e type operator:

\[
\int_{\Omega} [v^*(y, T_1) u(y, T_1) - v^*(y, T_2) u(y, T_2)] dy - \int_{T_1}^{T_2} \int_{\partial \Omega} [v^* \sigma u - (\tilde{\sigma} v)^* u] ds(y) dt = \int_{\Omega_{T_1, T_2}} \left( v^* L^*_{n+1} u - (L_{n+1})^* u \right) dt dy
\]

for all \( u, v \in C^{1,0}(\Omega_{T_1, T_2}) \) with \( L_{n+1}u, L_{n+1}v \in C(\Omega_{T_1, T_2}) \).

It follows from the definition of the fundamental solution, that

\[
(L_{n+1})_{x,t} \Phi(x, t, y, \tau) = \delta(x - y, t - \tau), \quad (L_{n+1}^*)_{y, \tau} \Phi(x, t, y, \tau) = \delta(x - y, t - \tau),
\]

\[
\Phi(x, t, y, \tau) = 0 \text{ for } \tau > t.
\]
Then, using the standard argument (see, for instance, [17, Ch. 6, §12] for the heat equation), we see that Green’s Formula (8) follows from (12) and Fubini Theorem.

**Theorem 3** (Uniqueness Theorem). *If \( \Gamma \) has at least one interior point (on \( \partial \Omega \)), and function \( u \in C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \overline{\Gamma_T}) \) satisfies \( L_{n+1}u \equiv 0 \) in \( \Omega \), \( u \equiv 0 \) on \( \Gamma_T \), \( \sigma u \equiv 0 \) on \( \Gamma_T \), then \( u \equiv 0 \) in \( \Omega_T \).

**Proof.** Under the hypothesis of the theorem there is an interior point \( x_0 \) on \( \Gamma \). Then there is such a number \( r > 0 \) that \( B(x_0, r) \cap \partial \Omega \subset \Gamma \) where \( B(x_0, r) \) is ball in \( U \subset \mathbb{R}^n \) with center at \( x_0 \) and radius \( r \). Fix an arbitrary point \((x', t') \in \Omega_T \). It is clear that there is a domain \( \Omega' \ni x' \) satisfying \( \Omega' \subset \Omega \) and \( \Omega' \cap \partial \Omega \subset \Gamma \cap B(x_0, r) \). Then \((x', t') \in \Omega'_{T_1, T_2} \) with some \( 0 < T_1 < T_2 < T \).

But \( u \in C^{2,1}(\Omega'_{T_1, T_2}) \cap C^{1,0}(\Omega'_{T_1, T_2}) \) and \( L_{n+1}u = 0 \) in \( \Omega'_{T_1, T_2} \) under the hypothesis of the theorem. Hence formula (8) implies:

\[
I_{\Omega'_{T_1, T_2}}u(x, t) + V_{\partial \Omega' \setminus \Gamma, T_2}u(x, t) + W_{\partial \Omega' \setminus \Gamma, T_1}u(x, t) = \begin{cases} u(x, t), & (x, t) \in \Omega'_{T_1, T_2}, \\ 0, & (x, t) \in U_T \setminus \Omega'_{T_1, T_2}, \end{cases}
\]

because \( u \equiv \sigma u \equiv 0 \) on \( \Gamma_T \).

Taking into account the character of the singularity of the kernel (see [4, Theorem 2.2]) \( \Phi(x, y, t, \tau) \) we conclude that the following properties are fulfilled for the integrals, depending on parameter, from the right hand side of identity (13):

\[
I_{\Omega'_{T_1, T_2}}(u) \in C^{2,1}(U_{T_1, T_2}),
\]

\[
W_{\partial \Omega' \setminus \Gamma, T_2}u, \quad V_{\partial \Omega' \setminus \Gamma, T_1}u \sigma u \in C^{2,1}((U \setminus (\partial \Omega' \setminus \Gamma))_{T_1, T_2})
\]

(see, for instance, [2, Ch. 4, §1], [15, Ch. 3, §10] or [3, Ch. 1, §3 and Ch. 5, §2]). Moreover, as \( \Phi \) is a fundamental solution to Lamé type operator then using (3) and Leibniz rule for differentiation of integrals depending on parameter we obtain:

\[
L_{n+1}I_{\Omega'_{T_1, T_2}}u = 0 \quad \text{in} \quad U_{T_1, T_2},
\]

\[
L_{n+1}V_{\partial \Omega' \setminus \Gamma, T_2}u \sigma u = L_{n+1}W_{\partial \Omega' \setminus \Gamma, T_1}u = 0 \quad \text{in} \quad (U \setminus (\partial \Omega' \setminus \Gamma))_{T_1, T_2}.
\]

Hence the function

\[
P(x, t) = I_{\Omega'_{T_1, T_2}}u(x, t) + V_{\partial \Omega' \setminus \Gamma, T_1}u(x, t) + W_{\partial \Omega' \setminus \Gamma, T_1}u(x, t),
\]

satisfies the Lamé type equation

\[
(L_{n+1}P)(x, t) = 0 \quad \text{in} \quad (U \setminus (\partial \Omega' \setminus \Gamma))_{T_1, T_2}.
\]

This implies that the function \( P(x, t) \) is real analytic with respect to the space variable \( x \in U \setminus (\partial \Omega' \setminus \Gamma) \) for any \( T_1 < t < T_2 \) (see, for instance, [16, Ch. VI, §1, Theorem 1]). In particular, by the construction the function \( P(x, t) \) is real analytic with respect to \( x \) in the ball \( B(x_0, r) \) and it equals to zero for \( x \in B(x_0, R) \setminus \overline{\Omega} \) for all \( T_1 < t < T_2 \). Therefore, the Uniqueness Theorem for real analytic functions yields \( P(x, t) \equiv 0 \) in \( (U \setminus (\partial \Omega' \setminus \Gamma))_{T_1, T_2} \), and in the cylinder \( \Omega'_{T_1, T_2} \) containing the point \((x', t')\). Now it follows from (13) that \( u(x', t') = P(x', t') = 0 \) and then, since the point \((x', t') \in \Omega_T \) is arbitrary we conclude that \( u \equiv 0 \) in \( \Omega_T \). The proof is complete. \( \square \)
Example 1. Let \( \mu = 1, \lambda = -1 \) and \( a_j = 0, 0 \leq j \leq n \). Then \( L_{n+1} \) reduces to the heat operator:

\[
L_{n+1} = \frac{\partial}{\partial t} - \Delta_n
\]

and corresponding fundamental solution is given by \( \Phi(x, y, t, \tau) = \varphi_0(x - y, t - \tau)I_n \) where

\[
\varphi_0(x, t) = \begin{cases} 
\frac{1}{(2\sqrt{\pi}t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{if } t > 0, \\
0 & \text{if } t \leq 0.
\end{cases}
\]

In this case \( \tilde{\sigma} = \sigma = \frac{\partial}{\partial \nu} \).

Example 2. Let \( \mu, \lambda \) be constant and \( a_j = 0, 0 \leq j \leq n \). Then \( L_{n+1} \) reduces to the parabolic Lamé operator

\[
L_{n+1} = \frac{\partial}{\partial t} - L_n
\]

and corresponding fundamental solution \( \Phi(x, y, t, \tau) \) is given by \((n \times n)\)-matrix with components \( \Phi_{i,j}(x, y, t, \tau) = \varphi_{i,j}(x - y, t - \tau) \) where

\[
\varphi_{i,j}(x, t) = \varphi_0(x, \mu t) \delta_{i,j} + \int_{\mu t}^{(2\mu + \lambda)t} \frac{\partial^2 \varphi_0(x, s)}{\partial x_j \partial x_i} ds,
\]

(see, for instance, [4]). In this case \( \tilde{\sigma} = \sigma = \mu \frac{\partial}{\partial \nu} + \mu \nu^* \otimes \nabla_n + \lambda \nu \text{ div}_n \).

2. The boundary problem

Green formula (8) and the Uniqueness Theorem 3 suggest us to consider two kind of problems for the parabolic Lamé type equation.

Let vector-functions

\[
u^{(0)}(x) \in [C(\overline{\Omega})]^n, f(x, t) \in [C_b(\overline{\Omega_T})]^n,
\]

\[
u^{(1)}(x, t) \in [C^{1,0}(\overline{\Omega_T}) \cap C_b(\overline{\Gamma_T})]^n, \nu^{(2)}(x, t) \in [C_b(\overline{\Gamma_T})]^n
\]

be given.

Problem 1. Find a vector-function \( u(x, t) \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T) \cup \overline{\Gamma_T})]^n \) satisfying the Lamé type equation

\[
L_{n+1}u = f \text{ in } \Omega_T
\]

and boundary conditions

\[
u(x, t) = \nu^{(1)}(x, t) \text{ on } \overline{\Gamma_T},
\]

\[
u\nu(x, t) = \nu^{(2)}(x, t) \text{ on } \overline{\Gamma_T}.
\]

Note that, if the surface \( \Gamma \) and the data of the problem are real analytic then the Cauchy-Kovalevsky Theorem implies that Problem 1 can not have more than one solution in the class of (even formal) power series. However the theorem does not imply the existence of solutions to Problem 1 because it grants the solution in a small neighborhood of the surface \( \Gamma_T \) only (but not in a given domain \( \Omega_T \)). In any case, we do not assume the real analyticity of \( \Gamma \) and the data \( \nu^{(1)}, \nu^{(2)} \) and \( f \).

Corollary 1. If \( \Gamma \) has at least one interior point (on \( \partial \Omega \)) then Problem 1 has no more that one solution.
Proof. Let \( v(x, t) \) and \( w(x, t) \) be two solutions to Problem 1. Then function
\[
\begin{align*}
    u = (v - w) & \in C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \Gamma_T) \cap C\left(\Omega_T \setminus (\partial \Omega \setminus \Gamma)_T\right)
\end{align*}
\]

is a solution to the corresponding problem with \( f = 0, \ u_1 = 0, \ u_2 = 0 \). Using 3 we conclude that \( u \) is identically zero in \( \Omega_T \).

Thus, the Uniqueness Theorem 3 implies that the data of Problem 1 are suitable in order to uniquely define its solution.

Easily, Problem 1 is ill-posed because this is the property of the Cauchy problem for elliptic systems in \( \mathbb{R}^n \) (see, for instance [13] or [16, Ch. 1, \S 2]). Of course, in this case the boundary data should be taken independent on \( t \). The Uniqueness Theorem clarify why the problem is ill-posed. The reason is the redundant data. Indeed, if \( \Gamma \) has at least one interior point (on \( \partial \Omega \)), then taking a smaller set \( \Gamma' \subset \Gamma \) we again obtain a problem with no more than one solution.

Another problem involves the initial data.

**Problem 2.** Find a vector-function \( u(x, t) \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \Gamma_T) \cap C_b(\Omega_T)]^n \) satisfying in \( \Omega_T \) Lamé type equation (14), boundary conditions (15), (16) and initial condition
\[
    u(x, 0) = u^{(0)}(x), \quad x \in \overline{\Omega}.
\]

Of course one should also take care on the compatibility of the data \( u^{(0)}, u^{(1)}, u^{(2)} \): at least
\[
    u^{(0)}(x) = u^{(1)}(x, 0) \quad \text{on} \quad \Gamma,
\]
and, if \( u^{(0)} \in C^1(\overline{\Omega}) \), even
\[
    \sigma u^{(0)}(x) = u^{(2)}(x, 0) \quad \text{on} \quad \Gamma.
\]

The motivation of Problems 1 and 2 is transparent. The space \( C_b(\Omega_T) \) is chosen because \( u \) represents the “velocity”. The first problem describes the situation where for some reasons at each time \( t \geq 0 \) only part \( \overline{\Gamma} \) of the solid surface \( \partial \Omega \) bounding the fluid is available for measurements. The second one describes the situation where the continuity up to \( \partial \Omega_T \) is postulated, the “velocity” \( u \) is known at every point \( x \in \overline{\Omega} \) at the initial time \( t = 0 \) but the data on \( \partial \Omega \setminus \Gamma \) were lost for \( t > 0 \).

Clearly, Problem 2 has no more than one solution, too, if \( \Gamma \) has at least one interior point (on \( \partial \Omega \)).

We note that in classical theory of (initial and) boundary problems for the parabolic equation (14), initial condition (17) and boundary condition \( \alpha u + \beta \sigma = u^{(3)} \) on the whole lateral surface \( \partial \Omega_T \) of the cylinder \( \Omega_T \) are usually considered. As a rule, such a problem is well-posed in proper spaces (Hölder spaces, Sobolev spaces etc.), see, for instance, [2].

Let us show that Problem 2 is ill-posed, too.

**Example 3.** Let the Lamé coefficients \( \mu, \lambda \) be constant and \( a_j = 0, \ 0 \leq j \leq n \).

Take a cube \( Q_n = \{ 0 < x_j < 1, 1 \leq j \leq n \} \) as base \( \Omega \) of the cylinder \( \Omega_T \). Let \( \Gamma \) be the face \( \{ x_n = 0 \} \) of the cube \( Q_n \). Then \( \Gamma_T = Q_{n-1} \times (0, T) \) and the stress tensor \( \sigma \) is given by the diagonal matrix with the non-zero entries
\[
    \sigma_{j,j} = \mu \frac{\partial}{\partial x_j}, 1 \leq j \leq n - 1, \ \sigma_{n,n} = (2 \mu + \lambda) \frac{\partial}{\partial x_n}.
\]

Fix \( N \in \mathbb{N} \) and consider the sequence of functions \( u(x, t, k, r) \in [C^\infty(\mathbb{R}^{n+1})]^n \) with the components:
\[
    u_1(x, t, k, r) = 0, \ldots, u_{n-1}(x, t, k, r) = 0, \ u_n(x, t, k, r) = \frac{e^{k^2(2\mu+\lambda)(t-r)+krx_n}}{k^{N}}.
\]
depending on a parameter $0 < r < +\infty$. Consider also the data $f(x,t,k,r)$, $u^{(0)}(x,t,k,r)$, $u^{(1)}(x,t,k,r)$, $u^{(2)}(x,t,k,r)$ having the following components:

$$f_j(x,t,k,r) = 0, \ 1 \leq j \leq n,$$

$$u_j^{(0)}(x,k,r) = 0, \ 1 \leq j \leq n - 1,$$

$$u_j^{(1)}(x_1,\ldots,x_{n-1},t,k,r) = 0, \ 1 \leq j \leq n - 1,$$

$$u_n^{(1)}(x_1,\ldots,x_{n-1},t,k,r) = e^{-k^2(2\mu+\lambda)r+krx_n},$$

$$u_j^{(2)}(x_1,\ldots,x_{n-1},t,k,r) = 0, \ 1 \leq j \leq n - 1,$$

$$u_n^{(2)}(x_1,\ldots,x_{n-1},t,k) = (2\mu+\lambda)\frac{e^{k^2(2\mu+\lambda)(t-T)}}{k^{N-1}}.$$ 

Then, for $0 < T < +\infty$, each function $u(x,t,k,T)$ is a solution to problem (14), (15), (16), (17) with the data $f(x,t,k,T)$, $u^{(0)}(x,t,k,T)$, $u^{(1)}(x,t,k,T)$, $u^{(2)}(x,t,k,T)$.

It is clear, that compatibility conditions (18), (19) hold and

$$f(x,t,k,T) \to 0 \text{ in } [C^\infty(\Omega_T)]^n, \quad u^{(0)}(x,k,T) \to 0 \text{ in } [C^\infty(\Omega_T)]^n,$$

$$u^{(1)}(x,t,k,T) \to 0 \text{ in } [C^\infty(\Omega_T)]^n, \quad u^{(2)}(x,t,k,T) \to 0 \text{ in } [C^\infty(\Omega_T)]^n,$$

if $N > 2s + 1$. On the other hand, for all $x_n > 0$ and all $N \in \mathbb{N}$ we have:

$$u_n(x,T,k) = \frac{e^{k^2(2\mu+\lambda)(T-T)+krx_n}}{k^N} \to +\infty \text{ as } k \to +\infty.$$ 

Thus, there is no continuity with respect to the data and hence Problem 2 is ill-posed for $0 < T < +\infty$.

Let now $T = +\infty$. Then we may consider the data with a fixed $0 < T_0 < +\infty$:

$$f(x,t,k,\infty) = 0 \in [C_b(\Omega_T)]^n, \quad u^{(0)}(x,k,\infty) = u^{(0)}(x,k,T_0) \in [C(\Omega_T)]^n,$$

$$u_j^{(i)}(x,t,k,\infty) = 0, \ 1 \leq j \leq n - 1, \ 1 \leq i \leq 2,$$

$$u_n^{(1)}(x,t,k,\infty) = \begin{cases} u_n^{(1)}(x,t,k,T_0), & t \leq T_0, \\ \frac{1}{k^N}, & t > T_0, \end{cases}$$

$$u_n^{(2)}(x,t,k,\infty) = \begin{cases} u_n^{(1)}(x,t,k,T_0), & t \leq T_0, \\ (2\mu+\lambda)\frac{e^{k^2(2\mu+\lambda)(t-T)}}{k^N}, & t > T_0. \end{cases}$$

Obviously, for $N \geq 2$,

$$f(x,t,k,\infty) \to 0 \text{ in } [C_b(\Omega_T)]^n, \quad u^{(0)}(x,k,\infty) \to 0 \text{ in } [C^\infty(\Omega_T)]^n,$$

$$u^{(1)}(x,t,k,\infty) \to 0 \text{ in } [C^1(\Omega_T)]^n, \quad u^{(2)}(x,t,k,\infty) \to 0 \text{ in } [C_b(\Omega_T)]^n.$$ 

The Uniqueness Theorem 3 for Problem 3 implies that

$$u_n(x,t,k,\infty) = u_n(x,t,k,T_0) \text{ for } 0 < t \leq T_0.$$ 

Then, for all $x_n > 0$ and all $2 \leq N \in \mathbb{N}$, we have $\lim_{k \to +\infty} u_n(x,T_0,k,\infty) = +\infty$. Thus, if the data $f$, $u^{(0)}$, $u^{(2)}$ admits the solution to (14) in $\Omega_T$ with boundary conditions (15), (16) for $T = +\infty$ and the initial condition (17) then there is no continuity with respect to the data in the chosen space. Otherwise there is no
solutions to the problem for some data in the data’s spaces. In any case, the problem is ill-posed for \( T = +\infty \), too.

As both Problems 1 and 2 are ill-posed, we will not study Problem 2 because in addition to (14)-(16) to investigate it one needs to know also the data related to initial condition (17). Besides, we will consider the case \( 0 < T < +\infty \) only.

3. Solvability Conditions

From now on we will study Problem 1 under the assumption that its data belong to Hölder spaces (cf., [3, Ch. 1, §1] for other boundary problems for parabolic equations). We recall that a function \( u(x) \), defined on a set \( M \in \mathbb{R}^m \), is called Hölder continuous with a power \( 0 < \lambda < 1 \) on \( M \), if there is such a constant \( C > 0 \) that

\[
|u(x) - u(y)| \leq C|x - y|^\lambda \quad \text{for all } x, y \in M
\]

(20) \(|x - y| = \sqrt{\sum_{j=1}^{m} (x_j - y_j)^2}\) being Euclidean distance between points \( x \) and \( y \) in \( \mathbb{R}^m \). Let \( C^\lambda(\Omega_T) \) stand for the set of Hölder continuous functions with a power \( \lambda \) over \( \Omega_T \). Besides, let \( C^{1+\lambda,\lambda}(\Omega_T) \) be the set of Hölder continuous functions with a power \( \lambda \) over \( \Omega_T \), having Hölder continuous derivatives \( u_{x_i}, 1 \leq i \leq n \), with the same power in \( \Omega_T \).

We choose a set \( \Omega^+ \) in such a way that the set \( D = \Omega \cup \Gamma \cup \Omega^+ \) would be a bounded domain with piece-wise smooth boundary. It is possible since \( \Gamma \) is an open connected set. It is convenient to set \( \Omega^- = \Omega \). For a function \( v \) on \( D_T \) we denote by \( v^+ \) its restriction to \( \Omega^+ \) and, similarly, we denote by \( v^- \) its restriction to \( \Omega \). It is natural to denote limit values of \( v \) on \( \Gamma_T \), when they are defined, by \( v^\pm_{|\Gamma_T} \).

**Theorem 4** (Solvability criterion). Let \( \Gamma \in C^{1+\lambda}, \quad f \in [C^\lambda(\Omega_T)]^n, u_1 \in [C^{1+\lambda,\lambda}(\Gamma_T)]^n, u_2 \in [C^\lambda(\Gamma_T)]^n \). Problem 1 is solvable if and only if there is a vector-function \( F \in [C^{2,1}(D_T)]^n \) satisfying the following conditions:

1) \( L_{n+1} F = 0 \) in \( D_T \),
2) \( F = G_{\Omega,0}(f) + V_{\Gamma,0}(u_2) + W_{\Lambda,0}(u_1) \) in \( \Omega_T^+ \).

**Proof.** Necessity. Let a function \( u(x,t) \in [C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \Gamma_T)]^n \) satisfies (14), (15), (16). Consider the function

\[
F = G_{\Omega,0}(f) + V_{\Gamma,0}(u_2) + W_{\Lambda,0}(u_1) - \chi_M u.
\]

in the domain \( D_T \), where \( \chi_M \) is a characteristic function of the set \( M \subset \mathbb{R}^{n+1} \). By the very construction condition 2) is fulfilled for it.

Clearly, the function \( u(x,t) \) belongs to the space \( [C^{1,2}(\Omega_T)]^n \) for each cylindrical domain \( \Omega_T' \) with such a base \( \Omega' \) that \( \Omega' \subset \Omega \) and \( \Omega' \cap \partial \Omega \subset \Gamma \). Besides, \( L_{n+1} u = f \in [C^\lambda(\Omega_T)]^n \). Without loss of the generality we may assume that the interior part \( \Gamma' \) of the set \( \Omega' \cap \partial \Omega \) is non-empty.

We note that \( \chi_{\Omega_T} u = \chi_{\Omega_T'} u \) in \( D'_T \), where \( D' = \Omega' \cup \Gamma' \cup \Omega^+ \). Then using Lemma 2 we obtain:

\[
F = G_{\Omega,\Gamma,0}(f) + V_{\Gamma',0}(u_2) + W_{\Lambda,\Gamma',0}(u_1) - I_{\Omega',0}(u) \text{ in } D'_T.
\]

(21) Arguing as in the proof of Theorem 3 we conclude that each of the integrals in the right hand side of (21) satisfies homogeneous Lamé type equation outside
the corresponding integration set. In particular, we see that $L_{n+1}F = 0$ in $D'_T$. 

Obviously, for any point $(x, t) \in D_T$ there is a domain $D'_T$ containing $(x, t)$. That is why $L_{n+1}F = 0$ in $D_T$, and hence $F$ belongs to the space $[C^{2,1}(D_T)]^n$. Thus this function satisfies condition 1), too.

**Sufficiency.** Let there be a function $F \in [C^{2,1}(D_T)]^n$, satisfying conditions 1) and 2) of the theorem. Consider on the set $D_T$ the function

$$U = G_{\Omega,0}(f) + V_{\Gamma,0}(u_2) + W_{T,0}(u_1) - F.$$  

As $f \in [C^\lambda(\Omega_T)]^n$ then the results of [3, Ch. 1, §3] imply

$$G_{\Omega,0}(f) \in [C^{2,1}(\Omega_T^+) \cap C^{1,0}(\partial \Omega_T^+) \cap C(\overline{D_T} \setminus (\partial \Gamma_T))]^n,$$

and, moreover,

$$L_{n+1}G_{\Omega,0}(f) = f \text{ in } \Omega_T, \quad L_{n+1}G_{\Omega,0}(f) = 0 \text{ in } \Omega_T^+.$$  

(22)

Since $u_2 \in [C^\lambda(\Omega_T)]^n$ then the results of [3, Ch. 5, §2] yield

$$V_{\Gamma,0}(u_2) \in [C^{2,1}(\Omega_T^+) \cap C^{1,0}((\Omega_T^+ \cup \Gamma)_T) \cap C(\overline{D_T} \setminus (\partial \Gamma_T))]^n,$$

and

$$L_{n+1}V_{\Gamma,0}(u_2) = 0 \text{ in } \Omega_T \cup \Omega_T^+.$$  

(25)

On the other hand, the behavior of the Double Layer Potential $W_{\Gamma,0}(u_1)$ is similar to the behavior of the normal derivative of Single Layer Potential $V_{\Gamma,0}(u_1)$. Hence

$$W_{\Gamma,0}(u_1) \in [C^{2,1}(\Omega_T^+) \cap C(\overline{\Omega_T} \setminus (\partial \Omega_T^+ \cup \Gamma_T))]^n,$$

(27)

and

$$L_{n+1}W_{\Gamma,0}(u_1) = 0 \text{ in } \Omega_T \cup \Omega_T^+.$$  

(28)

**Lemma 2.** Let $S \subset \Gamma \in C^{1+\lambda}$. If $u_1 \in [C^{1+\lambda,\lambda}(\Gamma_T)]^n$, then the potential $W_{\Gamma,0}(u_1)$ belongs to the space $[C^{1,0}(\Omega_T \cup S_T)]^n$ if and only if $W_{\Gamma,0}^+(u_2) \in [C^{1,0}((\Omega_T^+ \cup \Gamma_T)]^n$. Thus, formulas (22)–(28) and Lemma 2 imply that

$$U \in [C^{2,1}(\Omega_T^+) \cap C^{1,0}((\Omega_T^+ \cup \Gamma)_T) \cap C(\overline{\Omega_T} \setminus (\partial \Omega_T^+ \cup \Gamma_T))]^n,$$

and

$$L_{n+1}U = \chi_{D_T}f \text{ in } \Omega_T \cup \Omega_T^+.$$  

In particular, (14) is fulfilled for $U^-$.

Let us show that the function $U^-$ satisfies (15) and (16).

Since $F \in [C^{1,0}(D_T)]^n$ we see that $\partial^\alpha F^- = \partial^\alpha F^+$ on $\Gamma_T$ for $\alpha \in \mathbb{Z}_+$ with $|\alpha| \leq 1$ and

$$\partial^\alpha F^+_{|\Gamma_T} = (\partial^\alpha G^+_{\Omega,0}(f) + \partial^\alpha V_{\Gamma,0}^+(u_2) + \partial^\alpha W_{T,0}^+(u_1))_{|\Gamma_T}.$$  

It follows from formulas (23) and (25) that the Parabolic Volume Potential and the Single Layer Parabolic Potential are continuous if the point $(x, t)$ passes over the surface $\Gamma_T$. Then

$$U^-_{|\Gamma_T} = W_{\Gamma,0}^+(u_1)_{|\Gamma_T} - W_{T,0}^+(u_1)_{|\Gamma_T} = u_1.$$  

because of the theorem on jump behavior of the Parabolic Double Layer Potential (see, for instance, [3, Ch. 5, §2, theorem 1]), i.e. equality (15) is valid for $U^-$. 


Formula (23) means that the surface stress of the Parabolic Volume Potential is continuous if the point \((x, t)\) passes over the surface \(\Gamma_T\). Therefore

\[
\langle \sigma U \rangle^*_{\Gamma_T} = \left( \sigma W^-_{T,0} u_2 \right)_{\Gamma_T} - \left( \sigma W^+_T u_2 \right)_{\Gamma_T} + \left( \sigma W^-_{T,0} u_1 \right)_{\Gamma_T} - \left( \sigma W^+_T u_1 \right)_{\Gamma_T}.
\]  

(29)

By theorem on jump behavior of the stress of the Parabolic Single Layer Potential (see, for instance, [15, Ch. 3, §10, theorem 10.1])

\[
\left( \sigma W^-_{T,0} u_2 \right)_{\Gamma_T} - \left( \sigma W^+_T u_2 \right)_{\Gamma_T} = u_2.
\]  

(30)

Finally, we need the following lemma which is an analogue of the famous Theorem on jump behavior of the normal derivative of the Newton’s Double Layer Potential.

**Lemma 3.** Let \(\Gamma \in C^{1+\lambda}\) and \(u_2 \in [C^\lambda(\Gamma_T)]^n\). If \(W^-_{T,0}(u_1) \in [C^{1,0}(\Omega \cup \Gamma_T)]^n\) or \(W^+_T(u_1) \in [C^{1,0}(\Omega^+ \cup \Gamma_T)]^n\) then

\[
\left( \sigma W^-_{T,0} u_1 - \sigma W^+_T u_1 \right)_{\Gamma_T} = 0.
\]  

(31)

**Proof.** Really, let, for instance, \(W^-_{T,0}(u_1) \in [C^{1,0}(\Omega \cup \Gamma_T)]^n\). Then using Lemma 2 we obtain \(W^+_T u_1 \in [C^{1,0}(\Omega^+ \cup \Gamma_T)]^n\) and \(\left( \sigma W^+_T u_1 \right)_{\Gamma_T} \in [C(\Gamma_T)]^n\).

Let \(\phi \in \mathbb{C}_0^\infty(D_T)\) be a function with compact support in \(D_T\). Then formulas (9)–(11) yield:

\[
\int_{\Gamma_T} \phi^* \left( \sigma W^-_{T,0} u_1 - \sigma W^+_T u_1 \right) ds(x) dt =
\]

\[
\int_{\Omega_T \cup \Omega^+_T} \phi^* (L_n + a) W^-_{T,0} u_1 dx dt + \int_{T_1}^{T_2} D_{\Omega, \Omega^+} (W^-_{T,0} u_1, \phi) dt =
\]

\[
\int_{\Omega_T \cup \Omega^+_T} \phi^* \left( \frac{\partial}{\partial t} - A + a \right) W^-_{T,0} u_1 dx dt + \int_{T_1}^{T_2} D_{\Omega, \Omega^+} (W^-_{T,0} u_1, \phi) dt
\]

because \(L_{n+1} W^+_T u_1 = 0\) in \(\Omega^+\) according to (28).

Again, integrating by parts and using formulas (9)–(11) and Theorem on jump behavior of the Parabolic Double Layer Potential, we see that

\[
\int_{\Omega_T \cup \Omega^+_T} \phi^* \left( \frac{\partial}{\partial t} - A + a \right) W^-_{T,0} u_1 dx dt + \int_{T_1}^{T_2} D_{\Omega, \Omega^+} (W^-_{T,0} u_1, \phi) dt =
\]

\[
- \int_{\Omega_T \cup \Omega^+_T} \left( \frac{\partial \phi}{\partial t} \right)^* W^-_{T,0} u_1 dx dt - \int_{\Omega_T \cup \Omega^+_T} (L_n + A^*)^* W^-_{T,0} u_1 dx dt +
\]

\[
\int_{\Gamma_T} (\tilde{\sigma} \phi)^* (W^-_{T,0} u_1 - W^+_T u_1) ds(x) dt =
\]

\[
\int_{\Gamma_T} (\tilde{\sigma} \phi)^* u_1 ds(x) dt - \int_{\Omega_T \cup \Omega^+_T} (L_{n+1} \phi)^* W^-_{T,0} u_1 dx dt.
\]

But the kernel \(\Phi(x, y, t, \tau)\) is a fundamental solution of the backward parabolic operator \(L_{n+1}\) with respect to variables \((y, \tau)\). Hence

\[
\int_{D_T} (L_{n+1} \phi(x, t))^* \Phi(x, y, t, \tau) dx dt = \phi^*(y, \tau), \quad (y, \tau) \in D_T.
\]
Then the type of the singularity of the fundamental solution allows us to apply Fubini Theorem and to conclude that
\[
\int_{\Omega_T \cup \Omega^+} (L_{n+1}^* \phi)^* W_{\Gamma,0} u_1 dx dt = (34)
\]
\[
\int_{\Gamma_T} \tilde{\phi} \int_{D_T} (L_{n+1}^* \phi(x,t))^* \Phi(x, y, t, \tau) dx dt u_1 ds(y) d\tau = \int_{\Gamma_T} (\tilde{\phi} \phi)^* u_1 ds(y) d\tau.
\]
Finally, formulas (32)- (34) imply that
\[
\int_{\Gamma_T} \phi^* \left( \sigma W_{\Gamma,0}^- u_1 - \sigma W_{\Gamma,0}^+ u_1 \right) ds = 0
\]
for all \( \phi \in [C^\infty_0(D_T)]^n \). As such functions are dense in the Lebesgue space \([L^1(K)]^n\) for any compact \( K \subset \Gamma_T \) then formula (31) holds true. \( \square \)

Now using lemma 3 and formulas (29), (30), we conclude that \( (\sigma U)^- = u_2 \), i.e. (16) is fulfilled for \( U^- \).

Thus, function \( u(x, t) = U^-(x, t) \) satisfies conditions (14)–(16). The proof is complete. \( \square \)

It follows from formula (22) that properties of a solution to Problem 1 depend on properties of the extension \( F \) of the sum of the parabolic potentials, described in Theorem 4.

**Corollary 2.** Let \( S \subset \partial \Omega \setminus \Gamma \). Under the hypotheses of Theorem 4, Problem 1 is solvable in the space
\[
[C^{2,1}(\Omega_T) \cap C^{1,0}(\Omega_T \cup \Gamma_T) \cap C(\overline{\Omega_T \setminus S_T})]^n
\]
if and only if there exists a function
\[
F \in [C^{2,1}(D_T) \cap C^{1,0}(\Omega_T \cup \Gamma_T) \cap C(\overline{\Omega_T \setminus S_T})]^n,
\]
satisfying conditions 1) and 2) of Theorem 4.

In particular, if \( S = 0 \) then corollary 2 gives criterion for the existence of solution to Problem 1 in the space \([C(\Omega_T)]^n\).

We note that Theorem 4 is an analogue of Theorem by Aizenberg and Kytmanov [8]) describing solvability conditions of the Cauchy problem for the Cauchy–Riemann system (cf. also [14] in the Cauchy Problem for Laplace Equation or [12] in the Cauchy problem for general elliptic systems). Formula (22), obtained in the proof of Theorem 4, gives the unique solution to Problem 1. Clearly, if we will be able to write the extension \( F \) of the sum of potentials \( G_{\Omega,0}(f) + V_{\Gamma,0}(u_2) + W_{\Gamma,0}(u_1) \) from \( \Omega^+ \) onto \( D_T \) as a series with respect to special functions or a limit of parameter depending integrals then we will get a Carleman type formula for solutions to Problem 1 (cf. [8]). However this is a topic for another paper. In the sequel we will discuss polynomial and formal solutions for operators with constant coefficients only.

4. POLYNOMIAL SOLUTIONS AND DENSE SOLVABILITY

It is not difficult to prove dense solvability of Problem 1 in the case where \( \Gamma \) is an open connected set of the hyperplane \( \{x_n = 0\} \).

**Lemma 4.** If \( \Gamma \) is an open connected set if the hyperplane \( \{x_n = 0\} \) the Problem 1 is densely solvable.
Proof. First let us prove that if in this case the data of Problem 1 are polynomials then the problem is solvable and its solution is a polynomial.

Indeed, Problem 1 is easily can be reduced to the following one (see Example 3):

\[
L_{n+1}v = g \text{ in } \Omega_T
\]
\[
v(x_1, \ldots, x_{n-1}, 0, t) = 0 \text{ on } \Gamma_T,
\]
\[
\frac{\partial v}{\partial x_n}(x_1, \ldots, x_{n-1}, 0, t) = 0 \text{ on } \Gamma_T, 1 \leq j \leq n - 1,
\]
\[
(2\mu + \lambda) \frac{\partial v_n}{\partial x_n}(x_1, \ldots, x_{n-1}, 0, t) = 0 \text{ on } \Gamma_T, 1 \leq j \leq n - 1,
\]

with
\[
g(x, t) = f(x, t) - (L_n u_1)(x_1, \ldots, x_{n-1}, t) - x_n J(\mu, \lambda)(L_n u_2)(x_1, \ldots, x_{n-1}, t).
\]

where \(J(\mu, \lambda)\) is the diagonal matrix with the components
\[
J_{jj}(\mu, \lambda) = \mu^{-1}, 1 \leq j \leq n - 1, J_{nn}(\mu, \lambda) = (2\mu + \lambda)^{-1}.
\]

Besides, \(u(x, t) = v(x, t) + u_1(x_1, \ldots, x_{n-1}, t) + J(\mu, \lambda)x_n u_2(x_1, \ldots, x_{n-1}, t)\).

Now consider data \(g^{(j, \alpha)}(x, t) = t^j x^\alpha\) with a multi-index \(\alpha \in \mathbb{Z}_+^n\).

If \(0 \leq \alpha_1 + \ldots + \alpha_n = 1\), we easily obtain (unique) polynomial solutions
\[
v^{(j, \alpha)}(x, t) = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} w^{(j, \alpha)}(x_n, t), \alpha_n, j \in \mathbb{Z}_+, \quad (39)
\]
to problem (35)–(37) where
\[
w^{(0, k)}(y, t) = -\frac{y^{k+2}k!}{(k + 2)!}, \quad w^{(1, k)}(y, t) = -\frac{ty^{k+2}k!}{(k + 2)!} - \frac{y^{k+4}k!}{(k + 4)!}, k \in \mathbb{Z}_+, y \in \mathbb{R}
\]
and, by the induction with respect to \(j \in \mathbb{Z}_+\),
\[
w^{(j, k)}(y, t) = -\sum_{\mu=0}^j t^{j-\mu}y^{k+2\mu+2\mu+2}k!j! (k + 2\mu + 2)! (j-\mu)! \cdot k \in \mathbb{Z}_+, y \in \mathbb{R}. \quad (40)
\]

To finish the arguments we use the induction with respect to \(|\alpha'| \in \mathbb{Z}_+\) where
\(\alpha' = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}\). Namely, let for \(s \geq 2\) and all \(\alpha'\) with \(|\alpha'| = s\) the solutions to the problem are polynomial. If \(|\alpha'| = s + 1\) then
\[
L_{n+1}\left(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} w^{(j, \alpha)}(x_n, t)\right) = t^j x^\alpha - w^{(j, \alpha)}(x_n, t)\Delta_{n-1}\left(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}\right).
\]

Clearly, the degree of the polynomial \(p_{j, \alpha}(x, t) = w^{(j, \alpha)}(x_n, t)\Delta_{n-1}\left(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}\right)\) with respect to \(x' \in \mathbb{R}^{n-1}\) equals to \(s - 1\). Then, by the induction, problem (14)–(16) with data \(p_{j, \alpha}(x, t)\) admits a polynomial solution, say, \(r_{j, \alpha}(x, t)\). Therefore the solution \(v^{(j, \alpha)}(x, t)\) to problem (14)–(16) with data \(g^{(j, \alpha)}(x, t) = t^j x^\alpha, |\alpha'| = s + 1\), is given as follows:
\[
v^{(j, \alpha)}(x, t) = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} w^{(j, \alpha)}(x_n, t) + r_{j, \alpha}(x, t),
\]
i.e. it is a polynomial, too.

Now Problem 1 with zero boundary data in the case \(\Gamma \subset \{x_n = 0\}\) is densely solvable because any continuous function \(g\) on the compact set \(\Omega_T\) can be approximated by polynomials. But the reducing to zero boundary data was organized in such a way that one easily sees, in this case Problem 1 is densely solvable for non-zero boundary data, too. \(\square\)
The dense solvability of Problem 1 in general setting is natural to expect if the set $\partial \Omega \setminus \mathbf{T}$ has at least one interior point in $\partial \Omega$ (cf. [10] in the Cauchy Problem for elliptic equations).

Finally, we note that polynomial solutions indicated in the proof of Lemma 4 can be used in order to construct formal solutions to Problem 1.

5. Basis with double orthogonality

Denote by $\{h_\nu(s)\}$ the set of harmonic homogeneous polynomials (spherical harmonics) forming an orthonormal basis in $L^2(\partial B(0,1))$; here $\nu$ is the degree of the homogeneity and $1 \leq s \leq J(s)$ where $J(s) = \frac{(2n + 2s - 2)(2n + s - 3)}{s!(2n - 2)!}$ (see [18]).

Lemma 5. Let $\mu = 1$, $\lambda = -1$. Then the polynomials

$$H^{(s)}_{0,\nu}(x,t) = h^{(s)}_\nu(x), \nu \geq 0,$$

$$H^{(s,i)}_{N,0}(x,t) = \sum_{j=0}^{N} \frac{t^j x^{2N-2j}(2N)!}{j!(2N-2j)!}, 1 \leq i \leq n, N \geq 1,$$

$$H^{(s,i)}_{N,\nu}(x,t) = \sum_{j=0}^{N} \frac{t^j x^{2N-2j}h^{(s)}_\nu(x)(2N)!((n + 2N - 2 + 2\nu))!!}{j!(2N-2j)!!((n + 2N - 2j + 2\nu))!!}, \nu \geq 1, N \geq 1$$

are solutions to the heat equations in $\mathbb{R}^{n+1}$.

Besides, $H^{(s)}_{N,\nu}$ and $H^{(p)}_{N,M,\mu}$ are $L^2(B(0,R)T)$-orthogonal for all $R > 0$ and $T > 0$ if $(\nu, s) \neq (\mu, p)$.

Proof. Indeed, for $\nu \geq 0$, $N \geq 1$, and $1 \leq i \leq n$ we have:

$$L_{n+1}H^{(s)}_{0,\nu} = -\Delta_n h^{(s)}_\nu,$$

$$L_{n+1}H^{(s,i)}_{N,0} = \sum_{j=1}^{N} \frac{t^{j-1} x^{2N-2j}(2N)!}{(j-1)!(2N-2j)!} - \sum_{j=0}^{N-1} \frac{t^j x^{2N-2j-2}(2N)!}{j!(2N-2j-2)!} = 0.$$

On the other hand,

$$\frac{\partial |x|^{2k}}{\partial x_j} = 2kx_j |x|^{2k-2}, \quad \frac{\partial^2 |x|^{2k}}{\partial x_j^2} = 2k|x|^{2k-2} + 2k(2k - 2)x_j^2 |x|^{2k-4},$$

$$\Delta_n |x|^{2k} = 2k(n + 2k - 2)|x|^{2k-2}, k \geq 1.$$

Hence, for $\nu \geq 1$, $k \geq 1$,

$$\Delta_n (|x|^{2k} h_\nu(x)) = 2k(n + 2k + 2\nu - 2)|x|^{2k-2}h^{(s)}_\nu(x)$$

because of Euler’s formula

$$\sum_{j=1}^{n} x_j \frac{\partial h^{(s)}_\nu(x)}{\partial x_j} = \nu h^{(s)}_\nu(x).$$

Consider the polynomial $H = \sum_{j=0}^{N} c_j t^j |x|^{2N-2j} h^{(s)}_\nu(x)$ with constants $c_j$. Then

$$L_{n+1}H = \sum_{j=1}^{N} c_j t^{j-1} |x|^{2N-2j} h^{(s)}_\nu(x) -$$
\[
\begin{align*}
N-1 \sum_{j=0}^{N-1} c_j t^j (2N - 2j) (n + 2N - 2j + 2\nu - 2) |x|^{2N-2j} h_{\nu}^{(s)}(x) = \\
\sum_{j=1}^{N} t^{j-1} |x|^{2N-2j} h_{\nu}^{(s)}(x) [jc_j - c_{j-1} (2N - 2j + 2) (n + 2N - 2j + 2\nu)]
\end{align*}
\]

Thus, we get a recurrent formula

\[
c_j = j^{-1} c_{j-1} (2N - 2j + 2) (n + 2N - 2j + 2\nu)
\]

for the coefficients in the case \(L_{n+1} H = 0\). Choosing \(c_0 = 1\) we easily obtain

\[
c_j = \frac{(2N)! |n + 2N - 2 + 2\nu|!}{j!(2N - 2j)! (n + 2N - 2j + 2\nu)!}.
\]

Finally the statement on \(L^2(B(0,R)_{T})\)-orthogonality follows from Fubini Theorem and the homogeneity of the polynomials \(h_{\nu}^{(s)}(x)\).

This lemma suggests us to consider a function \(H_{\nu}(x, t) = \phi(|x|, t) h_{\nu}^{(s)}(x)\).

Easily

\[
\frac{\partial \phi(|x|, t)}{\partial x_j} = \frac{x_j}{|x|} \frac{\partial \phi(|x|, t)}{\partial z},
\]

\[
\frac{\partial^2 \phi(|x|, t)}{\partial x_j^2} = \frac{1}{|x|} \frac{\partial \phi(|x|, t)}{\partial z} - \frac{x_j^2}{|x|^2} \frac{\partial \phi(|x|, t)}{\partial z} + \frac{x_j^2}{|x|^2} \frac{\partial^2 \phi(|x|, t)}{\partial z^2},
\]

\[
\sum_{j=1}^{n} x_j \frac{\partial h_{\nu}}{\partial x_j} = \nu h_{\nu}.
\]

Hence

\[
L_{n+1} H_{\nu}(x, t) = \left( \frac{\partial \phi(|x|, t)}{\partial t} - \frac{\partial^2 \phi(|x|, t)}{\partial z^2} (|x|, t) - \frac{(n + 2\nu - 1)}{z} \frac{\partial \phi(|x|, t)}{\partial z} \right) h_{\nu}^{(s)}(x) = 0
\]

if \(\phi(z, t)\) is a solution to parabolic equation

\[
\frac{\partial \phi(z, t)}{\partial t} - \frac{\partial^2 \phi(z, t)}{\partial z^2} (z, t) - \frac{(n + 2\nu - 1)}{z} \frac{\partial \phi(z, t)}{\partial z} (z, t) = 0
\]

for \(z > 0, t > 0\)

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[18] Sobolev, S.L.,

(Roman Puzyrev) Siberian Federal University, Institute of Mathematics and Computer Science, pr. Svobodnyi 79, 660041 Krasnoyarsk, Russia
E-mail address: effervesce@mail.ru

(Alexander Shlapunov) Siberian Federal University, Institute of Mathematics and Computer Science, pr. Svobodnyi 79, 660041 Krasnoyarsk, Russia
E-mail address: ashlapunov@sfu-kras.ru