Abstract. This paper analyzes the existence of regularizations of integrals that apply to functions with a nonintegrable singularity at an endpoint of integration. This is a problem arising naturally in many contexts including solution of PDEs and singular ODEs. Regularizations, such as the classical Hadamard finite part (p.f.), share two fundamental properties: (i) the regularized integral of a function on an interval only depends on the values of the function in that interval, and of course, (ii) the regularized integral is an antiderivative.

Sufficient conditions for existence of regularizations are well known, and they require various types of smoothness. The following is a natural apparently open question: do regularizations exist beyond their known domain? We show that there exist regularizations of integrals satisfying the properties of p.f. which apply to singular functions without any conditions on the type or strength of the singularity.

However, the very existence of a regularization beyond p.f. satisfying only (i) and (ii) is independent of ZF (the usual ZFC axioms for mathematics without the Axiom of Choice AC), and even of ZFDC (ZF with the Axiom of Dependent Choice). This is established in §7.2, where we also show that such extensions cannot be naturally given - even if we are using the full axiom of choice. In particular, we show that there is no mathematical description that can be proved (within ZFC or even extensions of ZFC with large cardinal hypotheses) to uniquely define such a regularization. In a precise sense, the classical domain of p.f. is optimal.

Such results for a variety of spaces of functions are precisely formulated, and proved using methods from mathematical logic, descriptive set theory and analysis.

1. Introduction

In the local analysis of singular differential equations it is important to understand the behavior of solutions near singularities.

Perhaps the best understood singular initial condition problem is that of linear meromorphic linear ODEs with a regular singularity (a singularity is regular if it is a pole of low enough order, see [3]). Assume 0 is the singular point; the task is to find a fundamental system of solutions with conditions placed at zero. As a simple example the Euler equation \( f'' - 2x^{-2} f = 0 \) has the fundamental solution \( A/x + Bx^2 \). The conditions (at zero) \( (xf)'(0) = A \), \( (xf)''(0) = 6B \) select a unique solution. Frobenius theory handles general regular singularities. More difficult to characterize and to study locally are general irregular singularities. Their analysis triggered the development of powerful methods such as Borel-Ecalle summation.
and hyperasymptotics, see e.g. [3, 11, 13, 14, 16, 19, 30]. To take a simple example, the ODE \( y' = y + 1/x \) has an irregular singularity at infinity. There is exactly one solution \( y_0 \) which decays at \(+\infty\), and the general solution is \( y_0 + Ce^x \). Clearly, \( \lim_{x \to +\infty} e^{-x}f = C \) is an initial condition at infinity which selects exactly one solution for every \( C \). At \(-\infty\) the situation changes completely: any two solutions of this equation have the same asymptotic series. To resolve the solutions based on their behavior at \(-\infty\) for general ODEs, Ecalle-Borel summability and analyzability, a notion extending analyticity, see [11], [12] and references therein are needed. Analyzability extends to classes of PDEs, [12]; in special cases other methods of exponential asymptotics such as summation to the least term, [3] may suffice. We briefly discuss the implications of our results for this question in §4.

Singular integration arises in many other contexts, such as PDEs, pseudodifferential operators [20] and orthogonal polynomials. We will describe one of the most powerful and widely used one, the Hadamard finite part \( \text{p.f.} \); \( \text{p.f.} \) was introduced by Hadamard in [17] for dealing with divergent integrals of the type

\[
\Gamma(\alpha)I^\alpha = \int_a^b f(s)(b-s)^{\alpha-1}ds, \quad \alpha \leq 0
\]

where the singular term is the Green’s function for a hyperbolic PDE. Clearly, when \( 1 \leq \alpha \in \mathbb{N} \), then \( I^\alpha \) is the simply the \( \alpha \)th antiderivative of \( f \) (and for general noninteger \( \alpha \) it is by definition the derivative of order \(-\alpha\) of \( f \)).

The \( \text{p.f.} \) regularization has since become an important tool in a wide array of problems ranging from pseudodifferential operators [3], orthogonal polynomials with singular weights, [9] to mathematical physics [4]. The Cauchy principal part, in a sense a special case of \( \text{p.f.} \), is of central importance as well, for instance in the solution of Riemann-Hilbert problems or the solution of Laplace’s equation.

In orthogonal polynomials, singular initial value problems in ODEs and in mathematical physics, the interest is in regularizing integrals with \( x \)-independent integrand, see e.g. [4, 9] and references therein,

\[
\Gamma(\alpha)J^\alpha = \int_0^x s^{\alpha-1}f(s)ds
\]

For with more general algebraic-logarithmic singularities than \( s^{\alpha-1} \) [4]. The objective in [17] (p.134 and on) is to give a meaning to the right side of (1) for each \( b \); this problem \( \text{11} \) is equivalent to regularizing (2). Of course, if \( (b-s)^{\alpha-1} \) arises as a Green’s function, the interpretation of (1) in the sense of distributions suffices.

As shown by M. Riesz, cf. [31], a natural way to define the Hadamard finite part is by analytic continuation of (2), with respect to \( \alpha \), starting from a power of the kernel which is integrable. Analytic continuation from \( \text{Re} \alpha > 0 \) to \( \text{Re} \alpha > -n \) exists if

\[
f \in C^n((a,b]) \text{ and } f^{(n)}(s)s^{\alpha+n-1} \in L^1
\]

This is manifest after integration by parts:

\[
\Gamma(\alpha)(J^\alpha f)(x) = \sum_{k=0}^{n-1} \frac{\Gamma(\alpha)f^{(k)}(x)}{\Gamma(\alpha+k+1)} x^{\alpha+k} + \int_a^x s^{\alpha+n-1}f^{(n)}(s)ds
\]

\[\text{Any such pseudodifferential operator of noninteger order can be identified with some \( \text{p.f.} \), see [20] and also [4] and the comment below it.}\]
As shown by Hadamard, p.f. retains all properties of usual integration except for positivity (see however the note below). The analytic continuation interpretation makes this obvious, and shows that p.f. is a natural extension of integration.

**Note 1.** By inversion, \( x = 1/t \) we get a regularization at infinity of the integral \((t,g) \mapsto \int_t^\infty g\). The corresponding p.f. integral preserves positivity, in the sense that if \( f(t) \) is eventually positive as \( t \to \infty \), then \( \int_t^\infty f \) is also eventually positive. This is obvious from the asymptotic behavior of \( g \).

While \( \alpha = -n \) with conditions weaker than \( 3 \)? A nontrivial extension of \( J^\alpha \), \( \Re \alpha > -n \), with, say, \( a = 0 \) and \( b = 1 \) reduces, after \( n \) integration by parts, to the existence of a regularization \( P_0 \) as in the simple ODE above. The answer to the first one is that the necessary conditions for existence of well-behaved extensions are very weak: in particular differentiability of the integrand, as well as bounds at zero are not required. The answer to the second one is no. We will explain in detail what we mean by this. The proofs use a combination of analysis, descriptive set theory and mathematical logic.

The connection to a longstanding open question about Conway’s number field \( \mathbb{N} \) is explained in \( \S 3.2 \).

2. Setting

For our negative results, we impose only the properties of the integral from zero which, arguably, any extension should retain: \( [P_0(h)](x) = h(x) \) a.e. and that \( [P_0(h)](x) \) should only depend on the values of \( h \) on \((0, x]\). Consider the equivalence relation

\[ f \sim_0 g \text{ if } \exists \varepsilon > 0 \text{ s.t. } f = g \text{ on } (0, \varepsilon) \]

**Definition 2.** \( P \) is a property at zero if \( \forall \varepsilon > 0, f, g, f \sim_0 g \) implies \( P(f) \Leftrightarrow P(g) \).

An operator \( T \) based at zero, or simply at 0 is one for which \( \forall \varepsilon > 0, f, g \sim_0 g \) implies \( Tf \sim_0 Tg \). An initial condition at 0 is defined in a similar way. Properties and operators based at other points in \( \mathbb{R} \cup \{ \infty \} \) are analogously defined.

**Definition 3.** \( P_0 \) is an extension of the integral from zero on a set \( F \) of functions on \((0, 1]\) if

\( P_0 \) is an operator at zero with range in \( AC((0, 1]) \)

\( (P_0 f)' = f \) a.e.

**Note 4.** (1) Since \( (\int_0^x f)' = f \) and the range of both \( P_0 \) and \((x, f) \mapsto \int_1^x f \) is \( AC((0, 1]) \), for any \( f \) there is a constant \( C(f) \) s.t. \( \forall x \in (0, 1] \) we have \( [P_0 f](x) = \int_1^x f + C(f) \). This also implies that if \( f \in C((0, 1]) \), then \( (P_0 f)' = f \) everywhere and for any \( f \) in the domain of \( P_0 \), all \( y \in (0, 1] \) and all \( x \in (0, y) \) we have \( [P_0(f)](y) = [P_0(f)](x) + \int_x^y f \) where \( f \) is the usual Lebesgue integral.

\[ 4 \text{ Absolutely continuous functions, always italicized. In any case, there is little danger of confusion with the Axiom of Choice.} \]

\[ 4 \text{ Almost everywhere; for the negative results we work with continuous functions for which a.e. can be removed (see Note 3).} \]
(2) By (1) above, the range of \( P_0 \) is the same as that of \( \int_1^x \). In particular, if the domain is in \( C((0, 1]) \), then the range is in \( C^1((0, 1]) \).

(3) We are interested in proper generalizations of integration, and thus we assume from now on that

\[
\exists f \in F \text{ s.t. } \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} f(s)ds \text{ does not exist}
\]

**Note 5.** In the spaces we will work, one can construct from any weight giving the space a function \( \tau \) such that

\[
\tau \in F, \tau \geq 0 \text{ a.e. and } \int_0^1 \tau = \infty
\]

The domain of \( P_0 \) will consist of sets \( F \) of functions, which are \( L^1 \) away from zero, with norms which constrain the growth at zero without entailing regularity at 0. To avoid spurious singularities, the weights are required to satisfy

\[
w \in C((0, 1]), w(0) = 0 \text{ and } \forall \varepsilon \in (0, 1], \inf_{[\varepsilon, 1]} w > 0
\]

**Note 6.** The weight \( w \) can be assumed without loss of generality to be smooth. This can be arranged in a number of well-defined ways, see Lemma [17].

2.0.1. Analytic case. For functions with unique continuation, evidently \( \sim_0 \) is a vacuous restriction and \( \int_1^\infty \) would be an unintended \( P_0 \). A natural modification of (I), \( f \sim_0 g \) is defined as follows. For *any* decreasing sequence \( (\alpha_n)_{n \in \mathbb{N}} \) in \( (0, 1) \) s.t. \( \lim_{n \to \infty} \alpha_n = 0 \) we have

\[
\text{if } \forall n \in \mathbb{N} \left( \int_{\alpha_n+1}^{\alpha_n} f = \int_{\alpha_n+1}^{\alpha_n} g \right), \text{ then } \left( P_0 f)(\alpha_1) = (P_0 g)(\alpha_1) \right)
\]

(9) (where all the integrals on the left side exist of course in the usual sense). For weighted analytic functions spaces we will still denote by \( P_0 \) operators satisfying (9) and (II).

2.1. Spaces. All weighted \( L^p \) spaces, Orlicz spaces and other normed spaces commonly used in analysis which do not impose regularity have the lattice property: if \( \|f\| < \infty \) and \( |g| \leq |f| \) then \( \|g\| < \infty \). We do not need a Banach space structure, and the lattice property is used in a weaker form, WL. Let \( \chi_A \) be the indicator function of the set \( A \).

**Definition 7.** A set \( F \) of functions on \( (0, 1] \) has the WL property if for any \( f \in F \) and any open set \( O \subset (0, 1) \) the following is true: \( \exists g \in F \text{ s.t. } |g| \leq |f|, g = 1 \text{ in the complement of } O \text{ and } \int_O \left| g - f \chi_O \right| < 1 \) (where \( \int \) is the usual Lebesgue integral).

\[\text{[5]}\text{ WL holds for weighted } L^p \text{ spaces, as well as for the set of smooth functions in these weighted spaces, and it implies [7] (see Lemma [10] below).}\]

5 Meant, is the usual sense, as Hölder or Lipschitz continuity, differentiability, finiteness of nontrivial Sobolev norms, etc; cf. also [17], p. 11.
2.1.1. Sets of functions analyzed. We study subsets \( \tilde{\mathcal{F}} \) with WL of \( L_p^w \), \( p \in [1, \infty] \), weighted \( L^p \) spaces where the weight \( w \) satisfies \( (8) \). We consider subsets \( \mathcal{F} \) with various degrees of regularity, from measurability to \( C^\infty \) and real-analyticity. For the spaces consisting of real analytic functions, \( (II) \) is replaced by \( (9) \). Our results extend to other types of spaces. We denote by \( \tilde{\mathcal{F}} \) a collection of such spaces parameterized by a number of continuous functions.

Since our negative results become stronger if the domain of \( P_0 \) is narrowed, we restrict for simplicity to \( \mathcal{F} = \tilde{\mathcal{F}} \cap C^0((0,1]) \) (these are not necessarily closed subspaces, but closure is not needed). Then, “a.e.” can then be omitted from \( (II) \).

The collection of these spaces will be denoted by \( \mathcal{F} \).

Note 8. As specific examples we consider

\[
C^\infty := \{ f \in C^\infty((0,1]) : wf \in L^1((0,1]) \}
\]

\[
A_1 := \{ f \text{ real analytic on } (0,1) : wf \in L^1((0,1]) \}
\]

where \( w(x) = x\sqrt{\ln^2 x + 1} \) (arbitrarily specified weak weights that allow for non-\( L^1 \) growth can be chosen instead).

\[
A_2 := \{ f \text{ analytic in } D \setminus \{0\} : Wf \text{ bounded in } D \}
\]

where \( D \) is the unit disk and \( W \) is any specified weight decreasing faster than polynomially, say \( W = e^{-2|\ln |z||}^2 \) (any \( W \) s.t. \( 1/W \) grows super-polynomially would also work).

Note 9. (1) Condition \( (8) \) ensures that functions in \( \tilde{\mathcal{F}} \) are in \( L^1(\varepsilon,1) \forall \varepsilon > 0 \).

(2) The weighted \( L^p \) norm will be denoted \( \| \cdot \|_{p,w} \).

(3) Eq \( (6) \) holds iff \( w^{-1/p} \notin L^q, q^{-1} + p^{-1} = 1 \) for \( p \in (1, \infty) \), and \( w^{-1} \notin L^q \) for \( p \in \{0, \infty\} \) as expected by the \( (L^p, L^q) \) duality, see \( (8) \).

(4) The right endpoint 1 is arbitrary; the nonexistence of a good regularization in this space implies nonexistence in a space where the endpoint is any specified number.

3. Main results

Let \( \mathfrak{F} \) and \( \hat{\mathfrak{F}} \) be as discussed in \( \{2.1.1\} \). Let \( \mathfrak{J}, \hat{\mathfrak{J}} \) be the collection of all operators on the spaces \( \mathfrak{F} \) and \( \hat{\mathfrak{F}} \) respectively obey the conditions in Definition \( \{5\} \).

According to Theorem \( \{10\} \) below, ZFC proves that for each space \( \hat{\mathfrak{F}} \in \hat{\mathfrak{F}} \) there exists an operator \( \hat{I} \in \hat{\mathfrak{J}} \) acting on \( \hat{\mathfrak{F}} \) with some desirable properties. It is proved in \( \{7.1\} \) using the Axiom of Choice. Our negative results establish how unsatisfactory \( \{10\} \) really is.

Specifically, according to Theorem \( \{12\} \), there is no description for which it is provable in ZFC that there exists an \( \mathfrak{F} \) such that there exists an \( \mathfrak{J} \in \mathfrak{J} \) acting on \( \mathfrak{F} \). In particular, this rules out usable formulas for such operators that can be proved to work within the usual axioms for mathematics.

We emphasize Theorem \( \{12\} \) and Theorem \( \{11\} \) which is its direct consequence, over Theorem \( \{10\} \). The Axiom of Choice has long since moved from being controversial, to being accepted as part of the usual ZFC axiomatization for mathematics. However, the impossibility of giving explicit examples that can be verified to hold in ZFC represents a deeper and more serious impossibility than merely requiring the use of the axiom of choice to prove existence (beyond the
relatively benign DC). In practice, the two kinds of impossibility are closely related, although there are counterexamples to direct implications between the two.

**Theorem 10.** (a) ZFC proves the following. For any \( \tilde{F} \in \tilde{\mathcal{F}} \) there exist (uncountably many) \( \tilde{I} : \tilde{F} \to AC((0,1]) \) from \( \tilde{\mathcal{F}} \) which are also linear and coincide with standard integration from zero on \( L^1((0,1]) \).

(b) ZFDC proves the following. No element of \( \mathcal{J} \) is Borel measurable in the compact open topology.

(c) ZFDC proves the following. If \( \mathcal{J} \) is nonempty then there is a set of reals which is not Baire measurable.

(d) ZFDC does not prove that \( \mathcal{J} \) is nonempty.

**Theorem 11.** There is no mathematical description \( \beta \) such that the following is provable in ZFC. \( \beta \) uniquely defines an element of \( \mathcal{I} \) that acts on one of \( C^\infty, A_1 \) or \( A_2 \).

We now present a strong form of Theorem 11 that applies to the continuum many spaces in \( \tilde{\mathcal{F}} \).

**Theorem 12.** There is no mathematically described assignment \( \gamma \) such that the following is provable in ZFC. There exists \( \mathcal{F} \in \mathcal{F} \) such that \( \gamma(\mathcal{F}) : \mathcal{F} \to AC((0,1]) \) lies in \( \mathcal{I} \). This also holds for ZFC extended by the usual large cardinal hypotheses.

According to Theorem 12, there is no description for which it is provable in ZFC that for some \( \mathcal{F} \in \mathcal{F} \) the description uniquely defines, from \( \mathcal{F} \), an \( \mathcal{I} \in \mathcal{I} \) acting on \( \mathcal{F} \). Again, this rules out usable formulas for such operators that can be proved to work within the usual ZFC axioms for mathematics.

The proofs of Theorems 10 (b), (c) in §7.2 rely on the Interface Theorems from §5. The Interface Theorems show how to go explicitly from any element of \( \mathcal{J} \) to a corresponding summation operator which maps \( \{0,1\}^N \) into \( \mathbb{R}^N \). From the point of view of descriptive set theory and mathematical logic, it is easier to work with summation operators than elements of \( \mathcal{J} \). In §6 we establish the results about summation operators that we use in §7.2, using standard techniques from descriptive set theory.

Theorem 10 (d) is proved in §7.2 and Theorem 11 is proved in §7.3. Theorem 11 is an immediate consequence of Theorem 12. Theorems 10 (c) and 12 follow from Theorem 10 (c) using well known results from mathematical logic.

3.1. The analytic case and the optimality of p.f.

**Note 13.** Theorem 11 above implies in particular that

1. In \( A_1 \) there is no good generalization of the Hadamard regularization. The same conclusion holds if the weight in \( A_1 \) is replaced by any \( w \) defined so that \( \int_0^1 (1/w(s))ds \to +\infty \) as \( x \to 0 \).
   
   (a) If the weight \( w \) in \( A_2 \) is replaced with one satisfying \( (8) \) and defined so that in the modified space \( z^k g(z) \) is bounded for some \( k \) as \( z \to 0 \) then \( p.f. \) is a \( P_0 \), arguably the natural one.
   
   (b) In \( A_2 \), the conclusions in Theorem 11 apply to \( P_0 \). The same conclusions hold if the weight \( w \) in \( A_2 \) is replaced with one satisfying \( (8) \) and is specified so that \( \lim \sup_{z,|z|=0} |z^k g(z)| = \infty \) \( \forall k > 0 \).

The key to obtaining these negative mathematical logic results from the analytic questions is essentially the content of §5.
3.2. Further remarks. Also, our results appear to preclude the existence of an integration operator over a sufficiently large class of functions defined on No, the surreal numbers of J.H. Conway. Indeed, if, say continuous functions extended past the gap at $\infty$ and an integral existed for them, then $\int_{x}^{\infty} f := F f^{x}$ where $F$ is the finite part of a surreal number, would violate the conclusions of our theorems. This will be demonstrated elsewhere.

Note 14. It might be tempting to try to generalize $p.f.$ to functions that have convergent Laurent series in a punctured neighborhood of zero simply by termwise integrating their Laurent series. This, however does not work. Termwise integration of the negative part of general Laurent series convergent in, say, $C \setminus \{0\}$ fails. This follows of course from Note 13. The intuitive reason is more easily seen by first taking $x \to 1/x$: termwise integration of the Laurent series becomes termwise integration of the Taylor series at zero. There is no reason to expect that such an integral would only use information from a neighborhood of infinity (at least, not in the sense corresponding to (9)). Sums of general Laurent series also fail a condition that analytic (and more generally, analyzable functions satisfy): if $\alpha \to 0$ and $\forall n f(\alpha) = 0$, then $f = 0$.

Note 15. Condition (II) in Definition 3 is equivalent to requiring that for any $x \in (0, 1]$ and $f$ in the domain of $\mathcal{P}$ we have $[\mathcal{P}(f)](x) = [\mathcal{P}(f)](c) + \int_{c}^{x} f$ for some $c \in (0, x)$ where the integral is the usual Lebesgue integral. More conditions, for instance linearity if possible ($f > 0 \Rightarrow \mathcal{P} f > 0$) would be desirable for an integral regularized at zero (though positivity fails for the Hadamard finite part: $H \int_{0}^{1} s^{-1} ds = -1/x^{2}$). But for our negative result the weak condition (II) suffices, while for the positive results these extra properties of $p.f.$ would hold.

To avoid generalizations based on regularity, or trivial ones, we will use
(a) conditions on the functions which limit their size, but do not imply additional regularity at zero.
(b) spaces of functions which are rich enough (since for instance finite dimensional extensions are always possible).

4. Remarks About Singular Initial Value Problems

In the realm of ODEs with conditions placed at a singularity as mentioned in the introduction, arguably the simplest problem is the linear ODE $f' = g f$ or its inhomogeneous counterpart $y' = g$, $y = \ln f$, where $g$ is singular at zero (and, for simplicity), is smooth in $(0, 1]$. Can conditions at zero separate solutions? If $g$ is singular at zero, but $g \in L^{1}((0, 1])$, then the general solution is $y = (x \mapsto C + \int_{0}^{x} g)$ and $y(0) = C$ is such a condition. Even when $g \notin L^{1}$, solutions exist on $(0, 1]$, and they are of the form $\mathcal{P}(g) + C$ where $\mathcal{P}$ is some antiderivative. Now, to speak about conditions at zero we clearly must ensure that for $0 < x < 1$, $[\mathcal{P}(g)](x) + C$ does not depend on the values of $g$ outside $(0, x]$. This latter formulation brings the question to the form considered in 22. The question of existence of solutions with initial values placed at singular points can be brought, for some simple ODEs, to the form considered in [22] by elementary operations. For instance ODEs of the form $f'' + g(x) f = h$ where $g$ is continuous are amenable to the setting in [22] by the usual variation of parameters method and elementary substitutions. For operators of repeated integration of a non-$L^{1}$ function such as $(g, x) \mapsto f_{g}(x) := \int_{0}^{x} \int_{0}^{x} g(t) dtds$ it has to be ensured that $f'' = g$, thus $f$ needs to be $C^{1}$ with derivative in $AC[0, 1]$, ...
and \( f'_1(x) = f'_0 g(t)dt \); this brings the question to the form in (2). Of course, one can consider cases when both \( g \) and \( h \) are singular, or similar equations of higher order, linear or nonlinear. Comprehensive generalizations will be treated elsewhere.

5. INTERFACE THEOREMS

5.1. Informal discussion. Consider the Cantor set

\[ \{0,1\}^\mathbb{N} := \{(a_i)_{i \in \mathbb{N}} : a_i \in \{0,1\}\} \]

For each of the spaces \( \mathcal{F} \) and any nontrivial extension of integration we define a summation operator from \( n \) to infinity (based at infinity in the sense of Definition (2)) on \( \{0,1\}^\mathbb{N} \) with values in \( \mathbb{R}^\mathbb{N} \). Informally, this is a finite-valued summation operator with the property that for any two sequences which coincide eventually the sum also coincides eventually (see Proposition 22):

\[ (\exists N)(\forall n \geq N)(a_n = a'_n) \Rightarrow (\exists N)(\forall n \geq N) \left( \sum_{i=n}^{\infty} a_i = \sum_{i=n}^{\infty} a'_i \right) \]

Implausible as they might seem, such operators exist assuming AC. They are a byproduct of extensions of p.f. (e.g. to the whole of \( C^\infty((0,1]) \) with no growth or regularity condition at zero), which also exist assuming AC. As expected, such a summation is pathological and no formula can exist for it.

To formulate negative results about the summation operator and for proving them, descriptive set theory and mathematical logic are used non-trivially.

5.2. Detailed results.

**Lemma 16.** If \( \mathcal{F} = L^p_w((0,1]) \) or \( C^n((0,1]) \cap L^p_w((0,1]) \) where \( p \in [1,\infty], n \in \mathbb{N} \cup \{\infty\} \), then \( \mathcal{F} \) has the WL property.

**Proof.** In a space where no regularity beyond measurability is required, if \( f \in \mathcal{F} \), then \( X_\mathcal{F}f \in \mathcal{F} \) and there is nothing to prove. With more regularity, we will apply smoothing to \( X_\mathcal{F} \). An open set \( O \) in \( (0,1] \) is a countable union of relatively open subintervals, \( O = \cup_{n\in\mathbb{N}} I_n \); we will assume that \( I_n = (a_n, b_n) \) are the connected components of \( O \). We then define a \( C^\infty \) function \( \tilde{X}_n \) : which is \( \leq 1 \), that is one outside \( I_n \) and zero in the interval \( (a_n + m_n^{-1}(b_n - a_n), b_n - m_n^{-1}(b_n - a_n)) \), \( m_n \leq \varepsilon_n \) defined below. A concrete transition function which is 1 at \( x_1 \) and 0 at \( x_2 \) that can be continued by one at the left of \( x_1 \) and by zero to the right of \( x_2 \) is \( F = \exp(-(x-x_2)^2) \exp(-(x-x_1)^{-2}) \); we will denote by \( E(x_1, x_2) \) the function \( F \) above, extended by one and zero as described. On \( I_n \) we choose the least \( m_n := m > 4 \in \mathbb{N} \) with the property that

\[ \int_{a_n}^{b_n} |E(a_n, a_n + m^{-1}(b_n - a_n)) + 1 - E(b_n - m^{-1}(b_n - a_n), b_n)||f| < 2^{-n-1} \]

(This \( m_n \) is well defined since \( \int_a^{a+\delta} |f| \to 0 \) as \( \delta \to 0 \) for any \( a \in (0,1] \).) The sum over \( n \) of the integrals in (15) is \( < 1 \) and the result follows.

**□**

**Lemma 17.** Let \( w \) be a continuous weight with the property \( \mathbb{B} \). Then, \( \forall \varepsilon > 0 \) one can define a \( C^\infty \) norm \( w_\varepsilon \) s.t. \( 0 \leq w(x)/w_\varepsilon(x) \leq |1 - \varepsilon, 1 + \varepsilon| \), \( \forall x \) thus equivalent to \( w \).
Proof. This is standard, and can be done in a number of definable ways. One is similar to the construction in Lemma 16. Under the given assumptions ln w is also continuous on \((0, 1]\); on any interval \(J_n = [1/n, 1/(n + 1)]\) one defines the Chebyshev \(T_n\) polynomial of least degree s.t. \(|T_n - \ln w| < \varepsilon/2\). These polynomials can be glued together smoothly, as in Lemma 16.

Lemma 18. There is a definable operation that sends a weight \(w\) for which the associated \(F\) satisfies (8) to a function \(\tau\) s.t. (7) holds.

The proof of Lemma 18 is given in §8.

Lemma 19. For each set in \(F\) a decreasing sequence \((\alpha_k)\) in \((0, 1]\) can be defined in terms of any \(w, \tau\) for the set with the property (16)

\[
\int_{\alpha_k}^{\alpha_{k+1}} \tau = 1
\]

Proof. For the analytic cases, this is obvious from the construction, cf. §9. In the non-analytic spaces we consider, the function \(\tau\) constructed in Lemma 17 has the property that \(\theta(x) = \int_x^1 \tau\) is strictly decreasing, thus continuously invertible from \(\mathbb{R}^+\) into \(\mathbb{R}^+\). We let \(\alpha_0 = 1, \alpha_k = \theta^{-1}(k), k \geq 1\) and note that (10) holds.

Conventions

(1) From this point on we will assume, for each \(F \in \mathcal{F}\), without loss of generality that the weight \(w\) is smooth, that \(\tau = \tau_w\) or specified explicitly, and the sequence \((\alpha_k)\) has been constructed as above.

(2) If we work with a set of functions defined in terms of a \(\tau\) different from \(\tau_w\), then we will use that \(\tau\) as a parameter of the set.

Note 20. A summation operator acting on the sequence \(x_n\) is naturally defined as a solution of the recurrence \(s_{n+1} - s_n = x_n\). Clearly two solutions differ by a constant. This motivates the following.

Definition 21. Let \(x \in \{0, 1\}^\mathbb{N}\). The standard summation for \(x\), written \(\Sigma(x)\), is \((x_0, x_0 + x_1, x_0 + x_1 + x_2, \ldots)\). \(S\) is a summation operator if and only if \(S : \{0, 1\}^\mathbb{N} \to \mathbb{R}^\mathbb{N}\), where for all \(x \in \{0, 1\}^\mathbb{N}\) there exists \(c \in \mathbb{R}\) such that \(S(x) = \Sigma(x) + c\). I.e., \(S(x)\) is \(\Sigma(x)\) with \(c\) added to all terms of \(\Sigma(x)\).

Proposition 22. The existence of a \(P_0\) on some \(F \in \mathcal{F}\) implies the existence of a summation operator based at infinity on \(\{0, 1\}^\mathbb{N}\), where the sum of each element \(a \in \{0, 1\}^\mathbb{N}\) is defined in terms of \(a\) and \(w\) and \(\tau\).

Proof. For the open set \(O_a = \bigcup_{n:a_n=0} (\alpha_{n+1}, \alpha_n)\) and \(\tau\) as in (10) we construct a function \(\tau_a\) from \(O_a\) as in Lemma 16 with \(I_n = (\alpha_{n+1}, \alpha_n)\). It has the property \(\int_x^1 |f\chi_{O_a} - \tau_a| < \frac{1}{4}\). Define

\[
x_{n;a} = -\left( P_0 \tau_a + \int_0^{\alpha_n} |f\chi_{O_a} - \tau_a| \right)
\]

\(^6\)Of course, any weight giving the same topology defines the same \(F\).
where the last integrand is in $L^1$, and $(Sa)_j = \sum_{n=0}^{+\infty} a_j$. By construction, if $\forall n \geq n_0$ we have $a_n = \tilde{a}_n$, then $\forall n \geq n_0$ $\tau_n = \tilde{\tau}_n$ on $(0, \alpha_n)$. By Note 11 and (17), we have $x_{n+1} - x_n = \int_{\alpha_{n+1}}^{\alpha_n} f\chi_x = a_n$. Thus
\[ S = a \mapsto (x_{0,a}, x_{0,a} + x_{1,a}, x_{0,a} + x_{1,a} + x_{2,a}, \ldots) \]
is a summation operator on $[0, 1]^\mathbb{N}$.

\[ \Box \]

**Note 23.** The construction above is a bijection from the Cantor set onto its image in $C^\infty$, and is bicontinuous in the induced topology. Therefore it is Borel measurable.

5.3. **Parameterizations.** Our negative results hold not only for individual spaces like $C^\infty, A_1$ or $A_2$, but also for all of the continuumly many spaces in $\mathcal{F}$. This is reflected in the statements of Theorems 11 and 12. We accomplish this through our parameterization.

We parameterize $\mathcal{F}$ by real numbers, the real number containing all the information about the weight, regularity, and function $\tau$, see \[5.3\].

When working with collections of $\mathcal{F}$, in a first step, we parameterize the spaces by a collection of continuous functions (three in our examples, but we could allow for a countable sequence). An ad hoc codification is as follows. An $L^p$ space, $p \in [1, \infty]$ with weight $w$ vanishing at zero is uniquely associated with the pair $(1/p, w, 0)$ where $w$ is the weight and $p$ is the constant function. If $p = \infty$ we write $(0, w, 0)$. For a weighted $C^\infty$ set, we use the code $(2, w, 0)$ while for weighted $C^n$, $n \in \mathbb{N}$ the code is $(6(n + 1), w, 0)$. For a set of real analytic functions with weight $w$ and associated sequence $(\alpha_k)$, we proceed as follows. The sequence $\alpha_k$ is constructed from an entire function $G$ positive on the real line. We conjoin to associate to real-analytic functions the triple $(3, w, G)$ and for functions $F$ s.t. $F$ is analytic in $\mathbb{C} \setminus \{0\}$ we use the triple $(4, w, G)$. Sets where a continuous $\tau$ different from $\tau_x$ is chosen will be parameterized by $(5, 0, \tau)$. The equivalence condition is built in each of these spaces and needs no further notation. This code can be made bijectively into a single real number as shown below. We treat the first entry of the triples above, some real number, as a constant function, to simplify the presentation.

**Note 24** (Parameterization by a real number). Converting from a parameterization by three continuous functions to a one-real-number parameterization is standard and can be done in many ways, none of them particularly natural.

As an intermediate step, to each continuous function we associate two unique sequences of real numbers on $[0, 1]$, their Fourier sine/cosine coefficients, $s, c$ (since two continuous functions coincide iff their Fourier sine/cosine coefficients are the same). When we are dealing with an entire function $G$ real on the real line, we can restrict it to $[0, 1]$ and proceed as above, since the values of $G$ on any interval completely determine $G$. For three functions we get six sequences, $s_1, c_1, s_2, c_2, s_3, c_3$. We first associate to these a sequence of real numbers by interlacing them: $S_1 = s_{1,1}, c_{1,1}, s_{2,1}, c_{2,1}, s_{3,1}, c_{3,1}, s_{1,2}, \ldots$.

To $S_1$ one can associate a unique real number by the usual Cantor-type diagonal technique.

We note that the same space may be parameterized by more than one real number: this is the case when the norms induced by two different weights are equivalent.
When precise statements are not needed we will still call the spaces by their natural names.

The Interface Theorems provide two explicit functions.

i. An explicitly defined function from $\mathbb{R}$ onto $\mathcal{F}$ (the family of all spaces considered here).

ii. An explicitly defined function from $\mathbb{R}$ onto $C((0,1])$.

iii. An explicitly defined function from $\mathcal{I}$ (the family of all operators considered here) and suitable real numbers to summation operators based at infinity (formally defined below).

**Definition 25.** For any set $X$, $X^N$ is the set of all $f : N \to X$, which is the same as the set of all infinite sequences from $X$ indexed from 0. For $x, y \in X^N$, $x \sim_\infty y$ if and only if $(\exists n)(\forall m \geq n)(x_m = y_m)$. $F : X^N \to Y^N$ is based at infinity if and only if for all $x, y \in X^N$, $x \sim_\infty y \Rightarrow F(x) \sim_\infty F(y)$. We use $[x]_\infty$ for \{ $y \in X^N : x \sim_\infty y$ \}.

**Definition 26.** $\mathcal{S}$ is the collection of all summation operators on $\{0,1\}^N$ based at infinity.

**Theorem 27.** There is an explicitly defined $\Delta : \mathbb{R} \to \mathcal{F}$ with range $\mathcal{F}$. There is an explicitly defined $\Delta^0 : \mathbb{R} \to C((0,1])$ with range $C((0,1])$.

**Proof.** Take $\Delta(x)$ to be the space in $\mathcal{F}$ parameterized by $x$ in $\mathbb{R}$ such a space exists; $C^\infty$ otherwise. □

**Theorem 28.** There is an explicitly defined function $\Gamma$ such that the following holds. Let $\mathcal{I} \in \mathcal{I}$ and $x \in \mathbb{R}$, where $\Delta(x) = \text{dom}(\mathcal{I})$. Then $\Gamma(\mathcal{I}, x) \in \mathcal{S}$. Furthermore, if $\mathcal{I}$ is Borel measurable in the compact open topology, then $\Gamma(\mathcal{I}, x)$ is Borel measurable.

**Proof.** Let $\mathcal{I}, x$ be as given. Take $\Gamma(\mathcal{I}, x)$ to be the summation operator defined based to the weight $w$ in the space $\mathcal{F}$ provided by the parameter $x$ if such an $\mathcal{F}$ exists. If not, we take $\Gamma(\mathcal{I}, x)$ to be undefined. □

In §7.3 we give formal statements of the two Interface Theorems, Theorems 27 and 28 in order to give a detailed proof of Theorem 12. Theorem 27 and 28 are proved within ZFDC.

**Note 29.** We remark that all the proofs in §7.2 and §7.3 are carried out in ZFDC (in fact, ZF suffices).

### 6. Summation operators

Until the proof of Theorem 37 is complete, we fix a summation operator $\mathcal{S} : \{0,1\}^N \to \mathbb{R}^N$, see Definition 21, based at infinity, and prove that $\mathcal{S}$ is not Baire measurable. We assume that $\mathcal{S}$ is Baire measurable, and obtain a contradiction. The proof takes place within ZFDC, and is an application of a widely used technique from descriptive set theory. For useful information about Baire spaces and Baire category, we refer the reader to Kechris, section 8.

**Definition 30.** Let $f : X \to Y$, where $X, Y$ are topological spaces, and $E \subseteq X$. We say that $f$ is continuous over $E$ if and only if $f$ restricted to $E$ is a continuous function where $E$ is given the subspace (i.e., induced) topology.
Lemma 31 ([24], 8.38 p. 52). Let $X$ be a Baire space and $Y$ be a second countable space and assume $f : X \to Y$ is Baire measurable. Then $f$ is continuous over a comeager subset of $X$.

Lemma 32. Let $f : X \to X$ be a bicontinuous bijection, where $X$ is a Baire space. If $E \subseteq X$ is comeager then $f^{-1}(E)$ is comeager.

Proof. It suffices to observe that the forward image of any dense open set under $f$ is a dense open set. \hfill\Box

Lemma 33. Let $E \subseteq \{0,1\}^\mathbb{N}$ be comeager in $\{0,1\}^\mathbb{N}$. { $x \in \{0,1\}^\mathbb{N}$ : $[x]_\infty \subseteq E$ } is comeager in $\{0,1\}^\mathbb{N}$.

Proof. We apply Lemma 32 to the Baire space $\{0,1\}^\mathbb{N}$. For each nonempty finite sequence $\alpha$ from $\{0,1\}$, let $\alpha^* \in \{0,1\}^\mathbb{N}$ be $\alpha$ extended with all 0’s, and $f_\alpha$ be the bicontinuous bijection of $\{0,1\}^\mathbb{N}$ given by $f_\alpha(x) = x + \alpha^*$. Here $+$ is addition modulo 2. Obviously \{ $x \in \{0,1\}^\mathbb{N}$ : $[x]_\infty \subseteq E$ \} = $\bigcap f_\alpha^{-1}[E]$, which by Lemma 32 is the countable intersection of sets comeager in $\{0,1\}^\mathbb{N}$. \hfill\Box

Lemma 34. Let $F : \{0,1\}^\mathbb{N} \to \mathbb{R}$ be Baire measurable. There exists $x \in \{0,1\}^\mathbb{N}$ and a finite initial segment $\alpha$ of $x$ such that $(\forall y \in [x]_\infty \cap \{0,1\}^\mathbb{N}) (y \text{extends } \alpha \Rightarrow |F(x) - F(y)| < 1)$.

Proof. By Lemma 31 $F$ is continuous over a comeager set $E \subseteq \{0,1\}^\mathbb{N}$. By Lemma 33 we fix $[x]_\infty \subseteq E$, and let $F(x) = c \in \mathbb{R}$. $F^{-1}([c - \frac{1}{2}, c + \frac{1}{2}])$ is an open subset of $E$ (as a subspace of $\{0,1\}^\mathbb{N}$) that contains $x$. This open subset of $E$ must contain all elements of $[x]_\infty \cap \{0,1\}^\mathbb{N}$ that extend some particular finite initial segment $\alpha$ of $x$.

Definition 35. $S^* : \{0,1\}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ is defined by $S^*(x) = S(x) - \Sigma(x)$, which must be an element of $\mathbb{R}^\mathbb{N}$ whose terms are all the same. $S^{**}(x)$ is the unique term of $S^*(x)$.

Lemma 36. $S^*$ and $S^{**}$ are Baire measurable.

Proof. We first show that $S^*$ is Baire measurable. Let $J : \{0,1\}^\mathbb{N} \to \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}$ be given by $J(x) = (S(x), \Sigma(x))$. Then $S^*$ is the composition of $J$ with subtraction; i.e., to evaluate $S^*(x)$, first apply $J$, and then apply subtraction. Let $V \subseteq \mathbb{R}^\mathbb{N}$ be open. Then $(S^*)^{-1}[V] = J^{-1}[W]$, where $W \subseteq \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}$ is the inverse image of subtraction on $V$. By the continuity of subtraction, $W$ is open. Now $W$ is a countable union of finite intersections of Cartesian products of open subsets of $\mathbb{R}^\mathbb{N}$. Note that the inverse image of $J$ on the Cartesian product of any two open subsets of $\mathbb{R}^\mathbb{N}$ is Baire measurable. Hence the inverse image of $J$ on any open subset of $\mathbb{R}^\mathbb{N}$ is Baire measurable, as required. To see that $S^{**}$ is Baire measurable, note that $S^{**}$ is the composition of $S^*$ with the first projection function $\pi_1$; i.e., to evaluate $S^{**}(x)$, first apply $S^*$ and then apply $\pi_1$. Use the continuity of $\pi_1$. \hfill\Box

Theorem 37. The following is provable in ZFDC. There is no Baire measurable summation operator $\mathcal{S} : \{0,1\}_\infty \to \mathbb{R}^\mathbb{N}$ based at infinity.

Proof. We have only to complete the promised contradiction. Since $S^{**}$ is Baire measurable, by Lemma 33 fix $x \in \{0,1\}^\mathbb{N}$ and a finite initial segment $\alpha$ of $x$ such that $(\forall y \in [x]_\infty \cap \{0,1\}^\mathbb{N}) (y \text{extends } \alpha \Rightarrow |S^{**}(x) - S^{**}(y)| < 1)$. Let $y \in [x]_\infty \cap \{0,1\}^\mathbb{N}$ extend $\alpha$ and agree everywhere with $x$ except at exactly one argument.
(arguments are elements of $\mathbb{N}$). Obviously $|\Sigma(x) - \Sigma(y)|$ is eventually 1 or eventually $-1$. Since $x \sim_{\infty} y$, $S(x) \sim_{\infty} S(y)$, and so $S(x)$ and $S(y)$ eventually agree. Now $S^*(x) = S(x) - \Sigma(x)$ and $S^*(y) = S(y) - \Sigma(y)$. Hence $S^*(x) - S^*(y) = S(x) - S(y) + \Sigma(y) - \Sigma(x)$. Using the previous paragraph, $S^*(x) - S^*(y)$ is eventually of magnitude $< 1$, $S(x) - S(y)$ is eventually 0, and $\Sigma(y) - \Sigma(x)$ is eventually $-1$ or eventually 1. We have reached the required contradiction.

\[ \square \]

7. Proofs of the main results

In $L^p$, the “functions” are, as usual, equivalence classes of functions modulo sets of measure zero.

7.1. Theorem 10. ZFC proves the following. For any $\tilde{F} \in \mathcal{F}$ there exist (uncountably many) $\tilde{I} : \tilde{F} \rightarrow AC(\{0,1\})$ from $\mathcal{F}$ which are also linear and coincide with standard integration from zero on $L^1(\{0,1\})$.

Proof. (1) Let $\tilde{L}$ be the set of the equivalence classes induced by $\mathcal{F}$. Consider the vector space $\tilde{V}$ generated by $\tilde{L}$. Let $\tilde{V}_1$ be the equivalence classes induced by $\mathcal{F}$ of the $L^1$ functions in $\mathcal{F}$ and let $\tilde{B}_1$ be a Hamel basis in $V_1$. By the usual construction using Zorn’s Lemma let $\tilde{B}$ be a basis for $\tilde{L}$ containing $\tilde{B}_1$.

(2) For each $b \in B$ we choose a unique $b \in \tilde{b}$ with the convention that $b \in L^1$ if it is the representative of a $\tilde{b}_1 \in \tilde{V}_1$ and that 0 will represent the equivalence class of 0. The elements $b$ are linearly independent of each other. Indeed $\sum_{i \leq N} c_i b_i = 0$ implies $\sum_{i \leq N} c_i \tilde{b}_i \sim 0$, a contradiction.

(3) Let $\tilde{V}$ be the vector space generated by the $b$’s and $V_1$ be the vector space generated by $b_1$’s (the representatives of $\tilde{b}_1$).

(4) On $V_1$ we let $C$ be the linear functional $v_1 \rightarrow \int_0^1 v_1$. We write $V = V_1 \oplus V_2$; any $v$ can be uniquely written as $v = v_1 + v_2, v_{1,2} \in V_{1,2}$. We let $C(v) = C(v_1)$. This is obviously a linear functional on $V$.

(5) Let $f \in \mathcal{F}$. By assumption $f \in \tilde{F} \subset \tilde{L}$ for some $\tilde{f}$, and $\tilde{f}$ can be written uniquely in the form $\tilde{f} = \sum_{i=1}^N c_i \tilde{b}_i$, which is equivalent to $f = \sum_{i=1}^N c_i b_i + h, h \sim_0 0$. The decomposition is unique since

$$\sum_{i=1}^N c_i b_i + h = 0 \Leftrightarrow \sum_{i=1}^N c_i \tilde{b}_i \sim_0 0 \Leftrightarrow c_i = 0 \forall i \leq N$$

(6) Now we simply define

$$P_0 f = \int_1^x \sum_{i=1}^N c_i b_i + C \left( \sum_{i=1}^N c_i b_i \right) + \int_0^x h$$

where the last integral exists since $h$ is eventually zero. In the analytic case we write

$$P_0 f = \int_1^x \sum_{i=1}^N c_i b_i + C \left( \sum_{i=1}^N c_i b_i \right) + \int_{\alpha}^x h$$

where $i$ is arbitrary, since $\int_{\alpha}^{\alpha+1} h = 0$.

It is now straightforward to check that $P_0$ is a linear antiderivatives with the required properties. Eventual positivity comes from the fact that $P_0$ coincides with $\int_0^x$ in $L^1$ and the fact that $\int_1^x f \rightarrow -\infty$ otherwise. \[ \square \]
7.2. Proofs of Theorem 10 b, c, d. ZFC is the standard axiomatization for mathematics. ZF is ZFC without the axiom of choice. ZFDC is ZF extended with a weak form of the axiom of choice called Dependent Choice, abbreviated as DC. DC is a weak baseline form of the axiom of choice. DC is used to construct sequences, and is formulated as follows. Let $R$ be a binary relation (set of ordered pairs), where $(\forall x \in A)(\exists y \in A)(R(x, y))$. For each $x \in A$, there exists an infinite sequence $x = (x_0, x_1, \ldots)$ such that $x_0 = x$ and for all $i \in \mathbb{N}$, $R(x_i, x_{i+1})$. It has been shown that DC is provably equivalent, over ZF, to the Baire category theorem for complete metric spaces. See [2].

7.2.1. Theorem 10 b. ZFDC proves the following. No element of $\mathcal{I}$ is Borel measurable in the compact open topology.

Proof. Assume $I \in \mathcal{I}$ is Borel measurable in the compact open topology. By Theorems 27 and 28, proved in ZFDC, there is a summation operator based at infinity that is Borel measurable. However, according to Theorem 37, there is no summation operator based at infinity that is Baire measurable, and hence none that is Borel measurable. □

Note that Theorem 10 b does not involve provability or definability notions. Arguably, any subset of or function between Polish spaces, and more generally, $C((0, 1))$, that is not Borel measurable, is mathematically pathological or at least mathematically undesirable. The Borel measurable sets form a natural hierarchy of length the first uncountable ordinal, and it can be further argued that any subset of or function that does not lie in the first few levels of this hierarchy is pathological or at least mathematically undesirable. Borel measurability in Polish spaces is extensively investigated in descriptive set theory, particularly in connection with Borel equivalence relations and reductions between them. See [18].

7.2.2. Theorem 10 c. ZFDC proves the following. If $\mathcal{I}$ is nonempty then there is a set of reals which is not Baire measurable.

Proof. Assume that $\mathcal{I}$ is nonempty. By the first claim of Interface Theorem 28, there exists a summation operator $S$ based at infinity. By Theorem 37, $S$ is not Baire measurable. Hence there is a subset of $\{0, 1\}^\mathbb{N}$ that is not Baire measurable in $\{0, 1\}^\mathbb{N}$. Let $T \subseteq \{0, 1\}^\mathbb{N}$ consist of removing the elements of $\{0, 1\}^\mathbb{N}$ that are eventually constant. Then $T$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$. Also since we have removed only countably many points from $\{0, 1\}^\mathbb{N}$, there is a subset of $T$ that is not Baire measurable in $T$. Hence there is a subset of $\mathbb{R} \setminus \mathbb{Q}$ that is not Baire measurable in $\mathbb{R} \setminus \mathbb{Q}$. Hence there is a subset of $\mathbb{R}$ that is not Baire measurable in $\mathbb{R}$. □

Lemma 38. ZFDC does not prove the existence of a set of reals that is not Baire measurable.

Proof. This is proved in [35]. □

7.2.3. Theorem 10 d. ZFDC does not prove that $\mathcal{I}$ is nonempty.

Proof. Suppose ZFDC proves $\mathcal{I} \neq \emptyset$. By Theorem 10 b, ZFDC proves that there exists a set of reals that is not Baire measurable. This contradicts Lemma 38. □
7.3. **Proof of Theorem 12**  The most convincing negative results of this paper are Theorems 11 and 12, which are proved in this section. These involve explicit definability. In many contexts in descriptive set theory, we have non Borel measurability, yet we do have demonstrably explicit definability. The most direct example of this is by constructing an $A \subseteq \mathbb{R}^2$ such that every Borel measurable $B \subseteq \mathbb{R}$ is of the form $\{y \in \mathbb{R} : (c, y) \in A\}, c \in \mathbb{R}$. Then we can form the diagonal set $\{y \in \mathbb{R} : (y, y) \notin A\}$, which obviously differs from every Borel measurable $B \subseteq \mathbb{R}$.

A more mathematically interesting example is as follows. Consider the infinite product space $\mathbb{Q}^\mathbb{N}$, using the order topology on $\mathbb{Q}$. Then $\{x \in \mathbb{Q}^\mathbb{N} : \text{rng}(x) \text{ is a compact subset of } \mathbb{Q}\}$ is well known to be not Borel measurable.

Theorem 11 follows immediately from Theorem 12. Our proof of Theorem 12 uses the following formal versions of the two Interface Theorems 27, 28.

**Theorem 39.** There is a formula $\varphi$ of ZFC with exactly the free variables $x$ such that the following are provable in ZFDC.

i. $x \in \mathbb{R} \Rightarrow (\exists y)(\varphi)$.

ii. $\varphi \Rightarrow x \in \mathbb{R} \land y \in \mathfrak{g}$.

iii. $y \in \mathfrak{g} \Rightarrow (\exists x \in \mathbb{R})(\varphi)$.

**Theorem 40.** There is a formula $\rho$ of ZFC with exactly the free variables $x, y, z$, such that the following is provable in ZFDC. $x \in \mathfrak{G} \land \varphi(y, \text{dom}(x)) \land \rho(x, y, z) \Rightarrow z \in \mathfrak{G} \land (\exists z)(\rho(x, y, z)) \land (x \text{ is Borel measurable in the compact open topology} \Rightarrow z \text{ is Borel measurable})$.

**Lemma 41.** There is a model $M$ of ZFC such that any internal set of reals of $M$ that is $M$ definable with parameters from the internal reals of $M$, is internally Baire measurable. Let $\varphi$ be any standard large cardinal hypothesis, such as on the Chart of Cardinals in [22]. If ZFC + $\varphi$ is consistent then there is a model $M$ of ZFC such that any internal set of reals of $M$ that is $M$ definable with parameters from the internal reals of $M$, is internally Baire measurable.

**Proof.** Let $M$ be a model of ZFC + “there exists a strongly inaccessible cardinal”. Without loss of generality, we can assume that $M$ is countable. According to [35], any forcing extension $M^*$ of $M$ obtained by generically collapsing the first strongly inaccessible cardinal to $\omega_1$ is as required. (Solovay does this assuming $M$ is a countable transitive model, but it is well known that this can be done for any countable $M$).

For the second claim, we argue the same way starting with a model $M$ of ZFC extended by the large cardinal hypothesis. We obtain the forcing extension $M^*$, which still satisfies ZFC extended by the large cardinal hypothesis. This is because this notion of forcing is mild (of internal cardinality below the large cardinals), and mild extensions are well known to preserve standard large cardinal hypotheses (see [21], and the important case of measurable cardinals treated in [27]).

Note that for the first claim, we have used the consistency of ZFC + "there exists a strongly inaccessible cardinal" and not just the consistency of ZFC. The subsequent [36] shows that the consistency of ZFC is sufficient.

Here is the formal statement of Theorem 12

**Theorem 12 (formal).** There is no formula $\varphi$ of ZFC with exactly the free variables $x, y$, such that the following is provable in ZFC. $(\exists x \in \mathfrak{g})(\exists y)(\varphi) \land (\exists y)(\varphi \land$
y ∈ ℐ \land \text{dom}(y) = x)$. This also holds for ZFC extended by any of the usual large cardinal hypotheses, provided the extension results in a consistent system.

**Proof.** Let \( \varphi \) be such that the displayed statement is provable in ZFC. Let \( M \) be as given by Lemma \([11]\) in \( M \), let \( x ∈ ℐ \), \( y ∈ \text{dom}(y) = x \). By Theorem \([39]\) \( x \) is \( M \) definable from an internal real of \( M \). Hence \( y \) is \( M \) definable from an internal real of \( M \). By Theorem \([11]\) let \( z ∈ \mathcal{G} \) in the sense of \( M \), where \( z \) is \( M \) definable from \( y \). Hence \( z \) is \( M \) definable from a real internal to \( M \). By Theorem \([37]\) \( z \) is not Baire measurable in the sense of \( M \). By the explicit construction in the proof of Theorem \([10]\) \([33]\) that converts a non Baire measurable set in \((0,1)^N\) to a non Baire measurable set in \( \mathbb{R} \), we obtain a set of reals, internal to \( M \), which is non Baire measurable in the sense of \( M \), and also \( M \) definable from a real internal to \( M \). This contradicts Lemma \([11]\). \( \square \)

To prove the weaker Theorem \([11]\) it suffices to use Lemma \([11]\) with \( M \) definability without parameters. This is because the three spaces in question are explicitly defined. For this weakened form of Lemma \([11]\) we can adhere to \([35]\), merely generically collapsing \( \omega_1 \) to \( \omega \), and weaken the assumption of the consistency of ZFC + “there exists a strongly inaccessible cardinal” to the consistency of ZFC.

8. THE CONSTRUCTION OF \( \tau \)

**Proposition 42.**

1. In \( L^{p}_w \), \( p ∈ (1,∞) \) there is a \( \tau \) s.t. \([7]\) happens iff \( \int_0^1 w^{-\frac{1}{\tau + p}} = \infty \). If \( p = ∞ \) then \([7]\) holds iff \( \int_0^1 (1/w) = ∞ \), while if \( p = ∞ \), then \([7]\) holds iff \( w \) is unbounded below.

2. A smooth \( \tau \) can be defined in terms of \( w \).

**Proof.** Claim \([11]\) abstractly, follows from the Banach duality of the \( L^p \)s. We need to be more constructive, so we provide an explicit proof of \([11]\), \([2]\) simultaneously. Assume \( \int_0^1 w^{-\frac{1}{\tau + p}} < ∞ \) and let \( τ ≥ 0 ∈ L^{p}_w \). Then, by Hölder’s inequality we have

\[
\int_0^1 \tau = \int_0^1 [\tau w^{\frac{1}{\tau + p}}] w^{-\frac{1}{\tau + p}} ≤ \left( \int_0^1 w^{\frac{1}{\tau + p}} \right)^{\frac{\tau}{\tau + p}} \left( \int_0^1 w^{-\frac{1}{\tau + p}} \right)^{\frac{p}{\tau + p}} < ∞
\]

Conversely, assume that \( \int_0^1 w^{-\frac{1}{\tau + p}} = ∞ \). An explicit \( \tau \) with \( ||\tau||_1 = ∞ \) is thus \( w^{-\frac{1}{\tau - 1}} \).

If \( p = ∞ \) then clearly \( wτ ≤ 1 \) implies \( \int_0^1 wτ ≤ C \) if \( \int_0^1 (1/w) = C < ∞ \). If \( \int_0^1 (1/w) = ∞ \) then clearly \( \tau := 1/w ∈ L^∞_w \) while \( f τ = ∞ \). An explicit \( \tau \) with \( ||\tau||_1 = ∞ \) is now \( w^{-1} \).

In \( L^1 \) with \([5]\) we assume, without loss of generality, that \( w ≤ 1 \). Let \( A_k = \{x : 1/w(x) ∈ (1/k^2, 1/(k + 1)^2)\} \) and \( τ = \sum X_{A_k}/m(A_k) \) where \( m \) is the usual Lebesgue measure. Clearly, the intervals are disjoint and \( \int_{[0,1]} τ w ≤ \sum_k k^{-2} < ∞ \) while \( \int_0^1 τ > \frac{1}{2} ∑ k = 1 = ∞ \). \( \square \)

9. THE ANALYTIC CASE

Let \( G_1 : \mathbb{R}^+ → \mathbb{R}^+ \) be an increasing function with rapid growth. There is an entire function which grows even faster and is positive with all derivatives on \( \mathbb{R}^+ \). A simple construction is to take \( H_1(x) = \sup_{0,x+1} G(x) \) and \( H = [H_1] + 1 \) where
\[ \lfloor \cdot \rfloor \text{ is the integer part. Clearly } H : \mathbb{R}^+ \to \mathbb{N} \text{ grows faster than } G_1(x). \text{ One then takes} \]

\begin{equation}
G(x) = C + \sum_{k=1}^{\infty} \left( \frac{x}{k} \right)^{H(n)}
\end{equation}

which grows faster than \( H(x) \) and has the desired properties.

Let \( g = G^{-1} : \mathbb{R}^+ \to \mathbb{R}^+ \).

9.1. Analytic functions in \( \mathbb{C} \setminus \{0\} \).

(a) was proved after the statement.

(b) It is more intuitive to change variable to \( t = 1/x \psi(t) = \psi(1/t), \Psi(t) = \varphi(1/t) \) and work on \([1, \infty)\). This change of variables introduces an extra weight of \( x^{-2}, dx/x = -dt/t \).

**Definition 43.** Let \( A := \{(s_n)_{n \in \mathbb{N}}, s_n \in \mathbb{N} \text{ nondecreasing}\} \). Consider a continuous positive function \( G \) increasing, say, faster than \( 5^x \) as \( x \to \infty \) and s.t. \( G(1) > 1 \). Let \( g(x) = G^{-1}(x), z_k = G(k) \forall k \in \mathbb{N} \), and consider the Cantor space of functions

\begin{equation}
\mathcal{C}_1 := \left\{ F_a : F_a(z) = \sum_{k=1}^{\infty} B_k \prod_{j \neq k} \left( 1 - \frac{z}{z_j} \right)^2, \ a \in A \right\}; \ B_k := \frac{s_k}{\prod_{j \neq k} (1 - z_k/z_j)^2};
\end{equation}

**Note 44.** \( F_a(z_k) \) will be our \( \int_{z_k}^{z_{k+1}} f_a \), where \( f_a = F_a' \). By construction \( F_a(G(k)) = s_k \), and

We have the following straightforward estimate\[\]

\begin{equation}
|B_k| \lesssim s_k z_k^{-(2k-2)} \prod_{j=1}^{k-1} z_j^2 \leq s_k \frac{z_k^2 - 1}{z_k^2} \leq s_k 5^{-k}
\end{equation}

We first estimate the terms in the sum and the sum itself. Since by assumption \( z_k \) grow faster than \( 5^k \), the sum

\begin{equation}
\sum_{j=1}^{\infty} |z|^j |z_k|^{-2}
\end{equation}

converges, each infinite product in the sum converges (\[\], see also the estimates below). It is clear that for a given \( |z| \) all infinite products in \[\] are maximal when \( z = -|z| \).

We have

\begin{equation}
\sum_{k=1}^{N} B_k \prod_{j \neq k} \left( 1 + \frac{\rho}{z_j} \right)^2 = \prod_{j=1}^{\infty} \left( 1 + \frac{\rho}{z_j} \right)^2 \sum_{k=1}^{N} B_k \frac{1 + \rho/z_k}{(1 + \rho/z_k)^2} \leq \prod_{j=1}^{\infty} \left( 1 + \frac{\rho}{z_j} \right)^2 \sum_{k=1}^{N} B_k
\end{equation}

which converges by the assumption on \( z_k \), the estimate in \( \[\] \) and the convergence of \( \[\] \).

\[\]

\( ^7 \)For \( x \geq 0, y \geq 0 \), the notation \( x \lesssim y \) means as usual, \( x \leq Cy \) where \( C \geq 0 \) does not depend on \( x, y \) and other parameters relevant to the statements or proofs.
We then also have

\[
\sum_{k=1}^{\infty} B_k \prod_{j \neq k} \left(1 + \frac{\rho}{z_j}\right)^2 \leq \prod_{j=1}^{\infty} \left(1 + \frac{\rho}{z_j}\right)^2 \sum_{k=1}^{\infty} B_k
\]

\[
\leq \prod_{j=1}^{\infty} \left(1 + \frac{\rho}{z_j}\right)^2 = \prod_{z_j < \rho} \left(1 + \frac{\rho}{z_j}\right)^2 \prod_{z_j \geq \rho} \left(1 + \frac{\rho}{z_j}\right)^2 \leq 4^M \rho^{2M} \leq (4\rho^2)^{G^{-1}(\rho)}
\]

where \( M \) is the largest \( j \) s.t. \( z_j < \rho \) for \( j \leq M \) and we used \((1 + x)^2 < 4x^2\) for \( x > 1 \). The inequality implies, in particular that \( f \) is entire.

Eq. (26) implies that the growth of \( f \) can be made arbitrarily slow, while still super-polynomial, by choosing \( G \) appropriately. To show that the growth is super-polynomial, note that

\[
\prod_{j=1}^{\infty} \left(1 + \frac{\rho}{z_j}\right)^2 > \prod_{j=1}^{\infty} \frac{B_k}{(1 + \rho/z_k)^2} > \prod_{j=1}^{\infty} \frac{B_1}{(1 + \rho/z_1)^2} \geq \frac{\rho^{2q}}{z_2 \ldots z_q}
\]

for any \( q \), if \( \rho \) is large enough.

To estimate the derivative, for \(|z| \leq \rho\) we simply use Cauchy’s formula on a circle of radius \( 2\rho \):

\[
|F'(z)| = \left| \frac{1}{2\pi i} \oint_{|s| > 2\rho} \frac{F_a(s)}{(s-z)^2} \right| \leq \rho^{2g(\rho) - 1}
\]

By construction \( F_a(G(k)) = s_k \). With the sequence \( \alpha_k = 1/G(k) \) \( \forall k \in \mathbb{N} \) we have \( \int_{\alpha_k}^{\alpha_{k+1}} f = (s_{k+1} - s_k) = a_k \), the desired result.

### 9.2. Real-analytic functions.

We work at infinity, as before.

Let

\[
\mathcal{C}_2 = \{ \psi_a = \Psi'_a : \Psi_a(t) = F_a(u(t)) \ a = (s_n)_{n \in \mathbb{N}} \in A \}
\]

where \( F_a \) is as in (28) and \( u = U^{-1} \) where \( U \) is chosen to have entire arbitrarily fast growth, as in the previous subsection. The resulting rate of growth of \( \Psi'_a \), obtained from the previous section is \( \leq |u'(t)|g(u(t))^{2g(u(t))^{-1}} \), which can also be made smaller than any given nonintegrable positive real-analytic function \( \beta(x) \). Indeed, the ODE \( u'(x) = \beta(x)/g(u(x))^{2g(u(x))^{-1}} \) with a positive initial condition has a global solution since \( g \) is increasing, thus \( u \) is increasing and \( g(u) \) is increasing and non-vanishing, ensuring global existence of the solution of the ODE.

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