Quantization and isotropic submanifolds

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Abstract

We introduce the notion of an isotropic quantum state associated with a Bohr-Sommerfeld manifold in the context of Berezin-Toeplitz quantization of general symplectic manifolds, and we study its semi-classical properties using the near diagonal expansion of the Bergman kernel. We then show how these results extend to the case of complete orbifolds, and give an example of application to relative Poincaré series.

1 Introduction

Let \((X, \omega)\) be a compact symplectic manifold of dimension \(2n\), and let \((L, h^L)\) be a Hermitian line bundle over \(X\), endowed with a Hermitian connection \(\nabla^L\) such that its curvature \(R^L\) satisfies the following prequantization condition,

\[ \omega = \frac{\sqrt{-1}}{2\pi} R^L. \] (1.1)

Let \(J\) be an almost complex structure on \(TX\) compatible with \(\omega\), and let \(g^TX\) be the Riemannian metric on \(TX\) induced by \(\omega\) and \(J\). For any \(p \in \mathbb{N}^*\), we denote by \(L^p\) the \(p\)-th tensor power of \(L\). Then following [GU88, (1.7)], we consider the renormalized Bochner Laplacian acting on \(\mathcal{C}^\infty(X, L^p)\), given for any \(p \in \mathbb{N}^*\) by the formula

\[ \Delta^{L^p} = 2\pi np, \] (1.2)

where \(\Delta^{L^p}\) denotes the usual Bochner Laplacian. By analogy with the complex case, we define the finite dimensional space \(\mathcal{H}_p^o \subset \mathcal{C}^\infty(X, L^p)\) of almost holomorphic sections of \(L^p\) for any \(p \in \mathbb{N}^*\) as the direct sum of the eigenspaces associated with the small eigenvalues of (1.2) (see Section 2.1).

In fact, consider the special case of \(J\) being integrable, making \((X, J, \omega)\) into a Kähler manifold, together with a holomorphic Hermitian line bundle \((L, h^L)\) such that its Chern connection \(\nabla^L\) (that is its unique Hermitian connection compatible with the holomorphic structure) satisfies (1.1). For any \(p \in \mathbb{N}^*\), writing \(\bar{\partial}_p^o\) for the holomorphic \(\bar{\partial}\)-operator on forms with values in \(L^p\) and \(\bar{\partial}_p^o\) for its formal adjoint with respect to the \(L^2\)-Hermitian product, the Bochner-Kodaira formula tells us that the operator (1.2) is equal to \(2\bar{\partial}_p^o\bar{\partial}_p^o\). Then by a result of [BV89, Th. 1.1], this operator shows a spectral gap,
so that the small eigenvalues are all equal to 0. The space $\mathcal{H}_p$ of almost holomorphic sections considered above reduces then to the space $H^0(X, L^p)$ of holomorphic sections of $L^p$ in the Kähler case.

Given $(L, h^L, \nabla^L)$ over $(X, J, \omega)$ satisfying (1.1), the family $\{\mathcal{H}_p\}_{p \in \mathbb{N}^*}$ defined above is a natural generalization of its holomorphic quantization, where $p \in \mathbb{N}^*$ can be thought of as the inverse of the Planck constant and $\mathcal{H}_p$ is the associated space of quantum states. In this context, asymptotic results when $p$ tends to infinity are supposed to describe the so-called semi-classical limit, when the scale gets so large that we recover the laws of classical mechanics as an approximation of the laws of quantum mechanics.

On the other hand, in the framework of geometric quantization associated with a regular Lagrangian fibration on $X$, the quantum states of $X$ are represented by immersed Lagrangian submanifolds $\iota : \Lambda \hookrightarrow X$ satisfying a property called the Bohr-Sommerfeld condition, which asks for the existence of a non-vanishing section $\zeta \in \mathcal{C}^\infty(\Lambda, \iota^* L)$ parallel with respect to $\nabla^\iota L$ (see for example [Sm75]). We call the data of $(\Lambda, \iota, \zeta)$ a Bohr-Sommerfeld Lagrangian. The existence of a regular Lagrangian fibration on $X$ being too restrictive, we consider in general singular Lagrangian fibrations, in which we allow the dimension of the fibres to drop on a finite union of submanifolds of positive codimension in $X$. Removing the condition $\dim \Lambda = n$, we call the data of $(\Lambda, \iota, \zeta)$ a Bohr-Sommerfeld submanifold. The typical case of a singular Lagrangian fibration is the case of toric manifolds, where $X$ is endowed with an effective Hamiltonian action of $T^n = (S^1)^n$ and the fibres are given by the orbits of this action. For a comparison between holomorphic and real quantization in this context, see for example [BFMN11].

In this paper, we use the theory of the generalized Bergman kernel of Ma and Mariñescu in [MM08a] to study semi-classical properties of Bohr-Sommerfeld submanifolds in the context of the almost holomorphic quantization described above. Here, the quantization of a Bohr-Sommerfeld submanifold is represented by a sequence $\{s_p \in \mathcal{H}_p\}_{p \in \mathbb{N}^*}$, called an isotropic state, defined for any $p \in \mathbb{N}^*$ by the formula

$$s_p = \int_X P_p(x, \iota(y)) \zeta^p(y) dv_X(y),$$

where $dv_X$ is the Riemannian volume form of $(X, g^{TX})$, $\zeta^p \in \mathcal{C}^\infty(X, L^p)$ is the $p$-th tensor power of $\zeta$ and $P_p(\cdot, \cdot)$ is the generalized Bergman kernel, that is the Schwartz kernel with respect to $dv_X$ of the orthogonal projection $P_p$ from $\mathcal{C}^\infty(X, L^p)$ to $\mathcal{H}_p$ with respect to the natural $L^2$-Hermitian product. The expected behaviour of a quantum state in the semi-classical limit is to rapidly localize around the corresponding classical object, and we show in Proposition 3.5 that isotropic states indeed concentrate around the associated Bohr-Sommerfeld submanifold when $p$ tends to infinity. Furthermore, we establish in Theorem 3.6 the following estimate on the norm of these sections, which is the first main result of this paper and which we state here in its simplest form.

**Theorem 1.1.** Let $(\Lambda, \zeta, \iota)$ be a Bohr-Sommerfeld manifold of dimension $d = \dim \Lambda$. Then there exist $b_r \in \mathbb{R}$, $r \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,

$$\|s_p\|^2_p = p^{n-d/2} \sum_{r=0}^k p^{-r} b_r + O(p^{n-d/2-(k+1)}).$$

(1.4)
Furthermore, we have \( b_0 = 2^{d/2} \text{Vol}(\Lambda) \).

In Section 4 we study the \( L^2 \)-Hermitian product \( \langle \cdot, \cdot \rangle_p \) of two such sections for any \( p \in \mathbb{N}^* \). We show that this product tends to 0 rapidly as \( p \) tends to infinity whenever the two associated submanifolds do not intersect, and we establish Theorem 4.1 which is the second main result of this paper and which we state here in its simplest form.

**Theorem 1.2.** Let \((\Lambda_1, \iota_1, \zeta_1)\) and \((\Lambda_2, \iota_2, \zeta_2)\) be two Bohr-Sommerfeld submanifolds with clean and connected intersection, and let \( \{s_{j,p}\}_{p \in \mathbb{N}^*}, j = 1, 2, \) denote the associated isotropic states. Set \( l = \dim \Lambda_1 \cap \Lambda_2 \) and \( d_j = \dim \Lambda_j, \ j = 1, 2. \) Then there exist \( b_r \in \mathbb{C}, \ r \in \mathbb{N}, \) such that for any \( k \in \mathbb{N} \) and as \( p \to +\infty, \)

\[
\langle s_{1,p}, s_{2,p} \rangle_p = p^{n-d_1-d_2+\frac{k}{2}} \sum_{r=0}^k p^{-r} b_r + O(p^{n-d_1-d_2+\frac{k}{2}-(k+1)}),
\]

(1.5)

where \( \lambda \in \mathbb{C} \) is the value of the constant function on \( \Lambda_1 \cap \Lambda_2 \) defined for any \( x \in \Lambda_1 \cap \Lambda_2 \) by \( \lambda(x) = \langle \zeta_1(x), \zeta_2(x) \rangle_L \). Furthermore, if \( \dim \Lambda_1 = n, \) the following formula holds,

\[
b_0 = 2^{n/2} \int_{\Lambda_1 \cap \Lambda_2} \det^{-\frac{1}{2}} \left\{ \sqrt{-1} \sum_{k=1}^{n-l} h^{TX}(e_k, \nu_i) \omega(e_k, \nu_j) \right\}_{i,j=1}^{d_2-l} |dv|_{\Lambda_1 \cap \Lambda_2},
\]

(1.6)

where \( \{e_i\}_{i=1}^{n-l}, \{\nu_j\}_{j=1}^{d_2-l} \) are local orthonormal frames of the normal bundle of \( \Lambda_1 \cap \Lambda_2 \) in \( \Lambda_1, \Lambda_2 \) respectively, and \( |dv|_{\Lambda_1 \cap \Lambda_2} \) is the Riemannian density on \( \Lambda_1 \cap \Lambda_2 \) induced by \( g^{TX}. \)

Here the intersection of two immersed submanifolds is taken to be the fibred product over \( X \) of the immersions. We thus see that in the semi-classical limit, the Hermitian product of two isotropic states is closely related to the geometry of the intersection of the corresponding submanifolds. The left hand side of (1.5) is called the intersection product of \( s_{1,p} \) and \( s_{2,p} \), and can be thought as the cup product of some Lagrangian intersection theory (see [Tyu00] for a discussion on this idea).

To give the most general formulation of Theorem 1.1 and Theorem 1.2, we use the theory of Berezin-Toeplitz operators for the generalized Bergman kernel on symplectic manifolds of [LMM17], we consider any \( J \)-invariant Riemannian metric \( g^{TX} \) on \( TX \) and isotropic states taking values in an auxiliary Hermitian vector bundle \((E, h^E)\) with Hermitian connection \( \nabla^E \). In the case of non-connected intersection, the expansion (1.5) takes the form of a sum over the connected components. This is Theorem 3.1 and Theorem 4.1 respectively, in the case \( X \) smooth and compact.

In Section 5 we explain how the results of Section 3 extend to the case of \((X, g^{TX})\) complete non-compact orbifold, when the immersed isotropic submanifold \( \Lambda \) is compact and \((X, J, \omega, g^{TX})\) is Kähler. As an application to the case where \( X \) is the quotient of the Poincaré upper-half plane \( \mathbb{H} \) by a discrete subgroup \( \Gamma \) of \( SL_2(\mathbb{R}) \), we derive in Section 6 asymptotic results on relative Poincaré series in the theory of automorphic forms.

In the case \((X, J, \omega, g^{TX})\) compact Kähler manifold with \( c_1(TX) \) even, \( E = \mathbb{C} \) and \( \dim \Lambda_1 = \dim \Lambda_2 = n, \) Theorem 4.1 is the main result of Borthwick, Paul and Uribe in
with the expansion \( (1.5) \) given with half-integer powers of \( p \) instead of integer powers as in \( \text{(85)} \). This is explained in Remark 4.5, where we translate their use of the formalism of half-forms by taking for \( E \) a square root of the canonical bundle of \( X \). In the case where \( \Gamma \) acts freely on \( \mathbb{H} \) and where \( X = \mathbb{H}/\Gamma \) is compact, the application to relative Poincaré series in Section 6 is the result of \( \text{[BPU95, § 4]} \). In the case where \( (X, J, \omega, g^{TX}) \) is additionally equipped with an Hamiltonian action of a compact Lie group lifting to \( (L, h^L, \nabla^L) \), an equivariant version of the results of \( \text{[BPU95]} \) has been obtained by Debernardi and Paoletti \( \text{[DP06]} \). Semi-classical asymptotics on Lagrangian states have also been obtained by Charles in \( \text{[Cha03]} \) in the case of discrete intersections and in the same particular context than in \( \text{[BPU95]} \).

The theory of Berezin-Toeplitz operators was first developed by Bordemann, Meinrenken and Schlichenmaier in \( \text{[BMS94]} \) and Schlichenmaier in \( \text{[Sch00]} \) for the Kähler case, \( E = \mathbb{C} \) and \( g^{TX}(\cdot, \cdot) = \omega(\cdot, J \cdot) \). The approach of both \( \text{[BMS94]}, \text{[BPU95]}, \text{[Cha03]} \) and \( \text{[DP06]} \) is based on the work of Boutet de Monvel and Sjöstrand on the Szegö kernel in \( \text{[BS75]} \), and the theory of Toeplitz structures developed by Boutet de Monvel and Guillemin in \( \text{[BG81]} \). Note that the definitions of Section 3.1 extend in a straightforward way to the case of spin* quantization considered for example in \( \text{[MM08]} \), and the results of Section 3 and Section 4 certainly hold in this case. If \( (X, J, \omega, g^{TX}) \) is further endowed with an Hamiltonian action of a compact Lie group \( G \) lifting to \( (L, h^L, \nabla^L), (E, h^E, \nabla^E) \) such that \( 0 \in \text{Lie}(G)^* \) is a regular point of the associated moment map \( \mu : X \rightarrow \text{Lie}(G)^* \), and if \( \iota : \Lambda \rightarrow X \) intersects \( \mu^{-1}(0) \) cleanly in the sense of Definition 4.1, then one can use the full off-diagonal expansion of the \( G \)-invariant Bergman kernel of Ma and Zhang in \( \text{[MZ08, Th.0.2, Rem.0.3]} \) to prove a result analogous to Theorem 3.6 for the \( G \)-invariant part of the associated isotropic state.

A final motivation for this work is towards the program initiated by Witten in \( \text{[Wit89]} \) in holomorphic quantization of Chern-Simons theory, showing an asymptotic expansion for Lagrangian states associated to some special Bohr-Sommerfeld Lagrangians inside the moduli space of flat connections on a Riemann surface, defined in \( \text{[JW92, Prop. 7.2]} \) and \( \text{[Fre95, Prop. 3.27]} \). Bohr-Sommerfeld Lagrangians in this context have also been studied by Tyurin in \( \text{[Tyu00]} \), and in the more general context of the Abelian Lagrangian Algebraic Geometry program of Gorodentsev and Tyurin \( \text{[GT01]} \). In both cases, it is of particular importance to be able to consider orbifolds.

2 Generalized Bergman kernels on Symplectic Manifolds

In this section, we set the context and notations, and recall the results of \( \text{[MM02, MM08a]} \) and \( \text{[ILMM17]} \) we will need throughout the paper. We refer to the book \( \text{[MM07, Chap.4-8]} \) as a basic reference for the theory.
2.1 Setting

Let $(X, \omega)$ be a compact symplectic manifold of dimension $2n$ with tangent bundle $TX$, and let $J$ be an almost complex structure compatible with $\omega$. Take $g^{TX}$ to be any $J$-invariant Riemannian metric on $TX$, and let $\nabla^{TX}$ be the associated Levi-Civita connection.

For any Euclidean vector bundle $(E, g^E)$, we write $E_{\mathbb{C}}$ for its complexification and still write $g^E$ for the induced $\mathbb{C}$-bilinear product on $E_{\mathbb{C}}$. Let us then write $T_X = T^{(1,0)}X \oplus T^{(0,1)}X$ for the splitting of $T_X$ into the eigenspaces of $J$ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Then for any $x \in X$, $v, w \in T_x^{(1,0)}X$, we define the positive Hermitian endomorphism $R^L_x \in \text{End}(T_x^{(1,0)}X)$ by the formula

$$g^{TX}(R^L_x v, \bar{w}) = R^L(v, \bar{w}). \quad (2.2)$$

We denote by $K_X = \text{det}(T^*(1,0)X)$ the canonical line bundle of $(X, J)$, endowed with the Hermitian structure and connection $h^K_X$, $\nabla^K_X$ induced by $g^{TX}$, $\nabla^{TX}$ via $(2.1)$. We will consider as well the Riemannian metric $g^{TX}_\omega$ on $TX$ defined by the formula

$$g^{TX}_\omega(\cdot, \cdot) = \omega(\cdot, J \cdot), \quad (2.3)$$

and the Hermitian metric $h^{TX}_\omega$ on $(TX, J)$ defined by

$$h^{TX}_\omega = g^{TX}_\omega - \sqrt{-1} \omega. \quad (2.4)$$

Note that if $g^{TX} = g^{TX}_\omega$, then $R^L = 2\pi \text{Id}_{T_x^{(1,0)}X}$. For any submanifold $Y \subset X$, we will write $g^{TY}$, $g^{TY}_\omega$ for the Riemannian metrics on $Y$ induced by $g^{TX}$, $g^{TX}_\omega$ and $dv_Y, dv_{Y,\omega}$ for the induced Riemannian volume forms. In particular, we have

$$dv_{X,\omega} = \det \left( \frac{R^L}{2\pi} \right) dv_X. \quad (2.5)$$

Consider a Hermitian line bundle $(L, h^L)$ over $X$, together with a Hermitian connection $\nabla^L$ satisfying $(\text{III})$, and let $(E, h^E)$ be an auxiliary Hermitian vector bundle over $X$ with Hermitian connection $\nabla^E$ and curvature $R^E$. For any $p \in \mathbb{N}^*$, we write

$$E_p = L^p \otimes E, \quad (2.6)$$

endowed with the Hermitian metric and connection $h^{E_p}$, $\nabla^{E_p}$ induced by $h^L$, $h^E$, $\nabla^L$, $\nabla^E$.

**Definition 2.1.** The *Bochner Laplacian* $\Delta^{E_p}$ is the second order differential operator acting on $\mathcal{C}^\infty(X, E_p)$ by the formula

$$\Delta^{E_p} = -2n \sum_{j=1}^{2n} \left[ (\nabla_{e_j}^{E_p})^2 - \nabla_{e_j}^{E_p} \nabla_{\nabla^{TX} e_j}^{E_p} \right], \quad (2.7)$$

5
where \( \{e_j\}_{j=1}^{2n} \) is any local orthonormal frame of \( TX \) with respect to \( g^{TX} \).

For any \( p \in \mathbb{N}^* \) and any Hermitian smooth section \( \Phi \in \mathcal{C}^\infty(X, \operatorname{End}(E)) \), the renormalized Bochner Laplacian \( \Delta_{p,\Phi} \) is the second order differential operator acting on \( \mathcal{C}^\infty(X, E_p) \) by the formula
\[
\Delta_{p,\Phi} = \Delta^{E_p} - p \operatorname{Tr}[\hat{R}^L] + \Phi. \tag{2.8}
\]

From now on, we fix \( \Phi \in \mathcal{C}^\infty(X, \operatorname{End}(E)) \) and simply write \( \Delta_p \) for the associated renormalized Bochner Laplacian. In the Kähler case, if \( g^{TX} = g^E_\omega \) and if \( \Phi \) is equal to \( -\sqrt{-1}RE \) contracted with \( \omega \), we recover twice the Kodaira Laplacian of \( E_p \). On the other hand, if \( g^{TX} = g^E_\omega \) and \( E = \mathbb{C} \), we recover (1.2).

Let \( \langle \cdot, \cdot \rangle_{E_p} \) denote the Hermitian product on \( E_p \) induced by \( h^L \) and \( h^E \). The \( L^2 \)-Hermitian product \( \langle \cdot, \cdot \rangle_p \) on \( \mathcal{C}^\infty(X, E_p) \) is given for any \( s_1, s_2 \in \mathcal{C}^\infty(X, E_p) \) by the formula
\[
\langle s_1, s_2 \rangle_p = \int_X \langle s_1(x), s_2(x) \rangle_{E_p} dv_X(x). \tag{2.9}
\]

Let \( L^2(X, E_p) \) be the completion of \( \mathcal{C}^\infty(X, E_p) \) with respect to \( \langle \cdot, \cdot \rangle_p \). Then \( \Delta_p \) is a self-adjoint second order differential operator on \( L^2(X, E_p) \), and has discrete spectrum contained in \( \mathbb{R} \). Furthermore, we have the following refinement of [GU88, Th. 2.4].

**Theorem 2.2.** [MM02, Cor. 1.2] There exist \( \tilde{C}, \ C > 0 \) such that for all \( p \in \mathbb{N}^* \),
\[
\operatorname{Spec}(\Delta_p) \subset [-\tilde{C}, \tilde{C}] \cup [2\mu_0 p - C, +\infty[. \tag{2.10}
\]

where \( \mu_0 = \inf_{x \in X, v \in T_x^{1,0}} R^E_x(v, \overline{v})/g^{TX}_x(v, \overline{v}) \).

For any \( p \in \mathbb{N}^* \), we define the space of almost holomorphic sections \( \mathcal{H}_p \subset L^2(X, E_p) \) of \( E_p \) as the direct sum of the eigenspaces of \( \Delta_p \) associated with the eigenvalues in \( [-\tilde{C}, \tilde{C}] \). Then by standard elliptic theory, we have \( \mathcal{H}_p \subset \mathcal{C}^\infty(X, E_p) \) and \( \dim \mathcal{H}_p < +\infty \). Actually, by [MM02, Cor. 1.2], the dimension of \( \mathcal{H}_p \) is computed by the Riemann-Roch-Hirzebruch formula, and is in particular a polynomial of degree \( n \) in \( p \).

We write \( \pi_j : X \times X \to X \), \( j = 1, 2 \), for the first and second projections on \( X \). For any vector bundles \( E' \) and \( E'' \) over \( X \), we define a vector bundle over \( X \times X \) by the formula
\[
E' \otimes E'' = \pi_1^* E' \otimes \pi_2^* E''. \tag{2.11}
\]

The orthogonal projection \( P_p : \mathcal{C}^\infty(X, E_p) \to \mathcal{H}_p \) with respect to (2.9) has smooth Schwartz kernel \( P_p(\cdot, \cdot) \in \mathcal{C}^\infty(X \times X, E_p \otimes E_p^* \otimes E_p) \) with respect to \( dv_X \), defined for any \( \zeta \in \mathcal{C}^\infty(X, E_p) \) and \( x \in X \) by
\[
(P_p s)(x) = \int_X P_p(x, y) s(y) dv_X(y). \tag{2.12}
\]

For any \( F \in \mathcal{C}^\infty(X, \operatorname{End}(E)) \), we define the Berezin-Toeplitz quantization of \( F \) as the family \( \{T_{F, p}\}_{p \in \mathbb{N}^*} \) of operators acting on \( \mathcal{C}^\infty(X, E_p) \) for any \( p \in \mathbb{N}^* \) by
\[
T_{F, p} = P_p F P_p, \tag{2.13}
\]
where $F$ denotes the operator acting by pointwise multiplication by $F$. Then $T_{F,p}$ has smooth Schwartz kernel $T_p(\cdot, \cdot) \in \mathcal{C}^\infty(X \times X, E_p \otimes E_p^*)$, given for any $x, y \in X$ by

$$T_{F,p}(x, y) = \int_X P_p(x, w) F(w) P_p(w, y) dv_X(w). \quad (2.14)$$

For any $x_0 \in X$, we will write $\langle \cdot, \cdot \rangle_{x_0}$ and $|\cdot|_{x_0}$ for the Hermitian product and norm on $E_{x_0}$ induced by $h^E$. For any $\sigma > 0$, we use the notation $O(p^{-\sigma})$ as $p \to +\infty$ in the usual sense with respect to $|\cdot|_{x_0}$ and uniformly in $x_0 \in X$. The notation $O(p^{-\infty})$ means $O(p^{-\sigma})$ for any $\sigma > 0$. Unless otherwise stated, we also use the convention to sum on free indices appearing twice in a single term.

### 2.2 Local model

Let $(u, v) := (u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{R}^{2n}$ be the canonical symplectic coordinates associated with the standard symplectic form $\Omega$ on $\mathbb{R}^{2n}$ given by

$$\Omega = \sum_{j=1}^n du_j \wedge dv_j. \quad (2.15)$$

We write $\mathbb{R}^n \times \{0\} = \{(u, 0) \in \mathbb{R}^{2n} \mid u \in \mathbb{R}^n\}$ and $\{0\} \times \mathbb{R}^n = \{(0, v) \in \mathbb{R}^{2n} \mid v \in \mathbb{R}^n\}$ for the two canonical oriented Lagrangian subspaces of $(\mathbb{R}^{2n}, \Omega)$ and denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the canonical scalar product and norm of $\mathbb{R}^{2n}$. To match with the notations of [MM08a], we will write $Z := (u, v) \in \mathbb{R}^{2n}$, and use the same notation for the radial vector field of $\mathbb{R}^{2n}$. For any $\varepsilon > 0$, we denote by $B^{\mathbb{R}^{2n}}(0, \varepsilon)$ the ball of center 0 and radius $\varepsilon$ in $\mathbb{R}^{2n}$, and for any linear subspace $\Sigma \subset \mathbb{R}^{2n}$, we write $B^{\Sigma}(0, \varepsilon) := B^{\mathbb{R}^{2n}}(0, \varepsilon) \cap \Sigma$.

For any $m \in \mathbb{N}$, let $|\cdot|_{\mathbb{C}^m}$ denotes the $\mathcal{C}^m$-norm on $E_p \otimes E_p^*$ over $X \times X$ induced by $h^L, h^E, \nabla^L, \nabla^E$, and let $d^X(\cdot, \cdot)$ be the Riemannian distance on $(X, g^{TX})$.

**Proposition 2.3.** [MM08a, § 1.1] For any $m, k \in \mathbb{N}$, $\varepsilon > 0$ and $\theta \in [0, 1]$, there is $C_{m,k,\theta,\varepsilon} > 0$ such that for all $p \in \mathbb{N}^*$ and $x, x' \in X$ satisfying $d^X(x, x') > \varepsilon p^{-\theta/2}$,

$$|P_p(x, x')|_{\mathbb{C}^m} \leq C_{m,k,\theta,\varepsilon} p^{-k}. \quad (2.16)$$

Let us now take $x_0 \in X$, $\varepsilon_0 > 0$, $V \subset X$ open neighbourhood of $x_0$ and

$$\phi_{x_0} : B^{\mathbb{R}^{2n}}(0, \varepsilon_0) \subset \mathbb{R}^{2n} \to V \quad (2.17)$$

a diffeomorphism sending 0 to $x_0$, such that its differential at 0 identifies $\Omega$ and $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{2n}$ with $\omega$ and $g^\omega_{TX}$ on $T_{x_0}X$. Let us make such a choice of diffeomorphisms (2.17) for any $x_0$ in a small open set, smoothly in $x_0$. We cover $X$ with such open sets, and choose $\varepsilon_0 > 0$ which does not depend on $x_0 \in X$. As two Riemannian metrics induce equivalent distances in a continuous way with respect to parameters, there exist $0 < a < b$ such that for any $x_0 \in X$ and $Z, Z' \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$,

$$a|Z - Z'| < d^X(\phi_{x_0}(Z), \phi_{x_0}(Z')) < b|Z - Z'|. \quad (2.18)$$

Then by (2.18), we get the following corollary of Proposition 2.3.
**Corollary 2.4.** For any $\varepsilon > 0$, $m, k, \theta, \varepsilon > 0$ such that for all $x_0 \in X$, $p \in \mathbb{N}^*$ and $Z, Z' \in B^{2n}(0, \varepsilon_0)$ such that $|Z - Z'| > \varepsilon p^{-\theta/2}$,

$$|P_p(\phi_{x_0}(Z), \phi_{x_0}(Z'))|_{\mathcal{E}^m} \leq C_{m, k, \theta, \varepsilon} p^{-k}. \quad (2.19)$$

We use the following explicit local model on $\mathbb{R}^{2n}$ for the Bergman kernel, as defined in \cite{MM08b}, (3.25)] for any $Z, Z' \in \mathbb{R}^{2n}$,

$$\mathcal{P}_{x_0}(Z, Z') = \det \left( \hat{R}^L_{x_0}/2\pi \right) \exp \left( -\frac{\pi}{2} |Z - Z'|^2 - \pi \sqrt{-1} \Omega(Z, Z') \right). \quad (2.20)$$

Note that the difference of (2.20) with \cite{MM08b}, (3.25) comes from the fact that $\langle \cdot, \cdot \rangle$ is identified with $g^{TX}_x$ via (2.17) instead of $g^{TX}_x$ via the exponential map as in \cite{MM08b}, § 3.2.

Let $dZ$ be the canonical Lebesgue measure of $\mathbb{R}^{2n}$, and let $\kappa_{x_0} \in \mathcal{E}^\infty(B^{2n}(0, \varepsilon_0), \mathbb{R})$ be the smooth function satisfying, for any $Z \in B^{2n}(0, \varepsilon_0)$ in the chart (2.17),

$$du_{X, \omega}(Z) = \kappa_{x_0}(Z) dZ. \quad (2.21)$$

Then $\kappa_{x_0}(0) = 1$. In the chart (2.17), we identify $E, L$ over $B^{2n}(0, \varepsilon_0)$ with $E_{x_0}, L_{x_0}$ through parallel transport with respect to $\nabla^E, \nabla^L$ along radial lines of $B^{2n}(0, \varepsilon_0)$. For any $x_0$ in a small open set, we identify $L_{x_0}$ with $\mathbb{C}$ using any unit local frame of $L$.

For any $f \in \mathcal{C}^\infty(X, E)$, we write $f_{x_0} \in \mathcal{C}^\infty(B^{2n}(0, \varepsilon_0), E_{x_0})$ for the restriction of $f$ to $B^{2n}(0, \varepsilon_0)$ in this trivialization. Similarly, for any $T_p(\cdot, \cdot) \in \mathcal{C}^\infty(X \times X, E^*_p \otimes E^*_p)$, we denote by $T_p(x_0)(Z, Z') \in \text{End}(E_{x_0})$ its image evaluated at $Z, Z' \in B^{2n}(0, \varepsilon_0)$ in this trivialization. If $Q(Z, Z')$ is a polynomial in $Z, Z' \in \mathbb{R}^{2n}$, we write $Q \mathcal{P}_{x_0}(Z, Z') := Q(Z, Z') \mathcal{P}_{x_0}(Z, Z')$.

Recall that we chose a family of charts $\{\phi_{x_0}\}_{x_0 \in W}$ as in (2.17) smoothly in $x_0 \in W$, where $W$ is a small open set of $X$. Then $P_{p, x_0}(Z, Z')$ can be seen as a smooth section of $\pi^* \text{End}(E)$ over $W \times B^{2n}(0, \varepsilon_0) \times B^{2n}(0, \varepsilon_0)$ evaluated in $x_0 \in W$, $Z, Z' \in B^{2n}(0, \varepsilon_0)$, where $\pi : W \times B^{2n}(0, \varepsilon_0) \times B^{2n}(0, \varepsilon_0) \to W$ is the first projection. Let us write $| \cdot |_{\mathcal{E}^m(X)}$ for the $\mathcal{E}^m$-norm on $\pi^* \text{End}(E)$ induced by $h^E$ and derivation by $\nabla^\pi^* \text{End}(E)$ in the direction of $x_0 \in W$. We are now ready to state the following result, which was first proved in \cite{DLM06}, Th. 4.18’ in the case of the spin$^c$ Dirac operator, and which in the following form comes essentially from \cite{LMM16}, Th. 2.1.

**Lemma 2.5.** For any $m, k, \varepsilon > 0$ and $\delta \in [0, 1[$, there is $C > 0$ and $\theta \in ]0, 1[ \text{ such that for all } x_0 \in X, p \in \mathbb{N}^*$ and $|Z|, |Z'| < \varepsilon p^{-\theta/2}$,

$$|p^{-n} P_{p, x_0}(Z, Z') - \sum_{r=0}^{k} p^{-r/2} J_{r, x_0} \mathcal{P}_{x_0}(\sqrt{p} Z, \sqrt{p} Z') \kappa_{x_0}^{-1/2}(Z) \kappa_{x_0}^{-1/2}(Z')|_{\mathcal{E}^m(X)} \leq C p^{-l+\delta}, \quad (2.22)$$

where $\{J_{r, x_0}(Z, Z')\}_{r \in \mathbb{N}}$ is a family of polynomials in $Z, Z' \in \mathbb{R}^{2n}$ of the same parity as $r$ and with values in $\text{End}(E_{x_0})$, depending smoothly on $x_0 \in X$. Furthermore, we have

$$J_{0, x_0}(Z, Z') \equiv \text{Id}_{E_{x_0}}. \quad (2.23)$$
Parallel to Proposition 2.3 and Lemma 2.5, we have the following result on the asymptotic expansion in $p \in \mathbb{N}^*$ of the Berezin-Toeplitz operator (2.13). It was first proved in [MM08b, Lemma 4.6] in the spin$^c$ case, and in this form comes essentially from [ILMM17, Lemma 3.3].

**Lemma 2.6.** Let $F \in \mathcal{C}^\infty(X, \mathrm{End}(E))$. Then for any $0 < \varepsilon \leq \varepsilon_0$, $m, k \in \mathbb{N}$ and $\theta \in ]0,1[$, there is $C_{m,k,\theta,\varepsilon} > 0$ such that for all $x_0 \in X$, $p \in \mathbb{N}^*$, $Z, Z' \in \mathbb{R}^{2n}$, $|Z - Z'| > \varepsilon p^{-\theta/2}$,

$$|T_{F,p} (\phi_{x_0}(Z), \phi_{x_0}(Z'))|_{\varepsilon^m} \leq C_{m,k,\theta,\varepsilon} p^{-k}. \quad (2.24)$$

Furthermore, for any $m, k \in \mathbb{N}$, $\varepsilon > 0$ and $\delta \in ]0,1[$, there is $C > 0$ and $\theta \in ]0,1[$ such that for all $x_0 \in X$, $p \in \mathbb{N}^*$, $|Z|, |Z'| < \varepsilon p^{-\theta/2}$,

$$|p^{-n}T_{F,p,x_0}(Z, Z') - \sum_{r=0}^{k} p^{-r/2}Q_{r,x_0}(\sqrt{p}Z, \sqrt{p}Z')\kappa_{x_0}^{-1/2}(Z)\kappa_{x_0}^{-1/2}(Z')|_{\varepsilon^m(X)} \leq Cp^{-\frac{k+1}{2}+\delta}, \quad (2.25)$$

where $\{Q_{r,x_0}(Z, Z')\}_{r \in \mathbb{N}}$ is a family of polynomials in $Z, Z' \in \mathbb{R}^{2n}$ of the same parity as $r$ and with values in $\mathrm{End}(E_{x_0})$, depending smoothly on $x_0 \in X$. Furthermore, we have

$$Q_{0,x_0}(Z, Z') \equiv F_{x_0}. \quad (2.26)$$

### 2.3 Gaussian integrals

We now recall some well-known facts about Gaussian integrals, which will be used for local computations in the next sections. For any $k \in \mathbb{N}^*$, let $\langle \cdot, \cdot \rangle$ denote the canonical scalar product of $\mathbb{R}^k$. For any positive symmetric matrix $C$ acting on $\mathbb{R}^k$, we recall the following classical formula for the Gaussian integral,

$$\int_{\mathbb{R}^k} \exp(-\pi \langle Z, CZ \rangle) dZ = \det^{-\frac{1}{2}} C. \quad (2.27)$$

By analytic continuation, this formula is still valid when $C$ is a symmetric matrix with complex coefficients, providing the integral is well defined along a path in the space of symmetric matrices joining $C$ with a real positive symmetric matrix. Specifically, for $A$ positive symmetric matrix and $B$ real symmetric matrix, we will consider the path

$$\gamma : [0,1] \to \mathrm{GL}_k(\mathbb{C})$$

$$t \mapsto A + t\sqrt{-B}. \quad (2.28)$$

Then (2.27) holds for $C = A + \sqrt{-1}B$, with the determination of the square root given by continuation along the image of (2.28) by the application $\det^{-\frac{1}{2}} : GL_n(\mathbb{C}) \to \mathbb{C}$. Henceforth, we will always use this determination of the square root of the determinant for $C = A + \sqrt{-1}B$ as above.
3 Isotropic states

Through all this section, we use the context and notations of Section 2. In particular, recall that \((X, \omega)\) is a compact symplectic manifold of dimension \(2n\), and that the curvature of \(\nabla^L\) on \((L, h^L)\) over \(X\) satisfies \((\text{1.1})\).

3.1 Bohr-Sommerfeld submanifolds

An immersed submanifold \(\iota: \Lambda \to X\) is said to be isotropic if \(\iota^*\omega = 0\). If in addition \(\dim \Lambda = n\), it is said to be Lagrangian. We write \(\nabla^{\iota^*L}, |\cdot|_{\iota^*L}\) for the connection and norm induced by \(\nabla^L, h^L\) on the pullback line bundle \(\iota^*L\) over \(\Lambda\). Note that by \((\text{1.1})\), the condition \(\iota^*\omega = 0\) implies that \(\nabla^{\iota^*L}\) is flat. This observation motivates the following definition.

**Definition 3.1.** A properly immersed oriented isotropic submanifold \(\iota: \Lambda \to X\) is said to satisfy the **Bohr-Sommerfeld condition** if there exists a non-vanishing smooth section \(\zeta \in \mathcal{C}^\infty(\Lambda, \iota^*L)\) satisfying

\[
\nabla^{\iota^*L}\zeta = 0. \tag{3.1}
\]

Taking \(\zeta\) satisfying further \(|\zeta(x)|_{\iota^*L} = 1\) for any \(x \in \Lambda\), the data of \((\Lambda, \iota, \zeta)\) is called a **Bohr-Sommerfeld submanifold** of \(X\), or a **Bohr-Sommerfeld Lagrangian** if in addition \(\dim \Lambda = n\).

Note that this definition depends only on the symplectic structure on \((X, \omega)\) and the prequantization condition \((\text{1.1})\) on \((L, h^L, \nabla^L)\). Furthermore, as \(\nabla^L\) is Hermitian, up to renormalisation we can always assume that \(\zeta \in \mathcal{C}^\infty(\Lambda, \iota^*L)\) satisfying \((3.1)\) is such that \(|\zeta(x)|_{\iota^*L} = 1\) for any \(x \in \Lambda\). Finally, from the compactness of \(X\), the properness hypothesis on \(\iota\) is equivalent to the compactness of \(\Lambda\).

**Remark 3.2.** As noted above, if \(\iota: \Lambda \to X\) is isotropic, then \(\nabla^{\iota^*L}\) is flat over \(\Lambda\), hence determined by its holonomy \(\text{hol}_{\iota^*L}: \pi_1(\Lambda) \to S^1 \subset \mathbb{C}\). We can then reformulate \((3.1)\) by saying that \(\iota: \Lambda \to X\) satisfies the Bohr-Sommerfeld condition if and only if \(\text{hol}_{\iota^*L} = \{1\}\). Now if the order of \(\text{hol}_{\iota^*L}\) is finite, then there exists a finite covering \(j: \hat{\Lambda} \to \Lambda\) such that \(\text{hol}_{j \circ \iota^*L} = \{1\}\), so that \(\iota \circ j: \hat{\Lambda} \to X\) satisfies the Bohr-Sommerfeld condition. In particular, if there is \(k \in \mathbb{N}\) such that \(\iota: \Lambda \to X\) satisfies the Bohr-Sommerfeld condition for \(L^k\) instead of \(L\), then the order of \(\text{hol}_{\iota^*L}\) divides \(k\), thus is finite. Such a \(\iota: \Lambda \to X\) is called a **Bohr-Sommerfeld submanifold of order** \(k\), and up to finite covering, Definition 3.1 also accounts for these. In the same line of thought, if \(\iota: \Lambda \to X\) is not orientable, we can always work on the orientation double cover of \(\Lambda\).

Let us now set some notations. We write \(\iota^L, \iota^E\) and \(\iota_p\) for the natural maps covering \(\iota: \Lambda \to X\) on the respective total spaces of \(L, E\) and \(E_p\) for any \(p \in \mathbb{N}^*\). If \(\zeta\) is any section of \(\iota^*L\), we write \(\zeta^p\) for the \(p\)-th power of \(\zeta\) defined as a section of \(\iota^*L^p\). If additionally \(f\) is a section of \(\iota^*E\), we write \(\zeta^pf\) for the induced tensor product in \(\iota^*E_p\).

From now on, we fix an almost complex structure \(J\) on \(TX\) compatible with \(\omega\) and an auxiliary Hermitian vector bundle \((E, h^E)\) with Hermitian connection \(\nabla^E\).
**Definition 3.3.** The *isotropic state* associated to $(\Lambda, \iota, \zeta)$ and $f \in \mathcal{C}^\infty(\Lambda, \iota^*E)$ is the family of sections $\{s_{f, p} \in \mathcal{H}_p\}_{p \in \mathbb{N}^*}$ defined for any $x \in X$ by the formula

$$s_{f, p}(x) = \int_{\Lambda} P_p(x, \iota(y))t_p\zeta^p f(y)dv_\Lambda(y). \quad (3.2)$$

As $\iota$ is locally an embedding, when working locally we will often omit the mention of $\iota$, considering locally $\Lambda$ as a submanifold of $X$. With this convention, equation (3.2) writes

$$s_{f, p}(x) = \int_{\Lambda} P_p(x, y)\zeta^p f(y)dv_\Lambda(y). \quad (3.3)$$

We list the basic properties of isotropic states in the following proposition, which holds for any $p \in \mathbb{N}^*$.

**Proposition 3.4.** For any $f_1, f_2 \in \mathcal{C}^\infty(\Lambda, \iota^*E)$, we have the following additivity property,

$$s_{f_1 + f_2, p} = s_{f_1, p} + s_{f_2, p}. \quad (3.4)$$

For any $s \in \mathcal{H}_p$, we have the following reproducing property,

$$\langle s, s_{f, p} \rangle_p = \int_{\Lambda} \langle s(\iota(x)), t_p, \zeta^p f(x) \rangle_{E_p} dv_\Lambda(x). \quad (3.5)$$

For any $f \in \mathcal{C}^\infty(\Lambda, \iota^*E)$ and any $F \in \mathcal{C}^\infty(X, \text{End}(E))$, the action of $T_{F, p}$ on $s_{f, p}$ is given for any $x \in X$ by the formula

$$T_{F, p}s_{f, p} = \int_{\Lambda} T_{F, p}(x, \iota(y))t_p, \zeta^p f(y)dv_\Lambda(y). \quad (3.6)$$

**Proof.** First, the additivity property (3.4) is obvious from (3.2). Next, recall that $P_p$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle_p$ for any $p \in \mathbb{N}^*$, and restricts to the identity on $\mathcal{H}_p$. Then using (2.12), (3.3) and Fubini, we compute for any $s \in \mathcal{H}_p$,

$$\langle s, s_{f, p} \rangle_p = \int_X \langle s(y), \int_{\Lambda} P_p(y, \iota(x))t_p, \zeta^p f(x)dv_\Lambda(x) \rangle_{E_p} dv_X(y)$$

$$= \int_{\Lambda} \int_X P_p(\iota(x), y)s(y)dv_X(y), t_p, \zeta^p f(x) \rangle_{E_p} dv_\Lambda(x) \quad (3.7)$$

$$= \int_{\Lambda} \langle s(\iota(x)), t_p, \zeta^p f(x) \rangle_{E_p} dv_\Lambda(x).$$

The reproducing property (3.5) follows from (3.7). Finally, from (2.13), we get for any $f \in \mathcal{C}^\infty(\Lambda, \iota^*E)$ and $F \in \mathcal{C}^\infty(X, \text{End}(E))$ that $T_{F, p}s_{f, p} = P_pFs_{f, p}$. Then by (2.14), (3.2) and using Fubini, we get for any $x \in X$,

$$(T_{F, p}s_{f, p})(x) = \int_X \int_{\Lambda} P_p(x, w)F(w)P_p(w, \iota(y))t_p, \zeta^p f(y)dv_\Lambda(y)dv_X(w)$$

$$= \int_{\Lambda} T_{F, p}(w, \iota(y))t_p, \zeta^p f(y)dv_\Lambda(y). \quad (3.8)$$

From (3.8), we get (3.6).
3.2 Asymptotic expansion of isotropic states

In this section, we establish the first semi-classical properties of isotropic states. In particular, we show that the $L^2$-norm of an isotropic state admits an asymptotic expansion in $p \in \mathbb{N}^*$, and we compute the highest order term.

For any $p \in \mathbb{N}^*$, we write $| \cdot |_{E_p}$ for the norm on $E_p$ induced by $h^L$ and $h^E$. We show in the following proposition how an isotropic state concentrates around the image of the associated isotropic submanifold as $p$ tends to infinity.

**Proposition 3.5.** Let $f \in \mathcal{C}^\infty(\Lambda, \iota^*E)$. For any closed subset $K \subset X$ such that $K \cap \iota(\Lambda) = \emptyset$ and for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $x \in K$ and all $p \in \mathbb{N}^*$,

$$|s_{f,p}(x)|_{E_p} < C_k p^{-k}.$$  

(3.9)

**Proof.** This is a direct consequence of Proposition 2.3 and formula (3.2). □

For any $p \in \mathbb{N}^*$, we denote by $\| \cdot \|_p$ the norm on $\mathcal{C}^\infty(X, E_p)$ induced by $\langle \cdot, \cdot \rangle_p$, and by $| \cdot |_{\iota^*E}$ the norm on $\iota^*E$ over $\Lambda$ induced by $h^E$. The rest of the section is dedicated to the proof of the following theorem.

**Theorem 3.6.** Let $f \in \mathcal{C}^\infty(\Lambda, \iota^*E)$, and set $d = \text{dim } \Lambda$. Then there exist $b_r \in \mathbb{R}$, $r \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,

$$\|s_{f,p}\|_p^2 = p^{n-\frac{d}{2}} \sum_{r=0}^{k} p^{-r} b_r + O(p^{n-\frac{d}{2}-(k+1)}).$$  

(3.10)

Furthermore, we have

$$b_0 = 2^d/2 \int_{\Lambda} |f|^2_{\iota^*E} \det \left( \hat{R}^L_{x_0}/2\pi \right) \frac{dv_\Lambda}{dv_{\Lambda,\omega}} dv_\Lambda,$$  

(3.11)

and $b_0 = 2^n/2 \int_{\Lambda} |f|^2_{E_p} dv_{\Lambda,\omega}$ if $\text{dim } \Lambda = n$.

Additionally, for any $F \in \mathcal{C}^\infty(X, \text{End}(E))$, the product $\langle T_{F,p}s_{f,p}, s_{f,p} \rangle_p$ satisfies the expansion of (3.10) with $b_r \in \mathbb{C}$, $r \in \mathbb{N}$, and

$$b_0 = 2^d/2 \int_{\Lambda} \langle Ff, f \rangle_{\iota^*E} \det \left( \hat{R}^L_{x_0}/2\pi \right) \frac{dv_\Lambda}{dv_{\Lambda,\omega}} dv_\Lambda.$$  

(3.12)

**Proof.** Note first that the reproducing property (3.5) gives

$$\|s_{f,p}\|_p^2 = \int_{\Lambda} \langle s_{f,p}(\iota(x)), \zeta^p f(x) \rangle_{E_p} dv_\Lambda(x).$$  

(3.13)

Using (3.13), we are reduced to evaluate $s_{f,p}$ on the image of $\iota : \Lambda \to X$. Let then $x_0 \in X$ be in the image of $\iota$. As $\iota : \Lambda \to X$ is an immersion, there is an integer $m \in \mathbb{N}$ such that for any small enough connected neighbourhood $V$ of $x_0$ in $X$, there
are $m$ disjoint connected open sets $U_1, \ldots, U_m \subset \Lambda$ such that $\iota^{-1}(V) = \bigcup_{j=1}^m U_j$. Using Proposition \ref{prop2.3} we can then localize the problem in the following way,

$$s_{f,p}(x_0) = \int_{\Lambda} P_p(x_0, \iota(x)) \zeta^p f(x) dv_\Lambda(x)$$

$$= \sum_{j=1}^m \int_{U_j} P_p(x_0, \iota(x)) \zeta^p f(x) dv_\Lambda(x) + O(p^{-\infty}). \tag{3.14}$$

In view of \ref{eq3.10}, \ref{eq3.13} and \ref{eq3.14}, we can assume that $f$ has compact support around $\bigcup_{j=1}^m U_j$. Using \ref{eq3.14} and \ref{eq3.13}, we are reduced further to the case where $f$ has compact support around one of the $U_j$ for some $j$. As $U := U_j$ is embedded in $X$ through $\iota$, we can consider $U$ as a submanifold of $X$, and \ref{eq3.14} translates to

$$s_{f,p}(x_0) = \int_U P_p(x_0, x) \zeta^p f(x) dv_\Lambda(x) + O(p^{-\infty}). \tag{3.15}$$

Take $\varepsilon > 0$, $V \subset X$ and $\phi_{x_0} : B^{2n}(0, \varepsilon) \to V$ as in \ref{eq2.17}, identifying $U \subset V$ with $B^\Sigma(0, \varepsilon)$, where $\Sigma$ is a vector subspace of $\mathbb{R}^{2n}$. Then $\Sigma$ is an isotropic subspace of $(\mathbb{R}^{2n}, \Omega)$. We identify $E$, $L$ over $B^{2n}(0, \varepsilon)$ with $E_{x_0}$, $L_{x_0}$ as in Section \ref{sect2.2}. In particular, we use the unitary vector $\zeta(x_0)$ to identify $L_{x_0}$ with $\mathbb{C}$, where $\zeta \in \mathcal{C}^\infty(\Lambda, \iota^*L)$ is the section associated to $\langle \Lambda, \iota, \zeta \rangle$ as in Definition \ref{def3.1}. As $\zeta$ is parallel with respect to $\nabla^\iota L$ along $\Lambda$, it is identified with $1 \in \mathbb{C}$ over $B^\Sigma(0, \varepsilon)$ in this trivialization.

Let $du$ be the Lebesgue measure of $\Sigma$, and define the function $h \in \mathcal{C}^\infty(B^\Sigma(0, \varepsilon), \mathbb{R})$ for all $u \in B^\Sigma(0, \varepsilon)$ by

$$dv_\Lambda(u) = h(u)du. \tag{3.16}$$

Then $h(0) = (dv_\Lambda/dv_{\Lambda, \omega})(x_0)$. Using Corollary \ref{cor2.4} and Lemma \ref{lem2.5} for any $\delta \in ]0, 1[$, we get $\theta \in ]0, 1[$ such that

$$\langle s_{f,p}(x_0), \zeta^p f(x_0) \rangle_{E_p}$$

$$= \int_{B^\Sigma(0, \varepsilon, \theta/2)} \langle P_p(x_0, \phi_{x_0}(u)) \zeta^p f(\phi_{x_0}(u)), \zeta^p f(x_0) \rangle_{E_p} dv_\Lambda(u) + O(p^{-\infty})$$

$$= p^n \int_{B^\Sigma(0, \varepsilon, \theta/2)} \sum_{r=0}^k p^{-r/2}(J_{r, x_0} \mathcal{P}_{x_0}(0, \sqrt{p}u) f_{x_0}(u), f(x_0))_{x_0} \kappa_{x_0}^{-1/2}(u) dv_\Lambda(u)$$

$$+ p^n \int_{B^\Sigma(0, \varepsilon, \theta/2)} O(p^{-\frac{k+1}{2}+\delta}) dv_\Lambda(u) + O(p^{-\infty}) \tag{3.17}$$

$$= p^n \int_{B^\Sigma(0, \varepsilon, \theta/2)} \sum_{r=0}^k p^{-r/2}(J_{r, x_0} \mathcal{P}_{x_0}(0, \sqrt{p}u) f_{x_0}(u), f(x_0))_{x_0} \kappa_{x_0}^{-1/2}(u) h(u) du$$

$$+ p^n p^{-\frac{k+1}{2}} O(p^{-\frac{k+1}{2}+\delta}).$$

Let us write $g_{x_0} = h \kappa_{x_0}^{1/2} f_{x_0} \in \mathcal{C}^\infty(B^\Sigma(0, \varepsilon), E_{x_0})$. Then from \ref{eq2.21} and \ref{eq3.16}, we
get the following Taylor expansion in \( u \in \mathbb{R}^n \) up to order \( k \in \mathbb{N} \),

\[
g_{x_0}(u) = f(x_0)(dv_A/dv_{A,\omega})(x_0) + \sum_{1 \leq |\alpha| \leq k} \frac{\partial^n g_{x_0}}{\partial u^\alpha} \frac{u^\alpha}{\alpha!} + O(|u|^{k+1})
\]

\[
= f(x_0)(dv_A/dv_{A,\omega})(x_0) + \sum_{1 \leq |\alpha| \leq k} p^{-\alpha/2} \frac{\partial^n g_{x_0}}{\partial u^\alpha} \frac{\sqrt{p} u^\alpha}{\alpha!} + O(|\sqrt{p} u|^{k+1}).
\]  

(3.18)

On another hand, recall from Lemma 2.3 that \( J_{r,x_0}(0, \sqrt{p} u) \in \text{End}(E_{x_0}) \) is polynomial in \( \sqrt{p} u \) of the same parity as \( r \in \mathbb{N} \). Let \( M_k \) be the supremum of the degree of \( J_{r,x_0} \) for all \( 1 \leq r \leq k \), and write \( \delta' = \delta + (M_k + k + 1 + d)(1 - \theta)/2 \). We deduce from (3.17) and (3.18) the existence of a sequence \( \{G_r\}_{r \in \mathbb{N}} \) of polynomials in one variable of \( \mathbb{R}^n \) of the same parity as \( r \), with values in \( \mathbb{C} \), and with \( G_0(Z, Z') = |f(x_0)|_{x_0}(dv_A/dv_{A,\omega})(x_0) \), such that

\[
\langle s_{f,p}(x_0), \zeta^p f(x_0) \rangle_{E_p} = p^n \sum_{r=0}^{k} p^{-r/2} \int_{B^\Sigma(0, \varepsilon p^{(1-\theta)/2})} G_r(\sqrt{p} u) \mathcal{P}_{x_0}(0, \sqrt{p} u) du + O(p^{n-\frac{d(k+1)}{2}+\delta'})
\]  

(3.19)

Recall from (2.20) that

\[
\mathcal{P}_{x_0}(0, u) = \det \left( \hat{R}_{x_0}^{L}/2\pi \right) \exp \left( -\frac{\pi}{2} |u|^2 \right).
\]  

(3.20)

Thus as \( 1 - \theta > 0 \), we deduce from (3.20) that for any \( l \in \mathbb{N} \), there is \( C_l > 0 \) such that for any \( u \in \Sigma \) outside \( B^\Sigma(0, \varepsilon p^{(1-\theta)/2}) \),

\[
\mathcal{P}_{x_0}(0, u) \leq C_l p^{-l}.
\]  

(3.21)

We then deduce from (3.19) and (3.21) that

\[
\langle s_{f,p}(x_0), \zeta^p f(x_0) \rangle_{E_p} = p^{n-\frac{d}{2}} \sum_{r=0}^{k} p^{-r/2} \int_{\Sigma} G_r(u) \mathcal{P}_{x_0}(0, u) du + O(p^{n-\frac{d(k+1)}{2}+\delta'}).
\]  

(3.22)

As \( G_r \) is of the same parity as \( r \), we immediately deduce from (3.20) that for any \( m \in \mathbb{N} \),

\[
\int_{\Sigma} G_{2m+1}(u) \mathcal{P}_{x_0}(0, u) du = 0.
\]  

(3.23)

Finally, as \( G_0(Z, Z') = |f(x_0)|_{x_0} \), we get from (3.20) the following formula for the highest order term of (3.22),

\[
\int_{\Sigma} |f(x_0)|_{x_0}(dv_A/dv_{A,\omega})(x_0) \mathcal{P}_{x_0}(0, u) du
\]

\[
= |f(x_0)|_{x_0}(dv_A/dv_{A,\omega})(x_0) \det \left( \hat{R}_{x_0}^{L}/2\pi \right) \int_{\Sigma} \exp \left( -\frac{\pi}{2} |u|^2 \right) du
\]

\[
= 2^{d/2} |f(x_0)|_{x_0}(dv_A/dv_{A,\omega})(x_0) \det \left( \hat{R}_{x_0}^{L}/2\pi \right).
\]  

(3.24)
Then recalling that all the estimates above are uniform in $x_0 \in X$, and by (2.5), (3.13) and (3.23), it suffices to integrate (3.22) and (3.24) over $x_0 \in \Lambda$ with respect to $dv_\Lambda$ to get (3.10) and (3.11). Now if $\Lambda$ is Lagrangian, we know that

$$dv_{\Lambda,\omega} = \det^{1/2} \left( \hat{R}^L / 2\pi \right) dv_\Lambda.$$  

(3.25)

Using Lemma 2.6 and (3.6), the computation of $\langle T_{p,\Lambda} s_{f,\rho}, s_{j,\mu} \rangle_p$ is completely analogous to the one above. This achieves the proof of Theorem 3.6.

$$\square$$

4 Isotropic intersections

Let us consider two Bohr-Sommerfeld submanifolds $(\Lambda_j, \iota_j, \zeta_j)$ together with $f_j \in \mathcal{C}^\infty(\Lambda_j, \iota_j^* E)$, for $j = 1, 2$, and set $d_j = \dim \Lambda_j$. In this section, we establish the existence of an asymptotic expansion in $p \in \mathbb{N}^*$ of the Hermitian product $\langle s_{f,\Lambda}, s_{j,\mu} \rangle_p$ of the two associated isotropic states, and we compute the highest order term, which depends only on the geometry of the intersection. Note that the case $\{ s_{f,\Lambda} \}_{p \in \mathbb{N}^*} = \{ s_{f,\mu} \}_{p \in \mathbb{N}^*}$ is precisely the result of Theorem 3.6.

We will use the following definition of the intersection of immersions, which requires a natural regularity assumption, assumed throughout the section.

Definition 4.1. We say that two proper immersions $\iota_j : \Lambda_j \to X$, $j = 1, 2$ are intersecting cleanly if for any $x \in \iota_1(\Lambda_1) \cap \iota_2(\Lambda_2)$, any $y_j \in \Lambda_j$ such that $\iota_1(y_1) = \iota_2(y_2) = x$ and any small enough neighbourhoods $U_j \subset \Lambda_j$ of $y_j$, their intersection $\iota_1(U_1) \cap \iota_2(U_2)$ is a submanifold of $X$ satisfying $T_x \iota_1(U_1) \cap T_x \iota_2(U_2) = T_x(\iota_1(U_1) \cap \iota_2(U_2))$.

In that case, we define their intersection as the fibre product of $\iota_1 : \Lambda_1 \to X$ and $\iota_2 : \Lambda_2 \to X$ over $X$, which is given by the data of a manifold $\Lambda_1 \cap \Lambda_2$ together with two immersion $j_i : \Lambda_1 \cap \Lambda_2 \to \Lambda_i$, $i = 1, 2$, such that $\iota_1 \circ j_1 = \iota_2 \circ j_2$ and universal for this property.

4.1 Asymptotic expansion of discrete intersections

In this section, we deal with the case of discrete intersection. We consider first the easy case when the intersection is empty.

Proposition 4.2. Suppose that $\Lambda_1 \cap \Lambda_2 = \emptyset$, and let $F \in \mathcal{C}^\infty(X, \text{End}(E))$. Then for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $p \in \mathbb{N}^*$,

$$| \langle T_{p,\Lambda} s_{f,\rho}, s_{j,\mu} \rangle_p | < C_k p^{-k}. $$

(4.1)

Proof. Using the reproducing property (3.5), we get for any $p \in \mathbb{N}^*$,

$$\langle T_{p,\Lambda} s_{f,\rho}, s_{j,\mu} \rangle_p = \int_{\Lambda} \langle T_{p,\Lambda} s_{f,\rho}(\iota_2(x)), \zeta_2^* f(x) \rangle E_p dv_{\Lambda_2}(x).$$

(4.2)

In particular, choosing $K = \iota_2(\Lambda_2)$ in Proposition 3.5, we deduce (4.1) from (4.2). $\square$
In view of Proposition 4.2, we will assume from now on that \( \Lambda_1 \cap \Lambda_2 \) is not empty. In the statement of the following theorem, the immersions \( t_i : \Lambda_i \to X \) and \( j_i : \Lambda_1 \cap \Lambda_2 \to \Lambda_i \), \( i = 1, 2 \), are implicit, and we omit to mention them for simplicity.

**Theorem 4.3.** Suppose that \( (\Lambda_1, t_1, \zeta_1) \) and \( (\Lambda_2, t_2, \zeta_2) \) intersect cleanly, and that their intersection \( \Lambda_1 \cap \Lambda_2 \) in the sense above is discrete. Set \( m = \# \Lambda_1 \cap \Lambda_2 \) and write \( \Lambda_1 \cap \Lambda_2 = \{x_1, \ldots, x_m\} \). Then for any \( F \in \mathcal{C}^\infty(X, \text{End}(E)) \), there exist \( b_{q,r} \in \mathbb{C}, r \in \mathbb{N}, 1 \leq q \leq m, \) such that for any \( k \in \mathbb{N} \) and as \( p \to +\infty \),

\[
\langle T_{F,p}s_{f_1,p}, s_{f_2,p} \rangle_p = p^{n-\frac{d_1+d_2}{2}} \sum_{q=1}^{m} \lambda_q^p \sum_{r=0}^{k} p^{-r} b_{q,r} + O(p^{n-\frac{d_1+d_2}{2}-(k+1)}),
\]

(4.3)

where \( \lambda_q = \langle \zeta_1(x_q), \zeta_2(x_q) \rangle \). Furthermore, if \( \dim \Lambda_1 = n \), we have

\[
b_{q,0} = 2^{n/2} \langle F_{x_q}f_1(x_q), f_2(x_q) \rangle_{x_q} \det^{1/2} \left( R_{x_q}/2\pi \right) \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}(x_q) \det^{-\frac{1}{2}} \left\{ \sqrt{-1} \sum_{k=1}^{n} h^{TX}_\omega(e_k, \nu_i) \omega(e_k, \nu_j) \right\}_{i,j=1}^{d_2},
\]

(4.4)

where \( (e_i)_{i=1}^{n}, (\nu_j)_{j=1}^{d_2} \) are oriented orthonormal bases for \( g^{TX}_\omega \) of the tangent spaces of \( \Lambda_1, \Lambda_2 \) in \( X \) at \( x_q \), and the square root of the determinant is determined by \( (2.28) \).

**Proof.** We will prove Theorem 4.3 for \( F = \text{Id}_E \) (so that \( T_{F,p} = P_p \)), the proof of the general case being totally analogous by Lemma 2.6 and (3.6). First, using the reproducing property (3.5), we get for any \( p \in \mathbb{N}^* \),

\[
\langle s_{f_1,p}, s_{f_2,p} \rangle_p = \int_{\Lambda} \langle s_{f_1,p}(t_2(x)), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_2}(x).
\]

(4.5)

We can then reproduce the argument in the proof of Proposition 4.2 using Proposition 3.5 to reduce the proof to the case of \( f_2 \) with compact support in any neighbourhood of \( t_2^{-1}(t_1(\Lambda_1) \cap t_2(\Lambda_2)) = j_2(\Lambda_1 \cap \Lambda_2) \), which is a finite set by assumption. Symmetrically, using the reproducing property of \( s_{f_1,p} \) instead of \( s_{f_2,p} \), we can assume further that \( f_1 \) has compact support in any neighbourhood of \( t_1^{-1}(t_1(\Lambda_1) \cap t_2(\Lambda_2)) = j_2(\Lambda_1 \cap \Lambda_2) \). By (3.4) and (4.3), we are further reduced to the case of \( f_i \) with compact support in a neighborhood of only one point \( y_i \in j_i(\Lambda_1 \cap \Lambda_2) \) for any \( i = 1, 2 \). Using Proposition 3.5, we are finally reduced to the case \( t_1(y_1) = t_2(y_2) \), or equivalently, of \( j_i(y_1) = y_i \) for any \( i = 1, 2, x_q \in \Lambda_1 \cap \Lambda_2, 1 \leq q \leq m \). Set \( t_1(\Lambda_1) = t_2(\Lambda_2) =: x_0 \in X \).

Let \( U_j \) be so small that it is embedded in \( X \) by \( t_j \) for any \( j = 1, 2 \), so that we can consider them as submanifolds of \( X \) intersecting cleanly at \( x_0 \in X \) only. In particular, using (3.3), equation (4.5) becomes

\[
\langle s_{1,p}, s_{2,p} \rangle_p = \int_{U_2} \langle s_{f_1,p}(x), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_2}(x) + O(p^{-\infty})
\]

\[
= \int_{U_2} \int_{U_1} \langle P_{p}(x,y) \zeta_1^p f_1(y), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_1}(y) dv_{\Lambda_2}(x) + O(p^{-\infty}).
\]

(4.6)
Take \( \varepsilon > 0 \), \( V \subset X \) such that \( V \cup \Lambda_j = U_j \) and \( \phi_{x_0} : B^{\mathbb{R}^{2n}}(0, \varepsilon) \to V \) as in (2.17), identifying \( U_j \) with \( B^{\Sigma_j}(0, \varepsilon) \) for any \( j = 1, 2 \), where \( \Sigma_1 \) and \( \Sigma_2 \) are isotropic subspaces of \((\mathbb{R}^{2n}, \Omega)\). As \( U_1 \) and \( U_2 \) intersect cleanly at \( x_0 \) only, we have \( \Sigma_1 \cap \Sigma_2 = \{0\} \). We identify \( E, L \) over \( B^{\mathbb{R}^{2n}}(0, \varepsilon) \) with \( E_{x_0}, L_{x_0} \) as in Section 2.2 and use the unitary vector \( \zeta_1(x_0) \) to identify \( L_{x_0} \) with \( \mathbb{C} \). Then \( \zeta_1 \) is identified with \( 1 \in \mathbb{C} \) over \( B^{\Sigma_1}(0, \varepsilon) \) in this trivialization. As \( \zeta_2 \) is parallel with respect to \( \nabla^{1/2} \) over \( U_2 \), it is identified with \( \lambda \in \mathbb{C} \) over \( B^{\Sigma_2}(0, \varepsilon) \), where \( \lambda = \langle \zeta_1(x_0), \zeta_2(x_0) \rangle_L \).

Then for all \( p \in \mathbb{N}^* \), (4.6) writes

\[
\langle s_{1,p}, s_{2,p} \rangle_p = \lambda^p \int_{B^{\mathbb{R}^{2n}}(0, \varepsilon)} \int_{B^{\Sigma_1}(0, \varepsilon)} (P_p(\phi_{x_0}(Z), \phi_{x_0}(Z'))) f_{1,x_0}(Z'), f_{2,x_0}(Z))_x \, dv_{\Lambda_1}(Z') \, dv_{\Lambda_2}(Z) + O(p^{-\infty}) \tag{4.7}
\]

Let \( du \) and \( dw \) be the Lebesgue measures of \( \Sigma_1 \) and \( \Sigma_2 \) respectively. For any \( j = 1, 2 \), define the functions \( h_j \in \mathcal{C}^\infty(\Sigma^j(0, \varepsilon), \mathbb{R}) \) in the chart (2.17) for any \( u \in B^{\Sigma_1}(0, \varepsilon) \), \( w \in B^{\Sigma_2}(0, \varepsilon) \) by

\[
dv_{\Lambda_1}(u) = h_1(u) \, du \quad \text{and} \quad dv_{\Lambda_2}(w) = h_2(w) \, dw. \tag{4.8}
\]

Then for \( j = 1, 2 \), the functions \( h_j \) satisfy \( h_j(0) = (dv_{\Lambda_j}/dv_{\Lambda_j, \omega})(x_0) \). Recalling (2.18) and the fact that \( |P| = 1 \) for all \( p \in \mathbb{N}^* \), we can use Corollary 2.4 and Lemma 2.5 to get \( \theta \in ]0, 1[ \) for any \( k \in \mathbb{N} \), \( \delta \in ]0, 1[ \), such that (4.7) becomes

\[
\langle s_{1,p}, s_{2,p} \rangle_p = \lambda^p \int_{B^{\mathbb{R}^{2n}}(0, \varepsilon)} \int_{B^{\Sigma_1}(0, \varepsilon)} (P_p(\phi_{x_0}(Z), \phi_{x_0}(Z'))) f_{1,x_0}(Z'), f_{2,x_0}(Z))_x \, dv_{\Lambda_1}(Z') \, dv_{\Lambda_2}(Z) + O(p^{-\infty})
\]

\[
= \lambda^p \int_{B^{\mathbb{R}^{2n}}(0, \varepsilon, \varepsilon)} \int_{B^{\Sigma_1}(0, \varepsilon, \varepsilon)} (J_{r,x_0} \varphi_0(\sqrt{p}Z, \sqrt{p}Z')) f_{1,x_0}(Z'), f_{2,x_0}(Z))_x \, dv_{\Lambda_1}(Z') \, dv_{\Lambda_2}(Z) + O(p^{-\infty})
\]

\[
= \lambda^p \int_{B^{\mathbb{R}^{2n}}(0, \varepsilon, \varepsilon)} \int_{B^{\Sigma_1}(0, \varepsilon, \varepsilon)} (J_{r,x_0} \varphi_0(\sqrt{p}w, \sqrt{p}w)) f_{1}(u), f_{2}(w)_x \, dv_{\Lambda_1}(Z') \, dv_{\Lambda_2}(Z) + O(p^{-\infty})
\]

\[
\langle s_{1,p}, s_{2,p} \rangle_p = \lambda^p \int_{B^{\mathbb{R}^{2n}}(0, \varepsilon, \varepsilon)} \int_{B^{\Sigma_1}(0, \varepsilon, \varepsilon)} (J_{r,x_0} \varphi_0(\sqrt{p}w, \sqrt{p}w)) f_{1}(u), f_{2}(w)_x \, dv_{\Lambda_1}(Z') \, dv_{\Lambda_2}(Z) + O(p^{-\infty})
\]

Consider now the Taylor expansion up to order \( k \in \mathbb{N} \) of \( g_j = h_j k_{-1/2} f_{j,x_0} \in \mathcal{C}^\infty(B^{\Sigma_j}(0, \varepsilon), \mathbb{C}) \) for \( j = 1, 2 \) as in (3.18). By Lemma 2.5 and following the proof of Theorem 4.3 we get \( \delta' > 0 \) and a sequence \( \{G_r\}_{r \in \mathbb{N}} \) of polynomials in two variables of \( \mathbb{R}^{2n} \) with values in \( \mathbb{C} \), of the same parity as \( r \) with

\[
G_0(Z, Z') = \langle f_{1}(x_0), f_{2}(x_0) \rangle_{x_0} \frac{dv_{\Lambda_1}}{dv_{\Lambda_1, \omega}} \frac{dv_{\Lambda_2}}{dv_{\Lambda_2, \omega}}(x_0), \tag{4.10}
\]
such that (4.9) becomes

\[ \langle s_1, p, s_2, p \rangle_p = \lambda p^{n - \frac{d_1 + d_2}{2}} \sum_{r=1}^{k} p^{-r/2} \int_{B^{\Sigma_2}(0, \varepsilon p^{(1-\theta)/2})} \int_{B^{\Sigma_1}(0, \varepsilon p^{(1-\theta)/2})} G_r \mathcal{P}_{x_0}(w, u) du dw + O(p^{n - \frac{d_1 + d_2 + k + 1}{2} + \delta}). \] (4.11)

As \( \Sigma_1 \cap \Sigma_2 = \{0\} \) and as \( 1 - \theta > 0 \), we get from (2.20) the existence of \( C_l > 0 \) for any \( l \in \mathbb{N} \) such that for any \( u \in \Sigma_1, \ w \in \Sigma_2 \) outside \( B^{\Sigma_1}(0, \varepsilon p^{(1-\theta)/2}), \ B^{\Sigma_2}(0, \varepsilon p^{(1-\theta)/2}) \) respectively,

\[ \mathcal{P}_{x_0}(w, u) \leq C_l p^{-l}. \] (4.12)

From (4.12), equation (4.11) then becomes

\[ \langle s_1, p, s_2, p \rangle_p = \lambda p^{k} \sum_{r=1}^{k} p^{-r/2} \int_{\Sigma_2} \int_{\Sigma_1} G_r \mathcal{P}_{x_0}(w, u) du dw + O(p^{\frac{n}{2} + \delta}). \] (4.13)

Let us now evaluate the integrals in (4.13). Up to linear symplectic transformation, the canonical symplectic basis \( \{e_j, f_j\}_{j=1}^{n} \) of \( (\mathbb{R}^{2n}, \Omega) \) can be chosen such that \( \Sigma_1 = \langle e_1, \ldots, e_{d_1} \rangle \) as an oriented isotropic subspace. Let \( \nu_1, \ldots, \nu_{d_2} \in \Sigma_2 \) form an oriented orthonormal basis of \( \Sigma_2 \) for the metric induced by \( \langle \cdot, \cdot \rangle \). Consider the matrices \( A \) and \( B \) given by

\[ A = (a^j_i)_{1 \leq i \leq n, 1 \leq j \leq d_2} \quad \text{with} \quad a^j_i = \Omega(e_i, \nu_j), \]
\[ B = (b^j_i)_{1 \leq i \leq n, 1 \leq j \leq d_2} \quad \text{with} \quad b^j_i = \langle e_i, \nu_j \rangle. \] (4.14)

As \( \Omega(e_i, \nu_j) = \langle f_i, \nu_j \rangle \) for all \( 1 \leq i \leq d_1, 1 \leq j \leq d_2 \), we know that for any \( 1 \leq j \leq d_2 \),

\[ \nu_j = \sum_{i=1}^{n} b^j_i e_i + \sum_{i=1}^{n} a^j_i f_i. \] (4.15)

Let us write \( dt := dt_1 \ldots dt_{d_2} \) for the Lebesgue measure of \( \mathbb{R}^{d_2} \), and let \( \varphi \) be any measurable function with compact support on \( \mathbb{R}^{2n} \). Setting \( w = t_i \nu_i \) for any \( w \in \Sigma_2 \), integration of \( \varphi \) along \( \Sigma_2 \) for its Lebesgue measure \( dw \) becomes

\[ \int_{\Sigma_2} \varphi(w) dw = \int_{\mathbb{R}^{d_2}} \varphi \left( \sum_{j=1}^{d_2} t_j \nu_j \right) dt. \] (4.16)

Let us use the convention of Section 2.1 summing \( i \) from 1 to \( d_1 \) and \( k, j \) from 1 to \( d_2 \) whenever they appear as free indices. From the explicit expression (2.20), taking
Fourier transform and performing a change of variables, we compute

\[
\int_{\Sigma_2} \int_{\Sigma_1} G_r(w, u) \mathcal{P}_{x_0}(w, u) dudw = \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} G_r(t_j \nu_j, u_i e_i) \mathcal{P}_{x_0}(t_j \nu_j, u_i e_i) dudt
\]

\[
= \det \left( \hat{R}_{x_0}/2\pi \right) \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} G'_r(t) \exp \left( -\frac{\pi}{2} \sum_{i=d_1+1}^n (t_j b_i^j)^2 + (t_j a_i^j)^2 \right) dudt
\]

\[
= \det \left( \hat{R}_{x_0}/2\pi \right) \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} G''_r(t) \exp \left( -\frac{\pi}{2} \sum_{i=d_1+1}^n (t_j b_i^j)^2 + (t_j a_i^j)^2 \right) dudt
\]

(4.17)

\[
= 2^{d_2/2} \det \left( \hat{R}_{x_0}/2\pi \right) \int_{\mathbb{R}^{d_2}} G''_r(t) \exp \left( -\frac{\pi}{2} \sum_{i=d_1+1}^n (t_j b_i^j)^2 + (t_j a_i^j)^2 \right) dt,
\]

where \( G'_r(t), G''_r(t) \) are polynomials in \( t \in \mathbb{R}^{d_1} \) of the same parity as \( r \). Using that \( \Sigma_1 \cap \Sigma_2 = \{0\} \), we get the convergence of the integral in (4.17), and as the integrand is of the same parity as \( r \), the integral vanishes if \( r \) is odd. Together with (4.13), this proves (4.3).

Let us now compute the first coefficient of (4.13) in the case \( \dim \Lambda_1 = n \). From (4.17), we get

\[
\int_{\Sigma_2} \int_{\Sigma_1} \mathcal{P}_{x_0}(u, w) dudw
\]

\[
= 2^n/2 \det \left( \hat{R}_{x_0}/2\pi \right) \int_{\mathbb{R}^{d_2}} \exp \left( -\pi \sum_{i=1}^n (t_j a_i^j)^2 + \sqrt{-1} t_k b_i^k t_j \right) dt.
\]

(4.18)

As \( \langle \nu_1, \ldots, \nu_{d_2} \rangle \) is the basis of an isotropic submanifold, we get that \( \omega(\nu_j, \nu_k) = 0 \) for all \( 1 < j, k \leq d_2 \), which is equivalent through (4.15) to the fact that \( B^T A \) is symmetric. Then summing \( i \) from 1 to \( n \), the matrix \( (a_i^k a_i^j + \sqrt{-1} b_i^k a_i^j)_{k,j=1}^{d_2} = A^T A + \sqrt{-1} B^T A \) is symmetric, and its real part \( A^T A \) is strictly positive as \( A \) has maximal rank. Thus from (4.18) and using (2.27), we get

\[
\int_{\Sigma_2} \int_{\Sigma_1} \mathcal{P}_{x_0}(u, w) dudw = 2^{n/2} \det \left( \hat{R}_{x_0}/2\pi \right) \left( \sqrt{-1} (B - \sqrt{-1} A)^T A \right) \det \left( \hat{R}_{x_0}/2\pi \right).
\]

(4.19)

Then (4.1) follows from (2.4), (2.5), (3.25), (4.10), (4.14) and (4.18).
4.2 Asymptotic expansion of clean intersections

In this section, we deal with the case of general clean intersection in the sense of Definition \ref{def:clean}. As in Section \ref{sec:intersection}, the immersions \( \iota_1 : \Lambda_1 \to X \) and \( j_i : \Lambda_1 \cap \Lambda_2 \to \Lambda_i, \ i = 1, 2 \), are implicit in the statement of the following theorem, and we omit to mention them for simplicity.

**Theorem 4.4.** Suppose that \((\Lambda_1, \iota_1, \zeta_1)\) and \((\Lambda_2, \iota_2, \zeta_2)\) intersect cleanly. Let \( \Lambda_1 \cap \Lambda_2 = \bigcup_{q=1}^m Y_q \) be the decomposition into connected components of their intersection in the sense above, and set \( l_q = \dim Y_q \). Then for any \( F \in C^\infty(X, \text{End}(E)) \), there exist \( b_{q,r} \in \mathbb{C}, \ r \in \mathbb{N}, 1 \leq q \leq m, \) such that for any \( k \in \mathbb{N} \) and as \( p \to +\infty \),

\[
\langle T_{f_p} s_{f_1, p}, s_{f_2, p} \rangle_p = \sum_{q=1}^m b_q \frac{n-d_1-d_2}{2} + l_q \sum_{r=0}^k p^{-r} b_{q,r} + O(p^{n-d_1-d_2} + l_q - (k+1)), \tag{4.20}
\]

where \( \lambda_q \in \mathbb{C} \) is the value of the constant function on \( Y_q \) defined for any \( x \in Y_q \) by \( \lambda_q(x) = \langle \zeta_1(x), \zeta_2(x) \rangle \). If \( \dim \Lambda_1 = n \), we have

\[
b_{q,0} = 2^{n/2} \int_{Y_q} \langle F f_1(x), f_2(x) \rangle_E \det^{1/2} \left( \frac{R}{2\pi} \right) \frac{dv_{\Lambda_2}}{dv_{\Lambda_2\omega}}(x) \det^{-\frac{1}{2}} \left\{ \sqrt{-1} \sum_{k=1}^{n-l_q} h_{\omega}^{TX}(e_k, \nu_i) \omega(e_k, \nu_j) \right\}^{d_2-l_q}_{i,j=1}(x) |dv|_{Y_q, \omega}(x), \tag{4.21}
\]

where \( \langle e_i \rangle_{i=1}^{n-d_2}, \langle \nu_j \rangle_{j=1}^{n-d_4} \) are local orthonormal frames of the normal bundle of \( Y_q \) inside \( \Lambda_1, \Lambda_2 \) with respect to \( g_{\omega}^{TX_1}, g_{\omega}^{TX_2} \), and \( |dv|_{Y_q, \omega} \) is the Riemannian density of \( (Y_q, g_{\omega}^{TY_q}) \).

The square root of the determinant is determined by \( (2.28) \).

**Proof.** Let us set \( F = \text{Id}_E \), the proof of the general case being totally analogous by Lemma \ref{lem:locality} and \ref{lem:asymptotic}. Using Proposition \ref{prop:asymptotic} \ref{item:asymptotic} and \ref{item:asymptotic}, we can assume that \( \Lambda_1 \cap \Lambda_2 \) has a unique connected component \( Y \), and that \( f_j, \ j = 1, 2 \), have compact support in any given open sets. The following computations are then local on \( Y \), and we may assume \( Y \) oriented and embedded in \( \Lambda_2 \) by \( j_2 : Y \to \Lambda_2 \). We omit the mention of \( j_2 \) in the sequel. We set \( \set = \dim Y \).

Let \( N \) be the normal bundle of \( Y \) inside \( \Lambda_2 \), identified with the orthogonal complement of \( TY \) in \( (T\Lambda_2, g_{\omega}^{TX_2}) \), and let \( g_{\omega}^N \) be the induced metric on \( N \). Let \( \varepsilon > 0 \) be such that the exponential map \( \exp_{\omega}^\Lambda \) of \( (\Lambda_2, g_{\omega}^{TX_2}) \) restricted to \( B^N(0, \varepsilon) := \{ w \in N \ | \ |w|_{g_N^N} < \varepsilon \} \) is a diffeomorphism. Then for any \( x \in Y \) and with \( Y \) embedded in \( N \) as its zero section, the differential \( d\exp_{\omega,x} : T_xY \oplus N_x \to T_x \Lambda_2 \) is the identity map, and \( \exp_{\omega,x}^N(B^N(0, \varepsilon)) \) is a tubular neighbourhood of \( Y \) in \( \Lambda_2 \).

Let \( dw \) be an Euclidean volume form on the fibres of \( (N, g_{\omega}^N) \) such that the volume form \( dwdv_{Y, \omega} \) on the total space of \( N \) is compatible with the orientation of \( X \). Let \( h_2 \in C^\infty(B^N(0, \varepsilon), \mathbb{R}) \) be such that for any \( x \in \Lambda_2, \ w \in N \) with \( |w|_{g_{\omega,x}^N} < \varepsilon \),

\[
dv_{\Lambda_2}(x, w) = h_2(x, w) dv_{Y, \omega}(x). \tag{4.22}
\]
Then \( h_2(x, 0) = (dv_{\Lambda_2}/dv_{\Lambda_2,w})(x) \). Let us now define \( I(f_1, f_2) \in C^\infty (B^N(0, \varepsilon), \mathbb{C}) \) at \( x \in Y, w \in N_x \) with \( |w|_{g_{\omega,x}} < \varepsilon \), by the formula

\[
I(f_1, f_2)(x, w) = \int_{\Lambda_1} \langle P_p((x, w), \iota_1(y))\iota_{1,p}.\zeta^p_1 f_1(y), \zeta^p_2 f_2(x, w) \rangle_{E_p} h_2(x_0, w) dv_{\Lambda_1}(y). \tag{4.23}
\]

Using (3.2), (3.4), (3.5) and Proposition 3.5, we get from (4.22) and (4.23),

\[
\langle s_{1,p}, s_{2,p} \rangle_p = \int_{\Lambda_2} \int_{\Lambda_1} \langle P_p(\iota_2(x), \iota_1(y))\iota_{1,p}.\zeta^p_1 f_1(y), \iota_{2,p}.\zeta^p_2 f_2(x) \rangle_{E_p} dv_{\Lambda_1}(y)dv_{\Lambda_2}(x)
= \int_{\exp_{-1}(B^{N,0}(0, \varepsilon))} \int_{\Lambda_1} \langle P_p(\iota_1(y))\iota_{1,p}.\zeta^p_1 f_1(y), \zeta^p_2 f_2(x) \rangle_{E_p} dv_{\Lambda_1}(y)dv_{\Lambda_2}(x) + O(p^{-\infty})
= \int_{x \in Y} \int_{B^{N_0}(0, \varepsilon)} I(f_1, f_2)(x, w) dw dv_{\omega,x}(x) + O(p^{-\infty}) \tag{4.24}
\]

Fix now \( x_0 \in Y \). Take \( \varepsilon > 0 \), \( U \subset \Lambda_1 \) and a diffeomorphism \( \phi_{x_0}^\Lambda : B^{R^d}(0, \varepsilon) \to U \) sending 0 to \( x_0 \) and such that its differential at 0 identifies \( \langle \cdot , \cdot \rangle \) with \( g_{\omega_x}^\Lambda \). As \( \exp_{-1}(B^{N_0}(0, \varepsilon)) \) and \( \Lambda_1 \) intersect cleanly at \( x_0 \) only, for \( \varepsilon > 0 \) small enough we can extend the union \( \exp_{-1}(B^{N_0}(0, \varepsilon)) \cup \phi_{x_0}^\Lambda : B^{N_0}(0, \varepsilon) \cup B^{R^d}(0, \varepsilon) \to X \) to a diffeomorphism \( \phi_x : B^{R^2n}(0, \varepsilon) \to V \) as in (2.11), identifying \( U \) with \( B^{X}(0, \varepsilon) \), where \( \Sigma \) is an isotropic subspace of \((R^{2n}, \Omega)\) and where the fibre \((N_{x_0}, g_{\omega,x_0}^N)\) is seen as an Euclidean subspace of \((R^{2n}, \langle \cdot , \cdot \rangle)\).

Let us identify \( E, L \) over \( B^{R^2n}(0, \varepsilon) \) with \( E_{x_0}, L_{x_0} \) as in Section 2.2 and use \( \zeta_1(x_0) \) to identify \( L_{x_0} \) with \( \mathbb{C} \). Then \( \zeta_1, \zeta_2 \) are identified with \( 1, \mathcal{X} \in \mathbb{C} \) over \( B^{R^2n}(0, \varepsilon) \), where \( \lambda = \langle \zeta_1(x_0), \zeta_2(x_0) \rangle_L \). Let \( du \) be the Lebesgue measure of \( \Sigma \) and \( h_1 \in C^\infty(B^{X}(0, \varepsilon), \mathbb{R}) \) be such that for \( u \in B^{X}(0, \varepsilon) \),

\[
dv_{\Lambda_1}(u) = h_1(u) du. \tag{4.25}
\]

Then \( h_2(0) = (dv_{\Lambda_1}/dv_{\Lambda_1,w})(x_0) \). By Corollary 2.4 and Lemma 2.5 for any \( k \in \mathbb{N} \) and \( \delta \in [0, 1] \), we get \( \theta \in [0, 1] \) such that

\[
\int_{B^{N_0}(0, \varepsilon)} I(f_1, f_2)(x_0, w) dw
= \int_{B^{N_0}(0, \varepsilon)} \int_{B^{X}(0, \varepsilon)} \langle P_p(w, u)\zeta^p_1 f_1(u), \zeta^p_2 f_2(w) \rangle_{E_p} h_2(x_0, w) dv_{\Lambda_1}(u) dw
= \int_{B^{N_0}(0, \varepsilon)} \int_{B^{X}(0, \varepsilon)} \langle P_p(w, u)\zeta^p_1 f_1(u), \zeta^p_2 f_2(w) \rangle_{E_p} h_2(x_0, w) h_1(u) du dw + O(p^{-\infty})
= \lambda^p p^R \sum_{r=0}^{k} p^{-r/2} \int_{B^{N_0}(0, \varepsilon)} \int_{B^{X}(0, \varepsilon)} \langle J_r, x_0, \mathcal{X}_0, (\sqrt{p}w, \sqrt{p}u) \rangle_{f_1(x_0, u), f_2(x_0, w)} \kappa_{x_0}^{-1/2}(w) \kappa_{x_0}^{-1/2}(u) h_2(x_0, w) h_1(u) du dw + O(p^{-\frac{d_k+1}{2}+\frac{1}{2}}). \tag{4.26}
\]
Consider now the Taylor expansions up to order \( k \in \mathbb{N} \) of \( \frac{1}{2} h_{j}\kappa_{x_0}^{-1/2} f_j(x_0) \) for \( j = 1, 2 \) as in (3.18). As in the proof of Theorem 4.3 we get \( \delta' > 0 \) and a sequence \( \{F_{x_0}, r\} \in \mathbb{N} \) of polynomials in two variables of \( \mathbb{R}^{2n} \) with values in \( \mathbb{C} \), of the same parity as \( r \) and with \( F_{x_0,0}(Z, Z') = \langle f_1(x_0), f_2(x_0) \rangle x_0 (dv_{\Lambda_1}/dv_{\Lambda_1,\omega})(dv_{\Lambda_2}/dv_{\Lambda_2,\omega})(x_0) \), such that (4.11) becomes

\[
\int_{B^{N_{x_0}(0, r)}} I(f_1, f_2)(x_0, w) dw = p^{n - \frac{d_1 + d_2}{2} + \frac{1}{2}} \lambda^p \sum_{r=0}^{k} p^{-r/2} \int_{N_{x_0}} \sum_{l} F_{x_0, r}(w, u) \mathcal{P}_{x_0}(w, u) du dw + p^{n - \frac{d_1 + d_2}{2} + \frac{1}{2}} O(p^{-\frac{k+1}{2} + \delta'}). \tag{4.27}
\]

Thus writing

\[
b_r(x_0) = \int_{N_{x_0}} \sum_{l} F_{x_0, r}(w, u) \mathcal{P}_{x_0}(w, u) du dw,
\]

and recalling that the estimates are uniform in \( x_0 \in \mathbb{Y} \), we get from (4.23), (4.24) and (4.27),

\[
\langle s_1, p s_2, p \rangle_p = p^{n - \frac{d_1 + d_2}{2} + \frac{1}{2}} \lambda^p \sum_{r=0}^{k} p^{-r/2} \int_{T_{x_0}} b_r(x) dv_Y(x) + p^{n - \frac{d_1 + d_2}{2} + \frac{1}{2}} O(p^{-\frac{k+1}{2}}). \tag{4.29}
\]

Now, we can use (4.17) to compute (4.28) in general, and the argument of parity holds in the same way here, so that the coefficients \( b_r \) defined in (4.28) for \( r \in \mathbb{N} \) vanish identically for \( r \) odd. By (4.29), this gives (4.20).

Assume now \( \dim \, \Lambda_1 = n \), and let us compute

\[
b_0(x_0) = \frac{dv_{\Lambda_1}}{dv_{\Lambda_1,\omega}} \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}} (x_0) \langle f_1(x_0), f_2(x_0) \rangle x_0 \int_{N_{x_0}} \sum_{l} \mathcal{P}_{x_0}(w, u) du dw. \tag{4.30}
\]

In the same way than in the proof of Theorem 4.3 we can take the canonical symplectic basis \( \{e_j, f_j \}_{j=1}^n \) of \( (\mathbb{R}^{2n}, \Omega) \) such that \( \Sigma = \mathbb{R}^n \times \{0\} \) and such that \( \langle e_{n-l+1}, \ldots, e_n \rangle \) is an oriented orthonormal basis of \( (T_{x_0} Y, g_Y) \) in the identification of \( \mathbb{R}^{2n} \) with \( T_{x_0} Y \) via \( d\phi_{x_0} \). Let \( u_1, \ldots, u_{d_2-l} \in N_{x_0} \) be such that \( \langle u_1, \ldots, u_{d_2-l}, e_{n-l+1}, \ldots, e_n \rangle \) is an oriented orthonormal basis of the isotropic subspace \( \Sigma_2 := N_{x_0} \perp T_{x_0} Y \). Then for \( 1 \leq i \leq d_2 - l \) and \( n-l+1 \leq j \leq n \), we have that \( \langle u_i, f_j \rangle = -\omega(u_i, e_j) = 0 \). Thus setting

\[
A = (a^j_i)_{1 \leq i \leq n-l, 1 \leq j \leq d_2 - l} \quad \text{with} \quad a^j_i = \omega(e_i, f_j),
B = (b^j_i)_{1 \leq i \leq n-l, 1 \leq j \leq d_2 - l} \quad \text{with} \quad b^j_i = \langle e_i, f_j \rangle,
\]

we get for all \( 1 \leq j \leq d_2 - l \),

\[
\nu_j = \sum_{i=1}^{n-l} b^j_i e_i + \sum_{i=1}^{n-l} a^j_i f_i. \tag{4.32}
\]

Write \( dt := dt_1 \ldots dt_{d_2-l} \) for the Lebesgue measure of \( \mathbb{R}^{d_2-l} \). Using the summation convention of Section 2.1 with \( i \) from 1 to \( n-l \) and \( j, k \) from 1 to \( d_2-l \) whenever they
Using the explicit definition of $A$ as $\omega$ we get (4.21).

Remark 4.5. Suppose that the first Chern class $c_1(TX)$ of $(TX, J)$ is even in $H^2(X, \mathbb{Z})$. Then there exists a complex line bundle $K^{1/2}_X$ over $X$ such that its second tensor power is equal to the canonical line bundle $K_X$ of $X$. The choice of $K^{1/2}_X$ does not depend on $J$ compatible with $\omega$, and is called a metaplectic structure on $(X, \omega)$. Now if $\nu : \Lambda \to X$ is an immersed Lagrangian submanifold, then $\nu^* K_X$ is canonically isomorphic to $\det(T^*\Lambda_C)$ over $\Lambda$, and we call $\nu^* K^{1/2}_X$ the half-form bundle of $\Lambda$. We endow $K^{1/2}_X$ with the Hermitian structure induced by $h^X_\omega$ as in Section 2.1.

Consider now the setting of Theorem 4.1, with $\dim \Lambda_1 = \dim \Lambda_2 = n$. Via the isomorphism above, we define the angle of $i_j : \Lambda_j \to X$, $j = 1, 2$, as a function on any connected component $Y$ of their intersection by the formula

$$\det\{\Lambda_1, \Lambda_2\} = h^K_\omega (dv_{\Lambda_1}, dv_{\Lambda_2})^{-1}$$

$$= \det\left\{h^T_{\omega_X}(e_i, \nu_j)\right\}^{n-l}_{i,j=1}.$$  

(4.35)

On another hand, following [BPU95, Lemma 3.1], we can construct a sesquilinear pairing $\#: \nu^*_1 K^{1/2}_X|_Y \times \nu^*_2 K^{1/2}_X|_Y \to \det(T^*Y_C)$ over $Y$, depending only on the metaplectic structure of $(X, \omega)$, which at any $x \in Y$ takes two square roots $dv^{1/2}_{\Lambda_j, \omega, x}$ of $dv_{\Lambda_j, \omega, x}$.
for \( j = 1, 2 \) to
\[
dv_{\Lambda_1, \omega, x}^{1/2} \# dv_{\Lambda_2, \omega, x}^{1/2} = \det^{-1/2}\{\omega(e_i, \nu_j)\}^{n-i}_{i,j=1} dv_{\omega, x}, \tag{4.36}
\]
for an Euclidean volume form \( dv_{Y, \omega, x} \) of \((T_X Y, g_T X \omega, x)\) and some coherent choice of square root induced by \( dv_{\Lambda_1, \omega, x}^{1/2}, dv_{\Lambda_2, \omega, x}^{1/2}, dv_{Y, \omega, x} \). Then taking \( E = K_X^{1/2} \), Theorem 4.4 gives the following formula for \( b_0 \) on \( Y \) as in (4.21),
\[
b_0 = 2^{n-l} e^{-\sqrt{-1}(n-l)} \int_Y \det\{\Lambda_1, \Lambda_2\}^{-1} f_1 \# f_2. \tag{4.37}
\]

In the particular case of \((X, J, \omega)\) Kähler with \( c_1(TX) \) even, \( g_T X = g_T X \omega \) and \( \dim \Lambda_1 = \dim \Lambda_2 = n \), this formula can be compared with the one appearing in [BPU95, Prop. 3.16]. In particular, they get \( \det\{\Lambda_1, \Lambda_2\}^{-1/2} \) instead of \( \det\{\Lambda_1, \Lambda_2\}^{-1} \) as in (4.37). This discrepancy is due to the fact that even though they use half-forms, their Lagrangian states take values in \( L^p \) and not in \( L^p \otimes K_X^{1/2} \) as it is the case here.

Note that the proof of Theorem 4.4 delivers as well a formula for the first coefficient of (4.20) in the case \( \Lambda_1 \) and \( \Lambda_2 \) both not Lagrangian, although its geometric meaning is unclear, which is why we did not give it explicitly.

Note finally that without metaplectic structure on \((X, \omega)\), only the product of the square root of (4.35) with (4.36) make sense in general (see [Tu16] for related results).

5 Extensions to non-compact manifolds and orbifolds

In this section, we show how one can adapt the results of the previous Sections in the case of non-compact manifolds and orbifolds. We will work for simplicity in the case of \((X, J, \omega)\) Kähler and \( g_T X = g_T X \omega \). Then as underlined in the introduction, the renormalized Bochner Laplacian (2.8) reduces to the Kodaira Laplacian on sections.

Note further that the existence of an expansion of the form (2.25) is a straightforward consequence of the existence of an expansion as in [MM08b, (4.9)].

5.1 Non-compact case

Let \((X, J, \omega, g_T X)\) be a complete Kähler manifold with \( \omega(\cdot, \cdot) = g_T X(J \cdot, \cdot) \), let \((L, h^L)\) be a holomorphic line Hermitian bundle over \( X \) with Chern connection \( \nabla^L \) satisfying \((1.1)\), and let \((E, h^E)\) be an auxiliary holomorphic Hermitian bundle with Chern connection \( \nabla^E \). For any \( p \in \mathbb{N}^* \), let \( H^p_{(2)}(X, E_p) \) denote the space of holomorphic sections of \( E_p = L^p \otimes E \) which are square integrable with respect to the \( L^2 \)-Hermitian product defined as in (2.9). Let \( P_p \) denote the orthogonal projection from the space of \( L^2 \)-sections of \( E_p \) onto \( H^p_{(2)}(X, E_p) \) with respect to this product. Then as noticed in [MM07, Rem.1.4.3], \( P_p \) has smooth Schwartz kernel \( P_p(\cdot, \cdot) \in \mathcal{C}^\infty(X \times X, E_p \boxtimes E_p^*) \) with respect to the Riemannian volume form \( dv_X \) of \((X, g_T X)\), and \( P_p(\cdot, \cdot) \) is square integrable and holomorphic with respect to its first variable.
Let us write $R^{\text{det}}$ for the curvature of the Chern connection of $K^*_X$. Then we have the following result.

**Theorem 5.1.** [MM08d, Th. 5.2, 5.3] Suppose that there exists $C > 0$ such that for all $x \in X$ and $v \in T_x X$, the following inequality holds in the sense of endomorphisms of $E$,

$$\sqrt{-1}(R^{\text{det}}\text{Id}_E + R^E)(v, Jv) > -C\omega(v, Jv)\text{Id}_E.$$  \hfill (5.1)

Then for any compact set $K \subset X$, Proposition 2.3 holds uniformly for any $x, x' \in K$ and Lemma 2.5 holds uniformly for any $x_0 \in X$.

If $F \in \mathcal{C}^\infty(X, \text{End}(E))$ has compact support, then Lemma 2.6 holds uniformly for any $x_0 \in X$.

From now on, we suppose that (5.1) is verified for $X$. Then Definition 3.1 still makes sense in this context, provided $\Lambda$ is compact. Precisely, for $(\Lambda, \iota, \zeta)$ Bohr-Sommerfeld manifold as in Definition 3.1 with $\Lambda$ compact and for $f \in \mathcal{C}^\infty(\Lambda, \iota^*E)$, we define the associated isotropic state $\{s_{f,p}\}_{p \in \mathbb{N}}$ in the same way than in (3.2) for any $p \in \mathbb{N}^*$ and $x \in X$ by the formula

$$s_{f,p}(x) = \int_\Lambda P_p(\iota(y))\iota_p\zeta^p f(y) dv_\Lambda(y).$$  \hfill (5.2)

Then as $\Lambda$ is compact, we get that $s_{f,p} \in H^0(E_p)$. Furthermore, the following analogue of Proposition 3.4 holds.

**Lemma 5.2.** Suppose that $(X, J, \omega, g^T X)$ is a complete Kähler manifold satisfying (5.1), and let $(\Lambda, \iota, \zeta)$ be a compact Bohr-Sommerfeld submanifold of $X$. Then for any $s \in H^0(X, E_p)$, the following reproducing property holds,

$$\langle s, s_{f,p} \rangle_p = \int_\Lambda \langle s(\iota(x)), \iota_p\zeta^p f(x) \rangle_{E_p} dv_\Lambda(y).$$  \hfill (5.3)

Furthermore, for any $F \in \mathcal{C}^\infty(X, \text{End}(E))$ with compact support, property (3.6) holds.

**Proof.** As $\Lambda$ is compact, we can repeat the computations of (3.7), so that (5.3) holds. As $F \in \mathcal{C}^\infty(X, \text{End}(E))$ has compact support, we can repeat in the same way the computations of (3.8), and (3.6) holds as well in this context.

With these preliminaries, we can state the following generalization of the results of Section 3.2, Section 4.1 and Section 4.2.

**Theorem 5.3.** Suppose that $(X, J, \omega, g^T X)$ is a complete Kähler manifold satisfying (5.1). If $(\Lambda, \iota, \zeta)$ is a compact Bohr-Sommerfeld submanifold of $(X, \omega)$, then Theorem 3.6 holds.

Furthermore, if $(\Lambda_j, \iota_j, \zeta_j), \ j = 1, 2,$ are two compact Bohr-Sommerfeld submanifolds of $(X, \omega)$ intersecting cleanly, then and Theorem 4.4 hold.
Proof. Let \((\Lambda_j, \iota_j, \zeta_j), j = 1, 2,\) be two compact Bohr-Sommerfeld submanifolds of \(X,\) and consider \(f_j \in C^\infty(X, \iota_j^*E), j = 1, 2.\) By Theorem 5.1, we know that Proposition 3.5 is still true uniformly in any compact set \(K \subset X.\) Furthermore, using (5.2), (5.3) and omitting the immersions, we get for any \(p \in \mathbb{N}^*,\)

\[
\langle s f_1, p, s f_2, p \rangle_p = \int_{\Lambda_2} \langle \zeta_1^p f_1, \zeta_2^p f_2 \rangle_{E_p} dv_{\Lambda_2}(x) = \int_{\Lambda_2} \int_{\Lambda_1} \langle P_p(x, y) \zeta_1^p f_1(y), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_1}(y) dv_{\Lambda_2}(x).
\]

We can then choose the compact set \(K\) in Theorem 5.1 to contain \(\iota(\Lambda_1) \cup \iota(\Lambda_2),\) and the proof of Theorem 5.3 goes along the lines of the proofs of Theorem 3.6, Theorem 4.3 and Theorem 4.4. By the second part of Lemma 5.2, the case of \(\langle T_{F,p} s f_1, p, s f_2, p \rangle_p\) such that \(F \in C^\infty(X, \text{End}(E))\) has compact support is strictly analogous.

5.2 Orbifold case

In this section, we consider \((X, J, \omega, g^TX)\) complete Kähler orbifold satisfying (5.1), \((L, h^L)\) a holomorphic Hermitian proper orbifold line bundle over \(X\) with Chern connection \(\nabla^L\) satisfying (1.1), and \((E, h^E)\) a holomorphic Hermitian proper orbifold vector bundle over \(X\) endowed with its Chern connection \(\nabla^L.\) In order to give a precise meaning to these notions, we first state some notations and definitions from [MM07, § 5.4].

Definition 5.4. Let \(\mathcal{M}\) be the category whose objects are the pairs \((M, G),\) with \(M\) smooth connected manifold and \(G\) a finite group acting effectively on \(M,\) and whose morphisms \(\Phi : (M, G) \rightarrow (M', G')\) are families of open embeddings \(\varphi : M \rightarrow M'\) satisfying:

- For each \(\varphi \in \Phi,\) there is an injective group homomorphism \(\lambda_\varphi : G \rightarrow G'\) such that \(\varphi\) is \(\lambda_\varphi\)-equivariant.

- For \(g \in G'\) and \(\varphi \in \Phi,\) define \(g\varphi : M \rightarrow M'\) by \((g\varphi)(x) = g\varphi(x)\) for any \(x \in M.\) If \((g\varphi)(M) \cap \varphi(M) = \emptyset,\) then \(g \in \lambda_\varphi(G').\)

- For \(\varphi \in \Phi,\) we have \(\Phi = \{g\varphi \mid g \in G'\}.\)

Definition 5.5. Let \(X\) be a paracompact Hausdorff space and let \(\mathcal{U}_X\) be a covering of \(X\) consisting of connected open subsets, satisfying the condition

\[
\text{For any } U, U' \in \mathcal{U}_X \text{ and } x \in U \cap U', \\text{there is } U'' \in \mathcal{U}_X \text{ such that } x \in U'' \subset U \cap U'.
\]

An orbifold structure \(\mathcal{V}_X\) on \(X\) consists of the following datas:

- For any \(U \in \mathcal{U}_X,\) an object \((G_U, \bar{U})\) of \(\mathcal{M}\) and a ramified covering \(\tau_U : \bar{U} \rightarrow U\) which is \(G_U\)-invariant and induces a homeomorphism \(U \simeq \bar{U}/G_U.\)
• For any $U, V \in \mathcal{U}_X$ such that $U \subset V$, a morphism $\Phi_{UV} : (G_U, \tilde{U}) \to (G_V, \tilde{V})$ of 
\$\mathcal{M}\$, which covers the inclusion $U \subset V$ and satisfies $\Phi_{WU} = \Phi_{WV} \circ \Phi_{UV}$ for any $U, V, W \in \mathcal{U}_X$, with $U \subset V \subset W$.

If $\mathcal{U}_X'$ is a refinement of $\mathcal{U}_X$ satisfying the condition (\ref{3.5}), then there is an orbifold structure $\mathcal{V}_X'$ associated to $\mathcal{U}_X'$ such that $\mathcal{V}_X \cap \mathcal{V}_X'$ is again an orbifold structure. We then say that $\mathcal{V}_X$ and $\mathcal{V}_X'$ are equivalent. An equivalence class is called an orbifold structure on $X$. In particular, we can suppose that $\mathcal{U}_X$ is arbitrarily fine. In the sequel, we will always consider the unique maximal representative in the equivalence class.

In the above definitions, we can replace the objects of $\mathcal{M}$ by manifolds with specified structures together with a group preserving these structures, and morphisms preserving these structures. In the case in hand, by structure we mean an orientation, a Riemannian metric, a symplectic structure, an almost-complex structure or a complex structure. Furthermore, we can realise Cartesian products of orbifolds in the obvious way.

Let $(X, \mathcal{V}_X)$ be an orbifold. For each $x \in X$, up to refinement of $\mathcal{V}_X$, there exists $U_x \in \mathcal{U}_X$ containing $x$ and $\tilde{x} \in \tilde{U}$, $\tau_U(\tilde{x}) = x$, such that $\tilde{x}$ is a fixed point of $G_U$. Then by the second axiom of Definition (\ref{3.4}), such a group is unique up to isomorphism, and we denote it by $G^X_x$. If $|G^X_x| = 1$, then $X$ has a smooth structure in a neighbourhood of $x$, and we can call such an $x$ a smooth point of $X$. If $|G^X_x| > 1$, we call such an $x$ a singular point of $X$. We denote $X_{\text{sing}} = \{x \in X \mid |G^X_x| > 1\}$ the singular set of $X$, and $X_{\text{reg}} = \{x \in X \mid |G^X_x| = 1\}$ the regular set of $X$. In the sequel, we always denote by $\tilde{x} \in \tilde{U}$ a lift of $x \in U \in \mathcal{U}_X$.

The next set of definitions are generalisations of [Ma05, Def. 1.6, Def. 1.7].

**Definition 5.6.** An orbifold immersion $I : (Y, \mathcal{V}_Y) \to (X, \mathcal{V}_X)$ is a continuous map $i : Y \to X$, such that for any $V \in \mathcal{U}_X$ and any $U \in \mathcal{U}_Y$ connected component of $i^{-1}(V)$, there is a family $I_{UV}$ of immersions $i_{UV} : \tilde{U} \to \tilde{V}$ covering $i$ together with surjective group homomorphisms $\lambda_{UV} : G_U \to G_{\tilde{U}}$ such that $i_{UV}$ is $\lambda_{UV}$-equivariant. Furthermore, the families $I_{UV}$ satisfy $I_{UV} = \{g_i_{UV} \mid g \in G_U\}$ and are compatible with the orbifold structures in the obvious sense. In that case, we define the stabilizer of $V$ in $U$ by $K_{UV} = \text{Ker} \lambda_{UV}$. Then $m_{X,Y} := |K_{UV}|$ is locally constant on $Y$, and is called the relative multiplicity on $Y$.

A singular immersion $\tilde{I}$ from a smooth manifold $Y$ to an orbifold $(X, \mathcal{V}_X)$ is a continuous map $i : Y \to X$, together with immersions $\tilde{\iota}_V : U \to \tilde{V}$ covering $i$ for any $V \in \mathcal{U}_X$, such that $g \cdot \iota(U)$ intersects $\iota(U)$ cleanly in the sense of Definition $4.1$ for all $g \in G_V$. In that case, we define the stabilizer of $U$ in $V$ by the subgroup $K_{\tilde{I}V} \subset G_V$ fixing each point of $\tilde{\iota}_V(U)$. Then the relative multiplicity $m_{X,Y} = |K_{UV}|$ is again locally constant on $Y$.

An orbifold submersion $P : (M, \mathcal{V}_M) \to (X, \mathcal{V}_X)$ is a continuous map $\pi : M \to X$ such that $\pi(U) \in \mathcal{U}_X$ for any $U \in \mathcal{U}_M$, together with submersions $\pi_U : \tilde{U} \to \pi(U)$ covering $\pi$ and surjective group homomorphisms $\lambda_U : G_U \to G_{\pi(U)}$ for any $U \in \mathcal{U}_X$ making $\pi_U$ be $\lambda_U$-equivariant. Furthermore, we assume compatibility with the orbifold structures in the obvious sense.
Note that any \( x \in X \) can be seen as an immersed orbifold with \( m_{X,x} = |G_x| \). In both definitions of an immersion above, if \( \iota^{-1}(X_{\text{sing}}) \) has strictly positive measure for the density induced by any Riemannian metric, then \( G_V \) fixes \( \iota(U) \) and \( m_{X,Y} \) is strictly positive. The intersection of two orbifold immersions is still defined as in Definition 4.1 to be their fibred product over \( X \), which gets a natural orbifold structure making all maps into orbifold immersions.

Finally, note that we can easily combine the definitions above to get the notion of a singular orbifold immersion, and the results of this section hold in this case as well. For simplicity and clarity, we will keep both notions separated from each other.

**Definition 5.7.** An orbifold vector bundle is an orbifold submersion \( P : (E, \mathcal{V}_E) \to (X, \mathcal{V}_X) \) such that \( E_U := \pi^{-1}(U) \) belongs to \( \mathcal{U}_E \) for any \( U \in \mathcal{U}_X \) and \( \pi_{E_U} : \tilde{E}_U \to U \) are \( G_{E_U} \)-equivariant vector bundles. Furthermore, we ask the inclusions \( \Phi_{E_U} \) covering \( \Phi_{V_U} \) to be vector bundle maps, for any \( U, V \in \mathcal{U}_X \) such that \( U \subset V \).

If \( G_{E_U} \) acts effectively on \( U \) for all \( U \in \mathcal{U}_X \), that is the group morphisms \( \lambda_{E_U} : G_{E_U} \to G_U \) associated to \( P \) as in Definition 5.6 are isomorphisms, we say that \( E \) is *proper*.

We can then define the proper tangent orbifold bundle \( T^*X \) and the proper cotangent orbifold bundle \( T^*X \) over any orbifold \( (X, \mathcal{V}_X) \) in the obvious way. We can as well form tensor products of vector bundles by taking the tensor products locally over each orbifold chart, and we check easily that this operation preserves properness. If \( E \) is a proper orbifold bundle over \( X \) and if \( \Psi : (X, \mathcal{V}_X) \to (Y, \mathcal{V}_Y) \) is any of the orbifold maps of Definition 5.6, we can pullback \( E \) to \( Y \) by \( \Psi \) in the obvious way, and we write \( \Psi^*E \) for the pullback orbifold vector bundle, which is still proper.

If \( X \) is complete, we define a distance on \( X \) for any \( x, y \in X \) by

\[
d(x,y) = \inf_\gamma \left\{ \sum_j \int_{t_{j-1}}^{t_j} \frac{\partial}{\partial t} \tilde{\gamma}_j(t) | dt \bigm| \gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y, \right. \\
\left. \text{such that there exists } t_0 = 0 < t_1 < \cdots < t_k = 1, \gamma([t_{j-1}, t_j]) \subset U_j, \\
U_j \in \mathcal{U}_X, \text{ and a smooth map } \tilde{\gamma}_j : [t_{j-1}, t_j] \to \tilde{U}_j \text{ that covers } \gamma|_{[t_{j-1}, t_j]} \right\}, \quad (5.6)
\]

Let \( E \to X \) be an orbifold vector bundle. An orbifold section \( s : X \to E \) is called *smooth* if for each \( U \in \mathcal{U}_X \), the restriction of \( s \) to \( U \) is covered by a \( G_U^E \)-equivariant smooth section \( \tilde{s}_U : \tilde{U} \to \tilde{E}_U \). In the same way, if \( X \) is a complex orbifold and \( E \) is a holomorphic orbifold vector bundle, we say \( s \) is holomorphic if it is locally covered by holomorphic sections. The space of smooth (resp. holomorphic) sections of \( E \) is denoted by \( \mathcal{C}^\infty(X,E) \) (resp. \( \mathcal{H}^0(X,E) \)).

If \( X \) is oriented and \( \alpha \) is a smooth section of the exterior product orbifold bundle \( \Lambda(T^*X) \) with support in \( U \in \mathcal{U} \), we define

\[
\int_X \alpha := \frac{1}{|G_U|} \int_{\tilde{U}} \tilde{\alpha}_U, \quad (5.7)
\]
where \( \tilde{\alpha}_U \) is an invariant section covering \( \alpha \) over \( \tilde{U} \). We extend this definition for general \( \alpha \) using a partition of unity. In particular, if \( X \) is oriented and Riemannian, there is an induced Riemannian volume form \( dv_X \) on \( X \), so that we can integrate functions.

Let now \( (X, J, \omega) \) be a Kähler orbifold. As we can verify locally, for any Hermitian holomorphic proper orbifold bundle over \( X \), its Chern connection is well-defined and unique. Let then \( (L, h^L) \) be a holomorphic Hermitian proper orbifold line bundle, such that its Chern connection satisfies \( (1.1) \). We write \( g^{TX} \) for the Riemannian metric on \( X \) satisfying \( (2.3) \), and \( dv_X \) for the associated Riemannian volume form. Let \( (E, h^E) \) be an auxiliary holomorphic Hermitian proper orbifold vector bundle on \( X \).

We define the \( L^2 \)-Hermitian product associated with all the previous datas on \( C^\infty(X, E_p) \) by the formula \( (2.9) \), and the Bergman kernel \( P_p(\cdot, \cdot) \in C^\infty(X \times X, E_p \otimes E^*_p) \) is the Schwartz kernel with respect to \( dv_X \) of the orthogonal projection \( P_p \) from \( C^\infty(X, E_p) \) to \( H^0_{(2)}(X, E_p) \) as in \( (2.12) \). For any \( V \in \mathcal{U}_X \) and all \( p \in \mathbb{N}^* \), let \( \tilde{P}_p(\cdot, \cdot) \in C^\infty(\tilde{V} \times \tilde{V}, \tilde{E}_{p,V} \otimes \tilde{E}_{p,V}^*) \) be the \( G_V \times G_V \)-invariant lift of \( P_p(\cdot, \cdot) \in C^\infty(V \times V, E_p \otimes E^*_p) \). More generally, for any object on \( V \in \mathcal{U}_X \), we add a superscript \( \sim \) to denote the corresponding object on \( \tilde{V} \).

For any \( m \in \mathbb{N} \), let \( \cdot \mid_{C^m} \) denote the \( C^m \)-norm on \( E_p \otimes E^*_p \) over \( X \times X \) induced by \( h^L \), \( h^E \) and \( \nabla^L \), \( \nabla^E \). The following result is the version of Lemma \( (2.5) \) for orbifolds. It uses the fact, noticed in [Ma05], that the finite propagation speed of the wave equation on orbifolds holds.

**Proposition 5.8.** [MM08b, § 6.2], [MM07, Rem.5.4.12.b] Proposition \( (2.3) \) holds in the case of \( (X, J, \omega, g^{TX}) \) complete Kähler orbifold satisfying \( (6.1) \). Moreover, for any \( V \in \mathcal{U}_X \), there exists a section \( F(\tilde{D}_p)(\cdot, \cdot) \in C^\infty(\tilde{V} \times \tilde{V}, \tilde{E}_{p,V} \otimes \tilde{E}_{p,V}^*) \) satisfying the following properties:

For any \( \tilde{x}, \tilde{y} \in \tilde{V} \) and \( g \in G_V \),

\[
(g, 1)F(\tilde{D}_p)(g^{-1}\tilde{x}, \tilde{y}) = (1, g^{-1})F(\tilde{D}_p)(\tilde{x}, g\tilde{y}). \tag{5.8}
\]

For any \( m, l \in \mathbb{N} \), there is \( C_{m,l} > 0 \) such that for any \( \tilde{x}, \tilde{y} \in \tilde{V} \) and all \( p \in \mathbb{N}^* \),

\[
|\tilde{P}_p(\tilde{x}, \tilde{y})| = \sum_{g \in G_U} (1, g^{-1})F(\tilde{D}_p)(\tilde{x}, g\tilde{y})*|_{\tilde{x}} \leq C_{m,l}p^{-l}. \tag{5.9}
\]

\( F(\tilde{D}_p)(\cdot, \cdot) \) satisfies the expansion of Lemma \( (2.5) \) at any \( x_0 \in \tilde{V} \).

With all these prerequisites in hand, Definition \( (3.1) \) still makes sense in this context replacing the immersion \( \iota \) by an orbifold immersion or singular immersion \( I \) as in Definition \( (5.6) \). In the second case, we talk about a singular Bohr-Sommerfeld submanifold. In any case, if \( \Lambda \) is compact, the associated isotropic state as in \( (3.2) \) is well defined and Proposition \( (3.4) \) still holds. We will use the additivity property \( (3.4) \) to assume that the section \( f \) of Definition \( (3.3) \) has compact support in some given open set \( U \in \mathcal{U}_\Lambda \).

**Theorem 5.9.** Let \( (X, J, \omega, g^{TX}) \) be a complete Kähler orbifold satisfying \( (2.3) \), let \( (L, h^L) \) be a holomorphic Hermitian proper orbifold line bundle such that the curvature
of its Chern connection satisfies (1.1), and let \((E, h^E)\) be a holomorphic Hermitian proper orbifold vector bundle. Suppose that \((X, J, \omega, g^{TX})\) satisfies (5.1).

If \((\Lambda, I, \zeta)\) is a compact Bohr-Sommerfeld submanifold of \(X\) and \(F \in \mathcal{C}^\infty(X, \text{End}(E))\) has compact support, then Theorem 3.6 holds, with the following formula for the first coefficient of (3.12),

\[
b_0 = 2^{d/2} m_{X, \Lambda} \int_{\Lambda} \langle Ff, f \rangle_{T^* E} dv_\Lambda.
\]

(5.10)

If \((\Lambda_j, I_j, \zeta_j), j = 1, 2\), are two compact Bohr-Sommerfeld submanifolds of \(X\) intersecting cleanly and if \(F \in \mathcal{C}^\infty(X, \text{End}(E))\) has compact support, then the expansion of Theorem 4.4 holds. If \(\dim \Lambda_1 = n\), then the first coefficients \(b_{q,0}\) of (4.20) satisfy the formula (4.14) multiplied by

\[
m_{\Lambda_2, X}/m_{X, \Lambda_1}.
\]

(5.11)

Finally, the above holds for compact singular Bohr-Sommerfeld submanifolds of \(X\), provided their intersection locus is away from the singular set.

Proof. Let \((\Lambda, I, \zeta)\) be a compact Bohr-Sommerfeld submanifold, and let \(f \in \mathcal{C}^\infty(\Lambda, I^* E)\) have compact support in a sufficiently small open set \(U \subset \mathcal{U}_\Lambda\), connected component of \(\iota^{-1}(V)\) for some \(V \subset \mathcal{U}_X\). Then using (5.7) and (5.9), for any \(\bar{x} \in \bar{V}\) we have

\[
s_{f,p}(\bar{x}) = \frac{1}{|G_U|} \int_{\bar{U}} \tilde{P}_p(\bar{x}, \iota_{UV}(\bar{y})) \iota_{p,UV} \cdot \tilde{f} \tilde{\zeta}^p(\bar{y}) dv_U(\bar{y})
\]

\[
= \frac{1}{|G_U|} \int_{\bar{U}} \sum_{y \in G_U} (1, g^{-1}) F(\tilde{D}_p)(\bar{x}, \iota_{UV}(\bar{y})) \iota_{p,UV} \cdot \tilde{f} \tilde{\zeta}^p(\bar{y}) dv_U(\bar{y}) + O(p^{-\infty})
\]

\[
= \frac{1}{|G_U|} \int_{\bar{U}} \sum_{y \in G_U} F(\tilde{D}_p)(\bar{x}, \iota_{UV}(\bar{y})) \iota_{p,UV} \cdot (g \cdot \tilde{f} \tilde{\zeta}^p(\bar{y}^{-1}) \bar{y}) dv_U(\bar{y}) + O(p^{-\infty})
\]

\[
= \frac{|G_V|}{|G_U|} \int_{\bar{U}} F(\tilde{D}_p)(\bar{x}, \iota_{UV}(\bar{y})) \iota_{p,UV} \cdot \tilde{f} \tilde{\zeta}^p(\bar{y}) dv_U(\bar{y}) + O(p^{-\infty}).
\]

(5.12)

Here \(\iota_{UV} : \bar{U} \to \bar{V}\) is any member of the family of maps in \(I_{UV}\). Now by Definition 5.6, we have \(|G_V|/|G_U| = m_{X, \Lambda}\). By Proposition 5.8, \(F(\tilde{D}_p)(\cdot, \cdot)\) satisfies the expansion of Lemma 2.5 at any \(x_0 \in \bar{V}\), so that we can follow the proof of Theorem 3.6 to deduce from (5.12) an asymptotic expansion in \(p \in \mathbb{N}^*\) of the form (3.10) for the norm of \(s_{f,p}\), with highest coefficient given by (5.10) in the case \(F = \text{Id}_E\).

For any \(j = 1, 2\), let \((\Lambda_j, I_j, \zeta_j)\) be compact Bohr-Sommerfeld submanifolds and let \(f_j \in \mathcal{C}^\infty(\Lambda_j, I^* E)\) have compact support in a sufficiently small open set \(U_j \subset \mathcal{U}_\Lambda\), connected component of \(\iota^{-1}(V)\) for some \(V \subset \mathcal{U}_X\). Then as the reproducing property (3.5) still holds, analogous to (1.6), (5.12), using (5.7), (5.9), and omitting the immersion
Now for the general case, if $F = \text{Id}$ small enough. In the case of discrete intersection, we take $y$ to be a small enough neighbourhood of $X$ to get (5.13) in the case $F = \text{Id}_E$ and discrete intersection.

By Definition 5.6 we have $m_{X,\Lambda_2} = |G_Y|/|G_{U_2}|$, and then $m_{\Lambda_1,Y} = |G_{U_1}| = |G_{U_1}|$ for $U_1$. In the case of discrete intersection, we take $y \in \iota_2^{-1}(\Lambda_1 \cap \iota_2(\Lambda_2))$ and $V \in \mathcal{U}_X$ to be a small enough neighbourhood of $\iota_1(y) \in X$ to get (5.11) in the case $F = \text{Id}_E$ and discrete intersection.

Recall Definition 4.1. Let now $\tilde{W}$ be the lift of some open set $W \in \mathcal{U}_Y$, where is $Y$ the connected component of $\Lambda_1 \cap \Lambda_2$ such that its image by $j_1$ intersects the support of $f_1$, and set $l = \dim Y$. In the case of clean intersection, we can follow the proof of Theorem 4.4 until (4.20) to get an asymptotic expansion of the form (4.20), and get from (3.10), and suppose that $f$ has compact support in some $U \in \mathcal{U}_X$.

We can then go on to the proof of Theorem 4.4 to get (5.11) in the case $F = \text{Id}_E$. Now for the general case, if $F \in \mathcal{C}^\infty(X, \text{End}(E))$ has compact support, we can define its Berezin-Toeplitz quantization by (2.13), and it is shown in [MM08], Lemma 6.10] that it satisfies Lemma 2.6 as well. Furthermore, the formula (3.6) holds in the same way.

Finally, let us consider the case of singular Bohr-Sommerfeld submanifolds. Following (5.12)-(5.14), it suffices to prove the case $m_{X,Y} = 1$, and as we assumed the intersection locus away from the singular set, we need only to prove the analogue of (3.10), and suppose that $f$ has compact support in some $U \in \mathcal{U}_X$.

First recall that the reproducing property gives

$$\|s_{f,p}\|^2_p = \int_\Lambda \langle s_{f,p}(\iota(x)), \iota_p \zeta^p f(x) \rangle_{E_p} dv_\Lambda(x)$$

$$= \int_U \int_U \langle \tilde{P}_p \tilde{\iota}_V(x), \tilde{\iota}_V(y) \rangle \tilde{\iota}_p \zeta^p \tilde{f}(y), \tilde{\iota}_p \zeta^p \tilde{f}(x) \rangle_{E_p} dv_\Lambda(y)dv_\Lambda(x)$$

$$= \sum_{\gamma \in G_Y} \int_U \int_U \langle F(\tilde{D}_p)(\tilde{\iota}_V(x), g \tilde{\iota}_V(y)), \tilde{\iota}_p \zeta^p \tilde{f}(y), \tilde{\iota}_p \zeta^p \tilde{f}(x) \rangle_{E_p} dv_\Lambda(y)dv_\Lambda(x).$$

31
Now, as $G_V$ acts on $\tilde{V}$ preserving all the structures and by Definition 5.6, the immersion $g\tilde{\iota}_V$ is an isotropic immersion intersecting $\tilde{\iota}_V$ cleanly, for any $g \in G_V$. As $F(\tilde{D}_p)(\cdot,\cdot)$ satisfies the expansion of Lemma 2.5 we can then apply Theorem 4.4 to compute each term of the last line of (5.15). We then have an asymptotic expansion of the form (3.12).

To compute the first order term, note that if $g\tilde{\iota}_V$ and $\tilde{\iota}_V$ do not coincide, the highest order of the corresponding expansion (3.12) is strictly smaller than $n/2$. Thus we need only to consider the subgroup of $G_V$ fixing the image of $\iota$, which contains at least the identity element of $G_V$. Summing the contributions of all the elements of this subgroup and by (4.21), we get a function $b_U \in C^\infty(U, \mathbb{C})$, depending on $f$ only locally, such that the highest order term of (5.15) is given by integration of $b_U$ along $U$. Now, as $\iota^{-1}(X_{sing})$ is of measure 0, we can pick a sequence $U_n \subset U$, $n \in \mathbb{N}$, of open sets in $\mathcal{U}_A$ containing $\iota^{-1}(X_{sing})$ and whose measure tends to 0. We can then repeat (5.15) replacing $U$ by $U_n$ and use (5.10) on the regular part of $V$ to get the following formula for the highest order term, for all $n \in \mathbb{N}$,

\[ b_0 = 2^{d/2} \int_{\Lambda \setminus U_n} \langle F f(x), f(x) \rangle e^* dv_A(x) + \int_{U_n} b_U(x) dv_A(x). \] (5.16)

As the second term can be made arbitrarily small, we can take the limit of (5.16) at $n$ tends to infinity, so that formula (4.21) holds for singular Bohr-Sommerfeld submanifolds.

6 Application to relative Poincaré series

Recall that the special linear group

\[ SL_2(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| a, b, c, d \in \mathbb{R}, \ ad - bc = 1 \right\} \] (6.1)

acts on the Poincaré upper-half plane $\mathbb{H} := \left\{ z = x + \sqrt{-1}y \in \mathbb{C} \mid y > 0 \right\}$ by the formula

\[ g.z := \frac{az + b}{cz + d}. \] (6.2)

The induced action on the canonical line bundle $K_{\mathbb{H}} = T^{*,(1,0)}\mathbb{H}$ over $\mathbb{H}$ is given on the canonical section $dz$ by

\[ g.dz = (cz + d)^2 dz =: j(g, z)^2 dz. \] (6.3)

Let $g^{TH}$ be the hyperbolic metric on $\mathbb{H}$, defined by the formula

\[ g^{TH} = \frac{dx^2 + dy^2}{y^2}, \] (6.4)

32
so that the associated Kähler metric $\omega_\mathbb{H}$ satisfies

$$\omega_\mathbb{H} = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}.$$  \hfill (6.5)

Let us write $|\cdot|_{K_\mathbb{H}}$ for the Hermitian norm induced by $g^{T_H}$ on $K_\mathbb{H}$, which is given by

$$|dz|_{K_\mathbb{H}} = y.$$ \hfill (6.6)

Note that the group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by holomorphic isometries. Thus if $\Gamma$ is a discrete subgroup of $SL_2(\mathbb{R})$, the quotient $X := \mathbb{H}/\Gamma$ has an induced structure of a Kähler orbifold, and its canonical line bundle $K_X$ is the quotient of $K_\mathbb{H}$ by the induced action (6.3). We denote $g^{TX}$ and $\omega_X$ for the quotient metric and quotient Kähler form on $X$ respectively.

Let $F$ be a measurable fundamental domain of $\Gamma$ in $\mathbb{H}$. Through the natural identification $C^\infty(X, K^p_X) \cong C^\infty(\mathbb{H}, K_\mathbb{H})$ and trivializing $K_\mathbb{H}$ using its canonical section $dz$, we have from (6.3) and for any $p \in \mathbb{N}^*$ the following natural identification,

$$H^0(2)(X, K^p_X) \cong \{ f \in C^\infty(\mathbb{H}) \mid f \text{ holomorphic, } f(g.z) = f(z)j(g, z)^{2p}, \int_F |f(z)|^2 y^{2p-2} dx dy < \infty \}.$$ \hfill (6.7)

This identification will be used implicitly throughout the rest of this section.

**Remark 6.1.** Assume $\text{Vol}(X) < +\infty$, that is $\Gamma$ is a Fuchsian group of the first kind. As explained in [AMM16, § 6], the space $H^0(2)(X, K^p_X)$ is then identified through the identification (6.7) with the space $S_{2p}(\Gamma)$ of cusp forms of weight $2p$, the space of holomorphic functions on $\mathbb{H}$ satisfying the equivariance property of (6.7) and vanishing at infinity. Such spaces are of particular interest in arithmetic.

To set the previous discussion in the context of Section 2, we use the classical fact that the curvature of the Chern connection $\nabla^{K_\mathbb{H}}$ on $K_\mathbb{H}$ satisfies the condition (1.1) for the renormalized Kähler form $\omega_\mathbb{H}/2\pi$. As $SL_2(\mathbb{R})$ acts by holomorphic isometries, (1.1) holds for $K_X$ as well. Furthermore, as $R^{det} = -R^{K_X}$ is proportional to $\sqrt{-1}\omega_X$, it is easily seen that $K_X$ satisfies (5.1). Therefore, setting $L = K_X$ and $E = \mathbb{C}$, we are precisely in the context of the previous Sections for the renormalized Kähler form $\omega = \omega_X/2\pi$, with $g^{TX}_\omega = g^{TX}/2\pi$.

Recall that a smooth path $\gamma : [0, l] \to X$, $l > 0$, is said to be a closed loop if it induces a (singular) immersion $\tilde{\gamma} : S^1 \to X$ by identification of 0 with $L$. The following lemma describes the class of (singular) Bohr-Sommerfeld submanifolds we will be interested in.

**Lemma 6.2.** For $l > 0$, let $\gamma : [0, l] \to X$ be a closed loop in $X$ parametrized by arclength with respect to $g^{TX}$, and suppose that the holonomy of $K_X$ along $\gamma$ with respect to $\nabla^{K_\mathbb{H}}$ is trivial. Then the immersion $\tilde{\gamma} : S^1 \to X$, obtained from $\gamma$ by identification of 0 and $l$, satisfies the Bohr-Sommerfeld condition of Definition 3.1.
Proof. As $\omega_X$ is a 2-form, any smooth map $f : S^1 \to X$ satisfies $f^*\omega = 0$. Thus as dim $X = 2$, any immersion $\iota : S^1 \to X$ is Lagrangian. By Remark 3.2, it satisfies the Bohr-Sommerfeld condition if and only if the holonomy of the pullback connection is trivial, which is exactly the hypothesis of Lemma 6.2 by Remark 3.2.

In any case, such a path $\gamma : [0, l] \to X$, $l > 0$, is called a Bohr-Sommerfeld curve. The orientation on $\tilde{\gamma} : S^1 \to X$ is determined by the canonical vector field $\partial_t$ on $[0, l]$. Following Remark 3.2 if $\gamma : [0, l] \to X$, $l > 0$, is a smooth closed loop such that its holonomy is a $k$-th root of unity for some $k \in \mathbb{N}$, we can take a cover of degree $k$ of this loop to get a Bohr-Sommerfeld curve $\gamma_k : [0, kl] \to X$.

Note that as $X$ is a complex orbifold with dim$_\mathbb{C} X = 1$ and as $\Gamma$ acts on $\mathbb{H}$ holomorphically, the singular set $X_{\text{sing}}$ is necessarily a discrete set. By Definition 5.6 and as $S^1$ is a manifold, the stabilizer of $\tilde{\gamma}$ is then necessarily trivial in any case.

**Corollary 6.3.** A closed geodesic loop $\gamma : [0, l] \to X$, $l > 0$, parametrized by arclength, is a Bohr-Sommerfeld curve.

Proof. Recall that $K_X = T^*(1,0) X$ is equipped with the Hermitian metric and connection $h^{K_X}, \nabla^{K_X}$ induced by $g^{TX}, \nabla^{TX}$ via (2.1). For any $t \in [0, l]$, let $\tilde{\gamma}_{t} \in T_{\gamma(t)}X$ denote the vector tangent to the curve $\gamma : [0, l] \to X$, inducing $\tilde{\gamma}_{t}^{(1,0)} \in T^{(1,0)}X$ via (2.1). We write $\tilde{\gamma}_{t}^{(1,0),*} \in K_{X, \gamma(t)}$ for its metric dual. As $\gamma : [0, l] \to X$ is geodesic, we know that $\nabla^{TX}\tilde{\gamma} = 0$, so that $\nabla^{K_X}\tilde{\gamma}^{(1,0),*} = 0$, which means precisely that $\tilde{\gamma} : S^1 \to X$ satisfies the Bohr-Sommerfeld condition with associated section $\gamma^{(1,0),*} \in C^\infty(S^1, \tilde{\gamma}^*K_X)$.

Now if $X$ is an orbifold and if $z \in X$ is a singular point of $X$, then its associated group $G_z$ preserves the Riemannian structure, and sends a geodesic through $z$ to another geodesic through $\bar{z}$, which intersect transversally by unicity of the geodesics. Thus $\gamma : [0, l] \to X$ satisfies the definition of a singular immersion as in Definition 5.6.

Let $\gamma : [0, l] \to X$, $l > 0$, be a Bohr-Sommerfeld curve together with a unitary flat section $\zeta \in C^\infty([0, l], \gamma^*K_X)$, inducing a (possibly singular) Bohr-Sommerfeld submanifold $(S^1, \tilde{\gamma}, \zeta)$ as above. For any $p \in \mathbb{N}^*$, we define $s_{\gamma,p} \in H^{0}_{(2)}(X, K_{X}^{p})$ by

$$s_{\gamma,p}(x) = \int_{0}^{L} P_{p}^{X}(x, \gamma(t))\gamma_{p}^{\zeta}(t)dt, \quad (6.8)$$

for any $x \in X$, where $P_{p}^{X}(\cdot, \cdot)$ is the Bergman kernel with respect to $dv_X$ of the orthogonal projection on $H^{0}_{(2)}(X, K_{X}^{p})$. Then $s_{\gamma,p}$ is precisely the Lagrangian state associated to $(S^1, \tilde{\gamma}, \zeta)$ and $f = 1$, in the sense of Definition 3.3.

We can then apply Theorem 5.6 and Theorem 5.9 to get the following specialisation of (3.11) and (4.4), where we adopt the convention that $\sqrt{-a} = i\sqrt{a}$ if $a > 0$.

**Theorem 6.4.** Let $\gamma : [0, l] \to X$, $l > 0$, be a Bohr-Sommerfeld curve, and let $\{s_{\gamma,p}\}_{p \in \mathbb{N}^*}$ be as in (6.8). Then

$$\|s_{\gamma,p}\|^{2}_{L^2} = \left(\frac{p}{\pi}\right)^{1/2} + O(p^{-1/2}). \quad (6.9)$$
Furthermore, if $\gamma_1$ and $\gamma_2$ are two Bohr-Sommerfeld curves intersecting clean away from the singular set, we get
\[
\langle s_{\gamma_1,p}, s_{\gamma_2,p} \rangle = \sqrt{2} \sum_{z \in \gamma_1 \cap \gamma_2} \sum_{t_1, t_2 > 0, \quad \gamma_1(t_1) = \gamma_2(t_2) = z} \chi_{t_1, t_2}^p e^{\sqrt{-1}(\theta_z/2 - \pi/4)} \frac{1}{\sqrt{\sin(\theta_z)}} + O(p^{-1}), \quad (6.10)
\]
where $\theta_z \in [0, 2\pi]$ is the oriented angle from $\gamma_1$ to $\gamma_2$ at $z$ and where for all $t_1, t_2 > 0$ such that $\gamma_1(t_1) = \gamma_2(t_2)$, we define $\chi_{t_1, t_2} = \langle \gamma_1^L \cdot \zeta_1(t_1), \gamma_2^L \cdot \zeta_2(t_2) \rangle_{K_X}$.

Proof. In the case $X$ smooth and compact, $(6.9)$ and $(6.10)$ are standard computations from $(3.21)$ and $(4.4)$. We will indicate how to modify directly the argument to get the case $g^{TX} = 2\pi g^{TX}_\omega$ from the case $g^{TX} = g^{TX}_\omega$ in all generality.

For any $p \in \mathbb{N}^*$, let us write $P_{p, \omega}$ for the orthogonal projection to $H^0_p(X, K^p_X)$ with respect to the $L^2$-Hermitian product induced by $g^{TX}_\omega$. Then $P_{p, \omega} = P^X_p$, but $dv_{X, \omega} = dv_X/2\pi$, so that the associated Bergman kernel with respect to $dv_{X, \omega}$ satisfies $P_{p, \omega}(\cdot, \cdot) = 2\pi P^X_p(\cdot, \cdot)$. On another hand, the Riemannian volume form $dt_\omega$ on $[0, L]$ induced by $g^{TX}_\omega$ satisfies $dt_\omega = dt/\sqrt{2\pi}$. Thus, writing $\{s_{\omega, \gamma, p}\}_{p \in \mathbb{N}^*}$ for the Lagrangian state obtained replacing $g^{TX}$ by $g^{TX}_\omega$, we get from $(3.2)$ that $s_{\omega, \gamma, p} = \sqrt{2\pi} s_{\gamma, p}$ for any $p \in \mathbb{N}^*$.

Consider now two Bohr-Sommerfeld curves $\gamma_1$ and $\gamma_2$. Following the above notations, we get for any $p \in \mathbb{N}^*$,
\[
\langle s_{\gamma_1, p}, s_{\gamma_2, p} \rangle_p = \frac{1}{2\pi} \int_X \langle s_{\omega, \gamma_1, p}, s_{\omega, \gamma_2, p} \rangle_{K^p_X} dv_X = \langle s_{\omega, \gamma_1, p}, s_{\omega, \gamma_2, p} \rangle_{\omega, p}, \quad (6.11)
\]
where $\langle \cdot, \cdot \rangle_{\omega, p}$ denote the $L^2$-Hermitian product with respect to $g^{TX}_\omega$. Noticing finally that $\text{Vol}_\omega(\gamma) = l/\sqrt{2\pi}$ for any $\gamma : [0, l] \to X$, $l > 0$ parametrized by arclength with respect to $g^{TX}$, we recover $(6.9)$ and $(6.10)$ as in the case of $X$ smooth and compact.\]

In the case where $X$ is a compact Riemann surface, so that in particular $\Gamma$ acts freely on $\mathbb{H}$, Theorem 6.4 is the result of [BPU95, Th.4.4]. As shown in Proposition 6.6, formulas $(6.9)$ and $(6.10)$ are especially interesting in the case of curves $\gamma : \mathbb{R} \to \mathbb{H}$ such that there exists $l > 0$, $g_0 \in \Gamma$ satisfying $g_0 \cdot \gamma(t) = \gamma(t + l)$ for any $t \in \mathbb{R}$. We say that $\gamma$ is associated with $g_0$.

In particular, if $\gamma$ is a closed geodesic, then $\gamma$ is associated with an hyperbolic element $g_0 \in \Gamma$, that is satisfying $\text{Tr}(g_0) > 2$, unique up to conjugation. Closed geodesics belong to a larger class of hyperbolic curves called horocycles.

If $g_0 \in \Gamma$ is parabolic, that is satisfying $\text{Tr}(g_0) = 2$, then its action has no fixed points in $\mathbb{H}$, and it occurs in $\Gamma$ only in the case of $X$ non-compact. The most interesting associated curves in that case are the so-called horocycles, which are isometric to a horizontal line in $\mathbb{H}$.

If $g_0 \in \Gamma$ is elliptic, that is satisfying $\text{Tr}(g_0) < 2$, then $g_0$ fixes a unique point $z \in \mathbb{H}$, which descends to a singular point of $X$. The most interesting associated curves in that case are circles with center the fixed point of $g_0$ in $\mathbb{H}$. Note that $\Gamma$ acts freely on $\mathbb{H}$ if and only if it contains no elliptic elements.
Our next goal is to identify the Lagrangian states associated with such curves. The following result is classical and follows for instance from [Frei90, Prop.5.3, §II.1].

**Proposition 6.5.** For any \( p \in \mathbb{N}^* \), the Bergman kernel of \( H^0_{(2)}(\mathbb{H}, K^p_H) \) satisfies the formula

\[
P^F_p(z, w) = (-1)^p \frac{2^{2p-2}(2p-1)}{\pi} \frac{dz^p d\overline{w}^p}{(z - \overline{w})^{2p}},
\]

for any \( z, w \in \mathbb{H} \), where \( d\overline{w} \in \mathcal{K}_{H,w} \simeq K^*_H \) denotes the metric dual of \( dw \in K_{H,w} \).

Furthermore, through the identification \( (6.7) \), we have

\[
P^X_p(z, w) = \sum_{g \in \Gamma} P^F_p(z, g.w) j(g, w)^{2p},
\]

where the convergence of the right-hand side is absolute and uniform in \( z, w \) in any compact set of \( \mathbb{H} \).

The series \( (6.13) \) is an example of *Poincaré series*, and is a standard method to construct functions in \( S^2_p(\Gamma) \) as in Remark 6.1. A fundamental problem of the theory of cusp forms is to decide whether a given series vanishes identically or not.

If \( \Gamma_0 \subset \Gamma \) is a subgroup of \( \Gamma \), let us write \( \Gamma/\Gamma_0 \) for the set of equivalence classes \( \left[ g \right] := \{ gg_0 \in \Gamma \mid g_0 \in \Gamma_0 \} \) for all \( g \in \Gamma \). Recall that if \( g_0 \) is hyperbolic or parabolic, it generates a free group \( \Gamma_0 \subset \Gamma \), whereas if \( g_0 \) is elliptic, it generates a cyclic subgroup \( \Gamma_0 \subset \Gamma \).

Using Proposition 6.5 and a classical unfolding technique, we get explicit formulas for the Lagrangian states associated with remarkable curves. This is described in the next result.

**Proposition 6.6.** Let \( g_0 \in \Gamma \), and let \( \gamma : \mathbb{R} \to \mathbb{H} \) be a smooth curve on \( \mathbb{H} \) parametrized by arclength, together with a unitary flat section \( \zeta \in \gamma^* K_H \), such that there is an \( l > 0 \) satisfying \( g_0.\gamma(t) = \gamma(t+l) \) and \( g_0.\zeta(t) = \zeta(t+l) \) for all \( t \in \mathbb{R} \). Write \( \Gamma_0 \subset \Gamma \) for the subgroup generated by \( g_0 \).

If \( g_0 \) is hyperbolic or parabolic, then the Lagrangian state \( \{ s_{\gamma,p} \}_{p \in \mathbb{N}^*} \) associated to \( \gamma \) is given through \( (6.7) \) and for any \( p \in \mathbb{N}^* \) by

\[
s_{\gamma,p}(z) = (-1)^p \frac{2^{2p-2}(2p-1)}{\pi} \sum_{[g] \in \Gamma/\Gamma_0} \int_{-\infty}^{+\infty} \left( z - g.\gamma(t) \right)^{-2p} \zeta(t) j(g, \gamma(t))^{2p} \, dt. \tag{6.14}
\]

If \( g_0 \) is elliptic, then letting \( n \in \mathbb{N} \) be the order of \( \Gamma_0 \), the Lagrangian state \( \{ s_{\gamma,p} \}_{p \in \mathbb{N}^*} \) is given through \( (6.7) \) and for any \( p \in \mathbb{N}^* \) by

\[
s_{\gamma,p}(z) = (-1)^p \frac{2^{2p-2}(2p-1)}{\pi} \sum_{[g] \in \Gamma/\Gamma_0} \int_0^n \left( z - g.\gamma(t) \right)^{-2p} \zeta(t) j(g, \gamma(t))^{2p} \, dt. \tag{6.15}
\]

The convergence of the series in \( (6.14) \) and \( (6.15) \) are absolute and uniform in \( z \) in any compact set of \( \mathbb{H} \).
Proof. First note that by definition, for \(g, g' \in SL_2(\mathbb{R})\) and \(w \in \mathbb{H}\), we have \(j(gg', w) = j(g, g'w)\). Then from (6.8) and from the uniform convergence of (6.13), if \(g_0 \in \Gamma\) is hyperbolic or parabolic, we get

\[
\sum_{g \in \Gamma} \int_{\gamma} P_{\Gamma}(z, \gamma(t)) j(g, \gamma(t))^2 dt
\]

and we conclude by (6.12). Note that the sums in (6.16) do not depend on the choice of the representatives \(g \in \Gamma\) of any \([g] \in \Gamma/\Gamma_0\). The elliptic case (6.15) is strictly analogous.

The series (6.14) and (6.15) are called relative Poincaré series. We can now state our main theorem, which is a consequence of Theorem 6.4.

**Theorem 6.7.** If \(\gamma : \mathbb{R} \to \mathbb{H}\) satisfying the hypotheses of Proposition 6.6 descends to a Bohr-Sommerfeld curve, then there is a \(p_0 \in \mathbb{N}\) such that the associated series (6.14) or (6.15) do not vanish identically for \(p > p_0\). This holds in particular if \(\gamma : \mathbb{R} \to \mathbb{H}\) is a closed geodesic.

Proof. By (6.9), we know that there is \(p_0 \in \mathbb{N}\) such that \(s_{\gamma,p}(z)\) is non-vanishing for \(p > p_0\), so that we may conclude by Corollary 6.3 and Proposition 6.6.

In general, there are simple numerical criterions for horocycles, circles and hypercycles to satisfy the Bohr-Sommerfeld condition, and the integral in the sums (6.14) and (6.15) can be computed explicitly using Proposition 6.5 and elementary complex analysis. In particular, as computed in [BPU95, Th.4.11], if \(g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) is a hyperbolic element of \(\Gamma\), the series (6.14) for \(\gamma\) closed geodesic associated with \(g_0\) takes the form

\[
s_{\gamma,p}(z) = \sum_{[g] \in \Gamma/\Gamma_0} j(g, z)^{-2p} (c(g, z)^2 + (d - a)(g, z) - b)^{-p},
\]

where the convergence is uniform in \(z\) in any compact set of \(\mathbb{H}\), and we recover (up to normalisation) the relative Poincaré series associated to closed hyperbolic geodesics by Katok [K85, §1]. Furthermore, we get from Theorem 6.4 a formula for the highest order term in \(p \in \mathbb{N}\) of the intersection product of two closed geodesics, recovering a result of [K85, Th.3]. As showed in [K85, Th.1], if \(\Gamma\) is a Fuchsian group of the first kind, the series associated to the primitive hyperbolic elements of \(\Gamma\) as above generate the whole space \(S_{2p}(\Gamma)\).
Finally, note that there are many discrete subgroups $\Gamma \subset SL_2(\mathbb{R})$ of interest containing elliptic points and leading to non-compact quotients $X = \mathbb{H}/\Gamma$, even in the case of $\Gamma$ Fuchsian group of the first kind. The most famous examples are the classical modular curves.

References

[AMM16] H. Auvray, X. Ma and G. Marinescu, *Bergman kernels on punctured Riemann surfaces*, Arxiv e-prints, https://arxiv.org/abs/1604.06337, 2016.

[BFMN11] T. Baier, C. Florentino, J.M. Mourão, José and J.P. Nunes, *Toric Kähler metrics seen from infinity, quantization and compact tropical amoebas*, J. Differential Geom. 89 (2011), no. 3, 411–454.

[BG81] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Math. Studies, no. 99, Princeton Univ. Press, Princeton, NJ, 1981.

[BMS94] M. Bordemann, E. Meinrenken, and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and gl(N), N → ∞ limits*, Comm. Math. Phys. 165 (1994), 281–296.

[BPU95] D. Borthwick, T. Paul and A. Uribe, *Legendrian distributions with applications to relative Poincaré series*, Invent. Math. 122 (1995), no. 2, 359-402.

[BS75] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegö*, Journées: Équations aux Dérivées Partielles de Rennes (1975), Soc. Math. France, Paris, 1976, pp. 123–164. Astérisque, No. 34–35.

[BV89] J.-M. Bismut and E. Vasserot, *The asymptotics of the Ray–Singer analytic torsion associated with high powers of a positive line bundle*, Comm. Math. Phys. 125 (1989), 355–367.

[Cha03] L. Charles, *Quasimodes and Bohr-Sommerfeld conditions for the Toeplitz operators*, Comm. Partial Differential Equations 28 (2003), no. 9-10, 1527–1566.

[DLM06] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom., 72 (2006), no. 1, 1–41.

[DP06] M. Debernardi and R. Paoletti, *Equivariant asymptotics for Bohr-Sommerfeld Lagrangian submanifolds*, Comm. Math. Phys., 267 (2006), no. 1, 227–263.

[Fre95] D. S. Freed, *Classical Chern-Simons theory. I*, Adv. Math. 113 (1995), no. 2, 237–303.

[Frei90] E. Freitag, *Hilbert modular forms*, Springer-Verlag, Berlin, 1990.
[GU88] V. Guillemin and A. Uribe, *The Laplace operator on the n-th tensor power of a line bundle: eigenvalues which are bounded uniformly in n*, Asymptotic Anal. 1 (1988), 105–113.

[GT01] A. L. Gorodentsev and A. N. Tyurin, *Abelian Lagrangian algebraic geometry*, Izv. Ross. Akad. Nauk Ser. Mat., 65 (2001), no. 3, 15–50.

[ILMM17] L. Ioos, W. Lu, X. Ma and G. Marinescu, *Berezin-Toeplitz quantization for eigenstates of the Bochner-Laplacian on symplectic manifolds*, JGEA, 2017, DOI 10.1007/s12220-017-9977-y.

[JW92] L. C. Jeffrey and J. Weitsman, *Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula*, Comm. Math. Phys. 150 (1992), no. 2, 593-630.

[K85] S. Katok, *Closed geodesics, periods and arithmetic of modular forms*, Invent. Math. 80 (1985), no. 3, 469-480.

[LMM16] W. Lu, X. Ma and G. Marinescu, *Donaldson’s Q-operators for symplectic manifolds*, Science China Mathematics. 60 (2017), 1047-1056.

[Ma05] X. Ma, *Orbifolds and analytic torsions*, Trans. Amer. Math. Soc. 357 (2005), no. 6, 2205–2233.

[MM02] X. Ma and G. Marinescu, *The Spinc Dirac operator on high tensor powers of a line bundle*, Math. Z. 240 (2002), no. 3, 651-664.

[MM07] X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progress in Mathematics, vol. 254, Birkhäuser Boston, Inc., Boston, MA, 2007.

[MM08a] X. Ma and G. Marinescu, *Generalized Bergman kernels on symplectic manifolds*, Adv. in Math. 217 (2008), no. 4, 1756–1815.

[MM08b] X. Ma and G. Marinescu, *Toeplitz operators on symplectic manifolds*, J. Geom. Anal. 18 (2008), 565–611.

[MZ08] X. Ma and W. Zhang, *Bergman kernels and symplectic reduction*, Astérisque 318 (2008), viii+154.

[Sch00] M. Schlichenmaier, *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization*, Conférence Moshé Flato 1999, Vol. II (Dijon), Math. Phys. Stud., vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 289–306.

[Sni75] J. Sniatycki, *Wave functions relative to a real polarization*, Internat. J. Theoret. Phys. 14 (1975), no. 4, 277–288.
[Tuy16] G. M. Tuynman, *The metaplectic correction in geometric quantization*, J. Geom. Phys. 106 (2016), 401–426.

[Tyu00] A. N. Tyurin, *On the Bohr-Sommerfeld bases*, Izv. Ross. Akad. Nauk Ser. Mat. 64 (2000), no. 5, 163–196.

[Wit89] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys., 121 (1989), no. 3, 351–399.

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