THE CATEGORY $\mathcal{O}$ FOR A GENERAL COXETER SYSTEM

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Abstract. We study the category $\mathcal{O}$ for a general Coxeter system using a formulation of Fiebig. The translation functors, the Zuckerman functors and the twisting functors are defined. We prove the fundamental properties of these functors, the duality of Zuckerman functor and generalization of Verma’s result about homomorphisms between Verma modules.

1. Introduction

The Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ is introduced in [BGG76]. Roughly speaking, it is a full-subcategory of the category of modules of a semisimple Lie algebra which is generated by the category of highest weight modules. Soergel [Soe90] realized the endomorphism ring of the minimal progenerator of a block of $\mathcal{O}$ as the endomorphism ring of some module over the coinvariant ring of the Weyl group. As a corollary, a block of the category $\mathcal{O}$ depends only on the attached Coxeter system (the integral Weyl group) and the singularity of the infinitesimal character.

Generalizing this method, Fiebig [Fie08b] and Soergel [Soe07] construct some module over some algebra for any Coxeter system $(W, S)$. If we consider the case of a Weyl group, the endomorphism ring of this module is equal to that of the minimal progenerator of the deformed category $\mathcal{O}$. Specializing it, we get the category $\mathcal{O}$.

In this paper, we study the category $\mathcal{O}$ for a general Coxeter system. Let $(W, S)$ be a Coxeter system and take a reflection faithful representation $V$ of $(W, S)$ (see 2.5). After Braden-MacPherson [BM01], we consider the associated moment graph. Let $Z$ be the space of global sections of the structure algebra of this moment graph and $\{B(x)\}_{x \in W}$ the space of global sections of Braden-MacPherson sheaves. Then $Z$ is an $S(V^*)$-algebra and $B(x)$ is a $Z$-module. Consider a $C$-algebra $A = \text{End}_Z(\bigoplus_{x \in W} B(x)) \otimes_{S(V^*)} C$. If $(W, S)$ is the Weyl group of a semisimple Lie algebra, then the regular integral block of the BGG category is equivalent to the category of finitely generated right $A$-modules. However, in general case, the author does not know whether the algebra $A$ is Noetherian. Instead of this, we define a category $\mathcal{O}$ as the category of right $A$-modules. By the above reason, even if $(W, S)$ is the Weyl group of a semisimple Lie algebra, $\mathcal{O}$ is not equivalent to the ordinal BGG category.

We state our results. Put $P(x) = \text{Hom}_Z(\bigoplus_{y \in W} B(y), B(x)) \otimes_{S(V^*)} C$. Then $P(x)$ is a projective object of $\mathcal{O}$ and it has the unique irreducible quotient $L(x)$. In [Fie08a], the translation functor $\theta^x_s$ of the category of $Z$-modules are defined for a simple reflection $s$. Then the module $A' = \text{Hom}_Z(\bigoplus_{y} B(y), \bigoplus_{x} \theta^x_y B(x)) \otimes_{S(V^*)} C$ is an $A$-bimodule. Define a functor $\theta_s$ from $\mathcal{O}$ to $\mathcal{O}$ by $\theta_s(M) = \text{Hom}_A(A', M)$. Then we have the following theorem.

Theorem 1.1 (Proposition 3.14, Theorem 3.19). Let $s$ be a simple reflection and $x \in W$.

(1) The functor $\theta_s$ is self-adjoint and exact.

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(2) If $xs < x$, then $\theta_s(P(x)) = P(x)^{S_2}$.
(3) The module $\theta_sL(x)$ is zero if and only if $xs > x$.

Next, we consider the Zuckerman functor. Fix a simple reflection $s$ and let $O_s$ be a full-subcategory of $O$ consisting of a module $M$ such that $\text{Hom}_A(P(x), M) = 0$ for all $sx < x$. Then it is easy to see that the inclusion functor $\iota_s: O_s \to O$ has the left adjoint functor $\tau_s$. Put $\tau_s = \iota_s \circ \tau_s$ and let $L\tau_s$ be its left derived functor. Let $D^b(O)$ be the bounded derived category of $O$. We prove the following duality theorem.

**Theorem 1.2** (Theorem 4.10). \begin{enumerate}
\item For $i > 2$ and $M \in O$, we have $L^i\tau_s(M) = 0$. Hence $L\tau_s$ gives a functor from $D^b(O)$ to $D^b(O)$.
\item The functor $L\tau_s[-1]$ is self-adjoint.
\end{enumerate}

In the case of $g$-modules, this theorem is proved by Enright and Wallach [EW80] (in more general situation).

Next results is a generalization of Verma’s result about homomorphisms between Verma modules [Ver68]. Let $V(x)$ be a Verma $Z$-module [Fie08b, 4.5]. Put $M(x) = \text{Hom}_Z(\bigoplus_{y \in W} B(y), V(x)) \otimes S(V^*)C$. Then $M(x)$ gives a generalization of the Verma module. We prove the following theorem.

**Theorem 1.3** (Theorem 6.1). We have
$$\text{Hom}(M(x), M(y)) = \begin{cases} C & (y \leq x), \\ 0 & (y \nleq x). \end{cases}$$
Moreover, any nonzero homomorphism $M(x) \to M(y)$ is injective.

Final results are about the twisting functors [Ark97]. For a simple reflection $s$, we will define a generalization of the twisting functor $T_s$ (Section 5). We prove the following theorem.

**Theorem 1.4** (Proposition 5.5, Theorem 7.2, Theorem 7.3). Let $s$ be a simple reflection. We denote the derived functor of $T_s$ by $LT_s$. Let $D(O)$ be the derived category of $O$.
\begin{enumerate}
\item $L^iT_s = 0$ for $i > 1$.
\item The functor $LT_s$ gives an auto-equivalence of $D(O)$.
\item For a reduced expression $w = s_1 \cdots s_l$, $T_{s_1} \cdots T_{s_l}$ is independent of the choice of a reduced expression.
\end{enumerate}

In the case of the original BGG category, this is proved in [Ark97, AS03].

We summarize the contents of this paper. We recall results of Fiebig [Fie08a, Fie08b] in Section 2. The category $O$ and the translation functors are defined in Section 3, and the fundamental properties are proved. We also define an another functor $\varphi_s$. In Section 4, we prove Theorem 1.2. The definition of the twisting functors appears in Section 5, and fundamental properties are proved. Theorem 1.3 is proved in Section 6. We prove Theorem 1.4 in Section 7.

## 2. Preliminaries

In this section, we recall results of Fiebig [Fie08a, Fie08b].

### 2.1. Moment graphs and Sheaves

Throughout this paper, we consider $S(V^*)$ as a graded algebra for a vector space $V$ with grading $\deg V^* = 2$. We define the grading shifts $(k)$ by $(M(k))_n = M_{n-k}$ where $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a graded module.

**Definition 2.1.** Let $V$ be a vector space. A $V^*$-moment graph $G = (V, \mathcal{E}, h_G, t_G, l_G)$ is given by

• an ordered set \(\mathcal{V}\), called the set of vertices.
• a set \(\mathcal{E}\), called the set of edges.
• a map \(t_G, h_G: \mathcal{E} \rightarrow \mathcal{V}\) such that \(t_G(E) > h_G(E)\) for all \(E \in \mathcal{E}\).
• a map \(l_G: \mathcal{E} \rightarrow \mathbb{P}^1(V^*)\).

For \(E \in \mathcal{E}_G\), we denote \(l_G(E)\) by \(V^*_E\).

**Definition 2.2.** Let \(V\) be a vector space and \(G = (\mathcal{V}, \mathcal{E}, h_G, t_G, l_G)\) a \(V^*\)-moment graph.

1. A sheaf \(\mathcal{M} = ((\mathcal{M}_x)_{x \in \mathcal{V}}, (\mathcal{M}_E)_{E \in \mathcal{E}}, (\rho^G_{x, E}))\) on \(G\) is given by
   • a graded \(S(V^*)\)-module \(\mathcal{M}_x\).
   • a graded \(S(V^*)/V^*_E S(V^*)\)-module \(\mathcal{M}_E\).
   • an \(S(V^*)\)-module homomorphism \(\rho^G_{x, E}: \mathcal{M}_x \rightarrow \mathcal{M}_E\) for \(x \in \mathcal{V}\) and \(E \in \mathcal{E}\) such that \(x \in \{t_G(E), h_G(E)\}\).

2. Let \(\mathcal{M}, \mathcal{N}\) be sheaves on \(G\). A morphism \(f = ((f_x)_{x \in \mathcal{V}}, (f_E)_{E \in \mathcal{E}}): \mathcal{M} \rightarrow \mathcal{N}\) is given by
   • an \(S(V^*)\)-homomorphism \(f_x: \mathcal{M}_x \rightarrow \mathcal{N}_x\).
   • an \(S(V^*)\)-homomorphism \(f_E: \mathcal{M}_E \rightarrow \mathcal{N}_E\).
   • \(\rho^G_{x, E} \circ f_x = f_E \circ \rho^G_{x, E}\).

Define a sheaf \(\mathcal{A}_G\) on \(G\) by \(\mathcal{A}_G = ((S(V^*))_{x \in \mathcal{V}}, (S(V^*)/V^*_E S(V^*))_{E \in \mathcal{E}}, (\rho_x, E))\) where \(\rho_x, E\) is the canonical projection. This sheaf is called the *structure sheaf*.

For a sheaf \(\mathcal{M} = ((\mathcal{M}_x)_{x \in \mathcal{V}}, (\mathcal{M}_E)_{E \in \mathcal{E}}, (\rho^G_{x, E}))\) on \(G\), we can attach the space of its *global sections* by

\[
\Gamma(\mathcal{M}) = \left\{ ((m_x), (m_E)) \in \prod_{x \in \mathcal{V}} \mathcal{M}_x \oplus \prod_{E \in \mathcal{E}} \mathcal{M}_E \mid \rho^G_{x, E}(m_x) = m_E \right\}
\]

Put \(Z_G = \Gamma(\mathcal{A}_G)\). Then \(Z_G\) has the structure of a graded \(S(V^*)\)-algebra and \(\Gamma\) defines a functor from the category of sheaves on \(G\) to \(Z_G\)-mod, here \(Z_G\)-mod is the category of graded \(Z_G\)-modules. We also define the support of \(\mathcal{M}\) by \(\text{supp} \mathcal{M} = \{x \in \mathcal{V} \mid \mathcal{M}_x \neq 0\}\). The grading shifts for a sheaf is defined by \(\mathcal{M}(k) = ((\mathcal{M}_x)_{x \in \mathcal{V}}, (\mathcal{M}_E(k))_{E \in \mathcal{E}}, (\rho^G_{x, E}))\). Then we have \(\Gamma(\mathcal{M}(k)) = \Gamma(\mathcal{M})(k)\).

Let \(\mathcal{V}'\) be a subset of \(\mathcal{V}\). Put \(\epsilon' = \{E \in \mathcal{E} \mid h_G(E) \in \mathcal{V}', t_G(E) \in \mathcal{V}'\}\). Then \(\mathcal{G}' = (\mathcal{V}', \mathcal{E}', h_G|_{\mathcal{V}'}, t_G|_{\mathcal{V}'}, l_G|_{\mathcal{V}'})\) is also a \(V^*\)-moment graph. For a sheaf \(\mathcal{M} = ((\mathcal{M}_x)_{x \in \mathcal{V}}, (\mathcal{M}_E)_{E \in \mathcal{E}}, (\rho^G_{x, E}))\) on \(G\), \((\mathcal{M}_x)_{x \in \mathcal{V}'}, (\mathcal{M}_E)_{E \in \mathcal{E}'}, (\rho^G_{x, E})\) is a sheaf on \(\mathcal{G}'\).

We denote this sheaf by \(\mathcal{M}|_{\mathcal{G}'}\).

### 2.2. \(Z\)-module with Verma flags.

By the definition, we have \(Z_G \subset \prod_{x \in \mathcal{V}} S(V^*)\).

For \(\Omega \subset \mathcal{V}\), let \(Z^\Omega_G\) be the image of \(Z_G\) under the map \(\prod_{x \in \mathcal{V}} S(V^*) \rightarrow \prod_{x \in \Omega} S(V^*)\).

Let \(Z_G\)-mod\(^\Omega\) be the category of graded \(Z_G\)-modules that are finitely generated over \(S(V^*)\), torsion free over \(S(V^*)\) and the action of \(Z_G\) factors over \(Z^\Omega_G\) for a finite subset \(\Omega \subset \mathcal{V}\).

Let \(Q\) be the quotient field of \(S(V^*)\). Since \(Z_G \subset \prod_{x \in \mathcal{V}} S(V^*)\), we have \(Z_G \otimes_{S(V^*)} Q \subset \prod_{x \in \mathcal{V}} Q\). We also have \(Z^\Omega_G \otimes_{S(V^*)} Q \subset \prod_{x \in \Omega} Q\).

**Lemma 2.3** ([Fie08b, Lemma 3.1]). If \(\Omega\) is finite, then \(Z^\Omega_G \otimes_{S(V^*)} Q = \prod_{x \in \Omega} Q\).

For \(x \in \mathcal{V}\), put \(e_x = (\delta_{xy})_{y \in \mathcal{V}}\) where \(\delta\) is Kronecker's delta. Let \(M\) be an object of \(Z_G\)-mod\(^\Omega\) and take a finite subset \(\Omega \subset \mathcal{V}\) such that the action of \(Z_G\) on \(M\) factors over \(Z^\Omega_G\). For \(x \in \Omega\), put \(M^\Omega_Q = e_x(Q \otimes_{S(V^*)} M)\). Set \(M^\Omega_Q = 0\) for \(x \in \mathcal{V} \setminus \Omega\). Then we have \(M_Q = \bigoplus_{x \in \mathcal{V}} M^\Omega_Q\) where \(M_Q = Q \otimes_{S(V^*)} M\). These are independent of a choice of \(\Omega\). Since \(M\) is torsion-free, \(M \subset M_Q\).
Definition 2.4. For $M \in Z_G^f$, $\Omega \subset \mathcal{V}$, put

$$M_{\Omega} = M \cap \bigoplus_{x \in \Omega} M^x,$$

and set

$$M^\Omega = \mathrm{Im} \left( M \to M_Q \to \bigoplus_{x \in \Omega} M^x_Q \right).$$

A subset $\Omega \subset V$ is called upwardly closed if $x \in \Omega, y \geq x$ implies $y \in \Omega$.

Definition 2.5. We say that $M \in Z_G^f$ admits a Verma flag if the module $M^\Omega$ is a graded free $S(V^*)$-module for each upwardly closed $\Omega$.

Let $\mathcal{M}_G$ be a full-subcategory of $Z_G^f$ consisting of the object which admits a Verma flag.

Remark 2.6. Fiebig [Fie08a, Fie08b] uses a notation $\mathcal{V}$ for the category of modules which admits a Verma flag. Because we denote the set of vertices by $\mathcal{V}$, we use a different notation.

The category $\mathcal{M}_G$ is not an abelian category. However, $\mathcal{M}_G$ has a structure of an exact category [Fie08b, 4.1].

Definition 2.7. Let $M_1 \to M_2 \to M_3$ be a sequence in $\mathcal{M}_G$. We say that it is short exact if and only if for each upwardly closed subset $\Omega$ the sequence $0 \to M^\Omega_1 \to M^\Omega_2 \to M^\Omega_3 \to 0$ is an exact sequence of $S(V^*)$-modules.

2.3. Localization functor. Let $\mathcal{SH}(G)$ be the category of sheaves $\mathcal{M}$ on $G$ such that $\mathrm{supp}\mathcal{M}$ is finite and $\mathcal{M}_x$ is finitely generated and torsion free $S(V^*)$-module for each $x \in \mathcal{V}$. Then we have $\Gamma(\mathcal{SH}(G)) \subset Z^f$.

Proposition 2.8 (Fiebig [Fie08b]). The functor $\Gamma : \mathcal{SH}(G) \to Z^f$ has the left adjoint functor $\mathcal{L}$.

The functor $\mathcal{L}$ is called the localization functor.

For an image of $\mathcal{M}_G$ under $\mathcal{L}$, we have the following proposition. For a sheaf $\mathcal{M}$ on $G$ and $x \in \mathcal{V}$, put

$$\mathcal{M}^{[x]} = \mathrm{Ker} \left( \mathcal{M}_x \to \bigoplus_{h_G(E) = x} \mathcal{M}_E \right).$$

A sheaf $\mathcal{M}$ is called flabby if $\Gamma(\mathcal{M}) \to \Gamma(\mathcal{M}|_{\Omega})$ is surjective for all upwardly closed set $\Omega$.

Proposition 2.9 ([Fie08b]). (1) The functor $\mathcal{L}$ is fully-faithful on $\mathcal{M}_G$.

(2) For $M \in Z_G^f$, put $\mathcal{M} = \mathcal{L}(M)$. Then $M$ admits a Verma flag if and only if $\mathcal{M}$ is flabby and $\mathcal{M}^{[x]}$ is graded free for all $x \in \mathcal{V}$.

For $x \in \mathcal{V}$, define a sheaf $\mathcal{V}(x)$ by

$$\mathcal{V}(x)_y = \begin{cases} S(V^*) & (y = x), \\ 0 & (y \neq x), \end{cases}$$

$$\mathcal{V}(x)_E = 0.$$ 

The sheaf $\mathcal{V}(x)$ is called a Verma sheaf and its global section $V(x) = \Gamma(\mathcal{V}(x))$ is called a Verma module. The module $V(x)$ admits a Verma flag for all $x \in \mathcal{V}$. 
2.4. Projective object in $\mathcal{M}_G$. Let $\mathcal{G} = (V, E, h, l, t)$ be a $V^*$-moment graph. Since $\mathcal{M}_G$ is an exact category, we can define the notion of a projective object in $\mathcal{M}_G$. We can also define the notion of a projective object in $\mathcal{L}(\mathcal{M}_G)$ since $\mathcal{L}$ is fully-faithful on $\mathcal{M}_G$.

**Theorem 2.10** ([Fie08b, Theorem 5.2]). For each $x \in V$ there exists an indecomposable projective object $\mathcal{B}(X) \in \mathcal{L}(\mathcal{M}_G)$ such that $\mathcal{B}(x)_x \simeq S(V^*)$ and $\supp(\mathcal{B}(x)) \subseteq \{ y \mid y \leq x \}$.

Moreover, a projective object in $\mathcal{L}(\mathcal{M}_G)$ is a direct sum of $\{ \mathcal{B}(x)(k) \mid x \in V, k \in \mathbb{Z} \}$.

The sheaf $\mathcal{B}(x)$ is called a Braden-MacPherson sheaf $[BM01]$. 

2.5. Moment graph associated to a Coxeter system. Let $(W, S)$ be a Coxeter system such that $S$ is finite. We denote the set of reflections by $T$. A finite dimensional representation $V$ of $W$ is called a reflection faithful representation if for each $w \in W$, $V^w$ is a hyperplane in $V$ if and only if $w \in T$. By Soergel [Soe07], there exists a reflection faithful representation. Let $V$ be a reflection faithful representation. For each $t \in T$, let $\alpha_t \in V^*$ be a non-trivial linear form vanishing on the hyperplane $V^t$. If $s \neq t$, then $\alpha_s \neq \alpha_t$ [Fie08b, Lemma 2.2].

Let $S'$ be a subset of $S$ and $W'$ the subgroup of $W$ generated by $S'$. We attach a $V^*$-moment graph $\mathcal{G} = (V, E, h, l, t)$ to $((W, S), (W', S'))$ by

- $V = W/W'$, an order is induced by the Bruhat order.
- $E = \{ (xW', yW') \mid x \in T y W' \}$.
- If $x \in T y$, $x < y$, then $h_{al}(xW', yW') = xW'$, $t_{al}(xW', yW') = yW'$.
- $V_{(xW', yW')} = C_{al}$ for $xW' \in W/W', t \in T$.

In the rest of this paper, we fix a Coxeter system $(W, S)$ and a reflection faithful representation $V$. Let $\mathcal{G}$ be the $V^*$-moment graph associated to $((W, S), \{ \{ e \}, \emptyset \})$. Put $\mathcal{A} = \mathcal{A}_G$, $Z = Z_G$ and $\mathcal{M} = \mathcal{M}_G$.

2.6. Translation functor. We define an action of a simple reflection $s \in S$ on $\prod_{w \in W} S(V^w)$ by $s((z_w)_w) = (z_{ws})_w$. This action preserves $Z$. Put $Z^s = \{ z \in Z \mid s(z) = z \}$. Then $Z^s$ is an $S(V^s)$-subalgebra. For $M \in Z\text{-mod}^f$, put $\theta^2_s(M) = Z \otimes Z^s M(\Sigma)$. Let $\mathcal{B}(x)$ be the Braden-MacPherson sheaf and put $\mathcal{B}(x) = \mathcal{B}(x)(-\ell(x))$.

Set $B(x) = \Gamma(\mathcal{B}(x))$.

**Proposition 2.11** ([Fie08a, Proposition 5.5, Corollary 5.7]).

1. The functor $\theta^2_s$ preserves $\mathcal{M}$.

2. The functor $\theta^2_s$ is exact and self-adjoint.

3. For $M \in Z\text{-mod}^f$, supp($\mathcal{L}(\theta^2_s(M))$) $\subseteq$ supp($\mathcal{L}(M)$) $\cup$ supp($\mathcal{L}(M)$) $\otimes$.

4. Assume that $x > s$. There exists a projective object $P \in \mathcal{M}$ such that $\theta^2_s(B(x)) = B(\theta^2_s(x)) \otimes P$ and $\supp(\mathcal{L}(P)) \subseteq \{ y \in W \mid y \leq x \}$.

5. There exist degree zero canonical homomorphism $\text{Id}(1) \to \theta^2_s$ and $\theta^2_s \to \text{Id}(-1)$.

**Remark 2.12.** Set $c_s = (w(\alpha))_w$. The natural transformation $\text{Id}(1) \to \theta^2_s$ is given by $m \mapsto c_s \otimes m + 1 \otimes c_s m$ and $\theta^2_s \to \text{Id}(-1)$ is given by $z \otimes m \mapsto zm$.

3. THE CATEGORY $\mathcal{O}$

3.1. The functor $\varphi^2_s$. For a graded $S(V^*)$-module $M$ and $w \in W$, let $b_w(M)$ be an $S(V^w)$-module whose structure map is given by $S(V^*) \to S(V^*) \to \text{End}(M)$. We remark that if $M$ is annihilated by $\alpha_t$ for $t \in T$, then we have $b_t(M) \simeq M$ as a graded $S(V^*)$-module.
First we define a functor $a_S : \mathcal{SH}(\mathcal{G}) \to \mathcal{SH}(\mathcal{G})$ by the following. Let $\mathcal{M} \in \mathcal{SH}(\mathcal{G})$. Then the sheaf $a_S(\mathcal{M})$ is defined by

- $(a_S(\mathcal{M}))_x = b_{x^{-1}} M_{x^{-1}}$ for $x \in W$,
- $(a_S(\mathcal{M}))(E) = b_{x^{-1}}(M_{E'})$ where $x = h(E), h_E(E') = h_E(E)^{-1} = x^{-1}$ and $t_{g}(E') = t_{g}(E)^{-1} = (tx)^{-1}$,
- $\rho^{a_S(\mathcal{M})}_{x,E} = \rho^{a_S(\mathcal{M})}_{x^{-1},E'}$.

It is easy to see that these data define a sheaf $a_S(\mathcal{M})$ and functor $a_S : \mathcal{SH}(\mathcal{G}) \to \mathcal{SH}(\mathcal{G})$.

Let $a_Z : \prod_{w \in W} S(V^*) \to \prod_{w \in W} S(V^*)$ be an algebra homomorphism defined by $a((z_w)_w) = (w z_{w^{-1}})_w$. Then $a_Z$ preserves a subalgebra $Z$ and gives a $C$-algebra homomorphism. We remark that $a_Z$ is not an $S(V^*)$-algebra homomorphism. For a $Z$-module $M$, let $a_M(M)$ be a $Z$-module whose structure map is given by $Z \to Z \to \text{End}(M)$. This defines a functor $a_M : Z\text{-mod} \to Z\text{-mod}$.

**Lemma 3.1.**

1. We have $\text{supp}(a_S(\mathcal{M})) = \{x^{-1} \mid x \in \text{supp} \mathcal{M}\}$.
2. We have $a_S(\mathcal{SH}(\mathcal{G})^f) \subset \mathcal{SH}(\mathcal{G})^f$.
3. We have $a_M(Z\text{-mod}^f) \subset Z\text{-mod}^f$.
4. We have $\Gamma \circ a_S \simeq a_M \circ \Gamma$.
5. We have $\mathcal{L} \circ a_M \simeq a_S \circ \mathcal{L}$.

**Proof.** (1) and (2) is obvious from the definition.

(3) By the definition, we have $a_Z(Z^\Omega) = Z^{\Omega'}$ where $\Omega' = \{x^{-1} \mid x \in \Omega\}$. Hence if the action of $Z$ on $M$ factors over $Z^\Omega$, the action on $a_M(M)$ factors over $Z^\Omega'$.

(4) Let $\mathcal{M} \in \mathcal{SH}(\mathcal{G})$. By the definition, we have

$$\Gamma(a_S(\mathcal{M})) = \left\{ \left( (m_{x}), (m_{E}) \right) \in \prod_{x \in W} b_{x^{-1}} \mathcal{M}_{x^{-1}} \oplus \prod_{E \in E} b_{x^{-1}} \mathcal{M}_{E'} \mid \rho^{a_S(\mathcal{M})}_{x^{-1},E'}(m_{x}) = m_{E} \right\},$$

where $E'$ is the same as in the definition of $a_S$. Replace $x \mapsto x^{-1}$. Then $E'$ becomes $E$. Hence we get

$$\Gamma(a_S(\mathcal{M})) = \left\{ \left( (m_{x^{-1}}), (m_{E}) \right) \in \prod_{x \in W} b_{x} \mathcal{M}_{x} \oplus \prod_{E \in E} b_{x} \mathcal{M}_{E} \mid \rho^{a_M(\mathcal{M})}_{x,E}(m_{x}) = m_{E} \right\}.$$

From this formula, as a space, $\Gamma(a_S(\mathcal{M})) = \Gamma(\mathcal{M})$. The action of $z = (z_w) \in Z$ on $((m_{x}), (m_{E})) \in \Gamma(a_S(\mathcal{M}))$ is given by $((x(z_{x^{-1}})m_{x}), (x(z_{x^{-1}})m_{E}))$ where $t_{g}(E) = x$. This action coincide with the action of $z$ on $a_M(\Gamma(\mathcal{M}))$.

(5) Obviously, $a_S^2 = \text{Id}$ and $a_M^2 = \text{Id}$. In particular, $a_S : \mathcal{SH}(\mathcal{G})^f \to \mathcal{SH}(\mathcal{G})^f$ and $a_M : Z\text{-mod}^f \to Z\text{-mod}^f$ are self-adjoint. Hence, taking the left adjoint functor of the both sides in (4), we get (5). \hfill $\square$

**Proposition 3.2.** We have $a_M(M) = \mathcal{M}$.

**Proof.** Take $M \in \mathcal{M}$ and put $\mathcal{M} = \mathcal{L}(M), N = \mathcal{L}(a_M(M)) = a_S(\mathcal{M})$. We prove that $\mathcal{N}$ is flabby and $\mathcal{N}^{[x]}$ is graded free for all $x \in W$.

Let $\Omega$ be a upwardly closed subset and put $\Omega' = \{x^{-1} \mid x \in \Omega\}$. Then $\Omega'$ is also upwardly closed. Since $\mathcal{M}$ is flabby, $\Gamma(\mathcal{M}) \to \Gamma(\mathcal{M}|_{\Omega'})$ is surjective. Hence $\Gamma(\mathcal{N}) = a_M(\Gamma(\mathcal{M})) \to a_M(\Gamma(\mathcal{M}|_{\Omega'})) = \Gamma(\mathcal{N}|_{\Omega'})$ is surjective.

By the definition of $\mathcal{N}^{[x]}$, we have $\mathcal{N}^{[x]} = b_{x^{-1}}(\mathcal{M}^{[x^{-1}]})$. Since $\mathcal{M}^{[x^{-1}]}$ is graded free, $\mathcal{N}^{[x]}$ is graded free. \hfill $\square$

**Lemma 3.3.** We have $a_M(B(x)) = B(x^{-1})$. 

Proof. Since a gives an auto-equivalence of the category $\mathcal{M}$, $a_\mathcal{M}(B(x))$ is an indecomposable projective object. By Lemma 3.1 and the definition of $a_\mathcal{S}$, we have $\text{supp} \mathcal{L}(a_\mathcal{M}(B(x))) = \text{supp} a_\mathcal{S}(\mathcal{L}(B(x))) = \{y^{-1} | y \in \text{supp} \mathcal{L}(B(x))\}$ and $\mathcal{L}(a_\mathcal{S}(\mathcal{L}(B(x)))) = (a_\mathcal{S}(\mathcal{L}(B(x))))^{-1} = b_{x-1} \mathcal{L}(B(x)) = b_{x-1} S(V^*)(-\ell(x)) = b_{x-1} S(V^*)(-\ell(x)^{-1}) \simeq S(V^*)(-\ell(x)^{-1})$. Hence we get the lemma. \(\square\)

From Proposition 3.2, we can define the functor $\varphi_s^Z : \mathcal{M} \to \mathcal{M}$ by $\varphi_s^Z = a_\mathcal{M} \circ \theta_s^Z \circ a_\mathcal{M}$. Since $a_\mathcal{M}$ gives an equivalence of categories, the fundamental properties of $\varphi_s^Z$ follows from that of $\theta_s^Z$.

**Proposition 3.4.**
(1) The functor $\varphi_s^Z$ preserves $\mathcal{M}$.
(2) The functor $\varphi_s^Z$ is exact and self-adjoint.
(3) For $M \in Z\text{-mod}^I$, $\text{supp} \mathcal{L}(\varphi_s^Z(M)) \subset \text{supp} \mathcal{L}(M) \cup s(\text{supp} \mathcal{L}(M))$.
(4) Assume that $sx \geq x$. There exists a projective object $P \in \mathcal{M}$ such that $\varphi_s^Z(B(x)) = B(sx) \oplus P$ and $\text{supp} \mathcal{L}(P) \subset \{y \in W | y \leq x\}$.
(5) There exist degree zero canonical homomorphisms $\text{Id}(1) \to \varphi_s^Z$ and $\varphi_s^Z \to \text{Id}(-1)$.

We describe the functor $\varphi_s^Z$ more explicitly. We define an algebra homomorphism $\Gamma : \prod_{w \in W} S(V^*) \to \prod_{w \in W} S(V^*)$ by $\Gamma((z_w)_w) = (s(z_w))_w$. Note that this is not an $S(V^*)$-module homomorphism. The subalgebra $Z$ satisfies $\Gamma(Z) = Z$. Recall that the map $s : Z \to Z$ is defined by $s((z_w)_w) = ((sz_w)_w)$. Then it is easy to see that $rs \circ a_Z = a_Z \circ s$. Set $Zr^\ast = \{z \in Z | \Gamma(z) = z\}$. Then we have $\varphi_s^Z = Z \otimes Zr^\ast$. From this description, we get the following proposition.

**Proposition 3.5.** For simple reflections $s, t$, the functors $\theta_t^Z$ and $\varphi_s^Z$ commute with each other. Moreover, the natural transformation $\theta_t^Z(1) \to \varphi_s^Z \theta_t^Z$ (resp. $\varphi_s^Z(1) \to \theta_t^Z \varphi_s^Z$, $\varphi_s^Z \theta_t^Z \varphi_s^Z \to \varphi_s^Z(\theta_t^Z)$) can be identified with $\theta_t^Z(\text{Id}(1) \to \varphi_s^Z)$ (resp. $\varphi_s^Z(\text{Id}(1) \to \theta_t^Z)$, $\varphi_s^Z(\theta_t^Z \to \varphi_s^Z(\theta_t^Z \to \text{Id}(-1))))$.

Proof. First we remark that $t$ and $rs$ commute with each other. Put $Zr^\ast t = Z^\ast \cap Z_t$. We prove that $Z \otimes Zr^\ast t \simeq Z \otimes Zr^\ast, Z \otimes Zr^\ast M$ for a $Z$-module $M$. The same argument implies $Z \otimes Zr^\ast t \simeq Z \otimes Z_t, Z \otimes Zr^\ast M$.

Consider the map $\Xi : Z \otimes Zr^\ast t M \to Z \otimes Zr^\ast, Z \otimes Zr^\ast M$ defined by $\Xi(z \otimes m) = z \otimes 1 \otimes m$. This map is a $Z$-module homomorphism. Set $\alpha = \alpha_s$. We regard $\alpha$ as an element of $Z$ by the structure map $S(V^*) \to Z$. Put $c_t = \alpha t$. Then we have $Z = Z^\ast \oplus c_t Z^\ast$ [Fie08a, Lemma 5.1]. Since $a_Z(c_t) = \alpha_s$, we have $Z = Z^\ast \oplus \alpha Z^\ast$. Hence we get

$$Z \otimes Zr^\ast M = (1 \otimes 1 \otimes M) \oplus (\alpha \otimes 1 \otimes M) \oplus (1 \otimes c_t \otimes M) \oplus (\alpha \otimes c_t \otimes M).$$

Similarly, we get

$$Z \otimes Zr^\ast t M = (1 \otimes 1 \otimes M) \oplus (\alpha \otimes 1 \otimes M) \oplus (c_t \otimes M) \oplus (\alpha c_t \otimes 1 \otimes M).$$

Since $c_t \in Z^\ast$, $1 \otimes c_t \otimes M = c_t \otimes 1 \otimes M$ and $\alpha \otimes c_t \otimes M = \alpha c_t \otimes 1 \otimes M$. Hence $\Xi$ is an isomorphism.

We prove the second claim. We omit a grading. The map $Z \otimes Zt^\ast M \to Z \otimes Zr^\ast M$ is given by $1 \otimes m \mapsto 1 \otimes \alpha \otimes m + \alpha \otimes 1 \otimes m$ (Remark 2.12). Since $\alpha \in Z_t^\ast$, we have $1 \otimes \alpha \otimes m = 1 \otimes 1 \otimes \alpha m$. Under the isomorphism $Z \otimes Zt^\ast Z \otimes Zr^\ast M \simeq Z \otimes Zr^\ast Z \otimes Zt^\ast M \simeq Z \otimes Zr^\ast Z \otimes Z^\ast M$, $1 \otimes 1 \otimes m \in Z \otimes Zr^\ast Z \otimes Z^\ast M$ corresponds to $1 \otimes 1 \otimes m \in Z \otimes Zr^\ast Z \otimes Z^\ast M$. Hence the map $Z \otimes Zr^\ast M \to Z \otimes Zr^\ast Z \otimes Z^\ast M \simeq Z \otimes Z^\ast Z \otimes Z \otimes Zr^\ast M$ is given by $1 \otimes m \mapsto 1 \otimes 1 \otimes \alpha m + \alpha \otimes 1 \otimes m = 1 \otimes 1 \otimes \alpha m + 1 \otimes \alpha \otimes m$. This is equal to $\theta_t^Z(\text{Id} \to \varphi_s^Z)$. We can prove the other formulae by the same argument. \(\square\)

**Lemma 3.6.** Fix $s \in S$ and put $S' = \{s\}$, $W' = \{1, s\}$. Let $G'$ be the moment graph associated to $((W, S'), (W', S'))$, $\mathcal{F}(xW')$ the Braden-MacPherson sheaf and $B'(xW') = \Gamma(\mathcal{F}(xW')(-\ell(x)))$ for $x \in W$ such that $xs < x$. Using $Z_W' \simeq$
By Proposition 2.11, we have

\[ B(x) \]

Proof. Let \( \theta \) is contained in \( Z \). By the construction of the Braden-MacPherson sheaf \( [BM01, 1.4] \), \( L(\theta) \) is a direct summand of \( Z_{\theta} \). Take a projective object \( P \) such that \( Z_{\theta} = L^{\theta}(B(x)) \triangleq P \). We prove \( P = 0 \). In the rest of this proof, we omit a grading. By the construction of the Braden-MacPherson sheaf \( [BM01, 1.4] \), \( L(\theta)(\theta) = L(\theta)(\theta)(\theta) \) is not equivalent to the Bernstein-Gelfand-Gelfand (BGG) category \( \mathcal{O} \). Therefore, if \( P \neq 0 \), then \( \text{Res}_{Z_{\theta}}(L(\theta)) = B(x) \) and \( \text{Res}_{Z_{\theta}}(L(\theta)) = B'(x') \). Since \( L(\theta)(\theta) = \mathcal{O}(\theta)(\theta) \), we have \( \text{Res}_{Z_{\theta}}(L(\theta)) = B(x) \). Hence \( P \neq 0 \). This is a contradiction. Hence \( P = 0 \). \( \square \)

**Proposition 3.7.** Let \( s \) be a simple reflection and \( x \in W \).

1. If \( xs > x \), then \( \theta^{\sharp}_s B(x) = B(xs) \oplus \bigoplus_{y < x, y \neq y, k \in Z} B(y) \langle k \rangle^{m_{y,k}} \) for some \( m_{y,k} \in \mathbb{Z}_{\geq 0} \).
2. If \( xs < x \), then \( \theta^{\sharp}_s B(x) = B(x) \langle 1 \rangle \oplus B(x) \langle -1 \rangle \).
3. If \( sz > x \), then \( \varphi^{\sharp}_s B(x) = B(xs) \oplus \bigoplus_{y < x, y \neq y, k \in Z} B(y) \langle k \rangle^{m_{y,k}} \) for some \( m_{y,k} \in \mathbb{Z}_{\geq 0} \).
4. If \( sz < x \), then \( \varphi^{\sharp}_s B(x) = B(x) \langle 1 \rangle \oplus B(x) \langle -1 \rangle \).

Proof. Let \( W', S', B'(x') \) be as in the previous lemma.

1. Since \( \text{Res}_{Z_{\theta}}(B(x)) \) is a projective object and the support of \( \mathcal{L}(\text{Res}_{Z_{\theta}}(B(x))) \) is contained in \( \{ y \in W' \mid y \leq x \} \), we have \( \text{Res}_{Z_{\theta}}(B(x)) = \bigoplus_{k \in Z} B'(x) \langle k \rangle^{m_{k}} \oplus \bigoplus_{y < x, y \neq y, k \in Z} B'(y) \langle k \rangle^{m_{y,k}} \) for some \( m_{k} \) and \( m_{y,k} \). Then by the previous lemma, we get \( \theta^{\sharp}_s B(x) = \bigoplus_{k \in Z} B'(x) \langle k \rangle^{m_{k}} \oplus \bigoplus_{y < x, y \neq y, k \in Z} B(y) \langle k \rangle^{m_{y,k}} \). By Proposition 2.11, we have \( m_{k} = 0 \) if \( k \neq 1 \) and \( m_{1} = 1 \).
2. From \( [Fie08a, \text{Lemma } 5.1] \), we have \( \text{Res}_{Z_{\theta}}(Z \ominus Z_{\theta}, \cdot) = \text{Id} \oplus \text{Id}(2) \). Hence we have

\[ \theta_{s} B(x) = \theta^{\sharp}_s (Z \ominus Z_{\theta}, B'(xW')) = Z \ominus Z_{\theta}, (\text{Res}_{Z_{\theta}}(Z \ominus Z_{\theta}, B'(xW')))(1) \]

\[ \simeq Z \ominus Z_{\theta}, (B'(xW'))(1) \oplus B'(xW')(1) \simeq B(x)(1) \oplus B(x)(-1). \]

(3) and (4) follows from (1) and (2) and Lemma 3.3. \( \square \)

**3.2. Definition of the category \( \mathcal{O} \).** Set \( \tilde{A} = \text{End}_{Z}(\bigoplus_{x \in W} B(x)) \). This is an \( S(V^*) \)-algebra.

**Definition 3.8.** Put \( A = \tilde{A} \otimes S(V^*) \mathbb{C} \) where \( \mathbb{C} = S(V^*)/V^* S(V^*) \) is a one-dimensional \( S(V^*) \)-algebra. Define the category \( \mathcal{O} \) as the category of right \( A \)-modules.

**Remark 3.9.** Even if \( (W, S) \) is the Weyl group of some Kac-Moody Lie algebra, the category \( \mathcal{O} \) is not equivalent to the Bernstein-Gelfand-Gelfand (BGG) category since BGG category has some finiteness conditions. If \( (W, S) \) is a finite Weyl group, then the category of finitely generated right \( A \)-modules is equivalent to the regular integral block of the BGG category. More generally, if \( (W, S) \) is the Weyl group of some Kac-Moody Lie algebra, a block of the BGG category with positive level can be recovered from the algebra \( A \) \( [Fie08a] \).

Let \( \tilde{O} \) be the category of right \( \tilde{A} \)-modules. Since \( A = \tilde{A}/V^* \tilde{A} \) is a quotient of \( \tilde{A} \), we regard \( \mathcal{O} \) as a full-subcategory of \( \tilde{O} \).
Define the functor $\tilde{\Phi} : Z\text{-mod} \to \tilde{\mathcal{O}}$ by $\tilde{\Phi}(M) = \text{Hom}_Z(\bigoplus_{x \in W} B(x), M)$ and put $\tilde{\Phi}(M) = \tilde{\Phi}(M) \otimes_{S(V^\vee)} \mathbb{C}$.

**Lemma 3.10.** Let $P$ be a direct sum of $\{B(x) \mid x \in W\}$’s and $M \in \mathcal{M}$. Then the following canonical maps are isomorphisms:

- $\text{Hom}_Z(P, M) \to \text{Hom}_\mathfrak{A}(\tilde{\Phi}(P), \tilde{\Phi}(M))$.
- $\text{Hom}_Z(P, M) \otimes_{S(V^\vee)} \mathbb{C} \to \text{Hom}_\mathfrak{A}(\tilde{\Phi}(P), \tilde{\Phi}(M))$.

**Proof.** We may assume that $P = B(x)$ for some $x \in W$. Hence it is sufficient to prove when $P = \bigoplus_{x \in W} B(x)$. The lemma is obvious in this case. \hfill \Box

Set $\tilde{P}(x) = \tilde{\Phi}(B(x))$, $P(x) = \Phi(B(x)) = \tilde{P}(x) \otimes_{S(V^\vee)} \mathbb{C}$, $\tilde{M}(x) = \tilde{\Phi}(V(x))$ and $M(x) = \Phi(V(x)) = M(x) \otimes_{S(V^\vee)} \mathbb{C}$. The module $M(x)$ is called a Verma module. The module $P(x)$ has the unique irreducible quotient. The irreducible quotient is denoted by $L(x)$. This is a one-dimensional $A$-module and the unique irreducible quotient of $M(x)$. To summarize it, we get the following lemma.

**Lemma 3.11.**

1. $\tilde{P}(x)$ is a projective $\tilde{A}$-module.
2. $P(x)$ is a projective $A$-module.
3. $L(x)$ is a simple $A$-module (hence, simple $\tilde{A}$-module).
4. We have $\text{Hom}_\mathfrak{A}(P(x), L(y)) = \text{Hom}_\mathfrak{A}(\tilde{P}(x), L(y)) = \delta_{xy}$.

**Proof.** For (4), notice that we have $\text{Hom}_\mathfrak{A}(\tilde{M} \otimes_{S(V^\vee)} \mathbb{C}, N) = \text{Hom}_\mathfrak{A}(\tilde{M}, N)$ for $\tilde{M} \in \tilde{\mathcal{O}}$ and $N \in \mathcal{O}$. Hence we get $\text{Hom}_\mathfrak{A}(P(x), L(y)) = \text{Hom}_\mathfrak{A}(\tilde{P}(x), L(y))$. \hfill \Box

Since there exists a surjective morphism $B(x) \to V(x)$, we have a surjective map $P(x) \to M(x)$. Moreover, we get the following proposition.

**Proposition 3.12.** For $x \in W$, there exists a submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = P(x)$ such that $M_i/M_{i-1} \simeq M(x_i)$ for some $x_i \in W$. Moreover, we can take $\{M_i\}$ such that $x = x_n \geq x_{n-1} \geq \cdots \geq x_1$.

**Proof.** Consider the order filtration [Fie08b, 4.3] $\{N_i\}$ of $P(x)$. Then we have $N_{i(v)}/N_{i(v)-1} \simeq P(x)^{[v]}$. Since $P(x)^{[v]} = V(v)^{n_v}$ for some $n_v \in \mathbb{Z}_{\geq 0}$, we get the proposition. \hfill \Box

### 3.3. Translation functors.

In this subsection, we construct functors $\tilde{\theta}_s, \tilde{\varphi}_s : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$ using functors $\theta^Z_s$, $\varphi^Z_s$. Since the construction is the same, set $F^Z = \theta^Z_s$ or $\varphi^Z_s$ and we will construct a functor $\tilde{F} : \mathcal{O} \to \tilde{\mathcal{O}}$.

Put $\tilde{A}' = \tilde{\Phi}(\bigoplus_{y \in W} F^Z B(y))$. Then the module $\tilde{A}'$ is a right $\tilde{A}$-module and left $\text{End}(\bigoplus_{x \in W} F^Z B(x))$-module. Moreover, using a homomorphism $\text{End}(B(x)) \to \text{End}(F^Z B(x))$, $A'$ is an $A$-bimodule. Define $\tilde{F} : \mathcal{O} \to \tilde{\mathcal{O}}$ by $\tilde{F}(M) = \text{Hom}_\mathfrak{A}(A', \tilde{M})$ for $\tilde{M} \in \tilde{\mathcal{O}}$. Then $\tilde{F}(\tilde{M})$ is a right $\tilde{A}$-module. Since $F^Z B(y)$ is a direct summand of $(\bigoplus_{x \in W} B(x))^{\oplus m}$ for some $m$, $A'$ is a direct summand of $\tilde{A}^{\oplus m}$ for some $m$. Hence $A'$ is a projective right $A$-module. This implies that $\tilde{F}$ is an exact functor.

Set $B = \bigoplus_{y \in W} B(y)$. From Lemma 3.10, we have

$$\tilde{A}' \simeq \text{Hom}_\mathfrak{A}(\tilde{A}, \tilde{A}') = \text{Hom}_\mathfrak{A}(\tilde{\Phi}(B), \tilde{\Phi}(F^Z(B)))$$

$$\simeq \text{Hom}_Z(B, F^Z(B)) \simeq \text{Hom}_Z(F^Z(B), B) \simeq \text{Hom}_\mathfrak{A}(A', \tilde{A}).$$

So we have $\tilde{A}' \simeq \tilde{F}(\tilde{A})$.

Recall the following well-known lemma. For the sake of completeness, we give a proof.
Lemma 3.13. Let $R_1, R_2$ be an arbitrary ring, $C_1$ the category of right $R_i$-modules ($i = 1, 2$) and $G$ a right exact functor $C_1 \to C_2$. Then we have a functorial isomorphism $G(X) \simeq X \otimes_{R_1} G(R_1)$.

Proof. From an $R_1$-module homomorphism

$$X \simeq \text{Hom}_{R_1}(R_1, X) \to \text{Hom}_{R_2}(G(R_1), G(X)),$$

we have an $R_2$-module homomorphism $X \otimes_{R_1} G(R_1) \to G(X)$. If $X$ is free, this map is an isomorphism. For a general $X$, take an exact sequence $F_1 \to F_0 \to X \to 0$ such that $F_0, F_1$ are free. Then we have the following diagram:

$$\begin{array}{c}
F_1 \otimes_{R_1} G(R_1) \rightarrowtail F_0 \otimes_{R_1} G(R_1) \rightarrow X \otimes_{R_1} G(R_1) \rightarrow 0 \\
\downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
G(F_1) \rightarrowtail G(F_0) \rightarrow G(X) \rightarrow 0.
\end{array}$$

The left two homomorphisms are isomorphisms. Hence $X \otimes_{R_1} G(R_1) \to G(X)$ is also annihilated by an isomorphism. Hence we get (4).

Hence we have a homomorphism $\theta$ of $X$. Hence we have a homomorphism $\theta$.

We get the following proposition.

Proposition 3.14. (1) The functor $\widetilde{F}$ is self-adjoint. In particular, $\widetilde{F}$ is an exact functor.

(2) We have $\tilde{A}' \simeq \tilde{F}(\tilde{A})$.

(3) We have $\tilde{F}(\tilde{M}) \simeq \tilde{M} \otimes_{\tilde{A}} \tilde{F}(\tilde{A})$.

(4) We have $\Phi \circ F^Z \simeq F \circ \Phi$.

Proof. We already proved (1–3). We have

$$\begin{aligned}
\widetilde{F} \circ \tilde{\Phi}(M) &= \text{Hom}_{\tilde{A}}(\tilde{A}', \tilde{\Phi}(M)) = \text{Hom}_{\tilde{A}}(\tilde{\Phi}(\bigoplus_{y \in W} F^Z B(y)), \tilde{\Phi}(M)) \\
&\simeq \text{Hom}_Z(\bigoplus_{y \in W} F^Z B(y), M) \simeq \text{Hom}_Z(\bigoplus_{y \in W} B(y), F^Z M) = \tilde{\Phi}(F^Z(M)).
\end{aligned}$$

Hence we get (4).

Now we discuss the restriction of $\widetilde{F}$ to the full-subcategory $O$. First we consider $F^Z = \theta^Z$. For $M \in Z\text{-mod}$, $p \in S(V^*)$ induces a homomorphism $p: M \to M$. Hence we have a homomorphism $\theta^Z(p): \theta^Z(M) \to \theta^Z(M)$. From the construction of $\theta^Z$, this map is equal to the action of $\theta^Z(M) \to \theta^Z(M)$. Since $\tilde{A}'$ is an $A$-bimodule and $\tilde{A}$ is a $S(V^*)$-algebra, $\tilde{A}'$ is an $S(V^*)$-bimodule. From the above argument, the left and right $S(V^*)$-module structure of $\tilde{A}'$ coincide. Hence the action of $S(V^*)$ on $\tilde{M}(\tilde{A}', \tilde{M})$ coincides with the $S(V^*)$-action induced from that of $\tilde{M}$. In particular, if $\tilde{M}$ is annihilated by $V^*$ (i.e., $\tilde{M} \in \tilde{O}$), then $\tilde{\theta}_s(\tilde{M})$ is also annihilated by $V^*$. Hence $\tilde{\theta}_s$ gives a functor from $O$ to $\tilde{O}$ and satisfies the similar properties in Proposition 3.14. We denote this functor by $\tilde{\theta}_s$.

In the case of $\varphi^Z$, the situation is bad. In this case, a homomorphism $\varphi^Z(p)$ is not equal to $p$ for $p \in S(V^*)$ in general. Hence $\varphi_s$ does not give a functor from $O$ to $\tilde{O}$. Let $\varphi_s$ be the restriction of the functor $\varphi_s$ to $O$. This is a functor from $O$ to $\tilde{O}$.

Remark 3.15. By the same reason, we have $\theta_s(\tilde{M} \otimes_{S(V^*)} \mathbb{C}) \simeq (\tilde{\theta}_s(\tilde{M})) \otimes_{S(V^*)} \mathbb{C}$ for $\tilde{M} \in \tilde{O}$. The corresponding statement for $\varphi_s$ is false in general.
3.4. Natural transformations. We use the notation in the previous subsection. We start with the following lemma.

**Lemma 3.16.** For \( M \in \mathcal{M} \), the natural transformation \( M \to FZM \) is given by the self-adjointness of \( FZ \) and the natural transformation \( FZM \to M \).

**Proof.** We consider the case of \( FZ = \theta Z \). Using the functor \( aM \), we get the lemma in the case of \( FZ = \phi Z \).

In this case, \( FZM = Z \otimes Z, \). Since \( (\text{Res}_{Z^*}, Z \otimes Z^* \cdot), (Z \otimes Z^* \cdot, \text{Res}_{Z^*}) \) are adjoint pairs, we have

\[
\text{Hom}_Z(M, FZM) \simeq \text{Hom}_{Z^*}(M, M) \simeq \text{Hom}(FZM, M).
\]

The natural transformations \( M \to FZM \) (resp. \( FZM \to M \)) corresponds to \( \text{Id}: M \to M \) by the left (resp. right) isomorphism. Since these isomorphisms give a self-adjointness of \( FZ \), we get the lemma. \( \square \)

Since \( \tilde{\Phi} = \tilde{\Phi}(\bigoplus_{g \in W}(FZB(y))) \), we get a homomorphism \( \sigma: \tilde{\Phi} \to \tilde{\Phi} \cdot \) and \( \sigma^\prime: \tilde{\Phi} \to \tilde{\Phi} \). Then \( \sigma_{Z} = \text{Hom}(\sigma, \tilde{\Phi}) \) (resp. \( \sigma^\prime_{Z} = \text{Hom}(\sigma^\prime, \tilde{\Phi}) \)) gives a natural transformation \( \sigma: \tilde{\Phi} \to \text{Id} \) (resp. \( \sigma^\prime: \text{Id} \to \tilde{\Phi} \)).

Since we have an isomorphism \( \text{Res}_{Z^*} \tilde{\Phi}(\tilde{\Phi}) \simeq \tilde{\Phi} \otimes \tilde{\Phi} \), we can define another natural transformations by \( \text{id}_{Z} \otimes \sigma \) and \( \text{id}_{Z} \otimes \sigma^\prime \).

**Proposition 3.17.** We have \( \sigma_{Z} = \text{id}_{\tilde{\Phi}} \otimes \sigma^\prime \) and \( \sigma^\prime_{Z} = \text{id}_{\tilde{\Phi}} \otimes \sigma \). Moreover, we have the following commutative diagram for \( M, N \in O \):

\[
\begin{array}{ccc}
\text{Hom}(\tilde{\Phi}, \tilde{\Phi}) & \simeq & \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \\
\text{Hom}(\tilde{\Phi}, \tilde{\Phi}) & \simeq & \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \\
\text{Hom}(\tilde{\Phi}, \tilde{\Phi}) & \simeq & \text{Hom}(\tilde{\Phi}, \tilde{\Phi})
\end{array}
\]

**Proof.** In this proof, we omit the grading of objects of \( \mathcal{M} \).

First we prove the first claim for \( \tilde{\Phi} = \tilde{\Phi} \). Put \( B = \bigoplus_{g \in W} B(y) \). Recall that an isomorphism \( \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \simeq \tilde{\Phi} \) is induced from \( \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \simeq \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \) and \( \sigma \) (resp. \( \sigma^\prime \)) is induced from the natural transformation \( \text{Id} \to \tilde{\Phi} \) (resp. \( \tilde{\Phi} \to \text{Id} \)) in \( \mathcal{M} \). Hence we get the first claim for \( \tilde{\Phi} = \tilde{\Phi} \) from the corresponding statement in \( \mathcal{M} \) (Lemma 3.16).

To prove for a general \( \tilde{\Phi} \), take a free resolution \( \tilde{\Phi}_1 \to \tilde{\Phi}_0 \to \tilde{\Phi} \to 0 \). Since \( \tilde{\Phi} \) is exact, we have \( \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) = \text{Cok}(\text{Hom}(\sigma, \tilde{\Phi}) \to \text{Hom}(\sigma, \tilde{\Phi})) \). Since \( \tilde{\Phi}_i \) (\( i = 0, 1 \)) is free, we have \( \text{Hom}(\sigma, \tilde{\Phi}_i) = \text{id}_{\tilde{\Phi}} \otimes \sigma^\prime \). Hence we have \( \text{Hom}(\sigma, \tilde{\Phi}) = \text{id}_{\tilde{\Phi}} \otimes \sigma^\prime \).

The same argument implies \( \text{Hom}(\sigma^\prime, \tilde{\Phi}) = \text{id}_{\tilde{\Phi}} \otimes \sigma \).

We prove the second claim. We only prove the commutativity of the lower square. The same argument implies the proposition. An isomorphism \( \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \simeq \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \) is equal to

\[
\text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \simeq \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \simeq \text{Hom}(\tilde{\Phi}, \tilde{\Phi}(\tilde{\Phi})) = \text{Hom}(\tilde{\Phi}, \tilde{\Phi} \tilde{\Phi}).
\]

For \( f \in \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \), an image of \( f \) under \( \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \simeq \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \to \text{Hom}(\tilde{\Phi}, \tilde{\Phi}) \) is given by \( m \mapsto f(m \otimes (1)) \), namely, an image of \( f \) under the map \( \text{Hom}(\text{id}_{\tilde{\Phi}} \otimes \sigma, \tilde{\Phi}) \). We get the proposition from the first claim. \( \square \)
Theorem 3.18. Let $s, t$ be simple reflections. The functors $\tilde{\theta}_t$ and $\tilde{\varphi}_s$ from $\mathcal{O}$ to $\mathcal{O}$ commute with each other. Moreover, the natural transformation $\tilde{\theta}_t \rightarrow \tilde{\varphi}_s \tilde{\theta}_t$ (resp. $\tilde{\varphi}_s \rightarrow \tilde{\theta}_t \tilde{\varphi}_s$, $\tilde{\varphi}_s \tilde{\theta}_t \rightarrow \tilde{\theta}_t$, $\tilde{\theta}_t \tilde{\varphi}_s \rightarrow \tilde{\varphi}_s$) can be identified with $\tilde{\theta}_t(\text{Id} \rightarrow \tilde{\varphi}_s)$ (resp. $\tilde{\varphi}_s(\text{Id} \rightarrow \tilde{\theta}_t)$, $\tilde{\varphi}_s(\text{Id} \rightarrow \tilde{\varphi}_s)$, $\tilde{\theta}_t(\text{Id} \rightarrow \text{Id})$).

Proof. Since $\tilde{\varphi}_s(M) \simeq \tilde{M} \otimes \tilde{\varphi}_s(\tilde{A})$ and $\tilde{\theta}_t(M) \simeq \tilde{M} \otimes \tilde{\theta}_t(\tilde{A})$, we may assume that $\tilde{M} = \tilde{A}$. In this case, the theorem follows from the corresponding statement in $\mathcal{M}$, namely, Proposition 3.5.

3.5. Translation of projective modules and simple modules.

Theorem 3.19. \(\text{(1)}\) If $xs < x$, then $\tilde{\theta}_t \tilde{P}(x) = \tilde{P}(x)^{\oplus 2}$ and $\theta_t P(x) = P(x)^{\oplus 2}$.

\(\text{(2)}\) If $xs > x$, then $\tilde{\theta}_t \tilde{P}(x) = \tilde{P}(x) \oplus \bigoplus_{y < x, ys < y} \tilde{P}(y)^{m_y}$ and $\theta_t P(x) = P(x) \oplus \bigoplus_{y < x, ys < y} P(y)^{m_y}$ for some $m_y \in \mathbb{Z}_{\geq 0}$.

\(\text{(3)}\) $\tilde{\theta}_t L(x) = 0$ if and only if $xs > x$.

\(\text{(4)}\) If $sx < x$, then $\tilde{\varphi}_s \tilde{P}(x) = \tilde{P}(x)^{\oplus 2}$.

\(\text{(5)}\) If $sx > x$, then $\tilde{\varphi}_s \tilde{P}(x) = \tilde{P}(sx) \oplus \bigoplus_{y < x, sy < y} \tilde{P}(y)^{m_y}$ for some $m_y \in \mathbb{Z}_{\geq 0}$.

\(\text{(6)}\) $\varphi_s L(x) = 0$ if and only if $sx > x$.

Proof. The first statement of (1) and (2) follows from Proposition 3.7 and Proposition 3.14. We get the second statement of (1) (2) tensoring $\mathbb{C}$ to the first statement of (1) (2), respectively (see Remark 3.15).

From (1) and (2), we have $\theta_t A = \bigoplus_{ys < y} P(y)^{n_y}$ for some $n_y \geq 2$. Put $n_y = 0$ for $ys > y$. Then we have

$$\dim \theta_t L(x) = \dim \text{Hom}_A(\tilde{A}, \theta_t L(x)) = \dim \text{Hom}_A(\tilde{\varphi}_s \tilde{A}, L(x)) = \dim \text{Hom}_A\left(\bigoplus_y \tilde{P}(y)^{n_y}, L(x)\right) = n_y.$$ 

The proposition follows.

(4), (5) and (6) follow from the same argument. \qed

4. ZUCKERMANN FUNCTOR

4.1. Definition and commutativity with translation functors. Fix a simple reflection $s$. Let $\mathcal{O}_s$ be a full-subcategory of $\mathcal{O}$ consisting of a module $M$ such that $\text{Hom}_A(\tilde{P}(x), M) = 0$ for all $sx < x$. Let $\iota_s: \mathcal{O}_s \rightarrow \mathcal{O}$ be the inclusion functor. Then $\iota_s$ has the left adjoint functor $\tau_s$. It is defined by

$$\tau_s(M) = M/M'$$

where

$$M' = \bigcap_{\varphi: M \rightarrow M_1, M_1 \in \mathcal{O}_s} \text{Ker} \varphi.$$ 

Since $\tau_s$ has the right adjoint functor $\iota_s$, $\tilde{\tau}_s$ is a right exact functor. Put $\tau_s = \iota_s \tilde{\tau}_s$.

Lemma 4.1. Let $s$ be a simple reflection. For $M \in \mathcal{O}$, $M \in \mathcal{O}_s$ if and only if $\varphi_s M = 0$. In particular, $\theta_t$ preserves the category $\mathcal{O}_s$ for a simple reflection $t$.

Proof. From Theorem 3.19, we have $\tilde{\varphi}_s \tilde{A} = \bigoplus_{ys < y} \tilde{P}(y)^{m_y}$ for some $m_y \geq 2$. Hence, if $M \in \mathcal{O}_s$, then $\varphi_s M = \text{Hom}_A(\tilde{A}, \varphi_s M) = \text{Hom}_A(\tilde{\varphi}_s \tilde{A}, M) = 0$.

If $M \notin \mathcal{O}_s$, then $\text{Hom}(\tilde{P}(x), M) = \text{Hom}(P(x), M) \neq 0$ for some $x \in W$ such that $sx < x$. Hence $\text{Hom}(\tilde{P}(x), \varphi_s M) = \text{Hom}(\tilde{\varphi}_s \tilde{P}(x), M) = \text{Hom}(\tilde{P}(x)^{\oplus 2}, M) \neq 0$. Therefore, $\varphi_s M \neq 0$.

Take $M \in \mathcal{O}_s$. Then, by Theorem 3.18, $\varphi_s \theta_t M = \tilde{\theta}_t \varphi_s M = 0$. Hence $\theta_t M \in \mathcal{O}_s$. \qed
Proposition 4.2. The functors $\tau_s$ and $\theta_t$ commute with each other for simple reflections $s, t$.

Proof. From Lemma 4.1, the functor $\theta_t$ induces a self-adjoint functor from $O_s$ to $O_s$. We denote this functor by $\theta'_t$. Obviously, we have $\theta_t \theta'_t \simeq \theta'_t \theta_t$. Taking the left adjoint functor of the both sides, we get $\tau_s \theta_t \simeq \theta'_t \tau_s \simeq \tau_s \theta_t$. Hence we get $\theta_t \tau_s = \theta_t \theta'_t \tau_s \simeq \tau_s \theta_t$. $\square$

4.2. Translation of Verma modules. We consider $\varphi_s M(x)$. We start with two lemmas.

Lemma 4.3. Let $\{ M_\lambda \}$ be a family of $S(V^*)$-modules. Then we have an isomorphism $(\prod_\lambda M_\lambda) \otimes_{S(V^*)} C \simeq \prod_\lambda (M_\lambda \otimes_{S(V^*)} C)$.

Proof. Since $M \otimes_{S(V^*)} C = M/V^* M$ for an $S(V^*)$-module $M$, it is sufficient to prove that $V^*(\prod_\lambda M_\lambda) = \prod_\lambda (V^* M_\lambda)$. Notice that $V^*$ is finite-dimensional. Let $v_1, \ldots, v_r$ be a basis of $V^*$. Then $V^*(\prod_\lambda M_\lambda) = \sum_i v_i (\prod_\lambda M_\lambda) = \sum_i \prod_\lambda v_i M_\lambda = \prod_\lambda (V^* M_\lambda)$ $\square$

Lemma 4.4. Let $M_1 \to M_2 \to M_3$ be a sequence in $M$. If $\text{Hom}_Z(B(y), M_1) \otimes_{S(V^*)} C \to \text{Hom}_Z(B(y), M_2) \otimes_{S(V^*)} C \to \text{Hom}_Z(B(y), M_3) \otimes_{S(V^*)} C$ is exact for all $y$, then $\Phi(M_1) \to \Phi(M_2) \to \Phi(M_3)$ is exact.

Proof. From the previous lemma,

$$
\prod_{y \in W} (\text{Hom}_Z(B(y), M) \otimes_{S(V^*)} C) \simeq \prod_{y \in W} \text{Hom}_Z(B(y), M) \otimes_{S(V^*)} C
$$

$$
\simeq \text{Hom}_Z \left( \bigoplus_{y \in W} B(y), M \right) \otimes_{S(V^*)} C
$$

$$
= \Phi(M).
$$

We get the lemma. $\square$

Proposition 4.5. Let $s$ be a simple reflection and $x \in W$ such that $sx > x$.

1. We have an exact sequence $0 \to M(x) \to \Phi(\varphi_s^x V(sx)) \to M(sx) \to 0$, here the map $\Phi(\varphi_s^x V(sx)) \to M(sx)$ is the canonical map.
2. We have an exact sequence $0 \to M(x) \to \varphi_s M(sx) \to M(sx) \to 0$, here the map $\varphi_s M(sx) \to M(sx)$ is the canonical map.
3. We have an isomorphism $\tilde{\varphi}_s M(sx) \simeq \tilde{\varphi}_s M(x)$ and the map $M(x) \to \varphi_s M(sx)$ in (1) and $M(x) \to \tilde{\varphi}_s M(sx) \otimes_{S(V^*)} C$ is induced from the canonical map $\tilde{M}(x) \to \tilde{\varphi}_s M(sx)$.
4. For a $Z$-module $M$, the composition of the maps $\Phi(M) \to \varphi_s \Phi(M) \to \Phi(M)$ is equal to $0$.
5. We have an inclusion $M(sx) \to M(x)$.

Proof. Set $\alpha = \alpha_s$.

1. Put $\mathcal{M} = \mathcal{L}(\varphi_s V(sx))$. By [Fie08a, Lemma 5.4], we have

$$
\mathcal{M}_y = \begin{cases} 
S(V^*)\langle -1 \rangle & (y = x \text{ or } sx), \\
0 & (\text{otherwise}),
\end{cases}
$$

$$
\mathcal{M}_E = \begin{cases} 
S(V^*)/\alpha S(V^*)\langle -1 \rangle & (h_\varphi(E) = x, t_\varphi(E) = sx), \\
0 & (\text{otherwise}).
\end{cases}
$$
Hence we get an exact sequence $V(x)(-1) \to \varphi_s V(sx)(1) \to V(sx)$ (cf. [Fie08a, 3.4]). This implies an exact sequence

$$0 \to \text{Hom}_{\mathcal{Z}}(B(y), V(x)) \to \text{Hom}_{\mathcal{Z}}(B(y), \varphi_s^* V(sx)) \to \text{Hom}_{\mathcal{Z}}(B(y), V(sx)) \to 0$$

for all $y \in W$. Since $\text{Hom}_{\mathcal{Z}}(B(y), V(sx)) \simeq \text{Hom}_{S(V^*)}(B(y)s, S(V^*))$ and $\mathcal{B}(y)sx$ is free, we have that $\text{Hom}_{\mathcal{Z}}(B(y), V(sx))$ is free. Hence we get an exact sequence,

$$0 \to \text{Hom}_{\mathcal{Z}}(B(y), V(x)) \otimes_{S(V^*)} \mathcal{C} \to \text{Hom}_{\mathcal{Z}}(B(y), \varphi_s^* V(sx)) \otimes_{S(V^*)} \mathcal{C} \to \text{Hom}_{\mathcal{Z}}(B(y), V(sx)) \otimes_{S(V^*)} \mathcal{C} \to 0$$

for all $y \in W$. From the previous lemma, we get (1).

(2) For $\tilde{M} \in \hat{\mathcal{O}}$, we define a new $S(V^*)$-module structure on $\tilde{\varphi}_s(\tilde{M})$ as follows. The action of $p \in S(V^*)$ is given by $\varphi_s(p)$, here $p: \tilde{M} \to \tilde{M}$ is a $S(V^*)$-action on $\tilde{M}$. Then, in general, this action is different from the original $S(V^*)$-action (the action induced from the action of $\tilde{A}$). When we consider this $S(V^*)$-module structure, we denote $C(\tilde{\varphi}_s(\tilde{M}))$ instead of $\tilde{\varphi}_s(\tilde{M})$. By the definition, we get $C(\tilde{\varphi}_s(\tilde{M})) \otimes_{S(V^*)} \mathcal{C} = C(\tilde{\varphi}_s(\tilde{M} \otimes_{S(V^*)} \mathcal{C}))$. We define the $S(V^*)$-module structure on $\text{Hom}_{\mathcal{Z}}(B(y), \varphi_s^* V(sx))$ by the same way, and denote the resulting $S(V^*)$-module by $C^2(\text{Hom}_{\mathcal{Z}}(B(y), \varphi_s^* V(sx)))$. We have $C^2(\text{Hom}_{\mathcal{Z}}(B(y), \varphi_s^* V(sx))) = C(\tilde{\varphi}_s(\tilde{M}))$. Moreover, from the same argument in (1), we have an exact sequence

$$0 \to \text{Hom}_{\mathcal{Z}}(B(y), V(x)) \to C(\text{Hom}_{\mathcal{Z}}(B(y), \varphi_s^* V(sx))) \to \text{Hom}_{\mathcal{Z}}(B(y), V(sx)) \to 0$$

for all $x \in W$. Tensoring with $\mathcal{C}$, we get (2).

(3) Both $V(x)$ and $V(sx)$ are isomorphic to $S(V^*)$ as an $S(V^*)$-module. Let $z = (z_w)w \in Z \subset \bigoplus_{w \in W} S(V^*)$ and assume that $z \in Z^{r^*}$. Then we have $z_x = s(z_{sx})$. Hence the action of $z$ on $V(x)$ is given by the multiplication of $z_x$, while the action of $z$ on $V(sx)$ is given by the multiplication of $z_{sx} = s(z_x)$. Hence $S(V^*) \simeq V(x) \to V(sx) \simeq S(V^*)$ given by $p \mapsto s(p)$ is an isomorphism as $Z^{r^*}$-modules. Hence $\text{Res}_{Z^r, V(x)} \simeq \text{Res}_{Z^r, V(sx)}$. Therefore, $\varphi_s^* V(x) \simeq \varphi_s^* V(sx)$. Hence we get $\tilde{\varphi}_s \tilde{M}(x) \simeq \tilde{\varphi}_s \tilde{M}(sx)$. It is easy to see that the canonical map $M(x) \to \varphi_s M(x)$ is equal to the map we give in (1) and (2).

(4) The composition of the maps $M \to Z \otimes_{Z^r} M \to M$ is given by $m \mapsto 2am$. So the map $\text{Hom}_{\mathcal{Z}}(B, M) \to \text{Hom}_{\mathcal{Z}}(B, \varphi_s^* M) \to \text{Hom}_{\mathcal{Z}}(B, M)$ is given by $f \mapsto 2af$. If we tensor $\mathcal{C}$ over $S(V^*)$, this map becomes 0.

(5) This is a consequence of (1) and (4). \hfill \Box

4.3. Duality of Zuckerman functor.

**Lemma 4.6.** Let $f: M(s) \to M(e)$ be an injective map. Then we have $\tau_s(M(e)) = M(e)/f(M(s))$.

**Proof.** Put $M = \text{Ker}(M(e) \to \tau_s M(e))$. If $sx > x$, we have $\mathcal{B}(x)_e = \mathcal{B}(x)_s$ by Lemma 3.6 and [Fie08a, Lemma 5.4]. Hence

$$\text{rank} \text{Hom}_{\mathcal{Z}}(B(x), V(e)) = \text{rank} \text{Hom}_{S(V^*)}(\mathcal{B}(x)_e, S(V^*)) = \text{rank} \text{Hom}_{S(V^*)}(\mathcal{B}(x)_s, S(V^*)) = \text{rank} \text{Hom}_{\mathcal{Z}}(B(x), V(s)).$$

This implies $\text{dim} \text{Hom}_A(P(x), M(e)) = \text{dim} \text{Hom}_A(P(x), M(s))$. Therefore, we get $\text{Hom}_A(P(x), M(e)/f(M(s))) = 0$. Hence $M \subset f(M(s))$. Since $f(M(s)) \simeq M(s)$ has the unique irreducible quotient $L(s)$, we have $M = f(M(s))$. \hfill \Box

The module $\tau_s(A)$ is, of course, a right $A$-module. Using $A \simeq \text{End}_A(A, A) \to \text{End}_A(\tau_s(A), \tau_s(A))$, we also regard $\tau_s(A)$ as a left $A$-module. By the same argument, $\varphi_s(A)$ is a left $A$-module and right $\tilde{A}$-module.
Theorem 4.7. We have the following exact sequences, here all maps are canonical maps.

1. $0 \to A \to \varphi_s A \to A \to \tau_s A \to 0$ as left $A$- and right $\tilde{A}$-modules.
2. $0 \to A \to (\tilde{\varphi}_s \tilde{A}) \otimes_{S(V^*)} C \to A \to \tau_s A \to 0$ as left $\tilde{A}$- and right $A$-modules.

Proof. We only prove (1). The same argument implies (2).

We prove the exactness of $0 \to P(x) \to \varphi_s P(x) \to P(x) \to \tau_s P(x) \to 0$ by induction on $l(x)$.

First assume that $x = e$. Then $P(e) = M(e)$. By Proposition 4.5 (1) and (3), $0 \to M(e) \to \varphi_s M(e)$ is exact and its cokernel is isomorphic to $M(s)$. From Lemma 4.6, we have an exact sequence $0 \to M(s) \to M(e) \to \tau_s M(e) \to 0$. Hence $0 \to M(e) \to \varphi_s M(e) \to M(e) \to \tau_s M(e) \to 0$ is exact.

Assume that $x > e$ and take a simple reflection $t$ such that $xt < x$. Then by inductive hypothesis, the sequence $0 \to P(xt) \to \varphi_s P(xt) \to P(xt) \to \tau_s P(xt) \to 0$ is exact. By Theorem 3.18 and Proposition 4.2, we get the exact sequence $0 \to \theta_t P(xt) \to \varphi_s \theta_t P(xt) \to \theta_t P(xt) \to \tau_s \theta_t P(xt) \to 0$. Since $P(x)$ is a direct summand of $\theta_t P(xt)$, we get the theorem. \qed

Lemma 4.8. For $M \in \mathcal{O}$, we have the following.

1. We have $\varphi_s(M) \simeq M \otimes_A \varphi_s(A)$. Hence $\varphi_s(A)$ is a flat left $A$-module.
2. We have $\text{Hom}_A(\tilde{\varphi}_s(\tilde{A}) \otimes_{S(V^*)} C, M) \simeq \varphi_s(M)$. Hence $\tilde{\varphi}_s(\tilde{A}) \otimes_{S(V^*)} C$ is a projective right $A$-module.

Proof. (1) follows from Lemma 3.13. (2) is proved by the following equation:

$$\text{Hom}_A(\tilde{\varphi}_s(\tilde{A}) \otimes_{S(V^*)} C, M) = \text{Hom}_A(\tilde{\varphi}_s(\tilde{A}), M) \simeq \text{Hom}_A(\tilde{A}, \tilde{\varphi}_s M) \simeq \tilde{\varphi}_s(M) = \varphi_s(M)$$ \qed

Define a functor $\tau'_s : \mathcal{O} \to \mathcal{O}$ by $\tau'_s(M) = \text{Hom}_A(\tau_s(A), M)$. Since $\tau_s(M) \simeq M \otimes_A \tau_s(A)$, this functor is the right adjoint functor of $\tau_s$. Let $L\tau_s$ be the left derived functor of $\tau_s$, $R\tau'_s$ the right derived functor of $\tau'_s$. $D^b(\mathcal{O})$ the bounded derived category of $\mathcal{O}$.

Lemma 4.9. We have $R\tau'_s(A)[2] \simeq \tau_s(A)$ as $A$-bimodules.

Proof. We prove that $R^i\tau'_s(A) = 0$ for $i \neq 2$ and $R^2\tau'_s(A) = \tau_s(A)$. Let $k : D(\mathcal{O}) \to D(\overline{\mathcal{O}})$ be the functor induced from the inclusion functor $\mathcal{O} \to \overline{\mathcal{O}}$. It is sufficient to consider $k(R\tau'_s(A))$ since $k$ is an exact functor. We calculate $R\text{Hom}_A(\tau_s(A), M)$ using the projective resolution in Theorem 4.7 (2). (The reason why we calculate $k(R\tau'_s(A))$ is that a projective resolution in Theorem 4.7 is an exact sequence not of $A$-bimodules but of left $\tilde{A}$- and right $A$-modules.)

From Theorem 4.7 (2), $R\text{Hom}_A(\tau_s(A), A)$ is given by the complex

$$\cdots \to \text{Hom}_A(A, A) \to \text{Hom}_A(\tilde{\varphi}_s(\tilde{A}) \otimes_{S(V^*)} C, A) \to \text{Hom}_A(A, A) \to \cdots.$$ 

By Lemma 4.8, this complex is

$$\cdots \to A \to \varphi_s(A) \to A \to \cdots.$$ 

From Theorem 4.7 (1), this complex is equal to $\tau_s(A)[-2]$. \qed

Theorem 4.10. Let $s$ be a simple reflection.

1. We have $L^i\tau_s(M) = 0$ for $i > 2$ and $M \in \mathcal{O}$. Hence $L\tau_s$ gives a functor from $D^b(\mathcal{O})$ to $D^b(\mathcal{O})$.
2. The functor $L\tau_s[-1]$ is self-adjoint. More generally, for $M, N \in D^b(\mathcal{O})$, we have $R\text{Hom}(L\tau_s M[-1], N) = R\text{Hom}(M, L\tau_s N[-1]).$
Proof. Let \( k : D(\mathcal{O}) \to D(\tilde{\mathcal{O}}) \) be the functor induced from the inclusion functor \( \mathcal{O} \to \tilde{\mathcal{O}} \). We prove that \( H^i(k(L\tau_\alpha(M))) = 0 \) for \( i > 2 \). By Theorem 4.7 and isomorphism \( \tau_\alpha(M) \simeq \tau_\alpha(A) \otimes_A M, k(L\tau_\alpha(M)) \) is given by the complex \( (0 \to M \to M \otimes_A \varphi_\alpha(A) \to M \to 0) \). From this description, we get (1).

By the definition, \( \tau'_\alpha \) is the right adjoint functor of \( \tau_\alpha \). Hence we have an isomorphism \( R\text{Hom}(L\tau_\alpha(M), N) \simeq R\text{Hom}(M, R\tau'_\alpha(N)) \). To prove (2), it is sufficient to prove that \( R\tau'_\alpha[2] = L\tau_\alpha \). Since \( L\tau_\alpha(M) \simeq M \otimes_A^L \tau_\alpha(A) \), we have

\[
(L\tau_\alpha)^2(M) \simeq M \otimes_A^L \tau_\alpha(A) \otimes_A^L \tau_\alpha(A) \simeq M \otimes_A^L \tau_\alpha(\tau_\alpha(A))
\]

\[
\simeq M \otimes_A^L \tau_\alpha(\tau_\alpha(A)[2]) \to M \otimes_A^L A[2] = M[2],
\]

here the last map is induced from the adjointness of \( L\tau_\alpha \) and \( R\tau'_\alpha \). Hence using the adjointness again, we get the map \( L\tau_\alpha(M) \to R\tau'_\alpha[2] \). If \( A = M \), then this homomorphism is an isomorphism. For a general \( M \), taking a projective resolution, we can prove that the homomorphism is an isomorphism.

5. The functors \( T_s \) and \( C_s \)

5.1. Definition and adjointness. Let \( s \) be a simple reflection. Define a functor \( \tilde{T}_s : \mathcal{O} \to \tilde{\mathcal{O}} \) by \( \tilde{T}_s(M) = \text{Cok}(M \to \tilde{\varphi}_s(M)) \). The exactness of \( \tilde{\varphi}_s \) implies that \( \tilde{T}_s \) is right exact.

Lemma 5.1. For \( p \in S(V^*) \) and \( \tilde{M} \in \tilde{\mathcal{O}} \), we have \( s(p) = \tilde{T}_s(p) : \tilde{T}_s(M) \to \tilde{T}_s(\tilde{M}) \). In particular, we have \( \tilde{T}_s(\mathcal{O}) \subset \mathcal{O} \).

Proof. Since \( \tilde{T}_s \) is right exact, we have \( \tilde{T}_s(M) \simeq M \otimes_{\mathcal{A}} \tilde{T}_s(A) \). Hence we may assume that \( M = A \). Set \( B = \bigoplus_{y \in W} B(y) \). Then we have

\[
\varphi_s(A) = \text{Hom}_A(\Phi(\varphi_s^Z(B)), A)
\]

\[
= \text{Hom}_A(\text{Hom}_Z(B, \varphi_s^Z(B)), A)
\]

\[
\simeq \text{Hom}_A(\text{Hom}_Z(\varphi_s^Z(B), B), A)
\]

\[
= \text{Hom}_A(\text{Hom}_Z(Z \otimes_{Z^{\tau_s}} B, B), A).
\]

Take \( f \in \text{Hom}_Z(Z \otimes_{Z^{\tau_s}} B, B), z \in Z \) and \( b \in B \). Then \( p \in S(V^*) \) can acts on \( f \) by two ways. The first way is induced from the right \( \mathcal{A} \)-module structure, namely, \( f \mapsto ((z \otimes b) \mapsto f(z \otimes pb)) \), this induces a homomorphism \( p : \varphi_s(A) \to \varphi_s(A) \). The second way is induced from the left \( \mathcal{A} \)-module structure, namely, \( f \mapsto ((z \otimes b) \mapsto p(f(z \otimes b))) \), this induces a homomorphism \( \varphi_s(p) : \varphi_s(A) \to \varphi_s(A) \). We denote the first action by \( f \mapsto pf \) and section action by \( f \mapsto p \cdot f \). For \( p \in S(V^*) \subset Z \), we have \( r_s(p) = s(p) \). Hence if \( p \in S(V^*)^* \), then we have \( p \in Z^{\tau_s} \). So, in this case, we get \( pf = p \cdot f \). Hence \( p = \tilde{\varphi}_s(p) \). This implies \( p = \tilde{T}_s(p) \).

Set \( a = \alpha_s \). Since \( S(V^*) = S(V^*)^* \oplus \alpha S(V^*)^* \), it is sufficient to prove that \( T_s(\alpha) = -\alpha \). The natural transformation \( A \to \varphi_s(A) \) is induced from \( B \to Z \otimes_{Z^{\tau_s}} B \) and it is given by \( b \mapsto (\alpha \otimes b + 1 \otimes ab) \) (Remark 2.12). Hence \( A \simeq \text{Hom}_A(\tilde{T}_s(\tilde{M}), A) \) is given by

\[
a \mapsto (f \mapsto a(b \mapsto f(\alpha \otimes b + 1 \otimes ab))))
\]

where \( a \in A \simeq \text{Hom}_A(\text{Hom}_Z(Z \otimes_{Z^{\tau_s}} B, B), A) \) and \( f \in \text{Hom}(Z \otimes_{Z^{\tau_s}} B, B) \) and \( b \in B \).

Take \( a' \in \text{Hom}_A(\text{Hom}_Z(Z \otimes_{Z^{\tau_s}} B, B), A) \) and define \( a \in \text{Hom}_A(\text{Hom}_Z(B, B), A) \) by

\[
\text{Hom}_Z(B, B) \ni g \mapsto (a'(z \otimes b \mapsto g(zb))).
\]

Since \( B \to Z \otimes_{Z^{\tau_s}} B, b \mapsto (\alpha \otimes b + 1 \otimes ab) \) is a \( Z \)-module homomorphism, we have \( (\alpha \otimes zb + 1 \otimes ab) = (z \alpha \otimes b + z \otimes ab) \). Hence the image of \( a \) in \( \text{Hom}_A(\text{Hom}_Z(Z \otimes_{Z^{\tau_s}} B, B), A) \).
Let \( s \) be a simple reflection.

(1) We have \( L^iT_s = 0 \) for \( i > 1 \). Hence \( LT_s \) gives a functor \( D^b(O) \to D^b(O) \).

(2) We have a distinguished triangle \( LT_s \to \text{id} \to L\tau_s \) and \( \tau_s \to \text{id} \to RC_s \).

(3) We have \( R^iC_s = 0 \) for \( i > 4 \). Hence \( RC_s \) gives a functor \( D^b(O) \to D^b(O) \).

(4) We have a distinguished triangle \( L\tau_s[-2] \to \text{id} \to RC_s \).

(5) We have \( L^1T_sM = \text{Ker}(M \to \varphi_sM) \) and \( R^1C_sM = \text{Cok}(\varphi_sM \to M) \).

Proof. (1) follows from (2) and Theorem 4.10 (1). By Theorem 4.7, we have \( 0 \to T_s(A) \to A \to \tau_s(A) \to 0 \). Since \( T_s \) and \( \tau_s \) are right exact, we have \( T_s(M) = M \otimes_A T_s(A) \) and \( \tau_s(M) = M \otimes_A \tau_s(A) \). Hence (2) follows.

(3) follows from (4) and Theorem 4.10 (1). Since \( C_s \) is the right adjoint functor of \( T_s \), we have \( C_s(M) = \text{Hom}(A, C_s(M)) = \text{Hom}(T_s(A), M) \). Hence we have \( RC_s(M) = R\text{Hom}(T_s(A), M) \). By the exact sequence \( 0 \to T_s(A) \to A \to \tau_s(A) \to 0 \), we have a distinguished triangle \( R\text{Hom}(\tau_s(A), M) \to M \to RC_s(M) \).

We have \( R\text{Hom}(\tau_s(A), M) = R\text{Hom}(L\tau_s(A), M) = R\text{Hom}(A, L\tau_s(M)[-2]) = L\tau_s(M)[-2] \) by Theorem 4.10. Hence (4) follows. We prove (5). From (2) and
(4), we have $L^1T_s M = L^2\tau_s M = \text{Ker}(M \to C_s M) = \text{Ker}(M \to \varphi_s M)$. We also have $R^1C_s M = \tau_s M = \text{Cok}(T_s M \to M) = \text{Cok}(\varphi_s M \to M)$. □

Corollary 5.6. Assume that $(W, S)$ is the Weyl group of a semisimple Lie algebra $g$. From a result of Soergel [Soe90], the regular integral block of the BGG category $O^{BGG}$ of $g$ is equivalent to the category of finitely generated $A$-modules (Remark 3.9). We regard $O^{BGG}$ is a full subcategory of $O$. Then $T_s$ coincides with the twisting functor [Ark97] and $C_s$ coincides with the Joseph’s Enright functor [Jos82] on $O^{BGG}$.

Proof. Since $C_s$ is the right adjoint functor of $T_s$ (Theorem 5.3) and the Joseph’s Enright functor is the right adjoint functor of the twisting functor [KM05, Theorem 3], the statement for $C_s$ follows from that for $T_s$.

From Proposition 5.5 (2), for a projective object $P$, we have the following exact sequence:

$$0 \to T_s P \to P \to \tau_s P \to 0.$$  

The twisting functor $T'_s$ satisfies the same exact sequence [MS07, Proposition 2.4 (1)]. Hence $T_s P \simeq T'_s P$. Taking a projective resolution, we have $T_s M \simeq T'_s M$ for $M \in O'$. □

Proposition 5.7. Assume that $sx > x$. Then we have $T_s M(x) = M(sx)$ and $L^1T_s M(x) = 0$. Moreover, a natural transformation $M(sx) \to M(x)$ is injective.

Proof. This proposition follows from Lemma 4.6 and Proposition 5.5 (5). □

Proposition 5.8. We have

$$C_s M(x) = \begin{cases} M(sx) & (sx < x), \\ M(x) & (sx > x). \end{cases}$$

Proof. This proposition follows from Lemma 4.6. □

6. Homomorphisms between Verma modules

In this section, we prove the following theorem.

Theorem 6.1. We have

$$\text{Hom}(M(x), M(y)) = \begin{cases} \mathbb{C} & (y \leq x), \\ 0 & (y \nleq x). \end{cases}$$

Moreover, any nonzero homomorphism $M(x) \to M(y)$ is injective.

The surjective map $P(x) \to M(x)$ induces an injective map $\text{Hom}(M(x), M(y)) \to \text{Hom}(P(x), M(y))$. If $y \nleq x$, then

$$\text{Hom}(P(x), M(y)) = \text{Hom}(\Phi(B(x)), \Phi(V(y))) = \text{Hom}_Z(B(x), V(y)) \otimes_{S(V^*)} \mathbb{C} = \text{Hom}_{S(V^*)}(\mathcal{B}(x), S(V^*)) \otimes_{S(V^*)} \mathbb{C} = 0.$$  

Hence we get the theorem in the case of $y \nleq x$.

Next, we prove the ‘existence part’ of the theorem. Namely, we prove the following lemma.

Lemma 6.2. If $y \leq x$, then there exists an injective map $M(x) \to M(y)$.

If $x = sy$, this lemma follows from Proposition 5.7. Hence, to prove the lemma, it is sufficient to prove the following lemma (see the proof of [Dix96, 7.6.11. Lemma]).
Lemma 6.3. Let $s$ be a simple reflection and $x, y \in W$. Assume that there exists an injective map $f : M(x) \to M(y)$. If $sx > x$ then there exists an injective map $M(sx) \to M(sy)$.

Proof. By Proposition 5.7, there exists an injective map $M(sx) \to M(x)$. If $sy > y$, then there exists an injective map $M(y) \to M(sy)$. Hence the lemma follows.

We may assume that $sy < y$. By Proposition 5.7, we have $T_sM(x) = M(sx)$ and $T_sM(y) = M(sy)$. Hence we get the following diagram:

$$
\begin{array}{ccc}
M(x) & \xrightarrow{f} & M(y) \\
\uparrow & & \uparrow \\
M(sx) & \xrightarrow{T_sf} & M(sy).
\end{array}
$$

The vertical maps are the natural transformations and they are injective by Proposition 5.7. Hence $T_s f$ is injective.

To prove Theorem 6.1, it is sufficient to prove the following lemma.

Lemma 6.4. We have $\dim \text{Hom}(M(x), M(y)) \leq 1$.

Proof. We prove by induction on $\ell(x)$. If $x = e$, then $M(x) = M(e) = P(e) = \Phi(B(e))$. Hence we have

$$
\text{Hom}(M(e), M(y)) = \text{Hom}(\Phi(B(e)), \Phi(V(y))) = \text{Hom}_Z(B(e), V(y)) \otimes_{S(V)} \mathbb{C} = \text{Hom}_{S(V)}(\mathcal{B}(e)_y, V(y)) \otimes_{S(V)} \mathbb{C}.
$$

If $y \neq e$, then this space is zero. If $y = e$, then this space is $\mathbb{C}$.

Assume that $x \neq e$. Take a simple reflection $s$ such that $sx < x$. Then we have $M(x) = T_sM(sx)$ (Proposition 5.7). Since $C_s$ is the right adjoint functor of $T_s$, we have

$$
\text{Hom}(M(x), M(y)) = \text{Hom}(T_sM(sx), M(y)) = \text{Hom}(M(sx), C_s M(y)).
$$

If $sy > y$, then $C_s M(y) = M(sy)$. If $sy < y$, then $C_s M(y) = M(y)$ (Proposition 5.8). In each case, the dimension of this space is less than or equal to 1 by inductive hypothesis. \qed

7. More about the functors $T_s$ and $C_s$

Lemma 7.1. Let $s$ be a simple reflection and $x \in W$.

1. We have $L^1 T_s M(x) = 0$.
2. The natural transformation $M(x) \to RC_sLT_s M(x)$ is an isomorphism.

Proof. By Proposition 5.5 (5), we have $L^1 T_s M(x) = \text{Ker}(M(x) \to \varphi_s M(x))$. By Lemma 4.6, the last module is zero.

To prove (2), first we prove that $RC_sLT_s M(x) \simeq M(x)$. If $sx > x$, then $T_s M(x) = M(sx)$. Hence $C_s T_s M(x) = C_s M(sx) = M(x)$ by Proposition 5.8. By Proposition 5.5 (5) and Proposition 4.5, we have $R^1 C_s M(x) = \text{Cok}(\varphi_s M(sx) \to M(sx)) = 0$.

Next, assume that $sx < x$. First we prove that $R^1 C_s T_s M(x) = 0$. By Proposition 5.5 (4), we have $R^1 C_s T_s M(x) = \tau_x T_s M(x)$. To prove $\tau_x T_s M(x) = 0$, it is sufficient to prove that $\text{Hom}(T_s M(x), M) = 0$ for all $M \in \mathcal{O}_s$. Since $C_s$ is the right adjoint functor of $T_s$, we have $\text{Hom}(T_s M(x), M) = \text{Hom}(M(x), C_s M)$. By Lemma 4.1, we have $\varphi_s M = 0$. This implies $C_s M = 0$. Hence $\text{Hom}(T_s M(x), M) = 0$. \qed
Using the natural transformation $M(x) \simeq T_s M(sx) \to M(sx)$, we regard $M(x)$ as a submodule of $M(sx)$. By the definition of $T_s$ and Lemma 4.6, we have an exact sequence

$$0 \to M(sx)/M(x) \to T_s M(x) \to M(x) \to 0.$$  

Since $M(sx)/M(x) \in O_s$ (Lemma 4.6), $\varphi_s(M(sx)/M(x)) = 0$. From the definition of $C_s$ and Proposition 5.5 (5), $C_s(M(sx)/M(x)) = 0$ and $R^1C_s(M(sx)/M(x)) = M(sx)/M(x)$. Hence from the long exact sequence, we have

$$0 \to C_s T_s M(x) \to C_s M(x) \to M(sx)/M(x) \to 0.$$  

From Proposition 5.8, we have $C_s M(x) = M(sx)$. Hence $C_s T_s M(x) \simeq M(x)$.

Since $\text{End}(M(x)) = \mathbb{C}\text{id}$ by Theorem 6.1, the natural transformation $M(x) \to RC_s LT_s M(x)$ is zero or an isomorphism. Since this natural transformation comes from $id : T_s M(x) \to T_s M(x)$ and the adjointness, this is not zero.

\section*{Theorem 7.2.} The functor $LT_s$ gives an auto-equivalence of $D(O)$. Its quasi-inverse functor is $RC_s$.

\begin{proof}
We prove that the natural transformation $M \to RC_s LT_s M$ is an isomorphism for $M \in D(O)$. Taking a projective resolution, we may assume that $M$ is a projective module. Since a projective module has a filtration whose successive quotients are Verma modules, we may assume that $M$ is a Verma module. This is proved in the previous lemma.
\end{proof}

\section*{Theorem 7.3.} Let $w = s_1 \cdots s_t$ be a reduced expression of $w \in W$. Then $T_{s_1} \cdots T_{s_t}$ and $C_{s_1} \cdots C_{s_t}$ is independent of the choice of a reduced expression.

\begin{proof}
The statement for $C_s$ follows from the statement for $T_s$ (Theorem 5.3).

Put $F = T_{s_1} \cdots T_{s_t}$. Take another reduced expression $w = s_1' \cdots s_t'$ and put $G = T_{s_1'} \cdots T_{s_t'}$. We use (the dual of) the comparison lemma [KM05, Lemma 1]. Namely, for a projective module $P$, we prove the following statements.

1. The natural transformations $FP \to P$ and $GP \to P$ are injective.
2. $FP \cong GP$.
3. $\text{Im}(FP \to P) = \text{Im}(GP \to P)$.

We may assume $P = P(x)$ for some $x \in W$. We prove by induction on $\ell(x)$.

If $x = e$, then $P(x) = M(e)$. By Proposition 5.7, we have $FM(e) = GM(e) = M(w)$. Hence we get (2). We prove (1) by induction on $l$. Put $F' = T_{s_2} \cdots T_{s_t}$. The natural transformation $FP \to P$ is given by $FP = T_{s_1}F'P \to F'P \to P$.

The natural transformation $F'P \to P$ is injective by inductive hypothesis. Since $F'P = M(s_2 \cdots s_t)$, $T_{s_1}F'P \to F'P$ is injective (Proposition 5.7). Hence $FP \to P$ is injective. Since $\dim \text{Hom}(FM(e), M(e)) = \dim \text{Hom}(M(w), M(e)) = 1$ by Theorem 6.1, we get (3).

Assume that $x \neq e$ and take a simple reflection $t$ such that $xt < x$. Then $P = P(xt)$ satisfies (1–3). By Theorem 3.18, $T_s$ commutes with $\theta_e$. Hence $P = \theta_e P(xt)$ satisfies (1–3). Since $P(x)$ is a direct summand of $\theta_e P(xt)$, $P = P(x)$ satisfies (1–3).
\end{proof}

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