Event-triggered bipartite consensus for multiagent systems with general linear dynamics: An integral-type event-triggered control

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Abstract
We fill a gap in the proofs in the previous works (Wu X, Mu, X. Int J. Robust Nonlin Control. 2020;30:3753–3772; Zhang Z, Lunze J, Wang L. Int J Control. 2020;93:1005-1014; Zhang Z, Wang L. J Robust Nonlin Control. 2018;28:4175–4187; Dai, M-Z, Zhang C, Leung H, Dong P, Li B. IEEE Trans Syst, Man, Cybern: Syst. doi:10.1109/TSMC.2021.3119670) for the consensus using the integral-based event-triggered controls. More precisely, it was inferred for a Lyapunov function $V: [0, \infty) \to \mathbb{R}^+$ that $\dot{V}(t)$ is uniformly bounded by showing that $V(t)$ is uniformly bounded for $t \geq 0$. However, this argument may fail without further information while the boundedness of $\dot{V}(t)$ is crucially used for applying Barbalat’s lemma. The consequence of Barbalat’s lemma is that $\lim_{t \to \infty} V(t)$ which corresponds to the desired consensus result. To overcome this gap, Ma and Zhao (Inform Sci. 2018;457-458:208-221) put an extra condition about the boundedness of measurement error functions inside the proposed integral-based event-triggering protocol. In this article, we propose a new integral-based event-triggering protocol for bipartite consensus problems of the multi-agent systems whose dynamics are described by general linear systems without adding the uniform boundedness of measurement error functions as (Ma Y, Zhao J. Inform Sci. 2018;457-458:208-221) did. Via our new integral-based integral control strategy, we prove that the system achieves the bipartite consensus in asymptotic regime, and provide a complete solution of the freeness of both chattering and genuinely Zeno behaviors. Numerical results are provided supporting the effectiveness of the proposed controller.

KEYWORDS
bipartite consensus, integral-based event-triggered control, multi-agent systems with linear dynamics, Zeno behavior

1 | INTRODUCTION

The coordinated control of multi-agent systems (MASs) has received a lot of interest in the last decades, thanks to its wide applications in various fields containing engineering and computer science, biology, and social sciences. In particular, distributed controls for the coordinates of MASs over graphs have been studied intensively by variety of researchers.¹⁻⁸

Abbreviations: MIET, minimum inter-event time; MAS, multi-agent systems.
A fundamental problem in the coordinated control of multi-agent systems is the consensus problem which requires all agents to asymptotically achieve a common quantity during their cooperative interactions. The main task of the consensus problem in distributed MASs is to design the control so that all agents will obtain agreement using only interactions of neighbors.

Due to the limitation of sources in the MASs, we cannot assume that agents have continuous access to others’ states. Therefore, agents in the system should have strategy to take various actions in automatically schedules instead of doing so continuously. As an effective solution for the scheduling, event-triggered control designs have been developed for MASs. For event-triggered control systems, controller updates are activated only when a suitably designed event-triggering condition is satisfied. The event-triggered consensus problem of MASs is first studied by Dimarogonas, Frazzoli and Johansson and has been received tremendous attention after that. Numerous distributed event-triggered consensus protocols were introduced for MASs of first-order, of second-order, and of linear dynamics.

In certain practical problems, some agents collaborate while others are in another competitive group. These systems are represented by signed graphs and the weight of each edge will be positive/negative if the two agents are cooperative/competitive. The agents are said to exhibit a bipartite consensus if they reach agreement in modulus but not in sign. The distributed Laplacian-like control schemes were developed in Reference for the bipartite consensus of single-integrator agents, and extended to directed signed graph, general linear systems. Also, the bipartite consensus problems have been studied for general linear system with input-saturation and communication noises. The distributed control with event-triggered communications have been applied to the bipartite consensus problem for the single-integrator agents, the double-integrator systems, the general linear systems, and the heterogeneous systems. In addition, the distributed control with event-triggered communications were studied in the context of the bipartite consensus for the system with input time delay and the prescribed-time bipartite consensus problem. Bipartite consensus problems have been studied for applications such as opinion dynamics in social networks with hostile camps, quadrotor helicopters.

There are two common issues in designing the event-triggered controls for the consensus problems. One is to guarantee that the controlled system actually achieves the consensus in asymptotic regime or in a prescribed time. The other one is to find a lower bound for the difference between two consecutive triggering times. The larger the lower bound is, the better the system is in saving the energy for the communications. These two issues could be in conflict with each other, and so it is important to design a balanced controller satisfying both the two properties. In particular, it is a nontrivial issue to prove that the controlled system has a positive minimum inter-event time (MIET) between two consecutive triggering times, while the system achieves the consensus. The works of Garcia et al., Cheng and Ugrinovskii, Demir and Lunze designed event-triggered controllers which guarantee a positive MIET and the consensus is achieved up to a small error.

The idea of integral-based event-triggering condition yielding larger inter-event intervals was proposed by Mousavi, Ghodrat and Marquez. Later, this idea is applied to MASs in and it has been investigated further in References for the consensus problems using the integral-based event-triggered controls. In particular, Zhang, Lunze and Wang showed that the proposed controller system for the consensus problem has a positive MIET and the consensus is achieved asymptotically without a small error. However, there is a gap in the proofs of the consensus for the system in References as they applied Barbalat’s lemma without proving the uniform boundedness of the derivative of the consensus test function which is a nontrivial step. To overcome this gap, Ma and Zhao put an extra condition about the boundedness of measurement error functions inside their proposed integral-based event-triggering protocol. In this article, we propose a novel integral-based event triggering condition for our MASs with linear dynamics to solve the consensus problem without adding the extra boundedness condition as did. Furthermore, we establish a positive MIET for the system.

One more subtle issue in the event-triggered controls is the chattering Zeno problem, which means that where \( t_{k+1}^i = t_k^i \) is the \( k \)-th event-triggering instant of agent \( i \) (see (6)). To avoid this issue, we add an exponentially decaying term and show that it does not interrupt the system to achieve the bipartite asymptotic consensus. Our contributions are summarized as follows.

Statement of contributions:

1. We design a new integral-based event-triggered control for the bipartite consensus of general linear MASs and hence fill a gap in the proofs in the previous works for the consensus problems using the integral-based event-triggered controls. Our proposed protocol is even better than the integral-based event-triggered control introduced in as we do not need any assumption of uniformly boundedness of \( e(t) \), the error measurement function of the system, as added. The proof of the consensus uses a Lyapunov function \( V(t) \) associated with
function \( e(t) \) such that \( \|e(t)\|^2 \leq CV(t) \) holds with a constant \( C > 0 \) for all \( t \geq 0 \). The uniform boundedness of \( V(t) \) for \( t \geq 0 \) is crucial to apply Barbalat’s lemma to deduce that \( V(t) \) converges to zero as \( t \to \infty \). In the previous works, the boundedness of \( V(t) \) was deduced by the uniform boundedness of \( e(t) \) for \( t \geq 0 \). However, this argument may fail without further information. We provide a complete proof for the uniformly boundedness of \( e(t) \) (Lemma 5).

This fact is non-trivial and plays a crucial step to establish both the consensus and Zeno freeness of the systems. The assumption of uniformly boundedness of \( e(t) \) is added explicitly in Reference 42 or is implicitly used to apply Barbalat’s lemma in References 38–41.

2. Via a different approach, we extend the work of Yu et al.7 on Zeno-free analysis on event-triggered bipartite consensus for single-integrator multiagent systems to general linear dynamics. In addition to prove that the Zeno behavior is excluded, we obtain a uniform lower bound for the inter-event intervals for any agent.

Our article is organized as follows. In Section 2, we review our notations, basic definitions and properties of graphs, and present our problem formulation of bipartite event-triggered consensus of MASs with general linear dynamics. Our main results will be presented in Section 3. In Section 4.1, we will illustrate the efficiency and feasibility of our results by a numerical example. Then in Section 4.2, we compare the performance of our integral event-triggered control for bipartite problems with other types of the event-triggered controls. In Section 4.3, we present a numerical simulation of our method for a bipartite consensus problem for multi-agent vehicles. And in the last section, our conclusions are given.

## 2 PRELIMINARIES

### 2.1 Notations

Let \( \mathbb{R} \) be the set of all real numbers and \( \mathbb{N} \) be the set of all nonnegative integer numbers. We denote \( |s| \) the absolute value of a number \( s \in \mathbb{R} \). The Euclidean norm of a vector \( x \in \mathbb{R}^n \) is denoted by \( \|x\| \). We denote \( M_{m \times n}(\mathbb{R}) \) the set of all matrices of \( m \) rows and \( n \) columns with real-valued entries, and when \( m = n \) we write \( M_n(\mathbb{R}) \) instead. The transpose matrix of a matrix \( M \in M_{m \times n}(\mathbb{R}) \) is denoted by \( M^T \). Given \( M \in M_{m \times n}(\mathbb{R}) \), we denote by \( \|M\| \) its matrix operator norm. For every \( N \in \mathbb{N} \) we denote by \( 1_N \) the vector with \( 1 \) as entries. The sign function is denoted by \( \text{sgn}(\cdot) \). If \( A \in M_{m \times n}(\mathbb{R}) \) and \( B \in M_{p \times q}(\mathbb{R}) \), then \( A \otimes B \in M_{mp \times nq}(\mathbb{R}) \) is the Kronecker product.

### 2.2 Graphs

Let \( G = (\mathcal{V}, \mathcal{E}, \mathcal{W}) \) be a signed graph with a set of vertices \( \mathcal{V} = \{1, \ldots, N\} \), an edge set \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) and an adjacency matrix \( \mathcal{W} \in M_N(\mathbb{R}) \) of the signed weights of \( G \) with \( w_{ij} \neq 0 \) if \((j, i) \in \mathcal{E}\) and \( w_{ij} = 0 \) otherwise. We always assume that the graph has no self-loops, that is, \( w_{ii} = 0 \) for every \( i \in \mathcal{V} \). The Laplacian matrix \( L \) of \( G \) is defined by \( L = C - \mathcal{W} \), where

\[
C = \text{diag}\left\{ \sum_{j=1}^{N} |w_{1j}|, \sum_{j=1}^{N} |w_{2j}|, \ldots, \sum_{j=1}^{N} |w_{Nj}| \right\}.
\]

A connected signed graph \( G \) is \textit{structurally balanced} if there exists a bipartition \( \mathcal{V}_1, \mathcal{V}_2 \) of nodes such that \( w_{ij} \geq 0 \) for \( i, j \in \mathcal{V}_k, k \in \{1, 2\} \), and \( w_{ij} \leq 0 \) for \( i \in \mathcal{V}_k, j \in \mathcal{V}_l, l \neq k, l, k \in \{1, 2\} \).

**Definition 1** (43, Definition 6). For a connected graph \( G = (\mathcal{V}, \mathcal{E}, \mathcal{W}) \) with the Laplacian matrix \( L \), the general algebraic connectivity is defined by

\[
\alpha(L) = \min_{\psi(t) \neq 0, \psi(t) \neq \infty} \frac{\psi^T L \psi}{\psi^T \psi}.
\]

Furthermore, \( \alpha(L) = \lambda_2(L) \), where \( \lambda_2(L) \) is the smallest positive eigenvalue of the Laplacian matrix \( L \).

**Lemma 1** (4, Lemma 1). Let \( G = (\mathcal{V}, \mathcal{E}, \mathcal{W}) \) be a connected, structurally balanced signed graph. Then, there exists a matrix \( D = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_N) \) such that entries of \( DWD \) are all nonnegative, where \( \sigma_i \in \{1, -1\} \) for \( i \in \mathcal{V} \).
2.3 Problem formulation

Let \( A \in M_n(\mathbb{R}) \) and \( B \in M_{nxm}(\mathbb{R}) \). Consider a group of \( N \) agents with linear dynamics under the graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W}) \).

\[
\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i \in \mathcal{V} = \{1, \ldots, N\},
\]

where \( x_i(t) \in \mathbb{R}^n \) and \( u_i(t) \in \mathbb{R}^m \) are the state and the control input of agent \( i \), respectively. We assume that the signed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W}) \) is structurally balanced and connected.

The event-triggered consensus protocol is given as follows:

\[
u_i(t) = -K \sum_{j=1}^{N} |w_{ij}|(\dot{\bar{x}}_i(t) - \text{sgn}(w_{ij})\dot{\bar{x}}_j(t)), \quad i = 1, \ldots, N, \quad t \in [t_k^i, t_{k+1}^i) \text{ or } t \in [t_k^i, \infty) \text{ if } k = k_{\max}^i.
\]

where \( K \) is the control matrix which is designed later, \( \{t_k^{i,k_{\max}^i}_{k=0}\} \) is the non-decreasing sequence of triggering instants of agent \( i \) with the maximum number of triggering instants \( k_{\max}^i \) can be infinity, and \( \dot{\bar{x}}_i(t) := e^{(t-t_k^{i})A}x_i(t_k^i), a \in [t_k^i, t_{k+1}^i) \text{ or } t \in [t_k^i, \infty) \text{ if } k = k_{\max}^i \).

For every agent \( i \in \{1, \ldots, N\} \), we denote \( \Gamma_i \) the maximum time before which \( u_i(t) \) is well defined. From the boundedness, the forcing term \( Bu_i(t) \) (see Lemma 6), we get that the equation (1) always has solutions whenever \( u_i \) is defined.

Therefore, \( \Gamma_i \) is also the maximum time before which \( x_i(t) \) is well defined. Note that \( \Gamma_i \) can be infinity. We define \( \Gamma := \min(\Gamma_i : 1 \leq i \leq N) \), the maximum time before which all \( x_1(t), \ldots, x_N(t) \) are well defined.

We define the error measurements \( e_i(t) := \dot{\bar{x}}(t) - x_i(t) \).

\[
e_i(t) = [e_{i1}^T(t), \ldots, e_{iN}^T(t)]^T, x(t) = [x_1^T(t), \ldots, x_N^T(t)]^T, \dot{x}(t) = [\dot{x}_1^T(t), \ldots, \dot{x}_N^T(t)]^T.
\]

Using Kronecker product, we can write (1) as follows:

\[
\dot{x}(t) = (I_N \otimes A - L \otimes BK)x(t) - (L \otimes BK)e_x(t).
\]

For each agent \( 1 \leq i \leq N \), there are three possible cases for its triggering instants sequence \( \{t_k^{i,k_{\max}^i}_{k=0}\} \) and \( \Gamma_i \):

1. \( k_{\max}^i < \infty \) and \( \Gamma_i = \infty \);
2. \( k_{\max}^i = \infty \) and \( \Gamma_i = \lim_{k \to \infty} t_k^i < \infty \);
3. \( k_{\max}^i = \infty \) and \( \Gamma_i = \lim_{k \to \infty} t_k^i = \infty \).

If this case happens, then we say that the agent \( i \) has Zeno behavior.

There are two different types of Zeno behavior\(^{44}\) as follows. For an agent \( i \) which has Zeno behavior, agent \( i \) has:

1. Chattering Zeno if there exists \( M \in \mathbb{N} \) such that \( t_k^{i,k_{\max}^i} = t_k^i \) for every \( k \geq M \).
2. Genuinely Zeno if \( t_{k+1}^i - t_k^i > 0 \) for every \( k \in \mathbb{N} \).

Remark 1. To our knowledge, in the literature to prove the Zeno-freeness for multiagent systems, people often assumed in their arguments that for every \( k \in \mathbb{N} \), there exists \( t \in [t_k^i, t_{k+1}^i) \). This means they already assumed without proofs the freeness of chattering Zeno for the systems to show the exclusion of Zeno behavior. In the next section, we will present a triggered protocol excluding both chattering Zeno and genuinely Zeno behaviors for all agents of the system.

We say that the multi-agent system has Zeno behavior if there is an agent \( 1 \leq i \leq N \) having Zeno behavior. The system of agents is said to be bipartite consensus if \( \lim_{k \to \infty} \| \sigma_{x_i}(t) - \sigma_{x_j}(t) \| = 0 \) for every \( 1 \leq i, j \leq N \). We say that a multi-agent system has a positive minimum inter-event time if there exists \( \tau > 0 \) such that \( t_{k+1}^i - t_k^i \geq \tau \) for every \( i = 1, \ldots, N, 0 \leq k \leq k_{\max}^i \).

We need to find a triggering condition for each agent \( i \) such that the system achieves both bipartite consensus and Zeno-freeness.
3 | MAIN RESULTS

3.1 Bipartite consensus and the freeness of Zeno behavior

In this section, we will present an event-triggered design such that the system (1) achieves bipartite consensus. Let us introduce some intermediate variables.

We denote \( z_i(t) := \sigma_i x_i(t), \dot{z}_i(t) = \sigma_i \dot{x}_i(t), e_{z_i}(t) = z_i(t) - z_i(t), z(t) = [z_1^T(t), \ldots, z_N^T(t)]^T, \dot{z}(t) = [\dot{z}_1^T(t), \ldots, \dot{z}_N^T(t)]^T, e_z(t) = [e_{z_1}^T(t), \ldots, e_{z_N}^T(t)]^T. \)

We define a new Laplacian matrix \( L_D := DLD, \) where \( D \) is the matrix given in Lemma 1. Then from (3) we get

\[
\dot{z}(t) = (I_N \otimes A - L_D \otimes BK)z(t) - (L_D \otimes BK)e_z(t).
\]

Let \( h : [0, \infty) \rightarrow (0, \infty) \) be a continuous function such that \( C = \int_0^\infty h(s)ds < \infty. \) We define the triggering functions as

\[
f_i^k(t) := \int_{t_k^i}^t g_i(s)ds,
\]

where \( g_i(s) = ||e_{z_i}(s)||^2 - \beta_i \sum_{j=1}^N |w_{ij}|(\dot{z}_i(s) - \dot{z}_j(s))|^2 - \beta_i h(s) \) and \( \beta_i \) are constants which will be determined later.

The triggering instants for agent \( i \) are defined by

\[
t_{k+1}^i := \inf S_k^i,
\]

where \( S_k^i := \{ t > t_k^i : f_i^k(t) > 0 \}. \)

Remark 2. Now we discuss on a technical difficulty of proving the consensus when we use (6) instead of using the standard triggering condition \( s_k^i := \inf \{ s > s_k^i : g_i(s) > 0 \} \) (*). In (*), we can get the information that \( g_i(s) \leq 0 \) for every \( s \in [s_k^i, s_{k+1}^i] \) which helps us more useful properties of \( e_i(s) \) in this interval. From the protocol (6) we get lesser information of \( g_i(t) \) as well as \( e_i(t) \) on the interval \( [t_k^i, t_{k+1}^i] \). For example, the uniform boundedness of \( e_i(t) \), which is important to apply Barbalat’s lemma to prove bipartite consensus, is non-trivial if we use (5), see Lemma 5.

As we see from (6) that \( t_{k+1}^i \) is the infimum of the set \( \{ t > t_k^i : f_i^k(t) > 0 \} \) and hence it can coincide with \( t_k^i. \) In the literature, people often implicit assumed that \( t_{k+1}^i > t_k^i \) although the triggering instant \( t_{k+1}^i \) is defined as the infimum of a subset of the set \( \{ t > t_k^i \}. \) The following lemma shows that \( t_{k+1}^i \) is actually greater than \( t_k^i \), that is, the system does not have chattering Zeno behavior.

Lemma 2. For every \( i \in \{ 1, \ldots, N \} \) and \( 0 \leq k \leq k_{\text{max}} - 1 \) we have that \( t_{k+1}^i > t_k^i \).

Proof. For \( t \geq t_k^i \) we define \( g(t) := ||e_{z_i}(t)||^2 - \beta_i h(t). \) Then \( g(t_k^i) = -\beta_i h(t_k^i) < 0. \) As the continuity of \( g \) we get that there exists \( \varepsilon > 0 \) such that for every \( t \in \left[ t_k^i, t_k^i + \varepsilon \right] \) one has \( g(t) \leq -\frac{1}{2} \beta_i h(t_k^i) \). Hence, for every \( t \in \left[ t_k^i, t_k^i + \varepsilon \right] \) we get that

\[
f_k^i(t) = \int_{t_k^i}^t g_i(s)ds \\
\leq \int_{t_k^i}^{t_k^i + \varepsilon} g_i(s)ds \\
\leq -\frac{1}{2} \beta_i h(t_k^i) (t - t_k^i).
\]

This yields, \( f_k^i(t_k^i + \varepsilon) \leq -\frac{1}{2} \beta_i h(t_k^i) \varepsilon < 0. \) By the definition of \( t_{k+1}^i \), we obtain that \( t_{k+1}^i > t_k^i. \) \( \blacksquare \)

Now we provide some standard assumptions on the matrices \( A, B \) and the signed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, W) \).
**Assumption 1.** The pair \((A, B)\) stabilizable, i.e there exists \(K_1 \in M_n(\mathbb{R})\) such that all eigenvalues of \(A + K_1B\) has strictly negative real part. We also assume that the graph \(\mathcal{G}\) is structurally balanced and connected.

As \((A, B)\) is stabilizable, applying Theorem 9.5, there exists a positive matrix \(P \in M_n(\mathbb{R})\) and \(\kappa > 0\) such that

\[
A^TP + PA - \sigma(L)PBB^TP < -\kappa I_n. \tag{7}
\]

We choose the control matrix \(K := B^TP\) in (2). We define \(\bar{z}(t) := \frac{1}{N} \sum_{i=1}^{N} z_i(t)\). Then \(\dot{\bar{z}}(t) = A\bar{z}(t)\). To show this, we observe that

\[
\dot{z}_i(t) = \sigma_i \dot{x}_i(t)
= \sigma_i A x_i(t) + \sigma_i B u_i(t)
= A z_i(t) - \sigma_i B K \sum_{j=1}^{N} |w_{ij}| (\dot{x}_i(t) - \text{sgn}(w_{ij}) \dot{x}_j(t))
= A z_i(t) - \sigma_i B K \sum_{j=1}^{N} |w_{ij}| (\dot{x}_i(t) - \dot{x}_j(t)).
\]

Summing up this for \(1 \leq i \leq N\) we find \(\dot{\bar{z}}(t) = A\bar{z}(t)\).

Next we note that the difference \(e_i(t) = \dot{x}_i(t) - x_i(t)\) satisfies

\[
\dot{e}_i(t) = A(\dot{x}_i(t) - x_i(t)) - B u_i(t) = A e_i(t) - B u_i(t).
\]

Since \(e_{\bar{z},i}(t) = \sigma_i e_i(t)\) we have

\[
\dot{e}_{\bar{z},i}(t) = \sigma_i \dot{e}_i(t)
= \sigma_i (A e_i(t) - B u_i(t))
= A e_{\bar{z},i}(t) - \sigma_i B u_i(t). \tag{8}
\]

We also note that

\[
u_i(t) = -K \sum_{j=1}^{N} |w_{ij}| (\sigma_i \dot{z}_i(t) - \sigma_j \dot{z}_j(t))
= -K \sigma_i \sum_{j=1}^{N} |w_{ij}| (\dot{z}_i(t) - \dot{z}_j(t)).
\]

Inserting this to (8) we find

\[
\dot{e}_{\bar{z},i}(t) = A e_{\bar{z},i}(t) + K \sum_{j=1}^{N} |w_{ij}| (\dot{z}_i(t) - \dot{z}_j(t)). \tag{9}
\]

We set \(\delta_i(t) = z_i(t) - \bar{z}(t)\) and \(\delta(t) = [\delta_1(t)^T, \ldots, \delta_N(t)^T]^T\). Also we define the Lyapunov function \(V(t) := \delta^T(t)(I_N \otimes P)\delta(t)\). Recall that \(\Gamma_i\) is the maximum time before which \(x_i(t)\) is well defined, and \(\Gamma := \min\{\Gamma_i : 1 \leq i \leq N\}\) is the maximum time before which all \(x_1(t), \ldots, x_N(t)\) are well defined. It is non-trivial to get that \(\Gamma = \infty\), and we will show this fact later on Theorem 1.
Lemma 3. Let $c > 0$ such that $\beta := \kappa - c\|L_D \otimes PBB^TP\|^2 > 0$. Put $\beta_{\max} := \max\{\beta_1, \ldots, \beta_N\}$. Assume that $\beta_{\max} < \min\left\{\frac{1}{2\|L_D\|^2}, \frac{\beta c}{2\|L_D\|^2(\beta c + 1)}\right\}$. Then for every $t \in [0, \Gamma)$ we get that

$$V(t) - V(0) \leq -\beta\int_0^t \|\delta(s)\|^2ds + \frac{1}{c}\int_0^t \|e_z(s)\|^2ds$$

$$= -c_1\int_0^t \|\delta(s)\|^2ds + c_2,$$

where $c_1 = \beta - \frac{2\beta_{\max}\|L_D\|^2}{c(1 - 2\beta_{\max}\|L_D\|^2)} > 0$ and $c_2 = \frac{\sum_{i=1}^N\beta_{\max}C_{ij}}{c(1 - 2\beta_{\max}\|L_D\|^2)}$ with $C = \int_0^\infty h(s)ds < \infty$.

Proof. We have

$$\dot{V}(t) = \dot{\delta}^T(t)(I_N \otimes A^T - L_D \otimes KTB^T)(I_N \otimes P)\delta(t) - \dot{e}_z(t)\dot{\delta}^T(t)(I_N \otimes P)\delta(t)$$

$$+ \delta^T(t)(I_N \otimes P)((I_N \otimes A - L \otimes BK)\delta(t) - (L_D \otimes BK)e_z(t))$$

$$= \delta^T(t)(I_N \otimes (A^TP + PA))\delta(t) - 2\delta^T(t)(L_D \otimes PBB^TP)\delta(t)$$

$$- 2\dot{\delta}^T(t)(L_D \otimes PBB^TP)e_z(t).$$

From Definition 1 we get that

$$\delta^T(t)(L_D \otimes PBB^TP)\delta(t) \geq \alpha(L_D)\delta^T(t)(I_N \otimes PBB^TP)\delta(t).$$

Therefore

$$\dot{V}(t) \leq \delta^T(t)(I_N \otimes (A^TP + PA - \alpha(L)PBB^TP))\delta(t)$$

$$- 2\delta^T(t)(L_D \otimes PBB^TP)e_z(t)$$

$$\leq -\delta^T(t)(I_N \otimes I_N)\delta(t) - 2\delta^T(t)(L_D \otimes PBB^TP)e_z(t)$$

$$\leq -\kappa\delta^T(t)\delta(t) - 2\delta^T(t)(L_D \otimes PBB^TP)e_z(t).$$

On the other hand,

$$- 2\delta^T(t)(L_D \otimes PBB^TP)e_z(t)$$

$$\leq c\delta^T(t)(L_D \otimes PBB^TP)[\delta^T(t)(L_D \otimes PBB^TP)\delta(t)]$$

$$\leq c\|L_D \otimes PBB^TP\|^2\delta^T(t)\delta(t) + \frac{\|e_z(t)\|^2}{c}.$$  

Hence

$$\dot{V}(t) \leq -\beta\|\delta(t)\|^2 + \frac{\|e_z(t)\|^2}{c}.$$  

Therefore

$$V(t) - V(0) \leq -\beta\int_0^t \|\delta(s)\|^2ds + \frac{1}{c}\int_0^t \|e_z(s)\|^2ds.  \quad (10)$$

From the event triggered conditions (5) and (6), for every $i = 1, \ldots, N$, $k \in \mathbb{N}$ and $t \in [0, \Gamma)$, there exists $k \in \mathbb{N}$ such that $t \geq t_k^i$ and

$$\int_{t_k^i}^t \|e_z(s)\|^2ds \leq \beta_k\int_{t_k^i}^t \left(\sum_{j=1}^N w_{ij}(\hat{z}_j(s) - \tilde{z}_j(s))\right)^2ds + \beta_k\int_{t_k^i}^t h(s)ds.$$
Hence for every $t \in [0, \Gamma)$ we have
\[
\int_0^t \|e_x(s)\|^2 ds \leq \beta_i \int_0^t \left( \| \sum_{j=1}^N w_j(z_j(s) - z_j(s) + e_{z_j}(s) - e_{z_j}(s)) \| ds + \beta_i \int_0^1 h(s) ds \right.
\]
\[
\left. \leq \beta_i \int_0^t \left( \| \sum_{j=1}^N w_j(z_j(s) - z_j(s) + e_{z_j}(s) - e_{z_j}(s)) \| ds + \beta_i \int_0^{\infty} h(s) ds. \right) \right. \]

Let $L_{D,i}$ be the $i$th row of $L_D$ then for every $t \geq 0$ we get
\[
\int_0^t \|e_{z_i}(s)\|^2 ds \leq 2\beta_i \int_0^t (\|L_{D,i} \otimes I_n\|z(s))^2 + \|(L_D \otimes I_n)e_{z}(s)\|^2 ds + \beta_i C.
\]

Therefore, we obtain
\[
\int_0^t \|e_{z_i}(s)\|^2 ds \leq 2\beta_{\text{max}} \int_0^t (\|L_D \otimes I_n\|z(s))^2 + \|(L_D \otimes I_n)e_{z}(s)\|^2 ds + N \beta_{\text{max}} C.
\]

As $(L_D \otimes I_n)\delta(t) = (L_D \otimes I_n)\zeta(t)$, combining with $\beta_{\text{max}} < \frac{1}{2\|L_D\|^2}$ we obtain
\[
\int_0^t \|e_{z_i}(s)\|^2 ds \leq \frac{2\beta_{\text{max}}\|L_D\|^2}{1 - 2\beta_{\text{max}}\|L_D\|^2} \int_0^t \|\delta(s)\|^2 ds + \frac{N \beta_{\text{max}} C}{1 - 2\beta_{\text{max}}\|L_D\|^2}. \tag{11}
\]

Combining (10) with (11) we get that
\[
V(t) - V(0) \leq -\beta \int_0^t \|\delta(s)\|^2 ds + \frac{N \beta_{\text{max}} C}{c(1 - 2\beta_{\text{max}}\|L_D\|^2)}
\]
\[
+ \frac{2\beta_{\text{max}}\|L_D\|^2}{c(1 - 2\beta_{\text{max}}\|L_D\|^2)} \int_0^t \|\delta(s)\|^2 ds
\]
\[
= -c_1 \int_0^t \|\delta(s)\|^2 ds + c_2.
\]

The proof is done. 

\textbf{Lemma 4.} Suppose that $\dot{y}(t) = Ay(t)$ for $t \geq t_k$. Then we have
\[
\|y(T)\|^2_2 \leq \|y(t'_k)\|^2_2 + 2\|A\| \int_{t'_k}^T \|y(t)\|^2_2 dt
\]
for all $T \geq t'_k$, where $\|A\| := \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\|Av\|}{\|v\|}$.

\textbf{Proof.} We find
\[
\frac{d}{dt} \|y(t)\|^2 = 2 \langle y'(t), y(t) \rangle
\]
\[
= 2 \langle Ay(t), y(t) \rangle
\]
\[
\leq 2 \|A\| \|y(t)\|^2.
\]

Integrating this inequality over $[t'_k, T]$ we obtain
\[
\int_{t'_k}^T \frac{d}{dt} \|y(t)\|^2 dt \leq \int_{t'_k}^T 2 \|A\| \|y(t)\|^2 dt.
\]

This completes the proof. \hfill \blacksquare
Lemma 5. For every $i$ the function $\|e_{z_i}(t)\| = \|\dot{z}_i(t) - z_i(t)\|$ is uniformly bounded for $t \in [0, \Gamma)$.

Proof. From Lemma 3 we get that $V(t) \leq V(0) + c_2$ for every $t \in [0, \Gamma)$ and hence $\delta(t)$ is uniformly bounded on $t \in [0, \Gamma)$. Therefore, there exists $M > 0$ such that $\|z_i(t) - \bar{z}(t)\| \leq M$ for every $t \in [0, \Gamma)$ and every $i = 1, \ldots, N$. Therefore to prove $\|\dot{z}_i(t) - z_i(t)\|$ is bounded, it suffices to show that $\|\dot{z}_i(t) - \bar{z}(t)\|$ is bounded.

Since $P$ is a positive definite matrix, so is $IN \otimes P$. Hence, $V(t) = \delta^2(t)(IN \otimes P)\delta(t) \geq 0$. Therefore, from Lemma 3 we get that

$$\int_0^t \|\delta(s)\|^2 ds \leq \frac{V(0) + c_2}{c_1}. \tag{12}$$

Since (12) and (11), for every $i \in \{1, \ldots, N\}$, we obtain that

$$\int_0^t \|e_{z_i}(s)\|^2 ds \leq \int_0^t \|e_{z_i}(s)\|^2 ds \leq c_3, \tag{13}$$

where $c_3 := (2\beta_{\max}[L_P]_2^2(V(0) + c_2))/(c_1 - 2\beta_{\max}[L_P]_2^2) + c_2$.

Furthermore, for every $t \in [0, \Gamma)$ we have that

$$\int_0^t \|\dot{z}(s) - \bar{z}(s)\|^2 \leq 2 \int_0^t \|\dot{z}(s) - z(s)\|^2 ds + 2 \int_0^t \|z(s) - \bar{z}(s)\|^2 ds$$

$$= 2 \int_0^t \|e_{z_i}(s)\|^2 ds + 2 \int_0^t \|\delta(s)\|^2 ds.$$  

Using the bounds (12) and (13) for every $t \in [0, \Gamma)$, we obtain that

$$\int_0^t \|\dot{z}_i(s) - \bar{z}(s)\|^2 \leq 2 \left( c_3 + \frac{V(0) + c_2}{c_1} \right). \tag{14}$$

Assume that $\|\dot{z}_i(t) - \bar{z}(t)\|$ is not bounded on $t \in [0, \Gamma)$. Then there exists $M_1 > M^2 + 4\|A\|c_3 + (V(0) + c_2)/c_1$ such that $\|\dot{z}_i(T) - \bar{z}(T)\| > M_1$ for some $T \in [0, \Gamma)$. Choose $k \in \mathbb{N}$ such that $T \geq t_k$. Next, we define $y(t) := \dot{z}_i(t) - \bar{z}(t)$. Then we observe that $\bar{z}(t) = A\bar{z}(t)$ by averaging (4) and $\dot{z}_i(t) = A\dot{z}_i(t)$ due to the definition of $\dot{z}_i(t)$. Therefore we have $\dot{y}(t) = Ay(t)$, and so we may apply by Lemma 4 and (14) to deduce

$$M_1 < \|\dot{z}_i(T) - \bar{z}(T)\|^2$$

$$\leq \|\dot{z}_i(t_k^i) - \bar{z}(t_k^i)\|^2 + 2\|A\| \int_{t_k^i}^T \|\dot{z}_i(t) - \bar{z}(t)\|^2 dt$$

$$= \|z(t_k^i) - \bar{z}(t_k^i)\|^2 + 2\|A\| \int_{t_k^i}^T \|\dot{z}_i(t) - \bar{z}(t)\|^2 dt$$

$$\leq M^2 + 4\|A\| \left( c_3 + \frac{V(0) + c_2}{c_1} \right)$$

$$< M_1.$$

This is a contradiction. Hence, we get that $\|e_{z_i}(t)\|$ is bounded on $[0, \Gamma)$ for every $i = 1, \ldots, N$.  

We remark that the solution $x_i(t)$ to the linear ordinary differential equation (1) does not blow up in finite time if we have a uniform bound of the forcing term $Bu_i(t)$ for $t \geq 0$. Using the bounds of Lemma 3 and Lemma 5, we find the boundedness of $Bu_i(t)$ in the following lemma.

Lemma 6. The control $u_i(t)$ is uniformly bounded for $t \in [0, \Gamma)$ and $1 \leq i \leq N$. 

Proof. Using the definition (5) of \( u_i(t) \) and the triangle inequality, we get

\[
|u_i(t)| \leq K \sum_{j=1}^{N} |w_{ij}| |\dot{z}_i(t) - \dot{z}_j(t)|
\]

\[
\leq K \sum_{j=1}^{N} |w_{ij}| (|\dot{z}_i(t) - z_j(t)| + |\dot{z}_i(t) - z_i(t)| + |z_i(t) - z_j(t)|).
\]

(15)

Also we have \(|z_i(t) - z_j(t)| \leq |z_i(t) - \overline{z}(t)| + |\overline{z}(t) - z_j(t)| \leq 2M\) for the universal upper bound \(M > 0\) of \(\|z_i(t) - \overline{z}(t)\| \leq M\) defined in the proof of Lemma 5 for \(t \in [0, \Gamma)\) and \(1 \leq i \leq N\). Combining this bound and the uniform bound of Lemma 5 for the term \(\|e_{xi}(t)\| = \|\dot{z}_i(t) - z_i(t)\|\) in (15), we find that \(u_i(t)\) is uniformly bounded for \(t \in [0, \Gamma)\) for \(1 \leq i \leq N\).

\[
\text{Theorem 1. Let a multi-agent system as (1) such that the pair } (A, B) \text{ is stabilizable, and the graph } G = (V, E, W) \text{ is connected and structurally balanced. Let } P \in M_n(\mathbb{R}) \text{ be a positive matrix being a solution of } (7). \text{ We consider the event-triggered consensus protocol as (2) with the control matrix } K = B^TP. \text{ The event triggered times are determined by (5) and (6). Then}
\]

1. The multi-agent system excludes Zeno behavior. Furthermore, if \(k_{\max}^i \geq 1\) for every \(1 \leq i \leq N\) then the system has a positive minimum inter-event time, that is, there exists \(\tau > 0\) such that \(t_{k+1}^i - t_k^i \geq \tau\) for every \(i = 1, \ldots, N, 0 \leq k < k_{\max}^i\). More precisely,

\[
t_{k+1}^i - t_k^i \geq \log(1 + \beta_i(2\|A\| + \|BK\|^2))
\]

for all \(1 \leq i \leq N\), and \(0 \leq k < k_{\max}^i\).

2. \(\Gamma = \infty\) and the system achieves bipartite consensus.

Proof.

1. When \(k_{\max}^i = 0\) then there is no more triggering instant for the agent \(i\) except the initial triggering instant \(t_0^i\). Hence the agent \(i\) does not have Zeno behavior.

Now we consider the case \(k_{\max}^i \geq 1\) for every \(1 \leq i \leq N\). We note that

\[
\frac{d}{dt} \|e_{xi}(t)\|^2 = 2\langle e_{xi}(t), \dot{e}_{xi}(t) \rangle
\]

\[
\leq 2\|e_{xi}(t)\|\|\dot{e}_{xi}(t)\|
\]

\[
\leq 2\|BK\|\|e_{xi}(t)\| \left( \sum_{j=1}^{N} |w_{ij}| (|\dot{z}_i(t) - \dot{z}_j(t)| + |\dot{z}_i(t) - z_i(t)| + |z_i(t) - z_j(t)|) \right)
\]

\[
+ 2\|A\|\|e_{xi}(t)\|^2
\]

\[
\leq \left( \sum_{j=1}^{N} |w_{ij}| (|\dot{z}_i(t) - \dot{z}_j(t)| + |\dot{z}_i(t) - z_i(t)| + |z_i(t) - z_j(t)|) \right)^2
\]

\[
+ (2\|A\| + \|BK\|^2)\|e_{xi}(t)\|^2,
\]

where we used (9) in the second inequality. Given \(k \in \mathbb{N}\) and \(t \geq t_k^i\), we then have

\[
\|e_{xi}(t)\|^2 \leq \int_{t_k^i}^{t} e^{(2\|A\| + \|BK\|^2)(t-s)} \left( \sum_{j=1}^{N} |w_{ij}| (|\dot{z}_i(s) - \dot{z}_j(s)| + |\dot{z}_i(s) - z_i(s)| + |z_i(s) - z_j(s)|) \right)^2 ds
\]

\[
\leq e^{(2\|A\| + \|BK\|^2)(t-t_k^i)} \int_{t_k^i}^{t} \left( \sum_{j=1}^{N} |w_{ij}| (|\dot{z}_i(s) - \dot{z}_j(s)| + |\dot{z}_i(s) - z_i(s)| + |z_i(s) - z_j(s)|) \right)^2 ds.
\]
In this section, we present a numerical simulation of the bipartite consensus with event-triggered communication given by (6).

For the numerical simulation, the state error \( e_{C_i}(t) = \sigma_i \delta_i(t) - \sigma_j \chi_j(t) \) and the integration (5) of \( g_i(s) \) should be calculated for computing the event-triggering instant \( t_{k+1}^i \) defined in (6), where \( g_i(s) = \|e_{C_i}(s)\|^2 - \beta_1 \sum_{j=1}^N \|w_{ij}(\hat{z}_i(s) - \hat{z}_j(s))\|^2 \). In a real-application, the value can be checked only for digitalized values of the time \( t \in \mathbb{R} \). For example, we may consider a small sampling time \( h > 0 \) such that each agent checks the event-triggering condition at times of multiples of \( h \), that is, \( \{hk \in \mathbb{R} : k \in \mathbb{Z} \} \). Now we consider the event-triggering instants \( t_k^i = ha \) and \( t_{k+1}^i = hb \) for some integers.
a < b. Then, for \( t = hw \) with \( a < w < b \), we approximate the integral value \( f^k_i(t) \) using the quadrature method as follows:

\[
f^k_i(t) = \int_{t^k_i}^t g_i(s) ds \approx \sum_{l=a}^{w-1} h g_i(t^k_i + hl).
\] (17)

For \( a < w < b - 1 \), we have the identity \( f^k_i(hw + 1) = f^k_i(hw) + h g_i(t^k_i + hw) \). Using this, we may compute the value \( f^k_i(hw + 1) \) at time \( t = hw + 1 \) efficiently using the integral value \( f^k_i(hw) \) at the previous time instant \( t = hw \).

### 4.1 Basic example

We consider six agents and take the graph \( L \) defined in Reference 7 as follows:

\[
L = \begin{pmatrix}
3 & -1 & 2 & 0 & 0 & 0 \\
-1 & 5 & 4 & 0 & 0 & 0 \\
2 & 4 & 8 & -2 & 0 & 0 \\
0 & 0 & -2 & 6 & -1 & -3 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & -3 & 0 & 3
\end{pmatrix}
\]

We take \( A \) and \( B \) in\(^{22} \) given as

\[
A = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}.
\]

The matrix \( P \) satisfying (7) is computed as

\[
P = 5 \times \begin{pmatrix}
0.9862 & 0.0143 \\
0.0143 & 0.9212
\end{pmatrix}.
\]

The initial coordinates of agent 1 and agent 2 are selected randomly by a uniform distribution of interval \([0, 1/20]\) while those of other agents are selected randomly by a uniform distribution of interval \([-1/20, 0]\). We set the constant \( \beta_k = 0.008 \) for \( 1 \leq k \leq 6 \) and \( h(t) = 5/4e^{-t} \).

In Figures 1 we see that agent 1 and agent 2 converge to a same trajectory and the other agents converge to another same trajectory. For each \( 1 \leq k \) we let \( I_{ev}(k) \) be the minimum interval between two successive event-triggering times of agent \( k \). The values of \( I_{ev}(k) \) are given in Table 1.

Figure 2 indicates the event-triggering times of each agent. The interval lengths between two successive event-triggering times of each agent are given in Figures 3. The ranges of y-axis in the diagrams are fixed by \([0, S I_{ev}(k)]\) for each agent \( 1 \leq k \leq 6 \).

### 4.2 Comparison

Next we compare the performance of the integral event-triggered control (IETC) for bipartitle problems with other types of the event-triggered controls.

For notional simplicity, we set

\[
q_i(t) = \| \sum_{j=1}^N |w_{ij}|(\dot{z}_i(s) - \dot{z}_j(s)) \|.
\] (18)
TABLE 1 Minimum interval between triggering times.

| k | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $I_{kv}(k)(s)$ | 0.031 | 0.037 | 0.022 | 0.025 | 0.031 | 0.036 |

1. Integral event-triggered control with additional condition (IETCa) following the strategy of the work. \(^{42}\)

$$F_{k}^{i}(t) := \int_{t_{k}^{i}}^{t} G_{i}(s) ds,$$

(19)

where $G_{i}(s) = ||e_{z,i}(s)||^{2} - \beta_{i}||q_{i}(s)||^{2}$ and $Q_{j}(t) = ||e_{z,j}(s)||^{2} - \bar{w}_{i}$. The triggering instants for agent $i$ are defined by

$$t_{k+1}^{i} := \inf T_{k}^{i},$$

(20)

where $T_{k}^{i} := \{ t > t_{k}^{i} : F_{k}^{i}(t) > 0 \text{ or } Q_{j}(t) \geq 0 \}$. 

FIGURE 1 First and second coordinates of all agents.

FIGURE 2 Event-triggering instants.
2. Dynamic event-triggered control (DETC)\(^{47}\)

\[
\begin{align*}
t_{k+1}^i &= \inf \left\{ t > t_k^i : \left\| e_{z_i}(t) \right\|^2 - \delta_i \left\| q_i(t) \right\|^2 - \pi_i \eta_i(t) \geq 0 \right\}, \\
\dot{\eta}_i(t) &= -\beta_i \eta_i(t) + \theta_i \left( \delta_i \left\| q_i(t) \right\|^2 - \left\| e_{z_i}(t) \right\|^2 \right), \quad \eta_i(0) > 0. \tag{21}
\end{align*}
\]

3. Standard event-triggered control (SETC)\(^{15}\)

\[
\begin{align*}
t_{k+1}^i &= \inf \left\{ t > t_k^i : \left\| e_{z_i}(t) \right\|^2 - \delta_i \left\| q_i(t) \right\|^2 \geq \varphi_i(t) \right\}, \\
\end{align*}
\]

where \( \lim_{t \to \infty} \varphi_i(t) = 0. \)

We simulate the event-triggered distributed controls IETC, IETCa, DETC, and SETC. For each control, we selected parameters so that the bipartite consensus of states is attained well. Specifically, we set \( \beta_i = 0.008 \) and \( h_i(t) = (15/4)e^{-3t} \) for IETC. These parameters are also used for IETCa with choosing \( \bar{w}_i = 1 \). For DETC, we set \( \eta_i(0) = 0.02 \) and \( \delta = 0.2 \) with choosing \( \theta = 1, \beta = 0.7 \) and \( \pi_i = 1 \). For SETC, we set \( \delta_i = 0.0004 \) and \( \varphi_i(t) = 10^{-3}e^{-3t} \).

We set the initial states by choosing each coordinate from the normal distribution \( \mathcal{N}(0, 1/4) \). For comparison, we consider the following consensus error:

\[
E(t) = \sum_{k=1}^{N} \left\| z_k(t) - \bar{z}(t) \right\|^2. \tag{23}
\]

For each simulation, we find a minimal time \( \bar{t} > 0 \) such that \( E(\bar{t}) < 0.001 \) and we count the number of event-triggering for each agent up to time \( \bar{t} \).

We conduct the simulation for each type of controls for 50 times with generating the initial data randomly. The average of those counts are arranged in Table 2. The result shows that the triggering numbers for agents \( \{1, 2, 5, 6\} \) are lowest for IETC and those for agents \( \{3, 4\} \) are lowest for SETC.

We also compute the minimal value \( I_i > 0 \) among the length of intervals between two consecutive triggering times of each agent \( i \) and set \( I_{\min} = \min_{1 \leq i \leq N} I_i \). The average of those values in 50 tests are listed in Table 3. The result shows that IETC has the largest value of the average while IETCa and DETC have the second and third largest values of the average.

The graphs of the first coordinates with respect to time \( t \in [0, 3] \) are given in Figure 4 for IETC and IETCa, and Figure 5 for DETC and SETC. It turns out that the graphs of IETC and DETC oscillate more than the graphs of the other two controls. In conclusion, we find that IETC and DETC have small numbers of event-triggering times and large interval lengths between the consecutive triggering times while the graphs of coordinates may oscillate more than IETCa and
TABLE 2  Averaged number of event-triggering for each agent in 50 tests.

| Agent number | 1  | 2  | 3  | 4  | 5  | 6  |
|--------------|----|----|----|----|----|----|
| IETC         | 28.2 | 28.4 | 58.8 | 45.2 | 29.2 | 14.8 |
| IETCa        | 41.8 | 48.8 | 44.5 | 47.4 | 39.6 | 53.6 |
| DETC         | 32.8 | 32.6 | 58.4 | 52.8 | 43  | 17.2 |
| SETC         | 30.8 | 33.8 | 39  | 36.2 | 37.4 | 48.2 |

TABLE 3  Average of minimum interval between triggering times in 50 tests.

| Control | IETC | IETCa | DETC | SETC |
|---------|------|-------|------|------|
| $I_{min}$ | 0.01824 | 0.01184 | 0.01056 | 0.0026 |

FIGURE 4  The left (resp., right) graph corresponds to the first coordinates of all agents for IETC (resp., IETCa).

SETC. We also mention that the performance of the controls may depend on the parameters in the design of the triggering conditions and the control gain. An optimal choice for balancing between the performance and reducing the triggering times would be interesting to study further.

4.3  A vehicle problem

We consider a bipartite consensus problem for multi-agent vehicles\(^48\) whose dynamic is given as follows:

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \cos(\phi_i(t)), \\
\dot{y}_i(t) &= v_i(t) \sin(\phi_i(t)), \\
\dot{\phi}_i(t) &= w_i(t).
\end{align*}
\]

(24)

We follow the approach of the work\(^49\) to translate the above problem in a linear problem. Namely, we set $v_i^1(t) = v_i(t) \cos(\phi_i(t))$ and $v_i^2(t) = v_i(t) \sin(\phi_i(t))$. Then

\[
\begin{pmatrix}
\dot{v}_i^1(t) \\
\dot{v}_i^2(t)
\end{pmatrix}
= \begin{pmatrix}
\cos(\phi_i(t)) & -v_i(t) \sin(\phi_i(t)) \\
\sin(\phi_i(t)) & v_i(t) \cos(\phi_i(t))
\end{pmatrix}
\begin{pmatrix}
\dot{r}_i(t) \\
w_i(t)
\end{pmatrix}.
\]

(25)
We now take the control in the following form:

\[
\begin{pmatrix}
  v_i(t) \\
  w_i(t)
\end{pmatrix} = \begin{pmatrix}
  \cos(\phi_i(t)) & \sin(\phi_i(t)) \\
  -\sin(\phi_i(t))/v_i(t) & \cos(\phi_i(t))/v_i(t)
\end{pmatrix} \begin{pmatrix}
  u_i^x(t) \\
  u_i^y(t)
\end{pmatrix}.
\]

(26)

Then the state \( X_i(t) = (x_i(t), v_i^x(t), y_i(t), v_i^y(t)) \) and \( u_i(t) = (u_i^x(t), u_i^y(t)) \) satisfies (4) with replacing the notation \( x_i(t) \) by \( X_i(t) \) and

\[
A = \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad
B = \begin{pmatrix}
  0 & 0 \\
  1 & 0 \\
  0 & 0 \\
  0 & 1
\end{pmatrix}.
\]

(27)
Table 4 Minimum interval between triggering times.

| $k$ | 1   | 2   | 3   | 4   | 5   | 6   |
|-----|-----|-----|-----|-----|-----|-----|
| $I_{ev}(k)$ (s) | 0.1050 | 0.1345 | 0.0751 | 0.0749 | 0.0923 | 0.0950 |

We use the same matrix $L$ given in the first example, and the matrix $P$ is given as

$$
P = \begin{pmatrix}
1.9008 & 1.3065 & 0 & 0 \\
1.3065 & 2.4835 & 0 & 0 \\
0 & 0 & 1.9008 & 1.3065 \\
0 & 0 & 1.3065 & 2.4835
\end{pmatrix}
$$

and $K = B^TP$. Each coordinate of the initial state $X_i(0)$ is selected randomly from $\mathcal{N}(0, 1/2)$. Also we set the parameters as $\beta_k = 0.008$ for $1 \leq k \leq 6$ and $h(t) = 5/4e^{-t}$.

The $x - y$ trajectory of the vehicles are given in Figure 6. The result shows that the agents achieve the desired bipartite consensus and the minimal interval between the triggering times are positive constants as computed in Table 4.

5 | CONCLUSION

In this work, we designed an integral based event-triggered controller for bipartite consensus of the general linear system. We rigorously proved that the bipartite consensus is achieved and there is a positive MIET for the controlled system. The numerical simulation was provided supporting the theoretical results.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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