Exactly solvable models of supersymmetric quantum mechanics and connection to spectrum generating algebra

Asim Gangopadhyaya\textsuperscript{a,1}, Jeffry V. Mallow \textsuperscript{a,2}, Constantin Rasinariu \textsuperscript{b,3} and Uday P. Sukhatme\textsuperscript{b,4}

\textsuperscript{a) Department of Physics, Loyola University Chicago, Chicago, USA}
\textsuperscript{b) Department of Physics, University of Illinois at Chicago, Chicago, USA}

Abstract

For nonrelativistic Hamiltonians which are shape invariant, analytic expressions for the eigenvalues and eigenvectors can be derived using the well known method of supersymmetric quantum mechanics. Most of these Hamiltonians also possess spectrum generating algebras and are hence solvable by an independent group theoretic method. In this paper, we demonstrate the equivalence of the two methods of solution by developing an algebraic framework for shape invariant Hamiltonians with a general change of parameters, which involves nonlinear extensions of Lie algebras.

1 Introduction

Supersymmetric quantum mechanics (SUSYQM)\textsuperscript{[1]} provides an elegant and useful prescription for obtaining closed analytic expressions both for the energy eigenvalues and eigenfunctions of a large class of one dimensional problems. The main ingredients in SUSYQM are the supersymmetric partner Hamiltonians $H_- \equiv A^\dagger A$ and $H_+ \equiv AA^\dagger$. The $A$ and $A^\dagger$ operators used in this factorization are expressed in terms of the superpotential $W$ as follows:

$$A(x, a) = \frac{d}{dx} + W(x, a) \quad ; \quad A^\dagger(x, a) = -\frac{d}{dx} + W(x, a).$$

(1)

Here, $W$ is a real function of $x$ and $a$ is a parameter (or a set of parameters), which plays an important role in the approach of this paper. An interesting feature of SUSYQM is that for a shape

\textsuperscript{1}e-mail: agangop@luc.edu, asim@uic.edu
\textsuperscript{2}e-mail: jmallow@luc.edu
\textsuperscript{3}e-mail: costel@uic.edu
\textsuperscript{4}e-mail: sukhatme@uic.edu
invariant system \([2]\), i.e. a system satisfying an integrability condition
\[
W^2(x,a_0) + \frac{dW(x,a_0)}{dx} = W^2(x,a_1) - \frac{dW(x,a_1)}{dx} + R(a_0) ; \quad a_1 = f(a_0) ,
\]
the entire spectrum can be determined algebraically without ever referring to underlying differential equations \([1]\).

Several of these exactly solvable systems are also known to possess what is generally referred to as a spectrum generating algebra (SGA) \([3, 4]\). The Hamiltonian of these systems can be written as a linear or quadratic function of an underlying algebra, and all the quantum states of these systems can be determined by group theoretic methods.

One may naturally ask the question whether there is any connection between a general shape invariance condition and a spectrum generating algebra. In this paper we address the equivalence between the two approaches, considering a large class of change of parameters, including translations, scalings, projective transformations, as well as more complicated functions \(f(a_0)\).

In sec. 2, we start with a general shape invariant model. We make use of the operators \(A\) and \(A^\dagger\) to construct a three generator algebra. In particular, the shape invariance condition plays a crucial role in closing the algebra, which turns out to be either \(so(2,1)\) or a deformation of it. In sec. 3, several examples are presented. In particular, we discuss shape invariant potentials generated by a change of parameters corresponding to translation \(a_1 = a_0 + k\) and pure scaling \(a_1 = qa_0, q = \text{constant} (0 < q < 1)\). For the case of scaling, we find that the associated potential algebra is a nonlinear deformation of \(su(2)\). We also describe potential algebraic structure of cyclic potentials \([5]\) described as a series of shape invariant potentials which repeats after a cycle of \(k\) iterations. And finally, we discuss the potential algebra of Natanzon potentials \([6]\) and show that all translational shape invariant potentials can be generated from them.

## 2 The Algebraic Shape Invariant Model

To begin the construction of the operator algebra, let us express the shape invariance condition eq. \([2]\) in terms of \(A\) and \(A^\dagger\) :
\[
A(x,a_0)A^\dagger(x,a_0) - A^\dagger(x,a_1)A(x,a_1) = R(a_0) .
\]
This relation resembles a commutator structure. To obtain a closed \(su(2)\)-like algebra, we introduce an auxiliary variable \(\phi\) and define the following operators
\[
J_+ = e^{ip\phi} A^\dagger(x,\chi(i\partial\phi)) , \quad J_- = A(x,\chi(i\partial\phi)) e^{-ip\phi} ,
\]
where \(p\) is an arbitrary real constant and \(\chi\) is an arbitrary, real function. The operators \(A(x,\chi(i\partial\phi))\) and \(A^\dagger(x,\chi(i\partial\phi))\) are obtained from eq. \([3]\) with the substitution \(a_0 \rightarrow \chi(i\partial\phi)\). This generalization is analogous to the familiar spherical coordinate separation of variables scheme, in which
The last step in our construction is to define the operator \( J \) corresponding to several models. For example, we have

\[
\partial_\phi \Phi(\phi) \to \text{constant} \times \Phi(\phi); \text{ in this case the constant eigenvalue is } a_0. \]

From eq. (4), one obtains

\[
[J_+, J_-] = e^{i\rho a} A^\dagger(x, \chi(i\partial_\phi)) A(x, \chi(i\partial_\phi)) e^{-i\rho a} - A(x, \chi(i\partial_\phi)) A^\dagger(x, \chi(i\partial_\phi)).
\]

Eq. (6) can be easily cast into the following form

\[
[J_+, J_-] = - \{ A(x, \chi(i\partial_\phi)) A^\dagger(x, \chi(i\partial_\phi)) - A^\dagger(x, \chi(i\partial_\phi + p)) A(x, \chi(i\partial_\phi + p)) \}. \tag{6}
\]

At this point if we judiciously choose a function \( \chi(i\partial_\phi) \) such that \( \chi(i\partial_\phi + p) = f[\chi(i\partial_\phi)] \), the r.h.s. of eq. (6) can be simplified using shape invariance condition

\[
A(x, \chi(i\partial_\phi)) A^\dagger(x, \chi(i\partial_\phi + p)) A(x, \chi(i\partial_\phi + p)) = R(\chi(i\partial_\phi)),
\]

where we have identified

\[
a_0 \to \chi(i\partial_\phi); \quad a_1 = f(a_0) \to f[\chi(i\partial_\phi)] = \chi(i\partial_\phi + p). \tag{8}
\]

The last step in our construction is to define the operator \( J_3 \) as \( J_3 = -\frac{1}{\rho} \partial_\phi \). As a consequence, we obtain a deformed Lie algebra whose generators \( J_+, J_- \) and \( J_3 \) satisfy the commutation relations

\[
[J_3, J_\pm] = \pm J_\pm; \quad [J_+, J_-] = \xi(J_3), \tag{9}
\]

where \( \xi(J_3) \equiv -R(\chi(i\partial_\phi)) \) defines the deformation. Thus we see that shape invariance condition plays an indispensable role in the closing of this algebra.

Depending on the choice of the \( \chi \) function in eq. (8), we have different reparametrizations corresponding to several models. For example, we have

1. translational models: \( a_1 = a_0 + p \) for \( \chi(z) = z \) (in these models if \( R \) is a linear function of \( J_3 \) the algebra turns out to be \( so(2,1) \) or \( so(3) \) \footnote{Balantekin}, a similar conclusion was reached by \( \mu \) by using a somewhat different method);

2. scaling models: \( a_1 = e^p a_0 \equiv qa_0 \) for \( \chi(z) = e^z \),

3. cyclic models: \( a_1 = \frac{a_0 + \beta}{\gamma} a_0 + \beta \) for \( \chi(z) = \frac{(\lambda_1 - \delta) \lambda_1^z + (\lambda_2 - \delta) \lambda_2^z B(z)}{\gamma \lambda_1^z + \lambda_2^z}, \)

where \( \lambda_{1,2} \) are solutions of the equation \( (x - \alpha)(x - \delta) - \beta \gamma = 0 \) and \( B(z) \) is an arbitrary periodic function of \( z \) with period \( p \).

Other changes of parameters follow from more complicated choices for \( \chi(z) \). For example, if one takes \( \chi(z) = e^{e^z} \), one gets the change of parameters \( a_1 = a_0^2 \).

Note that the quantity \( J_+ J_- \) corresponds to the Hamiltonian

\[
H_-(x, i\partial_\phi + p) = A^\dagger(x, \chi(i\partial_\phi + p)) A(x, \chi(i\partial_\phi + p)). \tag{10}
\]
To find the energy spectrum of the Hamiltonian $H$ of eq. (10), we first construct the unitary representations of the deformed Lie algebra defined by eqs. (9). The technique proceeds as follows [9]. Define, up to an additive constant, a function $g(J_3)$ such that

$$\xi(J_3) = g(J_3) - g(J_3 - 1).$$  \hspace{1cm} (11)

The Casimir of this algebra is then given by $C_2 = J_+ J_- + g(J_3)$. It is known that in a basis in which $J_3$ and $C_2$ are diagonal, $J_+$ and $J_-$ play the role of raising and lowering operators, respectively. Operating on an arbitrary state $|h\rangle$ we have

$$J_3|h\rangle = h|h\rangle,$$
$$J_-|h\rangle = a(h) |h - 1\rangle,$$
$$J_+|h\rangle = a^*(h + 1) |h + 1\rangle.$$  \hspace{1cm} (12)

Using eqs. (9) and (12) we obtain

$$|a(h)|^2 - |a(h + 1)|^2 = g(h) - g(h - 1).$$  \hspace{1cm} (13)

The profile of $g(h)$ determines the dimension of the unitary representation. For example, let us consider the two cases presented in fig. 1. One obtains finite dimensional representations fig. 1a, by starting from a point on the $g(h)$ vs. $h$ graph corresponding to $h = h_{\text{min}}$, and moving in integer steps parallel to the $h$-axis till the point corresponding to $h = h_{\text{max}}$. At the end points, $a(h_{\text{min}}) = a(h_{\text{max}} + 1) = 0$, and we get a finite representation. (This is the case of $su(2)$ for example, where $g(h)$ is given by the parabola $h(h + 1)$.) If $g(h)$ is decreasing monotonically, fig. 1b, there exists only one end point at $h = h_{\text{min}}$. Starting from $h_{\text{min}}$ the value of $h$ can be increased in integer steps till infinity. In this case we have an infinite dimensional representation. As in the finite case, $h_{\text{min}}$ labels the representation. The difference is that here $h_{\text{min}}$ takes continuous values. Similar arguments apply for a monotonically increasing function $g(h)$.

Figure 1: Generic behaviors of $g(h)$. 

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Having the representation of the algebra associated with a characteristic model, we obtain (using eq. (10, 13)) the complete spectrum of the system. To illustrate how this mechanism works, we investigate few examples in the next section.

3 Examples

3.1 Self-Similar Potentials

The first example is for a scaling change of parameters $a_1 = qa_0$. Consider the simple choice $R(a_0) = r_1a_0$, where $r_1$ is a constant. This choice generates self-similar potentials studied in refs. [10, 11]. In this case, eqs. (9) become:

$$
\begin{align*}
\left[J_3, J_\pm \right] &= \pm J_\pm, \\
\left[J_+, J_- \right] &= \xi(J_3) \equiv -r_1 \exp(-pJ_3),
\end{align*}
$$

(14)

which is a deformation of the standard $so(2,1)$ Lie algebra.

For this case, from eqs. (14) and (11) one gets

$$
g(h) = r_1 \frac{e^{ph} - 1}{e^p - 1} = -r_1 \frac{1 - q^{h}}{1 - q} \quad ; \quad q = e^p.
$$

(15)

Note that for scaling problems [11], one requires $0 < q < 1$, which leads to $p < 0$. From the monotonically decreasing profile of the function $g(h)$, it follows that the unitary representations of this algebra are infinite dimensional. If we label the lowest weight state of the operator $J_3$ by $h_{\text{min}}$, then $a(h_{\text{min}}) = 0$. Without loss of generality we can choose the coefficients $a(h)$ to be real. Then one obtains from (13) for an arbitrary $h = h_{\text{min}} + n, \; n = 0, 1, 2, \ldots$

$$
a^2(h) = g(h - n - 1) - g(h - 1) = r_1 \frac{q^n - 1}{q - 1} q^{1-h}.
$$

(16)

The spectrum of the Hamiltonian $H_-(x, a_1)$ is given by

$$
H_\cdot |h\rangle = a^2(h)|h\rangle = r_1 \frac{q^n - 1}{q - 1} q^{1-h} |h\rangle.
$$

(17)

Therefore, the eigenenergies are

$$
E_n(h) = r_1 \alpha(h) \frac{q^n - 1}{q - 1} \quad ; \quad \alpha(h) \equiv q^{1-h}.
$$

(18)

To compare the above spectrum obtained using a group theoretic method with the results obtained from SUSYQM [1], we go to the $x$-representation. Here $|h\rangle \propto e^{ip\phi} \psi_{h_{\text{min}}, n}(x)$ and hence, the Schrödinger equation for the Hamiltonian $H_-$ reads

$$
\left\{-\frac{d^2}{dx^2} + W^2(x, e^{i\phi} + p) - W'(x, e^{i\phi} + p) - E\right\} e^{ip\phi} \psi_{h_{\text{min}}, n}(x) = 0,
$$

or

$$
\left\{-\frac{d^2}{dx^2} + W^2(x, \alpha(h)) - W'(x, \alpha(h)) - E\right\} \psi_{h_{\text{min}}, n}(x) = 0,
$$

(19)
which is exactly the Schrödinger equation appearing in ref. [11], with eigenenergies given by eq. (18). The elegant correspondence that exists between potential algebra and supersymmetric quantum mechanics for shape invariant potentials is further described in ref. [12].

For a more general case, we assume \( R(a_0) = \sum_{j=1}^{\infty} R_j a_0^j \). In this case

\[
g(h) = \sum_{j=1}^{\infty} \frac{R_j}{1 - e^{jph}} e^{-jph},
\]

and therefore one gets

\[
a_2^2(h) = g(h - n - 1) - g(h - 1) = \sum_{j=1}^{\infty} \alpha_j(h) \frac{1 - q^{jn}}{1 - q^j},
\]

where \( \alpha_j(h) = R_j e^{-j(h-1)} \). These results agree with those obtained in ref. [11].

### 3.2 Cyclic Potentials

Let us consider a particular change of parameters given by the following cycle (or chain):

\[
a_0, a_1 = f(a_0), a_2 = f(a_1), \ldots, a_{k-1} = f(a_{k-2}), a_k = f(a_{k-1}) = a_0,
\]

and choose \( R(a_i) = a_i \equiv \omega_i \). This choice generates cyclic potentials studied in ref. [5].

Cyclic potentials form a series of shape invariant potentials; the series repeats after a cycle of \( k \) iterations. In fig. 2 we show the first potential \( V(x, a_0) \) from a 3-chain \( (k = 3) \) of cyclic potentials, corresponding to \( \omega_0 = 0.15, \omega_1 = 0.25, \omega_2 = 0.60 \).

Such potentials have an infinite number of periodically spaced eigenvalues. More precisely, the level spacings are given by \( \omega_0, \omega_1, \ldots, \omega_{k-1}, \omega_0, \omega_1, \ldots, \omega_{k-1}, \omega_0, \omega_1, \ldots \).

In order to generate the change of parameters (22) the function \( f \) should satisfy \( f(f(\ldots f(x)\ldots)) \equiv f^k(x) = x \). The projective map

\[
f(y) = \frac{\alpha y + \beta}{\gamma y + \delta},
\]

with specific constraints on the parameters \( \alpha, \beta, \gamma, \delta \), satisfies such a condition [5].

The next step is to identify the Lie algebra behind this model. For this, we need to find the function \( \chi \) satisfying the equation

\[
\chi(z + p) = f(\chi(z)) \equiv \frac{\alpha \chi(z) + \beta}{\gamma \chi(z) + \delta}.
\]

It is a difference equation and its general solution is given by

\[
\chi(z) = \frac{(\lambda_1 - \delta) \lambda_1^{z/p} + (\lambda_2 - \delta) \lambda_2^{z/p} B(z)}{\gamma \left[ \lambda_1^{z/p} + \lambda_2^{z/p} B(z) \right]},
\]
where $\lambda_{1,2}$ are solutions of the equation $(x - \alpha)(x - \delta) - \beta\gamma = 0$. For simplicity $B(z)$ can be chosen to be an arbitrary constant. Plugging this expression in eqs. (9) we obtain:

\[
\begin{align*}
[J_3, J_{\pm}] &= \pm J_\pm; \\
[J_+, J_-] &= \xi(J_3) \equiv -\frac{1}{c} \frac{A(\lambda_1 - \delta)\lambda_1^{-J_3} + B(\lambda_2 - \delta)\lambda_2^{-J_3}}{A\lambda_1^{-J_3} + B\lambda_2^{-J_3}}.
\end{align*}
\]  

(26)

Applying our standard procedure to find the spectrum of the Hamiltonian $H_- = J_+ J_-$ we find that the ground state is at zero energy; the next $(k-1)$ eigenvalues are $E_l = \sum_{j=0}^l \omega_j$, $l = 0, 1, \ldots, (k-2)$, and all other eigenvalues are obtained by adding arbitrary multiples of the quantity $\Omega_k \equiv \omega_0 + \omega_1 + \cdots + \omega_{k-1}$. This result is in complete agreement with [3].

### 3.3 Natanzon Potentials

In sec. 2, we noted that for SIP’s with translationally related parameters (i.e. $a_1 = a_0 + 1$), the shape invariance condition helps in closing the algebra to the familiar $so(3)$ or $so(2,1)$, provided the $R(a_0)$ was linear in $a_0$ [3]. Several SIP’s belong to this category; among them are the Morse, Scarf I, Scarf II, and generalized Pöschl-Teller potentials. However, there are many important SIP’s (e.g., Coulomb), whose associated $R(a_0)$’s are not linear in $a_0$. Our method of the previous section would lead to deformed potential algebras for these systems. While we now know how to get representations of such algebras, in this section we shall take a different approach. We choose to generalize the structure of operators $J_\pm$ such that their algebra still remains linear. In fact, in this section, we generate shape invariant potentials from an underlying potential algebra instead of showing algebraic structure hidden in a shape invariant system.
Alhassid et al.\cite{3} had shown that the algebra associated with the \textit{general} potential of the Natanzon class is $so(2, 2)$. The Schrödinger equation for these potentials reduces in general to the hypergeometric equation. We show below that a further constraint generates all SIP’s with translational change of parameters. For the sake of completeness we will briefly examine the properties of $so(2, 2)$ algebra in this section, and show its connection to the Natanzon potentials \cite{6}. We then conjecture an additional constraint that would render them shape invariant. We find that this conjecture indeed produces all known SIP of the translational type. We shall find in fact that the subset of Natanzon potentials associated with the translational (additive) SIP’s has the simpler $so(2, 1)$ algebra.

We begin by describing Alhassid et al.’s realization of the $so(2, 2)$ algebra in terms of differential operators. For consistency, we use the formalism and the notations of refs. \cite{3}.

The differential realization can be written explicitly as

\begin{align*}
A_{\pm} &\equiv A_1 \pm A_2 = \frac{1}{2} e^{\pm i(\phi + \theta)} \left[ \mp \frac{\partial}{\partial \chi} + \tanh \chi (-i\partial_\phi) + \coth \chi (-i\partial_\theta) \right]; \\
A_3 &\equiv -\frac{i}{2} (\partial_\phi + \partial_\theta) ; \\
B_{\pm} &\equiv B_1 \pm B_2 = \frac{1}{2} e^{\pm i(\phi - \theta)} \left[ \mp \frac{\partial}{\partial \chi} + \tanh \chi (-i\partial_\phi) + \coth \chi (+i\partial_\theta) \right]; \\
B_3 &\equiv -\frac{i}{2} (\partial_\phi - \partial_\theta) .
\end{align*}

The $so(2, 1)$ algebra obeyed by these operators is

\[ [A_3, A_{\pm}] = \pm A_\pm, \quad [A_+, A_-] = -2A_3 , \]

and a similar one for the $B$’s. The Casimir operator $C_2$ is given by

\begin{equation}
C_2 = 2 \left( A_3^2 - A_+ A_- - A_3 \right) + 2 \left( B_3^2 - B_+ B_- - B_3 \right)
= \left[ \frac{\partial^2}{\partial \chi^2} + (\tanh \chi + \coth \chi) \frac{\partial}{\partial \chi} + \text{sech}^2 \chi (-i\partial_\phi)^2 - \text{cosech}^2 \chi (-i\partial_\theta)^2 \right]. \tag{28}
\end{equation}

Operators $A_3$, $B_3$ and $C_2$ can be simultaneously diagonalized, and their actions on their common eigenstate are given by

\begin{align*}
C_2|\omega, m_1, m_2\rangle &= \omega(\omega + 2) |\omega, m_1, m_2\rangle ; \\
A_3|\omega, m_1, m_2\rangle &= m_1 |\omega, m_1, m_2\rangle ; \\
B_3|\omega, m_1, m_2\rangle &= m_2 |\omega, m_1, m_2\rangle . \tag{29}
\end{align*}

(It is worth mentioning at this point that the Casimir operator given above is indeed self-adjoint with respect to a measure $\sinh \chi \cosh \chi d\chi d\phi d\theta$.)

Now we shall briefly describe a general Natanzon potential and show its connection to the above Casimir operator. A general Natanzon potential $U(r)$ is implicitly defined by \cite{6}

\begin{equation}
U[z(r)] = \frac{-fz(1-z) + h_0(1-z) + h_1z}{Q(z)} - \frac{1}{2} \{z, r\} , \tag{30}
\end{equation}

8
with $Q(z)$ quadratic in $z$: $Q(z) = az^2 + b_0 z + c_0 = a(1-z)^2 - b_1(1-z) + c_1$ and $f, h_0, h_1, a, b_0, b_1, c_0, c_1$ are constants. The Schwarzian derivative $\{z, r\}$ is defined by

$$\{z, r\} = \frac{d^3 z / dr^3}{dz / dr} - \frac{3}{2} \left[ \frac{d^2 z / dr^2}{dz / dr} \right]^2.$$  

(31)

The relationship between variables $z$ ($0 < z < 1$) and $r$ is implicitly given by

$$\left( \frac{dz}{dr} \right) = \frac{2z(1-z)}{\sqrt{Q(z)}}.$$  

(32)

To connect the Casimir operator $C_2$ of the $so(2, 2)$ algebra [eq. (28)] to the general Natanzon potential, we perform a similarity transformation on $C_2$ by a function $F$ and then follow that up by an appropriate change of variable $\chi = g(r)$. It has been shown [13] that to turn $C_2$ into the form of a Schrödinger Hamiltonian, one needs to choose $F \sim \left( \frac{\sinh(2g)}{g} \right)^{\frac{1}{2}}$, and choose $z = \tanh^2 g$. Then

$$U(z(r)) = E Q + \left[ -\frac{7}{4} + \frac{3}{2} z - \frac{7}{2} z^2 \right] - z(1-z) (-i\partial_\phi)^2 + (1-z) (-i\partial_\theta)^2 - \frac{1}{2} \{z, r\}$$

$$= \left[ - \left( aE - \frac{7}{4} + (-i\partial_\phi)^2 \right) z(1-z) + \left( c_0 E - \frac{7}{4} + (-i\partial_\theta)^2 \right) (1-z) \right.$$ 

$$+ ((a + b_0 + c_0) E - 1)] / Q(z) - \frac{1}{2} \{z, r\}.$$  

(33)

We have used

$$g' = \frac{dq}{dr} = \frac{dq}{dz} \frac{dz}{dr} = \frac{1}{2\sqrt{\zeta(1-z)}} \frac{2z(1-z)}{Q} = \sqrt{\frac{z}{Q}},$$

$g = \tanh^{-1} \sqrt{z}$ and $\frac{dz}{dr}$ from eq. (32). Now, with the following identification

$$f = aE - \frac{7}{4} + (-i\partial_\phi)^2,$$

$$h_0 = c_0 E - \frac{7}{4} + (-i\partial_\theta)^2,$$

$$h_1 = (a + b_0 + c_0) E - 1,$$  

(34)

the potential of eq. (33) indeed has the form of a general Natanzon potential [eq. (30)]. Further details are shown in ref. [13]. Finally, we are ready to explicitly demonstrate the connection between the Natanzon potential algebra and shape invariant potentials of supersymmetric quantum mechanics.

We now note that the similarity transformation can be rewritten: using $g = \tanh^{-1} \sqrt{z}$, $g' = \sqrt{\frac{z}{Q}}$ and eq. (32), we find $\left( \frac{\sinh(2g)}{g} \right) = \frac{\zeta}{g}$. At this point we go back to the operators $A_{\pm}$ [eq. (27)] and ask how they transform under the similarity transformation given by $F \sim \left( \frac{\sinh(2g)}{g} \right)^{\frac{1}{2}} \sim \sqrt{\frac{z}{Q}}$. This transformation carries operators $A_{\pm}$ to

$$A_{\pm} \rightarrow \tilde{A}_{\pm} = \frac{e^{\pm i(\phi + \theta)}}{2} \left[ \mp \left( \frac{d}{d\chi} + \frac{1}{2z'} \frac{dz'}{d\chi} - \frac{1}{2z} \frac{dz}{d\chi} \right) + \tanh \chi (-i\partial_\phi) + \coth \chi (-i\partial_\theta) \right].$$  

(35)
Except for the expression \( \left( \frac{1}{2z'} \frac{dz'}{d\chi} - \frac{1}{2z} \frac{dz}{d\chi} \right) \), this looks very much like eq. (27), which are in fact \( A_\pm \) of the shape invariant Pöschl-Teller potential\[1\]. Thus, if \( \left( \frac{1}{2z'} \frac{dz'}{d\chi} - \frac{1}{2z} \frac{dz}{d\chi} \right) \) were to be a linear combination of \( \tanh \chi \) and \( \coth \chi \), operators \( \tilde{A}_\pm \) could be cast in a form similar to the operators \( A_\pm \) of eq. (27), and we would get \( A_\pm \)'s that generate shape invariant Hamiltonians.

Hence to get shape invariant potentials, we require,

\[
\left( \frac{1}{2z'} \frac{dz'}{d\chi} - \frac{1}{2z} \frac{dz}{d\chi} \right) = \alpha \tanh \chi + \beta \coth \chi.
\]

(36)

This leads to

\[
z' = z^{1+\beta}(1-z)^{-\alpha-\beta},
\]

(37)

which is the second constraint on the relationship between variables \( z \) and \( r \). Since these variables are already constrained by eq. (32), only a handful of solutions would be compatible with both restrictions. The \( z(r) \)'s that are compatible with both eqs. (32) and (37) are given by

\[
z^{1+\beta}(1-z)^{-\alpha-\beta} = \frac{2z(1-z)}{\sqrt{Q(z)}},
\]

(38)

where \( Q(z) \) is a quadratic function of \( z \). After some computation, we find that there is only a finite number of values of \( \alpha, \beta \) which satisfy eq. (38). These values are listed in Table 1, and they exhaust all known shape invariant potentials that lead to the hypergeometric equation.

Furthermore, while the potential algebra of a general Natanzon system is \( so(2,2) \), and requires two sets of raising and lowering operators \( A_\pm \) and \( B_\pm \), all translational shape invariant potentials need only one such set. For all SIPs of Table 4.1 of ref. [1], one finds that all partner potentials are connected by change of just one independent parameter (although other parameters which don’t change are also present.) Thus there is a series of potentials that only differ in one parameter. From the potential algebra perspective, all these potentials differ only by the eigenvalue of an operator that is a linear combination of \( A_3 \) and \( B_3 \), and all are characterized by a common eigenvalue of \( C_2 \). Thus, these shape invariant potentials can be associated with a \( so(2,1) \) potential algebra generated by operators \( A_+ \), \( A_- \) and the same linear combination of \( A_3 \) and \( B_3 \).

Conclusion: In this paper, we have explored the reasons underlying the integrability of shape invariant Hamiltonians and shown that such systems naturally admit an algebraic structure known as potential algebra. We have derived these algebras for shape invariant systems with translational and scaling type change of parameters, as well as for cyclic potentials. In general, one finds deformations of the \( so(2,1) \) Lie algebra. Our approach links the group theoretic and supersymmetric quantum mechanics approaches for treating shape invariant potentials.

A.G. acknowledges a research leave and a grant from Loyola University Chicago which made his involvement in this work possible. Partial financial support from the U.S. Department of Energy is gratefully acknowledged.
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| $\alpha$ | $\beta$ | $\tilde{z}(r)$ | Superpotential | Potential |
|---------|---------|----------------|----------------|-----------|
| 0       | 0       | $z = e^{-r}$   | $\tilde{m}_1 \coth \frac{r}{2} + \tilde{m}_2$ | Eckart    |
| 0       | $-\frac{1}{2}$ | $z = \sin^2 \frac{r}{2}$ | $\tilde{m}_1 \cosec r + \tilde{m}_2 \cot r$ | Gen. Pöschl-Teller trigonometric |
| 0       | $-1$    | $z = 1 - e^{-r}$ | $\tilde{m}_1 \coth \frac{r}{2} + \tilde{m}_2$ | Eckart    |
| $-\frac{1}{2}$ | 0       | $z = \sech^2 \frac{r}{2}$ | $\tilde{m}_1 \cosech r + \tilde{m}_2 \coth r$ | Pöschl-Teller II |
| $-\frac{1}{2}$ | $-\frac{1}{2}$ | $z = \tanh^2 \frac{r}{2}$ | $\tilde{m}_1 \tanh \frac{r}{2} + \tilde{m}_2 \coth \frac{r}{2}$ | Gen. Pöschl-Teller |
| $-1$    | 0       | $z = 1 + \tanh \frac{r}{2}$ | $\tilde{m}_1 \tanh \frac{r}{2} + \tilde{m}_2$ | Rosen Morse |

Table 1: Shows all allowed value of $\alpha$, $\beta$ and the superpotentials that they generate.