Abstract

We study generalizations of Lorentzian warped products with one-dimensional base of the form $I \times_f X$, where $I$ is an interval, $X$ is a length space and $f$ is a positive continuous function. These generalized cones furnish an important class of Lorentzian length spaces in the sense of [KS18], displaying optimal causality properties that allow for explicit descriptions of all underlying notions. In addition, synthetic sectional curvature bounds of generalized cones are directly related to metric curvature bounds of the fiber $X$. The interest in such spaces comes both from metric geometry and from General Relativity, where warped products underlie important cosmological models (FLRW spacetimes). Moreover, we prove singularity theorems for these spaces, showing that non-positive lower timelike curvature bounds imply the existence of incomplete timelike geodesics.

Keywords: Length spaces, Lorentzian length spaces, causality theory, synthetic curvature bounds, triangle comparison, metric geometry, warped products

MSC2010: 51K10, 53C23, 53C50, 53B30, 53C80
1 Introduction

Warped products are of central importance to Riemannian geometry, in particular in the study of constant curvature geometries and as a rich source of examples and counterexamples (cf., e.g., [Pet16]). Generalized cones and warped products of metric spaces likewise play an important role in the theory of length metric spaces with synthetic curvature bounds (Alexandrov spaces). These spaces, while including Riemannian manifolds with curvature bounds, allow singularities, being closed, for example, under Gromov-Hausdorff limits or gluing operations. Alexandrov spaces have yielded major insights into classical Riemannian geometry ([Per93], cf. [Kap07, Gro01]). Generalized cones and warped products provide examples and counterexamples in Alexandrov geometry (cf. [Che99, AB98, AB04, AB16]). Moreover, the first-order structure is captured by the tangent cone; for instance, for Alexandrov spaces of curvature bounded below, the tangent cone at a point is homeomorphic to a small metric ball centered at the point (cf. [BBI01]).

In the smooth pseudo-Riemannian setting, Alexander and Bishop gave in [AB08] a characterization of sectional curvature bounds in terms of tri-
angle comparison, including applications to Lorentzian and general semi- 
Riemannian warped products and so-called Friedmann-Lemaître-Robertson- 
Walker (FLRW)-spacetimes. Lorentzian geometry enjoys a unique position 
in the smooth pseudo-Riemannian world: Pseudo-Riemannian metrics of 
signature (−, +, . . . , +) are central to the study of the physical theory of 
General Relativity and (apart from the Riemannian case) Lorentzian the-
ory is the most mathematically well explored case, providing many strong 
tools and results which are absent in more general signatures. Addition-
ally, the past decade has seen increasing interest from the mathematical 
physics community in the study of low regularity Lorentzian geometry and 
General Relativity. An intensive study of causality theory (cf. Min19b) 
for Lorentzian metrics of low regularity was initiated by P. Chruściel and 
J.D.E. Grant in [CG12] and then pursued by various authors, see Min15, 
KSS14, KSSV14, Sam16. In particular, Chruściel and Grant showed in 
[CG12] that for spacetimes with merely continuous metrics pathologies of 
the causal structure may occur, e.g. there are so-called causal bubbles, where 
the boundary of the lightcone is not a hypersurface but has positive measure 
(see also GKSS20). An important recent result for continuous Lorentzian 
metrics is the $C^0$-inextendibility of the Schwarzschild solution to the Ein-
stein equations, which was shown by J. Shierski in [Sh18] and has sparked 
further research into low regularity (in-)extendibility and causality (e.g. 
GLS18, GL17, DL17, GL18, GKS19). The importance of such low reg-
ularity (in-)extendibility results is rooted in the strong cosmic censorship 
conjecture (cf. e.g. Ise15), which, roughly, states that the maximal globally 
hyperbolic development of generic initial data for the Einstein equations 
is inextendible as a suitably regular Lorentzian manifold and which is inti-
mately related to the question of determinism in General Relativity. Another 
area of mathematical general relativity, where low regularity has recently 
come to the forefront of current research, is the study of singularities and 
in particular of so-called singularity theorems, predicting causal geodesic 
incompleteness under certain curvature and causality assumptions. The 
classical singularity theorems of Hawking and Penrose have only recently 
been successfully extended to the $C^{1,1}$-setting (KSSV15, KSV15, GGKS18), 
which is a natural regularity class to consider as curvature is still almost 
everywhere defined and locally bounded. Furthermore, these singularity 
theorems have recently been extended to the regularity class $C^1$ in Gra20. 
With the final results in Section 5 this paper will directly contribute to this 
line of research by proving singularity theorems for generalized cones. We 
also mention another natural generalization of smooth Lorentzian geometry, 
namely cone structures on differentiable manifolds and Lorentz-Finsler 
spacetimes, see FS12, BS18, Min19a, MS19, LNO19, which proved to be a 
significant extension of the field. Also, there currently is strong interest in 
bringing techniques from Riemannian geometry and optimal transport into 
the Lorentzian setting, cf. CM20, McC20, MS18. For extending these tech-
niques further to a synthetic setting it might prove useful to use generalized cones as introduced and studied in the present paper as a starting point.

Finally, we note that there have been several approaches to a synthetic or axiomatic description of (parts of) Lorentzian geometry and causality. We mention in particular the timelike spaces of Busemann [Bus67] and the causal set theory of quantum gravity [BLMS87] [Sur19]. For a more detailed discussion see the introduction and Subsection 5.3 of [KS18]. Another closely related direction of research is the recent approach of Sormani and Vega [SV16] and its further development by Allen and Burtscher in [AB19] of defining a metric on a spacetime that is compatible with the causal structure in case the spacetime admits a time function satisfying an anti-Lipschitz condition.

The importance of warped products, specifically, in General Relativity (cf., e.g., [O’N83] [Wal84] [Min07]) stems from the fact that the FLRW models of the universe in cosmology are particular examples of warped products with one-dimensional base, see e.g. [O’N83] Ch. 12. Such spaces have a very simple structure geometrically and provide a good starting point for trying to generalize the smooth theory. Our main object of study will be generalizations of Lorentzian warped products with one-dimensional base to the case where the fiber is merely a metric length space, but some of our results are new even if the fiber is a smooth Riemannian manifold. For example, we show that there is no causal bubbling in such spacetimes even if the warping function f, and hence the Lorentzian metric, is merely continuous and that maximizing causal curves of positive length have to be timelike. Moreover, any globally hyperbolic smooth spacetime (M, g) splits isometrically as (M, g) \cong (\mathbb{R} \times S, -\beta dt^2 + h_t), where S is a Cauchy hypersurface in M, \beta is a smooth positive function on \mathbb{R} \times S and h_t is a t-dependent family of Riemannian metrics on each level set \{t\} \times S (cf. [BS03] [BS05]).

Globally hyperbolic spacetimes can therefore be viewed as generalizations of warped products with one-dimensional base, so our methods may also find applications to such spaces in future research.

Both from the perspective of Lorentzian geometry and with a view to the fundamental importance of warped products in General Relativity it is therefore of interest to study generalizations of such geometries beyond the setting of smooth manifolds. A natural framework in which to carry out such an extension is the theory of Lorentzian length spaces ([KS18] [GKS19]), see Subsection 1.3 below for a brief introduction.

1.1 Main results and outline of the paper

The plan of the paper is as follows. In the remainder of this introduction we recall the basic notions of the theory of metric spaces with curvature bounds and Lorentzian length spaces. Section 2 introduces Minkowski-
cones, a Lorentzian analogue of cones over metric spaces, and a first instance of a generalized cone as defined in Section 3. The main result of Section 2 relates curvature bounds of the metric space \(X\) (the fiber) to timelike curvature bounds of the cone over \(X\). In fact, we prove

**Theorem 2.5.** Let \(Y = \text{Cone}(X)\) be the Minkowski cone over a geodesic length space \(X\). Then \(Y\) has timelike curvature bounded below (above) by 0 if and only if \(X\) is an Alexandrov space of curvature bounded below (above) by \(-1\).

In Section 3 we introduce the main object of this paper, a metric analogue of Lorentzian warped products with one-dimensional base and a length space as fiber. We then study timelike and causal curves, introduce a time-separation function and establish the main features of causality theory for these generalized cones. While there are a number of analogues to the metric theory of warped products (e.g., fiber independence of geodesics), these causality results require new methods. The main result in this section is that generalized cones display no causal pathologies, a fact that is used extensively later on. In more detail, we show that

**Proposition 3.22.** (Push-up and openness of \(I^{\pm}\)) Every generalized cone \(Y = I \times_f X\) such that \((X,d)\) is a length space has the property that \(p \ll q\) if and only if there exists a future directed causal curve from \(p\) to \(q\) of positive length, i.e., push-up holds. Moreover, \(I^{\pm}(p)\) is open for any \(p \in Y\).

The above result is then used in Section 4 to establish that generalized cones are examples of Lorentzian length spaces, without any additional assumptions on the causality or on the warping function \(f\) (continuous and positive):

**Theorem 4.8 & Corollary 4.9.** Any generalized cone \(I \times_f X\), where \((X,d)\) is a locally compact length space, is a strongly causal Lorentzian length space. If \(X\) is a locally compact geodesic length space, then \(I \times_f X\) is a regular strongly causal Lorentzian length space.

In particular we prove that if the fiber \(X\) is a geodesic length space that is proper then \(I \times_f X\) is globally hyperbolic. Section 5 is then devoted to relating synthetic curvature bounds (via triangle comparison) in generalized cones to corresponding bounds in the fiber. Here the main results are as follows:

**Theorem 5.3.** Let \(K,K' \in \mathbb{R}\) and let \((X,d)\) be a geodesic length space with curvature bounded below/above by \(K\). Then \(Y = I \times_f X\) has timelike curvature bounded below/above by \(K'\) if \(I \times_f \mathcal{M}^2(K)\) has timelike curvature bounded below/above by \(K'\).
Theorem 5.7. If $X$ is a geodesic length space, $Y = I \times f X$ has timelike curvature bounded below (above) by $K'$ and $Y' = I \times f \mathbb{M}^2(K)$ has timelike curvature bounded above (below) by $K'$ then $X$ has curvature bounded below (above) by $K$.

Moreover, the first result above allows us to generate an abundance of examples of Lorentzian length spaces with timelike curvature bounds.

We then apply our techniques in Section 6 to show that non-positive lower timelike curvature bounds imply the existence of incomplete timelike geodesics. That is, we provide synthetic singularity theorems for generalized cones. To be precise, we prove the following:

Corollary 6.4. Let $X$ be a geodesic length space, $Y = I \times f X$ with $I = (a,b)$, $f : I \to (0,\infty)$ smooth. Assume that $Y$ has timelike curvature bounded below by $K$. Then:

(i) If $K < 0$, then $a > -\infty$ and $b < \infty$ and hence the time separation function $\tau_Y$ of $Y$ is bounded by $b - a$. Thus any such $Y$ is timelike geodesically incomplete.

(ii) If $K = 0$ and $f$ is non-constant, then $a > -\infty$ or $b < \infty$ and hence $Y$ is past or future timelike geodesically incomplete.

These results may be viewed as sectional curvature analogues of the Lorentzian Bonnet-Myers’ theorem and of Hawking’s singularity theorem in the setting of generalized cones. Also, we relate timelike curvature bounds to big bang and big crunch singularities in Corollary 6.7.

In the appendix we describe a general approach to what we call Lorentzian length structures, analogous to the theory of length structures in metric geometry (cf. [BB19, Ch. 2]) based on which several basic results shown in Sections 2 and 3 can be shown to hold in greater generality.

1.2 Alexandrov spaces

We briefly recall the basic definitions of Alexandrov spaces, i.e., metric spaces with curvature bounds. For comprehensive introductions see [AKP19, BB19, BH99].

A metric space $(X, d)$ is a length space if for all $x, y \in X$ one has $d(x, y) = \inf \{ L^d(\gamma) : \gamma \text{ continuous and connects } x, y \}$, where $L^d(\gamma)$ is the length of $\gamma$. A geodesic in a metric space is a continuous curve $\gamma : [0, l] \to X$ such that $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in [0, l]$. A metric space is geodesic if any two points can be joined by a geodesic.
We use $\mathbb{M}^2(K)$ to denote the Riemannian model space of constant sectional curvature $K$, i.e.,

$$
\mathbb{M}^2(K) = \begin{cases} 
\mathbb{S}^2(r) & K = \frac{1}{r^2} \\
\mathbb{R}^2 & K = 0 \\
\mathbb{H}^2(r) & K = -\frac{1}{r^2}
\end{cases}.
$$

Moreover, a triangle $\Delta$ in a metric space $(X,d)$ is a triple of points $\Delta = (x,y,z)$ and a choice of geodesic segments joining $x,y,z$, i.e., its sides $[xy]$, $[xz]$ and $[yz]$. A comparison triangle of $\Delta$ is a triangle $\bar{\Delta} = (\bar{x},\bar{y},\bar{z})$ in some model space $\mathbb{M}^2(K)$ (for some $K \in \mathbb{R}$) that has the same side lengths as $\Delta$, i.e., $d(x,y) = \bar{d}(\bar{x},\bar{y})$, $d(x,z) = \bar{d}(\bar{x},\bar{z})$ and $d(y,z) = \bar{d}(\bar{y},\bar{z})$, where $\bar{d}$ is the metric on $\mathbb{M}^2(K)$.

A length space $(X,d)$ has curvature bounded below/above by $K \in \mathbb{R}$ if every point $x_0 \in X$ has a neighborhood $U$ such that for any triangle $\Delta = (x,y,z)$ in $U$ and any point $w$ on the side $[yz]$ the following holds: Let $\bar{\Delta} = (\bar{x},\bar{y},\bar{z})$ be a comparison triangle for $\Delta$ in $\mathbb{M}^2(K)$ and let $\bar{w}$ on the side $[\bar{y}\bar{z}]$ with the same distance to $y$ (or $z$), i.e., $d(y,w) = \bar{d}(\bar{y},\bar{w})$, where $\bar{d}$ is the metric on $\mathbb{M}^2(K)$. Then

$$
d(x,w) \geq \bar{d}(\bar{x},\bar{w}) / d(x,w) \leq \bar{d}(\bar{x},\bar{w}).
$$

### 1.3 Lorentzian length spaces

Here we give a very brief introduction to the theory of Lorentzian length spaces, as developed in [KS18], at the same time fixing some notations and terminology.

Let $Y$ be a set endowed with a preorder $\leq$ and a transitive relation $\ll$ contained in $\leq$. If $x \ll y$ or $x \leq y$, we call $x$ and $y$ timelike or causally related, respectively. If $Y$ is, in addition, equipped with a metric $d$ and a lower semicontinuous map $\tau: Y \times Y \to [0,\infty]$ that satisfies the reverse triangle inequality $\tau(x,z) \geq \tau(x,y) + \tau(y,z)$ (for all $x \leq y \leq z$), as well as $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0$ if $x \not\ll y$, then $(Y,d,\ll,\leq,\tau)$ is called a Lorentzian pre-length space and $\tau$ is called the time separation function (or Lorentzian distance) of $Y$.

A curve $\gamma: I \to Y$ (I an interval) that is non-constant on any subinterval of $I$ is called (future-directed) causal (timelike) if $\gamma$ is locally Lipschitz continuous and if for all $t_1, t_2 \in I$ with $t_1 < t_2$ we have $\gamma(t_1) \leq \gamma(t_2)$ ($\gamma(t_1) \ll \gamma(t_2)$). It is called null if, in addition to being causal, no two points on the curve are related with respect to $\ll$. Note that in General Relativity such curves are called achronal. For strongly causal continuous Lorentzian metrics, this notion of causality coincides with the usual one ([KS18, Prop. 5.9]). In analogy to the theory of metric length spaces, the length of a
causal curve is defined via the time separation function: For $\gamma: [a, b] \to Y$ future-directed causal we set

$$L_\tau(\gamma) := \inf \left\{ \sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b, \ N \in \mathbb{N} \right\}.$$ 

For smooth and strongly causal spacetimes $(M, g)$ this notion of length coincides with the usual one: $L_\tau(\gamma) = L_g(\gamma)$ ([KS18, Prop. 2.32]). A future-directed causal curve $\gamma: [a, b] \to Y$ is *maximal* if it realizes the time separation, i.e., if $L_\tau(\gamma) = \tau(\gamma(a), \gamma(b))$. Standard causality conditions can also be imposed on Lorentzian pre-length spaces, and substantial parts of the causal ladder ([MS08, Min19a]) continue to hold in this general setting, cf. [KS18, Subsec. 3.5].

Lorentzian length spaces are close analogues of metric length spaces in the sense that the time separation function can be calculated from the length of causal curves connecting causally related points. A Lorentzian pre-length space that satisfies some additional technical assumptions (cf. [KS18, Def. 3.22]) is called a *Lorentzian length space* if $\tau = T$, where for any $x, y \in Y$ we set

$$T(x, y) := \sup \{ L_\tau(\gamma) : \gamma \text{ future-directed causal from } x \text{ to } y \},$$

if the set of future-directed causal curves from $x$ to $y$ is not empty. Otherwise let $T(x, y) := 0$.

Any smooth strongly causal spacetime is an example of a Lorentzian length space. More generally, spacetimes with low regularity metrics and certain Lorentz-Finsler spaces [Min19a] provide further examples, cf. [KS18, Sec. 5].

Finally, by a *timelike geodesic triangle* we mean a triple $(x, y, z) \in Y^3$ with $x \ll y \ll z$ such that $\tau(x, z) < \infty$ and such that the sides are realized by future-directed causal curves (that is, there exist future directed causal curves $\alpha, \beta, \gamma$ from $x$ to $y$, from $y$ to $z$ and from $x$ to $z$, respectively, with $L_\tau(\alpha) = \tau(x, y)$, $L_\tau(\beta) = \tau(y, z)$ and $L_\tau(\gamma) = \tau(x, z)$). Curvature bounds are formulated by comparing such triangles with triangles of the same side lengths in one of the Lorentzian model spaces $L^2(K)$ of constant sectional curvature $K$. Here,

$$L^2(K) = \begin{cases} \tilde{S}^2_1(r) & K = \frac{1}{r^2} \\ \mathbb{R}^2_1 & K = 0 \\ \tilde{H}^2_1(r) & K = -\frac{1}{r^2} \end{cases},$$

where $\tilde{S}^2_1(r)$ is the simply connected covering manifold of the two-dimensional Lorentzian pseudosphere $S^2_1(r)$ (de Sitter space), $\mathbb{R}^2_1$ is two-dimensional Minkowski space, and $\tilde{H}^2_1(r)$ is the simply connected covering manifold of the
two-dimensional Lorentzian pseudohyperbolic space (anti-de Sitter space). In order to guarantee the existence of comparison triangles in one of the model spaces, one needs to impose size restrictions on the sides, see [KS18, Lem. 4.6].

Using this terminology, a Lorentzian pre-length space \((Y,d,\ll,\leq,\tau)\) is said to have timelike curvature bounded below (above) by \(K \in \mathbb{R}\) if every point in \(Y\) has a neighborhood \(U\) such that:

(i) \(\tau|_{U \times U}\) is finite and continuous.

(ii) Whenever \(x,y \in U\) with \(x \ll y\), there exists a causal curve \(\alpha\) in \(U\) with \(L_\tau(\alpha) = \tau(x,y)\).

(iii) If \((x,y,z)\) is a timelike geodesic triangle in \(U\), realized by maximal causal curves \(\alpha,\beta,\gamma\) whose side lengths satisfy the appropriate size restrictions, and if \((x',y',z')\) is a comparison triangle of \((x,y,z)\) in \(L^2(K)\) realized by timelike geodesics \(\alpha',\beta',\gamma'\), then whenever \(p,q\) are points on the sides of \((x,y,z)\) and \(p',q'\) are corresponding points\(^1\) of \((x',y',z')\), we have \(\tau(p,q) \leq \tau'(p',q')\) (respectively \(\tau(p,q) \geq \tau'(p',q')\)).

We call such a \(U\) a comparison neighborhood.

### 2 Minkowski cones over metric spaces

As a first explicit example we consider cones over metric spaces. Such spaces are very well-behaved and allow direct calculations even of spacelike distances. However, here we consider cones exclusively as Lorentzian length spaces, providing more details than in [Ale19], where such cones are considered in the setting of Lorentzian pseudometric spaces. In particular, they furnish instances of generalized cones as defined in Section 3 (cf. Example 3.31).

Proceeding by analogy with the metric geometry notion of cones over metric spaces (cf. [BB01, Subsec. 3.6.2]) we introduce the following: For \(X\) a geodesic length space, the Minkowski cone \(Y = \text{Cone}(X)\) is defined as the quotient of \([0,\infty) \times X\) resulting from identifying all points of the form \((0, p)\). We equip \(Y\) with the cone metric \(d_c\) as in [BB01, Def. 3.6.16] (however this choice is not important, as it suffices to pick some background metric on \(Y\) that induces the quotient topology on \([0,\infty) \times X\), to turn it into a Lorentzian pre-length space, see below). The equivalence class of \(\{0\} \times X\) in \(Y\) is called the vertex of \(Y\) and is denoted by \(0_Y\).

\(^1\)This means that \(p'\) lies on the side corresponding to the side containing \(p\) at the same time separation of the vertex (i.e., e.g. if \(p\) lies on the side \(xy\) then \(\tau(x,p) = \tau'(x',p')\), etc.). Similarly for \(q'\).
Remark 2.1. As a preparation for the following definition of the time separation function, consider $n$-dimensional Minkowski-space $\mathbb{R}^n_1$, with scalar product $(x,y) = -x_0y_0 + \sum_{i=1}^{n-1} x_iy_i$. Then $(n-1)$-dimensional hyperbolic space $\mathbb{H}^{n-1}$ is isometrically embedded into $\mathbb{R}^n_1$ as \{ $x \in \mathbb{R}^n_1 \mid \langle x,x \rangle = -1, \quad x_0 > 0$ \} = $\{ x \in \mathbb{R}^n_1 \mid \tau_{\mathbb{R}^n_1}(0,x) = 1, \quad x_0 > 0$ \} = $\Sigma$, where $\tau_{\mathbb{R}^n_1}$ is the time separation function on $\mathbb{R}^n_1$. Let us denote this embedding by $\psi$. The induced Riemannian distance function on $\mathbb{H}^{n-1}$ is uniquely determined by $\cosh$ $d_{\mathbb{H}^{n-1}}(x,y) = -\langle \psi(x), \psi(y) \rangle$ for $x, y \in \mathbb{H}^{n-1}$. Suppose now that $x, y \in \mathbb{H}^{n-1}$ and let $s, t > 0$. Then $\psi(x) = (\sqrt{1+|x'|^2}, x')$ for some $x' \in \mathbb{R}^{n-1}$, and analogously for $y$. Setting $\theta := d_{\mathbb{H}^{n-1}}(x,y)$ we calculate

$$
\langle t\psi(y) - s\psi(x), t\psi(y) - s\psi(x) \rangle = -t^2 - s^2 - 2st\langle \psi(x), \psi(y) \rangle = -(s^2 + t^2 - 2st \cosh \theta).
$$

This shows that the quotient $Y$ of $(0, \infty) \times \mathbb{H}^{n-1}$ modulo $(0, x) \sim (0, y)$ for all $x, y$ can be identified with the cone $I^+(0) \cup \{ 0 \} \subseteq \mathbb{R}^n_1$ via $(s, x) \mapsto sv(x)$, and restricting this identification, we see that $(0, \infty) \times \mathbb{H}^{n-1}$ corresponds to $I^+(0) \subseteq \mathbb{R}^n_1$. Pulling back the causal structure and time separation of $\mathbb{R}^n_1$ we make the following definitions: Two points $(s, x)$ and $(t, y)$ in $Y$ are said to satisfy $(s, x) \leq_Y (t, y)$ if and only if $s \leq t$ and $s^2 + t^2 - 2st \cosh \theta \geq 0$, which is equivalent to $sx \leq ty$ in $\mathbb{R}^n_1$. In addition, for $(s, x) \leq_Y (t, y)$ the Minkowski cone time separation function $\gamma_Y$ is defined by $\gamma_Y((s, x), (t, y)) = \tau_{\mathbb{R}^n_1}(sv(x), ty(y)) = \sqrt{s^2 + t^2 - 2st \cosh \theta}$, and otherwise $\gamma_Y((s, x), (t, y)) = 0$.

For future reference, let us briefly remark that the Minkowski cone time separation $\gamma_Y$ defined above induces a time separation $\gamma_C$ on $C := (0, \infty) \times \mathbb{H}^{n-1}$ via restriction and this restriction equals the time separation $\gamma_Y$ of the Lorentzian warped product manifold $W := (0, \infty) \times \mathbb{H}^{n-1}$, $g := -dt^2 + t^2 \langle \cdot, \cdot \rangle_{\mathbb{H}^{n-1}}$.

It suffices to show that the map $\varphi : W \rightarrow I^+(0) \subseteq \mathbb{R}^n_1$, $(s, x) \mapsto sv(x)$ is an isometry, as this will imply $\gamma_W((sv(x), t, y)) = \tau_{\mathbb{R}^n_1}(sv(x), ty(y)) = \tau_C((s, x), (t, y))$. We have $D\varphi|_{(r,z)} = rD\psi|_z \circ pr_{T\mathbb{H}^{n-1}} + \psi(z)pr_{T\mathbb{R}^+}$, hence for vectors $(S, X), (T, Y) \in T_r\mathbb{R}^+ \times T_z\mathbb{H}^{n-1}$ we get

$$
g((S, X), (T, Y)) = -ST + r^2\langle X, Y \rangle_{\mathbb{H}^{n-1}} = -ST + r^2\langle D\psi X, D\psi Y \rangle_{\mathbb{R}^n_1} = \langle S\psi(z), T\psi(z) \rangle_{\mathbb{R}^n_1} + \langle rD\psi|_zX, rD\psi|_zY \rangle_{\mathbb{R}^n_1} = \langle D\varphi(S, X), D\varphi(T, Y) \rangle_{\mathbb{R}^n_1},
$$

because $\langle \psi(z), D\psi|_z X \rangle_{\mathbb{R}^n_1} = 0$ for any $X \in T_z\mathbb{H}^{n-1}$. So $\varphi$ is an isometry, as claimed.

Coming back to the general case, to equip the cone $Y$ that results from identifying all points with first component 0 in $(0, \infty) \times X$ as defined above with a time separation function, we proceed analogously, with the metric
\( d_X \) of \( X \) taking over the role of \( \theta = d_{\mathbb{H}} \) from Remark 2.1. Thus we say that \((s,p) \leq (t,q)\) (resp. \( (s,p) \ll \ll (t,q) \)) if \( s \leq t \) and \( s^2 + t^2 - 2st \cosh d_X(p,q) \geq 0 \) (resp. \( > 0 \)) in both cases. Then
\[
\tau((s,p),(t,q)) := \sqrt{s^2 + t^2 - 2st \cosh d_X(p,q)},
\]
and \( \tau((s,p),(t,q)) := 0 \) otherwise.

**Proposition 2.2.** \((Y,d_c,\ll,\leq,\tau)\) is a Lorentzian pre-length space. Moreover, \( \tau \) is continuous.

**Proof.** Since the causal and timelike relations are defined via \( \tau \), and since \( \tau \) is clearly continuous with respect to \( d_c \), it only remains to check the reverse triangle inequality for \( \tau \). So let \((s,p),(t,q),(u,r) \in Y\) and fix three comparison points \( \tilde{p}, \tilde{q}, \tilde{r} \in \mathbb{H}^2 \subseteq \mathbb{R}^3 \) with \( d_{\mathbb{H}^2}(\tilde{p}, \tilde{q}) = d_X(p,q) \), and so on (note that some of these distances might be zero). Then from Remark 2.1 and the definition of \( \tau \) it follows that (denoting by \( \hat{\tau} \) the time separation function in \( \mathbb{R}^3 \)), \( \hat{\tau}(s\tilde{p},t\tilde{q}) = \tau((s,p),(t,q)) \), etc. The reverse triangle inequality for \( \tau \) therefore is immediate from that of \( \hat{\tau} \).

**Lemma 2.3.** Suppose that \( 0Y \neq (s,p) \ll (t,q) \in Y \).

(i) Let \( \gamma : [0,a] \to Y \), \( \gamma(\lambda) = (r(\lambda), \sigma(\lambda)) \) be a maximizing timelike curve from \((s,p)\) to \((t,q)\). Then \( \sigma \) is a minimizing geodesic from \( p \) to \( q \) in \( X \).

(ii) Conversely, suppose that \( \sigma \) is a minimizing geodesic from \( p \) to \( q \) in \( X \). Let \( \tilde{y}_0 \) and \( \tilde{y}_1 \) be points in \( I^+(0) \subseteq \mathbb{R}^2_1 \) with distance \( r(0) := s \) resp. \( r(a) := t \) from 0 and such that the hyperbolic angle \( \arccosh(-\frac{\tilde{y}_0 \cdot \tilde{y}_1}{\tilde{y}_0 \cdot \tilde{y}_1}) \) between them is \( d_X(p,q) \). For \( \lambda \in [0,a] \), let \( r(\lambda) \) be the distance of the intersection of the straight line connecting \( \tilde{y}_0 \) to \( \tilde{y}_1 \) with the half-ray in \( I^+(0) \) that has hyperbolic angle \( d_X(p,\sigma(\lambda)) = \lambda \) with the half-ray through \( \tilde{y}_0 \). Then \( \lambda \mapsto (r(\lambda), \sigma(\lambda)) \) is a \( \tau \)-realizing curve from \((s,p)\) to \((t,q)\) in \( Y \).

**Proof.** (i) Let \( \lambda_1 < \lambda_2 < \lambda_3 \in [0,a] \), let \( y_i := (r(\lambda_i), \sigma(\lambda_i)) \) and pick points \( \tilde{y}_i \) \((i = 1,2,3)\) in \( I^+(0) \subseteq \mathbb{R}^2_1 \) such that their distance from 0 is \( r(\lambda_i) \), the hyperbolic angle between \( \tilde{y}_1 \) and \( \tilde{y}_2 \) is \( d_X(\sigma(\lambda_1), \sigma(\lambda_2)) \), and the hyperbolic angle between \( \tilde{y}_2 \) and \( \tilde{y}_3 \) is \( d_X(\sigma(\lambda_2), \sigma(\lambda_3)) \). This means that \( \tau(y_1,y_2) = \hat{\tau}(\tilde{y}_1, \tilde{y}_2) \), as well as \( \tau(y_2,y_3) = \hat{\tau}(\tilde{y}_2, \tilde{y}_3) \). Now by assumption, \( \tau(y_1,y_3) = \tau(y_1,y_2) + \tau(y_2,y_3) \), and the ensuing equality for \( \hat{\tau} \) implies that the \( \tilde{y}_i \) must lie on a straight line in \( \mathbb{R}^2_1 \). Consequently, their hyperbolic angles must add up, i.e., \( d_X(\sigma(\lambda_1), \sigma(\lambda_2)) + d_X(\sigma(\lambda_2), \sigma(\lambda_3)) = d_X(\sigma(\lambda_1), \sigma(\lambda_3)) \). It follows that \( \sigma \) is indeed distance-realizing.

(ii) This is straightforward from the definition of \( \tau \) and Remark 2.1. \( \square \)
As an immediate consequence of Lemma 2.3 (ii) (and the obvious fact that \( s \mapsto (s, q) \) is a realizing geodesic from \( 0_Y = (0, q) \) to \( (t, q) \) for all \( t > 0, q \in X \)) we obtain:

**Corollary 2.4.** Any two causally related points in \( Y \) can be connected by a realizing geodesic, i.e., \( Y \) is geodesic.

The following result establishes, in the present setting, a relation between metric curvature bounds in the Alexandrov space \( X \) and timelike curvature bounds in the Minkowski cone \( Y \), foreshadowing analogous results for generalized cones in Section 5. In particular, the following theorem is a special case of Theorems 5.7 and 5.3 below (and is analogous to the result in the metric case, cf. [BBH01, Thm. 4.7.1]).

**Theorem 2.5.** Let \( Y = \text{Cone}(X) \) be the Minkowski cone over a geodesic length space \( X \). Then \( Y \) has timelike curvature bounded below (above) by 0 if and only if \( X \) is an Alexandrov space of curvature bounded below (above) by \(-1\).

**Proof.** We observe that timelike comparison triangles for \( Y \) and comparison triangles for \( X \) can be related in the following way: Let \((s, p) \ll (t, q) \ll (u, r) \in Y \) be the vertices of a timelike triangle \( \triangle \) in \( Y \). If \((s, p) \neq 0_Y \), choose three comparison points \( \tilde{p}, \tilde{q}, \tilde{r} \in \mathbb{H}^2 \subseteq \mathbb{R}^3_1 \) with \( d_{\mathbb{H}^2} (\tilde{p}, \tilde{q}) = d_X (p, q) \), and so on (note that \( \tilde{p}, \tilde{q}, \tilde{r} \) need not be pairwise distinct). If \((s, p) = 0_Y \), choose two points \( \tilde{q}, \tilde{r} \in \mathbb{H}^2 \) with \( d_{\mathbb{H}^2} (\tilde{p}, \tilde{q}) = d_X (p, q) \). Then by definition of \( \tau \), the points \( s \cdot \tilde{p}, t \cdot \tilde{q}, u \cdot \tilde{r} \) in \( \mathbb{R}^3_1 \) form a timelike comparison triangle \( \tilde{\triangle} \) for \( ((s, p), (t, q), (u, r)) \) (note that \( s \cdot \tilde{p} = 0 \) if \( (s, p) = 0_Y \) and that the points \( s \cdot \tilde{p}, t \cdot \tilde{q}, u \cdot \tilde{r} \) will always be pairwise distinct if \((s, p), (t, q), (u, r) \) are, even if \( \tilde{p}, \tilde{q}, \tilde{r} \) are not). Indeed, their \( \tau \)-side lengths in \( \mathbb{R}^3_1 \) are exactly the \( \tau \)-lengths of the original triangle in \( Y \), and so equivalently we may use the two-dimensional Minkowski space \( M \) spanned by \( s \cdot \tilde{p}, t \cdot \tilde{q}, u \cdot \tilde{r} \) as a (flat) comparison space for \( \tilde{\triangle} \). \( M \) is clearly totally geodesic in \( \mathbb{R}^3_1 \), so its time separation function is precisely the restriction of \( \tilde{\tau} \) to \( M \times M \). Thus \( \tilde{\Delta} \) can just as well be viewed as a subset of \( M \).

Suppose now, first, that \( Y \) has timelike curvature bounded below by 0. Let \( p_0 \in X \) and let \( V \subseteq Y \) be a neighborhood of \((1, p_0)\) on which timelike comparison holds. Then there exists \( \varepsilon > 0 \) and a neighborhood \( U \subseteq X \) of \( p_0 \) such that \( (1 - \varepsilon, 1 + \varepsilon) \times U \subseteq V \) and any triangle \((p, q, r)\) in \( U \) can be lifted to a timelike triangle \((s, p) \ll (t, q) \ll (u, r) \) in \( V \) (the last requirement follows from \((1)\) and Lemma 2.3 (ii) by shrinking \( U \) but keeping \( \varepsilon \) fixed). Let \((p, q, r)\) form a triangle in \( U \). Also, let \( m, n \) be points on the sides of \((p, q, r)\) and \( \tilde{m}, \tilde{n} \) be corresponding points on the sides of a comparison triangle \((\tilde{p}, \tilde{q}, \tilde{r})\) in \( \mathbb{H}^2 \). W.l.o.g. (renaming points if necessary) \( m \) lies on the side from \( p \) to \( q \) and \( n \) on the side from \( q \) to \( r \). Given realizing geodesics in \( U \) for the edges of \((p, q, r)\), by Lemma 2.3 (ii) we obtain corresponding realizing geodesics.
for the sides of the triangle $\Delta = ((s,p),(t,q),(u,r))$ in $V$. Note that the points $M = (r_{pq}(\lambda_m), m)$ and $N = (r_{qr}(\lambda_n), n)$ on these geodesics satisfy $(s,p) \leq M \leq (t,q) \leq N \leq (u,r)$ and are timelike related (or equal). Let $M = \ell_{\vec{m}} \vec{m}, N = \ell_{\vec{n}} \vec{n}$ be points in $\mathbb{R}^3$ on the sides of the comparison triangle $\tilde{\Delta} := (s \cdot \tilde{p}, t \cdot \tilde{q}, u \cdot \tilde{r})$ in $\mathbb{R}^3$. From the construction of $r(\lambda)$ in Lemma 2.3 (ii) we see that $r_{pq}(\lambda_m) = \ell_{\vec{m}}$ and $r_{qr}(\lambda_n) = \ell_{\vec{n}}$. So $M, N \in \Delta$ and $\tilde{M}, \tilde{N} \in \tilde{\Delta}$ are corresponding points and by (1) and the monotonicity of $\cosh$ it then follows that $d_X(m,n) \geq d_{\mathbb{H}^2}(\tilde{m}, \tilde{n})$, because $\tau(M,N) \leq \tilde{\tau}(\tilde{M}, \tilde{N})$.

Conversely, if $X$ has curvature bounded below by $-1$, then a similar (in fact, easier) argument, this time based on Lemma 2.3 (i), shows that $Y$ has timelike curvature bounded below by 0.

Bounds from above can be treated analogously. \qed

3 Generalized cones

In this section, we introduce a generalization of warped products of metric spaces to the Lorentzian setting.

**Definition 3.1.** For $(X,d)$ a metric space and $I \subseteq \mathbb{R}$ an open interval, set $Y := I \times X$ and put the product metric on $Y$, i.e., $D((t,x),(t',x')) = |t - t'| + d(x,x')$ for $(t,x),(t',x') \in Y$. Let $f: I \rightarrow (0, \infty)$ be continuous. Then $Y \equiv I \times f X$ is called a generalized cone and $f$ is called warping function. With this notation, i.e., $I \times f X$, we indicate that the Lorentzian structure (to be introduced below) on the product $I \times X$ can be thought of as "$-dt^2 + f^2 d_X^2$".

Alternatively, generalized cones can also be called (Lorentzian) warped products with one-dimensional base. Henceforth, all topological notions refer to the metric topology induced by $D$. Note, however that the concrete form of the metric on $I \times X$ plays no role as long as it induces the given metric structures on $I$ and $X$, respectively.

We first turn to the question of introducing an appropriate Lorentzian structure on a generalized cone. To this end, we have to define causal curves.

**Definition 3.2.** Let $Y = I \times f X$ be a generalized cone and let $\gamma: J \rightarrow Y$ be an absolutely continuous curve (with respect to $D$). Such a curve has components $\gamma = (\alpha, \beta)$, where $\alpha: J \rightarrow I$ and $\beta: J \rightarrow X$ are both absolutely continuous, and the metric derivative of $\beta, v_\beta$, exists almost everywhere (cf. [AGS05, Thm. 1.1.2]). We additionally require that $\alpha$ is strictly monotonous. The curve $\gamma$ is called

$$
\begin{align*}
\text{timelike} & \quad \alpha^2 + (f \circ \alpha)^2 v_\beta^2 < 0 \\
\text{null} & \quad \alpha^2 + (f \circ \alpha)^2 v_\beta^2 = 0 \\
\text{causal} & \quad \alpha^2 + (f \circ \alpha)^2 v_\beta^2 \leq 0,
\end{align*}
$$
almost everywhere. It is called future/past directed causal if \( \alpha \) is strictly monotonically increasing/decreasing, i.e., \( \dot{\alpha} > 0 \) or \( \dot{\alpha} < 0 \) almost everywhere.

**Remark 3.3.** So far in the development of the theory of Lorentzian length spaces, locally Lipschitz continuous curves were used as causal curves. However, as we shall establish in Lemma 3.13 below, every absolutely continuous causal curve has a parametrization as a Lipschitz curve. So using absolutely continuous curves is compatible with the previous works [KS18, GKS19]. Moreover, parametrizing a timelike curve with respect to arclength anyways only gives an absolutely continuous curve in general — an issue that also necessitated a special treatment in [KS18, Subsec. 3.7]. Analogous questions arise for spacetimes with continuous metrics, in which case we refer to [GKSS20].

**Definition 3.4.** (Length of a causal curve) Let \( \gamma = (\alpha, \beta) : [a, b] \to Y \) be a causal curve. Its length \( L(\gamma) \) is defined as

\[
L(\gamma) := \int_a^b \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2 v_\beta^2}.
\]

**Remark 3.5.** Note that \( \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2 v_\beta^2} \) is integrable as \( \alpha \) is absolutely continuous, \( f \) is bounded on the compact image of \( \alpha \) and the metric derivative is integrable by [AGS05, Thm. 1.1.2]. Moreover, from this it follows that the map \( t \mapsto L(\gamma|_{[a, t]}) = \int_a^t \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2 v_\beta^2} \) is absolutely continuous.

**Lemma 3.6.** Let \((Z, \rho)\) be a metric space, \( J, J' \) intervals, \( \lambda : J \to Z \) an absolutely continuous curve and \( \phi : J' \to J \) strictly monotonous and such that both \( \phi \) and \( \phi^{-1} \) are absolutely continuous. Then \( \xi := \lambda \circ \phi \) is absolutely continuous and

\[
v_\xi = (v_\lambda \circ \phi) |\phi'|.
\]

**Proof:** That \( \xi \) is absolutely continuous follows as in [Nat55] Thm. 3, Ch. IX, §1]. Moreover, the metric derivative of \( \lambda \) exists almost everywhere (cf. [AGS05 Thm. 1.1.2]), so let \( t \in J \) be such a point. Then for any \( h \in \mathbb{R} \) such that \( t + h \in J \) we have

\[
\rho(\lambda(t + h), \lambda(t)) = |h| v_\lambda(t) + r(h),
\]

where the remainder term satisfies \( r(h) = o(|h|) \). Next, let \( s \in J' \) such that both \( \phi'(s) \) and \( \lambda'(\phi(s)) \) exist. The set of all such \( s \) has full measure in \( J' \) because the absolutely continuous function \( \phi^{-1} \) maps sets of measure zero to sets of measure zero (Lusin’s property, cf. e.g. [AT04 Thm. 3.4.3]).
Furthermore, let $h \in \mathbb{R}$ with $s + h \in J'$. We conclude that

$$
\rho(\xi(s + h), \xi(s))
$$

$\rho(\lambda(\phi(s + h)), \lambda(\phi(s))) = |h\phi'(s) + r(h)|v_\lambda(\phi(s)) + r(h\phi'(s) + r(h))$

$= |h||\phi'(s)||v_\lambda(\phi(s)) + o(|h|)$

$= |h||\phi'(s)||v_\lambda(\phi(s)) + o(|h|)$,

yielding the claim.

A direct corollary of the above lemma is that the length of causal curves is invariant under reparametrizations.

**Corollary 3.7.** The length $L$ is reparametrization invariant, i.e., if $\gamma = (\alpha, \beta)$ is a causal curve defined on some interval $J$ and $\phi : J' \to J$ is strictly increasing and such that $\phi$ and $\phi^{-1}$ are absolutely continuous, then $\gamma \circ \phi$ is a causal curve of the same length and time orientation.

**Remark 3.8.** Note that this means that $Y$ with these future/past directed causal/timelike curves and this length functional is an example of a Lorentzian length structure as defined in the Appendix, see Definition A.2.

To establish that the length functional $L$ is upper semicontinuous (with respect to pointwise convergence) we need to describe the length in a variational way. As we show below, the variational length is the same as the length defined above via the (metric) derivative of the curve.

**Definition 3.9.** Let $\gamma = (\alpha, \beta) : [a, b] \to Y$ be a causal curve. For $s, t \in I$, $s \leq t$, set $m_{s,t} := \min_{r \in [s, t]} f(r) > 0$. Then the variational length of $\gamma$ is defined as

$$
L_{\text{var}}(\gamma) := \inf_{a = t_0 \leq t_1 \leq \ldots \leq t_N = b} \sum_{i=0}^{N-1} \sqrt{(\alpha(t_{i+1}) - \alpha(t_i))^2 - m^2_{\alpha(t_i), \alpha(t_{i+1})}d(\beta(t_i), \beta(t_{i+1}))^2}.
$$

To see that $L_{\text{var}}$ is well-defined we need the following lemma.

**Lemma 3.10.** Let $(X, d)$ be a metric space and let $\gamma = (\alpha, \beta) : [a, b] \to Y$ be a causal curve. Then for any $a \leq s \leq t \leq b$ we have:

$$
(\alpha(t) - \alpha(s))^2 - m^2_{\alpha(s), \alpha(t)}d(\beta(s), \beta(t))^2 \geq 0.
$$

**Proof:** Without loss of generality let $\gamma$ be future directed, i.e., $\dot{\alpha} > 0$ almost everywhere. Since $\gamma$ is causal we have $(f \circ \alpha)^2 v_\beta^2 \leq \dot{\alpha}^2$ and, as all
involved quantities are non-negative, in fact \((f \circ \alpha)v_\beta \leq \dot{\alpha}\). Denote by \(L^d\) the length functional of \((X, d)\), then

\[
m_{\alpha(s), \alpha(t)} L^d(\beta|[s, t]) = m_{\alpha(s), \alpha(t)} \int_s^t v_\beta \leq \int_s^t (f \circ \alpha)v_\beta \leq \int_s^t \dot{\alpha} = \alpha(t) - \alpha(s).
\]

Finally, as \(d(\beta(s), \beta(t)) \leq L^d(\beta|[s, t])\) we conclude that \(m_{\alpha(s), \alpha(t)}^2 d(\beta(s), \beta(t))^2 \leq (\alpha(t) - \alpha(s))^2\).

Also note that the variational length is invariant under reparametrizations as it is defined via partitions, cf. e.g. the proof of \cite[Prop. 1.1.8]{Pap14}. Additionally, \(L^{\text{var}}\) is additive which is easily inferred from the inequality (ii) in the next Lemma.

**Lemma 3.11.** Let \(a, b \in I\) with \(a \leq s \leq t \leq u \leq b\) and \(x, y, z \in X\) such that

\[
(t - s)^2 - m_{s,t}^2 d(x, y)^2 \geq 0, \quad (3)
\]

\[
(u - t)^2 - m_{t,u}^2 d(y, z)^2 \geq 0. \quad (4)
\]

Then

(i) \((u - s)^2 - m_{s,u}^2 d(x, z)^2 \geq 0\), and

(ii) \[
\sqrt{(t - s)^2 - m_{s,t}^2 d(x, y)^2} + \sqrt{(u - t)^2 - m_{t,u}^2 d(y, z)^2} \\
\leq \sqrt{(u - s)^2 - m_{s,u}^2 d(x, z)^2}.
\]

**Proof:** Clearly, \(m_{s,u} = \text{min}(m_{s,t}, m_{t,u})\) and without loss of generality we may assume that \(m_{s,u} = m_{s,t}\). Let \((X, Y, Z)\) be a comparison triangle of \((x, y, z)\) in the plane \(\mathbb{R}^2\), i.e., \(\|X - Y\| = d(x, y), \|X - Z\| = d(x, z)\) and \(\|Y - Z\| = d(y, z)\). For \(c > 0\) define the scaled Minkowski metric \(\eta_c\) on \(\mathbb{R}^3\) as \(\eta_c := -(dx^0)^2 + c((dx^1)^2 + (dx^2)^2)\). We claim that \((s, X) \leq (t, Y) \leq (u, Z)\) in \((\mathbb{R}^3, \eta_{m_{s,t}})\). That \((s, X) \leq (t, Y)\) follows directly from (4), and that \((t, Y) \leq (u, Z)\) follows from (3) and \(m_{s,t} \leq m_{t,u}\). Thus \((t, X) \leq (u, Z)\) by the transitivity of the causal relation \(\leq\) in \((\mathbb{R}^3, \eta_{m_{s,t}})\), giving (i).

To show (ii), denote by \(P\) the time separation function of \((\mathbb{R}^3, \eta_{m_{s,t}})\). Then

\[
P((s, X), (t, Y)) = \sqrt{(t - s)^2 - m_{s,t}^2 d(x, y)^2}, \quad \text{and}
\]

\[
P((t, Y), (u, Z)) = \sqrt{(u - t)^2 - m_{t,u}^2 d(y, z)^2} \geq \sqrt{(u - t)^2 - m_{t,u}^2 d(y, z)^2}.
\]

16
Consequently, by the reverse triangle inequality for $P$ we obtain
\[
\sqrt{(t-s)^2 - m_{s,t}^2 d(x,y)^2} + \sqrt{(u-t)^2 - m_{t,u}^2 d(y,z)^2} \\
\leq P((s,X),(t,Y)) + P((t,Y),(u,Z)) \leq P((s,X),(u,Z))
\]
as $m_{s,u} = m_{s,t}$.

**Remark 3.12.** The above Lemma 3.11 shows that the function
\[
T((t,x),(s,y)) := (t-s)^2 - m_{s,t}^2 d(x,y)^2,
\]
if non-negative, and $T((t,x),(s,y)) := 0$ otherwise, satisfies the reverse triangle inequality. So in principle it could also be used to define a Lorentzian (pre-)length space. However, as it only involves the minimum of $f$ on the interval $[s,t]$ it does not contain the full information of $f$ on this interval and it is not compatible with the smooth case (i.e., if $X$ is a smooth Riemannian manifold and $f$ is smooth). Despite this, it proves very useful when handling the variational length because the Lorentzian (pre-)length space definition of length in $(I \times X, \ll, \leq, T)$ coincides with the variational length $L^{\text{var}}$ defined above. We will show in Proposition 3.14 that the variational length equals the length defined in Definition 3.4.

**Lemma 3.13.** Every future directed causal curve has a reparametrization that is locally Lipschitz continuous. In particular, any future directed causal curve defined on a compact interval $\gamma : [a,b] \to Y$ has a reparametrization $\tilde{\gamma}$ such that $\tilde{\gamma}(t) = (t,\tilde{\beta}(t))$. Similarly, $\gamma$ has a reparametrization $\gamma' = (\alpha',\beta')$ such that $\beta' : [0,Ld(\beta')] \to X$ is parametrized with respect to arc length.

**Proof:** As this is a local question we may restrict to the case of compact intervals. Let $\gamma = (\alpha,\beta) : [a,b] \to Y$ be future directed causal, then $\dot{\alpha} > 0$ almost everywhere. Thus by a theorem of Zareckii (cf. [Nat55], p. 271), $\alpha^{-1}$ is absolutely continuous, hence can serve as an admissible parametrization for $\gamma$. Then $\tilde{\gamma} := \gamma \circ \alpha^{-1}$ satisfies $\tilde{\gamma}(t) = (t,\tilde{\beta}(t))$, where $\tilde{\beta} := \beta \circ \alpha^{-1}$.

By Corollary 3.7, $\tilde{\gamma}$ is future directed causal and so we have $f^2 v_\tilde{\beta}^2 \leq 1$. Thus $v_\tilde{\beta} \leq \frac{1}{C}$, where $C := \min_{r \in [\alpha(a),\alpha(b)]} f(r) > 0$. This implies that $\tilde{\beta}$ is Lipschitz continuous, due to
\[
d(\tilde{\beta}(s),\tilde{\beta}(t)) \leq L^d(\tilde{\beta}|_{[s,t]}) = \int_s^t v_\tilde{\beta} \leq \frac{1}{C}|t-s|,
\]
where $\alpha(a) \leq s < t \leq \alpha(b)$.

**Proposition 3.14.** Let $(X,d)$ be a metric space. Then the variational length of any causal curve $\gamma$ in $Y = I \times f X$ agrees with its length, i.e., $L(\gamma) = L^{\text{var}}(\gamma)$. 

17
Proof: Let $\gamma$ be a (without loss of generality) future directed causal curve. As $L^{\text{var}}$ and $L$ are invariant under reparametrizations, using Lemma 3.13 we may assume that $\gamma: [a, b] \to Y$ is parametrized as $\gamma(t) = (t, \beta(t))$ (where $a, b \in I$). Let $a \leq s < t \leq b$, then $d(\beta(s), \beta(t)) \leq L^d(\beta|_{[s, t]}) = \int_s^t v_\beta$ and so

\[
\frac{1}{(t-s)^2} d(\beta(s), \beta(t))^2 \leq \frac{1}{(t-s)^2} \left( \int_s^t v_\beta \right)^2 \leq \frac{1}{t-s} \int_s^t v_\beta^2,
\]

where in the last step we used Jensen’s inequality. Thus we obtain

\[
\frac{1}{t-s} d(\beta(s), \beta(t))^2 \leq \int_s^t v_\beta^2.
\]

(5)

Again using Jensen’s inequality we estimate

\[
\left( \frac{1}{t-s} \int_s^t \sqrt{1 - f^2 v_\beta^2} \right)^2 \leq \frac{1}{t-s} \int_s^t \left( 1 - f^2 v_\beta^2 \right) = 1 - \int_s^t f^2 v_\beta^2 = 1 - m_{s,t}^2 \int_s^t v_\beta^2 \leq 1 - m_{s,t}^2 \frac{d(\beta(s), \beta(t))^2}{(t-s)^2},
\]

which yields

\[
\int_s^t \sqrt{1 - f^2 v_\beta^2} \leq (t-s) \sqrt{1 - m_{s,t}^2 \frac{d(\beta(s), \beta(t))^2}{(t-s)^2}}.
\]

(6)

Now let $a = t_0 < t_1 < \ldots < t_N = b$ be a partition of $[a, b]$, then

\[
L(\gamma) = \int_a^b \sqrt{1 - f^2 v_\beta^2} = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \sqrt{1 - f^2 v_\beta^2} \leq \sum_{i=0}^{N-1} \sqrt{(t_{i+1} - t_i)^2 - m_{t_i,t_{i+1}}^2 d(\beta(t_i), \beta(t_{i+1}))^2},
\]

and taking the infimum over all partitions of $[a, b]$ gives $L(\gamma) \leq L^{\text{var}}(\gamma)$.

For the reverse inequality, note that we have by definition of $L^{\text{var}}$ as the infimum over all partitions of the interval that

\[
L^{\text{var}}(\gamma|_{[s, t]}) \leq \sqrt{(t-s)^2 - m_{s,t}^2 d(\beta(s), \beta(t))^2},
\]

(7)

for all $a \leq s < t \leq b$. Let $0 < \varepsilon < b - a$, set $\tilde{b} := b - \varepsilon > a$, $h := \frac{b-a}{N}$ where $N \in \mathbb{N}$ is such that $h \leq \varepsilon$ and set $t_i := a + ih$ for $i = 0, \ldots, N$. 

18
Then for \( t \in [a, b] \) we have \( t + h \in [a, b] \) and hence by Lemma 3.10 that \( h^2 - m_{t,t+h}^2 d(\beta(t), \beta(t+h))^2 \geq 0 \). Consequently we get

\[
\frac{1}{h} \int_a^b \sqrt{h^2 - m_{t,t+h}^2 d(\beta(t), \beta(t+h))^2} \, dt
= \frac{1}{h} \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \sqrt{h^2 - m_{t,t+h}^2 d(\beta(t), \beta(t+h))^2} \, dt
\]

\( \geq \frac{1}{h} \int_0^h L_{\text{var}}(\gamma|_{[a+t, b+q]}) \, dt \)

\( \geq \frac{1}{h} \int_0^h \left( L_{\text{var}}(\gamma) - L_{\text{var}}(\gamma|_{[a,a+q]}) - L_{\text{var}}(\gamma|_{[b,t,b]}) \right) \, dt \)

\( \geq \frac{1}{h} \int_0^h \left( L_{\text{var}}(\gamma) - L_{\text{var}}(\gamma|_{[a,a+b]}) - L_{\text{var}}(\gamma|_{[b,b]}) \right) \, dt \)

\( \geq \frac{1}{h} \int_0^h \sqrt{h^2 - m_{a,a+h}^2 d(\beta(a), \beta(a+h))^2} \, dt \)

\( \geq \frac{1}{h} \int_0^h \sqrt{h^2 - m_{a,a+h}^2 d(\beta(a), \beta(a+h))^2} \, dt \)

\( \geq \frac{1}{h} \int_0^h \sqrt{h^2 - m_{a,a+h}^2 d(\beta(a), \beta(a+h))^2} \, dt \)

where we used additivity of \( L_{\text{var}} \) (cf. Lemma 3.11) and, in (\( *) \), the substitution \( t' = t + a + ih = t + t_i \). As \( 0 \leq \frac{1}{h} \sqrt{h^2 - m_{t,t+h}^2 d(\beta(t), \beta(t+h))^2} \leq 1 \in L^1([a, b]) \) for all \( h \geq 0 \) such that \( t + h \in [a, b] \), we have by dominated convergence and the above inequality that

\[
\int_a^b \sqrt{1 - f^2(v^2)} \, dt = \lim_{h \to 0} \frac{1}{h} \int_a^b \sqrt{1 - m_{t,t+h}^2 \frac{d(\beta(t), \beta(t+h))^2}{h^2}} \, dt
\]

\( \geq L_{\text{var}}(\gamma) - \sqrt{\varepsilon^2 - m_{b-\varepsilon,b}^2 d(\beta(b - \varepsilon), \beta(b))^2} . \)

Thus

\[
L(\gamma) = L(\gamma|_{[a,b-\varepsilon]}) + L(\gamma|_{[b-\varepsilon,b]})
\]

\( \geq L_{\text{var}}(\gamma) - \sqrt{\varepsilon^2 - m_{b-\varepsilon,b}^2 d(\beta(b - \varepsilon), \beta(b))^2} + L(\gamma|_{[b-\varepsilon,b]}) . \)

and letting \( \varepsilon \searrow 0 \) finishes the proof.

**Proposition 3.15.** Let \((X, d)\) be a metric space and let \( \gamma_n, \gamma \ (n \in \mathbb{N}) \) be causal curves defined on the interval \([a, b]\) such that \( \gamma_n \to \gamma \) pointwise. Then

\[
\limsup_n L(\gamma_n) \leq L(\gamma) ,
\]

i.e., \( L \) is upper semicontinuous.
Proof: Let \( \sigma = (a = t_0 < t_1 < \ldots < t_N = b) \) be a partition of \([a,b]\), then the map \( \Phi_\sigma \) defined on the space of causal curves \( \lambda = (\alpha, \beta): [a,b] \to Y \) given by

\[
\Phi_\sigma(\lambda) := \sum_{i=0}^{N-1} \sqrt{(\alpha(t_{i+1}) - \alpha(t_i))^2 - m_{\alpha(t_i), \alpha(t_{i+1})}^2 d(\beta(t_i), \beta(t_{i+1}))^2}
\]

is clearly continuous with respect to pointwise convergence. Then \( L_{\text{var}}(\lambda) = \inf_\sigma \Phi_\sigma(\lambda) \) and so \( L = L_{\text{var}} \) (by Proposition \[3.14\]) is upper semicontinuous as the infimum of continuous functions (cf., e.g., [AB06, Lem. 2.41]).

**Theorem 3.16.** (Limit curve theorem) Let \((X, d)\) be a metric space and let \( \gamma_n = (\alpha_n, \beta_n) \ (n \in \mathbb{N}) \), \( \gamma = (\alpha, \beta): [a,b] \to Y \) be absolutely continuous curves such that each \( \gamma_n \) is future/past directed causal. Moreover, let \( \dot{\alpha} \neq 0 \) almost everywhere and let \( \gamma_n \to \gamma \) pointwise. Then \( \gamma \) is causal.

Proof: Let \( a \leq s < t \leq b \) be such that \( \dot{\alpha}(s) \) and \( v_\beta(s) \) exist. For every \( n \in \mathbb{N} \) we have by Lemma \[3.10\]

\[
(\alpha_n(t) - \alpha_n(s))^2 - m_{\alpha_n(s), \alpha_n(t)}^2 d(\beta_n(s), \beta_n(t))^2 \geq 0.
\]

Taking the limit \( n \to \infty \) yields

\[
(\alpha(t) - \alpha(s))^2 - m_{\alpha(s), \alpha(t)}^2 d(\beta(s), \beta(t))^2 \geq 0,
\]

so

\[
\left( \frac{\alpha(t) - \alpha(s)}{t-s} \right)^2 - m_{\alpha(s), \alpha(t)}^2 \frac{d(\beta(s), \beta(t))^2}{(t-s)^2} \geq 0.
\]

Now letting \( t \searrow s \) we get

\[
\dot{\alpha}(s)^2 - f(\alpha(s))^2 v_\beta(s)^2 \geq 0.
\]

Moreover, similarly one shows that \( \dot{\alpha} \geq 0 \) (if each \( \gamma_n \) is future directed) or \( \dot{\alpha} \leq 0 \) (if each \( \gamma_n \) is past directed), which yields \( \dot{\alpha} > 0 \) or \( \dot{\alpha} < 0 \) almost everywhere. Thus \( \gamma \) is a future or past directed causal curve.

At this point we define a natural time separation function on \( Y \), which directly generalizes the spacetime case. Some of the results could have been obtained in an even more general setting as they follow just from the existence of a causal structure and a length functional — a fact that was already indicated in [KS18, Rem. 5.11(i)]. For the interested reader we sketch this approach in Appendix \[A\] but it is not needed in the following.
Definition 3.17. (Time separation function) The time separation function (or Lorentzian distance) $\tau: Y \times Y \to [0, \infty]$ is defined as

$$\tau(y, y') := \sup \{ L(\gamma) : \gamma \text{ future directed causal curve from } y \text{ to } y' \},$$

if this set is non-empty, and $\tau(y, y') := 0$ otherwise.

Definition 3.18. (Causal relations) Let $y, y' \in Y$, then $y$ and $y'$ are chronologically related, denoted by $y \ll y'$, if there exists a future directed timelike curve from $y$ to $y'$. Moreover, $y$ and $y'$ are causally related, denoted by $y \leq y'$ if there exists a future directed causal curve from $y$ to $y'$ or $y = y'$.

Moreover, we define the chronological and causal future and past of a point as

$$I^+(y) := \{ y' \in Y : y \ll y' \}, \quad I^-(y) := \{ y' \in Y : y' \ll y \},$$

$$J^+(y) := \{ y' \in Y : y \leq y' \}, \quad J^-(y) := \{ y' \in Y : y' \leq y \}.$$

Lemma 3.19. The relations $\ll$ and $\leq$ are transitive, $\leq$ is reflexive and $\ll \subseteq \leq$.

Proof: Transitivity follows by concatenating curves. Reflexivity of the causal relation $\leq$ as well as the fact that every timelike curve is causal hold by definition. Thus $\ll \subseteq \leq$.

This can be summarized as:

Remark 3.20. The time separation function $\tau$ has the following properties:

(i) $\tau(y, y') = 0$ if $y' \not\leq y$ and

(ii) $\tau(y, y') > 0$ if $y \ll y'$.

Lemma 3.21. (Reverse triangle inequality) Let $y_1, y_2, y_3 \in Y$ with $y_1 \leq y_2 \leq y_3$, then

$$\tau(y_1, y_2) + \tau(y_2, y_3) \leq \tau(y_1, y_3).$$

Proof: This follows from the standard proof from Lorentzian geometry: Let $y_1, y_2, y_3 \in Y$ with $y_1 \leq y_2 \leq y_3$ and assume first that there are future directed causal curves from $y_1$ to $y_2$ and from $y_2$ to $y_3$. Then, given $\varepsilon > 0$ we can find future directed causal curves $\gamma_1$ from $y_1$ to $y_2$ and $\gamma_2$ from $y_2$ to $y_3$ such that $L(\gamma_i) > \tau(y_i, y_{i+1}) - \frac{\varepsilon}{2}$ for $i = 1, 2$. Consequently,

$$\tau(y_1, y_2) + \tau(y_2, y_3) < L(\gamma_1) + L(\gamma_2) + \varepsilon \leq \tau(y_1, y_3) + \varepsilon,$$

as the concatenation of $\gamma_1$ and $\gamma_2$ is a future directed causal curve from $y_1$ to $y_3$. Since $\varepsilon > 0$ was arbitrary the claim follows. In the remaining case where there are no future directed causal curves, say from $y_1$ to $y_2$, we have $\tau(y_1, y_2) = 0$ and $y_1 = y_2$, which implies the claim.
Spacetimes of low regularity (below Lipschitz) can exhibit the unwanted phenomenon of causal bubbling, as shown in [CG12] (cf. also [GKSS20]) for spacetimes with continuous metrics. However, the additional structure of a generalized cone excludes such pathologies.

For the formulation of the following result, we recall some terminology from [CG12]: $Y$ is said to possess the push-up property if the following holds: Whenever $\gamma : [a, b] \to M$ is a future/past directed causal curve from $p = \gamma(a)$ to $q = \gamma(b)$ with $L(\gamma) > 0$, there exists a future/past directed timelike curve connecting $p$ and $q$.

**Proposition 3.22.** (Push-up and openness of $I^\pm$) Every generalized cone $Y = I \times_f X$ such that $(X, d)$ is a length space has the property that $p \ll q$ if and only if there exists a future directed causal curve from $p$ to $q$ of positive length, i.e., push-up holds. Moreover, $I^\pm(p)$ is open for any $p \in Y$.

**Proof:** For each $p_0 \in I \equiv (a, b)$ we define the function $h_{p_0} : (a, b) \to (a, b)$ as the unique maximal solution of the ODE $\frac{d}{ds}h_{p_0} = f \circ h_{p_0}$ with $h_{p_0}(0) = p_0$ on $I$. Here $a_{p_0} = \int_{p_0}^a \frac{1}{f(s)} ds$ and $b_{p_0} = \int_{p_0}^b \frac{1}{f(s)} ds$, and $h_{p_0}$ is the inverse of $r \mapsto \int_{p_0}^r \frac{1}{f(s)} ds$. The function $h_{p_0}$ is strictly increasing, bijective and $C^1$. We are going to show that

$$I^+((p_0, \bar{p})) = \{(q_0, \bar{q}) \in Y : d(\bar{p}, \bar{q}) < b_{p_0} \text{ and } q_0 > h_{p_0}(d(\bar{p}, \bar{q}))\}, \quad (8)$$

which is clearly open. In proving this we will also see that $q \in I^+(p)$ if there exists a causal curve $\gamma$ from $p$ to $q$ with $L(\gamma) > 0$ (i.e., push-up holds).

We first show that $\Lambda(p) := \{(q_0, \bar{q}) \in Y : d(\bar{p}, \bar{q}) < b_{p_0 \, 0} \text{ and } q_0 > h_{p_0}(d(\bar{p}, \bar{q}))\} \subseteq I^+(p)$. Let $q \in \Lambda(p)$ and pick an almost minimizing unit-speed curve $\beta : [0, d(\bar{p}, \bar{q}) + \varepsilon] \to X$ from $\bar{p}$ to $\bar{q}$ in $X$, as well as $c > 0$ such that $d(\bar{p}, \bar{q}) + \varepsilon + c < b_{p_0}$ and $q_0 = h_{p_0}(d(\bar{p}, \bar{q}) + \varepsilon + c)$. We define $\alpha(s) := h_{p_0}(s + \frac{c}{d(\bar{p}, \bar{q}) + \varepsilon}\,s)$. Then $\gamma = (\alpha, \beta)$ is a future directed timelike curve from $p$ to $q$, since

$$\dot{\alpha}(s) = \left(1 + \frac{c}{d(\bar{p}, \bar{q}) + \varepsilon}\right) h_{p_0}(s + \frac{c}{d(\bar{p}, \bar{q}) + \varepsilon}\,s) > h_{p_0}\left(s + \frac{c}{d(\bar{p}, \bar{q}) + \varepsilon}\,s\right) = f\left(h_{p_0}\left(s + \frac{c}{d(\bar{p}, \bar{q}) + \varepsilon}\,s\right)\right) = f(\alpha(s)).$$

Now we show that if $q \notin \Lambda(p)$, then there cannot exist any future directed causal curve $\gamma$ from $p$ to $q$ with $L(\gamma) > 0$. Assume to the contrary that such a curve exists and is parametrized so that $\gamma : [p_0, q_0] \to Y$ and $\gamma(s) = (s, \gamma(s))$. We start with the case where $d(\bar{p}, \bar{q}) < b_{p_0}$ but $q_0 \leq h_{p_0}(d(\bar{p}, \bar{q}))$. Let $\beta_\varepsilon : [0, d(\bar{p}, \bar{q}) + \varepsilon] \to X (\varepsilon > 0)$ be an almost minimizing unit-speed curve in $X$ from $\bar{p}$ to $\bar{q}$ and set $\tilde{\beta}_\varepsilon := \beta_\varepsilon \circ h_{p_0}^{-1}[p_0, h_{p_0}(d(\bar{p}, \bar{q}) + \varepsilon)]$ and $n_\varepsilon(s) := (s, \tilde{\beta}_\varepsilon(s))$. Then $n_\varepsilon : [p_0, h_{p_0}(d(\bar{p}, \bar{q}) + \varepsilon)] \to Y$ is a null curve, or equivalently, $v_{\beta_\varepsilon}(s) =
Thus, we have
\[
L^d(\tilde{\gamma}_\epsilon) = \int_{p_0}^{h_{p_0}(d(\tilde{\gamma})+\epsilon)} v_{\tilde{\gamma}_\epsilon} = \int_{p_0}^{h_{p_0}(d(\tilde{\gamma})+\epsilon)} \frac{1}{f}.
\]
So letting \( \epsilon \to 0 \) gives \( d(\tilde{p}, \tilde{q}) = \int_{p_0}^{h_{p_0}(d(\tilde{p}, \tilde{q}))} \frac{1}{f} \geq \int_{p_0}^{q_0} \frac{1}{f} \). Since \( \gamma \) is causal, we have \( v_\tilde{\gamma} \leq \frac{1}{f} \) and furthermore since \( L(\gamma) > 0 \) it must be strictly less than \( \frac{1}{f} \) on some subset of \([p_0, q_0] \) having non-zero measure. So,
\[
d(\tilde{p}, \tilde{q}) = \int_{p_0}^{h_{p_0}(d(\tilde{p}, \tilde{q}))} \frac{1}{f} \geq \int_{p_0}^{q_0} \frac{1}{f} = \int_{p_0}^{q_0} v_\tilde{\gamma} = L^d(\tilde{\gamma}),
\]
a contradiction.

Finally, we treat the case where \( d(\tilde{p}, \tilde{q}) \geq b_{p_0} \). We again assume that \( \gamma : [p_0, q_0] \to Y \) with parametrization \( \gamma(s) = (s, \tilde{\gamma}(s)) \) is a future directed causal curve from \( p \) to \( q \). Since \( q_0 < b \) and \( h_{p_0}(s) \to b \) as \( s \to p_0 \) we can choose \( \epsilon > 0 \) such that \( q_0 < h_{p_0}(b_{p_0} - \epsilon) \). Let further \( x := \gamma(x_0) = (x_0, \tilde{x}) \) be the point on \( \gamma \) such that \( b_{p_0} - \epsilon = d(\tilde{p}, \tilde{x}) \). So \( \gamma|_{(0, x_0]} \) is a causal curve from \( p \) to \((x_0, \tilde{x})\) with \( d(\tilde{p}, \tilde{x}) < b_{p_0} \) and \( x_0 < q_0 < h_{p_0}(b_{p_0} - \epsilon) = h_{p_0}(d(\tilde{p}, \tilde{x})) \).

Hence, as above,
\[
d(\tilde{p}, \tilde{x}) = \int_{p_0}^{h_{p_0}(d(\tilde{p}, \tilde{x}))} \frac{1}{f} \geq \int_{p_0}^{q_0} \frac{1}{f} \geq \int_{p_0}^{q_0} v_\tilde{\gamma} = L^d(\tilde{\gamma})
\]
leads to a contradiction.

**Remark 3.23.** The preceding result can be understood as establishing that generalized cones are *causally plain*, i.e., there is no causal bubbling. This notion of causal plainness is however not the same as the one in [CG12] Def. 1.16] for spacetimes with continuous metrics. The reason is that we cannot speak about approximating smooth metrics, and hence have no notion of timelike curves for approximating metrics which have their lightcones inside those of the original Lorentzian metric. However, as shown in [GKSS20] our notion of causal plainness (i.e., the condition that push-up holds) is equivalent to the absence of *external bubbling*, cf. [GKSS20] Thm. 2.12]. Furthermore, if \( X \) is a Riemannian manifold with continuous metric \( h \) (i.e., if \( Y \) is a Lorentzian manifold with continuous metric \( g = -dt^2 + f(t)^2 h \)) it can be seen from the description [8] of \( I^+ \) that \( I_+ = I^+ \) and locally \( \partial I^+ = \partial J^+ \), so that in this case \( Y \) is indeed causally plain as defined in [CG12] Def. 1.16]. Moreover, the preceding result also sheds some light on the causality of the so-called *Colombini-Spagnolo metrics* (cf. [CG12] Sec. 2.1], [USS9]), i.e., metrics on \( \mathbb{R} \times S^1 \) of the form \( -dt^2 + f(t, x)dx^2 \), where \( f(t, x) = \frac{F(t)}{F(x)} \), for a specific continuous positive function \( F \).
Corollary 3.24. Let \( (X,d) \) be a length space. The following description, analogous to (8), holds for \( J^+ \):

\[
J^+((p_0,\bar{p})) = I^+((p_0,\bar{p})) \cup \{(q_0,\bar{q}) \in Y : \exists \text{ a minimizing curve in } X \text{ from } \bar{p} \text{ to } \bar{q} \text{ and } d(\bar{p},\bar{q}) < b_{p_0} \text{ and } q_0 = h_{p_0}(d(\bar{p},\bar{q}))\}.
\]

Further, if \( X \) is geodesic, then \( J^+ (p) \) is closed.

**Proof.** That \( J^+ (p) \) is closed if \( X \) is geodesic follows from (9): Let \( q_k = (q_{0k},\bar{q}_k) \) be elements of the right hand side of (9) that converge to \( q = (q_0,\bar{q}) \). To see that also \( q \) is an element of this set it suffices to exclude the case where \( d(\bar{p},\bar{q}_k) \to b_{p_0} = d(\bar{p},\bar{q}) \). However, in this case we would have \( q_0 = h_{p_0}(d(\bar{p},\bar{q}_k)) \to b \), resulting in \( q_0 = b \), a contradiction to \( q_0 \in I = (a,b) \).

It remains to show (9). First, let \( q = (q_0,\bar{q}) \) be an element of the right hand side of (9), let \( \beta : [0,d(\bar{p},\bar{q})] \to X \) be a minimizing unit speed curve and set \( \gamma : (0,\beta(0)) \to Y \), \( \gamma(s) := (h_{p_0}(s),\beta(s)) \). Then \( \gamma \) is a null curve connecting \( p \) and \( q \), so \( q \in J^+(p) \).

Conversely, we have to show that for any \( (q_0,\bar{q}) \in J^+(p) \setminus I^+(p) \) there must exist a minimizing curve in \( X \) from \( \bar{p} \) to \( \bar{q} \) with \( d(\bar{p},\bar{q}) < b_{p_0} \) and \( q_0 = h_{p_0}(d(\bar{p},\bar{q})) \). Let \( \gamma = (\alpha,\beta) \) be a causal curve from \( p \) to \( q \) with \( \beta : [0,d] \to X \) parametrized by arc-length. By Proposition 3.22, since \( q \notin I^+(p) \), we must have \( L(\gamma) = 0 \), i.e., \( \alpha^2 = f^2 v_\beta^2 = f^2 \) a.e., so \( \alpha = f \circ \alpha \). Since \( \alpha(0) = p_0 \) it follows that \( \alpha = h_{p_0} \). If \( \beta \) were not minimizing, i.e., if \( d(\bar{p},\bar{q}) < d \), there would exist a curve \( \bar{\beta} : [0,d] \to X \) from \( \bar{p} \) to \( \bar{q} \), parametrized proportional to arclength, which is strictly shorter than \( \beta \), hence satisfies \( v_\beta < 1 \) a.e. Then \( \bar{\gamma} := (\alpha,\bar{\beta}) \) is a timelike curve from \( p \) to \( q \), contradicting the fact that \( q \notin I^+(p) \). Consequently, \( \beta \) must be minimizing. Thus \( d = d(\bar{p},\bar{q}) \), so that \( q_0 = \alpha(d(\bar{p},\bar{q})) = h_{p_0}(d(\bar{p},\bar{q})) \). \( \square \)

Lemma 3.25. Let \( Y = I \times f X \) be a generalized cone, where \( (X,d) \) is a length space. Then the time separation function \( \tau \) is lower semi-continuous (with respect to \( D \)).

**Proof:** As the standard proof from Lorentzian geometry only uses openness of \( I^+ \) and the reverse triangle inequality it still works in our setting: Let \( y,y' \in Y \) and first assume that \( 0 < \tau(y,y') < \infty \) (in the case \( \tau(y,y') = 0 \) there is nothing to show). Let \( 0 < \varepsilon < \tau(y,y') \), then by definition of \( \tau \) there exists a future directed causal curve \( \gamma : [a,b] \to Y \) from \( y \) to \( y' \) with \( L(\gamma) \geq \tau(y,y') - \frac{\varepsilon}{2} > 0 \). By Remark 3.22 there are \( 0 < t_1 \leq t_2 < b \) such that \( 0 < L(\gamma|_{[a,t_1]}) < \frac{\varepsilon}{4} \) and \( 0 < L(\gamma|_{[t_2,b]}) < \frac{\varepsilon}{4} \). Setting \( y_1 := \gamma(t_1) \), \( y_2 := \gamma(t_2) \) and \( U := I^+(y_2) \), \( V := I^+(y_1) \) we obtain that \( \tau(y_1,y_2) \geq L(\gamma|_{[a,t_1]}) > 0 \), thus \( y \ll y_1 \) and hence \( y \in U \), which by Proposition 3.22 is an open neighborhood of \( y \). Analogously we get that \( y' \) is in the open set \( V \). At this point
let \((r, r') \in U \times V\), then \(r \ll y_1 \leq y_2 \ll r'\) and thus by the reverse triangle inequality (Lemma 3.21) we obtain

\[
\tau(r, r') = \tau(r, (y_1, y_2)) \geq \tau(y_1, (y_2, y_2)) + \tau(y_2, r') \geq \tau(y_1, y_2) + L(\gamma|_{[t_1, t_2]})
\]

which finishes this case. For the case \(\tau(y, y') = \infty\) the above construction shows the existence of arbitrarily long future directed causal curves from \(r\) to \(r'\), so \(\tau\) attains arbitrarily large values on suitable neighborhoods of \((y, y')\).

Proposition 3.26. Let \(Y = I \times_f X\) be a generalized cone, where \((X, d)\) is a length space. Then \((Y, D, \ll, \leq, \tau)\) is a Lorentzian pre-length space.

Proof: By Lemma 3.19 \((Y, \ll, \leq)\) is a causal space and by Lemma 3.25 the time separation function is lower semi-continuous. Finally, Remark 3.20 and Proposition 3.22 give the required properties of \(\tau\), cf. [KS18, Def. 2.8].

Example 3.27. Let \(f : I \to (0, \infty)\) be continuous and let \((X, h)\) be a Riemannian manifold. Then considered as a Lorentzian pre-length space, the warped product \(I \times_f X\), i.e., the product manifold \(I \times X\) endowed with the continuous Lorentzian metric \(-dt^2 + f^2 h\) coincides with the generalized cone \(I \times_f X\), as is immediate from Definitions 3.4 and 3.17. Therefore, there is no ambiguity in our notation.

Definition 3.28. Let \(Y = I \times_f X\) be a generalized cone and let \(\gamma = (\alpha, \beta) : [a, b] \to Y\) be a causal curve. Then the energy of \(\gamma\) is defined as

\[
E(\gamma) := \frac{1}{2} \int_a^b \dot{\alpha}^2 - (f \circ \alpha)^2 \beta^2.
\]

Contrary to the length, the energy of a curve depends on its parametrization. Nevertheless it will turn out to be a useful tool.

The following is an analogue of [AB98, Thm. 3.1] in the Riemannian case.

Theorem 3.29. Let \((X, d)\) be a geodesic length space and let \(\gamma = (\alpha, \beta) : [0, b] \to Y = I \times_f X\) be future directed causal and maximal. Then:

(i) The fiber component \(\beta\) is minimizing in \(X\).

(ii) Fiber independence holds, i.e., the base component \(\alpha\) is independent of \(\beta\), i.e., \(\alpha\) depends only on the length of \(\beta\). More precisely, let \((X', d')\)
be another geodesic length space, \( \beta' \) minimizing in \( X' \) with \( L^d(\beta') = L^d(\beta) \) and the same speed as \( \beta \), i.e., \( v_\beta = v_\beta' \). Then \( \gamma' := (\alpha, \beta') \) is a future directed maximal causal curve in \( Y' := I \times_f X' \), which is timelike if \( \gamma \) is timelike in \( Y \).

(iii) If \( \gamma \) is timelike, then it has an (absolutely continuous) parametrization with respect to arclength, i.e., \( -\dot{\alpha}^2 + (f \circ \alpha)^2v_\beta^2 = -1 \) almost everywhere.

(iv) If \( \gamma \) is timelike and parametrized with respect to arclength (so \( b = L(\gamma) \)), then the energy of \( \gamma \), \( E(\gamma) \), is minimal under all reparametrizations of \( \gamma \) on \( [0, b] \).

(v) If \( \gamma \) is timelike and parametrized with respect to arclength, then \( v_\beta \) is proportional to \( \frac{1}{(f \circ \alpha)^2} \).

(vi) If \( \gamma \) is timelike, it has an (absolutely continuous) parametrization proportional to arclength such that \( -\dot{\alpha}^2 + \frac{1}{(f \circ \alpha)^2} \) is constant.

Proof:

(i) Assume that \( \beta \) is not minimal (and hence not constant). We may suppose that \( \beta \) is parametrized with respect to arclength, i.e., \( v_\beta = 1 \) almost everywhere and \( \gamma : [0, b] \to X \), where \( b = L^d(\beta) \). Since \( \beta \) is not minimal, there exists another curve \( \bar{\beta} \) from \( \beta(0) \) to \( \beta(b) \), parametrized with respect to arclength, with \( 0 < L^d(\bar{\beta}) < L^d(\beta) = b \). We set \( T := L^d(\bar{\beta}) < b \) and define \( \bar{\gamma} = (\bar{\alpha}, \bar{\beta}) : [0, T] \to Y \) by setting \( \bar{\alpha}(s) := \alpha\left(\frac{s}{T}\right) \) for \( s \in [0, T] \). Then clearly \( \bar{\gamma} \) is timelike and using the reparametrization \( \bar{s} = \frac{b}{T}s \) we get

\[
L(\bar{\gamma}) = \int_0^T \sqrt{(\dot{\bar{\alpha}}(s))^2 - f(\bar{\alpha}(s))^2} \, ds
= \frac{T}{b} \int_0^b \sqrt{\left(\frac{b}{T}\dot{\alpha}(\bar{s})\right)^2 - f(\alpha(\bar{s}))^2} \, d\bar{s}
= \int_0^b \sqrt{\dot{\alpha}(\bar{s})^2 - \left(\frac{T}{b}f(\alpha(\bar{s}))\right)^2} \, d\bar{s} > L(\gamma),
\]

a contradiction.

(ii) Let \( \beta' \) be minimizing in \( X' \), defined on \( [0, b] \) with \( L^d(\beta') = L^d(\beta) \) and \( v_\beta = v_\beta' \). Set \( \gamma' := (\alpha, \beta') : [0, b] \to Y' \). Then \( \gamma' \) is future directed causal and

\[
L(\gamma') = \int_0^b \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2v_\beta'^2} = \int_0^b \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2v_\beta^2} = L(\gamma).
\]
It remains to show that $\gamma'$ is maximal in $Y'$ from $(\alpha(0), \beta'(0)) =: (t_0, x_0') =: y'_0$ to $(\alpha(b), \beta'(b)) =: (t_1, x_1') =: y'_1$. To this end assume to the contrary that there is a $\tilde{\gamma} = (\tilde{\alpha}, \tilde{\beta}) : [0, b] \to Y'$ that is future directed causal from $y'_0$ to $y'_1$ and longer than $\gamma'$, i.e., $L(\tilde{\gamma}) > L(\gamma')$. Without loss of generality we may assume that $\tilde{\gamma}$ is parametrized such that $\tilde{\beta}$ has speed $v_{\tilde{\beta}}$ proportional to $v_{\beta'}$. By minimality of $\beta'$ this implies that $v_{\tilde{\beta}} \geq v_{\beta'}$. Set $\gamma := (\tilde{\alpha}, \beta') : [0, b] \to Y'$, then $\gamma$ is future directed causal from $y'_0$ to $y'_1$ and $L(\gamma) \leq L(\tilde{\gamma})$. Furthermore, we obtain

$$L(\gamma) = \int_0^b \sqrt{\dot{\alpha}^2 - (f \circ \tilde{\alpha})^2 v_{\beta}^2} = \int_0^b \sqrt{\dot{\tilde{\alpha}}^2 - (f \circ \tilde{\alpha})^2 v_{\beta}^2} = L((\tilde{\alpha}, \beta)).$$

Consequently, $L((\tilde{\alpha}, \beta)) = L(\gamma) \geq L(\tilde{\gamma}) > L(\gamma') = L(\gamma)$, contradicting the maximality of $\gamma$, as $\tilde{\alpha}$ and $\alpha$ have the same endpoints.

(iii) Let $\gamma$ be timelike and define $\phi(s) := \int_0^s \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2 v_{\beta}^2}$ for $s \in [0, b]$. Then $\phi : [0, b] \to [0, L(\gamma)]$ is absolutely continuous and strictly monotonically increasing. Moreover, $\phi^{-1}$ exists and is absolutely continuous as $\dot{\phi} > 0$ almost everywhere, ([Nat55] p. 271). Consequently, $\gamma := \gamma \circ \phi^{-1} \equiv (\tilde{\alpha}, \tilde{\beta})$ is absolutely continuous by [Nat55] Thm. 3, Ch. IX and satisfies $-\dot{\tilde{\alpha}}^2 + (f \circ \tilde{\alpha})^2 v_{\beta}^2 = -1$ almost everywhere.

(iv) This follows directly from the Cauchy-Schwartz inequality applied to the length of any reparametrization of $\gamma$ on $[0, b]$.

(v) The claim follows as in the proof of [AB98] Thm. 3.1] by establishing that $\int_I (f \circ \tilde{\alpha})^2 v_{\beta} = \int_J (f \circ \alpha)^2 v_{\beta}$ for all intervals $I, J \subseteq [0, b]$ of the same length, where one uses point (iv) above.

(vi) By the previous two points we can assume that $\gamma$ is parametrized with respect to arclength, i.e., $-\dot{\alpha}^2 + (f \circ \alpha)^2 v_{\beta}^2 = -1$ almost everywhere and that $v_{\beta} = \frac{\phi}{(f \circ \alpha)^2}$ for some constant $c$. For $c \neq 0$, the reparametrization $\tilde{\gamma}(s) := \gamma(\frac{s}{c})$ does the job, and for $c = 0$ (i.e., $v_{\beta} = 0$) the reparametrization $\tilde{\gamma} := (\tilde{\alpha}, \tilde{\beta}) := \gamma \circ \phi^{-1}$, where $\phi(t) := \int_0^t f \circ \alpha$ yields $-\dot{\tilde{\alpha}}^2 + \frac{1}{(f \circ \alpha)^2} = 0$.

\[ \square \]

As a first consequence of fiber-independence we obtain:

**Corollary 3.30.** Let $X$ be a geodesic length space and let $Y = I \times_f X$. Then any maximizing causal curve $\gamma = (\alpha, \beta) : [-b, b] \to Y$ has a causal character, i.e., $\gamma$ is either timelike or null.
Proof: Denote by $Y'$ the Lorentzian warped product $I \times f \mathbb{R}$, i.e., the manifold $I \times \mathbb{R}$ endowed with the continuous metric $-dt^2 + f^2 dx^2$. Let $\beta$ be parametrized by arclength and set $\beta' : [-b, b] \to \mathbb{R}$, $\beta'(t) := t$. Then by Theorem 3.29 (i) and Example 3.27, $\gamma' := (\alpha, \beta')$ is a causal maximizer in $Y'$. We now use the same basic ideas as in the proof of [GL18 Thm. 1.1] (the difference being that the construction of the relevant curves is different, due to the metric being not locally Lipschitz but having a warped product $(the difference being that the construction of the relevant curves is different, due to the metric being not locally Lipschitz but having a warped product structure) to show that $\gamma'$ is either timelike or null. The same must therefore be true for $\gamma$.

Since we exclusively work in $Y'$ from now on, we will drop the $'$ from our notation. Assume $\gamma$ is neither null nor timelike. Without loss of generality, we may assume $\gamma(0) = (\alpha(0), 0) = (0, 0)$, $\dot{\gamma}(0)$ exists and is timelike and $N := \{s \in [-b, 0] : \gamma(s) \text{ is null}\}$ has non-zero measure. Let $\varepsilon$ be positive and define $\gamma_\varepsilon : [-b, 0] \to I \times \mathbb{R}$ by

$$\gamma_\varepsilon(s) = (\alpha(s), \beta_\varepsilon(s)) := (\alpha(s), \sqrt{1-\varepsilon} s + b \sqrt{1-\varepsilon} - b).$$

Then $\gamma_\varepsilon(-b) = \gamma(-b) = (\alpha(-b), -b)$ and $\gamma_\varepsilon(0) = (\alpha(0), \beta_\varepsilon(0)) = (0, b \sqrt{1-\varepsilon} - b)$. Note that for $\varepsilon < \frac{1}{2}$, there exists $C > 0$ such that $|\beta_\varepsilon(0)| = |b \sqrt{1-\varepsilon} - b| \leq C \varepsilon$. We estimate

$$L(\gamma_\varepsilon) = \int_{-b}^{0} \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2(1-\varepsilon)} + \int_{[-b, 0] \setminus N} \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2(1-\varepsilon)} + \sqrt{\varepsilon} \int_{N} f \circ \alpha \geq \int_{[-b, 0] \setminus N} \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2} + c \sqrt{\varepsilon} = L(\gamma|_{[-b, 0]}) + c \sqrt{\varepsilon}.$$

Next, note that there exist $\eta_0 > 0$ and $C_0 > 0$ such that $f(t) < C_0 < \dot{\alpha}(0)$ for $t \in [-\eta_0, \eta_0]$ and since $\alpha(s) = (\dot{\alpha}(0) + h(s)) s$ with $h(s) \to 0$ as $s \to 0$, there also exist $0 < \eta_1$ and $0 < C_0 < C_1 < \dot{\alpha}(0) < C_2$ such that $C_2 s > \alpha(s) > C_1 s$ for $s \in [0, \eta_1]$. Fix $k$ with $C_0 < k < C_1$ and set $s_\varepsilon := \frac{k}{C_1-k} \beta_\varepsilon(0)$. Then $0 < s_\varepsilon < \frac{C_k}{C_1-k} \varepsilon$ for $\varepsilon < \frac{1}{2}$. Let $\varepsilon$ small enough such that $s_\varepsilon + |\beta_\varepsilon(0)| < \min\{\eta_1, \frac{\eta_0}{C_0}\}$. Then the straight lines $h : s \mapsto (C_1 s, s)$ and $g_k : s \mapsto (ks - k \beta_\varepsilon(0), s)$ are timelike on $[\beta_\varepsilon(0), s_\varepsilon]$ and intersect each other in $s_\varepsilon > 0$. Further, $\alpha(s) > C_1 s$ on $(0, s_\varepsilon]$, so $(\alpha(0), 0) = (0, 0)$ lies strictly below $g_k$ but $(\alpha(s_\varepsilon), s_\varepsilon)$ lies strictly above $g_k$ (since it lies strictly above $h(s_\varepsilon)$ which is equal to $g_k(s_\varepsilon)$) and hence $s \mapsto (\alpha(s), s) = \gamma(s)$ intersects $g_k$ in some $0 < \bar{s} < s_\varepsilon$. Note that $g_k|_{[\beta_\varepsilon(0), s_\varepsilon]}$ is a future directed timelike curve from $\gamma_\varepsilon(0)$ to $\gamma(\bar{s})$. Now we estimate the length of the concatenation...
as follows:

\[
L(\gamma_\varepsilon + g_k |_{[\beta_*(0), \beta]}(0, s)}) > L(\gamma_\varepsilon) \geq c\sqrt{\varepsilon} + L(\gamma |_{[-b, 0]}) \\
= L(\gamma |_{[-b, s]}) + c\sqrt{\varepsilon} - L(\gamma |_{[0, s]}) \\
\geq L(\gamma |_{[-b, s]}) + c\sqrt{\varepsilon} - \alpha(s) \\
> L(\gamma |_{[-b, s]}) + c\sqrt{\varepsilon} - C_2 s \varepsilon \\
\geq L(\gamma |_{[-b, s]}) + c\sqrt{\varepsilon} - C_2 \frac{C_1}{k} \varepsilon > L(\gamma |_{[-b, s]})
\]

for \( \varepsilon \) small. This contradicts the maximality of \( \gamma \).

**Example 3.31.** (Minkowski cones as generalized cones.) Here we show that Minkowski cones as defined in Section 2 can equivalently be viewed as generalized cones. Let \( X \) be a geodesic length space, let \( Y := \text{Cone}(X) \) be the Minkowski cone over \( X \), with relations \( \ll_Y, \leq_Y \) and time separation function \( \tau_Y \). Let \( G := (0, \infty) \times_{id} X \) be the generalized cone with warping function \( f = id \) over \( X \). Since we did not explicitly treat generalized cones of the form \( I \times f X \) with a non-open interval \( I \) and a function \( f \) that might be zero at the endpoints (though, as indicated in Remark 3.32 below, these cases could be included relatively straightforwardly), we will compare the Lorentzian pre-length space \((G, D, \ll_G, \leq_G, \tau_G)\) with the Lorentzian pre-length space \( Y' := Y \setminus \{0\} = (0, \infty) \times X \) with metric \( D \) (which is equivalent to the restriction of the cone metric \( d_c \)), relations \( \ll_{Y'} := \ll_Y |_{Y' \times Y'}, \leq_{Y'} := \leq_Y |_{Y' \times Y'}, \) and time separation function \( \tau_{Y'} := \tau_Y |_{Y' \times Y'} \).

Let \( x = (x_0, \bar{x}), y = (y_0, \bar{y}) \in G \), then by the description of \( I^+ \) in (8) \( x \ll_G y \) if and only if for corresponding points \( x' = (x_0, \bar{x}') \in W := (0, \infty) \times_{id} \mathbb{H}^{n-1} \) and \( y' = (y_0, \bar{y}') \in W \) with \( d_X(\bar{x}, \bar{y}) = d_{\mathbb{H}^{n-1}}(\bar{x}', \bar{y}') \) one has \( x' \ll_{W} y' \). Similarly from (9) we see that, since both \( X \) and \( \mathbb{H}^{n-1} \) are geodesic, the same holds for \( \leq \). Lastly, by fiber independence (Theorem 3.29 (10)) we also have that \( \tau_G(x, y) = \tau_W(x', y') \). By the last two paragraphs in Remark 2.1 we have \( \leq_W = \ll_C, \leq_W = \ll_C \) and \( \tau_W = \tau_C \) (where \( C := (0, \infty) \times \mathbb{H}^{n-1} \) with the Minkowski cone structure as in Remark 2.1). Now since by definition \( \leq, \ll, \tau \) for Minkowski cones are clearly fiber independent as well, we have \( x \ll_{Y'} y \) if and only if \( x' \ll_C y' \) if and only if \( x \ll_G y \). The same holds for \( \leq \). Also clearly \( \tau_{Y'}(x, y) = \tau_C(x', y') = \tau_G(x, y) \). So the Lorentzian pre-length spaces \((G, D, \ll_G, \leq_G, \tau_G)\) and \((Y', D, \ll_{Y'}, \leq_{Y'}, \tau_{Y'})\) can be identified.

**Remark 3.32.** We have confined ourselves in this section to generalized cones \( I \times f X \) with \( I \) an open interval, but note that general intervals \( I \) could be treated in complete analogy. One could also consider the case where \( f \) has isolated zeros (either in the interior of the interval or at the interval boundaries) with the additional assumption that the improper integrals of
\[ \frac{1}{f} \] coming from both sides of each zero diverge. If \( f(t) = 0 \) for some \( t \in I \), we identify \( (t, x) \sim (t, x') \) for \( x, x' \in X \) to a point denoted by \( t_Y \). Defining all concepts analogously to the case where \( f > 0 \), it is easy to see that for any \( (x_0, x) \in Y \) with \( x_0 < t \) and \( x \in X \) arbitrary we have \( (x_0, x) \leq t_Y \) and \( \gamma : s \mapsto (x_0 + s, x) \) is a future directed timelike curve from \( (x_0, x) \) to \( t_Y \) with \( \tau((x_0, x), t_Y) = L(\gamma) \). Further, \( I^+(t_Y) = (I \cap (t, \infty)) \times X \) and \( I^-(t_Y) = (I \cap (-\infty, t)) \times X \). So any two points \( (x_0, x), (y_0, y), x_0 < y_0 \) with \( f \) having a zero on \( [x_0, y_0] \) are trivially timelike related. Therefore considering \( f \) having zeros in the interior of \( I \) largely reduces to the problem of allowing \( f \) to vanish at the boundary. Divergence of the integral of \( \frac{1}{f} \) as one approaches the zeroes of \( f \) ensures that \( I^\pm \) remains open (see Proposition 3.22) and thus, with some modifications, the main results in Sections 3 and 4 should remain valid, but this would need to be investigated more carefully.

4 Generalized cones as Lorentzian length spaces

We already established that every generalized cone \( Y = I \times_f X \), where \((X, d)\) is a length space, is a Lorentzian pre-length space in Proposition 3.26. Here we will show that such spaces are in fact Lorentzian length spaces if \( X \) is locally compact. To this end we need the following auxiliary results.

**Lemma 4.1.** Let \((X, d)\) be a metric space and let \( p = (t_0, \bar{p}), q = (t_1, \bar{q}) \in Y \), then for the causal diamond \( J(p, q) := J^+(p) \cap J^-(q) \) we have

\[
J(p, q) \subseteq \{(t, \bar{r}) \in Y : t_0 \leq t \leq t_1, \bar{r} \in \bar{B}^d_{\bar{m},t_0,\bar{t}}(\bar{p}) \cap \bar{B}^d_{\bar{m},t_1,\bar{t}}(\bar{q})\},
\]

where \( \bar{B}^d_\delta(x) = \{x' \in X : d(x, x') \leq \delta\} \) denotes the closed ball of radius \( \delta \) in \( X \).

**Proof:** Let \( r = (t, \bar{r}) \in J(p, q) \), \( p < r < q \), and let \( \gamma = (\alpha, \beta) : [0, b] \to Y \) be a future directed causal curve from \( p = \gamma(0) \) to \( r = \gamma(t^*) \) to \( q = \gamma(b) \). Then \( \alpha > 0 \) almost everywhere. We get \( t_0 = \alpha(0) \leq \alpha(t^*) = t \leq \alpha(b) = t_1 \). From the proof of Lemma 3.10 we conclude that

\[
t - t_0 = \alpha(t^*) - \alpha(0) \geq m_{t_0, t} d(\bar{p}, \bar{r}),
\]

and analogously \( t_1 - t \geq m_{t, t_1} d(\bar{r}, \bar{q}) \). \( \square \)

**Lemma 4.2.** Let \((X, d)\) be a metric space. Then any \( p = (p_0, \bar{p}) \in Y \) has a basis of open, causally convex neighborhoods, i.e., neighborhoods such that any causal curve with endpoints in that neighborhood is contained in it. This also shows that such a generalized cone is strongly causal. Moreover, the map \( Y \to I : (t, x) \mapsto t \) is a time function, i.e., \( t \) is continuous and strictly increasing along any future directed causal curve.
Proof. Using the same arguments as in the proof of the previous Lemma one easily checks that the family
\[
\left\{ (t, \bar{r}) \in Y : p_0 - \varepsilon < t < p_0 + \varepsilon, \bar{r} \in B^d_{\varepsilon} \cap B^d_{\varepsilon} \right\}_{\varepsilon > 0}
\]
satisfies the claim.

\[\text{Lemma 4.3.} \] Every generalized cone has the property that for every point \( y \) there is a neighborhood \( U \) of \( y \) and a constant \( C > 0 \) such that the (metric) \( D \)-arclength of every causal curve which is contained in \( U \) is bounded by \( C \), i.e., \( L^D(\gamma) \leq C \).

Proof: Let \( y = (t, p) \in Y \) and let \( I' \subseteq I \) be a compact interval containing \( t \). Set \( C' := \text{diam}(I') \) and \( C := \min_{r \in I'} f(r) > 0 \). Moreover, let \( \gamma = (\alpha, \beta) : [a, b] \to Y \) be a (without loss of generality) future directed causal curve that is contained in \( U := I' \times X \). Then since \( C^2 v_\beta^2 \leq (f \circ \alpha)^2 v_\beta^2 \leq \dot{\alpha}^2 \) we get
\[
L^D(\gamma) = \int_a^b \sqrt{\dot{\alpha}^2 + v_\beta^2} \leq \int_a^b \dot{\alpha} \sqrt{1 + \frac{1}{C^2}} \leq \left( 1 + \frac{1}{C^2} \right) C',
\]
as required.

We want to establish that generalized cones are Lorentzian length spaces. For this we first need to show that the different notions of causal curves and their length agree with the ones in the setting of Lorentzian length spaces.

In the following result (and thereafter), when comparing the different notions of causal curves, it will always be understood that parametrizations are chosen in which the respective curves are never locally constant (cf. [BB10, Ex. 2.5.3]).

Lemma 4.4. The notion of causal curves for a generalized cone agrees with the notion of a causal curves with respect to the relation \( \leq \) (cf. [KS18, Def. 2.18]).

Proof: Clearly, every future or past directed curve in the sense of Definition 3.2 is causal with respect to \( \leq \). For the converse, note that since this is a local question it suffices to consider segments of causal curves. So let \( \gamma = (\alpha, \beta) : [a, b] \to Y \) be a (without loss of generality) future directed causal curve with respect to \( \leq \), i.e., \( \forall a \leq s \leq t \leq b : \gamma(s) \leq \gamma(t) \). Thus for any \( a \leq s < t \leq b \) there is a future directed causal curve (in the sense of Definition 3.2) \( \gamma_{s,t} = (\alpha_{s,t}, \beta_{s,t}) : [0, 1] \to Y \) from \( \gamma(s) \) to \( \gamma(t) \). This implies that \( \alpha \) is strictly monotonically increasing as \( t \) is a time function (cf. Lemma 4.2).
We now want to construct a sequence of future directed causal curves (in the sense of Definition 3.2) that converges pointwise to \( \gamma \). For \( \sigma := (a = t_0 < t_1, \ldots, t_N = b) \) a partition of \([a, b]\), denote by \( \gamma^\sigma \) the future directed causal curve \( \gamma^\sigma := \gamma_{t_0, t_1} \ast \cdots \ast \gamma_{t_{N-1}, t_N} \) obtained by concatenating the curves \( \gamma_{t_i, t_{i+1}} \) \((0 \leq i \leq N - 1)\). Let \( \sigma_k \) be a sequence of such partitions whose norms (maximal length of a subinterval) tend to zero as \( k \to \infty \).

We show that \( \gamma_k := \gamma^{\sigma_k} \) converges pointwise to \( \gamma \). Let \( t \in [a, b] \) and let \( U \) be a neighborhood of \( \gamma(t) \). By Lemma 4.2 there exists a causally convex neighborhood \( V \) of \( \gamma(t) \) such that \( V \subseteq U \). As \( \gamma^{-1}(V) \) is a neighborhood of \( t \) in \([a, b]\), for \( k \) large any sub-interval of \( \sigma_k \) containing \( t \) lies entirely in \( V \), so in particular \( \gamma_k(t) \in U \). Consequently, \( \gamma_k \to \gamma \) pointwise and thus by the limit curve theorem 3.16 \( \gamma \) is a (future directed) causal curve in the sense of Definition 3.2.

**Proposition 4.5** (Local existence of maximal causal curves). Let \((X, d)\) be a locally compact metric space. Then every point in \( Y = I' \times_{\text{f}} X \) has a neighborhood \( U \) such that any two causally related points in \( U \) can be connected by a maximal causal curve.

**Proof:** Let \( p \in Y \), \( U' = I' \times X \) and \( C > 0 \) be given by Lemma 4.3 let \( W \subseteq X \) be a compact neighborhood of \( \bar{p} \) in \( X \) and set \( V := I' \times W \). Further, let \( U \subseteq V \subseteq U' \) be causally convex in \( V \) (cf. Lemma 4.2). Let \( y, z \in U \) with \( y < z \), and note that any causal curve from \( y \) to \( z \) has to be contained in \( U \). So local maximality in \( U \) implies global maximality. Let \( \gamma_n : [a, b] \to Y \) be a sequence of future directed causal curves from \( y \) to \( z \) such that \( L(\gamma_n) \to \tau(y, z) \). Then, by Lemma 4.3 \( L^D(\gamma_n) \leq C \) and so reparametrizing each \( \gamma_n \) proportional to \( D \)-arclength on \([a, b]\) yields a sequence of uniformly \( D \)-Lipschitz curves \( \tilde{\gamma}_n \) each of which is future directed causal. By the theorem of Arzela-Ascoli (the sequence \( \gamma_n \) is contained in the compact set \( V \)) we obtain a subsequence \( (\tilde{\gamma}_n)_k \) of \( (\tilde{\gamma}_n)_n \) that convergences uniformly to a Lipschitz curve \( \gamma \) from \( y \) to \( z \). As \( y < z \) this curve cannot be constant and so by possibly reparametrizing \( \gamma \) such that it is never locally constant we obtain a future directed causal curve \( \gamma \) from \( y \) to \( z \) that is contained in \( U \). Moreover, by Proposition 3.15 we get that

\[
L(\gamma) \leq \tau(y, z) = \limsup_k L(\gamma_{n_k}) \leq L(\gamma),
\]

so \( \gamma \) is maximal. \( \square \)

A similar argument gives that \( Y \) is locally causally closed (\([KS18\text{ ] Def. 3.4}]\):

**Lemma 4.6.** Let \((X, d)\) be a locally compact metric space. Then every point in \( Y \) has a neighborhood \( U \) such that for any \( y_n, z_n \in Y \) with \( y_n \to y \in \bar{U}, z_n \to z \in \bar{U} \) and \( y_n \leq z_n \) for all \( n \in \mathbb{N} \), it follows that \( y \leq z \).
The next step is to establish that the length of a causal curve agrees with the \( \tau \)-length introduced in [KS18, Def. 2.24]. Recall that the \( \tau \)-length, \( L(\gamma) \), is defined as

\[
L(\gamma) := \inf \{ \sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) : a = t_0 < t_1 < \ldots < t_N = b \},
\]

where \( \gamma \) is a future directed causal curve (and by Lemma 4.4 this is the same as causal with respect to \( \leq \)).

**Proposition 4.7.** Let \((X,d)\) be a locally compact metric space. If \(\gamma : [a,b] \to Y\) is a future directed causal curve, then \(L(\gamma) = L_\tau(\gamma)\).

**Proof:** Let \(a = t_0 < t_1 < \ldots < t_N = b\) be a partition of \([a,b]\), then

\[
\sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})) \geq \sum_{i=0}^{N-1} L(\gamma|_{[t_i,t_{i+1}]}) = L(\gamma),
\]

as \(L\) is additive. Taking the infimum over all partitions of \([a,b]\) yields \(L(\gamma) \leq L_\tau(\gamma)\).

For the reverse inequality we cover \(\gamma([a,b])\) by neighborhoods \(U_0, \ldots, U_N\) as in Proposition 4.4 and choose a partition \(\sigma := (a = t_0 < t_1 < \ldots < t_{N+1} = b)\) such that \(\gamma(t_{i+1}) \in U_i \cap U_{i+1}\) for every \(i = 0, \ldots, N - 1\). Consequently, there are future directed maximal causal curves \(\gamma_i^\sigma\) from \(\gamma(t_i)\) to \(\gamma(t_{i+1})\) for \(i = 0, \ldots, N\). The future directed causal curve \(\gamma^\sigma := \gamma_0^\sigma * \ldots * \gamma_N^\sigma\) has length

\[
L(\gamma^\sigma) = \sum_{i=0}^{N} L(\gamma_i^\sigma) = \sum_{i=0}^{N} \tau(\gamma(t_i), \gamma(t_{i+1})) \geq L_\tau(\gamma).
\]

By shrinking the cover \((U_i)_i\) and adapting the partition \(\sigma\) accordingly we get a sequence \(\gamma_k\) of future directed causal curves which, by an argument as in the proof of Lemma 4.4, converges pointwise to \(\gamma\), and satisfies \(L(\gamma_k) \geq L_\tau(\gamma)\) for all \(k \in \mathbb{N}\). As \(L\) is upper semicontinuous by Proposition 3.15 we get \(L(\gamma) \geq L_\tau(\gamma)\) and this finishes the proof. \(\square\)

Thus there is no need to distinguish between \(L\) and \(L_\tau\) and the different notions of causal curves also agree, so when applying the theory of Lorentzian length spaces to generalized cones we will always use the notions of the latter.

**Theorem 4.8.** Any generalized cone \(I \times_f X\), where \((X,d)\) is a locally compact length space, is a strongly causal Lorentzian length space.
Proof: By Proposition 3.26 and Lemma 4.6, \( Y \) is a locally causally closed Lorentzian pre-length space. Moreover, by definition of the causal relations it is causally path connected.

Since the different notions of causal curves agree by Lemma 4.4 and as \( L_{\tau} = L \) by Proposition 4.7, we directly obtain that \( \tau = T \), where

\[
T(y, z) = \sup \{ L_{\tau}(\gamma) : \gamma \text{ future-directed causal from } y \text{ to } z \},
\]

if the set is non-empty and \( T(y, z) = 0 \) otherwise.\(^2\)

It remains to show that \( Y \) is localizable (KS18, Def. 3.16), i.e., we need to show that every point \( y \in Y \) has an open neighborhood \( \Omega \) such that

- \( L^d(\gamma) \leq C \) for some \( C > 0 \) and all causal curves \( \gamma \) contained in \( \Omega \),
- there is a continuous \( \omega : \Omega \times \Omega \to [0, \infty) \) such that \( (\Omega, d|_{\Omega \times \Omega}, \ll_{\Omega \times \Omega}, \leq_{\Omega \times \Omega}, \omega) \) is a Lorentzian pre-length space with \( I^\pm(o) \cap \Omega \neq \emptyset \) for all \( o \in \Omega \), and
- for all \( o, o' \in \Omega \) with \( o < o' \) there is a future directed causal curve \( \gamma \) from \( o \) to \( o' \) that is maximal in \( \Omega \) and \( L(\gamma) = \omega(o, o') \leq \tau(o, o') \).

To this end we apply the argument of the proof of [GKS19, Lem. 4.3] to see that we can use \( \omega := \tau|_{U \times U} \) for a suitable neighborhood \( U \) of a point \( y = (t_0, x) \in Y \). Such a suitable neighborhood can be chosen by taking \( U \) to be one of the causally convex neighborhoods from Lemma 4.2 that is contained in the neighborhoods of Lemma 4.3 and Proposition 4.5.

Thus \( \omega \) is finite and lower semicontinuous. To see that \( \omega \) is also upper semicontinuous note that we can adapt the proof of [KS18, Thm. 3.28] to the simpler local situation in \( U \) by using the local existence of maximal causal curves (Proposition 4.5) and the upper semi-continuity of \( L \) (Proposition 3.15). Moreover, since \( U \) is open, one has \( I^\pm((t_0, x)) \cap U \neq \emptyset \).

This yields that \( Y \) is localizable and hence by the above is a Lorentzian length space. It also implies that \( Y \) is strongly causal in the sense of [KS18, Def. 2.35(iv)] by using the result for Lorentzian length spaces [KS18, Thm. 3.26(iv)] and Lemma 4.2.

Further, the Lorentzian length space \( I \times_f X \) is regular, i.e., maximal causal curves have a causal character (cf. [KS18, Def. 3.22]). Thus by Proposition 4.8 and Corollary 3.30 we immediately get the following:

Corollary 4.9. Any generalized cone \( I \times_f X \), where \( (X, d) \) is a geodesic locally compact length space, is a strongly causal and regular Lorentzian length space.

\(^2\)Note that this could also be inferred from the more general Theorem A.10.
Lemma 4.1 shows that if $X$ is proper the causal diamonds $J(p, q)$ are precompact. Moreover, by Lemma 4.2 any generalized cone is strongly causal. This is already close to the usual notion of global hyperbolicity. In the next Proposition we will show that generalized cones, where $X$ is proper and geodesic, are in fact globally hyperbolic (as defined for Lorentzian length spaces in [KS18, Def. 2.35(v)]).

**Proposition 4.10.** Let $I \times_f X$ be a generalized cone, where $(X, d)$ is a geodesic length space that is a proper metric space. Then $I \times_f X$ is globally hyperbolic.

**Proof:** From Theorem 4.8 we know that $Y$ is a strongly causal Lorentzian length space and hence non-totally imprisoning by [KS18, Thm. 3.26(iii)]. Moreover, from Corollary 3.24 we know that $J^\pm(p)$ is closed for every $p \in Y$ and Lemma 4.1 implies that for all $p, q \in Y$ the causal diamond $J(p, q)$ is contained in a compact set. Thus $J(p, q)$ is compact and so $Y$ is globally hyperbolic in the sense of [KS18, Def. 2.35(v)].

As any complete and locally compact length space is proper and geodesic (by the Hopf-Rinow-Cohn-Vossen theorem) we obtain the following corollary.

**Corollary 4.11.** Let $I \times_f X$ be a generalized cone, where $X$ is a locally compact, complete length space. Then $I \times_f X$ is globally hyperbolic.

Recall that a Lorentzian pre-length space is called *geodesic* ([KS18, Def. 3.27]) if any two causally related points can be joined by a maximal causal curve. As any globally hyperbolic Lorentzian length space is geodesic (Avez-Seifert, cf. [KS18, Thm. 3.30]), we conclude by the above that every generalized cone is geodesic if $X$ is proper and geodesic (in the metric space sense). This implies the following stronger result.

**Corollary 4.12.** Let $X$ be geodesic, then $I \times_f X$ is geodesic. Furthermore, any two timelike related points can be connected by a timelike geodesic.

**Proof.** Let $(x_0, \bar{x}), (y_0, \bar{y}) \in Y = I \times_f X$. Because $X$ is geodesic there exists a minimal curve $\beta : [0, d_X(\bar{x}, \bar{y})] \to X$, parametrized by arc-length, from $\bar{x}$ to $\bar{y}$. Let $X' = [0, d_X(\bar{x}, \bar{y})]$ (with the standard metric) and $Y' = I \times_f X'$. Since $X'$ is proper and geodesic, $Y'$ is geodesic and there exists a maximizing curve $\gamma' = (\alpha', \beta') : [0, d_X(\bar{x}, \bar{y})] \to Y'$, with $\beta'$ parametrized by arc-length, from $(x_0, 0)$ to $(y_0, d_X(\bar{x}, \bar{y}))$. Then $\gamma := (\alpha', \beta)$ is maximizing from $(x_0, \bar{x})$ to $(y_0, \bar{y})$ in $Y$ by Theorem 3.29 (ii). The second claim follows from Corollary 3.30.
5 Curvature bounds

In this Section we generalize the results of Section 2 to generalized cones, i.e., we relate (metric) curvature bounds of the fiber $X$ to timelike curvature bounds of the generalized cone $Y = I \times_f X$, and vice versa.

Lemma 5.1. Let $(X, d)$ and $(X', d')$ be two geodesic length spaces. Let $Y := I \times_f X$ and $Y' := I \times_f X'$. Then for any two pairs of points $x = (x_0, \bar{x}), y = (y_0, \bar{y}) \in Y$ and $x' = (x_0, \bar{x}'), y' = (y_0, \bar{y}') \in Y'$ with $d_X(\bar{x}, \bar{y}) \geq d_X(\bar{x}', \bar{y}')$ one has $\tau(x, y) \leq \tau'(x', y')$.

Proof. If $\tau(x, y) = 0$ this obviously holds, so assume $x \ll y$. Let $\beta : [a, b] \rightarrow X$ be a minimizing unit-speed geodesic from $\bar{x}$ to $\bar{y}$ in $X$. Then by Corollary 4.12 there exists a timelike curve $\gamma \equiv (\alpha, \beta) : [a, b] \rightarrow Y$ from $x$ to $y$ with $L(\gamma) = \tau(x, y)$. Further, let $\beta' : [a, b] \rightarrow X'$ be a curve from $\bar{x}'$ to $\bar{y}'$ in $X'$ such that $L_X(\beta') = d_X(\bar{x}', \bar{y}')$ and $v_{\beta'}$ is constant. Then $L(\beta) = d(\bar{x}, \bar{y}) \geq d_{X'}(\bar{x}', \bar{y}') = L(\beta')$ implies $v_{\beta'} \leq v_\beta = 1$, so the curve $\gamma' := (\alpha, \beta') : [a, b] \rightarrow Y'$ is timelike and

\[ \tau(x, y) = L(\gamma) = \int_a^b \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2} \leq \int_a^b \sqrt{\dot{\alpha}^2 - (f \circ \alpha)^2 v_{\beta'}^2} = L(\gamma') \leq \tau'(x', y'). \]

Moreover, for causally related points also strict inequalities are preserved in the above Lemma, i.e., if $d_X(\bar{x}, \bar{y}) < d_{X'}(\bar{x}', \bar{y}')$ then $\tau(x, y) < \tau'(x', y')$.

From this one obtains immediately the following converse:

Lemma 5.2. Let $(X, d)$ and $(X', d')$ be two geodesic length spaces. Let $Y := I \times_f X$ and $Y' := I \times_f X'$. Then for any two pairs of causally related points $x = (x_0, \bar{x}), y = (y_0, \bar{y}) \in Y$ and $x' = (x_0, \bar{x}'), y' = (y_0, \bar{y}') \in Y'$ with $\tau(x, y) \leq \tau'(x', y')$ one has $d_X(\bar{x}, \bar{y}) \leq d_{X'}(\bar{x}', \bar{y}')$.

We now turn to the relation between (metric) curvature bounds in the fiber $X$ (in the sense of [BB] Def. 4.6.2), cf. also Subsection 1.2 and timelike curvature bounds in the generalized cone $I \times_f X$ as defined in [KS] Def. 4.7, cf. also Subsection 1.3.

Theorem 5.3. Let $K, K' \in \mathbb{R}$ and let $(X, d)$ be a geodesic length space with curvature bounded below/above by $K$. Then $Y = I \times_f X$ has timelike curvature bounded below/above by $K'$ if $I \times_f M^2(K)$ has timelike curvature bounded below/above by $K'$.

Proof. As in the proof of Theorem 4.2, for any $w \in Y$ we can choose a causally convex neighborhood $U \subseteq Y$ according to Lemma 4.2 such
that $\tau|_{U \times U}$ is continuous and any two points $x, y \in U$ with $x \ll y$ can be connected by a maximal future-directed timelike curve $\gamma$ in $U$ with $L(\gamma) = \tau(x, y)$. We may further assume that $U$ was chosen small enough to satisfy the following conditions:

(i) There is an open set $V \subseteq X$ on which triangle comparison with $M^2(K)$ holds and such that for all $\bar{x} \in X$ for which there exists $x_0 \in \mathbb{R}$ such that $(x_0, \bar{x}) \in U$ we have $\bar{x} \in V$.

(ii) $U \subseteq [u_0, u_1] \times B^{d_X}_\varepsilon(\bar{w})$, where $\varepsilon$ and $|u_0-u_1|$ are so small that, for some (fixed) $\bar{w}' \in M^2(K)$ we have that $[u_0, u_1] \times B^{d_X}_\varepsilon(\bar{w}') \subseteq I \times f M^2(K)$ is contained in a neighborhood $U'$ on which timelike triangle comparison with $L^2(K')$ holds.

Let $\Delta = (x, y, z)$ be a timelike geodesic triangle in $U$, realized by maximal timelike curves $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ whose side lengths $a, b, c$ satisfy timelike size bounds for $K'$, i.e. $c \geq a + b$ and if $c = a + b$ and $K' > 0$ or $c > a + b$ and $K' < 0$ then $c < \frac{\pi}{\sqrt{|K'|}}$.

To establish that $Y$ has timelike curvature bounded below by $K'$ we have to show that if $\Delta'' = (x'', y'', z'')$ is a comparison triangle of $\Delta = (x, y, z)$ in $L^2(K')$, then for all points $p, q$ on the sides of $\Delta = (x, y, z)$ and corresponding points $p'', q''$ on the sides of $\Delta'' = (x'', y'', z'')$, we have $\tau(p, q) \leq \tau_{L^2(K')}(p'', q'')$. (To show that $Y$ has timelike curvature bounded above by $K'$ we have to show that $\tau(p, q) \geq \tau_{L^2(K')}(p'', q'')$.)

We do this in two steps. First, we construct a comparison triangle $\Delta' = (x', y', z')$ in $Y' := I \times f M^2(K)$ with $\tau(p, q) \leq \tau_{Y'}(p', q')$ (respectively $\geq$ in case of curvature bounded above).

The projection $\bar{\Delta} = (\bar{x}, \bar{y}, \bar{z})$ of $\Delta = (x, y, z)$ onto $X$ is a geodesic triangle $\bar{\Delta}$ in $X$ (which can be degenerate) and if $\gamma_{xy} = (\alpha_{xy}, \beta_{xy}), \gamma_{yz} = (\alpha_{yz}, \beta_{yz})$ and $\gamma_{zx} = (\alpha_{zx}, \beta_{zx})$ are the sides of $\Delta = (x, y, z)$, then the sides of $\bar{\Delta} = (\bar{x}, \bar{y}, \bar{z})$ are the minimizing curves $\beta_{xy}, \beta_{yz}, \beta_{zx}$ and are contained in $V$. Because $V$ was a neighborhood in $X$ on which triangle comparison with $M^2(K)$ holds, there exists a triangle $\bar{\Delta}' = (\bar{x}', \bar{y}', \bar{z}')$ in $M^2(K)$ such that $d_X(\bar{p}, \bar{q}) \geq d_{M^2(K)}(\bar{p}', \bar{q}')$ (respectively $\leq$ in case of curvature bounded above). This triangle $\bar{\Delta}'$ in $M^2(K)$ can be lifted to a triangle $\Delta' = (x', y', z')$ in $Y' = I \times f M^2(K)$ given by $x' := (x_0, \bar{x}'), y' := (y_0, \bar{y}'), z' := (z_0, \bar{z}')$. By fiber independence (Theorem 3.29 (ii)) $\Delta'$ is a triangle with the same side lengths as $\Delta$ and the points $p'$ and $q'$ corresponding to $p$ and $q$ are exactly $(p_0, \bar{p}')$ and $(q_0, \bar{q}')$. Thus, by Lemma 6.1 $\tau(p, q) \leq \tau_{Y'}(p', q')$ (respectively $\geq$ in case of curvature bounded above).

Now because $\Delta$ satisfies timelike size bounds, the same is true for $\Delta'$. Further, because of the symmetries of $M^2(K)$ we may additionally suppose that our comparison triangle $\Delta'$ was chosen such that $\bar{x}' = \bar{w}'$,
\[ \Delta' \subseteq U' \] by our choice of \( U \) (cf. item (ii)). So, by the timelike curvature bound of \( Y' = I \times_f M^2(K) \) there exists a timelike comparison triangle \( \Delta'' = (x'', y'', z'') \) of \( (x', y', z') \) in \( \mathbb{L}^2(K') \) such that for the points \( p'', q'' \) on \( \Delta'' \) corresponding to \( p', q' \), we have \( \tau_{Y'}(p', q') \leq \tau_{\mathbb{L}^2(K')} (p'', q'') \) (respectively, \( \tau_{Y'}(p', q') \geq \tau_{\mathbb{L}^2(K')} (p'', q'') \)) for a timelike curvature bound from above). Moreover, by construction and fiber independence (cf. Theorem 3.29) \( \Delta'' \) must be a comparison triangle for \( \Delta \).

Together with the inequality \( \tau(p, q) \leq \tau_{Y'}(p', q') \) (respectively \( \geq \)) from before we get \( \tau(p, q) \leq \tau_{\mathbb{L}^2(K')} (p'', q'') \) (respectively \( \geq \)), concluding the proof.

In the case of a smooth warping function \( f \) we can give sufficient conditions so that \( Y = I \times_f X \) has timelike curvature bounded by \( K' \).

**Corollary 5.4.** Let \( f : I \to (0, \infty) \) be smooth and let \( Y = I \times_f X, Y' = I \times_f M^2(K) \). If \( f \) is \( K' \)-concave (convex), i.e., \( f'' - K' f \leq 0 \) (respectively, \( f'' - K' f \geq 0 \)) and \( K = \sup(\mathbb{K}f^2 - (f')^2) \) (respectively, \( K = \inf(\mathbb{K}f^2 - (f')^2) \)) and \( X \) has curvature bounded below (above) by \( K \), then \( Y \) has timelike curvature bounded below (above) by \( K' \).

**Proof:** This follows directly from Theorem 5.3 and [AB08, Prop. 7.1] in the special case that the base is one-dimensional and from the fact that sectional curvature bounds in the sense of [AB08] imply timelike curvature bounds in the sense of Lorentzian length spaces (cf. [KS18, Ex. 4.9]).

We apply the preceding corollary to specific spaces and warping functions to obtain:

**Corollary 5.5.** Let \( X \) be a geodesic length space with curvature bounded below/above by \( K \) (third column). With the interval \( I \) given in the first column and the warping function given in the second column, \( I \times_f X \) has timelike curvature bounded below/above by \( K' \) (forth column):

| \( I \) | \( f \) | \( X \) | \( \text{CB b/a by } K \) | \( I \times_f X \) | \( \text{TLCB b/a by } K' \) |
|------|------|------|-----------------|-----------------|-----------------|
| \((0, \pi)\) | \( \sin \) | \( -1 \) | \( -1 \) | \( -1 \) |
| \((-\frac{\pi}{2}, \frac{\pi}{2})\) | \( \cos \) | \( -1 \) | \( -1 \) | \( 0 \) |
| \((0, \infty)\) | \( \text{id} \) | \( -1 \) | \( 0 \) | \( 1 \) |
| \( \mathbb{R} \) | \( \exp \) | \( 0 \) | \( 1 \) | \( 0 \) |
| \( \mathbb{R} \) | \( 1 \) | \( 0 \) | \( 0 \) | \( 1 \) |
| \( \mathbb{R} \) | \( \cosh \) | \( 1 \) | \( 1 \) | \( 1 \) |

A result in the converse direction holds as well, showing that if \( Y \) and \( Y' \) satisfy timelike curvature bounds then \( X \) has a curvature bound. To show
this, we first need the following Lemma that establishes that we can lift a geodesic triangle $\Delta$ in $X$ to a timelike geodesic triangle in $Y$, provided $\Delta$ is small enough.

**Lemma 5.6.** Let $Y = I \times_f X$ be a generalized cone, where $X$ is a geodesic length space. Let $(t_0, \bar{p}_0) \in Y$, then for all neighborhoods $V \subseteq Y$ of $(t_0, \bar{p}_0)$ there is a constant $C > 0$ (depending on $f$ and $V$) such that any convex neighborhood $U$ of $\bar{p}_0$ in $X$ with diam$(U) \leq C$ has the property that any geodesic triangle in $U$ can be lifted to a timelike geodesic triangle in $V$.

**Proof:** Let $(t_0, \bar{p}_0) \in Y$ and $V \subseteq Y$ a neighborhood of $(t_0, \bar{p}_0)$, and set $f(t_0) := m > 0$. Then there is an $\varepsilon > 0$ such that for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon] \subseteq I$ we have $f(t) \leq 2m$ and $[t_0 - \varepsilon, t_0 + \varepsilon] \times B_{\sqrt{2m}}(\bar{p}_0) \subseteq V$. Moreover, set $\alpha(t) := t$, $\alpha: [t_0 - \varepsilon, t_0 + \varepsilon] \to I$ and $t_\pm := \alpha(t_0 \pm \varepsilon) \in I$. Let $U$ be a convex neighborhood in $X$ with diam$(U) \leq \frac{\varepsilon}{2\sqrt{2m}} =: C$. Then $U \subseteq B_{\sqrt{2m}}(\bar{p}_0)$.

Let $\Delta = (\bar{x}, \bar{y}, \bar{z})$ be a geodesic triangle in $U$. Let $\bar{\beta} : [0, d(\bar{x}, \bar{y})] \to U$ be a minimizing unit-speed geodesic connecting $\bar{x}$ and $\bar{y}$. We now reparametrize $\bar{\beta}$ as follows: $\beta(s) := \bar{\beta}(\frac{s - d(\bar{x}, \bar{y})}{\varepsilon})$ for $s \in [t_0 - \varepsilon, t_0]$. Clearly, $\beta : [t_0 - \varepsilon, t_0] \to U$ is minimizing and $v_\beta = \frac{d(\bar{x}, \bar{y})}{\varepsilon}$ almost everywhere. We establish that $\gamma := (\alpha, \beta) : [t_0 - \varepsilon, t_0] \to V$ is future directed timelike from $x := (t_-, \bar{x})$ to $y := (t_0, \bar{y})$:

$$-\dot{\alpha}^2 + (f \circ \alpha)^2 v_\beta^2 = -1 + f(t) \left(1 + \frac{d(\bar{x}, \bar{y})^2}{\varepsilon^2} \right) \leq -1 + m^2 \frac{diam(U)^2}{\varepsilon^2} \leq -\frac{1}{2}.$$

Thus we have $(t_-, \bar{x}) \ll (t_0, \bar{y})$ and analogously $(t_0, \bar{y}) \ll (t_+, \bar{z}) =: z$. As $Y$ is geodesic by Corollary 4.12 there is a maximizing future directed timelike curve from $x$ to $y$, whose projection to $X$ is a minimizing curve in $X$ by Theorem 4.24 (i). Note that this projection is in general different from $\beta$. However, one can now proceed as in the proof of Corollary 4.12 and obtain a maximal timelike curve in $Y$ whose projection is $\beta$ or $\bar{\beta}$, respectively. Analogously, one can argue the existence of maximizing causal curves from $y$ to $z$ and from $x$ to $z$ for which the $X$-components are the minimizing geodesics from $\bar{y}$ to $\bar{z}$ and from $\bar{x}$ to $\bar{z}$. Thus $\Delta := (x, y, z)$ is a lift of $\Delta$ and by construction $\Delta$ and all its sides lie in $V$. \hfill $\square$

With this preparation we can show the following:

**Theorem 5.7.** If $X$ is a geodesic length space, $Y = I \times_f X$ has timelike curvature bounded below (above) by $K'$ and $Y' = I \times_f M^2(K)$ has timelike curvature bounded above (below) by $K'$ then $X$ has curvature bounded below (above) by $K$.

**Proof:** Fix $\bar{p}_0 \in X$ and $t_0 \in I$. Let $V \subseteq Y$ be a comparison neighborhood for $(t_0, \bar{p}_0)$ in $Y$ such that all timelike triangles in $V$ satisfy the size bounds.
Let $C > 0$ be given by Lemma $5.6$ and let $U$ be a convex neighborhood of $\bar{p}_0$ in $X$ with $\text{diam}(U) \leq C$. Let $\bar{\Delta} = (\bar{x}, \bar{y}, \bar{z})$ be a geodesic triangle in $U$ satisfying the appropriate size bounds with its sides realized by unit speed geodesics $\beta_{r,z}$ $(r, s \in \{x, y, z\})$. Let $\bar{p}, \bar{q}$ be points on $\bar{\Delta}$, say $\bar{p} \in \beta_{x,\bar{y}}$ and $\bar{q} \in \beta_{x,\bar{z}}$. Then by Lemma $5.6$ we can lift $\bar{\Delta}$ to a timelike geodesic triangle $\Delta = (x, y, z)$ in $I \times I \times X$, where $x = (x_0, \bar{x})$, $y = (y_0, \bar{y})$, $z = (z_0, \bar{z})$, with $x \ll y \ll z$, and whose sides are realized by future directed timelike maximal curves $\gamma_{r,s} = (\alpha_{r_0,s_0}(t_r), s)$ $(r, s \in \{x, y, z\})$. Let $\bar{\Delta}' = (\bar{x}', \bar{y}', \bar{z}')$ be a comparison triangle in $\mathbb{M}^2(K)$ of $\bar{\Delta}$, whose sides are realized by geodesics $\beta_{x,\bar{y}}$ in $\mathbb{M}^2(K)$. Again we lift $\bar{\Delta}'$ to a timelike geodesic triangle $\Delta'$ in $Y'$ of the same side lengths as $\Delta$. Moreover, let $\Delta''$ be a comparison triangle in $\mathbb{L}^2(K')$ of $\Delta$, which therefore is also a comparison triangle of $\Delta'$. Let $\bar{p}', \bar{q}'$ be points corresponding to $\bar{p}, \bar{q}$ on $\Delta'$. We can lift these points to points on $p, q \in \Delta$ and $p', q' \in \Delta'$ as follows: Let $t_{\bar{p}} \in [0, d_X(\bar{r}, \bar{s})]$, then $p := \gamma_{r,s}(t_{\bar{p}}) = (\alpha_{r_0,s_0}(t_{\bar{p}}), \bar{p})$, and analogously for $\bar{q}$ and $\bar{p}', \bar{q}'$. Then let $p', q''$ be corresponding points on $\Delta''$ with respect to $p, q$, which are also corresponding to $p', q'$. At this point we can use the curvature bounds to obtain

$$\tau(p, q) \leq \tau_{\mathbb{L}^2(K')}(p'', q'') \leq \tau'(p', q').$$

Since by construction $p \leq q$ and $p' \leq q'$, Lemma $5.2$ now gives $d_{\mathbb{L}^2(K)}(p', q') \leq d(\bar{p}, \bar{q})$, concluding the proof. □

### 6 Synthetic singularity theorems for generalized cones

**Lemma 6.1.** Let $K' > K$ (resp. $K' < K$). Then for small enough corresponding timelike triangles $\Delta = (x, y, z) \in \mathbb{L}^2(K)$ with $x \ll y \ll z$ such that $\tau_{\mathbb{L}^2(K)}(x, z) > \tau_{\mathbb{L}^2(K')}(x, y) + \tau_{\mathbb{L}^2(K)}(y, z)$ (i.e., $x, y, z$ don’t lie on a single maximizing geodesic) and $\Delta' = (x', y', z') \in \mathbb{L}^2(K')$, $x' \ll y' \ll z'$ and corresponding points $q \in yz \subseteq I^+(x')$, $q' \in y'z' \subseteq I^+(x')$ with $q \neq y, z$ (and consequently $q' \neq y', z'$) we have

$$\tau_{\mathbb{L}^2(K')}(x', q') > \tau_{\mathbb{L}^2(K)}(x, q') \quad \text{(resp.} < \text{)}$$

**Proof.** Note first that $K' \geq K$ implies that $\mathbb{L}^2(K)$ has sectional curvature bounded below by $K'$ in the sense of $[\text{AB08, Eq. (1.1)}]$ (i.e., $\mathcal{R} \geq K'$ if and only if spacelike sectional curvatures are bounded below by $K'$ and timelike ones bounded above by $K'$) and thus $[\text{AB08, Thm. 1.1}]$ shows $\tau_{\mathbb{L}^2(K')}(x', q') \geq \tau_{\mathbb{L}^2(K)}(x, q')$.

To show that strict inequality in the curvatures implies strict inequality of the time separations we follow the proof of $[\text{AB08, Thm. 1.1}]$, at times
referencing [Kir18] for additional details, following the notations therein except that our model space is now $\mathbb{L}^2(K')$, not $\mathbb{L}^2(K)$, and we use $'$ to denote the corresponding objects in the model space (as opposed to $\tilde{}$ in [AB08, Kir18]):

In this proof the curvature inequality is first used in Corollary 4.5, where the authors use $R_\sigma \geq R'_{\sigma'}$, and a general comparison result for modified shape operators along a geodesic, see [AB08, Thm. 4.3], to get that the modified shape operators satisfy $S \leq S'$ at corresponding points of corresponding non-null geodesic segments $\sigma$ and $\sigma'$. Note that by the proof of [AB08, Cor. 4.5], see also [Kir18, Lem. 5.2.2] for a more detailed argument, $S$ and $S'$ split into direct sums

$$S = S|_{\sigma} \oplus S|_{\sigma^\perp}, \quad S' = S'|_{\sigma'} \oplus S'|_{(\sigma')^\perp},$$

where the summands $S|_{\sigma}, S'|_{\sigma'}$ act on the 1-dimensional spaces tangent to the geodesics $\sigma, \sigma'$ whereas $S|_{\sigma^\perp}, S'|_{(\sigma')^\perp}$ act on their orthogonal complements. According to [AB08, Cor. 4.5] the summands acting on the 1-dimensional spaces tangent to the radial geodesics are actually equal, i.e., $S|_{\sigma} = S'|_{\sigma'}$ at corresponding points. Because of this it is not possible to get the strict inequality $S < S'$ even if $R_\sigma > R'_{\sigma'}$. However, the rigidity part of [AB08, Thm. 4.3] (resp. [Kir18, Thm. 2.2.3]) together with the proof of [AB08, Cor. 4.5] does imply that we get the strict inequality

$$S|_{\sigma^\perp} < S'|_{(\sigma')^\perp}.$$  \hspace{1cm} (11)

Indeed, if we had equality of these (restricted) shape operators at any $\sigma(t_0), \sigma'(t_0)$ then [AB08, Thm. 4.3] would imply that $R_{\sigma(t)} = R'_{\sigma'(t)}$ for all $t \leq t_0$. However, this cannot hold as $K' > K$ implies that $R_{\sigma} > R'_{\sigma'}$ for all parameter values.

It remains to argue that the strict inequality (11) carries through to a strict inequality in the time separations as long as the three points don’t lie on a common geodesic segment. Thus, let $x, y, z$ be as in the statement of the lemma and let $\gamma \equiv \gamma_{yz} : [0, 1] \to \mathbb{L}^2(K)$ be the geodesic from $y$ to $z$. For $q(s) = \gamma(s)$, let $\gamma_{q(s)} : [0, 1] \to \mathbb{L}^2(K)$ be the geodesic from $x$ to $q(s)$ and for each $s$ denote its tangent vector at the endpoint by $w(q(s))$, i.e. $w(q(s)) = \dot{\gamma}_{q(s)}(1)$. Then, because $x, y, z$ don’t lie on a single geodesic, $\dot{\gamma}(s) \not\parallel w(q(s))$ and hence, replacing [AB08, Cor. 4.5] by (11) in the proof of [AB08, Cor. 4.6] (and keeping in mind the direct sum decompositions (10) of $S, S'$ with $S|_{\tilde{\sigma}} = S'|_{\tilde{\sigma'}}$, we get $\langle S_{\gamma_{q(s)}}, (\dot{\gamma}(s)) \dot{\gamma}(s), \dot{\gamma}(s) \rangle < (1 - K'h_{K',x}(\gamma(s)))\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle$. Thus, following [AB08, Prop. 5.2], $(h_{K',x} \circ \gamma)^\prime(s) < (1 - K'h_{K',x}(\gamma(s)))\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle$.

Defining $\tilde{f} := h_{K',x} \circ \gamma_{yz} - h'_{K',x} \circ \gamma'_{yz}$ as in [AB08, Prop. 5.2], we get $\tilde{f} + E(\gamma)Kf < 0$ as contrasted with [AB08, Prop. 5.2]. From this $f \geq 0$ follows as in [AB08, Prop. 5.2], see [Kir18, Lem. 5.1.1] for details.
We now argue that \( f > 0 \) on the open interval \((0, 1)\). Assume there exists \( s_0 \in (0, 1) \) with \( f(s_0) = 0 \). Then \( f \) has a local minimum at \( s_0 \), so \( \dot{f}(s_0) \geq 0 \), contradicting the strict inequality \( \ddot{f}(s_0) < E(\gamma)Kf(s_0) = 0 \).

Thus, \( h_{K',x} \circ \gamma_{yz}(s) > h'_{K',x}(q) \circ \gamma'_{yz}(s) \) for all \( s \in (0, 1) \), and in particular \( h_{K',x}(q) > h'_{K',x}(q') \). Since \( h_{K',x}(q) \) is really only a function of the signed distance \( -\tau_{\mathbb{L}^2(K)}(x, q) \) and \( h'_{K',x}(q') \) is the same function of the signed distance \( -\tau_{\mathbb{L}^2(K')}((x', q')) \) and this function is strictly increasing (see [Kir18] Lem. 4.1.4) or simply by the definition of the \( h_{K,x} \) in [AB08, Eq. (1.6)]), this implies \( -\tau_{\mathbb{L}^2(K)}(x, q) > -\tau_{\mathbb{L}^2(K')}((x', q')) \), i.e.,

\[
\tau_{\mathbb{L}^2(K)}(x, q) < \tau_{\mathbb{L}^2(K')}((x', q'))
\]

and we are done with the case \( K' > K \).

The \( K' < K \) case follows from this immediately by simply switching \( K \) and \( K' \) above.

**Theorem 6.2.** Let \( X \) be a geodesic length space, \( Y = I \times_f X \) with \( f : I \to (0, \infty) \) smooth. Assume that \( Y \) has timelike curvature bounded below (above) by \( K \), then \( f \) is \( K \)-concave (convex), i.e., \( f'' - Kf \leq 0 \) (\( f'' - Kf \geq 0 \)).

**Proof.** We only treat the case where \( Y \) has timelike curvature bounded below by \( K \). The proof for a bound from above is analogous.

Assume to the contrary that \( f \) is not \( K \)-concave, i.e., there exists \( t_0 \in I \) such that \( f''(t_0) > Kf(t_0) \). Then there exists an interval \( J \subseteq I \), \( t_0 \in J \) and \( K' > K \) such that \( f'' > K'f \) on \( J \). Define \( Y' := J \times_f \mathbb{R} \), then \( Y' \) is a smooth two-dimensional Lorentzian manifold with (timelike) sectional curvature \( \mathcal{R} = \frac{f''}{f} > K' \). Since sectional curvature bounds in the sense of [AB08, Eq. (1.1)] (i.e., \( \mathcal{R} \geq K' \) if and only if spacelike sectional curvatures are bounded below by \( K' \) and timelike ones bounded above by \( K' \)) imply timelike curvature bounds in the sense of Lorentzian length spaces (cf. [KS18, Ex. 4.9], based on [AB08, Thm. 1.1]), we get that \( Y' \) has timelike curvature bounded above by \( K' > K \). Let \( x' = (x_0, 0), y' = (y_0, \tilde{y}), z' = (z_0, 0) \) be points in \( Y' \) forming a small timelike geodesic triangle. Let \( p' = x' \) and \( q' = (0, q_0) \) be corresponding points for \( x' \) and \( y' \) respectively. Then \( p'', q'' \) and \( p''', q''' \) be corresponding points for \( p', q' \) on the sides of the comparison triangle for \( \Delta(x', y', z') \) in \( \mathbb{L}^2(K) \) and \( \mathbb{L}^2(K) \), respectively. Then

\[
\tau_{Y'}(p', q') \geq \tau_{\mathbb{L}^2(K')}(p'', q'') > \tau_{\mathbb{L}^2(K)}(p''', q'''),
\]

where we used Lemma 6.1 to get the last strict inequality.

Now let \( \bar{x} \in X \) be arbitrary and set \( x := (x_0, \bar{x}), y := (y_0, \bar{y}), z := (z_0, \bar{x}) \), where \( \bar{y} \in X \) is chosen such that \( d_X(\bar{x}, \bar{y}) = d_{\mathbb{R}^2}(0, \tilde{y}) = |\tilde{y}| \). Then by fiber independence (Theorem 3.29(ii)), \( \Delta = (x, y, z) \) is a triangle in \( Y \) corresponding to \( (x', y', z') \) in \( Y' \). Let \( p = x \) and let \( q \) be the point

42
corresponding to \( q' \) on the side \( yz \) of \( \Delta \). Then, again by fiber independence, 
\( \tau_Y(p, q) = \tau_Y'(p', q') \) and using the assumption that \( Y \) has timelike curvature bounded below by \( K \) we obtain the contradiction

\[
\tau_{L^2(K)}(p'''', q''') \geq \tau_Y(p, q) = \tau_Y'(p', q') > \tau_{L^2(K)}(p'''', q''').
\]

\( \square \)

Remark 6.3. In the future it would also be interesting to examine if one might still obtain \( K \)-concavity (\( K \)-convexity) in the barrier sense for non-smooth warping functions \( f \).

We now relate non-positive lower timelike curvature bounds to the existence of singularities, i.e., incomplete causal geodesics. To this end we first recall some notions and results from [GKS19]. A geodesic in a Lorentzian length space \( X \) is a causal curve that is locally maximizing ([GKS19, Def. 4.1]) and for a smooth strongly causal spacetime \( (M, g) \) one has that causal pregeodesics of \( (M, g) \) are geodesics in the synthetic sense above and vice versa (of the same causal character) ([GKS19, Thm. 4.4]). Moreover, a geodesic \( \gamma : [a, b) \to X \) is extendible if there is a geodesic \( \bar{\gamma} : [a, b) \to X \) such that \( \bar{\gamma}|_{[a, b)} = \gamma \). Otherwise, \( \gamma \) is called inextendible. Also, we have an analogous notion of timelike geodesic completeness in the synthetic setting: A Lorentzian length space \( X \) is said to have property \( (TC) \) if all inextendible timelike geodesics have infinite \( \tau \)-length ([GKS19, Def. 5.1]). This notion is again compatible with the smooth spacetime case as a smooth and strongly causal spacetime is timelike geodesically complete if and only if it has property \( (TC) \) ([GKS19, Lem. 5.2]). Consequently, we call a timelike geodesic incomplete if it has finite \( \tau \)-length and the space \( X \) timelike geodesically incomplete if there is an inextendible timelike geodesic which is incomplete. Analogous notions are defined for past and future incompleteness.

Corollary 6.4. Let \( X \) be a geodesic length space, \( Y = I \times_f X \) with \( I = (a, b), f : I \to (0, \infty) \) smooth. Assume that \( Y \) has timelike curvature bounded below by \( K \). Then:

(i) If \( K < 0 \), then \( a > -\infty \) and \( b < \infty \) and hence the time separation function \( \tau_Y \) of \( Y \) is bounded by \( b - a \). Thus any such \( Y \) is timelike geodesically incomplete.

(ii) If \( K = 0 \) and \( f \) is non-constant, then \( a > -\infty \) or \( b < \infty \) and hence \( Y \) is past or future timelike geodesically incomplete.

Proof. We first show (i): Assume first that \( b = \infty \). Set \( u := \int f \). Theorem 6.2 implies \( f'' \leq Kf \), so \( u \) satisfies the differential inequality \( u' \leq -u^2 + K \leq \min\{-u^2, K\} \). Since \( K < 0 \) this shows that there exists some \( s_0 \in (a, \infty) \) with \( u(s_0) < 0 \). Let \( s_{\text{max}} \) be the maximal \( s > s_0 \) such that \( -\infty < u|_{[s_0, s)} < 0 \) and let \( s_1 := s_0 - \frac{1}{u(s_0)} > s_0 \). Then integrating \( u' \leq -u^2 \) shows that
\[ u \leq \frac{1}{s-s_1} = \frac{1}{s(s-s_0)} < u(s_0) < 0 \text{ on } (s_0, \min\{s_{\text{max}}, s_1\}) \]. But from this we see that \( s_{\text{max}} \leq s_1 < \infty \) and \( u \to -\infty \) as \( s \to s_{\text{max}} \) (otherwise it would contradict the maximality of \( s_{\text{max}} \)). Since \( u = \frac{f'}{f} \) this implies \( f \to 0 \) as \( s \to s_{\text{max}} \), contradicting \( f > 0 \) on \((a, \infty)\).

To show that \( a > -\infty \) we simply reverse the time orientation, i.e., we consider \( \tilde{f}(s) := f(-s) \) instead of \( f \).

For (ii), since \( f \) is non-constant we have \( f'(s_0) \neq 0 \) for some \( s_0 \). If \( f'(s_0) < 0 \), then \( u(s_0) < 0 \) and we get a contradiction to \( b = \infty \) as in (i). If \( f'(s_0) > 0 \), we can again just reverse the time orientation to get \( \tilde{f}'(-s_0) < 0 \) and a contradiction to \( a = -\infty \).

**Remark 6.5.** If \((X, h)\) is a smooth \( n \)-dimensional Riemannian manifold, then \( Y = I \times_f X \) with metric \( g = -dt^2 + f(t)^2 h \) is a smooth Lorentzian manifold and we may compare Corollary 6.4 with the Hawking singularity theorem and the Lorentzian Myers’ theorem (O’N83, Thm. 55A and 55B, BEC96, Thm. 11.9) applied to \( Y \). A key assumption in both these theorems is a bound on the timelike Ricci curvature, which is implied by certain sectional curvature bounds: In any smooth \((n+1)\)-dimensional Lorentzian manifold a bound on timelike sectional curvatures from below/above by \( K \) implies a bound on the timelike Ricci curvature from above/below by \(-nK\). However, even in the smooth setting it is not known, to the best of our knowledge, if a bound on merely the timelike sectional curvatures, which is strictly weaker than assuming a sectional curvature bound in the sense of [AB08, Eq. (1.1)], will imply triangle comparison for timelike triangles, i.e., a timelike curvature bound as in [KS18], or vice versa. To be more precise, let \((M, g)\) be a smooth Lorentzian manifold that has timelike sectional curvatures bounded below by some \( K \in \mathbb{R} \), then it is unclear if \((M, g)\) (viewed as a Lorentzian length space) has timelike curvature bounded below by \( K \) in the sense of triangle comparison with \( L^2(K) \). Thus, a timelike curvature bound as in [KS18] might in general not imply the corresponding timelike Ricci curvature bounds. However, in the specific warped product setting we are considering, the following simple relationship holds: For any \( V \in TX \) we have \( \text{Ric}(\partial_t, \partial_t) = -n\frac{f'}{f} = -n\mathcal{K}(\partial_t, V) \), where \( \mathcal{K}(\partial_t, V) \) denotes the sectional curvature of the timelike plane spanned by \( \partial_t \) and \( V \). So Theorem 6.2 shows that triangle comparison for timelike triangles implies both a bound on the sectional curvatures of all timelike planes orthogonal to \( X \) and on \( \text{Ric}(\partial_t, \partial_t) \). More specifically, timelike curvature bounded below by \( K \) implies that \( \text{Ric}(\partial_t, \partial_t) \geq -nK \).

Now the comparison with Hawking’s theorem is straightforward: It is well known that the key assumptions for Hawking’s theorem to hold boil down to \( \text{Ric}(\dot{\gamma}, \dot{\gamma}) \geq 0 \) for all timelike unit speed geodesics \( \gamma \) starting orthogonally to an initial Cauchy surface \( \Sigma \) with mean curvature \( H_\Sigma < \beta < 0 \). In our warped product we may take \( \Sigma \) to be of the form \( \{t_0\} \times X \). Then the
mean curvature $H_\Sigma$ equals $n{L'(t_0) \over f(t_0)}$ and $\dot{\gamma}(t) = \partial_t|_{\gamma(t)}$, so Hawking’s singularity theorem corresponds exactly to the $K = 0$ case of Corollary 6.4 (and whether one has a past or a future singularity is determined by the sign of $H_\Sigma = n{L'(t_0) \over f(t_0)}$, which equals the sign of ${L'(t_0) \over f(t_0)}$).

In the $K < 0$ case, Corollary 6.4 implies that the timelike diameter of $I \times f X$ is bounded by $b - a < \infty$, so this case corresponds to the Lorentzian Myers’ theorem. Note that while the standard formulation of the Lorentzian Myers’ theorem (as in [BEE96]) requires $\text{Ric}(W,W) \geq n|K| > 0$ for all unit timelike vectors $W$, one can use the techniques of the proof of the Hawking singularity theorem to show that to get a bound on the timelike diameter it is sufficient to assume $\text{Ric}(\dot{\gamma},\dot{\gamma}) \geq n|K| > 0$ for all timelike unit speed geodesics $\gamma$ starting orthogonally to a Cauchy surface $\Sigma$ (cf. [Gra16, Thm. 4.2 and Rem. 4.3]). However, the bound obtained in this way may be larger than $\pi \sqrt{|K|}$. With this in mind, we see that also the $K < 0$ case of Corollary 6.4 corresponds directly to its smooth counterpart.

Following [ON83, Def. 12.16] we define big bang and big crunch singularities as follows:

**Definition 6.6.** Let $Y = (a,b) \times f X$ be a generalized cone, where $f$ is smooth. Then

(i) the generalized cone $Y$ has a big bang singularity at $a$ if $f(t) \to 0$ and $f'(t) \to \infty$ as $t \searrow a$, and

(ii) the generalized cone $Y$ has a big crunch singularity at $b$ if $f(t) \to 0$ and $f'(t) \to -\infty$ as $t \nearrow b$.

**Corollary 6.7.** If $Y$ has a big bang or a big crunch singularity, then $Y$ cannot have timelike curvature bounded above by any $K \in \mathbb{R}$.

**Proof.** Assume that $f(s) \to 0$ and $f'(s) \to -\infty$ as $s \to b$ and that $Y$ has timelike curvature bounded above by $K$. First, if $K \geq 0$ then $f'' \geq K f \geq 0$, contradicting $f' \to -\infty$. So let $K < 0$ and let $s_0$ be such that $f \leq 1$ on $(s_0, b)$. Then on this interval $f'' \geq K f \geq K$, so $f'(s) \geq K(s - s_0) + f'(s_0)$, contradicting $f'(s) \to -\infty$. The big bang case follows by reversing the time orientation.

**7 Conclusion and outlook**

By transporting the notion of a (generalized) cone into the synthetic Lorentzian setting we achieved the following objectives:

- We established that the causality of generalized cones is optimal, even for fibers that are not necessarily manifolds.
• Generalized cones are instances of strongly causal Lorentzian length spaces.

• Timelike curvature bounds of a generalized cone can be related to the (metric) curvature bounds of its fiber and vice versa.

• Our results allow one to generate an abundance of examples of Lorentzian length spaces with timelike curvature bounds.

• We proved singularity theorems for generalized cones that are direct analogues of the Hawking singularity theorem and the Lorentzian Myers’ theorem. That is, we showed how non-positive lower timelike curvature bounds imply the existence of incomplete timelike geodesics.

Our methods are also expected to be applicable to more general spaces with one-dimensional fiber, like the Colombini-Spagnolo metrics discussed in Remark 3.23. More generally, topics of further investigation are the following:

• Generalize our methods and results to spaces of the form $I \times X$, where on each fiber $\{t\} \times X$ one has a $t$-dependent metric. This would generalize the metric splitting for globally hyperbolic spacetimes.

• Extend the results of Corollary 5.4 to non-smooth warping functions $f$ that are $K'$-convex or $K'$-concave in the support sense (cf. [AB03]).

• Put a Lorentzian structure on general warped products $B \times f F$, where the base $B$ is a Lorentzian length space and the fiber $F$ is a metric (length) space.

• Extend the singularity theorems of Section 6 to lower regularity (i.e., lower the regularity of the warping function $f$).

• Relate our approach to the theory of Sormani and Vega [SV16], in particular to the results on warped products of Allen and Burtscher [AB19].

Acknowledgments The authors are grateful to the anonymous referees for their detailed feedback and careful reading of the article.

This work was supported by research grants P28770 and J4305 of the Austrian Science Fund FWF and parts of this work were carried out while Melanie Graf was at the University of Tübingen.

A Appendix: Lorentzian length structures

In this appendix we want to outline that in the situation where one merely has
• a notion of causal curves and
• a notion of length of such curves,

one can already construct a Lorentzian pre-length space without push-up or topology, hence without lower semicontinuity of $\tau$, but such that nevertheless the time separation function is intrinsic. So in a certain sense the situation is even better than starting out with a Lorentzian pre-length space. In doing so we can reproduce some of the results of Section 3 but in greater generality. Also, this fact was already mentioned in [KS18], so here we expand on [KS18 Rem. 5.11(ii)] and provide some arguments to illustrate the matter.

In analogy with the metric case (cf. [BB10 Sec. 2.1]) we first define a Lorentzian length structure. To this end we also need to introduce notions of admissible curves and length functionals.

**Definition A.1.** Let $(X, d)$ be a metric space and denote by $A$ the class of absolutely continuous curves from an interval into $X$. Let $I^\pm \subseteq C^\pm \subseteq A$ be four families of of absolutely continuous curves. Then $(I^\pm, C^\pm)$ is called admissible if it satisfies the following axioms.

(C1) Every curve in $C^\pm$ (hence in $I^\pm$) is never constant, i.e., restrictions to non-trivial subintervals are non-constant.

(C2) The classes $I^\pm$ and $C^\pm$ are closed under (non-trivial) restrictions, e.g., if $\gamma: [a, b] \to X$ is in $I^\pm$ or in $C^\pm$, and $a \leq c < d \leq b$ then the restriction $\gamma|_{[c,d]}$ of $\gamma$ to $[c,d]$ is in $I^\pm$ or $C^\pm$, respectively.

(C3) The classes $I^\pm$ and $C^\pm$ are closed under concatenations, that is if $\gamma: [a, b] \to X$ is a curve such that the (non-trivial) restrictions $\gamma|_{[a,c]}$ and $\gamma|_{[c,d]}$ are in $I^\pm$ or in $C^\pm$ for some $a \leq c \leq b$ then so is their concatenation $\gamma$.

(C4) The classes $I^\pm$ and $C^\pm$ are closed under reparametrizations: Let $\gamma: J' \to X$ be in $I^\pm$ or in $C^\pm$ and let $\phi: J \to J'$ be a strictly increasing continuous map defined on an interval $J \subseteq \mathbb{R}$ such that it and its inverse are absolutely continuous. Then $\gamma \circ \phi \in I^\pm$ or $\in C^\pm$, respectively. Moreover, if $\phi$ is as above but orientation reversing, i.e., strictly decreasing, then $\gamma \circ \phi \in I^-\pm$ or $\in C^-\pm$, respectively.

We call the curves in $I^\pm$ future/past directed timelike curves and the ones in $C^\pm$ future/past directed causal curves. Moreover, set $I := I^+ \cup I^-$ and $C := C^+ \cup C^-$.

**Definition A.2.** Lorentzian length structure (LLStr) A Lorentzian length structure on a metric space $(X, d)$ is an admissible tuple $(I^\pm, C^\pm)$ of subsets of $A$ together with a function

$$L: C \to [0, \infty],$$

47
called the Lorentzian length functional, which satisfy the following list of properties:

(L1) $L$ is additive: If $\gamma: [a, b] \to X$ is in $\mathcal{C}$ then $L(\gamma) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,d]})$ for any $c \in (a, b)$.

(L2) $L$ is invariant under reparametrizations, i.e., for $\gamma$ and $\phi$ as in (C4) we require $L(\gamma \circ \phi) = L(\gamma)$.

(L3) For every $\gamma \in \mathcal{I}$ we have $L(\gamma) > 0$.

(L4) The length structure respects the topology of $X$ in the following sense: $L$ depends continuously on the parameter of the curve, e.g., if $\gamma: [a, b] \to X$ is in $\mathcal{C}$ we set $L(\gamma, a, t) := L(\gamma|_{[a,t]})$. Then $t \mapsto L(\gamma, a, t)$ is continuous.

We write $(X, d, \mathcal{I}^\pm, \mathcal{C}^\pm, L)$ for a Lorentzian length structure.

Recall that a causal space $(X, \ll, \leq)$ on a set $X$ is constituted by two transitive relations $\ll$ and $\leq$ on $X$, where $\leq$ is additionally reflexive and contains $\ll$, cf. [KS18, Def. 2.1] (this is a slightly stronger notion than the one introduced in [KP67]).

**Definition A.3.** Let $(X, d, \mathcal{I}^\pm, \mathcal{C}^\pm, L)$ be a Lorentzian length structure. For $x, y \in X$ define $x \leq y$ if $x = y$ or there exists a $\gamma \in \mathcal{C}^+$ from $x$ to $y$. Moreover, define $x < y$ if $x \leq y$ and $x \neq y$. Also, define $x \ll y$ if there exists a $\gamma \in \mathcal{I}^+$ from $x$ to $y$.

**Lemma A.4.** Let $(X, d, \mathcal{I}^\pm, \mathcal{C}^\pm, L)$ be a Lorentzian length structure and let the relations $\ll, \leq$ be as defined above. Then $(X, \ll, \leq)$ is a causal space.

**Proof:** This follows from axiom (C3) and $\mathcal{I}^+ \subseteq \mathcal{C}^+$.

**Definition A.5.** Let $(X, d, \mathcal{I}^\pm, \mathcal{C}^\pm, L)$ be a Lorentzian length structure. For $x, y \in X$ define the time separation of $x$ and $y$ by

$$\tau(x, y) := \sup\{L(\gamma) : \gamma \in \mathcal{C}^+ \text{ connects } x \text{ and } y\},$$

if the set of connecting future directed causal curves is non-empty, otherwise set $\tau(x, y) := 0$.

The time separation function $\tau$ has the following properties.

**Lemma A.6.** Let $(X, d, \mathcal{I}^\pm, \mathcal{C}^\pm, L)$ be a Lorentzian length structure and let the time separation function $\tau$ and the causal relations $\ll, \leq$ be as defined above. Then
(i) \( \tau(x,y) = 0 \) if \( x \not\leq y \) and

(ii) \( \tau(x,y) > 0 \) if \( x \ll y \).

**Proof:**

(i) This is immediate from the definitions.

(ii) Let \( x \ll y \), then there is a timelike curve \( \gamma \in I^+ \) from \( x \) to \( y \) and by axiom (L3) we have \( L(\gamma) > 0 \). Thus \( 0 < L(\gamma) \leq \tau(x,y) \).

The reverse triangle inequality holds:

**Lemma A.7.** Let \( (X,d,I^+,C^+,L) \) be a Lorentzian length structure with time separation function \( \tau \) and causal relations \( \ll, \leq \) as defined above. Then for all \( x,y,z \in X \) with \( x \leq y \leq z \)

\[ \tau(x,y) + \tau(y,z) \leq \tau(x,z) . \]

**Proof:** If \( x = y \) then either \( \tau(x,x) = 0 \) or there exists a causal curve in \( C^+ \) from \( x \) to \( x \). In the first case we are done. Analogously, if \( y = z \) either \( \tau(y,y) = 0 \) or there exists a causal curve in \( C^+ \) from \( y \) to \( y \). Again in the first case we are done. So we can without loss of generality assume that there are causal curves connecting \( x \) to \( y \) and connecting \( y \) to \( z \). Let \( \gamma, \lambda \in C^+ \) with \( \gamma \) from \( x \) to \( y \) and \( \lambda \) from \( y \) to \( z \). Then by axiom (C3) the concatenation \( \gamma * \lambda \) is in \( C^+ \) and connects \( x \) to \( z \). Thus \( L(\gamma) + L(\lambda) = L(\gamma * \lambda) \leq \tau(x,z) \), by (L1). Taking now the supremum over all future directed causal curves connecting \( x \) and \( y \) and the ones connecting \( y \) to \( z \) gives the claim.

To investigate only the algebraic properties and further consequences of having a set with a Lorentzian length structure we relax the notion of Lorentzian pre-length space as introduced in [KS18, Def. 2.8] and call the resulting generalization a **bare** Lorentzian pre-length space. In particular, we do not want to assume semi-continuity properties of the time separation function from the beginning.

**Definition A.8.** Let \( (X,\ll,\leq) \) be a causal space together with a function \( \tau: X \times X \to [0,\infty] \) that satisfy the following properties:

(i) If \( x \ll y \) then \( \tau(x,y) > 0 \).

(ii) The reverse triangle inequality holds: for all \( x,y,z \in X \) with \( x \leq y \leq z \) one has

\[ \tau(x,y) + \tau(y,z) \leq \tau(x,z) . \]

Then \( (X,\ll,\leq,\tau) \) is called a bare Lorentzian pre-length space.
This is a generalization of the notion of a Lorentzian pre-length space, where we do not assume that \( X \) comes with a metric (topology), push-up to hold or that \( \tau \) is lower semicontinuous with respect to this topology.

Given a bare Lorentzian pre-length space and a metric \( d \) on \( X \) one can still define causal curves, their length etc., and get the same basic properties (following the steps as in \([KS18\text{ Sec. 2}])\). For example, given now the causal relations defined in Definition A.3 a \( \leq \)-causal curve is a locally Lipschitz curve \( \gamma \) such that \( \gamma(s) \leq \gamma(t) \) for any two parameter values with \( s \leq t \) (cf. \([KS18\text{ Def. 2.18}])\). Note that any curve in \( C^+ \) is \( \leq \)-causal, and analogously for timelike curves.

Finally, we define

\[
\mathcal{T}(x,y) := \sup \{ L_\tau(\gamma) : \gamma \text{ future directed } \leq \text{-causal from } x \text{ to } y \},
\]

if the set of connecting future directed \( \leq \)-causal curves from \( x \) to \( y \) is non-empty. Otherwise, we set \( \mathcal{T}(x,y) := 0 \).

**Definition A.9.** A bare Lorentzian pre-length space \( (X, \ll, \leq, \tau) \) is called bare Lorentzian length space if \( \mathcal{T} = \tau \), i.e., if the time separation function \( \tau \) is intrinsic (for some background metric \( d \) on \( X \)).

At this point we are able to establish that a Lorentzian length structure gives rise to a bare Lorentzian length space.

**Theorem A.10.** Let \( (X, d, I^\pm, C^\pm, L) \) be a Lorentzian length structure and define the causal relations \( \leq, \ll \) and the time separation function \( \tau \) as above. Then \( (X, \ll, \leq, \tau) \) is a bare Lorentzian length space.

**Proof:** That \( (X, \ll, \leq, \tau) \) is a bare Lorentzian pre-length space follows from Lemmata A.4, A.6 and A.7.

As noted, any \( C^+ \)-causal curve is \( \leq \)-causal and analogously for \( I^+ \) and \( \ll \). Moreover, for any \( \leq \)-causal curve \( \gamma \) (hence also for any \( \gamma \in C^+ \)) from \( x \) to \( y \) one has \( L_\tau(\gamma) \leq \tau(x,y) \) by the definition of the \( \tau \)-length. Let \( (\gamma : [a, b] \rightarrow X) \in C^+ \) and let \( a = t_0 < t_1 < \cdots < t_N = b \) be a partition of \([a, b]\), then by (L1)

\[
L(\gamma) = \sum_{i=0}^{N-1} L(\gamma|_{[t_i, t_{i+1})}) \leq \sum_{i=0}^{N-1} \tau(\gamma(t_i), \gamma(t_{i+1})).
\]

\(^3\text{Of course, by endowing } X \text{ with the discrete metric, any function from } X \text{ to another topological space is continuous. Thus any such bare Lorentzian pre-length space without topology can be viewed as a Lorentzian pre-length space in the sense of \([KS18\text{ Def. 2.8]}\text{ (except that push-up need not hold). However, note that then there are no non-constant curves, hence no causal curves.}}\)
Taking the infimum over all partitions of \([a, b]\) yields that \(L(\gamma) \leq L_\tau(\gamma)\) for any \(\gamma \in C^+\) (defined on a compact interval). From this we immediately get \(\tau \leq T\) from the definitions by first taking the supremum on the right hand side and then on the left hand side.

It remains to show that \(T \leq \tau\). If \(T(x, y) = 0\) there is nothing to do, so let \(T(x, y) > 0\). This means that for any \(\varepsilon > 0\) there is a future directed \(\leq\)-causal curve \(\gamma\) from \(x\) to \(y\) with \(T(x, y) < L_\tau(\gamma) + \varepsilon \leq \tau(x, y) + \varepsilon\). Here we used that \(L_\tau(\gamma) \leq \tau(x, y)\), and since \(\varepsilon > 0\) was arbitrary we are done. 

References

- [AB98] S. B. Alexander and R. L. Bishop. Warped products of Hadamard spaces. *Manuscripta Math.*, 96(4):487–505, 1998.
- [AB03] S. B. Alexander and R. L. Bishop. \(\mathcal{FK}\)-convex functions on metric spaces. *Manuscripta Math.*, 110(1):115–133, 2003.
- [AB04] S. B. Alexander and R. L. Bishop. Curvature bounds for warped products of metric spaces. *Geom. Funct. Anal.*, 14(6):1143–1181, 2004.
- [AB06] C. D. Aliprantis and K. C. Border. *Infinite dimensional analysis*. Springer, Berlin, Third edition, 2006. A hitchhiker’s guide.
- [AB08] S. B. Alexander and R. L. Bishop. Lorentz and semi-Riemannian spaces with Alexandrov curvature bounds. *Comm. Anal. Geom.*, 16(2):251–282, 2008.
- [AB16] S. B. Alexander and R. L. Bishop. Warped products admitting a curvature bound. *Adv. Math.*, 303:88–122, 2016.
- [AB19] B. Allen and A. Y. Burtscher. Properties of the Null Distance and Spacetime Convergence. *Int. Math. Res. Not., to appear, arXiv:1909.04483 [math.DG]*, 2019.
- [AGS05] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.
- [AKP19] S. Alexander, V. Kapovitch, and A. Petrunin. Alexandrov geometry: preliminary version no. 1. *Preprint, arXiv:1903.08539 [math.DG]*, 2019.
- [Ale19] S. B. Alexander. Alexandrov Geometry for Lorentzian-Pseudometrics. 2019. in preparation.
[AT04] L. Ambrosio and P. Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

[BBI01] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[BEE96] J. K. Beem, P. E. Ehrlich, and K. L. Easley. *Global Lorentzian geometry*, volume 202 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, second edition, 1996.

[BH99] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BLMS87] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin. Space-time as a causal set. *Phys. Rev. Lett.*, 59(5):521–524, 1987.

[BS03] A. N. Bernal and M. Sánchez. On smooth Cauchy hypersurfaces and Geroch’s splitting theorem. *Commun. Math. Phys.*, 243(3):461–470, 2003.

[BS05] A. N. Bernal and M. Sánchez. Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. *Commun. Math. Phys.*, 257(1):43–50, 2005.

[BS18] P. Bernard and S. Suhr. Lyapounov Functions of Closed Cone Fields: From Conley Theory to Time Functions. *Comm. Math. Phys.*, 359(2):467–498, 2018.

[Bus67] H. Busemann. Timelike spaces. *Dissertationes Math. Rozprawy Mat.*, 53:52, 1967.

[CG12] P. T. Chruściel and J. D. E. Grant. On Lorentzian causality with continuous metrics. *Classical Quantum Gravity*, 29(14):145001, 32, 2012.

[Che99] C.-H. Chen. Warped products of metric spaces of curvature bounded from above. *Trans. Amer. Math. Soc.*, 351(12):4727–4740, 1999.

[CM20] F. Cavalletti and A. Mondino. Optimal transport in Lorentzian synthetic spaces, synthetic timelike Ricci curvature lower bounds and applications. *preprint, arXiv:2004.08934 [math.MG]*; 2020.
[CS89] F. Colombini and S. Spagnolo. Some examples of hyperbolic equations without local solvability. *Ann. Sci. École Norm. Sup. (4)*, 22(1):109–125, 1989.

[DL17] M. Dafermos and J. Luk. The interior of dynamical vacuum black holes I: The $C^0$-stability of the Kerr Cauchy horizon. *Preprint*, arXiv:1710.01722 [gr-qc], 2017.

[FS12] A. Fathi and A. Siconolfi. On smooth time functions. *Math. Proc. Cambridge Philos. Soc.*, 152(2):303–339, 2012.

[GGKS18] M. Graf, J. D. E. Grant, M. Kunzinger, and R. Steinbauer. The Hawking–Penrose Singularity Theorem for $C^{1,1}$-Lorentzian Metrics. *Comm. Math. Phys.*, 360(3):1009–1042, 2018.

[GKS19] J. D. E. Grant, M. Kunzinger, and C. Sämann. Inextendibility of spacetimes and lorentzian length spaces. *Annals of Global Analysis and Geometry*, 55(1):133–147, 2019.

[GKSS20] J. D. E. Grant, M. Kunzinger, C. Sämann, and R. Steinbauer. The future is not always open. *Lett. Math. Phys.*, 110(1):83–103, 2020.

[GL17] G. J. Galloway and E. Ling. Some remarks on the $C^0$-(in)extendibility of spacetimes. *Ann. Henri Poincaré*, 18(10):3427–3447, 2017.

[GL18] M. Graf and E. Ling. Maximizers in Lipschitz spacetimes are either timelike or null. *Classical Quantum Gravity*, 35(8):087001, 6, 2018.

[GLS18] G. J. Galloway, E. Ling, and J. Shbierski. Timelike completeness as an obstruction to $C^0$-extensions. *Comm. Math. Phys.*, 359(3):937–949, 2018.

[Gra16] M. Graf. Volume comparison for $C^{1,1}$-metrics. *Ann. Glob. Anal. Geom.*, 50(3):209–235, 2016.

[Gra20] M. Graf. Singularity theorems for $C^1$-Lorentzian metrics. *Comm. Math. Phys.*, 378(2):1417–1450, 2020.

[Gro01] K. Grove. Review of ”Metric structures for Riemannian and non-Riemannian spaces” by M. Gromov. *Bull. Amer. Math. Soc.*, 38, pages 353–363, 2001.

[Ise15] J. Isenberg. On strong cosmic censorship. In *Surveys in differential geometry 2015. One hundred years of general relativity*, volume 20 of *Surv. Differ. Geom.*, pages 17–36. Int. Press, Boston, MA, 2015.
[Kap07] V. Kapovitch. Perelman’s stability theorem. In Surveys in differential geometry. Vol. XI, volume 11 of Surv. Differ. Geom., pages 103–136. Int. Press, Somerville, MA, 2007.

[Kir18] M. Kirchberger. Lorentzian Comparison Geometry. Master’s thesis, University of Vienna. Available at http://othes.univie.ac.at/56285/, 2018.

[KP67] E. H. Kronheimer and R. Penrose. On the structure of causal spaces. Proc. Cambridge Philos. Soc., 63:481–501, 1967.

[KS18] M. Kunzinger and C. Sämann. Lorentzian length spaces. Ann. Glob. Anal. Geom., 54(3):399–447, 2018.

[KPS14] M. Kunzinger, R. Steinbauer, and M. Stojković. The exponential map of a $C^{1,1}$-metric. Differential Geom. Appl., 34:14–24, 2014.

[KSSV14] M. Kunzinger, R. Steinbauer, M. Stojković, and J. A. Vickers. A regularisation approach to causality theory for $C^{1,1}$-Lorentzian metrics. Gen. Relativity Gravitation, 46(8):Art. 1738, 18, 2014.

[KSSV15] M. Kunzinger, R. Steinbauer, M. Stojković, and J. A. Vickers. Hawking’s singularity theorem for $C^{1,1}$-metrics. Classical Quantum Gravity, 32(7):075012, 19, 2015.

[KSV15] M. Kunzinger, R. Steinbauer, and J. A. Vickers. The Penrose singularity theorem in regularity $C^{1,1}$. Classical Quantum Gravity, 32(15):155010, 12, 2015.

[LMO19] Y. Lu, E. Minguzzi, and S.-i. Ohta. Geometry of weighted Lorentz-Finsler manifolds I: Singularity theorems. Preprint, arXiv:1908.03832 [math.DG], 2019.

[McC20] R. McCann. Displacement concavity of Boltzmann’s entropy characterizes positive energy in general relativity. Camb. J. Math., 8(3):609—-681, 2020.

[Min07] E. Minguzzi. On the causal properties of warped product spacetimes. Classical Quantum Gravity, 24(17):4457–4474, 2007.

[Min15] E. Minguzzi. Convex neighborhoods for Lipschitz connections and sprays. Monatsh. Math., 177(4):569–625, 2015.

[Min19a] E. Minguzzi. Causality theory for closed cone structures with applications. Rev. Math. Phys., 31(5):1930001, 139, 2019.

[Min19b] E. Minguzzi. Lorentzian causality theory. Living Reviews in Relativity, 22(1):3, 2019.
[MS08] E. Minguzzi and M. Sánchez. The causal hierarchy of space-times. In Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., pages 299–358. Eur. Math. Soc., Zürich, 2008.

[MS18] A. Mondino and S. Suhr. An optimal transport formulation of the einstein equations of general relativity. Preprint, arXiv:1810.13309 [math-ph], 2018.

[MS19] E. Minguzzi and S. Suhr. Some regularity results for Lorentz-Finsler spaces. Ann. Global Anal. Geom., 56(3):597–611, 2019.

[Nat55] I. P. Natanson. Theory of functions of a real variable. Frederick Ungar Publishing Co., New York, 1955. Translated by Leo F. Boron with the collaboration of Edwin Hewitt.

[O’N83] B. O’Neill. Semi-Riemannian geometry with applications to relativity, volume 103 of Pure and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.

[Pap14] A. Papadopoulos. Metric spaces, convexity and non-positive curvature, volume 6 of IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society (EMS), Zürich, second edition, 2014.

[Per93] G. Y. Perel’man. Elements of Morse theory on Aleksandrov spaces. Algebra i Analiz, 5(1):232–241, 1993.

[Pet16] P. Petersen. Riemannian geometry, volume 171 of Graduate Texts in Mathematics. Springer, Cham, third edition, 2016.

[Säm16] C. Sämann. Global hyperbolicity for spacetimes with continuous metrics. Ann. Henri Poincaré, 17(6):1429–1455, 2016.

[Sbi18] J. Sbierski. The $C^0$-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry. J. Differential Geom., 108(2):319–378, 2018.

[Sur19] S. Surya. The causal set approach to quantum gravity. Living Reviews in Relativity, 22(5), 2019.

[SV16] C. Sormani and C. Vega. Null distance on a spacetime. Classical Quantum Gravity, 33(8):085001, 29, 2016.

[Wal84] R. M. Wald. General relativity. University of Chicago Press, Chicago, IL, 1984.