Nearly optimal bounds for the global geometric landscape of phase retrieval

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Abstract

The phase retrieval problem is concerned with recovering an unknown signal $x \in \mathbb{C}^n$ from a set of magnitude-only measurements $y_j = |\langle a_j, x \rangle|, j = 1, \ldots, m$. A natural least squares formulation can be used to solve this problem efficiently even with random initialization, despite its non-convexity of the loss function. One way to explain this surprising phenomenon is the benign geometric landscape: (1) all local minimizers are global; and (2) the objective function has a negative curvature around each saddle point and local maximizer. In this paper, we show that $m = O(n \log n)$ Gaussian random measurements are sufficient to guarantee the loss function of a commonly used estimator has such benign geometric landscape with high probability. This is a step toward answering the open problem given by Sun \textit{et al} (2018 \textit{Found. Comput. Math.} 18 1131–98), in which the authors suggest that $O(n \log n)$ or even $O(n)$ is enough to guarantee the favorable geometric property.

Keywords: phase retrieval, geometric landscape, nonconvex optimization

(Some figures may appear in colour only in the online journal)

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1. Introduction

1.1. Background

In x-ray crystallography \cite{13, 18, 27}, the signal that we are interested in is the electron density function of the crystal. This function can be assumed to be a periodic function with some unit cell given by

\[ x_c(t) = \sum_{l \in S} x(t - l), \]

where \( S \) is a finite subset of a lattice \( \Lambda \subset \mathbb{R}^D \) and \( x \) is a compactly supported motif, which will be defined more precisely later. The crystal is then irradiated with a planar beam, and the Fourier magnitudes of the scattered wave are recorded in the far field, namely,

\[ |\hat{x}_c(k)|^2 = \left| \int_{\mathbb{R}^D} x_c(t) e^{-2\pi i (t, k)} dt \right|^2 = \left| \sum_{l \in S} x(t - l) e^{-2\pi i (t, k)} dt \right|^2 = \left| \int_{\mathbb{R}^D} \sum_{l \in S} x(t - l) e^{-2\pi i (t, k)} dt \right|^2 = \sum_{l \in S} e^{-2\pi i (l, k)} \left| \int_{\mathbb{R}^D} x(t) e^{-2\pi i (t, k)} dt \right|^2 = |\hat{\delta}(k)|^2 |\hat{\delta}(k)|^2, \]

where \( \hat{x} \) and \( \hat{\delta} \) are the Fourier transforms of the signal \( x \) and a Dirac ensemble signal \( \delta \) supported on \( S \), respectively. As the size of the set \( S \) increases, the support of the function \( \hat{\delta} \) becomes increasingly concentrated on the dual lattice \( \Lambda^* \), giving rise to a discrete set of samples of \( |\hat{x}|^2 \) on \( \Lambda^* \). This phenomenon is known as the Bragg peaks. It also implies that the measurement data that we obtain are the Fourier magnitudes of the signal \( x \), where the signal \( x \) is defined by its Fourier series as:

\[ x(t) = \frac{1}{\text{Vol}(\Lambda)} \sum_{k \in \Lambda^*} \hat{x}(k) e^{2\pi i (k, t)}. \]

In this paper, we consider the discrete phase retrieval in a more general setting, where one is interested in how to recover an unknown signal \( x \in \mathbb{C}^n \) from a series of magnitude-only measurements

\[ y_j = |\langle a_j, x \rangle|, \quad j = 1, \ldots, m, \tag{1} \]

where \( a_j \in \mathbb{C}^n \) and \( m \) is the number of measurements. It is easy to see that the measurements \( a_j \) are the Fourier basis vectors for the x-ray crystallography, which is a special case of the setting (1). Besides the x-ray crystallography, the phase retrieval problem (1) also has applications in microscopy \cite{19, 26}, astronomy \cite{11}, coherent diffractive imaging \cite{17, 31} and optics \cite{24, 37}. Despite its simple mathematical formulation, it has been shown that reconstructing a finite-dimensional discrete signal from the magnitude of its Fourier transform is generally an NP-complete problem \cite{30}.

Due to the practical ubiquity of the phase retrieval problem, many algorithms have been designed for this problem. For example, based on the technique ‘matrix-lifting’, the phase retrieval problem can be recast as a low rank matrix recovery problem. By using convex relaxation one can show that the matrix recovery problem under suitable conditions is equivalent to a convex optimization problem \cite{6, 8, 36}. However, since the matrix-lifting technique involves semidefinite program for \( n \times n \) matrices, the computational cost is prohibitive for large scale problems. In contrast, many non-convex algorithms bypass the lifting step and operate directly
on the lower-dimensional ambient space, making them much more computationally efficient. Early non-convex algorithms were mostly based on the technique of alternating projections, e.g. Gerchberg–Saxton [17] and Fineup [14]. The main drawback, however, is the lack of theoretical guarantee. Later Netrapalli et al [28] proposed the AltMinPhase algorithm based on a technique known as spectral initialization. They proved that the algorithm linearly converges to the true solution with $O(n \log^3 n)$ resampling Gaussian random measurements. This work led further to several other non-convex algorithms based on spectral initialization. A common thread is first choosing a good initial guess through spectral initialization, and then solving an optimization model through gradient descent, such as [7, 9, 20, 34, 38–40]. We refer the reader to survey papers [1, 10, 21, 31] for accounts of recent developments in the theory, algorithms and applications of phase retrieval.

1.2. Prior arts and motivation

As stated earlier, producing a good initial guess using carefully-designed initialization seems to be a prerequisite for prototypical non-convex algorithms to succeed with good theoretical guarantee. A natural and fundamental question is:

Is it possible for non-convex algorithms to achieve successful recovery with a random initialization?

Recently, Sun et al carried out a deep study of the global geometric structure of the loss function:

$$F(z) = \frac{1}{m} \sum_{j=1}^{m} (|\langle a_j, z \rangle|^2 - y_j^2)^2,$$

where $y_j$ are measurements given in (1). They proved that the loss function does not possess any spurious local minima under $O(n \log^3 n)$ Gaussian random measurements. More specifically, it was shown in [32] that all minimizers of $F(z)$ coincide with the target signal $x$ up to a global phase, and $F(z)$ has a negative directional curvature around each saddle point. Thanks to this benign geometric landscape any algorithm which can avoid strict saddle points converges to the true solution with high probability. A trust-region method was employed in [32] to find the global minimizers with random initialization. The results in [32] require $m \geq O(n \log^3 n)$ samples to guarantee the favorable geometric property and efficient recovery. On the other hand, based on ample numerical evidences, the authors of [32] conjectured that the optimal sampling complexity could be $O(n \log n)$ or even $O(n)$ to guarantee the benign landscape of the loss function $F(z)$ (cf p. 1160 therein).

In this paper, we focus on this conjecture and prove that the loss function $F(z)$ possesses the favorable geometric property, as long as the measurement number $m \geq O(n \log n)$, by some sophisticated analysis. In other words, we prove that (1) all local minimizers of the loss function $F(z)$ are global; and (2) the objective function $F(z)$ has a negative curvature around each saddle point and local maximizer. This is a step toward proving the open problem.

We shall emphasize that if allowing some modifications to the loss function $F(z)$, the sampling complexity can be reduced to the optimal bound $O(n)$ [4, 5, 25]. In [25] the authors show that a combination of the loss function (2) with a judiciously chosen activation function also has the benign geometry structure under $O(n)$ Gaussian random measurements. Furthermore, in our recent work [3], we consider another new smoothed amplitude flow estimator which is based on a piece-wise smooth modification to the loss function

$$F(z) = \sum_{j=1}^{m} (|\langle a_j, z \rangle| - y_j)^2,$$
and we could also prove that the loss function (3) after some modifications has a benign geometric landscape under the optimal sampling threshold $m = O(n)$.

The emerging concept of a benign geometric landscape has also recently been explored in many other applications of signal processing and machine learning, e.g. matrix sensing [2, 29], tensor decomposition [15], dictionary learning [33] and matrix completion [16]. For general optimization problems there exist a plethora of loss functions with well-behaved geometric landscapes such that all local optima are also global optima and each saddle point has a negative direction curvature in its vicinity. Correspondingly several techniques have been developed to guarantee that the standard gradient based optimization algorithms can escape such saddle points efficiently, see e.g. [12, 22, 23].

1.3. Our contributions

In this paper, we focus on the open problem: what is the optimal sampling complexity to guarantee the loss function $F(z)$ given in (2) has favorable geometric landscape? We develop several new techniques and prove that $m \geq O(n \log n)$ Gaussian random measurements are enough. While we can not prove the optimality of this bound, it is an improvement over the result of $m \geq O(n^{3/2})$ given in [32]. The main result of our paper is the following theorem.

**Theorem 1.** Assume that $a_j \in \mathbb{R}^n, j = 1, \ldots, m$ are i.i.d. standard Gaussian random vectors and $0 \neq x \in \mathbb{R}^n$ is a fixed vector. There exist positive absolute constants $C, c$ and $C'$, such that if $m \geq C'n \log n$, then with probability at least $1 - cm^{-1} - 7 \exp(-c'm)$ the loss function $F(z)$ defined by (2) has no spurious local minimizers. In other words, the only local minimizer is $x$ up to a global phase and all saddle points are strict, i.e. each saddle point has a neighborhood where the function has negative directional curvature. Moreover, the loss function is strongly convex in a neighborhood of $\pm x$, and the point $z = 0$ is a local maximum point where the Hessian is strictly negative-definite.

**Remark 1.** To maintain conciseness and clarity in our paper, we focus solely on the geometric landscape in the real case. Nonetheless, the result in theorem 1 can also be extended to the complex case, as detailed in remark 3 in section 2.

**Remark 2.** Another interesting issue is to show the measurements are non-adaptive, i.e. a single realization of measurement vectors $a_j \in \mathbb{R}^n$ can be used to reconstruct all $0 \neq x \in \mathbb{R}^n$. However we shall not dwell on this refinement here for simplicity.

1.4. Notations

Throughout the paper, we write $z \in S^{n-1}$ if $z \in \mathbb{R}^n$ and $\|z\|_2 = 1$. We use $\chi$ to denote the usual characteristic function. For example $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. For any quantity $X$, we shall write $X = O(Y)$ if $|X| \leq CY$ for some universal constant $C > 0$. We write $X \lesssim Y$ if $X \leq CY$ for some universal constant $C > 0$. We shall write $X \ll Y$ if $X \leq cY$ where the constant $c > 0$ will be sufficiently small. We use $m \gtrsim n$ to denote $m \geq Cn$ where $C > 0$ is a universal constant. In this paper, we use $C, c$ and the subscript (superscript) form of them to denote universal constants whose values vary with the context.

1.5. Organization

The rest of the papers are organized as follows. In section 2, we divide the whole space $\mathbb{R}^n$ into several regions and investigate the geometric property of $F(z)$ on each region. In section 3,
we present the detailed justification for the technical lemmas given in section 2. Finally, the appendix collects some auxiliary estimates needed in the proof.

2. Proof of the main result

In the rest of this section we shall carry out the proof of theorem 1 in several steps. More specifically, we decompose $\mathbb{R}^n$ into several regions (not necessarily non-overlapping), on each of which $F(z)$ has certain property that will allow us to show that with high probability $F(z)$ has no local minimizers other than $\pm x$. Furthermore, we show $F(z)$ is strongly convex in a neighborhood of $\pm x$.

Without loss of generality, we assume $\|x\|_2 = 1$. Denote $\sigma = \sigma(z) := \langle z, x \rangle / \|z\| \|x\|$. Then we can decompose $\mathbb{R}^n$ into three regions as shown below.

$$\mathcal{R}_1 := \left\{ z \in \mathbb{R}^n : |\sigma| \leq \sqrt{\frac{\epsilon_0}{2}} - \epsilon_0 \right\},$$

$$\mathcal{R}_2 := \left\{ z \in \mathbb{R}^n : |\sigma| \geq 0.5 \text{ and } \text{dist}(z, x) \geq \delta_0 \right\},$$

$$\mathcal{R}_3 := \left\{ z \in \mathbb{R}^n : \text{dist}(z, x) \leq \delta_0 \right\},$$

where $\epsilon_0$ is an arbitrary small positive constant and $0 < \delta_0 < 1/4$ is a universal constant. Figure 1 visualizes the partitioning regions described above and gives the idea of how they cover the whole space.

The properties of $F(z)$ over these regions are summarized in the following three lemmas.

**Lemma 1.** For any $\epsilon_0 > 0$ there exists a constant $\delta_0 > 0$ such that with probability at least $1 - c m^{-2} - 5\exp(-c(\epsilon_0)m)$ it holds: any critical point $z \in \mathcal{R}_1 := \left\{ z \in \mathbb{R}^n : |\sigma| \leq \sqrt{\frac{\epsilon_0}{2}} - \epsilon_0 \right\}$ obeys

$$x^\top \nabla^2 F(z)x \leq -\delta_1$$

provided $m \geq C(\epsilon_0) n$. Here, $C(\epsilon_0)$ and $c(\epsilon_0)$ are positive constants depending only on $\epsilon_0$, and $c > 0$ is a universal constant.

**Lemma 2.** Assume that $m \geq C(\delta_0) n$. Then with probability at least $1 - c(\delta_0)m^{-2} - \exp(-c'(\delta_0)m)$ there is no critical point in the region $z \in \mathcal{R}_2 := \left\{ z \in \mathbb{R}^n : |\sigma| \geq 0.5 \text{ and } \text{dist}(z, x) \geq \delta_0 \right\}$. Here, $C(\delta_0), c(\delta_0)$ and $c'(\delta_0)$ are constants depending only on $\delta_0$.

**Lemma 3.** For any $0 < \delta_0 \leq 1/4$, assume that $m \geq C(\delta_0) n \log n$. There exists a constant $\delta_2$ such that with probability at least $1 - c(\delta_0)m^{-2} - \exp(-c'(\delta_0)m)$ we have $u^\top \nabla^2 F(z)u \geq \delta_2$ for all $z \in \mathcal{R}_3 := \left\{ z \in \mathbb{R}^n : \text{dist}(z, x) \leq \delta_0 \right\}$ and unit vectors $u \in \mathbb{R}^n$. In other words, $F(z)$ is strongly convex in $\mathcal{R}_3$. Here, $C(\delta_0), c(\delta_0)$ and $c'(\delta_0)$ are constants depending only on $\delta_0$.

The proofs of the above lemmas are given in section 3. Lemma 2 guarantees the gradient of $F(z)$ does not vanish in $\mathcal{R}_2$. Thus the critical points of $F(z)$ can only occur in $\mathcal{R}_1$ and $\mathcal{R}_3$. However, lemma 1 shows that at any critical point in $\mathcal{R}_1$, $F(z)$ has a negative directional curvature. Finally, lemma 3 implies that $F(z)$ is strongly convex in $\mathcal{R}_3$. Recognizing that $\nabla F(x) = 0$ and $x \in \mathcal{R}_3$, thus $x$ is the local minimizer. Putting it all together, we can establish theorem 1 as shown below.
Figure 1. Partition of $\mathbb{R}^2$: The target signal is $x = [1, 0]$ with constants $\epsilon_0 = 0.1$ and $\delta_0 = 0.2$. (a): The composition of all regions; (b): The region $R_1$; (c): The region $R_2$; (d): The region $R_3$.

Proof of theorem 1. For any $z$ being a possible critical point and satisfying $|\sigma| \leq \sqrt{\frac{1}{2} - 0.01}$, lemma 1 shows that $F(z)$ has a negative directional curvature. For any $z$ satisfying $|\sigma| \geq 0.5$ and $\text{dist}(z, x) \geq 0.01$, lemma 2 demonstrates that the gradient $\nabla F(z) \neq 0$. Finally, when $z$ is very close to the target solutions $\pm x$, $F(z)$ is strongly convex and $\pm x$ are the global solutions.
Remark 3. Using similar techniques, the result in theorem 1 can be extended to the complex case. Specifically, for any \( z \in \mathbb{C}^n \), if we define
\[
\phi(z) := \arg\min_{\theta \in [0,2\pi]} \|z - xe^{i\theta}\|_2 \quad \text{and} \quad \sigma(z) := \langle z, xe^{i\theta}(z) \rangle / \|z\| \|x\|,
\]
then lemmas 1–3 can be adapted to the following three results:

1. For any \( \varepsilon_0 > 0 \), there exists a constant \( \delta_1 > 0 \) such that with high probability it holds: any critical point \( z \in \mathcal{R}_1 := \left\{ z \in \mathbb{R}^n : |\sigma| \leq \sqrt{\frac{\delta_1}{\varepsilon_0}} \right\} \) obeys
\[
\left[ \begin{array}{c} xe^{i\theta}(z) \\ xe^{-i\theta}(z) \end{array} \right]^* \nabla^2 F(z) \left[ \begin{array}{c} xe^{i\theta}(z) \\ xe^{-i\theta}(z) \end{array} \right] \leq -\delta_1 \|x\|_2^2
\]
provided \( m \geq C(\varepsilon_0)n \). Here, \( \nabla^2 F(z) \) is the Wirtinger Hessian (see also [7, 32]) of \( F(z) \) defined as
\[
\nabla^2 F(z) = \frac{1}{m} \sum_{j=1}^{m} \left[ \begin{array}{cc} (2|a_j^* z|^2 - y_j^2) a_j a_j^\top & (a_j^* z)^2 a_j a_j^\top \\ (z^* a_j)^2 a_j a_j^\top & (2|a_j^* z|^2 - y_j^2) a_j a_j^\top \end{array} \right].
\]
So, all critical points in \( \mathcal{R}_1 \) are strict saddle points.

2. For any \( \delta_0 > 0 \), if \( m \geq C(\delta_0)n \) then with high probability there is no critical point in the region \( z \in \mathcal{R}_2 := \left\{ z \in \mathbb{R}^n : |\sigma| \geq 0.5 \right\} \) with \( \text{dist}(z,x) \geq \delta_0 \).

3. There exists a constant \( \delta_2 > 0 \) such that if \( m \geq C(\delta_0)n \log n \), then with high probability it holds: for all \( z \in \mathcal{R}_3 := \left\{ z \in \mathbb{R}^n : \text{dist}(z,x) \leq \delta_0 \right\} \), we have
\[
\left[ \begin{array}{c} u \\ \bar{u} \end{array} \right]^* \nabla^2 F(z) \left[ \begin{array}{c} u \\ \bar{u} \end{array} \right] \geq \delta_2,
\]
where
\[
u := \left\{ \begin{array}{ll}
\frac{z - xe^{i\theta}(z)}{\|z - xe^{i\theta}(z)\|_2}, & \text{if } z \neq xe^{i\theta}(z) \\
\left\{ h : \text{Im}(h^* z) = 0, \|h\|_2 = 1 \right\}, & \text{if } z = xe^{i\theta}(z)
\end{array} \right.
\]
In other words, \( ex \) with \( |e| = 1 \) are the only critical points and \( F(z) \) exhibits restricted strong convexity in \( \mathcal{R}_3 \).

With the above three results, we can also show that a geometric landscape similar to theorem 1 holds in the complex case. Specifically, the only local minimizer is \( x \) up to a global phase, all saddle points are strict, and the loss function is restricted strongly convex near the local minimizers.

3. Proofs of technical results in section 2

The basic idea of the proof is to show for each critical point except \( \pm x \) there is a negative curvature direction.

3.1. Proof of lemma 1

Proof. For any \( z \neq 0 \), denote
\[
z = \sqrt{\hat{R}} \hat{z} \quad \text{with} \quad \hat{z} \in \mathbb{S}^{n-1}.
\]
Recall the function

\[ F(z) = \frac{1}{m} \sum_{j=1}^{m} (|a_j, z|^2 - |a_j, x|^2)^2. \]  

Through a simple calculation, the Hessian of the function \( F(z) \) along the \( \xi \) direction can be denote by

\[ H_{\xi \xi}(z) := \xi^\top \nabla^2 F(z) \xi = \frac{4}{m} \sum_{j=1}^{m} (a_j^\top \xi)^2 (3(a_j^\top z)^2 - (a_j^\top x)^2). \]

We first show that \( z = 0 \) is a local maximum. Indeed, by corollary 2 in the appendix, if \( m \gtrsim n \) then it holds with probability at least \( 1 - \exp(-cm) \) that

\[ H_{\xi \xi}(0) = -\frac{4}{m} \sum_{j=1}^{m} (a_j^\top \xi)^2 (a_j^\top x)^2 \leq -c_1 < 0, \quad \forall \xi \in S^{n-1}. \]

Here, \( c \) and \( c_1 \) are universal positive constants. This means that with high probability the Hessian \( \nabla^2 f(0) \) is strictly negative definite.

Next, we consider the case where \( z \neq 0 \) and prove the loss function (5) has a negative curvature at each critical point in the regime \( R \leq |\hat{z}, x| \leq \sqrt{\frac{5}{4}} - \varepsilon_0 \). Through a simple calculation, we have

\[ \nabla F(z) = \frac{2}{m} \sum_{j=1}^{m} (|a_j, z|^2 - |a_j, x|^2) a_j a_j^\top z. \]

If at some \( z \neq 0 \) we have a critical point, then

\[ \langle \nabla F(z), z \rangle = \frac{2R}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 - \frac{2}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 = 0, \]

where \( R \) is defined by (4). By lemma 11, if \( m \gtrsim n \) then it holds with probability at least \( 1 - \exp(-c_2 m) \) that \( \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 \geq 1 \). Here, \( c_2 > 0 \) is a universal constant. Consequently, if at some \( z \neq 0 \) we have a critical point, then it holds

\[ R = \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 \geq \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^4. \]  

(6)

On the other hand, the Hessian at this point along the direction \( x \) is

\[ \frac{1}{4} H_{xx}(z) = 3R \cdot \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 - \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^4. \]

Using the equation (6), we obtain

\[ \frac{1}{4} H_{xx}(z) \cdot \left( \frac{m}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 \right) = 3 \left( \frac{m}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 \right)^2 - \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 \cdot \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^4. \]  

(7)
We claim that for any $0 < \epsilon < 1$, when $m \geq C(\epsilon)n$, with probability at least $1 - \frac{\epsilon}{m} - \exp(-c(\epsilon)m)$, the following holds:

$$
\frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2(a_j^\top x)^2 \leq 2\sigma^2 + 1 + 2\epsilon + 2\sqrt{\epsilon},
$$

(8)

where $C(\epsilon), c(\epsilon) > 0$ are constants depending only on $\epsilon$ and $c > 0$ is a universal constant. On the other hand, by lemma 11, when $m \geq C(\epsilon)n$, with probability at least $1 - 3\exp(-c(\epsilon)m)$ it holds

$$
\frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^4 \geq 3 - \epsilon.
$$

(9)

Putting (8) and (9) into (7), we obtain that for $m \geq C(\epsilon)n$, with probability at least $1 - \frac{\epsilon}{m} - 4\exp(-c(\epsilon)m)$, it holds

$$
\frac{1}{4}H_{xx}(z) - \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 \leq 3(2\sigma^2 + 1)^2 - 3 - 13\epsilon \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 + 60\sqrt{\epsilon} \sqrt{\frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4} + 24\epsilon.
$$

Since the term $\frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4$ is the sum of nonnegative random variables, the deviation below its expectation is bounded and the lower-tail is well-behaved. More concretely, by lemma 11, if $m \geq C(\epsilon)n$ then with probability at least $1 - 3\exp(-c(\epsilon)m)$ we have

$$
\frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 \geq 3 - \epsilon > 1, \quad \forall \hat{z} \in S^{d-1}.
$$

It immediately gives

$$
\frac{1}{4}H_{xx}(z) - \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 \leq 3(2\sigma^2 + 1)^2 - (3 - 13\epsilon - 60\sqrt{\epsilon}) \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 + 24\epsilon
$$

$$
\leq 3(2\sigma^2 + 1)^2 - 9 + 66\epsilon + 180\sqrt{\epsilon}
$$

$$
\leq -c_0\epsilon_0 + 66\epsilon + 180\sqrt{\epsilon}
$$

$$
< -\delta_1
$$

for some constant $\delta_1 > 0$ by taking $\epsilon$ to be sufficiently small (depending on $\epsilon_0$), provided $|\sigma| \leq \sqrt{\frac{3}{2} + 1} - \epsilon_0$. Here, $c_0$ is an absolute constant. This means the Hessian matrix has a negative curvature along the direction $x$, which proves the lemma.

Finally, it remains to prove the claim (8). Due to the heavy tail of fourth powers of Gaussian random variables, to prove the result with sampling complexity $m \gtrsim n$, we need to decompose it into several parts by a Lipschitz continuous truncated function. To do this, take $\phi \in C^\infty_c(\mathbb{R})$, $0 \leq \phi \leq 1$ for all $z \in \mathbb{R}$, $\phi \equiv 1$ for $|z| \leq 1$ and $\phi \equiv 0$ for $|z| > 2$. We can write
\[
\frac{1}{m} \sum_{j=1}^{m} (a_j^\top z)^2 (a_j^\top x)^2 = \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^2 \phi \left( \frac{a_j^\top x}{N} \right) (a_j^\top z)^2 \phi \left( \frac{a_j^\top z}{N} \right)
\]
\[
+ \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^2 \left( 1 - \phi \left( \frac{a_j^\top x}{N} \right) \right) (a_j^\top z)^2 \phi \left( \frac{a_j^\top z}{N} \right)
\]
\[
+ \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^2 (a_j^\top z)^2 \left( 1 - \phi \left( \frac{a_j^\top z}{N} \right) \right)
\]
\[=: B_1 + B_2 + B_3.
\]

Next, we give upper bounds for the terms \(B_1, B_2\) and \(B_3\). Thanks to the smooth cut-off, \(B_1\) can be well bounded. By lemma 12, for any \(0 < \epsilon < 1/2\), there exist constants \(C', c' > 0\) depending on \(N\) such that if \(m \geq C' \epsilon^{-2} \log(1/\epsilon)n\) then with probability at least \(1 - \exp(-c' \epsilon^2 m)\) it holds that

\[
\left| \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^2 \phi \left( \frac{a_j^\top x}{N} \right) (a_j^\top z)^2 \phi \left( \frac{a_j^\top z}{N} \right) - \mathbb{E}(a_1^\top x)^2 \phi \left( \frac{a_1^\top x}{N} \right) (a_1^\top z)^2 \phi \left( \frac{a_1^\top z}{N} \right) \right| \leq \epsilon,
\]

\[\forall z \in S^{n-1}.
\]

Moreover, note that

\[
\left| \mathbb{E}(a_1^\top x)^2 \phi \left( \frac{a_1^\top x}{N} \right) (a_1^\top z)^2 \phi \left( \frac{a_1^\top z}{N} \right) - \mathbb{E}(a_1^\top x)^2 (a_1^\top z)^2 \right|
\]
\[
\leq \mathbb{E}(a_1^\top x)^2 (a_1^\top z)^2 \cdot \left( \chi_{|a_1^\top x| \geq N} + \chi_{|a_1^\top z| \geq N} \right).
\]

Since \(\mathbb{E}(a_1^\top x)^2 (a_1^\top z)^2 = 2 \sigma^2 + 1\), it then follows from lemma 6 that for \(N\) sufficiently large (depending only on \(\epsilon\)) we have

\[
\mathbb{E}(a_1^\top x)^2 \phi \left( \frac{a_1^\top x}{N} \right) (a_1^\top z)^2 \phi \left( \frac{a_1^\top z}{N} \right) \leq \mathbb{E}(a_1^\top x)^2 (a_1^\top z)^2 + \epsilon = 2 \sigma^2 + 1 + \epsilon,
\]

which means

\[
B_1 := \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^2 \phi \left( \frac{a_j^\top x}{N} \right) (a_j^\top z)^2 \phi \left( \frac{a_j^\top z}{N} \right) \leq 2 \sigma^2 + 1 + 2 \epsilon.
\]

For the terms \(B_2\) and \(B_3\), when \(N\) is sufficiently large depending only on \(\epsilon\), applying lemma 10 gives

\[
\frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^2 \chi_{|a_j^\top z| \geq N} \leq \epsilon, \quad \forall z \in S^{n-1}
\]

with probability at least \(1 - \frac{2}{m} - \exp(-c'' \epsilon^2 m)\) provided \(m \geq C \epsilon^{-4} \log(1/\epsilon)n\). Here, \(C, c''\) and \(c''\) are universal positive constants. Thus for \(B_2\) and \(B_3\), by Cauchy-Schwarz inequality, we have
Without loss of generality we can assume \( (11) \).

Similarly, according to \( z \) at any potential critical point \( \hat{z} \in \mathbb{S}^{n-1} \), we have

\[
\frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 \leq 2\sigma^2 + 1 + 2\epsilon + 2\sqrt{\frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4},
\]

which completes the proof of claim \((8)\). \(\blacksquare\)

### 3.2. Proof of lemma 2

**Proof.** Without loss of generality we can assume \( \sigma := \langle \hat{z}, x \rangle \geq 0 \). For any \( z \neq 0 \), denote

\[
z = \sqrt{Rz} \quad \text{with} \quad \hat{z} \in \mathbb{S}^{n-1}.
\]

Recognize that

\[
\nabla F(z) = \frac{2}{m} \sum_{j=1}^{m} \left( |\langle a_j, \hat{z} \rangle|^2 - |\langle a_j, x \rangle|^2 \right) a_j a_j^\top z.
\]

At any potential critical point \( z = \sqrt{Rz} \), we should have \( \nabla F(z) = 0 \). Thus \( \langle \nabla F(z), z \rangle = 0 \) gives

\[
\frac{R}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 = \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2.
\]

Similarly, according to \( \langle \nabla F(z), x \rangle = 0 \) we have

\[
\frac{R}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^3 (a_j^\top x) = \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})(a_j^\top x)^3.
\]

Combining the above two equations leads to the following fundamental relation for any critical point \( z = \sqrt{Rz} \):

\[
\frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 \cdot \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)(a_j^\top \hat{z})^3 = \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4 \cdot \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})(a_j^\top x)^3.
\]

Observe that

\[
\mathbb{E} \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})(a_j^\top x)^3 = 3\sigma^2,
\]

11
where \( \sigma := \langle \hat{z}, x \rangle \geq 0 \). By corollary 3, for any \( 0 < \epsilon < 1/2 \), if \( m \geq C(\epsilon)n \) then with probability at least \( 1 - \frac{c(\epsilon)}{m^2} \) it holds

\[
\left| \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})(a_j^\top x)^3 - 3\sigma \right| \leq \epsilon. \tag{12}
\]

For the convenience, we denote

\[
A := \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^4, \quad B := \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})(a_j^\top x)^2 \quad \text{and} \quad A_1 := \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)(a_j^\top \hat{z})^3.
\]

We claim that for any \( 0 < \epsilon < 1/2 \), if \( m \geq C(\epsilon)n \) then with probability at least \( 1 - \frac{c(\epsilon)}{m^2} \) we have

\[
|B - 2\sigma^2 - 1| \leq 2\epsilon + 2\epsilon^2 A_1^\frac{1}{2} \tag{13}
\]

and

\[
|A_1 - 3\sigma| \leq 2\epsilon + \epsilon^2 A_1^\frac{3}{2}. \tag{14}
\]

Putting (12)-(14) into (11), we immediately have

\[
(2\sigma^2 + 1 + O(\epsilon)A_1^\frac{1}{2}) \cdot (3\sigma + O(\epsilon)A_1^\frac{1}{2}) = A(3\sigma + O(\epsilon)).
\]

with probability at least \( 1 - \frac{c(\epsilon)}{m^2} \) provided \( m \geq C(\epsilon)n \). By lemma 11, for \( m \geq C(\epsilon)n \) with probability at least \( 1 - \exp(-c(\epsilon)m) \) it holds that

\[
A \geq 3 - \epsilon. \tag{16}
\]

In particular, we have \( A \geq 1 \). Then we can simplify (15) as

\[
3\sigma(2\sigma^2 + 1) = A \cdot (3\sigma + O(\epsilon)) \geq (3 - \epsilon)(3\sigma + O(\epsilon)).
\]

(17)

Note that \( z \in \mathcal{R}_2 \), which means \( \sigma \geq 0.5 \). On the other hand, recall that \( \sigma \leq 1 \). By taking \( \epsilon > 0 \) to be sufficiently small, it then follows from (17) that \( \sigma \) must be sufficiently close to 1. It implies that for any \( 0 < \eta < 1 \), if \( m \geq C(\eta)n \) then with probability at least \( 1 - \frac{c(\eta)}{m^2} \) it holds

\[
\sqrt{1 - \eta} \leq \sigma \leq \sqrt{1 - \eta}. \tag{18}
\]

Furthermore, it follows from the equality in (17) that

\[
A \leq 2\sigma^2 + 1 + \frac{1}{\sigma} O(\epsilon) \leq 3 + \eta.
\]

Combining with (16) gives the desired two-way bound for \( A \) that

\[
3 - \eta \leq A \leq 3 + \eta.
\]

On the other hand, it follows from (13) that if \( m \geq C(\epsilon)n \) then with probability at least \( 1 - \frac{c(\epsilon)}{m^2} \) it holds

\[
B = 2\sigma^2 + 1 + O(\epsilon)A_1^\frac{1}{2}.
\]

This immediately means that the term \( B \) also has the desired two-way bounds

\[
3 - \eta \leq B \leq 3 + \eta.
\]
Finally if \( 0 \neq z = \sqrt{R} \hat{z} \) is a critical point, then by (6), we have

\[
R = \frac{B}{A}.
\]

Since we have already shown that \( B \) and \( A \) are well-bounded, it then follows that

\[
1 - \eta \leq R \leq 1 + \eta.
\]  

Combining (18) and (19), we obtain that if \( z := \sqrt{R} \hat{z} \) is a critical point then it holds

\[
\text{dist}(z, x) = \sqrt{R + 1 - 2\sqrt{R}\sigma} \leq \sqrt{3}\eta \leq \delta_0
\]

by taking \( \eta := \delta_0^2/3 \). This contradicts to the condition that \( \text{dist}(z, x) \geq \delta_0 \) for all \( z \in \mathcal{R}_2 \). Thus, the loss function \( F(z) \) has no critical point on \( \mathcal{R}_2 \). We arrive at the conclusion.

Finally, it remains to prove the claims (13) and (14). Let \( \phi \in C^\infty(\mathbb{R}) \) be such that \( 0 \leq \phi(x) \leq 1 \) for all \( x \in \mathbb{R} \), \( \phi(x) = 1 \) for \( |x| \leq 1 \) and \( \phi(x) = 0 \) for \( |x| \geq 2 \). Then we can write

\[
B = \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 \phi \left( \frac{a_j^\top \hat{z}}{N} \right) \phi \left( \frac{a_j^\top x}{N} \right) + r_B,
\]

\[
A_1 = \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^3 (a_j^\top x) \phi \left( \frac{a_j^\top \hat{z}}{N} \right) + r_1,
\]

where

\[
|r_B| \leq \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 (\chi_{|a_j^\top \hat{z}| \geq N} + \chi_{|a_j^\top x| \geq N});
\]

\[
|r_1| \leq \frac{1}{m} \sum_{j=1}^{m} |a_j^\top \hat{z}|^3 |a_j^\top x| \cdot \chi_{|a_j^\top \hat{z}| \geq N}.
\]

Through a simple calculation, we have

\[
\mathbb{E} \left\{ \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 \right\} = 2\sigma^2 + 1 \quad \text{and} \quad \mathbb{E} \left\{ \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z}) (a_j^\top x)^3 \right\} = 3\sigma.
\]

Using the same procedure as the claim (8), it is easy to derive from lemmas 12, 6 and 15 that for any \( 0 < \epsilon < 1 \) if \( m \geq C(\epsilon)n \) then with probability at least \( 1 - \frac{c_0}{m^\epsilon} - \exp(-c'(\epsilon)m) \) it holds

\[
\left| \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^2 (a_j^\top x)^2 \phi \left( \frac{a_j^\top \hat{z}}{N} \right) \phi \left( \frac{a_j^\top x}{N} \right) - 2\sigma^2 - 1 \right| \leq 2\epsilon
\]

and

\[
\left| \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \hat{z})^3 (a_j^\top x) \phi \left( \frac{a_j^\top \hat{z}}{N} \right) - 3\sigma \right| \leq 2\epsilon.
\]
To deal with the error terms, observe that
\[
\sum_{j=1}^{m} (a_j^T z)^2 \chi_{|a_j^T z| > N} (a_j^T x)^2 = \sum_{j=1}^{m} |a_j^T z|^4 \chi_{|a_j^T z| > N} \cdot (a_j^T x)^2 \cdot |a_j^T z|^4 \\
\leq \left( \sum_{j=1}^{m} |a_j^T z|^4 \chi_{|a_j^T z| > N} |a_j^T x|^3 \right) \left( \sum_{j=1}^{m} |a_j^T z|^4 \right)^{\frac{1}{4}};
\]
\[
\sum_{j=1}^{m} |a_j^T z|^3 \chi_{|a_j^T z| > N} |a_j^T x| = \sum_{j=1}^{m} |a_j^T z|^3 \chi_{|a_j^T z| > N} |a_j^T x| \cdot |a_j^T z|^4 \\
\leq \left( \sum_{j=1}^{m} |a_j^T z|^3 \chi_{|a_j^T z| > N} |a_j^T x|^3 \right) \left( \sum_{j=1}^{m} |a_j^T z|^4 \right)^{\frac{1}{4}};
\]
\[
\sum_{j=1}^{m} |a_j^T z|^2 \chi_{|a_j^T x| > N} (a_j^T x)^2 = \sum_{j=1}^{m} |a_j^T z|^2 \chi_{|a_j^T x| > N} (a_j^T x)^2 \cdot |a_j^T z|^4 \\
\leq \left( \sum_{j=1}^{m} |a_j^T z|^2 \chi_{|a_j^T x| > N} |a_j^T x|^3 \right) \left( \sum_{j=1}^{m} |a_j^T z|^4 \right)^{\frac{1}{4}}.
\]

Using the lemma 15 again, we obtain that when \( m \geq C(\epsilon)n \), with probability at least \( 1 - \frac{\epsilon c}{m^4} - \exp(-\epsilon^4 m) \) it holds
\[
|r_B| \leq 2 \epsilon^\frac{7}{2} A^\frac{1}{2} \quad \text{and} \quad |r_1| \leq \epsilon^\frac{7}{2} A^\frac{1}{2}.
\]

Thus, we complete the proofs of claim (13) and (14).

### 3.3. Proof of lemma 3

This section goes in the direction of showing the loss function is strongly convex in a neighborhood of \( \pm x \), as demonstrated in lemma 3.

**Proof.** Recall that along any direction \( u \in S^{n-1} \),
\[
 u^T \nabla^2 F(z) u = \frac{4}{m} \sum_{j=1}^{m} (a_j^T u)^2 \left( 3(a_j^T z)^2 - (a_j^T x)^2 \right).
\]

To prove the lemma, it suffices to give a lower bound for the first term and an upper bound for the second term. Indeed, for the second term \( \frac{1}{m} \sum_{j=1}^{m} (a_j^T u)^2(a_j^T x)^2 \), by lemma 17, for any \( 0 < \epsilon < 1/2 \) if \( m \geq C(\epsilon)n \log n \) then with probability at least \( 1 - \frac{\epsilon c}{m^2} \) it holds
\[
\frac{1}{m} \sum_{j=1}^{m} (a_j^T u)^2(a_j^T x)^2 \leq 1 + 2(u^T x)^2 + \epsilon \quad \text{for any} \quad u \in S^{n-1}.
\]
Here, we use the fact that $\mathbb{E}(a_1^T u)^2(a_1^T x)^2 = 1 + 2(u^T x)^2$. For the first term $\frac{1}{m} \sum_{j=1}^{m} (a_j^T u)^2(a_j^T z)^2$, denote $z := \sqrt{R} \hat{z}$ where $\hat{z} \in S^{n-1}$. Take $\phi \in C^\infty(\mathbb{R})$ such that $0 \leq \phi(x) \leq 1$ for all $x, \phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. It is easy to see
\[
\frac{1}{m} \sum_{j=1}^{m} (a_j^T u)^2(a_j^T z)^2 \geq R \frac{1}{m} \sum_{j=1}^{m} (a_j^T u)^2 \phi \left( \frac{a_j^T z}{N} \right) \left( \frac{a_j^T \hat{z}}{N} \right).
\]
By lemma 12, for any $0 < \epsilon < 1/2$ when $m \geq C(\epsilon)n$ with probability at least $1 - \exp(-c'(\epsilon)m)$, it holds that
\[
\left| \frac{1}{m} \sum_{j=1}^{m} (a_j^T u)^2 \phi \left( \frac{a_j^T z}{N} \right) \left( \frac{a_j^T \hat{z}}{N} \right) \right| - \mathbb{E} \left( (a_1^T u)^2 \phi \left( \frac{a_1^T z}{N} \right) \left( \frac{a_1^T \hat{z}}{N} \right) \right) \leq \epsilon, \quad \forall u, \hat{z} \in S^{n-1}.
\]
On the other hand, it follows from lemma 6 that there exists $N > 0$ sufficiently large (depending only on $\epsilon$) such that
\[
\left| \mathbb{E}(a_1^T u)^2 \phi \left( \frac{a_1^T z}{N} \right) \left( \frac{a_1^T \hat{z}}{N} \right) \right| - \mathbb{E}(a_1^T u)^2(a_1^T \hat{z})^2 \leq \epsilon
\]
Collecting the above estimators, we have that when $m \geq C(\epsilon)n\log n$, with probability at least $1 - \frac{\epsilon(\epsilon)}{m^2} - \exp(-c'(\epsilon)m)$, it holds
\[
\frac{1}{4} u^T \nabla^2 F(z) u \geq 3R \left( 1 + 2(u^T \hat{z})^2 - 2\epsilon \right) - 1 - 2(u^T x)^2 - \epsilon
\]
for all $u, \hat{z} \in S^{n-1}$. Here, we use the fact that $E(a_1^T u)^2(a_1^T \hat{z})^2 = 1 + 2(u^T \hat{z})^2$ in the first inequality. Recall that $z \in R_3$. It means
\[
dist(z, x) \leq \delta_0.
\]
Without loss of generality we assume $\sigma := \langle \hat{z}, x \rangle \geq 0$. It then follows from (21) that
\[
|\sqrt{R} - 1| \leq \|z - x\|_2 \leq \delta_0 \quad \text{and} \quad \|z + x\|_2 \leq 2 + \delta_0.
\]
Putting it into (20), we have
\[
\frac{1}{4} u^T \nabla^2 F(z) u \geq 3(1 - \delta_0)^2 - 1 - 6 \left( (1 + \delta_0)^2 + 1 \right) \epsilon - (2 + \delta_0)\delta_0.
\]
Note that $\delta_0 \leq 1/4$. By taking $\epsilon > 0$ sufficiently small we arrive at the conclusion. \qed
Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix. Preliminaries and supporting lemmas

In this section we shall adopt the following convention.

• For a random variable $Y$, we shall sometimes use ‘mean’ to denote $\mathbb{E}Y$. This notation is particularly handy when $Y$ is given by a sum of random variables involving various truncations and modifications.

• For a random variable $Y$, the sub-exponential Orlicz norm $\|Y\|_{\psi_1}$ is defined as
  \[ \|Y\|_{\psi_1} = \inf\{t > 0 : \mathbb{E}e^{\frac{|X|}{t}} \leq 2\}. \]
  In particular $\|Y\|_{\psi_1} \leq K \Leftrightarrow \mathbb{E}e^{\frac{|X|}{K}} \leq 2$. Similarly the sub-gaussian Orlicz norm $\|Y\|_{\psi_2}$ is
  \[ \|Y\|_{\psi_2} = \inf\{t > 0 : \mathbb{E}e^{\frac{|X|^2}{t}} \leq 2\}. \]

• We denote $\{a_j\}_{j=1}^m$ as a sequence of i.i.d. random vectors which are copies of a standard Gaussian random vector $a : \Omega \to \mathbb{R}^n$ satisfying $a \sim \mathcal{N}(0, I_n)$.

Lemma 4 (Hoeffding’s inequality, [35]). Let $X_i$, $1 \leq i \leq m$ be independent, mean zero, sub-gaussian random variables. Let $b = (b_1, \cdots, b_m) \in \mathbb{R}^m$. Then for every $t \geq 0$, we have
  \[ \mathbb{P}\left( \left| \sum_{i=1}^{m} b_i X_i \right| \geq t \right) \leq 2 \exp\left( -\frac{ct^2}{K^2 \|b\|_2^2} \right), \]
  where $\max_i \|X_i\|_{\psi_2} = K$.

Lemma 5 (Bernstein’s inequality, [35]). Let $X_i$, $1 \leq i \leq m$ be independent, mean zero, sub-exponential random variables. Let $b = (b_1, \cdots, b_m) \in \mathbb{R}^m$. Then for every $t \geq 0$, we have
  \[ \mathbb{P}\left( \left| \frac{1}{m} \sum_{i=1}^{m} b_i X_i \right| > t \right) \leq 2 \exp\left[ -c \cdot \min\left( \frac{m^2 t^2}{K^2 \|b\|_2^2}, \frac{mt}{K \|b\|_\infty} \right) \right], \]
  where $\max_i \|X_i\|_{\psi_1} = K$.

Lemma 6. For any $\epsilon > 0$, there exists $N_0 = N_0(\epsilon) > 0$, such that for any $N \geq N_0$, we have
  \[ \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}\left( (a_j^\top u)^2 (a_j^\top v)^2 X_{|a_j^\top v| \geq N} \right) \leq \epsilon, \]
  where $u \in \mathbb{S}^{n-1}$, $v \in \mathbb{S}^{n-1}$.
Proof. Since \( a_j \) are i.i.d. it suffices to prove the statement for a single random vector \( a \sim \mathcal{N}(0, I_n) \). Noting that \( a^\top u \sim \mathcal{N}(0, 1) \) and \( a^\top v \sim \mathcal{N}(0, 1) \), we have

\[
E\left( (a^\top u)^2 (a^\top v)^2 \chi_{|a^\top v| \geq N} \right) \leq \frac{1}{4} \epsilon^2 \cdot E((a^\top u)^4) + \frac{1}{\epsilon} \cdot E((a^\top v)^4 \chi_{|a^\top v| \geq N}) \leq \frac{3}{4} \epsilon + \frac{1}{\epsilon} \cdot \frac{1}{\sqrt{2\pi}} \int_{|y| \geq N} e^{-\frac{y^2}{2}} dy \leq \epsilon,
\]

if \( N \) is sufficiently large. Note that one can easily quantify \( N_0 \) in terms of \( \epsilon \). However we shall not dwell on this here. \( \square \)

Lemma 7. Let \( A = \frac{1}{m} \sum_{j=1}^{m} a_j a_j^\top \). For any \( 0 < \epsilon < 1 \), if \( m \geq C \epsilon^{-2} n \) then

\[
P(\|A - I\|_2 > \epsilon) \leq \exp(-c m \epsilon^2).
\]

In particular, for \( m \geq C \epsilon^{-2} n \), with probability at least \( 1 - \exp(-c \epsilon^2 m) \), we have

\[
\left| \frac{1}{m} \sum_{j=1}^{m} (a_j^\top u)(a_j^\top v) - \text{mean} \right| \leq \epsilon, \quad \forall u, v \in \mathbb{S}^{n-1}.
\]

Here, \( C \) and \( c \) are universal positive constants.

Proof. We briefly sketch the standard proof here for the sake of completeness. By using a \( \delta \)-net \( S_\delta \) on \( \mathbb{S}^{n-1} \) with \( 0 < \delta < \frac{1}{2} \) and \( \text{Card}(S_\delta) \leq (1 + \frac{2}{\delta})^n \), we have

\[
\|A - I\|_{op} \leq \frac{1}{1 - 2 \delta} \sup_{x, y \in S_\delta} \langle (A - I)x, y \rangle.
\]

Now for a pair of fixed \( x_0, y_0 \in S_\delta \), since \( \| (a_j^\top x)(a_j^\top y) \|_{\psi_1} \leq \| a_j^\top x_0 \|_{\psi_2} \| a_j^\top y_0 \|_{\psi_2} \lesssim 1 \), by using lemma 5, we have for any \( 0 < \epsilon_1 \leq 1 \),

\[
P(\| (A - I)x_0, y_0 \| > \epsilon_1) \leq 2 \exp(-c' \cdot m \cdot \epsilon_1^2).
\]

For any \( 0 < \epsilon < 1 \), taking \( \epsilon_1 := (1 - 2 \delta) \epsilon \) and \( \delta = \frac{1}{2} \), we have

\[
P(\|A - I\|_2 > \epsilon) \leq 2 \left( 1 + \frac{2}{\delta} \right)^{2n} \exp(-c' m (1 - 2 \delta^2) \epsilon^2) \leq 2 \exp(-c \epsilon^2 m),
\]

provided \( m \geq C \epsilon^{-2} n \). Here, \( C \) and \( c \) are universal positive constants. \( \square \)

Lemma 8. Suppose \( h : \mathbb{R} \to \mathbb{R} \) is a locally Lipschitz continuous function such that

\[
|h(z) - h(\bar{z})| \lesssim (1 + |z| + |\bar{z}|)|z - \bar{z}|, \quad \forall z, \bar{z} \in \mathbb{R}.
\]

Assume that \( \|h(Z)\|_{\psi_1} \lesssim 1 \) for a standard Gaussian random variable \( Z \sim \mathcal{N}(0, 1) \). For any \( 0 < \epsilon \leq \frac{1}{2} \), if \( m \geq C \epsilon^{-2} \log(1/\epsilon) n \), then with probability at least \( 1 - 3 \exp(-c \epsilon^2 m) \), it holds

\[
\left| \frac{1}{m} \sum_{j=1}^{m} h(a_j^\top u) - \text{mean} \right| \leq \epsilon, \quad \forall u \in \mathbb{S}^{n-1}.
\]

Here, \( C \) and \( c \) are universal positive constants.
Proof. Introduce a \( \delta \)-net \( S_{\delta} \) on \( \mathbb{S}^{n-1} \) with \( \text{Card}(S_{\delta}) \leq (1 + \frac{2}{\delta})^n \). Observe that by lemma 5, for any \( 0 < \epsilon_1 \leq 1 \), it holds

\[
P \left( \sup_{u \in S_{\delta}} \left| \frac{1}{m} \sum_{j=1}^m h(a_j^\top u) - \text{mean} \right| > \epsilon_1 \right) \leq \left( 1 + \frac{2}{\delta} \right)^n \cdot 2 \cdot \exp(-c_1 \cdot m \cdot \epsilon_1^2).
\]

By lemma 7, we have for \( m \geq Cn \) with probability at least \( 1 - \exp(-c_2 m) \), it holds that

\[
\frac{1}{m} \sum_{j=1}^m |a_j^\top u|^2 \leq 2, \quad \forall u \in \mathbb{S}^{n-1}.
\]

Thus with probability at least \( 1 - \exp(-c_2 m) \) and uniformly for \( u, v \in \mathbb{S}^{n-1} \), we have

\[
\left| \frac{1}{m} \sum_{j=1}^m h(a_j^\top u) - \frac{1}{m} \sum_{j=1}^m h(a_j^\top v) \right| \\
\leq \frac{1}{m} \sum_{j=1}^m |a_j^\top (u - v)| + \frac{1}{m} \sum_{j=1}^m |a_j^\top (u - v)||(|a_j^\top u| + |a_j^\top v|) \\
\leq \sqrt{\frac{1}{m} \sum_{j=1}^m |a_j^\top (u - v)|^2} \cdot \left( 1 + \sqrt{\frac{1}{m} \sum_{j=1}^m |a_j^\top u|^2} + \sqrt{\frac{1}{m} \sum_{j=1}^m |a_j^\top v|^2} \right) \\
\leq 10 \|u - v\|_2 \leq 10 \delta.
\]

Now we take \( \epsilon_1 = \frac{\epsilon}{2} \) and \( \delta = \frac{\epsilon}{20} \). It follows that for \( m \geq C \epsilon^{-2} \log(1/\epsilon) n \) with probability at least

\[
1 - \exp(-c_2 m) - \left( 1 + \frac{40}{\epsilon} \right)^n \cdot 2 \cdot \exp(-c_1 \cdot m \cdot \epsilon^2) \geq 1 - 3 \exp(-c_2 \epsilon^2 m),
\]

the desired inequality holds uniformly for all \( u \in \mathbb{S}^{n-1} \). \( \square \)

**Corollary 1.** Let \( 0 < \epsilon \leq 1/2 \). Assume \( m \geq C \epsilon^{-2} \log(1/\epsilon) n \). There exists \( N_0(\epsilon) = C_1 \sqrt{\log(1/\epsilon)} > 0 \) such that for any \( N \geq N_0 \), with probability at least \( 1 - \exp(-c_2 \epsilon^2 m) \), it holds

\[
\frac{1}{m} \sum_{j=1}^m \chi_{|a_j^\top u| > N} \leq \epsilon, \quad \forall u \in \mathbb{S}^{n-1}.
\]

**Proof.** We choose \( \phi \in C_c^\infty(\mathbb{R}) \) such that \( \phi(z) \equiv 1 \) for \( |z| \leq \frac{\epsilon}{2} \), \( \phi(z) = 0 \) for \( |z| \geq 1 \), and \( 0 \leq \phi(z) \leq 1 \) for all \( z \in \mathbb{R} \). Clearly then

\[
\frac{1}{m} \sum_{j=1}^m \chi_{|a_j^\top u| > N} \leq \frac{1}{m} \sum_{j=1}^m \left( 1 - \phi \left( \frac{a_j^\top u}{N} \right) \right).
\]

We can then apply lemma 8 with \( h(z) = 1 - \phi \left( \frac{z}{N} \right) \). Note that (below \( Z \sim \mathcal{N}(0, 1) \) is a standard normal random variable)

\[
\text{mean} = \mathbb{E} \left( 1 - \phi \left( \frac{Z}{N} \right) \right) \leq \mathbb{E} \chi_{|Z| > \frac{N}{2}} \leq O(e^{-N^2}) \leq \frac{1}{2} \epsilon,
\]

if \( N \geq N_0(\epsilon) \). \( \square \)
**Lemma 9.** Let $X_i$: $1 \leq i \leq m$ be independent random variables with
\[ \max_{1 \leq i \leq m} \mathbb{E}|X_i|^4 \lesssim 1. \]
Then for any $t > 0$,
\[ P\left( \left| \frac{1}{m} \sum_{j=1}^{m} X_j - \text{mean} \right| > t \right) \lesssim \frac{1}{m^2 t^2}. \]

**Proof.** Without loss of generality we can assume $X_i$ has zero mean. The result then follows from Markov’s inequality and the observation
\[ \mathbb{E} \left( \sum_{j=1}^{m} X_j \right)^4 \lesssim \mathbb{E} X_j^2 | X_j^2 | + \mathbb{E} X_i^4 \lesssim m^2. \]

**Lemma 10.** Let $0 < \epsilon \leq 1/2$. Assume $m \geq C \epsilon^{-4} \log(1/\epsilon)n$. There exists $N_0(\epsilon) = C \sqrt{\log(1/\epsilon)} > 0$ such that for any $N \geq N_0$, with probability at least $1 - \frac{C}{m} \exp(-c' \epsilon^2 m)$, we have
\[ \frac{1}{m} \sum_{j=1}^{m} (a_j^T x)^4 \mathbb{1}_{|a_j^T u| \geq \epsilon}, \quad \forall u \in \mathbb{S}^{n-1}. \]
Here, $C, c'$ and $c''$ are universal positive constants.

**Proof.** By using Cauchy-Schwartz, we have
\[ \frac{1}{m} \sum_{j=1}^{m} (a_j^T x)^4 \mathbb{1}_{|a_j^T u| \geq N} \lesssim \sqrt{\frac{1}{m} \sum_{j=1}^{m} (a_j^T x)^8} \sqrt{\frac{1}{m} \sum_{j=1}^{m} \mathbb{1}_{|a_j^T u| \geq N}}. \]
By using lemma 9, we have for any $t_1 > 0$,
\[ P\left( \left| \frac{1}{m} \sum_{j=1}^{m} (a_j^T x)^8 - \text{mean} \right| > t_1 \right) \lesssim \frac{1}{m^2 t_1^2}. \]
Choosing $t_1$ to be an absolute constant. The desired result then follows from corollary 1.

**Lemma 11.** For any $0 < \epsilon \leq 1/2$, there exist constants $C(\epsilon), c(\epsilon) > 0$ only depending on $\epsilon$ such that if $m \geq C(\epsilon)n$, then the following holds with probability at least $1 - \exp(-c(\epsilon)m)$:
\[ \frac{1}{m} \sum_{j=1}^{m} (a_j^T u)^4 \gtrsim 3 - \epsilon, \quad \forall u \in \mathbb{S}^{n-1}. \]

**Proof.** Let $\phi \in C^\infty(\mathbb{R})$ be such that $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}$, $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. Then for any $N \geq 1$,
\[ \frac{1}{m} \sum_{j=1}^{m} (a_j^T u)^4 \geq \frac{1}{m} \sum_{j=1}^{m} (a_j^T u)^4 \phi \left( \frac{a_j^T u}{N} \right). \]
We first take \( N \) sufficiently large (depending only on \( \epsilon \)) such that
\[
\frac{1}{m} \sum_{j=1}^{m} E(a_j^\top u)^4 \phi \left( \frac{a_j^\top u}{N} \right) = \mathbb{E}Z^4 \phi \left( \frac{Z}{N} \right) \geq 3 - \frac{\epsilon}{2},
\]
where \( Z \sim \mathcal{N}(0, 1) \). Then by using lemma 8, we have for \( m \geq C(\epsilon) \cdot n \),
\[
P \left( \left| \frac{1}{m} \sum_{j=1}^{m} (a_j^\top u)^4 \phi \left( \frac{a_j^\top u}{N} \right) - \text{mean} \right| > \frac{\epsilon}{2} \right) \leq \exp(-c(\epsilon)m).
\]

The desired result then easily follows. \( \square \)

**Lemma 12.** Suppose \( f_1, f_2 : \mathbb{R} \to \mathbb{R} \) are Lipschitz continuous functions such that
\[
|f_1(z)| + |f_2(z)| \leq L \cdot |1 + |z||, \quad \forall z \in \mathbb{R};
\]
\[
|f_k(z) - f_k(\tilde{z})| \leq L \cdot |z - \tilde{z}|, \quad \forall z, \tilde{z} \in \mathbb{R}, \ k = 1, 2,
\]
where \( L > 0 \) is a constant. Assume that \( \|f_1(Z)\|_{\psi_2} + \|f_2(Z)\|_{\psi_2} \leq 1 \) for a standard Gaussian random variable \( Z \sim \mathcal{N}(0, 1) \). For any \( 0 < \epsilon < 1/2 \), there exist constant \( C_1 > 0 \) only depending on \( L \) and universal constant \( c > 0 \) such that if \( m \geq C_1 \epsilon^{-2} \log(1/\epsilon)n \), then with probability at least \( 1 - \exp(-c\epsilon^2 m) \), we have
\[
\left| \frac{1}{m} \sum_{j=1}^{m} f_1(a_j^\top u)f_2(a_j^\top v) - \text{mean} \right| \leq \epsilon, \quad \forall u, v \in S^{n-1}.
\]

**Proof.** Introduce a \( \delta \)-net \( S_\delta \) on \( S^{n-1} \) with \( \text{Card}(S_\delta) \leq (1 + \frac{2}{\delta})^n \). By lemma 5, for any \( 0 < \epsilon_1 \leq 1 \), we have
\[
P \left( \sup_{u, v \in S_\delta} \left| \frac{1}{m} \sum_{j=1}^{m} f_1(a_j^\top u)f_2(a_j^\top v) - \text{mean} \right| > \epsilon_1 \right) \leq \left( 1 + \frac{2}{\delta} \right)^{2n} \cdot 2 \cdot \exp(-c_1\epsilon_1^2 m),
\]
where \( c_1 > 0 \) is a universal constant. Next by lemmas 7 and 8, if \( m \geq Cn \) then with probability at least \( 1 - \exp(-c_2 m) \), it holds
\[
\left| \frac{1}{m} \sum_{k=1}^{m} |a_j^\top w| - \text{mean} \right| \leq 0.01, \quad \forall w \in S^{n-1};
\]
\[
\left| \frac{1}{m} \sum_{k=1}^{m} |a_j^\top w|^2 - \text{mean} \right| \leq 0.01, \quad \forall w \in S^{n-1},
\]
where \( C, c_2 \) are universal positive constants. Consequently, for any \( u, v \in S^{n-1} \), there exist \( \tilde{u}, \tilde{v} \in S_\delta \) such that \( \|u - \tilde{u}\|_2 \leq \delta, \|v - \tilde{v}\|_2 \leq \delta \), and then
\[
\frac{1}{m} \sum_{j=1}^{m} |f_1(a_j^\top u)f_2(a_j^\top v) - f_1(a_j^\top \tilde{u})f_2(a_j^\top \tilde{v})|
\]
\[
\leq \frac{1}{m} \sum_{j=1}^{m} |f_1(a_j^\top u) - f_1(a_j^\top \tilde{u})| |f_2(a_j^\top v)| + \frac{1}{m} \sum_{j=1}^{m} |f_1(a_j^\top \tilde{u})| |f_2(a_j^\top v) - f_2(a_j^\top \tilde{v})|
\]
\[
\leq \frac{1}{m} \sum_{j=1}^{m} L^2 |a_j^\top (u - \tilde{u})| (1 + |a_j^\top v|) + \frac{1}{m} \sum_{j=1}^{m} L^2 (1 + |a_j^\top \tilde{u}|) |a_j^\top (v - \tilde{v})|
\]

If $\mathbf{u} \perp \mathbf{v}$, then with probability at least $1 - \exp(-c_2m)$, we have
\[
\frac{1}{m} \sum_{j=1}^{m} (\mathbf{a}_j^\top \mathbf{u})^2 (\mathbf{a}_j^\top \mathbf{v})^2 \geq c_1 > 0, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{S}^{n-1},
\]
where $C$ and $c_1$ are absolute positive constants.

**Proof.** Step 1. Write $\mathbf{v} = s\mathbf{u} + \sqrt{1-s^2} \mathbf{u}^\perp$, where $|s| \leq 1$, and $\mathbf{u}^\perp \in \mathbb{S}^{n-1}$ is such that $\langle \mathbf{u}^\perp, \mathbf{u} \rangle = 0$. Let $\mathbf{a} \sim \mathcal{N}(0, I_n)$ and denote $X = \mathbf{a}^\top \mathbf{u}$, $Y = \mathbf{a}^\top \mathbf{u}^\perp$. Clearly $X \sim \mathcal{N}(0, 1)$, $Y \sim \mathcal{N}(0, 1)$ and $X, Y$ are independent. Now let $N \geq 4$. We have
\[
\mathbb{E}(\mathbf{a}^\top \mathbf{u})^2 (\mathbf{a}^\top \mathbf{v})^2 \chi_{|X| \leq N, |X^\perp| \leq N} = \mathbb{E} \chi_{|X| \leq N, |X^\perp| \leq N} \mathbb{E} X^2 \left( sX + \sqrt{1-s^2} Y \right)^2 \chi_{|X| \leq N, |X^\perp| \leq N} \geq \mathbb{E} \chi_{|X| \leq N, |X^\perp| \leq N} \mathbb{E} \left( sX^4 + (1-s^2) X^2 Y^2 \right) \chi_{|X| \leq N, |X^\perp| \leq N} \geq 2c_1 > 0,
\]
where $c_1 > 0$ is an absolute constant, and $N$ is taken to be a sufficiently large absolute constant.

Step 2. Let $\phi \in C_c^\infty(\mathbb{R})$ be such that $\phi(x) = x^2$ for $|x| \leq N$ and $\phi(x) = 0$ for $|x| \geq N + 1$. Clearly if $m \geq Cn$, then with probability at least $1 - \exp(-cm)$, we have
\[
\left| \frac{1}{m} \sum_{j=1}^{m} \phi(\mathbf{a}_j^\top \mathbf{u}) \phi(\mathbf{a}_j^\top \mathbf{v}) - \text{mean} \right| \leq \frac{1}{2} c_1
\]
and thus
\[
\frac{1}{m} \sum_{j=1}^{m} (\mathbf{a}_j^\top \mathbf{u})^2 (\mathbf{a}_j^\top \mathbf{v})^2 \geq \frac{1}{m} \sum_{j=1}^{m} \phi(\mathbf{a}_j^\top \mathbf{u}) \phi(\mathbf{a}_j^\top \mathbf{v}) > c_1, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{S}^{n-1}.
\]

**Lemma 13.** Let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that
\[
\sup_{z \in \mathbb{R}} \frac{|h(z)|}{1 + |z|} \lesssim 1 \quad \text{and} \quad \sup_{z \neq 0} \frac{|h(z) - h(\tilde{z})|}{|z - \tilde{z}|} \lesssim 1.
\]
Define the set
\[
F = \{ \mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\|_2 \leq 1, \quad \mathbf{u}^\top \mathbf{x} = 0 \}.
\]
Suppose \( b = (b_1, \ldots, b_n) \in \mathbb{R}^n \) satisfies \( \|b\|_2 \lesssim \sqrt{m} \). For any \( 0 < \epsilon \leq 1/2 \), if \( m \geq C e^{-2} \log(1/\epsilon)n \), then with probability at least \( 1 - \exp(-cm^2) \), we have

\[
\left| \frac{1}{m} \sum_{j=1}^{m} b_j h(a_j^\top u) - \text{mean} \right| \leq \epsilon, \quad \forall u \in F.
\]

**Proof.** First it is easy to check that \( \max_j \|h(a_j^\top u)\|_2 \lesssim 1 \). By lemma 4, for each \( u \in F \), we have

\[
P \left( \left| \frac{1}{m} \sum_{j=1}^{m} b_j h(a_j^\top u) - \text{mean} \right| > \frac{\epsilon}{4} \right) \leq 2 \exp(-cm^2).
\]

Now let \( \delta > 0 \) and introduce a \( \delta \)-net \( S_\delta \) on the set \( F \). Note that the set \( F \) can be identified as a unit ball in \( \mathbb{R}^{n-1} \). We have \( \text{Card}(S_\delta) \leq (1 + \frac{\delta}{2})^n \). Thus

\[
P \left( \sup_{u \in S_\delta} \left| \frac{1}{m} \sum_{j=1}^{m} b_j h(a_j^\top u) - \text{mean} \right| > \frac{\epsilon}{4} \right) \leq 2 \left( 1 + \frac{2}{\delta} \right)^n \exp(-cm^2).
\]

By lemma 7, if \( m \geq Cn \), then with probability at least \( 1 - \exp(-cm) \), we have

\[
\frac{1}{m} \sum_{j=1}^{m} (a_j^\top u)^2 \leq 2\|u\|_2^2, \quad \forall u \in \mathbb{R}^n.
\]

Now if \( u \in S_\delta, v \in F \) with \( \|v - u\|_2 \leq \delta \), then with probability at least \( 1 - \exp(-cm) \), we have

\[
\left| \frac{1}{m} \sum_{j=1}^{m} b_j h(a_j^\top u) - \frac{1}{m} \sum_{j=1}^{m} b_j h(a_j^\top v) \right| \leq \frac{1}{m} \sum_{j=1}^{m} |b_j| \cdot |a_j^\top (u - v)|
\]

\[
\leq K_0 \sqrt{\frac{1}{m} \sum_{j=1}^{m} b_j^2} \sqrt{\frac{1}{m} \sum_{j=1}^{m} |a_j^\top (u - v)|^2}
\]

\[
\leq K_1 \|u - v\|_2,
\]

where \( K_0 > 0, K_1 > 0 \) are absolute constants. On the other hand

\[
\frac{1}{m} \sum_{j=1}^{m} |b_j| E(h(a_j^\top u) - h(a_j^\top v)) \lesssim \frac{1}{m} \sum_{j=1}^{m} |b_j| \cdot \|u - v\|_2 \lesssim \|u - v\|_2.
\]

It follows that for some absolute constant \( K_3 > 0 \),

\[
\left| \frac{1}{m} \sum_{j=1}^{m} b_j \cdot (h(a_j^\top u) - E(h(a_j^\top u))) - \frac{1}{m} \sum_{j=1}^{m} b_j \cdot (h(a_j^\top v) - E(h(a_j^\top v))) \right| \leq K_3 \|u - v\|_2.
\]
Now set $\delta = \frac{e}{10k + 10}$. The desired conclusion then follows with probability at least
\[ 1 - 2 \left( 1 + \frac{2}{\delta} \right)^n \exp(-cm^2) - \exp(-cm) \geq 1 - \exp(-c_1c_2m), \]
provided $m \geq Cc^{-2}\log(1/\epsilon)n$, where $c_1 > 0$ is an absolute constant.

\textbf{Lemma 14.} Let $h: \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that
\[ \sup_{z \in \mathbb{R}} \frac{|h(z)|}{1 + |z|} \leq 1 \quad \text{and} \quad \sup_{z \neq \tilde{z}} \frac{|h(z) - h(\tilde{z})|}{|z - \tilde{z}|} \leq 1. \]

Let $f_j : \mathbb{R} \to \mathbb{R}$ be such that
\[ \sup_{z \in \mathbb{R}} \frac{f_j(z)}{1 + |z|} \leq 1. \]

Define
\[ F = \{ u \in \mathbb{R}^n : \|u\|_2 \leq 1, \quad u^\top x = 0 \}. \]

Then for any $0 < \epsilon \leq 1/2$, there exist $C_1 = C_1(\epsilon) > 0$, $C_2 = C_2(\epsilon) > 0$, such that if $m \geq C_1 \cdot n$, then the following holds with probability at least $1 - \frac{C_2}{m}$:
\[ \left| \frac{1}{m} \sum_{j=1}^{m} f_j(a_j^\top x)h(a_j^\top u) - \text{mean} \right| \leq \epsilon, \quad \forall u \in F. \]

\textbf{Proof.} Step 1. Set $b_j = f_j(a_j^\top x)$. Denote
\[ K_0 = \mathbb{E}[b_j^2] \leq 1. \]

By lemma 9, we have
\[ \mathbb{P} \left( \left| \frac{1}{m} \sum_{j=1}^{m} b_j^2 - K_0 \right| > t \right) \leq \frac{1}{m^2 \pi^2}. \]

Then with probability at least $1 - \frac{1}{m^2}$, we have
\[ \frac{1}{m} \sum_{j=1}^{m} b_j^2 \leq B_0, \]

where $B_0 > 0$ is some absolute constant.

Step 2. Denote $\tilde{a}_j = a_j - (a_j^\top x)x$. An important observation is that $(a_j^\top x)_{1 \leq j \leq m}$ and $(\tilde{a}_j)_{1 \leq j \leq m}$ are independent. Note that for $u \in F$ we have $a_j^\top u = \tilde{a}_j^\top u$. Thus for every $b_j$ with the property $C \sum_{j=1}^{m} (b_j)^2 \leq B_0$, we have the following as a consequence of lemma 13: For any $0 < \epsilon \leq 1/2$, if $m \geq Cc^{-2}\log(1/\epsilon)n$, then with probability at least $1 - \exp(-c\epsilon^2m)$, we have
\[ \left| \frac{1}{m} \sum_{j=1}^{m} b_j \cdot (h(\tilde{a}_j^\top u) - \mathbb{E}(h(\tilde{a}_j^\top u))) \right| \leq \frac{1}{3} \epsilon, \quad \forall u \in F. \]

Step 3. By using the results from step 1 and step 2, with probability at least $1 - \frac{2}{m^2}$, we have
\[ \left| \frac{1}{m} \sum_{j=1}^{m} b_j \cdot (h(\tilde{a}_j^\top u) - \mathbb{E}(h(\tilde{a}_j^\top u))) \right| \leq \frac{1}{3} \epsilon, \quad \forall u \in F. \]
Now note that \(|E(h(\hat{a}_j^\top u))| = |E(h(\hat{a}_j^\top u))| \lesssim 1\). By lemma 9, we have
\[
P \left( \left| \frac{1}{m} \sum_{j=1}^{m} \left( b_j - E b_j \right) \right| > t \right) \lesssim \frac{1}{m^{2/3}}.
\]
Choosing \(t = \frac{\epsilon}{K_1}\), where \(K_1 > 0\) is a sufficiently large absolute constant such that
\[
\frac{1}{K_1} |E b_1| \leq \frac{\epsilon}{10}
\]
then yields the result.

**Corollary 3.** For any \(0 < \epsilon \leq 1/2\), there exist \(C_1 = C_1(\epsilon) > 0\), \(C_2 = C_2(\epsilon) > 0\), such that if \(m \geq C_1 \cdot n\), then the following holds with probability at least \(1 - \frac{C_2}{m}\):
\[
\left| \frac{1}{m} \sum_{j=1}^{m} (a_j^\top x)^3 (a_j^\top u) - \text{mean} \right| \leq \epsilon, \quad \forall u \in \mathbb{S}^{n-1}.
\]

**Proof.** We decompose \(u = (u^\top x) x + \tilde{u}\), where \(u^\top x = 0\). The result then easily follows from lemma 14.

**Lemma 15.** For any \(0 < \epsilon \leq 1/2\), there exists \(N_0 = N_0(\epsilon) > 0\), \(C_1 = C_1(\epsilon) > 0\), \(C_2 = C_2(\epsilon) > 0\), such that if \(m \geq C_1 n\), then the following hold with probability at least \(1 - \frac{C_2}{m}\): For any \(N \geq N_0\), we have
\[
\frac{1}{m} \sum_{j=1}^{m} |a_j^\top x|^3 |a_j^\top u| \chi_{|a_j^\top x| \geq N} \leq \epsilon, \quad \forall u \in \mathbb{S}^{n-1};
\]
\[
\frac{1}{m} \sum_{j=1}^{m} |a_j^\top x|^3 |a_j^\top u| \chi_{|a_j^\top u| \geq N} \leq \epsilon, \quad \forall u \in \mathbb{S}^{n-1}.
\]

**Proof.** We only sketch the proof. Write \(u = (u^\top x) x + \tilde{u}\), where \(\tilde{u} \in F = \{\tilde{u} \in \mathbb{R}^n : \|\tilde{u}\|_2 \leq 1, \tilde{u}^\top x = 0\}\). For the first inequality, note that (observe \(|u^\top x| \leq 1\))
\[
\frac{1}{m} \sum_{j=1}^{m} |a_j^\top x|^3 |a_j^\top u| \chi_{|a_j^\top x| \geq N} \leq \frac{1}{m} \sum_{j=1}^{m} |a_j^\top x|^4 \chi_{|a_j^\top x| \geq N} + \frac{1}{m} \sum_{j=1}^{m} |a_j^\top x|^3 |a_j^\top \tilde{u}| \chi_{|a_j^\top u| \geq N}.
\]
For the first term one can use lemma 9. For the second term one can use lemma 14.

Now for the second inequality, we write
\[
\frac{1}{m} \sum_{j=1}^{m} |a_j^\top x|^3 |a_j^\top u| \chi_{|a_j^\top u| \geq N}
\leq \frac{1}{m} \sum_{j=1}^{m} |a_j^\top x|^3 \chi_{|a_j^\top x| \leq M} |a_j^\top u| \chi_{|a_j^\top u| \geq N} + \frac{1}{m} \sum_{j=1}^{m} |a_j^\top x|^3 \chi_{|a_j^\top x| > M} |a_j^\top u|.
\]
For \(H_2\), by using the estimates already obtained in the beginning part of this proof, it is clear that we can take \(M\) sufficiently large such that \(H_2 \leq \epsilon/2\). After \(M\) is fixed, we return to the
estimate of \( H_1 \). Note that we can work with a smoothed cut-off function instead of the strict cut-off. The result then follows from lemma 12 by taking \( N \) sufficiently large.

**Lemma 16.** Define \( F = \{u \in \mathbb{R}^n : \|u\|_2 \leq 1, \ u^\top x = 0\} \). Suppose \( b = (b_1, \ldots, b_m) \in \mathbb{R}^m \) satisfies \( \|b\|_2 \lesssim \sqrt{m} \) and \( \|b\|_\infty \lesssim \log m \). For any \( 0 < \epsilon \leq 1/2 \), if \( m \geq C \epsilon^{-1} n \log n \), then with probability at least \( 1 - \exp(-c m / \log m) \), it holds that

\[
\left| \frac{1}{m} \sum_{j=1}^{m} b_j (a_j^\top u) (a_j^\top v) - \text{mean} \right| \leq \epsilon, \quad \forall u, v \in F.
\]

**Proof.** Let \( 0 < \delta < 1/2 \) and introduce a \( \delta \)-net \( S_\delta \) on the set \( F \). Note that the set \( F \) can be identified as a unit ball in \( \mathbb{R}^{n-1} \). We have \( \text{Card}(S_\delta) \leq (1 + \frac{3}{\delta})^n \). Introduce the operator

\[
A = \frac{1}{m} \sum_{j=1}^{m} b_j (a_j^\top - 1).
\]

We have

\[
\|A\|_2 = \sup_{x,y \in F} \langle Ax, y \rangle \leq \frac{1}{1 - 2\delta} \sup_{x,y \in S_\delta} \langle Ax, y \rangle.
\]

By lemma 5, for each \( u, v \in F \), we have

\[
P \left( \left| \frac{1}{m} \sum_{j=1}^{m} b_j (a_j^\top u) (a_j^\top v) - \text{mean} \right| > \frac{\epsilon}{4} \right) \leq 2 \exp \left( -c \min \left\{ m \epsilon^2 / (m \log m) \right\} \right) \leq 2 \exp \left( -c m \epsilon / \log m \right).
\]

Thus

\[
P \left( \sup_{u,v \in S_\delta} \left| \frac{1}{m} \sum_{j=1}^{m} b_j (a_j^\top u) (a_j^\top v) - \text{mean} \right| > \frac{\epsilon}{4} \right) \leq 2 \left( 1 + \frac{2}{\delta} \right)^n \exp \left( -c m \epsilon / \log m \right).
\]

Taking \( \delta = \frac{1}{4} \) and \( m \geq C \epsilon^{-1} n \log m \) then yields the result.

**Corollary 4.** Suppose \( h_1, h_2 : \mathbb{R} \to \mathbb{R} \) are locally Lipschitz continuous functions such that

\[
\sup_{z \in \mathbb{R}} \frac{|h_1(z)| + |h_2(z)|}{1 + |z|} \lesssim 1,
\]

\[
\sup_{z \neq 0} \frac{|h_i(z) - h_i(0)|}{|z|} \lesssim 1, \quad i = 1, 2.
\]

Define \( F = \{u \in \mathbb{R}^n : \|u\|_2 \leq 1, \ u^\top x = 0\} \). Suppose \( b = (b_1, \ldots, b_m) \in \mathbb{R}^m \) satisfies \( \|b\|_2 \lesssim \sqrt{m} \) and \( \|b\|_\infty \lesssim \log m \). For any \( 0 < \epsilon \leq 1/2 \), there exist a constant \( C_1 = C_1(\epsilon) > 0 \) such that if \( m \geq C_1 n \log n \), then with probability at least \( 1 - \exp(-c m / \log m) \), it holds that

\[
\left| \frac{1}{m} \sum_{j=1}^{m} b_j h_1(a_j^\top u) h_2(a_j^\top v) - \text{mean} \right| \leq \epsilon, \quad \forall u, v \in F.
\]

**Proof.** Let \( 0 < \delta < \frac{1}{4} \) and introduce a \( \delta \)-net \( S_\delta \) with \( \text{Card}(S_\delta) \leq (1 + \frac{3}{\delta})^n \) on the set \( F \). As we shall see momentarily, we will need to take \( \delta = O(\epsilon) \). By lemma 5, we have

\[
P \left( \sup_{u,v \in S_\delta} \left| \frac{1}{m} \sum_{j=1}^{m} b_j h_1(a_j^\top u) h_2(a_j^\top v) - \text{mean} \right| > \frac{\epsilon}{4} \right) \leq 2 \left( 1 + \frac{2}{\delta} \right)^n \exp \left( -c m \epsilon / \log m \right).
\]
Now for any $u, v \in F, \tilde{u}, \tilde{v} \in S_0$ with $\|u - \tilde{u}\|_2 \leq \delta, \|v - \tilde{v}\|_2 \leq \delta$, we have

$$\left| h_1(a_j^T u)h_2(a_j^T v) - h_1(a_j^T \tilde{u})h_2(a_j^T \tilde{v}) \right|$$

$$\leq |h_1(a_j^T u) - h_1(a_j^T \tilde{u})| \cdot |h_2(a_j^T v)| + |h_1(a_j^T \tilde{u})| \cdot |h_2(a_j^T v) - h_2(a_j^T \tilde{v})|$$

$$\leq K_0 \left( \frac{1}{\delta} m \sum_{j=1}^m |b_j| \left| (a_j^T u - \tilde{u}) \right|^2 + \left| (a_j^T v) \right|^2 + 2\delta + \frac{1}{\delta} \left| (a_j^T v - \tilde{v}) \right|^2 + \delta \left| (a_j^T \tilde{u}) \right|^2 \right),$$

where $K_0$ is an absolute constant. Now introduce the operator

$$A = \frac{1}{m} \sum_{j=1}^m |b_j| (a_j^T - 1).$$

By lemma 16, with probability at least $1 - \exp(-c \frac{m}{\log m})$, we have

$$\langle Ax, y \rangle \leq 1, \quad \forall x, y \in F.$$

Thus with the same probability, we have

$$\frac{1}{m} \sum_{j=1}^m |b_j| \left| h_1(a_j^T u)h_2(a_j^T v) - h_1(a_j^T \tilde{u})h_2(a_j^T \tilde{v}) \right|$$

$$\leq K_0 \left( \frac{1}{\delta} m \sum_{j=1}^m |b_j| \left| (a_j^T u - \tilde{u}) \right|^2 + \left| (a_j^T v) \right|^2 + 2\delta + \frac{1}{\delta} m \sum_{j=1}^m |b_j| \right)$$

$$\leq K_1 \delta,$$

where $K_1 > 0$ is another absolute constant. It is also not difficult to control the differences in expectation, i.e. for some absolute constant $K_2 > 0$,

$$\frac{1}{m} \sum_{j=1}^m |b_j| \left| \mathbb{E}h_1(a_j^T u)h_2(a_j^T v) - \mathbb{E}h_1(a_j^T \tilde{u})h_2(a_j^T \tilde{v}) \right| \leq K_2 \delta.$$

Now take $\delta = \frac{\epsilon}{4(K_1 + K_2)}$ and the desired result clearly follows by taking $\frac{m}{\log m} \gtrsim n$. \hfill \Box

**Lemma 17.** For any $0 < \epsilon \leq 1/2$, there are constants $C_1 = C_1(\epsilon) > 0, C_2 = C_2(\epsilon) > 0$, such that if $m \geq C_2 n \log n$, then with probability at least $1 - \frac{C_1}{m}$, it holds

$$\left| \frac{1}{m} \sum_{j=1}^m (a_j^T u)^2 (a_j^T x)^2 - \text{mean} \right| \leq \epsilon, \quad \forall u \in S^{n-1}.$$
Proof. Write $\mathbf{u} = (\mathbf{u}^\top \mathbf{x}) \mathbf{x} + \mathbf{u}^\perp$, where $\langle \mathbf{u}^\perp, \mathbf{x} \rangle = 0$. Then
\[
\frac{1}{m} \sum_{j=1}^{m} (a_j^\top \mathbf{u})^2 (a_j^\top \mathbf{x})^2 = (\mathbf{u}^\top \mathbf{x})^4 \cdot \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \mathbf{x})^4 + 2 (\mathbf{u}^\top \mathbf{x}) \cdot \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \mathbf{x})^3 \cdot (a_j^\top \mathbf{u}^\perp)
\]
\[
+ \frac{1}{m} \sum_{j=1}^{m} (a_j^\top \mathbf{u})^2 (a_j^\top \mathbf{u}^\perp)^2.
\]
Clearly the first two terms can be easily handled by lemmas 9 and 14 respectively. For these terms we actually only need $m \geq Cn$. To handle the last term we need $m \gtrsim n \log n$. The main observation is that $(a_j^\top \mathbf{x})$ and $(a_j^\top \mathbf{u}^\perp)$ are independent. Write $b_j = (a_j^\top \mathbf{x})^2$ and observe that with probability at least $1 - O\left(\frac{m}{n^2}\right)$, we have
\[
\frac{1}{m} \sum_{j=1}^{m} b_j \leq 100m, \quad \max_{1 \leq j \leq m} b_j \leq 100 \log m.
\]
For $m \gtrsim n \log n$, by using lemma 16, it holds with probability at least $1 - O(m^{-2}) - \exp\left(-\frac{cm}{\log m}\right) = 1 - O(m^{-2})$ that
\[
\left| \frac{1}{m} \sum_{j=1}^{m} b_j (a_j^\top \mathbf{u})^2 - \|\mathbf{u}\|_2^2 \right| \leq \frac{\epsilon}{2}, \quad \forall \mathbf{u} \in F = \{ v \in \mathbb{R}^n : \|v\|_2 \leq 1, v^\top \mathbf{x} = 0 \}.
\]
By lemma 9, we have with probability $1 - O(m^{-2})$,
\[
\|\mathbf{u}\|_2 \left| \frac{1}{m} \sum_{j=1}^{m} b_j - \text{mean} \right| \leq \left| \frac{1}{m} \sum_{j=1}^{m} b_j - \text{mean} \right| \leq \frac{\epsilon}{2}.
\]
The desired result then easily follows. \qed

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