Single-index models for extreme value index regression

TAKUMA YOSHIDA

1 Kagoshima University, Kagoshima 890-8580, Japan
E-mail: yoshida@sci.kagoshima-u.ac.jp

Abstract

Since the extreme value index (EVI) controls the tail behaviour of the distribution function, the estimation of EVI is a very important topic in extreme value theory. Recent developments in the estimation of EVI along with covariates have been in the context of nonparametric regression. However, for the large dimension of covariates, the fully nonparametric estimator faces the problem of the curse of dimensionality. To avoid this, we apply the single index model to EVI regression under Pareto-type tailed distribution. We study the penalized maximum likelihood estimation of the single index model. The asymptotic properties of the estimator are also developed. Numerical studies are presented to show the efficiency of the proposed model.

Keywords: Extreme value index; Heavy tail; Pareto-type model; Peak over threshold; Penalized spline; Single index model
MSC codes: 62G08, 62G20, 62G32

1 Introduction

Analyzing the probability of occurrence of a rare event in various fields such as meteorology, economics, sociology, ecology, life sciences, and so on. Here, a rare event is defined as one whose data is extremely high or low or located at the tail-end of the data distribution. Extreme value theory (EVT) is an efficient statistical tool to investigate the tail behavior of the distribution of the data. Many authors have developed the method, theory, and application of EVT, which are summarised in Beirlant et al. (2004), de Haan and Ferreira (2006), and Dey and Yan (2016). The tail behavior of the distribution can be classified into three types: heavy tail, light tail, and short tail. Such division is controlled by a parameter called extreme value index (EVI). If EVI is positive, the distribution has heavy tail. The light and short tails correspond to zero and negative EVI, respectively. In this paper, we focus on the case of heavy tail or positive EVI because the estimation of the positive EVI is more difficult rather than other cases. Then, the estimation of EVI is one of the most important topics in extreme value statistics. The Hill estimator, proposed by Hill (1975), is the fundamental estimator of positive EVI.

In recent years, there has been rapid development in the estimation of the conditional EVI with covariate information in the context of regression. The nonparametric estimator of conditional EVI was suggested by Gardes (2010), Stupfler (2013), Daouia et al. (2013), Stupfler and Gardes (2014), Goegebeur et al. (2014), Goegebeeur et al. (2015), and Ma et al. (2020). However, with the large dimension of covariates, such an estimator faces the problem of curse of dimensionality, and hence, the efficiency of a fully nonparametric estimator cannot be guaranteed. Therefore, for high dimensional covariates, we need to adopt flexible modelling of the target function to avoid the curse of dimensionality. For this, the linear model, proposed by
Wang and Tsai (2009), is the classical approach. However, the linear model cannot capture the behavior of the data having nonlinear structure. In other words, the linear model is too simple for some data. As the nonparametric based flexible modeling approach, Chavez-Demoulin and Davison (2005) and Youngman (2019) have used the generalized additive model. Li et al. (2020) carried out EVI regression with a partially linear model. Wang and Li (2012) and Wang et al. (2013) studied the conditional Hill estimator from linear extremal quantile regression. The varying coefficient model has been developed by Ma et al. (2019). Thus, the study of the EVI modelling has been a topic of interest in recent years. However, the single index model has not yet been imposed in EVI regression although the single index model is one of popular methods for flexible modeling as described below. This motivates us to introduce the single index model in the EVI estimation with large dimension of covariates.

The single index model was proposed by Ichimura (1993) and Härdle et al. (1993) in mean regression. This model is known as the semiparametric model and is structured as a hybrid of the linear transformation of covariates and one-dimensional nonparametric function. Hall (1989), Horowitz and Härdle (1993), Carrol et al. (1997), Yu and Ruppert (2002), Wang and Yang (2009), and Kuchibhotla and Patra (2016) have developed the single index model in mean regression. In quantile regression, Wu et al. (2010), Zu et al. (2012), and Ma and He (2015) have studied the single index model. Gardes (2018) and Xu et al. (2022) considered the usage of the single index model in extremal quantile regression. Evidently, the single index model has been widely researched. This motivated us to apply the single index model to EVI regression with large dimensional covariates. From the method of Gardes (2018) and Xu et al. (2022), we can estimate the EVI as the conditional Hill estimator using a conditional quantile with several quantile levels. However, the method of Gardes (2018) is complicated and has high computational cost. Further, the single index parameter depends on the quantile level, and hence, the obtained EVI estimator does not have the single index structure. Meanwhile, Xu et al. (2022) assumed the linear model as the conditional quantile. However, for the tail quantile, the linearity assumption is limited for some data. Accordingly, our aim is to directly apply the single index structure to EVI.

In single index models, we need to estimate the linear coefficient parameter vector and the one-dimensional nonparametric function. First, we assume that the Pareto-type-tailed model is the conditional distribution of the response variable as a function of the covariates. Then, the single index parameter and nonparametric function including EVI function is estimated via the maximum likelihood method after choosing extreme data using the peak over threshold (POT) method. For the nonparametric part in the single index model, we use the penalized spline method. Here, the estimator contains two tuning parameters: the threshold value in POT and the smoothing parameter in the penalized splines. These two tuning parameters are automatically selected discrepancy score, in the context of Pareto distribution approximation, as proposed by Wang and Tsai (2009). This score is more efficient and computationally expedient compared to that of cross-validation. We study the asymptotic distribution and the rate of convergence of the proposed estimator. From these results, we can verify whether the proposed single index model overcomes the problem of the curse of dimensionality. The finite sample performance of the proposed single index model is examined using a Monte Carlo simulation. We also report a real data example using motor bike insurance data from Ohlsson (see, R package insuranceData).

As a nonparametric method, we use splines instead of other methods such as kernel. According to Yu and Ruppert (2002) and Wang and Yang (2009), the spline method is computationally more efficient than the kernel method in the single index model. Furthermore, in the recent study of the regression with extreme value analysis, Youngman (2022) has proposed very useful R-package called evgam. The smoothing method used in evgam is mainly splines. Thus, the
demand for the methodology and the theory of the spline method is expected to increase in the field of EVT. This motivates us to focus on the spline method in this study.

The rest of the paper is organized as follows. Section 2 sets the single index model for EVI regression, the estimation procedure of the maximum likelihood method and tuning parameter selection. Asymptotic theory for the proposed estimator is established in Section 3. The simulation study is described in Section 4 while the real data example is given in Section 5. Section 6 concludes the paper. In Appendix, we review the important properties of splines and the technical lemmas and the proof of theorems are also provided.

2 Single index model

2.1 Model setting

For the random pair \((Y, X) \in \mathbb{R} \times \mathcal{X}, \mathcal{X} \subset \mathbb{R}^p\) with \(X = (X_1, \ldots, X_p) \in \mathcal{X}\), let \(F(y|\mathbf{x}) = P(Y < y|X = x)\) be the conditional distribution function of \(Y\) given \(X = x = (x_1, \ldots, x_p)\). We then consider the class of Pareto-type tailed distributions defined as

\[
P(Y > y|X = x) = 1 - F(y|\mathbf{x}) = y^{-1/\gamma(x)}L(y|\mathbf{x}),
\]

where \(\gamma(x) > 0\) is the EVI function and \(L\) is a slowly-varying function satisfying

\[
\lim_{y \to \infty} L(ay|x)/L(y|x) = 1
\]

for all \(x \in \mathcal{X}, a > 0\). Furthermore, the slowly-varying function \(L\) is assumed to be belonging to the Hall class (Hall 1982):

\[
L(y|x) = \ell_0(x) + \ell_1(x)y^{-\beta(x)/\gamma(x)} + \nu(y|x), \quad y \to \infty,
\]

where for any \(x \in \mathcal{X}\), \(\ell_0(x)\) and \(\beta(x)\) are positive, bounded and continuous functions, \(\ell_1\) is bounded and the continuous function, and \(\nu(y|x)\) is the remaining term satisfying

\[
\sup_{x \in \mathcal{X}} y^{\beta(x)/\gamma(x)}|\nu(y|x)| \to 0, \quad y \to \infty.
\]

Here, \(\beta(x)\) is the regression function version of the second-order parameter in EVT (de Haan and Ferreira 2006).

In this paper, we assume that the EVI can be expressed as the single-index model: \(\gamma(x) = \gamma(x^T\mathbf{\theta})\), where \(\gamma: \mathbb{R} \to \mathbb{R}_+\) is the univariate function and \(\mathbf{\theta} = (\theta_1, \ldots, \theta_p)^T \in \mathbb{R}^p\) is the single-index parameter vector. However, it is well known that the pair of the true structure \((\gamma, \theta)\) is not unique (see, Ichimura 1993, Kuchibhotla and Patra 2016). In our study, to identify the scale and signature of the single index parameter vector, we assume that \(\|\mathbf{\theta}\| = 1\) and \(\theta_1 > 0\), that is, \(\mathbf{\theta} \in \mathcal{S}_+^{p-1}\), where

\[
\mathcal{S}_+^{p-1} = \left\{(\theta_1, \ldots, \theta_p)^T \Big| \sum_{j=1}^p \theta_j^2 = 1, \theta_1 > 0 \right\}.
\]

Furthermore, for a fixed \(\theta \in \mathcal{S}_+^{p-1}\), \(\gamma\) depends on \(\theta\), writing \(\gamma(\cdot) = \gamma(\cdot|\theta)\) (see, Wang and Yang 2009). Thus, for a given \(x^T\mathbf{\theta} = z\), the single-index model satisfies \(g^*(x) = g(z|\theta)\).

The Pareto-type-tailed distribution with single-index model is defined as

\[
P(Y > y|X^T\mathbf{\theta} = x_0, X = x) = y^{-1/\gamma(x_0|\theta)}L(y|x).
\]
Let \( \{(Y_i, X_i) : i = 1, \ldots, n\}, X_i = (X_{i1}, \ldots, X_{ip})^T \) be i.i.d. random sample generated from similar distribution as \((Y, X)\). To estimate \((\gamma, \theta)\), we use the POT method. In other words, we introduce the threshold function \( w_n(x) \) and estimate \((\gamma, \theta)\) using all observations that exceed threshold function: \( \{(Y_i, X_i) : Y_i > w_n(X_i), i = 1, \ldots, n\} \). Actually, under given \( Y > w_n(x) \), \( X^T \theta = x_0 \) and \( X = x \), the transformed random variable \( Y/w_n(x) \) is approximately replaced with

\[
P \left( \frac{Y}{w_n(x)} > z \mid X^T \theta = x_0, X = x, Y > w_n(x) \right) \approx z^{-\gamma(x_0 | \theta)}, \quad z \geq 1,
\]

as \( w_n(x) \to \infty \). Thus, the Pareto-type tailed distribution can be approximately replaced with ordinary Pareto distribution. In the following, we continue the discussion using (4). The conditional density function \( f_{w_n}(\cdot | x) \) of \( Y/w_n(x) \) given \( Y > w_n(x) \) can be written by

\[
f_{w_n} \left( \frac{y}{w_n(x)} \mid X^T \theta = x_0, x \right) \approx \frac{1}{\gamma(x_0 | \theta)} \left( \frac{y}{w_n(x)} \right)^{\gamma(x_0 | \theta)} - 1.
\]

By using this, we have

\[
- \log f_{w_n} \left( \frac{y}{w_n(x)} \mid X^T \theta = x_0, x \right) \approx \frac{1}{\gamma(x_0 | \theta)} \log \left( \frac{y}{w_n(x)} \right) - \log \left( \frac{y}{w_n(x)} \right) + C
\]

\[
= \exp[\alpha(x_0 | \theta)] \log \left( \frac{y}{w_n(x)} \right) - \alpha(x_0 | \theta) + C,
\]

where \( \alpha(\cdot | \theta) = - \log \gamma(\cdot | \theta) \) and \( C \) is the independent term of \( \gamma \) and \( \theta \). In Section 2.2, we estimate \((\alpha, \theta)\) via the penalized maximum likelihood method based on the abovementioned logarithm of the approximated density function. The estimator of EVI for the point \( x \in \mathcal{X} \) is obtained by \( \hat{\gamma}(x^T \hat{\theta} | \theta) = \exp[\hat{\alpha}(x^T \hat{\theta} | \hat{\theta})] \), where \( \hat{\alpha} \) is the estimator of \( \alpha \) and \( \hat{\theta} \) is the estimator of \( \theta \). The proposed estimator can be seen as semiparametric version of linear estimator proposed by Wang and Tsai (2009).

**Remark 1**

In this paper, the single-index assumption is incorporated only for EVI \( \gamma \). We can extend this assumption to the conditional distribution as \( F(y|x) = F(y|x^T \theta) \) or \( L(y|x) = L(y|x^T \theta) \). However, we do not use the information of \( L \) to estimate the EVI function by POT. Therefore, the single-index assumption for \( L \) is not important. The condition of \( F(y|x) = F(y|x^T \theta) \) is somewhat stronger than our assumption. Thus, we use the single index model for only EVI.

### 2.2 Estimation procedure

We now estimate \((\alpha, \theta)\) from the data \( \{(Y_i, X_i) : i = 1, \ldots, n\} \). The nonparametric function \( \alpha \) is approximated by the spline method. In this paper, we assume that for any \( X \in \mathcal{X} \) and \( \theta \in \mathcal{S}_+^{p-1} \), there exist \( a, b \) such that \( a \leq X^T \theta \leq b \) (see, Section 3). Let \( C^d[a, b] \) be the class of functions with \( d \)-times continuously differentiable on \([a, b]\). We then define the set of knots \( \kappa = \{a = \kappa_0 < \kappa_1 < \ldots < \kappa_{K_0+1} = b\}, K_0 > 1 \) and the class of \( d \)th order spline:

\[
S(d, \kappa) = \{s \in C^{d-2}[a, b] : s \text{ is a polynomial of degree } (d-1) \text{ on each subinterval } [\kappa_j, \kappa_{j+1}] \} \quad d \geq 2.
\]

For \( d = 1 \), \( S(d, \kappa) \) is the set of step functions with jumps at each knot. We approximate \( \alpha(\cdot) \) by \( d \)th order spline function \( s \in S^d[a, b] \). For \( x \in [a, b] \), let \( B^{[d]}(x) = (B_0^{[d]}(x), \ldots, B_{K_0}^{[d]}(x))^T \) be the vector of \( d \)th order scaled Bspline basis with \( K = K_0 + d + 1 \) (see, Appendix A). For simplicity,
we write $B(x) = B^{[d]}(x)$ and $B_j(x) = B_j^{[d]}(x)$. From de Boor (2001), all $d$th order spline functions can be expressed as linear combinations of B-spline bases. In other words, for any $s \in S(d, \kappa)$, there exists $b = (b_1, \ldots, b_K) \in \mathbb{R}^K$ such that for any $z \in [a, b]$, $s(z) = B(z)^T b$. Thus, for a fixed $\theta \in S_{p-1}^+$, we obtain $\alpha(z; \theta) \approx B(z)^T b(\theta)$, where $b(\theta)$ means that the coefficients of $B$-spline model depends on the single-index parameter $\theta \in S_{p-1}^+$. Let

$$U_n(b, \theta|\lambda) = \frac{1}{n} \sum_{i=1}^{n} \left[ \exp[B(X_i^T \theta)^T b] \log \left( \frac{Y_i}{w_n(X_i)} \right) - B(X_i^T \theta)^T b \right] I(Y_i > w_n(X_i))$$

$$+ \frac{\lambda}{2} \int_a^b \left\{ \frac{d^n}{dx^n} B(x)^T b \right\}^2 dx$$

be the penalized (minus) log-likelihood loss function obtained from (5), where $\lambda > 0$ is the smoothing parameter. We aim to construct the estimator of $(b, \theta)$ by minimizing (6). First, for fixed $\theta$, we define the estimator of $b$ as

$$\hat{b}(\theta) = \arg \min_b U_n(b, \theta|\lambda).$$

Next, the estimator of $\theta$ is obtained by

$$\hat{\theta} = \arg \min_{\theta \in S_{p-1}^+} U_n(\hat{b}(\theta), \theta|0).$$

Lastly, we define the final estimator of $b$ as

$$\hat{b}(\hat{\theta}) = \arg \min_b U_n(b, \hat{\theta}|\lambda).$$

For any $x \in \mathcal{X}$, the EVI function is estimated by $\hat{\alpha}(x_0|\hat{\theta}) = B(x_0)^T \hat{b}(\hat{\theta})$ for $x_0 \in [a, b]$. Wang and Yang (2009) also considered the similar estimation procedure in mean regression. In (5), $U_n(\cdot, \cdot|0)$ implies that the penalty term in $U_n$ is removed for the estimation of $\theta$.

In practice, $(\hat{b}, \hat{\theta})$ are obtained by iterative algorithm. Let $\theta^{(k)} = (\theta_1^{(k)}, \ldots, \theta_p^{(k)})^T$ be the estimator of single index with $k$ steps iteration. For any vector $a$, we denote $||a|| = \sqrt{a^T a}$. As described above, the estimation is implemented by the following algorithm.

1. Set the tuning parameter $(w_n, \lambda)$ and the initial estimator of single index $\theta^{(0)}$ with $||\theta^{(0)}|| = 1$ and $\theta_1^{(0)} > 0$.

2. For given $k > 0$, calculate

$$b^{(k+1)} = \arg \min_b U_n(b, \theta^{(k)}|\lambda)$$

3. Update

$$\theta^{(k+1)} = \arg \min_{\theta} U_n(b^{(k+1)}, \theta|0)$$

and standardized as $\theta^{(k+1)} \leftarrow \theta^{(k+1)}||\theta^{(k+1)}||^{-1} \text{sign}(\theta_1^{(k+1)})$

4. Iterate Steps 2–3 until $||\theta^{(k+1)} - \theta^{(k)}||$ converges.

If the above iteration does not converge, Step 3 changes as the proximal gradient method:

$$\theta^{(k+1)} = \arg \min_{\theta} U_n(b^{(k+1)}, \theta|0) + \nu^{(k)}||\theta - \theta^{(k)}||^2,$$

where $\nu^{(k)}$ is the step size. In our experience, $\nu^{(0)} = 0.01$ and $\nu^{(k)} = 1.2\nu^{(k-1)}$ shows good estimation and fast convergence.
2.3 Tuning parameter selection

In the proposed estimator, we have the following three tuning parameters as the threshold function: $w_n(x)$, number of knots $K$, and smoothing parameter $\lambda$. According to the concept of Ruppert (2002), knots selection is not more important than $\lambda$. He showed that it is sufficient to use the equidistant knots with fixed large $K$. In our method, we choose $w_n(x)$ and $\lambda$ using the data-driven method. In this paper, we discuss the discrepancy measure provided by Wang and Tsai. (2009). Let $\hat{\theta} = \arg\min_{\theta} \{D(\lambda, w_n)\}$.

The tuning parameters ($\lambda, w_n$) are selected via minimizing $D(\lambda, w_n)$.

Since $w_n$ is the function of $x$, we can consider several functions as the candidate of $w_n$. Therefore, we introduce one easy choice of the class of $w_n(\cdot)$. We restrict the class of $w_n$ as the conditional quantile function $w_n(x) = Q(\tau|x)$ of $Y$ given $X = x$, where $\tau \in (0, 1)$ is the quantile level. Then, the pair of $(\lambda, \tau)$ is searched by minimizing $D(\lambda, \tau)$.

3 Asymptotic theory

In this section, we study the asymptotic property of the proposed estimator. We first define the true structure. We consider the true model (3) with $\gamma = \frac{\alpha}{\theta}$, where $\alpha : \mathbb{R} \to \mathbb{R}$, $\theta \in S_{\theta}^{p-1}$ and $L(y|x)$ has [2]. As described before, the true structure $(\gamma, \theta)$ or $(\alpha, \theta)$ is not unique. Therefore, to establish the asymptotic theory for the estimator, we need to define the unique pair of the target function and parameter. In following, we write the density function $f_{w_n}(\cdot|X^T \theta = x^T \theta, X = x)$ as $f_{w_n}((\cdot|\alpha(\cdot|\theta), \theta)$. We define that for fixed $\theta \in S^{p-1}_+$,

$$\alpha_0(\cdot|\theta) = \arg\min_{\alpha} E \left[-\log f_{w_n} \left(\frac{Y}{w_n(x)}|\alpha(\cdot|\theta), \theta\right)\right].$$

Next, we define the true single-index parameter vector as

$$\theta_0 = \arg\min_{\theta} E \left[-\log f_{w_n} \left(\frac{Y}{w_n(x)}|\alpha_0(\cdot|\theta), \theta\right)\right].$$

Finally, the target function of the nonparametric part is defined as $\alpha_0(\cdot|\theta_0)$. Wang and Yang (2009) also considered similar setting of true model in mean regression.

We provide the following conditions:

(C1) The marginal density function of $X$ is continuous and bounded away from 0 and $\infty$. The support $\mathcal{X}$ of $X$ is compact. In addition, there exists $a, b \in \mathbb{R}$ such that for any $X \in \mathcal{X}$ and $\theta \in S^{p-1}_+$ such that $a \leq X^T \theta \leq b$. Finally, the covariance matrix of $X$ is positive definite.

(C2) For any $\theta \in S^{p-1}_+$, $\alpha_0(\cdot|\theta) \in \mathcal{C}_q[a, b]$ for some positive $q$. The order of the difference penalty $m$ in [5] is smaller than $q$: $m < q$. 

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Thus, (C5) is very natural if \( B \) function in POT would be interpreted as the quantile function with some quantile level (see, function and this is the standard setting for \( Q = K \)). If (C7) is violated, the estimator may not be consistent threshold value. (C7) is important for penalized spline smoothing, which is related to Remark 5.3 (b) and Remark 6.6 of Xiao (2019). If (C7) is violated, the estimator may not be consistent estimator of true nonparametric function, which implies that \( \lambda \) should be chosen appropriately.

(C3) The functions \( \ell_0, \ell_1 \), and \( \beta \) are integrable and continuous with respect to \( x \in X \). Furthermore, \( \ell_0 \) and \( \beta \) are positive. There exists a constant \( \beta_{min} > 0 \) such that \( \beta_{min} \leq \beta(x) < \infty \) for all \( x \in X \).

(C4) \( \lim_{y \to \infty} \sup_{x \in X} \sup_{\theta \in S_{+}^{p-1}} \frac{y^{\beta(x)/\gamma} \ell(\theta)}{\nu(y|x)} \to 0. \)

(C5) The threshold function \( w_n(x) \) satisfies \( w_n(x) \to \infty \) for any \( x \in X \), and there exists a sequence \( \tau_n \) satisfying \( \tau_n \to 0 \) and constant \( c_w, C_w > 0 \) such that for any \( x \in X \) and \( \theta \in S_{+}^{p-1} \),

\[
\tau_n c_w \leq P(Y > w_n(x)|X^T \theta = x^T \theta, X = x) \leq \tau_n C_w.
\]

(C6) The knots sequence \( \kappa \) is quasi uniform. That is, \( c_{\ell} < \max_{j} \{\kappa_{j+1} - \kappa_j\} / \min_{j} \{\kappa_{j+1} - \kappa_j\} < c_u \) for some constant \( c_{\ell}, c_u > 0 \). The number of knots satisfies \( K \to \infty \), but \( K/(\tau_n n) \to 0 \).

(C7) The smoothing parameter \( \lambda = \lambda_n \) satisfies \( \lambda \to 0 \), \( \lambda/\tau_n \to 0 \) and \( K(\lambda/\tau_n)^{(1/2m)} = O(1) \) as \( n \to \infty \). Here, \( \tau_n \) is that defined in (C5).

Condition (C1) is a natural condition in regression. For the data points \( x_i (i = 1, \ldots, n) \) and any \( \theta \in S_{+}^{p-1} \), we have \( -||x_i|| \leq x_i^T \theta \leq ||x_i|| \). Therefore, for example, by centering and scaling the data \( x_i \), \( a \) and \( b \) can be found in practice. Wang and Yang (2009) and Wang and Tsai (2009) proposed another method to transform \( X \) that has finite support. (C2) is usual in a nonparametric function. In (5), the penalty is added to the mth derivative of \( c_0 \). Therefore, \( m < q \) is natural. (C3) and (C4) are standard conditions in Pareto-type-tail regression model with the Hall class. Roughly speaking, \( \tau_n \) in (C5) controls the rate of the data exceeds threshold. As total sample size increase, the sample size of the data exceeds threshold decreases. That is standard condition in extreme value theory (see, Hill 1975). We provide some example of choice of \( w_n \) concerned with (C5). Since

\[
P(Y > w_n(x)|X^T \theta_0 = x_0, X = x) \approx \ell_0(x)w_n(x)^{-1/\gamma_0(x_0|\theta_0)}(1 + o(1)),
\]

The \( \tau \)th conditional quantile of \( Y \) given \( X = x \) and \( X^T \theta_0 = x_0 \) is given as

\[
Q(\tau|x) = \inf_y \{y : P(Y > y|X^T \theta_0 = x_0, X = x) \leq \tau\}
\]

\[
= \tau^{-\gamma_0(x_0|\theta_0)}\ell_0(x)^{\gamma_0(x_0|\theta_0)}(1 + o(1)).
\]

Therefore, if the threshold function \( w_n(x) \) is set as the conditional quantile function: \( w_n(x) = Q(\tau|x) \), we obtain

\[
w_n(x)^{-1/\gamma_0(x^T \theta_0)} = \tau_n \ell_0(x)^{-1}(1 + o(1)) = O(\tau_n).
\]

Thus, (C5) is very natural if \( w_n(x) = Q(\tau_n|x) \) with \( \tau_n \to 0 \) as \( n \to \infty \). Actually, the threshold function in POT would be interpreted as the quantile function with some quantile level (see, Beirlant et al. 2004). (C6) is needed to obtain a good B-spline estimator of true nonparametric function and this is the standard setting for B-spline method (see, Xiao 2019). The condition \( K = o(n\tau_n) \) means that the number of knots cannot be larger than the sample size exceeding threshold value. (C7) is important for penalized spline smoothing, which is related to Remark 5.3 (b) and Remark 6.6 of Xiao (2019). If (C7) is violated, the estimator may not be consistent estimator of true nonparametric function, which implies that \( \lambda \) should be chosen appropriately.
3.1 The nonparametric part with a known single index parameter

We first develop the asymptotic property of the $\hat{\alpha}(-\theta)$ obtained from (7) with fixed $\theta \in \mathbb{S}^{p-1}$. Let $X_0 = X^T \theta \in [a, b]$ be the one-dimensional predictor with known $\theta$. In this section, for simplicity, we write $\alpha_0(\cdot) = \alpha_0(-\theta)$. We next prepare some symbols for the asymptotic bias and the variance of $\hat{\alpha}$. Let

$$s_0 = \arg\min_{s \in S(d, \kappa)} L(s),$$

where

$$L(s) = -E \{\exp[s(X_0)] \log(Y/w_n(X)) - s(X_0)\} I(Y > w_n(X)).$$

That is, $s_0$ is the best spline approximation to $\alpha_0$. Then, there exists $b_0 = b_0(\theta) \in \mathbb{R}^K$ such that $s_0(x) = B(x)^T b_0$. Define

$$b_{s,n}(x, q) = -\frac{\alpha_0(q)(x)}{q!K^q} B_q \left( \frac{x - \kappa_j}{\kappa_{j+1} - \kappa_j} \right), \quad x \in [\kappa_j, \kappa_{j+1}),$$

where $B_q$ is the $q$th degree Bernoulli polynomial function (see, Zhou et al. 1998). Lemma 1 in Appendix B shows that sup$_x |s_0(x) - \alpha_0(x) - b_{s,n}(x, q)|_\infty = o(K^{-q})$, which means that $b_{s,n}(\cdot, q)$ is the bias between $\alpha_0$ and the spline model. Further, we let $G = E[P(Y > w_n(X)|X)] B(X_0) B(X_0)^T$ and for $x \in [a, b]$, let

$$b_{\lambda,n}(x) = \lambda B(x)^T (G + \lambda \Delta_{m,K})^{-1} \Delta_{m,K} b_0,$$

where $\Delta_{m,K}$ is that given in Appendix A. Define

$$b_{\beta,n}(x) = B(x)^T (G + \lambda \Delta_{m,K})^{-1} E[P(Y > w_n(X)|X) r_n(X) B(X_0)],$$

and

$$r_n(x) = E \left[ \exp[B(x_0)^T b_0] \log \left( \frac{Y}{w_n(x)} \right) - 1 \left| X = x, Y > w_n(x) \right. \right],$$

where $x_0 = x^T \theta$. Finally, we define the smoother as

$$v_n(x) = \frac{1}{\sqrt{n}} E[P(Y > w_n(X)|X) B(X_0) B(X_0)^T]^{1/2} (G + \lambda \Delta_{m,K})^{-1} B(x), \quad x \in [a, b].$$

We first show the asymptotic normality of $\hat{\alpha}(\cdot) = \hat{\alpha}(-\theta)$ below.

**Theorem 1.** Suppose that (C1)-(C7). For any $x \in (a, b)$, as $n \to \infty$

$$\hat{\alpha}(x) - \alpha_0(x) = v_n(x)^T \varepsilon(1 + o(1)) + b_{s,n}(x, q)(1 + o_P(1)) + b_{\lambda,n}(x)(1 + o_P(1)) + b_{\beta,n}(x)(1 + o_P(1)),$$

where $\varepsilon$ is the $K$-variate standard normal random vector, $b_{s,n}(x, q) = O(K^{-q})$, $b_{\lambda,n}(x) = O((\lambda/\tau_n)^{1/2})$, $|b_{\beta,n}(x)| \leq O(\tau_n^{\beta_{\text{min}}})$ and $\|v_n(x)\|^2 = O((n\tau_n)^{-1}(\lambda\tau_n^{-1})^{-1/(2m)})$.

Here, $b_{s,n}$ is the asymptotic model bias between the best spline approximation and true function $\alpha_0$ (see, Lemma 1 in Appendix B). The term $b_{\lambda,n}$ arises by using penalty term concerned with $m$th derivative of spline function $s$. Under (C7), we have $(\lambda/\tau_n) = O(K^{-2m})$, which indicates that $b_{s,n}$ is dominated by $b_{\lambda,n}$, that is $b_{s,n} + b_{\lambda,n} = O(\lambda^{1/2}/\tau_n^{1/2})$. Further, $b_{\beta,n}$ is the approximation bias of the conditional pareto-type-tailed model (4). The asymptotic smoother $v_n(x)$ dominates the variance of $\hat{\alpha}$.

To obtain the rate of convergence, we next study the $L_2$-convergence of $\hat{\alpha}$.
Theorem 2. Suppose that (C1)-(C7). As $n \to \infty$,

$$E \left[ |\hat{\alpha}(X_0) - \alpha_0(X_0)|^2 \right] = O \left( \left( \frac{\lambda}{\tau_n} \right)^{-1/(2m)} \frac{1}{n\tau_n} \right) + O \left( \frac{\lambda}{\tau_n} \right) + O(\tau_n^{2\beta_{\min}}). \quad (10)$$

If we set $\lambda/\tau_n = (n\tau_n)^{-2m/(2m+1)}$, under $m \leq d + 1$,

$$E \left[ |\hat{\alpha}(X_0) - \alpha_0(X_0)|^2 \right] = O \left( (n\tau_n)^{-2m/(2m+1)} \right) + O(\tau_n^{2\beta_{\min}}). \quad (11)$$

In addition, if $\tau_n = n^{-m/((2\beta_{\min}+1)m+\beta_{\min})}$,

$$E \left[ |\hat{\alpha}(X_0) - \alpha_0(X_0)|^2 \right] = O \left( n^{-\frac{2\beta_{\min}}{2\beta_{\min}+1+m/m}} \right). \quad (12)$$

In Theorem 2, the first term of (10) is the variation of stochastic structure, which can be interpreted as the order of variance of the estimator. Other parts are asymptotic bias terms. Thus, the squared bias and variance are balanced by the smoothing parameter $\lambda$ and the order of threshold function $\tau_n$. In (11), we state the rate of convergence of $\hat{\alpha}$ by selecting the optimal $\lambda$. The term $O((n\tau_n)^{-2m/(2m+1)})$ can be seen as the optimal rate of the nonparametric estimator with sample size $n\tau_n$. Indeed, since $\tau_n$ concerns with the rate of the data exceeding threshold, $n\tau_n$ is close to the sample size of the data over the threshold function. However, (11) still has a trade-off with respect to $\tau_n$, which is dependent on the second-order parameter $\beta(x)$ of Pareto-type distribution. We now express $\tau_n = k/n$ with some $k$ satisfying $k \to \infty$ and $k/n \to 0$. Then, we have

$$E[|\hat{\alpha}(X_0) - \alpha(X_0)|^2] = O(k^{-2m/(2m+1)}) + O((k/n)^{2\beta_{\min}}). \quad (13)$$

We now consider the Pareto-type distribution with the Hall class of one dimensional data: $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} F$, where

$$P(Y > y) = 1 - F(y) = y^{-1/\gamma}\{\ell_0 + \ell_1 y^{-\beta/\gamma} + \nu(y)\}$$

with $\gamma, \ell_0, \beta > 0$, $\ell_1 \in \mathbb{R}$ and $\nu(y) = o(y^{-\beta/\gamma})$ as $y \to \infty$. Then, the typical estimator of $\gamma$ is the Hill estimator (Hill 1975)

$$\hat{\gamma}_{\text{Hill}} = \frac{1}{k} \sum_{i=1}^k \log \frac{Y_{(n+1-i)}}{Y_{(k)}},$$

where $Y_{(1)} \leq \ldots \leq Y_{(n)}$ is the order statistics of $Y_1, \ldots, Y_n$. It is well known from de Haan and Ferreira (2006) that the Hill estimator $\hat{\gamma}_{\text{Hill}}$ has an asymptotic rate of convergence as

$$E[|\hat{\gamma}_{\text{Hill}} - \gamma|^2] = O(k^{-1}) + O((k/n)^{2\beta}) \quad (14)$$

under some suitable conditions, $k \to \infty$ and $k/n \to 0$ as $n \to \infty$.

Compared with (13) and (14), the first terms of both estimators show the difference between the asymptotic order of the nonparametric estimator and the parametric estimator. On the other hand, the second terms are inherently similar. Drees (2001) shows the optimal rate of convergence of $\hat{\gamma}_{\text{HILL}}$ as

$$E[|\hat{\gamma}_{\text{HILL}} - \gamma|^2] = O \left( n^{-\frac{2\beta}{2\beta+1}} \right),$$

which is similar to (12) replacing with $\beta = \beta_{\min}$ and $m \to \infty$. This indicates that the results of Theorem 2 are a natural extension from the parametric method to the nonparametric regression.

Lastly, we organize the uniform rate of convergence of $\hat{\alpha}$. 

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Theorem 3. Suppose that (C1)-(C7). As $n \to \infty$,

$$
\sup_{x \in [a, b]} |\hat{\alpha}(x) - \alpha_0(x)| = O_P \left( \sqrt{\frac{(\lambda/\tau_n)^{-1/(2m)}}{n\tau_n}} \right) + O \left( \left(\frac{\lambda_n}{\tau_n}\right)^{1/2} \right) + O \left( \tau_n^{\beta_{\min}} \right).
$$

If $\lambda/\tau_n = O((n\tau_n/\log(n))^{-2m/(2m+1)})$,

$$
\sup_{x \in [a, b]} |\hat{\alpha}(x) - \alpha_0(x)| = O_P \left( \left(\frac{\log(n)}{n\tau_n}\right)^{-m/2m+1} \right) + O \left( \tau_n^{\beta_{\min}} \right).
$$

Further, if $\tau_n = O((\log(n)/n)^{-m/(2\beta_{\min}m+2m+\beta_{\min}))}$,

$$
\sup_{x \in [a, b]} |\hat{\alpha}(x) - \alpha_0(x)| = O \left( \left(\frac{\log(n)}{n}\right)^{-\frac{\beta_{\min}}{2\beta_{\min}+1}+1/m} \right).
$$

The rate of $L_\infty$ convergence is slightly slower than $L_2$ rate because of the term of $\log(n)$, which is a well-known result in nonparametric regression (e.g. Tsybakov 2009). If we write $h = (\lambda/\tau_n)^{1/(2m)}$, \eqref{15} can be expressed as

$$
\sup_{x \in [a, b]} |\hat{\alpha}(x) - \alpha_0(x)| = O \left( \sqrt{\frac{\log(n)}{n\tau_nh}} \right) + O(h^m) + O(\tau_n^{\beta_{\min}}).
$$

Under $\sqrt{n\tau_nh/\log(n)}\{h^{m+\tau_n^{\beta_{\min}}}\} \to 0$, we have $\sqrt{n\tau_nh/\log(n)}\sup_{x \in [a, b]} |\hat{\alpha}(x) - \alpha(x)| = O(1)$, which is related result to Theorem 2 of Goegebeur et al. (2015) with $d = 1$. Goegebeur et al. (2015) considers the Nadaraya-Watson-type kernel estimator. We can see from Theorem 3 that the spline estimator and the kernel estimator have similar asymptotic rates of convergence if we adjust the smonthing parameter well.

### 3.2 Parametric part

In this section, we establish the asymptotic theory of $\hat{\theta}$, which is defined as \eqref{S}. To this end, we need the asymptotics for $\hat{\alpha}(\cdot|\theta_0)$ discussed in the previous section. The single index parameter $\theta$ has two restrictions: $||\theta|| = 1$ and $\theta_1 > 0$. Therefore, we re-parametrize it. Let $S_c = \{ (\phi_1, \ldots, \phi_{p-1}) \in \mathbb{R}^{p-1} : ||\phi|| \leq c \}$ for some $0 < c < 1$. Then, we write $\theta = \theta(\phi) = (\sqrt{1 - ||\phi||^2}, \phi^T)^T$ for $\phi \in S_c$. Such transformation describes the condition $||\theta|| = 1$ and $\theta_1 > 0$ without any restriction. Then, suppose that there exists $\phi_0 \in S_c$ such that $\theta_0 = \theta(\phi_0)$. Such parametrization is used by Caroll et al. (1997). $S_c$ can be written as

$$
\hat{\phi} = \arg\min_{\phi \in S_c} U_n(\hat{b}(\theta(\phi)), \theta(\phi)|0).
$$

We will study the asymptotic property of $\hat{\phi}$. For $\alpha_0(\cdot) = \alpha_0(\cdot|\theta_0)$, let

$$
\Sigma = E \left[ \ell_0(X) \frac{\partial \alpha_0(X^T\theta(\phi_0))}{\partial \phi} \frac{\partial \alpha_0(X^T\theta(\phi_0))}{\partial \phi^T} \right]
$$
and let
\[ \psi_n = \Sigma^{-1} E \left[ \frac{\partial \alpha_0(X^T \theta_0)}{\partial \phi} \left\{ \exp[B(X^T \theta_0)^T b(\theta_0)] \log \left( \frac{Y_i}{w_n(X)} \right) - 1 \right\} I(Y > w_n(X)) \right]. \]

As the result, \( \Sigma \) and \( \psi_n \) are the asymptotic variance and the bias of \( \hat{\phi} \), respectively. We need some mathematical conditions.

(C8) \( U_{n}(\hat{b}(\theta(\phi)), \theta(\phi)|0) \) is the convex function with respect to \( \phi \in \mathcal{S}_c \).

(C9) \( \alpha(x^T \theta(\phi)|\theta(\phi)) \) is twice continuously differentiable with respect to \( \phi \). Further, \( \Sigma \) is a positive definite matrix.

**Theorem 4.** Suppose that (C1)–(C9). As \( n \to \infty \),
\[ \sqrt{n} \Sigma^{1/2} \{ \hat{\phi} - \phi_0 - \psi_n \} \overset{D}{\to} N(0, I) \]
and
\[ ||\psi_n|| \leq O \left( \sqrt{\left( \frac{\lambda}{\tau_n} \right)^{-1/2} \frac{\log(n)}{n \tau_n}} \right) + O((\lambda/\tau_n)^{1/2}) + O(\tau_n^{2\beta_{\min}}). \]

Under \( \lambda/\tau_n = O(n \tau_n/\log(n))^{-2m/(2m+1)} \),
\[ ||\hat{\phi} - \phi_0||^2 = O_P((n \tau_n)^{-1}) + O(\tau_n^{2\beta_{\min}}). \]

Further, under \( \tau_n = O(n^{-1/(2\beta_{\min}+1)}) \), the optimal rate of convergence of \( \hat{\phi} \) is
\[ ||\hat{\phi} - \phi_0||^2 = O(n^{-2\beta_{\min}}). \]

We see from Theorem 4 that the optimal rate of convergence of \( \hat{\phi} \) is inherently similar to that of the Hill estimator \( \hat{\gamma}_{\text{Hill}} \) discussed in Section 3.1. This result is quite natural as the Hill estimator is also a parametric estimator with \( p = 1 \) and \( X_1 = 1 \). It is easy to show that the asymptotic theory for \( \hat{\theta} \) can be established in a manner similar to \( \hat{\phi} \).

### 3.3 Nonparametric part with estimated single index

The coefficients of the \( B \)-spline model is estimated via \( \| \phi - \theta \| \leq O((n \tau_n)^{-1/2}) + O(\tau_n^{2\beta_{\min}}) \). Using this result, we can establish the asymptotic properties of the single index estimator \( \hat{\alpha}(x^T \hat{\theta}) \). Let \( b_{a,n}(x^T \theta) = \alpha_0^{(1)}(x^T \theta)x^T \psi_n \) be the additional asymptotic bias between \( \hat{\alpha}(x^T \hat{\theta}) \) and \( \alpha_0(x^T \theta_0) \), where \( \alpha_0^{(1)}(s) = \alpha_0^{(1)}(s|\theta_0) = \partial \alpha_0(s|\theta_0)/\partial s \). We see that \( b_{a,n}(x^T \theta_0) = O(\tau_n^{2\beta_{\min}}) \).

**Theorem 5.** Suppose that (C1)–(C9). For any \( x \in \mathcal{X} \), as \( n \to \infty \),
\[ \hat{\alpha}(x^T \hat{\theta}) - \alpha_0(x^T \theta_0) = v_n(x^T \theta_0)^T \varepsilon + b_{\lambda,n}(x^T \theta_0)(1 + o(1)) + b_{\beta,n}(x^T \theta_0)(1 + o(1)) + b_{a,n}(x^T \theta_0)(1 + o(1)), \]
where \( \varepsilon \sim N(0, I), v_n, b_{\lambda,n} \) and \( b_{\beta,n} \) are those given in Theorem 4 with \( b_0 = b_0(\theta_0) \).
The nonparametric part were good. When the proposed estimator for model (i). For worse, although it is not surprising. However, for of \( \hat{\theta} \).

The optimal rate of \( L_2 \)-convergence is similar to Theorem 6.

**Theorem 7.** Suppose that (C1)–(C9). As \( n \to \infty \),

\[
\sup_{x \in \mathcal{X}} |\hat{\alpha}(x^T \hat{\theta}) - \alpha_0(x^T \theta_0)| = O \left( \left( \frac{\lambda}{\tau_n} \right)^{-1/(2m)} \left( \frac{\log(n)}{n\tau_n} \right) + O \left( \frac{\lambda^{1/2}}{\tau_n^{1/2}} \right) + O(\tau_n^{2\beta_{min}}) \right).
\]

The optimal rate of \( L_\infty \)-convergence is similar to Theorem 3.

From Theorems 6 and 7 we see that the asymptotic rate of the single index estimator is similar to the estimator with fixed \( \theta_0 \). This result is not surprising because the asymptotic rate of the parametric estimator is faster than that of the nonparametric estimator. However, in Theorem 5 the new bias \( b_{\alpha,n} \) arises although its order is the same as \( b_{\beta,n} \).

## 4 Simulation

In this section, we investigate the finite sample performance of the proposed estimator by Monte Carlo simulation. For the response \( Y \in \mathbb{R} \) and the predictor \( X \in \mathbb{R}^p \), the true distribution is set as

\[
P(Y > y | X = x) = \frac{y^{-\gamma^*(x)}}{1 + \ell y^{-\gamma^*(x)}},
\]

where \( \ell \) is the positive constant and \( \gamma^* : \mathbb{R}^p \to \mathbb{R} \) is the EVI function. This model has the structure of Pareto-type-tailed distribution, where \( \gamma^* \) is determined by \( \ell_0(x) = 1 \), \( \ell_1(x) = \ell \) and \( \beta(x) = 1 \). We prepare the parameter \( \theta = (\theta_1, \ldots, \theta_p)^T \in \mathbb{R}^p \), where \( \theta_1 = 1, \theta_2 = 0.2 \) and \( \theta_3 = 0.5 \) and \( \theta_j = 0, j > 3 \). Then, the \( \theta \) is modified as \( \theta/||\theta|| \). The EVI is written by \( \gamma^*(\cdot) = \exp[-\alpha^*(\cdot)] \). We then consider the following four models: (i) \( \alpha^*(x) = 1.2 + 2x^T \theta \) and \( \ell_1 = 0 \), (ii) \( \alpha^*(x) = 1.2 + 2x^T \theta \) and \( \ell_1 = 0.5 \), (iii) \( \alpha^*(x) = 1.5 + 2\cos(2x^T \theta) \) and \( \ell_1 = 0.25 \), and (iv) \( \alpha^*(x) = -3 + \phi(x^T \theta; -\mu, \sigma) + \phi(x^T \theta; \mu, \sigma) \) with \( \mu = 0.3 \) and \( \sigma = 0.2 \), where \( \phi(z; \mu, \sigma) \) is the density function of Gaussian distribution with mean \( \mu \), standard deviation \( \sigma \), and \( \ell_1 = 0.25 \). Finally, we use (v) \( \alpha^*(x) = -1.2 - x_1(1-x_3) \sin(2\pi x_2) \) and \( \ell_1 = 0.25 \). The first four models have the structure of the single index model whereas the last model is a fully nonparametric model. Each predictor is independently generated by uniform distribution on an interval \([-1/\sqrt{3}, 1/\sqrt{3}] \).

We construct the proposed estimator of the single index model: \( \hat{\gamma}^*(x) = \hat{\gamma}(x^T \theta) \). Then, the threshold function is set as \( w_n(x) = Q(\tau) \), where \( Q(\tau) \) is the \( \tau \)th quantile of \( Y \). In practice, \( Q(\tau) \) is replaced with its sample version. We choose \( (\tau, \lambda) \) by using the discrepancy measure presented in Section 2.3. The proposed estimator is denoted by SIM-D.

We first show the performance of the proposed estimator SIM-D for model (i)–(iv) using sample size \( n = 2000 \) and 500 Monte Carlo iterations. We illustrate the box plot of each element of \( \hat{\theta} - \theta \) and \( \hat{\gamma}(z) \) with 20 points within the domain of \( X^T \theta \). Figure 1 shows the boxplot of the proposed estimator for model (i). For \( p = 3 \), the performances of both parametric part and the nonparametric part were good. When \( p \) increases, the behaviour of the estimator becomes worse, although it is not surprising. However, for \( p = 20 \), we can say that the estimators could capture the true structure. For \( p = 50 \), the parametric part \( \{\hat{\theta}_i - \theta_i\} (i = 1, 2, 3) \) and the
nonparametric estimator at the boundary have some bias incurred by including several irrelevant variables. Thus, large $p$ yields the underestimate, and in a future study, we expect to avoid these phenomena for very high-dimensional data. The boxplots of our estimator for model (ii) are illustrated in Figure 2. This model is also linear like model (i), but this model has distribution bias: $\beta(x) = 1$. Thus, the estimation of model (ii) is comparatively difficult than that of model (i). Nevertheless, the behaviour of the estimator is similar to that for model (i). Figure 3 shows the results of the boxplot of SIM-D for model (iii). Even when the true EVI is nonparametric form, the estimator has good efficiency. Similar to models (i) and (ii), for $p = 50$, the estimator is unstable and hence, for very high-dimensional data, some dimensional reduction is needed before applying the regression model. The behaviour of SIM-D for model (iv) can be confirmed by Figure 4. This model is more complicated than model (iii) since this is bi-modal, unlike (iii). Nevertheless, the proposed estimator captures the true structure of the model.

Next, we compare the proposed estimator with some of its competitors. The performance of the estimator is evaluated by Mean Integrated Squared Error:

$$MISE = \frac{1}{J} \sum_{j=1}^{J} E \left[ \left\{ \frac{\hat{\gamma}^*(X^*_j)}{\gamma^*(X^*_j)} - 1 \right\}^2 \right],$$

where $X^*$ is generated from the similar distribution of $X$, and $\hat{\gamma}^*$ is the estimator of $\gamma^*$. However, the data located outside of upper and lower 5%-quantile of $(X^*_j)^T \theta$ are removed. The total number of test data is adjusted as $J = 1000$. In our study, MISE is approximated by 500 Monte Carlo iterations. We also calculate the other estimators as competitors. First, we construct the proposed single index model, where the tuning parameter is selected by minimizing MISE. This estimator is denoted by SIM-M. The SIM-M is the optimal estimator from our model, but it can be calculated only in simulation since the information of true EVI is used. However, we can confirm the efficiency of the discrepancy score as the tuning parameter selection if SIM-D closes to SIM-M. Meanwhile, we use the linear model $\alpha(x) = \theta_0 + x^T \theta$, which is proposed by Wang and Tsai (2009). Next, consider the additive model $\alpha(x) = \alpha_0 + \alpha_1(x_1) + \cdots + \alpha_p(x_p), \alpha_j : \mathbb{R} \to \mathbb{R}$ (see, Youngman 2019). Each $\alpha_j$ is estimated by the smoothing spline method. All smoothing parameters for $\alpha_1, \ldots , \alpha_p$ are similar, and the threshold value for POT selected by the discrepancy score described in Section 2.3. We also considered the method of the quantile-based single index model proposed by Xu et al. (2022), which is denoted by SIMEXQ. Let $Q_Y(\tau|x)$ be the conditional quantile of $Y$ given $X = x$. Their method assumes that $Q_Y(\tau|x) = Q_Y(\tau|x^T \theta) = x^T \theta(\tau)$. Then, $\theta$ is estimated by linear quantile regression. The EVI is estimated by the conditional Hill estimator from the estimator of conditional quantile. The tuning parameters included in SIMEXQ are set as similar to Section 5.1 of Xu et al. (2022). Lastly, we also evaluate the fully nonparametric kernel estimator of EVI, denoted by Kernel (see, Goegebeur et al. 2015). In Kernel, the Gaussian kernel is used and the bandwidths of kernel are assumed to be equal.

In Tables 1, we report the MISE of the estimators for model (i) and (ii). The linear estimator is superior to other methods, but this result is reasonable since the true is also linear. Meanwhile, the results indicate that the performance of the proposed method is also good. In Table 2, we report the MISE of the estimators for model (iii) and (iv). These models have a single index structure, with our estimator giving the best performance. The SIMEXQ also demonstrated good behaviour although the linearity condition of the conditional quantile failed. Lastly, MISE of the estimators for (v) is also shown in Table 2. From the result, it is apparent that the single index model is one of efficient approaches even when fully nonparametric model is considered.
Figure 1: Simulation results for model (i) with $n = 2000$ and 500 Monte Carlo iterations. Upper panels: Box plot of each element of $\hat{\theta} - \theta$. First three boxplots are those for active variables. Lower panels: Box plot of $\hat{\gamma}(z)$ on 20 points for $z \in [0.1, 0.9]$. Left, middle, and right panels correspond to the model with $p = 3$, 20 and 50.

Table 1: The mean (sd) value of MISE of all estimators obtained via Monte Carlo simulation.

| Methods   | \(n = 1000\) | \(n = 2000\) |
|-----------|---------------|---------------|
|           | \(p = 3\)    | \(p = 20\)   | \(p = 50\)   | \(p = 3\)    | \(p = 20\)   | \(p = 50\)   |
| SIM-D     | 0.197 (0.23)  | 0.223 (0.28)  | 1.227 (1.45)  | 0.083 (0.06)  | 0.103 (0.12)  | 0.305 (0.28)  |
| SIM-M     | 0.115 (0.11)  | 0.144 (0.22)  | 0.733 (0.44)  | 0.053 (0.04)  | 0.086 (0.08)  | 0.278 (0.22)  |
| Linear    | 0.078 (0.07)  | 0.116 (0.10)  | 0.351 (0.19)  | 0.012 (0.01)  | 0.063 (0.02)  | 0.158 (0.10)  |
| Additive  | 0.964 (0.44)  | 0.253 (0.23)  | 0.526 (0.29)  | 0.102 (0.11)  | 0.113 (0.11)  | 0.251 (0.29)  |
| SIMEXQ    | 0.321 (0.24)  | 0.277 (0.33)  | 1.442 (2.01)  | 0.088 (0.07)  | 0.144 (0.16)  | 0.401 (0.31)  |
| Kernel    | 1.431 (1.22)  | 3.432 (3.44)  | 9.241 (21.1)  | 0.212 (0.55)  | 2.052 (3.82)  | 8.822 (4.42)  |

Model (ii): Linear model with bias

| Methods   | \(n = 1000\) | \(n = 2000\) |
|-----------|---------------|---------------|
|           | \(p = 3\)    | \(p = 20\)   | \(p = 50\)   | \(p = 3\)    | \(p = 20\)   | \(p = 50\)   |
| SIM-D     | 0.044 (0.08)  | 0.221 (0.34)  | 1.502 (0.71)  | 0.036 (0.03)  | 0.092 (0.10)  | 0.922 (0.53)  |
| SIM-M     | 0.039 (0.08)  | 0.192 (0.33)  | 1.552 (0.64)  | 0.028 (0.02)  | 0.088 (0.08)  | 0.844 (0.51)  |
| Linear    | 0.032 (0.02)  | 0.177 (0.25)  | 1.266 (0.39)  | 0.021 (0.01)  | 0.073 (0.05)  | 0.604 (0.45)  |
| Additive  | 0.043 (0.06)  | 0.199 (0.33)  | 1.877 (0.72)  | 0.034 (0.03)  | 0.121 (0.12)  | 1.254 (0.55)  |
| SIMEXQ    | 0.055 (0.08)  | 0.244 (0.35)  | 2.173 (0.83)  | 0.066 (0.04)  | 0.113 (0.10)  | 1.276 (0.52)  |
| Kernel    | 0.113 (0.11)  | 1.224 (1.11)  | 9.945 (3.14)  | 0.213 (0.24)  | 1.245 (2.11)  | 8.524 (4.22)  |
Figure 2: Simulation results for model (ii) with \( n = 2000 \) and 500 Monte Carlo iterations. The description is similar to Figure 1.

Table 2: The mean (sd) value of MISE of all estimators obtained via Monte Carlo simulation.

| Methods     | \( n = 1000 \) |  |  | \( n = 2000 \) |  |  |  |
|-------------|----------------|------------------|-----------------|------------------|------------------|------------------|
|             | \( p = 3 \)    | \( p = 20 \)     | \( p = 50 \)    | \( p = 3 \)     | \( p = 20 \)     | \( p = 50 \)     |
| SIM-D       | 0.097 (0.03)   | 0.279 (0.19)     | 1.732 (1.12)    | 0.024 (0.04)    | 0.201 (0.12)     | 1.106 (1.01)     |
| SIM-M       | 0.078 (0.03)   | 0.276 (0.17)     | 1.663 (1.01)    | 0.021 (0.03)    | 0.174 (0.13)     | 0.803 (0.92)     |
| Linear      | 0.335 (0.12)   | 1.752 (1.06)     | 3.123 (2.45)    | 0.137 (0.10)    | 0.666 (0.71)     | 2.822 (1.33)     |
| Additive    | 0.284 (0.11)   | 0.592 (0.22)     | 2.105 (1.55)    | 0.077 (0.08)    | 0.487 (0.55)     | 1.892 (1.23)     |
| SIMEXQ      | 0.121 (0.07)   | 0.363 (0.11)     | 1.911 (1.65)    | 0.043 (0.04)    | 0.286 (0.13)     | 1.216 (1.29)     |
| Kernel      | 0.221 (0.25)   | 2.212 (1.64)     | 8.226 (4.25)    | 0.071 (0.13)    | 1.236 (2.12)     | 6.632 (4.23)     |
|             |                |                  |                 | Model (iv): Single index model (bi-modal) |                |                  |
| SIM-D       | 0.102 (0.03)   | 0.382 (0.21)     | 2.128 (1.62)    | 0.055 (0.06)    | 0.247 (0.23)     | 1.211 (1.02)     |
| SIM-M       | 0.084 (0.03)   | 0.299 (0.20)     | 1.731 (1.11)    | 0.024 (0.05)    | 0.188 (0.13)     | 1.023 (0.96)     |
| Linear      | 1.231 (0.11)   | 2.282 (1.11)     | 4.313 (3.15)    | 1.027 (0.12)    | 1.126 (0.70)     | 3.212 (1.55)     |
| Additive    | 0.312 (0.13)   | 0.723 (0.26)     | 2.315 (1.82)    | 0.142 (0.07)    | 0.312 (0.44)     | 1.911 (1.21)     |
| SIMEXQ      | 0.172 (0.08)   | 0.388 (0.13)     | 2.201 (1.55)    | 0.066 (0.07)    | 0.244 (0.12)     | 1.341 (1.12)     |
| Kernel      | 0.731 (0.22)   | 2.551 (2.00)     | 9.742 (4.31)    | 0.174 (0.23)    | 1.541 (1.03)     | 7.213 (5.33)     |
|             |                |                  |                 | Model (v): Fully nonparametric model |                |                  |
| SIM-D       | 0.079 (0.05)   | 0.365 (0.18)     | 0.865 (0.61)    | 0.051 (0.03)    | 0.172 (0.14)     | 0.426 (0.26)     |
| SIM-M       | 0.059 (0.05)   | 0.193 (0.12)     | 0.734 (0.52)    | 0.038 (0.03)    | 0.126 (0.13)     | 0.354 (0.21)     |
| Linear      | 0.338 (0.23)   | 1.274 (0.72)     | 3.582 (2.33)    | 0.126 (0.06)    | 1.091 (0.92)     | 2.171 (1.43)     |
| Additive    | 0.049 (0.04)   | 0.286 (0.11)     | 0.936 (0.62)    | 0.027 (0.02)    | 0.095 (0.09)     | 0.331 (0.23)     |
| SIMEXQ      | 0.083 (0.12)   | 0.421 (0.21)     | 1.232 (1.21)    | 0.065 (0.04)    | 0.221 (0.16)     | 0.723 (0.43)     |
| Kernel      | 0.096 (0.03)   | 0.921 (1.62)     | 12.46 (6.35)    | 0.071 (0.13)    | 4.311 (2.12)     | 8.331 (6.14)     |
Figure 3: Simulation results for model (iii) with $n = 2000$ and 500 Monte Carlo iterations. The description is similar to Figure 1.

Figure 4: Simulation results for model (iv) with $n = 2000$ and 500 Monte Carlo iterations. The description is similar to Figure 1.
Remark 2 The tail dimension reduction (TDR), suggested by Gardes (2018), also employs the single or multi index model for the conditional quantile. Similar to Xu et al. (2022), we can construct the conditional Hill estimator as the estimator of EVI using conditional quantile with single index covariates obtained via TDR. However, the TDR requires extremely high computational cost, which is larger than $O(n2^p-1)$ with the exception of tuning parameter selection. Therefore, the construction of their estimator is very difficult for large $p$, and hence, we do not consider the TDR in this simulation. However, we confirmed that the TDR also had good performance for model (i)–(iv) with $p = 3$ in our simulation although these results are not cited in Tables 1–4.

5 Real data example

We apply the proposed single index model for EVI regression to the Ohlsson data, which is available in R package insuranceData as dataOhlsson. This data consists of 64548 motor cycle-related claims recorded from 1994–1998 by the Swedish insurer Wasa. Our purpose is to predict the $Y$: claim cost by using seven predictors including $X_1$:agarald, $X_2$: zon, $X_3$: mcklass, $X_4$: fordald, $X_5$:bonuskl, $X_6$: duration, and $X_7$: antskad. The description of each variable is detailed in the package’s site. We standardized each predictor. Furthermore, we remove the observations that $Y$ is zero. Thus, we use $n = 670$ observations.

To apply our method to this data, we use $K = 40$ equidistant knots on an interval $[-\min_i ||x_i||, \max_i ||x_i||]$. In addition, we need to select the threshold value $w_n(x)$ and the smoothing parameter $\lambda$. In this study, we use the $\tau$th marginal quantile of $Y$: $Q(\tau)$ as the threshold function, that is, $w_n(x) = w_n = Q(\tau)$. Similar to previous section, we use the sample version of $Q(\tau)$ in practice. We select $(\tau, \lambda)$ by using the discrepancy score mentioned in Section 2.3. As the result, we obtained $\hat{\theta} = (0.28, 0.33, -0.87, -0.21, -0.08, 0.01, 0.05)$ as the estimator of single index parameters. We can thus confirm that $X_3$ has a large impact on the tail behaviour of the claim cost. Indeed, the left panel of Figure 5 shows that the high $Y$ concentrates $X_3 = -0.722$, which indicates that the tail distribution of $Y$ is highly related to $X_3$. Compared with $X_5$, $X_6$, and $X_7$, the influence of $X_1$, $X_2$, and $X_4$ are highly affected by $Y$. We also obtained $\hat{\gamma}$ as the nonparametric part. To obtain an easy interpretation of the behaviour of the estimator of $\hat{\gamma}$, we constructed the estimator of the conditional quantile of $Y$ given $x^T \hat{\theta}$ as

$$\hat{Q}(\tau_E|x^T \hat{\theta}) = \left(\frac{1-\tau}{1-\tau_E}\right)^{-\hat{\gamma}(x^T \hat{\theta})} w_n,$$

where $\hat{Q}$ is known as the extrapolated estimator of conditional quantile (see, Weismann 1978, Xu et al. 2020) and $(w_n, \tau)$ is that given above. The overall shape of $\hat{\gamma}$ and $\hat{Q}(\tau_E|x^T \hat{\theta})$ are similar with respect to each $x^T \hat{\theta}$. We set $\tau_E = 0.99$. The behaviour of the extrapolated estimator is described in the right panel of Figure 5. We see from the result that estimators of the conditional quantile and EVI have a monotone-like nonlinear smooth curve.

We evaluate the prediction error of the estimator. Here, the prediction error is determined by

$$PE = E [\log \phi(Y^*, \mathbf{X}^*|\hat{\gamma}, w_n)]^{-1},$$

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Table 3: Prediction error (PE) of estimators for Ohlsson data. For each method, PE and sd are mean and standard deviation of PE from 100 Monte Carlo iterations.

| Method     | SIM  | Linear | Additive | Xu     | Kernel |
|------------|------|--------|----------|--------|--------|
| PE         | 0.473| 0.573  | 0.632    | 0.516  | 2.634  |
| sd         | 0.037| 0.069  | 0.033    | 0.023  | 0.125  |

where \((Y^*, X^*)\) is the test data that is generated from similar distribution to the data used to estimate EVI and

\[
\phi(Y, X|\gamma^*, w_n) = \frac{1}{\gamma^*(X)} \left( \frac{Y}{w_n(X)} \right)^{-1/\gamma^*(X)} Y^{-1} I(Y > w_n(X)),
\]

where \(\gamma^*: \mathbb{R}^p \rightarrow \mathbb{R}\) is the EVI function. In our method, we consider \(\gamma^*(X) = \gamma(X^T \theta)\). We note that the proposed prediction error (PE) is the inverse of the mean of log-likelihood function, which indicates that the good estimator yields small PE. The PE is approximated by five-fold cross validation. To remove the randomness of decomposition of the data by cross validation, PE was repeatedly calculated by 100 Monte Carlo iterations. We also obtained the PE of the linear model, additive model, the estimator of Xu et al. (2022), and the fully nonparametric estimator (see, Goegebeuer et al. 2015). The result is described in Table 3. The proposed single index model (SIM) shows the smallest PE. The result indicates that the linear model is too simple and the additive model is too complicated, which has seven nonparametric components. The performance of Xu et al. (2022) is satisfactory, but the linearity condition of the conditional quantile may be missing. The fully nonparametric kernel method is inferior to other estimators. Indeed, \(p = 7\) is too large to estimate the multivariate function appropriately. Consequently, the proposed single index model is easy to construct, including automatic tuning parameter selection, and has good performance as the method for the multivariate EVI regression.

6 Conclusion

In this paper, we applied the single index model to extreme value index (EVI) regression. By using the penalized maximum likelihood method for Pareto-type-tailed distribution approximation, the single index parameters and the one-dimensional nonparametric function included in
the single index model were estimated. We studied the asymptotic distribution and the rate of convergence of the proposed estimator. From these results, we can confirm that the single index model overcomes the curse of dimensionality. Simulation and real data application help describe the efficiency of the proposed model.

One of the most important future tasks is variable selection when the dimension of covariates \( p \) is quite large compared with sample size \( n \) or sample size exceeding threshold value. To the best of our knowledge, however, there is no result of sparse modelling or high-dimensional statistics in EVI regression. It would be interesting to investigate the hybrid method of high-dimensional statistics and extreme value theory.

In this paper, we focus on only positive EVI and use the Pareto-type-tailed distribution. The single index model can be extended to general EVI including negative \( \gamma \). Then, in such cases, we need to estimate not only EVI but also the scale function (see, de Haan and Ferreira 2006). Although the simultaneous estimation of the two target functions and establishing the asymptotic property of the estimator are quite difficult, it is very important to explore these aspects.

Appendix A: \( B \)-spline basis

We now describe the definition and the property of the \( B \)-spline basis. Let \( Z \) be a random variable with domain \([a,b]\). In this paper, we consider \( Z = X^T \theta \) for given \( \theta \in S^{p-1}_+ \). Again, we let \( \kappa = \{a = \kappa_0 < \kappa_1 < \ldots < \kappa_{K_0+1} = b\} \), \( K_0 > 1 \) be internal knots on an interval \([a,b]\). Furthermore, let \( \kappa_{-d+1} \leq \cdots \leq \kappa_{-1} < \kappa_0 \) and \( \kappa_{K_0+1} \leq \kappa_{K_0+2} \leq \cdots \leq \kappa_{K_0+d} \) be another set of knots. For \( j = -d + 1, \ldots, K_0 \) and \( Z = z \in [a,b] \), let

\[
\psi_j^{[0]}(z) = \begin{cases} 
1, & \kappa_j \leq z < \kappa_{j+1} \\
0, & \text{otherwise}
\end{cases}
\]

be the zero-degree \( B \)-spline basis. The \( d \)th degree \( B \)-spline basis can be defined recursively as

\[
\psi_j^{[d]}(z) = \frac{z - \kappa_j}{\kappa_{j+d-1} - \kappa_j} \psi_j^{[d-1]}(z) + \frac{\kappa_{j+d} - z}{\kappa_{j+d} - \kappa_{j+1}} \psi_{j+1}^{[d-1]}(z), \quad \forall z \in [a,b].
\]

For convenience, we treat \( 0/0 = 0 \). The scaled \( B \)-spline basis are defined as

\[
B_j^{[d]}(z) = \frac{\psi_j^{[d]}(z)}{\sqrt{V[\psi_j^{[d]}(Z)]}}, \quad j = -d + 1, \ldots, K_0.
\]

Since the part of \( \sqrt{V[\psi_j^{[d]}(Z)]} \) is scalar, the \( B_j^{[d]} \) and \( \psi_j^{[d]} \) have similar property as the function of \( z \in [a,b] \). The scaled \( B \)-spline basis is mathematically convenient than ordinary \( B \)-spline basis. Although the normalized \( B \)-splines (Liu et al. 2011) are also useful, but in our model, centerization of \( B \)-splines is meaningless, and hence, only scale is adjusted. In particular, as the property of the scaled \( B \)-spline bases, Lemma A.2 of Liu et al. (2011) is important in Appendix B–D below.

By the definition of \( B \)-spline basis, we find that the \( K = K_0 + d \) basis function is used. The \( d \)th degree \( B \)-spline is considered to be denoted by the \((d+1)\)th order of \( B \)-splines.

We see that the \( m \)th derivative of \( B \)-spline basis can be written by using \((d-m)\)th degree \( B \)-spline basis. Actually, we see that for \( m \geq 1 \),

\[
\frac{d^m B_j^{[d]}(x)^T b}{dx^m} = \frac{d^m}{dx^m} \sum_{j=1}^{K} B_j^{[d]}(x)b_j = \sum_{j=m+1}^{K} B_j^{[d-m]}(x)b_j^{(m)},
\]
where $b = (b_1, \ldots, b_K)^T \in \mathbb{R}^K$, 

$$b_j^{(1)} = \frac{b_j - b_{j-1}}{\kappa_j + d - \kappa_j}$$

and

$$b_j^{(m)} = (d + 1 - m) \frac{b_j^{(m-1)} - b_{j-1}^{(m-1)}}{\kappa_j + d + 1 - m - \kappa_j}.$$ 

This implies that the penalty term in (6) can be written as

$$\int \left[ \frac{d^m B^{\text{diff}}(x)^T b}{dx^m} \right]^2 dx = b^T \Delta_{m,K} b,$$

where $\Delta_{m,K} = D_{m,K}^T R_m D_{m,K}$. $R_m$ is the $(K - m)$th square matrix having $(i, j)$-entry

$$\int B_i^{[d-m]}(x) B_j^{[d-m]}(x) dx$$

and $D_{m,K}$ is the $(K - m) \times K$ matrix satisfying $b^{(m)} = (b_{m+1}^{(m)}, \ldots, b_K^{(m)})^T = D_{m,K} b$. If we use the equidistant knots $\kappa_j - \kappa_{j-1} = K^{-1}$, we obtain $D_m = K^{2m} D_{m,K}^{\text{diff}}$, where $D_{m,K}^{\text{diff}}$ is the $m$th difference order matrix, which is defined as $D_{m,K}^{\text{diff}} D_{m,K}^{\text{diff}} b = (b_2 - b_1, \ldots, b_K - b_{K-1})^T$ (see, Xiao 2019). Consequently, the penalty term has the quadratic form with respect to $b$.

**Appendix B: Lemmas and proofs of Theorems in Section 3.1**

We describe the technical lemmas and the proof of theorems in Section 3. In following, Lemmas 4–4 are used for results in Section 3.1. We next describe the proofs of Theorems 1–3.

First, we prepare the following symbols. Similar to $\alpha_0(\cdot) = \alpha_0(\cdot | \theta)$, we write $\gamma_0(\cdot) = \gamma_0(\cdot | \theta)$. For a square matrix $A$, let $\rho_{\min}(A)$ and $\rho_{\max}(A)$ be the minimum and maximum eigen value of $A$, respectively. For a vector $v = (v_1, \ldots, v_M)$, $M > 0$, $\|v\|_{\text{max}} = \max_{1 \leq i \leq M} |v_i|$. Furthermore, for simplicity, we write $P(Y > w_n(x)|X = x)$ as

$$P(Y > w_n(x)|X = x) = x^T \theta,$$

We note that $\theta \in \mathbb{S}_+^{p-1}$ is fixed in Section 3.1. For $s: \mathbb{R} \to \mathbb{R}$, let

$$L(s) = -E [\{\exp[s(X_0)] \log(Y/w_n(X)) - s(X_0)\} I(Y > w_n(X))] ,$$

where $X_0 = X^T \theta$ given in Section 3.1. We let

$$s_0 = \arg\min_{s \in \mathbb{S}(d,K)} |L(s) - L(\theta)|.$$ 

We now establish following lemmas.

**Lemma 1.** Suppose that (C1)–(C6). Then, as $K \to \infty$,

$$\sup_{x \in [a,b]} |\alpha_0(x) - s_0(x) - b_{s,n}(x)| = o(K^{-q}),$$

where $b_{s,n}(x)$ is that given in Section 3.1.
Proof of Lemma 1. From the property of \( B \)-spline, there exists the spline model \( s^* \) such that 
\[
\sup_{x \in [a, b]} |\alpha_0(x) - s^*(x)| \to 0 \quad \text{as} \quad K \to \infty \quad \text{(see, Barrow and Smith 1978).}
\]
By the definition of \( s_0 \), and continuously and convexity of \( L(\cdot) \), we have 
\[
|L(\alpha_0) - L(s_0)| \leq |L(\alpha_0) - L(s^*)| \to 0
\]
as \( K \to \infty \). This implies that \( \sup_{x \in \mathcal{X}} |\alpha_0(x) - s_0(x)| \to 0 \). Therefore, for some \( \eta \in (0, 1) \), 
\[
|L(s_0) - L(\alpha_0)| \leq |E [(\alpha_0(X_0) - s_0(X_0))\exp[\eta(\alpha_0(X_0) - s_0(X_0))] \exp[\alpha_0(X_0)] \log(Y/w_n(X)) - 1]I(Y > w_n(X))]|.
\]
From above, if we would like to minimize \( |L(s_0) - L(\alpha_0)|, |s_0(x) - \alpha_0(x)| \) should be small as much as possible. Meanwhile, under given \( X = x \) and \( Y > w_n(x) \), \( \exp[\alpha(x_0)] \log(Y/w_n(x)) \) is asymptotically distributed as standard exponential distribution. Therefore, in order to minimize \( |L(s_0) - L(\alpha_0)| \), \( s_0 \) needs to be determined such as \( \exp[\eta(s_0(\cdot) - \alpha_0(\cdot))] \) close to one. Consequently, we see that \( \alpha_0 \) is the best \( L_\infty \) \( B \)-spline approximation to \( \alpha \). This implies that \( s_0 \) is equal or very close to \( s^* \). Accordingly, from Zhou et al. (1998), we obtain
\[
||s_0 - \alpha_0 - b_n||_\infty = o(K^{-q}).
\]
This completes the proof. \( \square \)

For \( X_{0i} = X_i^T \theta (i = 1, \ldots, n) \), let
\[
\hat{U}_n(b) = \frac{1}{n} \sum_{i=1}^{n} B(X_{0i}) \left\{ \exp[B(X_{0i})^T b] \log \left( \frac{Y_i}{w_n(X_i)} \right) - 1 \right\} I(Y_i > w_n(X_i)) + \lambda \Delta_{m,K} b
\]
and let
\[
\tilde{U}_n(b) = \frac{1}{n} \sum_{i=1}^{n} B(X_{0i}) B(X_{0i})^T \exp[B(X_{0i})^T b] \log \left( \frac{Y_i}{w_n(X_i)} \right) I(Y_i > w_n(X_i)) + \lambda \Delta_{m,K}.
\]
Accordingly, we set \( \hat{U}(b) = E[\hat{U}_n(b)] \) and \( \tilde{U}(b) = E[\tilde{U}_n(b)] \).

Lemma 2. Suppose that (C1)–(C7). As \( n \to \infty \),
\[
\hat{U}_n(b_0) = \frac{1}{\sqrt{n}} \left\{ E[P(Y > w_n(X)|X)B(X_0)B(X_0)^T] \right\}^{1/2} \varepsilon + E[P(Y > w_n(X)|X)r_n(X)B(X_0)] + \lambda \Delta_{m,K} b_0,
\]
where \( \varepsilon \) is random variable with \( K \)-variate standard normal and \( \sup_{x \in \mathcal{X}} |r_n(x)| \leq O(K^{-q}) + O(r_n^{\beta_{\min}}) \).

Proof of Lemma 2. We note that
\[
\hat{U}_n(b_0) = \hat{U}_n(b_0) - E[\hat{U}_n(b_0)] + E[\hat{U}_n(b_0)].
\]
Here,
\[
E[\hat{U}_n(b_0)] = E \left\{ \exp[B(X_0)^T b_0] \log \left( \frac{Y}{w_n(X)} \right) - 1 \right\} I(Y > w_n(X))B(X_0) + \lambda \Delta_{m,K} b_0
\]
\[
= E[P(Y > w_n(X)|X)r_n(X)B(X_0)] + \lambda \Delta_{m,K} b_0,
\]

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where \( x_0 = x^T \theta \) and
\[
  r_n(x) = E \left[ \exp[B(x_0)^T b_0] \log \left( \frac{Y}{w_n(x)} \right) - 1 \right| X = x, Y > w_n(x) \].
\]
The above \( E[\dot{U}_n(b_0)] \) is the second term of right hand side of \( \dot{U} \) in Lemma 2. From the definition of he Pareto-type tailed model with Hall class, we have
\[
  E \left[ \frac{1}{\gamma_0(x_0)} \log \left( \frac{Y}{w_n(x)} \right) \right| X = x, Y > w_n(x) = \int_0^{\infty} P \left( \frac{1}{\gamma_0(x_0)} \log \left( \frac{Y}{w_n(x)} \right) > z \right| X = x, Y > w_n(x) \right) \, dz
\]
\[
= 1 - \frac{\ell_1(x)}{\ell_0(x)} \frac{\beta(x)}{\beta(x) + 1} w_n(x)^{-\beta(x)/\gamma(x)}(1 + o(1)) \tag{17}
\]
Lemma 1 shows that \( |B(x_0)^T b_0 - \alpha_0(x_0)| = O(K^{-q}) \). Therefore, we find that
\[
\sup_{x \in \mathcal{X}} |r_n(x)| \leq O(K^{-q}) + O(\tau_n^{\beta_{\min}}).
\]

We next show from Lyapnov’s central limit theorem and Cramér-Wold device that \( \dot{U}_n(b_0) - E[\dot{U}_n(b_0)] \) is asymptotically distributed as normal. It is sufficient to consider the leading term of \( \dot{U}_n(b_0) - E[\dot{U}_n(b_0)] \) since it has mean zero. For this, we use the fact that
\[
E_i = \frac{1}{\gamma_0(X_{0i})} \log \left( \frac{Y_i}{w_n(X_{0i})} \right)
\]
is asymptotically distributed as standard exponential distribution under given \( Y_i > w_n(x_i) \). For any vector \( u \in \mathbb{R}^K \), we consider
\[
(\dot{U}_n(b_0) - E[\dot{U}_n(b_0)])^T u = \sum_{i=1}^{n} (Z_i - E[Z_i]),
\]
where
\[
Z_i = \frac{1}{n} B(X_{0i})^T u (E_i - 1) I(Y_i > w_n(X_{0i}))(1 + o(1)).
\]
It is easy to show that \( \mathbb{V}[Z_i] = n^{-2} u^T E[P(Y > w_n(X)) B(X_0)] B(X_0)^T u (1 + o(1)) \). Condition (C5) and Lemma A.2 of Liu et al. (2011) implies that there exist constants \( c_1, c_2 > 0 \) such that
\[
c_1 \tau_n \leq u^T E[P(Y > w_n(X)) B(X_0)] B(X_0)^T u \leq c_2 \tau_n.
\]
Therefore, \( \sigma_n \equiv \sqrt{\sum_{i=1}^{n} \mathbb{V}[Z_i]} = O(n^{-1/2} \tau_n^{1/2}) \). Similarly, we have \( E||Z_i - E[Z_i]||^4 = n^{-4} E[P(Y > w_n(X)) (B(X)^T u)^4] \leq c_3 \tau_n n^{-4} \). Therefore, we have
\[
\sum_{i=1}^{n} E[||Z_i - E[Z_i]||^4] \leq O((n \tau_n)^{-1}) = o(1)
\]
as \( n \to \infty \). Thus, Lyapnov’s condition holds and hence,
\[
\sqrt{n}(\dot{U}_n(b_0) - E[\dot{U}_n(b_0)])^T u \overset{a.s.}{\sim} N(0, u^T E[P(Y > w_n(X)) B(X_0)] B(X_0)^T u).
\]
Here, for random variable \( A_n \) and \( B_n \), \( A_n \overset{a.s.}{\sim} B_n \) means that the distributions of \( A_n \) and \( B_n \) are asymptotically equivalent as \( n \to \infty \). Using Cramér-Wold device, we obtain
\[
\sqrt{n}E[P(Y > w_n(X)) B(X_0)] B(X_0)^T \dot{U}_n(b_0) - E[\dot{U}_n(b_0)] \overset{a.s.}{\sim} N_K(0, I_K),
\]
where \( I_K \) is the \( K \)-square identity matrix. This completes the proof. \( \square \)
Lemma 3. For any vector $\mathbf{v} \in \mathbb{R}^K$,
\[ \| \dot{U}(\mathbf{b}_0)^{-1} \mathbf{v} \|_{\max} \leq C \frac{\| \mathbf{v} \|_{\max}}{\tau_n} \]
for some constant $C > 0$. Furthermore, $\| (\dot{U}_n(\mathbf{b}_0)^{-1} - \dot{U}(\mathbf{b}_0)^{-1}) \mathbf{v} \|_{\max} = o(\| \mathbf{v} \|_{\max} \tau_n^{-1})$

Proof of Lemma 3

Let
\[ G_n(\mathbf{b}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{B}(X_{0i})\mathbf{B}(X_{0i})^T \exp[\mathbf{B}(X_{0i})^T \mathbf{b}] \log \left( \frac{Y_i}{w_n(X_i)} \right) I(Y_i > w_n(X_i)) \]
and $G(\mathbf{b}) = E[G_n(\mathbf{b})]$. Then, $\dot{U}(\mathbf{b}) = G(\mathbf{b}) + \lambda \Delta_{m,K}$. We have

\[
G(\mathbf{b}) = E \left[ \mathbf{B}(X_0)\mathbf{B}(X_0)^T \exp[\mathbf{B}(X_0)^T \mathbf{b}] \log \left( \frac{Y}{w_n(X)} \right) I(Y > w_n(X)) \right]
\]

\[
= E \left[ P(Y > w_n(X)|X) \mathbf{B}(X_0)\mathbf{B}(X_0)^T E \left[ \exp[\mathbf{B}(X_0)^T \mathbf{b}] \log \left( \frac{Y}{w_n(X)} \right) \bigg| Y > w_n(X), X \right] \right].
\]

Similar to [17], we have

\[
E \left[ \exp[\mathbf{B}(X_0)^T \mathbf{b}] \log \left( \frac{Y}{w_n(X)} \right) \bigg| Y > w_n(X), X \right] = 1 + o(1).
\]
Thus, for any vector $\mathbf{u} \in \mathbb{R}^K$, we obtain

\[
\mathbf{u}^T G(\mathbf{b}_0) \mathbf{u} = \mathbf{u}^T E \left[ P(Y > w_n(X)|X) \mathbf{B}(X_0)\mathbf{B}(X_0)^T \right] \mathbf{u}(1 + o(1)).
\]

Together with (C5) and Lemma A.2 of Liu et al. (2011), $\rho_{\min}(G(\mathbf{b}_0)) \geq c_1 \tau_n$ and $\rho_{\max}(G(\mathbf{b}_0)) \leq c_2 \tau_n$ with $c_1, c_2 > 0$. Here, we have

\[
\dot{U}(\mathbf{b}_0) = G(\mathbf{b}_0)^{1/2}(I + \lambda G(\mathbf{b}_0)^{-1/2} \Delta_{m,K} G(\mathbf{b}_0)^{-1/2})G(\mathbf{b}_0)^{1/2}
\]

From Proposition 4.2 of Xiao (2019), we have

\[
\rho_{\max}(I + \lambda G(\mathbf{b}_0)^{-1/2} \Delta_{m,K} G(\mathbf{b}_0)^{-1/2}) \leq 1 + c \frac{\lambda}{\tau_n} K^{2m}
\]

for some constant $c > 0$, and

\[
\rho_{\min}(I + \lambda G(\mathbf{b}_0)^{-1/2} \Delta_{m,K} G(\mathbf{b}_0)^{-1/2}) \geq 1.
\]

Therefore, we obtain $\rho_{\min}(\dot{U}(\mathbf{b})) > c_1 \tau_n$ and

\[
\rho_{\max}(\dot{U}(\mathbf{b})) < c_2 \tau_n (1 + c \lambda \tau_n^{-1} K^{2m}).
\]

Under the condition (C6), we have

\[
\frac{\rho_{\max}(\dot{U}(\mathbf{b}))}{\rho_{\min}(\dot{U}(\mathbf{b}))} \leq c_3 (1 + c \lambda \tau_n^{-1} K^{2m}) \leq c_4.
\]
for some positive constants $c_1, c_2, c_3$ and $c_4$. This and Theorem 2.2 of Demko (1977) yield that the $(i, j)$-element of $\bar{U}(b)^{-1}$ has

$$||\bar{U}(b)^{-1})_{ij}|| \leq \rho_{\text{max}}(\bar{U}(b)^{-1})Lt_{\|i-j\|}$$

for some $r \in (0, 1)$ and $L > 0$. For any square matrix $A$, $\rho_{\text{max}}(A^{-1}) = \rho_{\text{min}}(A)^{-1}$. This implies that

$$||\bar{U}(b)^{-1})_{ij}|| \leq \frac{L}{\tau_n}r_{\|i-j\|}$$

for some constant $L > 0$. Since $1 + r + r^2 + \cdots + r^K \leq (1 - r)^{-1}$, for any $v \in \mathbb{R}_K$,

$$||\bar{U}(b)^{-1}v||_{\text{max}} \leq C \frac{2}{1 - r}||v||_{\text{max}} \frac{1}{\tau_n}$$

for some constant $C > 0$.

We next evaluate

$$||\bar{U}_n(b)^{-1} - \bar{U}(b)^{-1})v||_{\text{max}} = ||\bar{U}(b)^{-1}(\bar{U}_n(b) - \bar{U}(b))\bar{U}_n(b)^{-1}v||_{\text{max}}.$$ First, we show

$$||\bar{U}_n(b_0) - \bar{U}(b_0)||_{\text{max}} = ||G_n(b_0) - G(b_0)||_{\text{max}} = o(\tau_n).$$

The $(j, k)$-element of $G_n(b_0) - G(b_0)$ is

$$G_{n, ij} = \sum_{i=1}^{n} \frac{1}{n} \{Z_i B_j(X_{0i}) B_k(X_{0i}) - E[Z_i B_j(X_{0i}) B_k(X_{0i})]\}$$

for $|j - k| \leq d$ and $G_{n, ij} = 0$ otherwise, where

$$Z_i = \exp[B(X_{0i})^T b_0] \log \left( \frac{Y_i}{w_n(X_i)} \right) I(Y_i > w_n(X_i)).$$

We note again that under $Y_i > w_n(X_i)$, $E_i = \gamma_0(X_{0i})^{-1} \log(Y_i/w_n(X_i))$ is approximately distributed as standard exponential distribution. Let $\mathcal{M} = \{\max_i E_i < M\}$ with some constant $M > 0$ and let

$$H_i(A) = \frac{1}{n} \{Z_i B_j(X_{0i}) B_k(X_{0i}) I(A) - E[Z_i B_j(X_{0i}) B_k(X_{0i}) I(A)]\}$$

for some set $A$. Then, we obtain $G_{n, ij} = \sum_{i=1}^{n} H_i(\mathcal{M}) + H_i(\mathcal{M}^c)$. Furthermore, we have $P(\mathcal{M}^c) = ne^{-M}$. This yields that

$$\sum_{i=1}^{n} H_i(\mathcal{M}^c) = O(n \sqrt{V[H_i(\mathcal{M}^c)]}) = O(n \tau_n e^{-M})$$

In later, we choose $M$ so that $O(ne^{-M}) = o(1)$. Therefore, it is sufficient to consider

$$G_{n, ij} = \sum_{i=1}^{n} H_i(\mathcal{M}) + o_P(\tau_n).$$
On the event $\mathcal{M}$, we see that $\max_i |H_i(\mathcal{M})| \leq c_1 n^{-1} M$ for some constant $c_1 > 0$. In addition, we have $V[H_i(\mathcal{M})] \leq c_2 n^{-2} \tau_n$ for some constant $c_2 > 0$. Therefore, Bernstein’s inequality examines that

$$P \left( \left| \sum_{i=1}^{n} H_i(\mathcal{M}) \right| > \delta \right) \leq \exp \left( -C \frac{n \tau_n^{-1} \delta^2}{1 + \delta \tau_n^{-1} M} \right)$$

for some constants $C > 0$. Putting $\delta = \tau_n \varepsilon_n$ and $M = 1/\varepsilon_n$ with $\varepsilon_n \to 0$, $\varepsilon_n \log(n) \to 0$ and $n \tau_n \varepsilon_n \to \infty$, we obtain

$$P \left( \left| \sum_{i=1}^{n} H_i(\mathcal{M}) \right| > \tau_n \varepsilon_n \right) \leq \exp \left[ -C n \tau_n \varepsilon_n \right] \to 0.$$

This means that (19) holds.

Similar to the proof of (18), $(j,k)$-element of $\hat{U}_n(b)^{-1}$ satisfies

$$|(\hat{U}_n(b)^{-1})_{jk}| \leq \frac{1}{\tau_n} Lq^{j-k}$$

for some $q \in (0,1)$ and $L > 0$. Thus, the property of inverse of band matrix examines $\|\hat{U}(b)^{-1}v\|_{\max} = \|v\|_{\max} O(1/\tau_n)$. Next, using (19) and the fact that $\hat{U}_n(b_0) - \hat{U}(b_0)$ is band matrix, $\|\{\hat{U}_n(b_0) - \hat{U}(b_0)\}\hat{U}(b)^{-1}v\|_{\max} = \|v\|_{\max} O(1)$. Finally, the inverse of band matrix property of $\hat{U}_n(b)^{-1}$ achieves

$$\|\hat{U}_n(b)(\hat{U}_n(b_0) - \hat{U}(b_0))\hat{U}(b)v\|_{\max} = \|v\|_{\max} O(\tau_n^{-1}),$$

which completes the proof. \(\square\)

In Lemma 4, we write $\hat{b}$ as $\hat{b}(\theta_0)$ defined in (7).

**Lemma 4.** Suppose that (C1)–(C7). For any $\delta_n > 0$ satisfying $\delta_n \to 0$ and

$$\frac{1}{\delta_n} \max \left\{ \{n \tau_n\}^{-1/4}, \sqrt{\frac{K \log(n)}{n \tau_n}}, \sqrt{\frac{\lambda}{n \tau_n}}, \tau_n^{\beta_{\min}} \right\} \to 0$$

as $n \to \infty$,

$$P(\|\hat{b} - b_0\| > \delta_n) \to 0.$$

**Proof of Lemma 4.** Let

$$L_0(b) = \mathbb{E} \left\{ \exp[B(X_0)^T b] \log \left( \frac{Y}{w_n(X)} \right) - B(X_0)^T b \right\} I(Y > w_n(X))$$

and

$$L(b) = U_n(b, \theta_0|\lambda)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left\{ \exp[B(X_0)^T b] \log \left( \frac{Y_i}{w_n(X_i)} \right) - B(X_0)^T b \right\} I(Y_i > w_n(X_i))$$

$$+ \frac{\lambda}{2} \int_a^b \left\{ \frac{d}{dx} B(x)^T b \right\}^2 dx.$$
We note that $L$ and $L_0$ are strictly convex functions. Therefore, $b_0 = \text{argmin}_b L_0(b)$ and $\hat{b} = \text{argmin}_b L(b)$. From Lemma 2 of Hijort and Pollard (1993), for $\delta_n > 0$,

$$P(||\hat{b} - b_0|| > \delta_n) \leq P \left( \sup_{||b - b_0|| \leq \delta_n} |L(b) - L_0(b)| \geq 2^{-1} \inf_{||b - b_0|| = \delta_n} |L_0(b) - L_0(b_0)| \right).$$

We now consider the vector $b \in \mathbb{R}^K$ satisfying $||b - b_0|| = \delta_n$. When $||b - b_0|| = \delta_n$, we can write $b = b_0 + \delta_n u, u \in \mathbb{R}^K$, where $||u|| = 1$. By the straightforward calculation, for some $\theta \in (0,1)$,

$$L_0(b) - L_0(b_0) = \delta_n E \left[ B(X_0)^T u \left\{ \exp[B(X_0)^T b_0] \log \left( \frac{Y}{w_n(X)} \right) - 1 \right\} I(Y > w_n(X)) \right] + \frac{\delta_n^2}{2} u^T E \left[ B(X_0) B(X_0)^T \exp[\theta \delta_n B(X_0)^T u] \exp[B(X_0)^T b_0] \log \left( \frac{Y}{w_n(X)} \right) I(Y > w_n(X)) \right] u.$$

We note that $B(x)^T u \neq 0$ for any $u$ with $||u|| = 1$. Furthermore, for any positive function $v(x) > 0$, the minimum eigen-value of $E[v(X_0)B(X_0)B(X_0)^T]$ has positive lower bound with $O(1)$ (see, Lemma A.2 of Liu et al. 2011). Together with (17) and $|B(X_0)^T b_0 - \alpha_0(X_0)| = O(K^{-q})$, we have

$$|L_0(b) - L_0(b_0)| \geq c^* \delta_n \tau_n \left( \frac{1}{K^q} + \delta_n \right) \geq 2c^* \delta_n \tau_n$$

for some constant $c^* \in (0,1)$. This $c^*$ is used later. Accordingly, we obtain

$$P(||\hat{b} - b_0|| > \delta_n) \leq P \left( \sup_{||b - b_0|| \leq \delta_n} |L(b) - L_0(b)| \geq 2^{-1} \inf_{||b - b_0|| = \delta_n} |L_0(b) - L_0(b_0)| \right) \leq P \left( \sup_{||b - b_0|| \leq \delta_n} |L(b) - L_0(b)| \geq c^* \delta_n^2 \tau_n \right).$$

For simplicity, we put $\zeta_n = c^* \delta_n^2 \tau_n$. Then, our purpose is to show

$$P \left( \sup_{||b - b_0|| \leq \delta_n} |L(b) - L_0(b)| \geq \zeta_n \right) \to 0. \quad (20)$$

Again, we consider $b = b_0 + \delta_n u, u \in \mathbb{R}^K$, where $||u|| \leq 1$. We then obtain

$$P \left( \sup_{||b - b_0|| \leq \delta_n} |L(b) - L_0(b)| \geq \zeta_n \right) \leq P \left( |L(b_0) - L_0(b_0)| \geq 2^{-1} \zeta_n \right) + P \left( \sup_{||b - b_0|| \leq \delta_n} |L(b) - L_0(b) - L_0(b_0)| \geq 2^{-1} \zeta_n \right) \equiv J_1 + J_2.$$

We evaluate $J_1$. Define

$$\ell(Y, X|b) = \left\{ \exp[B(X_0)^T b] \log \left( \frac{Y}{w_n(X)} \right) - B(X_0)^T b \right\} I(Y > w_n(X)).$$
Since \( \sup_{x \in [a,b]} |\alpha_0^{(m)}(x) - (d^m/dx^m)B(x)^T b_0| = O(K^{-(q-m)}) \) (see, Barrow and Smith 1978), we obtain
\[
L(b_0) - L_0(b_0) = \frac{1}{n} \sum_{i=1}^{n} \ell(Y_i, X_i | b_0) - E[\ell(Y_i, X_i | b_0)] + \lambda \int_a^b \{\alpha_0^{(m)}(x)\}^2 dx (1 + o(1)).
\]

Under the condition of Lemma 4, we have
\[
\frac{\lambda}{\zeta_n} \to 0.
\]
Therefore, it is sufficient to show
\[
P \left( \left| n^{-1} \sum_{i=1}^{n} \ell(Y_i, X_i | b_0) - E[\ell(Y_i, X_i | b_0)] \right| > 2^{-1} \zeta_n \right) \to 0.
\]

We let \( E_i = \gamma(X_{0i})^{-1} \log(Y_i/w_n(X_i)) \) under given \( Y_i > w_n(X_i) \). As we noted in the proof of Lemma 3, \( E_i \) is asymptotically distributed as standard exponential. We set \( M = \{\max_i |E_i| < M\} \) with \( M = \delta_n^{-2}. \) Since
\[
\ell(Y_i, X_i | b_0) - E[\ell(Y_i, X_i | b_0)] = \ell(Y_i, X_i | b_0)I(M) - E[\ell(Y_i, X_i | b_0)I(M)] + \ell(Y_i, X_i | b_0)I(M^c) - E[\ell(Y_i, X_i | b_0)I(M^c)]
\]
and
\[
\frac{1}{n} \sum_{i=1}^{n} \ell(Y_i, X_i | b_0)I(M^c) - E[\ell(Y_i, X_i | b_0)I(M^c)] = O(n\tau_ne^{-M}) = o(1)
\]
Next, we show that
\[
P \left( \left| n^{-1} \sum_{i=1}^{n} \ell(Y_i, X_i | b_0)I(M) - E[\ell(Y_i, X_i | b_0)I(M)] \right| > 2^{-1} \zeta_n \right) \to 0.
\]

We can then easily find that \( V[\ell(Y_i, X_i | b_0)I(M)] \leq c\tau_n \) for some constant \( c > 0 \). Furthermore, if \( \max_i |E_i| < M \) for a constant \( M > 0 \), we have \( \max |\ell(Y_i, X_i | b_0) - E[\ell(Y, X | b_0)]| < c_0 M \) with additional constant \( c_0 > 0 \). Therefore, using Bernstein’s inequality, we obtain that for some constant \( C^* > 0 \),
\[
P \left( n^{-1} \sum_{i=1}^{n} \ell(Y_i, X_i | b_0) - E[\ell(Y, X | b_0)] > 2^{-1} \zeta_n \right)
\]
\[
\leq c \exp \left[ -C^* n \zeta_n^2 \tau_n \right]
\]
\[
\leq c \exp \left[ -C^* n \tau_n \delta_n^2 \right]
\]
\[
\to 0
\]
from the condition \( \delta_n \{n\tau_n\}^{1/4} \to \infty. \)

Next, we focus on \( J_2. \) The Taylor expansion yields that for some \( \eta \in (0,1), \)
\[
L(b) - L(b_0)
\]
\[
= \delta_n \frac{1}{n} \sum_{i=1}^{n} B(X_{0i})^T u \left\{ \exp[\eta \delta_n B(X_{0i})^T u] \exp[B(X_{0i})^T b_0] \log \left( \frac{Y_i}{w_n(X_i)} \right) - 1 \right\} I(Y_i > w_n(X_i))
\]
\[
+ \lambda \left\{ \delta_n u^T \Delta_{m,K} b_0 + \delta_n^2 u^T \Delta_{m,K} u \right\}
\]
\[
= \delta_n \frac{1}{n} \sum_{i=1}^{n} \ell_1(X_i, Y_i | u) + \lambda \left[ \int_a^b \left\{ \frac{d^m}{dx^m} B^{[d]}(x)^T \{b_0 + \delta_n u\} \right\}^2 dx - \int_a^b \left\{ \frac{d^m}{dx^m} B^{[d]}(x)^T b_0 \right\}^2 dx \right],
\]

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where
\[ \ell_1(X, Y | u) = B(X_0)^T u \left\{ \exp[\eta \delta_n B(X_0)^T u] \exp[B(X_0)^T b_0] \log \left( \frac{Y}{w_n(X)} \right) - 1 \right\} I(Y > w_n(X)). \]

Under \( \delta_n \to 0 \), we obtain
\[ \lambda \left[ \int_a^b \left\{ \frac{d^n}{dx^n} B[\delta](x)^T \{b_0 + \delta_n u\} \right\}^2 \, dx - \int_a^b \left\{ \frac{d^n}{dx^n} B[\delta](x)^T b_0 \right\}^2 \, dx \right] = O(\lambda \delta_n). \]

This yields that
\[ \sup_{||u|| < 1} \left| L(b) - L(b_0) - L_0(b) + L_0(b_0) \right| \leq \delta_n \sup_{||u|| < 1} \left| \frac{1}{n} \sum_{i=1}^n \{\ell_1(X_i, Y_i | u) - E[\ell_1(X, Y | u)]\} \right| + O(\lambda \delta_n). \]

Under (C7), we have \( \lambda \delta_n / \zeta_n \to 0 \). Thus, the remaining proof is to show
\[ P \left( \sup_{||u|| < 1} \left| \frac{1}{n} \sum_{i=1}^n \{\ell_1(X_i, Y_i | u) - E[\ell_1(X, Y | u)]\} \right| \geq 2^{-1} \zeta_n \delta_n^{-1} \right) \to 0. \]

Let \( \mathcal{M}_1 = \{\max_i E_i \leq M_1\} \) with constant \( M_1 > 0 \). We then obtain
\[ \ell_1(X_i, Y_i | u) - E[\ell_1(X, Y | u)] = \ell_1(X_i, Y_i | u) I(M_1) - E[\ell_1(X, Y | u) I(M_1)] \]
\[ + \ell_1(X_i, Y_i | u) I(M_1^c) - E[\ell_1(X, Y | u) I(M_1^c)] \]

and it is easy to show that
\[ \frac{1}{n} \sum_{i=1}^n \ell_1(X_i, Y_i | u) I(M_1^c) - E[\ell_1(X, Y | u) I(M_1^c)] \leq O(\tau_n \exp[-M_1]) = o(\xi_n \delta_n) \]

if we put \( M_1 = 1 / \delta_n \). Next, we show
\[ P \left( \sup_{||u|| < 1} \left| \frac{1}{n} \sum_{i=1}^n \{\ell_1(X_i, Y_i | u) I(M_1) - E[\ell_1(X, Y | u) I(M_1)]\} \right| \geq 2^{-1} \zeta_n \delta_n^{-1} \right) \to 0. \]

Let \( \mathcal{U} = \{u \in \mathbb{R}^K : ||u|| < 1\} \) be the vector space and \( \mathcal{U}_1, \ldots, \mathcal{U}_N \) be a covering of \( \mathcal{U} \) with the diameter \( R_n = C \zeta_n / (4 \delta_n n^\nu) \) for some constant \( C > 0 \) and \( \nu > 0 \). That is, \( \mathcal{U} \subseteq \bigcup_{i=1}^N \mathcal{U}_i \). Then, Lemma 2.5 of van de Geer (2000) yields that it is sufficient to set \( N \leq C(n^\nu \delta_n / \zeta_n)^K \). Let \( u_k \in \mathcal{U}_k, k = 1, \ldots, N \). Then, we have
\[ P \left( \sup_{||u|| < 1} \left| \frac{1}{n} \sum_{i=1}^n \{\ell_1(X_i, Y_i | u) I(M_1) - E[\ell_1(X, Y | u) I(M_1)]\} \right| \geq 2^{-1} \zeta_n \delta_n^{-1} \right) \leq \sum_{k=1}^N P \left( \frac{1}{n} \sum_{i=1}^n \{\ell_1(X_i, Y_i | u_k) I(M_1) - E[\ell_1(X, Y | u_k) I(M_1)]\} \right) \]
\[ + \sup_{||u|| \in \mathcal{U}_k} \left| \frac{1}{n} \sum_{i=1}^n \ell_2(X_i, Y_i | u, u_k) \right| \geq 2^{-1} \zeta_n \delta_n^{-1} \]
\[ \leq \sum_{k=1}^N P \left( \frac{1}{n} \sum_{i=1}^n \{\ell_1(X_i, Y_i | u_k) I(M_1) - E[\ell_1(X, Y | u_k) I(M_1)]\} \right) \]
\[ + \sup_{||u|| \in \mathcal{U}_k} \left| \frac{1}{n} \sum_{i=1}^n \ell_2(X_i, Y_i | u, u_k) \right| \geq 2^{-1} \zeta_n \delta_n^{-1} \]
where
\[ \ell_2(X_i, Y_i | u, u_k) = \ell_1(X_i, Y_i | u)I(M_1) - E[\ell_1(X, Y | u)I(M_1)] - \{\ell_1(X_i, Y_i | u_k)I(M_1) - E[\ell_1(X, Y | u_k)I(M_1)]\}, \quad u \in U \]

We evaluate
\[ \sup_{\|u\| \in U_k} \left| \frac{1}{n} \sum_{i=1}^n \ell_2(X_i, Y_i | u, u_k) \right| . \]

By the Taylor expansion, there exists some constant \( c \in (0, 1) \) such that
\[
\ell_1(X_i, Y_i | u) - \ell_1(X_i, Y_i | u_k) = B(X_0)_1^T(u - u_k) \left\{ \exp[\eta \delta_n B(X_0)^T u_k] \exp[B(X_0)^T b_0] \log \left( \frac{Y_i}{w_n(X_i)} \right) - 1 \right\} I(Y_i > w_n(X_i)) \\
+ \eta \delta_n (u - u_k)^T B(X_0) B(X_0)^T u_k \\
\times \exp[\theta \delta_n B(X_0)^T u_k] \exp[\phi \delta_n B(X_0)^T u_k] \exp[B(X_0)^T b_0] \log \left( \frac{Y_i}{w_n(X_i)} \right) I(Y_i > w_n(X_i)).
\]

We see that under \( \delta_n \to 0 \),
\[
\exp[\theta \delta_n B(X_0)^T u_k] \exp[\phi \delta_n B(X_0)^T u_k] \exp[B(X_0)^T b_0] \log \left( \frac{Y_i}{w_n(X_i)} \right)
\]
is approximately distributed as \( E_i \). Therefore, under \( \max_i E_i \leq M_1 \), for some constant \( C \geq 0 \), we obtain
\[
\left| \frac{1}{n} \sum_{i=1}^n \ell_2(X_i, Y_i | u, u_k) \right| \leq 2CM_1 \delta_n R_n M_1 \leq C \delta_n M_1 \delta_n \tau_n
\]
and \( M_1 \delta_n \tau_n / n^\nu \to 0 \). This yields that
\[
\sup_{\|u\| \in U_k} \left| \frac{1}{n} \sum_{i=1}^n \ell_2(X_i, Y_i | u, u_k) \right| < 2^{-1} C \delta_n \tau_n.
\]

Lastly, we evaluate
\[
\sum_{k=1}^N P \left( \left| \frac{1}{n} \sum_{i=1}^n \{\ell_1(X_i, Y_i | u_k)I(M_1) - E[\ell_1(X, Y | u_k)I(M_1)]\} \right| \geq 2^{-1} \zeta_n \delta_n^{-1} \right) \\
= \sum_{k=1}^N P \left( \left| \frac{1}{n \tau_n} \sum_{i=1}^n \{\ell_1(X_i, Y_i | u_k)I(M_1) - E[\ell_1(X, Y | u_k)I(M_1)]\} \right| \geq c \delta_n \right) \\
\to 0
\]
for some constant \( c > 0 \). Under \( \max_i E_i \leq M_1 \), it is easy to find that \( \max_i \ell_1(X_i, Y_i | u_k) \leq c_0 \delta_n M_1 \) for some constant \( c > 0 \). Furthermore, we obtain \( V[\ell_1(X_i, Y_i | u_k)] \leq c_0 \delta_n^2 \tau_n \). This and Bernstein’s inequality yield that
\[
P \left( \left| \frac{1}{n \tau_n} \sum_{i=1}^n \{\ell_1(X_i, Y_i | u_k)I(M_1) - E[\ell_1(X, Y | u_k)I(M_1)]\} \right| \geq c \delta_n \right) \\
\leq C \exp \left[ -C \frac{\delta_n^2}{1/(n \tau_n) + M_1 \delta_n / (n \tau_n)} \right] \\
\leq C \exp \left[ -C \frac{\delta_n^2}{n \tau_n} \right].
\]
for some constant $C > 0$. Accordingly, we obtain

$$
\sum_{k=1}^{N} P \left( \frac{1}{n\tau_n} \sum_{i=1}^{n} \{ \ell_1(X_i, Y_i| u_k) I(\mathcal{M}_1) - E[\ell_1(X, Y| u_k) I(\mathcal{M}_1)] \} \geq c\delta_n \right) \\
\leq C \exp \left[ -Cn\tau_n\delta_n^2 + O(K \log n) \right] \\
\to 0
$$

from the condition $\delta_n \sqrt{n\tau_n/K \log(n)} \to \infty$. Consequently, $P(||\hat{\mathbf{b}} - \mathbf{b}_0|| > \delta_n) \to 0$ holds.

**Proof of Theorem 4.** From Lemma 4, we have $||\hat{\mathbf{b}} - \mathbf{b}_0|| \overset{P}{\to} 0$ as $n \to \infty$. This guarantees that

$$
0 = \mathbb{U}_n(\hat{\mathbf{b}}) = \hat{\mathbb{U}}_n(\mathbf{b}_0) = \hat{\mathbb{U}}_n(\mathbf{b}_0 + \Omega(\hat{\mathbf{b}} - \mathbf{b}_0))(\hat{\mathbf{b}} - \mathbf{b}_0) \\
\text{(21)}
$$

with $\Omega = \text{diag}(\omega_1, \ldots, \omega_K), \omega_i \in (0, 1)$. The continuity of $\hat{\mathbb{U}}_n$ implies that for any vector $\mathbf{v} \in \mathbb{R}^K, \||\hat{\mathbb{U}}_n(\mathbf{b}_0 + \Omega(\hat{\mathbf{b}} - \mathbf{b}_0)) - \hat{\mathbb{U}}_n(\mathbf{b}_0)\|| = o(||\hat{\mathbb{U}}_n(\mathbf{b}_0)\||)$. In addition, Lemmas 1 and 2 yield that

$$
\hat{\sigma}(x) - \sigma_0(x) = \hat{\sigma}(x) - s_0(x) + b_{s,n}(x) + o(K^{-q})
$$

and

$$
\hat{\sigma}(x) - s_0(x) = \mathbf{B}(x)^T \hat{\mathbb{U}}(\mathbf{b}) = \mathbf{B}(x)^T \hat{\mathbb{U}}_n(\mathbf{b}_0) = \mathbf{B}(x)^T \hat{\mathbb{U}}_n(\mathbf{b}_0)(1 + o_P(1)) \\
= \mathbf{B}(x)^T \hat{\mathbb{U}}_n(\mathbf{b}_0) - 1 \mathbb{E}[Y > w_n(X)] \mathbb{B}(X_0) \mathbb{B}(x) \mathbb{B}(x)^T / \sqrt{n} \varepsilon(1 + o_P(1)) \\
+ \mathbf{B}(x)^T \hat{\mathbb{U}}_n(\mathbf{b}_0) - 1 \mathbb{E}[Y > w_n(X)] \mathbb{B}(X_0)^T (1 + o_P(1)) \\
+ \lambda \mathbf{B}(x)^T \hat{\mathbb{U}}_n(\mathbf{b}_0) - 1 \delta_{n, K} \mathbf{b}_0 (1 + o_P(1)) \\
= \mathbf{v}_n(x)^T \varepsilon(1 + o_P(1)) + b_{\beta,n}(x)(1 + o_P(1)) + b_{\lambda,n}(x)(1 + o_P(1)). \\
\text{(22)}
$$

We have from Lemmas 2 and 3 that

$$
|b_{\beta,n}(x)| \leq O(\tau_n)^{\beta_{\min}}.
$$

Next, from the asymptotic order of scaled $B$-spline, Proposition 4.3 of Xiao (2019), Lemmas 4 and Condition (C7), we have

$$
|b_{\lambda,n}(x)| \leq O(K^{1/2} O(1/\tau_n) O(\lambda K^{m-1/2}) = O(\lambda K^{m}/\tau_n) = O \left( \frac{\lambda}{\tau_n} \right)^{1/2}.
$$

Finally, Condition (C7), property of $B$-spline and the repeated use of Lemma 4 yield that

$$
||\mathbf{v}_n(x)||^2 = \frac{1}{n} \mathbf{B}(x)^T \hat{\mathbb{U}}(\mathbf{b}_0) - 1 G(\mathbf{b}_0) \hat{\mathbb{U}}(\mathbf{b}_0) - 1 \mathbf{B}(x) \\
= O(K^{1/2} n^{-1} ||\hat{\mathbb{U}}(\mathbf{b}_0) - 1 G(\mathbf{b}_0) \hat{\mathbb{U}}(\mathbf{b}_0) - 1 \mathbf{B}(x)||_{\text{max}}) \\
= O(K^{1/2} / \{n\tau_n\}) O(||G(\mathbf{b}_0) \hat{\mathbb{U}}(\mathbf{b}_0) - 1 \mathbf{B}(x)||_{\text{max}}) \\
= O(K^{1/2} / \{n\tau_n\}) O(||\hat{\mathbb{U}}(\mathbf{b}_0) - 1 \mathbf{B}(x)||_{\text{max}}) \\
= O \left( \frac{K}{n\tau_n} \right) \\
= O \left( \frac{\lambda}{\tau_n} \right)^{-1/2m} \frac{1}{n\tau_n},
$$

which implies that Theorem 4 holds.
Proof of Theorem 2. Recall that \( X_0 = X^T \theta \), where \( \theta \) is the known single index parameter vector. From Lemma 1, we obtain
\[
E \left[ |\hat{\alpha}(X_0) - \alpha_0(X_0)|^2 \right] \leq O(K^{-2q}) + E \left[ |\hat{\alpha}(X_0) - s_0(X_0)|^2 \right] .
\]
Furthermore, scaled B-spline’s property (see, de Boor 2001) yields that
\[
E \left[ |\hat{\alpha}(X_0) - s_0(X_0)|^2 \right] \leq cE[||\hat{b} - b_0||^2]
\]
for some constant \( c > 0 \). From Lemma 2 and (21), we have \( \rho_{max}(\bar{U}(b_0) - 1 G(b_0)\bar{U}(b_0)^{-1}) \leq O(1/\tau_n) \). Therefore, we obtain
\[
\frac{1}{n} \text{trace}(\bar{U}(b_0))^{-1} G(b_0)\bar{U}(b_0)^{-1} \leq O \left( \frac{K}{\tau_n} \right) = O \left( \frac{\lambda}{\tau_n} \right)^{-1/2m} \frac{1}{n\tau_n} .
\]
Similar to the proof of Theorem 1 with constant \( c > 0 \),
\[
\lambda^2 ||\bar{U}(b_0)^{-1} \Delta_{m,K} b_0||^2 \leq c\lambda K ||\bar{U}(b_0)^{-1} \Delta_{m,K} b_0||_{max} \leq O \left( K^2 \lambda^2 K^{-2m-1} \right) = O \left( \frac{\lambda}{\tau_n} \right)
\]
and
\[
||\bar{U}(b_0)^{-1} E [P(Y > w_n(X)|X) r_n(X) B(X_0)] ||^2 \leq c_3 \frac{2\beta_{\min}}{\tau_n}.
\]
Consequently, under Condition (C7), we have
\[
E \left[ |\hat{\alpha}(X_0) - \alpha_0(X_0)|^2 \right] \leq O \left( \frac{\lambda}{\tau_n} \right)^{-1/2m} \frac{1}{n\tau_n} + O(K^{-2q}) + O \left( \frac{\lambda}{\tau_n} \right) + O \left( \frac{2\beta_{\min}}{\tau_n} \right) .
\]

Proof of Theorem 3. Similar to proofs of the Theorems 1 and 2
\[
\sup_{x \in [a,b]} |\hat{\alpha}(x) - \alpha_0(x)| \leq \sup_{x \in [a,b]} |B(x)^T \bar{U}_n(b_0) - E[\bar{U}_n(b_0)]|(1 + o(1))
\]
\[
+ O(K^{-q}) + O \left( \frac{\lambda_n^{1/2}}{\tau_n} \right) + O \left( \frac{2\beta_{\min}}{\tau_n} \right) .
\]

We now evaluate \( Z_n = \sup_{x \in [a,b]} |Z_n(x)| \), where
\[
Z_n(x) = B(x)^T \bar{U}_n(b_0)^{-1} \{ \bar{U}_n(b_0) - E[\bar{U}_n(b_0)] \}.
\]
We note that for any \( x \in [a,b] \), there exists \( j^* \) such that \( x \in [\kappa_j, \kappa_{j+1}) \). Therefore, we obtain
\[
\sup_{x \in [a,b]} |Z_n(x)| \leq \max_j \sup_{x \in [\kappa_j, \kappa_{j+1}]} |Z_n(x)|
\]
\[
\leq \max_j \sup_{x \in [\kappa_j, \kappa_{j+1}]} |Z_n(x) - Z_n(\kappa_j)| + \max_j |Z_n(\kappa_j)| .
\]

(24)
From the Lipschitz continuity of B-spline function (see, de Boor 2001), for kth element of \( B(x) \), we have that for \( x \in [\kappa_j, \kappa_{j+1}) \),

\[
|B_k(x) - B_k(\kappa_j)| \leq c\sqrt{K}|x - \kappa_j| \leq c_0\sqrt{K}
\]

with some constant \( c, c_0 > 0 \). Thus, the first term of (24) can be bounded by

\[
\max_{1 \leq j \leq K} \sup_{|x - \kappa_j| < |\kappa_{j+1} - \kappa_j|} |Z_n(x) - Z_n(\kappa_j)| \leq c_0\sqrt{K}||\hat{U}_n(b_0) - E[\hat{U}_n(b_0)]||_{\max}.
\]

Furthermore, Lemmas 3 yields that

\[
\left\| \hat{U}(b_0)^{-1}\{\hat{U}_n(b_0) - E[\hat{U}_n(b_0)]\} \right\|_{\max} \leq c_1 \frac{1}{\tau_n} \left\| \hat{U}_n(b_0) - E[\hat{U}_n(b_0)] \right\|_{\max}
\]

for some constant \( c_1 > 0 \). Next, we evaluate that for \( \varepsilon > 0 \),

\[
P\left( \left\| \hat{U}(b_0)^{-1}\{\hat{U}_n(b_0) - E[\hat{U}_n(b_0)]\} \right\|_{\max} > \varepsilon \right) \leq P\left( \left\| \hat{U}_n(b_0) - E[\hat{U}_n(b_0)] \right\|_{\max} > \frac{\varepsilon}{c_1} \right) = P\left( \left\| \hat{U}_n(b_0) - E[\hat{U}_n(b_0)] \right\|_{\max} > \tau_n \varepsilon\right).
\]

Let \( E_i = \exp[B(X_{0i})^T b_0] \log(Y_i/w_n(X_i)) \) and let \( \tilde{E}_i = E_i - 1 \). Again, we notice that under \( Y_i > w_n(X_i), i = 1, \ldots, n \), \( E_i \)'s are asymptotically distributed as standard exponential distribution. Thus, \( \tilde{E}_i \)'s are centered version of \( E_i \)'s. We let \( R_j(Y_i, X_i) = E_i B_j(X_{0i})I(Y_i > w_n(X_i)) \) and \( \mathcal{M} = \{\max_i E_i \leq M\} \) with \( M = 1/\varepsilon \). Then, the jth element of \( \hat{U}_n(b_0) - E[\hat{U}_n(b_0)] \) is

\[
\frac{1}{n} \sum_{i=1}^{n} R_j(Y_i, X_i) - E[R_j(Y_i, X_i)] = \frac{1}{n} \sum_{i=1}^{n} R_j(Y_i, X_i)I(\mathcal{M}) - E[R_j(Y_i, X_i)I(\mathcal{M})] + \frac{1}{n} \sum_{i=1}^{n} R_j(Y_i, X_i)I(\mathcal{M}^c) - E[R_j(Y_i, X_i)I(\mathcal{M}^c)]
\]

Since \( P(\mathcal{M}^c) = n \exp[-M] \), we have

\[
\frac{1}{n} \sum_{i=1}^{n} R_j(Y_i, X_i)I(\mathcal{M}^c) - E[R_j(Y_i, X_i)I(\mathcal{M}^c)] \leq O(\tau_n e^{-M}).
\]

Thus, if \( \varepsilon = \sqrt{\log(n)/\{n\tau_n\}} \), on the event \( \mathcal{M}^c \),

\[
\left\| \hat{U}_n(b_0) - E[\hat{U}_n(b_0)] \right\|_{\max} \leq O(K\tau_n e^{-M}) = o(\tau_n).
\]

Next, we show

\[
P\left( \max_{1 \leq j \leq K} \left\| \sum_{i=1}^{n} \frac{1}{n} R_j(Y_i, X_i)I(\mathcal{M}) - E[R_j(Y_i, X_i)I(\mathcal{M})] \right\| > \tau_n \varepsilon \right) \geq \sum_{j=1}^{k} P\left( \frac{1}{n} \sum_{i=1}^{n} R_j(Y_i, X_i)I(\mathcal{M}) - E[R_j(Y_i, X_i)I(\mathcal{M})] > \tau_n \varepsilon \right) \to 0 \quad (25)
\]
It is easy to show that $V[R_j(Y_i, X_i)] \leq n^{-1}\tau_n$ and $\max_j |R_j(Y_i, X_i)| \leq M/n$. From $M = \varepsilon^{-1}$, Bernstein’s inequality implies that

$$P \left( \left| \sum_{i=1}^{n} \frac{1}{n} R_j(Y_i, X_i) \right| > \tau_n \varepsilon \right) \leq C \exp \left[ -C n \tau_n \varepsilon^2 \right]$$

for some constant $C > 0$. This yields that

$$\sum_{j=1}^{K} P \left( \left| \sum_{i=1}^{n} \frac{1}{n} R_j(Y_i, X_i) \right| > \tau_n \varepsilon \right) \leq CK \exp \left[ -C n \tau_n \varepsilon^2 \right]$$

By putting $\varepsilon = \sqrt{\log(n)/n\tau_n}$, (25) can be shown. Consequently, we obtain

$$\left| \hat{U}(b_0)^{-1}\{\hat{U}_n(b_0) - E[U_n(b_0)]\} \right|_{\max} = O_P \left( \sqrt{\log(n)/n\tau_n} \right).$$

Thus,

$$\max_{1 \leq j \leq K} \sup_{|x - \kappa_j| < |\kappa_{j+1} - \kappa_j|} |Z_n(x) - Z_n(\kappa_j)| = O_P \left( \sqrt{K \log(n)/n\tau_n} \right).$$

Similarly, we can show that

$$\max_{1 \leq j \leq K} |Z_n(\kappa_j)| = O_P \left( \sqrt{K \log(n)/n\tau_n} \right),$$

which completes the proof of Theorem 3 by $K = O((\lambda/\tau_n)^{-1/2m}).$

\[\square\]

Appendix C: Lemmas and proof of Theorem in Section 3.2

We show the result in Section 3.2. Here, $\theta = \theta(\phi)$ is unknown parameter vector. Let

$$\eta_n = \max \{ \sqrt{K \log(n)/n\tau_n}, (\lambda/\tau_n)^{1/2}, \tau_{\min} \}. $$

From Theorem 3 for any $\theta \in S^{p-1}_{+}$,

$$\sup_{x \in [a, b]} |\hat{\alpha}(x|\theta) - \alpha_0(x)| = O(\eta_n).$$

We use this result in the proofs of Lemmas 5–6 and Theorem 4.

**Lemma 5.** Suppose that (C1)–(C9). Let $\delta_n$ be the positive sequence satisfying $\delta_n \to 0$, $(n\tau_n)^{1/2}\delta_n \to \infty$, $\delta_n \log(n) \to 0$ and $\eta_n/\delta_n^2 \to 0$. Then, as $n \to \infty$,

$$P(||\hat{\phi} - \phi_0|| > \delta_n) \to 0.$$
Proof of Lemma Let

\[ L_n(\phi) = \frac{1}{n} \sum_{i=1}^{n} \{ \exp[\alpha_0(X_i^T \theta(\phi) | \theta(\phi))] \log(Y_i/w_n(X_i)) - \alpha_0(X_i^T \theta(\phi) | \theta(\phi)) \} I(Y_i > w_n(X_i)) \]

and let

\[ \hat{L}_n(\phi) = \frac{1}{n} \sum_{i=1}^{n} \{ \exp[\hat{\alpha}(X_i^T \theta(\phi) | \theta(\phi))] \log(Y_i/w_n(X_i)) - \hat{\alpha}(X_i^T \theta(\phi) | \theta(\phi)) \} I(Y_i > w_n(X_i)). \]

Then, \( \hat{\phi} = \arg \min \hat{L}_n(\phi | \hat{\phi}(\theta(\phi))) \). From the theorems in Section 3.1, we see that for any \( \phi \in S_c \), \( |\hat{\alpha}(x^T \theta(\phi) | \theta(\phi)) - \alpha_0(x^T \theta(\phi) | \theta(\phi))| = O(\eta_n) \). Therefore, from the definition of \( L_n \), we have \( |\hat{L}_n(\phi) - L_n(\phi)| \leq C \eta_n \tau_n \) for some constant \( C_L > 0 \). Next, we define

\[ L(\phi) = E \{ \exp[\alpha_0(X^T \theta(\phi) | \theta(\phi))] \log(Y/w_n(X)) - \alpha_0(X^T \theta(\phi) | \theta(\phi)) \} I(Y > w_n(X)) \].

The true single index parameter vector \( \phi_0 \) is the minimizer of \( L_0(\phi) = -E[\log f(Y | \alpha_0(\cdot | \theta(\phi)), \theta(\phi))]. \)

By the definition of \( L_0 \) and \( \theta_0 = \theta(\phi_0) \), we have

\[ | - E[\log f_{w_n}(Y | \alpha_0(\cdot | \theta_0), \theta_0)] - L(\phi_0) | \leq C \tau_n^{\beta_{\min} + 1} \]

for some constant \( C \). We note that \( \alpha_0(\cdot) = \alpha_0(\cdot | \theta_0) \). Thanks to Hijort and Pollard (1993), for any sufficiently small \( \delta_n > 0 \) and some constant \( C > 0 \),

\[
P(\| \hat{\phi} - \phi_0 \| > \delta_n) \leq P \left( \sup_{\| \phi - \phi_0 \| < \delta_n} |\hat{L}_n(\phi) - L_n(\phi)| > 2^{-1} \inf_{\| \phi - \phi_0 \| = \delta_n} |L_n(\phi) - L_0(\phi_0)| \right) \leq P \left( \sup_{\| \phi - \phi_0 \| < \delta_n} |L_n(\phi) - L(\phi)| + C \tau_n \eta_n \right) > 2^{-1} \inf_{\| \phi - \phi_0 \| = \delta_n} |L(\phi) - L(\phi_0)| \right)
\]

Here, we write

\[ A_1(x | \phi) = \frac{d\alpha_0(x^T \theta(\phi) | \theta(\phi))}{d\phi} \]

and

\[ A_2(x | \phi) = \frac{d^2\alpha_0(x^T \theta(\phi) | \theta(\phi))}{d\phi d\phi^T}. \]

If \( \| \phi - \phi_0 \| = \delta_n \), there exists \( u \in \mathbb{R}^{p-1} \) such that \( \| u \| = 1 \) and \( \phi = \phi_0 + \delta_n u \). By the Taylor expansion, there exists \( \phi^* \) satisfying \( \| \phi_0 - \phi^* \| < \| \phi_0 - \phi \| = \delta_n \) such that \( \theta^* = \theta(\phi^*) \) and

\[
L(\phi) - L(\phi_0) = \delta_n u^T E \left[ A_1(X | \phi_0) \{ \exp[\alpha_0(X^T \theta_0)] \log(Y/w_n(X)) - 1 \} I(Y > w_n(X)) \right] + \delta_n^2 u^T E \left[ A_2(X | \phi_0) \{ \exp[\alpha_0(X^T \theta^* | \theta^*)] \log(Y/w_n(X)) - 1 \} I(Y > w_n(X)) \right] u \\
+ \delta_n^2 u^T E \left[ A_1(X, \phi_0)A_1(X, \phi^*)^T \exp[\alpha_0(X^T \theta^* | \theta^*)] \log(Y/w_n(X))I(Y > w_n(X)) \right] u \{1 + o_p(1)\}.
\]

Since

\[
| E \left[ \exp[\alpha_0(X^T \theta_0)] \log(Y/w_n(X)) - 1 \right] I(Y > w_n(x), X^T \theta_0 = x^T \theta_0, X = x) | = \frac{| \ell_1(x) |}{\ell_0(x)} \frac{\beta(x)}{\beta(x) + 1} w_n(x)^{-\beta(x)/\gamma_0(x^T \theta_0)} (1 + o(1)) = O(\tau_n^{\beta_{\max}}).
\]
for some constant $C > 0$, where $\beta_{\text{max}} = \sup_{x \in \mathcal{X}} \beta(x)$. Here, we are allowed to use $\beta_{\text{max}} = \infty$. Thus, under (C9), we have

$$\inf_{||\phi - \phi_0|| = \delta_n} |L(\phi) - L(\phi_0)| \geq C\delta_n \tau_n \left\{ \tau_n^{\beta_{\text{max}}} + \delta \right\} \geq C\delta_n^2 \tau_n.$$

By the assumption, we see that $\min\{\tau_n \eta_n, \tau_n^{\beta_{\text{min}} + 1}\}/\tau_n^2 \rightarrow 0$. Thus, our purpose is to show

$$P \left( \sup_{||\phi - \phi_0|| \leq \delta_n} |L_n(\phi|\theta(\phi)) - L(\phi|\theta_0)| > C\delta_n^2 \tau_n \right) \rightarrow 0.$$

We define $\phi = \phi_0 + \delta_n u$, where $||u|| \leq 1$. Then, we have

$$P \left( \sup_{||\phi - \phi_0|| \leq \delta_n} |L_n(\phi) - L(\phi)| > C\delta_n^2 \tau_n \right) \leq P \left( |L_n(\phi_0) - L(\phi_0)| > 2^{-1}C\delta_n^2 \tau_n \right) + P \left( \sup_{||\phi - \phi_0|| \leq \delta_n} |L_n(\phi) - L_n(\phi_0) - L(\phi) + L(\phi_0)| > 2^{-1}C\delta_n^2 \tau_n \right) \equiv J_1 + J_2.$$

Let

$$h(X_i, Y_i|u) = \{\exp[\alpha_0(X_i^T \theta(\phi_0 + \delta_n u)|\theta(\phi_0 + \delta_n u))] \log(Y_i/w_n(X_i)) - \alpha_0(X_i^T \theta(\phi_0 + \delta_n u)|\theta(\phi_0 + \delta_n u))\} I(Y_i > w_n(X_i))$$

and let $E_i = \exp[\alpha_0(X_i^T \theta_0)] \log(Y_i/w_n(X_i))$. It is known that $E_i$ is asymptotically distributed as standard exponential distribution. Then, using $\mathcal{M} = \{\max_i E_i\}$ with $M > 0$, we have

$$L_n(\phi_0) - L(\phi_0) = \frac{1}{n} \sum_{i=1}^{n} \{h(X_i, Y_i|0)I(\mathcal{M}) - E[h(X, Y|0)I(\mathcal{M})]\} + \frac{1}{n} \sum_{i=1}^{n} \{h(X_i, Y_i|0)I(\mathcal{M}^c) - E[h(X, Y|0)I(\mathcal{M}^c)]\}.$$

It is easy to show that $P(\mathcal{M}^c) = ne^{-M}$ and

$$\frac{1}{n} \sum_{i=1}^{n} \{h(X_i, Y_i|0)I(\mathcal{M}^c) - E[h(X, Y|0)I(\mathcal{M}^c)]\} \leq o(\delta_n^2 \tau_n)$$

if $M = 1/\delta_n$. Next, we show

$$P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \{h(X_i, Y_i|0)I(\mathcal{M}) - E[h(X, Y|0)I(\mathcal{M})]\} \right| > 2^{-1}C\delta_n^2 \tau_n \right) \rightarrow 0.$$

It is easy to show that $V[h(X_i, Y_i|0)] \leq C_1 \tau_n$ for some constant $C_1 > 0$ on the event $\mathcal{M}$. Under $\max_i E_i < M$, we have $\max_i |h_1(X_i, Y_i|0)| \leq C_2 M$ for some constant $C_2 > 0$. Then, if
\( M = 1/\delta_n \), the Bernstein’s inequality yields that

\[
P \left( \frac{1}{n} \sum_{i=1}^{n} \{ h(X_i, Y_i|0) I(M) - E[h(X, Y|0) I(M)] \} > C \delta_n^2 \tau_n \right)
\leq C_3 \exp \left[ -C_3 \frac{\delta_n^2 \tau_n}{\tau_n^2} \right]
\leq C_3 \exp \left[ -C_3 n \delta_n \right].
\]

for some constant \( C_3 > 0 \). Since \( \delta_n \) satisfies \( \log(n) < \delta_n^{-1} < n \), we have \( J_1 \to 0 \) as \( n \to \infty \).

We next evaluate \( J_2 \). From Taylor expansion, we obtain that there exists \( \xi \in (0, 1) \) such that

\[
h(X_i, Y_i|u) - h(X_i, Y_i|0)
= \delta_n u^T A_1(X_i|\phi_0 + \xi \delta_n u)
\times \{ \exp[\alpha_0(X_i^T \theta_0 + \xi \delta_n u)] \theta_0(\phi_0 + \xi \delta_n u)] \} \log(Y_i/w_n(X_i)) - 1 \} I(Y_i > w_n(X_i))
\equiv h_1(X_i, Y_i|u).
\]

Define \( M_1 = \{ \max_i E_i < M_1 \} \) with \( M_1 > 0 \). Then, in \( J_2 \), we can write

\[
L_n(\phi_0 + \delta u) - L_n(\phi_0) - L(\phi_0 + \delta u) + L(\phi_0)
= \frac{1}{n} \sum_{i=1}^{n} h_1(X_i, Y_i|u) - E[h_1(X, Y|u)]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} h_1(X_i, Y_i|u)I(M_1) - E[h_1(X, Y|u)I(M_1)]
+ \frac{1}{n} \sum_{i=1}^{n} h_1(X_i, Y_i|u)I(M_1^c) - E[h_1(X, Y|u)I(M_1^c)].
\]

Since \( P(M_1^c) = n \exp[-M_1] \), if \( M_1 = 1/\delta_n \), we have

\[
\left| \frac{1}{n} \sum_{i=1}^{n} h_1(X_i, Y_i|u)I(M_1^c) - E[h_1(X, Y|u)I(M_1^c)] \right| \leq o_P(\tau_n \delta_n^2)
\]

Let \( U = \{ u \in \mathbb{R}^{p-1} ||u|| \leq 1 \} \) be the vector space and \( U_1, \ldots, U_N \) be a covering of \( U \) with the diameter \( R_n = C \delta_n^2/n^\nu \) for some constant \( C > 0 \) and \( \nu > 0 \). Then, Lemma 2.5 of van de Geer (2000) yields that \( N \leq C(n^\nu/\delta_n^2 \tau_n)^{p-1} \). Let \( u_k \in U_k, k = 1, \ldots, N \) and let \( M_1 \) be positive constant. Define

\[
h_2(X_i, Y_i|u, u_k) = h_1(X_i, Y_i|u)I(M_1) - E[h_1(X, Y|u)I(M_1)]
- \{ h_1(X_i, Y_i|u_k)I(M_1) - E[h_1(X, Y|u_k)I(M_1)] \}.
\]
Then, we have
\[
P \left( \sup_{||u|| \leq 1} \left| \frac{1}{n} \sum_{i=1}^{n} h_1(X_i, Y_i|u) I(M_1) - E[h_1(X, Y|u) I(M_1)] \right| \geq 2^{-1} C \delta^2 \tau_n \right) \\
\leq \sum_{k=1}^{N} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} h_1(X_i, Y_i|u_k) I(M_1) - E[h_1(X, Y|u_k) I(M_1)] \right| \right) \\
+ \sup_{u \in U_k} \left| \frac{1}{n} \sum_{i=1}^{n} h_2(X_i, Y_i|u, u_k) \right| \geq 2^{-1} C \delta^2 \tau_n \\
\leq \sum_{k=1}^{N} P \left( \left| \frac{1}{n} \sum_{i=1}^{n} [h_1(X_i, Y_i|u_k) I(M_1) - E[h_1(X, Y|u_k) I(M_1)]] \right| \right) \\
+ \sup_{u \in U_k} \left| \frac{1}{n} \sum_{i=1}^{n} h_2(X_i, Y_i|u, u_k) \right| \geq 2^{-1} C \delta^2 \tau_n .
\]

We set \( M_1 = 1/\delta_n \). Since \( \delta_n \) satisfies \( \delta_n \log(n) \to 0 \), \( N n \exp[-M_1] \) converges to zero. Similar to the evaluation of \( \ell_2 \) in the proof of Lemma \[1\] under max, \( E_i \leq M_1 \), we obtain that for some constant \( c > 0 \),
\[
\sup_{u \in U_k} \left| \frac{1}{n} \sum_{i=1}^{n} h_2(X_i, Y_i|u, u_k) \right| < c M_1 \delta_n \tau_n R_n < 2^{-1} C \delta^2 \tau_n .
\]

Lastly, we will evaluate
\[
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} h_1(X_i, Y_i|u_k) I(M_1) - E[h_1(X, Y|u_k) I(M_1)] \right| \geq 2^{-1} C \delta^2 \tau_n \right) \to 0 .
\]

On the event \( M_1 \), we have \( \max_i |h_1(X_i, Y_i|u_k)| \leq C_1 \delta_n M_1 \) and \( V[h_1(X_i, Y_i|u_k)] \leq C_2 \delta^2 \tau_n \) for some constant \( C_1, C_2 > 0 \). Therefore, Bernsteins inequality realizes that
\[
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} h_1(X_i, Y_i|u_k) I(M_1) - E[h_1(X, Y|u_k) I(M_1)] \right| \geq 2^{-1} C \delta^2 \tau_n \right) \\
\leq C_3 \exp \left[ - \frac{\delta_n^{2} \tau_n^2}{C_1 \frac{\delta_n^2 \tau_n}{n} + C_2 \frac{M_1 \delta_n^2 \tau_n}{n} } \right] \\
\leq C_3 \exp \left[ - n \tau_n \delta_n^2 \right]
\]
for some constant \( C_3 > 0 \). Since \( \delta_n(n \tau_n)^{1/2} \to \infty \), we obtain \( J_2 \to 0 \) as \( n \to \infty \). This implies that \( P(||\hat{\phi} - \phi_0|| > \delta_n) \to 0 \) as \( n \to \infty \).

Proof of Theorem \[2\]: We write \( \hat{b}(\phi) = \hat{b}(\theta(\phi)) \). Define \( R(\phi) = U_n(\hat{b}(\phi), \theta(\phi)|0) \). Then, \( \hat{\phi} \) is minimizer of \( R(\phi) \). From Lemma \[3\] \( \hat{\phi} \) is close to \( \phi_0 \) when the sample size is large. This guarantees that the Taylor expansion
\[
0 = \frac{\partial R(\hat{\phi})}{\partial \phi} = \frac{\partial R(\phi_0)}{\partial \phi} + \frac{\partial^2 R(\phi_0^*)}{\partial \phi \partial \phi^T}(\hat{\phi} - \phi_0),
\]

37
where \( \phi^* \in \mathbb{R}^{p-1} \) is the midpoint between \( \hat{\phi} \) and \( \phi_0 \), i.e., \( ||\phi^* - \phi_0|| < ||\hat{\phi} - \phi_0|| \). This indicates that

\[
\hat{\phi} - \phi_0 = \left( \frac{\partial^2 R(\phi^*)}{\partial \phi \partial \phi^T} \right)^{-1} \left\{ \frac{\partial R(\phi_0)}{\partial \phi} - E \left[ \frac{\partial R(\phi_0)}{\partial \phi} \right] \right\} + \left( \frac{\partial^2 R(\phi^*)}{\partial \phi^2} \right)^{-1} E \left[ \frac{\partial R(\phi_0)}{\partial \phi} \right].
\]

Here, we obtain

\[
\frac{\partial R(\phi_0)}{\partial \phi} = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial B^{[d]}(X_i^T \theta_0)^T b(\phi_0)}{\partial \phi} \left[ \exp[B^{[d]}(X_i^T \theta_0)^T b(\phi_0)] \log \left( \frac{Y_i}{w_n(X_i)} \right) - 1 \right] I(Y_i > w_n(X_i)) = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial \alpha_0(X_i^T \theta_0 | \theta_0)}{\partial \phi} \left[ \exp[\alpha_0(X_i^T \theta_0)] \log \left( \frac{Y_i}{w_n(X_i)} \right) - 1 \right] I(Y_i > w_n(X_i)) + \frac{1}{n} \sum_{i=1}^{n} \frac{\partial \{ B^{[d]}(X_i^T \theta_0)^T b(\phi_0) - \alpha_0(X_i^T \theta_0 | \theta_0) \}}{\partial \phi} \left[ \exp[B^{[d]}(X_i^T \theta_0)^T b(\phi_0)] - \exp[\alpha_0(X_i^T \theta_0)] \right] \log \left( \frac{Y_i}{w_n(X_i)} \right) I(Y_i > w_n(X_i)) + \frac{1}{n} \sum_{i=1}^{n} \frac{\partial \{ B^{[d]}(X_i^T \theta_0)^T b(\phi_0) - \alpha_0(X_i^T \theta_0 | \theta_0) \}}{\partial \phi} \left[ \exp[B^{[d]}(X_i^T \theta_0)^T b(\phi_0)] - \exp[\alpha_0(X_i^T \theta_0)] \right] \log \left( \frac{Y_i}{w_n(X_i)} \right) I(Y_i > w_n(X_i))\]

\[\equiv R_1 + R_2 + R_3.\]

From Theorem 1, we see that \( E[R_2] = O(\tau_n \eta_n) \) and \( E[R_3] = o(\tau_n^{\beta_{\min}}) \). Similarly, the asymptotic orders of variances of \( R_2 \) and \( R_3 \) are dominated by the variance of \( R_1 \). The remaining is to show the asymptotic normality of \( R_1 \). From \( \{17\} \) and the fact that \( P(Y_i > w_n(x_i) | X_i = x_i) = O(\tau_n) \), we have

\[
E[R_1] = \tau_n E \left[ \frac{\partial \alpha_0(X_i^T \theta_0)}{\partial \phi} \ell_1(X) \frac{\beta(X)}{\beta(X) + 1} \tau_n \alpha(X) \right] (1 + o(1)).
\]

Similar to the proof of Lemma \( \{2\} \) we have

\[
V[R_1] = \frac{\tau_n}{n} E \left[ \ell_0(X) \frac{\partial \alpha_0(X_i^T \theta_0)}{\partial \phi} \frac{\partial \alpha_0(X_i^T \theta_0)}{\partial \phi^T} \right] (1 + o(1)).
\]

Therefore, by the central limit theorem, we have

\[
V[R_1]^{-1/2} \left\{ \frac{\partial R(b(\phi_0), \phi_0)}{\partial \phi} - E[R_1 + R_2] \right\} \xrightarrow{D} N(0,1).
\]
Next, we have
\[
\frac{\partial^2 R(\phi_0)}{\partial \phi \partial \phi^T} = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial^2 B[d](X_i^T \theta_0)^T \tilde{b}(\phi_0)}{\partial \phi \partial \phi^T} \left[ \exp[B[d](X_i^T \theta_0)^T b(\phi_0)] \log \left( \frac{Y_i}{w_n(X_i)} \right) - 1 \right] I(Y_i > w_n(X_i))
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} \frac{\partial B[d](X_i^T \theta_0)^T \tilde{b}(\phi_0)}{\partial \phi} \frac{\partial B[d](X_i^T \theta_0)^T b(\phi_0)}{\partial \phi^T} \times \left[ \exp[B[d](X_i^T \theta_0)^T b(\phi_0)] \log \left( \frac{Y_i}{w_n(X_i)} \right) \right] I(Y_i > w_n(X_i)).
\]

From Lemma 3 and Theorem 1, simple but tedious calculation yields that
\[
\tau_n^{-1} \frac{\partial^2 R(\phi_0)}{\partial \phi \partial \phi^T} \xrightarrow{P} E \left[ \ell_0(X) \frac{\partial \alpha_0(X_i^T \theta_0)}{\partial \phi} \frac{\partial \alpha_0(X_i^T \theta_0)}{\partial \phi^T} \right]
\]
as \(n \to \infty\). Since \(\phi^* \to \phi_0\) and \(R\) is continuous with respect to \(\phi\), we obtain
\[
\left( \frac{\partial^2 R(\phi^*)}{\partial \phi \partial \phi^T} \right)^{-1} E \left[ \frac{\partial R(\phi_0)}{\partial \phi} \right] = \psi_n(1 + o_P(1)) = O(\eta_n)
\]
and
\[
\sqrt{n \tau} \Sigma(\phi_0)^{1/2} \{ \hat{\phi} - \phi_0 - \psi_n \} \xrightarrow{D} N(0, 1).
\]
This implies that
\[
\| \hat{\phi} - \phi_0 \| \leq O((n \tau)^{-1/2}) + O(\tau^{\beta_{\min}}),
\]
which completes the proof.

\[\Box\]

Appendix D: Lemmas and proof of Theorems in Section 3.3

Finally, Lemma 7 and the proof of Theorems 5–7 are described for the results in Section 3.3. Let
\[
\eta_n^{-1} = \max \left\{ \sqrt{\left( \frac{\lambda}{\tau_n} \right)^{-1/2m} \frac{\log(n)}{n \tau_n} + \frac{\lambda^{1/2}}{\tau_n^{1/2}} + \tau_{min}^2} \right\}
\]
We write \(b_0 = b_0(\theta_0)\), which is defined in Section 3.1.

**Lemma 6.** Suppose that (C1)–(C9). Then, as \(n \to \infty\),
\[
B(x^T \theta_0)^T (\tilde{b}(\hat{\theta}) - b_0) = B(x^T \theta_0)^T \tilde{U}_n(b_0)^{-1} \tilde{U}_n(b_0)(1 + o_P(1)).
\]

**Proof of Lemma** Let
\[
Q_n(u) = \frac{1}{n} \sum_{i=1}^{n} \left\{ \exp[B[d](X_i^T \hat{\theta})^T (b_0 + u/\eta_n)] \log \left( \frac{Y_i}{w_n(X_i)} \right) - B[d](X_i^T \hat{\theta})^T (b_0 + u/\eta_n) \right\} I(Y_i > w_n(X_i))
\]
\[
+ \lambda \left\{ b_0 + u/\eta_n \right\}^T \Delta_{m,K} \left\{ b_0 + u/\eta_n \right\}.
\]
Then, the minimizer of $Q$ is obtained as

$$\hat{u} = \eta_n \{ \hat{b}(\theta) - b_0 \}.$$  

By the Taylor expansion, we have

$$Q_n(u) = Q_n(0) + \frac{1}{\eta_n} W_n(\hat{\theta})^T u + \frac{1}{\eta_n^2} u^T A_n(\hat{\theta}) u + \frac{\lambda}{\eta_n} u^T \Delta_m.K \{ b_0 + u/\eta_n \}$$

where

$$W_n(\hat{\theta}) = \frac{1}{n} \sum_{i=1}^{n} B[d](X_i^T \hat{\theta}) \exp[\frac{\partial B[d](X_i^T \hat{\theta}}{\partial \theta} b_0] \log \left( \frac{Y_i}{w_n(X_i)} \right) - 1 \} I(Y_i > w_n(X_i)), \quad ||u^*|| < ||u||$$

$$A_n(\hat{\theta}) = \frac{1}{n} \sum_{i=1}^{n} B[d](X_i^T \hat{\theta}) B[d](X_i^T \hat{\theta})^T \exp[\frac{\partial B[d](X_i^T \hat{\theta}}{\partial \theta} b_0] \log \left( \frac{Y_i}{w_n(X_i)} \right) I(Y_i > w_n(X_i))$$

From Theorem 4 we obtain

$$W_n(\hat{\theta}) = W_n(\theta_0) + B_n^T (\hat{\theta} - \theta_0),$$

where

$$B_n = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial B[d](X_i^T \theta^*)}{\partial \theta} u \exp[\frac{\partial B[d](X_i^T \theta^*)}{\partial \theta} b_0] \log \left( \frac{Y_i}{w_n(X_i)} \right) - 1 \} I(Y_i > w_n(X_i))$$

$$+ \frac{1}{n} \sum_{i=1}^{n} B[d](X_i^T \theta^*)^T \partial \frac{\partial B[d](X_i^T \theta^*)}{\partial \theta} b_0 \exp[\frac{\partial B[d](X_i^T \theta^*)}{\partial \theta} b_0] \log \left( \frac{Y_i}{w_n(X_i)} \right) I(Y_i > w_n(X_i))$$

and $||\theta^* - \theta_0|| < ||\hat{\theta} - \theta_0||$. From the proof of Theorem 4 we see that $||B_n|| = O_P(\tau_{\beta_{min}})$ and $||W_n(\hat{\theta}) - W_n(\theta_0)|| = O_P(\tau_{\beta_{min}})||\hat{\theta} - \theta_0|| = o_P(1)$ uniformly for $u \in \mathcal{U} \subset \mathbb{R}^K$, where $\mathcal{U}$ is any compact subset of $\mathbb{R}^K$. Similarly, simple each element of $A_n(\hat{\theta}) - A_n(\theta_0)$ has an order $o_P(1)$ uniformly for $u \in \mathcal{U}$. Therefore, from the convexity lemma of Pollard (1991), we have that $Q_n(u) - Q_n(0)$ is asymptotically equivalent to

$$\frac{1}{\eta_n} W_n(\theta_0)^T u + \frac{1}{\eta_n^2} u^T A_n(\theta_0) u + \frac{\lambda}{\eta_n} u^T \Delta_m.K \{ b_0 + u/\eta_n \}$$

uniformly for $u \in \mathcal{U} \subset \mathbb{R}^K$. We note that $\hat{U}_n(\theta_0, \theta_0|\lambda) = W_n(\theta_0)$ and $\tilde{U}_n(\theta_0, \theta_0|\lambda) = A_n(\theta_0) + \lambda \Delta_{m,K}$. This implies that

$$\eta_n B(x^T \theta_0)^T (\hat{b}(\hat{\theta}) - b_0) = \eta_n B(x^T \theta_0)^T (A_n(\theta_0) + \lambda \Delta_{m,K})^{-1} W_n(\theta_0)(1 + o_P(1))$$

$$= \eta_n B(x^T \theta_0)^T \tilde{U}_n(\theta_0, \theta_0|\lambda)^{-1} \tilde{U}_n(\theta_0, \theta_0|\lambda)(1 + o_P(1)).$$

Using Lemmas 2 and 3, we obtain

$$\eta_n B(x^T \theta_0)^T \tilde{U}_n(\theta_0, \theta_0|\lambda)^{-1} \tilde{U}_n(\theta_0, \theta_0|\lambda) = O_P(1).$$
Proof of Theorems 5–7. By the definition of the estimator and true function, we obtain
\[
\hat{\alpha}(x^T\hat{\theta}|\theta) - \alpha_0(x^T\theta_0) = \{B^{[d]}(x^T\hat{\theta}) - B^{[d]}(x^T\theta_0)\}^T\hat{b}(\hat{\theta}) + B^{[d]}(x^T\theta_0)^T(\hat{b}(\hat{\theta}) - b_0) + B^{[d]}(x^T\theta_0)^Tb_0 - \alpha_0(x^T\theta_0).
\]
From Lemma 6, we obtain
\[
\{B^{[d]}(x^T\hat{\theta}) - B^{[d]}(x^T\theta_0)\}^T\hat{b}(\hat{\theta}) = B^{[d-1]}(x^T\theta^*)^TD_1\hat{b}(\hat{\theta})x^T(\hat{\theta} - \theta_0) = \alpha_0'(x^T\theta_0)x^T(\hat{\theta} - \theta_0)(1 + o_P(1)) = \alpha_0'(x^T\theta_0)x^T\psi_n(1 + o(1)) + \frac{1}{\sqrt{n\tau_n}}\alpha_0'(x^T\theta_0)x^T\Sigma^{-1/2}\varepsilon(1 + o_P(1)),
\]
where \(\varepsilon \sim N(0, I)\). This yields that
\[
|\{B^{[d]}(x^T\hat{\theta}) - B^{[d]}(x^T\theta_0)\}^T\hat{b}(\hat{\theta})| \leq O(\tau_n^{1/2}) + O_P((n\tau_n)^{-1/2}).
\]
Next, Lemma 1 yields that
\[
B^{[d]}(x^T\theta_0)^Tb_0 - \alpha_0(x^T\theta_0) = b_{s,n}(x^T\theta_0,q)(1 + o(1)) = O(K^{-q}),
\]
which is the negligible order. Lastly, from Lemma 5 we have
\[
B^{[d]}(x^T\theta_0)^T(\hat{b}(\theta_0) - b_0) = B^{[d]}(x^T\theta_0)^T\tilde{U}_n(b_0,\theta_0|\lambda)^{-1}\tilde{U}_n(b_0,\theta_0|\lambda)(1 + o_P(1))
\]
and its asymptotic properties are obtained by from Theorems 1–3.

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