Harary polynomials

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Abstract. Given a graph property $P$, F. Harary introduced in 1985 $P$-colorings, graph colorings where each color class induces a graph in $P$. Let $\chi_P(G; k)$ counts the number of $P$-colorings of $G$ with at most $k$ colors. It turns out that $\chi_P(G; k)$ is a polynomial in $\mathbb{Z}[k]$ for each graph $G$. Graph polynomials of this form are called Harary polynomials. In this paper we investigate properties of Harary polynomials and compare them with properties of the classical chromatic polynomial $\chi(G; k)$. We show that the characteristic and Laplacian polynomial, the matching, the independence and the domination polynomial are not Harary polynomials. We show that for various notions of sparse, non-trivial properties $P$, the polynomial $\chi_P(G; k)$ is, in contrast to $\chi(G; k)$, not a chromatic, and even not an edge elimination invariant. Finally we study whether Harary polynomials are definable in Monadic Second Order Logic.

Keywords: Graph colorings · generalized chromatic polynomials · Courcelle’s Theorem · Monadic Second Order Logic.

1 Introduction and main results

1.1 Harary polynomials

Let $P$ be a graph property. In [19] F. Harary introduced the notion of $P$-coloring as a generalization of proper colorings, which he called conditional colorings. Let $G$ be a graph and $[k] = \{1, 2, \ldots, k\}$. A function $c : V(G) \to [k]$ is a $P$-coloring with at most $k$ colors if for every $i \in [k]$ the set $\{v \in V(G) : f^{-1}(i)\}$ induces a graph in $P$. If $P$ is the property that $E(G) = \emptyset$, this gives the proper colorings. Other properties of $P$ studied in the literature are $G$ is connected, $G$ is triangle-free or $G$ is a complete graph. F. Harary introduced $P$-colorings with the idea that they might behave in a similar way than proper colorings. $P$-colorings were further studied in [11].

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Let \( \chi_\mathcal{P}(G;k) \) be the number of \( \mathcal{P} \)-colorings of \( G \) with at most \( k \) colors, and \( \chi_\mathcal{P}(G) \) be the \( \mathcal{P} \)-chromatic number, which is the least \( k \) such that \( G \) has a \( \mathcal{P} \)-coloring.

\( \chi(G;k) \) is the chromatic polynomial, i.e., the Harary polynomial for \( \mathcal{P} \) the edgeless graphs. Generalizing Birkhoff’s Theorem from 1912 for \( \chi(G;k) \) it was noted in \([29]\) that for every finite graph \( G \) the counting function \( \chi_\mathcal{P}(G;k) \) is a polynomial in \( \mathbb{Z}[k] \), see also \([22]\). The family of polynomials \( \chi_\mathcal{P}(G;k) \) indexed by graphs \( G \) is a graph polynomial called a Harary polynomial in \([32]\), which can be written as

\[
\chi_\mathcal{P}(G;k) = \sum_{i \geq 1} b_i^\mathcal{P}(G)x(i),
\]

where \( b_i^\mathcal{P}(G) \) is the number of partitions of \( V(G) \) into \( i \) parts, where each part induces a graph in \( \mathcal{P} \).

**Facts 1** For every graph property \( \mathcal{P} \) and every graph \( G \) of order \( n \) we have

(i) \( b_0^\mathcal{P}(G) = 0 \), if the nullgraph \((\emptyset,\emptyset)\) is not in \( \mathcal{P} \).
(ii) \( b_1^\mathcal{P}(G) \in \{0,1\} \) and \( b_0^\mathcal{P}(G) = 1 \) iff \( G \in \mathcal{P} \).
(iii) \( b_0^\mathcal{P}(G) \in \{0,1\} \) and \( b_1^\mathcal{P}(G) = 1 \) iff \( K_1 \in \mathcal{P} \).
(iv) If \( k < n \) and \( b_k^\mathcal{P}(G) = 0 \) then for all \( 0 < \ell < k \) also \( b_\ell^\mathcal{P}(G) = 0 \).
(v) If \( G = G_1 \sqcup G_2 \) is the disjoint union of two graphs \( G_1 \) and \( G_2 \) then \( \chi_\mathcal{P}(G) = \max\{\chi_\mathcal{P}(G_1),\chi_\mathcal{P}(G_2)\} \).
(vi) If the single vertex graph \( K_1 \in \mathcal{P} \), the polynomial \( \chi_\mathcal{P}(G;k) \) is monic of degree \( |V(G)| \).

**Examples 2** Here are some \( \mathcal{P} \)-colorings and Harary polynomials studied in the literature.

(i) \( \text{mcc}_t \)-colorings. Here \( H \in \mathcal{P}_t \) is the graph property that the connected components of \( H \) have order at most \( t \). For \( t = 1 \) these are the proper colorings, for \( t = 2 \) these are the \( \mathcal{P}_2 \)-free colorings. They were introduced in \([26]\).
(ii) Let \( H \) be connected of order \( t \). \( \text{DU}(H) \) consist of non-empty disjoint unions of copies of \( H \). \( \text{DU}(H) \)-colorings are \( \text{mcc}_t \)-colorings. They are studied in \([18]\).
(iii) Let \( \text{Fr}(H) \) be class of \( H \)-free graphs. \( \text{Fr}(H) \)-colorings are studied in \([14,11]\). \( K_2 \)-free colorings are just the proper colorings. \( \mathcal{P}_2 \)-free colorings are the \( \text{mcc}_2 \)-colorings.

Note that \( \chi_{\text{Fr}(H)}(G;k) \) is multiplicative with respect to disjoint unions iff \( H \) is connected.
(iv) A graph property \( \mathcal{P} \) is additive if it is closed under forming disjoint unions. \( \mathcal{P} \) is hereditary, if it is closed under induced subgraphs. A coloring is \( \mathcal{AH} \) if it is a \( \mathcal{P} \)-coloring for some \( \mathcal{P} \) which is both additive and hereditary. \( \mathcal{AH} \)-colorings were studied in \([15]\).
(v) The adjoint polynomial \( A(G;x) = \chi_\mathcal{P}(G;k) \) is defined by taking \( \mathcal{P} \) to be the class of all complete graphs. It was introduced in \([27]\), see also \([8]\).
(vi) If $\mathcal{P}$ consists of all connected graphs, we speak of convex colorings, and put $C(G; x) = \chi_{\mathcal{P}}(G; k)$, see \[37\], \[38\], \[18\].

The purpose of this paper is to initiate the study of Harary polynomials by comparing them to the chromatic polynomial.

1.2 The chromatic polynomial and edge elimination invariants

One of the fundamental properties of the chromatic polynomial is its characterization via edge elimination properties. Given a graph $G$ and an edge $e \in E(G)$ we denote by $G-e, G/e$ and $G^e$ the graphs obtained from $G$ by deleting, contracting and extracting the edge $e$. Extraction deletes $e = (u, v)$ together with the vertices $u, v$ and all the edges incident with $u$ or $v$. A graph parameter $p(G)$ is an edge elimination (EE) invariant, see \[6\], if it can be written as a certain linear combination of $p(G-e), p(G/e), p(G^e)$.

It is well known that $\chi(G; k)$ is an EE-invariant even with using $p(G^e)$. Other EE-invariants are the matching polynomials, the Tutte polynomial and many others, \[40\], \[41\].

Theorem 1 (\[5\], \[6\]). There is a graph polynomial $\xi(G; x, y, z)$ such that

(i) $\xi(G; x, y, z)$ is an edge elimination invariant,
(ii) $\xi(G; x, y, z)$ is universal, i.e., every other graph parameter $p(G)$ which is an edge elimination invariant is a substitution instance of $\xi(G; x, y, z)$.

1.3 MSOL-definable graph polynomials

The language of graphs has one binary relation symbol for the edge relation. If we fix $k$ we note that $\chi(G; k) = 0$ iff $G$ is $k$-colorable. This can be expressed by formula in monadic second order logic MSOL in the language of graphs by the formula

$$\exists V_1 \exists V_2 \ldots \exists V_k (\text{Partition}(V_1, V_2, \ldots, V_k) \land \bigwedge_{i=1}^{k} \text{Indep}(V_i)).$$

$\text{Partition}(V_1, V_2, \ldots, V_k)$ and $\text{Indep}(V_i)$ are first order expressible in the language of graphs. The same works for Harary polynomials, provided $\mathcal{P}$ is MSOL-definable. Checking whether a graph $G$ is $k$-colorable is NP-complete. For the complexity of checking whether a graph is $\mathcal{P}$-colorable for various graph properties $\mathcal{P}$, the reader may consult \[10\].

However, using Courcelle’s celebrated Theorem, \[14\], Chapter 13] and \[16\], Chapter 11], MSOL-definability implies that checking whether a graph $G$ is $k$-colorable is fixed parameter tractable (FPT) for graphs of bounded tree-width, and even for graphs of bounded clique-width or rank-width.

For the chromatic polynomial one looks at the problem of computing the value of $\chi(G; k)$ for given $k$ as a function of $G$. For $k = 0, 1, 2$ this computable in polynomial time, whereas for $k \geq 3$ this is #P-complete, \[25\]. For graphs of
fixed tree-width \( w \), this is still in FPT. To see this one can use an extension of Courcelle’s Theorem to the class of MSOL-definable graph polynomials, [13].

The language of hypergraphs has two unary predicates \( V \) and \( E \) for vertices and edges which partition the universe, and a binary incidence relation \( R \) saying that vertices are connected to edges. We denote by MSOL\(_g\) (MSOL\(_h\)) the monadic second order logic in the language of graphs (hypergraphs).

**Proposition 1.** Let \( P \) be a graph property definable in MSOL\(_g\) (MSOL\(_h\)). Then checking whether a graph \( G \) is \( P \)-colorable with \( k \) colors is definable in MSOL\(_g\) (MSOL\(_h\)).

**Theorem 2 ([31]).** \( \chi(G; k) \) is not an MSOL\(_g\)-definable polynomial, but it is MSOL\(_h\)-definable.

**Theorem 3.** The most general EE-invariant \( \xi(G; x, y, z) \) is MSOL\(_h\)-definable.

To prove that \( \chi(G; k) \) is not MSOL\(_g\)-definable we use the method of connection matrices, explained in Section 4. To prove that \( \chi(G; k) \) is MSOL\(_h\)-definable we use that \( \chi(G; k) \) is an EE-invariant and Theorems 1 and 3. We do not know a direct method, without the use of an MSOL\(_h\)-definable EE-invariant, to show that \( \chi(G; k) \) is indeed MSOL\(_h\)-definable.

Theorem 2 still implies that evaluating \( \chi(G; k) \) is fixed parameter tractable (FPT) for graphs of tree-width at most \( w \), whereas for graphs of clique-width \( w \) this is still open, [33,7].

### 1.4 Main results

A graph property \( P \) is *trivial* if it is empty, finite (up to isomorphisms), or contains all finite graphs. Our main question in this paper asks whether Courcelle’s Theorem and its variations can be applied to Harary polynomials for non-trivial graph properties. This amounts to ask:

(i) Are there non-trivial graph properties \( P \) such that the Harary polynomial \( \chi_P(G, x) \) is MSOL\(_g\)-definable?

(ii) Are there non-trivial graph properties \( P \) such that the Harary polynomial \( \chi_P(G, x) \) is an EE-invariant and hence MSOL\(_h\)-definable?

A graph property \( P \) is *clique-free* if there is \( t \in \mathbb{N} \) such that no graph \( G \in P \) has an \( K_t \), i.e., a complete graph of order \( t \). Analogously, \( P \) is *biclique-free* if there is \( t \in \mathbb{N} \) such that no graph \( G \in P \) has \( K_{t,t} \) as a subgraph (not necessarily induced). \( K_{t,t} \) is the complete bipartite graph of order \( 2t \). Clearly, biclique-free implies clique-free, but not conversely.

**Theorem 4.** Let \( P \) be clique-free and \( \chi_P(G, x) \) the Harary polynomial associated with \( P \).

(i) \( \chi_P(G, x) \) is not MSOL\(_g\)-definable.

(ii) If additionally, \( P \) is monotone, hereditary or minor closed, \( \chi_P(G, x) \) is not an EE-invariant.
The same also holds for biclique-free graph properties.

Remark 1. If $\mathcal{P}$ consists of all complete graphs or all connected graphs, $\mathcal{P}$ is not clique-free, hence Theorem 3 does not apply to the Harary polynomials $A(G; x)$ and $C(G; x)$. Nevertheless, analogue results are presented in Sections 3 and 4.

1.5 Sparsity

For a systematic study of sparsity (and density) of graph properties see [34, 35].

**Theorem 5.** (i) Turán’s Theorem ([42] and [17, Chapter 8.3]):

Let $G$ be $K_t$-free. Then $|E(G)| \leq (1 - \frac{1}{t}) \frac{n^2}{2}$.

(ii) ([23]) Let $G$ be $K_{t,t}$-free. Then $|E(G)| = O(n^2 - \frac{1}{t})$.

(iii) ([39]) If a graph property $\mathcal{P}$ is nowhere dense or degenerate (or equivalently uniformly sparse) then $\mathcal{P}$ is biclique-free.

(iv) ([39]) There are graph properties $\mathcal{P}_1, \mathcal{P}_2$ which are both biclique-free but $\mathcal{P}_1$ is not degenerate and $\mathcal{P}_2$ is not nowhere dense.

In the light of Theorem 5, biclique-free is renamed to weakly sparse in [36]. However, clique-free graphs can be rather dense, with $c(t) \cdot n^2$ edges rather than $n^2 - \epsilon(t)$ edges.

Theorem 1 together with Theorem 5 shows that Harary polynomials which are EE-invariants or MSOL$_g$-definable have to be defined using dense properties $\mathcal{P}$ as required by Turán’s Theorem.

2 Graph polynomials which are not Harary polynomials

Many familiar graph polynomials are not Harary polynomials of the form $\chi_{\mathcal{P}}(G; x)$. We generalize here [32, Theorem 5.7].

**Lemma 1.** For every graph property $\mathcal{P}$ we have

\[
\chi_{\mathcal{P}}(G; 1) = \begin{cases} 
1 & G \in \mathcal{P}, \\
0 & G \notin \mathcal{P}.
\end{cases}
\]

Using Lemma 1 we get

**Proposition 2.** Let $F(G; x)$ be a graph polynomial and $G$ be a graph such that $F(G; 1) \neq 0$ and $F(G; 1) \neq 1$. Then there is no graph property $\mathcal{P}$ such that $\chi_{\mathcal{P}}(G; x) = F(G; x)$.

The characteristic polynomial $\text{char}(G; x)$ of a graph is the characteristic polynomial of its adjacency matrix, and the Laplacian polynomial $\text{Lap}(G; x)$ is the characteristic polynomial of its Laplace matrix, see [9].
The matching polynomials are defined using $m_i(G)$, the number of matchings of $G$ of size $i$.

$$M(G; x) = \sum_i m_i(G)x^i \quad \text{and} \quad \mu(G; x) = \sum_i (-1)^i m_i(G)x^{n-2i}.$$  

$M(G; x)$ is the generating matching polynomial and $\mu(G; x)$ is the matching defect polynomial, see [23].

Let $\text{Ind}_i(G)$ be the number of of independent sets of $G$ of size $i$, and $d_i(G)$ the number of dominating sets of $G$ of size $i$. We define the independence polynomial $\text{Ind}(G; x)$, [24], and the domination polynomial $\text{DOM}(G; x)$, [43, 20] as

$$\text{Ind}(G; x) = \sum_i \text{Ind}_i(G)x^i \quad \text{and} \quad \text{DOM}(G; x) = \sum_i d_i(G)x^i.$$

**Theorem 6.** The following are not Harary polynomials of the form $\chi_\mathcal{P}(G; x)$:

(i) The characteristic polynomial $\text{char}(G; x)$ and the Laplacian polynomial $\text{Lap}(G; x)$.

(ii) The generating matching polynomial $M(G; x)$ and the defect matching polynomial $\mu(G; x)$.

(iii) The independence polynomial $\text{Ind}(G; x)$

(iv) The domination polynomial $\text{DOM}(G; x)$.

**Proof.** We use Proposition [2] (i): $\text{char}(C_4; x) = (x-2)x^2(x+2)$ and $\text{Lap}(C_4; x) = x(x-4)(x-2)^2$.

(hence $\text{char}(C_4; 1) = \text{Lap}(C_4; 1) = -3$).

(ii): $M(C_4; x) = 4x + 2x^2$ and $\mu(C_4; x) = 1 + 4x + 2x^2$.

(hence $M(C_4; 1) = 6$ and $\mu(C_4; 1) = 7$).

(iii): $\text{Ind}(C_4; x) = 1 + 4x + 2x^2$, hence $\text{Ind}(C_4; x) = 7$.

(iv): $\text{DOM}(K_2; x) = 2x + x^2$, hence $\text{DOM}(K_2; 1) = 3$.

$\text{DOM}$ and $\text{Ind}$ are special cases graph polynomials of the form

$$\mathcal{P}_\phi(G; x) = \sum_{A \subseteq V(G) : \Phi(A)} x^{|A|}.$$  

Graph polynomials of this form are generating functions counting subsets $A \subseteq V(G)$ satisfying a property $\Phi(A)$, in the cases above, that $A$ is an independent, respectively a dominating set, see also [32]. We say that $\Phi$ determines $A$, if for every graph $G$ there is a unique $A \subseteq V(G)$ which satisfies $\Phi(A)$.

**Theorem 7.** Assume that $\Phi$ does not determine $A$, then there is no graph property $\mathcal{P}$ such that for all graphs $G$ $\chi_\mathcal{P}(G; x) = \mathcal{P}_\phi(G; x)$. Hence $\mathcal{P}_\phi(G; x)$ cannot be a Harary polynomial.

**Proof.** By Lemma [1], $\chi_\mathcal{P}(G; 1) \in \{0, 1\}$ for all graphs $G$. However, since $\Phi$ does not determine $A$, there is a graph $H$ with $\mathcal{P}_\phi(H; 1) \geq 2$. 


3 Are Harary polynomials edge elimination invariants?

3.1 Chromatic invariants

Following [2, Chapter 9.1], a function \( f \) which maps graphs into a polynomial ring \( R = \mathcal{K}[X] \) with coefficients in a field \( \mathcal{K} \) of characteristic 0 is called a chromatic invariant (aka Tutte-Grothendieck invariant) if the following hold.

(i) If \( G \) has no edges, \( f(G) = 1 \).
(ii) If \( e \in E(G) \) is a bridge, then \( f(G) = A \cdot f(G_{-e}) \).
(iii) If \( e \in E(G) \) is a loop, then \( f(G) = B \cdot f(G_{-e}) \).
(iv) There exist \( \alpha, \beta \in R \) such that for every \( e \in E(G) \) which is neither a loop nor a bridge we have \( f(G) = \alpha \cdot f(G_{-e}) + \beta \cdot f(G_{/e}) \).
(v) Multiplicativity: If \( G = G_1 \sqcup G_2 \) is the disjoint union of two graphs \( G_1, G_2 \) then \( f(G) = f(G_1) \cdot f(G_2) \).

Chromatic invariants have a characterization via the Tutte polynomial \( T(G; x, y) \), see [2, Chapter 9.1, Theorem 9.5].

**Theorem 8.** Let \( f \) be a chromatic invariant with \( A, B, \alpha, \beta \) as above. Then for all graphs \( G \)

\[
f(G) = \alpha^{|E| - |V| + k(G)}, \beta^{|V| - k(G)} \cdot T(G; A, B, \alpha, \beta).
\]

It follows by a counting argument that not all Harary polynomials are chromatic invariants. We give an explicit description of Harary polynomials which are not a chromatic invariant in Theorem 10 below.

3.2 Edge elimination invariants

The Tutte polynomial generalizes the chromatic, flow and other graph polynomials. It is natural to search for polynomials that generalize it, in turn. The *Most General Edge Elimination Invariant*, introduced in [6],[5] and also known as the \( \xi \) polynomial, generalizes the Tutte and the matching polynomials.

**Definition 1 (Edge Elimination Invariant).** Let \( F \) be a graph parameter with values in a ring \( R \). \( F \) is an EE-invariant if there exist \( \alpha, \beta, \gamma \in R \) such that

\[
F(G) = F(G_{-e}) + \alpha F(G_{/e}) + \beta F(G_{\mid e}) \quad (2)
\]

where \( e \in E(G) \), with the base conditions

\[
F(\emptyset) = 1, \quad F(K_1) = \gamma, \quad \text{and} \quad F(G \sqcup H) = F(G) \cdot F(H). \quad (3)
\]

**Theorem 9 ([5]).** Let \( \xi(G; x, y, z) \) be the graph polynomial

\[
\xi(G; x, y, z) = \sum_{A, B \subseteq E(G)} x^{|E(A \sqcup B)| - |c_{cov}(B)|} y^{|A| + |B| - |c_{cov}(B)|} z^{|c_{cov}(B)|},
\]

where the summation is over \( A, B \subseteq E(G) \) such that the vertex subsets \( V(A), V(B) \) covered by \( A \) and \( B \), respectively, are disjoint, \( c(A) \) is the number of connected components in \( V(G) \), \( A) \), and \( c_{cov}(B) \) is the number of covered connected component of \( B \), i.e. the number of connected components of \( (V(B), B) \).
(i) $\xi(G; x, y, z)$ is an EE-invariant.
(ii) Every EE-invariant is a substitution instance of $\xi(G; x, y, z)$.
(iii) Every chromatic invariant is a substitution instance of $\xi(G; x, y, z)$.
(iv) Both the matching polynomial and the Tutte polynomial are EE-invariants given by

$$T(G; x, y) = (x - 1)^{-|E(G)|} (y - 1)^{-|V(G)|} \xi(G; (x - 1)(y - 1), y - 1, 0),$$

and

$$M(G; w_1, w_2) = \xi(G; w_1, 0, w_2).$$

3.3 Are Harary polynomials EE-invariants?

**Lemma 2.** Let $\mathcal{P}$ be monotone (hereditary, minor closed) and $H$ be a graph of smallest order, and among those of smallest and size, such that $H$ is a forbidden subgraph (induced subgraph, minor) of $\mathcal{P}$. Assume further that $H$ has at least four vertices and one edge $e \in E(H)$. Then $\chi_\mathcal{P}(G; x)$ is not an EE-invariant.

**Proof.** As $\mathcal{P}$ is monotone (hereditary, minor closed) we note that deleting or contracting or extracting $e$ from $H$, we obtain a graph in $\mathcal{P}$. Hence we can compute:

$$\chi_\mathcal{P}(H, x) = x^{|V(H)|} - x \quad (4)$$
$$\chi_\mathcal{P}(H-e, x) = x^{|V(H)|} \quad (5)$$
$$\chi_\mathcal{P}(H_{/e}, x) = x^{|V(H)|-1} \quad (6)$$
$$\chi_\mathcal{P}(H_{\backslash e}, x) = x^{|V(H)|-2} \quad (7)$$

Now assume that $\chi_\mathcal{P}$ is a EE-invariant. Then we have

$$\chi_\mathcal{P}(H, x) = x^{|V(H)|} - x \quad (*)$$

$$= \chi_\mathcal{P}(H-e, x) + \alpha(x) \cdot \chi_\mathcal{P}(H_{/e}, x) + \beta(x) \cdot \chi_\mathcal{P}(H_{\backslash e}, x)$$

$$= x^{|V(H)|} + \alpha(x) \cdot x^{|V(H)|-1} + \beta(x) \cdot x^{|V(H)|-2} \quad (**)$$

for $\alpha(x), \beta(x) \in \mathbb{Z}[x]$ polynomials in $x$.

If $|V(H)| \geq 4$ the coefficient of $x$ in (*) is $-1$, and in (**) it is $0$ which is a contradiction.

**Theorem 10.** Assume that $\mathcal{P}$ is monotone (hereditary, minor closed) and clique-free, but contains no edgeless graph. Then $\chi_\mathcal{P}(G; x)$ is not an EE-invariant.

**Proof.** For every $G \in \mathcal{P}$ the set of edges $E(G) \neq \emptyset$, $K_3$ is a forbidden subgraph (induced subgraph, minor). If $K_4$ is a forbidden subgraph (induced subgraph, minor) of $\mathcal{P}$, we apply Lemma 2. Otherwise we compute $\chi_\mathcal{P}(K_3, x) = x^3 - x$. Let $H = K_1 \uplus (K_2 \sqcup K_1)$. $\chi_\mathcal{P}(H, x)$ has degree 4, since $\chi_\mathcal{P}(H, x)$ is monic and $H$ has order 4 (Facts 1).

Assuming that $\chi_\mathcal{P}(H, x)$ is an EE-invariant, $\chi_\mathcal{P}(K_3, x) = x^3 - x$ gives $\alpha = 0$ and $\beta = -1$. Computing $\chi_\mathcal{P}(H, x)$ we get $\chi_\mathcal{P}(H, x) = \chi_\mathcal{P}(K_3) + \alpha \chi_\mathcal{P}(K_3) + \beta \chi_\mathcal{P}(K_2)$. But this is a polynomial of degree 3, which is a contradiction.
The graph polynomials $C(G; x)$ and $A(G; x)$ are Harary polynomials where the property $P$ contains arbitrarily large cliques.

**Proposition 3.** Both $C(G; x)$ and $A(G; x)$ are not multiplicative, hence they are not EE-invariants.

**Proof.** $C(K_1, x) = A(K_1, x) = x$ and $C(K_1 \sqcup K_1, x) = A(K_1 \sqcup K_1, x) = x^2 - x \neq x^2$.

## 4 MSOL-definable graph polynomials

We assume the reader is familiar with Second and Monadic Second Order Logic for graphs. A good source is [12,31,21,32]. We distinguish between MSOL for the language of graphs, with one binary edge relation MSOL$_g$, and MSOL for the language of hypergraphs MSOL$_h$, with vertices and edges as elements and a binary incidence relation. We also refer to second order logic SOL$_g$, SOL$_h$ in a similar way.

A simple univariate MSOL$_g$-definable (MSOL$_h$, SOL$_g$, SOL$_h$-definable) graph polynomial $F(G; x)$ is a polynomial of the form

$$F(G; x) = \sum_{A \subseteq V(G): \phi(A)} \prod_{v \in I} x,$$

where $A$ ranges over all subsets of $V(G)$ satisfying $\phi(A)$ and $\phi(A)$ is a MSOL$_g$-formula. $F$ is MSOL$_h$-definable if $A$ ranges over $V(G) \cup E(G)$ and $\phi(A)$ is a MSOL$_h$-formula. $F$ is SOL$_g$-definable if $A$ ranges over $(V(G) \cup E(G))^m$. $F$ is SOL$_h$-definable if $A$ ranges over $(V(G) \cup E(G))^m$.

**Examples 3**

(i) The independence polynomial $\text{Ind}(G; x) = \sum_i \text{ind}(G, i) \cdot x^i$, can be written as

$$\text{Ind}(G, x) = \sum_{I \subseteq V(G)} \prod_{v \in I} x,$$

where $I$ ranges over all independent sets of $G$. To be an independent set is MSOL$_g$-definable.

(ii) The generating matching polynomial $M(G; x)$ is MSOL$_h$-definable, but unlikely to be MSOL$_g$-definable. Otherwise it would be fixed parameter tractable for clique-width at most $k$.

**Fact 4** If $P$ is MSOL$_g$-definable (MSOL$_h$-definable, SOL-definable) so is the Harary polynomial $\chi_P(G; x)$.

**Proof.** We only prove the case where $P$ is MSOL$_g$-definable, the other cases are similar.

Let $\phi$ be the MSOL$_g$-formula which defines $P$. Let $\Phi(X, E)$ be the formula which says that $X \subseteq V(G)^2$ is an equivalence relation on $V(G)$ such that each equivalence class induces a graph satisfying $\phi$. Now we can write

$$\chi_P(G; x) = \sum_{X \subseteq V(G)^2: \Phi(X, E)} x^{|X|}.$$
For the general case one allows several indeterminates \( x_1, \ldots, x_m \), and gives an inductive definition. One may also allow an ordering of the vertices, but the one requires the definition to be *invariant under the ordering*, i.e., different orderings still give the same polynomial.

**Examples 5** The Tutte polynomial is a bivariate MSOL\(_h\)-definable graph polynomial using an ordering on the vertices, \([30]\). Similarly, it can be shown that the polynomial \( \xi(G; x, y, z) \) is a trivariate MSOL\(_h\)-definable graph polynomial, \([4]\).

A univariate graph polynomial is MSOL\(_g\)-definable (MSOL\(_h\), SOL\(_g\), SOL\(_h\)-definable) if it is a substitution instance of a multivariate MSOL\(_g\)-definable (MSOL\(_h\), SOL\(_g\), SOL\(_h\)-definable) graph polynomial.

### 5 Connection Matrices

In this section we prove for many Harary -polynomials that they are not MSOL\(_g\)-polynomials.

We use tools from \([31]\). Let \( G_i \) be an enumeration of all finite graphs (up to isomorphisms). We denote by \( G_i \Join G_j \) the join of \( G_i \) and \( G_j \). Let \( F = F(G; x) \in \mathbb{Z}[x] \) be a graph polynomial. Let \( \mathcal{H}(\Join, F) \) be the infinite matrix where rows and columns are labeled by \( G_i \). Then we define

\[
\mathcal{H}(\Join, F)(G_i, G_j) = F(G_i \Join G_j; x).
\]

\( \mathcal{H}(\Join, F) \) is called a connection matrix aka Hankel matrix.

**Theorem 11** (\([31]\)). If \( F(G; x) \) is MSOL\(_g\)-definable, then \( \mathcal{H}(\Join, F) \) has finite rank over the ring \( \mathbb{Z}[x] \).

**Lemma 3.** Given a graph polynomial \( F \), and an infinite sequence of non-isomorphic graphs \( H_i, i \in \mathbb{N} \), let \( f : \mathbb{N} \to \mathbb{N} \) be an unbounded function such that for every \( k \in \mathbb{N} \), \( F(H_i \Join H_j, k) = 0 \) iff \( i + j > f(k) \).

Then the matrix \( \mathcal{H}(\Join, p) \) has infinite rank.

The same also holds when \( \Join \) is replace by the disjoint union \( \sqcup \).

Given a graph \( H \) we denote by \( \text{Forb}^{\text{sub}}(H) \) (\( \text{Forb}^{\text{ind}}(H) \)) the class of graphs which do not contain an (induced) subgraph isomorphic to \( H \). If \( H \) is a complete graph the two classes coincide, and we omit the superscript.

We now prove specific cases where we can apply Lemma 3.

**Lemma 4.** Let \( H_i = K_i \) the complete graph on \( i \) vertices.

(i) Let \( \mathcal{P}_1 \subseteq \text{Forb}(K_h) \). Then \( \chi_{\mathcal{P}_1}(K_i; k) = 0 \) iff \( i > hk \).

(ii) Let \( \mathcal{P}_2 \subseteq \text{Forb}^{\text{sub}}(H) \) for some connected graph \( H \) on \( h \) vertices. Hence \( \mathcal{P}_2 \) is monotone. Then \( \chi_{\mathcal{P}_2}(K_i; k) = 0 \) iff \( i > hk \).
Proof. If we partition a set of size \( i > h k \) into \( k \) disjoint sets, at least one of these sets has size \( i > h \). So of we partition \( K_i \), at least one of these sets induces a \( K_h \). So hence \( \chi_{P_1}(K_i; k) = 0 \). Since \( H \) is a subgraph of \( K_h \), \( \chi_{P_2}(K_i; k) = 0 \).

Theorem 12. Let \( P \) be a non-trivial graph property. If \( P \) is (i) monotone, (ii) clique-free, or (iii) almost sparse, the Harary polynomial \( \chi_P(G; x) \) is not MSOL\(_g\)-definable.

Proof. (i): If \( P \) is non-trivial and monotone there is a connected graph \( H \) with \( P \subseteq \text{Forb}^{\text{sub}}(H) \). By Lemma 4(ii) we get \( \chi_P(K_i; k) = 0 \) iff \( i > h k \). By Lemma 3, \( \mathcal{H}(\geq i, \chi_P(G; x)) \) has infinite rank. Now we use Theorem 11 (ii): \( P \subseteq \text{Forb}(K_h) \), hence by Lemma 4(i), we get again \( \chi_P(K_i; k) = 0 \) iff \( i > h k \). Then we proceed as in (i).

(ii): \( P \) is clique-free hence there is \( h \in \mathbb{N} \) with \( P \subseteq \text{Forb}(K_h) \). So we proceed as in (ii).

(iii): \( P \) is clique-free hence there is \( h \in \mathbb{N} \) with \( P \subseteq \text{Forb}(K_h) \). So we proceed as in (ii).

The graph polynomials \( C(G; x) \) and \( A(G; x) \) are Harary polynomials where the property \( P \) contains graphs of maximal density.

Proposition 4. Both \( C(G; x) \) and \( A(G; x) \) are not MSOL\(_g\)-definable.

Proof. In both cases we look at the graph \( M_n \) of order \( 2n \) which consists of \( n \) disjoint copies of \( K_2 \). We get \( C(G; k) = A(G; k) = 0 \) for \( n > 2k \), so we can apply Lemma 2 with the join replaced by the disjoint union.

6 Conclusions and Open Problems

We have initiated a systematic study of Harary polynomials.

In this paper we have shown that for monotone, hereditary and minor closed properties which are clique-free they are, unlike the chromatic polynomial, not edge elimination invariants. For the same properties they are, as is the case for the chromatic polynomial, not MSOL\(_g\)-definable. However, the chromatic polynomial is MSOL\(_h\)-definable.

Question 1. Is there a non-trivial Harary polynomial, different from the chromatic polynomial, which is MSOL\(_h\)-definable and/or an EE-invariant?

We suspect (but do not conjecture) that the chromatic polynomial is, up to equal distinguishing power, the only Harary polynomial which is an EE-invariant?

In future research we continue the study of the complexity of evaluating Harary polynomials, initiated in [13].

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