Higgs boson decay into bottom quarks and uncertainties of perturbative QCD predictions

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Abstract

Different methods for treating the results of higher-order perturbative QCD calculations of the decay width of the Standard Model Higgs boson into bottom quarks are discussed. Special attention is paid to the analysis of the $M_H$ dependence of the decay width $\Gamma(H \rightarrow \bar{b}b)$ in the cases when the mass of b-quark is defined as the running parameter in the $\overline{MS}$-scheme and as the quark pole mass. The relation between running and pole masses is taken into account in the order $\alpha_s^4$-approximation. Some special features of applications of Analytical Perturbation Theory (APT) are commented.

1 Introduction

The study of the Higgs boson decay width into bottom quarks is rather important for calculations of the branching ratios of this important ingredient of the Standard Model and its various extensions. Current LEP and Tevatron fits of the Standard Model parameters yield the value for the Higgs boson mass around $M_H = 76^{+33}_{-24}$ GeV C.L. 68%. With the direct LEP search limit $M_H \geq 115$ GeV the fits provide at C.L. 95% $M_H \leq 182$ GeV. In the case, if the scalar Higgs particle has the mass in the region $115 \text{ GeV} \leq M_H \leq 2M_W$, its decay width to bottom quarks $\Gamma(H \rightarrow bb)$ dominates over other channels. In particular, it is determining the branching ratio of $H \rightarrow \gamma\gamma$ process which is considered to be one of the most promising channel for searches of Higgs particles at LHC in the
mass region specified above (for reviews see [1]–[4]). There is also a possibility that the signal for \( H \to \bar{b}b \) process may be seen at Tevatron [5] through WH- and ZH- channels and at CMS-TOTEM [6] or/and FP420 [7] experimental proposals at LHC, aimed at searches of central exclusive \( H \) production, as discussed from theoretical point of view, e.g., in Refs. [8, 9]. The mentioned experimentally-oriented motivation is pushing ahead the intention to study in more detail the dominant theoretical effects to \( \Gamma(H \to \bar{b}b) \) in the region of relatively light Higgs boson. These effects are related to high-order perturbative QCD predictions with their intrinsic uncertainties. Moreover, the comparison of various representations for \( \Gamma(H \to \bar{b}b) = \Gamma_{H\bar{b}b} \) is rather important for planning the experimental program of high energy linear \( e^+e^- \)-colliders for measuring Higgs boson couplings [4].

2 QCD expressions for \( \Gamma_{H\bar{b}b} \)

Let us first consider QCD theoretical prediction for \( \Gamma_{H\bar{b}b} \) expressed in terms of running b-quark mass and the QCD coupling constant in the \( \overline{\text{MS}} \)-scheme as

\[
\Gamma_{H\bar{b}b} = \Gamma_0^b \frac{m_b^2(M_H)}{m_b^2} \left[ 1 + \sum_{i \geq 1} \Delta \Gamma_i a_s(M_H) \right]. \tag{1}
\]

Here \( \Gamma_0^b = 3\sqrt{2}/8\pi G_F m_H m_b^2 \), \( m_b \) and \( M_H \) are the pole b-quark and Higgs boson masses, \( a_s(M_H) = \alpha_s(M_H)/\pi \) and \( \overline{m}_b(M_H) \) are the QCD running parameters, defined in the \( \overline{\text{MS}} \)-scheme. The coefficients \( \Delta \Gamma_i \) are known analytically up to 4-th order correction of perturbation theory [10]. They consist of the positive contributions \( d_i^F \), calculated directly in the Euclidean region, and from the proportional to \( \pi^2 \) kinematic effects, which appear as the result of analytical continuation from the Euclidean space-like to the Minkowskian time-like region. This \( \pi^2 \)-term arises first in Eq. (1) at the \( a_s^2 \)-correction [11]. Its coefficient was corrected later in [12], [13], but the kinematic \( \pi^2 \)-term remained unaffected.

Using the notations of Ref. [14] one can write down the following relations:

\[
\Delta \Gamma_1 = d_1^F = \frac{17}{3}; \tag{2}
\]
\[
\Delta \Gamma_2 = d_2^F - \gamma_0(\beta_0 + 2\gamma_0)\pi^2/3; \tag{3}
\]
\[
\Delta \Gamma_3 = d_3^F - [d_1^F(\beta_0 + \gamma_0)(\beta_0 + 2\gamma_0) + \beta_1\gamma_0 + 2\gamma_1(\beta_0 + 2\gamma_0)]\pi^2/3; \tag{4}
\]
\[
\Delta \Gamma_4 = d_4^F - [d_2^F(\beta_0 + \gamma_0)(3\beta_0 + 2\gamma_0) + d_1^F(5\beta_0 + 6\gamma_0)/2 \\
+ 4d_1^F\gamma_1(\beta_0 + \gamma_0) + \beta_2\gamma_0 + 2\gamma_1(\beta_1 + \gamma_1) + \gamma_2(3\beta_0 + 4\gamma_0)]\pi^2/3 \\
+ \gamma_0(\beta_0 + \gamma_0)(\beta_0 + 2\gamma_0)(3\beta_0 + 2\gamma_0)\pi^4/30, \tag{5}
\]
where the $n_f$-dependence of $d_i^E$ $(i > 2)$ read

$$
d_2^E = \left[ \frac{10801}{144} - \frac{39}{2} \zeta_3 \right] - n_f \left[ \frac{65}{24} - \frac{2}{3} \zeta_3 \right]
\approx 51.567 - 1.907 n_f \approx 42.032 \quad (n_f = 5) ;
$$

$$
d_3^E = \left[ \frac{163613}{5184} - \frac{109735}{216} \zeta_3 + \frac{815}{12} \zeta_5 \right] - n_f \left[ \frac{46147}{486} - \frac{262}{9} \zeta_3 + \frac{5}{6} \zeta_4 + \frac{25}{9} \zeta_5 \right]
+ n_f^2 \left[ \frac{15511}{11664} - \frac{1}{3} \zeta_3 \right]
\approx 648.71 - 63.742 n_f + 0.92913 n_f^2 \approx 353.23 \quad (n_f = 5) ;
$$

$$
d_4^E = \left[ \frac{10811054729}{497664} - \frac{3887351}{324} \zeta_3 + \frac{458425}{432} \zeta_3^2 + \frac{265}{18} \zeta_4 + \frac{373975}{432} \zeta_5 \right]
- \frac{1375}{32} \zeta_6 - \frac{178045}{768} \zeta_7
+ n_f \left[ - \frac{1045811915}{373248} + \frac{5747185}{5184} \zeta_3 - \frac{955}{16} \zeta_3^2 - \frac{9131}{576} \zeta_4 + \frac{41215}{432} \zeta_5 \right]
+ \frac{2875}{288} \zeta_6 + \frac{665}{72} \zeta_7
+ n_f^2 \left[ \frac{220313525}{2239488} - \frac{11875}{432} \zeta_3 + \frac{5}{6} \zeta_3^2 + \frac{25}{96} \zeta_4 - \frac{5015}{432} \zeta_5 \right]
+ n_f^3 \left[ - \frac{520771}{559872} + \frac{65}{432} \zeta_3 + \frac{1}{144} \zeta_4 + \frac{5}{18} \zeta_5 \right]
\approx 9470.8 - 1454.3 n_f + 54.783 n_f^2 - 0.45374 n_f^3 \approx 3512.2 \quad (n_f = 5) .
$$

The term of Eq. (2) was evaluated in Ref. [11]. It is in agreement with the expressed in other ways results of previous studies, performed in [15–17]. The second coefficient was corrected in [12, 13]. The result of Eq. (7) was obtained in Ref. [18]. The exact value of the Euclidean coefficient of Eq. (8), analytically calculated in [10], turned out to be in reasonable agreement with the estimates, obtained within the used in Ref. [14] a variant of the effective charge approach (ECH) and the principle of minimal sensitivity (PMS) approach. The variant of the two approaches was developed in Ref. [19]. The coefficients $\beta_i$ and $\gamma_i$ enter the expansions of the QCD renormalization group (RG) $\beta$-function and anomalous dimension of mass function $\gamma_m$. The QCD $\beta$-function can be defined as

$$
\frac{da_s}{d \ln \mu^2} = \beta(a_s)
= -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 - \beta_4 a_s^6 + O(a_s^7) .
$$
Its $\overline{\text{MS}}$-scheme expressions were calculated analytically up to 4-loop corrections [20], confirmed recently in the work [21]. We present here the results of analytical evaluation of the coefficients of Eq. (9) ($\beta_0$ and $\beta_1$ are scheme-independent) in the $\overline{\text{MS}}$-scheme, supplemented by the numerical expressions, related to $n_f = 5$ number of active flavours:

\[
\beta_0 = \frac{1}{4} \left[ 11 - \frac{2}{3} n_f \right]
\approx 2.75 - 0.1667 n_f \approx 1.9167 \quad (n_f = 5) \quad (10)
\]

\[
\beta_1 = \frac{1}{16} \left[ 102 - \frac{38}{3} n_f \right]
\approx 6.375 - 0.7917 n_f \approx 2.4167 \quad (n_f = 5) \quad (11)
\]

\[
\beta_2 = \frac{1}{64} \left[ \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right]
\approx 22.32 - 4.3689 n_f + 0.09404 n_f^2 \approx 2.8267 \quad (n_f = 5) \quad (12)
\]

\[
\beta_3 = \frac{1}{256} \left[ \left( \frac{149753}{6} + 3564 \zeta_3 \right) - \left( \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_f 
+ \left( \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_f^2 + \frac{1093}{729} n_f^3 \right]
\approx 114.23 - 27.134 n_f + 1.5824 n_f^2 + 0.0059 n_f^3 \approx 18.852 \quad (n_f = 5) \quad (13)
\]

Throughout this work we fix $n_f = 5$ and neglect the contribution of lighter four quarks to the relation between running mass $\overline{m}_{b}(M_H)$ and pole (or on-shell) mass $m_b$ in Eq. (1) (more details will be given below). This is done for self-consistency of further analysis. Indeed, the similar contributions to the coefficient function of Eq. (1) are still unknown and are expected to be small. Notice an interesting fact: the growth of the coefficients of $\beta$-function at $n_f = 5$ is starting to manifest itself from the four-loop only (on the contrary to the case with $n_f = 3$ when the values of the coefficients of perturbative series for the $\beta$-function are monotonically increasing from the one-loop order). As to the five-loop coefficient $\beta_4$ in Eq. (9), it was estimated in Ref. [22] by means of Padé approximations approach. The input information, used in these estimates, is the analytical result for the $n_f^4$-contribution to $\beta_4$ calculated in [23]. In our normalization conditions it has the following form

\[
\beta_4^{[4]} = \frac{1}{1024} \left[ \frac{1205}{2916} - \frac{152}{81} \zeta_3 \right] n_f^4 = -0.0017969 n_f^4 \quad (14)
\]

The approximation of Ref. [22], which is taking it into account, reads:

\[
\beta_4 \approx \frac{10^5}{1024} \left[ 7.59 - 2.19 n_f + 20.5 n_f^2 - 49.8 \times 10^{-5} n_f^3 - 1.84 \times 10^{-5} n_f^4 \right]
\approx 741.2 - 213.87 n_f + 20.02 n_f^2 - 0.0483 n_f^3 - 0.0018 n_f^4 \approx 165.2 \quad (n_f = 5). \quad (15)
\]
We will use this estimate in our studies. The related to the \( \overline{\text{MS}} \)-scheme anomalous mass dimension function is defined as

\[
\frac{d \ln m_b}{d \ln \mu^2} = \gamma_m(a_s) = -\gamma_0 a_s - \gamma_1 a_s^2 - \gamma_2 a_s^3 - \gamma_3 a_s^4 - \gamma_4 a_s^5 + O(a_s^6).
\]  \tag{16}

The four-loop correction was calculated independently in Ref. [24] and in Ref. [25]. Let us present the explicitly known coefficients:

\[
\gamma_0 = 1; \tag{17}
\]

\[
\gamma_1 = \frac{1}{64} \left[ 1249 - \left( \frac{2216}{27} + \frac{160}{3} \zeta_3 \right) n_f - \frac{140}{81} n_f^2 \right] \approx 19.516 - 2.2841 n_f - 0.027006 n_f^2 \approx 7.42 \quad (n_f = 5); \tag{18}
\]

\[
\gamma_2 = \frac{1}{256} \left[ 4603055 \frac{162}{1249 - 324 \zeta_3 - 880 \zeta_4 - \frac{18400}{9} \zeta_5} n_f - \frac{91723}{27} + \frac{34192}{9} \zeta_3 - 880 \zeta_4 - \frac{18400}{9} \zeta_5 \right] \approx 98.933 - 19.108 n_f + 0.27616 n_f^2 + 0.005793 n_f^3 \approx 11.034 \quad (n_f = 5). \tag{19}
\]

In the same work [22] the following model for the five-loop coefficient of \( \gamma_m \) was proposed:

\[
\gamma_4 \approx 530 - 143 n_f + 6.67 n_f^2 + 0.037 n_f^3 - 8.54 \times 10^{-5} n_f^4 \approx -13.68 \quad (n_f = 5) \tag{21}
\]

It is based on application of the variant of the Padé approximation approach, used for getting Eq. (15) discussed above. The explicit analytical expression for the \( n_f^3 \) contribution to \( \gamma_4 \), extracted from the QED results of Ref. [26] and confirmed later on in [23], namely

\[
\gamma_4^{[3]} = \frac{1}{12} \left( \frac{65}{5184} + \frac{5 \zeta_3}{324} - \frac{\zeta_4}{36} \right) n_f^3 \tag{22}
\]

was used. Notice that the numerical value of Eq. (21) is negative. This means that the uncertainties of this Padé estimate are not small. Indeed, the analytical calculations of \( n_f^3 \) contribution to Eq. (21), performed in Ref. [27], gave

\[
\gamma_4^{[3]} = \frac{1}{12} \left[ \frac{331865}{124416} + \frac{803}{432} \zeta_3 + \frac{7}{12} \zeta_4 - \frac{20}{9} \zeta_5 \right] n_f^3 = 0.10832 n_f^3. \tag{23}
\]
It does not agree with the similar coefficient in Eq. (21). Substituting Eq. (23) into Eq. (21) one can see that for \( n_f = 5 \) the estimate for \( \gamma_4 \) is still negative, but rather small, namely \( \gamma_4 \approx -4.76595 \). We will incorporate this value in our further analysis just by fixing some parts of existing five-loop ambiguities. We hope that this expression, obtained by matching the explicit results of Eq. (22) and Eq. (23) and the Padé resummation technique, may be improved in the future.

3 Analytical continuation effects and APT approach

Consider now in more detail the contributions to \( \Gamma_{H\bar{b}\bar{b}} \) from the Minkowskian coefficients, defined in Eqs. (3)–(5). Let us remind, that they are composed from the Euclidean terms (see Eqs. (6)–(8)) and the combinations of the coefficients of the QCD RG functions \( \beta(a_s) \) and \( \gamma_m(a_s) \), which are defining proportional to \( \pi^2 \) analytical continuation effects in Eqs. (3)–(5). These kinematic effects turn out to be negative and quite sizable. Thus, the coefficients \( \Delta \Gamma_i (i \geq 2) \) are smaller, than their Euclidean “analogs” \( d^E_i \). Indeed, for \( n_f = 5 \) we have [12]

\[
\Delta \Gamma_2 \approx 29.147.
\]  

The numerical values of other terms obey the same pattern [10]:

\[
\Delta \Gamma_3 \approx 41.758; \quad (25)
\]

\[
\Delta \Gamma_4 \approx -825.75. \quad (26)
\]

This means that it is of interest how the effects of analytical continuation influence other results of perturbative QCD predictions. In the beginning of 80s this problem was discussed in the number of works on the subject (see, e.g., [28, 29]). At that time the problem of resummation of the \( \pi^2 \)-contributions to \( \Gamma_{H\bar{b}\bar{b}} \) was also considered in Ref. [11]. However, the real interest to resummations of the analytical continuation effects was attracted later on after appearance of Contour Improved Technique (CIT) [30, 31] and Analytic Perturbation Theory (APT), in particular. This method was proposed and developed by D. V. Shirkov and I. L. Solovtsov in the process of common investigations in Refs. [32]–[34] and in the separate publications as well (see works done with the decisive contribution of I. L. Solovtsov, [35, 36], and the ones created by inspiration of D.V. Shirkov [37, 38]). This QCD approach was already used in various applications (see, e.g., [39, 40]). Among them are the studies of Higgs boson decay into \( b\bar{b} \)-pair with the help of Fractional Analytical Perturbation Theory (FAPT) [41], which are complementary to definite considerations of Ref. [42]. Quite recently some new [43] and even a bit corrected [44] discussions of applications of FAPT to \( \Gamma_{H\bar{b}\bar{b}} \) and their comparisons with the results of Ref. [42], appeared in the literature. In view of this, we will focus ourselves here on the brief discussions of
several ideas of Ref. [42]. It is worth to stress, that this work was motivated in part by the desire to understand whether the bridge may be built between the renormalon approach (for a review see, e.g., [45]) and the resummation of the proportional to $\beta_0$-effects within Shirkov–Solovtsov APT. Note, that one of the main cornerstones of renormalon approach is $\beta_0$-resummation procedure as well. Today, thanks to the works of Refs. [41, 44] it is understood, that this bridge does exist. The essential point in the studies of Ref. [42] is that the CIT [30] and the concept of b-quark invariant mass are playing an important role. In view of the fact that the positive features of the invariant mass are sometimes not taken used to the total extent, let us remind the basic steps of definitions of this QCD parameter:

1. Define the running quark mass through the solution of the following RG equation for the anomalous dimension term (AD):

$$\tilde{m}_b^2(M_H) = \tilde{m}_b^2(m_b) \exp \left[-2 \int_{a_s(m_b)}^{a_s(M_H)} \frac{\gamma_m(x)}{\beta(x)} dx \right];$$  \hspace{1cm} (27)

2. Take the integral in the r.h.s. of Eq. (27):

$$\tilde{m}_b^2(M_H) = \tilde{m}_b^2(m_b) \left(\frac{a_s(M_H)}{a_s(m_b)}\right)^{2\gamma_0/\beta_0} \left(\frac{AD(a_s(M_H))}{AD(a_s(m_b))}\right)^2;$$  \hspace{1cm} (28)

3. Define

$$AD(a_s) = \left[1 + P_1 a_s + \left(P_1^2 + P_2\right) \frac{a_s^2}{2} + \left(\frac{1}{2} P_1^3 + \frac{3}{2} P_1 P_2 + P_3\right) \frac{a_s^3}{3} + \left(\frac{1}{6} P_1^4 + \frac{4}{3} P_1 P_3 + P_1^2 P_2 + P_4\right) \frac{a_s^4}{4}\right];$$ \hspace{1cm} (29)

4. Calculate its coefficients, expressed through

$$P_1 = -\frac{\beta_1 \gamma_0}{\beta_0^2} + \frac{\gamma_1}{\beta_0} \approx 1.17549, \hspace{0.5cm} P_2 = \frac{\gamma_0}{\beta_0^2} \left(\frac{\beta_1^2}{\beta_0^2} - \beta_2\right) - \frac{\beta_1 \gamma_1}{\beta_0^2} + \frac{\gamma_2}{\beta_0} \approx 1.16196;$$

$$P_3 = \left[\frac{\beta_1 \beta_2}{\beta_0} - \frac{\beta_1}{\beta_0} \left(\frac{\beta_1^2}{\beta_0^2} - \beta_2\right) - \beta_3\right] \frac{\gamma_0}{\beta_0^2} + \frac{\gamma_1}{\beta_0^2} \left(\frac{\beta_1^2}{\beta_0^2} - \beta_2\right) - \frac{\beta_1 \gamma_1}{\beta_0^2} + \frac{\gamma_3}{\beta_0} \approx -3.1505$$ \hspace{1cm} (30)

$$P_4 = \frac{\gamma_0}{\beta_0^2} \left[\frac{\beta_1^2}{\beta_0^2} - \beta_2\right] + \frac{\beta_2}{\beta_0^2} \left(\frac{\beta_1 \beta_2}{\beta_0} - \beta_3\right) - \beta_4\right] + \frac{\gamma_1}{\beta_0^2} \left[\frac{\beta_1 \beta_2}{\beta_0} - \frac{\beta_1}{\beta_0} \left(\frac{\beta_1^2}{\beta_0^2} - \beta_2\right) - \beta_3\right] + \frac{\gamma_2}{\beta_0^2} \left(\frac{\beta_1^2}{\beta_0^2} - \beta_2\right) - \frac{\gamma_3 \beta_1}{\beta_0^2} + \frac{\gamma_4}{\beta_0} \approx -33.2389;$$  \hspace{1cm} (31)
5. Define the invariant mass
\[
\hat{m}_b = \overline{m}_b(m_b) \left[ a_s(m_b) \overline{\gamma}_0 A(\alpha_s(m_b)) \right]^{-1} .
\] (32)

It should be stressed, that this definition seems to be simpler than one, introduced in Ref. [15], namely
\[
\hat{m}_b = \overline{m}_b(m_b) \left[ 2\beta_0 a_s(m_b) \overline{\gamma}_0 A(\alpha_s(m_b)) \right]^{-1} ,
\] (33)

which is often used in the literature.

In the large-\(\beta_0\) approximation the expression for \(\Gamma_{H\overline{b}b}\) can be expressed as [42]
\[
\Gamma_{H\overline{b}b} = \Gamma^0_0 \frac{\hat{m}_b^2(M_H)}{\overline{m}_b^2} (a_s(M_H))^{\nu_0} \left[ A_0 + \sum_{n \geq 1} \text{d}_n A_n(a_s(M_H)) \right]
\] (34)

\[
A_n = \frac{1}{\beta_0 \delta_n \pi} \left( 1 + \beta_0^2 \pi a_s^2 \right)^{-\delta_n/2} (a_s)^{n-1} \sin[\delta_n \arctan(\beta_0 \pi a_s)] ,
\] (35)

where \(\delta_n = n + \nu_0 - 1\), \(\nu_0 = 270/\beta_0\). Taking now into account that within large-\(\beta_0\) approximation, one has the following LO-expression
\[
a_s(M_H)^{\text{LO}} = \frac{1}{\beta_0 \ln(M_H^2/\Lambda^2)}
\] (36)

fixing \(n = 0\) and expanding \(A_0\) to first order in \(a_s\) we are getting the function:
\[
A_0 = \frac{1}{b \overline{H}_M} \frac{\sin(b \arctan(\pi/L_{M_H}))}{(1 + \pi^2/L_{M_H}^2)^{b/2}} ,
\] (37)

where \(b = \nu_0 - 1\), \(L_{M_H} = \ln(M_H^2/\Lambda^2)\). This expression was first derived in 80s in Ref. [11]. Unfortunately, its usefulness was not understood at this time. At the new stage of the development of QCD the analogs of this formula, are forming the basis of FAPT method [41], which is allowing to resum not only the terms proportional to \(\gamma_0\) and \(\beta_0\), but higher order corrections of RG functions as well. Thus, the extension of the Shirkov–Solovtsov method to the case of fractional powers [41] may be considered in part as the generalization for CIT resummation of “large-\(\beta_0\)” contributions, which also arises within renormalon approach [42]. In view of the appearance of the works of Refs. [41, 44], [43], it may be interesting to compare in the future the results of these two approaches in more detail.
4 On-shell and RG-resummation approach

4.1 On-shell parameterization

In the previous section we derived the relation between running and invariant b-quark masses. However, there is also the possibility to use on-shell approach and express the width $\Gamma_{H\bar{b}b}$ through the b-quark pole mass $m_b$ and the $\overline{\text{MS}}$-scheme coupling constant $\alpha_s(M_H)$ in different orders of perturbation theory and compare the results obtained with the running mass motivated RG-resummation approach. This analysis was done at the $\alpha_s^2$-level in Refs. [46, 47]. In these studies the effects of mass dependent $O(\alpha_s m_b^2/M_H^2)$-corrections, extracted from the calculations of Ref. [48], were also included. Among most important results of Refs. [46, 47] were the explicit demonstration of the importance of taking into account $\alpha_s^2$-corrections in the on-shell approach. Indeed, these effects turned out to be rather important for decreasing the difference between the $\Gamma_{H\bar{b}b}$ expressions, evaluated in the on-shell and RG-resummed approaches. The results of our studies were confirmed later on by the considerations of Ref. [49], where the expression of the $O(\alpha_s^2 m_b^2/M_H^2)$-contribution was evaluated and included. Note, however, that for the considered at present masses of Higgs boson these effects are less important, than higher order perturbative QCD corrections, and can be safely neglected. Keeping in mind the demands of Tevatron and LHC experiments and the ongoing discussion of the scientific program for International Linear Collider, in this section we will study the similar problem in more detail, taking into account the information on available at present higher-order QCD corrections to the RG-functions, coefficient function for $\Gamma_{H\bar{b}b}$ in Eq. (1) (see Sec. 2) and the relation between running and on-shell masses which we present in the following form:

$$\frac{m_b^2(m_b)}{m_b^2} = 1 - \frac{8}{3} a_s(m_b) - 18.556 a_s(m_b)^2 - 175.76 a_s(m_b)^3 - 1892 a_s(m_b)^4. \quad (38)$$

The $a_s^2$-correction is the result of calculations of Ref. [50], confirmed later on in [51]. The $a_s^3$-term comes from the analytical calculations of Ref. [52] and is confirming semi-analytical similar result, obtained in Ref. [53]. Note, that its coefficient turned out to be in a good agreement with the ECH/PMS estimate of Ref. [14]. This fact and the success of the ECH/PMS prediction for the value of $d_E^4$ term [10], we are supplementing the estimates of Ref. [14] by definite RG-inspired considerations and get our personal ECH-inspired number for the coefficient of $a_s^4$-term in Eq. (38). Proceeding further on with the help of the derived in Ref. [14] RG equations for the transformation of $m_b^2(M_H)$ to $m_b^2(m_b)$ and of $a_s(m_b)$ to $a_s(M_H)$, we get the following analog of Eq. (1):

$$\Gamma_{H\bar{b}b} = \Gamma_0 \left[ 1 + \Delta \Gamma_1 a_s(M_H) + \Delta \Gamma_2 a_s(M_H)^2 + \Delta \Gamma_3 a_s(M_H)^3 + \Delta \Gamma_3 a_s(M_H)^4 \right]. \quad (39)$$
where $\Gamma_0 = 3\sqrt{2}/(8\pi) G_F M_H m_b^2$ and

\[
\begin{align*}
\Delta\Gamma_1^b &= 3 - 2 L; \\
\Delta\Gamma_2^b &= -4.5202 - 18.139 L + 0.08333 L^2; \\
\Delta\Gamma_3^b &= -316.878 - 133.421 L - 1.15509 L^2 + 0.050926 L^3; \\
\Delta\Gamma_4^b &= -4366.17 - 1094.62 L - 55.867 L^2 - 1.8065 L^3 + 0.04774 L^4
\end{align*}
\]

and $L = \ln(M_H^2/m_b^2)$. We will define now the QCD coupling constant in different orders of perturbation theory as

\[
\begin{align*}
as(M_H)^{\text{NLO}} &= \frac{1}{\beta_0 \text{Log}} \left[ 1 - \beta_1 \ln(\text{Log}) \right]; \\
as(M_H)^{\text{N^2LO}} &= as(M_H)^{\text{NLO}} + \Delta as(M_H)^{\text{N^2LO}}; \\
as(M_H)^{\text{N^3LO}} &= as(M_H)^{\text{N^2LO}} + \Delta as(M_H)^{\text{N^3LO}}; \\
as(M_H)^{\text{N^4LO}} &= as(M_H)^{\text{N^3LO}} + \Delta as(M_H)^{\text{N^4LO}};
\end{align*}
\]

\[
\begin{align*}
\Delta as(M_H)^{\text{N^2LO}} &= \frac{1}{\beta_0^3 \text{Log}^2} \left( \beta_1^2 \ln^2(\text{Log}) - \beta_1^2 \ln(\text{Log}) + \beta_2 \beta_0 - \beta_1^2 \right); \\
\Delta as(M_H)^{\text{N^3LO}} &= \frac{1}{\beta_0^4 \text{Log}^4} \left[ \beta_1^3 \left( -\ln^3(\text{Log}) + \frac{5}{2} \ln^2(\text{Log}) + 2\ln(\text{Log}) - \frac{1}{2} \right) \\
&\quad - 3\beta_0 \beta_1 \beta_2 \ln(\text{Log}) + \beta_0^2 \frac{\beta_3}{2} \right]; \\
\Delta as(M_H)^{\text{N^4LO}} &= \frac{1}{\beta_0^5 \text{Log}^5} \left[ \beta_1^4 \left( \ln^4(\text{Log}) - \frac{13}{3} \ln^3(\text{Log}) - \frac{3}{2} \ln^2(\text{Log}) + 4\ln(\text{Log}) \\
&\quad + \frac{7}{6} \right) + 3\beta_1^2 \beta_2 \left( 2\ln^2(\text{Log}) - \ln(\text{Log}) - 1 \right) \\
&\quad - \beta_1 \beta_3 \left( 2\ln(\text{Log}) + \frac{1}{6} \right) + \frac{5}{3} \beta_2^2 + \frac{\beta_4}{3} \right],
\end{align*}
\]

where $\text{Log} = 2 \ln(M_H/\Lambda_{\text{MS}}^{(n_f=5)})$ and the additional terms $\Delta as(M_H)^{\text{N^3LO}}$ and $\Delta as(M_H)^{\text{N^4LO}}$ were obtained in Refs. [54, 55] with the corresponding $N^3$LO and $N^4$LO matching conditions, which are allowing to determine the values of $\Lambda_{\text{MS}}^{(n_f=5)}$ from $\Lambda_{\text{MS}}^{(n_f=4)}$ by passing threshold of production of heavy flavor (in our case on-shell mass $m_b$). The analytical results of Ref. [55] are in complete agreement with the mixture of previous analogous analytical and semi-analytical calculations of Refs. [56, 57]. These conditions generalize to higher orders the NLO and $N^2$LO formulae, derived in Ref. [58] (the corresponding $N^2$LO relation was corrected in Ref. [59]). To save the space, we will not present here the explicit form of these equations. An interested reader can consult Ref. [60], where the results of Refs. [58, 59] and [54] are presented for the case of considering b-quark pole.
mass as the matching point. The corresponding N$^4$LO version will be presented elsewhere. In order to perform the on-shell analysis we are using the results for the b-quark pole mass values from Ref. [61], which are increasing from LO to N$^4$LO, and the LO, NLO, N$^2$LO expressions for $\Lambda_{\text{MS}}^{(n_f=4)}$, related to one of the most recent parton distribution fits Ref. [62]. It can be shown, that NLO and N$^2$LO values of these parameters are in agreement with the central values of $\Lambda_{\text{MS}}^{(n_f=4)}$ obtained in the process of the performed in Ref. [60] fits of Tevatron $\nu N$ deep-inelastic scattering data. In view of this the N$^3$LO value of $\Lambda_{\text{MS}}^{(n_f=4)}$ will be taken from Ref. [60]. Leaving aside the discussions of the possible convergence of the fits, at the N$^4$LO we will use the same value of $\Lambda_{\text{MS}}^{(n_f=4)}$ as at the N$^3$LO and will make the guess about the value of the b-quark mass. Using now the matching conditions of Ref. [55] transformed into the form, given in Ref. [60], we obtain the following values of $\Lambda_{\text{MS}}^{(n_f=5)}$:

| order | $m_b$ GeV | $\Lambda_{\text{MS}}^{(n_f=4)}$ MeV | $\Lambda_{\text{MS}}^{(n_f=5)}$ MeV |
|-------|-----------|---------------------------------|---------------------------------|
| LO    | 4.74      | 220                             | 168                             |
| NLO   | 4.86      | 347                             | 254                             |
| N$^2$LO | 5.02     | 331                             | 242                             |
| N$^3$LO | 5.23     | 333                             | 243                             |
| N$^4$LO | 5.45     | 333                             | 241                             |

Table 1: The values of the parameters on the on-shell studies.

In order to study the effects of separate NLO, N$^2$LO, N$^3$LO and N$^4$LO corrections to different parameterizations, it is also useful to define the quantity

$$R_{H\bar{b}b} = \Gamma_{H\bar{b}b}/\Gamma_0$$

and study its $M_H$ dependence. At the final step it is necessary to substitute the results from Table 1 into Eq. (39)–(47) by iterative way and compare necessary curves. They are shown in Fig. 1 both for $\Gamma_{H\bar{b}b}$ and $R_{H\bar{b}b}$ quantities.

### 4.2 Renormalization group resummation approach

In order to apply RG-resummation approach to $\Gamma_{H\bar{b}b}$ it is necessary to combine definition of Eq. (1), the basic formulae (27)–(29), the concrete numbers of Eqs. (30)–(31), the transformation relation of Eq. (38) and the expressions for $a_s(M_H)$ defined through Eqs. (44)–(47) and the similar expressions for $a_s(m_b)$. The results are presented at Fig. 2.
Γ_{\bar{b}b} may be presented in the following form

\[
\Gamma_{\bar{b}b} = \Gamma_0^{(b)} \left( \frac{a_s(M_H)}{a_s(m_b)} \right)^{(24/23)} \frac{AD(a_s(M_H))^2}{AD(a_s(m_b))^2} \left[ 1 + \sum_{i \geq 1} \Delta \Gamma_i a_s^i(M_H) \right] \tag{49}
\]

\[
\times (1 - \frac{8}{3} a_s(m_b) - 18.556 a_s(m_b)^2 - 175.76 a_s(m_b)^3 - 1892.2 a_s(m_b)^4),
\]

where

\[
AD(a_s)^2 = 1 + 2.351 a_s^2 + 4.383 a_s^2 + 3.873 a_s^3 - 15.153 a_s^4 \tag{50}
\]
Different curves at Figures 1 and 2 are related to applications of the results for $\Gamma_{H\beta\beta}$ and $R_{H\beta\beta}$ of the step-by-step substitution into the definitions of the corresponding expressions for the QCD coupling constants $a_s(M_H)$ and $a_s(m_b)$ from Eqs. (36), (44)–(47). The considerations of Fig. 1 confirm the findings of Refs. [46, 47] on the importance of the 1- and 2-loop contributions to $\Gamma_{H\beta\beta}$ in parameterization with on-shell mass. On the other hand, the other two corrections are not so big, but they both have a tendency to decrease the value of $\Gamma_{H\beta\beta}$ to its RG-improved expression (see Fig. 2). Moreover, the RG-resummation approach demonstrate clearly, that the perturbative theory to this quantity are well under control. The r.h.s. parts of Fig. 1 and Fig. 2 demonstrate the stability of good convergence of the related perturbative approximations. Thus, the consideration of the order $\alpha_s^4$ corrections, calculated in Ref. [10], support the feature of minimizing the difference between on-shell and RG-resummed parameterizations, already observed at the $\alpha_s^3$-level in Ref. [63]. The l.h.s. plots of Fig. 1 and Fig. 2 are more interesting from phenomenological point of view. It will be the next task to compare these results for the Higgs decay width, obtained by using FAPT approach of Ref. [41] and resummed FAPT analysis of Ref. [43]. Thus, our considerations leave space for further studies of peculiar features (presumably, better convergence) of modified APT predictions.

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