Optimal Decompositions of Translations of $L^2$-functions

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Abstract. In this paper we offer a computational approach to the spectral function for a finite family of commuting operators, and give applications. Motivated by questions in wavelets and in signal processing, we study a problem about spectral concentration of integral translations of functions in the Hilbert space $L^2(\mathbb{R}^n)$. Our approach applies more generally to families of $n$ arbitrary commuting unitary operators in a complex Hilbert space $\mathcal{H}$, or equivalent the spectral theory of a unitary representation $U$ of the rank-$n$ lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. Starting with a non-zero vector $\psi \in \mathcal{H}$, we look for relations among the vectors in the cyclic subspace in $\mathcal{H}$ generated by $\psi$. Since these vectors $\{U(k)\psi | k \in \mathbb{Z}^n\}$ involve infinite “linear combinations,” the problem arises of giving geometric characterizations of these non-trivial linear relations. A special case of the problem arose initially in work of Kolmogorov under the name $L^2$-independence. This refers to infinite linear combinations of integral translates of a fixed function with $l^2$-coefficients. While we were motivated by the study of translation operators arising in wavelet and frame theory, we stress that our present results are general; our theorems are about spectral densities for general unitary operators, and for stochastic integrals.

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1. Introduction

The study of frames in Hilbert space serves as an operator theoretic context for a variety of wavelet problems (see e.g., [Dau92]), and more generally for the study of generalized bases in harmonic analysis. Loosely speaking, a frame is a generalized basis system involving the kind of “over completeness.” Since many wavelet constructions do not yield orthogonality, the occurrence of wavelet frames is common. In this paper, we will consider various spectral problems in frame and wavelet theory in the context of a single unitary operator $T$ in a fixed Hilbert space $\mathcal{H}$, or more generally a finite commuting family of unitary operators. This of course is equivalent with the consideration of a unitary representation of the group $\mathbb{Z}$ of all integers, or of $\mathbb{Z}^d$ in the Hilbert space $\mathcal{H}$. Since every such representation is unitarily equivalent to a representation by multiplication. We will take advantage of this fact, and account for over completeness in terms of spectrum in the sense of spectral multiplicity for unitary operators, see e.g., [Hel64, Hel86]. For general facts on operators in Hilbert space, the reader may refer to e.g., [Con90].

An arbitrary infinite configuration of vectors $(f_k)_{k \in \mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ can be quite complicated, and it will be difficult to make sense of finite and infinite linear combinations $\sum_{k \in \mathbb{Z}} c_k f_k$. However, in applications, a particular system of vectors $f_k$ may often be analyzed with the use of a single unitary operator $U$ in $\mathcal{H}$. This happens if there is a fixed vector $\varphi \in \mathcal{H}$ such that $f_k = U^k \varphi$ for all $k \in \mathbb{Z}$. When this is possible, the spectral theorem will then apply to this unitary operator. A key idea in our paper is to identify a spectral density function computed directly from the pair $(\varphi, U)$.

Hence the study of linear expressions $\sum_k c_k f_k$ may be done with the aid of the spectral function for this pair $(\varphi, U)$. A spectral function for a unitary operator $U$ is really a system of functions $(p_\varphi)$, one for each cyclic subspace $\mathcal{H}(\varphi)$. In each cyclic subspace, the function $p_\varphi$ is a complete unitary invariant for $U$ restricted to $\mathcal{H}(\varphi)$:
by this we mean that the function $p_{\varphi}$ encodes all the spectral data coming from the vectors $f_k = U^k \varphi$, $k \in \mathbb{Z}$. For background literature on the spectral function and its applications we refer to [BH05, JP05, JWW04, PSWX03, Sad06, TT07].

In summary, the spectral representation theorem is the assertion that commuting unitary operators in Hilbert space may be represented as multiplication operators in an $L^2$-Hilbert space. The understanding is that this representation is defined as a unitary equivalence, and that the $L^2$-Hilbert space to be used allows arbitrary measures, and $L^2$ will be a Hilbert space of vector valued functions, see e.g., [He86].

The term “frame” refers to a generalized basis system involving the kind of “over completeness” that arises in signal processing problems with redundancy, see e.g., [BCHL06, CKS06] and [Chr03]. It is our aim here to account for this kind of over completeness via representation by operators and vectors arising from an appropriately chosen spectral representation.

While our theorems apply to the general case of the spectral function for a finite family of commuting operators in Hilbert space, our motivation is from the case of wavelet operators in the Hilbert spaces $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^d)$. Hence section 3 begins with function theory. The general and axiomatic setting is introduced in sect 3, which also contains the main technical lemmas. The remaining of our paper deals with applications, and they are divided into three parts: operator theory, frame theory and wavelet resolution spaces (sect 5), and finally stochastic integration (sect 8). But our recurrent theme is the spectral function.

**Notation.** We will be using $\mathbb{Z}$ for the group of integers, $\mathbb{R}$ for the real numbers, $\mathbb{C}$ for the complexes, and $\mathbb{T}$ for the circle, or equivalently the one-torus, $\mathbb{T} = \{ z \in \mathbb{C} | |z| = 1 \}$. The terminology $L^2(\mathbb{R})$, or $L^2(\mathbb{R}^n)$ will refer to the usual $L^2$-spaces defined from Lebesgue measure. We will have occasion to consider other measures as well, but it will be understood that they are Borel measures. They may be singular or not. The terminology “absolutely continuous” will refer to the comparison with Lebesgue measure in the appropriate context.

To see the connection to spectral multiplicity for unitary operators, consider the unitary operator $T$ in the Hilbert space $\mathcal{H} := L^2(\mathbb{R})$ of translation by 1, i.e., $(Tf)(x) := f(x - 1)$. In this case, this unitary operator $T$ is represented by multiplication via the Fourier transform $W$, where we view $W$ as a unitary operator, $W : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. But we shall be interested in localizing the analysis of $T$, so in considering the action of $T$ on a single vector $\psi$ in $L^2(\mathbb{R})$, or on a finite family. The action of $T$ on a single vector is determined by a scalar measure, and the action on a finite family of vectors by a matrix valued measure, as we outline below.

In order to make a direct connection to the study of closed translation invariant subspaces in $L^2(\mathbb{R})$, or $L^2(\mathbb{R}^d)$ we begin in section 3 with the case when our Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R})$, and when the unitary operator $T$ is translation by 1, i.e., $(Tf)(x) := f(x - 1)$. In that case, of course, the powers of $T$ by $k \in \mathbb{Z}$, $(T^k f)(x) := f(x - k)$ for $k \in \mathbb{Z}$, $x \in \mathbb{R}$, and $f \in L^2(\mathbb{R})$. 
In the context of over completeness in the Hilbert space $L^2(\mathbb{R})$, the question of non-trivial infinite linear combinations arises naturally, for example in the study of translation invariant subspaces in $L^2(\mathbb{R})$, see [Dau92] and [Hel64]. In this context, we must study infinite sums $\sum_{k \in \mathbb{Z}} c_k T_k f$ of the translates $(T_k f)(x) := f(x - k)$ for $k \in \mathbb{Z}$, when $f$ is a fixed function. Since these translates are not assumed to be orthogonal, one must be careful in the consideration of such infinite combinations. For example should the coefficients $c_k$, $k \in \mathbb{Z}$, be chosen from $l^2$? Or from what sequence space?

The question of infinite linear dependencies of translates arises in Kolmogorov’s and Wiener’s prediction theory (see e.g., [MS80]), and it is referred to there as $L^2$-independence vs $L^2$-dependence. This framework will be naturally included in our considerations below. However, we will begin with $L^2(\mathbb{R})$-considerations.

As it turns out, our approach applies more generally to families of $n$ arbitrary commuting unitary operators in a complex Hilbert space $\mathcal{H}$, or equivalently to the spectral theory of a unitary representation $U$ of the rank-$n$ lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. Starting with a non-zero vector $\psi$ in $\mathcal{H}$, we look for relations among the vectors in the cyclic subspace $\mathcal{H}(\psi)$ in $\mathcal{H}$ generated by $\psi$. Since these vectors $\{U(k)\psi | k \in \mathbb{Z}^n\}$ involve infinite “linear combinations,” the problem arises of giving geometric characterizations of these non-trivial linear relations.

This is the setup in section 3 below. Our study of the cyclic subspace $\mathcal{H}(\psi)$ is done with the aid of an associated spectral measure $\mu = \mu_\psi$, and we begin in section 4 by introducing an isometric isomorphism from $L^2(\mu)$ onto the cyclic subspace $\mathcal{H}(\psi)$.

In the multivariable case, we must make use of function theory on the $n$-torus $\mathbb{T}^n$, and the reader is referred to [Rud69] [Rud86] for that.

2. A Lemma

In wavelet theory, there are natural choices of Hilbert spaces and of generating functions. For each such choice, one wishes to consider linear combinations for the purpose of decomposing general vectors. For this to be successful it helps that the vectors $f_k$ have the following representation as outlined in Introduction: with the use of a single unitary operator $U$ in $\mathcal{H}$, if there is a fixed vector $\varphi \in \mathcal{H}$ such that $f_k = U^k \varphi$ for all $k \in \mathbb{Z}$.

The following lemma gives a necessary and sufficient condition for this to work. The crux is two versions of stationarity.

**Lemma 2.1.** Let $\mathcal{H}$ be a Hilbert space, and let $\{f_k\}_{k \in \mathbb{Z}} \subset \mathcal{H} \setminus \{0\}$ be given. Then the following two conditions are equivalent:

(i) $\langle f_{j+n} | f_{k+n} \rangle_\mathcal{H} = \langle f_j | f_k \rangle_\mathcal{H}$, for all $j,k,n \in \mathbb{Z}$; and
(ii) There is a Gaussian probability space \((\Omega, P)\), and a stationary stochastic process \((X_k)_{k \in \mathbb{Z}}\), i.e., each \(X_k : \Omega \to \mathbb{C}\) a random variable, and an isometry \(W : \mathcal{H} \to L^2(\Omega, P)\) such that \(Wf_k = X_k\) for all \(k \in \mathbb{Z}\), and

\[
\langle f_j | f_k \rangle_{\mathcal{H}} = E(X_j X_k) = \int_{\Omega} X_j(\omega) X_k(\omega) dP(\omega).
\]

**Proof.** For every \(n \in \mathbb{N}\), consider the usual Gaussian density in \(\mathbb{C}^{2n+1}\), i.e., \(z = (z_{-n}, \ldots, z_0, z_1, \ldots, z_n)\) with mean \(E_n(z_j) = 0\) and covariance

\[
E_n(z_j z_k) = \langle f_j | f_k \rangle_{\mathcal{H}}, \quad -n \leq j, k \leq n.
\]

It is easy to see that this is a Kolgomorov consistent system \([\text{Jor06a, Nel69}]\). The existence of the infinite Gaussian space \((\Omega, P)\) with the desired properties now follows. In particular \(\Omega\) is a space of function \(\omega : \mathbb{Z} \to \mathbb{C}\), and \(X_k(\omega) = \omega(k)\), \(k \in \mathbb{Z}\).

**Remark 2.2.** In section \(\S\) we will be using an analogous family of Gaussian Hilbert spaces, and the associated spectral density functions.

### 3. Notation and Statement of the Problem

**Preliminaries**

While we will be stating our main results in the general context of unitary operators in Hilbert space (section \(\S\) below), and stochastic processes (section \(\S\)), it is helpful to first consider a prime example illustrating the kind of interplay between function theory and Hilbert space geometry. Such an example is afforded by the case of the unit-translation operator in the Hilbert space \(L^2(\mathbb{R})\), and we begin with this below.

We will study the interconnection between \(L^2(\mathbb{R})\)-functions and their restrictions to unit intervals \(I_n = [n, n+1)\) for \(n \in \mathbb{Z}\). Since the individual intervals \(I_n\) arise from \(I_0 = [0, 1)\) by integer translation, it will be convenient to compare the two Hilbert spaces \(L^2(0, 1)\) and \(L^2(\mathbb{R})\).

The \(L^2\)-norm on \(\mathbb{R}\) may be decomposed as

\[
\int_{\mathbb{R}} |\psi(x)|^2 dx = \sum_{n \in \mathbb{Z}} \int_{n}^{n+1} |\psi(x)|^2 dx = \int_{0}^{1} \sum_{n \in \mathbb{Z}} |\psi(x + n)|^2 dx.
\]

So if we introduce \(p_\psi(x) := \sum_{n \in \mathbb{Z}} |\psi(x + n)|^2 = \text{PER}_\mathbb{Z}|\psi|^2\), then \(p_\psi \in L^1(0, 1)\), and

\[
\int_{0}^{1} p_\psi(x) dx = \int_{\mathbb{R}} |\psi(x)|^2 dx = \|\psi\|_{L^2(\mathbb{R})}^2.
\]

We will study what properties of a given \(L^2(\mathbb{R})\)-function \(\psi\) can be predicted from the local \(p\)-version. The problem is intriguing, since the fixed function \(p \in L^1(0, 1)\) may be decomposed in many different ways as \(p(x) = \text{PER}_{\mathbb{Z}}|\psi(x)|^2\) with \(\psi \in L^2(\mathbb{R})\).

We will study the initial function \(p\) via \(L^2(\mu)\) where

\[
d\mu(x) = p(x) dx.
\]  

(3.1)
In fact, we will study $L^2(\mu)$ even if the measure $\mu$ is not assumed to be absolutely continuous with respect to Lesbegue measure. But in the absolutely continuous case, the function $p$ will pop up as a Radon-Nikodym derivative.

Moreover, we will need the following matrix version of (3.1). Let $N$ be a positive integer and let $M_N^+$ denote the complex $N \times N$ matrices $P$ satisfying $\text{spec}(P) \subset [0, \infty)$.

**Definition 3.1.** We say that a measurable function $[0, 1) \ni x \mapsto P(x) \in M_N^+$ is $L^1$ if for all vectors $v \in \mathbb{C}^N$ the functions $x \to \langle v | P(x) v \rangle$ is in $L^1(0, 1)$

**Notation.** For $v, w \in \mathbb{C}^N$, we set

$$\langle v | w \rangle := \sum_{k=1}^{N} v_k w_k.$$ 

If $\mu$ is any Borel measure supported on a subset of $\mathbb{R}$, and $f, g \in L^2(\mu)$, we set

$$\langle f | g \rangle_{L^2(\mu)} = \int f(x) g(x) d\mu(x).$$ 

Let $\mathcal{F} \subset L^2(\mathbb{R})$ be a finite subset of $N$ elements. We then define a function

$$P = P_{\mathcal{F}} : [0, 1] \to M_N^+$$ 

as follows. Let $\mathcal{F}$ index the rows and the columns in $P$ via $(P_{\phi, \psi})$, where

$$(P_{\phi, \psi}) = \text{PER}(\phi(x) \psi(x)).$$

An easy calculation shows that $P$ is $L^1$, and that

$$\int_0^1 P_{\phi, \psi}(x) dx = \langle \phi | \psi \rangle_{L^2(\mathbb{R})}, \text{ for all } \phi, \psi \in \mathcal{F}.$$

**Lemma 3.2.** Let $P : [0, 1] \to M_N^+$ be given and suppose that $x \mapsto \|P(x)\|_{2, 2} \in L^\infty$ then for all $v \in \mathbb{C}^N$, we have the estimate

$$\|P(x)v\|^2 \leq \lambda \langle v | P(x)v \rangle$$

where $\lambda := \sup_x \|P(x)\|_{2, 2}$

**Proof.** Since $\|P(x)v\|^2 = \langle v | P(x)^2 v \rangle$, the assertion is a statement about a single $P \in M_N^+$ referring now to the usual ordering on the Hermitian matrices. Let $P = \sum \lambda_i E_i$ be the spectral resolution with $(E_i)$ denoting orthogonal projections, i.e., $E_j E_k = \delta_{j,k} E_j$. Then

$$P^2 = \sum_j \lambda_j^2 E_j \leq \max_{k} \sum_j \lambda_j E_j = (\max_{k} \lambda_k) \ P.$$ 

The result is immediate from this. \qed
**Definition 3.3.** We shall use the following notation $e_k, k \in \mathbb{Z}$ for the Fourier basis in $L^2(0,1)$:

$$e_k(x) = e^{2\pi kx}, \ x \in [0,1].$$

With this convention, the Parseval identity in $L^2(0,1)$ reads,

$$\sum_{k \in \mathbb{Z}} |c_k|^2 = \int_0^1 \left| \sum_{k \in \mathbb{Z}} c_k e_k(x) \right|^2 dx, \text{ for all } c = (c_k) \in l^2(\mathbb{Z}). \quad (3.2)$$

**Definition 3.4.** Let $\mu$ be a positive Borel measure supported in the unit-interval $[0,1] \subset \mathbb{R}$. Assume $\mu([0,1]) < \infty$. Let $D \subset l^2(\mathbb{Z})$ of all finite sequence $(c_k)_{k \in \mathbb{Z}}$; i.e., $c_k = 0$ for all but a finite set of index values of $k$. We define a linear operator $F : D \rightarrow L^2(\mu)$ as follows:

$$F((c_k)) := \sum_k c_k e_k(x). \quad (3.3)$$

**Lemma 3.5.** If

$$\sum_{k \in \mathbb{Z}} \left\| \int e_k(x) d\mu(x) \right\|^2 < \infty, \quad (3.4)$$

then the operator $F$ is closable.

**Proof.** An operator is said to be closable if the closure of its graph $(c,Fc)_{c \in D}$ is again a graph of a linear operator. In that case, we say that the resulting operator is the closure of $F$. It is known that to test closability we only need to check that the domain of the adjoint operator $F^*$ is dense. The formula for $F^*$ is as follows:

$$\langle F^* f | c \rangle_{l^2} = \langle f | Fc \rangle_{L^2(\mu)}, \ f \in \text{dom}(F^*), \ c \in D.$$ 

It follows that

$$(F^* f)_k = \int_0^1 e_k(x) f(x) d\mu(x)$$

Hence, (3.4) is simply saying that the Fourier functions $e_k$ belong to $\text{dom}(F^*)$. Since the span of the functions $e_k$ is dense in $L^2(\mu)$, we conclude that $F$ is closable. \hfill \Box

We shall also need the following fundamental result from operator theory:

**Corollary 3.6.** Suppose (3.4) is satisfied, denote the closure of the operator $F$ by the same symbol. Then $F^* F$ is selfadjoint operator, and there is a partial isometry $U$ such that $F = U(F^* F)^{1/2} = (FF^*)^{1/2} U$. The partial isometry $U$ will be chosen with initial space equal to $l^2 \ominus \ker(F)$ and final space equal to the closure of the range of $F$. 

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**Optimal Decompositions of Translations of $L^2$-functions**
4. Unitary Operators

Let $\mathcal{H}$ be a (complex) Hilbert space and let $(T_1, \ldots, T_n)$ be a finite family of commuting unitary operators in $\mathcal{H}$. We introduce the following multi-index notation $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, i.e., $k_j \in \mathbb{Z}$ for $1 \leq j \leq n$, and

$$T^k := T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} \quad (4.1)$$

For vectors $\psi \in \mathcal{H} \setminus \{0\}$, we set

$$\psi_k := T^k \psi, \quad k \in \mathbb{Z}^d \quad (4.2)$$

The closed subspace in $\mathcal{H}$ generated by the vectors \{\(\psi_k | k \in \mathbb{Z}^d\)\} will be denoted $\mathcal{H}(\psi)$; and it is called the cyclic subspace generated by the vector $\psi$.

For points $z = (z_1, z_2, \ldots, z_n) \in \mathbb{T}^n$, we shall be using the $[0, 1)^n$ parametrization

$$z = (e^{2\pi i x_1}, e^{2\pi i x_2}, \ldots, e^{2\pi i x_n}), \quad (4.3)$$

and identifications:

$$z^k \longleftrightarrow (k_1 x_1, k_2 x_2, \ldots, k_n x_n) \text{ if } k \in \mathbb{Z}^n;$$

and

$$zw \longleftrightarrow (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad (4.4)$$

where the addition in the coordinates in (4.4) are addition mod 1, i.e., addition in the group $\mathbb{R}/\mathbb{Z}$. Hence $\mathbb{T}^n \simeq \mathbb{R}^n/\mathbb{Z}^n$.

Our present approach to the Spectral Representation Theorem for families of commuting unitary operators in Hilbert space is closest to that of [Nel69], a set of Lecture Notes by Ed Nelson; now out of print but available on URL [http://www.math.princeton.edu/~nelson/]. The multiplicity function is presented there as a complete invariant. Of course, in wavelet applications, there are also the additional consistency relations (see [BMM99]), but the notion of a multiplicity function is general.

This representation theoretic approach, adapted below to our present applications, fits best with how we use the Spectral Representation Theorem in understanding wavelets and generalized multiresolution analysis (GMRAs); see also [Bag00] [BJMP05] [BMM99]. The standard presentation of the Spectral Theorem in textbooks is typically different from the Spectral Representation Theorem, and here we spell out the connection; developing here a computational approach.

**Lemma 4.1.** Let $\mathcal{H}, T = (T_1, \ldots, T_n)$ and $\psi$ be as described above, and let $\mathcal{H}(\psi)$ be the cyclic subspace in $\mathcal{H}$ generated by \(\{\psi_k | k \in \mathbb{Z}^n\}\). Then there is a unique Borel measure $\mu$ on $[0, 1]^n$ and an isometric isomorphism

$$W : L^2([0, 1]^n, \mu) \to \mathcal{H}(\psi) \quad (4.5)$$

determined by

$$\sum_{k \in \mathbb{Z}^n} c_k z^k \mapsto \sum_{k \in \mathbb{Z}^n} c_k T^k \psi \quad (4.6)$$
Remark 4.2. The significant part of the lemma relates to the Spectral Theorem: It is the extension of the mapping in (4.10) from the trigonometric polynomials to the algebra of the measurable functions.

Proof. (of Lemma 4.1) We first elaborate the formula (4.6). Set

$z^k := e_k(x)$

$= e^{i2\pi k_1 x_1}e^{i2\pi k_2 x_2} \ldots e^{i2\pi k_n x_n}$

$= e^{i2\pi k \cdot x}$

with $k \cdot x := k_1 x_1 + \cdots + k_n x_n$, and for finite summations:

$$\sum_{k \in \mathbb{Z}} c_k z^k = \sum_{k \in \mathbb{Z}} c_k e_k(x).$$

(4.7)

This is the representation of the trigonometric polynomials as $\mathbb{Z}^n$-periodic functions. Set

$m_c(x) := \sum_{k \in \mathbb{Z}^n} c_k e_k(x)$

and

$m_c(T)\psi := \sum_{k \in \mathbb{Z}^n} c_k T^k \psi.$

(4.8)

By the Spectral Theorem, this is a projection valued measure on $[0,1]^n$ such that

$$m_c(T) = \int_{[0,1]^n} m_c(x) E(dx).$$

(4.9)

The measure $E$ is defined on the Borel sets $\mathcal{B}_n$ in $[0,1]^n$:

$E : \mathcal{B}_n \rightarrow \text{PROJ}(\mathcal{H})$

and

$$E(A_1 \cap A_2) = E(A_1) E(A_2), \text{ for all } A_1, A_2 \in \mathcal{B}_n.$$ (4.10)

A linear operator $E : \mathcal{H} \rightarrow \mathcal{H}$ is said to be a projection if and only if $E = E^* = E^2$. Note that condition (4.10) for a projection valued measure implies that for $A_1 \cap A_2 = \emptyset$, the subspaces $E(A_1)\mathcal{H}$ and $E(A_2)\mathcal{H}$ are orthogonal. The measure $\mu = \mu_\psi$ in the conclusion in the lemma is

$$\mu(A) := ||E(A)\psi||^2$$

$$= \langle \psi | E(A)\psi \rangle_\mathcal{H}.$$ (4.11)

Since

$$\mathbb{Z}^n \ni k \mapsto U(k) := T^k$$

$$= T_1^{k_1}T_2^{k_2} \ldots T_n^{k_n}$$

is a unitary representation, there is a projection valued measure

$$\mathcal{B}(\mathbb{T}^n) \ni A \mapsto E(A) \in \text{PROJ}(\mathcal{H})$$
such that
\[
I_{\mathcal{H}} = \int_{\mathbb{T}^n} E(dx), \quad \text{and}
\]
\[
U(k) = \int_{\mathbb{T}^n} e_k(x) E(dx), \quad \text{for all } k \in \mathbb{Z}^n.
\]

This means that for every measurable function \( m : \mathbb{T}^n \to \mathbb{C} \), the operator \( m(T) \) may be defined by the functional calculus

\[
m(T) = \int_{\mathbb{T}^n} m(x) E(dx).
\] (4.11)

Setting \( \mu = \mu_\psi \) we conclude that \( \mu \) is a scalar Borel measure, i.e.,

\[
\mu_\psi(A) := \langle \psi | E(A) \psi \rangle_{\mathcal{H}}
\] (4.12)

\[
= \| E(A) \psi \|^2_{\mathcal{H}}
\] (4.13)

We have used that

\[
E(A) = E(A)^* = E(A)^2, \quad \text{for all } A \in \mathcal{B}(\mathbb{T}^n).
\] (4.14)

**Lemma 4.3.** Let \( Q : \mathcal{H} \to \mathcal{H} \) be a linear operator. Then the following are equivalent:

(a) \( QU(k) = U(k)Q \), for all \( k \in \mathbb{Z}^n \); and
(b) \( QE(A) = E(A)Q \), for all \( A \in \mathcal{B}(\mathbb{T}^n) \).

In summary, a bounded operator \( Q \) commutes with the unitary representation \( U \) if and only if it commutes with the spectral projections.

**Proof.** Left to the reader. \( \square \)

We are now ready to define the isometry \( W = W_\psi \) from (4.15), where

\[
W_\psi : L^2(\mathbb{T}^n, \mu_\psi) \to \mathcal{H}(\psi).
\] (4.15)

Set

\[
W_\psi(m) := m(T)\psi
\] (4.16)

with \( m \) a measurable function on \( \mathbb{T}^n \), and with \( m(T) \) defined by the measurable functional calculus in (4.11).
The verification of the isometric property for $W_\psi$ is as follows:

\[ \| W_\psi(m) \|_H^2 = \| m(T)\psi \|^2 \]

\[ = \left\| \int_{\mathbb{T}^n} m(x)E(dx)\psi \right\|_H^2 \]

\[ = \langle \int_{\mathbb{T}^n} m(x)E(dx)\psi, \int_{\mathbb{T}^n} m(y)E(dy)\psi \rangle_H \]

\[ = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \overline{m(x)m(y)}E(dx)\psi E(dy)\psi \rangle_H \]

\[ \text{by (4.11)} \]

\[ = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \overline{m(x)m(y)}E(dx)\psi \| E(dy)\psi \|_H \]

\[ \text{by (4.10)} \]

\[ = \int_{\mathbb{T}^n} \overline{|m(x)|^2}E(dx)\psi \| \psi \|_H \]

\[ \text{by (4.12)} \]

\[ = \| m \|^2_{L^2(\mathbb{T}^n, \mu_\psi)} \]

While the calculation is done initially for $m \in L^\infty(\mu_\psi)$, after the isometric property

\[ \| m(T)\psi \|_H = \| m \|_{L^2(\mu_\psi)} \]

is verified, it follows that

\[ W_\psi(m) := m(T)\psi \]

is now well defined for all $m \in L^2(\mu_\psi)$; and the operator $W_\psi$ resulting by $L^2$-norm completion will be defined on all of $L^2(\mu_\psi)$.

To show that

\[ W_\psi L^2(\mu_\psi) = \mathcal{H}(\psi) \]

we check that if $\xi \in \mathcal{H}(\psi)$, and

\[ \langle \xi | m(T)\psi \rangle_H = 0, \text{ for all } m \in L^2(\mu_\psi), \]

then $\xi = 0$.

First note that for $A \in \mathcal{B}(\mathbb{T}^n)$ we have

\[ |\langle \xi | E(A)\psi \rangle|^2 \leq \| \xi \|^2 \mu_\psi(A). \]

Hence, by the Radon-Nikodym Theorem, there is a function $F_\xi \in L^1(\mu_\psi)$ such that

\[ \langle \xi | E(dx)\psi \rangle = F_\xi(x)d\mu_\psi(x), \]

and

\[ \int_{\mathbb{T}} m(x)F_\xi(x)d\mu_\psi(x) = 0 \]

for all functions $m$ as above.

As a result,

\[ \xi = \overline{F_\xi(T)}\psi. \]
But since (4.20) holds for all \(m\), we conclude that \(F_\xi = 0\), \(\mu_\psi\) a. e.

Substituting back into (4.21), we conclude that \(\xi = 0\). But \(W_\psi\) is isometric, so its range is closed. Since it is dense, the desired conclusion \(4.18\) now follows. \(\Box\)

We now turn to the case of matrix-value measures. The setting is as above: \(T = (T_1, ..., T_n)\) a given set of commuting unitary operators acting in a Hilbert space \(\mathcal{H}\), i.e.,

\[
T_j : \mathcal{H} \to \mathcal{H}, \quad 1 \leq j \leq n. \tag{4.22}
\]

The main difference is that we will be considering cyclic subspaces in \(\mathcal{H}\) generated by a fixed finite family \(\mathcal{F} = \{\psi_1, \psi_2, ..., \psi_N\}\) in \(\mathcal{H} \setminus \{0\}\). We let \(\mathcal{H}(\mathcal{F})\) denote the closed span of the vectors \(\{T^k\psi_j | k \in \mathbb{Z}^n, 1 \leq j \leq n\}\). \(\tag{4.23}\)

Let

\[
\mathcal{B}(\mathbb{T}^n) \ni A \mapsto E(A) \in \text{PROJ}(\mathcal{H})
\]

be the projection valued measure introduced in (4.11), i.e., satisfying

\[
T^k = \int_{\mathbb{T}^n} \varepsilon^k E(dz), \quad \tag{4.24}
\]

or in additive notation

\[
T^k = \int_{\mathbb{T}^n} e_k(x) E(dx). \quad \tag{4.25}
\]

Setting

\[
P_{r,s}(A) := \langle \psi_r | E(A) \psi_s \rangle, \quad \text{for } 1 \leq r, s \leq N, \ A \in \mathcal{B}, \tag{4.26}
\]

we note that \(P(\cdot)\) is a matrix-valued measure on \(\mathbb{T}^n\), i.e., taking values \(M^+_N\).

Specifically, let \(v \in \mathbb{C}^N\), and \(A \in \mathcal{B}(\mathbb{T}^n)\). Setting \(\psi := \sum_{r=1}^N v_r \psi_r\), we get:

\[
\langle v | P(A) v \rangle_{\mathbb{T}^n} = \sum_r \sum_s v_r \langle \psi_r | E(A) \psi_s \rangle v_s = \langle \sum_r v_r \psi_r | E(A) \sum_s v_s \psi_s \rangle_{\mathcal{H}} = \|E(A)\psi\|^2_{\mathcal{H}} \geq 0,
\]

as claimed.

**Definition 4.4.** Let \(P\) be the matrix-valued measure defined on \(\mathcal{B}(\mathbb{T}^n)\) as in (4.20).

The Hilbert space \(L^2(P)\) then consists of all measurable functions \(m : \mathbb{T}^n \to \mathbb{C}^N\) such that

\[
\|m\|^2_P := \int_{\mathbb{T}^n} \langle m | P(dx) m \rangle_{\mathcal{H}} < \infty \tag{4.27}
\]

where \(m^T = (m_1, ..., m_N)\), and

\[
\langle m | P(dx) m \rangle_{\mathcal{H}} := \sum_r \sum_s m_r(x) P_{r,s}(dx) m_s(x). \tag{4.28}
\]
Lemma 4.5. Let $\mathcal{H}$, $T = (T_1, ..., T_n)$ and $\mathcal{F} = \{\psi_1, \psi_2, ..., \psi_N\}$ be as described above. Let $P = P_{\mathcal{F}}$ be the corresponding matrix-value measure, and let $\mathcal{H}(\mathcal{F})$ denote the (closed) cyclic subspace in $\mathcal{H}$ generated by $\{T^k \psi | k \in \mathbb{Z}^n, \psi \in \mathcal{F}\}$. For $m \in L^2(P)$, set

$$W(m) := \sum_{r=1}^{N} m_r(T)\psi_r.$$  
(4.29)

Then $W$ defines an isometric isomorphism

$$W : L^2(P) \to \mathcal{H}(\mathcal{F})$$

mapping $L^2(P)$ onto $\mathcal{H}$. 

Proof. Except for technical modification, the proof of the lemma follows the idea in the proof of Lemma 4.1, which is the special case of $N = 1$.

Hence we restrict our present discussion to the verification that $W$ is isometric from $L^2(P)$ into $\mathcal{H}$.

For $m$ in $L^2(P)$, set $m^T := (m_1, ..., m_r)$, the scalar coordinate functions. Then

$$\|Wm\|^2_{\mathcal{H}} = \left\| \sum_{r=1}^{N} m_r(T)\psi_r \right\|^2_{\mathcal{H}}$$

$$= \left\| \sum_{r=1}^{N} \int_{\mathbb{T}^n} m_r(x)E(dx)\psi_r \right\|^2_{\mathcal{H}}$$

$$= \sum_{r=1}^{N} \sum_{s=1}^{N} \int_{\mathbb{T}^n} m_r(x)P_{r,s}(dx)m_s(x)$$

$$= \int_{\mathbb{T}^n} \langle m|P(dx)m \rangle$$

by $(4.28)$

$$\|m\|^2_{L^2(P)}.$$ 

The arguments from Lemma 4.1 shows that $W$ in fact maps onto $\mathcal{H}(\mathcal{F})$. 

5. Corollaries and Applications

In this section we return to the main application of the general spectral theory developed in section 4.

While the setting in section 4 applies, in the single variable case, to a general unitary operator $T$ in Hilbert space $\mathcal{H}$; and in the multivariable case to a finite commuting family $(T_1, T_2, ..., T_n)$ of unitary operators, the main application is to $\mathcal{H} = L^2(\mathbb{R})$, or to $\mathcal{H} = L^2(\mathbb{R}^n)$.

In the single variable case, the unitary operator $T$ will be translation by 1 to the right of functions on the real line $\mathbb{R}$; and in the multivariable case, the system
will consist of these translation operators but now referring to unit-translation in the \( n \) coordinate directions.

Our first Corollary of the general formulas for the spectral measure/function in section 4 will show that, in the translation case, the spectral measure \( \mu \) is absolutely continuous with respect to Lebesgue measure, and that the Radon-Nikodym derivative is the function introduced above in section 3.

While the results apply to the general multivariable setting, for clarity we will only give full details in the single variable case. But with the aid of section 3, the reader will be able to extend the formulas from \( n = 1 \) to \( n > 1 \).

We stress that there is a spectral measure \( \mu = \mu_\psi \) for every vector in the Hilbert space. But this section is concerned with the Hilbert spaces \( \mathcal{H} = L^2(\mathbb{R}^n) \).

Corollary 5.1. Let \((Tf)(x) := f(x - 1)\) be the translation in \( L^2(\mathbb{R}) \), and let

\[
\hat{f}(t) := \int_{\mathbb{R}} e^{-i2\pi tx} f(x) dx \tag{5.1}
\]

be the \( L^2 \)-Fourier transform. Let \( \psi \in L^2(\mathbb{R}) \setminus \{0\} \), and let \( \mu := \mu_\psi \) be the spectral measure introduced in Lemma 4.1. Then

(i) \( \mu \) is absolutely continuous with respect to Lebesgue measure \( dt \) on \([0,1]\);

(ii) The Radon-Nikodym derivative is as follows

\[
\frac{d\mu_\psi}{dt} = \sum_{n \in \mathbb{Z}} |\hat{\psi}(t + n)|^2 = PER|\hat{\psi}|^2(t); \tag{5.2}
\]

and, in particular,

(iii) \( PER|\hat{\psi}|^2 \) is in \( L^1(0,1) \).

Proof. Recall the functional calculus defined for the Borel functions \( m \) on \([0,1] \approx \mathbb{R}/\mathbb{Z} \) introduced in Lemma 4.5 by extending the following formula

\[
p(z) = \sum_k c_k z^k \mapsto \sum_k c_k T^k \tag{5.3}
\]

defined initially on the polynomials where the following notation and identification is used: \( p(t) := p(e^{i2\pi t}) \), and

\[
(T^k f)(x) = T \circ \cdots \circ T f(x) = f(x - k) \quad \text{for } x \in \mathbb{R}, \text{ and } k \in \mathbb{Z}.
\]

If \( m \) is a Borel function on \([0,1]\) we let \( m(T) \) denote the Borel functional calculus which arises from the extension of (5.3) via Lemma 4.1. If \( A \in \mathcal{B}([0,1]) \), we will use this for the function \( m := \chi_A \). In particular, \( \chi_A \) will automatically be extended from \([0,1]\) to \( \mathbb{R} \) by periodicity.
As a result
\[
\int_A \text{PER}(\hat{\psi})^2 dt = \int_0^1 \chi_A(t) \text{PER}(\hat{\psi})^2(t) dt \\
= \int_{-\infty}^{\infty} \chi_A(t) |\hat{\psi}(t)|^2 dt \\
= \|\chi_A(T)\psi\|^2 \\
= \mu_\psi(A).
\]
This proves that \(d\mu_\psi\) is absolutely continuous with respect to \(dt\) on \([0,1]\), and that the formula (5.2) holds for the Radon-Nikodym derivative \(\frac{d\mu_\psi}{dt}\). Since \(d\mu_\psi\) is in \(L^1(0,1)\) by the Radon-Nikodym theorem, we conclude that the almost everywhere defined function \(\text{PER}(\hat{\psi})^2\) is well defined, and in \(L^1(0,1)\). \(\square\)

We now turn to several generalizations, including the multivariable case, and some cases of unitary operators \(T\) different from the translation operator in \(L^2(\mathbb{R})\). There is an additional question which is motivated by applications, and which we will answer concern conditions for when the Radon-Nikodym derivative of the spectral measure \(\mu = \mu_\psi\) is in \(L^2\), and in \(L^\infty\).

We now turn to a variety of structural properties that may hold for a bilateral sequence of vectors obtained by the application of powers of a single unitary operator to a fixed non-zero vector \(\psi\) in a Hilbert space. These are conditions which arise in harmonic analysis [Chr03], in wavelets [BMM99, Bag00, BMP05], and in signal processing [CKS06], and they are denoted by the name “frame.” But there is a host of distinct frame conditions, and our next result shows that they are determined by specific properties of our associated measure \(\mu_\psi\).

In addition, we show that under certain conditions, the vector \(\psi\) may be replaced by a renormalized version which has the effect of turning the spectral Radon-Nikodym derivative into the indicator function for a certain Borel set.

**Definition 5.2.** Let \(T: \mathcal{H} \to \mathcal{H}\) be a unitary operator, and let \(\psi \in \mathcal{H} \setminus \{0\}\). For \(k \in \mathbb{Z}\), set
\[
\psi_k := T^k \psi. \quad (5.4)
\]
Let \(\mathcal{H}(\psi)\) be the cyclic subspace generated by \(\psi\), i.e., the closed space of the family \(\{\psi_k| k \in \mathbb{Z}\}\); see Lemma 4.1.

We shall be concerned with the following properties for the sequence of vectors \(\psi_k, k \in \mathbb{Z}\):

- **ONB:**
  \[
  \langle \psi_j | \psi_k \rangle_\mathcal{H} = \delta_{j,k}, \text{ for all } j, k \in \mathbb{Z}; \quad (5.5)
  \]
i.e., \(\{\psi_j\}\) is an orthonormal basis in \(\mathcal{H}(\psi)\).

- **Parseval:**
  \[
  \sum_{j \in \mathbb{Z}} |\langle \psi_j | f \rangle|^2 = \|f\|^2_\mathcal{H}, \text{ for all } \mathcal{H}(\psi); \quad (5.6)
  \]
i.e., \(\{\psi_j\}\) is a Parseval frame in \(\mathcal{H}(\psi)\).
• **Bessel:**
  There exists $B \in \mathbb{R}_+$ such that
  \[ \sum_{j \in \mathbb{Z}} |\langle \psi_j | f \rangle|^2 \leq B \|f\|_{H}^2, \text{ for all } f \in \mathcal{H}(\psi); \tag{5.7} \]
i.e., $\{\psi_j\}$ is a Bessel frame in $\mathcal{H}(\psi)$.

• **Frame:**
  There exists $A, B \in \mathbb{R}_+$, $A \leq B < \infty$ such that
  \[ A \|f\|_{H}^2 \leq \sum_{j \in \mathbb{Z}} |\langle \psi_j | f \rangle|^2 \leq B \|f\|_{H}^2, \text{ for all } f \in \mathcal{H}(\psi); \tag{5.8} \]
i.e., $\{\psi_j\}$ is a frame in $\mathcal{H}(\psi)$.

• **Riesz:** There exists $A, B \in \mathbb{R}_+$, $A \leq B < \infty$ such that
  \[ A \|c\|_2^2 \leq \left\| \sum_{k \in \mathbb{Z}} c_k \psi_k \right\|^2_{H} \leq B \|c\|_2^2, \text{ for all } c = (c_j) \in l^2(\mathbb{Z}); \tag{5.9} \]
i.e., $\{\psi_j\}$ is a Riesz basis (with bounds $(A, B)$) for $\mathcal{H}(\psi)$. A part of the definition is the convergence of $\sum_{k \in \mathbb{Z}} c_k \psi_k$ in $\mathcal{H}$.

• **Absolute Continuity:** We say that the measure $\mu_\psi$ is absolutely continuous if Radon-Nikodym derivative
  \[ \frac{d\mu_\psi}{dx} = p_\psi \in L^1(0,1) \] (5.10)
exists, where by $dx$, we mean the restriction to $[0,1]$ of Lebesgue measure. It will be convenient to use the isometric isomorphism $L^2(0,1) \rightarrow L^2(\mathbb{T})$ defined by: $f(x) \leftrightarrow F(e_1(x))$ where $e_1(x) := e^{i2\pi x}$.

The next result yields a gradation of the conditions in Definition 5.2. In its statement we will make use of the notation $\text{esssupp}$ for essential support, and $\text{esssup}$ for essential supremum. The abbreviation a.e. will be for almost every.

**Theorem 5.3.** We have the following implications: (5.7) Bessel $\implies$ (5.10) Absolute Continuity $\implies$ Renormalization.

By the last condition (renormalization) we mean this: There is a vector $\psi_{\text{REN}} \in \mathcal{H}$, and a Borel subset $A \subset [0,1]$ such that the following three conditions (a) - (c) hold:

(a) $\mathcal{H}(\psi) = \mathcal{H}(\psi_{\text{REN}})$, and
(b) $d\mu_{\psi_{\text{REN}}}(x) = \chi_A(x) dx$, and
(c) $A = \text{ess sup} \mu_\psi$.

**Proof.** Assume (5.7). Let $f \in \mathcal{H}(\psi)$ have the form $f = m(T)\psi$, see (4.17), as in the proof of Lemma 4.1. We then have
\[ \sum_{k \in \mathbb{Z}} |\langle \psi_k | f \rangle|^2 = \sum_{k \in \mathbb{Z}} \left| \int_0^1 \overline{e_k(x)} m(x) d\mu_\psi(x) \right|^2. \tag{5.11} \]
Using (5.7), we conclude that the Fourier coefficients of the measure $m(x)d\mu_\psi(x)$ are in $l^2(\mathbb{Z})$. Since this holds for all $m$, it follows that $d\mu_\psi$ has the form

$$d\mu_\psi(x) = p_\psi(x) dx \quad (5.12)$$

where $p_\psi \in L^1(0,1)$ is the Radon-Nikodym derivative. This is condition (5.10), which was asserted.

Now substituting back into (5.11) and (5.7), we get

$$\int_0^1 |m(x)p_\psi(x)|^2 dx \leq B \int_0^1 |m(x)|^2 p_\psi(x) dx \quad (5.13)$$

Rewriting this as

$$\int_0^1 |\sqrt{p_\psi}m|^2 p_\psi(x) dx \leq B \int_0^1 |m|^2 p_\psi dx$$

we see that (5.7) is equivalent to the boundedness of the following multiplication operator

$$Q_\psi : m \mapsto \sqrt{p_\psi}m \quad (5.14)$$

in the Hilbert space $L^2([0,1], p_\psi dx)$, i.e., to estimate

$$\|Q_\psi m\|^2_{L^2(p_\psi)} \leq B\|m\|^2_{L^2(p_\psi)}, \quad \text{for all } m. \quad (5.15)$$

Hence the function $p_\psi$ is essentially bounded (relative to Lebesgue measure), and $\text{esssup}(p_\psi) = \|p_\psi\|_\infty \leq B$. Let $A = A_\psi := \text{esssupp}(p_\psi) = \text{the essential support of } p_\psi$. Then the identities

$$p_\psi \chi_A = p_\psi \quad (5.16)$$

and

$$p_\psi \chi_{[0,1]\setminus A} = 0 \quad (5.17)$$

hold almost everywhere on $[0,1]$.

Using Lemma 4.1 and setting

$$\xi(x) := \chi_{A_\psi}(x)p_\psi(x)^{-1/2}, \quad (5.18)$$

we note that the functional calculus applied to $T$ yields a vector

$$\psi_{\text{REN}} := \xi(T)\psi \quad (5.19)$$

well defined in $\mathcal{H}$.

We now apply the functional calculus of $T$ to the modified vector, i.e., to $\psi_{\text{REN}}$, and by Lemma 4.1 we conclude that

$$\|m(T)\psi_{\text{REN}}\|^2_{\mathcal{H}} = \|m\xi(T)\psi\|^2_{\mathcal{H}}$$

by (4.17) \[ \int_0^1 |m\xi|^2 p_\psi dx \]

holds for all $m$. Conclusions (b) and (c) now follow. Conclusion (a) amounts to a third application of Lemma 4.1 to $\psi_{\text{REN}}$. \qed
Some of the conclusions in the next corollary are folklore, but we include them for completeness. Moreover, they arise as special cases of our more general theorems.

**Corollary 5.4.** Let $T : \mathcal{H} \to \mathcal{H}$ be a unitary operator, and let $\psi \in \mathcal{H} \setminus \{0\}$ be given. Suppose the measure $\mu_\psi$ is absolutely continuous, and let $d\mu_\psi(x) = p_\psi(x)dx$ with $p_\psi$ denoting the Radon-Nikodym derivative.

Then for the five frame properties in definition 5.2 concerning the subspace $\mathcal{H}(\psi)$ we have the following characterizations:

- **ONB (5.5):** $p_\psi \equiv 1$, i.e., $p_\psi(x) = 1$ for almost every $x \in [0, 1]$.

- **Parseval (5.6):** There exists $S \in \mathcal{B}([0, 1])$, $|S| > 0$ (referring to Lebesgue measure) such that $p_\psi(x) = \chi_S(x)$, $x \in [0, 1]$.

- **Bessel (5.7):** $p_\psi \in L^\infty(0, 1)$ and $\|p_\psi\|_\infty \leq B$.

- **Frame $(A, B)$ (5.8):** The estimate $A \leq p_\psi \leq B$ a.e. on ess supp $(p_\psi)$, i.e., holds on ess supp $(p_\psi)$ where ess supp stands for the essential support of $x \to p_\psi(x)$ on $[0, 1]$.

- **Riesz (5.9):** The estimate $A \leq p_\psi \leq B$ holds for almost every $x \in [0, 1]$.

**Remark 5.5.** So the distinction between the two conditions (5.8) and (5.8') is the question of whether the pointwise estimates are assumed only on the essential support, or everywhere; of course excepting Lebesgue measure 0.

**Proof.** The details of proofs are contained in the previous discussion except for the necessity of the condition stated in (5.6) above.

Suppose the Parseval identity (5.6) holds on $\mathcal{H}(\psi)$. Substituting $f = m(T)\psi$ into (5.6) we get the following identity

$$\int_0^1 |m(x)|^2 p_\psi(x)^2 dx = \int_0^1 |m(x)|^2 p_\psi(x) dx, \text{ for all } m.$$  

As a result $p_\psi(x)^2 = p_\psi(x)$ for almost every $x \in [0, 1]$ or $p_\psi(x)(p_\psi(x) - 1) = 0$, almost every $x \in [0, 1]$. Since $\psi \neq 0$ in $\mathcal{H}$, Lemma 4.1 shows that if $S = S_\psi$ denotes the essential support of $p_\psi$, then $|S_\psi| > 0$, and

$$p_\psi(x) = \chi_{S_\psi}(x) \text{ for almost every } x \in [0, 1],$$

as claimed. □

**Corollary 5.6.** Let $T : \mathcal{H} \to \mathcal{H}$ be a unitary operator, $\psi \in \mathcal{H} \setminus \{0\}$, and $d\mu_\psi(x) = p_\psi(x)dx$ as above. We assume that $p_\psi \in L^\infty(0, 1)$ with Bessel bound $B(< \infty)$.

Then the infinite series $\sum_{k \in \mathbb{Z}} c_k \psi_k$ is well defined and norm-convergent in $\mathcal{H}$ for all $c = (c_k)_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$. Moreover,

$$\sum_{k \in \mathbb{Z}} c_k \psi_k = 0, \text{ for some } c \in l^2 \setminus \{0\}$$

(5.21)
if and only if there is a
\[ E \in B([0,1]) \text{ such that } |E| > 0, \text{ and } p_\psi = 0 \text{ on } E. \] (5.22)

Proof. For \( c \in l^2 \), set \( m_c(x) = \sum_{k \in \mathbb{Z}} c_k e_k(x) \). Then
\[
\int_0^1 |m_c(x)|^2 \, dx = \sum_{k \in \mathbb{Z}} |c_k|^2; \tag{5.23}
\]
in particular \( m_c \in L^2(0,1) \).

By Corollary 5.4, functional calculus and Lemma 4.1, the vector \( m_c(T) \psi \) is then well defined for all \( c \in l^2 \); and
\[
\|m_c(T)\|_{\mathcal{H}}^2 = \left\| \sum_{k \in \mathbb{Z}} c_k \psi_k \right\|_{\mathcal{H}}^2
= \int_0^1 |m_c(x)|^2 p_\psi(x) \, dx
\leq B \int_0^1 |m_c(x)|^2 \, dx
= B \sum_{k \in \mathbb{Z}} |c_k|^2.
\]
Hence
\[
\lim_{n,m \to \infty} \left\| \sum_{k \in (-\infty,-m) \cup (n,\infty)} c_k \psi_k \right\|_{\mathcal{H}}^2 = 0,
\]
proving the first assertion.

Now suppose \( c \in l^2 \setminus (0) \), and (5.21) holds. Then
\[
\int_0^1 |m_c(x)|^2 p_\psi(x) \, dx = 0.
\]
If \( E_c := \text{esssup}\, |m_c(x)|^2 \), then \( |E_c| > 0 \), and \( p_\psi = 0 \) almost everywhere on \( E_c \).
This is the desired assertion (5.22).

Conversely, suppose (5.22) holds for some set \( E \in B([0,1]) \), i.e., that \( |E| > 0 \), and \( p_\psi = 0 \) on \( E \). Then, by (5.23) applied to \( \chi_E \), we get the \( L^2(0,1) \)-expansion
\[
\chi_E(x) = \sum_{k \in \mathbb{Z}} c_k e_k(x) \text{ almost everywhere on } [0,1], \quad |E| = \|\chi_E\|_{L^2}^2 = \sum_{k \in \mathbb{Z}} |c_k|^2 > 0.
\]
Hence \( \sum_{k \in \mathbb{Z}} c_k \psi_k = 0 \) in \( \mathcal{H} \), and \( c \neq 0 \) in \( l^2 \). This is (5.21), and the proof is completed. \( \square \)

Remark 5.7. The equivalence of the two assertions (5.21) and (5.22) in Corollary 5.6 above was conjectured in a talk in Chicago by Guido Weiss in the special case when the Hilbert space is \( L^2(\mathbb{R}) \) and when the unitary operator \( T \) is translation by 1.
Moreover, in this case, if \( \psi \) is the given non-zero \( L^2(\mathbb{R}) \)-function and if \( (5.21) \) holds for some non-zero \( c \) in the sequence space \( l^2 \), then we say that the integral translates of \( \psi \) satisfy an \( L^2 \)-dependency.

Since there is typically a substantial amount of redundancy in the representation in \( L^2(\mathbb{R}) \) of the sequence of translates \( \psi_k \), for \( k \in \mathbb{Z} \), this notion of \( L^2 \)-linear dependency, is more subtle than the familiar notion of linear dependency (involving only finite sums) from linear algebra. It was introduced by Kolmogorov, and it is used in prediction theory, see e.g., [MS80].

**Corollary 5.8.** Let \( T : \mathcal{H} \to \mathcal{H} \) be a unitary operator, \( \psi \in \mathcal{H}\setminus\{0\} \), and suppose there is some set \( S \in B([0,1]) \) such that

\[
0 < |S| < 1, \quad \text{and} \quad p\psi(x) = \chi_S(x), \quad \text{for all } x \in [0,1].
\]

Then there is some \( c \in l^2 \setminus \{0\} \) such that \( \sum_{k \in \mathbb{Z}} c_k T^k \psi = 0 \), i.e., there is a non-trivial linear relation in \( \mathcal{H}(\psi) \).

**Proof.** Note that \( (5.24)(5.25) \) and Corollary 5.4 imply that the vectors \( \{T^k \psi \mid k \in \mathbb{Z}\} \) form a Parseval frame in \( \mathcal{H}(\psi) \); and moreover that

\[
\|m(T)\psi\|^2_{H} = \int_S |m(x)|^2 \, dx \tag{5.26}
\]

holds for all \( m \in L^2(0,1) \). Now take \( m := \chi_{[0,1] \setminus S} \), and let \( (c_k) \in l^2 \setminus \{0\} \) be the corresponding Fourier coefficients; i.e.,

\[
1 - |S| = \sum_{k \in \mathbb{Z}} |c_k|^2 > 0.
\]

Recall \( m(x) = \sum_{k \in \mathbb{Z}} c_k e_k(x) \), and \( \|m(x)\|^2_{L^2} = \sum_{k \in \mathbb{Z}} |c_k|^2 \).

But then \( \sum_{k \in \mathbb{Z}} c_k T^k \psi = 0 \) in \( \mathcal{H} \) by virtue of \( (5.20) \). \( \square \)

### 6. Dyadic Wavelets

In this section we read off some corollaries regarding the two spectral functions that describe dyadic wavelets in \( L^2(\mathbb{R}) \).

**Lemma 6.1.** We state the following result for dyadic wavelet functions i.e.,

\[
m_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{C}, |m_0(t)|^2 + |m_0(t + \frac{1}{2})|^2 = 1,
\]

where \( |m_0(0)| = 1 \), the filter \( m_0 \) is assumed to satisfy the low-pass condition, passing at \( t = 0 \).

Let \( \varphi, \psi \in L^2(\mathbb{R}) \) be the two functions for a wavelet filter \( m_0 \). The consistency relation is the following identity which holds for all \( t \):

\[
p_\varphi(t) + p_\psi(t) = p_\varphi(\frac{t}{2}) + p_\varphi(\frac{t + 1}{2})
\]

**Proof.** [BJMP05]. \( \square \)
Wavelet functions

| Father function $\phi$ | Mother function $\psi$ |
|------------------------|------------------------|
| $\phi(x) = 1$ on $[0, 1]$ and 0 in $\mathbb{R} \setminus [0, 1]$ | $\psi(x) = 1$ in $(0, \frac{1}{2})$, $-1$ in $[\frac{1}{2}, 1)$, and 0 in $\mathbb{R} \setminus [0, 1]$. |

Given $k$ an odd integer, we now describe the STRETCHED HAAR WAVELET

$$
\phi_k = \frac{1}{k} \phi \left( \frac{t}{k} \right) \quad \psi_k = \frac{1}{k} \psi \left( \frac{t}{k} \right)
$$

$$
|\widehat{\phi}_k(t)|^2 = \frac{\sin^2(\pi kt)}{k^2(\pi t)^2} \quad |\widehat{\psi}_k(t)|^2 = \frac{\sin^4 \left( \frac{\pi kt}{2} \right)}{k^2(\pi t)^2}
$$

$$
\text{PER} |\widehat{\phi}|^2 = \frac{1}{k^2} \left( \frac{\sin(\pi kt)}{\sin(\pi t)} \right)^2 \quad \text{PER} |\widehat{\psi}|^2 = \frac{1}{(2k)^2} \left( \frac{\sin^4 \left( \frac{\pi kt}{2} \right)}{\sin^2(\pi t)^2} + \frac{\cos^4 \left( \frac{\pi kt}{2} \right)}{\cos^2(\pi t)^2} \right).
$$

Where we use the formula

$$
\sum_{n \in \mathbb{Z}} \frac{1}{(t + n)^2} = \frac{\pi^2}{\sin^2(\pi t)} = \pi^2 \csc^2(\pi t), \quad t \in \mathbb{R}.
$$

Spectral Density Functions on $[0, 1] \simeq \mathbb{R}/\mathbb{Z}$

$$
p_{\phi_1}(t) \equiv 1 \quad p_{\varphi_1}(t) \equiv 1
$$

If $k > 1$ then

$$
p_{\phi_k}(t) = \frac{1}{k^2} \left( \frac{\sin(\pi kt)}{\sin(\pi t)} \right)^2 \quad p_{\varphi_k}(t) = \frac{1}{(2k)^2} \left( \frac{\sin^4 \left( \frac{\pi kt}{2} \right)}{\sin^2(\pi t)^2} + \frac{\cos^4 \left( \frac{\pi kt}{2} \right)}{\cos^2(\pi t)^2} \right).
$$

---

**Figure 1.** Graph of $p_{\varphi_3}$. 
Hence we have consistency, but the peculiar thing is that in the stretched Haar case, we have a Parseval wavelet (i.e., for the system $(\psi_{j,k})$, i.e., $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, built on the interval from 0 to 3, instead of from 0 to 1). Specifically, $\sum_j \sum_k |\langle \psi_{j,k} | f \rangle|^2 = \| f \|^2$, for all $f \in L^2(\mathbb{R})$ but there is no Parseval property on $H(\phi)$ or on $H(\psi)$. Indeed, from [Jor06a], Chapter 6 we know that the mother function $\psi$ generates a Parseval wavelet frame in $H := L^2(\mathbb{R})$, but only if
we use both dyadic scaling and \( \mathbb{Z} \)-translations. However, in the 3-stretched Haar case, we don’t get Parseval frame systems for \( \mathbb{Z} \)-translations alone.

Specifically, if we consider the stretched father and mother functions, by themselves, i.e., the functions \( \varphi \) and \( \psi \) individually, then one may ask whether or not the system \( \{ \varphi(n + \cdot) | n \in \mathbb{Z} \} \) is Parseval in \( \mathcal{H}(\varphi) \); or \( \{ \psi(n + \cdot) | n \in \mathbb{Z} \} \) in \( \mathcal{H}(\psi) \), and the answer is ‘no’ for both. Neither of the two has the Parseval property. Using our Corollary, and our computations of the two spectral functions \( p_\psi \) and \( p_\varphi \) we can only ascertain that the two \( \mathbb{Z} \)-translation systems have the Bessel property with Bessel constant \( B = 1 \), but neither satisfies the stronger ones of the frame properties.

7. Cyclic Subspaces

**Definition 7.1.** Let \( \mathcal{A} \) be an abelian group (algebra) represented by bounded operators on a Hilbert space \( \mathcal{H} \). A closed subspace \( \mathcal{K} \subset \mathcal{H} \) is cyclic if there exists \( \varphi \in \mathcal{K} \setminus \{0\} \) such that \( [\mathcal{A}\varphi] = \mathcal{K} \). Here \([\cdot]\) denotes closed linear span.

**Lemma 7.2.** Let \( \mathcal{A} \) be as in definition 7.1 acting on a given Hilbert space \( \mathcal{H} \). Then there is an indexed family of cyclic vectors \( (\varphi_j) \) such that

\[
\mathcal{H} = \bigoplus_j [\mathcal{A}\varphi_j] \quad (7.1)
\]

The symbol \( \bigoplus \) in (7.1) indicates orthogonality, i.e.,

\[
[\mathcal{A}\varphi_j] \perp [\mathcal{A}\varphi_k] \text{ if } j \neq k.
\]

**Proof.** This is a standard application of Zorn’s lemma, see e.g., [Nel69] □

**Corollary 7.3.** Let \( T : \mathcal{H} \to \mathcal{H} \) be a unitary operator, and suppose \( p_\psi \in L^\infty \) for all \( \psi \in \mathcal{H} \). Let \( \mathcal{M} \subset \mathcal{H} \) be a closed \( T \)-invariant subspace. Then there are subsets \( [0,1] \supset S_1 \supset S_2 \supset \cdots \) in \( \mathcal{B}([0,1]) \), and vectors \( \psi_j \in \mathcal{M} \) such that \( p_{\psi_j} = \chi_{S_j} \) for \( j = 1, 2, \ldots \), and

\[
\mathcal{M} = \bigoplus_{j \geq 1} \mathcal{H}(\psi_j) \quad (7.2)
\]

where the summation in (7.2) indicates orthogonality of the spaces \( \mathcal{H}(\psi_j) \) corresponding to different index values for \( j \).

**Proof.** This follows from the Spectral-Theorem applied to the restricted operator \( T|_{\mathcal{M}} \), combined with the previous two corollaries; see also the references [BMM99] [Bag00] for special cases of this result. □
8. Stochastic Processes

In the next section we show that our spectral analysis of translations produces a particular class of stochastic processes. Hence our results for translations, and more generally, for unitary operators $T$ in Hilbert space have a probabilistic significance.

This is relevant to the way wavelets are used in the processing of signals with noise; see [Jor06a, Jor06b] and the following papers [JS07, Son07]. A popular method of spectral analyzing correlations in stochastic components in a signal or in an image is to introduce a spectral function, or spectral kernel, and then an associated selfadjoint operator. When this operator is then diagonalized, a substantial simplification occurs. The diagonalization goes under the name “Karhunen-Loève;” see [JS07, Son07]. But for the use of the Karhunen-Loève transform method, it will be significant that the stochastic processes involved be Gaussian. In this section we show that there is always such a Gaussian choice, and that it is canonical. Moreover we show (Corollary 8.4) that each of these Gaussian processes is naturally associated with a unitary operator so our results from sections 4 and 5 apply.

A stochastic process is an indexed system of random variables. By “random variables” we mean measurable functions on a probability space $(\Omega, M, P)$ where $\Omega$ is a set, $M$ is a fixed sigma-algebra of subsets in $\Omega$ (events), and $P$ is a probability measure, i.e., $P(S)$ is defined for $S \in M$.

We will make the starting assumption that our random variables $X$, referring to $(\Omega, M, P)$, will be in $L^2(\Omega, M, P)$, abbreviated $L^2(P)$, i.e., that

$$\int_{\Omega} |X(\omega)|^2 dP(\omega) = E_{\omega}(|X(\omega)|^2) = \langle X|X \rangle_{L^2(P)} < \infty. \quad (8.1)$$

In the present discussion, the index set for the random variables will be $[0, 1]$ or the Borel sets $B([0, 1])$. We will say that a process $X$ is Gaussian if $X_t$, for $t \in [0, 1]$, or $X_A$, for $A \in B([0, 1])$, is Gaussian.

Lemma 8.1. (Kolmogorov) Let $W$ be a set and let $R : W \times W \to \mathbb{C}$ be a positive (semi)-definite function, i.e.,

$$\sum \sum \tau_w R(v, w) c_w \geq 0, \quad (v, w) \in W \times W, \quad (8.2)$$

for all finite sequences $(c_w \in W)$, i.e., $c_w \in \mathbb{C} \setminus \{0\}$ for at most a finite subset in $W$, depending on $c$.

Then there is a Gaussian stochastic process $X, W, (\Omega, M, P)$ such that

$$E(X_v X_w) = R(v, w), \quad \text{for all } (v, w) \in W \times W. \quad (8.3)$$

Proof. See e.g., [Jor06a] or [Nel69].

Example. (Brownian motion.) For $s, t \in [0, 1]$, set $s \wedge t = \min(s, t)$, and

$$R(s, t) := s \wedge t. \quad (8.4)$$

Then $R$ is positive definite, and the Gaussian process $(X_t)$ with

$$E(X_{s} X_{t}) = s \wedge t, \quad \text{and} \quad (8.5)$$
is called the normalized Brownian motion on $[0, 1]$.

**Example.** (Set-indexed Gaussian.) Let $\mu$ be a finite Borel measure on $[0, 1]$. For $A, B \in \mathcal{B}([0, 1])$, the Borel sets in the unit interval, set

$$R(A, B) := \mu(A \cap B).$$

Then $R$ is positive definite, and the Gaussian process $(X_A)_{A \in \mathcal{B}([0, 1])}$ with

$$E(X_A X_B) = \mu(A \cap B)$$

and $E(X_A) = 0$, $A, B \in \mathcal{B}([0, 1])$; is called the $\mu$-Gaussian process. (Note that $E(\cdot) = \int_{\mathbb{R}} \cdot \, dP$, and that $P$ depends on $\mu$.)

**Remark 8.2.** An easy computation shows that Brownian motion is $\mu$-Gaussian with $\mu :=$ the Lebesgue measure restricted to $[0, 1]$.

**Lemma 8.3.** [Nel69] Let $(X_A)_{A \in \mathcal{B}([0, 1])}$ be a $\mu$-Gaussian process; and let $m \in L^2([0, 1], \mu)$. Then the stochastic integral

$$\int_0^1 m(t) \, dX_t \in L^2(\Omega, \mathcal{M}, P)$$

and

$$\left\| \int_0^1 m(t) \, dX_t \right\|_{L^2(P)}^2 = \int_0^1 |m(t)|^2 \, d\mu(t).$$

**Proof.** The idea in the proof is to construct the Gaussian probability space $(\Omega, \mathcal{M}, P)$ as in Lemma 2.1. The main difference between the two cases is that our stochastic processes $X$ is now indexed by Borel sets in contrast to the discrete index used in Lemma 2.1.

**Corollary 8.4.** Let $(X_A)_{A \in \mathcal{B}([0, 1])}$ be a $\mu$-Gaussian process; and define an operator $T : L^2(P) \to L^2(P)$ by

$$T(\int_0^1 m(t) \, dX_t) := \int_0^1 e_1(t) m(t) \, dX_t,$$

then $T$ is unitary.

**Proof.** By (8.10), it is enough to prove that

$$L^2(\mu) \ni m(t) \mapsto e_1(t) m(t) \in L^2(\mu),$$

is a unitary operator; but this last fact is obvious.

**Example.** (Non-Gaussian) Let $(Tf)(x) := f(x - 1)$ be translation in $L^2(\mathbb{R})$, and let $\psi \in L^2(\mathbb{R}) \setminus \{0\}$. Set

$$d\mu_\psi(t) := p_\psi(t) \, dt$$

with

$$p_\psi(t) := \sum_{n \in \mathbb{Z}} |\hat{\psi}(t + n)|^2, \quad t \in [0, 1];$$

and

$$E(X_t) = 0$$

is called the normalized Brownian motion on $[0, 1]$. 


and
\[ p_\psi(s, t) := \sum_{n \in \mathbb{Z}} \hat{\psi}(s + n) \hat{\psi}(t + n). \] (8.15)

Set
\[ X_t(n) := \hat{\psi}(t + n), \quad n \in \mathbb{Z}, \quad t \in [0, 1]. \] (8.16)

Then each \( X_t \) in \((X_t)_{t \in [0, 1]}\) is a random variable on \( \mathbb{Z} \) with counting measure, and
\[ E(X_s X_t) = p_\psi(s, t), \] for all \( s, t \in [0, 1]. \) (8.17)

**Proof.** See section 4 above. \( \square \)

Our final result also follows from these considerations and the lemmas in section 4. It is about the special case when function \( p_\psi(\cdot) \) in (8.14) is in \( L^2([0, 1]). \)

**Remark 8.5.** It is of interest to compare the realization of \( X_t(n) := \hat{\psi}(t + n) \) in \( L^2(\mathbb{R}); \) see (8.16) with its Gaussian version in \( L^2(\Omega_\psi, P_\psi), \) i.e., with the stochastic integral
\[ \int_0^1 m(t) dX_t \in L^2(\Omega_\psi, P_\psi), \]
and with \( P_\psi \) denoting the Gaussian measure.

The result is as follows:

For \( s, t \in [0, 1], \) set
\[ Q^\psi(s, t) := \sum_{n \in \mathbb{Z}} \hat{\psi}'(s + n) \hat{\psi}'(t + n), \]
and \( \hat{\psi}'(t) := \frac{d}{dt} \hat{\psi}(t). \) Then
\[ \left\| \int_0^1 m(t) dX_t \right\|_{L^2(\Omega_\psi, P_\psi)}^2 = \int_0^1 \int_0^1 m(s)m(t) Q^\psi(s, t) ds dt. \]

**Corollary 8.6.** Let \( T, L^2(\mathbb{R}) \) and \( \psi \) be as in Example (Non-Gaussian) above. Set
\[ \psi_k := T^k \psi = \psi(\cdot - k), \quad k \in \mathbb{Z}. \] (8.18)

On the dense subspace \( D \subset l^2 \) of finitely indexed sequences, set
\[ F((c_k)_{k \in \mathbb{Z}}) := \sum_{k \in \mathbb{Z}} c_k \psi_k. \] (8.19)

We have the following \((a) \iff (b) \implies (c), \) where the three affirmations 
(a)-(c) are as follows:
\begin{enumerate}
\item \( p_\psi \in L^2(0, 1), \)
\item \( (\langle \psi | \psi_k \rangle) \in l^2(\mathbb{Z}), \) and
\item \( l^2 \ni D \overset{F}{\longrightarrow} \mathcal{H}(\psi) \subset L^2(\mathbb{R}) \) is a closable operator, when \( F \) is defined by (8.17).\end{enumerate}
Proof. Computing the Fourier coefficients $\hat{p}_\psi(k)$, $k \in \mathbb{Z}$, we find

$$\hat{p}_\psi(k) = \int_0^1 \overline{e_k(t)} p_\psi(t) dt$$

$$= \int_0^1 \overline{c_k(t)} \sum_{n \in \mathbb{Z}} |\hat{\psi}(t + k)|^2 dt$$

$$= \int_\mathbb{R} \overline{c_k(t)} |\hat{\psi}(t)|^2 dt$$

$$= \text{by Parseval} \int_\mathbb{R} \psi(x) \psi(x-k) dx$$

$$= \langle \psi | \psi_k \rangle_{L^2}, \quad k \in \mathbb{Z},$$

which proves (a) $\iff$ (b).

Supposing (a), to prove that $F$ is a closable operator, note the formula

$$(F^* f)_k = \int_\mathbb{R} f(x-k) dx, \quad k \in \mathbb{Z}$$

for the adjoint operator. So condition (a) is the assertion that each $\psi_j$ is in the domain of $F^*$. Since $(\psi_j)_{j \in \mathbb{Z}}$ spans a dense subspace in $\mathcal{H}(\psi) \subset L^2(\mathbb{R})$, it follows that $F$ in (8.19) is a closable operator. \hfill \Box

9. A Connection to Karhunen-Loève Transforms

Suppose $X_t$ is a stochastic process indexed by $t$ in a finite interval $J$, and taking values in $L^2(\Omega, P)$ for some probability space $(\Omega, P)$. Assume the normalization $E(X_t) = 0$. Suppose the integral kernel $E(X_t X_s)$ can be diagonalized, i.e., suppose that

$$\int_J E(X_t X_s) \varphi_k(s) ds = \lambda_k \varphi_k(t)$$

with an ONB $(\varphi_k)$ in $L^2(J)$. If $E(X_t) = 0$ then

$$X_t(\omega) = \sum_k \sqrt{\lambda_k} \varphi_k(t) Z_k(\omega), \quad \omega \in \Omega$$

where $E(Z_j Z_k) = \delta_{j,k}$, and $E(Z_k) = 0$. The ONB $(\varphi_k)$ is called the KL-basis with respect to the stochastic processes $\{X_t : t \in I\}$.

The Karhunen-Loève-theorem [Ash90] states that if $(X_t)$ is Gaussian, then so are the random variables $(Z_k)$. Furthermore, they are $N(0,1)$ i.e., normal with mean zero and variance one, so independent and identically distributed. This last fact explains the familiar optimality of Karhunen-Loève method in transform coding.

The following result illustrates the significance of our spectral density functions in the analysis of Karhunen-Loève transforms.
Theorem 9.1. Let \((\Omega, P)\) be a probability space, \(J \subset \mathbb{R}\) an interval (possibly infinite), and let \((X_t)_{t \in J}\) be a stochastic process with values in \(L^2(\Omega, P)\). Assume \(E(X_t) = 0\) for all \(t \in J\). Then by the spectral density theorem for unitary operator (see Corollary 7.3), the Hilbert space \(L^2(J)\) splits as an orthogonal sum
\[
L^2(J) = \mathcal{H}_d \oplus \mathcal{H}_c
\] (9.1)
(d is for discrete and c is for continuous) such that the following data exists:
(a) \((\varphi_k)_{k \in \mathbb{N}}\) an ONB in \(\mathcal{H}_d\).
(b) \((Z_k)_{k \in \mathbb{N}}\) : independent random variables.
(c) \(E(Z_j Z_k) = \delta_{j,k}\), and \(E(Z_k) = 0\).
(d) \((\lambda_k) \subset \mathbb{R}_{\geq 0}\).
(e) \(\varphi(\cdot, \cdot) : \) a Borel measure on \(\mathbb{R}\) in the first variable, such that
   (i) \(\varphi(A, \cdot) \in \mathcal{H}_c\) for \(A\) an open subinterval of \(J\), and
   (ii) \(\langle \varphi(A_1, \cdot) | \varphi(A_2, \cdot) \rangle_{L^2(J)} = 0 \) whenever \(A_1 \cap A_2 = \emptyset\).
(f) \(Z(\cdot, \cdot) : \) a measurable family of random variables such that \(Z(A_1, \cdot)\) and \(Z(A_2, \cdot)\) are independent when \(A_1, A_2 \in \mathcal{B}_J\) and \(A_1 \cap A_2 = \emptyset\),
\[E(Z(\lambda, \cdot) Z(\lambda', \cdot)) = \delta(\lambda - \lambda'), \text{ and } E(Z(\lambda, \cdot)) = 0.\]

Finally, we get the following Karhunen-Loève expansions for the \(L^2(J)\)-operator with integral kernel \(E(X_t X_s)\):
\[
\sum_{k \in \mathbb{N}} \lambda_k |\varphi_k| \langle \varphi_k | \varphi_{d\lambda} \rangle + \int_J \lambda \langle \varphi(d\lambda, \cdot) | \varphi(d\lambda, \cdot) \rangle
\] (9.2)
Moreover, the process decomposes thus:
\[
X_t(\omega) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} Z_k(\omega) \varphi_k(t) + \int_J \sqrt{\lambda} Z(\lambda, \omega) \varphi(d\lambda, t).
\] (9.3)

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