Approximate inversion for Abel integral operators of variable exponent and applications to fractional Cauchy problems

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Abstract
We investigate the variable-exponent Abel integral equations and corresponding fractional Cauchy problems. The main contributions are twofold: We provide a novel inverse technique to convert the first-kind Volterra integral equation of variable exponent to a second-kind one, which, to our best knowledge, is not available in the literature; Based on this transformation, we carry out rigorous analysis to prove several theoretical results and their dependence on the variable exponent. In general, we conclude that by setting an integer limit of variable exponent $\alpha(t)$ at the initial time $t = 0$, the variable-exponent problems have similar properties as their integer-order analogues. Otherwise, they behave like their constant-exponent counterparts of order $\alpha \equiv \alpha(0)$. To be specific, we prove that the sensitive dependence of the well-posedness of classical Riemann-Liouville fractional differential equations on the initial value and the singularity of their solutions could be resolved by adjusting the variable exponent at the initial time, which demonstrates the advantages of introducing the variable exponent. The above findings suggest that the variable-exponent fractional problems may serve as a connection between integer-order and fractional models by adjusting the variable exponent at the initial time.

Keywords Abel integral equation · Variable-exponent · Fractional Cauchy problem · Approximate inversion

Mathematics Subject Classification 26A33 · 34A08 · 45D05

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1 Introduction

The Abel integral equation

$$I_0^t u(t) := \int_0^t \frac{u(s)}{(t-s)^\alpha} ds = f(t), \quad 0 < \alpha < 1,$$

has a long history with wide applications and has been extensively investigated, see e.g., [7, 9] and the references therein. However, variable-exponent problems, in which the exponent $\alpha$ may be a function of $s$ or $t$, are rarely studied [3, 6, 13, 18, 19, 24, 26] due to, e.g., the unavailability of the exact solutions. There is some recent progress on the analysis and numerical approximations to the second kind Volterra integral equations (VIEs) with variable exponents and the variable-exponent time or space fractional models that could be reduced to second kind VIEs via variable substitutions or spectral decompositions, [11, 22, 25]. But the variable-exponent Abel integral problems, which are indeed first kind VIEs with weak singularities, and the corresponding variable-exponent fractional Cauchy problems remain almost untreated in the literature to our best knowledge.

The crucial difficulty in analyzing such problems may lie in the needs to find their inversions or equivalent second kind VIEs, which is not encountered in [11, 22, 25] as the models in these works are naturally second kind VIEs.

In this paper we aim to provide a potential means to analyze the variable-exponent Abel integral equations and corresponding fractional differential equations involving the following variable-exponent Abel integral operators

$$\mathcal{I}^\alpha_{0^+} g(t) := \int_0^t \frac{g(s)}{(t-s)^\alpha(s)} ds, \quad \hat{\mathcal{I}}^\alpha_{0^+} g(t) := \int_0^t \frac{g(s)}{(t-s)^\alpha(t)} ds,$$

or their variants. The $\alpha(t)$ lies in between 0 and 1 and we will specify its range in each section. As applications for introducing the variable exponent, we find that the sensitive dependence of the well-posedness of the classical Riemann-Liouville fractional differential equations (R-L FDEs) on the initial value and the initial singularity of their solutions could be resolved by adjusting the initial value of the variable exponent. To be specific, the main contributions of this work are listed as follows:

- We find the approximate inverse operators of these variable-exponent Abel integral operators in the sense that they are reverted as the identity operator (or its multiple) added by a weak-singular integral operator. In other words, we indeed provide a novel means to convert the first-kind VIE of variable exponent to a second kind VIE, which, to our best knowledge, is not available in the literature. We base on this to analyze the well-posedness and smoothing properties of Abel integral equations and corresponding fractional Cauchy problems of variable exponent, which also remains untreated in the literature.

- It is known that the constant-exponent R-L fractional Cauchy problem of order $0 < \alpha < 1$ with the initial condition $u(0) = u_0$ is well-posed only in the case of $u_0 = 0$, which may be restrictive in real applications. Instead, in most literature, the fractional initial condition is proposed for this problem, which may be difficult
to determine in real applications. Different from the existing results, we prove that the variable-exponent R-L FDE of order $\alpha(t)$ is well-posed for any $u_0 \in \mathbb{R}$ by setting $\alpha(0) = 1$, which resolves the sensitive dependence of the well-posedness of classical R-L FDE on the initial value without using fractional initial conditions and thus extends the theory of R-L FDEs and their applications for non-zero initial values. Otherwise, if $\alpha(0) < 1$, then the variable-exponent R-L FDE has the same sensitivity issue on the initial value as in the constant-exponent case.

- The derivative of the solutions to the classical R-L FDEs of order $0 < \alpha < 1$ with the zero initial condition is in general unbounded at the initial time. We prove that this singularity could be eliminated by setting $\alpha(0) = 1$ and $(\partial_t \alpha)(0) = 0$ in the variable-exponent R-L FDEs of order $\alpha(t)$, which not only maintains the advantages of the fractional-order operators in characterizing the memory effects for $t > 0$, but resolves the initial singularity of the solutions that may not be physically relevant in practical problems [22]. Otherwise, if $\alpha(0) < 1$, then the solution to the variable-exponent R-L FDE has the same initial singularity as in the constant-exponent problem of order $\alpha \equiv \alpha(0)$.

Indeed, the conditions on the variable exponent proposed in these items suggest that by setting an integer limit of the variable exponent at the initial time, the properties of the FDEs tend to those of integer-order differential equations. Otherwise, if $\alpha(t)$ does not have the integer limit at the initial time, e.g. $\alpha(0) < 1$, then the variable-exponent R-L FDE has almost the same properties as in the constant-exponent case with order $\alpha \equiv \alpha(0)$. Therefore, the variable-exponent fractional problems may serve as a connection between integer-order and fractional models by adjusting the variable exponent at the initial time.

The rest of the paper is organized as follows: In Section 2 we present approximate inversions of several variable-exponent Abel integral operators, based on which we analyze the well-posedness of variable-exponent Abel integral equations in Section 3. In Section 4, we prove the smoothing properties of their solutions. In Section 5, we analyze the variable-exponent R-L FDEs and discuss the dependence of their well-posedness on the initial values and the singularity issue of their solutions. We finally address conclusions and potential future works in Section 6.

### 2 Approximate inversions of variable-exponent Abel integral operators

We introduce approximate inversion operators of the variable-exponent Abel integral operators in the sense to revert an operator to the identity operator (or its multiple) added by an integral operator with a weak-singular kernel. This could be employed to analyze the corresponding integral and differential equations in the rest of the paper. Let $C^m[0, T]$ for $0 \leq m \in \mathbb{N}$ be the space of $m$-th continuously differentiable functions on $[0, T]$, and $L^p(0, T)$ with $1 \leq p \leq \infty$ be the space of $p$-th Lebesgue integrable functions on $(0, T)$. These spaces are assumed equipped with the standard norms, [1]. We also use $C^m(0, T)$ to denote the space of functions that are $m$-th continuously differentiable on $[\varepsilon, T]$ for any $0 < \varepsilon \ll 1$. 


2.1 Approximate inversions of $\mathcal{I}_t^\alpha(t)$ and $\hat{\mathcal{I}}_t^\alpha(t)$

We consider these two operators with $0 < \alpha(t) < 1$ on $[0, T]$ and $\alpha \in C^1[0, T]$. Note that these conditions imply that $\alpha(t)$ lies in the interior of $[0, 1]$ for any $t \in [0, T]$ with lower and upper bounds $0 < \alpha_* \leq \alpha^* < 1$.

Theorem 1 The operators

$$\mathcal{D}_t^\alpha(t) := \partial_t \mathcal{I}_t^{1-\alpha(t)} \quad \text{and} \quad \mathcal{D}_t^\alpha(t) := \partial_t \mathcal{I}_t^{1-\alpha(t)}$$

serve as approximate inversion operators of $\mathcal{I}_t^\alpha(t)$ and $\hat{\mathcal{I}}_t^\alpha(t)$, respectively, in the sense that the following relations hold when both sides of the equations exist:

$$\mathcal{D}_t^\alpha(t) \mathcal{I}_t^\alpha(t) g(t) = \gamma(t) g(t) + \int_0^t \mathcal{K}(s, t) g(s) ds, \quad (2.1)$$

$$\mathcal{D}_t^\alpha(t) \hat{\mathcal{I}}_t^\alpha(t) g(t) = \gamma(t) g(t) + \int_0^t \mathcal{L}(s, t) g(s) ds, \quad (2.2)$$

where

$$\gamma(t) := \Gamma(\alpha(t)) \Gamma(1 - \alpha(t)),$$

the kernels $\mathcal{K}$ and $\mathcal{L}$ are defined as

$$\mathcal{K}(s, t) := \partial_t \left( B(\alpha(t), 1 - \alpha(s))(t - s)^{\alpha(t) - \alpha(s)} \right),$$

$$\mathcal{L}(s, t) := \partial_t \left( \int_0^1 \frac{1}{(1 - z)^{1-\alpha((t-s)z+s)} z^{\alpha((t-s)z+s)}} dz \right),$$

and $\Gamma(\cdot)$ and $B(\cdot, \cdot)$ refer to the standard Gamma and Beta functions, respectively,

$$\Gamma(p) := \int_0^\infty e^{-z} z^{p-1} dz, \quad p > 0; \quad B(p, q) := \int_0^1 (1 - z)^{p-1} z^{q-1} dz, \quad p, q > 0.$$

Furthermore, the following estimates hold for $0 \leq s \leq t \leq T$

$$|\mathcal{K}(s, t)| \leq Q(1 + |\ln(t - s)|), \quad |\mathcal{L}(s, t)| \leq Q. \quad (2.3)$$

Here $Q$ may depend on $\alpha_*, \alpha^*, \|\alpha\|_{C^1[0, T]}$ and $T$.

Proof Using the definitions of $\hat{\mathcal{I}}_t^{1-\alpha(t)}$ and $\mathcal{I}_t^\alpha(t)$, we exchange the order of double integrals to obtain

$$\hat{\mathcal{I}}_t^{1-\alpha(t)} \mathcal{I}_t^\alpha(t) g(t) = \int_0^t \frac{1}{(t-s)^{1-\alpha(t)}} \int_0^s \frac{g(y)}{(s-y)^{\alpha(y)}} dyds$$

$$= \int_0^t g(y) \int_y^t \frac{1}{(t-s)^{1-\alpha(t)}} \frac{1}{(s-y)^{\alpha(y)}} ds dy. \quad (2.4)$$
We use the variable substitution $z = (s - y)/(t - y)$, which implies

$$\begin{align*}
t - s &= (t - y)(1 - z), \\
s - y &= (t - y)z, \\
ds &= (t - y)dz,
\end{align*}$$

for the inner integral of (2.4) to get

$$\int_y^t \frac{1}{(t - s)^{1 - \alpha(t)}} \frac{1}{(s - y)^{\alpha(y)}} ds = (t - y)^{\alpha(t) - \alpha(y)} \int_0^1 \frac{1}{(1 - z)^{1 - \alpha(t)} z^{\alpha(y)}} dz. \tag{2.5}$$

Since the powers $1 - \alpha(t)$ and $\alpha(y)$ in (2.5) are independent from the variable $z$, we apply the definition of the Beta function $B(\cdot, \cdot)$ to obtain

$$\int_0^1 \frac{1}{(1 - z)^{1 - \alpha(t)} z^{\alpha(y)}} dz = B(\alpha(t), 1 - \alpha(y)). \tag{2.6}$$

Invoking (2.5)-(2.6) in (2.4) leads to

$$\hat{I}_t^{1-\alpha(t)} I_t^{\alpha(t)} g(t) = \int_0^t B(\alpha(t), 1 - \alpha(y))(t - y)^{\alpha(t) - \alpha(y)} g(y) dy. \tag{2.7}$$

As $\alpha \in C^1[0, T]$, we have

$$\lim_{y \to t^-} (t - y)^{\alpha(t) - \alpha(y)} = \lim_{y \to t^-} e^{(\alpha(t) - \alpha(y)) \ln(t - y)} = e^0 = 1.$$

Indeed, a general estimate on this function reads:

$$(t - y)^{\alpha(t) - \alpha(y)} \text{ has positive upper and lower bounds for } 0 \leq y \leq t \leq T. \tag{2.8}$$

Then we differentiate (2.7) and apply $B(\alpha(t), 1 - \alpha(t)) = \Gamma(\alpha(t)) \Gamma(1 - \alpha(t))$ ([15]) to obtain (2.1). Direct calculations show that

$$\mathcal{K}(s, t) = \left( \partial_t B(\alpha(t), 1 - \alpha(s)) \right) (t - s)^{\alpha(t) - \alpha(s)} + B(\alpha(t), 1 - \alpha(s)) \times (t - s)^{\alpha(t) - \alpha(s)} \left( \partial_t \alpha(t) \ln(t - s) + \frac{\alpha(t) - \alpha(s)}{t - s} \right). \tag{2.9}$$

Since $\alpha \in C^1[0, T]$ and it is bounded away from 0 and 1, the derivative and difference quotient of $\alpha$ as well as $B(\alpha(t), 1 - \alpha(s))$ and its partial derivatives are bounded, which leads to the first estimate of (2.3).
To prove (2.2) and the second estimate of (2.3), we employ the transformation

\[ s = (t - y)z + y \]

to obtain

\[ I_{t}^{1-\alpha(t)} \hat{I}_{t}^{\alpha(t)} g(t) = \int_{0}^{t} \frac{1}{(t - s)^{1-\alpha(s)}} \int_{0}^{s} \frac{g(y)}{(s - y)^{\alpha(s)}} dyds \]

\[ = \int_{0}^{t} g(y) \int_{y}^{t} \frac{1}{(t - s)^{1-\alpha(s)}} \frac{1}{(s - y)^{\alpha(s)}} dsdy \]

\[ = \int_{0}^{t} g(y) \int_{0}^{1} \frac{1}{(1 - z)^{1-\alpha((t-y)z+y)}} z^{\alpha((t-y)z+y)} dzdy. \]

Differentiating this equation yields

\[ D^{\alpha(t)}_{t} \hat{D}^{\alpha(t)}_{t} g(t) = g(t) \int_{0}^{1} \frac{1}{(1 - z)^{1-\alpha(t)} z^{\alpha(t)}} dz + \int_{0}^{t} g(y) \mathcal{L}(y, t) dy. \quad (2.10) \]

The first right-hand side term of (2.10) is exactly \( \Gamma(\alpha(t)) \Gamma(1 - \alpha(t)) g(t) \), and the kernel of the second right-hand side term could be bounded as

\[ |\mathcal{L}(y, t)| = \left| \int_{0}^{1} \frac{\partial_{t} \alpha((t-y)z+y)(\ln(1 - z) - \ln z)}{(1 - z)^{1-\alpha((t-y)z+y)} z^{\alpha((t-y)z+y)}} dz \right| \]

\[ \leq Q \int_{0}^{1} \frac{|\ln(1 - z)| + |\ln z|}{(1 - z)^{1-\alpha(t)} z^{\alpha(t)}} dz \leq Q, \]

which completes the proof. \( \square \)

### 2.2 Extensions

Based on the ideas as above, we could find approximate inversions for other commonly-used variable-exponent integral operators, and we list some representative examples in the following.

**Example 1** Variable-exponent R-L fractional integral operators. Such operators have attracted increased attention in recent years and is defined for \( 0 < \alpha(t) < 1 \), \[13, 14, 20, 21\]:

\[ I_{t}^{\alpha(t)} g(t) := \int_{0}^{t} \frac{1}{\Gamma(\alpha(s))} \frac{g(s)}{(t - s)^{1-\alpha(s)}} ds, \]

\[ \hat{I}_{t}^{\alpha(t)} g(t) := \int_{0}^{t} \frac{1}{\Gamma(\alpha(t))} \frac{g(s)}{(t - s)^{1-\alpha(t)}} ds. \]

The corresponding approximate inversion operators are the variable-exponent R-L fractional differential operators \( \hat{D}^{\alpha(t)}_{t} \) and \( D^{\alpha(t)}_{t} \), respectively, defined by \[13, 14, 20, \]
which, similar to the derivations as (2.4)–(2.10), satisfy

\[
\dot{I}^\alpha(I_t) g(t) := \partial_t I_t^{1-\alpha(t)} g(t) = \partial_t \left( \int_0^t \frac{1}{\Gamma(1-\alpha(t))} \frac{g(s)}{(t-s)^{\alpha(t)}} ds \right),
\]

\[
D_t^{\alpha(t)} g(t) := \partial_t \hat{I}_t^{1-\alpha(t)} g(t) = \partial_t \left( \int_0^t \frac{1}{\Gamma(1-\alpha(s))} \frac{g(s)}{(t-s)^{\alpha(s)}} ds \right),
\]

where

\[
\dot{D}_t^{\alpha(t)} I_t^{\alpha(t)} g(t) = g(t) + \int_0^t \partial_t \frac{(t-y)^{\alpha(y)-\alpha(t)}}{\Gamma(1-\alpha(t)+\alpha(y))} g(y) dy
\]

\[
D_t^{\alpha(t)} \hat{I}_t^{\alpha(t)} g(t) = g(t)
\]

\[
+ \int_0^t g(y) \partial_t \int_0^1 \frac{(1-z)^{-\alpha((t-y)z+y)} z^{\alpha((t-y)z+y)-1}}{\Gamma(1-\alpha((t-y)z+y)) \Gamma(\alpha((t-y)z+y))} dz dy.
\]

Example 2 A general variable-exponent R-L fractional integral operator. The definition of this operator is given for a \( C^1 \)-function \( 0 < \alpha(\cdot, \cdot) < 1 \), [13, 20],

\[
I_t^{\alpha(t, \cdot)} g(t) := \int_0^t \frac{1}{\Gamma(\alpha(t, s))} \frac{g(s)}{(t-s)^{1-\alpha(t, s)} ds}.
\]

The corresponding approximate inversion operator is, [13, 20],

\[
D_t^{\alpha(\cdot, t)} g(t) := \partial_t I_t^{1-\alpha(\cdot, t)} g(t) = \partial_t \left( \int_0^t \frac{1}{\Gamma(1-\alpha(s, t))} \frac{g(s)}{(t-s)^{\alpha(s, t)}} ds \right).
\]

which, by the substitution \( s = (t-y)z + y \), satisfies

\[
\dot{D}_t^{\alpha(\cdot, t)} I_t^{\alpha(\cdot, t)} g(t) = \partial_t \int_0^t \frac{(t-s)^{-\alpha(s, t)}}{\Gamma(1-\alpha(s, t))} \int_y^s (s-y)^{\alpha(s, y)-1} \Gamma(\alpha(s, y)) g(y) dy ds
\]

\[
= \partial_t \int_0^t g(y) \int_y^t \frac{(t-s)^{-\alpha(s, t)}}{\Gamma(1-\alpha(s, t))} \Gamma(\alpha(s, y)) ds dy
\]

\[
= \partial_t \int_0^t g(y) \mathcal{M}(y, t) dy = g(t) + \int_0^t \partial_t \mathcal{M}(y, t) g(y) dy
\]

where

\[
\mathcal{M}(y, t) := \int_0^1 \frac{(t-y)^{\alpha((t-y)z+y, y)-\alpha((t-y)z+y, t)}}{(1-z)^{\alpha((t-y)z+y, t)-1-\alpha((t-y)z+y, y)}} \frac{1}{\Gamma(1-\alpha((t-y)z+y, t)) \Gamma(\alpha((t-y)z+y, y))} dz.
\]
Though the kernel \( \partial_t \mathcal{M}(y, t) \) looks complicated, it is indeed bounded by \( Q(1 + | \ln(t - y)|) \) as \( K \) in Theorem 1.

**Example 3** Tempered variable-exponent R-L fractional integral operators. These kind of operators are defined for \( \sigma \geq 0 \) and \( 0 < \alpha(t) < 1 \), [4, 16],

\[
\sigma I^{\alpha(t)}_t g(t) := \int_0^t e^{-\sigma(t-s)} \frac{g(s)}{\Gamma(\alpha(s))} (t-s)^{1-\alpha(s)} ds,
\]

\[
\sigma \hat{I}^{\alpha(t)}_t g(t) := \int_0^t e^{-\sigma(t-s)} \frac{g(s)}{\Gamma(\alpha(t))} (t-s)^{1-\alpha(t)} ds.
\]

By virtue of the semigroup property of the exponential function, the corresponding approximate inversion operators could be given by the tempered variable-exponent R-L fractional differential operators \( \sigma \hat{D}^{\alpha(t)}_t \) and \( \sigma D^{\alpha(t)}_t \), respectively, defined by [4, 16]

\[
\sigma \hat{D}^{\alpha(t)}_t g(t) := \partial_t \sigma I^{1-\alpha(t)}_t g(t) = \partial_t \left( \int_0^t e^{-\sigma(t-s)} \frac{g(s)}{\Gamma(1-\alpha(s))} (t-s)^{\alpha(s)} ds \right),
\]

\[
\sigma D^{\alpha(t)}_t g(t) := \partial_t \sigma I^{1-\alpha(t)}_t g(t) = \partial_t \left( \int_0^t e^{-\sigma(t-s)} \frac{g(s)}{\Gamma(1-\alpha(s))} (t-s)^{\alpha(s)} ds \right).
\]

### 3 Well-posedness of variable-exponent Abel integral equations

Based on the previous section, we prove the well-posedness of the variable-exponent Abel integral equations with \( 0 < \alpha(t) < 1 \),

\[
\mathcal{I}^{\alpha(t)}_t u(t) = f(t), \quad t \in [0, T].
\]

By Theorem 1 and the fact that \( \gamma(t) \) is bounded and bounded away from 0, integral equation (3.1) could be transformed as a second kind VIE by applying the operator \( \hat{D}^{\alpha(t)}_t \) on both sides

\[
v(t) + \frac{1}{\gamma(t)} \int_0^t K(s, t) v(s) ds = \frac{\hat{D}^{\alpha(t)}_t f(t)}{\gamma(t)}. \]

Here we use \( v(t) \) in (3.2) instead of \( u(t) \) for the clarity of the notations in the following proof.

**Theorem 2** For \( \alpha, f \in C^1[0, T] \) with \( f(0) \neq 0 \), there exists a unique solution \( u \in C(0, T] \) to the variable-exponent Abel integral equation (3.1) such that \( \lim_{t \to 0^+} t^{1-\alpha(0)} u(t) \) exists, \( |u(0)| = \infty \) and

\[
\| t^{1-\alpha(0)} u \|_{C[0, T]} \leq Q \| f \|_{C^1[0, T]},
\]
where \( Q \) may depend on \( \alpha_s, \alpha^*, \| \alpha \|_{C^1[0,T]} \) and \( T \).

In particular, if \( f(0) = 0 \), then \( u \in C[0, T] \) with \( u(0) = 0 \) and the stability estimate

\[
\| u \|_{C[0,T]} \leq Q \| f \|_{C^1[0,T]}.
\]

**Proof** Instead of considering (3.1) directly, we first investigate (3.2). Define the space \( C_{\alpha(0)}(0, T) := \{ v \in C(0, T) : \lim_{t \to 0^+} t^{1-\alpha(0)} v(t) \text{ exists} \} \) equipped with the norm

\[
\| v \|_{C_{\alpha(0)}(0, T)} := \| e^{-\lambda t} t^{1-\alpha(0)} v \|_{C[0,T]} = \| e^{-\lambda t} t^{1-\alpha(0)} v \|_{L^\infty(0,T)}
\]

for some \( \lambda > \epsilon \approx 2.718 \) and a mapping \( S \) between this space by

\[
Sv(t) := -\frac{1}{\gamma(t)} \int_0^t K(s, t)v(s)ds + \hat{D}_t^{\alpha(t)} f(t) \gamma(t).
\]

We need first to show that \( S \) is well defined. Neglecting the \( \gamma(t) \) on the denominator, which is bounded away from 0, direct calculations show that

\[
\hat{D}_t^{\alpha(t)} f = \partial_t \int_0^t f(t-s) s^{1-\alpha(t)} ds
\]

\[
= \frac{f(0)}{t^{1-\alpha(t)}} + \int_0^t \frac{\partial_t f(t-s)}{s^{1-\alpha(t)}} ds + f(t-s) \partial_t \left( \frac{s^{1-\alpha(t)}}{s^{1-\alpha(t)}} \right) ds.
\]

As for any \( 0 < \epsilon \ll 1 \) such that \( 1 - \alpha(t) + \epsilon < 1 \), the following estimates hold

\[
\left| \partial_t \left( \frac{1}{s^{1-\alpha(t)}} \right) \right| = \left| \frac{\partial_t \alpha(t) \ln s}{s \ln s} \right| \leq Q \frac{\ln s}{s^{1-\alpha(t)}} \leq Q \frac{1}{s^{1-\alpha(t)+\epsilon}},
\]

\[
t^{1-\alpha(0)} = t^{\alpha(t)-\alpha(0)} = e^{(\alpha(t)-\alpha(0)) \ln(t-0)} \leq Q,
\]

we obtain \( \hat{D}_t^{\alpha(t)} f \in C_{\alpha(0)}(0, T) \). By (2.3) and \( v \in C_{\alpha(0)}(0, T) \), the first right-hand side term of (3.4) also belongs to \( C_{\alpha(0)}(0, T) \). Therefore, we conclude from (3.4) that \( Sv \in C_{\alpha(0)}(0, T) \) and thus \( S \) is well defined.

To prove the well-posedness of (3.4), it remains to show the contractivity of the integral operator in the first right-hand side term of (3.4) under the norm \( \| \cdot \|_{C_{\alpha(0)}(0,T)} \).
We apply the splitting $t^{1-\alpha(0)} = s^{1-\alpha(0)} + (t^{1-\alpha(0)} - s^{1-\alpha(0)})$ to bound this term by

$$
\left\| \frac{1}{\gamma(t)} \int_0^t K(s, t)v(s)ds \right\|_{C_0(0, T)} \\
\leq Q \int_0^t |K(s, t)e^{-\lambda(t-s)}s^{1-\alpha(0)}e^{-\lambda s}v(s)|ds \\
+ \int_0^t \frac{|s^{1-\alpha(0)}|}{s^{1-\alpha(0)}} |K(s, t)e^{-\lambda(t-s)}s^{1-\alpha(0)}e^{-\lambda s}v(s)|ds \\
= Q \int_0^t |K(s, t)e^{-\lambda(t-s)}v(s)|ds \\
+ \int_0^t \frac{(t-s)^{1-\alpha(0)}}{s^{1-\alpha(0)}} |K(s, t)e^{-\lambda(t-s)}|ds \\
\leq Q \left\| t^{1-\alpha(0)} e^{-\lambda t}v \right\|_{L^\infty(0, T)} \int_0^t |K(s, t)|e^{-\lambda(t-s)} ds \\
+ \int_0^t \frac{(t-s)^{1-\alpha(0)}}{s^{1-\alpha(0)}} |K(s, t)|e^{-\lambda(t-s)} ds \\
\leq Q \left\| t^{1-\alpha(0)} e^{-\lambda t}v \right\|_{L^\infty(0, T)} \int_0^t |K(s, t)|e^{-\lambda(t-s)} ds \\
+ \int_0^t \frac{(t-s)^{1-\alpha(0)}}{s^{1-\alpha(0)}} |K(s, t)|e^{-\lambda(t-s)} ds \\
\leq Q \left\| t^{1-\alpha(0)} e^{-\lambda t}v \right\|_{L^\infty(0, T)} \int_0^t |K(s, t)|e^{-\lambda(t-s)} ds \\
+ \int_0^t \frac{(t-s)^{1-\alpha(0)}}{s^{1-\alpha(0)}} |K(s, t)|e^{-\lambda(t-s)} ds.
$$

By (2.3) and the relations [8]

$$
\int_0^t (t-s)^{\mu-1}e^{-\lambda s} ds = B(\mu, 1)t^\mu_1 F_1(1; 1 + \mu, -\lambda t), \quad \mu > 0,
$$

$$
_1 F_1(1; \mu, x) = \frac{\Gamma(\mu)}{\Gamma(\mu - 1)} x \left(1 + \mathcal{O}(\frac{1}{x})\right), \quad x \to -\infty, \quad \mu \geq 1,
$$

where $\,_1 F_1$ is the Kummer hypergeometric function, and we have

$$
\left| \int_0^t |K(s, t)|e^{-\lambda(t-s)} ds \right| \leq Q \int_0^t (t-s)^{-\varepsilon}e^{-\lambda(t-s)} ds \\
= \frac{Q}{\lambda^1 - \varepsilon} \int_0^t y^{-\varepsilon} e^{-\lambda y} dy < \frac{Q}{\lambda^1 - \varepsilon}, \quad 0 < \varepsilon \ll 1,
$$

$$
\int_0^t \frac{(t-s)^{1-\alpha(0)}}{s^{1-\alpha(0)}} |K(s, t)|e^{-\lambda(t-s)} ds \leq Q \int_0^t \frac{e^{-\lambda(t-s)}}{s^{1-\alpha(0)}} ds \\
= \frac{QB(\alpha, 0, 1)}{\lambda^{\alpha(0)}} (\lambda t)^{\alpha(0)}_1 F_1(1; 1 + \alpha(0), -\lambda t) \leq \frac{Q}{\lambda^{\alpha(0)}}.
$$

Here we used the asymptotic property of $\,_1 F_1$ in (3.7) to achieve the boundedness of $(\lambda t)^{\alpha(0)}_1 F_1(1; 1 + \alpha(0), -\lambda t)$ for large $\lambda t$. We invoke these estimates in (3.6) to obtain

$$
\left\| \frac{1}{\gamma(t)} \int_0^t K(s, t)v(s)ds \right\|_{C_0(0, T)} \leq Q \left( \frac{1}{\lambda^1 - \varepsilon} + \frac{1}{\lambda^{\alpha(0)}} \right) \left\| v \right\|_{C_0(0, T)}.
$$

Then for $\lambda$ large enough, the integral operator in the first right-hand side term of (3.4) is a contraction under the norm $\left\| \cdot \right\|_{C_0(0, T)}$, which implies that there exists a unique solution in $C_0(0, T)$ to (3.2) with the stability estimate

$$
\left\| v \right\|_{C_0(0, T)} \leq Q \left\| \hat{\mathcal{T}}_f^{\alpha(t)} f \right\|_{C_0(0, T)} \leq Q \left\| f \right\|_{C_1[0, T]}.
$$
Now we turn to analyze (3.1). We rewrite (3.2) back to its original form as

$$\hat{D}_t^{\alpha(t)} I_t^{\alpha(t)} v(t) = \hat{D}_t^{\alpha(t)} f(t). \quad (3.8)$$

By (2.7), (2.8) and $v \in C_{\alpha(0)}(0, T]$, which implies $|t^{1-\alpha(0)}v(t)| \leq Q$ on $[0, T]$ for some constant $Q > 0$, we have

$$|\hat{D}_t^{1-\alpha(t)} I_t^{\alpha(t)} v| \leq \int_0^t B(\alpha(t), 1 - \alpha(y))(t - y)^{\alpha(t) - \alpha(y)} y^{\alpha(y) - 1} |y^{1-\alpha(0)}v(y)| dy$$

$$\leq Q \int_0^t y^{\alpha(0)-1} dy = \frac{Q}{\alpha(0)} t^{\alpha(0)} \to 0 \text{ as } t \to 0^+. $$

Thus we obtain

$$\lim_{t \to 0^+} \hat{D}_t^{1-\alpha(t)} I_t^{\alpha(t)} v(t) = 0,$$

based on which we integrate (3.8) from 0 to $t$ to get

$$\hat{D}_t^{1-\alpha(t)} (I_t^{\alpha(t)} v(t) - f(t)) = 0.$$

By the continuity of $I_t^{\alpha(t)} v(t) - f(t)$, we could prove by contradiction that $I_t^{\alpha(t)} v(t) = f(t)$, that is, $v(t)$ serves as a solution to model (3.1). In particular, we pass the limit $t \to 0^+$ for (3.2) and apply (3.5) to find that $|v(0)| = \infty$. The uniqueness of the solutions to (3.1) in $C_{\alpha(0)}(0, T]$ follows from that of (3.2).

Finally, we observe from (3.5) that if $f(0) = 0$, then the right-hand side of (3.5) and thus $\hat{D}_t^{\alpha(t)} f$ is continuous on $[0, T]$, and the above derivations could be performed in $C[0, T]$ directly without introducing the weight function $t^{1-\alpha(0)}$. In particular, we pass the limit $t \to 0^+$ for (3.2) and apply (3.5) to find that $v(0) = 0$. Thus we complete the proof of the theorem. \(\square\)

4 Smoothing properties of variable-exponent Abel integral equations

In this section we prove the estimate of $\partial_t u(t)$, where $u(t)$ is the solution to the variable-exponent Abel integral equation (3.1).

**Theorem 3** Suppose $\alpha, f \in C^2[0, T]$. Then the solution $u$ to the variable-exponent Abel integral equation (3.1) belongs to $C^1(0, T]$ with the estimate

$$|\partial_t u(t)| \leq Q \| f \|_{C^2[0, T] I_t^{\alpha(0) - 2}}, \ t \in (0, T]. \quad (4.1)$$

Here $Q$ may depend on $\alpha_*, \alpha^*, \| \alpha \|_{C^2[0, T]}$ and $T$.

If in addition $f(0) = 0$ but $(\partial_t f)(0) \neq 0$, $u \in C[0, T] \cap C^1(0, T]$ with the estimate

$$|\partial_t u(t)| \leq Q \| f \|_{C^2[0, T] I_t^{\alpha(0) - 1}}, \ t \in (0, T].$$
Otherwise, if \( f(0) = (\partial_t f)(0) = 0, u \in C^1[0, T] \) with the estimate
\[
|\partial_t u(t)| \leq Q\|f\|_{C^2[0, T]}, \quad t \in [0, T].
\]

**Proof** We replace \( v(t) \) by \( u(t) \) in (3.2), multiply \( t \) on both sides of the resulting equation, and split \( t = s + (t - s) \) to obtain
\[
tu(t) + \frac{1}{\gamma(t)} \int_0^t \mathcal{K}(s, t)su(s)ds + \frac{1}{\gamma(t)} \int_0^t \mathcal{K}(s, t)(t - s)u(s)ds = \frac{t\hat{D}_t^{\alpha(t)} f(t)}{\gamma(t)}.
\]
(4.2)

By (2.9) and
\[
\partial_s \left( B(\alpha(t), 1 - \alpha(s))(t - s)^{\alpha(t) - \alpha(s)} \right) = \left( \partial_s B(\alpha(t), 1 - \alpha(s)) \right)(t - s)^{\alpha(t) - \alpha(s)}
\]
\[
+ B(\alpha(t), 1 - \alpha(s))(t - s)^{\alpha(t) - \alpha(s)} \left( - \partial_s \alpha(s) \ln(t - s) - \frac{\alpha(t) - \alpha(s)}{t - s} \right),
\]
we obtain
\[
\mathcal{K}(s, t) = -\partial_s \left( B(\alpha(t), 1 - \alpha(s))(t - s)^{\alpha(t) - \alpha(s)} \right)
\]
\[
+ \left( (\partial_t + \partial_s) B(\alpha(t), 1 - \alpha(s)) \right)(t - s)^{\alpha(t) - \alpha(s)}
\]
\[
+ B(\alpha(t), 1 - \alpha(s))(t - s)^{\alpha(t) - \alpha(s)} (\partial_t \alpha(t) - \partial_s \alpha(s)) \ln(t - s)
\]
\[
=: -\partial_s \left( B(\alpha(t), 1 - \alpha(s))(t - s)^{\alpha(t) - \alpha(s)} \right) + \mathcal{R}(s, t),
\]
and we could apply \( \alpha \in C^2[0, T] \) to find for \( 0 \leq s \leq t \leq T \)
\[
|\mathcal{R}(s, t)| \leq Q, \quad |\partial_t \mathcal{R}(s, t)| \leq Q(1 + |\ln(t - s)|).
\]

We invoke these in the second left-hand side term of (4.2) and apply integration by parts to obtain
\[
\frac{1}{\gamma(t)} \int_0^t \mathcal{K}(s, t)su(s)ds
\]
\[
= -tu(t) + \frac{1}{\gamma(t)} \int_0^t B(\alpha(t), 1 - \alpha(s))(t - s)^{\alpha(t) - \alpha(s)} \partial_s (su(s))ds
\]
\[
+ \int_0^t \mathcal{R}(s, t) su(s)ds.
\]
(4.3)

Then we differentiate this equation to obtain
\[
\partial_t \left( \frac{1}{\gamma(t)} \int_0^t \mathcal{K}(s, t)su(s)ds \right)
\]
\[
= \int_0^t \partial_t \left( \frac{1}{\gamma(t)} B(\alpha(t), 1 - \alpha(s))(t - s)^{\alpha(t) - \alpha(s)} \right) \partial_s (su(s))ds
\]
\[
+ \frac{\mathcal{R}(s, t)}{\gamma(t)} tu(t) + \int_0^t \partial_t \left( \frac{\mathcal{R}(s, t)}{\gamma(t)} \right) su(s)ds.
\]
We use this relation to differentiate (4.2) as follows:

\[
\begin{align*}
\partial_t (tu(t)) + \int_0^t \partial_t \left( \frac{1}{\gamma(t)} B(\alpha(t), 1 - \alpha(s))(t - s)^{\alpha(t) - \alpha(s)} \right) \partial_s (su(s)) ds \\
+ \frac{\mathcal{R}(t, t)}{\gamma(t)} tu(t) + \int_0^t \partial_t \left( \frac{\mathcal{R}(s, t)}{\gamma(t)} \right) su(s) ds \\
+ \int_0^t \partial_t \left( \frac{\mathcal{K}(s, t)(t - s)}{\gamma(t)} \right) u(s) ds = \partial_t \left( \frac{t^\alpha f(t)}{\gamma(t)} \right). 
\end{align*}
\]

By (2.9) we have

\[
\left| \partial_t \left( \frac{\mathcal{K}(s, t)(t - s)}{\gamma(t)} \right) \right| \leq Q (1 + \ln(t - s)).
\]

Thus, we apply Theorem 2 to bound the last three left-hand side terms of (4.4) by \( Q \|f\|_{C^1[0, T]} \). We calculate \( t^\alpha f(t) \) by (3.5) as

\[
t^\alpha f(t) = f(0)t^\alpha + t \int_0^t \frac{\partial_t f(t - s)}{s^{1 - \alpha(t)}} ds + f(t - s) \frac{1}{s^{1 - \alpha(t)}} ds,
\]

the derivative of which could be bounded as

\[
|\partial_t (t^\alpha f(t))| \leq Q \|f\|_{C^2[0, T]} t^\alpha (0)^{-1}.
\]

We incorporate these estimates in (4.4) to obtain for some \( 0 < \varepsilon < 1 \)

\[
|\partial_t (tu(t))| \leq Q \int_0^t (1 + |\ln(t - s)|) |\partial_s (su(s))| ds + Q \|f\|_{C^2[0, T]} t^\alpha (0)^{-1}
\]

\[
\leq Q \int_0^t |\partial_s (su(s))| (t - s)^\varepsilon ds + Q \|f\|_{C^2[0, T]} t^\alpha (0)^{-1}.
\]

Then an application of the weak singular Gronwall inequality, see e.g., [23, Theorem 1.2], yields

\[
|\partial_t (tu(t))| \leq Q \|f\|_{C^2[0, T]} t^\alpha (0)^{-1}.
\]

As \( \partial_t (tu(t)) = u(t) + t \partial_t u(t) \), we immediately obtain from Theorem 2 that

\[
|t \partial_t u(t)| \leq Q \|f\|_{C^2[0, T]} t^\alpha (0)^{-1},
\]

which proves (4.1).

If \( f(0) = 0 \), then Theorem 2 implies that model (3.1) admits a unique continuous solution on \([0, T]\) satisfying \( u(0) = 0 \) and (3.3). Thus we could directly differentiate
(3.2) and perform estimates by similar techniques as above without multiplying \( t \) on both sides. In particular, differentiating the second right-hand side term of (3.5) yields
\[
\partial_t \int_0^t \frac{\partial_t f(t-s)}{s^{1-\alpha(t)}} ds = \frac{(\partial_t f)(0)}{t^{1-\alpha(t)}} + \int_0^t \frac{\partial_t (\partial_t f(t-s))}{s^{1-\alpha(t)}} ds.
\]
If \( (\partial_t f)(0) \neq 0 \), then \( \partial_t u \) has the singularity of \( t^{\alpha(t)-1} \leq Qt^{\alpha(0)-1} \). Otherwise, \( \partial_t u \) is continuous on \([0, T]\), which completes the proof.

\[\square\]

5 Analysis of Riemann-Liouville fractional differential equations

The classical R-L FDE of order \( 0 < \alpha < 1 \) reads ([5, 10])
\[
D^\alpha_t u(t) = h(t), \quad t \in (0, T]; \quad u(0) = u_0 \in \mathbb{R},
\]
the solution of which could be analytically obtained as
\[
u(t) = I^\alpha_t h(t).
\]
Note that if \( h \in C[0, T] \), taking \( t \rightarrow 0^+ \) on both sides of this equation leads to \( u(0) = 0 \). Thus, \( u_0 \) must be 0 in order to ensure the well-posedness of (5.1), which is restricted and extremely sensitive to the initial value, that is, any perturbation or noise on the initial value may lead to the ill-posedness of model (5.1). In most literature, the equation in (5.1) is usually equipped with the fractional initial condition, which may be difficult to determine in real applications.

Another observation is that if we differentiate (5.2), we obtain
\[
\partial_t u(t) = \partial_t I^\alpha_t h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} h(0) + I^\alpha_t \partial_t h(t),
\]
which indicates that the solution has an unbounded derivative at \( t = 0 \) if \( h(0) \neq 0 \). We will show that in variable exponent problems, this singularity could be removed by adjusting the variable exponent.

In this section we address these issues to study the following variable-exponent R-L FDE
\[
D^{\alpha(t)}_t u(t) = h(t), \quad t \in (0, T]; \quad u(0) = u_0.
\]
We discuss this model for two cases:
(i) \( 0 < \alpha(t) < 1 \) on \([0, T]\);
(ii) \( 0 < \alpha(t) < 1 \) on \((0, T]\) and \( \alpha(0) = 1 \).

We will draw conclusions that under the condition (i), (5.3) encounters the same sensitivity issue on the initial condition and the solution has a singular derivative as its constant-exponent analogue (5.1), while under the condition (ii), model (5.3) is well-posed for any \( u_0 \in \mathbb{R} \) and, if in addition \( (\partial_t \alpha)(0) = 0 \), the solution has a continuous
derivative, which resolves the sensitivity issue of the well-posedness of R-L FDEs on the initial conditions and the initial singularities of the solutions to FDEs.

5.1 Model (5.3) with condition (i)

We aim to seek its solution in $C[0, T]$ and formally integrate (5.3) from 0 to $t$ to obtain

$$I_t^{1-\alpha(t)} u(t) = \int_0^t h(s) ds + c_0. \tag{5.4}$$

Here $c_0$ is a constant that will be determined later. As $0 < \alpha(t) < 1$ on $[0, T]$, the results in Sections 3–4 could be employed since models (5.4) and (3.1) are almost equivalent (except for an additional Gamma function in $I_t^{1-\alpha(t)}$ that does not affect the analysis) if

$$f(t) = \int_0^t h(s) ds + c_0.$$  

If $c_0 \neq 0$, then $f(0) \neq 0$. By Theorem 2, the solution $u$ has the property $|u(0)| = \infty$, which could not satisfy the initial condition in (5.3). Therefore, we must set $c_0 = 0$ and then Theorem 2 implies that there exists a unique continuous solution to (5.4) with $u(0) = 0$, that is, $u_0$ must be 0 in order that model (5.3) is well-posed, which encounters the same sensitivity issue on the initial condition as its constant-exponent analogue (5.1). In this case ($u_0 = 0$), we could apply Theorem 3 to conclude that if $h(0) \neq 0$, then the solution $u$ to (5.4) and thus (5.3) has an initial singularity of $\mathcal{O}(t^{\alpha_0-1})$ as the constant-exponent case.

5.2 Model (5.3) with condition (ii)

By condition (ii), $\alpha(t)$ still has a positive lower bound $\alpha_*$ but is no longer bounded away from 1. We apply $\hat{D}_t^{1-\alpha(t)}$ on both sides of (5.4) and employ (2.11) to obtain

$$u(t) + \int_0^t \partial_t \left( \frac{(t-y)^{\alpha(t)-\alpha(y)}}{\Gamma(1+\alpha(t)-\alpha(y))} u(y) dy \right) = \hat{D}_t^{1-\alpha(t)} \left( \int_0^t h(s) ds + c_0 \right). \tag{5.5}$$

We base on this equation to analyze the well-posedness and smoothing properties of variable-exponent R-L FDE (5.3) in the following theorem.

**Theorem 4** Suppose $\alpha \in C^1[0, T]$, $h \in C[0, T]$ and the condition (ii) holds. Then the variable-exponent R-L FDE (5.3) admits a unique solution in $C[0, T]$ for any $u_0 \in \mathbb{R}$ with the stability estimate

$$\|u\|_{C[0,T]} \leq Q\left(\|h\|_{C[0,T]} + |u_0|\right).$$

Here $Q$ may depend on $\alpha_*$, $\|\alpha\|_{C^1[0,T]}$ and $T$.  

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If \( \alpha \in C^2[0, T] \) with \((\partial_t \alpha)(0) = 0, h \in C^1[0, T]\) and the condition (ii) holds, then \( u \in C^1[0, T]\) with the stability estimate

\[
\|u\|_{C^1[0, T]} \leq Q(\|h\|_{C^1[0, T]} + |u_0|).
\]

Here \( Q \) may depend on \( \alpha_*, \|\alpha\|_{C^2[0, T]} \) and \( T \).

**Proof** We apply the transformation \( s \to t - s \) to obtain

\[
\partial_t^{-\alpha(t)} \int_0^t h(s)ds = \partial_t^{\alpha(t)} \int_0^t h(t-s)ds = \partial_t \int_0^t \frac{s^{\alpha(t) - 1}}{\Gamma(\alpha(t))} \int_0^s h(y)dy ds
\]

\[
= \int_0^t \partial_t \left( \frac{s^{\alpha(t) - 1}}{\Gamma(\alpha(t))} \right) \int_0^t h(t-s)ds + \int_0^t \frac{s^{\alpha(t) - 1}}{\Gamma(\alpha(t))} h(t-s)ds.
\]

As \( \alpha(t) \geq \alpha_* > 0 \), we have

\[
\left| \partial_t \left( \frac{s^{\alpha(t) - 1}}{\Gamma(\alpha(t))} \right) \right| = \left| \frac{s^{\alpha(t) - 1}}{\Gamma(\alpha(t))} \left( \partial_t \left( \frac{s^{\alpha(t) - 1}}{\Gamma(\alpha(t))} \right) \ln s - \frac{\partial_t \Gamma(\alpha(t))}{\Gamma(\alpha(t))} \right) \right| \leq Q s^{\alpha_* - 1} |\ln s|.
\]

Therefore, the right-hand side of (5.6) is continuous on \([0, T]\). We could also evaluate \( \partial_t^{-\alpha(t)} c_0 \) exactly as follows

\[
\partial_t^{-\alpha(t)} c_0 = c_0 \partial_t \int_0^t \frac{1}{\Gamma(\alpha(t))} \frac{1}{(t-s)^{1-\alpha(t)}} ds = c_0 \partial_t \left( \frac{t^{\alpha(t)}}{\Gamma(1 + \alpha(t))} \right)
\]

\[
= c_0 \left( \frac{t^{\alpha(t) - 1}}{\Gamma(\alpha(t))} + \frac{(\partial_t \alpha(t)) t^{\alpha(t)} \ln t}{\Gamma(1 + \alpha(t))} + t^{\alpha(t)} \partial_t \left( \frac{1}{\Gamma(1 + \alpha(t))} \right) \right).
\]

As \( \alpha(0) = 1 \) and \( \alpha \in C^1[0, T] \), \( \lim_{t \to 0^+} t^{\alpha(t) - 1} = \lim_{t \to 0^+} e^{(\alpha(t) - \alpha(0)) \ln t} = 1 \).

Similar to the estimate of \( K(s, t) \), we have \( |\partial_t (t - y)^{\alpha(t) - \alpha(y)}| \leq Q(1 + |\ln(t - y)|) \).

Then by similar proofs as Theorem 2, integral equation (5.5) admits a unique solution in \( C[0, T] \) with stability estimate \( \|u\|_{C[0, T]} \leq Q(\|h\|_{C[0, T]} + c_0) \). Passing the limit \( t \to 0^+ \) on both sides of (5.5) and using the expressions (5.6) and (5.7) we obtain

\[
u(0) = c_0.
\]

Therefore, if we set \( c_0 = u_0 \), then the solution \( u \) to model (5.5) satisfies \( u(0) = u_0 \).

Finally we need to recover the differential equation (5.3) from the integral equation (5.5), the original form of which is

\[
\partial_t^{\alpha(t)} \left( I_t^{1-\alpha(t)} u(t) - \int_0^t h(s)ds - c_0 \right) = 0.
\]

As the content in \((\cdots)\) is continuous, we integrate this equation from 0 to \( t \) to obtain

\[
I_t^{\alpha(t)} \left( I_t^{1-\alpha(t)} u(t) - \int_0^t h(s)ds - c_0 \right) = 0,
\]
which, by contraction, immediately leads to

\[ I_t^{1-\alpha(t)} u(t) - \int_0^t h(s) ds - c_0 = 0. \]

As the second and the third left-hand side terms of this equation are differentiable, so does the first term. Thus we differentiate this equation to obtain (5.3), that is, the solution to the integral equation (5.5) also solves (5.3). The uniqueness of the solutions to (5.3) in $C[0, T]$ follows from that of (5.5).

To bound $\partial_t u(t)$, we may differentiate (5.5) and then perform estimates following the proof of Theorem 3. In particular, when we apply the integration by parts for the second left-hand side term of (5.5) as (4.3), the same term as the first right-hand side term of (5.7) with $c_0 = u_0$ will be generated, and they cancel each other. If we differentiate the numerator of the second right-hand side term of (5.7), we will encounter the factor

\[ (\partial_t \alpha(t)) \alpha(t) t^{\alpha(t)-1} \ln t. \]

In order to make this term continuous, we require $\alpha(0) = 1$, $(\partial_t \alpha)(0) = 0$ and $\alpha \in C^2[0, T]$ as assumed in the theorem. The rest of the proof is omitted as it is similar to that of Theorem 3 and thus we complete the proof.

\[ \square \]

6 Concluding remarks

We develop an approximate inversion technique for variable-exponent Abel integral operators, and analyze the corresponding integral and differential equations. In general, we conclude that by setting an integer limit of variable exponent $\alpha(t)$ at the initial time $t = 0$, the variable-exponent problems have similar properties as their integer-order analogues. Otherwise, they behave like their constant-exponent counterparts of order $\alpha \equiv \alpha(0)$. In particular, we prove that the sensitive dependence of the well-posedness of classical R-L FDEs on the initial value and the singularity of the solutions could be resolved by adjusting the initial value of the variable exponent, which demonstrates its advantages. The proposed approximate inversion technique provides a potential means to convert the intricate variable-exponent integral and differential problems to feasible forms and the derived results suggest that the variable-exponent fractional problems may serve as a connection between integer-order and fractional models by adjusting the variable exponent at the initial time.

There are other potential applications of this approximate inversion method. For instance, we could investigate analogous problems that replace $\alpha(s)$ by $\alpha(t)$ or $\alpha(s, t)$ in the aforementioned models via similar or extended techniques. Furthermore, since the variable-exponent Abel integral equations and corresponding fractional differential equations have been converted as second-kind VIEs, traditional treatments for the second-kind VIEs [2] could be employed to prove the high-order regularity of the solutions and to develop and analyze numerical schemes. Another important extension...
lies in using the proposed methods to investigate the variable-exponent analogue of subdiffusion equations [12, 17]. We will investigate these topics in the near future.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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