DIFFERENTIATION OF MEASURES ON COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. In this note we give a new proof of a version of the Besicovitch covering theorem, given in [EG1992], [Bogachev2007] and extended in [Federer1969], for locally finite Borel measures on finite dimensional complete Riemannian manifolds \((M, g)\). As a consequence, we prove a differentiation theorem for Borel measures on \((M, g)\), which gives a formula for the Radon-Nikodym density of two nonnegative locally finite Borel measures \(\nu_1, \nu_2\) on \((M, g)\) such that \(\nu_1 \ll \nu_2\), extending the known case when \((M, g)\) is a standard Euclidean space.

1. Introduction

The existence of the Radon-Nikodym derivative is one of the most frequently employed results in probability theory and mathematical statistics. In the general case, where \(\nu, \mu\) are locally finite measures on a general measurable space \(X\) and \(\nu \ll \mu\), classical proofs of the existence of the Radon-Nikodym derivative \(d\nu/d\mu\) are non constructive, see e.g. [Bogachev2007, p. 429, vol. 1], [BBT2008, §8.7, p. 336] for historical comments. For a class of metrizable measurable spaces \(X\), the theorem of differentiation of measures with a constructive proof [1] yields not only the existence of the Radon-Nikodym derivative, but also computes the Radon-Nikodym density based on an appropriate metric. As far as we know, that is the only way to get an explicit formula for the Radon-Nikodym derivative, see [SG1977, p. 189], [Panangaden2009, p. 56] for discussions on the relation between the Radon-Nikodym theorem and the theorem of differentiation of measures.

The main ingredient of all known proofs of the theorem of differentiation of measures is the construction (or the existence) of a differentiation basis, which is based on a covering theorem. All covering theorems are based on the same idea: from an arbitrary cover of a set in a metric space, one tries to select a subcover that is, in a certain sense, as disjointed as possible. According to [Heinonen2001, Chapter 1], there are three (types) of

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1The proof of the theorem of differentiation of measures on complete \(\sigma\)-finite measure spaces given in [BBT2008, Chapter 8] utilizes the existence of lifting, whose proof is non constructive.
covering theorems: the basic covering theorem, which is an extension of the classical Vitali theorem for $\mathbb{R}^n$ to arbitrary metric space, the Vitali covering theorem, which is an extension the classical Vitali theorem to the case of doubling metric measure spaces, and the Besicovitch-Federer theorem that has been first proved by Besicovitch [Besicovitch1945] for the case of $\mathbb{R}^n$ and then extended by Federer for directionally $(\varepsilon, M)$-limited subsets of a metric space $X$ [Federer1969, Theorem 2.8.14, p.150]. Examples of such subsets are compact subsets in a Riemannian manifold. The essence of Vitali theorems is that one finds a disjointed subcollection of the sets of a given cover that need not be a cover itself, but that when the radii are all enlarged by a fixed factor, covers everything. The essence of the Besicovitch theorems is to select a subcover so that each point is only covered a controlled number of times. Clearly, such theorems are useful when one has to estimate constants occurring in covering arguments.

The Besicovitch-Federer covering theorems has been revisited for the case of $\mathbb{R}^n$ [Sullivan1994, EG1992], and for any finite dimensional normed vector space, which results in a variation of the Besicovitch-Federer covering theorem for arbitrary metric spaces [Loeb1989], and extended in [Itoh2018] for non directionally limited subsets in $\mathbb{R}^n$.

In our note we give a new proof of the following version of the Besicovitch-Federer theorem.

**Theorem 1.1.** Assume that $\mathcal{F}$ is a collection of open 4-proper geodesic balls in a complete Riemannian manifold $(M, g)$ such that the set $A$ of the centers of the balls in $\mathcal{F}$ is bounded. Then one can find $N \in \mathbb{N}^*$ and subcollections $\mathcal{F}_1, \ldots, \mathcal{F}_N \subset \mathcal{F}$ each of which consists of at most countably many disjoint balls such that $A$ is covered by the balls from $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_N$.

Here, 4-proper means that the radius of the ball is at most $1/4$ of the injectivity radius of its center. A particular case of Theorem 1.1 is the version of the Besicovitch covering theorem for the standard Euclidean space $\mathbb{R}^n$ [Besicovitch1945], which has been formulated as Theorem 5.8.1 in [Bogachev2007, p. 361, vol. 1] based on the proof of [EG1992, Theorem 1.27]. There are three differences between Theorem 1.1 and Theorem 5.8.1 ibid.: firstly we make the assumption that $A$ is bounded, secondly, the geodesic balls are 4-proper, and thirdly, the balls are open instead of nondegenerate closed as in Theorem 5.8.1 ibid. Note that in the Besicovitch-Federer theorem [Federer1969, Theorem 2.8.14] the similar family $\mathcal{F}$ also consists of closed balls. (In fact Theorem 1.1 is also valid for closed balls, but we need to track and change, if necessary, several similar strict or non-strict inequalities in the proof.) The main idea of our proof is to use comparison theorems in Riemannian geometry to reduce the situation to the Euclidean one.

As a result, we shall prove a theorem of differentiation of measures for locally finite Borel measures on complete Riemannian manifolds [1.2], which
The balls

Lemma 2.1. Let \( \nu_1 \) and \( \nu_2 \) be locally finite Borel measures on \((M, g)\) such that \( \nu_2 \ll \nu_1 \). For \( x \in M \) we denote by \( D_r(x) \) the open geodesic ball of radius \( r \) in \( M \) with center in \( x \) and we set

\[
D_{\nu_1} \nu_2(x) := \lim_{r \to 0} \sup \frac{\nu_2(D_r(x))}{\nu_1(D_r(x))},
\]

\[
D_{\nu_1} \nu_2(x) := \lim_{r \to 0} \inf \frac{\nu_2(D_r(x))}{\nu_1(D_r(x))},
\]

where we set \( D_{\nu_1} \nu_2(x) = D_{\nu_1} \nu_2(x) = +\infty \) if \( \nu_1(D_r(x)) = 0 \) for some \( r > 0 \).

Furthermore if \( D_{\nu_1} \nu_2(x) = D_{\nu_1} \nu_2(x) \) then we denote their common value by

\[
D_{\nu_1} \nu_2(x) := \overline{D}_{\nu_1} \nu_2(x) = \underline{D}_{\nu_1} \nu_2(x)
\]

which is called the derivative of \( \nu_2 \) with respect to \( \nu_1 \) at \( x \).

Theorem 1.2. Let \( \nu_1 \) and \( \nu_2 \) be two nonnegative locally finite Borel measures on a complete Riemannian manifold \((M, g)\) such that \( \nu_2 \ll \nu_1 \). Then there is a measurable subset \( S_0 \subset M \) of zero \( \nu_1 \)-measure such that the function \( D_{\nu_1} \nu_2 \) is defined and finite on \( M \setminus S_0 \). Setting \( \overline{D}_{\nu_1} \nu_2(x) := 0 \) for \( x \in S_0 \) and \( \overline{D}_{\nu_1} \nu_2(x) := D_{\nu_1} \nu_2(x) \) for \( x \in M \setminus S_0 \), the function \( \overline{D}_{\nu_1} \nu_2 : M \to \mathbb{R} \) is measurable and serves as the Radon-Nikodym density of the measure \( \nu_2 \) with respect to \( \nu_1 \).

Theorem 1.2 is also different from Theorem 5.8.8 in [Bogachev2007, vol.1] in defining \( \overline{D}_{\nu_1} \nu_2 \), since we need to apply it to a family of Nikodym derivatives in our paper [JLT2020].

2. Proof of Theorem 1.1

Assume the conditions of Theorem 1.1. Let \( R := \sup\{r : D_r(a) \in \mathcal{F}\} \). We can find \( D_1 = D_{r_1}(a_1) \in \mathcal{F} \) with \( r_1 > 3R/4 \). The balls \( D_j, j > 1 \), are chosen inductively as follows. Let \( A_j := A \setminus \bigcup_{i=1}^{j-1} D_i \). If the set \( A_j \) is empty, then our construction is completed and, letting \( J = j - 1 \) we obtain \( J \) balls \( D_1, \ldots, D_J \). If \( A_j \) is nonempty, then we choose \( D_j := D_{r_j}(a_j) \in \mathcal{F} \) such that

\[
a_j \in A_j \text{ and } r_j > \frac{3}{4} \sup\{r : D_r(a) \in \mathcal{F}, a \in A_j\}.
\]

In the case of an infinite sequence of balls \( D_j \) we set \( J = \infty \).

Lemma 2.1. The balls \( D_j \) satisfy the following properties

(a) if \( j > i \) then \( r_j \leq 4r_i/3 \),

(b) the balls \( D_{r_j/3}(a_j) \) are disjoint and if \( J = \infty \) then \( r_j \to 0 \) as \( j \to \infty \),

(c) \( A \subset \bigcup_{j=1}^{\infty} D_j \).
Proof. Property (a) follows from the definition of \( r_i \) and the inclusion \( a_j \in A_j \subseteq A_i \).

Property (b) is a consequence of the following observation. If \( j > i \) then \( a_j \notin D_i \) and hence by (a) we have

\[
\rho_g(a_i, a_j) \geq r_i > \frac{r_i}{3} + \frac{r_j}{3}.
\]

Since \( A \) is bounded, \( r_j \) goes to 0 as \( j \to \infty \) if \( J = \infty \).

Finally (c) is obvious if \( J < \infty \). If \( J = \infty \) and \( \mathcal{D}_r(a) \in \mathcal{F} \) then there exists \( r_j \) with \( r_j < 3r/4 \) by (b). Hence \( a \in \bigcup_{i=1}^{J-1} D_i \) by our construction of \( r_j \). This completes the proof of Lemma 2.1.

\[ \square \]

We fix \( k > 1 \) and let

\[
I_k := \{ j : j < k, D_j \cap D_k \neq \emptyset \}, \quad M_k := I_k \cap \{ j : r_j \leq 3r_k \}.
\]

**Lemma 2.2.** There is a number \( c(A) \) independent of \( k \) such that \( \#M_k \leq c(A) \).

**Proof.** If \( j \in M_k \) and \( x \in D_{r_j/3}(a_j) \) then the balls \( D_j \) and \( D_k \) are open and have nonempty intersection and \( r_j \leq 3r_k \), hence

\[
\rho_g(x, a_k) \leq \rho_g(x, a_j) + \rho_g(a_j, a_k) < \frac{r_j}{3} + r_j + r_k < 5r_k.
\]

It follows that \( D_{r_j/3}(a_j) \subseteq D_{5r_k}(a_k) \). Denote by \( \text{vol}_g \) the Riemannian volume on \((M, g)\). By the disjointness of \( D(a_j, r_j/3) \) and the boundedness of \( A \), taking into account the Bishop volume comparison theorem [BC1964, Theorem 15, §11.10], see also [Le1993] for a generalization, there exists a number \( c_1(A) \) (depending on an upper bound for the Ricci curvature and on the local topology, but the latter will play no role for \( 4\)-proper balls) such that

\[
\text{vol}_g(D_{5r_k}(a_k)) \geq \sum_{j \in M_k} \text{vol}_g(D_{r_j/3}(a_j)) \geq c_1(A) \sum_{j \in M_k} \left( \frac{r_j}{3} \right)^n.
\]

Using property (a) in Claim 1, we obtain from (2.3)

\[
\text{vol}_g(D_{5r_k}(a_k)) \geq \sum_{j \in M_k} c_1(A) \left( \frac{r_j}{4} \right)^n = \#(M_k) c_1(A) \left( \frac{r_k}{4} \right)^n.
\]

By the Bishop comparison theorem there exists a number \( c_2(A) \) (depending on a lower bound for the Ricci curvature) such that \( \text{vol}_g(D_{5r_k}(a_k)) \leq c_2(A) \cdot (5r_k)^n \). In combination with (2.4) we obtain

\[
\#(M_k) \leq \frac{c_2(A)}{c_1(A)} 20^n.
\]

This completes the proof of Lemma 2.2.

\[ \square \]

**Lemma 2.3.** There exists a number \( d(A) \) independent of \( k \) such that \( \#(I_k \setminus M_k) \leq d(A) \).
**Proof.** Let us consider two distinct elements $i, j \in I_k \setminus M_k$. By (2.2) we have

$$1 < i, j < k, D_i \cap D_k \neq \emptyset, D_j \cap D_k \neq \emptyset, r_i > 3r_k, r_j > 3r_k.$$  

For notational simplicity we shall redenote $\rho_g(a_k, a_i)$ by $|a_i|$. Then (2.6) implies

$$|a_i| < r_i + r_k \text{ and } |a_j| < r_j + r_k.$$  

Let $\theta_{def}(a_i, a_j)\in[0, \pi]$ be the deformed angle between the two geodesic rays $(a_k, a_i)$ and $(a_k, a_j)$, connecting $a_k$ with $a_i$ and $a_j$ respectively, which is defined as follows

$$\theta_{def}(a_i, a_j) := \arccos \frac{|a_i|^2 + |a_j|^2 - \rho_g(a_i, a_j)^2}{2|a_i||a_j|}.$$  

**Figure 1.** The $\theta_{def}(a_i, a_j)$ vs $\theta_{ak}(a_i, a_j)$ (defined below).

We shall prove the estimate

$$\theta_{def}(a_i, a_j) \geq \theta_0 := \arccos61/64 > 0.$$  

By the construction, see also (2.1), we have $a_k \notin D_i \cup D_j$ and $r_i \leq |a_i|, r_j \leq |a_j|$. W.l.o.g. we assume that $|a_i| \leq |a_j|$. By (2.2) and (2.7) we obtain

$$3r_k < r_i \leq |a_i| < r_i + r_k, 3r_k < r_j \leq |a_j| < r_j + r_k, |a_i| \leq |a_j|.$$
We need two more claims for the proof of (2.8).

Claim 1. If \( \cos \theta_{\text{def}}(a_i, a_j) > 5/6 \) then \( a_i \in D_j \).

Proof of Claim 1. It suffices to show that if \( a_i \notin D_j \) then \( \cos \theta_{\text{def}}(a_i, a_j) \leq 5/6 \). Assume that \( a_i \notin D_j \). We shall consider two possibilities, first assume that \( \rho_g(a_i, a_j) \geq |a_j| \). Then our assertion follows from the following estimates

\[
\cos \theta_{\text{def}}(a_i, a_j) = \frac{|a_i|^2 + |a_j|^2 - \rho_g(a_i, a_j)^2}{2|a_i||a_j|} \leq \frac{|a_i|}{2|a_j|} \leq \frac{1}{2} < \frac{5}{6}.
\]

(2.10) Now assume that \( \rho_g(a_i, a_j) \leq |a_j| \). Then

\[
\cos \theta_{\text{def}}(a_i, a_j) = \frac{|a_i|^2 + |a_j|^2 - \rho_g(a_i, a_j)^2}{2|a_i||a_j|} \leq \frac{|a_i|}{2|a_j|} + \frac{(|a_j| - \rho_g(a_i, a_j))(|a_j| + \rho_g(a_i, a_j))}{2|a_i||a_j|}
\]

\[
\leq \frac{1}{2} + \frac{|a_j| - \rho_g(a_i, a_j)}{|a_j|} \leq \frac{1}{2} + \frac{r_j + r_k - r_j}{r_i} \leq \frac{5}{6}
\]

(2.11)

where in the second inequality we use the assumption \( |a_j| + \rho_g(a_i, a_j) \leq 2|a_j| \), in the third inequality we use \( |a_j| \leq r_j + r_k \) and taking into account \( a_i \notin D_j \) we have \( r_j \leq \rho_g(a_i, a_j) \), we also use \( r_i \leq |a_i| \) from (2.9), and in the last inequality we use \( 3r_k < r_i \) from (2.6). This completes the proof of Claim 1.

Claim 2. If \( a_i \in D_j \) then

\[
0 \leq \rho_g(a_i, a_j) + |a_i| - |a_j| \leq \frac{8}{3}(1 - \cos \theta_{\text{def}}(a_i, a_j))|a_j|.
\]

(2.12) Proof of Claim 2. We utilize the proof of [Bogachev2007, 5.8.3, p. 363, vol. 2]. Since \( a_i \in D_j \) we have \( i < j \). Hence \( a_j \notin D_i \) and therefore \( \rho_g(a_i, a_j) \geq r_i \). Keeping our convention that \( |a_i| \leq |a_j| \) we have

\[
0 \leq \frac{\rho_g(a_i, a_j) + |a_i| - |a_j|}{|a_j|} \leq \frac{\rho_g(a_i, a_j) + |a_i| - |a_j|}{|a_j|} = \frac{2|a_i|(1 - \cos \theta_{\text{def}}(a_i, a_j))}{\rho_g(a_i, a_j)}
\]

\[
\leq \frac{2(r_i + r_k)(1 - \cos \theta_{\text{def}}(a_i, a_j))}{r_i} \leq \frac{8}{3}(1 - \cos \theta_{\text{def}}(a_i, a_j)).
\]

Here in the inequality before the last we use the above inequality \( r_i < \rho_g(a_i, a_j) \) and \( |a_i| < r_i + r_k \) from (2.9). This completes the proof of Claim 2.

Continuation of the proof of (2.8). If \( \cos \theta_{\text{def}}(a_i, a_j) \leq 5/6 \), then \( \cos \theta_{\text{def}}(a_i, a_j) \leq 61/64 \). If \( \cos \theta_{\text{def}}(a_i, a_j) > 5/6 \) then \( a_i \in D_j \) by Lemma 2.1. Then \( i < j \) and hence \( a_j \notin D_i \). It follows that \( r_i \leq \rho_g(a_i, a_j) < r_j \).
Recall by Lemma 2.1 (a) \( r_j \leq 4r_i/3 \). Taking into account \( r_j > 3r_k \) from 2.2 we obtain

\[
\rho_g(a_i, a_j) + |a_i| - |a_j| \geq r_i + r_j - r_k \geq \frac{r_j}{2} - r_k \geq \frac{1}{8}(r_j + r_k) \geq \frac{1}{8}|a_j|
\]

which in combination with (2.12) yields \( |a_j|/8 < 8(1 - \cos \theta_{deF}(a_i, a_j))|a_j|/3 \). Hence \( \cos \theta_{deF}(a_i, a_j) \leq 61/64 \). This completes the proof of estimate (2.8).

In the next step we shall prove the existence of a lower bound for the angle \( \theta_{ak}(a_i, a_j) \) between the two geodesic rays \((a_k, a_i)\) and \((a_k, a_j)\), namely \( \theta_{ak}(a_i, a_j) \) is the angle between two vectors \( \vec{a}_i \) and \( \vec{a}_j \) on the tangent space \( T_{a_k}M^n \) provided with the restriction of the metric \( g \) to \( T_{a_k}M^n \), where \( \vec{a}_i \) (resp. \( \vec{a}_j \)) is the tangent vector at \( a_k \) of the geodesic \((a_k, a_i)\) (resp. of the geodesic \((a_k, a_j)\)).

**Claim 3.** There exists a positive number \( \alpha(A) \) independent of \( k, i, j \) such that \( \theta_{ak}(a_i, a_j) \geq \alpha(A) \).

**Proof of Claim 3.** Since \( A \) is bounded, by the Bishop-Crittenden comparison theorem [BC1964, Theorem 15, §11.10] that estimates the differential of the exponential map via the sectional curvature of the Riemannian manifold there exists a constant \( b(A) \) independent of \( a_i, a_j, a_k \) and sectional curvature bounds for \( A \subset M \) such that \( \theta_{ak}(a_i, a_j) > b(A) \cdot \theta_{deF}(a_i, a_j) \). Combining this with (2.8) implies Claim 3.

**Continuation of the proof of Lemma 2.3** Denote by \( \text{inj rad}_M(x) \) the injectivity radius of \( M \) at \( x \). Let \( r_A := \inf_{x \in A} \text{inj rad}_M(x) \). Since \( A \) is bounded, \( r_A > 0 \).

- Let \( \delta(A) \) be the largest positive number such that:
  - (i) \( \delta(A) \leq r_A/8 \),
  - (ii) For any \( x \neq y \neq z \neq x \in A \) satisfying the following relations
    
    \[
    \rho_g(x, y) \leq \frac{r_A}{4} \quad \text{and} \quad \rho_g(y, z) \leq \rho_g(x, y) \cdot \delta(A)
    \]

we have \( \theta_{ak}(y, z) \leq \alpha(A) \).

The existence of \( \delta(A) \) follows from the boundedness of \( A \) and the Bishop-Crittenden comparison theorem.

- Let \( d(A) \) be the smallest natural number such that for any \( x \in A \) and any \( r \in (0, r_A/4) \) we can cover the geodesic sphere \( S(x, r) \) of radius \( r \) centered at \( x \) by at most \( d(A) \) balls of radius \( r \cdot \delta(A) \). The existence of \( d(A) \) follows from the boundedness of \( A \) and Bishop’s comparison theorem (lower Ricci bound).

Claim 3 implies that \( \#(I_k \setminus M_k) \leq d(A) \). This completes the proof of Lemma 2.3.

**Completion of the proof of Theorem 1.1** Lemmas 2.2 and 2.3 imply that \( \#(I_k) \leq c(A) + d(A) \).
Now we make a choice of $F_i$ in the same way as in the proof of Theorem 5.8.1 ibid. Set $L(A) := c(A) + d(A)$. We define a mapping
\[ \sigma : \{1, 2, \cdots \} \to \{1, \cdots, L(A)\} \]
as follows: $\sigma(i) = i$ if $1 \leq i \leq L(A)$. If $k \geq L(A)$, we define $\sigma(k + 1)$ as follows. Since
\[ \#(I_{k+1}) = \#\{j | 1 \leq j \leq k, D_j \cap D_{k+1} \neq \emptyset\} < L(A) \]
there exists a smallest number $l \in \{1, \cdots, L(A)\}$ with $D_{k+1} \cap D_j = \emptyset$ for all $j \in \{1, \cdots, k\}$ such that $\sigma(j) = l$. Then we set $\sigma(k + 1) := l$. Finally, let
\[ F_j := \{D_i : \sigma(i) = j\}, j \leq L(A). \]
By definition of $\sigma$, every collection $F_j$ consists of disjoint balls. Since every ball $D_i$ belongs to some collection $F_j$, we have
\[ A \subset \bigcup_{j=1}^{L(A)} D_j = \bigcup_{j=1}^{L(A)} \bigcup_{D \in D_j} D. \]
This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

The proof of Theorem 1.2 uses the argument in the proof of [Bogachev2007, Theorem 5.8.8, p. 368, vol. 1], based on [EG1992], with a modification to deal with a general complete Riemannian metric $g$. Furthermore, assuming the conditions in Theorem 1.2, we modify $D_{\nu_1, \nu_2}$ a bit to get a function $\tilde{D}_{\nu_1, \nu_2}$ defined on $M$. This is necessary for dealing with a family of Radon-Nikodym derivatives, considered in [JLT2020].

First we shall show that $D_{\nu_1, \nu_2}(x)$ exists and is finite for $\nu_1$-a.e. Let $S := \{x : \tilde{D}_{\nu_1, \nu_2}(x) = +\infty\}$. We denote by $\mu^*$ the outer measure defined by a locally finite Borel measure $\mu$ on $M$. To show $\nu_1(S) = 0$ we need the following

**Proposition 3.1.** Let $0 < c < \infty$ and $A$ a subset of $M$.

(i) If $A \subset \{x : \tilde{D}_{\nu_1, \nu_2}(x) \leq c\}$ then $\nu_2^*(A) \leq c \nu_1^*(A)$.

(ii) If $A \subset \{x : \tilde{D}_{\nu_1, \nu_2}(x) \geq c\}$ then $\nu_2^*(A) \geq c \nu_1^*(A)$.

**Proposition 3.1** is an extension of [Bogachev2007, Lemma 5.8.7, vol. 1] and will be proved in a similar way based on Theorem 1.1 and Lemma 3.2 below. We shall say that an open geodesic ball $D_r(x) \subset (M, g)$ is $k$-proper, if $kr$ is at most the injectivity radius of $(M, g)$ at $x$.

**Lemma 3.2.** Let $\mu$ be a locally finite Borel measure on a complete manifold $(M, g)$. Suppose that $\mathcal{F}$ is a collection of open 4-proper geodesic balls in $(M, g)$ the set of centers of which is denoted by $A$, and for every $a \in A$ and every $\varepsilon > 0$, $\mathcal{F}$ contains an open 4-proper geodesic ball $D_r(a)$ with $r < \varepsilon$. If
A is bounded then for every nonempty closed set $U \subset M$, one can find an at most countable collection of disjoint balls $D_j \in \mathcal{F}$ such that

$$\bigcup_{j=1}^{\infty} D_j \subset U \text{ and } \mu^*((A \cap U) \setminus \bigcup_{j=1}^{\infty} D_j) = 0.$$  

**Proof of Lemma 3.2.** We prove Lemma 3.2 using Theorem 1.1 and the Bishop comparison theorem as well as arguments in the proof of [Bogachev2007, Corollary 5.8.2, p. 363]. Let $A$, $\mathcal{F}$ and $U$ be as in Lemma 3.2. By Theorem 1.1 there exist subcollections $\mathcal{F}_j$ such that $\mathcal{F}_j$ consists of at most countably many disjoint balls and

$$A \subset \bigcup_{j=1}^{L(A)} \bigcup_{D \in \mathcal{F}_j} D.$$  

Set

$$\mathcal{F}^1 := \{D \in \mathcal{F} | D \subset U\}.$$

Now we shall apply Theorem 1.1 to $A \cap U$ and $\mathcal{F}^1$. Then we have

$$(A \cap U) \subset \bigcup_{j=1}^{L(A \cap U)} \bigcup_{D \in \mathcal{F}_j^1} D.$$  

**Claim 4.** We can choose $L(A \cap U) \leq L(A)$.

**Proof of Claim 4.** Since $A \cap U \subset A$, we can choose the constant $c_1(A \cap U)$ (resp. $c_2(A \cap U)$, $d(A \cap U)$, $\alpha(A \cap U)$, $b(A \cap U)$) equal to $c_1(A)$ (resp. $c_2(A)$, $d(A)$, $\alpha(A)$, $b(A)$) such that the statements in the proof of Theorem 1.1 holds for $A \cap U$ with these (modified) constants. Since $A \cap U \subset A$, we have $r_{A \cap U} \geq r_A$, hence we can also choose $\delta(A \cap U) := \delta(A)$, and therefore $d(A \cap U) := d(A)$ such that the statements in the proof of Theorem 1.1 holds for $A \cap U$ with these (modified) constants. This proves Claim 4. \[\square\]

It follows that

$$\mu^*(A \cap U) \leq \sum_{j=1}^{L(A)} \mu^*((A \cap U) \cap \bigcup_{D \in \mathcal{F}_j} D).$$

Hence there exists $j \in \{1, \cdots, L(A)\}$ such that

$$\mu^*((A \cap U) \cap \bigcup_{D \in \mathcal{F}_j^1} D)) \geq \frac{1}{L(A)} \mu^*(A \cap U).$$

Therefore there exists a finite collection $D_1, \cdots, D_{k_1} \in \mathcal{F}_j^1$ such that

$$\mu^*((A \cap U) \cap \bigcup_{i=1}^{k_1} D_i) \geq \frac{1}{2L(A)} \mu^*(A \cap U).$$
Proof of Proposition 3.1. By the property of outer measures it suffices to prove Proposition 3.1 for bounded sets $A$.

Assume that $\nu$ of disjoint balls $D_{k_1+1}, \ldots, D_{k_2}$ from $\mathcal{F}^2$ and by (3.1) we have

$$
\mu^*((A \cap U) \setminus \bigcup_{j=1}^{k_1} D_j) = \mu^*((A \cap U) \setminus \bigcup_{j=k_1+1}^{k_2} D_j) 
\leq (1 - \frac{1}{2L(A)}) \mu^*(A \cap U_2) \leq (1 - \frac{1}{2L(A)})^2 \mu^*(A \cap U).
$$

Repeating this process we get for all $p \in \mathbb{N}^+$

$$
\mu^*((A \cap U) \setminus \bigcup_{j=1}^{k_p} D_j) \leq (1 - \frac{1}{2L(A)})^p \mu^*(A \cap U).
$$

Since $\mu^*(A) < \infty$ this proves Lemma 3.2 \hfill $\Box$

Proof of Proposition 3.1. By the property of outer measures it suffices to prove Proposition 3.1 for bounded sets $A$. We shall derive Proposition 3.1 from Lemma 3.2 as in the proof of [Bogachev2007, Lemma 5.8.7, p. 368, vol 1]. Assume that $A \subset \{ x : D_{a \nu_2} \leq c \}$. Let $\varepsilon > 0$ and $U$ be a closed set containing $A$. Denote by $\mathcal{F}$ the class of all open 4-proper geodesic balls $D_r(a) \subset U$ with $r > 0$, $a \in A$ and $\nu_2(D_r(a)) \leq (c + \varepsilon) \nu_1(D_r(a))$. By the definition of $D_{a \nu_2}$ we have inf$\{r : D_r(a) \in \mathcal{F}\} = 0$ for all $a \in A$. By Lemma 3.2 there exists an at most countable family of disjoint balls $D_j \in \mathcal{F}$ with $\nu_2^*(A \setminus \bigcup_{j=1}^{\infty} D_j) = 0$ and $\bigcup_{j=1}^{\infty} D_j \subset U$. Hence

$$
\nu_2^*(A) \leq \sum_{j=1}^{\infty} \nu_2(D_j) \leq (c + \varepsilon) \sum_{j=1}^{\infty} \nu_1(D_j) \leq (c + \varepsilon) \nu_1(U).
$$

Since $U \supset A$ is arbitrary, we obtain the desired estimate. The second assertion (ii) is proven similarly, one has only to take for $\mathcal{F}$ the class of balls that satisfy $\nu_2(D_r(a)) \geq (c - \varepsilon) \nu_1(D_r(a))$. This completes the proof of Proposition 3.1 \hfill $\Box$

Completion of the proof of Theorem 1.2. Proposition 3.1 implies that $\nu_1^*(S) = 0$.

Next let $0 < a < b$ and set

$$
S(a, b) : \{ x : D_{a \nu_2}(x) < a < b < D_{b \nu_2}(x) < +\infty \}.
$$

Proposition 3.1 implies that

$$
b \nu_1^*(S(a, b)) \leq \nu_2^*(S(a, b)) \leq a \nu_1^*(S(a, b)).
$$
Hence $\nu_1^*(S(a,b)) = 0$ because $a < b$. The union $S_1$ of $S(a,b)$ over all positive rational numbers $a, b$ also has zero $\nu_1^*$-measure. Hence there exists a measurable subset $S_0 \subset M$ of zero $\nu_1$-measure such that $S \cup S_1 \subset S_0$. This proves the first assertion of Theorem 1.2.

Now let us show that $\hat{D}_{\nu_1} \nu_2(x)$ is measurable. Clearly, it suffices to show that $D_{\nu_1} \nu_2 : M^n \setminus S_0 \to \mathbb{R}$ is measurable.

**Lemma 3.3.** For each $r > 0$ the function $f_r(x) := \nu_1(D_r(x)) : M^n \to \mathbb{R}$ is lower-semi continuous and hence measurable.

**Proof.** Since $\lim_{k \to \infty} \nu_1\{D_{r-1/k}(x)\} = \nu_1(D_r(x))$, taking into account that $D_{r-1/k}(x) \subset D_r(y)$ if $|x - y| < 1/k$, we obtain

$$\lim_{y \to x} \inf \nu_1(D_r(y)) \geq \nu_1(D_r(x))$$

which we needed to prove. \qed

Since $S_0$ is measurable, we obtain immediately from Lemma 3.3 the following

**Corollary 3.4.** For each $r > 0$ the restriction $f_r|_{M \setminus S_0}$ is a measurable function.

In the same way, the restriction of function $f'_r(x) := \nu_2(D_r(x))$ to $M \setminus S_0$ is measurable. For $k \in \mathbb{N}^+$ and $x \in M \setminus S_0$ we set

$$\tau_k(x) := \frac{\nu_2(D_{1/k}(x))}{\nu_1(D_{1/k}(x))}.$$ 

It follows that the function $\tau_k : M \setminus S_0 \to \mathbb{R}$ is measurable. Hence the function $D_{\nu_1} \nu_2(x) : M \setminus S_0 \to \mathbb{R}$ is measurable, which we had to prove.

Finally we prove that $D_{\nu_1} \nu_2$ serves as the Radon-Nikodym derivative of $\nu_2$ w.r.t. $\nu_1$. Equivalently we need to show that for any $A \in \Sigma_M$ we have

$$\nu_2(A) = \int_A D_{\nu_1} \nu_2 d(\nu_1).$$

Here we use the argument in [Bogachev2007] p. 368-369, vol.1. Let $t > 1$ and set for $m \in \mathbb{Z}$

$$A_m := A \cap \{x \in (M \setminus S_0)| t^m < D_{\nu_1} \nu_2(x) < t^{m+1}\}.$$ 

The union $\cup_{m=-\infty}^{\infty} A_m$ covers $A$ up to $\nu_2$-measure zero set, since $\nu_2$-a.e. we have $D_{\nu_1} \nu_2 > 0$. Hence we have

$$\nu_2(A) = \sum_{m=-\infty}^{\infty} \nu_2(A_m) \leq \sum_{m=-\infty}^{\infty} t^{m+1} \nu_1(A_m)$$

$$\leq t \sum_{m=-\infty}^{\infty} \int_{A_m} \hat{D}_{\nu_1} \nu_2 d\nu_1 = t \int_A D_{\nu_1} \nu_2 d\nu_1.$$
This is true for any $t > 1$. Hence

\[(3.3) \quad v_2(A) \leq \int_A D v_1 v_2 dv_1.\]

Using $v_2(A_m) \geq t^m v_1(A_m)$ we obtain

\[(3.4) \quad v_2(A) \geq \int_A D v_1 v_2 dv_1.\]

Clearly (3.2) follows from (3.3) and (3.4).

This completes the proof of Theorem 1.2.

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