NJL and QCD from String Theory

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We study a configuration of D-branes in string theory that is described at low energies by a four-dimensional field theory with a dynamically broken chiral symmetry. In a certain region of the parameter space of the brane configuration the low-energy theory is a non-local generalization of the Nambu-Jona-Lasinio (NJL) model. This vector model is exactly solvable at large $N_c$ and dynamically breaks chiral symmetry at arbitrarily weak ’t Hooft coupling. At strong coupling the dynamics is determined by the low-energy theory on D-branes living in the near-horizon geometry of other branes. In a different region of parameter space the brane construction gives rise to large $N_c$ QCD. Thus the D-brane system interpolates between NJL and QCD.

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1. Introduction

Quantum chromodynamics (QCD), the theory of the strong interactions, is weakly coupled at high energies but strongly coupled at the scale of typical hadron masses (\( \sim 1 \) GeV). It has proven difficult to use this theory to study analytically the properties of low-lying mesons and baryons.

In the approximation in which \( N_f \) flavors of quarks are taken to be massless, the Lagrangian of QCD has a chiral global symmetry \( U(N_f)_L \times U(N_f)_R \), acting on the left and right-handed quarks, \( q_L, q_R \). This symmetry is expected to be dynamically broken to the diagonal subgroup \( U(N_f)_{\text{diag}} \). Analyzing this breaking is difficult due to the strongly coupled nature of the theory.

An important early example of dynamical chiral symmetry breaking occurs in the Nambu-Jona-Lasinio (NJL) model \([1]\). This model contains fermions \( q_L, q_R \), which transform in the fundamental representation of a global \( U(N_c) \) symmetry, and interact via a local four Fermi coupling,

\[
\mathcal{L}_{\text{int}} = G q_L^\dagger \cdot q_R q_R^\dagger \cdot q_L ,
\]  

(1.1)

where the dot product denotes a contraction of the color indices. The fermions \( q_L \) and \( q_R \) also transform in the fundamental representation of \( U(N_f)_L \) and \( U(N_f)_R \) symmetries of \( \mathcal{L}_{\text{int}} \), respectively.

The four Fermi interaction \((1.1)\) provides an attractive force between the left and right-handed quarks. The idea of \([1]\), based on an analogy with the BCS treatment of superconductivity, was that this force may destabilize the trivial vacuum and lead to the breaking of chiral symmetry. To test whether this indeed occurs in the NJL model one must work with a finite UV cut-off, since the interaction \((1.1)\) is non-renormalizable. In \([1]\) it was shown that the resulting model breaks chiral symmetry when the coupling \( G \) exceeds a certain critical value.

The non-renormalizability of the NJL model implies that many of its predictions are sensitive to the precise nature of the UV cut-off. Thus, it is natural to look for a renormalizable field theory in which the ideas of \([1]\) can be tested in a more controlled setting. Such a setting is the two-dimensional analog of the NJL model, which is asymptotically free. Moreover, since it is a vector model\([2]\) it is exactly solvable at large \( N_c \) and finite \( N_f \). This model was analyzed by D. Gross and A. Neveu \([3]\), who showed that it breaks chiral symmetry and generates a mass scale via dimensional transmutation.

\[^1\] See \textit{e.g.} \([2]\) for a review of large \( N \) vector models.
The Gross-Neveu model provides a beautiful confirmation of the ideas of Nambu and Jona-Lasinio, and makes it interesting to look for more realistic four-dimensional models with similar properties. Restricting to asymptotically free theories one is naturally led to QCD, since there are no asymptotically free field theories in four dimensions without gauge fields. Unfortunately QCD, like most models that involve $N_c \times N_c$ matrices, is difficult to solve even in the limit $N_c \to \infty$.

In this paper we explore a different route. We study a configuration of D-branes in string theory that reduces at low energies to a field theory of left and right-handed fermions $q_L, q_R$, with an adjustable attractive interaction. The brane configuration we start with preserves a $U(N_f)_L \times U(N_f)_R$ global symmetry but, as we will see, this configuration (or vacuum) is unstable for all values of the coupling. The true vacuum has a non-zero condensate $\langle q_L^\dagger \cdot q_R \rangle$.

At weak coupling, the infrared dynamics of $q_L, q_R$ is captured by a non-local NJL model – a vector model that does not contain dynamical gauge fields and can be solved exactly in the limit $N_c \to \infty$, like the Gross-Neveu model. It breaks chiral symmetry for arbitrarily weak coupling and generates a mass for $q_L, q_R$. At strong coupling, the useful description is in terms of D-brane dynamics in curved spacetime, which can also be analyzed quite explicitly and exhibits similar properties.

The embedding of the NJL model in string theory that we describe provides another bonus. Despite its non-renormalizability, the NJL model (1.1) and its non-local generalizations have been used extensively in phenomenological studies of hadrons, and appear to give a rather accurate description of their properties (see e.g. [4-8] for reviews and further references). However, it is not understood to what degree these models can be thought of as effective low-energy descriptions of QCD. The realization of NJL as a low-energy limit of the theory on D-branes provides a new perspective on this problem. By varying the parameters of the D-brane configuration, one can interpolate between NJL and QCD. It is likely that the interpolation is smooth, i.e. the two models are in the same universality class.

The idea that embedding field theories in string theory makes it easier to understand otherwise mysterious phenomena in field theory is familiar from other contexts. For example, thinking of $N = 4$ SYM in four dimensions as the low-energy limit of the six-dimensional $(2,0)$ SCFT compactified on a two-torus makes manifest the $SL(2,\mathbb{Z})$ S-duality symmetry of $N = 4$ SYM, which acts as the modular group of the torus. Thinking of
$N = 1, 2$ SYM as low-energy limits of the $(2, 0)$ theory wrapped on other Riemann surfaces provides a geometric realization of the Seiberg duality of $N = 1$ SYM and of the Seiberg-Witten curves of $N = 2$ SYM (see e.g. [9] for a review). Similarly, in our case both NJL and QCD are realized as low-energy limits of the compactified $(2, 0)$ theory in the presence of defects. It is interesting that the $(2, 0)$ theory plays a central role in all of these constructions.

The plan of the paper is the following. In section 2 we describe the brane configuration and its massless excitations. We introduce the fundamental length scales in the problem and discuss the range of validity of the weak and strong-coupling approximations. In section 3 we study the low-energy dynamics of the branes at weak coupling and show that it is described by a non-local NJL model which exhibits chiral symmetry breaking at arbitrarily weak coupling (for large $N_c$).

In section 4 we study the low-energy dynamics at strong coupling using a description in terms of D-branes in curved spacetime. We find the chiral condensate and comment on the relation to the weak coupling analysis. In section 5 we discuss a generalization of the D-brane construction which interpolates between NJL and QCD. We conclude in section 6 with comments on our results and a discussion of possible extensions of our analysis. Our conventions and a few results used in the text are described in an appendix.

2. The D-brane configuration and some of its properties

Motivated by earlier studies of string theory duals of QCD in the large $N_c$ limit with the number of flavors $N_f$ held fixed (see e.g. [10-13]), we will consider a brane configuration in $\mathbb{R}^{9,1}$ that includes three kinds of D-branes: $D4$, $D8$ and $\overline{D8}$. The different branes are extended in the directions

$$
\begin{align*}
D4 : & \quad x \quad x \quad x \quad x \quad x \\
D8, \quad \overline{D8} : & \quad x \quad x \quad x \quad x \quad x \quad x \quad x \quad x \quad x
\end{align*}
$$

and are arranged as indicated in figure 1. Classically, the $N_f$ $D8$ and $\overline{D8}$-branes are parallel and separated by a distance $L$ in the $x^4$ direction. They intersect the $N_c$ $D4$-branes at the origin of the $\mathbb{R}^5$ labeled by $(x^5, \cdots, x^9)$ and at $x^4 = \pm L/2$. We will see that this classical picture is modified by quantum effects.

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2 This brane configuration was studied in [13], with the $x^4$ direction compact and with anti-periodic boundary conditions for the fermions. As we will see, the limit where the radius of $x^4$ goes to infinity is particularly instructive. We will return to the compact case in section 5.
Since the different branes share the directions \((x^0, x^1, x^2, x^3)\), it is natural to study the low-energy dynamics seen by a 3 + 1-dimensional observer living in the intersection region. In the following sections we will do this in different regimes of the parameter space of the brane configuration. Here, as preparation for this study, we start with some general observations on this problem.

As usual in D-brane physics, there are different sectors of open strings that we need to consider. \(p - p\) strings \((p = 4, 8, \overline{8})\) give rise to \(p + 1\)-dimensional gauge theories with gauge groups \(U(N_c), U(N_f)_L\) and \(U(N_f)_R\) respectively. The field content of these gauge theories is obtained by dimensional reduction of \(N = 1\) SYM in ten dimensions. The gauge couplings are (see e.g. [14])

\[
g_{p+1}^2 = (2\pi)^{p-2} g_s l_s^{p-3}. \tag{2.2}
\]

Since these theories live in more than 3 + 1 dimensions they are not of direct interest to us, as we are looking for modes localized in the extra dimensions. The gauge symmetry of the D-branes gives rise to a global symmetry of the 3 + 1-dimensional theory.

Normalizable modes in 3 + 1 dimensions are found in the \(4 - 8\) and \(4 - \overline{8}\) sectors, which contain states localized near the corresponding brane intersection. Open strings in these sectors have six Dirichlet-Neumann directions. The NS sector states are massive; the only massless modes are spacetime fermions. \(4 - 8\) strings give a left-handed Weyl fermion \(q_L\)
localized at $x^4 \simeq -L/2$, which transforms in the $(N_c, N_f, 1)$ of the global symmetry (2.3), while $4 - \overline{8}$ strings give a right-handed fermion localized at $x^4 \simeq L/2$, which transforms as $(N_c, 1, N_f)$. A nice feature of the brane setup of [13] is that the left and right-handed fermions are separated in the extra dimensions.

The low-energy dynamics of the $D4 - D8 - \overline{D8}$ system can be formulated in terms of $q_L$ and $q_R$. Alternatively, one can integrate out these fields and study the dynamics of the gauge fields in the presence of a localized source (see [15] for a related recent discussion). We will follow the former approach, which is closer in spirit to the standard field theoretic treatment of 3 + 1-dimensional dynamics.

Consider first the case of a single intersection, of $N_c$ $D4$-branes and $N_f$ $D8$-branes, which can be thought of as corresponding to the limit $L \rightarrow \infty$ of the configuration of figure 1. We would like to study this system in the limit $g_s \rightarrow 0$, $N_c \rightarrow \infty$ with $g_s N_c$ and $N_f$ held fixed, and at energies much below the string scale. In this limit the only interactions between the fermions $q_L$ are due to their non-derivative couplings to the $D4$-brane gauge fields. The leading effect of these couplings is a four-Fermi interaction whose strength is proportional to $g_5^2$, and whose precise form will be described in the next section.

While the fermions $q_L$ live in 3 + 1 dimensions, the gauge fields that they exchange are 4 + 1-dimensional. The 't Hooft coupling of the five-dimensional gauge theory

$$\lambda = \frac{g_5^2}{4\pi^2} N_c ,$$

has units of length. This length determines the range of the four-Fermi interaction. For example, one can show that this interaction gives rise to corrections to the propagator $\langle q_L^\dagger(x) q_L(y) \rangle$, which are suppressed relative to the leading, free field, result by powers of $\lambda/|x - y|$. Thus, at distance scales much larger than $\lambda$ these interactions can be neglected, while in the opposite regime $|x - y| \ll \lambda$ they are large. We will discuss both limits below.

We next turn to the case where both the $D8$ and $\overline{D8}$-branes are present. Since we would like to study the low-energy theory on the branes, we will take the ratio of the distance between the eight-branes and the string length, $L/l_s$, to be large but finite in the limit $g_s \rightarrow 0$. This will allow us to neglect effects such as the non-trivial dilaton and RR ten-form field strength created by the $D8$-branes.

For finite $L$ there are interactions between the left-handed and right-handed quarks which are due to exchange of $D4$-brane gauge bosons. These interactions are weak for small $\lambda/L$, and become stronger as one increases this parameter. As in other brane systems,
it is useful to label the different regimes by the value of the dimensionless parameter $g_s N_c = \lambda / l_s$. For small $g_s N_c$ the hierarchy of scales in the problem is the following

$$\lambda \ll l_s \ll L .$$

As mentioned above, we will only be interested in the physics at distance scales much larger than $l_s$ (but which may be larger or smaller than $L$). In this regime we can neglect stringy effects on the dynamics of the fermions, as well as most of the effects of the dynamics of the gauge fields on the $D4$ and $D8$-branes. The only effect we need to keep is the one gluon exchange between the left and right-handed quarks, since it gives a force between them which is absent at infinite $L$. Unlike the propagator corrections mentioned above, this is not a small correction to an existing effect, but rather the leading interaction in this channel.\footnote{One can ask whether we should include additional interactions between $q_L$ and $q_R$ due to exchange of the scalars on the $D4$-branes. These interactions provide small corrections to those due to gluon exchange, since the scalars are derivatively coupled to the quarks. For the scalars $\Phi_5, \cdots, \Phi_9$ this is due to the exact symmetry of the brane system $\Phi_i \rightarrow \Phi_i + \text{const.}$ $\Phi_4$ is a component of the $4 + 1$ dimensional gauge field, and is derivatively coupled to the quarks due to $4 + 1$ dimensional gauge invariance.}

Thus, in the limit \( (2.5) \) the dynamics is described by a field theory of the fermions $q_L$ and $q_R$, with a non-local interaction due to one (five-dimensional) gluon exchange. The theory contains a natural UV cut-off $l_s$. Since this is much larger than $\lambda$, the scale at which the non-linear dynamics of the five-dimensional gauge field becomes important, we do not have to include corrections to the Lagrangian obtained in the one gluon exchange approximation. In the next section we will study the resulting non-local NJL model, and show that its long distance dynamics is non-trivial.

As $g_s N_c$ increases, we get to the regime $l_s \ll \lambda \ll L$, in which we can still treat $q_L$ and $q_R$ as weakly interacting via single gluon exchange, but the natural UV cut-off is now $\lambda$ and not $l_s$. Further increasing $g_s N_c$ we get to $\lambda \sim L$, where multi-gluon exchange processes can no longer be neglected, and the single gluon exchange approximation breaks down. For

$$l_s \ll L \ll \lambda ,$$

five-dimensional gauge theory effects give rise to a strongly attractive interaction between $q_L, q_R$. This interaction can be studied using the results of \cite{16}, by analyzing the dynamics of $D8$-branes in the near-horizon geometry of the $D4$-branes. This will be discussed in section 4.
3. Weak coupling analysis

In this section we study the $D4 - D8 - \overline{D8}$ system in the weakly coupled regime (2.7).
We start by deriving the low-energy effective action for the fermions $q_L$, $q_R$. Consider first
the case of a single intersection, say that of $N_c$ $D4$-branes and $N_f$ $D8$-branes. Taking the
intersection to be at $x^4 = 0$, the effective action for the Weyl fermion $q_L$ and $U(N_c)$ gauge
field $A_M$ is given by

$$S = \int d^5 x \left[ -\frac{1}{4g_5^2} F_{MN}^2 + \delta(x^4) q_L^\dagger \not{\sigma}(i\partial_\mu + A_\mu) q_L \right]. \quad (3.1)$$

Note that while the first term (the gauge field Lagrangian) is integrated over the 4 + 1-
dimensional worldvolume of the $D4$-branes with coordinates $x^M$, $M = 0, 1, 2, 3, 4$, the
fermion Lagrangian is restricted to the 3 + 1-dimensional intersection at $x^4 = 0$ with
coordinates $x^\mu$, $\mu = 0, 1, 2, 3$.

Since the physical degree of freedom at the intersection is the fermion $q_L$, it is useful
to integrate out the five-dimensional gauge field $A_M$. Keeping only the quadratic terms in
the gauge field Lagrangian, and working in Feynman gauge, we find the following effective
action for $q_L$:

$$S_{\text{eff}} = \int d^4x q_L^\dagger \not{\sigma} \partial_\mu q_L - \frac{g_5^2}{16\pi^2} \int d^4x d^4y G(x - y, 0) \left[ q_L^\dagger(x) \not{\sigma} q_L(y) \right] \left[ q_L^\dagger(y) \not{\sigma} q_L(x) \right] \quad (3.2)$$

where $G(x^\mu, x^4)$ is proportional to the scalar propagator in 4 + 1 dimensions,

$$G(x^\mu, x^4) = \frac{1}{((x^4)^2 - x_\mu x^\mu)^2}, \quad (3.3)$$

The color indices in (3.2) are contracted in each term in brackets separately, while the
flavor ones are contracted between $q_L$ from one term and $q_L^\dagger$ from the other. Thus the
quantities in square brackets are color singlets and transform in the adjoint representation
of $U(N_f)_L$. To derive (3.2) we used a Fierz identity described in appendix A.

The action (3.2) can be treated using standard large $N$ techniques. The solution has
the following structure. For distance scales much larger than $\lambda$ (2.4), the field $q_L$ is free.
The effects of the interaction grow as the distance scale decreases, and become important
at distances of order $\lambda$.

At these distances there are two other types of interactions that are not taken into
account in (3.2). One is due to the non-linear terms in the gauge field Lagrangian (3.1).
Their contributions are not small at distances of order $\lambda$ and adding them brings back the full complexity of the large $N_c$ gauge theory. The other is due to string corrections. If the 't Hooft parameter $g_s N_c$ is small, the scale $\lambda$ (2.4) is much smaller than the string scale (see (2.5)). Thus, before we get to the point where the gauge theory effects become large, we reach a regime where the dynamics of $q_L$ is dominated by exchange of massive string states. In this section we will restrict the discussion to distances much larger than $l_s$ and $\lambda$, for which we can neglect all these effects, such that $q_L$ is free in this limit.

We are now ready to discuss the case of interest, which contains $D8$ and $\overline{D8}$-branes separated by a distance $L$. The analog of the Lagrangian (3.1) takes in this case the form

$$S = \int d^5x \left[ -\frac{1}{4g_5^2} F_{MN}^2 + \delta(x^4 + \frac{L}{2}) q_L^\dagger \sigma^\mu (i\partial_\mu + A_\mu) q_L + \delta(x^4 - \frac{L}{2}) q_R^\dagger \sigma^\mu (i\partial_\mu + A_\mu) q_R \right].$$

Integrating out the gauge field in the single gluon exchange approximation and neglecting interactions of the form in (3.2) we get

$$S_{\text{eff}} = i \int d^4x \left( q_L^\dagger \sigma^\mu \partial_\mu q_L + q_R^\dagger \sigma^\mu \partial_\mu q_R \right) + \frac{g_5^2}{4\pi^2} \int d^4xd^4y G(x - y, L) \left[ q_L^\dagger(x) \cdot q_R(y) \right] \left[ q_R^\dagger(y) \cdot q_L(x) \right],$$

where we again used a Fierz identity from appendix A. As mentioned above, (3.5) provides an accurate description of the dynamics of $q_L, q_R$ when $L$ and all other distance scales in the problem are much greater than $\lambda, l_s$.

To solve the non-local Nambu-Jona-Lasinio model (3.5) at large $N_c$ it is convenient to introduce a complex scalar field $T(x, y)$ which transforms as $(1, N_f, N_f)$ under the global symmetry (2.3), and rewrite the quartic interaction (3.5) as follows:

$$S_{\text{eff}} = i \int d^4x \left( q_L^\dagger \sigma^\mu \partial_\mu q_L + q_R^\dagger \sigma^\mu \partial_\mu q_R \right) + \int d^4xd^4y \left[ -\frac{N_c}{\lambda} \frac{T(x, y) T(y, x)}{G(x - y, L)} + T(x, y) q_L^\dagger(x) \cdot q_R(y) + T(x, y) q_R^\dagger(y) \cdot q_L(x) \right].$$

The equation of motion of $T$ is

$$T(x, y) = \frac{\lambda}{N_c} \frac{G(x - y, L) q_L^\dagger(x) \cdot q_R(y)}{\lambda}.$$  

Plugging it back into (3.6) (or, equivalently, integrating out $T, \overline{T}$), one recovers the original action (3.3). As usual in large $N$ theories, we would like instead to integrate out the
fermions and obtain an effective action for the scalars, whose dynamics becomes classical in the limit $N_c \to \infty$. Since we are mainly interested in the question of chiral symmetry breaking, we would like to compute the expectation value of $T(x, y)$ in the vacuum. Poincare symmetry implies that the latter must be a function of $(x - y)^2$. Thus, we will make this simplifying assumption and compute the effective action for $T(x, y) = T(|x - y|)$. 

To leading order in the $1/N_c$ expansion, the only corrections to the classical action of $T, \overline{T}$ come from one loop diagrams with an arbitrary number of external $T, \overline{T}$ fields. Computing the one-loop effective potential in the standard way, adding to it the classical term from (3.6) and dropping an overall factor of $N_c$ and the volume of spacetime leads to the effective potential

$$V_{\text{eff}} = \int d^4x T(x)\overline{T}(x)\frac{(x^2 + L^2)^{3/2}}{\lambda} - \int \frac{d^4k}{(2\pi)^4} \log \left(1 + \frac{T(k)\overline{T}(k)}{k^2}\right)$$

(3.8)

where $T(k)$ is the Fourier transform of $T(x)$ (see appendix A for conventions). In (3.8) we have also Wick rotated to Euclidean spacetime.

The equation of motion for $T$ that follows from (3.8) (the gap equation) is

$$\nabla^2 \left[ \frac{(x^2 + L^2)^{3/2}}{\lambda} T(x) \right] + T(x) = 0 ,$$

(3.9)

The trivial solution of this equation, $T(x) = 0$, corresponds to a vacuum with vanishing $\langle q_L^\dagger \cdot q_R \rangle$ and unbroken chiral symmetry. We will see that this solution is unstable. The true vacuum has non-zero $T$ and breaks chiral symmetry.

The potential $V_{\text{eff}}$ (3.8) contains a classical term, which is positive and favors $T = \overline{T} = 0$, and a negative one loop term, which becomes larger (i.e. more negative) as $|k|$ decreases. Thus, it is natural to expect that any non-trivial solution of (3.9) will be dominated by the low momentum modes of $T$.

To find such a solution it is useful to discuss separately two regimes. The first is the linear regime, in which

$$T(k)\overline{T}(k) \ll k^2 ,$$

(3.10)

so that we can expand the log in (3.8) and keep only the leading quadratic term and hence the linear term in the gap equation (3.9). In this regime (3.9) becomes

$$\nabla^2 \left[ \frac{(x^2 + L^2)^{3/2}}{\lambda} T(x) \right] + T(x) = 0 ,$$

(3.11)
where $\nabla^2$ is the four-dimensional Euclidean Laplacian.

As mentioned above, due to the relation with the chiral condensate, (3.7), we expect $T$ to depend only on $r = \sqrt{x^2}$. In terms of

$$F(r) = (r^2 + L^2)^{\frac{3}{2}} T(r) = \frac{\lambda}{N_c} (q_L^\dagger(x) \cdot q_R(0)) , \quad (3.12)$$

equation (3.11) takes the form

$$F''(r) + \frac{3}{r} F'(r) + \frac{\lambda F(r)}{(r^2 + L^2)^{3/2}} = 0 . \quad (3.13)$$

We now argue that the solution of (3.13) which is relevant in the linear regime (3.10) is simply $F(r) = C$ with $C$ a constant to be determined. To see this, note that (3.13) can be solved in closed form for $r \ll L$ in terms of the dimensionless coordinate $\rho = r \sqrt{\lambda/L}$ and for $r \gg L$ in terms of the dimensionless coordinate $\sigma = r/\lambda$. The solution for $r \ll L$ which is regular as $\rho \to 0$ is expressed in terms of Bessel functions as

$$F(\rho) \sim \frac{J_1(\rho)}{\rho} , \quad (3.14)$$

and approaches a constant for small $\rho$. The regime of validity of (3.14), $r \ll L$, corresponds to $\rho \ll \sqrt{\lambda/L} \ll 1$. Thus, to leading order in $\lambda/L$, the only part of the solution (3.14) we are sensitive to is its value at $\rho = 0$.

For $r \gg L$ the leading behavior of the solution at large $\sigma$ can also be expressed in terms of Bessel functions as

$$F(\sigma) \sim \frac{Y_2(2/\sqrt{\sigma})}{\sigma} , \quad (3.15)$$

and approaches a constant for large $\sigma$. Since $r \gg L$ implies $\sigma \gg 1$, we are again only sensitive to this constant.

Thus, $F(r)$ is constant both for $r \ll L$ and for $r \gg L$. One can show that it does not exhibit any non-trivial variation for $r \sim L$, i.e. the two constants are the same (to leading order in $\lambda/L$). This is a consequence of the fact that the coefficient of $F(r)$ in (3.13) is bounded from above by $\lambda/L^3$, which means that the scale of variation of the solution is $\sqrt{L^3/\lambda}$, a distance scale that is much larger than $L$ in the weak coupling regime. It can also be verified by a numerical solution of (3.13).

In terms of $T(r)$ (3.12), the solution is

$$T(r) = \frac{C}{(r^2 + L^2)^{3/2}} , \quad (3.16)$$

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with Fourier transform

$$T(k) = \frac{4\pi^2Ce^{-kL}}{k}.$$  \hspace{1cm} (3.17)

Since equation (3.11) is linear, the constant $C$ is not determined by the linear analysis. We will determine it below by matching to the non-linear regime.

As we anticipated from the form of the potential (3.8), the vacuum expectation value of $T$ (3.17) grows with decreasing momentum. The condition (3.10), which is necessary for the validity of the preceding discussion, is bound to be violated for sufficiently low momenta. We will see that this happens in the regime $kL \ll 1$. Thus, in the transition region we can neglect the exponential in (3.17), and find that the linearity assumption breaks down at $k \simeq \sqrt{C}$.

We saw before that in the linear regime (3.10) the gap equation (3.9) reduces to (3.11). In the opposite, strongly non-linear regime,

$$T(k)T(k) \gg k^2,$$  \hspace{1cm} (3.18)

one instead finds

$$\frac{1}{T(k)} = \int d^4x \frac{(x^2 + L^2)^{\frac{3}{2}}}{\lambda}T(x)e^{-ik \cdot x}.$$  \hspace{1cm} (3.19)

The solution of (3.19) is

$$T(x) = T(x) = A\delta^4(x),$$  \hspace{1cm} (3.20)

where the constant $A$ is determined by (3.19),

$$A = \sqrt{\frac{\lambda}{L^3}}.$$  \hspace{1cm} (3.21)

The $\delta$-function behavior of (3.20) means that $T(r)$ goes rapidly to zero for large $r$.

We can now determine the constant $C$ by matching the linear and non-linear regimes. Comparing (3.17) to the second line of (3.20) we see that the transition between the two occurs around the momentum $k^*$ which satisfies

$$\frac{4\pi^2C}{k^*} \simeq \sqrt{\frac{\lambda}{L^3}}.$$  \hspace{1cm} (3.22)

At that point we expect the parameter

$$\frac{T(k^*)T(k^*)}{(k^*)^2} \simeq \frac{A^2}{(k^*)^2} \simeq 1.$$  \hspace{1cm} (3.23)
This means that
\[ C \simeq \left( \frac{k^*}{2\pi} \right)^2 \simeq \frac{\lambda}{4\pi^2 L^3}. \]  

We are thus led to the picture shown in figure 2. For momenta \( k \gg \sqrt{\lambda/L^3} \) or, equivalently, distances \( |x| \ll \sqrt{L^3/\lambda} \), we are in the linear regime, in which the gap equation takes the form (3.11), and its solution is given by (3.16), (3.17). On the other hand, for momenta \( k \ll \sqrt{\lambda/L^3} \) the system is in the non-linear regime, where the gap equation is given by (3.19) and its solution by (3.20). The momentum scale
\[ k^* = \sqrt{\frac{\lambda}{L^3}} = \frac{1}{L} \sqrt{\frac{\lambda}{L}} \]  

at which the transition between the linear and non-linear regimes takes place is very low: \( k^* \ll 1/L \) for weak coupling, \( \lambda \ll L \). This provides an aposteriori justification for setting \( \exp(-k^*L) \simeq 1 \) in the analysis above.

**Fig. 2:** Behavior of \( T(k) \) as a function of momentum.

To show that the non-local NJL model (3.5) breaks chiral symmetry, it is important to establish that the non-trivial solution of the gap equation constructed above has lower energy than the trivial solution \( T = 0 \). This can be seen to be a property of any sufficiently well behaved solution of the gap equation.

Indeed, multiplying (3.9) by \( e^{ik\cdot y} \overline{T(y)} \) and integrating over \( k \) and \( y \) leads to
\[ \frac{1}{\lambda} \int d^4x (x^2 + L^2)^{\frac{3}{2}} T(x) \overline{T(x)} = \int \frac{d^4k}{(2\pi)^4} \frac{T(k) \overline{T(k)}}{k^2 + T(k) \overline{T(k)}}. \]  

(3.26)
Plugging this into (3.8) we find that $V_{\text{eff}}$ can be written as

$$V_{\text{eff}} = \int \frac{d^4k}{(2\pi)^4} \left[ \frac{T(k)\overline{T}(k)}{k^2 + T(k)\overline{T}(k)} - \log \left( 1 + \frac{T(k)\overline{T}(k)}{k^2} \right) \right]. \tag{3.27}$$

The integrand in (3.27) is non-positive\footnote{Indeed, defining $x = T(k)\overline{T}(k)/k^2$, the term in square brackets in (3.27) is given by $[\cdots] = \frac{x}{1+x} - \log(1+x)$, which is negative for all $x > 0$.} for any finite $T(k)$. Assuming that the integral over $k$ converges, which can be verified to be the case using the explicit form of $T(k)$ that we found, (3.17), (3.20), one concludes that $V_{\text{eff}} < 0$ for the non-trivial solution of the gap equation, i.e. the latter has lower energy than the trivial solution $T = 0$.

We finish this section with a few comments:

1. It is interesting to use the results above to calculate the quark anti-quark condensate $\langle q_L^\dagger(x) \cdot q_R(y) \rangle$. Using (3.12), (3.16) we see that in the linear regime

$$\langle q_L^\dagger(x) \cdot q_R(0) \rangle \simeq \frac{N_c L^3}{L^3}.$$

Thus, for $|x| = r \ll 1/k^*$ the chiral condensate is independent of $r$ and $\lambda$. For $r \gg 1/k^*$ it goes rapidly to zero. The full solution can in principle be obtained numerically to arbitrarily high precision.

2. We emphasize again that the results obtained in this section are valid to leading order in $\lambda/L$. One can compute higher order corrections to these results by including the non-linear terms in the gauge field Lagrangian and other corrections discussed above.

3. The chiral condensate (3.28) transforms in the $(N_f, N_f)$ of the $U(N_f)_L \times U(N_f)_R$ global symmetry (2.3). The flavor indices are suppressed above. The expectation value (3.28) breaks the chiral symmetry $U(N_f)_L \times U(N_f)_R \to U(N_f)_{\text{diag}}$.

4. The condensate $\langle q_L^\dagger \cdot q_R \rangle$ that we found in the non-local NJL model exhibits the type of behavior one might expect in QCD. There, the chiral condensate should presumably be roughly constant for $r$ smaller than $1/\Lambda_{QCD}$ and go to zero for large $r$.

4. **Strong coupling analysis**

In the last section we analyzed the dynamics of the chiral fermions $q_L$ and $q_R$ in the weakly coupled regime $\lambda \ll L$. As we increase $g_s N_c$, or bring the D8-branes closer, the effective coupling of the fermions due to exchange of D4-brane modes increases, and the approximations which we employed break down.
For \( L \ll \lambda \), there is another weakly coupled description of the dynamics. Instead of studying fermions coupled via five-dimensional gluon exchange we should consider the dynamics of \( D_8 \)-branes in the near-horizon geometry of the \( N_c \) \( D_4 \)-branes. In this section we will use this description to analyze the question of chiral symmetry breaking in this regime.

Using the conventions of [16] in which the radial coordinate \( U \) has dimensions of energy, the metric is given by

\[
ds^2 = \left( \frac{\alpha' U}{R} \right)^{3/2} (\eta_{\mu\nu} dx^\mu dx^\nu - (dx^4)^2) - \left( \frac{\alpha' U}{R} \right)^{-3/2} \left( (\alpha' dU)^2 + (\alpha' U)^2 d\Omega_4^2 \right),
\]

where \( \Omega_4 \) labels the angular directions in (56789). The fourbrane geometry also has a non-trivial dilaton background,

\[
e^\Phi = g_s \left( \frac{\alpha' U}{R} \right)^{3/4},
\]

The parameter \( R \) is given by

\[
R^3 = \pi g_s N_c (\alpha')^{3/2} = \frac{g_s^2}{4\pi} N_c \alpha' = \pi \lambda \alpha'.
\]

In this section we take \( x^4 \) to be non-compact. In the next section we will discuss the compact case studied in [13]. In what follows we set \( \alpha' = 1 \).

We now consider a probe \( D_8 \)-brane propagating in the geometry (4.1). The \( D_8 \)-brane wraps \( \mathbb{R}^{3,1} \times S^4 \) and forms a curve \( U = U(x^4) \) in the \( (U, x^4) \) plane, whose shape is determined by solving the equations of motion that follow from the DBI action on the \( D_8 \)-brane. In the background (4.1), (4.2) the action is

\[
S_{D8} = -T_8 V_{3+1} V_4 \int dx^4 U^4 \sqrt{1 + \left( \frac{R}{U} \right)^3 U'^2},
\]

where \( U' = dU/dx^4 \), \( V_{3+1} \) is the volume of 3 + 1 dimensional Minkowski spacetime, and \( V_4 \) the volume of a unit \( S^4 \). Since the integrand of (4.4) has no explicit \( x^4 \) dependence, \( U(x^4) \) satisfies the first order equation [13]

\[
\frac{U^4}{\sqrt{1 + \left( \frac{R}{U} \right)^3 U'^2}} = U_0^4.
\]

\[5\] The discussion of \( N_f \) coincident \( D_8 \)-branes is essentially identical.
An interesting solution of (4.5) is a U-shaped curve in the \((U,x^4)\) plane, which approaches \(x^4 = \pm \frac{L}{2}\) as \(U \rightarrow \infty\), and is equal to \(U_0\) at \(x^4 = 0\), the point of closest approach of the curve to \(U = 0\). The curve is symmetric under \(x^4 \rightarrow -x^4\).

Solving (4.5) for \(U'\) and integrating leads to

\[
x^4(U) = \int_{U_0}^U \frac{dU}{(U/R)^{3/2} \left( \frac{U^8}{U_0^8} - 1 \right)^{1/2}}
\]

which can be written in terms of complete and incomplete Beta functions as

\[
x^4(U) = \frac{1}{8} \frac{R^{3/2}}{U_0^{1/2}} \left[ B(9/16, 1/2) - B(U_0^8/U^8; 9/16, 1/2) \right].
\] (4.7)

From this we read off the asymptotic value \(x^4(\infty) = L/2\),

\[
L = \frac{1}{4} R^{3/2} U_0^{1/2} B(9/16, 1/2).
\] (4.8)

The incomplete Beta function has an expansion at small \(z\) given by

\[
B(z; a, b) = z^a \left[ \frac{1}{a} + \frac{1 - b}{a + 1} z + \cdots \right].
\] (4.9)

Keeping the first term in this expansion in (4.7) gives the form of the curve at large \(U\):

\[
U^4 \approx \frac{2}{9} \frac{R^{3/2} U_0^4}{L^{1/2} - x^4}.
\] (4.10)

The part of the D-brane that corresponds to \(x^4 < 0\) is determined by the symmetry \(U(x^4) = U(-x^4)\). The full D8-brane is shown in fig. 3.

![Fig. 3: A slice of the full D8-brane configuration showing the D8 and \(\overline{D8}\)-brane joined into a single D8-brane by a wormhole.](image)
The solution described above is not the near-horizon description of the brane configuration of figure 1. That configuration corresponds in the geometry (4.1) to a stack of \( N_f \) D8-branes stretched in \( U \) and localized at \( x^4 = -L/2 \), and a stack of \( N_f \) \( \overline{D}8 \)-branes stretched in \( U \) and localized at \( x^4 = L/2 \). In particular, unlike the solution (4.6), the D8 and \( \overline{D}8 \)-branes are not connected in this case. Thus, the configuration of figure 1 preserves a \( U(N_f)_L \times U(N_f)_R \) symmetry acting separately on the D8 and \( \overline{D}8 \)-branes, unlike the configuration (4.6) where the two are connected and the symmetry is only a single (diagonal) \( U(N_f) \).

It is a dynamical question whether for fixed \( L \) it is the separated parallel brane configuration or the connected curved one, with \( U_0 \) a function of \( L \) (4.8), that minimizes the energy density. The energy per unit volume is infinite, but the relative energy density of the two configurations is finite and can be computed by integrating the energy difference for each slice \( dU \). After substituting the curved configuration’s \( U' \) from (4.5) and rewriting the DBI action (4.4) as an integral over \( U \) rather than \( x^4 \), the difference \( \Delta E \equiv E_{\text{straight}} - E_{\text{curved}} \) is proportional to

\[
\Delta E \sim \int_0^{U_0} \left( U^{5/2} - 0 \right) dU + \int_{U_0}^{\infty} \left( U^{5/2} - \frac{U_0^8}{U^8} \right)^{-1/2} dU \\
= -\frac{1}{8} U_0^{7/2} B(-7/16, 1/2) \approx 0.052 U_0^{7/2}.
\]

Thus, \( E_{\text{straight}} > E_{\text{curved}} \), so the curved configuration is preferred. Physically, this is due to the attractive force between the D8 and \( \overline{D}8 \)-branes mediated by the D4-brane fields. In the previous section we studied its consequences in the weakly coupled regime. In the strongly coupled regime under consideration here, the attractive force leads to a large deformation of the D8-branes which can be seen in figure 3.

To check the validity of the supergravity approximation, it is convenient to rewrite (4.8) in terms of the variables \( \lambda \) (2.4) and \( L \). Omitting constants of order one, we have

\[
U_0 \simeq \frac{\lambda}{L^2}.
\]

As explained in [16], a necessary condition for the supergravity solution (4.1) to be valid is that the curvature in string units is small, which is the case when the effective ‘t Hooft coupling is large,

\[
\lambda \left( \frac{U}{R} \right)^{\frac{4}{3}} \gg 1,
\]
or, equivalently,

\[ \lambda U \gg 1 . \]  

(4.14)

\( U_0 \) (4.12) satisfies (4.14) if \( \lambda \gg L \). Fixing \( L \) and increasing \( \lambda \) pushes \( U_0 \) further into the regime of validity of supergravity. On the other hand, decreasing \( \lambda \) leads to smaller \( U_0 \), and eventually, when \( \lambda \simeq L \) the curvature at \( U_0 \) becomes of order one and the supergravity description breaks down.

In section 3 we saw that the weakly coupled non-local NJL description of the four-dimensional dynamics is valid for \( \lambda \ll L \). Now we see that the description in terms of a curved D8-brane is valid for \( \lambda \gg L \). This is an example of a bulk-boundary duality analogous to that between \( N = 4 \) SYM and supergravity in \( AdS_5 \times S^5 \). The analog of the \('t \)Hooft coupling of \( N = 4 \) SYM in our case is the dimensionless ratio \( \lambda/L \).

As emphasized in [13], the fact that what looks asymptotically like two disconnected stacks of D8 and \( D8 \)-branes is in fact part of a connected stack of curved D8-branes provides a nice geometric realization of chiral symmetry breaking. At high energies (which correspond to large \( U \) [16]) one sees an approximate \( U(N_f)_L \times U(N_f)_R \) symmetry while at low energies (small \( U \)) only \( U(N_f)_{\text{diag}} \) is manifest.

The energy scale associated with chiral symmetry breaking in the supergravity regime is \( U_0 \) (4.12). This scale can be thought of as the constituent quark mass of \( q_L, q_R \). Indeed, in the limit \( \lambda \gg L \) the spectrum contains free quark states which correspond to fundamental strings stretched between the curved D8-branes and \( U = 0 \) (the location of the \( N_c \) D4-branes). The energy of such strings is \( U_0 \). The analog of this scale at weak coupling is \( k^* \) (3.25). As in other bulk-boundary dualities, the mass goes like a different power of the coupling \( \lambda/L \) in the weak and strong coupling regimes.

To make the description of the 3 + 1-dimensional dynamics in terms of a curved D8-brane in the D4-brane geometry more precise, we need to find the map between bulk fields in the geometry (4.1) and boundary operators. Once this map is established, one can use the standard tools of holographic dualities to study the boundary theory. In particular, giving an expectation value to a non-normalizable operator in the bulk corresponds to adding the dual boundary operator to the Lagrangian. Normalizable bulk v.e.v.’s correspond to giving expectation values to the dual boundary operators.

\[ ^{6} \text{There is an upper bound on } U_0 \text{ coming from the requirement that } g_s(U_0) \ll 1, \text{ but it involves } N_c \text{ and we will not discuss it here.} \]
In our system, there are two kinds of bulk fields. One is closed string fields (such as the dilaton, graviton, etc) in the geometry (4.1), which exist even when the $D8$-branes are absent. The other is open string fields on the $D8$-branes. Both live in the bulk (i.e. at any $U$) and couple to $q_L, q_R$. We will briefly comment on the bulk-boundary map for some open string fields, leaving a more detailed analysis to the future.

In order to find the boundary operators corresponding to different open string modes we go back to the D-brane configuration we started with (figure 1). Consider first the open string tachyon stretched between the $D8$ and $\overline{D8}$-branes. This complex scalar field which transforms as $(N_f, N_f)$ under $U(N_f)_L \times U(N_f)_R$ couples to the fermions via a Yukawa-type interaction

$$\mathcal{L}_1 \simeq T q_R^\dagger \cdot q_L + \mathcal{T} q_L^\dagger \cdot q_R .$$

Thus, in the near-horizon geometry of the $D4$-branes, the open string tachyon $T, \overline{T}$ is dual to the boundary operators

$$T \leftrightarrow q_R^\dagger \cdot q_L ,$$

$$\mathcal{T} \leftrightarrow q_L^\dagger \cdot q_R .$$

In particular, turning on a mass for $q_L, q_R$ corresponds to giving a non-normalizable expectation value to the field $T$, while a v.e.v. for $q_R^\dagger \cdot q_L$ corresponds to a normalizable expectation value of $T$.

As we discussed above, the curved $D8$-brane (4.6) describes a vacuum with non-zero expectation value $\langle q_R^\dagger \cdot q_L \rangle$. Therefore, in addition to its curved shape, the D-brane (4.6) must have a non-zero normalizable condensate of the $8 - \overline{8}$ tachyon $T$. At first sight this may seem odd, but in fact something very similar is known to happen for the closely related hairpin D-brane [17-19]. In our case, the fact that $T$ has an expectation value can be seen from (4.15). Since $\langle q_R^\dagger \cdot q_L \rangle$ is non-zero, there is a tadpole for $T$ localized at the intersection. Therefore, it has a non-zero expectation value, which is also localized in the vicinity of $U = 0$.

Another mode that we can consider is the scalar $(X^4)_L$ that parametrizes the location of the $D8$-branes in the $x^4$ direction, and its $\overline{D8}$-brane counterpart $(X^4)_R$. Unlike the tachyon, $(X^4)_{L,R}$ do not have Yukawa-type couplings to $q_L, q_R$, as can be easily deduced from symmetry considerations. The lowest dimension coupling consistent with the symmetries has the form

$$\mathcal{L}_2 \simeq (X^4)_L (\partial_\mu J^\mu_L) + (X^4)_R (\partial_\mu J^\mu_R) .$$
where \( J_L^\mu, J_R^\mu \) are the \( U(N_f)_L \times U(N_f)_R \) currents. In the free theory at infinite \( L \) they are conserved, so the coupling (4.17) can be neglected. For finite \( L \), chiral symmetry breaking implies that \( \partial_\mu J_{L,R}^\mu \neq 0 \) and proportional to \( q_L^\dagger \cdot q_R q_R^\dagger \cdot q_L \) (integrated over some of the positions). Thus, (4.17) can be written as

\[
\mathcal{L}_2 \simeq (X^4)_L \left( q_L^\dagger \cdot q_R q_R^\dagger \cdot q_L \right)_{(N_f^2,1)} + (X^4)_R \left( q_L^\dagger \cdot q_R q_R^\dagger \cdot q_L \right)_{(1,N_f^2)}
\]

(4.18)

where the subscripts indicate that in the first term \( q_L^\dagger \cdot q_R q_R^\dagger \cdot q_L \) are coupled to an adjoint of \( U(N_f)_L \) and a singlet of \( U(N_f)_R \), and similarly for the second term.

In the near-horizon geometry (4.1), (4.18) implies that the bulk-boundary correspondence is

\[
\begin{align*}
(X^4)_L & \leftrightarrow (q_L^\dagger \cdot q_R q_R^\dagger \cdot q_L)_{(N_f^2,1)} \\
(X^4)_R & \leftrightarrow (q_L^\dagger \cdot q_R q_R^\dagger \cdot q_L)_{(1,N_f^2)}
\end{align*}
\]

(4.19)

The fact that the operators on the r.h.s. of (4.19) have non-zero expectation values in our brane configuration implies that the scalars \((X^4)_L, (X^4)_R\) must have a non-zero normalizable expectation value. This is nothing but the curved shape of the branes, which is given asymptotically by (4.10). It is normalizable since \( x^4 \) approaches \( L/2 \) as \( U \to \infty \) like \( L/2 - x^4 \sim U^{-\frac{9}{2}} \).

It would be interesting to study the correspondence between bulk fields and boundary operators in more detail, but we will not pursue that here. We finish this section with some comments on the \( 8-\overline{8} \) string, which seems to play an important role in this system.

In flat space the ground state of the \( 8-\overline{8} \) string is tachyonic for \( L \) less than the string length and has positive mass squared for \( L \) larger than the string length (as is the case here). One might be tempted to think that it can be decoupled from the dynamics because of its string scale mass. However we believe that this is incorrect.

One way to see this is to study the geometry of a fundamental string stretched between the \( D8 \) and \( \overline{D8} \)-branes in the near-horizon geometry of the \( D4 \)-branes. Naively, such a string is stretched in the \( x^4 \) direction at fixed \( U \). Let \( U^* \) be the fixed value of \( U \) and \( L^*/2 = x^4(U^*) \). Then the coordinate length of the string is \( L^* \), and including the warp factor in (4.1), its physical length is

\[
L(U) = L^* \left( \frac{U}{R} \right)^{\frac{3}{4}}.
\]

(4.20)
However, this is not the lowest energy configuration. As is familiar from the study of warped geometries [20-22], the string dips into the small $U$ region due to the warping. To find the amount of this dipping, we can proceed as follows.

The string is described by the Nambu-Goto action,

$$S_{NG} = - \int d^2 \xi \sqrt{-\det h_{ab}}$$

(4.21)

where

$$h_{ab} = \partial_a X^M \partial_b X^n G_{MN}$$

(4.22)

is the induced metric on the string and we omitted an overall numerical factor. Plugging (4.1) into the Nambu-Goto action and omitting a factor of the length of time, we find that

$$S_{NG} = - \int L^* d^4 x \sqrt{\left( \frac{U}{R} \right)^3 + U'^2}.$$  

(4.23)

The shape $U(x^4)$ satisfies the first order equation

$$\frac{\left( \frac{U}{R} \right)^{\frac{3}{2}}}{\sqrt{1 + \left( \frac{R}{U} \right)^{\frac{3}{2}} U'^2}} = \left( \frac{U_0^{(F)}}{R} \right)^{\frac{3}{2}}.$$  

(4.24)

Here $U_0^{(F)}$ is the minimal value of $U$ that the fundamental string attains, at $x^4 = 0$, where $U'$ vanishes. It is determined by $U^*$ (and $L^*$, but the latter is also determined by $U^*$ via the shape of the $D8$-brane, which we take as given).

Some algebra leads to the following relation between the various parameters:

$$\frac{L^*}{2U_0^{(F)}} = \left( \frac{R}{U_0^{(F)}} \right)^{\frac{3}{2}} \int_{L_0^{(F)}}^{U_0^{(F)}} d\xi \left( \frac{U}{R} \right)^{-\frac{3}{2}} (x^3 - 1)^{-\frac{3}{4}}.$$  

(4.25)

This equation is valid for all $U^*$, but it simplifies for large $U^*$, for which $L^*$ approaches (4.8), and one can assume that $U^* \gg U_0^{(F)}$ so that the upper limit of the integral (4.25) can be taken to infinity. This gives the following relation:

$$\frac{L}{2U_0^{(F)}} = \frac{1}{3} \left( \frac{R}{U_0^{(F)}} \right)^{\frac{3}{2}} B(2/3, 1/2).$$  

(4.26)

Note in particular that the amount by which the fundamental string descends into the $D4$-brane throat is independent of $U^*$ for large $U^*$.

We can also compare the minimal value of $U$ for the fundamental string and the $D8$-brane. The latter is given by (4.8); combining it with (4.26) we conclude that

$$\left( \frac{U_0^{(F)}}{U_0} \right)^{\frac{3}{2}} = \frac{8 B(2/3, 1/2)}{3 B(9/16, 1/2)} \approx 2.38.$$  

(4.27)

Thus, the fundamental string reaches the vicinity of the tip of the $D8$-brane, but stays away from it by a finite amount. This behavior is shown in figure 4.
Another interesting quantity to compute is the length $L_{\text{curved}}$ of the fundamental string whose shape is described by (4.24). Dividing this length by the length of the straight string (and omitting some numerical constants) one finds that the ratio is

$$\frac{L_{\text{curved}}}{L_{\text{straight}}} \sim \left( \frac{U^*}{U_0^{(F)}} \right)^{-\frac{1}{2}}.$$  \hspace{1cm} (4.28)

Hence, for large $U^*$ the curved string is much shorter than the straight one.

Thus, even a string that starts at a large value of $U$ and might be expected to naively decouple from the infrared physics at small $U$ in fact does not. To minimize its length, the fundamental string descends into the $D4$-brane throat, reaching the vicinity of the place where the $D8$-brane itself turns around, and proceeds to the other side in $x^4$.

5. NJL and QCD

An interesting generalization of the brane configuration we have been studying is obtained by compactifying the $x^4$ direction on a circle of radius $R_4$, with antiperiodic boundary conditions for the fermions living on the $D4$-branes. For finite $R_4$ the $U(N_c)$ gauge field on the $D4$-branes becomes dynamical. The antiperiodic boundary conditions give a mass to the adjoint fermions and scalars coming from $4 - 4$ strings, at tree level and one loop, respectively. Below the mass of the adjoint fermions and scalars the dynamical degrees of freedom are a $U(N_c)$ gauge field and $N_f$ fermions in the fundamental representation of the gauge group.
This brane configuration was studied in [13] as a description of QCD with $N_f$ flavors, generalizing the work of [23], where the system without D8-branes (and thus without fundamentals) was considered as a model for pure Yang-Mills theory. Its low-energy dynamics depends on two dimensionless parameters: $\lambda/L$ and $L/R_4$. In sections 2 – 4 we considered the case $L/R_4 = 0$ and studied the dependence of the dynamics on $\lambda/L$. In this section we will qualitatively describe the dependence of the dynamics on $L/R_4$, which varies over the range $[0, \pi]$. This will help us to relate the discussion of sections 2 – 4 of this paper to that of [13], and to QCD.

We start with the case where $x^4$ is compactified on a large circle of radius $R_4 \gg L$. As in the previous sections, the dynamics depends on the open string coupling $g_s N_c$. For weak coupling, $g_s N_c \ll 1$, the hierarchy of scales is (compare to (2.5))

$$\lambda \ll l_s \ll L \ll R_4 . \tag{5.1}$$

The fact that $R_4$ is finite means that the classical four-dimensional 't Hooft coupling $\lambda_4$

$$\lambda_4 = \frac{\lambda}{2\pi R_4} . \tag{5.2}$$

is finite as well. In the regime (5.1), the four-dimensional coupling $\lambda_4$ is small. Dimensional transmutation generates a dynamical scale $\Lambda_{QCD}$, at which four dimensional gauge interactions become large.

For small $\lambda_4$, $\Lambda_{QCD}$ is much smaller than the dynamically generated mass of the quarks, $k^*$ (3.25). Therefore, the dynamics splits into two essentially decoupled parts. The chiral symmetry breaking is still described by the non-local NJL model of section 3. The interactions of the quarks with four dimensional gauge fields provide a small correction to their properties at the scale $k^*$. In particular, $q_L$ and $q_R$ behave as free particles at distance scales of order $1/k^*$. The gauge fields introduce a confining potential for the quarks at the much larger distance scale $1/\Lambda_{QCD}$. Thus, for finite $R_4$ we expect to find a discrete spectrum of meson resonances with energies of order $2k^*$ and mass splittings of order $\Lambda_{QCD}$.

As the open string coupling $g_s N_c$ increases, the five-dimensional coupling and $\Lambda_{QCD}$ increase as well. For

$$l_s \ll L \ll \lambda \ll R_4 . \tag{5.3}$$

the four-dimensional coupling (5.2) is still small, but the five dimensional one is large. Thus, one can use the supergravity description of section 4 (with small corrections due to
the finiteness of $R_4$) to describe the dynamical mass generation of the quarks, while the confining interactions of the quarks with the gauge field are still described by Yang-Mills theory with a small 't Hooft coupling (5.2). In particular, the dynamics still splits into the chiral symmetry breaking part and the confining part, which occur at different energy scales.

Further increasing $g_s N_c$ we reach the regime

$$l_s \ll L \ll R_4 \ll \lambda .$$

(5.4)

Here, both the four-dimensional and the five-dimensional 't Hooft couplings are large, and one needs to use supergravity to study both chiral symmetry breaking and confinement.

The near-horizon metric of the $D4$-branes in this regime is given by

$$ds^2 = \left( \frac{\alpha' U}{R} \right)^{\frac{4}{3}} \eta_{\mu\nu} dx^\mu dx^\nu - f(U)(dx^4)^2 - \left( \frac{\alpha' U}{R} \right)^{-\frac{1}{3}} \left( \frac{(\alpha' dU)^2}{f(U)} + (\alpha' U)^2 d\Omega_4^2 \right),$$

(5.5)

where

$$f(U) = 1 - \left( \frac{U_{KK}}{U} \right)^3,$$

(5.6)

$R$ is given by (1.3) and

$$U_{KK} = \frac{4\pi \lambda}{9 R_4^2}.$$

(5.7)

As is familiar from studies of near-extremal $D4$-branes, the geometry (5.3) is smooth. The $U$ coordinate is restricted to $U \geq U_{KK}$, such that the warp factor in $\mathbb{R}^{3,1}$ as well as the volume of the four-spheres remain finite for all $U$. When (5.4) is valid, both $U_0$ (1.12), and $U_{KK}$ (5.7) are in the region where the curvature of the metric (5.5) is very small, so one can study low-lying open and closed strings using supergravity.

Equation (5.4) implies that $U_0 \gg U_{KK}$. Therefore, the dynamics associated with chiral symmetry breaking is still insensitive to the presence of dynamical four dimensional gauge fields. Indeed, in studying the shape of the $D8$-branes in section 4 we took $f(U) = 1$, rather than (5.6). However, this shape is only sensitive to $f(U \geq U_0)$. Thus, the analysis of section 4 is valid for finite $U_{KK}$, up to small corrections that go like $(U_{KK}/U_0)^3$. In particular, the dynamically generated mass of the quarks is still given by $U_0$ (1.8), (1.12).

What does change significantly in this case is the properties of constituent quarks, which for $U_{KK} = 0$ were described by fundamental strings going from $U_0$ down to $U = 0$. Indeed, consider a quark anti-quark pair, which corresponds to a fundamental string whose
ends lie on the $D8$-branes, and are separated by a distance $x$ in $\mathbb{R}^3$. For $R_4 \gg x \gg L$, this looks like a string going from the $D8$-brane down towards $U = 0$ and another one with opposite orientation a distance $x$ apart. Its shape is insensitive to the finite value of $R_4$. This implies that the quark and anti-quark are weakly interacting at these scales.

When the distance $x$ becomes of order $R_4$, the string notices that it can descend no further in $U$ and its properties change. This corresponds to the confining potential created by the four dimensional dynamics at the distance scale $R_4$. Thus, in this regime the quarks can be thought of as particles with the mass $U_0$ (4.12) which are free at short distances and form bound states whose size is of order $R_4$.

To summarize we find that if we compactify the system discussed in sections 2 – 4 on a large circle of radius $R_4 \gg L$, the chiral symmetry breaking dynamics of the fermions $q_L$ and $q_R$ remains essentially unchanged and is almost decoupled from that of the four dimensional gluons of $U(N_c)$. At the energy scales associated with chiral symmetry breaking the four-dimensional coupling is very small and one can neglect the dynamics of the gauge fields. The scale $\Lambda_{QCD}$ at which the gauge coupling becomes strong is well below the dynamically generated masses of the quarks. Gauge dynamics leads to the formation of bound states whose typical size is $1/\Lambda_{QCD}$.

To get a theory that looks more like QCD we need to take the parameter $L/R_4$ to be of order one. For example, [13] discussed the case in which the $D8$ and $\overline{D8}$-branes are maximally separated on the circle, which corresponds to $L = \pi R_4$. When $L$ and $R_4$ are comparable, the only dimensionless parameter that we can vary is $\lambda/L \sim \lambda/R_4$.

For $\lambda \ll L$, the low-energy theory on the branes is QCD with $N_f$ massless fundamentals, and a small four-dimensional ‘t Hooft coupling (5.2). The QCD scale is well below the other scales, $1/L, 1/R_4$. This is the theory one would like to solve, but unfortunately, there are no good analytical tools to study it in this regime.

For $\lambda \gg L$ the theory is not quite QCD, but it is expected to be in the same universality class. In particular, qualitative phenomena such as dynamical symmetry breaking, and the existence of a discrete spectrum of massive glueballs and mesons should not change as one varies $\lambda/L$, although the details of the spectrum may change. The limit $\lambda \gg L$ can be studied using the supergravity description (5.5). This was done in [13, 24] and we will not review the details here.

The only point we would like to mention is that unlike the limit $L/R_4 \rightarrow 0$ discussed in section 4, in the smooth spacetime (5.4) a configuration with separate $D8$ and $\overline{D8}$-branes terminating at the origin does not make sense, and only the “hairpin” shape joining the
asymptotic $D8$ and $\overline{D8}$-branes is possible. Thus, in contrast to the discussion of section 4, where chiral symmetry breaking was simply favored energetically, here it is mandatory. This is perhaps not surprising given that the ‘t Hooft matching conditions require chiral symmetry breaking for specific values of $N_f$ [23].

The discussion of this section leads us to one of the main points of this paper. In the case without quarks studied in [23], the dynamics depends only on one dimensionless parameter, $\lambda/R_4$. For small values of this parameter one finds pure Yang-Mills QCD, which is hard to treat analytically, while for large values one finds a system that differs from QCD but can be analyzed using supergravity.

In the $D4 \, - \, D8 \, - \, \overline{D8}$ system, which contains massless quarks, there are two dimensionless parameters, $\lambda/L$ and $L/R_4$. For $L/R_4 \sim 1$, the case discussed in [13,24], the situation is as in [23], but for small $L/R_4$ one gets a theory which is solvable both at weak coupling $\lambda \ll L$ and at strong coupling $\lambda \gg L$, by using the non-local NJL model of section 3 and the supergravity analysis of section 4, respectively. Moreover, we presented evidence that the system is in the same universality class (or phase) for all values of these parameters. Thus, we see that the brane construction interpolates between a regime in which the dynamics of the quarks is described by the non-local NJL model, and one where it is described by QCD. This might help explain why the NJL model is useful in studying mesons in QCD [4-8].

6. Discussion

Much of the early work describing large $N_c$ QCD using string theory has focused on pure Yang-Mills theory, without quarks. In such theories one can compute the spectrum of glueball states, but the results must then be compared to lattice QCD since glueballs have not been unambiguously identified experimentally. In trying to obtain more realistic models of QCD it is important to incorporate quarks and their meson bound states. The $D4 \, - \, D8 \, - \, \overline{D8}$ model studied in [13,24] and in this paper is a natural construction which incorporates the chiral symmetry of QCD with massless flavors, describes the spontaneous breaking of this symmetry, and leads to a spectrum of meson bound states.

Previous work on this model [13,24] focused on the region in the parameter space of brane configurations in which the separation between the $D8$-branes, $L$, is of the order of the radius of the extra dimension along the $D4$-branes, $R_4$. If the five-dimensional ‘t Hooft coupling $\lambda$ is small, $\lambda \ll L, R_4$, this brane configuration describes QCD, and is difficult to
analyze. For large \( \lambda \) the dynamics is not quite that of QCD but it can be analyzed using supergravity. There are reasons to believe that the theories one gets in the two limits are in the same universality class.

In this paper we studied the brane configuration of \([13,24]\) in the opposite limit \( R_4 \gg L \). Consideration of this limit and the space of brane configurations in general leads to some new insights into the dynamics of quarks and mesons.

One of the surprising results of our analysis is that there exist four-dimensional models of quarks without dynamical gauge fields which exhibit non-trivial infrared dynamics. In local quantum field theory this is believed to be impossible. Our model contains a non-local interaction between the left and right-handed quarks, but since this interaction arises in a D-brane system, we are assured that it does not lead to any pathologies.

The model in question is a non-local analog of the Nambu-Jona-Lasinio model, which exhibits dynamical symmetry breaking for arbitrarily small value of the coupling. While we obtained this model from string theory, one can construct it directly in field theory as follows. Consider a five-dimensional \( U(N_c) \) gauge theory with two codimension one defects separated by a distance \( L \). At the two defects there are \( N_f \) left and right-handed fermions in the fundamental representation of the gauge group, \( q_L \) and \( q_R \). The dynamics of the four-dimensional fermions and five-dimensional gauge field is governed by the action (3.4). The model has a \( U(N_f)_L \times U(N_f)_R \) global symmetry. Moreover, since the \( U(N_c) \) gauge fields are higher dimensional, \( U(N_c) \) acts as a global symmetry as well.

The dynamical degrees of freedom in the four-dimensional theory are the fermions \( q_L \) and \( q_R \). The five-dimensional gauge field provides an effective coupling between them. This coupling becomes weaker as the distance \( L \) increases, since the five dimensional gauge theory is infrared free. In the analogy to superconductivity that motivated \([1]\), \( q_L \) and \( q_R \) are analogous to the electrons, and the five-dimensional \( U(N_c) \) gauge bosons are analogous to the phonons.

To study the interaction among the fermions it is convenient to integrate out the five-dimensional gauge field. In the limit \( L \gg \lambda \), one can do that in the leading, single gluon exchange approximation. This gives rise to the non-local NJL model (3.5), which breaks chiral symmetry for arbitrarily large \( L \) (i.e. arbitrarily weak coupling).

Since the non-local NJL model (3.5) contains only vector degrees of freedom of \( U(N_c) \), it is exactly solvable in this limit. It can be thought of as a generalization to four dimensions of models such as the Gross-Neveu model [3], and the ‘t Hooft model [26] of two-dimensional QCD.
Compactifying $x^4$ on a circle of radius $R_4$ provides a continuous interpolation between the NJL model (3.5) and QCD. The latter is obtained in the limit $R_4 \approx L \gg \lambda$. Our analysis of the NJL model strongly suggests that it is in the same universality class as QCD.

Physically, the reason that the above construction simplifies the study of quark dynamics is the following. The basic mechanism for chiral symmetry breaking and the binding of quarks into mesons is the attractive interaction between the quarks due to exchange of $U(N_c)$ gauge bosons. Our model allows one to separate the two scales associated with chiral symmetry breaking and confinement. In the NJL limit, the mass scale associated with chiral symmetry breaking is much higher than that of confinement, and all the complications associated with the latter disappear in studying the former. In the QCD limit the two scales are comparable, which makes the model harder to analyze.

Many interesting questions remain to be addressed. As in previous discussions, we have dealt with the theory in the limit of zero bare quark mass. Since one can clearly make sense of the quark mass perturbation in the NJL model, we expect the same to be true on the supergravity side. It would be interesting to identify this perturbation and to study the theory with massive flavors in the different regimes. This is likely to involve a better understanding of the $8-\overline{8}$ string discussed in section 4. The detailed correspondence between bulk and boundary fields also remains to be worked out.

The NJL model has been widely used to study a host of phenomenological issues in QCD which are not amenable to perturbation theory, including the behavior of the theory at finite temperature and quark density. It would be interesting to address these issues in the non-local (gauged) NJL model that arises here as well as in its string theory dual.

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Appendix A. Conventions and useful results

We use a “mainly minus” metric convention. In particular, the four-dimensional flat space metric is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. We use the same conventions as [27] for the Weyl representation of gamma matrices in four dimensions. Thus

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$  \hspace{1cm} (A.1)

with $\sigma^\mu = (1, \vec{\sigma})$ and $\overline{\sigma}^\mu = (1, -\vec{\sigma})$.

We require two Fierz identities which rely on the two relations

$$ (\sigma^\mu)^\alpha (\sigma_\mu)^\beta = 2\epsilon_{\alpha\beta}\epsilon_{\alpha\beta} \hspace{1cm} (A.2)$$

the first of which is eqn (3.77) of [27] (which also holds with $\sigma^\mu$ replaced by $\overline{\sigma}^\mu$) and the second follows from the first using eqn (3.80) of [27].

Using these, we have for Grassman-valued Weyl spinors

$$ \left( \psi^{\dagger}_{1L} \overline{\sigma}^\mu \psi_{2L} \right) \left( \psi^{\dagger}_{3R} \sigma_\mu \psi_{4R} \right) = -2 \left( \psi^{\dagger}_{1L} \psi_{4R} \right) \left( \psi^{\dagger}_{3R} \psi_{2L} \right) $$  \hspace{1cm} (A.3)

and

$$ \left( \psi^{\dagger}_{1L} \overline{\sigma}^\mu \psi_{2L} \right) \left( \psi^{\dagger}_{3L} \sigma_\mu \psi_{4L} \right) = \left( \psi^{\dagger}_{1L} \overline{\sigma}^\mu \psi_{4L} \right) \left( \psi^{\dagger}_{3L} \sigma_\mu \psi_{2L} \right) .$$  \hspace{1cm} (A.4)

Fourier transforms in $d$ spacetime dimensions are given by

$$ \tilde{f}(k) = \int d^d x e^{ik \cdot x} f(x) , $$

$$ f(x) = \int \frac{d^dk}{(2\pi)^d} e^{-ik \cdot x} \tilde{f}(k) .$$  \hspace{1cm} (A.5)

Five-dimensional Fourier transforms of spherically symmetric functions of $k$ in Euclidean space,

$$ F(x) = \int \frac{d^5k}{(2\pi)^5} \tilde{F}(|k|) e^{ik \cdot x} , $$

can be computed using spherical coordinates to give

$$ F(x) = \frac{1}{4\pi^3} \int_0^\infty dk k^4 \tilde{F}(k) \left( \frac{\sin kx}{(kx)^3} - \frac{\cos kx}{(kx)^2} \right) .$$  \hspace{1cm} (A.6)

As an example, if $\tilde{F}(k) = 1/k^2$ we find the coordinate space propagator

$$ F(x) = \frac{1}{8\pi^2 (x^2)^{3/2}} .$$  \hspace{1cm} (A.8)
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