A Strong Rigidity Result on Complete Finsler Manifolds via a Second Order Differential Equation

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Abstract

We provide an extension of the Obata’s theorem to Finsler geometry and establish some rigidity results based on a second order differential equation. Mainly, we prove that: Every complete connected Finsler manifold of positive constant flag curvature is isometrically homeomorphic to an Euclidean sphere endowed with a certain Finsler metric and vice versa.

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1 Introduction

Rigidity describes quite different concepts in mathematics. Historically, one of the first rigidity theorems, proved by Cauchy in 1813, states that if the faces of a convex polyhedron were made of metal plates and the edges were replaced by hinges, the polyhedron would be rigid [1]. Although rigidity problems were of immense interest to engineers, the intensive mathematical study of these types of problems has occurred only in the late 20th century, see [10]. In geometry sometimes an object is considered as rigid if it has flexibility and not elasticity. In other words, a geometrical rigidity implies invariant with respect to isometries. In Riemannian geometry, the sectional curvature is invariant under isometries. Hence, a space of positive constant curvature is transformed into the same space by each isometry. This fact is sometimes described as the “strong rigidity” of a space of constant curvature.

In Finsler geometry, the encountered rigidity results are rather slightly weaker and they usually talk about under which assumptions on the flag curvature -analogous to the sectional curvature in Riemannian geometry- the underlying Finsler structure is either Riemannian or locally Minkowskian. A famous treatise in this area is by Akbar-Zadeh [1] where he established the following rigidity theorem for compact manifolds: Let $(M, F)$ be a compact without boundary Finsler manifold of constant flag curvature $K$. If $K < 0$, then $(M, F)$ is Riemannian. If $K = 0$, then $(M, F)$ is locally Minkowskian.

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There are several papers in Finsler geometry with results similar to Akbarzadeh’s rigidity theorem but by considering different assumptions. Foulon addressed the case of strictly negative flag curvature in Akbar-Zadeh’s theorem. He imposed the additional hypothesis that the curvature is covariantly constant along a distinguished vector field on the homogeneous bundle of tangent half lines to show that the Finsler structure is Riemannian, see [12]. Also, he presented a strong rigidity theorem for symmetric compact Finsler manifolds with negative curvature and proved that such manifolds are isometric to locally symmetric negatively curved Riemannian spaces [13]. This extends Akbar-Zadeh’s rigidity theorem to a so-called “strong rigidity” one. Shen [21] considered the case of negative but not necessarily constant flag curvature by adding the assumption that the $S$-curvature is constant and showed that the Akbar-Zadeh’s rigidity theorem still holds.

Following the several rigidity theorems in the two joint papers [16] and [17], Kim proved that: “Any compact locally symmetric Finsler manifold with positive constant flag curvature is Riemannian”, see [18]. Also, Bidabad [3] established some rigidity theorems as an application of connection theory in Finsler geometry. Another rigidity result is presented by Wu [25] who proved that any locally symmetric Finsler manifold with nonzero flag curvature must be Riemannian.

Here, we consider the missing part of the Akbar-Zadeh’s rigidity theorem and present some results for the case of $K > 0$. We also provide an extension of Obata’s theorem to Finsler geometry. This theorem in Riemannian geometry says (see [30] for more details):

**Theorem (Obata):** Let $(M, g)$ be a connected complete Riemannian manifold of dimension $n \geq 1$ which admits a non-constant smooth solution of Obata’s equation $\nabla dw + wg = 0$. Then $(M, g)$ is isometric to the $n$-dimensional round sphere $S^n$.

Finsler spaces of positive flag curvature have been studied and classified by several researchers and a number of results have been generalized from Riemannian spaces of positive sectional curvature to Finsler spaces of positive flag curvature, see for instance [6], [28], [26], and [24]. Also, Bidabad in [8], used the same idea of [5] and provided a classification of simply connected compact Finsler manifolds. In 2018, [9] Boonnam et al. proved that a complete Berwald manifold with nowhere vanishing flag curvature must be Riemannian. Also, several results and open problems about Finsler spaces with positive curvatures are addressed in [11].

Here, we apply an adapted coordinate system introduced in [5] to study the strong rigidity of Finsler manifolds of positive constant flag curvature. We show that: A complete $n$-dimensional Finsler manifold $(M, F)$ is of positive constant flag curvature if and only if $(M, F)$ is isometrically homeomorphic to an $n$-sphere equipped with a certain Finsler metric. Particularly, this result complements the Akbar-Zadeh’s rigidity theorem. Meanwhile, we extend Obata-Tanno’s theorem to Finsler geometry as follows: Let $(M, F)$ be a complete connected Finsler manifold of dimension $n \geq 2$. In order that there is a non-trivial solution of $\nabla^H \nabla^H \rho + C^2 \rho F = 0$ on $M$, it is necessary and sufficient that $(M, F)$ be isometric to an $n$-sphere of radius $1/C$. This result leads to provide a definition of a sphere in Finsler geometry.
2 Preliminaries

In this section, we review some definitions of Finsler geometry that we refer to through this paper. More details can be found in [22].

2.1 Finsler Manifolds

Let $M$ be a real $n$-dimensional manifold of class $C^\infty$ and $TM$ its tangent bundle, i.e. $TM = \bigcup_{x \in M} \{(x, y) : y \in T_x M\}$. A Finsler structure on $M$ is a function $F : TM \to [0, \infty)$, with the following properties: (I) $F$ is differentiable ($C^\infty$) on the tangent bundle of non-zero vectors $TM_0$; (II) $F$ is positively homogeneous of degree one in $y$, i.e. $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$, where $(x, y)$ is an element of $TM$. (III) The Hessian matrix of $F^2$, $(g_{ij}) := \left( \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \right)$, is positive definite on $TM_0$. A Finsler manifold is a pair consisting of a differentiable manifold $M$ and a Finsler structure $F$ on $M$. The tensor field $g$ with the components $g_{ij}$ is called the Finsler metric tensor and we denote a Finsler manifold by $(M, g)$. We denote the natural projection on $TM_0$ by $\pi$ and its differential by $\pi_*$, i.e. $\pi_* : TTM_0 \to TM$. The vertical vector bundle on $M$ is defined as $VTM := \bigcup_{\nu \in TM_0} V_\nu TM$ where $V_\nu TM = \ker \pi_*$ is the set of vectors tangent to $\nu \in TM_0$. The complementary decomposition $HTM$ where $HTM \oplus VTM = TTM_0$ is called the non-linear connection on $TM_0$. The coefficients of the non-linear connection are denoted by $C^i_j(x, y)$, where $C^i_j = \frac{\partial G^i}{\partial y^j}$ and $G^i = \frac{1}{4} g^{ik} \left( \frac{\partial^2 F^2}{\partial y^k \partial x^j} y^j - \frac{\partial F^2}{\partial x^k} \right)$. By using the local coordinates $(x^i, y^i)$ on $TM$, called the line elements, we have the local field of frames $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ on $TTM$. Given a non-linear connection, we can choose a local field of frames $\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}\}$ on $TTM_0$ where $\delta_{x^i} := \frac{\partial}{\partial x^i} - C^i_j \frac{\partial}{\partial y^j}$ and $\frac{\delta}{\delta y^i}$ are the set of vector fields on $HTM$ and $VTM$, respectively.

A $1$-form of the Cartan connection is given by $\omega^i_j = \Gamma^i_{jk} dx^k + C^i_{jk} dy^k$, where $\Gamma^i_{jk} = \frac{1}{2} g^{ir} \left( \frac{\delta g_{rk}}{\delta x^j} + \frac{\delta g_{rk}}{\delta x^j} - \frac{\delta g_{rk}}{\delta x^r} \right)$ and $C^i_{jk} = \frac{1}{2} g^{ir} \frac{\partial g_{rk}}{\partial y^j}$. The coefficients $\Gamma^i_{jk}$ and $C^i_{jk}$ are called coefficients of horizontal and vertical covariant derivatives of the Cartan connection, respectively. Given a tensor field $T$ with the components $T^i_{jk}(x, y)$ on $TM$, the components of the Cartan horizontal covariant derivative of $T$, $\nabla^H T$, are given by

$$\nabla^H T_{jk}^i := \frac{\delta}{\delta x^r} T_{jk}^i - T_{sk}^i \Gamma_{jr}^s - T_{js}^i \Gamma_{kr}^s + T_{jk}^s \Gamma_{sr}^i.$$

Assume that $\gamma : I \to M$ defined by $t \to x^i(t)$ be a smooth curve on $M$ and $\tilde{\gamma}(t) = (x^i(t), \frac{dx^i}{dt})$ its natural lift on $TM$. We say that $\gamma$ is a geodesic of the Finsler space $(M, F)$ if $\nabla^H \tilde{\gamma} = 0$. Here, $\tilde{\gamma}(t) = \frac{d}{dt} y^i + \frac{\partial y^i}{\partial x^j} \frac{dx^j}{dt}$, where $\frac{dy^i}{dt} := \frac{dy^i}{dt} + G^i_j(x(t), \frac{dx}{dt}) \frac{\partial y^j}{\partial x^i}$.\]
2.2 Finsler Manifolds with a Non-trivial Solution of $\nabla^H \nabla^H \rho = \phi g$

Let $\rho : M \to \mathbb{R}$ be a scalar function on $M$ that satisfies the following second order differential equation

$$\nabla^H \nabla^H \rho = \phi g,$$  \hspace{1cm} (2.1)

where $\nabla^H$ is the Cartan horizontal covariant derivative and $\phi$ is a function of $x$ alone. The connected component of a regular hypersurface defined by $\rho = \text{constant}$ is called a level set of $\rho$. We denote by $\text{grad}\rho$ the gradient vector field of $\rho$ which is locally written in the form $\text{grad}\rho = \rho^i \frac{\partial}{\partial x^i}$, where $\rho^i = g^{ij} \rho_j = \frac{\partial \rho}{\partial x^j}$ for $i, j, \ldots \in \{1, \ldots, n\}$. Note that the partial derivatives $\rho_j$ are defined on the manifold $M$ while $\rho^i$, the components of $\text{grad}\rho$, are defined on its slit tangent bundle $T M_0$. Hence, $\text{grad}\rho$ can be considered as a section of $\pi^*TM \to TM_0$, the pulled-back tangent bundle over $TM_0$, and its trajectories lie on $TM_0$. For more details see [5] and references therein.

One can easily verify that the canonical projection of the trajectories of the vector field $\text{grad}\rho$ are geodesic arcs on $M$ [5]. Therefore, we can choose local coordinates $(u^1 = t, u^2, \ldots, u^n)$ on $M$ such that $t$ is the parameter of the geodesic containing the projection of a trajectory of the vector field $\text{grad}\rho$ and the level sets of $\rho$ are given by $t = \text{constant}$. These geodesics are called $t$-geodesics. Since in this local coordinate system, the level sets of $\rho$ are given by $t = \text{constant}$, $\rho$ may be considered as a function of $t$ only. In the sequel we will refer to these level sets and these local coordinates as $t$-levels and adapted coordinates, respectively.

Let $(M, g)$ be a Finsler manifold and $\rho$ a non-trivial solution of Eq. (2.1) on $M$. Then, using the adapted coordinates, components of the Finsler metric tensor $g$ are given by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & g_{n2} & \cdots & g_{nn} \end{pmatrix},$$

and $t$ may be regarded as the arc-length parameter of $t$-geodesics. It can be easily verified that the Finsler metric form of $M$ is given by

$$ds^2 = (dt)^2 + \rho'^2 f_{\gamma\beta} du^\gamma du^\beta,$$  \hspace{1cm} (2.2)

where $f_{\gamma\beta}$ are components of a Finsler metric tensor on a $t$-level of $\rho$ and $\rho'^2 f_{\gamma\beta}$ is the induced metric tensor of this $t$-level. Here, prime denotes the ordinary differentiation with respect to $t$. In this paper, the Greek indices $\alpha, \beta, \gamma, \ldots$ run over the range $2, 3, \ldots, n$.

A point $o$ of $(M, g)$ is called a critical point of $\rho$ if the vector field $\text{grad}\rho$ vanishes at $o$, or equivalently if $\rho'(o) = 0$, see [5]. If a non-trivial solution of Eq. (2.1) has some critical points, then $M$ possess some interesting properties. For instance:

**Lemma 1.** [5] Let $(M, g)$ be an $n$-dimensional Finsler manifold which admits a non-trivial solution $\rho$ of Eq. (2.1) with one critical point. Then any $t$-level $M_t$ with Finsler metric form $ds^2 = f_{\gamma\beta} du^\gamma du^\beta$ has the positive constant flag curvature $\rho'^2(0)$.
3 A Special Solution of $\nabla^H \nabla^H \rho = \phi g$

Let $(M, g)$ be an $n$-dimensional Finsler manifold and $\rho : M \to \mathbb{R}$ a solution of Eq. (2.1). If $\phi$ is a linear function of $\rho$ with constant coefficients, then we say that $\rho$ is a special solution of Eq. (2.1). Hence, any special solution of Eq. (2.1) can be written in the form

$$\nabla^H \nabla^H \rho = (-K \rho + B)g,$$

where $K$ and $B$ are constants. The Eq. (3.1) along any geodesic with arc-length $t$ reduces to the ordinary differential equation

$$\frac{d^2 \rho}{dt^2} = -K \rho + B.$$  

(3.2)

Now for the special case $K = C^2 > 0$ and $B = 0$, we have

$$\frac{d^2 \rho}{dt^2} + C^2 \rho = 0.$$  

(3.3)

By a suitable choice of the arc-length $t$, a solution of Eq. (3.3) is given by

$$\rho(t) = A \cos(Ct),$$  

(3.4)

and its first derivative is

$$\rho'(t) = -AC \sin(Ct).$$  

(3.5)

So, we can see at a glance that Eq. (3.4) has two critical points corresponding to $t = 0$ and $t = \frac{\pi}{C}$ on $M$ which are repeated periodically. Hence, if $\rho$ is a non-trivial solution of Eq. (3.3), then it can be written in the following form

$$\rho(t) = -\frac{1}{C} \cos(Ct), \quad (A = -\frac{1}{C}).$$  

(3.6)

Taking Eq. (2.2) into account, the metric form of $M$ becomes

$$ds^2 = dt^2 + (\sin(Ct))^2 \overline{ds}^2,$$  

(3.7)

where $\overline{ds}^2$ is the metric form of a $t$-level of $\rho$ given by $\overline{ds}^2 = f_{\gamma\beta} du^\gamma du^\beta$. This is the polar form of a Finsler metric on a standard sphere of radius $\frac{1}{C}$, see [23].

4 Finsler Manifolds of Positive Constant Flag Curvature

Let $(x, y)$ be the line element of $TM$ and $P(y, X) \subset T_x M$ a 2-plane generated by the vectors $y$ and $X$ in $T_x M$. Then the flag curvature $K(x, y, X)$ with respect to the plane $P(y, X)$ at a point $x \in M$ is defined by

$$K(x, y, X) := \frac{g(R(X, y) y, X)}{g(X, X)g(y, y) - g(X, y)^2},$$

where $g$ is the metric function.
where \( R(X, y) y \) is the \( h \)-curvature tensor of Cartan connection. If \( K \) is independent of \( X \), then \((M, g)\) is called \textit{space of scalar curvature}. If \( K \) has no dependence on \( x \) or \( y \), then the Finsler manifold is said to be of \textit{constant (flag) curvature}, see for instance [2]. It can be easily verified that the components of the \( h \)-curvature tensor of Cartan connection in the adapted coordinate system are given by

\[
R^\alpha_{1\gamma} = -R^\alpha_{\gamma 1} = (\frac{\partial^m}{\partial y^\alpha})_{\gamma},
R^1_{1\gamma} = -R^1_{\gamma 1} = -\rho' \rho''' f_{\gamma},
R^\alpha_{\delta\gamma} = \overline{R}^\alpha_{\delta\gamma} = -(\rho''^2)(f_{\gamma}\delta^\alpha_{\delta} - f_{\delta\gamma}^\alpha),
\]

(4.1)

where \( \overline{R}^\alpha_{\delta\gamma} \) are components of \( h \)-curvature tensor related to the metric form \( ds^2 \) on a \( t \)-level of \( \rho \), see [5] for more details.

**Proposition 1.** The \( n \)-dimensional complete Finsler manifold \((M, g)\) is of constant flag curvature \( K = C^2 > 0 \), if and only if, there is a non-trivial solution of \( \nabla^H \nabla^H \rho = (-C^2 \rho + B)g \) on \( M \).

**Proof.** A Finsler manifold \((M, g)\) is of constant flag curvature \( K \) if and only if the components of the \( h \)-curvature tensor are given by the following, see [5] for more details.

\[
R_{ijk} = K(\delta_i g_{jk} - \delta_j g_{ik}),
\]

(4.2)

Using Eq. (4.2), we can easily drive the differential equation

\[
\ddot{A} + KA \dot{A} = 0,
\]

(4.3)

where \( A \) is the Cartan torsion tensor, \( \dot{A} := (\nabla_s A_{ijk}) y^s \) and \( \ddot{A} := (\nabla^H_s \nabla^H_{\dot{t}} A_{ijk}) y^s y^t \), see Section 1.4 of [7] for more details.

Assume that \( X, Y, Z \in \pi^*T M \) are fixed at \( v \in I_x M = \{ w \in T_x M, g(w, w) = 1 \} \). Let \( c : R \to M \) be the unit-speed geodesic on \((M, g)\) with \( \frac{dc}{dt}(0) = v \) and \( \dot{c} := \frac{dc}{dt} \) be the canonical lift of \( c \) to \( TM_0 \). Let \( X(t), Y(t) \) and \( Z(t) \) denote the parallel sections along \( \dot{c} \) with \( X(0) = X, Y(0) = Y \) and \( Z(0) = Z \). Put \( A(t) = A(X(t), Y(t), Z(t)), \dot{A}(t) = \dot{A}(X(t), Y(t), Z(t)) \) and \( \ddot{A}(t) = \ddot{A}(X(t), Y(t), Z(t)) \). Indeed along geodesics, we have \( \frac{dA}{dt} = \dot{A}, \frac{d\dot{A}}{dt} = \ddot{A} \) and Eq. (4.3) becomes

\[
\frac{d^2A(t)}{dt^2} + KA(t) = 0.
\]

(4.4)

The general solution of this differential equation is \( A(t) = A_0 \cos \sqrt{K}t + B_0 \sin \sqrt{K}t \). Therefore, Eq. (3.3) which represents a special case of Eq. (3.1) along geodesics, has a non-trivial solution on \( M \).

Conversely, let \( \rho \) given by Eq. (3.6) be a solution of Eq. (3.1) on \( M \). Then, there is an adapted coordinate system on \( M \) for which the components of \( h \)-curvature are given by (4.1).
Hence, first and second equations of (4.1) satisfy
\[
R_{hjk}^i = -\frac{\rho'''}{\rho'}(\delta_{h}g_{jk} - \delta_{j}g_{hk}).
\] (4.5)

Differentiate (3.6) with respect to \(t\) and replace the first and third derivatives of \(\rho\), we obtain
\[
-\frac{\rho'''}{\rho'} = C^2.
\]
Therefore, the first two equations of (4.1) satisfy Eq. (4.2).

For the third equation of (4.1), we recall that as we see in Section 3, \(\rho\) has critical points on \(M\). Thus, from Lemma 1, the \(t\)-levels of \(\rho\) are spaces of positive constant curvature \(\rho''(0) = C^2\). Therefore, the third equation of (4.1) becomes
\[
R_{\gamma\beta}^\alpha = (C^2 - \rho''(t))f_{\gamma\beta}\delta_{\delta}^\alpha - f_{\delta\beta}\delta_{\gamma}^\alpha.
\]

By substituting \(g_{\alpha\beta} = \rho^2f_{\alpha\beta}\) and the first and second derivatives of \(\rho\) in the above equation, we obtain
\[
R_{\gamma\beta}^\alpha = C^2(g_{\gamma\beta}\delta_{\delta}^\alpha - g_{\delta\beta}\delta_{\gamma}^\alpha).
\]
So, all three components of Cartan \(h\)-curvature tensor satisfy Eq. (4.2) and the Finsler manifold \((M, g)\) is of constant flag curvature \(K = C^2\).

Now, we are in a position to prove an extension of Obata-Tanno’s theorem to Finsler manifolds.

**Theorem 1.** Let \((M, g)\) be a complete connected Finsler manifold of dimension \(n \geq 2\). Then, \((M, g)\) is isometric to an \(n\)-sphere of radius \(\frac{1}{C}\) if and only if there is a non-trivial solution of the following equation on \(M\):
\[
\nabla^H\nabla^H\rho + C^2\rho g = 0.
\] (4.6)

**Proof.** Let \((M, g)\) be a Finsler manifold which admits a non-trivial solution of Eq. (4.6). According to Proposition 1, \((M, g)\) is of positive constant flag curvature \(C^2\). So, as we see in Section 3, the metric form of \((M, g)\) is given by (3.7) and so \((M, g)\) is isometric to an \(n\)-sphere of radius \(\frac{1}{C}\).

Conversely, if \((M, g)\) is isometric to an \(n\)-sphere of radius \(\frac{1}{C}\), then the metric form of \(M\) is given by \(ds^2 = (dt)^2 + \sin^2(Ct)\overline{ds}^2\), where \(\overline{ds}^2\) is the metric form of a hypersurface of \(M\). This is the polar form of a Finsler metric on an \(n\)-sphere in \(\mathbb{R}^{n+1}\) with the positive constant curvature \(C^2\), see [23]. Now by substituting the derivative of \(\rho(t) = -\frac{1}{C} \cos(Ct)\) in the metric form of \(M\), we obtain \(ds^2 = (dt)^2 + \rho^2(t)\overline{ds}^2\). Hence, \(\rho(t)\) is a non-trivial solution of the second order differential equation (3.3) or equivalently a non-trivial solution of Eq. (4.6) along geodesics.

Now, by considering the number of critical points of \(\rho\), we have the following result.
Corollary 1. Let \((M, g)\) be a complete connected Finsler manifold with dimension \(n \geq 2\). Then, \((M, F)\) is isometrically homeomorphic to an \(n\)-sphere if and only if \(\nabla^H \nabla^H \rho + C^2 \rho g = 0\) has a non-trivial solution.

Proof. Let \((M, g)\) admit a non-trivial solution of \(\nabla^H \nabla^H \rho + C^2 \rho g = 0\), then from Theorem 1 we know that it is isometric to an \(n\)-sphere of radius \(\frac{1}{C}\). On the other hand, since \(M\) is complete, Proposition 1 results in \((M, g)\) is of positive constant curvature. Therefore, by applying the extension of Meyers’s theorem to Finsler manifolds, see [1], we can conclude that \(M\) is compact. Thus, the function \(\rho\) admits its absolute maximum and minimum values on \(M\). Consequently, \(\rho\) has two critical points on \(M\) and an extension of Milnor theorem to Finsler geometry, [19], implies that \((M, g)\) is homeomorphic to an \(n\)-sphere.

Conversely, let \((M, g)\) be isometrically homeomorphic to an \(n\)-sphere of radius \(\frac{1}{C}\). Then, Theorem 1 implies that \(\nabla^H \nabla^H \rho + C^2 \rho g = 0\) has a non-trivial solution on \(M\). □ □

Following the Obata-Tanno theorem in Riemannian geometry a unit sphere is characterized by existence of a solution of the differential equation \(\nabla \nabla f + fg = 0\), where \(f\) is a certain function on Riemannian manifold \((M, g)\) and \(\nabla\) is the Levi-Civita connection associated to the Riemannian metric \(g\) [14]. Similarly, Theorem 1 implies that in Finsler geometry a unit sphere can be characterized by existence of a solution of \(\nabla^H \nabla^H \rho + \rho g = 0\), where \(\rho\) is a certain function on Finsler manifold \((M, g)\) and \(\nabla^H\) is the Cartan horizontal covariant derivative. In analogy with Riemannian geometry, this leads to a definition for an \(n\)-sphere in Finsler geometry as follows.

Definition 1. A Finslerian \(n\)-sphere is a complete connected Finsler manifold which admits a non-trivial solution of Eq. (4.6).

Equivalently, a Finslerian \(n\)-sphere is isometrically homeomorphic to an \(n\)-sphere endowed with a certain Finsler metric.

Theorem 2. Let \((M, g)\) be an \(n\)-dimensional complete connected Finsler manifold. Then, \((M, g)\) has positive constant flag curvature \(K = C^2\), if and only if, \((M, g)\) is isometrically homeomorphic to an \(n\)-sphere of radius \(\frac{1}{C}\) endowed with a certain Finsler metric.

Proof. Let \((M, g)\) be of positive constant flag curvature \(C^2\). As a consequence of Proposition 1 there is a non-trivial solution of Eq. (3.3) on \(M\). Thus, by means of Corollary 1 it is isometrically homeomorphic to an \(n\)-sphere of radius \(\frac{1}{C}\) equipped with the certain Finsler metric form \(ds^2 = (dt)^2 + \sin^2(Ct)ds'^2\).

Conversely, let \((M, g)\) be a Finsler manifold which is isometrically homeomorphic to an \(n\)-sphere of radius \(\frac{1}{C} > 0\). Then, Corollary 1 implies that \(M\) admits a non-trivial solution of Eq. (3.3). So, from Proposition 1 \((M, g)\) is of positive constant flag curvature \(C^2\). □ □
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