THE MINIMAL MODEL OF ROTA-BAXTER OPERAD WITH ARBITRARY WEIGHT

KAI WANG AND GUODONG ZHOU

Dedicated to Alexander Zimmermann on the occasion of his 60th birthday

ABSTRACT. This paper investigates Rota-Baxter algebras of arbitrary weight, that is, associative algebras endowed with Rota-Baxter operators of arbitrary weight, from an operadic viewpoint. Denote by $\mathcal{RB}_\lambda$ the operad of Rota-Baxter associative algebras of weight $\lambda$. A homotopy cooperad is explicitly constructed, which can be seen as the Koszul dual of $\mathcal{RB}_\lambda$ as it is proven that the cobar construction of this homotopy cooperad is exactly the minimal model of $\mathcal{RB}_\lambda$. This enables us to introduce the notion of homotopy Rota-Baxter algebras. The deformation complex of a Rota-Baxter algebra and the underlying $L_\infty$-algebra structure over it are exhibited as well.

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Introduction

A general philosophy of deformation theory of mathematical structures, as evolved from ideas of Gerstenhaber, Nijenhuis, Richardson, Deligne, Schlessinger, Stasheff, Goldman, Millson etc, is that the deformation theory of any given mathematical object can be described by a certain differential graded (=dg) Lie algebra or more generally an $L_\infty$-algebra associated to the mathematical object (whose underlying complex is called the deformation complex). This philosophy
has been made into a theorem in characteristic zero by Lurie \cite{47} and Pridham \cite{55}, expressed in terms of infinity categories. It is an important problem to construct explicitly the dg Lie algebra or $L_{\infty}$-algebra governing deformation theory of the mathematical object under consideration.

Another important problem about algebraic structures is to study their homotopy versions, just like $A_{\infty}$-algebras for usual associative algebras. From the perspective of operad theory, specifically, the task is to formulate a cofibrant resolution for the operad of an algebraic structure. The most desirable outcome would be providing a minimal model of the operad governing the algebraic structure. When this operad is Koszul, there exists a general theory, the so-called Koszul duality for operads \cite{26,25}, which defines a homotopy version of this algebraic structure via the cobar construction of the Koszul dual cooperad, which, in this case, is a minimal model. However, when an operad is NOT Koszul, essential difficulties arise and few examples of minimal models have been worked out. For instance, Gálvez-Carrillo, Tonks, and Vallette \cite{20} gave a cofibrant resolution of the Batalin-Vilkovisky operad using inhomogeneous Koszul duality theory. However, their cofibrant resolution is not minimal and in another paper of Drummond-Cole and Vallette \cite{17}, the authors succeeded in finding a minimal model which is a deformation retract of the cofibrant resolution found in the previous paper. Dotsenko and Khoroshkin \cite{15} constructed cofibrant resolutions for shuffle monomial operads by the inclusion-exclusion principle and for operads presented by a Gröbner basis \cite{14} they suggested a method to find cofibrant resolutions by deforming those of the corresponding monomial operads.

These two problems, say, describing controlling $L_{\infty}$-algebras and constructing homotopy versions, are closed related. In fact, given a cofibrant resolution, in particular a minimal model, of the operad in question, one can form the deformation complex of the algebraic structure and construct its $L_{\infty}$-structure as explained by Kontsevich and Soibelman \cite{38} and van der Laan \cite{62,63}. This method has been generalised to properads by Markl \cite{50}, Merkulov and Vallette \cite{52,53}, and to colored props by Frégier, Markl, and Yau \cite{19}. However, in practice, a minimal model or a small cofibrant resolution is not known a priori.

In this paper, we resolve completely the above two problems for Rota-Baxter associative algebras of arbitrary weight. We found the minimal model and the Koszul dual homotopy cooperad of the operad of Rota-Baxter algebras of arbitrary weight. Using the method given in Kontsevich and Soibelman \cite{38} and van der Laan \cite{62,63}, we exhibit the deformation complex as well as its $L_{\infty}$-structure.

Rota-Baxter algebras, formerly referred to as Baxter algebras, emerged from Baxter’s research in probability theory \cite{4}. Subsequent to Baxter’s work, scholars like Rota \cite{56}, and Cartier \cite{7}, among others, delved into this area, hence the appellation “Rota-Baxter algebras”. From 2000, renewed interest in the subject arose. Connes and Kreimer \cite{10} established a significant connection between Rota-Baxter algebras and mathematical physics by utilizing Hopf algebra techniques in the renormalization of quantum field theory, leading to an algebraic framework for the BPHZ renormalization process \cite{10}. Guo and Keigher \cite{30,31} realised free commutative Rota-Baxter algebras via mixable shuffles, which, under the name of quasi-shuffles, are closed related to multi-zeta values as shown by Hoffmann \cite{35}. Semenov-Tian-Shansky demonstrated in \cite{58} that a skew-symmetric solution to the classical Yang-Baxter equation within a Lie algebra is essentially a Rota-Baxter operator of weight zero on this Lie algebra. This correspondence extends to associative algebras, as illustrated by Aguiar \cite{1} and Bai \cite{3}. Moreover, Ebrahimi-Fard discovered that a Rota-Baxter algebra induces a dendriform algebra in the sense of Loday \cite{44}. Presently, Rota-Baxter algebras find myriad applications and connections across various mathematical domains such as combinatorics \cite{57}, multiple zeta values in number theory \cite{34},
operad theory [2, 5], and Hopf algebras [10]. For basic theory of Rota-Baxter algebras, readers are directed to the concise introduction [28] and the comprehensive monograph [29]. In this paper, we will introduce the concepts of cohomology of Rota-Baxter algebras and their natural homotopy versions. Exploring the applications of these concepts in the aforementioned areas related to Rota-Baxter algebras will be an intriguing question for the future.

The deformation theory and cohomology theory of Rota-Baxter algebras had been absent for a long time despite the importance of Rota-Baxter algebras. Recently there are some breakthroughs in this direction. Tang, Bai, Guo and Sheng [61] developed the deformation theory and cohomology theory of $O$-operators (also called relative Rota-Baxter operators) on Lie algebras, with applications to Rota-Baxter Lie algebras in mind. Das [11] developed a similar theory for Rota-Baxter algebras of weight zero. Lazarev, Sheng and Tang [42] succeeded in establishing deformation theory and cohomology theory of relative Rota-Baxter Lie algebras of weight zero and found applications to triangular Lie bialgebras. They determined the $L_\infty$-algebra that controls deformations of a relative Rota-Baxter Lie algebra and introduced the notion of a homotopy relative Rota-Baxter Lie algebra. The same group of authors also related homotopy relative Rota-Baxter Lie algebras and triangular $L_\infty$-bialgebras via a functorial approach to Voronov’s higher derived brackets construction [43]. Later Das and Misha also determined the $L_\infty$-structures controlling deformation of Rota-Baxter operators of weight zero on associative algebras [13]. These work all concern Rota-Baxter operators of weight zero.

A recent paper by Pei, Sheng, Tang and Zhao [54] considered cohomologies of crossed homomorphisms for Lie algebras and they found a dg Lie algebra controlling deformations of crossed homomorphisms. Another exciting progress in this subject is the introduction of the notion of Rota-Baxter Lie groups by Guo, Lang and Sheng [32]; as a successor to this work, Jiang, Sheng and Zhu considered cohomology of Rota-Baxter operators of weight 1 on Lie groups and Lie algebras and relationship between them [36, 37]. Das [12] investigated cohomology of Rota-Baxter operators of arbitrary weight on associative algebras. It seems that these are the only papers which investigates Rota-Baxter operators of nonzero weight (for a related work on differential algebras of nonzero weight, see [33]). In these papers, the authors dealt with the deformations of only the Rota-Baxter operators with the Lie algebra or associative algebra structure unchanged.

The goal of the present paper is to study simultaneous deformations of Rota-Baxter operators of nonzero weight and of associative algebra structures. One of the reasons is that when one structure remains undeformed, the homotopy version obtained could not be a minimal model of the operad of Rota-Baxter Lie algebras or Rota-Baxter associative algebras.

Several remarks are in order.

We worked out the minimal model for Rota-Baxter algebras in a way which is somehow converse to the classical approach. Grosso modo, our method is as follows: Given an algebraic structure on a space $V$ realised as an algebra over an operad, by considering the formal deformations of this algebraic structure, we first construct the deformation complex and using ad hoc method, find an $L_\infty$-structure on the underlying graded space of this complex such that the Maurer-Cartan elements are in bijection with the algebraic structures on $V$; when $V$ is graded, we define a homotopy version of this algebraic structure as Maurer-Cartan elements in the $L_\infty$-algebra constructed above; finally under suitable conditions, we could show that the operad governing the homotopy version is a minimal model of the original operad. However, this paper, as a completely new version of [67], is written in operadic language which does not reflect completely our original method. For instance, in the previous version [67], our proof of the constructed $L_\infty$-structure is elementary and ad hoc. For the reader less prepared in operad theory, we suggest to her/him the reading of [67].
It should be mentioned that there is another way to derive the $L_\infty$ structure in the literature, say, the derived bracket technique \cite{39, 64, 65}. The above mentioned papers of Sheng et al. use this method as a main tool.

It might be appropriate to explain here the relationship of our result with the paper of Dotsenko and Khoroshkin \cite{15}. In that paper, the authors tried to deform the minimal model of the corresponding monomial operads obtained by Gröbner basis of the Rota-Baxter operad and they got the generators of the operad of homotopy Rota-Baxter algebras. It seems that it is not easy to obtain all the relations. While our generators of homotopy Rota-Baxter algebras are the same, we could determine all the relations in an indirect way with the aid of the $L_\infty$-structure on the deformation complex found initially using our ad hoc method. However, it is fair to say that our method to verify the minimal model has been inspired from Dotsenko and Khoroshkin \cite{15}. Dotsenko kindly pointed out another proof based on the paper \cite{15}; see Remark 3.13.

There remain several problems unsolved so far. It is still open how to define homotopy morphisms between homotopy Rota-Baxter algebras and establish a homotopy version of the descendent property for Rota-Baxter algebras (see Propositions 1.19 and 1.20), that is, to found the minimal model of the corresponding coloured operad. We also need to define homotopy Rota-Baxter bimodules over homotopy Rota-Baxter algebras. These problems will be attacked in a forthcoming paper.

It would be an interesting problem to give a general approach for operated algebras in the sense of Guo \cite{27} and other algebraic structures. There are at least two interesting concrete problems. Loday \cite{45} asked to develop a Koszul duality theory for differential algebras with nonzero weight, whose operad is not Koszul. Using our method, we succeeded in finding the minimal model of the operad of differential algebras with nonzero weight \cite{8}, thus continuing \cite{33} and answering the question of Loday \cite{45}. Another problem is to show that the (coloured) operad of (relative) homotopy Rota-Baxter Lie algebras introduced by Lazarev, Sheng and Tang \cite{42} is the minimal model of that of (relative) Rota-Baxter Lie algebras. We are working on this project \cite{9}.

This paper is organised as follows. In the first section, we collection relevant notions and facts about $L_\infty$-algebras and homotopy (co)operads scattered in the literature in order to fix the notations and we also present an account about classical theory of Rota-Baxter algebras. In the second section a homotopy cooperad is introduced and in the third section, the cobar construction of this homotopy cooperad is shown to be the minimal model of the operad of Rota-Baxter algebras of arbitrary weight, so this homotopy cooperad can be considered as the Koszul dual homotopy cooperad of the operad of Rota-Baxter algebras. The notion of homotopy Rota-Baxter algebras of arbitrary weight is made explicit in the fourth section. In the fifth section, the deformation complex and its $L_\infty$-algebra structure are derived. In the sixth section, the cohomology theory of Rota-Baxter algebras introduced in \cite{11, 12, 13, 67} is recovered and an example is also included.

Throughout this paper, let $k$ be a field of characteristic $0$. All vector spaces are defined over $k$, all tensor products and Hom-spaces are taken over $k$. We assume that the reader is familiar with the theory of operads \cite{51, 46, 6}.

1. Preliminaries

1.1. $L_\infty$-algebras and Maurer-Cartan elements.

In this subsection, we will recall some preliminaries on differential graded Lie algebras and $L_\infty$-algebras. For more background on differential graded Lie algebras and $L_\infty$-algebras, we refer the reader to \cite{60, 61, 62, 24}.
Then

Remark 1.2. Let us consider the generalised Jacobi identity for
\[
\text{generalised anti-symmetry .}
\]

conditions: for arbitrary \( u_1 \cdots u_n \) \( \sum \). \( 1 \\leq i \leq n \)
family of graded linear operators \( l_n : L^{\otimes n} \to L, n \geq 1 \) with \( |l_n| = n - 2 \), subject to the following conditions: for arbitrary \( n \geq 1 \), \( \sigma \in S_n \) and \( x_1, \ldots, x_n \in L \),

(i) (generalised anti-symmetry)
\[
l_n(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}) = \chi(\sigma; x_1, \ldots, x_n) l_n(x_1 \otimes \cdots \otimes x_n);
\]

(ii) (generalised Jacobi identity)
\[
\sum_{i=1}^{n} \sum_{\sigma \in \text{Sh}(i, n - i)} \chi(\sigma; x_1, \ldots, x_n)(-1)^{i(n-i)} l_{n-i+1}(l_i(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}) \otimes x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(n)}) = 0,
\]

where \( \text{Sh}(i, n - i) \) is the set of \( (i, n - i) \) shuffles, that is, permutations \( \sigma \in S_n \) such that \( \sigma(1) < \cdots < \sigma(i) \) and \( \sigma(i+1) < \cdots < \sigma(n) \).

Then \( (L, \{l_n\}_{n \geq 1}) \) is called an \( L_\infty \)-algebra.

Definition 1.1. Let \( L = \bigoplus_{i \in \mathbb{Z}} L_i \) be a graded space over \( k \). Assume that \( L \) is endowed with a
family of graded linear operators \( l_n : L^{\otimes n} \to L, n \geq 1 \) with \( |l_n| = n - 2 \), subject to the following
conditions: for arbitrary \( n \geq 1 \), \( \sigma \in S_n \) and \( x_1, \ldots, x_n \in L \),

Remark 1.2. Let us consider the generalised Jacobi identity for \( n \leq 3 \) with the assumption of
generalised anti-symmetry.

(i) \( n = 1, l_1 \circ l_1 = 0 \), that is, \( l_1 \) is a differential,
(ii) \( n = 2, l_1 \circ l_2 = l_2 \circ (l_1 \otimes \text{Id} + \text{Id} \otimes l_1) \), that is \( l_1 \) is a derivation for \( l_2 \),
(iii) \( n = 3 \), for homogeneous elements \( x_1, x_2, x_3 \in L \)
\[
l_2(l_2(x_1 \otimes x_2) \otimes x_3) + (-1)^{|x_1||x_2|+|x_3|} l_2(l_2(x_2 \otimes x_3) \otimes x_1) + (-1)^{|x_3|(|x_1|+|x_2|)} l_2(l_2(x_3 \otimes x_1) \otimes x_2) = -l_2(l_2(x_1 \otimes x_2) \otimes x_3) - l_2(l_2(x_1 \otimes x_2) \otimes x_3) + (-1)^{|x_1|} l_2(x_1 \otimes l_2(x_2) \otimes x_3) + (-1)^{|x_1|} l_2(x_1 \otimes l_2(x_2) \otimes x_3)\]

that is, \( l_2 \) satisfies the Jacobi identity up to homotopy.

In particular, if all \( l_n = 0 \) with \( n \geq 3 \), then \( (L, l_1, l_2) \) is just a dg Lie algebra.

Definition 1.3. [H2] A weakly filtered \( L_\infty \)-algebra is a pair \( (L, \mathcal{F}_*L) \), where \( L \) is an \( L_\infty \)-algebra and
\( \mathcal{F}_*L \) is a descending filtration \( L = \mathcal{F}_1L \supset \cdots \supset \mathcal{F}_nL \supset \cdots \) of graded subspaces in \( L \) satisfying

(1) there exists \( n \geq 1 \) such that \( l_k(L^{\otimes k}) \subset \mathcal{F}_kL \) holds for all \( k \geq n \).
(2) the graded space \( L \) is complete with respect to this filtration, i.e., there is an isomorphism
of graded spaces \( L \cong \varinjlim_n(L/\mathcal{F}_nL) \).

One can also define Maurer-Cartan elements in weakly complete \( L_\infty \)-algebras.
Definition 1.4. [\[42\]] Let \((L, \{l_n\}_{n \geq 1}, \mathcal{F}.L)\) be a weakly filtered \(L_\infty\)-algebra. An element \(\alpha \in L_{-1}\) is called a Maurer-Cartan element if it satisfies the Maurer-Cartan equation:

\[
\sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}l_n(\alpha^\otimes n) = 0.
\]

Proposition 1.5 (Twisting procedure). [\[42\]] Given a Maurer-Cartan element \(\alpha\) in the weakly filtered \(L_\infty\)-algebra \(L\), one can introduce a new \(L_\infty\)-structure \(\{l_n^\alpha\}_{n \geq 1}\) on graded space \(L\), where \(l_n^\alpha : L^\otimes n \to L\) is defined as:

\[
l_n^\alpha(x_1 \otimes \cdots \otimes x_n) = \sum_{i=0}^{\infty} \frac{1}{i!}(-1)^{i+n+i(i-1)/2}l_{n+i}(\alpha^\otimes i \otimes x_1 \otimes \cdots \otimes x_n), \quad \forall x_1, \ldots, x_n \in L,
\]

The new \(L_\infty\)-algebra \((L, \{l_n^\alpha\}_{n \geq 1})\) is called the twisted \(L_\infty\)-algebra (by the Maurer-Cartan element \(\alpha\)).

Remark 1.6. (i) The signs in Definition 1.4 and Proposition 1.5 are different from those appearing in [\[42\]], as the conventions in [\[42\]] are essentially about \(L_\infty[1]\)-algebras [\[54, 59\]]. We refer the reader to [\[66\]] for the translation between \(L_\infty\)-structures and \(L_\infty[1]\)-structures.

(ii) The condition of being weakly filtered ensures the convergence of the infinite sums in Definition 1.4 and Proposition 1.5.

1.2. Homotopy (co)operads.

In this subsection, we collect some basics on non-symmetric homotopy (co)operads, as they are scattered in several references [\[42, 52, 53, 55\]], in particular, we also explain how to obtain \(L_\infty\)-structures from homotopy operads, in particular, from convolution homotopy operads.

Since we will work with non-symmetric homotopy (co)operads, we omit the adjective “non-symmetric” everywhere.

Recall that a graded collection \(O = \{O(n)\}_{n \geq 1}\) is a family of graded space indexed by positive integers, i.e., each \(O(n)\) itself being a graded space. For any \(v \in O(n)\), the index \(n\) is called the arity of \(v\). The suspension of \(O\), denoted by \(sO\) is defined to be the graded collection \(\{sO(n)\}_{n \geq 1}\). In the same way, one has the desuspension \(s^{-1}O\) of the graded collection \(O\).

We need some preliminaries about trees. For any tree \(T\), denote the weight (= number of vertices) and arity (= number of leaves) of \(T\) to be \(\omega(T)\) and \(\alpha(T)\) respectively. A planar tree is said to be reduced if each of its vertices has at least one leaf. As we only consider planar reduced trees, we also remove the adjectives “planar reduced” everywhere. Write \(\mathcal{X}\) to be the set of all trees with weight \(\geq 1\) and for arbitrary \(n \geq 1\), denote by \(\mathcal{X}(n)\) the set of trees of weight \(n\). Since trees are planar, each vertex in a tree has a total order on its inputs which will be drawn clockwise.

By the existence of the root, there is a natural induced total order on the set of all vertices of a given tree \(T \in \mathcal{X}\), which is given by counting the vertices starting from the root clockwise along the tree. We call this order the planar order.

Let \(T'\) be a divisor of \(T\). Define \(T/T'\) to be the tree obtained from \(T\) by replacing \(T'\) by a corolla (say, a tree with only one vertex) of arity \(\alpha(T')\). There is a natural permutation \(\sigma = \sigma(T, T') \in S_{\omega(T)}\) associated to the pair \((T, T')\) defined as follows. Assume the ordered set \(\{v_1 < \cdots < v_n\}\) to be the sequence of all vertices of \(T\) in the planar order and \(\omega(T') = j\). Let \(v'\) be the vertex in \(T/T'\) corresponding to the divisor \(T'\) in \(T\) and the serial number of \(v'\) in \(T/T'\) is \(i\) in the planar order (so there are \(i - 1\) vertices “before” \(T'\)). Then define \(\sigma = \sigma(T, T') \in S_n\) to be the unique permutation which does not permute the vertices \(v_1, \ldots, v_{i-1}\), and such that the
ordered set \( \{ v_{\sigma(i)} < \cdots < v_{\sigma(i+j-1)} \} \) is exactly the planar ordered set of all vertices of \( T' \) and the ordered set \( \{ v_1 < \cdots < v_{i-1} < v' < v_{\sigma(i+j)} < \cdots < v_\sigma(n) \} \) is exactly the planar ordered set of all vertices in the tree \( T/T' \).

Let \( \mathcal{P} = \{ \mathcal{P}(n) \}_{n \geq 1} \) be a graded collection. Let \( T \in \mathfrak{T} \) and \( \{ v_1 < \cdots < v_n \} \) be the set of all vertices of \( T \) in the planar order. Define \( \mathcal{P}^{\otimes T} \) to be \( \mathcal{P}(\alpha(v_1)) \otimes \cdots \otimes \mathcal{P}(\alpha(v_n)) \); morally an element in \( \mathcal{P}^{\otimes T} \) is a linear combination of decorated trees whose underlying tree is the tree \( T \) and each vertex \( v_i \) is decorated by an element of \( \mathcal{P}(\alpha(v_i)) \).

**Definition 1.7.** A homotopy operad structure on a graded collection \( \mathcal{P} = \{ \mathcal{P}(n) \}_{n \geq 1} \) consists of a family of operations

\[
\{ m_T : \mathcal{P}^{\otimes T} \to \mathcal{P}(\alpha(T)) \}_{T \in \mathfrak{T}}
\]

with \( |m_T| = \omega(T) - 2 \) such that the equation

\[
\sum_{T' \subset T} (-1)^{i-1+jk} \text{sgn}(\sigma(T, T')) m_{T/T'} \circ (\text{Id}^{\otimes i-1} \otimes m_{T'} \otimes \text{Id}^{\otimes k}) \circ r_{\sigma(T, T')} = 0
\]

holds for any \( T \in \mathfrak{T} \), where \( T' \) runs through the set of all subtrees of \( T \), \( i \) is the serial number of the vertex \( v' \) in \( T/T' \), \( j = \omega(T') \), \( k = \omega(T) - i - j \), and where \( r_{\sigma(T, T')} \) denoted the right action by \( \sigma = \sigma(T, T') \), that is,

\[
r_{\sigma}(x_1 \otimes \cdots \otimes x_n) = \varepsilon(\sigma; x_1, \ldots, x_n) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.
\]

Given two homotopy operads, a strict morphism between them is a morphism of graded collections compatible with all operations \( m_T, T \in \mathfrak{T} \).

Let \( I \) be the collection with \( I(1) = k \) and \( I(n) = 0 \) for any \( n \neq 1 \). The collection \( I \) is endowed with a homotopy operad structure in the natural way, that is, \( m_T : I(1) \otimes I(1) \to I(1) \) is given by the identity, when \( T \) is the tree with two vertices and one unique leaf, and \( m_T \) vanishes otherwise.

A homotopy operad \( \mathcal{P} \) is called strictly unital if there exists a strict morphism of homotopy operads \( \eta : I \to \mathcal{P} \) such that for each \( n \geq 1 \), the compositions

\[
\mathcal{P}(n) \cong \mathcal{P}(n) \otimes I(1) \xrightarrow{\text{Id} \otimes \eta} \mathcal{P}(n) \otimes \mathcal{P}(1) \xrightarrow{m_{T_{1,i}}} \mathcal{P}(n)
\]

and

\[
\mathcal{P}(n) \cong I(1) \otimes \mathcal{P}(n) \xrightarrow{\eta \otimes \text{Id}} \mathcal{P}(1) \otimes \mathcal{P}(n) \xrightarrow{m_{T_2}} \mathcal{P}(n)
\]

are identity maps on \( \mathcal{P}(n) \), where \( T_{1,i} \) with \( 1 \leq i \leq n \) is the tree of weight 2, arity \( n \) with its second vertex having arity 1 and connecting to the first vertex on its \( i \)-th leaf, and \( T_2 \) is the tree of weight 2, arity \( n \) whose first vertex has arity 1. Furthermore, for any tree \( T \) with \( \omega(T) \neq 2 \), \( m_T(\text{Id}^{\otimes i-1} \otimes \eta \otimes \text{Id}^{\otimes \omega(T)-i}) \) is required to be 0 for all \( 1 \leq i \leq \omega(T) \). A strictly unital homotopy operad \( \mathcal{P} \) is called augmented if there exists a strict morphism of homotopy operads \( \varepsilon : \mathcal{P} \to I \) such that \( \varepsilon \circ \eta = \text{Id}_T \).

If a homotopy operad \( \mathcal{P} \) satisfies \( m_T = 0 \) for all \( T \in \mathfrak{T} \) with \( \omega(T) \geq 3 \), then \( \mathcal{P} \) is just a nonunital dg operad in the sense of Markl [49]. Let’s recall the construction of free operads here. Let “|” represent the trivial tree with weight 0. For a graded collection \( \mathcal{P} \), define \( \mathcal{P}^{\otimes |} = I \). Denote \( \mathfrak{T}^+ = \{ | \} \cup \mathfrak{T} \). Then the unital free graded operad generated by \( \mathcal{P} \) is exactly the graded collection \( \bigoplus_{T \in \mathfrak{T}^+} \mathcal{P}^{\otimes T} \) equipped with the inherent composition operations induced by tree grafting, where \( \mathcal{P}^{\otimes T} \) serves as the operadic unit.

There is a natural \( L_\infty \)-algebra associated with a homotopy operad \( \mathcal{P} = \{ \mathcal{P}(n) \}_{n \geq 1} \). Denote \( \mathcal{P}^{\prod} := \prod_{n=1}^{\infty} \mathcal{P}(n) \). For each \( n \geq 1 \), define operations \( m_n = \sum_{T \in \mathfrak{T}(n)} m_T : (\mathcal{P}^{\prod})^\otimes n \to \mathcal{P}^{\prod} \) and \( l_n \) is
the anti-symmetrization of \( m_n \), i.e.,

\[
l_n(x_1 \otimes \cdots \otimes x_n) = \sum_{\sigma \in S_n} \chi(\sigma; x_1, \ldots, x_n)m_n(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}).
\]

**Proposition 1.10** ([22, 23, 25]). Let \( \mathcal{P} \) be a homotopy operad. Then \( (\mathcal{P})^{\otimes \infty}, \{l_n\}_{n \geq 1} \) is an \( L_\infty \)-algebra. In particular, if \( \mathcal{P} \) is a dg operad, \( \mathcal{P}^{\otimes \infty} \) is just a dg Lie algebra.

We shall need a fact about dg operads.

**Definition 1.9.** Let \( \mathcal{P} \) be a (nonunital) dg operad in the sense of Markl [19]. For any \( f \in \mathcal{P}(m) \) and \( g_1 \in \mathcal{P}(l_1), \ldots, g_n \in \mathcal{P}(l_n) \) with \( 1 \leq n \leq m \), define

\[
f\{g_1, \ldots, g_n\} = \sum_{i_j > j_{j-1} + j_{j-2} + \cdots + j_2 + j_1} \left( (f \circ_{i_1} g_1) \circ_{i_2} g_2 \cdots \right) \circ_{i_n} g_n.
\]

It is called the brace operation on \( \mathcal{P}^{\otimes \infty} \). For \( f \in \mathcal{P}(m), g \in \mathcal{P}(m) \), define

\[
[f, g]_G = f\{g\} - (-1)^{|g||f|} g\{f\} \in \mathcal{P}(m + n - 1),
\]

called the Gerstenhaber bracket of \( f \) and \( g \).

The operation \( l_2 \) in the dg Lie algebra \( \mathcal{P}^{\otimes \infty} \) is exactly the Gerstenhaber bracket defined above.

The brace operation in a dg operad \( \mathcal{P} \) satisfies the following pre-Jacobi identity:

**Proposition 1.10** ([22, 23, 25]). For any homogeneous elements \( f, g_1, \ldots, g_m, h_1, \ldots, h_n \) in \( \mathcal{P}^{\otimes \infty} \), we have

\[
(5) \left( f\{g_1, \ldots, g_m\}\right)\{h_1, \ldots, h_n\} = \sum_{0 \leq i_1 < j_1 \leq j_2 < j_3 < \cdots < i_m \leq j_m \leq n} (-1)^{\sum_{l=1}^m (|g_{i_l}|)(\sum_{j=1}^l (|h_j|))} f\{h_{1,i_1}, g_1\{h_{1,j_1+1}, h_{2,i_1+1,j_1}\}, \ldots, g_m\{h_{m,i_1+1,j_m}, h_{m+1,j_m}\}\}.
\]

In particular, we have

\[
(6) \quad \left( f\{g\}\right)\{h\} = f\{g\{h\}\} + f\{g, h\} + (-1)^{|g||h|} f\{h, g\}.
\]

Given a dg operad \( \mathcal{P} \), for each tree \( T \), one can define the composition \( m_{\mathcal{P}}^T : \mathcal{P}^{\otimes T} \to \mathcal{P}(\alpha(T)) \) in \( \mathcal{P} \) along \( T \) as follows: for \( \omega(T) = 1 \), \( m_{\mathcal{P}}^T = \text{Id} \); for \( \omega(T) = 2 \), \( m_{\mathcal{P}}^T = m_T \); for \( \omega(T) \geq 3 \), write \( T \) as the grafting of a subtree \( T' \), whose vertex set is that of \( T \) except the last one, with the corolla whose unique vertex is exactly the last vertex of \( T \) in the planar order, then define \( m_{\mathcal{P}}^T = m_{T/T'} \circ (m_{\mathcal{P}}^{T'} \otimes \text{Id}) \), where \( m_{\mathcal{P}}^{T'} \) is obtained by induction.

Dualizing the definition of homotopy operads, one has the notion of homotopy cooperads.

**Definition 1.11.** Let \( C = \{C(n)\}_{n \geq 1} \) be a graded collection. A homotopy cooperad structure on \( C \) consists of a family of operations \( \{\Delta_T : C(n) \to C^{\otimes T}\}_{T \in \mathcal{I}} \) with \( |\Delta_T| = \omega(T) - 2 \) such that for any \( c \in C, \Delta_T(c) = 0 \) for almost all but finitely many \( T \in \mathcal{I} \), and the family of operations \( \{\Delta_T\}_{T \in \mathcal{I}} \) satisfies the following identity:

\[
\sum_T \text{sgn}(\sigma(T, T'))(-1)^{i+1+j} r_{\sigma(T, T')} \circ (\text{Id}^{\otimes i} \otimes \Delta_T \otimes \text{Id}^{\otimes k}) \circ \Delta_T/T' = 0
\]

for any \( T \in \mathcal{I} \), where \( T', i, j, k \) have the same meanings as for homotopy operads.
The graded collection $I$ has a natural homotopy cooperad structure, that is, $\Delta_T : I(1) \to I(1) \otimes I(1)$ is given by the identity, when $T$ is the tree with two vertices and one unique leaf, and $\Delta_T$ vanishes otherwise. A homotopy cooperad $C$ is called strictly counital if there exists a strict morphism of homotopy cooperad $\varepsilon : C \to I$ such that

$$C(n) \xrightarrow{\Delta_{T_{1,i}}} C(n) \otimes C(1) \xrightarrow{\text{Id} \otimes \varepsilon} C(n) \otimes I(1) \cong C(n)$$

and

$$C(n) \xrightarrow{T_2} C(1) \otimes C(n) \xrightarrow{\varepsilon \otimes \text{Id}} I(1) \otimes C(n) \cong C(n)$$

are identity maps on $C(n)$, where $T_{1,i}$ with $1 \leq i \leq n$ and $T_2$ are the notations used before. Additionally, for any planar tree $T$ with $\alpha(T) = n$, $\omega(T) = m \neq 2, 1 \leq j \leq m$, the composition $C(n) \xrightarrow{\Delta_T} C(\alpha(v_1)) \otimes \ldots \otimes C(\alpha(v_m)) \xrightarrow{\text{Id}^j \otimes \varepsilon \otimes \text{Id}^{m-j}} C(\alpha(v_1)) \otimes \ldots \otimes C(\alpha(v_{j-1})) \otimes I(1) \otimes C(\alpha(v_{j+1})) \otimes \ldots \otimes C(\alpha(v_m))$

is required to be 0, where $v_1, \ldots, v_m$ are vertices of the tree $T$.

A homotopy cooperad $C$ is called coaugmented if there exists a strict morphism of homotopy cooperads $\eta : I \to C$ such that $\varepsilon \circ \eta = \text{Id}_T$. For a coaugmented homotopy cooperad $C$, the graded collection $\overline{C} = \text{Ker}(\varepsilon)$ endowed with operations $\{\overline{\Delta}_T\}_{T \in \mathcal{I}}$ is naturally a homotopy cooperad, where $\overline{\Delta}_T$ is the the restriction of operation $\Delta_T$ on $\overline{C}$.

A homotopy cooperad $E = \{E(n)\}_{n \geq 1}$ such that $\{\overline{\Delta}_T\}$ vanish for all $\omega(T) \geq 3$ is exactly a noncounital dg cooperad in the sense of Markl [55].

For a (noncounital) dg cooperad $E$, on can define the cocomposition $\Delta^H_E : C(\alpha(T)) \to E \otimes T$ along a tree $T$ in the dual way as the composition $m^T_P$ along $T$ for a dg operad $P$.

**Proposition-Definition 1.1.** [52] Let $C$ be a homotopy cooperad and $E$ be a dg cooperad. Then the graded collection $C \otimes E$ with $(C \otimes E)(n) := C(n) \otimes E(n)$, $\forall n \geq 1$ has a natural structure of homotopy cooperad as follows:

(i) For any tree $T \in \mathcal{I}$ of weight 1 and arity $n$,

$$\Delta^H_T(c \otimes e) = \Delta^C_T(c) \otimes e + (-1)^{|c|} c \otimes d_E e$$

for arbitrary homogeneous elements $c \in C(n), e \in E(n)$;

(ii) for any tree $T$ of weight $n \geq 2$, define

$$\Delta^H_T(c \otimes e) = (-1)^{\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} |e_k| |c_j|} (c_1 \otimes e_1) \otimes \ldots \otimes (c_n \otimes e_n) \in (C \otimes E)^{\otimes T},$$

with $c_1 \otimes \ldots \otimes c_n = \Delta^C_T(c) \in C^{\otimes T}$ and $e_1 \otimes \ldots \otimes e_n = \Delta^E_T(e) \in E^{\otimes T}$, where $\Delta^E_T$ is the cocomposition in $E$ along $T$.

The new homotopy cooperad is called the Hadamard product of $C$ and $E$, and denoted by $C \otimes_{\text{H}} E$.

Define $S = \text{End}^C_{ks}$ to be the graded cooperad whose underlying graded collection is given as $S(n) = \text{Hom}((ks)^{\otimes n}, ks), n \geq 1$. Denote $\delta_n$ to be the map in $S(n)$ which takes $s^{\otimes n}$ to $s$. The cooperad structure is given as

$$\Delta_T(\delta_n) = (-1)^{(j-1)(i-1)} \delta_{n-i+1} \otimes \delta_i \in (S)^{\otimes T}$$

for any tree $T$ of weight 2 whose second vertex is connected with the $j$-th leaf of its first vertex. We also define $S^{-1}$ to be the graded cooperad whose underlying graded collection is $S^{-1}(n) = \text{Hom}((ks^{-1})^{\otimes n}, s^{-1})$ for all $n \geq 1$ and the cooperad structure is given as

$$\Delta_T(\varepsilon_n) = (-1)^{(i-1)(n-i-1)} \varepsilon_{n-i+1} \otimes \varepsilon_i \in (S^{-1})^{\otimes T},$$
where $\varepsilon_n \in S^{-1}(n)$ is the map which takes $(s^{-1})^\otimes n$ to $s^{-1}$ and $T$ is the same as before. It is easy to see $S \otimes H S^{-1} \cong S^{-1} \otimes H S =: As^\vee$. Notice that for any homotopy cooperad $C$, we have $C \otimes_H As^\vee \cong C \cong As^\vee \otimes_H C$.

**Definition 1.12.** Let $C$ be a homotopy cooperad. Define the operadic suspension (resp. desuspension) of $C$ to be the homotopy cooperad $C \otimes_H S$ (resp. $C \otimes_H S^{-1}$), denoted as $\mathcal{P}C$ (resp. $\mathcal{P}^{-1}C$).

**Definition 1.13.** Let $C = \{C(n)\}_{n \geq 1}$ be a coaugmented homotopy cooperad. The cobar construction of $C$, denoted by $\Omega C$, is the free graded cooperad generated by the graded collection $s^{-1}C$, endowed with the differential $\partial$ which is lifted from $\partial : s^{-1}C \to \Omega C$ given by for any $f \in C(n)$,

$$\partial(s^{-1} f) = - \sum_{T \in \mathfrak{T}(n)} \sum_{(s^{-1})^n} \Delta_T(f),$$

This provides an alternative definition for homotopy operads. In fact, a graded collection $\mathcal{C} = \{\mathcal{C}(n)\}_{n \geq 1}$ is a homotopy cooperad if and only if the free graded cooperad generated by $s^{-1}C$ (also called the cobar construction of $C = \mathcal{C} \otimes \mathbb{I}$) is endowed with a differential such that it becomes a dg cooperad.

**Proposition 1.14.** [52, 53] Let $C$ be a homotopy cooperad and $\mathcal{P}$ be a dg operad. Then the graded collection $\text{Hom}(C, \mathcal{P}) = \{\text{Hom}(C(n), \mathcal{P}(n))\}_{n \geq 1}$ has a natural homotopy operad structure, which is defined in the following way:

(i) For any $T \in \mathfrak{T}$ with $\omega(T) = 1$,

$$m_T(f) = d_{\mathcal{P}} \circ f - (-1)^{|f|} f \circ \Delta_T$$

for $f \in \text{Hom}(C, \mathcal{P})(n)$;

(ii) for any $T \in \mathfrak{T}$ with $\omega(T) \geq 2$,

$$m_T(f_1 \otimes \cdots \otimes f_n) = (-1)^{\frac{n(n-1)+1+n}{2}} \sum_{k=1}^n (-1)^{|f_i|} m^T_{\mathcal{P}} \circ (f_1 \otimes \cdots \otimes f_i) \circ \Delta_T,$$

where $m^T_{\mathcal{P}}$ is the composition in $\mathcal{P}$ along $T$.

**Proposition 1.15.** [52, 53] Let $C$ be a coaugmented homotopy cooperad and $\mathcal{P}$ be a unital dg operad. Then there is a natural bijection:

$$\text{Hom}_{\text{algOp}}(\Omega C, \mathcal{P}) \cong MC\left(\text{Hom}(\mathcal{C}, \mathcal{P})^\Pi\right),$$

where the left-hand side is the set of morphisms of unital dg operads from $\Omega C$ to $\mathcal{P}$ and the right-hand side is the set of Maurer-Cartan elements in the $L_\infty$-algebra $\text{Hom}(\mathcal{C}, \mathcal{P})^\Pi$.

### 1.3. Rota-Baxter algebras and their bimodules.

**Definition 1.16.** Let $(A, \mu = \cdot)$ be an associative algebra over field $k$ and $\lambda \in k$. A linear operator $T : A \to A$ is said to be a Rota-Baxter operator of weight $\lambda$ if it satisfies

$$T(a) \cdot T(b) = T(a \cdot T(b) + T(a) \cdot b + \lambda a \cdot b)$$

for any $a, b \in A$, or in terms of maps

$$\mu \circ (T \otimes T) = T \circ (\text{Id} \otimes T + T \otimes \text{Id}) + \lambda T \circ \mu.$$

In this case, $(A, \mu, T)$ is called a Rota-Baxter algebra of weight $\lambda$. Denote by $\text{RBA}_\lambda$ the category of Rota-Baxter algebras of weight $\lambda$ with obvious morphisms.
Remark 1.17. The operad for Rota-Baxter algebras of weight $\lambda$, denoted by $\mathcal{RB}_\lambda$, is generated by a unary operator $T$ and a binary operator $\mu$ with operadic relations generated by

$$\mu \circ_1 \mu - \mu \circ_2 \mu \quad \text{and} \quad (\mu \circ_1 T) \circ_2 T - (T \circ_1 \mu) \circ_1 T - (T \circ_1 \mu) \circ_2 T - \lambda T \circ_1 \mu.$$  

It is obvious that the operad $\mathcal{RB}_\lambda$ is not a Koszul operad, not even a quadratic operad.

Recall that for a Koszul operad $\mathcal{P}$, the minimal model of $\mathcal{P}$ is exactly $\Omega \mathcal{P}^!$, the cobar construction of its Koszul dual cooperad $\mathcal{P}^!$. However, since the operad $\mathcal{RB}_\lambda$ is not Koszul, its Koszul dual $\mathcal{RB}_\lambda^!$ should be a homotopy cooperad, rather than a genuine cooperad; we will exhibit this homotopy cooperad in Section 1.15.

Definition 1.18. Let $(A, \mu, T)$ be a Rota-Baxter algebra and $M$ be a bimodule over associative algebra $(A, \mu)$. We say that $M$ is a bimodule over Rota-Baxter algebra $(A, \mu, T)$ or a Rota-Baxter bimodule if $M$ is endowed with a linear operator $T_M : M \to M$ such that the following equations

\begin{align}
T(a)T_M(m) &= T_M(aT_M(m) + T(a)m + \lambda am), \\
T_M(m)T(a) &= T_M(mT(a) + T_M(m)a + \lambda ma).
\end{align}

hold for any $a \in A$ and $m \in M$.

Of course, $(A, \mu, T)$ itself is a bimodule over the Rota-Baxter algebra $(A, \mu, T)$, called the regular Rota-Baxter bimodule.

Rota-Baxter algebras and Rota-Baxter bimodules have some descendent properties.

Proposition 1.19 ([\cite{25}, Theorem 1.17]). Let $(A, \mu, T)$ be a Rota-Baxter algebra. Define a new binary operation as:

$$a \star b := a \cdot T(b) + T(a) \cdot b + \lambda a \cdot b$$

for any $a, b \in A$. Then

(i) the operation $\star$ is associative and $(A, \star)$ is a new associative algebra;

(ii) the triple $(A, \star, T)$ also forms a Rota-Baxter algebra of weight $\lambda$ and denote it by $A_\star$;

(iii) the map $T : (A, \star, T) \to (A, \mu, T)$ is a morphism of Rota-Baxter algebras.

One can also construct new Rota-Baxter bimodules from old ones.

Proposition 1.20. Let $(A, \mu, T)$ be a Rota-Baxter algebra of weight $\lambda$ and $(M, T_M)$ be a Rota-Baxter bimodule over it. We define a left action “$\triangleright$” and a right action “$\triangleleft$” of $A$ on $M$ as follows: for any $a \in A$, $m \in M$,

\begin{align}
(a \triangleright m) &= T(a)m - T_M(am), \\
(m \triangleleft a) &= mT(a) - T_M(ma).
\end{align}

Then these actions make $M$ into a Rota-Baxter bimodule over $A_\star$ and denote this new bimodule by $\triangleright M_\triangleleft$.

The easy proof of the above result is left to the reader.
2. The Koszul dual homotopy cooperad

In this section, we will construct a homotopy cooperad which can be considered as the Koszul dual of the operad for Rota-Baxter algebras, and we will prove that the cobar construction of this homotopy cooperad is exactly the minimal model for the operad of Rota-Baxter algebras in Section 3.

Define a graded collection $\mathcal{S}(\mathcal{RB}^i)$ by

$$\mathcal{S}(\mathcal{RB}^i)(n) = ku_n \oplus kv_n, \text{ with } |u_n| = 0, |v_n| = 1$$

for $n \geq 1$. Now, we put a coaugmented homotopy cooperad structure on $\mathcal{S}(\mathcal{RB}^i)$. Firstly, consider trees of arity $n \geq 1$ in the following list:

(I) Trees of weight two: for each $1 \leq j \leq n$ and $1 \leq i \leq n - j + 1$, there exists such a tree, which can be visualized as

```
                      j
                     ...
  n - j + 1
```

(II) Trees of weight $k + 1 \geq 3$ and of “height” two: in these trees, the first vertex in the planar order has arity $k$, the vertex connected to the $i$-th leaf of the first vertex has arity $r_i$ for each $1 \leq i \leq k$ (so $2 \leq k \leq n$ and $r_1 + \cdots + r_k = n$); these trees have the following picture:

```
                      k
                     ...
  r_1 r_2 ... r_k
```

where $k \geq 2, r_i \geq 1$ for all $k \geq i \geq 1$.

(III) Trees of weight $q + 1 \geq 3$ and of “height” three and there exists a unique vertex in the first two levels: in these trees, there are numbers $2 \leq q \leq p \leq n$, $1 \leq k_1 < \cdots < k_{q-1} < p$, $1 \leq i \leq r_1$ and $r_j \geq 1$ for all $1 \leq j \leq q$ such that the first vertex has arity $r_1$, the second vertex has arity $p$ and is connected to the $i$-th leaf of the first vertex and the other vertices connected to the $k_1$-th (resp. $k_2$-th, ..., $k_{q-1}$-th) leaf of the second vertex and with arity $r_2$ (resp. $r_3, \ldots, r_q$), so $r_1 + \cdots + r_q + p - q = n$; these trees are drawn in this way:

```
                      p
                     ...
  r_2 r_3 ... r_q
```

Now, we define a family of operations $\{\Delta_T : \mathcal{S}(\mathcal{RB}^i) \to \mathcal{S}(\mathcal{RB}^i)^{\otimes T}\}_{T \in \mathbb{Z}}$ as follows:
(1) For a tree $T$ of type (I) with $1 \leq j \leq n, 1 \leq i \leq n - j + 1$, define $\Delta_T(u_n) = u_{n-j+1} \otimes u_j$, which can be drawn as

\[
\Delta_T(u_n) = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
u_{n-j+1}
\end{array}
\]

and

\[
\Delta_T(v_n) = \begin{cases}
v_n \otimes u_1, & j = 1, \\
\lambda^{-1}v_{n-j+1} \otimes u_j, & 2 \leq j \leq n-1, \\
n^{-1}v_1 \otimes u_n + u_1 \otimes v_n, & j = n,
\end{cases}
\]

which can be pictured as

when $j = 1, \Delta_T(v_n) = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
v_{n-j+1}
\end{array}
\]

when $2 \leq j \leq n-1, \Delta_T(v_n) = \lambda^{j-1} \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
v_{n-j+1}
\end{array}
\]

when $j = n, \Delta_T(v_n) = \lambda^{-1} \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
v_{n-j+1}
\end{array}
\]

+ \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
v_{n-j+1}
\end{array}
\]

(2) For a tree $T$ of type (II) with $2 \leq k \leq n, r_1 + \cdots + r_k = n, r_1, \ldots, r_k \geq 1$, define

\[
\Delta_T(v_n) = (-1)^{\frac{k(k-1)}{2}} \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
v_{r_1} \\
v_{r_2} \\
v_{r_3} \\
v_{r_4} \\
v_{r_5} \\
u_k
\end{array}
\]

(3) For a tree $T$ of type (III) with $2 \leq q \leq p \leq n, 1 \leq k_1 < \cdots < k_{q-1} \leq p, r_1 + \cdots + r_q + p - q = n, 1 \leq i \leq r_1, r_1, \ldots, r_q \geq 1$, define

\[
\Delta_T(v_n) = (-1)^{\frac{q(q-1)}{2}} \lambda^{p-q} \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
v_{r_1} \\
v_{r_2} \\
v_{r_3} \\
v_{r_4} \\
v_{r_5} \\
u_k
\end{array}
\]

(4) All other components of $\Delta_T, T \in \mathcal{T}$ vanish.
Proposition 2.1. The graded collection $\mathcal{S}(\mathcal{RBA}^i)$ endowed with the operations $\{\Delta_I\}_{I \in \Sigma}$ introduced above forms a coaugmented homotopy cooperad, whose strict counit is the natural projection $\epsilon : \mathcal{S}(\mathcal{RBA}^i) \rightarrow \mathbf{k}u_1 \cong I$ and the coaugmentation is just the natural embedding $\eta : I \cong \mathbf{k}u_1 \hookrightarrow \mathcal{S}(\mathcal{RBA}^i)$.

Proof. One needs to show that the induced derivation $\partial$ on the cobar construction of $\mathcal{S}(\mathcal{RBA}^i)$, i.e., the free operad generated by $s^{-1}\mathcal{S}(\mathcal{RBA}^i)$ is a differential, that is, $\partial^2 = 0$.

Denote $s^{-1}u_n, n \geq 2$ (resp. $s^{-1}v_n, n \geq 1$) by $x_n$ (resp. $y_n$) which are the generators of $\Omega(\mathcal{S}(\mathcal{RBA}^i))$. Notice that $|x_n| = -1$ and $|y_n| = 0$. By the definition of cobar construction of coaugmented homotopy cooperads, the action of differential $\partial$ on generators $x_n, y_n$ is given by the following formulas: for $n \geq 2$,

$$\partial(x_n) = -\sum_{j=2}^{n-1} x_{n-j+1}\{x_j\};$$  

and for $n \geq 1$,

$$\partial(y_n) = -\sum_{k=2}^{n} \sum_{\substack{r_1 + \cdots + r_k = n, \cr r_1, \ldots, r_k \geq 1}} x_k\{y_{r_1}, \ldots, y_{r_k}\} + \sum_{2 \leq p < n, \cr 1 < q < p} \sum_{\substack{r_1 + \cdots + r_q + p = n, \cr r_1, \ldots, r_q > 1}} \lambda^{p-q} y_1\{x_p\{y_{r_2}, \ldots, y_{r_q}\}\}.$$  

Note that

$$\partial(y_1) = 0.$$  

We just need to see that $\partial^2 = 0$ holds on generators $x_n, n \geq 2$ and $y_n, n \geq 1$, which can be checked by direct computations. Firstly, note that the suboperad in $\Omega(\mathcal{S}(\mathcal{RBA}^i))$ generated by the collection $\{x_n\}_{n \geq 2}$ is exactly the operadic suspension of the dg operad that governs $A_\infty$-algebras [46]. Equation (14) precisely represents the formulas for the differential on generators in this dg operad. Therefore, the equation $\partial^2(x_n) = 0$ holds true.

We also have

$$\partial^2(y_n) = \partial\left(-\sum_{k=2}^{n} \sum_{\substack{r_1 + \cdots + r_k = n, \cr r_1, \ldots, r_k \geq 1}} x_k\{y_{r_1}, \ldots, y_{r_k}\} + \sum_{2 \leq p < n, \cr 1 < q < p} \sum_{\substack{r_1 + \cdots + r_q + p = n, \cr r_1, \ldots, r_q > 1}} \lambda^{p-q} y_1\{x_p\{y_{r_2}, \ldots, y_{r_q}\}\}\right).$$

As $\partial$ is a derivation and by (16),

$$\partial^2(y_n) = -\sum_{k=2}^{n} \sum_{\substack{r_1 + \cdots + r_k = n, \cr r_1, \ldots, r_k \geq 1}} \partial(x_k)\{y_{r_1}, \ldots, y_{r_k}\} + \sum_{k=2}^{n-1} \sum_{\substack{r_1 + \cdots + r_k = n, \cr r_1, \ldots, r_k \geq 1}} x_k\{y_{r_1}, \ldots, \partial(y_{r_j}), \ldots, y_{r_k}\}$$

$$+ \sum_{2 \leq p < n, \cr 1 < q < p} \sum_{\substack{r_1 + \cdots + r_q + p = n, \cr r_1, \ldots, r_q > 1}} \lambda^{p-q} \partial(y_1)\{x_p\{y_{r_2}, \ldots, y_{r_q}\}\}$$

$$+ \sum_{2 \leq p < n, \cr 1 < q < p} \sum_{\substack{r_1 + \cdots + r_q + p = n, \cr r_1, \ldots, r_q > 1}} \lambda^{p-q} y_1\{\partial(x_p)\{y_{r_2}, \ldots, y_{r_q}\}\}. $$
\[- \sum_{2 \leq p \leq n-1, \ 1 \leq q < p} \sum_{r_1, \ldots, r_q \geq 1} \frac{1}{q!} \ \sum_{j=2}^q A^{p-q} y_{r_1} \{ x_p \{ y_{r_2}, \ldots, \partial(y_{r_j}), \ldots, y_r \} \} \]

\[- \sum_{k=2}^n \sum_{r_1, \ldots, r_k \geq 1} \left( \sum_{j=2}^{k-1} \sum_{i=2}^{r_j} \sum_{l_1, \ldots, l_{r_j} \geq 1} \left( x_{k-i+1} \{ x_i \} \right) \{ y_{r_1}, \ldots, y_{r_k} \} \right) \]

\[= \sum_{k=2}^n \sum_{r_1, \ldots, r_k \geq 1} \sum_{j=1}^k \sum_{u=2}^k \sum_{l_1, \ldots, l_{r_j} \geq 1} x_k \{ y_{r_1}, \ldots, x_u \{ y_{l_1}, \ldots, y_{l_u} \}, \ldots, y_{r_k} \} \]

\[+ \sum_{k=2}^n \sum_{r_1, \ldots, r_k \geq 1} \sum_{j=1}^k \sum_{u=2}^k \sum_{l_1, \ldots, l_{r_j} \geq 1} A^{p-q} x_k \{ y_{r_1}, \ldots, y_{l_u} \} \{ x_p \{ y_{l_1}, \ldots, y_{l_u} \} \}, \ldots, y_{r_k} \} \]

\[= \sum_{2 \leq p \leq n-1, \ 1 \leq q < p} \sum_{r_1, \ldots, r_q \geq 1} \sum_{k=2}^r \sum_{l_1, \ldots, l_{r_k} \geq 1} A^{p-q} \left( x_{k-l_1} \{ y_{l_1}, \ldots, y_{l_k} \} \right) \{ x_p \{ y_{r_2}, \ldots, y_{r_k} \} \} \]

\[+ \sum_{2 \leq p \leq n, \ 1 \leq q < p} \sum_{r_1, \ldots, r_q \geq 1} \sum_{k=2}^r \sum_{l_1, \ldots, l_{r_k} \geq 1} A^{p-q} \left( x_{k-l_1} \{ y_{l_1}, \ldots, y_{l_k} \} \right) \{ x_p \{ y_{r_2}, \ldots, y_{r_k} \} \} \]

\[= \sum_{2 \leq p \leq n-1, \ 1 \leq q < p} \sum_{r_1, \ldots, r_q \geq 1} \sum_{j=2}^q A^{p-q} y_{r_1} \{ x_p \{ y_{r_2}, \ldots, x_k \{ y_{l_1}, \ldots, y_{l_k} \}, \ldots, y_{r_q} \} \} \]

\[- \sum_{2 \leq p \leq n-1, \ 1 \leq q < p} \sum_{r_1, \ldots, r_q \geq 1} \sum_{j=2}^q A^{p-q} x_{k-j} \{ y_{l_1}, \ldots, y_{l_j} \} \{ x_p \{ y_{r_2}, \ldots, y_{r_q} \} \} \]
so we obtain $\partial^2(y_n) = 0$. □

We will justify the following definition by showing its cobar construction is exactly the minimal model of $\mathcal{RBV}$ in Section 3, hence the name “Koszul dual homotopy cooperad”.

**Definition 2.2.** The homotopy cooperad $\mathcal{S}(\mathcal{RBV}) \otimes S^{-1}$ is called the Koszul dual homotopy cooperad of $\mathcal{RBV}$, denoted by $\mathcal{RBV}^!$.

Precisely, the underlying graded collection of $\mathcal{RBV}^!$ is

$$\mathcal{RBV}^!(n) = ke_n \oplus ko_n, n \geq 1$$

with $e_n = u_n \otimes e_n$ and $o_n = v_n \otimes e_n$, thus $|e_n| = n - 1$ and $|o_n| = n$. The defining operations $\{\Delta_T\}_{T \in \Sigma}$ are given by the following formulae:

1. For a tree $T$ of type (I) with $1 \leq j \leq n, 1 \leq i \leq n - j + 1$,

   $$\Delta_T(e_n) = (-1)^{(j-1)(n-i+1)} e_n - j$$

   We distinguish three cases when introducing $\Delta_T(o_n)$. For $j = 1$,

   $$\Delta_T(o_n) = e_n - 1$$

   for $2 \leq j \leq n - 1$,

   $$\Delta_T(o_n) = (-1)^{(j-1)(n-i+1)}\lambda^{j-1} o_n - j$$

   and for $j = n$,

   $$\Delta_T(o_n) = \lambda^{n-1} o_n - 1 + e_n$$

2. For a tree $T$ of type (II) with $2 \leq k \leq n, r_1 + \cdots + r_k = n, r_1, \ldots, r_k \geq 1$.

   $$\Delta_T(o_n) = (-1)^{\frac{k(k-1)}{2}} o_{r_1} o_{r_2} \cdots o_{r_k}$$
Definition 3.3. The differential graded operad $(\mathbb{A})$ action of the differential on these generators in graded collection operad of homotopy Rota-Baxter algebras of weight $\mathbb{A}$ structure. More precisely, will prove that its cobar construction $\Omega$ will be a quasi-isomorphism of operads $F(M)$ to $\mathcal{R}(\mathbb{A})$. In the previous section, we constructed a coaugmented homotopy cooperad $\mathcal{R}(\mathbb{A})$. First, let’s recall the notion of minimal model of operads. $M$ admits a decomposition $M = \bigoplus_{i \geq 1} M(i)$ such that $\partial(M(k+1)) \subset \mathcal{F} \left( \bigoplus_{i = 1}^{k} M(i) \right)$ for any $k \geq 1$. Theorem 3.2 ([17]). When an operad $\mathcal{P}$ admits a minimal model, it is unique up to isomorphisms. Definition 3.3. The differential graded operad $\Omega(\mathcal{R}(\mathbb{A}))$, denoted by $\mathcal{R}(\mathbb{A})_{\infty}$, is called the operad of homotopy Rota-Baxter algebras of weight $\mathbb{A}$. Let us make this dg operad explicit. The dg operad $\mathcal{R}(\mathbb{A})_{\infty}$ is the free operad generated by the graded collection $s^{-1}\mathcal{R}(\mathbb{A})$ endowed with the differential induced from the homotopy cooperad structure. More precisely, $s^{-1}\mathcal{R}(\mathbb{A})_{\infty}(1) = ks^{-1}o_1$ and $s^{-1}\mathcal{R}(\mathbb{A})_{\infty}(n) = ks^{-1}e_n \oplus ks^{-1}o_n, n \geq 2$. Denote $m_n = s^{-1}e_n, n \geq 2$ and $T_n = s^{-1}o_n, n \geq 1$ respectively, so $|m_n| = n - 2, |T_n| = n - 1$. The action of the differential on these generators in $\mathcal{R}(\mathbb{A})_{\infty}$ is given by the following formulae:

$$\forall n \geq 2, \quad \partial m_n = \sum_{j=2}^{n-1} \sum_{i=1}^{n-j+1} (-1)^{i+j(n-i)} m_{n-j+1} \circ_i m_j$$
and for any \( n \geq 1 \),

\[
\delta T_n = \sum_{k=2}^{n} \sum_{l_1, \ldots, l_k \geq 1} (-1)^{\alpha} \left( \cdots (m_k \circ_l T_{l_k} \circ_{l_{k-1}} T_{l_{k-1}}) \cdots \right) \circ_{l_1, \ldots, l_{k-1}, l_k+1} T_k + \sum_{2 \leq p \leq n} \sum_{r_1, \ldots, r_q \geq 1} \sum_{1 \leq q \leq p} (-1)^{\beta} \lambda^{p-q} \left( T_{r_1} \circ_l \left( \cdots (m_p \circ_{k_1} T_{r_2} \circ_{k_2} T_{r_3}) \cdots \right) \circ_{k_{q-1}, \ldots, k_q, r_q+1} T_{r_q} \right),
\]

where the signs \((-1)^{\alpha}\) and \((-1)^{\beta}\) are respectively

\[
\alpha = 1 + \frac{k(k-1)}{2} + \sum_{j=1}^{k} (k-j)l_j = 1 + \sum_{j=1}^{k} (k-j)(l_j - 1),
\]

\[
\beta = 1 + i + (p + \sum_{j=2}^{q} (r_j - 1))(r_1 - i) + \sum_{j=2}^{q} (r_j - 1)(p - k_j).
\]

Let us display the elements in the dg operad \( \mathcal{RBA}_\infty \) using labelled planar rooted trees. The corolla with \( n \) leaves and a black vertex represents the generator \( m_n, n \geq 2 \), while the generators \( T_n, n \geq 1 \) by the corolla with \( n \) leaves and a white vertex:

\[
\begin{aligned}
&
1 \quad \cdots \quad n \\
&\quad m_n
\end{aligned}
\]

In this means, the action of the differential operator \( \delta \) on generators can be expressed by trees as follows:

\[
\begin{aligned}
&
1 \quad \cdots \quad n \\
&\quad m_n
\end{aligned}
= \sum_{j=2}^{n-1} \sum_{i=1}^{n-j+1} (-1)^{i+j(n-i)}
\]

\[
\begin{aligned}
&
1 \quad \cdots \quad n \\
&\quad \delta T_n
\end{aligned}
= \sum_{k=2}^{n} \sum_{l_1, \ldots, l_k \geq 1} (-1)^{\alpha} \left( \cdots \cdots \right) + \sum_{2 \leq p \leq n} \sum_{r_1, \ldots, r_q \geq 1} \sum_{1 \leq q \leq p} (-1)^{\beta} \lambda^{p-q} \left( T_{r_1} \circ_l \left( \cdots \cdots \right) \circ_{r_{q-1}, r_q+1} T_{r_q} \right)
\]
Remark 3.4. It should be noted that the collection spanned by the trees which only have black vertices is a dg suboperad of $\mathcal{RB}_{\infty}$, and this suboperad is exactly the $A_{\infty}$-operad.

The following result is the main result of this section, whose proof occupies the rest of this section.

Theorem 3.5. The dg operad $\mathcal{RB}_{\infty}$ is the minimal model of the operad $\mathcal{RB}$.

In order to prove Theorem 3.5, we are going to construct a quasi-isomorphism of dg operads $\mathcal{RB}_{\infty} \to \mathcal{RB}$, where $\mathcal{RB}$ is considered as a dg operad concentrated in degree 0.

Recall that the operad $\mathcal{RB}$ is generated by a unary operator $T$ and a binary operator $\mu$ with operadic relations:

$$\mu \circ_1 \mu - \mu \circ_2 \mu \quad \text{and} \quad (\mu \circ_1 T) \circ_2 T - (T \circ_1 \mu) \circ_1 T - (T \circ_1 \mu) \circ_2 T - \lambda T \circ_1 \mu.$$

Lemma 3.6. There exists a natural surjective map $\phi : \mathcal{RB}_{\infty} \to \mathcal{RB}$ of dg operads sending $m_2$ (resp. $T_1$, all other generators) to $\mu$ (resp. $T$, zero), which induces an isomorphism $H_0(\mathcal{RB}_{\infty}) \cong \mathcal{RB}$.

Proof. The degree zero part of $\mathcal{RB}_{\infty}$ is the free graded operad generated by $\{m_2, T_1\}$. The image of $\partial$ in this degree zero part is the operadic ideal generated by $\partial T_2$ and $\partial m_3$. By definition, we have

$$\partial(m_3) = m_2 \circ_1 m_2 - m_2 \circ_2 m_2,$$

$$\partial(T_2) = T_1 \circ_1 (m_2 \circ_1 T_1) + T_1 \circ_1 (m_2 \circ_2 T_1) + \lambda T_1 \circ_1 m_2 - (m_2 \circ_1 T_1) \circ_2 T_1.$$

Thus the map $\phi : \mathcal{RB}_{\infty} \to \mathcal{RB}$ induces the isomorphism $H_0(\mathcal{RB}_{\infty}) \cong \mathcal{RB}$. \qed

To show that $\phi : \mathcal{RB}_{\infty} \to \mathcal{RB}$ is a quasi-isomorphism, we just need to prove that $H_i(\mathcal{RB}_{\infty}) = 0$ for all $i > 0$. This will be achieved by constructing explicitly a homotopy. To this end, we need to establish a graded path-lexicographic ordering on $\mathcal{RB}_{\infty}$.

Each tree monomial gives rise to a path sequence; for details, see [23, Chapter 3]. More precisely, to any tree monomial $T$ with $n$ leaves (written as arity$(T) = n$), we can associate with a sequence $(x_1, \ldots, x_n)$ where $x_i$ is the word formed by generators of $\mathcal{RB}_{\infty}$ corresponding to the vertices along the unique path from the root of $T$ to its $i$-th leaf.

For two graded tree monomials $T, T'$, we compare $T, T'$ in the following way:

(i) If arity$(T) >$ arity$(T')$, then $T > T'$;
(ii) if arity$(T) =$ arity$(T')$, and $\deg(T) >$ $\deg(T')$, then $T > T'$, where $\deg(T)$ is the sum of the degrees of all generators of $\mathcal{RB}_{\infty}$ appearing in $T$;
(iii) if arity$(T) =$ arity$(T')(= n)$, $\deg(T) =$ $\deg(T')$, then $T > T'$ if the path sequences $(x_1, \ldots, x_n), (x'_1, \ldots, x'_n)$ associated to $T, T'$ satisfies $(x_1, \ldots, x_n) > (x'_1, \ldots, x'_n)$ with respect to the length-lexicographic order of words induced by

$$T_1 < m_2 < T_2 < m_3 < \cdots < T_n < m_{n+1} < T_{n+1} < \cdots.$$

It is ready to see that this is a well order. Under this order, the leading terms in the expansion of $\partial(m_n), \partial(T_n)$ are the following tree monomials respectively:

Let $S$ be a generator of degree $\geq 1$ in $\mathcal{RB}_{\infty}$. Denote the leading monomial of $\partial S$ by $\hat{S}$ and the coefficient of $\hat{S}$ in $\partial$ is written as $I_S$. A tree monomial of the form $\hat{S}$ is called typical, so all typical tree monomials are of the form

$$m_{n-1} \circ_1 m_2 \quad \text{and} \quad (T_{n-1} \circ_1 m_2) \circ_1 T_1,$$
which are illustrated above. It is easily seen that the coefficients $l_S$ are always $\pm 1$.

**Definition 3.7.** A tree monomial $T$ in $\mathcal{RBA}_\infty$ is called effective if $T$ satisfies the following conditions:

(i) There exists a typical divisor $T' = \hat{S}$ in $T$ such that on the path from the root of $T'$ to the leftmost leaf $l$ of $T$ above the root of $T'$, there are no other typical divisors, and there are no vertices of positive degree on this path except the root of $T'$ possibly.

(ii) For any leaf $l'$ of $T$ which lies on the left of $l$, there are no vertices of positive degree and no typical divisors on the path from the root of $T$ to $l'$.

The typical divisor $T'$ is called the effective divisor of $T$ and the leaf $l$ is called the typical leaf of $T$.

Morally, the effective divisor of a tree monomial $T$ is the left-upper-most typical divisor of $T$. It can be easily seen that for the effective divisor $T'$ in $T$ with effective leaf $l$, any vertex in $T'$ doesn’t belong to the path from root of $T$ to any leaf $l'$ located on the left of $l$.

**Example 3.8.** Consider three tree monomials as follows:

For the three trees displayed above, each has two typical divisors.

- $T_1$ is effective and the divisor in the blue dashed circle is its effective divisor and $l$ is its effective leaf.
- $T_2$ is not effective, since the first leaf is incident to a vertex of degree 1, say the root of $T_2$, which violates Condition (ii) in Definition 3.7.
- $T_3$ is not effective since there is a vertex of degree 1 on the path from the root of the typical divisor in the blue dashed circle to the leftmost leaf above it, which violates Condition (i) in Definition 3.7.

Now we are going to construct a homotopy map $\mathcal{H} : \mathcal{RBA}_\infty \to \mathcal{RBA}_\infty$, i.e., a graded map of degree 1, satisfying $\partial \mathcal{H} + \mathcal{H} \partial = 1d$ in positive degrees.

**Definition 3.9.** Let $T$ be an effective tree monomial in $\mathcal{RBA}_\infty$ and $T'$ be its effective divisor. Assume that $T' = \hat{S}$, where $S$ is a generator of positive degree. Then define

$$
\mathcal{H}(T) = (-1)^{\omega} \frac{1}{l_S} m_{T',S}(T),
$$
where \( m_{T',S}(T) \) is the tree monomial obtained from \( T \) by replacing the effective divisor \( T' \) by \( S \), \( \omega \) is the sum of degrees of all the vertices on the path from root of \( T' \) to the root of \( T \) (except the root vertex of \( T' \)) and on the left of this path.

Then we define a map \( \mathbf{S} \) of degree 1 on \( _{\mathcal{R}\mathcal{B}} \mathcal{A}_\infty \) as

(i) If \( T \) is not an effective tree monomial, then define \( \mathbf{S}(T) = 0; \)

(ii) If \( T \) is effective, denote by \( \overrightarrow{T} \) is obtained from \( T \) by replacing \( T' \) by \( T' - \frac{1}{l_s} \partial S \) with \( T' \) being the leading term of \( \partial S \). Define \( \mathbf{S}(T) = \overrightarrow{\mathbf{S}(T)} + \mathbf{S}(\overrightarrow{T}) \), where, since each tree monomial in \( \overrightarrow{T} \) is strictly smaller than \( T \), define \( \mathbf{S}(\overrightarrow{T}) \) by taking induction on leading terms (this can be done by Lemma 3.10).

Let’s explain more on the definition of \( \mathbf{S} \). Denote \( T \) by \( T_i \). By definition above, \( \mathbf{S}(T) = \overrightarrow{\mathbf{S}(T_i)} + \mathbf{S}(\overrightarrow{T_i}) \). Since \( \mathbf{S} \) vanishes on non-effective tree monomials, we have \( \mathbf{S}(T_i) = \mathbf{S}(\sum_{i_2 \in I_2} T_{i_2}) \) where \( \{T_{i_2}\}_{i_2 \in I_2} \) is the set of effective tree monomials together with their coefficients appearing in the expansion of \( \overrightarrow{T_i} \). Then by definition of \( \mathbf{S} \), \( \mathbf{S}(\sum_{i_2 \in I_2} T_{i_2}) = \overrightarrow{\mathbf{S}(\sum_{i_2 \in I_2} T_{i_2})} + \mathbf{S}(\sum_{i_2 \in I_2} \overrightarrow{T_{i_2}}) \), we have

\[
\mathbf{S}(T) = \overrightarrow{\mathbf{S}(T_i)} + \mathbf{S}(\sum_{i_2 \in I_2} T_{i_2}) + \mathbf{S}(\sum_{i_3 \in I_3} T_{i_3}) + \cdots + \mathbf{S}(\sum_{i_n \in I_n} T_{i_n}) + \ldots ,
\]

Taking induction on leading terms, \( \mathbf{S}(T) \) is the following series:

\[
\mathbf{S}(T) = \overrightarrow{\mathbf{S}(T_i)} + \mathbf{S}(\sum_{i_2 \in I_2} T_{i_2}) + \mathbf{S}(\sum_{i_3 \in I_3} T_{i_3}) + \cdots + \mathbf{S}(\sum_{i_n \in I_n} T_{i_n}) + \ldots ,
\]

where \( \{T_{i_n}\}_{i_n \in I_n} \) is the set of the effective tree monomials with their nonzero coefficients appearing in the expansion of \( \sum_{i_n-1 \in I_{n-1}} \overrightarrow{T_{i_{n-1}}} \).

**Lemma 3.10.** For any effective tree monomial \( T \), the expansion of \( \overrightarrow{\mathbf{S}(T)} \) in Equation (18) is always a finite sum, i.e., there exists some large integer \( n \) such that all tree monomials in \( \sum_{i_n \in I_n} T_{i_n} \) are not effective.

**Proof.** It is easy to see that \( \max\{T_{i_k} | i_k \in I_k\} > \max\{T_{i_{k+1}} | i_{k+1} \in I_{k+1}\} \) for all \( k \geq 1 \) (by convention, \( i_1 \in I_1 = \{1\} \), so the right-hand side of Equation (18) cannot be an infinite sum, as “>” is a well order.

**Lemma 3.11.** Let \( T \) be an effective tree monomial. Then \( \partial \overrightarrow{\mathbf{S}(T)} + \mathbf{S}(\overrightarrow{T - T}) = \overrightarrow{T} - \overrightarrow{T} \).

**Proof.** We can write \( T \) as a compositions in the following way:

\[
(\cdots (((((X_1 \circ_{i_1} X_2) \circ_{i_2} \cdots \circ_{i_{p-1}} X_p) \circ_{i_p} \widehat{S}) \circ_{j_1} Y_1) \circ_{j_2} Y_2) \cdots ) \circ_{j_q} Y_q ,
\]

where \( \widehat{S} \) is the effective divisor of \( T \) and \( X_1, \ldots, X_p \) are generators of \( _{\mathcal{R}\mathcal{B}} \mathcal{A}_\infty \) corresponding to the vertices which live on the path from root of \( T \) and root of \( \widehat{S} \) (except the root of \( \widehat{S} \)) and on the left of this path in the underlying tree of \( T \).

By definition,

\[
\partial \overrightarrow{\mathbf{S}(T)} = \frac{1}{l_s} \sum_{i_1} (-1)^{|X_i|} \partial \left(\cdots (((((X_1 \circ_{i_1} X_2) \circ_{i_2} \cdots \circ_{i_{p-1}} X_p) \circ_{i_p} \widehat{S}) \circ_{j_1} Y_1) \circ_{j_2} Y_2) \cdots ) \circ_{j_q} Y_q \right)
\]

\[
= \frac{1}{l_s} \left( \sum_{k=1}^p (-1)^{|X_i|} (\sum_{j=1}^k |X_j|) \right)
\]
(\cdots (((\cdots ((X_1 \circ_i X_2) \circ_{i_2} \cdots) \circ_{i_{k-1}} \partial X_k) \circ_{i_k} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{i_{j_1}} Y_1) \circ_{i_{j_2}} \cdots) \circ_{i_q} Y_q) \\
+ \frac{1}{l_S} \left( (\cdots (((\cdots ((X_1 \circ_i X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} \partial S) \circ_{i_{j_1}} Y_1) \circ_{i_{j_2}} \cdots) \circ_{i_q} Y_q) \\
+ \frac{1}{l_S} \left( \sum_{k=1}^q (-1)^{k-1} |S|^{\frac{k-1}{2}} \sum_{j=1}^{|S|} |Y_j| \right) \right)
(\cdots (((\cdots ((X_1 \circ_i X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} \partial S\circ_{i_{j_1}} Y_1) \circ_{i_{j_2}} \cdots) \circ_{i_q} Y_q)
)

and 

\mathfrak{S} \partial (\mathcal{U} - \overline{\mathcal{U}}) \\
= \frac{1}{l_S} \mathfrak{S} \partial \left( (\cdots (((\cdots ((X_1 \circ_i X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} \partial S) \circ_{i_{j_1}} Y_1) \circ_{i_{j_2}} \cdots) \circ_{i_q} Y_q) \\
+ \frac{1}{l_S} \mathfrak{S} \left( \sum_{k=1}^q (-1)^{k-1} \sum_{j=1}^{|S|} |Y_j| \right) \left( (\cdots (((\cdots ((X_1 \circ_i X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} \partial S) \circ_{i_{j_1}} Y_1) \circ_{i_{j_2}} \cdots) \circ_{i_q} Y_q) \right)

By the definition of the effective divisor in an effective tree monomial, it can be easily seen that each tree monomial in the expansion of 

(\cdots (((\cdots ((X_1 \circ_i \cdots) \circ_{i_{k-1}} \partial X_k) \circ_{i_k} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{i_{j_1}} Y_1) \circ_{i_{j_2}} \cdots) \circ_{i_q} Y_q

and of 

(\cdots (((\cdots ((X_1 \circ_i X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{i_{j_1}} Y_1) \circ_{i_{j_2}} \cdots) \circ_{i_q} Y_q

is still effective tree monomial whose effective divisor is still \( \mathfrak{S} \). Thus we have 

\begin{align*}
\mathfrak{S} \partial (\mathcal{U} - \overline{\mathcal{U}}) \\
= \frac{1}{l_S} \left( \sum_{k=1}^q (-1)^{k-1} \sum_{j=1}^{|S|} |Y_j| \right) \left( (\cdots (((\cdots ((X_1 \circ_i \cdots) \circ_{i_{k-1}} \partial X_k) \circ_{i_k} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{i_{j_1}} Y_1) \circ_{i_{j_2}} \cdots) \circ_{i_q} Y_q) \right)
+ \frac{1}{l_S} \left( \sum_{k=1}^q (-1)^{k-1} |S|^{\frac{k-1}{2}} \sum_{j=1}^{|S|} |Y_j| \right) \left( (\cdots (((\cdots ((X_1 \circ_i X_2) \circ_{i_2} \cdots) \circ_{i_{p-1}} X_p) \circ_{i_p} S) \circ_{i_{j_1}} Y_1) \circ_{i_{j_2}} \cdots) \circ_{i_q} Y_q) \right)
\end{align*}

Take sum of the above expansion, then we get 

\begin{align*}
\partial \overline{\mathfrak{S} (\mathcal{U})} + \mathfrak{S} \partial (\mathcal{U} - \overline{\mathcal{U}}) &= \mathcal{U} - \overline{\mathcal{U}}.
\end{align*}

\textbf{Proposition 3.12.} The degree 1 map \( \mathfrak{S} \) defined above satisfies \( \partial \mathfrak{S} + \mathfrak{S} \partial = \text{Id} \) in all positive degrees of \( \mathcal{R} \mathcal{B} \mathcal{M} \mathcal{S} \mathcal{S} \mathcal{I} \mathcal{O} \mathcal{S} \).

\textit{Proof.} Let \( \mathcal{U} \) be an effective tree monomial. Since the leading term of \( \overline{\mathcal{U}} \) is strictly smaller than \( \mathcal{U} \), by induction, we have 

\begin{align*}
\mathfrak{S} \partial (\overline{\mathcal{U}}) + \partial \mathfrak{S} (\mathcal{U}) &= \overline{\mathcal{U}}.
\end{align*}
By the definition of \( S \), \( S(T) = \overline{S}(T) + \overline{S}(-T) \) and we have \( \partial \overline{S}(T) = \overline{\partial \overline{S}(T)} + \overline{\partial \overline{S}(-T)} \). Thus,

\[
\partial \overline{S}(T) + \overline{S}\partial(T) = \overline{\partial \overline{S}(T)} + \overline{\partial \overline{S}(-T)} + \overline{\partial \overline{S}(-T)} + \overline{\partial \overline{S}(T)} = \overline{T - T + T} = T,
\]

where in the third equality we have used the induction hypothesis and

\[
\partial \overline{S}(T) + \overline{S}\partial(T - T) = T - T
\]

by Lemma \( \ref{lem:induction} \).

Next let’s prove that for a non-effective tree monomial \( T \), the equation \( \partial \overline{S}(T) + \overline{S}\partial(T) = T \) holds.

By the definition of \( S \), since \( T \) is not effective, \( S(T) = 0 \), thus we just need to check that \( \overline{S}\partial(T) = T \). Since \( T \) has positive degree, there must exist at least one vertex of positive degree. Let’s pick a special vertex \( S \) satisfying the following conditions:

(i) on the path from \( S \) to the leftmost leaf \( l \) of \( T \) above \( S \), there are no other vertices of positive degree;

(ii) for any leaf \( l' \) of \( T \) located on the left of \( l \), the vertices on the path from the root of \( T \) to \( l' \) are all of degree 0.

It is easy to see such a vertex always exists in \( T \). Morally, this vertex is the “left-upper-most” vertex of positive degree. Then the tree monomial \( T \) can be written as

\[
(\cdots (((\cdots (X_1 \circ_{i_1} X_2) \circ_{i_2} \cdots) \circ_{i_p} S) \circ_{j_1} Y_1) \circ_{j_2} \cdots) \circ_{j_q} Y_q,
\]

where \( X_1, \ldots, X_p \) correspond to the vertices located on the path from the root of \( T \) to \( S \) and on the left of this path in the plane.

By definition,

\[
\overline{S}\partial T = \overline{S} \sum_{k=1}^{p} (-1)^{r+1} |X_k| \cdots (((\cdots (X_1 \circ_{i_1} \cdots) \circ_{i_k} \overline{\partial}X_k) \circ_{i_{k+1}} \cdots) \circ_{i_p} S) \circ_{j_1} Y_1) \circ_{j_2} \cdots) \circ_{j_q} Y_q
\]

By the assumption, the divisor consisting of the path from \( S \) to \( l \) must be one of the following forms:

- \( m_n(n \geq 3) \) for \( n \geq 0 \)
- \( m_n(n \geq 3) \) for \( n = 1 \)
By the assumption that $T$ is not effective and the speciality of the position of $S$, one can see that the effective tree monomials in $\partial T$ will only appear in the expansion of

$$(-1)^{\sum_{\ell=1}^{p} |X_{\ell}|} \cdot (\cdots (((X_{1} \circ_{i_{1}} X_{2}) \circ_{i_{2}} \cdots) \circ_{i_{p-1}} X_{p}) \circ_{i_{p}} \partial S) \circ_{j_{1}} Y_{1} \circ_{j_{2}} \cdots) \circ_{j_{q}} Y_{q}.$$  

Consider the tree monomial

$$(\cdots (((X_{1} \circ_{i_{1}} X_{2}) \circ_{i_{2}} \cdots) \circ_{i_{p-1}} X_{p}) \circ_{i_{p}} \tilde{S}) \circ_{j_{1}} Y_{1} \circ_{j_{2}} \cdots) \circ_{j_{q}} Y_{q}$$

in $\partial T$. Then the path connecting root of $\tilde{S}$ and $l$ must be one of the following forms:

So the tree monomial

$$(\cdots (((X_{1} \circ_{i_{1}} X_{2}) \circ_{i_{2}} \cdots) \circ_{i_{p-1}} X_{p}) \circ_{i_{p}} \tilde{S}) \circ_{j_{1}} Y_{1} \circ_{j_{2}} \cdots) \circ_{j_{q}} Y_{q}$$

is effective and its effective divisor is exactly $\tilde{S}$ itself. Then we have

$$\begin{align*}
\hat{S} \partial T &= \hat{S}((-1)^{\sum_{\ell=1}^{p} |X_{\ell}|} \cdot (\cdots (((X_{1} \circ_{i_{1}} X_{2}) \circ_{i_{2}} \cdots) \circ_{i_{p-1}} X_{p}) \circ_{i_{p}} \partial S) \circ_{j_{1}} Y_{1} \circ_{j_{2}} Y_{2} \cdots) \circ_{j_{q}} Y_{q}) \\
&= l_{S} \hat{S}((-1)^{\sum_{\ell=1}^{p} |X_{\ell}|} \cdot (\cdots (((X_{1} \circ_{i_{1}} X_{2}) \circ_{i_{2}} \cdots) \circ_{i_{p-1}} X_{p}) \circ_{i_{p}} \tilde{S}) \circ_{j_{1}} Y_{1} \circ_{j_{2}} \cdots) \circ_{j_{q}} Y_{q}) \\
&\quad + \hat{S}((-1)^{\sum_{\ell=1}^{p} |X_{\ell}|} \cdot (\cdots (((X_{1} \circ_{i_{1}} X_{2}) \circ_{i_{2}} \cdots) \circ_{i_{p-1}} X_{p}) \circ_{i_{p}} (\partial S - l_{S} \tilde{S})) \circ_{j_{1}} Y_{1} \circ_{j_{2}} \cdots) \circ_{j_{q}} Y_{q}) \\
&= l_{S} \tilde{S}((-1)^{\sum_{\ell=1}^{p} |X_{\ell}|} \cdot (\cdots (((X_{1} \circ_{i_{1}} X_{2}) \circ_{i_{2}} \cdots) \circ_{i_{p-1}} X_{p}) \circ_{i_{p}} \tilde{S}) \circ_{j_{1}} Y_{1} \circ_{j_{2}} \cdots) \circ_{j_{q}} Y_{q})
\end{align*}$$
and it can be easily seen that the differential \( \partial \) prove that \( \partial \mathcal{S} / \partial S \)

Now, we define a map \( \mathcal{S} \):

It can be easily seen that the filtration is compatible with the differential \( \partial \):

Since all positive homologies of \( m_{\mathcal{S}} \mathcal{H}_{\mathcal{A}} \) vanish, by classical spectral sequence argument, we have that all positive homologies of \( m_{\mathcal{S}} \mathcal{H}_{\mathcal{A}} \) are trivial.

This provides another proof for Theorem 3.3.

According to the results in [15], the minimal model of the operad \( \mathcal{S} \mathcal{A} \) can be derived by employing a homotopical perturbation process on the minimal model \( m_{\mathcal{S}} \mathcal{H}_{\mathcal{A}} \) of the operad.
where the minimal model for an operad is exactly the desuspension of its Quillen homology, i.e., the expressions.

By the general theory of minimal models of operads, the space spanned by the generators of the minimal model for an operad is exactly the desuspension of its Quillen homology, i.e., the homology of the bar construction of the operad. So for the operad \( RBA \), denote by \( B(RBA) \) the bar construction of \( RBA \) and we have the following result.

**Corollary 3.14.** There exists a quasi-isomorphism of homotopy cooperads between \( RBA^i \) and \( B(RBA) \), which induces an isomorphism of graded collections

\[
H_\bullet(B(RBA)) \cong \lambda RBA^i.
\]

### 4. Homotopy Rota-Baxter algebras

Since we have found the operad of “homotopy Rota-Baxter algebras of weight \( \lambda \)”, we could now give the definition of homotopy Rota-Baxter algebras.

**Definition 4.1.** Let \((V, d_V)\) be a complex. Then a homotopy Rota-Baxter algebra of weight \( \lambda \) on \( V \) is defined to be a morphism of dg operads from \( RBA_\infty \) to the endomorphism operad \( \text{End}_V \).

Let \((V, d_V)\) be an algebra over the operad \( RBA_\infty \). Still denote by \( m_n : V^\otimes n \to V, n \geq 2 \) (resp. \( T_n : V^\otimes n \to V, n \geq 1 \)) the image of \( m_n \in \lambda RBA_\infty \) (resp. \( T_n \in \lambda RBA_\infty \)). We also rewrite \( m_1 = d_V \). Then Equations (17) and (1) give

\[
\sum_{i+j+k = n, i, k \geq 0, j \geq 1} (-1)^{i+j+k} m_{i+1+k} \circ (\text{Id}^{\otimes i} \otimes m_j \otimes \text{Id}^{\otimes k}) = 0
\]

and

\[
\sum_{i_1 + \cdots + i_k = n, i_1, \ldots, i_k \geq 1} (-1)^i m_k \circ \left( T_{i_1} \otimes \cdots \otimes T_{i_k} \right) = \sum_{1 \leq q \leq p} \sum_{r_1 + \cdots + r_p = n, r_1, \ldots, r_p \geq 1} \sum_{i+k=r_1, j_1 + \cdots + j_p = q} \sum_{i, k \geq 0, j_1, \ldots, j_p \geq 0} (-1)^{\eta} \lambda^{p-q} T_{r_1} \circ \left( \text{Id}^{\otimes i} \otimes m_p \circ (\text{Id}^{\otimes j_1} \otimes T_{r_2} \otimes \text{Id}^{\otimes j_2} \otimes \cdots \otimes T_{r_q} \otimes \text{Id}^{\otimes j_q}) \otimes \text{Id}^{\otimes k} \right),
\]

where

\[
\begin{align*}
\delta &= \frac{k(k-1)}{2} + \frac{n(n-1)}{2} + \sum_{j=1}^k (k-j)l_j, \\
\eta &= \frac{p(p-1)}{2} + \sum_{j=1}^q \frac{r_j(r_j-1)}{2} + k + \sum_{l=2}^q (r_l-1)(i + \sum_{r=1}^{l-1} j_r + \sum_{r=2}^{l-1} r_l) + pi
end{align*}
\]

\[
\begin{align*}
&= \frac{n(n-1)}{2} + i + (p + \sum_{j=2}^q (r_j-1))k + \sum_{l=2}^q (r_l-1)(\sum_{r=2}^q j_r + q-l)
end{align*}
\]

We obtain thus an equivalent definition of homotopy Rota-Baxter algebras.

**Definition 4.2.** Let \( V \) be a graded space. A homotopy Rota-Baxter algebra of weight \( \lambda \) on \( V \) consists of two families of graded maps \( m_n : V^\otimes n \to V, n \geq 1 \) and \( T_n : V^\otimes n \to V, n \geq 1 \) with \( |m_n| = n-2, |T_n| = n-1 \), subject to Equations (19) and (20).
Equation (\ref{eq:Stasheff}) is exactly the Stasheff identity defining $A_\infty$-algebras \cite{Stasheff}. In particular, the operator $m_1$ is a differential on $V$ and the operator $m_2$ induces an associative algebra structure on the homology $H_\ast (V, m_1)$.

**Example 4.3.** Expanding Equation \eqref{eq:Stasheff} for small $n$’s gives the following:

(i) When $n = 1$, $|T_1| = 0$ and
$$m_1 \circ T_1 = T_1 \circ m_1,$$
which implies that $T_1 : (V, m_1) \to (V, m_1)$ is a chain map;

(ii) when $n = 2$, $|T_2| = 1$ and
$$m_2 \circ (T_1 \otimes T_1) - T_1 \circ m_2 \circ (\text{Id} \otimes T_1) - T_1 \circ m_2 \circ (T_1 \otimes \text{Id}) + \lambda T_1 \circ m_2$$
$$= -\partial (T_2) = -(m_1 \circ T_2 + T_2 \circ (\text{Id} \otimes m_1) + T_2 \circ (m_1 \otimes \text{Id})), $$
which shows that $T_1$ is a Rota-Baxter operator of weight $\lambda$ with respect to $m_2$, but only up to homotopy given by the operator $T_2$.

Observe that for a homotopy Rota-Baxter algebra $(V; \{m_n\}_{n \geq 1}, \{T_n\}_{n \geq 1})$, its homology $H_\ast (V, m_1)$ endowed with the operators induced by $m_2$ and $T_1$ is a usual Rota-Baxter algebra.

5. FROM THE MINIMAL MODEL TO THE DEFORMATION COMPLEX AND ITS $L_\infty$-ALGEBRA STRUCTURE

In this section, we will use the minimal model $\mathcal{RBA}_\infty$, or more precisely the Koszul dual homotopy cooperad $\mathcal{RBA}^1$, to determine the deformation complex as well as the $L_\infty$-algebra structure on it for Rota-Baxter algebras of arbitrary weight.

**Definition 5.1.** Let $V$ be a graded space. Introduce an $L_\infty$-algebra $C_{RBA_1}(V)$ associated to $V$ as $C_{RBA_1}(V) := \text{Hom}(\mathcal{RBA}^1, \text{End}_V) \Pi$.

Now, let’s determine the $L_\infty$-algebra $C_{RBA_1}(V)$ explicitly. The sign rules in the homotopy cooperad $\mathcal{RBA}^1$ are complicated, so we need some transformations. Notice that there is a natural isomorphism of operads

$$\text{Hom}(S, \text{End}_S(V)) \cong \text{End}_V.$$ Explicitly, any $f \in \text{End}_V(n)$ corresponds to an element $\tilde{f} \in \text{Hom}(S, \text{End}_S(V))$ which is defined as $(\tilde{f}(\delta_n))(sv_1 \otimes \cdots \otimes sv_n) = (-1)^{\sum_{k=1}^{n-1} \sum_{j=k}^{n} |v_j|} (-1)^{n-1} |f|_s f(v_1 \otimes \cdots \otimes v_n)$ for any $v_1, \ldots, v_n \in V$.

Thus we have the following isomorphisms of homotopy operads:

$$\text{Hom}(\mathcal{RBA}^1, \text{End}_V) \cong \text{Hom}(\mathcal{RBA}^1, \text{Hom}(S, \text{End}_S(V)))$$
$$\cong \text{Hom}(\mathcal{RBA}^1 \otimes_H S, \text{End}_S(V))$$
$$\cong \text{Hom}(\mathcal{RBA}^1, \text{End}_S(V)).$$

We obtain

$$C_{RBA_1}(V) \cong \text{Hom}(\mathcal{RBA}^1, \text{End}_S(V)) \Pi.$$ Recall that $\mathcal{RBA}^1(n) = ku_n \oplus kv_n$ with $|u_n| = 0$ and $|v_n| = 1$. By definition

$$\text{Hom}(\mathcal{RBA}^1, \text{End}_S(V))(n) = \text{Hom}(ku_n \oplus kv_n, \text{Hom}((sv)^{\otimes n}, sv)).$$
Each $f \in \text{Hom}((sV)^{\otimes n}, sV)$ determines bijectively a map $\tilde{f}$ in $\text{Hom}(k u_n, \text{Hom}((sV)^{\otimes n}, sV))$ by imposing $\tilde{f}(u_n) = f$, and each $g \in \text{Hom}((sV)^{\otimes n}, V)$ is in bijection with a map $\hat{g}$ in $\text{Hom}(k v_n, \text{Hom}((sV)^{\otimes n}, sV))$ as $\hat{g}(v_n) = (-1)^{|g|} g$. Denote

$$\mathcal{C}_{\text{Alg}}(V) = \prod_{n \geq 1} \text{Hom}((sV)^{\otimes n}, sV) \quad \text{and} \quad \mathcal{C}_{\text{RBO}_i}(V) = \prod_{n \geq 1} \text{Hom}((sV)^{\otimes n}, V).$$

In this way, we identify $\mathcal{C}_{\text{RBA},i}(V)$ with $\mathcal{C}_{\text{Alg}}(V) \oplus \mathcal{C}_{\text{RBO},i}(V)$. By the general theory recalled in Subsection 12.2, a direct computation gives the $L_\infty$-algebra structure on $\mathcal{C}_{\text{RBA},i}(V)$:

(I) For homogeneous elements $sf, sh \in \mathcal{C}_{\text{Alg}}(V)$, define

$$l_2(sf \otimes sh) := [sf, sh]_G \in \mathcal{C}_{\text{Alg}}(V).$$

(II) (i) Let $n \geq 1$. For homogeneous elements $sh \in \text{Hom}((sV)^{\otimes n}, sV) \subset \mathcal{C}_{\text{Alg}}(V)$ and $g_1, \ldots, g_n \in \mathcal{C}_{\text{RBO},i}(V)$, define

$$l_{n+1}(sh \otimes g_1 \otimes \cdots \otimes g_n) \in \mathcal{C}_{\text{RBO},i}(V)$$

as:

$$l_{n+1}(sh \otimes g_1 \otimes \cdots \otimes g_n) = \sum_{\sigma \in S_n} (-1)^{\eta} \left( h \circ (sg_{\sigma(1)} \otimes \cdots \otimes sg_{\sigma(n)}) - (-1)^{|g_{\sigma(1)}|+1} s^{-1}(sg_{\sigma(1)}) \left\{ sh \{ sg_{\sigma(2)}, \ldots, sg_{\sigma(n)} \} \right\} \right),$$

where $(-1)^{\eta} = \chi(\sigma; g_1, \ldots, g_n)(-1)$.

(ii) Let $n \geq 2$. For homogeneous elements $sh \in \text{Hom}((sV)^{\otimes n}, sV) \subset \mathcal{C}_{\text{Alg}}(V)$ and $g_1, \ldots, g_m \in \mathcal{C}_{\text{RBO},i}(V)$ with $1 \leq m \leq n - 1$, define

$$l_{m+1}(sh \otimes g_1 \otimes \cdots \otimes g_m) \in \mathcal{C}_{\text{RBO},i}(V)$$

to be:

$$l_{m+1}(sh \otimes g_1 \otimes \cdots \otimes g_m) = \sum_{\sigma \in S_m} (-1)^{\xi} A^{n-m} s^{-1}(sg_{\sigma(1)}) \left\{ sh \{ sg_{\sigma(2)}, \ldots, sg_{\sigma(m)} \} \right\},$$

where $(-1)^{\xi} = \chi(\sigma; g_1, \ldots, g_m)(-1)$,

(III) Let $m \geq 1$. For homogeneous elements $sh \in \text{Hom}((sV)^{\otimes n}, sV) \subset \mathcal{C}_{\text{Alg}}(V)$, $g_1, \ldots, g_m \in \text{Hom}(T^c(sV), V) \subset \mathcal{C}_{\text{RBO},i}(V)$ with $1 \leq m \leq n$, for $1 \leq k \leq m$, define

$$l_{m+1}(g_1 \otimes \cdots \otimes g_k \otimes sh \otimes g_{k+1} \otimes \cdots \otimes g_m) \in \mathcal{C}_{\text{RBO},i}(V)$$

to be

$$l_{m+1}(g_1 \otimes \cdots \otimes g_k \otimes sh \otimes g_{k+1} \otimes \cdots \otimes g_m) = (-1)^{(k |g_j|+1)} l_{m+k+1}(sh \otimes g_1 \otimes \cdots \otimes g_m),$$

where the RHS has been introduced in (III) (i) and (ii).

(IV) All other components of operators $\{ l_n \}_{n \geq 1}$ vanish.

This is exactly the $L_\infty$-structure found in [57] by direct inspections. We define a filtration $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \supset \mathcal{F}_n \supset \cdots$ on $\mathcal{C}_{\text{RBA},i}(V)$ by setting

$$\mathcal{F}_n = \mathcal{C}_{\text{Alg}}^{\geq n}(V) \oplus \mathcal{C}_{\text{RBO},i}^{\geq n}(V), \forall n \geq 1,$$
where
\[ \mathcal{C}^{>n}_{\text{Alg}}(V) = \prod_{k \geq n} \text{Hom}((sV)^{\otimes k}, sV), \mathcal{C}^{>n}_{\text{RBA}}(V) = \prod_{k \geq n} \text{Hom}((sV)^{\otimes k}, V). \]

It is not difficult to see that the \( L_\infty \)-algebra \( \mathcal{C}_{\text{RBA}}(V) \) is weakly filtered with respect to the filtration \( \mathcal{T}_* \).

Proposition 5.2 gives immediately an alternative definition of homotopy Rota-Baxter algebras.

**Proposition 5.2.** A homotopy Rota-Baxter algebra structure of weight \( \lambda \) on a graded space is equivalent to a Maurer-Cartan element in the weakly filtered \( L_\infty \)-algebra \( \mathcal{C}_{\text{RBA}}(V) \). In particular, when \( V \) is concentrated in degree 0, a Maurer-Cartan element in \( \mathcal{C}_{\text{RBA}}(V) \) gives a Rota-Baxter algebra structure of weight \( \lambda \) on \( V \).

### 6. Cohomology theory of Rota-Baxter algebras

Now we introduce the cochain complex of a Rota-Baxter algebra with coefficients in a Rota-Baxter bimodule. We will see that this is exactly the underlying complex of Rota-Baxter algebras in Section 5. An explicit example is also included.

#### 6.1. Cohomology theory.

Let \((A, \mu)\) be an associative algebra and \(M\) be a bimodule over it. Recall that the Hochschild cochain complex of \(A\) with coefficients in \(M\) is

\[ C^n_{\text{Alg}}(A, M) := \bigoplus_{n=0}^{\infty} C^n_{\text{Alg}}(A, M), \]

where \( C^n_{\text{Alg}}(A, M) = \text{Hom}(A^{\otimes n}, M) \) and the differential \( \delta^n : C^n_{\text{Alg}}(A, M) \to C^{n+1}_{\text{Alg}}(A, M) \) is defined as:

\[ \delta^n(f)(a_{1,n+1}) = (-1)^{n+1}a_1 f(a_{2,n+1}) + \sum_{i=1}^{n} (-1)^{n-i+1} f(a_{1,i-1} \otimes a_i \cdot a_{i+1} \otimes a_{i+2,n+1}) + f(a_{1,n}) a_{n+1} \]

for all \( f \in C^n_{\text{Alg}}(A, M), a_1, \ldots, a_{n+1} \in A \). The cohomology of the Hochschild cochain complex \( C^\bullet_{\text{Alg}}(A, M) \) is called the Hochschild cohomology of \(A\) with coefficients in \(M\), denoted by \( \text{HH}^\bullet(A, M) \). When the bimodule \(M\) is the regular bimodule \(A\) itself, we just denote \( C^n_{\text{Alg}}(A, A) \) by \( C^n_{\text{Alg}}(A) \) and call it the Hochschild cochain complex of associative algebra \((A, \mu)\). Denote the cohomology \( \text{HH}^\bullet(A, A) \) by \( \text{HH}^\bullet(A) \), called the Hochschild cohomology of associative algebra \((A, \mu)\).

Let \((A, \mu, T)\) be a Rota-Baxter algebra and \((M, T_M)\) be a Rota-Baxter bimodule over it. Recall that Proposition 5.3 and Proposition 5.2 give a new associative algebra \(A_*\) and a new Rota-Baxter bimodule \(\triangleright M\) over \(A_*\). Consider the Hochschild cochain complex of \(A_*\) with coefficients in \(\triangleright M\):

\[ C^n_{\text{Alg}}(A_*, \triangleright M) = \bigoplus_{n=0}^{\infty} C^n_{\text{Alg}}(A_*, \triangleright M). \]

More precisely, for \(n \geq 0\), \( C^n_{\text{Alg}}(A_*, \triangleright M) = \text{Hom}(A^{\otimes n}, M) \) and its differential

\[ \partial^n : C^n_{\text{Alg}}(A_*, \triangleright M) \to C^{n+1}_{\text{Alg}}(A_*, \triangleright M). \]
is defined as:

\[ \partial^n(f)(a_{1,n+1}) = (-1)^{n+1}a_1 \triangleright f(a_{2,n+1}) + \sum_{i=1}^{n} (-1)^{n-i+1} f(a_{1,i-1} \otimes a_i \star a_{i+1} \otimes a_{i+2,n+1}) + f(a_{1,n}) \triangleleft a_{n+1} \]

\[ = (-1)^{n+1} \left( T(a_1)f(a_{2,n+1}) - T_M(a_1f(a_{2,n+1})) \right) \]

\[ + \sum_{i=1}^{n} (-1)^{n-i+1} \left( f(a_{1,i-1} \otimes a_iT(a_{i+1}) \otimes a_{i+2,n+1}) + f(a_{1,i-1} \otimes T(a_i)a_{i+1} \otimes a_{i+2,n+1}) \right) \]

\[ + \lambda f(a_{1,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,n+1}) \]

\[ + \left( f(a_{1,n})T(a_{n+1}) - T_M(f(a_{1,n})\eta_{n+1}) \right) \]

for any \( f \in C^n_{\text{Alg}}(\mathcal{A} \star, \triangleright M_\otimes) \) and \( a_1, \ldots, a_{n+1} \in A \).

**Definition 6.1.** Let \( A = (A, \mu, T) \) be a Rota-Baxter algebra of weight \( \lambda \) and \( M = (M, T_M) \) be a Rota-Baxter bimodule over it. Then the cochain complex \((C^\bullet_{\text{Alg}}(\mathcal{A} \star, \triangleright M_\otimes), \partial)\) is called the cochain complex of Rota-Baxter operator \( T \) with coefficients in \((M, T_M)\), denoted by \( C^\bullet_{\text{RBO}_A}(A, M) \). The cohomology of \( C^\bullet_{\text{RBO}_A}(A, M) \), denoted by \( H^\bullet_{\text{RBO}_A}(A, M) \), is called the cohomology of Rota-Baxter operator \( T \) with coefficients in \((M, T_M)\).

When \((M, T_M)\) is the regular Rota-Baxter bimodule \((A, T)\), we denote \( C^\bullet_{\text{RBO}_A}(A, A) \) by \( C^\bullet_{\text{RBO}_A}(A) \) and call it the cochain complex of Rota-Baxter operator \( T \), and denote \( H^\bullet_{\text{RBO}_A}(A, A) \) by \( H^\bullet_{\text{RBO}_A}(A) \) and call it the cohomology of Rota-Baxter operator \( T \).

Let \( M = (M, T_M) \) be a Rota-Baxter bimodule over a Rota-Baxter algebra of weight \( \lambda \) \( A = (A, \mu, T) \). Now, let’s construct a chain map

\[ \Phi^\bullet : C^\bullet_{\text{Alg}}(A, M) \rightarrow C^\bullet_{\text{RBO}_A}(A, M), \]

i.e., the following commutative diagram:

\[ \begin{array}{ccccccccc}
C^0_{\text{Alg}}(A, M) & \to & C^1_{\text{Alg}}(A, M) & \to & \cdots & \to & C^n_{\text{Alg}}(A, M) & \to & C^{n+1}_{\text{Alg}}(A, M) \\
\Phi^0 & & \Phi^1 & & \cdots & & \Phi^n & & \Phi^{n+1} \\
C^0_{\text{RBO}_A}(A, M) & \to & C^1_{\text{RBO}_A}(A, M) & \to & \cdots & \to & C^n_{\text{RBO}_A}(A, M) & \to & C^{n+1}_{\text{RBO}_A}(A, M) \\
\end{array} \]

Define \( \Phi^0 = \text{Id}_{\text{Hom}(k,M)} = \text{Id}_M \), and for \( n \geq 1 \) and \( f \in C^n_{\text{Alg}}(A, M) \), define \( \Phi^n(f) \in C^n_{\text{RBO}_A}(A, M) \) as:

\[ \Phi^n(f)(a_1 \otimes \cdots \otimes a_n) = f(T(a_1) \otimes \cdots \otimes T(a_n)) \]

\[ - \sum_{k=0}^{n-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} T_M \circ f(a_{1,i_1-1} \otimes T(a_{i_1}) \otimes a_{i_1+1,j_2-1} \otimes T(a_{i_2}) \otimes \cdots \otimes T(a_{i_k}) \otimes a_{i_k+1,n}). \]

**Proposition 6.2.** The map \( \Phi^\bullet : C^\bullet_{\text{Alg}}(A, M) \rightarrow C^\bullet_{\text{RBO}_A}(A, M) \) is a chain map.
This result is equivalent to the fact that the cochain complex \((C^\bullet_{\text{RBA}_A}(A, M), d^\bullet)\) of Rota-Baxter algebra \((A, \mu, T)\) with coefficients in \((M, T_M)\) in the following definition is a cochain complex, so it follows from Proposition 6.3.

**Definition 6.3.** Let \(M = (M, T_M)\) be a Rota-Baxter bimodule over a Rota-Baxter algebra of weight \(\lambda\) \(A = (A, \mu, T)\). We define the cochain complex \((C^\bullet_{\text{RBA}_A}(A, M), d^\bullet)\) of Rota-Baxter algebra \((A, \mu, T)\) with coefficients in \((M, T_M)\) to be the negative shift of the mapping cone of \(\Phi^\bullet\), that is, let

\[
C^0_{\text{RBA}_A}(A, M) = C^0_{\text{Alg}}(A, M) \quad \text{and} \quad C^n_{\text{RBA}_A}(A, M) = C^n_{\text{Alg}}(A, M) \oplus C^{n-1}_{\text{RBO}_A}(A, M), \forall n \geq 1,
\]

and the differential \(d^n : C^n_{\text{RBA}_A}(A, M) \to C^{n+1}_{\text{RBA}_A}(A, M)\) is given by

\[
d^n(f, g) = (\delta^n(f), -\delta^{n-1}(g) - \Phi^n(f))
\]

for any \(f \in C^n_{\text{Alg}}(A, M)\) and \(g \in C^{n-1}_{\text{RBO}_A}(A, M)\). The cohomology of \((C^\bullet_{\text{RBA}_A}(A, M), d^\bullet)\), denoted by \(H^\bullet_{\text{RBA}_A}(A, M)\), is called the cohomology of the Rota-Baxter algebra \((A, \mu, T)\) with coefficients in \((M, T_M)\). When \((M, T_M) = (A, T)\), we just denote \(C^\bullet_{\text{RBA}_A}(A, A), H^\bullet_{\text{RBA}_A}(A, A)\) by \(C^\bullet_{\text{RBA}}(A), H^\bullet_{\text{RBA}}(A)\) respectively, and call them the cochain complex, the cohomology of Rota-Baxter algebra \((A, \mu, T)\) respectively.

It is easy to see that the following proposition holds:

**Proposition 6.4.** There is a short exact sequence of complexes:

\[
0 \to s^{-1}C^\bullet_{\text{RBO}_A}(A, M) \overset{i}{\to} C^\bullet_{\text{RBA}_A}(A, M) \overset{p}{\to} C^\bullet_{\text{Alg}}(A, M) \to 0
\]

where \(i, p\) are the natural inclusion and projection respectively. Therefore we have a long exact sequence of cohomology groups

\[
0 \to H^0_{\text{RBA}_A}(A, M) \to \text{HH}^0(A, M) \to H^0_{\text{RBO}_A}(A, M) \to H^1_{\text{RBA}_A}(A, M) \to \text{HH}^1(A, M) \to \cdots
\]

\[
\cdots \to \text{HH}^p(A, M) \to H^p_{\text{RBO}_A}(A, M) \to H^{p+1}_{\text{RBA}_A}(A, M) \to \text{HH}^{p+1}(A, M) \to \cdots.
\]

**6.2. Maurer-Cartan characterisation of Rota-Baxter algebras.**

Now we verify easily by direct computation that the cohomology theory introduced above is exactly the deformation cohomology of Rota-Baxter algebras of arbitrary weight.

**Proposition 6.5.** Let \((A, \mu, T)\) be a Rota-Baxter algebra of weight \(\lambda\). Twist the \(L_\infty\)-algebra \(\mathfrak{C}_{\text{RBA}_A}(A)\) by the Maurer-Cartan element corresponding to the Rota-Baxter algebra structure \((A, \mu, T)\), then its underlying complex is exactly \(sC^\bullet_{\text{RBA}_A}(A)\), the shift of the cochain complex of Rota-Baxter algebra \((A, \mu, T)\), introduced in Definition 6.3.

Although \(\mathfrak{C}_{\text{RBA}_A}(A)\) is an \(L_\infty\)-algebra, the next result shows that once the associative algebra structure \(\mu\) over \(A\) is fixed, the graded space \(\mathfrak{C}_{\text{RBO}_A}(A)\), which, after twisting procedure, controls deformations of Rota-Baxter operators, is a genuine differential graded Lie algebra.

**Proposition 6.6.** Let \((A, \mu)\) be an associative algebra. Then the graded space \(\mathfrak{C}_{\text{RBO}_A}(A)\) can be endowed with a dg Lie algebra structure, and the set of its Maurer-Cartan elements is in bijection with the set of Rota-Baxter operators of weight \(\lambda\) on \((A, \mu)\). Given a Rota-Baxter operator \(T\) on associative algebra \((A, \mu)\), the underlying complex of the twisted dg Lie algebra \(\mathfrak{C}_{\text{RBO}_A}(A)\) by the corresponding Maurer-Cartan element is exactly the cochain complex of Rota-Baxter operator \(C^\bullet_{\text{RBO}_A}(A)\).
Proof. Consider $A$ as graded space concentrated in degree 0. Define $m = -s \circ \mu \circ (s^{-1} \otimes s^{-1}) : (sA)\otimes^2 \to sA$. Then $\alpha = (m, 0)$ is naturally a Maurer-Cartan element in $L_{\infty}$-algebra $C_{RBA,1}(A)$. By the construction of $l_n$ on $C_{RBA,1}(A)$, the graded subspace $C_{RBO,1}(A)$ is closed under the action of operators $\{l^\mu_n\}_{n \geq 1}$. Since the arity of $m$ is 2, the restriction of $l^\mu_n$ on $C_{RBO,1}(A)$ is 0 for $n \geq 3$. Thus $(C_{RBO,1}, \{l^\mu_n\}_{n=1,2})$ forms a dg Lie algebra. More explicitly, for $f \in \text{Hom}((sA)\otimes^n, A), g \in \text{Hom}((sA)\otimes^k, A)$,

$$l^\mu_1(f) = -l_2(m \otimes f) = -((-1)^{|f|+1} \lambda f \{m\}) = (-1)^n \lambda f \{m\}$$

$$l^\mu_2(f \otimes g) = l_3(m \otimes f \otimes g) = (-1)^{|f|} \left( (-1)^{|f|+1} m \circ (s f \otimes s g) - (-1)^{|f|+1} f \{m\} g \{s f\} \right)$$

$$= (-1)^{|f|} \left( s^{-1} m \circ (s f \otimes s g) + (-1)^{|f|+1} f \{m\} g \{s f\} \right)$$

Since $A$ is concentrated in degree 0, we have $C_{RBO,1}(A)_{-1} = \text{Hom}(sA, A)$. Take an element $\tau \in \text{Hom}(sA, A)_{-1}$. Then $\tau$ satisfies the Maurer-Cartan equation:

$$l^\mu_1(\tau) - \frac{1}{2} l^\mu_2(\tau \otimes \tau) = 0,$$

if and only if

$$-\lambda \tau \circ m + s^{-1} m \circ (s \tau \otimes s \tau) - \tau \circ (m \{s \tau\}) = 0.$$

Define $T = \tau \circ s : A \to A$. The above equation is exactly the statement that $T$ is a Rota-Baxter operator of weight $\lambda$ on associative algebra $(A, \mu)$.

Now let $T$ be a Rota-Baxter operator on associative algebra $(A, \mu)$. By the first statement, it corresponds to a Maurer-Cartan element $\beta$ in the dg Lie algebra $(C_{RBA,1}(A), l^\mu_1, l^\mu_2)$. More precisely, $\beta \in C_{RBO,1}(A)_{-1} = \text{Hom}(sA, A)$ is defined to be $\beta = T \circ s^{-1}$. For $f \in \text{Hom}((sA)\otimes^n, A)$, we compute $(l^\mu_1)^{\beta}(f)$. In fact,

$$(l^\mu_1)^{\beta}(f) = l^\mu_2(f) - l^\mu_2(\beta \circ f)$$

$$= (-1)^n \lambda f \{m\} + s^{-1} m \circ (s f \otimes s \beta) - \beta \{m\} f \{s f\}$$

$$+ s^{-1} m \circ (s f \otimes s \beta) + (-1)^n f \{m\} \{s \beta\},$$

which corresponds to $\partial^\mu(\hat{f})$ as defined in Definition 6.1. So the underlying complex of the twisted dg Lie algebra $C_{RBO,1}(A)$ by the corresponding Maurer-Cartan element $\beta$ is exactly the cochain complex of Rota-Baxter operator $C^{\bullet}_{RBO,1}(A)$.

□

Remark 6.7. The defining equation (5) of a Rota-Baxter operator is quadratic-linear in the Rota-Baxter operator if we consider the associative product as part of the underlying structure. This fact seems to suggest that Rota-Baxter operators are “relatively Koszul” with respect to the associative product, which may explain why the deformation complex of a Rota-Baxter operator in Proposition 6.1 is a differential graded Lie algebra instead of an $L_{\infty}$-algebra. However, to the best of our knowledge, there does not exist a theory of relative Koszul duality for operads in the literature and we are working on the project to develop such theory. While it is still out of reach, we give a computational proof of Proposition 6.1 instead of a conceptual one.
6.3. A curious example.

We conclude this section and also this paper by an example of Rota-Baxter algebra whose cohomology is explicitly computed.

Example 6.8. Let $A$ be the polynomial ring $k[x]$ in one variable and let $T$ be the indefinite integral operator on $A$, i.e., $T(x^n) = \frac{1}{n+1}x^{n+1}$ for any $n \geq 0$. Then the algebra $A$ endowed with the operator $T$ is a Rota-Baxter algebra of weight 0. We will show that

$$H^0_{RBA}(A) = 0, \forall n \geq 0.$$

By Proposition 1.11, the Rota-Baxter operator $T$ on $A$ induces a new multiplication $\star$:

$$x^m \star x^n = \left( \frac{1}{m+1} + \frac{1}{n+1} \right)x^{m+n+1}, \forall m, n \geq 0.$$

Actually, we have an isomorphism of non-unital associative algebras

$$A_\star \cong t k[t], x^n \mapsto \frac{1}{n+1}t^{n+1}$$

where $tk[t]$ is the set of all polynomials in a variable $t$ without constant terms, considered as a non-unital associative algebra via the usual multiplication. Thus $k \oplus A_\star \cong A$ as unital associative algebras. In this way, the cochain complex $C^\bullet_{RBO}(A)$ of the Rota-Baxter operator is exactly the normalized Hochschild cochain complex of $A$ with coefficients in the bimodule $\rightarrow_{A_{\emptyset}}$. Since $A$ is of global dimension 1, we have

$$H^0_{RBO}(A) = H^n_{Alg}(A_\star, \rightarrow_{A_{\emptyset}}) \cong \begin{cases} A & n = 0, 1 \\ 0 & n \geq 2, \end{cases}$$

We need to realise these isomorphisms on the complex $C^\bullet_{RBO}(A)$:

$$0 \to \text{Hom}(k, \rightarrow_{A_{\emptyset}}) \xrightarrow{\partial^0} \text{Hom}(A, \rightarrow_{A_{\emptyset}}) \xrightarrow{\partial^1} \text{Hom}(A \otimes k, \rightarrow_{A_{\emptyset}}) \to \cdots.$$ 

It is easy to see that $\partial^0$ vanishes, so $H^0_{RBO}(A) = \text{Hom}(k, \rightarrow_{A_{\emptyset}}) \cong A$ and $H^1_{RBO}(A) = \text{Ker}(\partial^1)$. Let’s make the isomorphism $H^1_{RBO}(A) \cong A$ explicit. In fact, for any $f \in C^1_{RBO}(A)$, $\partial^1(f) = 0$ if and only if $\partial(f)(x^m \otimes x^n) = 0$ for any $m, n \geq 0$. Then $f$ must fulfill the equation

$$T(x^m)f(x^n) - T(x^mf(x^n)) - f(x^m \star x^n) + f(x^m)T(x^n) - T(f(x^n)x^m) = 0, \forall m, n \geq 0.$$

Taking $m = 0$, we have

$$f(x^{n+1}) = \frac{n+1}{n+2} \left( xf(x^n) - T(f(x^n)) + \frac{1}{n+1}f(1)x^{n+1} - T(f(1)x^n) \right).$$

So the map $f$ is uniquely determined by $f(1) \in A$. Conversely, starting from any element $a \in A$ and using the above inductive formula, we can define a map $f_a : A \to A$ belonging to $\text{Ker}(\partial_1)$ via

$$f_a(1) = a, f_a(x^n) = x^n a - nT(x^{n-1}a), \forall n \geq 1.$$ 

Now we can compute the cohomology of the Rota-Baxter algebra $(A, \mu, T)$. Recall that $\Phi^0 = \text{Id}$, so $H^0_{RBA}(A) = 0$. According to Proposition 6.4, we have the long exact sequence:

$$0 = H^0_{RBA}(A) \to H^0_{Alg}(A) \xrightarrow{\text{Id}} H^0_{RBO}(A) \to H^1_{RBA}(A) \to HH^1(A) \xrightarrow{\Phi^1} H^1_{RBO}(A) \to H^2_{RBA}(A) \to H^2_{Alg}(A) = 0.$$
where $\Phi^1 : HH^1(A) \to H^1_{\text{RBO}}(A)$ is the induced map by $\Phi^1$, and $H^i_{\text{Alg}}(A) = H^i_{\text{RBO}}(A) = 0$ for all $i \geq 2$, we have

$$H^2_{\text{RBA}}(A) = \begin{cases} 
0 & n = 0 \text{ or } n \geq 3, \\
\ker \Phi^1, & n = 1, \\
\coker (\Phi^1), & n = 2
\end{cases}$$

Let’s describe $\ker (\Phi^1)$ and $\coker (\Phi^1)$ more precisely. Recall that there is an isomorphism

$$H^1_{\text{Alg}}(A) = \text{Der}(A) \cong A,$$

where $\text{Der}(A)$ is the space of derivations on $A$. Given a derivation $d$ on $A$, it is uniquely determined by $d(x) \in A$, providing the above isomorphism $\text{Der}(A) \cong A$. Given a derivation $d$ on $A$,

$$\Phi^1(d) = d \circ T - T \circ d \in \ker (\partial^1),$$

the corresponding element in $A$ for this cocycle is

$$(d \circ T - T \circ d)(1) = d \circ T(1),$$

as $d(1) = 0$. If $d \in \ker (\Phi^1)$, we must have $d \circ T(1) = d(x) = 0$, which implies that $d = 0$. So $\Phi^1$ is injective and $H^2_{\text{RBA}}(A) = \ker \Phi^1 = 0$. And $d(x)$ can be any element of $A$, so $\Phi^1$ is also full and $H^2_{\text{RBA}}(A) = \coker (\Phi^1) = 0$.

Therefore, we have proven that

$$H^2_{\text{RBA}}(A) = 0, \forall n \geq 0.$$

Our computation shows that this Rota-Baxter algebra has no nontrivial deformations when deforming simultaneously both the associative algebra structure and the Rota-Baxter operator, although there does exist nontrivial deformations when deforming only the Rota-Baxter operator.

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**School of Mathematical Sciences, Key Laboratory of Mathematics and Engineering Applications (Ministry of Education), Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai 200241, P.R.China**

*Email address:* wangkaimath@hotmail.com

*Email address:* gdzhou@math.ecnu.edu.cn