3D numerical modeling of the plasma beam expansion using the MHD-kinetic approach

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Abstract. In the current work a three-dimensional numerical model of a plasma beam expansion in the background plasma based on the kinetic description of an ion plasma component is proposed. To describe electrons the magneto-hydrodynamics equations are used. The Vlasov kinetic equations have been solved using the particle-in-cell (PIC) method. The finite-difference schemes have been utilized to solve the Maxwell system of equations and equations for electrons. The series of computations have been carried out to study collisionless mechanisms of the plasma flow interaction and a structure of the generated waves dealing with the conditions of laboratory experiments on the facility of ILP SB RAS.

1. Introduction

The 3D numerical modeling of the interaction of high-speed collisionless plasma flows is considered in the paper. Frequently, one can find these processes requiring the study in the astrophysics. The examples of such phenomena are, for instance, the Solar flares, the interaction of the Solar wind with the Earth’s magnetosphere, the Supernova explosions, etc. The interaction processes of collisionless plasma flows are studied in different experiments, such as the barium cloud experiments in the Earth’s magnetosphere tail (AMPTE) [1] and the experiments on the KI-1 facility (ILP SB RAS) [2]. KI-1 is the only experiment facility in Russia where such processes are studied. Another similar facility LAPD (LArge Plasma Device) [3] is in the University of California (UCLA).

Before, the processes of the plasma flow interaction had been modeled in the 2D axisymmetric geometry $(r, z)$, mostly [4]. At the same time some important physical processes, such as 3D instabilities, had been neglected. Moreover, it was impossible to take into account the constructional facility characteristics. However, the 3D computations, comparing with those in the 2D case, require more computational resources (such as a computer memory and speed), therefore, numerical algorithms have to admit a problem solution with large space and time steps. In particular, it is necessary to use the hybrid models where an ion component of plasma is described by the kinetic approach and an electron one is considered as a fluid.

In the current paper a mathematical model and a 3D numerical algorithm based on the particle in cell (PIC) method are proposed and, also, numerical computation results are given.
2. Numerical model

The 3D problem is considered in the following statement. The considered computational domain in the Cartesian coordinates \((X, Y, Z)\) has the parallelepiped shape and contains the stationary plasma background of a density \(n_0\). In the domain, there is a uniform magnetic field \(\vec{B} = (0, 0, B_z)\). The plasma flow is modeled by a plasma beam expanding in the cross to the magnetic field direction. At the initial time the beam is at the left domain boundary \(X = 0\) and has the shape of a sphere of a limited radius \(r_0\). Plasma density in the sphere is spread uniformly. The velocities of beam particles are distributed depending on angle \(\theta\) and an initial velocity on the beam front. The angle \(\theta\) is the angle between a radius vector \(r\) of a separate particle, which comes out from an injection point, and the axis passing through the beam center and the injection point (Fig. 1).

With time the plasma beam starts to expand in the cross direction to \(\vec{B}\) and to interact with the magnetic field as well as with the background plasma. In the problem statement

![Figure 1. Schematic of the calculation model.](image)

the boundary conditions are the same as the initial conditions on the domain boundary (the boundary conditions of the first kind). To model the process of the plasma and the magnetic field interaction, the 3D hybrid model based on the 2D numerical model described in [4], [5] is used. The considered physical process of the interaction of the collisionless plasma and the magnetic field is described by the Vlasov’s kinetic equation for ions and by the magnetic hydrodynamics (MHD) equations for electrons.

The equation for ions is

\[
\frac{\partial f_i}{\partial t} + \vec{v} \frac{\partial f_i}{\partial \vec{r}} + \vec{F_i} \frac{1}{m_i} \frac{\partial f_i}{\partial \vec{v}} = 0, \quad \vec{F_i} = e \left( \vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) - \vec{R}_e,
\]

where \(f_i\) is the ion distribution function, \(m_i\) is an ion mass, \(\vec{v}\) and \(\vec{\nu}\) are a coordinate vector, a velocity vector, \(\vec{F_i}\) is the Lorentz force. The ion-electron friction force is \(\vec{R}_e = m_e \kappa \left( \vec{V_i} - \vec{V_e} \right)\), \(m_e\) is an electron mass, \(\kappa\) is the collision frequency. The average velocities of ions and electrons are \(\vec{V_i}\) and \(\vec{V_e}\), \(\vec{E}\) and \(\vec{B}\) are electric and magnetic field intensities. The plasma density \(n\) and the average ion velocity \(\vec{V_i}\) are computed from the distribution function \(f_i\). In our case, we consider the hydrogen ions only.

For electrons, the MHD equations given below are used. The motion equation for the electron component is written as

\[
0 = -en \left( \vec{E} + \frac{1}{c} \vec{V}_e \times \vec{B} \right) - \nabla P_e + n\vec{R}_e,
\]
here $P_e$ is an electron pressure ($P_e = nT_e$) and $T_e$ is an electron temperature. Here the assumption $m_e/m_i \ll 1$ is used.

The equation for the electron temperature $T_e$ is

$$n \left( \frac{\partial T_e}{\partial t} + (\vec{V}_e \cdot \nabla) T_e \right) - (\gamma - 1) P_e \nabla \cdot \vec{V}_e = (\gamma - 1) (Q_e - \nabla \cdot q_e),$$

(3)

here $\gamma$ is the adiabatic index, $Q_e$ is the electron heating and $q_e$ is due to the plasma thermal conductivity. $Q_e = \vec{J}^2/\sigma$, $q_e = -\chi \nabla T_e$, where $\vec{J}$ are the plasma current, $\sigma$ is the plasma conductivity and $\chi$ is the thermal conductivity coefficient.

To determine the magnetic and electric fields, the Maxwell’s equations are used:

$$\nabla \times \vec{B} = \frac{4 \pi}{c} \vec{J},$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t},$$

$$\nabla \vec{E} = 0, \ \nabla \vec{B} = 0.$$  

(4)

In the paper, two important assumptions have been made. Firstly, it is the quasineutrality assumption, when the electron charge density is equal to the ion charge density $n_i = n_e = n$. Secondly, the displacement current is neglected in Maxwell equations. The displacement current is neglected as the process frequencies $\Omega \ll \omega_{pe} = \sqrt{4\pi ne^2/m_e}$ and $|\vec{V}_i|, |\vec{V}_e| \ll c$.

Finally, the sum of the friction forces has to be zero. If there is the friction force then there has to be the electron heating as the energy has to transfer into heating of the electron component. In this model, there are no influxes of energy through the boundaries, then the conservation of energy law has to be maintained. In the general form the energy conservation law is as follows

$$\sum_\alpha \frac{m_\alpha}{2} \int f_\alpha(\vec{r}, \vec{v}, t) v^2 d\vec{v} d\vec{r} + \int \left( n_e \frac{m_e v_e^2}{2} + \frac{B^2}{8\pi} + \frac{n_e T_e}{\gamma - 1} \right) d\vec{r} = \text{const},$$

(5)

where $\alpha$ is an ion particle sort (in the paper only one sort of hydrogen ions is used).

When solving the problems of the plasma physics, the energy conservation law is one of the criteria of the solution validity as many processes in plasma are instable. The difficulty of the 3D modeling is in high volume of processed data and large computation time. It leads to the necessity to debug the algorithm, essentially, in order to have opportunity to get high-quality results with the relatively large time and space steps.

3. Algorithms and solution methods

To solution the kinetic Vlasov’s equation (1), PIC method is used [4], [6], [7]. The motion equations for model particles are characteristics of the Vlasov’s equation.

$$\frac{d\vec{r}_j}{dt} = \vec{v}_j,$$

$$\frac{d\vec{v}_j}{dt} = \frac{q_j}{m_j} \left( \vec{E} + \frac{1}{c} \vec{v}_j \times \vec{B} \right) - \vec{R}_e,$$

(6)

where $j$ is a number of particle, $\vec{r}_j$ and $\vec{v}_j$ are a coordinate vector and a velocity vector, $q_j$ is a particle charge, $m_j$ is a particle mass.
The plasma density $n$ and the average velocity $\vec{V}_i$ are calculated by using model particles.

$$n = \frac{1}{r} \sum_j q_j R (r - r_j) R (z - z_j), \quad \vec{V}_i = \frac{1}{r n} \sum_j q_j r_j R (r - r_j) R (z - z_j).$$ (7)

The accuracy of PIC method depends on the number particle per cell. The form factor of the PIC method is used, when the particles are "smearing" into two cells:

$$\mathcal{R}(x) = \left\{ \begin{array}{ll}
0, & |x| > h; \\
\frac{1}{h} \left(1 - \frac{x}{h}\right), & |x| \leq h.
\end{array} \right.$$

The equation for the electron temperature and Maxwell equation are solved in the usual way on a uniform rectangular grid by using a finite-difference schemes first order accuracy in time and space $O(\tau, h)$, with specification of the boundary and the initial conditions [4], [6]. In the model, the law of energy conservation (5) is used to control the accuracy of the problem solutions.

The scheme of the program is shown in Fig. 2. Let us consider in more detail the individual blocks of the program.

**Block 0:** Setting up the initial data. The initial magnetic field intensity $\vec{B}_0$ and the initial particle velocity $\vec{v}_0$ are set on the fractional time step $m - 1/2$. Electric field intensity $\vec{E}_0$, the particle coordinate vectors $\vec{r}_0$ and the initial density $n_0$ are set on the time step $m$. In this block, the initial temperature $T_0$ and the number of particles $N_\text{part}$ are also set.

**Block 1:** Calculation of the magnetic field intensity $\vec{B}^m$. Since, at the initial moment of time, the electric field intensity $\vec{E}$, is set on the time step $m$, then from the Faraday law it possible to calculate magnetic field $\vec{B}$ also at the same time step $m$.

**Block 2:** Calculation of velocity and coordinate of particles. This is the so-called Lagrange stage. At this stage, from the motion particles equations (6) the velocities and the coordinates are computed on the time - shifted grids ($\vec{v}^{m+1/2}$, $\vec{r}^{m+1}$). To calculate the electric field intensity $\vec{E}$ and the magnetic field intensity $\vec{B}$ to the point of the particle position the linear interpolation is produced. At this stage, an implicit scheme is obtained for velocities, which is transformed into a system of linear algebraic equations at each time step. The scheme has a second order of accuracy. The plasma density $n$ at the time step $m + 1$ and the average particle velocities $\vec{V}$ at the fractional time step $m + 1/2$ are calculated by the found coordinates $\vec{r}^{m+1}$ and the velocities $\vec{v}^{m+1/2}$ of model particles, respectively.

**Block 3:** Calculation of the magnetic field intensity $\vec{B}^{m+1/2}$. The magnetic field intensity is calculated from Faradays law.

**Block 4:** Calculation of the currents $\vec{J}^{m+1/2}$ and the average electron velocities $\vec{V}_{e}^{m+1/2}$. The currents, at the fractional time step, are found from the Maxwell equations, the intensity on the desired time layer $m + 1/2$ is already calculated in block 3. The average electron velocities $\vec{V}_{e}$ on the layer $m + 1/2$ are determined from the equation $\vec{J}^{m+1/2} = \rho^{m+1/2} \left( \vec{V}_{i}^{m+1/2} - \vec{V}_{e}^{m+1/2} \right)$.

Due to neglect of the displacement current term, the electric field intensity $\vec{E}$ has to be determined from the MHD equation for the plasma electron component. The electric field intensity $\vec{E}$ is calculated on the time step $m + 1$.

**Block 5:** Calculation of the temperature and the energy. In this block the electron temperature is calculated at the time layer $T_{e}^{m+1}$ from the equation for temperature (3). Next, the kinetic energy of the beam $w_{\text{cloud}}$, the background particles energy $w_{\text{bg}}$, the magnetic field energy $w_{\text{mag}}$ and the total energy $w_{\text{mag}}$ also are calculated.

**Block 7:** Recording data. The data are recorded for further processing.
Checks the completion condition of the computational. Computational is terminated when preset time step is reached. If the preset step is not yet reached then the computational is continued from block 1 and the magnetic field intensity is started to calculate at the next time step.

Figure 2. Flowchart of the 3D hybrid code. The main steps of the computational are presented.

4. Results and discussion
To study the dynamics of the interaction of a plasma beam with a uniform magnetic field, numerical experiments were carried out with characteristic parameters close to the parameters of KI-1 facility (ILP SB RAS) [2], [9]. The aim of the numerical experiments was to demonstrate the possibility of numerical simulation of the processes observed at the KI-1 facility based on the presented 3D hybrid numerical model.

The characteristic parameters of the problem are given in Table 1. For the given Mach number $M_A > 1.8$, the deceleration radius $\tilde{R}$ is equal to the gas-dynamic radius $R_{gas}$ and is calculated by the follow formula:

$$\tilde{R} = R_{gas} = \left( \frac{3M}{4\pi n_0 m_i} \right)^{1/3},$$

where $M$ is the mass of a spherical plasma cloud, $n_0$ is a density of the background plasma. Mach number $M_A = V_{max}/V_A$, where again $V_{max}$ is a initial particle velocity at the beam front
and $V_A = B_0/\sqrt{4\pi n_s m_i}$ is Alfvén velocity. It should be noted that the formula (8) is derived under the assumption that the spherical cloud is expanding. To estimate the deceleration radius of the beam according to (8), it is assumed that the beam is a part of some large spherical plasma cloud, whose mass is eight times larger then the mass of the beam. The delta parameter is calculated as follows: $\delta = (R_{gas}/R_L)^2$. For the effective interaction of the beam particles with the magnetic field, it is necessary that the condition $\delta > 1$ be satisfied. Also, important characteristic dimensions are larmor radius $R_L = V_{max}/\omega_{iH}$ and the length of $c/\omega_{pi}$, where $\omega_{iH} = eB_0/(m_i c)$, $\omega_{pi}$ is the ion plasma frequency. Numerical parameters are given in Table 2.

### Table 1. Characteristic parameters of the problem

| name                     | designation | dimension | value     |
|--------------------------|-------------|-----------|-----------|
| Alfvén velocity          | $V_A$       | [cm/s]    | $3.2 \times 10^6$ |
| deceleration radius by field | $R_B$     | [cm]      | 100.00    |
| deceleration radius by backgr. | $R_{gas}$ | [cm]      | 42.00     |
| larmor radius             | $R_L$       | [cm]      | 27.00     |
| delta parameter           | $\delta$    | [-]       | 2.50      |
| Mach number               | $M_A$       | [-]       | 6.50      |
| characteristic length     | $c/\omega_{pi}$ | [cm]   | 4.16      |

### Table 2. Numerical parameters of the problem

| name                        | designation | dimension | value     |
|-----------------------------|-------------|-----------|-----------|
| number of cells by X, Y, Z  | $im \times lm \times km$ | [-]      | $32 \times 32 \times 32$ |
| size of calc. region by X, Y, Z | $xm \times ym \times zm$ | [cm]  | $115 \times 115 \times 115$ |
| number of background particles in cell | $lp$       | [-]       | 27.00     |
| number of beam particles    | $jmc$       | [-]       | $0.8 \times 10^6$ |

Numerical experiments were carried out when the number of physical particles $N_{H^+}$ and the initial radius of the cloud $r_0$ were varied. The beam plasma as well as background plasma consist of the hydrogen ions. At the initial time, the beam is located at the left boundary of calculation domain $X = 0$ and has the spherical shape of radius $r_0$.

Consider the dynamics of the plasma beam density when Mach number $M_A = 6.5$ and the initial beam radius $r_0 = 10[cm]$. The velocities of the model particles are distributed as follows:

$$V_r = V_{max} r \cos^3(\theta)/(2r_0),$$

where $V_r$ is the velocity of an individual particle directed to the the positive direction of the axes along the straight line connecting the injection point and the particle position point (radius vector of the particle). The initial beam velocity is $V_{max}$, $r$ is the module of particle radius vector, $r_0$ is the initial radius of the cloud, $\theta$ is the angle between the radius vector of the particle and the axis, passing through the injection point and the center of the beam (Fig.1).

Fig. 3 (a – c) shows the plasma density beam at the $XZ$ plane at consistent moments of time $4[\mu s]$, $6[\mu s]$ and $8[\mu s]$. At the initial moment of time, the plasma beam begins to expand...
in the transverse direction to the magnetic field, transferring its energy to the field and the background plasma (Fig. 3 (a – c)). In the process of expansion, the beam is decelerated by the magnetic field (Fig. 3 (b, c)). After reaching the deceleration radius $\tilde{R}$, the reverse process begins: the magnetic field begins to act on the beam, compressing it in the opposite direction (Fig. 3 (c)). As the plasma beam expands, a current layer forms at the beam boundary and also an ion concentration appears on its front (Fig. 3 (a – c)).

Fig. 4 (a – c) illustrates the density of a plasma beam at the $YX$ plane at consistent moments of time (4$[\mu s]$, 6$[\mu s]$ and 8$[\mu s]$). It can be seen from the figure that a dense current layer is formed (Fig. 4 (a)) with pronounced inhomogeneities (Fig. 4 (b, c)), which indicate a developing instability of the current layer.

Figures 5 (a – c) and 6 (a – c) show the background plasma density at the $XZ$ and $YX$ planes at successive times (4$[\mu s]$, 6$[\mu s]$, and 8$[\mu s]$). It can be seen that the particles of the background plasma are displaced from the region where the beam is located. This also indicates the displacement of the magnetic field and the formation of a magnetic cavity – a region of almost zero magnetic field. At the boundary of the magnetic cavity, a higher ion density concentration which has asymmetric shape is noticeable 6 (a – c).

The distributions of the ion density of the beam and the background plasma presented at the figures 3 – 6 provide a visual representation of the nature of the plasma flows and their interaction for the Mach numbers $M_A \gg 1$.

The figures 7 and 8 show the phase planes for the particles of the beam and the background particles along the rays that are drawn from the injection point and belong to the $XZ$ plane. Phase planes were constructed for the rays drawn at different angles (45° and 86°), measured...

**Figure 3.** Beam density at successive times in the $XZ$ plane.

**Figure 4.** Beam density at successive times in the $YX$ plane.
Figure 5. Background density at successive times in the $XZ$ plane.

Figure 6. Background density at successive times at the $YX$ plane.

from the $Z$ axis. From the figures 7 (a, b) it can be seen that the particles of the beam that move at the angles $45^\circ$ and $86^\circ$, begin to wrap around the deceleration radius $\tilde{R}$. The irregular pattern of the distribution of the beam and background particles on the phase plane (Fig. 7 (c) and 8 (c)), talks about developing instability. For the particles flying at an angle of $86^\circ$ (Fig. 7 (c) and 8 (c)), the irregular pattern is observed to a greater extent, since the expansion occurs strictly across the magnetic field.

Figure 7. The phase planes of the beam particles along the rays $r$, lying at angles $45^\circ$ and $86^\circ$, at successive times.

5. Conclusion
Thus, the 3D numerical model allowing study the dynamics of nonstationary nonlinear processes of the high-speed (super-Alfvénic) plasma flow interaction in the constant magnetic field
Figure 8. Phase planes of background particles along the rays $r$, lying at angles 45° and 86°, at successive times.

has been developed. When computing, there have been observed such effects as the current layer formation, its instability and the magnetic cavity generation. The familiar phenomena are observed in the laboratory experiments taking place on the KI-1 facility (ILP SB RAS) [9], [10].

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