Efficient quantification of non-Gaussian spin distributions

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We study theoretically and experimentally the quantification of non-Gaussian distributions via non-destructive measurements. Using the theory of cumulants, their unbiased estimators, and the uncertainties of these estimators, we describe a quantification which is simultaneously efficient, unbiased by measurement noise, and suitable for hypothesis tests, e.g., to detect non-classical states. The theory is applied to cold 87Rb spin ensembles prepared in non-gaussian states by optical pumping and measured by non-destructive Faraday rotation probing. We find an optimal use of measurement resources under realistic conditions, e.g., in atomic ensemble quantum memories.

Introduction - Non-Gaussian states are an essential requirement for universal quantum computation1,2 and several quantum communication tasks with continuous variables, including improving the fidelity of quantum teleportation3 and entanglement distillation4, 5. Optical non-Gaussian states have been demonstrated6–10 and proposals in atomic systems11–14 are being actively pursued. In photonic systems, histograms15 and state tomography6, 7, 9, 10 have been used to show non-Gaussianity, but require a large number of measurements. For material systems with longer time-scales these approaches may be prohibitively expensive. Here we demonstrate the use of cumulants, global measures of distribution shape, to show non-Gaussianity in an atomic spin ensemble. Cumulants can be used to show non-classicality16–18, can be estimated with few measurements and have known uncertainties, a critical requirement for proofs of non-classicality.

Approach - Quantification or testing of distributions has features not encountered in quantification of observables. For example, experimental measurement noise appears as a distortion of the distribution that cannot be “averaged away” by additional measurements. As will be discussed later, the theory of cumulants naturally handles this situation. We focus on the fourth-order cumulant κ4, the lowest-order indicator of non-Gaussianity in symmetric distributions such as Fock19 and “Schrödinger kitten” states20, 21. We study theoretically and experimentally the noise properties of Fisher’s unbiased estimator of κ4, i.e., the fourth “k-statistic” κ4, and find optimal measurement conditions. Because κ4 is related to the negativity of the Wigner function15, this estimation is of direct relevance to detection of non-classical states. We employ quantum non-demolition measurement, a key technique for generation and measurement of non-classical states in atomic spin ensembles20, 21 and nano-mechanical oscillators22.

Moments, cumulants and estimators - A continuous random variable X with probability distribution function P(X) is completely characterized by its moments μk ≡ ∫ X^k P(X)dX or cumulants κn = μn − ∑_{k=1}^{n-1} (n−k−1) μkn−kκk, where ( ) is the binomial coefficient.

Since Gaussian distributions have κn>2 = 0, estimation of κ4, (or κ3 for non-symmetric distributions), is a natural test for non-Gaussianity. In an experiment, a finite sample {X1...XN} from P is used to estimate the κ’s. Fisher’s unbiased estimators, known as “k-statistics” kn, give the correct expectation values (kn) = κn for finite N 23. Defining Sn = ∑_{i=1}^{N} X^i we have:

κ3 = (2S3 1 − 3NS1S2 + N2S3)/N (1)

κ4 = (−6S4 1 + 12NS2S2 − 3N(N − 1)S2

−4N(N − 1)S1S3 + N2(N + 1)S4)/N (2)

where N(m) ≡ N(N − 1)...(N − m).

We need the uncertainty in the cumulant estimation to test for non-Gaussianity, or to compare non-Gaussianity between distributions. For hypothesis testing and maximum-likelihood approaches, we need the variances of κ3, κ4 for a given P. These are found by combinatorial methods and given in reference24:

var(κ3) = κ6/N + 9N(κ2κ4 + κ3 2)/N(1) + 6N2κ3 2/N (3)

var(κ4) = κ8/N + 2N(8κ6κ2 + 24κ5κ3 + 17κ2 3)/N (1)

+72N2(κ4κ3 2 + 2κ3κ2 2)/N (2)

+24N2(N + 1)κ2 4/N (3). (4)

It is also possible to estimate the uncertainty in κ4 from data {X} using estimators of higher order cumulants25. The efficiency of cumulant estimation is illustrated in Fig.

Measurement noise - When the measured signal is Z = X + Y, where X is the true value and Y is uncorrelated noise, the measured distribution is the convolution P(Z) = P(X) ⊗ P(Y). The effect of this distortion on cumulants is the following: for independent variables, cumulants accumulate (i.e., add)26, so that
Ten realizations of $k_3$ versus $N$ drawn from each of the four distributions. Shaded regions show $κ_4 ± \sqrt{\text{var}(κ_4)}$, from Eqs 4. With $N = 1000$, $k_4$ distinguishes $p = 1/2$ (blue) from $p = 0$ (green, Gaussian) with $> 7σ$ significance, even though the histograms look similar “to the eye.”

$κ_n^{(Z)} = κ_n^{(X)} + κ_n^{(Y)}$, where $κ_n^{(Q)}$, $κ_n^{(Q)}$ indicate $κ_n$, $κ_n$ for distribution $P(Q)$. The extremely important case of uncorrelated, zero-mean Gaussian noise, $κ_n^{(Y)} = \sigma^2, \text{and other cumulants zero}$, is thus very simple: $κ_n^{(Z)} = κ_n^{(X)}$ except for $κ_n^{(Z)} = κ_n^{(X)} + \sigma^2$. Critically, added Gaussian noise does not alter the observed $κ_3, κ_4$.

Experimental system and state preparation - We test this approach by estimating non-Gaussian spin distributions in an atomic ensemble, similar to ensemble systems being developed for quantum networking with non-Gaussian states [24]. The collective spin component $F_z$ is measured by Faraday rotation using optical pulses. The detected Stokes operator is $S_y^{(\text{out})} = S_y^{(\text{in})} + GN_z F_z/2$, where $G$ is a coupling constant, $N_z$ is the number of photons, and $S_y^{(\text{in})}$ is the input Stokes operator, which contributes quantum noise. In the above formulation $X = F_z$, $Y = 2S_y^{(\text{out})}/(GN_z)$ and $Z = 2S_y^{(\text{out})}/(GN_L)$.

The experimental system is described in detail in references [21, 23, 26]. An ensemble of $\sim 10^6$ $^{87}$Rb atoms is trapped in an elongated dipole trap made from a weakly focused 1030nm beam and cooled to 25μK. A non-destructive measurement of the atomic state is made using pulses of linearly polarized light detuned 800MHz to the red of the $F = 1 \rightarrow F' = 0$ transition of the $D_2$ line and sent through the atoms in a beam matched to the transverse cloud size. The pulses are of 1μs duration, contain $3.7 \times 10^6$ photons on average, and are spaced by 10μs to allow individual detection. The 240:1 aspect ratio of the atomic cloud creates a strong paramagnetic Faraday interaction

$G \approx 6 \times 10^{-8}$ rad/spin. After interaction with the atoms, $S_y^{(\text{out})}$ is detected with a shot noise limited (SNL) balanced polarimeter in the $± 45^\circ$ basis. $N_L$ is measured with a beam-splitter and reference detector before the atoms. The probing-plus-detection system is shot-noise-limited above $3 \times 10^5$ photons/pulse. Previous work with this system has demonstrated QND measurement of the collective spin $F_z$ with an uncertainty of $∼ 500$ spins [21, 26].

We generate Gaussian and non-Gaussian distributions with the following strategy: we prepare a “thermal state” (TS), an equal mixture of the $F = 1, m_F = −1,0,1$ ground states, by repeated unpolarized optical pumping between the $F = 1$ and $F = 2$ hyperfine levels, finishing in $F = 1$ [26]. By the central limit theorem, the TS of $10^6$ atoms is nearly Gaussian with $⟨F_z⟩ = 0$ and $\text{var}(F_z) = σ^2 = 2N/3$. By optical pumping with pulses of circularly-polarized light we displace this to $⟨F_z⟩ = α$, with negligible change in $\text{var}(F_z)$ [27], to produce $P_α(F_z) = (e^{|α|^2})^{-1}\exp[-(|F_z − α|^2)/2σ^2]$. By displacing different TS alternately to $α_+ \text{ and } α_-$, we produce an equal statistical mixture of the two displaced states, $P_α^{(NG)}(F_z) = [P_+ (F_z) + P_-(F_z)]/2$. With properly-chosen $α_\pm$, $P_α^{(NG)}(F_z)$ closely approximates marginal distributions of mixtures of $n = 0,1$ Fock states and $m = N, N−1$ symmetric Dicke states. The experimental sequence is shown in Fig. 2.

Detection, Analysis and Results - For each preparation, 100 measurements of $F_z$ are made, with readings (i.e.,
estimated $F_2$ values by numerical integration of the measured signal $m_i = 2S_y^{\text{out}}(i)/N_L(i)$. Because the measurement is non-destructive and shot noise limited, we can combine $N_R$ readings in a higher-sensitivity multipulse with reading $M = \sum m_i$. This has the distribution $P_{\alpha}(M) = \exp[-(M - \alpha)^2/(2\sigma_M^2)]/(\sigma_M\sqrt{2\pi})$ where the variance $\sigma_M^2 = \sigma_A^2N'_R\sigma_A^2+N_R\sigma_R^2$ includes atomic noise $\sigma_A^2N'_R\sigma_A^2$ and readout noise, $\sigma_R^2 = N_R/N_L$ with $N'_R = N_A/N_A^X$. The variance $\sigma_A^2$ is determined from the scaling of $\text{var}(M)$ with $N_A$ and $N_R$, as in [26]. The readout noise can be varied over two orders of magnitude by appropriate choice of $N_R$. For one probe pulse and the maximum number of atoms we have $\sigma_A^2/\sigma_R^2 = 84.7$.

To produce a non-Gaussian distribution, we compute metapulses from $N_R$ samples drawn from displaced thermal state $\text{DM}[\alpha_+|\alpha_-]$ or $\text{DM}[\alpha_-|\alpha_+]$ preparations with equal probability, giving distribution $F_{\alpha}(M) = P_{\alpha}(M)$ with $P_{\alpha}(M)/2$. With $\alpha_M = (\alpha_+ - \alpha_-)/2$, the distribution has $k_2n+1 = 0$, $k_2 = \alpha_M^2 + \sigma_M^2$, $k_4 = -2\alpha_M^4$, $k_6 = 160\alpha_M^8$, $k_8 = -272\alpha_M^8$. Our ability to measure the non-Gaussianity is determined by $\langle k_4 \rangle = \kappa_4$ and from Eq. (5)

$$\text{var}(k_4) = 136N\alpha_M^8/N(1) - 144N^2\alpha_M^4(\alpha_M^2 + \sigma_M^2)^2/N(2) + 24N^2(N+1)(\alpha_M^2 + \sigma_M^2)^2/N(3).$$

As shown in Fig. 3 the experimentally obtained values agree well with theory, and confirm the independence from measurement noise.

The “signal-to-noise ratio” for $\kappa_4$, $S = \kappa_4^2/\text{var}(k_4)$, is computed using Eq. (4), $\kappa_4 = -2\alpha_M^4$, and experimental $\alpha_M$, $N_R$, $\sigma_R$, is shown as curves in Fig. 4. We can confirm this $S$ experimentally by computing $S_N = \langle k_4 \rangle^2/\text{var}(k_4)$ using $k_4$ values derived from several realizations of the experiment, each sampling $P_{\alpha}^{\text{NG}}N$ times. In the limit of many realizations $S_N \rightarrow S$. We employ a bootstrapping technique: From 100 samples of $P_{\alpha}^{\text{NG}}N$ for given parameters $\alpha_M$, $N_R$ and $N_A$, we derive thirty-three $N = 20$ realizations by random sampling without replacement, and compute $\langle k_4 \rangle$ and $\text{var}(k_4)$ on the realizations. As shown in Fig. 4, good agreement with theory is observed. Optimum estimation of non-Gaussian distributions - Finally, we note that in scenarios where measurements are expensive relative to state preparation (as might be the case for QND measurements of optical fields or for testing the successful storage of a single photon in a quantum memory), optimal use of measurement resources (e.g. measurement time) avoids both too few preparations and too few proplings.

We consider a scenario of practical interest for quantum networking: a heralded single-photon state is produced and stored in an atomic ensemble quantum memory. Assuming the ensemble is initially polarized in the $X$ direction, the storage process maps the quadrature components $X$, $P$ onto the corresponding atomic spin operators $X_A, P_A \propto F_z - F_y$, respectively. QND measurements of $F_z$ are used to estimate $X_A$, and thus the non-Gaussianity of the stored single photon. Due to imperfect storage, this will have the distribution of a mixture of $n = 0$ and $n = 1$ Fock states: $\rho = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$. For a quadrature $X$, we have the following probability distribution $P_\rho(X) = \exp[-x^2/(2\sigma_0^2)](px^2/\sigma_0^2 + 1 - p)/\sqrt{2\pi}\sigma_0$, where $\sigma_0$ is the width of the $n = 0$ state.

Taking into account the readout noise $\sigma_R^2$, the cumulants

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3}
\caption{(color online) Measured and predicted $k_4$ with residuals for non-Gaussian distributions of different $\alpha$. Readout noise is varied by the choice of $N_R$. Data is normalized to $N_R$ and $\sigma_A$. Top: Points show normalized $-k_4$, red line (top) shows $-k_4 + \sqrt{\text{var}(k_4)}$ calculated from the distribution parameters for the largest readout noise. Bottom: normalized residuals $(-k_4 + \kappa_4)/\sqrt{\text{var}(k_4)}$. The normalization is done with the expected $\text{var}(k_4)$ for each $N_R$. Measured $k_4$ agrees well with theory, in particular, measurement noise increases the observed variance, but not the expectation.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4}
\caption{(color online) Signal-to-noise in estimation of $\kappa_4$ versus readout noise for different $\alpha' = \alpha_M/(N_R\sigma_A)$. Points show measurement results, lines show theory. (details in the text)}
\end{figure}
are $\kappa_{\text{odd}} = 0$, $\kappa_2 = (2p + 1)\sigma_2^H + \sigma_2^H$, $\kappa_4 = -12p^2\sigma_4^H$, $\kappa_6 = 240p^6\sigma_6^H$, $\kappa_8 = -10080p^8\sigma_8^H$, where the readout noise $\sigma_2^H$ is included as above. Here $\kappa_4$ is directly related to the classicality of the state, since $p > 0.5$ implies a negative Wigner distribution \[19\].

For a fixed total number of measurement resources $N_M N_R$, an optimal distribution of resources per measurement $N_R$ exists as shown in Fig. 5. With increasing $N_R$, the signal-to-noise first increases due to the improvement of the measurement precision. Then, once the increased measurement precision no longer gives extra information about $k_4$, the precision decreases due to reduced statistics because of the limited total number of probes. For a large total number of measurements, we can derive a simplified expression of this optimum. We derive asymptotic expressions of $S$: $S_L (S_R)$ for $S_R < \sigma_0$ ($S_R \gg \sigma_0$).

The optimal $N_R$ is found by solving $S_L = S_R$ giving $\sigma_R^8 \approx \sigma_0^8 (1 + 8p - 12p^2 + 48p^3 - 24p^4)$. For this optimal $\sigma_R$, the measurement noise is in the same order of magnitude as the characteristic width of the non-Gaussian distribution.

**Conclusion** - The cumulant-based methods described here should be very attractive for experiments with non-Gaussian states of material systems such as atomic ensembles and nano-resonators, for which the state preparation time is intrinsically longer, and for which measurement noise is a greater challenge than in optical systems. Cumulant-based estimation is simultaneously efficient, requiring few preparations and measurements, accommodates measurement noise in a natural way, and facilitates statistically-meaningful tests, e.g., of non-classicality. Experimental tests with a cold atomic ensemble demonstrate the method in a system highly suitable for quantum networking, while the theory applies equally to other quantum systems of current interest.

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[26] M. Koschorreck, M. Napolitano, B. Dubost, and M. W. Mitchell, Phys. Rev. Lett. 104, 093602 (2010).

[27] G. Tóth and M. W. Mitchell, New Journal of Physics 12, 053007 (2010).