Explicit exact expression for the Thomas precession

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GAIA-CA-TN-LO-SK-004-2
issue 2, 14 March 2008

This work gives an explicit exact expression for the Thomas precession arising in the framework of Special Theory of Relativity as the spatial rotation resulting from two subsequence Lorentz boosts. The final result for the orthogonal matrix of Thomas precession is given by Eqs. (21)–(25). A trivial calculation leads to the compact formula (26) for the angle of rotation due to Thomas precession.

In the framework of Gaia the special-relativistic Thomas precession is an important step in the derivation of an aberrational formula with the Mansouri-Sexl parameters. The latter formula will be used to test the Local Lorentz Invariance with Gaia data as will be explained elsewhere.

PACS numbers: 03.30.+p
Keywords: special relativity, Thomas precession

I. INTRODUCTION

The Thomas precession naturally arises in Special Theory of Relativity as the additional rotation to be added to a Lorentz boost to represent the result of two subsequence Lorentz boosts. Although the derivation of the Thomas precession from the Lorentz transformations can be found in many textbooks (see, e.g., Jackson (1975) or Möller (1972)), it is normally done in the form of expansion in powers on $1/c$. Salingaros (1986) gives the exact expression for Thomas precession, but in a form that is not readily useful for further calculations. Sexl & Urbantke (2001) have also given the exact expression, but have not simplified it algebraically, leaving the reader with a rather lengthy calculations. The purpose of this short note is to derive the exact and fully simplified expression for the Thomas precession directly using two subsequence Lorentz transformations and representing them as a Lorentz transformation plus a spatial rotation. All calculations have been performed explicitly and in normal vector notations. The resulting formula for Thomas precession (Eqs. (21)–(25) below) is rather compact and is valid exactly. Using this expression for the Thomas precession a trivial calculation leads to Eq. (26) for the angle of rotation due to Thomas precession.

The exact formula for the Thomas precession is interesting by itself, but can also be considered as a step in the discussion of the Thomas precession in the framework of Mansouri-Sexl test theory (Mansouri & Sexl 1977). That latter discussion is important to interpret the results of various modern project performing high-accuracy directional measurements (e.g., Gravity Probe B, Gaia or SIM) in terms of the Local Lorentz Invariance (see, e.g., Klioner 2007, for the case of Gaia and SIM).
The notations of this paper are usual: \( c \) is the velocity of light in vacuum, lowercase Latin indices take values 1, 2, and 3 and refer to spatial components of corresponding quantities, index 0 is used for time components, Greek indices take values 0, 1, 2 and 3 and refer to all space-time components of corresponding quantities, repeated indices (both Latin and Greek ones) imply Einstein summation rule irrespective of their positions (e.g., \( a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \)), the spatial components of a quantity considered as a 3-vector are set in boldface (\( a = a^i \)), the absolute value (Euclidean norm) of a 3-vector \( a \) is denoted as \( a \) or \( |a| \) and is defined by \( a = |a| = (a^1 a^1 + a^2 a^2 + a^3 a^3)^{1/2} \), the scalar product of any two 3-vectors \( a \) and \( b \) with respect to the Euclidean metric \( \delta_{ij} \) is denoted as \( a \cdot b \) and defined as \( a \cdot b = \delta_{ij} a^i b^j = a^i b^i \), the Kronecker symbol (unit matrix) is denoted as \( \delta_{ij} \), parentheses surrounding a group of indices denote symmetrization (e.g., \( A_{(ij)} = \frac{1}{2} (A_{ij} + A_{ji}) \)), brackets surrounding two indices denote antisymmetrization (e.g., \( A_{[ij]} = \frac{1}{2} (A_{ij} - A_{ji}) \)).

II. TWO SUBSEQUENT LORENTZ TRANSFORMATIONS AND THE THOMAS PRECESSION

Let us consider three inertial reference systems: \((X^0 = cT, X^i)\), \((x^0 = ct, x^i)\) and \((\hat{x}^0 = c\hat{t}, \hat{x}^i)\). The velocity of \( x^\alpha \) with respect to \( X^\alpha \) is \( V^i \). The coordinates \( X^\alpha \) and \( x^\alpha \) are related by a Lorentz transformation of the form

\[
x^\alpha = \Lambda^\alpha_\beta X^\beta
\]

where

\[
\Lambda^0_0 = \Gamma,
\]
\[
\Lambda^0_a = -\Gamma K^a,
\]
\[
\Lambda^i_0 = -\Gamma K^i,
\]
\[
\Lambda^i_a = \delta^{ia} + \frac{\Gamma^2}{1 + \Gamma} K^i K^a,
\]
\[
\Gamma = (1 - K \cdot K)^{-1/2}, \quad \Gamma = \frac{1}{c} V.
\]

The inverse transformation reads

\[
X^\alpha = \tilde{\Lambda}^\alpha_\beta x^\beta,
\]

where \( \tilde{\Lambda}^\alpha_\beta \) is equal to \( \Lambda^\alpha_\beta \) with \(-K\) substituted for \( K \) (\( \Gamma \) remains the same after this substitution). The velocity of reference system \( \hat{x}^\alpha \) with respect to \( x^\alpha \) is \( v^i \), and one has

\[
\hat{x}^\alpha = \lambda^\alpha_\beta x^\beta,
\]

where \( \lambda^\alpha_\beta \) has the same form as \( \Lambda^\alpha_\beta \) with

\[
\gamma = (1 - k \cdot k)^{-1/2},
\]
\[
k = \frac{1}{c} v
\]
substituted for $\Gamma$ and $K$, respectively. Now, the velocity of $\hat{x}^\alpha$ relative to $X^\alpha$ is $\hat{V}$. Using standard considerations one gets the relation between the three velocities:

$$\hat{K} = \frac{1}{1 + p} \left[ \frac{1}{\Gamma} k + \left( 1 + \frac{\Gamma}{1 + \Gamma} p \right) K \right],$$  \hspace{1cm} (12)

where

$$\hat{K} = \frac{1}{c} \hat{V},$$ \hspace{1cm} (13)

$$p = k \cdot K.$$ \hspace{1cm} (14)

Combining (11) and (9) one has the relation between $\hat{x}^\alpha$ and $X^\alpha$:

$$\hat{x}^\alpha = \Sigma^\alpha_\beta X^\beta,$$ \hspace{1cm} (15)

$$\Sigma^\alpha_\beta = \lambda^\alpha_\rho \Lambda^\rho_\beta.$$ \hspace{1cm} (16)

Now, let us define matrix $\hat{\Lambda}^\alpha_\beta$ with the same structure as $\Lambda^\alpha_\beta$ but with

$$\hat{\Gamma} = \left( 1 - \hat{K} \cdot \hat{K} \right)^{-1/2}$$ \hspace{1cm} (17)

substituted for $\Gamma$ and $\hat{K}$ for $K$. According to (17), (10), (6) and (12) one gets

$$\hat{\Gamma} = \gamma \Gamma (1 + p).$$ \hspace{1cm} (18)

Straightforward calculations show that

$$\Sigma^0_\beta = \hat{\Lambda}^0_\beta,$$ \hspace{1cm} (19)

$$\Sigma^a_\beta = P^{ab} \hat{\Lambda}^b_\beta,$$ \hspace{1cm} (20)

where $P^{ab}$ is the orthogonal matrix describing the Thomas precession

$$P^{ab} = \delta^{ab} + A K^a K^b + B k^a K^b + C K^a k^b + D k^a k^b,$$ \hspace{1cm} (21)

$$A = \frac{(1 - \gamma) \Gamma^2}{(1 + \Gamma)(1 + \hat{\Gamma})},$$ \hspace{1cm} (22)

$$B = \frac{\gamma \Gamma}{1 + \hat{\Gamma}} \left( 1 + 2 \frac{\hat{\Gamma} - \gamma \Gamma}{(1 + \gamma)(1 + \Gamma)} \right),$$ \hspace{1cm} (23)

$$C = \frac{\gamma \Gamma}{1 + \hat{\Gamma}},$$ \hspace{1cm} (24)

$$D = \frac{\gamma^2 (1 - \Gamma)}{(1 + \gamma)(1 + \hat{\Gamma})}.$$ \hspace{1cm} (25)

Matrix $P^{ab}$ is orthogonal and satisfies the relation $P^{ac} P^{bc} = \delta^{ab}$. If $k$ is parallel to $K$ (that is, for $k = \alpha K$ with any $\alpha$), it is easy to check from (21)–(25) that the Thomas precession vanishes and $P^{ab} = \delta^{ab}$. Sexl & Urbantke (2001) have derived this result, but have not given it in explicit and fully simplified form.
The angle of rotation $\alpha$ due to Thomas precession can be directly computed from the trace of matrix $P^{ab}$ using the standard formula $1 + 2 \cos \alpha = P^{aa}$. Trivial calculation leads immediately to

$$1 + \cos \alpha = \frac{(1 + \gamma + \hat{\Gamma})^2}{(1 + \gamma)(1 + \Gamma)(1 + \hat{\Gamma})}. \quad (26)$$

The equivalent results have been derived after lengthy calculations by Macfarlane (1962, Eq. (124)) and Urbantke (1990, the last equation of the paper) and discussed also by Sexl & Urbantke (2001, Eq. (2.10.7)).

### III. IMPORTANT LIMITS FOR THE THOMAS PRECESSION

From this matrix one can easily restore all standard results concerning the Thomas precession. Expanding $P^{ab}$ in terms of $k = |\mathbf{k}|$ one gets

$$P^{ab} = \delta^{ab} + \frac{2 \Gamma}{1 + \Gamma} k^{[a} K^{b]} + \mathcal{O}(k^2), \quad (27)$$

where $A^{[i} B^{j]} = \frac{1}{2} (A^i B^j - A^j B^i)$ is the antisymmetric part of $A^i B^j$ for any two vectors $\mathbf{A}$ and $\mathbf{B}$. Defining $\delta \mathbf{K} = \hat{\mathbf{K}} - \mathbf{K}$ and using (12), Eq. (27) can be re-written as

$$P^{ab} = \delta^{ab} + \frac{2 \Gamma^2}{1 + \Gamma} \delta K^{[a} K^{b]} + \mathcal{O}(|\delta \mathbf{K}|^2). \quad (28)$$

This latter form can be found, e.g., in Jackson (1975) and Møller (1972). Finally, expanding (21)–(25) in powers of $1/c$ one gets

$$P^{ab} = \delta^{ab} + k^{[a} K^{b]} + \frac{1}{4} k^{[a} K^{b]} \left( k^2 + K^2 - p \right) - \frac{1}{8} \left( k^2 K^a K^b + K^2 k^a k^b - 2 p k^{(a} K^{b)} \right) + \mathcal{O}(c^{-6}). \quad (29)$$

This expansion can be conveniently used for modelling of high-accuracy directional data. Let us note that the symmetric part of the terms of order $\mathcal{O}(c^{-4})$ immediately follows from the antisymmetric terms of order $\mathcal{O}(c^{-2})$. Indeed, considering a general representation $P^{ab} = \delta^{ab} + \sum_{k=1}^{\infty} \epsilon^k f_k^{ab}$ with any formal parameter, the condition of orthogonality $P^{ac} P^{bc} = \delta^{ab}$ allows one to determine the symmetric term $f_k^{(ab)}$ at any order of $\epsilon$. In particular one has $f_1^{(ab)} = 0$, $f_2^{(ab)} = -\frac{1}{2} f_1^{[ac] f_1^{[bc]}$, $f_3^{(ab)} = -\frac{1}{2} \left( f_1^{[ac] f_2^{[bc]} + f_1^{[bc] f_2^{[ac]}} \right)$. With $\epsilon f_1^{[ab]} = k^{[a} K^{b]}$ one immediately restores the symmetric part of the terms of order $\mathcal{O}(c^{-4})$ given in the second line of (29): $\epsilon^2 f_2^{(ab)} = -\frac{1}{8} \left( k^2 K^a K^b + K^2 k^a k^b - 2 p k^{(a} K^{b)} \right)$. 
Acknowledgments

This work was partially supported by the BMWi grant 50 QG 0601 awarded by the Deutsche Zentrum für Luft- und Raumfahrt e.V. (DLR).

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