Particle trapping effects on transport in random electric field

O Cherniak and V I Zasenko
Bogolyubov Institute for Theoretical Physics, NAS of Ukraine, Kyiv 03143, Ukraine
E-mail: anchernyak@bitp.kiev.ua

Abstract. Test particle diffusion in a random electric field across uniform magnetic field is studied using analytical approach and direct numerical simulation. Our approximation describes particle trapping and is able to reproduce an asymptotically zero diffusion coefficient for infinite Kubo number, i.e. dimensionless correlation time. For large, but finite, correlation times the subdiffusive scaling relation for asymptotic diffusion coefficient is obtained. The considered approach also recovers the known scaling relation for small correlation times. The accurate method to take into account finite Larmor radius effects is demonstrated.

1. Introduction
Usually plasma is in non-equilibrium state and transport processes play significant role in it’s evolution. Intense fields are generated due to instabilities in such plasma. Particle interaction with these fields leads to anomalous transport, which can considerably exceed collisional one. We are interested in two-dimensional particle transport across magnetic field undergoing random electric field. The important feature of such transport is a particle trapping effect.

The particle trapping effect is most clearly manifested in a two-dimensional drift motion of particles in a static random field. The particle is trapped, when it moves along closed streamline, i.e. along equipotential line. Since all trajectories are closed in a static random field, all of the particles are trapped. Their motion is strongly correlated, therefore the approximation of Gaussian distribution of particle displacements is not relevant. The new approach for temporal evolution of statistical characteristics of trapped particles is needed. When analytical approach to particle drift transport in a random static field is constructed, it can be generalized to account for finite Larmor radius effects and to fields with a finite correlation time.

Here we present our analytical approximation [1], [2] for two-dimensional transport in a random electric field. The method was formulated for a field with infinite correlation time and it is able to recover the results of direct numerical simulation [2], [3]. The accurate way to account for finite Larmor radius effects using our approach is demonstrated in [4], [5]. We introduce the generalization of this method for transport in random fields with finite correlation time in this paper.
2. Numerical simulation

We study particle motion in a random electric field \( \mathbf{E}(x, t) \) superposed by uniform magnetic field \( \mathbf{B} = B \mathbf{e}_z \) using equations

\[
\frac{dx_d}{dt} = v_d = \frac{e}{m} \left[ \mathbf{E}(x_d + \mathbf{r}_L, t) \times \mathbf{e}_B \right] \Omega_B, \tag{1}
\]

\[
\frac{d\mathbf{r}_L}{dt} = -\frac{e}{m} \left[ \mathbf{E}(x_d + \mathbf{r}_L, t) \times \mathbf{e}_B \right] \Omega_B + \Omega_B [\mathbf{r}_L \times \mathbf{e}_B], \tag{2}
\]

where \( \Omega_B = \frac{eB}{mc} \) is cyclotron frequency, \( x_d = x - \mathbf{r}_L \) is coordinate of gyrocentre, \( x \) is exact particle’s position and \( \mathbf{r}_L = -[\mathbf{v} \times \mathbf{e}_B]/\Omega_B \) is Larmor radius. To study particle motion in a drift approximation, when \( \mathbf{r}_L(t) = 0 \), we use equation (1). To consider finite Larmor radius effects on particles transport both equations (1) and (2) are used.

Electric field \( \mathbf{E}(x, t) \) is given through statistically isotropic random potential \( \phi(x, t) \) taken as weighted sum of \( N = N_k N_\theta = 1440 \) harmonics

\[
\mathbf{E}(x, t) = -\frac{\partial}{\partial x} \phi(x, t) = A \phi_0 \sum_{s=1}^N k_s \exp \left( -\frac{1}{2} \left( \frac{k_s}{\Delta k} \right)^2 \right) \sin(-k_s x + \alpha_s(t)), \tag{3}
\]

where set of wave-vectors \( k_s \rightarrow k_i e_\theta \) is following

\[
k_i = \frac{ik_{\text{max}}}{N_k}, \quad i = 1, N_k, \tag{4}
\]

\[
e_\theta = (\cos \theta, \sin \theta), \quad \theta = \frac{2\pi j}{N_\theta}, \quad j = 1, N_\theta, \tag{5}
\]

and normalizing coefficient \( A \) obtained from condition \( \langle \mathbf{E}(x, t) \mathbf{E}(x, t) \rangle = E_0^2 = \phi_0^2 \Delta k^2/2 \) reads

\[
A = \left( \frac{4k_{\text{max}}}{\pi^{1/2} N_k N_\theta} \right)^{1/2}. \tag{6}
\]

The set of random phases \( \alpha_s \) defines a unique realization of the random electric field (3).

Numerical calculation of particles trajectories is done by Runge-Kutta method of the 5-th order. Particle trajectories are used to calculate mean square displacement, diffusion coefficient and correlation function of drift velocity components along particle trajectories. The correlation function of drift velocity components in fixed points of reference frame is obtained using Eq. (3) for a random field.

We consider a drift motion (1) of particles with the set of constant random phases \( \alpha_s(t) = \alpha_s \) to study particle transport in a random field with infinite correlation time. In this case the electric field (3) is static and potential is constant along the particle drift trajectory. Therefore particles move along closed trajectories, i.e. they are trapped.

The exact equation (2) is used when finite Larmor radius \( \mathbf{r}_L = x - x_d \) can not be neglected, for particles with considerable initial velocities. When particles gyrate they can change an equipotential line. Statistically this means, that exact particles motion is less correlated, than the drift one.

When considering the problem with finite correlation time we use the drift approximation (1) for particle motion, and take the electric field (3) with random phases \( \alpha_s(t) \) which jump with a frequency \( \nu \) and probability \( p \). These two parameters determine a correlation time of random field \( t_c = 1/(\nu \ln(1 - p)) \). The smaller correlation time we take, the less correlated motion of particles we obtain.

Found in numerical simulation mean square displacement, diffusion coefficient and Lagrangian correlation function are used to validate our analytical approach.
3. Analytical approach

We use the Taylor relation [6] for analytical approximation of magnetized particle transport in a random electric field. This gives running diffusion coefficient $D(t)$ and mean square displacement $\Delta(t)$ in terms of correlation function of drift velocity components along particles trajectories

$$D_x(t) = \frac{1}{2} \frac{d}{dt} \Delta_x(t) = \int_0^t d\tau C^L_{v_x v_x}(\tau).$$

This Lagrangian correlation function

$$C^L_{v_x v_x}(t) = \langle v_x(x(t + t_0)) v_x(x(t_0)) \rangle,$$

is unknown and its determination is a key problem in statistical theories. On the contrary, the correlation function at fixed points of reference frame

$$C^E_{v_x v_x}(x, t) = \langle v_x(x + x_0, t + t_0) v_x(x_0, t_0) \rangle,$$

is given and termed Eulerian. It is used to construct the approximated Lagrangian correlation function. There is no mathematically strict method to calculate Lagrangian correlation function from Eulerian one in general. Consequently, various approximations, i.e. closures of statistical equations, were proposed.

The most known approach is Corrsin approximation [7], that is valid for small Kubo numbers $K \ll 1$ ($K = V t_c/\lambda_c$ - a dimensionless correlation time, $V$ - an initial mean drift velocity $V^2 = C^E_{v_x v_x}(0, 0)$, $t_c$ and $\lambda_c$ - correlation time and length respectively) leads to asymptotic diffusion coefficient $D \equiv D(t) t \to \infty \sim K^2$. For large Kubo numbers $K \gg 1$ the Bohm scaling $D \sim K$ was obtained in [8]. But it is not validated by numerical simulation in [3], [9], where asymptotic diffusion coefficient is found to be $D \sim K^\gamma$, $\gamma < 1$.

We proposed the approximation for infinite correlation time $K \to \infty$ and isotropic random field [1], [3]. The mean displacement equals zero in this random field and therefore we use next non-vanishing moment, mean square displacement, to approximate particle dynamics. So Lagrangian correlation function along particle trajectories is approximated as Eulerian correlation function dependent on mean square displacement of particles, and due to statistically isotropic potential we obtain

$$C^E_{v_x v_x}(x) + C^E_{v_y v_y}(x) = C^E_{v_x v_x}(x) = C^E_{v_x v_x}(|x|),$$

$$C^L_{v_x v_x}(t) = C^E_{v_x v_x}(X(t)),$$

where $X(t) = \Delta^{1/2}(t)$ approximates particle dynamics through mean square displacement $\Delta(t) = \Delta_x(t) + \Delta_y(t)$. Generalized approximation [2] takes into account different dynamics of trapped particle groups, termed subensembles, which are characterized by the values of random potential in the initial particle positions

$$C^L_{v_x v_x}(t, \phi(0)) = \frac{\phi^2(0)}{\phi^2_0} C^E_{v_x v_x}(X(t)),$$

$$C^{L,L,S}_{v_x v_x}(t) = \int \frac{d\phi(0)}{(2\pi C^E_{\phi(0)})^{1/2}} \exp\left(-\frac{\phi(0)^2}{2 C^E_{\phi(0)}}\right) C^L_{v_x v_x}(t, \phi(0)).$$

The gyroaveraging of random potential is applied in [4], [5] to consider finite Larmor radius effects on particles transport

$$C^{L,R}_{v_x v_x}(t, r_L) = \frac{1}{4\pi^2} \int dk \exp(ikX(t) \cos \varphi_k) C^E_{v_x v_x}(k) J^2_0(kr_L).$$
For the finite correlation time the Lagrangian correlation function with explicit exponential decay is of the form
\[ C_{L,T}^{\nu_d}(t) = \exp(-t/t_c)C_{E}^{\nu_d}(X(t)). \] (15)

Substitution of correlation functions (11) - (15) in the Taylor relation (7) gives the final equation for mean square displacement \( \Delta(t) \), that is solved numerically.

4. Results
We start from particle drift motion in a random field with an infinite correlation time. Since particles move along the streamlines \( \phi(x) = \text{const} \) almost all of them, except the particles at zero potential, are trapped. These particles move along closed trajectories and travel for relatively small distances in compare with untrapped ones. The motion of trapped particles is strongly correlated and thus can not be approximated as random walk. This is reflected in Lagrangian correlation function by infinitely long negative tail as it is shown in Fig. 1 for the analytical approximation (11) and direct numerical simulation [1], [3]. Such evolution of correlation function gives asymptotically zero diffusion coefficient and subdiffusive evolution of mean square displacement.

Detailed comparison between Corrsin approximation [7], decorrelation trajectory method [10] and our approach [1] for infinite correlation time is presented in [3]. The mean square displacement is given in Fig. 2 and demonstrates qualitative agreement between moment approximation and direct numerical simulation.

The generalized approximation that takes into account different dynamics of particles grouped by initial values of random potential is developed in [2]. Such approach is more consistent with numerical simulation.

The account for finite Larmor radius leads to decorrelation of trajectories – particles with non-zero Larmor radius are not bound to particular streamline and thus travel for longer distances. Two methods of gyroaveraging and corresponded Eulerian correlation function of drift velocity components for analytical approximation were discussed in [4], [5]. The most satisfactory results [4] are obtained by gyroaveraging of random potential with further calculation of Eulerian correlation function (14). The evolution of diffusion coefficient for different initial values of Larmor radius is shown in the Fig. 3. It can be seen, that particles with smaller initial
Larmor radius have greater diffusion coefficient at the beginning, but it drops faster afterwards. Corresponding mean square displacement evolution [5] is shown in Fig. 4.

For particles in random fields with a finite correlation time the equation (15) with time-dependent electrical field (3) is used. Particle trapping in such fields is partial, i.e. limited in time. The negative values of correlation function vanishes with decrease of correlation time. In the limit of small Kubo numbers \( K \ll 1 \), i.e. short correlation times, particle motion is similar to Brownian one. Consequently the Corrsin approximation and asymptotic diffusion coefficient \( D \sim K^2 \) are valid.

In the model with jumping phase (3) evolution of Lagrangian correlation function with exponential decay (15) is shown in Fig. 5. It demonstrates negative values for \( t_c \geq 1 \), for \( t_c < 1 \) the Lagrangian correlation function is positive.

The dependence of asymptotic diffusion coefficient on Kubo number is given in Fig. 6. The approximation (15) reproduces the known result of the Corrsin approximation \( D \sim K^2 \) for small Kubo numbers \( K \ll 1 \) and for \( K \gg 1 \) gives \( D \sim K^7 \), \( \gamma \approx 0.8 \), that corresponds to a percolation regime.
5. Conclusions
We proposed the new method of statistical equations closure in order to describe two-dimensional diffusion of magnetized particles taking into account trapping effects. It is shown that closure by mean square displacement (11) recovers the results of direct numerical simulation [1], [3] for zero Larmor radius. The subensemble equations (12), (13) improve consistency with numerical modeling [2]. The generalization of our approach for finite Larmor radius gives agreement with simulation in a wide range of Larmor radius [4]. The dependence of diffusion coefficient on field correlation time is calculated. It recovers the limits of quasilinear scaling $D \sim K^2$, for small correlation times, and percolation regime $D \sim K^\gamma$, $\gamma \approx 0.8$ for large ones.

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