Vehicle load updating of roadway bridges based on Bayesian method and Monte Carlo simulation

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Abstract. For the life-cycle design of bridges, it is of significance to establish and update the probabilistic model. However, for the current situation of practical engineering, popularizing data acquisition equipments to collect long-term data could not meet the facts. The Monte Carlo simulation is utilized to simulate the prior samples and expand the sample size. The Bayesian method is adopted to update the sectional distribution. A mathematical example based on self programming is given to illustrate the method. The results show that: the cooperation of the Bayesian method and Monte Carlo simulation could update the short-term vehicle load model. The method is of good reliability and serviceability which could be references for further studies.

1. Introduction
With the development of economy, the number of bridges in China increases rapidly. For the research of bridge life-cycle design, whether the load model of bridges could be periodically predicted or updated is of great significance in this field.

The probabilistic model of highway bridges is mainly used to describe the vehicle load effects. One of the most important aspects in updating the probability model is to continuously update the sectional distributions. However, the current situation is that: the construction quality, maintenance, management and inspection level for different bridges are uneven. Based on this, how to continuously update the sectional distributions by periodically collecting less sample data becomes significant. The current studies show, the most economic and scientific method in solving the problem is the mathematical statistical theory[1].

In order to make this method more efficient, the log-normal distribution is chosen to simulate the vehicle load effects. In the first step, a small amount of data is collected and the distribution parameters are obtained by parameter estimation and fitting test. In the second step, a prior sample is obtained by the cooperation of the Monte Carlo method and the acceptance-rejection sampling, which is used to update the sectional distribution of the vehicle load effects. Then, the small amount of data collected during the required update period is adopted as the test samples for the updating of the sectional distribution. Based on the Bayesian principle, the extended Bayesian method is used to modify the distribution parameters, and the posterior samples of the updated sectional distribution of the vehicle load effects are obtained.

2. Monte Carlo based data augmentation

2.1. Monte Carlo method
Monte Carlo methods or Monte Carlo experiments are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. Their essential idea is using randomness to solve problems that might be deterministic in principle. They are often used in physical and mathematical problems and are most useful when it is difficult or impossible to use other approaches. Monte Carlo methods are mainly used in three problem classes: optimization, numerical integration, and generating draws from a probability distribution[2].

2.1.1. Basic principles of Monte Carlo methods. Let $X_1, X_2, \ldots, X_n$ be independent random variables and $f_{X_1}, f_{X_2}, \ldots, f_{X_n}$ be probability density functions. The limit state function could be expressed as:

$$Z = g(x_1, x_2, \ldots, x_n)$$

(1)

2.1.2. To solve the structural failure probability, the Monte Carlo based procedures can be expressed as random sampling, calculate the value of limit state function and calculate the number of failures. Let the sampling number be $N$ and failure number be $L$, the failure probability could be expressed as:

$$P_f = \frac{L}{N}$$

(2)

2.1.3. Sampling of basic variables. For a given distribution, the most important step in the Monte Carlo Simulation (MCS) is to obtain the random number of variables. Let the extreme value distribution type I for example, the random number of variables can be expressed as

$$x_i = F_x^{-1}(u_i)$$

(3)

where $F_x(x)$ is the cumulative distribution function and $u_i$ is the uniform random numbers from 0–1.

The distribution function of the extreme value distribution type I can be formulated as

$$F_x(x) = \exp\{-\exp[-\alpha(x_i - k)]\}$$

(4)

Then the random numbers $u_i$ can be expressed as

$$u_i = F_x(x_i) = \exp\{-\exp[-\alpha(x_i - k)]\}$$

(5)

The parameters $\alpha$ and $x_i$ can be solved as

$$x_i = k - \frac{1}{\alpha} \ln(-\ln u_i)$$

(6)

$$\alpha = 1.2825/\sigma_x k = mk - 0.450\sigma_x$$

(7)

The relationship between $x_i$ and $u_i$ can be expressed as

$$x_i = m_x - 0.450\sigma_x - 0.7797\sigma_x \ln(-\ln u_i)$$

(8)

2.2. Acceptance-rejection method[3]

For a given random variable $X$ with the finite closed interval of $[a, b]$, let the probability density be $f_x(x)$ and least upper bounds be $f_\theta = \max_{x \in [a, b]} f_x(x)$, the sampling processes of acceptance-rejection method can be summarized as

1. Generate the samples $u$ of uniform random variable $U$ in the interval $[a, b]$.
2. Generate the random numbers $r$ in the interval $[0, 1]$.
3. Accept $u$ if $r \leq f_x(u) / f_\theta$, otherwise repeat process (1).

2.3. Flowchart of Monte Carlo sampling

The mathematical software MATLAB is utilized to implement the method mentioned above. The procedure is shown in figure 1.
3. Promotion and application of Bayesian theorem

3.1. Bayesian theorem
Bayesian theorem is the application of probability and statistics of the observed phenomena on subjective judgement about the probability distribution standard correction method. Usually, the event A in the event B probability, probability and event B in the event A, conditions are not the same; however, these two are determined, the Bayesian theorem is this statement.

As a normative theory, Bayesian rule is effective for all probabilistic interpretation. However, frequency, and Bayesian principle regarding the probability is assigned a different view in the application. Frequency, according to the random events, or the number of total sample is assigned probability. Bayesian theory depends on the unknown proposition to assign probability[4-5].

3.2. Two types of Bayesian theorem promotion

3.2.1. Bayesian theorem promotion type 1
Let the prior distribution function of $\theta$ be $f'(\theta)$ and the posterior distribution function of $\theta$ be $f^*(\theta)$. Based on the Bayesian formula, the relationship between $f'(\theta)$ and $f^*(\theta)$ can be expressed as:

$$f^*(\theta)\Delta\theta = \frac{p(E|\theta_0) \cdot f'(\theta)\Delta\theta}{\sum_{i=1}^{n}p(E|\theta_i) \cdot f'(\theta)\Delta\theta}$$

$$p(E|\theta_0) = p(E|\theta_0 < \theta \leq \theta_0 + \Delta\theta)$$

where $p(E|\theta_0)$ is the test probability of occurrence for $\theta$ in the interval $[\theta_0, \theta_0 + \Delta\theta]$. $E$ is the test results, $n$ is the number of data segments and $\Delta\theta$ is the step length.

The mean value and variance of $\theta$ can be obtained:

$$\mu_0 = \sum_{i=1}^{n}\tilde{\theta}_i \cdot f^*(\theta)\Delta\theta$$

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Figure 1. Flowchart of Monte Carlo sampling
\[ \sigma^2_0 = \sum^n_{i=1} (\bar{\theta}_i - \mu_0)^2 \cdot f^*(\theta) \Delta \theta \]  \hspace{1cm} (12) 

where. \( \bar{\theta}_i \) is the median of \( \theta \) in the interval \([ \theta_i, \theta_i + \Delta \theta ]\)

3.2.2. Bayesian theorem promotion type 2

Let the normal distribution density function of \( x \) be \( f(x) \) and mean value of \( x \) be \( \mu \). \( x_1, x_2, \ldots, x_n \) are the test samples of \( x \), \( \bar{x} \) is the mean value of the samples, and \( \sigma \) is the standard deviation of the samples. The prior distribution function and posterior distribution of \( \mu \) can be expressed as:

\[ f^*(\mu) = \frac{\left[ \prod^n_{i=1} f(x_i | \mu) \right] \cdot f^*(\mu)}{\int_{-\infty}^{\infty} \left[ \prod^n_{i=1} f(x_i | \mu) \right] \cdot f^*(\mu) d\mu} = N\mu \left( \mu^*, \sigma^* \right) \]  \hspace{1cm} (13) 

\[ \mu^* = \frac{\bar{x} \cdot (\sigma')^2 + M' \cdot (\sigma^2 / n)}{(\sigma')^2 + (\sigma^2 / n)} \]  \hspace{1cm} (14) 

\[ \sigma^* = \sqrt{\frac{(\sigma')^2 - (\sigma^2 / n)}{(\sigma')^2 + (\sigma^2 / n)}} \]  \hspace{1cm} (15)

Let the prior distribution of \( x \) be \( N\mu \left( \mu', \sigma' \right) \), the prior distribution of \( \mu \) is in the form of \( N\mu \left( \mu', \sigma' \cdot \sqrt{m} \right) \). According to the reliability theory, \( \mu^* \) is the optimal estimation of the posterior distribution \( x \), can be expressed as:

\[ \mu^* = \frac{\bar{x} \cdot (\sigma')^2 / m + \mu' \cdot (\sigma^2) / n}{(\sigma')^2 / m + (\sigma^2) / n} \]  \hspace{1cm} (16) 

\[ \sigma^*_x \approx \sigma^* \cdot \sqrt{m+n} = \sqrt{\frac{(m+n) \cdot (\sigma')^2 / m + (\sigma^2) / n}{(\sigma')^2 / m + (\sigma^2) / n}} \]  \hspace{1cm} (17)

Let the log-normal distribution for example. If \( y = \ln x \), \( y \) obeys the normal distribution, the distributed parameters can be expressed as:

\[ \lambda^* = \mu^*_y = \frac{\bar{y} \cdot \xi^2 / m + \lambda \cdot (\sigma_y)^2 / n}{\xi^2 / m + (\sigma_y)^2 / n} \]  \hspace{1cm} (18) 

\[ \xi^* = \sigma^*_y = \sqrt{\frac{(m+n) \cdot \xi^2 / m + (\sigma_y)^2 / n}{\xi^2 / m + (\sigma_y)^2 / n}} \]  \hspace{1cm} (19)

3.3. Comparison and Selection of Two Extended Bayesian Methods

One of the extended Bayesian methods is to modify the distribution parameters of prior distribution by using the distribution parameters of test samples. The obtained posterior distribution is independent of the prior sample size and test samples. In the revision process of the extended Bayesian method II, not only the distribution parameters of the prior distribution and test samples are taken into account, but also the sample size of the both samples. The revised posterior distribution is obviously close to the prior distribution with the larger sample size.

According to the actual demands in updating the sectional distribution, the parameter correction in short period is carried out using the extended Bayesian method II. The posterior distribution could contain the information of the test samples obtained in a short period of time. Due to the sudden changes of the vehicle loads in a short period of time, the deviation of the correction results is small. Therefore, the updating of vehicle load effect section distribution should be based on the extended Bayesian method II.
4. Numerical example
To verify the applicability of the method, a numerical example is introduced. Let the sample size be 3000. The samples obey lognormal distribution with the parameters of $\mu = 8$ and $\sigma^2 = 1$. The parameters of the sample are estimated and tested. The fitting parameters of the sample are obtained as follows: $\mu = 8.0139$ and $\sigma^2 = 0.9931$ in the interval of $\mu \in (7.9784, 8.0495), \sigma^2 \in (0.9686, 1.0189)$. To make the update effects more intuitive, let the sample size of the experimental distribution be 3000, which accords with the lognormal distribution of parameter $\mu = 8.5$ and $\sigma^2 = 0.8$. Parameters of the example are shown in table 1.

| Sample type                      | Distribution type     | Sample size | $\mu$   | $\sigma^2$ |
|----------------------------------|-----------------------|-------------|---------|------------|
| Prior sample                     | Lognormal distribution| 3000        | 8       | 1          |
| Prior simulation extended sample | Lognormal distribution| 300000      | 8.0139  | 0.9931     |
| Test sample                      | Lognormal distribution| 3000        | 8.5     | 0.8        |
| Posterior sample                 | Lognormal distribution| 3000        | 8.0210  | 0.9942     |

Monte Carlo simulation is used to expand the number of samples to 300000. The simulated data are compared with the hypothetical measured data as shown in figure 2.

5. Conclusions
The Monte Carlo based vehicle loads simulation could effectively expand the prior distribution sample of a small amount of collected data. The posterior sample obtained by the extended Bayesian method II could meet the application prerequisites. Due to the sudden changes of vehicle loads in a short time, the deviation of the correction results is small.

It is concluded that: the short-term model updating for vehicle loads based on Bayesian principle and Monte Carlo simulation is feasible. For bridges of low accuracy of vehicle load model, the algorithm is of good efficiency.
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