Quantum Hamilton-Jacobi Equation and Broken Symmetry in Hydrodynamics of Liquid Helium II

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Abstract

Based on the quantum theory of Bohm and the phase coherence along with the mean field of Penrose and Onsager, it is shown that a free surface of He II behaves like a classical fluid. The broken symmetry of a macroscopic Bose system at the free surface in an external field is discussed in terms of the quantum fluctuations-dissipation. First, we apply this peculiarly universal behavior to explain a breakdown of superfluidity at a vortex core. Secondly, we resolve a long standing puzzle with Landau’s two-fluid model on a free surface of a rotating He II in a gravitational field.

PACS numbers: 03.65.-w, 03.75.Fi, 67.40-w, 67.55.Fa
The quantization of circulation in He II was suggested over 50 years ago by Onsager and Feynman [1, 2]. This prediction was confirmed for irrotational flow in multiply connected regions by Vinen [3] who has shown evidence for the quantization of circulation in units of $h/M$. A further detailed study of vortex dynamics by Rayfield and Reif using ions as microscopic probe particles [4] has revealed that, apart from the quantization of circulation on a macroscopic scale, the vortices behave like a normal fluid. Yet even now we know little about the core structure and the immediate region over which a study of collective excitations may provide clear evidence for the breakdown of superfluidity; it is, however, exceedingly difficult to observe an excitation spectrum in such a region which is smaller than the macroscopic scale of physical measurements. An extended study of the core structure was undertaken experimentally by Glaberson, Strayer, and Donnelly in 1968 [5, 6]. Their result showed that near the vortex line there is a high concentration of rotons and the breakdown of superfluidity at the core as observed earlier by Rayfield and Reif [4].

Our initial motivation for studying the vortex dynamics was to investigate the similarity between the nodal surface of a vortex core and that of rotating He II in a gravitational field. In the course of our study, it has become clear that one must address the question of whether there is any basic underlying mechanism that could be responsible for both the breakdown of superfluidity and the parabolic shape of rotating He II. More specifically, it is the purpose of this paper to show that the basic mechanism is the spontaneously broken gauge symmetry on the nodal surface. It is shown here for the first time that the unusual properties of a vortex core are closely related to those of a free surface of rotating He II.

The study of the free surface of rotating liquid He II began in 1950 by Osborne [7] to answer the question if the superfluid component in the two-fluid model [8] rotates with the normal component. He concluded that, at the rotational speed used, the superfluid came to a steady rotational state and the observed curvature of a rotating He II is incompatible with the two-fluid model.

It is generally accepted that the macroscopic rotation of He II can be achieved despite the condition $\nabla \times \mathbf{v}_s = 0$ if an array of vortices is aligned along the axis of rotation ($\Omega$) and satisfies Feynman’s criteria for the areal vortex number density $\Gamma_\Omega \geq \frac{2\Omega}{\kappa}$ with $\kappa = \frac{h}{M} = 0.997 \times 10^{-3} \text{cm}^2/\text{sec}$ [9]. We refer the reader Ref. [10] for details and the counterpoints of view in Refs. [9,10].

However, with further experimental evidence by Meservey [11] along the same lines as
Osborne in 1964, a question was raised if the theoretical model of the vortex lines is correct for the interpretation of Osborne’s observation. Meservey has also made a speculation on the role of a free surface energy associated with velocity discontinuities suggested by Mott. Such a free surface energy corresponding to the energy removed by a fixed number of vortices from the superfluid flow in a manner similar to that of the Meissner effect in a superconductor could explain how the free surface rotates in spite of the irrotational motion of He II.

However, the classical fluid like character of the free surface of rotating liquid helium II still remains as one of the most puzzling, unanswered questions in low temperature physics since the first experiment by Osborne. For a reason that we shall see in a moment the peculiarly universal property of the free surface of a superfluid in motion in the gravitational field can be explained by broken local symmetry; it will be shown here that, to the approximation to which interactions between pairs of atoms are considered (i.e., the hard sphere approximation), Meservey’s observation and his arguments are in agreement with our analysis.

An important step in the development of the vortex dynamics in He II was the introduction by Anderson of a concept of spontaneously broken gauge symmetry by which vortices in a superfluid are nucleated and leads to dissipation by $2\pi$-phase slippage at the walls. He further emphasized the main feature of a flow dissipation mechanism that, at the critical velocity $v_c$, which is much smaller than the Landau’s critical velocity $v_c \approx 60\text{ m/sec}$, a superfluid is unstable to a small perturbation and this instability drives the motion of a vortex line across orifice, crossing all the enclosed stream lines. In such a process a vortex changes the phase by $2\pi$ and a fixed amount of energy is removed from the flow - the fundamental flow dissipation mechanism which leads a breakdown of superfluidity. In an experiment by Avenel and Varoquax, Anderson’s phase-slip picture has been confirmed in a sub-micron orifice. Since a number of properties of He II and superconductors can be explained by his theory of broken symmetry, Anderson’s idea has great appeal.

Recently Su and Suzuki have shown a new derivation of the quantization of a vortex, and the similarity between the rotating He II and the Meissner effect in a superconductor based on the concept of off-diagonal long-range order (ODLRO) by Penrose and Onsager and on the notion of a regauged space translation. When a particle displacement is introduced into the Bose system, it invariably perturbs the density of the system. Moreover,
the gauge field is a dynamical variable which must be analytic unless it is spontaneously broken. We therefore find it difficult to believe the idea of the regauged space translation [16] is tenable. Moreover this idea led to the self-contradiction in their work as pointed out by Shi [18].

The purpose of the present paper is to show how Bohm’s quantum theory [19, 20] can be applied to specific problems in such a way as to incorporate Feynman’s atomic theory of the two-fluid model [21] and the concept of the phase coherence along with ODLRO. By symmetry breaking perturbations in Lagrangian coordinates [22, 23, 24, 25], we study the consequence of symmetry breaking at the free surface, and thereby extend Anderson’s idea of a symmetry breaking in He II. It will be shown here that the semiclassical method of symmetry breaking perturbations in Lagrangian coordinates is in fact a perfectly well-defined, accurate quantum mechanical approximation scheme. The unique feature of this perturbation method is that with Bohm’s quantum theory [19], one is able to separate a surface phenomenon from that of the bulk fluid.

More specifically, we will first show that the quantization of circulation is due to the spontaneously broken local symmetry that accompanies the excitations of highly concentrated rotons at the core of a vortex as observed in the experiments and that leads the breakdown of superfluidity at the vortex line as observed in the experiments by Glaberson, Strayer and Donnelly [4, 5, 6].

Second, it is shown here that the contour of a free surface of rotating He II is a necessary consequence of broken symmetry, which explains why the curvature of the free surface of a rotating He II remains parabolic [7, 11]. We also discuss the similarity between the exclusion of vortices from the bulk superfluid in a rotating He II below the critical velocity \( \Omega_c \) and the Meissner effect in a superconductor.

We begin with the concept of ODLRO in which the reduced density matrix of the condensed Bose system can be factorized,

\[
\rho(r, r') = \psi^\dagger(r)\psi(r) + \gamma(|r - r'|). \tag{1}
\]

where \( \gamma \to 0 \) as \( |r - r'| \to \infty \). The single particle wave function \( \psi(r) \) represents the condensed state in ODLRO and is viewed as macroscopic dynamical variables. With the hard sphere approximation of the inter-particle potential for Bose particles, one can show
that the mean field satisfies the nonlinear Schrödinger equation (Gross-Pitaevskii),

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \nabla^2 \psi + [V(\mathbf{x})_{\text{ext}} + g_1|\psi|^2]\psi,$$

(2)

where in the hard sphere approximation $g_1 = 4\pi\hbar^2a/M$ and $a$ is the s-wave scattering length [28, 29].

In our model of an imperfect Bose gas for a superfluid, between each pair of nearest particles there is a hard sphere repulsion of range $a$ and no other interaction. This pair-interaction brings about the Bose condensed state also ensures that the system possesses the longitudinal collective excitations (phonons). It is a self-consistent Hartree equation for the Bose condensed wave function. The whole meaning of an equation such as Eq. (2) depends on the physical meaning attached to $\psi(r, t)$, our definition of a superfluid in which the quantum field $\psi(r, t)$ is a function of macroscopic dynamical variables which represent a symmetry breaking due to the quantum fluctuations. Therefore, the whole problem of superfluid dynamics reduces to the question of how to treat $\psi(r, t)$. To introduce symmetry breaking perturbations in the atomic theory of two-fluid model [21], we employ Bohm’s interpretation of quantum theory which is essential to our discussion of both phase coherence and a symmetry breaking in He II.

The essence of Bohm’s theory [19, 20] is that we may write the wave function in the form $\psi(r, t) = f(r, t)\exp[i\frac{\hbar}{\bar{\hbar}}S(r, t)]$, where $S(r, t)$ is an action (or phase) and $\rho = |f(r, t)|^2$ may be interpreted as a number density for a system of $N$ Bose particles.

We then rewrite Eq. (2) to obtain,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \frac{\nabla S}{M}) = 0 \quad (3a)$$

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2M} + V(\mathbf{x}) - \frac{\hbar^2}{4M} \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} = 0, \quad (3b)$$

where $V(\mathbf{x}) \equiv V_{\text{ext}} + g_1 \rho$. Here $g_1 \rho$ term represents merely an internal order parameter in the sense of ODLRO of order-disorder systems [13] and can be treated as a pair-interaction potential [21].

The last term in Eq. (3b) is the effective quantum mechanical potential (EQMP) defined by

$$U_{\text{eqmp}} = -\frac{\hbar^2}{4M} \left[ \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right] = -\frac{\hbar^2 \nabla^2 f}{M f}. \quad (4)$$
In the limit $h \to 0$ (or less restrictively $\nabla \rho = 0$ for a homogeneous medium), $U_{eqmp} = 0$ and $S(r, t)$ is a solution of the Hamilton-Jacobi equation [30]. Eq. (3b) implies, however, that the particle moves under the action of the force that is not entirely derivable from the potential $V(x)$, but which also obtains a contribution from EQMP. The number density $\rho$ in Eq. (3a) is not entirely independent, but depends on $S(x)$. The solution of Eq. (3b) defines an ensemble of possible trajectories of a particle in the equation. We can obtain, in principle, the solution by integrating $v(x) = \nabla S(x)/M$. This is the basic idea of Bohm’s interpretation of quantum theory [31]. Finally, Eq. (3b) also suggests us the way of determining the most stable motion about which to linearize the dynamical equations, Eqs. [3].

Let us now consider to what degree the phase coherence can be defined in ODLRO for a many-body problem. Suppose if we set up a state in which a small number of atoms occupy in a ring around the vortex line and each atom is displaced by an infinitesimal position perturbation as the Feynman model suggests [2, 10], the phase coherence can be defined as a correlation between an individual particle phase change and that of the mean field by introducing perturbations in Lagrangian coordinates. The position perturbation of a particle is given by $x_i = x_{0,i} + \xi(x_{0,i}, t)$, where $\xi$ is a function of the initial position of a particle and time in the many-body wave function for $n$-particles in the ring, and remains attached to the particle in motion [22, 23, 24].

In order to show the phase coherence in a many-body problem in the ring, we introduce the wave function, $\psi(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots x_n) \exp[\imath S(x_1, x_2, \cdots, x_n)]$, where $n$ is the number of particles. $f^2$ is equal to the density of representative points $(x_1, x_2, \cdots, x_n)$ in 3n-dimensional ensemble in a Bose system. To the first-order in $\xi$,

$$S_i(x_1, x_2, \cdots, x_n) = S_i(x_{0,1}, x_{0,2}, \cdots, x_{0,n}) + \xi_i \cdot \nabla_{0,i} S_i(x_{0,1}, x_{0,2}, \cdots, x_{0,n}). \quad (5)$$

With the Lagrangian displacement vector $r = r_0 + \xi(r_0, t)$ which is now a position vector of a particle described by the one-particle wave function associated with the Bose condensed wave function (mean field) in the reduced density matrix in ODLRO, we may calculate the first-order in $S, v$ and $\rho$ for the mean field [22, 23, 24],

$$S(r, t) = S(r_0, t) + \xi \cdot \nabla_0 S(r_0, t) \quad (6a)$$
$$v(r, t) = v(r_0, t) + \frac{\partial}{\partial t} \xi + v(r_0, t) \cdot \nabla_0 \xi, \quad (6b)$$
$$\rho(r, t) = \rho(r_0) - \nabla_0 \cdot [\rho(r_0) \xi], \quad (6c)$$
where $\nabla_0$ denotes the partial derivative with respect to $r_0$ with $\nabla \rightarrow \nabla_0 - \nabla_0 \xi \cdot \nabla_0$. And $v(r_0, t) = \nabla S(r_0, t)/M$ is a solution of the Hamilton-Jacobi equation and $S(r_0, t)$ becomes a phase [19, 20]. Eq. (6c) was obtained from Eq. (6a) by using Eq. (6b) and then integrating over the time [23].

The first-order expansions in Eq. (6) are semiclassical since $S$ is a phase [20] and $\xi$ is a classical variable in a Lagrangian coordinate. The semiclassical perturbation method to the solution of the Hamilton-Jacobi equation is in fact a well defined, accurate quantum mechanical approximation scheme provided that $\psi(x, t)$ satisfies the nonlinear Schrödinger equation Eq. (2). Later we shall show this by deriving the Bogoliubov spectrum [32], but in the meantime we shall describe the scheme from semiclassical point of view.

It is useful to define the boundary conditions in terms of $\xi$ at a free surface, i.e., $\nabla \cdot \xi = 0$ and $\nabla \times \xi = 0$, which follow from Eq. (3) by the conditions of incompressibility and irrotational motion of a fluid at the free surface [33, 36, 39].

Our analysis of the quantization of circulation in He II requires the introduction of the phase coherence which is defined in ODLRO: $\xi \cdot \nabla S(x_0, t) = \sum_i \xi_i \cdot \nabla_i S_{0,i}(x_{0,i}, t)$, where the summation of the phase change by an individual atom on the right-hand side is not measurable, but the left-hand side in the mean-field is the one that can be measured in an experiment [20, 37]. Because of the phase coherence, the motion of one atom also implies the motion of others along the nodal surface. It must also be emphasized here that since the action $S$ is a macroscopic dynamic variable which represents symmetry breaking, it would, as in the case of electromagnetic field, depend to some extent on the actual location of the atom, and the perturbed terms become important in a physical process involving short distances (in Å).

Unlike the elementary excitations of quasi-particles (phonons and rotons) in He II, the macroscopic excitation of vortices takes place in a flow of a large amount of fluid at considerably higher energy [4, 38]. We follow closely Feynman’s picture of quantization of circulation [2, 10] and consider a ring of atoms that are located along the nodal surface for which the boundary conditions, $\nabla \times \xi = 0$ and $\nabla \cdot \xi = 0$ must be satisfied, and that the atoms are rotating under the gravitational field. The effect on the wave function is then given in the form, equivalent to Eq. (5). In ODLRO, the mean field $\psi(x, t)$ is a single-valued function, but the phase $S$ need not be single-valued in a multiply-connected system [13]; it need merely to return to its original value by traversing around a nodal surface.
With the spontaneously broken gauge symmetry, the Onsager-Feynman quantization can be stated in a more precise manner,

\[ \sum_{\text{cir}} \nabla S \cdot \xi = \oint \nabla S \cdot dl = nh. \]  

(7)

Here the summation is taken around the nodal surface and \( \xi \) is taken to be small since it is an atomic displacement. In He II, if one takes \( \nabla S(x_0, t) = Mv(x_0)s \), then \( \kappa = \oint v_s \cdot dl \), where \( \kappa = h/M = 0.997 \times 10^{-3} \text{cm}^2/\text{sec} \).

One sees that the spontaneously broken gauge symmetry is more fundamental for the creation of a vortex than that of the Bohr-Sommerfeld condition. The fact that the experimental data fit almost perfectly on the curve of a relation between the velocity \( v \) and energy \( E \) of a vortex ring with \( \kappa = h/M \) and the core radius \( a = (1.28 \pm 0.13) \hat{A} \) explains why the vortices behave entirely classically, since except for \( \kappa \) value both \( v \) and \( E \) are derived from classical hydrodynamics. It is therefore clear that the broken gauge symmetry leads to a breakdown of superfluidity.

The most definite demonstration of this concept was an observation of phase change by Inouye, et al., [37]. Moreover, it shows that \( \nabla \times \nabla S = 0 \) as it must be unless the gauge symmetry is spontaneously broken [14, 40, 41, 42].

There is no simple mathematical prescription for demonstrating that the symmetry of a Bose system is broken at a free surface of He II in a gravitational field. Therefore in order to demonstrate that the symmetry be always broken at a free surface of He II, it is necessary to introduce the collective excitations that take place simultaneously on both sides of the boundary for the solutions to the boundary-value problems in superfluid dynamics.

As a simple model which retains the essential features of the problem, we consider the free surface of a uniformly flowing superfluid. First, we consider the response of a free surface in a uniform superfluid flow to perturbations as a model for a reason that will become apparent shortly. Let us consider the superfluid in motion in the z-plane under the gravitational field. Here we assume the amplitude of a surface wave is small compared to the wavelength and thus take for the surface wave as the potential flow \( \nabla^2 \phi = 0 \) with \( \xi_s = -\nabla \phi \), which follows from the boundary conditions \( \nabla \cdot \xi = 0 \) and \( \nabla \times \xi = 0 \) [33, 36, 39]. In this connection, it is important to recall that the use of the Hamilton-Jacobi equation in solving for the motion of a particle is only a matter of convenience. Hence we employ Euler’s equation (with \( \nabla p = 0 \)
to describe the surface waves:

$$\rho \left( \frac{\partial}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\rho \mathbf{g} \cdot \nabla (r),$$

(8)

where \( \mathbf{g} \) is the gravitational acceleration.

In the rest frame of a fluid by Galilean transformation, we may take \( \rho_0 = \rho_0 \Theta(-z) \) and \( r = r_0 + \xi \) and linearize Eq. (8) along with \( \xi_s = -\nabla \phi \) as a displacement vector, and Eqs. (6b)-(6c). After a brief algebra we obtain the well-known dispersion relation for the gravity wave:

$$\omega^2 = gk,$$

(9)

where \( \phi = A \exp[kz] \cos(kx - \omega t) \) is assumed and \( k \) is the wave number.

To incorporate the capillary wave as a part of the surface wave, we must modify the potential \( \phi \) which is a solution of the Poisson equation. Furthermore, we note that the equation of motion for a particle at the free surface of the fluid is quite different in that the pressure difference between two sides of the free surface is not zero (i.e., \( \nabla p \neq 0 \) in Euler’s equation) and the potential \( \phi \) should instead satisfy the Laplace formula \[35, 39\],

$$\rho g \frac{\partial \phi}{\partial z} + \rho \frac{\partial^2 \phi}{\partial t^2} - \alpha \frac{\partial}{\partial z} \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right] = 0,$$

(10)

where \( g \) is the gravitational acceleration and \( \alpha \) is the surface tension, \( \alpha \approx 0.34 \text{erg/cm}^2 \) for He II. If we assume \( \phi = A \exp[kz] \cos(kx - \omega t) \), we are led immediately to the dispersion relation for the gravity wave,

$$\omega^2 = gk + \frac{\alpha k^3}{\rho_s}.$$  

(11)

The meaning of this dispersion relation is obvious: in the long wavelength limit, \( k \ll (g \rho/\alpha)^{1/2} \), we have a pure gravity wave; in the short wavelength limit (i.e., in \( A \)), the capillary waves obey the dispersion relation \( \omega^2 = \frac{\alpha k^3}{\rho_s} \) with the density \( \rho_0 \) at the free surface; the capillary waves have been observed in a recent experiment by Elliott, et al., \[43\].

The most remarkable feature of Eqs. (9) and (11) is that the dispersion relation is independent of internal dynamics of a superfluid, i.e., it is independent of the pair-interaction potential, and depends only on the external gravitational potential and the surface tension. The free surface thus behaves like a classical fluid, because it is independent of both \( \hbar \) and the speed of first sound \( c = [\frac{4\pi g \rho h^2}{M^2}]^{1/2} \).
With this tacit understanding, which will play an important role later, we shall next proceed with the calculation of the excitation spectrum for phonons in the fluid. Indeed, we have discovered this universal property of a free surface during the course of our study of collective excitations in a Bose condensate in trap in which the trapping force is comparable to the pair-interaction force of Bose particles in the system.

Inside of the free surface, the fluid is compressible, \( \nabla \cdot \boldsymbol{\xi} \neq 0 \), but it remains irrotational \( \nabla \times \boldsymbol{\xi} = 0 \) as it is a superfluid, which follows from Eq. (6). To obtain the phonon excitation spectrum, first note that the vortices are in an isolated region, so that we assume a uniform superfluid flow \( \mathbf{v}(x_0) \), and take the first-order terms \( \rho_1 \) and \( S_1 \) varying as \( C \exp\left[i(\mathbf{k} \cdot \mathbf{x} - \omega t)\right] \). It is then straightforward algebra to obtain the first-order linearized equations from Eq. (6),

\[
-i\omega \rho_1 - \frac{\rho_0}{M} k^2 S_1 = 0 \quad (12a)
\]
\[
-i\omega S_1 + \frac{4\pi\hbar^2 a}{M} \rho_1 + \frac{\hbar^2}{4M \rho_0} k^2 \rho_1 = 0. \quad (12b)
\]

Eqs. (12) yield at once the dispersion relation:

\[
\omega^2 = \frac{4\pi a \rho_0 \hbar^2 k^2}{M^2} + \frac{\hbar^2 k^4}{4M^2} \quad (13)
\]

This is the well-known Bogoliubov spectrum in He II [32]. It is evident why our semi-classical perturbation method is actually a perfectly well defined quantum mechanical approximation scheme.

We comment here briefly that, in the phonon regime \( k \to 0, \omega = \left[\frac{4\pi a \rho_0 \hbar^2 k^2}{M^2}\right]^{1/2} k \). The energy spectrum is characteristic of a sound wave with the speed of first sound \( c = \left[\frac{4\pi a \rho_0 \hbar^2 k^2}{M^2}\right]^{1/2} \) in an imperfect Bose gas. It is therefore natural to identify the underlying basic mechanism for the phenomena as a spontaneously broken gauge symmetry at the free surface in a Bose system (i.e., a transition from a superfluid to a normal fluid). The phonons may be interpreted as the massless Nambu-Goldstone bosons as \( k \to 0 \) in the symmetry breaking [40, 41, 42]. It should be emphasized that since in reality phonons are never confined to a strictly mathematical surface, the surface layer is thus composed of phonons and particles, which is consistent with the property of a free surface.

In connection with the two entirely different waves in He II under the gravitational field, it is worth while to note that Eq. (2) is the second-order partial differential equation. When it is linearized for an isolated system, it yields the second-order partial differential equation.
in $\xi$ which gives two particular solutions: one being the surface wave, the other the sound wave \[\text{[25, 26]}\]. In a linear regime, each wave satisfies its own dispersion relation. As shown above, the surface waves are always independent of internal dynamics - a classical fluid like behavior \[\text{[26]}\]. If in fact the gauge symmetry is not broken, we cannot hold the law of conservation of total energy in an isolated system \[\text{[26]}\].

The basic mechanism of symmetry breaking can be described as the following: as the sound wave propagates, it loses its phase coherence due to quantum fluctuations of particle trajectories driven by the effective quantum mechanical potential $U_{eqmp}$ and hence the sound wave dissipates by the interaction with a normal fluid at the free surface, which is also an irreversible process as the fluctuation-dissipation theory of Kubo \[\text{[48]}\]. It is more fundamental than that of Kubo’s theory of fluctuation-dissipation although both approaches are based on the dynamical variables (hidden variables in Bohm’s theory), since the fluctuation-dissipation is due to the broken gauge symmetry and the law of conservation of total energy in an isolated system can be preserved as a consequence of the broken gauge symmetry.

It should be stressed that the dissipation process gives rise the surface energy \[\text{[43]}\] which in turn raises the energy of particles in the surface layer; and therefore that the particles in the layer obey the Boltzmann statistics just like rotons in Landau’s two fluid model - a Bose quantum fluid to a normal fluid. Thus, we see that a complete picture of the two-fluid model emerges from this analysis \[\text{[50]}\].

The above analysis [Eq. (9) and Eq. (13)] is useful in the discussion of symmetry breaking at a free surface of a superfluid under the gravitational field [i.e., a breakdown of superfluidity at a vortex core and at the free surface of rotating He II]. It says two things: first, that the symmetry of a Bose system under an external field is broken for all wave numbers at the free surface. It is broken regardless of a local curvature of the free surface, for example, $k = m/r$ in cylindrical geometry, where $m$ is the mode number and $r$ is the radial distance of a point in cylindrical coordinates \[\text{[24]}\]. Secondly, the free surface always behaves like a classical fluid - a breakdown of superfluidity.

The first experimental observation on the collective excitations of rotons localized near the vortex core was reported in 1968 \[\text{[5]}\]. Keeping in mind the picture of a roton as the quantum analog of a smoke ring whose excitation energy is much higher than that of phonons, we limit, for the sake of simplicity, our discussion on the mechanism that drive roton excitations near the vortex line.
First we note that the free surface at the vortex core is a nodal surface and should therefore behave like a normal fluid. This explains why Rayfield and Reif have observed that a vortex responds like a normal fluid to an ion probe at the core. Moreover, as the superfluid density $\rho_s$ approaches to zero near the nodal surface of a vortex which rotates approximately with the speed of the first sound $c = \kappa/(\sqrt{8\pi\eta})$ with $\eta$ the coherence length, EQMP fluctuates rapidly to excite the rotons near the vortex line, because on the vortex line $U_{eqmp}$ is highly singular.

To show this explicitly, let us take a close look at EQMP. As emphasized by Bohm, the particle experiences a force from EQMP which fluctuates with the ensemble average energy and its particle momentum $p = \nabla S$ (e.g., $v_s = \kappa/2\pi r$ for a quantized vortex). If a particle happens to enter a region of space where $\rho$ becomes small, these fluctuations can become exceedingly large. Even with the vortex core model $\rho = \rho_0 r^2/r^2 + \eta^2$, EQMP fluctuates with the degree of divergence $\hbar^2/2\pi r^2$ as $r \to 0$. It is also obvious that the degree of fluctuations depends on the core radius and thus the local radius of a curvature. Hence it is apparent that the concentrated rotons near the vortex core are due to the spontaneously broken gauge symmetry that accompanies rotons (pseudo-Goldstone Bosons); the free (nodal) surface behaves like an ordinary fluid - normal fluid.

Just as in the case of the free surface He II under the gravitational field, the particles in the nodal surface of a vortex line obey the Boltzmann statistics due to the scattering with concentrated rotons; a remarkably simple picture which explains why Vinen’s observation of a vortex quantization based on the classical Magnus force remains correct. This picture, which is also in agreement with the observation of Rayfield and Reif, resolves the recent controversy over the Magnus force. While we have not made any attempt to derive the roton excitation spectrum, we feel our analysis captures the essence of the broken symmetry in a superfluid.

Because of the peculiarly universal classical behavior of the free surface, the breakdown of superfluidity observed in the experiment can be considered as a manifestation of the broken symmetry which accompanies pseudo-Goldstone bosons (rotons), i.e., a transition from a superfluid fluid to a normal fluid. This analysis also confirms Anderson’s insight on the role of quantum fluctuations in the symmetry breaking of the macroscopic Bose system.

We finally turn to the question of whether the superfluid component in a rotating He II
does in fact rotate. Osborne’s observation is believed to be incompatible with Landau’s two-fluid model from which he derived the following equation for the rotating He II:

\[ z = \frac{\rho_n \Omega^2}{\rho} \frac{r^2}{2g}, \tag{14} \]

where \( \rho = \rho_n + \rho_n \), \( \Omega \) is the angular velocity, \( g \) is the gravitational acceleration, and \( z \) and \( r \) are the vertical and radial coordinates of a point on the surface with respect to the vortex at the origin.

As shown above in a uniform flow of a superfluid, the free surface is independent of internal dynamics and behaves like a classical fluid. With this understanding, it is not difficult to derive the equation for the contour of a free surface of rotating He II in a steady state under the gravitational field:

\[ z = \frac{\Omega^2}{2g} r^2. \tag{15} \]

This is the expression confirmed in detail in experiments of a rotating He II both by Osborne and by Meservey. Here it is derived from the conditions that the density at the free surface is given by \( \rho_0 \) which follows from Eq. (6) with the boundary condition at the free surface. Also in the derivation of the above equation, we treated the superfluid as compressible (\( \nabla \cdot \boldsymbol{\xi} \neq 0 \)) outside the nodal surface (i.e., away from z-axis, a vortex line), yet remaining irrotational (\( \nabla \times \nabla S = 0 \) unless it is spontaneously broken) to be consistent with the quantum Hamilton-Jacobi equation. Here we wish to stress that the rotation of the bulk superfluid is not necessary in this derivation.

As emphasized by Mott, “the essential point in the two-fluid model is that when the normal fluid and superfluid are set in motion relative each other, there is no transfer of momentum from one to the other” [12]. The question arises then as to why the free surface of the superfluid does rotate with the angular velocity (\( \Omega \)) of the normal fluid to form a parabolic surface due to the centrifugal and gravitational forces on the surface. To answer this question we note that in reality a surface flow is not confined to a mathematical surface but to a surface layer with a finite thickness, which also contains the rotons, phonons, and He atoms with much higher energy than that of the bulk superfluid. Hence the free surface plays the role of a wall in Anderson’s analysis of a vortex nucleation [13, 14].

Therefore, we can see the vortices created at the solid base of a cylindrical container diffuse
upward to the free surface causing the rotation of the free surface through the exchange of the angular momenta of the vortices by $2\pi$ phase slippage that also accompanies dissipation process which gives rise to a free surface energy at the same time. As a result, the free surface rotates with the angular velocity $\Omega$ with the normal fluid which was examined by Meservey in his empirical study on the transient effects \[11\]. Another important point is that the free surface energy manifested in a recent observation of capillary waves at the free surface of He II \[43\] may indeed exclude vortices from the bulk superfluid in a manner similar that of the Meissner effect in a superconductor.

Our results are therefore just what we should expect on physical grounds - a picture of the quantum fluctuation-dissipation \[48\]. This picture emerges as a result of the broken symmetry and the phase slippage of vortices at the free surface, which is also consistent with Anderson’s insight into the role of quantum fluctuations on the symmetry breaking \[13, 40\]. More importantly, the present analysis confirms Meservey’s insight into the dynamics of upward diffusion of vortices from the solid base in a rotating He II \[11\].

Now it may be asked why this obvious inconsistency in the parabolic shape of rotating He II has long remained unresolved in Landau’s two-fluid model. To answer this question, it is helpful to consider Landau’s quantum hydrodynamics \[8\], in which he emphasized the superfluid velocity and its equation of motion in an infinite uniform fluid. Bogoliubov also developed a microscopic theory of superfluidity in He II using a model Hamiltonian in quantum field theory \[32\], which is valid only in a Hilbert space. Hence they have not addressed the surface phenomena. In 1966 Anderson introduced the concept of broken gauge symmetry in quantum fluids to explain the vortex nucleation \[13, 40\] after Josephson’s discovery of macroscopic interference phenomena \[49\]. However, the phenomena such as breakdown of superfluidity at a vortex core have remained unsolved in low temperature physics to date.

In summary, it is shown that the symmetry of a Bose system is broken at a free surface which accompanies quasi-particles (phonons and rotons) depending on the radius of curvature of the free surface and the flow velocity. This peculiarly universal classical fluid like behavior of a free surface in a rotating He II is a necessary consequence of the broken symmetry. One may thus interpret the breakdown of superfluidity at the vortex core as the spontaneously broken gauge symmetry that accompanies the pseudo-Goldstone bosons (rotons). The semiclassical perturbation method using the Lagrangian displacement vector
is indeed a perfectly well-defined quantum mechanical approximation scheme. As envisaged for the microscopic phenomena at short distances (i.e., several Å) in a many-body system by Bohm [19], EQMP plays a crucial role in the dynamics of a many-body interacting system at the quantum level of accuracy. It also explains why the vortices with an ion probe at the core move like a normal fluid [4]. We have also shown that the free surface energy due to surface tension in a rotating He II may indeed exclude the vortices from the bulk He II just like the Meissner effect in a superconductor, which explains Meservey’s observation on the contour of rotating He II.

I am grateful to Professor W. F. Vinen for pointing out the role of surface tension at the free surface of He II. I am also indebted to Dr. Robert Meservey for his helpful comments on the upward diffusion of vortices from the solid base in his rotating He II experiment. I also thank Professor G. B. Hess for sending me his paper on the geometrical effect of nucleation rate of vortices.

Note added in proof. After this paper was written, I learned about ’t Hooft’s conjecture [Massimo Blasone, Petr Jizba, Giuseppe Vitiello, Phys. Lett. A 287 (2001) 205]. The Letter describes the dissipation process of the 1-D classical oscillators with a damping term (coupled by time-reversal) by quantization of the orbits. In this paper, the symmetry of a Bose system is broken by the quantum fluctuation of the classical orbits defined by Hamilton-Jacobi equation near the free surface. The picture of fluctuation-dissipation in this paper is essentially equivalent to ’t Hooft’s conjecture.

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