1 Contextual Observables and Quantum Information.

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Abstract. In this short paper we present the main features of a new quantum programming language proposed recently by Peter Selinger which gives a good idea about the difficulties of constructing a scalable quantum computer. We show how some of these difficulties are related to the contextuality of quantum observables and to the abstract and statistical character of quantum theory (QT). We discuss also, in some detail, the statistical interpretation (SI) of QT and the contextuality of observables indicating the importance of these concepts for the whole domain of quantum information.

Keywords: Entanglement, qubits, quantum information, quantum computer, quantum measurement, quantum cryptography, foundations of quantum theory, contextual observables.

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1.1 Introduction

In classical information theory a fundamental unit is a bit taking the values 0 or 1 which can be easily implemented by presence or absence of a particular physical signal. In quantum information theory a fundamental unit is a qubit whose values are vectors in a Hilbert space $H_1=\mathbb{C}^2$ which are in one-to-one correspondence with the states of some two level physical system. Using two bits we may send 4 messages: 00, 01, 10, 11. With two qubits the messages are the vectors in $H_2=\mathbb{C}^2 \otimes \mathbb{C}^2$ with n qubits they are the vectors in $H_n$. This richness of the information together with long range correlations of the entangled quantum states stimulated a vigorous research on the quantum information theory and in particular on the quantum computer project.[10, 15, 30]

The quantum computation has been usually studied in terms of the Turing machines, or in terms of circuits and gates. Several detailed quantum programming languages have been recently developed[7, 23, 24, 31, 38]. In particular an attractive functional programming language was proposed by Selinger[35]. With all these developments understand better what kind of the quantum hardware is needed. This quantum device(computer) has to manipulate the states of the qubits in a well defined rapid and precise way. These states have to be implemented as the states of a particular quantum system. There are several proposed physical implementations of qubits and quantum gates using:
trapped ions[12, 22], cavity QED[40], nuclear spins[21], Josephson junctions[32] and quantum dots[17, 29, 41]. The states of interest of the quantum register are all pure states of n qubits. To make a quantum computer, with error correcting ability, the quantum register should contain from $10^4$ to $10^6$ qubits. One should be able to perform any unitary operation on the qubits using an appropriate sequence of some universal set of quantum gates [14, 30, 41]. One should be able to make a measurement in some chosen common measurement basis on each chosen qubit without disturbing the neighbouring qubits. The interactions of the system with the environment should be controlled in order to avoid a decoherence of the created pure states[4, 5, 20]. Finally the system should be scalable it means that the difficulty of performing gates, measurements, etc., should not grow too fast with the number of qubits in the register in this short. In spite of the magnificent technological progress in nanophysics[17, 22] and optics [10, 28, 42] we are still far from the engineering stage of the quantum computer and the goal of $10^{-5}$ errors in the functioning of the gates is far from being achieved experimentally[39, 20]. It is relatively easy task, on a paper, to manipulate entangled states of the qubits, to switch on and off the interactions, to apply a sequence of the unitary operators or to perform a particular measurement leading to the instantaneous collapse of the quantum state. To do it meaningfully and in a controllable way in the experiment is much more difficult because the quantum states are not the attributes of the individual quantum systems.

The states of the classical objects are characterized by well defined attributes which may be changed in the deterministic way quasi instantaneously. The states of the quantum system are described by the vectors $\Psi$ or by a density operator $\rho$ which give the statistical predictions for the repeated measurements of quantum observables performed after the identical preparatory manipulations on the same or on the "identical" quantum systems.[8, 9, 16].

The problem is that our knowledge about the microworld is always indirect [11, 25] and that we may only estimate the values of the physical observables from the empirical distributions of the experimental results. No experimental result in the microworld may be predicted with the certainty [9, 27] and the quantum theory supplemented sometimes by some stochastic ad hoc assumptions gives only the predictions on the statistical regularities observed in the experiments. This applies as well to all classic quantum experiments as to the experiments studying the interference effects using low intensity sources of entangled photons [33, 34, 42] or beams of heavy $C_{60}$ molecules[6]. This applies also to the experiments with the trapped ions and the quantum dots.

It seems to us that any successful device in quantum information must take into consideration the fact that the values of all quantum observables are contextual and the manipulation of the quantum state has nothing to do with the change of the attributes of some well defined and localised physical microsystem.

The paper is organized as follows. In the section 1 we give a short discussion of the attributive and contextual observables in physics. In the section 2 we recall a formalism and the statistical interpretation of the quantum theory. In the section 3 we talk about qubits and quantum gates. In the section 4 we
present a principle features of the Selinger’s quantum programming language. In section 5 we mention few implementation problems.

1.2 Contextual observables

Everyday observations taught us that the classical objects can be described and classified according to their characteristic properties, called attributes, such as their: form, size, dimension, colour etc. We found the effective classifications of plants, animals, chemical substances, stars and elementary particles. The static attributes are not sufficient to differentiate between different behaviour of the physical objects in various experiments this is why the contextual properties have been introduced.

An attributive property as a constant property of an object which can be observed and measured at any time and which is not modified by the measurement.

A contextual property as a property revealed only in a specific experiment and characterizing the interaction of the object with the measuring apparatus. Let us quote Accardi’s example of a colour of a chameleon which is green on the leaf and brown on the bark of the tree what resembles the behaviour of quantum systems [1, 2].

Another important example of contextual properties, inspired by some random experiments we discussed in [26], is the following. Let us imagine that we have several double sided coins $C_1, \ldots, C_k$ having two faces: red (“R”) and green (“G”). The coin may be flipped by different flipping devices $D_1, \ldots, D_n$. Each time if we flip a coin $C_i$ using the device $D_j$ we assume that we have an independent random Bernoulli trial with a probability of a success: $p(R) = p_{ij}$. If $p_{ij} \neq p_{kl}$ if $i \neq k$ and $j \neq l$ we clearly see that the probability $p_{ij}$ is neither an attribute of the coin $C_i$ nor the attribute of the device $D_j$. Each value $p_{ij}$ may be called a contextual property of the coin $C_i$. In general the values $p_{ij}$ are not known in advance and they can only be estimated within the error bars from the long runs of the corresponding random experiments.

A much more detailed discussion of the contextual observables may be found in [25] where we considered the sources of some hypothetical particle beams, detectors, counters, filters, transmitters and instruments and we obtained the following general results:

1) Properties of the beams depend on the properties of the devices and vice-versa and are defined only in terms of the observed interactions between them. For example a beam $b$ is characterized by the statistical distribution of outcomes obtained by passing by all the devices $d_i$. A device $d$ is defined by the statistical distributions of the results it produces for all available beams $b_i$. All observables are contextual and physical phenomena observed depend on the richness of the beams and of the devices.

2) In different runs of the experiments we observe the beams $b_k$ each characterized by its empirical probability distribution. Only if an ensemble $B$ of
all these beams is a pure ensemble of pure beams we can associate the estimated probability distributions of the results with the beams $b \in \beta$ and with the individual particles members of these beams.

3) A pure ensemble $\beta$ of pure beams $b$ is characterized by such probability distribution $s(r)$ which remains approximately unchanged:
   (i) for the new ensembles $\beta_i$ obtained from the ensemble $\beta$ by the application of the $i$-th intensity reduction procedure on each beam $b \in \beta$
   (ii) for all rich sub-ensembles of $\beta$ chosen in a random way

The quantum theory gives the probabilistic predictions thus it provides only a contextual description of the physical systems.

There is a wholeness in any physical experiment[8, 9]. The initial states are prepared and calibrated, they interact with the experimental arrangement and the modified final states and/or the final numerical results the measurements are found. The quantum theory (QT) does not give any intuitive spatio-temporal picture of what is physically happening, the QT gives only the predictions about the final states and about statistical distribution of the counts of the detectors.

1.3 Quantum formalism.

Let us recall now a standard description of the pure state of a quantum system with infinite number of degrees of freedom. We consider a Gelfand triplet of spaces $\Omega \subset H \subset \Omega'$ where $\Omega$ is for example a Schwartz space of rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^n$, $H=L^2(\mathbb{R}^n)$ and $\Omega'$ is a space of tempered distributions on $\Omega$. Let us consider two observables $A$ and $B$ measured in the mutually exclusive experiments I and II which are represented by self adjoint non commuting operators $\hat{A}$ and $\hat{B}$. If $\{\varphi^A_\lambda\}_\lambda \in \Lambda$ and $\{\varphi^B_\gamma\}_\gamma \in \Gamma$ are complete sets of the generalised eigenfunctions (tempered distributions) of $\hat{A}$ and $\hat{B}$ then we get two different eigenvalue expansions of the unit state vector $\Psi \in H$ of a studied physical system:

$$\Psi = \int_{\Lambda} \psi_A(\lambda) \varphi^A_\lambda d\lambda \quad (1)$$

and

$$\Psi = \int_{\Gamma} \psi_B(\gamma) \varphi^B_\gamma d\gamma \quad (2)$$

where $\psi_A(\lambda) = \langle \varphi^A_\lambda, \Psi \rangle$ and $\psi_B(\gamma) = \langle \varphi^B_\gamma, \Psi \rangle$ are square integrable complex value functions. The probabilities that the measured value of $A$ in the experiment I will fall into the interval $[a,b]$ and that the measured value of $B$ in the experiment II will fall in the interval $[c,d]$ are given by the following well known formulae:

$$P(a \leq A \leq b) = \int_a^b \lambda \overline{\psi_A(\lambda)} \psi_A(\lambda) d\lambda \quad (3)$$
\[ P(c \leq B \leq d) = \int_{c}^{d} \gamma \psi_B(\gamma) \psi_B(\gamma) d\gamma. \quad (4) \]

If a state of the system changes in time \( \Psi(t) = U(t)\Psi \), where \( U(t) \) is a unitary time evolution operator. The formulae (1) and (2) are mathematically equivalent but they describe completely different experiments. The predictions of QT for any experiment measuring the values of the observable \( O \) are always obtained using a couple ( a state \( \Psi \) and an operator \( \hat{O} \) ) and clearly all the information about the physical system is contextual and probabilistic. A value of a physical observable is not an attribute of the system but it is the characteristic of the pure ensemble of the experimental results created in the interaction of the system with the experimental device.

The fact that using the same state vector \( \Psi \) we may obtain the predictions on the results of all different measurements performed on the system led to the statements that the QT provides a complete description of each individual physical system. From this statement there is a one step to treat a state vector \( \Psi \) as an attribute of each single individual system what is incorrect and what leads to false paradoxes[9, 16, 27].

The most states in physics are not pure but mixed. In the case of a m-level system the most suitable is the formalism of the density operator \( \hat{\rho} \) acting in \( m \) dimensional hilbert space \( H_m \) such that the expectation value of of any observable \( O \) is given by

\[ \langle O \rangle = \text{Tr}(\hat{\rho}\hat{O}) . \quad (5) \]

The time evolution of \( \hat{\rho}(t) \) is given by:

\[ \hat{\rho}(t) = U(t)\hat{\rho}U(t)^* \quad (6) \]

where * denotes the hermitian conjugation.

The density operator formalism is more general since by coupling a system to the environment one may describe in terms of the reduced dynamics[3, 19] the passage from the pure to the mixed states allowing a description of the decoherence or of the measurement process. One gets simpler formalism by representing the operator \( \hat{\rho} \) by \( m \times m \) positive hermitian matrix \( \rho \). The density matrices and their transformations are building blocks of the quantum programming language[35]. In the following two sections we present the elements of this language.

### 1.4 Qubits and quantum gates

A pure state of a single qubit is a formal linear combination of two known states of some computational basis:\( q = \alpha |0\rangle + \beta |1\rangle \) which may be represented by a unit column vector \( u \) in \( C^2 \) and /or by \( 2 \times 2 \) complex density matrix \( uu^* \). The \( n \) qubit states are spanned by the tensor products of \( n \) single qubit states and all their states pure and mixed may be represented by \( 2^n \times 2^n \) complex density matrices \( M \).
A unitary operations $S$ on $n$ qubit states are called $n$-ary quantum gates and their action on the state is defined by equation (6): $S\rho S^*$. If $S$ is $n$-ary quantum gate, then the corresponding controlled gate $S_c$ is the $n+1$-ary gate defined by the $2^{n+1} \times 2^{n+1}$ matrix:

$$
S_c = \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix}
$$

(7)

where $I$ and $0$ are $2^n \times 2^n$ identity matrix and zero matrix respectively. In general a matrix with square brackets will be a matrix composed of 4 blocks each containing square matrices of the same dimensions.

The following set of four unary and 5 binary gates can be chosen to be built into the hardware[35]:

$$
N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{i} \end{bmatrix}
$$

$$
N_c, \quad H_c, \quad V_c, \quad W_c, \quad \text{and} \quad X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(8)

A set is not unique[30, 41] but it is sufficient to approximate the action of any unitary gate according to the theorem [14]which says that for any unitary matrix $S \in \mathbb{C}^{2^n \times 2^n}$ and $\epsilon > 0$ there exist a unitary matrix $S'$ and a unit complex number $\lambda$ such that

$$
\|S - \lambda S'\| < \epsilon
$$

(9)

and such $S' = I \otimes A \otimes J$ where $I, J$ are identity matrices of the appropriate dimensions, and $A$ is one of the gates $H, V_c$ or $X$.

In supplement to the unitary reversible operations, the operation " measure" is introduced:

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightarrow \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}
$$

(10)

where the first matrix corresponding to a classical bit 0 is obtained with the probability $\text{Tr} A$ and the second corresponding to the classical bit 1 is obtained with the probability $\text{Tr} B$. If the classical bits of information are ignored the final result is a normal sum of the outcome matrices(10):

$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightarrow \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}
$$

(11)
### 1.5 Quantum programming languages

Several proposals for the quantum programming languages have been made. Let us mention here the pseudo-cod formulation by Knill [23, 24], Omer’s rich procedural QCL[31], Bettelli’s quantum C++ [7] and Sanders and Zuliani qGCL [38]. All these quantum programming languages are so called imperative type languages because the quantum data are manipulated in terms of arrays of quantum bit references which requires the insertion of a number of run-time checks into the compiled code which must be executed at run-time not at the compile time.

The quantum programming language proposed by Selinger[35] is a functional programming language with a static type system which guarantees the absence of any run-time errors. The syntax and semantics of the language allow high-level features such as loops, recursive procedures, and structured data types. Each statement operates by transforming a specific set of inputs to outputs, and the principle of non-cloning of quantum data is enforced by the syntax. Both data flow and control flow are described using the paradigm “quantum data, classical control” in consistence with Shor’s factoring algorithm, Grover’s search algorithm and the Quantum Fourier Transform [13, 14, 30, 36, 37].

The models contains the classical and quantum flow charts. Quantum flow charts are similar to classical flow charts, except that a new type qbit of quantum bits, and two new operations: unitary transformations and measurements are added. The following notation is used for these new operations: q* = S for application of a unary unitary quantum gate S to a quantum bit q, p,q* = S for a binary quantum gate S applied to a pair p,q of quantum bits, <measure p>: for the branching statement giving a couple of outcomes after the measurement of p, ”◦” for the merge operation yielding the final mixed state etc.

In an example taken from [35] a flow chart below corresponds to the fragment of the program which: inputs two quantum bits p and q, measures p, and then performs one of two possible unitary transformations depending on the outcome of the measurement. The output is the modified pair p,q.

```
input p,q: qbit
       ↓
measure p
0     \downarrow 1
q* = N     p* = N
       \downarrow
merge (◦)
       ↓
output p,q: qbit
```

If the input is a general mixed state described by a 4×4 density matrix M written in the block notation the flow chart describes the following sequence of operations performed on the density matrix M.

\[(11)\]
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \rightarrow \begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & D
\end{bmatrix} \rightarrow
\]
\[
\begin{bmatrix}
\text{NAN*} & 0 \\
0 & 0
\end{bmatrix}, \begin{bmatrix}
D & 0 \\
0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
A' & 0 \\
0 & 0
\end{bmatrix}
\]
(12)

where \(A' = \text{NAN*} + D\). Let us note that by the usual normalisation convention the traces of all the density matrices are equal to the probabilities that the corresponding edge is reached. If the input is a pure state, then the states along each of the two branches of the measurement continue to be pure and a final mixed state is obtained by the merge operation.

1.6 Hardware requirements

The quantum computing language could be implemented on various devices but a real computation time gain could be only achieved using a quantum computer for example a QRAM type machine[23] which consists of a classical computer which controls a special quantum hardware device such that[35]:

a) The quantum device provides a potentially large number of individually addressable quantum bits.

b)Quantum bits can be manipulated via two fundamental operations: unitary transformations and measurements.

c)The quantum device will implement a fixed, finite set of unitary transformations which operate on one or two quantum bits at a time.

d)The classical controller communicates with the quantum device by sending a sequence of instructions, specifying which fundamental operations are to be performed.

e)The only output from the quantum device consists of the results of measurements, which are sent back to the classical controller.

The operating system has to keep a list of quantum bits that are currently in use by each process. When a process requests a new qubit, the operating system finds a qubit that is not currently in use, marks that qubit as being in use by the process, initializes the contents to 0, and returns the address of the newly allocated qubit. The process can then manipulate the qubit, for instance via operating system calls which take the qubit’s address as a parameter. The operating system ensures that processes cannot access qubits that are not currently allocated to them – this is very similar to classical memory management. Finally, when a process is finished using a certain qubit, it may deallocate it via another operating system call; the operating system will then reset the qubit to 0 and mark it as unused. In practice, there are many ways of making this scheme more efficient, for instance by dividing the available qubits into regions, and allocating and deallocating them in blocks, rather than individually.

Reseting or initializing a qubit to 0 is implemented by first measuring the qubit, and then by performing a conditional “not” operation \(N_c\) dependent on the outcome of the measurement.
In the introduction we reviewed various difficulties in constructing a scalable quantum computing device. Let us add here some additional problems related to the contextuality and SI of QT.

1) Concerning the measurements (10). Let us consider an entangled multiqubit state $\Psi$. The QT gives us statistical predictions on the results of various long runs of coincidence type measurements which may be performed by distant detectors on various qubits of a system. For any particular qubit we may obtain 0 or 1 with a given probability. Imagine now that in a given moment of time a single measurement performed on the qubit q gives the result 1 then according to SI we do not have any deterministic information about the results of other coincidence measurements performed in the same or future time on other qubits [9, 27]. The collapsed state vector gives us only the conditional probabilities for the results of the coincidence measurements performed on the remaining qubits corresponding to subensemble of all n qubit events in which a measurement performed on the qubit q gave 1.

2) Concerning the unitary gates (8,9). These transformations have to be implemented by sequences of: qubit-external field and/or qubit-qubit physical interactions. The ideal model requires an instantaneous switching on and off the various physical interactions without disturbing the other qubits. In practice one wants to use so called adiabatic switching which seems to be very difficult task.

3) The produced pure states of quantum register have to be protected against the decoherence due to the incontrollable influence of the environment.

1.7 Conclusions.

In our opinion any successful design of a quantum hardware for the purpose of quantum information must be consistent with the contextual and statistical character of quantum theory.

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