On the Spinor Representation of Surfaces in Euclidean 3-Space. *

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1 Introduction

The Weierstraß formula describes a conformal minimal immersion of a Riemann surface $M^2$ into the 3-dimensional Euclidean space $\mathbb{R}^3$. It expresses the immersion in terms of a holomorphic function $g$ and a holomorphic 1-form $\mu$ as the integral

$$f = \text{Re} \left( \int (1 - g^2, i(1 + g^2), 2g) \mu \right) : M^2 \to \mathbb{R}^3.$$

On the other hand let us consider the spinor bundle $S$ over $M^2$. This 2-dimensional vector bundle splits into

$$S = S^+ \oplus S^- = \Lambda^0 \oplus \Lambda^{1,0}.$$

Therefore the pair $(g, \mu)$ can be considered as a spinor field $\varphi$ on the Riemann surface. The Cauchy-Riemann equation for $g$ and $\mu$ is equivalent to the homogeneous Dirac equation

$$D(\varphi) = 0.$$

The choice of the Riemannian metric in the fixed conformal class of $M^2$ is not essential since the kernel of the Dirac operator is a conformal invariant.

A similar description for an arbitrary surface $M^2 \hookrightarrow \mathbb{R}^3$ is possible and has been pointed out probably for the first time by Eisenhardt (1909). This representation of any surface in $\mathbb{R}^3$ by a spinor field $\varphi$ on $M^2$ satisfying the inhomogeneous Dirac equation

$$D(\varphi) = H\varphi$$

involving the mean curvature $H$ of the surface has been used again in some recent papers (see [KS], [R], [Tai1], [Tai2], [Tai3]). However, the mentioned authors describe the relationship between surfaces in $\mathbb{R}^3$ and solutions of the equation $(\ast)$ in local terms in order to get explicit formulas. The aim of the present paper is to clarify the mentioned representation of surfaces in $\mathbb{R}^3$ by solutions of the equation $D(\varphi) =$

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$H\varphi$ in a geometrically invariant way. It turns out that the main idea leading to the description of a surface by a spinor field $\varphi$ is simple: Consider an immersion $M^2 \hookrightarrow \mathbb{R}^3$ and fix a parallel spinor $\Phi$ on $\mathbb{R}^3$. Then the restriction $\varphi = \Phi|_{M^2}$ of $\Phi$ to the surface is (with respect to the inner geometry of $M^2$) a non-trivial spinor field on $M^2$ and defines a spinor $\varphi^*$ of constant length which is a solution of the inhomogeneous Dirac equation

$$D(\varphi^*) = H\varphi^*.$$  

Conversely, given a solution $\varphi$ of the equation $(\ast)$ with constant length there exists a symmetric endomorphism $E : T(M^2) \to T(M^2)$ such that the spinor field satisfies a "twistor type equation"

$$\nabla_X \varphi = E(X) \cdot \varphi.$$  

The resulting integrability conditions for the endomorphism $E$ are exactly the Gauß and Codazzi equations. As a consequence, the solution $\varphi$ of the Dirac equation

$$D(\varphi) = H\varphi, \quad |\varphi| \equiv const > 0$$  

yields an isometric immersion of $M^2$ into $\mathbb{R}^3$. In a similar way one obtains the description of conformal immersions using the well-known formula for the transformation of the Dirac operator under a conformal change of the metric (see [BFGK]).

## 2 The Dirac Operator of a Surface immersed into a Riemannian 3-Manifold.

Let $Y^3$ be a 3-dimensional oriented Riemannian manifold with a fixed spin structure and denote by $M^2$ an oriented surface isometrically immersed into $Y^3$. Because the normal bundle of $M^2$ is trivial, the spin structure of $Y^3$ induces a spin structure on the Riemannian surface $M^2$. The spinor bundle $S$ of the 3-manifold $Y^3$ yields by restriction the spinor bundle of the surface $M^2$. Over $M^2$ this bundle decomposes into

$$S = S^+ \oplus S^-$$  

where the subbundles $S^\pm$ are defined by (see [F])

$$S^\pm = \{\varphi \in S : i \cdot e_1 \cdot e_2 \cdot \varphi = \pm \varphi\}.$$  

Here $\{e_1, e_2\}$ denotes an oriented orthonormal frame in $T(M^2)$ and $X \cdot \varphi$ means the Clifford multiplication of a spinor $\varphi \in S$ by a vector $X \in T(M^2)$. Since in the 3-dimensional Clifford algebra the relation

$$e_1 \cdot e_2 = e_3$$  

holds, we can replace the Clifford product $e_1 \cdot e_2$ by the normal vector $\vec{N}$ of $M^2 \hookrightarrow Y^3$:

$$S^\pm = \{\varphi \in S : i \cdot \vec{N} \cdot \varphi = \pm \varphi\}.$$  

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Consider a spinor field $\Phi$ defined on the 3-manifold $Y^3$. Its restriction $\varphi = \Phi|_{M^2}$ is a spinor field defined on $M^2$ and decomposes therefore into $\varphi = \varphi^+ + \varphi^-$ with

$$\varphi^+ = \frac{1}{2}(\varphi + i\vec{N} \cdot \varphi), \quad \varphi^- = \frac{1}{2}(\varphi - i\vec{N} \cdot \varphi).$$

We denote by $\nabla^{Y^3}$ and $\nabla^{M^2}$ the covariant derivatives in the spinor bundles on $Y^3$ and $M^2$ respectively. For any vector $X \in T(M^2)$ we have the well-known formula (see [BFGK])

$$\nabla^{Y^3}_X(\Phi) = \nabla^{M^2}_X(\varphi) - \frac{1}{2}(\nabla_X \vec{N}) \cdot \vec{N} \cdot \varphi.$$

The vector $(\nabla_X \vec{N})$ coincides with the second fundamental form $\mathrm{II} : T(M^2) \to T(M^2)$ of the submanifold $M^2 \hookrightarrow Y^3$. Since $\mathrm{II}$ is symmetric the Clifford product $e_1 \cdot \mathrm{II}(e_1) + e_2 \cdot \mathrm{II}(e_2)$ is a scalar and equals $(-2H)$ where $H$ denotes the mean curvature of the surface $M^2$. The Dirac operator $D$ of $M^2$ defined by the formula

$$D(\varphi) = e_1 \cdot \nabla^{M^2}_{e_1} \varphi + e_2 \cdot \nabla^{M^2}_{e_2} \varphi$$

can now be expressed by the covariant derivative $\nabla^{Y^3}$ and the mean curvature vector:

$$e_1 \cdot \nabla_{e_1}^{Y^3}(\Phi) + e_2 \cdot \nabla_{e_2}^{Y^3}(\Phi) = D(\varphi) + H \cdot \vec{N} \cdot \varphi.$$

Suppose that the spinor field $\Phi$ on $Y^3$ is a real Killing spinor, i.e. there exists a number $\lambda \in \mathbb{R}$ such that for any tangent vector $\vec{T} \in T(Y^3)$ the derivative of $\Phi$ in the direction of $\vec{T}$ is given by the Clifford multiplication:

$$\nabla^{Y^3}_\vec{T}(\Phi) = \lambda \cdot \vec{T} \cdot \Phi.$$

For the restriction $\varphi = \Phi|_{M^2}$ we obtain immediately the equation

$$D(\varphi) = -2\lambda \varphi - H \cdot \vec{N} \cdot \varphi.$$

Using the decomposition $\varphi = \varphi^+ + \varphi^-$ the last equation is equivalent to the pair of equations

$$D(\varphi^+) = (-2\lambda - iH)\varphi^-, \quad D(\varphi^-) = (-2\lambda + iH)\varphi^+.$$

We discuss two special cases.

**Proposition 1:** Let $M^2$ be a minimal surface in $Y^3$. Then the restriction $\varphi = \Phi|_{M^2}$ of any real Killing spinor $\Phi$ on $Y^3$ is an eigenspinor of constant length on the surface $M^2$:

$$D(\varphi) = -2\lambda \varphi.$$

On the other hand, suppose that $\Phi$ is a parallel spinor ($\lambda = 0$) on $Y^3$. Then we obtain

$$D(\varphi^+) = -iH \varphi^-, \quad D(\varphi^-) = iH \varphi^+.$$
If we introduce the spinor field $\varphi^* = \varphi^+ - i\varphi^-$, a simple calculation shows

$$D(\varphi^*) = H\varphi^*.$$ 

The spinor field $\varphi^*$ is given by

$$\varphi^* = \varphi^+ - i\varphi^- = \frac{1}{2}(\varphi + i \cdot \vec{N} \cdot \varphi) - \frac{i}{2}(\varphi - i \cdot \vec{N} \cdot \varphi) = \frac{1}{2}(1 - i)\varphi + \frac{1}{2}(-1 + i) \cdot \vec{N} \cdot \varphi.$$ 

Moreover, the length of $\varphi^*$ is constant. This construction yields the

**Proposition 2:** Let $\Phi$ be a parallel spinor field defined on the 3-manifold $Y^3$ and denote by $\varphi = \Phi|_{M^2}$ its restriction to $M^2$. Define the spinor field $\varphi^*$ on $M^2$ by the formula

$$\varphi^* = \frac{1}{2}(1 - i)\varphi + \frac{1}{2}(-1 + i) \cdot \vec{N} \cdot \varphi.$$ 

Then $\varphi^*$ is a spinor field of constant length on $M^2$ satisfying the Dirac equation

$$D(\varphi^*) = H\varphi^*$$

where $H$ denotes the mean curvature.

**Remark 1:** The map $\Phi \mapsto \varphi^*$ associating to any parallel spinor $\Phi$ on $Y^3$ a solution of the equation $D(\varphi^*) = H\varphi^*$ is injective.

**Remark 2:** We can apply the above mentioned formulas not only for Killing spinors. Indeed, for any spinor field $\Phi$ we have

$$D_{Y^3}(\Phi) = D(\varphi) + H \cdot \vec{N} \cdot \varphi + \vec{N} \cdot (\nabla_{\vec{N}}^{Y^3} \Phi)$$

where $D_{Y^3}$ is the Dirac operator of the 3-manifold $Y^3$. Suppose there exists a function $\kappa : M^2 \rightarrow \mathbb{C}$ such that the normal derivative $(\nabla_{\vec{N}}^{Y^3} \Phi)$ of the spinor field $\Phi$ is described by $\kappa$:

$$\left(\nabla_{\vec{N}}^{Y^3} \Phi\right) = \kappa \Phi.$$ 

Then we obtain

$$D_{Y^3}(\Phi) = D(\varphi) + (H + \kappa) \vec{N} \cdot \varphi.$$ 

This formula (in arbitrary dimension) has been used for the calculation of the spectrum of the Dirac operator on hypersurfaces of the Euclidean space (see [Bä], [Tr1], [Tr2]).
3 Solutions of the Dirac Equation with Potential on Riemannian Surfaces.

Let \((M^2, g)\) be an oriented, 2-dimensional Riemannian manifold with spin structure. \(H : M^2 \to \mathbb{R}^1\) denotes a given smooth, real-valued function defined on the surface. In this part we study spinor fields \(\varphi\) on \(M^2\) that are solutions of the differential equation

\[ D(\varphi) = H\varphi. \]

If we decompose the spinor field into \(\varphi = \varphi^+ + \varphi^-\) according to the splitting \(S = S^+ \oplus S^-\) of the spinor bundle the equation we want to study is equivalent to the system

\[ D(\varphi^+) = H\varphi^- , \quad D(\varphi^-) = H\varphi^+. \]

To any solution \(\varphi\) of this equation we associate two forms \(F_\pm\) defined for pairs \(X, Y \in T(M^2)\) of tangent vectors:

\[ F_+(X, Y) = \text{Re}(\nabla_X \varphi^+, Y \cdot \varphi^-) , \quad F_-(X, Y) = \text{Re}(\nabla_X \varphi^-, Y \cdot \varphi^+). \]

Proposition 3:

a.) \(F_\pm\) are symmetric bilinear forms on \(T(M^2)\).

b.) The trace of \(F_\pm\) is given by \(\text{Tr}(F_\pm) = -H|\varphi^\pm|^2\).

Proof: The symmetry of \(F_\pm\) is a consequence of the Dirac equation as well as the assumption that \(H\) is a real-valued function. Indeed, we have

\[
\text{Re}(\nabla_{e_1} \varphi^+, e_2 \varphi^-) = \text{Re}(e_1 \cdot \nabla_{e_1} \varphi^+, e_1 \cdot e_2 \cdot \varphi^-) = \text{Re}(H\varphi^- - e_2 \cdot \nabla_{e_2} \varphi^+, e_1 \cdot e_2 \cdot \varphi^-) = \\
= H \cdot \text{Re}(\varphi^-, e_1 \cdot e_2 \cdot \varphi^-) + \text{Re}(\nabla_{e_2} \varphi^+, e_2 \cdot e_1 \cdot e_2 \cdot \varphi^-) = \\
= 0 + \text{Re}(\nabla_{e_2} \varphi^+, e_1 \cdot \varphi^-).
\]

Moreover, we calculate the trace of \(F_\pm\):

\[
\text{Tr}(F_\pm) = \text{Re}(\nabla_{e_1} \varphi^\pm, e_1 \cdot \varphi^\mp) + \text{Re}(\nabla_{e_2} \varphi^\pm, e_2 \cdot \varphi^\mp) = -\text{Re}(D(\varphi^\pm), \varphi^\mp) = -H|\varphi^\pm|^2.
\]

We study now special solutions of the equation \(D(\varphi) = H\varphi\), i.e. solutions with constant length \(|\varphi| \equiv \text{const} \neq 0\). It may happen that the components \(\varphi^\pm\) have a non-empty zero set.

Proposition 4: Suppose that the spinor field \(\varphi\) defined on the Riemannian surface \(M^2\) is a solution of the equation

\[ D(\varphi) = H\varphi \quad \text{with} \quad |\varphi| \equiv \text{const} \neq 0. \]

Then the forms \(F_\pm\) are related by the equation
\[ |\varphi^+|^2 F_+ = |\varphi^-|^2 F_- . \]

**Proof:** In case one of the spinors \( \varphi^+ \) or \( \varphi^- \) vanishes at a fixed point \( m_0 \in M^2 \) the relation between \( F_+ \) and \( F_- \) is trivial. Otherwise there exists a neighbourhood \( V \) of the point \( m_0 \in M^2 \) such that both spinors \( \varphi^+ \) and \( \varphi^- \) are not zero at any point \( m \in V \). The spinors

\[
\frac{e_1 \cdot \varphi^-}{|\varphi^-|}, \quad \frac{e_2 \cdot \varphi^-}{|\varphi^-|}
\]

are an orthonormal base in \( S^+ \) with respect to the Euclidean scalar product \( \text{Re}(\cdot, \cdot) \).

Therefore we obtain (on \( V \))

\[
\nabla_X \varphi^+ = \text{Re} \left( \nabla_X \varphi^+, \frac{e_1 \cdot \varphi^-}{|\varphi^-|} \right) \frac{e_1 \cdot \varphi^-}{|\varphi^-|} + \text{Re} \left( \nabla_X \varphi^+, \frac{e_2 \cdot \varphi^-}{|\varphi^-|} \right) \frac{e_2 \cdot \varphi^-}{|\varphi^-|} =
\]

\[
= \frac{1}{|\varphi^-|^2} \{ F_+(X, e_1)e_1 + F_+(X, e_2)e_2 \} \cdot \varphi^- .
\]

A similar calculation yields the formula

\[
\nabla_X \varphi^- = \frac{1}{|\varphi^+|^2} \{ F_- (X, e_1)e_1 + F_- (X, e_2)e_2 \} \cdot \varphi^+ .
\]

We multiply the equations by \( \varphi^+ \) and \( \varphi^- \) respectively and sum up. Then we obtain

\[
\frac{1}{2} \nabla_X (|\varphi^+|^2 + |\varphi^-|^2) = \text{Re}(A(X) \varphi^-, \varphi^+)
\]

where the endomorphism \( A : T(M^2) \to T(M^2) \) is defined by

\[
A(X) = \left\{ \frac{F_+(X, e_1)}{|\varphi^-|^2} - \frac{F_-(X, e_1)}{|\varphi^+|^2} \right\} e_1 + \left\{ \frac{F_+(X, e_2)}{|\varphi^-|^2} - \frac{F_-(X, e_2)}{|\varphi^+|^2} \right\} e_2 .
\]

Since \( F_\pm \) are symmetric tensors, the endomorphism \( A \) is symmetric too. Moreover, the trace of \( A \) vanishes:

\[
\text{Tr} A = \frac{1}{|\varphi^-|^2} \text{Tr}(F_+) - \frac{1}{|\varphi^+|^2} \text{Tr}(F_-) = -H + H = 0 .
\]

The length of the spinor field \( \varphi \) is constant. This implies

\[
\text{Re}(A(X) \cdot \varphi^-, \varphi^+) = 0 .
\]

At any point \( m \in V \) of the set \( V \) the spinors \( \varphi^+, \varphi^- \) are non-trivial. Then the rank of the endomorphisms \( A : T(M^2) \to T(M^2) \) is not greater than one. All in all, \( A \) is symmetric, \( \text{Tr}(A) = 0 \) and \( \text{rg}(A) \leq 1 \), i.e. \( A \equiv 0 \).

We now consider the sum

\[
F = F_+ + F_- .
\]
At points with $\varphi^+ \neq 0$ (or $\varphi^- \neq 0$) we have

$$
\frac{F}{|\varphi|^2} = \frac{F_+ + F_-}{|\varphi^+|^2 + |\varphi^-|^2} = \left(\frac{|\varphi^-|^2}{|\varphi^+|^2} + 1\right) \frac{F_-}{|\varphi^-|^2} = \frac{F_-}{|\varphi^-|^2}
$$

as well as

$$
\frac{F}{|\varphi|^2} = \frac{F_+ + F_-}{|\varphi^+|^2 + |\varphi^-|^2} = \left(1 + \frac{|\varphi^+|^2}{|\varphi^-|^2}\right) \frac{F_+}{|\varphi^+|^2} = \frac{F_+}{|\varphi^+|^2}.
$$

The endomorphism $E : T(M^2) \to T(M^2)$, $E = \frac{F}{|\varphi|^2}$, is defined at all points of $M^2$ and the formulas derived in the proof of proposition 4 in fact prove the following

**Proposition 5:** Let $\varphi$ be a solution of the differential equation $D(\varphi) = H\varphi$ on a Riemannian surface $(M^2, g)$ with a real-valued function $H : M^2 \to \mathbb{R}^1$. Suppose that the length $|\varphi| \equiv \text{const} \neq 0$ of the spinor field $\varphi$ is constant. Then

$$
E(X) = \frac{1}{|\varphi|^2} \text{Re}(\nabla_X \varphi, Y \cdot \varphi)
$$

is a symmetric endomorphism $E : T(M^2) \to T(M^2)$ such that

a.) $\nabla_X \varphi^+ = E(X) \cdot \varphi^-$, $\nabla_X \varphi^- = E(X) \cdot \varphi^+$

b.) $\text{Tr}(E) = -H$.

For a given triple $(M^2, g, E)$ of a Riemannian surface and symmetric endomorphism the existence of a non-trivial solution $\varphi$ of the equation

$$
\nabla_X \varphi = E(X) \cdot \varphi
$$

implies certain integrability conditions. It turns out that in this way we obtain precisely the well-known Gauß and Codazzi equations of the classical theory of surfaces in Euclidean 3-space.

**Proposition 6:** Let $(M^2, g)$ be a 2-dimensional Riemannian surface with a fixed spin structure and suppose that $E : T(M^2) \to T(M^2)$ is a symmetric endomorphism. If there exists a non-trivial solution of the equation

$$
\nabla_X \varphi = E(X) \cdot \varphi
$$

then

a.) (Codazzi equation): $\nabla_X (E(Y)) - \nabla_Y (E(X)) - E([X, Y]) = 0$.

b.) (Gauß equation): $\det(E) = \frac{1}{4}G$, where $G$ is the Gaussian curvature of $(M^2, g)$.
Proof: We prove the two equations in a way similar to the derivation of the integrability conditions for the Riemannian metric in case the space admits a Killing spinor (see [BFGK]). We differentiate the equation
\[ \nabla_X \varphi = E(X) \cdot \varphi \]
and then we calculate the curvature tensor \( R^S \) of the spinor bundle \( S \):
\[
R^S(X,Y)\varphi = \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X,Y]} \varphi = \{(\nabla_X (E(Y)) - \nabla_Y (E(X)) - E([X,Y])) + E(Y)E(X) - E(X)E(Y)\} \cdot \varphi.
\]
On the other side, the curvature tensor \( R^S: S \to S \) is given by the formula
\[
R^S(e_1, e_2) = \frac{1}{2} R_{1212} e_1 \cdot e_2.
\]
Denote by \( A(X,Y) \) the differential of \( E \):
\[
A(X,Y) = \nabla_X (E(Y)) - \nabla_Y (E(X)) - E([X,Y]).
\]
A simple algebraic calculation in the spin representation then leads to the equations
\[
-A(X,Y)\varphi^- = \left(2\det(E) + \frac{R_{1212}}{2}\right)i\varphi^+
\]
\[
A(X,Y)\varphi^+ = \left(2\det(E) + \frac{R_{1212}}{2}\right)i\varphi^-.
\]
We multiply the first equation once by the vector \( A(X,Y) \):
\[
||A(X,Y)||^2 \varphi^- = - \left(2\det(E) + \frac{R_{1212}}{2}\right)^2 \varphi^-
\]
and then we conclude \( A(X,Y) \equiv 0 \) (Codazzi equation) as well as \( \det(E) = -\frac{1}{4}R_{1212} = \frac{1}{4}G \) (Gauß equation).

For a given triple \((M^2, g, E)\) consisting of a Riemannian spin surface \((M^2, g)\) and of a symmetric endomorphism \( E \) we will denote by \( \mathcal{K}(M^2, g, E) \) the space of all spinor fields \( \varphi \) satisfying the equation \( \nabla_X \varphi = E(X) \cdot \varphi \). It is invariant under the quaternionic structure \( \alpha: S \to S \), i.e. \( \mathcal{K}(M^2, g, E) \) is a quaternionic vector space (see §4). Denote by \((-H)\) the trace of \( E \),
\[
Tr(E) = -H.
\]
Then we have
\[
\mathcal{K}(M^2, g, E) \subset \ker(D - H).
\]
In this part of the paper we prove that any spinor field \( \varphi \in \ker(D - H) \) of constant length belongs to one of the subspaces \( \mathcal{K}(M^2, g, E) \) for a suitable symmetric endomorphism \( E, Tr(E) = -H \).
Finally, we consider the lengths
\[ L_+ = ||\varphi^+||^2, \quad L_- = ||\varphi^-||^2 \]
of a non-trivial solution \( \varphi \in \mathcal{K}(M^2, g, E) \). Using the integrability condition \( \det (E) = \frac{G}{4} \) (i.e. \( ||E||^2 = H^2 - \frac{G^2}{4} \)) as well as the well-known formula \( D^2 = \Delta + \frac{G}{2} \) for the square \( D^2 \) of the Dirac operator we can derive formulas for \( \Delta(L_\pm) \):

\[
\Delta(L_\pm) = 2(\Delta(\varphi^\pm), \varphi^\pm) - 2(\nabla(\varphi^\pm), \nabla(\varphi^\pm)) \\
= 2(D^2(\varphi^\pm), \varphi^\pm) - 2\left(\frac{G}{2}\right) \cdot ||\varphi^\pm||^2 - 2||E||^2||\varphi^\mp||^2 \\
= 2\left(H^2 - \frac{G}{2}\right)(L_\pm - L_+) + 2\text{Re}(\nabla H \cdot \varphi^\mp, \varphi^\pm)
\]

In particular, if \( H \equiv \text{const} \) is constant, the difference \( u = L_+ - L_- \) satisfies the differential equation
\[ \Delta(u) = 4\left(H^2 - \frac{G}{2}\right)u. \]

4 The Period Form of a Spinor with \( \nabla_X \varphi = E(X) \cdot \varphi \).

We consider a spinor field \( \varphi \) on a Riemannian surface \((M^2, g)\) such that
\[ \nabla_X \varphi = E(X) \cdot \varphi \]
for a fixed symmetric endomorphism \( E \). The spinor bundle \( S \) carries a quaternionic structure \( \alpha : S \to S \) commuting with Clifford multiplication and interchanging the decomposition \( S = S^+ \oplus S^- \) (see [F]). For any spinor field \( \varphi = \varphi^+ + \varphi^- \) we define three 1-forms by
\[ \xi^\varphi(X) = 2(X \cdot \varphi^+, \varphi^-) \]
\[ \xi^\varphi_+(X) = (X \cdot \varphi^+, \alpha(\varphi^+)) \quad \xi^\varphi_-(X) = (X \cdot \varphi^-, \alpha(\varphi^-)). \]
\( \xi^\varphi \) and \( \xi^\varphi_\pm \) are \( \Lambda^{1,0} \)-forms, \( \xi^\varphi_- \) is a \( \Lambda^{0,1} \)-form. Indeed, \( e_1 \cdot e_2 \) acts on \( S^+ \) (on \( S^- \)) by multiplication by \((-i)\) (by \(i\)). Now we obtain
\[ (\ast \xi^\varphi)(e_1) = -\xi^\varphi(e_2) = 2(-e_2 \cdot \varphi^+, \varphi^-) = (-ie_2 \cdot e_1 \cdot e_2 \cdot \varphi^+, \varphi^-) = -i\xi^\varphi_+(e_1), \]
i.e. \( \ast \xi^\varphi = -i\xi^\varphi_- \) holds. A similar calculation gives \( \ast \xi^\varphi_+ = -i\xi^\varphi_- \) and \( \ast \xi^\varphi_- = i\xi^\varphi_- \). We split the 1-form \( \xi^\varphi \) into its real and imaginary part:
\[ \xi^\varphi = w^\varphi + i\mu^\varphi. \]
Moreover, we introduce the 1-form \( \Omega^\varphi \)
\[ \Omega^\varphi = \xi^\varphi_+ - \xi^\varphi_- \]
Then we have

**Proposition 7:** Let \((M^2, g)\) be a Riemannian spin surface and \( E : T(M^2) \to T(M^2) \) a symmetric endomorphism of trace \(-H\). Suppose the spinor field \( \varphi \) is a solution of the equation \( \nabla_X \varphi = E(X) \cdot \varphi \). Then
Let us consider the case that $(M^2, g)$ is isometrically immersed into the Euclidean space $\mathbb{R}^3$, $\Phi$ is a parallel spinor on $\mathbb{R}^3$ and the spinor field $\varphi^*$ on $M^2$ defined by the formula

\[ d\varphi^* = 2H(|\varphi^-|^2 - |\varphi^+|^2)dM^2. \]

\[ d\Omega^\varphi = 0. \]

\textbf{Proof:} We calculate $dw^\varphi$:

\[
\frac{1}{2}dw^\varphi(X, Y) = X(\text{Re}(Y \cdot \varphi^+, \varphi^-)) - Y(\text{Re}(X \cdot \varphi^+, \varphi^-)) - \text{Re}([X, Y] \cdot \varphi^+, \varphi^-) = \\
= \{g(X, E(Y)) - g(Y, E(X))\}|\varphi^-|^2 + \{g(X, E(Y)) - g(Y, E(X))\}|\varphi^+|^2.
\]

Since $E$ is symmetric, we obtain $dw^\varphi = 0$. A similar calculation shows the formula for $d\mu^\varphi$. For the proof of $d\Omega^\varphi = 0$ we first remark that the quaternionic structure $\alpha : S \to S$ and the hermitian product $(\cdot ,\cdot )$ on $S$ are related by

\[
(\varphi_1, \alpha(\varphi_2)) = -\overline{(\alpha(\varphi_1), \varphi_2)}. 
\]

Using this formula we can transform $d\xi^-\varphi$ in the following way:

\[
 d\xi^-\varphi(X, Y) = (Y \cdot E(X) \cdot \varphi^+, \alpha(\varphi^-)) + (Y \cdot \varphi^-, \alpha(E(X) \cdot \varphi^+)) \\
- (X \cdot E(Y) \cdot \varphi^+, \alpha(\varphi^-)) - (X \cdot \varphi^-, \alpha(E(Y) \cdot \varphi^+)) = \\
= -\overline{(\alpha(Y \cdot E(X) \cdot \varphi^+), \varphi^-)} - (E(X) \cdot Y \cdot \varphi^-, \alpha(\varphi^+)) \\
+ \overline{(\alpha(X \cdot E(Y) \cdot \varphi^+), \varphi^-)} + (E(Y) \cdot X \cdot \varphi^-, \alpha(\varphi^+)) = \\
= -\overline{(E(X) \cdot Y \cdot \varphi^-, \alpha(\varphi^+))} - (E(X) \cdot Y \cdot \varphi^-, \alpha(\varphi^+)) \\
+ (E(Y) \cdot X \cdot \varphi^-, \alpha(\varphi^+)) + (E(Y) \cdot X \cdot \varphi^-, \alpha(\varphi^+)).
\]

On the other hand we calculate $d\xi^+\varphi$:

\[
 d\xi^+\varphi(X, Y) = (Y \cdot E(X) \cdot \varphi^-, \alpha(\varphi^+)) + (Y \cdot \varphi^+, \alpha(E(X) \cdot \varphi^-)) \\
- (X \cdot E(Y) \cdot \varphi^-, \alpha(\varphi^+)) - (X \cdot \varphi^+, \alpha(E(Y) \cdot \varphi^-)) = \\
= (Y \cdot E(X) \cdot \varphi^-, \alpha(\varphi^+)) - (E(X) \cdot Y \cdot \varphi^+, \alpha(\varphi^-)) \\
- (X \cdot E(Y) \cdot \varphi^-, \alpha(\varphi^+)) + (E(Y) \cdot X \cdot \varphi^+, \alpha(\varphi^-)).
\]

Finally we obtain

\[
 d(\xi^-\varphi - \xi^+\varphi)(X, Y) = -\{E(X) \cdot Y + Y \cdot E(X)\}\varphi^-, \alpha(\varphi^+) \\
+ \{E(Y) \cdot X + X \cdot E(Y)\}\varphi^-, \alpha(\varphi^+) = \\
= 2\{g(E(X), Y) - g(E(Y), X)\}(\varphi^-, \alpha(\varphi^+))
\]

and $d(\xi^-\varphi - \xi^+\varphi) = 0$ follows again by the symmetry of $E$. \hfill \blacksquare
$$\varphi^* = \frac{1}{2} (\Phi|_{M^2} + i \cdot \bar{N} \cdot \Phi|_{M^2}) + \frac{i}{2} (i \cdot \bar{N} \cdot \Phi|_{M^2} - \Phi|_{M^2})$$

(see §2). In this case the forms $w^{\varphi^*}$ and $\Omega^{\varphi^*}$ are given by the expressions

$$w^{\varphi^*}(X) = -\text{Im}(X \cdot \Phi, \Phi), \quad \Omega^{\varphi^*}(X) = (X \cdot \Phi, \alpha(\Phi)),$$

and are exact 1-forms. Indeed, we defined functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{C}$ by

$$f(m) = -\text{Im}(m \cdot \Phi, \Phi), \quad g(m) = \langle m \cdot \Phi, \alpha(\Phi) \rangle$$

and then we have $df = w^{\varphi^*}$, $dg = \Omega^{\varphi^*}$. We remark that $f$ and $g$ describe in fact the isometric immersion $M^2 \hookrightarrow \mathbb{R}^3$ we started with. The 3-dimensional spinor $\Phi \in \Delta_3$ defines a real 3-dimensional subspace $\Delta_3(\Phi)$ by

$$\Delta_3(\Phi) = \{ \Psi \in \Delta_3 : \text{Re}(\Psi, \Phi) = 0 \}.$$

The map $\Psi \mapsto (-\text{Im}(\Psi, \Phi), (\Psi, \alpha(\Phi)))$ is an isometry between $\Delta_3(\Phi)$ and $\mathbb{R}^1 \oplus \mathbb{C} = \mathbb{R}^3$. Clearly, the immersion $M^2 \hookrightarrow \mathbb{R}^3$ is given by

$$M^2 \ni m \mapsto m \cdot \Phi \in \Delta_3(\Phi),$$

i.e. by the functions $f|_{M^2}$ and $g|_{M^2}$. With respect to $d(f|_{M^2}) = w^{\varphi^*}$ and $d(g|_{M^2}) = \Omega^{\varphi^*}$ we obtain a formula for the isometric immersion $M^2 \hookrightarrow \mathbb{R}^3$:

$$\oint (w^{\varphi^*}, \Omega^{\varphi^*}) : M^2 \to \mathbb{R}^3.$$

(Weierstraß representation of the surface.)

In general, we call a solution $\varphi$ of the differential equation $\nabla_X \varphi = E(X) \cdot \varphi$ exact iff the corresponding forms $w^\varphi, \Omega^\varphi$ are exact 1-forms. Using the definition

$$\text{Hess} \ (h)(X, Y) = \frac{1}{2} \{ g(\nabla_X (\text{grad} (h)), Y) + g(X, \nabla_Y (\text{grad} (h))) \}$$

of the Hessian of a smooth function $h$ defined on a Riemannian manifold we obtain the

**Proposition 8:** Let $\varphi \in \mathcal{K}(M^2, g, E)$ be an exact solution of the differential equations $\nabla_X \varphi = E(X) \cdot \varphi$ with $df = w^\varphi$, $dg = \Omega^\varphi$. Then

a.) $\text{Hess} \ (f) = 2 \left( |\varphi^+|^2 - |\varphi^-|^2 \right) E$.

b.) $|\text{grad} \ f|^2 = 4 |\varphi^+|^2 |\varphi^-|^2$.

c.) $\text{Hess} \ (g) = -4 (\varphi^-, \alpha(\varphi^+)) E$.

d.) $|\text{grad} \ (g)|^2 = (|\varphi^+|^2 - |\varphi^-|^2)^2$. 

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In particular, the determinant of the Hessian of the function \( f \) is given by

\[
\det(Hess(f)) = 4(|\phi^+|^2 - |\phi^-|^2)^2 \det(E) = (|\phi^+|^2 - |\phi^-|^2)^2 G.
\]

Here we used proposition 6, i.e. \( \det(E) = \frac{1}{4} G \).

**Corollary:** Let \( M^2 \) be a compact Riemannian spin-manifold and suppose that \( \phi \in K(M^2, g, E) \) is an exact, non-trivial solution. Then the spinors \( \phi^+ \) or \( \phi^- \) vanish at least at one point. Moreover, there exists \( m_0 \in M^2 \) such that \( G(m_0) \geq 0 \).

**Proof:** At a maximum point \( m_0 \in M^2 \) of \( f \) we have

\[
\text{grad}(f)(m_0) = 0, \quad \det(Hess(f)(m_0)) \geq 0.
\]

Recall that for any 2-dimensional Riemannian manifold \( (M^2, g) \) and any function \( h : M^2 \to \mathbb{R}^1 \) the 2-form

\[
\{2 \det(Hess(h)) - |\text{grad}(h)|^2 G\} dM^2 = d\mu^1
\]

is exact (see [S], page 47). Using this formula in case of an exact solution \( \phi \in K(M^2, g, E) \) we obtain

\[
\int_{M^2} (|\phi^+|^2 - |\phi^-|^2)^2 G = 2 \int_{M^2} |\phi^+|^2 |\phi^-|^2 G.
\]

**Corollary:** Let \( M^2 \) be a compact Riemannian spin manifold and suppose that \( \phi \in K(M^2, g, E) \) is an exact solution. Then

\[
\int_{M^2} (|\phi|^4 - 6|\phi^+|^2|\phi^-|^2)G = 0.
\]

We again discuss the last formula in case of an isometrically immersed surface \( M^2 \hookrightarrow \mathbb{R}^3 \) and a given parallel spinor \( \Phi \) on \( \mathbb{R}^3 \). We apply the integral formula to the spinor \( \phi^* = \phi^*_+ + \phi^*_- \) where

\[
\phi^*_+ = \frac{1}{2} \left( \Phi + i \vec{N} \cdot \Phi \right), \quad \phi^*_- = \frac{1}{2} \left( -\Phi + i \cdot \vec{N} \cdot \Phi \right).
\]

In this case we have

\[
|\phi^*_+|^2 = \frac{1}{2} |\Phi|^2 + \frac{1}{4} (i \vec{N} \cdot \Phi, \Phi), \quad |\phi^*_-|^2 = \frac{1}{2} |\Phi|^2 - \frac{1}{2} (i \vec{N} \cdot \Phi, \Phi)
\]

and \( (|\Phi| \equiv 1) \) therefore

\[
1 - 6|\phi^*_+|^2 |\phi^*_-|^2 = -\frac{1}{2} + \frac{3}{2} (i \cdot \vec{N} \cdot \Phi, \Phi)^2.
\]
Consequently the integral formula yields
\[ \int_{M^2} G = 3 \int_{M^2} (i \bar{N} \Phi, \Phi)^2 G. \]

The spinors \( i \Phi \) as well as \( \bar{N} \cdot \Phi \) belong to \( V(\Phi) \subset \Delta_3 \), the space of the immersion \( M^2 \hookrightarrow \mathbb{R}^3 = V(\Phi) \). The last formula means therefore
\[ \int_{M^2} G = 3 \int_{M^2} (\bar{N}, a_3)^2 G \]
for the unit vector \( a_3 = i \Phi \in V(\Phi) = \mathbb{R}^3 \).

5 The Spin Formulation of the Theory of Surfaces in \( \mathbb{R}^3 \).

An oriented, immersed surface \( M^2 \hookrightarrow \mathbb{R}^3 \) inherits from \( \mathbb{R}^3 \) an inner metric \( g \), a spin structure and a solution \( \varphi \) of the Dirac equation
\[ D(\varphi) = H \varphi \]
of constant length \( |\varphi| \equiv 1 \) where \( H \) denotes the mean curvature of the surface. The spinor field \( \varphi \) on \( M^2 \) is the restriction of a parallel spinor field \( \Phi \) of the Euclidean space \( \mathbb{R}^3 \). The period forms \( w^\varphi \) and \( \Omega^\varphi \) are exact and the immersion \( M^2 \hookrightarrow \mathbb{R}^3 \) is given by integration of the \( \mathbb{R}^1 \oplus \mathbb{C} = \mathbb{R}^3 \) valued form \( (w^\varphi, \Omega^\varphi) \). At least locally the converse is true: Given an oriented, 2-dimensional Riemannian manifold \( (M^2, g) \) with a fixed spin structure and a solution of constant length of the Dirac equation \( D(\varphi) = H \varphi \) for some smooth function \( H: M^2 \rightarrow \mathbb{R}^1 \), there exists a symmetric endomorphism \( E: T(M^2) \rightarrow T(M^2) \) such that \( \varphi \in K(M^2, g, E) \). Moreover, \( 2E \) is the second fundamental form of an isometric immersion \( (M^2, g) \rightarrow \mathbb{R}^3 \). We formulate this description of the theory of surfaces in \( \mathbb{R}^3 \) in the following

Theorem 1: Let \( (M^2, g) \) be an oriented, 2-dimensional Riemannian manifold and \( H: M^2 \rightarrow \mathbb{R}^1 \) a smooth function. Then there is a correspondence between the following data:

1. an isometric immersion \( (\tilde{M}^2, g) \rightarrow \mathbb{R}^3 \) of the universal covering \( \tilde{M}^2 \) into the Euclidean space \( \mathbb{R}^3 \) with mean curvature \( H \).
2. a solution \( \varphi \) with constant length \( |\varphi| \equiv 1 \) of the Dirac equation \( D(\varphi) = H \cdot \varphi \).
3. a pair \( (\varphi, E) \) consisting of a symmetric endomorphism \( E \) such that \( \text{Tr}(E) = -H \) and a spinor field \( \varphi \) satisfying the equation \( \nabla_X \varphi = E(X) \cdot \varphi \).

We apply now the well-known formulas for the change of the Dirac operator under a conformal change of the metric. Suppose that \( \tilde{g} = \sigma g \) are two conformally equivalent metrics on \( M^2 \) where \( \sigma: M^2 \rightarrow (0, \infty) \) is a positive function. Denote by \( D \) and \( \tilde{D} \) the Dirac operator corresponding to the metric \( g \) and \( \tilde{g} \) respectively. Then
\[ \tilde{D}(\tilde{\varphi}) = \sigma^{-3/4} D(\sigma^{1/4} \varphi) \]

holds (see [BFGK]). Let us consider a solution \( \varphi \) of the Dirac equation

\[ D(\varphi) = \lambda \varphi \]

on \((M^2, g)\) and suppose that \( \varphi \) never vanishes. We introduce the Riemannian metric \( \tilde{g} = |\varphi|^4 g \) as well as the spinor field \( \varphi^* = \frac{\varphi}{|\varphi|} \). Then we obtain

\[ \tilde{D}(\varphi^*) = \frac{\lambda}{|\varphi|^2} \varphi^* \quad , \quad |\varphi^*| \equiv 1, \]

and thus an isometric immersion \((\tilde{M}^2, |\varphi|^4 g) \hookrightarrow \mathbb{R}^3\) with mean curvature \( H = \frac{\lambda}{|\varphi|^2} \).

**Theorem 2:** Let \((M^2, g)\) be an oriented, 2-dimensional Riemannian manifold. Any spinor field \( \varphi \) without zeros that is a solution of the equation

\[ D(\varphi) = \lambda \varphi \]

defines an isometric immersion \((\tilde{M}^2, |\varphi|^4 g) \hookrightarrow \mathbb{R}^3\) with mean curvature \( H = \frac{\lambda}{|\varphi|^2} \).

**Remark 1:** Consider the case that \( M^2 \hookrightarrow S^3 \) is a minimal surface in \( S^3 \). Let \( \Phi \) be a real Killing spinor on \( S^3 \), i.e.

\[ \nabla_{\vec{T}}(\Phi) = \frac{1}{2} \vec{T} \cdot \Phi. \]

The restriction \( \varphi = \Phi|_{M^2} \) is an eigenspinor of the Dirac operator on \( M^2 \) with constant length (proposition 1). Therefore \( \varphi \) defines an isometric immersion of \((\tilde{M}^2, g) \hookrightarrow \mathbb{R}^3\) with mean curvature \( H \equiv -1 \). This transformation associates to any minimal surface \( M^2 \hookrightarrow S^3 \) a surface of constant mean curvature \( H \equiv -1 \) in \( \mathbb{R}^3 \), a well-known construction (see [La]).

**Remark 2:** Using the described correspondence between isometric immersions of surfaces into \( \mathbb{R}^3 \) and solutions of the Dirac equation \( D(\varphi) = H \cdot \varphi \) one can immediately remark that several statements of the elementary theory of surfaces are equivalent to several statements concerning solutions of the twistor equation (see [BFGK]). For example, in [Li] (see also proposition 6) one can find the following theorem: if \( f : M^2 \rightarrow \mathbb{R}^1 \) is a real-valued function such that the equation

\[ \nabla_{\vec{T}}(\varphi) + \frac{1}{2} f \cdot \vec{T} \cdot \varphi = 0 \]

admits a non-trivial solution then \( f \) is constant and \( f^2 = G \). In the theory of surfaces this statement corresponds to the fact that an umbilic surface is a part of the sphere or the plane. Indeed, an umbilic surface \( M^2 \hookrightarrow \mathbb{R}^3 \) admits a spinor field \( \varphi \) such that
\[ \nabla \vec{T}(\varphi) + \frac{1}{2} H \vec{T} \cdot \varphi = 0 \]

and therefore \( H^2 = G = \text{const} \), i.e. the second fundamental form is proportional to the metric. In a similar way one can translate other facts of the theory of surfaces into properties of solutions of the equation \( \nabla_X \varphi = E(X) \cdot \varphi \).

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