Kirillov’s conjecture and $\mathcal{D}$-modules

Esther Galina and Yves Laurent

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Introduction

Let $G = Gl_n(\mathbb{R})$ or $G = Gl_n(\mathbb{C})$ and let $P$ be the subgroup of matrices whose last row is $(0,0,\ldots,0,1)$. Kirillov [6] made the following conjecture:

**Conjecture** If $\pi$ is an irreducible unitary representation of $G$ on a Hilbert space $H$ then $\pi|_P$ is irreducible.

The proof of this conjecture has a long story, we refer to the introduction of Baruch [1] for details about it. A first proof for the complex case was done by Sahi [8]. The complete proof, that includes the real and complex case, was given by Baruch [1]. He uses an argument of Kirillov to show that the conjecture is an easy corollary of the following theorem:

**Theorem 1.** Let $T$ be a $P$-invariant distribution on $G$ which is an eigendistribution with respect to the center of the universal enveloping algebra associated with $G$. Then there exists a locally integrable function $f$ on $G$ which is $G$-invariant and real analytic on the regular set $G'$ such that $T = f$. In particular $T$ is $G$-invariant.

Baruch’s proof of theorem [1] uses standard methods to reduce the problem to nilpotent points and then needs a rather long and detailed study of the nilpotent $P$-orbits of the adjoint representation of $P$ on the Lie algebra $\mathfrak{g}$ of $G$.

If we replace ”$P$-invariant” by ”$G$-invariant” in theorem [1] we get a well known result of Harish-Chandra that we proved in [3] by means of $\mathcal{D}$-modules. We defined a class of $\mathcal{D}$-modules that we called ”tame”: a $\mathcal{D}$-module is tame if it satisfies a condition on the roots of a family of polynomials, the $b$-functions (see [1,1]). The main property of these $\mathcal{D}$-modules is that their solutions are always locally integrable. Then we proved that in
the Harish-Chandra case, the distribution $T$ is solution of a $\mathcal{D}$-module, i.e. a system of partial differential equations, which is tame.

In this paper, we want to prove theorem [1] by the same method. In fact our proof will be simple as we will not have to calculate the roots of the $b$-functions as in [3] but use only geometric considerations on the characteristic variety of the $\mathcal{D}$-module. We don’t need neither a concrete characterization of nilpotent $P$-orbits in $\mathfrak{g}$, we only use the stratification of $\mathfrak{g}$ in $G$-orbits and the parametrization by the dimension of $P$-orbits in a single $G$-orbit.

Our theorem is purely complex, its is a result for $\mathcal{D}$-modules on $\text{Gl}_n(\mathbb{C})$. So it gives results for distributions on any real form of $\text{Gl}_n(\mathbb{C})$. In the real form is $\text{Gl}_n(\mathbb{R})$ or $\text{Gl}_n(\mathbb{C})$ it gives theorem [1]. For other real forms it gives a result on distributions which are not characterized by the action of a group $P$ and does not seem to have an easy interpretation.

From the theorem with $G = \text{Gl}_n(\mathbb{C})$ we deduce easily the same theorem for $G = \text{Sl}_n(\mathbb{C})$ and $P$ a maximal parabolic subgroup. This gives the analog of theorem [1] for $\text{Sl}_n(\mathbb{C})$ and $\text{Sl}_n(\mathbb{R})$.

In section 1, we recall the definition of tame $\mathcal{D}$-modules and we define precisely the modules $\mathcal{M}_{F,p}$ that we want to consider. Then in section 1.3, we state our main results. In section 2, we study the very simple but illuminating case of $\mathfrak{sl}_2$.

In section 3, we prove general theorems on $\mathcal{D}$-modules defined on semi-simple Lie groups which will be used later to reduce the dimension of the Lie algebra. Then we give the proof of the main results in section 4.

1 Notations and definitions.

1.1 Tame $\mathcal{D}$-modules.

Let $\Omega$ be a complex analytic manifold. We denote by $\mathcal{O}_\Omega$ the sheaf of holomorphic functions on $\Omega$ and by $\mathcal{D}_\Omega$ the sheaf of differential operators on $\Omega$ with coefficients in $\mathcal{O}_\Omega$. If $(x_1, \ldots, x_n)$ are local coordinates for $\Omega$, we denote by $D_{x_i}$ the derivation $\frac{\partial}{\partial x_i}$. We refer to [2] for the theory of $\mathcal{D}_\Omega$-modules.

In this paper, we will consider coherent cyclic $\mathcal{D}$-modules that is $\mathcal{D}$-modules $\mathcal{M} = \mathcal{D}_\Omega/I$ quotient of $\mathcal{D}_\Omega$ by a locally finite ideal $I$ to $\mathcal{D}_\Omega$. Then the characteristic variety of $\mathcal{M}$ is the subvariety of $T^*\Omega$ defined by the principal symbols of the operators in $I$.

A $\mathcal{D}_\Omega$-module is said to be holonomic if its characteristic variety $Ch(\mathcal{M})$ has dimension $n = \dim \Omega$. Then $Ch(\mathcal{M})$ is homogeneous lagrangian and there exists a stratification $\Omega = \bigcup \Omega_\alpha$ such that $Ch(\mathcal{M}) \subset \bigcup \mathfrak{t}_{\Omega_\alpha}^* \Omega$ [5, Ch. 5].

Here a stratification of a manifold $\Omega$ is a locally finite union $\Omega = \bigcup \Omega_\alpha$ such that

- For each $\alpha$, $\overline{\Omega_\alpha}$ is an analytic subset of $\Omega$ and $\Omega_\alpha$ is its regular part.
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- $\Omega_\alpha \cap \Omega_\beta = \emptyset$ for $\alpha \neq \beta$.
- If $\overline{\Omega_\alpha} \cap \Omega_\beta \neq \emptyset$ then $\overline{\Omega_\alpha} \supset \Omega_\beta$.

Let $Z$ be a submanifold of $\Omega$ given in coordinates by $Z = \{(x, t) \in \Omega \mid t_1 = \cdots = t_p = 0\}$. The polynomial $b$ is a b-function for $M$ along $Z$ if there exists in the ideal $I$ an equation $b(\theta) + Q(x, t, D_x, D_t)$ where $\theta = t_1D_{t_1} + \cdots + t_pD_{t_p}$ and $Q$ is of degree $-1$ for the $V$-filtration. This means that $Q$ may be written as $\sum t_i Q_i(x, t, D_x, [t_k D_{t_j}])$. This b-function is said to be tame if the roots of the polynomial $b$ are strictly greater than $-p$.

A more precise and intrinsic definition is given in [3] and [7], the definition is also extended to "quasi" or "weighted" b-functions” where $\theta$ is replaced by $n_1 t_1 D_{t_1} + \cdots + n_p t_p D_{t_p}$ for integers $(n_1, \ldots, n_p)$. In the definition of tame the codimension $p$ of $Z$ is replaced by $\sum n_i$. As this definition will not be explicitly used here, we refer to [3] for the details.

**Definition 1.1.1.** [3] The cyclic holonomic $\mathcal{D}_\Omega$-module $M$ is tame if there is a stratification $\Omega = \bigcup \Omega_\alpha$ such that $\text{Ch}(M) \subset \bigcup \Omega_\alpha T^*_{\Omega_\alpha} \Omega$ and, for each $\alpha$, $\Omega_\alpha$ is open in $\Omega$ or there is a tame quasi-b-function associated to $\Omega_\alpha$.

The definition extends as follows:

**Definition 1.1.2.** [3] The cyclic holonomic $\mathcal{D}_\Omega$-module $M$ is weakly tame if there is a stratification $\Omega = \bigcup \Omega_\alpha$ such that $\text{Ch}(M) \subset \bigcup \Omega_\alpha T^*_{\Omega_\alpha} \Omega$ and, for each $\alpha$ one of the following is true:

(i) $\Omega_\alpha$ is open in $\Omega$,

(ii) there is a tame quasi-b-function associated with $\Omega_\alpha$,

(iii) no fiber of the conormal bundle $T^*_{\Omega_\alpha} \Omega$ is contained in $\text{Ch}(M)$.

In (iii), the fibers of $T^*_{\Omega_\alpha} \Omega$ are relative to the projection $\pi : T^* \Omega \rightarrow \Omega$. When $\Omega_\alpha$ is invariant under the action of a group compatible with the $\mathcal{D}$-module structure - which will be the case here, (iii) is equivalent to:

(iii)' $T^*_{\Omega_\alpha} \Omega$ is not contained in $\text{Ch}(M)$.

The following property of a weakly tame $\mathcal{D}_\Omega$-module has been proved in [3]:

**Theorem 1.1.3.** If the holonomic $\mathcal{D}_\Omega$-module $M$ is weakly tame it has no quotient with support in a hypersurface of $\Omega$.

If $\Lambda$ is a real analytic manifold and $\Omega$ its complexification, we also proved:

**Theorem 1.1.4.** Let $M$ be a holonomic weakly tame $\mathcal{D}_\Omega$-module, then $M$ has no distribution solution on $\Lambda$ with support in a hypersurface.

We proved that under some additional conditions, the distribution solutions of a tame holonomic $\mathcal{D}$-module are locally integrable that is in $L^1_{loc}$.
1.2 $\mathcal{D}$-modules associated to the adjoint action.

Let $G$ be a complex reductive Lie group, $P$ a Lie subgroup, $\mathfrak{g}$ and $\mathfrak{p}$ their Lie algebras.

The differential of the adjoint action of $G$ on $\mathfrak{g}$ defines a morphism of Lie algebra $\tau$ from $\mathfrak{g}$ to $\operatorname{Der}\mathcal{O}[\mathfrak{g}]$ the Lie algebra of derivations on $\mathcal{O}[\mathfrak{g}]$ by:

$$(\tau(Z)f)(X) = \frac{d}{dt} f(\exp(-tZ).X)|_{t=0} \quad \text{for } Z, X \in \mathfrak{g}, f \in \mathcal{O}[\mathfrak{g}]$$

(1.1)
i.e. $\tau(Z)$ is the vector field on $\mathfrak{g}$ whose value at $X \in \mathfrak{g}$ is $[X, Z]$. We denote by $\tau(\mathfrak{g})$ the set of all vector fields $\tau(Z)$ for $Z \in \mathfrak{g}$. It is the set of vector fields on $\mathfrak{g}$ tangent to the orbits of the adjoint action of $G$ on $\mathfrak{g}$. In the same way, $\tau(\mathfrak{p})$ is the set of all vector fields $\tau(Z)$ for $Z \in \mathfrak{p}$ and is the set of vector fields on $\mathfrak{g}$ tangent to the orbits of $P$ acting on $\mathfrak{g}$.

The group $G$ acts on $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. The space $\mathcal{O}[\mathfrak{g}^*]$ of polynomials on $\mathfrak{g}^*$ is identified with the symmetric algebra $S(\mathfrak{g})$. We denote by $\mathcal{O}[\mathfrak{g}^*]^G = S(\mathfrak{g})^G$ the space of invariant polynomials on $\mathfrak{g}^*$ and by $\mathcal{O}_+[\mathfrak{g}^*]^G = S_+(\mathfrak{g})^G$ the subspace of polynomials vanishing at $\{0\}$. The common roots of the polynomials in $\mathcal{O}_+[\mathfrak{g}^*]^G$ are the nilpotent elements of $\mathfrak{g}^*$.

Let $\mathcal{D}_\mathfrak{g}^G$ be the sheaf of differential operators on $\mathfrak{g}$ invariant under the adjoint action of $G$. The principal symbol $\sigma(R)$ of such an operator $R$ is a function on $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$ invariant under the action of $G$. If $F$ is a subsheaf of $\mathcal{D}_\mathfrak{g}^G$, we denote by $\sigma(F)$ the sheaf of the principal symbols of all elements of $F$.

**Definition 1.2.1.** [7] A subsheaf $F$ of $\mathcal{D}_\mathfrak{g}^G$ is of (H-C)-type if $\sigma(F)$ contains a power of $\mathcal{O}_+[\mathfrak{g}^*]^G$ considered as a subring of $\mathcal{O}_+[\mathfrak{g} \times \mathfrak{g}^*]^G$. A (H-C)-type $\mathcal{D}_\mathfrak{g}$-module is the quotient $\mathcal{M}_F$ of $\mathcal{D}_\mathfrak{g}$ by the ideal $\mathcal{I}_F$ generated by $\tau(\mathfrak{g})$ and by a subsheaf $F$ of (H-C)-type.

As described in [7, Examples 2.1.3. and 2.1.4], there are two main examples of (H-C)-type $\mathcal{D}_\mathfrak{g}$-module:

**Example 1.2.2.** An element $A$ of $\mathfrak{g}$ defines a vector field with constant coefficients on $\mathfrak{g}$ by:

$$(A(D_x)f)(x) = \frac{d}{dt} f(x + tA)|_{t=0} \quad \text{for } f \in S(\mathfrak{g}^*), x \in \mathfrak{g}$$

By multiplication, this extends to an injective morphism from the symmetric algebra $S(\mathfrak{g})$ to the algebra of differential operators with constant coefficients on $\mathfrak{g}$; we will identify $S(\mathfrak{g})$ with its image and denote by $P(\mathcal{D}_x)$ the image of $P \in S(\mathfrak{g})$. If $F$ is a finite codimensional ideal of $S(\mathfrak{g})^G$, its graded ideal contains a power of $S_+(\mathfrak{g})^G$ hence when it is identified to a set of differential operators with constant coefficients, $F$ is a subsheaf of $\mathcal{D}_\mathfrak{g}$ of (H-C)-type and $\mathcal{M}_F$ is a $\mathcal{D}_\mathfrak{g}$-module of (H-C)-type.

If $\lambda \in \mathfrak{g}^*$, the module $\mathcal{M}_\mathcal{F}^\lambda$ defined by Hotta and Kashiwara [4] is the special case where $F$ is the set of polynomials $Q - Q(\lambda)$ for $Q \in S(\mathfrak{g})^G$.

**Example 1.2.3.** The enveloping algebra $U(\mathfrak{g})$ is the algebra of left invariant differential operators on $G$. It is filtered by the order of operators and the associated graded algebra
is isomorphic by the symbol map to $S(g)$. This map is a $G$-map and defines a morphism from the space of bi-invariant operators on $G$ to the space $S(g)^G$. This map is a linear isomorphism, its inverse is given by a symmetrization morphism \cite[Theorem 3.3.4.]{B2}. Then, through the exponential map a bi-invariant operator $P$ defines a differential operator $\tilde{P}$ on the Lie algebra $g$ which is invariant under the adjoint action of $G$ (because the exponential intertwines the adjoint action on the group and on the algebra) and the principal symbol $\sigma(\tilde{P})$ is equal to $\sigma(P)$.

An eigendistribution $T$ is a distribution on an open subset of $G$ which is an eigenvector for all bi-invariant operators $Q$ on $G$, that is satisfies $QT = \lambda T$ for some $\lambda$ in $\mathbb{C}$.

Let $U$ be an open subset of $g$ where the exponential is injective and $U_G = \exp(U)$. Let $T$ be an invariant eigendistribution on $U_G$ and $\tilde{T}$ the distribution on $U$ given by $\langle T, \varphi \rangle = \langle \tilde{T}, \varphi \circ \exp \rangle$. As $T$ is invariant and eigenvalue of all bi-invariant operators, $\tilde{T}$ is solution of an (H-C)-type $D_g$-module.

In this paper, we fix a (H-C)-type subsheaf $F$ of $D_g^G$. We denote by $M_{F,g}$ the quotient of $D_g$ by the ideal $I_F$ generated by $\tau(g)$ and $F$. We denote by $M_{F,p}$ the quotient of $D_g$ by the ideal $J_F$ generated by $\tau(p)$ and $F$. We have a canonical surjective morphism whose kernel will be denoted by $K_p$:

$$0 \rightarrow K_p \rightarrow M_{F,p} \rightarrow M_{F,g} \rightarrow 0 \quad (1.2)$$

By example \cite{B2} the distribution of theorem \cite{B1} is solution of such a module $M_{F,p}$ (modulo transfer by the exponential map).

The Killing form is a non-degenerate invariant bilinear form on the semi-simple Lie algebra $[g, g]$ satisfying $B([X, Z], Y) = B([X, Y], Z)$. We extend it to a non-degenerate invariant bilinear form on $g$. This defines an isomorphism between $g$ and its dual $g^\ast$.

The cotangent bundle to $g$ is equal to $g \times g^\ast$ identified to $g \times g$ by means of the Killing form. Then it is known \cite[Prop 4.8.3.]{B1} that if $\mathfrak{N}$ is the nilpotent cone of $g$, the characteristic variety of $M_{F,g}$ is equal to

$$\{(X, Y) \in g \times g \mid Y \in \mathfrak{N}, [X, Y] = 0 \} \quad (1.3)$$

In the same way:

**Lemma 1.2.4.** The characteristic variety of $M_{F,p}$ is contained in

$$\{(X, Y) \in g \times g \mid Y \in \mathfrak{N}, [X, Y] \in p^\perp \} \quad (1.4)$$

**Proof.** Let us first consider that variety as a subset of $g \times g^\ast$. The characteristic variety of $M_{F,p}$ is contained in the variety defined by $F$ that is the nilpotent cone of $g^\ast$. On the other hand, it is contained in the variety defined by $\tau(p)$ that is

$$\{(X, \xi) \in g \times g^\ast \mid \forall Z \in p \quad \langle [X, Z], \xi \rangle = 0 \}$$
The isomorphism defined by the Killing form exchanges the nilpotent cone of $\mathfrak{g}$ and that of $\mathfrak{g}^*$, hence after this isomorphism the characteristic variety is a subset of $\mathfrak{g} \times \mathfrak{g}$ contained in
\[
\{(X,Y) \in \mathfrak{g} \times \mathfrak{g} | Y \in \mathfrak{N}, \forall Z \in \mathfrak{p} \ B([X,Z], Y) = 0\}
\]
But we have $B([X,Z], Y) = B([X,Y], Z)$ which gives the result.

Remark 1.2.5. Using theorem 3.3.1 it is not difficult to show that the characteristic variety of $\mathcal{M}_{F,p}$ is in fact equal to the set (1.4).

The variety (1.3) is lagrangian [4] hence the module $\mathcal{M}_{F,\mathfrak{g}}$ is always holonomic but in general the variety (1.4) is not lagrangian and $\mathcal{M}_{F,p}$ is not holonomic. We will see that it is the case when $G = \text{Gl}_n(\mathbb{C})$ and $P$ is the set of matrices fixing a non zero vector in $\mathbb{C}^n$, or $G = \text{Sl}_n(\mathbb{C})$ and $P$ a maximal parabolic group.

1.3 Main Result

To state the main results, we restrict to the following cases:

- $G$ is the group $\text{Gl}_n(\mathbb{C})$ acting on $\mathbb{C}^n$ by the usual action and $P$ is the stability subgroup of $G$ at $v_0 \in \mathbb{C}^n$, that is $P = \{ g \in G \mid g.v_0 = v_0 \}$.
- $G$ is the group $\text{Sl}_n(\mathbb{C})$ acting on the projective space $\mathbb{P}_{n-1}(\mathbb{C})$ and $P$ is a maximal parabolic subgroup, that is the stability group of a point in $\mathbb{P}_{n-1}(\mathbb{C})$.
- $G$ is a product of several groups $\text{Gl}_n(\mathbb{C})$ and $\text{Sl}_n(\mathbb{C})$ and $P$ is the corresponding stability group.

In the first two cases, all subgroups $P$ are conjugated (except the trivial case $v_0 = 0$). The third case will be useful during the proof.

Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of $G$ and $P$. Our main result which will be proved in [3] is the following:

**Theorem 1.3.1.** For any subsheaf $F$ of $\mathcal{D}_\mathfrak{g}^G$ of $(H-C)$-type, the $\mathcal{D}_\mathfrak{g}$-module $\mathcal{M}_{F,p}$ is holonomic and weakly tame.

Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}$ and $G_{\mathbb{R}}$ be the corresponding group. Theorem 1.3.1 and theorem 1.5.7. in [3] implies:

**Corollary 1.3.2.** $\mathcal{M}_{F,p}$ has no singular distribution (or hyperfunction) solution on an open set of $\mathfrak{g}_{\mathbb{R}}$.

In [3 corollary 1.6.3] we proved that $\mathcal{M}_{F,\mathfrak{g}}$ has a stronger property: all its solutions are $L^1_{\text{loc}}$. Here we prove only that $\mathcal{M}_{F,p}$ is weakly tame but we will still be able to show that all solutions are $L^1_{\text{loc}}$.
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Let $g_\mathbb{R}$ be $\mathfrak{gl}_n(\mathbb{R})$, then $G_\mathbb{R}$ is equal to $Gl_n(\mathbb{R})$. Let $v_0$ be a non zero vector of $\mathbb{R}^n$ and $P_\mathbb{R}$ be the stability group of $v_0$, $p_\mathbb{R}$ its Lie algebra. Remark that $p_\mathbb{R}$ is a real form for $p$. The same is true if $g_\mathbb{R}$ is $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{gl}_n(\mathbb{C})$ viewed as a real form of $\mathfrak{gl}_2n(\mathbb{C})$.

**Theorem 1.3.3.** Assume that $g_\mathbb{R}$ is $\mathfrak{gl}_n(\mathbb{R})$, $\mathfrak{gl}_n(\mathbb{C})$, $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{C})$ or any direct sum of them. Let $G_\mathbb{R}$ be the corresponding Lie group and $P_\mathbb{R}$ the stability group of a real point.

Then any distribution solution of $\mathcal{M}_{F, p}$ which is invariant under the action of $P_\mathbb{R}$ is a $L^1_{loc}$ function invariant under the action of $G_\mathbb{R}$.

**Proof of theorem** From example 1.2.3, we know that a distribution satisfying the conditions of theorem is a solution of a module $\mathcal{M}_{F, p}$ hence is $G$-invariant. So as in Baruch [1], this theorem is an easy consequence of theorem 1.3.3.

However if $g_\mathbb{R}$ is a real form of $Gl_n(\mathbb{C})$ different from $Gl_n(\mathbb{R})$ or $Gl_n(\mathbb{C})$, the intersection of $p$ and $g_\mathbb{R}$ is not a real form for $p$. Then corollary 1.3.2 is still true but the solutions of $\mathcal{M}_{F, p}$ do not correspond to the action of a group.

**2 Example: the $\mathfrak{sl}_2$-case**

We consider the canonical base of $\mathfrak{sl}_2$:

$$
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
$$

and the general matrix of $\mathfrak{sl}_2$ is written as $Z = xH + yX + zY$. Let $g = \mathfrak{sl}_2$ and define $p$ as the subspace generated by $H$ and $X$.

In coordinates $(x, y, z)$ we have:

$$
\tau(H) = 2(zD_z - yD_y) \\
\tau(X) = -zD_x + 2xD_y \\
\tau(Y) = yD_x - 2xD_z
$$

By definition, the value of $x\tau(H) + y\tau(X) + z\tau(Y)$ at the point $Z = xH + yX + zY$ is $[xH + yX + zY, xH + yX + zY] = 0$ hence we have $x\tau(H) + y\tau(X) + z\tau(Y) = 0$.

Here $\tau(p)$ is generated by $(\tau(H), \tau(X))$ while $\tau(g)$ is generated by $(\tau(H), \tau(X), \tau(Y))$ hence the kernel of $\mathcal{M}_{F, p} \to \mathcal{M}_{F, g}$ is the submodule of $\mathcal{M}_{F, p}$ generated by $\tau(Y)$. This defines an exact sequence:

$$
0 \to K_F \xrightarrow{\tau(Y)} \mathcal{M}_{F, p} \longrightarrow \mathcal{M}_{F, g} \longrightarrow 0
$$

The module $K_F$ is the quotient of $\mathcal{D}_g$ by the ideal $J = \{ Q \in \mathcal{D}_g \mid Q \tau(Y) \in J_F \}$. Here $J_F$ is the ideal of $\mathcal{D}_g$ generated by $\tau(p)$ and a subsheaf $F$ of $\mathcal{D}_g^G$ is of (H-C)-type.
The equations
\[ z\tau(Y) = -x\tau(H) - y\tau(X) \quad [\tau(X), \tau(Y)] = \tau(H) \quad [\tau(H), \tau(Y)] = -2\tau(Y) \]
and \([Q, \tau(Y)] = 0\) for any \(Q \in F \subseteq D^G_g\)
show that the \(z, \tau(X), \tau(H) + 2\) and \(F\) are contained in \(J\).

The characteristic variety of \(K_F\) is thus contained in the set defined by \(z\) and \(F\), that is
\[ \{(x, y, z, \xi, \eta, \zeta) \in g \times g^* \mid z = 0, \xi = 0, \eta = 0\} \]
that is the conormal bundle to \(S = \{z = 0\}\). This implies that \(K_F\) is isomorphic to a power of \(B_{S|g} = D_g/D_gz + D_gD_x + D_gD_y\).

For example, if \(F = \{D_x^2 + 4D_yD_z - \lambda\}\), then \(J\) is generated by \((z, D_y, D_z^2)\).

Consider now distribution solutions of these modules, they make an exact sequence:
\[
\begin{align*}
0 \rightarrow \text{Sol}(\mathcal{M}_g) & \rightarrow \text{Sol}(\mathcal{M}_p) \rightarrow \text{Sol}(\mathcal{K}_F) \\
& \uparrow^{\tau(Y)} \\
& \rightarrow \text{Sol}(\mathcal{K}_F)
\end{align*}
\]
A solution of \(\mathcal{K}_F\) is canceled by \(z\) and solution of a system isomorphic to a power of \(B_{S|g}\) hence it is of the form \(\varphi(x, y)\delta(z)\) where \(\varphi(x, y)\) is analytic and \(\delta(z)\) is the Dirac distribution.

**Proposition 2.0.4.** The module \(\mathcal{M}_p\) has no singular distribution solution.

**Proof.** Let \(T\) be a singular distribution solution of \(\mathcal{M}_p\). Outside of \(\{z = 0\}\), a solution of \(\mathcal{M}_p\) satisfies \(\tau(Y)T = 0\) hence is a solution of \(\mathcal{M}_g\). From [3, cor 1.6.3] this implies that \(T\) vanishes outside of \(\{z = 0\}\) hence is of the form \(T = T_1(x, y)\delta(z)\).

We must have \(\tau(Y)T(x, y, z) = \varphi(x, y)\delta(z)\) hence \(yD_xT_1(x, y)\delta(z) - 2xT_1(x, y)\delta'(z) = \varphi(x, y)\delta(z)\). So \(T_1(x, y)\) satisfy \(yD_xT_1(x, y) = \varphi(x, y)\) and \(xT_1(x, y) = 0\). As \(\varphi\) is analytic this implies that \(\varphi = 0\).

So \(T(x, y, z) = \delta(x)\delta(y)\delta(z)\), but then we would have \(\tau(Y)T(x, y, z) = 0\) and thus \(T\) would be a singular solution of \(\mathcal{M}_g\) which is impossible. \(\square\)

### 3 General results on inverse image by invariant maps.

In the section, we will prove some general results on the \(\mathcal{D}\)-module associated to an action of a group \(G\) on a manifold.
3.1 Inverse image of a $\mathcal{D}$-module.

We begin with elementary properties of inverse images that can be found for example in [2].

Let $\Phi : U \to V$ be a holomorphic map between two complex analytic manifolds. The inverse image of a coherent $\mathcal{D}_V$-module $M$ by $\Phi$ is, by definition, the $\mathcal{D}_U$-module:

$$\Phi^* M = \mathcal{O}_U \otimes_{\mathcal{O}_V} \Phi^{-1} M$$

The module $\Phi^* M$ is not always coherent but this is the case if $M$ is holonomic or if $\Phi$ is a submersion.

When $\Phi$ is the canonical projection $U \times V \to V$, the module $\Phi^* M$ is the external product $\mathcal{O}_U \hat{\otimes} M$ hence if $M = \mathcal{D}_V / I$ where $I$ is a coherent ideal of $\mathcal{D}_V$ then $\Phi^* M = \mathcal{D}_{U \times V} / J$ where $J$ is the ideal of $\mathcal{D}_{U \times V}$ generated by $I$.

Suppose now that $\Phi : U \to V$ is a submersion and let $I$ be a coherent ideal of $\mathcal{D}_V$. We consider the subset $J_0$ of $\mathcal{D}_U$ defined in the following way:

An operator $Q$ defined on an open subset $U'$ of $U$ is in $J_0$ if and only if there exists some differential operator $Q'$ on $\Phi(U')$ belonging to $I$ and such that for any holomorphic function $f$ on $V$ we have $Q(f \circ \Phi) = Q'(f) \circ \Phi$.

Then $\Phi^* M = \mathcal{D}_U / J$ where $J$ is the ideal of $\mathcal{D}_U$ generated by $J_0$. The problem being local on $U$, this is easily deduced from the projection case.

Let $G$ be a group acting on a manifold $U$. To an element $Z$ of the Lie algebra $\mathfrak{g}$ of $G$ we associate a vector field $\tau_U(Z)$ on $U$ defined as in (1.1) by:

$$\tau_U(Z)(f)(x) = \frac{d}{dt}f(exp(-tZ).x)|_{t=0} \quad (3.1)$$

**Lemma 3.1.1.** Let $\Phi : U \to V$ be a submersive map of $G$-manifolds satisfying $\Phi(g.x) = g.\Phi(x)$ for any $(g,x) \in G \times U$. Let $I$ be a coherent ideal of $\mathcal{D}_V$ and $M$ be the coherent $\mathcal{D}_V$-module $\mathcal{D}_V / I$. Then the inverse image $\Phi^* M$ of $M$ by $\Phi$ is a coherent $\mathcal{D}_U$-module $\mathcal{D}_U / J$ such that:

For any $Z \in \mathfrak{g}$, $\tau_V(Z)$ belongs to $I$ if and only if $\tau_U(Z)$ belongs to $J$.

**Proof.** An direct calculation shows that $\tau_V(Z)(f \circ \Phi) = \tau_U(Z)(f) \circ \Phi$ which shows immediately the lemma. $\square$

3.2 Equivalence.

Let $G$ be a complex Lie group acting transitively on a complex manifold $\Omega$. Let $v_0 \in \Omega$ and let $P = G^{v_0}$ be the stability subgroup at $v_0$, hence $\Omega$ is isomorphic to the quotient $G/P$. 
We denote by \((g,v) \mapsto g.v\) the action of \(G\) on \(\Omega\) and by \((g,X) \mapsto g.X\) the adjoint action of \(G\) on its Lie algebra \(\mathfrak{g}\). Then \(G\) acts on \(\mathfrak{g} \times \Omega\) by \(g.(X,v) = (g.X, g.v)\). The group \(P\) acts on \(\mathfrak{g}\) by restriction of the action of \(G\).

Let \(U\) be an open subset of \(\Omega\) containing \(v_0\) and \(\varphi\) a holomorphic map \(\varphi : U \rightarrow G\) such that \(\varphi(v).v_0 = v\) for all \(v\) in \(U\).

This defines a submersive morphism \(\Phi : \mathfrak{g} \times U \rightarrow \mathfrak{g}\) by \(\Phi(X,v) = \varphi(v)^{-1}.X\). The subsets of \(\mathfrak{g} \times U\) invariant under \(G\) are exactly the sets \(\Phi^{-1}(S)\) where \(S\) is an orbit of \(P\) on \(\mathfrak{g}\).

**Remark:** It is known that \(\Phi\) defines an equivalence between distributions on \(\mathfrak{g} \times U\) invariant under \(G\) and distributions on \(\mathfrak{g}\) invariant under \(P\) (see Baruch [1] for example). We will prove a similar result for \(\mathcal{D}\)-modules. However, in the case of distributions the map \(\Phi\) is of class \(C^\infty\) hence may be globally defined. Here we need a holomorphic map and such a section is not defined globally on an open set \(U\) stable under \(G\). This is of no harm as long as we consider locally the vector fields tangent to the orbits. In this section, when we speak of \(G\)-orbits on \(\mathfrak{g} \times U\), it means the intersection of \(\mathfrak{g} \times U\) with a \(G\)-orbit of \(\mathfrak{g} \times \Omega\).

For \(X \in \mathfrak{g}\) the action of \(G\) on \(\mathfrak{g} \times \Omega\) and on \(\mathfrak{g}\) defines vector fields \(\tau_{\mathfrak{g} \times \Omega}(X)\) on \(\mathfrak{g} \times \Omega\) and \(\tau_{\mathfrak{g}}(X)\) on \(\mathfrak{g}\) through formula (5.1).

Let \(\mathfrak{p}\) be the Lie algebra of \(P\) and denote by \(\tau(\mathfrak{p})\) the set of vector fields \(\tau_{\mathfrak{g}}(X)\) for \(X \in \mathfrak{p}\). Let us denote by \(\tau_*(\mathfrak{g})\) the set of vector fields \(\tau_{\mathfrak{g} \times \Omega}(X)\) for \(X \in \mathfrak{g}\). Define now \(\mathcal{N}_{\tau_*}(\mathfrak{g})\) as the quotient of \(\mathcal{D}_{\mathfrak{g} \times \Omega}\) by the ideal generated by \(\tau_*(\mathfrak{g})\) and \(\mathcal{M}_{\tau(\mathfrak{p})}\) as the quotient of \(\mathcal{D}_{\mathfrak{g}}\) by the ideal generated by \(\tau(\mathfrak{p})\).

**Lemma 3.2.1.** *The map \(\Phi\) defines an isomorphism between the restrictions to \(\mathfrak{g} \times U\) of the \(\mathcal{D}_{\mathfrak{g} \times \Omega}\)-modules \(\mathcal{N}_{\tau_*}(\mathfrak{g})\) and \(\Phi^*\mathcal{M}_{\tau(\mathfrak{p})}\).*

**Proof.** Let \(\Psi\) be the map \(\mathfrak{g} \times U \rightarrow \mathfrak{g} \times U\) given by \(\Psi(X,v) = (\Phi(X,v), v)\). It is an isomorphism which exchanges the \(G\)-orbits on \(\mathfrak{g} \times U\) with the product by \(U\) of the \(P\)-orbits on \(\mathfrak{g}\). Hence it exchanges the vector fields tangent to the \(G\)-orbits that is \(\tau_*(\mathfrak{g})\) with the product of the set \(\tau(\mathfrak{p})\) of vector fields on \(\mathfrak{g}\) tangent to the \(P\)-orbits by the set \(\mathcal{T}_U\) of all vector fields on \(U\) that is \(\tau(\mathfrak{p}) \otimes \mathcal{T}_U\).

The quotient of \(\mathcal{D}_{\mathfrak{g} \times U}\) by \(\tau(\mathfrak{p}) \otimes \mathcal{T}_U\) is precisely \(p^*\mathcal{M}_{\tau}\) where \(p : \mathfrak{g} \times U \rightarrow \mathfrak{g}\) is the canonical projection \(p(X,v) = X\) (see the previous section). As \(\Phi = p \circ \Psi\), we are done. \(\square\)

We may also define the module \(\mathcal{M}_{\tau(\mathfrak{g})}\) as the quotient of \(\mathcal{D}_{\mathfrak{g}}\) by the ideal generated by \(\tau(\mathfrak{g})\). Then we have:

**Lemma 3.2.2.** *The map \(\Phi\) defines an isomorphism between the restrictions to \(\mathfrak{g} \times U\) of the \(\mathcal{D}_{\mathfrak{g} \times \Omega}\)-modules \(\mathcal{M}_{\tau(\mathfrak{g})} \otimes \mathcal{O}_U\) and \(\Phi^*\mathcal{M}_{\tau(\mathfrak{g})}\).*
Proof. The inverse image by $\Phi$ of a $G$-orbit is the product of that $G$-orbit by $U$ hence the proof is the same than the proof of lemma 3.2.1.

Let $Q$ be a differential operator on $\mathfrak{g}$, then $Q \otimes 1$ is a differential operator on $\mathfrak{g} \times U$ and as $\Psi$ is an isomorphism, this defines $\Psi^*(Q \otimes 1)$ as a differential operator on $\mathfrak{g} \times U$. If $Q$ is $P$-invariant, then $\Psi^*(Q \otimes 1)$ is $G$-invariant on $\mathfrak{g} \times U$ and if $Q$ is $G$-invariant on $\mathfrak{g}$ then $\Psi^*(Q \otimes 1)$ is equal to $Q \otimes 1$. We denote $\tilde{\Psi}(Q) = \Psi^*(Q \otimes 1)$.

Let $F$ be a set of differential operators on $\mathfrak{g}$ invariant under the $P$-action, we consider four $\mathcal{D}$-modules:

- $M_{F,p}$ is the quotient of $\mathcal{D}_{\mathfrak{g}}$ by the ideal generated by $F$ and $\tau_{\mathfrak{g}}(p)$
- $M_{F,\mathfrak{g}}$ is the quotient of $\mathcal{D}_{\mathfrak{g}}$ by the ideal generated by $F$ and $\tau_{\mathfrak{g}}(\mathfrak{g})$
- $N_{F,\mathfrak{g}}$ is the quotient of $\mathcal{D}_{\mathfrak{g} \times \Omega}$ by the ideal generated by $\tilde{\Psi}(F)$ and $\tau_{s}(\mathfrak{g}) = \tau_{\mathfrak{g} \times \Omega}(\mathfrak{g})$
- the product $M_{F,\mathfrak{g}} \hat{\otimes} \mathcal{O}_\Omega$

As a consequence of Lemma 3.2.1 and Lemma 3.2.2 we have the following result:

Proposition 3.2.3. The $\mathcal{D}_{\mathfrak{g} \times U}$-modules $N_{F,\mathfrak{g}}$ and $\Phi^*(M_{F,p})$ are isomorphic as well as $M_{F,\mathfrak{g}} \hat{\otimes} \mathcal{O}_\Omega$ and $\Phi^*(M_{F,\mathfrak{g}})$.

These isomorphism are compatible with the morphisms $M_{F,p} \rightarrow M_{F,\mathfrak{g}}$ and $N_{F,\mathfrak{g}} \rightarrow M_{F,\mathfrak{g}} \hat{\otimes} \mathcal{O}_\Omega$.

If the operators of $F$ are $P$-invariant the operators of $\tilde{\Psi}(F)$ are $G$-invariant and if they are $G$-invariant then those of $\tilde{\Psi}(F)$ are $G$-invariant and independent of $v \in U$.

3.3 Reduction to a subalgebra

We assume now that $G$ is a reductive Lie group operating on a manifold $\Omega$ hence on $\mathfrak{g} \times \Omega$. The algebra $\mathfrak{g}$ is reductive hence $[\mathfrak{g}, \mathfrak{g}]$ is a semi-simple Lie algebra with a non-degenerate Killing form $B$. We extend the form $B$ to a non-degenerate invariant bilinear form on $\mathfrak{g}$ that we still denote by $B$.

Let $S \in \mathfrak{g}$ be a semi-simple element and $\mathfrak{m} = \mathfrak{g}^S$, the reductive Lie subalgebra of elements commuting with $S$. Let $\mathfrak{q} = \mathfrak{m}^\perp$ the orthogonal for the form $B$ and $\mathfrak{m}'' = \{ Y \in \mathfrak{m} \mid \det(ad Y)|_\mathfrak{q} \neq 0 \}$, let $M = G^S$ the associated Lie group.

We consider the map $\Psi : G \times \mathfrak{m}'' \times \Omega \rightarrow \mathfrak{g} \times \Omega$ defined by $\Psi(g, Y, v) = (gY, g_v)$. As $\mathfrak{g} = \mathfrak{m} \oplus [\mathfrak{g}, \mathfrak{s}]$, $\Psi$ is a submersion onto the open set $G\mathfrak{m}'' \times \Omega$. If $U$ is a $G$-invariant open subset of $\mathfrak{g} \times \Omega$, $\Psi^{-1}(U)$ is equal to $G \times U'$ for some open subset $U'$ of $\mathfrak{m}'' \times \Omega$ invariant under the action of $M$. 
Let $F$ be a (H-C)-type subsheaf of $\mathcal{D}_g^G$ defined on $U$. According to definition (1.2.1), $F$ is a subsheaf of $\mathcal{D}_g^G$ such that $\sigma(F)$ contains a power of $\mathcal{O}_+[g]^G$. Let $\tau_*(g)$ be the sheaf of vector fields tangent to the orbits of $G$ on $g \times \Omega$ as in section 3.2.

Let $\mathcal{N}_{F,g}$ be the coherent $\mathcal{D}_{g \times \Omega}$-module defined on $U$ as the quotient of $\mathcal{D}_{g \times \Omega}$ by the ideal generated by $F$ and $\tau_*(g)$.

Remark that here we assume that the operators of $F$ are $G$-invariant. For such operators $Q$ we have $\tilde{\Psi}(Q) = Q \otimes 1$ hence we may confuse $\tilde{\Psi}(F)$ and $F$.

**Theorem 3.3.1.** There exists a (H-C)-type subsheaf $F'$ of $\mathcal{D}_m^M$ on $U'$ such that $\Psi^*\mathcal{N}_{F,g} \simeq \mathcal{O}_G \otimes \mathcal{N}_{F',m}$ on $\Omega$.

**Proof.** The map $\Psi$ is a submersion hence $\Psi^*\mathcal{N}_{F,g}$ is coherent and canonically a quotient of $\mathcal{D}_{G \times m'^\times \Omega}$ by an ideal $J$.

Consider the action of $G$ on $G \times m'^\times \Omega$ given by $g' \cdot (g, A, v) = (g'g, A, v)$. The map $\Psi$ is compatible with this action of $G$ hence we may apply lemma 3.1.1 to the inverse image $\Psi^*\mathcal{N}_{F,g}$. We get that $J$ is an ideal containing the vector fields $\tau_G(X)$ for all $X \in g$ that is all vector fields on $G$. This shows that $J$ is the product of $\mathcal{D}_G$ by an ideal of $\mathcal{D}_{U'}$. Hence $\Psi^*\mathcal{N}_{F,g} = \mathcal{O}_G \otimes \mathcal{N}$ where $\mathcal{N}$ is some holonomic module on $\Omega$.

Consider now the action of $M$ on $G \times m'^\times \Omega$ given by

$$m.(g, A, v) = (m g m^{-1}, m.A, m.v)$$

and on $g \times \Omega$ induced by that of $G$. We may again apply lemma 3.1.1. We get that $\mathcal{N}$ is equal to the quotient of $\mathcal{D}_{m\times \Omega}$ by an ideal $I$ which contains the vector fields $\tau_{m \times \Omega}(X)$ for any $X \in m$.

We will now define the set $F'$ from $F$. As $S$ is semi-simple we have $g = m \oplus [g, S]$ hence a local isomorphism $\psi : [g, S] \otimes m'' \otimes g$ given by $\psi(X, m) = exp(X).m$. In coordinates $(x, t)$ induced by this isomorphism, all derivations in $x$ are in the ideal generated by the vector fields tangent to the $G$-orbits.

After division by these derivations an operator $Q$ invariant under $G$ depends only on $(t, D_t)$ i.e. is a differential operator on $m$ invariant under the action of $M$. Denote by $\psi^*Q$ this operator. If the principal symbol of $Q$ is a function of $\mathcal{O}[g]^G$, the principal symbol of $\psi^*Q$ is its restriction to $\mathcal{O}[m]^M$. Hence if $F$ is an (H-C)-type subsheaf of $\mathcal{D}_g^G$, $F' = \psi^*F$ is an (H-C)-type subsheaf of $\mathcal{D}_m^M$. Then the ideal $I$ is generated by $F'$ and $\tau_{m \times V}(m)$ which shows the theorem.

$\square$
4 The $Gl_n(\mathbb{C})$ and $Sl_n(\mathbb{C})$ cases

4.1 Main proof

WAssume now that $G$ is the linear group $Gl_n(\mathbb{C})$ acting on $V = \mathbb{C}^n$ by the standard action. Then $P$ is the subgroup of matrices which leave invariant a point $v_0 \in V = \mathbb{C}^n$ and its Lie algebra $\mathfrak{p}$ is the set of matrices which cancel $v_0$. If $v_0 = 0$ then $P = G$ and everything is trivial otherwise $v_0 \in V^* = \mathbb{C}^n - \{0\}$ and all subgroups $P$ are conjugate.

It is known [10] that a $G$-orbit in $\mathfrak{g}$ splits into a finite number of $P$-orbits. More precisely, let $\mathfrak{g}^{(d)}$ be the set of matrices $A$ such that the vector space generated by $(A^p v_0)_{p=0,\ldots,n-1}$ is $d$-dimensional. Then the $P$-orbits are exactly the intersections of the $G$-orbits with the varieties $\mathfrak{g}^{(d)}$. In particular, $\mathfrak{g}^{(n-1)}$ is a Zarisky open subset of $\mathfrak{g}$ where $P$-orbits and $G$-orbits coincide.

Remark 4.1.1. Let $\Sigma$ be the complementary of $\mathfrak{g}^{(n-1)}$. It is a hypersurface of $\mathfrak{g}$. Outside of $\Sigma$, $P$- and $G$-orbits coincide, hence the vector fields $\tau(\mathfrak{p})$ and $\tau(\mathfrak{g})$ are the same. So the kernel $K_\mathfrak{p}$ of $M_{F,p} \to M_{F,\mathfrak{g}}$ is supported by $\Sigma$.

More generally, we will consider a product

$$G = \prod_{k=1}^N Gl_{n_k}(\mathbb{C}) \text{ acting on } V = \prod_{k=1}^N \mathbb{C}^{n_k} \quad (4.1)$$

Let $F$ be a (H-C)-type subset of $\mathcal{D}_G$, we may consider the $\mathcal{D}$-modules $M_{F,\mathfrak{g}}, M_{F,p}$ and $N_{F,\mathfrak{g}}$ as in section 3.2. We will show:

**Proposition 4.1.2.** There is a stratification $\mathfrak{g} = \bigcup \mathfrak{g}_\alpha$ such that

1. The characteristic variety of $M_{F,p}$ is contained in the union of the conormals to the strata $\mathfrak{g}_\alpha$.

2. For each $\alpha$, if the conormal to $\mathfrak{g}_\alpha$ is contained in the characteristic variety of $M_{F,p}$, then $M_{F,p}$ admits a tame quasi-$b$-function along $\mathfrak{g}_\alpha$.

By definition this shows that the module $M_{F,p}$ is holonomic and weakly tame (theorem 1.3.1). In the proof we will encounter three situations:

a) the conormal to $\mathfrak{g}_\alpha$ is not contained in the characteristic variety of $M_{F,p}$

b) the module $M_{F,p}$ is isomorphic to $M_{F,\mathfrak{g}}$ in a neighborhood of $X_\alpha$ which implies the existence of a tame $b$-function because $M_{F,\mathfrak{g}}$ is tame.

c) the module $M_{F,p}$ is a power of the module associated to a normal crossing divisor and is trivially tame.

Remark that we will never need to explicit the definition of a tame $b$-function here. We will get it from results of [3] concerning the module $M_{F,\mathfrak{g}}$. 


By proposition $3.2.3$ proposition $4.1.2$ is equivalent to the following:

**Proposition 4.1.3.** There is a stratification $\mathfrak{g} \times V = \bigcup X_\alpha$ such that

1. The characteristic variety of $\mathcal{N}_{F,\mathfrak{g}}$ is contained in the union of the conormals to the strata $X_\alpha$.

2. For each $\alpha$, if the conormal to $X_\alpha$ is contained in the characteristic variety of $\mathcal{N}_{F,\mathfrak{g}}$, then $\mathcal{N}_{F,\mathfrak{g}}$ admits a tame quasi-$b$-function along $X_\alpha$.

### 4.2 Stratification

Let us first recall the stratification that we defined in [3] on any semi-simple algebra $\mathfrak{g}$.

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system associated to $\mathfrak{h}$. For each $\alpha \in \Delta$ we denote by $\mathfrak{g}_\alpha$ the root subspace corresponding to $\alpha$ and by $\mathfrak{h}_\alpha$ the subset $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{h}$.

Let $\mathcal{F}$ be the set of the subsets $\pi$ of $\Delta$ which are closed and symmetric that is such that $(\pi + \pi) \cap \Delta \subset \pi$ and $\pi = -\pi$. For each $\pi \in \mathcal{F}$ we define $\mathfrak{h}_\pi = \sum_{\alpha \in \pi} \mathfrak{h}_\alpha$, $\mathfrak{g}_\pi = \sum_{\alpha \in \pi} \mathfrak{g}_\alpha$, $\mathfrak{h}^\perp_\pi = \{ H \in \mathfrak{h} \mid \alpha(H) = 0 \text{ if } \alpha \in \pi \}$, $(\mathfrak{h}^\perp_\pi)' = \{ H \in \mathfrak{h}^\perp_\pi \mid \alpha(H) \neq 0 \text{ if } \alpha \notin \pi \}$ and $\mathfrak{q}_\pi = \mathfrak{h}_\pi + \mathfrak{g}_\pi$. $\mathfrak{q}_\pi$ is a semisimple Lie subalgebra of $\mathfrak{g}$.

**Remark 4.2.1.** With the notations of [3] we have $\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{g}_P$ and $\mathfrak{m}'' = (\mathfrak{h}^\perp_\pi)' \oplus \mathfrak{h}_P \oplus \mathfrak{g}_P$.

To each $\pi \in \mathcal{F}$ and each nilpotent orbit $\mathcal{O}$ of $\mathfrak{q}_\pi$ we associate a conic subset of $\mathfrak{g}$

$$S(\pi, \mathcal{O}) = \bigcup_{x \in (\mathfrak{h}^\perp_\pi)'} G.(x + \mathcal{O})$$

It is proved in [3] that these sets define a finite stratification of $\mathfrak{g}$ independent of the choice of $\mathfrak{h}$.

If $\mathfrak{g}$ is a reductive Lie algebra, we get a stratification of $\mathfrak{g}$ by adding the center $\mathfrak{c}$ of $\mathfrak{g}$ to any stratum of the semi-simple algebra $[\mathfrak{g}, \mathfrak{g}]$:

$$\bar{S}(\pi, \mathcal{O}) = S(\pi, \mathcal{O}) \oplus \mathfrak{c}$$

This applies in particular to $\mathfrak{gl}_n(\mathbb{C})$. For a matrix $X$ of $\mathfrak{gl}_n(\mathbb{C})$ and a vector $v$ of $\mathbb{C}^n$, we denote by $d(X, v)$ the dimension of the vector space generated by $(v, Xv, X^2v, \ldots, X^{n-1}v)$ where $Xv$ denotes the usual action. If $X = X_1 + \cdots + X_q$ is an element of $\bigoplus_{i=1}^q \mathfrak{gl}_n(\mathbb{C})$, $d(X, v)$ is the sum $\sum d(X_i, v_i)$.

Let $v_0$ be a non-zero vector of $\mathbb{C}^n$. To each $\pi \in \mathcal{F}$, each nilpotent orbit $\mathcal{O}$ of $\mathfrak{q}_\pi$ and each integer $p \in [0 \ldots n - 1]$ we associate:

$$S(\pi, \mathcal{O}, p) = \{ X \in \bar{S}(\pi, \mathcal{O}) \mid d(X, v_0) = p \}$$

The sets $\{ X \in \mathfrak{g} \mid d(X, v_0) = p \}$ form a finite family of closed algebraic subsets of $\mathfrak{g}$ hence the sets $S(\pi, \mathcal{O}, p)$ define a new stratification of $\mathfrak{g}$. 

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In the same way, we define a stratification of $g \times V$ by

$$T_{(\pi, D, p)} = \{ (X, v) \in g \times V \mid X \in \tilde{S}_{(\pi, D)}, \ d(X, v) = p \}$$

If $\Phi$ is the map $\Phi: g \times U \to g$ defined by a map $\varphi: U \to G$ as in section 3.2, we have $\Phi^{-1}(S_{(\pi, D, p)} = T_{(\pi, D, p)})$.

The stratification $(\tilde{S}_{(\pi, D)})$ has been associated to $\mathcal{M}_{F, g}$ in 3. We will associate $(S_{(\pi, D, p)})$ to $\mathcal{M}_{F, p}$ and $(T_{(\pi, D, p)})$ to $\mathcal{N}_{F, g}$.

To end this section let us calculate the characteristic variety of the module $\mathcal{N}_{F, g}$ when $G = GL_n(\mathbb{C})$. On $g = gl_n(\mathbb{C})$ we consider the scalar product $(A, B) \mapsto \text{trace}(AB)$ which extends the Killing form of $\mathfrak{sl}_n(\mathbb{C})$. This identifies $g$ and $g^*$ and in the same way the usual hermitian product $(u, v) \mapsto <u, \bar{v}>$ on $\mathbb{C}^n$ identifies $V$ and $V^*$.

If $u$ and $v$ are two vectors of $V = \mathbb{C}^n$ we denote by $u \wedge \bar{v}$ the $(n, n)$-matrix whose entry $(i, j)$ is $u_i \bar{v}_j$.

**Proposition 4.2.2.** The characteristic variety of $\mathcal{N}_{F, g}$ is contained in

$$\{ (X, u, Y, v) \in g \times V \times g \times V \mid Y \in \mathfrak{h}, [X, Y] = u \wedge \bar{v} \} \quad (4.2)$$

**Proof.** The proof is similar to the proof of lemma 4.2.4. The vector field $\tau_{g \times V}(Z)$ has value $([X, Z], Zu)$ at the point $(X, u) \in g \times V$ hence the characteristic variety of $\mathcal{N}_{F, g}$ is contained in the set of points $(X, u, Y, v) \in g \times V \times g \times V$ satisfying $B([X, Z], Y) + <Zu, \bar{v}> = 0$ for any $Z \in g$.

But we have $<Zu, \bar{v}> = \sum Z_{ij} u_j \bar{v}_i = B(Z, u \wedge \bar{v})$ hence $B(Z, [X, Y] - u \wedge \bar{v}) = 0$ for any $Z$ which means that $[X, Y] = u \wedge \bar{v}$. \hfill \square

### 4.3 Nilpotent points

In this section, we take $G = GL_n(\mathbb{C})$, $g = gl_n(\mathbb{C})$, $v_0$ is a non zero vector of $\mathbb{C}^n$, $P = G^{v_0}$ and $p$ its Lie algebra.

**Lemma 4.3.1.** Let $X$ be a regular nilpotent element of $g$. Then if the orbit $P.X$ is not open dense in the orbit $G.X$, there exists a semi-simple element $Y$ in $g$ which is not in the center of $g$ and such that $[X, Y] \in p^\perp$.

**Proof.** Let $g$ act on the vector space $V = \mathbb{C}^n$ by $(X, v) \in g \times V \to Xv$. If $X$ is nilpotent regular, its Jordan form has only one block, we deduce easily the following statements:

- the kernel $H$ of $X^{n-1}$ is a hypersurface
- the image of $V$ by $X$ is $H$
- if $v \notin H$, $(v, Xv, X^2v, \ldots, X^{n-1}v)$ is a basis of $V$
So, there is a unique integer \( p \in [0, \ldots, n-1] \) and some \( w \notin H \) such that \( v_0 = X^p w \). Then \((w, Xw, \ldots, v_0 = X^p w, Xv_0, \ldots, X^{n-p-1}v_0)\) is a basis of \( V \).

If \( X \) and \( X' \) are two regular nilpotent matrices with the same characteristic integer \( p \), the matrix of \( GL_n(\mathbb{C}) \) which sends \((w, Xw, \ldots, v_0 = X^p w, Xv_0, \ldots, X^{n-p-1}v_0)\) on \((w', X'w', \ldots, v_0 = X'^p w', X'v_0, \ldots, X'^{n-p-1}v_0)\) sends \( v_0 \) on itself hence is an element of \( P \) which conjugates \( X \) and \( X' \).

The \( P \)-orbits in the \( G \)-orbit of nilpotent regular matrices are thus given by this integer \( p \). We have \( p = 0 \) if and only if \( v_0 \notin H \) hence the \( P \)-orbit given by \( p = 0 \) is open in the \( G \)-orbit, that is the first alternative of the lemma.

Consider now the case \( p \geq 1 \). Let \( V_1 \) be the span of \((w, Xw, \ldots, X^{p-1}w)\) and \( V_2 \) be the span of \((v_0, Xv_0, \ldots, X^{n-p-1}v_0)\). We have \( V = V_1 \oplus V_2, XV_1 \subset V_1 \oplus \mathbb{C}v_0 \) and \( XV_2 \subset V_2 \).

Let \((a, b) \in \mathbb{C}^2, a \neq b\) and \( \Phi_{ab} = aI_{V_1} + bI_{V_2} \). (\( I_{V_i} \) is the identity morphism on \( V_i \)). As \( \Phi_{ab} \) is semi-simple, we are done if we prove that \( [\Phi_{ab}, X] \) is an element of \( p^\perp \). This is equivalent to the fact that \( [\Phi_{ab}, X] \) sends any \( u \) of \( V \) into \( \mathbb{C}v_0 \).

Let \( u = u_1 + u_2 \) the decomposition of \( u \in V \) with \( u_1 \in V_1 \) and \( u_2 \in V_2 \). Let \( Xu_1 = w_1 + \lambda v_0 \) with \( w_1 \in V_1 \) and \( Xu_2 = w_2 \) with \( w_2 \in V_2 \). Then we have:

\[
[\Phi_{ab}, X]u = \Phi_{ab}Xu_1 + \Phi_{ab}Xu_2 - X\Phi_{ab}u_1 - X\Phi_{ab}u_2 = \Phi_{ab}(w_1 + \lambda v_0 + w_2) - aXu_1 - bXu_2 = aw_1 + b\lambda v_0 + bw_2 - aw_1 - a\lambda v_0 - bw_2 = (b - a)\lambda v_0
\]

Consider for a while \( G = Sl_n(\mathbb{C}) \) acting by the adjoint representation on its Lie algebra \( sl_n(\mathbb{C}) \). The conormal to the orbit \( G.X \) is the set of points

\[
\{ (Y, Z) \in \mathfrak{g} \times \mathfrak{g} \mid [Y, Z] = 0, \exists g \in G, Y = g.X \}
\]

If \( Y \) is nilpotent regular, all \( Z \) such that \([Y, Z] = 0\) are nilpotent and the conormal to the orbit is contained in the variety \([1.3]\). If \( X \) is nilpotent non regular, there exists always \( Z \) semi-simple such that \([X, Z] = 0\) and the conormal to the orbit is not contained in the variety \([1.3]\).

Consider again \( G = GL_n(\mathbb{C}) \) acting on \( gl_n(\mathbb{C}) \). In the stratification \( \left( \tilde{S}(\pi, \Sigma) \right) \), the stratum of a nilpotent \( X \) is the direct sum of the orbit \( G.X \) and of the center \( \mathfrak{c} \) of \( \mathfrak{g} \). The conormal to the stratum of \( X \) is the direct sum of the center of \( \mathfrak{g} \) and of the conormal to the orbit in \( sl_n(\mathbb{C}) \). So, the conormal to the stratum of \( X \) is contained in the set \([1.3]\) if and only if \( X \) is regular nilpotent.

Let \( P \) be as before the stability subgroup of \( v_0 \in \mathbb{C}^n \). The same calculation than the proof of lemma \([1.2.4]\) shows that the conormal to the \( P \)-orbit is the set

\[
\{ (Z, Y) \in \mathfrak{g} \times \mathfrak{g} \mid Z = g.X, g \in P, \text{ and } [Z, Y] \in p^\perp \}
\]
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while the conormal to the stratum of $X$ is the set

$$\{(Z,Y) \in \mathfrak{g} \times \mathfrak{g} \mid Z = g.X + X_0, g \in P, X_0 \in \mathfrak{c}, Y \notin \mathfrak{c}, [Z,Y] \in p^\perp\}$$

This set is contained in the characteristic variety of $\mathcal{M}_{F,p}$ that is the set (1.4) if and only if all non nilpotent points commuting with $X$ are in the center $\mathfrak{c}$.

So we have three options:

1) If the $P$-orbit of $X$ is dense in the $G$-orbit this means that the tangent vector fields are the same hence that $\mathcal{M}_{F,p}$ and $\mathcal{M}_{F,\mathfrak{g}}$ are isomorphic in a neighborhood of $X$. As $\mathcal{M}_{F,\mathfrak{g}}$ is tame (3 corollary 1.6.3) the same is true for $\mathcal{M}_{F,p}$.

2) If $X$ is nilpotent regular and the orbit $P.X$ is not dense in $G.X$, lemma 4.3.1 shows that the conormal to the stratum of $X$ is not contained in the characteristic variety of $\mathcal{M}_{F,p}$.

3) If $X$ is nilpotent non regular, the stratum of $X$ is not contained in the characteristic variety of $\mathcal{M}_{F,p}$ because the same was true for $\mathcal{M}_{F,\mathfrak{g}}$.

We proved:

**Corollary 4.3.2.** Let $X$ be a nilpotent point of $\mathfrak{g}$. If the conormal to the direct sum of the center of $\mathfrak{g}$ and of the $P$-orbit is contained in the characteristic variety of $\mathcal{M}_{F,p}$, then $\mathcal{M}_{F,p}$ is isomorphic to $\mathcal{M}_{F,\mathfrak{g}}$ near $X$ and $\mathcal{M}_{F,p}$ admits a tame $b$-function.

This was proved for $G = GL_n(\mathbb{C})$ but extends immediately to the case where $G$ is a product $\prod GL_{n_k}(\mathbb{C})$.

By the isomorphism $\Phi^*$ of section 3.2, this result gives an analogous result for $\mathcal{N}_{F,\mathfrak{g}}$ and in the next two sections we will consider the case of $\mathcal{N}_{F,\mathfrak{g}}$.

### 4.4 Commutative algebra

As a second step of the proof, we assume that the rank of $[\mathfrak{g}, \mathfrak{g}]$ is 0 which means that $\mathfrak{g}$ is commutative. Hence $G = (\mathbb{C}^*)^N$ acting on $\mathbb{C}^n$ by componentwise multiplication. Then the action of $G$ on $\mathfrak{g} \times V = \mathbb{C}^n \times \mathbb{C}^n$ is the multiplication on the second factor.

**Lemma 4.4.1.** If $G = (\mathbb{C}^*)^N$ the module $\mathcal{N}_{F,\mathfrak{g}}$ is holonomic and tame.

*Proof.* Let us fix coordinates $(x_1, \ldots, x_n; y_1, \ldots, y_n)$ of $\mathfrak{g} \times V = \mathbb{C}^n \times \mathbb{C}^n$. The orbits of $G$ on $\mathfrak{g} \times V$ are given by the components of the normal crossing divisor $\{y_1y_2 \ldots y_n = 0\}$ and the vector fields tangent to the orbits are generated by $y_1D_{y_1}, y_2D_{y_2}, \ldots, y_nD_{y_n}$.

On the other hand, the set $F$ is a set of differential operators on $\mathfrak{g}$ whose principal symbols define the zero section of the cotangent space to $\mathfrak{g}$. So the characteristic variety of the module $\mathcal{N}_{F,\mathfrak{g}}$ is the set:

$$\{(x,y,\xi,\eta) \in T^*(\mathbb{C}^n \times \mathbb{C}^n) \mid \xi_1 = \cdots = \xi_n = 0, \ y_1\eta_1 = \cdots = y_n\eta_n = 0\}$$
and the module is holonomic.

Define a stratification of $\mathbb{C}^n \times \mathbb{C}^n$ by the sets $\mathbb{C}^n \times S_\alpha$ where the sets $S_\alpha$ are the smooth irreducible components of $\{y_1 \ldots y_n = 0\}$ that is the sets $S_p = \{y_1 = \cdots = y_p = 0, \ y_{p+1} \ldots y_n \neq 0\}$ and all the sets deduced by permutation of the $y_i$'s.

The characteristic variety of $N_{F,g}$ is contained in the union of the conormals to the strata and the operator $y_1 D_1 + y_2 D_2 + \cdots + y_p D_p$ is a b-function for $S_p$ which is tame by definition. So the module $N_{F,g}$ is tame.

**Definition 4.4.2.** If $\Sigma$ is a normal crossing divisor on a manifold $\Omega$, we denote by $B_\Sigma$ the $D$-module quotient of $D_\Omega$ by the ideal generated by the vector fields tangent to $\Sigma$.

As the principal symbols of the differential operators of $F$ defines the zero section of the cotangent space to $\mathfrak{g}$ the $D_\mathfrak{g}$-module $D_\mathfrak{g}/D_\mathfrak{g}F$ is isomorphic to a power of $\mathcal{O}_\mathfrak{g}$ [2] and $N_{F,g}$ is isomorphic to a power of the module $B_\Sigma$ associated to $\{y_1 \ldots y_n = 0\}$.

### 4.5 Proof of the main theorem

We will now prove theorem [1.3.1] by induction on the dimension of the semi-simple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. More precisely, we will show the corresponding theorem for $N_{F,g}$ which we know to be equivalent.

If the dimension $p$ of $[\mathfrak{g}, \mathfrak{g}]$ is 0, the result has been proved in section 4.3. So we may assume that $p$ is positive and that the result has been proved when the dimension is strictly lower than $p$.

Let $X = S + N$ be the Jordan decomposition of a point $X \in \mathfrak{g}$. If $S = 0$ that is if $X$ is nilpotent, it has been proved in section 4.3 that the module $N_{F,g}$ is weakly tame along the stratum going through $X$ that is the orbit of $X$ plus the center.

So we may assume that $S \neq 0$ and consider the algebra $\mathfrak{g}^S$ that is the commutator of $S$. As $S$ is not zero, $\mathfrak{g}^S$ is a reductive Lie algebra which is a direct sum of algebras $\mathfrak{gl}_{n_k}$. As the dimension of $[\mathfrak{g}^S, \mathfrak{g}^S]$ is strictly lower than $p$ the result is true for $\mathfrak{g}^S$.

We apply theorem [3.3.1] to get a submersive map $\Psi : G \times \mathfrak{m}'' \times V \to \mathfrak{g} \times V$ such that $\Psi^* N_{F,g}$ is equal to $\mathcal{O}_G \otimes N_{F',m}$. Here $\mathfrak{m}''$ is an open subset of $\mathfrak{g}^S$ hence by the induction hypothesis $N_{F',m}$ is weakly tame and thus $\Psi^* N_{F,g}$ is weakly tame.

As $\Psi$ is submersive, this implies that $N_{F,g}$ itself is weakly tame in a neighborhood of $S$. As it was remarked in the proof of [3] Proposition 3.2.1, the stratum of $X = S + N$ meets any neighborhood of $S$ hence the result is true in a neighborhood of $X$. This concludes the proof.

The hypersurface $\Sigma$ of $\mathfrak{g}$ was defined in remark 4.1.1 and by definition $M_{F,p}$ is isomorphic to $M_{F,g}$ on $\mathfrak{g} - \Sigma$.

**Proposition 4.5.1.** On the set $\mathfrak{g}_{rs}$ of regular semi-simple points, $\Sigma$ is a normal crossing divisor and $M_{F,p}$ is isomorphic to a power of $B_\Sigma$. 

Lemma 4.7.1. \( \text{function of the open set } U \) on \( \Sigma \) of remark 4.1.1 and proposition 4.5.1.

\( U \) into a finite number of connected components \( \text{gl} \) and by the isomorphism of \( \mathbf{3.2} \) \( \mathcal{M}_{F,p} \) are powers of the module associated to a normal crossing divisor. 

The variety \( \Sigma \) is the set of matrices \( X \) such that \( v_0, Xv_0, \ldots, X^{n-1}v_0 \) are linearly dependent. For example, if \( v_0 = (0, \ldots, 0, 1) \), the equation of \( \Sigma \) is given by the determinant obtained by taking the last row of \( I, X, \ldots, X^{n-1} \).

4.6 The \( Sl_n(\mathbb{C}) \) case

We consider \( sl_n(\mathbb{C}) \) as a component of the direct sum \( \mathfrak{gl}_n(\mathbb{C}) = sl_n(\mathbb{C}) \oplus \mathbb{C} \). When \( Gl_n(\mathbb{C}) \) acts on \( \mathfrak{gl}_n(\mathbb{C}) \) the action is trivial on the center \( c \simeq \mathbb{C} \) hence the set of vector fields \( \tau(\mathfrak{g}) \) are in fact defined on \( sl_n(\mathbb{C}) \) and are identical to the vectors induced by the action of \( Sl_n(\mathbb{C}) \). In the same way, if \( P \) is the stability group in \( Gl_n(\mathbb{C}) \) of \( v_0 \in \mathbb{C}^n \) and \( P' \) the stability group in \( Sl_n(\mathbb{C}) \) of the corresponding point of \( gl_{n-1}(\mathbb{C}) \), \( P' \) is the image of \( P \) under the map \( X \mapsto (detX)^{-1}X \). So they define the same vector fields on \( sl_n(\mathbb{C}) \).

Let \( F_0 \) be the set of all vector fields on \( c \). If \( F' \) is a \((\text{H-C})\)-type subset of \( D^G_{\mathfrak{g}} \) for \( \mathfrak{g} = sl_n(\mathbb{C}) \), the set \( F = F' \otimes F_0 \) is a \((\text{H-C})\)-type for \( Gl_n(\mathbb{C}) \) and we have

\[ \mathcal{M}_{F,p} = \mathcal{M}_{F',p'} \otimes O_c \]

So the theorem 1.3.1 for \( Gl_n(\mathbb{C}) \) induces immediately the same theorem for \( Sl_n(\mathbb{C}) \). The same argument works for a product of copies of \( Gl_n(\mathbb{C}) \) and \( Sl_n(\mathbb{C}) \).

Remark that a \((\text{H-C})\)-type subset of \( D^G_{\mathfrak{g}} \) for \( \mathfrak{g} = gl_n(\mathbb{C}) \) is not the product of a \((\text{H-C})\)-type subset for \( sl_n(\mathbb{C}) \) by \( F_0 \) so we could not deduce the result for \( Gl_n(\mathbb{C}) \) from the corresponding result for \( Sl_n(\mathbb{C}) \). For the same reason if theorem 1.3.1 is true for two Lie algebras this does not immediately implies the result for their direct sum.

4.7 Application to real forms

Let \( \mathfrak{g}_\mathbb{R} \) be \( sl_n(\mathbb{R}) \), \( gl_n(\mathbb{R}) \), \( sl_2(\mathbb{C}) \) or \( gl_2(\mathbb{C}) \) and \( \mathfrak{g} \) be a complexification of \( \mathfrak{g}_\mathbb{R} \), that is \( sl_n(\mathbb{C}) \), \( gl_n(\mathbb{C}) \), \( sl_{2n}(\mathbb{C}) \) or \( gl_{2n}(\mathbb{C}) \) respectively. Let \( \Sigma_\mathbb{R} \) be the intersection of \( \mathfrak{g}_\mathbb{R} \) with the variety \( \Sigma \) of remark 4.1.1 and proposition 4.3.1.

If \( U \) is an open subset of \( gl_{2n}^a_\mathbb{R} \), the set of semisimple regular points of \( \mathfrak{g}_\mathbb{R} \), \( \Sigma_\mathbb{R} \) divides \( U \) into a finite number of connected components \( U_1, \ldots, U_N \). Let \( Y_i \) be the characteristic function of the open set \( U_i \).

Lemma 4.7.1. Let \( U \) be a simply connected open subset of \( gl_{2n}^a_\mathbb{R} \). Any distribution \( T \) solution on \( U \) of \( \mathcal{M}_{F,p} \) is equal to a finite sum \( \sum f_i(x)Y_i(x) \) where \( f_i \) is an analytic function defined on \( U \) and solution of \( \mathcal{M}_{F,p} \).
Proof. On \( U - \Sigma_R \), \( M_{F,p} \) is isomorphic to \( M_{F,\mathfrak{g}} \) hence \( T \) is a solution of \( M_{F,\mathfrak{g}} \). By [3] we know that \( M_{F,\mathfrak{g}} \) is elliptic on \( \mathfrak{g}^{rs}_{\mathfrak{g}} \). Thus for each connected component \( U_i \), \( T|_{U_i} \) is an analytic function solution of \( M_{F,\mathfrak{g}} \). Hence it extends to a solution of \( M_{F,\mathfrak{g}} \) on the whole of \( U \).

This shows that \( T \) is equal to \( \sum f_i(x)Y_i(x) \) plus a distribution \( S \) supported by \( \Sigma_R \). But \( M_{F,p} \) is weakly tame hence has no solutions supported by a hypersurface. So \( S = 0 \) and \( T = \sum f_i(x)Y_i(x) \) \( \square \)

Let us now prove theorem 1.3.3.

Let \( T \) be a distribution on an open subset of \( \mathfrak{g}_R \) which is solution of \( M_{F,p} \) and invariant under \( P_R \). By the previous lemma the restriction of \( T \) to \( \mathfrak{g}^{rs}_{\mathfrak{g}} \) is a sum \( \sum f_i(x)Y_i(x) \) where \( f_i \) is an analytic function defined on \( U \) and solution of \( M_{F,\mathfrak{g}} \). But on the complement of \( \Sigma_R \) the orbits of \( P_R \) and \( G_R \) are the same. Hence if \( T \) is invariant under \( P_R \) all functions \( f_i \) are equal and \( T \) is an analytic solution of \( M_{F,\mathfrak{g}} \).

By [3] corollary 1.6.3, \( T_{\mathfrak{g}^{rs}_{\mathfrak{g}}} \) extends to a \( L^1_{\text{loc}} \) function \( T' \) on \( \mathfrak{g}_R \) solution of \( M_{F,\mathfrak{g}} \). The distribution \( S = T - T' \) is a distribution solution of \( M_{F,p} \) supported by the hypersurface \( \mathfrak{g}_R - \mathfrak{g}_{\mathfrak{g},rs} \) hence vanishes. This shows that \( T \) is a \( L^1_{\text{loc}} \)-function on \( \mathfrak{g}_R \) solution of \( M_{F,\mathfrak{g}} \), hence that \( T \) is \( G \)-invariant.

Remark 4.7.2. Solutions of \( M_{F,p} \) which are not globally invariant by \( P_R \) may not be solution of \( M_{F,\mathfrak{g}} \). As an example, in the case of \( \mathfrak{sl}_2 \) with the notations of §2 the Heaviside function \( Y(z) \) equal to 0 if \( z < 0 \) and to 1 if \( z \geq 0 \) is a solution of \( M_{F,p} \) but not of \( M_{F,\mathfrak{g}} \).

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