MULTIFRACTAL ANALYSIS OF WEIGHTED ERGODIC AVERAGES

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Abstract. We propose to study the multifractal behavior of weighted ergodic averages. Our study in this paper is concentrated on the symbolic dynamics. We introduce a thermodynamical formalism which leads to a multifractal spectrum. It is proved that this thermodynamical formalism applies to different kinds of dynamically defined weights, including stationary ergodic random weights, uniquely ergodic weights etc. But the validity of the thermodynamical formalism for very irregular weights, like Möbius function, is an unsolved problem. The paper ends with some other unsolved problems.

1. Introduction

For a given topological dynamical system \((X,T)\), a continuous function \(f \in C(X)\) and a sequence of weights \(w = (w_n) \subset \mathbb{C}\) such that \(\sum_{n=0}^\infty |w_n| = \infty\), we would like to study the asymptotic behavior of the weighted Birkhoff sums

\[
S_N^{(w)} f(x) = \sum_{n=0}^{N-1} w_n f(T^n x).
\]

One of associated problems is the multifractal analysis of \(S_N^{(w)} f(x)\). This is a difficult problem even for simple dynamical systems when the sequence of weights is irregular like the Möbius function \(\mu : \mathbb{N} \to \{-1, 0, 1\}\). Recall that \(\mu(1) = 1; \mu(n) = (-1)^k\) if \(n = p_1 \cdots p_k\) a product of \(k\) distinct primes; \(\mu(n) = 0\) otherwise.

The usual Birkhoff sums (with constant weight \(w_n = 1\)) were extensively studied in the literature for different dynamical systems ([3], [4], [5], [10], [20], [21], [25], [29], [38], [39], [41], [47], [52], [53], [54], [55], [56], [60], [66], [71], [72], [73], [76]). Some variants or generalizations of the usual Birkhoff sums were also well studied ([1], [2], [6], [22], [30], [31], [32], [33], [34], [40], [61], [64]). For surveys on the topic, see [11], [19], [59].

In this paper we consider the symbolic dynamics \((X,T)\) where \(X := S^N\), \(S\) being a finite set of \(q\) elements \((q \geq 2\) being an integer), and \(T\) is the shift transformation defined by \((Tx)_n = x_{n+1}\) for \(x = (x_n) \in S^N\). Let us assume that \((w_n)\) is bounded so that \(w_n f(T^n x)\) is a bounded
sequence of functions in \(C(X)\). So, more generally, we consider

\[
S_N f(x) := \sum_{n=0}^{N-1} f_n(T^n x) = \sum_{n=0}^{N-1} f_n(x_n, x_{n+1}, \cdots)
\]

where \(f = (f_n) \subset C(X)\) is a sequence of continuous functions such that \(\|f_n\|_\infty = O(1)\), where \(\|\cdot\|_\infty\) denotes the supremum norm in \(C(X)\).

Define the lower and upper weighted Birkhoff averages by

\[
\underline{A}(x) := \liminf_{N \to \infty} \frac{S_N f(x)}{N} \quad \overline{A}(x) := \limsup_{N \to \infty} \frac{S_N f(x)}{N}.
\]

For \(-\infty < a \leq b < +\infty\), we define the level set

\[
E([a, b]) := \{x \in X : a \leq \underline{A}(x) \leq \overline{A}(x) \leq b\}.
\]

If \([a, b]\) reduces to a singleton \([a]\), we write \(E(a)\) instead of \(E([a, a])\).

The space \(X\) is equipped with its natural distance defined by

\[
d(x, y) = q^{-n}
\]

where \([x_0, x_1, \cdots, x_{n-1}]\) is the cylinder set consisting of all points \(y\) such that \(y_k = x_k\) for \(0 \leq k < n\). This Bernoulli measure, which is \(T\)-invariant, is our reference measure. For any real number \(\lambda \in \mathbb{R}\), define

\[
P_n(x) := \exp \left( \sum_{k=0}^{n-1} f_k(x_k, x_{k+1}, \cdots) \right) \quad Z_n(\lambda) := \mathbb{E}P_n^\lambda(x)
\]

where the expectation \(\mathbb{E}\) is relative to the Bernoulli measure \(\sigma\). In the sequel, we will make the following assumptions

(H1) \(\forall \lambda \in \mathbb{R}, \quad \phi(\lambda) := \lim_{n \to \infty} \frac{1}{n} \log Z_n(\lambda) \) exists.

(H2) \(\sup_{N \geq 1} \sup_{x_0=y_0, \cdots, x_{N-1}=y_{N-1}} \sum_{k=0}^{N-1} |f_k(x_{k+1}, \cdots) - f_k(y_{k+1}, \cdots)| < \infty\).

The function \(\phi\) is convex then continuous. It is called the **pressure function**, associated to \((f_n)\). Recall that the **subderivative** of \(\phi\) at \(\lambda\), denoted \(\partial \phi(\lambda)\), is the set of real numbers \(d\)'s such that

\[
\forall \eta \in \mathbb{R}, \quad \phi(\lambda + \eta) - \phi(\lambda) \geq d\eta.
\]
Notice that \( \partial \phi(\lambda) \) is a closed interval. The conjugate of a convex function \( \phi \) on \( \mathbb{R} \) is defined by

\[
\phi^*(\beta) = \sup_{\alpha \in \mathbb{R}} (\beta \alpha - \phi(\alpha)) \quad (\forall \beta \in \mathbb{R}),
\]

which is a convex function too. For all these notions and facts on convex functions we can refer to [14]. Let

\[
\psi(\lambda) := \phi(\lambda) + \log q.
\]

It would be better to call \( \psi \) the pressure function, because, when \( f_n(x) = f(T^nx) \) for all \( n \geq 1 \) (\( f \) being fixed), \( \psi(\lambda) \) is exactly the pressure of \( \lambda f \) in the usual sense (see [8]) defined by

\[
\lim_{N \to \infty} \frac{1}{N} \log \sum_{a_0, a_1, \ldots, a_{N-1}} \sup_{x \in [a_0, a_1, \ldots, a_{N-1}]} e^{\lambda \sum_{k=0}^{N-1} f(T^kx)}.
\]

One of our main results is the following theorem.

**Theorem 1.1.** Suppose that the assumptions (H1) and (H2) are satisfied. For \( \lambda > 0 \), we have

\[
-\frac{\psi^*(\max \partial \psi(\lambda))}{\log q} \leq \dim E(\partial \psi(\lambda)) \leq \Dim E(\partial \psi(\lambda)) \leq -\frac{\psi^*(\min \partial \psi(\lambda))}{\log q}.
\]

For \( \lambda < 0 \), we have similar estimates but we have to exchange the roles of \( \min \partial \psi(\lambda) \) and \( \max \partial \psi(\lambda) \).

The ideal case is when \( \psi \) is differentiable. Then \( \partial \psi(\lambda) \) reduces to a singleton and we get equalities instead of inequalities in Theorem 1.1. In other words, for \( \alpha = \psi'(\lambda) \) we have

\[
(2) \quad \dim E(\alpha) = \Dim E(\alpha) = -\frac{\psi^*(\alpha)}{\log q} = \frac{\psi(\lambda) - \lambda \alpha}{\log q}.
\]

Thus, a natural problem is to prove the differentiability of \( \psi \) in concrete cases. We will prove this in some cases.

Let us apply Theorem 1.1 to \( f_n(x) = w_nf(T^nx) \) where the weights \( (w_n) \) are dynamically defined. We say that \( f \) is of bounded variation, if \( \sum_{n=1}^{\infty} \text{var}_n(f) < \infty \) with

\[
\text{var}_n(f) := \sup_{x_0 = y_0, \ldots, x_{n-1} = y_{n-1}} |f(x) - f(y)|.
\]

First we apply Theorem 1.1 to the case of ergodic random stationary weights. Especially in the case that \( f \) depends only on a finite number of coordinates, the pressure function be will proved to be analytic.

**Theorem 1.2.** Consider the case \( f_n(x) = \omega_n f(T^nx) \). Suppose that \( (\omega_n) \) is an ergodic sequence of real random variables with \( \|\omega_n\|_\infty = O(1) \) and that \( f \) is of bounded variation. Then

(a) almost surely the assumptions (H1) and (H2) are satisfied and the function \( \phi \) is independent of \( \omega \);
(b) if, furthermore, \( f \) depends only on a finite number of coordinates, then \( \phi \) is an analytic function of \( \lambda \in \mathbb{R} \).

As we will see, the first assertion of Theorem 1.2 follows from Kingman’s subadditive ergodic theorem and the second assertion follows from Ruelle’s theorem \[69\]. If \( f \) depends only on a finite number of coordinates and if \((\omega_n)\) is a sequence of independent and identically distributed random variable taking a finite number of values, Pollicott’s method in \[62\] can allow us to numerically compute \( \psi \) and then to numerically find the multifractal spectrum presented by the formula (2).

If the weight \((w_n)\) is realized by a uniquely ergodic dynamical system, it is natural to ask if the pressure exists for every such realization. The answer is confirmative under the extra condition that \( f \) depends only on a finite number of coordinates. The problem is actually converted to the existence of maximum Liapounov exponent of matrix-valued cocycles.

**Theorem 1.3.** Consider the case \( f_n(x) = \phi(\Theta^nw)f(T^nx) \), where \( \Theta : \Omega \to \Omega \) is a uniquely ergodic dynamical system and \( \phi \in C(\Omega) \) is a continuous function and \( f \in C(X) \) is a function depending only on a finite number of coordinates. Then for every \( \omega \in \Omega \), the pressure function \( \phi \) is well defined and independent of \( \omega \), and is an analytic function of \( \lambda \).

This follows essentially from a result due to Furman \[35\] and from Theorem 1.2.

Now assume that \((w_n)\) is a sequence taking a finite number of real values, say \( A \subseteq \mathbb{R} \). The shift \( \Theta : A^\omega \to A^\omega \) acts on the closed orbit \( \{\Theta^kw\} \). We will assume that the subsystem \((\{\Theta^kw\}, \Theta)\) is minimal and uniquely ergodic (then we will simply say that \( w \) is minimal and uniquely ergodic). Then the condition imposed in Theorem 1.3 that \( f \) depends on the first coordinates can be dropped for the function \( \phi \) to be well defined. Let us point out that all primitive substitutive sequences are minimal and uniquely ergodic \[51\].

**Theorem 1.4.** Consider \( f_n(x) = w_nf(T^nx) \). Suppose that \((w_n) \in A^\omega \) is minimal and uniquely ergodic and that \( f \) is of bounded variation. Then the assumptions (H1) and (H2) are satisfied.

The proof of Theorem 1.4 is based on the notion of return word (see \[13\]).

Now let us look at some particular cases for which we can find an explicit formula for the function \( \phi \) and then an explicit formula for the spectrum given by (2). For simplicity, just consider the case \( S = \{-1,1\} \). A typical example of \( f_n(x_n, x_{n+1}, \cdots) \) is of the form

\[
(3) \quad w_n(a + bx_n + cx_{n+1} + dx_{n+1} + ex_{n+2} + fx_{n+2}x_{3n})
\]
where $a, b, c, d, e, f$ are fixed constants. Special cases include

\begin{align}
(4) \quad f_n(x_n, x_{n+1}, \ldots) &= w_n x_n x_{n+1}, \\
(5) \quad f_n(x_n, x_{n+1}, \ldots) &= w_n x_n x_{2n}, \\
(6) \quad f_n(x_n, x_{n+1}, \ldots) &= w_n x_n x_{2n} x_{3n}.
\end{align}

When $w_n = 1$ for all $n$, the first case defined by (4) is classical and well studied (for example, see [20, 21]), and the cases defined by (5) and (6) are studied in [24] using Riesz product measures. Notice that we cannot apply Theorem 1.1 to the last two cases because the assumption (H2) is not satisfied.

In the following we discuss the case $f_n(x) = w_n x_n x_{n+1}$ with some more or less regular weights $(w_n)$. By the way, we will also discuss some generalizations of $w_n x_n x_{n+1}$. An explicit formula for $\phi$ will be obtained.

As a corollary of Theorem 1.1, we can then prove the following result. But we can and we will provide a direct proof of the following theorem too.

**Theorem 1.5.** Let $S = \{-1, 1\}$. Assume that $(w_n)$ takes a finite number of values $v_0, v_1, \ldots, v_m$ and that $(f_n)$ are of the form

\[ f_n(x) = x_n g_n(x_{n+1}, x_{n+2}, \ldots). \]

Suppose further that

(C1) all $g_n$ take values in $\{-1, 1\}$ and there is an integer $L \geq 1$ such that $g_n(x_n, \ldots)$ depends only on $x_n, x_{n+1}, \ldots, x_{n+L}$;

(C2) the following frequencies exist

\begin{equation}
(7) \quad p_j := \lim_{N \to \infty} \frac{\#\{1 \leq n \leq N : w_n = v_j\}}{N} \quad (0 \leq j \leq m).
\end{equation}

Then for $\alpha \in (-\sum p_j |v_j|, \sum p_j |v_j|)$ we have

\[ \dim E(\alpha) = \frac{1}{\log 2} \sum_{j=0}^{m} p_j \left( \log(e^{\lambda_\alpha v_j} + e^{-\lambda_\alpha v_j}) - \lambda_\alpha v_j e^{\lambda_\alpha v_j} + e^{-\lambda_\alpha v_j} \right) \]

where $\lambda_\alpha$ is the unique solution of the equation

\[ \sum_{j=0}^{m} p_j v_j e^{\lambda_\alpha v_j} - e^{-\lambda_\alpha v_j} = \alpha. \]

Notice that the result doesn’t depend on the form of $g_n$, but only on the weights $(w_n)$. The key point for this independence is that $g_n$ only takes $-1, 1$ as its values.

As a corollary of Theorem 1.5, we have the following result for

\[ F(\alpha) = \left\{ x \in \{-1, 1\}^N : \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) x_n x_{n+1} = \alpha \right\} \]

where $\mu$ is the Möbius function.
Theorem 1.6. For any $\alpha \in (-6/\pi^2, 6/\pi^2)$, we have
\[
\dim F(\alpha) = 1 - \frac{6}{\pi^2} + \frac{6}{\pi^2 \log 2} H \left( \frac{1}{2} + \frac{\pi^2}{12} \alpha \right)
\]
where $H(x) = -x \log x - (1-x) \log(1-x)$.

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2. Thermodynamical formalism: Proof of Theorem 1.1

We present here a thermodynamic formalism proposed in [18], adapted to our setting in the present paper. This formalism would work in other cases. As we will point out in the last section, there will be works to do with the limit defining the pressure and with the differentiability of the pressure.

2.1. Fundamental inequalities. For $m \leq n$, denote
\[
P_{m,n}(x) := \frac{P_n(x)}{P_m(x)}; \quad Z_{m,n}(\lambda) := \mathbb{E} P_{m,n}^\lambda(x).
\]

The following inequalities are fundamental. In [18] (p. 1318), the inequalities are stated in a more general case and proved in a different way.

Lemma 2.1 (Fundamental inequalities). For any real number $\lambda \in \mathbb{R}$, there exists a positive constant $C(\lambda) > 0$ such that for all integers $0 \leq l \leq m \leq n$ we have
\[
\frac{1}{C(\lambda)} \leq \frac{Z_{l,n}(\lambda)}{Z_{l,m}(\lambda) Z_{m,n}(\lambda)} \leq C(\lambda).
\]
The constant $C(\lambda)$ grows at most exponentially as function of $\lambda$, i.e. $C(\lambda) = e^{O(|\lambda|)}$.

Proof. For all integers $0 \leq m \leq n$, let
\[
S_{m,n} f(x) = \sum_{k=m}^{n-1} f_k(T^k x) = \sum_{k=m}^{n-1} f_k(x_k, x_{k+1}, \cdots).
\]
Then $P_{m,n} = e^{S_{m,n}}f$ (by convention $S_{n,n}f = 0$ so that $P_{n,n} = 1$). First we notice that for any $(a_0, a_1, \cdots, a_{m-1}) \in S^m$ we have
\[ Z_{m,n}(\lambda) = q^m \int_{[a_0, a_1, \cdots, a_{m-1}]} e^{\lambda S_{m,n}f(x)} d\sigma(x). \]
Indeed, the map $T^m : [a_0, a_1, \cdots, a_{m-1}] \to X$ is bijective and it maps the probability measure $q^m \sigma|_{[a_0, a_1, \cdots, a_{m-1}]}$ to the probability measure $\sigma$. So, by a change of variables, the member at the right hand side of (9) is equal to
\[ q^m \int_{[a_0, a_1, \cdots, a_{m-1}]} e^{\lambda \sum_{k=m}^{n-1} f_k(T^k x)} d\sigma(x) = \int_X e^{\lambda \sum_{k=m}^{n-1} f_k(T^k x)} d\sigma(x). \]
Then, by the $T$-invariance of $\sigma$, we get
\[ \int_X e^{\lambda \sum_{k=m}^{n-1} f_k(T^k x)} d\sigma(x) = \int_X e^{\lambda \sum_{k=m}^{n-1} f_k(T^k x)} d\sigma(x) = Z_{m,n}(\lambda). \]
Thus, (9) is proved. Now write
\[ Z_{l,n}(\lambda) = \sum_{a_0, a_1, \cdots, a_{m-1}} \int_{[a_0, a_1, \cdots, a_{m-1}]} e^{\lambda S_{l,m}f(x)} d\sigma(x). \]
By the distortion hypothesis (H2), we have
\[ Z_{l,n}(\lambda) \approx \sum_{a_0, a_1, \cdots, a_{m-1}} q^{-m} e^{\lambda S_{l,m}f(a_0, a_1, \cdots, a_{m-1}, \ast)} \cdot q^m \int_{[a_0, a_1, \cdots, a_{m-1}]} e^{\lambda S_{m,n}f(x)} d\sigma(x). \]
where $\ast$ represents any fixed sequence, and the constant involved in "$\approx"$ is $e^{O(\lambda)}$. By (9), the above expression reads as
\[ Z_{l,n}(\lambda) \approx \sum_{a_0, a_1, \cdots, a_{m-1}} q^{-m} e^{\lambda S_{l,m}f(a_0, a_1, \cdots, a_{m-1}, \ast)} \cdot Z_{m,n}(\lambda). \]
Using once more the hypothesis (H2), we get that the last sum is equal to $Z_{l,m}(\lambda)$ up to a multiplicative constant $e^{O(\lambda)}$.

We emphasize that the equality (9) is a key point.

2.2. Construction of Gibbs measure. Let
\[ d\mu_{n,\lambda} := \frac{P_n^\lambda(x)}{Z_n(\lambda)} dx. \]
It is a probability measure on $X$.

**Lemma 2.2** (Gibbs property). All weak limits of the sequence of probability measures $(\mu_{n,\lambda})$ are equivalent. For any such a limit, denoted by $\mu_\lambda$, we have
\[ \mu_\lambda([x_0, x_1, \cdots, x_{n-1}]) \approx \frac{P_n^\lambda(x)}{q^n Z_n(\lambda)}. \]
The constant involved in "$\approx"$ depends on $\lambda$ but is independent of $n$ and $x$, and is of the size $e^{O(\lambda)}$. 

Proof. The proof is already in ([18], p.1319). It is simpler in the present case. For completeness, we include it here. Let $C_n(x)$ be the cylinder $[x_0, x_1, \cdots, x_{n-1}]$. Assume that $\mu_\lambda$ is the weak limit of $(\mu_{N_j}, \lambda)$ for some sequence of integers $(N_j)$. We have

$$
\mu_\lambda(C_n(x)) = \lim_{j \to \infty} \frac{1}{Z_{N_j}(\lambda)} \int_{C_n(x)} P_{N_j}^\lambda(y)dy
$$

$$
\leq C' \limsup_{j \to \infty} \frac{1}{Z_n(\lambda)Z_{n,N_j}(\lambda)} \int_{C_n(x)} P_n^\lambda(y)P_{n,N_j}^\lambda(y)dy
$$

$$
\leq C'' \limsup_{j \to \infty} \frac{P_n^\lambda(x)}{q^nZ_n(\lambda)} \cdot \frac{q^n}{Z_{n,N_j}(\lambda)} \int_{C_n(x)} P_{n,N_j}^\lambda(y)dy
$$

$$
= C'' \frac{P_n^\lambda(x)}{q^nZ_n(\lambda)}.
$$

The first inequality above is a consequence of the fundamental inequalities (8); the second inequality is a consequence of the distortion hypothesis (H2) and the last equality is because of (9).

The inverse inequality can be proved in the same way, because we have both side estimates in our fundamental inequalities. □

The measure $\mu_\lambda$ will be called Gibbs measure associated to $(f_k)$ and $\lambda$. Fix $n \geq 1$. Define

$$
\forall k \geq 0, \quad g_k(x) = f_{n+k}(x)
$$

The Gibbs measure associated to $(g_k)$ and $\lambda$ will be denoted by $\mu_\lambda^{(n)}$. This measure depends on the tail from $n$ on of $(f_k)$.

Notice that $\mu_\lambda = \mu_\lambda^{(0)}$. The family $(\mu_\lambda^{(n)})$ has the following quasi-Bernoulli property, which is a direct consequence of the above Lemma 2.1 and Lemma 2.2.

**Lemma 2.3** (Quasi-Bernoulli property). For all integers $n$ and $m$ and for all sequences $I \in S^n$ and $J \in S^m$, we have

$$
\mu_\lambda([IJ]) \approx \mu_\lambda([I])\mu_\lambda^{(m)}([J])
$$

where the constants involved in “$\approx$” are independent of $n, m$ and $I, J$.

2.3. Large deviation. We are going to present a law of large numbers with respect to our Gibbs measures. It is a consequence of a well known result on large deviation. The large deviation was used in multifractal analysis in early works (see [9], for example). The following result on convex functions and their conjugates will be useful.

**Proposition 2.4** ([14], p.221). For any convex function defined on $\mathbb{R}$, we have

(i) $\alpha\beta \leq \phi(\beta) + \phi^*(\alpha), \quad (\forall \alpha, \beta \in \mathbb{R}).$
(ii) $\alpha \beta = \phi(\beta) + \phi^*(\alpha) \iff \alpha \in \partial \phi(\beta)$.

(iii) $\alpha \in \partial \phi(\beta) \iff \beta \in \partial \phi^*(\alpha)$.

(iv) $\phi^{**}(\beta) = \phi(\beta), \ (\forall \beta \in \mathbb{R})$.

Let $(W_n)$ be a sequence of random variables on a probability space $(\Omega, \mathcal{A}, \nu)$ and $(a_n)$ be a sequence of positive real numbers tending to the infinity. Suppose that the following limit exits

$$c(\beta) := c_W(\beta) := \lim_{n \to \infty} \frac{1}{a_n} \log \mathbb{E} e^{\beta W_n}, \ \forall \beta \in \mathbb{R}.$$ 

We call $c(\beta)$ the free energy function of $(W_n)$ with respect to $\nu$ and weighted by $(a_n)$. By the upper large deviation bound theorem ([14], p. 230), for any non empty compact set $K \subset \mathbb{R}$ we have

$$\limsup_{n \to \infty} \frac{1}{a_n} \log \sigma \{a_n^{-1} W_n \in K\} \leq - \inf_{\alpha \in K} c^*(\alpha).$$

Notice that $c(0) = 0$. By Proposition 2.4 (i), we have $c^*(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$. By Proposition 2.4 (ii), we have $c^*(\alpha) = 0$ iff $\alpha \in \partial c(0)$. Let

$$\Delta^- := \min \partial c(0), \ \Delta^+ := \max \partial c(0).$$

For any compact set $K$ disjoint from $[\Delta^-, \Delta^+]$, we have $\eta := \inf_K c^*(\alpha) > 0$. By the upper large deviation bound theorem, for large $n$ we have

$$\nu \{a_n^{-1} W_n \in K\} \leq e^{-\eta a_n/2}.$$

Suppose that $\sum e^{-\epsilon a_n} < \infty$ for all $\epsilon > 0$ (it is the case when $a_n = n$). By the Borel-Cantelli lemma, we get

$$\nu - a.e \min \partial c(0) \leq \lim \inf_{n \to \infty} \frac{W_n}{a_n} \leq \lim \sup_{n \to \infty} \frac{W_n}{a_n} \leq \max \partial c(0).$$

Now fix a Gibbs measure $\mu_\lambda$. We consider the free energy of $S_n f(x)$ relative to $(\mu_\lambda, \{n\})$ defined as follows

$$c_\lambda(\beta) := \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{\mu_\lambda} P^\beta_n(x).$$

Lemma 2.5 ([13], p.1322). Suppose the limits defining $\phi(\lambda)$ exist. Then the limit defining $c_\lambda(\beta)$ exists and we have

$$c_\lambda(\beta) = \phi(\lambda + \beta) - \phi(\lambda).$$

It follows that $c_\lambda(0) = 0$ so that $c^*_\lambda(\alpha) \geq 0$ for all $\alpha$. Also $c^*_\lambda(\alpha) = 0$ iff $\alpha \in \partial \phi(\lambda)$. Then we can apply [17] to get the following law of large numbers.

Lemma 2.6 (Law of large numbers). For $\mu_\lambda$-almost all $x$, we have

$$\min \partial \phi(\lambda) \leq A(x) \leq \overline{A}(x) \leq \max \partial \phi(\lambda).$$
It will be more practical to work with

\[ \psi(\lambda) := \phi(\lambda) + \log q. \]

The above inequalities in Lemma 2.6 also hold with \( \psi \) replacing \( \phi \).

2.4. Dimensions of Gibbs measures. The local lower and upper dimensions of a measure \( \mu \) are respectively defined by

\[
\underline{D}(\mu, x) = \liminf_{n \to \infty} \frac{\log \mu([x_0, x_1, \ldots, x_{n-1}])}{\log q^n},
\]

\[
\overline{D}(\mu, x) = \limsup_{n \to \infty} \frac{\log \mu([x_0, x_1, \ldots, x_{n-1}])}{\log q^n}.
\]

The lower and upper Hausdorff dimensions of \( \mu \) are respectively defined by

\[
dim_* \mu = \inf \{ \dim E : \mu(E) > 0 \},
\]

\[
dim^* \mu = \inf \{ \dim E : \mu(E^c) = 0 \}.
\]

The lower packing dimension \( \text{Dim}_* \mu \) and the upper packing dimension \( \text{Dim}^* \mu \) are similarly defined by using the packing dimension \( \text{Dim} E \) instead of the Hausdorff dimension \( \dim E \).

A systematic study of the Hausdorff dimensions \( \dim_* \mu \) and \( \dim^* \mu \) was carried out in [16, 17] when \( X \) is a homogeneous space. Later, the packing dimensions \( \text{Dim}_* \mu \) and \( \text{Dim}^* \mu \) were studied independently by Tamashiro [73] and Heurteaux [42]. Let us just state the following result.

**Proposition 2.7** ([16, 17, 42, 73]). For the Hausdorff dimensions we have

\[
dim_* \mu = \text{essinf}_x D(\mu, x), \quad \dim^* \mu = \text{esssup}_x D(\mu, x).
\]

Similar formulas hold for the packing dimensions \( \text{Dim}_* \mu \) and \( \text{Dim}^* \mu \) if we replace \( D(\mu, x) \) by \( \text{Dim}(\mu, x) \).

From the Gibbs property (Lemma 2.2), we get immediately the following relation between the local dimensions of a Gibbs measure and the averages \( A(x) \) and \( \overline{A}(x) \).

**Lemma 2.8** (Local dimensions of Gibbs measures). For all \( x \in X \) we have

\[
D(\mu_\lambda, x) = \frac{\psi(\lambda) - \lambda A(x)}{\log q} \text{ if } \lambda > 0; \quad D(\mu_\lambda, x) = \frac{\psi(\lambda) - \lambda \overline{A}(x)}{\log q} \text{ if } \lambda < 0.
\]

Similar equalities hold when \( D(\mu_\lambda, x) \) is replaced by \( \text{Dim}(\mu_\lambda, x) \) and \( \overline{A}(x) \) by \( A(x) \).

The measure \( \mu_0 \) is the symmetric Bernoulli measure and its dimension is equal to 1. The dimension of the Gibbs measures are estimated as follows.
Lemma 2.9 (Dimensions of Gibbs measures).

1. If $\lambda > 0$, we have
\[
-\frac{\psi^*(\max \partial \psi(\lambda))}{\log q} \leq \dim_* \mu_\lambda \leq \dim_* \mu_\lambda \leq -\frac{\psi^*(\min \partial \psi(\lambda))}{\log q}.
\]

2. If $\lambda < 0$, we have similar estimates but we have to exchange the positions of $\max \partial \psi(\lambda)$ and $\min \partial \psi(\lambda)$.

3. We have exactly the same estimates for the packing dimension $\dim_* \mu_\lambda$ and $\dim_* \mu_\lambda$.

Proof. (1) From Lemma 2.8, Lemma 2.6 and Proposition 2.7, we get
\[
\psi(\lambda) - \lambda \max \partial \psi(\lambda) \leq \dim_* \mu_\lambda \leq \dim_* \mu_\lambda \leq \psi(\lambda) - \lambda \min \partial \psi(\lambda).
\]

But, by Proposition 2.4 (ii), we have
\[
\psi(\lambda) - \lambda \min \partial \psi(\lambda) = -\psi^*(\min \partial \psi(\lambda)),
\]
\[
\psi(\lambda) - \lambda \max \partial \psi(\lambda) = -\psi^*(\max \partial \psi(\lambda)).
\]

(2) It is the same argument, but we have to exchange the roles of $\max \partial \psi(\lambda)$ and $\min \partial \psi(\lambda)$ in the above inequalities. Notice that $A(x)$ and $\mathbf{A}(x)$ have the same bounds in Lemma 2.6.

(3) It is the exactly the same argument as in (1) and (2).

2.5. Proof of Theorem 1.1. Now we are ready to prove Theorem 1.1. Let $\partial \phi(\lambda) = [\alpha_-, \alpha^+]$.

Assume $\lambda > 0$. The fact $A(x) \geq \alpha_-$ for $x \in E([\alpha_-, \alpha^+])$ implies that the set $E([\alpha_-, \alpha^+])$ is contained in
\[
\bigcap_{\epsilon > 0} \bigcup_{N \geq 1} \bigcap_{n > N} \left\{ n(\alpha_- - \epsilon) \leq \sum_{k=0}^{n-1} f_k(x) \right\}.
\]

By the $\sigma$-stability of the packing dimension, to upper bound the packing dimension of $E([\alpha_-, \alpha^+])$ it suffices to estimate the Minkowski dimension of the last set of intersection. i.e. $\cap_{n \geq N} \{ n(\alpha_- - \epsilon) \leq S_n f(x) \}$.

Consider the family $C_n$ of all the $n$-cylinders intersecting that set of intersection. For any $d > 0$ we have
\[
\sum_{[x_0, x_1, \ldots, x_{n-1}] \in C_n} q^{-nd} \leq \sum_{[x_0, x_1, \ldots, x_{n-1}] \in C_n} q^{-nd} \cdot \frac{e^{\lambda \sum_{k=0}^{n-1} f_k(x)}}{e^{\lambda n(\alpha_- - \epsilon)}} \leq \sum_{[x_0, x_1, \ldots, x_{n-1}]} q^{-nd} \cdot \frac{e^{\lambda \sum_{k=0}^{n-1} f_k(x)}}{q^n Z_n(\lambda)} \cdot \frac{q^n Z_n(\lambda)}{e^{\lambda n(\alpha_- - \epsilon)}}.
\]
By the Gibbs property of $\mu_\lambda$ (Lemma 2.2), we have
\[
\sum_{[x_0, x_1, \ldots, x_{n-1}] \in C_n} q^{-nd} \leq C q^{-nd} \cdot \frac{q^n Z_n(\lambda)}{e^{\lambda n(\alpha_\lambda - \epsilon)}} \sum \mu_\lambda([x_0, \ldots, x_{n-1}])
\leq C q^{-nd} q^n e^{n(\phi(\lambda) + \epsilon) - n(\alpha_\lambda - \epsilon)}
= C q^{-nd} q^n e^{n \left( \frac{1}{\log q} (\psi(\lambda) - \lambda \alpha_\lambda + \epsilon + \lambda \epsilon) \right)} \leq C
\]
if $d > \frac{1}{\log q} (\psi(\lambda) - \lambda \alpha_\lambda + \epsilon + \lambda \epsilon)$. It follows that the Minkowski dimension of the set in question is smaller than $\frac{1}{\log q} (\phi(\lambda) - \lambda \alpha_\lambda + \epsilon + \lambda \epsilon)$. Let $\epsilon \to 0$, we get
\[
(12) \quad \dim E(\partial \psi(\lambda)) \leq \frac{\psi(\lambda) - \alpha_\lambda}{\log q} = - \frac{\psi^*(\min \partial \psi(\lambda))}{\log q}.
\]
If $\lambda < 0$, we can similarly prove
\[
(13) \quad \dim E(\partial \psi(\lambda)) \leq \frac{\psi(\lambda) - \alpha_\lambda}{\log q} = - \frac{\psi^*(\max \partial \psi(\lambda))}{\log q},
\]
but we must start with the fact that $E[\alpha_- , \alpha_+]$ is contained in
\[
(14) \quad \bigcap_{\epsilon > 0, N \geq 1} \bigcap_{n > N} \left\{ n(\alpha_\lambda + \epsilon) \geq \sum_{k=0}^{n-1} f_k(x) \right\}.
\]
Notice that we have opposite inequalities in (11) and (14).

Prove now the lower bound. By Lemma 2.6, $E(\partial \psi(\lambda))$ is of full $\mu_\lambda$-measure. In particular, $A(x) \leq \max \partial \psi(\lambda)$ for $\mu_\lambda$-almost every $x$. If $\lambda > 0$, by Lemma 2.2 this implies
\[
\mu_\lambda - a.e. x \quad D(\mu_\lambda, x) \geq \frac{\psi(\lambda) - \lambda \max \partial \psi(\lambda)}{\log q}.
\]
Thus
\[
(15) \quad \dim E(\partial \psi(\lambda)) \geq - \frac{\psi^*(\max \partial \psi(\lambda))}{\log q}.
\]
When $\lambda < 0$, we use the fact $A(x) \geq \min \partial \psi(\lambda)$ for $\mu_\lambda$-almost every $x$ to get
\[
(16) \quad \dim E(\partial \psi(\lambda)) \geq - \frac{\psi^*(\min \partial \psi(\lambda))}{\log q}.
\]
The four inequalities (12), (13) (15) and (16) are what we have to prove.

2.6. $\tau$-function of the Gibbs measure $\mu_\lambda$. For the Gibbs measure $\mu_\lambda$, we define the function
\[
\tau_\lambda(\beta) = \lim_{n \to \infty} \frac{1}{\log q^n} \log \sum_{[x_0, x_1, \ldots, x_{n-1}]} \mu_\lambda([x_0, x_1, \ldots, x_{n-1}])^\beta.
\]
The function $\tau_\lambda$ and the function $\phi$ have a simple explicit relation. The differentiability of $\phi$ at $\lambda$ is equivalent to the differentiability of $\tau_\lambda$ at $1$.

**Lemma 2.10** (Relation between $\tau$ and $\phi$). Under the assumptions (H1) and (H2), the limit defining $\tau_\lambda(\beta)$ exists for all $\beta \in \mathbb{R}$ and we have

$$ -\tau_\lambda(\beta) = 1 + \frac{\phi(\beta \lambda) - \beta \phi(\lambda)}{\log q}. $$

Consequently, $\phi$ is differentiable at $\lambda$ iff $\tau_\lambda$ is differentiable at $1$. In this case we have

$$ \tau'_\lambda(1) = \frac{\phi(\lambda) - \lambda \phi'(\lambda)}{\log q}. $$

**Proof.** The equality (17) follows from the Gibbs property:

$$ \sum_{x_0, x_1, \ldots, x_{n-1}} \mu_\lambda([x_0, x_1, \ldots, x_{n-1}])^\beta \approx q^n \int_X e^{\beta \lambda S_n f(x)} \frac{dx}{Z_n(\lambda)^\beta} \approx q^n \frac{Z_n(\beta \lambda)}{Z_n(\lambda)^\beta}. $$

☐

3. **Stationary weights: Proof of Theorem 1.2**

3.1. **Proof of Theorem 1.2** Assume that $\omega = (\omega_n)$ takes values in an interval $I \subset \mathbb{R}$. Let $\Omega = I^n$ and let $\Theta$ denote the shift map on $\Omega$ so that $\Theta^n \omega = (\omega_{n+k})_{k \geq 0}$. To express clearly the dependence on $\omega$, denote by $\mu_\lambda^\omega$ the Gibbs measure corresponding to the weight $\omega \in \Omega$ and write

$$ Z_n(\lambda, \omega) = \int_X e^{\lambda S_n^\omega f(x)} dx $$

where

$$ S_n^\omega f(x) = \sum_{k=0}^{n-1} \omega_k f(T^k x). $$

The fundamental inequalities (8) read as

$$ \frac{1}{C(\lambda)} \leq \frac{Z_{n+m}(\lambda, \omega)}{Z_n(\lambda, \omega) Z_{n+m}(\lambda, \Theta^n \omega)} \leq C(\lambda). $$

The condition (H2) is clearly satisfied. By Kingman’s ergodic theorem, almost surely the condition (H1) is also satisfied, i.e. almost surely the following limit exists

$$ \phi^\omega(\lambda) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\lambda, \omega) $$

and $\phi^\omega(\lambda)$ is almost surely equal to the function $\widetilde{\phi}(\lambda)$ defined by

$$ \widetilde{\phi}(\lambda) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log Z_n(\lambda, \omega). $$
We actually have
\[ \left| \tilde{\phi}(\lambda) - \frac{1}{n} \mathbb{E} \log Z_n(\lambda, \omega) \right| \leq \frac{|\lambda| \log C(\lambda)}{n}. \]

Now suppose that \( f \) depends only on the first \( r \geq 2 \) coordinates \( (\tilde{\phi}(\lambda) \) is easy to compute when \( r = 1 \), i.e. \( f \) takes the form \( f(x) = f(x_0, x_1, \cdots, x_{r-1}) \).

For fixed \( \lambda \in \mathbb{R} \) and fixed \( w \in \Omega \), let us define a \( q^{r-1} \times q^{r-1} \)-matrix \( A_w(\lambda) = (a_{u,v})_{(u,v) \in S^{r-1} \times S^{r-1}} \) as follows: if the \((r-2)\)-suffix of \( u \) is equal to the \((r-2)\)-prefix of \( v \), i.e. \( u = x_0 x_1 \cdots x_{r-2} \) and \( v = x_1 \cdots x_{r-1} \) for some \((x_0, x_1, \cdots, x_{r-1}) \in S^r \), then \( a_{u,v} = e^{\lambda w f(x_0, x_1, \cdots, x_{r-1})} \); otherwise \( a_{u,v} = 0 \). Since \( S^r(\omega) f(x) \) is locally constant on cylinders of length \( n + r \), it is easy to see that
\[ Z_n(\lambda, \omega) = \frac{1}{q^{n+r-1}} \| A^{(0)}(\lambda) A^{(1)}(\lambda) \cdots A^{(n-1)}(\lambda) \| \]
where \( \| A \| \) denotes the norm defined by the sum of all the entries of a non-negative matrix \( A \). Observe that our matrices \( A^{(n)}(\lambda) \) are non-negative and that the product of any \( r \) consecutive matrices are strictly positive. So,
\[ Z_{nr}(\lambda, \omega) = \frac{1}{q^{nr+r-1}} \| B^{(0)}(\lambda) B^{(1)}(\lambda) \cdots B^{(n-1)}(\lambda) \| \]
where
\[ B^{(n)}(\lambda) = A^{(0)}(\lambda) A^{(1)}(\lambda) \cdots A^{(r-1)}(\lambda) \]
which is a strictly positive matrix. So,
\[ \tilde{\phi}(\lambda) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log \| B^{(0)}(\lambda) B^{(1)}(\lambda) \cdots B^{(n-1)}(\lambda) \| - \log q. \]
The above limit is the largest characteristic exponent of the random positive matrix \( B^{(r)}(\lambda) \). Since \( \lambda \mapsto B^{(r)}(\lambda) \) analytic and \( B^{(r)}(\lambda) \) is positive, the exponent is an analytic function of \( \lambda \), by Ruelle’s theorem (Theorem 3.1. in [69]).

4. Uniquely ergodic weights: Proof of Theorem 1.3

Let us borrow the notation and the argument from the above proof of Theorem 1.2. Assume \( f(x) = f(x_0, x_1, \cdots, x_{r-1}) \). For fixed \( \lambda \in \mathbb{R} \) and fixed \( \omega \in \Omega \), let us define a \( q^{r-1} \times q^{r-1} \)-matrix
\[ A^{(r)}(\lambda) = (a_{u,v})_{(u,v) \in S^{r-1} \times S^{r-1}} \]
as follows: if the \((r - 2)\)-suffix of \(u\) is equal to the \((r - 2)\)-prefix of \(v\), i.e. \(u = x_0x_1 \cdots x_{r-2}\) and \(v = x_1 \cdots x_{r-1}\) for some \((x_0, x_1, \cdots, x_{r-1}) \in S^r\), then

\[
a_{u,v} = e^{\lambda\phi(x_0x_1\cdots x_{r-1})},
\]

otherwise \(a_{u,v} = 0\). \(B_\omega\) is similarly defined as above. Since the function \(\omega \mapsto A_\omega\) is eventually positive (i.e. \(B_\omega\) is strictly positive), by the part 3 of Theorem 3 from Furman \[35\], the following limit exists

\[
\lim_{n \to \infty} \frac{1}{n} \log \|A_{\omega}(\lambda)A_{\Theta\omega}(\lambda) \cdots A_{\Theta^{n-1}\omega}(\lambda)\|
\]

for all \(\omega\) (and all \(\lambda\)) and the limit is actually uniform in \(\omega\). The independence of \(\omega\) of the limit is due to the ergodicity and the analyticity of the limit as function of \(\lambda\) follows from Theorem 1.2.

But notice that Theorem 3 in \[35\] requires that \(A_\omega\) belongs to \(GL_{q_{r-1}}(\mathbb{R})\). It is not the case in general for our \(A_\omega\). However the part 3 of Theorem 3 in \[35\] doesn’t need this condition. This is because the entries of the positive matrices \(B_{\Theta^{n}\omega}\) are bounded from below by a constant \(\delta > 0\) and from above by \(\delta^{-1}\) (\(\delta\) being independent of \(\omega\) and of \(n\)).

Let us state Furman’s result, that we have used above, by dropping the invertibility of the matrix: Let \((X, \mu, T)\) be a uniquely ergodic system and suppose that \(A\) is a continuous real \(d \times d\)-matrix function defined on \(X\) and that there exists an integer \(p \geq 1\) such that

\[
A(T^{p-1}x) \cdots A(Tx)A(x) > 0
\]

meaning that all entries are positive for all \(x\). Then for every \(x \in X\) the following limit exists:

\[
\lim_{n \to \infty} \frac{1}{n} \log \|A(T^{n-1}x) \cdots A(Tx)A(x)\|.
\]

The limit is actually uniform in \(x \in X\).

Lemma 5 in \[35\] which was used in the proof of the above result can be modified as follows without requiring the invertibility: let \((B_n)\) be a sequence of positive \(d \times d\)-matrices with entries in the interval \([\delta, \delta^{-1}]\) (\(\delta > 1\) being a constant). Let

\[
\Delta := \left\{ (x_i) \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1, x_i \geq 0 \right\}
\]

and \(\overline{\Delta}\) be the corresponding set of \(\Delta\) in the projective space \(P^{d-1}\). Then there exists a unique point \(\overline{u} \in P^{d-1}\) such that

\[
\bigcap_{n=1}^{\infty} \overline{B_n}\overline{\Delta} = \{\overline{u}\}
\]

where \(\overline{B_n}\) is the projective transformation associated to \(B_n\). Here is a proof. Let \(K\) be the cone \(\{(x_1, \cdots, x_d) : x_i \geq 0 (1 \leq i \leq d)\}\). The
Hilbert projective metric defined in $\mathbb{R}^d$ is equal to
\[ d(x, y) = \log \max_{1 \leq i \leq d} \frac{x_i}{y_i}. \]
See [12]. Then for any positive matrix $B = (b_{i,j})$, we have
\[ d(Bx, By) = \log \max_{1 \leq i \leq d} \frac{\sum_{j=1}^{d} b_{i,j}x_j}{\sum_{j=1}^{d} b_{i,j}y_j}. \]
It easy to see that if $\delta \leq b_{i,j} \leq \delta^{-1}$ for all $i$ and $j$, we have
\[ d(Bx, By) \leq 4 \log \frac{1}{\delta} < \infty. \]
Thus, the hypothesis on $B_n$’s implies that the projective diameters of $B_n$’s are bounded, so that the operators $B_n$ are contractive with a uniform contracting ratio $\tanh(\log \delta^{-1}) < 1$ ([7], see also [12] p. 333).

Based on Lemma 5 in [35], it is proved in [35] that there exists a function $\overline{u} : X \to [\Delta]$ such that $\overline{u}(Tx) = A(x)\overline{u}(x)$ namely
\[ A(x)\overline{u}(x) = \overline{u}(Tx)\|A(x)\overline{u}(x)\|. \]
Using this, we get
\[ A(n, x)\overline{u}(x) = A(T^{n-1}x)A(T^{n-2}x)\cdots A(Tx)A(x)\overline{u}(x) \]
\[ = A(T^{n-1}x)A(T^{n-2}x)\cdots A(Tx)\overline{u}(Tx)\|A(x)\overline{u}(x)\|. \]
Inductively we get
\[ A(n, x)\overline{u} = \overline{u}(T^{n}x)\|A(T^{n-1}x)\overline{u}(T^{n-1}x)\| \cdots \|A(Tx)\overline{u}(Tx)\|\|A(x)\overline{u}(x)\|. \]
So,
\[ \frac{\log \|A(n, x)\overline{u}(x)\|}{n} = \frac{\log \|\overline{u}(T^{n}x)\|}{n} + \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^{k}x) \]
where $\phi(x) = \log \|A(x)\overline{u}(x)\|$, which is a continuous function on $X$. Then we can conclude by the unique ergodicity of $T$.

The above argument repeats that in [35] with some modifications and details (it seems that there is something wrong at the bottom of page 807 in [35]).

5. Minimal and uniquely ergodic weights: Proof of Theorem 1.4

We first recall some useful facts on orbital systems, especially orbital systems generated by primitive substitutive sequences and the notion of return word [13], which is the key for proving Theorem 1.4. The reference [65] is a good source for primitive substitutive sequences.
5.1. **Minimal and uniquely ergodic sequences.** Let $\mathcal{A}$ be a finite set, called **alphabet**. Elements of $\mathcal{A}$ are called **letters**. A **word** of $\mathcal{A}$ is an element $x = x_0 x_1 \cdots x_{n−1}$ of $\mathcal{A}^n$, where $n$ is denoted $|x|$ and is called the **length** of $x$. The length of the empty-word $\emptyset$ is 0. Let $\mathcal{A}^+ = \bigcup_{n=1}^{\infty} \mathcal{A}^n$ and $\mathcal{A}^* = \mathcal{A}^+ \cup \{\emptyset\}$. Sequences in $\mathcal{A}^\infty$ are called infinite words.

With concatenation, $\mathcal{A}^*$ becomes a monoid. Let $\mathcal{B}$ be another alphabet. By concatenation, every map $\varphi : \mathcal{A} \to \mathcal{B}^+$ induces a map $\varphi : \mathcal{A}^* \to \mathcal{B}^*$, and a map from $\mathcal{A}^\infty$ into $\mathcal{B}^\infty$, which is still denoted by $\varphi$.

A **substitution** is a triple $(\zeta, \mathcal{A}, \alpha)$ where $\mathcal{A}$ is an alphabet, $\zeta : \mathcal{A} \to \mathcal{A}^+$ is a map and $\alpha \in \mathcal{A}$, such that

(S1) the first letter of $\zeta(\alpha)$ is $\alpha$;
(S2) $\lim_{n \to \infty} |\zeta^n(\alpha)| = \infty$.

The limit $u_\zeta := \lim_{n \to \infty} \zeta^n(\alpha) \in \mathcal{A}^\infty$ exists, and it is characterized by $\zeta(u_\zeta) = u_\zeta$ (i.e. $u_\zeta$ is a fixed point of $\zeta$) and the first letter of $u_\zeta$ is $\alpha$. If $\varphi : \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is another alphabet. We define $w_\zeta = \varphi(u_\zeta)$. Such a sequence is called a **substitutive sequence**.

A substitution $(\zeta, \mathcal{A}, \alpha)$ is said to be **primitive** if there exists an integer $k$ such that for all letters $\beta \in \mathcal{A}$ and $\gamma \in \mathcal{A}$, $\beta$ is a letter in $\zeta^k(\gamma)$. In this case, the corresponding sequence $w_\zeta$ is said to be **primitive**.

Let $x = x_0 x_1 \cdots \in \mathcal{A}^\infty$, let $m$ and $n$ be two integers with $m \leq n$. We write $x_{[m,n]}$ for the word $x_m \cdots x_n$, called a **factor** of $x$. The index $m$ is called the **recurrence** of $x_{[m,n]}$. Factors $x_{[0,n]}$ are called **prefixes**. For a word $x = x_0 x_1 \cdots x_{\ell−1}$, we can also define its factors and prefixes. **Suffixes** of this word $x$ are defined to be the words $x_k x_{k+1} \cdots x_{\ell−1}$ for $0 \leq k < \ell$.

An infinite sequence $x = x_0 x_1 \cdots \in \mathcal{A}^\infty$ is said to be **minimal** if for every integer $\ell \geq 1$, there exists an integer $L \geq 1$ such that each factor of $x$ of length $\ell$ occurs as factor of every factor of $x$ of length $L$. Let $\Theta : \mathcal{A}^\infty \to \mathcal{A}^\infty$ be the shift transformation. That $x$ is minimal means that the orbital system $(\{\Theta^n x\}, \Theta)$ is minimal. If the system $(\{\Theta^n x\}, \Theta)$ is minimal and uniquely ergodic, we say $x$ is minimal and uniquely ergodic.

Let $x$ be a minimal sequence over an alphabet $\mathcal{A}$ and $u$ be a non-empty prefix of $x$. We call **return word over** $u$ every factor $x_{[i,j−1]}$ where $i$ and $j$ are two successive occurrences of $u$ in $x$. We use $\mathcal{R}_u(x)$ to denote the set of all return words over $u$.

Let $x$ be a minimal sequence. For every prefix $u$ of $x$, the sequence has a unique decomposition

\[(21) \quad x = m_0 m_1 m_2 \cdots \in \mathcal{R}_u(x)^\infty;\]

The following lemma contains the key facts for us.

**Lemma 5.1.** Suppose that $x \in \mathcal{A}^\infty$ is minimal and uniquely ergodic. For each prefix $u$ of $x$,

(a) the set of return word $\mathcal{R}_u(x)$ is finite.
(b) the following frequencies exist:

\[
p_v = \lim_{k \to \infty} \frac{\#\{0 \leq i < k : m_i = v\}}{k} \quad (\forall v \in R_u(w))
\]

where \( m_j \)'s are the factors in the decomposition \((27)\) of \( x \).

Every primitive substitutive sequence is minimal and uniquely ergodic \((51)\).

Let us look at the Thue-Morse sequence \((t_n)\) defined by the substitution \(0 \mapsto 01, 1 \mapsto 10:\)

\[
0110100110010110100101101001011001101001 \cdots
\]

If we take the prefix \( u = 0 \), then we get the following decomposition

\[
011 01 0 011 0 011 0 011 01 0 011 01 0 011 01 0 01 \cdots
\]

It is known that there is no cubes in \((t_n)\). It is easy to see that

\[
R_0((t_n)) = \{0, 01\}.
\]

If we take the prefix \( u = 01 \), then we get the following decomposition

\[
011 010 0110 011 010 011 010 01 0110 011 010 010 0110 011 010 01 \cdots
\]

In this case we have

\[
R_{01}((t_n)) = \{01, 010, 011, 0110\}.
\]

5.2. **Proof of Theorem 1.4.** The assumption \((H2)\) is easy to check by using the hypothesis of the bounded variation of \( f \). In the following, we check the assumption \((H1)\).

If \( w \) is periodic, then \( \phi \) is well defined. So, in the following, we assume that \( w \) is not periodic.

Let \( n \geq 1 \) be a fix integer and let \( u \) be the prefix of \( w \) having length \( n \). Since \( w \) is aperiodic, so is \( u \). Therefore every return word \( v \in R_u(w) \) has length \( |v| \geq \frac{n}{2} \). Assume that

\[
w = v_1v_2 \cdots v_k \cdots \quad \text{with} \quad v_j \in R_u(w).
\]

Such a decomposition exists and is unique, see \((21)\). For any word \( b = b_0b_1 \cdots b_{m-1} \in A^+ \), we introduce the notation

\[
Z_b := Z_b(\lambda) = \int \exp \left( \lambda \sum_{j=0}^{m-1} b_j f(T^j x) \right) dx.
\]

Notice that

\[
Z_{m,n}(\lambda) = Z_{w_m w_{m+1} \cdots w_{m+n-1}}(\lambda).
\]

\[
(23)
\]
Indeed, by the definition of $Z_{m,n}(\lambda)$ and the invariance of $dx$, we have

\[
Z_{m,n}(\lambda) = \int \exp \left( \lambda \sum_{j=m}^{m+n-1} w_j f(T^j x) \right) dx
= \int \exp \left( \lambda \sum_{j=m}^{m+n-1} w_j f(T^{j-m} x) \right) dx.
\]

Thus, by Lemma (2.1), we have

\[
C^{-k} \leq \frac{Z_{v_1 v_2 \cdots v_k}}{Z_{v_1} Z_{v_2} \cdots Z_{v_k}} \leq C^k.
\]

It follows that

\[
\log \frac{Z_{v_1 v_2 \cdots v_k}}{|v_1 v_2 \cdots v_k|} = \frac{1}{k} \sum_{i=1}^{k} \log Z_{v_i} \pm \frac{k}{|v_1 v_2 \cdots v_k|} \log C.
\]

By Lemma 5.1 (b), the following frequencies exist:

\[
p_v := \lim_{k \to \infty} \frac{\#\{1 \leq i \leq k : v_i = v\}}{k} \quad (\forall v \in R_u(w)).
\]

Therefore, by Lemma 5.1 (a),

\[
\lim_{k \to \infty} \frac{1}{|v_1 v_2 \cdots v_k|} = \frac{1}{\sum_{v \in R_u(w)} p_v |v|},
\]

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \log Z_{v_i} = \sum_{v \in R_u(w)} p_v \log Z_v.
\]

Let

\[
L_u = \lim inf_{k} \frac{\log Z_{v_1 v_2 \cdots v_k}}{|v_1 v_2 \cdots v_k|}, \quad T_u = \lim sup_{k} \frac{\log Z_{v_1 v_2 \cdots v_k}}{|v_1 v_2 \cdots v_k|},
\]

and

\[
A_u = \frac{\sum_{v \in R_u(w)} p_v \log Z_v}{\sum_{v \in R_u(w)} p_v |v|}.
\]

Notice that $\frac{k}{|v_1 v_2 \cdots v_k|} \leq \frac{2}{n}$. From (24), we get

\[
A_u - \frac{2 \log C}{n} \leq L_u \leq T_u \leq A_u + \frac{2 \log C}{n}.
\]

Then

\[
0 \leq T_u - L_u \leq \frac{4 \log C}{n}.
\]

For any $N$, there exists a unique integer $k$ such that

\[
|v_1 v_2 \cdots v_k| \leq N < |v_1 v_2 \cdots v_k v_{k+1}|.
\]

So, by the definition of $Z_N(\lambda)$, we have

\[
\log Z_N(\lambda) = \log Z_{v_1 v_2 \cdots v_k} \pm |\lambda| M \|w\|_{\infty} \|f\|_{\infty}
\]
where $M = \max_{v \in R_u(w)} |v|$. It follows that

\begin{equation}
\phi = L_u, \quad \overline{\phi} = \overline{L}_u,
\end{equation}

where

\begin{equation}
\phi(\lambda) = \liminf_{N \to \infty} \frac{\log Z_N(\lambda)}{N}, \quad \overline{\phi}(\lambda) = \limsup_{N \to \infty} \frac{\log Z_N(\lambda)}{N}.
\end{equation}

From (26) and (27), we get

\begin{equation}
0 \leq \overline{\phi}(\lambda) - \phi(\lambda) \leq 4 \log C_n.
\end{equation}

Observe that both $\phi$ and $\overline{\phi}$ are independent of $n$. Letting $n \to \infty$, we get $\phi(\lambda) = \overline{\phi}(\lambda)$. The theorem is thus proved.

Notice that from (25) and (27), we get the following approximation of $\phi$ by the real analytic functions $A_u$:

\begin{equation}
\phi(\lambda) = \sum_{v \in R_u(w)} \frac{p_v \log Z_v(\lambda)}{\sum_{v \in R_u(w)} p_v |v|} \pm 2 \log C_n.
\end{equation}

Recall that $n$ is the length of the prefix $u$. This approximation is uniform on any compact set of $\lambda$ because $C = e^{O(\lambda)}$.

6. Proof of Theorem 1.5

The condition (C1) in Theorem 1.5 implies the condition (H1) in Theorem 1.1. So, in order to apply Theorem 1.1 to prove Theorem 1.5, it suffices to compute the pressure function.

6.1. Computation of the pressure function. The observation stated in the following lemma will allow us to compute the pressure function.

**Lemma 6.1** (Bernoullicity). Let

\[ f_n(x) = x_n g_n(x_{n+1}, \ldots, x_{n+p}, \ldots) \]

where $g_n$ is a Borel function taking values in $\{-1, 1\}$. Then $f_n$’s are independent symmetric Bernoulli variables, in other words

\[ \forall (t_1, \ldots, t_k) \in \mathbb{R}^k, \quad \mathbb{E} [e^{t_1 f_1 + \cdots + t_n f_n}] = \frac{1}{2^n} \prod_{k=1}^n (e^{t_k} + e^{-t_k}). \]

**Proof.** First remark that for any $r \in \mathbb{R}$, we have $\mathbb{E} e^{t r} = \frac{1}{2} (e^r + e^{-r})$, which depends only on the absolute value of $r$. Write

\[ \mathbb{E} [e^{t_1 f_1 + \cdots + t_n f_n}] = \mathbb{E} [e^{t_2 f_2 + \cdots + t_n f_n} \mathbb{E} (e^{t_1 f_1} | x_2, \ldots, x_n, \ldots)]. \]

Observe that $x_1$ is independent of $x_2, x_3, \ldots$. By the above remark we have

\[ \mathbb{E} (e^{t_1 f_1} | x_2, \ldots, x_n, \ldots) = \frac{1}{2} (e^{t_1} + e^{-t_1}), \]
because the conditional expectation is equal to the expectation with respect to $x_1$ with $x_2, x_3, \cdots$ being fixed. Thus, by induction, we get

$$E e^{t_1 f_1 + \cdots + t_n f_n} = \frac{1}{2^n} \prod_{k=1}^{n} (e^{t_k} + e^{-t_k}).$$

□

It follows that the pressure function $\phi$ is independent of the form of the functions $g_n$’s. So, in the following, without of loss of generality we continue our discussion with $g_n(x_{n+1}, x_{n+2}, \cdots) = x_{n+1}$. According to Theorem 1.1 the result on $\dim H(\alpha)$ depends only on the function

$$\phi(\lambda) = \lim_{N \to \infty} \frac{1}{N} \log E e^{\lambda \sum_{n=1}^{N} w_n x_n x_{n+1}}$$

provided that the limit exists. The limit does exist and is computable.

Lemma 6.2 (Pressure function). Suppose that $f_n$’s satisfy the assumption made in Lemma 6.1 and that $(w_n)$ take values $v_0, v_1, \cdots, v_m$ such that the frequencies $p_j$’s defined by (7) exist. Then

$$\phi(\lambda) = \sum_{j=0}^{m} p_j \log(e^{\lambda v_j} + e^{-\lambda v_j}) - \log 2.$$  

Proof. Lemma 6.1 gives

$$E e^{\lambda \sum_{n=1}^{N} w_n x_n x_{n+1}} = 2^{-N} \prod_{n=1}^{N} (e^{\lambda w_n} + e^{-\lambda w_n}).$$

Then (29) follows immediately if we use the hypothesis on $(w_n)$.

We would like to give another proof of (30). One reason is to get rid of Lemma 6.1 which could be mysterious for some readers. The other reason is that this method will allow us to treat other cases.

Since the function $\sum_{n=1}^{N} w_n x_n x_{n+1}$ is constant on cylinders of length $N + 1$, we have

$$E e^{\lambda \sum_{n=1}^{N} w_n x_n x_{n+1}} = \frac{1}{2^{N+1}} \sum_{x_1, \cdots, x_{N+1} \in \{-1, 1\}} e^{\lambda \sum_{n=1}^{N} w_n x_n x_{n+1}}.$$  

Let

$$A_n = \begin{pmatrix} e^{\lambda w_n} & e^{-\lambda w_n} \\ e^{-\lambda w_n} & e^{\lambda w_n} \end{pmatrix}.$$  

It is clear that

$$E e^{\lambda \sum_{n=1}^{N} w_n x_n x_{n+1}} = \frac{1}{2^{N+1}} \|A_1 \cdots A_N\|_S$$

References:
[1] Theorem 1.1

(30)

(31)
where \(\|B\|_S\) denotes the norm of a matrix \(B\), the sum of all elements of \(B\). Notice that all \(A_n\)'s are of the form

\[
A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad (a, b \in \mathbb{R})
\]

which commute each other. Indeed, they can be simultaneously diagonalized as follows

\[
(32) \quad R^{-1}AR = D := \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix},
\]

where

\[
R = \sqrt{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad R^{-1} = \sqrt{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},
\]

which are independent of \(a\) and \(b\). Apply (32) to \(A = A_n (1 \leq n \leq N)\) to get

\[
R^{-1}A_nR = D_n = \begin{pmatrix} e^{\lambda w_n} + e^{-\lambda w_n} & e^{-\lambda w_n} - e^{\lambda w_n} \\ e^{-\lambda w_n} - e^{\lambda w_n} & e^{\lambda w_n} + e^{-\lambda w_n} \end{pmatrix}.
\]

Then \(A_1 \cdots A_n = RD_1 \cdots D_NR^{-1}\). Notice that

\[
\|A_1 \cdots A_n\|_S = 2 \prod_{n=1}^{N} (e^{\lambda w_n} + e^{-\lambda w_n}).
\]

This, together with (31), leads to (30). \(\square\)

### 6.2. Proof of Theorem 1.5

By Lemma 6.2, we have

\[
\psi(\lambda) = \sum_{j=0}^{m} p_j \log(e^{\lambda v_j} + e^{-\lambda v_j}).
\]

According to Theorem 1.1, we have to compute the conjugate function \(\psi^*\). By simple calculations we get that

\[
\psi'(\lambda) = \sum_{j=0}^{m} p_j v_j \frac{e^{\lambda v_j} - e^{-\lambda v_j}}{e^{2\lambda v_j} + 1}, \quad \psi''(\lambda) = 4 \sum_{j=0}^{m} \frac{p_j v_j^2 e^{2\lambda v_j}}{(e^{2\lambda v_j} + 1)^2} > 0.
\]

\[
\phi'(+\infty) = \sum_{j=0}^{m} p_j |v_j|, \quad \phi'(-\infty) = -\sum_{j=0}^{m} p_j |v_j|.
\]
Then, for any \( \alpha \in (\psi'(-\infty), \psi'(+\infty)) \), there exists a unique \( \lambda_\alpha \) such that \( \psi'(\lambda_\alpha) = \alpha \), i.e.

\[
\sum_{j=0}^{m} p_j v_j e^{\lambda_\alpha v_j} - e^{-\lambda_\alpha v_j} = \alpha.
\]

So, \( \psi^*(\alpha) = \alpha \lambda_\alpha - \psi(\lambda_\alpha) \), which gives the formula

\[
\dim E(\alpha) = \frac{1}{\log 2} \sum_{j=0}^{m} p_j \left( \log(e^{\lambda_\alpha v_j} + e^{-\lambda_\alpha v_j}) - \lambda_\alpha v_j e^{\lambda_\alpha v_j} - e^{-\lambda_\alpha v_j} e^{\lambda_\alpha v_j} + e^{-\lambda_\alpha v_j} \right).
\]

6.3. Gibbs measures are Markovian measures. In the case

\[
f_n(x_n, x_{n+1}) = w_n x_n x_{n+1},
\]

we can directly prove Theorem 1.5 without using Theorem 1.1. Because we can directly construct the Gibbs measures as inhomogeneous Markov measures and compute their dimensions without using Lemma 2.9.

Consider the stochastic matrix

\[
P_n = \frac{1}{e^{\lambda w_n} + e^{-\lambda w_n}} \begin{pmatrix} e^{\lambda w_n} & e^{-\lambda w_n} \\ e^{-\lambda w_n} & e^{\lambda w_n} \end{pmatrix}.
\]

We denote it by \((p_{n,j}^{(j)})\). It is clear that \((\frac{1}{2}, \frac{1}{2})\) is a left invariant probability vector of all \(P_n\). Let us define the inhomogeneous Markov measure \(\mu_\lambda\) by

\[
\mu_\lambda([x_0, x_1, \ldots, x_n]) = \frac{1}{2} p_{0}^{(0)} x_0 x_1 \cdots p_{x_{n-1}, x_n}^{(n-1)}.
\]

In other words,

\[
\mu_\lambda([x_0, x_1, \ldots, x_n]) = \frac{1}{2 Z_n(\lambda)} \exp \left( \lambda \sum_{k=0}^{n-1} w_k x_k x_{k+1} \right),
\]

where \(Z_n(\lambda) = \prod_{k=0}^{n-1} (e^{\lambda w_k} + e^{-\lambda w_k})\).

Lemma 6.3 (Law of large numbers). For every \(n \geq 0\) we have

\[
\int x_n x_{n+1} d\mu_\lambda(x) = \frac{e^{\lambda w_n} - e^{-\lambda w_n}}{e^{\lambda w_n} + e^{-\lambda w_n}}.
\]

For \(\mu_\lambda\)-almost all \(x\) we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n x_n x_{n+1} = \sum_{j=0}^{m} p_j v_j e^{\lambda v_j} - e^{-\lambda v_j} e^{\lambda v_j} + e^{-\lambda v_j}.
\]

Proof. By the definition of \(\mu_\lambda\) we have

\[
\int x_n x_{n+1} d\mu_\lambda(x) = \sum_{x_0, x_1, \ldots, x_{n+1}} x_n x_{n+1} \cdot \frac{1}{2} p_{x_0, x_1}^{(0)} \cdots p_{x_{n}, x_{n+1}}^{(n)}.
\]
Since \( \sum_i p_{i,j}^{(k)} = 1 \), we have
\[
\int x_n x_{n+1} d\mu_\lambda(x) = \frac{1}{2} \sum_{x_n, x_{n+1}} x_n x_{n+1} p_{x_n, x_{n+1}} = \frac{e^{\lambda w_n} - e^{-\lambda w_n}}{e^{\lambda w_n} + e^{-\lambda w_n}}
\]
where the last equality follows from the definition of \( P_n \).
Let \( Y_n = x_n x_{n+1} \). Similar computation shows that \( Y_n - \mathbb{E}_{\mu_\lambda} Y_n \) are orthogonal. Then (34) follows from the Menshov theorem (see [43]) and the Kronecker lemma (see [70]).

**Lemma 6.4** (Dimensions of Markov-Gibbs measures).
\[
\dim \mu_\lambda = 1 \log 2 \sum_{j=0}^m p_j \log(e^{\lambda v_j} + e^{-\lambda v_j}) - \lambda \sum_{k=0}^{n-1} w_k x_k x_{k+1} + o(1)
\]

*Proof.* From the definition (33) of \( \mu_\lambda \), we have
\[
\frac{\log \mu_\lambda([x_0 x_1 \cdots x_n])}{\log 2^{-n}} = \frac{\log 2 Z_n(\lambda)}{n} - \frac{\lambda}{2n} \sum_{k=0}^{n-1} w_k x_k x_{k+1} + o(1)
\]
Then by Lemma 6.3 \( \mu_\lambda \)-a.e. we have
\[
D(\mu_\lambda, x) = \frac{1}{\log 2} \sum_{j=0}^m p_j \log(e^{\lambda v_j} + e^{-\lambda v_j}) - \frac{\lambda}{2n} \sum_{j=0}^m p_j v_j \frac{e^{\lambda v_j} - e^{-\lambda v_j}}{e^{\lambda v_j} + e^{-\lambda v_j}}.
\]

\[
7. \text{Proof of Theorem 1.6}
\]
In the case of M"obius weights \( w_n = \mu(n) \), by Lemma 6.2 we have
\[
\psi(\lambda) = f_0 \log 2 + (1 - f_0) \log(e^\lambda + e^{-\lambda})
\]
where \( 1 - f_0 = \frac{6}{\pi^2} \), because it is well known that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \mu(n) = 0, \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N |\mu(n)| = \frac{6}{\pi^2}
\]
(see [74], Theorem 3.8 and Theorem 3.10). For \( \alpha \in (-6/\pi^2, 6/\pi^2) \) we can solve the equation \( \psi'(\lambda) = \alpha \), i.e.
\[
\frac{6 e^\lambda - e^{-\lambda}}{\pi^2 (e^\lambda + e^{-\lambda})} = \alpha.
\]
Indeed, let \( \alpha' = \frac{\alpha}{6/\pi^2} \in (0, 1) \). Then we get the solution \( \lambda_\alpha \):
\[
e^{\lambda_{\alpha}} = \sqrt{\frac{1 + \alpha'}{1 - \alpha'}}, \quad \text{i.e.} \quad \lambda_{\alpha} = \frac{1}{2} \log \frac{1 + \alpha'}{1 - \alpha'}.
\]
Then
\[ -\psi^*(\alpha) = \psi(\lambda_\alpha) - \alpha \lambda_\alpha \]
\[ = f_0 \log 2 + (1 - f_0) \log \left( \sqrt{\frac{1 + \alpha'}{1 - \alpha'}} + \sqrt{\frac{1 - \alpha'}{1 + \alpha'}} \right) - \frac{\alpha}{2} \log \frac{1 + \alpha'}{1 - \alpha'}. \]

Notice that
\[ 2 \log \left( \sqrt{\frac{1 + \alpha'}{1 - \alpha'}} + \sqrt{\frac{1 - \alpha'}{1 + \alpha'}} \right) = \log \left( \frac{1 + \alpha'}{1 - \alpha'} + \frac{1 - \alpha'}{1 + \alpha'} + 2 \right) \]
\[ = \log \frac{4}{(1 + \alpha')(1 - \alpha')} \]
So, letting \( p = \frac{1 + \alpha'}{2} \) and \( p' = \frac{1 - \alpha'}{2} \), we get
\[ -\psi^*(\alpha) = f_0 \log 2 + \frac{1 - f_0}{2} \log \frac{4}{(1 + \alpha')(1 - \alpha')} - (1 - f_0) \frac{\alpha'}{2} \log \frac{1 + \alpha'}{1 - \alpha'} \]
\[ = f_0 \log 2 - (1 - f_0) \frac{1}{2} \log(pp') - (1 - f_0) \frac{\alpha'}{2} \log(p/p') \]
\[ = f_0 \log 2 - (1 - f_0) \left( \frac{1 + \alpha'}{2} \log p + \frac{1 - \alpha'}{2} \log p' \right) \]
\[ = f_0 \log 2 + (1 - f_0) H(p). \]
So, by Theorem 1.5, we get
\[ \dim F(\alpha) = 1 - \frac{6}{\pi^2} + \frac{6}{\pi^2 \log 2} H \left( \frac{1}{2} + \frac{\pi^2}{12} \frac{\alpha}{1 - f_0} \right). \]

The above proof, without any change, actually proves the following more general result.

**Theorem 7.1.** Assume that \((w_n)\) is a sequence taking \(-1, 0, 1\) as values and having \(f_0\) as the frequency of 0’s. For any \(\alpha \in \left( (1 - f_0), 1 - f_0 \right)\), we have
\[ \dim F(\alpha) = f_0 + \frac{1 - f_0}{\log 2} H \left( \frac{1}{2} + \frac{\alpha}{2(1 - f_0)} \right), \]
where
\[ F(\alpha) = \left\{ x \in \{-1, 1\}^\mathbb{N} : \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} w_n x_n x_{n+1} = \alpha \right\}. \]

**8. Final remarks**

R.1. We can consider complex valued or vector valued functions \(f_n\). Assume that \(f_n\)’s take values in \(\mathbb{R}^d\). Then we have to change the definition of \(Z_n(\lambda)\) as follows
\[ Z_n(\lambda) = \mathbb{E} \exp \left( \lambda \sum_{k=0}^{n-1} f_k(x_k, x_{k+1}, \ldots) \right), \quad (\lambda \in \mathbb{R}^d) \]
where $\lambda \cdot a$ denotes the inner product in $\mathbb{R}^d$.

R.2. Theorem 1.1 is not applicable to the case

$$f_n(x_n, x_{n+1}, \cdots) = w_n x_n x_{2n}.$$ 

Because, although the condition (H1) is satisfied (see Lemma 6.2), but the condition (H2) is not satisfied. New ideas are needed to study this case. The special case where $w_n = 1$ for all $n$ was treated in [28, 58]. The more general case $f_n(x) = f(x_n, x_{2n})$ was studied in [28] and the method of [28] could be used to treat a general $(w_n)$ which takes a finite number of values and admits frequencies for all possible values. See [26, 49, 57, 77] for related works. A form of non-linear thermodynamic formalism based on solutions to a nonlinear equation was useful for such a problem [46, 27, 28]. The idea comes from Kenyon, Peres and Solomyak [46]. Pollicott [63] considered a more general setting of nonlinear transfer operator. For a nonlinear Perron-Frobenius theory, see [48].

R.3. If $f_n(x_n, x_{n+1}, \cdots)$ is of the form $w_n f(x_n, x_{n+1})$ where $f : S \times S \to \mathbb{R}$ is an arbitrary function, we can apply Theorem 1.1 to this case. But we have to make sure that the limit defining $\phi$ exists. Better is to ensure the differentiability of $\phi$. However both are questionable and works are to be done for a given weight $(w_n)$, except for the dynamically produced weights considered in Theorem 1.2, Theorem 1.3 and Theorem 1.4.

Problem 1. Find conditions on $(w_n)$ and on $f$ such that $\phi$ is well defined and differentiable.

In the following, we discuss some sub-problems.

R.4. A very special case of Problem 1 is as follows.

Problem 2. Suppose that $w_n$ is the M"{o}bius function $\mu(n)$ and $f : \{0, 1\} \times \{0, 1\} \to \{0, 1\}$ is defined by $f(x, y) = xy$. Is $\phi$ well defined and differentiable? We emphasize that it is $\{0, 1\}$ but not $\{-1, 1\}$. If we would like to work with $\{-1, 1\}$, the problem arises for

$$f(x, y) = axy + bx + cy \quad (a, b, c \text{ being constants}).$$

Here is an idea to attack such a problem, which was used for proving Theorems 1.2 and Lemma 6.2. Assume $f_n(x_n, x_{n+1}, \cdots) = w_n f(x_n, x_{n+1})$ where $f : S \times S \to \mathbb{R}$ and $(w_n)$ are given. For any $\lambda \in \mathbb{R}$, define a $S \times S$-matrix

$$A_n := A_n(\lambda) := (e^{\lambda f(i, j)})_{(i,j) \in S \times S}.$$ 

By the same argument as in the proof of (31), we can obtain

$$Z_n(\lambda) = \mathbb{E} e^{\lambda \sum_{n=1}^N w_n f(x_n, x_{n+1})} = \frac{1}{q^{N+1}} \| A_1 \cdots A_N \|$$
where $\|B\|$ denotes the sum of all elements of a matrix $B$. So, we are led to prove the existence of the following Liapounov exponent

$$
L(\lambda) = \lim_{N \to \infty} \frac{1}{N} \log \|A_1 \cdots A_N\|.
$$

Notice that $L(\lambda)$ is nothing but $\psi(\lambda)$, if the limit in (36) exists. Remark that for the case concerned by Problem 2, the matrix $A_n(\lambda)$ takes a simple form

$$
A_n(\lambda) = \begin{pmatrix} 1 & 1 \\ 1 & e^{\lambda \mu(n)} \end{pmatrix}.
$$

R.5. Keep the same notation as in R.4. Replace $(w_n)$ by is a sequence of independent and identically distributed random variables $(\omega_n)$ taking a finite number of values. By Theorem 1.2, $L(\lambda)$ is analytic. The method presented by Pollicott in [62] can be used to numerically compute $L(\lambda)$.

R.6. If $(w_n)$ is a primitive substitutive sequence, we have proved that $L(\lambda)$ is well defined and is analytic (Theorem 1.3).

**Problem 3.** Suppose that $(w_n)$ is a primitive substitutive sequence. Then $L(\lambda)$ differentiable, by Theorem 1.3. Is it possible to get a closed form for $L(\lambda)$? How about Thue-Morse sequence or other specific sequences?

R.7. Following Katzenelson and Weiss [15], Furman (35, Theorem 1) proved that on any unique ergodic system $(\Omega, \Theta)$, there exist continuous subadditive cocyles $(f_n)$ such that the limit of $n^{-1} f_n(\omega)$ doesn’t exist for some $\omega$. Our pressures considered in Theorem 1.3 are defined by the limit for special cycles. By Theorem 1.3, the limit defining the pressure does exist under the condition that $f$ depends only on a finite number of coordinates.

**Problem 4.** Can we drop this condition of dependence on finite coordinates but we assume that $f$ is of bounded variation?

Finally, let us repeat that we are interested in evaluating

$$
\lim_{n \to \infty} \frac{1}{n} \log \left\| \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & e^{\lambda w_1} \\ 1 & e^{\lambda w_2} \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ 1 & e^{\lambda w_n} \end{pmatrix} \right\|
$$

for different weights $(w_n)$. Three questions are associated: does the limit exist? is the limit differentiable as function of $\lambda$? is it possible to compute the limit?

**Addendum** B. Bárány, M. Rams and R. X. Shi have obtained some results similar to Theorem 1.2 and Theorem 1.5 with a different approach, which will be presented in a forthcoming paper.
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