THE HARDY-RELLICH INEQUALITY AND UNCERTAINTY PRINCIPLE ON THE SPHERE

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Abstract. Let $\Delta_0$ be the Laplace-Beltrami operator on the unit sphere $S^{d-1}$ of $\mathbb{R}^d$. We show that the Hardy-Rellich inequality of the form

$$\int_{S^{d-1}} |f(x)|^2 \, d\sigma(x) \leq c_d \min_{e \in S^{d-1}} \int_{S^{d-1}} (1 - \langle x, e \rangle) \left| (-\Delta_0)^{\frac{1}{2}} f(x) \right|^2 \, d\sigma(x)$$

holds for $d = 2$ and $d \geq 4$ but does not hold for $d = 3$ with any finite constant, and the optimal constant for the inequality is $c_d = \frac{8}{(d-3)^2}$ for $d = 2, 4, 5$ and, under additional restrictions on the function space, for $d \geq 6$. This inequality yields an uncertainty principle of the form

$$\min_{e \in S^{d-1}} \int_{S^{d-1}} (1 - \langle x, e \rangle) |f(x)|^2 \, d\sigma(x) \int_{S^{d-1}} |\nabla_0 f(x)|^2 \, d\sigma(x) \geq c'_d$$

on the sphere for functions with zero mean and unit norm, which can be used to establish another uncertainty principle without zero mean assumption, both of which appear to be new.

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1. Introduction

The purpose of this paper is to establish an analogue of the Hardy-Rellich inequality and the uncertainty principle on the sphere $S^{d-1} := \{ x \in \mathbb{R}^d : \|x\| = 1 \}$, where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. To motivate our results, we first recall these inequalities on $\mathbb{R}^d$.

Let $\Delta$ denote the usual Laplace operator on $\mathbb{R}^d$. For $\alpha > 0$, $(-\Delta)^{\alpha}$ denotes the fractional power of $-\Delta$. The inequality of the type

$$(1.1) \quad \int_{\mathbb{R}^d} |f(x)|^2 \|x\|^\mu \, dx \leq c \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{1}{2}} f(x) \right|^2 \|x\|^{\mu+2\alpha} \, dx,$$

is called the Hardy-Rellich-type inequality. It is the classical Hardy inequality when $\alpha = 1$, and the Rellich inequality when $\alpha = 2$. There are many papers devoted to the study of this inequality and its various generalizations. In particular, the best constant in (1.1) was calculated in [3, 6, 15] under some assumptions on the parameters; see also [10]. The uncertainty principle is a fundamental result in quantum mechanics and it can be formulated, in the form of the classical Heisenberg...
inequality, as

\begin{equation}
\inf_{a \in \mathbb{R}^d} \int_{\mathbb{R}^d} \|x - a\|^2 |f(x)|^2 dx \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx \geq \frac{d^2}{4} \left( \int_{\mathbb{R}^d} |f(x)|^2 dx \right)^2.
\end{equation}

The uncertainty principle has been widely studied and extended; see, for example, \cite{4, 4} and the references therein.

Our main results in this paper are analogues of such results on the unit sphere \(S^{d-1}\), in which we work with the Laplace-Beltrami operator \(\Delta_0\) and the spherical gradient \(\nabla_0\), which are the restriction of \(\Delta\) and \(\nabla\) on the sphere, respectively. Let \(d\sigma(x)\) be the usual rotation-invariant measure on \(S^{d-1}\). For smooth functions \(f\) on \(S^{d-1}\) that satisfy \(\int_{S^{d-1}} f(x)d\sigma = 0\), our main result on the Hardy-Rellich inequality states that

\begin{equation}
\int_{S^{d-1}} |f(x)|^2 d\sigma(x) \leq c_d \min_{e \in S^{d-1}} \int_{S^{d-1}} (1 - \langle x, e \rangle)(-\Delta_0)^{\frac{1}{2}} f(x)^2 d\sigma(x),
\end{equation}

where the constant \(c_d\) satisfies \(c_d \geq 8/(d - 3)^2\), which shows, in particular, a surprising result that the inequality (1.4) holds for all dimensions but \(d = 3\), that is, except for \(S^2\). We will also show that the best constant in the inequality is \(c_d = 8/(d - 3)^2\) for all \(f\) if \(d = 2, 4, 5\), and for \(f\) in a subspace if \(d \geq 6\). We then use the inequality (1.4) to establish an uncertainty principle, which states that

\begin{equation}
\min_{e \in S^{d-1}} \int_{S^{d-1}} (1 - \langle x, e \rangle)|f(x)|^2 d\sigma \int_{S^{d-1}} |\nabla_0 f(x)|^2 d\sigma \geq c'_d \left( \int_{S^{d-1}} |f(x)|^2 d\sigma \right)^2
\end{equation}

for smooth functions \(f\) satisfying \(\int_{S^{d-1}} f(x)d\sigma = 0\). The proof, however, is not applicable for \(d = 3\). The gap prompted us to search for a different approach. A second proof shows that (1.4) does hold for \(d = 3\).

Recall that the geodesic distance on the sphere is defined by \(d(x, y) = \arccos \langle x, y \rangle\), so that

\[1 - \langle x, y \rangle = 2 \sin^2 \frac{d(x, y)}{2},\]

which shows that (1.4) can be regarded as a close analogue of (1.2). Given the numerous extensions of the uncertainty principles on a wide range of settings, it is somewhat surprising that this formulation of the uncertainty principle has not appeared, as far as we know, in the literature. The inequality that carries the name of the uncertainty principle on the sphere in the literature is (8, 9, 11)

\begin{equation}
(1 - \|\tau(f)\|^2) \int_{S^{d-1}} |\nabla_0 f|^2 d\sigma \geq c|\tau(f)|^2
\end{equation}

for smooth functions \(f\) satisfying \(\|f\|_2 = 1\), where \(\tau(f)\) is the vector defined by

\[\tau(f) := \int_{S^{d-1}} x|f(x)|^2 d\sigma(x)\]

The inequality (1.5), however, is stronger than (1.5), since it implies

\begin{equation}
(1 - \|\tau(f)\|^2) \left( \int_{S^{d-1}} |\nabla_0 f|^2 d\sigma \right) \geq c \|\tau(f)\|,
\end{equation}

and we know that \(\|\tau(f)\| \leq 1\) and \(1 - \|\tau(f)\|^2 \leq 1 - \|\tau(f)\|^2\). Thus, our uncertainty principle (1.4) appears to be not only a close analogue of the classical result on \(\mathbb{R}^d\), but also stronger than what is known in the literature.
Since the zonal functions $f((x, \cdot))$ in $L^2(S^{d-1})$ can be identified with functions in $L^2(w_\lambda, [-1, 1])$ with $w_\lambda(t) = (1 - t^2)^{\lambda-1/2}$ and $\lambda = (d - 2)/2$, both the Hardy-Rellich inequality and the uncertainty principle can be stated for functions in $L^2(w_\lambda, [-1, 1])$ for $\lambda = (d - 2)/2$, where the operator $\Delta_0$ is replaced by the second order differential operator that has the Gegenbauer polynomial as the eigenfunctions. Furthermore, these inequalities can be formulated more generally for all $\lambda > -1/2$, as we shall do in most of our statements.

The paper is organized as follows. The next section is devoted to the orthogonal expansions in spherical harmonics, which will be our main tool. The Hardy-Rellich inequalities are discussed and proved in Section 3, with the assumption of a technical lemma that will be proved in the Section 5. The inequalities of uncertainty principle are established in Section 4.

2. Spherical harmonic expansions

Throughout this paper, all functions are assumed to be real valued and Lebesgue measurable on $S^{d-1}$ whenever $d \geq 3$. Let $L^2(S^{d-1})$ denote the space of functions of finite norm

$$\|f\|_2 := \left(\frac{1}{\omega_d} \int_{S^{d-1}} |f(x)|^2 \, d\sigma\right)^{1/2} \quad \text{with} \quad \omega_d := \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

where $\omega_d$ is the surface area of the sphere $S^{d-1}$ and $d\sigma(x)/\omega_d$ is the normalized Lebesgue measure on $S^{d-1}$.

A spherical polynomial of degree $n$ on $S^{d-1}$ is the restriction of an algebraic polynomial of total degree at most $n$ in $d$-variables on $S^{d-1}$. We denote by $\Pi^d_n$ the space of real spherical polynomials of degree at most $n$ on $S^{d-1}$. A spherical harmonic of degree $n$ in $d$-variables is the restriction of a homogeneous harmonic polynomial of degree $n$ on $S^{d-1}$. We denote by $\mathcal{H}^d_n$, $n = 0, 1, \ldots$, the space of spherical harmonics of degree $n$ on $S^{d-1}$, which has dimension

$$a^d_n := \dim \mathcal{H}^d_n = \frac{(2n + d - 2)\Gamma(n + d - 1)}{(n + d - 2)\Gamma(n + 1)\Gamma(d - 1)}, \quad n = 0, 1, \ldots. \tag{2.1}$$

These spaces are known to be mutually orthogonal with respect to the inner product of $L^2(S^{d-1})$. Since the space of spherical polynomials is dense in $L^2(S^{d-1})$, we have the orthogonal decomposition

$$L^2(S^{d-1}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^d_n : \quad f = \sum_{n=0}^{\infty} \text{proj}_n f,$$

where $\text{proj}_n$ is the orthogonal projection of $L^2(S^{d-1})$ onto the space $\mathcal{H}^d_n$.

The restriction of the Laplace operator on the sphere is the Laplace-Beltrami operator $\Delta_0$, which is defined by

$$\Delta_0 f := \Delta F|_{S^{d-1}}, \quad \text{where} \quad F(x) = f\left(\frac{x}{\|x\|}\right).$$

For each $n = 0, 1, \ldots$, the space of spherical harmonics $\mathcal{H}^d_n$ is the eigenfunction-space of $\Delta_0$ with the eigenvalue $-n(n + d - 2)$, that is,

$$\mathcal{H}^d_n = \{ f \in C^2(S^{d-1}) : \Delta_0 f = -n(n + d - 2)f \}, \quad n = 0, 1, \ldots.$$
For \( r \in \mathbb{R} \setminus \{0\} \), the fractional Laplace-Beltrami operator \((-\Delta_0)^r\) is defined in a distributional sense through \( \text{proj}_0 \left[ (-\Delta_0)^r f \right] = 0 \) and
\[
(2.3) \quad \text{proj}_n \left[ (-\Delta_0)^r f \right] = (n(n + d - 2))^r \text{proj}_n(f), \quad n = 1, 2, \ldots.
\]
Let \( \nabla \) denote the usual gradient operator of \( \mathbb{R}^d \). Then the tangential gradient \( \nabla_0 f \) of a function \( f \in C^1(\mathbb{S}^{d-1}) \) is defined by
\[
\nabla_0 f = \nabla F \big|_{\mathbb{S}^{d-1}}, \quad \text{where } F(x) = f \left( \frac{x}{\|x\|} \right).
\]
It is known (\cite[p.80, Lemma 1]{7}) that, for \( f, g \in C^2(\mathbb{S}^{d-1}) \),
\[
\langle \Delta_0 f, g \rangle_{L^2(\mathbb{S}^{d-1})} = - \int_{\mathbb{S}^{d-1}} \langle \nabla_0 f, \nabla_0 g \rangle \, d\sigma(x),
\]
which, in particular, implies, since \( \Delta_0 \) is self-adjoint in \( L^2(\mathbb{S}^{d-1}) \), that
\[
(2.4) \quad \|(-\Delta_0)^{1/2} f\|_2 = \|\nabla_0 f \|_{2}^{1/2} =: \|\nabla_0 f\|_2.
\]
When \( d = 2 \), we parametrize \( \mathbb{S}^1 \) by \( x = e^{i\theta} \) for \( \theta \in [0, 2\pi) \) and identify \( f(e^{i\theta}) \) with \( f(\theta) \). Choosing \( \{e^{in\theta}, e^{-in\theta}\} \) as a basis of \( H^2 \), the function \( f \in L^2(\mathbb{S}^1) \) has the usual Fourier series
\[
(2.5) \quad f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{in\theta}, \quad \text{where } \hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-int} \, dt.
\]
In this case \( \text{proj}_n f = \hat{f}_n e^{in\theta} + \hat{f}_{-n} e^{-in\theta} \), \( \nabla_0 = \partial / \partial \theta \) and \( \Delta_0 = \partial^2 / \partial \theta^2. \)

For \( d > 2 \), we will need an explicit form of an orthonormal basis for \( H^d_n \), parametrized by \( x = (x, \xi) \in \mathbb{S}^{d-1} \), where \( \xi \in \mathbb{S}^{d-2} \) and \( 0 \leq \theta \leq \pi \). This basis can be derived from the usual basis in spherical coordinates; see, for example, (\cite[p. 35]{2}). For completeness, we give an independent derivation below. For \( \lambda > -1/2 \) and \( n \in \mathbb{N}_0 \), let \( C^\lambda_n \) denote the Gegenbauer polynomial of degree \( n \).
The polynomials \( C^\lambda_n \) satisfy the orthogonal relation (\cite[(4.7.15)]{13})
\[
(2.6) \quad c_n \int_{-1}^{1} C^\lambda_n(t) C^\lambda_m(t)(1-t^2)^{\lambda-1/2} \, dt = h_n \delta_{m,n}, \quad h_n^\lambda := \frac{(2\lambda)_n}{(n + \lambda)n!},
\]
where \((a)_n = a(a+1) \cdots (a+n-1)\) is the Pochhammer symbol and \( c_\lambda \) is the normalization constant \( c_\lambda = 1/\int_{-1}^{1} (1-t^2)^{\lambda-1/2} \, dt = \frac{\Gamma(\lambda+1)}{\sqrt{\pi \Gamma(\lambda+1/2)}} \).

**Proposition 2.1.** Let \( \lambda = \frac{d-2}{2} \) and \( d > 2 \). For \( m \in \mathbb{N}_0 \), let \( \{Y^m_j(\xi) : 1 \leq j \leq a^d_m \} \) be an orthonormal basis of \( H^{(d-1)_m} \). For \( x = (\cos \theta, \sin \theta, \xi) \in \mathbb{S}^{d-1} \) with \( 0 \leq \theta \leq \pi \) and \( \xi \in \mathbb{S}^{d-2} \), we define
\[
P^n_{j,k}(x) = C^{\lambda+n-k}_k(\cos \theta)(\sin \theta)^{n-k} Y^{n-k}_j(\xi), \quad 1 \leq j \leq a^d_{n-k}, \quad 0 \leq k \leq n.
\]
Then \( \{P^n_{j,k} : 1 \leq j \leq a^d_{n-k}, \quad 0 \leq k \leq n\} \) is an orthogonal basis of \( H^d_n \) and
\[
(2.7) \quad H^n_k := \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} \left[ P^n_{j,k}(x) \right]^2 \, d\sigma(x) = h_k^{n-k+\lambda}.
\]
**Proof.** Using the integral formula
\[
\frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} f(x) \, d\sigma_d(x) = c_\lambda \int_0^\pi \left[ \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-2}} f(\cos \theta, \xi \sin \theta) \, d\sigma_{d-1}(\xi) \right] (\sin \theta)^{2\lambda} \, d\theta,
\]
where \( c_\lambda = 
\]
and the orthonormality of \( Y_j^{-k} \), we obtain that
\[
\langle P_{j,k}^n, P_{j,k'}^n \rangle_{L^2(S^{d-1})} = c_\lambda \delta_{j,j'} \delta_{n-k,n-k' \pm 1} 
\times \int_0^\pi C_k^{n-k+\lambda} (\cos \theta) C_{k'}^{n-k'+\lambda} (\cos \theta) (\sin \theta)^{2n-k-k'+2\lambda} \, d\theta,
\]
from which the mutual orthogonality of \( P_{j,k}^n \) follows, so is the formula of \( H_k^n \).

Since each \( Y_j^{-k} \) is the restriction to \( S^{d-2} \) of a homogeneous polynomial in \( d-1 \) variables of degree \( n-k \), it follows readily that, for \( x = (x_1, \ldots, x_d) \in S^{d-1} \),
\[
P_{j,k}^n(x) = C_k^{n-k+\lambda} (x_1) Y_j^{-k}(x_2, \ldots, x_d),
\]
which shows that \( P_{j,k}^n \) is a homogeneous polynomial. Furthermore, it is easy to verify that \( \sum_{k=0}^n a_{n-k}^{d-1} = a_n^d = \dim H_n^d \). Since the orthogonality determines the spherical harmonics, \( \{ P_{j,k}^n : 1 \leq j \leq a_{n-k}^{d-1}, 0 \leq k \leq n \} \) is a basis of \( H_n^d \).

Definition 2.2. For \( d > 2 \), we define the Fourier coefficients of \( f \in L^2(S^{d-1}) \) with respect to the mutually orthogonal basis \( \{ P_{j,k}^n(x) \} \) by
\[
\hat{f}_{j,k} := \frac{1}{\omega_d} \int_{S^{d-1}} f(y) P_{j,k}^n(y) \, d\sigma(y), \quad 0 \leq k \leq n, \ 1 \leq j \leq a_{n-k}^{d-1}.
\]
As a direct consequence of Proposition 2.1, the projection operator can be expressed as the following:

Lemma 2.3. For each \( f \in L^2(S^{d-1}) \), \( d > 2 \), and \( n \in \mathbb{N}_0 \),
\[
\operatorname{proj}_n f(x) = \sum_{k=0}^n \sum_{j=1}^{a_{n-k}^{d-1}} \hat{f}_{j,k}^n \left[ H_k^n \right]^{-1/2} P_{j,k}^n(x),
\]
and
\[
\| \operatorname{proj}_n f \|^2 = \sum_{k=0}^n \sum_{1 \leq j \leq a_{n-k}^{d-1}} \left| \hat{f}_{j,k}^n \right|^2.
\]
The reason for our choice of the particular basis in Proposition 2.1 lies in the following result.

Lemma 2.4. Let \( d > 2 \). If \( f \in L^2(S^{d-1}) \) and \( \int_{S^{d-1}} f(x) \, d\sigma(x) = 0 \), then
\[
\frac{1}{\omega_d} \int_{S^{d-1}} x_1 |f(x)|^2 \, d\sigma(x) = \sum_{n=1}^\infty \sum_{k=0}^n \gamma_k^n \sum_{1 \leq j \leq a_{n-k}^{d-1}} \hat{f}_{j,k}^n \hat{f}_{j,k}^{n+1},
\]
where
\[
\gamma_k^n := \sqrt{\frac{(2n-k+2\lambda)(k+1)}{(n+k)(n+\lambda+1)}}.
\]
Proof. Firstly, we note that, by the three term relation of the Gegenbauer polynomials (see [15], p.81, (4.7.17)), for \( x = (\cos \theta, \xi \sin \theta) \) with \( \xi \in S^{d-2} \) and \( \theta \in [0, \pi] \),
\[
x_1 P_{j,k}^n(x) = [A_k^n C_{k+1}^{n-k+\lambda} (\cos \theta) + B_k^n C_{k-1}^{n-k+\lambda} (\cos \theta)] (\sin \theta)^{n-k} Y_j^{-k}(\xi)
= A_k^n P_{j,k+1}^{n+1}(x) + B_k^n P_{j,k-1}^{n-1}(x),
\]
and
where the coefficients are given by
\[ A_n^k := \frac{k + 1}{2(n + \lambda)} \quad \text{and} \quad B_n^k := \frac{2n - k + 2\lambda - 1}{2(n + \lambda)}, \]
and we assume that \( P_{j,k}^{n-1}(x) = 0 \). In particular, this implies
\[ x_1 \text{proj}_n f(x) = \sum_{k=0}^{n} \sum_{1 \leq j \leq a_{n-k}^{d-1}} [H_k^{n}]^{-\frac{1}{2}} \hat{f}_{j,k}^n \left[ A_n^k P_{j,k+1}^{n+1}(x) + B_n^k P_{j,k-1}^{n-1}(x) \right]. \]
Consequently, by the orthogonality of \( P_{j,k}^n \), it follows that
\[ \int_{S^{d-1}} x_1 \text{proj}_n f(x) \text{proj}_m f(x) d\sigma = 0, \quad \text{unless } |m - n| = 1, \]
and that
\[
\frac{1}{\omega_d} \int_{S^{d-1}} x_1 |f(x)|^2 d\sigma = 2 \sum_{n=1}^{\infty} \frac{1}{\omega_d} \int_{S^{d-1}} x_1 \text{proj}_n f(x) \text{proj}_{n+1} f(x) d\sigma(x)
\]
\[
= \sum_{n=1}^{\infty} \frac{\int_{S^{d-1}} f(y)Z_n(x,y) d\sigma(y)}{\omega_d} \sum_{k=0}^{n} \gamma_k^n \sum_{1 \leq j \leq a_{n-k}^{d-1}} \hat{f}_{j,k}^n \hat{f}_{j,k+1}^{n+1},
\]
where the first step uses the assumption that \( \text{proj}_n f(x) = \frac{1}{\omega_d} \int_{S^{d-1}} f(x) d\sigma = 0 \). This completes the proof.

A zonal function on the sphere is a function that depends only on \( \langle x, y \rangle \), that is, a function of the form \( f_0(\langle x, y \rangle) \). It is well known that the reproducing kernel \( P_n(\cdot, \cdot) \) of \( H_n^d \) in \( L^2(S^{d-1}) \) is given by a zonal polynomial
\[ Z_n(x,y) = \frac{n + \lambda}{\lambda} C_n^\lambda(\langle x, y \rangle), \quad \lambda = \frac{d - 2}{2}, \]
which is the integral kernel of \( \text{proj}_n f \), that is,
\[ \text{proj}_n f(x) = \frac{1}{\omega_d} \int_{S^{d-1}} f(y)Z_n(x,y) d\sigma(y), \quad \forall x \in S^{d-1}. \]
For a function \( f \) defined on \([-1, 1] \), it is well known that the spherical harmonic expansion of a zonal function \( x \mapsto f(\langle x, y \rangle) \) agrees with the Gegenbauer expansion of \( f \) in \( C_n^\lambda \) with \( \lambda = \frac{d - 2}{2} \).

The connection to the Gegenbauer expansions holds for general parameters of \( \lambda \). For \( f \in L^2(w_\lambda, [-1, 1]) \) with \( w_\lambda(t) = (1 - t^2)^{\lambda - 1/2} \), the Gegenbauer expansion of \( f \) is given by
\[ f(t) = \sum_{n=0}^{\infty} \hat{f}_n^\lambda(h_n^\lambda)^{-\frac{1}{2}} C_n^\lambda(t), \quad \hat{f}_n^\lambda := (h_n^\lambda)^{-\frac{1}{2}} c_\lambda \int_{-1}^{1} f(s)C_n^\lambda(s)w_\lambda(s) ds, \]
where \( c_\lambda \) denotes the normalization constant of \( w_\lambda \), which follows from the fact that \( (h_n^\lambda)^{-\frac{1}{2}} C_n^\lambda(t) \) is orthonormal and the identity holds in the \( L^2 \) sense. As in the
proof of Lemma 2.4, we can deduce from the three-term relation of the Gegenbauer polynomials the following result:

**Proposition 2.5.** For \( \lambda > -\frac{1}{2} \) and \( f \in L^2(\omega_\lambda, [-1, 1]) \),

\[
(2.13) \quad c_\lambda \int_{-1}^{1} s|f(s)|^2 w_\lambda(s) \, ds = \sum_{n=1}^{\infty} c_n^\lambda \hat{f}_n^\lambda \hat{f}_{n+1}^\lambda.
\]

For \( \lambda = 0 \), the Gegenbauer polynomials become the Chebyshev polynomials of the first kind, or the cosine functions upon setting \( t = \cos \theta \), which correspond to the zonal functions in the case of \( S^1 \). For the Fourier series in (2.5), we have

\[
(2.14) \quad \frac{1}{2\pi} \int_0^{2\pi} (\cos \theta)|f(\theta)|^2 \, d\theta = \sum_{n=-\infty}^{\infty} \hat{f}_n \hat{f}_{n+1},
\]

which can be easily verified upon using \( \cos \theta = (e^{i\theta} + e^{-i\theta})/2 \).

### 3. The Hardy-Rellich-type inequality

Let us start with the simple case of \( S^1 \), the proof of which nevertheless indicates what is needed in the higher dimension. What we need is an inequality that can be deduced from the classical Hardy inequality. The Hardy inequality (cf. [5, p. 239, (9.8.1)]) states that for \( 1 < p < \infty \) and any sequence of real numbers \( b_n \),

\[
(3.1) \quad \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} b_k \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \left| b_n \right|^p.
\]

**Lemma 3.1.** If \( \{a_k\}_{k=1}^{\infty} \) is a sequence of real numbers, then

\[
(3.2) \quad \sum_{n=1}^{\infty} |a_n a_{n+1}| \leq \sum_{n=1}^{\infty} \left( 1 - \frac{1}{8n^2} \right) a_n^2.
\]

**Proof.** Without loss of generality, we may assume that \( a_n \geq 0 \) for all \( n \in \mathbb{N} \), and that \( \sum_{n=1}^{\infty} a_n^2 < \infty \). Setting \( a_0 = 0 \) and \( b_n = a_n - a_{n-1} \) for \( n \geq 1 \), we can rewrite (3.1) in the following equivalent form:

\[
\sum_{n=1}^{\infty} n^{-p} a_n^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} (a_n - a_{n-1})^p,
\]

which, upon setting \( p = 2 \) and using \( (a_n - a_{n-1})^2 = a_n^2 + a_{n-1}^2 - 2a_n a_{n-1} \), can be rearranged to give the desired inequality \( \Box \).

Recall that for \( f \) defined on \( S^1 \), we identify \( f(e^{i\theta}) \) with \( f(\theta) \) for \( \theta \in [0, 2\pi) \). The Hardy-Rellich inequality on \( S^1 \) takes the following form:

**Theorem 3.2.** Let \( f \in L^2(S^1) \) satisfy \( f' \in L^2(S^1) \) and \( \int_0^{2\pi} f(\theta) d\theta = 0 \). Then

\[
(3.3) \quad \int_0^{2\pi} (1 - \cos \theta)|f'(\theta)|^2 \, d\theta \geq \frac{1}{8} \int_0^{2\pi} |f(\theta)|^2 \, d\theta.
\]

Furthermore, the constant \( 1/8 \) is sharp.
Proof. The assumption implies that \( \hat{f}_0 = 0 \). Applying the inequality (3.2) to (2.14) shows that
\[
\sum_{n=-\infty}^{\infty} |\hat{f}_n \hat{f}_{n+1}| = \sum_{n=1}^{\infty} |\hat{f}_n \hat{f}_{n+1}| + \sum_{n=-\infty}^{\infty} |\hat{f}_n \hat{f}_{-n+1}| \leq \sum_{n=-\infty}^{\infty} \left( 1 - \frac{1}{8n^2} \right) |\hat{f}_n|^2,
\]
which implies, by the Parseval identity and (2.14), that
\[
\frac{1}{2\pi} \int_0^{2\pi} (1 - \cos \theta)|f(\theta)|^2 d\theta = \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2 - \sum_{n=-\infty}^{\infty} \hat{f}_n \hat{f}_{n+1} \geq \frac{1}{8} \sum_{n=-\infty}^{\infty} \frac{1}{n^2} |\hat{f}_n|^2.
\]
Applying the above inequality with \( f \) replaced by \( f' \), the stated result follows from the fact that \( \tilde{f}_n = n \hat{f}_n \) and the Parseval identity. That the constant 1/8 is sharp is proved later in Theorem 3.6. \( \square \)

We note that the condition \( \int_0^{2\pi} f(\theta)d\theta = 0 \) is necessary for the inequality (3.5), as it can be seen by setting \( f(\theta) = 1 \). Such a condition is also necessary for the Hardy-Rellich inequality on \( S^{d-1} \) for \( d \geq 2 \).

For \( d > 2 \) and \( \alpha > 0 \), we define the Sobolev space \( W^\alpha_2 \) on \( S^{d-1} \) by
\[
W^\alpha_2 := \left\{ f \in L^2(S^{d-1}) : (-\Delta_0)^{\alpha/2} f \in L^2(S^{d-1}) \right\}.
\]

**Theorem 3.3.** If \( d \geq 4 \), \( f \in W^1_2(S^{d-1}) \) and \( \int_{S^{d-1}} f(x) d\sigma(x) = 0 \), then
\[
\int_{S^{d-1}} |f(x)|^2 d\sigma(x) \leq c_d \min_{e \in S^{d-1}} \int_{S^{d-1}} (1 - \langle x, e \rangle)(-\Delta_0)^{1/2} f(x)^2 d\sigma(x),
\]
where the positive constant \( c_d \) depends only on \( d \).

**Proof.** By rotation invariance of the Lebesgue measure \( d\sigma(x) \), without loss of generality, we may assume that \( e = (1, 0, \cdots, 0) \). Let
\[
J(f) := \frac{1}{\omega_d} \int_{S^{d-1}} (1 - x_1) \left| (-\Delta_0)^{1/2} f(x) \right|^2 d\sigma.
\]
Using Lemma 2.3
\[
\frac{1}{\omega_d} \int_{S^{d-1}} x_1 \left| (-\Delta_0)^{1/2} f(x) \right|^2 d\sigma = \sum_{n=1}^{\infty} \sum_{k=0}^{n} \gamma_k^n \sum_{1 \leq j \leq a_{n-k}^d} \sqrt{n(n+2\lambda)} \hat{g}^n_{j,k} \hat{g}^{n+1}_{j,k+1},
\]
where \( \hat{g}^n_{j,k} = \sqrt{n(n+2\lambda)} \hat{f}^n_{j,k} \). The constants \( \gamma_k^n \) for \( 1 \leq k \leq n \) and \( \gamma^n_n \) can be rewritten as follows:
\[
\gamma_k^n = \sqrt{(n+\lambda+\frac{1}{2})^2 - (n+\lambda-\frac{1}{2} - k)^2}/(n+\lambda(n+\lambda+1)) \quad \text{and} \quad \gamma^n_n = \sqrt{1 - \frac{\lambda(n-1)}{(n+\lambda)(n+\lambda+1)}},
\]
which shows that \( \gamma_k^n \) is an increasing function in \( k \) and \( \gamma^n_n \) is an increasing function in \( n \) if \( \lambda \geq 1 \), or equivalently, \( d \geq 4 \). Using these facts and \( 2|\hat{g}^n_{j,k} \hat{g}^{n+1}_{j,k+1}| \leq |\hat{g}^n_{j,k}|^2 + |\hat{g}^{n+1}_{j,k+1}|^2 \), we conclude that for \( d \geq 4 \),
\[
\int_{S^{d-1}} x_1 \left| (-\Delta_0)^{1/2} f(x) \right|^2 d\sigma \leq \frac{1}{2} \sum_{n=1}^{\infty} \sum_{k=0}^{n} \gamma_k^n \sum_{j=1}^{a_{n-k}^d} \left( |\hat{g}^n_{j,k}|^2 + |\hat{g}^{n+1}_{j,k+1}|^2 \right).
\]
Consequently, we deduce easily that
\[ J(f) \geq \sum_{n=1}^{\infty} \sum_{k=0}^{n} (1 - \gamma_n^2) \sum_{j=1}^{a_{d-k}^{d-1}} |\widehat{g}_{j,k}|^2. \]

It follows from the expression
\[ (3.6) \quad 1 - \gamma_n^2 = \frac{1}{1 + \sqrt{(n+2\lambda)(n+1)}} \frac{(\lambda - 1)\lambda}{(n+\lambda)(n+\lambda+1)} \]

that \((1 - \gamma_n^2)n(n + \lambda)\) is bounded below by a constant \(c > 0\) for \(\lambda > 1\). Consequently, if \(d > 4\) then
\[ J(f) \geq c \sum_{n=1}^{\infty} \sum_{k=0}^{n} \sum_{j=1}^{a_{d-k}^{d-1}} |\widehat{f}_{j,k}|^2 = c\|f\|_{2}^2. \]

If \(d = 4\), then \(\lambda = 1\) and \(\gamma_n^2 = 1\), we use \(\gamma_k^2 \leq \gamma_n^2 = 1\) and the Cauchy-Schwartz inequality, followed by Lemma 3.1, to conclude that
\[ \frac{1}{\omega_d} \int_{S^{d-1}} x_1 \left| (-\Delta_0)^{\frac{1}{2}} f(x) \right|^2 d\sigma \leq \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n} \sum_{j=1}^{a_{d-k}^{d-1}} |\widehat{g}_{j,k}|^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{n} \sum_{j=1}^{a_{d-k}^{d-1}} |\widehat{g}_{j,k+1}|^2 \right)^{\frac{1}{2}} \]
\[ \leq \sum_{n=1}^{\infty} \left( 1 - \frac{1}{8n^2} \right) \sum_{k=0}^{n} \sum_{j=1}^{a_{d-k}^{d-1}} |\widehat{g}_{j,k}|^2, \]

which implies immediately that
\[ J(f) \geq \frac{1}{8} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n} \sum_{j=1}^{a_{d-k}^{d-1}} |\widehat{g}_{j,k}|^2 \geq \frac{1}{8} \sum_{n=1}^{\infty} \sum_{k=0}^{n} \sum_{j=1}^{a_{d-k}^{d-1}} |\widehat{f}_{j,k}|^2 = \frac{1}{8}\|f\|_{2}^2, \]

by the definition of \(\widehat{g}_{j,k}\) and the Parseval identity. \(\square\)

The above proof does not produce an optimal constant for the inequality for \(d > 4\), although we can deduce explicit expression for the constant from the proof. The case \(d = 4\) is more delicate than the case \(d > 4\), as it requires the Hardy inequality, just as the case of \(d = 2\) in Theorem 3.2. The case \(d = 3\) is left open in the above two theorems. In the following we will address the problem of optimal constant, which also answers the question on \(d = 3\). The key step lies in the case of \(L^2(w_{\lambda}, [-1,1])\), which corresponds to the zonal functions in \(L^2(S^{d-1})\) when \(\lambda = \frac{d-2}{2}\), which we consider first.

For \(\lambda > -1/2\), the norm of the space \(L^2(w_{\lambda}, [-1,1])\) is defined by
\[ \|f\|_{\lambda,2} := \left( c_{\lambda} \int_{-1}^{1} |f(t)|^2 w_{\lambda}(t) dt \right)^{1/2}. \]

The differential operator that has the Gegenbauer polynomials as eigenfunctions is defined by
\[ D_{\lambda} := (1 - t^2) \frac{d^2}{dt^2} - (2\lambda + 1) \frac{d}{dt}, \]
which is the restriction of \(\Delta_0\) on functions of the form \(f(x) = f(x_1)\) with \(x = (x_1, \ldots, x_d) \in S^{d-1}\), and
\[ D_{\lambda} C_n^{\lambda}(t) = -n(n + 2\lambda) C_n^{\lambda}(t), \quad n = 0, 1, 2, \ldots. \]
Let us also define, for $\alpha \in \mathbb{R}$,
\[
W^2_\alpha(w, [-1, 1]) := \{ f \in L^2(w, [-1, 1]) : (-D_\alpha)^\alpha f \in L^2(w, [-1, 1]) \}.
\]
We start with the following theorem.

**Theorem 3.4.** For $\lambda > -1/2$, let $f \in L^2(w, [-1, 1]) \cap W^2_\alpha(w, [-1, 1])$ satisfy $\int_{-1}^{1} f(t)w(t)dt = 0$. If $\lambda \neq \frac{1}{2}$, then
\[
\int_{-1}^{1} |f(t)|^2 w(t) dt \leq C_\lambda \int_{-1}^{1} (1 - t) \left| (-D_\lambda)^{\frac{1}{2}} f(t) \right|^2 w(t) dt,
\]
where $C_\lambda$ is a positive constant depending only on $\lambda$, and in the case when $0 \leq \lambda \leq 1$ and $\lambda \neq \frac{1}{2}$, $C_\lambda = \frac{8}{(2\lambda-1)^2}$, and it is optimal. The inequality (3.8) fails when $\lambda = \frac{1}{2}$.

**Proof.** First, we prove the result for the cases of $\lambda > 1$ and $-\frac{1}{2} < \lambda < 0$, where the optimal constant is not known and hence the proof is much easier. Let $\tilde{g}_n^\lambda = \sqrt{n(n+2\lambda)} f_n^\lambda$. Using (2.13) and $2ab = a^2 + b^2 - (a-b)^2$, we obtain
\[
c_\lambda \int_{-1}^{1} \left| (-D_\lambda)^{\frac{1}{2}} f(s) \right|^2 w(s) ds = \frac{1}{2} \sum_{n=1}^{\infty} \gamma_n^\lambda \left( |\tilde{g}_n^\lambda|^2 + |\tilde{g}_{n+1}^\lambda|^2 - |\tilde{g}_n^\lambda - \tilde{g}_{n+1}^\lambda|^2 \right)
\]
\[
= \sum_{n=1}^{\infty} \frac{\gamma_{n-1}^\lambda + \gamma_n^\lambda}{2} |\tilde{g}_n^\lambda|^2 - \frac{1}{2} \sum_{n=1}^{\infty} \gamma_n^\lambda |\tilde{g}_n^\lambda|^2 - \frac{1}{2} \sum_{n=1}^{\infty} \gamma_n^\lambda |\tilde{g}_{n+1}^\lambda|^2 \leq \sum_{n=1}^{\infty} \gamma_n^\lambda |\tilde{g}_n^\lambda|^2,
\]
where the last step uses the fact that $\gamma_n^\lambda$ is nonnegative and increasing in $n$ when $\lambda(\lambda - 1) > 0$. This implies that
\[
J_\lambda(f) := c_\lambda \int_{-1}^{1} (1 - t) \left| (-D_\lambda)^{\frac{1}{2}} f(t) \right|^2 w(t) dt
\]
\[
\geq \sum_{n=1}^{\infty} (1 - \gamma_n^\lambda) |\tilde{g}_n^\lambda|^2 = \sum_{n=1}^{\infty} \gamma_n^\lambda |\tilde{g}_n^\lambda|^2,
\]
where $\gamma_n^\lambda := (1 - \gamma_n^\lambda)n(n+2\lambda)$. Using (3.9), we may write
\[
\gamma_n^\lambda = \frac{\lambda(\lambda-1)x_n}{\sqrt{n+1}} - \frac{n}{\sqrt{n+1}},
\]
with
\[
x_n := \frac{(n+2\lambda)(n+1)}{(n+\lambda)(n+\lambda+1)} = 1 - \frac{\lambda(\lambda-1)}{(n+\lambda)(n+\lambda+1)}.
\]
Note that $x_n$ is an increasing function in $n$ when $\lambda(\lambda-1) > 0$. Since $x/(1 + \sqrt{x})$ is an increasing function for $x > 0$, it follows that
\[
\gamma_n^\lambda \geq \frac{1}{2} \frac{\lambda(\lambda-1)x_1}{1 + \sqrt{x_1}} =: C_\lambda > 0.
\]
This together with (3.9) implies the desired estimate (3.8) in the case when $\lambda > 1$ or $-\frac{1}{2} < \lambda < 0$.

Next, we prove the estimate (3.8) with the optimal constant $C_\lambda := \frac{8}{(2\lambda-1)^2}$ for $\lambda \in [0, 1]$ and $\lambda \neq \frac{1}{2}$. The proof is quite involved. It relies on an observation that
\( \gamma_n^n \) admits a factorization in the form of \( \alpha_n^n \alpha_{n+1}^\lambda \); namely, \( \gamma_n^n := \alpha_n^n \alpha_{n+1}^\lambda \), where

\[
\alpha_{2n+1}^\lambda := \sqrt{\frac{2\Gamma(n + \frac{3}{2})\Gamma(n + 1 + \lambda)}{(2n + \lambda + 1)\Gamma(n + 1)\Gamma(n + \lambda + 1)}}, \quad n = 0, 1, \ldots,
\]

\[
\alpha_{2n}^\lambda := \sqrt{\frac{2(n + \lambda + 1)}{(2n + \lambda + 1)\Gamma(n + 1)\Gamma(n + \lambda + 1)}}, \quad n = 1, 2, \ldots.
\]

Using (2.13), we have

\[
c_n \int_{-1}^1 s \left| (-D\lambda)^2 f(s) \right|^2 w_\lambda(s) ds = \sum_{n=1}^\infty \gamma_n^n \tilde{g}_n^n \tilde{g}_n^n = \sum_{n=1}^\infty \alpha_n^n \alpha_{n+1}^\lambda \tilde{g}_n^n \tilde{g}_n^n + 1 \leq \sum_{n=1}^\infty \alpha_n^n \alpha_{n+1}^\lambda \tilde{g}_n^n \tilde{g}_n^n + 1 \leq \sum_{n=1}^\infty \left( 1 - \frac{1}{8n^2} \right) |\alpha_n^n|^2 |\tilde{g}_n^n|^2.
\]

Let us define, for \( n \in \mathbb{N} \), and \( \beta_n := \alpha_n^n \),

\[
(3.12) \quad \beta_n(n) := \left(1 - \alpha_n^n - \frac{\alpha_n^n}{8n^2}\right)n(n + 2\lambda).
\]

It follows that

\[
\sum_{n=1}^\infty |\tilde{g}_n^n|^2 \leq \sum_{n=1}^\infty \sum_{n=1}^\infty \left(1 - \alpha_n^n - \frac{\alpha_n^n}{8n^2}\right) |\tilde{g}_n^n|^2 = \sum_{n=1}^\infty \beta_n(n) |\tilde{f}_n^n|^2 \geq \left( \inf_{n \geq 1} \beta_n(n) \right) \|f\|_{L^2, \lambda}^2.
\]

However, by Lemma 3.5 below,

\[
\inf_{n \geq 1} \beta_n(n) = \beta_\lambda(\infty) := \frac{(2\lambda - 1)^2}{8}, \quad \lambda \in [0, 1].
\]

This completes the proof of (3.8) for the case of \( \lambda \in [0, 1] \).

Finally, we point out that the optimality of the constant \( C_\lambda := \frac{8}{(2\lambda - 1)^2} \) and the fact that (3.8) fails for \( \lambda = \frac{1}{2} \) are contained in Theorem 3.6 below.

For convenience, we define \( n(\lambda) \) to be the smallest positive integer such that

\[
\min\{\beta_n(n) : n \geq n(\lambda)\} = \beta_\lambda(\infty).
\]

**Lemma 3.5.** The following statements hold:

(i) \( \gamma_n^n = \alpha_n^n \alpha_{n+1}^\lambda \) for all \( n \in \mathbb{N} \).
(ii) The sequences \( \{\alpha_{2n+1}^\lambda\}_{n=0}^\infty \) and \( \{\alpha_{2n}^\lambda\}_{n=1}^\infty \) are decreasing when \( 0 \leq \lambda \leq 1 \) and increasing when \( \lambda > 1 \) or \( \lambda < 0 \).
(iii) \( \lim_{n \to \infty} \beta_n(n) = \beta_\lambda(\infty) := (2\lambda - 1)^2/8 \).
(iv) For \( n \geq 3\lambda^2/2 \), \( \{\beta_n(2n+1)\}_{n=n_0}^\infty \) and \( \{\beta_n(2n)\}_{n=n_0}^\infty \) both decrease to \( \beta_\lambda(\infty) \); in particular, \( n(\lambda) \leq 3\lambda^2/2 \).
(v) \( n(1/2) = n(1) = n(2/3) = 0 \), and \( n(2) = 4 \).

The proof of this lemma quite technical and therefore is delayed till the appendix.

For convenience, we set, for a given integer \( k \in \mathbb{N} \),

\[
L_k^2(w_\lambda, [-1, 1]) := \left\{ f \in L^2(w_\lambda, [-1, 1]) : \hat{f}_j = 0, \quad 0 \leq j \leq k \right\}.
\]
Theorem 3.6. If for some \( n_0 \in \mathbb{N}_0 \) the inequality
\[
(3.13) \quad \int_{-1}^{1} |f(t)|^2 w_\lambda(t) dt \leq C \int_{-1}^{1} (1 - t) \left| (-D_\lambda)^{\frac{1}{2}} f(t) \right|^2 w_\lambda(t) dt
\]
holds for all \( f \in L^2_{n_0} \cap W^1_2(w_\lambda, [-1, 1]) \), then
\[
(3.14) \quad C \geq C_\lambda := \frac{8}{(2\lambda - 1)^2}.
\]
In particular, the inequality \((3.13)\) does not hold with a finite constant if \( \lambda = 1/2 \). Furthermore, the equality \( C = C_\lambda \) is attained if \( n_0 = n(\lambda) \).

Proof. Assume that \((3.13)\) were not true, then there would be an \( \varepsilon > 0 \) such that
\[
C^{-1} - \varepsilon > \frac{(2\lambda - 1)^2}{8} = \lim_{n \to \infty} \beta_\lambda(n),
\]
which implies that there exists a positive integer \( N_0 > n_0 \) such that
\[
\beta_\lambda(n) = n(n + 2 \lambda) \left( 1 - \alpha_n^2 + \frac{1}{8n^2} \alpha_n^2 \right) < C^{-1} - \varepsilon, \quad \forall n \geq N_0.
\]
Here and in what follows, we write \( \alpha_n \) for \( \alpha_n^\lambda \) whenever it causes no confusion. Since \( \alpha_n \sim 1 \) for \( n \) sufficiently large and \( \alpha_n \rightarrow 1 \) when \( n \to \infty \), we may choose \( N_0 \) sufficiently large so that
\[
(3.15) \quad \left( 1 - \frac{1}{C n(n + 2 \lambda)} \right) \frac{1}{\alpha_n^2} \leq \left( 1 - \frac{C^{-1} - \varepsilon}{n(n + 2 \lambda)} \right) \frac{1}{\alpha_n^2} - \frac{\varepsilon}{8n^2} \leq 1 - \frac{1}{1 + \varepsilon}
\]
whenever \( n \geq N_0 \).

Let \( \hat{b}_n \) be a sequence of nonnegative numbers such that \( \sum_{n=N_0}^{\infty} \hat{b}_n^2 < \infty \). We consider the function
\[
f(t) = \sum_{n=N_0}^{\infty} \hat{b}_n [h_n^\lambda]^{-\frac{1}{2}} C_n^\lambda(t)(n(n + 2 \lambda))^{-\frac{1}{2}}, \quad -1 \leq t \leq 1.
\]
On the one hand, \( [h_n^\lambda]^{-\frac{1}{2}} C_n^\lambda(t) \) is orthonormal in \( L^2(w_\lambda, [-1, 1]) \),
\[
\|f\|_2^2 = \sum_{n=N_0}^{\infty} \hat{b}_n^2 (n(n + 2 \lambda))^{-1}.
\]
On the other hand, since \( (-D_\lambda)^{1/2} f(t) = \sum_{n=N_0}^{\infty} \hat{b}_n [h_n^\lambda]^{-\frac{1}{2}} C_n^\lambda(t) \), using \((2.11)\) and the fact that \( \gamma_n = \alpha_n \alpha_{n+1} \), we obtain that
\[
c_\lambda \int_{-1}^{1} (1 - t) \left| (-D_\lambda)^{\frac{1}{2}} f(t) \right|^2 dt = \sum_{n=N_0}^{\infty} \hat{b}_n^2 - \sum_{n=N_0}^{\infty} \alpha_n \alpha_{n+1} \hat{b}_n \hat{b}_{n+1}.
\]
Therefore, if \((3.13)\) holds, we conclude that
\[
\sum_{n=N_0}^{\infty} \hat{b}_n^2 - \sum_{n=N_0}^{\infty} \alpha_n \alpha_{n+1} \hat{b}_n \hat{b}_{n+1} \geq C^{-1} \sum_{n=N_0}^{\infty} \frac{1}{n(n + 2 \lambda)} \hat{b}_n^2,
\]
or equivalently, setting \( \bar{g}_n = \alpha_n \hat{b}_n \), that
\[
\sum_{n=N_0}^{\infty} \left( 1 - \frac{1}{C n(n + 2 \lambda)} \right) \alpha_n^{-2} |\bar{g}_n|^2 \geq \sum_{n=N_0}^{\infty} \bar{g}_n \bar{g}_{n+1}.
\]
By (3.15), this implies that
\[ \sum_{n=N_0}^{\infty} \hat{g}_n \hat{g}_{n+1} \leq \sum_{n=N_0}^{\infty} \left( 1 - \frac{1 + \varepsilon}{8n^2} \right) |\hat{g}_n|^2, \]
which becomes, upon rearranging terms,
\[ (3.16) \quad \sum_{n=N_0}^{\infty} \frac{1 + \varepsilon}{4n^2} |\hat{g}_n|^2 \leq |\hat{g}_{N_0}|^2 + \sum_{n=N_0}^{\infty} (\hat{g}_n - \hat{g}_{n+1})^2. \]

By the definition of \( \hat{b}_n \) and the assumption on \( \hat{b}_n \), using the fact that \( \alpha_n \sim 1 \) for \( n \) sufficiently large, the inequality (3.16) holds for an arbitrary sequence of nonnegative numbers \( \hat{g}_n \) satisfying \( \sum_{n=N_0}^{\infty} |\hat{g}_n|^2 < \infty \).

Now for a given sufficiently large integer \( N \geq 2N_0 \), we define
\[ \hat{g}_n := \begin{cases} \sqrt{n}, & \text{if } N_0 \leq n \leq N; \\ \sqrt{N - \frac{n}{N} + 1}, & \text{if } N < n \leq N^2 + N; \\ 0, & \text{if } n > N^2 + N \text{ or } n < N_0. \end{cases} \]

Then, on the one hand, a direct calculation shows that
\[ \sum_{n=N_0}^{\infty} \frac{1 + \varepsilon}{4n^2} |\hat{g}_n|^2 \geq \frac{1 + \varepsilon}{4} \sum_{n=N_0}^{N} \frac{1}{n} = \frac{1 + \varepsilon}{4} \log N + O(1), \quad \text{as } N \to \infty, \]
whereas on the other hand,
\[ \sum_{n=N_0}^{\infty} (\hat{g}_n - \hat{g}_{n+1})^2 \leq \sum_{n=N_0}^{N-1} (\sqrt{n} - \sqrt{n+1})^2 + \sum_{k=0}^{N^2-1} \left( \sqrt{N - \frac{k}{N} - \sqrt{N - \frac{k+1}{N}}} \right)^2 \]
\[ \leq \sum_{n=N_0}^{N-1} \left( \frac{1}{2\sqrt{n}} \right)^2 + \sum_{k=0}^{N^2-1} \left( \frac{1}{N} \right)^2 \left( \sqrt{\frac{N - k}{N}} \right)^2 \]
\[ = \frac{1}{4} \log N + O(N^{-1} \log N) \]
as \( N \to \infty \). Therefore, by (3.16), we conclude that
\[ \frac{1 + \varepsilon}{4} \log N \leq \frac{1}{4} \log N + O(1), \]
which, however, cannot hold for sufficiently large \( N \).

We now prove sufficiency. Using the fact that \( \gamma_n^n = \alpha_n \alpha_{n+1} \), we derive from (2.13) and Lemma 3.1 that
\[ c_\lambda \int_{-1}^{1} s \left| (-D_\lambda)^{\frac{1}{2}} f(s) \right|^2 w_\lambda(s) ds = \sum_{n=n_0+1}^{\infty} \alpha_n \alpha_{n+1} \hat{g}_n \hat{g}_{n+1} \]
\[ \leq \sum_{n=n_0+1}^{\infty} \left( 1 - \frac{1}{8n^2} \right) \alpha_n^2 |\hat{g}_n|^2 \]
where \( \hat{g}_n = \hat{f}_n \sqrt{n(n+2\lambda)}. \) Consequently, for \( J_\lambda(f) \) as in (3.3),
\[ J_n(f) \geq \sum_{n=n_0+1}^{\infty} |\hat{g}_n|^2 \left( 1 - \frac{1}{8n^2} \right) \alpha_n^2 |\hat{g}_n|^2 = \sum_{n=n_0+1}^{\infty} \beta_\lambda(n) |\hat{f}_n|^2. \]
Consequently, by Lemma 3.5
\[ J_n(f) \geq \beta_\lambda(\infty) \sum_{n=n_0+1}^{\infty} |\hat{f}_n|^2 = \frac{1}{8}(2\lambda - 1)^2 \|f\|^2, \]
which is the desired inequality (3.13) with $C = C_\lambda$. \qed

**Remark 3.7.** By Theorem 3.4, Lemma 3.6 and Theorem 3.8, the Hardy-Rellich inequality (3.13) holds for $n(\lambda) = 0$ and optimal constant if $0 < \lambda \leq 1$ and $\lambda = 3/2$. The numerical computation suggests that this should be true for $1 < \lambda < \lambda_0$, where $\lambda_0 \approx 1.8258$, which requires strengthening (v) of Lemma 3.5 to $n(\lambda) = 0$ for $1 < \lambda \leq \lambda_0$.

We are now in a position to discuss the optimal constant in the Hardy-Rellich inequality on the sphere. For convenience, we set, for a given integer $k \in \mathbb{N}$,
\[ L_k^p(S^{d-1}) := \left\{ f \in L^2(S^{d-1}) : \int_{S^{d-1}} f(x)P(x)\,d\sigma(x) = 0, \quad \forall P \in \Pi_k^1 \right\}. \]

**Theorem 3.8.** The following assertion holds:

(i) For $d \geq 4$, there exists a positive integer $n(d)$, $n(d) \leq 3(d-2)^3/16$, such that for all $f \in L^2_n(S^{d-1}) \cap W^1_2(S^{d-1})$,
\[ \int_{S^{d-1}} |f(x)|^2\,d\sigma(x) \leq C_d \min_{e \in S^{d-1}} \int_{S^{d-1}} (1 - \langle x,e \rangle)(-\Delta_0)^{\frac{3}{2}} f(x)\,d\sigma(x), \]
where $C_d = \frac{8}{(d-3)^3}$ is optimal.

(ii) $n(2) = n(4) = n(5) = 0$ and $n(6) = 4$.

(iii) For $d = 3$, the inequality (3.17) fails to hold for any finite constant $C_d$.

**Proof.** As in the proof of Theorem 3.3 we may assume that $e = (1, 0, \ldots, 0)$. Since $f \in L^2_n(S^{d-1})$, $\hat{f}^n_{j,k} = 0$ for $n \leq n(d)$. Using Lemma 2.4 and the fact that $\gamma^n_k \leq \gamma^n_0$ for $0 \leq k \leq n$, we obtain
\[ \frac{1}{\omega_d} \int_{S^{d-1}} x_1 \left| (-\Delta_0)^{\frac{3}{2}} f(x) \right|^2\,d\sigma(x) \leq \sum_{n=n(d)+1}^{\infty} \sum_{k=0}^{n} \gamma^n_k \sum_{j \leq \min(n,k)} |\hat{g}^n_{j,k}||\hat{g}^n_{j,k+1}|, \]
with $\hat{g}^n_{j,k} = \sqrt{n(n+2\lambda)}\hat{f}^n_{j,k}$. In analogy to (3.7), we use $\gamma^n_n = \alpha_n \alpha_{n+1}$, the Cauchy-Schwartz inequality and Lemma 3.1 to conclude
\[ \frac{1}{\omega_d} \int_{S^{d-1}} x_1 \left| (-\Delta_0)^{\frac{3}{2}} f(x) \right|^2\,d\sigma(x) \leq \sum_{n=n(d)+1}^{\infty} \alpha_n \alpha_{n+1} \left( \sum_{k=0}^{n} \sum_{j=1}^{n-k} |\hat{g}^n_{j,k}|^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{n} \sum_{j=1}^{n-k} |\hat{g}^n_{j,k+1}|^2 \right)^{\frac{1}{2}} \leq \sum_{n=n(d)+1}^{\infty} \alpha_n^2 \left( 1 - \frac{1}{8n^2} \right) \sum_{k=0}^{n} \sum_{j=1}^{n-k} |\hat{g}^n_{j,k}|^2. \]
where Lemma 3.1 is applied on \( a_n = \alpha_n \left( \sum_{k=0}^{n} \sum_{j=1}^{\alpha_n} |g_{j,k}^n|^{2} \right)^{\frac{1}{2}} \). Hence, for \( J(f) \) defined in (3.5), we obtain

\[
J(f) \geq \sum_{n=n(d)}^{\infty} \left[ 1 - \alpha_n^2 \left( 1 - \frac{1}{8n^2} \right) \right] \sum_{k=0}^{n} \sum_{j=1}^{\alpha_n} |g_{j,k}^n|^{2}.
\]

We choose \( n(d) \) to be the integer \( n(\lambda) \) with \( \lambda = (d - 2)/2 \) in Lemma 3.5. By the definition of \( \beta_\lambda(n) \), we conclude then

\[
J(f) \geq \beta_\lambda(\infty) \sum_{n=1}^{\infty} \sum_{k=0}^{n} \sum_{j=1}^{\alpha_n} |\hat{f}_{j,k}^n|^{2} = \frac{1}{8} (d - 3)^2 \|f\|_2^2,
\]

which proves (3.17). Applying to functions of the form \( f(x_1) \) for \( x = (x_1, \ldots, x_d) \in \mathbb{S}^{d-1} \), the inequality (3.17) becomes the inequality (3.13) for the Gegenbauer weight function with \( \lambda = (d - 2)/2 \), from which the optimality of the constant follows from Theorem 3.6. This completes the proof of (i). While (ii) follows immediately from Lemma 3.9, the same argument for the optimal constant in (i) also proves (iii) by Theorem 3.6.

The proof of the above theorem can also be used to determine a constant in the Hardy-Rellich inequality. Indeed, it yields the following corollary:

**Corollary 3.9.** Let \( d \geq 4 \). If \( \tau_d := \min_{n \geq 1} \tau_\lambda(n) > 0 \), where \( \lambda = (d - 2)/2 \), then the Hardy-Rellich inequality (3.13) holds for all \( f \in L_0^2(\mathbb{S}^{d-1}) \cap W_1^2(\mathbb{S}^{d-1}) \) with \( C = \tau_d^{-1} \). In particular, \( \tau_6 = \frac{141}{128} \) and

\[
\tau_d = \beta_\lambda(1) = (d - 1) \left( 1 - \frac{7\sqrt{\pi\Gamma(\frac{d}{2})}}{4d\Gamma(\frac{d+1}{2})} \right),
\]

for \( d = 7, 8, 9, 10 \).

In fact, we only need to verify that \( \tau_d \) has the stated value. By Lemma 3.5, we only need to compare the values of \( \beta_n(\lambda) \) for \( n \leq 3\lambda^2/2 \) with that of \( \beta_\lambda(\infty) \), which can be verified numerically for small \( d \). The result shows that

\[
\tau_6 = \beta_2(2) = \frac{141}{128} < \frac{9}{8} = \beta_2(\infty),
\]

and for \( d \geq 7 \), \( \tau_d = \beta_\lambda(1) \).

We expect that the corollary holds for all \( d \geq 10 \). However, a more interesting question is that if

\[
C_d = \frac{8}{(d - 3)^2} = (\beta_\lambda(\infty))^{-1} < \tau_d^{-1}, \quad d \geq 6,
\]

is the optimal constant for the Hardy-Rellich inequality with \( f \in L_0^2(\mathbb{S}^{d-1}) \cap W_1^2(\mathbb{S}^{d-1}) \). We have proved that it is for \( d = 2, 4, 5 \). Thus, the question of finding the optimal constant remains open for \( d \geq 6 \).

4. **Uncertainty Principles**

Our uncertainty principle follows as an application of the Hardy-Rellich inequality in the previous section.
\textbf{Theorem 4.1.} Let $f \in W^1_2(S^{d-1})$ be such that $\int_{S^{d-1}} f(\sigma(y))d\sigma(y) = 0$ and $\|f\|_2 = 1$. If $d \geq 2$ then
\begin{equation}
\min_{e \in S^{d-1}} \left[ \frac{1}{\omega_d} \int_{S^{d-1}} (1 - \langle x, e \rangle)|f(x)|^2 \, d\sigma(x) \right] \|\nabla_0 f\|_2^2 \geq B_d
\end{equation}
where the constant $B_d$ is given by
\begin{equation}
B_d = (d - 1) \left( 1 - \frac{2}{\sqrt{d + 3}} \right), \quad d \geq 3,
\end{equation}
and, alternatively, for $d \neq 3$, $B_d = C_d^{-1}$ with $C_d$ being the constant in the Hardy-Rellich inequality. In particular, $B_2 = 1/8$ and $1/8$ is sharp.

\textit{Proof.} Since $\int_{S^{d-1}} f(y) \, d\sigma(y) = 0$, $(-\Delta_0)^{\frac{1}{2}}(-\Delta_0)^{-\frac{1}{2}}f = f$. Thus, using the Cauchy-Schwartz inequality, we have that
\begin{align*}
1 &= \|f\|_2^2 = \frac{1}{\omega_d} \int_{S^{d-1}} [(-\Delta_0)^{\frac{1}{2}}f(x)][(-\Delta_0)^{-\frac{1}{2}}f(x)] \, d\sigma(x) \\
&\leq \|(-\Delta_0)^{-\frac{1}{2}}f\|_2 \|(-\Delta_0)^{\frac{1}{2}}f\|_2,
\end{align*}
which, by (3.17) applied to $(-\Delta_0)^{\frac{1}{2}}f$ instead of $f$, is estimated by
\begin{equation}
C_d\|(-\Delta_0)^{\frac{1}{2}}f\|_2 \min_{e \in S^{d-1}} \int_{S^{d-1}} (1 - \langle x, e \rangle)|f(x)|^2 \, d\sigma(x), \quad d \neq 3.
\end{equation}
This together with (2.4) implies the desired inequality for $d \neq 3$. For the sharpness of the constant $B_2 = 1/8$, see (1.21) below.

Next we give a different proof of (4.1) that covers the case of $d = 3$ as well. Define the differential operators
\begin{equation}
D_{i,j} = x_i \partial_j - x_j \partial_i, \quad 1 \leq i \neq j \leq d.
\end{equation}
We shall use the following two identities about these differential operators:
(i) For $f, g \in C^1(S^{d-1})$, and $1 \leq i \neq j \leq d$,
\begin{equation}
\int_{S^{d-1}} D_{i,j} f(x) g(x) \, d\sigma(x) = -\int_{S^{d-1}} f(x) D_{i,j} g(x) \, d\sigma(x).
\end{equation}
(ii) For $f \in C^1(S^{d-1})$,
\begin{equation}
\|\nabla_0 f(x)\|^2 = \sum_{1 \leq i < j \leq d} |D_{i,j} f(x)|^2, \quad x \in S^{d-1}.
\end{equation}
These two identities can be found in \cite[Chapter 1]{1}, and they can be also easily verified by straightforward calculations.

Without loss of generality, we may assume that the minimum is achieved at $e = (1, 0, \ldots, 0)$. For convenience, we set
\begin{equation}
\begin{aligned}
r := \frac{1}{\omega_d} \int_{S^{d-1}} (1 - x_1)|f(x)|^2 \, d\sigma(x) \quad \text{and} \quad Lf := r\|\nabla_0 f\|_2^2.
\end{aligned}
\end{equation}
Using (4.3) and the fact that $D_j x_j = x_1$ for $j \geq 2$, it follows readily that

\[
\frac{1}{\omega_d} \int_{S^{d-1}} \left( \sum_{j=2}^{d} x_j D_1, j f(x) \right) f(x) \, d\sigma(x) = -\frac{d-1}{2} \frac{1}{\omega_d} \int_{S^{d-1}} x_1 |f(x)|^2 \, d\sigma(x) = -\frac{d-1}{2} (1 - r).
\]

Using (4.4) and the fact that $\|x\| = 1$, we see that

\[
\left| \sum_{j=2}^{d} x_j D_1, j f(x) \right|^2 \leq \left( \sum_{j=2}^{d} x_j^2 \right) \left( \sum_{j=2}^{d} |D_1, j f(x)|^2 \right) \leq (1 - x_1^2) \|\nabla_0 f(x)\|^2,
\]

which implies, by (4.5) and the Cauchy-Schwartz inequality,

\[
\frac{(d-1)^2}{4} (1 - r)^2 \leq \left( \frac{1}{\omega_d} \int_{S^{d-1}} \left| \sum_{j=2}^{d} x_j D_1, j f(x) \right|^2 \frac{1}{1 - x_1^2} \, d\sigma(x) \right) \times \left( \frac{1}{\omega_d} \int_{S^{d-1}} |f(x)|^2 (1 - x_1^2) \, d\sigma(x) \right) \leq \|\nabla_0 f\|^2 \frac{1}{\omega_d} \int_{S^{d-1}} |f(x)|^2 (1 - x_1^2) \, d\sigma(x).
\]

Using again $\|f\|_2^2 = 1$, the Cauchy-Schwartz inequality shows that

\[
\frac{1}{\omega_d} \int_{S^{d-1}} |f(x)|^2 x_1^2 \, d\sigma(x) \geq \left| \frac{1}{\omega_d} \int_{S^{d-1}} |f(x)|^2 x_1 \, d\sigma(x) \right|^2 = (1 - r)^2,
\]

from which it follows that

\[
\frac{1}{\omega_d} \int_{S^{d-1}} |f(x)|^2 (1 - x_1^2) \, d\sigma(x) \leq 1 - (1 - r)^2 = (2 - r) r.
\]

Thus, by (4.6), we conclude that

\[
\frac{(d-1)^2}{4} (1 - r)^2 \leq (2 - r) r \|\nabla_0 f\|^2_2 = (2 - r) Lf,
\]

or equivalently,

\[
Lf \geq \frac{(d-1)^2}{4} \frac{(1 - r)^2}{2 - r}.
\]

On the other hand, by (2.2), (4.3) and the assumption that $\int_{S^{d-1}} f(x) \, d\sigma(x) = 0$,

\[
1 = \|f\|^2_2 = \sum_{n=1}^{\infty} \|\text{proj}_n f\|^2_2 \leq \frac{1}{d-1} \sum_{n=1}^{\infty} n(n + d - 2) \|\text{proj}_n f\|_2^2 = \frac{1}{d-1} \|\nabla_0 f\|^2_2.
\]

Hence, it follows that $Lf = r \|\nabla_0 f\|^2_2 \geq (d-1) r$. Together with (4.8), we have shown that

\[
Lf \geq (d-1) \max \left\{ \frac{d-1}{4} \frac{(1 - r)^2}{2 - r}, r \right\} \geq (d-1) \min_{t \in (0, 2)} \left( \frac{d-1}{4} \frac{(1-t)^2}{2-t}, t \right).
\]

Finally, choosing $t \in (0, 2)$ such that $\frac{d-1}{4} \frac{(1-t)^2}{2-t} = t$, we obtain (4.2).
Remark 4.2. The constant $B_d$ obtained via the Hardy-Rellich inequality is $(d-3)^2/8$ for $d = 2, 4, 5$ and for the restricted class of $L^2_{n(d)}(S^{d-1}) \cap W^2_{2}(S^{d-1})$. For $d = 4, 5$ this is worse than the constant $B_d$ in \cite{14}. On the other hand, when $d \to \infty$, $B_d = d-1 + O(\sqrt{d})$ in \cite{14}, which can be improved to $B_d = n(d)d + O(\sqrt{d})$ in the restricted class of $L^2_{n(d)}(S^{d-1}) \cap W^2_{2}(S^{d-1})$, and it is worse in the order of magnitude for large $d$.

The same idea of this proof also yields the following inequality in $L^2(w_\lambda, [-1, 1])$.

Corollary 4.3. Let $\lambda > -1/2$. For $f \in W^1_{2}([-1, 1])$ such that $\int_{-1}^{1} f(y)w_\lambda(y)dy = 0$ and $\|f\|_{1, 2} = 1$, there is a positive constant $B_\lambda$ such that

\begin{equation}
\int_{-1}^{1} (1-t)|f(t)|^2w_\lambda(t)dt \geq B_\lambda,
\end{equation}

where $B_\lambda = 2 - \frac{2\sqrt{\lambda}}{\lambda}$ for $\lambda = \frac{1}{2}$, and $B_\lambda = C_{\lambda}^{-1}$ for $\lambda \neq \frac{1}{2}$ with $C_{\lambda}$ being the constant in the Hardy-Rellich inequality. In particular, for $0 \leq \lambda \leq 3/2$ and $\lambda \neq 1/2$, $B_\lambda = (2\lambda - 1)^2/8$.

The quantity on the left hand side of (4.11) is related to the following vector in $\mathbb{R}^d$.

$$
\tau(f) := \int_{S^{d-1}} x|f(x)|^2 d\sigma(x).
$$

The norm of the vector $\tau(f)$ in $\mathbb{R}^d$ is denoted by $\|\tau(f)\|$. We observe that

\begin{equation}
\|\tau(f)\| \leq \int_{S^{d-1}} |f(x)|^2 d\sigma(x) = \|f\|_2^2.
\end{equation}

Corollary 4.4. Let $f \in W^1_{2}(S^{d-1})$ be such that $\int_{S^{d-1}} f(y) d\sigma(y) = 0$ and $\|f\|_2 = 1$. If $d \geq 2$, then

\begin{equation}
(1 - \|\tau(f)\|)\|\nabla_0 f\|_2^2 \geq C_{d}^{-1}.
\end{equation}

Proof. Since $\|z\| = \max_{e \in \mathbb{R}^d} \langle z, e \rangle$ for all $z \in \mathbb{R}^d$, $\|\tau(f)\| = \max_{e \in S^{d-1}} \langle \tau(f), e \rangle$, which shows that

\begin{equation}
\|\tau(f)\| = \max_{e \in S^{d-1}} \left[ \frac{1}{\omega_d} \int_{S^{d-1}} \langle x, e \rangle |f(x)|^2 d\sigma(x) \right].
\end{equation}

Since $\|f\|_2 = 1$, it follows that

\begin{equation}
1 - \|\tau(f)\| = \min_{e \in S^{d-1}} \left[ \frac{1}{\omega_d} \int_{S^{d-1}} (1 - \langle x, e \rangle) |f(x)|^2 d\sigma(x) \right].
\end{equation}

Thus, (4.11) is an equivalent form of (4.1).

As in the case of the Hardy-Rellich inequality, the condition $\int_{S^{d-1}} f(x)d\sigma = 0$ is necessary for the uncertainty principle inequalities stated above, as can be seen by setting $f(x) = 1$. This restriction, however, can be removed to give the following new version of uncertainty principle.

Theorem 4.5. Assume that $d \geq 2$ and let $f \in W^1_{2}(S^{d-1})$ be such that $\|f\|_2 = 1$. Then

\begin{equation}
(1 - \|\tau(f)\|)\|\nabla_0 f\|_2^2 \geq cd\|\tau(f)\|.
\end{equation}
Proof. We first prove (4.14) for the case of \( d \geq 4 \). Let \( m_f \) denote the mean value of \( f \), that is, \( m_f := \frac{1}{\omega_d} \int_{S^{d-1}} f(x) d\sigma \). Then \( m_f \leq \|f\|_2 \leq 1 \). By definition, \( m_f = \text{proj}_0 f \). By Cauchy-Schwartz inequality,

\[
m_f^2 \leq \frac{1}{\omega_d} \int_{S^{d-1}} |f(x)|^2 (1 - \langle x, e \rangle) d\sigma(x) \frac{1}{\omega_d} \int_{S^{d-1}} (1 - \langle x, e \rangle)^{-1} d\sigma(x) \tag{4.15}
\]

since, for \( d \geq 4 \),

\[
\frac{1}{\omega_d} \int_{S^{d-1}} d\sigma(x) \frac{1}{1 - \langle x, e \rangle} = \frac{1}{\omega_d} \int_{-1}^1 (1 - t^2) \frac{dt}{1 - t} = \frac{2d^{-3} \Gamma(\frac{d}{2}) \Gamma(\frac{d-3}{2})}{\Gamma(d-2)} = \frac{d - 2}{d - 3}.
\]

Now define \( I_f := m_f + (-\Delta_0)^{-\frac{1}{2}} f \). Since \( \int_{S^{d-1}} (-\Delta_0)^{-\frac{1}{2}} f d\sigma = 0 \) by definition, we have

\[
\frac{1}{\omega_d} \int_{S^{d-1}} I_f(x) (m_f + (-\Delta_0)^{-\frac{1}{2}} f) d\sigma(x) \geq \|f\|_2^2 = 1.
\]

Applying the Hardy-Rellich inequality on \((-\Delta_0)^{-\frac{1}{2}} f\) and using (4.15), we deduce that

\[
1 \leq \|f\|_2^4 \leq \|I_f\|_2^4 \left\|m_f + (-\Delta_0)^{1/2} f\right\|_2^2
\]

\[
\leq c \min_{e \in S^{d-1}} \left( \frac{1}{\omega_d} \int_{S^{d-1}} (1 - \langle x, e \rangle) |f(x)|^2 d\sigma(x) \right)^{\frac{1}{2}} \left( \|\nabla_0 f\|_2^2 + m_f^2 \right). \tag{4.16}
\]

Thus, if \( |m_f| \leq 4 \|\nabla_0 f\|_2 \), then desired inequality (4.12) follows directly from (4.10) and (4.16). Thus, it remains to prove (4.14) under the additional assumption that \( |m_f| > 4 \|\nabla_0 f\|_2 \). To this end, we write \( f = m_f + g \). Since \( m_f = \text{proj}_0 f \),

\[
\|g\|_2 = \left( \sum_{n=1}^{\infty} n(n + \lambda) \|\text{proj}_n f\|_2^2 \right)^{1/2} \leq \left( \sum_{n=1}^{\infty} n(n + \lambda) \|\text{proj}_n f\|_2^2 \right)^{1/2} \leq \frac{1}{4} |m_f|,
\]

which implies that \( |m_f| = \|f - g\|_2 \geq \|f\|_2 - \|g\|_2 \geq 1 - \frac{1}{4} |m_f| \), so that \( 1 \geq |m_f| \geq \frac{1}{5} \). Since \( |f|^2 = |m_f|^2 + 2m_f g + |g|^2 \), it follows from (4.13) that

\[
1 - \|\tau(f)\| = \min_{e \in S^{d-1}} \int_{S^{d-1}} (1 - \langle x, e \rangle) |f(x)|^2 d\sigma
\]

\[
= m_f^2 + \min_{e \in S^{d-1}} \left[ -2m_f \int_{S^{d-1}} \langle x, e \rangle g(x) d\sigma + \int_{S^{d-1}} (1 - \langle x, e \rangle) |g(x)|^2 d\sigma \right]
\]

since \( \int_{S^{d-1}} g(x) d\sigma(x) = 0 \), from which it follows that

\[
1 - \|\tau(f)\| \geq m_f^2 - 2|m_f| \|g\|_2 \geq \frac{1}{2} m_f^2 \geq \frac{8}{25}.
\]
A similar argument also yields
\[
\|\tau(f)\| = \max_{e \in S^{d-1}} \int_{S^{d-1}} \langle x, e \rangle (m_f^2 + g^2 + 2m_f g) \, d\sigma
\]
\[
= \max_{e \in S^{d-1}} \left( \int_{S^{d-1}} \langle x, e \rangle g(x) d\sigma(x) + 2m_f \int_{S^{d-1}} \langle x, e \rangle g(x) d\sigma(x) \right)
\]
\[\leq (2|m_f| + 1)\|g\|_2^2 \leq 3\|\nabla_0 f\|_2^2.
\]
Thus, combining these two inequalities, we conclude that
\[
(1 - \|\tau(f)\|)\|\nabla_0 f\|_2^2 \geq \frac{8}{25}\|\nabla_0 f\|_2^2 \geq \frac{8}{25} \cdot \frac{1}{3}\|\tau(f)\|.
\]
This proves (4.14) for \(d \geq 4\). Note that the only place in the above proof where the condition \(d \geq 4\) is needed is the inequality (4.15).

Thus, it remains to prove that (4.14) holds for \(d = 2, 3\). We shall consider the case of \(d = 3\) only, as the same proof below works equally well for the case \(d = 2\).

If
\[
m_f^2 \leq 25(1 - \|\tau(f)\|),
\]
then by the remark at the end of the last paragraph, the proof for \(d \geq 4\) with slight modifications works equally well for the case \(d = 3\). Thus, it suffices to prove the assertion for \(d = 3\) under the additional assumption that
\[
(4.17) \quad m_f^2 \geq 25(1 - \|\tau(f)\|).
\]
Without loss of generality, we may assume that the supremum in (4.12) is achieved at the point \(e = (1, 0, 0) \in S^2\) so that \(1 - \|\tau(f)\| = \frac{1}{25} \int_{S^2} |f(x)|^2 (1 - x_1) \, d\sigma(x)\). Thus, (4.17) implies that
\[
1 - \|\tau(f)\| = \frac{1}{4\pi} \int_{S^2} (1 - x_1)|f(x)|^2 \, d\sigma(x)
\]
\[\leq \frac{1}{25} m_f^2 \leq \frac{1}{25}\|f\|_2^2 \leq \frac{1}{25}.
\]
By (4.8) in the proof of Theorem 4.1 with \(r = 1 - \|\tau(f)\| \leq \frac{1}{25}\), which does not require the condition that \(\int_{S^2} f(x) \, d\sigma(x) = 0\), we deduce that
\[
(1 - \|\tau(f)\|)\|\nabla_0 f\|_2^2 = Lf \geq \min_{t \in (0, \frac{1}{25})} \frac{(1 - t)^2}{2 - t} \geq c \|\tau(f)\|.
\]
This completes the proof. \(\square\)

Since, by (4.10), \(1 - \|\tau(f)\| \geq 1 - \|\tau(f)\|\) and \(\|\tau(f)\|^2 \leq \|\tau(f)\|\), it follows as a corollary of Theorem 4.5 that
\[
(4.18) \quad (1 - \|\tau(f)\|)\|\nabla_0 f\|_2^2 \geq c_d\|\tau(f)\|^2.
\]
This inequality was called the uncertainty principle on the sphere and was discussed in several papers in the literature [8, 9, 11]. The inequality (4.18) is weaker than (4.14) since it can be deduced from the latter. In fact, a simple proof of this inequality follows from our proof of Theorem 4.1.

**Corollary 4.6.** If \(f \in W^2_2(S^{d-1})\), and \(\|f\|_2 = 1\), then
\[
(4.19) \quad (1 - \|\tau(f)\|)\|\nabla_0 f\|_2^2 \geq \left(\frac{d - 1}{2}\right)^2\|\tau(f)\|^2.
\]
Proof. Using (4.12), we can assume that \( \| \tau(f) \| = \frac{1}{\omega_d} \int_{\mathbb{R}^{d-1}} x_1 |f(x)|^2 d\sigma(x) \) without loss of generality. With \( r = 1 - \| \tau(f) \| \), we can rewrite (4.8) as
\[
(2 - r)\| \nabla \tau f \|_2^2 \geq \frac{(d - 1)^2}{4} (1 - r)^2,
\]
which is the desired inequality (4.19).

The constant \( (\frac{d-1}{2})^2 \) in (4.19) was shown to be optimal in [9] by using the heat kernel defined by
\[
(4.20) \quad q^\lambda(t) := \sum_{n=1}^\infty e^{-n(n+2\lambda)t} \frac{n + \lambda}{\lambda} C_n^\lambda(s).
\]
Indeed, the computation in [9] shows that \( \tau(q^\lambda_t(s)) \| q^\lambda_t(s) \|_2 \rightarrow 1 \) as \( t \rightarrow 0^+ \), where \( \| \cdot \|_2 \) denotes the \( L^2(w_\lambda; [-1, 1]) \) norm, and
\[
\lim_{t \rightarrow 0^+} \frac{\| \sqrt{1 - \{t\}} q^\lambda_t \|_2}{\| q^\lambda_t \|_2} = \frac{1}{2} \left( \lambda + \frac{1}{2} \right) \quad \text{and} \quad \lim_{t \rightarrow 0^+} \frac{\| (-D_\lambda)^{\frac{q}{2}} q^\lambda_t \|_2}{\| q^\lambda_t \|_2} = \lambda + \frac{1}{2}.
\]
Setting \( f(x) = q^\lambda_t((x, e)) \) then shows the optimality of the constant in (4.19).

We end up this section with the following remark. Our proof of Theorem 4.1 does not lead to the optimal constants in these inequalities, since the proof based on the Hardy-Rellich inequality as well as the Hölder inequality with \( F = (-\Delta_0)^{\frac{q}{2}} f \) and \( G = (-\Delta_0)^{-\frac{q}{2}} f \), whereas the constant in the second proof is discussed in Remark 4.2. If we set \( f = q^\lambda_t/\| q^\lambda_t \|_2 \) in (4.19) and letting \( t \rightarrow 0^+ \), then we obtain \( B_\lambda \leq (2\lambda + 1)^2/8 \). In particular, for the optimal constant in \( B_d \) in (4.1), we conclude, together with Theorem 4.4, that
\[
(4.21) \quad \frac{(d - 3)^2}{8} \leq B_d \leq \frac{(d - 1)^2}{8}
\]
for \( d = 2, 4, 5 \). In particular, this shows that the constant \( B_2 = 1/8 \) is optimal for the inequality (4.1) for \( d = 2 \). Furthermore, setting \( f(x) = q^\lambda_t((x, e)) \) and letting \( t \rightarrow 0^+ \) in (4.14) shows that that the constant in (4.14) satisfies \( c_d \leq (d - 1)^2/8 \).

5. Appendix: Proof of Lemma 3.5

The item (i) of the lemma follows from a straightforward calculation. For (ii), we let
\[
\Phi_\lambda(x) = \frac{\Gamma(x + 1)\Gamma(x + 1/2 + \lambda)}{(x + \lambda/2)\Gamma(x + 1/2)\Gamma(x + \lambda)}.
\]
Then it is easy to verify that \( \Phi_\lambda(n) = \alpha_{2n}^2 \) and \( \Phi_\lambda(n + 1/2) = \alpha_{2n+1}^2 \). A direct computation shows that
\[
\frac{\Phi_\lambda(x + 1)}{\Phi_\lambda(x)} = 1 + \frac{\lambda(\lambda - 1)}{(x + \lambda)(2x + 1)(2x + \lambda + 2)},
\]
from which the monotonicity of \( \alpha_{2n} \) and \( \alpha_{2n+2} \) follows readily.

For the proof of (iii), we define
\[
(5.1) \quad \Psi_\lambda(x) = \left( 1 - \Phi_\lambda(x) + \frac{1}{32x^2} \Phi_\lambda(x) \right) x(x + \lambda).
\]
It is easy to verify then that
\[
\beta_\lambda(2n) = 4\Psi_\lambda(n) \quad \text{and} \quad \beta_\lambda(2n + 1) = 4\Psi_\lambda(n + 1/2).
\]
Proof. We consider the difference operator $\Delta$

and the following proposition.

where the function $\lambda$ is given by

Substituting this asymptotic formula into (5.1), the limit in (iii) follows readily.

To prove (iv), we rewrite, after a direct computation, that

where

Using the following formula with $c = \frac{1}{2}$ and $z = n + \frac{1+\lambda}{2}$,

\[ \frac{\Gamma(z + a + c)}{\Gamma(z + a)} = 1 + \frac{c(2a + c - 1)}{2z} + \frac{c(c - 1)[3(2a + c - 1)^2 - c - 1]}{24z^2} + O(z^{-3}) \]
as $z \to \infty$, a straightforward calculation shows that

\[ \Phi(x) = 1 + \frac{\lambda - \lambda^2 + \frac{1}{2}}{8x^2} + O(x^{-3}). \]

Substituting this asymptotic formula into (5.1), the limit in (iii) follows readily.

To prove (iv), we rewrite, after a direct computation, that

where the function $G_\lambda$ is given by

\[ G_\lambda(x) = \frac{\Gamma(x+1/2+\lambda)}{\Gamma(x+1/2)\Gamma(x+\lambda)} = \, _2F_1 \left( \frac{-1}{4}, -\frac{\lambda}{4}, 1 \right) \]
in terms of the hypergeometric function $_2F_1$. Then (iv) is a consequence of the following proposition.

**Proposition 5.1.** For $x \geq 3\lambda^3$, $\Psi_x(x + 1) < \Psi_x(x)$. In particular, $\{\beta_x(2n)\}$ and $\{\beta_x(2n + 1)\}$ are both decreasing for $n \geq 3\lambda^3/2$.

**Proof.** We consider the difference operator $\Delta f(x) = f(x + 1) - f(x)$ and $\Delta^{r+1} = \Delta^r \Delta$ for $r = 2, 3, \ldots$. From the definition, it shows

\[ \Delta^3 \Psi_x(x) = 2x + \lambda + 1 + A(x)G_\lambda(x), \]

where

\[ A(x) = \frac{(x + \lambda)(32x^2 - 1)}{16(2x + \lambda)} - \frac{x(x + \lambda + 1)(2x + 2\lambda + 1)(32x + 1)^2 - 1}{16(x + \lambda)(2x + 1)(2x + \lambda + 2)}. \]

Taking two more differences gives, with the help of a computer algebra system (we used the Mathematica), that

\[ \Delta^3 \Psi_x(x) = \frac{F_x(x)}{128(x + 2x)(2x + 2x)(4 + 2x + 2x)(6 + 2x + 2x)\Gamma(x + \lambda + 3)}, \]

where

\[ F_x(x) = -\lambda(1 + \lambda)(2 + \lambda)(4 + \lambda)(37 - 77\lambda + 37\lambda^2) \]
\[ + (-568 + 308\lambda + 1346\lambda^2 + 325\lambda^3 - 574\lambda^4 - 501\lambda^5)x \]
\[ + 4(72 + 386\lambda + 3\lambda^2 - 280\lambda^3 - 227\lambda^4 + 48\lambda^5 + 8\lambda^6)x^2 \]
\[ + 4(270 - 97\lambda - 394\lambda^2 - 251\lambda^3 + 80\lambda^4 + 104\lambda^5)x^3 \]
\[ + 16(-10 - 79\lambda - 49\lambda^2 - 8\lambda^3 + 48\lambda^4)x^4 \]
\[ + 128(-4 - 3\lambda - 2\lambda^2 + 3\lambda^3)x^5 - 128x^6. \]
We show that if $x \geq 3\lambda^3$, then $F_\lambda(x) \leq 0$ so that $\Delta^3 \Psi_\lambda(x) \leq 0$. This relies on the following expression of $F_\lambda$, computed by the Mathematica,

$$F_\lambda(x) = -128(x - 3\lambda^3)x^5 - 128(4 + 3\lambda + 2\lambda^2)(x - 3\lambda^3)x^4 - 16(10 + 79\lambda + 49\lambda^2 + 104\lambda^3 + 24\lambda^4 + 48\lambda^5)(x - 3\lambda^3)x^3 - 4(-270 + 97\lambda + 394\lambda^2 + 371\lambda^3 + 868\lambda^4 + 484\lambda^5 + 1248\lambda^6 + 288\lambda^7 + 576\lambda^8)(x - 3\lambda^3)x^2 - 4(-72 - 386\lambda - 3\lambda^2 - 530\lambda^3 + 1134\lambda^4 + 1105\lambda^5 + 2604\lambda^6 + 1452\lambda^7 + 3744\lambda^8 + 864\lambda^9 + 1728\lambda^{10})(x - 3\lambda^3)x - (568 - 308\lambda - 1346\lambda^2 - 1189\lambda^3 - 4058\lambda^4 + 465\lambda^5 - 6360\lambda^6 + 6216\lambda^7 + 13608\lambda^8 + 13260\lambda^9 + 31248\lambda^{10} + 17424\lambda^{11} + 44928\lambda^{12} + 10368\lambda^{13} + 20736\lambda^{14})x - \lambda(1 + \lambda)(2 + \lambda)(4 + \lambda)(37 - 77\lambda + 37\lambda^2).

If $x \geq 3\lambda^3$, then every term in the right hand side of the above expression is negative, so that $F_\lambda(x)$, hence $\Delta^2 \Psi_\lambda(x)$, is negative if $x \geq 3\lambda^3$. By the definition of $\Delta$, it follows that $\Delta^2 \Psi_\lambda(x) \geq \Delta\Delta^2 \Psi_\lambda(x + 1)$ for $x \geq 3\lambda^3$. Since the limit of $\Psi_\lambda(x)$ as $x \to \infty$ is finite, $\Delta^2 \Psi_\lambda(x) \to 0$ as $x \to \infty$. In particular, $\lim_{x \to \infty} \Delta^2 \Psi_\lambda(x) = 0$, so that $\Delta^2 \Psi_\lambda(x) \geq 0$ for $x \geq 3\lambda^3$. The same argument implies then $\Delta^3 \Psi_\lambda(x) \leq \Delta\Delta^2 \Psi_\lambda(x + 1) \leq 0$, which shows, in turn, that $\Psi_\lambda(x + 1) \leq \Psi_\lambda(x)$ for $x \geq 3\lambda^3$ as desired.

We further conjecture that the condition $n \geq 3\lambda^3/2$ in the above proposition is not needed for $1/2 \leq \lambda \leq 3/2$. For $\lambda = 1/2, 1, 3/2, 2$, this can be verified by evaluating $b_3(n)$ numerically, which proves (v) of Lemma 5.3.

Let us note that a more careful computation of the Proposition 5.1 shows that we could improve the condition $x \geq 3\lambda^3$ somewhat, say to $x \geq 3\lambda^3 - c\lambda^2$ for some $c > 0$. However, the region on which $\Delta^3 \Psi_\lambda(x) < 0$ is a subset of the region on which $\Psi_\lambda(x)$ is monotonically decreasing. Determining the cut-off point $x_0$ so that $\Psi_\lambda(x)$ is decreasing for $x \geq x_0$ appears to be not so easy.

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Erratum: The Hardy-Rellich Inequality and Uncertainty Principle on the Sphere

Abstract. The text below is the erratum submitted to Constructive Approximation.

Several forms of uncertainty principles on the unit sphere are established in [1]. When stated in terms of the vector

\[ \tau(f) := \frac{1}{\omega_d} \int_{S^{d-1}} x|f(x)|^2 \, d\sigma(x) \]

of \( \mathbb{R}^d \) (normalization constant \( 1/\omega_d \) was missing in [1]), our main result is in

Corollary 4.4 Let \( f \in W^1_2(S^{d-1}) \) be such that \( \int_{S^{d-1}} f(y) \, d\sigma(y) = 0 \) and \( \|f\|_2 = 1 \). If \( d \geq 2 \), then

\[ (1 - \|\tau(f)\|)\|\nabla_0 f\|^2_2 \geq C_d^{-1}. \]

Here \( C_d \) is a constant given in Theorem 4.1. We next attempted to remove the condition that \( \int_{S^{d-1}} f(y) \, d\sigma(y) = 0 \) and stated

Theorem 4.5 Assume that \( d \geq 2 \) and let \( f \in W^1_2(S^{d-1}) \) be such that \( \|f\|_2 = 1 \). Then

\[ (1 - \|\tau(f)\|)\|\nabla_0 f\|^2_2 \geq c_d \|\tau(f)\|. \]

This theorem, however, is incorrect. This was pointed out to us by Stefan Steinerberger who showed that the inequality (4.14) does not hold for the function \( f(\cos \theta, \sin \theta) = 1 + \epsilon \sin \theta \) for small enough \( \epsilon \) when \( d = 2 \). The mistake in the proof appeared on the line 6 of page 166, which states that \( \|\tau(f)\| \leq (2|m_f| + 1)\|g\|^2_2 \) but it should have been \( \|\tau(f)\| \leq \|g\|^2_2 + 2|m_f|\|g\|_2 \). As a consequence, the right hand side of (4.14) has to be replaced by \( c_d \|\tau(f)\|^2 \). Since \( \|\tau(f)\| \leq \|f\|^2_2 \), the resulted inequality is then equivalent to

\[ (1 - \|\tau(f)\|^2)\|\nabla_0 f\|^2_2 \geq c_d \|\tau(f)\|^2, \]

which was already known in the literature; see the discussion in [1] and references therein.

Since (4.14) no longer holds, an immediate question is whether the uncertainty principle in (4.11) and that in (1) are equivalent, assuming \( \int_{S^{d-1}} f(y) \, d\sigma(y) = 0 \). The following proposition shows that they are not equivalent and (4.11) is stronger than (1) for a large class of functions.

**Proposition 1.** For \( n \geq 3 \) let \( Y \in H^d_n \), a real spherical harmonic of degree \( n \) on \( S^{d-1} \), and let \( Q \) be a real polynomial of degree at most \( n-2 \) such that \( \int_{S^{d-1}} Q(x) \, d\sigma = 0 \). Assume that both \( |Y(x)|^2 \) and \( |Q(x)|^2 \) are even in every coordinate. Let

\[ f = b(Y + Q), \quad \text{where} \quad b^{-1} := \|Y + Q\|_2 > 0. \]

Then \( \tau(f) = 0 \). In particular, (1) becomes the trivial inequality \( \|\nabla_0 f\|^2_2 \geq 0 \) whereas (4.11) shows that \( \|\nabla_0 f\|^2_2 \geq c > 0 \).
Proof. Since the degree of $Q$ is at most $n - 2$, it follows from the orthogonality of $Y$ and the even parity of $Y^2$ and $Q^2$ that
\[
\int_{S^{d-1}} x_i |f(x)|^2 d\sigma = \int_{S^{d-1}} x_i \left( Y(x)^2 + 2Y(x)Q(x) + Q(x)^2 \right) d\sigma(x) = 0
\]
for $1 \leq i \leq d$. Hence, $\tau(f) = 0$. By its definition, $\|f\|_2 = 1$ and, by the orthogonality of $Y$ and the zero mean of $Q$, we see that $\int_{S^{d-1}} f(x)d\sigma = 0$ so that (4.11) is applicable to $f$. □

As a simple example of the function $f$, we can choose $Q(x) = x_1^k$ and $Y(x) = C_n^\lambda(x_1)$ for $x = (x_1, \ldots, x_d) \in S^{d-1}$, where $\lambda = (d - 2)/2$ and $1 \leq k \leq n - 2$.

Acknowledgement. The authors thank Stefan Steinerberger for pointing out the mistake in [1].

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