Near action-degenerate periodic-orbit bunches: A skeleton of chaos

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Abstract. Long periodic orbits of hyperbolic dynamics do not exist as independent individuals but rather come in closely packed bunches. Under weak resolution a bunch looks like a single orbit in configuration space, but close inspection reveals topological orbit-to-orbit differences. The construction principle of bunches involves close self-“encounters” of an orbit wherein two or more stretches stay close. A certain duality of encounters and the intervening “links” reveals an infinite hierarchical structure of orbit bunches. — The orbit-to-orbit action differences $\Delta S$ within a bunch can be arbitrarily small. Bunches with $\Delta S$ of the order of Planck’s constant have constructively interfering Feynman amplitudes for quantum observables, and this is why the classical bunching phenomenon could yield the semiclassical explanation of universal fluctuations in quantum spectra and transport.

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Introduction

Extremely unstable motion, so sensitive to perturbation that long-term prediction is impossible, such is the common notion of chaos, deterministic laws à la Newton notwithstanding. As a contrasting feature of chaos, within the continuum of unstable trajectories straying through the accessible space, there is a dense set of periodic orbits\(^1\), and these are robust against perturbations. Here we report that long periodic orbits associate to hierarchically structured bunches. — Orbits in a bunch are mutually close everywhere and yet topologically distinct. The construction principle for bunches is provided by close self-encounters where two or more stretches of an orbit run mutually close over a long (compared to the Lyapounov length) distance. The self-encounter stretches are linked by orbit pieces (“links”) of any length. Different orbits in a bunch are hardly distinct geometrically along the links in between the self-encounters, but the links are differently connected in self-encounters. Under weak resolution a bunch looks like a single orbit, and the orbits in a bunch may have arbitrarily small action differences. — Orbit bunches are a new and largely unexplored topic in classical mechanics but also spell fascination by their strong influence on quantum phenomena: Bunches with orbit-to-orbit action differences smaller than Planck’s constant are responsible for universal fluctuations in energy spectra\(^2^-^6\), as well as for universal features of transport through chaotic electronic devices\(^7\),\(^8\).

The simplest orbit bunch, exhibited in Fig. 1, is a pair of orbits differing in an encounter of two stretches (a “2-encounter”). Arrows on the two orbits indicate the sense of traversal. That elementary bunch was discovered by Sieber and Richter\(^2\),\(^3\) who realized that each orbit with a small-angle crossing is “shadowed” by one with an avoided crossing.

Bunches may contain many orbits. Anticipating the discussion below we show a multi-orbit bunch in Fig. 2 besides a 2-encounter, two 3-encounters (each involving three stretches) are active; the various intra-encounter connections are resolved only in the inset blow-ups.

As an interesting phenomenon in bunches we meet “pseudo-orbits”; Fig. 3 depicts the prototype where the replacement of a crossing by an avoided crossing entails decomposition of the original orbit into two shorter orbits; the latter are then said to form a pseudo-orbit.

The distinction of genuine periodic orbits and their pseudo-orbit partners is further illustrated in Fig. 4 for the bunch of Fig. 2.

We shall show how orbit bunches come about, illustrating our ideas for a particle moving in a two dimensional chaotic billiard, like the cardioid of Fig. 5. The particle moves on a straight line with constant velocity in between bounces and is specularly reflected at each bounce.

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**Figure 1. Cartoon of simplest orbit bunch.** One orbit has small-angle crossing which the other avoids. Difference between orbits grossly exaggerated.
Figure 2. Bunch of 72 orbits looking like a single orbit: Only blow-ups resolve intra-encounters connections distinguishing orbits; each inter-encounter link is 72-fold, with separations yet smaller than in encounters.

Figure 3. Simplest pseudo-orbit, a partner of an orbit with a 2-encounter.

Figure 4. Different intra-encounter connections of the bunch of Fig. 2
(a) One of the 24 genuine orbits, (b) one of the 48 pseudo-orbits.

Unstable initial value problem vs stable boundary value problem

To prepare for our explanation of orbit bunches it is well to recall some basic facts about chaos. We consider hyperbolic dynamics where all trajectories, infinite or periodic, are unstable (i.e. have non-zero Lyapounov rate $\lambda$): Tiny changes of the initial data (coordinate and velocity) entail exponentially growing deflections both towards the future and the past.

Turning from the initial value problem to a boundary value problem we may specify initial and final positions (but no velocity) and ask for the connecting trajectory piece during a prescribed time span. No solution need exist, and if one exists it need not be the only one. However, hyperbolicity forces a solution to be locally unique. Of foremost interest are time spans long compared to the inverse of the Lyapounov
rate. Then, slightly shifted boundary points yield a trajectory piece approaching the original one within intervals of duration $\sim 1/\lambda$ in the beginning and at the end, like $\sim \lambda^{-1}$; towards the “inside” the distance between the perturbed and the original trajectory decays exponentially. (That fact is most easily comprehended by arguing in reverse: only an exponentially small transverse shift of position and velocity at some point deep inside the original trajectory piece can, if taken as initial data, result in but slightly shifted end points.)

When beginning and end points for the boundary value problem are merged each solution in general has a cusp (i.e. different initial and final velocities) there. If the cusp angle is close to $\pi$ one finds, by a small shift of the common beginning/end, a close by periodic orbit smoothing out the cusp and otherwise hardly distinguishable from the cusped loop, like in either loop of Fig. 5.

**Self-encounters**

Fig. 5 depicts a long periodic orbit. It appears to behave ergodically, i.e. it densely fills the available space. Fig. 5 also indicates that a long orbit crosses itself many times. The smaller the crossing angle the longer the two crossing stretches remain close; if the closeness persists through many bounces we speak of a 2-encounter.

A general close self-encounter of an orbit has two or more stretches close to one another. We speak of an $l$-encounter when $l$ stretches are all mutually close throughout many bounces. For a precise definition one may pick one of the $l$ encounter stretches as a reference and demand that none of the $(l - 1)$ companions be further away than some distance $d_{\text{enc}}$; the latter “encounter width” must be chosen small compared to the billiard diameter $D$ and such that the ensuing “encounter length” $L_{\text{enc}}$ is much larger than $D$. To sum up, a close encounter is characterized by the following order of the various length scales,

$$d_{\text{enc}} \ll D \ll L_{\text{enc}} \ll L,$$

with $L$ the orbit length.

**Orbit bunches**

In the setting of Fig. 1 we can formulate an important insight into chaotic dynamics\textsuperscript{2,3}. The equations of motion allowing for the self-crossing orbit also allow for a partner
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orbit which has the crossing replaced by a narrowly avoided crossing, as shown by
the dashed line in Fig. 1. The existence of the partner follows from the shadowing
theorem\(^9\). Arguing more explicitly, we invoke the exponential stability of the boundary
value problem mentioned above. Namely, the two loops of the orbit with a crossing
may be regarded as solutions of the boundary value problem with the beginning and
end at the point of crossing. We may slightly shift apart beginning and end for each
loop while retaining two junctions, as \(\infty \rightarrow \infty\), with nearly no change for those
loops away from the junctions. By tuning the shifts we can smooth out the cusps in
the junctions, \(\infty \rightarrow \infty\), and thus arrive at the partner orbit with an avoided crossing
and reversed sense of traversal of one loop.

Throughout the links outside the encounter the two orbits are exponentially close.
The length (and thus action) difference is the smaller the narrower the encounter\(^2,3\);
its is quadratic in the crossing angle, \(\Delta L \propto \epsilon^2\).

The foregoing mechanism for generating partner orbits works for \(l\)-encounters
with any integer \(l\). Each encounter serves as a “switch”: Its \(l\) orbit stretches allow
for \(l\) different connections of the nearly fixed links outside. For a given orbit a single
\(l\)-encounter may and in a certain sense does give rise to \((l! - 1)\) partner orbits. The
“certain sense” refers to the already mentioned fact that a partner so generated may
be a pseudo-orbit, i.e. decompose into two or more shorter orbits (see Fig. 3).

A long orbit has many close self-encounters, some with \(l = 2\), some with \(l = 3\),
etc. Every such encounter gives rise to partner (pseudo-)orbits whereupon many-orbit
bunches come about; the number of orbits within a bunch acquires from each close
self-encounter the pertinent factor \(l!\). Fig. 2 illustrates a multi-orbit bunch; among the
\((3!)^2 2! = 72\) constituents there are 24 genuine periodic orbits and 48 pseudo-orbits.
In the various (pseudo-)orbits of a bunch, links are traversed in different order, and
even the sense of traversal of a link may change if time reversal invariance holds (see
Fig. 1).

Hierarchies of bunches

We would like to mention two ramifications of the concept of orbit bunches. First, orbit
bunches form hierarchical structures, due to the near indistinguishability of different
orbits of a bunch within links. Every link of a bunch may thus be considered as
an extremely close encounter of the participating (pseudo-)orbits, and reconnections
therein produce new longer (pseudo-)orbits; the length (and action) of the new
(pseudo-)orbit is approximately a multiple of that of the original one. (The Sieber-
Richter pair of Fig. 1 makes for a pedagogical example: Considering, say, the two
left links as stretches of an encounter we may switch these and thereby merge the
two orbits.) This “process” of creating ever longer orbits by selecting a link of a
bunch and treating the orbit links therein as inter-orbit encounters to switch stretches
can be continued. Each step produces action differences between (pseudo-)orbits
exponentially smaller than the previous step. A sequence of steps establishes an
infinite hierarchical structure, and we may see a duality of encounters and links as its
basis. — Second, when an orbit closely encounters an orbit from another bunch, the
encounter stretches may be switched to merge the two orbits such that the original
lengths are approximately added; clearly, the associated bunches then also unite.
Quantum signatures of bunches

The new perspective on classical chaos arose as a byproduct from work on discrete energy spectra of quantum dynamics with chaotic classical limits. As first discovered for atomic nuclei and later found for atomic, molecular, and many mesoscopic dynamics, the sequence of energy levels displays universal fluctuations on the scale of the mean level spacing. For instance, each such spectrum reveals universal statistical variants of repulsion of neighboring levels which depend on no other properties of the dynamics than presence or absence of certain symmetries, most notably time reversal invariance; correlation functions of the level density also fall in symmetry classes. A successful phenomenological description was provided by the Wigner/Dyson theory of random matrices (RMT); that theory employs averages over ensembles of Hermitian matrices modelling Hamiltonians, rather than dealing with any specific dynamical system.

Proving universal spectral fluctuations for individual chaotic dynamics was recognized as a challenge in the 1980’s. Only quite recently it has become clear that orbit bunches generated by switching stretches of close self-encounters provide the clue, within the framework of Gutzwiller’s periodic-orbit theory. The quantum mechanically relevant bunched orbits have, as already mentioned, action differences of the order of Planck’s constant. Using such bunches the validity of RMT predictions for universal spectral fluctuations of individual chaotic dynamics has been demonstrated recently\(^2\sim^6\).

Concurrently with the developing understanding of spectral fluctuations just sketched, it was realized that the role of encounters as switches is not restricted to periodic orbits but also arises for long entrance-to-exit trajectories between different leads of chaotic cavities\(^7\sim^8\),\(^10\sim^12\). Bunches of trajectories connecting entrance and exit leads could be invoked to explain universal conductance fluctuations of conductors. Like the semiclassical work on spectra, that explanation is a welcome step beyond RMT inasmuch as it applies to individual conductors rather than ensembles.

Conclusion

To conclude, bunches of periodic orbits are a hitherto unnoticed phenomenon in classical chaos, in close correspondence to universal quantum phenomena. System specific behavior in mesoscopic situations is also amenable to the new semiclassical methods\(^10\sim^12\) and we may expect further application there. A semiclassical theory of localization phenomena stands out as a challenge. — Classical applications of orbit bunches comprise action correlations among periodic orbits\(^5\),\(^13\); others could arise in a theory of the so-called Frobenius-Perron resonances which describe the approach of ergodic equilibrium for sets of trajectories. Similarly, orbit bunches can be expected to become relevant for the well known cycle expansions\(^14\) of classical observables (where an important role of pseudo-orbits was first noticed long since). — The hierarchical structure of bunches, a beautiful phenomenon in its own right, deserves further study.

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