Universal Parametric Correlations of Eigenfunctions in Chaotic and Disordered Systems

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Abstract

This paper establishes the universality of parametric correlations of eigenfunctions in chaotic and weakly disordered systems. We demonstrate this universality in the framework of the gaussian random matrix process and obtain predictions for a number of parametric correlators, one of them analytically. We present numerical evidence from different models that verifies our predictions.

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The statistical fluctuations of spectra and wave functions in complex systems are well known to conform to the predictions of random matrix theory (RMT) [1]. These systems, whose common feature is their non-integrability, range from single-particle systems exhibiting chaotic behavior [2], such as ballistic quantum dots with an irregular confining potential [3], to interacting many-particle systems, such as strongly correlated electron models [4] and atomic nuclei [5]. RMT predictions hold also for electron systems with a random impurity potential which is sufficiently weak to allow for diffusion. According to RMT, the eigenfunctions at a given point in space are gaussian random variables [1,6], whereas if the energy levels are scaled by the mean spacing, the spectral correlations become independent of the details of the system and obey the Wigner-Dyson spacing distribution.

It has recently been discovered that when these systems are allowed to depend on a parameter (e.g. an external field), the correlations between spectra belonging to different parameter values become universal upon an appropriate scaling of the parameter [7], [8], [9]. The scaling factor turned out to be the RMS of the level velocity divided by the mean spacing.

The purpose of this paper is to establish the universality of parametric correlations of the eigenfunctions in these systems. This universality is deduced within a general framework which we find suitable for this discussion: the gaussian process (GP) [10], a random matrix process corresponding to each one of Dyson’s three gaussian ensembles (GE). We then concentrate on the case of conserved time-reversal symmetry and present two parametric correlators involving the eigenfunctions. We demonstrate the universality of these correlators by comparing their RMT predictions, obtained from an appropriate GP, with results of numerical simulations for a chaotic system and a disordered system. Finally we introduce a third correlator, which we are able to calculate analytically, and thus provide an explicit example to the general discussion of scaling. The universality in this case is again verified by numerical simulations.

Dyson [11] showed that there are only three possible types of gaussian ensembles that can describe a physical system, depending on its symmetry: orthogonal (GOE), unitary (GUE)
and symplectic (GSE). When the system depends on a parameter $x$, and respects the same symmetry for all values of $x$, it has been proposed \cite{10} that its statistical properties may be described by corresponding gaussian processes termed respectively gaussian orthogonal process (GOP), gaussian unitary process (GUP), and gaussian symplectic process (GSP).

A GP is a set of random $N \times N$ matrices $H(x)$ whose elements are distributed at each $x$ according to the appropriate GE with a prescribed correlation among elements at different values of $x$:

$$H_{ij}(x) = 0,$$
$$
\overline{H_{ij}(x)H_{kl}(x')} = \frac{a^2}{2\beta} f(x, x') g^{(\beta)}_{ij,kl},
$$

(1)

where $g^{(\beta=1)}_{ij,kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$ and $g^{(\beta=2)}_{ij,kl} = 2\delta_{il}\delta_{jk}$. We will be concerned with stationary processes for which the process correlation function $f(x, x') = f(|x-x'|)$ and is normalized such that $f(0) = 1$. A GP is completely determined by its first two moments defined in (1) and has the useful property that the joint probability distribution of any finite number of matrices $H(x), H(x'), H(x''), \ldots$ is gaussian. In particular, at any $x$ we have $P[H(x)] \propto \exp[-\beta \text{Tr} H(x)^2/2a^2]$. The case $\beta = 1$ corresponds to conserved time-reversal symmetry and $H(x)$ are real symmetric, whereas for $\beta = 2$ this symmetry is broken and $H(x)$ are complex hermitean. As usual in applications of RMT we are interested in the limit $N \to \infty$.

An example of a gaussian process is Dyson’s Brownian motion model \cite{12} where $f(x) = \exp(-\gamma |x|)$. This model has been used to relate the above-mentioned spectral correlators to the spacetime correlations of the Sutherland-Calogero-Moser system \cite{13,14}.

Considering now an arbitrary two-point correlation function $c(x, x') = \overline{O(x)O(x')}$ where $O(x)$ is some observable involving the spectrum and eigenfunctions of $H(x)$, it can be shown \cite{10,13} that $c(x-x')/c(0)$ depends, apart from on $a$, on the combination $N[1-f(x-x')]$. To make a correspondence with a particular system we note that $a$ sets the mean level spacing $\Delta$ near the center of the spectrum through $a/\Delta = \sqrt{2N}/\pi$. Absorbing $a$ into $H(x)$ in (1) is thus achieved by scaling the energies $E_i$ by $\Delta$ to get the “unfolded” energies $\epsilon_i = E_i/\Delta$, leaving $c(x-x')/c(0)$ independent of $a$. Next, for $x'$ near $x$ we expand
\[ f(x - x') \approx 1 - \kappa (x - x')^2 \] (2)

and, following [8], we consider the variance of the level velocity \((\partial \epsilon_i / \partial x)^2\). Writing

\[ \partial E_i / \partial x \approx \langle \psi_i(x) \mid H(x') - H(x) \mid \psi_i(x) \rangle / (x' - x) \]

where \(| \psi_i(x) \rangle\) are the eigenstates of \(H(x)\), we calculate its variance using the joint two-matrix distribution \(P[H(x), H(x')]\). The calculation is performed in two steps. First, using the conditional distribution for \(H(x')\) given \(H(x)\) which can be shown to be

\[ P[H(x') \mid H(x)] \equiv P[H(x), H(x')] / P[H(x)] \]

\[ \propto \exp \left\{ -\beta \text{Tr} \left[ H(x') - fH(x) \right]^2 / 2a^2 (1 - f^2) \right\} \] (3)

with \(f \equiv f(x - x')\), we average over \(H(x')\) keeping \(H(x)\) fixed to get

\[ \left( \frac{\partial E_i}{\partial x} \right)^2 \approx \frac{1}{(x - x')^2} \left[ \frac{a^2}{\beta} (1 - f^2) + (1 - f)^2 E_i(x)^2 \right] . \] (4)

Second, taking the limit \(x' \to x\) and using (2), the second term on the r.h.s. of (4) vanishes and we obtain the following expression for the non-universal quantity \(\kappa\):

\[ \kappa = \beta \pi^2 \left( \frac{1}{N} \frac{\partial \epsilon_i}{\partial x} \right)^2 . \] (5)

Thus after the scaling

\[ x \to \bar{x} = \left( (\partial \epsilon_i / \partial x)^2 \right)^{1/2} x \] (6)

we get

\[ f \approx 1 - \beta \frac{\pi^2}{4} \frac{(\bar{x} - \bar{x}')^2}{N} , \] (7)

and all correlators, being determined by the combination \(N(1 - f)\), become universal as functions of \(\bar{x} - \bar{x}'\).

A few remarks are in order. First, the scaling (3) is identical to that found in [8] for spectral correlators. Second, this scaling was derived here under the assumption that the second derivative of \(f\) is the first non-vanishing one (see [2]). This is not always the case.
as exemplified by Dyson’s Brownian motion model for which \( f \approx 1 - \gamma \ | \ x \ | \). The more general case is discussed elsewhere [10]. Third, the form (6) of \( f \) implies that the typical correlation length \( x \) scales like \( \frac{1}{\sqrt{N}} \) in the GP, as was shown in [14] and will be seen explicitly in the analytical expression for the correlator \( \tilde{c}(\omega, x - x') \) derived below. Fourth, and most importantly, we note that previous treatments of parametric correlations had to invoke the supersymmetry method [8] or Dyson’s Brownian motion model [13] and could demonstrate universality of spectral correlators only (however, see [16]). In the GP framework, on the other hand, the universality of all correlators emerges quite simply. In particular, correlators involving the eigenfunctions are universal.

We shall demonstrate the last point by investigating two such quantities, the averaged parametric overlap

\[
o(x - x') = | \langle \psi_i(x) | \psi_i(x') \rangle |^2
\]

and the projection correlator

\[
p(x - x') = \langle \phi | \psi_i(x) \rangle \langle \phi | \psi_i(x') \rangle.
\]

Both measure the decorrelation of wave functions as their separation along \( x \) increases. \( o(x - x') \) gives the overlap of wavefunctions at different \( x \), whereas \( p(x - x') \) provides the correlation between their components along a fixed normalized vector \( | \phi \rangle \). To see that \( p(x - x') \) is independent of the choice of \( | \phi \rangle \), notice that had we chosen instead \( | \phi' \rangle = U | \phi \rangle \) for some unitary \( U \), we could have rotated \( H(x) \) by \( U^\dagger \) at each \( x \) without affecting the probability measure (see (3)), thereby recovering the original result for \( p(x - x') \). In particular, for \( | \phi \rangle = | r \rangle \) the projection correlator describes the correlation between eigenfunctions belonging to different values of \( x \) at a given space point \( r \): \( p(x - x') = \overline{\psi_i^\tau(r) \psi_i^\tau(r)} \). For the orthogonal case \( \psi_i^\tau(r) \) are real and determined up to a sign, which is easy to keep fixed as \( x \) varies.

We cannot provide analytical expressions for these correlators. However, since they are universal we can use any gaussian process to obtain a theoretical prediction for them from a random matrix simulation. A simple GOP is generated by \( H(x) = H_1 \cos x + H_2 \sin x \),
where $H_1, H_2$ are independent GOE matrices. This process is stationary with $f(x - x') = \cos(x - x')$ and the scaling (9) is readily found to be $x \to \bar{x} = (\sqrt{2N}/\pi)x$. The theoretical curves, generated by a simulation of random matrices with $N = 150$, are given by the dashed lines in Fig. 1.

To verify the universality we studied $o(x - x')$ and $p(x - x')$ in both a chaotic system and a disordered system. The first is the interacting boson model (IBM), a many-body system used to describe collective states of medium and heavy mass nuclei [17]. Its constituents are bosons which model nucleon pairs coupled to angular momentum of 0 or 2. Depending on two parameters $\chi, \eta$ the IBM can be integrable or non-integrable. The time-dependent mean field equations, obtained in the limit of an infinite number of bosons and which constitute the classical limit, are correspondingly regular or chaotic [18]. We have calculated the above correlators as a function of $\chi$ in the regime $\eta = 0, -0.8 < \chi < -0.5$ where the system is almost fully chaotic. Since the total spin $J$ of the nucleus is a conserved quantum number, we can study the correlations for different values of the spin. Results for $J = 2$ and $J = 6$ with 25 bosons are displayed in the right panel of Fig. 1.

The second system we studied was the Anderson model, a two-dimensional lattice Hamiltonian with on-site disorder and nearest-neighbor hopping. The site energies $W_i$, measured in units of the hopping term, are uniformly distributed in $[-W/2, W/2]$ where $W$ controls the transition between the diffusive and localized regimes. We considered the cases of cylindrical geometry with $W = 4$ and toroidal geometry with $W = 2$, introducing a parametric dependence by adding a step potential of strength $x$ along one of the lattice directions [8]. Results for a $27 \times 27$ lattice are presented on the left panel of Fig. 1. Both correlators in both systems are in excellent agreement with the GOP prediction.

The last part of this paper introduces a third correlator which we are able to calculate analytically using Efetov’s supersymmetry method [19]. This correlator, related to the parametric overlap (8) but involving both energies and eigenfunctions, is given by

$$\tilde{o}(\omega, x - x') = \frac{\sum_{ij} \left| \langle \psi_i(x) | \psi_j(x') \rangle \right|^2 \delta(\epsilon_i(x) + \omega - \epsilon_j(x'))}{\sum_{ij} \delta(\epsilon_i(x) + \omega - \epsilon_j(x'))}. \quad (10)$$
It measures the averaged parametric overlap of eigenfunctions whose corresponding energies are separated by \( \omega \) in units of the mean spacing. We now outline the calculation of the numerator of (10), denoted \( o(\omega, x - x') \) (details will be published elsewhere [15]). In order to employ the supersymmetry method we first express it in terms of Green functions: 

\[
o(\omega, x - x') = \frac{1}{2\pi} \text{ReTr} \left[ G(\epsilon^-, x)G(\epsilon^+ + \omega, x') - G(\epsilon^-, x)G(\epsilon^-, \omega, x') \right]
\]

where \( \epsilon^\pm = \epsilon \pm i\delta \). Each Green function is then written as an integral over \( 4N \)-dimensional graded vector \( \Psi \). After performing the GOP averaging, a Hubbard-Stratonovich transformation and a subsequent integration over the \( \Psi \)-variables result in an integral over a \( 16 \)-dimensional graded matrix \( R \). This integration is carried out in the saddle-point approximation, which is exact in the infinite-\( N \) limit. In this limit the quadratic corrections decouple and we are left with an integral over the saddle-point manifold:

\[
o(\omega, x - x') = \frac{1}{2\Delta^2} \text{Re} \int D[Q] P[Q] \exp(F[Q])
\]  

(11)

where \( Q \) is a \( 8 \times 8 \) graded matrix and

\[
P[Q] = \text{Trg}(QS_{bb}^{12})\text{Trg}(QS_{bb}^{21})
\]

\[
F[Q] = i\pi(\omega + 2i\delta)\text{Trg}(Q\Lambda) + \frac{1}{16}N\kappa(x - x')^2\text{Trg}[Q, \Lambda]^2
\]

(12)

Using Efetov’s parametrization of \( Q \) [19] most of the integrals are elementary and the result is given by

\[
o(\omega, x - x') = \frac{1}{2\Delta^2} \text{Re} \int D[\lambda] P[\lambda] \exp(F[\lambda])
\]

(13)

where

\[
\int D[\lambda] \equiv \int_1^\infty \int_1^\infty \int_{-1}^1 \frac{(1 - \lambda^2)d\lambda_1 d\lambda_2 d\lambda}{(\lambda_1^2 + \lambda_2^2 + \lambda^2 - 2\lambda_1 \lambda_2 \lambda - 1)^2}
\]

\[
P[\lambda] = 2\lambda_1^2 \lambda_2^2 - \lambda_1^2 - \lambda_2^2 - \lambda^2 + 1
\]

\[
F[\lambda] = i\pi(\omega + 2i\delta)(\lambda_1 \lambda_2 - \lambda) - N\kappa(x - x')^2(2\lambda_1^2 \lambda_2^2 - \lambda_1^2 - \lambda_2^2 - \lambda^2 + 1)
\]

(14)

Note the combination \( N(x - x')^2 \) in \( o(\omega, x - x') \) which explicitly suggests the \( \sqrt{N} \) scaling of \( x \) mentioned above. The scaling (3) removes the dependence on the system-specific \( \kappa \) through
the substitution $N\kappa(x - x')^2 = \frac{x^2}{4}(\bar{x} - \bar{x}')^2$. Our result for the denominator of (10), after scaling, becomes identical to the level density correlator calculated in [8]. An expression for $\tilde{o}(\omega, x - x')$ can similarly be derived for the case of broken time-reversal symmetry by GUP-averaging the Green function product [15].

We verified our derivation of $\tilde{o}(\omega, x - x')$ by a comparison with a GOP simulation and confirmed the universality of this correlator by studying it in the above two cases of the Anderson model. We remark that the expression in (13)-(14) corresponds to a regularization of (10) by convoluting both the numerator and the denominator with a Lorentzian of width $\delta$. The same operation, which amounts to smearing the $\delta$-functions, was performed in the numerical computations. The results are displayed in Fig. 2 for $\delta = 0.05$ in units of the mean spacing and various values of $\omega$ and are in excellent agreement with the GOP prediction.

In conclusion, we have shown that the concept of the gaussian process is particularly suitable for a discussion of parametric correlations in chaotic and disordered systems and for demonstrating their universality. We established the existence of universal parametric correlations of eigenfunctions in these systems, provided predictions for three such correlators, and verified them in different models.

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FIGURES

FIG. 1. The eigenfunction correlators $o(\bar{x} - \bar{x'})$ (top) and $p(\bar{x} - \bar{x'})$ (bottom) as a function of the scaled parameter (see Eqs. (6),(8) and(9)). The dashed lines are the GOP prediction obtained from simulations of $H(x) = H_1 \cos x + H_2 \sin x$ with $N = 150$, using the middle third of the spectrum. On the left are calculations in the IBM in its chaotic regime, including 80 states out of 117 for $J = 2$ (pluses) and 130 states out of 184 for $J = 6$ (squares). On the right are calculations in the Anderson model in its diffusive regime, including the middle 200 states out of 729. Results are given for a cylindrical geometry with $W = 4$ (circles) and for a toroidal geometry with $W = 2$ (crosses).

FIG. 2. The eigenfunction correlator $\tilde{o}(\omega, \bar{x} - \bar{x'})$ (see Eq. (10)) as a function of the scaled parameter for several values of $\omega$ measured in units of the mean spacing. The solid lines are the analytical results (see Eqs. (13)-(14)), the dashed lines are the GOP simulations, and the circles and crosses are the results for the same cases of the Anderson model as in Fig. 1. The $\delta$-functions in (10) are regularized with a Lorentzian of width $\delta = 0.05$ in units of the mean spacing.
