ON AN APPARENTLY BILINEAR INEQUALITY
FOR THE FOURIER TRANSFORM

MICHAEL CHRIST

Abstract. A bilinear inequality of Geba et al. for the Fourier transform is shown to be
equivalent to a simpler linear inequality, and the range of exponents is extended. Related
mixed-norm inequalities are discussed.

This note is a comment on a bilinear inequality of Geba et al. [2]. Our thesis is that a
simpler and purely linear inequality underlies the bilinear inequality. This reasoning extends
the bilinear inequality to the full range of allowable exponents. The linear inequality itself
is not new; indeed, it is proved in [2] as a corollary of the bilinear one.

1. Linear versus bilinear

View $\mathbb{R}^d$ as $\mathbb{R}^{d'} \times \mathbb{R}^{d''}$ with coordinates $x = (x', x'')$ and $\xi = (\xi', \xi'')$. Consider the Fourier
transform

$$\hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) \, dx = \int e^{-2\pi i x' \cdot \xi'} \int e^{-2\pi i x'' \cdot \xi''} F(x', x'') \, dx'' \, dx'. $$

Consider mixed-norm spaces $L^p L^s = L^p_{x'} L^s_{x''}$ with norms $\|F\|_{p,s} = \|F\|_{L^p_{x'} L^s_{x''}}$ defined by
first taking the $L^s$ norm with respect to $x''$ for each fixed $x'$, then the $L^p$ norm of the
resulting function of $x'$. The spaces $L^s_{x''} L^p_{x'}$, in which the order in which the individual
norms is taken is reversed, also play a part in our discussion. All inequalities in this note
can be regarded as a priori inequalities for Schwartz functions.

Define

$$C_r = r^{1/2} t^{-1/2t}$$

where $t = r' = r/(r - 1)$ is the exponent conjugate to $r$. Then $C_r < 1$ for all $r \in (1, 2)$. Beckner [1] has shown that $C^n_r$ is the optimal constant in the Hausdorff-Young inequality.

That is, $\|\hat{f}\|_{L^p(\mathbb{R}^n)} \leq C^p_r \|f\|_{L^p(\mathbb{R}^n)}$ for all $p \in [1, 2]$ and all $n \geq 1$, and the inequality holds
with no smaller constant factor on the right-hand side.

The linear inequality in question is as follows.

**Proposition 1.** For all $p \in [1, 2]$ and all $F$,

$$\|\hat{F}\|_{L^p(\mathbb{R}^d)} \leq C^p_r \|F\|_{p,1}. $$

**Proof.** Define

$$f(x') = \int F(x', x'') \, dx''.$$

If $F \in L^p L^1$ then $f \in L^p(\mathbb{R}^{d'})$ and $\|f\|_{L^p} \leq \|F\|_{p,1}$. Moreover

$$\hat{F}(\xi', 0) = \int e^{-2\pi i x' \cdot \xi'} F(x', x'') \, dx'' \, dx' = \int e^{-2\pi i x' \cdot \xi'} f(x') \, dx' = \hat{f}(\xi').$$

Date: August 25, 2015.

Research supported by NSF grant DMS-1363324.
It now suffices to invoke the Hausdorff-Young inequality. □

The bilinear inequality of [2] states that for all Schwartz functions,
\[(2) \quad \|\hat{F}G\|_{L^r} \leq C\|F\|_{p,s}\|G\|_{q,t}\]
for all \((p, s; q, t; r)\) satisfying
\[(3) \quad s^{-1} + t^{-1} = 1 \quad \text{and} \quad r^{-1} = 1 - p^{-1} - q^{-1}\]
with the supplementary restrictions \(p, q, r \geq 2\) and \(s = t = 2\). The \(L^r\) norm is that of \(\mathbb{R}^d\). The relations \((3)\) are necessary for the inequality to hold, as observed in [2]. It is likewise necessary that \(r \geq 2\).

The next result states that inequality \((2)\), for the full range of possible exponents satisfying \((3)\), with no supplementary restrictions, follows from the linear inequality \((1)\). Conversely, \((1)\) follows from \((2)\) for the more limited range of exponents treated in [2], as already noted in [2].

**Corollary 2.** If \(p, q \in [1, \infty), \quad r \in [2, \infty), \quad s, t \in [1, \infty),\) and \((p, s, q, t, r)\) satisfies \((3)\) then the inequality \((2)\) holds with \(C = C_{p,q}^r\).

**Proof.** If \(F \in L^pL^s\) and \(G \in L^qL^t\) then by Hölder’s inequality, \(FG \in L^uL^v\) where \(u^{-1} = p^{-1} + q^{-1}\) and \(v^{-1} = s^{-1} + t^{-1}\). The assumption \(r \geq 2\) is equivalent to \(u \in [1, 2]\), while \(s^{-1} + t^{-1} = 1\) means that \(v = 1\). □

2. Mixed norm inequalities

A mixed norm inequality \(\|\hat{F}\|_{p', \infty} \leq C\|F\|_{p, 1}\) would not only imply \((1)\), but would be dramatically stronger. Therefore it is natural to ask whether such inequalities are valid. We next discuss inequalities with mixed norms for \(\hat{F}\) on the left-hand side.

Given a function of \((y', y'')\), one can form mixed norms \(\|F\|_{L^p_{y'}L^t_{y''}}\), in which an \(L^t\) norm is taken of \(y'' \mapsto F(y', y'')\) for fixed \(y'\), and then the \(L^q\) norm of the resulting function of \(y''\) is taken. One can equally well reverse the order, \(\|F\|_{L^p_{y''}L^q_{y'}}\), taking the \(L^q\) norm first with respect to \(y''\) and then the \(L^t\) norm with respect to \(y'\). These two quantities are related by Minkowski’s integral inequality:
\[(4) \quad \|F\|_{L^p_{y'}L^t_{y''}} \leq \|F\|_{L^t_{y'}L^q_{y''}}, \quad \text{whenever} \quad t \geq q.\]

**Proposition 3.** For all \(p, s \in [1, 2]\),
\[(5) \quad \|\hat{F}\|_{L^p_{\xi'\xi}}L^s_{\xi''} \leq C_{p}^d \mathbf{C}^d_{s} \|F\|_{L^p_{x'x}L^s_{x'x}}.\]

Note that the order of the norms on the left-hand side is reversed; the \(L^p_{x'x}\) norm is taken first.

**Proof.** Regard the Fourier transform for \(\mathbb{R}^d\) as the composition of a partial Fourier transform in the second variable, that is, in \(\mathbb{R}^{d'}\), followed by a partial Fourier transform in the first variable. The first operation maps \(L^p_{x'}L^s_{x''}\) to \(L^p_{x}L^s_{x'}\) with operator norm \(\mathbf{C}^d_{s}\). Since \(p \leq 2 \leq s'\), \((1)\) says that the space \(L^p_{x}L^s_{x'}\) is contained in \(L^p_{x'}L^s_{x}\) with a contractive inclusion map. The second operation maps this last mixed-norm space to \(L^s_{\xi''}L^p_{\xi'}\), with operator norm \(\mathbf{C}^d_{p}\). □

\(^1\) There are typographical inaccuracies in the formulas relating exponents in inequalities (1.10) and (1.11) of [2].
It is likewise natural to seek inequalities with \( \| \hat{F} \|_{L^{p'}_\xi L^{s'}_\eta} \) on the left-hand side; now there is no reversal of the ordering. Such an inequality with \( s = 1 \) would be far stronger than (I). For certain pairs of exponents, such inequalities do hold, as direct consequences of Proposition 3.

**Corollary 4.** If \( p, s \in [1, 2] \) and \( p \leq s \) then

\[
\| \hat{F} \|_{p', s'} \leq C_p^d C_s^{d''} \| F \|_{p, s}.
\]

**Proof.** \( p' \geq s' \), so by (I), \( \| \hat{F} \|_{L^{p'}_\xi L^{s'}_\eta} \leq \| \hat{F} \|_{L^{s'}_\xi L^{p'}_\eta} \). Invoke Proposition 3.

However, the extension of Corollary 4 to \( s < p \) breaks down, no matter how large a constant factor is allowed on the right-hand side.

**Proposition 5.** Let \( d', d'' \) both be strictly positive integers, and let \( s < p \). Then the Fourier transform fails to map \( L^p_d L^s_d(\mathbb{R}^d \times \mathbb{R}^{d''}) \) boundedly to \( L^{p'}_\xi L^{s'}_\eta(\mathbb{R}^d \times \mathbb{R}^{d''}) \).

**Sketch of proof.** First consider \( s = 1 \). Then \( s' = \infty \) and if such an inequality were valid then by a limiting argument one could choose \( F(x', x'') = f(x') \delta_{x''=\psi(x')}, \) for an arbitrary measurable function \( \psi : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d''} \), to conclude that

\[
\| \int_{\mathbb{R}^{d''}} e^{-2\pi i \phi(x') \psi(x'')} e^{-2\pi i \xi' \cdot x'} f(x') \, dx' \|_{p'} \leq C \| f \|_p,
\]
uniformly for all \( f \in L^p(\mathbb{R}^{d'}) \) and all Lebesgue measurable functions \( \phi, \psi : \mathbb{R}^{d'} \rightarrow \mathbb{R}^{d''} \). This is absurd, as choosing \( \phi(x') \equiv \xi' \) and \( \psi(x') \equiv -x' \) makes plain.

Next consider the case \( 1 < s < p \leq 2 \). To simplify notation consider the case \( d' = d'' = 1 \). Let \( f, g : \mathbb{R}^1 \rightarrow \mathbb{C} \) be Schwartz functions that do not vanish identically. For \( t \in \mathbb{R}^+ \) consider \( F(x, y) = f_t(x)g(y - x) \) where \( f_t(x) = t^{1/p} f(tx) \). Then

\[
\hat{F}(\xi, \eta) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f_t(x) \int_{\mathbb{R}} e^{-2\pi i \eta y} g(y - x) \, dy \, dx
\]

\[
= \hat{g}(\eta) \int_{\mathbb{R}} e^{-2\pi i (\xi - \eta) t} t^{1/p} f(tx) \, dx
\]

\[
= \hat{g}(\eta) t^{-1/p'} \hat{f}(t^{-1}(\xi - \eta)).
\]

It is straightforward to verify that as \( t \rightarrow 0 \), the \( L^{p'}_\xi L^{s'}_\eta \) norm of this function has order of magnitude \( t^{-1/p'} t^{1/s'} \). If \( s < p \), \( 1/s' < 1/p' \) and consequently \( t^{-1/p'} t^{1/s'} \rightarrow \infty \) as \( t \rightarrow 0 \).

The case of arbitrary dimensions can be reduced to this construction for \( \mathbb{R}^1 \times \mathbb{R}^1 \) by consideration of functions that are products of the individual coordinates, with \( f_t \) defined by dilation with respect to a single coordinate and \( g(y - x) \) replaced by \( g(x'' - v x'') \) for some nonzero vector \( v \in \mathbb{R}^{d''} \) and some coordinate \( x'' \) of \( x' \).

The constant factors on the right-hand sides of the inequalities in the first four results stated above are all optimal. To demonstrate this, it suffices to consider product functions \( F(x', x'') = g(x') h(x'') \) and to invoke the optimality of the constants in Beckner’s inequality.

The formulations and reasoning extend to a product of any two locally compact Abelian groups, with norms defined with respect to Haar measure and Fourier transforms normalized so as to be unitary from \( L^2 \) to \( L^2 \) of the dual groups, and contractive from \( L^1 \) to \( L^\infty \) of the dual groups. The constants \( C_p^d \) and \( C_p^{d''} C_{d''}^{s'} \) in the above results are replaced by the
optimal constant in the Hausdorff-Young inequality for the first factor, and by the product of the optimal constants for the two factor groups, respectively.

References

[1] W. Beckner, *Inequalities in Fourier analysis*, Ann. of Math. (2) 102 (1975), no. 1, 159–182

[2] D. Geba, A. Greenleaf, A. Iosevich, E. Palsson and E. Sawyer, *Restricted convolution inequalities, multilinear operators and applications*, Math. Res. Lett. 20 (2013), no. 4, 675-694.

Michael Christ, Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

E-mail address: mchrist@berkeley.edu