An estimation of the greedy algorithm’s accuracy for a set cover problem instance

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Abstract. For the set cover problem, by modifying the approach that leads to the proof of the logarithmic approximation guarantee for the greedy algorithm, we obtain an estimation of the greedy algorithm’s accuracy for a given input. We show that, for a wide share of the problem instances, the accuracy of the greedy algorithm may be estimated much better than the common logarithmic approximation guarantee suggests.

Keywords: set cover problem, greedy algorithm.

1 The set cover problem

In the set cover problem (SCP), we have a set $U = \{1, \ldots, m\} = [m]$, and such a collection of its subsets $S = \{S_1, \ldots, S_n\}$ that

$$\bigcup_{i=1}^{n} S_i = U.$$ 

The collection of sets $S' = \{S_{i_1}, \ldots, S_{i_l}\}$, $S_{i_j} \in S$, is called a cover of $U$ if

$$\bigcup_{j=1}^{l} S_{i_j} = U.$$ 

We have a weight function $w : S \to \mathbb{R}_+$ ($\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$), $w_i = w(S_i)$ is a weight of the set $S_i$. The weight of the collection of sets $S' = \{S_{i_1}, \ldots, S_{i_l}\}$ is equal to the sum of weights of the sets that it contains:

$$w(S') = \sum_{j=1}^{l} w(S_{i_j}).$$

To solve the problem, we must find an optimal cover of $U$, i.e. we need to find the cover of $U$ that has minimum weight.

An instance $\mathcal{P}$ of SCP is an SCP with predefined $U$, $S$ and $w$. Let $A$ be an approximate algorithm for SCP. An approximation guarantee of $A$ is the value $\rho_A(m)$ such that, for every instance $\mathcal{P}$ that may be defined on $U = [m]$, we have

$$\frac{w(Cvr)}{w(Opt)} \leq \rho_A(m),$$
where \( Cvr \) is a cover obtained by \( A \) for the instance \( P \) and \( Opt \) is an optimal solution of \( P \).

SCP is \( NP \)-hard \([1]\). In \([2]\), it has been shown that, whenever \( P \neq NP \) holds, \( \rho_A(m) > (1 - o(1)) \ln m \) for any approximate algorithm \( A \) with polynomial complexity. In \([3,4]\), there have been presented another inapproximability results for SCP which exclude the possibility of a polynomial time approximation with better than logarithmic approximation guarantee.

There is the greedy algorithm for polynomial time approximation of SCP with complexity \( O(m^2n) \). For positive real valued weights, it holds \([5]\) that

\[
\frac{w(Gr)}{w(Opt)} \leq H(m) \leq \ln m + 1, \tag{1}
\]

where \( Gr \) is the cover that the algorithm produces and \( H(m) = \sum_{k=1}^{\ln m} 1/k \). For a particular instance of the problem, we have

\[
\frac{w(Gr)}{w(Opt)} \leq H(\tilde{m}), \tag{2}
\]

where \( \tilde{m} = \max\{|S_i| : S_i \in Opt\} \) is specified by the instance. Since SCP is \( NP \)-hard, instead of \( \tilde{m} \), we must use the value of \( \bar{m} = \max\{|S_i| : S_i \in S\} \) in order to obtain more tight upper bound on the ratio \( w(Gr)/w(Opt) \). For non-weighted case of SCP, the upper and lower bounds on the approximation guarantee have been obtained in \([6]\). It has been shown that, for the worst case \( P \) that may be specified on \( U = [m] \), we have

\[
T_l(m) < \frac{w(Gr)}{w(Opt)} < T_u(m), \tag{3}
\]

where \( T_l(m) = \ln m - \ln \ln m - 0.31, T_u(m) = \ln m - \ln \ln m + 0.78 \).

Considering a particular instance of the problem, we show how to estimate the ratio \( w(Gr)/w(Opt) \) more precisely than the common logarithmic approximation guarantee suggests. The presented estimation is more accurate than \( H(\tilde{m}) \) for majority of SCP instances for which the value of \( \tilde{m} \) is large enough. The estimation may be used to obtain the lower bound on the optimal cover weight and so it can be applied to branch-and-bound strategies for the problem.

2 The estimation of the greedy algorithm’s accuracy for an instance of the problem

Implementing the greedy algorithm, we take the sets from \( S \) into \( Gr \) relying on the values of charged weights \( w_i/|S_i| \), i.e. using the values of the weights that \( S_i \) charges to yet uncovered elements of \( U \) at the moment we choose a set to include into \( Gr \).
The greedy algorithm for SCP

Step 0. \( Gr := \varnothing \).

Step 1. If \( Gr \) is a cover of \( U \), then stop the algorithm, else go to the step 2.

Step 2. Choose such \( k \) that

\[
\frac{w_k}{|S_k|} = \min \left\{ \frac{w_i}{|S_i|} \mid S_i \in S, S_i \not\subset \bigcup_{S_j \in Gr} S_j \right\},
\]

(4)

\( Gr := Gr \cup \{S_k\} \). \( S_i := S_i \setminus S_k \) for all \( S_i \not\in Gr \). Go to the Step 1.

Let it takes \( l \) iterations of the greedy algorithm to cover \( U = [m] \) for the instance \( P \) of SCP. Let the algorithm covers \( s_k \) elements of \( U \) on its \( k \)-th iteration and let \( m_k \) denotes the number of yet uncovered elements of \( U \) after the \( k \)-th iteration is completed, \( m_0 = m \), and let \( s = \{s_1, \ldots, s_l\} \).

Theorem.

\[
\frac{w(Gr)}{w(Opt)} \leq G(s) = H(m) - \Delta(P),
\]

(5)

where

\[
G(s) = \sum_{k=1}^l \frac{s_k}{m_{k-1}}.
\]

\( \Delta(P) \geq 0 \). \( \Delta(P) = 0 \) if and only if it takes \( m \) iterations of the greedy algorithm to obtain a cover of \( U \).

The estimation \( G(s) \) is a refinement of the estimation (1). We prove (5) by modifying the well known proof of the logarithmic approximation guarantee of the greedy algorithm. For example, it is presented in [7]. Instead of majorization of the weights that sets from \( Gr \) charge to distinct elements of \( U \), we majorize the weights of sets from \( Gr \) itself. Doing this, we obtain the estimation (5) for an instance of SCP and this estimation appears to be more accurate than \( H(\bar{m}) \) for a wide share of SCP instances.

Suppose that, before implementing the \( k \)-th iteration of the greedy algorithm, we have the subset \( U_k \) of the elements of \( U \) that are yet uncovered, \( m_k = |U_k| \). After implementing the Step 2 on the previous iterations, all of the modified sets in \( S \) contain only elements of \( U_k \). Let us suppose that, in accordance with (4), the greedy algorithm chooses the set \( S_k \) at the \( k \)-th iteration. Then, as it is will shown further,

\[
\frac{w(S_k)}{|S_k|} \leq \frac{w(Opt_k)}{m_k},
\]

where \( Opt_k \) is an optimal cover of \( U_k \) that we may obtain using the modified sets from \( S \). We use this inequality to prove (5). It also used to prove (1) and (2),
but we prove the different statement that deals with another subject. We don’t searching for the worst case instance of the problem to obtain an estimation of the greedy algorithm’s accuracy but we estimate the accuracy of the greedy algorithm for a particular instance of the problem taking into account the algorithm operating on the instance. And, as a result, we obtain the different value of the bound on $w(Gr)/w(Opt)$.

For an instance $P$ of SCP with $U = [m]$, let the ordered collection of sets $\{S_1, S_2, \ldots, S_l\}$ be the cover $Gr$. Let $Opt = \{A_1, A_2, \ldots, A_r\}$, $A_i \in S$, be an optimal cover and let $a_i = w(A_i)$.

**Lemma.**

$$\frac{w(S_1)}{|S_1|} \leq \frac{w(Opt)}{m}.$$

**Proof.** Let

$$A'_1 = A_1, \quad A'_2 = A_2 \setminus A_1, \ldots, \quad A'_r = A_r \setminus \bigcup_{j=1}^{r-1} A_j.$$

The sets $A'_j$ are pairwise disjoint. Let us renumber the sets $A'_j$ in accordance with nondecreasing order of ratios $a_j/|A'_j|$. Thus we have

$$\frac{a_1}{|A'_1|} \leq \frac{a_2}{|A'_2|} \leq \ldots \leq \frac{a_r}{|A'_r|}.$$

Since, for positive $a, b, c, d$, holds

$$\frac{a}{c} \leq \frac{b}{d} \Rightarrow \frac{a}{c} \leq \frac{a + b}{c + d},$$

then

$$\frac{a_1}{|A'_1|} \leq \frac{\sum_{j=1}^{r} a_j}{\sum_{j=1}^{r} |A'_j|} = \frac{w(Opt)}{m}.$$

Taking into account (4), we have

$$\frac{w(S_1)}{|S_1|} \leq \frac{a_1}{|A_1|} = \frac{a_1}{|A'_1|} \leq \frac{w(Opt)}{m}.$$

Let us prove the Theorem now.

**Proof.** Let us consider a selection of a set in $Gr$ at the $k$-th iteration as a selection of the first set to cover the part of $U$ that is yet uncovered before the $k$-th iteration. That is to say that we consider a new instance of SCP with
modified sets. The weights of the sets are the same as initially. Let the greedy algorithm choose the set \( \hat{S}_k \) on the \( k \)-th iteration of its operating, where

\[ \hat{S}_k = S_k \setminus \bigcup_{j=1}^{k-1} S_j. \]

Since \( s_k = |\hat{S}_k| \), using the proven Lemma, for every iteration, we have

\[ w_1 \leq \frac{s_1}{m_0} w(\text{Opt}), \quad w_2 \leq \frac{s_2}{m_1} w(\text{Opt}_2), \ldots, \quad w_l \leq \frac{s_l}{m_{l-1}} w(\text{Opt}_l). \]

Since \( w(\text{Opt}_1) \leq \ldots \leq w(\text{Opt}_2) \leq w(\text{Opt}) \), it holds that

\[ w(\text{Gr}) = \sum_{k=1}^{l} w_k \leq \frac{s_1}{m_0} w(\text{Opt}) + \frac{s_2}{m_1} w(\text{Opt}_2) + \ldots + \frac{s_l}{m_{l-1}} w(\text{Opt}_l) \leq \]

\[ \left( \frac{s_1}{m_0} + \frac{s_2}{m_1} + \ldots + \frac{s_l}{m_{l-1}} \right) w(\text{Opt}) = \]

\[ = \left( H(m) - H(m) + \frac{s_1}{m_0} + \frac{s_2}{m_1} + \ldots + \frac{s_l}{m_{l-1}} \right) w(\text{Opt}) = (H(m) - \Delta(\mathcal{P})) w(\text{Opt}), \]

where

\[ \Delta(\mathcal{P}) = H(m) - \left( \frac{s_1}{m_0} + \frac{s_2}{m_1} + \ldots + \frac{s_l}{m_{l-1}} \right). \]

Assuming \( m_l = 0 \) and, since \( m_k > m_{k+1} \) for \( k < l \), summing over \( i \) in decreasing order, we obtain

\[ \Delta(\mathcal{P}) = \sum_{k=1}^{l} \left( \sum_{i=m_{k-1}}^{m_k+1} \frac{1}{i} \right) - \frac{s_k}{m_{k-1}}. \]  \hspace{1cm} (6)

Since

\[ \Delta(\mathcal{P}) = \sum_{k=1}^{l} \left( \sum_{i=m_{k-1}}^{m_k+1} \frac{1}{i} \right) = \sum_{k=1}^{l} \left( \sum_{i=m_{k-1}}^{m_k+1} \frac{1}{i} \right) - \frac{m_k+1}{m_k-1} \]

\[ = \sum_{k=1}^{l} \sum_{i=m_{k-1}}^{m_k+1} \left( \frac{1}{i} - \frac{1}{m_k-1} \right), \]

we have \( \Delta(\mathcal{P}) \geq 0 \) and \( \Delta(\mathcal{P}) = 0 \) if and only if \( m_{k-1} = m_k + 1 \), i.e. if \( s_k = 1 \) for all of \( k = 1, l \). \( \blacksquare \)

3 Estimating of the greedy algorithm’s accuracy for an instance of the set cover problem

Let \( \mathcal{P} \) be an instance of SCP with \( U = [m], S \subseteq 2^U \). By the Theorem, for the instance \( \mathcal{P} \), we may estimate the ratio \( w(\text{Gr})/w(\text{Opt}) \) using \( G(s) \), where \( s \) is a
Let us compare \( G(s) \) to \( H(m) \) for all of the instances of the SCP that may be defined on \( U = [m] \) for \( m = 10, 35 \).

For every sequence \( s = \{s_1, \ldots, s_l\} \), there exists such a class \( C_s \) of instances of SCP that the greedy algorithm produces \( s \) on these instances. For example, the greedy algorithm produces a particular sequence \( s \) if the instance of SCP has the following form. Let \( q_i = \sum_{j=1}^l s_j, i = 2, l, q_1 = 0 \). Let \( S = \{S_1, \ldots, S_l, A\} \), where \( S_i = \{q_i + 1, \ldots, q_i + s_i\}, A = \{1, 2, \ldots, m\}. \) Let \( w(S_i) = |S_i| = s_i \) for \( i = 1, l-1 \), \( w(S_l) = |S_l| + 1, w(A) = m + \varepsilon, 0 < \varepsilon < 1 \). Using the greedy algorithm, we obtain the cover which consists of sets \( S_1, \ldots, S_l \) for the SCP. The weight of this cover is equal to \( m + 1 \) while the weight of the optimal cover (the weight of the set \( A \)) is equal to \( m + \varepsilon \). So, as a result of the greedy algorithm operating, we have the sequence \( s \).

| \( m \) | \( \mu \in (0, 0.2] \) | \( \mu \in (0.2, 0.4] \) | \( \mu \in (0.4, 0.6] \) | \( \mu \in (0.6, 0.8] \) | \( \mu \in (0.8, 1] \) |
|---|---|---|---|---|---|
| 10 | 0 | 13.5 | 64.8 | 100 | 100 |
| 11 | 0 | 12.4 | 54.7 | 92.5 | 100 |
| 12 | 0 | 9.9 | 52.9 | 97.5 | 100 |
| 13 | 0 | 13.9 | 64.5 | 93.3 | 100 |
| 14 | 0 | 9.1 | 56.1 | 96.2 | 100 |
| 15 | 0 | 14.5 | 69.6 | 99.0 | 100 |
| 16 | 0 | 12.9 | 62.6 | 96.2 | 100 |
| 17 | 0 | 11.6 | 58.7 | 98.3 | 100 |
| 18 | 0 | 12.4 | 65.0 | 95.7 | 100 |
| 19 | 0 | 11.0 | 61.2 | 97.7 | 100 |
| 20 | 0 | 19.5 | 67.4 | 99.1 | 100 |
| 21 | 0 | 15.5 | 63.2 | 97.2 | 100 |
| 22 | 0 | 11.8 | 59.6 | 98.7 | 100 |
| 23 | 0 | 11.1 | 64.7 | 96.7 | 100 |
| 24 | 0 | 10.1 | 61.3 | 98.1 | 100 |
| 25 | 0 | 18.6 | 68.5 | 99.2 | 100 |
| 26 | 0 | 16.9 | 64.2 | 97.8 | 100 |
| 27 | 0 | 15.1 | 59.7 | 98.8 | 100 |
| 28 | 0 | 14.1 | 65.9 | 97.4 | 100 |
| 29 | 0 | 12.0 | 62.8 | 98.5 | 100 |
| 30 | 0.5 | 17.2 | 67.2 | 99.2 | 100 |
| 31 | 0.2 | 15.8 | 64.0 | 98.1 | 100 |
| 32 | 0.1 | 14.5 | 61.0 | 98.9 | 100 |
| 33 | 0 | 13.5 | 56.0 | 97.8 | 100 |
| 34 | 0 | 12.5 | 62.2 | 98.6 | 100 |
| 35 | 0.9 | 15.8 | 66.9 | 99.3 | 100 |

*Table 1.* The share of sequences \( s \) for which \( G(s) < H(m(s)) \).
| $m$ | $\mu \in (0, 0.2]$ | $\mu \in (0.2, 0.4]$ | $\mu \in (0.4, 0.6]$ | $\mu \in (0.6, 0.8]$ | $\mu \in (0.8, 1]$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
|     | mean $\Delta$  | max $\Delta$    | mean $\Delta$  | max $\Delta$    | mean $\Delta$  | max $\Delta$    |
| 10  | 0.0           | 12.3           | 18.4           | 21.3           | 42.9           | 30.9           | 55.8           | 53.3           | 65.9           |
| 11  | 0.0           | 10.9           | 8.9            | 14.9           | 40.6           | 27.9           | 53.2           | 45.5           | 66.9           |
| 12  | 0.0           | 11.0           | 12.0           | 19.1           | 45.4           | 28.6           | 55.8           | 47.3           | 67.8           |
| 13  | 0.0           | 10.9           | 8.9            | 14.9           | 40.6           | 27.9           | 58.0           | 48.8           | 68.6           |
| 14  | 0.0           | 10.9           | 8.9            | 14.9           | 40.6           | 27.9           | 59.8           | 50.1           | 69.2           |
| 15  | 0.0           | 13.9           | 30.6           | 20.7           | 50.5           | 31.1           | 61.3           | 51.3           | 69.9           |
| 16  | 0.0           | 13.1           | 28.6           | 19.7           | 49.2           | 28.3           | 59.7           | 45.8           | 70.4           |
| 17  | 0.0           | 11.7           | 26.8           | 19.1           | 51.8           | 29.7           | 61.2           | 47.0           | 70.9           |
| 18  | 0.0           | 12.8           | 33.6           | 20.0           | 50.7           | 28.0           | 62.4           | 48.0           | 71.4           |
| 19  | 0.0           | 12.4           | 32.0           | 19.5           | 52.9           | 29.2           | 63.5           | 48.9           | 71.8           |
| 20  | 0.0           | 12.7           | 37.5           | 20.8           | 54.9           | 30.4           | 64.5           | 49.8           | 72.2           |
| 21  | 0.0           | 12.2           | 36.0           | 19.8           | 54.0           | 28.2           | 63.4           | 45.5           | 72.6           |
| 22  | 0.0           | 12.3           | 34.8           | 19.2           | 55.7           | 29.2           | 64.3           | 46.4           | 72.9           |
| 23  | 0.0           | 13.5           | 39.3           | 19.9           | 54.9           | 27.6           | 65.2           | 47.1           | 73.2           |
| 24  | 0.0           | 13.2           | 38.1           | 19.4           | 56.4           | 28.6           | 65.9           | 47.8           | 73.5           |
| 25  | 0.0           | 12.9           | 42.0           | 19.8           | 57.8           | 29.6           | 66.6           | 48.5           | 73.8           |
| 26  | 0.0           | 12.3           | 40.9           | 19.2           | 57.1           | 27.7           | 65.8           | 45.0           | 74.1           |
| 27  | 0.0           | 12.0           | 39.9           | 19.0           | 58.4           | 28.6           | 66.5           | 45.7           | 74.3           |
| 28  | 0.0           | 12.3           | 43.2           | 19.2           | 57.7           | 27.1           | 67.1           | 46.3           | 74.5           |
| 29  | 0.0           | 12.2           | 42.3           | 18.7           | 58.9           | 27.9           | 67.7           | 46.9           | 74.8           |
| 30  | 2.7           | 6.8            | 13.1           | 45.2           | 19.4           | 59.9           | 28.7           | 68.2           | 47.4           | 75.0           |
| 31  | 2.2           | 5.5            | 12.7           | 44.4           | 18.8           | 59.4           | 27.2           | 67.5           | 44.5           | 75.2           |
| 32  | 1.7           | 4.3            | 12.4           | 43.6           | 18.4           | 60.4           | 28.0           | 68.1           | 45.0           | 75.4           |
| 33  | 1.3           | 3.1            | 12.4           | 46.2           | 18.9           | 59.9           | 26.6           | 68.6           | 45.5           | 75.5           |
| 34  | 0.9           | 2.0            | 12.2           | 45.4           | 18.5           | 60.8           | 27.3           | 69.0           | 46.0           | 75.7           |
| 35  | 5.5           | 11.9           | 13.3           | 47.7           | 18.8           | 61.6           | 28.0           | 69.4           | 46.5           | 75.9           |

Table 2. Improvement of the greedy algorithm’s accuracy estimation.

Since $m(s) \leq \tilde{m}$ for any $P \in C_s$, the value of $H(m(s))$ is not larger than the value of $H(\tilde{m})$. Thus, counting the number of the instances of $s$ for which $G(s) < H(\tilde{m})$ holds, we obtain a numeric lower bound on the number of instances $s$ for which $G(s) < H(m(s))$ for all $P \in C_s$. And thus, for given $m$, we estimate the share of the classes $C_s$ such that $G(s) < H(\tilde{m})$ for $P \in C_s$.

Let $\mu(s) = m(s)/m$. In the Table 1, according to $\mu(s)$ that belongs to predefined intervals, we show the shares of the classes $C_s$ for which the estimation $G(s)$ appears to be more accurate than the estimation $H(m(s))$. Going through all of the possible instances of $s$ for $m = 10, 35$, we have found that the share of such classes grows as the value of $\mu(s)$ grows. And it also shows that such a share tends to grow as $m$ grows.

The Table 2 shows how much the estimation $G(s)$ may be more accurate than $H(m(s))$ for different values of $\mu(s)$. For such $s$ that $G(s) < H(\tilde{m})$, let

$$\Delta(s) = (H(m(s)) - G(s))/H(m(s)) \times 100,$$
i.e. $\Delta(s)$ is an improvement of $G(s)$ over $H(m(s))$ in percents. We present the mean and the maximum values of the improvements for $m = 10, 35$ and for different values of $\mu(s)$. While the value of $H(m(s))$ belongs to the interval $[2.93, 4.15]$ for all of the sequences $s$ that may be defined on $[m]$, $10 \leq m \leq 35$, it may be seen that, for large enough values of $\mu(s)$, the improvement can be tens of percents.

Conclusions

Considering the set cover problem, we estimate the accuracy of the greedy algorithm for the given input. We show that a logarithmic approximation guarantee may be too generic for an instance of the problem since it supposes the worst case while the instance may be not that hard. The accuracy of the greedy algorithm may be estimated more precisely if we take into account the algorithm operating on the instance.

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