Many-body $q$-exponential distribution prescribed by factorization of joint probability

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The factorization problem of $q$-exponential distribution within nonextensive statistical mechanics is discussed on the basis of Abe’s general pseudoadditivity for equilibrium systems. It is argued that the factorization of compound probability into product of the probabilities of subsystems is nothing but the consequence of existence of thermodynamic equilibrium in the interacting systems having Tsallis entropy. So the factorization does not needs independent noninteracting systems and should be respected in all exact calculations concerning interacting nonextensive subsystems. This consideration makes it legitimate to use $q$-exponential distribution either for composite system or for single body in many-body systems. Some known results of ideal gases obtained with additive energy are reviewed.

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I. INTRODUCTION

In thermostatistics, the factorization of the compound probability into product of single body probabilities

$$\rho = \prod_{n=1}^{N} \rho_n,$$  \hspace{1cm} (1)

where $\rho$ is density operator and $N$ is the number of bodies in the system, is crucial for applications of statistical mechanics to many-body systems and for the statistical interpretation of thermodynamics. Boltzmann-Gibbs statistics ($BGS$), in considering only short
range interactions, makes this factorization a natural result of its exponential distribution with additive energy. In nonextensive statistical mechanics (NSM) \[1\], intended originally to describe complex systems with long range interactions or fractal structure of space-time, this factorization is not as evident as in BGS. From the beginning of this nonextensive theory, in order to show the nonextensive character of NSM, one has supposed Eq.(1) for a system containing statistically independent subsystems and obtained for the total entropy \(S_q\) \[1\]:

\[
\ln[1 + (1 - q)S_q] = \sum_{n=1}^{N} \ln[1 + (1 - q)S_q(n)]
\]

where \(q\) is the parameter of Tsallis entropy \(S_q = -\text{Tr} \rho \frac{\rho^q}{1-q}\) (Boltzmann constant \(k_B = 1\)) [for \(N = 2\), \(S_q = S_q(1) + S_q(2) + (1 - q)S_q(1)S_q(2)\) as one often finds in the literature]. As indicated by Tsallis \[1\], Eq.(2) expresses in fact the additivity of Rényi entropy \(S_R = -\frac{\text{Tr} \rho^q}{1-q}\) if Eq.(1) applies. Due to this supposed “independence” of subsystems for NSM, it has been believed by many that exact calculations \[3–8\] should use the additive hamiltonian \(H_0\) given by

\[
H_0 = \sum_{n=1}^{N} H_n,
\]

where \(H_n\) is the hamiltonian of \(n^{th}\) subsystem. However, this additive hamiltonian is compatible with neither Eq.(1) nor Eq.(2) since these relations applied to \(q\)-exponential distribution (qed) \(\rho \propto [1 - (1 - q)\beta H]^{-\frac{1}{q}}\), as given by maximization of Tsallis entropy under some constraints \[1,7,9\], imply \[1,10,11\]:

\[
H = \sum_{n=1}^{N} H_n + \sum_{k=2}^{N} [(q - 1)\beta]^{k-1} \sum_{n_1 < n_2 < \ldots < n_k} \prod_{j=1}^{k} H_{n_j}
\]

\[
= H_0 + H_c,
\]

where \(\beta\) is the inverse temperature. So in order to keep simultaneously Eq.(3) and Eq.(1) and to apply NSM to many-body systems using one-body qed, a so called factorization approximation is proposed \[12\] with the assumption that the second term on the right hand side of Eq.(3) may be neglected. This approximation has been, explicitly or not, employed in
the applications of NSM to the cases like, among others, quantum particle systems [12,14], turbulent flows [11,15], the polytropic model of galaxies, solar neutrinos, peculiar velocity of galaxy clusters, electron plasma (for updated comments on some of these works, we refer to reference [13] and the references there-in). In these calculations, it is admitted that one-body qed applies. Although these applications clearly shows the usefulness and necessity of one-body qed, the approximation [12] neglecting the correlation energy with sometimes weak interacting dilute particles [17] is, on the contrary, not a reassuring basis. Recently, it has been argued [10] that the correlation energy ($H_c$) given by the second term of Eq.(4) is in general not negligible. A numerical result for $N$-oscillator system [8] shows that the partition function given by using Eq.(3) is completely different from that given by using Eq.(1) when $N$ is large. We are facing with a difficult question : which one of Eq.(1) and Eq.(3) should be related to independence of subsystems? In a recent work, Beck proposed a nontrivial idea to define an “independence” according to Eq.(4) [17] so we can write Eq.(1) and one-particle qed without any approximation. This means that we address correlated “independent systems” satisfying Eq.(4). This idea surely needs to be justified.

In this letter, the probability factorization problem of NSM is discussed from a different consideration inspired by the general pseudoadditivity of entropy [20] and energy [21] required by the existence of thermodynamic equilibrium. It will be argued that Eq.(1) needs not independence of noninteracting systems and is to be considered as a basic assumption of NSM. Eq.(3) is only a kind of extensive approximation and should be employed carefully. On this basis, some basic applications of NSM to classic and quantum gases are revisited.

II. THERMAL EQUILIBRIUM AND FACTORIZATION OF COMPOUND PROBABILITY

Apart from one-particle qed, another issue tightly related to probability factorization is the establishment of zeroth law and the definition of temperature for NSM. This is obviously of central importance for the theory. Eq.(3) and noninteracting model due to Eq.(1) are
so deeply accepted that it was even believed that the zeroth law of thermodynamics was absent within NSM [18], since Eq.(2) does not hold with additive Hamiltonian, and without Eq.(2), we can not talk about the zero law and temperature! Recently, a series of works have been published on this issue [19] claiming the establishment of zeroth law and the definition of a generalized temperature on the basis of additive Hamiltonian Eq.(3) and, I stress it, Eq.(2). If this zeroth law is established by neglecting the correlation energy $H_c$, it is merely approximate. If it is exact for noninteracting systems, then the paradox between Eq.(3) and Eq.(1) [thus Eq.(2)] persists due to qed. In addition, other questions arise: if the zeroth law holds only for, e.g. noninteracting particle systems, why do we have to discard BGS? And what is the origin of the nonextensivity?

This confused situation and the above questions are, in our opinion, simply due to the fact that Eq.(1) is not clearly founded for NSM. It certainly implies independence of noninteracting systems for BGS. But does it mean the same thing for NSM?

Very recently, Abe [20] proposed a general pseudoadditivity for entropy required by the existence of thermal equilibrium in composite nonextensive systems: $f(S) = f(S_1) + f(S_2) + \lambda f(S_1)f(S_2)$ where $f$ is certain differentiable function satisfying $f(0) = 0$ and $\lambda$ a constant depending on the nature of the system of interest. So for a system containing $N$ subsystems, the thermal equilibrium requires following pseudoadditivity:

$$\ln[1 + \lambda f(S)] = \sum_{n=1}^{N} \ln[1 + \lambda f(S_n)].$$

On the other hand, Eq.(3) applied to Tsallis entropy means $f(S) = S$ and $\lambda = 1 - q$ [20], which directly leads to $\ln Tr p^g = \sum_{i=1}^{N} \ln Tr p_i^q$ or Eq.(1) (i.e. $(p_ip_j)^q = p_{ij}^q$ means $p_ip_j = p_{ij}$ where $p_i$ is the probability of state $i$). So this probability factorization must be regarded as a condition or a consequence of thermodynamic equilibrium, instead of statistical independence. It should be raised to the rank of basic assumption for equilibrium thermodynamics with Tsallis entropy and must be rigorously respected by all exact calculations. In this way, the zeroth law becomes evident and a temperature can be straightforwardly defined with maximum entropy and minimum energy [3,10,22]. According to above discussions, all cal-
calculations based on Eq.(1) or using one-body qed are, as a matter of fact, exact applications of NSM. And all calculations based on additive energy Eq.(3) should now be considered as a kind of “extensive approximation” and should be employed with great care.

Energy has been proved [21] to satisfy the same kind of pseudoadditivity at equilibrium as Eq.(5). If we choose \( f(H) = H \) and \( \lambda = (q - 1)\beta \), we get

\[
\ln[1 + (q - 1)\beta H] = \sum_{n=1}^{N} \ln[1 + (q - 1)\beta H_n]
\]

which is just Eq.(4) satisfying Eq.(1).

Here I would like to mention that, Eq.(3) or Eq.(4) contains inverse temperature \( \beta \). This surely has something to do with the fact that this relation is subject to thermal equilibrium. The dependence of the correlation energy \( H_c \) on temperature is something of nontrivial and needs explanation. A possible argument is that \( H \) is only an effective Hamiltonian so the effects of interactions or correlations represented by \( q \) perhaps depends on the state of the system and thus on temperature. If this is true, we may expect \( T \)-dependent \( q \) values.

In the following section, I will discuss some interesting consequences of Eq.(3).

### III. ADDITIVE Q-DEFORMED ENERGY

Let us begin by introducing an additive “energy” for NSM for the sake of convenience. We refer to

\[
h = \frac{\ln(1 + (q - 1)\beta H)}{(q - 1)\beta}
\]

as \( q \)-deformed hamiltonian and recast Eq.(3) into

\[
h = \sum_{n=1}^{N} h_n.
\]

This means following transformations:

\[
H = \frac{e^{(q-1)\beta h} - 1}{(q - 1)\beta}, \quad H_n = \frac{e^{(q-1)\beta h_n} - 1}{(q - 1)\beta}
\]

and
\[
\rho \propto [1 + (q - 1) \beta H]^{1/(1-q)} = e^{-\beta h} = \prod_n e^{-\beta h_n}. \tag{10}
\]

It is interesting to see that the \textit{qed} can be transformed into Boltzmann-Gibbs exponential distribution with a deformed energy. The reader may wonder why we refer to the additive hamiltonian \( h \) or \( h_n \), instead of the nonadditive one \( H \) or \( H_n \), as deformed hamiltonian. It should be noticed that, when addressing a system of \( N \) particles, we have to write \( H_n = \frac{p_n^2}{2m} + V_n \) for single particle so that \( h_n = \frac{\ln[1 + (q - 1) \beta (\frac{p_n^2}{2m} + V_n)]}{(q - 1) \beta} \). It is clear that \( H_n \), instead of \( h_n \), is the physical energy. According to this consideration, the total hamiltonian of a system of "ideal gas" (\textit{free particles in the sense that we do not write the correlation energy between the particles in Hamiltonian and let it be represented by} \( q \neq 1 \) \textit{in the deformed energy}) should be written as

\[
H = \frac{e^{(q-1)\beta h} - 1}{(q - 1) \beta} = \frac{e^{(q-1)\beta \sum_{n=1}^{N} h_n} - 1}{(q - 1) \beta} = \frac{e^{\sum_{n=1}^{N} \ln[1 + (q - 1) \beta (\frac{p_n^2}{2m} + V_n)]} - 1}{(q - 1) \beta} = \sum_{n=1}^{N} \frac{p_n^2}{2m} + H_c \tag{11}
\]

When \( q = 1 \) \((H_c = 0)\), we recover \( H = \sum_{n=1}^{N} \frac{p_n^2}{2m} \).

I would like to emphasize here that, as shown in Eq.(10), the deformed hamiltonian \( h \) may be employed, if we want, to recover the mathematical structure of BGS in introducing a \( q \)-deformed information measure \( I_q = -\ln \rho \) and an entropy \( S = -\text{Tr} \rho \ln \rho \) where \( \rho = \frac{e^{-\beta h}}{Z} \) \((Z = \text{Tr} e^{-\beta h})\). We straightforwardly obtain \( \frac{\partial S}{\partial u} = \beta = 1/T \) and a deformed first law \( du = dw + T dS \), where \( u = \text{Tr} \rho h \) is the deformed internal energy and \( dw \) the work on the system.

**IV. REVIEW OF SOME RESULTS ABOUT IDEAL GASES**

Now I will show some consequences of Eq.(11) on ideal gases.

**A. Classical ideal gas**

For classical ideal gas, the total hamiltonian should be given by Eq.(11). So the total partition function \( Z \) is given by [in the formalism of complete distribution (\( \text{Tr} \rho = 1 \)) with unnormalized average energy \( U = \text{Tr} H \rho^0 \)]:
\[ Z = \text{Tr}[1 - (1 - q)\beta H]^{1/q} \]  
\[ = \prod_{n=1}^{N} \text{Tr}[1 - (1 - q)\beta H_n]^{1/q} \]  
\[ = \{\text{Tr}[1 - (1 - q)\beta H_n]^{1/q}\}^N = z^N \]

where \( z = \text{Tr}[1 - (1 - q)H_n]^{1/q} \) is the single particle partition function. The calculation of \( Z \) in reference [3][4] can be adopted for single particle. For \( q > 1 \), we obtain:

\[ z = \left(\frac{2\pi m}{k}\right)^{d/2} \frac{\Gamma\left(\frac{1}{q-1} - \frac{d}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)(q-1)^{d/2}} \beta^{-d/2} = \Delta \beta^{-d/2} \]  

where \( d \) is the dimension of volume \( V \). The total internal energy is

\[ U = -\frac{\partial}{\partial \beta} \left( z_q^{1-q} - 1 \right) \]  
\[ = -\frac{\partial}{\partial \beta} z_q^{N(1-q)} - 1 \]  
\[ = -\frac{\partial}{\partial \beta} z_q^{N(1-q)} - 1 \]  
\[ = \frac{dN}{2\Delta^{N(q-1)}} (kT)^{dN(1-q)/2+1}. \]

The specific heat is so given by

\[ C_v = \frac{\partial U}{\partial T} = \frac{k dN}{2\Delta^{N(q-1)}} (kT)^{dN(1-q)/2}. \]

We notice that the \( T \)-dependence of \( C_v \) is the same as given by the extensive approximation but the \( q- \) and \( N \)-dependences are changed [3]. This point is also noticed in the case of \( N \)-harmonic-oscillator system [8]. Another interesting point is that, in the present case, the variation interval of \( q \) is enlarged to \( 1 < q < 1 + \frac{2}{dN} \) from \( 1 < q < 1 + \frac{2}{dN} \) given by the extensive approximation [3][4].

For \( q < 1 \), similar discussions can be made with the extension of Hilhorst formula given by Prato [4].

**B. Quantum gas**

For nonextensive quantum gas, from the qed for grand canonical ensemble [12][5] with complete distribution and normalized average \( (U = \text{Tr} H \rho) \), we can write:

\[ Z = \text{Tr}[1 - (1 - q)\beta H]^{1/q} \]

\[ = \prod_{n=1}^{N} \text{Tr}[1 - (1 - q)\beta H_n]^{1/q} \]

\[ = \{\text{Tr}[1 - (1 - q)\beta H_n]^{1/q}\}^N = z^N \]
\[ [1 + (1 - q)\beta(H - \mu N)]^{1/q} = \prod_{n=1}^{N} [1 + (1 - q)\beta(H_n - \mu)]^{1/q} \] (16)

Let this equation be replaced by

\[ e^{-\beta h'} = \prod_{n=1}^{N} e^{-\beta h'_n} \] (17)

where \( h' = \frac{\ln[1+(1-q)\beta(H-\mu N)]}{\beta(1-q)} \) and \( h'_n = \frac{\ln[1+(1-q)\beta(H_n-\mu)]}{\beta(1-q)} \). The total partition function \( Z \) is then given by

\[ Z = \text{Tr}[1 - (q - 1)\beta(H - \mu N)]^{1/q} \] (18)

\[ = \text{Te}^{-\beta h'} \]

\[ = \text{Te}^{-\beta \sum_n h'_n} \]

\[ = \prod_k \sum_{n_k} e^{-n_k \beta e'_k} \]

where \( e'_k \) is the eigenvalue of \( h'_n \) and \( n_k \) the occupation number of the state \( k \) of single particle. For boson and fermion, we obtain, respectively,

\[ Z = \prod_k \sum_{n_k=0}^{\infty} e^{-n_k \beta e'_k} = \prod_k \frac{1}{1 - e^{-\beta e'_k}} \quad \text{and} \quad Z = \prod_k \sum_{n_k=0}^{1} e^{-n_k \beta e'_k} = \prod_k (1 + e^{-\beta e'_k}). \] (19)

Then, it is straightforward to get :

\[ \bar{n}_l = \text{Tr} \rho n_l = -\frac{1}{\beta} \frac{\partial (\ln Z)}{\partial e'_l} = \frac{1}{e^{\beta e'_l} \pm 1} = \frac{1}{[1 + (q - 1)\beta(e_l - \mu)]^{1/q} \pm 1} \] (20)

where \( e_l \) is the eigenvalue of the one particle hamiltonian \( H_n \). “+” and “-” correspond to fermions and bosons, respectively. These are just the standard quantum distributions given by Büyükkılıç et al with the so called factorization approximation [12] and unnormalized average.

On the other hand, in the present formalism of NSM, unnormalized average does not give simple quantum distribution similar to Eq.(20) for standard bosons and fermions. This is due to the fact that \( Z = \prod_k \sum_{n_k} e^{-n_k \beta e'_k} \neq \prod_k [\sum_{n_k} e^{-q n_k \beta e'_k}]^{1/q} \). Here I only show that, in this case, Eq.(20) becomes

\[ \bar{n}_l = \text{Tr} \rho^q n_l = \frac{Q}{[1 + (q - 1)\beta(e_l - \mu)]^{1/q} \pm 1} \] (21)
where $Q = \text{Tr} \rho^q$ can be regarded as a parameter depending on $q$. $Q > 1$ $Q = 1$ and $Q < 1$ for $q < 1$, $q = 1$ and $q > 1$, respectively. These distributions seem interesting because they allow intermediate occupation number between that of bosons and fermions. In particular, at absolute zero, for “fermionlike” particle with “+”, $\bar{n}_l = Q$ when $e_l < \mu$ and $\bar{n}_l = 0$ when $e_l > \mu$. This means that it is possible for several “fermions” to occupy an one-particle quantum state if $Q > 1$ or $q < 1$. Consequently, the Fermi surface $\epsilon_F$ at $T = 0$ or $\beta = \infty$ changes as a function of $Q$ or of the interaction between the particles: $\epsilon_F = \frac{\epsilon_F}{Q^{2/3}}$ where $\epsilon_F$ is the conventional Fermi energy at $q = 1$ or $Q = 1$. This result can be compared to that of the fractional exclusion statistics (FES) \cite{23} for intermediate particles different from bosons and fermions. It is not surprising to see that nonextensive statistics has similar effect to that of FES describing low dimension quasi-particles or elementary excitations \cite{23,24}, because qed is just a result of interactions which are perhaps the origin of the quasi-particles. However, the fact that only the NSM formalism with unnormalized average can give the intermediate quantum distributions seems to deserve further investigation. Quite interesting efforts have been made by some authors \cite{13,25,26} to relate nonextensive statistics and the quantum distributions given by Eq.(20) to FES.

**V. CONCLUSION**

Summing up, it is argued that the nonextensive thermostatistics should be based on the factorization of compound probability suggested, not by “independence” of noninteracting systems, but by Abe’s pseudoadditivity for equilibrium interacting systems having Tsallis entropy. So Eq.(11) should be viewed as a fundamental hypothesis of NSM and has to be rigorously satisfied by all exact calculations relative to equilibrium systems. All calculations based on additive hamiltonian Eq.(3) are to be considered as extensive energy approximation which should be employed with great care. In this formalism, the applications of NSM to ideal gases are revisited. The results are different from those given by extensive approximation. The standard quantum distributions can be given by using normalized average.
calculus. The unnormalized average leads to a kind of fractional exclusion distributions.

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