MATSAEV’S TYPE THEOREMS FOR SOLUTIONS OF THE
STATIONARY SCHRÖDINGER EQUATION AND
ITS APPLICATIONS

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ABSTRACT. Our aim in this paper is to give lower estimates for solutions of the
stationary Schrödinger equation in a cone, which generalize and supplement the
result obtained by Matsaev’s type theorems for harmonic functions in a half
space. Meanwhile, some applications of this conclusion are also given.

1. Introduction. Let \( \mathbb{R} \) and \( \mathbb{R}_+ \) be the set of all real numbers and the set of all
positive real numbers, respectively. We denote by \( \mathbb{R}^n \) (\( n \geq 2 \)) the n-dimensional
Euclidean space. A point in \( \mathbb{R}^n \) is denoted by \( P = (X, x_n) \), where \( X = (x_1, x_2, \ldots, x_{n-1}) \).
The Euclidean distance between two points \( P \) and \( Q \) in \( \mathbb{R}^n \) is denoted by
\( |P - Q| \). Also \( |P - O| \) with the origin \( O \) of \( \mathbb{R}^n \) is simply denoted by
\( |P| \). The boundary and the closure of a set \( E \) in \( \mathbb{R}^n \) are denoted by
\( \partial E \) and \( E \), respectively.

We introduce a system of spherical coordinates \( (r, \Theta) \), \( \Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1}) \), in
\( \mathbb{R}^n \) which are related to cartesian coordinates \( (x_1, x_2, \ldots, x_{n-1}, x_n) \) by
\[
    x_1 = r (\prod_{j=1}^{n-1} \sin \theta_j) \ (n \geq 2), \quad x_n = r \cos \theta_1,
\]
and if \( n \geq 3 \), then
\[
    x_{n-m+1} = r (\prod_{j=1}^{m-1} \sin \theta_j) \cos \theta_m \ (2 \leq m \leq n-1),
\]
where \( 0 \leq r < +\infty, -\frac{1}{2} \pi \leq \theta_{n-1} < \frac{3}{2} \pi \), and if \( n \geq 3 \), then \( 0 \leq \theta_j \leq \pi \ (1 \leq j \leq n-2) \).

The unit sphere and the upper half unit sphere in \( \mathbb{R}^n \) are denoted by \( S^{n-1} \) and
\( S_+^{n-1} \), respectively. For simplicity, a point \( (1, \Theta) \) on \( S^{n-1} \) and the set \( \{ \Theta; (1, \Theta) \in \Omega \} \) for a set \( \Omega \), \( \Omega \subset S^{n-1} \), are often identified with \( \Theta \) and \( \Omega \), respectively. For two sets \( \Xi \subset \mathbb{R}_+ \) and \( \Omega \subset S^{n-1} \), the set \( \{(r, \Theta) \in \mathbb{R}^n; r \in \Xi, (1, \Theta) \in \Omega \} \) in \( \mathbb{R}^n \) is simply
denoted by \( \Xi \times \Omega \). In particular, the half space \( \mathbb{R}_+ \times S_+^{n-1} = \{(X, x_n) \in \mathbb{R}^n; x_n > 0 \} \)
will be denoted by \( \mathbb{T}_n \). We denote the sets \( I \times \Omega \) and \( I \times \partial \Omega \) with an interval on \( \mathbb{R} \).

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by $C_n(\Omega; I)$ and $S_n(\Omega; r)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$ which is $\partial C_n(\Omega) - \{O\}$. For $P \in \mathbb{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at $P$ and radius $r$ in $\mathbb{R}^n$. $S_r = \partial B(O, r)$

We use the standard notations $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$. Further, we denote by $w_n$ the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbb{S}^{n-1}$, by $\partial/\partial Q$ denotes the differentiation at $Q$ along the inward normal into $C_n(\Omega)$, by $dS_n$ the $(n - 1)$-dimensional volume elements induced by the Euclidean metric on $S_r$. For positive functions $h_1$ and $h_2$, we say that $h_1 \lesssim h_2$ if $h_1 \leq M h_2$ for some constant $M > 0$. If $h_1 \lesssim h_2$ and $h_2 \lesssim h_1$, we say that $h_1 \approx h_2$.

Let $\mathcal{A}_a$ denote the class of nonnegative radial potentials $a(P)$, i.e. $0 \leq a(P) = a(r), P = (r, \Theta) \in C_n(\Omega)$, such that $a \in L^1_{loc}(C_n(\Omega))$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

This article is devoted to the stationary Schrödinger equation

$$\text{Sch}_a u(P) = -\Delta_n u(P) + a(P) u(P) = 0 \text{ for } P \in C_n(\Omega),$$

(1)

where $\Delta_n$ is the Laplace operator and $a \in \mathcal{A}_a$. Note that solutions of equation (1) are the (classical) harmonic functions in a cone in the case $a = 0$. Under these assumptions the operator $\text{Sch}_a$ can be extended in the usual way from the space $C_0^\infty(C_n(\Omega))$ to an essentially self-adjoint operator on $L^2(C_n(\Omega))$ (see [8]). We will denote it $\text{Sch}_a$ as well. This last one has a Green-Sch function $G^a_{\Omega}(P, Q)$. Here $G^a_{\Omega}(P, Q)$ is positive on $C_n(\Omega)$ and its inner normal derivative $\partial G^a_{\Omega}(P, Q)/\partial u_Q \geq 0$, where $\partial/\partial Q$ denotes the differentiation at $Q$ along the inward normal into $C_n(\Omega)$.

We denote this derivative by $\mathcal{P}T^a_{\Omega}(P, Q)$, which is called the Poisson-Sch kernel with respect to $C_n(\Omega)$. We remark that $G^a_{\Omega}(P, Q)$ and $\mathcal{P}T^a_{\Omega}(P, Q)$ are the Green function and Poisson kernel of the Laplacian in $C_n(\Omega)$ respectively.

Let $\Omega$ be a domain on $\mathbb{S}^{n-1}$ with smooth boundary $\partial \Omega$. Consider the Dirichlet problem

$$(\Lambda_n + \lambda)\varphi = 0 \text{ on } \Omega,$$

$$\varphi = 0 \text{ on } \partial \Omega,$$

where $\Lambda_n$ is the spherical part of the Laplace operational $\Delta_n$. We denote the least positive eigenvalue of this boundary value problem by $\lambda$ and the normalized positive eigenfunction corresponding to $\lambda$ by $\varphi(\Theta)$, $\int_{\Omega} \varphi^2(\Theta) dS_1 = 1$. In order to ensure the existence of $\lambda$ and a smooth $\varphi(\Theta)$. We put a rather strong assumption on $\Omega$: if $n \geq 3$, then $\Omega$ is a $C^{2, \alpha}$-domain $(0 < \alpha < 1)$ on $\mathbb{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces. Then $\varphi \in C^2(\overline{\Omega})$ and $\partial \varphi/\partial n > 0$ on $\partial \Omega$ (here and below, $\partial/\partial n$ denotes differentiation along the interior normal).

Let $V(r)$ and $W(r)$ stand for solutions of the equation

$$-Q''(r) - \frac{n - 1}{r} Q'(r) + \left(\frac{\lambda}{r^2} + a(r)\right) Q(r) = 0, \quad 0 < r < \infty.$$  

(2)

It is known (see, for example, [17]) that if the potential $a \in \mathcal{A}_a$, then equation (2) has a fundamental system of positive solutions $\{V, W\}$ such that $V$ is nondecreasing with (see [8, 11, 12, 14])

$$0 \leq V(0+) \leq V(r) \nearrow \infty \text{ as } r \to +\infty,$$

and $W$ is monotonically decreasing with

$$+\infty = W(0+) > W(r) \searrow 0 \text{ as } r \to +\infty.$$ 

For the sake of brevity, we shall write $\chi$ instead of $W(1)V'(1) - V(1)W'(1)$.
We will also consider the class $\mathcal{B}_a$, consisting of the potentials $a \in \mathcal{A}_a$ such that there exists the finite limit $\lim_{r \to \infty} r^2a(r) = k \in [0, \infty)$, and moreover, $r^{-1}|r^2a(r) - k| \in L(1, \infty)$. If $a \in \mathcal{B}_a$, then solutions of the equation (1) are continuous (see [16]).

In the rest of the paper, we assume that $a \in \mathcal{B}_a$ and we shall suppress this assumption for simplicity. Further, We use $H(a, \Omega)$ to denote the class of solutions of equation (1), which are continuous in $C_n(\Omega)$. We remark that $H(0, \Omega)$ denote the class of functions harmonic in $C_n(\Omega)$ and continuous in $C_n(\Omega)$.

Let

$$\mathcal{A}_a = \{a \in \mathcal{A}_a \mid a \text{ is continuous, } a \geq 0 \text{ on } \Omega \}.$$

Then

$$\mathcal{B}_a = \{a \in \mathcal{A}_a \mid a \text{ is continuous, } a \geq 0 \text{ on } \Omega \}.$$

Throughout this paper, let $M$ denote various constants independent of the variables in questions, which may be different from line to line. Let $\rho(t)$ be a function on a segment $[1, +\infty)$ and $\rho(t) > \pi^2$. If $a = 0$ and $\Omega = S^{n-1}_a$, then $\pi^2 = 1, \pi^0 = 1$, $\pi^0 = 1 - n$ and $\varphi(\Theta) = (2\pi\rho^{-1})^{1/2} \cos \theta_1$.

The estimate we deal with has a long history which can be traced back to Mat-saev’s estimate of harmonic functions from below (see, for example, Matsaev [7, p. 209]).

**Theorem 1.1.** Let $A_1$ be a constant and $u(z) \in H(0, S^{1}_a)$ ($|z| = R$). Suppose that

$$u(z) \leq A_1 R^\rho, \quad z \in T_2, \quad R > 1, \quad \rho > 1$$

and

$$|u(z)| \leq A_1, \quad R \leq 1, \quad z \in T_2.$$

Then

$$u(z) \geq -\frac{A_1 M(1 + R^\rho)}{\sin \theta},$$

where $z = Re^{i\theta} \in T_2$ and $M$ is a constant independent of $A_1, R, \theta$ and the function $u(z)$.

Further versions and refinements of Theorem 1.1 may be found in the monograph by Nikol’ski (see [9, Ch. 1]) and in the paper Krasichkov-Ternovski (see [6]).

Recently, Zhang, Kou and Deng (see [18]) consider Theorem 1.1 in the $n$-dimensional ($n \geq 2$) case and obtain the following result. For related results in a cone, we refer the reader to the paper by Qiao and Pan (see [13]).

**Theorem 1.2.** Let $A_2$ be a constant and $u(P) \in H(0, S^{n-1}_a)$ ($|P| = R$). If

$$u(P) \leq A_2 R^\rho, \quad P \in T_n, \quad R > 1, \quad \rho > n - 1$$

and

$$|u(P)| \leq A_2, \quad R \leq 1, \quad P \in T_n,$$

then

$$u(P) \geq -A_2 M(1 + R^\rho) \cos^{1-n} (\theta_1),$$

where $P \in T_n$ and $M$ is a constant independent of $A_2, R, \theta_1$ and the function $u(P)$.

Our aim is to prove the following result. For a related result with respect to Schrödinger operator, we refer the reader to the paper by Kheyfits (see [5]).
Theorem 1.3. Let \( u(P) \in H(a, \Omega) \ (P = (R, \Theta)) \). Suppose that the following (I) and (II) are satisfied:

(I) For any \( P = (R, \Theta) \in C_n(\Omega; (1, \infty)) \), we have

\[
V(R) \int_{S_n(\Omega; (1, R))} u^{-}(t, \Phi) W(t) \frac{\partial \varphi(\Phi)}{\partial n_\Phi} d\sigma_Q \leq A_2 \rho(R) R^{\rho(R)} \ (\rho(R) > \aleph^+_0)
\]

and

\[
\int_{S_n(\Omega; R)} u^{-}(R, \Theta) \varphi(\Theta) dS_R \leq A_2 \rho(R) R^{\rho(R)},
\]

where \( d\sigma_Q \) is the surface area element on \( S_n(\Omega) \).

(II) For any \( P = (R, \Theta) \in C_n(\Omega) \) and \( R \leq 1 \), we have

\[
u(P) \geq -A_2.
\]

Then

\[
u(P) \geq -A_2 M(1 + \rho(R) R^{\rho(R)}) \varphi^{1-n}(\Theta),
\]

where \( P \in C_n(\Omega) \) and \( M \) is a constant independent of \( A_2, R, \varphi(\Theta) \) and the function \( u(P) \).

Corollary 1. Let \( u(P) \in H(0, \Omega) \ (P = (R, \Theta)) \). The conclusion of Theorem 1.3 remains valid if (7) in Theorem 1.3 is replaced by

\[
\int_{S_n(\Omega; (1, R))} u^{-}(t, \Phi) t^n \frac{\partial \varphi(\Phi)}{\partial n_\Phi} d\sigma_Q \leq A_2 \rho(R) R^{\rho(R)-n} \ (\rho(R) > \aleph^+_0).
\]

If, in addition, \( \Omega = S_n^{n-1} \) and \( \rho(R) \equiv \rho = const \) in Corollary 1, we immediately obtain

Corollary 2. The conclusion of Theorem 1.2 remains valid if (4) and (5) in Theorem 1.2 are replaced by the following (I) and (II) respectively.

(I) For any \( P = (R, \Theta) \in T_n \), \( R > 1 \), \( \rho > 0 \), we have

\[
\int_{S_n(S_n^{n-1}; (1, R))} u^{-}(t, \Phi) t^{-n} d\sigma_Q \leq A_2 R^{\rho+1},
\]

and

\[
\int_{S_n(S_n^{n-1}; R)} u^{-}(R, \Theta) \cos \theta_1 dS_R \leq A_2 R^{\rho}.
\]

(II) For any \( P = (R, \Theta) \in T_n \) and \( R \leq 1, \ (9) \) holds.

The applications of Theorem 1.3 are listed as follows. Application I

Theorem 1.4. The conclusion of Theorem 1.3 remains valid if (7) and (8) in Theorem 1.3 are replaced by

\[
u(P) \leq A_2 R^{\rho(R)}.
\]

By taking \( a = 0 \) and \( \rho(R) \equiv \rho = const \) in Theorem 1.4, we obtain the following Corollary, which generalizes Theorem 1.2 to the conical case.

Corollary 3. Let \( u(P) \in H(0, \Omega) \ (P = (R, \Theta)) \). If

\[
u(P) \leq A_2 R^{\rho}, \ P \in C_n(\Omega), \ R > 1, \ \rho > \aleph^+_0
\]
and 
\[ u(P) \geq -A_2, \quad R \leq 1, \quad P \in C_n(\Omega), \]
then 
\[ u(P) \geq -A_2 M(1 + R^\rho) \varphi^{1-n}(\Theta), \quad (13) \]
where \( P \in C_n(\Omega) \) and \( M \) is a constant independent of \( K, R, \varphi(\Theta) \) and the function \( u(P) \).

By taking \( \Omega = S^{n-1} \) in Corollary 3, we obtain

**Corollary 4.** Obviously, (4) and (5) in Theorem 1.2 may be replaced by weaker conditions
\[ u(P) \leq A_2 R^\rho, \quad P \in T_n, \quad R > 1, \quad \rho > 1 \quad (14) \]
and (II) in Corollary 2 respectively.

**Remark 1.** On the one hand, by the proofs of Theorems 1.3 and 1.4, we know that (I) in Corollary 2 is weaker than (14) in Corollary 4, which is weaker than (4) in Theorem 1.2. On the other hand, (II) in Corollary 2 is weaker than (5) in Theorem 1.2. So Corollary 2 generalizes Corollary 4, which is also a generalization of Theorem 1.2.

**Application II**

**Theorem 1.5.** The conclusion of Theorem 1.3 remains valid if (7) and (8) in Theorem 1.3 are replaced by
\[ V(R) \int_{S_n(\Omega; (1, R))} u^+(t, \Phi) W(t) \frac{\partial \varphi(\Phi)}{\partial n_\Phi} d\sigma_Q \leq A_2 \rho(R) R^{\rho(\Phi)} (\rho(R) > N^+) \quad (15) \]
and
\[ \int_{S_n(\Omega; R)} u^+(R, \Theta) \varphi(\Theta) dS_R \leq A_2 \rho(R) R^{\rho(\Phi)} \quad (16) \]
respectively.

**Remark 2.** From the proofs of Theorems 1.4 and 1.5, we can easily show that (7) and (8) in Theorem 1.3 are equivalent to (15) and (16) in Theorem 1.5.

**Corollary 5.** Let \( u(P) \) be a harmonic function of order \( \rho (= \text{const} > N^+) \) and finite type on \( C_n(\Omega) \), continuous on \( C_n(\Omega) \). If (II) in Theorem 1.3 is satisfied, then (13) holds.

The following result is due to Masaev (see [7, p. 212, Th. 3]).

**Theorem 1.6.** Let \( u(z) (z = Re^{i\alpha}) \) be a subharmonic function on \( T_2 \), which satisfies the estimate
\[ u(z) \leq M(1 + R^\rho) \sin^{-l}(\alpha), \quad (17) \]
where \( \rho > 1 \) and \( l \geq 0 \). Then \( u(z) \) is of order \( \rho \) and finite type.

Masaev’s inequalities like (17) are crucial in many problems, since they are intrinsically connected with the estimates of the Cauchy type integrals. Specifically, Theorem 1.6 has found important applications in operator theory, thus it may be of interest to extend these results onto more general classes of functions. Indeed,
Govorov and Zhuravleva (see [3]) proved that Theorem 1.6 can be extended to analytic functions in a half plane, if in addition, they are continuous up to, and satisfy a similar upper bound at the boundary.

**Theorem 1.7.** Let \( u(z) (z = Re^{i\alpha}) \) be analytic on \( T_2 \) and continuous on \( \overline{T}_2 \). If 
\[
\log |u(Re^{i\alpha})| \leq MR^\rho \sin^{-l} \alpha, \quad 0 < \alpha < \pi, \ R > R_0 > 0
\]
and 
\[
\log |u(t)| \leq M|t|^\rho, \quad |t| > R_0,
\]
where \( \rho > 1 \) and \( l \geq 0 \). Then
\[
\max_{0 < \alpha < \pi} \log |u(Re^{i\alpha})| \leq M(1 + R^\rho).
\]

For harmonic functions on \( T_2 \), we refer readers to the following Theorem (see [7]), where the bounded condition is replaced by the weaker integral condition.

**Theorem 1.8.** Let \( u(z) \in H(0, S^1_+) \) \( (z = Re^{i\alpha}) \). Suppose that the following (I) and (II) are satisfied:

(I) If \( R \leq 1 \), then \( u(z) \leq M \).

(II) If \( R > 1 \) and \( \rho > 1 \), then
\[
\int_1^R u^+(t)t^{-2}dt \leq MR^{\rho-1} \text{ and } \int_{S^1_+} u^+(t)\sin \alpha d\alpha \leq MR^\rho.
\]

Then
\[
\max_{0 < \alpha < \pi} u(Re^{i\alpha}) \leq M(1 + R^\rho).
\]

Application III

**Theorem 1.9.** Let \( u(P) \in H(0, \Omega) \) \( (P = (R, \Theta)) \). Suppose that the following (I) and (II) are satisfied:

(I) If \( R \leq 1 \) and \( P = (R, \Theta) \in \overline{C_\alpha(\Omega)} \), then \( u(P) \leq M \).

(II) If \( R > 1 \), \( \rho(R) > \rho_0^+ \) and \( P = (R, \Theta) \in C_\alpha(\Omega; (1, \infty)) \), then
\[
V(R) \int_{S_\alpha(\Omega; (1, R))} u^+(t, \Phi)W(t)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\sigma_Q \leq M\rho(R)R^{\rho(R)},
\]
\[
\int_{S_\alpha(\Omega; R)} u^+(R, \Theta)\varphi(\Theta)dS_R \leq M\rho(R)R^{\rho(R)}.
\]

Then
\[
\max_{\Theta \in \Omega} u(R, \Theta) \leq M(1 + \rho(R)R^{\rho(R)}).
\]

When \( \Omega = S^n_+ \) and \( \rho(R) \equiv \rho = \text{const} \), we have the following result, which generalize Theorem 1.8 to the higher dimensional half space.

**Corollary 6.** Let \( u(P) \in H(0, S^{n-1}_+) \) \( (P = (R, \Theta)) \). Suppose that the following (I) and (II) are satisfied:

(I) If \( R \leq 1 \) and \( P = (R, \Theta) \in \overline{T}_n \), then \( u(P) \leq M \).

(II) If \( R > 1 \), \( \rho > 1 \) and \( P = (R, \Theta) \in C_\alpha(S^{n-1}_+; (1, \infty)) \), then
\[
\int_{S_\alpha(S^{n-1}_+; (1, R))} u^+(t, \Phi)t^{-n}d\sigma_Q \leq MR^{n-1},
\]
(18)
\[
\int_{S_n(R^2; R)} u^+(R, \Theta) \cos \theta dS_R \leq MR^\rho.
\]

Then

\[
\max_{\Theta \in S_n^{-1}} u(R, \Theta) \leq M(1 + R^\rho).
\]  \hspace{1cm} (19)

**Remark 3.** When \( n = 2 \), Corollary 6 reduces to Theorem 1.8. From (19) we know that \( u \) is at most of the growth order \( \rho \) and normal type in \( T_n \).

2. **Lemmas.** In our discussions, the following estimates for the Poisson-Sch kernel \( P_{\Omega}^\alpha(P, Q) \) are fundamental, which follow from Kheyfits (see [8, p. 353]) and Azarin (see [1, Lemma 4 and Remark]).

**Lemma 2.1.**

\[
P_{\Omega}^\alpha(P, Q) \leq MV(r) W(R) \frac{\partial \varphi(\Phi)}{\partial n},
\]  \hspace{1cm} (20)

for any \( P = (r, \Theta) \in C_n(\Omega) \) and any \( Q = (t, \Phi) \in S_n(\Omega) \) satisfying \( 0 < \frac{t}{r} \leq \frac{4}{5} \).

\[
P_{\Omega}^0(P, Q) \leq MV(t) W(R) \frac{\partial \varphi(\Phi)}{\partial n} + M \frac{r^\varphi(\Theta)}{|P - Q|^n} \frac{\partial \varphi(\Phi)}{\partial n} \phi(\Phi),
\]  \hspace{1cm} (21)

for any \( P = (r, \Theta) \in C_n(\Omega) \) and any \( Q = (t, \Phi) \in S_n(\Omega; (\frac{4}{5}r, \frac{4}{5}r)) \).

Let \( G_{\Omega,R}^\alpha(P, Q) \) be the Green-Sch function in \( C_n(\Omega, (0, R)) \). Then

\[
\frac{\partial G_{\Omega,R}^\alpha(P, Q)}{\partial R} \leq MV(r) W(R) \frac{\varphi(\Theta) \varphi(\Phi)}{R},
\]  \hspace{1cm} (22)

where \( P = (r, \Theta) \in C_n(\Omega) \) and \( Q = (R, \Phi) \in S_n(\Omega; R) \).

The following Carleman formula with respect to the Schrödinger operator plays an important role in our discussions, which is due to Levin and Kheyfits (see [8, p. 356]). Carleman formula for harmonic functions in a cone and its applications, we refer readers to the papers by Qiao (see [10]), Rashkovskii and Ronkin (see [15]).

**Lemma 2.2.** If \( R > 1 \) and \( u(t, \Phi) \) is the solution of the equation (1) on \( C_n(\Omega; (1, R)) \), then

\[
0 = \chi \int_{S_n(\Omega; R)} \frac{W(R)}{R} u(R, \Phi) \varphi(\Phi) dS_R + \int_{S_n(\Omega; (1, R))} u(t, \Phi) \frac{\partial \varphi}{\partial n} \Psi(t) d\sigma_Q + d_1 + d_2 \frac{W(R)}{V(R)},
\]  \hspace{1cm} (23)

where

\[
\Psi(t) = W(t) - \frac{W(R)}{V(R)} V(t),
\]

\[
d_1 = \int_{S_n(\Omega; 1)} u(\varphi(\Phi) W(1) - W(1) \varphi(\Phi) \frac{\partial u}{\partial n} dS_1,
\]

\[
d_2 = \int_{S_n(\Omega; 1)} V(1) \varphi(\Phi) \frac{\partial u}{\partial n} - u(\varphi(\Phi) V(1) dS_1.
\]
3. **Proof of Theorem 1.3.** By the Riesz Decomposition Theorem (see [15]), for any \( P = (r, \Theta) \in C_n(\Omega; (0, R)) \) we have

\[
-u(P) = \int_{S_n(\Omega; (0, R))} \mathcal{P} \mathcal{T}_\Omega^a(P, Q)(-u(Q))d\sigma_Q + \int_{S_n(\Omega; R)} \frac{\partial G_{\Omega, R}(P, Q)}{\partial R}(-u(Q))dS_R.
\]  

(24)

Now we distinguish three cases.

**Case 1.** \( P = (r, \Theta) \in C_n(\Omega; (\frac{5}{4}, \infty)) \) and \( R = \frac{5}{4}r \).

Since \(-u(x) \leq u^-(x)\), we obtain

\[
-u(P) \leq \sum_{i=1}^{4} I_i(P)
\]  

(25)

from (24), where

\[
I_1(P) = \int_{S_n(\Omega; (0, 1))} \mathcal{P} \mathcal{T}_\Omega^a(P, Q)u^-(Q)d\sigma_Q,
\]

\[
I_2(P) = \int_{S_n(\Omega; (1, \frac{5}{4}r))} \mathcal{P} \mathcal{T}_\Omega^a(P, Q)u^-(Q)d\sigma_Q,
\]

\[
I_3(P) = \int_{S_n(\Omega; (\frac{5}{4}r, R))} \mathcal{P} \mathcal{T}_\Omega^a(P, Q)u^-(Q)d\sigma_Q,
\]

\[
I_4(P) = \int_{S_n(\Omega; R)} \mathcal{P} \mathcal{T}_\Omega^a(P, Q)(-u(Q))d\sigma_Q.
\]

Then from (3), (7) and (20), we have

\[
I_1(P) \leq A_2 M \varphi(\Theta)
\]  

(26)

and

\[
I_2(P) \leq MW(r) \frac{V(\frac{5}{4}r)}{W(\frac{5}{4}r)} \varphi(\Theta) \int_{S_n(\Omega; (1, \frac{5}{4}r))} u^-(Q)W(t) \frac{\partial \varphi(\Phi)}{\partial n_\Phi} d\sigma_Q 
\]

\[
\leq A_2 M \rho(R) R^\alpha(\varphi(\Theta)).
\]  

(27)

By (21), we consider the inequality

\[
I_3(P) \leq I_{31}(P) + I_{32}(P),
\]  

(28)

where

\[
I_{31}(P) = M \int_{S_n(\Omega; (\frac{5}{4}r, R))} \frac{u^-(Q)\varphi(\Theta) \partial \varphi(\Phi)}{t^{n-1}} \frac{\partial \varphi(\Phi)}{\partial n_\Phi} d\sigma_Q,
\]

\[
I_{32}(P) = Mr \varphi(\Theta) \int_{S_n(\Omega; (\frac{5}{4}r, R))} \frac{u^-(Q)r \varphi(\Theta) \partial \varphi(\Phi)}{|P-Q|^n} \frac{\partial \varphi(\Phi)}{\partial n_\Phi} d\sigma_Q.
\]

We first have

\[
I_{31}(P) \leq MV(r) \varphi(\Theta) \int_{S_n(\Omega; (\frac{5}{4}r, R))} u^-(Q)W(t) \frac{\partial \varphi(\Phi)}{\partial n_\Phi} d\sigma_Q 
\]

\[
\leq A_2 M \rho(R) R^\alpha(\varphi(\Theta)).
\]  

(29)
from (7). Next, we shall estimate $I_{32}(P)$. Take a sufficiently small positive number $k$ such that

$$S_n(\Omega; \frac{4}{5}r, \frac{5}{4}r) \subset \bigcup_{P=(r,\Theta) \in \Pi(k)} B(P, \frac{1}{2}r),$$

where

$$\Pi(k) = \{ P = (r, \Theta) \in C_n(\Omega); \inf_{(1, \Theta) \in \Theta} |(1, \Theta) - (1, z)| < k, \ 0 < r < \infty \},$$

and divide $C_n(\Omega)$ into two sets $\Pi(k)$ and $C_n(\Omega) - \Pi(k)$. If $P = (r, \Theta) \in C_n(\Omega) - \Pi(k)$, then there exists a positive $k'$ such that $|P - Q| \geq k'r$ for any $Q \in S_n(\Omega)$, and hence

$$I_{32}(P) \leq M \int_{S_n(\Omega; \frac{4}{5}r, R)} u^{-}(Q) \frac{\partial \varphi(\Theta)}{\partial n_{\Phi}} d\sigma_Q$$

$$\leq A_2 M \rho(R) R^{\rho(R)} \varphi(\Theta),$$

(30)

which is similar to the estimate of $I_{31}(P)$.

We shall consider the case $P = (r, \Theta) \in \Pi(k)$. Now put

$$H_i(P) = \{ Q \in S_n(\Omega; \frac{4}{5}r, R); 2^{i-1} \delta(P) \leq |P - Q| < 2^i \delta(P) \},$$

where $\delta(P) = \inf_{Q \in \partial C_n(\Omega)} |P - Q|$. Since $S_n(\Omega) \cap \{ Q \in \mathbb{R}^n : |P - Q| < \delta(P) \} = \emptyset$, we have

$$I_{32}(P) = M \sum_{i=1}^{i(P)} \int_{H_i(P)} u^{-}(Q) r \varphi(\Theta) \frac{\partial \varphi(\Phi)}{\partial n_{\Phi}} d\sigma_Q,$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1} \delta(P) \leq \frac{r}{2} < 2^{i(P)} \delta(P)$.

Since $r \varphi(\Theta) \leq M \delta(P)$ ($P = (r, \Theta) \in C_n(\Omega)$), similar to the estimate of $I_{31}(P)$ we obtain

$$\int_{H_i(P)} u^{-}(Q) r \varphi(\Theta) \frac{\partial \varphi(\Phi)}{\partial n_{\Phi}} d\sigma_Q$$

$$\leq \int_{H_i(P)} r \varphi(\Theta) \left( \frac{\partial \varphi(\Phi)}{2^{i-1} \delta(P)} \right) \frac{u^{-}(Q)}{P - Q} d\sigma_Q$$

$$\leq M 2^{(i-1)n} \varphi^{1-n}(\Theta) \int_{H_i(P)} -u(Q) \frac{V(t) W(t)}{t} \frac{\partial \varphi(\Phi)}{\partial n_{\Phi}} d\sigma_Q$$

$$\leq A_2 M \rho(R) R^{\rho(R)} \varphi^{1-n}(\Theta)$$

for $i = 0, 1, 2, \ldots, i(P)$.

So

$$I_{32}(P) \leq A_2 M \rho(R) R^{\rho(R)} \varphi^{1-n}(\Theta).$$

(31)

From (28), (29), (30) and (31) we see that

$$I_{3}(P) \leq A_2 M \rho(R) R^{\rho(R)} \varphi^{1-n}(\Theta).$$

(32)

On the other hand, we have from (8) and (22) that

$$I_{4}(P) \leq M V(r) \varphi(\Theta) \int_{S_n(\Omega; R)} u^{-}(Q) \frac{W(R)}{R} \varphi(\Phi) dS_R$$

$$\leq A_2 M \rho(R) R^{\rho(R)} \varphi(\Theta).$$

(33)
We thus obtain (26), (27), (32) and (33) that
\[-u(P) \leq A_2 M \rho(R) (1 + R^\rho(R)) \phi^{1-n}(\Theta).\] (34)

**Case 2.** \(P = (r, \Theta) \in C_n(\Omega; (\frac{4}{5}, \frac{5}{4}]) \) and \(R = \frac{5}{4} r\).

From (24) we have
\[-u(P) = I_1(P) + I_5(P) + I_4(P),\]
where \(I_1(P)\) and \(I_4(P)\) are defined in Case 1 and
\(I_5(P) = \int_{S_n(\Omega; (1, R))} \mathcal{P} \mathcal{I}_\Omega(P, Q) u^- d\sigma_Q.\)

Similar to the estimate of \(I_3(P)\) in Case 1 we have
\(I_5(P) \leq A_2 M \rho(R) R \rho(R) \phi^{1-n}(\Theta),\)
which together with (26) and (33), gives (34).

**Case 3.** \(P = (r, \Theta) \in C_n(\Omega; (0, \frac{4}{5}]).\)

It is evident from (9) we have
\[-u \leq A_2,\]
which also gives (34).

From (34) we finally have (10), which is the conclusion of Theorem 1.3.

4. **Proof of Theorem 1.4.** Applying Lemma 2.2 to \(u = u^+ - u^-\), we have
\[
\chi \int_{S_n(\Omega; R)} \frac{u^+ \varphi}{R} W(R) dS_R + \int_{S_n(\Omega; (1, R))} u^+ \Psi(t) \frac{\partial \varphi}{\partial n} d\sigma_Q + d_1 + \frac{W(R)}{V(R)} d_2

= \chi \int_{S_n(\Omega; R)} \frac{u^- \varphi}{R} W(R) dS_R + \int_{S_n(\Omega; (1, R))} u^- \Psi(t) \frac{\partial \varphi}{\partial n} d\sigma_Q
\] (35)
for a fixed \(R > 1\).

It immediately follows from (12) that
\[
V(R) \int_{S_n(\Omega; R)} \frac{W(R)}{R} u^+ \varphi dS_R \leq A_2 M \rho(R) R^\rho(R)
\] (36)
and
\[
V(R) \int_{S_n(\Omega; (1, R))} u^+ \Psi(t) \frac{\partial \varphi}{\partial n} d\sigma_Q

\leq V(R) \int_{S_n(\Omega; (1, R))} Kt^\rho(t) V(t) \left( \frac{W(t)}{V(t)} - \frac{W(R)}{V(R)} \right) \frac{\partial \varphi}{\partial n} d\sigma_Q

\leq A_2 MV(R) \int_1^R r^\rho(r) V(R) W(R) \left( 1 - \frac{W(R)}{V(R) W(R)} \right) \frac{\partial \varphi}{\partial n} dr

\leq A_2 MV(R) \int_1^R \frac{r^\rho(r) - 1}{V(r)} V(R) W(R) \frac{\partial \varphi}{\partial n} dr
\] (37)

Notice that
\[
d_1 + d_2 \frac{W(R)}{V(R)} \leq A_2 M \rho(R) R^\rho(R).\] (38)

Hence from (35), (36), (37) and (38) we have
\[
V(R) \int_{S_n(\Omega; R)} \frac{W(R)}{R} u^- \varphi dS_R \leq A_2 M \rho(R) R^\rho(R)
\] (39)
and
\[ V(R) \int_{\Omega; (1, R)} u^- \Psi(t) \frac{\partial \varphi}{\partial n} d\sigma_Q \leq A_2 M \rho(R) R^{\rho(R)}. \] (40)

Hence, (40) gives
\[ V(R) \int_{\Omega; (1, R)} u^- W(t) \frac{\partial \varphi}{\partial n} d\sigma_Q \]
\[ \leq \int_{\Omega; (1, \frac{\rho(R)+1}{\rho(R)} R)} u^- \left( W(t) - \frac{V(t) W(\frac{\rho(R)+1}{\rho(R)} R)}{V(\frac{\rho(R)+1}{\rho(R)} R)} \right) \frac{\partial \varphi}{\partial n} d\sigma_Q \]
\[ \times \frac{V(\rho(R)+1)}{V(\frac{\rho(R)+1}{\rho(R)} R)} \frac{V(R)}{V(\rho(R)+1)} \frac{V(\frac{\rho(R)+1}{\rho(R)} R)}{V(\rho(R))} \]
\[ \leq A_2 M \frac{V(\rho(R)+1)}{V(\frac{\rho(R)+1}{\rho(R)} R)} \frac{V(R)}{V(\rho(R)+1)} \frac{V(\frac{\rho(R)+1}{\rho(R)} R)}{V(\rho(R))} \]
\[ \leq A_2 M \frac{(\rho(R)+1)^x}{(\rho(R)+1)^x - (\rho(R))^x} R^{\rho(R)} \]
\[ \leq A_2 M \frac{V(2) \rho(R) R^{\rho(R)}}{V(2)} \]
\[ \leq A_2 M \rho(R) R^{\rho(R)}, \]
which together with (39) and Theorem 1.3, gives the conclusion of Theorem 1.4.

5. Proof of Theorem 1.5. By (15), we get
\[ V(R) \int_{\Omega; (1, R)} u^+(t, \Phi) \Psi(t) \frac{\partial \varphi}{\partial n} d\sigma_Q \]
\[ \leq V(R) \int_{\Omega; (1, R)} u^+(t, \Phi) W(t) \frac{\partial \varphi}{\partial n} d\sigma_Q \]
\[ \leq A_3 \rho(R) R^{\rho(R)}. \] (42)

Notice that \( d_1 + d_2 \frac{W(R)}{V(R)} = O(1) \) in (35), as \( R \to \infty \). We have (8) holds and
\[ V(R) \int_{\Omega; (1, R)} u^- (t, \Phi) \Psi(t) \frac{\partial \varphi}{\partial n} d\sigma_Q \leq A_3 \rho(R) R^{\rho(R)} \] (43)
from (8), (35) and (42). Hence, (43) implies (7) holds. The proof of it is similar to (41). So I omit it here. And then we obtain the result from Theorem 1.3.

6. Proof of Theorem 1.9. Consider the function \( u_1(P) = -u(P) \) for any \( P = (R, \Theta) \). Applying Theorem 1.3 to \( u_1(P) \), we obtain the conclusion of Theorem 1.9 from the boundedness of \( \varphi(\Theta) \).

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