Secure Coding via Gaussian Random Fields

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Abstract—Inverse probability problems whose generative models are given by strictly nonlinear Gaussian random fields show the all-or-nothing behavior: There exists a critical rate at which Bayesian inference exhibits a phase transition. Below this rate, the optimal Bayesian estimator recovers the data perfectly, and above it the recovered data becomes uncorrelated. This study uses the replica method from the theory of spin glasses to address information-theoretic problems, such as channel coding and data encryption. The notation \( \log(\cdot) \) indicates the natural logarithm, and \( \mathbb{E} \{ \cdot \} \) denotes mathematical expectation. The real axis is shown by \( \mathbb{R} \). For sake of brevity, \( \{1, \ldots, N\} \) and \( \{N_0, N_0 + 1, \ldots, N_1\} \) are abbreviated as \( [N] \) and \( [N_0 : N_1] \), respectively. The capacity of a real Gaussian channel with SNR \( x \) is denoted by

\[
C(x) = \frac{1}{2} \log (1 + x) .
\]

A. Contributions and Related Work

The main contribution of this study is as follows: We show that the secrecy capacity of the Gaussian wiretap channel given in (2) is achieved by employing a strictly nonlinear Gaussian random field as the encoder. The motivation comes from a recent work by Fyodorov [14], where a Gaussian random field is used for signal encryption. In particular, Fyodorov shows that encryption via a purely quadratic Gaussian random field shows an asymptotic phase transition at a threshold signal-to-noise ratio (SNR), below which recovery via the method of least-squares becomes uncorrelated. Our initial investigations in Section V] shows that combining Fyodorov’s encryption with a simple sphere coding technique can achieve a perfect secrecy rate close to the secret capacity of the wiretap channel. This interesting finding motivates a new coding scheme to achieve reliable and perfectly secure transmission over the Gaussian wiretap channel.

Intuitively, the proposed encoder can be seen as a combination of Sourlas’ coding [12] with the random linear binning technique [17]. Decoding is performed via a standard Bayesian decoder. Using the replica method, we show that the Bayesian decoder asymptotically exhibits the all-or-noting behavior [18]–[20] when a strictly nonlinear Gaussian random field is used as the encoder. This finding is then used to show that the proposed coding scheme asymptotically achieves the secrecy capacity of the Gaussian wiretap channel.

B. Notation

Scalars, vectors and matrices are represented with non-bold, bold lower-case and bold upper-case letters, respectively. The transposed of \( A \) is indicated by \( A^T \). The Euclidean norm of \( x \) is denoted by \( \|x\| \). For \( K \)-dimensional vectors \( x \) and \( y \), we define the normalized inner-product as

\[
\langle x; y \rangle = \frac{x^T y}{K} .
\]

C. Gaussian Random Fields

Gaussian random fields are key components in this paper. We hence define them at this point: In a nutshell, a Gaussian random field is a randomized mapping whose output entries are Gaussian. The mapping is not necessarily linear, i.e. a random matrix, and can be of higher orders. It can hence be observed as an extension of Gaussian random matrices. The exact definition is given below.

Definition 1: The mapping \( \mathcal{V}(\cdot) : \mathbb{R}^K \rightarrow \mathbb{R}^N \) is a Gaussian random field with covariance function \( \Phi(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \), if for any pair of vectors \( s_1, s_2 \in \mathbb{R}^K \), the entries of

\[
\mathcal{V}(s_i) = [\mathcal{V}_1(s_i), \ldots, \mathcal{V}_N(s_i)]^T ,
\]

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for $i \in \{1, 2\}$, are distributed Gaussian and satisfy
\[
\mathbb{E} \{ \mathcal{V}_m (s_1) \mathcal{V}_n (s_2) \} = \mathbb{I} \{ m = n \} \Phi (s_1; s_2),
\]  
for $m, n \in [N]$, with $\mathbb{I} \{ m = n \}$ being the indicator function.

A basic example of a Gaussian random field is the linear field with covariance function $\Phi (u) = u$. The Gaussian field in this example can be represented as $\mathcal{V} (s) = \mathbf{A} s$, where $\mathbf{A} \in \mathbb{R}^{N \times K}$ is an independent and identically distributed (i.i.d.) Gaussian matrix whose entries are zero-mean with variance $1/K$. More examples can be found in [16].

II. STATEMENT OF MAIN RESULT

Consider a Gaussian wiretap channel in which a transmitter intends to securely send a message of $K = NR$ bits to a legitimate receiver over an additive white Gaussian noise (AWGN) channel by $N$ uses of the channel. An eavesdropper overhears the transmitted signal through an independent AWGN channel.

Let $M \in [2^{NR}]$ denote the secret message. The transmitter encodes $M$ via a secure encoder $f_N (\cdot) : [2^{NR}] \mapsto \mathbb{R}^N$ into the codeword $x = [x_1, \ldots, x_N]^T$, such that
\[
\frac{1}{N} \sum_{n=1}^{N} \mathbb{E} \{ |x_n|^2 \} \leq P,
\]  
for some average transmit power $P$. It then transmits the codeword over the AWGN channel using $N$ subsequent transmission time intervals. The legitimate receiver therefore receives
\[
y = x + w_B,
\]  
for an i.i.d. Gaussian noise vector $w_B$ whose entries are zero-mean with variance $\sigma_B^2$. The legitimate receiver employs the decoder $g_N (\cdot) : \mathbb{R}^N \mapsto [2^{NR}]$ to estimate the secret message, i.e., $\hat{M} = g_N (y)$.

The eavesdropper overhears the transmitted sequence $x$ over an independent AWGN channel and receives
\[
y_E = x + w_E,
\]  
for i.i.d. Gaussian noise $w_E$ whose entries are zero-mean with variance $\sigma_E^2$. It is further assumed that the encoder and decoder are publicly known to all parties.

A. Reliable and Secure Transmission

The secret message $M$ contains $K = NR$ information bits and is transmitted within $N$ channel uses. It is hence concluded that the transmission rate in this setting is $R$ bits per channel use. We now focus on a sequence of encoder-decoder pairs indexed by $N$ which transmit secret messages with rate $R$. The transmission is said to be reliable and secure with this encoder-decoder sequence if the following conditions are satisfied [21]:

- The recovery error $\Pr \{ M \neq \hat{M} \}$ tends to zero as the dimension $N$ grows large.
- The information leakage to eavesdropper, i.e.,
\[
L_N = \frac{1}{N} I (M; y_E),
\]  
tends to zero as the dimension $N$ grows large.

The secrecy capacity of the wiretap channel is defined as the maximum reliable and secure transmission rate. Using the random binning approach of Wyner [22], the secrecy capacity of this Gaussian wiretap channel is shown to be [13]
\[
C_S = \left[ \mathcal{C} \left( \frac{P}{\sigma_B^2} \right) - \mathcal{C} \left( \frac{P}{\sigma_E^2} \right) \right]^+, \tag{9}
\]  
where $[x]^+ = \max \{0, x\}$.

B. Secure Coding via Gaussian Random Fields

Invoking Gaussian random fields, we now propose a secure coding scheme. Our main result shows that under some heuristic assumptions, this scheme achieves the secrecy capacity. The proposed scheme uses a strictly nonlinear Gaussian random field to securely encode the secret message. The decoder then recovers the message via the optimal Bayesian estimator.

To state the proposed coding scheme, let $s \in \{ \pm 1 \}^K$ be the bipolar representation of the message $M$, i.e., each information bit of $M$ is shown by $\pm 1$. For secure encoding, the transmitter follows the following steps:

1) It generates at random
\[
\tilde{K} = \left\lceil \frac{N}{\log 2 \mathcal{C} \left( \frac{P}{\sigma_E^2} \right)} \right\rceil \tag{10}
\]  
i.d. uniform bipolar symbols, i.e., $k = [k_1, \ldots, k_\tilde{K}]^T$.

2) It adds $k$ as a prefix to $s$ and finds $\tilde{s}$ as $\tilde{s} = \Pi [k^T, s^T]^T$, where $\Pi$ is a $(K + \tilde{K}) \times (K + \tilde{K})$ random permutation whose permuted vector $\tilde{s}$ has the following property: Let the $\ell$-th bin of $\tilde{s}$ with size $B = [1 + K/\tilde{K}]$ be
\[
[\tilde{s}]_\ell = \{ s_k : k \in [\ell - 1]B + 1 : \ell B \}. \tag{11}
\]  
Then, for every choice of $\ell \in [\tilde{K}]$, there exists only one symbol of $k$ in $[\tilde{s}]_\ell$.

3) It determines the codeword as $x = \mathcal{V} (\tilde{s})$, for a Gaussian field $\mathcal{V} (\cdot) : \mathbb{R}^{K + \tilde{K}} \mapsto \mathbb{R}^N$ whose covariance function is $\Phi (u) = Pu^A$ for some $\lambda \geq 3$.

For recovery, the legitimate receiver applies minimum mean squared error (MMSE) estimation to recover the message. To this end, the legitimate receiver follows the following steps:

1) It calculates the sufficient statistic $\tilde{r} = \mathbb{E} [\tilde{s} | y_B, \mathcal{V}]$. With uniform prior distribution, the MMSE estimator is given by
\[
\tilde{r} = \frac{1}{Z} \sum_{u \in \{ \pm 1 \}^{K + \tilde{K}}} u \exp \left\{ -\frac{||y_B - \mathcal{V} (u)||^2}{2\sigma_B^2} \right\}, \tag{12}
\]  
for normalization $Z$ defined as
\[
Z = \sum_{u \in \{ \pm 1 \}^{K + \tilde{K}}} \exp \left\{ -\frac{||y_B - \mathcal{V} (u)||^2}{2\sigma_B^2} \right\}. \tag{13}
\]  

2) It sets $r = \Pi^T \tilde{r}$, finds $\tilde{s}_k = \text{sgn} \left( r_{K+k} \right)$ for $k \in [K]$ with $\text{sgn} (\cdot)$ being the sign operator, and finds $\hat{M}$ from $\tilde{s}$.

One can see the connection between $\Pi$ and the random binning technique [17, 23]; see also [24, 25] and references therein for more details on the random binning technique and its applications.
C. The Main Result

The main result of this study states that the proposed coding scheme achieves the secrecy capacity of the wiretap channel. We present this result explicitly below:

Result 1: Consider the sequence \( \{ (f_N, g_N) : N \in \mathbb{Z} \} \), where \( f_N \) and \( g_N \) denote the encoder and decoder in Section II-B respectively. Set \( K = NC_S / \log 2 \), where \( C_S \) is the secrecy capacity of the wiretap channel. Then, the transmission via this sequence is asymptotically reliable and secure.

Proof. The proof is given in Section V.\end{proof}

Considering Result 1 it is worth mentioning few remarks:

- We first note the constraint on the covariance of \( \mathcal{V} (\cdot) \), i.e., \( \lambda \geq 3 \). It indicates that the codewords are generated via a mapping whose total number of coefficients grows with the message length at least cubically. In general, using a Gaussian field with covariance function \( \Phi (w) = w^2 \), the codewords are generated via an order \( \lambda \) polynomial from a set of \( NK^\lambda \) coefficients.

- The proposed scheme is randomized, as the mapping \( \mathcal{V} (\cdot) \) is generated at random. Nevertheless, unlike the classical random coding which generates all \( N2^K \) components of the codebook at random, the proposed scheme uses \( NK^\lambda \) random components to specify \( \mathcal{V} (\cdot) \). It then constructs the codewords from the messages using \( \mathcal{V} (\cdot) \).

- The key property of the proposed scheme which leads to achieving the secrecy capacity of the wiretap channel is the so-called non-linearity of the Gaussian field. In fact, by adding a linear term to the encoding function, one can observe that the information leakage to the eavesdropper does not vanish asymptotically. This is justified by a rather known property of the linear model: Linear models always carry information about the model parameter.

III. ASYMPTOTICS OF THE CODING SCHEME

The derivation of Result 1 relies on the asymptotic characterization of a class Bayesian algorithms used for unsupervised learning in nonlinear generative models. The detailed derivations are given in the extended manuscript [16]. In the sequel, we state a particular form of the generic result in [16] which describes the asymptotic properties of the decoder when it is employed to decode a message encoded via a Gaussian random field. This result is then utilized to sketch a proof for Result 1.

A. Asymptotic Characterization of the Bayesian Decoder

The statistics of \( \mathbf{s} \) are asymptotically described via the results of [16]. We illustrate the asymptotic characterization through the following setting: Consider a vector of uniform bipolar symbols \( \mathbf{s}_0 \in \{ \pm 1 \}^K \) which is mapped via a Gaussian random field \( \mathcal{V} (\cdot) \) into \( \mathbf{x}_0 \in \mathbb{R}^N \). The mapped vector \( \mathbf{x}_0 \) is observed as \( \mathbf{y}_0 = \mathbf{x}_0 + \mathbf{w}_0 \) for Gaussian noise \( \mathbf{w}_0 \) whose entries are zero-mean with variance \( \sigma^2_w \). Let \( \mathbf{r}_0 \) be the optimal MMSE estimation of \( s_0 \) from the observation \( \mathbf{y}_0 \) with side information \( \mathcal{Y} (\cdot) \), i.e.,

\[
\mathbf{r}_0 = \mathbb{E} \{ s_0 | \mathbf{y}_0, \mathcal{Y} \} = \sum_{s_0 \in \{ \pm 1 \}^K} s_0 \mathbb{P} (s_0 | \mathbf{y}_0, \mathcal{Y}) ,
\]

where the posterior distribution \( \mathbb{P} (s_0 | \mathbf{y}_0, \mathcal{Y}) \) is determined for the true prior belief on \( s_0 \) and the true noise variance \( \sigma^2_w \). From the asymptotic results of [16], we can derive the following metrics of this setting in the asymptotic regime, i.e., \( N, K \uparrow \infty \) with bounded \( R = K/N \):

1) The asymptotic information rate which is defined as

\[
\mathcal{I} (s_0) = \lim_{N \uparrow \infty} \frac{1}{N} \mathbb{I} (s_0; \mathbf{y}_0 | \mathcal{Y}) .
\]

2) The asymptotic joint distribution of an encoded-decoded pair, i.e., \( (s_0k, r_0k) \) for \( k \in [K] \).

The final expressions for these metrics are given in closed forms in terms of the so-called decoupled setting. In the sequel, we first define the decoupled setting and then give the closed-form expression for the above metrics.

B. Decoupled Setting

Corresponding to the vector-valued setting in Section II-A we define the decoupled setting as follows: Consider the uniformly distributed \( s \in \{ \pm 1 \} \). This symbol is passed through an AWGN channel whose noise variance is controlled by the parameter \( m \in [0, 1] \). Namely, the observation is given by

\[
y (m) = s + \frac{w}{\sqrt{E (m)}} ,
\]

where \( w \) is zero-mean and unit-variance Gaussian noise, and

\[
E (m) = \frac{\Phi (m)}{\Phi (1)(-\Phi (m))},
\]

with \( \Phi (\cdot) \) denoting the covariance function of \( \mathcal{V} (\cdot) \).

In this setting, the MMSE estimator of \( s \) is given by

\[
r (m) = \tanh (E (m) y (m)) .
\]

The input-output mutual information, i.e.,

\[
I_D (m) = I (s; y (m)) ,
\]

is further given by

\[
I_D (m) = E (m) - \mathbb{E} w \left\{ \log \cosh \left( E (m) + \sqrt{E (m) w} \right) \right\} .
\]

We now define a new metric which is controlled by \( m \): The energy function is defined as

\[
\mathcal{L} (m) = RI_D (m) + C_D (m) + (1 - m) C_D^* (m) ,
\]

for the function \( C_D (m) \) which is given by

\[
C_D (m) = C \left( \frac{\Phi (1) - \Phi (m)}{\sigma^2_0} \right) .
\]

Proposition 1: Let \( m^* \) be the minimizer of the energy function \( \mathcal{L} (m) \) over \( [0, 1] \). Then, \( (s_{0k}, r_{0k}) \) for \( k \in [K] \) converges in distribution to the decoupled pair \( (s, r (m^*)) \), and the asymptotic information rate is given by \( \mathcal{I} (s_0) = \mathcal{L} (m^*) \).

Proof. The proof is directly concluded from Corollary 1 and Proposition 2 in the extended manuscript [16].\end{proof}
IV. ALL-OR-NOTHING PHENOMENON

The parameter $m^*$ in Proposition [1] is often referred to as the overlap. This appellation comes from the fact that $m^*$ gives the expected normalized inner product between $s_0$ and $r_0$: As $m^*$ is a minimizer of the energy function $\mathcal{L}(m^*) = 0$ as a necessary condition. This leads to

$$m^* = \mathbb{E}_w \left\{ \tanh \left( \sqrt{E(m^*)} w + E(m^*) \right) \right\}, \quad (23a)$$

$$= \mathbb{E}(r(m^*)s) \overset{(a)}{=} \mathbb{E}\{\langle s_0; r_0 \rangle\}, \quad (23b)$$

where $(a)$ follows from Proposition [1] considering the convergence of $\langle s_{0k}; r_{0k} \rangle$ to $\langle s, r(m) \rangle$ in distribution. As the result, $m^*$ bounds the fraction of bipolar symbols in $s_0$ which are correctly decoded after hard-thresholding $r_0$.

Proposition [1] gives an interesting finding about the overlap. To illustrate this finding, we focus on the special form of fields used in the proposed coding scheme, i.e., a Gaussian field with covariance function $\Phi(u) = Pu^\lambda$. For these Gaussian fields, we use Proposition [1] to investigate the behavior of $m^*$ against $R$. This leads to the following conclusions:

- For $\lambda = 1$, i.e., linear fields, the overlap starts from $m^* = 1$ at small rates, i.e., $R \downarrow 0$, and continuously decreases by growth of $R$. It however never reaches zero, i.e., $m^* \neq 0$ for any choice of $R$.

- For $\lambda = 2$, i.e., purely quadratic fields, the overlap starts from $m^* = 1$ at small rates, i.e., $R \downarrow 0$, and shows a first-order phase transition at a critical rate $R^*$ at which the overlap jumps from $m^* = 1$ to $0 \neq m^* < 1$ discontinuously. It then shows a second-order phase transition at a threshold rate $R_{\text{Th}}$ at which $m^* = 0$ for $R \geq R_{\text{Th}}$.

- For $\lambda \geq 3$, the overlap shows a first-order phase transition at the critical rate $R^*$ at which the overlap jumps from $m^* = 1$ to $m^* = 0$ and remains zero for $R \geq R^*$.

The detailed proof of the above findings is out of the scope of a conference paper and can be followed in [16, Section IV].

The above results lead to this conclusion: For $\lambda \geq 3$, the energy function $\mathcal{L}(m)$ has two local minima at $m = 0$ and $m = 1$ and a local maximum at some $0 < m_0 < 1$. For $R < R^*$, the global minimum is at $m = 1$, leading to $m^* = 1$. This means that for $R < R^*$, $r_0$ perfectly recovers $s_0$. For $R \geq R^*$, the overlap is zero meaning that $r_0$ and $s_0$ are uncorrelated. This behavior is often called all-or-nothing phenomenon and is observed in various inference problems, e.g., see [18–20].

A. Heuristic Derivation of the Critical Rate $R^*$

A rigorous derivation of the critical rate faces complications. However, using the all-or-noting phenomenon, the rate can be derived heuristically: Noting that the global minimum occurs at $m = 0$ or $m = 1$, one can compare the energy function at these two points.

As $\Phi(0) = \Phi'(0) = 0$, we have $\mathcal{L}(0) = \mathcal{C}(P/\sigma_0^2)$; however, an explicit calculation of the energy function at $m = 1$ is not trivial. We hence use the following approximation: For $x \gg 0$, we can approximately write $\log \cosh(x) \approx x - \log 2$.

Now, let us define random variable $\tilde{w} = E(1) + \sqrt{E(1)}w$. Since $\tilde{w}$ is a Gaussian random variable with mean and variance $E(1) > 0$, the probability of $\tilde{w}$ being close to zero or negative is negligible. We hence use the above approximation and write

$$\mathbb{E}_{\tilde{w}} \{ \log \cosh(\tilde{w}) \} \approx \mathbb{E}\{\tilde{w}\} - \log 2 = E(1) - \log 2. \quad (24)$$

Hence $\mathcal{L}(1) \approx R \log 2$. This leads to the conclusion that $R^* = \frac{1}{\log 2} \mathcal{C}\left(\frac{P}{\sigma_0^2}\right)$.\quad (25)

The all-or-nothing phenomenon is further observed in terms of the information rate: At $R^*$, the information rate changes from $I(\sigma_0) = R \log 2$ (at $m^* = 1$) to $I(\sigma_0) = \mathcal{C}\left(P/\sigma_0^2\right)$ (at $m^* = 0$). In the former case, the end-to-end channel is noiseless and hence the information rate equals to the entropy rate of the message; however, by the phase transition, the end-to-end channel becomes noisy and the information rate is restricted by the channel capacity. This finding indicates that by using a strictly nonlinear Gaussian field as the encoder and an MMSE decoder, the coding scheme shows a sharp phase transition at the channel capacity. This result confirms the earlier findings reported by Sourlas in [12] and [26].

B. Numerical Validations

To validate our heuristic derivations, we numerically determine the overlap and the asymptotic information rate, given by Proposition [1] and compare them with the derivations in Section IV-A. To this end, we set $\sigma_0^2 = 0.1$ and $P = 1$ and plot the overlap and information rate against $R$ for various choices of $\lambda$. The results are shown in Figs. [1] and [2]. As the figures show, for $\lambda = 3$ the first-order phase transition occurs exactly at the heuristically derived rate. By plotting the figures for larger choices of $\lambda$, one can observe that the figures are not numerically distinguishable from the one given for $\lambda = 3$; see [16]. We hence conjecture that for $\lambda \geq 3$ the overlap jumps at $R^*$ from exactly being one to exactly being zero. A rigorous proof of this conjecture is however skipped at this point.

V. DERIVATION OF THE MAIN RESULT

We now invoke our asymptotic derivations to sketch a proof for Result [1]. To this end, we consider the proposed coding schemes in Section [1-B] and show that it leads to a reliable and secure transmission, when we set $R = K/N = C_2/\log 2$. For brevity, we focus on scenarios with non-zero secrecy rates, i.e., $\sigma_B^2 > \sigma_E^2$. Noting that the proof invokes derivations based on the replica method, it is natural that it contains some heuristics. For brevity, we use the notation $C_i = \mathcal{C}(P/\sigma_i^2)$ for $i \in \{E, B\}$.\quad (More precisely, a Gaussian random field whose covariance function has a polynomial expansion with cubic and/or larger terms.)
We now consider the end-to-end channel from $\tilde{f}$ probability is given by the other hand, we have
\[ I \{ \tilde{f} \} = 0 \]
where
\[ I \{ \cdot \} \]
denotes the indicator function.

Noting that $|\tilde{r}| \leq 1$, we have
\[ \langle \tilde{s}; \text{sgn}(\tilde{r}) \rangle = 1 - 2f. \]  
\[ \text{Note that } |\tilde{r}_k| \leq 1, \text{ we can write} \]
\[ \langle \tilde{s}; \text{sgn}(\tilde{r}) \rangle \geq 1 - \langle \tilde{s}; \tilde{r} \rangle - f. \]
This concludes that $f \leq 1 - \langle \tilde{s}; \tilde{r} \rangle$.

From Proposition 1, we know that $E \{ \langle \tilde{s}; \tilde{r} \rangle \}$ in the large-system limit is determined by the overlap. We hence conclude that in the asymptotic limit, $E \{ f \} \leq 1 - m^\ast$. As the result, $m^\ast = 1$ guarantees the existence of a Gaussian field by which a reliable transmission is possible.

The all-or-nothing phenomenon of Gaussian fields with $\lambda \geq 3$ indicates that $m^\ast = 1$ is achievable when $R + C_E/\log 2 \leq C_S/\log 2$. This proves the reliability for any $R \leq C_S/\log 2$.

A. Proof of Reliability

We start the derivation by noting that $\langle \tilde{s}; \text{sgn}(\tilde{r}) \rangle = 1 - 2f$. On the other hand, we have
\[ \langle \tilde{s}; \text{sgn}(\tilde{r}) \rangle = 1 - 2f. \]  
\[ \text{Noting that } |\tilde{r}_k| \leq 1, \text{ we can write} \]
\[ \langle \tilde{s}; \text{sgn}(\tilde{r}) \rangle \geq 1 - \langle \tilde{s}; \tilde{r} \rangle - f. \]
This concludes that $f \leq 1 - \langle \tilde{s}; \tilde{r} \rangle$.

From Proposition 1, we know that $E \{ \langle \tilde{s}; \tilde{r} \rangle \}$ in the large-system limit is determined by the overlap. We hence conclude that in the asymptotic limit, $E \{ f \} \leq 1 - m^\ast$. As the result, $m^\ast = 1$ guarantees the existence of a Gaussian field by which a reliable transmission is possible.

The all-or-nothing phenomenon of Gaussian fields with $\lambda \geq 3$ indicates that $m^\ast = 1$ is achievable when $R + C_E/\log 2 \leq C_S/\log 2$. This proves the reliability for any $R \leq C_S/\log 2$.

B. Proof of Security

We start the security proof by writing
\[ I (s; y_E | V) = I (\tilde{s}; y_E | V) - I (k; y_E | s, V). \]  
Since for $R > 0, R + C_E/\log 2 > C_E/\log 2$, we conclude that
\[ \lim_{N \to \infty} \frac{1}{N} I (\tilde{s}; y_E | V) = C_E, \]  
based on the all-or-nothing phenomenon.

To calculate the second term, we first note that
\[ I (k; y_E | s, V) = I (\tilde{s}; y_E | s, V). \]  
The right hand side is the mutual information in a genie-aided setting in which the eavesdropper recovers $\tilde{s}$ while the message $s$ is revealed to it: Let $s = v$ be a particular realization of the message being known to the eavesdropper. We note that after permutation there exists only one unknown bipolar symbol in each bin of $\tilde{s}$, i.e., the symbol of $k$ available in the bin. Thus, each bin describes a binary unknown in the inference problem. The effective transmission rate over the channel from $\tilde{s}$ to $y_E$ in this case is $K/N = C_E/\log 2$. We hence use Proposition 1 and write
\[ \lim_{N \to \infty} \frac{1}{N} I (\tilde{s}; y_E | s = v, V) = C_E. \]  
As the right hand side does not depend on $v$, we have
\[ \lim_{N \to \infty} \frac{1}{N} I (\tilde{s}; y_E | s, V) = C_E. \]  
This concludes the security proof.
VI. CONCLUSIONS

Bayesian estimation over a channel with AWGN whose generative model is described by a strictly-nonlinear Gaussian random field shows a first-order phase transition at the capacity of the AWGN channel: By setting the rate arbitrarily close to the channel capacity, perfect recovery via MMSE estimation is asymptotically guaranteed; however, by exceeding the channel capacity, the Bayesian estimation becomes uncorrelated. This all-or-nothing phenomenon is used to establish a secure communication in a Gaussian wiretap channel. Our investigations demonstrate that a coding scheme, whose encoder constructs the codewords by passing the message through a Gaussian random field, asymptotically achieves the secrecy capacity of the Gaussian wiretap channel.

A natural direction for future work is to develop an approximate message passing algorithm for Bayesian decoding. Using such implementation, the performance of the proposed scheme can be further investigated for finite-length transmissions. The work in this direction is currently ongoing.

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