Chiral Expansion from Renormalization Group Flow Equations

Lars Jendges,1 Bertram Klein,2 Hans-Jürgen Pirner,1,3 and Kai Schwenzer4

1Institute for Theoretical Physics, University of Heidelberg, Philosophenweg 19, 69120 Heidelberg
2GSI, Planckstrasse 1, 64291 Darmstadt
3Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg
4Institute for Physics, University of Graz, Universitätsplatz 5, A-8010 Graz

(Dated: March 26, 2022)

We explore the influence of the current quark mass on observables in the low energy regime of hadronic interactions within a renormalization group analysis of the Nambu–Jona-Lasinio model in its bosonized form. We derive current quark mass expansions for the pion decay constant and the pion mass, and we recover analytically the universal logarithmic dependence. A numerical solution of the renormalization group flow equations enables us to determine effective low energy constants from the model. We find values consistent with the phenomenological estimates used in chiral perturbation theory.

I. INTRODUCTION

Chiral Perturbation Theory (χPT) is an effective field theory which describes the low energy limit of the physics of the strong interaction where its theoretical description, Quantum Chromodynamics (QCD), is entirely non-perturbative. It is based on the expansion of an effective Lagrangian in terms of small external momenta and the pion mass. A finite pion mass arises from the small explicit breaking of chiral symmetry. As an effective field theory, χPT is defined for scales much smaller than the chiral symmetry breaking scale $\Lambda_{\chi} \sim 1$ GeV, and it is renormalizable order by order in the small momentum expansion [1, 2, 3]. The physics at low momenta is completely dominated by the light pions. Other fluctuations are separated from the light Goldstone bosons by a mass gap and therefore suppressed. Due to the constraints of chiral symmetry, all quantum corrections are due to pions and depend on the pion mass parameter. However, going beyond leading order in the chiral expansion, the number of undetermined coupling constants in the chiral Lagrangian increases rapidly.

The chiral expansion in small pion masses and momenta gives rise to terms that are non-
analytic in the pion mass, i.e logarithmic corrections appear at the one-loop level. In leading order, these are universal corrections and entirely determined by the symmetry. Therefore, they should be reproduced by any theory that describes the low energy domain within a suitable approximation. On the lattice, the appearance of these chiral logarithms has not yet been unambiguously demonstrated, but recent results remain compatible with the existence of such corrections \cite{4}. It is our purpose to show that the logarithmic terms are recovered in the pion sector of the bosonized Nambu–Jona-Lasinio (NJL) model \cite{5,6}, and to determine them in a Renormalization Group (RG) approach.

An effective field theory can be considered as the limit of a more fundamental theory at low energy. Only fields relevant in this energy range are retained, while all other degrees of freedom are integrated out. The effects of these higher modes enter into low energy couplings (LECs) which multiply the remaining operators in the low energy theory. These LECs have to be obtained either from experiment \cite{3}, or by non-perturbative methods from the underlying gauge dynamics of QCD. Because of the strong gauge coupling at low energies and the emergence of collective hadronic degrees of freedom this remains a difficult task.

Recent progress in a non-perturbative description has been made using Dyson-Schwinger Equations (DSE) \cite{7,8,9}, or RG flow equations \cite{10,11,12,13,14}. Numerical lattice simulations are limited by the computational effort and require extrapolations to large volumes, small quark masses, and small lattice spacing. The extrapolation to small quark mass is actively pursued \cite{4} and an important topic for \chi PT. Finite volume effects have been addressed in \chi PT as well, both for the nucleon \cite{15} and for the pion \cite{16,17,18,19}, and in the framework of the quark meson model with RG methods \cite{20,21,22}.

Due to the importance of the low energy couplings on one hand, and the difficulty of obtaining them from QCD on the other, there has been a strong interest in phenomenological models \cite{23,24,25,26,27,28,29,30,31,32,33,34,35}.RG methods are uniquely suited to interpolate between the low energy effective theory and QCD \cite{33}.

There are bottom-up or top-down models. In the bottom-up approach, the starting point is \chi PT. Additional vector, axial-vector and scalar meson resonances may be included. In ref. \cite{24}, it was argued that the coupling constants of the chiral Lagrangian in next-to-leading order could be accounted for mainly by contributions of vector meson resonances. In \cite{34} it was found that in a linear meson model the low energy couplings are dominated by the exchange of scalar mesons. Such descriptions rely entirely on a hadronic picture. On the
other hand, the top-down approach starts from models with quark degrees of freedom, which generate mesonic degrees of freedom by quark dynamics. These approaches are essentially based on the NJL model \[5, 6\], which includes a fermionic self-interaction. Meson fields arise from a bosonization of the quark interaction with a Hubbard-Stratonovich transformation \[36\]. Pions appear explicitly as the Goldstone bosons of spontaneous breaking of the chiral symmetry through a quark condensate. For a review of this class of models, see e.g. \[29, 33\]. A comparison with $\chi$PT has been undertaken e.g. in \[26, 27, 28, 31, 32\]. More recently, a third approach based on a weak form of the ADS/CFT correspondence \[37\] shows a remarkable agreement with hadron phenomenology \[38\].

In the RG treatment \[39, 40, 41, 42\] of NJL-based models, chiral symmetry is dynamically broken, and the pions emerge naturally in the RG flow of the bosonized model. In order to capture the low energy dynamics dominated by the pions, the meson dynamics has to be included in the RG flow. This goes beyond the leading order of the often employed large-$N_c$ approximation \[43, 44, 45\]. Only the inclusion of such $1/N_c$ corrections \[42\] makes it possible to derive the leading-order logarithmic corrections in the quark mass expansion of low energy observables. In particular, our results show that the wave function renormalization $Z_\phi$ for the derivative term of the mesonic fields is essential, since it enters into the expression for the pion decay constant $f_\pi$, the coupling constant for the lowest-order derivative term in $\chi$PT (compare \[34\]).

In a comparison of RG results to $\chi$PT, one would like to calculate the same effective couplings. Already the lowest order $\chi$PT Lagrangian contains information about arbitrary pion $n$-point functions, which makes an order-by-order comparison with the results of RG methods difficult, since the effective RG action is expanded in $n$-point functions. Alternatively, one can proceed in the same way as in the comparison with experimental results. By calculating observables and identifying the influence of the low energy constants one can compare effective values for these couplings. In this way one does not actually have to project the RG flow onto the operators used in the chiral expansion. We will employ this method in the present paper. Our comparison between $\chi$PT and the NJL model will be limited to the two flavor case and of course to only those observables that appear in both theories. In the specific case of the bosonized NJL model, which we define at a large UV scale, there are several free parameters. The UV parameters are fixed by requiring that the model should reproduce the physical values for the pion decay constant $f_\pi$ and the pion mass $m_\pi$, after
all quantum fluctuations have been integrated out. While the actual values of $f_\pi$ and $m_\pi$ are input parameters to the model and not predicted, their predicted dependence on the current quark mass tests whether the model as a low energy description of meson physics is compatible with other approaches. Since the low energy constants depend on the underlying short distance theory, they differentiate whether the model contains the relevant dynamics for the emergence of the low energy physics. Such a consistency check can put the model on a more sound footing, even without a direct connection to the gauge dynamics of QCD.

The main part of this paper is organized as follows: After a short review of some results of chiral perturbation theory at one-loop order in section II we will present the RG approach to the bosonized NJL model and introduce the RG flow equations in section III. Analytic results for the non-analytic dependence of the pion mass and the pion decay constant on the symmetry breaking parameter are given in section IV. Numerical results on the same quantities as function of the current quark mass are given in section V. We will demonstrate that the results from the NJL model for the couplings that appear in the pion mass and the pion decay constant are compatible with the values used in $\chi$PT calculations. We present a summary and our conclusions in section VI. Details of the derivation of the RG flow equations can be found in the appendix.

II. RESULTS FROM CHIRAL PERTURBATION THEORY

In this section we give a short summary of some principles and results of $\chi$PT at the one-loop level following the work of Gasser and Leutwyler and the recent more pedagogic article. Starting point is the Lagrangian $L_{\chi PT}$ of the non-linear sigma model in the O(4) chiral field $\hat{U} = (\hat{U}^0, \hat{U}^i)$ of unit length expanded up to fourth order in momentum. We give this Lagrangian with the 10 phenomenological coupling constants ($l_i, h_i$) here for later comparison with the RG-results

$$L_{\chi PT} = \frac{f_\pi^2}{2} \nabla_\mu \hat{U}^T \nabla^\mu \hat{U} + 2Bf_\pi^2 (s^0 \hat{U}^0 + p^i \hat{U}^i) + l_1 (\nabla^\mu \hat{U}^T \nabla_\mu \hat{U})^2 + l_2 (\nabla^\mu \hat{U}^T \nabla^\nu \hat{U})(\nabla_\mu \hat{U}^T \nabla_\nu \hat{U}) + l_3 (\chi^T \hat{U})^2 + l_4 (\nabla^\mu \chi^T \nabla_\mu \hat{U}) + l_5 (\hat{U}^T F^{\mu \nu} F_{\mu \nu} \hat{U}) + l_6 (\nabla^\mu \hat{U}^T F_{\mu \nu} \nabla^\nu \hat{U}) + l_7 (\hat{\chi}^T \hat{U})^2 + h_1 \chi^T \chi + h_2 tr(F_{\mu \nu} F^{\mu \nu} + h_3 \hat{\chi}^T \hat{\chi}),$$ (1)
with

\[ \nabla_\mu \hat{U}^0 = \partial_\mu \hat{U}^0 + a_i^\mu \hat{U}^i \]
\[ \nabla_\mu \hat{U}^i = \partial_\mu \hat{U}^i + \varepsilon^{ikl} v_\mu^k \hat{U}^l - a_i^\mu \hat{U}^0 \]
\[ \chi = 2B(s^0, p^i), \]
\[ \tilde{\chi} = 2B(p^0, -s^i) \]
\[ F_{\mu\nu} \hat{U} = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \hat{U}, \]

where \( v_\mu(x), a_i^\mu(x), s(x) \) and \( p^i(x) \) are external vector, axialvector, scalar and pseudoscalar fields. Without isospin breaking the constants \( l_7 \) and \( h_3 \) are zero.

The constant \( f_{\pi 0} \) is the pion decay constant in the chiral limit, which in the literature of chiral perturbation theory usually is called \( F \), and \( B \) parametrizes the chiral condensate in the chiral limit,

\[ B = -\frac{1}{2} \frac{\langle \bar{q}q \rangle_0}{f_{\pi 0}^2} = -\frac{1}{2} \frac{\langle \bar{u}u + \bar{d}d \rangle_0}{f_{\pi 0}^2}. \]

In the two flavor case treated in this work, we consider the masses of the two lightest flavors to be equal. In order to take the quark mass term into account one expands the scalar external field around \( s = m_c \), where \( m_c \) is the average current quark mass:

\[ m_c = \frac{1}{2}(m_u + m_d) \]

One obtains the Gell-Mann–Oakes–Renner relation, where the parameter \( M \) is the pion mass in lowest order in the current quark mass

\[ M^2 = 2m_c B = -m_c \frac{\langle \bar{q}q \rangle_0}{f_{\pi 0}^2}. \]

The non-linear sigma model is not renormalizable, but it is possible to make the theory finite at every loop order by introducing appropriate counter terms. In chiral perturbation theory this renormalization is usually done by dimensional regularization. Up to one-loop order, one needs the above ten low energy constants \( l_i \) and \( h_i \) which depend logarithmically on the renormalization scale \( \mu \),

\[ l_i^r = \frac{\gamma_i}{32\pi^2} (\bar{l}_i + \log \frac{M^2}{\mu^2}) \quad i = 1, \ldots, 7 \]
\[ h_i^r = \frac{\delta_i}{32\pi^2} (\bar{h}_i + \log \frac{M^2}{\mu^2}) \quad i = 1, 2, 3. \]
The low energy constants \( \bar{l}_i \) are independent of the renormalization scale and are related to physical observables. The constants \( \bar{h}_i \) have no direct physical relevance, but follow from the renormalization procedure. To go to higher loop order, an increasingly larger number of constants is needed. By computing Greens functions from chiral perturbation theory in the limit of very small external momenta, one derives physical observables which are expanded in \( M^2 \). As shown above, \( M^2 \) is related to the current quark mass \( m_{c} \). For future reference, we quote the chiral expansions for the pion decay constant, the chiral quark condensate and the pion mass,

\[
\begin{align*}
  f_\pi &= f_{\pi 0} \left(1 - \frac{M^2}{16\pi^2 f_\pi^2} \log \frac{M^2}{\mu^2} + \frac{l_4^r}{f_\pi^2} M^2 + O(M^4)\right) \\
  \langle \bar{q}q \rangle &= \langle \bar{q}q \rangle_0 \left(1 - \frac{3M^2}{32\pi^2 f_\pi^2} \log \frac{M^2}{\mu^2} + \frac{2(h_1^r + l_3^r)}{f_\pi^2} M^2 + O(M^4)\right) \\
  m_{\pi}^2 &= M^2 \left(1 + \frac{M^2}{32\pi^2 f_\pi^2} \log \frac{M^2}{\mu^2} + 2\frac{l_3^r}{f_\pi^2} M^2 + O(M^4)\right).
\end{align*}
\]

We will compare our RG results with these expansions.

The expansions are characterized by the so-called chiral logarithms and by higher order terms in the expansion parameter \( M^2 \). It is possible to combine the chiral logarithms with the linear term in \( M^2 \) (cf. equations (5) and (6)), so that the constants which appear do not depend on the renormalization scale \( \mu \) any more, but only on \( M^2 \),

\[
\begin{align*}
  f_\pi &= f_{\pi 0} \left(1 + \frac{M^2}{16\pi^2 f_\pi^2} \bar{l}_4 + O(M^4)\right) \\
  \langle \bar{q}q \rangle &= \langle \bar{q}q \rangle_0 \left(1 + \frac{M^2}{32\pi^2 f_\pi^2} \bar{l}_3 + O(M^4)\right) \\
  m_{\pi}^2 &= M^2 \left(1 - \frac{M^2}{32\pi^2 f_\pi^2} \bar{l}_3 + O(M^4)\right).
\end{align*}
\]

The renormalization scale-independent constants \( \bar{l}_i \) are parametrized in terms of the constants \( \Lambda_i^2 \),

\[
\begin{align*}
  \bar{l}_i &= \log \frac{\Lambda_i^2}{M^2} \\
  \Lambda_3^{\chi PT} &= 0.59^{+1.40}_{-0.41} \text{ GeV}, \; \Lambda_4^{\chi PT} = 1.26 \pm 0.14 \text{ GeV}
\end{align*}
\]

The \( M^2 \)-dependence is contained in the logarithms, \( \Lambda_3^{\chi PT} \) and \( \Lambda_4^{\chi PT} \) are scale-independent constants, as given in (46), while \( \bar{h}_1 \) is not determined in chiral perturbation theory since \( \langle \bar{q}q \rangle \) is not a physical quantity.
III. RENORMALIZATION GROUP FLOW EQUATIONS FOR THE
BOSONIZED NJL MODEL

In this section, we give a brief overview of the renormalization group approach to the NJL
model. The detailed approximation scheme and the derivation of the flow equations are
presented in [47, 48]. Here we only review the main results and give the flow equations that
are used for the analysis of the chiral low energy expansion in the next section.

The basic idea of the renormalization group is to describe the dependence of an effective
action on an infrared cutoff scale, and to follow the evolution of the couplings under a change
of this cutoff scale. In this way, quantum fluctuations are integrated out in a systematic
way, and a theory with bare couplings defined at a given UV scale is transformed into an
effective low energy theory in which all quantum fluctuations with large momenta are already
included in the couplings [49, 50].

Our starting point is the NJL model [5, 6] with quark fields interacting via a point-like four-
fermion interaction. We use this model of dynamical chiral symmetry breaking to describe
the transition from quark fields to hadronic degrees of freedom for scales $k \lesssim 1$ GeV. The
Lagrangian reads in Euclidean spacetime

$$L_{\text{NJL}} = \bar{q}(\partial + m_c)q - g_{\text{NJL}} (\bar{q}q + (\bar{q}i\gamma_5 q)^2).$$

This Lagrangian contains the dominant chirally symmetric four-fermion interaction. Chiral
symmetry is preserved by the interaction term, but broken explicitly by a finite current quark
mass $m_c$. By bosonization of the quark fields at the UV scale, we obtain the associated linear
$\sigma$-model [51], where the chiral symmetry of the quark fields $(\bar{q}, q)$ is for two quark flavors
reflected in an $O(4)$-symmetry of the meson fields $\Phi = (\sigma, \vec{\pi})$,

$$L^{\text{UV}} = \bar{q}(\partial + m_c)q + g_{\text{UV}} [(\bar{q}q + (\bar{q}i\vec{\pi}5 q)^2) + U_{\text{UV}}(\Phi^2, \sigma, k_{\text{UV}})$$

with

$$U_{\text{UV}}(\Phi^2, \sigma, k_{\text{UV}}) = \frac{m_{\text{UV}}^2}{2}(\sigma^2 + \vec{\pi}^2) - \delta \sigma,$$

$$\delta = \frac{m_{\text{UV}}^2}{g_{\text{UV}}} m_c.$$

The parameter $\delta$ depends on the current quark mass and describes explicit chiral symmetry
breaking.
We sketch the derivation of renormalization group flow equations for the effective action \( \Gamma \) of the bosonized NJL model in the Schwinger proper-time regularization scheme \[52\]. The effective action is the generating functional of the one-particle irreducible Greens functions of quarks and meson fields. The derivative expansion of the effective action up to next-to-leading order is given by

\[
\Gamma[\bar{q}, q, \Phi] = \int d^4x \left( U(\Phi^2, \sigma) + \frac{1}{2} Z_{ab}^a(\Phi^2)(\partial_{a}\Phi^a)(\partial_{b}\Phi^b) + G(\Phi^2) \bar{q}(\sigma + i\bar{\tau}\gamma_5) q + Z_q(\Phi^2) \bar{q}\bar{q}q \right) \tag{19}
\]

In general, the effective wave-function renormalization factors \( Z_{ab}^a(\Phi^2) \), \( Z_q(\Phi^2) \), as well as the effective Yukawa coupling \( G(\Phi^2) \) in this action, are functions of the bosonic fields \( \Phi \) and thereby include arbitrary many “elementary” couplings - in full analogy to the effective potential \( U(\Phi^2, \sigma) \). In ref. \[47\] a solution is given for this general case. In this work we only treat the effective potential \( U(\Phi^2, \sigma) \) as field-dependent, while we approximate the other couplings by taking only their first, \( \Phi \)-independent term into account, i.e.

\[
Z_{ab}^a(\Phi^2) \equiv Z_{ab}(\sigma_0^2) \delta^{ab}, \quad G(\Phi^2) \equiv G(\sigma_0^2), \quad Z_q(\Phi^2) \equiv Z_q(\sigma_0^2), \tag{20}
\]

where \( \sigma_0 \) denotes the vacuum expectation value from the effective potential \( U \). We use the same wavefunction renormalization \( Z_{ab}(\sigma_0^2) \) for all mesonic fields. The flow equations for the bosonized NJL model in the proper-time renormalization group scheme are derived in ref. \[47\]. These equations for the individual couplings of the effective action take the following general form:

\[
k \frac{\partial U(\Phi^2,k)}{\partial k} = - \frac{k^6}{32\pi^2} \left( 4N_c N_f \frac{1}{k^2 + M_q^2} - 3 \frac{1}{k^2 + M_\pi^2} - \frac{1}{k^2 + M_\sigma^2} \right) \tag{21}
\]

\[
k \frac{\partial Z_{ab}(\sigma_0^2,k)}{\partial k} = - \frac{k^6}{16\pi^2} Z_{\Phi} \left( \frac{4N_c N_f G^2}{(k^2 + M_q^2)^3 Z_q^2 Z_{\Phi}} + 4\Lambda^2 \frac{F_f^2}{(k^2 + M_q^2)^2 (k^2 + M_\pi^2)^2} \right) \bigg|_{\sigma_0^2} \tag{22}
\]

\[
k \frac{\partial G(\sigma_0^2,k)}{\partial k} = - \frac{k^6}{16\pi^2} Z_q^2 Z_{\Phi} \left( \frac{6k^2 + 3M_q^2 + 3M_\pi^2}{(k^2 + M_q^2)^2 (k^2 + M_\pi^2)^2} - \frac{2k^2 + M_q^2 + M_\sigma^2}{(k^2 + M_q^2)^2 (k^2 + M_\sigma^2)^2} \right) \bigg|_{\sigma_0^2} \tag{23}
\]

\[
k \frac{\partial Z_q(\sigma_0^2,k)}{\partial k} = - \frac{k^6 G^2}{32\pi^2} Z_q Z_{\Phi} \left( \frac{9M_q^2 + M_\pi^2}{(k^2 + M_q^2)^2 (k^2 + M_\pi^2)^2} - \frac{6k^2}{(k^2 + M_q^2)^2 (k^2 + M_\pi^2)^2} \right) \bigg|_{\sigma_0^2} + 3 \frac{M_q^2 + M_\sigma^2}{(k^2 + M_q^2)^2 (k^2 + M_\pi^2)^2} - 2k^2 \frac{M_\sigma^2}{(k^2 + M_q^2)^3 (k^2 + M_\pi^2)^3} \bigg|_{\sigma_0^2}. \tag{24}
\]

The flow equations describe the evolution of the effective potential \( U \), the Yukawa coupling \( G \), and the wave function renormalizations \( Z_{ab} \) and \( Z_q \) under a change of the renormalization
scale \( k \). The effective masses in the flow equations are given by

\[
M_q^2(\Phi^2, k) = \frac{G^2}{Z_\Phi^2} \Phi^2, \tag{25}
\]

\[
M_\pi^2(\Phi^2, k) = \frac{2}{Z_\Phi} \frac{\partial U}{\partial \Phi^2}, \tag{26}
\]

\[
M_\sigma^2(\Phi^2, k) = \frac{2}{Z_\Phi} \left( \frac{\partial U}{\partial \Phi^2} + \frac{\partial^2 U}{\partial \Phi^2} \Phi^2 \right), \tag{27}
\]

the effective pion decay constant \( F_\pi(\Phi^2) \) by

\[
F_\pi^2 = Z_\Phi \Phi^2, \tag{29}
\]

and the effective four-boson coupling \( \Lambda(\Phi^2) \) by

\[
\Lambda = \frac{2}{Z_\Phi^2} \frac{\partial^2 U(\Phi^2)}{(\partial \Phi^2)^2} = \frac{M_\sigma^2 - M_\pi^2}{2 Z_\Phi \Phi^2}. \tag{30}
\]

They depend on the field \( \Phi^2 \) and the scale \( k \). Evaluated at the vacuum expectation value \( \Phi = \sigma_0 \), the masses reduce to the physical masses at the scale \( k \)

\[
\begin{align*}
m_q(k) &= M_q(\Phi^2, k) \big|_{\Phi^2 = \sigma_0^2}, \\
m_\pi(k) &= M_\pi(\Phi^2, k) \big|_{\Phi^2 = \sigma_0^2}, \\
m_\sigma(k) &= M_\sigma(\Phi^2, k) \big|_{\Phi^2 = \sigma_0^2}. \tag{31}
\end{align*}
\]

A flow equation for the vacuum expectation value \( \langle \sigma \rangle = \sigma_0 \) can be derived from the minimum condition of the effective potential,

\[
\frac{\partial}{\partial \sigma} U(\Phi^2, \sigma, k) \bigg|_{\sigma = \sigma_0, \Phi^2 = 0} = 0. \tag{32}
\]

Details of the derivation can be found in appendix A, here we only quote the result

\[
k \frac{\partial}{\partial k} \sigma_0 = -\frac{2}{Z_\Phi M_\sigma^2(\sigma_0^2)} k \frac{\partial}{\partial k} U_0'(\Phi^2, k) \bigg|_{\Phi^2 = \sigma_0^2}, \tag{33}
\]

where \( U_0(\Phi^2, k) \) denotes the symmetric part of the effective potential, without the symmetry breaking term proportional to \( \delta \), and the prime denotes a derivative in the symmetric variable \( \Phi^2 \). Note that the set of RG flow equations \( (21)-(24) \) depends only on the symmetric variable \( \Phi^2 \). Explicit symmetry breaking introduced by the linear term \( \delta \sigma \) affects only the initial condition \( \sigma_0(k_{UV}) \) in eq. (33).

Two generic solutions of the renormalization group flow equations are shown in Fig. 1. For zero current quark mass \( m_c = 0 \), spontaneous chiral symmetry breaking sets in at the
FIG. 1: The scale-dependent constituent quark mass $m_q$ and the meson masses ($m_\sigma, m_\pi$) as functions of the infrared cutoff parameter $k$. Shown are results for simplified solutions of the RG flow equations with constant wave function normalizations $Z_\Phi = Z_q \equiv 1$ and constant Yukawa coupling $G$. Dashed lines show the chiral limit $m_c \to 0$, while solid lines give the solutions with explicit symmetry breaking. The grey bars indicate the bands in which the masses are approximated as constant in the analytical calculation of section IV.

chiral symmetry breaking scale $k_{\chi_{SB}}$ and then develops fully in the infrared. This picture is characterized by a non-vanishing quark mass $m_q$ and a sigma mass $m_\sigma \approx 2m_q$ in the infrared. Below the chiral symmetry breaking scale, the evolution of the vacuum expectation value $\sigma_0$ quickly freezes. For a finite current quark mass $m_c$, see full lines in Fig. 1. The transition from the UV-dynamics to the IR-dynamics is smooth, the meson masses never become zero.

IV. ANALYTICAL LOW ENERGY EXPANSIONS FROM THE RENORMALIZATION GROUP

In this section, we show that the RG flow equations of the bosonized NJL model generate the proper leading order pion contributions to physical observables. Due to the long-range fluctuations of the pions, a logarithmic term (chiral log) appears in the quark mass expansion. The RG flow equations generate such a term. Their full solution is best suited to sum up all long-range fluctuations, as we know from the physics of second order phase transitions. Since the RG flow equations go beyond an expansion in powers of the coupling constants,
they cannot be easily mapped onto a perturbative expansion. However, we can identify
terms with the same analytical behavior in the lowest order pion mass, the parameter
$M$, and make a direct comparison to $\chi$PT results. In particular, the one-loop perturbative result
can be recovered from RG flow equations by replacing all running masses by constant masses
in lowest order [53]. Treating the small pion mass as constant in the RG flow equations is
justified only below the chiral symmetry breaking scale $k_{\chi SB}$, where the pion exists as a
low mass and low momentum degree of freedom and pion fluctuations are the dominating
effects. The chiral symmetry breaking scale $k_{\chi SB}$ in the RG approach appears as a UV cutoff.
Although $k_{\chi SB}$ sets the scale for the logarithmic dependence on the pion mass parameter
$M^2$, additional contributions may modify the value.

As a consequence of fixing the masses, the flow equations (21) and (22) decouple from
equations (23) and (24). This can be seen by substituting equations (29) and (30) in equation
(22) and replacing the ratio $G^2 Z_q^2$ by $\frac{m^2_q}{\sigma_0^2(k)}$. Therefore the evolution of $G$ and $Z_q$ is not relevant
for the solution of equations (21) and (22). The system reduces to only two relevant equations
for the quantities we consider, one for the effective potential $U$ and one for the wave function
renormalization $Z_\phi$,

$$
\frac{d}{dk} \frac{dU}{dk} = -\frac{k^6}{32\pi^2} \left( \frac{4N_c N_f}{k^2 + m^2} - 3 \frac{1}{k^2 + m^2} - \frac{1}{k^2 + m^2} \right) \quad (34)
$$

$$
\frac{d}{dk} \frac{dZ_\phi}{dk} = -\frac{k^6}{16\pi^2\sigma_0^2} \left( \frac{4N_c N_f}{(k^2 + m^2)^3} m^2_q + \frac{(m^2_q - m^2)^2}{(k^2 + m^2)^2 (k^2 + m^2)^2 (k^2 + m^2)^2} \right). \quad (35)
$$

As pointed out in the introduction, the inclusion of the wave function renormalization $Z_\phi$ in
the RG flow is crucial to obtain the correct behavior for the low $k$-dynamics. We emphasize
that the masses that enter on the right hand side of the flow equations (34) and (35) are set
constant in the $k$-flow and depend only on the parameter $M^2$. This is indicated by using
small letters for the masses.

To compare this approach with chiral perturbation theory, we expand the pion decay con-
stant, the chiral quark condensate and the pion mass in the current quark mass $m_c$. We
distinguish between the chiral symmetric and the explicitly broken system by keeping the
current quark mass $m_c$ in all expressions. The parameter $M^2$ in the $\chi$PT-expansion is
directly related to $m_c$ by the Gell-Mann–Oakes–Renner relation. In lowest order we identify

$$
M^2 = 2m_c B, \quad (36)
$$

where $B$ is one of the low energy constants that appear in the chiral Lagrangian [1].
For explicit symmetry breaking the effective potential \( U(\Phi^2, \sigma) \) depends on \( M^2 \) via the sigma field

\[
U(\Phi^2, \sigma) = U_0(\Phi^2) - \delta\sigma, \tag{37}
\]

\[
U_0(\Phi^2) = \frac{\lambda}{4}\left(\Phi^2 - \Phi_0^2\right)^2, \tag{38}
\]

where we expanded the potential up to fourth order. We parametrize the mesonic fields and the symmetry breaking parameter as

\[
\Phi = \begin{pmatrix} \sigma \\ \vec{\pi} \end{pmatrix} \quad \text{and} \quad \delta = \frac{m_{\text{UV}}^2}{g_{\text{UV}}} m_c. \tag{39}
\]

The chiral and physical minima of the effective potential are defined by

\[
\Phi_0 = \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix} \quad \text{and} \quad \Phi_{\text{phys}} = \begin{pmatrix} \sigma_0 \\ 0 \end{pmatrix}, \tag{40}
\]

which determines the pion decay constants in the chiral limit and the physical case as

\[
f_{\pi 0} = \sqrt{Z_\Phi} \phi_0 \quad \text{and} \quad f_\pi = \sqrt{Z_\Phi} \sigma_0. \tag{41}
\]

We will denote the difference between the physical pion decay constant and the decay constant in the chiral limit by \( \Delta f_\pi \equiv f_\pi - f_{\pi 0} \). The renormalized coupling constant is \( \lambda_R = \frac{\lambda}{Z_\Phi} \).

Starting from the mass equations (25), (26) and (27), we identify the \( M^2 \)-dependence of the quark and meson masses. For the pion mass we find

\[
m_{\pi 0}^2 = \frac{1}{Z_\Phi} \left. \frac{\partial^2 U_0(\Phi^2)}{\partial \pi^2} \right|_{\Phi_0} = 0, \tag{42}
\]

\[
m_{\pi}^2 = \frac{1}{Z_\Phi} \left. \frac{\partial^2 U_0(\Phi^2)}{\partial \pi^2} \right|_{\Phi_{\text{phys}}} = 2\lambda_R f_{\pi 0} \Delta f_\pi + \mathcal{O}(\Delta f_\pi^2). \tag{43}
\]

As in chiral perturbation theory, the parameter \( M \) is the pion mass in lowest order of the chiral expansion

\[
m_{\pi}^2 = M^2 + \mathcal{O}(M^4), \tag{44}
\]

which enables us to identify

\[
M^2 = 2\lambda_R f_{\pi 0} \Delta f_\pi. \tag{45}
\]
Analogously, we derive the expansions for the sigma mass and the constituent quark mass:

\[
m^2_{\sigma_0} = \frac{1}{Z_\Phi} \left. \frac{\partial^2 U_0(\Phi^2)}{\partial \sigma^2} \right|_{\Phi_0} = 2\lambda_R f_{\pi0}^2, \tag{46}
\]

\[
m^2_{\sigma} = \frac{1}{Z_\Phi} \left. \frac{\partial^2 U_0(\Phi^2)}{\partial \sigma^2} \right|_{\Phi_{\text{phys}}} = m^2_{\sigma_0} + 3M^2, \tag{47}
\]

\[
m_{q0} = \frac{G}{Z_q} |\Phi| \left. \right|_{\Phi_0} = \frac{G}{Z_q} \phi_0, \tag{48}
\]

\[
m_q = \frac{G}{Z_q} |\Phi| \left. \right|_{\Phi_{\text{phys}}} = \frac{G}{Z_q \sqrt{Z_\Phi}} (f_{\pi0} + \Delta f_{\pi}) = m_{q0} + AM^2, \tag{49}
\]

where the coefficient of the \(M^2\)-term of the constituent quark mass is given by

\[
A = \frac{m_{q0}}{2f_{\pi0}^2\lambda_R} = \frac{m_{q0}}{m^2_{\sigma_0}}. \tag{50}
\]

Once the values \(m_{\sigma_0}\) and \(m_{q0}\) for the sigma mass and the constituent quark mass in the chiral limit are fixed, the \(M^2\)-correction terms in \(m_{\sigma}\) and \(m_q\) enter the flow equations. Although these terms do not contribute to the logarithmic corrections in \(m_{\pi}^2\) and \(f_{\pi}\), they influence the \(M^2\)-corrections.

We first discuss the results for the pion decay constant \(f_{\pi} = \sigma_0 \sqrt{Z_\Phi}\). To derive its flow we need the evolution equations for \(\sigma_0\) and \(Z_\Phi\). With the parametrization of the effective potential \(U(\Phi^2, \sigma)\) (eq. (37)), we can express the evolution of the expectation value through the evolution of the effective potential. The expectation value satisfies

\[
\frac{\partial}{\partial \sigma} U(\Phi^2, \sigma) \bigg|_{\sigma = \sigma_0, \bar{\pi}^2 = 0} = \frac{\partial U_0(\Phi^2)}{\partial \Phi^2} \bigg|_{\Phi^2 = \sigma_0^2} 2\sigma_0 - \delta = U'_0(\sigma_0^2) \frac{1}{2} \sigma_0 - \delta = 0, \tag{51}
\]

since necessarily \(\bar{\pi} = 0\). The prime denotes the derivative of \(U_0(\Phi^2)\) with respect to \(\Phi^2\).

While the coefficient of the symmetry breaking term \(\delta\) does not evolve under the RG flow, all other coefficients in the potential \(U_0(\Phi^2)\) evolve, so that \(\sigma_0 = \sigma_0(k)\) is a function of the RG scale. Details of the derivation can be found in appendix \(A\). Combining eq. (51) with the flow equation for the derivative coupling \(Z_\Phi\), we find

\[
k \frac{\partial}{\partial k} f_{\pi}^2 = \frac{k^6}{8\pi^2} \left\{ \frac{1}{(k^2 + m_{\pi}^2)^2} + \frac{1}{(k^2 + m_{\sigma}^2)^2} + \frac{1}{(k^2 + m_{\pi}^2)(k^2 + m_{\sigma}^2)} \\
- \frac{3}{2} \frac{m_{\pi}^2}{m_{\sigma}^2} \left( \frac{1}{(k^2 + m_{\pi}^2)^2} + \frac{1}{(k^2 + m_{\sigma}^2)^2} \right) \\
- \frac{1}{2} 4N_cN_f m_q^2 \left( \frac{2}{m_{\sigma}^2} \left( \frac{1}{(k^2 + m_{\pi}^2)^2} + \frac{1}{(k^2 + m_{\sigma}^2)^3} \right) \right) \right\} \tag{52}
\]
Integrating this equation, we find a logarithmic correction to $f_\pi$
\[
f_\pi = f_{\pi 0} \left( 1 - \frac{M^2}{16\pi^2 f_{\pi 0}^2} \log \frac{M^2}{k^2_{\chi_{SB}}} + \mathcal{O}(M^2) \right). \tag{53}
\]

The structure of the result is exactly the same as in $\chi$PT (cf. eq. (1)). Crucial for this result is the inclusion of the bosonic wavefunction renormalization, which enters at order $1/N_c$ and is not recovered in the standard large $N_c$-limit. While the generic form of the logarithmic term arises entirely from IR pion fluctuations and is universal, the scale inside the logarithm as well as all other analytic corrections depend on the detailed dynamics at larger scales and cannot be trusted in this simplified analytic approximation. These contributions have to be studied in a detailed numerical analysis of the non-perturbative flow equations performed in the next section.

Analogously to the pion decay constant, we can derive a flow equation for the square of the pion mass $m_{\pi}^2$. As can be seen from equations (26) and (31), the physical pion mass is derived from the potential $U(\Phi^2)$,
\[
m_{\pi}^2 = \left. \frac{2}{Z_\Phi} \frac{\partial U}{\partial \Phi^2} \right|_{\Phi = \Phi_{\text{phys}}}. \tag{54}
\]

Differentiating the flow equation for the effective potential (34) with respect to $\Phi^2$ and combining it with the equation for $Z_\Phi$ (eq. (35)) leads us to the flow equation for $m_{\pi}^2$
\[
k \frac{\partial}{\partial k} m_{\pi}^2 = \frac{2}{Z_\Phi} k \frac{\partial U'}{\partial \Phi^2} - \frac{2}{Z_\Phi^2} k \frac{\partial Z_\Phi}{\partial k} + \frac{2 U''(\sigma_0^2)}{Z_\Phi} k \frac{\partial \sigma_0^2}{\partial k}
\]
\[
= \frac{k^6 m_{\pi}^2}{16\pi^2 f_{\pi 0}^2} \left( \left( -\frac{1}{2} + \frac{3}{2} m_{\pi}^2 \right) \left( \frac{1}{(k^2 + m_{\pi}^2)^2} + \frac{1}{(k^2 + m_{\sigma}^2)^2} \right) - \frac{2}{(k^2 + m_{\pi}^2)^2} \right)
\]
\[
+ \frac{4N_c N_f}{(k^2 + m_{\sigma}^2)^2} m_{\pi}^2 \left( \frac{1}{k^2 + m_{\pi}^2} + \frac{1}{m_{\sigma}^2} \right) \tag{55}
\]

We remind the reader that in this equation all values on the right hand side are considered constant in $k$ and in lowest order in the chiral expansion, i.e. in particular $m_{\pi}^2 \equiv M^2$ and $f_{\pi} \equiv f_{\pi 0}$. Similarly to the case for $f_\pi$, integration gives us the logarithmic correction to $m_{\pi}^2$
\[
m_{\pi}^2 = M^2 \left( 1 + \frac{M^2}{32\pi^2 f_{\pi 0}^2} \log \frac{M^2}{k^2_{\chi_{SB}}} + \mathcal{O}(M^2) \right). \tag{56}
\]

As before for $f_\pi$, we also find for $m_{\pi}^2$ the correct logarithmic behavior compared with $\chi$PT. Again we do not calculate the contributions linear in $M^2$, which we will compute numerically in the following section. To conclude, we emphasize that the RG approach to the NJL model
shows analytically the same low energy behavior as $\chi$PT for the two observables $f_\pi$ and $m_\pi^2$ considered here. It should be mentioned that the values of the zeroth order terms $f_{\pi 0}$ and $M^2$ depend on the specific values of the integration boundary $k_{\chi SB}$. With the approximation of constant masses we cannot obtain reasonable zeroth order values with the same scale $k_{\chi SB}$ for $f_\pi$ and $m_\pi^2$. For completeness, we have also derived the chiral expansion of the quark condensate $\langle \bar{q} q \rangle$ in appendix B. It also reproduces the logarithmic term known from $\chi$PT.

V. NUMERICAL EVALUATION OF THE LOW ENERGY CONSTANTS

In this section we present numerical results for the pion decay constant and the pion mass from the NJL model as functions of the current quark mass which parametrizes explicit chiral symmetry breaking. As demonstrated in the previous section, we reproduce the logarithmic dependence of $f_\pi$ and $m_\pi^2$ on the symmetry breaking parameter, since the fluctuations of the pions are correctly included. In chiral perturbation theory, the low energy coupling constants of the next-to-leading order chiral Lagrangian manifest themselves as scales $\Lambda_i$ in the logarithmic terms of the chiral expansion. These scales contain modifications due to additional terms proportional to $\sim M^2$.

For the numerical evaluation, we expand each coupling of the effective action (eq. (19)) in a Taylor series around the physical minimum $\Phi_{\text{phys}} = (\sigma_0(k), \vec{0})$. As discussed in section III, we take for the wavefunction renormalization functions $Z_\Phi$ and $Z_q$ and the Yukawa coupling $G$ only the first, $\Phi$-independent term into account. Since the effective potential $U$ determines the vacuum expectation value $\sigma_0$ and therefore is the most important coupling, we expand $U$ as follows:

$$U(\Phi^2, \sigma, k) = \sum_{i=1}^{4} a_i(k)(\Phi^2 - \sigma_0(k)^2)^i - \delta \sigma$$  \hspace{1cm} (57)

where as usual $\Phi^2 = \sigma^2 + \vec{\pi}^2$. The minimum condition for $\sigma_0(k)$,

$$\frac{\partial}{\partial \sigma} U(\Phi^2, \sigma, k) \bigg|_{\sigma = \sigma_0(k), \vec{\pi}^2 = 0} = 0,$$  \hspace{1cm} (58)

gives as an additional constraint

$$a_1(k) = \frac{\delta}{2\sigma_0(k)}.$$  \hspace{1cm} (59)

We solve the coupled set of four flow equations (21)-(24) numerically for different values of the UV cutoff and for a wide range of current quark masses $m_c$. To obtain its bosonized
form we perform a Hubbard-Stratonovich transformation of the initial NJL model. In this bosonized Lagrangian the kinetic term for the mesonic fields and higher meson interaction terms are zero, so that the meson potential is characterized only by $m_{UV}$. This gives the full set of initial conditions at the UV scale:

$$U_{UV}(\Phi^2, \sigma) = \frac{1}{2} m_{UV}^2 \Phi^2 - \frac{m_{UV}^2}{G_{UV}} m_c \sigma$$

$$Z_{\Phi,UV} \equiv 10^{-9}$$

$$G_{UV} \equiv 1$$

$$Z_{q,UV} \equiv 1$$

(60)

For a given cutoff $\Lambda_{UV}$, we determine the parameter $m_{UV}$ and the physical current quark mass $m_{phys}$ by requiring that the values $f_\pi = 92.4 \pm 0.3$ MeV and $m_\pi = 138.0 \pm 1.0$ MeV are reproduced within these tolerances. This condition is the most natural choice and allows a calculation independent from $\chi$PT. As a consequence, the value of the physical current quark mass $m_{phys}$ varies with the UV cutoff $\Lambda_{UV}$. To compare our results with chiral perturbation theory it is necessary to connect our symmetry breaking parameter $m_c$ with the parameter $M^2$ used in $\chi$PT. As already explained in section IV, we set

$$M^2 = 2m_c B.$$  

In Figs. 2 and 3 we compare our results for $f_\pi$ and $m_\pi^2$ as functions of the current quark mass $m_c$ with those of $\chi$PT, obtained from [46]. We use a scaled current quark mass, since the absolute value of the quark mass does not have an immediate physical meaning and depends on a chosen scale. In ref. [46], the $\chi$PT results are plotted against the dimensionless variable $m_c/m_s$, where $m_c = m_u = m_d$ is the mass of the u- or d-quark, and $m_s$ is the mass of the strange quark. The physical pion decay constant and pion mass are obtained for $m_c/m_s = 1/26$.

In order to be able to compare our results with $\chi$PT, we introduce a scale $m_{phys}$ such that $m_c/m_{phys} = 1$ for the physical value $f_\pi = 92.4$ MeV, and we rescale the $\chi$PT results accordingly. Over the considered range of UV cutoffs, the value of $m_{phys}$ varies from $m_{phys} = 12.9$ MeV for $\Lambda_{UV} = 1.0$ GeV to $m_{phys} = 6.0$ MeV for $\Lambda_{UV} = 1.5$ GeV. By construction, the RG gives the physical value for $f_\pi$ at $m_c = m_{phys}$ for all UV cutoffs $\Lambda_{UV}$. As shown in Fig. 2, the curves $f_\pi(m_c/m_{phys})$ become steeper with increasing cutoff $\Lambda_{UV}$. For the cutoff value $\Lambda_{UV} = (1.00 \pm 0.05)$ GeV, the RG result falls within the band given by $\chi$PT.
FIG. 2: The pion decay constant $f_\pi$ is shown as a function of the scaled current quark mass $m_c/m_{\text{phys}}$ for different values of the UV cutoff. The shaded region is given by $\chi$PT, obtained from ref. [46]. The individual lines correspond to the full numerical RG results for different cutoffs from $\Lambda_{UV} = 1.5$ GeV down to $\Lambda_{UV} = 1.0$ GeV in steps of 0.1 GeV. The curve with the largest slope belongs to $\Lambda_{UV} = 1.5$ GeV, the curve with the smallest slope corresponds to $\Lambda_{UV} = 1.0$ GeV. All curves are fixed to agree at the physical point ($m_c/m_{\text{phys}} = 1$), which corresponds to $f_\pi = 92.4$ MeV.

chiral limit $m_c \to 0$, the pion decay constant from the RG and from $\chi$PT do not necessarily match. Depending on the value of the UV cutoff, our result for $f_{\pi 0} = f_\pi(m_c \to 0)$ varies between 79 MeV for a cutoff of $\Lambda_{UV} = 1.5$ GeV and 87 MeV for a cutoff of $\Lambda_{UV} = 1.0$ GeV. In $\chi$PT, the chiral limit of the pion decay constant is $f_{\pi 0} = 86.2 \pm 0.5$ MeV [17].

We note that the slope of $f_\pi$ as a function of $m_c/m_{\text{phys}}$ increases with increasing value of $\Lambda_{UV}$. In the NJL model, this is due to the increase of $m_{UV}$ with the cutoff $\Lambda_{UV}$, which can be seen from Tab. [11]. As a result, for a given current quark mass $m_c$, the linear symmetry breaking term $\delta = \frac{m_{UV}^2}{\Lambda_{UV}} m_c$ also increases with the cutoff. This leads to the increase of the slope of $f_\pi(\frac{m_c}{m_{\text{phys}}})$ with $\Lambda_{UV}$ observed in Figure 2.

The results for the pion mass $m_\pi^2$ are shown in Fig. 3 for the same set of UV cutoffs. In contrast to the pion decay constant, the slope of $m_\pi^2$ as a function of the symmetry breaking parameter decreases with increasing $\Lambda_{UV}$. The RG result falls within the band of the $\chi$PT result for a much larger range of cutoffs $\Lambda_{UV}$, compared with the result of the pion decay
FIG. 3: The pion mass is shown as function of the scaled current quark mass $m_c/m_{\text{phys}}$ for different UV cutoffs $\Lambda_{UV}$. The shaded region is given by $\chi$PT, obtained from ref. [46]. The lines correspond to the RG results for different cutoff choices, from $\Lambda_{UV} = 1.5$ GeV (smallest slope) down to 1.0 GeV (largest slope) in steps of 0.1 GeV.

constant. Since the corrections to $m_\pi^2$ are already of order $M^4$, and thus one order higher than the corrections to $f_\pi$, $m_\pi^2$ increases almost linearly with $m_c/m_{\text{phys}}$. We find that for $\Lambda_{UV} \approx 1.0$ GeV the results for $f_\pi$ and $m_\pi^2$ are consistent with those of $\chi$PT, see Figs. 2 and 3. However, if one does not consider any additional physical observable, there is a priori no reason to favor any particular value of the UV cutoff for the NJL model, as long as the cutoff varies between $1$ GeV < $\Lambda_{UV} < 1.5$ GeV. A cutoff much smaller than 1 GeV is not justified, since the phenomenological $\Lambda_4$-parameter is around 1 GeV. Consistent with this, ref. [47] finds that for a cutoff slightly below $\Lambda_{UV} < 1$ GeV the RG equations cannot reproduce the physical $f_\pi$. A much larger cutoff would extend the effective four-fermion interaction into a region where dynamical gluon effects become important.

While the leading-order behavior of $f_{\pi 0}$ and the slope of $m_\pi^2$ as a function of $m_c$ depend on the implementation of the symmetry breaking term in the model Lagrangian, the ratios $\frac{f_\pi}{f_{\pi 0}}$ and $\frac{m_\pi^2}{M^2}$ are more indicative of pion fluctuations (cf. Figs. 4). They are determined by the pion mass in lowest order $M$ and the low energy constants $\Lambda_3$ and $\Lambda_4$.

The pion mass $m_\pi^2$ is normalized with $M^2 = 2Bm_c$, where the value for $2B$ has been obtained
from the leading order fit to the pion mass. In the chiral limit, we have

$$\lim_{m_c \to 0} \frac{m_c^2}{M^2} = 1. \quad (61)$$

With this normalization, corrections to the linear $m_c$-dependence can be compared with results of $\chi$PT \cite{46}, see shaded regions in Figs. [4]. In general, the cutoff sensitivity is enhanced in the expansion around $m_c = 0$. At the maximum value of our calculation $m_c/m_{\text{phys}} = 3.4$ it amounts to 20%.

In order to obtain the low energy constants, we fit the pion decay constant $f_\pi(m_c)$ and the pion mass $m_\pi^2(m_c)$ (given in Figs. [4]) as functions of $m_c$ to the parametrization from $\chi$PT at one-loop order \cite{46}

$$f_\pi = f_{\pi 0} \left(1 - \frac{1}{16\pi^2} \frac{2Bm_c}{f_{\pi 0}^2} \log \frac{2Bm_c}{\Lambda_4^2} + \mathcal{O}(m_c^2)\right) \quad (62)$$

$$\frac{m_\pi^2}{m_c} = 2B \left(1 + \frac{1}{32\pi^2} \frac{2Bm_c}{f_{\pi 0}^2} \log \frac{2Bm_c}{\Lambda_3^2} + \mathcal{O}(m_c^2)\right). \quad (63)$$

We use current quark masses up to $m_c = 20$ MeV, which corresponds to pion masses up to $\sim 250$ MeV, depending on the exact value of the UV cutoff.

In Table [I] we show the pion decay constant in the chiral limit $f_{\pi 0}$ and the leading term of the pion mass $M(m_{\text{phys}}) \equiv \sqrt{2Bm_{\text{phys}}}$ as functions of the UV cutoff $\Lambda_{\text{UV}}$, $m_{\text{UV}}$ and $m_{\text{phys}}$.

As explained before, a fixed cutoff $\Lambda_{\text{UV}}$ leaves the two free parameters $m_{\text{UV}}$ and $m_{\text{phys}}$.

![Plot of ratios $f_\pi/f_{\pi 0}$ and $m_\pi^2/M^2$ as functions of the scaled current quark mass $m_c/m_{\text{phys}}$. The $\chi$PT results of ref. \cite{46} give the shaded regions, the RG flow equations yield the individual lines. For $f_\pi/f_{\pi 0}$, the RG results are shown from $\Lambda_{\text{UV}} = 1.5$ GeV (top line) to $\Lambda_{\text{UV}} = 1.0$ GeV (bottom line), and for $m_\pi^2/M^2$, from $\Lambda_{\text{UV}} = 1.0$ GeV (top line) to $\Lambda_{\text{UV}} = 1.5$ GeV (bottom line), in steps of 0.1 GeV.]

FIG. 4: We plot the ratios $f_\pi/f_{\pi 0}$ and $m_\pi^2/M^2$ as functions of the scaled current quark mass $m_c/m_{\text{phys}}$. The $\chi$PT results of ref. \cite{46} give the shaded regions, the RG flow equations yield the individual lines. For $f_\pi/f_{\pi 0}$, the RG results are shown from $\Lambda_{\text{UV}} = 1.5$ GeV (top line) to $\Lambda_{\text{UV}} = 1.0$ GeV (bottom line), and for $m_\pi^2/M^2$, from $\Lambda_{\text{UV}} = 1.0$ GeV (top line) to $\Lambda_{\text{UV}} = 1.5$ GeV (bottom line), in steps of 0.1 GeV.
TABLE I: Values for the pion decay constant in the chiral limit $f_{\pi 0}$, $M(m_{\text{phys}})$, the scale-independent low energy constants $\Lambda_3$ and $\Lambda_4$ and the scale-dependent quantities $\bar{l}_3$ and $\bar{l}_4$ at scale $M(m_{\text{phys}})$ from the fits to the RG results, dependent on the UV values $\Lambda_{UV}$ and $m_{UV}$. The values of $m_{\text{phys}}$ are determined by the condition that our result should reproduce the physical values of $f_{\pi}$ and $m_{\pi}$ at $m_{\text{phys}}$.

which are then fixed from $m_{\pi}^2$ and $f_{\pi}$. The different values from Tab. I for different cutoffs lead to the following intervals for $f_{\pi 0}$ and $M(m_{\text{phys}})$. We find over the range of UV cutoffs considered

$$79.4 \text{ MeV} \leq f_{\pi 0} \leq 86.9 \text{ MeV}, \quad 136.3 \text{ MeV} \leq M(m_{\text{phys}}) \leq 141.0 \text{ MeV}. \quad (64)$$

Fits to our full numerical RG results with equations (62) and (63) give the low energy constants $\Lambda_3$ and $\Lambda_4$, which are also listed in Tab. I. We extract from the table the ranges

$$0.04 \text{ GeV} \leq \Lambda_3 \leq 1.22 \text{ GeV}, \quad -2.29 \leq \bar{l}_3 \leq 4.31, \quad (66)$$

$$1.25 \text{ GeV} \leq \Lambda_4 \leq 1.30 \text{ GeV}, \quad 4.37 \leq \bar{l}_4 \leq 4.52. \quad (67)$$

These ranges are obtained by varying the UV cutoff between the maximal and minimal values in the model, determined by the momentum region for which the NJL model can be considered as a suitable phenomenological description. Changing the cutoff, one finds from Table I for example a large value for $f_{\pi 0}$ correlated with a small value of $\Lambda_3$. The values of
the low energy constants used in $\chi$PT \cite{46} are

\begin{align}
0.08 \text{ GeV} & \leq \Lambda_3^{\chi PT} \leq 1.99 \text{ GeV}, \quad 0.5 \leq \overline{l}_3^{\chi PT} \leq 5.3, \quad (68) \\
1.12 \text{ GeV} & \leq \Lambda_4^{\chi PT} \leq 1.40 \text{ GeV}, \quad 4.2 \leq \overline{l}_4^{\chi PT} \leq 4.6. \quad (69)
\end{align}

The constant $\Lambda_4 (\overline{l}_4)$ reflects the correction of the leading order result of $f_\pi$ due to quantum fluctuations, and thus is less dependent on the particular implementation of chiral symmetry breaking than $\Lambda_3 (\overline{l}_3)$, which determines the next-to-leading order term of the expansion of $m_\pi^2$.

FIG. 5: Comparison between the pion mass $m_\pi$ and the constituent quark mass $m_q$, obtained from the RG flow equations, as functions of the scaled current quark mass $m_c/m_{\text{phys}}$. A large mass gap exists between $m_\pi$ and $m_q$ for small current quark masses $m_c/m_{\text{phys}}$. The plot is made exemplarily for a UV cutoff $\Lambda_{UV} = 1.0$ GeV.

Deviations between the results from $\chi$PT and the RG treatment of the bosonized NJL model are to some degree expected, in particular for large values of the pion mass. In the NJL model the low momentum regime contains free constituent quarks with masses of roughly $\sim 300$ MeV (see Fig. 5) for realistic values of the pion mass and the pion decay constant. In contrast to the assumption in $\chi$PT, the mass gap between the light pseudo-Goldstone boson $m_\pi$ and the more massive constituent quark $m_q$ becomes smaller for larger current quark masses, and the dominance of the pion field fluctuations is lost. This suggests that effects of quark loops or higher ($\overline{q}q$) bound states may be enhanced for larger $m_c/m_{\text{phys}}$. Therefore we have investigated the range of $m_c/m_{\text{phys}}$ up to approximately 13. For cutoffs
FIG. 6: The plots show the pion decay constant $f_\pi$ and the pion mass $m_\pi$ for large values of the scaled quark masses $m_c/m_{\text{phys}}$ for the two cutoff choices $\Lambda_{UV} = 1.0$ GeV($m_{\text{phys}} = 12.9$ MeV) and $\Lambda_{UV} = 1.5$ GeV($m_{\text{phys}} = 6.0$ MeV). The shaded regions represent the results of $\chi$PT, while the bold curves give the result from RG flow equations. The curves are shown up to a quark mass $m_c \sim 170$ MeV and $m_c \sim 80$ MeV, respectively. Results from lattice calculations with Wilson fermions from [4] are plotted for comparison and represented by the bold dots, while the grey dots give results from [54] with staggered fermions.

$\Lambda_{UV} = 1.0$ GeV and $\Lambda_{UV} = 1.5$ GeV, this amounts to current quark masses up to $m_c \sim 170$ MeV and $m_c \sim 80$ MeV, respectively. The results for $f_\pi(m_c)$ and $m_\pi(m_c)$ are shown in Figs. 6. For $\Lambda_{UV} = 1.0$ GeV, the expected variations for large quark masses do not leave the band predicted by $\chi$PT [46]. For $\Lambda_{UV} = 1.5$ GeV, the value of $f_\pi$ for large quark masses deviates more strongly from $\chi$PT. In the same figure, we show results of recent lattice results with Wilson fermions, taken from [4], which were obtained with a new preconditioned Hybrid Monte-Carlo algorithm, and from [54] with staggered fermions.

While the description of these lattice results does not require chiral logarithms, they do not rule out such terms, either. A comparison of large quark-mass lattice simulations with the curves of Fig. 6 allows an extrapolation of lattice data to small quark masses, as long as they lie in the region covered by our calculation and their slope is compatible with our results. A set of values for $f_\pi$ and $m_\pi$ for large quark masses from our calculation can be found in table II in appendix D.
VI. CONCLUSIONS

We have studied the quark mass dependence of low energy observables within the NJL model, using renormalization group flow equations. We have shown that these equations analytically generate the non-analytic terms in the chiral expansion known from chiral perturbation theory. Our result confirms that the pion contributions to the low energy constants are treated correctly in the RG flow equations. For this it is essential to consider the renormalization of the kinetic term of the pion fields. While it is difficult in general to map non-perturbative RG results on a perturbative expansion, it is much easier to identify non-analytic parts. To derive these results, we treat the pion mass as a constant in the RG flow equation below the chiral symmetry breaking scale. Integrating the RG flow is then equivalent to performing a perturbative one-loop calculation. The resulting chiral logarithms $\sim M^2 \log M^2$ are accompanied by an analytic power series in $M^2$, where $M$ is the pion mass to lowest order. Going beyond the analytical results, we find that the numerical solution of the full set of the RG flow equations gives low energy constants which are compatible with the phenomenological values from $\chi$PT. As input to these equations, in the bosonized NJL model one needs the triple of an UV cutoff $\Lambda_{UV}$, a mass parameter $m_{UV}$ and a current quark mass $m_{phys}$. For a given value of the UV cutoff, the remaining input parameters are adjusted to reproduce the physical pion decay constant $f_\pi$ and the pion mass $m^2_\pi$ at the physical point. Therefore there is some freedom in choosing such a triple. Output of the RG approach are the values of the pion decay constant in the chiral limit $f_{\pi0}$, the pion mass in lowest order $M(m_{phys}) = \sqrt{2BM_{phys}}$, and the two low energy constants $\Lambda_3$ and $\Lambda_4$ (see Table I). When the UV cutoff $\Lambda_{UV}$ is varied in a reasonable range $1.0 \text{ GeV} < \Lambda_{UV} < 1.5 \text{ GeV}$, the chiral limit of the pion decay constant varies as $79.4 \text{ MeV} \leq f_{\pi0} \leq 86.9 \text{ MeV}$ and the pion mass in lowest order as $136.3 \text{ MeV} \leq M(m_{phys}) \leq 141.0 \text{ MeV}$. The corresponding low energy constants are $0.04 \text{ GeV} \leq \Lambda_3 \leq 1.22 \text{ GeV}$ and $1.25 \text{ MeV} \leq \Lambda_4 \leq 1.30 \text{ GeV}$. For the renormalization scale-independent quantities we find $-2.29 \leq \bar{l}_3 \leq 4.31$ and $4.37 \leq \bar{l}_4 \leq 4.52$ at the mass scale $M(m_{phys})$, see equation (13).

The NJL model lacks confinement, and therefore our model contains free quarks. As a consequence, the mass gap between the light pseudo-Goldstone bosons and the other hadronic states is reduced to a mass gap between the constituent quarks and pions. Nevertheless, our results show that the values of the low energy constants which we obtained from the
model are compatible with chiral perturbation theory. On the other hand it is well known that the linear $\sigma$-model without quarks cannot reproduce chiral perturbation theory: Quark loops are essential to mimic effects of higher mass meson resonances like the $\rho$ meson.

The topic of low energy constants in phenomenological models has been addressed in many publications, see e.g. [26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. Our results complement an earlier study [34] that aimed at the low energy expansion within the linear $\sigma$-model and gave estimates for the low energy constants $L_4 - L_8$ in the $SU(3) \otimes SU(3)$ parametrization, which correspond to $l_1 - l_3$, $l_5$ and $l_6$ in the $SU(2) \otimes SU(2)$ case. These results showed a qualitative agreement with phenomenology, but relied on a mean field approximation and introduced an arbitrary matching scale. Since these assumptions are not necessary in our approach, it would be an interesting challenge to study also the low energy expansion of $\chi PT$ at NLO. This can in principle be done without additional assumptions by an extension of our effective action to NNLO in the derivative expansion.

The results presented here demonstrate that the NJL model gives a reasonable description of the infrared chiral dynamics and its quark mass dependence. By construction, it is based on the same representation of the chiral flavor symmetry as QCD. Therefore, it involves degrees of freedom that go beyond the low energy dynamics of the pions that are fixed by the symmetry. The requirement that a model must be consistent with the physical low energy constants provides constraints on the full set of possible dynamical theories at large momentum scales. We find that including quark effects, the results from the model are compatible with $\chi PT$, whereas a linear $O(N)$-model without quarks is not [2]. In this respect the NJL model passes an important test for applicability to chiral physics in the non-perturbative regime. This is also relevant for dense quark matter in the deconfined phase. Due to the difficulty of performing lattice gauge simulations in this regime, the NJL model is widely used as a tool for investigations of color-superconducting quark matter, see e.g. [55], and its phase structure, which is sensitive to the quark masses [56].

Finally, a quark mass expansion is essential to extrapolate lattice gauge simulations to the physical regime. The lowest-order expansion of chiral perturbation theory will be sufficient as long as the lattice masses are sufficiently small, but it may become unreliable for large masses which are required for studies with chiral fermions [57, 58]. A description that explicitly includes quark dynamics beyond the infrared regime may give a better account of the large quark-mass behavior. We have given results for $f_\pi$ and $m_\pi$ for large quark
masses in the curves in Fig. 6 and the table (Tab. III) which can be used to extrapolate to the physical regime. The NJL model can help to bridge the gap between lattice simulations with heavy chiral fermions and the physical regime.

Acknowledgments

We would like to thank J. Braun for many useful discussions. This work is supported in part by the Helmholtz association under grant no. VH-VI-041, in part by the EU Integrated Infrastructure Initiative Hadron Physics (I3HP) under contract RII3-CT-2004-506078 and funded in part by the German Research Foundation (DFG) under grant no. AL 279/5-1.

APPENDIX A: FLOW EQUATION FOR THE PION DECAY CONSTANT

In the appendix, we have collected some technical details of the derivation of the various flow equations used in the approximate one-loop calculations. In this first section, we give the details for the derivation of the flow equation for the pion decay constant. We start by considering the minimum condition eq. (51) and take the derivative with respect to the renormalization scale. Note that $\delta$ is independent of the RG scale:

$$k \frac{\partial}{\partial k} (U'_0(\sigma^2_0, k) - \delta) \equiv 0$$

$$k \frac{\partial}{\partial k} U'_0 \bigg|_{\Phi^2 = \sigma^2_0} = 2\sigma_0 + U''_0(\sigma^2_0) \left( k \frac{\partial}{\partial k} \sigma^2_0 \right) 2\sigma_0 + U'_0(\sigma^2_0) 2 \left( k \frac{\partial}{\partial k} \sigma_0 \right) \equiv 0$$

$$k \frac{\partial}{\partial k} U'_0 \bigg|_{\Phi^2 = \sigma^2_0} = 2\sigma_0 + \frac{4\sigma^2_0 U''_0(\sigma^2_0) + 2 U'_0(\sigma^2_0)}{2\sigma_0} \left( k \frac{\partial}{\partial k} \sigma^2_0 \right) \equiv 0. \quad (A1)$$

We can identify the term multiplying the derivative of $\sigma_0$ as the mass of the sigma meson, evaluated at the expectation value $\sigma_0$,

$$Z_\Phi M^2_\sigma(\sigma^2_0) = 2U'_0(\sigma^2_0) + 4\sigma^2_0 U''_0(\sigma^2_0). \quad (A2)$$

In this way, we can express the evolution of the expectation value in terms of the evolution of the derivative of the potential $U_0(\Phi^2)$ as

$$k \frac{\partial}{\partial k} \sigma^2_0 = - \frac{4\sigma^2_0}{Z_\Phi M^2_\sigma(\sigma^2_0)} k \frac{\partial}{\partial k} U'_0(\Phi^2) \bigg|_{\Phi^2 = \sigma^2_0}. \quad (A3)$$
To get the desired differential equation for $U'_0(\Phi^2)$, we differentiate the flow equation for the potential (34) and insert the derivatives of the masses from equations (25)-(26):

$$k \frac{\partial}{\partial k} U'_0(\Phi^2) = \frac{\partial}{\partial \Phi^2} \left( k \frac{\partial}{\partial k} U_0(\Phi^2) \right)$$

$$= \frac{k^6}{32\pi^2} \left\{ \frac{4N_c N_f}{(k^2 + M_q^2)^2} \frac{\partial M_q^2}{\partial \Phi^2} - \frac{3}{(k^2 + M_q^2)^2} \frac{\partial M_{\pi}^2}{\partial \Phi^2} - \frac{1}{(k^2 + M_q^2)^2} \frac{\partial M_{\pi}^2}{\partial \Phi^2} \right\}$$

$$= \frac{k^6}{32\pi^2} \left\{ \frac{4N_c N_f}{(k^2 + M_q^2)^2} \frac{G^2}{Z_q^2} - \frac{3}{2 \Phi^2} \left( M_{\pi}^2 - M_q^2 \right) \left( \frac{1}{(k^2 + M_q^2)^2} + \frac{1}{(k^2 + M_{\pi}^2)^2} \right) \right\}. \quad (A4)$$

For the flow equation for the expectation value of the field $\Phi$, we evaluate this equation at $\Phi^2 = \sigma_0^2$. In order to obtain a flow equation for the pion decay constant, we now need to include the flow equation for the coupling $Z_{\Phi}$:

$$k \frac{\partial}{\partial k} f_\pi^2 = k \frac{\partial}{\partial k} (Z_{\Phi} \sigma_0^2) = \left( k \frac{\partial}{\partial k} Z_{\Phi} \right) \sigma_0^2 + Z_{\Phi} \left( k \frac{\partial}{\partial k} \sigma_0^2 \right)$$

$$= \frac{k^6}{16\pi^2} \left\{ - \frac{4N_c N_f}{(k^2 + M_q^2)^2} \frac{G^2}{Z_q^2} \sigma_0^2 \left( \frac{1}{(k^2 + M_q^2)} + \frac{2}{M_{\pi}^2} \right) \right.$$  

$$\left. + \left( \frac{3 M_{\pi}^2 - M_q^2}{M_{\pi}^2} - 1 \right) \left( \frac{1}{(k^2 + M_q^2)^2} + \frac{1}{(k^2 + M_{\pi}^2)^2} \right) \right\}$$

$$+ \frac{2}{(k^2 + M_q^2)(k^2 + M_{\pi}^2)} \right\}. \quad (A5)$$

**APPENDIX B: THE CHIRAL CONDENSATE $\langle \bar{q}q \rangle$**

In principle it should be possible to derive the quark mass expansion of the chiral quark condensate $\langle \bar{q}q \rangle$ analytically in a manner analogous to the pion decay constant $f_\pi$ and the pion mass $m_\pi^2$ in section \[\[. For this purpose one has to compute the chiral condensate from the partition function $Z$,

$$Z[J, \bar{\eta}, \eta] = e^{-W[J,\bar{\eta},\eta]} = \int \mathcal{D}q \mathcal{D}\bar{q} \mathcal{D}\Phi \exp \left( -S[\bar{q},q,\Phi] + \int d^4x (J \Phi + \bar{\eta}q + \bar{q}\eta) \right), \quad (B1)$$

by differentiating it with respect to the additional source term $m_c \bar{q}q$

$$\langle \bar{q}q \rangle = \frac{\partial}{\partial m_c} \log Z(m_c) = \frac{\partial}{\partial m_c} U(\Phi_0^2, m_c). \quad (B2)$$

Using renormalization group methods, this would lead to a flow equation for $\langle \bar{q}q \rangle$ similar to equations \[\[ and \[\[ for $f_\pi$ and $m_\pi^2$. Unfortunately, it turns out that we probably miss some relevant terms in equation \[\[. This may be related to a lack of rebosonization
of the evolving four-fermion term \[59\]. Therefore we restrict ourselves to show that the \(\langle \bar{q}q \rangle\)-expansion follows from the other two expansions. We derive \(\langle \bar{q}q \rangle\) from our results for \(f_\pi\) and \(m_\pi^2\) from section IV by using the Gell-Mann–Oakes–Renner relation, \(m_\pi^2 = \frac{\langle \bar{q}q \rangle}{f_\pi^2} m_c\), and then replace \(m_c\) by the lowest-order relation in the chiral expansion, \(m_c = M^2 f_\pi^2 \langle \bar{q}q \rangle_0\). This gives

\[
\langle \bar{q}q \rangle = \frac{m_\pi^2 f_\pi^2}{m_c} \langle \bar{q}q \rangle_0 = \langle \bar{q}q \rangle_0 \left(1 - \frac{3 M^2}{32 \pi^2 f_\pi^2} \log \frac{M^2}{k_{\chi SB}} + \mathcal{O}(M^2)\right). \tag{B3}
\]

As for the pion decay constant and the pion mass, we reproduce exactly the logarithmic term of \(\chi\)PT (cf. eq.(8)). In the way employed here, this is a consistency check of our expansions for \(f_\pi\) and \(m_\pi^2\) (eqs. 53 and 56) with the Gell-Mann–Oakes–Renner relation.

**APPENDIX C: REGULARIZATION INDEPENDENCE OF THE CHIRAL LOGARITHMS**

In general, for finite values of the renormalization scale \(k\), the results for the RG flow depend on the choice of regularization function. However, for the family of cutoff functions that we consider in this paper,

\[
f^{(a)}(\tau k^2) = \left(\sum_{j=0}^{a} \frac{1}{j!} (\tau k^2)^j \right) \exp(-\tau k^2), \tag{C1}
\]

we can show that the logarithmic contributions involving the pion mass are independent of the choice of function within this family. The general flow equation for the potential, which results from a cutoff function of this type for \(a \geq 2\), is

\[
k \partial_k U_k(\Phi^2) = \left(\frac{k^2}{16 \pi^2 a (a-1)} \right) \frac{1}{(k^2 + M_\pi^2(\Phi^2))^{a-1}} + \frac{3}{(k^2 + M_\pi^2(\Phi^2))^{a-1}} - \frac{4 N_c N_f}{(k^2 + M_\sigma^2(\Phi^2))^{a-1}}. \tag{C2}
\]

As an example, consider the pion contributions to the flow equation for the expectation value of the field, \(\sigma_0(k)\):

\[
[k \partial_k \sigma_0(k)]_{\text{pion}} = \frac{2 \lambda \sigma_0(k)}{M_\pi^2(\sigma_0(k)^2)} \frac{3}{16 \pi^2} k^2 \frac{1}{a} \frac{(k^2)^a}{(k^2 + M_\pi^2(\sigma_0(k)^2))^{a-1}}. \tag{C3}
\]

For simplicity we neglect the couplings beyond the four-point couplings as in equation 38 and approximate \(\frac{\partial M^2}{\partial \Phi^2} = \lambda\).
The term relevant for the logarithmic contribution is

\[
\frac{1}{a} \frac{(k^2)^a}{(k^2 + M^2_{\pi})^a} = \frac{1}{a} \frac{(k^2 + M^2_{\pi} - M^2_{\pi})^a}{(k^2 + M^2_{\pi})^a} = \frac{1}{a} \frac{(k^2 + M^2_{\pi})^a - (a) M^2_{\pi} (k^2 + M^2_{\pi})^{a-1} + \left(\frac{a}{2}\right) (M^2_{\pi})^2 (k^2 + M^2_{\pi})^{a-2} + \cdots}{(k^2 + M^2_{\pi})^a} = \frac{1}{a} - M^2_{\pi} \frac{1}{(k^2 + M^2_{\pi})^2} + \frac{(a - 1)}{2} (M^2_{\pi})^2 \frac{1}{(k^2 + M^2_{\pi})^2} - + \cdots
\]

(C4)

We note that the second term is independent of the parameter \(a\). After setting \(M^2_{\pi} = M^2\) and on integration over the scale \(k\), this term yields

\[-\frac{1}{2} M^2 \log(k^2 + M^2),\]

(C5)

and is thus responsible for the emergence of the chiral log. We conclude that in the current approach, the logarithmic contribution is independent of the regularization scheme.

In general, the expansion of the observables in terms of the pion mass parameter \(M^2\) is dependent on the choice of the cutoff function in our perturbative approximation. However, the leading logarithmic term in the pion mass parameter (the chiral log) is independent of the choice of \(a\) for the cutoff function.

**APPENDIX D: RESULTS FOR LARGE QUARK MASSES**

In the following, we list selected results for \(f_{\pi}\) and \(m_{\pi}\) from the RG calculations for large values of the current quark mass. We give results for several different UV cutoff values.
| $m_c$ | $f_\pi$ | $m_\pi$ | $f_\pi$ | $m_\pi$ | $f_\pi$ | $m_\pi$ |
|------|---------|---------|---------|---------|---------|---------|
| 10   | 91.4    | 120.9   | 93.1    | 144.8   | 97.9    | 175.9   |
| 20   | 94.7    | 172.3   | 99.3    | 204.3   | 108.0   | 244.6   |
| 40   | 99.4    | 247.1   | 107.7   | 289.1   | 120.9   | 341.1   |
| 60   | 102.9   | 306.9   | 113.5   | 355.9   | 129.7   | 416.2   |
| 80   | 105.5   | 359.6   | 117.9   | 414.2   | 136.3   | 481.0   |
| 100  | 107.5   | 408.1   | 121.3   | 467.4   | 141.4   | 539.7   |
| 120  | 109.1   | 454.2   | 124.0   | 517.4   | 145.7   | 594.4   |

TABLE II: Values for the pion decay constant $f_\pi$ and the pion mass $m_\pi$ for large current quark masses $m_c$ from the RG calculation. The data are given for three different UV cutoff choices $\Lambda_{UV}$. All values are given in [MeV].
[16] G. Colangelo, S. Dürr, and R. Sommer, Nucl. Phys. Proc. Suppl. 119, 254 (2003), hep-lat/0209110.
[17] G. Colangelo and S. Dürr, Eur. Phys. J. C33, 543 (2004), hep-lat/0311023.
[18] G. Colangelo and C. Haefeli, Phys. Lett. B590, 258 (2004), hep-lat/0403025.
[19] G. Colangelo, S. Dürr, and C. Haefeli (2005), hep-lat/0503014.
[20] J. Braun, B. Klein, and H. J. Pirner, Phys. Rev. D71, 014032 (2005), hep-ph/0408116.
[21] J. Braun, B. Klein, and H. J. Pirner, Phys. Rev. D72, 034017 (2005), hep-ph/0504127.
[22] J. Braun, B. Klein, H. J. Pirner, and A. H. Rezaeian (2005), hep-ph/0512274.
[23] D. Ebert and H. Reinhardt, Nucl. Phys. B271, 188 (1986).
[24] G. Ecker, J. Gasser, A. Pich, and E. de Rafael, Nucl. Phys. B321, 311 (1989).
[25] T. H. Hansson, M. Prakash, and I. Zahed, Nucl. Phys. B335, 67 (1990).
[26] C. Schüren, E. Ruiz Arriola, and K. Goeke, Nucl. Phys. A547, 612 (1992).
[27] E. Ruiz Arriola, Phys. Lett. B253, 430 (1991).
[28] E. Ruiz Arriola, Phys. Lett. B264, 178 (1991).
[29] S. P. Klevansky, Rev. Mod. Phys. 64, 649 (1992).
[30] C. Schüren, F. Doring, E. Ruiz Arriola, and K. Goeke, Nucl. Phys. A565, 687 (1993).
[31] J. Mueller and S. P. Klevansky, Phys. Rev. C50, 410 (1994).
[32] H. J. Hippe and S. P. Klevansky, Phys. Rev. C52, 2172 (1995).
[33] J. Bijnens, Phys. Rept. 265, 369 (1996), hep-ph/9502335.
[34] D. U. Jungnickel and C. Wetterich, Eur. Phys. J. C2, 557 (1998), hep-ph/9704345.
[35] F. J. Llanes-Estrada and P. D. A. Bicudo, Phys. Rev. D 68, 094014 (2003).
[36] J. Hubbard, Phys. Rev. Lett. 3, 77 (1959).
[37] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rept. 323, 183 (2000), hep-th/9905111.
[38] L. Da Rold and A. Pomarol, Nucl. Phys. B721, 79 (2005), hep-ph/0501218.
[39] D. U. Jungnickel and C. Wetterich, Phys. Rev. D53, 5142 (1996), hep-ph/9505267.
[40] J. Berges, D. U. Jungnickel, and C. Wetterich, Phys. Rev. D59, 034010 (1999), hep-ph/9705474.
[41] B.-J. Schaefer and J. Wambach, Nucl. Phys. A757, 479 (2005), nucl-th/0403039.
[42] J. Meyer, K. Schwenzer, H.-J. Pirner, and A. Deandrea, Phys. Lett. B526, 79 (2002), hep-ph/0110279.
[43] G. ’t Hooft, Nucl. Phys. **B72**, 461 (1974).

[44] G. ’t Hooft, Nucl. Phys. **B75**, 461 (1974).

[45] E. Witten, Nucl. Phys. **B160**, 57 (1979).

[46] H. Leutwyler, Czech. J. Phys. **52**, B9 (2002), hep-ph/0212325.

[47] L. Jendges, J. Meyer, H.-J. Pirner, and K. Schwenzer (in preparation).

[48] K. Schwenzer, Ph.D. thesis, University of Heidelberg (2003).

[49] C. Wetterich, Phys. Lett. **B301**, 90 (1993).

[50] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rept. **363**, 223 (2002), hep-ph/0005122.

[51] G. Ripka, *Quarks Bound by Chiral Fields* (Oxford University Press, Oxford, 1997).

[52] S.-B. Liao, Phys. Rev. **D53**, 2020 (1996), hep-th/9501124.

[53] D. F. Litim and J. M. Pawlowski, Phys. Rev. **D65**, 081701 (2002), hep-th/0111191.

[54] C. e. a. Davies, Phys.Rev.Lett. **92**, 022001 (2004).

[55] M. Buballa, Phys. Rept. **407**, 205 (2005), hep-ph/0402234.

[56] S. B. Ruster, V. Werth, M. Buballa, I. A. Shovkovy, and D. H. Rischke, Phys. Rev. **D72**, 034004 (2005), hep-th/0503184.

[57] D. B. Kaplan, Phys. Lett. **B288**, 342 (1992), hep-lat/9206013.

[58] H. Neuberger, Phys. Lett. **B417**, 141 (1998), hep-lat/9707022.

[59] H. Gies and C. Wetterich, Phys. Rev. **D65**, 065001 (2002), hep-th/0107221.