BRST quantization of Matrix Chern-Simons Theory

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Abstract

The BRST quantization of matrix Chern-Simons theory is carried out, the symmetries of the theory are analyzed and used to constrain the form of the effective action.

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1 Introduction

The cubic matrix models\[1, 2, 3, 4\] were invented as a possible approach to the problem of formulating string or $\mathcal{M}$ theory in a background invariant framework. They also provide a matrix formulation of certain quantum deformed extensions of loop quantum gravity\[2, 5, 6\]. The basic philosophy that motivates these theories is that quantum and classical theories of gravity which are background independent can, in most if not all of the known cases, be constructed by modifying topological field theories. The idea is then to construct matrix models that extend a matrix form of Chern-Simons theory\[1, 2\].

The quantization of these theories faces certain issues due to the fact that the action is presented in a first order form, which means that they define theories on phase spaces rather than configuration spaces. These theories also have gauge symmetries and constraints that must be taken into account correctly. In this letter we present an approach to quantization based on the standard BRST method. We carry out the quantization in detail for the case of Matrix Chern-Simons theory, with and without couplings to fermions, and show that it leads to results which are direct extensions of the usual results for Chern-Simons theory. These results should be directly extendable to the cubic matrix models.

While these results are encouraging, we must, however, mention another important issue that we do not solve in this paper. This is to give a genuinely background independent quantization of the cubic matrix models. What we describe here is instead a background dependent quantization of a theory whose classical formulation is background independent. This is because in a BRST formalism the quantum theory is defined relative to a given background, which is a solution to the classical equations of motion of the theory. Whether there is a form of the quantum theory that is well defined at the more fundamental, background independent, level, which unifies the particular background dependent quantum theories, remains an open question\[4\].

In the next section we review Matrix Chern-Simons theory. We discuss the symmetries of the theory and give several examples including a possible matrix version of $2 + 1$ gravity and a supersymmetric model. The main technical work of the paper is in section 3, where we discuss gauge fixing, ghosts, the BRST and anti-BRST transformations. In section 4 we show that pure Matrix Chern-Simons theory has also a vector supersymmetry, and use this to discuss the form of the effective action, with and without fermions.

2 Matrix Chern-Simons theory

2.1 The Cubic matrix model

We consider an action $S[A]$ where $A = A_a \tau^a$, the $A_a$ being $N \times N$ matrices and the $\tau^a$ the generators of a Lie algebra $\mathcal{G}$ in a given representation. We can also use a superLie algebra, in which case we need to use the supertrace instead of the trace and the supercommutators between elements of the $\mathcal{G}$ algebra. We will use the indices $i, j, k$ for the $N \times N$ indices and $\alpha, \beta, \gamma$ for the $\mathcal{G}$ representation. We introduce the cubic matrix model:

$$S[A] = Tr_{N \times N} A^\alpha_a [A^\beta_\beta, A^\gamma_\gamma] = Tr_{\mathcal{G}} A^\alpha_i [A^\beta_j, A^\gamma_k]$$

(1)

where $Tr_{\mathcal{G}}$ is the trace for the $\mathcal{G}$ representation and $Tr_{N \times N}$ the trace for $N \times N$ matrices. The action can also be written as

$$S[A] = (Tr_{N \times N} Tr_{\mathcal{G}} \tau^a \tau^b \tau^c) (Tr_{N \times N} A_a [A_b, A_c]) = \frac{1}{3} \varphi^{abc} Tr_{N \times N} A_a [A_b, A_c]$$

(2)

where $\varphi^{abc} = 3/2 Tr_{\mathcal{G}} \tau^a \tau^b \tau^c$ is the usual structure constant $f^{abc}$ with the third index raised by the metric $\eta^{ab} = Tr(\tau^a \tau^b)$. $\varphi^{abc} = \eta^{ad} f^{bcd}$

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1One of us has explored an approach to a background independent quantum theory based on a hidden variable theory, and stochastic quantization\[5\]. Other authors have also noted that matrix models are in some sense automatically hidden variables theories, in that quantum theory can be defined from their classical statistical mechanics\[6, 7\]. Whether there is a less radical approach to background independent quantization of matrix models is an open question.
This cubic action $S$ has two global symmetries. First, it is invariant under the global action of the group $G$ generated by $G$:

$$A_i^j \rightarrow R^{-1}(g)A_i^j R(g)$$

where $R(g)$ is the representation matrix of the group element $g \in G$. $g$ acts on the generators $\tau^a$ by conjugation. The infinitesimal version of this symmetry is

$$\delta A_i^j = \epsilon [A_i^j, u] \text{ with } u \in G$$

The second symmetry is an invariance under rotation by the group $G_N = \text{GL}_N(\mathbb{R})$:

$$A^\alpha_\beta \rightarrow M^{-1}A^\alpha_\beta M \text{ or equivalently } A_a \rightarrow M^{-1}A_a M$$

for $M \in G_N$. The infinitesimal variation is then given by a $N \times N$ matrix $m$:

$$\delta A_a = \epsilon [A_a, m]$$

Next, we can look at the classical solutions $X$ of the action $S$. They are given by the equation

$$\varphi^{abc}[X_b, X_c] = 0$$

Let point out that the set of solutions is invariant under both $G$ and $G_N$ rotations. We can study the fluctuations of our matrix around the new background given by $X$ by introducing the new action

$$S_X[A] = S[X + A] - S[X] = \varphi^{abc} \left( T_N \cdot N A_a[X_b, A_c] + \frac{1}{3} T_N \cdot N A_a[A_b, A_c] \right)$$

The $G_N$ symmetry now reads

$$M^{-1}(X_a + A_a)M = M^{-1}X_a M + M^{-1}A_a M = X_a + A^{(M)}_a$$

so that $S_X$ has a $G_N$ gauge symmetry given by

$$A_a \rightarrow A^{(M)}_a = M^{-1}A_a M + M^{-1}[X_a, M]_N$$

which shows that the background $X_a$ takes the role of a derivation, similarly to the differential calculus in Non-Commutative Geometry. In this setting, $A_a$ behaves like $G_N$ gauge field. Let nevertheless point out that if $X_a = 0$ then it is not a gauge field anymore but behaves simply like a $G_N$ matter field.

The action $S_X$ also has a $G$ gauge symmetry given by

$$A_i^j \rightarrow A_i^{j(g)} = g^{-1}A_i^j g + g^{-1}[X_i^j, g]$$

Here too, $A_i$ behaves like a $G$ gauge field when $X_i^j \neq 0$ and like a matter field otherwise. An interesting configuration is when $(X_a)_i^j$ is a diagonal $N \times N$ matrix, so that the diagonal fields $A_i^j$ are gauge fields and the off-diagonal elements $A_i^j$ are the matter fields.

### 2.2 Examples

Matrix Chern-Simons theory is the particular case when we choose $G = SU(2)$. Then the generators are

$$\tau_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \tau_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then the structure constants are given by the antisymmetric tensor $\varphi^{abc} = i\epsilon^{abc}$ and the metric $\eta^{ab} = \delta^{ab}$ is trivial. The classical solutions are sets of three matrices $X_1, X_2, X_3$ which commute with each other. Then
the action $S_X$ looks like a discrete version of the usual Chern-Simons action. More precisely, after a triple compactification \([1]\) achieved through a special choice of background solutions $X$ in a large matrix limit, the trace reproduces the integration over a 3-torus and the matrix Chern-Simons action exactly reproduces the usual Chern-Simons field theory.

Another similar example is given by the choice $G = SU(2)$. Then the generators would be

\[
\tau_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \tau_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Then the structure constants are once again given by the antisymmetric tensor $\varphi^{abc} = \epsilon^{abc}$ but the metric now has a $(-,+,+)$ signature instead of the previous $(+,+,+)$. One can also investigate a supersymmetric extension of $SU(2)$ by considering the superalgebra $\mathcal{G} = osp(1|2)$. This introduces odd-Grassmann generators and a spinor degree of freedom. This adds the fermionic term:

\[
S^\text{fermions} = Tr_{N\times N}[\phi_B[A, \phi_A]](\tau^a)_A^{-B}
\]

where $\phi$ is the spinor and $A, B = \pm 1/2$ the spinor indices. We can choose the classical solution to have components only in the $su(2)$ generators so that we get fermionic degrees of freedom behaving like matter fields. Let us point out that this is not the classical supersymmetric extension of the Chern-Simons action. Indeed, the usual supersymmetry is coupled to the Poincare group whereas, in our case, it is an extension of the Lorentz symmetry: we couple it to the frame rotations and not to the translations.

We can also turn to groups larger than $SU(2)$. An interesting example is given by $SL(2, C)$, which is the complexification of $SU(2)$. Indeed its generators are the $J^a = \tau^a$ and the $K^a = i\tau^a$. Let’s take $A = A_a \tau^a + iE_a \tau^a$ and write the cubic action choosing the fundamental $2 \times 2$ representation of $SL(2, C)$:

\[
S[A] = -\epsilon^{abc} (Tr_{N\times N}(E_a[A_b, A_c]) + Tr_{N\times N}(E_a[E_b, E_c])) + i\epsilon^{abc} (Tr_{N\times N}(E_a[A_b, E_c]) + Tr_{N\times N}(A_a[E_b, A_c]))
\]

The first term here looks very much like the $EF + EEE$ action of $2 + 1$d gravity with cosmological constant (obtained by rescaling the matrix $E$). Indeed after (triple) compactification, we indeed find back exactly that action. The $i$ term is the extra term coming in Witten’s reformulation of $2 + 1$d gravity as a $SL(2, C)$ Chern-Simons theory \([1]\).

### 2.3 Symmetries and the physical Interpretation

We can notice that in addition to the global gauge symmetries, there are local versions of the gauge symmetries \([3]\) and \([3]\). The action $S$ is further invariant under the transformations

\[
A_i^j \rightarrow g_i^{-1} A_i^j g_j
\]

and

\[
A_\alpha^\beta \rightarrow M_\alpha^{-1} A_\alpha^\beta M_\beta
\]

One has a nice interpretation of the local gauge symmetry \([3]\) in the context of M-theory \([1]\). The matrix $(A_i^j)_{1 \leq i, j \leq N}$ represents the interactions (due to open strings) between $N$ D0-branes (or equivalently $N$ points). And one has a local gauge symmetry $G$ at each of these points, which gives \([16]\).

One also has “translation” symmetries:

\[
A_a \rightarrow A_a + \lambda_a Id_N \quad A_i^j \rightarrow A_i^j + \lambda_i^j Id_G
\]

The link between the matrix models and the usual physical actions goes usually through compactification procedures \([1]\) which creates dimensions and a space-time out the matrix in the limit $N \rightarrow \infty$. Then, one
finds back for example the usual Chern-Simons theory on a 3d manifold out of the Matrix CS model described above [1, 2]. In this context, the translation symmetries [18] really become the symmetry by translation in the emerging space-time.

Nevertheless, it would be interesting to give a meaning to the matrix models for $N$ finite without talking about the possible infinite matrix limit. In sight of the expression (3), one automatically thinks about a potential link with non-commutative geometry (see [12] for example) with $X$ being the Dirac operator governing differential calculus. However, the $A = A_a \tau^a$ do not form an algebra whose product could help us construct the actions $S$ or $S_X$. Still, there is some hope in using the algebra of $2 \times 2$ matrices to translate the matrix model into the spectral triple language. We could then make the gauge group $SU(2)$ appear as the unitary part of $\mathcal{M}_2(C)$. Or we could say that using the algebra $\mathcal{M}_2(R)$ in the cubic matrix model defined is equivalent to using the algebra $sp(2)$. These possibilities will be investigated in future work.

3 Gauge fixing the cubic action

One interesting and necessary step in studying the actions (1) or (8) is to gauge fix them. In our case, we study the gauge fixing of the $G_N$ group since it is the apparent gauge symmetry of the action (8). Nevertheless, the same techniques work perfectly for the gauge fixing of the $G$ symmetry.

The first gauge fixing procedure which comes at one’s mind is choosing a representant for each orbit $\theta_a A_a$ of the orbits of $\theta^a A_a$, where $\theta^a$ is a fixed vector. Then the gauged fixed action is

$$\bar{S} = \frac{1}{6} Tr(A_a [A_b, A_c]) + Tr(W [X_a, A^a]) + Tr(U[X_a, [A^a, V]])$$

where the even-Grassmann $N \times N$ matrix $W$ enforces the gauge fixing condition. Through this procedure, we can introduce a background $X$ in the background independent action $S$.

The gauge fixing of the action $S_X$ is very similar to the one of $S$. Indeed, the variation of the gauge fixing condition is now

$$\delta_m[X_a, A^a] = [X_a, [X^a + A^a, m]]$$

2One can carry out the BRST analysis in that case the same way as in the case of the Landau gauge which we present. Let us choose a section $s$ of the orbits of $\theta^a A_a$, where $\theta^a$ is a fixed vector. Then the gauged fixed action is

$$\bar{S} = \frac{1}{6} Tr(A_a [A_b, A_c]) + Tr(W (\theta^a A_a - s(\theta^a A_a))) + Tr(U[\theta^a A_a, V])).$$

The residual BRST symmetry is the same as in equation (20) and ensures invariance of the path integral under change of section.
so that it is sufficient to replace $A$ by $X + A$ in the previous calculations:

$$S^\text{ghost}_X [A, W, U, V] = Tr_{N \times N} (W[X^a, A_a]) + Tr_{N \times N} (U[X^a, [X_a + A_a, V]])$$

(24)

and

$$S_X = S^\text{ghost}_X = -\frac{1}{2} \epsilon_{abc} \left( Tr(A_a[X_b, A_c]) + \frac{1}{3} Tr(A_a[A_b, A_c]) \right) + Tr(W[X_a, A^a]) + Tr(U[X_a, [X^a + A^a, V]])$$

(25)

The resulting action $S_X$ has the exact same structure as the gauge-fixed Chern-Simons action [13] and we can similarly find the BRST transformations under which $S_X$ is invariant. In the following paragraph, we are going to write down the BRST generators in the case of $\tilde{S}$, keeping in mind that they can be easily generalized to $\tilde{S}_X$ by changing $A$ into $X + A$ in the different formulas.

### 3.2 BRST transformations

The gauge fixed action $\tilde{S}$ is invariant under the following BRST transformations where $\epsilon$ is a odd-Grassmann valued number:

$$\begin{align*}
\delta A_a &= [A_a, V] \epsilon \\
\delta U &= W \epsilon \\
\delta V &= V^2 \epsilon \\
\delta W &= 0
\end{align*}$$

(26)

It is simply a gauge transformation for the initial action $S$ and it is easy to check the ghost part $S^\text{ghost}$ is also invariant under these transformations. Thus, we can introduce the BRST charge $Q_{+1}$ acting as:

$$\begin{align*}
Q_{+1} A_a &= -[A_a, V] \\
Q_{+1} U &= W \\
Q_{+1} V &= V^2 \\
Q_{+1} W &= 0
\end{align*}$$

(27)

We can then rewrite the ghost part of the action as

$$S^\text{ghost} = Tr_{N \times N} (-U[X^a, Q_{+1} A_a]) - Tr_{N \times N} (A_a[X^a, Q_{+1} U])$$

(28)

from which it is straightforward to check the invariance under $Q_{+1}$.

We can also write down the action as if we had done an integration by parts on the ghost term. Then, the gauge fixing part of the action reads:

$$S^\text{ghost} = Tr_{N \times N} ((W - \{U, V\})[X^a, A_a]) - Tr_{N \times N} (V[X^a, [A_a, U]])$$

(29)

The action written in this form has a similar BRST invariance as above. It is generated by what we call the anti-BRST operator $Q_{-1}$:

$$\begin{align*}
Q_{-1} A_a &= -[A_a, U] \\
Q_{-1} U &= U^2 \\
Q_{-1} V &= -W + \{U, V\} \\
Q_{-1} W &= [U, W]
\end{align*}$$

(30)

We have the following commutation relations between the BRST charges:

$$Q_{+1}^2 = 0 \quad Q_{-1}^2 = 0 \quad \{Q_{+1}, Q_{-1}\} = 0$$

(31)

Moreover, we can assign the ghost number 0 to $A$ and $W$, -1 to $U$ and +1 to $V$. Then, $Q_{+1}$ increases the ghost number by an unit, and $Q_{-1}$ decreases it by an unit.
4 The effective action

Once we have gauged fixed the action, one would like to compute its loop expansion and the resulting effective action. In this section, we will restrict ourself to the study of the Matrix Chern-Simons model \( G = SU(2) \). One can not apply the usual techniques of perturbative expansion for the gauge fixed action \( \tilde{S} \) for it doesn’t have any quadratic term. On the other hand, one can use the action \( \tilde{S}_X \) since the background \( X \) introduces propagators for the matrix \( A \) and the ghost matrices \( U,V \). Moreover, the introduction of the matrix \( W \) allows to invert the quadratic terms in order to derive the propagator of the \( A \) matrices. More precisely, let’s note \( x_a = [X_a,] \). As the \( X_a \) matrices commute, the morphisms \( x_a \) also commute. Then, the matrix correlating the \( A \) and \( W \) is:

\[
C = \begin{pmatrix}
0 & x_3 & -x_2 & x_1 \\
-x_3 & 0 & x_1 & x_2 \\
x_2 & -x_1 & 0 & x_3 \\
-x_1 & -x_2 & -x_3 & 0
\end{pmatrix}
\]

whose inverse is the propagator \( P = C^{-1} = -RC \) where we have introduce the matrix \( R = (x_1^2 + x_2^2 + x_3^2)^{-1} \). The propagator of the ghost is simply the matrix \( R \) and we have two types of 3-vertices \( AAA \) and \( UVA \). Then, one can easily check by hand that the only 1-loop and 2-loop corrections are only of the type \( A[A, A] \) and that all other possible terms are canceled. This comes from an additional symmetry of the matrix Chern-Simons action, similar to the so-called vector supersymmetry (VSUSY) of the ordinary Chern-Simons theory. This symmetry is special to the case \( G = SU(2) \) (and also \( G = SU(1,1) \) for which \( \varphi^{abc} = \epsilon^{abc} \) \((a, b, c = 1, 2, 3) \). In the continuum limit \( N \to \infty \) in which we recover the full Chern-Simons theory, it protects the theory from infrared effects and contributes to the finiteness (or disappearing depending of the regularization scheme) of the quantum corrections. The symmetry for \( \tilde{S}_X \) \((22) \) reads with \( \alpha = 1, 2, 3 \):

\[
\begin{align*}
\delta_\alpha A_a &= \epsilon_{a\alpha b}[X_b, V] \epsilon_\alpha \\
\delta_\alpha U &= A_\alpha \epsilon_\alpha \\
\delta_\alpha V &= 0 \\
\delta_\alpha W &= [X_\alpha + A_\alpha, V] \epsilon_\alpha
\end{align*}
\]

In fact, this supersymmetry also exists for \( \tilde{S} \) \((23) \) and reads:

\[
\begin{align*}
\delta_\alpha A_a &= \epsilon_{a\alpha b}[X_b, V] \epsilon_\alpha \\
\delta_\alpha U &= A_\alpha \epsilon_\alpha \\
\delta_\alpha V &= 0 \\
\delta_\alpha W &= [A_\alpha, V] \epsilon_\alpha
\end{align*}
\]

It is not difficult to see that, as in ordinary Chern-Simons theory, the only term that can appear in the effective action which is invariant under local gauge, \( BRST \) and vector supersymmetry transformations is the original action itself. The result can then only be a correction in the coupling \( k \).

We may also consider the inclusion of fermions, through a term such as

\[
I^\Psi = Tr \left[ \Psi^4 [X_a, \Psi^B] \right] \tau^A_{AB}.
\]

Such a term can be introduced by considering the cubic matrix model associated to the superalgebra \( osp(1|2) \). It is not hard to see that it breaks the vector supersymmetry. Then, the theory also knows about a background metric, formed by \( q^{ab} = Tr r^a r^b \). The remaining \( BRST \) invariance then allows the appearance of a Yang-Mills like term in the effective action, of the form,

\[
S_{1\text{loop}} = c Tr ([X_a, X_b][X_c, X_d]) q^{ac} q^{bd}
\]

So long as \( N \) is finite this term need not be considered part of the fundamental action, but only as a part of the effective theory governing low energy phenomena. But if we take \( N \to \infty \) then it may become
necessary to introduce this as a fundamental term in the action, as in continuum Chern-Simons-Yang-Mills theory [14].

5 Conclusions

The results of this, fairly straightforward technical paper, are interesting first of all for the project of basing \( \mathcal{M} \) theory on a cubic matrix model; we see that the BRST quantization does suffice to base a perturbative quantization of these theories around backgrounds defined by classical solutions to their field equations. Moreover the fact that, as expected, the results mirror those of ordinary Chern-Simons theory, with and without coupling to fermions confirms the physical picture behind the loop/string duality postulated in [14]. The basic physical idea there is that, as in the case of topologically massive gauge theories in 2+1 dimensions, these theories will have two phases, one background dependent and one background independent. In the latter the degrees of freedom on the toroidal compactifications will be Chern-Simons like, which means that the perturbation theory of the matrix models is independent of the metric structure defined by the toroidal compactification. As it results it generates an extension of loop quantum gravity of the kind described in [14,15]. However, in the other phase, the degrees of freedom include (topologically) massive quanta of 3d Yang-Mills theories which, in the case of compactifications of the kind described in [1,2] become modes of strings. The possibility that both behaviors can arise as different phases of a single matrix theory is the dynamical basis of the conjecture that loop quantum gravity, at least with certain choices of representation labels, and string theory, may be dual descriptions of the same theory.

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\(^3\)We note that this is similar to the physical picture of Kogan et al. but realized precisely in the context of a matrix model.
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