MODULATED SEMI-IN vars

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Abstract. We prove the basic properties of determinantal semi-invariants for presentation spaces over any finite dimensional hereditary algebra over any field. The results include the virtual generic decomposition theorem, stability theorem and the c-vector theorem, the last says that the c-vectors of a cluster tilting object are, up to sign, the determinantal weights of the determinantal semi-invariants defined on the cluster tilting objects. Applications of these theorems are given in several concurrently written papers.

Introduction

There is a rich theory of semi-invariants for representations of quivers [S91], [KI], [DW], [SW], [SVdB] and its relation to cluster categories and cluster algebras [IOTW09], [Ch], [BHIT]. In this paper, we show how this theory and its relation to cluster algebras can be extended to finite dimensional hereditary algebras over a field, which in particular include all modulated acyclic quivers over any field. Furthermore, we prove the relationship between c-vectors and semi-invariants.

Over a fixed field K, a K-modulated quiver is a triple \((Q, \{F_i\}_{i \in Q_0}, \{M_{ij}\}_{i \rightarrow j \in Q_1})\) where \(Q\) is a quiver (directed graph) without oriented cycles, \(F_i\) is a finite dimensional division algebra for each vertex \(i \in Q_0\) and \(M_{ij}\) is an \(F_i-F_j\) bimodule for every arrow \(i \rightarrow j\) in \(Q_1\). The standard modulation of a simply laced quiver \(Q\) is given by taking each \(F_i = K\) and each \(M_{ij} = K\). A representation \(V\) of a modulated quiver with dimension vector \(\alpha = (\alpha_1, \cdots, \alpha_n)\) consists of \(F_i\)-modules \(V_i\) of dimension \(\alpha_i\) at each vertex \(i \in Q_0\) and \(F_j\)-linear maps \(V_i \otimes_{F_i} M_{ij} \rightarrow V_j\) for each arrow \(i \rightarrow j\) in \(Q_1\).

We study representation and presentation spaces of modulated quivers. When \(Q\) is a simply laced quiver, the standard definition of the representation space of \(Q\) with dimension vector \(\alpha \in \mathbb{N}^n\) is

\[
\text{Rep}(Q, \alpha) = \bigoplus_{\alpha \in \mathbb{N}^n} \text{Hom}_K(K^{\alpha_1}, K^{\alpha_2}).
\]

When \(K\) is algebraically closed, any finite dimensional hereditary algebra is Morita equivalent to the path algebra \(KQ\) of a quiver \(Q\). Choosing an element of \(\text{Rep}(Q, \alpha)\) is equivalent to choosing a \(KQ\)-module structure on the \(KQ_0\)-module \(\bigoplus \alpha_i\).

Over the modulated quiver \((Q, \{F_i\}_{i \in Q_0}, \{M_{ij}\}_{i \rightarrow j \in Q_1})\) the representation space for dimension vector \(\alpha \in \mathbb{N}^n\) is

\[
\text{Rep}(Q, \{F_i\}_{i \in Q_0}, \{M_{ij}\}_{i \rightarrow j \in Q_1}, \alpha) = \bigoplus_{\alpha \in \mathbb{N}^n} \text{Hom}_{F_j}(M^{\alpha_1}_{ij}, F_j^{\alpha_2}).
\]

Each element of \(\text{Rep}(Q, \{F_i\}_{i \in Q_0}, \{M_{ij}\}_{i \rightarrow j \in Q_1}, \alpha)\) gives the right \(\bigoplus \alpha_i\)-module the structure of a right module over the tensor algebra of \((Q, \{F_i\}_{i \in Q_0}, \{M_{ij}\}_{i \rightarrow j \in Q_1})\). In each case, the representation space is an affine space over \(K\).

2010 Mathematics Subject Classification. 16G20; 20F55.

The first author is supported by NSA Grant #H98230-13-1-0247.

The second author is supported by Simons Foundation Grant #209082.

The third author is supported by NSF Grant #DMS-1103813 and #DMS-0901185.

The fourth author is supported by NSF Grant #DMS-1400740.
In this paper we deal with arbitrary finite dimensional hereditary algebras over any field, not necessarily tensor algebras of modulated quivers. Notice that, if \( K \) is not perfect, there may be finite dimensional hereditary algebras over \( K \) which are not Morita equivalent to the tensor algebras of modulated quivers. (Appendix A, Sec 3)

For an arbitrary finite dimension hereditary algebra \( \Lambda \) we define (in (2.1.8)) the representation space \( \text{Rep}(\Lambda, \alpha) \) to be a certain subspace of the space \( \text{Hom}_\Lambda(\text{rad} P(\alpha), P(\alpha)) \), where \( P(\alpha) \) denotes \( \bigoplus P^\alpha_i \). This is isomorphic to \( \text{Rep}(Q, \{ F_i \}_{i \in Q_+}, \{ M_j \}_{j \in Q_0}) \) in the modulated case. We identify each element \( f \in \text{Rep}(\Lambda, \alpha) \) with the \( \Lambda \)-module which is the cokernel of the homomorphism \( f : \text{rad} P(\alpha) \rightarrow P(\alpha) \).

We consider \( \text{Rep}(\Lambda, \alpha) \) as an affine space over \( K \). At the beginning we assume that \( K \) is infinite so that nonempty open subsets of this space are dense. (We extend to arbitrary fields later, in Section 3.3) The first theorem of this paper is Theorem 2.1.9. If there exists a \( \Lambda \)-module \( M \) which is rigid meaning \( \text{Ext}_\Lambda^1(M, M) = 0 \) with \( \dim M = \alpha \), then the elements of \( \text{Rep}(\Lambda, \alpha) \) which are isomorphic to \( M \) form an open dense subset. We call this the generic representation of dimension \( \alpha \) and denote it by \( M_\alpha \). If \( M_\alpha \) is indecomposable then \( \alpha \) is a real Schur root of \( \Lambda \). As a consequence we have:

**Theorem 0.0.1** (Generic Decomposition Theorem 2.1.8). Let \( \Lambda \) be a finite dimensional hereditary algebra over an infinite field, and let \( \beta_1, \cdots, \beta_k \) be real Schur roots so that \( M_{\beta_i} \) do not extend each other. Then for any nonnegative integer linear combination \( \alpha = \sum_{i=1}^k n_i \beta_i \), the generic representation in \( \text{Rep}(\Lambda, \alpha) \) is isomorphic to \( \bigoplus_{i=1}^k M^n_{\beta_i} \).

Representation spaces are defined for \( \alpha \in \mathbb{N}^n \). Next, we generalize the construction to arbitrary integer vectors \( \alpha \in \mathbb{Z}^n \) by constructing presentation spaces and considering their direct limit which we call virtual representation space. We choose vectors \( \gamma_0, \gamma_1 \in \mathbb{N}^n \) so that \( \dim P(\gamma_0) - \dim P(\gamma_1) = \alpha \). We call \( \text{Hom}_\Lambda(\gamma_1, P(\gamma_0)) \) a presentation space and denote it by \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) and view it as a generalization of \( \text{Rep}(\Lambda, \alpha) \). However, there are an infinite number of choices for \( \gamma_0, \gamma_1 \) for each \( \alpha \in \mathbb{Z}^n \). To define a single space for each \( \alpha \in \mathbb{Z}^n \) which contains all of these presentation spaces, we take their direct limit (colimit):

\[
\text{Vrep}(\Lambda, \alpha) := \text{colim} \text{Pres}_\Lambda(\gamma_1, \gamma_0)
\]

where the colimit is over all pairs \( \gamma_0, \gamma_1 \) so that \( \dim P(\gamma_0) - \dim P(\gamma_1) = \alpha \). Representatives of \( \text{Vrep}(\Lambda, \alpha) \) are presentations \( p : P(\gamma_1) \rightarrow P(\gamma_0) \) which we denote \( P(\gamma_s) \). The next theorem in this paper is:

**Theorem 0.0.2** (Virtual Generic Decomposition Theorem 2.3.11). Let \( \{ \beta_i \} \) be a partial cluster tilting set (Definition 2.3.10). Let \( \alpha = \sum r_i \beta_i \in \mathbb{Z}^n \) where \( r_i \in \mathbb{Q} \). Then \( r_i \in \mathbb{Z} \) and the general virtual representation in \( \text{Vrep}(\Lambda, \alpha) \) is isomorphic to \( \bigoplus_i P(\gamma_i)^{r_i} \) where \( P(\gamma_i) \) are rigid objects in \( \text{Vrep}(\Lambda, \beta_i) \). In other words, the set of all elements of \( \text{Vrep}(\Lambda, \alpha) \) isomorphic to \( \bigoplus_i P(\gamma_i)^{r_i} \) is open and dense.

The groups \( \text{Aut}_\Lambda(P(\gamma_0)), \text{Aut}_\Lambda(P(\gamma_1)) \) act on presentation space \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \). A semi-invariant on \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) is a polynomial function \( \sigma : \text{Pres}_\Lambda(\gamma_1, \gamma_0) \rightarrow K \) so that, for any \( (g_0, g_1) \in \text{Aut}_\Lambda(P(\gamma_0)) \times \text{Aut}_\Lambda(P(\gamma_1))^{op} \) and \( f : P(\gamma_1) \rightarrow P(\gamma_0) \) we have \( \sigma(g_0 f g_1) = \chi_0(g_0) \chi_1(g_1) \sigma(f) \), where \( \chi_0, \chi_1 \) are characters \( \text{Aut}_\Lambda(P(\gamma_s)) \rightarrow K^* \) for \( s = 0, 1 \) where by character we mean a regular (polynomial) function which is a homomorphism of groups. Every group homomorphism \( \text{Aut}_\Lambda(P(\alpha)) \rightarrow K^* \) factors through the group \( \prod_{i=1}^n \text{Aut}_\Lambda(P^{\alpha_i}) = \prod_{i=1}^n GL(\alpha_i, F_i) \). Since \( \sigma \) is defined on the affine space \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \), these characters extend to the endomorphism rings of \( P(\gamma_0), P(\gamma_1) \) (by \( g \mapsto \sigma(g f)/\sigma(f) \) for a fixed \( f \in \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) on which \( \sigma(f) \neq 0 \)). In Appendix B Theorem 0.2.41 we show that every character \( \text{End}_F(F^m) \rightarrow K \) is a power of the “reduced norm” (and thus a fractional power of the determinantal character given by taking the determinant of an \( F \)-endomorphism of \( F^m \) considered as a linear map over \( K \). See Definition 6.1.6. Therefore, the characters
associated to any semi-invariant on the presentation space $\text{Pres}_\Lambda(\gamma_1, \gamma_0)$ are nonnegative integer powers of the reduced norm for each division algebra $F_i$. This gives a vector weight in $\mathbb{N}^n$. The weights coming from $P(\gamma_0)$ and $P(\gamma_1)$ are equal when defined.

In this paper we do not use the reduced norm weights. We use determinantal (det-) weights. The coefficients of the det-weights are in general fractions. They are integers if and only if the characters are powers of the determinantal character. We also consider only weights $\alpha$ for any isomorphism. We call the set of such $\alpha \in \mathbb{Z}^n$ the (integral) domain of the semi-invariant of det-weight $\beta$ and denote it by $D_{\mathbb{Z}}(\beta)$.

In Section 3 we prove the virtual stability theorem which states that these domains of semi-invariants are given by “stability conditions”.

**Theorem 0.0.3** (Virtual Stability Theorem 3.1.1). Let $\Lambda$ be a finite dimensional hereditary algebra over a field with $n$ simple modules. Let $\alpha \in \mathbb{Z}^n$ and $\beta$ a real Schur root. Then, the following are equivalent:

1. There exists a morphism of projective modules $f : P \rightarrow Q$ so that $\dim Q - \dim P = \alpha$ and $f$ induces an isomorphism $f^* : \text{Hom}_\Lambda(Q, M_\beta) \cong \text{Hom}_\Lambda(P, M_\beta)$.
2. Stability conditions for $\alpha$ and $\beta$ hold: $\langle \alpha, \beta \rangle = 0$ and $\langle \alpha, \beta' \rangle \leq 0$ for all real Schur subroots $\beta' \subseteq \beta$ where $\langle \cdot, \cdot \rangle$ is the Euler-Ringel form defined in Proposition 1.2.2.
3. There is a semi-invariant of det-weight $\beta$ on the presentation space $\text{Hom}_\Lambda(P, Q)$.

We prove this first in the case when $\beta$ is sincere and $K$ is infinite (subsection 3.4), then for any $\beta$ (subsection 3.5), then for any field $K$ (subsection 3.6).

In Section 3 we prove the c-vector theorem below which states that, up to a precisely given sign, the det-weights $\beta_i$ associated to a cluster tilting object are equal to the c-vectors associated to the cluster tilting object.

**Theorem 0.0.4** (c-vector Theorem 4.3.1). Let $T = \bigoplus_{i=1}^n T_i$ be a cluster tilting object for $\Lambda$ and let $f_i = \dim K \text{End}_\Lambda(T_i)$.

1. There exist unique real Schur roots $\beta_1, \ldots, \beta_n$ so that $\dim T_i \in D(\beta_j)$ for $i \neq j$.
2. The c-vectors associated to the cluster tilting object are equal to $\beta_i$ up to sign: $c_i = \pm \beta_i$. More precisely, $c_i = (-f_i/\dim T_i, \beta_i)$ which are the reduced weights $\beta_i$ associated to a cluster tilting object.
3. $\dim T_i, c_i = -f_i$ for each $i = 1, \ldots, n$.

The c-vector theorem implies the sign coherence of c-vectors since weight vectors always lie in $\mathbb{N}^n$. Sign coherence of c-vectors has been shown in many cases [DWZ], [P] and in general in [GHKK]. We end with an example of a semi-invariant picture (Figure 1) illustrating some of our theorems and the important properties of the picture used in other papers [BHIT], [IOTW], [ITT], [ITT].

There are also two appendices. Appendix A (Sec 5) discusses when an hereditary algebra is Morita equivalent to the tensor algebra of a modulated quiver and gives an example when this is not true. Appendix B (Sec 6) reviews the basic properties of reduced norm and shows that every character $M_k(D) \rightarrow K$ is a power of the reduced norm. Thus every semi-invariant on presentation space has a weight vector $w \in \mathbb{N}^n$ so that, under automorphisms of $P(\gamma_0), P(\gamma_1)$, the semi-invariant changes by the product of $\pi_i^{w_i}$, where $\pi_i$ are the reduced norms of the $GL(F_i)$ blocks of the automorphisms. We call $w$ the reduced weight. We define the reduced norm semi-invariants $\overline{\sigma}_\beta$ and show that their reduced weights $\beta$ are the c-vectors associated to a reduced exchange matrix $\overline{B}_\Lambda = ZB\Lambda Z^{-1}$. 


1. Basic definitions

In this paper Λ will be a basic finite dimensional hereditary algebra over any field K. Basic means that, as a right module over itself, the summands of Λ are pairwise nonisomorphic. Finite dimensional hereditary algebras share many important properties with the tensor algebra of their associated modulated quiver. For example they have the same Euler matrix, the same real Schur roots, the same semi-invariant domains and the same c-vectors, which are the topics we study in this paper. So, we begin with modulated quivers which are slightly easier to understand than the general case. Then we extend the definitions of presentation spaces and semi-invariants on presentation spaces to general finite dimensional hereditary algebras.

1.1. Modulated quivers. By a modulated quiver $(Q, M)$ over $K$ we mean a finite quiver $Q$ without oriented cycles together with

1. a finite dimensional division algebra $F_i$ over $K$ at each vertex $i$ of $Q$ and
2. a finite dimensional $F_iF_j$ bimodule $M_{ij}$ for every arrow $i \rightarrow j$ in $Q$.

The absence of multiple arrows is not a restriction. If we have a quiver with more than one arrow $i \rightarrow j$ then these are combined into one arrow with the associated bimodule being the direct sum of the bimodules on the original arrows. For example, the quiver $1 \rightarrow 2$ is equivalent to $1 \rightarrow 2$ with bimodule $K^2$ on the arrow.

Definition 1.1.1. The valuation on $Q = (Q_0, Q_1)$ given by the modulation $M$ is defined to be the sequence of positive integers $f_i, i \in Q_0$ and $d_{ij}, d_{ji}$ for $i \rightarrow j$ in $Q_1$ given as follows.

1. $f_i = \dim_K F_i$ for each $i \in Q_0$.
2. $d_{ij} = \dim_{F_i} M_{ij}, d_{ji} = \dim_{F_j} M_{ij}$ for each $i \rightarrow j$ in $Q_1$.

Proposition 1.1.2. (DR) For any sequence of positive integers $f_i, i \in Q_0$ and pairs of positive integers $(d_{ij}, d_{ji})$ for every arrow $i \rightarrow j$ in $Q_1$ there exists a modulation of $Q$ having these numbers as valuation if and only if $d_{ij}f_j = f_id_{ji}$ for all $i, j$.

Proof. Let $K$ be any finite field, $K = \mathbb{F}_q$. For each $i$ let $F_i$ be the field with $q^{f_i}$ elements. For each arrow $i \rightarrow j$, let $M_{ij}$ be the field with $d_{ij}f_j$ elements. \qed

A (finite dimensional) representation $V$ of a modulated quiver $Q$ is given by

1. a finite dimensional $F_i$-vector space $V_i$ at each vertex $i$ in $Q_0$ and
2. an $F_j$ linear map $V_i \otimes_{F_i} M_{ij} \rightarrow V_j$ for every arrow $i \rightarrow j$ in $Q_1$.

A (finite dimensional) representation of a modulated quiver is the same as a finite dimensional module over the tensor algebra $T(Q, M)$ of $(Q, M)$ which is defined to be the direct sum of all tensor paths:

$$T(Q, M) := \bigoplus M_{j_0,j_1} \otimes_{F_{j_1}} M_{j_1,j_2} \otimes_{F_{j_2}} \cdots \otimes_{F_{j_{r-1}}} M_{j_{r-1},j_r},$$

including paths of length zero ($F_j$), with multiplication given by concatenation of paths. Since the quiver $Q$ has no oriented cycles this algebra is a finite dimensional hereditary algebra over $K$.

Definition 1.1.3. Given a finite dimensional hereditary algebra $\Lambda$ over a field $K$, the associated modulated quiver $(Q, M)$ is given as follows. Fix an ordering of the simple $\Lambda$-modules $S_1, \ldots, S_n$. Let $P_i$ be the projective cover of $S_i$.

1. Let $Q$ be the quiver with $Q_0 = \{1, \ldots, n\}$ and arrows $i \rightarrow j$ when $\operatorname{Ext}^1_{\Lambda}(S_i, S_j) \neq 0$.
2. Let $F_i = \operatorname{End}_\Lambda(S_i)$ for each $i \in Q_0$.
3. For each $i \rightarrow j$ in $Q_1$ let $M_{ij} = \operatorname{Hom}_\Lambda(P_j, rP_i/r^2P_i)$.

There are examples of hereditary algebras which are not equivalent to their associated modulated quiver. We discuss these pathologies in Appendix A (Sec 5). Our results are general and hence include these pathological cases as well.
1.2. Euler matrix. The underlying valued quiver of an hereditary algebra $\Lambda$ is the valued quiver of its associated modulated quiver. However, it is useful to go directly from $\Lambda$ to its underlying valued quiver, i.e., $f_i = \dim_K F_i$ where $F_i = \text{End}_\Lambda(S_i)$ for each $i \in Q_0$, $d_{ij} = \dim_{F_j} \text{Hom}_\Lambda(P_j, r P_i/r^2 P_i)$, $d_{ji} = \dim_{F_i} \text{Hom}_\Lambda(P_j, r P_i/r^2 P_i)$ for each $i \to j$ in $Q_1$.

The dimension vector $\dim V$ of $V$ is defined to be the vector whose $i$-th coordinate is $\dim_{F_i} V_i$. We also have the dimension vector over $K$ given by

$$\dim_K V = D \dim V$$

where $D$ is the diagonal matrix with diagonal entries $f_i = \dim_K F_i$. Let $E, L, R$ be the $n \times n$ matrices with $ij$ entries

$$E_{ij} = \dim_K \text{Hom}_\Lambda(S_i, S_j) - \dim_K \text{Ext}_\Lambda^1(S_i, S_j)$$

$$L_{ij} = \dim_{F_j} \text{Hom}_\Lambda(S_i, S_j) - \dim_{F_j} \text{Ext}_\Lambda^1(S_i, S_j)$$

$$R_{ij} = \dim_{F_i} \text{Hom}_\Lambda(S_i, S_j) - \dim_{F_i} \text{Ext}_\Lambda^1(S_i, S_j)$$

Then $E_{ij} = L_{ij}f_j = f_iR_{ij}$ or, equivalently,

$$E = LD = DR.$$ 

We call $E$ the Euler matrix of $\Lambda$, $L$ the left Euler matrix of $\Lambda$ and $R$ the right Euler matrix of $\Lambda$. The underlying valued quiver of $\Lambda$ has vertices $1 \leq i \leq n$ corresponding to the simple modules $S_i$ and an arrow $i \to j$ if $E_{ij} < 0$ with valuations $f_i$ on vertex $i$ and $(d_{ij}, d_{ji}) = (-L_{ij}, -R_{ij})$ for every arrow $i \to j$.

**Example 1.2.1.** For example, for the valued quiver $f_2=3, f_1=2$ we have:

$$LD = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = E = \begin{bmatrix} 2 & 0 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = DR.$$ 

The matrices $L, R$ are always unimodular and $\det E = \det D$ is always the product of the dimensions $f_i$ of $F_i = \text{End}_\Lambda(P_i) = \text{End}_\Lambda(S_i)$.

We also use, in the subsection on $c$-vectors (sec 4.2), the exchange matrix $B = L^t - R$. Since $DB = DL^t - DR = E^t - E$, $DB$ is always skew symmetric. In the example this is:

$$B = L^t - R = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}, \quad DB = E^t - E = \begin{bmatrix} 0 & -6 \\ 6 & 0 \end{bmatrix}.$$ 

**Proposition 1.2.2.** Let $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the Euler-Ringel pairing given by $\langle x, y \rangle = x^tEy$. Then, for any two $\Lambda$-modules $M, N$ we have:

$$\langle \dim M, \dim N \rangle = \dim_K \text{Hom}_\Lambda(M, N) - \dim_K \text{Ext}_\Lambda^1(M, N).$$

For example, pairing $(P_1, \cdots, P_n)$ with $(S_1, \cdots, S_n)$ gives

$$\langle \dim P_i, \dim S_j \rangle = \dim_K \text{Hom}_\Lambda(P_i, S_j) = f_i\delta_{ij}.$$ 

This equation can be written as $PEI_n = D$ where the $i$-th row of the matrix $P$ is $\dim P_i$. Furthermore $PE = D$ and $E = LD$ imply that $P = L^{-1}$.

**Proposition 1.2.3.** Suppose that $\text{End}_\Lambda(M)$ is a division algebra. Then $(\dim M, \dim N)$ and $(\dim N, \dim M)$ are divisible by $f_M = \dim_K \text{End}_\Lambda(M)$.

**Proof.** $(\dim M, \dim N) = \dim_K \text{Hom}_\Lambda(M, N) - \dim_K \text{Ext}_\Lambda^1(M, N)$ which is divisible by $f_M$ since $\dim_K \text{Hom}_\Lambda(M, N)$ and $\text{Ext}_\Lambda^1(M, N)$ are vector spaces over $\text{End}_\Lambda(M)$. 

**Notation 1.2.4.** For each $\alpha \in \mathbb{N}^n$ we use the notation $P(\alpha) = \bigoplus_i P_i^{\alpha_i}$. For example, $\Lambda = P(1,1,\cdots,1)$ if $\Lambda$ is basic.
With this notation, we have the following. Suppose $\dim M = \beta$ and $\gamma \in \mathbb{N}^n$. Then

$$\langle \dim P(\gamma), \dim M \rangle = \sum_{i=1}^{n} \gamma_i \dim K \hom_{\Lambda}(P_i, M) = \sum_{i=1}^{n} \gamma_i f_i \beta_i.$$  

1.3. Exceptional sequences. We review the definition and basic properties of exceptional sequences. See [CB93], [Rin94] for details.

**Definition 1.3.1.** Let $\Lambda$ be a finite dimensional hereditary algebra over any field $K$. Then a $\Lambda$-module $M$ is called *exceptional* if $\ext^1_{\Lambda}(M, M) = 0$ and $\end_{\Lambda}(M)$ is a division algebra. In particular $M$ is indecomposable. A sequence of modules $(X_1, \cdots, X_k)$ is called an *exceptional sequence* if all objects are exceptional and

$$\hom_{\Lambda}(X_j, X_i) = \ext^1_{\Lambda}(X_j, X_i) = 0 \text{ for all } j > i.$$  

An exceptional sequence is called *complete* if it is of maximal length. By 1.3.3(1) below, the maximal length is equal to the number of nonisomorphic simple modules.

The following are standard examples of complete exceptional sequences.

**Proposition 1.3.2.** Let $\Lambda$ be a finite dimensional hereditary algebra with admissible order given by $\hom_{\Lambda}(P_j, P_i) = 0$ for all $j > i$. Then:

1. The simple modules $(S_n, S_{n-1}, \cdots, S_1)$ form an exceptional sequence.
2. The projective modules $(P_1, P_2, \cdots, P_n)$ form an exceptional sequence.
3. The injective modules $(I_1, I_2, \cdots, I_n)$ form an exceptional sequence where $I_i$ is the injective envelope of the simple module $S_i$ for $i = 1, \cdots, n$.

Exceptional sequences have many nice properties. We list here only those properties that we use to prove the main theorems in this paper.

**Proposition 1.3.3.** Let $n$ be the number of simple $\Lambda$ modules.

1. Exceptional sequences are complete if and only if they have $n$ objects.
2. Every exceptional sequence can be extended to a complete exceptional sequence.
3. If $(X_1, \cdots, X_n)$ is an exceptional sequence, $\{\dim X_i\}$ generates $\mathbb{Z}^n$ as a $\mathbb{Z}$-module.
4. Given an exceptional sequence $(X_1, \cdots, X_{n-1})$ of length $n-1$ and any $j = 1, \cdots, n$, there are modules $Y_j$, unique up to isomorphism, so that

$$(X_1, \cdots, X_{j-1}, Y_j, X_j, \cdots, X_{n-1})$$

is a (complete) exceptional sequence.
5. $\end_{\Lambda}(Y_j) \cong \end_{\Lambda}(Y_{j'})$ for all $j, j'$ in (4) above.
6. Let $(X_1, \cdots, X_n)$ be an exceptional sequence. Then:

- If $X_n$ is non-projective, then $(\tau X_n, X_1, \cdots, X_{n-1})$ is an exceptional sequence.
- If $X_n = P_k$ is projective, then $(I_k, X_1, \cdots, X_{n-1})$ is an exceptional sequence.

Condition (4) implies that there is an action of the braid group on $n$ strands on the set of (isomorphism classes of) complete exceptional sequences. For example, the braid generator $\sigma_i$ which moves the $i$-th strand over the $(i+1)$-st strand acts by:

$$(1.3.1) \quad \sigma_i(X_1, \cdots, X_n) = (X_1, \cdots, X_{i-1}, X'_{i+1}, X_i, X_{i+2}, \cdots, X_n)$$

where, by property (4), $X'_{i+1}$ is the unique exceptional module which fits in the indicated location in the exceptional sequence given the other objects. One of the very important theorems about exceptional sequences used in this paper is the following result proved in [CB93] in the algebraically closed case and [Rin94] in general.

**Theorem 1.3.4** (Crawley-Boevey, Ringel). The braid group on $n$ strands acts transitively on the set of all complete exceptional sequences.
In the case when \( K \) is algebraically closed, or, more generally when \( \Lambda = KQ \) is the path algebra of a simply laced quiver without oriented cycles, it follows that \( \text{End}_\Lambda(M) = K \) for every exceptional \( \Lambda \)-module \( M \). In general, the endomorphism rings of the \( X_i \) are division algebras which remain the same after any braid move by Proposition [1.3.3(5)]. So, Theorem [1.3.4] implies the following.

**Corollary 1.3.5.** For any exceptional sequence \( (X_1, \ldots, X_n) \), there is a permutation \( \pi \) of \( n \) so that \( \text{End}_\Lambda(X_i) \cong \text{End}_\Lambda(S_{\pi(i)}) \) for all \( i \).

Another important consequence of this theorem is the following.

**Proposition 1.3.6.** Suppose that \( (\beta_1, \ldots, \beta_n) \) is the set of dimension vectors of a complete exceptional sequence. Then each vector \( \beta_j \) is uniquely determined by the other vectors together with the requirements that

1. \( \langle \beta_k, \beta_i \rangle = 0 \) for \( k > i \).
2. The vectors \( \beta_i \) additively generate \( \mathbb{Z}^n \).

Note that these conditions depend only on \( n \) and the Euler form \( \langle \cdot, \cdot \rangle \). This implies that there is an action of the braid group on \( n \)-strands on the set of dimension vectors of exceptional sequences which is given by

\[
\sigma_i(\beta_1, \ldots, \beta_n) = (\beta_1, \ldots, \beta_{i-1}, \beta_i^*+1, \beta_i, \beta_{i+2}, \ldots, \beta_n)
\]

where \( \beta_i^*+1 \) is the unique vector in \( \mathbb{N}^n \) satisfying the conditions of the proposition above.

**Corollary 1.3.7.** An exceptional \( \Lambda \)-module is uniquely determined up to isomorphism by its dimension vector. Furthermore, a vector \( \beta \in \mathbb{N}^n \) is the dimension vector of an exceptional module if and only if it appears in \( \sigma(\alpha_1, \ldots, \alpha_n) \) for some element \( \sigma \) of the braid group on \( n \) strands where \( \alpha_i = \dim S_i \) are the unit vectors in \( \mathbb{Z}^n \). In particular, the set of such dimension vectors depends only on the underlying valued quiver of \( \Lambda \).

This is a restatement of another important theorem of Schofield [S91]: The dimension vectors of the exceptional \( \Lambda \)-modules are the real Schur roots of \( \Lambda \). In this paper we will not need the original definition of a real Schur root [K]. Following [S91, S92], we use the characterizing property of real Schur roots given by the above corollary as the definition.

**Definition 1.3.8.** [S92] A real Schur root of \( \Lambda \) is a vector \( \beta \in \mathbb{N}^n \) with the property that there exists an exceptional module \( M_\beta \) with \( \dim M_\beta = \beta \).

As a special case of Corollary [1.3.7] above we have the following.

**Corollary 1.3.9.** The real Schur roots of an hereditary algebra are the same as those of the associated modulated quiver.

1.4. **Extension to arbitrary fields.** The main results of this paper hold for arbitrary fields. The proofs are first done for infinite fields and they are extended to all fields using the following arguments.

Recall that if \( K \) is a finite field and \( F \) is a finite field extension of \( K \) then

\[
F \otimes_K K(t) \cong F(t)
\]

is a finite field extension of \( K(t) \). For any \( K \)-algebra \( \Lambda \) we will use the notation \( \Lambda(t) \) to denote \( \Lambda \otimes_K K(t) \). This is a finite dimensional hereditary algebra over \( K(t) \). For any \( \Lambda \)-module \( M \), let \( M(t) \) denote the \( \Lambda(t) \)-module \( M \otimes_K K(t) \). Recall that the dimension vector of \( M(t) \) as a \( \Lambda(t) \)-module is the vector whose \( i \)-th coordinate is \( \dim_{F(t)} \text{Hom}_{\Lambda(t)}(P_i(t), M(t)) \).

**Theorem 1.4.1.** Let \( \Lambda \) be a finite dimensional hereditary algebra over a finite field \( K \) and let \( M \) be an exceptional \( \Lambda \)-module with \( \dim M = \beta \). Then, \( \Lambda(t) \) is a finite dimensional hereditary algebra over \( K(t) \) and \( M(t) \) is an exceptional \( \Lambda(t) \)-module with the same dimension vector \( \beta \). Furthermore, every exceptional \( \Lambda(t) \)-module is isomorphic to \( M(t) \) for a unique exceptional \( \Lambda \)-module \( M \).
Proof. Since tensor product over $K$ with $K(t)$ is exact we get:

$$\Hom_{\Lambda}(X,Y) \otimes_{K} K(t) \cong \Hom_{\Lambda(t)}(X(t),Y(t))$$

$$\Ext_{\Lambda}^{2}(X,Y) \otimes_{K} K(t) \cong \Ext_{\Lambda(t)}^{2}(X(t),Y(t))$$

for any two $\Lambda$-modules $X, Y$. Therefore, a $\Lambda$-module $M$ is exceptional if and only if $M(t) = M \otimes_{K} K(t)$ is an exceptional $\Lambda(t)$-module. The rest follows from Corollary 1.3.7. □

2. Virtual representations and semi-invariants

Throughout this section we consider representations of a finite dimensional hereditary algebra $\Lambda$ over an infinite field $K$.

2.1. Generic decomposition theorem. We first recall the Happel-Ringel Lemma \[HR\].

Lemma 2.1.1 (Happel-Ringel). Suppose that $T_1, T_2$ are indecomposable modules over an hereditary algebra $\Lambda$ so that $\Ext_{\Lambda}^{1}(T_1, T_2) = 0$. Then any nonzero morphism $T_2 \to T_1$ is either a monomorphism or an epimorphism.

An important consequence of this lemma is the following observation of Schofield.

Lemma 2.1.2 (Schofield). Suppose that $\{M_i\}$ is a set of nonisomorphic indecomposable modules so that $\Ext_{\Lambda}^{1}(M_j, M_i) = 0$ for all $i, j$. Then $\{M_i\}$ can be renumbered so that $\Hom_{\Lambda}(M_j, M_i) = 0$ for $j > i$, i.e., so that it forms an exceptional sequence.

Proof. If not, there is an oriented cycle of nonzero morphisms between the $M_i$ which are monomorphisms or epimorphisms. In this oriented cycle there must be an epimorphism followed by a monomorphism, hence the composition is neither a monomorphism nor an epimorphism which contradicts the Happel-Ringel Lemma. □

We give a definition of the representation space $\Rep(\Lambda, \alpha)$ for any finite dimensional hereditary algebra $\Lambda$ and $\alpha \in \mathbb{N}^n$ which will agree with the classical definition of $\Rep(\Lambda, \alpha)$ when $\Lambda = KQ$, the path algebra of a quiver, or when $\Lambda = T(Q, \mathcal{M})$, the tensor algebra of any modulated quiver. In the definition we need to choose a decomposition of the radical of each projective module, however, if different decompositions are chosen, there is an isomorphism of the affine varieties which induce isomorphisms on cokernel modules.

Definition 2.1.3. For each $\alpha \in \mathbb{N}^n$ consider the following space

\[
H(\Lambda, \alpha) := \prod_{j \in Q_0} \Hom_{\Lambda}(\bigoplus_{i \in S_j} P_i^{j,i}, \bigoplus_{j \in S_j} P_i^{j,i}) \subseteq \Hom_{\Lambda}(\bigoplus_{i,j} P_i^{j,i}, \alpha(\Lambda))
\]

where $d_{ij} = \dim_{K_i} \Ext_{\Lambda}^{1}(S_j, S_i)$ for $i \neq j$. For each choice of decompositions

\[
\{\varphi_j : \bigoplus_{i \in Q_0} P_i^{j,i} \cong \text{rad} P_j\}_{j \in Q_0}
\]

we get an isomorphism $\varphi_{\alpha} = \bigoplus_{i,j} P_i^{j,i} \cong \text{rad} P(\alpha)$ and we define the representation space $\Rep_{\alpha}(\Lambda, \alpha)$ to be

$$\Rep_{\alpha}(\Lambda, \alpha) := \{f : \text{rad} P(\alpha) \to P(\alpha) \mid f \circ \varphi_{\alpha} \in H(\Lambda, \alpha)\}$$

Let $N_f := \text{coker}(f - \text{inc} : \text{rad} P(\alpha) \to P(\alpha))$ for each $f \in \Rep_{\alpha}(\Lambda, \alpha)$.

Remark 2.1.4. (1) $\dim N_f = \alpha$ for every $f \in \Rep_{\alpha}(\Lambda, \alpha)$.

(2) Every $\Lambda$-module $M$ with $\dim M = \alpha$ is isomorphic to $N_f$ for some $f \in \Rep_{\alpha}(\Lambda, \alpha)$.

(3) If $\psi_{\alpha}$ is obtained by another choice of decompositions (2.1.2) then there is a linear isomorphism $\lambda : \Rep_{\alpha}(\Lambda, \alpha) \cong \Rep_{\psi}(\Lambda, \alpha)$ with the property that $N_f = N_{\lambda f}$.

(4) When $\Lambda = KQ$ is the path algebra of a quiver or $\Lambda = T(Q, \mathcal{M})$ the tensor algebra of a modulated quiver, there are canonical choices for the isomorphisms $\varphi_j$ of (2.1.2) and the resulting representation space agrees with the classical definition of $\Rep(\Lambda, \alpha)$. 

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(5) We will write $\text{Rep}_\varphi(\Lambda, \alpha) = \text{Rep}(\Lambda, \alpha)$ suppressing the choice of $\varphi$.

The following was originally proved by Kac [Ka] for quivers over an algebraically closed field $K$. However, the proof that we give below works over any infinite field.

**Lemma 2.1.5.** Let $\Lambda$ be an hereditary algebra over an infinite field. Suppose that $M$ is a rigid $\Lambda$-module with projective presentation $0 \to P \xrightarrow{\varphi} P' \to M \to 0$. Then the set of all $f \in \text{Hom}_\Lambda(P, P')$ with $\text{coker}(f) \cong M$ is a nonempty Zariski open set in $\text{Hom}_\Lambda(P, P')$.

**Proof.** Since $\text{Hom}_\Lambda(P, P')$ contains a monomorphism, the set of all monomorphisms $P \to P'$ is open. For any two monomorphisms $f_1, f_2 : P \hookrightarrow P'$, $\text{Ext}_\Lambda^1(\text{coker} f_1, \text{coker} f_2)$ is the cokernel of the map

$$\psi(f_1, f_2) : \text{End}_\Lambda(P) \oplus \text{End}_\Lambda(P') \to \text{Hom}_\Lambda(P, P')$$

which sends $(g, g')$ to $g' \circ f_1 + f_2 \circ g$. Since $\psi(p, p)$ is surjective, the subset $U \subseteq \text{Hom}_\Lambda(P, P')$ of all monomorphisms $f : P \to P'$ so that $\psi(f, p), \psi(p, f)$ and $\psi(f, f)$ are all surjective is open. This implies that $\text{coker } f \oplus M$ is rigid for all $f \in U$. We will show that $\text{coker } f$ is isomorphic to $M$ for any $f \in U$. This will imply that $\{f \in \text{Hom}_\Lambda(P, P') : \text{coker } f \cong M\} = U$ is open.

For any $f \in U$, let $\{N_j\}$ be the components of $\text{coker } f$. Let $\{M_i\}$ be the components of $M$. Then $\{M_i, N_j\}$ form a collection of indecomposable modules which do not extend each other. So, by Schofield’s observation, we can arrange them into an exceptional sequence, possibly with repetitions. Take the last object in the sequence. By symmetry, suppose it is $N_j$. Since $\dim K = \dim \text{coker } f$ we have

$$\dim_K \text{Hom}_\Lambda(N_j, M) = (\dim N_j, \dim M) = \dim_K \text{Hom}_\Lambda(N_j, \text{coker } f) \neq 0.$$

So, there is a nonzero morphism $N_j \to M_i$ for some $i$. Since $N_j$ is last in the exceptional sequence, this can happen only if $N_j \cong M_i$. Then $\text{coker } f/N_j, M/M_i$ are rigid modules of the same dimension vector. So, $\text{coker } f/N_j \cong M/M_i$ by induction on dimension. We conclude that $\text{coker } f \cong N_j \oplus \text{coker } f/N_j \cong M_i \oplus M/M_i \cong M$ as claimed. □

**Theorem 2.1.6.** Let $\Lambda$ be an hereditary algebra over an infinite field. Suppose that $\alpha \in \mathbb{N}^n$ and $M$ is a rigid module with $\dim M = \alpha$. Then the set of all $f \in \text{Rep}(\Lambda, \alpha)$ so that $N_f \cong M$ forms an open dense subset of $\text{Rep}(\Lambda, \alpha)$.

**Proof.** Let $\psi : \text{Rep}(\Lambda, \alpha) \hookrightarrow \text{Hom}_\Lambda(\text{rad } P(\alpha), P(\alpha))$ be the affine linear embedding given by $\psi(f) = f - \text{inc}$. Let $V$ be the set of all $f \in \text{Rep}(\Lambda, \alpha)$ so that $N_f \cong M$. Then $V$ is open since it is the inverse image under $\psi$ of the open subset of $\text{Hom}_\Lambda(\text{rad } P(\alpha), P(\alpha))$ of all maps with cokernel isomorphic to $M$. □

Since exceptional modules are rigid, we have the following immediate consequence.

**Corollary 2.1.7.** Suppose that $M_\alpha$ is an exceptional $\Lambda$-module with $\dim M_\alpha = \alpha$. Then the set of all $f \in \text{Rep}(\Lambda, \alpha)$ so that $N_f \cong M_\alpha$ forms an open and thus dense subset of $\text{Rep}(\Lambda, \alpha)$. In particular $M_\alpha$ is uniquely determined up to isomorphism by $\alpha$.

Another important consequence of Theorem 2.1.6 is the following.

**Corollary 2.1.8** (Generic Decomposition Theorem for rigid modules in modulated case). Suppose that $\alpha_1, \cdots, \alpha_k$ are real Schur roots so that $\text{Ext}_\Lambda^1(M_{\alpha_i}, M_{\alpha_j}) = 0$ for all $i, j$. Let $\gamma = \sum_{i=1}^k n_i \alpha_i$ be a nonnegative integer linear combination of these roots. Then the generic representation with dimension vector $\gamma$ is isomorphic to $\bigoplus_{i=1}^k M_{\alpha_i}^{n_i}$.

**Proof.** Apply Theorem 2.1.6 to the module $M = \bigoplus_{i=1}^k M_{\alpha_i}^{n_i}$ with $\dim M = \gamma$. □
2.2. Presentation Spaces and Semi-invariants. Let $\gamma_0, \gamma_1$ be vectors in $\mathbb{N}^n$. We define the presentation space $\text{Pres}_\Lambda(\gamma_1, \gamma_0)$ to be

$$\text{Pres}_\Lambda(\gamma_1, \gamma_0) := \text{Hom}_\Lambda(P(\gamma_1), P(\gamma_0))$$

where we use the notation $P(\alpha) = \bigoplus P_i^{\alpha_i}$. Presentation spaces are affine spaces over $K$. They are related to representation spaces as follows. Suppose that $\alpha \in \mathbb{N}^n$. Then $\varphi : \text{rad}P(\alpha) \cong P(\gamma)$ for $\gamma \in \mathbb{N}^n$ and we have the $K$-linear embedding:

$$\text{Rep}(\Lambda, \alpha) \hookrightarrow \text{Pres}_\Lambda(\gamma, \alpha)$$

sending $f : \text{rad}P(\alpha) \to P(\alpha)$ to $(f - \text{inc}) \circ \varphi^{-1}$. The elements of $\text{Rep}(\Lambda, \alpha)$ and their images in $\text{Pres}_\Lambda(\gamma, \alpha)$ represent the same module $N_f$. The algebraic group $\text{Aut}(P(\gamma_1))^\text{op} \times \text{Aut}(P(\gamma_0))$ acts on presentation space by composition: $(a, b)f = baf$. This is a generalization of what happens in the algebraically closed case.

**Definition 2.2.1.** A semi-invariant on $\text{Pres}_\Lambda(\gamma_1, \gamma_0) = \text{Hom}_\Lambda(P(\gamma_1), P(\gamma_0))$ is defined to be a regular function

$$\sigma : \text{Pres}_\Lambda(\gamma_1, \gamma_0) \to K$$

for which there exist characters $\eta_s : \text{Aut}(P(\gamma_s)) \to K^*$, $s = 0, 1$, so that, for all $(g_0, g_1) \in \text{Aut}(P(\gamma_0)) \times \text{Aut}(P(\gamma_1))$ and $f \in \text{Pres}_\Lambda(\gamma_1, \gamma_0)$, we have $\sigma(g_0fg_1) = \sigma(f)\eta_0(g_0)\eta_1(g_1)$. The pair of characters $(\eta_0, \eta_1)$ is called the weight of $\sigma$.

The following lemma shows that such characters are products of character on $\text{GL}(\alpha_i, F_i)$.

**Lemma 2.2.2.** Every group homomorphism $\text{Aut}_\Lambda(P(\alpha)) \to K^*$ factors through the group $\prod_i \text{Aut}_\Lambda(P_i^{\alpha_i})$.

**Proof.** When $K$ has only two elements, the lemma holds trivially. So, we may assume $K$ has at least three elements. Then every element of $K$ can be written as $a - b$ where $a, b \neq 0$. So, the elementary matrix

$$\begin{bmatrix} 1 & 0 \\ a - b & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$$

is a commutator. We write automorphisms of $P(\alpha)$ in block matrix form. Since $\text{Hom}_\Lambda(P_j, P_i) = 0$ for $i < j$, the matrix is lower triangular with diagonal blocks in $\text{Aut}_\Lambda(P_i^{\alpha_i})$. So, every element in the kernel of the homomorphism $\pi : \text{Aut}_\Lambda(P(\alpha)) \to \prod \text{Aut}_\Lambda(P_i^{\alpha_i})$, when written in matrix form, is lower triangular with 1s on the diagonal. But all such matrices are products of elementary matrices such as the one above. So ker $\pi$ lies in the commutator subgroup of $\text{Aut}_\Lambda(P(\alpha))$. Since $K^*$ is abelian, any homomorphism $\varphi : \text{Aut}_\Lambda(P(\alpha)) \to K^*$ is trivial on commutators. Therefore ker $\pi \subseteq$ ker $\varphi$ which implies that $\varphi$ factors through $\pi$ proving the lemma. \(\square\)

**Remark 2.2.3.** (1) By Lemma 2.2.2 every character $\chi : \text{Aut}_\Lambda(\bigoplus P_j^{\alpha_j}) \to K^*$ is a product of component characters $\text{Aut}_\Lambda(P_i^{\alpha_i}) \to K^*$.

(2) Since $\text{End}_\Lambda(P_i) = F_i$ is a division algebra, we have $\text{Aut}_\Lambda(P_i^{\alpha_i}) \cong \text{GL}(n_i, F_i)$. This has a character given by the determinant of the induced automorphism $S_i^{\alpha_i} \to S_i^{n_i}$ considered as a $K$-linear map:

$$\chi_i = \text{det}_K : \text{Aut}_\Lambda(P_i^{\alpha_i}) \to K^*.$$ 

Then $\chi_i$ is a polynomial of degree $n_if_i$ where $f_i = \dim_K S_i = \dim_K F_i$.

(3) We will only consider special characters which we call “determinantal” (Definition 2.2.4 below). There may be other characters called “reduced norms” which are explained in detail in Appendix B, Sec 6.
Definition 2.2.4. We call a character \( \chi : \text{Aut}_\Lambda(\bigoplus P_j^{n_j}) \to K^* \) \emph{determinantal} if its components characters \( \text{Aut}_\Lambda(P_i^{n_i}) \to K^* \) are integer powers of the determinant \( \chi_i \), i.e., there exists a vector \( \alpha \in \mathbb{Z}^n \) so that \( \chi = \prod_i \chi_i^{\alpha_i} \). The coordinate \( \alpha_i \) is uniquely determined by \( \chi \) if and only if \( n_i \neq 0 \) (and \( K = \text{infinite} \)).

The following proposition is analogous to Proposition 3.3.3 from [10TW09] which was proved for simply laced quivers. But the same proof works in general.

Proposition 2.2.5. Let \( \Lambda \) be a finite dimensional hereditary algebra over an infinite field \( K \). Suppose that \( \sigma \) is a nonzero semi-invariant on \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) with weights \( \eta_0, \eta_1 \) which are determinantal characters given by \( \eta_0 = \prod_i \chi_i^{\alpha_i}, \eta_1 = \prod_i \chi_i^{\beta_i} \) where \( \alpha, \beta \in \mathbb{Z}^n \). Then \( \alpha_i = \beta_i \) whenever the \( i \)-th coordinates of \( \gamma_0, \gamma_1 \) are both nonzero. \( \square \)

Definition 2.2.6. We say that a semi-invariant \( \sigma \) on \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) has determinantal \emph{weight vector} \( (\det-weight) \beta \in \mathbb{Z}^n \) if both of its weights can be written as \( \chi_i^{\beta_i} \). In other words, for any \( f : P(\gamma_1) \to P(\gamma_0), h \in \text{Aut}(P(\gamma_1)), g \in \text{Aut}(P(\gamma_0)) \) we have:

\[
\sigma(gfh) = \sigma(f) \prod_i \chi_i(g)^{\beta_i} \chi_i(h)^{\beta_i}
\]

We also say that \( f : P(\gamma_1) \to P(\gamma_0) \) admits a semi-invariant of det-weight \( \beta \) if there exists a semi-invariant \( \sigma \) of det-weight \( \beta \) on \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) so that \( \sigma(f) \neq 0 \).

Example 2.2.7. The following example of a modulated quiver illustrates many of these concepts. Let \( K = \mathbb{R} \) and consider the following \( \mathbb{R} \)-modulated quiver.

\[
F_3 = \mathbb{R} \xrightarrow{M_{a2}=H} F_2 = \mathbb{C} \xrightarrow{M_{21}=C} F_1 = \mathbb{R}
\]

Then \( F_2 \) is the representation \( 0 \to \mathbb{C} \to \mathbb{R}^2 \) with radical \( 0 \to 0 \to \mathbb{R}^2 \) which is \( S^2 = P_2^2 = \text{rad } P_2 \). The structure map of the module \( F_2 \) gives an \( \mathbb{R} \)-linear isomorphism \( \mathbb{C} \cong \mathbb{R}^2 \). Let \( \gamma_0 = e_2 = (0,1,0), \gamma_1 = e_1 = (1,0,0) \). Then \( P(\gamma_0) = P_2, P(\gamma_1) = P_1 \) and the presentation space \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) = \text{Hom}_\Lambda(P_1, P_2) \cong \mathbb{R}^2 \). Thus any homomorphism \( f : P_1 \to P_2 \) is given by two real numbers \( (x,y) \). Then \( \sigma(f) = x^2 + y^2 \) is a semi-invariant on \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) of weight \( \beta = (2,1,*) \) with \( \beta_3 \) being undefined. To see this consider the group which acts on the presentation space. The group is \( G = GL(1, \mathbb{R})^{op} \times GL(1, \mathbb{C}) \times GL(0, \mathbb{R}) \). If \( g = (r, a+bi, 1) \in G \) then \( g \mathbf{f} = (a+bi) \mathbf{f} = (axr - byr, ayr + bxr) \) with \( \sigma(gf) = (a^2 + b^2)(x^2 + y^2)r^2 = \chi_1(g)^2 \chi_2(g)^1 \chi_3(g)^m \sigma(f) \).

Since \( \chi_3(g) = 1 \), this equation is true for any \( m \in \mathbb{Z} \). Thus, \( \sigma \) is a determinantal semi-invariant on \( \text{Pres}_\Lambda(e_1, e_2) \) with det-weight \( (2,1,m) \) for any \( m \in \mathbb{Z} \). The third coordinate is not well-defined since \( \gamma_{0,3} = 0 = \gamma_{1,3} \).

We now show that the coordinates of \( \beta \) are nonnegative when they are well-defined.

Proposition 2.2.8. If a semi-invariant on \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) has well-defined det-weight \( \beta \) then \( \beta \in \mathbb{N}^n \), i.e., \( \beta_i \geq 0 \) for all \( i \).

Proof. Since \( \beta \) is well-defined, for each \( i \), either \( \gamma_{0,i} \neq 0 \) or \( \gamma_{1,i} \neq 0 \). By symmetry assume \( n_i = \gamma_{0,i} = 0 \). Choose \( f \in \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) so that \( \sigma(f) \neq 0 \). Let \( \psi : \text{End}_\Lambda(P_i^{n_i}) \to \text{End}_\Lambda(\bigoplus P_j^{n_j}) \) be the embedding which sends \( g \in \text{End}_\Lambda(P_i^{n_i}) \) to the endomorphism of \( \bigoplus P_j^{n_j} \) which is \( g \) on \( P_i^{n_i} \) and the inclusion map on \( P_j^{n_j} \) for every \( j \neq i \). Then \( g \to \sigma(\psi(g)f) \) is a regular function \( \text{End}_\Lambda(P_i^{n_i}) \to K \) which extends the map \( g \to \chi_i(g)^{\beta_i} \sigma(f) \) on \( \text{Aut}_\Lambda(P_i^{n_i}) \) and sends \( 0 \) to \( 0 \). This is impossible for \( \beta_i < 0 \). Therefore, \( \beta_i \geq 0 \) for every \( i \). \( \square \)

The following proposition is one of the motivations for the uniform notation \( V_{\text{rep}}(\Lambda, \alpha) \) introduced in the next section in Definition 2.3.1

Proposition 2.2.9. Suppose that \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) has a semi-invariant of det-weight \( \beta \) and \( (L^i)^{-1}(\gamma_0 - \gamma_1) = \alpha \) where \( L \) is the left Euler matrix and \( K \) is infinite. Then \( \langle \alpha, \beta \rangle = 0 \).
Proof. Consider the automorphisms of $P(\gamma_1) = \bigoplus_i P^{\gamma_1,i}_i$ and $P(\gamma_0) = \bigoplus_i P^{\gamma_0,i}_i$ given by multiplication by $\lambda \in K^*$. The character of this automorphism of $P^{\gamma_1,i}_i$ is $\chi_i(\lambda) = \det(\lambda^{\gamma_1,i}) = \lambda^{\gamma_1,i}$, since $\lambda f = f \lambda$ for all $f \in \text{Pres}_\Lambda(\gamma_1, \gamma_0)$ we conclude that $\lambda \sum \gamma_{i,j} i_i \beta_i = \lambda \sum \gamma_{i,j} i_i \beta_i$.

Since this polynomial equation holds for all $\lambda \in K^*$ which is infinite, we conclude that $\sum \gamma_{i,j} i_i \beta_i = \sum \gamma_{i,j} i_i \beta_i$. So,

$$0 = \sum (\gamma_{0,i} - \gamma_{1,i}) i_i \beta_i = \langle \alpha, \beta \rangle$$

since $(L^t)^{-1}(\gamma_0 - \gamma_1) = \alpha$ and $\langle \alpha, \beta \rangle = \alpha^t E \beta = \alpha^t LD \beta = (\gamma_0 - \gamma_1)D \beta$. □

As a corollary of this proof we have the following.

Corollary 2.2.10. A semi-invariant $\sigma$ on $\text{Pres}_\Lambda(\gamma_1, \gamma_0)$ with det-weight $\beta$ is a homogeneous polynomial function of degree $\sum_i \gamma_{i,j} i_i \beta_i$, which is also equal to $\sum_i \gamma_{0,i} i_i \beta_i$ assuming $K$ is infinite. In particular, $\beta = 0$ if and only if $\sigma$ is constant. □

When $f \in \text{Pres}_\Lambda(\gamma_1, \gamma_0)$ is a monomorphism $P(\gamma_1) \hookrightarrow P(\gamma_0)$, we have:

$$\dim \text{coker } f = \dim P(\gamma_0) - \dim P(\gamma_1) = (L^t)^{-1}(\gamma_0 - \gamma_1)$$

which is $\alpha$ in the proposition above. We want to view different presentations of the same module as being equivalent. To make this precise we make the following definitions.

2.3. Virtual representations. “Virtual representations” will be given by “stabilizing” presentation $f : P(\gamma_1) \rightarrow P(\gamma_0)$. These will form the objects of the “virtual representation category” and the elements of the “virtual representation space.” First, note that

$$P(\gamma + \delta) = P(\gamma) \oplus' P(\delta)$$

where $\oplus'$ denotes the “shuffle sum” given by collecting isomorphic summands together. We use this to make the equality strict. For example $(P_1 \oplus P_2) \oplus' P_1$ denotes $P_1 \oplus P_1 \oplus P_2$. Given any three dimension vectors $\gamma_0, \gamma_1, \delta \in \mathbb{N}^n$, consider the linear monomorphism

$$\text{Pres}_\Lambda(\gamma_1, \gamma_0) \hookrightarrow \text{Pres}_\Lambda(\gamma_1 + \delta, \gamma_0 + \delta)$$

given by sending $f : P(\gamma_1) \rightarrow P(\gamma_0)$ to $f \oplus' 1_{P(\delta)} : P(\gamma_1) \oplus' P(\delta) \rightarrow P(\gamma_0) \oplus' P(\delta)$. We call this map stabilization. This gives a directed system whose objects are all presentation spaces $\text{Pres}_\Lambda(\delta_1, \delta_0)$ having the property that $\delta_0 - \delta_1 = \gamma_0 - \gamma_1$. This implies $\dim P(\delta_0) - \dim P(\delta_1) = \alpha = \dim P(\gamma_0) - \dim P(\gamma_1) \in \mathbb{Z}^n$. Equivalently, $\gamma_0 - \gamma_1 = L^t \alpha$.

Definition 2.3.1. For any $\alpha \in \mathbb{Z}^n$ we define the virtual representation space $Vrep(\Lambda, \alpha)$ to be the direct limit (colimit):

$$Vrep(\Lambda, \alpha) := \text{colim } \text{Pres}_\Lambda(\gamma_1, \gamma_0) = \text{colim } \text{Hom}_\Lambda(P(\gamma_1), P(\gamma_0))$$

where the colimit is taken over all pairs $\gamma_0, \gamma_1 \in \mathbb{N}^n$ so that $\gamma_0 - \gamma_1 = L^t \alpha$. Elements of $Vrep(\Lambda, \alpha)$ will be called virtual representations of $\Lambda$ of dimension vector $\alpha \in \mathbb{Z}^n$. We take the direct limit topology on $Vrep(\Lambda, \alpha)$. Since each presentation space is irreducible, it follows that $Vrep(\Lambda, \alpha)$ is irreducible, i.e., any nonempty open subset is dense.

The main purpose of the virtual representation space is to make the weights of semi-invariants well-defined. See Definition 2.4.6 below.

We now construct the category $Vrep(\Lambda)$ of all virtual representations of $\Lambda$. The object set of this category will be the disjoint union

$$\text{Ob}(Vrep(\Lambda)) := \bigsqcup_{\alpha \in \mathbb{Z}^n} Vrep(\Lambda, \alpha).$$
Representatives of $Vrep(\Lambda, \alpha)$ are presentations $p : P(\gamma_1) \to P(\gamma_0)$ which we denote $P(\gamma_*)$. A morphism in $Vrep(\Lambda)$ can be defined on representatives as in the following diagram

$$
P(\xi_\ast) = \begin{array}{c} P(\xi_0) \\
f=(f_0,f_1) \\
\downarrow f_1 \\
P(\eta_\ast) = \begin{array}{c} P(\eta_1) \\
\downarrow q \\
P(\eta_0) \end{array}
\end{array}
$$

In other words, $(f_0, f_1)$ gives a chain map $P(\xi_\ast) \to P(\eta_\ast)$. Two such chain maps are equivalent $(f_0, f_1) \sim (f'_0, f'_1)$ if they are homotopic, i.e., if there is a map $h : P(\xi_0) \to P(\eta_1)$ so that $f'_1 = f_1 + hp$ and $f'_0 = f_0 + qh$. We define a morphism $X \to Y$ to be an equivalence class of such chain maps under the equivalence relation generated by homotopy as explained above and stabilization which means $(f_0, f_1) \sim (f_0 \oplus \eta 1_p, f_1 \oplus \eta 1_p)$ for any projective module $P = P(\xi)$.

Since direct sum does not commute with stabilization, to define direct sums in $Vrep(\Lambda)$ we define the category $Pres(\Lambda)$ and show that it is equivalent to $Vrep(\Lambda)$. $Pres(\Lambda)$ is the category whose objects are all chain complexes of finitely generated projective modules in degrees 0 and 1: $P(\gamma_*) = (p : P(\gamma_1) \to P(\gamma_0))$ and whose morphisms are homotopy classes of degree 0 chain maps. Objects of $Pres(\Lambda)$ will be called presentations.

**Proposition 2.3.2.** The stabilization map $P(\gamma_*) \mapsto (P(\gamma_*))$ is an equivalence of categories $Pres(\Lambda) \cong Vrep(\Lambda)$.

*Proof.* As a chain complex, every presentation is homotopy equivalent to each of its stabilizations. Therefore, any two representatives of the same virtual representation are canonically isomorphic as objects of $Pres(\Lambda)$. Given any two objects $X, Y$ in $Vrep(\Lambda)$, a morphism $f : X \to Y$ is represented by a morphism $\hat{f} = (f_0, f_1) : P(\xi_\ast) \to P(\eta_\ast)$ in $Pres(\Lambda)$. Since these representatives are unique up to canonical isomorphism in $Pres(\Lambda)$, $\hat{f}$ is unique. So, $	ext{Hom}_{Vrep(\Lambda)}(X, Y) \cong \text{Hom}_{Pres(\Lambda)}(P(\xi_\ast), P(\eta_\ast))$. In other words the stabilization functor is full, faithful and dense. So, it is an equivalence. \qed

Since the kernel of $p : P(\gamma_1) \to P(\gamma_0)$ splits off of $P(\gamma_1)$, we get the following.

**Proposition 2.3.3.** The indecomposable objects of $Pres(\Lambda)$ and $Vrep(\Lambda)$ are

1. projective presentations of indecomposable $\Lambda$-modules and
2. shifted indecomposable projective $\Lambda$-modules $P[1]$, i.e., $P \to 0 \in Pres(\Lambda)$.

**Definition 2.3.4.** The underlying module of a presentation $P(\gamma_*) = (P(\gamma_1) \xrightarrow{p} P(\gamma_0))$ is defined to be $|P(\gamma_*)| := \ker p \oplus \coker p$. In particular, $|P[1]| = P$.

**Remark 2.3.5.** Let $P(\xi_\ast) = (f : P(\xi_1) \to P(\xi_0))$ and $P(\eta_\ast) = (g : P(\eta_1) \to P(\eta_0))$ be objects in $Pres(\Lambda)$ and representatives of objects in $Vrep(\Lambda)$. The following are equivalent.

1. $P(\xi_\ast) \cong P(\eta_\ast)$ in $Pres(\Lambda)$.
2. $P(\xi_\ast) \cong P(\eta_\ast)$ in $Vrep(\Lambda)$.
3. $\ker f \cong \ker g$ and $\coker f \cong \coker g$ in $mod-\Lambda$.
4. $f, g$ are homotopy equivalent.
5. If, in addition, $\xi_0 = \eta_0$ then $f, g$ are chain isomorphic.

For two objects $P(\xi_\ast), P(\eta_\ast)$ of $Pres(\Lambda)$ (or $Vrep(\Lambda)$) we define $\text{Ext}^1_{Pres(\Lambda)}(P(\xi_\ast), P(\eta_\ast))$ in the usual way as the space of homotopy classes of chain maps $P(\xi_\ast) \to P(\eta_\ast)[1]$.

**Corollary 2.3.6.** $Pres(\Lambda)$ is equivalent to the full subcategory of the bounded derived category of $mod-\Lambda$ with objects all $P(\gamma_*)$ so that $\text{Hom}_{\mathcal{D}^b}(P(\gamma_*), Y[k]) = 0$ for all $Y \in mod-\Lambda$ and all $k \neq 0, 1$. Furthermore, $\text{Ext}^1_{Pres(\Lambda)}(P(\gamma_*), P(\delta_*)) = \text{Ext}^1_{\mathcal{D}^b}(P(\gamma_*), P(\delta_*))$ for all $P(\gamma_*), P(\delta_*) \in Pres(\Lambda)$.
Proof. It is clear that all $P(\gamma_s) \in \text{Pres}(\Lambda)$ satisfy this condition. Conversely, suppose that $P(\gamma_s)$ satisfies the condition. Then $P(\gamma_s) \in \text{mod-}\Lambda$ or $P(\gamma_s) = Z[1]$ where $Z \in \text{mod-}\Lambda$. In the second case we have $\text{Hom}_{\text{D}_b}(Z[1], Y[2]) = 0$ for all modules $Y$. This implies that $Z$ is projective. $\square$

Recall that the cluster category $\mathcal{C}_\Lambda$ of $\Lambda$ is the orbit category of the bounded derived category $D^b(\text{mod-}\Lambda)$ under the functor $F = \tau^{-1}[1]$ (see [BMRRT]). Recall that a partial cluster tilting object is an object $T$ of $\mathcal{C}_\Lambda$ so that $\text{Ext}^1(\Lambda, T, T) = 0$ and if it has $n$ nonisomorphic summands it is called a cluster tilting object. The fundamental domain of the functor $F$ consists of $\Lambda$-modules and shifted projective modules. Therefore we get the following.

Corollary 2.3.7. The functor $\Psi : \text{Pres}(\Lambda) \to \mathcal{C}_\Lambda$ which sends each object to its $F$-orbit is a faithful functor which induces a bijection between isomorphism classes of objects. Furthermore, $\text{Ext}^1_{\text{D}_b}(P(\xi_s), P(\eta_s)) = 0 = \text{Ext}^1_{\text{D}_b}(P(\eta_s), P(\xi_s))$ if and only if $\text{Ext}^1(\Psi P(\xi_s), \Psi P(\eta_s)) = 0 = \text{Ext}^1(\Psi P(\eta_s), \Psi P(\xi_s))$ for all $P(\xi_s), P(\eta_s)$ in $\text{Pres}(\Lambda)$.

Definition 2.3.8. The dimension vector of a presentation $P(\gamma_s) = (p : P(\gamma_1) \to P(\gamma_0))$ is defined to be $\text{dim } P(\gamma_s) := \dim P(\gamma_0) - \dim P(\gamma_1) = \dim \text{coker } p - \dim \text{ker } p$. This is the unique integer vector $\alpha \in \mathbb{Z}^n$ satisfying $L^t \alpha = 70 - 71$ where $L$ is the left Euler matrix of $\Lambda$.

Theorem 2.3.9. Let $P(\gamma_s) = (p : P(\gamma_1) \to P(\gamma_0))$ be a presentation with dimension vector $\text{dim } P(\gamma_s) = \alpha$ so that $\text{Ext}^1(\text{Pres}(\Lambda))(P(\gamma_s), P(\gamma_s)) = 0$. Then the set of all presentations isomorphic to $P(\gamma_s)$ is an open dense subset of the $K$-affine space $\text{Pres}_\Lambda(\gamma_1, \gamma_0)$.

Proof. Let $P = \text{ker } p$. Then $P(\gamma_s) = P[1] \oplus P(\gamma_0')$ where $P(\gamma_0') = (P(\gamma') \xrightarrow{q} P(\gamma_0))$ is a projective presentation of a $\Lambda$-module $M$ with $\text{Ext}^1(\Lambda, M, M) = 0$ and $\dim M = \beta$. By assumption, $0 = \text{Ext}^1_{\text{D}_b}(P[1], P(\gamma_0')) = \text{Ext}^1_{\text{D}_b}(\text{Pres}(\Lambda))(P[1], P(\gamma_0')) = \text{Hom}_{\text{D}_b}(P, P(\gamma_0')) = \text{Hom}_{\Lambda}(P, M)$. Let $f : P(\gamma_1) \to (\gamma_0)$ be a general morphism. Restrict $f$ to the components of $P(\gamma_1)$ to get $f_1 : P \to P(\gamma_0)$ and $f_2 : P(\gamma_1') \to P(\gamma_0')$. Since $q$ is a monomorphism and $f_2$ is a general map, it follows from Lemma 2.1.5 that $f_2$ is a monomorphism with cokernel isomorphic to $M$. So, $f_2$ is homotopy equivalent and thus isomorphic to $q : P(\gamma_1') \to P(\gamma_0)$. Since $\text{Hom}_{\Lambda}(P, M) = 0$, $f_1 = f_2 \circ s$ for some $s : P(\gamma_0) \to P(\gamma_1)$). Then presentation $(f_1 - f_2 \circ s, f_2) = (0, f_2)$ is isomorphic to $f : P(\gamma_1) \to P(\gamma_0)$ and to $P[1] \oplus P(\gamma_0') = P(\gamma_s)$. Thus the general presentation $f : P(\gamma_1) \to P(\gamma_0)$ is isomorphic to $P(\gamma_s)$ in $\text{Pres}_\Lambda(\gamma_1, \gamma_0)$. $\square$

Recall that a partial cluster tilting set is a set $\{\beta_i\}$ of distinct real Schur roots and negative projective roots which are the dimension vectors of components of a partial cluster tilting object in the cluster category $\mathcal{C}_\Lambda$. If the partial cluster tilting set has exactly $n$ elements it is called a cluster tilting set.

Definition 2.3.10. A partial cluster tilting object in $\text{Pres}(\Lambda)$ is defined to be $\bigoplus P(\gamma_1^i)$ such that $\{\text{dim } P(\gamma_1^i)\}$ is a partial cluster tilting set. If this object has $n$ nonisomorphic summands it is called a cluster tilting object in $\text{Pres}(\Lambda)$.

Theorem 2.3.11 (Virtual Generic Decomposition Theorem). Let $\{\beta_i\}$ be a partial cluster tilting set. Let $\alpha = \sum r_i \beta_i \in \mathbb{Z}^n$ where $r_i \in \mathbb{Q}$. Then $r_i \in \mathbb{Z}$ and the general virtual representation in $\text{Vrep}(\Lambda, \alpha)$ is isomorphic to $\bigoplus_i P(\gamma_i)^{r_i}$ where $P(\gamma_i)$ are rigid objects in $\text{Vrep}(\Lambda, \beta_i)$. In other words, the set of all elements of $\text{Vrep}(\Lambda, \alpha)$ isomorphic to $\bigoplus_i P(\gamma_i)^{r_i}$ is open and dense.

Proof. The underlying modules $|P(\gamma_i^j)|, i = 1, \cdots, k$ form an exceptional sequence by Lemma 2.1.2 if we put the shifted projectives last. This can be extended to a complete exceptional sequence, say $\{M_j\}_{j=1}^n$. (See section 1.3.) The dimension vectors $\text{dim } M_j$ generate $\mathbb{Z}^n$ by Proposition 1.3.3 (3). Therefore, the integer vectors in the $\mathbb{Q}$-span of the vectors in the
subset \( \{ \dim M_i \}_{i=1}^k \) lie in the \( \mathbb{Z} \)-span of these vectors. By Theorem \[2.3.9\] the virtual representations isomorphic to \( \bigoplus_{i=1}^k P(\gamma_i)^{n_i} \) form an open dense subset of each presentation space and therefore of the colimit \( \text{Vrep}(\Lambda, \gamma) \).

\[ \square \]

### 2.4. Virtual semi-invariants.

We return to the discussion of semi-invariants. We consider direct sums of presentations.

**Lemma 2.4.1.** Let \( f : P(\gamma_1 + \delta_1) \to P(\gamma_0 + \delta_0) \) be a direct sum of two projective presentations \( f = f_1 \oplus f_2 \) where \( f_1 : P(\gamma_1) \to P(\gamma_0) \) and \( f_2 : P(\delta_1) \to P(\delta_0) \). If \( f \) admits a semi-invariant of det-weight \( \beta \) then so does each \( f_i \).

**Proof.** Consider the composition:

\[
\text{Pres}_\Lambda(\gamma_1, \gamma_0) \times \text{Pres}_\Lambda(\delta_1, \delta_0) \xrightarrow{\sigma} \text{Pres}_\Lambda(\gamma_1 + \delta_1, \gamma_0 + \delta_0) \xrightarrow{\beta} K
\]

where \( \sigma \) is a semi-invariant of det-weight \( \beta \) on \( \text{Pres}_\Lambda(\gamma_1 + \delta_1, \gamma_0 + \delta_0) \) so that \( \sigma(\iota(f_1, f_2)) \neq 0 \).

Then, semi-invariants on \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) and \( \text{Pres}_\Lambda(\delta_1, \delta_0) \) can be defined by \( \sigma(\iota(-, f_2)) : \text{Pres}_\Lambda(\gamma_1, \gamma_0) \to K \) and analogously for \( \text{Pres}_\Lambda(\delta_1, \delta_0) \). It is easy to see that these are regular functions and they are semi-invariants of det-weight \( \beta \). Indeed, suppose that \( g_1, g_2, h_1, h_2 \) are automorphisms of \( P(\gamma_1), P(\delta_1), P(\gamma_0), P(\delta_0) \). Then \( g = g_1 \oplus g_2 \) and \( h = h_1 \oplus h_2 \) are automorphisms of \( P(\gamma_0) \oplus P(\delta_0) \) and \( P(\gamma_1) \oplus P(\delta_1) \) respectively so that \( \chi_i(g) = \chi_i(g_1)\chi_i(g_2) \) and \( \chi_i(h) = \chi_i(h_1)\chi_i(h_2) \). Therefore

\[
\sigma(g_1 f_1 h_1, g_2 f_2 h_2) = \sigma(gfh) = \sigma(\iota(f_1, f_2)) \prod \chi_i(g_1)^{\beta_i} \chi_i(g_2)^{\beta_i} \chi_i(h_1)^{\beta_i} \chi_i(h_2)^{\beta_i}
\]

So, \( \sigma(\iota(-, f_2)) \) is a semi-invariant on \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) of det-weight \( \beta \) which is nonzero on \( f_1 \) and similarly with \( \sigma(\iota(f_1, -)) \).

Let \( \text{Pres}(\Lambda, \alpha) = \bigsqcup \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) denote the disjoint union of presentation spaces \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) over all pairs \( \gamma_0, \gamma_1 \in \mathbb{N}^n \) so that \( \Lambda^t \alpha = \gamma_0 - \gamma_1 \).

**Proposition 2.4.2.** Suppose that \( \alpha_1, \ldots, \alpha_n \in \mathbb{Z}^n \) are linearly independent. Suppose that \( f_i \in \text{Pres}(\Lambda, \alpha_i) \). Then \( \bigoplus f_i \in \text{Pres}(\Lambda, \sum \alpha_i) \) does not admit a semi-invariant with nonzero det-weight.

**Proof.** If \( f = \bigoplus f_i \) admits a semi-invariant of det-weight \( \beta \) then, by Lemma \[2.4.1\] so does every \( f_i \). By Proposition \[2.2.9\] we conclude that \( \langle \alpha_i, \beta \rangle = 0 \) for all \( i \). So, \( \beta = 0 \).

**Remark 2.4.3.** For any semi-invariant \( \sigma \), there is a power of \( \sigma \) which has determinant weight. This follows from the fact that \( \text{det} \) is a power of the reduced norm \( \mathbf{7} : M_k(\mathbb{F}) \to K \). We refer the reader to Appendix B, Sec \[B\] for the definition of reduced norm and the proof of the theorem (Theorem \[6.2.1\]) that all characters are powers of the reduced norm. Consequently, in Proposition \[2.4.2\] above, \( \bigoplus f_i \) does not admit a semi-invariant of any weight since, if it did, then some power of that semi-invariant would be a semi-invariant with nonzero determinant weight.

**Definition 2.4.4.** By a semi-invariant on \( \text{Pres}(\Lambda, \alpha) \) we mean a semi-invariant on one of the presentation spaces \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) in the disjoint union. Such a semi-invariant \( \sigma \) will be called **determinantal** if there is a module \( M \) so that, for all \( f : P(\gamma_1) \to P(\gamma_0) \) in \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \), \( \sigma(f) \) is the determinant of the induced map

\[
\text{Hom}_\Lambda(f, M) : \text{Hom}_\Lambda(P(\gamma_0), M) \to \text{Hom}_\Lambda(P(\gamma_1), M).
\]

We denote this by \( \sigma_M \). It is easy to see that \( \sigma_M \) is a semi-invariant of det-weight \( \dim M \). In case the det-weight of \( \sigma_M \) is not well-defined, we take it to be \( \dim M \) by definition.

When the ground field \( K \) is algebraically closed then Schofield \[S91\] showed that the determinantal semi-invariants generate the ring of all semi-invariants in the Dynkin case and this theorem was extended in \[DW\] to all quivers over an algebraically closed field.
Corollary 2.4.5. Let \( \alpha = \sum n_i \beta_i \) be an integer linear combination of the vectors \( \beta_i \) in a cluster tilting set. If \( n_i > 0 \) for all \( 1 \leq i \leq n \), then \( \text{Pres}(\Lambda, \alpha) \) has no semi-invariant with nonzero determinantal weight.

Proof. If there is a nonzero semi-invariant \( \sigma \) on \( \text{Hom}_\Lambda(P, Q) \) with \( \text{dim}(Q) - \text{dim}(P) = \alpha \) then \( \sigma \) will be nonzero on the generic element of \( \text{Hom}_\Lambda(P, Q) \). By Corollary 2.3.11, the generic element splits as a direct sum of \( n \) objects with linearly independent dimension vectors. But this contradicts Proposition 2.4.2. Our Corollary follows. \( \square \)

Definition 2.4.6. A virtual semi-invariant of det-weight \( \beta \) on \( \text{Vrep}(\Lambda, \alpha) \) is a mapping

\[ \sigma : \text{Vrep}(\Lambda, \alpha) \rightarrow K \]

whose restriction to each \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \subset \text{Vrep}(\Lambda, \alpha) \) is a semi-invariant of det-weight \( \beta \).

By definition of direct limit, a virtual semi-invariant on \( \text{Vrep}(\Lambda, \alpha) \) is the same as a system of semi-invariants one on each \( \text{Pres}_\Lambda(\gamma_1, \gamma_0) \) which are compatible with stabilization. One example is the determinantal semi-invariant \( \sigma_M \) defined above. Since each coordinate of \( \gamma_0 \) and \( \gamma_1 \) become arbitrarily large, the weight of a virtual semi-invariant is well-defined when \( K \) is infinite.

3. Virtual stability theorem

In this section we will prove the Virtual Stability Theorem (3.1.1) which states that the domain \( D_{\mathbb{Z}}(\beta) \) of the semi-invariant with det-weight \( \beta \) defined in 3.1.3 is the subset of \( \mathbb{Z}^n \) given by the stability conditions of 3.1.1(2). We also give a description of all elements of this set (Proposition 3.5.2).

3.1. Statements of the theorem. Let \( \beta, \beta' \) be real Schur roots. We say that \( \beta' \) is a real Schur subroot of \( \beta \) if \( M_{\beta'} \) contains a submodule isomorphic to \( M_{\beta} \).

Theorem 3.1.1 (Virtual Stability Theorem). Let \( K \) be any field. Let \( \Lambda \) be a finite dimensional hereditary \( K \)-algebra with \( n \) non-isomorphic simple modules. Let \( \alpha \in \mathbb{Z}^n \) and \( \beta \) a real Schur root. Then, the following are equivalent:

1. There exists a virtual representation \( f : P \rightarrow Q \) so that \( \text{dim} Q - \text{dim} P = \alpha \) and \( f \) induces an isomorphism

\[ f^* : \text{Hom}_\Lambda(Q, M_{\beta}) \cong \text{Hom}_\Lambda(P, M_{\beta}). \]

2. Stability conditions for \( \alpha \) and \( \beta \) hold: \( \langle \alpha, \beta \rangle = 0 \) and \( \langle \alpha, \beta' \rangle \leq 0 \) for all real Schur subroots \( \beta' \subseteq \beta \).

3. There is a nonzero determinantal semi-invariant of det-weight \( \beta \) on the virtual representation space \( \text{Vrep}(\Lambda, \alpha) \).

Remark 3.1.2. In the previous paper [IOTW09], the authors proved the Virtual Stability Theorem for hereditary algebras over an algebraically closed field and vectors \( \alpha \in \mathbb{Z}^n \), which may have negative coordinates. This was an extension of the results of [DW] and [Ki] from \( \alpha \in \mathbb{N}^n \) to \( \alpha \in \mathbb{Z}^n \). Here we extend this theorem to hereditary algebras over any field. We also note that Condition (2) is weaker and thus the theorem is stronger than the original theorems of [DW] and [Ki] since the condition \( \langle \alpha, \beta' \rangle \leq 0 \) is only required for real Schur subroots \( \beta' \) of \( \beta \) and not for all subroots of \( \beta \).

We now restate the theorem in terms of the following sets, usually referred to as various “domains of virtual semi-invariants” or “supports of virtual semi-invariants”.

Definition 3.1.3. Let \( \beta \) be a real Schur root. We define the following:

\[ D_{\mathbb{Z}}(\beta) := \{ \alpha \in \mathbb{Z}^n : \text{Condition(1) holds} \} = \text{integral support} \text{ of det-weight} \beta, \]

\[ D(\beta) := \text{convex hull of} \ D_{\mathbb{Z}}(\beta) \text{ in} \mathbb{R}^n = \text{real support of det-weight} \beta, \]
\[ D^s(\beta) := \{ x \in \mathbb{R}^n : \langle x, \beta \rangle = 0 \text{ and } \langle x, \beta' \rangle \leq 0 \text{ for all real Schur subroots } \beta' \subseteq \beta \} \]

= support of real semi-stability conditions,

\[ D^s_\Lambda(\beta) := \{ \alpha \in \mathbb{Z}^n : \langle \alpha, \beta \rangle = 0 \text{ and } \langle \alpha, \beta' \rangle \leq 0 \text{ for all real Schur subroots } \beta' \subseteq \beta \} \]

= support of integral semi-stability conditions.

The Virtual Stability Theorem 3.1.1 can now be restated as:

**Theorem 3.1.4 (Virtual Stability Theorem').** Let \( K \) be any field, \( \Lambda \) a finite dimensional hereditary \( K \)-algebra with \( n \) simple modules. Let \( \beta \) be a real Schur root. Then

\[ D^s_\Lambda(\beta) = D_\Lambda(\beta). \]

The proof of the theorem occupies the rest of this section. We will first prove the theorem for infinite fields and in subsection 3.6 we will extend the proof to all fields. We start with the simple lemma showing the equivalence of conditions (1) and (3) in the Virtual Stability Theorem 3.1.1 hence reducing the proof to showing that \( D^s_\Lambda(\beta) = D_\Lambda(\beta) \), i.e. Virtual Stability Theorem 3.1.4.

**Lemma 3.1.5.** Let \( K \) be an infinite field, \( \Lambda \) a finite dimensional hereditary \( K \)-algebra with \( n \) simple modules, \( \alpha \in \mathbb{Z}^n \) and \( \beta \) a real Schur root. The following are equivalent:

1. There exists a virtual representation \( f : P \to Q \) with \( \dim Q - \dim P = \alpha \) so that \( f \) induces an isomorphism \( f^* : \text{Hom}_\Lambda(Q, M_\beta) \cong \text{Hom}_\Lambda(P, M_\beta) \).

2. There is a nonzero determinantal semi-invariant of det-weight \( \beta \) on virtual representation space \( \text{Vrep}(\Lambda, \alpha) \).

**Proof.** (1) \( \implies \) (2) It follows from (1) that \( \dim_K \text{Hom}_\Lambda(Q, M_\beta) = \dim_K \text{Hom}_\Lambda(P, M_\beta) \) and therefore determinant of \( h^* \) is defined for all \( h \in \text{Pres}(\Lambda, \alpha) \), is non-zero for \( h = f \) and is compatible with stabilization. Hence \( \sigma = \text{determinant} \) is a (determinantal) semi-invariant on \( \text{Vrep}(\Lambda, \alpha) \).

(2) \( \implies \) (1) Given a determinantal virtual semi-invariant of det-weight \( \beta \), we have an isomorphism \( f^* : \text{Hom}_\Lambda(Q, M) \cong \text{Hom}_\Lambda(P, M) \) for some \( M \) with \( \dim M = \beta \). Since being an isomorphism is an open condition, \( f^* \) must also be an isomorphism for \( M = M_\beta \). \( \square \)

**Lemma 3.1.6.** Let \( \beta \) be a real Schur root. Then \( D_\Lambda(\beta) \subseteq D^s_\Lambda(\beta) \).

**Proof.** Let \( \alpha \in D_\Lambda(\beta) \). Then there is \( f : P \to Q \) in \( \text{Pres}(\Lambda, \alpha) \) such that \( f^* : \text{Hom}(Q, M_\beta) \to \text{Hom}(P, M_\beta) \) is an isomorphism. So, \( \langle \alpha, \beta \rangle = \dim_K \text{Hom}_\Lambda(Q, M_\beta) - \dim_K \text{Hom}_\Lambda(P, M_\beta) = 0 \). The induced map \( \text{Hom}(Q, M_{\beta'}) \to \text{Hom}(P, M_{\beta'}) \) is a monomorphism for all real Schur subroots \( \beta' \subseteq \beta \). Therefore \( \langle \alpha, \beta' \rangle \leq 0 \), i.e. stability condition (2) holds and \( \alpha \in D^s_\Lambda(\beta) \). \( \square \)

3.2. Perpendicular categories of \( M_\beta \) and associated exceptional sequences. To a real Schur root \( \beta \) we associate an exceptional sequence \( (M_\beta, E_1, \ldots, E_{n-1}) \) which will play a crucial role in the proof of the theorem. We identify the \( \Lambda \)-module \( M_\beta \) with its projective presentation in \( \text{Vrep}(\Lambda, \beta) \).

**Definition 3.2.1.** (a) For any \( X \) in \( \text{Vrep}(\Lambda) \) let \( ^\perp X \) be the left \( \text{Hom}_{\text{Vrep}(\Lambda)} \)-, \( \text{Ext}^1_{\text{Vrep}(\Lambda)} \)-perpendicular category of \( X \) in \( \text{Vrep}(\Lambda) \), i.e., \( ^\perp X \) is the full subcategory of \( \text{Vrep}(\Lambda) \) with objects \( Y \) so that \( \text{Hom}_{\text{Vrep}(\Lambda)}(Y, X) = 0 = \text{Ext}^1_{\text{Vrep}(\Lambda)}(Y, X) \). \( X^{\perp} \) is defined similarly.

(b) For any \( M \) in \( \text{mod-} \Lambda \), let \( ^\perp M \) be the left \( \text{Hom}_\Lambda \)-, \( \text{Ext}^1_\Lambda \)-perpendicular category of \( M \) in \( \text{mod-} \Lambda \) with objects \( N \) so that \( \text{Hom}_\Lambda(N, M) = 0 = \text{Ext}^1_\Lambda(N, M) \). \( M^{\perp} \) is defined similarly.

The following lemma will relate perpendicular categories in \( \text{Vrep}(\Lambda) \) and in \( \text{mod-} \Lambda \) allowing us to use some well known theorems for module categories.

**Lemma 3.2.2.** (a) For any \( \Lambda \)-module \( M \), \( ^\perp M \cap \text{mod-} \Lambda = ^\perp M \).

(b) For any \( X \) in \( \text{Vrep}(\Lambda) \), \( X^{\perp} \cap \text{mod-} \Lambda = |X|^{\perp} \).
Definition 3.2.3. A wide subcategory of $\text{mod-}\Lambda$ for any hereditary algebra $\Lambda$ is defined to be an extension closed full subcategory $\mathcal{W} \subseteq \text{mod-}\Lambda$ which is abelian and exactly embedded (a sequence in $\mathcal{W}$ is exact in $\mathcal{W}$ if and only if it is exact in $\text{mod-}\Lambda$). A wide subcategory is said to have rank $k$ if it is isomorphic to the module category of an hereditary algebra with $k$ simple objects.

Remark 3.2.4. We need the following well-known properties of wide subcategories. [InTh]

(1) Every finitely generated wide subcategory of $\text{mod-}\Lambda$ is isomorphic to the module category of an hereditary algebra.

(2) If $M$ is a $\Lambda$-module whose components form an exceptional sequence with $k$ terms, the right $\text{Hom}_\Lambda$- perpendicular category $M^\perp$ of $M$ is a finitely generated wide subcategory of $\text{mod-}\Lambda$ of rank $n - k$. The same holds for $\perp M$.

(3) For any finitely generated wide subcategory $\mathcal{W}$ of $\text{mod-}\Lambda$ we have $(\perp \mathcal{W})^\perp = \mathcal{W}$ and $\perp (\mathcal{W}^\perp) = \mathcal{W}$.

In our case $\perp M_\beta$ is a wide subcategory of rank $n - 1$ since $M_\beta$ is indecomposable. Let $E_1, \cdots, E_{n-1}$ be the simple objects of $\perp M_\beta$. These objects are exceptional and using Proposition [1.3.2](1) can be ordered in such a way that the following sequence is an exceptional sequence in $\text{mod-}\Lambda$:

$$(3.2.1) \quad (M_\beta, E_1, \cdots, E_{n-1}).$$

By Lemma 3.2.2 we also have:

$$ (E_1 \oplus \cdots \oplus \widehat{E}_k \oplus \cdots \oplus E_{n-1})^\perp \cap \text{mod-}\Lambda = (E_1 \oplus \cdots \oplus \widehat{E}_k \oplus \cdots \oplus E_{n-1})^\perp, $$

and we use this to define, for each $k = 1, \ldots, n - 1$, the following subcategories of $\text{mod-}\Lambda$ ,

$$(3.2.2) \quad \mathcal{W}_k := (E_1 \oplus \cdots \oplus \widehat{E}_k \oplus \cdots \oplus E_{n-1})^\perp \subset \text{mod-}\Lambda.$$ 

These are wide subcategories of $\text{mod-}\Lambda$ of rank 2 which contains $M_\beta$ by definition of the $E_i$s.

Lemma 3.2.5. Let $\beta$ be a real Schur root. Let $(E_1, \cdots, E_{n-1})$ be an exceptional sequence of simple objects of $\perp M_\beta$ and let $P'_k$ be the projective cover of $E_k$ in $\perp M_\beta \subset \text{mod-}\Lambda$. Then $P'_k$ is a projective $\Lambda$-module if and only if $M_\beta$ is a simple object in $\mathcal{W}_k$.

Proof. Several exceptional sequences will be created out of $(E_1, \cdots, E_{n-1})$ and will be used in the proof. Since all $E_i$ are simple objects and $P'_k$ is projective in $\perp M_\beta \subset \text{mod-}\Lambda$ it follows that $\text{Hom}_{\perp M_\beta}(P'_k, E_i) = 0$ for $i \neq k$ and also $\text{Ext}^1_{\perp M_\beta}(P'_k, E_i) = 0$.

Therefore:

(a) $(E_1, \cdots, \widehat{E}_k, \cdots, E_{n-1}, P'_k)$ is an exceptional sequence in $\perp M_\beta$.

Since $\perp M_\beta \subset \text{mod-}\Lambda$ is exact embedding it follows that $\text{Hom}_\Lambda(P'_k, E_i) = 0 = \text{Ext}^1_\Lambda(P'_k, E_i)$ for all $i \neq k$. This together with the fact that $(E_1, \cdots, \widehat{E}_k, \cdots, E_{n-1}, P'_k) \subset \perp M_\beta$ implies:

(b) $(M_\beta, E_1, \cdots, \widehat{E}_k, \cdots, E_{n-1}, P'_k)$ is an exceptional sequence in $\text{mod-}\Lambda$.

After applying Proposition [1.3.3](6) to the exceptional sequence (a), one obtains:

(c) $(I'_k, E_1, \cdots, \widehat{E}_k, \cdots, E_{n-1})$ is an exceptional sequence in $\perp M_\beta$, and

(d) $(M_\beta, I'_k, E_1, \cdots, \widehat{E}_k, \cdots, E_{n-1})$ is an exceptional sequence in $\text{mod-}\Lambda$.

After applying Proposition [1.3.3](6) to the exceptional sequence (b), one obtains:

(e) $(X, M_\beta, E_1, \cdots, \widehat{E}_k, \cdots, E_{n-1})$ is an exceptional sequence in $\text{mod-}\Lambda$, where $X = \tau_\Lambda P'_k$ if and only if $P'_k$ is not a projective $\Lambda$-module, and $X$ is the injective envelope of $P'_k/\text{rad} P'_k$ if and only if $P'_k$ is a projective $\Lambda$-module.

(f) $(M_\beta, I'_k)$ and $(X, M_\beta)$ are exceptional sequences in $\mathcal{W}_k$. This follows from (d), (e) and definition of $\mathcal{W}_k$.

(g) There is a $\mathcal{W}_k$-irreducible map $M_\beta \rightarrow I'_k$ if and only if $M_\beta \oplus I'_k$ is not semi-simple, since $\text{rank}(\mathcal{W}_k) = 2$. 18
(h) There is a $W_k$-irreducible map $X \to M_\beta$ if and only if $X \oplus M_\beta$ is not semi-simple, since \( \text{rank}(W_k) = 2 \).

**Claim 1:** If $M_\beta$ is not simple in $W_k$ then $P'_k$ is not a projective $\Lambda$-module.

**Proof:** Since $M_\beta$ is not simple it follows from (g) and (h) that there is an almost split sequence $X \to M^n_\beta \to I'_k$ in $W_k$. Since $P'_k \in \perp M_\beta$ we have $\text{Ext}^1(\Lambda(P'_k, X) \cong \text{Hom}(P'_k, I'_k) \neq 0$. So, $P'_k$ is not projective in $\mod\Lambda$.

**Claim 2:** If $M_\beta$ is simple in $W_k$ then $P'_k$ is a projective $\Lambda$-module.

**Proof:** If $M_\beta$ is simple in $W_k$, either $M_\beta$ is simple injective or simple projective.

**Case 2a:** If $M_\beta$ is a simple injective object in $W_k$, then there is no $W_k$-irreducible map $M_\beta \to I'_k$. So it follows by (g) that $I'_k$ is a simple projective object in $W_k$. Since $W_k$ has rank=2, there are only two simple objects. Therefore $X$ is not simple and by (h) there is a $W_k$-irreducible map $X \to M_\beta$. Since $M_\beta$ is simple injective in $W_k$ it follows that $X$ is injective envelope of the simple object $I'_k$ in $W_k$. If $X$ is injective $\Lambda$-module it follows by (e) that $P'_k$ is a projective $\Lambda$-module. If $X$ is not injective $\Lambda$-module, then $X = \tau_\Lambda P'_k$. But in this case there is a non-zero composition of $W_k$-maps, and therefore $\Lambda$-maps $P'_k \to S_k = I'_k \to X = \tau_\Lambda P'_k$. However this would imply $\text{Ext}^1(\Lambda(P'_k, P'_k)) \neq 0$ giving a contradiction to the fact that $P'_k$ is rigid. Hence, $P'_k$ is a projective $\Lambda$-module.

**Case 2b:** If $M_\beta$ is simple projective in $W_k$ then there is no $W_k$-irreducible map $X \to M_\beta$ and therefore $X$ must be simple $W_k$ object by (h), hence simple injective. Since the rank of $W_k$ is 2, it follows that $I'_k$ is not simple, hence there is a $W_k$ irreducible map $M_\beta \to I'_k$. Therefore $I'_k$ is a projective $W_k$ object, projective cover of $X$. So, there is a short exact sequence in $W_k$: $0 \to M^n_\beta \to I'_k \to X \to 0$ where $m \geq 1$. Since $\text{Hom}(\Lambda(P'_k, M_\beta) = 0$ and $\text{Hom}(\Lambda(P'_k, I'_k) \neq 0$ we have a nonzero $\Lambda$ morphism $P'_k \to X$. Therefore, $P'_k$ is a projective $\Lambda$-module by (e). This proves the proof of the lemma. □

3.3. **Subsets** $\Delta^+(\beta) \subseteq \Delta(\beta) \subseteq D_\mathbb{Z}(\beta) \subseteq D_\mathbb{Z}^\text{ss}(\beta)$. In this subsection we define two new subsets, which will be used in the proof of the Virtual Stability Theorem [19, 1]. i.e., we will prove $D_\mathbb{Z}(\beta) = D_\mathbb{Z}^\text{ss}(\beta)$.

**Definition 3.3.1.** Let $\beta$ be a real Schur root and let $(M_\beta, E_1, \ldots, E_{n-1})$ be the exceptional sequence as defined in Equation (3.2.1). Let:

$$\Delta^+(\beta) := \left\{ \sum_{1 \leq i \leq n-1} k_i \dim E_i : k_i \in \mathbb{N} \right\}.$$

**Lemma 3.3.2.** The set $\Delta^+(\beta)$ contains all integer points in its convex hull in $\mathbb{R}^n$.

**Proof.** Since $(M_\beta, E_1, \ldots, E_{n-1})$ is an exceptional sequence, by Proposition 1.3.3 every vector in $\mathbb{Z}^n$ can be expressed uniquely as an integer linear combination of the vectors $\dim E_i$ and $\beta$. Since all elements in the convex hull of $\Delta^+(\beta)$ can be written as nonnegative real linear combinations of the vectors $\dim E_i$ it follows that when the integer points in this convex hull are written in this way, the nonnegative coefficients are necessarily integers. So, they are nonnegative integers. □

**Remark 3.3.3.** The following are simple useful facts about perpendicular categories and the set $D_\mathbb{Z}(\beta)$.

1. Let $P(\gamma_+) \in \text{Vrep}(\Lambda)$. If $P(\gamma_+) \in \perp M_\beta$ then $\dim P(\gamma_+) \in D_\mathbb{Z}(\beta)$. (1)
2. Let $N \in \mod\Lambda$. If $N \in \perp M_\beta$ then $\dim N \in D_\mathbb{Z}(\beta)$. (2)
3. Let $P_j$ be an indecomposable projective $\Lambda$-module. Then $P_j \in \perp M_\beta \iff P_j \in \perp v M_\beta \iff P_j[1] \in \perp v M_\beta \iff \beta_j = 0$. (3)
Definition 3.3.4. Let $J_{\beta} = \{ j \in \mathbb{Z} \mid \beta_j = 0 \}$. Then
\[
\Delta(\beta) := \{ \sum_{1 \leq i \leq n-1} k_i \dim E_i + \sum_{j \in J_{\beta}} \ell_j \dim P_j : k_i \in \mathbb{N}, \ell_j \in \mathbb{Z} \} \subseteq D_{\mathbb{Z}}(\beta).
\]

Proposition 3.3.5. $\Delta^+(\beta) \subseteq \Delta(\beta) \subseteq D_{\mathbb{Z}}(\beta) \subseteq D_{\mathbb{Z}}^+(\beta)$.

Proof. The first inclusion follows from the definitions. The second follows from Remark 3.3.3. The third is Lemma 3.1.6.

To prove the stability theorem we will show that $D_{\mathbb{Z}}^+(\beta) \subseteq \Delta(\beta)$ and therefore the last three sets in 3.3.5 are equal. To do this we first consider the case when $J_{\beta}$ is empty, i.e., when $\beta$ is sincere.

3.4. Sincere case. When $\beta$ is sincere we have $\Delta^+(\beta) = \Delta(\beta)$. Thus we are reduced to showing that $D_{\mathbb{Z}}^+(\beta) \subseteq \Delta^+(\beta)$. Note that $\frac{1}{2} M_{\beta} \subseteq \text{mod-}\Lambda$ when $\beta$ is sincere.

Lemma 3.4.1. If $\beta$ is sincere then $D_{\mathbb{Z}}^+(\beta) = \Delta^+(\beta)$ and therefore $D_{\mathbb{Z}}^+(\beta) = D_{\mathbb{Z}}(\beta)$.

Proof. Since $\beta$ is sincere, it follows that there are no projective $\Lambda$-modules in $\frac{1}{2} M_{\beta}$ and therefore none of the $P_k$ are projective $\Lambda$-modules. This implies that $M_{\beta}$ is not a simple object in $W_k$ (as defined in equation 3.2.2) for each $k = 1, \ldots, n-1$ by Lemma 3.2.5. We will use this fact to construct certain Schur subroots $\gamma_k \subseteq \beta$.

Claim 1. For each $k = 1, 2, \ldots, n-1$ there is a real Schur subroot $\gamma_k$ of $\beta$ so that
\[
(1) \quad \langle \dim E_i, \gamma_k \rangle = 0 \text{ if } i \neq k,
\]
\[
(2) \quad \langle \dim E_k, \gamma_k \rangle < 0.
\]

Construction of $\gamma_k$: For each $k$, the category $W_k = (E_1 \oplus \cdots \oplus \hat{E}_k \oplus \cdots \oplus E_{n-1})^\perp \subseteq \text{mod-}\Lambda$ is a wide subcategory of rank 2 which contains $M_{\beta}$ by definition of the $E_i$s. Let $R_k, S_k$ be the simple objects of $W_k$. Since $M_{\beta}$ is not simple, there exists a nontrivial extension of $S_k$ by $R_k$ (or $R_k$ by $S_k$):
\[
S_k^p \rightarrow M_{\beta} \rightarrow R_k^q
\]
where $p, q \geq 1$. Let $\gamma_k = \dim S_k$. Then $\gamma_k$ is a proper real Schur subroot of $\beta$.

Properties of $\gamma_k$:
\[
(1) \quad \langle \dim E_i, \gamma_k \rangle = \langle \dim E_i, \dim S_k \rangle = 0 \text{ for all } i \neq k \text{ since } S_k \in W_k = (\oplus_{i \neq k} E_i)^\perp.
\]
\[
(2) \quad \langle \dim E_k, \gamma_k \rangle < 0. \text{ We prove this in two steps:}
\]
Step 1: Since $\text{Hom}_{\Lambda}(E_k, M_{\beta}) = 0$ we must also have $\text{Hom}_{\Lambda}(E_k, S_j) = 0$. Therefore
\[
\langle \dim E_k, \gamma_k \rangle = \langle \dim E_k, \dim S_k \rangle \leq 0.
\]
Step 2: $\langle \dim E_k, \dim S_k \rangle \neq 0$ since all vectors $z$ satisfying $\langle \dim E_i, z \rangle = 0$ for all $i$ are scalar multiples of $\beta$ which is not possible since $\gamma_k \subseteq \beta$. This finishes the proof of Claim 1.

Claim 2: $D_{\mathbb{Z}}^+(\beta) \subseteq \Delta^+(\beta)$.

Proof of claim 2: Since $(M_{\beta}, E_1, \ldots, E_{n-1})$ is an exceptional sequence, by Proposition 1.3.3 every vector in $\mathbb{Z}^n$ can be expressed uniquely as an integer linear combination of $\beta$ and the vectors $\dim E_i$. Let $\alpha \in D_{\mathbb{Z}}^+(\beta)$. Then $\alpha$ is an integer linear combination of the roots $\dim E_i$, say $\alpha = \sum_{i=1}^{n-1} a_i \dim E_i$. The stability conditions which define $D_{\mathbb{Z}}^+(\beta)$ (Definition 3.1.3) and $\langle \dim E_i, \gamma_k \rangle = 0$ imply
\[
\langle \alpha, \gamma_k \rangle = a_k \langle \dim E_k, \gamma_k \rangle \leq 0.
\]

Since $\langle \dim E_k, \gamma_k \rangle < 0$ by (2), this implies that $a_k \geq 0$ for each $k$. Since all $a_i$ are integers it follows that $\alpha \in \Delta^+(\beta)$.

This finishes the proof that $D_{\mathbb{Z}}^+(\beta) = D_{\mathbb{Z}}(\beta)$ for $\beta$ a sincere Schur root.
3.5. Non-sincere case. (Proof of Theorem 3.1.4) In this subsection we will prove $D_Z^ss(\beta) = D_\mathbb{Z}(\beta)$ for all real Schur roots $\beta$. In order to deal with non-sincere roots, we need the following lemma.

**Lemma 3.5.1.** Let $Q$ be a quiver, $\beta$ a real Schur root which is not sincere, with $\beta_j = 0$. Let $Q_{(j)}$ be the quiver obtained from $Q$ by deleting vertex $j$ and all adjacent edges. Then:

1. $D_Z^ss(Q, \beta) = \{\alpha + m \dim P_j \mid \alpha \in D_Z^ss(Q_{(j)}, \beta), m \in \mathbb{Z}\}$
2. $D_\mathbb{Z}(Q, \beta) = \{\alpha + m \dim P_j \mid \alpha \in D_\mathbb{Z}(Q_{(j)}, \beta), m \in \mathbb{Z}\} = D_Z(Q_{(j)}, \beta) + Z \dim P_j$.

**Proof.** (1) Since $P_j$ is one dimensional (over $F_j$) at vertex $j$, for any integer vector $\alpha \in \mathbb{Z}^n$, $\alpha - \alpha_j \dim P_j$ lies in $\mathbb{Z}^{n-1}$. Since $(\dim P_j, \beta_j) = 0$ for all subroots $\beta' \subseteq \beta$, it follows that $\alpha \in D_Z^ss(\beta)$ if and only if $\alpha - \alpha_j \dim P_j$ lies in $D_Z^ss(Q_{(j)}, \beta)$.

(2) Since $P_j$ and $P_j[1]$ are in the perpendicular category $\perp M_\beta$, the same is true for $D_Z(Q, \beta)$: $\alpha \in D_Z(Q, \beta)$ if there is a virtual representation $P(\gamma_j)$ of dimension $\alpha$ which lies in $\perp M_\beta$. Then $P(\gamma_j) \oplus P_j^{|\alpha_j|}$ and $P(\gamma_j) \oplus P_j[1]^{|\alpha_j|}$ are virtual representations in $\perp M_\beta$ and one of them has dimension $\alpha - \alpha_j \dim P_j$ which lies in $D_Z(Q_{(j)}, \beta)$. Conversely, $D_Z(Q_{(j)}, \beta) + Z \dim P_j$ is contained in $D_Z(\beta)$. So, they are equal. \hfill \Box

**Proof of Virtual Stability Theorem 3.1.4** (when the field $K$ is infinite). The proof is by induction on the number of vertices of the quiver $Q$. Let $\beta$ be a real Schur root. If $\beta$ is sincere then $D_Z^ss(\beta) = D_\mathbb{Z}(\beta)$ by Lemma 3.4.1.

If $\beta$ is not sincere then by Lemma 3.5.1(1): $D_Z^ss(Q, \beta)$, is the set of all integer vectors of the form $\alpha + m \dim P_j$ where $\alpha$ lies in $D_Z^ss(Q_{(j)}, \beta)$ and $m \in \mathbb{Z}$. Since $Q_{(j)}$ has $n-1$ vertices, by induction $D_Z^ss(Q_{(j)}, \beta) = D_Z(Q_{(j)}, \beta)$. Then by Lemma 3.5.1(2) it follows that $D_Z^ss(\beta) = D_Z(\beta)$. This finishes the proof of Theorem 3.1.4. \hfill \Box

The extended version of the stability theorem includes also the equality $D_Z^ss(\beta) = \Delta(\beta)$ which was proved for $\beta$ sincere in Lemma 3.4.1 which we now extend to the general real Schur root $\beta$.

**Proposition 3.5.2.** Let $\beta$ be a real Schur root. Then

$$D_Z^ss(\beta) = D_\mathbb{Z}(\beta) = \Delta(\beta).$$

**Proof.** When $\beta$ is sincere, this is Lemma 3.4.1. So, suppose $\beta$ is not sincere. Let $j \in J_\beta$ and let $\Lambda_{(j)} = \Lambda / \Lambda e_j \Lambda$. Then the quiver $Q_{(j)}$ as in Lemma 3.5.1 is the quiver of $\Lambda_{(j)}$. Then, by induction on $n$ we have

$$D_Z^ss(Q_{(j)}, \beta) = \Delta(Q_{(j)}, \beta) = \{ \sum_{1 \leq i \leq n-2} a_i \dim E'_i + \sum_{k \in J_\beta, k \neq j} b_k \dim P_k : a_i \in \mathbb{N}, b_k \in \mathbb{Z} \},$$

where $E'_i$ are the simple objects of $\perp M_\beta$ in $\text{mod-}\Lambda_{(j)}$. By Lemma 3.5.1 we conclude that

$$D_Z^ss(\beta) = \Delta(Q_{(j)}, \beta) + Z \dim P_j = \{ \sum_{1 \leq i \leq n-2} a_i \dim E'_i + \sum_{k \in J_\beta} b_k \dim P_k : a_i \in \mathbb{N}, b_k \in \mathbb{Z} \}.$$

Since each $E'_i$ and each $P_k$ is a module in $\perp M_\beta$, their dimension vectors are nonnegative $\mathbb{Z}$-linear combinations of the dimension vectors of the simple objects $E_i$ of $\perp M_\beta$. Therefore,

$$D_Z^ss(\beta) \subseteq \{ \sum_{1 \leq i \leq n-1} a'_i \dim E_i + \sum_{k \in J_\beta} b'_k \dim P_k : a'_i \in \mathbb{N}, b'_k \in \mathbb{Z} \} = \Delta(\beta).$$

By Proposition 3.3.5 the opposite inclusion $\Delta(\beta) \subseteq D_\mathbb{Z}(\beta) \subseteq D_Z^ss(\beta)$ holds. \hfill \Box
3.6. Extension to arbitrary fields $K$.  (Proof of Theorem 3.1.1) Suppose that the ground field $K$ is finite. Then, we still have the trivial implication $(1)_\Lambda \Rightarrow (3)_\Lambda$. Since $-\otimes_K K(t)$ is an exact functor, $(3)_\Lambda \Rightarrow (3)_{\Lambda(t)}$ which we have shown to be equivalent to $(1)_{\Lambda(t)}$ and $(2)$ which does not refer to $K$. It remains to show that these imply $(1)_\Lambda$.

Recall that, for every real Schur root $\beta$, the simple objects $E_i$ and projective objects $P_j$ of $^A \mathcal{M}_\beta \cap \text{mod-} \Lambda(t)$ are exceptional $\Lambda(t)$-modules. By Theorem 1.4.1 these are isomorphic to $E_i(t), P_j(t), M_j'(t)$ for unique exceptional $\Lambda$-modules $E_i', P_j', M_j'$. Then, for any $\alpha \in D^b_{Z}(\beta) = \Lambda(t)(\beta)$, we have

$$\alpha = \sum k_i \dim E_i'(t) + \sum \ell_j \dim P_j'(t) = \sum k_i \dim E_i + \sum \ell_j \dim P_j'$$

where $k_i \in \mathbb{N}$ and $\ell_j \in \mathbb{Z}$. So, the direct sum of the $\Lambda$-modules $E_i^{k_i}$, projective modules $P_j^{\ell_j}$ for $\ell_j \geq 0$ and the shifted projective modules $P_j^{\ell_j}[1]$ for $\ell_j \leq 0$ give a virtual representation $P(\gamma_s) = (f : P(\gamma_1) \to P(\gamma_0))$ so that, $\text{Hom}_\Lambda(f, M_j') : \text{Hom}_\Lambda(P(\gamma_0), M_j') \cong \text{Hom}_\Lambda(P(\gamma_1), M_j')$. This shows that $(2) \Rightarrow (1)_\Lambda$. So, Theorem 3.1.1 holds for finite $K$. This completes the proof of Theorem 3.1.1 for all finite dimensional hereditary algebras over any field.

**Remark 3.6.1.** When $K$ is any perfect field, $\Lambda \otimes_K \overline{K}$ is an hereditary algebra over the algebraically closed field $\overline{K}$ and it should be possible to extend the Virtual Stability Theorem from [IOTW09] to $\Lambda$. However, if $K$ is not perfect and $F_i = K(a^{1/p})$ is the division algebra at vertex $i$ then the socle of $S_i \otimes_K \overline{K}$ has infinite projective dimension. So, $\Lambda \otimes_K \overline{K}$ is not hereditary in that case. That is the reason we did not take this approach.

4. C-Vectors and Semi-Invariants

In this section we use the Virtual Stability Theorem for semi-invariants (3.1.1) to prove two fundamental theorems relating determinantal weights of semi-invariants, cluster tilting objects and $c$-vectors corresponding to a cluster tilting object. Theorem 4.1.5 gives the relation between semi-invariants and cluster tilting objects. Theorem 4.3.1 relates the semi-invariants of a cluster tilting object to the corresponding $c$-vectors. These theorems (4.1.5 and 4.3.1) are inspired by work of Speyer and Thomas [ST]. In type $A$ these theorems are re-interpreted in terms of finite and infinite trees in [IOs], [ITW].

4.0. Preview. We illustrates Theorems 4.4.5 and 4.3.1 in an example.

**Remark 4.0.1.** Since $c$-vectors come from cluster theory, we use the well-known language of cluster categories [BMRRT]. We recall from Corollary 2.3.7 that there is a bijection between presentations in $\text{Pres}(\Lambda)$ and objects of the cluster category $\mathcal{C}_\Lambda$ so that the presentation of any $\Lambda$-module is sent to that module, and the shifted projective $P[1]$ is sent to the same shifted projective object. Objects of $\text{Pres}(\Lambda)$ are Ext-orthogonal if and only if the corresponding objects of the cluster category $\mathcal{C}_\Lambda$ are Ext-orthogonal since:

$$\text{Ext}^1_{\mathcal{C}_\Lambda}(X, Y) \cong \text{Ext}^1_{\text{Pres}(\Lambda)}(X, Y) \oplus D\text{Ext}^1_{\text{Pres}(\Lambda)}(Y, X)$$

where $D = \text{Hom}_K(\cdot, K)$.

**Example 4.0.2.** The figure below illustrates the relation between cluster tilting objects, domains of semi-invariants and $c$-vectors for the quiver $Q = 1 \leftarrow 2 \leftarrow 3$ of type $A_3$. The picture indicates the unit sphere $S^2$ in $\mathbb{R}^3$ stereographically projected to $\mathbb{R}^2$ with the center being the dimension vector of $\Lambda = P_1 \oplus P_2 \oplus P_3$. The 9 vertices are the (normalized dimension vectors of) indecomposable objects of the cluster category $\mathcal{C}_\Lambda$ labeled as objects of $\text{Pres}(\Lambda) \cong \text{Vrep}(\Lambda)$. The 6 lines are $D(\beta) \cap S^2$, where $\beta$ are the 6 positive roots, dimension vectors of indecomposable modules. There are 14 regions which are spherical triangles whose vertices are components of cluster tilting objects. For example, the upper
left triangle has vertices \( T_1 = S_1 = P_1, T_2 = P_2[1], T_3 = P_3[1] \) with walls \( D(\beta_i) \), where \( \beta_i \) are given by Theorem 4.1.1. The second theorem of this section, Theorem 4.3.1, shows that the \( c \)-vectors corresponding to each cluster tilting object \( \bigoplus T_i \) are, up to sign, equal to the det-weights \( \beta_i \) of the semi-invariants defined on the walls \( D(\beta_i) \) of the conical simplex spanned by \( \dim T_i \). For example, in the upper left spherical simplex, the \( c \)-vectors are \( e_1 = -\beta_1 = -e_1, e_2 = \beta_2 = e_2, e_3 = \beta_3 = e_3 \). The sign of \( c \)-vectors is positive on the outside of each curve and negative on the inside.

4.1. Structure of semi-invariant domains. If \( v_1, \ldots, v_k \) are vectors in \( \mathbb{R}^n \), the conical polyhedron spanned by the \( v_i \) is the set of all nonnegative linear combinations of the \( v_i \). We will denote it by \( C(v_1, \ldots, v_k) \). If the vectors \( v_i \) are linearly independent we call the conical polyhedron a conical simplex.

For each real Schur root \( \beta \), the domain \( D(\beta) \) of the determinantal semi-invariant with det-weight \( \beta \) is equal to the codimension-one conical polyhedron \( \Delta(\beta) \) in \( \mathbb{R}^n \) by Proposition 3.5.2. This is a conical polyhedron since it is the set of nonnegative linear combinations of the vectors \( \dim E_i, \dim P_j \) and \(-\dim P_j \).

**Remark 4.1.1.** (1) We consider objects of \( C_\Lambda \) as objects in \( \text{Pres}(\Lambda) \cong \text{Vrep}(\Lambda) \) following Remark 4.0.1 Corollary 2.3.1 and Proposition 2.3.2 and, for \( X \) in \( \text{Pres}(\Lambda) \), denote by \( X^{\perp_v}, \perp_v X \) perpendicular categories of \( X \) in \( \text{Pres}(\Lambda) \).

(2) Recall that \( |X|^{\perp} = X^{\perp_v} \cap \text{mod-}\Lambda \) by Lemma 3.2.2 where \( M^{\perp} \) and \( \perp M \) are \( \text{Hom}_\Lambda^{-\text{Ext}} \), parallel categories of \( \Lambda \)-module \( M \) in \( \text{mod-}\Lambda \) where \( [M] = M \) for modules \( M \) and \( [P[1]] = P \) for shifted projective modules \( P[1] \).

(3) Let \( R \) be a rigid object in \( \text{Pres}(\Lambda) \). Then

\[
M_\beta \in |R|^{\perp} \iff M_\beta \in R^{\perp_v} \iff \dim R \in D(\beta).
\]

This follows from Theorem 2.3.11, the definition of \( D(\beta) \) and (2) above.

(4) Let \( v \in D(\beta) \). Then, by the Virtual Stability Theorem 3.1.1, \( v \) lies in the interior of \( D(\beta) \) if and only if \( \langle v, \beta \rangle < 0 \) for all proper real Schur subroots \( \beta' \subsetneq \beta \).

If \( T_0 = T_1 \oplus \cdots \oplus T_k \) is a partial cluster tilting object in \( C_\Lambda \), the dimension vectors \( \dim T_i \) are linearly independent. So they span the conical simplex \( C(\dim T_1, \ldots, \dim T_k) \) which we abbreviate by \( C(T_0) \).
Lemma 4.1.2. Let \( T_0 = T_1 \oplus \cdots \oplus T_k \) be any partial cluster tilting object. Then
(a) \( |T_0|^\perp \) is isomorphic to \( \text{mod-} \Gamma \) where \( \Gamma \) is an hereditary algebra with \( n-k \) simple objects.
(b) The conical simplex \( C(T_0) \) is contained in \( D(\beta) \) if and only if the single vector \( \dim T_0 = \sum_{i=1}^k \dim T_i \) lies in \( D(\beta) \).
(c) Let \( M_\beta \) be an exceptional module in \( |T_0|^\perp \). Then \( \dim T_0 \) lies in the interior of \( D(\beta) \) if and only if \( M_\beta \) is a simple object of \( |T_0|^\perp \).

Proof. (a) follows from Remark 4.1.2.
(b) If \( \dim T_i \in D(\beta) \) then \( \dim T_0 \in D(\beta) \) since \( D(\beta) \) is convex. Conversely, suppose \( \dim T_0 \in D(\beta) \) then there exists an object \( V \in \text{Pres}(\Lambda) \) with \( \dim V = \dim T_0 \) which admits a determinantal semi-invariant of det weight \( \beta \). Since this is an open conditions, the generic object of this dimension has the same property. This is \( T_0 \). So, \( T_0 \in \perp V M_\beta \) which implies that each \( T_i \in \perp V M_\beta \). So, \( C(T_0) \subseteq D(\beta) \).
(c) Suppose \( M_\beta \) is a simple object of \( |T_0|^\perp \). Suppose that \( \dim T_0 \) does not lie in the interior of \( D(\beta) \). Then \( \dim T_0 \in \partial D(\beta) \). So, \( \langle \dim T_0, \beta' \rangle = 0 \) for some proper real Schur subroot \( \beta' \subseteq \beta \). Then, by the Virtual Stability Theorem 3.1.1 \( \dim T_0 \) lies in \( D(\beta') \) and therefore \( M_{\beta'} \) is an object of \( |T_0|^\perp \) by (b). But \( M_{\beta'} \) is a subobject of \( M_\beta \) contradicting the assumption that \( M_\beta \) is simple in \( |T_0|^\perp \). So, \( \dim T_0 \) is in the interior of \( D(\beta) \).

Suppose \( M_\beta \) is not simple in \( |T_0|^\perp \). Then \( M_\beta \) contains a simple submodule \( M_{\beta''} \in |T_0|^\perp \). Any such \( \beta'' \) is a real Schur root. So, \( \langle \dim T_0, \beta'' \rangle = 0 \). Therefore, \( \dim T_0 \in D(\beta) \cap D(\beta'') \subseteq \partial D(\beta) \). So, \( \dim T_0 \in \partial D(\beta) \) when \( M_\beta \) is not simple in \( |T_0|^\perp \).

\( \square \)

Remark 4.1.3. Let \( M_{\alpha_1}, \cdots, M_{\alpha_{n-k}} \) be the simple objects of \( |T_0|^\perp \). Then, the vector \( \dim T_0 \) lies in the interior of exactly \( n-k \) semi-invariant domains \( D(\alpha_1), \cdots, D(\alpha_{n-k}) \).

The following proposition will be used in the proof of the c-vector theorem.

Proposition 4.1.4. Let \( T_0 = T_1 \oplus \cdots \oplus T_{n-2} \) be a partial cluster tilting object with \( n-2 \) summands. Let \( M_\beta \) be an exceptional object of \( |T_0|^\perp \). Then
1. If \( M_\beta \) is a non-simple object of \( |T_0|^\perp \) there is, up to isomorphism, only one object \( T(\beta) \) so that \( T_0 \oplus T(\beta) \) is a partial cluster tilting object and \( \dim T(\beta) \in D(\beta) \).
2. If \( M_\beta \) is a simple object in \( |T_0|^\perp \) there are two nonisomorphic objects \( T', T'' \) in the cluster category \( C_\Lambda \) of \( \text{mod-} \Lambda \) so that \( T_0 \oplus T' \) and \( T_0 \oplus T'' \) are partial cluster tilting objects and so that \( T', T'' \) lie in \( \perp V M_\beta \).

Proof. In Case (1), by the lemma, the partial cluster tilting object \( T_0 \) lies on the boundary of the polyhedral region \( D(\beta) \). Therefore, there is at most one way to complete it to a cluster tilting object in \( D(\beta) \). Thus, it suffices to show the existence of a nonzero object \( T(\beta) \in \perp V M_\beta \) so that \( T_0 \oplus T(\beta) \) is a partial cluster tilting object and, in Case (2), we need to show that there are two objects \( T', T'' \) in this cluster category which complete the cluster tilting object. At least one of them, say \( T' \), is a module in \( \perp V M_\beta \). Letting \( T(\beta) = T' \), this proves Case (1).

Case (2) \( M_\beta \) is simple in \( |T_0|^\perp \). If both \( T', T'' \) are modules we are done. So, suppose that \( T'' = P[1] \) for some projective object \( P \in \perp M_\beta \). Then we claim that \( P \) is projective in \( \text{mod-} \Lambda \) making \( T'' = P[1] \) an object of the cluster category of \( \text{mod-} \Lambda \).

Suppose that \( P \) is the projective cover of the simple object \( E_k \in \perp M_\beta \). Then the dimension vectors of the objects \( T_1, \cdots, T_{n-2} \) lie on the face of the positive simplex \( \Delta^+ (\beta) \) opposite the vertex \( E_k \). By Lemma 4.1.2(c), \( D(\beta) \) contains a small neighborhood of the point \( \dim T_0 \) inside the hyperplane \( H_\beta \). After rescaling, any such neighborhood contains an integer point having negative \( E_k \)-coordinate. By the Virtual Stability Theorem 3.1.1 such a
point has the form \[ \sum k_i \dim E_i + \sum \ell_j \dim P_j, \] where \( E_i \) are the simple objects of \( \frac{1}{\dim}M_\beta \) and \( P_j \) are the projective objects of \( \frac{1}{\dim}M_\beta \) which are also projective in \( \text{mod}-\Lambda \). By construction, at least one of these \( P_j \) must have \( E_k \) in its composition series. But then the projective cover \( P \) of \( E_k \) in \( \frac{1}{\dim}M_\beta \) is a submodule of \( P_j \) which is projective in both \( \frac{1}{\dim}M_\beta \) and \( \text{mod}-\Lambda \). So, \( T'' = P[1] \) lies in the cluster category of \( \text{mod}-\Lambda \) as claimed. \( \square \)

**Theorem 4.1.5.** Let \( T = T_1 \oplus \cdots \oplus T_n \) be a cluster tilting object for \( \Lambda \). Then:

(a) The dimension vectors \( \dim T_i \) span a conical simplex in \( \mathbb{R}^n \) whose walls are \( D(\beta_i) \) for uniquely determined real Schur roots \( \beta_i \).

(b) \( \text{End}_\Lambda(M_{\beta_i}) \cong \text{End}_{\mathcal{C}_\Lambda}(T_i) \) for each \( i \).

(c) The interior of the conical simplex spanned by \( \{ \dim T_i \}_{i=1}^n \) does not meet any \( D(\beta) \).

(d) The objects \( T_i \) can be numbered in such a way that \( \text{End}_{\mathcal{C}_\Lambda}(T_i) \cong \text{End}_\Lambda(S_i) = F_i \) where \( S_i \) are the simple \( \Lambda \)-modules.

(e) Furthermore, \( \langle \dim T_i, \beta_j \rangle = \delta_{ij} \varepsilon_j f_j \), where \( f_j = \dim K F_j = \dim K \text{End}_{\mathcal{C}_\Lambda}(T_j) = f_{\beta_j} \) with the notation \( f_{\beta} = \dim K \text{End}_\Lambda(M_{\beta}) \) and \( \varepsilon_j = \pm 1 \) is the sign of \( \langle \dim T_j, \beta_j \rangle \).

**Proof.** (a) Each face of the conical simplex is spanned by \( \dim T_i \) with one \( T_j \) deleted. Then \( |T/T_j|^{\frac{1}{\dim}} \) has a unique simple object, say, \( M_{\beta_j} \) and \( \dim T_i \in D(\beta_j) \) for \( i \neq j \) by Remark 4.1.3.

(b) By Schofield (2.1.2), the \( T_i \) can be renumbered to form an exceptional sequence \( (T_1, \ldots, T_n) \). For each \( j \) we have another exceptional sequence \( (M_{\beta_j}, T_1, \ldots, T_j, \ldots, T_n) \). By Proposition 1.3.3 (5), this implies (b). By 1.3.3 (3), this also implies that \( \dim T_j = \pm \dim M_{\beta_j} \) plus a linear combination of \( \dim T_i \) for \( i \neq j \). We need this to prove (e).

(c) The interior of the conical simplex \( \sigma \) cannot lie in any \( D(\alpha) \). If it did, then \( D_{\mathbb{Z}}(\alpha) \) would contain a integer point, say, \( v \), in the interior of \( \sigma \). But then the general virtual representation \( P \rightarrow Q \) with dimension vector \( v \) would lie in \( D_{\mathbb{Z}}(\alpha) \). However, by the virtual canonical decomposition theorem this representation is a direct sum of the representations \( T_i \) and each \( T_i \) occurs. So, \( \dim T_i \in D_{\mathbb{Z}}(\alpha) \) for all \( i \). This would make \( D_{\mathbb{Z}}(\alpha) \) \( n \) dimensional contradicting the fact that it has codimension one.

(d) follows from Corollary 1.3.5.

(e) follows from (a) and (b): Since \( T_i \in D(\beta_j) \) for \( i \neq j \), we have \( \langle \dim T_i, \beta_j \rangle = 0 \) for \( i \neq j \). And \( f_j = f_{\beta_j} \) by (b). In the proof of (b) we observed that \( \dim T_i = \pm \dim M_{\beta_j} \) plus a linear combination of \( \dim T_j \) for \( i \neq j \). Since \( \langle \dim T_i, \beta_j \rangle = 0 \) for all \( i \neq j \), this implies

\[ \langle \dim T_j, \beta_j \rangle = \pm \langle \beta_j, \beta_j \rangle = \pm f_{\beta_j} = \pm f_j \]

We denote the sign by \( \varepsilon_j \). This completes the proof of (e). \( \square \)

Using this theorem we can now define the matrices \( \Gamma_T \) whose columns are equal, by definition, to det-weights of semi-invariants up to sign and will be shown to be equal to the \( c \)-vectors of the cluster tilting object \( T_i \) up to sign, by Theorem 1.3.1 below.

**Definition 4.1.6.** For any cluster tilting object \( T = \bigoplus_{i=1}^n T_i \) for \( \Lambda \), let \( \Gamma_T \) be the \( n \times n \) integer matrix with columns \( \gamma_i = \varepsilon_i \beta_i \) where \( \beta_i \) are the unique real Schur roots so that \( \dim T_i \in D(\beta_j) \) for \( i \neq j \) and \( \varepsilon_i = \pm 1 \) is the sign of \( \langle \dim T_i, \beta_i \rangle \).

**Corollary 4.1.7.** Let \( V \) be the \( n \times n \) matrix with columns \( \dim T_i \). Then

\[ V^t E \Gamma_T = D \]

where \( E \) is the Euler matrix and \( D \) is the diagonal matrix with diagonal entries \( f_i \).

One very important observation about the significance of the sign \( \varepsilon_i \) is the following.

**Proposition 4.1.8.** Suppose that \( \varepsilon_k = \text{sgn} \langle \dim T_k, \beta_k \rangle > 0 \). Then \( T_k \) is a module and there does not exist any epimorphism \( B \twoheadrightarrow T_k \) where \( B \) is a module in \( \text{add} T/T_k \).
Consider the set of all \( n \times X \) sign. We write \( v \rangle \) and Zelevinsky which can be phrased as follows.

Definition 4.2.2. [FZ07] Let \( \langle FZ07 \rangle \) be a skew-symmetrizable matrix with symmetrizer \( D \) and let \( \langle FZ07 \rangle \) be a diagonal matrix \( D \) with positive integer diagonal entries so that \( DB \) is skew-symmetric. \( D \) is called the symmetrizer of \( B \). An extended exchange matrix is defined to be a \( 2n \times n \) matrix \( \tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix} \) whose top half \( B \) is skew-symmetrizable.

Definition 4.2.1. [FZ07] For any extended exchange matrix \( \tilde{B} = (b_{ij}) \) and any \( 1 \leq k \leq n \), the mutation \( \mu_k \tilde{B} \) of \( \tilde{B} \) in the \( k \)-direction is defined to be the matrix \( \tilde{B}' = (b'_{ij}) \) defined by

\[
(4.2.1) \quad b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k \\
 b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik}b_{kj} > 0 \\
b_{ij} & \text{otherwise}
\end{cases}
\]

For any finite sequence of positive integers \( k_1, k_2, \cdots, k_r \leq n \) we have the iterated mutation \( \mu_{k_r} \cdots \mu_{k_2} \mu_{k_1} \tilde{B} \) of \( \tilde{B} \).

Consider the set of all \( n \times n \) matrices \( C \) which appear at the bottom of matrices \( \tilde{B} = \begin{bmatrix} B \\ C \end{bmatrix} \) given by iterated mutation of the initial extended exchange matrix. The columns of all such matrices \( C \) are called the c-vectors of \( B_0 \). The matrices \( C \) are called the c-matrices of \( B_0 \).

We recall that a vector \( v \) is called sign coherent if its nonzero coordinates have the same sign. We write \( v > 0 \) if this sign is positive and \( v \neq 0 \). We will use a theorem of Nakanishi and Zelevinsky which can be phrased as follows.

Theorem 4.2.3. [NZ] Let \( B_0 \) be a skew-symmetrizable matrix with symmetrizer \( D \) and let \( X \) be a set of \( n \times n \) integer matrices \( C \) with the following properties.

1. \( I_n \in X \)
2. For any \( C \in X \), the columns of \( C \) are sign coherent and nonzero.
3. For any \( C \in X \), the matrix \( BC := D^{-1}C^TDB_0C \) has integer entries \( b_{ij} \).
4. Let \( C \in X \) and \( 1 \leq k \leq n \), then \( X \) contains the matrix \( C' = \mu_kC \) with columns \( c'_j \) given as follows, where \( b_{kj} \) are entries of \( BC \).
   \[
   c'_j = \begin{cases} 
-c_k & \text{if } j = k \\
c_j + |b_{kj}|c_k & \text{if } b_{kj}c_k > 0 \\
c_j & \text{otherwise}
\end{cases}
\]
5. \( X \) is minimal with the above properties.
Then $X$ is the set of $c$-matrices of $B_0$ and the columns of $C \in X$ are the $c$-vectors of $B_0$.

To specify the $c$-vectors corresponding to a cluster tilting object, we need to choose an initial cluster tilting object. Let $B_0 = L^t - R$, where $L, R$ are the left and right Euler matrices (Section 1.1). Then $B_0$ is an $n \times n$ skew-symmetric matrix since $DB_0 = E^t - E$ is skew-symmetric, where $E$ is the Euler matrix. We will use $B_0$ as the initial exchange matrix and $\tilde{B}_0 = \begin{bmatrix} B_0 \\ I_n \end{bmatrix}$ as the initial extended exchange matrix. The following easy observation will be useful.

**Lemma 4.2.4.** For any two vectors $x, y \in \mathbb{R}^n$ we have $\langle y, x \rangle - \langle x, y \rangle = x^t DB_0 y$.

### 4.3. $c$-vector theorem.

**Theorem 4.3.1** ($c$-vector theorem). Let $\Lambda$ be any finite dimensional hereditary algebra over any field. Let $C_\Lambda$ be the cluster category of $\Lambda$. Let the initial cluster tilting object in $C_\Lambda$ be $\Lambda[1] = \bigoplus_{i=1}^n P_i[1]$. Then the $c$-vectors associated to the cluster tilting object $T = \bigoplus_{i=1}^n T_i$ are $c_i = -\varepsilon_i b_i$ where $b_i$ are the associated det-weights and each $\varepsilon_i$ is the sign of $(\dim T_i, b_i)$.

The plan for the proof of this theorem is as follows. Let $X$ denote the set of matrices

$$X := \{ -\Gamma_T = -[\gamma_1, \ldots, \gamma_n] \mid \gamma_i = \varepsilon_i b_i \}.$$ 

where $\varepsilon_i$ is the sign of $(\dim T_i, b_i)$. We will show that $X$ satisfies the conditions of Theorem 4.2.3 (1), (2) are Lemma 4.3.2, (3) is Lemma 4.3.3, (4) is Proposition 4.3.4 and (5) follows from the fact that mutation acts transitively on the set of cluster tilting objects (Hu).

Therefore, by Theorem 4.2.3 $X$ is equal to the set of all $c$-matrices of the initial exchange matrix $B_0 = L^t - R$.

**Lemma 4.3.2.** (1) $X$ contains the identity matrix $I_n$.

(2) The columns of $X$ are sign coherent.

**Proof.** (1) For $T = \Lambda[1]$, $\Gamma_\Lambda[1] = -I_n$. This follows from Definition 4.1.6 since $S_i \in P_j^t$ for $i \neq j$ which implies that $-\dim P_j = D(e_i)$ where $e_i = \dim S_i$ is the $i$-th unit vector and $(\dim P_i[1], e_i) = -1$. Therefore $I_n \in X$.

(2) Since the columns $\gamma_i$ of $\Gamma_T$ are, up to sign, dimension vectors of indecomposable modules $M_{\beta_i}$, they are sign coherent. \hfill \qed

**Lemma 4.3.3.** Let $T = T_1 \oplus \cdots \oplus T_n$ be a cluster tilting object and $\Gamma_T$ the associated matrix with columns $\gamma_i = \varepsilon_i b_i$. Then the matrix $B_T = B_{-\Gamma} = D^{-1} T^t DB_0 \Gamma$ has integer entries. Hence $X$ satisfies Condition (3) in Theorem 4.2.3.

**Proof.** By Lemma 4.2.4, the entries of $B_T$ are $b_{ij} = f_i^{-1}(\langle \gamma_j, \gamma_i \rangle - \langle \gamma_i, \gamma_j \rangle)$. The columns $\gamma_i$ of $\Gamma_T$ are, up to sign, dimension vectors of exceptional modules $M_{\beta_i}$ and $\text{End}_\Lambda(M_{\beta_i}) \cong \text{End}_{C_\Lambda}(T_i) \cong F_i$. Also, $f_i = \dim_K F_i$. Therefore, $b_{ij}$ are integers by Proposition 4.2.3. \hfill \qed

We need to show that the set $X$ satisfies condition (4) in Theorem 4.2.3 which is the following proposition whose proof will occupy the rest of this section.

**Proposition 4.3.4.** Under the mutation $\mu_k$ of $T$, the matrix $\Gamma_T$ changes to $\Gamma'_T$ with columns $\gamma'_j$ given as follows where $b_{ij}$ are the entries of $B_T$.

$$\gamma'_j = \begin{cases} -\gamma_k & \text{if } j = k \\ \gamma_j + |b_{kj}| \gamma_k & \text{if } b_{kj} \gamma_k < 0 \\ \gamma_j & \text{otherwise} \end{cases}$$

The inequality $b_{kj} \gamma_k < 0$ is reversed from Theorem 4.2.3 (4) since $\gamma_j, \gamma_k$ will turn out to be negative $c$-vectors. We will prove Proposition 4.3.3 first in the special case when $n = 2$. We will then show that the general case follows from the special case.
4.4. Consecutive roots. In order to set up the reduction to the rank 2 case, we need to rephrase Proposition 4.3.4 in terms of the “consecutive roots” \( -\gamma_j, \gamma_k, \gamma'_j \).

**Definition 4.4.1.** Let \( T_0 \in \text{Pres}(\Lambda) \) be a partial cluster tilting object with \( n-2 \) summands. Define \( S_\Lambda(T_0) \) to be the set of all ordered pairs \( (\gamma, U) \) where

1. \( U \) is an exceptional object of \( \text{Pres}(\Lambda) \) so that \( T_0 \oplus U \) is rigid.
2. \( \gamma = \pm \beta \) where \( M_\beta \) is the unique exceptional module in \( |T_0 \oplus U| \).

**Remark 4.4.2.** Note that, by Proposition 4.1.4, \( U \) is uniquely determined by \( \gamma \) except when \( M_\gamma \) is a simple object of \( |T_0| \) in which case there are exactly two possibilities for \( U \).

**Proposition 4.4.3.** For each \( (\gamma, U) \in S_\Lambda(T_0) \), there is a unique \( (\gamma', U') \in S_\Lambda(T_0) \) so that

1. \( T_0 \oplus U \oplus U' \) is a cluster tilting object in \( \text{Pres}(\Lambda) \).
2. \( \langle \dim U', \gamma' \rangle > 0 \).
3. \( \langle \dim U, \gamma \rangle < 0 \).

**Proof.** There are two objects \( U', U'' \) so that \( T_0 \oplus U \oplus U', T_0 \oplus U \oplus U'' \) are cluster tilting objects. These objects must lie on opposite sides of the hyperplane \( H_\gamma = \{ x \in \mathbb{R}^n \mid \langle x, \gamma \rangle = 0 \} \). Therefore, up to reordering, we have: \( \langle \dim U', \gamma \rangle < 0, \langle \dim U'', \gamma \rangle > 0 \). So, \( U' \) is uniquely determined by \( S_1 \) and \( S_3 \).

Let \( M_\beta \) be the unique exceptional object in \( |T_0 \oplus U'| \). Then \( (\beta', U'), (-\beta', U') \) are the elements of \( S_\Lambda(T_0) \) with second entry \( U' \). Let \( \gamma' = \text{sgn} \langle \dim U, \beta' \rangle \beta' \). Then \( \langle \gamma', U' \rangle \) is the unique pair satisfying \( S_1, S_2, S_3 \).

**Definition 4.4.4.** Let \( \rho(\gamma, U) \) denote the unique pair \( (\gamma', U') \) given by Proposition 4.4.3. A sequence of pairs \( (\gamma_1, U_1), (\gamma_2, U_2), (\gamma_3, U_3), \ldots \in S_\Lambda(T_0) \) will be called consecutive pairs if \( \rho(\gamma_i, U_i) = (\gamma_{i+1}, U_{i+1}) \). And \( \gamma_1, \gamma_2, \gamma_3, \ldots \) will be called consecutive roots if there exist \( \{U_i\} \) so that \( \{\gamma_i, U_i\} \) is a sequence of consecutive pairs.

**Corollary 4.4.5.** \( \rho : S_\Lambda(T_0) \to S_\Lambda(T_0) \) is a bijection.

**Proof.** If \( \rho(\gamma, U) = (\gamma', U') \) then one sees easily that \( \rho(-\gamma', U') = (-\gamma, U) \). So, \( s \circ \rho \circ s \) is the inverse of \( \rho \) where \( s(\gamma, U) = (-\gamma, U) \).

**Lemma 4.4.6.** Suppose that \( T = T_0 \oplus T_j \oplus T_k \) and \( T' = \mu_k T = T_0 \oplus T_j \oplus T'_k \). Let \( \gamma_j = \varepsilon_j \beta_j \) and \( \gamma_k = \varepsilon_k \beta_k \) be the corresponding \( \gamma \) vectors of \( T \). Let \( \gamma'_j = \varepsilon'_j \beta'_j, \gamma'_k = \varepsilon'_k \beta'_k \) be the corresponding \( \gamma \) vectors of \( T' \). Then \( \gamma'_j = -\gamma_k \) and \( -\gamma_j, \gamma_k, \gamma'_j \) are consecutive pairs in \( S_\Lambda(T_0) \). In particular, \( -\gamma_j, \gamma_k, \gamma'_j \) are consecutive roots.

**Proof.** By definition of \( \varepsilon_j, \varepsilon_k \) we have \( \langle \dim T_k, \gamma_k \rangle > 0 \) and \( \langle \dim T_j, \gamma_j \rangle > 0 \). Therefore, \( \rho(-\gamma_j, T_k) = (\gamma'_k, T'_j) \). For \( T', \beta'_k \) is by definition the unique positive root so that \( T_0 \oplus T_j \in D(\beta'_k) \). So, we must have \( \beta'_k = \beta_k \). The signs \( \varepsilon_k, \varepsilon'_k \) must be opposite since \( T_k, T'_k \) must lie on opposite sides of the set \( D(\beta_k) \). (If they were on the same side, the cones spanned by \( T \) and \( T' \) would overlap.) So, \( \gamma'_k = -\gamma_k \). Then, \( \rho(\gamma_k, T_j) = \rho(-\gamma_k, T_j) = (\gamma'_j, T'_k) \).

**Lemma 4.4.7.** Proposition 4.3.4 follows from the following equation for all triples of consecutive roots: \( \gamma, \gamma', \gamma'' \):

\[
\gamma'' = \begin{cases} 
-\gamma + |b|\gamma' & \text{if } \gamma' < 0 \\
-\gamma & \text{otherwise}
\end{cases}
\]

where \( b = f^{-1}_{\gamma'}((\gamma', \gamma) - (\gamma, \gamma')) \) and \( f_{\gamma'} = \dim K \text{End}_{\Lambda}(M_{\gamma'}) \).

**Proof.** Suppose that the formula above for \( \gamma'' \) holds for all triples of consecutive roots. Then it holds in the particular case \( \gamma = -\gamma_j, \gamma' = \gamma_k \) and \( \gamma'' = \gamma'_j \). Substituting these values of \( \gamma, \gamma', \gamma'' \) transforms the given equation into the formula for \( \gamma'_j \) given in Proposition 4.3.4 except for the missing statement \( \gamma'_j = -\gamma_k \) which was shown in Lemma 4.4.6 above.
4.5. Proof of Proposition 4.3.4 in rank 2 case. The results of this section are well-known. We include them for clarity. Let \( H \) be a finite dimensional hereditary algebra of rank 2. Then \( H \) will be Morita equivalent to the tensor algebra of a modulated quiver. (See Appendix A.) So, we assume that

\[
H = \begin{bmatrix} F_1 & 0 \\ F_2 & M \end{bmatrix}
\]

where \( F_1, F_2 \) are division algebras over \( K \) and \( M \) is an \( F_2 \)-bimodule. A (right) \( H \)-module can be viewed as a representation \( V = (V_1, V_2, f : V_2 \otimes F_2 M \to V_1) \) of the modulated quiver

\[
F_1 \overset{M}{\to} F_2.
\]

Recall that \( \dim V = (\dim_{F_1} V_1, \dim_{F_2} V_2) \) and \( f_i = \dim_{K} F_i \). Let \( d_i = \dim_{F_i} M \). Then

\[
\dim_{K} M = m = f_1 d_1 = f_2 d_2.
\]

The projective \( H \)-modules are \( P^H_1 = (F_1, 0, 0) \) and \( P^H_2 = (M, F_2, id : F_2 \otimes M \to M) \). The injective \( H \)-modules are given by a dual construction, \( I^H_1 = (0, F_2, 0) \), \( I^H_2 = (F_1, M^*, ev) \) where \( M^* = \text{Hom}_{F_1}(M, F_1) \) and \( ev : M^* \otimes M \to F_1 \) is the evaluation map. The simple \( H \)-modules are \( P^H_1, I^H_2 \). These have dimension vectors

\[
(4.5.1) \quad \dim P^H_1 = (1, 0), \quad \dim P^H_2 = (d_1, 1), \quad \dim I^H_1 = (1, d_2), \quad \dim I^H_2 = (0, 1)
\]

It is well-known that \( H \) has finite type, i.e., has only finitely many exceptional representations up to isomorphism, if and only if \( d_1 d_2 \leq 3 \). These are the quivers \( A_1 \times A_1, A_2, B_2, G_2 \). We let \( s \) denote the number of indecomposable modules. So, \( s = 2, 3, 4, 6 \) or \( \infty \).

All exceptional \( H \)-modules are either preprojective or preinjective. We denote the preprojective modules \( Y_i \) and the preinjective modules \( Z_j \) keeping in mind that \( Z_j = Y_{s-j+1} \) in the finite case. The preprojective and preinjective component(s) of the Auslander-Reiten quiver of \( H \) is given by:

\[
\begin{array}{cccccc}
Y_1 & \rightarrow & Y_2 & \rightarrow & \cdots & \rightarrow & Y_{s-1} \\
P_1^H & \rightarrow & P_2^H & \rightarrow & \cdots & \rightarrow & P_{s-1}^H \\
Y_3 & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & Y_s \\
Z_1 & \rightarrow & Z_2 & \rightarrow & \cdots & \rightarrow & Z_{s-1} \\
I_2^H & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & I_{s-1}^H \\
\end{array}
\]

The arrows denote irreducible maps \( Y_i \to Y_{i+1} \) and \( Z_j \to Z_{j+1} \). The Auslander-Reiten translation \( \tau_H \) acts by “shifting two spaces to the left” and we have \( H \)-almost split sequences (subscripts of \( d \) should be taken modulo 2).

\[
Y_i \to Y_{d_{i+1}} \to Y_{i+2}, \quad Z_{j+2} \to Z_{d_{j+1}} \to Z_{j} \quad (i, j \geq 1).
\]

So, the dimension vectors of \( Y_i, Z_j \) are given recursively using (4.5.1) by

\[
\begin{align}
\dim Y_i &= d_{i-1} \dim Y_{i-1} - \dim Y_{i-2} \quad i \geq 3 \\
\dim Z_j &= d_{j-1} \dim Z_{j-1} - \dim Z_{j-2} \quad j \geq 3.
\end{align}
\]

The Auslander-Reiten quiver of the cluster category \( \mathcal{C}_H \) of \( H \) [BMRRT] has two more exceptional objects, \( Y_1[1] = P_1^H[1] \) and \( Y_2[1] = P_2^H[1] \), which come between \( Z_1 \) and \( Y_1 \):

\[
\begin{array}{cccccc}
\cdots & \rightarrow & Z_1 & \rightarrow & Y_2 & \rightarrow & \cdots \\
\cdots & \rightarrow & Y_1 & \rightarrow & Y_1[1] & \rightarrow & Y_2[1] \\
Z_2 & \rightarrow & \cdots & \rightarrow & \cdots & \rightarrow & Y_3 \\
\end{array}
\]

This is called the transjective component of the Auslander-Reiten quiver of \( \mathcal{C}_H \).

Proposition 4.5.1. The cluster tilting objects of \( \mathcal{C}_H \) are sums of pairs of consecutive objects in the above quiver: \( Z_{i+1} \oplus Z_i, \quad Z_1 \oplus Y_1[1], \quad Y_1[1] \oplus Y_2[1], \quad Y_2[1] \oplus Y_1, \quad Y_1 \oplus Y_{i+1} \).
Lemma 4.5.2. For $1 < i < s$, $\gamma_i = add Y_{i+1}$ and $\gamma_{i+1} = add Z_i$. For the simple objects $Y_1, Z_1, \gamma_1 = add (Y_2 \oplus Y_2[1])$ and $\gamma_1 = add (Y_1 \oplus Y_1[1])$.

Proposition 4.5.3. (a) The elements of $S_H(0)$ are $(\pm \dim Z_j, Z_j)$, $(\pm \dim Y_i, Y_{i+1})$ for $i, j \geq 1$ and the 6 pairs $(\pm \dim Y_1, Y_2[1])$, $(\pm \dim Z_1, Y_1)$ and $(\pm \dim Z_1, Y_1[1])$.

(b) The action of $\rho$ on these pairs is given by the following list and by the reverse of the list (given by changing the sign of the first entries and reversing the order).

1. $\rho (\dim Z_{j+2}, Z_{j+1}) = (\dim Z_{j+1}, Z_j)$ for $j = 1, \ldots, s - 2$.
2. $\rho (\dim Z_2, Z_1) = (\dim Z_1, Y_1[1])$
3. $\rho (\dim Z_1, Y_1[1]) = (\dim Y_1, Y_2[1])$
4. $\rho (\dim Z_1, Y_1) = (\dim Z_1, Y_1)$
5. $\rho (\dim Z_1, Y_1) = (\dim Y_1, Y_2)$
6. $\rho (\dim Y_1, Y_{i+1}) = (\dim Y_{i+1}, Y_{i+2})$ for $i = 1, \ldots, s - 2$.

Proof. (a) By Remark 4.4.2, $\gamma$ uniquely determines $U$ in the pair $(\gamma, U)$ when $M_\gamma$ is not a simple object. By definition $U \in \perp M_\gamma$. So, $U$ must be the object after $M_\gamma$ in the Auslander-Reiten quiver of $H$. This gives the pairs $(\pm \dim Z_1, Z_{j-1})$, $(\pm \dim Y_i, Y_{i+1})$. When $\gamma$ is a simple root, $U = P$ or $P[1]$ where $P$ is the projective which does not map to $M_\gamma$. This gives $(\pm \dim Y_1, Y_2)$ and the remaining 6 pairs.

(b) The computation of $\rho(\gamma, U)$ in (1), (2), (3) and (6) are examples of the general formula: $\rho (\dim Z_{j+2}, Z_{j+1}) = (\dim Z_{j+1}, Z_j)$ which holds in rank 2. (4) and (5) follow from (3) by change of sign. □

Proof of Proposition 4.3.4 in rank 2. By Proposition 4.5.3, there are two sequences of consecutive roots in $S_H(0)$:

$\cdots, \dim Z_3, \dim Z_2, \dim Z_1, - \dim Y_1, - \dim Z_1, \dim Y_1, \dim Y_2, \dim Y_3, \cdots$

$\cdots, - \dim Y_3, - \dim Y_2, - \dim Y_1, \dim Z_1, \dim Y_1, - \dim Z_1, - \dim Z_2, - \dim Z_3, \cdots$

We consider only the first. The second is similar. The calculations are summarized in the following chart.

| $\gamma$ | $\gamma'$ | $\gamma''$ | $f_{\gamma'}$ | $\langle \gamma', \gamma \rangle$ | $\langle \gamma, \gamma' \rangle$ | $b$ | $sgn(b\gamma')$ | formula for $\gamma''$ |
|---------|-----------|-----------|--------------|----------------|----------------|---|----------------|------------------|
| $\dim Z_3$ | $\dim Z_2$ | $\dim Z_1$ | $f_1$ | 0 | $m$ | $-d_1$ | $-d_1$ | $\dim Z_2 - \dim Z_3$ |
| $\dim Z_2$ | $\dim Z_1$ | $- \dim Y_1$ | $f_2$ | 0 | $m$ | $-d_2$ | $-d_2$ | $\dim Z_1 - \dim Z_2$ |
| $\dim Z_1$ | $- \dim Y_1$ | $- \dim Z_1$ | $f_1$ | 0 | $m$ | $-d_1$ | $-d_1$ | $- \dim Z_1$ |
| $- \dim Y_1$ | $- \dim Z_1$ | $\dim Y_1$ | $f_2$ | 0 | $m$ | $-d_2$ | $-d_2$ | $\dim Y_1$ |
| $- \dim Z_1$ | $\dim Y_1$ | $\dim Z_1$ | $f_2$ | $-m$ | 0 | $-d_1$ | $-d_1$ | $\dim Y_1 + \dim Z_1$ |
| $\dim Y_1$ | $\dim Y_2$ | $\dim Z_2$ | $f_2$ | 0 | $m$ | $-d_2$ | $-d_2$ | $\dim Y_2 - \dim Z_1$ |

That $\gamma''$ agrees with the formula from Lemma 4.4.7 follows from the formulas (4.5.1), (4.5.2), (4.5.3). So, Proposition 4.3.4 holds in the case $n = 2$ by Lemma 4.4.7. □

4.6. Proof of Proposition 4.3.4 in general case. Let $T_0$ be a partial cluster tilting object for $\Lambda$ with $n - 2$ summands. Let $L_1, L_2$ be the indecomposable injective objects of the rank 2 hereditary abelian subcategory $|T_0|^\perp$ of mod-$\Lambda$. Let $H = \text{End}_\Lambda(L)^{op}$ where $L = L_1 \oplus L_2$.

Proposition 4.6.1. The functor $F = \text{Hom}_\Lambda(\cdot, L) : \text{mod}-\Lambda \to \text{mod}-H$ induces an isomorphism of categories $|T_0|^\perp \cong \text{mod}-H$.

The functor $F$ does not have the properties that we need on objects outside the subcategory $|T_0|^\perp$. So, we will replace it with a mapping $\eta$ which, unfortunately, is defined only on objects. We now set up the notation for this mapping.

Let $\Psi_+$ be the set of all dimension vectors of exceptional objects in $|T_0|^\perp$, let $\Psi = \Psi_+ \cup -\Psi_+$ and let $V \subset \mathbb{R}^n$ be the two dimensional subspace spanned by $\Psi$. Then every
vector $v \in V$ is given uniquely as $v = x_1 \alpha_1 + x_2 \alpha_2$ where $x_1, x_2 \in \mathbb{R}$ and $\alpha_1, \alpha_2 \in \Psi_+$ are the dimension vectors of the simple objects $M_{\alpha_1}, M_{\alpha_2}$ of $|T_0|^{\perp}$ corresponding to $L_1, L_2$.

**Remark 4.6.2.** (a) The Euler-Ringel pairing on $\mathbb{R}^2$ is given by:
\[
\langle (x_1, x_2), (y_1, y_2) \rangle_H = (x_1 \alpha_1 + x_2 \alpha_2, y_1 \alpha_1 + y_2 \alpha_2)
\]
(b) The linear isomorphism $\pi : V \to \mathbb{R}^2$ given by $\pi(x_1 \alpha_1 + x_2 \alpha_2) = (x_1, x_2)$ is also an isometry, i.e., $(v, w) = \langle \pi(v), \pi(w) \rangle_H$ for all $v, w \in V$.
(c) Furthermore, $\pi(\Psi_+)$ is the set of all dimension vectors of exceptional $H$-modules.
(d) Let $\lambda_i = \dim L_i$ and $f_2 = \dim_k \text{End}_\Lambda(M_{\alpha_j}) = \dim_k \text{Hom}_\Lambda(M_{\alpha_j}, L_j)$. Then $\langle \alpha_i, \lambda_j \rangle = f_2 \delta_{ij}$.

Let $\theta : \mathbb{R}^n \to V$ be the linear mapping given by
\[
\theta(x) = \sum_{i=1,2} f_i^{-1}(x, \lambda_i) \alpha_i.
\]

**Lemma 4.6.3.** (a) $\theta$ is a projection, i.e., $\theta(v) = v$ for all $v \in V$.
(b) $\langle \theta(x), \beta \rangle = \langle x, \beta \rangle$ for all $x \in \mathbb{R}^n$, $\beta \in \Psi$.
(c) $\langle \pi \theta(x), \pi(\beta) \rangle_H = \langle x, \beta \rangle$ for all $x \in \mathbb{R}^n$, $\beta \in \Psi$.
(d) $\theta(\dim T_0) = 0$.

**Proof.** (a) follows from the observation that $\theta(\alpha_i) = \alpha_i$. (b) follows from the calculation: $\langle \theta(x), \lambda_j \rangle = \langle x, \lambda_j \rangle$ and the fact that $\lambda_1, \lambda_2$ span $V$. (c) follows from (b) and the fact that $\pi : V \to \mathbb{R}^2$ is an isometry. (d) follows from the fact that $L_i \in |T_0|^{\perp}$.

Suppose that $\overline{\beta} \in \pi \Psi_+ \subset \mathbb{Z}^2$. Then the hyperplane $\{ \pi \in \mathbb{R}^2 | \langle \pi, \overline{\beta} \rangle_H = 0 \}$ is a line through the origin in $\mathbb{R}^2$. And $D_H(\overline{\beta})$ is a closed subset of this line given by:
\[
D_H(\overline{\beta}) = \{ x \in \mathbb{R}^2 | \langle x, \overline{\beta} \rangle_H = 0 \text{ and } \langle x, \overline{\beta} \rangle_H \leq 0 \text{ for all } \overline{\beta} \}\]

**Lemma 4.6.4.** (a) For any $\beta \in \Psi_+$, $\pi \theta(D_\Lambda(\beta)) \subseteq D_H(\pi(\beta))$.
(b) If $\pi \theta(x) \in D_H(\pi(\beta))$, $x \in \mathbb{R}^n$, then $\dim T_0 + \delta x \in D_\Lambda(\beta)$ for sufficiently small $\delta > 0$.

**Proof.** (a) Suppose $x \in D_\Lambda(\beta)$. Then $\langle \pi \theta(x), \pi(\beta) \rangle_H = \langle x, \beta \rangle = 0$ and $\langle \pi \theta(x), \pi(\beta') \rangle_H = \langle x, \beta' \rangle \leq 0$ for all $\beta' \subseteq \beta$ in $\Psi_+$. So, $\pi \theta(x) \in D_H(\pi(\beta))$.

(b) Suppose that $\pi \theta(x) \in D_H(\pi(\beta))$. Then, for any $\delta > 0$ we have:
\[
\langle \dim T_0 + \delta x, \beta', \beta' \rangle = \langle \pi \theta(\dim T_0 + \delta x), \pi(\beta') \rangle_H = \delta \langle \pi \theta(x), \pi(\beta') \rangle \leq 0
\]
For any subroot $\beta' \subseteq \beta$ with $\beta' \in \Psi_+$ we have:
\[
\langle \dim T_0 + \delta x, \beta' \rangle = \langle \pi \theta(\dim T_0 + \delta x), \pi(\beta') \rangle_H = \delta \langle \pi \theta(x), \pi(\beta') \rangle \leq 0
\]
For any subroot $\beta'' \subseteq \beta$ with $\beta'' \notin \Psi_+$ we have $\langle \dim T_0, \beta'' \rangle < 0$.

We need the following criterion equivalent to $S1$ from Proposition 4.4.3.

**Lemma 4.6.5.** Suppose that $U, U'$ are exceptional objects of $\text{Pres}(\Lambda)$ so that $T_0 \oplus U, T_0 \oplus U'$ are rigid. Then $T_0 \oplus U \oplus U'$ is a cluster tilting objects of $\Lambda$ only if $S1'$ holds:
$S1' : \forall a, b, c \in \mathbb{R}_{>0}, \forall \beta \in \Phi_+(\Lambda)$, $a \dim T_0 + b \dim U + c \dim U' \notin D_\Lambda(\beta)$.

**Proof.** The necessity of $S1'$ was shown in Corollary 2.4.15.

To show sufficiency, suppose $T_0 \oplus U \oplus U'$ is not a cluster tilting object. Let $V, V'$ be the two objects making $T = T_0 \oplus U \oplus V$ and $T' = T_0 \oplus U \oplus V'$ into cluster tilting objects in $\text{Pres}(\Lambda)$. Then $\dim(T_0 \oplus U)$ lies in the interior of $C(T) \cup C(T')$ where $C(T)$ is the conical...
simplex spanned by \( \{ \dim T_i \} \), the components of \( T \). Since \( U' \neq V, V' \), by the virtual generic decomposition theorem \([2.3.11]\) it follows that \( U' \notin C(T) \cup C(T') \). So, the straight line from \( \dim (T_0 \oplus U) \) to \( \dim U' \) goes through the boundary of \( C(T) \cup C(T') \). But, 
\[
\partial (C(T) \cup C(T')) \subset \partial C(T) \cup \partial C(T')
\]
is a union of domains \( D(\alpha) \). So, there exist \( a, b > 0 \) so that
\[
a \dim T_0 + b \dim U' \in D(\alpha)
\]
for some \( \alpha \) contradicting \( S1' \). □

**Proposition 4.6.6.** Let \( (\gamma, U) \in S_\Lambda(T_0) \). Then
(a) There is a unique object \( \eta(U) \in \text{Pres}(H) \) so that \( \dim \eta(U) = \pi \theta (\dim U) \).
(b) \( (\pi(\gamma), \eta(U)) \in S_H(0) \).
(c) If \( \rho(\gamma, U) = (\gamma', U') \) then \( \rho_H(\pi(\gamma), \eta(U)) = (\pi(\gamma'), \eta(U')) \).

**Proof.** (a) Let \( |\gamma| = \beta \in \Psi_+ \). Since \( \dim U \in D_\Lambda(\beta) \), we have, by Lemma \( 4.6.4 \) that \( \pi \theta (\dim U) \in D_H(\pi(\beta)) \). Let \( M_\pi \) be the object which comes right after \( M_{\pi(\beta)} \) in the Auslander-Reiten quiver of \( H \) except in the case when \( M_{\pi(\beta)} \) is the simple injective object in which case we let \( M_\pi \) be the simple projective object. Then \( \pi \) is the unique positive root of \( H \) in \( D_H(\pi(\beta)) \). Since \( \pi, \pi \theta (\dim U) \in \mathbb{Z}^2 \) are collinear and the coordinates of \( \pi \) are relatively prime, \( \pi \theta (\dim U) \) must be an integer multiple of \( \pi \). But \( \rho(\gamma, U) = (\gamma', U') \) implies that \( \dim (U, \gamma') > 0 \). And \( \dim U, \gamma' \) is \( \pm f_{\gamma'} \) by Theorem \( 4.1.3(c) \). So, by Lemma \( 4.6.3(c) \) this implies
\[
\langle \pi \theta (\dim U), \pi (\gamma') \rangle_H = \langle \dim U, \gamma' \rangle = f_{\gamma'}.
\]
But \( \langle \pi, \pi (\gamma') \rangle_H \) is also an integer multiple of \( f_{\gamma'} \). So, \( \pi \theta (\dim U) = \pm \pi \). If \( \pi (\dim U) = -\pi \in D_H(\pi(\beta)) \) then \( M_\pi \) must be projective in \( \text{mod-H} \) and \( \pi \theta (\dim U) = M_\pi \). Otherwise, \( \pi \theta (\dim U) = \dim M_\pi \). So, either \( \eta(U) = M_\pi[1] \) or \( \eta(U) = M_\pi \) is the unique object in \( \text{Pres}(H) \) with \( \dim \eta(U) = \pi \theta (\dim U) \in D_H(\pi(\beta)) \).

(b) Since \( \dim \eta(U) = \pi \theta (\dim U) \in D_H(\pi(\beta)) \), \( (\pi(\beta), \eta(U)) \in S_H(0) \) which implies \( (\pi(\gamma), \eta(U)) \in S_H(0) \) since \( \gamma = \pm \beta \) and, therefore, \( \pi(\gamma) = \pm \pi(\beta) \).

(c) To show that \( \rho_H(\pi(\gamma), \eta(U)) = (\pi(\gamma'), \eta(U')) \) we will verify \( S2, S3, S1' \) from Proposition \( 4.4.3 \) and Lemma \( 4.6.5 \).

**Remark 4.6.7.** The mapping \( \eta \) is not a functor. However, it has very nice properties. The mapping \( \eta \) gives a bijection between the set of all objects \( U \) which are Ext-orthogonal to \( T_0 \) and the set of all exceptional objects in \( \text{Pres}(H) \). The fact that \( \eta \) is surjection onto this set follows from Proposition \( 4.6.6(c) \) and the fact that \( \text{Pres}(H) \) has only two orbits of the action of \( \rho \). To show that \( \eta \) is 1-1, suppose \( \eta(U) = \eta(U') \). Then \( \dim \eta(U) = \dim \eta(U') \) lie in the same \( D_H(\pi(\beta)) \). This implies \( \dim U, \dim U' \) lie in \( D(\beta) \) which implies \( \beta \) is simple and \( \dim \eta(U) = -\dim \eta(U') \), a contradiction. For more details, see [IT16].

**Proof of Proposition 4.3.4 in general.** Suppose that \( (\gamma, U), (\gamma', U'), (\gamma'', U'') \) are consecutive pairs in \( S_\Lambda(T_0) \). Then, by Proposition \( 4.6.6 \) \( \pi(\gamma), \pi(\gamma'), \pi(\gamma'') \) are consecutive roots for the rank 2 hereditary algebra \( H \). Therefore, by the calculation in the last subsection, the formula in Lemma \( 4.4.7 \) holds for \( \pi(\gamma), \pi(\gamma'), \pi(\gamma'') \).

But \( \pi \) is an isometry. So, the formula also holds for \( \gamma, \gamma', \gamma'' \). By Lemma \( 4.4.7 \) this implies Proposition \( 4.3.4 \). □
We can now prove the \(c\)-vector theorem.

**Proof of Theorem 4.3.1.** The statement is that the \(c\)-vectors of a cluster tilting object \(T\) are \(-\gamma_i\). This holds for the initial cluster tilting object \(\Lambda[1]\) by definition. It is well-known that cluster mutation acts transitively on the set of cluster tilting objects. (See \[Hu\].) Therefore, it suffices to show that the equation \(c_i = -\gamma_i\) remains true under mutation. But this is what was shown in Proposition 4.3.4 with the aid of Theorem 4.2.3. □

### 4.7. Example

Figure 1 illustrates several concepts discussed in the paper. Take the modulated quiver

\[
F_1 = \mathbb{C} \xleftarrow{M_{21} = \mathbb{C}} F_2 = \mathbb{R} \xleftarrow{M_{22} = \mathbb{R}} F_3 = \mathbb{R}
\]

The tensor algebra of this quiver is of finite type with 9 indecomposable objects:

\[
\begin{align*}
P_3 & \xleftarrow{} S_2 \xleftarrow{} S_3 \\
P_2 & \xleftarrow{} X \xleftarrow{} Z_1 \\
P_1 & \xleftarrow{} Y \xleftarrow{} Z_2
\end{align*}
\]

Consider \(L\), the intersection with the unit sphere \(S^2 \subseteq \mathbb{R}^3\) with the union \(\bigcup D(\beta)\) of all nine semi-invariant domains. Figure 1 shows the stereographic projection of \(L\) onto the plane.

![Figure 1](image)

**Figure 1.** The three circles are domains of semi-invariants with simple det-weights. Other det-weights are dimension vectors of other representations. For example, edges \(e_1, e_2, e_3\) are domains of \((0,1,1), (1,2,2), (1,1,1)\). The four dark vertices \(S_1, Z_2, Y, P_1[1]\) indicate the objects with endomorphism ring \(\mathbb{C}\). The semi-invariant domains \(D(1,0,0), D(1,2,0)\) and \(e_2 = D(1,2,2)\) which correspond to \(S_1, Y, Z_2\) by Proposition 4.7.1 are also darkened.

In reading Figure 1 the following easy observation is helpful.

**Proposition 4.7.1.** Let \(T = \bigoplus T_i\) be a cluster tilting object with associated matrix \(\Gamma_T = (\gamma_i)\). Suppose that \(\gamma_k = \beta_k\) is positive and all other columns of \(\Gamma_T\) are negative. Then \(\dim T_k = \beta_k\). In other words, when the following triangle appears in a picture, \(T_2 = M_{\beta_2}\)
Proof. As we say in the proof of the $\alpha$-vector theorem, rank 2 case, for any $j \neq k$, the modules $M_{\gamma_j}$ and $M_{\gamma_k}$ are consecutive objects in the Auslander-Reiten quiver of the rank 2 perpendicular category $|T_0|$ where $T_0 = \bigoplus_{i \neq j,k} T_i$. Therefore, $\langle \gamma_k, \gamma_j \rangle = 0$ for all $j \neq k$. We also have $\langle \dim T_k, \gamma_j \rangle = 0$ for all $j \neq k$. Since $\Gamma_T$ is an invertible matrix (by Corollary 4.1.7), this implies that $\dim T_k$ is a scalar multiple of $\gamma_k$. So, $\dim T_k = \beta_k$. \hfill $\square$

Example 4.7.2. Examples of Proposition 4.7.1 in Figure 1.

1. $\dim Z_1 = (0,1,1)$ and $e_1 = D(0,1,1)$
2. $\dim Z_2 = (1,2,2)$ and $e_2 = D(1,2,2)$ which extends from $S_3$ to $S_2$
3. $\dim P_3 = (1,1,1)$ and $e_3 = D(1,1,1)$ which extends from $S_3$ through $Z_2$, $X$ to $Y$.
4. $\dim X = (1,2,1)$ and $D(1,2,1)$ is the edge connecting $Z_2$ and $S_2$.

Figure 1 also illustrates the following concepts used in the paper. For $\beta = (1,1,1)$, the simple objects of the category $\perp M_\beta$ are $S_3$ and $Y$ with dimension vectors $\alpha_1 = (0,0,1)$ and $\alpha_2 = (0,2,1)$. These form the corners (endpoints in this dimension) of the convex region $D(\beta)$. The other roots in this region are positive integer linear combinations: $\dim Z_2 = 2\alpha_1 + \alpha_2$ and $\dim X = \alpha_1 + \alpha_2$.

4.8. Applications. In concurrently written papers we use the results of this paper to:

1. Develop the theory of signed exceptional sequences and show they are in bijection with ordered cluster tilting objects [IT16]. We have seen a special case: $S_A(T_0)$ is the set of all signed exceptional sequences for $|T_0|^\perp$.
2. Develop the theory of semi-invariant picture groups and compute their cohomology in type $A_n$ [OTW3].
3. Show that, for acyclic modulated quivers of finite type, the maximal green sequences are in bijection with the positive expressions for the Coxeter element in the picture group [IT17].
4. For any acyclic modulated quiver with a bimodule $M_{ij} : i \rightarrow j$ of infinite type, show that any maximal green sequence mutates at $j$ before $i$ [BHIT].

Finally, we point out that Theorem 4.3.1 implies the sign coherence of $\alpha$-vectors (that in each $\alpha$-vector the coordinates have the same sign) a theorem which has been proven many times and in fact the present version of this paper grew out of a desire to understand the proof given by Speyer-Thomas [ST]. Proposition 2.2.8 gives the conceptual proof of this fact. Namely, semi-invariants defined on presentation spaces are necessarily sign coherent.

In future work, we plan to extend the results of this paper to modulated quivers with oriented cycles.

5. Appendix A: Associated modulated quiver

In this appendix we discuss the problem of when a finite dimensional hereditary algebra over a field $K$ is Morita equivalent to the tensor algebra of its associated modulated quiver.

Theorem 5.0.1. $\Lambda$ is Morita equivalent to $T(Q, M)$ if and only if, for each arrow $i \rightarrow j$, the $F_i\cdot F_j$-bimodule epimorphism

$$\text{Hom}_\Lambda(P_j, rP_i) \twoheadrightarrow M_{ij} = \text{Hom}_\Lambda(P_j, rP_i/r^2P_i)$$

has a section. (Recall that $F_i = \text{End}_\Lambda(P_i)$.)
Proof. This condition is necessary since it holds on the category of representations of $T(Q, M)$. Conversely, suppose the condition holds on $\text{mod}-\Lambda$. Choose a section $\sigma_{ij} : M_{ij} \to \text{Hom}_{\Lambda}(P_j, rP_i)$ of (5.0.1) for every $i \to j$ in $Q_1$. For every $\Lambda$-module $X$, let $X_i = \text{Hom}_{\Lambda}(P_i, X)$. This is a right $F_i$-module. For each arrow $i \to j$ in $Q_1$, define the morphism $X_i \otimes_{F_i} M_{ij} \to X_j$ to be the composition:

$$X_i \otimes_{F_i} M_{ij} \xrightarrow{r \otimes \sigma_{ij}} \text{Hom}_{\Lambda}(rP_i, rX) \otimes_{F_i} \text{Hom}_{\Lambda}(P_j, rP_i) \xrightarrow{c} \text{Hom}_{\Lambda}(P_j, rX) \hookrightarrow \text{Hom}_{\Lambda}(P_j, X) = X_j$$

where $r : \text{Hom}_{\Lambda}(P_i, X) \to \text{Hom}_{\Lambda}(rP_i, rX)$ is the restriction map and $c$ is composition. Since each morphism in this sequence is natural in $X$, this defines a functor

$$\varphi : \text{mod}-\Lambda \to \text{Rep}(Q, M)$$

which is clearly exact and faithful since it takes nonzero objects to nonzero objects.

We claim that $\varphi P_i$ is the projective cover $P_i^T$ of $S_i$ in $\text{Rep}(Q, M)$. This follows by induction on the length of $P_i$ and the fact that the structure maps $c(r \otimes \sigma_{ij}) : M_{ij} \to \text{Hom}_{\Lambda}(P_j, rP_i)$ of $\varphi P_i$ are, together, adjoint to the isomorphism $\bigoplus_j M_{ij} \otimes_{F_i} P_j \cong rP_i$.

Thus, $\text{Hom}_{\Lambda}(P_i, X) = X_i = \text{Hom}_{\text{mod}}(P_i^T, \varphi X)$ and it follows that $\varphi$ is an equivalence between the full subcategories of projective objects of $\text{mod}-\Lambda$ and $\text{Rep}(Q, M)$. Being exact, $\varphi$ extends to an equivalence of the module categories.

\[\square\]

Example 5.0.2. Let $L = \mathbb{F}_2(t)$ with subfields $K = \mathbb{F}_2(t^4) \subset F = \mathbb{F}_2(t^2) \subset L$. We have a short exact sequence of $L$-bimodules:

$$0 \to L \otimes_F L \xrightarrow{j} L \otimes_K L \xrightarrow{p} L \otimes_F L \to 0$$

where $j$ sends $1 \otimes 1$ to $t^2 \otimes 1 + 1 \otimes t^2$ and $p$ takes $1 \otimes 1$ to $1 \otimes 1$. This sequence does not split since $L \otimes_K L$ is indecomposable as an $L$-bimodule. This follows from the $L$-algebra isomorphism $\varphi : L[X]/(X^4) \to L \otimes_K L$ given by $\varphi(X) = t \otimes 1 + 1 \otimes t$ where we consider $L \otimes_K L$ as an $L$-algebra using $L \otimes 1$.

Let $\Lambda$ be the tensor algebra of the modulated quiver

\[\xymatrix{ F_1 \ar[rr]^{M_{12}} & & F_2 \ar[rr]^{M_{23}} & & F_3 \ar[ll]^{M_{13}} \ar[dd]^{L \otimes_K L} \ar[ll]_{L} \ar[rr]_{F} & & L }\]

modulo the relation that the composition $L \otimes_F L$ of the top two arrows is identified with the image of $j$ in $L \otimes_K L$. Then $\Lambda$ is hereditary since the radical of each projective module is projective, e.g., $rP_1 \cong P_2 \oplus P_3^3$. However, the bimodule morphism $M_{13} = \text{Hom}_{\Lambda}(P_3, rP_1) \to M_{13}$ is not split because it is equal to the map $p$ in (5.0.2). By Theorem 5.0.1, $\Lambda$ is not Morita equivalent to the tensor algebra of its associated modulated quiver.

6. Appendix B: Reduced norm

This appendix reviews the definition and properties of the reduced norm $[J]$ and uses them to compare the determinantal weight with the “true weight” of a semi-invariant on presentation spaces as claimed in Remark 2.4.3. We assume that $K$ is an infinite field.

6.1. Definitions. For $A$ a finite dimensional algebra over $K$, the general element of $A$ is

$$a(\xi) = \sum \xi_i u_i \in A \otimes_K K(\xi)$$

where $u_1, \ldots, u_n$ is a vector space basis for $A$ over $K$ and $\xi_1, \ldots, \xi_n$ are a transcendence basis for $K(\xi) = K(\xi_1, \ldots, \xi_n)$. Let

$$m_{a(\xi)}(\lambda) = \lambda^m + c_1(\xi)\lambda^{m-1} + \cdots + c_m(\xi) \in K(\xi)[\lambda]$$
be the minimal polynomial of \(a(\xi)\) over \(K(\xi)\). The degree \(m\) of \(m_a(\xi)(\lambda)\) is called the degree of \(A\) over \(K\). We call it the reduced degree in cases where the word “degree” is already defined as in the case of field extensions.

It is easy to see that the reduced degree of a finite separable extension of \(K\) is equal to its vector space dimension over \(K\) (the usual notion of degree). However, this is not true in general for inseparable extensions and division algebras.

If \(D\) is a finite dimensional division algebra over its center \(C\) then \(\dim_C D = d^2\) where \(d\) is the degree of \(D\) over \(C\). Furthermore, there is an open dense subset of \(D\) consisting of all elements \(b \in D\) so that \(C(b)\) is a separable field extension of \(C\) of degree \(d\). Each of these is called a maximal separable subfield of \(D\).

**Example 6.1.1.** Let \(A = \mathbb{H}\) and \(K = \mathbb{R}\). The minimal polynomial of the general element \(a = t + xi + yj + zk \in \mathbb{H}\) is \(m_a(\lambda) = \lambda^2 - 2t\lambda + t^2 + x^2 + y^2 + z^2\). So, \(\mathbb{H}\) has degree 2 over \(\mathbb{R}\). For any \(b \in \mathbb{H}\) which is not in \(\mathbb{R}\), \(\mathbb{R}(b) \cong \mathbb{C}\) is a maximal (separable) subfield of \(\mathbb{H}\).

**Lemma 6.1.2.** [J] \(m_a(\xi)(\lambda)\) is a polynomial in \(\xi_1, \cdots, \xi_n, \lambda\) and \(c_j(\xi) \in K[\xi]\) is a homogeneous polynomial of degree \(j\) in the variables \(\xi_i\).

The reduced characteristic polynomial of \(b \in A\) is the specialization of \(m_a(\xi)(\lambda)\) given by

\[
m_b(\lambda) = \sum_{i=0}^{m} c_i(b_1, \cdots, b_n)\lambda^{m-i} \in K[\lambda]
\]

where \(b = \sum b_iu_i, \ b_i \in K\) and \(c_0 = 1\). We will use the notation \(c_i(b) = c_i(b_1, \cdots, b_n)\).

**Proposition 6.1.3.** [J]

- (0) \(m_b(\lambda)\) depends only on \(b \in A\). (The coefficients \(c_i(b)\) are independent of the choice of basis \(u_1, \cdots, u_n\).)
- (1) \(m_b(b) = 0\). Equivalently, the minimal polynomial \(\mu_b(\lambda)\) of \(b\) is a factor of \(m_b(\lambda)\).
- (2) Every root of \(m_b(\lambda)\) is a root of \(\mu_b(\lambda)\).
- (3) The set of all \(b \in A\) for which \(m_b(\lambda)\) is the minimal polynomial of \(b\) is an open dense subset of \(A\).
- (4) \(m_b(\lambda)\) is invariant under extension of scalars, i.e., \(m_b(\lambda) = m_{b \otimes 1}(\lambda)\) if \(b \otimes 1 \in A \otimes_K L\) is the image of \(b\) for any extension field \(L\) of \(K\).

The following observation follows easily from Properties (2) and (3).

**Lemma 6.1.4.** The reduced degree of a finite purely inseparable extension \(F\) of \(K\) is the smallest power \(q = p^e\) of \(p = \text{char} K\) so that \(F^q \subseteq K\). Furthermore the reduced characteristic polynomial is \(m_b(\lambda) = \lambda^q - b^q\) for every \(b \in F\).

**Example 6.1.5.** Let \(A = \mathbb{F}_p(s, t)\) and \(K = \mathbb{F}_p(s^p, t^p)\). Then \(a^p \in K\) for any \(a \in A\) and the minimal polynomial of the general element \(a \in A\) is \(m_a(\lambda) = \lambda^p - a^p\). So, the reduced degree of \(A\) over \(K\) is \(q\) although \(A\) is a field extension of \(K\) of degree \(p^2\).

**Definition 6.1.6.** The reduced norm \(\overline{\pi} : A \to K\) is defined to be the homogeneous polynomial function of degree \(m\), the degree of \(A\) over \(K\), given on any \(a \in A\) by \(\overline{\pi}(b) = (-1)^m c_m(b)\).

The main properties of the reduced norm are the following.

\[
\overline{\pi}(ab) = \overline{\pi}(a)\overline{\pi}(b), \quad \overline{\pi}(1) = 1.
\]

Any polynomial function \(\chi : A \to K\) satisfying these two properties will be called a character on \(A\). Another easy consequence of Properties (2) and (3) is the following. If \(A, B\) are finite dimensional algebras over \(K\) and \((a, b) \in A \times B\), then \(m_{(a,b)}(\lambda) = m_a(\lambda)m_b(\lambda)\). This implies in particular that the degree of \(A \times B\) over \(K\) is the sum of the degrees of \(A, B\) over \(K\). Also the reduced norm over \(A \times B\) is the product:

\[
\overline{\pi}_{A \times B}(a, b) = \overline{\pi}_A(a)\overline{\pi}_B(b).
\]
6.2. Theorems related to this paper.

**Theorem 6.2.1.** Let $D$ be a finite dimensional division algebra over $K$ which has degree $d$ over its center $C$ and suppose that $C$ has reduced degree $c$ over $K$. Then

(a) For any $k \geq 1$, $M_k(D)$ has degree $dk$.

(b) Any character $M_k(D) \rightarrow K$ is a nonnegative power of the reduced norm.

**Proof.** We first compute the degree of $M_k(D)$ over $K$. Let $L$ be the maximal separable subfield of $D$. Then $L$ is separable over $C$ of degree $d$ and it is well-known that $M_k(D) \otimes_C L \cong M_{dk}(L)$. Let $E$ be the separable closure of $K$ in $L$. Then $F = E \cap C$ is the separable closure of $K$ in $C$ and $L = CE$. Let $s = [F : K]$. Then $c = qs$ where $q = p^m$ is the reduced degree of $C$ over $F$. By Lemma 6.1.4, the reduced degree of $L$ over $E$ is also $q$ and $q$ is minimal so that $L^t \subseteq E$. Let $S$ be the splitting field of $E$ over $K$. Then $C \otimes_F S \cong CS$ is a separable field extension of $CE = L$. So,

$$M_k(D) \otimes_F S = M_k(D) \otimes_C C \otimes_F S = M_k(D) \otimes_C L \otimes_L CS \cong M_{dk}(CS).$$

**Claim** The degree of $M_{dk}(CS)$ over $S$ is $qdk$ and the reduced norm $M_{dk}(CS) \rightarrow S$ is the $q$-th power of the determinant over $CS$.

**Proof:** Any $a \in M_{dk}(CS)$ satisfies its characteristic polynomial $f(\lambda) = \det(\lambda - a) \in CS[\lambda]$ with degree $dk$. Then $f(\lambda)^q$ is a polynomial in $S[\lambda]$ of degree $qdk$ satisfied by $a$. So, the degree of $M_{dk}(CS)$ over $S$ is $\leq qdk$. Now consider the inclusion of the diagonal matrices:

$$CS^{dk} = CS \times \cdots \times CS \hookrightarrow M_{dk}(CS)$$

Since the general element of $CS$ has degree $q$ over $S$, the general element of $CS^{dk}$ has degree $qdk$ over $S$. So, the degree of $M_{dk}(CS)$ over $S$ is $\geq qdk$. So, it is equal to $qdk$. Furthermore, the reduced characteristic polynomial is $\det(\lambda - a)^q$ and the reduced norm is $\det(a)^q$.

(a) Since $S$ is the splitting field of $E$ over $K$ and $F$ is an intermediate field, we have $F \otimes_K S \cong S^s$ where $s = [F : K]$. Since (reduced) degree is invariant under extension of scalars, the degree of $M_k(D)$ over $K$ is equal to the degree of $M_k(D) \otimes_K S$ over $S$. But

$$M_k(D) \otimes_K S = M_k(D) \otimes_F F \otimes_K S = M_k(D) \otimes_F S^s = M_{dk}(CS)^s$$

which has degree $s$ times the degree of $M_{dk}(CS)$ over $S$. By the claim above this is $s$ times $qdk$ which is $dkqs = dk$ proving (a).

(b) Consider any character $\chi : M_k(D) \rightarrow K$. We note that arbitrary (polynomial) characters must be homogeneous polynomials. By extending scalars we get a character

$$\chi_S : M_k(D) \otimes_K S \cong M_{dk}(CS)^s \rightarrow S$$

which must be a product of $s$ characters $(\chi_S)_i : M_{dk}(CS) \rightarrow S$. By symmetry given by the action of $Gal(S/K)$, these $s$ characters are equal. By restriction to diagonal matrices we get a character $CS^{dk} \rightarrow S$. But a character on $CS^{dk}$ is a product of characters one for each factor. By symmetry, these characters must all be equal: $\chi_S|CS^{dk} = (\chi_0)^{dk}$. But each character $\chi_0 : CS \rightarrow S$ is a power of the reduced norm $\pi_{CS} : CS \rightarrow S$ since $\chi_0(x) = x^m$ and this lies in $S$ only when $m$ is a multiple of $q$, say $m = qt$, $\chi_0 = \pi_{CS}^t$. Therefore,

$$\chi_S|CS^{dk} = (\chi_0)^{dk} = \pi_{CS}^{dk}$$

which has degree equal to $qt^{dk}$. When $\chi$ is the reduced norm $\pi$ we get $t = 1$. Therefore, in general we get $(\chi_S)_t = \pi^t$ when restricted to the diagonal matrices where $\pi$ is the reduced norm of $M_{dk}(CS)$ over $S$. However, any invertible matrix is equivalent to a diagonal matrix under row and column operations which are given by multiplication by elements of the commutator subgroup of $GL(dk,CS)$. Since $S^s$ is abelian, each group homomorphism $(\chi_S)_t : GL(dk,CS) \rightarrow S^s$ is uniquely determined by its restriction to diagonal invertible matrices. So, $\chi_S = \pi^t$ for all elements of $GL(dk,CS)^s$. Since this is an open dense subset
of $M_{dk}(CS)^*$, $\chi_S = \overline{\pi}'$ as homogeneous polynomials over $S$. But both polynomials have coefficients in $K$. So, they give $\chi = \overline{\pi}'$ as characters $M_k(D) \to K$. \hfill \square

**Remark 6.2.2.** Theorem 6.2.1 implies that every character $M_k(D) \to K$ is a nonnegative fractional power of the $K$-determinant $\det_K$: $\det_K = \overline{\pi}'^d/c$ where $f = \dim_K D$. Thus, the “true weight” of a semi-invariant with determinantal weight $\beta$ is the vector whose $i$-th coordinate is $\beta_i f_i/d_i c_i$ where $d_i c_i$ is the (reduced) degree of $f_i$ over $K$. In particular, if $m$ is the least common multiple of the integers $f_i/d_i c_i$ then the $m$-th power $\sigma^m$ of any semi-invariant on a presentation space $\text{Hom}_\Lambda(P(\gamma_1), P(\gamma_0))$ has determinantal weight.

**Corollary 6.2.3.** Suppose that $F_1, F_2$ are division algebras over $K$ of dimensions $f_1, f_2$ and degrees $n_1, n_2$ over $K$. Let $M$ be an $F_1$-$F_2$-bimodule with $\dim_K M = m$. Then

$$\frac{mn_1}{f_1 n_2}, \frac{mn_2}{f_2 n_1} \in \mathbb{Z}.$$  

**Proof.** The reduced norm gives a character

$$\text{End}_{F_1}(M) \cong M_{mn/f_1}(F_1) \overset{\overline{\pi}_1}{\to} K$$

which is polynomial of degree $mn_1/f_1$. Composing with the inclusion $F_2 \hookrightarrow \text{End}_{F_1}(M)$ we get a character $\chi : F_2 \to K$ of degree $mn_1/f_1$. By Theorem 6.2.1 $\chi$ is an integer power of the reduced norm $\overline{\pi}_2 : F_2 \to K$ which has degree $n_2$. Therefore $n_2$ divides $mn_1/f_1$ making $mn_1/f_1 n_2$ an integer. The other case is similar. \hfill \square

**Definition 6.2.4.** Let $\Lambda$ be a finite dimensional hereditary algebra over a field $K$. Let $B_\Lambda = L^1 - R$ be the exchange matrix of $\Lambda$. Define the reduced exchange matrix of $\Lambda$ to be

$$\overline{B}_\Lambda = ZB\Lambda Z^{-1}$$

where $Z$ is the diagonal matrix with entries $z_i = f_i/n_i$ where $n_i$ is the degree of $F_i$ over $K$ and $f_i = \dim_K F_i$. The entries of $\overline{B}_\Lambda$ are

$$\overline{b}_{ij} = \frac{n_j}{f_j n_i}(\langle e_j, e_i \rangle - \langle e_i, e_j \rangle)$$

where $e_i$ are the unit vectors. Since $|\langle e_j, e_i \rangle|$ is the dimension of an $F_i$-$F_j$-bimodule, $\overline{b}_{ij}$ are integers by Corollary 6.2.3. Given a cluster tilting object $T$ with exchange matrix $B_T$ and $c$-matrix $C_T$, we define the reduced exchange matrix and the matrix of reduced $c$-vectors by $\overline{B}_T = ZB_T Z^{-1}$ and $\overline{C}_T = ZC_T Z^{-1}$.

Since mutation of exchange matrices and extended exchange matrices commutes with conjugation, $\overline{B}_T$ and $\overline{C}_T$ have integer coordinates and are obtained from $\begin{bmatrix} B_\Lambda \\ I_0 \end{bmatrix}$ by mutation. We claim that the reduced $c$-vectors are the reduced weights of the reduced norm semi-invariants which we now define.

**Definition 6.2.5.** The reduced norm semi-invariant $\overline{\sigma}_\beta$ is the polynomial function

$$\text{Pres}_\Lambda(\gamma_1, \gamma_0) = \text{Hom}_\Lambda(P(\gamma_1), P(\gamma_0)) \to K$$

which sends $f : P(\gamma_1) \to P(\gamma_0)$ to the reduced norm of

$$\text{Hom}(f, 1) : \text{Hom}_\Lambda(P(\gamma_0), M_\beta) \to \text{Hom}_\Lambda(P(\gamma_1), M_\beta)$$

considered as a linear map of $F_\beta$-vector spaces. We define the reduced weight of a semi-invariant $\sigma$ on presentation space $\text{Pres}_\Lambda(\gamma_1, \gamma_0)$ to be the vector $w \in \mathbb{N}^n$ so that $\sigma(g h) = \prod \overline{\pi}_i(g)^{w_i} \sigma(f) \overline{\pi}_i(h)^{w_i}$ where $\overline{\pi}_i(g)$ is the reduced norm of the $GL(\gamma_i, F_i)$-component of $g \in \text{Aut}_\Lambda(P(\gamma_0))$ and similarly for $\overline{\pi}_i(h)$.
Lemma 6.2.6. The reduced weight of the reduced norm semi-invariant $\overline{\sigma}_\beta$ is

$$\overline{\sigma}_\beta = \frac{1}{z_\beta}(z_1\beta_1, z_2\beta_2, \cdots, z_n\beta_n).$$

Proof. By Remark 6.2.2 we have $\sigma_\beta = \overline{\sigma}_\beta^g$ where $z_\beta = \dim_K F_\beta/\deg_K F_\beta$. Since the det-weight of $\sigma_\beta$ is $\overline{\beta}$ we have:

$$\overline{\sigma}_\beta(gfh) = \sigma_\beta(gfh)^{1/z_\beta} = \prod \chi_i(g)^{\beta_i/z_\beta} \sigma_\beta(f)^{1/z_\beta} \chi_i(h)^{\beta_i/z_\beta} = \prod \overline{\chi}_i(g)^{n_i\beta_i/z_\beta} \overline{\sigma}_\beta(f)^{n_i\beta_i/z_\beta}$$

where $\chi_i(g) = \overline{\chi}_i(g)^{z_i}$ is the det-weight of the $GL(\gamma_0, F_i)$-component of $g \in \text{Aut}_\Lambda(P(\gamma_0))$. So, the reduced weight of $\overline{\sigma}_\beta$ is $(n_i\beta_i/z_\beta) = \overline{\beta}$. □

In the notation of Corollary 4.1.7 we have the following.

Lemma 6.2.7. For any cluster tilting object $T$ of $\Lambda$ we have

$$V^t \overline{E}_T T = N$$

where $N = DZ^{-1}$ is the diagonal matrix with entries $n_i$, $\overline{E} = EZ^{-1} = LN$, $\overline{T} = Z\Gamma_T Z^{-1}$.

Proof. By Corollary 4.1.7 we have: $V^t \overline{E}_T T = V^t \overline{E}_T T Z^{-1} = DZ^{-1} = N$. □

We can now restate the $c$-vector theorem in terms of reduced $c$-vectors.

Theorem 6.2.8 (Reduced Norm $c$-vector Theorem). The reduced $c$-vectors associated to a cluster tilting object $T$ are

$$\overline{c}_j = -\varepsilon_j \overline{\beta}_j$$

where $\overline{\beta}_j$ is the reduced weight of the reduced norm semi-invariant $\overline{\sigma}_{\beta_j}$.

Proof. Since conjugation of exchange matrices and $c$-matrices commutes with mutation, given that $c_j$ is the $j$-th $c$-vector of the object $T$, the reduced vector $\overline{c}_j$ is the $j$-th $c$ vector of $T$ using $\overline{B}_\Lambda$ as initial exchange matrix. Since $c_j = -\varepsilon_j \beta_j$ by Theorem 4.3.1 reduction of both sides, using the fact that $z_j = \beta_j$, gives $\overline{c}_j = -\varepsilon_j \overline{\beta}_j$. □

Example 6.2.9. Consider the $\mathbb{R}$-modulated quiver $\mathbb{H} \leftarrow \mathbb{C} \leftarrow \mathbb{C}$. This has 9 indecomposable modules giving the same picture as Figure 1. Using the same label for these modules as in Figure 1 we have:

| label | $\beta$ | $z_\beta$ | $\overline{\beta}$ |
|-------|---------|-----------|---------------------|
| $S_1$ | $1,0,0$ | 2         | $(1,0,0)$           |
| $P_2$ | $1,1,0$ | 1         | $(2,1,0)$           |
| $P_3$ | $1,1,1$ | 1         | $(2,1,1)$           |
| $Y$   | $1,2,0$ | 2         | $(1,1,0)$           |
| $X$   | $1,2,1$ | 1         | $(2,2,1)$           |
| $S_2$ | $0,1,0$ | 1         | $(0,1,0)$           |
| $Z_2$ | $1,2,2$ | 2         | $(1,1,1)$           |
| $Z_1$ | $0,1,1$ | 1         | $(0,1,1)$           |
| $S_3$ | $0,0,1$ | 1         | $(0,0,1)$           |

As an example, take the injective module $Z_1$. This has a determinantal semi-invariant of det-weight $\beta = (1,2,2)$ since a presentation for $Z_1$ is $P_2 \oplus P_1 \rightarrow P_3 \oplus P_2 \rightarrow Z_1$. We take homomorphisms to $Z_2$ to get:

$$\text{Hom}_\Lambda (P_3 \oplus P_2, Z_2) = \mathbb{C}^2 \oplus \mathbb{C}^2 \rightarrow \text{Hom}_\Lambda (P_2 \oplus P_1, Z_2) = \mathbb{C}^2 \oplus \mathbb{H}$$
The determinantal semi-invariant $\sigma_\beta$ is given by considering this as an isomorphism of 8-dimensional real vector spaces and taking determinant. This has determinantal weight $(1,2,2)$ since the automorphism of $P_3$ given by $z = a + bi \in \mathbb{C}^*$ has real determinant
\[
\begin{vmatrix}
a & b \\
-b & a
\end{vmatrix} = a^2 + b^2
\]
and multiplies the $8 \times 8$ determinant by $(a^2 + b^2)^2$ (since it multiplies the first two $\mathbb{C}$ coordinates) which is the second power of the det-weight $|z|^2$ of $z$. Similarly any $z \in \text{Aut}(P_2)$ also changes the $8 \times 8$ determinant by $|z|^4$ making the det-weight of $\sigma_\beta$ equal to $(?,2,2)$. The first coordinate of the det-weight is 1 since $h \in \text{Aut}(P_1)$ changes the $8 \times 8$ determinant by $|h|^4$ which is the det-weight of $h$.

The reduced norm semi-invariant $\sigma_\beta$ is given by considering (6.2.1) as an isomorphism of 2-dimensional vector spaces over $\mathbb{H}$ and taking the reduced norm over $\mathbb{H}$ which is the square root of the real determinant. So, any automorphism of $P_3$ or $P_2$ given by $z \in \mathbb{C}^*$ will change the reduced norm semi-invariant by $|z|^2$ which is the norm of $z$. Also, any automorphism of $P_1$ given by $h \in \mathbb{H}^*$ will change $\sigma_\beta$ by $|h|^2 = \pi(h)$. So, the reduced weight is $(1,1,1)$.

Acknowledgements

The last two authors gratefully acknowledge the support of National Science Foundation. The first author was supported by the National Security Agency. The second author was supported by the Simons Foundation. The first two authors also acknowledge support of the NSF at the beginning of this project many years ago. The first and third authors thank Faculty of Mathematics and Computer Science of Nicolaus Copernicus University in Torun, Poland and the University of Iowa for their hospitality in September 2013 and November 2014, where the results were announced. The third author thanks University of Syracuse, University of Barcelona and Centro de Investigación en Matemáticas (CIMAT) in Guanajuato, Mexico for the invitation to present the results of this paper and application on April 11, May 26 and June 26, 2015. The first, third and fourth authors are very grateful to the University of Connecticut for hosting (and to National Science Foundation for sponsoring) a very enjoyable and productive International Conference in Representation Theory and Commutative Algebra (ICRTCA) in honor of the fourth author on April 24-27, 2015. The first and third authors thank the Centre de Recerca Matemàtica (CRM) at the University of Barcelona for their hospitality during May 2015 where first versions of the appendices of this paper were written. The second author gratefully acknowledges the support provided by the SFB 1085 Higher Invariants at the University of Regensburg while on sabbatical in the fall of 2014, and funded by the Deutsche Forschungsgemeinschaft (DFG). The authors also had very useful conversations and communications with Hugh Thomas, Nathan Reading, Calin Chindris, Helmut Lenzing, Stephen Hermes, Thomas Brüstle, Alex Dugas and Ernst Dieterich.

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