Trace cohomology revisited

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Abstract. We use a cohomology theory coming from the canonical trace on a $C^*$-algebra of the projective variety to prove an analog of the Riemann Hypothesis for the Kuga-Sato varieties.

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1. Introduction

The aim of our note is a proof of the Riemann Hypothesis for a class of projective varieties over finite fields using the notion of a trace cohomology introduced in [13]. To define such a cohomology, recall that the Serre $C^*$-algebra $\mathcal{A}_V$ of an $n$-dimensional complex projective variety $V_C$ is the norm-closure of a self-adjoint representation of the twisted homogeneous coordinate ring of $V_C$ by the bounded linear operators on a Hilbert space $H$. We shall write $\tau : \mathcal{A}_V \otimes \mathcal{K} \rightarrow \mathbb{R}$ to denote the canonical normalized trace on the stable $C^*$-algebra $\mathcal{A}_V \otimes \mathcal{K}$, i.e. a positive linear functional of norm 1, such that $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{A}_V \otimes \mathcal{K}$, see [Blackadar 1986] [2], p. 31. Applying the Chern character formula to the algebra $\mathcal{A}_V \otimes \mathcal{K}$, one obtains an injective homomorphism $\tau_* : H^i(V_C) \rightarrow \mathbb{R}$. The $i$-th trace cohomology group $\{H^i(V_C) \mid 0 \leq i \leq 2n\}$ of $V_C$ is an additive abelian subgroup $\tau_*(H^i(V_C))$ of the real line $\mathbb{R}$. We refer the reader to Section 2 for the details.

It was shown in [14] that all of Weil’s Conjectures, except for an analog of the Riemann Hypothesis (RH), follow from simple properties of the trace cohomology. Recall that the Kuga-Sato variety is a fiber product of the modular curves; we refer the reader to Section 2.4 for an exact definition. The $i$-th cohomology group of such a variety is related to the space of cusp forms of weight $i + 1$ [Deligne 1969] [3] and [Scholl 1985] [15]. In this note we use the trace cohomology to prove the RH for the Kuga-Sato varieties with a lifting from characteristic $p$ to characteristic zero [Hartshorne 2010] [8, Theorem 22.1]. Namely, let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements and $V(\mathbb{F}_q)$ be a smooth $n$-dimensional Kuga-Sato variety over $\mathbb{F}_q$. 
Theorem 1.1. The roots $\alpha_{ij}$ of polynomials $P_i(t)$ in the zeta function $Z_V(t) = \frac{P_1(t) \cdots P_{n-1}(t)}{P_0(t) \cdots P_{2n}(t)}$ of $V(\mathbb{F}_q)$ are algebraic numbers of the absolute value $|\alpha_{ij}| = q^{\frac{i}{2}}$.

Theorem 1.1 is not new. It has been proved in full generality and different methods in the classical work [Deligne 1974] [4]. The novelty of our approach are concepts of noncommutative geometry, e.g. the Serre $C^*$-algebras and trace cohomology. The latter provide pathways and tools to some open problems of algebraic geometry and number theory inaccessible otherwise.

The article is organized as follows. The Serre $C^*$-algebras and trace cohomology are introduced in Section 2. Theorem 1.1 is proved in Section 3. The trace cohomology of an algebraic curve is calculated in Section 4.

2. Preliminaries

An excellent survey of noncommutative algebraic geometry is written by [Stafford & van den Bergh 2001] [18]. For an introduction to the $C^*$-algebras and their $K$-theory we refer the reader to [Murphy 1990] [11] and [Blackadar 1986] [2], respectively. The Serre $C^*$-algebras were defined in [13] and the trace cohomology in [14]. The Weil’s Conjectures were introduced in [Weil 1949] [19].

2.1. Serre $C^*$-algebra

Let $V$ be a projective scheme over a field $k$ and let $\mathcal{L}$ be an invertible sheaf of linear forms on $V$. If $\sigma$ is an automorphism of $V$, then the pullback of $\mathcal{L}$ along $\sigma$ will be denoted by $\mathcal{L}^\sigma$, i.e. $\mathcal{L}^\sigma(U) := \mathcal{L}(\sigma U)$ for every $U \subset V$. Consider the graded $k$-algebra

$$B(V, \mathcal{L}, \sigma) = \bigoplus_{i \geq 0} H^0(V, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{i-1}})$$

(2.1)

called a twisted homogeneous coordinate ring of $V$; notice that such a ring is non-commutative unless $\sigma$ is the trivial automorphism. Recall that multiplication of sections of $B(V, \mathcal{L}, \sigma)$ is defined by the rule $ab = a \otimes b^{\sigma^m}$, where $a \in B_m$ and $b \in B_n$. Given a pair $(V, \sigma)$ consisting of a Noetherian scheme $V$ and an automorphism $\sigma$ of $V$, an invertible sheaf $\mathcal{L}$ on $V$ is called $\sigma$-ample, if for every coherent sheaf $\mathcal{F}$ on $V$, the cohomology group $H^q(V, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} \otimes \mathcal{F})$ vanishes for $q > 0$ and $n >> 0$. Notice, that if $\sigma$ is trivial, this definition is equivalent to the usual definition of ample invertible sheaf [Serre 1955] [16]. If $\mathcal{L}$ is a $\sigma$-ample invertible sheaf on $V$, then

$$\text{Mod } (B(V, \mathcal{L}, \sigma)) / \text{Tors} \cong \text{Coh } (V),$$

(2.2)

where $\text{Mod}$ is the category of graded left modules over the ring $B(V, \mathcal{L}, \sigma)$, $\text{Tors}$ is the full subcategory of $\text{Mod}$ of the torsion modules and $\text{Coh}$ is the category of quasi-coherent sheaves on a scheme $V$, see [M. Artin & van den Bergh 1990] [1]. In view of (2.2) the ring $B(V, \mathcal{L}, \sigma)$ is indeed a coordinate ring of $V$, see [Serre 1955] [16].
Remark 2.1. Suppose that $R$ is a commutative graded ring, such that $V = Spec (R)$ is a projective variety. Denote by $R[t, t^{-1}; \sigma]$ the ring of skew Laurent polynomials defined by the commutation relation $b^\sigma t = tb$ for all $b \in R$, where $b^\sigma$ is the image of $b$ under automorphism $\sigma : V \to V$. Then $R[t, t^{-1}; \sigma] \cong B(V, L, \sigma)$, see [M. Artin & van den Bergh 1990] [11].

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For a ring of skew Laurent polynomials $R[t, t^{-1}; \sigma]$, we shall consider a homomorphism

$$\rho : R[t, t^{-1}; \sigma] \to \mathcal{B}(\mathcal{H}).$$

(2.3)

Recall that the algebra $\mathcal{B}(\mathcal{H})$ is endowed with a $*$-involution coming from the scalar product on the Hilbert space $\mathcal{H}$. We shall call representation (2.3) $*$-coherent if (i) $\rho(t)$ and $\rho(t^{-1})$ are unitary operators, such that $\rho^*(t) = \rho(t^{-1})$ and (ii) for all $b \in R$ it holds $(\rho^*(b))^{\sigma(\rho)} = \rho^*(b^\sigma)$, where $\sigma(\rho)$ is an automorphism of $\rho(R)$ induced by $\sigma$. Whenever $B := R[t, t^{-1}; \sigma]$ admits a $*$-coherent representation, $\rho(B)$ is a $*$-algebra; the norm-closure of $\rho(B)$ yields a $C^*$-algebra, see e.g. [Murphy 1990] [11], Section 2.1.

Definition 2.2. By a Serre $C^*$-algebra of $V$ we understand the norm-closure of $\rho(B)$; such a $C^*$-algebra will be denoted by $\mathcal{A}_V$.

Remark 2.3. Each Serre $C^*$-algebra $\mathcal{A}_V$ is a crossed product $C^*$-algebra, see e.g. [Williams 2007] [20], pp 47-54 for the definition and details; namely, $\mathcal{A}_V \cong C(V) \times_\sigma \mathbb{Z}$, where $C(V)$ is the $C^*$-algebra of all continuous complex-valued functions on $V$ and $\sigma$ is a $*$-coherent automorphism of $V$.

Remark 2.4. Let $\mathcal{K}$ be the $C^*$-algebra of all compact operators on a Hilbert space $\mathcal{H}$. The stable Serre $C^*$-algebra $\mathcal{A}_V \otimes \mathcal{K}$ is endowed with the unique normalized trace (tracial state) $\tau : \mathcal{A}_V \otimes \mathcal{K} \to \mathbb{R}$, i.e. a positive linear functional of norm 1 such that $\tau(yx) = \tau(xy)$ for all $x, y \in \mathcal{A}_V \otimes \mathcal{K}$, see [Blackadar 1986] [2], p. 31.

2.2. Trace cohomology

Let $k$ be a number field. Let $V(k)$ be a smooth $n$-dimensional projective variety over $k$, such that variety $V := V(\mathbb{F}_q)$ is the reduction modulo $q$ of $V(k)$ for a fixed choice of integral model [Hartshorne 2010, Theorem 22.1] [8]. In other words, $V(k)$ is defined by polynomial equations for $V$ over the field of complex numbers. Because the Serre $C^*$-algebra $\mathcal{A}_V$ of $V(k)$ is a crossed product $C^*$-algebra of the form $\mathcal{A}_V \cong C(V(k)) \times \mathbb{Z}$ (Remark 2.3), one can use the Pimsner-Voiculescu six term exact sequence for the crossed products, see e.g. [Blackadar 1986] [2], p. 83 for the details. Thus one gets the short exact sequence of the algebraic $K$-groups: $0 \to K_0(C(V(k))) \xrightarrow{i} K_0(\mathcal{A}_V) \to K_1(C(V(k))) \to 0$, where map $i_*$ is induced by the natural embedding of $C(V(k))$ into $\mathcal{A}_V$. We have $K_0(C(V(k))) \cong K^0(V(k))$ and $K_1(C(V(k))) \cong K^{-1}(V(k))$, where $K^0$ and $K^{-1}$ are the topological $K$-groups of the variety $V(k)$, see [Blackadar 1986] [2], p. 80. By the Chern character formula, one gets $K^0(V(k)) \otimes \mathbb{Q} \cong H^{even}(V(k); \mathbb{Q})$ and $K^{-1}(V(k)) \otimes \mathbb{Q} \cong H^{odd}(V(k); \mathbb{Q})$. 


where $H_{\text{even}}$ ($H_{\text{odd}}$) is the direct sum of even (odd, resp.) cohomology groups of $V(k)$. Notice that $K_0(A_V \otimes \mathcal{K}) \cong K_0(A_V)$ because of stability of the $K_0$-group with respect to tensor products by the algebra $\mathcal{K}$, see e.g. [Blackadar 1986] [2], p. 32. One gets the commutative diagram in Figure 1, where $\tau_*$ denotes a homomorphism induced on $K_0$ by the canonical trace $\tau$ on the $C^*$-algebra $A_V \otimes \mathcal{K}$. Recall that $H_{\text{even}}(V(k)) := \bigoplus_{i=0}^{n} H^{2i}(V(k))$ and $H_{\text{odd}}(V) := \bigoplus_{i=1}^{n} H^{2i-1}(V(k))$, where $H^*(V(k))$ is the singular cohomology of $V(k)$. One gets for each $0 \leq i \leq 2n$ an injective homomorphism

$$\tau_* : H^i(V(k)) \longrightarrow \mathbb{R}. \quad (2.4)$$

**Definition 2.5.** By an $i$-th trace cohomology group $H^i_{\text{tr}}(V)$ of $V$ one understands the abelian subgroup of $\mathbb{R}$ defined by map (2.4).

**Remark 2.6.** The abelian group $H^i_{\text{tr}}(V)$ is called a pseudo-lattice, see [Manin 2004] [9], Section 1. The endomorphisms in the category of pseudo-lattices are given by multiplication of its points by the real numbers $\alpha$ such that $\alpha H^i_{\text{tr}}(V) \subseteq H^i_{\text{tr}}(V)$. It is known that the ring $\text{End} (H^i_{\text{tr}}(V)) \cong \mathbb{Z}$ or $\text{End} (H^i_{\text{tr}}(V)) \otimes \mathbb{Q}$ is a real algebraic number field. In the latter case $H^i_{\text{tr}}(V) \subset \text{End} (H^i_{\text{tr}}(V)) \otimes \mathbb{Q}$, see [Manin 2004] [9], Lemma 1.1.1 for the case of quadratic fields. Notice that one can write multiplication by $\alpha$ in a matrix form by fixing a basis in the pseudo-lattice; thus the ring $\text{End} (H^i_{\text{tr}}(V))$ is a commutative subring of the matrix ring $M_{b_i}(\mathbb{Z})$, where $b_i$ is equal to the rank of pseudo-lattice, i.e. the cardinality of its basis.

**Remark 2.7.** Notice that the trace cohomology $H^i_{\text{tr}}(V)$ is an abelian group with order, see [Goodearl 1986] [5] for an introduction. The total order is defined by an order-preserving homomorphism $H^i_{\text{tr}}(V) \rightarrow \mathbb{R}$ given by formula (2.4).

2.3. Weil’s Conjectures

Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements and $V := V(\mathbb{F}_q)$ be a smooth $n$-dimensional projective variety over $\mathbb{F}_q$. The famous Weil conjectures establish a deep relation between the arithmetic of $V$ and topology of the variety $V_{\mathbb{C}}$ defined by the polynomial equations over the field of complex numbers [Weil
Namely, let $N_m$ be the number of rational points of $V$ over the field $\mathbb{F}_{q^m}$ and

$$Z_V(t) = \exp \left( \sum_{m=1}^{\infty} \frac{N_m t^m}{m} \right)$$  \hspace{1cm} (2.5)

the corresponding zeta function. Weil conjectured that: (i) $Z_V(t)$ is a quotient of polynomials with rational coefficients; (ii) $Z_V(q^{-n}t^{-1}) = \pm q^n \frac{\chi}{t^\chi} Z_V(t)$, where $\chi$ is the Euler-Poincaré characteristic of $V_C$; (iii) $Z_V(t)$ satisfies an analog of the Riemann Hypothesis, i.e.

$$Z_V(t) = \prod_{1 \leq i \leq 2n-1} P_i(t)^{2n-i} P_0(t),$$  \hspace{1cm} (2.6)

so that $P_0(t) = 1 - t^r$, $P_{2n}(t) = 1 - q^nt$ and for each $1 \leq i \leq 2n-1$ the polynomial $P_i(t)$ has integer coefficients and can be written in the form $P_i(t) = \prod (1 - \alpha_{ij} t)$, where $|\alpha_{ij}| = q^{\frac{i}{2}}$; (iv) the degree of polynomial $P_i(t)$ is equal to the $i$-th Betti number of variety $V_C$. The properties (i)-(iv) are true for algebraic curves (i.e. for $n = 1$) and Weil pointed out that they follow from a cohomology theory of the variety $V$. Such a cohomology was constructed by Grothendieck and called the $\ell$-adic cohomology; all but conjecture (iii) can be deduced from basic properties of the $\ell$-adic cohomology [Grothendieck 1968] [6].

### 2.4. Kuga-Sato varieties

Let $\Gamma(N)$ be the principal congruence subgroup of level $N \geq 3$. Denote by $X(N) = \mathbb{H}/\Gamma(N)$ the corresponding modular curve, where $\mathbb{H} := \{ z = x + iy \in \mathbb{C} \mid y > 0 \}$ is the Lobachevsky half-plane. The Kuga-Sato variety of level $N$ is the $k$-th power of the universal elliptic curve $\mathcal{E}$ over the modular curve, i.e.

$$V_N = \underbrace{\mathcal{E} \times_{X(N)} \cdots \times_{X(N)}}_{k \text{ times}} \mathcal{E}.$$  \hspace{1cm} (2.7)

In what follows, we assume that the variety $V_N$ is compact. Such a compactification is described in [Deligne 1969] [3, Lemma 5.4]. It is known, that

$$H_{\text{et}}^i(V_N; \mathbb{Q}_\ell) \cong S_{i+1}(\Gamma(N)),$$  \hspace{1cm} (2.8)

where $H_{\text{et}}^i(V_N; \mathbb{Q}_\ell)$ is the $\ell$-adic cohomology of $V_N$ and $S_{i+1}(\Gamma(N))$ is the space of cusp forms for the group $\Gamma(N)$ [Deligne 1969] [3, Definition 2.8 and Theorem 2.10] and [Scholl 1985] [15, Section 2.5].

### 3. Proof of theorem [1.1]

For the sake of clarity, let us outline main ideas. The trace cohomology $H^i_{tr}(V)$ will be used to construct a positive-definite Hermitian form $\varphi(x, y)$ on the cohomology group $H^i(V(k))$; see also remark [2.7]. Such a construction involves the Deligne-Scholl theory linking the $\ell$-adic cohomology $H_{\text{et}}^i(V; \mathbb{Q}_\ell)$ of the Kuga-Sato variety $V$ with the space of cusp forms $S_{i+1}(\Gamma)$ of weight $i + 1$ for a finite index subgroup $\Gamma \subset SL_2(\mathbb{Z})$, see [Deligne 1969] [3] and [Scholl...
1985] [15]. It is proved that the Petersson inner product on \( S_{i+1} \) defines, via the trace cohomology, the required form \( \varphi(x, y) \). Since the regular maps of \( V \) preserve the form \( \varphi(x, y) \) modulo a positive constant, one obtains an analog of the Riemann hypothesis for the zeta function of \( V \). We shall split the proof in a series of lemmas.

**Lemma 3.1. (Deligne-Scholl)** If \( V \) is the Kuga-Sato variety, then there exists a finite index subgroup \( \Gamma \) of the modular group \( SL_2(\mathbb{Z}) \), such that

\[
\dim H^i_{tr}(V) = 2 \dim_{\mathbb{C}} S_{i+1}(\Gamma),
\]

where \( S_{i+1}(\Gamma) \) is the space of cusp forms of weight \( i + 1 \) relatively group \( \Gamma \).

**Proof.** This lemma follows from the results of [Deligne 1969] [3, Definition 2.8 and Theorem 2.10] and [Scholl 1985] [15, Section 2.5]. Namely, let \( \Gamma \) be a finite index subgroup of \( SL_2(\mathbb{Z}) \), such that the modular curve \( X_\Gamma := \mathbb{H}/\Gamma \) can be defined over the field \( \mathbb{Q} \). It was proved that for each prime \( \ell \) there exists a continuous homomorphism

\[
\rho : Gal (\overline{\mathbb{Q}} | \mathbb{Q}) \rightarrow \text{End} (W),
\]

where \( W \) is a 2\( d \)-dimensional vector space over \( \ell \)-adic numbers \( \mathbb{Q}_\ell \) and \( d = \dim_{\mathbb{C}} S_{i+1}(\Gamma) \), see [Scholl 1985] [15]. It was proved earlier, that \( W \cong H^i_{et}(V; \mathbb{Q}_\ell) \) for a variety \( V \) over the field \( \mathbb{Q} \) and the (arithmetic) Frobenius element of the Galois group \( Gal (\overline{\mathbb{Q}} | \mathbb{Q}) \) corresponds to the (geometric) Frobenius endomorphism of the \( \ell \)-adic cohomology \( H^i_{et}(V; \mathbb{Q}_\ell) \), see [Deligne 1969] [3] for \( \Gamma \) being a congruence group.

Let \( V(k) \) be a variety over the complex numbers associated to \( V \). By the comparison theorem

\[
H^i_{et}(V; \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong H^i(V(k); \mathbb{C}),
\]

see e.g. [Hartshorne 1977] [7, p. 454]. On the other hand, from definition 2.5 we have \( \dim H^i_{tr}(V) = \dim H^i(V(k); \mathbb{C}) \) and, therefore,

\[
\dim H^i_{tr}(V) = \dim H^i_{et}(V).
\]

By \( \dim H^i_{et}(V; \mathbb{Q}_\ell) = 2 \dim_{\mathbb{C}} S_{i+1}(\Gamma) \) by the Deligne-Scholl theory and, therefore,

\[
\dim H^i_{tr}(V) = 2 \dim_{\mathbb{C}} S_{i+1}(\Gamma).
\]

Lemma 3.1 follows. \( \square \)

**Lemma 3.2.** The trace cohomology \( H^i_{tr}(V) \) defines a \( \mathbb{Z} \)-module embedding

\[
H^i(V(k)) \hookrightarrow S_{i+1}(\Gamma).
\]

**Proof.** Recall that the Petersson inner product

\[
(x, y) : S_{i+1}(\Gamma) \times S_{i+1}(\Gamma) \rightarrow \mathbb{C}
\]

on the space \( S_{i+1}(\Gamma) \) is given by the integral

\[
(f, g) = \int_{X_\Gamma} f(z) \overline{g(z)} (\Im z)^{i+1} dz.
\]
The product is linear in $f$ and conjugate-linear in $g$, so that $(g, f) = \overline{(f, g)}$ and $(f, f) > 0$ for all $f \neq 0$, see e.g. [Milne 1997] [10, p. 57].

Fix a basis $\{\alpha_1, \ldots, \alpha_d; \beta_1, \ldots, \beta_d\}$ in the $\mathbb{Z}$-module $H^i_{tr}(V)$. In view of the standard properties of scalar product $(x, y)$, there exists a unique cusp form $g \in S_{i+1}(\Gamma)$, such that

$$(f_j, g) = \alpha_j + i\beta_j,$$  \hspace{1cm} (3.9)

where $\{f_1, \ldots, f_d\}$ is the orthonormal basis in $S_{i+1}(\Gamma)$ consisting of the Hecke eigenforms. But $\alpha_j$ and $\beta_j$ are the image of generators of the $\mathbb{Z}$-module $H^i(V(k))$ under the trace map $\tau_*$, see definition [2.5]. Using formula (3.9), one defines an embedding

$$\iota: H^i(V(k)) \hookrightarrow S_{i+1}(\Gamma),$$  \hspace{1cm} (3.10)

whose image $\iota(H^i(V(k)))$ is a $\mathbb{Z}$-module generated by the real and imaginary parts of the Hecke eigenforms $f_j \in S_{i+1}(\Gamma)$. Lemma 3.3 follows.

**Corollary 3.3.** There exists a unique positive-definite Hermitian form

$$\varphi(x, y): H^i(V(k)) \times H^i(V(k)) \to \mathbb{C},$$  \hspace{1cm} (3.11)

on the $\mathbb{Z}$-module $H^i(V(k))$ coming from the Petersson inner product on the space $S_{i+1}(\Gamma)$.

**Proof.** The Petersson inner product is a Hermitian form because $(g, f) = \overline{(f, g)}$ and a positive-definite form because $(f, f) > 0$ for all $f \neq 0$. It is easy to see, that such a form is unique. In view of lemma 3.2 one gets the conclusion of corollary 3.3.

**Lemma 3.4.** The ring $\text{End} (H^i_{tr}(V))$ is isomorphic to the ring $\mathbb{T}_{i+1}(\Gamma)$ of the Hecke operators on the space $S_{i+1}(\Gamma)$ and it is a commutative subring of the matrix ring $\text{End} (H^i(V(k)))$.

**Proof.** In view of lemma 3.1 and remark 2.6, the ring $\text{End} (H^i_{tr}(V))$ is generated by the eigenvalues of Hecke operators corresponding to their (common) eigenform $f_j \in S_{i+1}(\Gamma)$; notice that $\mathbb{T}_{i+1}(\Gamma)$ is always a non-trivial ring if $\Gamma$ is a congruence subgroup and extends to such for the non-congruence subgroups of finite index as shown in [Scholl 1985] [15]. On the other hand, it is known that the Hecke ring $\mathbb{T}_{i+1}(\Gamma)$ is isomorphic to a commutative subring of the matrix ring $\text{End} (H^i(V(k)))$ represented by the symmetric matrices with positive integer entries, see e.g. [Milne 1997] [10]. Lemma 3.4 follows.

**Lemma 3.5.** Each regular map $f : V \to V$ induces a linear map $f^*_i : H^i(V(k)) \to H^i(V(k))$ of degree $\deg(f^*_i)$, whose characteristic polynomial $\text{char}(f^*_i)$ has integer coefficients and roots of the absolute value $|\lambda| = [\deg(f^*_i)]^{1/n}$.

**Proof.** Consider a regular map $f : V \to V$ obtained by the reduction modulo $q$ of an algebraic map $\tilde{f} : V(k) \to V(k)$ of the corresponding variety over the field of complex numbers. Such a map always exists, see [Hartshorne 2010] [8, Theorem 22.1]. Let us show that the linear map $f^*_i : H^i(V(k)) \to H^i(V(k))$ induced by $\tilde{f}$ on the integral cohomology $H^i(V(k))$ must preserve, up to a
constant multiple, the positive-definite Hermitian form $\varphi(x,y)$ on $H^i(V(k))$ given by formula (3.11).

Indeed, let $\tilde{f}(V(k)) \subseteq V(k)$ be a constructible subset; clearly, such a subset carries the structure of an algebraic variety. We repeat the trace cohomology construction for the variety $\tilde{f}(V(k))$; thus one gets a positive-definite Hermitian form $\bar{\varphi}(x,y)$ on $H^i(\tilde{f}(V(k)))$. But $H^i(\tilde{f}(V(k))) \subseteq H^i(V(k))$ and therefore one gets yet another positive-definite Hermitian form $\varphi(x,y)$ on $H^i(\tilde{f}(V(k)))$. Since such a form is unique (see lemma 3.2), one concludes that $\bar{\varphi}(x,y)$ coincides with $\varphi(x,y)$ modulo a positive factor $C$. It is easy to see, that $C = [\deg(f_{*})]^{1/n}$. Indeed, the volume form can be calculated by the formula $v = |\det(f_{*})|v_0 = \deg(f_{*})v_0$; on the other hand, the multiplication by $C$ map gives the volume $v = C^nv_0$, where $n$ is the dimension of variety $V$.

Let $\lambda$ be a root of the characteristic polynomial $\text{char}(f_{*}) := \det(\lambda I - f_{*})$. Since the kernel of the map $\lambda I - f_{*}$ is non-trivial, let $x \in \text{Ker}(\lambda I - f_{*})$ be a non-zero element; clearly, $f_{*}^i x = \lambda x$. Consider the value of scalar product $(x,y) = \varphi(x,y)$ on $x = y = f_{*}^i x$, i.e.

\[(f_{*}^i x, f_{*}^i x) = (\lambda x, \lambda x) = \lambda \overline{\lambda}(x,x). \tag{3.12}\]

On the other hand,

\[(f_{*}^i x, f_{*}^i x) = [\deg(f_{*})]^{1/n}(x,x). \tag{3.13}\]

Because $(x,x) \neq 0$, one can cancel it in (3.12) and (3.13), so that

\[\lambda \overline{\lambda} = [\deg(f_{*})]^{1/n} \text{ or } |\lambda| = [\deg(f_{*})]^{1/n}. \tag{3.14}\]

Note that $\text{char}(f_{*}) \in \mathbb{Z}[\lambda]$ because $H^i(V(k))$ is a $\mathbb{Z}$-module; lemma 3.5 follows. \hfill $\square$

Remark 3.6. Note that any non-trivial map $f_{*}^i \in \text{End} (H^i_{tr}(V)) \subseteq \text{End} (H^i(V(k)))$ corresponds to a non-algebraic (transcendental) map $\tilde{f} : V(k) \to V(k)$, because the roots of $\text{char}(f_{*})$ are real numbers in this case. Of course, there are many other examples of the non-algebraic maps $\tilde{f} : V(k) \to V(k)$.

Lemma 3.7. $\deg(f_{*}) = [\deg(f)]^i$.

Proof. It is well known, that the cusp forms $g(z) \in S_{i+1}(\Gamma)$ are bijective with the holomorphic differentials

\[g(z)dz^{i+1} \tag{3.15}\]

on the Riemann surface $X_{\Gamma} = \mathbb{H}/\Gamma$. To prove lemma 3.7 one can use the Riemann-Hurwitz formula:

\[2g(Y) - 2 = m [2g(X) - 2] + \sum_P (e_P - 1), \tag{3.16}\]

where $e_P$ is the multiplicity at the point $P$ of an $m$-fold holomorphic map $Y \to X$ between the Riemann surfaces of genus $g(Y)$ and $g(X)$, see e.g. [Milne 1997] [10, p. 17]. Because the differential (3.15) is locally defined, one
can substitute in (3.16) \( g(X) = g(Y) = 0 \) and assume \( P = 0 \) to be a unique ramification point. Thus
\[
m = \frac{e_P + 1}{2}
\] (3.17)
and the \( m \)-fold differential (3.15) implies \( e_P = i \), i.e. the holomorphic map \( Y \to X \) is given by the formula
\[
z \mapsto z^i.
\] (3.18)
On the other hand, for a regular map \( f : V \to V \) it holds \( \deg(f) = \deg(\tilde{f}) = \deg(f_*) \). Since degree is a multiplicative function on composition of maps, one gets from (3.18) and the link between \( i \)-th cohomology of \( V(k) \) and the space \( S_{i+1}(\Gamma) \), that
\[
\deg(f_*)^i = [\deg(f_*)]^i = [\deg(f)]^i.
\] (3.19)

Lemma 3.7 follows.

Corollary 3.8. (Riemann Hypothesis) The roots \( \alpha_{ij} \) of polynomials \( P_i(t) \) in formula (2.6) are algebraic numbers of the absolute value \( |\alpha_{ij}| = q^{\frac{i}{2}} \).

Proof. It is easy to see, that the Frobenius map \( f : (z_1, \ldots, z_n) \mapsto (z_1^q, \ldots, z_n^q) \) of variety \( V \) is regular and \( \deg(f) = q^n \). Therefore, one can apply lemmas 3.5 and 3.7 to such a map and get the equality \( |\alpha_{ij}| = q^{\frac{i}{2}} \) for each \( 0 \leq i \leq 2n - 1 \). Corollary 3.8 follows.

Corollary 3.8 finishes the proof of theorem 1.1.

4. Examples

The groups \( H^i_{tr}(V) \) are truly concrete and simple; in this section we calculate the trace cohomology for \( n = 1 \), i.e. when \( V \) is a smooth algebraic curve. In particular, we find the cardinality of the set \( \mathcal{E}(\mathbb{F}_q) \) obtained by the reduction modulo \( q \) of an elliptic curve with complex multiplication.

Example. The trace cohomology of smooth algebraic curve \( C(\mathbb{F}_q) \) of genus \( g \geq 1 \) is given by the formulas:
\[
\begin{align*}
H^0_{tr}(C) & \cong \mathbb{Z}, \\
H^1_{tr}(C) & \cong \mathbb{Z} + \mathbb{Z}\theta_1 + \cdots + \mathbb{Z}\theta_{2g-1}, \\
H^2_{tr}(C) & \cong \mathbb{Z},
\end{align*}
\] (4.1)
where \( \theta_i \in \mathbb{R} \) are algebraically independent integers of a number field of degree \( 2g \).
Proof. It is known that the Serre $C^*$-algebra of the (generic) complex algebraic curve $C$ is isomorphic to a toric $AF$-algebra $A_\theta$, see [12] for the notation and details. Moreover, up to a scaling constant $\mu > 0$, it holds

$$\tau_*(K_0(A_\theta \otimes \mathcal{K})) = \begin{cases} \mathbb{Z} + \mathbb{Z} \theta_1, & \text{if } g = 1 \\ \mathbb{Z} + \mathbb{Z} \theta_1 + \cdots + \mathbb{Z} \theta_{6g-7}, & \text{if } g > 1, \end{cases}$$

(4.2)

where constants $\theta_i \in \mathbb{R}$ parametrize the moduli (Teichmüller) space of curves $C$ [12]. If $C$ is defined over a number field $k$, then each $\theta_i$ is algebraic and their total number is equal to $2g - 1$. (Indeed, since $\text{Gal} (\bar{k} \mid k)$ acts on the torsion points of $C(k)$, it is easy to see that the endomorphism ring of $C(k)$ is non-trivial. Because such a ring is isomorphic to the endomorphism ring of the Jacobian $\text{Jac} C$ and $\dim C \text{Jac} C = g$, one concludes that $\text{End} C(k)$ is a $\mathbb{Z}$-module of rank $2g$ and each $\theta_i$ is an algebraic number.) After scaling by a constant $\mu > 0$, one gets

$$H^1_{tr}(C) := \tau_*(K_0(A_\theta \otimes \mathcal{K})) = \mathbb{Z} + \mathbb{Z} \theta_1 + \cdots + \mathbb{Z} \theta_{2g-1}$$

(4.3)

Because $H^0(C) \cong H^2(C) \cong \mathbb{Z}$, one obtains the rest of formulas (4.1). □

Remark 4.1. If $k \cong \mathbb{Q}$, then $\Gamma \cong \Gamma(N)$ is the principal congruence subgroup of level $N$, since $C(\mathbb{Q}) \cong X_{\Gamma(N)}$ for some integer $N$. As explained, the Petersson inner product on $S_2(\Gamma(N))$ gives rise to a positive-definite Hermitian form $\varphi$ on the cohomology group $H^1(C) \cong \mathbb{Z}^{2g}$. Note that the form $\varphi$ can be obtained from the classical Riemann’s bilinear relations for the periods of curve $C$; this yields Weil’s proof of the Riemann hypothesis for function $Z_C(t)$.

Remark 4.2. Notice the cardinality of the set $C(\mathbb{F}_q)$ is given to the formula

$$|C(\mathbb{F}_q)| = 1 + q - tr (\omega) = 1 + q - \sum_{i=1}^{2g} \lambda_i,$$

(4.4)

where $\lambda_i$ are the eigenvalues of the Frobenius endomorphism $\omega \in \text{End} (H^1_{tr}(C))$.

Example. The case $g = 1$ is particularly instructive; for the sake of clarity, we shall consider elliptic curves having complex multiplication. Let $\mathcal{E}(\mathbb{F}_q)$ be the reduction modulo $q$ of an elliptic with complex multiplication by the ring of integers of an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, see e.g. [Silverman 1994] [17], Chapter 2. It is known, that in this case the trace cohomology formulas (4.1) take the form

$$\begin{cases}
H^0_{tr}(\mathcal{E}(\mathbb{F}_q)) \cong \mathbb{Z}, \\
H^1_{tr}(\mathcal{E}(\mathbb{F}_q)) \cong \mathbb{Z} + \mathbb{Z} \sqrt{-d}, \\
H^2_{tr}(\mathcal{E}(\mathbb{F}_q)) \cong \mathbb{Z}.
\end{cases}$$

(4.5)

We shall denote by $\psi(\mathfrak{P}) \in \mathbb{Q}(\sqrt{-d})$ the Grössencharacter of the prime ideal $\mathfrak{P}$ over $p$, see [Silverman 1994] [17], p. 174. It is easy to see, that in this case
the Frobenius endomorphism $\omega \in \text{End}(H^1_{tr}(E(F_q)))$ is given by the formula
\[
\omega = \frac{1}{2} \left[ \psi(P) + \overline{\psi(P)} \right] + \frac{1}{2} \sqrt{\left( \psi(P) + \overline{\psi(P)} \right)^2 + 4q}
\tag{4.6}
\]
and the corresponding eigenvalues
\[
\begin{cases}
\lambda_1 &= \omega = \frac{1}{2} \left[ \psi(P) + \overline{\psi(P)} \right] + \frac{1}{2} \sqrt{\left( \psi(P) + \overline{\psi(P)} \right)^2 + 4q}, \\
\lambda_2 &= \bar{\omega} = \frac{1}{2} \left[ \psi(P) + \overline{\psi(P)} \right] - \frac{1}{2} \sqrt{\left( \psi(P) + \overline{\psi(P)} \right)^2 + 4q}.
\end{cases}
\tag{4.7}
\]
Using formula (4.4), one gets the following equation
\[
|E(F_q)| = 1 - (\lambda_1 + \lambda_2) + q = 1 - \psi(P) - \overline{\psi(P)} + q,
\tag{4.8}
\]
which coincides with the well-known expression for $|E(F_q)|$ in terms of the Grössencharacter, see e.g. [Silverman 1994] [17, p. 175].

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References

[1] M. Artin and M. van den Bergh, Twisted homogeneous coordinate rings, J. of Algebra 133 (1990), 249-271.
[2] B. Blackadar, $K$-Theory for Operator Algebras, MSRI Publications, Springer, 1986.
[3] P. Deligne, Formes modulaires et représentations $\ell$-adiques, Sem. Bourbaki, exp 355. Lecture Notes in Mathematics 179, pp. 139-172, Springer 1969.
[4] P. Deligne, La conjecture de Weil. I, Publications Mathématiques de l’IHÉS 43 (1974), 273-307.
[5] K. R. Goodearl, Partially Ordered Abelian Groups with Interpolation, Math. Surveys and Monographs 20, AMS, Providence, 1986.
[6] A. Grothendieck, Standard conjectures on algebraic cycles, in: Algebraic Geometry, Internat. Colloq. Tata Inst. Fund. Res., Bombay, 1968, pp. 193-199.
[7] R. Hartshorne, Algebraic Geometry, GTM 52, Springer, 1977.
[8] R. Hartshorne, Deformation Theory, GTM 257, Springer, 2010.
[9] Yu. I. Manin, Real multiplication and noncommutative geometry, in “Legacy of Niels Hendrik Abel”, 685-727, Springer, 2004.
[10] J. S. Milne, Modular Functions and Modular Forms, Lecture Notes, Univ. of Michigan, 1997.
[11] G. J. Murphy, C$^*$-Algebras and Operator Theory, Academic Press, 1990.
[12] I. Nikolaev, Noncommutative geometry of algebraic curves, Proc. Amer. Math. Soc. 137 (2009), 3283-3290.
[13] I. Nikolaev, On traces of Frobenius endomorphisms, Finite Fields and their Applications 25 (2014), 270-279.
I. Nikolaev, *Remark on Weil’s conjectures*, Adv. Pure Appl. Math. 7 (2016), 213-221.

A. J. Scholl, *Modular forms and de Rham cohomology; Atkin-Swinnerton-Dyer congruences*, Invent. Math. 79 (1985), 49-77.

J. P. Serre, *Fasceaux algébriques cohérents*, Ann. of Math. 61 (1955), 197-278.

J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, GTM 151, Springer 1994.

J. T. Stafford and M. van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc. 38 (2001), 171-216.

A. Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. 55 (1949), 497-508.

D. P. Williams, *Crossed Products of C*-Algebras*, Math. Surveys and Monographs, Vol. 134, Amer. Math. Soc. 2007.

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