Pedestrian Solution of the Two-Dimensional Ising Model

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The partition function of the two-dimensional Ising model with zero magnetic field on a square lattice with \( m \times n \) sites wrapped on a torus is computed within the transfer matrix formalism in an explicit step-by-step approach inspired by Kaufman’s work. However, working with two commuting representations of the complex rotation group \( \text{SO}(2n, \mathbb{C}) \) helps us avoid a number of unnecessary complications. We find all eigenvalues of the transfer matrix and therefore the partition function in a straightforward way.

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I. INTRODUCTION

Since Onsager’s solution [1] in the transfer matrix approach [2] of the two-dimensional Ising model [3] with vanishing magnetic field and its subsequent simplification by Kaufman [4], there have been a number of related as well as alternative solutions, see e.g. Baxter’s book on exactly solved models in statistical mechanics [5] and references therein. Among the transfer matrix solutions are the ones by Schultz, Mattis and Lieb, by Thompson, by Kaufman [4], there have been a number of related approaches [2] of the two-dimensional Ising model [3] with complications. We find all eigenvalues of the transfer matrix and therefore the partition function in a straightforward way.

We work with a square lattice with \( m \times n \) rows and \( n \) columns subject to toroidal boundary conditions. The transfer matrix is expressed in terms of the generators of two commuting representations of the complex rotation group \( \text{SO}(2n, \mathbb{C}) \). These representations naturally arise from projected bilinears of \( 2^n \times 2^n \) spin matrices. Conservatively speaking, we reduce Kaufman’s approach to its essential steps, avoiding in particular the doubling of the number of eigenvalues of the transfer matrix and subsequent rather involved arguments for the choice of the correct ones. We also need not investigate the transformation properties of the spin matrices. In our opinion, the value of this work is not that it gives the most concise solution, but that it provides a clear line of arguments and requires only rather basic tools from mathematics and mathematical physics.

We proceed in a step-by-step fashion, exhibiting also those steps that closely parallel the corresponding steps in Kaufman’s treatment [4] or Huang’s corresponding textbook write up [8]. On the one hand this is necessary because of our slightly different (but, in our opinion, more natural) conventions, in particular for \( \gamma_0 \). On the other hand, we would like to keep this work as self-contained as possible.

Finally, this work may help sharpen the view on the difficulties encountered when attempting to solve the two-dimensional model with magnetic field or the three-dimensional model and might be useful for simplifying other problems in statistical mechanics for which an exact solution is attempted.

Our notation is in the spirit of [8], with sans serif capitals reserved for \( 2^n \times 2^n \) matrices. The structure of this work is as follows: In section II the model and its transfer matrix \( T \) are defined. In section III we express the transfer matrix in terms of \( 2^n \times 2^n \) spin matrices \( X, Y, Z \). A rescaled transfer matrix \( V \) is defined whose eigenvalues are, up to a trivial factor, the eigenvalues of \( T \). In section IV, we define further spin matrices \( \Gamma \) and two commuting projection classes \( J_{\alpha \beta}^+ \) and \( J_{\alpha \beta}^- \) of their bilinears. After investigating the relevant properties of the \( J_{\alpha \beta}^\pm \), we express \( V \) in terms of them. In section V, we introduce \( 2n \times 2n \) matrices \( J_{\alpha \beta} \) whose algebra is identical to that of \( J_{\alpha \beta}^\pm \) and define matrices \( V^\pm \) in terms of the \( J_{\alpha \beta} \) such that the relation between \( V^\pm \) and \( J_{\alpha \beta} \) is closely related to that between \( V \) and \( J_{\alpha \beta}^\pm \). The \( V^\pm \) are subsequently diagonalized. In section VI, the analogy between \( V^\pm \) and \( V \) is exploited for the diagonalization of \( V \). The eigenvalues of \( V \) and the partition function are found explicitly. In section VII we comment on the technical difficulties encountered when attempting to solve the two-dimensional model with magnetic field or the three-dimensional model.

II. THE MODEL AND ITS TRANSFER MATRIX \( T \)

We work with a square lattice with \( m \) rows and \( n \) columns and consequently \( m \times n \) sites. The energy is
given by
\[ E = E_a + E_b, \] (1)
\[ -\beta E_a = a \sum_{(ij)_a} s_i s_j = \sum_{\mu=1}^{n} a s_{\mu} s_{\mu+1}, \] (2)
\[ -\beta E_b = b \sum_{(ij)_b} s_i s_j = \sum_{\mu=1}^{n} b s_{\mu} s_{\mu+1}, \] (3)
with \( a = -\beta J_a \), \( b = -\beta J_b \) and \( \beta^{-1} = kT \), where \( J_a \) and \( J_b \) are temperature-independent interaction energy parameters which by convention are negative (other sign combinations do not change the partition function, as can be seen by appropriate redefinitions of half of the signs of the spin variables). Consequently, \( a \) and \( b \) are real and positive. \( (ij)_a \) and \( (ij)_b \) indicate summation over neighboring spins in the respective directions. We identify rows 1 and \( m+1 \) and columns 1 and \( n+1 \), i.e., the lattice is wrapped on a torus, see also Fig. 1. The \( s_{\mu} \) can take the values \( \pm 1 \). The partition function is then given by
\[ Z(a,b) = \sum_{s_{11}} \cdots \sum_{s_{mn}} \exp(-\beta E) \]
\[ = \sum_{s_{11}} \cdots \sum_{s_{mn}} \exp \left[ \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} (a s_{\mu} s_{\mu+1} + b s_{\mu} s_{\mu+1}) \right] \]
\[ = \sum_{s_{11}} \cdots \sum_{s_{mn}} \prod_{\mu=1}^{m} \prod_{\nu=1}^{n} \exp(as_{\mu} s_{\mu+1} + bs_{\mu} s_{\mu+1}). \] (4)
It can be expressed with the help of a \( 2^n \times 2^n \) transfer matrix \( T \),
\[ Z(a,b) = \sum_{\{\pi_i\}} \langle \pi_1 | T | \pi_{m} \rangle \langle \pi_{m-1} | T | \pi_{m-1} \rangle \cdots \langle \pi_3 | T | \pi_2 \rangle \langle \pi_2 | T | \pi_1 \rangle = \text{Tr} \ T^m, \] (5)
where \( \pi_\mu = \{s_{\mu}, \ldots, s_{\mu+n}\} \) for \( \mu = 1, \ldots, m \) and \( T \) is defined by its elements \( (s_{n+1} \equiv s_1) \),
\[ \langle \pi | T' | \pi' \rangle = \prod_{\nu=1}^{n} \exp(as_{\nu} s_{\nu}' + bs_{\nu} s_{\nu+1}). \] (6)
We can split \( T \) into a product of two matrices
\[ T = V_a V_b, \] (7)
defining \( V_a' \) and \( V_b \) by their elements
\[ \langle \pi | V_a' | \pi' \rangle = \prod_{\nu=1}^{n} \exp(as_{\nu} s_{\nu}') \] (8)
and
\[ \langle \pi | V_b | \pi' \rangle = \prod_{\nu=1}^{n} \delta s_{\nu} s_{\nu}' \exp(bs_{\nu} s_{\nu+1}). \] (9)

### III. \( T \) AND SPIN MATRICES \( X_\nu, Y_\nu, Z_\nu \)

With the help of the Pauli matrices
\[ \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] (10)
and the \( 2 \times 2 \) unit matrix \( 1 \), define Hermitian \( 2^n \times 2^n \) spin matrices by the direct products
\[ X_\nu = \underbrace{1 \otimes \cdots \otimes 1}_{\nu-1} \otimes \sigma_x \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-\nu}, \] (11)
and analogously for \( Y_\nu \) and \( Z_\nu \). For any \( \nu \) and \( \nu' \), they obey the commutation relations
\[ [X_\nu, X_{\nu'}] = [Y_\nu, Y_{\nu'}] = [Z_\nu, Z_{\nu'}] = 0, \] (12)
while only for \( \nu \neq \nu' \) holds
\[ [X_\nu, Y_{\nu'}] = [Y_\nu, Z_{\nu'}] = [Z_\nu, X_{\nu'}] = 0. \] (13)
For any \( \nu \) we have
\[ X_\nu^2 = Y_\nu^2 = Z_\nu^2 = 1, \] (14)
\[ \{X_\nu, Y_\nu\} = \{Y_\nu, Z_\nu\} = \{Z_\nu, X_\nu\} = 0, \] (15)
\[ X_\nu Y_\nu = i Z_\nu, \quad Y_\nu Z_\nu = i X_\nu, \quad Z_\nu X_\nu = i Y_\nu, \] (16)
where \( 1 \) is now the \( 2^n \times 2^n \) unit matrix.
Define \( \tilde{a} \) by
\[ \sinh(2\tilde{a}) \sinh(2a) = 1, \] (17)
so that \( \tilde{a} > 0 \),
\[ \tanh \tilde{a} = \exp(-2a), \quad \tanh a = \exp(-2\tilde{a}). \] (18)
and
\[
\begin{pmatrix}
e^{+a} & e^{-a} \\
e^{-a} & e^{+a}
\end{pmatrix} = [2 \sinh(2a)]^{1/2} \exp(a\sigma_x). \tag{19}
\]
Then we can write
\[
V'_a = [2 \sinh(2a)]^{n/2} V_a \tag{20}
\]
with
\[
V_a = \prod_{\nu=1}^{n} \exp(aX_\nu), \tag{21}
\]
and
\[
V_b = \prod_{\nu=1}^{n} \exp(bZ_\nu Z_{\nu+1}), \tag{22}
\]
where we have identified \(Z_{n+1} = Z_1\). The transfer matrix (7) may then be expressed as
\[
T = [2 \sinh(2a)]^{n/2} V_b V_a. \tag{23}
\]
Due to the cyclic property of the trace, we may rewrite the partition function (4) as
\[
Z(a, b) = [2 \sinh(2a)]^{mn/2} \text{Tr} V^m, \tag{24}
\]
where \(V\) is defined by the Hermitian matrix
\[
V = V_{a/2} V_b V_{a/2} \tag{25}
\]
with
\[
V_{a/2} = \prod_{\nu=1}^{n} \exp(aX_\nu/2), \tag{26}
\]
so that \(V_{a/2}^2 = V_a\). If \(A_k\) are the \(2^n\) eigenvalues of \(V\), we have
\[
Z(a, b) = [2 \sinh(2a)]^{mn/2} \sum_{k=1}^{2^n} A_k^m. \tag{27}
\]
Our task is therefore to find the eigenvalues of \(V\).

IV. SPIN MATRICES \(\Gamma_\nu\) AND ALGEBRAS OF THEIR PROJECTED BILINEARS \(J_{\alpha\beta}^\pm\)

Define the \(2n\) matrices \((\nu = 1, \ldots, n)\)
\[
\Gamma_{2\nu - 1} = X_1 \cdots X_{\nu - 1} Z_\nu, \tag{28}
\]
\[
\Gamma_{2\nu} = X_1 \cdots X_{\nu - 1} Y_\nu, \tag{29}
\]
which obey
\[
\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}. \tag{30}
\]
Define further the matrix
\[
U_X = X_1 \cdots X_n = i^n \Gamma_1 \Gamma_2 \cdots \Gamma_{2n}, \quad U_X^2 = 1, \tag{31}
\]
which anticommutes with every \(\Gamma_\nu\),
\[
\{\Gamma_\mu, U_X\} = 0. \tag{32}
\]
Now we can write for the matrices appearing in the exponents of (22) and (26)
\[
X_\nu = -\frac{i}{2} [\Gamma_{2\nu}, \Gamma_{2\nu - 1}], \quad \nu = 1, \ldots, n, \tag{33}
\]
\[
Z_\nu Z_{\nu + 1} = -\frac{i}{2} [\Gamma_{2\nu + 1}, \Gamma_{2\nu}], \quad \nu = 1, \ldots, n - 1, \tag{34}
\]
\[
Z_n Z_1 = \frac{i}{2} U_X [\Gamma_1, \Gamma_{2n}]. \tag{35}
\]
So far our treatment has been rather similar to Huang’s write up [8] of Kaufman’s approach [4]. Our subsequent treatment rests on the observation that the formulation (25) of \(V\) with (22) and (26) involves only the product of \(U_X\) of all \(\Gamma_\nu\) and bilinears \(\Gamma_\alpha \Gamma_\beta\), see (31), (33)-(35). This will allow us to find two commuting algebras of projected bilinears of the \(\Gamma_\nu\).

Define the projectors
\[
P^\pm = \frac{1}{2} (1 \pm U_X), \tag{36}
\]
for which hold
\[
P^+ + P^- = 1, \quad P^+ P^+ = P^+, \quad P^- P^- = P^-, \tag{37}
\]
\[
P^+ P^- = P^- P^+ = 0, \quad P^\pm U_X = U_X P^\pm = \pm P^\pm \tag{38}
\]
and
\[
[P^\pm, \Gamma_\alpha \Gamma_\beta] = 0. \tag{39}
\]
With their help, define the matrices
\[
J_{\alpha\beta} = -\frac{i}{4} [\Gamma_\alpha, \Gamma_\beta], \quad J_{\alpha\beta}^\pm = P^\pm J_{\alpha\beta}, \tag{40}
\]
so that
\[
J_{\alpha\beta} = J_{\alpha\beta}^+ + J_{\alpha\beta}^-, \quad U_X J_{\alpha\beta}^\pm = \pm J_{\alpha\beta}^\pm. \tag{41}
\]
Since
\[
J_{\alpha\beta}^\pm = -J_{\beta\alpha}^\pm, \tag{42}
\]
there are \(n(2n - 1)\) such independent matrices of each kind \(J_{\alpha\beta}^+\) and \(J_{\alpha\beta}^-\). It is straightforward to show that their algebra decomposes into two commuting parts,
\[
[J_{\alpha\beta}^+, J_{\gamma\delta}^-] = 0, \tag{43}
\]
which obey identical algebras
\[
[J_{\alpha\beta}^+, J_{\gamma\delta}^+] = i(\delta_{\alpha\gamma} J_{\beta\delta}^+ + \delta_{\beta\delta} J_{\alpha\gamma}^+ - \delta_{\alpha\delta} J_{\beta\gamma}^+ - \delta_{\beta\gamma} J_{\alpha\delta}^+). \tag{44}
\]
Next note that with (33)-(35), (39) and (40) we can write
\[
X_\nu = 2(J_{2\nu,2\nu - 1}^+ + J_{2\nu,2\nu - 1}^-), \quad \nu = 1, \ldots, n, \tag{45}
\]
\[
Z_\nu Z_{\nu + 1} = 2(J_{2\nu + 1,2\nu + 2}^+ + J_{2\nu + 1,2\nu + 2}^-), \quad \nu = 1, \ldots, n - 1, \tag{46}
\]
\[
Z_n Z_1 = -2U_X (J_{1,2n}^+ + J_{1,2n}^-) = -2(J_{1,2n}^+ - J_{1,2n}^-). \tag{47}
\]
This allows us to express \( V_{a/2} \) from (26) and \( V_b \) from (22) in terms of the \( J_{\alpha\beta}^\pm \),
\[
V_{a/2} = \prod_{\nu=1}^n \exp[\tilde{a}(J_{2\nu,2\nu-1}^+ + J_{2\nu,2\nu-1}^-)] = V_{a/2}^+ V_{a/2}^- \tag{47}
\]
with
\[
V_{a/2}^\pm = \prod_{\nu=1}^n \exp(\tilde{a}J_{2\nu,2\nu-1}^\pm), \tag{48}
\]
and
\[
V_b = \exp[-2b(J_{1,2n}^+ - J_{1,2n}^-)] \times \prod_{\nu=1}^{n-1} \exp[2b(J_{2\nu+1,2\nu}^+ + J_{2\nu+1,2\nu}^-)]
= V_b^+ V_b^- \tag{49}
\]
with
\[
V_b^\pm = \exp(\mp 2bJ_{1,2n}^\pm) \prod_{\nu=1}^{n-1} \exp(2bJ_{2\nu+1,2\nu}^\pm). \tag{50}
\]
The rescaled transfer matrix \( V \) defined in (25) reads then
\[
V = V^+ V^- \tag{51}
\]
with
\[
V^\pm = V_{a/2}^\pm V_b^\pm V_{a/2}^\pm, \quad [V^+, V^-] = 0. \tag{52}
\]

V. THE MATRICES \( V^\pm \) AND THEIR DIAGONALIZATION

In this section we formulate matrices \( V^\pm \) whose definitions are similar to the expressions of \( V^\pm \) through (48), (50) and (52). However, the \( V^\pm \) have convenient periodicity properties which allow for explicit diagonalization. Our treatment closely parallels that of Kaufman [4] as written up by Huang [8]. However, we will be more explicit and also carefully analyze some subtleties of the analysis concerning the representation of the relevant similarity transformation matrices \( S_{\pm} \) and the sign of \( \gamma_0 \), see below.

Define \( N \times N \) matrices \( J_{\alpha\beta} \) by their elements
\[
(J_{\alpha\beta})_{ij} = -i(\delta_{\alpha i}\delta_{\beta j} - \delta_{\beta i}\delta_{\alpha j}), \tag{53}
\]
where Greek and Latin indices run from 1 to \( N \). Since \( J_{\alpha\beta} = -J_{\beta\alpha} \), there are \( N(N-1)/2 \) such independent matrices. As can be easily checked, they obey the algebra
\[
[J_{\alpha\beta}, J_{\gamma\delta}] = i(\delta_{\alpha\gamma}J_{\beta\delta} + \delta_{\beta\delta}J_{\alpha\gamma} - \delta_{\beta\gamma}J_{\alpha\delta} - \delta_{\alpha\delta}J_{\beta\gamma}), \tag{54}
\]
which is, if we set \( N = 2n \), identical to the algebras (43) of \( J^+_{\alpha\beta} \) and \( J^-_{\alpha\beta} \). Now consider the matrices
\[
S = \exp(ic_{\alpha\beta}J_{\alpha\beta}), \tag{55}
\]
for which holds
\[
S^T = \exp(ic_{\alpha\beta}J_{\alpha\beta})^T = \exp(ic_{\alpha\beta}J_{\alpha\beta}^T)
= \exp(-ic_{\alpha\beta}J_{\alpha\beta}) = S^{-1}. \tag{56}
\]
Since
\[
(det S)^2 = det S \det S^T = det S \det S^{-1} = 1, \tag{57}
\]
and because \( S \) is smoothly connected to the unit matrix, we have \( det S = 1 \). For real parameters \( c_{\alpha\beta} \), the \( S \) are also real and form the group SO(\( N \)) of \( N \times N \) orthogonal matrices with unit determinant. The algebra (54) is therefore called the Lie algebra of SO(\( N \)). Here, we let the \( c_{\alpha\beta} \) be arbitrary complex numbers, so the matrices \( S \) form the group SO(\( N, C \)) of complex \( N \times N \) matrices with
\[
S^T = S^{-1}, \quad det S = 1. \tag{58}
\]

Define the SO(\( 2n, C \)) matrices
\[
V^\pm = V_{a/2} V_b^\pm V_{a/2} \tag{59}
\]
with
\[
V_{a/2} = \prod_{\nu=1}^n \exp(\tilde{a}J_{2\nu,2\nu-1}) \tag{60}
\]
and
\[
V_b^\pm = \exp(\mp 2bJ_{1,2n}) \prod_{\nu=1}^{n-1} \exp(2bJ_{2\nu+1,2\nu}), \tag{61}
\]
in analogy with \( V^\pm, V_{a/2}^\pm \) and \( V_b^\pm \) in (52), (48) and (50). Since \( \tilde{a} \) and \( b \) are real, the matrices \( V_{a/2}, V_b^\pm \) and \( V^\pm \) are not only orthogonal, but also Hermitian, so the \( V^\pm \) have only real eigenvalues and in each case a complete set of orthonormal eigenvectors. Since the algebras of the three sets of matrices \( J_{\alpha\beta}^+, J_{\beta\alpha}^+ \) and \( J_{\alpha\beta}^\pm \) are identical and because \( V^\pm \) will turn out to be explicitly diagonalizable, we will be able to find all eigenvalues of \( V \).

The structure of \( V_{a/2} \) and \( V_b^\pm \) is given by
\[
V_{a/2} = \begin{pmatrix}
R_\tilde{a} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & R_\tilde{a} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots R_\tilde{a}
\end{pmatrix} \tag{62}
\]
with
\[
R_\tilde{a} = \begin{pmatrix}
cosh \tilde{a} & i \sinh \tilde{a} \\
-i \sinh \tilde{a} & \cosh \tilde{a}
\end{pmatrix} \tag{63}
\]
and
\[
V_b^\pm = \begin{pmatrix}
\cosh(2b) & 0 & 0 & \cdots & 0 & 0 & \pm i \sinh 2b \\
0 & R_{2b} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & R_{2b} & 0 \\
\mp i \sinh 2b & 0 & 0 & \cdots & 0 & \cosh(2b) \\
\end{pmatrix}
\] (64)
with
\[
R_{2b} = \begin{pmatrix}
\cosh 2b & i \sinh 2b \\
- i \sinh 2b & \cosh 2b \\
\end{pmatrix}
\] (65)

Carrying out the matrix multiplication in (59) is straightforward and gives
\[
V^\pm = \begin{pmatrix}
A & B & 0 & \cdots & 0 & \mp B^\dagger \\
B^\dagger & A & B & \cdots & 0 & 0 \\
0 & B^\dagger & A & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & A & B \\
\mp B & 0 & 0 & \cdots & B^\dagger & A \\
\end{pmatrix}
\] (66)
with
\[
A = \cosh 2b \begin{pmatrix}
\cosh 2\tilde{a} & i \sinh 2\tilde{a} \\
- i \sinh 2\tilde{a} & \cosh 2\tilde{a} \\
\end{pmatrix}
\] (67)
and
\[
B = \sinh 2b \begin{pmatrix}
\pm \frac{1}{2} \sinh 2\tilde{a} & - i \sinh^2 \tilde{a} \\
i \cosh^2 \tilde{a} & - \frac{1}{2} \sinh 2\tilde{a} \\
\end{pmatrix}
\] (68)

Let us make the following ansatz for the normalized eigenvectors of \( V^\pm \),
\[
\psi = \frac{1}{\sqrt{n}} \begin{pmatrix}
z u \\
z^2 u \\
z^3 u \\
\vdots \\
z^n u \\
\end{pmatrix}
\] (69)
where \( z \) is a complex number and \( u \) is a two-component vector that we assume to be normalized, \( u^\dagger u = 1 \). The condition
\[
V^\pm \psi = \lambda \psi
\] (70)
leads to the \( n \) equations
\[
(zA + z^2B \mp z^n B^\dagger)u = z\lambda u,
\] (71)
\[
(z^2 A + z^3 B + z B^\dagger)u = z^2 \lambda u,
\] (72)
\[
(z^3 A + z^4 B + z^2 B^\dagger)u = z^3 \lambda u,
\] (73)
\[
(z^{n-1} A + z^n B + z^{n-2} B^\dagger)u = z^{n-1} \lambda u,
\] (74)
\[
(z^n A \mp z B + z^{n-1} B^\dagger)u = z^n \lambda u.
\] (75)

The equations with \( z^2 \) through \( z^{n-1} \) on the right hand side are identical, which leaves three independent equations
\[
(A + zB \mp z^{n-1} B^\dagger)u = \lambda u,
\] (76)
\[
(A + zB + z^{-1} B^\dagger)u = \lambda u,
\] (77)
\[
(A \mp z^{1-n}B + z^{-1} B^\dagger)u = \lambda u.
\] (78)

These equations can simultaneously be solved when
\[
z^n = \mp 1.
\] (79)
since then they become identical,
\[
(A + zB + z^{-1} B^\dagger)u = \lambda u.
\] (80)
The sign \( \mp \) in (79) is associated with \( V^\pm \). Altogether, there are \( 2n \) values of \( z \) that solve (79),
\[
z_k = e^{i\pi k/n}, \quad k = 0, \ldots, 2n-1.
\] (81)

Even \( k \) lead to \( z^n = +1 \) while odd \( k \) lead to \( z^n = -1 \), i.e.
\[
k = 1, 3, 5, \ldots, 2n - 1 \quad \text{for} \ V^+,
\] (82)
\[
k = 0, 2, 4, \ldots, 2n - 2 \quad \text{for} \ V^-.
\] (83)

For each \( k = 0, \ldots, 2n - 1 \), we still have to find the associated two eigenvalues \( \lambda^+_k \) and \( \lambda^-_k \) and their corresponding eigenvectors \( u^+_k \) and \( u^-_k \) in (80), i.e. in
\[
M_k u^+_k = \lambda^+_k u^+_k
\] (84)
with the Hermitian matrix
\[
M_k = A + e^{i\pi k/n} B + e^{-i\pi k/n} B^\dagger = \begin{pmatrix}
d_k & o_k \\
o_k^* & d_k \\
\end{pmatrix}
\] (85)
with
\[
d_k = \cosh 2\tilde{a} \cosh 2b - \cos \frac{\pi k}{n} \sinh 2\tilde{a} \sinh 2b
\] (86)
and
\[
o_k = - \sin \frac{\pi k}{n} \sinh 2b
\] (87)
\[
+ i \left( \sinh 2\tilde{a} \cosh 2b - \cos \frac{\pi k}{n} \cosh 2\tilde{a} \sinh 2b \right).
\] (87)

Explicit evaluation gives
\[
\det M_k = d_k^2 - o_k o_k^* = 1,
\] (88)

implying [up to an overall sign, fixed by tr\( M_k = \lambda^+_k + \lambda^-_k = 2d_k > 0 \), see (86) above] eigenvalues of the form
\[
\lambda^+_k = e^{+\gamma_k}, \quad \lambda^-_k = e^{-\gamma_k}
\] (89)
with real \( \gamma_k \). The value of \( \gamma_k \) can be found from
\[
\text{tr} \ M_k = e^{+\gamma_k} + e^{-\gamma_k} = 2 \cosh \gamma_k.
\] (90)
so that
\[ \cosh \gamma_k = \cosh 2\bar{a} \cosh 2b - \cos \frac{\pi k}{n} \sinh 2\bar{a} \sinh 2b, \] (91)
and from (88) follows then that we may write
\[ o_k = i e^{i\delta_k} \sinh \gamma_k \] (92)
with some phase \( \delta_k \) to be considered further below.

If \( \gamma_k \) is a solution, then also \(-\gamma_k\) is one, but this has already been taken into account in (89). Let us fix the sign of \( \gamma_0 \) by defining \( \gamma_k = 2\bar{a} \) for \( b = 0 \) and then analytically continuing to other values of \( b \). For \( k = 1, \ldots, 2n - 1 \), this means \( \gamma_k > 0 \). On the other hand, for \( \gamma_0 \) this means
\[ \gamma_0 = 2(\bar{a} - b). \] (93)

Our sign convention for the \( \gamma_k \) and in particular for \( \gamma_0 \) will allow us to treat all \( \gamma_k \) on an equal footing for both \( \bar{a} > b \) and \( \bar{a} < b \), i.e. irrespective of the temperature.

It is obvious from (86) that for \( k = 1, \ldots, n - 1 \)
\[ \gamma_k = \gamma_{2n-k}, \] (94)
i.e. the eigenvalues \( \lambda_k^{\pm} \) with \( k = 1, \ldots, n - 1 \) occur in pairs. Since for \( 0 < k < n \) [extending for the moment (91) to non-integer values of \( k \)],
\[ \frac{\partial \gamma_k}{\partial k} = \pi \sinh 2\bar{a} \sinh 2b \frac{\sin \frac{\pi k}{n}}{n \sinh \gamma_k} > 0, \] (95)
we have
\[ 0 < |\gamma_0| < \gamma_1 < \cdots < \gamma_n. \] (96)

This means
\[ 0 < +\gamma_0 < \gamma_1 < \cdots < \gamma_n, \quad \bar{a} > b, \] (97)
\[ 0 < -\gamma_0 < \gamma_1 < \cdots < \gamma_n, \quad \bar{a} < b. \] (98)

We have plotted two examples with \( a = b \) in Fig. 2.

Comparison of (87) with (92) gives
\[ \cos \delta_k \sinh \gamma_k = \sinh 2\bar{a} \cosh 2b - \cos \frac{\pi k}{n} \cosh 2\bar{a} \sinh 2b, \] (99)
\[ \sin \delta_k \sinh \gamma_k = \sin \frac{\pi k}{n} \sinh 2b. \] (100)

The \( \delta_k \) are smooth functions of \( \bar{a} \) and \( b \). For \( k = 0 \), (99) and (100) give
\[ \cos \delta_0 \sinh \gamma_0 = \sinh 2(\bar{a} - b), \] (101)
\[ \sin \delta_0 \sinh \gamma_0 = 0, \] (102)
so that with (93) we have
\[ \delta_0 = 0, \] (103)
while from (94), (99) and (100) follows for \( k = 1, \ldots, n \)
\[ \delta_{2n-k} = -\delta_k, \] (104)
implying
\[ \delta_n = 0. \] (105)

The normalized eigenvectors of \( M_k \) may now be written as
\[ u_k^\uparrow = \frac{1}{\sqrt{2}} \left( e^{i\delta_k}, -ie^{-\frac{i\delta_k}{2}} \right), \quad u_k^\downarrow = \frac{1}{\sqrt{2}} \left( -ie^{\frac{i\delta_k}{2}}, e^{-i\delta_k} \right), \] (106)
as may be checked by inserting them into (84). The eigenvectors \( \psi_k^\pm \) of \( V \), defined according to (69) and (81) by
\[ \psi_k^{\pm} = \frac{1}{\sqrt{n}} \begin{pmatrix} e^{ik\pi/n} u_k^{\uparrow} \\ e^{2ik\pi/n} u_k^{\uparrow} \\ \vdots \\ e^{-nik\pi/n} u_k^{\uparrow} \end{pmatrix}, \] (107)
are all orthogonal. For different eigenvalues this is clear, while for the degenerate eigenvalues \( \gamma_k \) and \( \gamma_{2n-k}, \ k =
1, \ldots, n - 1, this is easily seen by explicitly computing the scalar product of \( \psi_k^{\dagger \dagger} \) and \( \psi_{2n-k}^{\dagger \dagger} \),

\[
\psi_k^{\dagger \dagger} \psi_{2n-k}^{\dagger \dagger} = \frac{1}{n} u_{k}^{\dagger \dagger} u_{2n-k}^{\dagger \dagger} \sum_{l=1}^{n} \left( e^{-ik\pi/n} e^{(2n-k)\pi/l} \right)^l
= \frac{1}{n} u_{k}^{\dagger \dagger} u_{2n-k}^{\dagger \dagger} \left( 1 - e^{-2\pi i(n+1)k/n} \right)
= 0.
\]

From here on, we will be considerably more explicit than [4] (the following discussion is omitted altogether in [8]). This is because, without the following steps, the solution is incomplete.

Let us define

\[
R_+^{-1} = \left( \psi_1^{\dagger \dagger}, \psi_2^{\dagger \dagger}, \psi_3^{\dagger \dagger}, \psi_4^{\dagger \dagger}, \ldots, \psi_{2n-3}^{\dagger \dagger}, \psi_{2n-1}^{\dagger \dagger}, \psi_{1}^{\dagger \dagger} \right)
\]

and

\[
R_-^{-1} = \left( \psi_0^{\dagger \dagger}, \psi_1^{\dagger \dagger}, \psi_2^{\dagger \dagger}, \psi_3^{\dagger \dagger}, \psi_4^{\dagger \dagger}, \ldots, \psi_{2n-4}^{\dagger \dagger}, \psi_{2n-2}^{\dagger \dagger}, \psi_{2}^{\dagger \dagger} \right).
\]

(109)

The \( R_+^{-1} \) are then unitary and diagonalize \( V \) according to

\[
R_+ V^{\dagger} R_+^{-1} =
\begin{pmatrix}
\exp(\gamma_1) & 0 & \cdots & 0 \\
0 & \exp(\gamma_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \exp(\gamma_{2n-1})
\end{pmatrix}
\]

and

\[
R_- V^{\dagger} R_-^{-1} =
\begin{pmatrix}
\exp(\gamma_0) & 0 & \cdots & 0 \\
0 & \exp(\gamma_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \exp(\gamma_{2n-2})
\end{pmatrix}
\]

(110)

where (94) has been used.

The diagonal form above is not the most useful for the following considerations. Let us instead apply a further similarity transformation to obtain again a form that is part of SO\( (N,C) \). First, there is a more useful way of writing \( R_\pm^{-1} \). For this purpose, define the \( 2 \times 2 \) matrices

\[
D_{kl} = (z^k_l u^\dagger_{1}, z^k_{2n-l} u^\dagger_{2n-l})
\]

(108)

and organize them as

\[
R_+^{-1} = \frac{1}{\sqrt{n}} \begin{pmatrix} D_{11} & D_{31} & \cdots & D_{2n-1,1} \\
D_{12} & D_{32} & \cdots & D_{2n-1,2} \\
\vdots & \vdots & \ddots & \vdots \\
D_{1n} & D_{3n} & \cdots & D_{2n-1,n} \end{pmatrix}
\]

(114)

and

\[
R_-^{-1} = \frac{1}{\sqrt{n}} \begin{pmatrix} D_{01} & D_{21} & \cdots & D_{2n-2,1} \\
D_{02} & D_{22} & \cdots & D_{2n-2,2} \\
\vdots & \vdots & \ddots & \vdots \\
D_{0n} & D_{2n} & \cdots & D_{2n-2,n} \end{pmatrix}
\]

(115)

Now, define \( R_X \) and its inverse by

\[
R_X^{-1} = \begin{pmatrix}
\exp(\mp i \pi \sigma_x/4) & \cdots & \exp(\mp i \pi \sigma_x/4) \\
\exp(\mp i \pi \sigma_x/4) & \cdots & \exp(\mp i \pi \sigma_x/4) \\
\exp(\mp i \pi \sigma_x/4) & \cdots & \exp(\mp i \pi \sigma_x/4) \\
\end{pmatrix}
\]

(111)

and apply the further similarity transformation

\[
V_{S}^{\pm} = R_X R_\pm V^{\dagger} R_\pm^{-1} R_X^{-1} \equiv S_{\pm} V^{\dagger} S_{\mp}^{-1}
\]

(116)

with

\[
\exp(\mp i \pi \sigma_x/4) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mp i \\
\mp i & 1 \end{pmatrix}
\]

(117)

and apply the further similarity transformation

\[
V_{S}^{\pm} = R_X R_\pm V^{\dagger} R_\pm^{-1} R_X^{-1} \equiv S_{\pm} V^{\dagger} S_{\mp}^{-1}
\]

to (111) and (112).
\[ = \exp \left( \sum_{\nu=1}^{n} \gamma_{(\frac{i}{2})} J_{2
u,2
u-1} \right), \]  

while on the other hand we have

\[ S_+^{-1} = R_+^{-1} R_X^{-1} = \frac{1}{\sqrt{n}} \begin{pmatrix} \tilde{D}_{11} & \tilde{D}_{13} & \cdots & \tilde{D}_{1,2n-1} \\ \tilde{D}_{21} & \tilde{D}_{23} & \cdots & \tilde{D}_{2,2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}_{n1} & \tilde{D}_{n3} & \cdots & \tilde{D}_{n,2n-1} \end{pmatrix}, \]  

(120)

\[ S_-^{-1} = R_-^{-1} R_X^{-1} = \frac{1}{\sqrt{n}} \begin{pmatrix} \tilde{D}_{10} & \tilde{D}_{12} & \cdots & \tilde{D}_{1,2n-2} \\ \tilde{D}_{20} & \tilde{D}_{22} & \cdots & \tilde{D}_{2,2n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}_{n0} & \tilde{D}_{n2} & \cdots & \tilde{D}_{n,2n-2} \end{pmatrix}, \]  

(121)

with

\[ \tilde{D}_{kl} = \begin{pmatrix} \cos \left( \frac{kl\pi}{n} + \frac{\delta_k}{2} \right) & -\sin \left( \frac{kl\pi}{n} + \frac{\delta_k}{2} \right) \\ \sin \left( \frac{kl\pi}{n} - \frac{\delta_k}{2} \right) & \cos \left( \frac{kl\pi}{n} - \frac{\delta_k}{2} \right) \end{pmatrix}. \]  

(122)

Let us show now that the \( S_\pm \) are elements of SO(2n, C). Since \( R_\pm \) and \( R_X \) are unitary, the \( S_\pm \) are also unitary. Since the \( S_\pm \) are also real, they are orthogonal. We still need to show that \( \det S_\pm = 1 \). Since \( S_\pm \) are orthogonal, we have

\[ (\det S_\pm)^2 = \det S_\pm \det S_\pm^T = \det S_\pm \det S_-^{-1} = 1 \]  

(123)

and the question is only if \( \det S_\pm = +1 \) or \( \det S_\pm = -1 \). Since \( S_\pm \) are analytic in \( b \), we may work with \( b = 0 \) and therefore \( \delta_k = 0 \) and then analytically continue to non-zero \( b \), upon which \( \det S_\pm \) cannot change discontinuously and therefore not change at all. We have

\[ \det S_-^{-1} = \det R_X^{-1} S_-^{-1} R_X = \det R_X^{-1} R_-^{-1}, \]  

(124)

where for \( \delta_k = 0 \)

\[ R_X^{-1} R_-^{-1} = \frac{1}{\sqrt{n}} \begin{pmatrix} \tilde{D}_{11} & \tilde{D}_{13} & \cdots & \tilde{D}_{1,2n-1} \\ \tilde{D}_{21} & \tilde{D}_{23} & \cdots & \tilde{D}_{2,2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}_{n1} & \tilde{D}_{n3} & \cdots & \tilde{D}_{n,2n-1} \end{pmatrix}, \]  

(125)

\[ R_X^{-1} R_-^{-1} = \frac{1}{\sqrt{n}} \begin{pmatrix} \tilde{D}_{10} & \tilde{D}_{12} & \cdots & \tilde{D}_{1,2n-2} \\ \tilde{D}_{20} & \tilde{D}_{22} & \cdots & \tilde{D}_{2,2n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}_{n0} & \tilde{D}_{n2} & \cdots & \tilde{D}_{n,2n-2} \end{pmatrix}, \]  

(126)

with

\[ \tilde{D}_{kl} = \begin{pmatrix} \exp(+ikl\pi/n) & \exp(-ikl\pi/n) \\ \exp(-ikl\pi/n) & \exp(+ikl\pi/n) \end{pmatrix}. \]  

(127)

For the purpose of computing their determinants, we may reorganize \( R_X^{-1} R_-^{-1} \) into block matrices

\[ \begin{pmatrix} R_+^T & R_-^T \end{pmatrix}, \]  

(128)

such that all \( \exp(+ik\pi/n) \) are in the \( n \times n \) matrices \( R_+^T \) and all \( \exp(-ik\pi/n) \) in the \( n \times n \) matrices \( R_-^T \), keeping the ordering of the \( \exp(+ik\pi/n) \) among themselves and of the \( \exp(-ik\pi/n) \) among themselves. Such a reorganization involves equal numbers of exchanges of rows and columns, respectively, and does therefore not change the determinant of \( R_X^{-1} R_-^{-1} \). Also, we have \( R_\pm = R_\pm^* \) and therefore

\[ \det S_\pm^{-1} = \det R_X^{-1} R_-^{-1} = \det R_+^T \det R_-^T = |\det R_\pm^T|^2 > 0, \]  

(129)

which leaves only

\[ \det S_\pm = 1. \]  

(130)

This means that the matrices \( S_\pm \) are elements of SO(2n, C) and can be represented as

\[ S_\pm = \exp(ic_{\alpha\beta} J_{\alpha\beta}) \]  

(131)

with certain unknown complex parameters \( c_{\alpha\beta} \).

Let us summarize the crucial result of this section: Applying similarity transformations \( V_\pm = S_\pm V \) with certain matrices \( S_\pm = \exp(ic_{\alpha\beta} J_{\alpha\beta}) \) to \( V = V_{a/2} V_b V_{a/2} \) with \( V_{a/2} = \prod_{\nu=1}^{n} \exp(\bar{a}J_{2\nu,2\nu-1}) \) and \( V_b^{\pm} = \exp(\tau b J_{1,2n}) \prod_{\nu=1}^{n} \exp(2b J_{2\nu+1,2\nu}) \), we obtain \( V_\pm = \exp \left( \sum_{\nu=1}^{n} \gamma_{(\frac{i}{2})} J_{2\nu,2\nu-1} \right) \) with \( \gamma_k \) as defined in (91) and the subsequent sign convention. This is what we will use in the next section to determine all eigenvalues of \( V \) and therefore the partition function \( Z(a,b) \).

VI. DIAGONALIZATION OF \( V \)

Now use the same parameters \( c_{\alpha\beta}^{\pm} \) from the last section to define the \( 2^n \times 2^n \)-dimensional transformation matrix

\[ S = S_+ S_- \]  

(132)

and write

\[ V_\pm = S V S^{-1} = S_+ V S_+^{-1} S_- V S_-^{-1} \equiv V_\pm S \]  

(133)

The factors

\[ V_\pm = \exp(ic_{\alpha\beta}^{\pm} J_{\alpha\beta}) V_{a/2}^{\pm} V_{b}^{\pm} \exp(-ic_{\alpha\beta}^{\pm} J_{\alpha\beta}) \]  

(134)
have the same structure as

\[ V^\pm_S = \exp(i\gamma_{\alpha\beta} J_{\alpha\beta}) V_{a/2}^\pm V_{a/2}^\pm \exp(-i\gamma_{\alpha\beta} J_{\alpha\beta}) \quad (135) \]

from the last section. Now imagine using the Baker-Campbell-Hausdorff formula [9]

\[
\exp(A) \exp(B) = \exp \left( A + B - \frac{1}{2} [B, A] \right) + \frac{1}{12} \left( [A, [A, B]] + [B, [B, A]] \right) + \cdots
\]

(136)

to work out all products of exponentials in \( V^\pm_S \). Since the \( J^\pm_{\alpha\beta}, J^-_{\alpha\beta} \) and \( J_{\alpha\beta} \) obey identical algebras, the result is

\[
V^\pm_S = \exp \left( \sum_{\nu=1}^n \gamma_{(\nu-1)/2} J_{2\nu,2\nu-1}^\pm \right),
\]

(137)

so that

\[
V_S = \exp \left( \sum_{\nu=1}^n \gamma_{2\nu-1} J_{2\nu,2\nu-1}^+ + \gamma_{2\nu-2} J_{2\nu,2\nu-1}^- \right)
\]



= \exp \left[ \frac{1}{4} \sum_{\nu=1}^n \gamma_{2\nu-1} (1 + U_X) X_{\nu} \right]

\[
+ \frac{1}{4} \sum_{\nu=1}^n \gamma_{2\nu-2} (1 - U_X) X_{\nu} \]

(138)

with the same \( \gamma_k \) as defined in (91) and the subsequent sign convention.

To diagonalize \( V_S \), define another similarity transformation

\[
V_Y = R_Y V_S R_Y^{-1}
\]

(139)

with \( R_Y \) and its inverse given by

\[
R_Y^\pm = 2^{-n/2} \prod_{\nu=1}^n (1 \pm i Y_{\nu}).
\]

(140)

Since

\[
R_Y X_{\nu} R_Y^{-1} = Z_{\nu},
\]

(141)

this transformation takes \( V_S \) into

\[
V_Y = R_Y V_S R_Y^{-1} = \exp \left[ \frac{1}{4} \sum_{\nu=1}^n \gamma_{2\nu-1} (1 + U_Z) Z_{\nu} \right]

\[
+ \frac{1}{4} \sum_{\nu=1}^n \gamma_{2\nu-2} (1 - U_Z) Z_{\nu} \]

(142)

with

\[
U_Z = Z_1 \cdots Z_n.
\]

(143)

The matrix \( V_Y \) is diagonal, but we still have to determine its elements. \( U_Z \) is a diagonal matrix with elements +1 and −1 occurring in equal numbers. For each element holds: If an even (odd) number of \( Z_{\nu} \) provides a factor −1, the matrix element of \( U_Z \) is +1 (−1). This means: (i) A matrix element of \( (1 + U_Z)/2 \) is 1 (0) if an even (odd) number of \( Z_{\nu} \) provides a factor −1; (ii) A matrix element of \( (1 - U_Z)/2 \) is 1 (0) if an odd (even) number of \( Z_{\nu} \) provides a factor −1. It follows that the \( 2^n \) eigenvalues of \( V \) split into \( 2^{n-1} \) eigenvalues of the form

\[
\exp \left( \frac{1}{2} \sum_{\nu=1}^n (\pm) \gamma_{2\nu-1} \right),
\]

(144)

and \( 2^{n-1} \) eigenvalues of the form

\[
\exp \left( \frac{1}{2} \sum_{\nu=1}^n (\pm) \gamma_{2\nu-2} \right),
\]

(145)

where in the first (second) case all sign combinations with an even (odd) number of minus signs occur. This is reflected by the indices “e” and “o” in our result for the partition function,

\[
Z(a, b) = [2 \sinh(2a)]^{mn/2} \left[ \sum_e \exp \left( \frac{m}{2} \sum_{\nu=1}^n (\pm) \gamma_{2\nu-1} \right) \right]

\[
+ \sum_o \exp \left( \frac{m}{2} \sum_{\nu=1}^n (\pm) \gamma_{2\nu-2} \right)
\]

\[
\times \left\{ \prod_{k=1}^n \left[ 2 \cosh \left( \frac{m}{2} \gamma_{2k-1} \right) \right] + \prod_{k=1}^n \left[ 2 \sinh \left( \frac{m}{2} \gamma_{2k-1} \right) \right] \right\}.
\]

(146)

The last term within the braces has a sign differing from that in [4]. This is due to our different sign convention for \( \gamma_0 \). The eigenvalues of \( T \) are of course obtained by multiplying (144) and (145) with the trivial factor

\[ [2 \sinh(2a)]^{n/2}. \]

The results for the eigenvalues of \( T \) and the partition function are the starting point for the analysis of the thermodynamic properties of the two-dimensional Ising
model, the most interesting case being the thermodynamic limit \(m, n \to \infty\). Such analyses can now proceed as usual (see e.g. [1, 4, 8]) and will not be repeated here.

VII. REMARKS ABOUT MAGNETIC FIELD AND THREE DIMENSIONS

Let us briefly remark on the difficulties encountered when trying to solve the two-dimensional model with magnetic field or the three-dimensional model in our approach. In the first case, we need besides \(V_a\) and \(V_b\) another matrix

\[
V_c = \prod_{\nu=1}^{n} \exp(cZ_{\nu}),
\]

(147)

and in the second case another matrix

\[
V_d = \prod_{\nu=1}^{n} \exp(dZ_{\nu}Z_{\nu+n'}),
\]

(148)

if the three-dimensional model is obtained from the two-dimensional one by letting sites \(\nu\) and \(\nu+n'\) for \(\nu = 1, \ldots, n\) interact with each other (assuming we have chosen \(n\) such it can be divided by \(n'\); the lattice is then \(m \times n' \times n/n'\) and has in the \(n'\) and \(n/n'\) directions the character of a screw on a torus, while in the \(m\) direction it remains periodic). For the three-dimensional model with magnetic field, we need both \(V_c\) and \(V_d\).

Both the \(Z_{\nu}\) and the \(Z_{\nu}Z_{\nu+n'}\) (with \(n' > 1\)) are not part of the algebra of \(J_{\alpha\beta}^+\) and \(J_{\alpha\beta}^-\). In fact, it turns out that amending the Lie algebra with either of these classes of matrices makes the number of elements in the new algebra grow proportional to \(2^n\) instead of \(n^2\). This eliminates the possibility of using an extended version of the algebra of matrices \(J_{\alpha\beta}\) with side length growing proportional to \(n\). But this was essential for the convenient periodicity properties of the matrices \(V^{\pm}\).

[1] L. Onsager, Phys. Rev. 65, 117 (1944).
[2] H.A. Kramers and G.H. Wannier, Phys. Rev. 60, 252 (1941); 60, 263 (1941).
[3] W. Lenz, Phys. Z. 21, 613 (1920); E. Ising, Z. Phys. 31, 253 (1925).
[4] B. Kaufman, Phys. Rev. 76, 1232 (1949).
[5] R.J. Baxter, Exactly Solved Models in Statistical Mechanics (Academic Press, London, 1982).
[6] T.D. Schultz, D.C. Mattis, and E.H. Lieb, Rev. Mod. Phys. 36, 856 (1964); C.J. Thompson, J. Math. Phys. 6, 1392 (1965); R.J. Baxter, Ann. Phys. (N.Y.) 70, 193 (1972); M.J. Stephen and L. Mittag, J. Math. Phys. 13, 1944 (1972).
[7] B. Kastening, Phys. Rev. E 64, 066106 (2001), cond-mat/0111380.
[8] K. Huang, Statistical Mechanics, 2nd ed. (Wiley, New York, 1987).
[9] J.E. Campbell, Proc. London Math. Soc. 28, 381 (1897); 29, 14 (1898); H.F. Baker, ibid., 34, 347 (1902); 3, 24 (1905); F. Hausdorff, Ber. Verh. Saechs. Akad. Wiss. Leipzig, Math.-Naturwiss. Kl. 58, 19 (1906).