1/\(N_C\) Expansion of QCD Amplitudes*

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This talk comments the main features of a hadronic description of QCD in the limit of large number of colours. We derive a quantum field theory for mesons based on chiral symmetry and a perturbative expansion in 1/\(N_C\). Some large–\(N_C\) and next-to-leading order results are reviewed.

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1. Introduction

The success of large–\(N_C\) determinations based on 't Hooft’s limit of QCD [1] has led to an extraordinary development of the field and has naturally raised the question about subleading effects [2]–[7]. The analysis beyond leading order in 1/\(N_C\) (LO) is essential to validate the large–\(N_C\) limit. A formally well defined 1/\(N_C\) expansion can be easily obtained by implementing the proper \(N_C\) scalings of the hadronic couplings and masses [1, 3, 4, 5]. On the other hand, naively, loops with heavy resonance are expected to produce corrections of the form \(M_R^2/M_{\pi}^2\), order 1/\(N_C\) but numerically large. However, previous phenomenological analyses have shown that the next-to-leading order contributions (NLO) remain under control [3, 5]. Rewriting the resonance parameters in terms of widths (\(\Gamma_R\)) and masses (\(M_R\)), one observes that the 1/\(N_C\) expansion is actually an expansion around the narrow-width limit. The corrections to the large–\(N_C\) amplitudes are suppressed by \(\Gamma_R/M_R \sim 1/N_C\). However, it is not yet clear how broad-width states like the \(\sigma\) meson fit in this pattern [8, 9].

Since the QCD action is chirally invariant, one needs to construct a chiral theory for resonances (\(R\chi\)T) that preserves the symmetry. This feature, common to several phenomenological lagrangians [7, 10, 11, 12, 13], ensures the recovery of Chiral Perturbation Theory (\(\chi\)PT) [14] at low energies even

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at the loop level. Likewise, the validity of the $1/N_C$ expansion at all energies allows to match QCD at short distances, where it is described by the operator product expansion (OPE) [15].

2. Large–$N_C$, next-to-leading order and resummations

In the large–$N_C$ limit, QCD contains an infinite tower of hadronic states, the resonances ($R$) and the Goldstones from the spontaneous chiral symmetry breaking ($\phi$). The Green-functions are provided by the tree-level exchanges [1]. Other observables like the form-factors are derived from them and they are also given by the tree-level diagrams. Their absorptive contribution is a series of delta functions, so the amplitudes are determined by the masses (position of the real poles) and the corresponding couplings (residues).

At NLO in $1/N_C$, the perturbative amplitudes contain two-meson absorptive cuts together with single and double real poles coming, respectively, from diagrams with one and two tree–level propagators [3, 5]. Pure perturbation theory, i.e., without resummation, is valid when no intermediate particle is near its mass-shell.

However, the perturbative expansion breaks down in the neighbourhood of the resonance poles at any finite order in $1/N_C$ and a Dyson-Schwinger summation is required [16]. This shifts the real resonance poles into unphysical Riemann sheets. The particles gain a finite width and the amplitude becomes finite.

In the past, the attention has been focused either on the large-$N_C$ limit or on resummed descriptions. However, the previous step to any resummation is the perturbative calculation and only a few examples of it exist by the moment [2]–[7]. By-passing this intermediate point may lead to strongly model dependent resummations and, therefore, incorrect determinations.

3. Resonance Chiral Theory

3.1. Leading order in $1/N_C$

In general, one is forced to work within a minimal hadronical approximation with a finite number of states (MHA) [17]. This is an acceptable approximation in the case of amplitudes that are chiral-order parameter, provided that we include a minimal number of light states, enough to reproduce the short-distance QCD power behaviour.

Since we work within a large–$N_C$ framework, the particles are classified in $U(n_f)$ multiplets [18]. The Goldstones from the spontaneous chiral symmetry breaking $\phi = (\pi, K, \eta_8, \eta_0)$ are incorporated through covariant tensors $G(\phi)$ [10, 11, 14]. The lightest $1^{--}$, $1^{++}$, $0^{++}$, $0^{--}$ resonance fields are included, being the spin–1 mesons represented through antisymmetric tensors
The last ingredient of $R\chi T$ relies on the assumption that operators with a large number of derivatives are forbidden and only $O(p^2)$ chiral tensors are to be considered. The addition of higher power operators is expected to lead to a wrong growing behaviour of the Green-functions at large momenta. These ingredients yield the general lagrangian,

$$\mathcal{L}_{R\chi T} = \mathcal{L}_G(\phi) + \sum_R \mathcal{L}(R, \phi) + \sum_{R,R'} \mathcal{L}(R, R', \phi) + ... \tag{1}$$

The operators with just Goldstones are given by $\chi PT$ at $O(p^2)$ \([14]\):

$$\mathcal{L}_G(\phi) = \frac{F^2}{4} \langle u_{\mu} u^{\mu} + \chi_+ \rangle. \tag{2}$$

The operators linear in the resonance fields were derived in Ref. \([10]\):

$$\sum_R \mathcal{L}(R, \phi) = \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f^\mu_+ \rangle + \frac{iG_V}{\sqrt{2}} \langle V_{\mu\nu} u^{\mu} u^{\nu'} \rangle + c_d \langle S\chi_+ \rangle + ... \tag{3}$$

The analysis of three-point functions and form-factors have requires the introduction of operator with two and three resonance fields \([5, 11, 19]\).

In order to make the theory dual to QCD, it must be matched at the regions where it is calculable, this is, at low and high energies. The recovery of the low energy QCD structure, described by $\chi PT$, is trivial once chiral symmetry has been properly incorporated. On the other hand, $R\chi T$ must reproduce the OPE at short distances \([15]\). For instance, the matching of the $V - A$ correlator yields the well known Weinberg sum-rules and establishes a relation between resonance parameters at LO in $1/N_C$ \([20]\).

### 3.2. Next-to-leading order

The one-loop diagrams give place to NLO contributions. They produce ultraviolet (UV) divergences that require new operators with NLO couplings in order to be renormalised. Many of these operators can be actually removed through the use of the equations of motion \([3, 4]\). Furthermore, it has been proved that some matrix elements do not need local $\chi PT$ operators to fulfill the renormalisation \([6]\). We will refer here to the example of the correlator $\Pi(t) \equiv \Pi_{SS}(t) - \Pi_{PP}(t)$ \([5]\).

The first step is to examine the absorptive part of the amplitude by means of the optical theorem. The contribution from a two-particle intermediate state $M_1 M_2$ to the spectral function, shown in Fig. \([11]\), is in general proportional to some squared form factors \([5]\):

$$\text{Im}\Pi(t)_{M_1 M_2} \propto \left| \mathcal{F}_{M_1 M_2}(t) \right|^2. \tag{4}$$
Fig. 1. One-loop absorptive contributions to $\Pi(t)$. All the lines stand for tree-level meson propagators in our perturbative calculation.

The vanishing of $F_{\pi\pi}(t)$ at infinite momentum [21] makes $\text{Im}\Pi(t)_{\pi\pi} \rightarrow 0$ when $t \rightarrow \infty$. Demanding a similar vanishing behaviour for each separate absorptive contribution $\text{Im}\Pi(t)_{M_1M_2}$ leads to a series of constraints for the form-factors at LO in $1/N_C$ [5].

After the preliminary large–$N_C$ analysis of the absorptive subdiagrams, one is ready to aboard the renormalisation of the one-partic le-irreducible vertex-functions entering in our amplitude [3, 4]. Imposing the proper UV asymptotic conditions in the absorptive part of $\Pi(t)$ leads to the absence of new UV divergent structure [6]. No new operators are needed for the renormalisation, just a $1/N_C$ shift of the parameters existing at LO.

An alternative way to calculate NLO amplitudes [2, 3, 7] is the use of dispersion relations [5]. In our case, it is possible to write down an unsubtracted dispersive integral for $\Pi(t)$. The two-meson cuts contribute to the correlator with a finite part, $\Delta\Pi(t)_{M_1M_2}$, and $\Pi(t)$ depends now on the renormalised resonance masses and couplings:

$$\Pi(t) = 2B_0^2 \left[ \frac{8c_{\pi}^2}{M_S^2 - t} - \frac{8d_{\pi}^2}{M_P^2 - t} + \frac{F^2}{t^2} + \sum_{M_1M_2} \Delta\Pi(t)_{M_1M_2} \right]. \quad (5)$$

The NLO contribution $\Delta\Pi(t)_{M_1M_2}$ only depends on $\text{Im}\Pi(t)_{M_1M_2}$ [5] and, at high energies, behaves like

$$\Delta\Pi(t) = \frac{F^2}{t} \delta^{(1)}_{\text{NLO}} + \frac{F^2 M_S^2}{t^2} \left( \delta^{(2)}_{\text{NLO}} + \tilde{\delta}^{(2)}_{\text{NLO}} \ln \frac{-t}{M_S^2} \right) + O \left( \frac{1}{t^3} \right). \quad (6)$$

The matching of the one-loop $\chi_T$ correlator to the OPE yields a NLO generalisation of the first and second Weinberg sum-rules [5]:

$$8c_{\pi}^2 - 8d_{\pi}^2 - F^2 (1 + \delta^{(1)}_{\text{NLO}}) = 0,$$

$$-8c_{\pi}^2 M_S^2 - 8d_{\pi}^2 M_P^2 + F^2 M_S^2 \delta^{(2)}_{\text{NLO}} = -8 \tilde{\delta}, \quad (7)$$

where $\tilde{\delta} \equiv 3\pi\alpha_s F^4/4$ turns out to be numerically negligible [20]. The matching is completed by demanding that the $\frac{1}{t^2} \ln (-t/M_S^2)$ term also vanishes, this is, $\tilde{\delta}^{(2)}_{\text{NLO}} = 0$. These relations fix the renormalised resonance couplings $c_{\pi}^r$, $d_{\pi}^r$ in terms of the renormalised resonance masses $M_R^2$ [5].
One-loop $R\chi T$ reproduces at low energies the one–loop $\chi$PT structure, generating the proper running for the chiral couplings. Thanks to this, it is possible to provide predictions for the renormalised $L^R_8(\mu)$ at any $\mu$ in terms of the renormalised $R\chi T$ parameters. In our example, the short-distance matching of the form-factors at large–$N_C$ and the correlator at NLO fixes the chiral coupling $L^R_8(\mu)$ in terms of the renormalised masses $M^R_R$, yielding the prediction \[5\]:
\[
L^R_8(\mu_0) = (0.6 \pm 0.4) \cdot 10^{-3}, \quad \text{for } \mu_0 = 770 \text{ MeV}. \tag{8}
\]
The main error, also present at LO, comes from the scalar and pseudoscalar masses. The uncertainty on the saturation scale is completely removed. One must keep in mind that any large–$N_C$ estimate of the LECs contains an inherent theoretical error due to the NLO running from the loops. There is no particular saturation scale for all the $\chi$PT couplings. This uncertainty can be only removed by taking the calculation up to the one-loop level.

4. Open questions

Although it is possible to extract some information about the couplings of highly excited mesons ($M_R \gg \Lambda_{\text{QCD}}$) \[22\]–\[25\], one still needs to specify the structure of the spectrum at high energies. It can be solved in some models \[25\]–\[27\] but, in general, the QCD spectrum is badly known in the ultraviolet. This forces to work under a MHA \[17\], introducing uncertainties \[22\]–\[24\] that are reflected in some problems in the short-distance matching of three-point Green functions \[28\]. An improved way to perform the matching would be desirable. In addition to making MHA a complete and self-consistent description, it would allow the exploration of Green-functions that are not order parameter and are actually dominated by the high part of the infinite series of resonances \[22\]–\[25\].

A last standing problem is the equivalence between large–$N_C$ lagrangians. The spin–1 mesons can be described through different formulations \[7\]–\[12\], \[13\]–\[29\]. However, the equivalence between representations has been only proven at $O(p^4)$ \[12\]–\[30\] and higher orders have not been explored. Likewise, a general proof forbidding operators of order higher than $O(p^2)$ in the lagrangian is still missing. Nevertheless, the slow but firm advances in the field are establishing solid foundations where to base the $1/N_C$ hadronic phenomenology.

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