A Gribov equation for the photon Green’s function

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Abstract. We present a derivation of the Gribov equation for the gluon/photon Green’s function $D(q)$. Our derivation is based on the second derivative of the gauge-invariant quantity $\text{Tr} \ln D(q)$, which we interpret as the gauge-boson ‘self-loop’. By considering the higher-order corrections to this quantity, we are able to obtain a Gribov equation which sums the logarithmically enhanced corrections. By solving this equation, we obtain a non-perturbative running coupling in both QCD and QED. In the case of QCD, $\alpha_S$ has a singularity in the space-like region corresponding to super-criticality which is argued to be resolved in Gribov’s light-quark confinement scenario. For the QED coupling in the UV limit, we obtain a $\propto Q^2$ behaviour for space-like $Q^2 = -q^2$. This implies the decoupling of the photon and an NJLVL-type effective theory in the UV limit.

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1 Introduction

Gribov’s programme [1][2] for dealing with the problem of strongly interacting quarks is based on the picture of super-critical binding of light quarks, leading to chiral-symmetry breaking and confinement through the reorganization of the quark (the Dirac-sea) states in the vacuum [3].

The central analytical tool used in this approach is the Gribov equation.

In order to derive the Gribov equation, let us consider the (all-order) self-energy correction, $\Sigma(q)$, to the quark propagator $G(q)$. We denote the 4-momentum, $q_\mu$, derivative $\partial/\partial q_\mu$ by $\partial_\mu$. The raising and lowering of indices is implicit. The 4-dimensional Laplacian operator is then defined as $\partial^2 \equiv \partial_\mu \partial^\mu$, using the Einstein summation convention.

The application of $\partial^2$ to the all-order $\Sigma(q)$ yields an infinite series of perturbative diagrams, in which some quark or gluon propagator is replaced by its derivative.

The leading, logarithmically-enhanced, contribution to $\Sigma(q)$ comes from phase-space regions with large hierarchies of internal momenta. Corresponding to this region, the leading contribution to the second derivative of $\Sigma(q)$ comes from diagrams in which both derivatives act on the same propagator, i.e., $\partial^2 D(q - q')$ where $D(q - q')$ is the internal gluon (in this case, bare) propagator carrying 4-momentum $q - q'$. Neglecting the vacuum polarization contributions, it is possible to arrange the internal momenta in such a way that derivatives of the quark propagator $G$ do not occur. This simplifies things (in the Feynman gauge) since we have the relation:

$$\frac{\partial^2}{(q - q')^2 + i\epsilon} = -4\pi^2i\delta^{(4)}(q - q'),$$

which removes a loop integration and make it formally equivalent to the amplitude for the emission of two zero-momentum gluons.

The leading contribution to this double emission is obviously $\Gamma_\mu G(q)\Gamma_{\mu'}$, i.e., the successive emission of two gluons from a quark line. The 0-momentum emission vertex is related to the Green’s function by the Ward–Takahashi identity:

$$\Gamma_\mu(q, q, 0) = -\partial_\mu G^{-1}(q).$$

Since $\Sigma(q)$ gives the running of $G^{-1}(q)$, the end result of this discussion is the following Gribov equation which sums all the leading logarithmically enhanced contributions:

$$\partial^2 G^{-1}(q) = g\partial_\mu G^{-1}(q)\partial^\mu G^{-1}(q) + O(g^2).$$

The coupling $g$ is $\alpha/\pi$ for QED and $C_F\alpha_S/\pi$ for QCD. Because of the removal of the loop integration, this equation is local in the momentum space, and benefits from the absence of the momentum space integration over the dangerous IR (infra-red) region.

As mentioned above, the renormalization of the gluon, or the photon in QED, propagator has been neglected in this discussion, though it is possible to incorporate this effect partly by replacing $g$ in eqn. [3] by a running coupling, $g(q)$. 


Leaving aside the problem of the coupled evolution of the quark and gluon Green’s functions for now, the problem is to write down an analogous Gribov equation for the gluon/photon sector.

In this case, the vertex \( \Gamma(q, q', q - q') \) is evidently no longer constant at the tree level, and is linear in the momenta. Therefore the simple application of the above method, of applying \( \partial^2 \) to the vacuum-polarization operator \( \Pi_{\lambda\alpha} \), is insufficient in the sense that it yields extra contributions due to the derivatives of the vertex. Gribov’s solution to this problem \([1]\) was to employ the linear Duffin–Kemmer formalism, in which the interaction is constant, so that we can analyze the problem in a similar way to that of the fermion sector. However, the resulting equation, being a coupled second-order differential equation, is hardly manageable, and furthermore suffers from artificial divergences which plague the Duffin–Kemmer formalism \([2]\).

On the other hand, the analysis based on the Duffin–Kemmer formalism is illuminating at least in the sense that it illustrates how to recover the Ward–Takahashi identity, eqn. \([2]\) in the zero-momentum limit. This problem and the problem of gauge fixing are related. These comprise the main difficulty in formulating a Gribov equation for the gluon/photon sector.

On the other hand, for the problem of the UV (ultraviolet) evolution of the QED coupling, Gribov postulated \([1,2,4]\) a formulation based on the third derivative of the polarization operator \( \Pi_{\lambda\alpha} \). Three differentiations are necessary to remove the UV divergence. In this case, Gribov obtained:

\[
\left( \frac{d^2}{d\xi^2} + 2\frac{d}{d\xi} \right) \frac{1}{g} \approx \frac{1}{6} \left[ 1 - 2\Gamma_f \right]^2 - 3. \tag{4}
\]

\( \xi = \ln |q^2|, \) and \( \Gamma_f \) is the fermion anomalous dimension \( d\ln Z_f/d\xi \). In the weak-coupling limit, we have \( \Gamma_f = 0 \) and so the right-hand side gives \( -2/3 \). The ordinary RGE (renormalization-group equation) evolution is recovered in this case. In the strong-coupling limit, \( \Gamma_f \) tends to \((1 + \sqrt{3})/2\), and so the right-hand side tends to zero. In this case, the running coupling continues to evolve, but more slowly than in the RGE evolution which suffers from the problem of the UV Landau pole.

In the UV limit, the right-hand side of eqn. \((4)\) tends to \(-2/g^2\). This leads to a logarithmic evolution, \( g \to \xi \). Hence the QED (actually \( U(1)_Y \)) coupling grows large but finite for finite \( \xi \) and, as postulated by Gribov \([3]\), causes the formation of super-critical states, which could be identified with the Higgs and Goldstone bosons in EWSB (electroweak symmetry breaking).

In the derivation of both eqn. \((4)\) and \((3)\), the running-coupling effects due to the renormalization of the internal boson lines are neglected. Furthermore, in the QED case, the right-hand side of eqn. \((4)\) is a derivative of a phase-space integral, and so the formulation is not strictly local.

The purpose of this paper is to present an approach to deal with these problems. Starting from a gauge invariant expression which corresponds to the gauge-boson ‘self-loop’, and collecting the logarithmically-enhanced terms as in the derivation of eqn. \((3)\), we obtain a Gribov equation for gauge bosons, and an evolution equation for the running coupling, in a closed form.

It is found that the equation can be integrated analytically. We analyze the solution for both QCD and QED.

In QCD, the solution exhibits a branch-cut singularity at space-like momentum \( Q^2 = -q^2 = A^2_{\text{QCD}} \), indicating vacuum instability due to the formation of the critical state. The problem of vacuum instability is resolved in Gribov’s scenario due to the reorganization of the quark states which is best understood using the Dirac-sea picture. We argue that this will remove the singularity and give rise to a Green’s function for the gluon which has the analytical properties that are expected for confined particles.

If, on the other hand, there are no quark states which become super-critical, there is a problem as to how to stabilize the vacuum, and we believe that this is not possible within the framework of ordinary field theory.

In the case of QED at high energy, we found that the coupling grows linearly with \( Q^2 = -q^2 \). This corresponds to the decoupling of the photon, and an effective high-energy theory described by a contact interaction term, as in the NJLVL model \([6]\). This evolution of the coupling is faster than the logarithmic behaviour found by Gribov. We show that provided that one includes the contribution due to internal photon renormalization in eqn. \((3)\) the \( \propto Q^2 \) behaviour of the coupling is obtained also in this latter case.

This paper is organized as follows. We describe the framework of our approach in sec. \([2]\) We derive the main equation in sec. \([3]\). Its solution is presented in sec. \([4]\). We discuss its application in QCD and QED in secs. \([5]\) and \([6]\) respectively. The conclusions are stated at the end.

2 General framework

Let us begin with the gluon/photon Green’s function:

\[
D(g, \zeta) = Z(q^2)^{-g\alpha g} + (1 - \zeta^{-1})q\alpha q_0/q^2, \tag{5}
\]

where \( \zeta \) is the gauge-fixing parameter and \( Z(q^2) \) is the renormalization coefficient. The running coupling, \( \alpha(q^2) \), is proportional to \( Z(q^2) \).

The central quantity which we adopt in the following discussion is the trace of the logarithm of \( D(q, \zeta) \), which we denote by \( \Xi_\zeta \):

\[
\Xi_\zeta(q) = \text{Tr} \ln D(q, \zeta). \tag{6}
\]

Although this quantity will be shown to be essentially gauge invariant, we introduced the index \( \zeta \) to indicate that \( \Xi_\zeta \) includes an unphysical polarization contribution.

We define the logarithm in eqn. \((6)\) by its Taylor series expansion, i.e.:

\[
\ln |\lambda(I + M)| = I \ln \lambda + M^2/2 + M^3/3 - \ldots, \tag{7}
\]
where $\lambda$ is scalar and $M$ is a $d \times d$ matrix. $I$ is the $d$-
-dimensional identity matrix. Applying this to eqn. 6 we
obtain:
\[ \Xi_{\zeta}(q) = 4 \ln \left( \frac{-Z(q^2)}{q^2 + i\varepsilon} \right) + \ln \zeta^{-1}. \]  
(8)
This result is for the 4-dimensional vector boson, but is
easily generalizable to arbitrary particles. Since the $\zeta$
dependence can be absorbed in the definition of $Z(q^2)$ or the
scale of $q^2$, $\Xi_{\zeta}$ is essentially gauge invariant.
\[ \Xi_{\zeta} \] corresponds diagrammatically to the self-loop of
the gauge boson, i.e., just a circle, without the phase-space
integration. This assignment is natural because when we
take its derivative, we obtain the trace of a propagator
integration. This assignment is natural because when we
take its derivative, we obtain the trace of a propagator
\[ \delta\mu \Xi_{\zeta}(q) = \text{Tr} \left[ -\partial\mu D^{-1}(q) D(q) \right], \]  
(9)
which follows directly from eqn. 8 We represent this relationship
diagrammatically as:
\[ \begin{array}{c}
\text{circle} \\
\delta\mu \\
\text{circle}
\end{array} \]
(10)
The thick solid line represents $D(q, \zeta)$ and the dashed line
represents $-\partial D^{-1}(q, \zeta)$.
We would like to note at this point that the quantity
inside the trace on the right-hand side of eqn. 8 has anal-
ogous form to $A_\mu(q)$ introduced in ref. [1]:
\[ A_\mu(q) = \partial_\mu G^{-1}(q) G(q). \]  
(11)
Since we have:
\[ \partial_\mu G(q) \equiv -G(q)\partial_\mu G^{-1}(q)G(q) \equiv -G(q)A_\mu(q), \]  
(12)
the derivative of a propagator is equivalent to its multi-
plication by $-A_\mu(q)$ and is, by the Ward–Takahashi iden-
tity, equivalent to the insertion of the 0-momentum photo-
tron/gluon emission vertex. It is hence appropriate to rep-
resent $\Xi_{\zeta}(q)$ as a self-loop.
In ref. [1], $A_\mu(q)$ was introduced to reduce eqn. 3 to a first-order equation, namely:
\[ \partial_\mu A_\mu(q) + (1 - g)A_\mu(q)A_\mu(q) = 0. \]  
(13)
We reproduce this equation here for comparison with the
photonic/gluonic Gribov equation, which we shall write in a
similar form later.
As for the physical interpretation of $\Xi_{\zeta}$, the logarithm of
the propagator has the interpretation as the density of
states [7]. Indeed, we see that the phase-difference:
\[ \frac{1}{\pi} \text{Im} \left[ \Xi(q_2) - \Xi(q_1) \right], \]  
(14)
by the Levinson theorem, is the number of states in that
phase-space interval. However, as stated earlier, $\Xi_{\zeta}$ in-
cludes an unphysical polarization contribution, and the
number of states should in fact be $3/4$ of this. This can be
achieved by eliminating the scalar polarization component in
eqn. 8.
\[ \Xi(q) = \text{Tr} P(q) \ln D(q, \zeta) = 3 \ln \left( \frac{-Z(q^2)}{q^2 + i\varepsilon} \right). \]  
(15)
Here, $P$ is the transverse projection operator:
\[ P_{\lambda\sigma} = g_{\lambda\sigma} - \frac{g_{\lambda\sigma}q_\sigma}{q^2}. \]  
(16)
$\Xi(q)$ is gauge invariant.
We may equally have chosen the derivative of $\Xi$ as the
starting point of our discussion. Let us define:
\[ A_\mu(q, \zeta) = \partial_\mu D^{-1}(q, \zeta) D(q, \zeta). \]  
(17)
We also define its transverse component:
\[ A_\mu^T(q) = PA_\mu(q, \zeta)P, \]  
(18)
which is gauge invariant. We can then define $\Xi$ as the
definite integral of $A_\mu^T$, i.e.:
\[ \Xi(q) = -\text{Tr} \int_0^q \! \! A_\mu^T(q', \zeta) dq', \]  
(19)
Note that this is a line integral and not a phase-space
volume integral which $\Xi(q)$ certainly is not. In general,
because of the singularities in $D(q, \zeta)$, the value of $\Xi$
depends on the path of integration in the complex plane. In
particular, if we adopt a closed contour for the integration,
we have an Aharonov–Bohm-type phase corresponding to
the ‘gauge field’ $\text{Tr} A_\mu^T(q)$. The value of the phase is $2\pi$
times the number of states enclosed by the contour.
Using the above tools, the discussion of the Ward–
Takahashi identity becomes relatively simple. To see this,
let us write the 0-momentum limit of the bare 3-point
vertex using the usual Feynman rules as:
\[ \Gamma_{\mu,\text{bare}}(q, q, 0) = 2q_\mu g_{\lambda\sigma} - q_\lambda g_{\mu\sigma} - q_\sigma g_{\mu\lambda}. \]  
(20)
We have omitted the colour matrix $f^{ABC}$ for the sake of
simplicity. This will be reintroduced later, in eqns. 48 and
50 in the form of the colour factor $C_A$ inherent in
the beta-function coefficient $b_0$. $D^{-1}$, whose derivative
we shall now compare with eqn. 20, is given by:
\[ D^{-1}(q, \zeta) = Z^{-1}(q') \left[ -q'^2 g_{\lambda\sigma} + (1 - \zeta)q_\lambda q_\sigma \right]. \]  
(21)
Taking the bare propagator $D_0^{-1}$, i.e., without considering
the contribution due to the renormalization $Z(q^2)$, the
derivative is given by:
\[ \partial D_0^{-1}(q, \zeta) = -2q_\mu g_{\lambda\sigma} + (1 - \zeta)(q_\lambda g_{\mu\sigma} + q_\sigma g_{\mu\lambda}). \]  
(22)
The expressions 20 and 22 are related by the following
tree-level identity:
\[ \Gamma_{\mu}(q, q, 0) = -\left( \partial_\mu D^{-1}(q, \zeta) - \zeta \frac{\partial}{\partial \zeta} D^{-1}(q, \zeta) \right). \]  
(23)
exactly as found by Gribov [1]. The first term on the right-hand side gives the Ward–Takahashi identity, but the second is the complication arising from Slavnov–Taylor identity. The latter term vanishes only for \( \zeta = 0 \), but the transverse parts are the same.

Now let us consider operations on \( \mathcal{Z}(q) \). Since this is gauge invariant, we may choose the transverse gauge, \( \zeta^{-1} = 0 \). In this case, the insertion of a zero-momentum vertex anywhere in the all-order diagram for \( \mathcal{Z}(q) \), i.e., including all vacuum-polarization contributions, is accompanied by two transverse projectors, and therefore the non-transverse part of the vertex do not contribute. We hence seem to recover the Ward–Takahashi identity.

This is not the end of the story, because even though the quantity \( D(q)\Gamma(q, q, 0)D(q) \) is transverse, the quantity \( \lim_{\zeta^{-1} \to 0} \left[ D(q)\partial D^{-1}(q)D(q) \right] \) is not necessarily so. The trick is to invoke the transversality of \( \mathcal{Z}(q) \). For instance, we have:

\[
- \text{Tr} \left[ D(q, \zeta)\partial D^{-1}(q, \zeta) \right] = \partial_\mu \mathcal{Z}_\mu(q) \tag{24}
\]

which differs from

\[
\text{Tr} \left[ D(q, \zeta)\Gamma_\mu(q, q, 0) \right] = \partial_\mu \mathcal{Z}(q), \tag{25}
\]

but so long as we make use of \( \mathcal{Z}(q) \), i.e., proceed by taking the transverse component of the expression inside the trace in eqn. 24 the non-transverse component does not contribute and the two expressions become identical. In summary, the Ward–Takahashi identity:

\[
\Gamma_\mu(q, q, 0) = -\partial_\mu D^{-1}(q, \zeta), \tag{26}
\]

holds provided that we work with gauge-invariant and transverse quantities. For transversality, we look for the vanishing of the diagram when any gluon in it is assigned scalar polarization, \( \epsilon_\mu(k) \to k_\mu \). This holds for \( \mathcal{Z}(q) \) because of its symmetry: the graph looks the same (and transverse) from anywhere in (the perturbative expansion of) the diagram.

Unfortunately, in this discussion, we have lost gauge invariance: the transversality argument works only when we utilize a transverse gauge, \( \zeta^{-1} = 0 \) with transverse \( \mathcal{Z}(q) \) and \( A^\mu_T(q) \).

A simpler working rule, which reproduces the same result provided that we are only interested in products of \( D \) and \( D^{-1} \), is simply to work with the Feynman gauge and impose transversality at the end. To illustrate this point, \( A_\mu(q, 1) \), which is written in the Feynman gauge, can be converted to \( A^\mu_T(q) \) simply by multiplying on either side by the transverse projection operator \( P \) as can be easily verified:

\[
A^\mu_T(q) \equiv PA_\mu(q, 1) \equiv A_\mu(q, 1)P. \tag{27}
\]

We have only discussed the tree-level relationship up to now, but the loop corrections decorate both sides of eqn. 26 in the same way, and so eqn. 26 is an all-order relation.

### 3 Derivation of the equation

In the Feynman gauge, the zeroth-order Gribov equation comes out immediately from eqn. 1:

\[
\frac{\partial^2 D(q, 1)}{\partial \ln q^2} \approx 0 \quad (q \neq 0). \tag{28}
\]

In the rest of the discussion, we omit the second argument, \( \zeta = 1 \), for simplicity. The small contribution to the right-hand side is due to the running of the coupling, so that it is \( \mathcal{O}(\Gamma) \), where \( \Gamma \) is the anomalous dimension defined by:

\[
\Gamma(q^2) = \frac{\partial \ln Z}{\partial \ln q^2}. \tag{29}
\]

This is related to the usual RGE beta function \( \beta \) by \( \Gamma = \beta/\alpha = d\ln \alpha/d\ln q^2 \). There is a difference of factor 1/2 between our convention and that of ref. [1].

Eqn. 28 can be established in a more gauge invariant manner by using \( \mathcal{Z}_\zeta \) introduced in eqn. 6. Because of its gauge invariance, its derivative, \( -\text{Tr} A_\mu(q, \zeta) \) is also gauge invariant. We are thus justified in using the Feynman gauge. Omitting the trace for simplicity, we can then write:

\[
\partial_\mu A_\mu(q) = \partial (\partial D^{-1}(q)D(q)) = -\partial (D^{-1}(q)\partial D(q)). \tag{30}
\]

Since:

\[
\partial D(q) = -D(q)\partial D^{-1}(q)D(q), \tag{31}
\]

and making use of eqn. 28 we obtain:

\[
\partial_\mu A_\mu(q) = A_\mu(q)A_\mu(q) + \mathcal{O}(\Gamma). \tag{32}
\]

For the \( \mathcal{O}(\Gamma) \) term, since it is due to the running of the coupling, we need to analyze the polarization operator \( \Pi(q^2) = -\Pi_{\mu\mu}(q)/3q^2 \). Let us see how this enters our framework.

The expansion of \( \mathcal{Z}(q) \) yields the following:

\[
\begin{array}{c}
\text{O} \\
\text{1} \\
\text{+} \\
\text{+} \\
\text{+} \\
\end{array} = \begin{array}{c}
\text{O} \\
\text{1} \\
\text{+} \\
\text{+} \\
\text{+} \\
\end{array} \tag{33}
\]

The thin lines represent the bare propagator \( D_0(q) \), and the blobs represent the insertion of the polarization operator. The factors \( 1/n \) are the symmetry factors for the circular symmetry. The contribution of the polarization operator, i.e, the sum of the contributions from all but the first term of eqn. 33 is proportional to:

\[
\Pi(q^2) + \frac{\Pi^2(q)}{2} + \frac{\Pi^3(q)}{3} + \cdots = -\ln (1 - \Pi(q^2)). \tag{34}
\]

Including the first term of eqn. 33 we thus obtain:

\[
\mathcal{Z}(q) = 3\ln \left( \frac{-(1 - \Pi(q^2))^{-1}}{q^2 + i\varepsilon} \right). \tag{35}
\]
Comparing with eqn. \( \text{35} \) we obtain:
\[
Z(q^2) \propto (1 - \Pi(q^2))^{-1} = \alpha(q^2),
\]
which is as expected and consistent with the usual renormalization considerations.

In order to find the sub-leading term in eqn. \( \text{32} \) we take the derivative of eqn. \( \text{33} \). Taking the first derivative, we have a series of diagrams, approximately half of which involving the derivatives of the line and the rest involving the derivatives of the blob. The derivatives of the line are already taken account of in the leading-order term of eqn. \( \text{32} \) and so we are left with the following series of diagrams:

\[
\begin{align*}
\text{1st order:} & \quad \ \ \\
\text{2nd order:} & \quad \ \ \\
\text{3rd order:} & \quad \\
\text{...} & \quad \\
\end{align*}
\]

The combinatorial factor at each order cancels with the symmetry factor present in eqn. \( \text{33} \).

When we take the next derivative, we again have some derivatives of the line, and these are already included in eqn. \( \text{32} \) In addition, we have cases in which the two derivatives act on separate blobs, such as:

\[
\begin{align*}
\text{1st order:} & \quad \ \ \\
\text{2nd order:} & \quad \\
\text{3rd order:} & \quad \\
\text{...} & \quad \\
\end{align*}
\]

These give rise to the renormalization of eqn. \( \text{32} \) and therefore are already included. With this consideration, the remaining second-derivative terms are:

\[
\begin{align*}
\text{1st order:} & \quad \ \ \\
\text{2nd order:} & \quad \\
\text{3rd order:} & \quad \\
\text{...} & \quad \\
\end{align*}
\]

As before, the thick solid line represents the renormalized propagator. As for the blob, it is essentially \( \Pi_{\lambda\sigma} \). However, evaluating \( \partial^2 \Pi_{\lambda\sigma} \) will not yield a sensible answer, because there are cancellations between the propagators \( \sim 1/q^2 \) and \( \Pi_{\lambda\sigma} \sim q^2 \) which make the theory renormalizable and make eqn. \( \text{32} \) valid.

This \( \sim q^2 \) factor, in the Feynman gauge and at the, dressed if necessary, one-loop level (which is all that is needed), arises from the linear momentum dependence of the external vertices, i.e., the vertices which are connected with the external propagators. Thus we need an operation which extracts only the contributions to \( \partial^2 \Pi_{\lambda\sigma} \) that arises from the derivatives of the (dressed) internal lines and not the \( \propto q \) linear external-momentum dependence of the vertices attached to external propagators. Let us denote this operation by \( \partial^2_R \), where the subscript \( R \) stands for renormalization. With this understanding, we may write the equation in terms of \( \Pi(q^2) \), as:
\[
\frac{1}{4} \text{Tr} [\partial^2_R \Pi(q) - \partial^2_R \Pi(q)] = \alpha \partial^2_R \Pi(q^2).
\]

The left-hand side of this equation is not gauge invariant, and the Feynman gauge is implied. It is a trivial matter to convert this into a transverse expression, just by substituting \( \partial^T \Pi(q) \) for \( \partial^R \Pi(q) \) and averaging factor \( 1/3 \) for \( 1/4 \).

It is interesting to explicitly evaluate \( \alpha \partial^2 R \Pi(q^2) \) and demonstrate that this does not equal to the left-hand side of eqn. \( \text{40} \). To do so, let us first write out the explicit form of \( A_{\mu}(q) \). For general choice of \( \zeta \), we have:
\[
A_{\mu}(q, \zeta) = \partial_{\mu} D^{-1}(q, \zeta) D(q, \zeta) = \frac{2q_{\mu}}{q^2} (1 - \Gamma) g_{\lambda\sigma} + \frac{q_{\lambda}}{q^2} (\zeta - 1) P_{\mu\sigma} + \frac{q_{\sigma}}{q^2} (1 - \zeta^{-1}) P_{\mu\lambda}.
\]

We see that the gauge dependent terms, which are proportional to \( q_{\lambda} \) and \( q_{\sigma} \), vanish when either we take the trace of \( A_{\mu} \), or in the Feynman gauge. In the Feynman gauge, we have:
\[
A_{\mu}(q) = \frac{2q_{\mu}}{q^2} (1 - \Gamma) g_{\lambda\sigma}.
\]

It is a simple matter to work out \( A_{\mu}(q) \) and \( \partial^2_R A_{\mu}(q) \). These are given by:
\[
A_{\mu}(q) A_{\mu}(q) = (1 - \Gamma)^2 \frac{4g_{\lambda\sigma}}{q^2},
\]
and
\[
\partial^2_R A_{\mu}(q) = - \left[ \hat{\Gamma} - (1 - \Gamma)^2 \frac{4g_{\lambda\sigma}}{q^2},
\right.
\]
respectively. Here, \( \hat{\Gamma} \) refers to the derivative by \( \ln q^2 \). We may also show that exactly the same relations hold for \( A_{\mu}(q) \), provided that we replace \( g_{\lambda\sigma} \) on the right-hand side by \( P_{\lambda\sigma} \).

From eqns. \( \text{33} \) and \( \text{44} \) we obtain:
\[
\frac{1}{4} \text{Tr} [\partial^2_R A_{\mu}(q) - A_{\mu}(q) A_{\mu}(q)] = \frac{4}{q^2} \left[ -\hat{\Gamma} + \Gamma - \Gamma^2 \right].
\]

On the other hand, \( \alpha \partial^2 \Pi(q^2) \) yields, using eqn. \( \text{36} \):
\[
\alpha \partial^2 \Pi(q^2) = -Z(q^2) \partial^2 Z^{-1}(q^2) = \frac{4}{q^2} \left[ -\hat{\Gamma} + \Gamma - \Gamma^2 \right],
\]
i.e., similar to eqn. \( \text{45} \) but with a different sign for \( \hat{\Gamma} \). Therefore \( \partial^2_R \Pi(q^2) \) and \( \partial^2 \Pi(q^2) \) differ.

Now turning our attention to \( \partial^2_R \Pi(q^2) \), and temporarily adopting the language of perturbation theory with bare propagators, \( \Pi(q^2) \) has an expansion containing, in general, many gluon lines.

Following Gribov's argument mentioned in the introduction, the logarithmically enhanced contribution to the polarization operator comes from regions in the phase
space which involve large hierarchy of internal momenta. The greatest contribution to \( \partial_R^2 \Pi(q^2) \) comes from terms in which both derivatives act on the same line, and so we may apply eqn. 11 which removes a momentum integration.

This operation converts the amplitude into that of the emission of two zero-momentum gluons, so that the leading contribution will be given by:

\[
\partial_R^2 \Pi(q^2) \propto \text{Tr} [A_\mu(q)A_\mu(q)] + \mathcal{O}(\Gamma),
\]

including all the renormalization correction to the vertices and propagators.

At the one-loop order, we have, independently of the gauge parameter:

\[
\Pi_1(q) = -4b_0 \int \frac{d^4k}{-4\pi^2} \frac{1}{(k^2 + i\epsilon)((q - k)^2 + i\epsilon)},
\]

where the subscript in \( \Pi_1 \) refers to the perturbative order. \( b_0 \) is the first beta function coefficient, which we define to be positive for asymptotically-free theories. Applying \( \partial_R^2 \) to this expression, we obtain:

\[
\partial_R^2 \Pi_1(q) = \partial^2 \Pi_1(q) = -4b_0 \frac{1}{q^2 + i\epsilon}.
\]

Comparing with the form of eqn. 43 this fixes the constant of proportionality in eqn. 47 to be \(-b_0/4\).

Now let us work out the same constant of proportionality in an all-order approach based on dressed propagators. Inserting (some of the) Dyson–Schwinger type all-order corrections, as shown in fig. 1 into eqn. 48 we have:

\[
\Pi_{SD}(q) = -\frac{b_0}{3g^2} \text{Tr} \int \frac{d^4k}{-4\pi^2} \Gamma_{\lambda, \text{bare}}D(k)\Gamma_\lambda D(q-k).
\]

As in eqn. 20 \( \Gamma_{\text{bare}} \) represents the bare vertex.

---

**Fig. 1.** The Dyson–Schwinger-type correction to the one-loop gluonic vacuum-polarization operator. The thick lines and the vertex to the right are renormalized. The thin line and the vertex to the left are bare.

As it stands, eqn. 50 is problematic in the sense that it is not explicitly gauge invariant, and if it is, we should also include the contribution due to the ghost.

Since we would like to make use of the Ward–Takahashi identity of eqn. 26 it is necessary to work with transverse quantities. As discussed in sec. 2 a simple way to do this is to work in the Feynman gauge and impose transversality in the end. It is in fact not even necessary to reintroduce transversality at the end, but as a penalty, the spin-averaging factor will be replaced by 1/4.

We may thus replace \( \Gamma_\lambda \) by \(-\partial_\lambda D^{-1}\) (Feynman gauge), with the understanding that the phase-space region giving rise to large logarithms has \( q - k \ll q, k \) and hence the zero-momentum emission vertex is a good approximation to the full vertex. In this case, the simplest choice of scale would be \( D^{-1}(k) \). The integrand in eqn. 50 then becomes:

\[
\partial_\lambda D_0^{-1}(k)D(k)\partial_\lambda D^{-1}(k)D(q-k).
\]

When we apply \( \partial_R^2 \) to eqn. 50 the leading renormalization contribution comes from the double derivative, \( \partial^2 \), of \( D(q-k) \). The application of \( \partial^2 \) to \( D(q-k) \) yields a delta function in the approximation that the coupling is constant. If not, we have a correction term proportional to \( \Gamma \). The first derivative of \( D(q-k) \) yields:

\[
\partial_\mu D(q) = \frac{2Z(1-\Gamma)g_\mu}{((q-k)^2 + i\epsilon)^2} g_{\lambda\sigma}.
\]

Hence \( \partial^2 D(q-k) \) is approximately multiplied by factor \(-Z(1-\Gamma)\) as compared with eqn. 11 Factors of \( Z \) cancel in the expression for \( \partial_R^2 \Pi_{SD} \), which now reads:

\[
-\frac{b_0(1-\Gamma)}{4} \text{Tr} \left[ \partial_\lambda D_0^{-1}(q)D_0(q)\partial_\lambda D^{-1}(q)D(q) \right].
\]

Let us make the scale choice as \( \Gamma(q^2) \). The bare propagator \( D_0(q) \) and the renormalized propagator \( D(q) \) differ only by the factor \( Z \). Similarly, the difference between \( \partial_\lambda D_0^{-1} \) and \( \partial_\lambda D^{-1} \) is, by eqn. 42 \( Z^{-1}(1-\Gamma) \). Thus we have:

\[
\partial_R^2 \Pi_{SD}(q) = -\frac{b_0}{4} \text{Tr} \left[ \partial_\lambda D^{-1}(q)D(q)\partial_\lambda D^{-1}(q)D(q) \right].
\]

We have now shown that by applying Dyson–Schwinger-type corrections to the vacuum-polarization graph, we are able to reproduce the form which is expected by the logarithmic enhancement argument of Gribov which implies eqn. 47 without the need of such algebraic manipulations. Although this correspondence between the two approaches may seem intuitive and natural, we are not sure about how one would go about formulating such a correspondence in the case of the fermionic Gribov equation, eqn. 4.

In any case, together with eqn. 10 we have now established the following equation:

\[
\partial_\mu A_\mu(q) = (1-b_0\alpha) A_\mu(q)A_\mu(q) + \mathcal{O}(\Gamma^2).
\]

According to eqn. 52 the term in eqn. 53 that is proportional to \( b_0\alpha \), which corrects eqn. 52 is \( \mathcal{O}(\Gamma) \). This is true, since \(-b_0\) is the first expansion coefficient of the beta function. We have:

\[
-b_0\alpha = \Gamma + \mathcal{O}(\Gamma^2).
\]

Taking the first term, we obtain a rather compact expression with no parameter dependence:

\[
\partial_\mu A_\mu(q) = (1 + \Gamma(q^2)) A_\mu(q)A_\mu(q) + \mathcal{O}(\Gamma^2),
\]

from which we expect \( b_0 \) to reappear as a constant of integration. We shall see in the paragraph following eqn. 76...
that this reproduces the magnitude of the next-order coefficient, $b'$, of the beta function expansion. Therefore the error in eqn. 57 is practically $O(\Gamma^2)$. It is actually difficult to modify eqn. 56 without introducing unphysical fixed points. This point will be discussed further in the paragraph leading up to eqn. 60.

4 Solution of the equation

Let us now solve our Gribov equation, eqn. 57:

Substituting eqns. 43 and 44 into eqn. 57, we obtain:

\[ \dot{\Gamma} - (1 - \Gamma) + (1 + \Gamma)(1 - \Gamma)^2 \equiv \dot{\Gamma} - \Gamma^2(1 - \Gamma) = 0. \] (58)

That is,

\[ \frac{d(\beta/\alpha)}{d\ln q^2} = (\beta/\alpha)^2(1 - \beta/\alpha). \] (59)

\[ \Gamma = \beta/\alpha \] is positive for QED and negative for QCD. The equation is nominally not applicable to the case of QED beyond the one-loop order, or to the vacuum polarization due to the quark loop in QCD. On the other hand, we believe that the formalism is still useful in describing the UV behaviour of QED because of the fixed points inherent in eqn. 59. Note that the zeros of the right-hand side of eqn. 59 represent fixed points. The $\Gamma = 0$ fixed point corresponds to the trivial vacuum whereas $\Gamma = 1$ corresponds to the QED UV fixed point. In this limit, the running of the QED coupling cancels the $1/q^2$ propagator factor, and so the photon decouples.

Even though it is not obvious that the photon should necessarily decouple in the UV limit, a limit in which the photon decouples is almost necessarily a fixed point of the theory, since in this limit there is no longer photon propagation and hence no further evolution of the photon propagator. In this regard, the $\Gamma = 1$ fixed point of eqn. 59 seems physical.

On the other hand, the equation will certainly break down when discussing effects due to fermion masses or fermionic super-critical state formation. Related to this point, we do not expect eqn. 59 to be valid when the expansion parameter, i.e., in this case, $\Gamma$, is large.

In the case of UV QED, we think that the evolution is barely permissible because of the presence of the physical fixed point, but in the case of IR QCD, as there is no fixed point for negative $\Gamma$, $\Gamma$ diverges at, as we shall show, $\Lambda_{\text{QCD}}$. In this case, we do not believe that the equation is quantitatively correct. However, the behaviour of the solution is, we believe, nevertheless physical and, in any case, Gribov’s super-critical state formation occurs before this singularity. A measure of the quantitative accuracy of the equation is provided by the comparison with the perturbative beta-function expansion, and this will be presented in the next section.

We would like to note, to avoid confusion, that we also expect the $\Gamma = 1$ fixed point of eqn. 59 to be correct in the case of QCD because of the decoupling behaviour which it represents. However, the limit indicated by this fixed point does not arise in asymptotically-free theories.

Before proceeding, we note that, had we chosen a different expression for eqn. 56, we would, in general, end up with extra fixed points with unphysical power-like behaviour of the coupling. Because of this, it is difficult to introduce a simple higher-order modification to eqn. 56 without affecting its physical behaviour. On the other hand, if there arises a need to create a toy model for the running coupling with some particular power-like fixed-point behaviour, it is easy to artificially modify eqn. 59 to serve this purpose. For instance, one may like to introduce an $\alpha_S$ which is finite in the space-like region. A possibility would then be the substitution:

\[ - b_0\alpha = \Gamma - \frac{\Gamma^3}{1 - \Gamma}. \] (60)

In this case, corresponding to eqn. 59, we obtain:

\[ \dot{\Gamma} = \Gamma^2(1 - \Gamma^2), \] (61)

which is self-dual under $\ln q^2 \leftrightarrow - \ln q^2$ and $\Gamma \leftrightarrow - \Gamma$. In the low-energy limit of QCD, this gives us a $\alpha \propto 1/q^2$ behaviour, i.e., a simple toy model for the long-distance linear potential.

Resuming our discussion of eqn. 59, let us first confirm that it leads to the ordinary result for the running coupling when $\Gamma$ is small. We would like to calculate the evolution in the space-like region, i.e., for positive $Q^2 = -q^2$. Omitting the sub-leading term, we obtain:

\[ \beta/\alpha = [\ln Q^2 + \text{const.}]^{-1}. \] (62)

The constant on the right-hand side is $\ln \Lambda^2$. The left-hand side is the logarithmic derivative of $\ln \alpha$, so we obtain:

\[ \ln \alpha = \int \frac{1}{2} \ln(\Lambda^2/Q^2) \, d\ln Q^2 = \ln(\Lambda^2/Q^2) + \text{const.} \] (63)

This second integration constant gives the leading-order beta-function coefficient $b_0$, so we finally obtain:

\[ \alpha(Q^2) = \frac{1}{b_0 \ln(Q^2/\Lambda^2)}. \] (64)

As before, $b_0$ is, in our convention, positive for QCD and negative for QED.

More generally, the integration of eqn. 59 yields:

\[ \ln(\Lambda^2/Q^2) = \Gamma^{-1} + \ln |\Gamma^{-1} - 1|. \] (65)

The modulus is a shorthand for writing $\Gamma^{-1} - 1$ for QED and $1 - \Gamma^{-1}$ for QCD. Even though it is not possible to invert this equation, there is a trick to integrate it. We have:

\[ \log \alpha = \int \frac{d\ln Q^2}{\Gamma(1 - \Gamma)} = \log \left( \frac{-b_0^{-1} \Gamma}{1 - \Gamma} \right). \] (67)
$b_0^{-1}$ is the constant of integration. Thus we have:

$$(b_0 \alpha)^{-1} = 1 - \Gamma^{-1}, \quad (68)$$

which implies the following beta-function expansion:

$$\beta = \alpha \Gamma = -b_0 \alpha^2 (1 + b_0 \alpha + b_0^2 \alpha^2 + \cdots). \quad (69)$$

We shall compare this with the QCD beta-function coefficients in the paragraph following eqn. (68). We now substitute eqn. (68) in eqn. (65) to obtain:

$$\ln(Q^2/A_0^2) = (b_0 \alpha)^{-1} + \ln |b_0| \alpha, \quad (70)$$

or:

$$Q^2/A_0^2 = |b_0| \alpha \exp(1/b_0 \alpha). \quad (71)$$

We have defined $A_0 = \Lambda e^{-1/2}$. In the following, let us discuss the properties of this solution.

5 The case of QCD

Eqn. (59) and its solution, eqns. (70) and (71), are general both to QED and QCD. However, eqn. (59) has no zeros for negative $\Gamma$, and so its solution becomes non-analytical at $Q^2 = \Lambda^2 = A_0^2$. At this point, $b_0 \alpha$ reaches 1.

To see the behaviour of eqn. (71) below $A_0^2$, let us write:

$$b_0 \alpha = a e^{i \phi}. \quad (72)$$

Then by taking the imaginary part of eqn. (71) we obtain

$$a \phi = \sin \phi. \quad (73)$$

At $Q^2 = \Lambda^2, a = 1$ and $\phi = 0$. Below $Q^2 = \Lambda^2$, depending on whether we move the singularity above or below the real axis, $\phi$ becomes positive or negative. Adopting positive $\phi$, $a$ gradually decreases, and by the form of eqn. (73) $\phi$ gradually increases. In the limit $Q^2 \rightarrow 0$, $a$ vanishes, and we end up with $\phi = \pi$, i.e., $\alpha$ becomes negative. This rotation of phase implies that there is one gluonic supercritical state between $Q^2 = 0$ and $Q^2 = \Lambda^2$, with negative mass $b_0 \alpha$.

Taking the real part of eqn. (71) we obtain the other constraint:

$$\frac{\Lambda_0^2}{Q^2} = \frac{\phi}{\sin \phi} \exp \left(-\frac{\phi}{\tan \phi} \right). \quad (74)$$

We can now make a plot of $b_0 \alpha$ against $Q^2/\Lambda^2$ for both real and imaginary regions. In the real region, we use eqn. (74) to evaluate $Q^2/\Lambda^2$ as a function of $b_0 \alpha$, whereas in the imaginary region, we make use of eqns. (73) and (74) to evaluate $a$ and $\Lambda_0^2/Q^2$ as a function of $\phi$. Fig. 2 shows the plot obtained in this way.

Above $Q^2 = \Lambda^2$, we see that there is a modification to the one-loop perturbative evolution, which persists up to considerably high energy, but most of the modification can be absorbed by a shift in $A_0^2$, found numerically to be about $e^2 \approx 7$. This implies that the measured $A_{QCD}$ is about three times smaller than the true $A_{QCD}$ that is obtained in the high-energy limit.

To study this effect, let us consider the iterative inversion of eqn. (70):

$$(b_0 \alpha)^{-1} = \log \left( \frac{Q^2}{\Lambda^2} \log \left( \frac{Q^2}{\Lambda^2} \log \left( \frac{Q^2}{\Lambda^2} \log(\cdots) \right) \right) \right). \quad (75)$$

At the first level of truncation we obtain $A^2 \rightarrow A^2 b_0^2$ where $b_0$ is the value of $\alpha$ at some relevant scale, and so on.

In fact, such a shift in $A_{QCD}$ is known to be present already at the perturbative level. One expression for $\alpha_S$ at the two-loop order reads [8]:

$$\alpha_S^{-1}(Q^2) + b' \ln \left( \frac{b' \alpha_S(Q^2)}{1 + b' \alpha_S(Q^2)} \right) = b_0 \ln(Q^2/\Lambda^2), \quad (76)$$

where $b'$ is the ratio of the first and second beta-function coefficients:

$$\beta_{\text{perturbative}} = -b_0 \alpha^2 (1 + b' \alpha + b'' \alpha^2 + \cdots). \quad (77)$$

We see that eqn. (70) has almost the same form as eqn. (76). Since the presence of an extra $\alpha$ inside the denominator of the second term is a higher-order effect, and so is the choice of $\Lambda$, the two equations differ only by the difference between $b_0$ and $b'$, as can be inferred from eqn. (69). In the real-world QCD, this difference is given by:

$$\frac{b'}{b_0} = \frac{6(153 - 19 n_f)}{(33 - 2 n_f)^2} = 0.790 \cdots, \quad (78)$$

for $n_f = 3$. On the other hand, the large-$N_C$, or $n_f = 0$, limit of the same quantity is 0.843 $\cdots$. In either case, it is reasonably close to unity.

To analyze the higher-order contributions [8][10], let us introduce an alternative and common notation for the beta-function coefficients:

$$\beta/\alpha = -\sum_{n=1}^{\infty} \beta_{n-1} (\alpha/4\pi)^n. \quad (79)$$
The beta function coefficients and their ratios. The three-loop \([9]\) and four-loop \([10]\) contributions are calculated.

| \(n_f = 3\) | \(n_f = 0\) | \(N_C = \infty\) |
|----------------|----------------|------------------|
| \(\beta_0 = 4\pi b_0\) | 9               | 11               |
| \(\beta_1 = (4\pi)^2 b_0 b'\) | 64              | 102              |
| \(\beta_2 = (4\pi)^3 b_0 b''\) | 3863/6          | 2859/2           |
| \(\beta_3 = (4\pi)^4 b_0 b'''\) | 120904          | 292430           |
| \(\beta_1/\beta_0\) | 7.11            | 9.27             |
| \(\beta_2/\beta_1\) | 10.06           | 14.00            |
| \(\beta_3/\beta_2\) | 18.78           | 20.47            |
| \((\beta_2/\beta_1)/(\beta_1/\beta_0)\) | 1.415           | 1.510            |
| \((\beta_3/\beta_2)/(\beta_2/\beta_1)\) | 1.867           | 1.462            |

Table 1. The beta function coefficients and their ratios. The three-loop \([9]\) and four-loop \([10]\) contributions are calculated in the \(\text{MS}\) scheme.

These coefficients are shown in tab. 1 for the three cases: three light-quark flavours, zero flavours, and for \(N_C = \infty\) which implies zero flavours. The zero-flavour case starts to differ from the \(N_C = \infty\) case only at the four-loop order, where the non-planar contributions which are absent up to the three-loop order arise. \(\beta_2\) and \(\beta_3\) are dependent on the renormalization scheme, and the numbers quoted in tab. 1 correspond to the \(\text{MS}\) scheme. The choice of the renormalization scheme will affect our discussion here, at least in principle.

One notices that even though the ratio between two successive coefficients, \(\beta_{n+1}/\beta_n\), deviates away from \(\beta_0\) at higher orders, it seems to tend to a constant, when the number of flavours is zero. Of course, it is dangerous to draw any conclusions through knowing only these four coefficients, and our knowledge about the higher order coefficients is limited, but let us proceed with this tentative discussion for now.

To quantify this statement about the ratio of two successive coefficients tending to a constant, we have also tabulated the ratio:

\[
(\beta_{n+1}/\beta_n)/(\beta_n/\beta_{n-1}) = \beta_{n+1}/\beta_{n-1}/\beta_n^2, \quad \text{(80)}
\]

in tab. 1. For the beta-function expansion to have a finite radius of convergence, by d’Alembert’s ratio test, it is necessary, though not sufficient, for this ratio to tend to one. The point at which the beta-function expansion first diverges is the point at which super-critical behaviour arises. We expect the ratio to be positive, since its being negative would imply super-criticality for negative \(\alpha_S\), that is, for repulsive strong interaction, and this is unphysical.

The values of \(\beta_{n+1}/\beta_n\) being different from \(\beta_0\) implies that eqn. (83) is obviously not literally correct, but is an approximation in the sense that up to the three-loop order, these ratios are actually close to \(\beta_0\), even with finite (small) number of flavours. This gives partial assurance about the validity of our approach.

At higher order than three loops, in our opinion, the validity of the approach lies not in the numerical accuracy of the Gribov-equation evolution but rather in the physics it describes, namely super-criticality at finite \(\alpha_S\).

As for the formal presence of the \(\Gamma = 1\) fixed point in eqn. (66) this means that in the limit of large \(\alpha_S\), when the \(\beta\)-function is analytically continued beyond its formal radius of convergence, we expect \(\beta/\alpha\) to tend to 1.

Let us return to the discussion of the branch-point singularity mentioned at the beginning of this section. The formation of the gluonic super-critical state, indicated by this singularity, would make the vacuum unstable, since there is energy gain in, for example, the pair production of such states from the vacuum. As far as we can see, there is no way to stop or saturate this decay, either statistically or dynamical, and so pure QCD has to be, in our opinion, inconsistent.

However, the situation becomes different when we have light quarks. According to Gribov, the bound states of a pair of light quarks becomes super-critical when the coupling exceeds the critical coupling \(\alpha_c\) given by:

\[
\frac{\alpha_c}{\pi} C_F = 1 - \frac{\sqrt{2}}{3} = 0.183\cdots. \quad \text{(81)}
\]

This is smaller than the gluonic branch point which occurs at \(b_0\alpha = 1\):

\[
\frac{\alpha}{\pi} C_F \bigg|_{b_0\alpha = 1} = \frac{16}{33 - 2n_f} \approx 0.592\cdots. \quad \text{(82)}
\]

As before, we have taken \(n_f = 3\). We note that, potentially, the effective coupling which appears in the Gribov equation of eqn. (3) should be corrected by \((1 - \Gamma)\), which arises from eqn. (82). This is a small effect for the value of \(\alpha_S\) given by eqn. (81), but it reduces the critical coupling. In addition, the actual values of the beta function coefficients, or the ratios of them, tabulated in tab. 1 imply that the gluonic super-criticality occurs at much lower values of \(\alpha_S\), probably a half or so, than is implied by \(b_0\alpha = 1\). Explicitly, by the ratio test for the convergence of the beta function expansion, gluon super-criticality occurs at \(\alpha_S = \alpha_S^{\text{glu}}\) given by:

\[
\frac{\alpha_S^{\text{glu}}}{\pi} = \lim_{N \to \infty} \frac{\beta_N}{\beta_{N+1}}. \quad \text{(83)}
\]

It is not clear per se whether this limit is actually finite, or that it is renormalization-scheme independent. However, our work suggests that it is at most finite and is probably non-zero. If it is zero, the implication would be that there is some form of super-criticality at any value of the coupling, and this appears unlikely to us.

Returning to the discussion of the presence of the light quarks, since it is a non-trivial matter to consider the coupled evolution of the gluon and quark Green’s functions, let us assume that Gribov’s argument leading to the formation of light-quark super-critical states is essentially unmodified by the running of \(\alpha_S\). Obviously this is questionable if the singularity at any value of the coupling, and this appears unlikely to us.

Firstly, due the formation of the light-quark super-critical state, the quark Green’s function becomes complex, and this corresponds to the decay of the light quarks by emitting the super-critical state, or more strictly the decay of the vacuum. When this occurs, the gluon Green’s function also becomes complex, indicating the decay of...
the gluon by emitting the super-critical state through the
quarks. This would move the singularity off the real axis,
and the evolution of $\alpha_S$ will continue to $Q^2 = 0$ without
any further singularities.

$$Q^2 = -q^2$$

Fig. 3. The singularities of the gluon Green’s function. The
branch-point at $Q^2 = 0$ is labelled 0, that at $\Lambda^2$ is labelled
1. 2 corresponds to 1 moved off axis due to the decay into
light-quark super-critical states via light quarks.

This situation is shown in fig. 3. In one-loop perturbative
QCD, there is a branch-point at $q^2 + i\varepsilon = 0$, as well
as a simple pole at $Q^2 = -q^2 = \Lambda^2_{\text{QCD}}$. However, the non-
perturbative effects inherent in eqn. (57) tame this and make
it a branch point, corresponding to gluonic super-critical
singularity. The decay into light-quark super-critical states
move this singularity off the real axis.

This is when the negative-energy super-critical states
are not completely filled. When they are filled, these de-
cays become forbidden, and so $\alpha_S$ becomes real again.
When this occurs, the singularity near $Q^2 = \Lambda^2_{\text{QCD}}$ should
disappear, but there will be a new branch cut starting at
$q^2 - i\varepsilon = 0$ which, as in ref. [1], arises as a result of what
is best described as the reorganization of the Dirac sea.

To describe the evolution of the Green’s functions with
the new Dirac sea, according to the approach adopted in
ref. [1], it is sufficient to modify the evolution equations
by including the contribution of the Goldstone boson (the
pion). Since the Goldstone boson only couples to the gluon
through quarks, the inclusion of the Goldstone boson con-
tribution would be through the light-quark loops in the
vacuum polarization operator.

As stated above, the coupled evolution of quark and
gluon Green’s function is a non-trivial matter, but so long
as the modification is local in the momentum space as indi-
cated by ref. [1], the IR behaviour of $\alpha_S(Q^2)$ is governed
by eqn. (57). Even if the locality does not hold, in general,
we expect the quark contribution to the running of the QCD
coupling to be small, the evolution in the IR-free region
is almost entirely due to the gluons. Thus it may seem
strange that the gluons screen the coupling here rather
than yield the usual anti-screening behaviour. However,
this is as expected, because in the low-energy effective
theory, as a result of the forbidden decay to the super-
critical states (or in other words, the decay into the new
states created by Dirac-sea reorganization), there is a new
cut with the branch point at $q^2 - i\varepsilon = 0$. This means that
the Wick rotation has to be performed in the opposite
direction to the usual one, and so the direction of the
running is inverted.

This scenario is as discussed by Gribov [1]. The new
cuts, which apparently violate causality, correspond to
the presence of the positive-energy super-critical states
of negative-energy quarks. Causality is not violated, but it
appears as if positive-energy quarks are travelling back-
wards in time.

An IR-free gluon is decoupled at $Q^2 = 0$. Thus it
cannot be found as a free particle, though bound states
with finite radii can contain it. The masses of these bound
states would be of the order $\Lambda_{\text{QCD}}$ purely by dimensional
considerations. Hence the energy scale, or the energy gap,
for gluon decoupling is $\mathcal{O}(\Lambda_{\text{QCD}})$. This is soft confinement,
meaning that the gluons are bound together only by finite-
distance dynamics.

6 The case of QED

Let us now turn our attention to the case of the running
coupling in QED.

We show the plot of eqn. (71) for negative $b_0$ in fig. 4.
This plot is generated by evaluating $Q^2/\Lambda^2$ as a function
of $b_0\alpha$
The high-energy behaviour is governed by the $\Gamma = 1$ fixed point of eqn. \[5\]. The large-coupling limit of eqn. \[7\] is given by:

$$\lim_{\alpha \to \infty} \frac{\alpha(Q^2)}{b_0} = \frac{Q^2}{\Lambda^2}.\quad (84)$$

This corresponds to the straight line shown in fig. 5. We also show the ordinary one-loop perturbative result in the same plot.

As in QCD, we expect that the value of $\Lambda$ is highly sensitive to various factors, including the higher-order corrections. Therefore we do not trust the constant of proportionality in eqn. \[5\] to be accurate. On the other hand, we believe that the $\propto Q^2$ behaviour to be correct since, as explained earlier, the photon decouples from the theory only in this case.

This $\propto Q^2$ behaviour for the UV coupling yields a current–current contact interaction, similar to the NJL model \[6\]. The coupling constant, $M_N^2$, by eqn. \[3\], is given by $4\pi b_0 \Lambda^2$. The interaction Lagrangian density is, for space-like exchange:

$$\mathcal{L}_I = \frac{1}{M_N^2} \bar{\psi} \gamma^\mu \psi \bar{\psi} \gamma^\mu \psi.\quad (85)$$

Ref. \[6\] reports the formation of a massless Goldstone mode. This is consistent with Gribov’s EWSB mechanism \[5\] based on top quark condensation \[11\] due to the $U(1)_Y$ Landau pole. There are other bosonic modes such as the Higgs boson and the massive axial vector boson reported in ref. \[7\]. Although the photon decouples at $Q^2 = \Lambda^2$, these bosons remain physical even above this scale\[8\].

Our results are supported by a numerical study of high-energy QED \[12\], which yielded exactly the same conclusions, namely that the photon decouples, yielding a contact interaction which gives rise to chiral-symmetry breaking.

We believe that the effect of gravity does not spoil the applicability of this EWSB mechanism, because the relevant gravitational coupling remains small above the Planck scale \[13\].

There are no fixed points for time-like running, and therefore eqn. \[6\] remains a valid description for negative $Q^2 = -q^2$. In this case, the real part of the coupling changes its sign at $q^2 = \Lambda^2$, indicating that the photon becomes a ghost, or unphysical, above this value of $q^2$.

Let us now compare our results with eqn. \[4\] due to Gribov. As mentioned in the introduction, the strong-coupling limit of this equation is given by:

$$\left( \frac{d^2}{d\xi^2} + 2 \frac{d}{d\xi} \right) \frac{1}{g} = -\frac{2}{g^2}.\quad (86)$$

where $\xi = \ln(Q^2/\Lambda^2)$. If we omit the $g^{-2}$ term, the general solution of this equation is:

$$g^{-1} = C_1 + C_2 e^{-2\xi} = C_1 + C_2 \xi^4/Q^4.\quad (87)$$

$C_1$ and $C_2$ are the constants of integration. The coupling asymptotically tends to $C_1^{-1}$ at high energy.

With the inclusion of the $g^{-2}$ term, this is no longer constant, and we obtain $g^{-1} \to \xi^{-1}$, or:

$$g \to \ln(Q^2/\Lambda^2),\quad (88)$$

which is the behaviour obtained by Gribov \[4\].

Let us consider the modification to eqn. \[86\] due to the running of the photon propagator, viz. eqn. \[52\] for $g_{\text{eff}}$:

$$g_{\text{eff}} = g(1 - \Gamma).\quad (89)$$

We make this substitution because the $g$ on the right-hand side of eqns. \[4\] and \[86\] arises from the solution of eqn. \[3\]. By exactly the same argument as that in the paragraph following eqn. \[52\] there is an extra renormalization effect which multiplies $g$ by $(1 - \Gamma)$.

Since the left-hand side of eqn. \[86\] is derived by manipulating the derivatives of $\Pi(q^2)$, it is unaffected by this effect. We then have:

$$\left( \frac{d^2}{d\xi^2} + 2 \frac{d}{d\xi} \right) \frac{1}{g} = -\frac{2}{g^2(1 - \Gamma)^2}.\quad (90)$$

We have assumed that the strong-coupling approximation is valid in the sense that $g_{\text{eff}}$ is greater than approximately 1. An asymptotic solution, this time, is:

$$\Gamma \to 1 - \sqrt{2\xi^2/Q^2}, \quad g \to Q^2/\Lambda^2.\quad (91)$$

This choice of $\Gamma$ makes $g(1 - \Gamma)^2 \to 2$ asymptotically constant, and so $g \to Q^2/\Lambda^2 = \xi^2$ solves eqn. \[90\].

Hence we believe that our results are consistent with that of Gribov, provided that one takes into account the
effect due to photon renormalization inside the vacuum-polarization operator.

Before concluding this section, we would like to mention one property of eqn. 71 which does not seem particularly useful, but we think is worth mentioning.

Obviously eqn. 71 cannot be inverted to obtain $\alpha$ as some elementary function of $Q^2$, but certain moments of it can be evaluated in a closed form and are finite. Let $x = Q^2/\Lambda^2$ and $a = |b_0|\alpha$. Then:

$$
\int_0^\infty x^n a^{-m} da = \int_0^\infty e^{-n/a} a^{n-m} da.
$$

(92)

Then by the definition of the Euler Gamma function, this becomes:

$$
n^{1+n-m} \Gamma(m - n - 1).
$$

(93)

The moments of $x^n$ under $a$ may not be useful at all, but they are related to the moments of $a^{-m}$ under $x$, which may seem slightly more useful.

7 Conclusions

We derived a local Gribov equation for the gluon/photon Green’s function $D(q)$, and solved it for both QCD and QED.

Our derivation is based on taking the second derivative of $\text{Tr} \ln D(q)$. We separated out the parts due to the running coupling from the parts due to the renormalization of the leading, two-gluon(photon)-insertion term. Both using Gribov’s logarithmic-enhancement argument and using a Dyson–Schwinger-type expression, the part due to the running coupling is shown also to be of the form which corresponds to the double emission of zero-momentum gluons/photons, and so the Gribov equation can be written down in a compact form.

The Gribov equation gives an equation for the running of the coupling in a closed form. We obtained the solution of this equation in an analytical form, for both QCD and QED. Although we do not expect our equation to be fully applicable to QED, we argued that it has sensible UV behaviour.

In the case of QCD, we obtain an $\alpha_S$ which is finite, but has a branch-point singularity at $Q^2 = \Lambda^2_{\text{QCD}}$. We interpret this as being due to the formation of gluonic super-critical states, which makes the vacuum unstable. However, adopting Gribov’s scenario, the singularity is moved off the axis due to the decay of the gluon into light-quark super-critical states. Furthermore, with the reorganization of the (Dirac-sea) vacuum, these decays become forbidden, giving rise to a coupling which only has singularities along the time-like axis. However, there are singularities on both $+i\varepsilon$ and $-i\varepsilon$ sides of the time-like axis, giving rise to, in the low-energy limit, a coupling which is IR-free. The gluon is then softly confined.

In the case of QED, the Gribov equation has both an IR fixed point given by $\Gamma = 0$ and UV fixed point given by $\Gamma = 1$. The latter gives a UV $\propto Q^2$ limiting behaviour for the coupling. The photon decouples from the theory. The high energy limit of QED is then given by the contact interaction. This supports Gribov’s scenario of EWSB by top-quark condensation due to the strong $U(1)_Y$ interaction near the would-be Landau pole.

We believe that our formalism can be applied also to the problem of scale generation in gravity.

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