GENERALIZED FUBINI TRANSFORM WITH TWO VARIABLES

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ABSTRACT. In the present paper, we define the generalized Kwang-Wu Chen matrix. Basic properties of this generalization, such as explicit formulas and generating functions are presented. Moreover, we focus on a new class of generalized Fubini polynomials. Then we discuss their relationship with other polynomials such as Fubini, Bell, Eulerian and Frobenius-Euler polynomials. We have also investigated some basic properties related to the degenerate generalized Fubini polynomials.

1. INTRODUCTION

The $n$th Bernoulli numbers $B_n$ are defined by the generating function

\begin{equation}
\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.
\end{equation}

The rational numbers $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30$ and $B_{2n+1} = 0$ for $n > 0$, have many beautiful properties. The most basic recurrence relation is

\begin{equation}
\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0.
\end{equation}

In 2001, Kwang-Wu Chen [5] gave an algorithm for computing Bernoulli numbers, with

\begin{equation}
a_{0,m} = \frac{1}{m+1}; \quad a_{n+1,m} = -(m+1)a_{n,m+1} + ma_{n,m}.
\end{equation}

The primary purpose of this paper is to extend the Fubini transform for generalizing Fubini polynomials and studying its properties. We first generalize (1.3). The idea is to construct an infinite matrix $M := (a_{n,m})_{n,m \geq 0}$ in which the first row $a_{0,m} := \alpha_m$ of the matrix is the initial sequence and the first column $a_{n,0} := \beta_n$ is the final sequence. More precisely, for nonzero complex numbers $x$ and $y$, we propose to study the following three-term recurrence relation

\begin{equation}
a_{n+1,m}(x,y) = x(m+1)a_{n,m+1}(x,y) + yma_{n,m}(x,y).
\end{equation}

By setting $x = -1$ and $y = 1$ in (1.4), we get (1.3). More directly, we propose to generalize the Fubini transformation.

The Fubini transform of a sequence $(\alpha_n)_{n \geq 0}$ is the sequence $(\beta_n)_{n \geq 0}$ given by

$$
\beta_n = \sum_{k=0}^{n} k! \binom{n}{k} t^k \alpha_k
$$

and the inverse transform is

$$
\alpha_n = \frac{1}{n!t^n} \sum_{k=0}^{n} s(n,k) \beta_k.
$$

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2. Definitions and notation

In this section, we introduce some definitions and notations which are useful in the rest of the paper. Following the usual notations [7].

The falling factorial $x^\underline{n}$ $(x \in \mathbb{C})$ is defined by

$$x^\underline{n} = x(x-1) \cdots (x-n+1), \ x^\underline{0} = 1$$

and the rising factorial denoted by $x^\bar{n}$, is defined by

$$x^\bar{n} = x(x+1) \cdots (x+n-1), \ x^\bar{0} = 1.$$

The (signed) Stirling numbers of the first kind denoted $s(n,k)$ are the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} s(n,k) x^k. \quad (2.1)$$

The exponential generating function is

$$\frac{1}{k!} (\ln(1+t))^k = \sum_{n \geq k} s(n,k) \frac{t^n}{n!}, \quad (2.2)$$

and $s(n,k)$ satisfy the following recurrence relation:

$$s(n+1,k) = s(n,k-1) - ns(n,k) \quad (2.3)$$

and that

$$s(n,0) = \delta_{n,0} \ (n \in \mathbb{N}), \ s(n,k) = 0 \ (k > n \text{ or } k < 0),$$

where $\delta_{n,m}$ denoted Kronecker symbol.

The Stirling numbers of the second kind denoted $\{n\choose k}$ count the number of ways to partition a set of $n$ things into $k$ nonempty subsets. Explicitly $\{n\choose k}$ are the coefficients in the expansion

$$x^n = \sum_{k=0}^{n} \{n\choose k\} x^k.$$

The $r$-Stirling numbers [3] denotes $\{n\choose k\}_r$, for any positive $r \in \mathbb{N}$, the number of partitions of a set of $n$ objects into exactly $k$ nonempty, disjoint subsets, such that the first $r$ elements are in distinct subsets. These numbers obey the recurrence relation

$$\{n\choose k\}_r = 0, \quad n < r, \quad (2.4)$$

$$\{n\choose k\}_r = \delta_{k,r}, \quad n = r,$$

$$\{n\choose k\}_r = k \{n-1\choose k\}_r + \{n-1\choose k-1\}_r, \quad n > r$$

and

$$\{n\choose k\}_r = \{n\choose k\}_{r-1} - (r-1) \{n-1\choose k\}_{r-1}. \quad (2.5)$$

The exponential generating function is given by

$$\frac{1}{k!} e^r (e^t - 1)^k = \sum_{n \geq k} \{n+r\choose k+r\} \frac{t^n}{n!}. \quad (2.6)$$
3. The Generalized Fubini Transform

Theorem 1. Given an initial sequence \((a_{0,m})_{m \geq 0}\) define the matrix \(\mathcal{M}\) associated with the initial sequence by (1.4) then

1. The entries of the matrix \(\mathcal{M}\) are given by

\[
a_{n,m}(x,y) = \frac{1}{m!} \sum_{k=0}^{n} \binom{n+m}{k+m} (k+m)! x^{n-k} y^{k} a_{0,m+k}.
\]

2. Suppose that the initial sequence \(a_{0,m+r}\) has the following ordinary generating function \(A_r(t) = \sum_{k=0}^{\infty} a_{0,k+r} t^k\). Then, the sequence \((a_{n,r}(x))_{n \geq 0}\) of the \(r\)th columns of the matrix \(\mathcal{M}\) has an exponential generating function \(B_r(t;x,y) = \sum_{n=0}^{\infty} a_{n,r}(x) t^n / n!\), given by

\[
B_r(t;x,y) = \frac{e^{txy}}{r!} \left( e^{-ty} \frac{d}{dt} \right)^r \left( \frac{e^{txy} - 1}{y} \right) A_r \left( \frac{x}{y} (e^{txy} - 1) \right).
\]

Proof. (1) We prove the relation (3.1) by induction on \(n\). The result clearly holds for \(n = 0\), we now show that the formula for \(n + 1\) follows from (1.4) and induction hypothesis

\[
a_{n+1,m}(x,y) = \frac{1}{m!} \sum_{k=0}^{n} \binom{n+m+1}{k+m+1} (k+m+1)! x^{n-k} y^{k} a_{0,m+k+1}
+ m \left\{ \binom{n+m}{m} y^{n+1} a_{0,m} + \frac{1}{(m-1)!} \sum_{k=1}^{n} \binom{n+m}{k+m} (k+m)! x^{n-k+1} y^{k} a_{0,m+k+1} \right.
+ \frac{1}{m!} \binom{n+m+1}{n+m+1} (n+m+1)! x^{n+1} a_{0,m+n+1}.
\]

After some rearrangements, we get

\[
a_{n+1,m}(x,y) = \frac{1}{m!} \sum_{k=1}^{n} \left\{ \binom{n+m}{k+m} (k+m)! x^{k} y^{n-k+1} a_{0,m+k} + m \binom{n+m}{k+m} (k+m)! x^{k} y^{n-k+1} a_{0,m+k} + \frac{1}{(m-1)!} \sum_{k=1}^{n} \binom{n+m}{k+m} (k+m)! x^{k} y^{n-k+1} a_{0,m+k} + \frac{1}{m!} \binom{n+m+1}{n+m+1} (n+m+1)! x^{n+1} a_{0,m+n+1}.\]

From (2.3) and (2.5), and after some rearrangements, we get

\[
a_{n+1,m}(x,y) = \frac{1}{m!} \sum_{k=0}^{n} \left\{ \binom{n+m+1}{k+m} (k+m)! x^{k} y^{n-k+1} a_{0,m+k} + m \binom{n+m}{k+m} (k+m)! x^{k} y^{n-k+1} a_{0,m+k} + \frac{1}{(m-1)!} \sum_{k=1}^{n} \binom{n+m}{k+m} (k+m)! x^{k} y^{n-k+1} a_{0,m+k} + \frac{1}{m!} \binom{n+m+1}{n+m+1} (n+m+1)! x^{n+1} a_{0,m+n+1}.\]
\]

\[
= \frac{1}{m!} \sum_{k=0}^{n+1} \binom{n+m+1}{k+m} (k+m)! x^{k} y^{n-k+1} a_{0,m+k}.
\]
(2) The verification of (3.2) follows by induction on \( n \). By using (3.1), we obtain
\[
B_r(t; x, y) = \sum_{n \geq 0} \left( \frac{1}{n!} \sum_{k=0}^{n} \binom{n + r}{k} (k + r)! x^k y^{n-k} a_{0,r+k} \right) t^n n!
= \sum_{k \geq 0} \frac{(k + r)!}{r!} \left( \frac{x}{y} \right)^k a_{0,r+k} \sum_{n \geq k} \binom{n + r}{k} \frac{(ry)^n}{n!}.
\]
From the relation (2.6), we obtain
\[
B_r(t; x, y) = \sum_{k \geq 0} \frac{(k + r)!}{r!} \left( \frac{x}{y} \right)^k a_{0,r+k} \left( x (e^{ry} - 1) \right)^k.
\]
Since
\[
\frac{(k + r)}{r} \left[ \frac{x}{y} (e^{ry} - 1) \right]^k = \frac{1}{r! x^r} \left( e^{-ty} \frac{d}{dt} \right)^r \left[ \frac{x}{y} (e^{ry} - 1) \right]^{k+r},
\]
we get
\[
B_r(t; x, y) = e^{ryt} \frac{1}{r!} \left( e^{-ty} \frac{d}{dt} \right)^r \left[ \left( \frac{e^{ry} - 1}{y} \right) A_r \left( \frac{x}{y} (e^{ry} - 1) \right) \right].
\]
This evidently completes the proof of Theorem.

\[\square\]

The following corollary represents another expression for the generating function \( B_r \).

**Corollary 2.**

(3.3) \[ B_r(t; x, y) = \frac{1}{r!} \sum_{k=0}^{r} \binom{r}{k} \frac{(e^{ry} - 1)}{y} A_r \left( \frac{x}{y} (e^{ry} - 1) \right). \]

To prove formula (3.3) using
\[
\left( e^{-ty} \frac{d}{dt} \right)^r F(t) = e^{-ryt} \frac{1}{r!} \sum_{k=0}^{r} \binom{r}{k} \frac{d^k}{dt^k} F(t),
\]
with
\[
F(t) = \left( \frac{e^{ry} - 1}{y} \right) A_r \left( \frac{x}{y} (e^{ry} - 1) \right).
\]

**Theorem 3.** Given final sequence \((a_n, 0)_{n \geq 0}\), define the matrix \( \mathcal{M} \) associated with the final sequence by

(3.4) \[ a_{n+1}(x, y) = \frac{1}{x(m+1)} \left( a_{n+1,m}(x, y) - yma_{n,m}(x, y) \right), \]

then

(1) The entries of the matrix \( \mathcal{M} \) are given by

(3.5) \[ a_{n,m}(x, y) = \frac{y^m}{x^m m!} \sum_{k=0}^{m} x^{-k} s(m, k) a_{n+k,0}. \]
(2) Suppose that the final sequence \( a_{n+r,0} \) has the following exponential generating function 
\[ \hat{\mathcal{B}}_r(t) = \sum_{k \geq 0} a_{k+r,0} \frac{t^k}{k!}. \]
Then, the sequence \((a_{r,m}(x))_{m \geq 0}\) of the \( r \)th row of the matrix \( \mathcal{M} \) has
an ordinary generating function \( \hat{\mathcal{F}}_r(t; x, y) = \sum_{m \geq 0} a_{r,m}(x,y)t^m \), given by
\[
(3.6) \quad \hat{\mathcal{F}}_r(t; x, y) = \hat{\mathcal{B}}_r \left( y^{-1} \ln \left( 1 + \frac{ty}{x} \right) \right).
\]

**Proof.**

(1) We prove by induction on \( m \), the result clearly holds for \( m = 0 \). By induction hypothesis and (3.4), we have

\[
a_{n,m+1}(x,y) = \frac{\gamma^m}{x^{m+1} (m+1)!} \left( \sum_{k=0}^{m} y^{-k}s(m,k)a_{n+k,0} + y^m \sum_{k=0}^{m} y^{-k}s(m,k)a_{n+k,0} \right)
\]

After some rearrangements, we get

\[
a_{n,m+1}(x,y) = \frac{\gamma^m}{x^{m+1} (m+1)!} \left( \sum_{k=0}^{m} y^{-k}s(m,k-1)a_{n+k,0} + y^m \sum_{k=0}^{m} y^{-k}s(m,k)a_{n+k,0} \right)
\]

From (2.3) and after some rearrangements, we get

\[
a_{n,m+1}(x,y) = \frac{\gamma^m}{x^{m+1} (m+1)!} \left( \sum_{k=0}^{m} y^{-k}s(m,k-1)a_{n+k,0} + y^m \sum_{k=0}^{m} y^{-k}s(m,k)a_{n+k,0} \right)
\]

which completes the proof.

(2) According to (3.3), we have

\[
\hat{\mathcal{F}}_r(t; x, y) = \sum_{m \geq 0} \left( \frac{\gamma^m}{x^m m!} \sum_{k=0}^{m} y^{-k}s(m,k)a_{r+k,0} \right) t^m
\]

From the relation (2.2), we obtain
the sequence $\beta$ Fubini polynomials, by means of the generating function

\[ \alpha \left( t; x, y \right) = \sum_{k \geq 0} a_{r+k,0} y^{-k} \frac{1}{k!} \left( \ln \left( 1 + \frac{tx}{y} \right) \right)^k \]

\[ = \beta \left( y^{-1} \ln \left( 1 + \frac{tx}{y} \right) \right), \]

which completes the proof.

**Corollary 4.** For $n, m \geq 0$, we have

\[ \sum_{k=0}^{n} \binom{n+m}{k+m} (k+m)! x^k y^{n-k} a_{0,m+k} = \sum_{k=0}^{m} s(m,k) x^{-m} y^{m-k} a_{n,k,0}. \]

The identity (3.7) can be viewed as the generalized Fubini transform which can be reduced, for $m = 0$ to the Fubini transform of the sequence $\alpha$, and for $n = 0$ to the inverse Fubini transform of the sequence $\beta$.

4. ON GENERALIZED FUBINI POLYNOMIALS

Setting the initial sequence $a_{0,m} = 1$ in (1.4), we get the following matrix

\[ M = \begin{pmatrix}
1 & 1 & 1 & \cdots \\
x & y + 2x & 2y + 3x & 3y + 4x & \cdots \\
2x^2 + xy & 6x^2 + 6xy + y^2 & 12x^2 + 15xy + 4y^2 & \cdots \\
6x^3 + 6x^2y + xy^2 & 24x^3 + 36x^2y + 14xy^2 + y^3 & \cdots \\
24x^4 + 36x^3y + 14x^2y^2 + xy^3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} \]

Since $A_r(t) = \frac{1}{e^t}$, it follows from (3.2) and (3.3) that the final sequence has an exponential generating function given by

\[ B_r(t; x, y) = e^{ty} \frac{d^r}{dt^r} \left( e^{-ty} \frac{d}{dt} \left[ \left( \frac{e^y - 1}{y} \right)^r \frac{1}{1 - \frac{1}{y} (e^y - 1)} \right] \right) \]

\[ = \frac{1}{r!} \sum_{k=0}^{r} s(r,k) \frac{d^k}{dt^k} \left[ \left( \frac{e^y - 1}{y} \right)^r \frac{1}{1 - \frac{1}{y} (e^y - 1)} \right]. \]

In particular for $r = 0$, we have

\[ B_0(t; x, y) = \frac{1}{1 - \frac{1}{y} (e^y - 1)}. \]

**Definition 5.** We defined a sequence of polynomials $\tilde{\alpha}_n(x,y)$ of two variables $x,y$, called generalized Fubini polynomials, by means of the generating function

\[ \frac{1}{1 - \frac{1}{y} (e^y - 1)} = \sum_{n \geq 0} \tilde{\alpha}_n(x,y) \frac{t^n}{n!}. \]

The explicit formula for $\tilde{\alpha}_n(x,y)$ is given by

\[ \tilde{\alpha}_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} k! x^k y^{n-k}. \]
By setting \( y = 1 \) in (4.1), we get
\[
\frac{1}{1 - x(e^t - 1)} = \sum_{n \geq 0} \hat{F}_n(x, 1) \frac{t^n}{n!},
\]
(4.3)
\[
= \sum_{n \geq 0} \omega_n(x) \frac{t^n}{n!},
\]
where \( \omega_n(x) \) denotes the Fubini polynomials \([1, 8, 11]\), defined by
\[
\omega_n(x) = \sum_{k=0}^{n} \binom{n}{k} k! x^k.
\]

By (4.1) and (4.3), we can write the relation between \( \omega_n(x) \) and \( F_n(x, y) \), given by the following two formulas
\[
F_n(x, y) = y^n \omega_n \left( \frac{x}{y} \right)
\]
(4.4)
and
\[
\omega_n(x) = y^{-n} \hat{F}_n(xy, y).
\]
(4.5)

The Fubini polynomials \( \omega_n(x) \) are related to the geometric series in the following way \([1, 2]\)
\[
\left( \frac{x}{1 - x} \right)^n = \sum_{k \geq 0} x^k = \frac{1}{1 - xe^{t}} = \sum_{n \geq 0} \left( \sum_{k \geq 0} \frac{1}{k!} (e^t)^k \right) t^n.
\]
(4.6)

This relation can be extended to a more general form depending on two variables \( x \) and \( y \).

**Theorem 6.** For \( x \) different to \( y \), the polynomials \( \hat{F}_n(x, y) \) have the following property
\[
\frac{y}{y-x} \hat{F}_n \left( \frac{xy}{y-x}, y \right) = \sum_{k \geq 0} \left( \frac{x}{y} \right)^k (yk)^n = y^n \left( \frac{x}{1-x} \right)^n \frac{y}{y-x}.
\]
(4.7)

**Proof.** We have
\[
\frac{y}{y-x} \sum_{n \geq 0} \hat{F}_n \left( \frac{xy}{y-x}, y \right) \frac{t^n}{n!} = \frac{y}{y-x} \left( \frac{1}{1 - ye^{t} - (e^t - 1)} \right)
\]
\[
= \frac{1}{1 - ye^{t}}
\]
\[
= \sum_{k \geq 0} \left( \frac{x}{y} \right)^k (e^t)^k.
\]
Then
\[
\frac{y}{y-x} \sum_{n \geq 0} \hat{F}_n \left( \frac{xy}{y-x}, y \right) \frac{t^n}{n!} = \sum_{k \geq 0} \frac{x}{y} \sum_{n \geq 0} (ky)^n \frac{t^n}{n!}
\]
\[
= \sum_{n \geq 0} \sum_{k \geq 0} \frac{x^k}{y} (ky)^n \frac{t^n}{n!}.
\]
Equating the coefficients of $\frac{r^n}{n!}$, we get
\[
\frac{y}{y-x} \mathcal{S}_n \left( \frac{xy}{y-x}, y \right) = \sum_{k \geq 0} \left( \frac{x}{y} \right)^k (ky)^n.
\]

On the other hand, we apply the formula (4.1) in [1], we get
\[
\sum_{k \geq 0} \left( \frac{x}{y} \right)^k (yk)^n = y^n \sum_{k \geq 0} \left( \frac{x}{y} \right)^k \sum_{k \geq 0} \left( \frac{x}{y} \right)^k
\]
\[
= y^n \left( \frac{x}{y} \right)^n \frac{y}{y-x}.
\]
(4.8)

This evidently completes the proof of the theorem. □

**Remark 1.** By setting $y = 1$ in (4.7) we get (4.6).

Now, recall that the exponential generating function for Bell polynomials $\phi_n(x)$, is given by
\[
e^{(e^t-1)} = \sum_{n \geq 0} \phi_n(x) \frac{t^n}{n!}
\]
and given explicitly by
\[
\phi_n(x) = \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\} x^k.
\]
(4.10)

In the following result, we will give the integral representation for $\mathcal{S}_n(x, y)$ and the link with $\phi_n(x)$.

**Theorem 7.** For $n \geq 0$, we have
\[
\mathcal{S}_n(x, y) = y^n \int_{0}^{\infty} \phi \left( \frac{x}{y} \lambda \right) e^{-\lambda} d\lambda
\]
(4.11)

and
\[
\sum_{n \geq 0} \mathcal{S}_n(x, y) \frac{t^n}{n!} = \int_{0}^{\infty} e^{-\lambda(1-\frac{y}{x} (e^\lambda-1))} d\lambda.
\]
(4.12)

**Proof.** Replacing $x$ by $\frac{x}{y} \lambda$ in (4.10) and multiplying both sides by $y^n e^{-\lambda}$ and integrating for $\lambda$ from zero to infinity, we have
\[
y^n \int_{0}^{\infty} \phi \left( \frac{x}{y} \lambda \right) e^{-\lambda} d\lambda = y^n \int_{0}^{\infty} \left( \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\} \left( \frac{x}{y} \lambda \right)^k \right) e^{-\lambda} d\lambda
\]
\[
= y^n \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\} \frac{x^k}{y^k} \int_{0}^{\infty} e^{-\lambda} \lambda^k d\lambda
\]
\[
= y^n \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\} \frac{x^k}{y^k} k!.
\]

By comparing with (4.2) we get (4.11).
Now to prove (4.12), using (4.11), we have
\[
\sum_{n \geq 0} \mathcal{F}_n(x, y) \frac{t^n}{n!} = \sum_{n \geq 0} y^n \int_0^{+\infty} \phi \left( \frac{x}{y} \lambda \right) e^{-\lambda} \frac{(ty)^n}{n!} d\lambda = \int_0^{+\infty} e^{-\lambda} \sum_{n \geq 0} \phi \left( \frac{x}{y} \lambda \right) \frac{(ty)^n}{n!} d\lambda.
\]
we apply (4.9), we get
\[
\sum_{n \geq 0} \mathcal{F}_n(x, y) \frac{t^n}{n!} = \int_0^{+\infty} e^{-\lambda} \sum_{n \geq 0} \phi \left( \frac{x}{y} \lambda \right) y^n \omega_{n-k} \left( \frac{x}{y}, y \right) \frac{t^n}{n!} d\lambda = \int_0^{+\infty} \frac{e^{-\lambda} e^{\frac{x}{y}(e^\lambda - 1)}}{1 - x(e^\lambda - 1)} d\lambda.
\]

Remark 2. By setting y = 1 in (4.11) and (4.12), respectively, we get (3.11) and (3.13) in [1].

The Fubini polynomials of two variables \(\omega_n(x, y)\) are defined in [8, 9, 11] by the following generating function
\[
e^{ty} \frac{1}{1 - x(e^y - 1)} = \sum_{n \geq 0} \omega_n(x, y) \frac{t^n}{n!}.
\]
The next result represents the relation between \(\mathcal{F}_n(x, y)\) and \(\omega_n(x, y)\).

Theorem 8. For \(n \geq 0\), we have
\[
\mathcal{F}_n(x, y) = y^n \sum_{k=0}^{n} y^k \binom{n}{k} (-1)^k \omega_{n-k} \left( \frac{x}{y}, y \right).
\]

Proof. From (4.1) and (4.13), we have
\[
\sum_{n \geq 0} \mathcal{F}_n(x, y) \frac{t^n}{n!} = e^{-ty} \frac{e^{ty^2}}{1 - x(e^y - 1)} = \left( \sum_{n \geq 0} \frac{(-ty^2)^n}{n!} \right) \left( \sum_{n \geq 0} \omega_n \left( \frac{x}{y}, y \right) \frac{(ty)^n}{n!} \right) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} (-y^2)^k \omega_{n-k} \left( \frac{x}{y}, y \right) \frac{t^n}{n!} \right) = \sum_{n \geq 0} \left( y^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k y^k \omega_{n-k} \left( \frac{x}{y}, y \right) \frac{t^n}{n!} \right),
\]
that is to say
\[
\mathcal{F}_n(x, y) = y^n \sum_{k=0}^{n} y^k \binom{n}{k} (-1)^k \omega_{n-k} \left( \frac{x}{y}, y \right).
\]

In addition to the above properties of \(\mathcal{F}_n(x, y)\) polynomials, we now present some recurrence relations. The following lemma will be useful for the proof of the next theorem.
Lemma 1. For nonzero complex numbers $x$ and $y$, we have

$$
\frac{e^{ty}}{1 - \frac{x}{y} (e^{ty} - 1)} = \left( \frac{1}{x} - \frac{1}{y} (e^{ty} - 1) \right) \frac{d}{dt} \left( \frac{1}{1 - \frac{x}{y} (e^{ty} - 1)} \right).
$$

Theorem 9. For $n \geq 0$, we have

$$
\mathfrak{F}_{n+1} (x, y) = \left( \frac{xy}{x+y} \right) \sum_{k=0}^{n} \binom{n}{k} y^{n-k} \left( \mathfrak{F}_k (x, y) + \frac{1}{y} \mathfrak{F}_{k+1} (x, y) \right).
$$

Proof. Using the above lemma, then (4.15) is equivalent to

$$
\sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} y^{n-k} \mathfrak{F}_k (x, y) \right) \frac{t^n}{n!} = \left( \frac{1}{x} - \frac{1}{y} \sum_{n \geq 0} \frac{(ty)^n}{n!} \right) \sum_{n \geq 0} \mathfrak{F}_{n+1} (x, y) \frac{t^n}{n!}.
$$

Then,

$$
\sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} y^{n-k} \mathfrak{F}_k (x, y) \right) \frac{t^n}{n!} = \frac{1}{x} \sum_{n \geq 0} \mathfrak{F}_{n+1} (x, y) \frac{t^n}{n!} - \frac{1}{y} \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} y^{n-k} \mathfrak{F}_{k+1} (x, y) \right) \frac{t^n}{n!}.
$$

Equating the coefficients of $\frac{t^n}{n!}$, we get

$$
\sum_{k=0}^{n} \binom{n}{k} y^{n-k} \mathfrak{F}_k (x, y) = \frac{1}{x} \mathfrak{F}_{n+1} (x, y) - \frac{1}{y} \sum_{k=0}^{n} \binom{n}{k} y^{n-k} \mathfrak{F}_{k+1} (x, y) + \frac{1}{y} \mathfrak{F}_{n+1} (x, y).
$$

and after some rearrangements, we obtain the result.

Remark 3. As a special case, we get the formula (24) in [6] by setting $y = 1$ in (4.16).

Theorem 10. For $n \geq 0$, we have

$$
\mathfrak{F}_{n+1} (x, y) + y \mathfrak{F}_n (x, y) = (x + y) \sum_{k=0}^{n} \binom{n}{k} \mathfrak{F}_k (x, y) \mathfrak{F}_{n-k} (x, y).
$$
**Proof.** Considering the derivative of the generating function of the polynomials $\mathcal{F}_n(x,y)$ (4.1), we have

\[
\sum_{n \geq 0} \mathcal{F}_{n+1}(x,y) t^n \frac{n!}{n!} = x e^{ty} \frac{1}{1 - \frac{e^y - 1}{y}} = \left( \frac{x + y}{1 - \frac{e^y - 1}{y}} - y \right) \frac{1}{1 - \frac{e^y - 1}{y}}
\]

\[
= (x+y) \sum_{n \geq 0} \mathcal{F}_n(x,y) \frac{t^n}{n!} \sum_{n \geq 0} \mathcal{F}_n(x,y) \frac{t^n}{n!} - y \sum_{n \geq 0} \mathcal{F}_n(x,y) \frac{t^n}{n!}
\]

\[
= (x+y) \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} \mathcal{F}_k(x,y) \mathcal{F}_{n-k}(x,y) - y \mathcal{F}_n(x,y) \right) \frac{t^n}{n!}.
\]

Equating the coefficients of $\frac{t^n}{n!}$, and after some rearrangements, we obtain the result. □

For $y = 1$, we get the result of the Theorem 1 in [8].

**Theorem 11.** For $n \geq 0$ and for $x_1$ different to $x_2$, we have

\[
\sum_{n \geq 0} \binom{n}{k} \mathcal{F}_k(x_1,y) \mathcal{F}_{n-k}(x_2,y) = \frac{x_2 \mathcal{F}_n(x_2,y) - x_1 \mathcal{F}_n(x_1,y)}{x_2 - x_1}.
\]

**Proof.** The proof of (4.18) becomes as follows

\[
\frac{1}{1 - \frac{x_1}{y} (e^y - 1)} \frac{1}{1 - \frac{x_2}{y} (e^y - 1)} = \frac{x_2}{x_2 - x_1} \frac{1}{1 - \frac{x_2}{y} (e^y - 1)} - \frac{x_1}{x_2 - x_1} \frac{1}{1 - \frac{x_1}{y} (e^y - 1)}.
\]

□

Now, in this part of the paper, we will connect the polynomials $\mathcal{F}_n(x,y)$ with Eulerian polynomials and Frobenius-Euler polynomials. It is known that for $x \neq 1$ and $n \geq 0$, the Eulerian polynomials $A_n(x)$ and the Frobenius-Euler polynomials $H_n(x;y)$ are defined respectively by the following generating functions [12, 13]

\[
\frac{1 - x}{e^x - 1} - x = \sum_{n \geq 0} A_n(x) \frac{t^n}{n!},
\]

\[
\frac{1 - x}{e^x - 1} \frac{e^{ty}}{e^y - 1} = \sum_{n \geq 0} H_n(x;y) \frac{t^n}{n!}.
\]

**Theorem 12.** For $n \geq 0$, and for nonzero complex numbers $x$ and $y$, we have

\[
\mathcal{F}_n(x,y) = x^n A_n \left( 1 + \frac{y}{x} \right)
\]

and for $t \neq 1$, we have

\[
A_n(t) = \left( \frac{t-1}{y} \right)^n \mathcal{F}_n \left( \frac{y}{t-1}, y \right) = \left( \frac{1}{x} \right)^n \mathcal{F}_n \left( x, x(t-1) \right).
\]
Proof. The generating functions (4.1) and (4.19) can be rewritten as
\[
\frac{1}{e^y - (1 + \frac{1}{x})} = \sum_{n \geq 0} \frac{\delta_n(x, y) t^n}{n!}
\]
and for \(x \neq 1\)
\[
\frac{1}{e^y - x} = -\sum_{n \geq 0} \frac{A_n(x) t^n}{(x-1)^{n+1} n!}.
\]
Then,
\[
\frac{x}{y} \delta_n(x, y) = y^n A_n \left( \frac{1 + \frac{y}{x}}{y} \right).
\]
Which is equivalent to (4.21).

Now, for \(t = 1 + \frac{y}{x}\) in (4.21), we obtain (4.22).

\[\square\]

Theorem 13. For \(n \geq 0\), we have
\[
\mathfrak{F}_n(x, y) = y^n \sum_{k=0}^n \binom{n}{k} (-1)^k H_{n-k} \left( 1 + \frac{y}{x} \right).
\]

Proof. From the generating functions (4.1) and (4.20), we have,
\[
\sum_{n \geq 0} \mathfrak{F}_n(x, y) \frac{t^n}{n!} = e^{-ty} \left( 1 - \frac{1 + \frac{y}{x}}{y} \right) e^{ty} = e^{-ty} \sum_{n \geq 0} H_n \left( 1 + \frac{y}{x} \right) \frac{(ty)^n}{n!}.
\]
In the same way as the proof of Theorem 8, we get the result.

\[\square\]

5. PROBABILITY REPRESENTATION

We consider a geometric distributed random variable \(X\). The probability density function, for \(k \in \mathbb{N}^*\) and two parameters \(p\) and \(q\), such that \(q = 1 - p\), as follows:
\[
P(X = k) = pq^{k-1}.
\]
The higher moment of \(X\) is given by
\[
E(X^n) = \sum_{k=1}^\infty k^n p(1 - p)^{k-1}.
\]
In the next paragraph, we show that \(\mathfrak{F}_n(x, y)\) can be viewed as the \(n\)th moment of a random variable \(X - 1\) where \(X\) follows the geometric law.

Theorem 14. Let \(X\) be a random variable follows the geometric law and for \(p = \frac{x}{x+y} > 0\), we have
\[
\mathfrak{F}_n(x, y) = y^n \sum_{k=0}^\infty \binom{x}{y} (yk)^n (X - 1)^n.
\]

Proof. From the generating functions (4.1) and (4.20), we have,
\[
\sum_{n \geq 0} \mathfrak{F}_n(x, y) \frac{t^n}{n!} = e^{ty} \left( 1 - \frac{1 + \frac{y}{x}}{y} \right) e^{-ty} = e^{-ty} \sum_{n \geq 0} H_n \left( 1 + \frac{y}{x} \right) \frac{(ty)^n}{n!}.
\]
In the same way as the proof of Theorem 8, we get the result.
**Proof.** From (4.1), we have

\[
\sum_{n \geq 0} \mathfrak{F}_n(x, y) \frac{t^n}{n!} = \frac{y}{x+y} \left( 1 - \frac{x+y}{e^{x+y}} \right) \sum_{k \geq 0} \left( \frac{x}{x+y} \right)^k (e^{x+y})^k
\]

\[
= \frac{y}{x+y} \sum_{k \geq 0} \left( \frac{x}{x+y} \right)^k \sum_{n \geq 0} \frac{(ky)^n}{n!} t^n
\]

\[
= \frac{y}{x+y} \sum_{n \geq 0} \left( \sum_{k \geq 0} \left( \frac{x}{x+y} \right)^k (ky)^n \right) \frac{t^n}{n!}
\]

Equating \( \frac{t^n}{n!} \) and by comparing with (5.1), we obtain the result. \( \square \)

### 6. Degenerate Generalized Fubini Polynomials

For any nonzero real number \( \lambda \), we define the degenerate generalized Fubini polynomials as

\[
(6.1) \quad \frac{1}{\lambda} \left( 1 - \frac{1}{\lambda}(1 + \lambda ty)^{\frac{1}{\lambda}} - 1 \right) = \sum_{n \geq 0} \mathfrak{F}_{n, \lambda}(x, y) \frac{t^n}{n!}
\]

It is clear that \( \lim_{\lambda \to 0} (1 + \lambda ty)^{\frac{1}{\lambda}} = e^{ty} \) and therefore \( \lim_{\lambda \to 0} \mathfrak{F}_{n, \lambda}(x, y) = \mathfrak{F}_n(x, y) \).

Now, recall that the degenerate Stirling numbers of the second kind \( \left\{ \binom{n}{k} \right\}_\lambda \), are defined by the following generating function [4]

\[
(6.2) \quad \frac{1}{k!} \left( 1 - \frac{1}{\lambda}(1 + \lambda ty)^{\frac{1}{\lambda}} - 1 \right)^k = \sum_{n \geq k} \binom{n}{k} \frac{t^n}{\lambda n!}
\]

In the next result, we will give the explicit formula for \( \mathfrak{F}_{n, \lambda}(x, y) \).

**Theorem 15.** For \( n \geq 0 \), we have

\[
(6.3) \quad \mathfrak{F}_{n, \lambda}(x, y) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\lambda} k! x^ky^{n-k}
\]

**Proof.** From (6.1), we note that

\[
(6.4) \quad \sum_{n \geq 0} \mathfrak{F}_{n, \lambda}(x, y) \frac{t^n}{n!} = \frac{1}{\lambda} \left( 1 - \frac{1}{\lambda}(1 + \lambda ty)^{\frac{1}{\lambda}} - 1 \right)
\]

\[
= \sum_{k \geq 0} \left( \frac{x}{y} \right)^k \left( 1 + \lambda ty \right)^{\frac{1}{\lambda}} - 1 \right)^k
\]

\[
= \sum_{k \geq 0} \left( \frac{x}{y} \right)^k k! \sum_{n \geq k} \binom{n}{k} \frac{(ty)^n}{n!}
\]

\[
= \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} \frac{k! x^ky^{n-k}}{\lambda n!} \right) t^n
\]

Equating \( \frac{t^n}{n!} \), we obtain the result. \( \square \)
Remark 4. Now, by setting $y = 1$ in (6.1), we get

$$
\frac{1}{1 - x((1 + \lambda t)^{1/\lambda} - 1)} = \sum_{n \geq 0} \tilde{\beta}_{n, \lambda}(x, 1) \frac{t^n}{n!} = \sum_{n \geq 0} \omega_{n, \lambda}(x) \frac{t^n}{n!},
$$

where $\omega_{n, \lambda}(x)$ denotes the Fubini polynomials [10], defined by

$$
\omega_{n, \lambda}(x) := \prod_{k=0}^{n} \binom{n}{k} \frac{x^k}{\lambda^k}.
$$

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