ON THE GEOMETRY OF FORMS ON SUPERMANIFOLDS

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Abstract. This paper provides a rigorous account on the geometry of forms on supermanifolds, with a focus on its algebraic-geometric aspects. First, we introduce the de Rham complex of differential forms and we compute its cohomology. We then discuss three intrinsic definitions of the Berezinian sheaf of a supermanifold - as a quotient sheaf, via cohomology of the super Koszul complex or via cohomology of the total de Rham complex. Further, we study the properties of the Berezinian sheaf, showing in particular that it defines a right $\mathcal{D}$-module. Then we introduce integral forms and their complex and we compute their cohomology, by providing a suitable Poincaré lemma. We show that the complex of differential forms and integral forms are quasi-isomorphic and their cohomology computes the de Rham cohomology of the reduced space of the supermanifold. The notion of Berezin integral is then introduced and put to the good use to prove the superanalog of Stokes’ theorem and Poincaré duality, which relates differential and integral forms on supermanifolds. Finally, a different point of view is discussed by introducing the total tangent supermanifold and (integrable) pseudoforms in a new way. In this context, it is shown that a particular class of integrable pseudoforms having a distributional dependence supported at a point on the fibers are isomorphic to integral forms. Within the general overview several new proofs of results are scattered.

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1. Introduction - A Tale of Forms and Berezinians

Supergeometry is the study of supermanifolds, i.e. manifolds characterized by sheaves of $\mathbb{Z}_2$-graded (commutative) algebras, called superalgebras, whose “functions” might commute or anticommute, depending on their even or odd $\mathbb{Z}_2$-degree \textsuperscript{[50, 51, 53]}. Supermanifolds can also be endowed with a tighter structure, roughly speaking a symmetry that exchange even and odd directions \textsuperscript{[77]}. This
provides a geometric realization of what physicists call *supersymmetry transformations* in the context of modern quantum field theories, thus making supergeometry into the right mathematical environment to set and study these physical theories - one such being superstring theory [28].

In some sense, the characterizing $\mathbb{Z}_2$-graded commutativity of supergeometry is the lowest possible degree of non-commutativity. For this reason several constructions from ordinary purely commutative algebra and geometry can be easily generalized to a supergeometric setting. But this is not really the point. Indeed, supergeometry features new notions that resist such a trivial extension, challenging instead our geometric intuition.

One such notion is of *global* nature, and related to the holomorphic type. Indeed, complex holomorphic supermanifolds can be *non-projected* or *non-split*, meaning that they cannot be directly reconstructed from their underlying purely commutative manifolds [38, 53]. As such, they are genuinely new geometric objects, having “a life of their own”. Interest into these non-split / non-projected geometries has remarkably grown within the last years [3, 7, 13], prompted by Donagi and Witten’s paper [30], where it is showed that the *supermoduli space* of super Riemann surfaces is indeed non-projected for genus at least 5.

But issues are also of *local* nature. Indeed, whereas on an ordinary manifold differential forms anticommute, thus forcing the de Rham complex to terminate at the dimension of the manifold, on a supermanifold differentials of odd functions do instead commute, so that the de Rham complex is not bounded [28, 53]. This easy fact has very far reaching consequences. Poincaré duality - as known from ordinary commutative geometry - breakdowns, and there exists a new complex, which is “dual” to the de Rham complex. This is the so-called complex of *integral forms*, where the Berezinian bundle - a characteristic supergeometric construction which controls integration on supermanifolds - sits and plays the role the canonical bundle plays in the ordinary de Rham complex [11]. Accordingly, integration theory on supermanifolds is quite peculiar and highly non-trivial [53, 74]. Most notably, once again, its subtleties are related to important physical questions, regarding both the foundations of supersymmetric theories as manifestly invariant theory on superspaces [17] and actual computations of quantities of physical interests, such as scattering amplitudes in superstring theory [78].

This paper is mostly concerned with the algebraic-geometric aspects of this rich and beautiful theory of forms (and integration) on supermanifolds. Before we start, though, as to provide the reader with some perspective and context - and also to make justice to the researchers who has contributed the most to build and shape this branch of mathematics -, we will briefly go through and comment the historical development of the theory.

The concept of differential forms on supermanifolds originated from the astounding creativity of the work Felix Berezin - the “mastermind of *super-mathematics*” as in [65] - and his collaborators around 1970 [10], years before the mathematical formalization of the notion of *supermanifold* was even available [50, 51]. As in the ordinary theory, differential forms on supermanifolds can be structured into a complex, the de Rham complex, which - differently from a purely commutative setting - is not bounded from above. Also, allow for integration on reduced, or *bosonic*, submanifolds of a supermanifold by their pull-back, but they do not control integration on supermanifolds. A meaningful notion of integration on a space with odd, or *fermionic*, directions indeed requires the notion of *Berezinian* bundle (after Berezin) - probably one of the most peculiar construction in the theory of supermanifolds -, which plays the same role the determinant or canonical bundle play on an ordinary manifold. The fact that the Berezinian bundle does not appear in (the generalization of) de Rham complex on supermanifolds - and, as such, its sections are not differential forms - marks a significative departure from the ordinary integration theory and its relation to differential forms. Later on, in 1977, in the pioneering [11, 12] Bernstein and Leites introduced the complex of *integral* (or, perhaps more appropriately, integrable) *forms*, which is bounded from above (but not from below) and whose “top” bundle is given by the Berezinian of the supermanifold. Whereas the complex of differential forms controls integration on ordinary bosonic submanifolds of a supermanifold, the complex of integral forms controls integration on sub-supermanifolds of the same odd (or fermionic) dimension of the ambient supermanifold. In this integration procedure fermionic variables are somehow “frozen” and integrated over in the Berezin sense. Bernstein and Leites exploited further the characteristic geometry of supermanifolds where even and odd variables coexist as to introduced also *pseudodifferential forms* [12, 13]. These newly-defined type of forms generalize
the notion of differential forms allowing for more general, non-polynomial, dependences on the even 1-forms - while nilpotency constraints the odd 1-forms to have a polynomial dependence. In particular pseudodifferential forms can be \textit{integrable}, provided that they vanish fast enough at infinity: it is this particular kind of pseudoforms that allow for integration on sub-supermanifolds of any codimension, thus supplementing integral forms in the integration theory on supermanifolds, as they can only be integrated on codimension \( k/0 \) sub-supermanifolds. Nonetheless, differently than differential and integral forms, pseudodifferential forms do not carry any grading. This problem has been addressed in the Eighties and early Nineties by Voronov and Zorich \cite{Voronov1989, Voronov1990} and later on by Voronov \cite{Voronov1992}. Building on earlier work by Gaiduk, Khudaverdian and Schwarz \cite{Gaiduk1990}, these authors were able to develop a theory of forms on supermanifolds graded by superdimension, which are now referred to as \textit{Voronov-Zorich forms}. Remarkably, Voronov-Zorich forms admit different descriptions: a quite surprising one being via \textit{Lagrangians} related to parametrized supersurfaces \cite{Manin1991}. In more recent years, this intriguing point of view has been further push forward by Voronov \cite{Voronov2008, Voronov2009}, who provided an extension of the de Rham complex with forms carrying a \textit{negative} degree, something possible only due to the peculiar geometry of forms on supermanifolds. Quite remarkably, integral forms, (integrable) pseudodifferential forms and Voronov-Zorich forms are all related via \textit{integral transformations}. For example, it is possible to define an integral transform \cite{Manin1991}, called \textit{odd Fourier transform}, which maps isomorphically integral forms to a particular class of pseudodifferential forms having a \textit{Dirac delta} distributional dependence on the even 1-forms, which are thus supported at a single point.

Foundational problems in the theory of supermanifolds - and in particular in the theory of forms on supermanifolds - have been addressed during the Eighties also by other groups of researchers. Manin, together with the students at his school of algebraic geometry, worked extensively on problems related to complex supermanifolds (and their deformations) in relation to the back then new-born \textit{superstring theory}, which was polarizing the attention of the high-energy physics community. The fundamental intrinsic definition of the Berezinian bundle as arising from the homology of a very non-trivial generalization of the \textit{Koszul complex} appeared first in \cite{Ogievetsky1989}, where Ogievetsky and Penkov, students of Manin at that time, introduced \textit{Serre duality} for projective supermanifolds, showing that the Berezinian bundle provides the right super-analog of the dualizing sheaf in (commutative) algebraic geometry. Subsequently, this important construction has been briefly reported by Manin in his beautiful book \cite{Manin1991} - but the paper \cite{Ogievetsky1989} was inadvertently not acknowledged, thus causing a bit of confusion regarding the attribution of the result \cite{Ogievetsky1989}. Quite recently a self-contained thorough discussion of the super Koszul complex and its relation with the Berezinian has been given in \cite{Bartocci2009}. Further crucial properties of differential and integral forms on supermanifolds was discussed by Penkov in relation to the theory of \( \mathcal{D} \)-modules on supermanifolds in the very beautiful paper \cite{Penkov1992}: in this work, among many other things, the author shows that the Berezinian bundle carries a natural structure of right \( \mathcal{D} \)-module, showing once again similarities with the canonical bundle of an ordinary manifold. Beside the “Russian school”, other groups of researchers worked on “supermathematics” starting from the Eighties. Bartocci, Bruzzo and Hernández Ruipérez were very active in the research on foundational problems in the theory of supermanifolds. In particular, it is due to Hernández Ruipérez and Muñoz Masque one of the neatest intrinsic construction of the Berezin bundle for smooth supermanifolds \cite{Hernandez2001}, to be compared to the aforementioned contemporary construction via super Koszul complex by Ogievetsky and Penkov \cite{Ogievetsky1989}, which looks instead to the algebraic category.

The theory of forms and integration on supermanifolds and superspaces has many applications in modern physics. \textit{Supersymmetry} - a building pillar of contemporary high-energy physics, which relates \textit{bosonic} fields to \textit{fermionic} fields \cite{Bagger1981} - can only be realized geometrically at the cost of upgrading the ambient manifold of the theory to a supermanifold. In this context, the action of the physical theory is written in term of a Berezin integral of a section of the Berezinian of the supermanifold, which plays the role of the \textit{Lagrangian density} of the theory, and the supersymmetry transformations are generated by particular \textit{odd} vector fields. The machinery of the Berezin integral and the property of the Berezinian bundle make possible for the Lie derivative of the action of the physical theory to integrate to zero, thus making supersymmetry into a manifest symmetry. In other words, to set a supersymmetric theory on a supermanifold, with its peculiar notion of integration and Berezinian, has the same meaning as set a Lorentz-invariant theory on a semi-Riemannian Lorentzian manifold: only in this way symmetries of the physical theory becomes...
geometric transformations, uncovering mathematical structures and making the symmetry of the
theory manifest.
It is fair to say, though, that physics has probably not yet absorbed the elegance and subtleness of
the theory of forms on supermanifolds: integration on superspaces is regarded as an algebraic for-
mal machinery and the theory of integral forms - in its various realizations - is mostly ignored. As
a result, the full power of the formalism has not been exploited yet in its applications. There exist
remarkable exceptions and efforts in this direction though. In [6, 7] Belopolsky used a variation
on the theme of Voronov-Zorich forms in relation to the physical problem of computing scattering
amplitudes in superstring perturbation theory, putting forward a supergeometric version of the so-called picture changing operators introduced in conformal field theory in [34, 55]. In more
recent years, Catenacci, Castellani, Grassi - together with several other collaborators - realized
that the so-called rheonomic principle [17], that lies at the basis of the geometric formulation
of supergravity theories on supermanifolds, needed to be lifted to the integral forms complex of
the supermanifold to make sense. This opened up to the formulation of many supergravity and
supersymmetric theories via integral forms and a new understanding of their structures and super-
symmetries [18, 21, 22, 24, 26]. Whereas high-energy physics and string theory communities have
been in their greatest parts pretty insensitive to the subtleties related to forms and integration
theory on supermanifolds, in a totally opposite fashion the development of the theory of Batalin-
Vilkovisky (BV) quantization has been highly influenced by supergeometry and prompted several
advances in the field. In particular, integral forms, in their incarnation as semi-densities on odd
symplectic supermanifolds, play a major role in BV quantization, see [55] for a supergeometric-
aware detailed review of the topic. In this context have to be cited the very influential work [63]
by Schwarz and the seminal contributions [43–46] by Khudaverdian and Neressian, regarding the
relations between BV geometry and forms and integration on supermanifolds - in particular, it is
due to Khudaverdian the first supergeometric definition of the BV Laplacian [17, 48]. Later on,
building upon Khudaverdian works, in [63] ˇSevera provided a wonderful homological construction
of the BV Laplacian via a (quite surprising) spectral sequence related to a “deformed” de Rham
operator by the (odd) symplectic form of the ambient odd symplectic supermanifold: notably, the
construction resides on the cohomology of the aforementioned super Koszul complex, introduced
by Ogievetsky and Penkov. Finally, in recent years the theory of integral forms and the related
integration theory on supermanifolds has been revitalized and drawn again to physicists’ attention
by the review [76] by Witten, which was written with an eye to applications to superstrings [78].
The point of view of the author emphasizes the relation of differential and integral forms with
Clifford-Weyl (super)algebras and their representations. This has been recently prompted other
studies, also in the realm pure mathematics, see for example the interesting [66] in relation with
Lie superalgebras.
The books on the topics deserve a separate mention. First off, Voronov’s [74] provides a thorough
discussion on integration on supermanifolds, emphasizing the author’s construction of $r|s$-forms as
variation of Lagrangians and the related integration theory. This is the only dedicated book on
the topic to this day - we warn the reader, though, that this paper takes a different perspective.
Further, Manin’s [53] features a deep chapter on supergeometry which, among many other things,
introduces integral forms, Berezin integral and densities. The exposition leans toward an algebraic
geometric point of view, which is the one taken also the present paper: as such, [53] could be
considered as a main reference.

The paper is structured as follows. In section 2 the basic constructions in supergeometry are
introduced. In particular, the notion of supermanifold is discussed from the point of view or locally-
ringed space and certain natural sheaves are introduced. Section 3 is dedicated to differential forms
on supermanifolds and their (de Rham) cohomology, in particular Poincaré lemma is discussed.
Section 4 is dedicated to one of the most peculiar construction in supergeometry, that of Berezian.
More in details, we will define the Berezinian bundle via three constructions: as a quotient sheaf,
as constructed via the super Koszul complex and as resulting from the cohomology of the so-called
total de Rham complex, which mediates between the first two and it is substantially new. The
relations between these constructions are commented. In section 5 we will study some properties of
the Berezinian bundle, in particular we will see that it allows a right $\mathcal{D}$-module structure, such as the
canonical bundle of an ordinary manifold. Section 6 deals with integral forms and their (Spencer)
cohomology, in particular we will prove that the complexes of differential and integral forms are quasi-isomorphic, i.e. they compute the same cohomology, actually the de Rham cohomology of the reduced manifolds. In section 7 the Berezin integral is introduced. We will prove Stokes’ theorem for supermanifolds (without boundaries) and, as an application, we will see how it allows manifest supersymmetry invariance in physics. In section 8 we will see how the notion of Poincaré duality gets modified when working on supermanifolds. Finally, in section 9 we will present a different point of view, by introducing pseudoforms on supermanifolds as functions, endowed with certain properties, which are defined on the total space of the tangent bundle of a supermanifold, whose geometry is discussed in details in a new way. We will show that for a specific class of these forms - namely those having a distributional dependence supported at zero in the fiber directions -, there is an isomorphism with the previously defined integral forms. The approach taken in this last section differs from the available literature, in that - consistently with the spirit of the paper - only algebraic-geometric inspired ideas and methods have been employed, in place of the traditional analytic approach via integral transforms.

We stress that this paper does not aim to be fully encompassing - and indeed some point of views which we have hinted upon in this introduction are not discussed here - for these, we refer for example to [24]. Instead, we have chosen to provide a - hopefully - conceptually clear and mathematically rigorous exposition, keeping our focus on the algebraic-geometric aspects of the theory. The most important and peculiar constructions - which are not well-known outside a public of experts - have been spelled out in details, trying to provide a firmly founded systematization of the results, alongside with new comprehensive proofs which are often not available in the literature. Finally, efforts have been put to make the exposition as self contained as possible and in the hope to provide a readable, but not overwhelming, reference to the subject.

2. ELEMENTS OF GEOMETRY OF SUPERMANIFOLDS

In this section we briefly recall the main definitions in the theory of supermanifolds, see for example the classical [50, 51] or [2, 53]. We start with one of the most fundamental concept in supergeometry, that of superspace.

**Definition 2.1 (Superspace).** A superspace is a pair \((|\mathcal{M}|, \mathcal{O}_\mathcal{M})\), where \(|\mathcal{M}|\) is a topological space and \(\mathcal{O}_\mathcal{M}\) is a sheaf of \(\mathbb{Z}_2\)-graded supercommutative rings over \(|\mathcal{M}|\), such that the stalks \(\mathcal{O}_{\mathcal{M},x}\) at every point of \(|\mathcal{M}|\) are local rings. Analogously, a superspace is a locally ringed space having structure sheaf given by a sheaf of \(\mathbb{Z}_2\)-graded supercommutative rings.

Note that the requirement about the stalks being local rings reduces to ask that the even component of the stalk is a usual commutative local ring. Indeed if \(A = A_0 \oplus A_1\) a super ring, then \(A\) is local if and only if its even part \(A_0\) is, see for example [67].

Given two superspaces we can define a morphism between them in the usual fashion.

**Definition 2.2 (Morphisms of Superspaces).** Given two superspaces \(\mathcal{M}\) and \(\mathcal{N}\) a morphism \(\varphi : \mathcal{M} \to \mathcal{N}\) is a pair \(\varphi := (\phi, \phi^\sharp)\) where

1. \(\phi : |\mathcal{M}| \to |\mathcal{N}|\) is a continuous morphism of topological spaces;
2. \(\phi^\sharp : \mathcal{O}_\mathcal{N} \to \phi_* \mathcal{O}_\mathcal{M}\) is a morphism of sheaves of \(\mathbb{Z}_2\)-graded rings, having the properties that it preserves the \(\mathbb{Z}_2\)-grading and that given any point \(x \in |\mathcal{M}|\), the homomorphism \(\phi_\sharp^\sharp : \mathcal{O}_{\mathcal{N},\phi(x)} \to \mathcal{O}_{\mathcal{M},x}\) is local, i.e. it preserves the (unique) maximal ideal, i.e. \(\phi_\sharp^\sharp (m_{\phi(x)}) \subseteq m_x\).

It is easy to see that superspaces together with their morphisms forms a category, we call it \(\mathbf{SSp}\). Before we go on, some remarks on the previous definitions are in order.

**Remark 2.3.** With an eye to the ordinary theory of schemes in algebraic geometry, we stress that the request that the morphism \(\phi_\sharp^\sharp : \mathcal{O}_{\mathcal{N},\phi(x)} \to \mathcal{O}_{\mathcal{M},x}\) preserves the maximal ideal in the second point of the definition above is of particular significance in supergeometry. Indeed it is important to notice that the structure sheaf \(\mathcal{O}_\mathcal{M}\) of a superspace is in general not a sheaf of functions. As long as the structure sheaf \(\mathcal{O}_\mathcal{M}\) of a certain space or, more in general, of a scheme, is a sheaf of functions, then a section \(s\) of \(\mathcal{O}_\mathcal{M}\) takes values in the field of fractions \(k(x) = \mathcal{O}_{\mathcal{M},x}/m_x\) that depends on the point \(x \in |\mathcal{M}|\), as a function \(x \mapsto s(x) \in k(x)\), and the maximal ideal \(m_x\) contains the germs of functions that vanish at \(x \in |\mathcal{M}|\). In the case of superspaces, nilpotent sections - and thus in particular all of the odd sections - would be identically equal to zero as functions on
points, and indeed the maximal ideal \( m_x \) contains the germs of all the nilpotent sections in \( \mathcal{O}_{M,x} \).

In this context, the request that \( \phi^*_x : \mathcal{O}_{N,\phi(x)} \rightarrow \mathcal{O}_{M,x} \) is local becomes crucial, while in the case of a genuine sheaf of functions the locality is automatically achieved. In particular, locality implies that a non unit element in the stalk \( \mathcal{O}_{N,\phi(x)} \), such as a germ of a nilpotent section, can only be mapped to another non unit element in \( \mathcal{O}_{M,x} \), such as another germ of a nilpotent section. In other words, nilpotent elements cannot be mapped to invertible elements.

Anyway, we fell like we have to advice the reader that we will often abuse the notation by keep denoting \( \phi \) instead of \( \phi^* \) the morphism of sheaves related to a superspace morphism \( \varphi : M \rightarrow N \).

**Remark 2.4.** It is crucial to observe that one can always construct a superspace out of two “classical” data: a topological space, call it again (by abuse of notation) \( |M| \), and a vector bundle over \( |M| \), call it \( E \) (analogously: a locally-free sheaf of \( \mathcal{O}_{|M|} \)-modules). We denote \( \mathcal{O}_{|M|} \) the sheaf of continuous functions (with respect to the given topology) on \( |M| \) and we put \( \wedge^* E^* = \mathcal{O}_{|M|} \). Then the sheaf of sections of the bundle of exterior algebras \( \wedge^* E^* \) has an obvious \( \mathbb{Z}_2 \)-grading (by taking its natural \( \mathbb{Z} \)-grading mod 2) and therefore in order to realize a superspace it is enough to take the structure sheaf \( \mathcal{O}_M \) of the superspace to be the sheaf of \( \mathcal{O}_{|M|} \)-valued sections of the bundle of exterior algebras of \( E \). This construction is so important to bear its own name \([30]\).

**Definition 2.5 (Local Model \( \mathcal{S}(\{M\}, E) \)).** Given a pair \( (|M|, E) \), where \( |M| \) is a topological space and \( E \) is a vector bundle over \( |M| \), we call \( \mathcal{S}(\{M\}, E) \) the superspace modelled on the pair \( (|M|, E) \), where the structure sheaf is given by the \( \mathcal{O}_{|M|} \)-valued sections of the exterior algebra \( \wedge^* E^* \).

Note that we have given a somehow minimal definition of local model, indeed we have let \( |M| \) to be no more than a topological space and as such we are only allowed to take \( \mathcal{O}_{|M|} \) to be the sheaf of continuous functions on it. Clearly, we can also work in richer and more structured category, such as the real smooth, complex analytic or algebraic category as we will do in this paper. This amount to consider local models based on the pair \( (\mathcal{M}_{\text{red}}, E) \), where \( \mathcal{M}_{\text{red}} \) is a smooth or complex manifold or an algebraic variety - we keep denoting its underlying topological space with \( |M| \), as above - with \( \mathcal{O}_{\mathcal{M}_{\text{red}}} \) being its sheaf of smooth, holomorphic of algebraic functions and \( E \) being a smooth, holomorphic of algebraic vector bundle.

We will call smooth, holomorphic or algebraic local model a local model \( \mathcal{S}(\mathcal{M}_{\text{red}}, E) \) which is constructed from the above data. This leads to the definition of supermanifold in the appropriate category, which we explicitly give in the real smooth or complex analytic category.

**Definition 2.6 (Real / Complex Supermanifold).** A real (complex) supermanifold \( M \) of dimension \( n|m \) is a superspace that is locally isomorphic to some smooth (holomorphic) local model \( \mathcal{S}(\mathcal{M}_{\text{red}}, E) \), where \( \mathcal{M}_{\text{red}} \) is a smooth (complex) manifold of dimension \( n \) and \( E \) is a smooth (holomorphic) vector bundle on \( \mathcal{M}_{\text{red}} \) of rank \( m \).

In other words, if \( \mathcal{M}_{\text{red}} \) is covered by an atlas \( \{ U_i \}_{i \in I} \), the structure sheaf \( \mathcal{O}_M = \mathcal{O}_{M,0} \oplus \mathcal{O}_{M,1} \) of the supermanifold \( M \) is described via a collection \( \{ \psi_{U_i} \}_{i \in I} \) of local isomorphisms of sheaves

\[
U_i \rightarrow \psi_{U_i} : \mathcal{O}_M|_{U_i} \xrightarrow{i} \wedge^* E^*|_{U_i}
\]  

where we have denoted with \( \wedge^* E^* \) the sheaf of \( \mathcal{O}_{\mathcal{M}_{\text{red}}} \)-valued sections of the exterior algebra of \( E \) considered with its \( \mathbb{Z}_2 \)-gradation. Also, notice that a morphism of supermanifolds is nothing but a morphism of superspaces, so that one has the related category of supermanifolds, that we denote with \( \mathbb{S}\text{Man} \).

The special case in which there is a single global isomorphism instead of a family of local isomorphisms deserves a name by its own.

**Definition 2.7 (Split Supermanifold).** We say that a supermanifold \( M \) is a split supermanifold if it is globally isomorphic to its local model. That is, if we have a sheaf isomorphism \( \mathcal{O}_M \cong \wedge^* E^* \).

Clearly, split supermanifolds are the easiest supermanifolds to deal with, as their structure sheaves are simply sheaves of \( \mathcal{O}_{\mathcal{M}_{\text{red}}} \)-valued sections of exterior algebras, and as such they are locally-free sheaves of \( \mathcal{O}_{\mathcal{M}_{\text{red}}} \)-modules.

In order to see how real and complex supermanifold might differ from the point of view of their global geometry, we need to introduce some further pieces of informations related to a supermanifold.
Definition 2.8 (Nilpotent Sheaf). Let $\mathcal{M}$ be a real (complex) supermanifold with structure sheaf $\mathcal{O}_\mathcal{M}$. We call the nilpotent sheaf $\mathcal{J}_\mathcal{M}$ of $\mathcal{M}$ the sheaf of ideals of $\mathcal{O}_\mathcal{M} = \mathcal{O}_{\mathcal{M},0} \oplus \mathcal{O}_{\mathcal{M},1}$ generated by all of the nilpotent sections in $\mathcal{O}_\mathcal{M}$, i.e. we put $\mathcal{J}_\mathcal{M} := \mathcal{O}_{\mathcal{M},1} \oplus \mathcal{O}^2_{\mathcal{M},1}$.

It is crucial to note that modding out all of the nilpotent sections from the structure sheaf $\mathcal{O}_\mathcal{M}$ of the supermanifold $\mathcal{M}$ we recover the structure sheaf $\mathcal{O}_{\mathcal{M}_{red}}$ of the underlying ordinary manifold $\mathcal{M}_{red}$, the local model was based on.

Definition 2.9 (Reduced Space). Let $\mathcal{M}$ be a real (complex) supermanifold with structure sheaf $\mathcal{O}_\mathcal{M}$. We call the reduced space of $\mathcal{M}$ the smooth (complex) manifold $\mathcal{M}_{red}$ with structure sheaf given by the quotient $\mathcal{O}_{\mathcal{M}_{red}} := \mathcal{O}_\mathcal{M}/\mathcal{J}_\mathcal{M}$.

Loosely speaking, the reduced manifold in $\mathcal{M}_{red}$ arises by setting all the nilpotent sections in $\mathcal{O}_\mathcal{M}$ to zero. In other words, more invariantly, attached to any real or complex supermanifold there is a short exact sequence that relates the supermanifold to its reduced manifold

$$0 \rightarrow \mathcal{J}_\mathcal{M} \rightarrow \mathcal{O}_\mathcal{M} \rightarrow \mathcal{O}_{\mathcal{M}_{red}} \rightarrow 0. \quad (2.2)$$

The surjective sheaf morphism $\iota^2 : \mathcal{O}_\mathcal{M} \rightarrow \mathcal{O}_{\mathcal{M}_{red}}$ corresponds to the existence of an embedding $\mathcal{M}_{red} \hookrightarrow \mathcal{M}$ of the reduced manifold $\mathcal{M}_{red}$ inside the supermanifold $\mathcal{M}$. Notice that $\mathcal{J}_\mathcal{M} = \ker(\iota)$, where $\iota : \mathcal{O}_\mathcal{M} \rightarrow \mathcal{O}_{\mathcal{M}_{red}}$ is the surjective sheaf morphism in $(2.2)$.

Using the nilpotent sheaf associated to a supermanifold $\mathcal{M}$, say of odd dimension $m$ we can construct a descending filtration of length $m$ of $\mathcal{O}_\mathcal{M}$ as follows,

$$\mathcal{O}_\mathcal{M} \supset \mathcal{J}_\mathcal{M} \supset \mathcal{J}_\mathcal{M}^2 \supset \cdots \supset \mathcal{J}_\mathcal{M}^m \supset \mathcal{J}_\mathcal{M}^{m+1} = 0.$$  

(2.3)

This allows us to give the following definition.

Definition 2.10 (Gr $\mathcal{O}_\mathcal{M}$ and Gr $\mathcal{M}$). Let $\mathcal{M}$ be a supermanifold having odd dimension $m$ together with the filtration of its structure sheaf $\mathcal{O}_\mathcal{M}$ as in $(2.3)$. We define the following sheaf of supercommutative algebras

$$\text{Gr} \mathcal{O}_\mathcal{M} := \bigoplus_{i=0}^{m} \text{Gr}^{(i)} \mathcal{O}_\mathcal{M} = \mathcal{O}_{\mathcal{M}_{red}} \oplus \mathcal{J}_\mathcal{M}^2 \oplus \cdots \mathcal{J}_\mathcal{M}^{m} \oplus \mathcal{J}_\mathcal{M}^{m+1}/\mathcal{J}_\mathcal{M}^2 \oplus \cdots \mathcal{J}_\mathcal{M}^{m}/\mathcal{J}_\mathcal{M}^{m+1}.$$  

(2.4)

where $\text{Gr}^{(i)} \mathcal{O}_\mathcal{M} := J^i_\mathcal{M}/J^{i+1}_\mathcal{M}$ and the $\mathbb{Z}_2$-grading is obtained by taking the obvious $\mathbb{Z}$-grading mod 2. We call the split supermanifold associated to $\mathcal{M}$ the supermanifold arising from the super-subspace $(\mathcal{M}, \text{Gr} \mathcal{O}_\mathcal{M})$ and we denote it by $\text{Gr} \mathcal{M}$.

If the supermanifold $\mathcal{M}$ has odd dimension $m$, the quotient $\text{Gr}^{(1)} \mathcal{O}_\mathcal{M} = J^1_\mathcal{M}/J^2_\mathcal{M}$ in $(2.4)$ is a locally-free sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules of rank $0|\mathfrak{m} - i.e. it is locally generated by $m$ odd sections - and it plays a special role so that for notational convenience we denote it $\mathcal{F}_\mathcal{M} := J^1_\mathcal{M}/J^2_\mathcal{M}$, as to recall its “fermionic” behavior.

The importance of $\mathcal{F}_\mathcal{M}$ lies in that - up to parity - it is isomorphic to $E^*$, the vector bundle appearing in the local model $\mathcal{S}(\mathcal{M}_{red}, E)$ whose the supermanifold $\mathcal{M}$ is based upon, i.e. one has $E^* \cong \Pi \mathcal{F}_\mathcal{M}$, where $\Pi : \mathcal{SH}_{\mathcal{M}_{red}} \rightarrow \mathcal{SH}_{\mathcal{M}_{red}}$ is the so-called parity changing or parity shifting functor, which maps a locally-free sheaf of rank of rank $p/q$ to one of rank $q/p$, by reversing its parity $\mathbb{Z}_2$. In view of this one has that $\text{Gr} \mathcal{M} = \mathcal{S}(\mathcal{M}_{red}, \Pi \mathcal{F}_\mathcal{M})$ and the local isomorphisms characterizing the structure sheaf $\mathcal{O}_\mathcal{M}$ can be rewritten, over an open set $U \subset \mathcal{M}$, as

$$\mathcal{O}_\mathcal{M}|_U \cong \wedge^\bullet E^*|_U \cong S^\bullet \mathcal{F}_\mathcal{M}|_U,$$  

(2.5)

where $S^i : \mathcal{SH}_{\mathcal{M}_{red}} \rightarrow \mathcal{SH}_{\mathcal{M}_{red}}$ is the $i$-th supersymmetric power functor $\mathbb{Z}_2$, so that in particular $\text{Gr} \mathcal{O}_\mathcal{M} = \bigoplus_{i=0}^{m} S^i \mathcal{F}_\mathcal{M}$.

We conclude this section by stressing out the major difference between the realm of real and complex supermanifolds, that lies in the fact that real supermanifold are always split and actually all isomorphic to $\text{Gr} \mathcal{M}$. This result - in a slightly different form - was first proved by Marjorie Batchelor in $[3]$.

Theorem 2.11 (Batchelor). Let $\mathcal{M}$ be a smooth supermanifold. Then its structure sheaf is non-canonically isomorphic to a sheaf of exterior algebras for some smooth vector bundle $E$ over $\mathcal{M}_{red}$.

In particular, $\mathcal{M}$ is split and one has $\mathcal{M} \cong \text{Gr} \mathcal{M}$. 
and non-projected (or non-split) supermanifolds. Indeed, considering the reduced space can see the difference in terms of the structure of transition functions between split supermanifolds. Notice that in this case classes $\omega$ notice that in this case examples the easiest supergeometric generalization of a conic in the complex projective superspace is mostly rare phenomenon, but as a characterizing and quite common phenomenon instead. For instance the supermanifold “has a life of its own.”

In complex supergeometry the absence of a projection should not be looked at as an exotic and fundamental obstruction where the obstruction theory can indeed be non-zero, thus leading to the peculiar - and very interesting - complex supergeometry of supermanifolds, and as such a non-projected supermanifold cannot be “reconstructed” easily from its underlying ordinary complex manifold, in the words of Donagi and Witten in [30], a non-projected supermanifold $\mathcal{M}_{\text{red}}$ is characterized up to isomorphism by the triple $\omega$. This mean that it guarantees the existence of a certain covering of open sets together with charts such that the isomorphism is realized, but it is not constructive: in other words it does not tell how to concretely realize such an isomorphism.

Remark 2.12. It is important to note that the theorem states the existence of a non-canonical isomorphism. This mean that it guarantees the existence of a certain covering of open sets together with charts such that the isomorphism is realized, but it is not constructive: in other words it does not tell how to concretely realize such an isomorphism.

Remark 2.13. Notice also that, globally, a real supermanifold can be seen as the total space of a sheaf of exterior algebras $\wedge^{*}\mathcal{E}^{*}$ over $\mathcal{M}_{\text{red}}$, whose fibers are nilpotent. In particular, this means that charts $\{U_{i}, x^{i}_{I}, \theta^{I}_{\alpha}\}_{I \in I}$ for $i = 1, \ldots, n$ and $\alpha = 1, \ldots, m$ can always be found such that if $x^{i}_{I}$ and $x^{j}_{I}$ are local coordinates in any two open sets $U_{j}$ and $U_{k}$ in $\{U_{I}\}_{I \in I}$ having non-empty intersection $U_{k} \cap U_{j} \neq \emptyset$, then we will have

$$x^{k}_{\ell} = x^{k}_{\ell}(x^{1}_{I}, \ldots, x^{n}_{I}), \quad \theta^{I}_{\alpha} = \sum_{\beta=1}^{m}[g^{I}_{\ell j}(x)]_{\alpha \beta} \theta^{\beta},$$

(2.6)

where $[g^{I}_{\ell j}(x)]_{\alpha \beta} \in \mathcal{Z}^{1}(U_{k} \cap U_{j}, GL(q, \mathbb{R}))$ are the transition functions of the vector bundle $\mathcal{E}^{*}$.

That is, when changing charts, the even local coordinates $x$’s transform as the coordinates of the ordinary smooth manifolds $\mathcal{M}_{\text{red}}$ and the odd local coordinates $\theta$’s transform linearly, as the generating sections of the vector bundle $\mathcal{E}^{*}$.

Remark 2.14. Loosely speaking, a general obstruction theory to split the structure sheaf of a supermanifold can be constructed by filtering $\mathcal{O}_{\mathcal{M}}$ as in (2.4). This was first done by Green in [35]. For a supermanifold based on the local model $\mathfrak{S}(\mathcal{M}_{\text{red}}, \mathcal{E})$, obstructions are given by cohomology classes $\omega_{i}$ lying in the cohomology groups

$$\omega_{2i} \in H^{i}(\mathcal{M}_{\text{red}}, \mathcal{T}_{\mathcal{M}_{\text{red}}} \otimes \wedge^{2} \mathcal{E}^{*}), \quad \omega_{2i+1} \in H^{i}(\mathcal{M}_{\text{red}}, \mathcal{E} \otimes \wedge^{2i+1} \mathcal{E}^{*})$$

(2.7)

for $i \geq 1$ and where $\mathcal{T}_{\mathcal{M}_{\text{red}}}$ is the tangent sheaf of the reduced space $\mathcal{M}_{\text{red}}$. The subtlety here is that whereas the fundamental obstruction, i.e. $\omega_{2}$, which is controlled by the group $H^{1}(\mathcal{M}_{\text{red}}, \mathcal{T}_{\mathcal{M}_{\text{red}}} \otimes \wedge^{2} \mathcal{E}^{*})$ is always defined, instead the higher obstructions, i.e. $\omega_{i}$ for $i \geq 3$, are only defined if all the lower ones vanish [38, 55].

In the smooth category all the sheaves are fine as a consequence of the existence of smooth partitions of unity, then these cohomology groups are automatically zero for a real supermanifold and there are no obstruction to split $\mathcal{O}_{\mathcal{M}}$ as $\mathfrak{g} \mathfrak{r} \mathcal{M}$, providing another proof of Batchelor theorem.

On the other hand things change dramatically for complex supermanifolds, where the above cohomology groups can indeed be non-zero, thus leading to the peculiar - and very interesting! - complex supergeometry of non-split and non-projected supermanifolds, i.e. those complex supermanifolds that do not admit a retraction or projection for their defining exact sequence (2.2)

$$0 \to \mathcal{T}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}} \overset{\pi}{\to} \mathcal{O}_{\mathcal{M}_{\text{red}}} \to 0,$$

(2.8)

where $\pi : \mathcal{G}_{\mathcal{M}_{\text{red}}} \to \mathcal{O}_{\mathcal{M}}$ is such that $\pi \circ \pi^{2} = id_{\mathcal{O}_{\mathcal{M}_{\text{red}}}}$ and corresponds to a map $\pi : \mathcal{M} \to \mathcal{M}_{\text{red}}$. Notice that, in particular, the structure sheaf of a non-projected supermanifold is not a sheaf of $\mathcal{O}_{\mathcal{M}_{\text{red}}}$-algebras, and as such a non-projected supermanifold cannot be “reconstructed” easily from its underlying ordinary complex manifold, in the words of Donagi and Witten in [30], a non-projected supermanifold “has a life of its own”.

In complex supergeometry the absence of a projection should not be looked at as an exotic and mostly rare phenomenon, but as a characterizing and quite common phenomenon instead. For example the easiest supergeometric generalization of a conic in the complex projective superspace $\mathbb{C}P^{2|2}$ cut out by the equation

$$X_{0}^{2} + X_{1}^{2} + X_{2}^{2} + \Theta_{1}\Theta_{2} = 0 \subset \mathbb{C}P^{2|2},$$

(2.9)

turns out to be a $1|2$-dimensional non-projected supermanifold, see for example [30] or [53], which is characterized up to isomorphism by the triple

$$C_{\omega} := (\mathcal{M}_{\text{red}} = \mathbb{C}P^{1}, \quad \mathcal{E}^{*} = \mathcal{O}_{\mathbb{C}P^{1}}(-2)^{\oplus 2}, \quad H^{1}(\mathcal{M}_{\text{red}}, \mathcal{T}_{\mathcal{M}_{\text{red}}} \otimes \wedge^{2} \mathcal{E}^{*}) \ni \omega_{2} \neq 0),$$

(2.10)

notice that in this case $\omega_{2} \in H^{1}(\mathbb{C}P^{1}, \mathcal{O}_{\mathbb{C}P^{1}}(-2)) \cong \mathbb{C}$. In the light of Batchelor theorem [4], one can see the difference in terms of the structure of transition functions between split supermanifolds and non-projected (or non-split) supermanifolds. Indeed, covering the reduced space $\mathbb{C}P^{1}$ by the
standard (two) open sets, and considering the related systems of local coordinates to be given by 
\( (z|\theta_1, \theta_2) \) and \( (w|\psi_1, \psi_2) \) respectively, for the above super conic \( C_w \), one finds
\[
w = \frac{1}{z} + \frac{\theta_1 \theta_2}{z^3}, \quad \psi_1 = \frac{\theta_1}{z^2}, \quad \psi_2 = \frac{\theta_2}{z^2}.
\] (2.11)
The non-vanishing obstruction class \( \omega_2 \neq 0 \) in the cohomology guarantees that there are no choices of coordinates or redefinitions of charts that bring the even transition back to the form of that of the reduced space \( CP^1 \), i.e. \( w = 1/z \). On the contrary, for a non-projected supermanifold the even transition functions depend crucially also from the odd part of the geometry, as one can see in the 2.11.

2.1. Derivations, Differential Operators and \( D \)-modules. Having introduced the concept of supermanifolds - in particular in the smooth real and complex analytic category -, we now briefly discuss the most important natural sheaves that can be defined on them and that will be used the most in this paper.

In particular, working for instance over a complex supermanifold \( \mathcal{M} \), the structure sheaf \( O_M \) can be looked at as a subsheaf of the sheaf of its \( \mathbb{C} \)-endomorphisms, \( \text{End}_{\mathbb{C}}(O_M) \), taking \( g \mapsto \sigma_f(g) := fg \) for any sections \( f, g \in O_M \). Likewise, the tangent sheaf \( T_M \) of \( \mathcal{M} \) - or, analogously, the sheaf of derivations \( \text{Der}_{\mathbb{C}}(O_M) \) - can also be looked at as a subsheaf of \( \text{End}_{\mathbb{C}}(O_M) \) defining
\[
T_M := \{ X \in \text{End}_{\mathbb{C}}(O_M) : X(fg) = X(f)g + (-1)^{|f||g|} fX(g), \ f, g \in O_M \}.
\] (2.12)
Since we only consider smooth supermanifolds, then \( T_M \) is locally-free of rank \( n/m = \dim_{\mathbb{C}} \mathcal{M} \) and if \( x_i|\theta_\alpha \) are local coordinates for a chart \( U \subset \mathcal{M} \), then a section \( X \in T_M \) over \( U \) is given by
\[
X_U = \bigoplus_{i=1}^n O_M(U) \cdot \partial x_i \oplus \bigoplus_{\alpha=1}^m O_M(U) \cdot \partial \theta_\alpha.
\] (2.13)
This means that the tangent sheaf is freely locally-generated by the even and odd derivations
\[
T_M(U) = O_M(U) \cdot \{ \partial x_1, \ldots, \partial x_n, \partial \theta_1, \ldots, \partial \theta_m \}.
\] (2.14)
The above point of view, aimed at relating the structure sheaf and the tangent sheaf with the sheaf of endomorphisms of \( O_M \) is particularly useful when one is interested into introducing the sheaf of differential operators on \( \mathcal{M} \), which will play an important role in what follows. We give the definition for a complex supermanifold, but the same can be done for a real and also algebraic supermanifold.

**Definition 2.15 (The Sheaf \( D_M \)).** Let \( \mathcal{M} \) be a complex supermanifold. We define the sheaf of differential operators of \( \mathcal{M} \) to be the subsheaf of \( \text{End}_{\mathbb{C}}(O_M) \) generated by \( O_M \) and \( T_M \), and we denote it by \( D_M \).

If \( x_i|\theta_\alpha \) is a coordinate system over an open set \( U \), then \( \{ x_i|\alpha, \partial x_i|\partial \theta_\alpha \}_{i=1,\ldots,n,\alpha=1,\ldots,m} \) gives a local trivialization of \( D_M|_U \), where \( x_i|\alpha \in O_M|_U \) and \( \partial x_i|\partial \theta_\alpha \in T_M|_U \) satisfy the following defining relations
\[
[x_i, x_j] = 0, \quad [\partial x_i, \partial x_j] = 0, \quad [x_i, \partial x_j] = \delta_{ij}, \quad \{ \theta_\alpha, \partial \theta_\beta \} = 0, \quad \{ \partial \theta_\alpha, \partial \theta_\beta \} = 0, \quad \{ \theta_\alpha, \theta_\beta \} = \delta_{\alpha\beta}
\] (2.15)
where \( [\cdot, \cdot] \) denotes a commutator and \( \{ \cdot, \cdot \} \) denotes an anticommutator. Notice that, locally, these relations define the Weyl superalgebra \( D_{\mathbb{C}|n|m} \), so that posing
\[
U \mapsto D_M(U) := \{ D_U : D_U \text{ is a differential operator on } O_M(U) \},
\] (2.16)
then one has
\[
D_U = \bigoplus_{\ell \in \mathbb{N}, \varepsilon \in \mathbb{Z}_+^{m}} \bigoplus_{i \in \mathbb{N}} O_M|_U \partial x_i^\ell \partial \theta_\alpha^\varepsilon \quad \text{where} \quad \left\{ \begin{array}{l} \partial x_i^\ell := \partial x_i^{\ell_1} \partial x_i^{\ell_2} \cdots \partial x_i^{\ell_n}, \\ \partial \theta_\alpha^\varepsilon := \partial \theta_\alpha^{\varepsilon_1} \partial \theta_\alpha^{\varepsilon_2} \cdots \partial \theta_\alpha^{\varepsilon_m} \end{array} \right.
\] (2.17)
In particular we define \( \deg(D_U) := \max(|\ell| + |\varepsilon|) \), if \( |\ell| = \sum_{i=1}^n \ell_i \) and \( |\varepsilon| = \sum_{\alpha=1}^m \varepsilon_\alpha \) and we call \( \deg(D_U) \) the degree of the differential operator \( D_U \in D_M|_U \).

The above local description of equation (2.17) leads to a natural filtration of \( D_M \). We define
\[
F^i D_M(U) := \{ D_U \in D_M(U) : \deg(D_U|_U) \leq i \text{ for all } V \subseteq U \}.
\] (2.18)
Notice that the above filtration is increasing, i.e. $F^i \mathcal{D}_M \subseteq F^{i+1} \mathcal{D}_M$, and exhaustive, i.e. $\cup_i F^i \mathcal{D}_M = \mathcal{D}_M$. More in general one has the following relations

$$F^i \mathcal{D}_M : F^j \mathcal{D}_M \subseteq F^{i+j} \mathcal{D}_M, \quad [F^i \mathcal{D}_M, F^j \mathcal{D}_M] \subseteq F^{i+j-1} \mathcal{D}_M,$$

and we define the associated graded module $\text{gr}^*_F(\mathcal{D}_M)$ with respect to the above filtration:

$$\text{gr}^*_F(\mathcal{D}_M) = \bigoplus_{i=0}^{\infty} F^i \mathcal{D}_M / F^{i-1} \mathcal{D}_M,$$

where we have defined $\text{gr}^i_F(\mathcal{D}_M) := F^i \mathcal{D}_M / F^{i-1} \mathcal{D}_M$.

**Remark 2.16.** It has to be stressed that, in particular, $F^1 \mathcal{D}_M$ is a Lie sub-superalgebra of $\mathcal{D}_M$, since indeed one has that $[\cdot, \cdot] : F^1 \mathcal{D}_M \times F^1 \mathcal{D}_M \to F^1 \mathcal{D}_M$. Further, $F^0 \mathcal{D}_M$ is a Lie ideal of this Lie superalgebra: indeed if $f \in F^0 \mathcal{D}_M$, then $[f, G] \in F^0 \mathcal{D}_M$ for any $G \in F^1 \mathcal{D}_M$. It thus makes sense to consider the quotient of $F^1 \mathcal{D}_M$ by $F^0 \mathcal{D}_M$, and it is not hard to realize that

$$\mathcal{T}_M \cong F^1 \mathcal{D}_M / F^0 \mathcal{D}_M,$$

which in turn leads to

$$\text{gr}^*_F(\mathcal{D}_M) = S^*_\mathcal{T}_M. \quad (2.22)$$

**Remark 2.17.** One could think about this in analogy with the Poincaré-Birkhoff-Witt (PBW) theorem for the universal enveloping algebra $U(\mathfrak{g})$ of a certain Lie (super)algebra $\mathfrak{g}$. Indeed, given a filtration as above, the quotient operation does not just reduce to the leading terms, but it does remarkably more: it sets all the commutators in $\text{gr}^*_F(\mathcal{D}_M)$ to zero. Indeed, if two elements $F, G$ are such that their product is in $\text{gr}^*_F(\mathcal{D}_M)$, then their commutator is in $\text{gr}^{i-1}_F(\mathcal{D}_M)$ so that it is set to zero in the quotient. In other words, elements surviving the quotient are those that do not come from commutators: in particular, all of the elements commute and this leads naturally to the symmetric (super)algebras, where all of the (super)commutators vanish. Notice that, similarly, $\text{gr}^* U(\mathfrak{g}) \cong S^* (\mathfrak{g})$, which is the meaning of PBW theorem. In general, we might think about the following analogies between a Lie (super)algebra and its universal enveloping algebra (together with its PBW filtration) and the derivations on a (super)manifolds, and the differential operators on $\mathcal{M}$:

$$\mathcal{T}_M \cong \mathfrak{g} \quad (2.23)$$

$$\mathcal{D}_M \cong U(\mathfrak{g}) \quad (2.24)$$

**Remark 2.18.** Finally, it is worth to observe that the associated graded module $\text{gr}^*_F(\mathcal{D}_M)$ is naturally isomorphic to $\pi_* \mathcal{O}_{T^* \mathcal{M}}$, where $\pi : T^* \mathcal{M} \to \mathcal{M}$ is the cotangent bundle of the supermanifold $\mathcal{M}$. This expresses the fact that “functions” on a certain space are given by the symmetric algebra over the dual space. We will use this fact explicitly in the last section of this paper.

**Remark 2.19.** As made clear by the previous discussion, the sheaf $\mathcal{D}_M$ is a sheaf of non-commutative algebras, as it follows from the non-trivial relations $[x_i, \partial_j] = \delta_{ij}$ and $\{\theta_{\alpha}, \partial_{\beta}\} = \delta_{\alpha\beta}$. Actually, the sheaf of differential operators over a certain (super)manifold, defined as above, is the prototypical example of sheaf of non-commutative algebras. Clearly, this leads to the fact that when an action involving $\mathcal{D}_M$ is concerned, then it is important to distinguish between left and right actions, as in the following definition.

**Definition 2.20 (D_M-Modules).** Let $\mathcal{M}$ be a complex supermanifold and let $\mathcal{E}$ be a sheaf over $\mathcal{M}$. We say that $\mathcal{E}$ is a sheaf of left/right $\mathcal{D}_M$-modules, or simply a left/right $\mathcal{D}_M$-module, if $\mathcal{E}(U)$ is endowed with a left/right $\mathcal{D}_M(U)$-module structure for any open set $U$, which is compatible with the restriction morphisms of the sheaf.

It is to be observed that a left $\mathcal{D}_M$-module can not at all be endowed with the structure of right $\mathcal{D}_M$-module and viceversa. In this paper we will indeed come across left $\mathcal{D}_M$-module which are not right $\mathcal{D}_M$-modules and viceversa.
3. Differential Forms and de Rham Cohomology of Supermanifolds

Given a smooth real supermanifold $\mathcal{M}$ of dimension $n|m$ with structure sheaf $\mathcal{O}_M$ and having introduced its tangent sheaf $\mathcal{T}_M = \mathcal{D}_{\mathcal{E}X}(\mathcal{O}_M)$, it is immediate to consider its dual sheaf $\mathcal{H}om_{\mathcal{O}_M}(\mathcal{T}_M, \mathcal{O}_M) \cong \mathcal{T}^*_M$. This is a locally-free sheaf of $\mathcal{O}_M$-module of rank $n|m$ and one has the usual duality pairing $\mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{O}_M$ defined as $dy_a(\partial_b) = \delta_{ab}$ if $y_a = x_i|\theta_a$. Notice that for this to be a perfect pairing one needs to assign a $\mathbb{Z}_2$-parity to the generators in a way such that $|dx_i| = 0$ and $|d\theta_a| = 1$. When reducing to the underlying real manifold $\mathcal{M}_{\text{def}}$, with an eye to the associated de Rham algebra of differential forms with respect to the wedge product, this would lead to the awkward situation of having a basis of commuting $dx$'s.

In order to restore the correspondence with the usual de Rham theory on the underlying reduced space, it is therefore customary to define the sheaf $1$-forms $\Omega^1_{\mathcal{M}}$ on a supermanifold $\mathcal{M}$ to be the parity shifted of $\mathcal{T}^*_M$, i.e. we take $\Omega^1_{\mathcal{M}} := \mathcal{H}om_{\mathcal{O}_M}(\Pi \mathcal{T}_M, \mathcal{O}_M)$. This is a locally-free sheaf of $\mathcal{O}_M$-modules of rank $m|n$ and if $x_i|\theta_a$ are local coordinates over a certain open set $U$, then one has

$$\Omega^1_{\mathcal{M}}(U) = \mathcal{O}_M(U) \cdot \{d\theta_1, \ldots, d\theta_m, dx_1, \ldots, dx_n\}, \tag{3.1}$$

where now the parity is assigned such that $|d\theta_a| = 0$ and $|dx_i| = 1$ for any $\alpha = 1, \ldots, m$ and $i = 1, \ldots, n$. It is important to stress that now $\Omega^1_{\mathcal{M}}$ has (a perfect) duality pairing with the parity shifted tangent sheaf $\Pi \mathcal{T}_M$ instead of with the tangent sheaf. Again $\Pi \mathcal{T}_M$ is a locally-free sheaf of $\mathcal{O}_M$-modules of rank $m|n$ and if $x_i|\theta_a$ are local coordinates over an open set $U$ as above, one has

$$\Pi \mathcal{T}_M(U) = \mathcal{O}_M(U) \cdot \{\pi \partial_{\theta_1}, \ldots, \pi \partial_{\theta_m}, |\pi \partial_{x_1}, \ldots, \pi \partial_{x_n}\}, \tag{3.2}$$

where $|\pi \partial_{x_i}| = |\partial_{x_i}| + 1 = 1$ and $|\pi \partial_{\theta_a}| = |\partial_{\theta_a}| + 1 = 0$ for any $i = 1, \ldots, n$ and $\alpha = 1, \ldots, m$. With this convention, if $y_a = x_i|\theta_a$ is a local system of coordinates, the duality pairing $\langle \cdot, \cdot \rangle : \Omega^1_{\mathcal{M}} \otimes \Pi \mathcal{T}_M \to \mathcal{O}_M$ reads

$$\langle dy_a, \pi \partial_{\theta_b} \rangle = \delta_{ab}. \tag{3.3}$$

Notice that in what follows we will often simply write $dy_a(\pi \partial_{\theta_b}) = \delta_{ab}$ as indeed $dy_a$ and $\pi \partial_{\theta_b}$ are generating sections of $\mathcal{H}om(\Pi \mathcal{T}_M, \mathcal{O}_M)$ and $\Pi \mathcal{T}_M$ respectively on a certain open set. Once given these basic definitions and fixed our conventions, we introduce the following $\Omega^k_{\mathcal{M}}$.

**Definition 3.1** (de Rham Superalgebra). Let $\mathcal{M}$ be a real superdimension of dimension $n|m$ with structure sheaf $\mathcal{O}_M$. We call the de Rham algebra of $\mathcal{M}$ the sheaf of $\mathcal{O}_M$-superalgebras given by

$$\Omega^\bullet_{\mathcal{M}} := \bigoplus_{k=0}^{\infty} \mathcal{S}^k \mathcal{O}_{\mathcal{O}_M}^{\ast} \Omega^1_{\mathcal{M}}, \tag{3.4}$$

where $\mathcal{S}^k : \mathcal{S}h_{\mathcal{O}_M} \to \mathcal{S}h_{\mathcal{O}_{\mathcal{O}_M}}$ is the $k$-supersymmetric functor and $\Omega^1_{\mathcal{M}}$ is the sheaf of $1$-forms of $\mathcal{M}$ defined as above, where the $\mathbb{Z}_2$-grading is induced by that of $\Omega^1_{\mathcal{M}}$.

**Remark 3.2.** Notice that the above algebra is readily made into a $\mathbb{Z}$-graded algebra by assigning $\text{deg}(\mathcal{O}_M) = 0$ and $\text{deg}(\Omega^1_{\mathcal{M}}) = 1$: this is the obvious degree induced by the (local) polynomial superalgebra

$$\mathcal{S}^k \mathcal{O}_M(U) = \mathcal{O}_M(U) \cdot \{d\theta_1, \ldots, d\theta_m, dx_1, \ldots, dx_n\}. \tag{3.5}$$

The sections of the locally-free $\mathcal{O}_M$-submodule $\mathcal{S}^k \Omega^1_{\mathcal{M}}$ of degree $k$ are called $k$-forms, as in the ordinary commutative setting.

**Remark 3.3.** It is crucial to observe the difference with respect to the usual de Rham or exterior algebra on an ordinary manifold $X$, whose non-zero sections can be of degree $n = \dim X$ at most, due to anti-commutativity of the exterior product. Over a supermanifold, instead, a system of local generator obeys the supercommutation relation of the supersymmetric algebra, i.e. if $y_a = x_i|\theta_a$ then $[dy_a, dy_b]_{+} = 0$. This leads in particular, for any $\alpha, \beta = 1, \ldots, m$ and $i, j = 1, \ldots, n$ to

$$[dx_i, dx_j]_{+} := dx_i dx_j + dx_j dx_i \rightsquigarrow dx_i dx_j = -dx_j dx_i. \tag{3.6}$$

where, for notational convenience, we have left the supersymmetric product of two elements in $\Omega^\bullet_{\mathcal{M}}$ understood. While the first of these relations are just the characterizing relations of an exterior
algebra over an ordinary manifold, thus stating the anticommutativity of two 1-forms, the second of these relations, instead, implies for example that $d\theta^n_{\alpha} \neq 0$ for any $n \geq 1$ and for any $\alpha = 1, \ldots, m$.

It follows that, provided that the supermanifold has odd dimension greater or equal than 1, there are non-zero forms of any degree.

We now introduce the following odd homomorphism acting on the de Rham superalgebra.

**Definition 3.4** (de Rham differential). Let $\mathcal{M}$ be a real supermanifold of dimension $n|m$ with structure sheaf $\mathcal{O}_M$. Given a section $\omega$ of the de Rham superalgebra $\Omega^*_M$ represented as $\omega = dy^l \otimes f_I$ in a certain trivialization $y_a = x_i|\theta_a$ for some multi-index $I$ with $f_I \in \mathcal{O}_M$, we define the de Rham differential $d : \Omega^*_M \to \Omega^*_M$ to be the odd derivation $d := \sum_{a} dy_a \otimes \partial_{y_a}$. More precisely we have

$$d : \Omega^*_M \to \Omega^*_M$$

$$dy^l \otimes f_I \mapsto d(dy^l \otimes f_I) = \sum_{a} (-1)^{|y_a||dy^l|} dy_a dy^l \otimes \partial_{y_a}(f_I).$$

**Remark 3.5.** First of all it is easy to see that the $d$ is well-defined, i.e. it is globally defined - this follows easily from the transformation properties of the $dy_a$‘s and the $\partial_{y_a}$’s -, it is odd since $|dy_a| = |\partial_{y_a}| + 1$ and it is a derivation on the de Rham superalgebra, i.e. it satisfies the $\mathbb{Z}_2$-graded Leibniz rule in the form

$$d(\omega \eta) = (d\omega)\eta + (-1)^{|\omega|}\omega(d\eta),$$

for $\omega$ and $\eta$ any two forms in the de Rham superalgebra, and where $|\omega|$ is the parity of $\omega$.

Let indeed $\omega$ and $\eta$ be represented as $\omega = dy^l \otimes f_I$ and $\eta = dy^j \otimes g_J$, for some multi-indices $I$ and $J$, then we have

$$d(\omega \eta) = d((dy^l \otimes f_I)(dy^j \otimes g_J)) = (-1)^{|dy^l||f_I|}d(dy^l \otimes f_I g_J)$$

$$= \sum_{a} (-1)^{|dy^l||f_I|+|y_a||dy^l|+|f_I||dy^l|} dy_a dy^l \otimes \partial_{y_a}(f_I g_J)$$

$$= \sum_{a} (-1)^{|y_a||dy^l|} dy_a dy^l \otimes \partial_{y_a}(f_I) + \sum_{a} (-1)^{|y_a||dy^l|+|f_I||dy^l|+1} dy_a dy^l \otimes \partial_{y_a}(g_J)$$

$$= \sum_{a} (-1)^{|y_a||dy^l|} dy_a dy^l \otimes \partial_{y_a}(f_I) + \sum_{a} (-1)^{|y_a||dy^l|+|f_I||dy^l|+1} dy_a dy^l \otimes \partial_{y_a}(g_J)$$

$$= (d\omega)\eta + (-1)^{|\omega|}\omega(d\eta).$$

where we have used that $\partial_{y_a}$ is a (super)derivation of the structure sheaf $\mathcal{O}_M$ and that $|d| = 1$. We are now in the position to prove the following lemma.

**Lemma 3.6.** The pair $(\Omega^*_M, d)$ defines a differential graded superalgebra (DGSa).

**Proof.** We have already shown that $d$ is a derivation of the de Rham superalgebra. We are left to prove that $d$ is nilpotent, i.e. $d^2 = 0$. To this end, we simply observe that (up to a constant)

$$d^2 = \sum_{a,b} (dy_a \otimes \partial_{y_a})(dy_b \otimes \partial_{y_b}) + \sum_{a,b} (dy_b \otimes \partial_{y_b})(dy_a \otimes \partial_{y_a})$$

$$= \sum_{a,b} (-1)^{|y_a||dx_i|} dy_a dy_b \otimes \partial_{y_a} \partial_{y_b} + \sum_{a,b} (-1)^{|y_b||dx_i|} dy_b dy_a \otimes \partial_{y_a} \partial_{y_b}$$

$$= \sum_{a,b} \left((-1)^{|y_a||y_b|+|y_a|} + (-1)^{|y_a||y_b|+|y_a|+1}\right) dy_a dy_b \otimes \partial_{y_a} \partial_{y_b}$$

$$= 0,$$

where we have used that $|dy_a| = |y_a| + 1$ for any $a$ even and odd.  \qed
The de Rham complex is a (right) resolution of the sheaf $R$ on a supermanifold, namely a generalization of the ordinary Poincaré lemma. We are now interested in proving the main result concerning cohomology of the de Rham complex on a supermanifold, namely a generalization of the ordinary Poincaré lemma.

**Theorem 3.9 (Poincaré Lemma for Differential Forms).** Let $M$ be a real supermanifold and let $(\Omega^*_M, d)$ be the de Rham complex of $M$. Then one has

$$H^k_d(\Omega^*_M) \cong \begin{cases} \mathbb{R}_M & k = 0 \\ 0 & k > 0. \end{cases}$$

(3.14)

where $\mathbb{R}_M$ is the sheaf of locally constant function on $M$. In particular:

1. The de Rham complex is a (right) resolution of the sheaf $\mathbb{R}_M$, (3.15)

2. Any closed form is locally exact on a real supermanifold.

**Proof.** We start observing that a form of zero degree is section $f \in \mathcal{O}_M$ and, as in the ordinary setting it is immediate to see that the request $df = 0$ forces $f$ to be locally constant, so that one finds indeed $H^0_d(\Omega^*_M) \cong \mathbb{R}_M$.

We now let $\omega \in \Omega^k_M$ for $k \geq 1$ and we show that for any $k \geq 1$ there exists a homotopy $h^k : \Omega^k_M \to \Omega^{k-1}_M$ for the differential $d$, i.e. a map such that

$$h^{k+1} \circ d^k + d^{k-1} \circ h^k = id_{\Omega^k_M},$$

(3.16)

where we have specified the degree involved for the sake of clarity, and where the maps go as follows

$$\cdots \Omega^{k-1}_M \xrightarrow{h^k} \Omega^k_M \xrightarrow{d^k} \Omega^{k+1}_M \xrightarrow{h^{k+1}} \cdots.$$ (3.17)

Let us consider the following homotopy between the identity and the zero map in $M$:

$$G : [0, 1] \times M \xrightarrow{\omega} M.$$ (3.18)

$$t, y_a := x_1, \ldots, x_p, \theta_1, \ldots, \theta_q) \longrightarrow t y_a := (t x_1, \ldots, t x_p | t \theta_1, \ldots, t \theta_q).$$

This induces a map at the level of the de Rham complex via its pull-back

$$G^* : \Omega^k_M \xrightarrow{\omega} \Omega^k_{[0,1] \times M}.$$ (3.19)

Now, writing the map $G$ as a family of maps parametrized by $t \in [0, 1]$, i.e. $G_t : M \to M$, we can rewrite the pull-back above as a family of pull-back maps $G^*_t : \Omega^k_M \to \Omega^k_M$.

We define the homotopy $h^k$ to be the map

$$\Omega^k_M \ni \omega \longmapsto \int_{t=0}^{t=1} dt \{ t_0 G^*_t(\omega) \} \in \Omega^{k-1}_M.$$ (3.20)
where $\iota_{\partial_t}$ is the contraction with respect to the vector field $\partial_t$ on the interval $[0,1]$. Now we compute
\[
(h^{k+1} \circ d^k + d^{k-1} \circ h^k)(\omega) = \int_0^1 dt \left( \iota_{\partial_t} G_t^* (d^k \omega) \right) + \int_0^1 dt \left( d^{k-1} \iota_{\partial_t} G_t^* (\omega) \right) \\
= \int_0^1 dt \left( \iota_{\partial_t} d^k + d^{k-1} \iota_{\partial_t} \right) G_t^* (\omega) \\
= \int_0^1 dt \partial_t G_t^* (\omega) \\
= G_t^* (\omega) - G_{t_0}^* (\omega).
\]
(3.21)
We now observe that, by definition, $G_t^* = 0$ and $G_t^* = \omega$, so that $(h^{k+1} \circ d^k + d^{k-1} \circ h^k)(\omega) = \omega$ for any $\omega \in \Omega_{k>0}^M$. In particular, one has that $H^{k>0}_{dR}(\Omega_M^*) = 0$ for $k > 0$, and the cohomology is concentrated in degree zero.

**Remark 3.10.** This result says something remarkable, but in some sense predictable: we cannot expect new topological invariants arising from the odd part of the geometry of a supermanifold. Instead, the topology is fully encoded in the reduced manifold $M_{\text{red}}$. Intuitively, this is to be ascribed to the characterizing nilpotentcy of the odd part of geometry, which is not "strong enough" to modify rather rough invariants related to a geometric space, such as the topological ones. In some circles this goes under the slogan "fermions are very small" - whatever it means.

**Definition 3.11 (de Rham Cohomology of $M$).** Let $M$ be a real supermanifold. We define the de Rham cohomology of $M$ to be the cohomology of the global sections of the (sheaf of) differentially graded superalgebras $(\Omega_M^*, \delta, d)$, i.e.
\[
H^k_{dR}(M) := H^k_{\delta}(\tilde{H}^0_{\delta}(\Omega_M^*)).
\]
(3.22)
where $\tilde{H}^0_{\delta}(\Omega_M^*)$ is 0-Čech cohomology group of $\Omega_M^*$, i.e. the global sections or $\Omega_M^*$.

**Remark 3.12.** Note that this is the usual definition of de Rham cohomology of a real manifold upon considering an ordinary manifold instead of a supermanifold $M$. In particular, we have the following easy consequence of the previous Poincaré lemma.

**Theorem 3.13 (Quasi-Isomorphism 1).** Let $M$ be a real supermanifold and let $M_{\text{red}}$ be its reduced manifold. Then the de Rham complex of $M$ is quasi-isomorphic to the de Rham complex of $M_{\text{red}}$.

In particular, one has that
\[
H^*_{dR}(M) \cong H^*_{dR}(M_{\text{red}}).
\]
(3.23)

**Proof.** The theorem is an easy consequence of the Čech-to-de Rham spectral sequence for the double complex $(\Omega_M^*, \delta, d)$, where $\delta$ is the Čech differential and $d$ is the de Rham differential, see [14]. More in detail, the generalized Mayer-Vietoris short exact sequence (hence the existence of a partition of unity) and Poincaré lemma yield $H^*_{dR}(M_{\text{red}}) \cong \check{H}^*([M_{\text{red}}], \mathbb{R}_M)$ in the ordinary setting and, analogously, $H^*_{dR}(M) \cong \check{H}^*([M_{\text{red}}], \mathbb{R}_M)$ in the supergeometric setting respectively, where $\check{H}^*([M_{\text{red}}], \mathbb{R}_M)$ is the Čech cohomology of the sheaf of locally-constant functions $\mathbb{R}_M$. It follows that
\[
H^*_{dR}(M) \cong \check{H}^*([M_{\text{red}}], \mathbb{R}_M) \cong H^*_{dR}(M_{\text{red}}),
\]
(3.24)
thus concluding the proof.

In particular, one has a generalization of the ordinary Poincaré Lemma to the “local model” real supermanifold $\mathbb{R}^{p,q}$.

**Theorem 3.14 (Poincaré Lemma for $\mathbb{R}^{p,q}$).** The de Rham cohomology of the supermanifold $\mathbb{R}^{p,q}$ is concentrated in degree zero, i.e.
\[
H^k_{dR}(\mathbb{R}^{p,q}) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}
\]
(3.25)
Proof. Follows immediately from the above 3.9 and 3.13.

Remark 3.15. Analogous results hold true for the compactly supported de Rham cohomology on a supermanifold. More in details one finds that

\[ H^\bullet_{\mathcal{R},c}(\mathcal{M}) \cong H^\bullet_{\mathcal{R},c}(\mathcal{M}_{\text{red}}), \]

so that in particular, one gets the compactly supported Poincaré lemma for \( \mathbb{R}^{p|q} \), which reads

\[ H^k_{\mathcal{R},c}(\mathbb{R}^{p|q}) \cong \begin{cases} \mathbb{R} & k = p \\ 0 & k \neq p. \end{cases} \]

As can be imagined, a representative is given by the lift of the top form on \( \mathcal{M}_{\text{red}} \) on the supermanifold, i.e. \( dz_1 \ldots dz_p \mathcal{B}(z_1, \ldots z_p) \), where \( \mathcal{B} \) is a bump function.

Remark 3.16. The above theorem 3.13 guarantees that even if the de Rham complex of a supermanifold \( \mathcal{M} \) of dimension \( n|m \) is not bounded from above, its de Rham cohomology groups \( H^k_{\mathcal{R}}(\mathcal{M}) \) can only be non-zero up to degree \( n \), i.e. up to the degree which equals the even dimension of the supermanifold or analogously the dimension of its reduced manifold \( \mathcal{M}_{\text{red}} \). All the other higher degree do not contributes to the de Rham cohomology or \( \mathcal{M} \). In other words, the de Rham cohomology of the supermanifold can be non-zero only in the framed part of the complex

\[ 0 \longrightarrow \Omega^0_{\mathcal{M}}(\mathcal{M}) \longrightarrow \Omega^1_{\mathcal{M}}(\mathcal{M}) \longrightarrow \ldots \longrightarrow \Omega^{n-1}_{\mathcal{M}}(\mathcal{M}) \longrightarrow \Omega^n_{\mathcal{M}}(\mathcal{M}) \longrightarrow \Omega^{n+1}_{\mathcal{M}}(\mathcal{M}) \longrightarrow \ldots \]

where we have denoted \( \Omega^k_{\mathcal{M}}(\mathcal{M}) \) the global sections of the sheaf \( \Omega^k_{\mathcal{M}} \) for any \( k \).

Remark 3.17. Finally, it is to be noted that the fact that the de Rham complex of a supermanifold is not bounded from above implies - among things - that there is no tensor density playing the role of the top exterior bundle \( \Omega^k_{\mathcal{M}}X \) over an ordinary manifold \( X \). This is the starting point of the quite unique integration theory on supermanifold, whose main character is introduced in the following section.

4. The Berezinian Sheaf: Constructions and Geometry

In this section we introduce one of the crucial and most peculiar notion arising in supergeometry, that of Berezinian sheaf of a supermanifold, which we will denote as \( \text{Ber}(\mathcal{M}) \). An “operative” first definition can be given by characterizing \( \text{Ber}(\mathcal{M}) \) in terms of its transition functions.

Definition 4.1 (Berezinian - via Transition Functions). Let \( \mathcal{M} \) be a real or complex supermanifold of dimension \( n|m \) and let \( \{ (U_i, x_{ij}) | \theta_{U_i} \} \) be an atlas of charts covering \( \mathcal{M} \). Then we define the Berezinian sheaf \( \text{Ber}(\mathcal{M}) \) of \( \mathcal{M} \) as the locally-free sheaf of right \( O_{\mathcal{M}} \)-modules of rank \( \delta_{0,n+m} | \delta_{0,n+m+1} \), whose local generators \( \{ D_{U_i} (x_{ij}) | \theta_{U_i} \} \) transforms as

\[ D_{U_j|U_i \cap U_j} (x_{ij}) | \theta_{U_j} = D_{U_i|U_i \cap U_j} (x_{ij}) | \theta_{U_i} \text{Ber}(\text{Jac}(\varphi_{ij})) \]

with

\[ \text{Ber}(\text{Jac}(\varphi_{ij})) = \det(A - BD^{-1}C) \det(D)^{-1}, \]

where \( \text{Jac}(\varphi_{ij}) \) is the super Jacobian of the change of coordinates

\[ \varphi_{ij} : U_i|U_i \cap U_j \longrightarrow U_j|U_i \cap U_j \]

\[ x_{ij} | \theta_{U_i} \longrightarrow \varphi_{ij,0}(x | \theta) = x_{ij} | \varphi_{ij,1}(x | \theta) = \theta_{U_j}, \]

and where we have posed

\[ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} \partial_x \varphi_{ij,0} & \partial_x \varphi_{ij,1} \\ \partial_\theta \varphi_{ij,0} & \partial_\theta \varphi_{ij,1} \end{array} \right) = \text{Jac}(\varphi_{ij}). \]

Beside its importance in relation to the quite unique and highly non-trivial integration theory on supermanifolds \([74, 76]\), the Berezinian sheaf deserves a special attention on its own. In what follows we will present and review three of its constructions, all of them inspired to algebraic geometric methods.

The first one, due to Hernández Ruipérez and Muñoz Masque \([62]\), is a beautiful realization of the Berezinian sheaf of a real smooth supermanifold as a certain quotient of natural sheaves, which has the merit of being relatively easy and to make apparent its relation with integration theory.
4.1. Berezinian Sheaf as a Quotient Sheaf. Working on a real smooth supermanifold of dimension $n|m$, the construction of the Berezinian sheaf of $\mathcal{M}$ as a quotient sheaf is obtained starting from the sheaf $\Omega^n_{\mathcal{M},c} \otimes_{\mathcal{O}_{\mathcal{M}}^c} \mathcal{D}^{(m)}_{\mathcal{M}}$ of differential operators $\mathcal{D}^{(m)}_{\mathcal{M}}$ of degree $m$ on $\mathcal{O}_{\mathcal{M}}$ taking values in (the sheaf of) compactly supported differential forms $\Omega^n_{\mathcal{M},c}$ of degree $n$. We denote this (locally-free) sheaf of $\mathcal{O}_{\mathcal{M}}$-modules as $\mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})$, for short, and its elements will be written as $\omega \otimes F \in \mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})$. Notice that, locally over an open set $U$ with coordinates $y_a = x_i|\theta_\alpha$, for $i = 1,\ldots,n$ and $\alpha = 1,\ldots,m$ a system of generators over $\mathcal{O}_{\mathcal{M}}$ is given by

$$\mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})(U) = \left\{ \frac{d y_{j_1} \cdots d y_{j_n}}{n} \otimes \frac{\partial}{\partial y_{k_1}} \cdots \frac{\partial}{\partial y_{k_m}} \right\} \cdot \mathcal{O}_{\mathcal{M}}(U) = \left\{ dy_{J_1} \otimes \frac{\partial}{\partial y_{K}} \right\} \cdot \mathcal{O}_{\mathcal{M}}(U) \quad (4.4)$$

where $I_j$ and $I_k$ are two multi-indices such that $|I_j| = n$ and $|I_k| = m$ and where $\mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})$ has been considered with the structure of right $\mathcal{O}_{\mathcal{M}}$-module.

The key object in the construction is a pretty simple sub-sheaf of $\mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})$. If $i: \mathcal{M}_{red} \to \mathcal{M}$ is the embedding of the reduced manifold into the supermanifold $\mathcal{M}$ as in (2.2), we have a corresponding pull-back map $i^*: \Omega^n_{\mathcal{M},c} \to \Omega^n_{\mathcal{M}_{red},c}$ on differential forms. We introduce a sheaf of $\mathcal{O}_{\mathcal{M}}$-modules $\mathcal{K}_\mathcal{M}$ as the sub-sheaf of sections of $\mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})$ having the following property

$$\mathcal{K}_\mathcal{M}(U) = \left\{ \omega \otimes F \in \mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})(U) : \exists \eta \in \Omega^{n-1}_{\mathcal{M}_{red},c}(U) : i^*(\omega \otimes F(f)) = d\eta \forall f \in \mathcal{O}_{\mathcal{M},c} \right\}, \quad (4.5)$$

where $U$ is an open set and $f \in \mathcal{O}_{\mathcal{M},c}(U)$ and $\eta \in \Omega^{n-1}_{\mathcal{M}_{red},c}(U)$ are a compactly supported function and a compactly supported form respectively. Notice that the above is well-defined as $F(f) \in \mathcal{O}_{\mathcal{M},c}$. Then one can prove the following theorem, [6], [2].

**Theorem 4.2** (Berezinian as a Quotient). Let $\mathcal{M}$ be a real supermanifold of dimension $n|m$ and let $\mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})$ and $\mathcal{K}_\mathcal{M}$ be defined as above. Then the sheafification of the quotient pre-sheaf $\mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})/\mathcal{K}_\mathcal{M}$ is a locally-free sheaf of (right) $\mathcal{O}_{\mathcal{M}}$-module of rank $\delta_{0,n+m}\delta_{1,n+m+1}$, whose generator reads on an open set $U$ with coordinates $x_i|\theta_\alpha$ for $i = 1,\ldots,n$ and $\alpha = 1,\ldots,m$

$$\mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})/\mathcal{K}_\mathcal{M}(U) \cong \left[ dx_1 \cdots dx_n \otimes \frac{\partial}{\partial \theta_1} \cdots \frac{\partial}{\partial \theta_m} \right] \cdot \mathcal{O}_{\mathcal{M}}(U), \quad (4.6)$$

where the square bracket stays for the class of the form-valued differential operator modulo $\mathcal{K}_\mathcal{M}$.

In particular, the above quotient is naturally isomorphic to the Berezinian sheaf of the supermanifold, i.e.

$$\text{Berez}(\mathcal{M}) \cong \mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})/\mathcal{K}_\mathcal{M}. \quad (4.7)$$

**Proof.** We consider $\omega \otimes F \in \mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})$ and we work in a coordinate system $x_i|\theta_\alpha$ over an open set $U$, such that $\mathcal{D}^{(m)}_{\mathcal{M}}(\Omega^n_{\mathcal{M},c})$ has a basis as above in (4.4). Let us consider the following instances. If it appears a term of the form $d\theta_{\alpha}^n$ for any $\alpha$ and any $n \geq 1$ in $\omega$, then the corresponding $\omega \otimes F$ goes to zero under $i^*$, and as such it is in $\mathcal{K}_\mathcal{M}$. So this force $\omega$ to be of the kind $dx_1 \ldots dx_n$. Now consider $\omega \otimes F$ to be of the kind $\omega \otimes F = dx_1 \ldots dx_n \otimes \partial_{g_{ij}}|\partial_{\theta_\alpha}$ for some multi-indices $I$ and $J$ such that $|I| + |J| = m$. If $I \neq 0$ then $\omega \otimes F \in \mathcal{K}_\mathcal{M}(U)$, indeed consider for example $dx_1 \ldots dx_n \otimes \partial_{g_{ij}}|\partial_{\theta_\alpha}|$; the crucial case is that of $f$ of the form $f(x|\theta) = g_i(x)\theta_\alpha \ldots \theta_m$, with $g_i$ a compact supported function on $U$ to get $\omega \otimes F(f) = dx_1 \ldots dx_n \partial_{x_i}g_i(x) = d(x_1dx_2 \ldots dx_ng_i(x))$, which implies $\omega \otimes F \in \mathcal{K}_\mathcal{M}$. This is enough to prove that the class $\mathcal{D}(x|\theta) := [dx_1 \ldots dx_n \otimes \partial_{\theta_\alpha} | \partial_{\theta_\alpha}]$ defines a generator for the above quotient sheaf. Also, this can indeed be identified with the Berezinian sheaf, upon checking that the class $\mathcal{D}(x|\theta)$ transform indeed as in (4.2) under a change of coordinates.
local coordinates. This is a local check; a careful computation is postponed to the next subsection, in a slightly different context.

### Remark 4.3.
As said above, the previous intrinsic construction of the Berezinian as the sheafification of a quotient pre-sheaf of differential operators valued into differential forms has the unquestionable merit of being relatively easy and, at the same time, as explained at the end of the second section of [62], it makes the relation with the Berezinian sheaf and the related integration theory apparent. A minor drawback of the construction is that it only holds true for real supermanifolds, as the existence of compactly supported function is crucial to the above proof.

4.2. **Berezinian Sheaf from Koszul Complex.** Homological algebra comes in help to provide an intrinsic construction of the Berezinian sheaf on complex or algebraic supermanifolds, where compactly supported functions are not available and therefore the above quotient construction breakdowns. As hinted in the introduction to this section, the idea of a suitable generalization to a supergeometric setting of the Koszul complex originally appeared in [58] and was subsequently in [53]. Very recently an encompassing construction has been given in [31].

We let \( E \) be any locally-free sheaf of rank \( p/q \) on a supermanifold \( M \). We define

\[
\mathcal{R} := \bigoplus_{k \geq 0} \mathcal{R}^k \quad \text{with} \quad \mathcal{R}^k := S^k \mathcal{E}
\]

(4.8)

\[
\mathcal{R}^\Pi := \bigoplus_{k \geq 0} \mathcal{R}^\Pi_k \quad \text{with} \quad \mathcal{R}^\Pi_k := S^k \Pi \mathcal{E},
\]

(4.9)

and in turn we consider the following sheaf of \( \mathcal{O}_M \)-superalgebras given by the tensor product

\[
\mathcal{K}^\mathcal{E} := \bigoplus_{k \geq 0} \mathcal{K}^\mathcal{E}_{-k} = \mathcal{R} \otimes_{\mathcal{O}_M} \mathcal{R}^\Pi = \mathcal{R} \otimes_{\mathcal{O}_M} \mathcal{R}^\Pi.
\]

(4.10)

Further, let us consider two basis of local generators for \( \mathcal{E} \) and \( \Pi \mathcal{E} \) respectively given by \( \{v_i|\chi_\alpha\} \) and a \( \{\pi\chi_\alpha|\pi\nu\} \). Using these we can define the following operator acting on \( \mathcal{K}^\mathcal{E} \):

\[
\delta : \mathcal{K}^\mathcal{E} = \mathcal{R} \otimes_{\mathcal{O}_M} \mathcal{R}^\Pi \xrightarrow{\delta} \mathcal{K}^\mathcal{E} = \mathcal{R} \otimes_{\mathcal{O}_M} \mathcal{R}^\Pi
\]

(4.11)

\[
\begin{align*}
\delta : & \mathcal{K}^\mathcal{E} = \mathcal{R} \otimes_{\mathcal{O}_M} \mathcal{R}^\Pi \\
& \xrightarrow{\delta} \left( \sum_{i=1}^p v_i \otimes \partial_{\pi v_i} + \sum_{j=1}^q \chi_j \otimes \partial_{\pi \chi_j} \right) (r \otimes r^\Pi),
\end{align*}
\]

where \( r \in \mathcal{R} \) and \( r^\Pi \in \mathcal{R}^\Pi \). Since the derivations can be seen as the dual bases to the bases of \( \mathcal{E} \) and \( \Pi \mathcal{E} \) it is not hard to see that the above is globally well-defined and independent of the choice of local bases. Further, \( \delta \) is homogenous of degree \(-1\) with respect to the \( \mathbb{Z}_2\)-gradation of \( \mathcal{K}^\mathcal{E} \) seen as a sheaf of \( \mathcal{R}\)-modules and it is odd with respect to the \( \mathbb{Z}_2\)-graded structure on \( \mathcal{K}^\mathcal{E} \); more in particular it is not hard to show that it is nilpotent, i.e. \( \delta \circ \delta = 0 \), so that the pair \( (\mathcal{K}^\mathcal{E}, \delta) \) defines a differentially graded sheaf of \( \mathcal{R}\)-algebras. We can thus give the following definition.

**Definition 4.4** (Super Koszul Complex). Let \( M \) be a real, complex or algebraic supermanifold. Given any locally-free sheaf of \( \mathcal{O}_M \)-modules \( \mathcal{E} \), we call the pair \( (\mathcal{K}^\mathcal{E}, \delta) \) defined as above the **super Koszul complex** associated to \( \mathcal{E} \):

\[
\cdots \xrightarrow{\delta} \mathcal{R} \otimes S^k \Pi \mathcal{E} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{R} \otimes S^2 \Pi \mathcal{E} \xrightarrow{\delta} \mathcal{R} \otimes \Pi \mathcal{E} \xrightarrow{\delta} \mathcal{R} \xrightarrow{\delta} 0.
\]

(4.12)

One of the main result of [57] is concerned with the homology of this complex of sheaves.

**Theorem 4.5** (Homology of the Super Koszul Complex). Let \( M \) be a real, complex or algebraic supermanifold with structure sheaf \( \mathcal{O}_M \), let \( \mathcal{E} \) be a locally-free sheaf of \( \mathcal{O}_M \)-modules on \( M \) and let \( (\mathcal{K}^\mathcal{E}, \delta) \) be the super Koszul complex associated to \( \mathcal{E} \), defined as above. Then the super Koszul complex \( (\mathcal{K}^\mathcal{E}, \delta) \) is an exact resolution of \( \mathcal{O}_M \) endowed with the structure of sheaf of \( \mathcal{R}\)-modules, i.e. the homology \( \mathcal{H}_i((\mathcal{K}^\mathcal{E}, \delta)) \) is concentrated in degree 0,

\[
\mathcal{H}_i((\mathcal{K}^\mathcal{E}, \delta)) \cong \begin{cases} \mathcal{O}_M & i = 0 \\ 0 & i \neq 0. \end{cases}
\]

(4.13)

**Remark 4.6.** With reference to the above result, notice that \( \mathcal{O}_M \) is indeed a \( \mathcal{R}\)-module thanks to the following short exact sequence of sheaves

\[
0 \longrightarrow \mathcal{I}_\mathcal{R} \longrightarrow \mathcal{R} \longrightarrow \mathcal{O}_M \longrightarrow 0,
\]

(4.14)
where $\mathcal{I}_R := \bigoplus_{k \geq 1} S^k \mathcal{E}$ is the sheaf of ideals of $\mathcal{R}$ generated by $\mathcal{E} \subset R$ so that $\mathcal{O}_M \cong R / \mathcal{I}_R \mathcal{R}$.

**Remark 4.7.** The Koszul resolution of theorem 4.4 allows to compute other derived functors in a supergeometric context. In particular, given the Koszul super complex $K^*_E$ as above, one can introduce the dual construction via the functor $\operatorname{Hom}_R(-, \mathcal{R})$, which yields the pair $(K^*_E, \delta^*) := (\operatorname{Hom}_R(K^*_E, \mathcal{R}), \operatorname{Hom}_R(\delta, \mathcal{R}))$. Defining

$$R_k^{\Pi^*} := \bigoplus_{k \geq 0} R_k^* \quad \text{with} \quad R_k^{\Pi^*} := S^k \Pi^*$$

(4.15)

the sheaf of $\mathcal{O}_M$-superalgebras generated by $\{\partial_{\pi\chi_i} | \partial_{\pi v_i}\}$ for $i = 1, \ldots, p$ and $\alpha = 1, \ldots, q$ in $\Pi^*$, it is easy to see that

$$K^*_E := \bigoplus_{k \geq 0} K^*_E = R \otimes \mathcal{O}_M \bigoplus_{k \geq 0} R_k^{\Pi^*} = R \otimes \mathcal{O}_M R^{\Pi^*}. \quad (4.16)$$

The fundamental observation is that $K^*_E$ is acted by an operator $\delta^*$, whose definition is formally identical to that of $\delta$ given above in (4.11). But here $\delta^*$ has to be looked at as a multiplication operator by the element $\sum_{i=1}^p v_i \otimes \partial_{\pi v_i} + \sum_{j=1}^q \chi_j \otimes \partial_{\chi_j}$ in $K^*_E$. This is enough to guarantee that $\delta^* \circ \delta^* = 0$, so that we can introduce the following.

**Definition 4.8** (Dual of the Super Koszul Complex). Let $\mathcal{M}$ be a real, complex or algebraic supermanifold. Given any locally-free sheaf of $\mathcal{O}_M$-modules $\mathcal{E}$, we call the pair $(K^*_E, \delta^*)$ defined as above the **dual of the super Koszul complex** associated to $\mathcal{E}$:

$$0 \rightarrow \mathcal{R} \xrightarrow{\delta^*} \mathcal{R} \otimes \Pi^* \xrightarrow{\delta^*} \mathcal{R} \otimes S^2 \Pi^* \xrightarrow{\delta} \cdots \xrightarrow{\delta^*} \mathcal{R} \otimes S^k \Pi^* \xrightarrow{\delta^*} \cdots$$

(4.17)

Now, we aim at computing the (co)homology of the dual of the super Koszul complex. Recalling that by definition we have $K^*_E := \operatorname{Hom}_E(K^*_E, \mathcal{R})$, then

$$\operatorname{Ext}^*_R(\mathcal{O}_M, \mathcal{R}) = \mathcal{H}^i((K^*_E, \delta^*)). \quad (4.18)$$

The homology (sheaf) of dual of the Koszul complex is computed in the following theorem, see [57].

**Theorem 4.9** (Homology of the dual of the Super Koszul Complex). Let $\mathcal{M}$ be a real, complex or algebraic supermanifold with structure sheaf $\mathcal{O}_M$, let $\mathcal{E}$ be a locally-free sheaf of $\mathcal{O}_M$-modules on $\mathcal{M}$ and let $(K^*_E, \delta)$ be the dual of the super Koszul complex associated to $\mathcal{E}$, defined as above. Then its homology is concentrated in degree $p$ and locally-generated over $\mathcal{O}_M$ by the class

$$\operatorname{Ext}^*_R(\mathcal{O}_M, \mathcal{R}) \cong [\chi_1 \ldots \chi_q \otimes \partial_{\pi v_1} \ldots \partial_{\pi v_p}] : \mathcal{O}_M \quad (4.19)$$

where $\chi_1 \ldots \chi_q \in S^q \mathcal{E}$ and $\partial_{\pi v_1} \ldots \partial_{\pi v_p} \in S^p \Pi^*$.\n
**Proof.** It is immediate to observe that the element $\mathcal{D} := \chi_1 \ldots \chi_q \otimes \partial_{\pi v_1} \ldots \partial_{\pi v_p} \in S^q \mathcal{E} \otimes S^p \Pi^*$ belong to the kernel of $\delta^*$. We can then observe that locally $\mathcal{R} \otimes \mathcal{O}_M \mathcal{R}^{\Pi^*}$ is generated over $\mathcal{O}_M$ by the elements $(v_1, \ldots, v_p, \partial_{\pi\chi_1}, \ldots, \partial_{\pi\chi_q}, \partial_{\pi v_1}, \ldots, \partial_{\pi v_p}, \chi_1, \ldots, \chi_q)$. Posing $N := p + q$, we can redefine the generators as $(s_1, \ldots, s_N) := (v_1, \ldots, v_p, \partial_{\pi\chi_1}, \ldots, \partial_{\pi\chi_q})$ and $(\psi_1, \ldots, \psi_N) := (\partial_{\pi v_1}, \ldots, \partial_{\pi v_p}, \chi_1, \ldots, \chi_q)$, so that in particular one has that $\delta^* = \sum_{i=1}^N u_i \psi_i$ and $\mathcal{D} = \prod_{j=1}^N \psi_i$ and $\mathcal{R} \otimes \mathcal{R}^{\Pi^*}$ becomes a sheaf of exterior algebras

$$(\mathcal{R} \otimes \mathcal{R}^{\Pi^*})(U) \cong \bigwedge_{\mathcal{O}(U)\{s_1, \ldots, s_N\}} (\psi_1, \ldots, \psi_N) \quad (4.20)$$

the over the commutative ring $\mathcal{O}(U)\{s_1, \ldots, s_N\}$. This is the dual of the ordinary commutative Koszul complex, whose cohomology is concentrated in top-degree, in this case $N$, and generated over $\mathcal{O}_M$ by the element $\mathcal{D} = \psi_1 \ldots \psi_N$, see [31].\n
**Remark 4.10.** The crucial point is now that the class singled out by the dual of the Koszul complex transforms by the multiplication by the Berezinian of an automorphism of the sheaf $\mathcal{E}$. More precisely, we prove the following result.
Theorem 4.11. Let $\mathcal{M}$ be a real, complex or algebraic supermanifold with structure sheaf $\mathcal{O}_M$, let $\mathcal{E}$ be a locally-free sheaf of $\mathcal{O}_M$-modules on $\mathcal{M}$ and let $\phi \in \text{Aut}(\mathcal{E})$ be an automorphism of $\mathcal{E}$. Then the induced automorphism $\hat{\varphi} \in \text{Aut}(\mathcal{E} \text{xt}^p_\mathcal{R}(\mathcal{O}_M, \mathcal{R}))$ is given by the multiplication by the inverse of the Berezinian of the automorphism, i.e.

$$\hat{\varphi} : \mathcal{E} \text{xt}^p_\mathcal{R}(\mathcal{O}_M, \mathcal{R}) \rightarrow \mathcal{E} \text{xt}^p_\mathcal{R}(\mathcal{O}_M, \mathcal{R})$$

\[D \rightarrow D \cdot \text{Ber}(\varphi)^{-1}\]  

(4.21)

Proof. Fixing a local system of generator for $\mathcal{E}$ given by $\{v_1, \ldots, v_p | \chi_1, \ldots, \chi_q\}$, then $\varphi \in \text{Aut}(\mathcal{E})$ is represented by a matrix $[M] \in GL(p|q, \mathcal{O}_M(U))$

$$[\varphi]_{\alpha\beta} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} a_{ki} & b_{kj} \\ \hline c_{ki} & d_{kj} \end{array}\right)$$

(4.22)

with $A, B$ even and $C, D$ odd submatrices. Now, if $\varphi_i$ for $i = 1, 2$ are automorphisms of $\mathcal{E}$, contravariant functoriality of the construction, see [53], implies that the product of two matrices $[\varphi_1] \cdot [\varphi_2]$ corresponds to the product $\hat{\varphi}([\varphi_2]) \cdot \hat{\varphi}([\varphi_1])$, which acts an an automorphisms of $\mathcal{E} \text{xt}^p_\mathcal{R}(\mathcal{O}_M, \mathcal{R})$. If follows that we can use the standard decomposition

$$(\begin{array}{c|c} A & B \\ \hline C & D \end{array}) = \left(\begin{array}{c|c} 1 & BD^{-1} \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} A - BD^{-1}C & 0 \\ \hline 0 & D \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline D^{-1}C & 1 \end{array}\right)$$

(4.23)

and considering separately the cases.

(1) : $[\varphi] = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)$

(2) : $[\varphi] = \left(\begin{array}{c|c} 1 & 0 \\ \hline * & 1 \end{array}\right)$

(3) : $[\varphi] = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right)$

(4.24)

It is easy to see that only in the first case we have an induced transformation of the homology class $\mathcal{D} = [\chi_1 \ldots \chi_q \otimes \delta_{v_1} \ldots \delta_{v_p}]$ given by the multiplication by $\det(D) \cdot \det(A)^{-1}$. In the two remaining cases the homology class $\mathcal{D}$ is invariant. It follows from the above decomposition that

$$\hat{\varphi}_{\mathcal{M}, \mathcal{E}}(\varphi) = \det(D) \det(A - BD^{-1}C)^{-1},$$

which is indeed $\text{Ber}([\varphi])^{-1}$ as claimed. □

Remark 4.12. Notice that what plays a crucial role in the previous proof is the fact that $\mathcal{D}$ is a cohomology class, so that parts of the transformation on the representative yield exact elements, which do not contribute to the induced automorphism.

Remark 4.13. Clearly, it is enough to consider $\mathcal{E}^* = \mathcal{H}om(\mathcal{E}, \mathcal{O}_M)$ instead of $\mathcal{E}$ and the related homology of the dual of the super Koszul complex as to obtain the expected transformation

$$\hat{\varphi}_{\mathcal{E}^*} : \mathcal{E} \text{xt}^p_{\mathcal{E}^*}(\mathcal{O}_M, S^* \mathcal{E}^*) \rightarrow \mathcal{E} \text{xt}^p_{\mathcal{E}^*}(\mathcal{O}_M, S^* \mathcal{E}^*)$$

\[D^* \rightarrow D^* \cdot \text{Ber}(\varphi_{\mathcal{E}^*})\]  

(4.25)

where $\varphi_{\mathcal{E}^*}$ is an automorphism of $\mathcal{E}^*$. This suggests that given a locally-free sheaf $\mathcal{E}$ on a real, complex or algebraic supermanifold, one can see the previous construction as a defining one for the notion of Berezinian sheaf of $\mathcal{E}$, by posing

$$\text{Ber}(\mathcal{E}) := \mathcal{E} \text{xt}^p_{\mathcal{E}^*}(\mathcal{O}_M, S^* \mathcal{E}^*)$$

(4.26)

In particular, as to make contact with the previous subsection, we give the following definition with agrees also with that of Manin [53] and Witten [76].

Definition 4.14 (Berezinian of a Supermanifold). Let $\mathcal{M}$ be a real, complex or algebraic supermanifold with structure sheaf $\mathcal{O}_M$. We call the Berezinian sheaf of $\mathcal{M}$ and we denote it by $\text{Ber}(\mathcal{M})$ the locally-free sheaf of $\mathcal{O}_M$-modules of rank $\delta_{0,n+m}|\delta_{1,n+m}$ defined by

$$\text{Ber}(\mathcal{M}) := \text{Ber}(\Omega_\mathcal{M}^1),$$

(4.27)

where $\text{Ber}(\Omega_\mathcal{M}^1) = \mathcal{E} \text{xt}^p_{S^* \mathcal{O}_\mathcal{M}^1}(\mathcal{O}_\mathcal{M}, S^* \Omega_\mathcal{M}^1)^\ast$.

Remark 4.15. Notice that, as above, the Berezinian sheaf of $\mathcal{M}$ is locally generated by the class $[dx_1 \ldots dx_p \otimes \theta_1 \ldots \theta_q]$ and that it can be equivalently defined as $\text{Ber}(\mathcal{M}) = \text{Ber}(\mathcal{T}_\mathcal{M})$, since if $A$ is an automorphism, then $\text{Ber}(A^\ast) = \text{Ber}(A), \text{Ber}(A^{-1}) = \text{Ber}(A)^{-1}$ and $\text{Ber}(\Pi A) = \text{Ber}(A)^{-1}$.
4.3. Berezinian Sheaf from Cohomology of Forms and Operators. The last construction of the Berezinian sheaf that we are to discuss stands somewhat in between the previous ones, as it employs the sheaf $D_M$ to “deform” in a non-commutative fashion the previous Koszul complex construction \[\mathfrak{D}.\] In particular we consider the following tensor product of sheaves.

**Definition 4.16** (Universal de Rham Sheaf). Let $\mathcal{M}$ be a real or complex supermanifold. We call the tensor product sheaf $\Omega^\bullet \mathcal{M} = \Omega^\bullet_{\text{er}} \otimes_{\mathcal{O}_M} D_M$ the universal de Rham sheaf of $\mathcal{M}$.

**Remark 4.17.** It is easy to see that $\Omega^\bullet \mathcal{M}$ is both $\mathbb{Z}$-graded and $\mathbb{Z}_2$-graded. Further, it is a sheaf of left $\Omega^\bullet_{\text{er}}$-modules - and hence also left $\mathcal{O}_M$-modules - and a sheaf of right $D_M$-modules. Notice, by the way, that the structure of right $\mathcal{O}_M$-module induced by $D_M$ does not coincide with that of left $\mathcal{O}_M$-module, since $D_M$ is non-commutative.

**Remark 4.18.** There is an obvious operator acting on $\Omega^\bullet_{\mathcal{M}}$, whose action is given by

\[
\mathcal{D} : \Omega^\bullet_{\mathcal{M}} \to \Omega^\bullet_{\mathcal{M}} \quad \omega \otimes F \mapsto \mathcal{D}(\omega \otimes F) := \sum_a (-1)^{|\omega||x_a|} dx_a \omega \otimes \partial_{x_a} \circ F,
\]

for any $\omega \otimes F \in \Omega^\bullet_{\mathcal{M}}$. It is not hard to prove that $\mathcal{D}$ is globally well-defined - as the $dx_a$’s and the $\partial_{x_a}$’s transform dually - and that for any $f \in \mathcal{O}_M$ one has that indeed $\mathcal{D}(\omega f \otimes F) = \mathcal{D}(\omega \otimes f F)$. Further, it is immediate to observe that the operator $\mathcal{D}$ is nilpotent, since it can be see as the multiplication by the odd element $\sum_a dx_a \otimes \partial_{x_a} \in \Omega^\bullet_{\mathcal{M}}$, i.e.

\[
\mathcal{D}(\omega \otimes F) := \left( \sum_a dx_a \otimes \partial_{x_a} \right) \cdot (\omega \otimes F). \tag{4.29}
\]

We thus have that the pair $(\Omega^\bullet_{\mathcal{M}}, \mathcal{D})$ defines a complex of sheaves whose $\mathbb{Z}$-grading is induced by the one of $\Omega^\bullet_{\text{er}}$. The cohomology of this complex provides another construction of the Berezinian sheaf of $\mathcal{M}$.

**Theorem 4.19** (Cohomology of $\Omega^\bullet_{\mathcal{M}}$). Let $\mathcal{M}$ be a real or complex supermanifold of dimension $\dim \mathcal{M}$. Then the homology of the complex $(\Omega^\bullet_{\mathcal{M}}, \mathcal{D})$ is naturally isomorphic to the Berezinian sheaf of $\mathcal{M}$, i.e.

\[
H^p\left( (\Omega^\bullet_{\mathcal{M}}, \mathcal{D}) \right) \cong \text{Ber}(\mathcal{M}) \tag{4.30}
\]

and $H^i\left( (\Omega^\bullet_{\mathcal{M}}, \mathcal{D}) \right) \cong 0$ for any $i \neq p$.

**Proof.** We need to construct a homotopy for the operator $\mathcal{D}$. We work in a local chart $(U, x_a)$ so that the sheaf $\Omega^\bullet_{\mathcal{M}} := \Omega^\bullet_{\text{er}} \otimes_{\mathcal{O}_M} D_M$ is given by the sheaf of vector spaces generated by monomials having the form $\omega \otimes F$, with $\omega = \sum_i x_I$ and $F = \partial_J f$ for multi-indices $I$ and $J$ and some function $f \in \mathcal{O}_M\vert_U$. We claim that the homotopy is given by the (local) operator

\[
\mathcal{H}(\omega \otimes F) := \sum_a (-1)^{|\omega||x_a|+|\theta^f|+1} t_{\pi \partial_{x_a}} dx_I \otimes [\partial_J, x_a] f. \tag{4.31}
\]

where the derivation $t_{\pi \partial_{x_a}} := \partial_{dx_a}$ is the contraction with respect to the coordinate field $\pi \partial_{x_a}$, so that $t_{\pi \partial_{x_a}}(dx_a) = \partial_{dx_a}$. A lengthy but not too hard computation gives

\[
(\mathcal{H} \mathcal{D} + \mathcal{D} \mathcal{H})(\omega \otimes F) = (p + q + \text{deg}_0(\omega) + \text{deg}_0(\partial_J) - \text{deg}_1(\omega) - \text{deg}_1(\partial_J))(\omega \otimes F), \tag{4.32}
\]

where $\text{deg}_0$ and $\text{deg}_1$ are the degree with respect to the even and odd generators. We have that $\text{deg}_1(\omega) \leq p$ and $\text{deg}_1(\partial_J) \leq q$, therefore the homotopy fails if and only if $\text{deg}_0(\omega) = 0 = \text{deg}_0(\partial_J)$ and $\text{deg}_1(\omega) = p$, $\text{deg}_1(\partial_J) = q$ : the monomial $\omega \otimes F$ is given by $dz_1 \ldots dz_p \otimes \partial_1 \ldots \partial_q f$ for $f \in \mathcal{O}_M\vert_U$. This element is clearly in the kernel of $\mathcal{D}$ and it generates the Berezinian sheaf $\text{Ber}(\mathcal{M})$ as $\mathcal{O}_M$-module.

**Remark 4.20.** Notice, once again, that also this construction holds true in the smooth and holomorphic category, but also in the algebraic category.
Having defined the Berezinian sheaf in the previous section, we are now interested in studying its properties. In the first subsection, in theorem 5.1, we will show how the Berezinian sheaf of a supermanifold $\mathcal{M}$ is related to the canonical sheaf of its underlying manifold $\mathcal{M}_{\text{red}}$; this will prove crucial in the definition of the Berezin integral. Further, in the second subsection, we will show that the Berezinian sheaf is a *right* $\mathcal{D}_\mathcal{M}$-module. This theory has been first developed by Penkov in the marvelous \cite{53}, where Serre duality - see also \cite{58} - and Mebkhout duality for complex supermanifolds are also proved. Subsequently, results in this direction appeared also in the book \cite{53}, however the relation with $\mathcal{D}_\mathcal{M}$-module theory is left somewhat hidden. In this section we make this connection apparent, by spelling out all of its details and stressing differences and similarities with the ordinary commutative theory. Indeed, the presence of a right $\mathcal{D}_\mathcal{M}$-module structure on the Berezinian sheaf is another striking analogy between the Berezinian sheaf and its commutative counterpart, the canonical sheaf on an ordinary manifold $X$, which carries as well the structure of right $\mathcal{D}_X$-module. More precisely we will see that in a similar fashion as in ordinary commutative theory, the right $\mathcal{D}_\mathcal{M}$-module structure on $\text{Ber}(\mathcal{M})$ is related to the action of the Lie derivative on it.

**Remark 5.1.** As it should be clear from the previous section, it has to be noticed that the Berezinian sheaf is *not* a sheaf of differential forms, and as such it does not appear in the de Rham complex $\Omega^*_\mathcal{M}$ of $\mathcal{M}$. It follows that it is not trivial to define a notion of Lie derivative acting on sections of $\text{Ber}(\mathcal{M})$. Indeed, the so-called Cartan formula $\mathcal{L}_X \omega = \{d, i_X\}(\omega)$ which holds true for differential forms $\omega \in \Omega^*_X$ for any $i = 0, \ldots, \dim X$ and can be readily generalized to the de Rham complex of a supermanifold, does not apply to the Berezinian.

### 5.1. Berezinian and Canonical Sheaf

In this first subsection we prove an easy, yet very important isomorphism, which establishes a crucial relation between the Berezinian sheaf $\text{Ber}(\mathcal{M})$ of a supermanifold $\mathcal{M}$ of dimension $pq$ and the canonical sheaf $\Omega^p_{\text{red}}$ of the reduced space of $\mathcal{M}_{\text{red}}$. We start by proving the following ancillary result, see for example \cite{53}.

**Lemma 5.2.** Let $\mathcal{M}$ be a real or complex supermanifold. Then we have the following isomorphism of sheaves of $\mathcal{O}_{\mathcal{M}_{\text{red}}}$-modules

$$\mathcal{F}_\mathcal{M} \cong \Pi \left( \Omega^1_{\mathcal{M}} / \mathcal{J}_\mathcal{M} \Omega^1_{\mathcal{M}} \right)_0$$

where the subscript 0 refers to the $\mathbb{Z}_2$-grading of the quotient sheaf $\Omega^1_{\mathcal{M}} / \mathcal{J}_\mathcal{M} \Omega^1_{\mathcal{M}}$.

**Proof.** First of all, notice that locally, a basis of $\mathcal{F}_\mathcal{M} = \mathcal{J}_\mathcal{M} / \mathcal{J}_\mathcal{M}^2$ is given by $\theta_\alpha \mod \mathcal{J}_\mathcal{M}^2$ for $\alpha = 1, \ldots, q$ where $q$ is the odd dimension of $\mathcal{M}$. Further, locally, a basis of $(\Omega^1_{\mathcal{M}} / \mathcal{J}_\mathcal{M} \Omega^1_{\mathcal{M}})_1$ read $d\theta_\alpha \mod \mathcal{J}_\mathcal{M} \Omega^1_{\mathcal{M}}$, again for $\alpha = 1, \ldots, q$ where $m$ is the odd dimension of $\mathcal{M}$. We claim that the isomorphism reads $\theta_\alpha \mod \mathcal{J}_\mathcal{M}^2 \mapsto \theta_\alpha \mod \mathcal{J}_\mathcal{M} \Omega^1_{\mathcal{M}}$, and we prove that it is well-defined and independent of the charts. Indeed, let $x'_\alpha = z_i \theta_\alpha$ be another local system of coordinates, then the transformation for the transformation of $\mathcal{F}_\mathcal{M} = \mathcal{J}_\mathcal{M} / \mathcal{J}_\mathcal{M}^2$ one has that $\theta_\alpha' \equiv \sum \beta f_{\alpha \beta}(x) \theta_\beta \mod \mathcal{J}_\mathcal{M}^2$. It follows that

$$d\theta_\alpha' = \sum_b dx_b \frac{\partial \theta_\alpha'}{\partial x_b} + \sum \beta d\theta_\beta \frac{\partial \theta_\alpha'}{\partial \theta_\beta}$$

$$= \sum_b dx_b \frac{\partial}{\partial x_b} \left( \sum_{\gamma} f_{\alpha \gamma}(x) \theta_\gamma \mod \mathcal{J}_\mathcal{M}^2 \right) + \sum \beta d\theta_\beta \frac{\partial}{\partial \theta_\beta} \left( \sum_{\gamma} f_{\alpha \gamma}(x) \theta_\gamma \mod \mathcal{J}_\mathcal{M}^2 \right)$$

$$= \sum_{b, \gamma} dx_b \frac{\partial f_{\alpha \gamma}(x)}{\partial x_b} \theta_\gamma \mod \mathcal{J}_\mathcal{M}^2 + \sum \beta d\theta_\beta f_{\alpha \beta}(x) \mod \mathcal{J}_\mathcal{M}^2$$

$$\equiv \sum \beta d\theta_\beta \mod (\mathcal{J}_\mathcal{M} \Omega^1_{\mathcal{M}}) f_{\alpha \beta}(x),$$

since $\sum_{b, \gamma} dx_b \frac{\partial f_{\alpha \gamma}(x)}{\partial x_b} \theta_\gamma \mod \mathcal{J}_\mathcal{M}^2 \equiv 0 \mod \mathcal{J}_\mathcal{M} \Omega^1_{\mathcal{M}}$. Reversing the parity of the local generators $d\theta_\beta$ concludes the proof. \[\square\]

Using the above lemma we prove the result we claimed at the beginning of the subsection.
Theorem 5.3. Let $\mathcal{M}$ be a real or complex supermanifold and let $\mathcal{J}_\mathcal{M} \subset \mathcal{O}_\mathcal{M}$ be its nilpotent sheaves of $\mathcal{O}_\mathcal{M}$, $\mathcal{E}$ be a sheaf of $\mathcal{O}_\mathcal{M}$-modules

\[ \varphi : \mathcal{J}_\mathcal{M}^q \Ber(\mathcal{M}) \cong \Omega^q_{\mathcal{M}_{\text{red}}}, \]

where $\Ber(\mathcal{M})$ is the Berezinian sheaf of $\mathcal{M}$ and $\Omega^q_{\mathcal{M}_{\text{red}}}$ is the canonical sheaf of $\mathcal{M}_{\text{red}}$. In local coordinates $x_a = z_i|\theta_a$ the above isomorphism reads

\[ \mathcal{D}(x)\theta_1 \ldots \theta_q \leftrightsquigarrow dz_1, \ldots, dz_q, \]

where $\mathcal{D}(x) \in \Ber(\mathcal{M})$, $\theta_1 \ldots \theta_q \in J^q_\mathcal{M}$ and $\det(f)_{\text{red}}$ and $\det(f)_{\text{red}}$ are $\mathcal{O}_\mathcal{M}$ and $\mathcal{O}_{\mathcal{M}_{\text{red}}}$, respectively.

Proof. First of all we observe that $\mathcal{J}_\mathcal{M}^q \Ber(\mathcal{M})$ is obviously a sheaf of $\mathcal{O}_{\mathcal{M}_{\text{red}}}$-modules, as $\mathcal{J}_\mathcal{M}^q \cong \mathcal{J}_\mathcal{M}/\mathcal{J}_\mathcal{M}^{q+1}$ is a sheaf of $\mathcal{O}_{\mathcal{M}_{\text{red}}}$-modules. Also, it is clear that if a generating section $\theta_a \in J_\mathcal{M}/J_\mathcal{M}^2$ lies such that $\theta_a \equiv \sum f_{\alpha\beta}(z)\theta_\beta$ mod $J_\mathcal{M}^2$ then

\[ \mathcal{J}_\mathcal{M}^q \ni \theta_1 \ldots \theta_q = \det(f_{\alpha\beta})\theta_1 \ldots \theta_q = \det(f_{\alpha\beta},_\text{red})\theta_1 \ldots \theta_q. \]

Now, considering local coordinates $x_a = z_i|\theta_a$, an index $a$ running on both even and odd coordinates, for a generic change of coordinates $x'_a = z'_i|\theta'_a$ one has that

\[ \mathcal{D}(x')\theta'_1 \ldots \theta'_q = \mathcal{D}(x)\Ber \left( \frac{\partial x'}{\partial x} \right)_{\text{red}} \det(f_{\alpha\beta})_{\text{red}} \theta_1 \ldots \theta_q \]

\[ = \mathcal{D}(x) \left( \frac{\partial z'}{\partial z} \right)_{\text{red}}^{-1} \det(f_{\alpha\beta})_{\text{red}} \theta_1 \ldots \theta_q. \]

It follows from the previous lemma that $\det(\partial z'_{\theta})_{\text{red}} = \det(f_{\alpha\beta})_{\text{red}}$, which concludes the theorem since $\det(\partial z'_{\theta})_{\text{red}}$ is indeed the transformation of a generating section of the canonical sheaf $\Omega^q_{\mathcal{M}_{\text{red}}}$ of the reduced manifold.

Remark 5.4. The above theorem holds true in exactly the same forms considering the compactly supported case instead: this will prove crucial to define a meaningful notion of integral on supermanifolds, as we shall see later on in this paper.

5.2. $\mathcal{D}_\mathcal{M}$-modules and Connections. It is an easy yet fundamental result of $\mathcal{D}$-module theory that giving a $\mathcal{D}$-module structure on a sheaf corresponds to define a flat connection on it. Nonetheless, as observed above, attention must be paid due to the non-commutativity of the sheaf $\mathcal{D}$ so that one has to distinguish between left and right action and thus left and right $\mathcal{D}$-module structures: accordingly, we will introduce left and right connections on sheaves and the related notion of flatness.

Let us start from left $\mathcal{D}$-modules. In this case the needed left action is induced by a left connection: this is nothing but the standard notion of affine, or Koszul connection on a sheaf introduced in ordinary differential geometry. In particular we will work over a real or complex supermanifold, and thus the base field will be either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The notations employed mostly follow $\mathcal{M}$, in particular for any $f \in \mathcal{O}_\mathcal{M}$ and $X \in T_\mathcal{M}$ we denote the commutator $[X,f] = X(f)$ as

\[ [X,f] := X \circ f - (-1)^{|X||f|} f \circ X, \]

as to stress that we are considering the operator product $\circ$ in $\mathcal{D}_\mathcal{M}$.

Definition 5.5 (Left Connection on $\mathcal{E}$). Let $\mathcal{M}$ be a real or complex supermanifold with structure sheaf $\mathcal{O}_\mathcal{M}$ and let $\mathcal{E}$ be a sheaf of $\mathcal{O}_\mathcal{M}$-modules. Then we say that a left connection on $\mathcal{E}$ is $\mathcal{E}$-bilinear morphism $\Delta_L : F^1\mathcal{D}_\mathcal{M} \otimes \mathcal{E} \to \mathcal{E}$ such that the following are satisfied for any $f \in \mathcal{O}_\mathcal{M}$, $X \in T_\mathcal{M}$ and $e \in \mathcal{E}$:

1. $\Delta_L(f \otimes e) = f e$,
2. $\Delta_L(f \circ X \otimes e) = f \Delta_L(X \otimes e)$,
3. $\Delta_L(X \circ f \otimes e) = \Delta_L(X \otimes f e)$.

In particular, we say that the left connection $\Delta_L$ is flat if for any $e \in \mathcal{E}$ and $X,Y \in T_\mathcal{M}$ it satisfies

\[ \Delta_L([X,Y] \otimes e) = \Delta_L(X \otimes \Delta_L(Y \otimes e)) - (-1)^{|X||Y|} \Delta_L(Y \otimes \Delta_L(X \otimes e)). \]
Remark 5.6. It is to be noted that the previous definition is adapted as to serve the needs of \( \mathcal{D} \)-module theory. In particular, notice that since \( F^1\mathcal{D}_M = \mathcal{O}_M \oplus \mathcal{T}_M \) generates \( \mathcal{D}_M \), the definition has been extended as to include also the action of the structure sheaf \( \mathcal{O}_M \) in \( F^1\mathcal{D}_M \) on \( \mathcal{E} \) in the first point. The second point is the usual \( \mathcal{O}_M \)-linearity in the first entry of the connection and the third point is the Leibniz rule, since
\[
\Delta_L(X \circ f \circ e) = \Delta_L\left(\left(\nabla(f) + (-1)^{|f||X|}f \circ X\right) \otimes e\right) \\
= \Delta_L(X(f) \otimes e) - (-1)^{|f||X|}\Delta_L(f \circ X \otimes e) \\
= X(f)e + (-1)^{|f||X|}f\Delta_L(X \otimes e).
\]
Finally, the condition of flatness, or integrability for a connection can be rephrased by saying that the connection commutes with the operation of commutator, i.e.
\[
\Delta_L([X,Y] \otimes -) = [\Delta_L(X \otimes -), \Delta_L(Y \otimes -)],
\]
for any \( X,Y \in \mathcal{T}_M \).

We can now state the following crucial result.

**Theorem 5.7** (Left \( \mathcal{D}_M \)-Modules & \( \mathcal{O}_M \)-Modules). Let \( \mathcal{M} \) be a real or complex supermanifold and let \( \mathcal{E} \) be a sheaf on \( \mathcal{M} \). Then \( \mathcal{E} \) is a sheaf of left \( \mathcal{D}_M \)-module on \( \mathcal{M} \) if and only if \( \mathcal{E} \) is a sheaf of \( \mathcal{O}_M \)-modules endowed with a flat left connection.

**Remark 5.8.** The proof of the above theorem is obvious and it simply amounts to check that associativity of the left \( \mathcal{D}_M \)-action on \( \mathcal{E} \) is reproduced by the flat connection. We will prove instead the analogous in case of right \( \mathcal{D}_M \)-modules, which is equally simple but less ordinary, and might cause some confusion.

**Remark 5.9.** Notice also that the above theorem does not require the sheaf \( \mathcal{E} \) of \( \mathcal{O}_M \)-modules to be locally-free of finite rank, therefore one can look at a \( \mathcal{D}_M \)-module as a generalization of a vector bundle endowed with a flat connection.

The next corollary gives an obvious example of sheaf of left \( \mathcal{D}_M \)-modules.

**Corollary 5.10** (Structure Sheaf \( \mathcal{O}_M \)). Let \( \mathcal{M} \) be a real or complex supermanifold and let \( \mathcal{O}_M \) be its structure sheaf. Then \( \mathcal{O}_M \) is a sheaf of left \( \mathcal{D}_M \)-modules.

**Proof.** Obviously, \( \mathcal{O}_M \) can be endowed with a flat left connection, which is nothing but the exterior derivative, seen as a map \( d : \mathcal{T}_M \otimes \mathcal{O}_M \to \mathcal{O}_M \) via the \( X \otimes f \mapsto (df)(X) \) and where the action \( \mathcal{O}_M \otimes \mathcal{O}_M \to \mathcal{O}_M \) is given by the superalgebra structure of \( \mathcal{O}_M \). Flatness is obvious. \( \square \)

Let us now pass to the case of right \( \mathcal{D}_M \)-module. In order to prove the analogous result of theorem 5.7 for right \( \mathcal{D}_M \)-modules we need to introduce a different kind of connection, which is to be related to a right action. As in [53], employing the same notation as above we have the following.

**Definition 5.11** (Right Connection on \( \mathcal{E} \)). Let \( \mathcal{M} \) be a real or complex supermanifold with structure sheaf \( \mathcal{O}_M \) and let \( \mathcal{E} \) be a sheaf of \( \mathcal{O}_M \)-modules. Then we say that a right connection on \( \mathcal{E} \) is a \( \mathbb{K} \)-bilinear morphism \( \Delta_R : \mathcal{E} \otimes \mathcal{O}^1 \mathcal{D}_M \to \mathcal{E} \) such that the following are satisfied for any \( f \in \mathcal{O}_M \), \( X \in \mathcal{T}_M \) and \( e \in \mathcal{E} \):

\[
(1) \quad \Delta_R(e \otimes f) = ef;  \\
(2) \quad \Delta_R(e \otimes X \circ f) = \Delta_R(e \otimes X) f;  \\
(3) \quad \Delta_R(e \otimes f \circ X) = \Delta_R(e f \otimes X),
\]

In particular, we say that the right connection \( \Delta_R \) is flat if for any \( e \in \mathcal{E} \) and \( X,Y \in \mathcal{T}_M \) it satisfies
\[
\Delta_R(e \otimes [X,Y]) = \Delta_R(\Delta_R(e \otimes X) \otimes Y) - (-1)^{|X||Y|} \Delta_R(\Delta_R(e \otimes Y) \otimes X). \quad (5.11)
\]

**Remark 5.12.** Notice that the third point in the definition is \( \mathcal{O}_M \)-linearity and that we have a modified Leibniz rule, adapted to right structures. Indeed, one has
\[
\Delta_R(e \otimes X \circ f) = \Delta_R(e \otimes (X(f) + (-1)^{|X||f|} f \circ X)) \\
= \Delta_R(e \otimes X(f)) + (-1)^{|X||f|} \Delta_R(e \otimes f \circ X) \\
= e \nabla(f) + (-1)^{|X||f|} \Delta_R(e f \otimes X), \quad (5.12)
\]
so that the second property above can be rewritten as
\[ \Delta_X(e f \otimes X) = (-1)^{|X||f|}(\Delta_X(e \otimes X)f - eX(f)). \] (5.13)

Using right connections we can prove the following.

**Theorem 5.13** (Right \( \mathcal{D}_M \)-Modules & \( \mathcal{O}_M \)-Modules). Let \( \mathcal{M} \) be a real or complex supermanifold and let \( \mathcal{E} \) be a sheaf on \( \mathcal{M} \). Then \( \mathcal{E} \) is a sheaf of right \( \mathcal{D}_M \)-module on \( \mathcal{M} \) if and only if \( \mathcal{E} \) is a sheaf of \( \mathcal{O}_M \)-modules endowed with a flat right connection.

**Proof.** The right \( \mathcal{D}_M \)-module structure on \( \mathcal{E} \) corresponds to a right action
\[ \sigma_X : \mathcal{E} \times \mathcal{D}_M \to \mathcal{E} \] (5.14)
\[ (e, F) \mapsto e \cdot F := \sigma_X(e, F) \]
In particular, associativity reads
\[ \sigma_X(\sigma_X(e, D), H) = \sigma_X(e, D \circ H) \] (5.15)
or analogously \( (e \cdot D) \cdot H = e \cdot (D \circ H) \) for \( e \in \mathcal{E} \) and \( D, H \in \mathcal{D}_M \). Let us define \( \sigma_X \) as acting by right multiplication on functions. Then for \( X \in \mathcal{T}_M \) and \( f \in \mathcal{O}_M \) we have that associativity reads
\[ \sigma_X(e, f \circ X) = \sigma_X(\sigma_X(e, f), X) = \sigma_X(e f, X), \] (5.16)
which is last of the defining condition for a right connection. Furthermore, we have
\[ \sigma_X(e, X(f)) = \sigma_X(e, (X \circ f - (-1)^{|X||f|} f \circ X)) \]
\[ = \sigma_X(e, X \circ f) - (-1)^{|X||f|} \sigma_X(e f, X) \]
\[ = \sigma_X(\sigma_X(e, X), f) - (-1)^{|X||f|} \sigma_X(e f, X) \]
\[ = \sigma_X(e, X)f - (-1)^{|X||f|} \sigma_X(e f, X), \] (5.17)
where we have used linearity and associativity. On the other hand, by definition \( \sigma_X(e, X(f)) = eX(f) \), so that
\[ \sigma_X(e f, X) = (-1)^{|X||f|}(\sigma_X(e, f)X - eX(f)), \] (5.18)
which is the modified Leibniz rule and it corresponds to second defining condition of a right connection. Lastly,
\[ \sigma_X(e, [X, Y]) = \sigma_X(e, X \circ Y - (-1)^{|X||Y|} Y \circ X) \]
\[ = \sigma_X(\sigma_X(e, X), Y) - (-1)^{|X||Y|} \sigma_X(\sigma_X(e, Y), X) \] (5.19)
which is the requirement of flatness. Notice that, conversely, starting from a right connection, it is enough to prove that it defines an associative right action on the generating \( F^1 \mathcal{D}_M \cong \mathcal{O}_M \oplus \mathcal{T}_M \), with \( [X, f] = X(f) \) to have a right action of \( \mathcal{D}_M \).

Just like above for left \( \mathcal{D}_M \)-modules, the above theorem give an alternative characterization of right \( \mathcal{D}_M \)-modules. Notice, also, that the notion of right connection is not at all exotic as it might sound at first: namely working over a ordinary real or complex manifold \( X \) it is easy to prove that - up to a sign - the Lie derivative defines a right connection on the canonical sheaf \( \omega_X := \wedge^{\dim X} \mathcal{T}_X^* \), which is therefore a sheaf of right \( \mathcal{D}_X \)-modules.

**Lemma 5.14** (\( \omega_X \) is a Right \( \mathcal{D}_X \)-module). Let \( X \) be a real or complex manifold and let \( \Omega_X^{\dim X} \) be its canonical sheaf. Then \( \Omega_X^{\dim X} \) is a sheaf of right \( \mathcal{D}_M \)-module.

The proof of this theorem, together with a detailed discussion about the relation between the Lie derivative and the right \( \mathcal{D}_X \)-module structure of the canonical sheaf of an ordinary manifold, is deferred to the appendix, as to keep the focus on the case of supermanifolds in the main text.

In light of the previous section and the definition of the Berezinian sheaf of a supermanifold as the correct super-analog of the notion of canonical sheaf for an ordinary manifold, it is natural to ask if the \( \mathcal{D}_M \)-module property of lemma 5.14 goes through to the super setting and also the Berezinian sheaf is a sheaf of right \( \mathcal{D}_M \)-modules. We will prove that this is indeed the case in the next section.
5.3. Lie Derivative of $\text{Ber}(\mathcal{M})$ and Right $\mathcal{D}_\mathcal{M}$-module Structure. Before we actually compute the action of the Lie derivative on a section of the Berezian sheaf of a supermanifold, we start with some remark concerning the relation between left and right structure on a supermanifold. In particular, let $\mathcal{E}$ be a locally-free sheaf of left $\mathcal{O}_\mathcal{M}$-modules of rank $pq$ which is generated over an open set $U$ by a set $\{e_a\}_{a \in \mathcal{I}}$ of $p$ even and $q$ odd generators. Then $\mathcal{E}$ is naturally also a locally-free sheaf of right $\mathcal{O}_\mathcal{M}$-modules simply taking into account the sign rule, i.e. given a local section $s_U \in \mathcal{E}$ over an open set $U$ such that $s_U = \sum_a f^a e_a$ for $f^a \in \mathcal{O}_\mathcal{M}(U)$, then

$$s_U = \sum_a f^a e_a = \sum_a (-1)^{|e_a||f^a|} e_a f^a.$$  

(5.20)

In particular, the tangent sheaf $\mathcal{T}_\mathcal{M}$ can be seen as a sheaf of left $\mathcal{O}_\mathcal{M}$-modules. Accordingly, one has that a vector fields acts as usual from the left, i.e.

$$\vec{X}(f) := \sum_a X^a \partial_a (f).$$  

(5.21)

As seen above, this satisfies the Leibniz rule in the form

$$\vec{X}(fg) = \vec{X}(f)g + (-1)^{|X||f|} f \vec{X}(g),$$  

(5.22)

for $f, g \in \mathcal{O}_\mathcal{M}$. On the other hand it makes sense to consider a right action of a vector field on a function, when $\mathcal{T}_\mathcal{M}$ is seen as a sheaf of right $\mathcal{O}_\mathcal{M}$-modules, and we write

$$(f)\vec{X} := \sum_a (f) \partial_a X^a.$$  

(5.23)

This is what is usually called right derivative in theoretical physics. In this case the Leibniz rule reads

$$(fg)\vec{X} = f((g)\vec{X}) + (-1)^{|X||g|} ((f)\vec{X})g.$$  

(5.24)

for any $f, g \in \mathcal{O}_\mathcal{M}$. The connection by left and right derivation is given, as expected, by the sign rule, i.e.

$$\vec{X}(f) = (-1)^{|X||f|} (f)\vec{X},$$  

(5.25)

which proves to be in agreement with the Leibniz rule as indeed

$$\vec{X}(fg) = (-1)^{|X||f||g|} (fg)\vec{X}.$$  

(5.26)

Having these considerations in mind, we look for an action of the Lie derivative on sections of Berezian sheaf of a supermanifold in a similar fashion as in $[16]$ for the canonical sheaf on an ordinary manifold. We remark, though, that this is not at all trivial. Indeed, sections of the Berezian sheaf are not differential forms, and as such the usual Cartan calculus and Cartan homotopy formula for the Lie derivative does not apply. On a very general ground, following for example $[28]$ or $[74]$, one can give a definition of Lie derivative using the flow along a vector field.

**Definition 5.15 (Lie Derivative).** Let $\mathcal{M}$ be a real or complex supermanifold, let $\mathcal{E}$ be a sheaf canonically associated to $\mathcal{M}$ and let $X \in \mathcal{T}_\mathcal{M}$ be a vector field. Then we define the Lie derivative of a section $S \in \mathcal{E}$ as

$$\mathcal{L}_X(S) := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} [\varphi^X_\varepsilon]^* S,$$  

(5.27)

where $\varphi^X_\varepsilon : \mathcal{M} \rightarrow \mathcal{M}$ is the flow of the vector field $X$ and where the parity of the parameter $\varepsilon$ is the same as the parity of the field $X$.

**Remark 5.16.** Clearly, as in the ordinary context, this formalizes the idea of infinitesimal variation induced on a certain section by the action of certain vector field. We now apply this definition to recover an expression for the action of the Lie derivative on sections of the Berezian sheaf.

**Theorem 5.17 (Lie Derivative of Sections of Berezian Sheaf).** Let $\mathcal{M}$ be a real or complex supermanifold. Let $\mathcal{D}$ be a section of the Berezian sheaf $\text{Ber}(\mathcal{M})$ and $X$ be a vector field in $\mathcal{T}_\mathcal{M}$, such that they have trivializations $\mathcal{D} = \mathcal{D}(x)f(x)$ and $X = \sum a X^a \partial_a$ in a certain local chart
(U, x_a). Then the Lie derivative \( \mathcal{L}_x(D) \) defined as in (5.24) of \( D \in \text{Ber}(\mathcal{M}) \) along the field \( X \in T_M \) is given in \( U \) by

\[
\mathcal{L}_x(D) = (-1)^{|x||D|}|D(x) \sum_a (f X^a \frac{\partial}{\partial x^a},
\]

for \( a = 1, \ldots n \). Let \( 1, \ldots, m \) even and odd coordinates.

**Proof.** We use (5.27) and directly compute the variation of the sections \( D = D(x) f(x) \) in the chart \( U \subset X \) for an infinitesimal diffeomorphism controlled by a parameter \( \varepsilon \), having parity \( |\varepsilon| = |X| \), so that it makes sense to consider the even transformation \( x^a \mapsto x^a \). On the one hand one has

\[
D(x^a + \varepsilon X^a) = D(x^a) \text{Ber}(x^a + \varepsilon X^a) \frac{\partial}{\partial x^a} = D(x^a) \text{Ber}(\delta_a^a + \varepsilon \partial_a X_a).
\]

The Berezinian of the coordinates transformation can be rewritten as

\[
\text{Ber}(\delta_a^a + \varepsilon \partial_a X_a) = 1 + \varepsilon \sum_a (-1)^{|x_a||X_a|} \partial_a X_a,
\]

where the sign is due to the super trace of the Jacobian matrix. Equivalently, acting from the left instead, one has

\[
D(x^a + \varepsilon X^a) = D(x^a) \text{Ber}(\partial_b(x^a + \varepsilon X^a)) = D(x^a) \text{Ber}(\delta_b^a + \varepsilon \partial_b X_a),
\]

so that this gives

\[
D(x^a + \varepsilon X^a) = D(x^a)(1 + \varepsilon \sum_a (-1)^{|x_a||X_a|} \partial_a X_a)
\]

\[
= D(x^a) \left( 1 + \sum_a (-1)^{|x_a||X_a|} \partial_a \left( \varepsilon X^a \right) \right)
\]

\[
= D(x^a \left( 1 + \varepsilon \sum_a (-1)^{|x_a||x_a| + |x_a||X_a|} \partial_a X_a \right)
\]

\[
= D(x^a(1 + \varepsilon \sum_a (-1)^{|x_a||X_a|} \partial_a X_a)
\]

where we have used that \( |x_a| = |x_a||x_a| \) and that \( |\varepsilon| = |X| \). Considering the expansion of the local function, one finds

\[
D(x^a + \varepsilon X^a) f(x^a + \varepsilon X^a) = D(x^a) \left( 1 + \varepsilon \sum_a (-1)^{|x_a||X_a|} \partial_a X_a \right) \left( f(x^a) + \varepsilon \sum_a X^a \partial_a f \right).
\]

In turn, this gives

\[
D(x^a + \varepsilon X^a) f(x^a + \varepsilon X^a) = D(x^a) f(x^a) +
\]

\[
+ D(x) \varepsilon \sum_a \left( -1 \right)^{|x_a||X_a|} \left( \partial_a X_a \right) f + X^a (\partial_a f).
\]

We rearrange the summands inside the parentheses as follows:

\[
(-1)^{|x_a||X_a|} \left( \partial_a X_a \right) f + X^a (\partial_a f) = (-1)^{|x_a||X^a| + |f||X^a|} \left( \partial_a X_a \right) f + (-1)^{|X^a|(|x_a| + |f|)} (\partial_a f) X^a
\]

\[
= (-1)^{|X^a|(|x_a| + |f|)} \partial_a (f X^a).
\]

This leads to the following expression:

\[
D(x^a + \varepsilon X^a) f(x^a + \varepsilon X^a) = D(x^a) f(x^a) + \sum_a \left( -1 \right)^{|X^a|(|x_a| + |f|)} \partial_a (f X^a)
\]

\[
= D(x^a) f(x^a) + \varepsilon \sum_a \left( -1 \right)^{|X^a|(|x_a| + |f|)} \partial_a (f X^a)
\]

\[
= D(x^a) f(x^a) + \varepsilon (-1)^{|D(x)||X^a|} D(x) \sum_a \left( -1 \right)^{|X^a|(|x_a| + |f|)} \partial_a (f X^a)
\]
The only summand that matters in the computation of the Lie derivative is the one which is linear in the parameter $\varepsilon$ and indeed we have that
\[
\lim_{t \to 0} \frac{(\Phi^t)^* \mathcal{L} - \mathcal{L}}{t} = (-1)^{|\mathcal{L}(x)|} |\mathcal{L}(x)| \sum_a (-1)^{|X^a|} (|x_a| + |f|) \partial_a (f X^a).
\]
Finally, notice that the right-hand side of the above expression for the Lie derivative can be rewritten as
\[
\mathcal{L}_X (\mathcal{D}) = (-1)^{|\mathcal{D}(x)|} |\mathcal{D}(x)| \sum_a (-1)^{|X^a| (|x_a| + |f|) + x_a (|f| + |X^a|)} (f X^a) \partial_a
\]
\[
= (-1)^{|\mathcal{D}(x)|} |\mathcal{D}(x)| \sum_a (f X^a) \partial_a
\]
(5.38)
taking into account that $|\mathcal{D}| = |\mathcal{D}(x)| + |f|$, thus concluding the proof. □

Remark 5.18. As already observed the resemblance with the expression (B.3) for the Lie derivative of the canonical sheaf is apparent. Nonetheless, in order to get the supergeometric analog of lemma B.3 it is useful to introduce the following re-definition of the Lie derivative as to get rid of an inconvenient sign
\[
\mathcal{L}_X (\mathcal{D}) := (-1)^{|\mathcal{D}(x)|} \mathcal{L}_X (\mathcal{D}).
\]
In local coordinates, for sections $\mathcal{D} = \mathcal{D}(x) f$ and $X = \sum_a X^a \partial_a$, one has the following action
\[
\mathcal{L}_X (\mathcal{D}) = \mathcal{D}(x) \sum_a (f X^a) \partial_a = \mathcal{D}(x) \sum_a (-1)^{|x_a| (|f| + |X^a|)} \partial_a (f X^a).
\]
(5.40)
We can thus finally prove the following theorem.

Theorem 5.19 (Ber(\mathcal{M}) is a Right \mathcal{D}_\mathcal{M}-module). Let \mathcal{M} be a real or complex supermanifold and let $\Delta^\text{Ber}_\mathcal{R} : \text{Ber}(\mathcal{M}) \otimes_\mathbb{R} F^1 \mathcal{D}_\mathcal{M} \to \text{Ber}(\mathcal{M})$ be defined as
\[
\Delta^\text{Ber}_\mathcal{R} (\mathcal{D} \otimes f) := \mathcal{D} f,
\]
(5.41)
\[
\Delta^\text{Ber}_\mathcal{R} (\mathcal{D} \otimes X) := - \mathcal{L}_X (\mathcal{D}),
\]
(5.42)
for any $\mathcal{D} \in \text{Ber}(\mathcal{M})$, $f \in \mathcal{O}_\mathcal{M} \subset F^1 \mathcal{D}_\mathcal{M}$ and $X \in \mathcal{T}_\mathcal{M} \subset F^1 \mathcal{D}_\mathcal{M}$. Then the followings hold true.
1. $\Delta^\text{Ber}_\mathcal{R}$ defines a flat right connections on the Berezinian sheaf Ber(\mathcal{M}).
2. In any system of coordinates $\Delta^\text{Ber}_\mathcal{R}$ is the unique right connection on Ber(\mathcal{M}) satisfying
\[
\Delta^\text{Ber}_\mathcal{R} (\mathcal{D}(x) \otimes \partial_a) := - \mathcal{L}_{\partial_a} (\mathcal{D}(x)) = 0 \quad \forall a = 1, \ldots, n, 1, \ldots, m,
\]
(5.43)
where $\mathcal{D}(x)$ is a local generating section of Ber(\mathcal{M}) and $\partial_a \in \mathcal{T}_\mathcal{M}$ is a coordinate vector field.

In particular $\Delta^\text{Ber}_\mathcal{R}$ endows Ber(\mathcal{M}) with a right $\mathcal{D}_\mathcal{M}$-module structure.

Proof. We start using (5.40) to show that the axioms of a right connection holds true. The only non trivial verification to carry out is that $\Delta^\text{Ber}_\mathcal{R} (\mathcal{D} \otimes X \circ f) = \Delta^\text{Ber}_\mathcal{R} (\mathcal{D} \otimes X) f$. We recall that $X \circ f = X(f) + (-1)^{|X||f|} f \circ X$. Upon using this, one computes
\[
\Delta^\text{Ber}_\mathcal{R} \left( \mathcal{D}(x) g \otimes \sum_a X^a \partial_a \circ f \right) = \Delta^\text{Ber}_\mathcal{R} \left( \mathcal{D}(x) g \otimes \sum_a X^a \partial_a f \right) + \Delta^\text{Ber}_\mathcal{R} \left( \mathcal{D}(x) g \otimes \sum_a (-1)^{|f||X^a|} f X^a \partial_a \right)
\]
\[
= \mathcal{D}(x) \sum_a g X^a \partial_a f - \mathcal{D}(x) \sum_a (-1)^{|f|(|X^a| + |x_a|) + x_a (|f| + |X^a|)} \partial_a (g X^a)
\]
\[
= \mathcal{D}(x) \sum_a g X^a \partial_a f - \mathcal{D}(x) \sum_a (-1)^{|x_a| (|X^a| + |g|)} \partial_a (g X^a) f + (-1)^{|x_a| (|X^a| + |g|)} g X^a \partial_a f
\]
\[
= - \mathcal{D}(x) \sum_a (-1)^{|x_a| (|X^a| + |g|)} \partial_a (g X^a) f,
\]
(5.44)
upon observing that the first term and the last term cancel pairwise in the semi-last equality above. This gives
\[
\Delta^\text{Ber}_\mathcal{R} (\mathcal{D}(x) g \otimes \sum_a X^a \partial_a) f = - \mathcal{D}(x) \sum_a (-1)^{|x_a| (|X^a| + |g|)} \partial_a (g X^a) f,
\]
(5.45)
thus proving that \( \Delta^\text{Ber}_R(D \otimes X \circ f) = \Delta^\text{Ber}_R(D \otimes X)f \).

Also, it follows trivially from equation (5.44) that the action of \( \Delta^\text{Ber}_R \) satisfies the third defining property of a right connection, while the first one is just the right multiplication.

Let us now prove that the right connection defined by \( \Delta^\text{Ber}_R \) is flat, i.e. it satisfies

\[
\Delta^\text{Ber}_R(D \otimes [X,Y]) = \Delta^\text{Ber}_R(\Delta^\text{Ber}_R(D \otimes X) \otimes Y) - (-1)^{|X||Y|} \Delta^\text{Ber}_R(\Delta^\text{Ber}_R(D \otimes Y) \otimes X),
\]

for any pair of fields \( X, Y \in T_M \). On the one hand, recalling that the supercommutator \([\cdot, \cdot]\) is given by

\[
[X, Y] = \left( \sum_a X^a \partial_a, \sum_b Y^b \partial_b \right) = \sum_a \left( X^b \partial_a Y^a - (-1)^{|X||Y|} Y^b \partial_b X^a \right) \partial_a,
\]

one computes

\[
\Delta^\text{Ber}_R(D \otimes [X,Y]) = - \mathcal{D}(x) \sum_{a,b} \left( (-1)^{|x_a|(|f|+|X^a|)} \partial_a (f X^b \partial_b (Y^a)) + (-1)^{|X||Y|+|x_a|(|f|+|Y^a|)} \partial_a (f Y^b \partial_b (X^a)) \right),
\]

where we have used that \( |X^b \partial_b Y^a| = |X| + |Y^a| \) and \( |Y^b \partial_b X^a| = |Y| + |X^a| \). On the other hand, one has

\[
\Delta^\text{Ber}_R(\Delta^\text{Ber}_R(D \otimes X) \otimes Y) = \Delta^\text{Ber}_R \left( - \mathcal{D}(x) \sum_a (-1)^{|x_a|(|f|+|X^a|)} \partial_a (f X^a \otimes \sum_b Y^b \partial_b) \right) = + \mathcal{D}(x) \sum_{a,b} (-1)^{|x_a|(|f|+|X^a|)+|x_a|(|f|+|X^a|+|X^b|)} \partial_a (f X^a Y^b) + (-1)^{|x_a|(|f|+|X^a|)+|x_a|(|f|+|X^a|+|Y^a|)} \partial_b (f X^a Y^b),
\]

Now note that \( \partial_a (f X^a Y^b) = \partial_a (f X^a Y^b) - (-1)^{|x_a|(|f|+|X^a|)} f X^a \partial_b Y^b \), so that plugging this into the above one

\[
\Delta^\text{Ber}_R(\Delta^\text{Ber}_R(D \otimes X) \otimes Y) = + \mathcal{D}(x) \sum_{a,b} (-1)^{|x_a|(|f|+|X^a|)+|x_a|(|f|+|X^a|+|Y^a|)} \partial_b (f X^a Y^b) + (-1)^{|x_a|(|f|+|X^a|)} \partial_a (f X^a Y^b) + \mathcal{D}(x) \sum_{a,b} (-1)^{|x_a|(|f|+|X^a|)+|x_a|(|f|+|X^a|)} \partial_a (f X^a Y^b).
\]

The same holds true for the other part, that is one finds

\[
\Delta^\text{Ber}_R(\Delta^\text{Ber}_R(D \otimes Y) \otimes X) = \Delta^\text{Ber}_R \left( - \mathcal{D}(x) \sum_a (-1)^{|x_a|(|f|+|Y^a|)} \partial_b (f Y^b \otimes \sum_a X^a \partial_a) \right) = + \mathcal{D}(x) \sum_{a,b} (-1)^{|x_a|(|f|+|Y^a|)+|x_a|(|f|+|Y^a|+|X^a|)} \partial_a (f Y^b X^a) + (-1)^{|x_a|(|f|+|Y^a|)+|x_a|(|f|+|Y^a|+|X^a|)} \partial_b (f Y^b X^a),
\]

It follows that, by Leibniz rule, one gets

\[
\Delta^\text{Ber}_R(\Delta^\text{Ber}_R(D \otimes Y) \otimes X) = + \mathcal{D}(x) \sum_{a,b} (-1)^{|x_a|(|f|+|Y^a|)+|x_a|(|f|+|Y^a|+|X^a|)} \partial_b (f Y^b X^a) + \mathcal{D}(x) \sum_{a,b} (-1)^{|x_a|(|f|+|Y^a|)+|x_a|(|f|+|Y^a|+|X^a|)} \partial_a (f Y^b X^a).
\]

Let us now consider the full expression

\[
\Delta^\text{Ber}_R(\Delta^\text{Ber}_R(D \otimes X) \otimes Y) = (-1)^{|X||Y|} \Delta^\text{Ber}_R(\Delta^\text{Ber}_R(D \otimes Y) \otimes X) =
\]

\[
- \mathcal{D}(x) \sum_{a,b} \left( (-1)^{|x_a|(|f|+|X^a|+|Y^a|)} \partial_a (f X^b \partial_b (Y^a)) - (-1)^{|X||Y|+|x_a|(|f|+|Y^a|+|X^a|)} \partial_a (f Y^b \partial_b (X^a)) \right)
\]

\[
+ \mathcal{D}(x) \sum_{a,b} \left( (-1)^{|x_a|(|f|+|X^a|)+|x_a|(|f|+|X^a|+|Y^a|)} \partial_a (f X^b Y^a) + (-1)^{|X||Y|+|x_a|(|f|+|Y^a|)+|x_a|(|f|+|Y^a|+|X^a|)} \partial_a (f X^a Y^a) \right).
\]

We need the last two terms to cancel pairwise. We observe that

\[
\partial_a (f Y^b X^a) = (-1)^{|x_a|(|f|+|X^a|)} \partial_b (f X^a Y^a).
\]

(5.53)
Renaming the indexes in the last addendum one finds that
\[
\mathcal{D}(x) \sum_{a,b} \partial_a \partial_b (f X^a Y^b) \left( (-1)^{|x_a|(f|+|X^a|)+|x_b|(f|+|Y^b|)} + 
- (-1)^{|X^a|Y^b+x_a((f|+|Y^a|)+|x_b|(f|+|X^b|)+|x_a|X^b|Y^a|) + 
\right) \tag{5.54}
\]
This leads to the following equality
\[
|x_a|(f|+|X^a|) + |x_a|(f|+|X^a|) = 
\]
which is easily verified by rearranging the right-hand side. This tells that term in (5.54) is identically zero, thus proving flatness of $\Delta^{Ber}_X$ and concluding the proof of the first point.

For the second point, uniqueness of $\Delta^{Ber}_X$ follows from the fact that $\mathcal{D}(x)$ is a local generator for $\text{Ber}^r(\mathcal{M})$ and that the $\partial_a$'s give a system of local generators for $\mathcal{T}_\mathcal{M}$. Also, choosing a system of coordinates, $\Delta^{Ber}_X (\mathcal{D}(x) \otimes \partial_a) = - \mathcal{L}_{\partial_a} (\mathcal{D}(x)) = 0$ is readily verified simply applying (5.40). That this remains zero in any system of coordinates $x' = x'(x)$ is a (lengthy) local check, which is carried out in [53]. □

Remark 5.20. The above result establishes that there exists a right action of vector fields on the Berezinian sheaf of a supermanifold given - up to a sign - by the Lie derivative $\mathcal{D} \cdot X := -\varphi(x) \sum_a (-1)^{|x_a|(f|+|f|)} \partial_a (f X^a)$. It is worth noticing that the above construction can be obtained in a rather more neat and economic way in a purely algebraic fashion, by using the third construction of the Berezinian sheaf that we have discussed in the previous section in theorem 4.19, which introduces the sheaf $\mathcal{D}_\mathcal{M}$ from the very beginning. In particular, one prove the following.

Corollary 5.21 (Ber($\mathcal{M}$) is a Right $\mathcal{D}_\mathcal{M}$-Module - Cohomological Version). Let $\mathcal{M}$ be a real or complex supermanifold. Then right action
\[
\mathcal{H}^p ((\mathcal{D}^{\bullet}_{\mathcal{M}}, \mathcal{D})) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_\mathcal{M} \longrightarrow \mathcal{H}^p ((\mathcal{D}^{\bullet}_{\mathcal{M}}, \mathcal{D})) \tag{5.58}
\]
is uniquely characterized by the condition $\mathcal{D}(x) \cdot \partial_a = 0$ for any $a$ even and odd, where $\mathcal{D}(x) \in \mathcal{H}^p ((\mathcal{D}^{\bullet}_{\mathcal{M}}, \mathcal{D})) \cong \text{Ber}^r(\mathcal{M})$. More in particular, the action is given (up to a overall sign) by the Lie derivative on $\text{Ber}^r(\mathcal{M})$.

Proof. With reference to Theorem 4.19 one has that in cohomology $\mathcal{D}(x) \cdot \partial_a = [dz_1 \ldots dz_p \otimes \partial_{\theta_1} \otimes \ldots \partial_{\theta_l} \partial_a] = 0$ for any $a$, which characterizes the right action of $\mathcal{D}_\mathcal{M}$ on $\text{Ber}^r(\mathcal{M})$. In particular, one sees that considering a section $\mathcal{D} = dz_1 \ldots dz_p \otimes \partial_{\theta_1} \ldots \partial_{\theta_l} f \in \text{Ber}^r(\mathcal{M})$ and a vector fields $X = \sum a X^a \partial_a$, one finds
\[
\mathcal{D} \cdot X = \mathcal{D}(x) \sum_a (-1)^{|x_a|(f|+|f|)} (-\partial_a (f X^a) + \partial_a \cdot f X^a)
\]
where we have used that $\mathcal{D}(x) \cdot \partial x_a = 0$ is zero in the cohomology. This matches (5.57). □

This shows the relevance of the construction of the Berezinian via the total de Rham algebra $\Omega^* \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_\mathcal{M}$.

Remark 5.22. Finally, it is worth to observe the similarities between these results and constructions and those elucidated in appendix B regarding the right $\mathcal{D}_X$-module structure and its relation with the Lie derivative for the canonical sheaf of an ordinary manifold $X$. 


6. Integral Forms and Spencer Cohomology of Supermanifolds

In the previous section we have shown that, given a real or complex supermanifold \( M \), there exists a flat right connection which endows \( \text{Ber}(M) \) with the structure of a right \( D_M \)-module. In analogy with the ordinary case of the canonical sheaf of a real or complex manifold, we want to study the structure of the so-called Spencer complex related to \( \text{Ber}(M) \). This is particularly relevant in the context of supermanifolds since the Berezinian sheaf does not appear in the de Rham complex - which instead it is the case for the canonical sheaf of ordinary manifolds - thus giving rise to a genuinely new complex. Also, as we shall see shortly, sections of the Berezinian sheaf are the objects to look at for a meaningful notion of integration over a supermanifold. The following definition is related to this aspect.

**Definition 6.1 (Integral Forms).** Let \( M \) be a real or complex supermanifold of dimension \( p|q \). We call integral forms of degree \( p - i \) the sections of the sheaves

\[
\Sigma^{p-i}_M := \text{Ber}(M) \otimes_{\mathcal{O}_M} S^i \Pi T_M
\]

for any \( i \geq 0 \).

Notice that, equivalently, one can define integral forms of degree \( n - i \) to be sections of the sheaves \( \Sigma^{n-i}_M := \text{Hom}_{\mathcal{O}_M}(\Omega^n_M, \text{Ber}(M)) \).

**Remark 6.2.** The degree is assigned so that sections of the Berezinian sheaf \( \Sigma^p_M = \text{Ber}(M) \) are top-integral forms in degree \( p \), which equals the even dimension of the supermanifold, mimicking what happen for the canonical sheaf in the ordinary setting. Further, it is to be stressed that integral forms can have any negative degree: this is in some sense “dual” to what happen in the case of differential forms on a supermanifolds, as seen above.

We are now interested in structuring the sheaves of integral forms into a complex: to this end we need to introduce a suitable differential. This is where the (flat) right connection \( \Delta^\text{Ber}_M: \text{Ber}(M) \otimes T_M \rightarrow \text{Ber}(M) \) discussed in the previous section serves its purposes. In particular, we define a morphism of sheaves as follows.

\[
\delta: \Sigma^{p-1} = \text{Ber}(M) \otimes \Pi T_M \rightarrow \Sigma^p = \text{Ber}(M)
\]

\[
\mathcal{D} \otimes \pi X \rightarrow \delta(\mathcal{D} \otimes \pi X) := (-1)^{|\mathcal{D}|+|X|} \Delta^\text{Ber}_M(\mathcal{D} \otimes X).
\]

By the previous section this is well-defined and it does not depend on the choice of local coordinates. Also, notice that it is \( \mathcal{O}_M \)-linear since

\[
\delta(\mathcal{D}f \otimes \pi X) = (-1)^{|\mathcal{D}|+|f|+|X|} \Delta^\text{Ber}_M(\mathcal{D}f \otimes X) = (-1)^{|\mathcal{D}|+|f|+|X|} \Delta^\text{Ber}_M(\mathcal{D} \otimes f X) = \delta(\mathcal{D} \otimes f \pi X),
\]

by the property of the right connection \( \Delta^\text{Ber}_M \). Moreover, since \( \Delta^\text{Ber}_M(- \otimes X) = -\mathcal{L}_X(-) \), the above can also be rewritten as

\[
\delta(\mathcal{D} \otimes \pi X) = (-1)^{|\mathcal{D}|+|X|+1} \mathcal{L}_X(\mathcal{D}) = (-1)^{|\mathcal{D}|+|X|} \mathcal{L}_X(\mathcal{D}),
\]

which stress the dependence from the parity of the II-vector field \( \pi X \in \Pi T_M \). Using the explicit expression of the Lie derivative \( \mathcal{L}_X \), in local coordinates, for \( \mathcal{D} = \mathcal{D}(x) f \) and \( \pi X = \sum_a X^a \pi \partial_a \) the action of \( \delta \) is therefore given by

\[
\delta(\mathcal{D} \otimes \pi X) = (-1)^{|X|+|\mathcal{D}|} \Delta^\text{Ber}_M(\mathcal{D} \otimes X) = (-1)^{|X|+|\mathcal{D}|+1} \mathcal{L}_X(\mathcal{D}) = (-1)^{|X|+|\mathcal{D}|} \mathcal{D}(x) \sum_a (-1)^{|X^a|+|\mathcal{D}^a|} \partial_a (f X^a),
\]

where we have used \( \Delta^\text{Ber}_M \).

**Remark 6.3.** In order to extend the above morphism to integral forms of any degree, in similar fashion as in ordinary differential geometry, we introduce the contraction operator \( \iota_\omega: S^* \Pi T_M \rightarrow S^* \Pi T_M \) where \( \omega \in \Omega^1_M \), which is a derivation of the supersymmetric algebra \( S^* \Pi T_M \) characterized by the properties that \( \iota_\omega(f) = 0 \) for any \( f \in \Omega_M \) and \( \iota_\omega(\pi X) = \omega(\pi X) \) for any \( \pi X \in \Pi T_M \). In particular, on the bases of \( \Omega^1_M \) and \( \Pi T_M \) we set \( \iota_{dx_a}(\pi \partial_b) = \delta_{ab} \). We will employ the notation...
for any $\pi X$ acts indeed as a derivation. In a certain trivialization, for sections $\omega \in \Omega^1_M$ and $\pi X \in \Pi T_M$ one that

$$\omega(\pi X) = \sum_a \omega_a \partial_a (\sum_b X^b \pi \partial_b) = \sum_a (-1)^{a+1} \omega_a X^a. \quad (6.6)$$

Using this, we prove the following lemma.

**Lemma 6.4 (Representation Lemma).** Let $\mathcal{M}$ be a real or complex supermanifold. Then the action of $\delta : \text{Ber}(\mathcal{M}) \otimes \Pi T_M \to \text{Ber}(\mathcal{M})$ defined as in (6.2) can be given the following operator representation

$$\delta = \sum_a (-1)^{|a|+1} \xi \partial_a \otimes \tau d x_a, \quad (6.7)$$

where $\xi \partial_a$ is the Lie derivative with respect to the coordinate vector field $\partial_a$ acting on sections of the Berezinian sheaf $\text{Ber}(\mathcal{M})$ and $\tau d x_a$ is the contraction with respect to the coordinate 1-form $d x_a$ acting on section of the II-tangent sheaf $\Pi T_M$.

**Proof.** Let us choose a chart with $\pi X = \sum_a X^a \partial_a$ and $\mathcal{D} = \mathcal{D}(f)$ Then one computes

$$\left( \sum_a (-1)^{|a|+1} \xi \partial_a \otimes \tau d x_a \right) \left( \mathcal{D}(f) \otimes \sum_b X^b \pi \partial_b \right)$$

$$= \sum_a \xi \partial_a (\mathcal{D}(f) X^a)(-1)^{|D|+|X^a|+1}(d x_a \otimes \tau d x_a)(\pi \partial_b)$$

$$= \mathcal{D}(f) \sum_a (-1)^{|D|+|X^a|+1}(-1)^{|D|+|X^a|+1}(d x_a \otimes \tau d x_a)(\pi \partial_b)$$

$$= \mathcal{D}(f) \sum_a (-1)^{|D|+|X^a|+1}(-1)^{|D|-|X^a|+1}(d x_a \otimes \tau d x_a)(\pi \partial_b), \quad (6.8)$$

which matches $\delta(D \otimes \pi X)$ as in (6.5). \qed

**Remark 6.5.** The above lemma is convenient to extend the action of $\delta$ to integral forms of any degree. More precisely, it can be proved that $\delta$ extends to a derivation $\delta : \Sigma^{p-1} \to \Sigma^{p-1}$ for any $i \geq 1$, where $p$ is the even dimension of the supermanifold, in the sense that

$$\delta(D \otimes \pi X^a \pi Y^b) = \delta(D \otimes \pi X^a) \pi X^b + (-1)^{|X^a||Y^b|} \delta(D \otimes \pi X^a) \pi X^b \quad (6.9)$$

for any $\pi X^a \in S^n \Pi T_M$ and $\pi X^a \in S^n \Pi T_M$ for any $m, n \geq 1$, and where we have the supersymmetric product understood for notational convenience. More in particular, using (6.7), one computes

$$\delta(D \otimes \pi X \pi Y) = \sum_a \xi \partial_a (D X^a \otimes \tau d x_a)(\pi \partial_b)$$

$$+ (-1)^{|X^a||Y^b|} \sum_a (-1)^{|D|+|X^a|+1} \xi \partial_a (D Y^a \otimes \tau d x_a)(\pi \partial_b)$$

$$= \sum_a (-1)^{|D|+|X^a|+1} \xi \partial_a (D X^a) \left( \sum_b X^b \pi \partial_b \right)$$

$$+ (-1)^{|X^a||Y^b|} \sum_a (-1)^{|D|+|X^a|+1} \xi \partial_a (D Y^a) \left( \sum_b X^b \pi \partial_b \right)$$

$$= \delta(D \otimes \pi X) \pi Y + (-1)^{|X^a||Y^b|} \delta(D \otimes \pi Y) \pi X. \quad (6.10)$$

This proves that Leibniz formula in the form (6.3) holds true, so that $\delta : \Sigma^{p-1} \to \Sigma^{p-1}$ is indeed a derivation. Working by induction on the degree of the integral forms one gets to the conclusion. Notice that this shows that $\delta : \Sigma^{p-1} \to \Sigma^{p-1}$ indeed extends to a derivation $\delta : \Sigma^{p+1} \to \Sigma^{p+1}$ for any $i \geq 0$ and that it is globally well-defined, i.e. it does not depends on the choice of coordinates. We can thus prove the following lemma.
Lemma 6.6. The pair \((\Sigma^\bullet_M, \delta)\) defines a differential graded supermodule (DGsM).

Proof. We are only left to prove that \(\delta\) is nilpotent, i.e. \(\delta^2 = 0\). First we note that, for any \(a\) even and odd, one has that \(L_{\partial_a}\) and \(\iota_{dx}\) have opposite parity so that \(\delta\) is odd. Let us prove that \(L_{\partial_a}\) and \(\iota_{dx}\) satisfies the same commutation relations, in particular they (super)commute pairwise.

On the one hand, clearly \([\iota_{dx}, \iota_{dx}] = 0\). On the other hand

\[
[L_{\partial_a}, L_{\partial_b}]D = L_{\partial_b}L_{\partial_a}(D) - (-1)^{|x_a||x_b|}L_{\partial_b}L_{\partial_a}(D)
\]

\[
= (-1)^{|D||x_a|+1}L_{\partial_b}(\Delta^\text{Ber}(D \otimes \partial_b)) - (-1)^{|x_a||x_b|}(-1)^{|D||x_a|+1}L_{\partial_b}(\Delta^\text{Ber}(D \otimes \partial_b))
\]

\[
= (-1)^{|D||x_a|+|x_b|+|x_a||x_b|}(\Delta^\text{Ber}(D^\text{Ber}(D \otimes \partial_b) \otimes \partial_a) - (-1)^{|x_a||x_b|}(\Delta^\text{Ber}(D^\text{Ber}(D \otimes \partial_a) \otimes \partial_b))
\]

\[
= (-1)^{|D||x_a|+|x_b|+|x_a||x_b|}\Delta^\text{Ber}(D \otimes [\partial_a, \partial_b]) = 0,
\]

where we have used the flatness of the right connection in the semilast equality and that \([\partial_a, \partial_b] = 0\).

Thanks to the previous lemma 6.6 we can now give the following definition.

Definition 6.7 (Spencer Complex / Complex of Integral Forms of \(M\)). We call the differential graded supermodules \((\Sigma^\bullet_M, \delta)\) the Spencer complex of \(M\) or complex of integral forms of \(M\)

\[
\ldots \longrightarrow \Sigma_M^{p-n} \overset{\delta}{\longrightarrow} \Sigma_M^{p-1} \overset{d}{\longrightarrow} \Sigma_M^p \overset{d}{\longrightarrow} 0,
\]

where \(p\) is the even dimension of \(M\).

Remark 6.8. It is important to stress the following important difference between differential and integral forms - which emerges also from the statement of the previous lemma, if compared to the analogous lemma 3.6 for differential forms: integral forms are not structured into a sheaf of superalgebras, i.e. it does not make sense to multiply two integral forms. Indeed, if on the one hand the supersymmetric product of two a \(\Pi\)-polynomials yields a \(\Pi\)-polynomials since \(S^\Pi \Pi T_M\) is a sheaf of superalgebras, the multiplication of two sections of the Berezinian sheaf \(\text{Ber}(M)\) is not a section of the Berezinian sheaf, but instead a section of \(\text{Ber}(M)^{\otimes 2}\), which never appears in the definition of the sheaves \(\Sigma^p_M\).

Remark 6.9. Also, it is crucial to observe the difference between the de Rham complex / complex of differential forms and the Spencer complex / complex of integral forms on a supermanifold \(M\). Whereas the first one is not bounded from above, the second one is not bounded from below instead.

\[
\ldots \longrightarrow \Omega^0_M \longrightarrow \Omega^1_M \longrightarrow \ldots \longrightarrow \Omega^n_M \longrightarrow \ldots
\]

We now state the analog of the Poincaré lemma in the context of integral forms. This appears without an actual proof in [53]. The following proof is adapted from the very recent [16].

Theorem 6.10 (Poincaré Lemma for Integral Forms). Let \(M\) be a real supermanifold and let \((\Sigma^\bullet_M, \delta)\) be the Spencer complex of \(M\). Then one has

\[
H^k_\delta(\Sigma^\bullet_M) \cong \begin{cases} \mathbb{R}_M & k = 0 \\ 0 & k > 0. \end{cases}
\]

where \(\mathbb{R}_M\) is the sheaf of locally constant function on \(M\). In particular, \(H^0_\delta(\Sigma^p_M)\) is generated by the section \(\pi_0 = D(x) \theta_1 \ldots \theta_q \otimes \pi \partial_{x_1} \ldots \pi \partial_{x_p}\), where \(D(x)\) is a generating section of the Berezinian sheaf and \(\pi \partial_{x_1} \ldots \pi \partial_{x_p}\) is the totally anti-symmetric \(\Pi\)-polyvector field in \(S^\Pi T_M\).

Proof. We show for any \(k \neq 0\) a homotopy \(h^k: \text{Ber}(M) \otimes S^{p-k}\Pi T_M \rightarrow \text{Ber}(M) \otimes S^{p-k-1}\Pi T_M\) for \(\delta\), i.e. a map such that

\[
h^k+1 \circ \delta^k + \delta^{k-1} \circ h^k = id_{\text{Ber}(M) \otimes S^{p-k}\Pi T_M},
\]

(6.15)
where we have specified the degree and where the maps go as follows

\[
\cdots \xrightarrow{\cdots} \text{Ber}(\mathcal{M}) \otimes S^{n-k+1} T_{\mathcal{M}} \xrightarrow{\cdots} \text{Ber}(\mathcal{M}) \otimes S^{n-k} T_{\mathcal{M}} \xrightarrow{\delta^k} \text{Ber}(\mathcal{M}) \otimes S^{n-k-1} T_{\mathcal{M}} \xrightarrow{\cdots} (6.16)
\]

\[
\cdots \xrightarrow{\cdots} \text{Ber}(\mathcal{M}) \otimes S^{n-k+1} T_{\mathcal{M}} \xrightarrow{\delta^{k+1}} \text{Ber}(\mathcal{M}) \otimes S^{n-k} T_{\mathcal{M}} \xrightarrow{\delta^{k+1}} \text{Ber}(\mathcal{M}) \otimes S^{n-k-1} T_{\mathcal{M}} \xrightarrow{\cdots} .
\]

We claim that the homotopy is given by

\[
h^k(D(x) f \otimes \pi X) := (-1)^{|f|+|\pi X|} D(x) \sum_b (-1)^{|f|(|x_b|+1)} \left( \int_0^1 dt t^{K_a x_b G_t^* f} \right) \otimes \pi \partial_b(\pi X), \quad (6.17)
\]

where \( D(x) \) is a generating section of the Berezinian, \( f \) is a section of the structure sheaf and \( \pi X \) is a \( \Pi \)-polyfield of degree \( k \) in the form \( \pi X = \pi \theta^I \) for some multi-index \( |I| = k \) and \( K_a \) is a constant, dependent on the integral form \( s = D(x) f \otimes \pi X \) chosen which will be fixed in what follows.

On the one hand we have

\[
h^{k+1} \circ \delta^k(D(x) f \otimes \pi X) = D(x) \sum_{a,b} (-1)^{|x_a|+|x_b|} \left( \int_0^1 dt t^{K_a x_b G_t^* f} \right) \otimes \pi \partial_b \cdot \partial_{\pi \partial_a} \pi X.
\]

(6.18)

On the other hand, the action of \( \delta^{k-1} \circ h^k \) is more complicated, namely made out of four summands:

\[
\delta^{k-1} \circ h^k(D(x) f \otimes \pi X) = +D(x) \sum_a \int_0^1 dt t^{K_a G_t^* f} \otimes \pi X
\]

(6.19)

\[
+ D(x) \sum_a (-1)^{|x_a|} \int_0^1 dt t^{K_a x_b G_t^* f} \otimes \pi X.
\]

(6.20)

\[
+ D(x) \sum_a (-1)^{|x_a|+1} \int_0^1 dt t^{K_a G_t^* f} \otimes \pi \partial \cdot \partial_{\pi \partial_a} \pi X.
\]

(6.21)

\[
- D(x) \sum_{a,b} (-1)^{|f|+|x_a|+1} \left( \int_0^1 dt t^{K_a x_b G_t^* f} \right) \otimes \pi \partial_b \cdot \partial_{\pi \partial_a} \pi X.
\]

(6.22)

For the last line \( 6.22 \) to cancel \( 6.18 \) we need \( K_{\delta a} = Q_a + 1 \), by chain-rule. For the summand \( 6.19 \) it is immediate to observe

\[
\varphi \sum_a \int_0^1 dt t^{Q_a G_t^* f} \otimes F = (p+q) \varphi \left( \int_0^1 dt t^{Q_a G_t^* f} \right) \otimes F.
\]

(6.23)

For the summand \( 6.20 \), assuming without loss of generality that \( f \) is homogeneous of degree \( \text{deg}_a(f) \) in the theta’s we have

\[
D(x) \sum_a (-1)^{|x_a|} \int_0^1 dt t^{K_a x_b G_t^* f} \otimes \pi X = D(x) f \otimes \pi X - \delta_{K_a+1+\text{deg}_a(f)} (D(x) f(0) \otimes \pi X) + (\delta_{K_a+1+\text{deg}_a(f)}) D(x) \left( \int_0^1 dt t^{K_a G_t^* f} \right) \otimes \pi X,
\]

(6.24)
upon integration by parts. For the summand \(6.21\), we define \(\deg_{\pi \partial_\theta}(\pi X)\) and \(\deg_{\pi \partial_\delta}(\pi X)\) the degree of \(\pi X\) in its even and odd monomials. We have
\[
D(x) \sum_a (-1)^{|x_a|+1} \int_0^1 dt K^*_t f \otimes \pi \partial_a \cdot \partial_{\pi \partial_\delta} \pi X =
\]
\[
= (\deg_{\pi \partial_\delta}(\pi X) - \deg_{\pi \partial_\delta}(\pi X)) D(x) \left( \int_0^1 dt K^*_t f \right) \otimes \pi X.
\]
(6.25)
Altogether one gets
\[
(\delta^{k-1} \circ h^k + h^{k+1} \circ \delta^k)(D(x) f \otimes \pi X) = D(x) f \otimes \pi X - \delta K + 1 + \deg_{\pi \partial_\theta}(f) \cdot 0 \varphi(0) \otimes \pi X +
\]
\[
+ (p + q + \deg_{\pi \partial_\theta}(\pi X) - \deg_{\pi \partial_\delta}(\pi X) - 2 \deg_{\pi \partial_\theta}(f) - K - 1) D(x) \int_0^1 dt K^*_t f \otimes \pi X.
\]
(6.26)
This implies that in order to have a homotopy \(K_s\) must be such that
\[
K_s = p + q + \deg_{\pi \partial_\theta}(\pi X) - \deg_{\pi \partial_\delta}(\pi X) - 2 \deg_{\pi \partial_\theta}(f) - 1,
\]
(6.27)
so one is led to consider
\[
(\delta^{k-1} \circ h^k + h^{k+1} \circ \delta^k)(D(x) f \otimes \pi X) = D(x) f \otimes \pi X +
\]
\[
- \delta (p + q + \deg_{\pi \partial_\theta}(\pi X) - \deg_{\pi \partial_\delta}(\pi X) - \deg_{\pi \partial_\theta}(f)) \cdot 0 D(x) f (0\theta) \otimes \pi X.
\]
(6.28)
It is easy to see that the above homotopy fails if \(\deg_{\pi \partial_\theta}(F) = 0\), \(\deg_{\pi \partial_\delta}(F) = p\) and \(\deg_{\pi \partial_\theta}(f) = q\), so that one identifies the generator of the cohomology in the element \(\sigma_0 := D(x) \theta_1 \ldots \theta_q \otimes \pi \partial_{z_1} \ldots \pi \partial_{z_2}\), which is indeed obviously closed under the action of \(\delta\) by inspection.

This result allows us to compute the cohomology of integral forms and, in turn, to make contact with the cohomology of differential forms. Namely, we give the following definition.

**Definition 6.11** (Spencer Cohomology of \(\mathcal{M}\)). Let \(\mathcal{M}\) be a real supermanifold. Then we define the Spencer cohomology of \(\mathcal{M}\) to be the cohomology of the *global sections* of the (sheaf of) differentially graded supermodules \((\Sigma^*_\mathcal{M}, \delta)\), i.e.
\[
H^k_{Sp}(\mathcal{M}) := H^k_{\tilde{\mathcal{H}}^0(\Sigma^*_\mathcal{M})},
\]
(6.29)
where \(\tilde{\mathcal{H}}^0(\Sigma^*_\mathcal{M})\) is the Čech cohomology group of \(\Sigma^*_\mathcal{M}\), i.e. the global sections or \(\Sigma^*_\mathcal{M}\).

Having proved Lemma 6.10, the Spencer cohomology of \(\mathcal{M}\) is easily computed in exactly the same fashion as in Theorem 6.13.

**Theorem 6.12** (Quasi-Isomorphism II). Let \(\mathcal{M}\) be a real supermanifold and let \(\mathcal{M}_{\text{red}}\) be its reduced manifold. Then the Spencer complex of \(\mathcal{M}\) is quasi-isomorphic to the de Rham complex of \(\mathcal{M}_{\text{red}}\). In particular, one has that
\[
H^*_{Sp}(\mathcal{M}) \cong H^*_{Sp}(\mathcal{M}_{\text{red}}).
\]
(6.30)

**Proof.** Once again, the theorem is an easy consequence of the Čech-to-de Rham spectral sequence for the double complex \((\Sigma^*_\mathcal{M}, \delta, d)\), where \(\delta\) is the Čech differential and \(d\) is the integral forms differential. On the one hand, generalized Mayer-Vietoris short exact sequence (hence the existence of a partition of unity) and Poincaré lemma yield \(H^*_\delta(\mathcal{M}_{\text{red}}) \cong H^*(|\mathcal{M}_{\text{red}}|, \mathbb{R}_\mathcal{M})\) in the ordinary setting, on the other hand lemma 6.10 gives that \(H^*_\delta(\mathcal{M}) \cong \tilde{H}^*(|\mathcal{M}_{\text{red}}|, \mathbb{R}_\mathcal{M})\) in the supergeometric setting, where \(\tilde{H}^*(|\mathcal{M}_{\text{red}}|, \mathbb{R}_\mathcal{M})\) is the Čech cohomology of the sheaf of locally-constant functions \(\mathbb{R}_\mathcal{M}\). It follows that
\[
H^*_\delta(\mathcal{M}) \cong \tilde{H}^*(|\mathcal{M}_{\text{red}}|, \mathbb{R}_\mathcal{M}) \cong H^*_\delta(\mathcal{M}_{\text{red}}),
\]
(6.31)
thus concluding the proof. \(\square\)

Just like in the case of differential forms, we have in particular the Poincaré lemma for integral forms for the model supermanifold \(\mathbb{R}^{p|q}\).
Theorem 6.13 (Poincaré Lemma for $\mathbb{R}^{p|q}$). The Spencer cohomology of the supermanifold $\mathbb{R}^{p|q}$ is given by

$$H_{sp}^k(\mathbb{R}^{p|q}) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0. \end{cases}$$  \hspace{1cm} (6.32)$$

Proof. Follows immediately from the above 6.10 and 6.12 □

The above theorem 6.12 and theorem 6.13 have an obvious yet important corollary: the cohomology of differential and integral forms compute exactly the same invariants, i.e. the de Rham cohomology of the reduced space of the supermanifold.

Corollary 6.14 ($H^\bullet_{sp}(\mathcal{M}) \cong H^\bullet_{dR}(\mathcal{M})$). Let $\mathcal{M}$ be a real supermanifold. Then the de Rham cohomology of differential forms of $\mathcal{M}$ is naturally isomorphic to the Spencer cohomology of integral forms of $\mathcal{M}$, i.e.

$$H^\bullet_{sp}(\mathcal{M}) \cong H^\bullet_{dR}(\mathcal{M}).$$  \hspace{1cm} (6.33)$$

Proof. It follows immediately from theorem 6.13 and theorem 6.12 □

Remark 6.15. An analogous result hold true for the compactly supported Spencer cohomology. One finds that

$$H^\bullet_{sp,c}(\mathcal{M}) \cong H^\bullet_{dR,c}(\mathcal{M}),$$  \hspace{1cm} (6.34)$$

where $H^\bullet_{sp,c}(\mathcal{M}_{ad})$ is in turn isomorphic to $H^\bullet_{dR,c}(\mathcal{M}_{ad})$. In particular, the compactly supported Poincaré lemma for integral forms on $\mathbb{R}^{p|q}$ reads

$$H_{sp,c}^k(\mathbb{R}^{p|q}) \cong \begin{cases} \mathbb{R} & k = p \\ 0 & k \neq p. \end{cases}$$  \hspace{1cm} (6.35)$$

A representative is given by $\mathcal{D}(x)\theta_1 \ldots \theta_q\mathcal{R}(z_1, \ldots, z_p)$ for a compactly supported bump function $\mathcal{R}$, which integrate to one on the reduced space. Compactly supported integral forms will play a crucial role in the next section, when integration on supermanifold will be introduced.

Remark 6.16. As an addendum to the above remark 3.16, the previous corollary 6.14 says that, despite both of the complex of differential and integral forms are actually not bounded either from above or below, their cohomology can indeed be non-zero (and isomorphic) only in the framed part of the diagram below - where the degree of differential forms matched the degree of integral forms - from zero to the even dimension of the supermanifold.

$$\begin{array}{c}
0 \rightarrow \Omega^0_{\mathcal{M}}(\mathcal{M}) \rightarrow \Omega^1_{\mathcal{M}}(\mathcal{M}) \rightarrow \ldots \rightarrow \Omega^p_{\mathcal{M}}(\mathcal{M}) \rightarrow 0.
\end{array}$$

Once again, here we have denoted $\Omega^k_{\mathcal{M}}(\mathcal{M})$ and $\Sigma^k_{\mathcal{M}}(\mathcal{M})$ the global section of the sheaves of differential and integral forms.

7. Berezin Integral and Stokes’ Theorem on Supermanifolds

In this section we introduce the notion of Berezin integral for real supermanifolds [10, 12, 24, 76], following the philosophy and exposition given in [53], that underlies the role of integral forms of their cohomology as introduced above. In particular, we denote with $Ber(\mathcal{M})$ the Berezinian sheaf whose sections have compact support on $\mathcal{M}$ and with $\Sigma^\bullet$ the Spencer complex of compactly supported integral forms on $\mathcal{M}$. Further, we assume our supermanifold always has a finite good cover. We start with the following preparatory lemma, see [53].

Lemma 7.1. Let $\mathcal{M}$ be a real supermanifold of dimension $p|q$. Then one has the following isomorphism of sheaves of $\mathcal{O}_{\mathcal{M},c}$-modules

$$Ber(\mathcal{M}) \cong \mathcal{J}_M^p Ber(\mathcal{M}) + \delta(\Sigma^{p-1}_{\mathcal{M},c}).$$  \hspace{1cm} (7.1)$$

More in particular the intersection $\mathcal{J}_M^p Ber(\mathcal{M}) \cap \delta(\Sigma^{p-1}_{\mathcal{M},c})$ is such that

$$\varphi(\mathcal{J}_M^p Ber(\mathcal{M}) \cap \delta(\Sigma^{p-1}_{\mathcal{M},c})) \cong d(\Omega^{p-1}_{\mathcal{M},c}),$$  \hspace{1cm} (7.2)$$

where $\varphi : \mathcal{J}_M^p Ber(\mathcal{M}) \rightarrow \Omega^{p}_{\mathcal{M},c}$ is the isomorphism of Theorem 5.5.
Proof. Let \( D_i \in \mathcal{B}er_i(\mathcal{M}) = \Sigma^p_{M,i} \) be a section of the Berezinian sheaf. Then, using a partition of unity \( \{ \rho_k \}_{k \in I} \) for \( \mathcal{M} \), we represent \( D \) as a (locally) finite sum

\[
D = \sum_{i=1}^n D_i^{(i)}
\]  

(7.3)

where \( \text{supp}(D_i^{(i)}) \) is compact and it is contained in a certain open set \( U^{(i)} \) locally described by the coordinate system \( x_\alpha = z_1, \ldots, z_p | \theta_1, \ldots, \theta_q \), where the dependence of \( i \) is understood. Then one can in turn decompose \( D_i^{(i)} \) as follows

\[
D_i^{(i)}(x) = D_i^{(i)}(x) + D_i^{(i)}(x) \theta_1 \ldots \theta_q,
\]  

(7.4)

such that every monomial in the \( \theta \)-expansion of \( D_i(x) \) contains at most \( q-1 \) theta’s, i.e. it is of the form

\[
D_{\epsilon,0}^{(i)}(x) = \mathcal{D}(x)^{\theta_1^1 \ldots \theta_q^q f_L}(x)
\]  

(7.5)

for \( \epsilon = (\epsilon_1, \ldots, \epsilon_q) \) with \( \epsilon_i \in \{0,1\} \) and \( |\epsilon| < q \) and \( \mathcal{D}(x) = [dz_1 \ldots dz_p \otimes \partial_{\theta_1} \ldots \partial_{\theta_q}] \) a generating section. It is easy to see that any such section is in the image of \( \delta^{p-1} : \Sigma^p_{M,i} \rightarrow \Sigma^p_{M,i} \). Indeed, let for example be \( \epsilon_j = 0 \), then, by definition of \( \delta \), one sees that up to sign

\[
\delta(\mathcal{D}(x)^{\theta_1^1 \ldots \theta_j^q f_L}(x) \otimes \pi_{\partial_{\theta_j}}) = \mathcal{D}(x)^{\theta_1^1 \ldots \theta_{j-1}^1 \theta_{j+1}^1 \ldots \theta_q^q f_L}(x).
\]  

(7.6)

Notice that since every real supermanifold real is split, the \( \theta \)-degree of the expansion is invariant. It follows that

\[
\sum_i \delta(D_i^{(i)}) \in \delta(\Sigma^{p-1}_{M,i}),
\]  

(7.7)

and hence the difference \( D_i - \sum_i D_i^{(i)} \) lies in \( \mathcal{F}^p_{M,i} \mathcal{B}er_i(\mathcal{M}) \), which proves the first statement.

For the second statement, let first be \( D_i \in \mathcal{F}^p_{M,i} \mathcal{B}er_i(\mathcal{M}) \cap \delta(\Sigma^{p-1}_{M,i}) \), with \( D_i = \delta \omega_i \) for some \( \omega_i \in \Sigma^{p-1}_{M,i} \). Let us decompose \( \omega_i \in \Sigma^{p}_{M,i} \) as in (7.3),

\[
\omega_i = \sum_j \omega_j^{(i)},
\]  

(7.8)

for \( \text{supp}(\omega_j^{(i)}) \) compact and contained in some open set \( U^{(i)} \) locally described by the coordinate system \( x_\alpha = z_1, \ldots, z_p | \theta_1, \ldots, \theta_q \), where again we have left the dependence of \( i \) understood. Now, without loss of generality, any \( \omega_j^{(i)} \) can be taken of the form

\[
\omega_j^{(i)} = D(x)^{\theta_1 \ldots \theta_j f}(z) \otimes \sum_a \pi_{\partial_{x_a}}
\]  

(7.9)

with \( f \), depending on the even coordinates only, since we have seen that any monomial which does not have all the theta’s is the image of \( \delta \), and since \( \delta^2 = 0 \) it will not contribute to \( D_i \in \Sigma^{p}_{M,i} \).

Further there should be at least one of the \( \pi_{\partial_{\theta_j}} \)’s, since otherwise either \( \delta \omega_j^{(i)} \notin \mathcal{F}^p_{M,i} \mathcal{B}er_i(\mathcal{M}) \) or \( \delta \omega_j^{(i)} = 0 \) in \( \mathcal{F}^p_{M,i} \mathcal{B}er_i(\mathcal{M}) \). But then, one concludes that \( \delta \omega_j^{(i)} \) should be of the form

\[
\delta(\omega_j^{(i)}) = \sum_{j=1}^p D(x) \theta_1 \ldots \theta_j \partial_{x_a} f_l(z),
\]  

(7.10)

so that if follows

\[
\varphi(\delta(\omega_j^{(i)})) = \sum_{j=1}^p dz_1 \ldots dz_p \partial_{x_a} f_l(z) \in d(\Omega^p_{\mathfrak{m}_i,d}).
\]  

(7.11)

proving that \( \varphi(\mathcal{F}^p_{M,i} \mathcal{B}er_i(\mathcal{M}) \cap \delta(\Sigma^{p-1}_{M,i}) \) \( \subset \mathfrak{m}_i,d \). 

Viceversa, let us take \( D_i \in \mathcal{F}^p_{M,i} \mathcal{B}er_i(\mathcal{M}) \) such that \( \varphi(D_i) = d \eta_{ad,i} \), for \( \eta_{ad,i} \in \Omega_{\mathfrak{m}_i,d}^{p-1} \) and let us prove that \( D_i \in \delta(\Sigma^{p-1}_{M,i}) \). To this end, once again we decompose \( \omega_{ad,i} \) as follows

\[
\eta_{ad,i} = \sum_i \eta_{ad,i}^{(i)},
\]  

(7.12)
as above, where now \( \text{supp}(\eta_{\text{red},c}) \) is compact and contained in some open set \( U^{(i)} \) locally described by the coordinate system \( z_1, \ldots, z_p \) for \( \mathcal{M}_{\text{red}} \), so that in these coordinates one has

\[
\eta_{\text{red},c}^{(i)} = \sum_k d z_1 \cdots d z_k \cdot d z_p f_k^{(i)}(z)
\]  

(7.13)

Accordingly, one can lift \( \eta_{\text{red},c}^{(i)} \) to the integral form in \( \Sigma_{\mathcal{M},c}^{p-1} \) given by

\[
\omega_c^{(i)} = \sum_k \mathcal{D}(x) \theta_{1} \cdots \theta_{q} f_k^{(i)}(z) \otimes \pi \partial z_k
\]  

(7.14)

so that up to sign one gets \( \varphi(\delta \omega_c^{(i)}) = d \eta_{\text{red},c}^{(i)} \). Notice that the support of \( \eta_{\text{red},c}^{(i)} \) is the same as the one of \( \omega_c^{(i)} \) and that, with abuse of notation, we have denoted the even local coordinates in the same way on \( \mathcal{M}_{\text{red}} \) and \( \mathcal{M} \). Summing over the index \( i \), defining \( \omega_c := \sum_i \omega_c^{(i)} \), one has that

\[
\varphi(\mathcal{D}_c - \delta \omega_c) = \varphi(\mathcal{D}_c) - \varphi(\delta \omega_c) = \varphi(\mathcal{D}_c) - d \eta_{\text{red},c} = 0,
\]  

(7.15)

by hypothesis. Let us set \( \tilde{\mathcal{D}}_c := \mathcal{D}_c - \delta \omega_c \), choose a partition of unity \( \rho_j \) subordinate to the above open sets, so that it commutes with the isomorphism \( \varphi \) and posing so that \( \varphi(\tilde{\mathcal{D}}_c^{(i)}) = \rho_i \tilde{\mathcal{D}}_c^{(i)} \), one gets \( \varphi(\tilde{\mathcal{D}}_c^{(i)}) = 0 \). Then, it follows that in the above domain \( \tilde{\mathcal{D}}_c^{(i)} \) does not contain all of the \( \varphi \)-s, otherwise we would get \( \varphi(\tilde{\mathcal{D}}_c^{(i)}) \neq 0 \). In turn, this implies that \( \tilde{\mathcal{D}}_c^{(i)} \) is in the image of \( \varphi \), i.e. there exists \( \tilde{\omega}_c^{(i)} \in \Sigma_{\mathcal{M},c}^{p-1} \) such that \( \delta(\tilde{\omega}_c^{(i)}) = \tilde{\mathcal{D}}_c^{(i)} \) and the support of \( \tilde{\omega}_c^{(i)} \) is at most the same as the one of \( \tilde{\mathcal{D}}_c^{(i)} \). Posing \( \omega_c := \sum_i \tilde{\omega}_c^{(i)} \) and summing over \( i \), recalling that \( \mathcal{D}_c = \mathcal{D}_c - \delta \omega_c \), we have \( \mathcal{D}_c = \delta(\tilde{\omega}_c + \omega_c) \), proving that \( \mathcal{D}_c \in \delta(\Sigma_{\mathcal{M},c}^{p-1}) \).

The previous lemma has the following immediate consequence, that we state as a theorem.

**Theorem 7.2.** Let \( \mathcal{M} \) be a real supermanifold. Then the following natural isomorphism of sheaves of \( \mathcal{O}_{\mathcal{M}_{\text{red}}} \)-modules holds true

\[
\frac{\text{Ber}_c(\mathcal{M})}{\text{Im}(\delta^{p-1})} \cong \frac{\Omega^{p}_{\mathcal{M}_{\text{red},c}}}{\text{Im}(d^{p-1})}.
\]  

(7.16)

where \( \delta^{p-1} : \Sigma_{\mathcal{M},c}^{p-1} \rightarrow \Sigma_{\mathcal{M},c}^{p-1} \) and \( d^{p-1} : \Omega^{p-1}_{\mathcal{M},c} \rightarrow \Omega^{p-1}_{\mathcal{M},c} \).

**Proof.** The isomorphism is induced by the map \( \varphi : \mathcal{J}_M^{p} \text{Ber}_c(\mathcal{M}) \rightarrow \Omega^{p}_{\mathcal{M},c} \) and it follows immediately from lemma 7.1 since \( \text{Ber}_c(\mathcal{M}) \) decomposes as \( \text{Ber}_c(\mathcal{M}) \cong \mathcal{J}_M^{p} \text{Ber}_c(\mathcal{M}) + \delta(\Sigma_{\mathcal{M},c}^{p-1}) \) and \( \varphi(\mathcal{J}_M^{p} \text{Ber}_c(\mathcal{M}) \cap \delta(\Sigma_{\mathcal{M},c}^{p-1}) \cong \text{Im}(d^{p-1}) \). \( \square \)

We can thus finally define the Berezin integral of a section of \( \text{Ber}_c(\mathcal{M}) \).

**Definition 7.3 (Berezin Integral).** Let \( \mathcal{M} \) be a real supermanifold of dimension \( p/q \) such that \( \mathcal{M}_{\text{red}} \) is oriented and let \( x_a = z_1, \ldots, z_p, \theta_1, \ldots, \theta_q \) is a local system of coordinate on an open set \( U \). Let \( \mathcal{D}_c \in \text{Ber}_c(\mathcal{M}) \) be a compactly supported section of the Berezinian sheaf which reads

\[
\mathcal{D}_c = \sum_k \mathcal{D}(x) \theta_{1}^{(1)} \cdots \theta_{q}^{(2)} f_k(z_1, \ldots, z_p),
\]  

(7.17)

with \( \xi = (\epsilon_1, \ldots, \epsilon_q) \) for \( \epsilon_j \in \{0, 1\} \) in the above local coordinates in \( (U, x_a) \). Then we define the Berezin integral of \( \mathcal{D}_c \) as the map

\[
\int_{\mathcal{M}} : \text{Ber}_c(\mathcal{M}) \rightarrow \mathbb{R}
\]  

(7.18)

(7.19)

given in the coordinate domain \( (U, x_a) \) by

\[
\int_U \mathcal{D}_c := \int_{U_{\text{red}}} d z_1 \cdots d z_p f_1^{(1)}(z_1, \ldots, z_p),
\]  

(7.20)

and we extend the definition to all \( \mathcal{M} \) by additivity via a partition of unity.
Remark 7.4. The above definition is well-given. Indeed, it does not depend on the choice of local coordinates as a consequence of the previous lemma 7.1 and theorem 7.2. Indeed, more invariantly, the Berezin integral is the map given by composition of the isomorphism of theorem 7.2 with the ordinary integral, i.e.

$$\text{Ber}_c(M)/\delta(\Sigma_{\mathcal{M},c}^{p-1}) \xrightarrow{\bar{\varphi}} \Omega^{p-1}_{\mathcal{M},c}/d(\Omega_{\mathcal{M},c}^{p-1}) \xrightarrow{\int} \mathbb{R},$$   (7.21)

which induces the isomorphism $H^p_{\text{sp},c}(M_{\text{ad}}) \cong \mathbb{R}$ in compactly supported de Rham cohomology. More in particular, the above constructions allows to prove an analog of Stokes theorem for supermanifolds.

Theorem 7.5 (Stokes Theorem for Supermanifolds). Let $M$ be a real supermanifold of dimension $pq$ with $M_{\text{ad}}$ oriented and let $D_c \in \text{Ber}_c(M)$. Then the following are true.

1. There exists $\omega_c \in \Sigma_{\mathcal{M},c}^{p-1}$ such that $D_c = \delta \omega_c$ if and only if

$$\int_{\mathcal{M}} D_c = 0.$$  (7.22)

In other words a compactly supported integral form of degree $p$ is exact, i.e. $[D_c] \equiv 0 \in H^0_{\text{sp},c}(M)$ if and only if it has vanishing Berezin integral.

2. If $M$ is connected, the Berezin integral defines an isomorphism

$$\int_{\mathcal{M}} : H^0_{\text{sp},c}(M) \xrightarrow{\cong} \mathbb{R}.$$  (7.23)

In particular, a representative of $H^0_{\text{sp},c}$ is given by $\sigma_c := D(x)\theta_1 \ldots \theta_q \mathcal{B}_i(z_1, \ldots, z_p)$, where $\mathcal{B}_i$ is any bump function which integrate to one on $M_{\text{ad}}$.

Proof. It is enough to use (7.21). More in particular, on the one hand we have observed that if $D_c \notin \mathcal{J}_d \text{Ber}_c(M)$, i.e. does not have all the theta’s, then it is in the image of $\delta$ and, by definition, its Berezin integral yields zero. On the other hand, if $D_c \in \mathcal{J}_d \text{Ber}_c(M) \cap \delta(\Sigma_{\mathcal{M},c}^{p-1})$, i.e. it has all of the theta’s and it is in the image of $\delta$, then if one has $D_c = D(x)\theta_1 \ldots \theta_q f(z)$ then $f(z)$ is a divergence, and $D_c$ gets mapped to an element in $d(\Omega_{\mathcal{M},c}^{p-1})$, which integrates to zero by ordinary Stokes theorem. The second point follows immediately from (7.21) and the previous theorem 7.2. \hfill \Box

Remark 7.6. Explicitly one has the following isomorphism between local representatives

$$H^0_{\text{sp},c}(M) \ni D(x)\theta_1 \ldots \theta_q \mathcal{B}_i(z_1, \ldots, z_p) \xrightarrow{\bar{\varphi}} dz_1 \ldots dz_p \mathcal{B}_i(z_1, \ldots, z_p) \in H^0_{\text{sp},c}(M_{\text{ad}}),$$  (7.24)

where again $\mathcal{B}_i(z_1, \ldots, z_p)$ is a bump function which integrate to one over $M_{\text{ad}}$. More in general, as in remark 6.15 it can be proved that the compactly supported Spencer cohomology of integral forms is isomorphic to the compactly supported de Rham cohomology, so that one has $H^0_{\text{sp},c}(M) \cong H^0_{\text{sp},c}(M_{\text{ad}})$.

Remark 7.7. We will not deal with the subtle case of supermanifold with boundaries and the related Stokes’ Theorem. More on this can be found in [53] and [72, 76].

7.1. Supersymmetry and the Berezin Integral. The foremost application of the theory of integration on supermanifolds is related with high-energy physics, in particular with modern supersymmetric field theories [17, 28]. Very roughly speaking, physical elementary particles are described mathematically via irreducible (projective and unitary) representations of the Poincaré group, whose Casimir invariants are in turn related to the mass and the spin (or helicity in the zero-mass case) of the particles. In this context, a supersymmetry is a physical symmetry which relates particles characterized by integer spins (bosons, physically describing “interactions”) to particles characterized by half-integer spins (fermions, physically describing “matter”): such symmetries are building pillars of the most far-reaching theories in contemporary physics, string theory being an example.

Brieﬂy, a physical theory on an unspecified space(time) $M$, where $M$ is an ordinary manifold, is
described by an action functional $A : \mathcal{F}_M \to \mathbb{R}$ where $\mathcal{F}_M$ is the space of fields $\varphi^i$ of the theory. One usually writes the action as an integral over $M$

$$A := \int_M \mathcal{L}(\varphi^i)$$

(7.25)

where $\mathcal{L}$ is the so-called Lagrangian density of the theory, which - for an ordinary space-time manifold $M$ - is a (compactly supported) section of the canonical sheaf of $M$, i.e. $\mathcal{L} \in \Omega^{d-1,1}_{\text{can}}$, so that the integral makes sense (we consider $M$ to be oriented). We say that the physical theory described by $A$ is invariant under the (infinitesimal) transformation generated by a vector field $X \in T_M$ if

$$\delta_X A := \int_M \mathcal{L}_X \mathcal{L} = 0,$$

(7.26)

where $\mathcal{L}_X \mathcal{L}$ is the Lie derivative of $\mathcal{L}$ with respect to the field $X$. In this case one says that the theory is a symmetry of the theory and the field $X$ is the generator of the symmetry. The simplest way to make a physical theory manifestly invariant under a given transformation is to construct the theory in a space whose isometry group contains such a transformation: supermanifolds do this job for supersymmetry transformations. More precisely, supersymmetric field theories can be made into manifestly invariant theories under supersymmetry if they can be constructed as theories on particular supermanifolds called superspacetimes, which are defined as homogenous superspaces for the action of Poincaré Lie supergroups and whose reduced manifolds are ordinary physical spacetimes (e.g. the Minkowski spacetime $\mathbb{R}^{1,d-1}$ in $D$ dimensions) $[29, 32, 33, 67]$.

Superspacetimes are constructed out of three pieces of data,

1. a real quadratic vector space $(V, Q)$ of dimension $\dim V = D$, with $Q : V \to \mathbb{R}$ whose associated symmetric bilinear form $\mathcal{B} : V \times V \to \mathbb{R}$ has signature $(1, D - 1)$;
2. a real spinorial representation $\mathcal{S} : \text{Spin}(V) \to \text{Aut}(V)$ of dimension $\dim \mathcal{S} = q$ of the group $	ext{Spin}(V)$;
3. a $\text{Spin}(V)$-equivariant symmetric non-zero bilinear map $\gamma : S \times S \to V$.

Notice that the map $\gamma : S \times S \to V$ always exists in the given setting $[67]$. In the above data $V$ is called (abelian) translation algebra, as it is appear as the summand in the Poincaré Lie algebra $\mathfrak{iso}(V) := V \times \mathfrak{so}(V)$ corresponding to spacetime-translation. The related Poincaré Lie superalgebra $\mathfrak{so}(V) = g_0 \oplus g_1$ is constructed out of the direct product of vector spaces

$$\mathfrak{so}(V) = (V + \mathfrak{so}(V)) \oplus S,$$

(7.27)

where $g_0 := V + \mathfrak{so}(V)$ and $g_1 := S$. Here $S$ is looked at as a $g_0$-module with a trivial action

$$V \cdot s := [v, s] = 0$$

(7.28)

for any $v \in V$ and $s \in S$. Defining further

$$[s_1, s_2] := \gamma(s_1, s_2)$$

(7.29)

for any $s_1, s_2 \in S$, one obtains a Lie superalgebra, indeed $[s, [s, s]]$ is zero since $[s, [s, s]] = [s, \gamma(s, s)] = 0$ by (7.28). Letting $S$ be irreducible with basis given by $Q_a$ for $a = 1, \ldots, q$ and $P_\mu$ with $\mu = 1, \ldots D$ the standard basis of $V \cong \mathbb{R}^{1,D-1}$, one can write - upon a suitable normalization of $\gamma$ -

$$[Q_a, Q_b] = \sum_{\mu = 1}^D \gamma_{ab}^{\mu} P_\mu,$$

(7.30)

where $[Q_a, Q_b] := \gamma(Q_a, Q_b)$ so that $\gamma_{ab}^{\mu} = \gamma_{ba}^{\mu}$: this is the way the crucial commutation relation of Poincaré Lie superalgebra appears in physics.

The Poincaré Lie supergroup $\mathbb{P} \mathfrak{so}(V)$ is obtained exponentiating this construction, and the related superspacetime is the quotient supermanifold (or coset superspace, as most frequently called in physics) obtained by modding out the Lie group $\text{SO}(V)$ of Lorentz transformations $[67]$.

$$\mathcal{M} := \mathbb{P} \mathfrak{so}(V) / \text{SO}(V).$$

(7.31)

Equally, one can notice that $V \oplus S$ is also a Lie superalgebra, with $g_0 = V$, the ordinary translation algebra and $g_1 = S$: this is called translation superalgebra and the superspacetime is given by its
related Lie supergroup. Notice that $\dim(M) = \dim(V) \dim(S) = D|q$, and if $(x^\mu|\theta^a)$ and $(y^\mu|\psi^a)$ are coordinates for $M$, then the group law $(z|\lambda) := (x|\theta) \cdot (y|\psi)$ reads
\[ z^\mu = x^\mu \cdot y^\mu = x^\mu + y^\mu - \frac{1}{2} \sum_{a,b} \gamma_{ab}^\mu \theta^a \psi^b, \quad \lambda^a = \theta^a + \psi^a. \tag{7.32} \]
Correspondingly, a set of even left invariant vector fields is given by $\{Q_a\}_{a=1,\ldots,q}$, while a set of odd left invariant vector fields are given by $\{Q_a\}_{a=1,\ldots,q}$, with
\[ Q_a := \frac{\partial}{\partial \theta^a} + \frac{1}{2} \sum_{\mu,b} \gamma_{ab}^\mu \theta^b \frac{\partial}{\partial x^\mu}, \tag{7.33} \]
and it is not hard to verify that $[Q_a, Q_b] = \sum_{\mu=1}^D \gamma_{ab}^\mu \partial_{\mu}$, just like the above (7.30), upon identifying $P_\mu := \partial_{\mu}$, as customary, since the $\partial_{\mu}$'s generate space-time translations. In view of this, supersymmetry transformations are generated by the vector fields $Q_a$.

In light of the previous sections, one can generalize the definition (7.23) and (7.26) on a superspacetime. The action of the physical theory is now given by an integral on the superspacetime
\[ A' := \int_M \mathcal{L}'(\varphi'), \tag{7.34} \]
where now the Lagrangian density is a section of the compactly supported Berezinian sheaf of the superspacetime $M$, i.e. $\mathcal{L}' \in \text{Ber}_s(M)$. It can be trivialized as
\[ \mathcal{L}'(\varphi') = D(x|\theta) \Phi(\varphi'(x), \theta^a), \tag{7.35} \]
with $D(x|\theta) = [dx_1, \ldots, dx_D \otimes \partial_{\theta_1}, \ldots \partial_{\theta_q}]$ a generating section for the Berezinian and where $\Phi(\varphi', \theta^a) \in \mathcal{O}_M$ is a so-called superfield, containing the original physical fields $\varphi' \in \mathcal{O}_M$ - plus auxiliary fields - in its component expansion. The component fields transform one into another under supersymmetry, forming a so-called supersymmetry multiplet.

**Lemma 7.8** (Supersymmetry Invariance). Let $A'$ be an action on a connected superspacetime $M$ as in (7.34) and let $Q$ be any supersymmetry generator of the form (7.33). Then
\[ \delta_Q A' = \int_M \mathcal{L}_Q(\mathcal{L}') = 0. \tag{7.36} \]
In particular, the action is invariant under supersymmetry.

**Proof.** The result follows from the action of the Lie derivative on sections of the Berezinian sheaf as in equation (7.37) and from Stokes theorem (7.35) for supermanifolds.

More precisely, the part of the Lie derivative with respect to the odd coordinate vector fields $\partial_{\theta_a} \in Q_a$ integrate to zero since $\mathcal{L}_{\partial_{\theta_a}}(\mathcal{L}') \notin \mathcal{J}'_q \text{Ber}_s(M)$ and, as such, it is $\delta$-exact as an integral form.

Similarly, the part of the Lie derivative with respect to the odd vector field $\sum_{\mu,b} \gamma_{ab}^\mu \theta^b \frac{\partial}{\partial x^\mu} \in Q$ yields a divergence, which is an element in $\mathcal{J}'_q \text{Ber}_s(M) \cap \delta(\Sigma^{D-1})$ that integrate to zero again by Stokes theorem. \[ \square \]

**Remark 7.9.** In general, verifying that a theory is indeed invariant under supersymmetry is not trivial matter, and it often requires going through delicate and lengthy calculations. The above lemma shows that upon using an adequate and mathematically aware formalism - based on supermanifolds, integral forms and the related integration theory -, supersymmetry invariance becomes apparent and virtually no checks are required. Nonetheless, it is fair to say that writing a superspace action is in general not an easy task and indeed there exist theories of great physical interest for which action on superspacetime is not known.

**Remark 7.10.** More in general, Lagrangian densities on superspacetimes might also involve also covariant derivatives, i.e. differential operators acting on superfields which commutes with the supersymmetry generators: this is the case for example of the kinetic term (field strength) of supersymmetric gauge theories. Geometrically, these can be constructed as right invariant vector fields on the superspacetime $M$, in opposition with $Q$ being left invariant.
8. Poincaré Duality on Supermanifolds

Having available a notion of integration on supermanifolds via the Berezin integral, we can prove the analog of Poincaré duality for supermanifolds. Whereas Poincaré duality on ordinary manifolds yields a perfect pairing between the Rham cohomology groups of the manifolds, we will see instead that on a supermanifold Poincaré duality defines a perfect pairing between the cohomology of two different complexes, that of differential and that of integral forms: this is rooted in the peculiar geometry of forms and, in turn, integration theory on supermanifolds. We start with some preliminary remarks.

**Remark 8.1.** As discussed in the first section around equation (3.3), there is an obvious pairing $\Pi T_M \times \Omega^1_M \to \Omega^0_M$, given by the contraction of $\Pi$-vector fields and forms on $M$. This can be extended to higher supersymmetric powers of $\Pi T_M$ and $\Omega^1_M$

$$\langle \cdot, \cdot \rangle : S^n \Pi T_M \times \Omega^m_M \to S^{n-m} \Pi T_M$$

for any $n, m \geq 1$ with $n \geq m$ so that associativity is met in the form $\langle \langle \pi X^{(n)}, \omega_1 \rangle, \omega_2 \rangle = \langle \pi X^{(n)}, \omega_1 \omega_2 \rangle$, for any $\pi X^{(n)} \in S^n \Pi T_M$ and $\omega_1, \omega_2 \in \Omega^m_M$ whose sum of degrees does not exceed $n$. This pairing induces a *right multiplication* between integral forms in $\Sigma^{p-n}_M = \text{Ber}(M) \otimes S^n \Pi T_M$ and differential forms in $\Omega^m_M$, that we write as

$$\cdot : \Sigma^{p-n}_M \times \Omega^m_M \to \Sigma^{p-(n-m)}_M,$$

for $n, m \geq 0$, $n \geq m$ and $n + m$ and where $p$ is the even dimension of $M$, so that $\Sigma^p_M = \text{Ber}(M)$. Notice that, at this stage, this does not define a structure of right $\Omega^*_M$-module on integral forms, since the multiplication is defined only for certain degrees. The crucial observation is that the above multiplication is compatible with the differential, i.e. we have a Leibniz rule in the form

$$\delta(\sigma \cdot \omega) = \delta \sigma \cdot \omega + (-1)^{\lvert \sigma \rvert} \sigma \cdot d\omega,$$

for any $\sigma \in \Sigma^{p-n}_M$ and $\omega \in \Omega^m_M$ and where $d$ is the de Rham differential. This can be proved in local coordinates by expanding the tensor $\pi X^{(n)}$ over a base of $S^n \Pi T_M$ of supersymmetric products of the $\pi \partial x_a$’s and use (8.1). This is an extension of the bookkeeping of the signs involved in the contraction of $\Pi$-vector fields and forms, where the signs play a crucial role: we check it in the case $\sigma \in \Sigma^{p-2}_M$ and $\omega \in \Omega^1_M$, for $\sigma = \sum_{a,b} D(x) f \otimes X_{ab} \pi \partial_a \pi \partial_b$ and $\omega = \sum_a g_a dx_a$. One has

$$\delta(\sigma \cdot \omega) = \sum_{a,b} (-1)^{\lvert x_a \rvert + 1 + 1 + \lvert X_{ab} \rvert + \lvert D(x) \rvert + 1 + 1 + 1} \partial_a (f X_{ab} g_b + f X_{ab} \partial_a g_b) \pm (a \leftrightarrow b)$$

and

$$\sum_{a,b} (-1)^{\lvert x_a \rvert + 1 + 1 + \lvert X_{ab} \rvert + 1 + \lvert D(x) \rvert + 1 + 1 + 1 + 1 + 1 + 1} \partial_a (f X_{ab} g_b) \pm (a \leftrightarrow b).$$

On the other hand, one computes directly that

$$\delta(\sigma \cdot \omega) = \sum_{a,b} (-1)^{\lvert x_a \rvert + 1 + 1 + \lvert X_{ab} \rvert + \lvert D(x) \rvert + 1 + 1 + 1 + 1 + 1 + 1} \partial_a (f X_{ab} g_b) \pm (a \leftrightarrow b),$$

matching (8.4). The previous Leibniz rule (8.3) is the key to prove Poincaré duality for supermanifolds.

**Theorem 8.2** (Poincaré Duality for Supermanifolds). Let $M$ be a real supermanifold of dimension $p|q$ with $M_{ad}$ oriented. Then, for any $n \geq 0$, the Berezin integral defines a perfect pairing in cohomology

$$H^{p-n}_\text{Sp,c}(M) \times H^n_{\text{dR}}(M) \to \mathbb{R}$$

$$([\sigma_i], [\omega]) \mapsto \int_M \sigma \cdot \omega$$

In particular, there is a natural isomorphism

$$(H^{p-n}_\text{Sp,c}(M))^* \cong H^n_{\text{dR}}(M).$$
Proof. First of all let us observe that the map is well-defined. Indeed if \( \sigma_x \in \Sigma_{p,n}^p \) and \( \omega \in \Omega_{p}^n \) are two representative, then \( \sigma_x \cdot \omega \in \Sigma_{p,n}^p = \text{Ber}_x(M) \), so that it can indeed be integrated in the Berezin sense. In other words the map is given by the following composition:

\[
\begin{array}{ccc}
\Sigma_{p,n}^p \times \Omega_{p}^n & \longrightarrow & \Sigma_{p,n}^p \\
(\sigma_x, \omega) & \longrightarrow & \sigma_x \cdot \omega \\
& \longrightarrow & \int \sigma \cdot \omega.
\end{array}
\]

(8.8)

Further, let \( \sigma_x \) and \( \omega \) be closed, i.e. \( \delta \sigma = 0 \) and \( d \omega = 0 \). It follows immediately from the \([S.3]\) that \( \sigma_x \cdot \omega \) is closed, i.e. \( \delta (\sigma_x \cdot \omega) = 0 \).

Now let \( \sigma_x \) be exact, i.e. \( \sigma_x = \delta \eta \) for some \( \eta \in \Sigma_{p,n}^{p(n-1)} \). Then, using again \([S.3]\) one has that

\[
\sigma_x \cdot \omega = \delta \eta \cdot \omega = \delta (\eta \cdot d \omega),
\]

proving that \( \sigma_x \cdot \omega \) is exact. If instead \( \omega \in \Omega_{p}^n \) is exact, i.e. \( \omega = d \gamma \) for some \( \gamma \in \Omega_{p}^{n-1} \), then by \([S.3]\) one has

\[
\sigma_x \cdot \omega = \sigma_x \cdot d \gamma = (-1)^{|\sigma_x|} \delta (\sigma_x \cdot \gamma),
\]

proving that \( \sigma_x \cdot \omega \) is exact. We thus have that the composition of maps \([8.8]\) descends to cohomology as to give

\[
\begin{array}{ccc}
H_{p,n}^p(M) \times H_{dR}^n(M) & \longrightarrow & H_{p,n}^p(M) \\
(\sigma_x, \omega) & \longrightarrow & \sigma_x \cdot \omega \\
& \longrightarrow & \int \sigma \cdot \omega.
\end{array}
\]

(8.11)

Now let us work by induction on the cardinality of the good covering, namely let us start proving the result for the pivotal case of a covering of cardinality one. This corresponds to the Poincaré lemma for \( \mathbb{R}^{p,q} \), which follows straightforwardly from the Poincaré lemmas for differential forms and compactly supported integral forms, i.e.

\[
H_{dR}^k(\mathbb{R}^{p,q}) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & k \geq 0 \end{cases}, \quad H_{dR}^k(\mathbb{R}^{p,q}) \cong \begin{cases} \mathbb{R} & k = p \\ 0 & k \neq p. \end{cases}
\]

(8.12)

The only non-zero pairing reads

\[
\begin{array}{ccc}
H_{dR}^k(\mathbb{R}^{p,q}) \times H_{dR}^k(\mathbb{R}^{p,q}) & \longrightarrow & H_{dR}^k(\mathbb{R}^{p,q}) \\
([D(x)\theta_1 \ldots \theta_q B_i], [1]) & \longrightarrow & [D(x)\theta_1 \ldots \theta_q B_i] \cdot [\omega] \\
& \longrightarrow & \int_{\mathbb{R}^{p,q}} D(x)\theta_1 \ldots \theta_q B_i.
\end{array}
\]

(8.13)

Upon choosing \( B_i(z_1, \ldots, z_p) \) a bump functions which integrates to one on \( \mathbb{R}^p \), one gets

\[
\int_{\mathbb{R}^{p,q}} D(x)\theta_1 \ldots \theta_q B_i = \int_{\mathbb{R}^p} dz_1 \ldots dz_p B_i(z_1, \ldots, z_p) = 1,
\]

(8.14)

concluding the proof of the isomorphism \( (H_{dR}^p(\mathbb{R}^{p,q}))^* \cong H_{dR}^0(\mathbb{R}^{p,q}) \).

Now, if \( \mathcal{M} \) is covered by two open sets \( U \) and \( V \) so that with abuse of notation we can write \( \mathcal{M} = U \cup V \) as supermanifolds, thanks to the Mayer-Vietoris sequences for ordinary and compactly supported cohomology we have the following (sign-)commutative diagram \([14]\)

\[
\begin{array}{cccccccccccc}
\ldots & H_{dR}^{p-k+1}(U)^* \oplus H_{dR}^{p-k+1}(V)^* & \longrightarrow & H_{dR}^{p-k+1}(U \cap V)^* & \longrightarrow & H_{dR}^{p-k+1}(U \cup V)^* & \longrightarrow & H_{dR}^{p-k+1}(U)^* \oplus H_{dR}^{p-k+1}(V)^* & \longrightarrow & \ldots \\
& a_{U \cap V}^{p-k-1} \downarrow & & a_{U \cup V}^{p-k-1} \downarrow & & a_{U \cup V}^{p-k} \downarrow & & a_{U \cup V}^{p-k} \downarrow & & a_{U \cup V}^{p-k+1} \downarrow \\
\ldots & H_{dR}^{k-1}(U) \oplus H_{dR}^{k-1}(V) & \longrightarrow & H_{dR}^{k-1}(U \cap V) & \longrightarrow & H_{dR}^{k}(U \cup V) & \longrightarrow & H_{dR}^{k}(U) \oplus H_{dR}^{k}(V) & \longrightarrow & \ldots
\end{array}
\]

(8.15)
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Notice that the above is induced by the pairing of the two Mayer-Vietoris long exact sequence, so that one has

\[
\int_{U \cup V} \delta \sigma_i \cdot \omega = \pm \int_{U \cup V} \sigma_i \cdot d\omega. \tag{8.17}
\]

By Poincaré duality for \( \mathbb{R}^{p|q} \) the maps \( \alpha_{U + \nu}^{k-1}, \alpha_{U - \nu}^k \) and \( \alpha_{U \cap V}^k \) are isomorphisms. It follows from five lemma that also \( \alpha_{U \cap V}^k \) is an isomorphism. The proof is then concluded working by induction on the cardinality of the covering. Namely, suppose Poincaré duality holds for supermanifolds admitting a covering by \( n \) opens sets at most and consider a manifold with a covering of \( n + 1 \) open sets, \( \{U_1, \ldots, U_{n+1}\} \): then by hypothesis Poincaré duality holds for the supermanifolds \( (\bigcup_{i=1}^n U_i) \cap U_{n+1}, U_{n+1} \) and \( U_{n+1} \cap U_i \), so reasoning exactly as above one conclude that Poincaré duality holds true also for \( \bigcup_{i=1}^n U_i \).

\( \square \)

**Remark 8.3.** The same is true also switching the compact support from integral to differential forms, i.e. we have a perfect pairing

\[
H_{Sp}^{p-n}(\mathcal{M}) \times H_{Sk}^{n}(\mathcal{M}) \rightarrow \mathbb{R} \tag{8.18}
\]

where we have used that \( \pi_0 \) \( \mathcal{M} \) is a bump function which integrate to one on \( \mathbb{R}^P \). The pairing integral then reads

\[
\int_{\mathbb{R}^{p|q}} \sigma \cdot \omega = \int_{\mathbb{R}^{p|q}} D(x) \theta_1 \ldots \theta_q \otimes \pi \partial_{z_1} \ldots \partial_{z_p} \cdot (dz_1 \ldots dz_p) \mathcal{B}_i(z_1, \ldots, z_p)
\]

\[
= \int_{\mathbb{R}^{p|q}} D(x) \theta_1 \ldots \theta_q \mathcal{B}_i(z_1, \ldots, z_p)
\]

\[
= \int_{\mathbb{R}^P} dz_1 \ldots dz_p \mathcal{B}_i(z_1, \ldots, z_p) = 1, \tag{8.19}
\]

9. **Different Perspectives: Forms and Integration on Total Space**

Given a supermanifold \( \mathcal{M} \), there exists a different point of view on the theory of forms and integration on \( \mathcal{M} \), which highlights the role of the total space \( TTot_{\mathcal{M}} \) of the (parity-shifted) tangent bundle \( TTot_{\mathcal{M}} \rightarrow \mathcal{M} \) of \( \mathcal{M} \). This point of view has been introduced since the early days by Bernstein and Leites [12], Voronov and Zorich [66], Gaiduk, Khudaverdian and Schwarz [36]. Recently Witten has provided a terse review with applications to string theory in sight in [74], inspired by the work of Belopolsky [7]. Further, Castellani, Catenacci, Grassi and collaborators have worked extensively on the formalism and its applications to supergravity and superstrings [18–20]. In this
transition functions of \( \Omega^1 \Omega \)

Let us now study the geometry of the cotangent sheaf \( \Omega \)

forms are indeed a specific subclass - actually a subalgebra - of pseudodifferential forms.

for short.

as above. A section of \( T \) differential forms on \( \text{Pseudodifferential Forms} \)

Definition 9.4

even \( X \) be the corresponding charts on \( \text{Definition 9.4} \)

notice that asking that sections of the structure sheaf \( \mathcal{O} \)

functions are allowed. In particular, it makes sense to consider -

dependence on the fiber coordinates.

Remark 9.3. Let \( \mathcal{M} \) be a real supermanifold. The above
description of the (real) supermanifold \( \text{Remark 9.3} \)

section we provide a different exposition, which an eye to the global
gometry of the total tangent space supermanifold.

Definition 9.1 (Tot(\( \Pi \text{T}_{\mathcal{M}} \)) and Pseudoforms). Let \( \mathcal{M} \) be a smooth supermanifold of dimension \( p|q \) and let \( \Pi T_{\mathcal{M}} \) be its parity-shifted tangent sheaf. We define \( \mathcal{T}_{\mathcal{M}} := \text{Tot}(\Pi T_{\mathcal{M}}) \) as the \( p + q|p + q \)
dimensional supermanifold given as a ringed space by the pair \( ([\mathcal{T}_{\mathcal{M}}, \mathcal{O}_{\mathcal{T}_{\mathcal{M}}}]) \), where \( [\mathcal{T}_{\mathcal{M}}, \mathcal{O}_{\mathcal{T}_{\mathcal{M}}}] := \bigcup_{x \in \mathcal{M}}(\Pi T_{\mathcal{M}}, x) \), i.e. the total space of the even part of the (parity-shifted) tangent sheaf, and where sections of \( \mathcal{O}_{\mathcal{T}_{\mathcal{M}}} \) allows any dependence on the fiber coordinates.

Remark 9.2. Locally, the supermanifold \( \mathcal{T}_{\mathcal{M}} \) admits the following description. Let \( (U, x_a) \) be a local chart for the \( p|q \)-dimensional manifold \( \mathcal{M} \), where the index \( a \) runs on even and odd local coordinates. Then \( (\pi^{-1}(U), x_a, X_a) \) where \( X_a := dx_a \), gives a local chart for \( \mathcal{T}_{\mathcal{M}} \). We stress that \( X_a \) is now seen as a local coordinate for \( \mathcal{T}_{\mathcal{M}} \), better than a section of a vector bundle on \( \mathcal{M} \).

Given two charts \( (U, x_a) \) and \( (V, z_b) \) on \( \mathcal{M} \) with \( U \cap V \neq \emptyset \), we let \( (\pi^{-1}(U), x_a) \) and \( (\pi^{-1}(V), z_b) \) be the corresponding charts on \( \mathcal{T}_{\mathcal{M}} \). Then we have the obvious transition functions for \( \mathcal{T}_{\mathcal{M}} \):

\[
x_a = z_b(x), \quad X_a = Z_b \left( \frac{\partial x_a}{\partial z_b} \right).
\]

(9.1)

In the following, if no confusion occurs and as to conform with recent literature, we will denote the local coordinates \( X_a \) of the supermanifold \( \mathcal{T}_{\mathcal{M}} \) simply as \( dx_a \), see [76].

Remark 9.3. Let \( \mathcal{M} \) be a real supermanifold. The above local description of the (real) supermanifold \( \mathcal{T}_{\mathcal{M}} \) via charts allows to represent local sections of the sheaf \( \mathcal{O}_{\mathcal{T}_{\mathcal{M}}} \) over an open set \( \pi^{-1}(U) \) in terms of the local coordinates \( x_a \) and \( X_a \) as a function \( f(x_a, X_a) \). Notice that in the smooth category also generalized and transcendental functions are allowed. In particular, it makes sense to consider - for example - a transcendental dependence on the even coordinates \( X_a \)'s. For example, considering the \( 0|1 \)-dimensional real supermanifold \( \mathbb{R}^{0|1} \) described by an odd coordinate \( \theta \), it makes sense for the \( 1|1 \)-dimensional supermanifold \( \mathcal{T}\mathbb{R}^{0|1} \) to consider sections of its structure sheaf \( \mathcal{O}_{\mathcal{T}\mathbb{R}^{0|1}} \) of the form

\[
f(\theta, d\theta) = \exp(d\theta), \quad g(\theta, d\theta) = \log(d\theta), \quad h(\theta, d\theta) = \sin(d\theta),
\]

(9.2)

where \( d\theta = X \) is the even coordinate of the \( 1|1 \) dimensional supermanifold \( \mathcal{T}\mathbb{R}^{0|1} \). In this respect, notice that asking that sections of the structure sheaf \( \mathcal{O}_{\mathcal{T}_{\mathcal{M}}} \) have polynomial dependence on the even \( X_a \) is equivalent to set \( \mathcal{O}_{\mathcal{T}_{\mathcal{M}}} := \Omega^*_\mathcal{M} \), i.e. the sections of the structure sheaf of \( \mathcal{T}_{\mathcal{M}} \) are ordinary differential forms on \( \mathcal{M} \). This remarks leads to the following definition [12, 53, 76].

Definition 9.4 (Pseudodifferential Forms). Let \( \mathcal{M} \) be a real supermanifold and let \( \mathcal{T}_{\mathcal{M}} \) be defined as above. A section of \( \mathcal{O}_{\mathcal{T}_{\mathcal{M}}} \) is said to be a pseudodifferential forms on \( \mathcal{M} \), or a pseudoform on \( \mathcal{M} \) for short.

The above definition is justified by the previous remark [43], which shows how ordinary differential forms are indeed a specific subclass - actually a subalgebra - of pseudodifferential forms.

Let us now study the geometry the geometry of the cotangent sheaf \( \Omega^1 \mathcal{T}_{\mathcal{M}} \) of the supermanifold \( \mathcal{T}_{\mathcal{M}} \). This is a locally-free sheaf of \( \mathcal{O}_{\mathcal{T}_{\mathcal{M}}} \)-modules of rank \( p + q|p + q \). The transition functions of \( \Omega^1 \mathcal{T}_{\mathcal{M}} \) are easily characterized, thanks to [9, 14].

Lemma 9.5 (Transition Functions of \( \Omega^1 \mathcal{T}_{\mathcal{M}} \)). Let \( \mathcal{T}_{\mathcal{M}} \) be defined as above and let \( (dx_a, dp_a) \) and \( (dz_b, dq_b) \) two local bases of \( \Omega^1 \mathcal{T}_{\mathcal{M}} \) on the open sets \( \pi^{-1}(U) \) and \( \pi^{-1}(V) \) on \( \mathcal{M} \) with \( U \cap V \neq \emptyset \). The transition functions of \( \Omega^1 \mathcal{T}_{\mathcal{M}} \) read

\[
dx_a = dz_b \left( \frac{\partial x_a}{\partial z_b} \right), \quad dX_a = dz_b \left( \frac{\partial X_a}{\partial z_b} \right) + (-1)^{|z_b|+1} z_b d \left( \frac{\partial x_a}{\partial z_b} \right).
\]

(9.3)

(9.4)

Proof. The transition functions of the \( dx_a \)'s are obvious. For the transition functions of the \( dX_a \)'s one first observe that

\[
dx_a = dz_b \left( \frac{\partial X_a}{\partial z_b} \right) + dZ_b \left( \frac{\partial X_a}{\partial Z_b} \right).
\]

(9.5)
The first summand in (9.5) reads
\[ dz_b \left( \frac{\partial X_a}{\partial z_b} \right) = dz_b \frac{\partial}{\partial z_b} \left( Z_c \frac{\partial x_a}{\partial z_c} \right) = (-1)^{|z_b|+1} Z_b d \left( \frac{\partial x_a}{\partial z_b} \right), \]
(9.6)
The second summands in (9.5) reads
\[ dz_b \left( \frac{\partial X_a}{\partial Z_b} \right) = dz_b \frac{\partial}{\partial Z_b} \left( Z_c \frac{\partial x_a}{\partial z_c} \right) = dZ_b \left( \frac{\partial x_a}{\partial z_c} \right), \]
(9.7)
which concludes the proof.

The previous lemma has an immediate consequence on the global geometry of \( \Omega^1_{T_M} \).

**Lemma 9.6 (\( \Omega^1_{T_M} \) as Extension).** Let \( T_M \) be defined as above. Then the canonical exact sequence
\[
\begin{array}{c}
0 \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \end{array}
\pi^*\Omega^1_{T_M} \rightarrow \Omega^1_{T_M} \rightarrow \Omega^1_{T_M/M} \rightarrow 0
\]
(9.8)
induces the isomorphism of locally-free sheaves \( \Omega^1_{T_M/M} \cong \pi^*\tau^{\ast}_{M} \). In particular, \( \Omega^1_{T_M} \) is defined as the extension of locally-free sheaves
\[
\begin{array}{c}
0 \\
\rightarrow \\
\rightarrow \\
\rightarrow \end{array}
\pi^*\Omega^1_{M} \rightarrow \Omega^1_{T_M} \rightarrow \pi^*\tau^{\ast}_{M} \rightarrow 0.
\]
(9.9)

**Proof.** It follows immediately from the form of the transition functions given in the previous Lemma upon noticing that \( dx_a \) and \( dX_a \) have opposite parity, by the very definition of \( X_a \).

**Remark 9.7.** On a general ground, extensions as in (9.5) are classified by the first Ext-group
\[
\text{Ext}^1(\pi^*\tau^{\ast}_{M}, \pi^*\Omega^1_{M}) \cong H^1(|\mathcal{M}|, \mathcal{H}\text{om}(\pi^*\tau^{\ast}_{M}, \pi^*\Omega^1_{M})).
\]
(9.10)
In the case \( \mathcal{M} \) is smooth and \( T_M \) is the associated smooth supermanifold defined as above, the extension is always (non-canonically) split. More in particular, one has a non-canonical isomorphism \( \Omega^1_{T_M} \cong \pi^*\Omega^1_{M} \oplus \pi^*\tau^{\ast}_{M} \). Locally, this means that it is always possible to find a covering by open sets in which the second summand in equation (9.4) is zero. This result follows from the fact that in the smooth category every sheaf is fine, thus soft and acyclic, and from an application of Leray-Serre spectral sequence.

On the other hand, in the case of a complex supermanifold \( \mathcal{M} \) and the related complex supermanifold \( T_M \), the extension might indeed be non-split. Studying the related cohomology class is a non-trivial and interesting problem.

In any case, both for smooth and complex supermanifold, the above characterization of \( \Omega^1_{T_M} \) as an extension allows to easily prove a fundamental property of \( T_M \).

**Lemma 9.8 (\( \text{Ber}(T_M) \cong \pi^*\mathcal{O}_M \)).** Let \( T_M \) be defined as above. Then there is a canonical isomorphism
\[
\text{Ber}(T_M) \cong \pi^*\mathcal{O}_M.
\]
(9.11)

**Proof.** From the extension exact sequence (9.3) and the definition of the Berezinian it follows that
\[
\text{Ber}(T_M) \cong \pi^* (\text{Ber}(\mathcal{M}) \otimes \text{Ber}^*(T_M^\ast)).
\]
(9.12)
Since for any locally-free sheaf \( \mathcal{E} \) one has \( \text{Ber}(\Omega\mathcal{E}) \cong \text{Ber}^*(\mathcal{E}) \), then in particular \( \text{Ber}^*(T_M^\ast) \cong \text{Ber}(\Omega\Omega\mathcal{T}_M^\ast) \). It follows that, by definition of Berezinian sheaf of a supermanifold, one has \( \text{Ber}(\Omega\Omega\mathcal{T}_M^\ast) \cong \text{Ber}^*(\mathcal{M}) \). Then one has
\[
\text{Ber}(T_M) \cong \pi^* (\text{Ber}(\mathcal{M}) \otimes \text{Ber}^*(\mathcal{M})) \cong \pi^*\mathcal{O}_M,
\]
(9.13)
concluding the proof.

**Remark 9.9.** The above result could have also been proved upon using the explicit form of the transition functions for \( \Omega^1_{T_M} \), as given in Lemma 9.5.
Remark 9.10 (TM is a “Calabi-Yau” Supermanifold). While in general there is no natural choice for a section of the Berezinian sheaf on the supermanifold $\mathcal{M}$, instead it is a crucial consequence of lemma 9.8 that the associated supermanifold $TM$ comes endowed with a canonical Berezinian or canonical volume form, which is independent on the choice of coordinates on $\mathcal{M}$, and therefore on $TM$. When working in the complex holomorphic category one would say that the complex supermanifold $TM$ is a Calabi-Yau supermanifold. We will denote the canonical volume form on $TM$ with

$$D_{TM}(x, X) \in Ber(TM).$$

(9.14)

Remark 9.11 (Integration on $TM$). Thanks to the existence of a canonical volume form for $TM$, any function on $TM$ can be mapped naturally to the Berezin integral on the supermanifold $TM$, i.e.

$$O_{TM} \ni f(x, X) \longmapsto \int_{TM} f(x, X)D_{TM}(x, X) \in \mathbb{R}.$$  

(9.15)

It is important to notice, though, that the above integral can be divergent, thus making the mapping ill-defined. This is related to the fact the supermanifold $TM$ is in general not compact, since the fibers of $TM$ above every point of the $p|q$-dimensional supermanifold $\mathcal{M}$ are isomorphic to $\mathbb{R}^{p|q}$, see also [53].

An important class of functions in $O_{TM}$ which are in general not integrable over $TM$ are differential forms on $\mathcal{M}$, i.e. functions on $TM$ which have polynomial dependence on the fiber coordinates. To see this consider the example in Remark 9.3. We take the superpoint $\mathbb{R}^{0|1}$ and the related 1\|1-dimensional supermanifold $T\mathbb{R}^{0|1}$. The differential form $P(\theta, d\theta) = \theta d\theta \in \Omega^{1}_{1\|1}$ is indeed an element of $O_{T\mathbb{R}^{0|1}}$ having polynomial dependence on the fibers, and whose integral is divergent since the ordinary Riemann-Lebesgue integral on the even variable $d\theta$ is clearly divergent,

$$\int_{T\mathbb{R}^{0|1}} D(\theta)d\theta \theta d\theta = \int_{\mathbb{R}^{0|1}} D(\theta)d\theta \int_{\mathbb{R}^{1\|0}} D(d\theta)d\theta.$$  

(9.16)

On the other hand, the function $f(\theta, d\theta) = \theta e^{-(d\theta)^2}$ is integrable on $T\mathbb{R}^{0|1}$, and indeed one has

$$\int_{T\mathbb{R}^{0|1}} D(\theta)d\theta \theta e^{-(d\theta)^2} = \int_{\mathbb{R}^{0|1}} D(\theta)d\theta \int_{\mathbb{R}^{1\|0}} D(d\theta)e^{-(d\theta)^2} = \sqrt{\pi}.$$  

(9.17)

This justifies the following definition, which distinguish among classes of functions on $TM$ in relation with their Berezin integral [11, 53, 76].

**Definition 9.12 (Integrable Pseudodifferential Form).** Let $\mathcal{M}$ be a real supermanifold and let $TM$ be defined as above. We say that the pseudodifferential form $f \in O_{TM}$ is integrable if its Berezin integral on $TM$ is convergent, i.e. $\int_{TM} D_{TM}f < \infty$. We will denote the sheaf of integrable pseudodifferential forms with $O^\text{int}_{TM}$.

Remark 9.13 (Integral over the Fibers). Let us now restrict to integrable pseudodifferential forms so that the integral (9.15) makes sense, i.e. we have a well-defined map

$$O^\text{int}_{TM} \ni f_{st}(x, X) \longmapsto \int_{TM} f_{st}(x, X)D_{TM}(x, X) \in \mathbb{R}.$$  

(9.18)

This integral can be understood as a (Berezin) integral along the fibers $f_{st} := \pi^{-1}(x)$ of the fibration $TM \xrightarrow{\pi} \mathcal{M}$, i.e.

$$\int_{f_{st}} \equiv \pi_{st} : O^\text{int}_{TM} \longrightarrow Ber(\mathcal{M}),$$  

(9.19)

followed by the “usual” Berezin integral on the base supermanifold $\mathcal{M}$, so that one has a map

$$O^\text{int}_{TM} \xrightarrow{\pi_{st}} Ber_{\pi_{st}}(\mathcal{M}) \xrightarrow{f_{st}} \mathbb{R}$$  

(9.20)

Loosely speaking, defining integration on the total space $TM$ via the composition of the above maps, i.e. $\int_{TM} := \int_{\mathcal{M}} \circ \pi_{st}$, has to be seen as a sort of factorization of the integral over the total space as an integral in the fiber, or vertical directions followed by an integral over the base [14, 76].
Nonetheless, it is important to stress that the map $\pi_*$ corresponds to a Berezin integral. We choose a certain trivialization over an open set $U$ in the base of a $p$-dimensional real supermanifold $M$ and define the corresponding even and odd coordinate $x = y_i | \theta_\alpha$ and $X := d\theta_\alpha | dy_i$ on the $q$-dimensional fibers over $U$ for $i = 1, \ldots, p$ and $\alpha = 1, \ldots, q$. By anticommutativity of the $dx$'s a function $f_{\text{int}}(x, X) \in \mathcal{O}_{\text{T}M}^\ast(U)$ can be written as

$$f_{\text{int}}(x, X) = dx_{i_1} \ldots dx_{i_\ell} F_i(x_i, d\theta_\alpha | \theta_\alpha),$$

for $\ell \leq p$ and some compactly supported, integrable function $F_i$ in the coordinates $x$'s and $d\theta$'s. The map $\pi_*$ is thus defined so that it acts as a true Berezin integral on the odd fiber coordinates $dx$'s, i.e., it yields

$$\pi_*(f_{\text{int}}(x, X)) = \begin{cases} 0 & \ell < p \\ \left(\int_{\mathbb{R}^q} F_i(x_i, d\theta_\alpha | \theta_\alpha) d\mu(d\theta_\alpha)\right) \otimes \mathcal{D}_M(x_i | \theta_\alpha) & \ell = p, \end{cases}$$

where $d\mu(d\theta_\alpha)$ is a (Lebesgue) measure for the real even variables $d\theta$'s. Notice the result of the integral is indeed a function in $\mathcal{O}_{\text{M}, \ast}(U)$, depending on the coordinates $x$'s and $\theta$'s on the base manifold.

It can be proved that given a pair of open sets $U, V$ in $M$ with $U \cap V \neq \emptyset$ and defining $f_{\text{int}}^\ell \in \mathcal{O}_{\text{T}M}^\ast(U)$ and $f_{\text{int}}^\ell \in \mathcal{O}_{\text{T}M}^\ast(V)$, then $\pi_* f_{\text{int}}^\ell = \pi_*(f_{\text{int}}^\ell)$ in the intersection. It follows that for a certain open covering $\{U_i\}_{i \in I}$ and the related trivializations, one finds that $\{\pi_* f_{\text{int}}^\ell\}_{i \in I}$ glue together, yielding a section $\pi_* f_{\text{int}} \in \text{Ber}_r(M)$, see [14, 76].

Remark 9.14. In light of the recent physics-oriented literature, a brief remark on notation is in order here. In the influential review [76] Witten denotes a section of the Berezinian of a supermanifold by $\text{Ber}_r(M)$, see [14, 76]. Witten denotes a section of the Berezinian of a supermanifold $M$ as $[dx d\theta] \in \text{Ber}_r(M)$ for a choice of coordinate $x|\theta$ on $M$. Accordingly, the canonical volume form on $\text{T}M$ is denoted with

$$[dx d(d\theta) d\theta d(dx)] \in \text{Ber}(\text{T}M),$$

as to reminds that $d\theta$'s and the $dx$'s are now seen as coordinates for $\text{T}M$. In other words the expressions $d(dx)$'s and $d(d\theta)$'s are just the symbols corresponding to the $dx$'s in the notation of lemma 9.7. As such, one should not read them as the application of the de Rham differential $d$ on a local basis of the cotangent bundle, which would be clearly vanishing as $d \circ d = 0$.

9.1. Special Class of Integrable Pseudoforms: Distributions on the Fibers. In the spirit of supersymmetric localization [61] it is convenient to focus on a particular class of pseudoforms in $\mathcal{O}_{\text{T}M}^\ast$, admitting only a particular dependence of the fiber coordinates $X = d\theta|dx_i$. These are the pseudoforms Belopolsky [5], Castellani, Catenacci and Grassi [18] and Witten [76] focus on, thought they have been introduced in the early days of the theory, see for example [12, 69].

Definition 9.15 (Delta Forms). Let $M$ be a real supermanifold and let $\text{T}M$ be defined as above. We call delta forms the class of integrable pseudoforms in $\mathcal{O}_{\text{T}M}^\ast$ whose dependence of the even fiber coordinates of $\text{T}M$ is distributional and supported at the origin. We denote the sheaf of delta forms by $\mathcal{O}_{\text{T}M}^\delta$. Given an open set $U$ in $M$, in the local trivialization of $\text{T}M$ over $U$ with coordinates $x = x_i | \theta_\alpha$ and $X = d\theta_\alpha | dx_i$, a delta form $\omega \in \mathcal{O}_{\text{T}M}^\delta$ can be written as

$$\omega_U(x, X) = \sum_{\ell = 0}^{\infty} F_{\ell}(x_i | \theta_\alpha)(dx_1)^{\ell_1} \ldots (dx_p)^{\ell_p} \delta^{\ell_1}(d\theta_1) \ldots \delta^{\ell_p}(d\theta_q),$$

for $\ell_j \geq 0$ and $\ell_k = (0, 1)$ and where $F_{\ell \in \mathcal{O}} \in \pi^\ast \mathcal{O}_M$. The expressions $\delta^{\ell_i}(d\theta_j)$ are Dirac’s delta distributions [18, 76] - and their derivative, when $\ell_j > 0$ - supported at the origin in the even real variable $d\theta_j$ of $\text{T}M$. They can be seen as linear functional acting on differential forms, instead of on ordinary functions: these kind of mathematical object are called de Rham current [33]. In this paper they will be treated in a formal algebraic fashion. The sheaf $\mathcal{O}_{\text{T}M}^\delta$ is endowed with several different structures, which interplay with the notion of integration on this particular class of pseudoforms. These structures are spelled out in the following remarks.
Remark 9.16 ($\mathcal{O}_M$-module structure of $\mathcal{O}^\delta_T M$). We first note that the sheaf $\mathcal{O}^\delta_T M$ is not a sheaf of algebras. This comes from the well-known fact from analysis that the product of two distributions is not well-defined. On the other hand, $\mathcal{O}^\delta_T M$ carries the structure of a sheaf of $\pi^*\mathcal{O}_M$-modules, as the multiplication of delta forms by sections coming from the supermanifold $M$ is well-defined.

Remark 9.17 ($\mathcal{D}^\delta _T M$-module structure of $\mathcal{O}^\delta_T M$). The sheaf of delta forms $\mathcal{O}^\delta_T M$ is a $\mathcal{D}$-module. More precisely, it carries the structure of $\mathcal{D}^\delta _T M$-module. Indeed, locally, the Clifford-Weyl superalgebra $\mathcal{C}\mathcal{W}_{\pi \mathbb{J}}(\mathbb{R})$ generated by $(d\theta_\alpha,\partial \theta_\alpha, dx_i, \partial_x_i)$, for $i=1,\ldots,p$ and $\alpha=1,\ldots,q$ with non trivial (super)commutation relations given by

$$\frac{\partial}{\partial \partial \theta_1}, \frac{\partial}{\partial \partial \theta_2} = \delta_{12}, \quad \left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\} = \delta_{ij}, \quad \text{(9.25)}$$

acts on sections in $\mathcal{O}^\delta_T M(U)$ according to the following definitions (given on monomials)

$$dx_i \cdot (dx_1)^{i_1} \ldots (dx_1)^{i_q} (dx_p)^{p_1} \delta^{q_1}(d\theta_1) \ldots \delta^{q_\ell}(d\theta_\ell)$$

$$= \delta_{x_i} \cdot (dx_1)^{i_1} \ldots (dx_1)^{i_q} (dx_p)^{p_1} \delta^{q_1}(d\theta_1) \ldots \delta^{q_\ell}(d\theta_\ell)$$

$$\frac{\partial}{\partial dx_i} \cdot (dx_1)^{i_1} \ldots (dx_1)^{i_q} (dx_p)^{p_1} \delta^{q_1}(d\theta_1) \ldots \delta^{q_\ell}(d\theta_\ell)$$

$$= \delta_{x_i} \cdot (dx_1)^{i_1} \ldots (dx_1)^{i_q} (dx_p)^{p_1} \delta^{q_1}(d\theta_1) \ldots \delta^{q_\ell}(d\theta_\ell)$$

$$d\theta_\alpha \cdot (dx_1)^{i_1} \ldots (dx_p)^{p_1} \delta^{q_1}(d\theta_1) \ldots \delta^{q_\ell}(d\theta_\ell)$$

$$= \delta_{x_i} \cdot (dx_1)^{i_1} \ldots (dx_p)^{p_1} \delta^{q_1}(d\theta_1) \ldots \delta^{q_\ell}(d\theta_\ell)$$

$$(dx_1)^{i_1} \ldots (dx_p)^{p_1} \delta^{q_1}(d\theta_1) \ldots \delta^{q_\ell}(d\theta_\ell)$$

$$= \delta_{x_i} \cdot (dx_1)^{i_1} \ldots (dx_p)^{p_1} \delta^{q_1}(d\theta_1) \ldots \delta^{q_\ell}(d\theta_\ell)$$

where again $\epsilon_i = \{0,1\}$ and $\ell_\alpha \geq 0$, for any $i=1,\ldots,p$ and any $\alpha=1,\ldots,q$. Notice that the action [28] can be seen as arising from “integration by parts”, and in particular one defines

$$d\theta_\alpha \cdot (dx_1)^{i_1} \ldots (dx_p)^{p_1} \delta^{q_1}(d\theta_1) \ldots \delta^{q_\ell}(d\theta_\ell)$$

This is can be seen as the “de Rham current” analog of usual relation that characterizes a Dirac delta distribution in functional analysis: working for simplicity over the real line $\mathbb{R}$, for $\delta_{x_0} \in \mathcal{D}'(\mathbb{R})$, we have

$$\delta_{x_0}(f) \equiv \int_{\mathbb{R}} f(x) \delta_{x_0} dx = f(x_0), \quad \text{(9.31)}$$

for any point $x_0 \in \mathbb{R}$ and any test function $f \in \mathcal{S}(\mathbb{R}^n)$ of Schwartz class. The algebraic way to $\{\delta^{\ell}(\cdot)\}_{\ell \geq 0}$ consist into endowing the $\mathcal{O}_M$-module $\{\delta^{\ell}(\cdot)\}_{\ell \geq 0}$ of Dirac delta and its derivatives with a $\mathcal{D}$-module structure, by defining the $\mathcal{D}$-action by

$$\frac{d}{dx} \cdot \delta^{\ell}(\cdot) = \delta^{\ell+1}(\cdot), \quad (x - a) \cdot \delta^{\ell}(\cdot) = (x_0 - a) \cdot \delta^{\ell}(\cdot) = 0. \quad \text{(9.32)}$$

In light of these, the previous [20]-[29] should not be surprising, as they are exactly the same relations, but given in the context of forms (or de Rham current) on a supermanifold.

Remark 9.18 ($\mathbb{Z}_2$-grading of $\mathcal{O}^\delta_T M$). There is a twist in the description of the $\mathbb{Z}_2$-grading of the sheaf $\mathcal{O}^\delta_T M$. Indeed, whereas the $d\theta'$s are even, the $\delta(d\theta')$'s and their derivatives are defined to be odd, so that their product has parity $q$. It follows that a generic monomial in the expression [24] for $\omega_U$ above has parity given by $q + |f| + \sum_{k=1}^p c_k$, where $f$ is assumed homogeneous in its $\mathbb{Z}_2$-degree. The reason behind the odd parity of the delta's can be seen by looking at the following integral

$$I = \int_{\mathbb{R}^{0|2}} \theta_1 \theta_2 \delta(d\theta_1) \delta(d\theta_2) \mathcal{D}(d\theta_\theta) \quad \text{(9.33)}$$

Thanks to the delta's, when applying $\pi_*$ as to integrate along the fibers $d\theta'$s, the integral is localized to the ordinary Berezin integral of $\theta_1 \theta_2$ over the superpoint $\mathbb{R}^{0|2}$, which yields 1. Exchanging $\theta_1 \leftrightarrow \theta_2$, would yield an integral equal to $-1$ instead, unless the transformation of $\delta(d\theta_1) \delta(d\theta_2)$
would correct it with $-1$. Formally, we thus say the delta’s anticommute, e.g. $\delta(d\theta_1)\delta(d\theta_2) = -\delta(d\theta_2)\delta(d\theta_1)$.

Remark 9.19 (Z-gradation of $\mathcal{O}_M^\delta$). A Z-gradation can also be introduced on $\mathcal{O}_M^\delta$. Considering a section $\omega \in \mathcal{O}_M^\delta$ trivialized as in (9.24), its Z-degree is given by

$$\deg_Z(\omega) = \sum_{k=1}^p e_k - \sum_{j=1}^q \ell_j,$$

(9.34)

which implies that the Z-degree of a delta form is such that $-\infty < \deg_Z(\omega) < p$. This is stable under change of coordinates, and therefore the definition of the Z-degree is well-posed, as we shall see shortly. We will call the sub-sheaf the degree-$k$ delta forms $\mathcal{O}_M^{\delta(k)}$.

Notice that delta forms in degree $k \leq p$ can be generated via the $\mathcal{D}_\Pi_\omega$-module structure introduced above starting from the (unique, up to a multiplication by a section of $\mathcal{O}_M$) delta form in degree $p$, given by

$$\omega_{(p)}(x, d\theta|\theta, dx) = dx_1 \ldots dx_p \delta(d\theta_1) \ldots \delta(d\theta_q),$$

(9.35)

in a certain choice of coordinates. Then, sections $\omega_{(k)} \in \mathcal{O}_M^{\delta(k)}$ of degree $k < p$ are obtained by acting with differential operators of order $k$ on $\omega_{(p)}$. These are locally constructed from the $\partial_d\theta$’s and $\partial_d\psi$’s. In particular, we will have that locally

$$\frac{\partial^{|I|}}{\partial d\theta^I \partial dx^K} \cdot \omega_{(p)}(x, d\theta|\theta, dx) \in \mathcal{O}_M^{\delta(p-|I|)},$$

(9.36)

where $I, J$ and $K$ are multi-indices such that $I = (J, K)$ so that $|I| = |J| + |K|$. Notice that $K$ is such that $|K| \leq p$, since the $\partial_d\psi$’s are anticommuting, while $J$ can be of any order $|J| \geq 0$. On the other hand, given a form in degree $n < p$, differential forms of degree $k$ acts via the $\mathcal{D}_\Pi_\omega$-action defined above by raising the degree of the delta form by $k$. This means that

$$d\theta^I dx^K \cdot \omega_{(n)}(x, d\theta|\theta, dx) \in \mathcal{O}_M^{\delta(n+|I|)},$$

(9.37)

if again $|I| = |J| + |K|$, for some multi-indices $I = (J, K)$. Clearly, the (local) $\mathcal{D}_\Pi_\omega$-action can also result in annihilating the delta form, both in the case of (9.36) and (9.37), as it is clear from the (9.26)-(9.29).

Notice that in the above picture delta forms can be seen locally as element of a Fock space, which is constructed via the $\mathcal{D}_\Pi_\omega$-action starting from a pivot $\omega_{(p)} \in \mathcal{O}_M^{\delta(p)}$ “state”, see (9.24).

Further, it is to be observed that, given the above definition, delta forms exist in the very same degrees as integral forms in $\Sigma_M$, where again $-\infty < n < p$. This does not happen by chance, indeed it is possible to prove that there exists an isomorphism between integral and delta forms. Before we see this, a remark on the transformation properties of the delta’s is in order.

Remark 9.20 (Transformations Properties of the Delta’s). It has to be stressed that an expression involving any number of delta’s which is lesser than the odd dimension $q$ of base supermanifold $M$, e.g. $\delta(d\theta_1) \ldots \delta(d\theta_{q-1})$ does not make sense, as its transformation properties are not well-defined. To see this in an informal way, let us consider a generic 1/2 dimensional supermanifold and look at the transformation properties of a single delta $\delta(d\theta)$ under a change or coordinates $d\theta' = \alpha d\theta + \beta d\psi + \gamma dx$, for $\alpha, \beta, \gamma \neq 0$, $\alpha, \beta$ even and $\gamma$ odd. First, we observe that since $dx$ is “infinitesimal” (as it is nilpotent) we can formally expand about it in the following fashion

$$\delta(d\theta') = \delta(\alpha d\theta + \beta d\psi + \gamma dx) = \delta(\alpha d\theta + \beta d\psi) + \gamma dx \delta'((\alpha d\theta + \beta d\psi)).$$

(9.38)

Now the problem is to make sense out of the expression $\delta(\alpha d\theta + \beta d\psi)$ and its derivative. Keep working formally, focusing on the first summand, one could get to the following expression

$$\delta(\alpha d\theta + \beta d\psi) = \delta \left( \alpha \left( d\theta + \frac{\beta}{\alpha} d\psi \right) \right) = \frac{1}{\alpha} \left( \delta(d\theta) + \frac{b}{\alpha} d\psi \delta'(d\theta) + \ldots \right)$$

$$= \sum_{k=0}^{\infty} \frac{\beta^k}{k! \alpha^{k+1}} (d\psi)^k \delta^k(d\theta),$$

(9.39)

which would suggest that, if $\delta(d\theta)$ was as a section, the corresponding sheaf would not be locally-free of finite rank. But clearly, this is just the tip of the iceberg, as there are more inconsistencies:
in the first place $d\theta$ and $d\psi$ are honest even variables on $\mathbb{T}\mathcal{M}$, so that the expression $\alpha d\theta$ and $\beta d\psi$ are not at all nilpotent nor infinitesimal. Further, also forgetting about this, one might have chosen to expand about $d\psi$ instead of $d\theta$.

On the other hand, by definition, all of the $q$ delta’s (or their derivatives) are required to appear in sections of $\mathcal{O}^k_{\mathbb{T}\mathcal{M}}$. We shall see in the next theorem that this requirements leads to well-defined transformation properties.

**Theorem 9.21** (Delta Forms are Isomorphic to Integral Forms). Let $\mathcal{M}$ be a real supermanifold of dimension $p|q$. Then the sheaf $\mathcal{O}^k_{\mathbb{T}\mathcal{M}}$ of delta forms of degree $k$ is isomorphic to the sheaf $\Sigma^k_\mathcal{M}$ of integral forms of degree $k$ for any $k \leq p$.

**Proof.** To prove the statement is enough to verify that the transformation properties of the generating sections do coincide. Let us start from degree $p$, corresponding to $\Sigma^p_\mathcal{M} \cong \text{Ber}(\mathcal{M})$. Without loss of generality, we can restrict ourselves to consider coordinate transformations $\varphi$ of $\mathcal{M}$ of the split type, i.e. $x'_i = f_i(x)$ and $\theta'_\alpha = \sum_\beta g_{\alpha\beta}(x)\theta_\beta$. The only degree $p$ delta form in $\mathcal{O}^k_{\mathbb{T}\mathcal{M}}$ is given in a certain trivialization by

$$\omega_{(p)} = dx_1 \ldots dx_p \delta(d\theta_1) \ldots \delta(d\theta_q).$$

(9.40)

The part $dx_1 \ldots dx_p$ contributes with the determinant of the Jacobian of the change of coordinates $x'_i = f_i(x)$, while the Dirac-delta part contributes in the following way

$$\delta(d\theta'_1) \ldots \delta(d\theta'_q) = \delta \left( \sum_{\beta=1}^q (\partial_{\theta_\beta} f'_i) d\theta_\beta + \sum_{i=1}^p (\partial_{x_i} f'_i) dx_i \right) \ldots \delta \left( \sum_{\beta=1}^q (\partial_{\theta_\beta} f'_q) d\theta_\beta + \sum_{i=1}^p (\partial_{x_i} f'_q) dx_i \right).$$

The part proportional to $dx$’s does not contribute by (9.38), due to the presence of $dx_1 \ldots dx_p$, so that one is left with

$$\delta(d\theta'_1) \ldots \delta(d\theta'_q) = \delta \left( \sum_{\beta=1}^q (\partial_{\theta_\beta} f'_q) d\theta_\beta \right) \ldots \delta \left( \sum_{\beta=1}^q (\partial_{\theta_\beta} f'_q) d\theta_\beta \right)$$

$$= \left( \det \left( \frac{\partial f'_i}{\partial \theta_\beta} \right) \right)^{-1} \delta(d\theta'_1) \ldots \delta(d\theta'_q).$$

(9.41)

upon using the properties of the Dirac’s delta distributions. Putting the pieces together, we find

$$\omega'_{(p)} = \det \left( \frac{\partial f'}{\partial x} \right) \det \left( \frac{\partial f'_q}{\partial \theta_\beta} \right)^{-1} \omega_{(p)} = \text{Ber}(\text{Jac}(\varphi)) \omega_{(p)}.$$ (9.42)

This settle the degree $p$ case. For degree lower than $p$ it is enough to observe that, following remark 9.19, delta forms in $\mathcal{O}^{(p-1)}_{\mathbb{T}\mathcal{M}}$ are obtained via the $D_{\Omega^1_{\mathbb{T}\mathcal{M}}}$-action of differential operators of order $k$ on the above section $\omega_{(p)} \in \Sigma^p_\mathcal{M} \cong \mathcal{O}^{(p)}_{\mathbb{T}\mathcal{M}}$. In particular for degree $p - 1$, working locally, we have an action of $\partial_{x_i}$’s and $\partial_{\theta_\beta}$’s: these can be as linear maps acting on $\pi^*\Omega^1_{\mathcal{M}}$, hence they belong to $(\pi^*\Omega^1_{\mathcal{M}})^* \cong \pi^*\Pi\mathcal{T}\mathcal{M}$. It follows that the transformation of a section $\omega_{(p-1)} \in \mathcal{O}^{(p-1)}_{\mathbb{T}\mathcal{M}}$ will be given by

$$\omega'_{(p-1)} = \text{Jac}(\varphi)^{\Pi} \otimes \text{Ber}(\text{Jac}(\varphi)) \omega_{(p)}.$$ (9.43)

where $\text{Jac}(\varphi)^{\Pi}$ is the parity-transpose of the Jacobian of the change of coordinates, which identifies the transition functions of the locally-free sheaf $\Pi\mathcal{T}\mathcal{M}$. This yields the isomorphism $\mathcal{O}^{(p-1)}_{\mathbb{T}\mathcal{M}} \cong \text{Ber}(\mathcal{M}) \otimes \Pi\mathcal{T}\mathcal{M} = \Sigma^{p-1}_\mathcal{M}$ : explicitly $\delta_{\partial_{x_i}} \mapsto \pi \partial_{x_i}$ and $\delta_{\partial_{\theta_\beta}} \mapsto \pi \partial_{\theta_\beta}$, for any $i = 1, \ldots, p$ and $\alpha = 1, \ldots, q$. In the very same fashion, higher degree differential operators are identified with sections on $\mathbb{S}^k\Pi\mathcal{T}\mathcal{M}$, thus showing that $\mathcal{O}^{(p-k)}_{\mathbb{T}\mathcal{M}} \cong \text{Ber}(\mathcal{M}) \otimes \mathbb{S}^k\Pi\mathcal{T}\mathcal{M} = \Sigma^{p-k}_\mathcal{M}$.

**Remark 9.22** (Integration theory). It follows from the previous theorem that integration on supermanifolds via integral forms $\Sigma^k_\mathcal{M}$ parallels integration theory via delta forms $\mathcal{O}^{(k)}_{\mathbb{T}\mathcal{M}}$ on tangent supermanifolds. While this is clear in the case of Berezinian, which accounts for integration on the full supermanifold, it might be helpful to consider a codimension $1|0$ example. To this end, let us consider the supermanifold $\mathbb{R}^{1|2}$, with a system of coordinates given by $x|\theta_1, \theta_2$. The integral form
\( \sigma_0 = D(\theta_1, \theta_2) \theta_1 \theta_2 \otimes \pi \partial_x \in H^0_{\delta}(M) \) can be integrated over a codimension 1|0 sub-supermanifold of \( M \). In particular, we consider

\[
\mathcal{N} := \{ x| \theta_1, \theta_2 \in \mathbb{R}^{1|2} : x = 0 \} \subset \mathbb{R}^{1|2}.
\]  

The integral of \( \sigma_0 \) over \( \mathcal{N} \) is defined via the Poincaré dual of \( \mathcal{N} \) and using the pairing \( \langle, \rangle \) between integral and differential forms, i.e.

\[
\int_{\mathcal{N}} \sigma_0 := \int_M \sigma_0 \cdot \omega_{\mathcal{N}},
\]

where \( \omega_{\mathcal{N}} \in H^1_{\delta}(M) \) is given by \( \omega_{\mathcal{N}} = \delta(x)dx \), with \( \delta(x) \) a Dirac delta distribution centered in zero. Plugging these into (9.45) one gets

\[
\int_{\mathcal{N} \subset \mathbb{R}^{1|2}} \sigma_0 = \int_{\mathbb{R}^{1|2}} D(x| \theta_1 \theta_2 \theta_1 \theta_2 \otimes \pi \partial_x \cdot \delta(x)dx = \int_{\mathbb{R}^{1|2}} D(x| \theta_1, \theta_2) \theta_1 \theta_2 \delta(x) = 1,
\]

where we have used the duality pairing between \( \pi \partial_x \) and \( dx \), given by \( dx(\pi \partial_x) = 1 \).

In a similar fashion, via delta forms, one first observe that \( \sigma_0 \) corresponds to the delta form \( \theta_1 \theta_2 \delta(d \theta_1) \delta(d \theta_2) \). Now, instead of duality, the integral uses the \( \Omega^*_{M} \)-module structure (or \( D\Omega^*_M \)-structure), multiplying \( \sigma_0 \in \mathcal{O}_{\mathcal{T}M}^{(0)} \) in the delta representation by the Poincaré dual form \( \omega_{\mathcal{N}} \) of \( \mathcal{N} \), which yields a delta form in \( \mathcal{O}_{\mathcal{T}M}^{(1)} \). More precisely, one has

\[
\int_{\mathcal{N} \subset \mathbb{R}^{1|2}} \sigma_0 = \int_{\mathbb{R}^{1|2}} D(x| \theta_1 \theta_2 \theta_1 \theta_2, dx) \theta_1 \theta_2 \delta(d \theta_1) \delta(d \theta_2) \delta(x)dx = \int_{\mathbb{R}^{1|2}} D(x| \theta_1, \theta_2) \theta_1 \theta_2 \delta(x) = 1.
\]

Notice that the integral of \( \sigma_0 \) on \( \mathcal{N} \) only depends on the (co)homology on \( \mathcal{N} \). Let us indeed consider the sub supermanifold

\[
\tilde{\mathcal{N}} := \{ x| \theta_1, \theta_2 \in \mathbb{R}^{1|2} : x + \theta_1 \theta_2 = 0 \} \subset \mathbb{R}^{1|2},
\]

which is in the same homology class of \( \mathcal{N} \). We have \( \omega_{\tilde{\mathcal{N}}} = \delta(x)dx + d \theta_1 \theta_2 - \theta_1 d \theta_2 \) and it is easy to see that

\[
\int_{\tilde{\mathcal{N}}} \sigma_0 = \int_M \sigma_0 \cdot \omega_{\tilde{\mathcal{N}}} = \int_M \sigma_0 \cdot (\omega_{\mathcal{N}} + d \eta) = \int_M \sigma_0 \cdot \omega_{\mathcal{N}} = \int_{\mathcal{N}} \sigma_0,
\]

where \( d \eta = d(\theta_1 \theta_2) = d \theta_1 \theta_2 - \theta_1 d \theta_2 \). Notice that this consideration is totally general - and not limited to the present example, as it relies on Stokes theorem \[76\] - indeed any summand of the kind \( \sigma_0 \cdot d \eta \) is exact and does not contribute to the Berezin integral - and Poincaré duality \[77\].

**Remark 9.23** (Delta Forms and Pseudoforms). Delta forms have been defined by requiring that all of the coordinate \( d \theta \)'s have distributional dependence of Dirac delta type. It was this very requirement that allowed to prove theorem 9.21 above, thus showing that the formalism of delta forms is equivalent to that of integral forms that have previously defined.

Nonetheless, the above requirement can be relaxed to a less stringent one, allowing, for example, a “mixed setting”, in which some of the \( d \theta \)'s have distributional dependence (hence of the kind of a delta-integral form) and the remaining have a polynomial dependence (hence of a kind of differential form) \[78\]. Even if there are important mathematical problems related to this framework - as we shall see - , this particular kind of (generally non-integrable) pseudoforms is that which is considered in superstring perturbation theory \[79\]. In this context the number of localized variable \( d \theta \)'s is referred to as picture number \( p \) of the (pseudo)form. In particular, forms having picture number \( p = 0 \) are differential forms and forms having maximal picture number \( p = q \) (which equal the odd dimension \( q \) of the supermanifold) are integral forms. An example of pseudoform having middle dimensional - i.e. non minimal and non maximal - picture number can be given considering again
the easy case of $\mathbb{R}^{1|2}$. The most general pseudoform of picture $p = 1$ is given by

$$
\omega(x, d\theta \mid \theta, dx) = \sum_{k_1, k_2 = 0}^{\infty} f_{k_1, k_2}(x \mid \theta) (d\theta_1)^{k_1} \delta^{(k_2)} (d\theta_2) + \sum_{\ell_1, \ell_2 = 0}^{\infty} g_{\ell_1, \ell_2}(x \mid \theta) (d\theta_2)^{k_2} \delta^{(k_1)} (d\theta_1) 
$$

$$
+ \sum_{i_1, i_2 = 0}^{\infty} h_{i_1, i_2}(x \mid \theta) dx (d\theta_1)^{i_1} \delta^{(i_2)} (d\theta_2) + \sum_{i_1, i_2 = 0}^{\infty} c_{i_1, i_2}(x \mid \theta) dx (d\theta_2)^{i_2} \delta^{(i_1)} (d\theta_1),
$$

(9.50)

where $f, g, h, c \in \mathcal{O}_{\mathbb{R}^{1|2}}$ for any choice of indices. The degree of these kind forms is defined as to agree with the definition given for differential and integral forms: a $k$-derivative of any delta’s counts $-k$. It is thus easy to see that there exists pseudoforms of middle picture $1 < p < q$ at any degree, whereas differential forms have non-negative degree and we have defined integral forms so that they have degree lower or equal than the even dimension of the supermanifold. Further, working in the same way as above, one sees that any module of pseudoforms of a certain middle dimensional picture $1 < p < q$ at a fixed degree $k \in (-\infty, +\infty)$ - we call it $\Omega^{k,p}_{\mathcal{M}}$ - has an infinite number of generators, hence cannot be described as a vector bundles or locally-free sheaves of $\mathcal{O}_{\mathcal{M}}$-modules of finite rank. For example, pseudoforms of picture 1 and degree $k$ on $\mathbb{R}^{1|2}$ would be generated by

$$
\Omega^{k,1}_{\mathbb{R}^{1|2}} = \mathcal{O}_{\mathbb{R}^{1|2}} \cdot \{(d\theta_1)^{\ell_1} \delta^{(\ell_2)} (d\theta_2), dx (d\theta_1)^{i_1} \delta^{(i_2)} (d\theta_2), 1 \leftrightarrow 2\},
$$

(9.51)

for any $\ell_1 - \ell_2 = k$ and $j_1 - j_2 = k - 1$. On the other hand, the crucial problem which prevents from having a well-given mathematical definition of these modules of pseudoforms having a middle dimensional picture $0 < p < q$ is rooted in the ill-defined transformation of a single delta (and its derivative) $\delta^{(k)} (d\theta_1)$. Indeed, as explained above in the discussion around equation (9.39), a single delta $\delta (d\theta)$ does not define a section of any a vector bundle on $\mathcal{M}$ and in particular it does not survive a change of coordinate. As it stands, a single $\delta (d\theta)$ - and more in general an expression consisting of a non-maximal number of delta’s - is no more than a symbol: only delta forms - where all of the $q \, d\theta$’s have distributional dependence of the Dirac delta type - do indeed yield a well-defined section of a vector bundle as seen in theorem 9.21.

It is to be noted, though, that when restricted to a specific immersed sub-supermanifold of the right codimension in $\mathcal{M}$, pseudoforms of non-maximal picture are well-behaved, as they define the integral forms of the sub-supermanifold. As such, pseudoforms of non-maximal picture $0 < p < q$ are seen in relation to integration over general sub-supermanifolds of codimension $k = q - p$, for any $k = 0, \ldots, p$, depending on the degree of the pseudoform. It would be interesting to elucidate the relations between pseudoforms of non-maximal picture $p$ and the $d$-densities defined by Manin in [53], chapter 4, paragraph 7.
ON THE GEOMETRY OF FORMS ON SUPERMANIFOLDS 53

APPENDIX A. NILPOTENT OPERATORS IN SUPERALGEBRA

In this appendix we report an easy yet very useful result that gives a criterion to establish the nilpotency of an operator. This appears as lemma 3 in [53], chapter 3, section 4 - the reader is advised of little confusing misprint in the proof. Let us consider the following generic setting: let $S$ and $T$ be $A$-modules, for $A$ a supercommutative ring and let $\sigma_a : S \to S$ and $\tau_a : T \to T$ two families of homogeneous homomorphisms for a finite set of indices $a = 1, \ldots, n$.

We say that $\sigma_a$ and $\sigma_b$ commute if

$$[\sigma_a, \sigma_b] := \sigma_a \sigma_b - (-1)^{|\sigma_a||\sigma_b|}\sigma_b \sigma_a = 0 \quad (A.1)$$

We say that $\sigma_a$ and $\sigma_b$ anticommute if

$$\{\sigma_a, \sigma_b\} := \sigma_a \sigma_b + (-1)^{|\sigma_a||\sigma_b|}\sigma_b \sigma_a = 0 \quad (A.2)$$

Also we assume the following:

1. $|\sigma_a| + |\tau_a|$ does not depends on $a$;
2. the pairs $(\sigma_a, \sigma_b)$ and $(\tau_a, \tau_b)$ either commute or anticommute.

Then, it make sense to define the following operator

$$d := \sum_a \sigma_a \otimes \tau_a : S \otimes_A T \to S \otimes_A T$$

$$f = s \otimes t \mapsto d(f) := \sum_a (-1)^{|\tau_a||s|}\sigma_a(s) \otimes \tau_a(t).$$

We note that it has a well-defined parity (even or odd), since we have assumed that the parity of $|\sigma_a| + |\tau_a|$ does not depend on $a$. We have the following lemma.

**Lemma A.1.** Let $d := \sum_a \sigma_a \otimes \tau_a$ be as above, then if either one of the following is satisfied

1. $d$ is even and the pair $(\sigma_a, \sigma_b)$ and $(\tau_a, \tau_b)$ have opposite commutation rules
2. $d$ is odd and the pair $(\sigma_a, \sigma_b)$ and $(\tau_a, \tau_b)$ have the same commutation rules

then $d \circ d = 0$.

**Proof.** Let us consider the expression for $d^2 := d \circ d$. One finds the sum

$$d^2 \supset (\sigma_a \otimes \tau_a)(\sigma_b \otimes \tau_b) + (\sigma_b \otimes \tau_b)(\sigma_a \otimes \tau_a) \quad (A.3)$$

for some $a, b$. Commuting one has

$$(\sigma_a \otimes \tau_a)(\sigma_b \otimes \tau_b) + (\sigma_b \otimes \tau_b)(\sigma_a \otimes \tau_a)$$

$$= (1)^{|\sigma_a||\tau_a| + |\sigma_b||\tau_b| + |\sigma_b||\tau_a| + |\sigma_a||\tau_b|} \epsilon_{\sigma \tau} \epsilon_{\sigma \tau} (\sigma_a \sigma_b \otimes \sigma_a \tau_b) \quad (A.4)$$

where $\epsilon_{\sigma} = \pm 1$ and $\epsilon_{\tau} = \pm 1$ depending on the commutation relations of $\sigma$ and $\tau$.

Say $d$ is even. Then this implies that $|\sigma_a| = |\tau_a|$ and $|\sigma_b| = |\tau_b|$ and $\epsilon_{\sigma \tau} \epsilon_{\sigma \tau} = +1$ since one pair of morphisms commutes and the other anticommutes. Then one finds that

$$-1)^{|\sigma_a||\sigma_b| + |\sigma_b||\sigma_a| + |\sigma_a||\sigma_b| + |\sigma_b||\sigma_a|} = (1)^{|\sigma_a||\sigma_b|} - (1)^{|\sigma_a||\sigma_b|} = 0 \quad (A.5)$$

Say $d$ is odd. Then this implies that $|\sigma_a| = |\tau_a| + 1$ and $|\sigma_b| = |\tau_b| + 1$ and $\epsilon_{\sigma \tau} \epsilon_{\sigma \tau} = -1$ since the two pairs of morphisms both commutes or anticommutes. Then one finds

$$-1)^{|\sigma_a||\sigma_b| + |\sigma_b||\sigma_a| + |\sigma_a||\sigma_b| + |\sigma_b||\sigma_a| + 1} = (1)^{|\sigma_a||\sigma_b| + |\sigma_b||\sigma_a| + 1}$$

$$= (1)^{|\sigma_a||\sigma_b| + |\sigma_b||\sigma_a| + 1} \quad (A.6)$$

Therefore, in both cases the sum in the parenthesis in $A.3$ vanishes. \[\square\]

Notice that, even if it is given in the context of superalgebra, the above lemma applies to a broad range of geometrical constructions - for example, the ordinary de Rham differential can be immediately proved to be nilpotent as a consequence of lemma A.1 as $\sigma_a = dx_a$ and $\tau_a = \partial_a$. 
Appendix B. Right \( \mathcal{D} \)-modules and Canonical Sheaf

In this appendix we prove lemma [5.13] and we comment further on the relations between the Lie derivative and the \( \mathcal{D}_X \)-module structure of the canonical sheaf of an ordinary manifold \( X \). The interested reader is invited to compare and appreciate the similarities of these constructions with those of section [3] where the right \( \mathcal{D}_M \)-module structure on the Berezinian sheaf is discussed.

Lemma B.1 (\( \omega_X \) is a Right \( \mathcal{D}_X \)-module). Let \( X \) be a real or complex manifold and let \( \Omega^\text{dim}_X \) be its canonical sheaf. Then \( \Omega^\text{dim}_X \) is a sheaf of right \( \mathcal{D}_M \)-module.

Proof. By the previous theorem [5.13], it is enough to show that we can define a flat right connection on \( \omega_X \). For sections \( \omega^{top} \in \Omega^\text{dim}_X \) and \( f \in \mathcal{O}_X \) and \( X \in \mathcal{T}_X \) we give the following definition

\[
\Delta_R(\omega^{top} \otimes f) := \omega^{top} f
\]

(B.1)

First, observe that

\[
\Delta_R(\omega^{top} \otimes f \circ X) = L_X(f^{\omega^{top}}) = d \circ \iota_X^{\omega^{top}} = d(f \iota_X^{\omega^{top}})
\]

(B.2)

On the other hand, one has

\[
\Delta_R(\omega^{top} f \otimes X) = L_X(\omega^{top} f) = \iota_X \circ d(\omega^{top} f) + d(\iota_X(\omega^{top} f))
\]

(B.3)

\[
= \iota_X(df \wedge \omega^{top} + f \wedge d\omega^{top}) + d(\iota_X(\omega^{top} f))
\]

since \( df \wedge \omega^{top} = 0 = d\omega^{top} \), so that \( \Delta_R(\omega^{top} \otimes f \circ X) = \Delta_R(\omega^{top} f \otimes X) \), which is the third defining property of a right connection. Further, we have that

\[
\Delta_R(\omega^{top} \otimes X) f = -L_X(\omega^{top} f),
\]

(B.4)

but also

\[
\Delta_R(\omega^{top} \otimes X \circ f) = -L_X(f^{\omega^{top}}) + \omega^{top} X(f) = -L_X(\omega^{top} f) + \omega^{top} X(f)
\]

(B.5)

so that we have indeed \( \Delta_R(\omega^{top} \otimes X \circ f) = \Delta_R(\omega^{top} \otimes X) f \), which proves the second defining property for a right connection. Finally, it is an obvious property of the Lie derivative that \( L_{[X,Y]} = [L_X, L_Y] \), which settles flatness.

Remark B.2. Notice that it is crucial for the above to hold true that \( \omega^{top} \) is really a section of the canonical sheaf. In other words, \( \Omega_X \) is not a right \( \mathcal{D}_M \)-module unless \( i = \text{dim} X \). Also, notice that working locally in a chart \( U \subset X \) with local coordinates \( x_1, \ldots, x_n \) such that a section of the canonical sheaf \( \Omega^\text{dim}_X \) over \( U \) reads \( \omega^{top}_U = \omega(x) f \) for some functions \( f \in \mathcal{O}_X(U) \) and \( \omega(x) = dx_1 \wedge \ldots \wedge dx_n \) and considering a vector fields over \( U \) such that \( X_U = \sum_i X_i \partial_{x_i} \), then one easily has that

\[
L_X(\omega^{top}) = \omega(x) \sum_{i=1}^n \partial_{x_i}(X^i f),
\]

(B.6)

indeed

\[
L_X(\omega^{top}) = L_X(\omega(x) f) + \omega(x) L_X(f) = dx_X(\omega(x)) f + \omega(x) X(f)
\]

(B.7)

Starting from the above \( B.6 \), it can be seen that there exists a unique right connection on \( \Omega^\text{dim}_X \) satisfying the condition \( \Delta_R(\omega(x) \otimes \partial_{x_i}) \) for all \( i = 1, \ldots, n \) in any coordinate system. In particular, the following holds true.

Lemma B.3. Let \( X \) be a real or complex manifold and of dimension \( n \) let \( \Omega^\text{top}_X \) be its canonical sheaf. Then there exists a unique right connection on \( \Omega^\text{top}_X \) such that

\[
\Delta_R(\omega(x) \otimes \partial_{x_i}) = 0
\]

(B.8)
for any $i = 1, \ldots, n$ and for all system of local coordinates $(U, x_1, \ldots, x_n)$, with $\omega(x) = dx_1 \wedge \ldots \wedge dx_n$ a generating section of $\Omega^n_U$ and $\partial_{x_i}$ is a coordinate vector field over $U$.

Proof. It is immediate using (B.6) to see that indeed $\Delta_X(\omega(x) \otimes \partial_{x_i}) = -\mathcal{L}_{\partial_{x_i}}(\omega(x)) = 0$. Uniqueness follows from the fact that $\{\partial_{x_i}\}_{i=1,\ldots,n}$ is a system of generators for $\mathcal{T}_X$ and $\omega(x) = dx_1 \wedge \ldots \wedge dx_n$ is a generator for $\Omega^n_X$. It is an exercise to check that changing coordinates to $x'_i = x'_i(x)$ one still gets $\Delta_X(\omega(x') \otimes \partial_{x'_i}) = 0$ for any $i$, thus concluding the proof. $\square$

Remark B.4. Notice that the above Lemma B.3 can be rephrased in terms of $\mathcal{D}_X$-module theory by saying that the right $\mathcal{D}_X$-module structure on $\Omega^n_X$ is uniquely characterized by the right action

$$\omega(x) \cdot \partial_{x_i} = 0.$$  \hspace{1cm} (B.9)

for any $i = 1, \ldots, \dim X$. This is to be related to theorems 5.19 and corollary 5.21.

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