Some result on integrality of several matrix representation of complete $r$-uniform hypergraph

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Abstract. The eigenvalues of matrix representation hypergraph have the possibility that all of it can be an integer or otherwise. The Laplacian integral hypergraphs are those hypergraphs whose Laplacian spectrum consists entirely of integers likewise the definition of signless Laplacian integral and Seidel integral. This research focuses on the properties of the entry matrix and formulates the spectrum of the Laplacian matrix, Signless Laplacian matrix, and Seidel Matrix of complete $r$-uniform hypergraphs. Hence, it can determine the integrality of the representation matrix respectively. The result concludes that complete $r$-uniform hypergraphs are Laplacian integral, signless Laplacian integral, and Seidel integral.

1. Introduction

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set. A hypergraph on $V$ is a family $\mathcal{E} = \{E_1, E_2, \ldots, E_m\}$ of subsets of $V$ such that

$$E_i \neq \emptyset \forall i \in \{1, 2, \ldots, m\} \quad \text{and} \quad \bigcup_i E_i = V.$$ 

The elements of $V$ are called vertices and the sets $\{E_1, E_2, \ldots, E_m\}$ are the hyperedges of the hypergraph. Let $r$ and $n$ are integer with $3 \leq r \leq n$. A hypergraph whose $n$ order is called complete $r$-uniform hypergraphs if all of its hyperedges have size $r$ and every $r$ vertices forms a hyperedge \[2\]. The number of its hyperedge is $\binom{n}{r}$. Complete $r$-uniform hypergraphs is denoted by $K_n^r$. Several studies about $r$-uniform hypergraphs found in the topic about Codegree Turan density, Ramsey Theorem, and Coding Theory. Codegree densities of Fano Plane studied by Alan et al. [1]. The Fano Plane is $r$-uniform hypergraphs on seven vertices and seven hyperedges. Furthermore, Mark Budden et al. [3] proof that that several known Ramsey number inequalities can be extended to the setting of $r$-uniform hypergraphs.

This research study about how to determine Laplacian integral of Complete $r$ uniform hypergraph. Also, this work emphasize another matrix representation such as signless Laplacian, and Seidel matrix. The related work done by Zakiyyah [9] that finite projective plane order $n$ is Laplacian integral. The notion of Laplacian, signless Laplacian and Seidel matrix can be found in a graph theory. Merris [4] studied about determined Laplacian integral from its degree sequences. Kirkland also studies about the several graph with maximum degree are Laplacian intergral [4]. Further research about Seidel matrix has been done by Mei et al. [8], complete bipartite and complete multipartite graph are Seidel integral. The studies about signless Laplacian can be found in Mirzakhah et al. [6].
2. Laplacian matrix of complete $r$-uniform hypergraphs

Let $\mathcal{H}$ denote the hypergraph and $M = M(\mathcal{H}) = (m_{ij})$ denote the adjacency matrices, Rodriguez \[7\] defined Laplacian degree of a vertex $v_i \in V(\mathcal{H})$ as

$$\delta_i(v_i) = \sum_{j=1}^{n} m_{ij}.$$ 

Laplacian matrix of hypergraph is denoted by $L(\mathcal{H})$ and defined by $L(\mathcal{H}) = D(\mathcal{H}) - M(\mathcal{H})$ with $M(\mathcal{H})$ and $D(\mathcal{H})$ are adjacency matrix and $D(\mathcal{H}) = \text{diag}(\delta_1(v_1), \delta_1(v_2), ..., \delta_1(v_n))$, for every $v_i \in V$, $i = 1, ..., n$, respectively.

Let matrix $L(K_r^n) = (l_{ij}) = \delta_i(K_r^n)I_n - M(K_r^n)$ with matrix $L(K_r^n)$, $\delta_i(K_r^n)$, and, $M(K_r^n)$ denote Laplacian matrix, Laplacian degree and adjacency matrix of $K_r^n$, respectively. Here the entries of Laplacian matrix of $K_r^n$.

$$l_{ij} = \begin{cases} -(n-2) \over r-2, & \text{if } i \neq j, \\ (n-1)(n-2) \over r-2, & \text{if } i = j. \end{cases}$$

A hypergraph is called Laplacian integral if all eigenvalues of its Laplacian spectrum are integers. The following theorem states the spectrum of complete $r$-uniform hypergraphs and denoted by $\text{Spec}_L$ with the first row represent the eigenvalues and its multiplicity in second row.

**Theorem 1.** Let complete $K_r^n$ denote $r$-uniform hypergraphs with $n$ order and $3 \leq r \leq n$. The spectrum of Laplacian matrix of $K_r^n$ is

$$\text{Spec}_L(K_r^n) = \left( \begin{array}{cc} n \over r-2 & 0 \\ n-1 & 1 \end{array} \right).$$

**Proof.** Consider Laplacian matrix of complete $r$-uniform hypergraph as follow :

$$L(K_r^n) = n \left( \frac{n-2}{r-2} \right) I_n - \left( \frac{n-2}{r-2} \right) J_n.$$  \hspace{1cm} (1)

By equation \[1\] we split into two part. Let $A = \left( n \over n-2 \right) I_n$. The eigenvalues of $A$ are $\mu_1 = ... = \mu_n = n \over n-2$ and it correspondence with eigenvector $v = (v_1, ..., v_n) \in \mathbb{R}^n$ and $v \neq 0$. Then, let $B = \left( r-2 \right) J_n$. The eigenvalues of $B$ are $0$ and $n \over n-2$ with $E_0$ and $E_n \over n-2$ denote eigenspaces, respectively. Here we have $L(K_r^n) = A - B = \left(n \over n-2 \right) I_n - \left(n \over n-2 \right) J_n$. Consider that eigenvector of matrix $A$ is all nonzero vector in $\mathbb{R}^n$. Let $Ax = \mu_A x$ and $Bx = \mu_B x$. Hence

$$Ax - Bx = \mu_A x - \mu_B x = (\mu_A - \mu_B)x.$$  \hspace{1cm} (2)

Let mention from the equation \[2\] that the eigenvalues of matrix $L(K_r^n)$ can be reproduced from subtraction of the eigenvalues matrix $A$ and $B$ which have common eigenvector. The final result that the eigenvalues of $L(K_r^n)$ are $\mu_1 = n \over n-2$ dan $\mu_2 = 0$, with the multiplicity $m(\mu_A) = n - 1$ and $m(\mu_B) = 1$ respectively.

The theorem \[4\] states that all eigenvalues of the spectrum of Laplacian matrix $L(K_r^n)$ are integer. Another result, this theorem give new methods to find the eigenvalues of the spectrum by consider the order of the hypergraphs and the number of vertex from each hyperedge.

**Corollary 1.1.** Complete $r$-uniform hypergraphs $K_r^n$ are Laplacian integral.
3. Signless Laplacian matrix of complete r-uniform hypergraph

In the previous section, study about Laplacian matrix with \( L(H) = D(H) - M(H) \). According definition of signless Laplacian matrix in graph and use it into hypergraph that \( X(H) = D(H) + M(H) \), with \( X(H) \), \( D(H) \), \( M(H) \) denote signless Laplacian, Laplacian and adjacency matrix of hypergraph, respectively. Let matrix \( X(K^r_n) = (x_{ij}) = \delta_i(K^r_n)I_n + M(K^r_n) \) with matrix \( X(K^r_n) \), \( \delta(K^r_n) \), and \( M(K^r_n) \) denote signless Laplacian matrix, Laplacian degree and adjacency matrix of \( K^r_n \), respectively. Here the entries of signless Laplacian matrix of \( K^r_n \).

\[
x_{ij} = \begin{cases} 
    \binom{n-2}{r-2}, & \text{if } i \neq j, \\
    (n-1)\binom{n-2}{r-2}, & \text{if } i = j. 
\end{cases}
\]

A hypergraph is called signless Laplacian integral if all eigenvalues of its signless Laplacian spectrum are integers.

**Theorem 2.** Let complete \( r \)-uniform hypergraphs \( K^r_n \) with \( n \) order and \( 3 \leq r \leq n \). The spectrum of signless Laplacian matrix of \( K^r_n \) is

\[
\text{Spec}_{\text{X}}(K^r_n) = \begin{pmatrix} 2(n-1)\binom{n-2}{r-2} & (n-2)\binom{n-2}{r-2} \end{pmatrix}.
\]

Theorem 2 gives information that all eigenvalues of spectrum from signless Laplacian \( K^r_n \) are integers. The largest eigenvalue of the spectrum of signless Laplacian matrix of \( K^r_n \) is \( 2(n-1)\binom{n-2}{r-2} \).

**Proof.** Consider signless Laplacian matrix \( K^r_n \) as follow:

\[
X(K^r_n) = (n-2)\binom{n-2}{r-2} I_n + \binom{n-2}{r-2} J_n.
\]

Step solution to determine spectrum of signless Laplacian matrix \( K^r_n \) likewise solved problem in Theorem 1. The eigenvalues of \( (n-2)\binom{n-2}{r-2} I_n \) are \( (n-2)\binom{n-2}{r-2} \) and the eigenvalues of \( \binom{n-2}{r-2} J_n \) are 0 and \( n\binom{n-2}{r-2} \). Hence, the eigenvalues of signless Laplacian matrix \( K^r_n \) are \( 2(n-1)\binom{n-2}{r-2} \) and \( (n-2)\binom{n-2}{r-2} \).

**Corollary 2.1.** Complete \( r \)-uniform hypergraphs \( K^r_n \) are signless Laplacian integral.

4. Seidel Matrix of r-Uniform Hypergraph

We generalize the definition of Seidel matrix into hypergraph. Let \( J \) denote matrix each of whose entries is 1. Seidel matrix \( S = J - I - 2M(H) \) with \( I \) and \( M(H) \) are identity matrix and adjacency matrix of hypergraph, respectively. A hypergraph is called Seidel integral if all eigenvalues of its Seidel spectrum are integers. Following theorem state about the spectrum of Seidel matrix of \( r \)-uniform hypergraph.

**Theorem 3.** Let complete \( r \)-uniform hypergraphs \( K^r_n \) with \( n \) order and \( 3 \leq r \leq n \). The spectrum of Seidel matrix \( K^r_n \) is

\[
\text{Spec}_{\text{S}}(K^r_n) = \begin{pmatrix} 2\binom{n-2}{r-2} - 1 & (n-1)(1-2\binom{n-2}{r-2}) \end{pmatrix}.
\]

All eigenvalues from Seidel matrix of complete \( r \)-uniform hypergraphs are integers hence complete \( r \)-uniform hypergraphs \( K^r_n \) are Seidel integral. This theorem gives the value of the largest and smallest of the its eigenvalues.
Proof. Let $M(H) = (m_{ij})$, the entries of adjacency matrix of complete tripartite hypertrees are $m_{ij} = m = n - 2r - 2$ for $i \neq j$ and $m_{ij} = 0$ for $i = j$.

Consider Seidel matrix of hypergraph, $S = J - I - 2M(H)$ with $J$, $I$, and $M(H)$ denote matrix with whose entries is 1, identity matrix and adjacency matrix, respectively.

Seidel matrix of complete tripartite hypertrees, denoted by $S(K_n^r)$, satisfy:

$$S(K_n^r) = \begin{pmatrix}
0 & 1 - 2m & \ldots & 1 - 2m \\
1 - 2m & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 - 2m \\
1 - 2m & 1 - 2m & \ldots & 0
\end{pmatrix}$$

(4)

By Equation 4, the diagonal entries of $S(K_n^r)$ are 0 and $1 - 2m$ or $1 - 2\binom{n-2}{r-2}$ for otherwise. Such that $S(K_n^r)$ can be written as $\left(1 - 2\binom{n-2}{r-2}\right)J - \left(1 - 2\binom{n-2}{r-2}\right)I$. Finally, the eigenvalues of $S(K_n^r)$ are $2\binom{n-2}{r-2} - 1$ and $(n - 1)\left(1 - 2\binom{n-2}{r-2}\right)$.

All eigenvalues from Seidel matrix of complete $r$-uniform hypergraphs are integers hence complete $r$-uniform hypergraphs $K_n^r$ are Seidel integral.

5. Conclusion
This research focus on the matrix representation of complete $r$-uniform hypergraph such as Laplace, Seidel and signless Laplacian. Complete $r$-uniform hypergraph contain $n$ vertices in which every $r$-subset of the vertices represents a hyperedge. The result of this work give description on how to find the spectrum from the order hypergraph and the number of vertex from each hyperedge. For further research, this method will be applied to another hypergraph. The eigenvalues of Laplacian spectrum of complete $r$ uniform hypergraph consist entirely of integers. Hence for the next work will be determine in another matrix representation of incidence matrix and what the relation with the eigenvalues of Laplacian spectrum.

Acknowledgments
This work is supported by Research and Technology Transfer Office, Bina Nusantara University as a part of Penelitian Terapan Binus Aplikasi Hipergraf pada Clustering Data with contract number: No.025/VR.RTT/IV/2020 and contract date: 6 April 2020.

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