COMPLEXITY AND INVARIANT MEASURE
OF THE PERIOD-DOUBLING SUBSHIFT

MIROSLAVA POLÁKOVÁ

Abstract. Explicit formulas for complexity and unique invariant measure
of the period-doubling subshift can be derived from those for the Thue-Morse
subshift, obtained by Brlek, De Luca and Varricchio, and Dekking. In this note
we give direct proofs based on combinatorial properties of the period-doubling
sequence. We also derive explicit formulas for correlation integral and other
recurrence characteristics of the period-doubling subshift. As a corollary we
obtain that the determinism of this subshift converges to 1 as the distance
threshold approaches 0.

1. Introduction

The period-doubling sequence
\[ \omega = \omega_1 \omega_2 \ldots = 0100 0101 0100 0100 \ldots \]
can be defined in various ways. First, its \( n \)-th member is 0 if and only if the largest
\( k \) such that \( k \)-th power of 2 divides \( n \), is odd; otherwise it is 1. Second, \( \omega \) is a
unique fixed point of the primitive substitution \( 0 \mapsto 01, 1 \mapsto 00 \). Third, \( \omega \) is the
Toeplitz sequence defined by patterns \((0*)\) and \((1*)\); for the general definition of
Toeplitz sequences see [12, 8].

The induced subshift, again called period-doubling, is strictly ergodic (i.e. it
is minimal and has a unique invariant measure) and has zero topological entropy.
Dynamical properties of this subshift were studied already in 50s and 60s, see the
book [10] by Gottschalk and Hedlund and the article [12] by Jacobs and Keane; for
some recent references see e. g. [5, 1, 4]. In the book [13], period-doubling subshift
(called Feigenbaum subshift therein) is mentioned many times as an example with
interesting dynamics.

The period doubling sequence is tightly connected with the Thue-Morse se-
quence, which is a unique fixed point of the primitive substitution \( 0 \mapsto 01, 1 \mapsto 10 \)
which starts with 0. Complexity of this sequence was studied in [3, 6] and the
invariant measure was considered in [7].

The period-doubling sequence \( \omega \) is a 2-to-1 image of the Thue-Morse sequence
[10, Definition 12.51]; every subword \( w = w_1 \ldots w_n \) of \( \omega \) corresponds to exactly two
subwords \( u = u_1 \ldots u_{n+1} \) of the Thue-Morse sequence such that \( u_i = u_{i+1} \) if and
only if \( w_i = 1 \). This relation and the results from [3, 6, 7] yield formula (1.1) for the
complexity of \( \omega \), and a description of the unique invariant period-doubling measure

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μ; namely for every allowed m-word u (m ≥ 1) we have

\[ \mu([u]) = \begin{cases} 
2 & \text{if } k = 0; \\
3 \cdot 2^{k-1} + 2q & \text{if } k \geq 1 \text{ and } q \leq 2^{k-1}; \\
4 \cdot 2^{k-1} + q & \text{if } k \geq 1 \text{ and } q > 2^{k-1}; 
\end{cases} \]

where \( k \geq 0 \) is such that \( 2^k \leq m < 2^{k+1} \).

These results are well-known, but cannot be easily found in the literature. Since the period-doubling substitution is of constant length, it is possible to study the complexity of it using a general method from [10]; however, it yields a set of non-trivial recurrent formulas and it seems difficult to derive (1.1) from them.

Dekking [7] has described factor frequencies in the Thue-Morse sequence and the Fibonacci sequence. Factor frequencies in generalized Thue-Morse words were studied in [2]. Frid [9] has obtained a precise description of factor frequencies in the Fibonacci sequence. Factor frequencies in generalized Thue-Morse words were trivial recurrent formulas and it seems difficult to derive (1.1) from them.

**Theorem 1.1.** (Complexity of the period-doubling sequence) Let \( u \) be the least integer such that \( \mu([u]) \) allows us to derive formulas for correlation integrals (for corresponding definitions see Section 2). For \( 0 \leq q < 2^k \) and \( k \geq 0 \) be such that \( 2^k \leq m < 2^{k+1} \), then the set of all \( m \)-words is equal to the set of first \((3/2)m\) subwords of \( \omega \).

**Theorem 1.2.** Let \( u \) be an allowed m-word \((m \geq 1)\), \( k \geq 0 \) be such that \( 2^k \leq m < 2^{k+1} \) and \( i \) be the least integer such that \( u = w_i^{(m)} \). Then \( 1 \leq i \leq 3 \cdot 2^k \), and

1. if \( i \leq 2^k - q \), or \( q < 2^{k-1} \) and \( 2^k < i \leq 2^k + 2^{k-1} - q \), then \( \mu([u]) = 2/(3 \cdot 2^k) \);

2. otherwise \( \mu([u]) = 1/(3 \cdot 2^k) \).

**Corollary 1.3.** Let \( m = 2^k + q \) with \( k \geq 0 \) and \( 0 \leq q < 2^k \). Denote by \( r(m) \) the number of m-words \( u \) such that \( \mu([u]) = 2/(3 \cdot 2^k) \). Then

\[ r(m) = \begin{cases} 
1 & \text{if } k = 0; \\
3 \cdot 2^{k-1} - 2q & \text{if } k \geq 1 \text{ and } q < 2^{k-1}; \\
2^k - q & \text{if } k \geq 1 \text{ and } q \geq 2^{k-1}; 
\end{cases} \]
\( m_\varepsilon \in \mathbb{N} \) as follows: if \( \varepsilon \geq 1 \) then \( m_\varepsilon = 0 \); otherwise \( m_\varepsilon \) is a unique positive integer such that
\[
2^{-m_\varepsilon} \leq \varepsilon < 2^{-m_\varepsilon + 1}.
\] (1.2)

**Theorem 1.4.** Let \( \varepsilon > 0 \). Then the correlation integral of the unique invariant measure \( \mu \) of the period-doubling subshift is
\[
c(\mu, \varepsilon) = \lim_{n \to \infty} C(\omega, n, \varepsilon) = \begin{cases} 
1 & \text{if } m_\varepsilon = 0; \\
5/9 & \text{if } m_\varepsilon = 1; \\
(3 \cdot 2^{k+1} - 4q)/(3 \cdot 2^k)^2 & \text{if } m_\varepsilon \geq 2 \text{ and } q < 2^{k-1}; \\
(5 \cdot 2^k - 2q)/(3 \cdot 2^k)^2 & \text{if } m_\varepsilon \geq 2 \text{ and } q \geq 2^{k-1};
\end{cases}
\]
where \( k \geq 0 \) and \( 0 \leq q < 2^k \) are integers such that \( m_\varepsilon = 2^k + q \).

For simple inequalities for \( c(\mu, \varepsilon) \) see Corollary [5.1]. Theorem 1.4 together with the results from [11] yield asymptotic values for two of the basic measures of recurrence quantification analysis: recurrence rate (RR) and determinism (DET).

**Theorem 1.5** (Recurrence rate of \( \omega \)). Let \( \ell \geq 1 \) and \( \varepsilon > 0 \). Then the recurrence rate \( \text{RR}_\ell(\omega, \varepsilon) \) exists and
\[
\text{RR}_\ell(\omega, \varepsilon) = \begin{cases} 
1 & \text{if } m_\varepsilon = 0; \\
5/9 & \text{if } m_\varepsilon = 1 \text{ and } \ell = 1; \\
(3 \cdot 2^{k+1} - 4q + 4\ell - 4)/(3 \cdot 2^k)^2 & \text{if } m_\varepsilon + \ell \geq 3 \text{ and } q < 2^{k-1}; \\
(5 \cdot 2^k - 2q + 2\ell - 2)/(3 \cdot 2^k)^2 & \text{if } m_\varepsilon + \ell \geq 3 \text{ and } q \geq 2^{k-1};
\end{cases}
\]
there, for \( m_\varepsilon + \ell \geq 3 \), \( k \geq 1 \) and \( 0 \leq q < 2^k \) are unique integers such that \( m_\varepsilon + \ell - 1 = 2^k + q \).

**Theorem 1.6** (Determinism of \( \omega \)). Let \( \ell \geq 2 \) and \( \varepsilon > 0 \). Then \( \text{DET}_\ell(\omega, \varepsilon) \) exists,
\[
\text{DET}_\ell(\omega, \varepsilon) = \frac{\text{RR}_\ell(\omega, \varepsilon)}{\text{RR}_1(\omega, \varepsilon)}
\]
and
\[
\lim_{\varepsilon \to 0} \text{DET}_\ell(\omega, \varepsilon) = 1.
\]
Moreover, \( \text{DET}_\ell(\omega, \varepsilon) = 1 \) if and only if one of the following three cases happens:
(a) \( \varepsilon \geq 1 \);
(b) \( 2^k \leq m_\varepsilon < m_\varepsilon + \ell - 1 < 2^k + 2^{k-1} \) for some \( k \in \mathbb{N} \);
(c) \( 2^k + 2^{k-1} \leq m_\varepsilon < m_\varepsilon + \ell - 1 < 2^k + 2^{k+1} \) for some \( k \in \mathbb{N} \).

Figure 1 illustrates \( \text{RR}_2 \) and \( \text{DET}_2 \) of the period-doubling sequence.

**Remark 1.7.** We trivially have that, for every \( \varepsilon < 1 \),
\[
\lim_{\ell \to \infty} \text{DET}_\ell(\omega, \varepsilon) = 0.
\]

Theorems 1.4, 1.5 and 1.6 are stated for embedding dimension 1. For general embedding dimension, see Subsection 5.1. See also [20] for formulas for other recurrence quantifiers.

This paper is organized as follows. Preliminaries are given in Section 2. Complexity of the period-doubling sequence (Theorem 1.1) is derived in Section 3 as a consequence of some other properties of this sequence. In Section 4 we give the proof of Theorem 1.2. In Section 5 we apply these results to prove Theorems 1.4
Moreover, we consider a generalization of our results to arbitrary embedding dimension.

2. Preliminaries

The set of positive integers \( \{1, 2, \ldots \} \) is denoted by \( \mathbb{N} \). The set \( A = \{0,1\} \) is called an alphabet. Put \( A^* = \bigcup_{k \geq 0} A^k \); \( A^* \) endowed with concatenation is a monoid. Members of \( A^* \) are called words. A \textit{word of length} \( m \), or an \textit{\( m \)-word} \((m \geq 1)\), is any \textit{\( m \)-word} \((m \geq 0); v_i \) is the \( i \)-th letter of \( v \). The \textit{empty word} \((the\ unique\ word\ of\ length\ 0)\) is denoted by \( \varepsilon \). A subword of \( v = v_1 \ldots v_n \) \textit{starting at the \( i \)-th letter} is any word \( v_i v_{i+1} \ldots v_{n'} \) with \( i \leq n' \leq n \).

The \textit{period-doubling} substitution \( \zeta \) is defined as follows:

\[
\zeta : A \to A^*, \quad \zeta(0) = 01, \quad \zeta(1) = 00. \tag{2.1}
\]

The substitution \( \zeta \) induces a morphism (denoted also by \( \zeta \)) of the monoid \( A^* \) by putting \( \zeta(\varepsilon) = \varepsilon \) and \( \zeta(w) = \zeta(w_1)\zeta(w_2)\ldots\zeta(w_n) \) for any nonempty word \( w = w_1 w_2 \ldots w_n \). Likewise, \( \zeta \) induces a map (again denoted by \( \zeta \)) from \( A^\mathbb{N} \) to \( A^\mathbb{N} \) by

\[
\zeta(x) = \zeta(x_1)\zeta(x_2)\ldots \quad \text{for} \quad x = (x_n)_{n \in \mathbb{N}} \in A^\mathbb{N}.
\]

The iterates \( \zeta^k (k \geq 1) \) of \( \zeta \) are defined inductively by \( \zeta^1 = \zeta \) and \( \zeta^k = \zeta \circ \zeta^{k-1} \) for \( k \geq 2 \).

\textit{Period-doubling sequence} \( \omega = 01000101010001000100 \ldots \) is the unique fixed point of \( \zeta : A^\mathbb{N} \to A^\mathbb{N} \). Recall that, for every \( i \in \mathbb{N} \), \( \omega_i \) is equal to \( k_i \mod 2 \), where \( k_i \) is the largest integer such that \( 2^{k_i} \) divides \( i \). For every integers \( m, i \geq 1 \), the \( m \)-word starting at the position \( i \) is denoted by \( w_i^{(m)} \):

\[
w_i^{(m)} = w_i w_{i+1} \ldots w_{i+m-1}.
\]

For \( m = 2^k (k \geq 1) \) put

\[
0^{(m)} = \zeta^k(0) \quad \text{and} \quad 1^{(m)} = \zeta^k(1);
\]
note that both \(0^{(m)}\) and \(1^{(m)}\) are words of length \(m\).

Any subword of \(\omega\) (including the empty one) is called allowed. The language \(\mathcal{L}_\omega\) of \(\omega\) is the set of all allowed words. The set of all allowed \(m\)-words is denoted by \(\mathcal{L}_\omega^m\). Complexity function of \(\omega\) is the map \(p = p_\omega : \mathbb{N} \to \mathbb{N}\) such that, for every \(m \in \mathbb{N}\), \(p(m) = \#\mathcal{L}_\omega^m\) is the number of allowed \(m\)-words.

Note that for every \(m = 2^k (k \geq 0)\) we have \(0^{(1)} = 0, 1^{(1)} = 1, \) and
\[
0^{(2m)} = 0^{(m)}1^{(m)}, \quad 1^{(2m)} = 0^{(m)}0^{(m)}. \quad (2.2)
\]

A measure-theoretical dynamical system is a system \((X, \mathcal{B}, \mu, f)\), where \(X\) is a set, \(\mathcal{B}\) is \(\sigma\)-algebra over \(X\), \(\mu\) is a probability measure and \(f : X \to X\) is a \(\mu\)-measurable and \(f\)-invariant transformation, i.e. \(f^{-1}(B) \in \mathcal{B}\) and \(\mu(f^{-1}(B)) = \mu(B)\) for every \(B \in \mathcal{B}\). The system \((X, \mathcal{B}, \mu, f)\) is ergodic if \(\mu(B) = 0\) or \(\mu(B) = 1\) for every \(B \in \mathcal{B}\) with \(f^{-1}(B) = B\).

A pair \((X, f)\) is called a topological dynamical system if \(X\) is a compact metric space and \(f : X \to X\) is a continuous map. A dynamical system \((X, f)\) is minimal if there is no proper subset \(M \subset X\) which is nonempty, closed and \(f\)-invariant (a set \(M\) is \(f\)-invariant if \(f(M) \subset M\)). Let \(\mathcal{B}_X\) denote the system of all Borel subsets of \(X\). A probability measure \(\mu\) is said to be invariant if \(\mu(f^{-1}(A)) = \mu(A)\) for every \(A \in \mathcal{B}_X\); that is, \((X, \mathcal{B}_X, \mu, f)\) is a measure-theoretical dynamical system. By Krylov-Bogolyubov theorem, for every \((X, f)\) there exists an invariant measure \(\mu\). System \((X, f)\) is called uniquely ergodic if such a measure \(\mu\) is unique. Moreover, if \((X, f)\) is also minimal, we call it strictly ergodic.

Metric \(\rho\) on \(\Sigma = A^\mathbb{N}\) is defined for every \(\alpha, \beta \in \Sigma\) by \(\rho(\alpha, \beta) = 0\) if \(\alpha = \beta\), and \(\rho(\alpha, \beta) = 2^{-k+1}\) if \(\alpha \neq \beta\), where \(k = \min\{i : \alpha_i \neq \beta_i\}\). Note that \((\Sigma, \rho)\) is a compact metric space. For an \(m\)-word \(v\) we define the cylinder \([v]\) by \([v] = \{\alpha \in \Sigma : \alpha_i = v_i\ \text{for}\ i \leq m\}\). Cylinders form a basis of the topology and \([v] = B(x, \varepsilon)\) for every \(x \in [v]\) and \(\varepsilon = 2^{-|v|}\), where \(B(x, \varepsilon)\) denotes the closed ball with the center \(x\) and radius \(\varepsilon\). A shift is the map \(\sigma : \Sigma \to \Sigma\) defined by \(\sigma(\alpha_1\alpha_2\alpha_3\ldots) = \alpha_2\alpha_3\ldots\). For each nonempty closed \(\sigma\)-invariant subset \(Y \subset \Sigma\), the restriction of \((\Sigma, \sigma)\) to \(Y\) is called a subshift. The closure of the orbit \((\sigma^n(\alpha))_{n \geq 0}\) of any \(\alpha \in \Sigma\) defines a subshift, as it is always nonempty, closed and \(\sigma\)-invariant set. Period-doubling subshift is the orbit closure of the period-doubling sequence.

Let \((X, \sigma)\) be a subshift over \(A\), \(\rho\) be the metric defined above and \(\mu\) be a \(\sigma\)-invariant measure. Correlation integral of \(\mu\) is defined for \(\varepsilon > 0\) as follows:
\[
c(\mu, \varepsilon) = \mu \times \mu \{ (x, y) \in X \times X : \rho(x, y) \leq \varepsilon \}.
\]
If \(2^{-m} \leq \varepsilon < 2^{-m+1}\) then clearly
\[
c(\mu, \varepsilon) = \sum_{v \in \sigma^m} \mu([v])^2.
\]
For \(x \in X, n \in \mathbb{N}\) and \(\varepsilon > 0\), correlation sum is defined by
\[
C(x, n, \varepsilon) = \frac{1}{n^2} \# \{ (i, j) : 0 \leq i, j < n, \ \rho(\sigma^i(x), \sigma^j(x)) \leq \varepsilon \}.
\]
For uniquely ergodic systems, \(\lim_n C(x, n, \varepsilon) = c(\mu, \varepsilon)\) for every but countably many \(\varepsilon > 0\) and every \(x \in X\) [13].

For any \(\ell \geq 1\) consider Bowen’s metric
\[
\rho_\ell(\alpha, \beta) = \max_{0 \leq k < \ell} \rho(\sigma^k(\alpha), \sigma^k(\beta)) \quad .
\]
An easy computation gives that we always have

$$\rho(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha_i \neq \beta_i \text{ for some } 1 \leq i \leq \ell, \\ 2^{\ell-1} \rho(\alpha, \beta) & \text{if } \alpha_i = \beta_i \text{ for all } 1 \leq i \leq \ell. \end{cases}$$  \hfill (2.3)

We can now define

$$C_\ell(x, n, \varepsilon) = \frac{1}{n^2} \# \{(i, j) : 0 \leq i, j < n, \rho(\sigma^i(x), \sigma^j(x)) \leq \varepsilon\}. \hfill (2.4)$$

Recurrence quantification analysis ([22], see also [14, 21]) gives several complexity measures quantifying structures in recurrence plots, which are useful for visualization of recurrence. Two of them are recurrence rate (RR) and determinism (DET). By [11, Proposition 1], recurrence rate and determinism can be expressed by correlation sums as follows:

$$RR_\ell = \ell \cdot C_\ell - (\ell - 1) \cdot C_{\ell + 1} \quad \text{and} \quad \text{DET}_\ell = \frac{RR_\ell}{RR_1}, \hfill (2.5)$$

where $\ell$ is the minimal required line length; arguments $x, n, \varepsilon$ are omitted and we consider embedding dimension 1. For general embedding dimension $d$ see Subsection 5.1.

If the limit of $C_\ell(x, n, \varepsilon)$ for $n \to \infty$ exists, it is denoted by $C_\ell(x, \varepsilon)$. Analogously we define $RR_\ell(x, \varepsilon)$ and $\text{DET}_\ell(x, \varepsilon)$.

3. Complexity of the period-doubling sequence

3.1. Length $m = 2^k$. In this section, we prove Theorem 1.1 in the special case when the length $m$ is a power of 2. We start with two lemmas. The first one follows by induction using (2.2) and the second one is a direct consequence of $\zeta^k(\omega) = \omega$.

Lemma 3.1. For any $m = 2^k$ ($k \geq 0$), the $m$-words $0^{(m)}$, $1^{(m)}$ differ exactly at the $m$-th letter:

$$(0^{(m)})_i = (1^{(m)})_i \quad \text{for } i < m, \quad (0^{(m)})_m \neq (1^{m})_m.$$  

Moreover, if $k$ is even then $(0^{(m)})_m = 0$ and $(1^{(m)})_m = 1$, and if $k$ is odd then $(0^{(m)})_m = 1$ and $(1^{(m)})_m = 0$.

Lemma 3.2. Let $m = 2^k$ ($k \geq 0$). Then the period-doubling sequence $\omega$ can be written in the form $\omega = (\omega_1)^{(m)}(\omega_2)^{(m)} \ldots$. That is, for every $i \in \mathbb{N}$,

$$w_{(i-1)m+1}^{(m)} = \begin{cases} 0^{(m)} & \text{if } \omega_i = 0, \\ 1^{(m)} & \text{if } \omega_i = 1. \end{cases}$$

Lemma 3.3. For the period-doubling sequence $\omega$, $p(1) = 2$ and $p(2) = 3$. Moreover, the allowed 1-words are $w_1^{(1)} = 0$ and $w_2^{(1)} = 1$, and the allowed 2-words are $w_1^{(2)} = 01$, $w_2^{(2)} = 10$, and $w_3^{(2)} = 00$.

Proof. We only need to prove that the word 11 is not allowed. But this immediately follows from the fact that $\omega_{2i-1} = 0$ for every $i$. \hfill $\square$

Lemma 3.4. Let $m = 2^k$ ($k \geq 1$). Then the words $w_i^{(m)}$ ($1 \leq i \leq \frac{3}{2}m$) are pairwise distinct.
Proof. We start by showing that, for \( \frac{3}{2}m < i \leq 4m, \)
\[
    w_i^{(m)} = \begin{cases} 
    w_{i-m/2}^{(m)} & \text{if } \frac{3}{2}m < i \leq 2m; \\
    w_{i-2m}^{(m)} & \text{if } 2m < i \leq 3m; \\
    w_{i-3m}^{(m)} & \text{if } 3m < i \leq 4m.
    \end{cases}
\] (3.1)

To see this, realize that \( \omega = 0^{(m)}1^{(m)}0^{(m)}0^{(m)}\ldots \) by Lemma 3.2. Hence, by Lemma 3.1, \( w_i^{(m)} = w_{i-2m}^{(m)} \) for \( 2m < i \leq 3m \) and \( w_i^{(m)} = w_{i-3m}^{(m)} \) for \( 3m < i \leq 4m. \)
Furthermore, \( \omega = 0^{(n)}1^{(n)}0^{(n)}0^{(n)}1^{(n)}\ldots \) where \( n = m/2. \) So analogously, \( w_i^{(m)} = w_{i-n}^{(m)} \) for \( \frac{3}{2}m < i < 2m. \)

We now proceed by induction on \( k. \) For \( k = 1, \) the claim follows from Lemma 3.3. Assume now that the claim is valid for some \( k \geq 1; \) we are going to show that it is valid for \( k + 1. \) Put \( m = 2^k. \) Since \( w_i^{(2m)} = w_i^{(m)}w_{i+m}^{(m)} \) (3.1) and the induction hypothesis yield that the words \( w_i^{(2m)} \) for \( 1 \leq i \leq 3m \) are pairwise distinct. \( \square \)

**Lemma 3.5.** Let \( m = 2^k \) \((k \geq 1)\) and \( v \) be any allowed \( m \)-word. Then exactly one of the following is true:

1. \( v \) is a subword of \( 0^{(m)}1^{(m)} \) starting at the \( i \)-th letter with \( i \leq m; \)
2. \( v \) is a subword of \( 1^{(m)}0^{(m)} \) starting at the \( i \)-th letter with \( i \leq m/2. \)

Proof. We start by showing that at least one of (1), (2) is true. If \( v \in \{0^{(m)}, 1^{(m)}\}, \) we are done. Otherwise, by Lemma 3.2, \( v \) is a subword of \( 0^{(m)}0^{(m)} \) or \( 0^{(m)}1^{(m)} \) or \( 1^{(m)}0^{(m)} \), starting at an index \( j \leq m. \) By Lemma 3.1, \( v \) is a subword of \( 0^{(m)}1^{(m)} \) or \( 1^{(m)}0^{(m)}. \) In the former case we have (1). In the latter case, we have (2) since \( 1^{(m)}0^{(m)} = 0^{(m)}0^{(m)}0^{(m)}1^{(n)} \) by (2.2), where \( n = m/2. \)

Moreover, \( \omega \) starts with \( 0^{(m)}1^{(m)}0^{(m)}, \) so \( v = w_i^{(m)} \) for some \( 1 \leq i \leq \frac{3}{2}m. \) By Lemma 3.4, the words \( w_i^{(m)} \) \((1 \leq i \leq \frac{3}{2}m)\) are pairwise distinct, so only one of (1) and (2) is true. \( \square \)

**Proposition 3.6.** Let \( m = 2^k \) \((k \geq 1). \) Then \( p(m) = \frac{3}{2}m \) and \( L_m^\omega = \{w_i^{(m)} : 1 \leq i \leq \frac{3}{2}m\}. \)

Proof. Lemma 3.5 gives \( p(m) \leq \frac{3}{2}m. \) On the other hand, \( p(m) \geq \frac{3}{2}m \) by Lemma 3.4. The description of \( L_m^\omega \) now follows from Lemma 3.4. \( \square \)

**Remark 3.7.** For \( m = 2^k \) \((k \geq 1)\) we also have \( L_m^\omega = \{w_i^{(m)} : \frac{3}{2}m < i \leq 3m\}; \) this follows from (3.1). \( \square \)

### 3.2. General length \( m. \)

**Lemma 3.8.** Let \( m = 2^k + q, \) where \( k \geq 1 \) and \( 1 \leq q < 2^k. \) Let \( 1 \leq i < j \leq 3 \cdot 2^k. \) Then \( w_i^{(m)} = w_j^{(m)} \) if and only if exactly one of the following conditions holds:

1. \( 1 \leq i \leq 2^k - q \) and \( j = i + 2^{k+1}; \)
2. \( q < 2^{k-1}, 2^{k+1} + 1 \leq i \leq 3 \cdot 2^{k-1} - q, \) and \( j = i + 2^{k-1}. \)

Consequently, for every \( 1 \leq i < 3 \cdot 2^k \) there is at most one \( j \) such that \( i < j \leq 3 \cdot 2^k \) and \( w_i^{(m)} = w_j^{(m)}. \)

Proof. For \( 1 \leq i < j \leq 3 \cdot 2^k \) put
\[
    \varphi(i, j) = \min\{1 \leq h \leq 2^{k+1} : \omega_{i+h-1} \neq \omega_{j+h-1}\};
\]
It is clear that $w_i^{(m)} = w_j^{(m)}$ if and only if $\varphi(i, j) > m$. \hspace{1cm} (3.2)

It is clear that

$$\text{if } \varphi(i, j) \geq 2 \text{ then } \varphi(i + 1, j + 1) = \varphi(i, j) - 1.$$ \hspace{1cm} (3.3)

Fix $1 \leq i < j \leq 3 \cdot 2^k$ and assume that $w_i^{(m)} = w_j^{(m)}$; we are going to show that either $[1]$ or $[2]$ is true. Since $m \geq 2^k$ we have that $w_i^{(2^k)} = w_j^{(2^k)}$ and, by (3.1), exactly one of the following is true:

(a) $1 \leq i \leq 2^k$ and $i = j + 2^k + 1$;
(b) $2^k < i \leq 3 \cdot 2^k - 1$, and $j = i + 2^k - 1$.

Assume that $[a]$ is true. Since $\varphi(1, 2^{k+1} + 1) = 2^{k+1}$ and $j = i + 2^k \leq 3 \cdot 2^k$, (3.3) implies

$$\varphi(i, j) = 2^{k+1} - (i - 1). \hspace{1cm} (3.4)$$

Since $w_i^{(m)} = w_j^{(m)}$ by assumption, (3.2) implies $2^{k+1} - (i - 1) > m$, that is $i \leq 2^k - q$. So we have [1].

If $[b]$ is true then, by Lemma 3.2, $w_i^{(2^k + 1)} = 0(n)0(n)1^{(n)}(n)$ and $w_i^{(2^k + 1)} = 0(n)0(n)1^{(n)}(n)$ for $n = 2^{k-1}$. From Lemma 3.1 it follows that $\varphi(1 + 2^k, 1 + 2^k - 1) = 3 \cdot 2^{k-1}$. Since $2^k < i \leq 3 \cdot 2^k - 1$ and $j = i + 2^k - 1$, (3.3) yields

$$\varphi(i, j) = \varphi(p + 2^k, p + 2^k + 2^k - 1) = 3 \cdot 2^{k-1} - (p - 1),$$

(3.5)

(notice that $0 < p \leq 2^{k-1}$). By the assumption $w_i^{(m)} = w_j^{(m)}$ and so, by (3.2), $\varphi(i, j) > m = 2^k + q$. Now (3.5) gives $3 \cdot 2^{k-1} - q \geq i$, so we have [2].

Now assume that one of the conditions [1], [2] holds. If [1] holds we have $w_i^{(m)} = w_j^{(m)}$, since (3.4) implies $\varphi(i, j) > m$. Similarly, if [2] is true then $\varphi(i, j) > m$ by (3.5), so again $w_i^{(m)} = w_j^{(m)}$.

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** It is clear from Lemma 3.3 that [1] is true for $k = 0$, so we may assume that $k > 0$. Let $n = 2^{k+1}$. By Proposition 3.6

$$p(m) = p(n) - \#\{(i, j) : 1 \leq i < j \leq 3 \cdot 2^k, w_i^{(m)} = w_j^{(m)}\}.$$ 

If $q \geq 2^{k-1}$ then only [1] from Lemma 3.8 occurs, consequently, $p(m) = p(n) - (2^k - q) + 4 \cdot 2^{k-1} + 2q$. Otherwise, both [1] and [2] from Lemma 3.8 occur and so $p(m) = p(n) - (2^k - q) - (2^{k-1} - q) = 3 \cdot 2^{k-1} + 2q$. \hspace{1cm} $\square$

From Theorem 1.1 we immediately have that

$$p(m + 1) - p(m) \in \{1, 2\} \text{ for every } m$$

and

$$\frac{3}{2} m \leq p(m) \leq \frac{5}{3} m \text{ for every } m \geq 2.$$
4. Invariant measure of the period-doubling subshift

Let $(X, \sigma)$ be the period-doubling subshift; i.e. $X$ is the orbit closure of $\omega$ and $\sigma : X \to X$ is the left shift. By [15] (see also [19, Proposition 5.2 and Theorem 5.6]), $(X, \sigma)$ is strictly ergodic.

Denote the unique invariant measure of $(X, \sigma)$ by $\mu$. By [17],

$$\mu([v]) = \lim_{n \to \infty} \frac{1}{n} \# \{1 \leq i \leq n : w_i^{(m)} = v\}$$

for every $v \in \mathcal{L}^m$. In this section we prove Theorem 1.2 which gives an explicit formula for measures of cylinders $[v]$. We follow [19, Sections 5.3-5.4]. Fix an integer $m \in \mathbb{N}$ and recall that $\mathcal{L}^m$ is the set of all $m$-words in $\omega$. Define a substitution $\zeta^{(m)}$ over alphabet $\mathcal{L}^m$ as follows: for $u \in \mathcal{L}^m$, write $\zeta(u) = y_1 y_2 \ldots y_{2m}$, and define $\zeta^{(m)}(u) = (y_1 \ldots y_m)(y_2 \ldots y_{m+1})$. Let $M^m$ be the composition matrix of $\zeta^{(m)}$, that is $M^m$ is a $p(m) \times p(m)$ non-negative matrix such that, for $u, v \in \mathcal{L}^m$, $(M^m)_{uv}$ is the number of occurrences of $v$ in $\zeta^{(m)}(u)$. Trivially every member of $M^m$ belongs to $\{0, 1, 2\}$.

By [19, Corollary 5.2], the Perron-Frobenius eigenvalue of $M^m$ is $\lambda = 2$. Furthermore, if $d^m = (d^m_{u,v})_{u \in \mathcal{L}^m}$ is the unique normalized eigenvector of $M^m$ corresponding to $\lambda$, then $\mu([u]) = d^m_{\cdot u}$ by [19, Corollary 5.4], see also [9, Proposition 1].

**Lemma 4.1.** Let $m = 2^k$ $(k \geq 1)$. Then $d^m = \frac{2}{3m}(1, 1, \ldots, 1)$. Consequently, $\mu([v]) = \frac{2}{3m}$ for every allowed $m$-word $v$.

**Proof.** It is enough to show that every row sum of $M_m$ is equal to 2. For $m = 2$ it is easy. So assume that $m \geq 4$. By (3.1) we have

$$\zeta^{(m)}(w_{\lfloor i/2 \rfloor}^{(m)}) = \sum_{j=1}^{\lfloor i/4 \rfloor} \left(\frac{w_{\lfloor i/4 \rfloor}^{(m)}w_{\lfloor j/2 \rfloor}^{(m)}}{w_{\lfloor i/4 \rfloor}^{(m)}} \right) \text{ for } i \leq \frac{3}{4}m;$$

$$(\text{4.1})$$

$$\zeta^{(m)}(w_{\lfloor i/4 \rfloor}^{(m)}) = \frac{w_{\lfloor (i-1)/2 \rfloor}^{(m)}w_{\lfloor (i-1)/4 \rfloor}^{(m)}}{w_{\lfloor (i-1)/2 \rfloor}^{(m)}} \text{ for } \frac{3}{4}m < i \leq m;$$

$$(\text{4.2})$$

$$\zeta^{(m)}(w_{\lfloor i/4 \rfloor}^{(m)}) = \frac{w_{\lfloor (i-1)/2 \rfloor}^{(m)}w_{\lfloor (i-1)/2 \rfloor}^{(m)}}{w_{\lfloor (i-1)/2 \rfloor}^{(m)}} \text{ for } m < i \leq \frac{3}{2}m.$$}

Hence, for $1 \leq j \leq m$, the word $w_{\lfloor j/2 \rfloor}^{(m)}$ occurs in $\zeta^{(m)}(w_{\lfloor i/2 \rfloor}^{(m)})$ for $i = \lfloor \frac{j}{4} \rfloor$ and $i = \lfloor \frac{j}{2} \rfloor + m$. Further, for $m < j \leq \frac{3}{4}m$, the word $w_{\lfloor j/4 \rfloor}^{(m)}$ occurs in $\zeta^{(m)}(w_{\lfloor i/4 \rfloor}^{(m)})$ for $i = \lfloor \frac{j}{2} \rfloor$ and $i = \lfloor \frac{j}{2} \rfloor + \frac{m}{2} + \frac{m}{4}$. The proof is complete.

**Proof of Theorem 1.3.** Theorem 1.2 holds for $q = 0$ by the previous lemma, so let $q \geq 1$. Put $n = 2^{k+1}$. If (1) is true then, by Lemma 3.8, there is exactly one index $j$ such that $i < j \leq 3 \cdot 2^k$ and $w_{\lfloor i/4 \rfloor}^{(m)} = w_j^{(m)}$; in this case $[v] = [w_i^{(n)}] \cup [w_j^{(n)}]$. Otherwise, $[v] = [w_i^{(n)}]$. Now the theorem follows from Lemma 4.1.

5. Correlation integral and RQA measures

**Proof of Theorem 1.4.** By [18], modified to uniquely ergodic systems, $\lim C(\omega, n, \varepsilon) = c(\mu, \varepsilon)$ provided $c(\mu, \varepsilon)$ is continuous at $\varepsilon$. Since the metric $\rho$ attains only values from $2^{2n0} \cup \{0\}$, $C(\omega, n, \varepsilon)$ and $c(\mu, \varepsilon)$ are constant on $\varepsilon \in [2^{-m}, 2^{-m+1})$ for every $m$. This easily implies $\lim_n C(\omega, n, \varepsilon) = c(\mu, \varepsilon)$ for every $\varepsilon$. Since

$$c(\mu, \varepsilon) = \sum_{v \in \mathcal{L}^m} \mu([v])^2,$$

Theorem 1.2 and Corollary 1.3 yield the desired result.

□
Corollary 5.1. Let $0 < \varepsilon < \frac{1}{2}$ and $m_\varepsilon$ be defined as in (1.2). Then

$$\frac{2}{3m_\varepsilon} \leq c(\mu, \varepsilon) \leq \frac{25}{36m_\varepsilon}.$$ 

Moreover, if $m_\varepsilon \in \{2^k, 2^k + 2^{k-1}, k \geq 1\}$ then $c(\mu, \varepsilon) = \frac{2}{3m_\varepsilon}$, and if $m_\varepsilon \in \{2^k + 2^{k-1}, k \geq 1\}$ then $c(\mu, \varepsilon) = \frac{25}{36m_\varepsilon}$.

Proof. Write $m_\varepsilon = 2^k + q$ with $k \geq 1$ and $0 \leq q < 2^k$. Let $x = \frac{q}{m_\varepsilon} \in [0, \frac{1}{2})$. Using Theorem 1.4 and substituting $2^k = m_\varepsilon - q$ into $m_\varepsilon c(\mu, \varepsilon)$ we get

$$m_\varepsilon c(\mu, \varepsilon) = \frac{6 - 10x}{9(1-x)^2} \quad \text{for } 0 \leq q < 2^{k-1},$$

$$m_\varepsilon c(\mu, \varepsilon) = \frac{5 - 7x}{9(1-x)^2} \quad \text{for } 2^{k-1} \leq q < 2^k.$$

Using elementary calculus we obtain that $\frac{2}{3} \leq m_\varepsilon c(\mu, \varepsilon) \leq \frac{25}{36}$ if $0 \leq q < 2^{k-1}$ and $\frac{2}{3} \leq m_\varepsilon c(\mu, \varepsilon) \leq \frac{25}{36} < \frac{25}{36}$ if $2^{k-1} \leq q < 2^k$. Moreover, minimum is attained at the points $x = 0$ and $x = \frac{3}{7}$, corresponding to $q = 0$ and $q = 2^{k-1}$, and maximum is attained at the point $x = \frac{1}{5}$ corresponding to $q = 2^{k-2}$. \hfill \Box

Proof of Theorem 1.3. If $\varepsilon \geq 1$ then, by (2.5) and Theorem 1.4, $RR_\ell(\omega, n, \varepsilon) = 1$ for every $n$, hence $RR_\ell(\omega, \varepsilon) = 1$. So assume that $0 < \varepsilon < 1$. By (2.3), for every $x, y \in X$ we have $\rho(x, y) \leq \varepsilon$ if and only if $\rho(x, y) \leq 2^{-\ell+1}\varepsilon$. So

$$C_\ell(\omega, n, \varepsilon) = C(\omega, n, 2^{-\ell+1}\varepsilon).$$

Thus, by (2.5) and Theorem 1.4

$$RR_\ell(\omega, \varepsilon) = \lim_{n \to \infty} RR_\ell(\omega, n, \varepsilon) = \ell c(\mu, 2^{-\ell+1}\varepsilon) - (\ell - 1) c(\mu, 2^{-\ell}\varepsilon). \quad (5.1)$$

Notice that $m_{2^{-\ell+1}\varepsilon} = m_\varepsilon + \ell$ and $m_{2^{-\ell+1}\varepsilon} = m_\varepsilon + \ell - 1$, since $\varepsilon < 1$ and $\ell \geq 1$. Put $m = m_\varepsilon + \ell - 1$. If $m = 1$ (i.e., $m_\varepsilon = 1$ and $\ell = 1$), then $RR_\ell(\omega, \varepsilon) = 5/9$ by (5.1) and Theorem 1.4. So we may assume that $m \geq 2$ (i.e. $m_\varepsilon + \ell \geq 3$) and hence we may write $m = 2^k + q$ with $k \geq 1$ and $0 \leq q < 2^k$.

Now we consider four cases: $q < 2^{k-1} - 1$, $q = 2^{k-1} - 1$, $2^{k-1} \leq q < 2^k - 1$, and $q = 2^k - 1$. In the first and third cases we have $m_{2^{-\ell+1}\varepsilon} = m + 1 = 2^k + (q + 1)$ with $q + 1 < 2^{k-1}$ and $2^{k-1} \leq q < 2^k$, respectively. So (5.1) and Theorem 1.4 give the formulas for $RR_\ell(\omega, \varepsilon)$.

In the second case ($q = 2^{k-1} - 1$) we can write $m_{2^{-\ell+1}\varepsilon} = 2^k + 2^{k-1}$ and in the fourth case ($q = 2^k - 1$) we can write $m_{2^{-\ell+1}\varepsilon} = 2^{k+1} + 0$; as above, (5.1) and Theorem 1.4 yield the formula for $RR_\ell(\omega, \varepsilon)$. \hfill \Box

Proof of Theorem 1.6. From (2.5) and the definition of determinism, we have

$$DET_\ell(\omega, n, \varepsilon) = \frac{RR_\ell(\omega, n, \varepsilon)}{RR_1(\omega, n, \varepsilon)}.$$ 

Using (5.1) and the fact that $RR_1(\omega, \varepsilon) = c(\mu, \varepsilon) > 0$, we obtain

$$DET_\ell(\omega, \varepsilon) = \lim_{n \to \infty} DET_\ell(\omega, n, \varepsilon) = \frac{RR_\ell(\omega, \varepsilon)}{RR_1(\omega, \varepsilon)}. \quad (5.2)$$
It is clear that $\text{DET}_\ell(\omega, \varepsilon) = 1$ for $\varepsilon \geq 1$, so assume that $\varepsilon < 1$. Let $m_\varepsilon = 2^k + q$ and $m_\varepsilon + \ell - 1 = 2^{k'} + q'$, where $k, k' \geq 0$, $0 \leq q < 2^k$ and $0 \leq q' < 2^{k'}$. We now compute $\text{DET}_\ell(\omega, \varepsilon)$ using Theorems 4.4, 1.5, and (5.2). We distinguish three cases.

(a) Let $\varepsilon \in (0, 1)$ be such that $k' = k$; then $q' = q + \ell - 1$; we write $\varepsilon \in E_a$. If $0 \leq q < q' < 2^{k-1}$ or $2^{k-1} \leq q < q' < 2^k$, we immediately have $\text{DET}_\ell(\omega, \varepsilon) = 1$. Otherwise $0 \leq q < 2^{k-1} \leq q' < 2^k$

\[
\text{DET}_\ell(\omega, \varepsilon) = \frac{5 \cdot 2^k - 2q}{6 \cdot 2^k - 4q} < 1.
\]

Here $2^{k-1} - \ell + 1 \leq q < 2^{k-1}$ and so $q^{2-k} \rightarrow 1/2$ for $\varepsilon \rightarrow 0$. Thus we have

\[
\lim_{\varepsilon \rightarrow 0} \text{DET}_\ell(\omega, \varepsilon) = 1.
\]

(b) Let $\varepsilon \in (0, 1)$ be such that $k' = k+1$; we write $\varepsilon \in E_b$. Then $q' = q + \ell - 1 - 2^k$, and so

\[
\text{DET}_\ell(\omega, \varepsilon) = \frac{3 \cdot 2^k + \Delta}{3 \cdot 2^k + 2\Delta} < 1, \quad \text{where } \Delta = -q' + l - 1 \in \{1, \ldots, \ell - 1\}.
\]

Clearly

\[
\lim_{\varepsilon \rightarrow 0} \text{DET}_\ell(\omega, \varepsilon) = 1.
\]

(c) If $\varepsilon \in (0, 1) \setminus (E_a \cup E_b)$, then $k' \geq k + 2$ and we again have $\text{DET}_\ell(\omega, \varepsilon) < 1$. Since this can happen only for large enough $\varepsilon$, this case does not affect the limit $\lim_{\varepsilon \rightarrow 0} \text{DET}_\ell(\omega, \varepsilon)$. (In fact, if $\varepsilon < \min(2^{-(\ell-2)}, 1)$ then $m_\varepsilon \geq \ell - 1$, and so $2^{k'} + q' = m_\varepsilon + \ell - 1 \leq 2m_\varepsilon = 2(2^k + q)$. From this we immediately have $k' \leq k + 1$.)

Thus we have proved that $\text{DET}_\ell(\omega, \varepsilon) = 1$ if and only if one of (a)–(c) happens (otherwise $\text{DET}_\ell(\omega, \varepsilon) < 1$) and that $\lim_{\varepsilon \rightarrow 0} \text{DET}_\ell(\omega, \varepsilon) = 1$.

\[
\square
\]

5.1. General embedding dimension. Up to now we considered recurrence characteristics without embedding. The results can be easily generalized to arbitrary embedding dimension $d \geq 1$.

If $x$ is a sequence over $A = \{0, 1\}$, then the embedded sequence $x^d$ is a sequence over $A^d = \{0, 1\}^d$ defined by

\[
x^d = x_1^d x_2^d \ldots = (x_1 x_2 \ldots x_d)(x_{2d} x_{3d} \ldots x_{d+1}) \ldots
\]

A metric $\rho^d$ in the embedding space $(A^d)^N$ is defined as in Section 2 that is,

\[
\rho^d(x^d, y^d) = \begin{cases} 2^{-k+1} & \text{if } x^d \neq y^d, \text{ where } k = \min\{i : x_i^d \neq y_i^d\}, \\ 0 & \text{if } x^d = y^d. \end{cases}
\]

If $k > 1$ then trivially

\[
\rho^d(x^d, y^d) = 2^{-k+1} \quad \text{if and only if} \quad \rho(x, y) = 2^{-(d+k-2)}.
\]

So for correlation sums $C^d_\ell$, defined by (2.4) with $\rho_\ell$ replaced by $\rho^d_\ell$, it holds that

\[
C^d_\ell(x^d, n, \varepsilon) = C(x, n, 2^{-(l-1)-(d-1)} \varepsilon)
\]
for every $x \in A^N, \varepsilon \in (0, 1)$ and $n \in \mathbb{N}$. This together with Theorem 1.4 yield an explicit formula for (embedded) correlation integrals $c^d_\ell(\omega^d, \varepsilon)$ for the period-doubling sequence $\omega$. To obtain formulas for $RR^d_\ell(\omega^d, \varepsilon)$ and $DET^d_\ell(\omega^d, \varepsilon)$ it suffices to use (2.5):

$$RR^d_\ell(\omega^d, n, \varepsilon) = \ell \cdot C^d_\ell(\omega^d, n, \varepsilon) - (\ell - 1) \cdot C^d_{\ell + 1}(\omega^d, n, \varepsilon),$$

$$DET^d_\ell(\omega^d, n, \varepsilon) = \frac{RR^d_\ell(\omega^d, n, \varepsilon)}{RR^d_1(\omega^d, n, \varepsilon)}.$$

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Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, Banská Bystrica, Slovakia

E-mail address: miroslava.sartorisova@umb.sk