Random nilpotent groups of maximal step

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Abstract. Let $G$ be a random torsion-free nilpotent group generated by two random words of length $\ell$ in $U_n(\mathbb{Z})$. Letting $\ell$ grow as a function of $n$, we analyze the step of $G$, which is bounded by the step of $U_n(\mathbb{Z})$. We prove a conjecture of Delp, Dymarz, and Schaffer-Cohen, that the threshold function for full step is $\ell = n^2$.

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1. Introduction

A group $G$ is nilpotent if its lower central series,

$$G = G_0 \geq G_1 \geq \cdots \geq G_r = \{1\}$$

defined by $G_{i+1} = [G, G_i]$, eventually terminates. The first index $r$ for which $G_r = \{1\}$ is called the step of $G$. One may ask what a generic nilpotent group looks like, including its step. Questions about generic properties of groups can be answered with random groups, first introduced by Gromov [5]. Since Gromov’s original few relators and density models are nilpotent with probability 0, they cannot tell us about generic properties of nilpotent groups. Thus, there is a need for new random group models that are nilpotent by construction.

Delp, et al. (2019) [3] introduced a model for random nilpotent groups, motivated by the observation that any finitely generated torsion-free nilpotent group can be embedded in the group $U_n(\mathbb{Z})$ of $n \times n$ upper triangular integer matrices with ones on the diagonal [4]. Note that, since any finitely generated nilpotent group contains a torsion-free subgroup of finite index, we are not losing much by restricting our attention to torsion-free groups. (Another model is considered in [2]).

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We construct a random subgroup of $U_n(Z)$ as follows. Let $E_{i,j}$ be the elementary matrix with 1’s on the diagonal, a 1 at position $(i, j)$ and 0’s elsewhere. Then $S = \{E^1_{i,j+1} : 1 \leq i < n\}$ forms the standard generating set for $U_n(Z)$. We call the entries at positions $(i, i+1)$ the superdiagonal entries. Define a random walk of length $\ell$ to be a product

$$V = V_1 V_2 \ldots V_\ell$$

where each $V_i$ is chosen independently and uniformly from $S$. Let $V$ and $W$ be two independent random walks of length $\ell$. Then $G = \langle V, W \rangle$ is a random subgroup of $U_n(Z)$. We have step($G$) $\leq$ step($U_n(Z)$), and it is not hard to check that step($U_n(Z)$) $= n - 1$. If step($G$) $= n - 1$, we say $G$ has full step.

Now let $n \to \infty$ and $\ell = \ell(n)$ grow as a function of $n$. We say a proposition $P$ holds asymptotically almost surely (a.a.s.) if $P_n \to 1$ as $n \to \infty$. Delp et al. (2019) gave results on the step of $G$, depending on the growth rate of $\ell$ with respect to $n$. Recall that $f = o(g(n))$ means $f(n)/g(n) \to 0$ as $n \to \infty$ and $f = \omega(g(n))$ means $f(n)/g(n) \to \infty$ as $n \to \infty$.

**Theorem 1.1** (Delp-Dymarz-Schaffer-Cohen). Let $n, \ell(n) \to \infty$ and $G = \langle V, W \rangle$ where $V, W$ are independent random walks of length $\ell$. Then:

1. If $\ell(n) = o(\sqrt{n})$ then a.a.s. step($G$) $= 1$, i.e. $G$ is abelian.
2. If $\ell(n) = o(n^2)$ then a.a.s. step($G$) $< n - 1$.
3. If $\ell(n) = \omega(n^2)$ then a.a.s. step($G$) $= n - 1$, i.e. $G$ has full step.

In this paper we close the gap between cases 2 and 3.

**Theorem 1.2.** Let $n, \ell(n) \to \infty$ and $G = \langle V, W \rangle$. If $\ell(n) = \omega(n^2)$ then a.a.s. $G$ has full step.

To prove this requires a careful analysis of the nested commutators that generate $G_{n-1}$. In Section 1, we give a combinatorial criterion for a nested commutator of $V$’s and $W$’s to be nontrivial. In Section 2, we show this criterion is satisfied asymptotically almost surely when $V, W$ are random walks.

**2. Nested commutators**

Let $G = G_0 \geq G_1 \geq \ldots$ be the lower central series of $G$. We have

$$G_i = [G, G_{i-1}] = [G, [G, \ldots, [G, G] \ldots]]$$

In particular, $G_i$ includes all $i + 1$-fold nested commutators of elements of $G$. We restrict our attention to commutators where each factor is $V$ or $W$.

Let $\{0, 1\}^d$ be the $d$-dimensional cube, or the set of all length $d$ binary vectors. For $x \in \{0, 1\}^d, y \in \{0, 1\}^e$ we define the norm $N(x) = \sum_{1 \leq i \leq d} x_i$ and the concatenation $xy \in \{0, 1\}^{d+e}$. For example if $x = (1, 0, 0)$ and $y = (0, 1)$ then $xy = (1, 0, 0, 0, 1) = 1031$. 

We define a family of maps $C_d : \{0, 1\}^d \rightarrow G_d$ as follows.
\[
    C_1(1) = V \\
    C_1(0) = W \\
    C_d(1x) = [V, C_{d-1}(x)] \\
    C_d(0x) = [W, C_{d-1}(x)]
\]

Thus for example, $C_2(1011) = C_2(10001) = [V, [W, [W, [W, V] ]]]$. We omit the subscript $d$ when it is obvious. To prove $G$ has full step, it suffices to find an $x \in \{0, 1\}^{n-1}$ such that $C(x)$ is nontrivial. We begin with Lemma 2.3 from [3], which gives a recursive formula for the entries of a nested commutator.

**Lemma 2.1.** Let $a \in \{0, 1\}$, $x \in \{0, 1\}^{d-1}$. Then $C(ax) \in G_d$ and we have
\[
    C(ax)_{i,j+d} = C(a)_{i,j+1}C(x)_{i+1,j+1} - C(a)_{i+d-1,j+1}C(x)_{i,j+d-1}
\]
and furthermore $C(ax)_{i,j} = 0$ for $j < i + d$.

In particular, for $d = n - 1$ only the upper rightmost entry $C(ax)_{1,n}$ can be nonzero.

From the formula, it is clear that $C(ax)_{i,j+d}$ is a degree-$d$ polynomial in the superdiagonal entries of $V$ and $W$. Let us state this more precisely and analyze the coefficients of the polynomial.

**Lemma 2.2.** Let $d \geq 1$. There exists a function $K_d : \{0, 1\}^d \times \{0, 1\}^d \rightarrow \mathbb{Z}$ such that for $1 \leq i \leq n - d$ we have
\[
    C(x)_{i,i+d} = \sum_{y \in \{0, 1\}^d} K_d(x, y) \prod_{i \leq j < i+d} V_{y_{j+1}}^{y_j} W_{j+1}^{1-y_j} (1)
\]

Furthermore, setting $K_d(x, y) = 0$ for $N(x) \neq N(y)$ we have a recursion
\[
    K_d(ax, byc) = K_1(a, b)K_{d-1}(x, yc) - K_1(a, c)K_{d-1}(x, by)
\]
with base cases
\[
    K_1(0, 0) = K_1(1, 1) = 1 \\
    K_1(0, 1) = K_1(1, 0) = 0
\]

Note that $K_d(x, y)$ does not depend on $i$. We also drop the subscript $d$ since it can be inferred from $x$ and $y$.

**Proof.** Abbreviate
\[
    U(i, d, y) := \prod_{i \leq j < i+d} V_{y_{j+1}}^{y_j} W_{j+1}^{1-y_j}
\]

We first prove inductively that there exist coefficients $K_d : \{0, 1\}^d \times \{0, 1\}^d \rightarrow \mathbb{Z}$ such that
\[
    C(x)_{i,i+d} = \sum_{y \in \{0, 1\}^d} K_d(x, y) U(i, d, y)
\]
The case \( d = 1 \) is trivial. Assume it holds for \( d - 1 \). Let \( a \in \{0, 1\} \) and \( x \in \{0, 1\}^{d-1} \), then we have
\[
C(ax)_{i,j+d} = C(a)_{i,j+1} C(x)_{i+1,j+d} - C(a)_{i+d-1,j+d} C(x)_{i,j+d-1}
\]
Expanding \( C(a)_{i,j+1} \) and \( C(x)_{i+1,j+d} \), the first term is
\[
= \left[ K_1(a, 1)V_{i+1} + K_1(a, 0)W_{i+1} \right] \left[ \sum_{y \in \{0, 1\}^{d-1}} K_{d-1}(x, y)U(i + 1, d - 1, y) \right]
\]
\[
= \sum_{y \in \{0, 1\}^{d-1}} K_1(a, 1)K_{d-1}(x, y)U(i, d, 1y) + K_1(a, 0)K_{d-1}(x, y)U(i, d, 0y)
\]
\[
= \sum_{b,c \in \{0, 1\}} K_1(a, b)K_{d-1}(x, y')U(i, d, by')
\]
Similarly, the second term is
\[
= \sum_{b,c \in \{0, 1\}} K_1(a, c)K_{d-1}(x, by')U(i, d, by')
\]
Combining, we get
\[
C(ax)_{i,j+d} = \sum_{b,c \in \{0, 1\}} [K_1(a, b)K_{d-1}(x, yc) - K_1(a, c)K_{d-1}(x, by)] U(i, d, byc)
\]
and setting \( K_d(ax, byc) = K_1(a, b)K_{d-1}(x, yc) - K_1(a, c)K_{d-1}(x, by) \), the lemma is proved for \( d \). It is also easy to see inductively that \( K_d(x, y) = 0 \) for \( N(x) \neq N(y) \), so we may add the condition \( N(x) = N(y) \) under the sum.

We now have a strategy for choosing \( x \in \{0, 1\}^{n-1} \) such that \( C(x) \) is nontrivial. In the random model, it may happen that \( V_{i,i+1} = 0 \) for some \( i \). Define the vector \( v \in \{0, 1\}^{n-1} \) by \( v_i = 1 \) if \( V_{i,i+1} \neq 0 \) and \( v_i = 0 \) otherwise. For now assume \( 0 < N(v) < n - 1 \). If we choose \( x \) such that \( N(x) = N(v) \), then Equation 1 simplifies to
\[
C_{n-1}(x)_1,n = K_d(x, v) \prod_{1 \leq j < n} V_{i,j+1}^{\nu_j} W_{i,j+1}^{1-\nu_j}.
\]
If we assume there is no \( i \) such that \( V_{i,i+1} = W_{i,i+1} = 0 \), the product of matrix entries is nonzero. So, we just need to choose \( x \) such that \( K_d(x, v) \neq 0 \). We can do this with some additional assumptions on \( v \).

**Lemma 2.3.** Let \( v \in \{0, 1\}^{n-1} \) with \( 0 < N(v) < n - 1 \). Write \( v = 1^{a_1}01^{a_2} \ldots 1^{a_k-1}01^{a_k} \). Assume that \( a_i \geq 1 \) for all \( i \), i.e., there are no adjacent 0’s, and that \( a_1 \neq a_k \). Then there exists \( x \in \{0, 1\}^{n-1} \) such that \( K(x, v) \neq 0 \).

We will prove in section 2 that all assumptions used hold asymptotically almost surely.
Proof. Using the recursion from Lemma 2.2, the following identities are easily verified by induction:

1. If $a, b \geq 0$, then
   \[ K(1^{a+b}0^1, 1^a01^b) = \binom{a+b}{a}(-1)^b \]

2. If $a, b \geq 1$, $c \geq 0$ with $c < \min(a, b)$, then
   \[ K(1^c0x, 1^a y^1b) = 0 \]

3. Let $a, b \geq 0$. If $a < b$ then
   \[ K(1^a0x, 1^a0y1^b) = K(x, y1^b) \]
   If $b < a$ then
   \[ K(1^b0x, 1^a0y1^b) = K(x, 1^a y)(-1)^{b+1} \]

4. If $a, b \geq 0$ then
   \[ K(1^{a+b}0^2x, 1^a01y10^1b) = 2\binom{a+b}{a}(-1)^bK(x, 1y1) \]

Let $\nu = 1^a 01^a ... 01^a$. First assume $k = 2\ell$ is even. We set
\[ x = 1^{a_1+a_{2\ell}0^2}1^{a_2+a_{2\ell-1}0^2} ... 1^{a_{\ell}+a_{\ell+1}0} \]

Then applying identity 4 repeatedly followed by identity 1, we obtain
\[ K(x, \nu) = 2\ell(-1)^{a_{2\ell}+a_{2\ell-1}+\cdots+a_{\ell+1}}\binom{a_1+a_2+\cdots+a_{\ell+1}}{a_1} \binom{a_2+\cdots+a_{2\ell}}{a_2} \cdots \binom{a_\ell+a_{\ell+1}}{a_\ell} \]

If $k$ is odd, we apply identity 3 once and proceed as before. \qed

3. Asymptotics

In Section 1, we derived a combinatorial condition on the superdiagonal entries of $V$ and $W$ sufficient for $G$ to have full step. Define
\[ V = \{i : 1 \leq i < n, V_{i,i+1} = 0\} \]
\[ W = \{i : 1 \leq i < n, W_{i,i+1} = 0\} \]

Then, to apply Lemma 2.3, we need that

1. $V$ and $W$ are nonempty.
2. $V \cap W = \emptyset$.
3. $V$ has no adjacent elements.
4. $\min V \neq n - \max V$.

If condition (1) does not hold, then Theorem 1.2 follows by a modification of Lemma 5.4 in [3].

We now show that in the random model, if $\ell = \omega(n^2)$, then the superdiagonal entries satisfy conditions (2)-(4) asymptotically almost surely. Recall that $V$ and $W$ are random walks
\[ V = V_1V_2...V_\ell \]
\[ W = W_1W_2...W_\ell \]
where each $V_i, W_i$ is chosen independently and uniformly from the generating set $S = \{E_i^{i+1} : 1 \leq i < n\}$.

Define

$$
\sigma_j(Z) = \begin{cases} 
1 & \text{if } Z = E_{j,j+1} \\
-1 & \text{if } Z = E_{j+1,j}^{-1} \\
0 & \text{otherwise}
\end{cases}
$$

Then we have

$$
V_{i,i+1} = \sum_{j=1}^\ell \sigma_j(V_j).
$$

When $\ell \gg n$, the superdiagonal entries $V_{i,i+1}$ behave roughly like independent random walks on $\mathbb{Z}$. We restate Corollary 3.2 from [3].

**Lemma 3.1.** Suppose $\ell = o(n)$. Then uniformly for $1 \leq k_1 < k_2 < \cdots < k_d < n$ we have

$$
P[k_i \in V \cap W \text{ for all } i] \sim \left( \frac{n}{2\pi \ell} \right)^d
$$

By the union bound, we have $P[V \cap W \neq \emptyset] \ll n^2/\ell \to 0$. Thus, condition (2) holds a.a.s. For conditions (3) and (4), we will need a bound on the size of $V$.

**Lemma 3.2.** Fix $\epsilon > 0$. Then $P[|V| > \epsilon\sqrt{n}] \to 0$ as $n \to \infty$.

**Proof.** Define random variables

$$
X_i = \begin{cases} 
1 & V(i,i+1) = 0 \\
0 & V(i,i+1) \neq 0
\end{cases}
$$

So $|V| = \sum_i X_i$. From Lemma 3.1 we have $E[X_i] \ll \sqrt{n/\ell}$ and $E[X_i X_j] \ll n/\ell$ for $1 \leq i < j < n$. Hence $E[|V|] \ll \sqrt{n^3/\ell}$ and $\text{Var}[|V|] \ll n^3/\ell$. By Chebyshev’s inequality,

$$
P[|V| \geq \epsilon\sqrt{n}] \leq \frac{\|V| - \sqrt{n^3/\ell} \geq \sqrt{n}(\epsilon - \sqrt{n^2/\ell})\]}
\leq \frac{1}{(\epsilon - \sqrt{n^2/\ell})^2(\ell/n^2)} \to 0
$$

Observe that the distribution of $V$ is invariant under permutation. In other words, for a fixed set $S \subseteq \{1, \ldots, n-1\}$ and a permutation $\pi$ on $\{1, \ldots, n-1\}$ we have

$$
P[V = S] = P[V = \pi S]
$$
and hence,
\[ P(V = S) = \frac{1}{\binom{n-1}{|S|}} P(|V| = |S|) \]
Let \( A(k) \) be the number of sets \( S \subseteq \{1, \ldots, n-1\} \) of size \( k \) with at least one pair of adjacent elements. We have
\[ A(k) \leq (n-2)\binom{n-3}{k-2}. \]
Let \( B(k) \) be the number of sets \( S \) for which \( \min S = n - \max S \). Summing over the possible values of \( \min S \) we have
\[ B(k) \leq \sum_{1 \leq a \leq n/2} \binom{n-1-2a}{k-2}. \]
One easily checks
\[ \frac{A(k) + B(k)}{\binom{n-1}{k}} \leq 2k^2/n. \]
For \( k \leq \epsilon \sqrt{n}, \) this is \( \leq 2 \epsilon^2 \). On the other hand, \( P(|V| > \epsilon \sqrt{n}) \to 0, \) so we are done.

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