Proof of Weierstrass gap theorem

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Abstract

In this expository note we give proof of the Weierstrass gap theorem in Cohomology terminology. We analyze gap sequence for finding possible gaps and non-gaps on $X$.

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1 Introduction:

Let $X$ be a compact Riemann surface of genus $g$. One of the important questions is the existence of meromorphic functions having pole at a single point $p$ on $X$. By the Riemann Roch theorem, we can show that there always exists a non-constant meromorphic function $f \in \mathcal{M}(X)$ which has a pole of order $\leq g + 1$ at $p$ and is holomorphic in $X \setminus \{p\}$.

One of the basic results in this topic is Weierstrass gap theorem, which states that

Theorem 1 For a surface of genus $g \geq 1$ there are precisely $g$ integers

\begin{equation}
1 = n_1 < n_2 < \cdots < n_g < 2g
\end{equation}

such that there does not exist a meromorphic function on $X$ with a pole of order $n_k$ at $p$.

The numbers $n_j$, for $j = 1, \ldots, g$ are called “gaps” at $p$ and their complement in $\mathbb{N}$ are called “non-gaps”. Further, the sequence is uniquely determined by the point $p$. 

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The case $g = 0$ is trivial as there is always a function on the sphere with a simple pole. The case $g = 1$ is clear since there is no meromorphic function with a single simple pole.

In this note we furnish the proof of Theorem 1 in the language of sheaf cohomology, in the spirit of proof given in Springer [4]. The proof of Noether’s theorem was given in Farkas and Kra [1] from which the proof of Weierstrass gap theorem was deduced as a special case. The proofs are generally given as an application of Riemann-Roch Theorem. In section 2, we mention some of the consequences of Riemann-Roch and Serre Duality theorems. The proofs may be found in [2]. The proof of Weierstrass gap theorem is given in Section 3. In Section 4, we form a combinatorial problem which appears to be a by product from the statement of Weierstrass gap theorem and analyze gaps and non-gaps.

2 Some Consequences of Riemann-Roch and Serre Duality Theorems

Let $D$ be a divisor on $X$.

If $deg D < 0$ then $H^0(X, \mathcal{O}_D) = 0$.

We can use Serre-Duality theorem to obtain equality of dimensions:

\[
\dim H^0(X, \Omega_D) = \dim H^1(X, \mathcal{O}_D)
\]

For $D = 0$ we obtain

\[
\dim H^0(X, \Omega) = g = \dim H^1(X, \mathcal{O})
\]

Here $H^0(X, \Omega) = \Omega(X)$ which denotes the sheaf of holomorphic 1-forms on $X$.

The following equality can be obtained as an application of Serre Duality Theorem:

\[
\dim H^1(X, \Omega) = \dim H^0(X, \mathcal{O}) = 1
\]

Note 2 This is a known fact again saying there are no non-constant holomorphic maps on a compact Riemann surface.

Suppose $X$ is a compact Riemann surface of genus $g$. Let $K$ be the canonical divisor on $X$. Then $deg K = 2g - 2$. 

3 Proof of Theorem 1 (Weierstrass Gap Theorem)

Proof. Suppose $P \in X$. If $D$ is a zero divisor, then $\dim H^1(X, \mathcal{O}) = g$ and $\deg(D) = 0$.

By the Riemann Roch theorem $\dim H^0(X, \mathcal{O}) = 1$. Therefore, there are no non-constant holomorphic function on $X$.

Define the divisor $D_P$ such that

$$D_P(P) = \begin{cases} 
1 & \text{if } x = P \\
0 & \text{if } x \neq P 
\end{cases}$$

$\deg(D_P) = 1$. Once again by the Riemann -Roch theorem,

$$\dim H^0(X, \mathcal{O}_{D_P}) = 2 - g + \dim H^0(X, \Omega_{-D_P})$$

If $\dim H^0(X, \Omega_{-D_P}) = g$, then $\dim H^0(X, \mathcal{O}_{D_P}) = 2$, hence there exists $f \in \mathcal{M}(X)$ which has a simple pole at $P$ and is holomorphic in $X \setminus \{P\}$.

If $\dim H^0(X, \Omega_{-D_P}) = g - 1$, then $\dim H^0(X, \mathcal{O}_{D_P}) = 1$, hence there is no meromorphic function which has a simple pole at $P$ and is holomorphic in $X \setminus \{P\}$.

Now we want to see the effect of changing $D_P = (n-1)D$ to $D = nD_P$. By the Riemann - Roch Theorem

$$\dim H^0(X, \Omega_{-D(n-1)P}) = n - g + \dim H^0(X, \Omega_{-D(n-1)P})$$

and

$$\dim H^0(X, \mathcal{O}_{-D(n-1)P}) = n + 1 - g + \dim H^0(X, \Omega_{-D(n-1)P})$$

If

$$\dim H^0(X, \Omega_{-D(n-1)P}) = \dim H^0(X, \Omega_{-D(n-1)P})$$

then

$$\dim H^0(X, \mathcal{O}_{-D(n-1)P}) = \dim H^0(X, \mathcal{O}_{-D(n-1)P}) + 1.$$ 

So there exists a meromorphic function $f \in \mathcal{M}(X)$ with a pole of order $n$ at $P$ and is holomorphic in $X \setminus \{P\}$.

If

$$\dim H^0(X, \Omega_{-D(n-1)P}) = \dim H^0(X, \Omega_{-D(n-1)P}) - 1$$

then

$$\dim H^0(X, \mathcal{O}_{-D(n-1)P}) = \dim H^0(X, \mathcal{O}_{-D(n-1)P}).$$
So there will not exist a function with a pole of order \( n \) at \( P \) and holomorphic in \( X \setminus \{ P \} \).

So if \( \dim H^0(X, \Omega_{-D_nP}) \) remains the same as \( n \) increases by 1, a new linearly independent function is added in going from the sheaf \( \mathcal{O}_{D_nP} \) to \( \mathcal{O}_{D(P+1)} \).

From the Eq. 3 we have \( \dim H^0(X, \Omega) = g \). We have already seen that if \( D \) is the divisor of a non-vanishing meromorphic 1-form on a compact Riemann surface of genus \( g \), then \( \deg (\omega) = 2g - 2 \).

Let \( K \) be its canonical divisor. Then we can deduce that

\[
\dim H^0(X, \Omega_{-D(2g-1)P}) = 0.
\]

Therefore, the number of times \( \dim H^0(X, \Omega_{-D_nP}) \) does not remain the same must be \( g \) times and at each change it decreases by 1.

It completes the proof. 

4 Analyzing gaps and non-gaps

Suppose \( p \in X \). If \( f \) has a pole of order \( s \) at \( p \), and \( g \) has a pole of order \( t \) at \( p \), then \( fg \) has a pole of order \( s + t \) at \( p \). Therefore, the set of non-gaps forms an additive sub-semi group of \( \mathbb{N} \).

Let \( d \) be the least non-gap value at the point \( p \), if \( n > d \) is a gap then \( n - d \) is again a gap value. Therefore, all the gaps occur in finite arithmetical sequences of the form \( j, j + d, j + 2d, \ldots, j + \lambda_j d \), where \( j = 1, 2, \ldots, d - 1 \) and \( \lambda_j = 0, 1, 2, \ldots \). See pp. 124 [3]. A point \( p \) is called a hyperelliptic Weierstrass point if the non-gap sequence starts with 2 and the hyperelliptic Riemann surfaces are characterized by the gap sequence:

\[
P = \{1, 3, \ldots, 2g - 1\}
\]

hence the non-gaps are \( Q = \{2, 4, \ldots, 2g\} \).

For the Exceptional Riemann surfaces the gap sequence \( P \) and the non-gap sequence \( Q \) are given by

\[
P = \{1, 2, 3, \ldots, g - 1, g + 1\} \quad \text{and} \quad Q = \{g, g + 2, \ldots, 2g - 1\}.
\]

For example, if \( g = 3 \) we see the possible gap sequences are

\{1, 3, 5\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}

and corresponding non-gaps are:

\{2, 4, 6\}, \{4, 5, 6\}, \{3, 5, 6\}, \{3, 4, 6\}.

The following problem may be of some interest to see possible gaps and non-gaps at a given point.
Question 1:
Write the numbers 2 to $2g-1$ in to two (disjoint) parts $P = \{n_1, n_2, \ldots, n_{g-1}\}$ and $Q = \{m_1, m_2, \ldots, m_{g-1}\}$ such that no number in $P$ is a sum of any combination of numbers in $Q$. How many pairs of such $P$ and $Q$ exist?

Clearly then $\{1\} \cup P$ gives possible gaps and $Q \cup \{2g\}$ gives possible non-gaps.

References

[1] H.M. Farkas and I. Kra, Riemann surfaces, GTM, Springer, Second Edition, 1991.

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[3] R.C. Gunning, Lectures on Riemann surfaces, Princeton University Press, second printing 1968.

[4] George Springer, Introduction to Riemann surfaces, Addison-Wesley Publishing Company, Inc, 1957.