DECONSTRUCTIBILITY AND THE HILL LEMMA IN GROTHENDIECK CATEGORIES

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ABSTRACT. A full subcategory of a Grothendieck category is called deconstructible if it consists of all transfinite extensions of some set of objects. This concept provides a handy framework for structure theory and construction of approximations for subcategories of Grothendieck categories. It also allows to construct model structures and t-structures on categories of complexes over a Grothendieck category. In this paper we aim to establish fundamental results on deconstructible classes and outline how to apply these in the areas mentioned above. This is related to recent work of Gillespie, Enochs, Estrada, Guil Asensio, Murfet, Neeman, Prest, Trlifaj and others.

INTRODUCTION

The main subject of this paper is the notion of a deconstructible class in a Grothendieck category. Roughly speaking, these are classes whose all objects can be formed by transfinite extensions from a set of objects. Such classes arise in plenitude in structure theory of modules and in homological algebra, the classes of abelian $p$-groups, projective modules or flat quasi-coherent sheaves serving as examples. We aim to establish easy to use yet strong enough properties which cover a considerable part of technicalities tackled in the recent work of Gillespie, Estrada, Guil Asensio, Murfet, Neeman, Prest, Trlifaj and others; we refer to [13, 16, 17, 18, 23, 26, 29, 28].

In the last decade, deconstructible classes in module categories have been implicitly used in analyzing structure of modules (e.g. [19]), to build up theory for approximations and cotorsion pairs [19], which was among others successfully applied to solve the Baer splitting problem [3] and to prove finite type of tilting modules [5, 6]. As explained in [30, §4], it turned out recently that analogous ideas apply in a much broader setting. For example in:

1. Gillespie’s construction [16, 17, 18] of flat monoidal model structures on module and sheaf categories (see [23] for a nice overview).
2. Neeman’s [29] and Murfet’s [26, 3.16] proof of existence of certain triangulated adjoint functors without using Brown representability (see [27] for an overview and motivation).

As we explain in a moment, a key point in both settings is the fact that certain classes of chain complexes over modules or sheaves are deconstructible. Here we get to the main aim of the paper: We show that the notion of deconstructibility in

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Grothendieck categories is very well-behaved. The main reason for the favorable properties is a so-called Generalized Hill Lemma (Theorem 2.1 in this paper) which originated in the theory of abelian $p$-groups [22]. It roughly says that given one expression of an object in a Grothendieck category as a transfinite extension with prescribed factors, there are typically many other such expressions. Though being somewhat technical, the Hill Lemma has several nice consequences.

First of all, if $\mathcal{F}$ is a deconstructible class in a Grothendieck category and $X \subseteq F$ is a “small” subobject of $F \in \mathcal{F}$, there is always a “small” subobject $Y$ such that $X \subseteq Y \subseteq F$ and $Y, F/Y \in \mathcal{F}$. Such a property was very important for instance in Gillespie’s considerations in [18]. Giving it a precise meaning and a name in Definition 2.6, we get:

**Corollary 2.7.** Any deconstructible class in a Grothendieck category is a $\kappa$-Kaplansky class for arbitrarily large regular cardinals $\kappa$. On the other hand any Kaplansky class which is closed under taking direct limits is deconstructible.

Next, deconstructible classes are closed under some natural constructions:

**Proposition 2.9.** The following assertions hold for any Grothendieck category $\mathcal{G}$:

1. If $\mathcal{E}$ is a deconstructible class in $\mathcal{G}$ and $\mathcal{F}$ is the class consisting of all direct summands of objects from $\mathcal{E}$, then $\mathcal{F}$ is deconstructible, too.

2. The intersection of any set indexed collection of deconstructible classes is deconstructible again.

A central requirement for homological algebra and many applications, such as those mentioned above, is existence of approximations. Given a full subcategory $\mathcal{F}$ of a category $\mathcal{G}$, we say that a morphism $f : F \rightarrow X$ is an $\mathcal{F}$-precover of $X$ if $F \in \mathcal{F}$ and any other morphism $f' : F' \rightarrow X$ with $F' \in \mathcal{F}$ factors through $f$. The class $\mathcal{F}$ is called precovering if each $X \in \mathcal{G}$ admits an $\mathcal{F}$-precover. Preenvelopes and preenveloping classes are defined dually. Now we have:

**Theorem.** ([30, 2.14] and proof of [30, 4.19]) Let $\mathcal{F}$ be a deconstructible class in a Grothendieck category. Then $\mathcal{F}$ is precovering. If in addition $\mathcal{F}$ is closed under products, it is also preenveloping.

For the results by Gillespie, Neeman and Murfet mentioned above, one needs certain classes of chain complexes of quasi-coherent sheaves to be precovering or preenveloping, which one proves via deconstructibility. To start with, many classes of modules are well-known to be deconstructible (consult [19]). If one wishes to deal with sheaves, good news coming from [13] is that one can prove deconstructibility of a class of sheaves locally:

**Theorem.** ([13, 3.7 and (proof of) 3.8]) Assume that $\mathcal{X}$ is a scheme with the structure sheaf $\mathcal{O}$, and let $V$ be a family of open sets which covers $\mathcal{X}$ as well as all intersections $u \cap v$ for each $u, v \in V$. Assume further that we are given for each $v \in V$ a deconstructible class $\mathcal{F}_v \subseteq \text{Mod-}\mathcal{O}(v)$. Then the class of quasi-coherent sheaves defined as $\mathcal{F} = \{ \mathcal{X} \in \text{Qco}(\mathcal{X}) \mid \mathcal{X}(v) \in \mathcal{F}_v \text{ for each } v \in V \}$ is deconstructible in $\text{Qco}(\mathcal{X})$.

The passage to complexes is also made straightforward by the Hill Lemma. Given a Grothendieck category $\mathcal{G}$ and a deconstructible class $\mathcal{F} \subseteq \mathcal{G}$, various classes of complexes constructed from $\mathcal{F}$ are automatically deconstructible in $\text{C}(\mathcal{G})$. Here we use Notation 4.1 originating in [16]:
Theorem 4.2. The following hold for any Grothendieck category $G$ and a deconstructible class $F \subseteq G$:

1. $C(F)$ is deconstructible in $C(G)$.
2. $\tilde{F}$ is deconstructible in $C(G)$.
3. If $F$ is a generating class in $G$, then $dg-\tilde{F}$ is deconstructible in $C(G)$.

Finally, we mention a general structure result which may be of interest by itself. It was observed (though not yet published at the time of writing of this paper) by Enochs for certain deconstructible classes and used to give an alternative proof that these classes were precovering. The result roughly says that we can in some sense limit the length of the transfinite extensions (see §3 for more explanation). Here, we denote by $\text{Sum} S$ the class of all coproducts of copies of objects in $S$.

Theorem 3.1. Let $G$ be a Grothendieck category, $S$ a set of objects and $F$ the closure of $S$ under transfinite extensions. Then there exists an infinite regular cardinal $\kappa = \kappa(G,S)$ such that any $X \in F$ is a transfinite extension of objects from $\text{Sum} S$ of length $\leq \kappa$.

After giving (hopefully) motivating relations of deconstructible classes to some remarkable recent work of other authors, we are going to prove in the rest of the paper the results above for which we did not give a reference.

Acknowledgments. This work was closely related to and stimulated by work on [30]. I would like to thank Manuel Saorín, my co-author in [30], for useful discussions and especially for suggesting Proposition 4.4. I would also like to thank Jan Trlifaj for communicating the unpublished result by Enochs in §3 and for stimulating discussions.

1. Preliminaries

In this paper, we reserve the notation $G$ for a Grothendieck category, that is, an abelian category with exact direct limits and a generator. Given an infinite regular cardinal $\kappa$, recall that a direct limit in $G$ is called $\kappa$-direct if the indexing set $I$ of the direct system is $\kappa$-directed. That is, each subset of $I$ of cardinality $< \kappa$ has an upper bound in $I$.

An object $X \in G$ is called $< \kappa$-presentable if the functor $\text{Hom}_G(X,-) : G \to \text{Ab}$ preserves $\kappa$-direct limits. An object $X \in G$ is called $< \kappa$-generated if the functor $\text{Hom}_G(X,-)$ preserves all $\kappa$-direct limits with all morphisms in the direct system being monomorphisms. We refer to [11, 15, 32] or [18, App. A] for more information on these notions, and point out that the distinction between direct limits and filtered colimits used in the references is inessential because of [11, 1.5]. A Grothendieck category $G$ is called locally $< \kappa$-presentable if it has a generating set $S$ consisting of $< \kappa$-presentable objects. In all the cases above, we will use the word “finitely” instead of “$< \aleph_0$.” Using this terminology, it is well-known that for a unital ring $R$ and $G = \text{Mod}-R$, the notions of $< \kappa$-presentable and $< \kappa$-generated objects coincide with usual $< \kappa$-presented and $< \kappa$-generated modules, respectively (see [35, §§24.10 and 25.2] for $\kappa = \aleph_0$), and $G$ is locally finitely presentable.

Given an object $X \in G$ and monomorphisms $i : Y \to X$ and $i' : Y' \to X$, we call $i$ and $i'$ equivalent if there is a (unique) isomorphism $f : Y \to Y'$ such that $i = i'f$. Equivalence classes of monomorphisms $Y \to X$ are called subobjects of $X$ and, abusing the notation as usual, denoted by $Y \subseteq X$. If $i : Y \to X$ and $j : Z \to X$
Lemma 1.1. Let \( \mathcal{G} \) be a Grothendieck category. Then the category \( \mathbf{C}(\mathcal{G}) \) is also a Grothendieck category.

Proof. Clearly, \( \mathbf{C}(\mathcal{G}) \) is abelian with exact filtered colimits, since limits and colimits in \( \mathbf{C}(\mathcal{G}) \) are computed componentwise. Suppose \( G \in \mathcal{G} \) is a generator for \( \mathcal{G} \). Then the complexes of the shape

\[
\ldots \to 0 \to 0 \to G \xrightarrow{1_G} G \to 0 \to 0 \to \ldots,
\]

form a generating set for \( \mathbf{C}(\mathcal{G}) \).

There is more structure on \( \mathbf{C}(\mathcal{G}) \) which we will need. Namely, instead of all short exact sequences \( 0 \to X \to Y \to Z \to 0 \) of complexes, we sometimes consider only the sequences for which \( 0 \to X^n \to Y^n \to Z^n \to 0 \) splits in \( \mathcal{G} \) for each \( n \in \mathbb{Z} \). These exact sequences make \( \mathbf{C}(\mathcal{G}) \) an exact category in the sense of \([24\text{ App. A}]\), with the componentwise split exact structure, and allow us to define a corresponding variant of the Yoneda Ext which we denote by \( \text{Ext}^1_{\mathbf{C}(\mathcal{G}), c.s.} \), to distinguish it from the usual Ext-functor on \( \mathbf{C}(\mathcal{G}) \) which we denote by \( \text{Ext}^n_{\mathbf{C}(\mathcal{G})} \). We refer to \([24\text{ App. A}]\) and \([25\text{ XII.4 and XII.5}]\) for details.

Let us stress two facts here. First, for each pair \( Z, X \in \mathbf{C}(\mathcal{G}) \), the group \( \text{Ext}^1_{\mathbf{C}(\mathcal{G}), c.s.}(Z, X) \) is naturally a subgroup of \( \text{Ext}^1_{\mathbf{C}(\mathcal{G})}(Z, X) \). Second, we have the following well-known lemma:

Lemma 1.2. \([20\text{ §1}]\) Assigning to each chain complex morphism \( f : Z[-1] \to X \) the componentwise split exact sequence \( 0 \to X \to C_f \to Z \to 0 \), where \( C_f \) is the mapping cone of \( f \), induces a natural epimorphism

\[
\text{Hom}_{\mathbf{C}(\mathcal{G})}(Z[-1], X) \to \text{Ext}^1_{\mathbf{C}(\mathcal{G}), c.s.}(Z, X),
\]

whose kernel is formed precisely by the null-homotopic morphisms \( Z[-1] \to X \).

Next, we turn to the central concept—filtrations and filtered objects. Generalizing the corresponding concepts from \([19\text{ §3.1}]\), we can define them as follows:
Definition 1.3. Let \( \mathcal{S} \) be a class of objects of \( \mathcal{G} \). An object \( X \in \mathcal{G} \) is called \( \mathcal{S} \)-filtered if there exists a well-ordered direct system \((X_\alpha, i_{\alpha\beta} \mid \alpha < \beta \leq \sigma)\) indexed by an ordinal number \( \sigma \) such that

1. \( X_0 = 0 \) and \( X_\sigma = X \),
2. for each limit ordinal \( \mu \leq \sigma \), the colimit of the subsystem \((X_\alpha, i_{\alpha\beta} \mid \alpha < \beta < \mu)\) is precisely \( X_\mu \), the colimit morphisms being \( i_{\alpha\mu} : X_\alpha \to X_\mu \),
3. \( i_{\alpha\beta} : X_\alpha \to X_\beta \) is a monomorphism in \( \mathcal{G} \) for each \( \alpha < \beta \leq \sigma \),
4. \( \text{Coker } i_{\alpha,\alpha+1} \in \mathcal{S} \) for each \( \alpha < \sigma \).

The direct system \((X_\alpha, i_{\alpha\beta})\) is then called an \( \mathcal{S} \)-filtration of \( X \). The class of all \( \mathcal{S} \)-filtered objects in \( \mathcal{G} \) is denoted by \( \text{Filt-} \mathcal{S} \).

Roughly speaking, \( \text{Filt-} \mathcal{S} \) is the class of all transfinite extensions of objects of \( \mathcal{S} \). We will often consider all \( X_\alpha \) as subobjects of \( X \), in which case condition (2) translates to: \( X_\mu = \bigcup_{\mu \leq \sigma} X_\alpha \) for each limit ordinal \( \mu \leq \sigma \). The key notion here comes up when \( \mathcal{S} \) is a set of objects rather than just a class.

Definition 1.4. A class \( \mathcal{F} \) of objects in \( \mathcal{G} \) is called deconstructible if there is a set \( \mathcal{S} \) such that \( \mathcal{F} = \text{Filt-} \mathcal{S} \).

Remark 1.5. In the literature, it is sometimes only required that there be a set \( \mathcal{S} \subseteq \mathcal{F} \) such that each object of \( \mathcal{F} \) is \( \mathcal{S} \)-filtered, that is, \( \mathcal{F} \subseteq \text{Filt-} \mathcal{S} \). We refer for example to [24]. However, we need the equality between \( \mathcal{F} \) and \( \text{Filt-} \mathcal{S} \) for some results. Luckily, deconstructible classes occurring in practice usually seem to have this property.

Let us establish some elementary closure properties of deconstructible classes:

Lemma 1.6. Let \( \mathcal{F} \) be a deconstructible class in \( \mathcal{G} \). Then any \( \mathcal{F} \)-filtered object of \( \mathcal{G} \) belongs to \( \mathcal{F} \), so \( \mathcal{F} = \text{Filt-} \mathcal{F} \). In particular, \( \mathcal{F} \) is closed under taking coproducts and extensions.

Proof. Assume that \( \mathcal{F} = \text{Filt-} \mathcal{S} \) for some set \( \mathcal{S} \) and that \( X \in \mathcal{G} \) is an \( \mathcal{F} \)-filtered object. That is, there is an \( \mathcal{F} \)-filtration \((X_\alpha, i_{\alpha\beta} \mid \alpha < \beta \leq \sigma)\) with \( X_\sigma = X \). We claim that it is possible to refine this filtration to an \( \mathcal{S} \)-filtration. To see this, fix \( \alpha < \sigma \), denote \( F_\alpha = \text{Coker } i_{\alpha,\alpha+1} \), and consider an \( \mathcal{S} \)-filtration \((G_\gamma, j_{\gamma\delta} \mid \gamma < \delta \leq \tau)\) of \( F_\alpha \). Using the pull back diagrams

\[
\begin{array}{c}
0 \to X_\alpha \xrightarrow{m_{\alpha\gamma}} Y_\gamma \xrightarrow{g_\gamma} G_\gamma \xrightarrow{j_{\gamma\delta}} 0 \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
0 \to X_\alpha \xrightarrow{i_{\alpha,\alpha+1}} X_{\alpha+1} \xrightarrow{F_\alpha} 0.
\end{array}
\]

it is straightforward to construct a direct system \((Y_\gamma, m_{\gamma\delta} \mid \gamma < \delta \leq \tau)\) satisfying (2)–(4) of Definition 1.3 and such that \( Y_0 = X_\alpha \) and \( Y_\tau = X_{\alpha+1} \). Now, we can for each \( \alpha < \tau \) insert such a direct system between \( X_\alpha \) and \( X_{\alpha+1} \). In this way, we obtain an \( \mathcal{S} \)-filtration for \( X \). Hence \( X \in \mathcal{F} \).

The fact that \( \mathcal{F} \) is closed under extensions is then clear. Finally, given a family \((F_\gamma' \mid \gamma \in I)\) of objects of \( \mathcal{F} \), we can assume that \( I = \sigma \) is an ordinal. Then there is a filtration \((X_\alpha, i_{\alpha\beta} \mid \alpha < \beta \leq \sigma)\) for \( \bigoplus_{\gamma \in I} F_\gamma' \), where \( X_\alpha = \bigoplus_{\gamma < \alpha} F_\gamma' \) and the morphisms \( i_{\alpha\beta} \) are the canonical split monomorphisms. \( \square \)
To finish the section, we state a connection between filtrations and the Ext-functor. The proposition below is usually used in connection with so-called cotorsion pairs, a concept which we will not need as such in this paper. The adapted version is then as follows:

**Proposition 1.7.** Let $\mathcal{G}$ be a Grothendieck category and $\mathcal{S}$ be a generating set of objects. Let us define the following classes in $\mathcal{G}$:

- $C = \{ C \in \mathcal{G} \mid \text{Ext}^1_\mathcal{G}(S, C) = 0 \text{ for each } S \in \mathcal{S} \}$,
- $F = \{ F \in \mathcal{G} \mid \text{Ext}^1_\mathcal{G}(F, C) = 0 \text{ for each } C \in C \}$.

Then $\mathcal{F}$ coincides with the class of all direct summands of objects of $\text{Filt}\mathcal{S}$.

**Proof.** The statement is a direct consequence of [15, Lemma 3.6] and [13, Lemma 4.3]. Alternatively, a more general version for certain exact categories is given in [30, Corollary 2.14], while a more specialized version for module categories has been proved in [19, Corollary 3.2.4]. □

**Remark 1.8.** A non-trivial consequence proved later on in Proposition 2.9(1) is that $\mathcal{F}$ is a deconstructible class.

2. **The Hill Lemma for Grothendieck Categories**

An important technical tool for dealing with transfinite filtrations in module categories is known as the Generalized Hill Lemma, cf. [19, Theorem 4.2.6] or [33, Theorem 6]. This result, whose idea is due to Hill [22] and versions of which appeared in [8, 14], roughly says that if we have a single filtration of a module, we automatically get a large family of filtrations. The key point here is that when overcoming some technical details, a completely analogous result holds for all Grothendieck categories.

**Theorem 2.1** (Generalized Hill Lemma). Let $\kappa$ be an infinite regular cardinal and $\mathcal{G}$ a locally $< \kappa$-presentable Grothendieck category. Suppose that $\mathcal{S}$ is a set of $< \kappa$-presentable objects and that $X$ is the union of an $\mathcal{S}$-filtration

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_\alpha \subseteq X_{\alpha+1} \subseteq \cdots \subseteq X_\sigma = X$$

for some ordinal $\sigma$. Then there is a complete sublattice $\mathcal{L}$ of $(\mathcal{P}(\sigma), \cup, \cap)$ and

$$\ell : \mathcal{L} \longrightarrow \text{Subobj}(X)$$

which assigns to each $S \in \mathcal{L}$ a subobject $\ell(S)$ of $X$, such that the following hold:

- (H1) For each $\alpha \leq \sigma$ we have $\alpha = \{ \gamma \mid \gamma < \alpha \} \in \mathcal{L}$ and $\ell(\alpha) = X_\alpha$.
- (H2) If $(S_i)_{i \in I}$ is any family of elements of $\mathcal{L}$, then $\ell(\bigcup S_i) = \sum \ell(S_i)$ and $\ell(\bigcap S_i) = \bigcap \ell(S_i)$. In particular, $\ell$ is a complete lattice homomorphism from $(\mathcal{L}, \cup, \cap)$ to the lattice $(\text{Subobj}(X), \cup, \cap)$ of subobjects of $X$.
- (H3) If $S, T \in \mathcal{L}$ are such that $S \subseteq T$, then the object $N = \ell(T)/\ell(S)$ is $\mathcal{S}$-filtered. More precisely, there is an $\mathcal{S}$-filtration $(N_\beta \mid \beta \leq \tau)$ and a bijection $b : T \setminus S \rightarrow \tau (= \{ \beta \mid \beta < \tau \})$ such that $X_{\alpha+1}/X_\alpha \cong N_{b(\alpha)+1}/N_{b(\alpha)}$ for each $\alpha \in T \setminus S$.
- (H4) For each $< \kappa$-generated subobject $Y \subseteq X$, there is $S \in \mathcal{L}$ of cardinality $< \kappa$ (so $\ell(S)$ is $< \kappa$-presentable by (H3)) such that $Y \subseteq \ell(S) \subseteq X$.

**Remark 2.2.** Although somewhat lengthy, the statement is very natural. To understand the motivation, imagine the following simple model situation. Let $\mathcal{G} = \text{Mod}-R$, $\mathcal{S} = \{ R \}$ and $X$ be a module with an $\mathcal{S}$-filtration $(X_\alpha \mid \alpha \leq \sigma)$. Clearly,
X \cong R^{(\sigma)} is a free module and we can identify \(X_\alpha = R^{(\alpha)}\) for each \(\alpha \leq \sigma\). In this situation, we have the obvious complete lattice homomorphisms \(\ell\) from \((P(\sigma), \cup, \cap)\)
(= the power set of \(\sigma\), when \(\sigma\) is viewed as the set of all smaller ordinals) to the lattice of submodules of \(X\) which is given by \(\ell(S) = R(S)\). The relation between elements \(S \in P(\sigma)\) and \(\ell(S) \subseteq X\) is very well understood and, in particular, allows us to construct many other \(S\)-filtrations of \(X\). The somewhat surprising fact is that much of this setup is preserved for general filtrations in Grothendieck categories.

We devote a considerable part of this section to the proof, which is strongly inspired by the arguments used for [19, Theorem 4.2.6] or [33, Theorem 6]. Let us define \(L\) and \(\ell\) first. Given the \(S\)-filtration \((X_\alpha \mid \alpha \leq \sigma)\) of \(X\), it is not difficult to see that we can fix a family \((A_\alpha \mid \alpha < \sigma)\) of \(<\kappa\)-generated subobjects of \(X\) such that \(X_{\alpha+1} = X_\alpha + A_\alpha\) for each \(\alpha < \sigma\). Then one calls a subset \(S \subseteq \sigma\) closed if every \(\alpha \in S\) satisfies

\[X_\alpha \cap A_\alpha \subseteq \sum_{\gamma < \alpha} A_\gamma.\]

Next we define

\[L = \{S \subseteq \sigma \mid S \text{ is closed}\}\]

and for any subset \(S \subseteq \sigma\) we denote \(\ell(S) = \sum_{\alpha \in S} A_\alpha\). The restriction of \(\ell\) to \(L\) will be the map \(\ell\) from the statement of Theorem 2.3. Let us establish a few basic properties of the just defined concepts.

**Lemma 2.3.** If \(S \subseteq \sigma\) is closed, then \(\ell(S) \cap X_\alpha = \ell(S \cap \alpha)\) for each \(\alpha < \sigma\).

**Proof.** We prove by induction that for each \(\beta \leq \sigma\) we have

\[\ell(S \cap \beta) \cap X_\alpha = \ell(S \cap \beta) \cap X_\alpha.\]

This is clear for \(\beta \leq \alpha\), and for limit ordinals \(\beta\) it follows from the fact that the lattice of subobjects of \(X\) is upper continuous; see Lemma A.6. Assume finally that \(\beta = \delta + 1\) and \(\delta \geq \alpha\). Since \(\ell(S \cap \alpha) \subseteq X_\alpha \subseteq X_\delta\), we are left to show

\[(\ell(S \cap \beta) \cap X_\alpha) \cap X_\delta = \ell(S \cap \alpha) \cap X_\delta.\]

Here we distinguish two cases. Either \(\delta \notin S\) and we simply use the inductive hypothesis, or \(\delta \in S\). In the latter case, we have

\[(\ell(S \cap \beta) \cap X_\alpha) \cap X_\delta = \left((\ell(S \cap \delta) + A_\delta) \cap X_\delta\right) \cap X_\alpha =\]

\[= \left(\ell(S \cap \delta) + (A_\delta \cap X_\delta)\right) \cap X_\alpha = \ell(S \cap \delta) \cap X_\alpha = \ell(S \cap \alpha) = \ell(S \cap \alpha) \cap X_\delta,\]

using that Subobj\((X)\) is modular, that \(S\) is closed, and the inductive hypothesis. □

Next we observe that under some conditions, \(\ell\) commutes with intersections.

**Lemma 2.4.** If \((S_i \mid i \in I)\) is a family of elements of \(L\), then \(\ell\left(\bigcap_{i \in I} S_i\right) = \bigcap_{i \in I} \ell(S_i)\).

**Proof.** Clearly \(\ell\left(\bigcap_{i \in I} S_i\right) \subseteq \bigcap_{i \in I} \ell(S_i)\). Suppose for the moment that the other inclusion does not hold and let \(\beta \leq \sigma\) be minimal such that

\[\ell\left(\bigcap_{i \in I} S_i\right) \cap X_\beta \not\subseteq \bigcap_{i \in I} \ell(S_i) \cap X_\beta\]
Obviously $\beta > 0$ and $\beta$ cannot be a limit ordinal since $\text{Subobj}(X)$ is upper continuous. Thus, $\beta = \delta + 1$ for some $\delta$ and $\ell(\bigcap_{i \in I} S_i) \cap X_\delta = \bigcap_{i \in I} \ell(S_i) \cap X_\delta$. It follows that $\bigcap_{i \in I} \ell(S_i) \cap X_\delta \subseteq \bigcap_{i \in I} \ell(S_i) \cap X_\beta$, so using Lemma 2.5 we get

$$\ell(S_i \cap \delta) = \ell(S_i) \cap X_\delta \subseteq \ell(S_i) \cap X_\beta = \ell(S_i \cap \beta)$$

for each $i \in I$. In particular, $\delta$ must belong to $S_i$ for each $i \in I$. Hence $A_\delta \subseteq \ell(\bigcap_{i \in I} S_i) \subseteq \bigcap_{i \in I} \ell(S_i)$ and we have

$$\left( \bigcap_{i \in I} S_i \right) \cap X_\beta + X_\delta = X_\beta = \left( \bigcap_{i \in I} \ell(S_i) \cap X_\beta \right) + X_\delta.$$  

Since $\text{Subobj}(X)$ is modular, the last equality together with $\ell(\bigcap_{i \in I} S_i) \cap X_\delta = \bigcap_{i \in I} \ell(S_i) \cap X_\delta$ implies that $\ell(\bigcap_{i \in I} S_i) \cap X_\beta = \bigcap_{i \in I} \ell(S_i) \cap X_\beta$, which is a contradiction.

□

Now we are able to prove (H2) of Theorem 2.1.

**Lemma 2.5.** Let $(S_i \mid i \in I)$ be a family of elements of $\mathcal{L}$. Then both $\bigcup_{i \in I} S_i$ and $\bigcap_{i \in I} S_i$ belong to $\mathcal{L}$. In particular, $\ell: \mathcal{L} \to \text{Subobj}(X)$ is a complete lattice homomorphism.

**Proof.** It is easy to see that $S = \bigcup_{i \in I} S_i \in \mathcal{L}$, since for any $\alpha \in S$ there is $i_0 \in I$ such that $\alpha \in S_{i_0}$ and we have

$$X_\alpha \cap A_\alpha \subseteq \bigcup_{\gamma \leq \alpha} A_\gamma \subseteq \bigcup_{\gamma \in S_{i_0}} A_\gamma.$$  

If on the other hand $T = \bigcap_{i \in I} S_i$ and $\alpha \in T$, then the assumptions and Lemma 2.4 yield

$$X_\alpha \cap A_\alpha \subseteq \bigcap_{i \in I} \ell(S_i \cap \alpha) = \ell(T \cap \alpha) = \bigcup_{\gamma \in T} A_\gamma,$$

so $T \in \mathcal{L}$, too. Finally, $\ell: \mathcal{L} \to \text{Subobj}(X)$ is easily seen to be a complete lattice homomorphism using Lemma 2.4. □

At this point we are in a position to complete the proof of Theorem 2.1. Before doing so, we point out that the only assumption we used to prove Lemmas 2.3 to 2.5 was that $\text{Subobj}(X)$ was a complete modular upper continuous lattice.

**Proof of Theorem 2.1.** Lemma 2.5 tells us that $\mathcal{L}$ is a complete sublattice of the lattice $(\mathcal{P}(\sigma), \cup, \cap)$ and that $\ell: \mathcal{L} \to \text{Subobj}(X)$ is a complete lattice homomorphism. This proves (H2). Clearly, $\alpha \in \mathcal{L}$ and $\ell(\alpha) = X_\alpha$ for each $\alpha \leq \sigma$, so (H1) follows.

To establish (H3), we proceed exactly as in the proof of [33 Theorem 6(H3)]. Namely, we consider the filtration $(F_\alpha \mid \alpha \leq \lambda)$ of $\ell(T)/\ell(S)$, where $F_\alpha$ is defined for each $\alpha \leq \sigma$ by

$$F_\alpha = \ell(S \cup (T \cap \alpha)) \quad \text{and} \quad \bar{F}_\alpha = F_\alpha/\ell(S).$$

It follows that for given $\alpha < \sigma$, either $\alpha \in T \setminus S$ and

$$\bar{F}_{\alpha+1}/\bar{F}_\alpha \cong (F_\alpha + A_\alpha)/F_\alpha \cong A_\alpha/(A_\alpha \cap F_\alpha) = A_\alpha/(A_\alpha \cap X_\alpha) \cong X_{\alpha+1}/X_\alpha,$$

or $\bar{F}_{\alpha+1} = \bar{F}_\alpha$. The filtration $(N_\beta \mid \beta \leq \tau)$ is obtained from $(F_\alpha \mid \alpha \leq \sigma)$ just by removing repetitions, and $b: T \setminus S \to \tau$ is defined in the obvious way.
Finally, (H4) is proved similarly as in [33, Theorem 6] on pages 310/311. Given a $<\kappa$-generated subobject $Y \subseteq X$, it is not difficult to see that there is a (not necessarily closed) subset $S' \subseteq \sigma$ of cardinality $<\kappa$ such that $Y \subseteq \ell(S')$. We will prove that each $S' \subseteq \sigma$ of cardinality $<\kappa$ is contained in a closed subset $S \subseteq \sigma$ of cardinality $<\kappa$. In view of Lemma 2.5, it suffices to prove this for singletons $S = \{\beta\}$. This is achieved by induction on $\beta < \sigma$. If $\beta < \kappa$, we simply take $S = \beta + 1$. Otherwise, Lemma A.2 applied on the short exact sequence

$$0 \to X_\beta \cap A_\beta \to A_\beta \to X_{\beta+1}/X_\beta \to 0$$

tells us that $X_\beta \cap A_\beta$ is $<\kappa$-generated. By induction, there is $S_0 \in \mathcal{L}$ of cardinality $<\kappa$ such that $X_\beta \cap A_\beta \subseteq \ell(S_0)$. We claim that $S = S_0 \cap \{\beta\}$ is a set we want.

It is enough to show that $S$ is closed and it suffices to check the definition only for $\beta$. However, we have $X_\beta \cap A_\beta \subseteq X_\beta \cap \ell(S_0) = \ell(S_0 \cap \beta) \subseteq \sum_{\gamma \in S, \gamma < \beta} A_\gamma$ and the claim is proved. To finish, note that $\ell(S)$ is $<\kappa$-presentable by (H3) and Corollary A.5.

Before giving applications, let us point out that the image of $\ell$ is a complete distributive sublattice of Subobj($X$), while Subobj($X$) itself is usually only modular. Now we start illustrating the potential of the theorem by proving certain non-trivial consequences.

We start with a relation to so-called Kaplansky classes (see [12, 18]). Kaplansky classes have been explicitly or implicitly used for proving approximation properties of flat modules or sheaves in various settings [2, 7, 9, 10, 11, 12], a fact which Gillespie [18] and Hovey [23] later applied in a crucial way to constructing monoidal model structures for complexes of sheaves.

**Definition 2.6.** Let $F \subseteq G$ be a class of objects and $\kappa$ a regular cardinal. Then $F$ is said to be a $\kappa$-Kaplansky class if for any $F \in F$ and a $<\kappa$-generated subobject $X \subseteq F$, there exists a $<\kappa$-presentable subobject $Y$ of $F$ such that $X \subseteq Y \subseteq F$ and $Y, F/Y \in F$. We say that $F$ is a Kaplansky class if it is $\kappa$-Kaplansky for some regular cardinal $\kappa$.

We prove as an easy corollary of Theorem 2.1 that a deconstructible class is always a Kaplansky class, a result which for instance considerably simplifies Gillespie’s arguments (especially those in [18, §4]). For module categories an analogous result has been stated as [21, Lemmas 6.7 and 6.9], while for categories of quasi-coherent sheaves this was implicitly proved in [13].

**Corollary 2.7.** Let $F$ be a class of objects in a Grothendieck category $G$. Then the following hold:

1. If $F$ is a deconstructible class, then there is an infinite regular cardinal $\kappa$ such that $F$ is a $\lambda$-Kaplansky class for each $\lambda \geq \kappa$.
2. If $F$ is a Kaplansky class and closed under taking direct limits, then it is deconstructible.

**Proof.** (1) is an easy consequence of Theorem 2.1(H4) and (H3) and Corollary A.5 while part (2) is rather standard and we refer to the proof of [18, Lemma 4.3] for a detailed argument.

**Remark 2.8.** The condition of $F$ being closed under direct limits in Corollary 2.7(2) cannot be easily removed. A recent result of Herbera and Trlifaj [21, Example 6.8]
shows that the class of flat Mittag-Leffler modules over the endomorphism ring of an infinite dimensional vector space is Kaplansky, but not deconstructible.

Further, we show that deconstructibility is kept under some natural operations on classes—under the closure under direct summands and under set-indexed intersections.

**Proposition 2.9.** Given an uncountable regular cardinal $\kappa$ and a locally $<\kappa$-presentable Grothendieck category $\mathcal{G}$, the following hold:

1. Let $\mathcal{E} = \text{Filt-} \mathcal{S}$ in $\mathcal{G}$, where $\mathcal{S}$ is a set of $<\kappa$-presentable objects, and let $\mathcal{F}$ be the class consisting of all direct summands of objects from $\mathcal{E}$. Then there is a set $\mathcal{S}'$ of $<\kappa$-presentable objects such that $\mathcal{F} = \text{Filt-} \mathcal{S}'$.

2. Let $(\mathcal{E}_i \mid i \in I)$ be a collection of classes of objects of $\mathcal{G}$ such that $|I| < \kappa$. Suppose that for each $i \in I$, there is a set $\mathcal{S}_i$ of $<\kappa$-presentable objects such that $\mathcal{E}_i = \text{Filt-} \mathcal{S}_i$. Then there is a set $\mathcal{S}'$ of $<\kappa$-presentable objects such that $\bigcap_{i \in I} \mathcal{E}_i = \text{Filt-} \mathcal{S}'$.

**Proof.** (1) The argument is analogous to [33 Lemma 9]. Suppose that $X \in \mathcal{F}$, that is, there is $Y \in \mathcal{G}$ such that $Z = X \oplus Y$ is $\mathcal{S}$-filtered. We denote by $\pi_{X,Y} : Z \to X$ and $\pi_Y : Z \to Y$ the corresponding split projections and by $\mathcal{S}'$ a representative set of all $<\kappa$-presentable objects of $\mathcal{F}$, and we show that $X$ is $\mathcal{S}'$-filtered.

Let $\ell : \mathcal{L} \to \text{Subobj}(Z)$ be a complete lattice homomorphism as in Theorem 2.1 and let $\mathcal{H} = \{\ell(S) \mid S \in \mathcal{L}\}$. We first show that there is a filtration $(Z_\alpha \mid \alpha < \sigma)$ of $Z$ such that

1. $Z_\alpha \in \mathcal{H}$,
2. $Z_\alpha = \pi_{X,Y}(Z_{\alpha}) + \pi_Y(Z_{\alpha})$, and
3. $Z_{\alpha+1}/Z_\alpha$ is $<\kappa$-presentable for each $\alpha < \sigma$.

To this end, we put $Z_0 = 0$ and $Z_\alpha = \bigcup_{\gamma < \alpha} Z_\gamma$ for limit ordinals. Suppose we have constructed $Z_\alpha \subseteq Z$ and wish to construct $Z_{\alpha+1}$. Note that $\mathcal{G}$ being locally $<\kappa$-presentable ensures that there is a $<\kappa$-generated subobject $W \subseteq Z$ such that $W \not\subseteq Z_\alpha$. Using Theorem 2.1(H4), we find $S \in \mathcal{L}$ of cardinality $<\kappa$ such that $W \subseteq \ell(S)$. Denoting $W' = \ell(S)$, $Q_0 = Z_\alpha + W'$ and combining (H2) and (H3) with Corollary A.3, we observe that $Q_0 \in \mathcal{H}$, $Z_\alpha + W \subseteq Q_0$ and $W', Q_0/Z_\alpha$ are $<\kappa$-presentable. We also have:

$Q_0 \subseteq \pi_X(Q_0) + \pi_Y(Q_0) = \pi_X(Z_\alpha + W') + \pi_Y(Z_\alpha + W') = Z_\alpha + (\pi_X(W') + \pi_Y(W')).$

Since $\pi_X(W') + \pi_Y(W')$ is $<\kappa$-generated, we can use the same argument as above to find $Q_1 \in \mathcal{H}$ such that $\pi_X(Q_0) + \pi_Y(Q_0) \subseteq Q_1$ and $Q_1/Z_\alpha$ is $<\kappa$-presentable.

By proceeding inductively, we find a chain of subobjects $Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \ldots$ from $\mathcal{H}$ such that $Q_i \subseteq \pi_X(Q_{i-1}) + \pi_Y(Q_{i-1}) \subseteq Q_{i+1}$ and $Q_i/Z_\alpha$ is $<\kappa$-presentable for each $i < \omega$. It is easy to see that $Z_{\alpha+1} = \bigcup_{i < \omega} Q_i$ satisfies conditions (i)–(iii) above.

Having constructed the filtration $(Z_\alpha \mid \alpha \leq \sigma)$, it is not difficult to prove that

$Z_{\alpha+1}/Z_\alpha \cong \pi_X(Z_{\alpha+1})/\pi_X(Z_\alpha) \oplus \pi_Y(Z_{\alpha+1})/\pi_Y(Z_\alpha)$

for each $\alpha < \sigma$; we refer to [33 p. 314]. It follows that $(\pi_X(Z_\alpha) \mid \alpha \leq \sigma)$ is an $\mathcal{S}'$-filtration of $X$, as desired.

(2) First we claim that there is a limit ordinal $\tau < \kappa$ and a map $b : \tau \to I$ such that $b^{-1}(i)$ is an unbounded subset in $\tau$ for each $i \in I$. Without loss of generality, we may assume that $I = \mu$ is an infinite cardinal number. Let $\tau$ be the ordinal type of $\omega \times \mu$ with the lexicographical ordering, and denote by $c : \tau \to \omega \times \mu$ the order
isomorphism. Then we can define $b : \tau \to \mu = I$ as the composition of $c$ with the projection $\omega \times \mu \to \mu$ onto the second component. It is straightforward to check that $b$ has the required properties and the claim is proved.

Suppose now that $X \in \bigcap_{i \in I} \mathcal{E}_i$. Then we can for each $i \in I$ fix an $\mathcal{S}$-filtration of $X$ and, using Theorem 2.1 a corresponding complete lattice homomorphism $\ell_i : \mathcal{L}_i \to \text{Subobj}(X)$. Denote $\mathcal{H}_i = \{\ell_i(S) \mid S \in \mathcal{L}_i\}$ for each $i \in I$, and let $\mathcal{H} = \bigcap_{i \in I} \mathcal{H}_i$. It is easy to see from Theorem 2.1(H2) and (H3) that $\mathcal{H}$ is closed under taking arbitrary sums and intersections, and that for each $N, P \in \mathcal{H}$ with $N \subseteq P$ we have $P/N \in \bigcap_{i \in I} \mathcal{E}_i$. Next we will inductively construct a filtration $(X_\alpha \mid \alpha < \sigma)$ of $X$ such that

(a) $X_\alpha \in \mathcal{H}$, and
(b) $X_{\alpha+1}/X_\alpha$ is $\kappa$-presentable for each $\alpha < \sigma$.

If we succeed to do so, it will easily follow that $\bigcap_{i \in I} \mathcal{E}_i = \text{Filt}-\mathcal{S}'$ for a representative set $\mathcal{S}'$ of all $\kappa$-presentable objects in $\bigcap_{i \in I} \mathcal{E}_i$ (cf. also Lemma 1.6).

Again, we put $X_0 = 0$ and $X_\alpha = \bigcup_{\gamma < \alpha} X_\gamma$ for limit ordinals $\alpha$. Suppose now we have constructed $X_\alpha \not\subseteq X$. Then there is a $\kappa$-generated subobject $W \subseteq X$ such that $W \not\subseteq X_\alpha$. Using Theorem 2.1(H2)–(H4) and Corollary A.5 we obtain $Q_0 \in \mathcal{H}_{b(0)}$ such that $X_\alpha + W \subseteq Q_0$ and $Q_0/X_\alpha$ is $\kappa$-presentable. Therefore, there is $W' \subseteq X$ which is $\kappa$-generated and such that $X_\alpha + W' = Q_0$. Using the same argument, we get $Q_1 \in \mathcal{H}_{b(1)}$ such that $Q_0 = X_\alpha + W' \subseteq Q_1$ and $Q_1/X_\alpha$ is $\kappa$-presentable. Proceeding further in this way, we construct a chain $(Q_\iota \mid \iota < \tau)$ of subobjects of $X$ such that $Q_\iota \in \mathcal{H}_{b(\iota)}$ and $Q_\iota/X_\alpha$ is $\kappa$-presentable for each $\iota < \tau$. It easily follows that $X_{\alpha+1} = \bigcup_{\iota < \tau} Q_\iota$ satisfies conditions (a) and (b) above. This finishes the construction and also the proof of the theorem.

3. Bounding the length of a filtration

Given a set $\mathcal{S}$ in a Grothendieck category $\mathcal{G}$, let us denote by $\text{Sum}\mathcal{S}$ the class of all coproducts of copies of objects from $\mathcal{S}$. If we now have an $\mathcal{S}$-filtration $(X_\alpha \mid \alpha \leq \sigma)$ of some $X \in \mathcal{G}$, there is of course no obvious bound on $\sigma$. However, we show that we can always rearrange the filtration to have a bound on its length independent of $X$, if we allow the factors to be in $\text{Sum}\mathcal{S}$ instead of $\mathcal{S}$. This extends a result (unpublished at the time of writing of this paper) by Enochs.

**Theorem 3.1.** Let $\kappa$ be an infinite regular cardinal and $\mathcal{G}$ a locally $\kappa$-presentable Grothendieck category. Given a class $\mathcal{S}$ of $\kappa$-presentable objects and an object $X \in \text{Filt}-\mathcal{S}$, then $X$ has a $(\text{Sum}\mathcal{S})$-filtration of the form $(X'_\beta \mid \beta \leq \kappa)$.

**Remark 3.2.** It is worthwhile to look closer at the case $\kappa = \aleph_0$. The theorem says that given a locally finitely presentable Grothendieck category $\mathcal{G}$, a set $\mathcal{S} \subseteq \mathcal{G}$ of finitely presentable objects and an $\mathcal{S}$-filtered object $X$, then $X$ is the union of a countable chain $0 = X'_0 \subseteq X'_1 \subseteq X'_2 \subseteq X'_3 \subseteq \cdots$ such that $X'_{n+1}/X'_n \in \text{Sum}\mathcal{S}$ for each $n \geq 0$. As a simple model situation, we can think of a right noetherian ring, where any semiartinian right module has a countable filtration by semisimple modules.

**Proof of Theorem 3.1.** Let $(X_\alpha \mid \alpha \leq \sigma)$ be an $\mathcal{S}$-filtration of $X$ and $\ell : \mathcal{L} \to \text{Subobj}(X)$ be a complete lattice homomorphism as in Theorem 2.1. Let us further for each $\alpha < \sigma$ fix a $\kappa$-generated subobject $A_\alpha \subseteq X$ such that $X_{\alpha+1} = X_\alpha + A_\alpha$. 


Theorem [2.4(H4)] allows us to fix for each $\alpha < \sigma$ a set $S_\alpha \in \mathcal{L}$ such that $\alpha \in S_\alpha$, $|S_\alpha| < \kappa$ and $A_\alpha \subseteq \ell(S_\alpha)$. By (H1) and (H2), we may also assume that $S_\alpha \subseteq (\alpha + 1)$, by possibly passing from $S_\alpha$ to $S_\alpha \cap (\alpha + 1)$.

With the notation above, we inductively define a map $\text{lev}: \sigma \to \kappa$ by putting

$$\text{lev}(\alpha) = \sup\{\text{lev}(\gamma) + 1 \mid (\gamma < \alpha) \land (\gamma \in S_\alpha)\},$$

and call $\text{lev}(\alpha)$ the level of $\alpha$. Here, $\sup \emptyset = 0$ by definition. Note that $\text{lev}(\alpha)$ is well-defined since $\kappa$ is regular and $|S_\alpha| < \kappa$ for each $\alpha < \sigma$. Let us denote $T_\beta = \{\gamma \mid (\gamma < \sigma) \land (\text{lev}(\gamma) < \beta)\}$; one readily checks that $T_\beta = \bigcup\{S_\gamma \mid (\gamma < \sigma) \land (\text{lev}(\gamma) < \beta)\}$. We claim that

$$X_\beta' = \ell(T_\beta) = \left(\sum_{\gamma < \beta, \text{lev}(\gamma) < \beta} A_\gamma\right)$$

yields a $\langle \text{Sum} \mathcal{S}\rangle$-filtration $(X_\beta' \mid \beta \leq \kappa)$ of $X$, as desired. The only non-trivial part is proving that for any fixed $\beta < \kappa$ we have $X_\beta'/X_\beta' \in \text{Sum} \mathcal{S}$. We will show more, namely that

$$X_{\beta + 1}'/X_\beta' \cong \bigoplus_{\gamma < \beta} X_{\gamma + 1}/X_\gamma.$$

Since for each $\delta < \sigma$ we have $X_{\delta + 1}'/X_\delta \cong A_\delta/(A_\delta \cap X_\delta)$, it is easy to see that it suffices to show for each $\delta < \sigma$ with $\text{lev}(\delta) = \beta$ that

$$A_\delta \cap X_\delta' = A_\delta \cap X_\delta \quad \text{and} \quad (A_\delta + X_\delta') \cap \sum_{\gamma < \delta, \text{lev}(\gamma) = \beta} A_\gamma \subseteq X_\delta'.$$

For the first part, note that we have $A_\delta \cap X_\delta \subseteq \ell(S_\delta) \cap X_\delta = \ell(S_\delta \cap \delta) \subseteq X_\delta'$ and on the other hand, using Theorem [2.4(H1)] and (H2) and the fact that $\delta \not\in T_\beta$, also:

$$A_\delta \cap X_\delta' \subseteq \ell(\delta + 1) \cap \ell(T_\beta) = \ell((\delta + 1) \cap T_\beta) \subseteq \ell(\delta) = X_\delta.$$

Hence $A_\delta \cap X_\delta' = A_\delta \cap X_\delta$. The second part is similar. Namely, given any $\gamma < \sigma$ of level $\beta$, the only element in $S_\gamma$ whose level is at least $\beta$ is $\gamma$ itself. It follows that $S_\delta \cap S_\gamma \subseteq T_\beta$ for any such $\gamma < \delta$, and

$$(A_\delta + X_\delta') \cap \sum_{\gamma < \delta, \text{lev}(\gamma) = \beta} A_\gamma \subseteq \ell(S_\delta \cup T_\beta) \cap \sum_{\gamma < \delta, \text{lev}(\gamma) = \beta} \ell(S_\gamma) =$$

$$\ell\left((S_\delta \cup T_\beta) \cap \bigcup_{\gamma < \delta, \text{lev}(\gamma) = \beta} S_\gamma\right) \subseteq \ell(T_\beta) = X_\beta'.$$

This finishes the proof of the claim and also of the theorem. \qed

4. Deconstructibility for complexes

In this section, we discuss a crucial point of papers [16, 17] by Gillespie on construction of certain nice model structures and also papers [28, 29] by Neeman which are focused on existence of certain triangulated adjoint functors. Namely, given a Grothendieck category $\mathcal{G}$ and a deconstructible class $\mathcal{F} \subseteq \mathcal{G}$, one requires that certain classes in $\text{C}(\mathcal{G})$ determined by $\mathcal{F}$ also be deconstructible. There are various candidates for classes determined by $\mathcal{F}$. In the simplest case, we may take $\text{C}(\mathcal{F})$, the full subcategory of $\text{C}(\mathcal{G})$ formed by the complexes with components in $\mathcal{F}$. In the above mentioned papers, other classes played an important role, too:
Notation 4.1. Let $\mathcal{F}$ be a class of objects in a Grothendieck category $\mathcal{G}$, and let $\mathcal{C} = \{ C \in \mathcal{G} \mid \text{Ext}^1_{\mathcal{G}}(F, C) = 0 \text{ for each } F \in \mathcal{F} \}$. Then we denote:

1. by $\hat{\mathcal{F}}$ the class of all acyclic complexes $X \in \mathbf{C}(\mathcal{G})$ such that $Z^n(X) \in \mathcal{F}$ for each integer $n$;
2. by $\text{dg-}\hat{\mathcal{F}}$ the class of all complexes $X \in \mathbf{C}(\mathcal{F})$ such that every chain complex morphism $X \to Y$ with $Y \in \mathcal{C}$ is null-homotopic. Here, $\mathcal{C}$ follows the notation from (1) for $\mathcal{C}$ in place of $\mathcal{F}$.

Although the definition of $\text{dg-}\hat{\mathcal{F}}$ may seem mysterious at the first sight, it only generalizes Spaltenstein’s concept of $K$-projective complexes [31], which is reconstructed by taking $\mathcal{G} = \text{Mod-}R$ and for $\mathcal{F}$ the class of all projective modules. A slight modification of [16, Lemma 3.9] shows that for an extension closed class $\mathcal{F}$ we always have the inclusions

$$\hat{\mathcal{F}} \subseteq \text{dg-}\hat{\mathcal{F}} \subseteq \mathbf{C}(\mathcal{F}).$$

Our main result in this direction then says:

Theorem 4.2. Let $\mathcal{G}$ be a Grothendieck category and $\mathcal{F}$ be a deconstructible class of objects of $\mathcal{G}$. Then the following assertions hold:

1. $\mathbf{C}(\mathcal{F})$ is deconstructible in $\mathbf{C}(\mathcal{G})$. More precisely, there is a set $Q$ of bounded below complexes from $\mathbf{C}(\mathcal{F})$ such that $\mathbf{C}(\mathcal{F}) = \text{Filt-}Q$.
2. $\hat{\mathcal{F}}$ is deconstructible in $\mathbf{C}(\mathcal{G})$. More precisely, there is a set $U$ of bounded above complexes from $\hat{\mathcal{F}}$ such that $\hat{\mathcal{F}} = \text{Filt-}U$.
3. If $\mathcal{F}$ is a generating class in $\mathcal{G}$, then $\text{dg-}\hat{\mathcal{F}}$ is deconstructible in $\mathbf{C}(\mathcal{G})$. In this case, each $X \in \text{dg-}\hat{\mathcal{F}}$ is a summand of a complex filtered by the stalk complexes of the form $F[n]$ with $F \in \mathcal{F}$ and $n \in \mathbb{Z}$.

We will prove the theorem in a few steps, giving a corresponding argument for each of the parts (1), (2) and (3) separately. We start with Theorem 4.2(1).

Proposition 4.3. Let $\kappa$ be an infinite regular cardinal and $\mathcal{G}$ a locally $\kappa$-presentable Grothendieck category. Suppose that $\mathcal{F} \subseteq \mathcal{G}$ is a deconstructible class such that $\mathcal{F} = \text{Filt-}S$, where $S$ is a representative set of all $\kappa$-presentable objects of $\mathcal{G}$ contained in $\mathcal{F}$. Then each $X \in \mathbf{C}(\mathcal{F})$ is filtered by bounded below complexes with components in $S$. In particular, $\mathbf{C}(\mathcal{F})$ is deconstructible.

Proof. Let $\kappa$, $\mathcal{G}$, $\mathcal{F}$ and $S$ be as above, and denote by $Q$ the class of all bounded below complexes over $\mathcal{G}$ with all components in $S$. Given $X \in \mathbf{C}(\mathcal{F})$, we fix for each component $X^n$ an $S$-filtration and for this filtration a complete lattice morphism $\ell_n : L_n \to \text{Subobj}(X^n)$ provided by Theorem 2.1. For each integer $n$, we further put $H_n = \{ \ell_n(S) \mid S \in L_n \}$.

Using this data, we will construct by induction a $Q$-filtration $(X_\alpha \mid \alpha \leq \sigma)$ of the complex $X$ such that $X_\alpha \in H_n$ for each $n \in \mathbb{Z}$ and $\alpha \leq \sigma$. We are required to put $X_0 = 0$ and take direct unions of subcomplexes at limit steps. For a non-limit step, assume we have constructed $X_\alpha \subseteq X$ and we take an integer $n$ and a $\kappa$-generated subobject $W \subseteq X^n$ such that $W \not\subseteq X_\alpha^n$. Then we put $X_{\alpha+1}^m = X_\alpha^m$ for $m < n$ and take $X_{\alpha+1}^n \in H_n$ such that $X_\alpha^n + W \subseteq X_{\alpha+1}^n$ and $X_{\alpha+1}^n/X_\alpha^n$ is $\kappa$-presentable and $S$-filtered. Note that we can always do this using Theorem 2.1(H4) and (H3) and Corollary 4.5. Further note that, up to isomorphism, $X_{\alpha+1}^n/X_\alpha^n \in S$ by Lemma 1.4.

For $m > n$ we proceed by induction. Suppose we have already constructed $X_{\alpha+1}^{m-1}$ such that $X_{\alpha+1}^{m-1}/X_\alpha^{m-1} \in S$, up to isomorphism. Then there is an $\alpha$-generated
subobject $W' \subseteq X_{\alpha+1}^m$ such that $X_{\alpha+1}^m = X_{\alpha}^m + W'$. Since $d_{X_{\alpha}^m}^m(W')$ is a $\kappa$-generated subobject of $X_{\alpha}^m$, we can again use Theorem 2.1 to find $X_{\alpha}^m \in \mathcal{H}_\alpha$ such that $X_{\alpha}^m + d_{X_{\alpha}^m}^m(W') \subseteq X_{\alpha+1}^m$ and $X_{\alpha+1}^m/X_{\alpha}^m$ is isomorphic to an object from $\mathcal{S}$. This finishes the induction. It is then easy to check that $X_{\alpha+1}^m$ is a subcomplex of $X$ and $X_{\alpha+1}^m/X_{\alpha}^m$ is isomorphic to an element of $\mathcal{Q}$.

The case of $\tilde{F}$ (see Notation 1.1) is similar, but technically more involved.

**Proposition 4.4.** Let $\kappa$ be an infinite regular cardinal and $\mathcal{G}$ a locally $< \kappa$-presentable Grothendieck category. Suppose that $\mathcal{F} \subseteq \mathcal{G}$ is a deconstructible class such that $\mathcal{F} = \text{Filt-} \mathcal{S}$ for a set $\mathcal{S}$ of $< \kappa$-presentable objects. Then each $X \in \tilde{F}$ is filtered by bounded above complexes from $\tilde{F}$ with $< \kappa$-presentable components. In particular, $\tilde{F}$ is deconstructible.

**Proof.** Let $\mathcal{U}$ be a representative set for all bounded above complexes in $\tilde{F}$ with $< \kappa$-presentable components. We must prove that $\tilde{F} = \text{Filt-} \mathcal{U}$. In fact, it is easy to see using Lemma 1.6 that $\text{Filt-} \mathcal{U} \subseteq \tilde{F}$, so we are left with proving that any $X \in \tilde{F}$ is $\mathcal{U}$-filtered.

Given such $X$, we fix for each cycle object $Z^n(X)$ an $\mathcal{S}$-filtration and for this filtration a complete lattice morphisms $\ell_n : \mathcal{L}_n \to \text{Subobj}(Z^n(X))$ provided by Theorem 2.1. For each integer $n$, we further put $\mathcal{H}_n = \{ \ell_n(S) \mid S \in \mathcal{L}_n \}$. Now we will inductively construct a filtration $(X_\alpha \mid \alpha \leq \sigma)$ of $X$ such that for each $\alpha < \sigma$:

1. $X_\alpha$ is an acyclic complex,
2. $X_{\alpha+1}/X_\alpha$ is bounded above,
3. $Z^n(X_\alpha) \in \mathcal{H}_n$ for each $n \in \mathbb{Z}$, and
4. $Z^n(X_{\alpha+1})/Z^n(X_\alpha)$ is $< \kappa$-presentable for each $n \in \mathbb{Z}$.

Assume for the moment we have such a filtration. Then $X_{\alpha+1}/X_\alpha$ is an acyclic complex for each $\alpha$ by the $3 \times 3$ lemma. Since each $Z^n(X_{\alpha+1})/Z^n(X_\alpha)$ is $< \kappa$-presentable, Lemma 1.3 tells us that each component of $X_{\alpha+1}/X_\alpha$ is $< \kappa$-presentable. Using (2), (3) and Theorem 2.1(H3), it immediately follows that $X_{\alpha+1}/X_\alpha \in \mathcal{U}$, up to isomorphism. Hence $(X_\alpha \mid \alpha \leq \sigma)$ is a $\mathcal{U}$-filtration and we are done.

To construct the filtration, we put $X_0 = 0$ and $X_\alpha = \bigcup_{\gamma < \alpha} X_\gamma$ for limit ordinals $\alpha \leq \sigma$. Note that $X_\alpha$ is acyclic in the latter case because direct limits are exact in $\mathcal{G}$. For non-limit steps, assume we have constructed $X_\alpha \nsubseteq X$ for some $\alpha$ and we wish to construct $X_{\alpha+1}$. It is easy to see that there is $n \in \mathbb{Z}$ such that $Z^n(X_\alpha) \nsubseteq Z^n(X)$. We put $X_{\alpha+1}^m = X_\alpha^m$ for all $m > n$ and construct $X_{\alpha+1}^m$ inductively for $m \leq n$. To this end, suppose that $X_{\alpha+1}^{m+1}$ has been already constructed and denote $Z_{\alpha+1}^{m+1} = Z_{\alpha+1}^m \cap Z^{m+1}(X)$. Since $Z_{\alpha+1}^{m+1}/Z_{\alpha+1}^{m+1}(X_\alpha)$ is $< \kappa$-presentable (see requirement (4) above), there is a $< \kappa$-generated subobject $W \subseteq X_\alpha^m$ such that $Z_{\alpha+1}^{m+1} = Z^{m+1}(X_\alpha) + d_{X_\alpha^m}^{m+1}(W)$. If $m = n$, we in addition take $W$ so that $W \nsubseteq Z^n(X_\alpha)$, which will ensure further in the construction that $X_\alpha^m \nsubseteq X_{\alpha+1}^m$. Denoting $K = (X_\alpha^m + W) \cap Z^n(X)$, we get the following diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & Z^n(X_\alpha) & \longrightarrow & X_\alpha^m & \longrightarrow & Z_{\alpha+1}^{m+1}(X_\alpha) \\
& \downarrow{\subseteq} & \downarrow{\subseteq} & \downarrow{\subseteq} & \downarrow{\subseteq} & \downarrow{\subseteq} \\
0 & \longrightarrow & K & \longrightarrow & X_\alpha^m + W & \longrightarrow & Z_{\alpha+1}^{m+1}(X_\alpha) \\
\end{array}
\]
Using the $3 \times 3$ lemma, we get an exact sequence

$$0 \to K/Z^m(X_\alpha) \to (X^m_\alpha + W)/X^m_\alpha \to Z^{m+1}(X_\alpha) \to 0.$$ 

Since $Z^{m+1}(X_\alpha)$ is $< \kappa$-presentable and $(X^m_\alpha + W)/X^m_\alpha \cong W/(X^m_\alpha \cap W)$ is $< \kappa$-generated, the factor $K/Z^m(X_\alpha)$ is $< \kappa$-generated and such that $K = Z^m(X_\alpha) + W$. By Theorem 2.1 (H4), there is a set $S \subseteq L_m$ of cardinality $< \kappa$ such that $W' \subseteq \ell_m(S)$. Now we set $Z^{m+1}_\alpha = Z^m(X_\alpha) + \ell_m(S)$ and $X^{m+1}_\alpha = X^m_\alpha + W + \ell_m(S)$. Observe that using modularity of the lattice of subobjects of $X^m$, we have

$$X^{m+1}_\alpha \cap Z^m(X) = K + \ell_m(S) = Z^m(X_\alpha) + \ell_m(S) = Z^m\alpha+1.$$ 

Further, using Theorem 2.1 and Corollary A.3 one readily infers that

$$Z^m_\alpha = Z^m(X_\alpha) + \ell_m(S) \in \mathcal{H}_m$$

and $Z^{m+1}_\alpha/Z^m(X_\alpha)$ is $S$-filtered and $< \kappa$-presentable.

Finally, since $\ell_m(S) \subseteq Z^m(X)$, we have $\text{Im}(d^m_X|_{X^{m+1}_\alpha}) = Z^{m+1}_\alpha$. Therefore, $X^{m+1}_\alpha \subseteq X^m$ has all the required properties. This finishes the induction step for $m$ and also the construction. 

To finish the proof of Theorem 4.2, we focus on part (3). Our exposition here uses ideas of Gillespie from [16, §3], which were also nicely presented by Hovey in [23, §7].

**Proposition 4.5.** Let $\mathcal{G}$ be a Grothendieck category, and $\mathcal{F} \subseteq \mathcal{G}$ be a class of objects which is generating and deconstructible. Then each $X \in \text{dg-} \mathcal{F}$ is a summand of an object filtered by stalk complexes of the form $F[n]$ with $F \in \mathcal{F}$ and $n \in \mathbb{Z}$. In particular, $\text{dg-} \mathcal{F}$ is deconstructible.

**Proof.** Let us fix a set $S \subseteq \mathcal{F}$ which is generating in $\mathcal{G}$ and such that $\mathcal{F} = \text{Filt}-S$, and denote

$$C = \{C \in \mathcal{G} | \text{Ext}^1_\mathcal{G}(S, C) = 0 \text{ for each } S \in \mathcal{S}\}.$$ 

Let $Q$ be the set of all stalk complexes of the form $S[n]$ for an integer $n$ and $S \in \mathcal{S}$, and denote by $\mathcal{F}'$ the class of all direct summands of objects from $\mathcal{F}$. With this notation (and using Notation 1.1), we will prove that

(1) $C = \{Y \in C(\mathcal{G}) | \text{Ext}^1_{C(\mathcal{G})}(Q, Y) = 0 \text{ for each } Q \in \mathcal{Q}\}$, and

(2) $\text{dg-} \mathcal{F}' = \{X \in C(\mathcal{G}) | \text{Ext}^1_{C(\mathcal{G})}(X, Y) = 0 \text{ for each } Y \in \mathcal{C}\}$.

The first statement of the proposition will then become an immediate consequence of Proposition 1.7. Indeed, note that $\text{Filt}-Q$ is a generating class in $C(\mathcal{G})$ since it contains complexes of the form

$$\ldots \to 0 \to 0 \to S \xrightarrow{1_\mathcal{S}} S \to 0 \to 0 \to \ldots$$

with $S \in \mathcal{S}$, and $S$ is assumed to be generating in $\mathcal{G}$. Then $\text{dg-} \mathcal{F}'$ is the closure of Filt-$Q$ under direct summands by Lemmas 1.1 and 1.6, and $\text{dg-} \mathcal{F} = C(\mathcal{F}) \cap \text{dg-} \mathcal{F}'$ directly from the definition.

Let us prove (1) and (2). For (1), assume first that $Y \in C(\mathcal{G})$ is such that $\text{Ext}^1_{C(\mathcal{G})}(Q, Y) = 0$. To begin with, we show that $Y$ is acyclic. Suppose it is not, so $H^n(Y) \neq 0$ for some $n \in \mathbb{Z}$. Then there is a chain complex morphism $f : S[-n] \to Y$ with $S \in \mathcal{S}$, which is not null-homotopic. Indeed, there is an epimorphism in $\mathcal{G}$ of the form $\bigoplus_{i \in I} S_i \to Z^n(Y)$ with $S_i \in \mathcal{S}$ for each $i \in I$. If $H^n(Y) \neq 0$, at least one of the components $S_i \to Z^n(Y)$ cannot factor through the differential map $Y^{n-1} \to Z^n(Y)$ of $Y$, and we can take $S = S_i$ and the induced chain complex
morphism $S[-n] \to Y$ for $f$. Having such $f$ and using Lemma [12] the mapping cone $C_f$ of $f$ fits into a componentwise split but non-split exact sequence

$$0 \to Y \to C_f \to S[-n + 1] \to 0,$$

which contradicts $\text{Ext}^1_{C(\mathcal{G})}(Q, Y) = 0$. Hence $X$ is acyclic.

Next we show that $Z^n(Y) \in \mathcal{C}$ for all $n \in \mathbb{Z}$. If not, then we would have a non-split extension $0 \to Z^n(Y) \to E \to S \to 0$ in $\mathcal{G}$ for some integer $n$ and $S \in \mathcal{S}$. This would induce, using the pushout diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & Z^n(Y) & \longrightarrow & Y^n & \longrightarrow & Z^{n+1}(Y) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E & \longrightarrow & W^n & \longrightarrow & Z^{n+1}(Y) & \longrightarrow & 0,
\end{array}$$

a non-split extension

$$0 \to Y \to W \to S[-n] \to 0$$

in $C(\mathcal{G})$, which is a contradiction again. Hence $Z^n(Y) \in \mathcal{C}$ for each integer $n$ and consequently $Y \in \hat{\mathcal{C}}$.

Assume on the other hand that $Y \in \hat{\mathcal{C}}$. Since $\mathcal{C}$ is extension closed, any extension of the form

$$0 \to Y \to W \to S[n] \to 0$$

with $S \in \mathcal{S}$ is componentwise split. In particular, we have the equality

$$\text{Ext}^1_{C(\mathcal{G})}(S[n], Y) = \text{Ext}^1_{C(\mathcal{G}), c.s.}(S[n], Y).$$

To prove that the Ext-groups above vanish, we must in view of Lemma [12] show that any morphism $f : S'[n-1] \to Y$ is null-homotopic. To see this, note that such $f$ is given by a morphism $f' : S \to Z^{n-1}(Y)$ in $\mathcal{G}$, and $f'$ factors through the epimorphism $Y^{n-2} \to Z^{n-1}(Y)$ since $Z^{n-2}(Y) \in \mathcal{C}$. It follows that $\text{Ext}^1_{C(\mathcal{G})}(Q, Y) = 0$ and the proof of (1) is finished.

For (2), first assume that $X$ is a complex over $\mathcal{G}$ such that $\text{Ext}^1_{C(\mathcal{G})}(X, \hat{C}) = 0$. We claim that $X$ must belong to $C(\mathcal{F}')$. Equivalently by Proposition [17], we must show that $\text{Ext}^1_{\mathcal{G}}(X^n, \mathcal{C}) = 0$ for each component $X^n$ of $X$. To this end, we employ a similar argument as in [10] Proposition 3.6. Namely, note that if there is a non-split extension $0 \to C' \to V^n \to X^n \to 0$ for an integer $n$ and $C' \in \mathcal{C}$, then the pullback diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & C' & \longrightarrow & V^{n-1} & \longrightarrow & X^{n-1} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C' & \longrightarrow & V^n & \longrightarrow & X^n & \longrightarrow & 0
\end{array}$$

induces a non-split extension

$$0 \to C \to V \to X \to 0$$

in $C(\mathcal{G})$, where $C \in \hat{\mathcal{C}}$ is the correspondingly shifted complex of the form

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow C' \overset{1_c}{\longrightarrow} C' \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots.$$

This would contradict our assumption, so the claim is proved. Next, note that if $C \in \hat{\mathcal{C}}$, then any extension of $C$ by $X$ must be componentwise split, so

$$\text{Ext}^1_{C(\mathcal{G})}(X, C) = \text{Ext}^1_{C(\mathcal{G}), c.s.}(X, C).$$
Hence, \( X \in \text{dg-} \tilde{\mathcal{F}} \) directly by Lemma 1.2 and the definition; see Notation 4.1.

If on the other hand \( X \in \text{dg-} \tilde{\mathcal{F}} \), we just retrace the steps. Namely, we observe that for each \( C \in \mathcal{C} \) we have \( \text{Ext}^1_{\mathcal{C}(G),c.s.}(X,C) = 0 \) because of Lemma 1.2 and \( \text{Ext}^1_{\mathcal{C}(G),c.s.}(X,C) = \text{Ext}^1_{\mathcal{C}(G),c.s.}(X,C) \) because \( X \in \mathcal{F}' \). Hence \( \text{Ext}^1_{\mathcal{C}(G),c.s.}(X,C) = 0 \) and the proof of (2) is complete.

Finally, to prove the deconstructibility of \( \text{dg-} \tilde{\mathcal{F}} \), note that we have proved that \( \text{dg-} \tilde{\mathcal{F}} \) consists precisely of direct summands of complexes from \( \text{Filt-Q} \). Thus, both \( \text{dg-} \tilde{\mathcal{F}} \) and \( \text{dg-} \tilde{\mathcal{F}} = \mathcal{C}(\mathcal{F}) \cap \text{dg-} \tilde{\mathcal{F}} \) are deconstructible by Propositions 2.9 and 4.3. \( \square \)

**Proof of Theorem 4.2.** The theorem now follows directly by Propositions 4.3–4.5, when taking \( \kappa \) appropriately large. \( \square \)

**Appendix A. Properties of Grothendieck categories**

In the text we need some basic properties of \(< \kappa \)-presentable and \(< \kappa \)-generated objects in Grothendieck categories. We also need to know that the lattice of sub-objects of an object in a Grothendieck category is always modular and upper continuous. These facts can be mostly found in the monographs [32] by Stenström and [15] by Gabriel and Ulmer. Here we give a short overview of the necessary properties together with appropriate references or short proofs. We start with a basic property of any Grothendieck category.

**Lemma A.1.** Any Grothendieck category \( \mathcal{G} \) is locally \(< \kappa \)-presentable for some infinite regular cardinal \( \kappa \). In particular, for any \( X \in \mathcal{G} \) there is an infinite regular cardinal \( \lambda = \lambda(X) \) such that \( X \) is \(< \lambda \)-presentable.

**Proof.** This is a well-known consequence of the Popescu-Gabriel Theorem [32 X.4.1]. Namely, if \( G \in \mathcal{G} \) is a generator and \( R = \text{End}_{\mathcal{G}}(G) \), then \( H = \text{Hom}_{\mathcal{G}}(G,-) : \mathcal{G} \to \text{Mod-R} \) induces an equivalence of \( \mathcal{G} \) with a full subcategory \( \mathcal{G}' \subseteq \text{Mod-R} \). By its definition on [32] pp. 198–199], \( \mathcal{G}' \) is closed under \( \kappa \)-direct limits in \( \text{Mod-R} \) for some infinite regular cardinal \( \kappa \) and \( H(G) = R \) is clearly \(< \kappa \) presentable in \( \mathcal{G}' \). \( \square \)

Next we have an important characterization of \(< \lambda \)-presentable objects for \( \lambda \) large enough.

**Lemma A.2.** Let \( \mathcal{G} \) be a Grothendieck category and \( \kappa \) an infinite regular cardinal such that \( \mathcal{G} \) is locally \(< \kappa \)-presentable. Then the following are equivalent for an object \( X \in \mathcal{G} \) and a regular cardinal \( \lambda \geq \kappa \):

1. \( X \) is \(< \lambda \)-presentable.
2. \( X \) is \(< \lambda \)-generated and, whenever \( 0 \to K \to E \to X \to 0 \) is a short exact sequence in \( \mathcal{G} \) such that \( E \) is \(< \lambda \)-generated, \( K \) is also \(< \lambda \)-generated.

**Proof.** A straightforward generalization of the proof for [32 Proposition V.3.4] (which is given for \( \kappa = \aleph_0 \)) applies. See also [15 6.6(e)]. \( \square \)

The previous proposition provides us with a way to recognize \(< \lambda \)-generated and presentable objects using their presentations by a fixed family of \(< \kappa \)-presentable generators.

**Lemma A.3.** [15 7.6 and 9.3] Let \( \mathcal{G} \) be a Grothendieck category, \( \kappa \) be an infinite regular cardinal, and assume \( \mathcal{G} \) has a generating set \( \mathcal{S} \) consisting of \(< \kappa \)-presentable objects. Then the following hold for an object \( X \in \mathcal{G} \) and a regular cardinal \( \lambda \geq \kappa \):
(1) $X$ is $< \lambda$-generated if and only if there exists an exact sequence
\[ \bigoplus_{i \in I} S_i \rightarrow X \rightarrow 0 \]
with $|I| < \lambda$ and $S_i \in \mathcal{S}$ for all $i \in I$.

(2) $X$ is $< \lambda$-presentable if and only if there exists an exact sequence
\[ \bigoplus_{j \in J} S_j \rightarrow \bigoplus_{i \in I} S_i \rightarrow X \rightarrow 0 \]
with $|I|, |J| < \lambda$ and $S_i, S_j \in \mathcal{S}$ for all $i \in I$ and $j \in J$.

Proof. The ‘if’ part of (1) follows from the fact that $< \lambda$-generated objects are closed under factors and coproducts with $< \lambda$ summands. For the ‘only if’ part of (1), take an epimorphism $p : \bigoplus_{u \in U} S_u \rightarrow X$ for some set $U$ and denote by $I$ be the set of all subsets of $U$ of cardinality $< \lambda$. Then $X = \lim_{I \in I} \operatorname{Im}(p \lceil \bigoplus_{i \in I} S_i)$ is a $\lambda$-directed limit of monomorphisms, so $\operatorname{Im}(p \lceil \bigoplus_{i \in I} S_i) = X$ for some $I \in I$.

Part (2) is an easy consequence of (1) and Lemma A.2. □

It is well-known for module categories that the classes of $< \lambda$-generated and $< \lambda$-presented modules are closed under extensions. We have an analogue for Grothendieck categories.

Lemma A.4. Let $\mathcal{G}$ be a Grothendieck category and $\kappa$ an infinite regular cardinal such that $\mathcal{G}$ is locally $< \kappa$-presentable. Then for any regular cardinal $\lambda \geq \kappa$, the classes of $< \lambda$-generated and $< \lambda$-presentable objects are closed under extensions.

Proof. For the class of $< \lambda$-generated objects, an obvious generalization of [32, Lemma V.3.1(ii)] applies. We do not even use the assumption that $\mathcal{G}$ is locally $< \kappa$-presentable.

Suppose we have a short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ such that $X, Z$ are $< \lambda$-presentable. Then $Y$ is $< \lambda$-presentable. In order to apply Lemma A.2 assume that $0 \rightarrow K \rightarrow E \rightarrow Y \rightarrow 0$ is any short exact sequence such that $E$ is $< \lambda$-generated and consider the commutative diagram with exact rows, where $L = \operatorname{Ker}(pq) :
\begin{array}{ccccc}
0 & \rightarrow & L & \rightarrow & E & \rightarrow & Z & \rightarrow & 0 \\
& & | & | & | & | & | & | \\
0 & \rightarrow & X & \rightarrow & Y & \rightarrow & Z & \rightarrow & 0
\end{array}
$ Since $Z$ is $< \lambda$-presentable, $L$ is $< \lambda$-generated by Lemma A.2. Since $X$ is $< \lambda$-presentable, $K \cong \operatorname{Ker} t$ is $< \lambda$-generated. Applying Lemma A.2 for the third time, it follows that $Y$ is $< \lambda$-presentable. □

In the proof of the Generalized Hill Lemma (Theorem 2.1), the following consequence is used:

Corollary A.5. Let $\mathcal{G}$ be a locally $< \kappa$-presentable Grothendieck category for some infinite regular cardinal $\kappa$. If $X$ is an object with a filtration $(X_\alpha \mid \alpha \leq \sigma)$ by $< \kappa$-presentable objects and such that $\sigma < \kappa$, then $X$ is $< \kappa$-presentable.

Proof. This is easily proved by induction on $\sigma$ using Lemma A.2 and the fact that the class of $< \kappa$-presentable objects is closed under taking colimits of diagrams of size less than $\kappa$ (see [15, Satz 6.2]). □
Finally, we establish some properties of lattices of subobjects. Recall that if $(\mathcal{L}, \vee, \wedge)$ is a complete lattice, we call $\mathcal{L}$ upper continuous (cf. [32, §III.5]) if

$$\left( \bigvee_{d \in D} d \right) \wedge a = \bigvee_{d \in D} (d \wedge a)$$

whenever $a \in \mathcal{L}$ and $D \subseteq \mathcal{L}$ is a directed subset. Now we have:

**Lemma A.6.** Let $\mathcal{G}$ be a Grothendieck category. If $X \in \mathcal{G}$ is an object, the lattice $(\text{Subobj}(X), \Sigma, \cap)$ of subobjects is a complete modular upper continuous lattice.

**Proof.** This is proved in [32, Propositions IV.5.3 and V.1.1(c)]. □

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