Abstract

A distinguishing \( r \)-vertex-labelling (resp. \( r \)-edge-labelling) of an undirected graph \( G \) is a mapping \( \lambda \) from the set of vertices (resp. the set of edges) of \( G \) to the set of labels \( \{1, \ldots, r\} \) such that no non-trivial automorphism of \( G \) preserves all the vertex (resp. edge) labels. The distinguishing number \( D(G) \) and the distinguishing index \( D'(G) \) of \( G \) are then the smallest \( r \) for which \( G \) admits a distinguishing \( r \)-vertex-labelling or \( r \)-edge-labelling, respectively. The distinguishing chromatic number \( D\chi(G) \) and the distinguishing chromatic index \( D\chi'(G) \) are defined similarly, with the additional requirement that the corresponding labelling must be a proper colouring.

These notions readily extend to oriented graphs, by considering arcs instead of edges. In this paper, we study the four corresponding parameters for oriented graphs whose underlying graph is a path, a cycle, a complete graph or a bipartite complete graph. In each case, we determine their minimum and maximum value, taken over all possible orientations of the corresponding underlying graph, except for the minimum values for unbalanced complete bipartite graphs \( K_{m,n} \) with \( m = 2, 3 \) or \( 4 \) and \( n > 3, 6 \) or \( 13 \), respectively, or \( m \geq 5 \) and \( n > 2^{m} - \left\lceil \frac{m}{2} \right\rceil \), for which we only provide upper bounds.

Keywords: Distinguishing number; Distinguishing index; Distinguishing chromatic number; Distinguishing chromatic index; Automorphism group; Oriented graph; Complete bipartite graph.

MSC 2010: 05C20, 20B25.

1 Introduction

All graphs considered in this paper are simple. For a graph \( G \), we denote by \( V(G) \) its set of vertices, and by \( E(G) \) its set of edges.

An \( r \)-vertex-labelling of a graph \( G \) is a mapping \( \lambda \) from \( V(G) \) to the set of labels \( \{1, \ldots, r\} \). An \( r \)-vertex-colouring of \( G \) is a proper \( r \)-vertex-labelling of \( G \), that is, an \( r \)-vertex-labelling \( \lambda \) such that \( \lambda(u) \neq \lambda(v) \) for every edge \( uv \) of \( G \). The chromatic number \( \chi(G) \) of \( G \) is then the smallest number of labels (called colours in that case) needed for a vertex-colouring of \( G \). Similarly, an \( r \)-edge-labelling of \( G \) is a mapping \( \lambda' \) from \( E(G) \) to the set of labels \( \{1, \ldots, r\} \), and...
| Graph $G$ | $\chi(G)$ | $D(G)$ | $D_\chi(G)$ | $OD^-(G)$ | $OD^+(G)$ | $OD^-_\chi(G)$ | $OD^+_\chi(G)$ |
|-----------|----------|--------|-------------|----------|----------|----------------|----------------|
| 1. $P_{2n}$, $n \geq 1$ | 2 | 2 | 2 | 1 (Th. 5) | 1 (Th. 6) | 2 (Th. 5) | 2 (Th. 6) |
| 2. $P_{2n+1}$, $n \geq 1$ | 2 | 2 | 3 | 1 (Th. 5) | 2 (Th. 6) | 2 (Th. 5) | 3 (Th. 6) |
| 3. $C_4$ | 2 | 3 | 4 | 1 (Th. 5) | 2 (Th. 7) | 2 (Th. 5) | 4 (Th. 7) |
| 4. $C_5$ | 3 | 3 | 4 | 1 (Th. 5) | 2 (Th. 7) | 3 (Th. 5) | 3 (Th. 7) |
| 5. $C_6$ | 2 | 2 | 4 | 1 (Th. 5) | 2 (Th. 7) | 2 (Th. 5) | 3 (Th. 7) |
| 6. $C_{2n}$, $n \geq 4$ | 2 | 2 | 3 | 1 (Th. 5) | 2 (Th. 7) | 3 (Th. 5) | 3 (Th. 7) |
| 7. $C_{2n+1}$, $n \geq 3$ | 3 | 2 | 3 | 1 (Th. 5) | 2 (Th. 7) | 3 (Th. 5) | 3 (Th. 7) |
| 8. $K_n$, $n \geq 3$ | $n$ | $n$ | $n$ | 1 (Th. 5) | 2 (Th. 8) | $n$ (Th. 5) | $n$ (Th. 8) |
| 9. $K_{1,n}$, $n \geq 2$ | 2 | $n$ | $n+1$ | $\left\lceil \frac{n}{2} \right\rceil$ (Th. 9) | $n$ (Th. 9) | 1+ $\frac{1}{2}$ (Th. 9) | $n+1$ (Th. 9) |
| 10. $K_{n,n}$, $n \geq 2$ | 2 | $n+1$ | 2$n$ | 1 (Th. 12) | $n$ (Th. 12) | 2 (Th. 12) | 2$n$ (Th. 12) |
| 11. $K_{m,n}$, $n > m \geq 2$ | 2 | $n$ | $m+n$ | Th. 15 and Cor. 22 | $n$ (Th. 15) | Th. 15 and Cor. 22 | $m+n$ (Th. 15) |

Table 1: Table of results for $OD^-(G)$, $OD^+(G)$, $OD^-_\chi(G)$ and $OD^+_\chi(G)$.

An $r$-edge-colouring of $G$ is a proper $r$-edge-labelling of $G$, that is, an $r$-edge-labelling $\lambda'$ such that $\lambda'(e) \neq \lambda'(e')$ for every two adjacent edges $e$ and $e'$ (that is, such that $e$ and $e'$ have one vertex in common). The chromatic index $\chi'(G)$ of $G$ is then the smallest number of labels (or colours) needed for an edge-colouring of $G$.

An automorphism $\phi$ of a graph $G$ is an edge-preserving mapping from $V(G)$ to $V(G)$, that is, such that $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(G)$. For a given vertex or edge-labelling of $G$, an automorphism $\phi$ of $G$ is $\lambda$-preserving if $\lambda(\phi(u)) = \lambda(u)$ for every vertex $u$ of $G$, or $\lambda(\phi(uv)) = \lambda(uv)$ for every edge $uv$ of $G$, respectively. A vertex or edge-labelling $\lambda$ of $G$ is distinguishing if the only $\lambda$-preserving automorphism of $G$ is the identity, that is, the labelling $\lambda$ breaks all the symmetries of $G$. Such a distinguishing vertex or edge-labelling is optimal if $G$ does not admit any vertex or edge-labelling using less colours.

The distinguishing number, distinguishing chromatic number, distinguishing index and distinguishing chromatic index of a graph $G$, denoted by $D(G)$, $D_\chi(G)$, $D'(G)$ and $D'_\chi(G)$, respectively, are then defined as the smallest $r$ for which $G$ admits a distinguishing $r$-vertex-labelling, a distinguishing $r$-vertex-colouring, a distinguishing $r$-edge-labelling or a distinguishing $r$-edge-colouring, respectively. Distinguishing numbers and distinguishing chromatic numbers have been introduced by Albertson and Collins in [3] and Collins and Trenk in [9], respectively, while distinguishing indices and distinguishing chromatic indices have been introduced by Kalinowski and Pilśniak in [15] (these two parameters are often denoted $\chi_D$ and $\chi'_D$ instead of $D_\chi$ and $D'_\chi$, respectively).

A graph $G$ is rigid (or asymmetric) if the only automorphism of $G$ is the identity. Therefore, $D(G) = 1$ if and only if $G$ is rigid and, similarly, $D'(G) = 1$ if and only if $G$ is rigid. Moreover, for every such graph $G$, $D_\chi(G) = \chi(G)$ and $D'_\chi(G) = \chi'(G)$. Note here that being rigid is not a necessary condition for any of these two equalities to hold (consider the path of order 2 or the path of order 3, respectively).
Our aim in this paper is to study the distinguishing number, distinguishing chromatic number, distinguishing index and distinguishing chromatic index of several classes of oriented graphs. By oriented graphs, we mean here antisymmetric digraphs, that is, digraphs with no directed cycle of length at most 2, or, equivalently, digraphs obtained from undirected graphs by giving to each of their edges one of its two possible orientations. All the notions of vertex-labelling, vertex-colouring, edge-labelling, edge-colouring, automorphism, distinguishing labelling, distinguishing number, distinguishing chromatic number, distinguishing index and distinguishing chromatic index, readily extends to oriented graphs by simply considering arcs instead of edges. For each undirected graph $G$ with $m$ edges, and each distinguishing parameter, we will study both the minimum and maximum possible value of the parameter, taken over all of the $2^m$ possible orientations of $G$.

Distinguishing numbers of digraphs have been studied in a few papers (see [2, 17, 18, 19, 20, 21]), while distinguishing numbers or indices of various classes of undirected graphs have attracted a lot of attention (see for instance [1, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16]). Up to our knowledge, distinguishing chromatic number, distinguishing index and distinguishing chromatic
index of digraphs have not been considered yet.

Our paper is organised as follows. In Section 2, we formally introduce the main definitions and give some basic results. We then consider simple classes of graphs, namely paths, cycles and complete graphs in Section 3, and complete bipartite graphs in Sections 4 and 5. We finally propose some directions for future work in Section 6.

Our results concerning distinguishing numbers and distinguishing indices are summarized in Tables 1 and 2, respectively, where $P_n$, $C_n$ and $K_n$ denote the path, the cycle and the complete graph of order $n$, respectively, and $K_{m,n}$ denotes the complete bipartite graph whose parts have size $m$ and $n$.

2 Preliminaries

An oriented graph is a digraph with no loops and no pairs of opposite arcs. For an oriented graph $\overrightarrow{G}$, we denote by $V(\overrightarrow{G})$ and $A(\overrightarrow{G})$ its set of vertices and its set of arcs, respectively. Let $\overrightarrow{G}$ be an oriented graph and $u$ a vertex of $\overrightarrow{G}$. The out-degree of $u$ in $\overrightarrow{G}$, denoted $d^+_\overrightarrow{G}(u)$, is the number of arcs in $A(\overrightarrow{G})$ of the form $uv$, and the in-degree of $u$ in $\overrightarrow{G}$, denoted $d^-\overrightarrow{G}(u)$, is the number of arcs in $A(\overrightarrow{G})$ of the form $vu$. The degree of $u$, denoted $d_\overrightarrow{G}(u)$, is then defined by $d_\overrightarrow{G}(u) = d^+_\overrightarrow{G}(u) + d^-\overrightarrow{G}(u)$. If $uv$ is an arc in $\overrightarrow{G}$, $u$ is an in-neighbour of $v$ and $v$ is an out-neighbour of $u$. We denote by $N^+_\overrightarrow{G}(u)$ and $N^-\overrightarrow{G}(u)$ the set of out-neighbours and the set of in-neighbours of $u$ in $\overrightarrow{G}$, respectively. Hence, $d^+_\overrightarrow{G}(u) = |N^+_\overrightarrow{G}(u)|$ and $d^-\overrightarrow{G}(u) = |N^-\overrightarrow{G}(u)|$. A source vertex is a vertex with no in-neighbours, while a sink vertex is a vertex with no out-neighbours. Let $v$ and $w$ be two neighbours of $u$. We say that $v$ and $w$ agree on $u$ if either both $v$ and $w$ are in-neighbours of $u$, or both $v$ and $w$ are out-neighbours of $u$, and that $v$ and $w$ disagree on $u$ otherwise. For a subset $S$ of $V(\overrightarrow{G})$, we denote by $G[S]$ the sub-digraph of $\overrightarrow{G}$ induced by $S$, which is obviously an oriented graph.

For any finite set $\Omega$, $\text{Id}_\Omega$ denotes the identity permutation acting on $\Omega$. Since the set $\Omega$ will always be clear from the context, we will simply write $\text{Id}$ instead of $\text{Id}_\Omega$ in the following.

An automorphism of an oriented graph $\overrightarrow{G}$ is an arc-preserving permutation of its vertices, that is, a one-to-one mapping $\phi : V(\overrightarrow{G}) \rightarrow V(\overrightarrow{G})$ such that $\phi(u)\phi(v)$ is an arc in $\overrightarrow{G}$ whenever $uv$ is an arc in $\overrightarrow{G}$. The set of automorphisms of $\overrightarrow{G}$ is denoted $\text{Aut}(\overrightarrow{G})$. The order of an automorphism $\phi$ is the smallest integer $k > 0$ for which $\phi^k = \text{Id}$. An automorphism $\phi$ of an oriented graph $\overrightarrow{G}$ is non-trivial if $\phi \neq \text{Id}$. A vertex $u$ of $\overrightarrow{G}$ is fixed by $\phi$ if $\phi(u) = u$.

We now introduce the distinguishing parameters we will consider. Let $\lambda$ be a vertex-labelling of an oriented graph $\overrightarrow{G}$. Recall first that an automorphism $\phi$ of $\overrightarrow{G}$ is $\lambda$-preserving if $\lambda(\phi(u)) = \lambda(u)$ for every vertex $u$ of $\overrightarrow{G}$, and that a vertex-labelling $\lambda$ of $\overrightarrow{G}$ is distinguishing if the only $\lambda$-preserving automorphism of $\overrightarrow{G}$ is the identity. A distinguishing vertex-colouring is then a distinguishing proper vertex-labelling. Similarly, an automorphism $\phi$ of an oriented graph $\overrightarrow{G}$ is $\lambda$-preserving, for a given arc-labelling $\lambda$ of $G$, if $\lambda(\phi(\overrightarrow{uv})) = \lambda(\overrightarrow{uv})$ for every arc $\overrightarrow{uv}$ of $\overrightarrow{G}$, and an arc-labelling $\lambda$ of $\overrightarrow{G}$ is distinguishing if the only $\lambda$-preserving automorphism of $\overrightarrow{G}$ is the identity. A distinguishing arc-colouring is then a distinguishing proper arc-labelling. We then define the four following distinguishing parameters of an oriented graph $\overrightarrow{G}$.

1. The oriented distinguishing number of $\overrightarrow{G}$, denoted $OD(\overrightarrow{G})$, is the smallest number of labels needed for a distinguishing vertex-labelling of $\overrightarrow{G}$.
2. The oriented distinguishing chromatic number of \( \overrightarrow{G} \), denoted \( OD_{\chi}(\overrightarrow{G}) \), is the smallest number of labels needed for a distinguishing vertex-colouring of \( \overrightarrow{G} \).

3. The oriented distinguishing index of \( \overrightarrow{G} \), denoted \( OD'(\overrightarrow{G}) \), is the smallest number of labels needed for a distinguishing arc-labelling of \( \overrightarrow{G} \).

4. The oriented distinguishing chromatic index of \( \overrightarrow{G} \), denoted \( OD_{\chi}'(\overrightarrow{G}) \), is the smallest number of labels needed for a distinguishing arc-colouring of \( \overrightarrow{G} \).

Using these parameters, we can define eight new distinguishing parameters of an undirected graph \( G \).

1. The minimum oriented distinguishing number of \( G \), denoted \( OD^- (G) \), is the smallest oriented distinguishing number of its orientations.

2. The maximum oriented distinguishing number of \( G \), denoted \( OD^+ (G) \), is the largest oriented distinguishing number of its orientations.

3. The minimum oriented distinguishing chromatic number of \( G \), denoted \( OD^- _\chi (G) \), is the smallest oriented distinguishing chromatic number of its orientations.

4. The maximum oriented distinguishing chromatic number of \( G \), denoted \( OD^+ _\chi (G) \), is the largest oriented distinguishing chromatic number of its orientations.

5. The minimum oriented distinguishing index of \( G \), denoted \( OD'^- (G) \), is the smallest oriented distinguishing index of its orientations.

6. The maximum oriented distinguishing index of \( G \), denoted \( OD'^+ (G) \), is the largest oriented distinguishing index of its orientations.

7. The minimum oriented distinguishing chromatic index of \( G \), denoted \( OD'^-_\chi (G) \), is the smallest oriented distinguishing chromatic index of its orientations.

8. The maximum oriented distinguishing chromatic index of \( G \), denoted \( OD'^+_\chi (G) \), is the largest oriented distinguishing chromatic index of its orientations.

Let \( G \) be an undirected graph, \( \overrightarrow{G} \) be any orientation of \( G \), and \( \lambda \) be any vertex or edge-labelling of \( G \) (which can also be be considered as a vertex or arc-labelling of \( \overrightarrow{G} \), respectively). Observe that every automorphism of \( \overrightarrow{G} \) is an automorphism of \( G \). From this observation and the definition of our distinguishing parameters, we directly get the following result.

**Proposition 1** For every undirected graph \( G \),

1. \( OD^- (G) \leq OD^+(G) \leq D(G) \),
2. \( OD'^-(G) \leq OD'^+(G) \leq D'(G) \),
3. \( \chi(G) \leq OD^- _\chi (G) \leq OD^+ _\chi (G) \leq D_\chi (G) \),
4. \( \chi'(G) \leq OD'^-_\chi (G) \leq OD'^+ _\chi (G) \leq D'_\chi (G) \),
5. \( D(G) \leq D_\chi (G) \), \( OD^- (G) \leq OD^- _\chi (G) \), and \( OD^+(G) \leq OD^+ _\chi (G) \),
6. \( D'(G) \leq D'_\chi (G) \), \( OD'^-(G) \leq OD'^-_\chi (G) \), and \( OD'^+(G) \leq OD'^+_\chi (G) \).

As observed in the previous section, for every undirected graph \( G \), \( D(G) = 1 \) if and only if \( G \) is rigid, and \( D_\chi (G) = \chi(G) \) if \( G \) is rigid. This property obviously also holds for oriented graphs: for every oriented graph \( \overrightarrow{G} \), \( OD(\overrightarrow{G}) = 1 \) if and only if \( \overrightarrow{G} \) is rigid, and \( OD_\chi (\overrightarrow{G}) = \chi(\overrightarrow{G}) \) if \( \overrightarrow{G} \) is rigid. We thus have the following result.

**Proposition 2** If \( G \) is an undirected graph that admits a rigid orientation, then \( OD^- (G) = 1 \), \( OD'^-(G) = 1 \), \( OD^- _\chi (G) = \chi(G) \) and \( OD'^-_\chi (G) = \chi'(G) \).
3 Paths, cycles and complete graphs

The distinguishing number of undirected paths, cycles and complete graphs has been determined by Albertson and Collins in [3], while their distinguishing chromatic number was given by Collins and Trenk in [9]. The following theorem summarizes these results.

**Theorem 3 ([3], [9])**

1. For every integer \( n \geq 1 \), \( D(P_{2n}) = D'_\chi(P_{2n}) = 2 \).
2. For every integer \( n \geq 1 \), \( D(P_{2n+1}) = 2 \) and \( D'_\chi(P_{2n+1}) = 3 \).
3. \( D(C_4) = 3 \) and \( D'_\chi(C_4) = 4 \).
4. \( D(C_5) = 3 \) and \( D'_\chi(C_5) = 3 \).
5. \( D(C_6) = 2 \) and \( D'_\chi(C_6) = 4 \).
6. For every integer \( n \geq 7 \), \( D(C_n) = 2 \) and \( D'_\chi(C_n) = 3 \).
7. For every integer \( n \geq 1 \), \( D(K_n) = D'_\chi(K_n) = n \).

On the other hand, the distinguishing index and distinguishing chromatic index of undirected paths, cycles and complete graphs, have been determined by Kalinowski and Pilśniak in [15], and Alekhani and Soltani in [5]. The following theorem summarizes these results.

**Theorem 4 ([15], [5])**

1. \( D'(P_2) = D'_\chi'(P_2) = 1 \).
2. For every integer \( n \geq 2 \), \( D'(P_{2n}) = 2 \) and \( D'_\chi'(P_{2n}) = 3 \).
3. For every integer \( n \geq 1 \), \( D'(P_{2n+1}) = D'_\chi'(P_{2n}) = 2 \).
4. \( D'(C_4) = 3 \) and \( D'_\chi'(C_4) = 4 \).
5. \( D'(C_5) = 3 \) and \( D'_\chi'(C_5) = 3 \).
6. \( D'(C_6) = 2 \) and \( D'_\chi'(C_6) = 4 \).
7. For every integer \( n \geq 7 \), \( D'(C_n) = 2 \) and \( D'_\chi'(C_n) = 3 \).
8. \( D'(K_3) = 3 \) and \( D'_\chi'(K_3) = 3 \).
9. \( D'(K_4) = D'(K_5) = 3 \) and \( D'_\chi'(K_4) = D'_\chi'(K_5) = 5 \).
10. For every integer \( n \geq 3 \), \( D'(K_{2n}) = 2 \) and \( D'_\chi'(K_{2n}) = 2n - 1 \).
11. For every integer \( n \geq 3 \), \( D'(K_{2n+1}) = 2 \) and \( D'_\chi'(K_{2n+1}) = 2n + 1 \).

It is not difficult to observe that every path, cycle or complete graph, admits a rigid orientation. We thus get the following result, which proves columns \( OD^-(G) \) and \( OD^-_\chi(G) \) of Table 1 for lines 1 to 8, and columns \( OD'^-(G) \) and \( OD'^-_\chi(G) \) of Table 2 for lines 1 to 12. We say that an oriented path, or an oriented cycle, is directed if all its arcs have the same direction.

**Theorem 5** If \( G \) is an undirected path, cycle, or complete graph, then \( G \) admits a rigid orientation. Therefore, \( OD^-(G) = OD'^-(G) = 1 \), \( OD^-_\chi(G) = \chi(G) \) and \( OD'^-_\chi(G) = \chi'(G) \).

**Proof.** Since all directed paths and transitive tournaments are rigid, and every orientation of the cycle \( C_n \), \( n \geq 3 \), obtained from the directed cycle by reversing exactly one arc is rigid, the first statement holds. The second statement then directly follows from Proposition 2. \( \square \)

We now consider the parameters \( OD^+(G) \), \( OD^+_\chi(G) \), \( OD'^+(G) \) and \( OD'^+_\chi(G) \) for \( G \) being a path, a cycle, or a complete graph. For paths, we have the following result, which proves columns \( OD^+(G) \) and \( OD^+_\chi(G) \) of Table 1 and columns \( OD'^+(G) \) and \( OD'^+_\chi(G) \) of Table 2 for lines 1 and 2.
Figure 1: Distinguishing vertex and edge-colourings of non-rigid orientations of $C_6$.

**Theorem 6** Let $P_n$ denote the path of order $n$. We then have $OD^+(P_n) = OD_{\chi'}^+(P_n) = 1$ and $OD^+_\chi(P_n) = OD^+\chi'(P_n) = 2$ if $n$ is even, and $OD^+(P_n) = OD^+_\chi(P_n) = OD^+\chi'(P_n) = 2$ and $OD^+_\chi(P_n) = 3$ otherwise.

**Proof.** Let us denote by $P_n = v_1 \ldots v_n$, $n \geq 2$, the undirected path of order $n$. Note that the only non-trivial automorphism of $P_n$ is the permutation $\pi$ that exchanges vertices $x_i$ and $x_{n-i+1}$ for every $i$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

If $n$ is even, every orientation of $P_n$ is rigid, which gives $OD^+(P_n) = OD^+\chi'(P_n) = 1$ and $OD^+_\chi(P_n) = OD^+\chi'(P_n) = 2$ by Proposition 2.

Suppose now that $n$ is odd. In that case, every orientation such that the edges $v_iv_{i+1}$ and $v_{n-i}v_{n-i+1}$ have opposite directions is not rigid, which implies that the value of the four considered distinguishing parameters is strictly greater than 1. Since $D(P_n) = D'(P_n) = D_{\chi'}(P_n) = 2$, we get $OD^+(P_n) = OD^+\chi'(P_n) = OD^+\chi'(P_n) = 2$ by Proposition 1. Finally, since every 2-vertex-colouring of $P_n$ is preserved by $\pi$, we have $OD^+_\chi(P_n) > 2$ and thus, since $D_{\chi'(P_n)} = 3$, $OD^+_\chi(P_n) = 3$ by Proposition 1.

For cycles, we have the following result, which proves columns $OD^+(G)$ and $OD^+_\chi(G)$ of Table 1 and columns $OD^+\chi'(G)$ and $OD^+_\chi'(G)$ of Table 2 for lines 3 to 7 (the case of the 3-cycle is covered by Theorem 8).

**Theorem 7**

1. For every integer $n \geq 4$, $OD^+(C_n) = OD^+\chi(C_n) = 2$.

2. $OD^+_\chi(C_4) = 4$ and, for every integer $n \geq 5$, $OD^+_\chi(C_n) = 3$.

3. For every integer $n \geq 4$, $OD^+_\chi'(C_n) = 3$.

**Proof.** Let us denote by $C_n = v_1 \ldots v_nv_1$, $n \geq 4$, the undirected cycle of order $n$. Since the directed cycle $C_n$ is not rigid, the value of the four considered distinguishing parameters is strictly greater than 1.

Let us first consider distinguishing labellings and let $\overrightarrow{C}$ be any orientation of $C_n$. The 2-vertex-labelling $\lambda$ of $\overrightarrow{C}$ defined by $\lambda(v_1) = \lambda(v_2) = 1$ and $\lambda(v_i) = 2$ for every $i$, $3 \leq i \leq n$, is clearly distinguishing since $v_1$ and $v_2$ must be fixed by every $\chi$-preserving automorphism of $\overrightarrow{C}$. We thus get $OD^+(C_n) = 2$ for every $n \geq 4$. Similarly, the 2-edge-labelling $\chi'$ of $\overrightarrow{C}$ that assigns colour 1 to exactly one arc of $\overrightarrow{C}$ is distinguishing since the end-vertices of the arc coloured with 1 must be fixed by every $\chi$-preserving automorphism of $\overrightarrow{C}$. Therefore, $OD^+\chi'(C_n) = 2$ for every $n \geq 4$.

Let us now consider distinguishing colourings. Suppose first that $n$ is odd. In that case, since $\chi(C_n) = \chi'(C_n) = 3$, we get $OD^+_\chi(C_n) = D(\chi(C_n) = 3$ and $OD^+_\chi'(C_n) = D_{\chi'}(C_n) = 3$ by Proposition 1 and Theorem 3 or 4.
Suppose now that \( n \) is even. Note that every 2-vertex or 2-arc-colouring of the directed cycle \( C_n \) is \( \rho^2 \)-preserving, where \( \rho \) denotes the automorphism of \( C_n \) defined by \( \rho(v_i) = v_{i+1} \) for every \( i \), \( 1 \leq i \leq n-1 \), and \( \rho(v_n) = v_1 \). This implies \( OD_\rho^+(C_n) > 2 \) and \( OD_\rho^+(C_n) > 2 \). We thus get \( OD_\rho^+(C_n) = D_\chi(C_n) = 3 \) and \( OD_\rho^+(C_n) = D_\chi'(C_n) = 3 \) for every even \( n \geq 8 \), thanks to Proposition 1 and Theorem 3 or 4.

Consider now the cycle \( C_4 = v_1v_2v_3v_4v_1 \). Note that, up to symmetries, \( C_4 \) has only two non-rigid orientations, namely the directed cycle \( C_4^\prime \) and the orientation \( C_4^\prime \) with arcs \( \overrightarrow{v_1v_2}, \overrightarrow{v_3v_4}, \overrightarrow{v_3v_4}, \) and \( \overrightarrow{v_3v_4} \). Moreover, observe that \( \text{Aut}(C_4) \) has three non-trivial automorphisms, one exchanging \( v_1 \) and \( v_3 \), one exchanging \( v_2 \) and \( v_4 \), and the third one exchanging both these pairs of vertices. This implies that every distinguishing vertex-colouring of \( C_4 \) must use four colours, and thus \( OD_\chi^+(C_4) = 3 \). To see that \( OD_\chi^+(C_4) = 3 \), it suffices to observe that, by colouring the arcs of \( C_4 \) or \( C_4^\prime \) cyclically with colours 1213, we obtain a distinguishing 3-arc-colouring.

Consider finally the cycle \( C_6 \). Up to symmetries, \( C_6 \) has four distinct non-rigid orientations, depicted in Figure 1. It is not difficult to check that the vertex and arc-colourings described in the same figure are all optimal, which gives \( OD_\chi^+(C_6) = OD_\chi^+(C_6) = 3 \).

This concludes the proof. \( \square \)

The distinguishing number of tournaments has been studied by Albertson and Collins in [2], where it is proved in particular that \( D(T_n) \leq 1 + \left\lceil \frac{\log n}{2} \right\rceil \) for every tournament \( T_n \) of order \( n \). Moreover, they conjectured that \( D(T) \leq 2 \) for every tournament \( T \). As observed by Godsil in 2002 (this fact is mentioned by Lozano in [20]), this conjecture follows from Gluck’s Theorem 12.

For every integer \( n \geq 3 \), the values of \( OD^-(K_n) \), \( OD_\chi^-(K_n) \), \( OD^+(K_n) \) and \( OD_\chi^+(K_n) \) are already given by Theorem 3. The values of the other parameters are given by the following result, which proves columns \( OD^+(G) \) and \( OD_\chi^+(G) \) of Table 1 line 8, and columns \( OD^+(G) \) and \( OD_\chi^+(G) \) of Table 2 lines 8 to 12.

**Theorem 8** For every integer \( n \geq 3 \), \( OD^+(K_n) = OD_\chi^+(K_n) = 2 \), \( OD_\chi^+(K_n) = n \), and \( OD_\chi^+(K_n) = \chi'(K_n) \).

**Proof.** Since every complete graph \( K_n \), \( n \geq 3 \), admits a non rigid orientation (consider for instance the orientation \( \overrightarrow{K_n} \) of \( K_n \) obtained from the transitive orientation of \( K_n \) with directed path \( x_1 \ldots x_n \) by reversing the arc \( x_{n-2}x_n \), so that \( x_{n-2}, x_{n-1} \) and \( x_n \) form a directed cycle; the permutation \( (x_{n-2}, x_{n-1}, x_n) \) is then clearly a non trivial automorphism of \( \overrightarrow{K_n} \)), we get \( OD^+(K_n) \geq 2 \) and \( OD^+(K_n) \geq 2 \).

Since, as mentioned above, \( D(T) \leq 2 \) for every tournament \( T \), we get, by Proposition 1 \( OD^+(K_n) \leq 2 \) and thus \( OD^+(K_n) = 2 \). By Proposition 1 we also have \( OD^+(K_n) \leq D'(K_n) \), and thus \( OD^+(K_n) = 2 \) if \( n \geq 6 \). If \( n = 3 \), then the only non rigid orientation of \( K_3 \) is the directed cycle. By assigning colour 1 to one arc and colour 2 to the two other arcs, we get a distinguishing arc-labelling, so that \( OD_\chi^+(K_3) = 2 \). If \( n \in \{4,5\} \), observe that each non rigid orientation of \( K_n \) contains a transitive triangle. By assigning colour 1 to the three arcs of this triangle and colour 2 to all the other arcs, we get a distinguishing arc-labelling, so that \( OD_\chi^+(K_4) = OD_\chi^+(K_5) = 2 \).

Finally, by Proposition 1 \( \chi(K_n) \leq OD_\chi^+(K_n) \leq D_\chi(K_n) \) and \( \chi'(K_n) \leq OD_\chi^+(K_n) \leq D_\chi'(K_n) \), which gives, thanks to Theorems 3 and 4, \( OD_\chi^+(K_n) = n \), and \( OD_\chi^+(K_n) = \chi'(K_n) \) if \( n \neq 4 \), respectively. Now, there are only two orientations of \( K_4 \) that admit a non trivial automorphism, one with a source vertex and the other one with a sink vertex, and the three other vertices in both of them inducing a directed 3-cycle. In each case, every 3-edge-colouring is clearly distinguishing, which gives \( OD_\chi^+(K_4) = 3 = \chi'(K_4) \). \( \square \)
4 Complete bipartite graphs: easy cases

In this section, we consider “easy cases” of complete bipartite graphs, namely $K_{1,n}$ and $K_{n,n}$ for every $n \geq 3$ (the cases $n \in \{1,2\}$ correspond to $P_2$, $P_3$ or $C_4$, already considered in the previous section).

Concerning stars $K_{1,n}$, it is not difficult to get the following result, which proves line 9 of Table 1 and line 13 of Table 2.

**Theorem 9** For every integer $n \geq 3$, $OD^-(K_{1,n}) = OD^-(K_{1,n}) = \left[\frac{n}{2}\right]$, $OD^+(K_{1,n}) = OD^+(K_{1,n}) = n$, $OD_\chi^-(K_{1,n}) = n + 1$, and $OD_\chi^+(K_{1,n}) = OD_\chi^-(K_{1,n}) = n$.

**Proof.** Let $x$ be the central vertex of $K_{1,n}$ and $\{y_1, \ldots, y_n\}$ the set of neighbours of $x$. Let $\overrightarrow{K}$ denote any orientation of $K_{1,n}$. Observe that every automorphism of $\overrightarrow{K}$ can only permute in-neighbours of $x$, and out-neighbours of $x$, $x$ being fixed. Hence, by considering the orientation of $K_{1,n}$ for which $d^+(x) = n$, we get $OD^+(K_{1,n}) = OD^+(K_{1,n}) = n$.

On one other hand, the minimum value of $OD^-(K_{1,n})$ and $OD^-(K_{1,n})$ is attained when the number of in-neighbours and the number of out-neighbours of $x$ differ by at most one, which gives $OD^-(K_{1,n}) = OD^+(K_{1,n}) = \left[\frac{n}{2}\right]$. Similarly, since all in-neighbours (resp. all out-neighbours) must have distinct colours in every distinguishing vertex-colouring of $K_{1,n}$, and these colours have to be distinct from the colour of $x$, we get $OD_\chi^-(K_{1,n}) = 1 + \left[\frac{n}{2}\right]$ (when the number of in-neighbours and out-neighbours of $x$ differ by at most one), and $OD_\chi^+(K_{1,n}) = n + 1$ (when all arcs are out-going arcs from $x$).

Finally, since all arc-colourings of every orientation of $K_{1,n}$ are distinguishing, we get $OD_\chi^-(K_{1,n}) = OD_\chi^+(K_{1,n}) = \chi'(K_{1,n}) = n$. \qed

The distinguishing number, distinguishing chromatic number, distinguishing index and distinguishing chromatic index of balanced complete bipartite graphs $K_{n,n}$ have been studied by Collins and Trenk [9], Kalinowski and Pilśniak [15], and Alikhani and Soltani [5]. The following theorem summarizes these results.

**Theorem 10** ([9], [15], [5])

1. $D(K_{3,3}) = 4$, $D_\chi(K_{3,3}) = 6$, $D'(K_{3,3}) = 3$ and $D_\chi'(K_{3,3}) = 5$.
2. For every $n \geq 4$, $D(K_{n,n}) = n + 1$, $D_\chi(K_{n,n}) = 2n$, $D'(K_{n,n}) = 2$ and $D_\chi'(K_{n,n}) = n + 1$.

Let us denote by $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ the bipartition of $K_{n,n}$. It is a well-known fact the every proper edge-colouring of $K_{n,n}$ corresponds to a Latin square of order $n$, the colour of the edge $x_iy_j$, $1 \leq i, j \leq n$, corresponding to the symbol in row $i$ and column $j$. A Latin square $L$ is asymmetric [23] if, for every three permutations $\alpha$, $\beta$ and $\gamma$ on the set $\{1, \ldots, n\}$, the Latin square $L'$ defined by $L'[\alpha(i), \beta(j)] = \gamma(L[i,j])$ for every $i, j, 1 \leq i, j \leq n$ is equal to $L$ if and only if $\alpha$, $\beta$ and $\gamma$ are all the identity. In other words, if an edge-colouring $\lambda$ of $K_{n,n}$ corresponds to an asymmetric Latin square, then the only $\lambda$-preserving automorphism of $K_{n,n}$ that preserves the bipartition is the identity. Therefore, if $\overrightarrow{K}$ is the orientation of $K_{n,n}$ defined by $N^+(\overrightarrow{K})(x_i) = Y$ for every $i$, $1 \leq i \leq n$, and $\lambda$ is any $n$-arc-colouring of $\overrightarrow{K}$ corresponding to an asymmetric Latin square, then none of the non-trivial automorphisms of $\overrightarrow{K}$ is $\lambda$-preserving.

The following result, concerning asymmetric Latin squares and due to K.T. Phelps [22], will be useful for our next result.

**Theorem 11** ([22]) For every integer $n \geq 7$, there exists an asymmetric Latin square of order $n$. Moreover, the smallest asymmetric Latin squares have order 7.
For balanced complete bipartite graphs, we have the following result, which proves line 10 of Table 1 and lines 14, 15 and 16 of Table 2.

**Theorem 12** For every integer \( n \geq 3 \), \( OD^-(K_{n,n}) = OD^+(K_{n,n}) = 1 \), \( OD^+(K_{n,n}) = n \), \( OD^+(K_{n,n}) = 2 \), \( OD^+(K_{n,n}) = 2 \), \( OD^+(K_{n,n}) = n \) and \( OD^+(K_{n,n}) = 2n \). Moreover, \( OD^+(K_{n,n}) = n + 1 \) if \( 3 \leq n \leq 6 \), and \( OD^+(K_{n,n}) = n \) if \( n \geq 7 \).

**Proof.** Let \( X = \{x_1, \ldots, x_n\} \) and \( Y = \{y_1, \ldots, y_n\} \) denote the bipartition of \( K_{n,n} \), \( n \geq 3 \).

Observe first that the orientation of \( K_{n,n} \) given by \( N^+(x_i) = \{y_j : j \leq i\} \) for every \( i, 1 \leq i \leq n \), is rigid, since all vertices in \( X \) have distinct in-degrees. Therefore, thanks to Proposition 2, we get \( OD^-(K_{n,n}) = OD^-(K_{n,n}) = 1 \), \( OD^-(K_{n,n}) = \chi(K_{n,n}) = 2 \) and \( OD^+(K_{n,n}) = \chi(K_{n,n}) = n \) for every \( n \geq 3 \).

Consider now the orientation \( \vec{K} \) of \( K_{n,n} \) given by \( N^+(x_i) = Y \) for every \( i, 1 \leq i \leq n \). Clearly, for every two permutations \( \pi_X \) of \( X \) and \( \pi_Y \) of \( Y \), the product \( \pi_X \pi_Y \) is an automorphism of \( \vec{K} \). Therefore, all vertices of \( \vec{K} \) must be assigned distinct colours by every distinguishing vertex colouring, which gives \( OD^+(K_{n,n}) = 2n \).

The \( n \)-vertex-labelling \( \varphi \) of \( K_{n,n} \) defined by \( \varphi(x_i) = \varphi(y_i) = i \) for every \( i, 1 \leq i \leq n \), is distinguishing for every orientation of \( K_{n,n} \) since every label is used on two vertices that are connected by an arc. Hence, \( OD^+(K_{n,n}) \leq n \). For the orientation \( \vec{K} \) of \( K_{n,n} \) defined in the previous paragraph, every distinguishing vertex-labelling must assign distinct labels to every vertex in each part, and thus \( OD^+(K_{n,n}) = n \).

Consider now the 2-edge-labelling \( \varphi' \) of \( K_{n,n} \) defined by \( \varphi'(x_iy_j) = 1 \) if and only if \( j < i \), for every \( i, j, 1 \leq i, j \leq n \). Note that every two vertices in each part have a distinct number of incident edges with label 1. Therefore, reasoning similarly as in the previous paragraph, we get \( OD^+(K_{n,n}) = 2 \).

If \( n \geq 7 \), it follows from Theorem 11 that the orientation \( \vec{K} \) of \( K_{n,n} \) admits a distinguishing \( n \)-arc-colouring, and thus \( OD^+(K_{n,n}) = \chi(K_{n,n}) = n \). Finally, if \( 3 \leq n \leq 6 \), consider the \( (n+1) \)-edge-colouring \( \lambda \) of \( K_{n,n} \) defined by \( \lambda(x_iy_j) = i+j \) \( \pmod{n+1} \) for every \( i, j, 1 \leq i, j \leq n \). Since, in each case, the set of colours used on the incident edges of every two vertices are distinct (the colour \( i \) does not appear on the edges incident with \( x_i \), and the colour \( j \) does not appear on the edges incident with \( y_j \)), we get that \( \lambda \) is a distinguishing arc-colouring of every orientation of \( K_{n,n} \), and thus \( OD^+(K_{n,n}) \leq n+1 \). On the other hand, it follows from Theorem 11 that for every \( n \)-edge-colouring \( \lambda \) of \( K_{n,n} \), there exists a non trivial \( \lambda \)-preserving automorphism of \( K_{n,n} \) such that every vertex is mapped to a vertex in the same part. Considering the orientation \( \vec{K} \) of \( K_{n,n} \) defined above, this gives \( OD^+(K_{n,n}) > n \), and thus \( OD^+(K_{n,n}) = n+1 \).

This completes the proof. \( \Box \)

## 5 Complete bipartite graphs: other cases

We consider in this section the remaining cases of complete bipartite graphs, that is, unbalanced complete bipartite graphs \( K_{m,n} \) with \( 2 \leq m < n \).

The distinguishing number and the distinguishing chromatic number of unbalanced complete bipartite graphs have been determined by Collins and Trenk [9], while their distinguishing chromatic index has been given by Alekhani and Soltani [5]. The following theorem summarizes these results.

**Theorem 13** ([9], [5]) For every two integers \( m \) and \( n \), \( 2 \leq m < n \),

\[
D(K_{m,n}) = n, \quad D_\chi(K_{m,n}) = m + n \quad \text{and} \quad D_\chi'(K_{m,n}) = n.
\]
Figure 2: An orientation of the complete bipartite graph $K_{3,5}$.

As recalled in the previous section, the distinguishing index of balanced complete bipartite graphs $K_{n,n}$ has been given by Kalinowski and Pilśniak [15] (see Theorem 10). The distinguishing index of complete bipartite graphs $K_{m,n}$ with $n > m$ is less easy to determine. This has been done by Fisher and Isaak [11], and independently by Imrich, Jerebic and Klavžar [13].

**Theorem 14 ([11], [13])** Let $m$, $n$ and $r$ be integers such that $r \geq 2$ and $(r - 1)^m < n \leq r^m$. We then have

$$D'(K_{m,n}) = \begin{cases} \ r, & \text{if } n \leq r^m - \lfloor \log_r m \rfloor - 1, \\ r + 1, & \text{if } n \geq r^m - \lfloor \log_r m \rfloor + 1. \end{cases}$$

Moreover, if $n = r^m - \lfloor \log_r m \rfloor$, then $D'(K_{m,n})$ is either $r$ or $r + 1$ and can be computed recursively in time $O(\log^4(n))$.

Before considering the eight distinguishing parameters for complete bipartite graphs we are interested in, we introduce some notation and definitions. For the complete bipartite graph $K_{m,n}$, $m < n$, we denote by $(X,Y)$ the partition of its vertex set and let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. Moreover, for every orientation $\vec{K}$ of $K_{m,n}$ and every integer $d$, $0 \leq d \leq n$, we will denote by $Y_d$ the set of vertices in $Y$ with out-degree $d$, that is, $Y_d = \{y \in Y \mid d_\vec{K}^+(y) = d\}$.

For example, in the orientation of $K_{3,5}$ depicted in Figure 2, we have $Y_0 = \{y_1, y_2\}$, $Y_1 = \emptyset$, $Y_2 = \{y_3, y_5\}$ and $Y_3 = \{y_4\}$.

For general complete bipartite graphs, we have the following result.

**Theorem 15** For every two integers $m$ and $n$, $2 \leq m < n$, the following holds.

1. $OD^+(K_{m,n}) = OD^+_{\chi}(K_{m,n}) = n$, $OD^+(K_{m,n}) = D'(K_{m,n})$ and $OD^+_{\chi}(K_{m,n}) = m + n$.

2. If $K_{m,n}$ admits a rigid orientation, then $OD^-(K_{m,n}) = OD^-(K_{m,n}) = 1$, $OD^\chi(K_{m,n}) = 2$ and $OD^\chi_{\chi}(K_{m,n}) = n$.

3. If $K_{m,n}$ does not admit any rigid orientation, then $OD^-(K_{m,n}) \leq \left\lceil \frac{n}{m-1} \right\rceil$, $OD^\chi(K_{m,n}) \leq 1 + \left\lceil \frac{n}{m-1} \right\rceil$, $OD^\chi(K_{m,n}) \leq D'(K_{m,n})$ and $OD^\chi_{\chi}(K_{m,n}) = n$.

**Proof.** Consider the orientation $\vec{K}$ of $K_{m,n}$ given by $N^+(x_i) = Y$ for every $i$, $1 \leq i \leq m$. In that case, since $m \neq n$, we have $Aut(\vec{K}) = Aut(K_{m,n})$, which implies $OD^+(K_{m,n}) = D(K_{m,n})$, $OD^+_{\chi}(K_{m,n}) = D^\chi(K_{m,n})$, $OD^+(K_{m,n}) = D'(K_{m,n})$ and $OD^+_{\chi}(K_{m,n}) = D^\chi(K_{m,n})$. Equalities
in the first item then follows from Theorem \[13\]. On one other hand, equalities in the second item directly follow from Proposition \[2\].

Suppose now that $K_{m,n}$ does not admit any rigid orientation. We first claim that we necessarily have $m < \left\lceil \frac{n}{m-1} \right\rceil$. Indeed, this follows from Lemmas \[17\], \[19\] and \[20\] proved later: if $m = 2, 3$ or $4$, then $n$ is at least $4, 7$ or $14$, respectively (see Lemmas \[17\] and \[19\]), and the claim holds, while $n > 2^m - \left\lceil \frac{m}{2} \right\rceil$ if $m \geq 5$ (see Lemma \[20\]). It is then easy to check that, for every $m \geq 5$,

\[
\left\lceil \frac{n}{m-1} \right\rceil > \left\lceil \frac{2^m - \left\lceil \frac{m}{2} \right\rceil}{m-1} \right\rceil > m.
\]

For every non-trivial automorphism $\phi$ of $K_{m,n}$, and every two vertices $u$ and $v$ of $K_{m,n}$ with $\phi(u) = v$, we have $d^+(u) = d^+(v)$, and either $u, v \in X$ or $u, v \in Y$. Consider now any orientation $\overrightarrow{K}'$ of $K_{m,n}$ satisfying the two following properties:

P1. $d^+_R(\gamma_j) = 1 + (j \mod m - 1)$ for every $j$, $1 \leq j \leq n$, and

P2. $N^+_R(\gamma_i(m-1)) = \{x_i\}$ for every $i$, $1 \leq i \leq m$.

Thanks to Property P1, we have $1 \leq d^+_R(\gamma_j) \leq m - 1$ for every $j$, $1 \leq j \leq n$, and the cardinality of each set $Y_d$, $1 \leq d \leq m - 1$, is at most $\left\lceil \frac{n}{m-1} \right\rceil$. Moreover, Property P2 ensures that the sets of in-neighbours of every two vertices in $X$ are distinct, which implies that every automorphism of $\overrightarrow{K}'$ that permutes vertices in $X$ must also permute vertices in $Y$.

Consider now the $\left\lceil \frac{n}{m-1} \right\rceil$-vertex-labelling $\lambda$ of $\overrightarrow{K}'$ defined by with $\lambda(x_i) = 1$ for every $i$, $1 \leq i \leq m$, and $\lambda(\gamma_j) = 1 + \left\lfloor \frac{j-1}{m-1} \right\rfloor$ for every $j$, $1 \leq j \leq n$. This labelling assigns distinct labels to the vertices of each set $Y_d$, $1 \leq d \leq m - 1$, and is thus distinguishing, which gives $OD^-(K_{m,n}) \leq \left\lceil \frac{n}{m-1} \right\rceil$. Moreover, by using one additional label for the vertices of $X$ instead of label $1$, the labelling $\lambda$ becomes a distinguishing vertex-colouring, and thus $OD^-(K_{m,n}) \leq 1 + \left\lceil \frac{n}{m-1} \right\rceil$.

Let us now consider distinguishing arc-labellings of $\overrightarrow{K}'$. Observe that if the restriction of any arc-labelling $\lambda$ of $\overrightarrow{K}'$ to the subgraphs $\overrightarrow{K}'[X \cup Y_d]$ induced by $X \cup Y_d$, $1 \leq d \leq m - 1$, is distinguishing for this subgraph, then $\lambda$ is a distinguishing arc-labelling of $\overrightarrow{K}'$. Since for every $d$, $1 \leq d \leq m - 1$, we have $OD^-(\overrightarrow{K}'[X \cup Y_d]) \leq OD^-(K_{m,\left\lceil \frac{m}{m-1} \right\rceil})$, we get $OD^-(K_{m,n}) \leq OD^-(K_{m,\left\lceil \frac{m}{m-1} \right\rceil})$.

Finally, from Proposition \[1\] and Table \[2\] we get

\[
n = \chi'(K_{m,n}) \leq OD^-(K_{m,n}) \leq D_\chi'(K_{m,n}) = n,
\]

and thus $OD^\chi'(K_{m,n}) = n$, which completes the proof of the third item. \[\Box\]

Our goal now is thus to determine for which values of $m$ and $n$, $K_{m,n}$ admits a rigid orientation. Let us first introduce a few more notation. For a given orientation $\overrightarrow{K}$ of $K_{m,n}$, we associate with each vertex $y_i$ from $Y$ the word $w_i = b_i^1 \cdots b_i^m$ on the alphabet $\{0, 1\}$, defined by $b_i^j = 0$ if $x_jy_i$ is an arc, and $b_i^j = 1$ otherwise. Figure \[2\] gives the word associated with each vertex from $Y$ for the depicted orientation of $K_{5,5}$.

For every integer $m \geq 2$, we will denote by $\overrightarrow{K}K_m$ the (unique, up to isomorphism) orientation of the complete bipartite graph $K_{m,2^m}$ for which all the words associated with the vertices in $Y$ are distinct, and by $(X^*, Y^*)$ the corresponding bipartition of $V(\overrightarrow{K}K_m)$. This orientation will be called the canonical orientation of $K_{m,2^m}$. Observe that every vertex $x$ in $X^*$ has exactly $2^{m-1}$ in-neighbours and $2^{m-1}$ out-neighbours in $\overrightarrow{K}K_m$.\[\overline{12}\]
Let $\overrightarrow{K}$ be any orientation of $K_{m,n}$, $n \geq m \geq 2$. We say that two vertices $u$ and $v$ are full twins in $\overrightarrow{K}$ if $N^+(u) = N^+(v)$ (which implies $N^-(u) = N^-(v)$). For example, $y_1$ and $y_2$ are full twins in the orientation of $K_{3,5}$ depicted in Figure 2. Observe that the existence of full twins in an orientation of a complete bipartite graph ensures that this orientation is not rigid.

**Proposition 16** Let $\overrightarrow{K}$ be any orientation of $K_{m,n}$, $n > m \geq 2$. If there exist two full twins $u$ and $v$ in $\overrightarrow{K}$, then $\overrightarrow{K}$ is not rigid. In particular, if $n > 2^m$, then $K_{m,n}$ does not admit any rigid orientation.

**Proof.** It suffices to observe that the transposition $(u,v)$ of $V(\overrightarrow{K})$ is an automorphism of $\overrightarrow{K}$. The second statement follows from the fact that the maximum number of distinct orientations of the edges incident with a vertex in $Y$ is $2^m$, so that every orientation of $K_{m,n}$, $n > 2^m$, necessarily contains a pair of full twin vertices. \[\square\]

Similarly, we say that $u$ and $v$ are full antitwins in $\overrightarrow{K}$ if $N^+(u) = N^-(v)$ (which implies $N^-(u) = N^+(v)$).

Let now $\{x,x\}'$ be a pair of vertices from $X$. We say that $y$ and $y'$ are $\{x,x\}'$-antitwins in $\overrightarrow{K}$ if (i) the set of vertices $\{x,x',y,y'\}$ induces a directed 4-cycle, and (ii) $y$ and $y'$ agree on every vertex $x''$ from $X \setminus \{x,x'\}$. In particular, it means that there is no other directed 4-cycle containing both $y$ and $y'$. Note that any two such $\{x,x\}'$-antitwins have the same out-degree, and thus belong to the same subset $Y_d$ of $Y$, for some integer $d$, $1 \leq d \leq m - 1$ (in particular, a source vertex or a sink vertex cannot have an $\{x,x\}'$-antitwin). Moreover, observe that if $\{y,y'\}$ and $\{y,y''\}$ are both pairs of $\{x,x\}'$-antitwins in $\overrightarrow{K}$, then $y'$ and $y''$ are necessarily full twins in $\overrightarrow{K}$.

For example, in the orientation of $K_{3,5}$ depicted in Figure 2, $y_1$ and $y_2$ are full twins, $y_1$ and $y_4$ (or $y_2$ and $y_4$) are full antitwins, while $y_3$ and $y_5$ are $\{x_1,x_3\}$-antitwins. Observe also that the canonical orientation $\overrightarrow{KK}_m^*$ of $K_{m,2m}$ contains no pair of full twins, that every vertex $y$ from $Y^*$ has one full antitwin, and that every vertex $y$ from $Y^*$ which is neither a source not a sink has an $\{x,x\}'$-antitwin for exactly $d^-(y) \times d^+(y)$ pairs of vertices $\{x,x\}'$.

For $m = 2$, we have the following result.

**Lemma 17** The complete bipartite graphs $K_{2,4}$ does not admit any rigid orientation, while $K_{2,3}$ admits a rigid orientation.

**Proof.** Let $\overrightarrow{K}$ be any orientation of $K_{2,4}$. By Proposition 16, we can assume that $\overrightarrow{K}$ has no full twins, that is, $\overrightarrow{K}$ is the orientation of $K_{2,4}$ depicted in Figure 3. It is then easy to check
that the permutation $(x_1, x_2)(y_2, y_3)$ is an automorphism of $\vec{K}$. A rigid orientation of $K_{2,3}$ is depicted in Figure 3. □

We now want to characterize the values of $m$ and $n$, with $n > m \geq 2$, for which $K_{m,n}$ admits a rigid orientation. For every such graph, by Proposition 2, we will then have $OD^-(K_{m,n}) = OD^-(K_{m,n}) = 1, OD^-(K_{m,n}) = 2$ and $OD^-(K_{m,n}) = n$.

Each orientation $\vec{K}$ of a complete bipartite graph $K_{m,n}$, $n > m \geq 2$, having no full twins is a subdigraph of the canonical orientation $\vec{K}_{m,n}$, obtained by deleting some vertices of $Y^*$.

In the following, such an orientation $\vec{K}$ will be described by the set $W(\vec{K})$ of words associated with these deleted vertices. Note that the cardinality of the set $W(\vec{K})$ is precisely $2^m - n$. Let $W(\vec{K}) = \{w_1, \ldots, w_{2^m-n}\}$. Note that for every vertex $x_i$ in $X$, $1 \leq i \leq m$, we have

$$d^- \left( x_i \right) = 2^{m-1} - \sum_{j=1}^{2^m-n} w_j^i. \quad (1)$$

For example, $W(\vec{K}_{3,5}^*) = \emptyset$, while the set of words describing the orientation $\vec{K}$ of $K_{3,5}$ depicted in Figure 4 is $W(\vec{K}) = \{010, 011, 111\}$, and the in-degree of $x_3$ is $2^2 - (0 + 1 + 1) = 2$.

Since every orientation $\vec{K}$ of $K_{m,n}$, $3 \leq m < n \leq 2^m$, having no full twins is a subdigraph of $\vec{K}_{m,n}^*$, we get that for every two distinct vertices $y$ and $y'$ in $Y$, there exists a vertex $x$ in $X$ such that $y$ and $y'$ disagree on $x$. Therefore, every non-trivial automorphism of $\vec{K}$ must act on $X$.

Observation 18. For every orientation $\vec{K}$ of $K_{m,n}$, $3 \leq m < n \leq 2^m$, having no full twins, the only automorphism of $\vec{K}$ that fixes every vertex $x$ of $X$ is the identity. Therefore, if all vertices in $X$ have distinct in-degrees (or, equivalently, distinct out-degrees), then $\vec{K}$ is rigid.

Thanks to this observation, we can solve the cases $m = 3$ and $m = 4$.

Lemma 19. For every integer $n$, $4 \leq n \leq 6$, $K_{3,n}$ admits a rigid orientation. For every integer $n$, $5 \leq n \leq 13$, $K_{4,n}$ admits a rigid orientation.

Proof. Consider the orientations $\vec{K}$, $\vec{K}'$ and $\vec{K}''$ of $K_{3,4}$, $K_{3,5}$ and $K_{3,6}$, respectively, given by $W(\vec{K}) = \{110, 010, 000, 111\}$, $W(\vec{K}') = \{110, 010, 000\}$ and $W(\vec{K}'') = \{110, 010\}$. In each
The set $W_7$

| 1 2 3 4 5 6 7 | $j$ |
|--------------|-----|
| 0 0 1 1 1 1 1 |     |
| 0 1 0 0 1 1 1 |     |
| 0 0 0 1 0 0 1 |     |
| 0 0 0 0 0 1 0 |     |

The sums $S_j(W_7)$

| 0 1 1 2 2 3 3 | $j$ |
|---------------|-----|

The set $W_8$

| 0 0 1 1 1 1 1 1 |
| 0 1 0 0 1 1 1 1 |
| 0 0 0 1 0 0 1 1 |
| 0 0 0 0 0 1 0 1 |

The sums $S_j(W_8)$

| 0 1 1 2 2 3 3 4 | $j$ |
|----------------|-----|

Figure 5: The sets $W_7$ and $W_8$ for the proof of Lemma 20.

We are now able to prove the following result.

**Lemma 20** For every two integers $m$ and $k$, $m \geq 5$, $\left\lceil \frac{m}{2} \right\rceil \leq k < 2^m - m$, $K_{m,2^m-k}$ admits a rigid orientation.

**Proof.** For every set of words $W = \{w_1, \ldots, w_k\} \subseteq \{0,1\}^m$, we denote by $S^i(W)$, $1 \leq i \leq m$, the sum of the $i$-th symbols of the words in $W$, that is, $S^i(W) = \sum_{\ell=1}^k w_{i\ell}^\ell$.

We first consider the case $k = \left\lceil \frac{m}{2} \right\rceil$. We will construct a particular set $W_m = \{w_1, \ldots, w_k\}$ of $k$ words such that the orientation $\vec{K}$ of $K_{m,2^m-k}$ with $W(\vec{K}) = W_m$ is rigid. These $k$ words are defined as follows (see Figure 5 for the sample sets $W_7$ and $W_8$):

- $w_1 = 0.1^{m-2}$,
- for every $i$, $2 \leq i \leq \left\lceil \frac{m}{2} \right\rceil - 1$, $w_i = 0^{2i-3}1001.1^{m-2i}$,
- $w_{\left\lceil \frac{m}{2} \right\rceil} = 0^{m-2}.10$ if $m$ is odd, or $w_{\left\lceil \frac{m}{2} \right\rceil} = 0^{m-3}.101$ if $m$ is even.

Observe that $S^1(W_m) = 0$, $S^{2i}(W_m) = S^{2i+1}(W_m) = i$ for every $i$, $1 \leq i \leq \left\lfloor \frac{m-1}{2} \right\rfloor$, and $S^m(W_m) = \frac{m}{2}$ if $m$ is even. By Equation (1), this implies that the set of pairs of vertices in $X$ having the same in-degree is $\{(x_{2i}, x_{2i+1}) : 1 \leq i \leq \left\lceil \frac{m-1}{2} \right\rceil\}$. Therefore, the restriction to $X$ of every non-trivial automorphism of $\vec{K}$ must be the product of at least one transposition corresponding to these pairs of vertices.
We claim that such a situation cannot occur, so that $\overrightarrow{K}$ does not admit any non-trivial automorphism. Indeed, we cannot exchange $x_{2i}$ and $x_{2i+1}, 1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$, since the pair $\{y, y'\}$ of $\{x_{2i}, x_{2i+1}\}$-antitwins, where $y$ is the vertex associated with the word $w_i$, is such that $y \in W_m$, and thus $y$ has been deleted, while $y' \notin W_m$, and thus $y'$ still belongs to $\overrightarrow{K}$ (hence, $x_{2i}$ and $x_{2i+1}$ disagree on $y'$). The orientation $\overrightarrow{K}$ of $K_{m, 2m-k}$ given by $W(\overrightarrow{K}) = W_m$ is thus rigid.

Let us now consider the case $\lceil \frac{m}{2} \rceil < k < 2^m - m$, and let $p = k - \lceil \frac{m}{2} \rceil$. We then construct a set of words $W_k$, by adding to the set $W_m$ previously defined $\lfloor \frac{k}{2} \rfloor$ pairs of full antitwins distinct from the pair $\{0^m, 1^m\}$, together with the word $0^m$ if $p$ is odd. Note that this is always possible since $W_m$ contains $\lfloor \frac{m}{2} \rfloor$ words, and thus $\frac{1}{2}(2^m - 2 \lfloor \frac{m}{2} \rfloor)$ pairs of antitwins are available while we need at most $\frac{1}{2}(2^m - m - \lfloor \frac{m}{2} \rfloor)$ such pairs. Doing so, the sums $S^j(W_k), 1 \leq j \leq m$, satisfy the same properties as in the previous case so that, again, the set of pairs of vertices in $X$ having the same in-degree is $\{(x_{2i}, x_{2i+1}) : 1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor\}$. The same argument as before then allows us to conclude that the orientation $\overrightarrow{K}$ of $K_{m, 2m-k}$ given by $W(\overrightarrow{K}) = W_k$ is also rigid. This completes the proof. $\square$

In order to complete the study of complete bipartite graphs, we still have to consider the graphs $K_{m, 2m-k}$ for every $k, 0 \leq k < m$. The following result is useful for small values of $k$.

**Lemma 21** Let $\overrightarrow{K}$ be any orientation of $K_{m,n}, 3 \leq m < n \leq 2^m$, not containing full twins. If there exist two indices $i$ and $i'$, $1 \leq i < i' \leq m$, such that $w_j^i = w_j^{i'}$ for every word $w_j \in W(\overrightarrow{K})$, then $\overrightarrow{K}$ is not rigid. In particular, if $m \geq 3$ and $0 \leq k < \log_2(m)$, then $K_{m, 2m-k}$ does not admit any rigid orientation.

**Proof.** Observe that there are exactly $p = 2^{m-2}$ pairs of $\{x_i, x_{i'}\}$-antitwins in $Y$ in the canonical orientation $K_{m,n}^*$ of $K_{m,n}$. Let us denote the set of these pairs by $A_{x_i, x_{i'}} = \{(y_{j1}, y_{j1}'), \cdots, (y_{j_p}, y_{j_p}')\}$. Since all vertices corresponding to words in $W(\overrightarrow{K})$ agree on $x_i$ and $x_{i'}$, all vertices belonging to a pair of $A_{x_i, x_{i'}}$ belong to $\overrightarrow{K}$. Therefore, the permutation $(x_i, x_{i'}) (y_{j1}, y_{j1}'), \cdots, (y_{j_p}, y_{j_p}')$ is a non-trivial automorphism of $\overrightarrow{K}$, and thus $\overrightarrow{K}$ is not rigid.

Finally, if $m \geq 3$ and $0 \leq k < \log_2(m)$, then, for every orientation $\overrightarrow{K'}$ of $K_{m, 2m-k}$, there necessarily exist two indices $i$ and $i'$, $1 \leq i < i' \leq m$, such that $w_j^i = w_j^{i'}$ for every word $w_j \in W(\overrightarrow{K'})$, which implies that $\overrightarrow{K'}$ is not rigid. $\square$

From Theorem 15(2) and Lemmas 17, 19 and 20, we finally get the following corollary.

**Corollary 22** For every two integers $m$ and $n$, $m < n$, with either $m \geq 5$ and $n \leq 2^m - \lceil \frac{m}{2} \rceil$, or $(m, n) \in \{(2, 3), (3, 4), (3, 5), (3, 6)\} \cup \{(4, p), 5 \leq p \leq 13\}$, we have

$$OD^-(K_{m,n}) = OD^-(K_{m,n}) = 1, \quad OD^-(K_{m,n}) = 2 \quad \text{and} \quad OD^-(K_{m,n}) = n.$$ 

## 6 Discussion

In this paper, we have studied the distinguishing number, the distinguishing chromatic number and the distinguishing chromatic index of oriented graphs.

We have determined the minimum and maximum values, taken over all possible orientations of the corresponding underlying graph, of these parameters for paths, cycles, complete graphs and bipartite complete graphs, except for the minimum values for unbalanced bipartite complete graphs $K_{m,n}$ not covered by Corollary 22 in which case we were only able to provide upper bounds (see Theorem 15(3)).
Following our work, and apart the question of considering other graph classes, the main question is thus to determine the minimum values of the distinguishing parameters of unbalanced bipartite complete graphs not covered by Corollary 22, that is, of $K_{m,n}$ with $m \geq 5$ and $n > 2^m - \lceil \frac{m}{2} \rceil$. In particular, it would be interesting to know which of those complete bipartite graphs admit a rigid orientation. By Lemma 21 we know that $K_{m,n}$, $m < n$, does not admit any rigid orientation when $n > 2^m \log_2(m)$. We can also prove that $K_{m,n}$ does not admit any rigid orientation when $n = 2^m - 2^p$, for some $p \geq 2$. However, we do not have any complete characterization of these graphs yet.

References

[1] Michael O. Albertson. Distinguishing Cartesian powers of graphs. *Electron. J. Combin.* 12 (2005), #N17
[2] Michael O. Albertson and Karen L. Collins. A Note on Breaking the Symmetries of Tournaments. *Proc. 13th Southeastern Int. Conf. on Combinatorics, Graph Theory, and Computing. Congr. Numer.* 136 (1999), 129–131.
[3] Michael O. Albertson and Karen L. Collins. Symmetry Breaking in Graphs. *Electron. J. Combin.* 3(1) (1996), #R18.
[4] Saeid Alikhani and Samaneh Soltani. The distinguishing number and the distinguishing index of the lexicographic product of two graphs. *Discuss. Math. Graph Theory* 38 (2018), 853–865.
[5] Saeid Alikhani and Samaneh Soltani. The chromatic distinguishing index of certain graphs. *AKCE Int. J. Graphs and Combin.*, in press.
[6] Vikraman Arvind, Christine T. Cheng and Nikhil R. Devanur. On computing the distinguishing numbers of planar graphs and beyond: a counting approach. *SIAM J. Discrete Math.* 22(4) (2008), 1297–1324.
[7] Bill Bogstad and Lenore J. Cowen. The distinguishing number of the hypercube. *Discrete Math.* 283(1–3) (2004), 29–35.
[8] Christine T. Cheng. On computing the distinguishing and distinguishing chromatic numbers of interval graphs and other results. *Discrete Math.* 309(16) (2009), 5169–5182.
[9] Karen L. Collins and Ann N. Trenk. The Distinguishing Chromatic Number. *Electron. J. Combin.* 13 (2006), #R16.
[10] Ehsan Estaji, Wilfried Imrich, Rafal Kalinowski, Monika Piłśniak and Thomas Tucker. Distinguishing Cartesian products of countable graphs. *Discuss. Math. Graph Theory* 37 (2017), 155–164.
[11] Michael Fisher and Garth Isaak. Distinguishing colorings of Cartesian products of complete graphs. *Discrete Math.* 308 (2008), 2240–2246.
[12] David Gluck. Trivial set-stabilizers in finite permutation groups. *Can. J. Math.* 35(1) (1983), 59–67.
[13] Wilfried Imrich, Janja Jerebic and Sandi Klavžar. The distinguishing number of Cartesian products of complete graphs. *European J. Combin.* 29(4) (2008), 922–929.
[14] Wilfried Imrich and Sandi Klavžar. Distinguishing Cartesian powers of graphs. *J. Graph Theory* 53(3) (2006), 250–260.
[15] Rafal Kalinowski and Monika Piłśniak. Distinguishing graphs by edge-colourings. *European J. Combin.* 45 (2015), 124–131.
[16] Sandi Klavžar and Xuding Zhu. Cartesian powers of graphs can be distinguished by two labels. *European J. Combin.* 28(1) (2007), 303–310.

[17] Claude Laflamme, L. Nguyen Van Thé and Norbert Sauer. Distinguishing Number of Countable Homogeneous Relational Structures. *Electron. J. Combin.* 17 (2010), #R20.

[18] Andrew Lazowski and Stephen M. Shea. Finite Factors of Bernoulli Schemes and Distinguishing Labelings of Directed Graphs. *Electron. J. Combin.* 19 (2012), #P1.

[19] Antoni Lozano. Symmetry Breaking in Tournaments. *Electron. J. Combin.* 20(1) (2013), #P69.

[20] Antoni Lozano. Distinguishing Tournaments with Small Label Classes. *Acta Math. Univ. Comenianae* 88(3) (2019), 923–928.

[21] Kahina Meslem and Éric Sopena. On the Distinguishing Number of Cyclic Tournaments: Towards the Albertson-Collins Conjecture. *Discrete Applied Math.* 266 (2019), 219–236.

[22] Kevin T. Phelps. Automorphism free Latin square graphs. *Discrete Math.* 31 (1980), 193–200.

[23] Douglas S. Stones. Symmetries of partial Latin squares. *European J. Combin.* 34 (2013), 1092–1107.