OBSERVABLE-GEOMETRIC PHASES AND QUANTUM COMPUTATION

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Abstract. This paper presents an alternative approach to geometric phases from the observable point of view. Precisely, we introduce the notion of observable-geometric phases, which is defined as a sequence of phases associated with a complete set of eigenstates of the observable. The observable-geometric phases are shown to be connected with the quantum geometry of the observable space evolving according to the Heisenberg equation. They are indeed distinct from Berry’s phase \[4, 14\] as the system evolves adiabatically. It is shown that the observable-geometric phases can be used to realize a universal set of quantum gates in quantum computation. This scheme leads to the same gates as the Abelian geometric gates of Zhu and Wang \[21, 22\], but based on the quantum geometry of the observable space beyond the state space.

1. Introduction

The geometric phase can be used in quantum computing by implementing each of quantum gates in terms of geometric phases only \[19\]. This scheme for computing is called geometric quantum computation. Geometric quantum computation involves adiabatic or nonadiabatic, as well as Abelian or non-Abelian characteristics of the underlying quantum evolution for the quantum state. It is shown in \[21, 22\] that the non-adiabatic Abelian geometric phases can be used to realize a universal set of quantum gates and thus are sufficient for universal all-geometric quantum computation. Furthermore, the non-adiabatic non-Abelian geometric phases are shown in \[18\] to achieve universality as well. We refer to \[16\] for a up-date review and background.

Here, we present an alternative approach to the geometric phase from the observable point of view. Precisely, we introduce the concept of observable-geometric phases, which is defined as a sequence of phases associated with a complete set of eigenstates of the observable. The observable-geometric phases are shown to be connected with the geometry of the observable space evolving according to the Heisenberg equation. In particular, the observable-geometric phases obtained are distinct from Berry’s phase \[3, 14\] as the system evolves adiabatically. As application, the observable-geometric phases can be used to realize a universal set of quantum gates in quantum computation. The quantum gate based on the observable-geometric phases can be interpreted as the Abelian geometric (ZW) gates of Zhu and Wang \[21, 22\]; indeed the two schemes lead to the same gates. However, our gates are connected with a quantum geometric structure over the observable space induced by a complete measurement, distinct from that the ZW gates are related with classical (Riemannian) geometry of the state space of the system.

The paper is organized as follows. In Section 2 we will define the notion of the observable-geometric phase using the propagator of a time-dependent Hamiltonian, which is shown to be independent of the choice of the Hamiltonian as long as the Heisenberg equations involving the Hamiltonian. Also, we show that the observable-geometric phase does not coincide with Berry’s phase as the system evolves adiabatically. In Section 3, the observable-geometric phase of a qubit system is computed in details, and moreover, we will show how to make use of the Lewis-Resenfeld invariant theory \[10\] computing the observable-geometric phase of a general quantum system. As application, in Section 4 we show that the observable-geometric phases can be used to realize a universal set of quantum gates in quantum computation. We will give a summary in Section 5. Finally, we include an appendix as Section 6 on a quantum-geometric structure over the observable space consisting of all complete orthogonal decompositions.

For the sake of simplicity, we only consider finite quantum systems, namely the associated Hilbert spaces \(\mathbb{H}\) have a finite dimension. This is sufficient for quantum computation. In what follows, we
always denote by $B(\mathbb{H})$ the algebra of all bounded operators on $\mathbb{H}$, by $\mathcal{O}(\mathbb{H})$ the set of all self-adjoint operators on $\mathbb{H}$, and by $U(\mathbb{H})$ the group of all unitary operators on $\mathbb{H}$. Without specified otherwise, the integer $d$ always denotes the dimension of $\mathbb{H}$, and $I$ the identity operator on $\mathbb{H}$.

2. Observable-geometric phase

Consider a finite quantum system with a time-dependent Hamiltonian $h(t)$. Since the associated Hilbert space $\mathbb{H}$ has finite dimension $d$, each Hamiltonian $h(t)$ has a discrete spectrum as well as all observables considered in the following. Indeed, the spectrum of $h(t)$ at any given time will not be of importance. Instead, we shall consider the time evolution operator, as called propagator, generated by $h(t)$ (see [13] Theorem X.69]). This is a two-parameter continuous family of unitary operators such that for any $t, r, s \in \mathbb{R}$,

\begin{equation}
U(t, t) = I, \quad U(t, r)U(r, s) = U(t, s),
\end{equation}

and

\begin{equation}
\frac{id}{dt}U(t, s) = h(t)U(t, s),
\end{equation}

that is, for any $s \in \mathbb{R}$ and $\phi \in \mathbb{H}$, $\phi_s(t) = U(t, s)\phi$ is the unique solution of the time-dependent Schrödinger equation

\begin{equation}
\frac{id}{dt}\phi_s(t) = h(t)\phi_s(t)
\end{equation}

with $\phi_s(s) = \phi$. Given any observable $X_0$, namely a self-adjoint operator on $\mathbb{H}$, by (2.2) we conclude that $X(t) = U(0, t)X_0U(0, t)$ is the unique solution of the time-dependent Heisenberg equation

\begin{equation}
\frac{dX(t)}{dt} = [X(t), \hat{h}(t)]
\end{equation}

with $X(0) = X_0$, where $\hat{h}(t) = U(0, t)h(t)U(t, 0)$. If there exists $T > 0$ such that $X(T) = X(0)$, the time evolution of observable $(X(t) : t \in \mathbb{R})$ is then called cyclic with period $T$, and $X_0 = X(0)$ is said to be a cyclic observable.

Given an observable $X_0$ with non-degenerate eigenstates $\psi_n$ for all $1 \leq n \leq d$, assume that $X(t) = U(0, t)X_0U(0, t)$ is cyclic with period $T$, namely $X(T) = X_0$ so that $U(0, T)\psi_n = e^{i\theta_n}\psi_n$ with some $\theta_n \in [0, 2\pi)$ for every $n = 1, \ldots, d$. Denoted by $\bar{\psi}_n(t) = U(0, t)\psi_n$ for $1 \leq n \leq d$, which are the eigenstates of $X(t)$, we conclude that $\bar{\psi}_n(t)$ satisfies the skew (time-dependent) Schrödinger equation

\begin{equation}
\frac{id}{dt}\bar{\psi}_n(t) = -\hat{h}(t)\bar{\psi}_n(t)
\end{equation}

with $\bar{\psi}_n(0) = \psi_n$, due to the fact that $U(0, t) = U(t, 0)^{-1}$. Define

\begin{equation}
|\tilde{\psi}_n(t)\rangle = e^{-i\int_0^t [\bar{\psi}_n(s), h(s)]ds}|\bar{\psi}_n(t)\rangle
\end{equation}

and

\begin{equation}
|\tilde{\psi}_n(t)\rangle = e^{-i\alpha_n(t)}|\bar{\psi}_n(t)\rangle
\end{equation}

with $\alpha_n(t) \in [0, 2\pi]$ being continuous in $t$ such that $\alpha_n(T) - \alpha_n(0) = \theta_n$, i.e., $|\tilde{\psi}_n(T)\rangle = |\tilde{\psi}_n(0)\rangle$ for every $n \geq 1$. Note that for any $n \geq 1$,

\begin{equation}
|\tilde{\psi}_n(t)\rangle = e^{-i\int_0^t [\psi_n(0), h(s)]ds}|\psi_n(t)\rangle
\end{equation}

because $\hat{h}(t) = U(0, t)h(t)U(t, 0)$, and $|\tilde{\psi}_n(T)\rangle = e^{i\beta_n}|\psi_n(0)\rangle$ with

\begin{equation}
\beta_n = \theta_n - \int_0^T \langle \psi_n(0) | h(t) | \psi_n(0) \rangle dt.
\end{equation}

Now, from (2.5) we conclude

\begin{equation}
\langle \tilde{\psi}_n(t) | \frac{d}{dt} | \tilde{\psi}_n(t) \rangle = 0,
\end{equation}

and

\begin{equation}
\beta_n = \int_0^T i\langle \tilde{\psi}_n(t) | \frac{d}{dt} | \tilde{\psi}_n(t) \rangle dt
\end{equation}
since $|\tilde{\psi}_n(T)| = |\psi_n(0)|$. This leads to the notion of the geometric phase for observable as follows.

**Definition 2.1.** Using the above notations, we define the geometric phases of the periodic evolution of observable $X(t)$ by

$$
\beta_n = \theta_n - \int_0^T \langle \psi_n(0)|h(t)|\psi_n(0)\rangle dt
$$

which is uniquely defined up to $2\pi k$ $(k$ is integer$)$ for every $n = 1, \ldots, d$. We simply call $\beta_n$’s the observable-geometric phases.

**Remark 2.1.** 1) Note that for every $n = 1, \ldots, d$, \(\theta_n = \arg(\psi_n(0)|U(T,0)|\psi_n(0))\) and

$$
\langle \psi_n(0)|h(t)|\psi_n(0)\rangle = \langle \psi_n(t)|\hat{h}(t)|\psi_n(t)\rangle
$$

for all $t$. Thus, the observable-geometric phases $\beta_n$’s are obtained by removing the dynamical part from the total phases of eigenstates of observable according to the evolution of the Heisenberg equation (2.4) or equivalent (2.5). This is in spirit the same as the definition of the Aharonov-Anandan (AA) geometric phase [1].

2) When some eigenvalues of the initial observable $X_0$ are degenerate as eigenstates, this would lead to the notion of non-Abelian observable-geometric phase as similar to the usual non-Abelian geometric phase (cf. [2, 18]). This could be done as defining the geometric phases for mixed states as in [17, 15]). We will discuss it elsewhere.

Although the observable-geometric phases in (2.10) are defined as similar to the AA geometric phase, the geometric interpretation is completely different. Indeed, (2.10)’s are obtained by removing the dynamical part from all complete orthogonal decompositions (see Section 6 for the notations). The equation (2.8) can be expressed as a condition (see 6.17 below) for quantum parallel transportation along $C_W$. Therefore, the geometric interpretation of $\beta_n$’s defined in (2.10) is related with the geometric properties of $C_W$ determined by a quantum geometric structure over the observable space $W(\mathbb{H})$, distinct from that the AA geometric phase is associated with classical (Riemannian) geometry of the state space. We will give the details of this geometric interpretation in Section 6.5.

If the system evolves adiabatically (cf. [4, Chapter 2]), $\hat{h}(t)$ varies slowly with $h(t)|n(t)) = E_n(t)|n(t))$, for a complete set $\{|n(t))\}$, such that the state remains an eigenstate of $h(t)$ at all time $t$ with the same energy quantum number $n$, namely the evolution operator

$$
U(t,0) \simeq \sum_n e^{i\int_0^t \langle \psi(s)|\frac{d}{dt}|\psi(s)\rangle - E_n(s)ds}|n(t)(n(0))|
$$

to a good approximation, see [11, (5)-(6)] or [3, (2.37)-(2.39)] for the details. For an adiabatic cyclic evolution with period $T$, i.e., $h(0) = h(T)$, such that all the eigenvalues of $h(0)$ are non-degenerate, the observable-geometric phases of the cyclic evolution of observable $X(t) = U(0,t)h(0)U(t,0)$ are

$$
\beta_n = -\int_0^T \langle \psi(s)|\frac{d}{dt}|\psi(s)\rangle - E_n(s)ds dt - \int_0^T \langle \psi(0)|h(t)|n(t)\rangle dt
$$

because

$$
U(0,T)|n(0)) = e^{-i\int_0^T \langle \psi(s)|\frac{d}{dt}|\psi(s)\rangle - E_n(s)ds}|n(0)).
$$

In this case, in general, the observable-geometric phases $\beta_n$’s do not coincide with Berry’s phase

$$
\gamma_n = \int_0^T \langle \psi(t)|\frac{d}{dt}|\psi(t)\rangle dt
$$

as define in [11, 14] (cf. 11). But the observable-geometric phases $\beta_n$’s, defined by (2.10), do not depend on any adiabatic evolution restriction, and can be defined for any period evolution of observable in a general quantum system, not just Hamiltonian $h(t)$ in an adiabatic cyclic system.
3. Examples

For illustrating the observable-geometric phases, we first consider a qubit case, namely the Hilbert space $\mathbb{H} = \mathbb{C}^2$ and the observable space is $\mathcal{W}(\mathbb{C}^2)$. We then show how to make use of the Lewis-Riesenfeld invariant theory\cite{10} computing the observable-geometric phase of a general quantum system. In what follows, we will apply \eqref{2.10} to qubit systems, respectively subject to an orientated magnetic field and a rotating magnetic field.

Recall that the Pauli matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are defined by
$$
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

**Example 3.1.** (An orientated magnetic field) Consider that a spin-$\frac{1}{2}$ particle with a magnetic moment is in a homogeneous magnetic field $\vec{B} = (0, 0, B)$ along the $z$ axis, whose Hamiltonian is $H = -\mu B \sigma_z/2$ where $\mu$ is the Bohr magneton. Given a spin observable $X_0$ with two non-degenerate eigenstates
\begin{equation}
\psi_1 = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} \sin \phi \\ \cos \phi \end{pmatrix}
\end{equation}
in $\mathbb{C}^2$, $X(t) = U(0, t)X_0U(t, 0)$ satisfies Eq.\eqref{2.4} with $\hat{h}(t) = h(t) = -\mu B \sigma_z/2$ and $U(t, 0) = e^{i\mu B t \sigma_z/2}$ so that $\psi_n(t) = U(0, t)\psi_n$ ($n = 1, 2$), where
$$
\psi_1(t) = e^{-i\mu B t \sigma_z/2}\psi_1 = \begin{pmatrix} e^{-i\mu B t^2/2} \cos \phi
\end{pmatrix}, \quad \psi_2(t) = e^{-i\mu B t \sigma_z/2}\psi_2 = \begin{pmatrix} e^{-i\mu B t^2/2} \sin \phi
\end{pmatrix},
$$
satisfies the skew Schrödinger equation \eqref{2.5}, namely
$$
\frac{d\psi_n(t)}{dt} = \mu B \frac{\sigma_z}{2}\psi_n(t), \quad n = 1, 2.
$$
The evolution $X(t)$ is periodic with period $T = \frac{2\pi}{\mu B}$, namely $X(T) = X_0$ and precisely
$$
\psi_1(T) = e^{i\pi}\psi_1, \quad \psi_2(T) = e^{i\pi}\psi_2.
$$

By \eqref{2.10} we have
$$
\beta_1 = \pi(1 + \cos \phi), \quad \beta_2 = \pi(1 - \cos \phi),
$$
which are the geometric invariants of the curve $C_W$:

$$
(0, \frac{\pi}{\mu B}) \ni t \mapsto \left\{ \begin{pmatrix} \cos^2 \frac{\phi}{2} & \frac{1}{2}e^{-2i\mu B t} \sin \phi \\ \frac{1}{2}e^{2i\mu B t} \sin \phi & \cos^2 \frac{\phi}{2} \end{pmatrix}, \begin{pmatrix} \sin^2 \frac{\phi}{2} & \frac{1}{2}e^{-2i\mu B t} \cos \phi \\ \frac{1}{2}e^{2i\mu B t} \cos \phi & \sin^2 \frac{\phi}{2} \end{pmatrix} \right\}
$$
in $\mathcal{W}(\mathbb{C}^2)$.

Indeed, taking a fixed point $O_0 = \{\langle e_1 | e_1 \rangle, \langle e_2 | e_2 \rangle\} \in \mathcal{W}(\mathbb{C}^2)$, and letting
$$
\tilde{U}(t) = e^{it\mu B \cos \phi |\psi_1(t)\rangle\langle e_1| + e^{-it\mu B \cos \phi |\psi_2(t)\rangle\langle e_2|}
$$
for $t \in (0, \frac{2\pi}{\mu B})$, we have that $\tilde{C}_p : (0, \frac{\pi}{\mu B}) \ni t \mapsto \tilde{U}(t)$ is the horizontal $O_0$-lift of $C_W$ with respect to $\hat{\Omega}$ in the principal bundle $\xi_{O_0}$ (the definitions of the canonical connection $\hat{\Omega}$ and the bundle $\xi_{O_0}$ refer to Section 6) so that
$$
\tilde{U}(T) = e^{-i\beta} |\psi_1\rangle\langle e_1| + e^{i\beta} |\psi_2\rangle\langle e_2|,
$$
with $\beta = \beta_2 = \pi(1 - \cos \phi)$ (because $\beta_1 = 2\pi - \beta_2$) is the holonomy element associated with the connection $\hat{\Omega}$, $C_W$, and $U_0 = \sum_{n=1}^2 |\psi_n\rangle\langle e_n|$ in $\xi_{O_0}$.

**Example 3.2.** (A rotating magnetic field) Consider a spin-$\frac{1}{2}$ particle subject to a rotating background magnetic field
$$
\mathbf{B}(t) = (B_0 \cos \omega t, B_0 \sin \omega t, B_1)
$$
with a constant angular velocity $\omega$, where $B_0$ and $B_1$ are constants, whose Hamiltonian is given by
$$
h(t) = -\frac{\mu}{2}\mathbf{B}(t) \cdot \vec{\sigma} = -\frac{1}{2}(\omega_0 \sigma_x \cos \omega t + \omega_0 \sigma_y \sin \omega t + \omega_1 \sigma_z)
$$
with $\omega_i = \mu B_1$ ($i = 0, 1$). Since
$$
\sigma_x \cos \omega t \sigma_z + \sigma_y \sin \omega t \sigma_z = e^{-i\omega t \sigma_z/2} \sigma_x e^{i\omega t \sigma_z/2},
$$
the Hamiltonian takes the form \( h(t) = e^{-i\omega t} \frac{\sigma_z}{2} h_0 e^{i\omega t} \frac{\sigma_z}{2} \) with

\[
h_0 = -\frac{1}{2} \omega_0 \sigma_x - \frac{1}{2} \omega_1 \sigma_z.
\]

If \( \phi(t) \) satisfies the time-dependent Schrödinger equation \((2.3)\), then \( \psi(t) = e^{i\omega t} \frac{\sigma_z}{2} \phi(t) \) satisfies the Schrödinger equation

\[
\frac{d\psi(t)}{dt} = (h_0 - \frac{1}{2} \omega \sigma_z) \psi(t).
\]

Because

\[
H = h_0 - \frac{1}{2} \omega \sigma_z = -\frac{1}{2} \omega_0 \sigma_x - \frac{1}{2} (\omega_1 + \omega) \sigma_z
\]
is time-independent, we have \( \psi(t) = e^{-itH} \psi(0) \) and thus

\[
\phi(t) = e^{-i\omega t} \frac{\sigma_z}{2} e^{-itH} \psi(0)
\]
so that \( U(t, 0) = e^{-i\omega t} \frac{\sigma_z}{2} e^{-itH} \) is the propagator associated with \( h(t) \).

Having obtained the exact expression of the propagator \( U(t, 0) \) we can next identify the cyclic evolution of observable, namely the solutions which satisfy \((2.4)\) with

\[
\tilde{h}(t) = U(0, t) h(t) U(t, 0) = e^{itH} h_0 e^{-itH}.
\]

Given a spin observable \( X_0 \) with non-degenerate eigenstates \( \psi_1 \) and \( \psi_2 \), \( X(t) = U(0, t) X_0 U(t, 0) \) is by definition cyclic with period \( T \) if and only if \( \{ \psi_1, \psi_2 \} \) is a complete set of eigenstates of \( U(0, T) \). The corresponding eigenvalues \( e^{i\theta_n} \) are the total phase factors, namely \( U(0, T) \psi_n = e^{i\theta_n} \psi_n \). The corresponding observable-geometric phases are

\[
\beta_n = \theta_n + \frac{\omega_0}{2\omega} \langle \psi_n | \sigma_x | \psi_n \rangle \sin T + \frac{\omega_0}{2\omega} \langle \psi_n | \sigma_y | \psi_n \rangle (1 - \cos \omega T) + \frac{\omega_1}{2} \langle \psi_n | \sigma_z | \psi_n \rangle T
\]
for every \( n = 1, 2 \). As shown in Theorem 6.1, \( \beta_1, \beta_2 \) are two geometric invariants of the curve

\[
C_W : [0, T] \ni t \mapsto \{ U(0, t) | \psi_n \rangle | U(t, 0) : n = 1, 2 \}
\]
in \( W(\mathbb{C}^2) \), and

\[
\tilde{U}(T) = e^{i\beta_1} | \psi_1 \rangle \langle e_1 | + e^{i\beta_2} | \psi_2 \rangle \langle e_2 |
\]
is the holonomy element associated with the connection \( \tilde{\Omega}, C_W \), and \( U_0 = \sum_{n=1}^2 | \psi_n \rangle \langle e_n | \) in \( \xi_{O_0} \), given an arbitrary point \( O_0 = \{ | e_1 \rangle, | e_2 \rangle \} \) in \( W(\mathbb{C}^2) \).

A special case is the period \( T = 2\pi/\omega \), namely the same period as the Hamiltonian \( h(t) \). In this case, cyclic evolutions are obtained by finding the simultaneous eigenvectors of the operator

\[
U(0, 2\pi/\omega) = e^{i\frac{\Omega}{T} H} e^{i\frac{\phi}{2} \sigma_z} = -e^{i\frac{\phi}{2} \sigma_z}.
\]
These eigenvectors are consequently also ones of the operator

\[
H = -\frac{1}{2} \omega_0 \sigma_x - \frac{1}{2} (\omega_1 + \omega) \sigma_z,
\]

a complete set of whose eigenvectors is

\[
\psi_+ = \left( \frac{\cos \frac{\phi}{2}}{\sin \frac{\phi}{2}} \right), \quad \psi_- = \left( -\frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}} \right),
\]

namely

\[
H \psi_{\pm} = \pm \frac{1}{2} \sqrt{\omega_0^2 + (\omega_1 + \omega)^2} \psi_{\pm},
\]

where

\[
\phi = 2 \arctan \frac{\omega_0}{\omega_1 + \omega + \sqrt{\omega_0^2 + (\omega_1 + \omega)^2}}.
\]

Then, for the cyclic evolution \( X(t) = U(0, t) X_0 U(t, 0) \) starting from the initial observable \( X_0 \) with two non-degenerate eigenstates \( \psi_{\pm} \), the total phase factors are

\[
\theta_{\pm} = \pi \pm \frac{\pi}{\omega} \sqrt{\omega_0^2 + (\omega_1 + \omega)^2}
\]
because \( U(0, 2\pi/\omega) \psi_{\pm} = e^{i\theta_{\pm}} \psi_{\pm} \). Thus, by \((3.2)\) the corresponding observable-geometric phases are

\[
\beta_{\pm} = \pi \pm \frac{\pi}{\omega} \sqrt{\omega_0^2 + (\omega_1 + \omega)^2} + \omega_1 \cos \phi,
\]

which are two geometric invariants of the curve

\[
C_W : [0, T] \ni t \mapsto \{ U(0, t) | \psi_+ \rangle | U(t, 0), U(0, t) | \psi_- \rangle | U(t, 0) \}
\]
in \( W(\mathbb{C}^2) \). Moreover,

\[
\tilde{U}(T) = e^{i\beta} | \psi_+ \rangle \langle e_1 | + e^{-i\beta} | \psi_- \rangle \langle e_2 |
\]
with $\beta = \beta_+ \,(\text{because} \, \beta_- = 2\pi - \beta_+)\,\text{is the holonomy element associated with the connection} \, \hat{\Omega}, C_W,$
and $U_0 = |\psi_+e_1| + |\psi_-e_2| \,\text{in} \, \xi_{O_0},\,\text{given an arbitrary point} \, O_0 = \{|e_1\rangle, |e_2\rangle\} \,\text{in} \, W(\mathbb{C}^2)$.

Generally speaking, it is a difficult task for getting a explicit expression of the evolution operator of a quantum system with a time-dependent Hamiltonian. To this end, Lewis and Riesenfeld \cite{10} developed a dynamical invariant theory for computing the evolution operator. Recall that a dynamical of a quantum system with a time-dependent Hamiltonian. To this end, Lewis and Riesenfeld \cite{10}
(Lewis-Riesenfeld phase for a qubit) Consider a qubit system with time-dependent
as define in \cite{1} (cf. \cite{8}). In what follows, we apply the formula (3.6) to the qubit system.

with the corresponding eigenstates $\lambda_{\pm}$

$$\lambda_{\pm} = \pm \sqrt{\Delta_{\pm} + (\Omega_1 - \omega)^2}$$

where $\Delta_{\pm} = \frac{1}{\Omega_0}(\lambda_{\pm} + \Omega_1 - \omega)$. By direct computation, the corresponding Lewis-Riesenfeld phases of
these vectors are

$$\alpha_{\pm}(t) = \frac{1}{2}(\omega \mp \lambda)t$$

so that

$$U(t, 0) = e^{\frac{t}{2}(\omega - \lambda)t}|\varphi_-(0)\rangle$$

as define in \cite{1} (cf. \cite{8}). In what follows, we apply the formula (3.6) to the qubit system.

Example 3.3. (The Lewis-Riesenfeld phase for a qubit) Consider a qubit system with time-dependent Hamiltonian given by

$$h(t) = \frac{1}{2} (\Omega_0 \sigma_x \cos \omega t + \Omega_0 \sigma_y \sin \omega t + \Omega_1 \sigma_z),$$

where $\Omega_0, \Omega_1, \text{and} \, \omega$ are all constants, see Example 3.2 for the details. The corresponding dynamical
invariant for this system is

$$I(t) = \Omega_0 \sigma_x \cos \omega t + \Omega_0 \sigma_y \sin \omega t + (\Omega_1 - \omega) \sigma_z$$

(cf. \cite{8}), whose eigenvalues

$$\varphi_{\pm}(t) = \frac{e^{-\frac{\omega}{2}t\Delta_{\pm}}}{\sqrt{1 + \Delta_{\pm}}}$$

with the corresponding eigenstates

where $\Delta_{\pm} = \frac{1}{\Omega_0}(\lambda_{\pm} + \Omega_1 - \omega)$. By direct computation, the corresponding Lewis-Riesenfeld phases of
these vectors are

$$\alpha_{\pm}(t) = \frac{1}{2}(\omega \mp \lambda)t$$

so that

$$U(t, 0) = e^{\frac{t}{2}(\omega - \lambda)t}|\varphi_+(0)\rangle + e^{\frac{t}{2}(\omega + \lambda)t}|\varphi_-(0)\rangle.$$
Clearly, $I(t)$ is cyclic in time with period $T = \frac{2\pi}{\omega}$, so does $h(t)$. By (3.6), the observable-geometric phases of the cyclic evolution $X(t) = U(0,t)I(0)U(t,0)$ are

\begin{equation}
\beta_\pm = \pi \pm \frac{\pi \lambda}{\omega} - \frac{\Omega_1(\Delta^2_\pm - 1)}{2(1 + \Delta^2_\pm)}.
\end{equation}

4. Geometric quantum computation

In this section, we shall show that the observable-geometric phases can be used to realize a set of universal quantum gates. As is well-known, for a universal set of quantum gates, we need to achieve two kinds of noncommutative 1-qubit gates and one nontrivial 2-qubit gate (cf. [6] [11]). Thus, we need to realize two noncommutative 1-qubit gates and a nontrivial 2-qubit gate as the geometric evolution operators on loops in the observable spaces of a qubit and 2-qubits, respectively.

**Proposition 4.1.** The two 1-qubit gates $e^{i\theta|1\rangle\langle 1|}$ and $e^{i\sigma_z}$ can be respectively realized as geometric quantum gates associated with the geometry of the observable space $\mathcal{V}(\mathbb{C}^2)$ of a qubit, which are two well-known gates constituting a universal set of quantum gates for a single-qubit system.

**Proof.** As in Example 3.2, a general time-dependent Hamiltonian for a qubit has only one term $h(t) = -\mu B(t) \cdot \vec{\sigma}/2$, where $B(t) = (B_0 \cos \omega t, B_0 \sin \omega t, B_1)$ denotes the total magnetic field felt by the qubit, $B_0, B_1$ and $\omega$ are constants, and $\mu$ is the Bohr magneton. Given a spin observable $X_0$ with two non-degenerate eigenstates

$$\psi_+ = \cos \frac{\phi}{2} |0\rangle + \sin \frac{\phi}{2} |1\rangle, \quad \psi_- = -\sin \frac{\phi}{2} |0\rangle + \cos \frac{\phi}{2} |1\rangle,$$

where $|0\rangle$ and $|1\rangle$ constitute the computational basis for the qubit, the observable evolution $t \mapsto U(0,t)X_0U(t,0)$ is periodic with period $T = 2\pi/\omega$ provided

$$\phi = 2 \arctan \frac{\omega_1}{\omega_1 + \omega + \sqrt{\omega_0^2 + (\omega_1 + \omega)^2}},$$

with $\omega_i = \mu B_i$ ($i = 0, 1$), where $U(t,0) = e^{-i\omega t \varphi_0 t} e^{-itH}$ with $H = -[\omega_0 \sigma_z + (\omega_1 + \omega)\sigma_x]/2$. This cyclic process determines the evolution operator $U(T,0) = e^{-i\theta} |\psi_+\rangle \langle \psi_+| + e^{i\theta} |\psi_-\rangle \langle \psi_-|$ with $\theta = \pi + \frac{\pi}{\omega} \sqrt{\omega_0^2 + (\omega_1 + \omega)^2}$. If we can remove the dynamical phase from the total phase $\theta$, then the evolution operator becomes a geometric quantum gate. This can be done by using a two-loop method as in the case of the AA geometric phases (cf. [21] [22]).

To this end, we first allow the time-dependent Hamiltonian $h(t)$ to go through cyclic evolution with period $T = 2\pi/\omega$. Precisely, we consider the process where a spin observable $X_0$ with two non-degenerate eigenstates $\psi_\pm$ can evolve cyclically. We first decide the cyclic evolution observable $X_0$ by choosing $\phi$ as above. The phases of eigenstates $\psi_\pm$ acquired in this way would contain both a geometric and a dynamical component as described in (2.10). In order to remove the dynamical phase accumulated in the above process, we allow $X_0$ to evolve along the time-reversal path of the first-period loop during the second period. This process can be realized by reversing the effective magnetic field with $B(t + T) = -B(T - t)$ on the same loop of the first-period $[0, T]$, and thus $h(t + T) = -h(T - t)$ for $t \in [0, T]$. As a result, the total phases accumulated in the two periods will be just the observable-geometric phases, because the dynamical phase $\gamma^{(d)}_n = \int_0^T \langle \psi_n | h(t) | \psi_n \rangle dt$ appearing in (2.10) will be canceled.

Indeed, the dynamical phase $\gamma^{(2,d)}_n$ for the second period is equal to that $\gamma^{(1,d)}_n$ for the first period with the opposite sign, namely $\gamma^{(2,d)}_n = -\gamma^{(1,d)}_n$, since

$$\gamma^{(2,d)}_n = \int_0^{2T} \langle \psi_n | h(t) | \psi_n \rangle dt = \int_0^T \langle \psi_n | h(t + T) | \psi_n \rangle dt$$

$$= -\int_0^T \langle \psi_n | h(T - t) | \psi_n \rangle dt = -\int_0^T \langle \psi_n | h(t) | \psi_n \rangle dt = -\gamma^{(1,d)}_n.$$

On the other hand, the propagator for the second period is $U(t,0) = e^{i\omega(t - 2T)\sigma_z/2} e^{itH}$ for $t \in (T, 2T]$. Thus, the observable-geometric phases are

$$\beta_\pm = \pm \frac{\pi}{\omega} \sqrt{\omega_0^2 + (\omega_1 + \omega)^2},$$
namely \( U(2T, 0) = e^{i\beta}|\psi_+\rangle\langle\psi_+| + e^{-i\beta}|\psi_-\rangle\langle\psi_-| \) with \( \beta = \beta_- \).

Therefore, by removing the dynamical phases, we obtain the evolution operator \( U(2T, 0) = U_{\phi, \beta} \) with the matrix representation
\[
U_{\phi, \beta} = \left( \begin{array}{cc}
  e^{i\beta} & e^{-i\beta} \\
  e^{-i\beta} & e^{i\beta}
\end{array} \right).
\]

As noted above, \( U_{\phi, \beta} \) depends only on the geometric property of the curve \( C_W \) and \( \phi \), and thus is a geometric quantum gate. We write an input state as \( |\psi_{\text{in}}\rangle = \alpha_+|0\rangle + \alpha_-|1\rangle \) with \( \alpha_{\pm} = \langle \psi_{\pm}(0)|\psi_{\text{in}}\rangle \). Then \( U_{\phi, \beta} \) is a single-qubit gate such that \( |\psi_{\text{out}}\rangle = U_{\phi, \beta}|\psi_{\text{in}}\rangle \).

By computing, two operations \( U_{\phi, \beta} \) and \( U_{\phi', \beta'} \) do not commute if and only if \( \phi \neq \phi' + k\pi \) and \( \beta, \beta' \neq k\pi \), where \( k \) is an integer. By choosing \( B_0, B_1 \) and \( \omega \) so that \( \phi = 0, \phi' = \frac{\pi}{2} \), and \( \beta = \beta' = \frac{\pi}{2} \), we get the two single-qubit gates
\[
U_{0, \pi/2} = i e^{i\pi/2(1)} \quad U_{\pi, \pi/2} = e^{i\pi/4},
\]
This completes the proof. \( \square \)

Next, we turn to a nontrivial 2-qubit gate, namely the controlled-NOT gate (c-NOT), which is defined as \( |k\rangle|m\rangle \rightarrow |k\rangle|k \oplus m\rangle \) for \( k, m = 0, 1 \), where \( \oplus \) denotes the addition modulo 2.

**Proposition 4.2.** The controlled-NOT gate can be realized as a geometric quantum gate associated with the geometry of the observable space \( W(\mathbb{C}^2 \otimes \mathbb{C}^2) \) of the two-qubits system.

**Proof.** Consider a 2-qubit quantum system with a time-dependent Hamiltonian \( h(t) \), where the first qubit is the control qubit while the second the target one. Suppose \( h(t) \) be of the form
\[
h(t) = \begin{pmatrix} h_0(t) & 0 \\ 0 & h_1(t) \end{pmatrix}.
\]
This is the case when the control qubit is far away from the resonance condition for the operation of the target qubit. Given an initial 2-qubit observable \( X_0 \) with four non-degenerate eigenstates:
\[
|0\rangle \otimes \left( \cos \frac{\phi(0)}{2}|0\rangle + \sin \frac{\phi(0)}{2}|1\rangle \right),
|0\rangle \otimes \left( -\sin \frac{\phi(0)}{2}|0\rangle + \cos \frac{\phi(0)}{2}|1\rangle \right),
|1\rangle \otimes \left( \cos \frac{\phi(1)}{2}|0\rangle + \sin \frac{\phi(1)}{2}|1\rangle \right),
|1\rangle \otimes \left( -\sin \frac{\phi(1)}{2}|0\rangle + \cos \frac{\phi(1)}{2}|1\rangle \right),
\]
where \( \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \) is the computational basis, the evolution \( t \rightarrow U(0, t)X_0U(t, 0) \) is cyclic with period \( T \) under the suitable choices of \( \phi(0) \) and \( \phi(1) \) so that \( |\psi_{\text{in}}\rangle \) are eigenstates of \( U(T, 0) \) for \( k = 0, 1 \). In this case, the evolution operator to describe the two-qubit gate is given by
\[
U(\phi(0), \beta(0); \phi(1), \beta(1)) = \begin{pmatrix} U_{\phi(0), \beta(0)} & 0 \\ 0 & U_{\phi(1), \beta(1)} \end{pmatrix},
\]
where \( \beta^{(k)}(\phi^{(k)}) \) denotes the geometric phase (the cyclic initial observable) of the target bit when the state of the control qubit corresponds to \( |k\rangle \) with \( k = 0, 1 \), respectively.

To achieve the controlled-NOT gate, we take \( h_0 = 0 \) and \( h_1(t) = -\mu B(t) \cdot \sigma/2 \) with \( B(t) = (B_0 \cos \omega t, B_0 \sin \omega t, B_1) \) as in Proposition 4.1. Then the gate in this case with \( T = 2\pi/\omega \) is given by
\[
U(\frac{\pi}{2}, 0; \phi, \beta) = \begin{pmatrix} I & 0 \\ 0 & U_{\phi, \beta} \end{pmatrix},
\]
where \( \beta \) is the total phase accumulated in the evolution when the control qubit is in the state \( |1\rangle \), and \( \phi \) is chose as in Proposition 4.1. By a two-loop method as above, we can remove the dynamical phase and obtain the geometric quantum gate
\[
U(\frac{\pi}{2}, 0; \frac{\pi}{2}, \frac{\pi}{2}) = \begin{pmatrix} I & 0 \\ 0 & i\sigma_x \end{pmatrix}.
\]
This gate is equivalent to the controlled-NOT gate, up to an overall phase factor \( i \) for the target qubit. Therefore, we get a nontrivial 2-qubit gate in terms of the observable-geometric phases. \( \square \)
Remark 4.1. (1) Universal quantum computing has been achieved by Zhu and Wang \cite{21, 22} using the gates associated with the AA geometric phase, as well by Sjöqvist et al \cite{13} using the non-Abelian geometric phase (cf. \cite{2}). As shown in \cite{10}, the non-Abelian geometric gates of Sjöqvist et al can be interpreted as the Abelian geometric gates of Zhu and Wang.

(2) The quantum gate based on the observable-geometric phases can be also interpreted as the Abelian geometric gates of Zhu and Wang; indeed the two schemes lead to the same gates as shown in Propositions 4.1 and 12. However, there are two differences between them: 1) the ZW gates are related with the geometry of the state space of the system, while ours are connected with the geometry of the observable space; 2) the gates in the ZW scheme are all based at the simultaneous evolutions of two orthogonal basic vectors, while in our setting all gates are based at the evolution of a single observable.

5. Conclusions

Geometric phases provide a new way of looking at quantum mechanics. The usual theory of the geometric phase (cf. \cite{14, 5}) is based on the Schrödinger picture, that is, the geometric phase is defined for the state. Here, we define the observable-geometric phases in the Heisenberg picture. This provides a new way of studying the geometry for the quantum system from the viewpoint of the observable. In particular, the observable-geometric phases can be used to realize a universal set of quantum gates in quantum computation. Therefore, it may not be unreasonable to hope that this new insight may have heuristic value.

6. Appendix: Quantum geometry over the observable space

6.1. Observable space. Recall that a complete orthonormal decomposition in $\mathbb{H}$ is a set of projections of rank one $\{|n\rangle\langle n| : n \geq 1\}$ satisfying

\[
\sum_{n \geq 1} |n\rangle\langle n| = I, \quad \langle n|m\rangle = \delta_{nm}.
\]

We denote by $\mathcal{W}(\mathbb{H})$ the set of all complete orthonormal decompositions in $\mathbb{H}$. Note that a complete orthonormal decomposition $O = \{|n\rangle\langle n| : n \geq 1\}$ determines uniquely a basis $\{|n\rangle\}_{n \geq 1}$ up to phases for basic vectors. Conversely, a basis uniquely defines a complete orthonormal decomposition in $\mathbb{H}$. Therefore, a complete orthonormal decomposition can be regarded as a complete measurement. Since every observable has a spectral decomposition corresponding to (at least) a complete orthonormal decomposition, the evolution of a quantum system by the Heisenberg equation is

\[
i \frac{dX}{dt} = [X, H]
\]

for the observable $X$, gives rise to a curve in $\mathcal{W}(\mathbb{H})$. This is the reason why $\mathcal{W}(\mathbb{H})$ can be regarded as the observable space, whose geometry and topology induce geometric and topological phases for quantum systems.

For $O, O' \in \mathcal{W}(\mathbb{H})$, we define the distance $D_{\mathcal{W}}(O, O')$ by

\[
D_{\mathcal{W}}(O, O') = \inf\{\|I - U\| : U^{-1}O'U = O, \quad U \in \mathcal{U}(\mathbb{H})\}.
\]

It is easy to check that $\mathcal{W}(\mathbb{H})$ is a complete metric space together with the distance $D_{\mathcal{W}}$. Thus, $\mathcal{W}(\mathbb{H})$ is a topological space as a metric space.

To obtain more information on $\mathcal{W}(\mathbb{H})$, we define $\mathcal{X}(\mathbb{H})$ to be the set of all ordered sequences $\{(n)\langle n|\}_{n \geq 1}$, where $\{|n\rangle\langle n| : n \geq 1\}$'s are all complete orthonormal decompositions in $\mathbb{H}$. A distance on $\mathcal{X}(\mathbb{H})$ is defined as follows: For $\{(n)\langle n|\}_{n \geq 1}, (n')\langle n'|\}_{n \geq 1} \in \mathcal{X}(\mathbb{H})$,

\[
D_{\mathcal{X}}(\{(n)\langle n|\}_{n \geq 1}, (n')\langle n'|\}_{n \geq 1}) = \inf\{\|I - U\| : U \in \mathcal{U}(\mathbb{H}), |n\rangle = U|n\rangle, \forall n\}.
\]

Then it is easy to check that $D_{\mathcal{X}}$ is a distance such that $\mathcal{X}(\mathbb{H})$ is a complete metric space and

\[
\mathcal{W}(\mathbb{H}) \cong \frac{\mathcal{X}(\mathbb{H})}{\Pi(d)},
\]
where $\Pi(d)$ denotes the permutation group of $d$ objects, and the notation $\cong$ indicates the isometric isomorphism of two metric spaces. From this identification, we find that $\mathcal{W}(\mathbb{H})$ is topologically non-trivial as its fundamental group is isomorphic to $\Pi(d)$, since $\mathcal{X}(\mathbb{H})$ can be considered as a compact, simply connected manifold.

**Proposition 6.1.** For an arbitrary fixed basis $\{|e_n\rangle\}_{n \geq 1}$ of $\mathbb{H}$,
\[
\mathcal{W}(\mathbb{H}) \cong \{\mathcal{G}(U) : U \in \mathcal{U}(\mathbb{H})\}
\]
with
\[
(6.4) \quad \mathcal{G}(U) = \left\{ \sum_{n \geq 1} e^{i\theta_n} |\sigma(n)\rangle\langle e_n| : \forall \sigma \in \Pi(d), \forall \theta_n \in [0, 2\pi) \right\},
\]
where $|n\rangle = U|e_n\rangle$ for any $n \geq 1$, and the distance between two elements is defined by
\[
d(\mathcal{G}(U), \mathcal{G}(U')) = \inf\{\|K - G\| : K \in \mathcal{G}(U), G \in \mathcal{G}(U')\}.
\]

**Proof.** At first, we prove that
\[
\mathcal{X}(\mathbb{H}) \cong \frac{\mathcal{U}(\mathbb{H})}{\mathcal{U}(1)^d},
\]
from which we conclude the result, since $\mathcal{W}(\mathbb{H}) \cong \frac{\mathcal{X}(\mathbb{H})}{\Pi(d)}$.

Indeed, for an arbitrary fixed basis $\{|e_n\rangle\}_{n \geq 1}$ of $\mathbb{H}$, we have that $\frac{\mathcal{U}(\mathbb{H})}{\mathcal{U}(1)^d} = \{[U] : U \in \mathcal{U}(\mathbb{H})\}$ with
\[
[U] = U \cdot \mathcal{U}(1)^d = \left\{ \sum_{n \geq 1} e^{i\theta_n} |n\rangle\langle e_n| : \forall \theta_n \in [0, 2\pi) \right\},
\]
where $|n\rangle = U|e_n\rangle$ for $n \geq 1$. Define $T : \mathcal{X}(\mathbb{H}) \rightarrow \frac{\mathcal{U}(\mathbb{H})}{\mathcal{U}(1)^d}$ by
\[
T(|(n)\rangle|n\rangle)_{n \geq 1} \mapsto [U]
\]
for any $|(n)\rangle|n\rangle \in \mathcal{X}(\mathbb{H})$, where $U$ is the unitary operator so that $|n\rangle = U|e_n\rangle$ for $n \geq 1$. Then, $T$ is surjective and isometric, and so the required assertion follows. This completes the proof. \qed

For illustration, we consider the topology of $\mathcal{W}(\mathbb{H})$ in the qubit case of $\mathbb{H} = \mathbb{C}^2$. Indeed, we have
\[
\mathcal{X}(\mathbb{C}^2) \cong \frac{\mathcal{U}(2)}{\mathcal{U}(1)^2} \cong \frac{SU(2)}{U(1)} \cong S^2,
\]
and so
\[
\mathcal{W}(\mathbb{C}^2) \cong \frac{S^2}{\mathbb{Z}_2},
\]
where we have used the fact $\Pi(2) = \mathbb{Z}_2$. This has a simple geometrical interpretation, since every element in $\mathcal{X}(\mathbb{C}^2)$ has the form $X_{\vec{n}} = |\vec{n}\rangle \langle \vec{n}|$ with $\vec{n} = (n_x, n_y, n_z) \in S^2$, where
\[
|\pm \vec{n}\rangle = \frac{1}{2} (I \pm \vec{n} \cdot \vec{e}).
\]
Although $X_{\vec{n}} \neq X_{-\vec{n}}$ in $\mathcal{X}(\mathbb{C}^2)$, they both correspond to the same element in $\mathcal{W}(\mathbb{C}^2)$. This implies that $\mathcal{W}(\mathbb{C}^2)$ may have non-trivial topology: There are exactly two topologically distinct classes of loops in $\mathcal{W}(\mathbb{C}^2)$, one corresponds to the trivial class $X_{\vec{n}} \mapsto X_{\vec{n}}$ while the other to the non-trivial class $X_{\vec{n}} \mapsto X_{-\vec{n}}$. Then the first fundamental group $\pi_1(\mathcal{W}(\mathbb{C}^2)) \cong \mathbb{Z}_2$, and thus the topology of the observable space $\mathcal{W}(\mathbb{C}^2)$ for the qubit system is nontrivial.

6.2. **Fibre bundles over the observable space.** Following [3], a bundle is a triple $(E, \pi, B)$, where $E$ and $B$ are two Hausdorff topological spaces, and $\pi : E \rightarrow B$ is a continuous map which is always assumed to be surjective. The space $E$ is called the total space, the space $B$ is called the base space, and the map $\pi$ is called the projection of the bundle. For each $b \in B$, the set $\pi^{-1}(b)$ is called the fiber of the bundle over $b$. Given a topological space $F$, a bundle $(E, \pi, B)$ is called a fiber bundle with the fiber $F$ provided every fiber $\pi^{-1}(b)$ for $b \in B$ is homeomorphic to $F$. For a topological group $G$, a bundle $(E, \pi, B)$ is called a $G$-bundle, denoted by $(E, \pi, B, G)$, provided $G$ acts on $E$ from the
right preserving the fibers of $E$ such that the map $f$ from the quotient space $E/G$ onto $B$ defined by $f(xG) = \pi(x)$ for $xG \in E/G$ is a homeomorphism, namely

$$
\begin{array}{c}
E \\ id \\
\downarrow P_G \\
E/G \\ f \; \triangledown \\
\downarrow \pi \\
B
\end{array}
$$

where $P_G$ is the usual projection. A $G$-bundle $(E, \pi, B, G)$ is principal if the action of $G$ on $E$ is free in the sense that $xg = x$ for some $x \in E$ and $g \in G$ implies $g = 1$, and the group $G$ is then called the structure group of the bundle $(E, \pi, B, G)$ (in physical literatures $G$ is also called the gauge group, cf. [5]). Note that in a principal $G$-structure group of the bundle $(E, \pi, B, G)$, every fiber $\pi^{-1}(b)$ for $b \in B$ is homeomorphic to $G$ by the freedom of the $G$-action, hence it is a fiber bundle $(E, \pi, B, G)$ with the fiber $G$ and is simply called a principal fiber bundle with the structure group $G$.

Next, we construct principal fiber bundles over the observable space $\mathcal{W}(\mathbb{H})$. To this end, fix a point $O_0 = \{|e_n\rangle\langle e_n| : n \geq 1\} \in \mathcal{W}(\mathbb{H})$. For any $O \in \mathcal{W}(\mathbb{H})$, we write

$$
\mathcal{F}_{O_0}^O = \{U \in \mathcal{U}(\mathbb{H}) : U^\dagger OU = O_0\},
$$

that is, $U \in \mathcal{F}_{O_0}^O$ if and only if $\{U|e_n\rangle : n \geq 1\}$ is a basis such that $O = \{U|e_n\rangle\langle e_n|U^\dagger : n \geq 1\}$.

Indeed, if $O = \{|n\rangle\langle n| : n \geq 1\}$, then

$$
\mathcal{F}_{O_0}^O = \mathcal{G}(U) = \left\{ \sum_{n \geq 1} e^{i\theta_n} |\sigma(n)\rangle\langle e_n| : \forall \sigma \in \Pi(d), \forall \theta_n \in [0, 2\pi) \right\},
$$

where $U$ is a unitary operator so that $|n\rangle = U|e_n\rangle$ for $n \geq 1$. Also, define

$$
\mathcal{G}_{O_0} = \left\{ \sum_{n \geq 1} e^{i\theta_n} |e_{\sigma(n)}\rangle\langle e_n| : \forall \sigma \in \Pi(d), \forall \theta_n \in [0, 2\pi) \right\}.
$$

We note that $\mathcal{U}(1)^d$ has a unitary representation

$$
\mathcal{U}(1)^d = \left\{ U_\theta = \sum_{n = 1}^d e^{i\theta_n} |e_n\rangle\langle e_n| : \forall \theta = (\theta_1, \ldots, \theta_d) \in [0, 2\pi)^d \right\},
$$

while $\Pi(d)$ has a unitary representation

$$
\Pi(d) = \left\{ U_\sigma = \sum_{n = 1}^d |e_{\sigma(n)}\rangle\langle e_n| : \forall \sigma \in \Pi(d) \right\}.
$$

Thus, $\mathcal{G}_{O_0}$ is a (non-abelian) subgroup of $\mathcal{U}(\mathbb{H})$ generated by $\mathcal{U}(1)^d$ and $\Pi(d)$.

The (right) action of $\mathcal{G}_{O_0}$ on $\mathcal{F}_{O_0}^O$ is defined as: For any $G \in \mathcal{G}_{O_0}$,

$$(G, U) \mapsto UG$$

for all $U \in \mathcal{F}_{O_0}^O$. Evidently, this action is free and invariant, namely $\mathcal{F}_{O_0}^O \cdot G = \mathcal{F}_{O_0}^O$ for any $G \in \mathcal{G}_{O_0}$ and every $O \in \mathcal{W}(\mathbb{H})$. Note that

$$
\mathcal{U}(\mathbb{H}) = \bigcup_{O \in \mathcal{W}(\mathbb{H})} \mathcal{F}_{O_0}^O,
$$

and $\mathcal{F}_{O_0}^O$ is homeomorphic to $\mathcal{G}_{O_0}$ as topological spaces since $\mathcal{F}_{O_0}^O = \mathcal{G}[U]$ for some $U$ such that $O = \{U|e_n\rangle\langle e_n|U^\dagger : n \geq 1\}$.

The following is then principal fiber bundles over the observable space.

**Definition 6.1.** Given $O_0 \in \mathcal{W}(\mathbb{H})$, a principal fiber bundle over $\mathcal{W}(\mathbb{H})$ associated with $O_0$ is defined to be

$$
\xi_{O_0}(\mathbb{H}) = (\mathcal{U}(\mathbb{H}), \mathcal{W}(\mathbb{H}), \Pi_{O_0}, \mathcal{G}_{O_0}),
$$

where $\mathcal{U}(\mathbb{H})$ is the total space, and the bundle projection $\Pi_{O_0} : \mathcal{U}(\mathbb{H}) \mapsto \mathcal{W}(\mathbb{H})$ is defined by

$$
\Pi_{O_0}(U) = O
$$

provided $U \in \mathcal{F}_{O_0}^O$ for (unique) $O \in \mathcal{W}(\mathbb{H})$, namely $\Pi^{-1}(O) = \mathcal{F}_{O_0}^O$ for every $O \in \mathcal{W}(\mathbb{H})$.

We simply denote this bundle by $\xi_{O_0} = \xi_{O_0}(\mathbb{H})$. 
Remark 6.1. In the sequel, we will see that the fixed point \( O_0 \in \mathcal{W}(\mathbb{H}) \) physically plays the role of a complete measurement. Indeed, it corresponds to a von Neumann measurement. On the other hand, the point \( O_0 \) induces a differential structure over the base space \( \mathcal{W}(\mathbb{H}) \) and determines the geometric structure of \( \xi_{O_0} \), namely quantum connection and parallel transportation.

For any two points \( O_0, O'_0 \in \mathcal{W}(\mathbb{H}) \) with \( O_0 = \{ |e_n\rangle\langle e_n| : n \geq 1 \} \) and \( O'_0 = \{ |e'_n\rangle\langle e'_n| : n \geq 1 \} \), we define a unitary operator \( U_0 \) by \( U_0|e_n\rangle = |e'_n\rangle \) for \( n \geq 1 \). Then the map \( T : \xi_{O_0} \rightarrow \xi_{O'_0} \) defined by \( TU = UU_0^{-1} \) for all \( U \in \mathcal{U}(\mathbb{H}) \) is an isometric isomorphism on \( \mathcal{U}(\mathbb{H}) \) such that \( T \) maps the fibers of \( \xi_{O_0} \) onto the fibers of \( \xi_{O'_0} \) over the same points in the base space \( \mathcal{W}(\mathbb{H}) \), namely the following diagram is commutative:

![Diagram](image)

that is, \( \Pi_{O_0} = \Pi_{O'_0} \circ T \). Thus, \( \xi_{O_0} \) and \( \xi_{O'_0} \) are isomorphic as principal fiber bundles (cf. [9]).

6.3. Quantum connection. Since \( d = \dim(\mathbb{H}) < \infty \), \( \xi_{O_0} \) has a natural differential structure, namely \( \mathcal{U}(\mathbb{H}) \) and \( \mathcal{W}(\mathbb{H}) \) are both differential manifolds. In fact, \( \mathcal{U}(\mathbb{H}) \) is a Lie group, while \( \mathcal{W}(\mathbb{H}) \) can be identified as a submanifold of the Grassmannian manifold of \( d \)-dimensional subspaces of \( \mathcal{B}(\mathbb{H}) \) (cf. [20]). However, the classical differential structure over \( \mathcal{W}(\mathbb{H}) \) seems not to be applicable in the geometric interpretation of observable-geometric phases. To define the suitable concepts of quantum connection and parallel transportation over the principal fiber bundle \( \xi_{O_0} \), we need to introduce a differential structure over \( \mathcal{W}(\mathbb{H}) \) associated with each fixed \( O_0 \in \mathcal{W}(\mathbb{H}) \). Indeed, we will introduce a geometric structure over \( \xi_{O_0} \) in a certain operator-theoretic sense, where the differential structure on \( \mathcal{W}(\mathbb{H}) \) is different from the one given in [20].

Let us begin with the definition of tangent vectors for \( \mathcal{G}_{O_0} \) in the operator-theoretic sense.

Definition 6.2. Fix \( O_0 \in \mathcal{W}(\mathbb{H}) \). For a given \( U \in \mathcal{G}_{O_0} \), an operator \( Q \in \mathcal{B}(\mathbb{H}) \) is called a tangent vector at \( U \) for \( \mathcal{G}_{O_0} \), if there is a curve \( \chi : (-\varepsilon, \varepsilon) \ni t \mapsto U(t) \in \mathcal{G}_{O_0} \) with \( \chi(0) = U \) such that for every \( h \in \mathbb{H} \), the limit

\[
\lim_{t \to 0} \frac{U(t)(h) - U(h)}{t} = Q(h)
\]

in \( \mathbb{H} \), denoted by \( Q = \frac{d\chi(t)}{dt} \bigg|_{t=0} \). The set of all tangent vectors at \( U \) is denoted by \( T_U \mathcal{G}_{O_0} \), and \( T \mathcal{G}_{O_0} = \bigcup_{U \in \mathcal{G}_{O_0}} T_U \mathcal{G}_{O_0} \). In particular, we denote \( g_{O_0} = T_I \mathcal{G}_{O_0} \) if \( U = I \).

It is easy to check that given \( U \in \mathcal{G}_{O_0} \) with the form \( U = \sum_{n=1}^{d} e^{i\theta_n} |e_{\sigma(n)}\rangle\langle e_n| \) for some \( \sigma \in \Pi(d) \), for every \( Q \in T_U \mathcal{G}_{O_0} \) there exists a sequence of complex number \((\alpha_n)_{n \geq 1}\) such that

\[
Q = \sum_{n \geq 1} \alpha_n |e_{\sigma(n)}\rangle\langle e_n|.
\]

In particular, each element \( Q \in g_{O_0} \) is of form

\[
Q = \sum_{n \geq 1} \alpha_n |e_n\rangle\langle e_n|,
\]

where \((\alpha_n)_{n \geq 1}\) is a sequence of complex number. Thus, \( T_U \mathcal{G}_{O_0} \) is a linear subspace of \( \mathcal{B}(\mathbb{H}) \).

The following is the tangent space for the base space \( \mathcal{W}(\mathbb{H}) \) in the operator-theoretic sense.

Definition 6.3. 1) Fix \( O_0 \in \mathcal{W}(\mathbb{H}) \). A continuous curve \( \chi : [a, b] \ni t \mapsto O(t) \in \mathcal{W}(\mathbb{H}) \) is said to be differential at a fixed \( t_0 \in (a, b) \) relative to \( O_0 \), if there is a nonempty subset \( \mathcal{A} \) of \( \mathcal{B}(\mathbb{H}) \) satisfying that for any \( Q \in \mathcal{A} \) there exist \( \varepsilon > 0 \) so that \( (t_0 - \varepsilon, t_0 + \varepsilon) \subset [a, b] \) and a strongly continuous curve \( \gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \ni t \mapsto U_t \in \mathcal{F}_{O_0}^{(t)} \) such that the limit

\[
\lim_{t \to t_0} \frac{U_t(h) - U_{t_0}(h)}{t - t_0} = Q(h)
\]

for any \( h \in \mathbb{H} \). In this case, \( \mathcal{A} \) is called a tangent vector of \( \chi \) at \( t = t_0 \) and denoted by

\[
\mathcal{A} = \frac{dO(t)}{dt} \bigg|_{t=t_0} = \frac{d\chi(t)}{dt} \bigg|_{t=t_0}.
\]
We can define the left (or, right) tangent vector of $\chi$ at $t = a$ (or, $t = b$) in the usual way.

2) Fix $O_0 \in \mathcal{W}(\mathbb{H})$. Given $O \in \mathcal{W}(\mathbb{H})$, a tangent vector of $\mathcal{W}(\mathbb{H})$ at $O$ relative to $O_0$ is defined to be a nonempty subset $A$ of $\mathcal{B}(\mathbb{H})$, provided $A$ is a tangent vector of some continuous curve $\chi$ at $t = 0$, where $\chi : (-\varepsilon, \varepsilon) \ni t \mapsto O(t) \in \mathcal{W}(\mathbb{H})$ with $\chi(0) = O$, i.e., $A = \frac{dO(t)}{dt} |_{t=0}$. We denote by $T_O^W\mathcal{W}(\mathbb{H})$ the set of all tangent vectors at $O$, and write $T\mathcal{W}(\mathbb{H}) = \bigcup_{O \in \mathcal{W}(\mathbb{H})} T_O^W\mathcal{W}(\mathbb{H})$.

Therefore, the tangent vectors for the base space $\mathcal{W}(\mathbb{H})$ is strongly dependent on the choice of a measurement point $O_0$.

**Definition 6.4.** 1) A strongly continuous curve $\gamma : [a, b] \ni t \mapsto U(t) \in \mathcal{U}(\mathbb{H})$ is said to be differential at a fixed $t_0 \in (a, b)$, if there is an operator $Q \in \mathcal{B}(\mathbb{H})$ such that the limit

$$\lim_{t \to t_0} \frac{U(t) - U(t_0)}{t - t_0} = Q(h)$$

for all $h \in \mathbb{H}$. In this case, $Q$ is called the tangent vector of $\gamma$ at $t = t_0$ and denoted by

$$Q = \left. \frac{d\gamma}{dt} \right|_{t=t_0} \equiv \left. \frac{dU(t)}{dt} \right|_{t=t_0}.$$

We can define the left (or, right) tangent vector of $\gamma$ at $t = a$ (or, $t = b$) in the usual way.

Moreover, $\gamma$ is called a smooth curve, if $\gamma$ is differential at each point $t \in [a, b]$, and for any $h \in \mathbb{H}$, the $\mathbb{H}$-valued function $t \mapsto \frac{d\gamma}{dt}(h)$ is continuous in $[a, b]$.

2) Fix $O_0 \in \mathcal{W}(\mathbb{H})$. For a given $P \in \mathcal{U}(\mathbb{H})$, an operator $Q \in \mathcal{B}(\mathbb{H})$ is called a tangent vector of $\xi_{O_0}$ at $P$, if there exists a strongly continuous curve $\gamma : (-\varepsilon, \varepsilon) \ni t \mapsto P_t \in \mathcal{U}(\mathbb{H})$ with $\gamma(0) = P$, such that $\gamma$ is differential at $t = 0$ and $Q = \frac{d\gamma}{dt} |_{t=0} = \frac{dP_t}{dt} |_{t=0}$. Denote by $T_P\xi_{O_0}(\mathcal{H})$ the set of all tangent vectors of $\xi_{O_0}$ at $P$ relative to $O_0$, and write

$$T\xi_{O_0}(\mathcal{H}) = \bigcup_{P \in \mathcal{U}(\mathbb{H})} T_P\xi_{O_0}(\mathcal{H}).$$

3) Fix $O_0 \in \mathcal{W}(\mathbb{H})$. Given $P \in \mathcal{U}(\mathbb{H})$, a tangent vector $Q \in T_P\xi_{O_0}(\mathbb{H})$ is said to be vertical, if there is a strongly continuous curve $\gamma : (-\varepsilon, \varepsilon) \ni t \mapsto P_t \in \mathcal{U}(\mathbb{H})$ with $\gamma(0) = P$ such that $\gamma$ is differential at $t = 0$ and $Q = \frac{d\gamma}{dt} |_{t=0}$. We denote by $V_P\xi_{O_0}(\mathbb{H})$ the set of all vertically tangent vectors at $P$.

**Remark 6.2.** For any $P \in \mathcal{U}(\mathbb{H})$, the tangent space $T_P\xi_{O_0}(\mathbb{H})$ is the same for all measurement points $O_0$ since it is the tangent space of $\mathcal{U}(\mathbb{H})$ at $P$ in the operator-theoretic sense. However, the vertically tangent space $V_P\xi_{O_0}(\mathbb{H})$ is different from each other for distinct measurement points. In particular, every $Q \in V_P\xi_{O_0}(\mathbb{H})$ with $O_0 = \{ |e_n\rangle \langle e_n| : 1 \leq n \leq d \}$ has the form

\begin{equation}
Q = \sum_{n \geq 1} \alpha_n P |e_n\rangle \langle e_n|,
\end{equation}

where $(\alpha_n)_{n \geq 1}$ is a sequence of complex number.

For any fixed $O_0 \in \mathcal{W}(\mathbb{H})$, recall that for each $G \in \mathcal{G}_{O_0}$, the right action $R_G$ of $\mathcal{G}_{O_0}$ on $\xi_{O_0}$ is defined by

$$R_G(U) = UG, \quad \forall U \in \mathcal{U}(\mathbb{H}).$$

This induces a map $(R_G)_* : T_P\xi_{O_0}(\mathbb{H}) \mapsto T_{RG(P)}\xi_{O_0}(\mathbb{H})$ for each $P \in \mathcal{U}(\mathbb{H})$ such that

\begin{equation}
(R_G)_*(Q) = QG, \quad \forall Q \in T_P\xi_{O_0}(\mathbb{H}).
\end{equation}

Since $R_G$ preserves the fibers of $\xi_{O_0}$, then $(R_G)_*$ maps $V_P\xi_{O_0}(\mathbb{H})$ into $V_{RG(P)}\xi_{O_0}(\mathbb{H})$.

Now, we are ready to define the concept of quantum connection over the observable space.

**Definition 6.5.** Fix $O_0 \in \mathcal{W}(\mathbb{H})$. A connection on the principal fiber bundle $\xi_{O_0} = (\mathcal{U}(\mathbb{H}), \mathcal{W}(\mathbb{H}), \Pi_{O_0}, \mathcal{G}_{O_0})$ is a family of linear functionals $\Omega = \{ \Omega_P : P \in \mathcal{U}(\mathbb{H}) \}$, where for each $P \in \mathcal{U}(\mathbb{H})$, $\Omega_P$ is a linear functional in $T_P\xi_{O_0}(\mathbb{H})$ with values in $\mathcal{G}_{O_0}$, satisfying the following conditions:

1) For any $P \in \mathcal{U}(\mathbb{H})$ and for all vertically tangent vectors $Q \in V_P\xi_{O_0}(\mathbb{H})$, one has

\begin{equation}
\Omega_P(Q) = P^{-1} Q.
\end{equation}

2) $\Omega_P$ depends continuously on $P$, in the sense that if $P_n$ converges to $P$ as well as $Q_n \in T_{P_n}\xi_{O_0}(\mathbb{H})$ converges $Q_0 \in T_P\xi_{O_0}(\mathbb{H})$ in the operator topology of $\mathcal{B}(\mathbb{H})$, then $\lim_{n \to \infty} \Omega_{P_n}(Q_n) = \Omega_P(Q_0)$ in $\mathcal{G}_{O_0}$. 
(3) Under the right action of $G_{O_0}$ on $\xi_{O_0}(\mathbb{H})$, $\Omega$ transforms according to

\[
\Omega_{R_G}(P)[(R_G)_*(Q)] = G^{-1}\Omega_P(Q)G,
\]

for $G \in G_{O_0}$, $P \in U(\mathbb{H})$, and $Q \in T_P\xi_{O_0}(\mathbb{H})$.

Such a connection is simply called an $O_0$-connection.

Next, we present a canonical example of such quantum connections, which plays a crucial role in the expression of observable-geometric phases.

**Example 6.1.** Given a fixed $O_0 = \{e_n\rangle\langle e_n| : 1 \leq n \leq d\}$ in $W(\mathbb{H})$, we define $\hat{\Omega} = \{\Omega_P : P \in U(\mathbb{H})\}$ as follows: For each $P \in U(\mathbb{H})$, $\hat{\Omega}_P : T_P\xi_{O_0}(\mathbb{H}) \mapsto g_{O_0}$ is defined by

\[
\Omega_P(Q) = P^{-1} \ast Q
\]

for any $Q \in T_P\xi_{O_0}(\mathbb{H})$, where

\[
P^{-1} \ast Q = \sum_{n=1}^{d} \langle e_n|P^{-1}Q|e_n\rangle\langle e_n|e_n\rangle.
\]

By (6.10), one has $P^{-1} \ast Q = P^{-1}Q \in g_{O_0}$ for any $Q \in V_P\xi_{O_0}(\mathbb{H})$, namely $\hat{\Omega}_P$ satisfies (6.11). The conditions (2) and (3) of Definition 6.5 are clearly satisfied by $\hat{\Omega}$. Hence, $\hat{\Omega}$ is an $O_0$-connection on $\xi_{O_0}$. In this case, we write $\Omega_P = P^{-1} \ast dP$ for any $P \in U(\mathbb{H})$.

### 6.4. Quantum parallel transportation

The quantum parallel transportation in the state space was introduced in [14, 1] and studied in [3] in details. This section is devoted to the study of quantum parallel transport over the observable space.

**Definition 6.6.** Fix a point $O_0 \in W(\mathbb{H})$. For a continuous curve $C_W : [a, b] \ni t \mapsto O(t) \in W(\mathbb{H})$, a lift of $C_W$ with respect to $O_0$ is defined to be a continuous curve

\[
C_P : [a, b] \ni t \mapsto U(t) \in U(\mathbb{H})
\]

such that $U(t) \in F_{O_0}^{O(t)}$ for any $t \in [a, b]$.

**Remark 6.3.** Note that, a lift of $C_W$ depends on the choice of the point $O_0$; for the same curve $C_W$, lifts are distinct for different points $O_0$. For this reason, such a lift $C_P$ is called a $O_0$-lift of $C_W$.

**Definition 6.7.** Fix a point $O_0 \in W(\mathbb{H})$. A continuous curve $C_W : [a, b] \ni t \mapsto O(t) \in W(\mathbb{H})$ is said to be smooth, if it has a $O_0$-lift $C_P : [a, b] \ni t \mapsto U(t) \in U(\mathbb{H})$ which is a smooth curve. In this case, $C_P$ is called a smooth $O_0$-lift of $C_W$.

Note that, if a continuous curve $C_W : [a, b] \ni t \mapsto O(t) \in W(\mathbb{H})$ is smooth, then it is differential at every point $t \in [a, b]$. Indeed, suppose that $C_P : [a, b] \ni t \mapsto U(t) \in U(\mathbb{H})$ is a smooth $O_0$-lift of $C_W$. For each $t \in [a, b]$, we have $\frac{dC_P(t)}{dt} \in \mathcal{B}(\mathbb{H})$, namely $\frac{dO(t)}{dt}$ is a nonempty subset of $\mathcal{B}(\mathbb{H})$, and hence $C_W$ is differential.

**Definition 6.8.** Fix $O_0 \in W(\mathbb{H})$ and let $\Omega$ be an $O_0$-connection on $\xi_{O_0}(\mathbb{H})$. Let $C_W : [0, T] \ni t \mapsto O(t) \in W(\mathbb{H})$ be a smooth curve. If $C_P : [0, T] \ni t \mapsto \bar{U}(t)$ is a smooth $O_0$-lift of $C_W$ such that

\[
\Omega_{\bar{U}(t)} \left[ \frac{d\bar{U}(t)}{dt} \right] = 0
\]

for every $t \in [0, T]$, then $C_P$ is called a horizontal lift of $C_W$ with respect to $O_0$ and $\Omega$.

In this case, $C_P$ is simply called the horizontal $O_0$-lift of $C_W$ associated with $\Omega$. And, the curve $C_P : t \mapsto \bar{U}(t)$ is called the parallel transportation along $C_W$ with the starting point $C_P(0) = \bar{U}(0)$ with respect to the connection $\Omega$ on $\xi_{O_0}(\mathbb{H})$.

The following proposition shows the existence of the horizontal lifts.

**Proposition 6.2.** Fix $O_0 \in W(\mathbb{H})$ and let $\Omega$ be an $O_0$-connection on $\xi_{O_0}(\mathbb{H})$. Let $C_W : [0, T] \ni t \mapsto O(t) \in W(\mathbb{H})$ be a smooth curve. For any $U_0 \in F_{O_0}^{O(0)}$, there exists a unique horizontal $O_0$-lift $\tilde{C}_P$ of $C_W$ with the initial point $\tilde{C}_P(0) = U_0$. 
Proof. Let \( \Gamma : [0, T] \ni t \mapsto U(t) \in \mathcal{U}(\mathbb{H}) \) be a smooth \( O_0 \)-lift of \( C_W \) with \( \Gamma(0) = U_0 \). To prove the existence, note that the condition (2) of Definition 6.5 implies that the function \( t \mapsto \Omega(t) \left[ \frac{d\Gamma(t)}{dt} \right] \) is continuous in \([0, T]\). Then,

\[
(6.15) \quad \frac{dG(t)}{dt} = -\Omega(t) \left[ \frac{d\Gamma(t)}{dt} \right] \cdot G(t)
\]

with \( G(0) = I \) has the unique solution in \([0, T]\). Therefore, \( \tilde{C}_P(t) = \Gamma(t) \cdot G(t) \) is the required horizontal \( O_0 \)-lift of \( C_W \) for the initial point \( U_0 \in \mathcal{F}^{O_0}_{O_0} \).

To prove the uniqueness, suppose \( \hat{C}_P : [0, T] \ni t \mapsto \hat{U}(t) \in \mathcal{U}(\mathbb{H}) \) be another horizontal \( O_0 \)-lift of \( C_W \) for the initial point \( U \in \mathcal{F}^{O_0}_{O_0} \). Then \( \hat{C}_P(t) = \hat{C}_P(t) \cdot \tilde{G}(t) \) for all \( t \in [0, T] \), where \( \tilde{G}(0) = I \).

Since \( \hat{G}(t) = I \) for all \( t \in [0, T] \). Hence, the horizontal \( O_0 \)-lift of \( C_W \) is unique for the initial point \( U \in \mathcal{F}^{O_0}_{O_0} \).

\[\square\]

Example 6.2. Let \( C_P : [0, T] \ni t \mapsto U(t) \in \mathcal{U}(\mathbb{H}) \) be a unitary evolution satisfying the Schrödinger equation

\[
(6.16) \quad i \frac{dU(t)}{dt} = H(t)U(t)
\]

where \( H(t) \) are time-dependent Hamiltonian operators in \( \mathbb{H} \). Given a fixed \( O_0 = \{ |\psi_n\rangle \langle \psi_n | : 1 \leq n \leq d \} \in \mathcal{W}(\mathbb{H}) \), define \( O(t) = \{ |\psi_n(t)\rangle \langle \psi_n(t)| : 1 \leq n \leq d \} \) for all \( t \in [0, T] \) by \( O(t) = U(t)O(t)U(t)^{-1} \).

We define \( \hat{C}_P : [0, T] \ni t \mapsto \hat{U}(t) \in \mathcal{U}(\mathbb{H}) \) by

\[
\hat{U}(t) = \sum_{n=1}^{d} \exp \left( i \int_{0}^{t} \langle \psi_n | [U'(t') \cdot i \frac{dU(t')}{dt'} | e_n] dt' \right) \cdot U(t) |\psi_n\rangle \langle e_n|
\]

for every \( t \in [0, T] \), along with the initial point \( \hat{U}(0) = U(0) \in \mathcal{F}^{O_0}_{O_0} \). Then \( \hat{C}_P \) is a smooth \( O_0 \)-lift of \( C_W \) such that

\[\hat{\Omega}_{\hat{C}_P} \left[ \frac{d\hat{U}(t)}{dt} \right] = 0 \]

for all \( t \in [0, T] \), where \( \hat{\Omega} \) is the canonical \( O_0 \)-connection introduced in Example 6.1. Thus, \( \hat{C}_P \) is the horizontal \( O_0 \)-lift of \( C_W \) with respect to \( \hat{\Omega} \), namely \( \hat{C}_P \) is the parallel transportation along \( C_W \) with the starting point \( \hat{C}_P(0) = U(0) \) with respect to the connection \( \hat{\Omega} \) on \( \xi_{O_0}(\mathbb{H}) \).

6.5. Geometric interpretation of observable-geometric phases. We are now ready to give a geometric interpretation of \( \beta_n \)'s defined as in \( (2.10) \) in Section 2. Indeed, given a point \( O_0 = \{ |\psi_n\rangle \langle \psi_n | : n \geq 1 \} \in \mathcal{W}(\mathbb{H}) \), using the notations involved in Section 2 we define \( \hat{U}(t) \in \mathcal{U}(\mathbb{H}) \) for \( 0 \leq t \leq T \) by

\[
\hat{U}(t) = \sum_{n=1}^{d} |\tilde{\psi}_n(t)\rangle \langle \psi_n|,
\]

where \( |\tilde{\psi}_n(t)\rangle \)'s are defined in \( (2.6) \). Then,

\[\hat{C}_P : [0, T] \ni t \mapsto \hat{U}(t) \in \mathcal{U}(\mathbb{H}) \]

is a smooth \( O_0 \)-lift of \( C_W : [0, T] \ni t \mapsto O(t) = \{ |\psi_n(t)\rangle \langle \psi_n(t)| : 1 \leq n \leq d \} \) by \( (2.8) \), we have

\[
(6.17) \quad \hat{\Omega}_{\hat{C}_P} \left[ \frac{d\hat{U}(t)}{dt} \right] = 0
\]

for all \( t \in [0, T] \), where \( \hat{\Omega} \) is the canonical connection (cf. Example 6.1). This means that \( [0, T] \ni t \mapsto \hat{U}(t) \) is the parallel transportation along \( C_W \) associated with the canonical connection \( \hat{\Omega} \) on \( \xi_{O_0} \). Therefore, \( \hat{C}_P \) is the horizontal \( O_0 \)-lift of \( C_W \) with respect to \( \hat{\Omega} \) in the principal bundle \( \xi_{O_0} \) such that \( \hat{U}(T) |\psi_n\rangle = |\tilde{\psi}_n(T)\rangle = e^{i\beta_n} |\psi_n\rangle \), and so

\[
(6.18) \quad \hat{U}(T) = \sum_{n=1}^{d} e^{i\beta_n} |\psi_n\rangle \langle \psi_n|
\]
is the holonomy element associated with the connection $\hat{\Omega}, C_W$, and $U_0 = \sum_{n=1}^{d} |\psi_n\rangle\langle e_n|$ in $\xi_{O_0}$.

In conclusion, we have the following theorem.

**Theorem 6.1.** (1) For every $n = 1, \ldots, d$, the geometric phase $\beta_n$ defined in (2.10) is given by

\begin{equation}
\beta_n = \langle e_n| \int_0^T \hat{\Omega}_U(t) \left[ \frac{dU(t)}{dt} \right] dt |e_n\rangle = \langle e_n| \int_{C_W} U^{-1} \ast dU |e_n\rangle,
\end{equation}

where $C_P : [0, T] \ni t \mapsto U(t) \in \mathcal{F}_{O_{\Omega}}^{\text{lift}}(0)$ corresponds to any of the closed smooth $O_{\Omega}$-lifts of $C_W$ with $U(0) = U_0$, and $\hat{\Omega}_U = U^{-1} \ast dU$ is the canonical connection on $\xi_{O_0}(\mathbb{H})$. Thus, $\beta_n$'s are independent of the choice of the time parameterization of $U(t)$, namely the speed with which $U(t)$ traverses its closed path. It is also independent of the choice of the Hamiltonian as long as Heisenberg equations (2.4) involving these Hamiltonians describe the same closed path $\hat{C}_W$ in $\mathcal{W}(\mathbb{H})$.

(2) The set $\{ \beta_n : 1 \leq n \leq d \}$ is independent of the choice of the starting point $U_0$.

(3) The set $\{ \beta_n : 1 \leq n \leq d \}$ is independent of the choice of the measurement point $O_0$. Therefore, this number set is considered to be a set of geometric invariants for $C_W$.

**Remark 6.4.** In the definition (2.10), the $\beta_n$'s are in fact independent of the choice of measurement points $O_0$ and background geometry over the fiber bundle $\xi_{O_0}$.

**Proof.** (1) Let $C_P : [0, T] \ni t \mapsto U(t) \in \mathcal{F}_{O_{\Omega}}^{\text{lift}}(0)$ be a smooth $O_{\Omega}$-lift of $C_W$ such that $U(T) = U(0) = U_0$. By Proposition 6.2 we have $\tilde{U}_{\theta}(t) = U(t)G(t)$ for every $0 \leq t \leq T$, where $G(t)$ satisfies (6.15) with $\Gamma(t) = U(t)$. Then there are continuously differential functions $\psi_n$ with $\psi_n(0) = 0$ and $\psi_n(T) = \beta_n$ such that $\tilde{G}(\epsilon_n) = e^{i\psi_n(t)}\epsilon_n$ for all $n \geq 1$. By (6.11) and (6.12), one has

$$\int_0^T |e_n\rangle \hat{\Omega}_U(t) \left[ \frac{dU(t)}{dt} \right] |e_n\rangle dt = \int_0^T |e_n\rangle G(t) \left[ \frac{dG^{-1}(t)}{dt} \right] |e_n\rangle dt = -i\beta_n.$$

This follows (6.19).

(2) For any $\tilde{U}_0 \in \mathcal{F}_{O_{\Omega}}^{\text{lift}}(0)$ there exists some $G = \sum_{n=1}^{d} e^{i\theta_n}|\sigma_{\sigma(n)}\rangle\langle e_n|$ in $\mathcal{G}_{O_{\Omega}}$ with $\sigma \in \Pi(d)$ and $\theta_n \in \mathbb{R}$ such that $\tilde{U}_0 = U_0 G$. Then $\tilde{C}_P : [0, T] \ni t \mapsto \tilde{U}(t) = \tilde{U}(t)G$ is the horizontal $O_{\Omega}$-lift of $C_W$ with the starting point $\tilde{U}(0) = U_0 G$ such that $\tilde{U}(T)|\epsilon_n\rangle = e^{i\beta_n}\tilde{U}(0)|\epsilon_n\rangle$ for all $1 \leq n \leq d$. Thus, the set $\{ \beta_n : n \geq 1 \}$ is invariant for any starting point $U_0 \in \mathcal{F}_{O_{\Omega}}^{\text{lift}}(0)$. Combining this fact with (6.19) yields

$$\{ \beta_n : n \geq 1 \} = \left\{ i\langle e_n| \int_{C_P} \hat{\Omega}_U [d\hat{U}] |e_n\rangle : n = 1, \ldots, d \right\}$$

for any closed smooth $O_{\Omega}$-lift $\tilde{C}_P$ of $C_W$. Therefore, the observable-geometric phases are independent of the choice of the starting point and only depends on the geometry of the curve $C_W$ with respect to the $O_{\Omega}$-connection $\hat{\Omega}$.

(3) Let $\tilde{C}_P : [0, T] \ni t \mapsto \tilde{U}(t)$ be the horizontal $O_{\Omega}$-lift of $C_W$ with respect to $\Omega$ with the starting point $\tilde{U}(0) = U_0$. For any $O_P = \{ |e'_n\rangle\langle e'_n| : 1 \leq n \leq d \} \in \mathcal{W}(\mathbb{H})$ there exists some $U' \in \mathcal{U}(\mathbb{H})$ such that $O_P = U' O_{\Omega} U'^{-1}$ with $|e'_n\rangle = U' |e_n\rangle$ for $n = 1, \ldots, d$. Then $\Omega' = \{ \{ \Omega_P : P \in \mathcal{U}(\mathbb{H}) \} \}$ is an $O_{\Omega}'$-connection on $\xi_{O_{\Omega}'}$, where $\Omega_P (Q) = U' \Omega_{P U'} (Q U')^{-1} U'^{-1}$ for any $P \in \mathcal{U}(\mathbb{H})$ and for all $Q \in T_P \xi_{O_{\Omega}'}$. By computation, we conclude that $\tilde{C}_P' : [0, T] \ni t \mapsto \tilde{U}'(t) = \tilde{U}(t) U'^{-1}$ is the horizontal $O_{\Omega}'$-lift of $C_W$ with respect to $\Omega'$ with the starting point $\tilde{U}'(0) = U_0 U'^{-1}$. Therefore,

$$\tilde{U}'(T)|e'_n\rangle = \tilde{U}(T) U'^{-1} |e'_n\rangle = \tilde{U}(T)|e_n\rangle = e^{i\beta_n} \tilde{U}(0)|e_n\rangle = e^{i\beta_n} \tilde{U}'(0)|e'_n\rangle,$$

and hence the set of the geometric phases of $C_W$ with respect to $\Omega'$ is the same as that of $\Omega$. \hfill $\square$

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