Market price competition in networks: How do monopolies impact social welfare?

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Abstract

We study the efficiency of allocations in two-sided markets where each seller controls the price of a divisible good, and every buyer purchases the cheapest bundle of goods that satisfies their demands. Different buyers may have different values for the same bundle, and the distribution of these values can be summarized by a buyer demand function. While it is known that stable allocations (Nash Equilibrium) in such settings need no longer coincide with Walrasian Equilibrium, can be very inefficient, and may not even exist, the exact properties of equilibria are not fully understood. In this work, we show that for a large class of natural buyer demand functions, equilibrium always exists and allocations can often be close to optimal. In the process, we obtain tight bounds on the price of stability for many important classes of demand functions. Although our results extend to general markets, we especially focus on networked markets, i.e., sellers own edges in a graph, buyers desire paths, and thus goods are a mix of substitutes and complements. For this setting, we essentially characterize the efficiency of stable allocations based on the demand function and on $M$, the number of monopolies present in the market.

While it is known that monopolies can cause large inefficiencies in general, our main results indicate that for many natural demand functions the efficiency only drops linearly with $M$. For example, for concave demand we prove that the price of stability is at most $1 + \frac{M}{2}$, for demand with monotone hazard rate it is at most $1 + M$, and for polynomial demand the efficiency decreases logarithmically with $M$. Unlike much of existing work, we do not assume that monopolies do not exist. On the contrary, our main contribution is showing how the quality of stable outcomes changes as the number of monopolies grows. Finally, we present several important special cases of such markets for which complete efficiency is achieved: the solution maximizing social welfare admits Nash equilibrium pricing.

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1 Introduction

We study the interaction between buyers and sellers in markets with the following features: i) Each seller controls a unique, divisible good and incurs a cost based on how much of it he produces, ii) A large number of buyers have access to the entire market, each requiring some specific bundles of goods. For example, each seller may control a link in a network, while each buyer seeks to purchase capacity on paths connecting some pair of nodes. Assuming that the sellers’ goal is to maximize revenue and buyers buy the cheapest bundles to maximize utility, the main question that we then ask is: “How efficient are the equilibrium allocations in such markets?”.

The most commonly used notion of a stable allocation in such settings is Walrasian Equilibrium: a set of prices such that if both the buyers and sellers act as if they can purchase or sell ‘any amount’ of goods at these prices, then the market clears. Walrasian equilibria have very nice properties: not only are they guaranteed to exist under somewhat minimal assumptions \[1\], but the allocations are always socially optimal. In many markets, however, there is no central authority or “invisible hand” that fixes these Walrasian prices. Instead we consider a model where the sellers control their own prices. Assuming that the seller can anticipate the demand for their good, it is then reasonable to expect the sellers to set prices so as to maximize their profits, as opposed to clearing the market. Are the equilibrium outcomes still optimal when sellers act as price-setters and not price-takers?

The following example illustrates that this is no longer the case even in single-item markets.

Example 1: Consider a single good controlled by a single seller who can produce \(x\) amount of this good at a cost of \(x^2\). Let there be a large number of infinitesimal buyers in the market who in total desire one unit of this good, such that a fraction \((1-p)\) of the buyers value the good at price \(p\) or more. The unique Walrasian price is \(p = 2/3\) where exactly one-third of the buyers purchase the good and the seller produces the same amount. The seller’s profit is then \(p^* \frac{1}{3} - \left(\frac{1}{3}\right)^2 = 1/9\). However, suppose that the seller increases her price to \(p' = 3/4\): the demand drops to one-fourth but the seller’s profit is now \(p'^* \frac{1}{4} - \left(\frac{1}{4}\right)^2 = 1/8\), which is strictly larger than the original profit. So, if the seller can increase her price and anticipate the resulting demand, then she stands to benefit by breaking the Walrasian Equilibrium.

Thus when the sellers act as price-setters, the equilibrium outcome is no longer optimal. In this paper, we attempt to bound this loss in efficiency from Walrasian outcomes. We begin by modeling the market interaction as a two-stage pricing game.

Two-stage market game

Sellers and Buyers: We consider a market with a set \(E\) of available goods such that each seller controls exactly one infinitely divisible good \(e \in E\). The seller incurs a cost of \(C_e(x)\) for producing \(x\) amount of this good. The market has a large population of non-atomic buyers\(^1\). Each individual buyer \(i\) has a set \(B_i \subseteq 2^E\) of bundles that meet her needs, but is indifferent between those bundles: the buyer value is \(v_i\) if she receives any bundle in \(B_i\) and zero otherwise. The sellers’ goal is to set prices so as to maximize profit, i.e, if \(p_e\) is the unit price of \(e\) and buyers buy \(x_e\) amount of this good, then the seller’s profit is \(p_e x_e - C_e(x_e)\). Once prices are fixed, buyers always buy the cheapest bundle in \(B_i\) as long as its price is not larger than \(v_i\).

Our two-sided market captures several settings of interest. For instance, \(e\) could be a link in a physical network and \(C_e(x)\) the seller’s cost for transiting \(x\) units of traffic, with buyers purchasing

\[^1\) Each individual buyer’s demand is negligible compared to the total population of buyers in the market. We can equivalently consider atomic buyers with concave utilities, see Section 2.\]
bandwidth on paths. Alternatively, we could represent a market of computer components \( E \) where each \( e \in E \) denotes a different computer part (e.g., a type of processor or a video card), with buyers wanting to buy complete computer systems. Many other settings may be represented in this way, including advertisers buying ads on websites, where \( C_e(x_e) \) would represent the negative externalities experienced by a website showing too many ads [31,43].

**Equilibrium:** In the first stage of the game, each seller chooses a price \( p_e \) per unit amount of his good. In the second stage, buyers buy bundles (sets of items) to maximize their utility. We assume a full information model as sellers are often aware of market demand; thus the sellers anticipate how their change in price will affect the demand for their good. A Nash Equilibrium for this game is a set of prices and an allocation of goods such that

1. all buyers receive one of their utility-maximizing bundles given the prices
2. no seller can increase his profit by unilaterally changing the price even with the most optimistic anticipation of demand (see Section 2 for details).

**Monopolies and Bertrand Competition**

Our two-stage game is essentially a generalization of the classic model of competition proposed by Bertrand where sellers set prices and buyers choose quantities to purchase. It has been studied in many settings (e.g., [6,13,25,26,33]), but most of this work does not consider the kind of combinatorial markets that we are interested in, where some bundles are substitutes and some may be complementary to each other. In fact, most of the existing work only looks at settings where all the items are similar and competition is almost perfect. On the other hand, markets with combinatorial structure have generated great interest among the Computer Science community (see e.g., [15,21,32] and their references).

Our work is most closely related to the model of Bertrand Competition in Networks first studied by Chawla and Roughgarden [20] and later extended in [19]. They also look at a two-stage pricing game similar to ours: the main difference is that instead of production costs, sellers are limited in their inventory of the item. The convex production costs that we consider in our paper are a generalization of such capacitated networks (See Section 8). [19,20] show tight worst-case bounds on the efficiency of equilibrium. For instance, even in markets with a few items where buyers desire the same bundles, the welfare can be arbitrarily bad compared to the social optimum.

More specifically, [20] shows that in the worst case, the efficiency of equilibrium drops exponentially as the number of monopolies increases. In fact, monopolies are often the main culprit of inefficiency in markets. Papers studying markets often assume that no monopolies exist at all when showing efficiency results (e.g., [4,23,46]), and as we see from Example 1, even a single monopoly can cause inefficiency, as compared to classical Bertrand competition where no monopolies exist. However, monopolies are quite real and economists have long attempted to quantify the loss in the US GNP due to monopolies in single-item markets, a parameter termed as deadweight loss [24]. In fact, for markets such as ours, several monopolies may be operating simultaneously. For instance, in a computer market, one provider may monopolize the supply of processors, and one the operating system. Nevertheless, we are able to prove that for a large class of demand functions the efficiency of equilibrium scales linearly with the number of monopolies present: thus the question to ask is not “Are there monopolies?” but is instead “How many monopolies are there?” To the best of our knowledge, no work other than [20] has shown such an explicit characterization of equilibrium quality in terms of the number of monopolies present in the market.

**Similar-minded buyers with asymmetric demand.** Although we consider more general models in Section 7 our primary focus will be on markets where all buyers desire the same bundles
of goods, i.e., $B_i$ is the same for all players. However, each buyer may hold a different valuation $v_i$ for the same set of bundles. In such scenarios, it is convenient to aggregate the buyer demand into a single inverse demand function $\lambda(x)$. For a given $x$, $\lambda(x) = v$ is defined such that exactly $x$ of the buyer population holds a value of $v$ or more for the satisfactory bundles. If $x$ amount of buyers buy their utility maximizing bundles, then the total value derived by the buyers can be given by $\Lambda(x) = \int_{t=0}^{x} \lambda(t) dt$. Since prices are intrinsic to the system, the total social welfare depends only the allocation and is equal to the value derived by the buyers minus the cost incurred by the sellers.

Alternatively, an inverse demand function could model the setting where a single buyer requires a large quantity of good but her utility for the good is concave, i.e., as the buyer receives more amount of the good, her (marginal) value decreases.

The Market as a Graph. We are particularly interested in markets where the items exhibit some complex combination of substitutes and complements. One convenient way to express large markets with some combinatorial structure is via a graph where each edge is an item controlled by a different seller and each bundle is a path in the graph. Our similar-minded buyer case then boils down to a networked market with a source $s$ and destination $t$, such that all the buyers want to buy any path between these nodes, although they may value this differently. This is a natural way to model users buying bandwidth, or even more traditional markets. For instance, the edges could represent electronic components or pre-assembled hardware, and each $s-t$ path is a set of components required to build a fully functional computer. We also remark that although our main results concern networked markets, we show in Section 7 how to extend them to more general markets.

1.1 Our contributions

For most of our results, we only look at “single-source, single-sink” markets with asymmetric-valued buyers as defined above. Our aim in this paper is to characterize the quality of equilibrium in terms of the demand and market structure, and specifically show the effect of monopolies on efficiency. Therefore, all our efficiency bounds depend only on $M$, the number of monopolies in the market. For the graph model, these are exactly the edges that are present in all the $s-t$ paths.

Our results are mostly constructive: we characterize the equilibrium solution by explicitly deriving the prices set by the sellers and using the properties of minimum cost flows. These prices are natural and fair (see Section 3); it is therefore reasonable to believe that this equilibrium would actually arise. Alternately, our pricing scheme can be suggested by a central authority interested in outcomes with high social welfare. Once these equilibrium prices are adopted, no one would deviate from them, and thus stable pricing with good social welfare can be implemented. All our efficiency results are therefore bounds on the Price of Stability (PoS): the ratio of the welfare of the social optimum with that of the best Nash equilibrium. This is in contrast with the price of anarchy, which considers the worst Nash equilibrium. Price of Anarchy in most of our games can be very bad: however this is for trivial reasons and such equilibrium outcomes are highly unrealistic as all players receive zero utility and no allocations are made. As in [19], we call such solutions with no goods being allocated trivial equilibrium.

Existence

Our first result concerns existence. Although non-trivial Nash Equilibrium may not exist even for simple markets with two complementary items, we show that there exists a Nash equilibrium for all inverse demand functions where the price elasticity is monotone (MPE functions, see Appendix A for details). This is a very natural assumption on the demand curve, which is obeyed by most of the
demand functions considered in the literature and all the functions listed in Figure 1. In addition, we also show how to compute the equilibrium prices and allocations efficiently.

**Efficiency for general classes of demand functions**

| Classes of MPE functions | Price of Stability |
|--------------------------|--------------------|
| Polynomial Demand $\lambda(x) = a - x^\alpha$ (PoS $\approx 1 + \log(M^\alpha)\alpha$) | $1 + (M\alpha)^{1/\alpha}$ |
| CED with small elasticity $\lambda(x) = (a - x)^\alpha$ | $1 + \frac{M}{\alpha+1}$ |
| Exponential Demand $\lambda(x) - r(x) = |\log(x)|^{\alpha}$ | $1 + M$ |
| $e^\frac{M}{M-1}$ | $\frac{M}{M-1}e^{(M-1)}$ |
| Unbounded | |

Figure 1: Our bounds on the Price of Stability for different types of inverse demand functions $\lambda(x)$. As the hazard rate $h(x) = |\lambda'(x)|/\lambda(x)$ of the function changes from highly increasing (left) to quickly decreasing (right), the quality of equilibrium solutions changes from perfectly efficient to infinitely worse than the optimum. Equilibrium exists for all functions in the figure, since they all satisfy the MPE property, and it is easy to show that it is perfectly efficient when $M = 0$.

Our main contribution is showing that for a large class of demand functions, the efficiency drops only linearly as the number of monopolies increases. In particular, we prove the following

- When all buyers have symmetric valuations (uniform capacitated demand), there exists Nash Equilibrium maximizing social welfare.

- If the inverse demand function $\lambda(x)$ has a monotone hazard rate (MHR), the Price of Stability is $1 + M$.

- When $\lambda(x)$ is concave (a special class of MHR functions), the PoS is $1 + \frac{M}{2}$.

These efficiency bounds are tight and involve minimal assumptions on graph structure and production costs. Both concave and MHR inverse demand assumptions are quite general and include several popular demand functions considered previously (see Section A for details and examples).

**Filling the gaps: The Price of Stability for popular demand functions**

In addition, we almost completely characterize the efficiency for all demand functions obeying the MPE condition. This characterization is summarized in Figure 2. For instance, for concave but polynomial demand, the efficiency only drops logarithmically with $M$ whereas for more general MPE functions, it can grow exponentially fast. Further, the efficiency of markets with highly inelastic demand is close to that of markets with concave demand ($1 + \frac{M}{2}$).
The main conclusion to draw from this is that the presence of monopolies does not completely destroy efficiency: it crucially depends on the properties of the demand curve and the number of these monopolies. We supplement this result by showing that in several settings without any monopolies, there exist welfare-maximizing Nash Equilibrium. This generalizes Bertrand’s classic maxim of ‘competition causes efficiency’ to combinatorial markets.

All of our efficiency bounds hold for more general markets (without a graph structure) having similar-minded buyers with one important difference: $M$ now stands for the number of virtual monopolies (VM) \[20\]. These are the goods that gain monopoly-like power at equilibrium because the alternatives are too expensive (see Section \[7\]). While the number of virtual monopolies can be as large as the cardinality of the largest bundle, for many reasonable market structures it is small.

Finally, we tackle a question of special interest: when do the allocations in our market game coincide with the Walrasian Equilibrium? In other words, what conditions cause Nash Equilibria be fully efficient? For this we also consider markets where different buyers may desire different bundles and identify the following cases where Price of Stability equals one.

- **Similar-Minded buyers**
  - Markets where all buyers have identical valuations, or
  - Sellers have a fixed inventory (\(\implies\) no costs) and buyer demand is highly elastic

- **Not Similar-Minded Buyers** (Different buyers value different bundles)
  - When buyer demand is fully elastic or each buyer type has a last-mile monopoly
  - For combinatorial markets with series-parallel structure and no monopolies

### 1.2 Related Work

**Bertrand Competition.** As mentioned earlier, our work is a generalization of classic Bertrand competition, and more specifically of \[20\] and \[19\]. This work looked at Bertrand competition in networks, and showed that stable allocations can be very inefficient. As our model is more general than theirs, we cannot hope to do better over all instances, however we show that for many important demand functions this inefficiency is bounded. As we show in Section \[3\] the behavior with production costs can be quite different than with capacities; nevertheless some of our results (specifically Theorem \[4.7\]) are essentially generalizations of results from \[19\].

The negative results in \[20\] and \[19\] have led some researchers to consider more sophisticated pricing mechanisms and other notions of equilibrium: see \[23, 38, 39, 44\]. For example, \[39\] and \[23\] consider non-linear pricing, where the unit price of a good increases with demand. While complex pricing mechanisms do sometimes lead to improvement in efficiency, it imposes additional complications on the buyers as they now have to anticipate the change in price due to the behavior of other buyers. Thus even though fixed pricing (as we consider) is more natural, its effects are not really well-understood beyond the fact that competition leads to better outcomes and monopolies cause a loss in welfare. Our work captures both these maxims, but attempts to provide additional insight on the structure and quality of equilibrium.

Work such as \[3, 4\] has also considered two-sided markets where buyers pay the price on each edge, but also incur a cost due to the congestion on the edge; such settings are essentially a combination of the type of market we consider and the classic selfish routing games \[47\]. Unfortunately, most of the results in this settings are only known for simple structures such as parallel links or parallel paths. One exception is \[46\], which considers a unique one-sided model where the routing decisions are taken locally by sellers who also set prices on their edges. They show that in the absence of monopolies, local decisions by sellers can result in a PoS of one. However, their model
is different from ours in many respects, as the buyers do not choose their routes and are always required to send their flow.

**Other Related Work.** There has been a lot of nice work studying Walrasian equilibrium in combinatorial auctions, see [30, 34, 41] for some recent examples. Such papers usually focus on markets with indivisible goods, in which Walrasian equilibrium may not exist, and analyze various special cases and equilibrium notions [15, 30, 34, 41]. In contrast, we look at settings where Walrasian equilibrium always exists, but is not the appropriate solution concept for price-setting sellers. Closest to our work in this area is the paper on price competition [11] with many sellers and where a single buyer has arbitrary combinatorial valuations. [11] prove that for indivisible goods, and for a single buyer who wants exactly 1 unit of a bundle, the Price of Stability is one. This result is very close in spirit to our Theorem 7.6; while we consider divisible goods and more general cost functions, as well as more than a single buyer (or alternatively, a single atomic buyer with much more general demand), we conjecture that our results could also be extended to their setting.

Finally, our two-stage game bears some similarity to first price procurement or path auctions (see [35, 45] and the references therein). Such auctions are very useful for modeling competition between sellers, but usually ignore the buyer side of the market by assuming that there is a single buyer who wants to purchase exactly one bundle (instead of having some price-dependent demand). Contrary to our setting, path auctions become uninteresting in the presence of monopolies; existing work has mostly focused on concepts like frugality [35, 45] and not social welfare.

## 2 Definitions and Preliminaries

An instance of our two-stage game is specified by a directed graph, a source and a sink \((s, t)\), an inverse demand function \(\lambda(x)\) and a cost \(C_e(x)\) on each edge. There is a population \(T\) of infinitesimal buyers; every buyer wants to purchase edges on some \(s\)-\(t\) path and \(x\) amount of buyers hold a value of \(\lambda(x)\) or more for these paths. Equivalently, we could consider a single atomic buyer with demand \(\Lambda(x) = \int_{t=0}^{x} \lambda(t)dt\) is her utility for a total of \(x\) units of the bundles corresponding to \(s\)-\(t\) paths. We define \(M\) to be the number of monopolies in the market: an edge \(e\) is a monopoly if removing it disconnects the source and sink. We make the following standard assumptions on the demand and the cost functions.

1. The inverse-demand function \(\lambda(x)\) is continuous on \([0, T]\) and non-increasing. The latter assumption simply means that the total demand should not increase if sellers increase their price.

2. \(C_e(x)\) is non-decreasing and convex \(\forall e\), a standard assumption for production or congestion costs. Moreover, \(C_e(x)\) is continuous, twice differentiable, and its derivative \(c_e(x) = \frac{d}{dx}C_e(x)\) satisfies \(c_e(0) = 0\). We show how to relax the final assumption in Section 7.

**Nash Equilibrium.** A solution of our two-stage game is a vector of prices on each item \(\vec{p}\) and an allocation or flow \(\vec{x}\) of the amount of each bundle purchased, representing the strategies of the sellers and buyers respectively. The total flow or market demand is equal to the number of buyers with non-zero allocation \(x = \sum_{P \in \mathcal{P}} x_P\), where \(\mathcal{P}\) is the set of \(s - t\) paths. We can also decompose this flow \(\vec{x}\) into the amount of each edge purchased by the buyers \((x_e)\). Given such a solution, the total utility of the sellers is \(\sum_{e \in E} (p_e x_e - C_e(x_e))\) and the aggregate utility of the buyers is \(\int_{t=0}^{x} \lambda(t)dt - \sum_{e \in E} p_e x_e\). The total social welfare is simply \(\int_{t=0}^{x} \lambda(t)dt - \sum_{e} C_e(x_e)\).

We now formally define the equilibrium states of our game. An allocation \(\vec{x}\) is said to be a best-response by the buyers to prices \(\vec{p}\) if buyers only buy the cheapest paths and for any cheapest
path $P$, $\lambda(x) = \sum_{e \in P} p_e$. That is, buyers act as price-takers and any buyer whose value is at least the price of the cheapest path will purchase some such path. A solution $(\vec{p}, \vec{x})$ is a Nash Equilibrium if $\vec{x}$ is a best-response allocation to the prices and, $\forall e$ if the seller unilaterally changes his price from $p_e$ to $p'_e$, then for every feasible best-response flow $(x'_e)$ for the new prices, seller $e$’s profit cannot increase, i.e., $p_e x_e - C_e(x_e) \geq p'_e x'_e - C_e(x'_e)$. Our notion of equilibrium is quite strong as the seller does not have to predict exactly the resulting flow: for every best-response by the buyers, the seller’s profit should not increase.

Classes of inverse demand functions that we are interested in

We now define some classes of demand functions that we consider in this paper. For the sake of notational convenience, we assume that the inverse demand function is continuously differentiable, and thus $\lambda'(x)$ is well-defined (and not positive as $\lambda(x)$ is non-increasing). However, all our results hold exactly even without this assumption (see Section 7). The reader is asked to refer to the Appendix for additional discussion and interpretation of each class of functions.

**Uniform buyers:** $\lambda(x) = \lambda_0 > 0$ for $x \leq T$. In other words, a population of $T$ buyers all have the same value $\lambda_0$ for the bundles.

**Concave Demand:** $\lambda'(x)$ is a non-increasing function of $x$. This includes the popular linear inverse demand case ($\lambda(x) = a - x$) where the demand drops linearly as price increases.

**Monotone Hazard Rate (MHR) Demand:** $\lambda'(x) / \lambda(x)$ is non-decreasing or $h(x) = |\lambda'(x)| / \lambda(x)$ is non-decreasing in $x$. This is equivalent to the class of log-concave functions where $\log(\lambda(x))$ is concave, and essentially captures inverse demand functions without a heavy tail. Example function: $\lambda(x) = e^{-x}$.

**Monotone Price Elasticity (MPE):** $x h(x) = x |\lambda'(x)| / \lambda(x)$ is a non-decreasing function of $x$ which tends to zero as $x \to 0$. This is equivalent to functions where the price elasticity of demand is non-decreasing as the price increases. Price elasticity measures the responsiveness of the market demand over its sensitivity to price, and exactly equals $\lambda(x) / x \lambda'(x)$ using our notation. See Appendix A for detailed discussion of related economic concepts.

Each class of demand function defined above strictly contains all the classes defined previously, i.e., Uniform Demand $\subset$ Concave $\subset$ MHR $\subset$ MPE.

**Min-Cost Flows and the Social Optimum:** Since an allocation vector on a graph is equivalent to a $s-t$ flow, we briefly dwell upon minimum cost flows. Formally, we define $R(x)$ to be the cost $\sum_e C_e(x_e)$ of the min-cost flow of magnitude $x \geq 0$ and $r(x)$, its derivative, i.e., $r(x) = \frac{d}{dx} R(x)$. Both the flow and its cost can be computed via a simple convex program given a graph and cost functions. Clearly, $R(x)$ is non-decreasing since increasing the amount of flow can only lead to an increase in cost, as the production costs $C_e$ are non-decreasing. We prove in the Appendix that:

**Proposition 2.1.** $R(x)$ is continuous, differentiable, and convex for all $x \geq 0$.

It is also easy to see from the KKT conditions that for a min-cost flow $\vec{x}$, we have $r(x) = \sum_{e \in P} c_e(x_e)$ for any path $P$ with non-zero flow (for a full proof see the Appendix). Given an instance of our game, the optimum solution is an allocation or flow which maximizes the social welfare $\Lambda(x) - \sum_e C_e(x_e)$. Since the buyers’ utility depends only on the magnitude of the flow, welfare is maximized when the flow is of minimal cost. The optimum solution therefore maximizes $\Lambda(x) - R(x)$ and must satisfy the following condition:

**Proposition 2.2.** The solution maximizing social welfare is a min-cost flow of magnitude $x^*$ satisfying $\lambda(x^*) \geq r(x^*)$. Moreover, $\lambda(x^*) = r(x^*)$ unless $x^* = T$.  

7
3 Existence and Computation of Equilibrium Prices

In this section, we show that for a large class of demand functions, we are always guaranteed the existence of at least one Nash Equilibrium. Moreover, we gain insight into equilibrium structure, by showing that these Nash Equilibria follow a specific pricing rule, defined below. This allows us to characterize the structure of Nash Equilibrium in the later sections and show bounds on their quality.

The largest class of functions that we consider are MPE demand functions: this class includes many reasonable demand functions (see Figure 1 for examples). As we mentioned MPE stands for monotone price elasticity, that is the elasticity of price for this demand function is non-decreasing with \( x \). In other words, the market cannot be more elastic at a lower price as compared to its price elasticity at a higher price.

Our pricing rule is very natural: it is obtained as a result of a simple ascending-price algorithm where each item’s starting price is its marginal cost. For the purposes of computation, we also show that an equilibrium following our pricing rule can be computed in polynomial time using a binary search algorithm.

Our pricing rule: We now define our pricing rule. Let \( E \) be the set of edges and \( c_e(x_e) \) be the differential cost function on each edge. Throughout this section \( \vec{x} \) will refer to a valid flow and \( x \) to the magnitude of this flow. Consider any instance with \( M \) monopolies. For any given minimum cost flow \( \vec{x} \) of magnitude \( x \), we will use the following pricing rule:

\[
p_e(\vec{x}) = \begin{cases} 
\frac{\lambda(x) - r(x)}{M} + c_e(x_e) & \text{if } e \text{ is a Monopoly} \\
c_e(x_e) & \text{otherwise}
\end{cases}
\]

This pricing rule has a simple interpretation. Edges are priced at their marginal cost. In addition, for any flow \( \lambda(x) - r(x) \) is the total available slack or surplus, which is divided equally among all the monopolies. For example, if all edges are monopolies having the same cost function, then \( r(x) = M c_e(x) \) and \( p_e(x) = \frac{\lambda(x)}{M} \). Moreover, for a given demand \( (x) \), the prices returned by our pricing rule are obtained as the result of the following ascending-price process

Every item is priced at its marginal cost. All monopolies increase their price gradually until they meet the market demand.

We begin by showing the not surprising result that when there are no monopolies \( (M = 0) \), then pricing at marginal cost makes the optimum flow stable. This case is quite easy: it is the addition of monopolies that leads to interesting behavior of equilibrium. This result holds for any arbitrary demand and cost functions.

Lemma 3.1. At any flow \( \vec{x} \), an edge priced at \( p_e = c_e(x_e) \) can never increase its profit by lowering its price.

Proof. Suppose an edge decreases its price from \( p_e = c_e(x_e) \) to \( p'_e \) and its flow increases from \( x_e \) to \( x'_e \), then we need to show that \( p_e x_e - C_e(x_e) \geq p'_e x'_e - C_e(x'_e) \). Consider the function \( p_e x - C_e(x) \).

For a fixed \( p_e = c_e(x_e) \), its derivative is negative for \( x > x_e \). Therefore, for any \( x'_e > x_e \), we have \( p_e x_e - C_e(x_e) \geq p_e x'_e - C_e(x'_e) \). But since \( p'_e \leq p_e \), we know \( p_e x_e - C_e(x_e) \geq p_e x'_e - C_e(x'_e) \geq p'_e x'_e - C_e(x'_e) \). This completes the proof.

Claim 3.2. If the graph contains no monopolies, then there always exists a Nash equilibrium which maximizes social welfare.
Proof. The solution is quite straight-forward. We compute the optimum solution \( x^* \) and price edges according to our pricing rule above, which translates to each edge being priced at \( p_e = c_e(x_e) \) since the instance has no monopolies. By Lemma [B.4] and Proposition [2.2], we know that for every flow carrying path \( P_i \), \( \lambda(x^*) \geq r(x^*) = \sum_{e \in P_i} c_e(x_e) = \sum_{e \in P_i} p_e \). So all flow carrying paths have the same total price, call it \( p^* \). We claim that sending a flow of magnitude \( x^* \) is a best-response by the buyers to this price. First consider the case when \( \lambda(x^*) > r(x^*) \). According to Proposition [2.2], \( x^* = T \) is the total flow available in the market. Thus all the buyers have a value of at least \( \lambda(x^*) \) for the paths and since the price is not larger than this, all of them purchase one of the \( s-t \) paths. When \( \lambda(x^*) = r(x^*) = p^* \), it means that exactly \( x^* \) of the buyers value the paths at \( p^* \) or more, so these many buyers send the flow. In both cases, buyer behavior is indeed a best-response to the prices.

Now we show that sellers cannot change their price unilaterally and increase their profit. We claim that for every edge \( e \) with flow, if the edge is removed from the graph, then there exists at least one \( s-t \) path with a total price of \( p^* \). Then, no edge would wish to unilaterally increase its price as the flow would switch to the alternative path. Since the graph has no monopolies, there exists at least one \( s-t \) path not containing \( e \) for every edge \( e \). Let \( P_i = v_1v_2 \cdots v_kv_{k+1} \cdots v_r \) be a flow carrying path with \( e \) where \( v_1 = s \) and \( v_r = t \). If the flow on \( e \) is \( x_e < x^* \), then there exists at least one flow carrying path without \( e \) and the price of this path is \( p^* \), so we are done. If not, then for some \( i \leq k \) and some \( j \geq k + 1 \), there must exist a path \( \gamma \) between \( v_i \) and \( v_j \) that only passes through edges with no flow on them. The price on these edges without flow must be \( c_e(0) = 0 \). Consider the new path \( P' = v_1 \cdots v_i \gamma v_j \cdots v_r \). The price of this path is no larger than \( p^* \), since the price of edges in \( \gamma \) is zero. So the price of this path must be exactly \( p^* \). Therefore, we conclude that no single edge can increase its price and still retain some flow.

By Lemma [3.1] we already know that no edge can priced at its marginal can decrease its price and make more profit, no matter how much the flow increases by. For the \( \lambda(x^*) > r(x^*) \) case, the whole market demand is satisfied so decreasing the price has no additional impact anyway. The edges without flow are already priced at 0, so they cannot decrease their price. This completes the proof.

Indeed, the above claim shows that in the absence of monopolies, the practice of marginal pricing always results in a fully efficient Nash Equilibrium. Note our notion of a “no monopoly” graph is weaker than what has been considered in some other papers [23,16] and therefore, our result, much stronger. However, our main goal lies in finding out the effect of monopolies on social welfare and on existence of equilibrium.

In many settings, trivial equilibria exist that result from all prices being very large, and thus no flow being routed (i.e., an empty allocation with \( x = 0 \)) such that result from all prices being very large, and thus no flow being routed (i.e., an empty allocation with \( x = 0 \)). Such equilibria are unsatisfying, and we are mainly interested in non-trivial equilibrium where the allocation is non-empty. Before showing our existence result, we restate our assumptions on the demand functions, some of which we relax in a later section.

Assumptions on the Demand. We take the inverse demand function to be monotone non-decreasing and continuous in the domain \([0, T]\) such that \( \lambda(T) \geq 0 \) and \( \lambda(x) > 0 \) for all \( x < T \). We also assume that \( \lambda(x) \) is continuously differentiable in \((0, T)\), i.e., \( \lambda'(x) \) exists and is continuous. However, this assumption is merely for the purpose of improving exposition, as we show in Section 7, all the existence and efficiency results hold even when \( \lambda(x) \) is just continuous. To simplify notation, we use \( \lambda'(T) \) to refer the limit \( \lim_{x \to T^-} \lambda'(x) \).

We are now in a position to prove our main existence result. We make absolutely no assumption on how much the buyers value the item (no assumption on how large \( \lambda(0) \) can be), which separates both our existence and efficiency results from those in [20]. Since we have already shown that in
the absence of monopolies, (efficient) equilibrium always exists, the following theorems implicitly assume the existence of one or more monopolies operating in the market.

Recall that a solution of our game is given by \( (\vec{p}, \vec{x}) \). Clearly, once prices are fixed, buyers always buy best-response bundles so we only consider solutions of this form. Finally for any edge, we define \( p_e \) to be the increase in the price of an edge from its marginal cost. That is, given a solution, \( p_e = c_e(x_e) + p_e' \). We will now show some sufficient conditions that this ‘increased’ price must obey at a Nash Equilibrium and then use this to explicitly construct one such solution via our pricing rule. The proofs of the following lemmas are somewhat technical, so in order to improve readability, they are presented in the appendix.

**Lemma 3.3.** Given a solution \( (\vec{p}, \vec{x}) \) with \( \vec{x} \) a best-response flow to prices \( \vec{p} \), with \( \lambda(\vec{x}) > 0 \) and \( \lambda(\vec{x}) \geq \vec{x}|\lambda'(\vec{x})| \), we have that no seller \( e \) can increase his price and improve profits as long as either one of the following conditions hold,

1. The good \( e \) is tight, i.e., \( \exists \) some s-t path that does not contain the edge \( e \) and has the same total price as the flow-carrying s – t paths that do contain \( e \) (or)
2. \( p_e \geq \vec{x}|\lambda'(\vec{x})| \)

We now show sufficient conditions on the other half of seller behavior, namely we give conditions on when a seller cannot decrease his price.

**Lemma 3.4.** Given a solution \( (\vec{p}, \vec{x}) \) with \( \vec{x} \) a best-response flow to prices \( \vec{p} \), satisfying \( \lambda(\vec{x}) \geq \vec{x}|\lambda'(\vec{x})| \), no seller \( e \) can decrease his price and improve profits as long as any one of the following conditions hold,

1. \( \vec{x}_e = \vec{x} = T \) (i.e., the edge receives the total flow or demand available in the market) or
2. \( p_e = 0 \) or
3. \( p_e \leq \vec{x}|\lambda'(\vec{x})| \) and \( \vec{x}_e = \vec{x} \)

Now combining both of the above lemmas, we identify sufficient conditions for given vectors of prices and allocations to be a Nash Equilibrium. The essence of the following corollary is that either the equilibrium must contain the entire flow available in the market (no buyers left unsatisfied) or that the increased price \( \vec{p}_e \) for the monopoly items must equal the quantity \( x|\lambda'(x)| \). Notice that we have already reasoned in Claim 3.2 that non-monopoly edges do not have the power to increase or decrease their price if the equilibrium flow is a minimum cost flow of the given magnitude.

**Corollary 3.5.** Any given solution \( (\vec{p}, \vec{x}) \) with \( \lambda(\vec{x}) > 0 \) and \( \vec{x} \) a best-response flow to prices \( \vec{p} \) is a Nash Equilibrium if the following conditions are met

1. \( \vec{x} \) is a minimum cost flow of magnitude \( x \).
2. All non monopoly edges are priced at their marginal cost, i.e, \( p_e = 0 \).
3. For all monopoly edges, \( p_e \geq \vec{x}|\lambda'(\vec{x})| \) and one of the following is true,
   (a) \( p_e \) is strictly equal to \( \vec{x}|\lambda'(\vec{x})| \)
   (b) \( p_e = 0 \) or
   (c) \( \vec{x} = T \).
Figure 2: The magnitude of the optimal flow $x^*$ and the equilibrium flow $\tilde{x}$.

Proof. The cost of every flow-carrying path $P$ is exactly $\sum_{e \in P} c_e(\tilde{x}_e) + \sum_{e \in \mathcal{M}} b_e$ where $\mathcal{M}$ is the set of monopolies. Since $\tilde{x}$ is a best-response flow, it must be that $\lambda(\tilde{x})$ equals the cost of these path, so for every monopoly $e$, $\lambda(\tilde{x}) \geq \tilde{x} \lambda'(\tilde{x})$. So the requirement for our previous lemmas that $\lambda(\tilde{x}) \geq \tilde{x} \lambda'(\tilde{x})$ is satisfied. No monopoly edge will want to change its price due to Lemmas 3.3 and 3.4.

Now coming to the non-monopoly edges, since $\tilde{p}_e = 0$, no non-monopoly edge would ever wish to decrease its price. Moreover, since $\tilde{x}$ is a minimum-cost allocation, we claim that for every non-monopoly edge $e$, $\exists$ a s-t path not containing $e$ with the same price. The proof is very similar to that of Claim 3.2 and involves showing that for every edge $e$, there exists a shortcut or ear not containing this edge and having the same price. So we conclude that no edge can increase or decrease its price. Therefore this is a Nash Equilibrium.

Now, in order to show that there exists an equilibrium following our pricing rule, all we need to show is that there exists a point where the pricing rule satisfies the above conditions.

**Theorem 3.6.** For any MPE demand function $\lambda$, there exists a Nash equilibrium with flow $\tilde{x} > 0$ obeying our pricing rule.

Proof. Recall that at any given minimum cost allocation vector $\tilde{x}$, our pricing rule prices all the non-monopoly edges at $\tilde{p}_e = 0$ or equivalently the total price $p_e = c_e(\tilde{x}_e)$. The monopoly edges are priced at $\tilde{p}_e = \frac{\lambda(\tilde{x}) - r(\tilde{x})}{M}$. First, note that our pricing rule always leads to a best-response flow: the cost of every path $P$ is exactly $\sum_{e \in P} c_e(\tilde{x}_e) + \lambda(\tilde{x}) - r(\tilde{x})$, and as we showed in Appendix B, this is exactly $\lambda(\tilde{x})$ for every flow-carrying path, and at least $\lambda(\tilde{x})$ for every other path. Thus, the flow $\tilde{x}$ is a best-response flow to our prices.

To show that the prices are stable, paraphrasing Corollary 3.5, all we need to show is that for any given instance with a MPE demand, we need to show that some $\tilde{x} > 0$ with $\lambda(\tilde{x}) > 0$ satisfies the following conditions

- $\lambda(\tilde{x}) - r(\tilde{x}) \geq M\tilde{x} |\lambda'(\tilde{x})|$ (no price increase) and any one of
  1. $\lambda(\tilde{x}) - r(\tilde{x}) = M\tilde{x} |\lambda'(\tilde{x})|$ or
  2. $\lambda(\tilde{x}) - r(\tilde{x}) = 0$ or
  3. $\tilde{x} = T$. 

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Let $x^*$ be any optimal flow of magnitude $x^*$. Assume for now that $\lambda(x^*) > 0$. If this point satisfies $\lambda(x^*) - r(x^*) \geq Mx^*|\lambda'(x^*)|$, then we claim to have found a point that satisfies all our requirements. Clearly the ‘no price increase’ condition is met by definition. Moreover, either $x^* = T$ in which case Condition (3) is met or $x^* < T$, which implies that $\lambda(x^*) = r(x^*)$ by Proposition 2.2 and Condition (2) is met. In this case, we set $\tilde{x} = x^*$ this is a Nash Equilibrium. Observe that if the optimum point satisfies this condition, then $|\lambda'(x^*)|$ is bounded because $\lambda(x^*) - r(x^*)$ is bounded. So in the event that $x^* = T$, $\lambda'(T)$ exists and is bounded.

Now suppose that the optimal point does not meet $\lambda(x^*) - r(x^*) \geq Mx^*|\lambda'(x^*)|$ (see Figure 2 for an illustration). Then we claim that $\exists$ some $\tilde{x} \in (0, x^*]$, which satisfies Condition (1) above, and therefore the no price increase condition as well. Since the optimum point does not satisfy the “No Price Increase” condition, we have $\lambda(x^*) - r(x^*) < \lim_{x \to x^*} Mx|\lambda'(x)|$ (the limit is necessary for the case that $x^* = T$ and this $\lambda'(x^*)$ is not defined). Then, we claim that $\exists$ some $\tilde{x}$ obeying the following condition.

$$\lambda(\tilde{x}) - r(\tilde{x}) = M\tilde{x}|\lambda'(\tilde{x})|$$

(1)

For MPE functions as $x \to 0$, the ratio $\frac{Mx|\lambda'(x)|}{\lambda(x)}$ also tends to zero. This means that $\exists$ some $\epsilon_0 > 0$ such that for all $0 \leq x \leq \epsilon_0$, $\lambda(x) > Mx|\lambda'(x)|$ since $M$ is finite. Moreover $r(x)$ goes to zero as $x$ tends to zero. This means that $\lim_{x \to 0} (\lambda(x) - r(x) - Mx|\lambda'(x)|) = \lim_{x \to 0} (\lambda(x) - Mx|\lambda'(x)|) > 0$ since the term $\lambda(x) - Mx|\lambda'(x)|$ is strictly positive for sufficiently small $x$ and $r(x) \to 0$. But since the “no price increase” condition is not true at $x^*$, $\lambda(x^*) - r(x^*) - Mx^*|\lambda'(x^*)| < 0$. As $\lambda(x)$, $r(x)$ and $\lambda'(x)$ are all continuous by assumption (see Section 7 for the case when $\lambda'(x^*)$, $r(x^*)$ may not be continuous), this means there is an intermediate point where $\lambda(x) - r(x) - Mx|\lambda'(x)| = 0$. So there exists at least one point $\tilde{x} \in (0, x^*]$, satisfying equation 1. Since $\lambda$ is non-increasing, this means that $\lambda(\tilde{x}) \geq \lambda(x^*) > 0$. By the above arguments a min-cost flow of this magnitude with our pricing rule forms a Nash equilibrium, as desired.

Finally, we have made the assumption that $\lambda(x^*) > 0$, since our Lemmas 3.3 and 3.4 need this condition. Suppose that $\lambda(x^*) = 0$ and therefore $r(x) = 0$ everywhere (Proposition 2.2). We claim that in this case, there must exist some $x_1$ sufficiently close to $x^*$ where $\lambda(x_1) > 0$ but is still strictly smaller than $Mx_1|\lambda'(x_1)|$. We formally prove this claim in the Appendix (See claim C.3). In this case, by the properties of MPE functions, there must exist some $\tilde{x} > 0$ with $\lambda(\tilde{x}) > 0$ satisfying Equation 1. By the same argument as above, this is a Nash Equilibrium.

Notice that in the cases where the equilibrium satisfies the no price increase condition and $\bar{p}_e = 0$ or $\tilde{x} = T$, the equilibrium solution is optimal and therefore maximizes social welfare. This means that for MPE demand functions, which satisfy these conditions, the Price of Stability is one. So in order to obtain upper bounds on the efficiency, we only need to show bounds on cases where this condition is not met. Therefore, in the rest of the paper, we will focus only on the case where the equilibrium must satisfy $\bar{p}_e = \tilde{x}|\lambda'(\tilde{x})|$. The following corollary explicitly characterizes the equilibrium prices and allocations.

**Corollary 3.7.** For any inverse demand function belonging to the class MPE, we are guaranteed the existence of a Nash Equilibrium with a min-cost flow $(\tilde{x}_e)$ of size $\tilde{x} \leq x^*$ such that,

1. All non-monopoly edges are priced at $c_e(\tilde{x}_e)$.

2. Either $\frac{\lambda(\tilde{x}) - r(\tilde{x})}{M} = \tilde{x}|\lambda'(\tilde{x})|$ or $\tilde{x} = x^*$.
Uniqueness of Equilibria

We have established that for MPE demand there exists at least one \( \tilde{x} > 0 \) satisfying our desiderata. We now move on to our uniqueness and computation result. Namely, we show that if the equilibrium is not optimal, then there exists a unique equilibrium obeying our pricing rule. Then, we show that such an equilibrium can be computed using an intuitive binary search based approach. Notice that the equilibrium obeying our pricing rule is quite natural: it can also be computed by a natural price-ascending algorithm where we slowly decrease the flow in the system beginning at the optimal solution \( x^* \) while applying our pricing rule, until the conditions in the above corollary are met.

We first present some simple properties about the equilibria that obey our pricing rule which we prove formally in the appendix. The following hold as long as the demand functions are MPE and the cost function is non-zero.

1. If a sub-optimal Nash equilibrium with a flow of magnitude \( \tilde{x} \) obeys our pricing rule, then \( \tilde{x} < x^* \).
2. If a Nash Equilibrium obeying our pricing rule is sub-optimal, then it must satisfy \( \lambda(\tilde{x}) - r(\tilde{x}) = M\tilde{x}|\lambda'(\tilde{x})| \).
3. Suppose that \( \exists \tilde{x} < x^* \) satisfying \( \lambda(\tilde{x}) - r(\tilde{x}) = M\tilde{x}|\lambda'(\tilde{x})| \). Then, all \( x < \tilde{x} \) must satisfy \( \lambda(x) - r(x) > Mx|\lambda'(x)| \) and all \( x > \tilde{x} \) must satisfy \( \lambda(x) - r(x) < Mx|\lambda'(x)| \).

Now we have the necessary tools to prove our uniqueness result. The full proof is given in the Appendix.

**Theorem 3.8.** For \( \lambda \) in MPE, either the equilibrium admits the optimal flow, or there is a unique equilibrium obeying our pricing rule as long as production costs are non-zero.

**Does uniqueness really matter?** Although we only analyze a specific pricing rule in this paper, it is still useful to show uniqueness of equilibrium for two reasons. First, our pricing rule is natural and fair, with each edge priced at its marginal cost and all the monopolies sharing the market surplus. Thus it is reasonable to assume that the prices at an actual equilibrium follow our pricing rule. Second, if a central authority cannot regulate the exact prices as in a Walrasian Equilibrium but can instead set some rules/suggestions on how items can be priced, then we may expect exactly the equilibrium that we consider to be formed. Since there is only one equilibrium that obeys this pricing rule, then our analysis on price of stability in the later sections of this paper would tell us exactly what the quality of this equilibrium would be. Since Linear \( \subset \) Concave \( \subset \) MHR \( \subset \) MPE, the uniqueness result applies to all these smaller classes as well.

**Computing the equilibrium**

We now show that the ‘unique’ equilibrium obeying our pricing rule can be computed efficiently up to any required precision. As always in the case of real-valued settings (e.g., for convex programming, other equilibrium concepts, etc), “computing” a solution means getting within arbitrary precision of the desired solution; the exact solution may not be computable efficiently or even be irrational.

**Theorem 3.9.** A Nash equilibrium from Theorem 3.6 can be computed by a simple binary search for any \( \lambda \in \text{MPE} \).
Proof. The first step in our procedure involves computing the optimal flow using a convex program and checking if it satisfies the condition of Lemma 3.3, i.e., \( \lambda(x^*) - r(x^*) \geq M x^* |\lambda'(x^*)| \). If that is the case, then by Corollary 3.5, we are done. The following algorithm therefore, is for the case when \( \lambda(x^*) - r(x^*) < M x^* |\lambda'(x^*)| \).

According to Theorem 3.8, since the equilibrium is not optimal, it is a unique point \( \tilde{x} < x^* \) satisfying \( \lambda(\tilde{x}) - r(\tilde{x}) = M \tilde{x} |\lambda'(\tilde{x})| \). Furthermore, by Lemma C.6, it is true that for all \( x > \tilde{x} \), \( \lambda(\tilde{x}) - r(\tilde{x}) < M \tilde{x} |\lambda'(\tilde{x})| \) and \( \forall x < \tilde{x}, \lambda(\tilde{x}) - r(\tilde{x}) > M \tilde{x} |\lambda'(\tilde{x})| \). This motivates a binary search approach.

The algorithm is straightforward. We maintain a window \([x_1, x_2]\) at every iteration. Initially set \( x_1 = 0 \) and \( x_2 = x^* \). Let \( x = \frac{1}{2}(x_1 + x_2) \), and compute \( \alpha(x) = \lambda(x) - r(x) - M x |\lambda'(x)| \). If \( \alpha(x) > 0 \), update the window to \([x, x_2]\) and if \( \alpha(x) < 0 \), update the window to \([x_1, x]\). If \( \alpha = 0 \), then we are done. Repeat this until the difference between the two extreme points falls within the desired precision and return (say) the mid-point. Since the size of the window is halved at every point, the algorithm terminates in time proportional to \( O(\log(x^*)) \).

The proof of correctness follows immediately from the uniqueness theorem and Lemma C.6.

4 Effects of Demand Curves and Monopolies on Efficiency of Nash Equilibrium

In this paper, we are interested in settings where socially efficient outcomes are reached despite the presence of self-interested sellers with monopolizing power over the market. While for arbitrary functions \( \lambda(x) \) belonging to the class MPE, the price of stability can be unbounded, we show that for many natural classes of functions it is relatively small, even in the presence of a limited number of monopolies. These good Nash equilibria can be computed by a central authority and suggested to the sellers, or can simply be arrived at by the sellers themselves, since the pricing rule implementing them is quite natural.

We begin by stating the various efficiency bounds that we show in this section. Following this, we discuss each class of demand functions in detail and prove the Price of Stability bound for each of these classes.

(Theorem)

The social welfare of Nash equilibrium from Section 3 is always within a factor of:

- 1 of the optimum for uniform (capacitated) demand \( \lambda \). In other words, PoS=1.
- \( 1 + \frac{M}{2} \) of the optimum for concave \( \lambda \).
- \( 1 + M \) of the optimum for Log-concave (i.e., MHR) \( \lambda \).
- \( \frac{M}{M-1} e^{M-1} \) for MPE \( \lambda \) with \( |x\lambda'(x)| \) non-decreasing and \( M \geq 2 \).

Moreover, all but the last of these bounds are tight.

4.1 Uniform (Capacitated) Buyer Demand

We begin with the simplest buyer demand function \( \lambda(x) = \lambda_0 \) for \( x \leq T \) and \( \Lambda(x) = \lambda_0 x \) for \( x = 0 \) to \( T \). The function is defined for \( x = 0 \) to \( T \) and continuity and differentiability hold in one direction at \( x = T \), so it trivially belongs to the class MPE and existence is guaranteed. If we consider a continuum of buyers at the source, then the interpretation is that all the buyers have the exact same valuation for the bundles, with the total number of buyers being \( T \). If we consider
a single buyer with a concave utility, then this means that the buyer desires $T$ units of any $s-t$ bundle and receives a utility of $\lambda_0$ per unit of the bundle.

It is important to mention here that the Nash Equilibrium that we construct may not necessarily be a Walrasian Equilibrium although the allocations are the same in both cases. Consider the simplest example with just one good, $\lambda_0 = 1$, $T = 1$ and $C_e(x) = \frac{x^2}{T}$. The Walrasian equilibrium has a price of $p_N = 0.5$ on the edge and all buyers purchase the bundle. This price is not stable for price-setting sellers. On the other hand, any (efficient) Nash Equilibrium must have a price of $p_N = 1$ on the edge and the total flow is still $\bar{x} = 1$.

**Theorem 4.1.** *The Price of Stability is 1 for any instance in which the buyers have a uniform (capacitated) demand, i.e., there exists a Nash Equilibrium with the optimal flow. Moreover, this holds even when \( \exists \) edges with $c_e(0) > 0$.***

Here, we prove the first half of the theorem. The intuition for the $c_e(0) > 0$ case involves some additional terminology involving goods known as virtual monopolies and the proof for this case is in the Appendix.

**Proof of the first half of the theorem.** If there exist several optimum solutions, then denote by $\bar{x}^*$ the optimum solution with the maximum possible flow. In other words, any solution with a total flow of size $x' > x^*$ is non-optimal. We show that there exist a set of prices such that this optimal flow forms a Nash Equilibrium. We begin with the case where $c_e(0) = 0$ for all edges. For this case, we claim that our pricing rule always returns a Nash Equilibrium.

First recall that our pricing rule ensures all flow carrying paths are priced at exactly $\lambda(x^*) = \lambda_0$, which for uniform demand is the value for the paths held by all buyers. This means that if any edge increases its price, all paths containing that edge would now have a price strictly larger than $\lambda_0$. No buyer would be able to afford such a path and therefore, the flow on the edge would drop to zero and its profit cannot increase. So we only need to show that sellers cannot decrease their price and improve profits.

Also recall from Lemma 3.1 that edges priced at their marginal cost can never decrease their price and increase profits whatever be the resulting flow. Since our pricing rule ensures that all edges are priced at least at their marginal cost, we only need to worry about edges which have a price strictly larger than their marginal cost. The only edges for which this is possible are the monopolies, which are priced at $p_e = c_e(x^*) + \frac{\lambda(x^*) - r(x^*)}{M}$. For $p_e > c_e(x^*)$, we can conclude that $\lambda(x^*) > r(x^*)$ if any edge is to be priced above its marginal cost. However, by Proposition 2.2 this implies that $x^* = T$ and therefore at the optimum flow all the buyers (population of $T$) have non-zero allocation. In such a case, decreasing prices would have no effect since there are no more unallocated buyers left in the market. We therefore conclude that no edge can increase or decrease its price and thus for the $c_e(0) = 0$ case, our pricing rule returns a Nash Equilibrium with flow $x^*$.

**4.2 Monotone Hazard Rate**

Our main result in this section is showing a tight bound on the efficiency of markets where the (inverse) demand has a monotone hazard rate. As mentioned previously, these correspond exactly to the case when the inverse demand function is log-concave, i.e., $\log(\lambda(x))$ is concave. Apart from being a strict generalization of Concave functions, log-concave inverse demand also captures market settings where the inverse demand function does not have a heavy-tail (is not ‘too convex’). Thus log-concavity is a very natural assumption on the inverse demand function and it is not surprising to see that such functions have received considerable attention in Economics literature [7, 12, 40].
We now formally define the class \textit{MHR}. In Appendix \[A\] we provide a detailed discussion of this class of demand functions and show that it is equivalent to the log-concavity of the demand.

\textbf{Definition} Class \textit{MHR}. An inverse demand function is said to belong to the class \textit{MHR} if it has a monotone hazard rate, i.e., \( -\frac{\lambda'(x)}{\lambda(x)} = \frac{|\lambda'(x)|}{\lambda(x)} \) is a non-decreasing function of \( x \).

Commonly used inverse demand (other than concave) functions belonging to this class include exponential functions similar to \( \lambda(x) = e^{-x} \) \[22,42\] and relatively inelastic functions of the form \( \lambda(x) = (a-x)^\alpha \) for \( \alpha > 1 \) (refer Section \[A\] for more details). The following lemma, which we prove in the Appendix, establishes that as long as a function \( \lambda(x) \) has a monotone hazard rate, the function \( \frac{|\lambda'(x)|}{\lambda(x)-r(x)} \) is also non-decreasing as \( x \) increases.

\begin{lemma}
Let \( \lambda(x) \) be any inverse demand function satisfying Monotone Hazard Rate. Given an instance specifying a graph \( G \) and cost functions \( C_e(x) \), then the function \( \frac{|\lambda'(x)|}{\lambda(x)-r(x)} \) is also non-decreasing \( \forall x \leq x^* \), where \( x^* \) is the size of the optimum flow for that instance.
\end{lemma}

We can now show our main result. Notice that this bound only relies on \( \lambda \) being MHR, it does not need any additional conditions for the cost functions \( C_e \).

\textbf{Theorem 4.3.} The Price of Stability (PoS) for all MHR functions \( \lambda \) is \( 1 + M \) where \( M \) is the number of monopolies for a given instance.

Many natural scenarios feature settings where even if there are many sellers, the number of complementary sellers independently and simultaneously monopolizing the market is limited. Our theorem indicates that the efficiency drops linearly only when the number of such sellers increases as opposed to the total number of sellers operating in the market: it is one of very few bounds interpolating between perfect competition and complete monopoly structure.

\textbf{Proof.} Let \( \vec{x}^* \) be the optimum solution with a total flow of magnitude \( x^* \). We use \( \vec{\tilde{x}} \) to denote the Nash Equilibrium with flow of size \( \tilde{x} \) as guaranteed by our existence result and which obeys the conditions of Corollary 3.5. Recall from our algorithm that among all feasible flows of magnitude \( \tilde{x} \), \( \vec{\tilde{x}} \) is the flow with minimum cost, and that \( \tilde{x} \leq x^* \). Thus, the social welfare of the optimum flow is \( \Lambda(x^*) - R(x^*) \), and the social welfare of our equilibrium solution is \( \Lambda(\tilde{x}) - R(\tilde{x}) \). Thus the Price of Stability is given by,

\begin{equation}
\frac{\int_0^{x^*} [\lambda(x) - r(x)]dx}{\int_0^x [\lambda(x) - r(x)]dx} = \frac{\int_0^{\tilde{x}} [\lambda(x) - r(x)]dx}{\int_0^\tilde{x} [\lambda(x) - r(x)]dx} + \frac{\int_{\tilde{x}}^{x^*} [\lambda(x) - r(x)]dx}{\int_0^{\tilde{x}} [\lambda(x) - r(x)]dx}
\end{equation}

\begin{equation}
= 1 + \frac{\int_{\tilde{x}}^{x^*} [\lambda(x) - r(x)]dx}{\int_0^{\tilde{x}} [\lambda(x) - r(x)]dx}
\end{equation}

We know from Lemma 4.2 that \( \frac{|\lambda'(x)|}{\lambda(x)-r(x)} \) is non-decreasing in \( x \). So \( \forall x \geq \tilde{x} \), we have \( \frac{\lambda(x)-r(x)}{|\lambda'(x)|} \leq \frac{\lambda(\tilde{x})-r(\tilde{x})}{|\lambda'(\tilde{x})|} \). Equivalently, this implies that \( \lambda(x) - r(x) \leq |\lambda'(x)| \frac{\lambda(\tilde{x})-r(\tilde{x})}{|\lambda'(\tilde{x})|} \).

The term following \( |\lambda'(x)| \) is a constant with respect to \( x \). So substituting this in the integral in the numerator, we have,

\[ \text{PoS} \leq 1 + \frac{\lambda(\tilde{x}) - r(\tilde{x})}{|\lambda'(\tilde{x})|} \frac{\int_{\tilde{x}}^{x^*} |\lambda'(x)|}{\int_0^{\tilde{x}} [\lambda(x) - r(x)]dx} \].

\|16\|
From Corollary 3.5, we know that \( \lambda(\tilde{x}) - r(\tilde{x}) = M \tilde{x} |\lambda'(\tilde{x})| \) at the equilibrium point \( \tilde{x} \). Substituting this in the above upper bound for the PoS, we can further simplify it as

\[
\text{PoS} \leq 1 + M\tilde{x} \frac{\int_{\tilde{x}}^{x^*} |\lambda'(x)| dx}{\int_{0}^{\tilde{x}} |\lambda(x) - r(x)| dx}.
\]

Now, look at the denominator in the above expression. \( \lambda(x) - r(x) \) is a non-increasing function of \( x \). So, we can lower bound \( \int_{0}^{\tilde{x}} |\lambda(x) - r(x)| dx \) by \( \tilde{x}(\lambda(\tilde{x}) - r(\tilde{x})) \). Finally, we can bound the PoS as follows,

\[
\text{PoS} \leq \left( 1 + M\tilde{x} \frac{\int_{\tilde{x}}^{x^*} |\lambda'(x)| dx}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))} \right) \leq 1 + M\tilde{x} \frac{\int_{\tilde{x}}^{x^*} |\lambda'(x)| dx}{\lambda(\tilde{x}) - r(\tilde{x})} \leq 1 + M\tilde{x} \frac{\int_{\tilde{x}}^{x^*} -\lambda'(x) dx}{\lambda(\tilde{x}) - r(\tilde{x})} = 1 + M\tilde{x} \frac{\lambda(\tilde{x}) - \lambda(x^*)}{\lambda(\tilde{x}) - r(\tilde{x})} \leq 1 + M
\]

The last step comes from the fact that \( r(\tilde{x}) \leq r(x^*) \leq \lambda(x^*) \) since at the optimum, \( \lambda(x^*) \geq r(x^*) \) and we know from Proposition 2.1 that \( r \) is non-decreasing. This completes our proof. \( \Box \)

**Claim 4.4.** The PoS bound of \( 1 + M \) for the family of MHR functions is tight.

**Proof.** As is standard with such examples, we consider a function at the boundary of MHR, \( \lambda(x) = e^{-x} \) for \( x \geq \frac{1}{M} \) and \( \lambda(x) = 0 \) for \( 0 \leq x \leq 1/M \). We just sketch the tightness proof here, the full explanation is given in the Appendix.

There are \( M \) sellers forming a path of length \( M \) between the source and the sink (\( M \) complementary goods). The production costs for all sellers are identical and are very close to 0 for \( x \) smaller than \( x^* \), for some large \( x^* > 1/M \), at which point the production costs increase very rapidly, so that \( r(x^*) = \lambda(x^*) \), and thus \( x^* \) is the size of the flow at optimum. The social welfare is \( \int_{0}^{x^*} \lambda(x) dx = e^{-1/M} - e^{-x^*} + e^{-1/M} \) since the production cost is essentially zero.

We claim that \( \tilde{x} = \frac{1}{M} \) is the only equilibrium point and every edge must have price exactly equal to \( e^{-M}/M \). The social welfare at \( \tilde{x} \) is \( e^{-1/M}/M \), so the Price of Stability tends to \( 1 + M \). Notice that for this family of instances, this is the unique non-trivial equilibrium. \( \Box \)

### 4.2.1 Concave Demand Functions

We now show that the Price of Stability for \( \lambda(x) \) being a concave function is \( 1 + M/2 \). Recall that an inverse demand function \( \lambda(x) \) is said to be concave if its derivative \( \lambda'(x) \) is decreasing or equivalently \( |\lambda'(x)| \) is increasing. Although concave functions belong to the family \( MHR \), they have a strictly increasing hazard rate, which is why we obtain an improved bound on the Price of Stability. The Price of Anarchy when the market demand is concave has received some attention before \([8, 37]\), albeit for slightly different models. More importantly, concave functions contain the special case of linear inverse demand functions of the form \( \lambda(x) = a - x \), which are common in modeling market demand. These linear functions lie at the boundary of the concave functions, and our PoS bound of \( 1 + M/2 \) is tight for these functions.
Theorem 4.5. For concave $\lambda(x)$, the Price of Stability is at most $1 + \frac{M}{2}$. Further, there exist instances where no equilibrium can achieve a better Price of Stability.

Proof. The proof uses the same notation as the proof of Theorem 4.3 and we build upon some of the arguments made there. We know from the proof of theorem 4.3 that the following is an upper bound on the PoS,

$$1 + \frac{\int_{\tilde{x}}^{x^*} [\lambda(x) - r(x)] dx}{\int_{0}^{\tilde{x}} [\lambda(x) - r(x)] dx}.$$

![Figure 3: Plot of an arbitrary concave inverse demand function](image)

In terms of the Figure 3, the term in numerator of the PoS expression is exactly the shaded area in the left figure, i.e., the area between $\lambda(x)$ and $r(x)$ in the desired region. Now since $\lambda(x)$ is concave, we know in the region $\tilde{x}$ to $x^*$, the function decreases at a rate of $|\lambda'(\tilde{x})|$ or faster. Therefore we define an auxiliary function $g(x)$ (dashed lines in the right figure), such that $g(\tilde{x}) = \lambda(\tilde{x})$ and $g(x)$ decreases linearly at a rate of $|\lambda'(\tilde{x})|$. Since $\lambda(x)$ is concave, we conclude that for all $x \in [\tilde{x}, x^*].$

Now, we can say that the shaded area in the left figure which we need to evaluate is upper bounded by the shaded area in the right figure. That is, the shaded triangle represents the area between $g(x)$ and $r(\tilde{x})$ from $\tilde{x}$ to some point $x_{max} \geq x^*$, where $g(x) = r(\tilde{x})$. The area under this triangle is given by,

$$\Delta = \frac{1}{2}(x_{max} - \tilde{x})(\lambda(\tilde{x}) - r(\tilde{x})) = \frac{1}{2} \frac{\lambda(\tilde{x}) - r(\tilde{x})}{|\lambda'(\tilde{x})|}(\lambda(\tilde{x}) - r(\tilde{x})).$$

Going back to our original PoS expression, we have

$$\text{PoS} \leq 1 + \frac{\int_{\tilde{x}}^{x^*} [\lambda(x) - r(x)] dx}{\int_{0}^{\tilde{x}} [\lambda(x) - r(x)] dx} \leq \frac{\Delta}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))} = 1 + \frac{1}{2} \frac{\lambda(\tilde{x}) - r(\tilde{x})}{|\lambda'(\tilde{x})|} = 1 + \frac{1}{2}M.$$
Proof sketch of Tight Example. (Full analysis in the Appendix)

Consider a simple path of $M$ links, having the following inverse demand function.

$$\lambda(x) = \begin{cases} 
2M + 1 & 0 \leq x \leq 1 \\
(2M + 1) - 2(x - 1) & 1 \leq x
\end{cases}$$

Every edge has a cost function given by $C_e(x) = \frac{1}{M}x$. The following figure shows both $\lambda(x)$ and $r(x)$ for a sample instance with $M = 4$.

![Figure 4: Tight example for concave inverse demand functions.](image)

From the figure, at the optimum solution $\lambda(x)$ and $r(x)$ meet ($x^* = M + 1$). The social welfare of the solution is equal to $(2M + 1 - 1) + \frac{1}{2}(2M + 1 - 1) * (M + 1 - 1) = 2M + M^2$. We claim that the point $\tilde{x} = 1$ is a NE when the edges are priced according our pricing rule. Note that every edge is priced at $p_e = 2 + \frac{1}{M}$ so the total price on the path is $2M + 1$ so a flow of $\tilde{x} = 1$ is a valid best-response. Sellers cannot increase their price as they would lose their entire flow. We prove that if any seller decreases his price, then the resulting flow can only lead to a decrease in profit.

The social welfare at this equilibrium is just $(2M + 1 - 1) = 2M$ and the efficiency for this instance equals $1 + M/2$. It is not hard to show that any equilibrium for this instance must admit a total flow of $\tilde{x} = 1$ and thus the bound is tight. 

**Monotone Price Elasticity (MPE) with $|x\lambda'(x)|$ increasing**

Recall the definition of the MPE functions: $x\frac{|\lambda'(x)|}{\lambda(x)}$ is non-decreasing in $x$ and further tends to zero as $x$ tends to zero. It is not hard to see that MHR functions also fall under this class: if $h(x) = \frac{|\lambda'(x)|}{\lambda(x)}$ is increasing then $xh(x)$ is also increasing with $x$. So MPE is a generalization of MHR.

We know that MHR functions have a Price of Stability of $1 + M$. Now we show bounds on the price of stability for another family of functions that belong to MPE but may not necessarily have a monotone hazard rate. First we state without proof a simple claim that incorporates the cost functions into the hazard rate. The proof of this is identical to the proof of Lemma 4.2, replacing $|\lambda'(x)|$ with $x|\lambda'(x)|$.

**Claim 4.6.** If $\lambda(x)$ is a MPE function, i.e., $x\frac{|\lambda'(x)|}{\lambda(x)}$ is non-decreasing, then $x\frac{|\lambda'(x)|}{\lambda(x) - r(x)}$ is also non-decreasing, where $r(x)$ is the differential min-cost function.
Now, we can show our main theorem in this section. We show a bound on the price of stability for another large class of MPE functions: ones in which \( x|\lambda'(x) | \) is non-decreasing. As an example for such functions, consider \( \lambda(x) = | \log(\frac{a}{x}) | \) for \( x = 0 \) to \( a \), which is not MHR but is MPE.

**Theorem 4.7.** If \( \lambda(x) \) has \( x|\lambda'(x) | \) non-decreasing, then the Price of Stability for any given instance with this demand is at most \( \frac{M}{M-1}e^{M-1} \).

Chawla and Niu [19] showed a bound of \( e^k \) on the Price of Anarchy for instances with non-decreasing \( x|\lambda'(x) | \), when each edge has a capacity instead of a production cost. Here \( k \) is the length of the longest \( s-t \) path. Our result shows that if we consider the Price of Stability, then it is exponential in only the number of monopolies and is slightly tighter than just \( e^M \). Moreover, we extend their results to instances with production costs which generalize capacities.

**Proof.** Let \( \tilde{x} \) be the equilibrium flow of size \( \tilde{x} \) obeying Corollary 3.5 and let \( x^* \) be the size of the optimum flow maximizing social welfare. From Proposition C.5, we know that \( x^* \leq \tilde{x} \). We first show that \( \frac{x^*}{\tilde{x}} \leq e^M \). Note that this immediately gives us a bound of \( e^M \) on the PoS since the flow at equilibrium always has a higher value than the flow after that. Following this, we tighten the bound to \( \frac{M}{M-1}e^{M-1} \). For ease of exposition, the proof of the following claim has been moved to the Appendix.

**Claim 4.8.** For any given instance with \( x|\lambda'(x) | \) being non-decreasing, it must be that \( \frac{x^*}{\tilde{x}} \geq e^{-M} \).

Now for the rest of the proof. Define \( f(x) = \ln (\lambda(x) - r(x)) \). Then \( f'(x) = \frac{\lambda'(x)}{\lambda(x) - r(x)} \leq \frac{\lambda'(x)}{\lambda(x) - r(x)} \) since \( r'(x) \) is always non-negative as \( r(x) \) is non-decreasing. So, by integrating \( f'(x) \) between \( \tilde{x} \) and \( x > \tilde{x} \), we get

\[
f(x) - f(\tilde{x}) = \int_{\tilde{x}}^{x} f'(x) dx \leq \int_{\tilde{x}}^{x} \frac{\lambda'(x)}{\lambda(x) - r(x)} dx.
\]

Now since \( \lambda(x) \) is a MPE function, we have for \( x \geq \tilde{x} \), \( x|\frac{|\lambda'(x)|}{\lambda(x) - r(x)} \geq \frac{|\lambda'(\tilde{x})|}{\lambda(\tilde{x}) - r(\tilde{x})} \) or equivalently since \( \lambda'(x) \) is negative, \( \frac{\lambda'(x)}{\lambda(x) - r(x)} \leq \frac{|\lambda'(\tilde{x})|}{\lambda(\tilde{x}) - r(\tilde{x})} \). This is due to our equilibrium conditions from Corollary 3.5. Putting this in the above integral, we get

\[
\ln (\lambda(x) - r(x)) \leq \int_{\tilde{x}}^{x} \frac{\lambda'(x)}{\lambda(x) - r(x)} dx \leq \int_{\tilde{x}}^{x} -1 M \frac{1}{Mx} dx = -1 M \ln(\frac{x}{\tilde{x}})
\]

Since \( \ln(x) \) is a monotone increasing function, we can directly say,

\[
\lambda(x) - r(x) \leq (\lambda(\tilde{x}) - r(\tilde{x}))(\frac{x}{\tilde{x}})^{\frac{1}{M}}.
\]

From our previous proofs, we know that the PoS is bounded from above by \( 1 + \int_{\tilde{x}}^{x^*} \frac{\lambda(x) - r(x)}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))} dx \).

Substituting our bound for \( \lambda(x) - r(x) \) from the above inequality, we get

\[
\text{PoS} \leq 1 + \frac{\int_{\tilde{x}}^{x^*} (\frac{x}{\tilde{x}})^{\frac{1}{M}} dx}{\tilde{x}} = 1 + \frac{M}{M-1} (\frac{x^*}{\tilde{x}})^{\frac{M-1}{M}} - (\frac{x}{\tilde{x}})^{\frac{M-1}{M}}.
\]
Figure 5: Demand function of the form $\lambda(x) = x^{-1/r}$ truncated at some small $\epsilon$.

We already know that $\bar{x}/x \leq e^M$ from the claim proven above. So putting this in the above inequality gives us,

$$\text{PoS} \leq 1 + \frac{M}{M-1} \left( e^M \right)^{\frac{M-1}{M}} - \frac{M}{M-1} \leq \frac{M}{M-1} e^{M-1}.$$  

\[\blacksquare\]

**Claim 4.9.** The price of stability can be arbitrarily bad for MPE functions.

**Proof Sketch** (Refer Appendix for rest of the proof)
Consider a two-link path ($M = 2$). The same example can generalized for any value of $M$. The demand function has the following structure: $\lambda(x) = x^{-1/r}$ for $r > M$ and is truncated at some very small $\epsilon > 0$ as shown in Figure 5.

We assume that each edge has a cost function given by $C_e(x) = c_0 x$ for some positive $c_0$. The unique optimum flow is the solution with $(\bar{x})^{-1/r} = 2c_0$. Now, the most general condition that any equilibrium point must obey is given by Lemma G.3. Applying the fact that $x\lambda'(x) = -\frac{1}{r} x^{-1/r}$, we get the equilibrium condition as $\tilde{x}^{-1/r} (\frac{1}{2} - \frac{1}{r}) = c$.

Using some basic algebra and computing the social welfare of both these solutions, we see that as $r \to M$, the ratio of the social welfares becomes arbitrarily large. At the same time, Theorem 8.1 implies that if the edges have capacities instead of costs then for any $r < M$, the Price of Stability is one. \[\blacksquare\]

5 **Specific Demand Functions: A partial characterization of the Price of Stability**

The above classes of inverse demand functions encapsulate many different types of functions, but there is a large gap between a PoS of 1 and a PoS of $1 + \frac{M}{2}$, for example. We attempt to interpolate between our bounds above by considering very specific (but important and commonly studied) classes of functions defined below, and obtain the following results. We begin by defining three natural classes of demand functions characterized by a single parameter before showing PoS bounds.
based on the parameter.

**Definition**

- Let \( F_p \) denote concave inverse demand functions that decrease polynomially with degree \( \alpha \), i.e., \( \lambda(x) = \lambda_0 (1 - x^\alpha) \) for any \( \lambda_0 \geq 0 \) and \( \alpha \geq 1 \).
- Let \( F_{ced} \) denote the class of MHR functions with inverse demand function \( \lambda(x) = \lambda_0 (1 - x^\alpha) \) for any \( \lambda_0 \geq 0 \) and \( \alpha \geq 1 \).
- Let \( F_{exp} \) denote the MPE functions and cost functions which can be represented as \( \lambda(x) - r(x) = |\ln(x/a)|^{\frac{1}{\alpha}} \) for \( x \leq a \), and \( \alpha \geq 1 \).

**Discussion about the demand functions.**

\( F_p : \lambda(x) = \lambda_0 (1 - x^\alpha) \). Polynomial demand is the most typical form of concave inverse demand \([16]\). \( \alpha = 1 \) denotes the extremely popular linear inverse demand functions \([2, 37]\). We show that the Price of Stability increases only logarithmically with \( M \) for highly polynomial demand as opposed to linear for general concave demand.

\( F_{ced} : \lambda(x) = \lambda_0 (1 - x^\alpha) \). CED stands for constant elastic demand (see for example \([28]\)), where the elasticity of demand is \( \frac{1}{\alpha} - 1 \) and denotes how much the demand changes with the changes in price. Our result indicates that when the demand is relatively inelastic, the Price of Stability can still be much better than \( 1 + M \). It is well-known that several essential commodities have such a relatively inelastic demand: only a large increase in price can lead to a lot of consumers dropping out. For instance, the elasticity of the market for electricity is believed to be somewhere around \(-0.3\) \([29]\); our results would guarantee a Nash equilibrium within a factor of \( \approx 1 + 0.59M \) of optimum for such functions.

\( F_{exp} : \lambda(x) - r(x) = |\ln(x/a)|^{\frac{1}{\alpha}} \). Finally, logarithmic inverse demand functions seem to be quite common in literature \([16, 49]\) because they represent the case when the direct demand is exponentially dropping with price. The following theorem shows that by varying \( \alpha \) in all the functions above, we can interpolate between our results in the previous sections, and quantify how the quality of equilibrium changes as the functions become less or more concave.

**Theorem 5.1.** 1. For \( \lambda \in F_p \), the Price of Stability is at most \( (1 + M\alpha)^{\frac{1}{\alpha}} \). When \( \alpha \geq M \), this quantity is approximately \( 1 + \frac{\log(M\alpha)}{\alpha} \).

2. For \( \lambda \in F_{ced} \), the Price of Stability is at most \( 1 + M \frac{\alpha}{\alpha + 1} \) for \( \alpha \geq 1 \).

3. For \( (\lambda, r) \in F_{exp} \), the Price of Stability is at most \( e^{\frac{M}{\alpha}} \) for \( \alpha \geq 1 \).

Notice that our first two results hold irrespective of the nature of the cost functions, while the last bound depends on the cost functions \( C_e \) as well as \( \lambda \). The proof of this theorem is somewhat technical and located in the Appendix in order to improve readability.

6 Bad Examples for Multiple-Source Networks.

We show two bad examples that illustrate the difficulty in extending our results for the single-source case to multiple-sources, i.e., when different buyers have different desired bundles. The first result is somewhat familiar: it is a variation of the left-shoe right-shoe example commonly used for showing non-existence of Walrasian Equilibrium. In our case, Walrasian prices *do* exist, but no Nash Equilibrium does.
Claim 6.1. There exists an instance of our two-stage pricing game with two sources and a single sink that does not admit a Nash Equilibrium even when the buyers have linear capacitated demand functions.

Example Let the two source nodes be $s_1$ and $s_2$. There are two paths between $s_1$ and $t$, a direct edge $e_1 = (s_1, t)$, and a simple path $s_1 \rightarrow s_2 \rightarrow t$. In other words, $s_1$ has two preferred bundles $\{(e_1), (e_2, e_3)\}$, and $s_2$ has exactly one bundle $e_3$ that it desires. Source $s_1$ has a demand for one unit of flow which it values at $\lambda_1 = 100$, and $s_2$ also has demand for one unit of flow, which it values at $\lambda_2 = 25$. We assume that $e_1$ has a cost function $C_1(x) = 3x$ and $e_3$ has a cost $C_3(x) = 2x$. $e_2$ has no production cost. The same example can actually be shown to hold for more convex cost functions, although for simplicity we stick to these linear cost functions.

Proof. The following points establish that no Nash Equilibrium exists. We will let $p_i$ denote the price of edge $e_i$.

1. In any Nash equilibrium, $e_3$ makes a profit of at least 23, else it would change its price to $25 - \epsilon$ and receive at least one unit of flow, obtaining a profit of at least $25 - \epsilon - 2(1)$. Thus, since the total flow possible on $e_3$ is at most size 2, it must be that $p_3 \geq 13.5$ in every Nash equilibrium (since then $2p_3 - 2(2) = 23$).

2. In any Nash equilibrium, all of $s_1$’s flow must be on edge $e_1$. If no flow is being sent on $e_1$, then clearly $e_1$ would have incentive to lower its price and obtain a positive profit; we know it can outbid the alternate path since $p_3 \geq 13.5$. If some, but not all, of $s_1$’s flow is sent on $e_1$, then it must be that $p_1 = p_2 + p_3 \geq 13.5$. In this case, edge $e_1$ can change its price to $p_1 - \epsilon$ for a small enough $\epsilon$, receive all of the flow from $s_1$, and increase its profit.

3. Since $e_3$ is only receiving flow from $s_2$ in a Nash equilibrium, it must be that $p_3 = 25$. Otherwise $e_3$ would benefit from changing its price: if $p_3 < 25$ it could raise the price by some tiny amount and still receive the same flow; if $p_3 > 25$ then its profit is 0 since it receives no flow. Therefore $p_1 \leq 25 + p_2$: if it were larger then $e_1$ would receive 0 flow because it would take the $e_2, e_3$ path.

4. Now consider the prices $p_1 \leq 25 + p_2$ and $p_3 = 25$. If $p_1 > 25$, then $e_2$ could lower its price to be small enough until $25 + p_2 < p_1$, and then $e_2$ would receive positive profit because it would receive flow from $s_1$. Thus, $p_1 \leq 25$. $e_1$ is receiving all of $s_1$’s flow, so it could raise its price unless $p_1 = 25 + p_2$; since $p_1 \leq 25$ this implies that $p_1 = p_3 = 25$ and $p_2 = 0$ at every Nash equilibrium.

5. $p_1 = p_3 = 25$ and $p_2 = 0$ is not a Nash equilibrium, however: $e_3$ has incentive to lower its price to $25 - \epsilon$, and receive utility $2(25 - \epsilon) - 4$ instead of $25 - 2$.

Previously, we saw that a weaker version of Bertrand’s Paradox holds on arbitrary networks, namely that there always exists a fully efficient equilibrium due to competition when there are no monopolies. The following example shows that this is no longer the case with multiple sources. Even if every source has no monopoly edge, there exist instances where the Price of Stability is strictly larger than one, i.e., while competition may drive down prices, solutions are no longer optimal.

Claim 6.2. There exists an instance with two sources, one sink, and no monopoly edges for either source, such that the Price of Stability is strictly greater than 1.
Figure 6: Instance with two sources $s_1, s_2$ and one sink $t$. The cost function of each edge is given as the edge weight.

Proof Sketch.

Consider the instance shown in the figure with two sources $s_1$ and $s_2$. Let the demand of $s_1$ be $\lambda(x) = 1 - x$ for $x \leq 1$ and 0 afterward, and the demand of $s_2$ be $\lambda(x) = 4 - x$ for $x \leq 4$ and 0 afterward. At the unique optimum point $s_1$ sends a flow of $1/3$ on its direct link (edge $e_1$) to the sink and $s_2$ sends 2 units of flow, one on each of its paths.

We claim that there exist no set of prices stabilizing the optimal flow and give an overview of the proof here. The full proof is in the Appendix. All of $s_1$’s flow paths must be priced at $1 - \frac{1}{3} = \frac{2}{3}$ and for $s_2$ at $p = 2$. But since the edge $e_2 = (i_1, t)$ incurs a cost production cost of 1, its price must be at least one. This violates $s_1$’s price constraints and therefore the flow cannot be supported.

7 More General Markets and Production Costs

All the results above hold for the somewhat limited case in which the demand and cost functions are continuously differentiable and the market structure is that of a graph. However, what if all the buyers still have an identical set $B$ of valid combinatorial bundles (with each buyer possibly having different valuations), but this set $B$ did not correspond to the set of $s$-$t$ paths in a graph? This case can become a lot more intractable, since there can be sellers $e$ which are not true monopolies (they do not belong to all bundles in $B$), but still hold much more power than other sellers. In this section, we now extend the results from the previous theorems to more general market structures and demand functions.

We begin by showing that when the demand and cost functions are no longer differentiable, all of our results hold exactly. We then move on to the case when the market can no longer be represented by a graph. For this case, the efficiency bounds hold as long as $M$ is the total number of sellers monopolizing the market at equilibrium which may be strictly larger than the number of actual monopolies but is still bounded by the size of the largest consumer bundle.

7.1 Dropping Differentiability in Demand

We now generalize our model to consider the case when $\lambda(x)$ is continuous, monotone non-increasing, but not necessarily differentiable. Assuming that $\lambda(x)$ is continuous is reasonable as long as the market is sufficiently large: in order to gain a few additional consumers, a seller need only change his price by a small amount. At the same time, inverse demand functions of the type seen in Figure [8] are quite common and may be due to some underlying belief of the population or that
the consumers can be divided into several types, each following its own sub-demand function. For such general cases, the derivative $\lambda'(x)$ may not be defined at some points. However, recall the well-known result that a monotone function is differentiable almost everywhere (a.e.).

For almost-everywhere differentiable $\lambda(x)$, we define the class MPE to be as follows. Define $\lambda'_-(x) = \lim_{x \to x^-} \lambda'(x)$ and $\lambda'_+(x) = \lim_{x \to x^+} \lambda'(x)$. A function is MPE if these limits exist and for all $x_1 < x_2$ we have that $\frac{x_1 \lambda'_-(x_1)}{\lambda(x_1)} \leq \frac{x_1 \lambda'_+(x_1)}{\lambda(x_1)} \leq \frac{x_2 \lambda'_+(x_2)}{\lambda(x_2)} \leq \frac{x_2 \lambda'_-(x_2)}{\lambda(x_2)}$ and if $\frac{x_1 \lambda'_-(x_1)}{\lambda(x_1)}$ tends to 0 as $x \to 0$. In particular, the property also implies that for any $x$, we have $|\lambda'_-(x)| \leq |\lambda'_+(x)|$. The proof of the following theorem is quite technical but mostly similarly to that of Theorem 3.6 save a few minor points. and therefore we defer to the Appendix its full details.

**Theorem 7.1.** For continuous but non-differentiable $\lambda(x)$ belonging to the class MPE, there always exists a Nash equilibrium with a non-trivial flow of $\tilde{x} > 0$ obeying our pricing rule. Furthermore this equilibrium is unique as long as the production costs are non-zero at this point.

As with the continuously differentiable case, the following corollary characterizes the equilibrium prices and allocation.

**Corollary 7.2.** Any given solution $(\bar{p}, \bar{x})$ with $\bar{x}$ being a best-response allocation to prices $\bar{p}$ and $\lambda(\bar{x}) > 0$ is a Nash Equilibrium if the following conditions are met

1. $\bar{x}$ is a minimum cost allocation of magnitude $x$.
2. All non monopoly edges are priced at their marginal cost, i.e, $\bar{p}_e = 0$.
3. For all monopoly edges, $\bar{p}_e \geq \bar{x}|\lambda'_-(\bar{x})|$ and one of the following is true,
   (a) $\bar{p}_e \leq \bar{x}|\lambda'_+(\bar{x})|$ or
   (b) $\bar{p}_e = 0$ or
   (c) $\bar{x} = T$.

From the corollary and the proof of the above theorem, we can glean that any non-optimal equilibrium obeying our conditions must satisfy

$$M \bar{x}|\lambda'_-(\bar{x})| \leq \lambda(\bar{x}) - r(\bar{x}) \leq M \bar{x}|\lambda'_+(\bar{x})|$$  \hspace{1cm} (4)

All of our other theorems on the Price of Stability follow almost immediately. For instance, for MHR functions,

$$\int_{\bar{x}}^{x^*} (\lambda(x) - r(x))dx \leq \frac{\lambda(\bar{x}) - r(\bar{x})}{\bar{x}|\lambda'_+(\bar{x})|} \int_{\bar{x}}^{x^*} (|\lambda'(x)|)dx \leq M \int_{\bar{x}}^{x^*} (|\lambda'(x)|)dx.$$  

Note that the integration has to be performed piecewise. The same idea applies to the proof of Concave and MPE functions.

**Piecewise Cost Functions**

We now move on to the case when the production cost for any item $C_e(x)$ is continuous but not necessarily differentiable. It is extremely common in network flow literature \cite{5} and even in markets of continuous goods \cite{23} to consider such cost functions. Piecewise cost functions are useful to model settings where the cost of production depends on the technology used: for instance a manufacturing company may possess the latest technology capable of producing some $x_1$ units of their item for
a given time period. However, if their demand exceeds $x_1$, then they may be forced to switch to
older and more expensive technologies. In such cases, the non-differentiable points of the piecewise
cost function represents the change in underlying technology.

Formally, for a given instance, we assume that $C_e(x)$ is convex, continuous but not (necessarily)
differentiable for all $e$. However, we assume that it is piecewise differentiable, i.e., if at some $x = x_0$,
$\frac{d}{dx} C_e(x_0) = c_e(x_0)$ does not exist, then both $\lim_{x \to x_0^-} c_e(x) = c_e^-(x_0)$ and $\lim_{x \to x_0^+} = c_e^+(x_0)$ exist
and are finite. Once again, if we take $R(x)$ to be the min-cost flow function as defined previously,
then it is not too hard to see that $R(x)$ is also continuous, convex but not differentiable. We define
$r^-(x)$ and $r^+(x)$ accordingly wherever $R(x)$ is not differentiable as the left hand and right hand
limits of the derivative. Since the functions are convex, we can immediately infer that $c_e^-(x) \leq c_e^+(x)$
and $r^-(x) \leq r^+(x)$ always hold. We still assume that $\forall e, c_0(0) = 0$, which we relax in the following
section.

Before showing that all our results extend to this general case, it is useful to first highlight
the difficulties encountered when facing piecewise cost functions and why our techniques do not
directly apply here. Recall from Corollary 3.5 that in order for a solution pair $(\tilde{p}, \tilde{x})$ to be a Nash
Equilibrium, (i) all $s-t$ paths with flow should have a price of $\lambda(x)$ (or $x = T$ and the price is
less than $\lambda(x)$, and (ii) $\lambda(x) - r(x) = Mx |\lambda'(x)|$ (or $\lambda(x) - r(x) \geq Mx |\lambda'(x)|$ and $x = x^*$. In order to
achieve the first required condition, our pricing rule priced all edges at their marginal cost and then
raised the price on all monopoly edges to ensure that the price on all paths increases uniformly.
Our pricing rule is not applicable here because it is not clear what the marginal cost even means
since for every edge there is now a $c_e^+$ and $c_e^-$. Even if we price our edges at $c_e^-$ or $c_e^+$, it is not
necessary that all $s-t$ flow carrying paths have the exact same price.

The second condition is also not directly applicable here. Since at the equilibrium point $\tilde{x}$, $r(\tilde{x})$
may not be defined, we need to redefine this condition in terms of $r^-(\tilde{x})$ and $r^-(\tilde{x})$ and show that
it is still an equilibrium. In the Appendix, we formalize a series of claims that allow us to overcome
these difficulties. Here, we simplify these claims into a sufficient condition that gives us a handle
on the equilibrium prices.

Given a min-cost flow $\tilde{x}$ for an instance with $M \geq 1$ and a quantity $\tilde{c}_e$ for all $e$ such that,

1. $r^-(\tilde{x}) \leq \lambda(\tilde{x}) - M\tilde{x} |\lambda'(\tilde{x})| \leq r^+(\tilde{x})$.
2. $\forall e, c_e^-(\tilde{x}_e) \leq \tilde{c}_e \leq c_e^+(x_e)$.
3. For any flow carrying path $P$, $\sum_{e \in P} \tilde{c}_e = \lambda(\tilde{x}) - M\tilde{x} |\lambda'(\tilde{x})|$.

Then, pricing all non-monopoly edges at $p_e = \tilde{c}_e$ and all monopoly edges at $p_e = \tilde{c}_e + \tilde{x} |\lambda'(x)|$
results in a Nash Equilibrium.

The above condition depends heavily on these mysterious $\tilde{c}_e$ components in order to balance the
prices on all the paths. Indeed, once these components are obtained for some $\tilde{x}$ satisfying Condition 1,
it is easy to derive a Nash Equilibrium. We now show how to compute these $\tilde{c}_e$’s for any given
instance satisfying all of the requirements above. The proof uses a primal-dual like approach but is
combinatorial and constructive. We actually show something stronger in the following lemma, for
any given $p^*$ lying between $r^-(x)$ and $r^+(x)$, we can always compute a set of prices on the edges
such that all desired paths have a cost of $p^*$.

**Claim 7.3.** For any given min-cost flow $\tilde{x}$ and a price $p^*$ satisfying $r^-(x) \leq p^* \leq r^+(x)$, we can
always compute a vector of $\tilde{c}_e$ for each edge $e$ obeying the following requirements

1. For all $e$, $c_e^-(x_e) \leq \tilde{c}_e \leq c_e^+(x_e)$.
2. For any flow carrying path $P$, $\sum_{e \in P} \tilde{c}_e = p^*$ and for any non flow-carrying path $P'$, $\sum_{e \in P'} \tilde{c}_e \geq p^*$.

The proof of the above claim is rather involved and therefore we only provide a brief sketch of the algorithm used to compute the $\tilde{c}_e$'s for each edge here. The full proof of Claim 7.3 can be found in the Appendix. The approach is somewhat similar to primal-dual algorithms. We first compute the minimum cost flow and define positive and negative node potentials on each node to be the minimum cost of sending flow to that node and maximum gain by removing flow from the node respectively. It is not hard to see that for the sink node, the negative potential equals $r^-(x)$ and the positive potential equals $r^+(x)$.

We then select a cut of the (residual) graph separating nodes with equal and unequal positive and negative potentials, and increase the $c^-_e$ for edges going across the cut in order to obtain a new set of node potentials. We repeat this process until the marginal cost of removing flow from the sink $p^*$ satisfies the conditions in the claim. The vector $\tilde{c}_e$'s are simply the final negative marginal cost on each edge. In the appendix, we show that the above procedure terminates after a finite number of such iterations.

Now, we can immediately obtain our existence and PoS results.

**Claim 7.4.** For any instance without monopolies, we can obtain a Nash Equilibrium with the optimal flow.

**Proof.** At the optimum $x^*$, it must be true that either $r^+(x^*) \geq \lambda(x^*) \geq r^-(x^*)$ or $\lambda(x^*) \geq r^+(x^*) \geq r^-(x^*)$ and $x^* = T$. In the first case, we simply set $p^* = \lambda(x^*)$, use Claim 7.3 and set the price on every edge $p_e = \tilde{c}_e$ as per that claim.

What if $\lambda(x^*) > r^+(x^*)$. In this case, we set $p^* = r^+(x^*)$ and use Claim 7.3 and once again price edges to be $p_e = \tilde{c}_e$. Even though the flow carrying paths may be priced less than $\lambda(x^*)$, the buyer behavior is still a best-response since $x^* = T$ and there are no buyers left in the market. The proof that sellers cannot decrease their price proceeds exactly as we mentioned for the continuous cost case.

**Claim 7.5.** For instances with MPE demand $\lambda$ and $M \geq 1$, $\exists$ a Nash Equilibrium which is either optimal or satisfies

$$\lambda(\tilde{x}) - r^+(\tilde{x}) \leq M\tilde{x}|\lambda'(\tilde{x})| \leq \lambda(\tilde{x}) - r^-(\tilde{x}).$$

**Proof Sketch** The full proof is in the Appendix. In case the optimum solution is not a Nash Equilibrium, then it must satisfy, $Mx^*|\lambda'(x^*)| > \lambda(x^*) - r^-(x^*)$. Then, we claim that for MPE functions, there must exist some $\tilde{x} > 0$ such that $\lambda(\tilde{x}) - r^+(\tilde{x}) \leq M\tilde{x}|\lambda'(\tilde{x})| \leq \lambda(\tilde{x}) - r^-(\tilde{x})$. This is true because as $x \to 0$, $Mx|\lambda'(x)| < \lambda(x) - r^+(x)$ and so there must exist an intermediate $\tilde{x} > 0$ satisfying the requirements. Now, we set $p^* = \lambda(\tilde{x}) - M\tilde{x}|\lambda'(\tilde{x})|$ and run the algorithm of Claim 7.3 to obtain quantities $\tilde{c}_e$ on all edges.

All our bounds for the price of stability hold since the the only property that we require about the equilibrium is given by

$$\lambda(\tilde{x}) - r^+(\tilde{x}) \leq M\tilde{x}|\lambda'(\tilde{x})| \leq \lambda(\tilde{x}) - r^-(\tilde{x}).$$

While calculating the social welfare of the Nash Equilibrium, we have so far used the fact that the integral of $\lambda(x) - r(x)$ from 0 to $\tilde{x}$ is at least $\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))$. Here in this case, we can use the fact that the integral is at least $\tilde{x}(\lambda(\tilde{x}) - r^-(\tilde{x}))$ since $r(x)$ is still non-decreasing so we can do the integration piecewise. We do not reprove all our theorems here as the proofs are essentially the same.
7.2 Looking beyond graphs: Extension to General Markets

**Uniform demand markets with arbitrary combinatorial valuations** Recall that uniform demand represents the case when all the buyers value the desired bundles exactly the same amount. For the single-source single-sink case, this can be captured by two parameters: a value \( \lambda_0 \) that buyers receive per unit of the bundle and a population \( T \) indicating the total number of buyers with this demand. For a given set \( E \) of goods, we now extend this uniform demand case to buyers who hold arbitrary combinatorial valuations. As with the single-source setting, each buyer has the same valuation (say \( v(S) \)) for any bundle \( S \).

Formally, consider a function \( v(S) \) for every set \( S \subseteq E \) of goods which denotes the buyer’s value for one unit of this specific bundle. We only impose the free-disposal requirement: \( v(S) \) is monotone. The demand is capacitated, i.e., the population of the buyers having this requirement is \( T \). Clearly, this is a generalization of our previous case where \( v(S) = 0 \) for all non-desired bundles and \( v(S) = \lambda_0 \) for desired bundles and their supersets. The total value derived by the buyers for receiving \( x \) units of a bundle \( S \) is given by \( v(S) \cdot x \). There are no restrictions on the production costs.

A Nash equilibrium of this game is specified by a price vector \( \vec{p} \), an allocation \( \vec{x} \) of the bundles such that

1. All bundles with non-zero amount purchased are utility maximizing and bundles which are not purchased cannot give larger utility than the purchased bundles.
2. Sellers cannot change their price and improve their utility for any anticipated best-response demand.

**Theorem 7.6.** *The Price of Stability for the above Generalized Pricing Game for uniform demand buyers with arbitrary combinatorial valuations is one.*

The proof of this theorem (in the Appendix) is very similar to that of Theorem 4.1 and uses a specialized version of Algorithm 1 which is described in the following section.

7.3 General markets with non-combinatorial bundles

We begin by formally redefining our two-stage pricing game to the case where buyers may desire arbitrary bundles and not necessarily sets having some combinatorial structure (\( s-t \) paths). Once again, let \( E \) be any set of items, each controlled by a different seller facing a convex, continuously differentiable production cost \( C_e(x) \). We use \( B \subseteq 2^E \) to represent the bundles desired by all the buyers. Buyers are indifferent between these bundles so we can capture the market demand using an inverse demand function \( \lambda(x) \) such that at least \( x \) of the population holds a value of \( \lambda(x) \) or more for the bundles. We no longer assume that \( c_e(0) = 0 \) for any of the items.

Previously we had assumed that for all items \( c_e(0) = 0 \). What this really means is that sellers have similar entry costs, i.e., the cost of producing a very small amount of good is small for all sellers. Such a notion encourages competition. On the contrary if we consider two items (say substitutes) such that \( c_e(0) \) for one item is much smaller than the other, then it would mean that if the total demand at equilibrium is small enough, the item with the larger value of \( c_e(0) \) would not even enter the market and the other item can (virtually) monopolize the market. We do not argue here whether or not it is realistic for such asymmetric production costs among sellers; instead we show that whatever be the relationship between the production costs, the effect of monopolies does not change.
**Algorithm 1** Ascending Price Algorithm for Prices

**Require:** A min-cost allocation $\vec{x}$ of magnitude $x$, $\lambda(x)$, $c_e(x_e)$.

1: Let the initial price on each item be $p^0_e = c_e(x_e)$.
2: Let $M_A$ be the set of active monopolies initialized to $\mathcal{M}$ and $M_I$ be the set of inactive monopolies, initially empty.
3: Let $B$ be the set of bundles with non-zero allocation and $B'$ be the bundles with zero allocation.
4: Increase the price on all active monopolies uniformly until
   1. One or more of the monopolies become tight, i.e., there now exists some monopoly $e \in M_A$ such that some bundle in $B'$ that does not contain $e$ now has the same price as all bundles in $B$.
   2. The price of all optimal bundles is now exactly $\lambda(x)$. In this case, exit the algorithm.
5: In the first case, remove all the ‘tight’ monopolies from $M_A$ and add them to $M_I$ and repeat the above step.
6: The algorithm terminates when either the total price on the bundles equals $\lambda(x)$ or the set $M_A$ is empty.

**Virtual Monopolies (VM)** A small discussion on virtual monopolies is in order here. Given any solution to our two-stage pricing game $(\vec{p}, \vec{x})$, an item $e \in E$ is called a virtual monopoly if it belongs to all the bundles consumed by the buyers, i.e., $e \in B_i$ for all $B_i \in \mathcal{B}$ with $x_{B_i} > 0$. Note that $e$ need not necessarily belong to all buyer bundles in $B$, however due to the difference in production costs and the nature of the bundles, the item belongs to all the bundles that are actually consumed by the buyers. Therefore, in that particular solution of the market game the item has the power of a monopoly: it can increase its price (up to some extent) and not lose its flow due to the absence of competition. We call such items “Virtual Monopolies” keeping with the notation used in [20].

Clearly, any ‘pure monopoly’ (item belonging to all desired bundles) is also a virtual monopoly. For markets with graph structure obeying $c_e(0) = 0$, the set of pure and virtual monopolies coincide. This is no longer true for general markets.

Before showing that all our techniques extend to general markets, we first redefine our pricing rule to apply for the case with virtual monopolies. However, we do not directly define a closed form expression as in Section 3. Instead for any given minimum cost allocation $\vec{x}$, we define an ascending price process (Algorithm 1) beginning with all items priced at their marginal cost. The difficulty in obtaining a closed form expression is that the price of a VM cannot be arbitrarily increased since there are usually higher priced alternative bundles that do not contain this item. The following ascending price process returns the prices on each item given by $p_e(x)$. Let $\mathcal{M}$ be the set of items monopolizing the allocation $\vec{x}$ and let $M$ be the cardinality of this set. Our complete existence proof depends on a series of properties we prove about our algorithm culminating in showing that the prices returned vary continuously with the input $x$ for every single good. For better readability, the proofs of some of these properties are located in the Appendix.

**Lemma 7.7.** The following is true for our algorithm

1. At any time step $t$, all bundles belonging to $B$ have the same price (say $p^t$).
2. At any time step $t$, no bundle belonging to $B'$ can have a price smaller than that of any bundle in $B$. 

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3. For every good \( e \) that belongs to \( M_I \) at some time step \( t \), there exists some bundle in \( B' \) not containing \( e \) that has the same price as all bundles in \( B \) at that time step.

Notice that when we refer to a time step \( t \), it is the instant after all the tight monopolies have been transferred to \( M_I \). The next corollary follows from applying the lemma at the termination step.

**Corollary 7.8.** For a given \( x \), when the algorithm terminates, the following must hold.

1. All bundles with non-zero allocation have the same price (Let’s call this \( P(x) \)).
2. No bundle with zero allocation has a price smaller than \( P(x) \).

Now that we seem to have a fair understanding of how the pricing rule functions for a fixed minimum cost flow of a given magnitude, we turn our attention to the fact that there may exist several minimum cost flows of a given magnitude if the cost functions are not purely convex. However, upon close examination of the algorithm it appears that the prices depend only the marginal cost of a good and not on the exact allocation. The following lemma establishes that the value of the differential cost function \( c_e(x_e) \) is the same for all min-cost allocations of a given magnitude \( x \). Using this, we show that our algorithm returns the same price for every min-cost allocation of a given magnitude.

**Lemma 7.9.** For any two minimum-cost allocations of magnitude \( x \), say \( \vec{a} \) and \( \vec{b} \), and any good \( e \), \( c_e(a_e) = c_e(b_e) \), where \( c_e \) denotes the marginal cost. Moreover, for every single good this uniquely defined marginal cost varies continuously as we vary \( x \).

**Claim 7.10.** Our pricing rule returns the same set of prices \( (\vec{p}) \) on every minimum cost allocation of magnitude \( x \).

*Proof Sketch* Intuitively this is true because for all minimum cost flows of a given magnitude and any given edge \( e \), \( c_e(x_e) \) is uniquely defined. It is not hard to see that for any fixed value of \( x \), the price of any given good can be expressed as a function of only the marginal costs. The full proof can be found in the Appendix.

Now we know that the prices returned by our pricing rule do not really depend on which flow is input and only depends on the magnitude of \( x \), i.e., we can use the term \( p_e(x) \) without any ambiguity. We also define a quantity \( \bar{p}_e(x) \) to be the increase in the price of an item from its marginal cost, i.e., \( p_e(x) = c_e(x_e) + \bar{p}_e(x) \). In order to show our main existence result, we require the fact that as we vary \( x \) and compute the prices returned by our algorithm, \( \bar{p}_e(x) \) and hence \( p_e(x) \) varies continuously. In a first step towards showing this, we define some properties that a single good can obey at a given value of \( x \) and show that as long as these properties are fixed in a region, the pricing rule behaves nicely.

**Definition (Monopolies and Non-monopolies)** A good \( e \) is said to be a virtual monopoly for a given value of \( x \) if there exists at least one min-cost allocation where the total allocation of \( e \), \( x_e = x \).

A good \( e \) is said to be a non-monopoly for a given value of \( x \) if the price of the good using our pricing rule is \( c_e(x_e) \), its marginal cost.

The second definition is essentially equivalent to saying that for every min-cost allocation of magnitude \( x \), there exists a bundle not containing the good that has the same (or lower) marginal cost as the cheapest bundles containing it. Notice that the set of monopolies and non-monopolies need not be disjoint, for a given \( x \), a single good could belong to both.
**Definition (Active Monopolies and Inactive monopolies)** For a given $x$, a monopoly $e$ is said to be an active monopoly if upon running our pricing rule, it belongs to the active set when the algorithm terminates.

For a given $x$, a monopoly $e$ is said to be an inactive monopoly if given the prices returned by our algorithm, there exists a bundle $B_1$ not containing $e$ such that the price of this bundle equals the price of any optimal bundle containing $e$.

Once again, these two sets can overlap: this can happen for example if the termination condition for the algorithm occurs simultaneously with $e$ becoming tight.

**Definition (Rank and tight bundles)** For a given $x$, and a set of inactive monopolies, the position or rank of each monopoly $e$ in the set is a number greater than or equal to one that denotes when in the process of the algorithm, it becomes active. If several items become inactive at the same time, then the rank includes all of those positions. For instance if three monopolies become tight at the beginning, then they all have rank 1, 2, and 3; just as $e$ can simultaneously be both active and inactive, it can also have several simultaneous ranks.

For any given inactive monopoly $e$, a bundle $B_i \not\ni e$ is said to make $e$ tight if at the point in the algorithm where $e$ first becomes inactive, the price of the bundle equals the price of any bundle with non-zero allocation containing $e$.

**Profile Space** For a given value of $x$, an item could have several valid possible property sets. For instance it could be a monopoly, an inactive monopoly, have a rank $k$ and some bundle that makes it inactive. For the same value of $x$, it could be considered a monopoly, an active monopoly and have no rank or bundle. We aggregate the valid properties of all items into a single profile vector, namely a valid profile vector is a 4-tuple $\vec{v} = (M, M_A, R, B)$, where $M$ is the set of monopolies, $M_A \subseteq M$ is the set of active monopolies, $R$ is a set of monopolies and the order in which they become tight, and for every inactive monopoly, there must be one bundle listed in $B$ that makes it inactive. Here $R$ consists exactly of $M \setminus M_A$, the set of inactive monopolies. The set of all such profiles is the profile space.

A profile vector $\vec{v}$ is said to be consistent at a given value of $x$ if there exists a valid min-cost allocation $\vec{x}$ with a corresponding set of prices $\vec{p}$ returned by our pricing rule where,

1. For every $e \in M$, we have $x_e = x$ and for every $e \not\in M$, we have $p_e = c_e(x_e)$.
2. Every $e \in M_A$ belongs to $M$ and it satisfies the active monopoly property. For every $e \in M \setminus M_A$, there must exist a bundle not containing $e$ and having the same price as the allocated bundles.
3. For every $e \in M \setminus M_A$, its rank in $R$ is one of its valid ranks during the run of the algorithm. In other words, there is some tie-breaking for when several monopolies become tight simultaneously which will make $e$ become tight in the position given by $R$.
4. For every $e \in M \setminus M_A$, the certificate bundle given by $B$ really does make it tight during the run of our algorithm (even though other bundles may have become tight at the same time).

We define $S_x$ to be the set of all consistent profile vectors for an allocation of magnitude $x$. Now, finally we can define sets of points where our pricing rule shows nice properties. Formally, define the set $I_v$ with respect to a profile vector $v$ as the set of all $x$ such that $\vec{v} \in S_x$. We also define an additional quantity $\Gamma(x)$ to be the quantity $\vec{p}(x)$ for all the active monopolies in $M_A$ for a given interval $I_v$. Notice that this quantity is exactly the same for all $e \in M_A$. This is true because the increase in price is uniform from time step 0 until time step $K$ for these items.
Now, in order to prove our existence result, we need to show that the prices returned by the algorithm vary continuously with $x$. As a first step towards this goal, we show some nice properties obeyed by the prices inside each $I_v$ and the fact that the $I_v$’s are closed. We formalize these claims in the Appendix.

1. **(Continuity)** The price of every item is continuous within $I_v$. Formally, for a given profile vector $\vec{v}$, suppose $\exists$ an infinite sequence of points $(x_1, x_2, \ldots, x_n)$ converging to $X$ as $n \to \infty$ all belonging to $I_v$. Moreover if $X \in I_v$, then $\lim_{n \to \infty} p_e(x_n) = p_e(X)$ for all items.

2. **(Closed)** For any given profile vector, the set $I_v$ is closed. Formally suppose there $\exists$ a sequence $(x_1, \ldots, x_n)$ all of which belong to $I_v$ for some profile vector $\vec{v}$. If $\lim_{n \to \infty} x_n = X$, then $X \in I_v$.

The proofs of both the above properties rely on the fact that for a single $x$, given a valid profile $\vec{v}$, we can obtain a closed form expression for the price of every good $e$ that is simply an additive function of the marginal costs. Now we use both these properties to show that in the required range of $[0, x^*]$, the prices are actually continuous everywhere.

**Claim 7.11.** For every good $e$, the increased price on that good $\bar{p}_e$ is continuous for all $x$ in $[0, x^*]$. Moreover, the increased price on the active monopolies $\Gamma(x)$ is also a continuous function of $x$ in any region where $\Gamma(x)$ is non-empty.

**Proof.** Fix any value of $x$, say $x_0$. We want to show that $\lim_{x \to x_0} \bar{p}_e(x) = \lim_{x \to x_0^+} \bar{p}_e(x) = \bar{p}_e(x_0)$ for every single good $e$. Let’s pick any one direction, say $x \to x_0^+$.

Let $V$ represent the minimal but complete set of profile vectors that are all active as $x$ approaches $x_0$. Notice that $V$ is finite and non-empty since at least one profile vector has to be active at every point in the limit and there are only a finite number of profile vectors. When $|V| = 1$, then only a single profile vector $\vec{v}$ is active and using Lemma G.16, $\vec{v} \in S_{x_0}$. Further, using the continuity argument in Lemma G.14, we see that $\bar{p}_e(x)$ is continuous as $x$ approaches $x_0$ from the right. The case where $|V| > 1$ is slightly more tricky and so we will resort to a more fundamental definition of continuity.

For every $\vec{v} \in V$, define $X_v$ to be the set of points $x$ in $(x_0, x_1)$ such that $v \in S_x$ (and in decreasing order). By definition, $X_v$ should be an infinite set of points converging to $x_0$. Using Lemma G.16, we can say that every $\vec{v} \in V$ also belongs to $S_{x_0}$. Suppose we represent $X_v$ in the form of $(x^{(1)}, \ldots, x^{(n)}, \ldots)$ so that $\lim_{n \to \infty} x^{(n)} = x_0$. Then Lemma G.14 tells us that $\lim_{n \to \infty} \bar{p}_e(x^{(n)}) = \bar{p}_e(x_0)$ for each good $e$.

This means (applying the $\epsilon - \delta$ definition of continuity), for every $\vec{v} \in V$ and every $\epsilon > 0$, there must exist a positive integer $n_v$ such that for all $n > n_v$, $\bar{p}_e(x) - \epsilon \leq \bar{p}_e(x^{(n)}) \leq \bar{p}_e(x_0) + \epsilon$. Define,

$$\delta = \min_{\vec{v} \in V} (x^{(n_v)} - x_0).$$

Now we can say that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x_0 \leq x < x_0 + \delta$, we have

$$\bar{p}_e(x) - \epsilon \leq \bar{p}_e(x_0) \leq \bar{p}_e(x) + \epsilon.$$

Therefore, we can conclude that for every good $e$, $\bar{p}_e(x)$ is continuous at every $x$. Moreover, this would imply that the total price $p_e(x) = c_e(x) + \bar{p}_e(x)$ is also continuous since the marginal costs are also continuous. Finally, the term $\Gamma(x)$ is simply the increased price of some active monopoly at $x$, which also has to be continuous. This is indeed true, because the set active monopolies is closed and all active monopolies have the exact same increased price. 

\[\square\]
Lemma 7.12. If the set of active monopolies $M_A(x)$ at some $x$ is non-empty, then for $e \in M_A(x)$, $\bar{p}_e(x) \geq \frac{\lambda(x) - r(x)}{M}$, where $M$ is the number of virtual monopolies at the minimum cost allocation of magnitude $x$.

Proof. This is not hard to see since we begin by pricing all edges at the marginal cost and distribute the slack among the remaining virtual monopolies. Further, if there are still active monopolies at termination, then it means that the final price of $\lambda(x)$ has been reached. The total slack must therefore equal $\lambda(x) - r(x)$. Moreover, for some items, we stop increasing in the middle, so among all goods, the active monopolies must have the largest value of $\bar{p}_e(x)$ which is at least $\frac{\lambda(x) - r(x)}{M}$ since the prices are increased uniformly.

Now we can show that for MPE functions, $\exists$ at least one non-trivial equilibrium point via the following theorem.

Theorem 7.13. As long as the demand function is continuously differentiable and belongs to the class MPE, there exists at least one $\bar{x} \leq x^*$ which is a Nash Equilibrium.

Proof Sketch. The proof is very similar to the proofs of the other existence theorems so we only sketch the high level ideas here. For a full proof, the reader is asked to refer to the Appendix. Consider the region in which the set of active monopolies $M_A(x)$ is non-empty and let $x_0 \leq x^*$ be the largest point belonging to this region such that $\exists x < x_0$ does not belong to this region. Then, we show that $\exists x_0 \leq \bar{x} \leq x^*$ where $\Gamma(\bar{x}) = \bar{x}\lambda'(\bar{x})$ or else $x_0$ is a Nash Equilibrium.

Recall that the main tool that enabled us to show all the PoS results was the Equilibrium functions, $\exists$. We show that similar conditions apply here when the equilibrium is not optimal.

Corollary 7.14. There exists some $M' \leq M(\bar{x})$ such that for any demand function $\lambda$ in the class MPE, there exists a non-trivial equilibrium obeying

1. Either $\lambda(\bar{x}) - r(\bar{x}) = M'\bar{x}|\lambda'(\bar{x})|$ or $\bar{x}$ is optimal.
2. Every non-monopoly and non-VM edge is priced at its marginal cost.

Proof. This statement is not hard to see. Notice from Lemma 7.12 that at the equilibrium, any active monopoly $e$ must have $\bar{p}_e(\bar{x}) = \bar{x}\lambda'(\bar{x})$ and $\bar{p}_e(\bar{x})$ lies between $\lambda(\bar{x}) - r(\bar{x})$ and $\frac{\lambda(\bar{x}) - r(\bar{x})}{\bar{x}}$. If there are no active monopolies at equilibrium, then let $e$ be an edge which becomes inactive at termination. Clearly, this edge must satisfy $\frac{\lambda(\bar{x}) - r(\bar{x})}{\bar{x}} \leq \bar{p}_e(\bar{x}) \leq \bar{x}|\lambda'(\bar{x})|$, so for some $M'$, the above equality must hold.

All our previous efficiency bounds therefore hold, with $M$ being the number of virtual monopolies. It is important to recall here that even in the absence of ‘pure’ monopolies, the Price of Stability is no longer 1. This is because the equilibrium may be dominated by virtual monopolies. Indeed, it was the presence of some structure (graph market) and equal entry costs ($c_e(0) = c_e(0)$) that allowed us to exactly determine how many monopolies exist.

Theorem 7.15. For buyers with identical sets of valid bundles $B_i$, all the results from Section 3 and 4 hold, but with $M$ being the number of virtual monopolies of the equilibrium solution $\bar{x}$, instead of the number of monopolies.

The above theorem also holds for production costs with $c_e(0) \neq 0$. While in the worst case, the number of virtual monopolies can be as large as the number of sellers, for markets with reasonable (but not perfect) competition it is likely to be a lot less.
Thus, for the case when all buyers agree on which bundles are “good”, our results above show that the price of stability behaves nicely for many important classes of demand functions. What about the truly general case, in which every buyer may have a different set $B_i$ of valid bundles (we call this the multi-parameter case)? Or in the network setting, what if there were many sources and many sinks, with different buyers desiring to buy different types of paths? Unfortunately, as we have shown in Section 6, Nash equilibrium may not exist even for two sources and a single sink. Perhaps more surprisingly, we give relatively simple examples in which perfect efficiency is no longer achieved, even with complete absence of monopolies! Nevertheless, we prove in the next section that for some interesting special cases, fully efficient Nash equilibrium still exists even when buyers desire different sets of bundles.

8 When does competition lead to complete efficiency?

A special case of interest is identifying scenarios in which efficient equilibrium (i.e., as good as the optimal solution) exists. In the previous sections, we identified two such cases: single-parameter networks with linear (capacitated) demand, and markets with no monopolies. Here, we identify a few other interesting settings where Price of Stability is one. It is interesting to note that while Walrasian Equilibrium always exists for most settings that we consider, in some of these settings the Walrasian prices are not Nash equilibrium prices.

8.1 Single-parameter markets with ‘somewhat’ elastic demand and capacitated sellers

So far, we have not made any assumptions on the production cost of the sellers other than the fact that it is convex, twice differentiable, and $c_e(0) = 0$. However, a natural scenario arises when each edge $e$ has a capacity $c_e$ as opposed to a cost function. The capacity models the case when the seller possesses exactly $c_e$ units of the item which he wishes to sell and there is no production cost. Since the seller has a hard capacity constraint which cannot be violated, we assume that in any solution where buyers purchase more than $c_e$ units from that seller, the seller’s cost becomes infinite. Recall that this is equivalent to a convex cost function $C_e(x)$ satisfying:

$$C_e(x) = \begin{cases} 0 & 0 \leq x \leq c_e \\ \infty & c_e < x \end{cases}$$

Such hard capacity constraints are common in networks like the Internet [20] where each service provider owns an edge (or a router) and there are hard capacity constraints on the bandwidth or peak traffic. A model with capacitated sellers was considered in [20] and the follow-up paper [19] where several non-trivial bounds were established on the efficiency of equilibrium. However, even for single-source single-sink networks, the efficiency loss can be arbitrarily high as the number of monopolies increases or the inequality in the market (ratio between the highest and lowest valued consumer) increases. Here, we identify an important class of demand functions which lead to an efficient equilibrium irrespective of the market structure (number of monopolies) and the consumer differences – there can be consumers with high and low valuations for the same set of goods.

The demand functions that lead to PoS=1 have the form $\lambda(x) = ax^{-1/r}$. It is not hard to see that this represents the case when the elasticity of demand is constant and quite large. That is, increasing the price by a small amount leads to a large decrease in the total flow (see Figure 8.1). We also remark here that if we truncate this function at some small value of $x$ (where $\lambda(x)$ is still finite), the demand function now belongs to the class MPE. For instance, Figure 8.1 shows
a truncated demand function where $\lambda(x)$ has the form $ax^{-1/r}$ until some small value of $x$ (here $x = 0.1$). For $x < 0.1$, $\lambda(x)$ remains constant at $\lambda(0.1)$. Since this function belongs to the class MPE, we are guaranteed the existence of a Nash Equilibrium. However, on our characterization curve in Figure 1, this function belongs to the class with unbounded Price of Stability as illustrated by Claim 4.9.

Surprisingly, if we make one additional assumption for these instances: that edges have fixed capacities instead of costs, we can show that Price of Stability is one. This strongly highlights that the behavior of instances with convex costs is not necessarily the same or even an extension of the behavior with capacities. Our next theorem establishes how elastic these demand functions can be for a given instance in order to ensure a PoS of one. We believe that this theorem is independently interesting as it establishes a connection between the elasticity of demand and how many monopolies control the market. Specifically we show that for the above demand functions, when $r > M$, then PoS = 1.

**Theorem 8.1.** For any single-source single-sink instance with edge capacities, $M$ monopolies, and demand function $\lambda(x) = ax^{-1/r}$, the Price of Stability is one as long as $r > M$.

The claim holds even when the demand function is truncated at some $x$ or maximum price. This theorem essentially states that as the number of monopolies increases, the demand function needs to become more and more linear to guarantee complete efficiency of Nash equilibrium. We give some intuition for the proof here deferring its full details to the Appendix.

**Proof Sketch.** Without loss of generality, suppose that $\lambda(x) = x^{-1/r}$. Since the demand function is defined for all $x > 0$, at the optimal flow $x^*$, the minimum cut of the network is saturated. If the cut includes at least one monopoly, then price the unsaturated monopolies (set $U$) at $p_e^* = \frac{1}{r} \lambda(x^*)$, and distribute the remaining price equally on the saturated monopolies (set $S$).

Note that since $|S| + |U| = M$ and $r > M$, then the price of each saturated monopoly $p_e^* > \frac{1}{M} \lambda(x^*)$. The saturated monopolies cannot decrease their price as their capacity has been reached. Suppose one of them increases its price so that the new flow is $x < x^*$. Then, the new profit of that edge is $\pi(x) = px = [p_e^* + \lambda(x) - \lambda(x^*)]x$ and its derivative $\pi'(x) = p_e^* - \lambda(x^*) + \lambda(x) + x\lambda'(x) > \frac{1}{M} \lambda(x^*) - \lambda(x^*) + \lambda(x) - \frac{1}{r} \lambda(x)$. Indeed, from the definition of $\lambda(x)$, it is clear that $x\lambda'(x) > -\frac{1}{r} \lambda(x)$.

Now, we conclude that $\pi'(x)$ is positive for all $x < x^*$ since $\lambda(x) > \lambda(x^*)$ and so $\pi(x)$ can only
be smaller than the original profit \( \pi(x^*) \). By a similar argument, we can show that unsaturated monopolies have no incentive to increase or decrease their price. If no monopoly is saturated, we price all unsaturated monopolies as mentioned above and distribute the remaining part of \( \lambda(x) \) among any minimum cut. The rest of the proof is the same. ■

8.2 Multiple Source Networks

So far we have only looked at single-parameter networks, where every buyer desires the exact same bundles but may value them differently. We now move on to markets where different buyers may have different preferred bundles and different demand functions. This can occur, for example, in a multiple-source, multiple-sink network with markets located at different sources, and requiring different paths. Our results do not immediately extend to this case. Recall from the example of Claim 6 that even a simple instance with two sources need not admit any Nash Equilibrium. Similarly Claim 6.2 illustrates the surprising result that even if every item is substitutable, there may not be any efficient Nash equilibrium. This is due to the presence of Virtual Monopolies. In what follows, we identify some ‘nice instances’ with multiple sources where Price of Stability is one.

Large Uniform Demand and Strictly Convex Costs

We consider an arbitrary market modeled as a multiple-source multiple-sink network with each source representing one or several buyers. Each type of buyer has a particular set of bundles which it values equally; all other bundles are valued at 0. We assume that each buyer has a fixed utility for receiving every additional unit of her bundle, and the amount demanded is very large. That is for every buyer \( i \), the inverse demand function can be represented as,

\[
\lambda_i(x) = \lambda_i \quad 0 \leq x \leq L_i \quad L_i \text{ is large.}
\]

The demand is fully elastic but large. Such a “uncapacitated linear” demand has been considered before for applications in Internet routing [9] where the source has to send a large number of packets each of which is equally important. We also assume that the cost functions are strictly convex, i.e. for every given edge \( C_e(x) \) is strictly convex or \( \frac{d}{dx} C_e(x) = c_e(x) \) is increasing. We now claim that in this setting, the Price of Stability is one.

Claim 8.2. In multiple-source multiple-sink markets with Linear Elastic Demand and strictly convex edge costs, there exist a set of prices such that the optimum flow \( (x^*_e) \) is also a Nash Equilibrium.

Remark: The claim holds for actually much weaker assumptions. We do not require that the demand is very large, we just need that if \( x^*_i \) is the flow of buyer \( i \) at the optimum, then \( \lambda_i(x) = \lambda_i \) for \( x \leq x^*_i \). Similarly, we do not require that the costs are strictly increasing, just that \( c_e(x) > \max_i \{\lambda_i\} \) for a large enough \( x \) for all edges.

Proof. The proof is relatively straightforward. If the costs are convex, then there is a unique optimum flow \( (x^*_e) \). Price each edge at \( p_e = c_e(x^*_e) \), where \( x^*_e \) is the total flow on that edge at the optimum. We claim that this marginal pricing results in a Nash Equilibrium. First, no source will send more than \( L_i \) units of flow at the optimum since this additional flow will not result in any utility. Now, observe that by the properties of optimal flows, for any buyer \( i \) with a non-zero flow
\( x^*(i) \) at the optimum and for any \( s_i-t_i \) paths \( P_i \) with non-zero flow and \( P_j \) with no flow, we have

\[
\lambda_i = \sum_{e \in P_i} c_e(x^*_e) = \sum_{e \in P_i} p_e \tag{5}
\]

\[
\lambda_i \leq \sum_{e \in P_j} c_e(x^*) = \sum_{e \in P_j} p_e \tag{6}
\]

Clearly, the buyers are acting as price-takers here as each buyer is only choosing her best-response paths and all paths on which she’s sending flow have a price of \( \lambda_i \), so she has no incentive to increase or decrease her flow. Similarly, buyers who are not sending any flow also do not have any incentive to deviate as their paths are all more expensive than their value for the flow. So no buyer wants to deviate: this is a best-response flow for our prices.

Consider any seller \( e \). By Claim 3.1, if they reduce their price, whatever be the resulting flow, the deviation will not be profitable (this claim still holds even in the presence of multiple buyer types, i.e., multiple sources). Suppose that such a seller increases its price. First assume that there is a non-zero flow \( x^*_e \) on the edge. Then, from Equations 5, 6, for every source \( i \) for whom there is a bundle \( (s-t \text{ path}) \) containing \( e \), and every such bundle \( P_i \), we know \( \sum_{e \in P_i} p_e \geq \lambda_i \). This means that if the seller increase his price, its flow would drop to zero as it is not profitable for any buyer to send flow on a path costing strictly more than \( \lambda_i \). The same applies to edges with no flow on them since they are priced at \( p_e = c_e(0) \). This concludes our proof.

\[\square\]

**Uniform Capacitated Demand where Sources are leaves**

For the previous case with linear elastic demand, we made assumptions on the demand and cost functions but not on the market structure. We now consider a multiple-source multiple-sink market with linear but capacitated demand (each source need no longer have a large flow to send). We do not make assumptions on the cost functions. However, we assume that the market structure is such that every source is a unique leaf node in the graph. In other words, for a given source \( i \), there exists an item \( e \) such that \( e \) belongs to every single bundle that source \( i \) desires. Moreover for any other source \( j \neq i \), \( e \) does not belong to any of \( j \)’s preferred bundles. Informally, the edge is a local monopoly for exactly one source.

Such a network structure is quite common in several real-world networks including Internet, postal and transportation networks \[1, 48\]. The commonality between these physical networks is that there exists a well-connected, easily-accessible central network that links major hubs, but the last-mile between the hub and the final user is often controlled by a local monopoly. It is not feasible for competitors to compete at the last-mile due to the heavy infrastructure costs and minimal returns. Formally, the setting is defined as follows.

There exists a graph \( G \) with several \( s_i-t_i \) pairs representing the buyers. Each buyer is located at a different leaf node \( s_i \) of the graph, i.e., \( s_i \) has a (out)degree of one and \( s_i \neq s_j \) for all \( i, j \). Each buyer has an inverse demand function given by

\[
\lambda(x) = \lambda_i \quad 0 \leq x \leq l_i,
\]

with \( \lambda(x) = 0 \) for \( x > l_i \).

**Claim 8.3.** For instances with multiple types of linear capacitated markets where the sources are all leaf nodes, the Price of Stability is one.

**Proof Sketch** The proof is almost identical to the proof of Claim 8.2 except that we take the optimal flow, price each edge at its marginal and then distribute the remaining surplus price (if any) to the leaf controlling each source.
Series-Parallel networks with no monopolies

In some sense, Claim 3.2 embodies the very essence of the Bertrand Paradox, the fact that competition drives prices to their marginal cost. So it is perhaps surprising that this does not hold in general networks as illustrated by Example 6.2. However, we now show that for a large class of markets which have the series-parallel structure, the absence of monopolies still gives us a PoS of 1. We conjecture that this result is tight in the sense that no larger class of graphs without monopolies give us a PoS of 1 for every possible demand function.

We now extend the definition of a series-parallel graph [27] to one with multiple sources. Series-parallel graphs are commonly used [23,39] to model the substitute and complementary relationship that exists between various products in arbitrary combinatorial markets. Series-parallel graphs also make sense for modeling the combinatorial structure that exists in markets, because often a series-parallel graph can be decomposed down into simpler sub-market structures as in [23].

For any given multiple-source network, the super graph of the given network is the same graph with an additional “super-source” node which has exactly one outgoing edge to each source node $s_i$.

**Definition Multiple-Source Single-Sink Series-Parallel Networks.** A multiple-source single-sink directed network is said to be a series-parallel graph if the corresponding supergraph forms a series-parallel DAG with the super-source as source node and common sink as the sink node.

Observe that any existing single-source series-parallel network trivially falls under this class as adding a super-source still retains the series-parallel behavior of the network. The notion of “no-monopolies” for a complex network has the same idea as a single-source network: there is no single item that monopolizes all the bundles for any given source. In other words, there is no edge in the graph such that its removal would disconnect any source from the sink. We are now in a position to show our main result extending the Bertrand Paradox to arbitrary series-parallel graphs.

**Theorem 8.4.** A multiple-source (MS) series-parallel directed network with no monopolies admits a Nash Equilibrium that has the same social welfare as the optimum solution for any given instance where all edges have $c_e(0) = 0$. Therefore, the Price of Stability is one.

Note that we have made no assumptions on the demand functions, number of sources, or the cost functions. So our result is purely a condition on the market structure.

**Proof Sketch.** The proof relies on some simple properties of series-parallel graphs that we show in the full proof (refer Appendix). We take the optimum flow and price every edge at its marginal cost, i.e., $p_e = c_e(x_e)$. It is not hard to see that for the buyers the paths are indeed best-response paths and that no seller would wish to decrease his price. We show that for every edge $e$ there is a predecessor node $u$ and a successor node $v$ in every $s_i - t$ path through $e$ such that there is another $u-v$ path not containing $e$. So if any seller increases his price, he stands to lose his entire flow and therefore the solution is a Nash Equilibrium. ■.

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A The Inverse Demand Function

Our main aim in this paper is to understand the efficiency of equilibrium allocations for different types of demand. In order to characterize the structure of demand in the market, we used an inverse demand function $\lambda(x)$. Intuitively, the inverse demand function gives us a value $\lambda(x) = v$ such that exactly $x$ of the non-atomic buyers hold a value of $v$ or more for the desired bundles. It
is not hard to see that the total value derived by the buyers when \( x \) buyers buy the desired bundles
is the integral of \( \lambda(x) \) from 0 to \( x \). Alternatively, we could define the inverse demand function for
a market with one atomic buyer with a large demand.

Atomic buyers: While we defined the model above with non-atomic buyers who each desire
an infinitesimal amount of any bundle in \( B_i \), we could equivalently have defined the market as
consisting of a single buyer with a large demand, so that this buyer’s utility is \( \Lambda(x) \) for obtaining \( x \)
amount of any combination of bundles in \( B_i \). Note that this is very different from simply assuming
a single buyer who wants one unit of a good at a fixed value; properties of the utility function \( \Lambda(x) \)
can greatly change the properties of the system.

Direct Demand The inverse demand function captures the price that the market can pay as a
function of the total demand. Such functions are very commonly used in Economics while studying
the efficiency of markets with divisible goods as they provide a simple way to relate social welfare
and prices. On the other hand, in markets with several atomic buyers, it is also natural to model
the distribution of demand using a ‘direct demand function’, which is essentially the inverse of the
inverse demand function. Mathematically, we can represent the direct demand by a non-increasing
function \( D(p) \), such that exactly \( D(p) \) buyers hold a value of \( p \) or more for the bundles.

Let us also define a quantity \( d(p) = -\frac{d}{dp}D(p) \), which in some sense tells us the amount of buyers
who hold a value of exactly \( p \) for the bundles.

We now define some commonly used quantities in terms of the direct and inverse demand
functions.

1. The value \( \frac{dp}{dx} \) is the change in the value or ‘lowest-valued’ buyers when you add or remove
   some buyers. It is not hard to see that this is equal to \( \lambda'(x) \) at some value of \( x \). Similarly,
   the quantity \( \frac{dx}{dp} = -d(p) \).

2. Elasticity of Demand Informally, this is the relative ‘responsiveness’ of the market over the
   sensitivity to price. Mathematically \( \epsilon = \frac{dx/x}{dp/p} \). The elasticity can be rewritten as a function
   of demand \( (x) \) as \( \epsilon(x) = \frac{\lambda(x)}{x\Lambda'(x)} \) because \( p \) as a function of \( x \) is exactly equal to \( \lambda(x) \). The
   elasticity of demand is negative for the vast majority of goods.

3. Constant Elasticity of Demand (CED) This represents the class of (inverse demand
   functions) such that the elasticity of demand is constant, i.e., \( \epsilon(x) = \epsilon \leq 0 \) for all \( x \). It known
that these functions can be represented in the form of \( d(p) = Ap^\epsilon \) where \( \epsilon < 0 \) is referred to
as the elasticity of demand. When \( -1 < \epsilon < 0 \), the demand is said to highly inelastic. As
\( \epsilon \to -\infty \), the demand becomes more and more elastic. Using some simple algebra, the same
inverse demand functions for the CED case are

\[
\lambda(x) = (a - x)^{1/\epsilon} - 1 < \epsilon < 0 \quad \text{Relatively inelastic demand}
\]
\[
\lambda(x) = e^{-x} \quad \epsilon = -1 \quad \text{Monotone Hazard Rate}
\]
\[
\lambda(x) = x^{1/\epsilon} \quad -\infty < \epsilon < -1 \quad \text{Elastic Demand}
\]

Usually the demand for essential commodities (food, electricity) are considered to be inelastic
whereas the demand for non-essentials are somewhat elastic depending on the case.

We defined several classes of demand functions in the paper and show bounds on the Price
of Stability for many of these classes. We now discuss each of these demand functions, their
interpretations and give examples. Although for ease of presentation, we assume that \( \lambda(x) \) is
continuously differentiable, this assumption is not necessary for any of our results, as we argue in Section 7.

Uniform buyers $\lambda(x) = \lambda_0 > 0$ for $x \leq T$. In other words, a population of $T$ buyers all have the same value $\lambda_0$ for the bundles. Alternatively, one or more atomic buyers exist in the market and the total amount demanded by all atomic buyers is $T$. And each atomic buyer receives a value of $\lambda_0$ for every unit of the good(s).

From a traditional Economics point of view, the quantity $\frac{dx}{dp} = 0$ for $p < \lambda_0$.

Concave Demand $\lambda'(x)$ is a non-increasing function of $x$. Informally, $\lambda(x)$ is decreasing faster and faster as $x$ increases. This means that if it so happens that the buyers are in agreement over the value of the bundles, then more buyers are clustered around a higher price than a lower price.

This includes the popular linear inverse demand case ($\lambda(x) = a - x$) [36,49] where the demand drops linearly as price increases. In this case $\frac{dx}{dp}$ is a constant and so changing the price always leads to the same (addition) or removal of buyers. That is buyers at every price are equally sensitive to a change in price. Another example function that we examine later is concave and polynomially decreasing, i.e., $\lambda(x) = a - x^\alpha$ for $\alpha \geq 1$.

The classic way of representing such functions would be to consider $|\frac{dx}{dp}|$ is non-decreasing with $p$.

Monotone Hazard Rate (MHR) Demand This is a strict generalization of Concave Demand. Mathematically, it is the class of functions where $\frac{\lambda'(x)}{\lambda(x)}$ is non-increasing or $h(x) = \frac{|\lambda'(x)|}{\lambda(x)}$ is non-decreasing in $x$. This is equivalent to the class of log-concave functions [7, 12, 40] and essentially captures inverse demand functions without a heavy tail.

While uniformly decreasing demand (linear inverse demand, $\lambda(x) = a - x$) has been assumed more commonly due to its tractable nature, it is more likely that the elasticity of demand is not constant across different prices. Indeed different segments of the market may react differently to a change in price. MHR functions capture a very interesting class of such functions where the responsiveness of a market relative to the value of the buyers is non-decreasing. More concretely if at some price $2p$, an increase of $dp$ in the price leads to a reduction in $dx$ number of buyers. Then at a price $p$, in order to make the same number of consumers ($dx$) drop out, the increase in price has to be at least $\frac{1}{2}dp$. In simple terms, the market cannot be ‘overly sensitive’ at smaller prices compared to its sensitivity at a larger price. Example function: $\lambda(x) = e^{-x}$.

Such functions denote the case where $\frac{dx}{dp/p}$ is non-decreasing with $p$.

Monotone Price Elasticity (MPE) $xh(x) = \frac{x|\lambda'(x)|}{\lambda(x)}$ is a non-decreasing function of $x$ which tends to zero as $x \to 0$. This is equivalent to functions where the price elasticity of demand is non-decreasing as the price increases. Notice that $\epsilon(x) = \frac{1}{xh(x)}$. So since $xh(x)$ is non-decreasing, this means that $\epsilon(x)$ is also non-decreasing with $x$. If we look at the absolute value of $\epsilon(x)$, it is decreasing in $x$, which means that the market cannot be too elastic at a lower price as compared to a higher price. Or buyers who value the item more are more sensitive.

In addition to all papers that capture MHR, Concave and Uniform demand, several other papers have also looked at these kind of functions [2,8,14,17,19,39].
While uniformly decreasing demand (linear inverse demand, \(\lambda(x) = a - x\)) has been assumed more commonly due to its tractable nature, it is more likely that the elasticity of demand is not constant across different prices. Indeed different segments of the market may react differently to a change in price. MHR functions capture a very interesting class of such functions where the responsiveness of a market relative to the value of the buyers is non-decreasing. More concretely if at some price \(2p\), an increase of \(dp\) in the price leads to a reduction in \(dx\) number of buyers. Then at a price \(p\), in order to make the same number of consumers \((dx)\) drop out, the increase in price has to be at least \(\frac{1}{2}dp\). In simple terms, the market cannot be ‘overly sensitive’ at smaller prices compared to its sensitivity at a larger price.

Each class of demand functions contains all the previously defined classes, i.e., Uniform \(\subseteq\) Concave \(\subseteq\) MHR \(\subseteq\) MPE. We also extend these definitions to functions which may not be differentiable everywhere. However, it is well known that if the \(\lambda(x)\) is monotone, then its derivative exists ‘almost everywhere’. In this case, we define the MPE class to be obey monotonicity where the derivative is defined and at any \(x_0\) where it is not defined, we require

\[
\lim_{x \to x_0} x \frac{|\lambda'(x)|}{\lambda(x)} \leq \lim_{x \to x_0^+} x \frac{|\lambda'(x)|}{\lambda(x)}.
\]

We also consider specific demand functions that have more specific forms. \(\lambda(x) = a - x^\alpha\) for \(\alpha \geq 1\) which are concave but essentially more ‘concave’ than simple linear inverse demand. \(\lambda(x) = (a - x)^\alpha\), which belong to the class MHR but are relatively more inelastic. Notice that linear inverse demand functions form some sort of a border between these two classes.

Finally, we look at functions of the form \(\lambda(x) - r(x) = |\ln(x/a)|^{1/\alpha}\) where \(\alpha \geq 1\). Ignoring the structure of the cost function, this essentially means that the direct demand is exponentially decreasing.

### B Min-Cost Flows

In this section we prove Proposition 2.1 establishing the properties of the min-cost flow value \(R(x)\).

**Flow Allocations:** We now discuss flow allocations, which we use only in our proofs. Given a flow \(\vec{x}\) whose total magnitude is \(x\), we represent the flow by an allocation vector \(\alpha\), such that the flow on edge \(e\) is \(\alpha_e x = x_e\) for \(\alpha_e \leq 1\). Then, we can define the total cost as a function of just \(x\), i.e., \(C^\alpha(x) = \sum e C_e(\alpha_e x)\) for a fixed allocation rule \(\alpha\). We also define a total differential cost function, the marginal cost of sending additional flow according to a given allocation.

\[c^\alpha(x) = \frac{d}{dx} C^\alpha(x) = \sum e \frac{d}{dx} C_e(\alpha_e x) = \sum e \alpha_e C_e(\alpha_e x).\]

We are now in a position to prove some properties of the min-cost function \(R(x)\). Intuitively, it also seems like \(R(x)\) must be continuous since increasing the flow by a small amount should not lead to a large increase in the cost. We begin by showing this formally.

**Proposition B.1.** \(R(x)\) is continuous \(\forall x\).

**Proof.** Since \(R\) is a non-decreasing function, we know the following inequality must hold: \(\lim_{\epsilon \to 0} R(x-\epsilon) \leq R(x) \leq \lim_{\epsilon \to 0} R(x+\epsilon)\). Let \(R(x) = C^\alpha(x)\), i.e., \(\alpha\) is the optimal allocation for the min-cost flow of value \(x\). Then for any \(\epsilon > 0\), \(R(x+\epsilon) \leq C^\alpha(x+\epsilon)\), simply because \(R(x+\epsilon)\) is the cost of the best allocation, which includes the allocation \(\alpha\). Taking the limit on both sides as \(\epsilon\) tends to zero, we get that \(\lim_{\epsilon \to 0} R(x+\epsilon)\) cannot be any larger than \(C^\alpha(x) = R(x)\). Thus we get that \(\lim_{\epsilon \to 0} R(x+\epsilon) = R(x)\) for all \(x\).
Moving on to the other limit, suppose that $\lim_{\epsilon \to 0} \mathcal{R}(x - \epsilon) = r_1 < \mathcal{R}(x) = \lim_{\epsilon \to 0} \mathcal{R}(x + \epsilon)$. For some sufficiently small $\epsilon$, let $\vec{x}_1$ be the min-cost flow corresponding to $\mathcal{R}(x - \epsilon)$ and $\vec{x}_2$ be the flow corresponding to $\mathcal{R}(x + \epsilon)$. Consider the flow $\vec{x} = \frac{1}{2}\vec{x}_1 + \frac{1}{2}\vec{x}_2$. Clearly, this is a feasible flow of magnitude $x$. Moreover, since $C$ is convex, we know that $C(\vec{x}) \leq \frac{1}{2}C(\vec{x}_1) + \frac{1}{2}C(\vec{x}_2) < \mathcal{R}(x)$. The last inequality holds because for some sufficiently small $\epsilon$, the convex combination $\frac{1}{2}C(\vec{x}_1) + \frac{1}{2}C(\vec{x}_2)$ lies in between $r_1$ and $\mathcal{R}(x)$. This is a contradiction since $\mathcal{R}(x)$ is the minimum cost of a flow of magnitude $x$. Thus $\mathcal{R}(x)$ is continuous at $x$.

Recall the derivative of the min-cost flow $r(x) = \frac{d}{dx}\mathcal{R}(x)$. Although $\mathcal{R}(x)$ is continuous, it is not clear that it is differentiable, so we define the appropriate left and right hand derivatives $r^-(x)$, $r^+(x)$ according to the first principles. We now show that these two are the same. Then, we show that the derivative $r(x)$ is continuous and non-decreasing in $x$.

**Proposition B.2.** $\mathcal{R}(x)$ is differentiable for all $x$.

**Proof.** Recall the differential cost function for a fixed allocation $c^\alpha(x) = \sum c_{\alpha} c_{\epsilon} (c_{\epsilon} x)$. Since $C_{\epsilon}$ is differentiable, then $c^\alpha$ is continuous. More precisely, $c^\alpha(x) = \lim_{\epsilon \to 0} \frac{C^\alpha(x + \epsilon) - C^\alpha(x)}{\epsilon}$.

Let $\alpha(x)$ denote the optimal flow allocation for a flow of size $x$ (i.e., $\mathcal{R}(x) = C^\alpha(x)(x)$). Then, we know that

$$r^-(x) = \lim_{\epsilon \to 0} \frac{\mathcal{R}(x) - \mathcal{R}(x - \epsilon)}{\epsilon} = \lim_{\epsilon \to 0} \frac{C^\alpha(x) - C^\alpha(x - \epsilon)(x - \epsilon)}{\epsilon},$$

$$r^+(x) = \lim_{\epsilon \to 0} \frac{\mathcal{R}(x + \epsilon) - \mathcal{R}(x)}{\epsilon} = \lim_{\epsilon \to 0} \frac{C^\alpha(x + \epsilon)(x + \epsilon) - C^\alpha(x)}{\epsilon}.$$

From the above definitions, we see that $r^-(x)$ is always greater than $c^\alpha(x)(x)$. This is true because $\mathcal{R}(x - \epsilon) \leq C^\alpha(x)(x - \epsilon)$ since the former is the min-cost flow of that magnitude, and the latter may be using a suboptimal allocation. Similarly, $\mathcal{R}(x + \epsilon) \leq C^\alpha(x)(x + \epsilon)$, so we have $r^+(x) \leq c^\alpha(x)(x)$. This gives us our first bound,

$$r^+(x) \leq c^\alpha(x)(x) \leq r^-(x).$$

Now, suppose that $r^+(x) < r^-(x)$. This would imply that

$$\lim_{\epsilon \to 0} \frac{\mathcal{R}(x) - \mathcal{R}(x - \epsilon)}{\epsilon} > \lim_{\epsilon \to 0} \frac{\mathcal{R}(x + \epsilon) - \mathcal{R}(x)}{\epsilon}.$$

Thus there exists a sufficiently small $\epsilon_0$ such that $\mathcal{R}(x) - \mathcal{R}(x - \epsilon_0) > \mathcal{R}(x + \epsilon_0) - \mathcal{R}(x)$ which implies $\mathcal{R}(x) > \frac{1}{2} (\mathcal{R}(x + \epsilon_0) + \mathcal{R}(x - \epsilon_0))$.

Let the flows corresponding to $\mathcal{R}(x + \epsilon)$ and $\mathcal{R}(x - \epsilon)$ be $\vec{x}_1$ and $\vec{x}_2$ respectively. Once again, take the average flow $\vec{x} = \frac{1}{2}(\vec{x}_1 + \vec{x}_2)$. This flow has a magnitude of $x$ and its cost $C(\vec{x}) \leq \frac{1}{2}(C(\vec{x}_1) + C(\vec{x}_2)) = \frac{1}{2}(\mathcal{R}(x + \epsilon) + \mathcal{R}(x - \epsilon)) < \mathcal{R}(x)$. However, this is a contradiction since no feasible flow of magnitude $x$ can have a cost less than $\mathcal{R}(x)$. So we conclude that $r^+(x) \geq r^-(x)$. So finally, we have

$$r^+(x) = c^\alpha(x)(x) = r^-(x).$$

Moreover, since all $C_{\epsilon}$ are differentiable, then $c^\alpha(x)(x)$ is always finite, thus proving the desired claim that $r(x)$ always exists and is continuous. □
**Proposition B.3.** $R(x)$ is convex.

*Proof.** Consider arbitrary flow values $x_1$ and $x_2$, and let $x = \frac{1}{2}x_1 + \frac{1}{2}x_2$. Suppose to the contrary that $R(x) > \frac{1}{2}R(x_1) + \frac{1}{2}R(x_2)$. Now consider the flow vector $\bar{x} = \frac{1}{2}x_1' + \frac{1}{2}x_2'$ where $x_1', x_2'$ are the flow vectors corresponding to the minimum cost flows at $x_1$, $x_2$ respectively. Clearly $\bar{x}$ has magnitude $x$. Moreover since $C$ is convex, $C(\bar{x}) \leq \frac{1}{2}(C(x_1') + C(x_2')) = \frac{1}{2}(R(x_1) + R(x_2)) < R(x)$. This is a contradiction as no flow of magnitude $x$ can have a cost less than $R(x)$, and so $R(x)$ must be convex. \qed

Now that we have a good understanding of the min-cost function $R(x)$, we show that if we take the min-cost flow at any value $x$, the cost of sending an additional infinitesimal flow $dx$ equals the marginal cost of any one path with flow on it. First we show some simple lemmas that yield some insight on the allocation on the min-cost flow.

**Lemma B.4.** For any given $x$, let $\bar{x}$ be the minimum cost flow vector and $P_i, P_j$ be any two paths with non-zero flow in $\bar{x}$. Then, $\sum_{e \in P_i} c_e(x_e) = \sum_{e \in P_j} c_e(x_e)$. If $P_i$ is a path with no flow and $P_j$ has non-zero flow in the minimum cost flow, then $\sum_{e \in P_i} c_e(x_e) \geq \sum_{e \in P_j} c_e(x_e)$.

*Proof.** The lemma can be formally proved using the Karush-Kuhn-Tucker conditions for the min-cost flow optimization program. However, observe any cost minimizing flow must also be a local optimum (for a convex program, the local and global optima coincide). If the above lemma were not true, and for two such paths suppose that $\sum_{e \in P_i} c_e(x_e) > \sum_{e \in P_j} c_e(x_e)$, without loss of generality. Then, consider a new flow with the flow on $P_i$ reduced by some sufficiently small $\epsilon$ and the flow on $P_j$ increased by the same amount. If the cost of the old flow is $z$, then the cost of the new flow is

$$z + \sum_{e \in P_j} [C_e(x_e + \epsilon) - C_e(x_e)] + \sum_{e \in P_i} [C_e(x_e) - C_e(x_e - \epsilon)]$$

$$= z + \epsilon \left( \sum_{e \in P_j} c_e(x_e) - \sum_{e \in P_i} c_e(x_e) \right)$$

$$< z,$$

which is a contradiction because $z$ is minimum cost among all feasible flows supporting a flow magnitude of $x$. The same argument works for switching a small amount of flow to a path with zero flow. \qed

**Lemma B.5.** For a minimum cost flow vector $\bar{x}$ of magnitude $x$, and for any path $P_i$ with $x_{P_i} > 0$, we have $r(x) = \sum_{e \in P_i} c_e(x_e)$.

*Proof.** From the proof of $R$ being differentiable, we know that if $\alpha$ is the min-cost allocation for a flow of magnitude $x$, then $e^\alpha(x) = r(x)$. So we just have to prove that $e^\alpha(x) = \sum_{e \in P_i} c_e(x_e)$ for any path with non-zero flow.

For every path $P_i$ with non-zero flow, we know by Lemma B.4 that $\sum_{e \in P_i} c_e(x_e)$ is the same. Let the value of this quantity by $y$. We also know that for any given edge $e$, $\alpha_e = \sum_{P_i \ni e} \alpha_{P_i}$ by
definition. So if \( P \) is the set of paths with non-zero flow, we have

\[
\sum_{e} \alpha_e c_e(x_e) = \sum_{P_i \in P} \sum_{e \in P_i} \alpha_{P_i} c_e(x_e)
\]

\[
= \sum_{P_i \in P} \alpha_{P_i} \sum_{e \in P_i} c_e(x_e)
\]

\[
= y \sum_{P_i \in P} \alpha_{P_i} = y
\]

\[ \]

C Proofs from Section 3: Existence and Computation of Equilibrium Prices

Lemma 3.3. Given a solution \( (\vec{p}, \vec{x}) \) with \( \vec{x} \) a best-response flow to prices \( \vec{p} \), with \( \lambda(\vec{x}) > 0 \) and \( \lambda(\vec{x}) \geq \vec{x} \mid \lambda'(\vec{x}) \), we have that no seller \( e \) can increase his price and improve profits as long as either one of the following conditions hold,

1. The good \( e \) is tight, i.e., \( \exists \) some \( s-t \) path that does not contain the edge \( e \) and has the same total price as the flow-carrying \( s-t \) paths that do contain \( e \) (or)

2. \( \bar{p}_e \geq \vec{x} \mid \lambda'(\vec{x}) \)

Proof. The first part of the lemma is fairly trivial: let \( P_1 \) be any flow-carrying path that contains \( e \) and \( P_2 \) be a path not containing \( e \) but having the same overall price as \( P_1 \). Since buyers always buy the best-response bundles, if seller \( e \) increases his price, the price of \( P_1 \) (or any other flow carrying path containing \( e \)) would become strictly larger than that of \( P_2 \). All the buyers that originally purchased from this seller would now shift to using \( P_2 \) or any other path not containing \( e \), and seller \( e \)'s profit would become zero which cannot be strictly larger than his original profit.

Suppose an edge \( e \) is not tight and a seller who obeys \( \bar{p}_e \geq \vec{x} \mid \lambda'(\vec{x}) \) can increase his price (perhaps to some extent) and still have the cheapest paths pass through this edge. In this case, it is necessary that the entire flow \( \vec{x} \) must be using this edge. To see why, suppose if the flow on this edge \( x_e < \vec{x} \), then there must be at least one other \( s-t \) path not containing \( e \) with non-zero flow on it. Since all the paths with non-zero flow must have the same price \( \lambda(\vec{x}) \), this boils down to our first condition of \( e \) being tight since there is an alternative path with the same price. Therefore, \( x_e = \vec{x} \) and every flow carrying path contains \( e \).

Now suppose that seller \( e \) increases his price from \( p_e \) to \( p'_e \), and the resulting flow on the edge is \( x \). Then, it is not hard to see that \( x < \vec{x} \), because the price of every flow carrying path has increased by a non-zero amount \( (p'_e - p_e) \) and if the flow remained the same, then it would mean that \( \lambda(x) \) is not uniquely defined at \( \vec{x} \). We first establish the relation between \( p'_e \) and \( \lambda(x) \), namely that \( p'_e - p_e = \lambda(x) - \lambda(\vec{x}) \). To see why, notice that the flow of size \( x \) is a best-response flow to the prices in our original solution, except with price \( p'_e \) on edge \( e \). As we argued above, all cheapest paths in our given solution \( \vec{x} \) have cost \( \lambda(\vec{x}) \); thus all cheapest paths in this new pricing have cost \( \lambda(\vec{x}) - p_e + p'_e \). Since this pricing results in a flow of size \( x \), it must be that \( \lambda(x) = \lambda(\vec{x}) - p_e + p'_e \); the
buyers who purchase the cheapest bundles are exactly the ones who value them more than \( \lambda(x) \). Thus, we know that \( p_e' = p_e + \lambda(x) - \lambda(\hat{x}) \).

Now consider seller \( e \)'s original profit, \( p_e\hat{x} - C_e(\hat{x}) \). After \( e \) changes its price to \( p_e' \), this profit become \( p_e'x - C_e(x) \), which by the above argument equals \( [p_e + \lambda(x) - \lambda(\hat{x})]x - C_e(x) \). We want to show that this new profit is at most the old profit as long as Condition (2) is obeyed. Define \( \pi(x) = [p_e + \lambda(x) - \lambda(\hat{x})]x - C_e(x) \). We will prove that in the domain \( (0, \hat{x}] \), \( \pi(x) \) is maximized when \( x = \hat{x} \), thus implying that no matter what amount the seller increases his price by, at the resulting flow of magnitude \( x \), his profit cannot be strictly larger than the original profit.

We now proceed to prove that \( \pi(x) \) is maximized at \( \hat{x} \). Specifically, we look at the derivative of \( \pi(x) \) and show that it is non negative for \( x \leq \hat{x} \). Recall that the seller obeys Condition (2) which implies,

\[
p_e = \bar{p}_e + c_e(\hat{x}) \geq \hat{x}|\lambda'(x)| + c_e(\hat{x}).
\]

Since

\[
\pi(x) = [p_e + \lambda(x) - \lambda(\hat{x})]x - C_e(x).
\]

Thus the derivative of \( \pi(x) \) is equal to

\[
\pi'(x) = p_e + \lambda(x) - \lambda(\hat{x}) + x\lambda'(x) - c_e(x)
= \bar{p}_e + c_e(\hat{x}) + \lambda(x) - \lambda(\hat{x}) + x\lambda'(x) - c_e(x)
\geq \lambda(x) - \lambda(\hat{x}) + \hat{x}|\lambda'(\hat{x})| - x|\lambda'(x)| + (c_e(\hat{x}) - c_e(x))
\]

Since the last term in the parenthesis, \( c_e(\hat{x}) - c_e(x) \) is non-negative (\( c_e \) is non-decreasing), in order to show that \( \pi'(x) \geq 0 \) for all \( x \leq \hat{x} \), it suffices if we show the first terms are non-negative. The following proposition implies that for MPE functions, this is indeed true.

**Proposition C.1.** For \( \lambda \in MPE \) and \( M \geq 1 \), we have that \( \lambda(\hat{x}) - \lambda(x) \leq \hat{x}|\lambda'(\hat{x})| - x|\lambda'(x)| \) for \( x < \hat{x} \).

**Proof.** First suppose that \( |\lambda'(\hat{x})| > 0 \). Let \( A = - (\lambda(\hat{x}) - \lambda(x)) \) and \( B = - (\hat{x}|\lambda'(\hat{x})| - x|\lambda'(x)|) \); we want to show that \( A \geq B \). Suppose to the contrary that \( A < B \). \( A \) is non-negative since \( \lambda \) is non-increasing, and \( B \) is nonnegative since \( B > A \). We know by the property of MPE functions that

\[
\frac{\lambda(\hat{x})}{\hat{x}|\lambda'(\hat{x})|} \leq \frac{\lambda(x)}{x|\lambda'(x)|} = \frac{\lambda(\hat{x}) + A}{\hat{x}|\lambda'(\hat{x})| + B}.
\]

Let \( C = \lambda(\hat{x}) \) and \( D = \hat{x}|\lambda'(\hat{x})| \), and consider how \( C/D \) compares with \( (C + B)/(D + B) \). First, notice that we have at \( \hat{x} \),

\[
C = \lambda(\hat{x}) \geq \lambda(\hat{x}) - r(\hat{x}) \geq M\hat{x}|\lambda'(\hat{x})| \geq \hat{x}|\lambda'(\hat{x})| = D.
\]

Therefore, \( C \geq D \). Thus we know that \( C(D + B) \geq D(C + B) \), and thus \( \frac{C}{D} \geq \frac{C + B}{D + B} \). By our assumption that \( A < B \), this implies that \( \frac{C}{D} > \frac{C + A}{D + B} \), i.e.,

\[
\frac{\lambda(\hat{x})}{\hat{x}|\lambda'(\hat{x})|} > \frac{\lambda(\hat{x}) + A}{\hat{x}|\lambda'(\hat{x})| + B}.
\]

This contradicts Inequality (7) above, thus proving that \( A \geq B \).

What if \( \lambda'(\hat{x}) = 0 \)? Since MPE functions must have non-decreasing \( \frac{x|\lambda'(x)|}{\lambda(x)} \), we can only conclude that for all \( 0 < x \leq \hat{x} \), \( \lambda'(x) = 0 \). In this case, it is not hard to see that the proposition trivially holds since \( \lambda(\hat{x}) = \lambda(x) \) and the RHS is zero as well. \( \square \)
We have therefore shown that for all $0 < x < \bar{x}$, $\pi'(x)$ is non-negative. This means that $\pi(x)$ is non-decreasing in this region and therefore maximized at $x = \bar{x}$ in the domain $[0, \bar{x}]$. Therefore, no ‘monopoly edge’ can benefit by changing its price, as desired. A small remark on how much the monopoly can actually increase its price by is in order here. It may be possible (as we show later for the case with Virtual monopolies in Section 7) that a monopoly may only be able to increase his price up to some amount and so may not be able to obtain every possible flow size $x < \bar{x}$. That said, the above proof is still valid for this case because we have shown that in every possible domain $[x, \bar{x}]$, $\pi(x)$ is non-decreasing. So our statement is much stronger. Also notice that if $\bar{x} = T$, then $\lambda'(T)$ has to be finite if the prices are finite; the above argument works for this case as well.

**Lemma 3.4** Given a solution $(\vec{p}, \vec{x})$ with $\vec{x}$ a best-response flow to prices $\vec{p}$, satisfying $\lambda(\vec{x}) \geq \vec{x} |\lambda'(\vec{x})|$, no seller $e$ can decrease his price and improve profits as long as any one of the following conditions hold,

1. $\bar{x}_e = \bar{x} = T$ or
2. $p_{e} = 0$ or
3. $p_{e} \leq \bar{x} |\lambda'(\bar{x})|$ and $\bar{x}_e = \bar{x}$

**Proof.** Once again the first two conditions are easy to prove. If $\bar{x}_e = T$, it just means that there are no more buyers left in the market. So for any such seller, a decrease in price is not going to lead to any additional flow. In the second case, $p_{e} = 0$ simply means that a seller is priced at its marginal price and so by Lemma 3.1 no such seller would wish to decrease his price. For the final condition, suppose that $\bar{x} < T$ and $p_{e} > 0$.

Notice that as with the previous proof, any decrease in price from $p_{e}$ to $p'_{e}$ will result in a flow of magnitude $x > \bar{x}$. As we argued previously, since buyers will only indulge in best-response behavior, it is necessary that $p'_{e} = p_{e} + \lambda(\bar{x}) - \lambda(x)$.

We want to show that the new profit a seller makes is at most the old profit as long as Condition (3) is obeyed. Consider seller $e$’s profit, $\pi(x) = [p_{e} + \lambda(x) - \lambda(\bar{x})]x - C_{e}(x)$. We will prove that in the domain $[\bar{x}, T]$, $\pi(x)$ is maximized when $x = \bar{x}$, thus implying that no matter what amount the seller decreases his price by, at the resulting flow of magnitude $x$, his profit cannot be strictly larger than the original profit.

We now proceed to prove that $\pi(x)$ is maximized at $\bar{x}$. Specifically, we look at the derivative of $\pi(x)$ and show that it is not positive for $x \geq \bar{x}$. Recall that the seller obeys Condition (3) which implies,

$$p_{e} = \bar{p}_{e} + c_{e}(\bar{x}) \leq \bar{x} |\lambda'(\bar{x})| + c_{e}(\bar{x}).$$

Since

$$\pi(x) = [p_{e} + \lambda(x) - \lambda(\bar{x})]x - C_{e}(x).$$

Thus the derivative of $\pi(x)$ is equal to

$$\pi'(x) = p_{e} + \lambda(x) - \lambda(\bar{x}) + x\lambda'(x) - c_{e}(x)$$

$$= \bar{p}_{e} + c_{e}(\bar{x}) + \lambda(x) - \lambda(\bar{x}) + x\lambda'(x) - c_{e}(x)$$

$$\leq \lambda(x) - \lambda(\bar{x}) + \bar{x} |\lambda'(\bar{x})| - x |\lambda'(x)| + (c_{e}(\bar{x}) - c_{e}(x))$$

Since the last term in the parenthesis, $c_{e}(\bar{x}) - c_{e}(x)$ is not positive ($c_{e}$ is non-decreasing), in order to show that $\pi'(x) \leq 0$ for all $x \geq \bar{x}$, it is sufficient to show the first terms are not positive either. The following proposition implies that for MPE functions, this is indeed true.
Lemma C.2. For $\lambda \in MPE$ and $M \geq 1$, we have that $\lambda(\bar{x}) - \lambda(x) \geq \bar{x}|\lambda'(\bar{x})| - x|\lambda'(x)|$ for $x > \bar{x}$.

Proof. First notice that if $\lambda'(\bar{x}) = 0$, the inequality is trivially true. $\lambda'(\bar{x})$ is also bonded because the only case where it is not is when $\bar{x} = T$ and for that case we already know that no seller will decrease the price. Now, let $A = \lambda(\bar{x}) - \lambda(x)$ and $B = \bar{x}|\lambda'(\bar{x})| - x|\lambda'(x)|$; we want to show that $A \geq B$. Suppose to the contrary that $A < B$. We know by the property of MPE functions that

$$\frac{\lambda(\bar{x})}{\bar{x} |\lambda'(\bar{x})|} \geq \frac{\lambda(x)}{x |\lambda'(x)|} = \frac{\lambda(\bar{x}) - A}{\bar{x} |\lambda'(\bar{x})| - B}. \quad (8)$$

Let $C = \lambda(\bar{x})$ and $D = \bar{x}|\lambda'(\bar{x})|$, and consider how $C/D$ compares with $(C - A)/(D - A)$. First, notice that $C \geq D$: this is because by our choice of $\bar{x}$, we have that $\lambda(\bar{x}) - r(\bar{x}) = M\bar{x}|\lambda'(\bar{x})|$, and we can assume that $M \geq 1$ since $e$ is a monopoly edge. Also, $D \geq B > A$, so we know that $C(D - A) \leq D(C - A)$, and thus $C/D \leq C - A < C - A$. By our assumption that $A < B$, this implies that $C/D \leq C - A < C - A$, i.e.,

$$\frac{\lambda(\bar{x})}{\bar{x} |\lambda'(\bar{x})|} \leq \frac{\lambda(\bar{x}) - A}{\bar{x} |\lambda'(\bar{x})| - B}.$$ 

This contradicts Inequality (8) above, thus proving that $A \geq B$.

Therefore, we have shown that for MPE functions, as long as condition (3) is obeyed, $\pi'(x)$ is positive for all $\bar{x} \leq x \leq T$. So it means that $\pi(x)$ is non increasing in this domain and therefore it is maximized at $x = \bar{x}$. Also notice that $\pi(x)$ is a continuous function of $x$. So even if $\lambda'(T)$ is not defined, we know that for all $x < T$, $\pi(x) \leq \pi(\bar{x})$. This means that $\pi(T) = \lim_{x \to T} \pi(x) \leq \pi(\bar{x})$ as well.

Claim C.3. Suppose that for all $x$ in the interval $[x_1, x^*]$, $Mx|\lambda'(x)| \leq \lambda(x)$ and $\lambda(x_1) > 0$, then $\lambda(x^*) > 0$ as long as $x_1$ is sufficiently close to $x^*$. Therefore, if $\lambda(x^*) = 0$, in any such small interval, there must exist some $x$ satisfying $Mx|\lambda'(x)| > \lambda(x)$.

Proof. Since $Mx|\lambda'(x)| \leq \lambda(x)$ for all $x$ in the interval, we have

$$\lambda(x^*) = \lambda(x_1) - \int_{x=x_1}^{x^*} |\lambda'(x)| dx$$

$$\geq \lambda(x_1) - \int_{x=x_1}^{x^*} \frac{\lambda(x)}{Mx} dx$$

$$\geq \lambda(x_1) - \int_{x=x_1}^{x^*} \frac{\lambda(x_1)}{Mx} dx$$

$$= \lambda(x_1) - \frac{1}{M} \lambda(x_1) \log \left( \frac{x^*}{x_1} \right)$$

$$= \lambda(x_1) \left( 1 - \frac{1}{M} \log \left( \frac{x^*}{x_1} \right) \right)$$

So what this tells us is that if $\lambda(x^*) = 0$, then if we take some reasonably close point where $\lambda(x_1) > 0$, then in the domain $[x_1, x^*]$ at least some $x$ must satisfy $Mx|\lambda'(x)| > \lambda(x)$.

Lemma C.4. For every equilibrium of magnitude $\bar{x}$ obeying our pricing rule for MPE demand, either the equilibrium is optimal or it satisfies $\lambda(\bar{x}) - r(\bar{x}) = M\bar{x}|\lambda'(\bar{x})|$. 

50
Proof. Suppose that the equilibrium is not optimal, satisfies our pricing rule but does not meet the above condition, i.e., \( \lambda(\tilde{x}) - r(\tilde{x}) \neq M\tilde{x}|\lambda'(\tilde{x})| \). Then, every monopoly must be priced at an increased price of \( p_e = \frac{\lambda(\tilde{x}) - r(\tilde{x})}{M} \) by our pricing rule. First assume that \( \lambda(\tilde{x}) - r(\tilde{x}) < M\tilde{x}|\lambda'(\tilde{x})| \). Then we claim that every monopoly can increase its price by some small finite amount and improve its profit. Let us return to the derivative of our profit function,

\[
\pi'(x) = p_e + \lambda(x) - \lambda(\tilde{x}) + x\lambda'(x) - c_e(x).
\]

Since \( p_e = \frac{\lambda(\tilde{x}) - r(\tilde{x})}{M} + c_e(\tilde{x}) < \tilde{x}|\lambda'(\tilde{x})| + c_e(\tilde{x}) \), \( \pi'(x) \) is strictly negative. Moreover since \( \pi'(x) \) is continuous since \( \lambda(x) \) is continuous, it is clear that at some \( x < \tilde{x} \), \( \pi(x) > \pi(\tilde{x}) \) and so some monopoly can increase its price and also its profit.

Similarly, consider \( \lambda(\tilde{x}) - r(\tilde{x}) > M\tilde{x}|\lambda'(\tilde{x})| \), we can show that \( \pi'(x) \) is positive so \( \pi(x) > \pi(\tilde{x}) \) for some \( x \) arbitrarily close to \( \tilde{x} \) but larger. So it means that the edge can decrease its price by a small amount and improve its profit.

\[\square\]

**Proposition C.5.** Let \( x^* \) denote the optimal flow and \((\tilde{p}, \tilde{x}) \) be a Nash Equilibrium that obeys our pricing rule. Then, \( \tilde{x} \leq x^* \).

Proof. We have already established that \( \tilde{x} = x^* \) in the absence of monopolies so we can now focus on instances containing at least one monopoly edge. Assume by contradiction that \( x^* < \tilde{x} \). Without loss of generality, also assume that the optimal solution is unique.

By our pricing rule, we know that any monopoly edge \( e \) has a price \( p_e = \frac{\lambda(\tilde{x}) - r(\tilde{x})}{M} + c_e(\tilde{x}) \). However, since \( \tilde{x} \) is strictly non optimal, it means that \( \lambda(\tilde{x}) < r(\tilde{x}) \) and so the first term in \( p_e \) is negative, and therefore \( p_e < 0 \). That is, the monopoly priced below its marginal cost. Also note that at this point, \( \lambda(\tilde{x}) - r(\tilde{x}) < 0 \leq M\tilde{x}|\lambda'(\tilde{x})| \). So by Lemma C.4, any monopoly edge can increase its price and make a larger profit.

\[\square\]

**Lemma C.6.** Let \( \tilde{x} < x^* \) be some point satisfying \( \lambda(\tilde{x}) - r(\tilde{x}) = M\tilde{x}|\lambda'(\tilde{x})| \). Then, all \( x < \tilde{x} \) must satisfy \( \lambda(x) - r(x) > M\tilde{x}|\lambda'(\tilde{x})| \) and all \( x > \tilde{x} \) must satisfy \( \lambda(x) - r(x) < M\tilde{x}|\lambda'(\tilde{x})| \) as long as the demand is MPE and the production costs are non-zero.

Proof. First assume that at \( x = \tilde{x} \), \( \lambda'(\tilde{x}) < 0 \), i.e., \( \lambda(x) \) is strictly decreasing at this point. Consider \( x > \tilde{x} \), we show that \( \lambda(x) - r(x) < Mx|\lambda'(x)| \). Since \( \lambda(x) \) is strictly decreasing at \( \tilde{x} \), \( \forall x > \tilde{x} \), \( \lambda(x) < \lambda(\tilde{x}) \). Now since \( \lambda(x) \in \text{MPE} \),

\[
\frac{\lambda(x)}{x|\lambda'(x)|} < \frac{\lambda(\tilde{x})}{\tilde{x}|\lambda'(\tilde{x})|}.
\]

We claim that since \( r(x) \geq r(\tilde{x}) > 0 \),

\[
\frac{\lambda(x) - r(x)}{x|\lambda'(x)|} \leq \frac{\lambda(x) - r(\tilde{x})}{x|\lambda'(x)|} < \frac{\lambda(\tilde{x}) - r(\tilde{x})}{\tilde{x}|\lambda'(\tilde{x})|} = M.
\]

To see why the last inequality is strict in the above equation, first suppose that \( x|\lambda'(x)| \leq \tilde{x}|\lambda'(\tilde{x})| \). In this case, the last inequality trivially holds due to equation 9 and the fact that \( \lambda(x) < \lambda(\tilde{x}) \). If this is not the case and that \( x|\lambda'(x)| > \tilde{x}|\lambda'(\tilde{x})| \). Assume by contradiction that the last equality fails to hold strictly. This must mean that

\[
r(\tilde{x})(\frac{1}{\tilde{x}|\lambda'(\tilde{x})|} - \frac{1}{x|\lambda'(x)|}) \geq \frac{\lambda(\tilde{x})}{\tilde{x}|\lambda'(\tilde{x})|} - \frac{\lambda(x)}{x|\lambda'(x)|} > \lambda(\tilde{x})(\frac{1}{\tilde{x}|\lambda'(\tilde{x})|} - \frac{1}{x|\lambda'(x)|}),
\]

which in turn implies that \( r(\tilde{x}) > \lambda(\tilde{x}) \), a contradiction. This completes the proof that \( \lambda(x) - r(x) < Mx|\lambda'(x)| \) holds for all \( x > \tilde{x} \).
Similarly suppose that $x < \tilde{x}$, we can show that $\lambda(x) - r(x) > Mx|\lambda'(x)|$ in the same fashion. Now, what if $\lambda'(\tilde{x}) = 0$? The equilibrium satisfies the condition $\lambda(\tilde{x}) - r(\tilde{x}) = M\tilde{x}|\lambda'(\tilde{x})|$ which in turn equals zero. This means that $\lambda(\tilde{x}) = r(\tilde{x})$ and so $\tilde{x}$ has to be one of the solutions maximizing social welfare. However, we have assumed $\tilde{x} < x^*$ for any optimal solution and so this case is not possible.

Theorem 3.8. For $\lambda$ in MPE, either the equilibrium admits the optimal flow, or there is a unique equilibrium obeying our pricing rule as long as production costs are non-zero.

Proof. First, we have already shown existence in Theorem 3.6 so we know there exists at least one equilibrium obeying our pricing rule. Now suppose that this equilibrium is not optimal, then by Proposition C.5 it is true that $\tilde{x} < x^*$. By Lemma C.4 this equilibrium satisfies $\lambda(\tilde{x}) - r(\tilde{x}) = M\tilde{x}|\lambda'(\tilde{x})|$. Finally, since $\tilde{x} < x^*$, we know for all $x \neq \tilde{x}$, $\lambda(x) - r(x) \neq M\tilde{x}|\lambda'(\tilde{x})|$ by Lemma C.6. Thus, all these other points where $x \neq \tilde{x}$ cannot be equilibria. In conclusion, if we know that no optimal equilibrium exists, then there is a unique equilibrium following our pricing rule.

D Proofs from Section 4: Effects of Demand Curves and Monopolies on Efficiency

Theorem 4.1. The Price of Stability is 1 for any instance in which the buyers have a uniform (capacitated) demand, i.e., there exists a Nash Equilibrium with the optimal flow. Moreover, this holds even when $\exists$ edges with $c_e(0) > 0$.

Proof for the $c_e(0) > 0$ case. The proof for the $c_e(0) > 0$ case is slightly different. There will now be some additional edges monopolizing the flow at the optimum; we call them Virtual Monopolies (VM) (see also Section 7). Formally, given a solution $\tilde{x}^*$, VMs are the edges which contain the total flow $x^*$, but may not be contained in all $s-t$ paths. The set of VMs includes the pure monopolies but also edges monopolizing flow at $x^*$. For this case, we use a generalization of our pricing rule as described in Algorithm 4 from Section 7. Specifically, we run the algorithm for the optimum flow of magnitude $x^*$ as described above. Now we show that the prices returned by this process is a Nash Equilibrium.

We show in Section 7 that the allocation returned by the algorithm is a best-response to the prices, therefore we only need to show that the sellers cannot increase or decrease their prices in order to increase revenue. Now, at the end of this process, we have reached a stage where either all paths with flow have a price $\lambda_0$ or no virtual monopoly can increase its price without losing all its flow to some path without flow originally. If the price on all paths is $\lambda_0$, then in that case no edge can increase its price since this is the maximum price that buyers will pay. So no edge can increase its price, virtual monopoly or otherwise.

Finally, as with our previous case the only edges which can decrease their price and ‘potentially’ make a larger profit are those priced at $p_e > c_e(x_e^*)$. The existence of such edges would imply that $\lambda(x^*) > r(x^*)$, since if $\lambda_0 = \lambda(x^*) = r(x^*)$, then our process would terminate trivially at the first iteration since we cannot increase the price on any edge without losing the flow. For this case, no VM will decrease its price because there are no more unallocated buyers left, and so it would not gain any new flow by decreasing its price. This completes the proof. ■.

Lemma 4.2. Let $\lambda(x)$ be any inverse demand function satisfying Monotone Hazard Rate. Given an instance specifying a graph $G$ and cost functions $C_e(x)$, then the function $\frac{|\lambda'(x)|}{\lambda(x) - r(x)}$ is also
non-decreasing $\forall x \leq x^*$, where $x^*$ is the size of the optimum flow for that instance.

**Proof.** Recall from Proposition 3.3 that for a given graph and convex cost functions, $r(x)$ is non-decreasing in $x$. Also recall that as long as $x \leq x^*$, $\lambda(x) \geq r(x)$, since $\lambda(x^*) = r(x^*)$ and $\lambda$ is non-increasing. Consider $x_1 \leq x_2$. Since $\lambda(x)$ is MHR, we know $\frac{|\lambda'(x_1)|}{\lambda(x_1)} \leq \frac{|\lambda'(x_2)|}{\lambda(x_2)}$, which implies $|\lambda'(x_1)|\lambda(x_2) \leq |\lambda'(x_2)|\lambda(x_1)$. Let us consider two cases

**Case I:** $|\lambda'(x_1)| > |\lambda'(x_2)|$

We need to show

\[
\frac{|\lambda'(x_1)|}{\lambda(x_1) - r(x_1)} \leq \frac{|\lambda'(x_2)|}{\lambda(x_2) - r(x_2)}
\]

\[\iff |\lambda'(x_1)|\lambda(x_2) \leq |\lambda'(x_2)|\lambda(x_1) + |\lambda'(x_1)|r(x_2) - |\lambda'(x_2)|r(x_1) \tag{10}\]

But we already know that $|\lambda'(x_1)|\lambda(x_2) \leq |\lambda'(x_2)|\lambda(x_1) + |\lambda'(x_1)|(r(x_2) - r(x_1))$, where the term on the LHS is less than the first term on the RHS due to the MHR assumption and the second term on the RHS is always positive as $r(x)$ is non-decreasing. Since $|\lambda'(x_1)| > |\lambda'(x_2)|$, we know $|\lambda'(x_1)|(r(x_2) - r(x_1)) \leq |\lambda'(x_1)|r(x_2) - |\lambda'(x_2)|r(x_1)$, which gives us the required result.

**Case II:** $|\lambda'(x_1)| \leq |\lambda'(x_2)|$

In this case, we have

\[
\frac{|\lambda'(x_1)|}{\lambda(x_1) - r(x_1)} \leq \frac{|\lambda'(x_2)|}{\lambda(x_1) - r(x_1)} \leq \frac{|\lambda'(x_2)|}{\lambda(x_2) - r(x_2)}. \tag{11}\]

The last step comes from the fact that $\lambda(x_2) - r(x_2) \leq \lambda(x_1) - r(x_1)$ since $\lambda(x)$ is a non-increasing function and $r(x)$ is non-decreasing.

**Claim 4.4.** The PoS bound of $1 + M$ for the family of MHR functions is tight.

**Proof.** As is standard with such examples, we consider a function at the boundary of MHR, $\lambda(x) = e^{-x}$ for $x \geq \frac{1}{M}$ and $\lambda(x) = e^{-1/M}$ for $0 \leq x \leq 1/M$.

Note that this function is not differentiable at $1/M$, although it still obeys our more general definition of MHR functions for piecewise-differentiable functions (the hazard rate is 1 for $x > 1/M$ and 0 for $x < 1/M$, and thus non-decreasing). Alternatively, we can consider a “smoothed” version of this function which behaves the same way outside of the neighborhood of $1/M$ but is continuously differentiable in that neighborhood as well; the result still holds for such a function.

There are $M$ sellers forming a path of length $M$ between the source and the sink ($M$ complementary goods). The production costs for all sellers are identical and are very close to 0 for $x$ smaller than $x^*$, for some large $x^* > 1/M$, at which point the production costs increase very rapidly, so that $r(x^*) = \lambda(x^*)$, and thus $x^*$ is the size of the flow at optimum. The social welfare is $\int_0^{x^*} \lambda(x)dx = e^{-1/M} - e^{-x^*} + e^{-1/M}$ since the production cost is essentially zero; this welfare converges to $\frac{M}{e} - e^{-1/M}$ and $x^*$ approaches infinity. It is easily verifiable that the optimum flow is not a Nash Equilibrium because sellers can increase their price and improve profits.

Now consider any non-trivial equilibrium (with non-zero flow $\tilde{x}$) with prices $\tilde{p}$. First we claim that $\tilde{x} \geq \frac{1}{M}$. Indeed if $\tilde{x} < \frac{1}{M}$, then the total price of the path equals $\lambda(\tilde{x}) = e^{-1/M}$. Any edge can decrease its price infinitesimally and receive a flow of $\frac{1}{M}$ and increase its profit (since there are $\frac{1}{M}$ buyers who value the path at least $e^{-1/M}$). So we only consider the case when $\tilde{x} \geq \frac{1}{M}$.

First consider the possibility of equilibrium at some point where $\lambda'(x)$ is defined. By Lemma 3.3 when $\lambda'_+$ and $\lambda'_-$ are equal, the equilibrium condition reduces to every monopoly edge satisfying $p_e + \tilde{x}\lambda'(\tilde{x}) = 0$. Since the second term depends only the total flow, we conclude that all the edges in the path must have the same price at any equilibrium. Moreover, the total price on the path
must be $\lambda(\tilde{x})$ and so every edge must be priced at $\frac{\lambda(\tilde{x})}{M}$. Thus it must be that $\lambda(\tilde{x}) = M\tilde{x}|\lambda'(\tilde{x})|$. But there is no such point satisfying this when $x > \frac{1}{M}$.

Now consider $\tilde{x} = \frac{1}{M}$. Can this point be an equilibrium? Once again by the conditions of Lemma G.3, every edge must satisfy $p_e \leq \tilde{x}|\lambda'_e(\tilde{x})| = \frac{e^{-M}}{M}$. So the sum of edge prices is no larger than $e^{-M}$. But in order to induce a flow of $\tilde{x} = \frac{1}{M}$, the total price of path must be exactly $e^{-M}$. So we conclude that at the only possible equilibrium point, all edges have an equal price of $\frac{e^{-M}}{M}$.

The social welfare at $\tilde{x}$ is $\frac{e^{-1/M}}{M}$, so the Price of Stability tends to $1 + M$. Notice that for this family of instances, this is the unique non-trivial equilibrium as well.

**Theorem 4.5**. For concave $\lambda(x)$, the Price of Stability is at most $1 + \frac{M}{2}$. Further, there exist instances where no equilibrium can achieve a better Price of Stability.

**Proof. Example.** (Full analysis in the Appendix)

Consider a simple path of $M$ links, having the following inverse demand function

$$
\begin{align*}
\lambda(x) &= 2M + 1 & 0 \leq x \leq 1 \\
\lambda(x) &= (2M + 1) - 2(x - 1) & 1 \leq x
\end{align*}
$$

This function is only piecewise concave, but a “smoothed” version of this function where the neighborhood of $x = 1$ is continuously differentiable still gives the desired bound (although our upper bound also holds for piecewise concave functions; see Section 7). Every edge has a cost function given by $C_e(x) = \frac{1}{M}x$. The following figure shows both $\lambda(x)$ and $r(x)$ for a sample instance with $M = 4$. Note that $\lambda(x)$ is continuous and decreases linearly from $x = 1$ to $x = M + 1$.

![Figure 8: Tight example for concave inverse demand functions.](image)

We remark here that the cost functions do not satisfy $c_e(0) = 0$. Once again in Section 7, we show that our results extend to this case too. For this example, it is enough to consider a function with $c_e(0) = 0$, but which increases rapidly for $x > 0$ until it becomes a constant function $c_e(x) = 1/M$.

From the figure, at the optimum solution $\lambda(x)$ and $r(x)$ meet ($x^* = M + 1$). The social welfare of the solution is the total area under the curve minus the area under the dashed line, which equals $(2M + 1 - 1) + \frac{1}{2}(2M + 1 - 1) \ast (M + 1 - 1) = 2M + M^2$. We claim that the point $\tilde{x} = 1$ is a
NE when the edges are priced according our pricing rule. To see why this is true, note that every edge is priced at \( p_e = 2 + \frac{1}{M} \), so the total price on the path is \( 2M + 1 \) so a flow of \( \tilde{x} = 1 \) is a valid best-response. Each edge makes a profit of \( \pi_e = (2 + \frac{1}{M}) \cdot 1 - \frac{1}{M} = 2 \). Note that \( 2M + 1 \) is the maximum value held by the buyers for this bundle, so an increase in price by any seller would push the total price of the path to a number strictly larger than \( 2M + 1 \) and the flow would drop to zero. Thus, no edge has any incentive to increase its price.

We now consider the profit of an edge which decreases its price. Suppose any one seller reduces his price by \( \Delta \) and the flow increases by \( \epsilon \), then \( \Delta \epsilon = 2 \), which is the slope of the linear part of the curve.

The new profit of the edge is given by
\[
\pi'_e = (2 + \frac{1}{M} - 2\epsilon)(1 + \epsilon) - \frac{1}{M}(1 + \epsilon) \\
= (1 + \epsilon)(2 + \frac{1}{M} - 2\epsilon - \frac{1}{M}) \\
= (1 + \epsilon)(2 - 2\epsilon) \\
= (2 - 2\epsilon^2) \\
\leq 2.
\]

So the new profit is never larger than the old profit. So clearly this pricing gives an equilibrium. The social welfare at this equilibrium is just \( (2M + 1) - 1 = 2M \).

We now claim that for every equilibrium of this instance, the total flow is 1, i.e., \( \exists \) no equilibrium where the flow \( \tilde{x} \neq 1 \). Suppose this is not the case and that \( \exists \) an equilibrium where \( \tilde{x} < 1 \). The total price of the path is still \( 2M + 1 \) and so any one edge (say the edge with the largest price) can decrease its price infinitesimally and increase its flow to 1, thereby leading to an increase in profit. So, there cannot exist an equilibrium with flow smaller than 1.

Suppose there is an equilibrium with flow \( x \) strictly larger than 1. The total price on the path must be strictly less than \( 2M + 1 \) and so at least one edge has a price \( p_e < \frac{2M + 1}{M} = 2 + \frac{1}{M} \). The profit made by this edge at this equilibrium is \( \pi_e = p_e x - \frac{1}{M} x \).

Since the total flow is \( x \), the price on the path must be \( p = (2M + 1) - 2(x - 1) < 2M + 1 \). Suppose this particular edge increases his price so that the new flow is 1. That is, the increase in gap must fill the price difference between \( 2M + 1 \) and \( 2M + 1 - 2(x - 1) \), which is just \( 2(x - 1) \). The new price of this edge is therefore, \( p_e + 2(x - 1) > p_e \). The new profit made by this edge is \( \pi'_e = (p_e + 2(x - 1))1 - \frac{1}{M} \). Some basic algebra tells us that \( \pi' > \pi \) as long as \( p_e < 2 + \frac{1}{M} \). So, we conclude that every equilibrium solution has to have a flow of \( \tilde{x} = 1 \) and has a social welfare of \( 2M \). The Price of Stability for this instance is therefore,
\[
\text{PoS} = \frac{2M + M^2}{2M} = 1 + \frac{M}{2}.
\]

Claim \[4.8\]. For any given instance with \( x|\lambda'(x)| \) being non-decreasing, it must be that \( \frac{\tilde{x}}{x} \geq e^{-M} \).

**Proof.** Following our pricing rule, at equilibrium, we have \( \frac{\lambda(\tilde{x}) - r(\tilde{x})}{M} = \tilde{x}|\lambda'(\tilde{x})| \). Since \( x|\lambda'(x)| \) is non-decreasing, it must be that for all \( x \geq \tilde{x}, x|\lambda'(x)| \geq \tilde{x}|\lambda'(\tilde{x})| = \frac{\lambda(\tilde{x}) - r(\tilde{x})}{M} \). Equivalently for
\[ x \geq \hat{x}, \]
\[ |\lambda'(x)| = -\lambda'(x) \geq \frac{\lambda(\hat{x}) - r(\hat{x})}{Mx} \]

Integrating both sides from \( x = \hat{x} \) to \( x^* \), we get
\[ \lambda(\hat{x}) - \lambda(x^*) \geq \frac{\lambda(\hat{x}) - r(\hat{x})}{M} \ln\left(\frac{x^*}{\hat{x}}\right). \]

This implies
\[ \ln\left(\frac{x^*}{\hat{x}}\right) \leq M \frac{\lambda(\hat{x}) - \lambda(x^*)}{\lambda(\hat{x}) - r(\hat{x})}. \]

Finally, we get \( \frac{x^*}{\hat{x}} \leq \exp\left(\frac{\lambda(\hat{x}) - \lambda(x^*)}{\lambda(\hat{x}) - r(\hat{x})}\right) \leq e^M. \) The last inequality follows from the fact that at the optimum flow \( x^* \), \( \lambda(x^*) \geq r(x^*) \geq r(\hat{x}). \)

**Claim [4.9]**. The price of stability can be arbitrarily bad for MPE functions.

**Proof.** Consider a two-link path (\( M = 2 \)). The same example can generalized for any value of \( M \), however assuming a specific value of \( M \) allows us to derive the equilibrium and optimum flows analytically. The demand function has the following structure: \( \lambda(x) = x^{-1/r} \) for \( r > M \). However, this function does not belong to the class MPE since \( \lim_{x \to 0} \frac{2|\lambda'(x)|}{\lambda(x)} \neq 0. \) Therefore, we truncate this function at some very small \( \epsilon > 0 \) such that \( \lambda(x) = x^{-1/r} \) for \( x \geq \epsilon \) and \( \lambda(x) = (\epsilon)^{-1/r} \) for \( x < \epsilon \). \( \lambda'(x) \) is defined at every point other than \( x = \epsilon \), but since our results carry over to non-differentiable \( \lambda(x) \), we use this function. A sample demand function of this form is represented in Figure 9.

We assume that each edge has a cost function given by \( C_e(x) = c_0x \) for some positive \( c_0 \). The unique optimum flow is the solution with \( (x^*)^{-1/r} = 2c_0 \). First it is easy to see that this is not a Nash Equilibrium because for any set of prices the edge with the lower price can increase its price and improve profits. Now, the most general condition that any equilibrium point must obey is given by Lemma G.3. For points where \( \lambda'(x) \) is defined which in our case is everywhere except \( x = \epsilon \), the condition reduces to \( p_e + \hat{x}\lambda'(\hat{x}) - c_0(\hat{x}) = 0. \) Since both edges have the same \( c_e \), this...
means that in any equilibrium at a point other than \( \epsilon \), both the edges need to have the exact same price \( p_e \).

Since the sum of prices is \( \lambda(\tilde{x}) \), the price on each edge must be equal to \( \frac{\lambda(\tilde{x})}{2} \). First let’s see if any \( \tilde{x} \geq \epsilon \) satisfies the equilibrium conditions. Substituting the value for \( p_e \) and the fact that \( x\lambda'(x) = \frac{1}{r}x^{-1/r} \), we get the equilibrium condition as

\[
\tilde{x}^{-1/r}(\frac{1}{2} - \frac{1}{r}) = c.
\]

It is not hard to see that if we price the edges according to our pricing rule at this point, which once again just reduces to \( p_e = \lambda(\tilde{x})/2 \), this point satisfies the conditions of Lemmas \[3.3\] and \[3.4\] for \( x \) in \([\epsilon, x^+]\). We do not care about the region between \([0, \epsilon]\) since \( \lambda(x) \) is constant in this region so the profit is no larger than \( \pi(\epsilon) \).

Now we have a closed form expression for both the optimum point \( x^* \) and NE \( \tilde{x} \). Using some basic algebra and computing the social welfare of both these solutions, we see that as \( r \to M \), the ratio of the social welfare becomes arbitrarily large. At the same time, Theorem \[8.1\] implies that if the edges have capacities instead of costs then for any \( r < M \), the Price of Stability is one.

Finally, we remark that if we take a family of examples where \( r > M \) but approaches \( M \) and moreover, if for each member of the family, we take \( 0 < \epsilon < \tilde{x} \) as defined above, then \( \tilde{x} \) will be the unique equilibrium point. We have already established that \( \tilde{x} \) is the unique equilibrium point where \( \lambda'(x) \) is defined. So only the other point we need to consider is \( \epsilon \). However, in order for this to be a Nash Equilibrium, no edge should be able to decrease his price, and for both the edges \( p_e \leq \epsilon|\lambda'_+(\epsilon)| \). But then, the total price on the path would be \( 2p_e + 2c_\epsilon(\epsilon) \leq 2\epsilon|\lambda'_+(\epsilon)| + 2c_0 \), which we can show is strictly smaller than \( \lambda(\epsilon) \). So this is not a Nash Equilibrium.

\[
\square
\]

E Proofs from Section 5: Specific Demand Functions

Theorem \[5.1\]

1. For \( \lambda \in F_p \), the Price of Stability is at most \((1 + M\alpha)^{\frac{1}{\alpha}}\). When \( \alpha \geq M \), this quantity is approximately \( 1 + \frac{\log(M\alpha)}{\alpha} \).

2. For \( \lambda \in F_{ced} \), the Price of Stability is at most \( 1 + M\alpha^{\frac{\alpha}{\alpha-1}} \) for \( \alpha \geq 1 \).

3. For \((\lambda, r) \in F_{exp} \), the Price of Stability is at most \( e^{\frac{M}{\alpha}} \) for \( \alpha \geq 1 \).

Notice that our first two results hold irrespective of the nature of the cost functions, while the last bound depends on the cost functions \( C_e \) as well as \( \lambda \). The proof of this theorem is somewhat technical and located in the Appendix in order to improve readability.

\textbf{Proof. Proof of Statement 1.} Let \( \lambda(x) = (1 - x^\alpha) \), where \( \alpha \geq 1 \). The exact same proof holds when there are scaling factors and \( \lambda(x) \) has the form \( \lambda_0(1 - (\frac{x}{\theta})^\alpha) \). The function is concave so we know \( \exists \) a Nash Equilibrium obeying the conditions of Corollary \[3.5\]. Let \( r(x) \) be the differential min-cost function. At the equilibrium point \( \tilde{x} \), we have \( \lambda(\tilde{x}) - r(\tilde{x}) - M\tilde{x} |\lambda'(\tilde{x})| = 0 \). Substituting \( \lambda(x) = (1 - x^\alpha) \) and \( \lambda'(x) = -\alpha x^{\alpha-1} \), we get \( (1 - \tilde{x}^\alpha - M\alpha \tilde{x}^\alpha) = r(\tilde{x}) \). At the optimum point \( x^* \), we know that \( \lambda(x^*) \geq r(x^*) \) by Proposition \[2.2\]. Since \( r(x) \) is non-decreasing, we have, \( 1 - (x^*)^\alpha \geq r(x^*) \geq r(\tilde{x}) = 1 - \tilde{x}^\alpha (1 + M\alpha) \). Therefore, we get \( (\frac{\tilde{x}^*}{2})^\alpha \leq (1 + M\alpha) \) or equivalently, \( \tilde{x}^* \leq (1 + M\alpha)^{\frac{1}{\alpha}} \).

Recall that social welfare of a min-cost flow is size \( x \) is equal to \( \int_0^x [\lambda(t) - r(t)] dt \). Now, since \( \lambda(x) - r(x) \) is non-increasing, and the social welfare of \( \tilde{x} \) equals \( \int_0^{\tilde{x}} [\lambda(t) - r(t)] dt \) since it is a min-cost
flow, then we know that the Price of Stability is bounded by \( \frac{x^*}{x} \). So for functions in \( F_p \), we have the \( \text{PoS} \leq \frac{x^*}{x} \leq (1 + Ma) \frac{1}{\alpha} \). For a fixed value of \( M \), as \( \alpha \) increases the bound tends to \( 1 + \frac{\log(Ma)}{\alpha} \).

Therefore, the PoS also tends to this value. ■

Proof of Statement 2. Once again, we consider \( \lambda(x) = (1 - x)^{\alpha} \) for \( \alpha \geq 1 \). It is easy to verify that the proof is valid when this function is scaled and/or shifted. The function belongs to MHR, so Corollary 3.3 holds. Moreover, it is convex.

So at equilibrium, we have \( (1 - \tilde{x})^{\alpha} - r(\tilde{x}) = M\alpha \tilde{x}(1 - \tilde{x})^{\alpha - 1} \) or \( r(\tilde{x}) = (1 - \tilde{x})^{\alpha} - M\alpha \tilde{x}(1 - \tilde{x})^{\alpha - 1} \). So now the PoS is bounded by

\[
\text{PoS} \leq 1 + \frac{\int_{\tilde{x}}^{y} \lambda(x)dx - r(\tilde{x})(y - \tilde{x})}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))},
\]

using the same argument as in our previous proofs of upper bounds.

Consider the following function \( f(x) = \lambda(x) - r(\tilde{x}) \). Since \( r(x) \) is non-decreasing, we have that \( \lambda(x) - r(x) \leq \lambda(x) - r(\tilde{x}) \), \( \forall x \geq \tilde{x} \). So we can bound the integral in the numerator above by \( \int_{\tilde{x}}^{y} f(x)dx \). Furthermore, let \( y \geq x^* \) be the point such that \( \lambda(y) = (1 - y)^{\alpha} = r(\tilde{x}) \); this point is larger than \( x^* \) since at the optimum solution \( \lambda(x^*) \geq r(x^*) \geq r(\tilde{x}) \) and \( r(x) \) is non-decreasing. For each \( x \leq y \), we have that \( f(x) \geq 0 \), and so the numerator above can be bounded by \( \int_{\tilde{x}}^{y} f(x)dx \). Thus our bound on the PoS becomes,

\[
\text{PoS} \leq 1 + \frac{\int_{\tilde{x}}^{y} \lambda(x)dx - r(\tilde{x})(y - \tilde{x})}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))} = 1 + \frac{\int_{\tilde{x}}^{y} (1 - \tilde{x})^{\alpha + 1} - (1 - y)^{\alpha + 1} - r(\tilde{x})(y - \tilde{x}) - \alpha r(\tilde{x})(y - \tilde{x})}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))} = 1 + \frac{\int_{\tilde{x}}^{y} (1 - \tilde{x})^{\alpha + 1} - (1 - y)^{\alpha + 1} - r(\tilde{x})(y - \tilde{x}) - \alpha r(\tilde{x})(y - \tilde{x})}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))}
\]

Now, we substitute \( r(\tilde{x})(y - \tilde{x}) = r(\tilde{x})(1 - \tilde{x} - (1 - y)) = r(\tilde{x})(1 - \tilde{x} - r(\tilde{x})(1 - y) \). We know that \( r(\tilde{x})(1 - \tilde{x}) = (1 - \tilde{x})^{\alpha + 1} - M\alpha \tilde{x}(1 - \tilde{x})^{\alpha} \) from the equilibrium conditions and that \( r(\tilde{x})(1 - y) = (1 - y)^{\alpha + 1} \) due to our choice of \( y \). So, \( r(\tilde{x})(y - \tilde{x}) = (1 - \tilde{x})^{\alpha + 1} - M\alpha \tilde{x}(1 - \tilde{x})^{\alpha} - (1 - y)^{\alpha + 1} \). Substituting this in the bound for the PoS, we get

\[
\text{PoS} \leq 1 + \frac{M\alpha \tilde{x}(1 - \tilde{x})^{\alpha} - r(\tilde{x})(y - \tilde{x})}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))} = 1 + \frac{M\tilde{x}(1 - \tilde{x})^{\alpha} - r(\tilde{x})(M\tilde{x})}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))} = 1 + \frac{M\tilde{x}(1 - \tilde{x})^{\alpha} - r(\tilde{x})(M\tilde{x})}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))} = 1 + \frac{M\tilde{x}(1 - \tilde{x})^{\alpha} - r(\tilde{x})(M\tilde{x})}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))}
\]

Now, we use the idea that since \( \lambda(x) \) is convex, \( y - \tilde{x} \geq M\tilde{x} \), which we will prove later. So substituting this in the above inequality, we get

\[
\text{PoS} \leq 1 + \frac{\alpha}{\alpha + 1} \frac{M\tilde{x}(1 - \tilde{x})^{\alpha} - r(\tilde{x})(M\tilde{x})}{\tilde{x}(\lambda(\tilde{x}) - r(\tilde{x}))} = 1 + \frac{\alpha}{\alpha + 1} M,
\]

as desired. Now, we prove the required sub-claim.

Lemma E.1. If \( \lambda(x) \) is convex and \( \lambda(y) = r(\tilde{x}) \), then \( y - \tilde{x} \geq M\tilde{x} \).

Proof. Let \( r(x) = r(\tilde{x}) \). We know at the equilibrium point that \( |\lambda'(\tilde{x})| = \frac{\lambda'(\tilde{x}) - r'(\tilde{x})}{M\tilde{x}} \). Since \( \lambda(x) \) is convex, it means that \( |\lambda'(x)| \) is non-increasing, which means that for any \( x \geq \tilde{x} \), \( |\lambda'(x)| \leq |\lambda'(\tilde{x})| \). So, we have \( \lambda(x) \geq \lambda(\tilde{x}) - |\lambda'(\tilde{x})|(x - \tilde{x}) \). Consider the following point \( x_2 = \tilde{x} + M\tilde{x} \). At \( x_2 \),
we have \( \lambda(x_2) - r(\tilde{x}) \geq \lambda(\tilde{x}) - |\lambda'(\tilde{x})|(x_2 - \tilde{x}) - r(\tilde{x}) = \lambda(\tilde{x}) - r(\tilde{x}) - \frac{\lambda(\tilde{x}) - r(\tilde{x})}{M\tilde{x}} \times M\tilde{x} = 0. \) Since at \( x_2, \lambda(x_2) \geq r(\tilde{x}) = \lambda(y), \) and \( \lambda \) is non-increasing, then it must be that \( y \geq x_2. \) So, we get \( y - \tilde{x} \geq x_2 - \tilde{x} = M\tilde{x}. \)

\[ \blacksquare \]

**Proof of Statement 3.** Since \( \lambda(x) - r(x) = |\ln(\tilde{x})|^{\frac{1}{x}} \), at the optimum flow \( x^* \), we have \( \lambda(x^*) - r(x^*) \geq 0 \) so \( x^* \leq a \) and \( \tilde{x} < a. \) Since \( r(x) \) is non-decreasing, \( r'(x) \) is non-negative. This gives us \( \lambda'(x) \geq \lambda'(x^*) - r'(x^*) = -\frac{1}{x^*|\ln(\tilde{x})|^{\frac{1}{x^*}} - 1} \). At equilibrium, we know that \( \lambda(\tilde{x}) - r(\tilde{x}) = M\tilde{x}|\lambda'(\tilde{x})| \leq M\tilde{x}|\lambda'(x) - r'(x)|. \) Thus, \( |\ln(\tilde{x})|^{\frac{1}{\tilde{x}}} \leq M\tilde{x}|\ln(\tilde{x})|^{\frac{1}{x}} - 1 \). Canceling out the common terms, we get at equilibrium \( |\ln(\tilde{x})| \leq \frac{M}{\alpha} \) or \( \tilde{x} \geq a e^{-\frac{M}{\alpha}} \). So \( \frac{\tilde{x}}{x} \leq e^{-\frac{M}{\alpha}} \), which gives us the desired bound on the Price of Stability using the arguments from the proof of Statement 1 above.

**F Proofs from Section 6: Bad Examples for Multiple Source Networks**

**Claim 6.2.** There exists an instance with two sources, one sink, and no monopoly edges for either source, such that the Price of Stability is strictly greater than 1.

**Proof.** Consider the instance shown in the figure with two sources \( s_1 \) and \( s_2 \). Let the demand of \( s_1 \) be \( \lambda(x) = 1 - x \) for \( x \leq 1 \) and 0 afterward, and the demand of \( s_2 \) be \( \lambda(x) = 4 - x \) for \( x \leq 4 \) and 0 afterward. At the unique optimum point \( s_1 \) sends a flow of 1/3 on its direct link (edge \( e_1 \)) to the sink and \( s_2 \) sends 2 units of flow, one on each of its paths.

We claim that there exist no set of prices stabilizing a flow of \( \frac{1}{3} \) for \( s_1 \) and 2 units for \( s_2 \) divided equally on both the paths. Since \( s_1 \) is sending a total of a third of a unit of flow, the price on any flow path must be \( 1 - \frac{1}{3} = \frac{2}{3}. \) Similarly, the price on any flow-containing path for \( s_2 \) must be 2. Any set of prices stabilizing this flow must therefore have \( e_1 \) being priced at \( p = \frac{2}{3}. \) Consider the flow carrying path for \( s_2 \) consisting of the edges \( (s_2, i_1) \) and \( (i_1, t) \). We claim that in any Nash Equilibrium with this flow, the edge \( e_2 = (i_1, t) \) must be priced at \( p_2 \geq 1. \) Indeed, if the price on this edge is less than 1, then its profit is \( p_2 \times 1 - 1 < 0, \) which means that the edge is no longer stable.

At the current prices, look at the two paths available for \( s_1 \) to send flow on. It’s direct edge \( e_1 \) is priced at \( p_1 = \frac{2}{3} \) and the other path has a price no smaller than \( p_2 \geq 1. \) So the best-response is
to always send flow on the edge \( e_1 \) and so \( e_1 \) is a virtual monopoly for this source (refer Section 7 for a formal definition). Can these prices be a Nash Equilibrium? We claim that this is not the case. Suppose \( e_1 \) increases his price to \( p'_1 = \frac{3}{4} \). Note that for \( s_1 \), the \( e_1 \) path is still strictly cheaper than the other path at the current prices, so its best-response for this price would be to send a flow of \( \frac{1}{4} \) on this edge. The seller’s new profit is \( \frac{3}{4} \times \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{8} \) which is greater than its old profit of \( \frac{1}{9} \). The original solution is therefore not stable for the sellers.

It is not hard to verify that this is a unique optimum solution (since costs are convex and \( \lambda(x) \) is strictly decreasing) so we conclude that there exists no efficient Nash Equilibrium. However, unlike the previous example, this instance does admit a suboptimal equilibrium. Consider a price of \( p_1 = \frac{3}{4} \) on \( s_1 \)'s direct link, \( p_4 = 2 \) on \( s_2 \)'s direct link and both the other links priced at \( p_2 = 2 \) and \( p_3 = 0 \). Suppose that \( s_1 \) sends a flow of \( \frac{1}{4} \) on its direct link and \( s_2 \) sends one unit on each of its path. It is not hard to verify that this is a Nash Equilibrium for both the buyers and sellers. \( \square \)

G Proofs from Section 7: More General Markets and Production Costs

G.1 Dropping Differentiability in Demand

**Theorem 7.1.** For continuous but non-differentiable \( \lambda(x) \) belonging to the class MPE, there always exists a Nash equilibrium with a non-trivial flow of \( \tilde{x} > 0 \) obeying our pricing rule. Furthermore this equilibrium is unique as long as the production costs are non-zero at this point.

**Proof.** As with our continuously differentiable case, we give a series of lemmas that capture some sufficient conditions for equilibrium. We then show that for MPE functions, there must always exist a point satisfying these conditions and also obeying our pricing rule. Recall that the quantity \( p_e \) is defined to be the increased price on an edge from its marginal cost, i.e., the total price \( p_e = \tilde{p}_e + c_e(x_e) \). Usually when we refer to \( \tilde{p}_e \), we refer to the equilibrium prices.

First we show that Lemma 3.3 still holds if \( \lambda \) is not continuously differentiable, with the condition in the lemma being \( \tilde{x}|\lambda'_-(\tilde{x})| \) instead of \( \tilde{x}|\lambda'(\tilde{x})| \). The proof is very similar to that of Lemma 3.3 so we only sketch the differences here. Define the profit function \( \pi(x) = [p_e + \lambda(x) - \lambda(\tilde{x})]x - C_e(x) \). We will prove that in the domain \((0, \tilde{x})\), \( \pi(x) \) is maximized when \( x = \tilde{x} \), thus implying that no matter what amount the seller increases his price by, at the resulting flow of magnitude \( x \), his profit cannot be strictly larger than the original profit. Once again, consider the derivative of this function. Since \( \lambda'(x) \) may not exist at all points, we simply define the derivative in the limit as well, i.e., \( \pi'_-(x) \) and \( \pi'_+(x) \) at points where the derivative does not exist.

\[
\pi'_-(x) = p_e + \lambda(x) - \lambda(\tilde{x}) + x\lambda'_+(x) - c_e(x) = \tilde{p}_e + c_e(\tilde{x}) + \lambda(x) - \lambda(\tilde{x}) + x\lambda'_+(x) - c_e(x) \\
\geq \lambda(x) - \lambda(\tilde{x}) + \tilde{x}|\lambda'_-(\tilde{x})| - x|\lambda'_+(x)| + (c_e(\tilde{x}) - c_e(x))
\]

Since the last term in the parenthesis, \( c_e(\tilde{x}) - c_e(x) \) is non-negative (\( c_e \) is non-decreasing), in order to show that \( \pi'(x) \geq 0 \) for all \( x \leq \tilde{x} \), it suffices if we show the first terms are non-negative. The following proposition implies that for MPE functions, this is indeed true.

**Proposition G.1.** For \( \lambda \in MPE \) and \( M \geq 1 \), we have that \( \lambda(\tilde{x}) - \lambda(x) \leq \tilde{x}|\lambda'_-(\tilde{x})| - x|\lambda'_+(x)| \) for \( x < \tilde{x} \).
The proof of this proposition is very similar to that of Proposition [C.1] and we use the fact that,
\[
\frac{\lambda(\tilde{x})}{\tilde{x}|\lambda'(\tilde{x})|} \leq \frac{\lambda(x)}{x|\lambda'_+(x)|}.
\]
We have argued now that for all \( x \), \( \pi'_+(x) \geq 0 \). Now for any given \( x \), \( \pi'_-(x) \geq \pi'_+(x) \), since \( |\lambda'_-(x)| \geq |\lambda'_+(x)| \), so \( \pi'_-(x) \geq \pi'_+(x) \geq 0 \) for all \( x \leq \tilde{x} \). This completes the proof of Lemma [3.3].

Now we show that Lemma [4.4] still holds, with the condition in the lemma being \( \tilde{x} |\lambda'_-(\tilde{x})| \) instead of \( \tilde{x} |\lambda'(\tilde{x})| \). We want to show that the new profit a seller makes is at most the old profit as long as Condition (3) is obeyed. Consider seller \( e \)'s profit, \( \pi(x) = [p_e + \lambda(x) - \lambda(\tilde{x})]|x - C_e(x) | \). We will prove that in the domain \([\tilde{x}, T]\), \( \pi(x) \) is maximized when \( x = \tilde{x} \), thus implying that no matter what amount the seller decreases his price by, at the resulting flow of magnitude \( x \), his profit cannot be strictly larger than the original profit. As with the previous lemma, we define the derivatives \( \pi'_-(x) \) and \( \pi'_+(x) \).

\[
\pi'_-(x) = p_e + \lambda(x) - \lambda(\tilde{x}) + x\lambda'_-(x) - c_e(x) = \bar{p}_e + c_e(\tilde{x}) + \lambda(x) - \lambda(\tilde{x}) + x\lambda'_-(x) - c_e(x)
\]

Since the last term in the parenthesis, \( c_e(\tilde{x}) - c_e(x) \) is not positive \( (c_e \text{ is non-decreasing}) \), in order to show that \( \pi'(x) \leq 0 \) for all \( x \geq \tilde{x} \), it is sufficient to show the first terms are not positive either. The following proposition implies that for MPE functions, this is indeed true.

**Lemma G.2.** For \( \lambda \in \text{MPE} \) and \( M \geq 1 \), we have that \( \lambda(\tilde{x}) - \lambda(x) \geq \bar{p}_e - x|\lambda'_-(x)| \) for \( x > \tilde{x} \).

**Proof.** The proof is slightly trickier than all the other cases. The following equation however is the main step
\[
\frac{\lambda(\tilde{x})}{\bar{p}_e} \geq \frac{\lambda(\tilde{x})}{\tilde{x}|\lambda'_-(\tilde{x})|} \geq \frac{\lambda(x)}{x|\lambda'_-(x)|}
\]
The equation is true because \( \bar{p}_e \leq \tilde{x} |\lambda'_-(\tilde{x})| \). Now since we have defined all the prices to be positive, it is not possible for any edge in a flow carrying path to be priced larger than \( \lambda(\tilde{x}) \). This implies that \( \lambda(\tilde{x}) \geq p_e \geq \bar{p}_e \). Now our previous analysis for Proposition [C.2] carries over to this case. \( \square \)

Similarly, \( \pi'_+(x) \leq \pi'_-(x) \leq 0 \). This completes the proof that Lemma [3.4] still holds. As with the continuously differentiable case, Corollary [7.2] characterizes the equilibrium prices and allocations. So all we need to show now is that there exists a pair of prices and allocation satisfying the conditions of that corollary.

To show that such a solution exists, once again we price edges according to our pricing rule and so for monoplies \( \bar{p}_e = \frac{\lambda(\tilde{x}) - r(\tilde{x})}{M} \) at some \( \tilde{x} \) where we have taken the minimum cost flow. We claim that either the optimum point \( x^* \) satisfies \( \lambda(x^*) - r(x^*) = Mx^*|\lambda'_-(x^*)| \) or there exists \( \tilde{x} > 0 \) that is a Nash Equilibrium. Clearly if the optimum point satisfies the desired condition, we are done from the corollary, just as in the argument in Section [3]. Suppose that is not true, then \( \lambda(x^*) - r(x^*) < Mx^*|\lambda'_-(x^*)| \). But since the function belongs to the class MPE, \( \lim_{x \to 0} \frac{\lambda(x) - r(x)}{Mx|\lambda'_+(x)|} > \lim_{x \to 0} Mx|\lambda'_+(x)| = \lim_{x \to 0} Mx|\lambda'_-(x)| \). Since \( \lambda(x) - r(x) \) is continuous, the function \( \frac{Mx|\lambda'_-(x)|}{\lambda(x)} \) is monotone and \( \frac{Mx|\lambda'_-(x)|}{\lambda(x)} \leq \frac{Mx|\lambda'_+(x)|}{\lambda(x)} \), we claim there must exist some intermediate value of \( x \), where \( Mx|\lambda'_-(x)| \leq \lambda(x) - r(x) \leq Mx|\lambda'_+(x)| \). Why is this true?

First, if \( \lambda'(x) \) is defined in some region (say) \([x_1, x_2]\) such that \( \lambda(x) - r(x) > Mx_1|\lambda'(x_1)| \), and \( \lambda'(x_2) - r(x_2) < Mx_2|\lambda'(x_2)| \), then there must exist some \( x \in [x_1, x_2] \) where \( \lambda(x) - r(x) = Mx|\lambda'(x)| \).
Indeed, recall that if a function is differentiable in some region $[x_1, x_2]$, the derivative cannot have jump discontinuities in this region. Moreover since this is MPE function, the derivative is finite everywhere and the limit at any point exists, and so the curve $\lambda(x) - r(x) - Mx|\lambda'(x)|$ is continuous and equal 0 at some point in $[x_1, x_2]$.

So the only place where the analysis breaks is for points where $\lambda'(x)$ is not defined. Since $\lambda(x)$ is differentiable almost everywhere, we only have to worry about point discontinuities. But in this case, we will have that $|\lambda_-(x)| > |\lambda_+(x)|$, and since this is an MPE function, we will obtain a point where $\lambda(x) - r(x) - Mx|\lambda_+(x)| < 0 < \lambda(x) - r(x) - Mx|\lambda_-(x)|$, as desired.

Coming to the uniqueness, the proof is quite similar to that of Theorem 3.8. The following is the key lemma required to show uniqueness.

**Lemma G.3.** At any Nash Equilibrium that is not optimal, no monopoly edge can increase or decrease its price. Therefore, such an edge must satisfy,

$$\ddot{x}|\lambda_-(\ddot{x})| \leq \ddot{p}_e \leq \ddot{x}|\lambda_+(\ddot{x})|.$$

The proof follows from the fact that no monopoly can increase or decrease its price by an infinitesimal amount. For our pricing this just boils down to satisfying

$$\ddot{x}|\lambda_-(\ddot{x})| \leq \frac{\lambda(\ddot{x}) - r(\ddot{x})}{M} \leq \ddot{x}|\lambda_+(\ddot{x})|.$$

Once again we can show that, when $\lambda(x)$ is strictly decreasing at $x = \ddot{x}$, this $S$ is a single point. The additional case arises when it is strictly decreasing in one direction only, i.e., $|\lambda_-(\ddot{x})| = 0$. In this case, as long as $\ddot{x}$ is not optimal, for this function to belong to the class MPE, it is necessary that $\lambda'(x) = 0$ for all $x$ in $[0, \ddot{x})$. This means that a monopoly would want to increase its price since $\ddot{p}_e \leq x|\lambda_+(x)| = 0$ is no longer true here. Therefore we conclude that once again as long as the equilibrium is not optimal, it is unique and as with the previous case, we can show that it can be computed using a binary search algorithm.

**G.2 Piecewise Cost functions**

**Claim G.4.** Given a solution vector pair $(\vec{p}, \vec{c})$ for an instance with $M \geq 1$, where $\vec{c}$ is a best-response flow and all flow carrying paths are priced at $\lambda(\vec{c}) \geq \vec{x}|\lambda'(\vec{x})|$ and non-flow paths are priced at or above $\lambda(\vec{c})$, no seller $e$ can change his price and improve profits as long as

1. $p_e \geq \ddot{x}|\lambda'(\ddot{x})| + c_e^-(x)$.
2. $p_e \leq \ddot{x}|\lambda'(\ddot{x})| + c_e^+(x)$ (or) $\ddot{x} = T$.

**Proof.** Clearly, we have assumed that the buyer behavior is a best-response. The proof is quite similar to that of Lemmas 3.3 and 3.4. We define a profit function $\pi(x) = [p_e + \lambda(x) - \lambda(\ddot{x})]x - C_e(x)$. The derivative of this function can be defined in the limit as $\pi_+(x)$ and $\pi_-(x)$ where $c_e(x)$ is not defined. We show that for all $x > \ddot{x}$, $\pi_+(x) \leq \pi_+(x) \leq 0$ and for all $x < \ddot{x}$, $\pi_-(x) \geq \pi_-(x) \geq 0$. This can be shown by simply differentiating the expression for $\pi(x)$ and using $p_e \leq \ddot{x}|\lambda'(\ddot{x})| + c_e^+(x)$ and $p_e \geq \ddot{x}|\lambda'(\ddot{x})| + c_e^-(x)$ respectively. Notice that when $\ddot{x} = T$, no seller would decrease his price since there are no buyers left in the market.

While the previous claim gave us sufficient conditions that the prices should obey so that sellers cannot increase or decrease their price, the following two lemmas give us an idea of what exactly these prices should be.
Lemma G.5. Given a min-cost flow $\tilde{x}$ for an instance with $M \geq 1$ and a quantity $\bar{c}_e$ for all $e$ such that,

1. $r^-(\tilde{x}) \leq \lambda(\tilde{x}) - M\tilde{x}|\lambda'(\tilde{x})| \leq r^+(\tilde{x})$.
2. $\forall e, c\tilde{e}_e(\tilde{x}) \leq \bar{c}_e \leq c^\dagger(x_e)$.
3. For any flow carrying path $P$, $\sum_{e \in P} \bar{c}_e = \lambda(\tilde{x}) - M\tilde{x}|\lambda'(\tilde{x})|$.

Then, pricing all non-monopoly edges at $p_e = \bar{c}_e$ and all monopoly edges at $p_e = \bar{c}_e + \tilde{x}|\lambda'(x)|$ results in a Nash Equilibrium.

Proof. First notice that there are $M$ monopolies and every path has $\sum_{e \in P} \bar{c}_e = \lambda(\tilde{x}) - M\tilde{x}|\lambda'(\tilde{x})|$. So once the price on the monopoly is set to $\bar{c}_e$, and then raised by $\tilde{x}|\lambda'(\tilde{x})|$, every flow carrying path is now priced at exactly $\lambda(\tilde{x})$ so buyer behavior is still a best-response. These mysterious $\bar{c}_e$’s for every edge somehow balance the price of all the flow carrying paths so that when we increase the price on monopolies, the price of all flow paths increase to the desired amount of $\lambda(\tilde{x})$.

Now all non-monopoly edges are priced below $c\tilde{e}(x_e)$ so no edge can decrease its price. Further non-monopoly edges cannot increase their price as they would lose all their flow. Moving on to monopolies, the price on each monopoly clearly satisfies Conditions (1) and (2) of Lemma G.4 so no monopoly seller can increase or decrease their price and we conclude that this is a Nash Equilibrium.

Lemma G.6. Suppose that the optimal flow satisfies $\lambda(x^*) - Mx^*|\lambda'(x^*)| \geq r^-(x^*)$ and we are given $\bar{c}_e$ for all edges satisfying the following

1. $c\tilde{e}_e(x_e) \leq \bar{c}_e \leq c^\dagger(x_e)$
2. For all flow carrying paths $P$, $\sum_{e \in P} \bar{c}_e = r^-(x^*) \leq r^+(x^*) \leq \lambda(x^*)$.

Then $x^* = T$. Further, the solution where all non-monopoly edges are priced at $p_e = \bar{c}_e$ and monopoly edges are priced at $p_e = \bar{c}_e + \frac{\lambda(x^*) - r^-(x^*)}{M}$ is a Nash Equilibrium.

The proof follows almost directly from Lemma G.4.

Claim 7.3. For any given min-cost flow $\tilde{x}$ and a price $p^*$ satisfying $r^-(x) \leq p^* \leq r^+(x)$, we can always compute a vector of $\bar{c}_e$ for each edge $e$ obeying the following requirements

1. For all $e$, $c\tilde{e}_e(x_e) \leq \bar{c}_e \leq c^\dagger(x_e)$.
2. For any flow carrying path $P$, $\sum_{e \in P} \bar{c}_e = p^*$ and for any non flow-carrying path, $P'$, $\sum_{e \in P'} \bar{c}_e \geq p^*$.

Proof. Observe that if $\forall$ edges, $c\tilde{e}_e(x_e) = c^\dagger(x_e)$, the proof of this claim is trivial. We just set $\bar{c}_e = c\tilde{e}_e(x_e) = c^\dagger(x_e)$ for all edges. It is not hard to see that for this case, $r^+(x) = r^-(x)$. Since what we have is a min-cost flow, by Lemma B.4 for all flow paths $\sum c\tilde{e}_e(x_e) = \sum \bar{c}_e = r(x) = p^*$. Indeed, when all the cost functions are twice differentiable, the way to ensure equal price on all paths is to set the price on every edge to be its marginal cost. What about the general case, where for each edge the marginal cost of removing and adding flow does not coincide?

We begin by extending Lemma B.4 to piecewise cost functions. If $P$ is the set of paths with non-zero flow in $\tilde{x}$ and $P'$ is the set of paths with zero flow, then it is not hard to see the following.

$$\max_{P \in P'} \sum_{e \in P} c\tilde{e}_e(x_e) \leq r^-(x) \leq p^* \leq r^+(x) \leq \min_{P \in P \cup P'} \sum_{e \in P} c^\dagger(x_e).$$
In the other words, \( r^-(x) \) is at least the marginal cost of removing one unit of flow from any flow carrying path and \( r^+(x) \) is at most the marginal cost of adding one unit of flow to any path. So even if we price edges at either of the two marginals, all the paths may not have the same price of \( p^* \). In order to balance the price on all the paths, we turn to the concept of node potentials which are usually obtained from the dual to the min-cost flow LP. First, given the marginal cost of removing an infinitesimal unit of flow \( (k_e^-) \) and the marginal cost of adding an infinitesimal unit of flow \( (k_e^+) \) for every single edge, we formally define two node potentials as follows.

**Definition** The negative potential \( \pi_v^- \) is defined on every node to be the maximum cost ‘saved’ by removing an infinitesimal unit of flow from the source \( s \) to \( v \).

**Definition** The positive potential \( \pi_v^+ \) is defined on every node to be the minimum cost incurred for sending an infinitesimal unit of flow from the source \( s \) to \( v \).

Given a min-cost flow \( \tilde{x} \), these two marginal costs are well defined for every edge as \( c_e^-(x_e) \) and \( c_e^+(x_e) \) respectively. In this case, the node potentials obey some nice properties which we now show. First, we redefine these node potentials formally in terms of the residual graph obtained for a min-cost flow, although the (re)definition extends to any negative and positive marginal costs as long as the residual graph does not have negative cycles.

**Alternative definition of node potentials.** Define \( G^R \) to have the same set of nodes as the original graph but for every \((u, v)\) in the original graph, the residual graph has two directed edges \((u, v)\) and \((v, u)\). The weight on the forward edge \((u, v)\) is said to be \( k_e^+ = c_e^+(x_e) \) and the reverse edge has a negative weight of \( -k_e^- = -c_e^-(x_e) \). Now for any given node \( v \), it is not hard to verify that there exists at least one \( v-s \) path with a negative total weight denoting the change in cost by removing an infinitesimal unit of flow along that path. Then, \( \pi_v^- \) is the absolute value of the total cost of the minimum cost \( v-s \) path. Similarly \( \pi_v^+ \) denotes the cost of sending some flow along the minimum cost \( s-v \) path, which is positive.

We begin by making the following observations about the node potentials when the marginal costs are derived from a min-cost flow.

1. For the sink node \( t \), \( \pi_t^- = r^-(x) \) and \( \pi_t^+ = r^+(x) \). Indeed \( r^-(x) \) and \( r^+(x) \) denote the cheapest possible way of removing and sending one infinitesimal unit of flow respectively, which coincide with the definition of node potentials for the sink node \( t \).

2. For every node \( v \), \( \pi_v^- \leq \pi_v^+ \). If this is not true, then we would have identified a negative cycle in the residual graph.

3. For every edge \( e = (u, v) \) in the original graph, \( \pi_v^- \geq \pi_u^- + c_e^-(x_e) \) and \( \pi_v^- \geq \pi_v^- - c_e^+(x_e) \).

4. For every edge \( e = (u, v) \) in the original graph, \( \pi_v^+ \leq \pi_u^+ + c_e^+(x_e) \) and \( \pi_u^+ \leq \pi_v^+ - c_e^-(x_e) \).

The third observation is true because the path from \( v \) to \( s \) via \( u \) is a candidate for the path which provides the maximum ‘gain’ in cost from \( v \) to \( s \). So the cost along this path is just \( c_e^-(x_e) + \pi_u^- \) which provides a lower bound for the actual maximum savings path from \( v \) to \( s \). Similarly, returning to the residual graph interpretation, flow can be removed from \( u \) to \( s \) by sending some additional flow along \((u, v)\) and then removing the sent flow from \( v \) to \( s \). So, this path acts as a lower bound for \( \pi_u^- \). A similar argument can be used to prove Observation (4). Before giving our algorithm to adjust prices along the paths, we first show how these node potentials can be applied to derive edge prices.
Lemma G.7. Given any set of node potentials obeying Observations (2), (3) and (4) above, if we define a quantity \( \tilde{k}_e \) on every flow carrying edge \( e = (u,v) \) to be \( p_v - p_u \) and on non-flow-carrying edges to be \( c_e^+(0) = 0 \), then the vector of \( \tilde{k}_e \)'s satisfies the following conditions

1. For all \( e \), \( c_v^-(x_e) \leq \tilde{k}_e \leq c_e^+(x_e) \).

2. For any flow-carrying path \( P \), \( \sum_{e \in P} \tilde{k}_e = \pi^+_t \). For any non-flow-carrying path, this quantity cannot be smaller.

Proof. The first part of the lemma follows almost directly from Observation (3). First notice that any flow carrying is a simple path and thus each node must appear only once on this path. So \( \sum_{e \in P} \tilde{k}_e = \sum_{e = (u,v) \in P} p_v - p_u = \pi^+_t - \pi^+_s = \pi^+_t \). Now what about non-flow carrying simple paths (paths with cycles will clearly have a larger cost)? Let \( P' \) be some such path. There must be several contiguous sequences in this path consisting only of non-flow carrying edges. Let \( P'' = e_1, e_2, \ldots, e_{|P''|} \) be any such contiguous segment connecting two nodes \( v_i \) and \( v_{i+1} \). Using the second half of Observation (3) for all nodes in this sequence, we get \( \pi^-_{v_i} \geq \pi^-_{v_{i+1}} - 0 \) since \( c_e^+(x_e) = 0 \) for all these edges as they do not carry flow.

Let us decompose the path \( P'' \) into flow-carrying and non-flow carrying segments. In particular, let \( u_1, v_1, u_2, v_2, \ldots, u_k, v_k \) denote the starting and ending nodes of every flow carrying segment in that order so that \( u_1 = s \) and \( v_k = t \). Then \( \sum_{e \in P''} \tilde{k}_e = \sum_{i=1}^{k} (\pi^-_{v_i} - \pi^-_{u_{i+1}}) \). Using, our previous observation that \( \pi^-_{v_i} \geq \pi^-_{u_{i+1}} \), this becomes,

\[
\sum_{e \in P''} \tilde{k}_e = \sum_{i=1}^{k-1} (\pi^-_{v_i} - \pi^-_{u_{i+1}}) + \pi^-_t \geq \pi^-_t.
\]

This completes the proof that all non-flow-carrying paths are priced no smaller than \( \pi^-_t \). \( \square \)

Notice that for the given instance, \( \pi^-_t \leq p^* \leq \pi^+_t \). If \( \pi^-_t = p^* \), then by Lemma G.7, we have proved our main claim since setting \( \tilde{c}_e = \tilde{k}_e \) for every edge satisfies the requirements of the claim. Suppose this is not the case and we have, \( \pi^-_t < p^* \leq \pi^+_t \), then somehow need to transform the instance into a new instance satisfying all the observations above such that \( \pi^-_t = p^* \). We do this via the following high-level procedure.

Consider the same graph with some given marginal costs \( k^-_e \) and \( k^+_e \) on the edges. At every iteration of the procedure, we choose an edge \( e \) and either increase \( k^-_e \) or decrease \( k^+_e \). However, we ensure that at each step some invariants similar to the observations above are maintained. Changing the marginal costs on the edges in turn changes the node potentials but we ensure that no negative cycles are created in the residual graph. We terminate the procedure when \( \pi^-_t = p^* \).

We now describe the procedure exactly. Consider the same network as before and initialize \( \forall e, k^-_e = c^-_e(x_e) \) as the negative marginal cost and \( k^+_e = c^+_e(x_e) \) as the positive marginal cost. That is if we call this iteration 0, then \( k^-_e(0) = c^-_e(x_e) \) and \( k^+_e(0) = c^+_e(x_e) \). Since the node potentials depend only on the negative and positive marginal costs, we can define the node potentials for this graph. At the beginning of some iteration \( i + 1 \), let \( (S, \bar{S}) \) denote a cut of this graph such that \( S \) is the set of nodes \( v \) satisfying \( \pi^-_v(i) = \pi^+_v(i) \). Clearly at initialization, \( s \in S \) and \( t \in \bar{S} \). Each iteration of the algorithm is described by the following steps,

1. Beginning of iteration \( i + 1 \)

2. Pick any edge \( e = (u,v) \) of the graph going across the cut

   - If \( u \in S \) and \( v \in \bar{S} \), then increase \( k^-_e \) until either \( \pi^-_v(i + 1) = \pi^+_v(i) \) or \( \pi^-_t(i + 1) = p^* \). In the latter case, we terminate the algorithm.
• If \( v \in S \) and \( u \in \bar{S} \), then decrease \( k_e^+ \) until either \( \pi_e^-(i + 1) = \pi_e^+(i) \) or \( \pi_e^-(i + 1) = p^* \).
In the latter case, we terminate the algorithm.

3. Recompute the \((S, \bar{S})\)-cut as defined above for the new values of \( k_e^- \) and \( k_e^+ \).

We now show some invariants that the algorithm maintains. Notice that the end of any iteration (say \( i \)) coincides with the beginning of the next iteration \((i + 1)\). When we say \( \pi_e^-(i) \) or \( k_e^-(i) \), we refer to the values at the end of iteration \( i \).

**Lemma G.8.** The above algorithm maintains the following invariants at the end of every iteration

1. \( c_e^+(x_e) \leq k_e^- \leq k_e^+ \leq c_e^-(x_e) \) for all \( e \).
2. For all \( v \), \( \pi_v^- \leq \pi_v^+ \).
3. For every edge \( e = (u, v) \), \( \pi_v^- \geq \pi_u^- + k_e^- \) and \( \pi_u^- \geq \pi_v^- - k_e^+ \).
4. For every edge \( e = (u, v) \), \( \pi_v^+ \leq \pi_u^+ + k_e^+ \) and \( \pi_u^+ \leq \pi_v^+ - k_e^- \).

**Proof.** We prove this by induction on the steps of the algorithm. Since we initialized \( k_e^-(0) = c_e^-(x_e) \) and \( k_e^+(0) = c_e^-(x_e) \), the invariants are definitely true at the base step. At each iteration, we pick a single edge going across the cut and either increase \( k_e^- \) or decrease \( k_e^+ \), so we just have to show that increasing or decreasing these as defined in the algorithm will not violate the invariants. Suppose that the invariants above are true until the end of the \( i \)th iteration. First consider the case when in the \( i + 1 \) iteration we pick some edge \( e = (u, v) \) such that \( u \in S \), \( v \in \bar{S} \) and so we increase \( k_e^- \).

At the beginning of the iteration \( k_e^-(i) \) and \( k_e^+(i) \) still satisfy the requirements before we increase \( k_e^- \). Since we do not alter \( k^- \) and \( k^+ \) for any other edge, we only need to show that the first invariant is not violated for \( e \). Recall that by definition, \( \pi_u^- (i) = \pi_u^+ (i) \) and \( \pi_v^- (i) < \pi_v^+ (i) \). During this iteration, we increase \( k_e^- \) such that the path from \( v \) to \( s \) along which the maximum cost can be saved must pass through \( e \) and moreover, the cost saved cannot be larger than \( \pi_v^+ (i) \). We begin by making three simple observations that must be obeyed after the increase in \( k_e^- \).

1. **The new value of \( k_e^- \), i.e., \( k_e^-(i + 1) \) cannot be larger than \( \pi_v^+(i) - \pi_u^-(i) \).**
   This is not hard to see since when \( k_e^-(i + 1) = \pi_v^+(i) - \pi_u^-(i) \), we have a flow removing path from \( v \) to \( s \) that removes flow on edge \( e \) providing a gain of \( \pi_v^+(i) - \pi_u^-(i) \) and then recursively removes flow from \( u \) along the maximum gain path of \( u \), which provides a gain of at least \( \pi_u^-(i) \). The total gain along this path is therefore \( \pi_v^+(i) \) and so the algorithm must halt here.

2. **At the end of iteration \( i + 1 \), the residual graph cannot have any negative cycles and therefore all the maximum gain or minimum cost paths are simple paths.**
   Assume recursively that the residual graph does not have any negative cycles up to iteration \( i \). Now, the only edge whose marginal cost we’ve changed is the reverse edge \((v, u)\) whose cost went from \( -k_e^-(i) \) to \( -k_e^-(i + 1) \). So any new negative cycle, if at all it exists now must use this edge \((v, u)\). But we claim that the rest of the cycle which is essentially some path from \( u \) to \( v \) cannot have a cost smaller than \( \pi_v^+(i) - \pi_u^+(i) \). This is easy to see because if such a path with smaller cost exists then at the end of iteration \( i \), while computing \( \pi_v^+ \), we could have sent flow from \( s \) to \( u \) and then along this path from \( u \) to \( v \), which is smaller than \( \pi_v^+(i) \), a contradiction. So the total cost along the cycle after the increase in \( k_e^- \) is not smaller than \( -k_e^-(i + 1) + \pi_v^+(i) - \pi_u^+(i) \geq 0 \).

3. **At the end of iteration \( i + 1 \), \( \pi_v^+(i + 1) = \pi_v^+(i) \), \( \pi_u^+(i + 1) = \pi_u^+(i) \).**
   The first statement that \( \pi_v^+(i + 1) = \pi_v^+(i) \) is easy to see. \( \pi_v^+ \) is the minimum cost of sending
flow from $s$ to $v$ and such a path cannot send flow to $v$ and then remove flow along $e$. At iteration $i + 1$, the maximum gain path for $u$ cannot remove flow along the edge $e$ since it would violate the simple path condition, so $\pi_u^-(i) = \pi_u^+(i)$. Suppose that $\pi_u^+(i + 1)$ is no longer equal to $\pi_u^+(i)$. Then, it could have only decreased since looking at the residual graph, the only change is that $k_e^{-}$ became more negative. Then, it is the case that,

$$\pi_u^+(i) > \pi_u^+(i + 1) = \pi_u^+(i + 1) - k_e^{-}(i + 1) \geq \pi_u^+(i).$$

This is a contradiction since at iteration $i$, $\pi_u^-(i) = \pi_u^+(i)$.

Now we can show that Invariant (1) is true at the end of iteration $i + 1$ as well. Using the fact that $\pi_u^-(i) = \pi_u^+(i)$, we can derive a bound for $k_e^{-}(i + 1)$ as,

$$k_e^{-}(i + 1) \leq \pi_v^+(i) - \pi_u^-(i)$$

$$= \pi_v^+(i) - \pi_u^+(i)$$

$$\leq \pi_v^+(i)$$

By Invariant (4), $\pi_v^+(i) \leq \pi_u^+(i) + k_e^+(i)$

$$= \pi_v^+(i)$$

So, the first invariant is not violated at the end of the current iteration. We also make a small remark about the node potentials here. Since the maximum gain path for $v$ at the end of the iteration uses edge $e$, we have

$$\pi_v^-(i + 1) = k_e^-(i + 1) + \pi_u^-(i) \leq \pi_u^+(i) - \pi_u^-(i) + \pi_u^+(i) = \pi_v^+(i).$$ \hspace{1cm} (12)

Moving on to invariant (2), Equation [12] shows that $\pi_v^-(i + 1) \leq \pi_v^+(i) = \pi_v^+(i + 1)$. Invariant (2) is therefore, not violated for vertex $v$ or $u$ (Observation 3). What about the other nodes? First, we show that the increase in $k_e^{-}$ does not lead to a change in $\pi_w^+$ for any node $w$. In other words, the cost of sending an infinitesimal unit of flow from $s$-$w$ does not change. Assume by contradiction that for some $w$, the quantity $\pi_w^+(i + 1) < \pi_w^+(i)\textsuperscript{2}$. Then, since we changed the marginal cost for only one edge, the minimum cost $s$-$w$ path for sending flow must contain $e = (u, v)$. In particular, since we only increased the negative marginal cost of $e$, the minimum cost $s$-$w$ path must be removing flow along $e$. So, $\pi_w^+(i + 1) = \pi_v^+(i + 1) - k_e^-(i + 1) + q$, where $q$ represents the cost of the path from $u$ to $w$. Since $\pi_v^+(i + 1) = \pi_v^+(i)$ and $k_e^-(i + 1) \leq (\pi_v^+(i) - \pi_u^+(i))$, we have

$$\pi_w^+(i + 1) = \pi_v^+(i) + q$$

So, we have shown that $\pi_w^+(i) > \pi_v^+(i) + q$. However, this contradicts the fact that we have sent flow on the minimum cost path at time $i$, since we might have as well directly sent flow from $s$ to $u$ on a path costing $\pi_u^+(i)$. This means that $\pi_w^-(i) \leq \pi_u^+(i) + q$, and so the positive potential on any node cannot change. What about the negative potential? Since we are increasing $k_e^-$, it might increase for some nodes. However, we show that Invariant (2) is still obeyed and so for every $w$, $\pi_w^-(i + 1) \leq \pi_w^+(i + 1) = \pi_w^+(i)$. Suppose the negative potential increases for some node $w$. Then the path from $w$ to $s$ must be removing flow on the edge $e$. So this gives us, $\pi_w^+(i + 1) = q_1 + k_e^-(i + 1) + q_2 \leq q_1 + (\pi_v^+(i) - \pi_u^+(i)) + \pi_u^+(i + 1)$, where $q_1$ is the amount gained by removing flow from $w$ to $v$ and $q_2$ from $u$ to $s$. Now let us go back to time $i$, one possible way

\footnote{We have already seen from the residual graph interpretation that this cannot increase}.
of sending flow from \( s \) to \( v \) is to send flow from \( s \) to \( w \) and then remove flow from the \( w-v \) path gaining cost of \( q_1 \). Mathematically, this gives us \( \pi^+_v(i) \leq \pi^+_w(i) - q_1 \). Now we can show our desired bound, namely

\[
\pi^+_w(i + 1) \leq q_1 + (\pi^+_v(i) - \pi^-_u(i)) + \pi^-_u(i + 1) \\
= (q_1 + \pi^+_v(i)) \\
\leq \pi^+_w(i) \\
= \pi^+_w(i + 1)
\]

This completes the proof of invariant (2).

Invariants (3) and (4) must hold, because they are a property of node potentials and do not depend on the exact value of \( k^-_e \) or \( k^+_e \). In particular, given any edge \( e' = (w, y) \), the maximum gain from removing an infinitesimal unit of flow from \( s \) to \( y \) is not smaller than the gains from removing a unit of flow along the \( (w, y) \) edge and then removing the flow along the path providing maximum gains for \( w \). Mathematically, this translates to \( \pi^-_y \geq \pi^-_w + k^-_w \). Similarly given the same edge, flow from \( s \) to \( w \) can be removed by sending an infinitesimal amount of flow from \( w \) to \( y \) and then removing flow along the maximum gains path for \( y \). Mathematically, this means that \( \pi^-_w \geq \pi^-_w - k^+_e \). Invariant (4) can be argued similarly.

**What if \( u \in S \) and \( v \in S \).**

The proof for this case is similar so we only sketch it. By definition, \( v \) must satisfy \( \pi^-_v(i) = \pi^+_v(i) \). In this case, we decrease \( k^+_e \) until \( \pi^-_u(i + 1) = \pi^+_u(i) \). As with our previous proof, it is not hard to see that at iteration \( i + 1 \), \( \pi^-_v(i) \geq \pi^-_u(i + 1) = \pi^-_u(i) - k^+_e(i + 1) \). This means that, \( k^+_e(i + 1) \geq \pi^-_v(i) - \pi^+_u(i) \geq k^+_e(i + 1) \). The last inequality comes from Invariant (4) which states that \( \pi^-_u(i) \leq \pi^+_u(i) - k^+_e(i) \). Invariant (1) is therefore not violated.

Once again, \( \pi^+_w(i + 1) = \pi^+_w(i) \) for every node \( w \), since we can show as we did previously that the minimum cost path for sending flow from \( s \) to \( w \) will not add any flow on the \( (u, v) \) edge. Now consider \( \pi^-_w(i + 1) \). It will change only if the new ‘maximum savings’ path will add flow on \( e \). This means that (if \( q_1 \) is as defined before)

\[
\pi^-_w(i + 1) = q_1 - k^+_e(i + 1) + \pi^-_w(i) \\
\leq q_1 - (\pi^-_v(i) - \pi^+_u(i)) + \pi^-_v(i) \\
= q_1 + \pi^+_u(i) \\
\leq \pi^+_w(i) = \pi^+_w(i + 1)
\]

Since \( \pi^+_u(i) \leq \pi^+_w(i) - q_1 \) by Invariant (4)

\[\square\]

**Lemma G.9.** As long as \( \pi^-_i < p^* \), there exists a cut \((S, \bar{S})\) as defined above with \( t \in \bar{S} \). Moreover, at every iteration of the algorithm, for some edge \( e \), there is either a non-zero increase in \( k^-_e \) or a non-zero decrease in \( k^+_e \).

**Proof.** The first part is easy to see. Recall from the proof of Lemma G.5 that \( \pi^+_v \) does not change for any node \( v \). Therefore \( \forall v, \) at any iteration \( i, \pi^+_v(i) = \pi^+_v(0) \). This means for the sink node \( \pi^+_v(i) = r^+(x) \) at every iteration. In order for \( t \) to be in the \( S \) side of the cut at the end of some iteration \( i \), it must be that \( \pi^-_i(i) = r^+_i(i) = r^+(x) \geq p^* \). So the contraposition of this statement is that as long as \( \pi^-_i(i) < p^*, t \in \bar{S} \). So a non-trivial cut must exist at every iteration of the algorithm.

Suppose at the beginning of some iteration \( i + 1, \pi^-_i(i) < p^* \), and we pick some edge \( e = (u, v) \).

Is it possible that \( k^-_e = k^+_e \) and therefore there is a zero change in the marginal costs in this iteration? Assume by contradiction that this is the case. First suppose that \( v \in \bar{S} \) and \( u \in S \) so
that $\pi_u^-(i) = \pi_u^+(i)$ and $\pi_u^-(i) > \pi_v^-(i)$. We know from Invariants (3) that the following must be true,

$$\pi_v^-(i) \geq \pi^- u(i) + k_e^-(i) = \pi^- u(i) + k_e^+(i)$$

and by invariant (4),

$$\pi_v^+(i) \leq \pi_u^+(i) + k_e^+(i) \leq \pi_u^+(i) + (\pi^- v(i) - \pi_u^-(i)) = \pi_e^-(i).$$

This is clearly a contradiction since $\pi_v^+(i) > \pi_u^-(i)$. Similarly suppose the edge $e = (u, v)$ is directed such that $u \in \bar{S}$. Then, assuming to the contrary that $k_e^-(i) = k_e^+(i)$, gives us a contradiction via the exact same inequalities mentioned above.

Now, it is easy to why the algorithm must terminate when $\pi^- v(K) = p^*$ at each iteration, for some edge $e$, the difference $k_e^+ - k_e^-$ strictly decreases without violating $k_e^+ \geq k_e^-$. And as long as $\pi^- e < p^*$, we are guaranteed an edge satisfying $k_e^- < k_e^+$. This cannot happen forever and so the algorithm must terminate when $\pi^- t = p^*$.

Once this set of node potentials are reached, we can just apply Lemma 7.7 and obtain a $\tilde{c}_e = \bar{k}_e$ for each as defined in the lemma. That is, the quantity $\tilde{c}_e$ on each edge $e = (u, v)$ is simply $\pi^- v(K) - \pi_u^-(K)$. By Lemma 7.7, we are guaranteed that for all flow carrying paths these quantities sum up to $\tilde{c}_e$ and that $c_e^-(x_e) \leq \tilde{c}_e c_e^+(x_e)$ since our node potentials satisfy the required invariants.

Claim 7.5. For instances with MPE demand $\lambda$ and $M \geq 1$, $\exists$ a Nash Equilibrium which is either optimal or satisfies

$$\lambda(\tilde{x}) - r^+(\tilde{x}) \leq M\tilde{x}|\lambda'(\tilde{x})| \leq \lambda(\tilde{x}) - r^-(\tilde{x}).$$

Proof. First suppose that the optimum solution $x^*$ satisfies, $M x^*|\lambda'(x^*)| \leq \lambda(x^*) - r^-(x^*)$ and that $r^-(x^*) \leq r^+(x^*) \leq \lambda(x^*)$. In the first case, we set $p^* = r^-(x^*)$ and use Claim 7.3 to obtain $\tilde{c}_e$ on all edges. These quantities satisfy the requirements of Lemma 7.6 and so we can set prices as mentioned in that Lemma to obtain a Nash Equilibrium.

Next if the optimum still satisfies $M x^*|\lambda'(x^*)| \leq \lambda(x^*) - r^-(x^*)$ but $r^-(x^*) \leq \lambda(x^*) \leq r^+(x^*)$. Then we set $p^* = \lambda(x^*)$, run the algorithm in claim 7.3 and price each edge as mentioned in Lemma 7.5. This clearly is a Nash Equilibrium.

Finally, what if $M x^*|\lambda'(x^*)| > \lambda(x^*) - r^-(x^*)$? Then, we claim that for MPE functions, there must exist some $\tilde{x} > 0$ such that $\lambda(\tilde{x}) - r^+(\tilde{x}) \leq M\tilde{x}|\lambda'(\tilde{x})| \leq \lambda(\tilde{x}) - r^-(\tilde{x})$. This is true because as $x \to 0$, $M x|\lambda'(x)| < \lambda(x) - r^+(x)$ and so there must exist an intermediate $\tilde{x} > 0$ satisfying the requirements. Now, we set $p^* = \lambda(\tilde{x}) - M\tilde{x}|\lambda'(\tilde{x})|$ and run the algorithm of Claim 7.3 to obtain quantities $\tilde{c}_e$ on all edges. These quantities clearly fulfill the requirements of Lemma 7.5 so we can price edges as mentioned in the lemma to obtain a Nash Equilibrium. This completes our existence proof.

G.3 General markets with non-combinatorial bundles

Uniform Demand Markets with arbitrary combinatorial valuations

Theorem 7.6. The Price of Stability for the above Generalized Pricing Game for uniform demand buyers with arbitrary combinatorial valuations is one.
Proof. Let $\vec{x}^*$ be the optimum allocation vector with the maximum possible allocation $x^*$. The following must be true for any bundle $S$ with $x^*_S > 0$ and any other bundle $S'$ with $x^*_{S'} \geq 0$:

$$v(S) - \sum_{e \in S} c_e(x^*_e) \geq v(S') - \sum_{e \in S'} c_e(x^*_e).$$

Let $M$ be the set of monopolies and virtual monopolies (VM) at the optimum: these are the items that belong to all bundles with non-zero allocation at the optimum. In particular, we claim that the following simple algorithm applied for the optimal allocation $\vec{x}^*$ gives us a Nash Equilibrium. Let $B$ be the set of bundles with non-zero consumption and $B'$ be the bundles with zero consumption. Begin with a price of $p^0_e = c_e(x^*_e)$ on each item. We now describe the ascending price process.

1. Pick any item $e$, which belongs to all optimal bundles $B$.

2. Increase the price on $e$ such that for all bundles $S$ in $B$, either $v(S) - \sum_{e \in S} p_e = 0$ or $\exists$ some $S' \in B'$, such that $v(S') - \sum_{e \in S'} p_e = v(S) - \sum_{e \in S} p_e$, i.e., $S'$ is now as good as any optimal bundle.

3. Repeat this process until you cannot increase the price on any VM anymore.

We claim that the above algorithm along with the flow $\vec{x}^*$ gives us a Nash Equilibrium. The following observations are pertinent here: (i) At any step of the algorithm, all bundles in $B$ give the same utility (ii) At any step in the above algorithm, no bundle in $B'$ gives more utility than any bundle in $B$ (iii) Non-monopoly items are priced at $c_e(x^*_e)$ throughout. The first point is true because the VMs belong to all bundles in $B$, so increasing the price on such an edge leads to an equal increase in the price of any such bundle. Moreover, initially all such bundles are priced such that the utility $v(S) - \sum_{e \in S} p^0_e$ is the same. For any item $e$, we stop the increase when some bundle $S'$ in $B'$ will become better than the bundles in $B$. This means that $e \notin S'$.

Now once again, it is obvious by Lemma 3.1 that no non-monopoly edge will increase or decrease its price. Suppose that the final price on any item $e$ is $p^*_e$. By point (ii) above, no bundle with zero allocation provides more utility than the bundles in $B$, so buyers are choosing best-response bundles. We only need to focus on seller behavior. We know $\forall S, S' \in B, v(S) - \sum_{e \in S} p^*_e = v(S') - \sum_{e \in S} p^*_e \geq 0$. Suppose this quantity equals 0, then no seller will increase the price on his item, because the bundles containing that item will give strictly negative utility and buyer will never purchase such a bundle. If the utility is strictly larger than zero, then it means the algorithm terminated because no VM could increase its price further without losing all its buyers to some bundle in $B'$, so in this case sellers would not increase their price, or they would lose everything.

Can sellers lower their price? We claim that this cannot happen. Suppose that the final price of all sellers is $p^*_e = p^0_e$. In this case, all items are priced at their marginal so no one would wish to lower their price as per Lemma 3.1. If even one VM is priced at $p^*_e > p^0_e$, this means that initially $v(S) - \sum_{e \in S} c_e(x^*_e) > 0$ and $x^* = T$, all the buyers have received some bundle so no buyer remains unallocated. This is true because if $v(S) - \sum_{e \in S} c_e(x^*_e) > 0$ for some bundle $S$ and there are still some buyers with no allocation, then if we allocate the bundle $S$ to an infinitesimal amount of buyers, we could receive additional non-negative utility of $v(S) - \sum_{e \in S} c_e(x^*_e)$ (Since $c_e(x)$ are all continuous). This violates the optimality of $x^*$ and therefore it must be that there are no unallocated buyers left.

So, decreasing the price has no effect. We therefore conclude that sellers have no incentive to change their price and ergo the prices given the algorithm along with the optimal flow $\vec{x}^*$ is a Nash Equilibrium optimizing social welfare. \qed
Lemma 7.7. The following is true for our algorithm

1. At any time step $t$, all bundles belonging to $B$ have the same price (say $p')$.

2. At any time step $t$, no bundle belonging to $B'$ can have a price smaller than that of any bundle in $B$.

3. For every good $e$ that belongs to $M_I$ at some time step $t$, there exists some bundle in $B'$ not containing $e$ that has the same price as all bundles in $B$ at that time step.

Proof. We prove these claims inductively. All the claims are true at the initial time step $t = 0$ when edges are priced at their marginal cost. Indeed all bundles belonging to $B$ must necessarily have the same marginal cost (optimality conditions). Moreover, no bundle in $B'$ can have a smaller marginal cost since these bundles are not consumed. If any virtual monopoly is inactive, then clearly there must exist some bundle in $B'$ not containing $e$ with the same marginal cost as all bundles in $B$.

Suppose that all the claims are true up to some time $t - 1$. After this time step we increase the price uniformly for all VMs still belonging to $M_A$. Note that since these are virtual monopolies, they belong to all bundles in $B$ with non-zero allocation. Therefore, the price of all the bundles in $B$ increases uniformly and thus our first claim must be true at time $t$. Now is it possible that increasing the price for VMs in $M_A$ can make some bundle in $B'$ cheaper than the optimal bundles? This is not the case. By definition any bundle in $B'$ that has the same price as the bundles in $B$ must contain all $e \in M_I$. If not, then the VM $e$ would have become tight and we would have shifted it to $M_I$. So this means that increasing the price on the active monopolies also leads to an uniform increase in the price of bundles belonging to $B'$ that have the same price as the optimal bundles. Moreover, we stop the increase when some new bundle in $B'$ has the same price as bundles in $B$ but does not contain some $e \in M_I$. This becomes time step $t$. So claim 2 is also true.

Notice that for every monopoly $e \in M_I$ at time $t - 1$, there must exist some bundle $B_1 \in B'$ such that the price of $B_1$ at $t - 1$ equals the price of any bundle in $B$. By the same reasoning as before, all VMs in $M_A$ at time $t - 1$ are present in $B_1$. Therefore, since these are the only goods whose prices are increased, the price of $B_1$ at time $t$ is still the same as the price of the optimal bundles so $e$ is still ‘tight’. Also, if some VM newly becomes tight at time $t$, then by definition, we transfer it to $M_I$ and there must exist some bundle not containing this item with the same price as the optimal bundles. So claim 3 is true as well.

Lemma G.10. As long as the production costs are convex, the set of min-cost allocations of magnitude $x$ is closed and convex.

Lemma 7.9. For any two minimum-cost allocations of magnitude $x$, say $\vec{a}$ and $\vec{b}$, and any good $e$, $c_e(a_e) = c_e(b_e)$, where $c_e$ denotes the marginal cost.

Proof. If for some edge $e$, $a_e = b_e$, then the lemma trivially holds. So without l.o.g, we assume that $a_e < b_e$. First consider any $\vec{x} = \alpha\vec{a} + (1 - \alpha)\vec{b}$ for some $\alpha > 0$. Clearly $\vec{x}$ is also a minimum cost allocation. Moreover, we know $C(\vec{x}) = \alpha C(\vec{a}) + (1 - \alpha)C(\vec{b})$ and so we have,

$$\sum_{e \in E} C_e(\alpha a_e + (1 - \alpha) b_e) = \sum_{e \in E} \alpha C_e(a_e) + (1 - \alpha) C_e(b_e).$$

But each term in the left hand side is no larger than the corresponding term in the right hand side for convex $C_e$. This means that the equality can hold iff $\forall e, C_e(\alpha a_e + (1 - \alpha) b_e) = \alpha C_e(a_e) + (1 - \alpha) C_e(b_e)$. Moreover this must be true for all $\alpha \in (0, 1)$. So this is only possible if the derivative of
$c_e$ is constant in this region. In other words, $c_e(x_e)$ must have the same value for all $x$ in $[a_e, b_e]$. The range is closed because $c_e$ is a continuous function.

We have established that the value of the differential cost function $c_e(x_e)$ is unique at any given allocation magnitude $x$ for a good $e$. We now show that this differential cost of a good varies continuously as we vary $x$.

**Claim G.11.** For any $e$, the marginal cost of the item at the min-cost allocation is a continuous function of $x$, i.e., $c_e(x_e)$ is continuously changing as we vary $x$ and compute the min-cost allocation. Therefore, $c_{B_i}(x)$ is also continuous for all bundles $B_i$.

The second part of the claim follows trivially from the first since the marginal cost of a bundle is simply the sum of marginal costs of the goods constituting it. Furthermore, the sum of a finite number of continuous functions is continuous. So we only prove the first part here.

**Proof.** We begin with a simple but fundamental result about minimum cost allocations that we will need later, namely that the limit of a sequence of minimum cost allocations of converging magnitude is also a min-cost allocation. i.e,

**Lemma G.12.** Let $(\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n)$ be a sequence of min-cost allocations of magnitude $x_1, x_2, \ldots, x_n$ approaching $X$ as $n \to \infty$. If $\lim_{n \to \infty} \vec{X}_n = \vec{X}$, then $\vec{X}$ is a min-cost allocation of magnitude $X$.

**Proof.** Without loss of generality, let us assume that that $x_i \geq X$ for all $i$ or else we can divide the sequence into two subsequences. Now for every $x_i$, consider the flow $\vec{X}_i$ but with the allocation on every bundle scaled down by a factor $\frac{x_i}{\vec{X}_i}$, i.e., the new allocation is $\frac{\vec{X}}{x_i} \vec{X}_i$. We know that for every $i$, the scaled down allocation has a cost that is no more than $R(x_i)$ since it is scaled down and no less than $R(X)$, which is the min-cost allocation of magnitude $X$. Now, as $n \to \infty$, the cost of the scaled down allocation approaches $C(\vec{X})$ and at the same time is still sandwiched between $R(x_n)$ and $R(X)$. Since $\lim_{n \to \infty} R(x_n) = R(X)$, it means that $C(\vec{X}) = R(X)$ and therefore $\vec{X}$ is indeed a min-cost allocation of magnitude $X$.

Now, we prove yet another fundamental result: for any given sequence of positive real numbers approaching $X$, there must exist a subsequence also approaching $X$ such that the min-cost allocations at each point along this subsequence approaches a min-cost allocation of magnitude $X$. As a first step towards this, we prove a simpler lemma that implies that for any single good $e$, there must be a subsequence along which the total allocation of that good converges to a finite value.

**Lemma G.13.** Let $(x_1, \ldots, x_n)$ be a sequence of positive real numbers (denoting the total allocation) approaching $X$ as $n \to \infty$. Then for any given good $e$, there must exist

1. a value $X_e$,
2. a subsequence $(y_1, \ldots, y_n)$ also approaching $X$ as $n \to \infty$,
3. min-cost allocations $(\vec{Y}_1, \ldots, \vec{Y}_n)$ where $\vec{Y}_n$ is a min-cost allocation of magnitude $y_n$ such that $\lim_{n \to \infty} (Y_n)_e = X_e$, where $(Y_n)_e$ is the allocation of good $e$ in $\vec{Y}_n$.

We also assume that the sequence $(x_1, \ldots, x_n)$ is bounded from above by (say) the magnitude of the optimal allocation $x^*$. This is a reasonable assumption which we will not violate when we invoke this lemma.
Claim 7.10. Our pricing rule returns the same set of prices \( (\vec{p}) \) on every minimum cost allocation of magnitude \( x \).

Proof. Intuitively this is true because for all minimum cost flows of a given magnitude and any given edge \( e \), \( c_e(x_e) \) is uniquely defined. Suppose that \( S \) is the set of all minimum cost allocations of this given magnitude. First if for all \( \vec{x} \in S \), an item \( e \) does not belong to \( M \), its price is \( c_e(x_e) \) (which is a fixed for a given \( x \)). So for such items, the pricing rule returns the same price for all allocations.

For every min-cost allocation, the marginal cost of every bundle remains the same. This is because the marginal cost of a bundle is simply the sum total of the marginal cost of all the items belonging to that bundle. Therefore, every iteration of our algorithm must be exactly the same.
for all such minimum cost allocations. Namely, any good \( e \) that becomes inactive at time step 0 for any one allocation \( \vec{x} \) must do so for all allocations. This is because for \( \vec{x} \), there exists a bundle not containing \( e \) with the same marginal cost as the bundles \( e \). Moreover, the marginal costs are constant across all min-cost allocations.

Notice now that the set of active monopolies for which we increase the price remains the same for all allocations. Moreover, the initial price (marginal cost) of all bundles are also the same across allocations. This means that (inductively) at ever time step \( t \leq K \), all goods and bundles have the same price and the same set of active monopolies become inactive. Therefore, we conclude that the algorithm terminates at the exact same set of prices for all minimum cost allocations of a given magnitude.

\[ \text{Claim G.14.} \quad \text{The price of every item is continuous within } I_v. \text{ Formally, for a given profile vector } \vec{v}, \text{ suppose } \exists \text{ an infinite sequence of points } (x_1, x_2, \ldots, x_n) \text{ converging to } X \text{ as } n \to \infty \text{ all belonging to } I_v. \text{ Moreover if } X \in I_v, \text{ then } \lim_{n \to \infty} p_e(x_n) = p_e(X) \text{ for all items.} \]

\[ \text{Proof.} \quad \text{Since the set of monopolies remains consistent in this sequence for all non-monopoly goods } e, \text{ the price on the non-monopolies is } c_e(x_i) \text{ which is continuous by Corollary G.11. Notice also that in this sequence all inactive monopolies become tight in the exact same order and the same active monopolies remain active. Now, notice that for a fixed } x_i, \text{ we can express the price of any good } e \text{ as a simple linear combination of the marginal costs of various bundles. For instance if } e_1 \text{ is the item that becomes inactive first (position 1) and the bundle not containing } e_1 \text{ that has the same price as the optimum bundles at time } t = 1 \text{ is } B_1, \text{ its given certificate. Suppose that } B^* \text{ is a bundle with non-zero allocation. Then, the increased price of } e_1 \text{ as a function of } x_i \text{ in this sequence is simply, } \bar{p}_{e_1}(x_i) = \frac{c_{B_1}(x_i) - c_{B^*}(x_i)}{|M| - |M \cap B_1|}, \text{ where } |M| \text{ is the cardinality of the set of monopolies.} \]

\[ \text{Notice that since } \mathcal{M} \text{ is fixed in this interval, the price depends only on the marginal costs, which are all continuous. That is, for all bundles } \lim_{n \to \infty} c_B(x_n) = c_B(X). \text{ So, } \bar{p}_{e_1}(x) \text{ is continuous for this sequence. Similarly if } e_2 \text{ is some item that becomes tight at position 2, then we can express } \bar{p}_{e_2}(x_i) \text{ as } \bar{p}_{e_1}(x_i) \text{ plus a function of the marginal costs, which is just a more complex expression but still a linear function of the marginal costs. Continuing this way, it is not difficult to see that all prices are continuous as } x_i \to X. \]

What if there are active monopolies? Then, suppose the algorithm proceeds for \( K - 1 \) rounds and in the last iteration, the price of the active monopolies are increased until \( P(x_i) \) becomes equal to \( \lambda(x_i) \). In that case, if the price of the optimal bundles at the beginning of round \( K \) is \( p_{B^*}^{K-1}(x_i) \), then the increase in the final iteration for any one active monopoly \( e \) is simply \( \frac{\lambda(x_i) - p_{B^*}^{K-1}(x_i)}{M_{K-1}} \). Note that \( M_{K-1} \) is just the set of active monopolies plus the ones which become inactive at the very end. \( \lambda(x_i) \) tends to \( \lambda(X) \) as \( i \to \infty \) by definition and the second term can be broken down into a linear function of the marginal costs and the increase in each previous iteration, which are all continuous.

The total price of any VM is simply \( c_e(x_i) + \bar{p}_e(x_i) \) which is a continuous function since the marginal costs are assumed to be continuous. Since the price on every good is continuous, the price on every bundle is also continuous. Finally since the set of active monopolies at termination \( M_A(x) \) is the same in this interval (or at least there is a tie-breaking rule which will make it the same), then the increased price on these fixed items \( \Gamma(x) \) is clearly continuous.

\[ \text{For the rest of this section, we will use } M^k \text{ to denote the set of active monopolies at the end of round (time step) } k \text{ of our algorithm and } p^k_B(x) \text{ to represent the price of any bundle } B \text{ at the same time instant. The following corollary gives us a closed form expression for the increase in price of all monopolies during any one round. The reasoning follows from the arguments made in Claim G.14.} \]
Corollary G.15. Suppose that for a fixed $x$, a good $e$ becomes tight in round $k+1$ during the course of our algorithm. Also suppose that $B_{k+1}$ is its witness bundle and $B^*$ is any optimal bundle. Then the price increase of all active monopolies during round $k+1$ is given by

$$p_{B_{k+1}}^k(x) - p_{B^*}^k(x) \frac{M^k - M^k \cap B_{k+1}}{M^k - M^k \cap B_{k+1}}.$$

Claim G.16. For any given profile vector, the set $I_v$ is closed. Formally suppose there $\exists$ a sequence $(x_1,\ldots,x_n)$ all of which belong to $I_v$ for some profile vector $\vec{v}$. If $\lim_{n \to \infty} x_n = X$, then $X \in I_v$.

Proof. We break this proof down into several parts, each identifying one new boundary event that could occur at $X$, potentially causing $X$ to not belong to $I_v$. However, we show that whatever this boundary event can be, the profile vector $\vec{v}$ still has to be active at $X$ by continuity arguments. We will be using the closed form prices given by Corollary G.15 throughout this proof. Since each profile vector $\vec{v}$ can be characterized as a 4-tuple, the only possible new events are

1. An item $e$ stops being a monopoly or a new item becomes a monopoly.
2. The order in which some (inactive) monopolies become tight changes.
3. A new witness bundle makes a given inactive monopoly tight.
4. The set of active monopolies changes.

- **Monopolies** We want to show that if $e \in M$ in profile $v$ which is a valid profile along the sequence, then there exists some valid flow at $X$, where $e \in M$. At every point in the infinite sequence $(x_1,\ldots,x_n,\ldots)$, $e \in M$ and at each of these points there exist a set of valid allocations where $e$ is a monopoly. From Lemma G.10 we know that the limiting case of a minimum cost allocation is also a minimum cost allocation. Therefore, the allocation of item $e$ at some minimum cost allocation of magnitude $X$ is given by $X_e = \lim_{n \to \infty} x_n e = \lim_{n \to \infty} x_n = X$. So for the given sequence of min-cost allocations, the limiting min-cost allocation also has $e$ to be a monopoly.

- **Non-Monopoly** This half of the claim is easier. If $e \notin M$ in this sequence, it means that for every $x_n$, $\exists$ some bundle whose marginal cost is equal to the marginal cost of any min-cost bundle carrying $e$. If $e$ has zero allocation in the sequence, then the marginal cost of optimal bundles can be strictly larger than that of the bundles containing $e$.

Consider the bundle $B'$ minimizing $\lim_{n \to \infty} c_{B_e}(x_n)$ over all such bundles that do not contain $e$. Let $B_e$ be some bundle with non-zero allocation at $X$ containing $e$. Clearly $\lim_{n \to \infty} c_{B_e}(x_n) \geq \lim_{n \to \infty} c_{B'}(x_n)$ since $e$ is not a monopoly at any point in the sequence. So $c_{B_e}(X) \geq c_{B'}(X)$, which implies that by our definition, $e$ is a non-monopoly at $x_1$ as well since there exists some bundle with equal price.

The following simple lemma helps us to establish that at the limit $X$, as long as the order does not change, the prices remain continuous.

**Lemma G.17.** Let $M$ be the set of monopolies for the sequence $(x_1,\ldots,x_n)$ converging to $X$ which become tight in the order $e_1,e_2,\ldots,e_M$. Let $k \leq M$ denote the largest index such that at $X$, the monopolies become tight in the order $e_1,e_2,\ldots,e_k$. Then for all $1 \leq i \leq k$, $\lim_{n \to \infty} p_{e_i}(x_n) = p_{e_i}(X)$.
We can prove this inductively. Suppose that $B_1$ is the bundle that is $e_1$’s witness at each $x_n$. Then $\bar{p}_{e_1}(x_n) = \frac{e_B(x_n) - e_{B^*}(x_n)}{|M - |M^c|B_1|}$, where $B^*$ is some bundle with non-zero allocation. The denominator is the same at $X$ and the numerator is a continuous function. Therefore $\bar{p}_{e_1}(x)$ is continuous. Similarly we can show that $\bar{p}_{e_i}(x)$ for all $i \leq k$ is also continuous. Now, using this we prove that at $x$, both the order in which the monopolies become tight and the witness bundles remain the same for some tie-breaking rule.

Once again assume that at $X$, $(e_1, \ldots, e_k)$ are the first few monopolies that become tight in the same order as in the sequence $(x_1, \ldots, x_n)$ as defined in the above lemma.

- **Order** Suppose the monopoly which becomes tight after $e_k$ is $e \neq e_{k+1}$. In the limit $x_n \to X$, $e_{k+1}$ becomes tight right before $e$. Suppose $B$ is the bundle which makes $e$ tight at $X$ and $B_{k+1}$ is the witness bundle for $e_{k+1}$ at this instant. Consider the instant at which $e$ becomes tight for a magnitude of $X$ and let $M^k$ be the set of active monopolies at this instant including $e$. $\bar{p}_e(X) = \bar{p}_{e_1}(X) + \frac{p^k_{B}(X) - p^k_{B^*}(X)}{M^k - M^k \cap B}$. We also introduce two new quantities $\Delta_e(x)$ and $\Delta_{e_{k+1}}(x)$ for $e$ and $e_{k+1}$ at every $x$, which are defined as,

$$
\Delta_e(x) = \frac{p^k_B(x) - p^k_{B^*}(x)}{M^k - M^k \cap B},
\Delta_{e_{k+1}}(x) = \frac{p^k_{B_{k+1}}(x) - p^k_{B^*}(x)}{M^k - M^k \cap B_{k+1}}.
$$

These quantities refer to the price increase of every active monopoly required after $e_k$ became inactive to make the goods $e$ and $e_{k+1}$ tight respectively. Recall that the price of every active monopoly is increased uniformly so during any one round, the price increase of the active monopolies is the same.

In particular, since for the $x_i$’s $e_{k+1}$ becomes tight before $e$, this means that $\Delta_{e_{k+1}}(x_i)$ is exactly equal to $\bar{p}_{e_{k+1}}(x_i) - \bar{p}_e(x_i)$. This is the price increase of $e_{k+1}$ after $e_k$ becomes tight. This quantity has to be smaller than $\Delta_e(x)$ since $e_{k+1}$ becomes tight first. Similarly at $X$, $\Delta_e(X)$ gives us the price increase of $e$ after $e_k$ becomes tight. However, since $\Delta_{e_{k+1}}(X)$ is the increase in price of active monopolies required to make $e_{k+1}$ tight after $e_k$ became inactive, this means that $\Delta_{e_{k+1}}(X) > \Delta_e(X)$.

We can now infer that

$$
\lim_{n \to \infty} \Delta_e(x_n) = \Delta_e(X) < \Delta_{e_{k+1}}(X) = \lim_{n \to \infty} \Delta_{e_{k+1}}(x_n).
$$

This is however a contradiction since we know that for all $x_n$, $\Delta_e(x_n) \geq \Delta_{e_{k+1}}(x_n)$ because $e_{k+1}$ becomes tight before $e$. This way, we can inductively prove that the order in which the monopolies become inactive at $X$ is the same as in the sequence $(x_1, \ldots, x_n)$.

- **Bundle** For an inactive monopoly $e_k$, suppose a new bundle $B'_k$ makes it tight at $X$ as opposed to the limit where the certificate is $B_k$. Without loss of generality, we can assume that for all $e$ in the order before $e_k$, the certificate bundle remains the same (or just take the first such $k$ for which the certificate changes strictly for $e_k$). Clearly the price of the bundle $B'_k$ as $x_n$ approaches $X$ at the discrete instant $k$ is smaller than or equal to that of the bundle $B_k$. Mathematically, this means that in the limit,

$$
\Delta_B(x_n) := \frac{p^{k-1}_{B_k}(x_n) - p^{k-1}_{B^*}(x_n)}{M^{k-1} - M^{k-1} \cap B_k} \leq \frac{p^{k-1}_{B'_k}(x_n) - p^{k-1}_{B^*}(x_n)}{M^{k-1} - M^{k-1} \cap B'_k} := \Delta_{B'_k}(x_n),
$$

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where $\Delta_{B_k}(x)$ and $\Delta_{B_k^e}(x)$ are defined as mentioned above, analogous to our previous definitions of $\Delta_e(x)$. But since $B_k$ is not a witness bundle at $X$, this means $\Delta_{B_k}(X) > \Delta_{B_k^e}(X)$, which is a contradiction since these are all continuous functions. And so $B_k$ has to be a witness bundle at $X$.

- **Active Monopoly.** Is it possible that $e \in M_A$ as $x_n \to X$ but this is not true at $X$? We have already shown that at $X$, all the inactive monopolies become tight in the same order with the same witness bundle. Is it possible that a monopoly which was previously active becomes inactive in the final iteration at $x = X$? Let this monopoly be $e_{k+1}$ and $(e_1, \ldots, e_k)$ represent the monopolies which became inactive before $e_{k+1}$ (in that order). Let $B_{k+1}$ be the witness bundle that makes $e_{k+1}$ tight at $X$. Notice that since this bundle does not make $e_{k+1}$ tight for $x_n$, this can only mean

$$
\sum_{i=1}^k \bar{p}_{e_i}(x_n) + M_A \left( \bar{p}_{e_k}(x_n) + \frac{p_{B_{k+1}}^k(x_n) - p_B^k(x_n)}{M^k - M^k \cap B_{k+1}} \right) \geq \lambda(x_n) - r(x_n),
$$

where $M_A$ is the number of active monopolies at each $x_n$ as per the given profile vector. We now explain the above inequality: the first term in the LHS refers to the total increased price for monopolies $e_1$ through $e_k$. The second term inside the parenthesis is the total price increase required after $e_k$ becomes inactive if $e_{k+1}$ is to become tight with bundle $B_{k+1}$. Since at $x_n$, $e_{k+1}$ is an active monopoly, the term inside the parenthesis acts as an upper bound for $e_{k+1}$'s increased price. This because because for an active monopoly, the price of the monopoly cannot become tight before the algorithm terminates, by definition.

All the $M_A$ active monopolies have the same increased price and therefore, the second term in the LHS is simply an upper bound for the increased price of all active monopolies. The inequality implies that an upper bound on the total increased price of all monopolies cannot be smaller than the available slack $\lambda(x_n) - r(x_n)$.

Suppose that at $X$, $e_{k+1}$ is to be an inactive monopoly. We have already established the continuity of all the functions in the above inequality. Then taking the limit of the above equation as $x_n$ tends to $X$, would give us a jump discontinuity at $X$. This is a contradiction and therefore, by our definition of an active monopoly $e_{k+1}$ has to remain active at $x = X$.

- **Inactive Monopoly** Suppose that an inactive monopoly $e_k$ becomes active at $X$. Without loss of generality, let $k$ be the smallest index for which an inactive monopoly becomes active and let $B_k$ be $e_k$’s certificate as $x_n$ approaches $X$. Since the monopoly is inactive for all $x_n$, we have the following inequality analogous to Inequality 13 for active monopolies.

$$
\sum_{i=1}^{k-1} \bar{p}_{e_i}(x_n) + (M - (k - 1)) \left( \bar{p}_{e_{k-1}}(x_n) + \frac{p_{B_k}^{k-1}(x_n) - p_{B_k^e}^{k-1}(x_n)}{M^{k-1} - M^{k-1} \cap B_k} \right) \leq \lambda(x_n) - r(x_n).
$$

The first term in the left hand side of the above inequality is the increased price for the monopolies $e_1$ through $e_k$ and the second term inside the large parenthesis is the increased price of monopoly $e_k$, which acts as a lower bound for the price of the other monopolies. What the inequality represents is that the sum of the increased prices of all monopolies cannot exceed the total slack $\lambda(x) - r(x)$. This inequality should hold as in the limiting case of $x_n$ tending to $X$ and therefore at $X$ since these are all continuous functions. However,
that would mean that by definition $e_k$ cannot be a strictly active monopoly at $X$. So $B_k$ still remains $e_k$'s certificate at that point.

Now, it is not hard to reason that no combination of these events can lead to $\vec{v}$ becoming an invalid profile vector at $X$.

**Theorem 7.13.** As long as the demand function is continuously differentiable and belongs to the class MPE, there exists at least one $\tilde{x} \leq x^*$ which is a Nash Equilibrium.

**Proof.** First consider the case when running the algorithm at the optimum point $x^*$ returns a non-empty set $M_A(x^*)$. Define $x_0$ to be the smallest $x \leq x^*$ such that for all $x \geq x_0$, the set of active monopolies is non-empty. Notice that such a $x_0$ must exist since at $x^*$, $M_A(x^*)$ is non-empty. Moreover, since the region is closed, we know that $x_0$, all the monopolies belonging to $M_A(x_0)$ are both active and in-active and therefore the total price of all optimal bundles at $x_0$ is $P(x_0) = \lambda(x_0)$.

Recall that $\Gamma(x)$ is the quantity $\bar{p}_e(x)$, the increased price for all active monopolies.

Since Lemmas 3.3 and 3.4 did not depend on flows anywhere, the same argument applies for general markets as well. First suppose that at $x^*$, $\Gamma(x^*) \geq x^*|\lambda'(x^*)|$, then we set $\tilde{x} = x^*$ and we are done. By Lemma 3.3 no seller would wish to increase his price. This is because both the non-monopolies and the inactive monopolies satisfy the first condition of the Lemma, i.e., they are tight. The active monopolies satisfy the second condition that $\bar{p}_e \geq x|\lambda'(x)|$ at the given value. Further, we claim that no seller would wish to decrease his price. This is true since we are at the optimum, either $\lambda(x^*) = r(x^*)$ or $x^* = T$. In the first case, every good is priced at its marginal so no seller would wish to decrease prices. In the second case, $x^* = T$ and therefore there are no buyers left in the market so reducing the price has no effect. Now what if $\Gamma(x^*) < x^*\lambda'(x^*)$?

Suppose that at $x_0$, $\Gamma(x) \geq x_0|\lambda'(x_0)|$. Then we claim that there exists a $\tilde{x} \geq x_0$ satisfying the equilibrium condition, namely $\Gamma(\tilde{x}) = \tilde{x}\lambda'(\tilde{x})$. This follows because $\Gamma(x) - x|\lambda'(x)|$ is a continuous function so if it is positive at one point ($x_0$) and negative at the other ($x^*$), then clearly there must exist a root somewhere in that interval. It is easy to see that if such a point does indeed exist, it satisfies all our equilibrium conditions. First no active monopoly can increase or decrease the price due to Lemmas 3.3 and 3.4. Look at any inactive monopoly $e$. Clearly for this item, we know $\bar{p}_e(\tilde{x}) \leq \Gamma(x)$ because active monopolies have the largest increase in price. So this satisfies condition (3) of Lemma 3.4. Moreover, inactive monopolies cannot increase their price because by definition, there exists some other bundle not containing this and having the same price as all the used bundles.

Finally, what about $\Gamma(x_0) < x|\lambda'(x_0)|$? In this case, we set $\tilde{x} = x_0$ and claim that it is a Nash Equilibrium. First, it is not too hard to see that if this condition is true, then $x_0 > 0$. Let $N$ be the size (cardinality) of the largest bundle. Clearly at any value of $x$, $\Gamma(x) \geq \frac{\lambda(x) - r(x)}{N}$ according to Lemma 7.12. For $x_0$, it is true that $\frac{\lambda(x_0) - r(x_0)}{N} \leq \Gamma(x_0) < x_0|\lambda'(x_0)|$. If it is the case that $x_0 = 0$, then the function $\frac{x|\lambda'(x)|}{\lambda(x)}$ tends to a non-zero number as $x \to 0$ which violates the requirement for MPE functions. And so, $x_0 > 0$.

We now claim that if at $x_0$, $\Gamma(x_0) < x|\lambda'(x_0)|$, then this point along with the prices returned by our algorithm form a Nash Equilibrium. First for every monopoly edge $e$, $\bar{p}_e(x_0) \leq \Gamma(x_0) \leq x_0|\lambda'(x_0)|$, so no edge would wish to decrease its price as per Condition (3) of Lemma 3.4. The monopolies cannot increase their price either because they are all inactive. And so we are done.

Finally, the case when at $x^*$, there are no active monopolies is actually quite simple. We claim that our pricing rule at $x^*$, returns a Nash Equilibrium. First of all, there are no active monopolies, so no good can increase its price. Second, if $\lambda(x^*)$ was equal to $r(x^*)$, then simply pricing all the goods at their marginal cost suffices and this case does have active monopolies. Moreover, we
know from Proposition 2.2 (which in no way is only limited to flows) that if \( \lambda(x^*) > r(x^*) \), then \( x^* = T \). This means that there is no use in decreasing the price since the entire market has been allocated.

**Theorem 8.4.** A multiple-source (MS) series-parallel directed network with no monopolies admits a Nash Equilibrium that has the same social welfare as the optimum solution for any given instance where all edges have \( c_e(0) = 0 \). Therefore, the Price of Stability is one.

**Proof.** We begin by showing two simple structural properties on MS series-parallel graphs. If \( P \) is a given path on a graph, we use the notation \( P_{uv} \) to refer to the sub-path of \( P \) between nodes \( u \) and \( v \).

**Lemma G.18.** Let \( E_1 \) and \( E_2 \) be two node-disjoint paths in the graph between nodes \( u \) and \( v \) and \( w \in E_2 \). Then any \( w-t \) path must go through \( v \).

**Proof.** The series-parallel nature of the directed graph precludes the existence of bridges. Formally, if \( P_1 \) and \( P_2 \) are node-disjoint paths between two given nodes, then for any \( i \in P_1 \) and \( j \in P_2 \), there cannot exist a path between \( i \) and \( j \) (and vice-versa) \[18\]. For our given lemma, assume by contradiction that there exists a \( w-t \) path \( P' \) not containing \( v \). Let \( P \) be any \( v-t \) path and suppose that \( P' \) first intersects \( P \) at some node \( z \).

Consider the following two paths between the nodes \( u \) and \( z \): \( E_1 \rightarrow P_{uv} \) and \( E_{2uw} \rightarrow P'_{uw} \). Clearly, these two paths are mutually disjoint. But there exists a bridge between \( w \) and \( v \), namely \( E_{2uw} \), which contradicts the series-parallel property.

**Lemma G.19.** Let \( E_1 \) and \( E_2 \) be any two node-disjoint paths between two nodes \( u \) and \( v \) and \( w \in E_2 \). Then any \( s_j-w \) path must go through \( u \), where \( s_j \) is any source node.

In other words, this means that if there exists two sub-bundles which are complements of each other and a buyer \( s_i \) has access to one of them, then she must have access to the other one as well.

**Proof.** The series-parallel nature of the directed graph precludes the existence of bridges. Formally, if \( P_1 \) and \( P_2 \) are mutually disjoint paths between two given nodes. Then, for any \( i \in P_1 \) and \( j \in P_2 \), there cannot exist a path between \( i \) and \( j \) (and vice-versa) \[18\]. Now we can prove our lemma.

Suppose that the premise is false and there is some \( s_j-w \) path \( P' \) not containing \( u \). Let \( P \) be some \( s-u \) path in the graph (such a path must necessarily exist in a series-parallel graph). Among all the nodes on \( P \) that the path \( P' \) intersects, let \( z \) be the one the one closest to \( u \) (along path \( P \)). In other words, there exist a path from \( z \) to \( u \) along \( P \), and \( z \) to \( w \) along \( P' \) that do not have any nodes in common. Now consider the following two mutually disjoint paths, \( P_{zu} \rightarrow E_1 \) and \( P'_{zw} \rightarrow E_{2uw} \). There exists a bridge between \( u \) and \( w \) along \( E_2 \), which contradicts the series-parallel assumption. Therefore, any \( s_j-w \) path must contain \( u \).

Now consider any optimal solution \( x^* \). Suppose that we price each edge \( e \) at its marginal cost \( c_e(x^*) \). We claim that these prices along with the optimal flow give us a Nash Equilibrium. One half of the proof is fairly easy: Lemma 3.1 tells us that no edge would ever decrease its price from the marginal cost irrespective of the amount of flow it gains. So we only need to show that sellers cannot increase their price. We show something stronger namely that if any edge \( e \) increases its price from \( p_e = c_e(x^*) \), then it would lose all of its flow. This would imply that for every edge \( e \) and every source \( s_i \) sending some non-zero flow on that edge, there exists a \( s_i-t \) path not containing \( e \) with some flow on it (sent by \( s_i \)).
Consider any edge $e$ with some flow on it (or else its price is zero), and some source $s_i$ which has flow on that edge. The total price of any $s_i$-$t$ path with $s_i$-flow on it is $\lambda_i(x^*)$, where $x^*$ is the amount being sent by source $s_i$, and so all we need to show is that there exists some $s_i$-$t$ path not containing $e$ and having exactly this price. Let $P$ be any flow-carrying $s_i$-$t$ path that contains the edge $e$. Since no single edge monopolizes any source, this means that there must exist a $s_i$-$t$ path not containing $e$. More specifically, there is some predecessor $u$ and successor $v$ of $e$ along the path $P$, such that there exists an alternative $P'$ that has no edge in common with $P_{uv}$. That is $P'$ is a shortcut as we used for no monopolies case with just one source and sink (See Claim 3.2).

If we show that (upon marginal pricing), $P_{uv}$ and $P'$ have the same cost, then we are done since the new path $P_{s_iu} \to P' \to P_{tv}$ and the original flow carrying path $P$ would have the same price, and thus $e$ would lose everything if it were to increase its price. Suppose that $s_i$ sends non-zero flow on some edge $e_0 = (w_1, w_2)$ in $P'$, then we claim that $s_i$ must send some flow on a $u$-$v$ path that has no edge in common with $P_{uv}$. Since $e \in P_{uv}$ and $e_0 \in P'$ and the two paths are disjoint, then there cannot be any bridge connecting these two edges. More specifically, source $s_i$ is sending flow on $e_0$ through some path that does not contain $e$. All flow carrying paths must have the same marginal cost and therefore some price of $\lambda_i(x^*)$ and so for this case, $e$ cannot increase its price as it would lose its flow to the $s_i$-$t$ path not containing $e$. What if $s_i$ does not send flow on any edge belonging to $P'$?

In this case, we would like to show that $P_{uv}$ and $P'$ have exactly the same marginal cost or there exists some other path $P_j$ that is also mutually disjoint from $P_{uv}$ with the same price. Assume that $P_{uv}$ has a strictly smaller price $\sum_{e \in P_{uv}} c_e(x^*_e)$ than $P'$’s price of $\sum_{e \in P'} c_e(x^*_e)$. Note that $P'$ cannot have the smaller price because $s_i$ has access to both these sub-paths and if $P'$ has a smaller marginal cost, we could have easily shifted some flow on $P'$ and reduced the cost of the optimal solution.

Since $\sum_{e \in P'} c_e(x^*_e) > \sum_{e \in P_{uv}} c_e(x^*_e)$, there must be some flow carrying edge $e_1 = (w_3, w_4)$ on $P'$ that contains flow from some source $s_j$. By Lemma G.19, all $s_j$-$w_3$ paths must pass through $u$. And By Lemma G.18 all $w_3$-$t$ paths must go through $v$. Consider some path on which source $j$ sends non-zero flow throughout which also contains $w_3$ (say $P_j$). $P_j$ passes through both $u$ and $v$ but is disjoint from $P_{uv}$ because $P_j$ contains $w_3$ which also contained in $P'$. This means that if $P_{uv}$ and $P_{juv}$ are not disjoint, then there exists two internal nodes $u$ and $v$ in $P_{uv}$ and $P_{juv}$ (or vice-versa) which are connected which cause a bridge between $P_{uv}$ and $P'$. Therefore, it must be the case that $P_{uv}$ and $P_{juv}$ have no edge in common.

Now we know that source $i$ sends flow on $P_{uv}$ and source $j$ on $P_{juv}$. We claim now that $\sum_{e \in P_{uv}} c_e(x^*_e) = \sum_{e \in P_{juv}} c_e(x^*_e)$. If this is not the case, then one of $s_i$ or $s_j$ could shift their flow between these two sub-paths and reduce the overall cost, which is a contradiction. Therefore, for any edge $e$, there exists an alternative path not containing that edge. Therefore, marginal cost pricing results in a Nash Equilibrium.

\[\square\]

H Proofs from Section 8: When does competition lead to complete efficiency?

\textbf{Theorem 8.1}. For any single-source single-sink instance with edge capacities, $M$ monopolies, and demand function $\lambda(x) = ax^{1/r}$, the Price of Stability is one as long as $r > M$.

\textit{Proof.} Consider any optimal flow $(x^*_e)$ of size $x^*$. Clearly, since the demand function is defined for all $x > 0$, at the optimal flow $x^*$, the minimum cut of the network is saturated. First assume that at least one monopoly edge is saturated. That is, for some monopoly $c_e = x^*$. For ease of exposition,
we prove the claim when \( \lambda(x) = x^{-1/r} \), the same proof applies when the function is scaled by a constant. Observe that \( x\lambda'(x) = -\frac{1}{r}x^{-1/r} = -\frac{1}{r}\lambda(x) \). We now price each edge as follows. If it is a non-monopoly edge, then set \( p_e^* = 0 \); if it is a monopoly edge which is unsaturated, set \( p_e^* = \frac{1}{r}\lambda(x^*) \), and distribute the remaining price equally on the saturated monopolies. That is, if \( U \) denotes the set of unsaturated monopolies and \( S \) the set of saturated monopolies, then

\[
\begin{align*}
  p_e^* &= \frac{1}{r}\lambda(x^*) & e &\in U \\
  p_e^* &= \frac{1}{|S|}(\lambda(x^*) - \frac{|U|}{r}\lambda(x^*)) & e &\in S
\end{align*}
\]  

Note that since \(|S| + |U| = M\) and \( r > M \), then the price of each saturated monopoly \( p_e^* > \frac{1}{M}\lambda(x^*) \). We now claim that these prices along with the flow of \((x^*)\) form a Nash equilibrium for the given instance. Since the flow is optimal, this immediately gives us a Price of Stability of one. First observe that the cost of all paths exactly equals \( \lambda(x^*) \), so this is indeed a best-response flow to the prices.

Now we show that no non-monopoly edge has any incentive to raise or lower its price. The price on each such edge is 0, so they cannot lower costs. Similar to the proof of Claim 3.2 if these non-monopolies increase their cost, then all the flow will shift to an alternative \( s - t \) path so their profit is not increasing. Thus, it only remains to be shown that no monopoly edge can increase or decrease its price.

First consider the saturated monopolies. Clearly, these monopolies do not wish to decrease their price as they are already saturated and cannot sustain a larger flow. Suppose one of these monopolies \( e \) increases its price to \( \rho \), resulting in a flow of \((x^*)\). Define \( \pi(x) = px = [p_e^* + \lambda(x) - \lambda(x^*)]x \) to be the profit of \( e \) after changing its price to \( \rho \). To show that this deviation is not beneficial, we will show that \( \pi(x) \) is maximized at \( x = x^* \).

The derivative of \( \pi(x) \) is

\[
\pi'(x) = p_e^* - \lambda(x^*) + \lambda(x) + x\lambda'(x) \geq \frac{1}{M}\lambda(x^*) - \lambda(x^*) + \lambda(x) - \frac{1}{r}\lambda(x),
\]

due to our observations about \( p_e^* \) and \( x\lambda'(x) \) above. Thus, to show that \( \pi(x^*) \geq \pi(x) \) for all \( x \leq x^* \), we only need to show that the above is at least 0, i.e.,

\[
\frac{M - 1}{M}\lambda(x^*) \leq \frac{r - 1}{r}\lambda(x).
\]

This is true since \( r > M \) and \( \lambda \) is non-increasing, as desired. Thus, all saturated monopolies have no incentive to change their price.

Consider any unsaturated monopoly \( e \), with a price of \( p_e^* = \frac{1}{r}\lambda(x^*) \). Note that if \( p_0 \) is the total price of all the other monopolies, then \( p_0 = \frac{r - 1}{r}\lambda(x^*) \). Suppose this monopoly changes its price to \( \rho \) and the resulting flow is \( x \). Then, by the same argument as above, we know that its profit is \( \pi(x) = px = [p_e^* + \lambda(x) - \lambda(x^*)]x \), and so

\[
\pi'(x) = p_e^* - \lambda(x^*) + \lambda(x) + x\lambda'(x) = \frac{r - 1}{r}(\lambda(x) - \lambda(x^*)).
\]

Since \( \lambda(x) \) is non-increasing, we know that for all \( x \leq x^* \), \( \pi'(x) \geq 0 \), and for all \( x \geq x^* \), \( \pi'(x) \leq 0 \). Thus, \( \pi(x) \) is maximized at \( x^* \), and so there is no incentive for \( e \) to change its price.
The above argument of stability was only for the case when at least one monopoly is saturated. Now, suppose that no monopoly is saturated. Then we price all monopolies at \( p_e^* = \frac{1}{r} \lambda(x^*) \). Since the minimum cut must be saturated in an optimum flow, this cut consists only of non-monopoly edges. Distribute the remaining surplus of \( p = \lambda(x^*) - \frac{M}{r} \lambda(x^*) \) among any minimum cut. That is for any edge in the minimum cut, set the price equal to \( p \). Clearly every \( s-t \) path with flow is now priced at \( \lambda(x^*) \), while all other paths are more expensive. Thus, \( x^* \) is a best-response flow for these prices. By the same argument as above, the monopoly edges will not wish to increase or decrease their price and neither will the non-monopoly edges priced at zero. Now consider a non-monopoly edge priced at \( p \): it cannot increase its price as it will lose all its flow to an alternate path still priced at \( \lambda(x^*) \). If the edge decreases its price, then the flow on the edge can only increase, which is not desired as the edge is already saturated. This completes the proof that for capacitated networks with elastic demand, the Price of Stability is one.

Claim 8.3. For instances with multiple types of linear capacitated markets where the sources are all leaf nodes, the Price of Stability is one.

Proof. The proof is almost identical to the proof of Claim 8.2 so we only focus on the differences. Once again consider the optimal flow \( x^* \) and price each edge at its marginal cost. If the conditions in equations 5, 6 hold, then marginal pricing gives us a Nash Equilibrium. Suppose that is not the case, then the condition of Equation 5 may not hold. That is for some source \( i \) and every \( s_i-t_i \) path with non-zero flow \( P_i \), it may be the case that

\[
\lambda_i > \sum_{e \in P_i} c_e(x_e^*)
\]  

(16)

This happens because of the finite demand that the source has, so the source has exhausted its demand \( l_i \) of flow. In this case, since \( s_i \) is a leaf node, the sole edge \( e \) connecting \( s_i \) to the rest of the graph is a monopoly for source \( i \). Suppose the current price of this edge is \( p_e \), increase the price to \( p'_e = p_e + \lambda_i - \sum_{e \in P_i} c_e(x_e^*) \), where \( P_i \) is any \( s_i-t_i \) path with non-zero flow. Since the flow is a min-cost flow all such paths have equal marginal cost, so the increase in price is unambiguous. Repeat this for all sources which satisfy equation 16. Note that with the new prices \((x_e^*)\) is still a best-response flow for each buyer since the price of each \( s_i-t_i \) path for buyer \( i \) is now exactly \( \lambda_i \). Also note that we have increased prices on only the leaf edges, so the flow of other sources is not affected by this change.

Now, the proof of Claim 8.2 holds. No node in the network can increase its price as it would lose all the flow. Look at the leaf monopoly edges with increased price: if they decrease their price, then their flow cannot increase since that source node is already sending its full quota of \( l_i \) units of flow. Leaf monopolies which are still priced at the marginal have no incentive to decrease their price either. So this set of prices forms a Nash Equilibrium supporting the optimal flow. \( \square \)