Topological Parameters in Gravity

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Topological parameters in

- Canonical gravity
  (Boundary terms arising in the variation of Lagrangian can be ignored)

- Generic manifolds with boundaries
  (Boundary terms arising in the variation of Lagrangian cannot be ignored)
Quantization of gravity

- **Perturbative** quantization of Einstein-Hilbert gravity based on an expansion around a fixed metric $g_{\mu\nu}^{(0)}$ leads to uncontrollable infinities.

- **Nonperturbative** approaches are based on canonical quantization.

- However, a complete canonical quantization of gravity with metric is difficult; field equations (**Hamiltonian constraint**) are too complicated.

- The aim to find a simpler Hamiltonian constraint motivated a gauge theoretic description of gravity- the **SU(2)** formulation.
Gravity as a gauge theory

- The theory of (Lorentzian) gravity in 4-d can be described as a gauge theory, based on the gauge group $\text{SO}(3,1)$ (first-order formulation).

- The corresponding (Hilbert-Palatini) Lagrangian density:

\[
L(e^I_\mu, \omega^{IJ}_\mu) = \frac{1}{2} e e^\mu_\mu e^\nu_\nu R^{IJ}_{\mu\nu}(\omega)
\]
Gravity as a SU(2) gauge theory

- The action corresponding to the real SU(2) formulation was originally given in terms of Holst Lagrangian:

\[ L(e, \omega) = \frac{1}{2} ee^I e^J R^{IJ}_{\mu\nu}(\omega) + \frac{\eta}{2} ee^I e^J \tilde{R}^{IJ}_{\mu\nu}(\omega) \]

where \( \tilde{R}^{IJ}_{\mu\nu} = \frac{1}{2} \epsilon^{IJKL} R_{\mu\nu K L} \)

- The second term (Holst) involves the Barbero-Immirzi parameter \( \eta \); does not affect the classical EOM

- But it changes the canonical structure in a way that leads to a SU(2) gauge theoretic formulation for gravity theory

- Note that \( \eta \) is an additional coupling constant in first order gravity; does not appear in the standard metric formulation
Nice features of Holst formulation

- The phase-space variables are the SU(2) gauge fields $A$ and their conjugate $E$, analogous to the Yang-Mills phase space with the identifications
  \text{Grav:}(A^i_a \rightarrow \text{Connection fields}, \ E^a_i \rightarrow \text{Triads})
  \text{YM:}(A^i_a \rightarrow \text{Vector potential}, \ E^a_i \rightarrow \text{Electric field})

- The generator of the SU(2) gauge symmetry is Gauss’ law: $D.E \approx 0$
Unsatisfactory features of Holst Lagrangian

• Does not explain the origin of Immirzi parameter

• Does not provide a satisfactory prescription for the inclusion of matter couplings; Holst term needs non-universal modifications in presence of matter

Puzzles:

• What is $\eta$’s exact origin?
• Does the quantum theory see this parameter? (Yes! Area spectrum in LQG)
• Is there a more general SU(2) formulation with more such parameters?
A new Lagrangian for the SU(2) formulation  
(Date, Kaul and Sengupta, PRD 2009)

• Holst term is replaced by the Nieh-Yan topological density

\[ L(e, \omega) = \frac{1}{2} ee^\mu_I e^\nu_J R_{\mu\nu}^{IJ}(\omega) + \frac{\eta}{2} I_{NY} \]

• \( I_{NY} \) involves torsion, locally a total divergence:

\[ I_{NY} = \epsilon^{\mu\nu\alpha\beta} D_\mu e^I_\nu D_\alpha e_I^\beta + \frac{1}{2} ee^\mu_I e^\nu_J \tilde{R}^{IJ}_{\mu\nu} \]
\[ = \partial_\mu [\epsilon^{\mu\nu\alpha\beta} e^I_\nu D_\alpha e_I^\beta] \]

• \( I_{NY} \) is topological since it depends on the global properties of the manifold
• Does it lead to a real SU(2) theory of gravity as before (Holst)? (Hamiltonian analysis)

• Does it cure any of the problems plaguing the Holst formulation? (Yes!)
• SU(2) formulation with new Lagrangian

\[ L(e, \omega) = \frac{1}{2} e_I^\mu e_J^\nu R_{\mu \nu}^{IJ}(\omega) + \frac{\eta}{2} I_{NY} \]

Hamiltonian decomposition of tetrads

\[ e_t^I = N M^I + N^a V_a^I \quad \text{(time components)} \]
\[ e_a^I = V_a^I \quad \text{(space components)} \]
\[ [e_I^\mu (16) \equiv V_a^I, M^I, N^a, N \ (9 + 3 + 3 + 1)] \]

Hamiltonian decomposition of spin connections

\[ \omega_t^{IJ} \text{(time components)} \]
\[ \omega_a^{IJ} \text{(space components)} \]
\[ [\omega_{IJ}^\mu (24) \equiv \omega_t^{IJ}, \omega_a^{IJ} (6 + 18)] \]
• Redefinition of fields

Tetrads \((16=9+3+3+1)\) : \((E^a_i, \chi_i, N_a, N)\)

\[
E^a_i = 2\epsilon\epsilon_t[0e^a_i]
\]
\[
\chi_i = \frac{M_i}{M_0}
\]

Spin-connections \((24=9+9+6)\) : \((A^i_a, K^i_a, \omega^{IJ}_t)\)

\[
A^i_a = \omega^{(\eta)0i}_a = \omega^{0i}_a + \eta \tilde{\omega}^{0i}_a
\]
\[
K^i_a = \omega^{0i}_a
\]
Full Lagrangian density

\[ L = \frac{1}{2} e e^\mu_I e_\nu^J R_{\mu\nu}^{IJ}(\omega) + \frac{\eta}{2} I_{NY} \]

\[ = \hat{E}_i^a \partial_t A_a^i + F_i^a \partial_t K_a^i + t_I^a \partial_t V_I^a \]

\[ - NH - N^a H_a - \frac{1}{2} \omega^{IJ} G_{IJ} \]

where,

\[ \hat{E}_i^a = E_i^a + \frac{1}{\eta} \epsilon^{ijk} E_j^a \chi_k, \]

\[ F_i^a = \left( \eta + \frac{1}{\eta} \right) \left[ -\epsilon^{ijk} E_j^a \chi_k \right] \]

- No velocities for \( N, N^a, \omega_I^J \) implying the constraints: \( H \approx 0, H_a \approx 0, G_{IJ} \approx 0 \)
  (1 Hamiltonian, 3 Diffeomorphism and 6 Rotation+Boost constraints)

- The pairs \( (K_i^a, F_i^a) \) and \( (V_a^I, t_I^a) \) are not independent, leading to second-class constraints
  (their solution implies vanishing of torsion)
\[ L = \hat{E}^a_i \partial_t A_a^i + F^a_i \partial_t K_a^i + t_I^a \partial_t V_I^a - NH - N^a H_a - \frac{1}{2} \omega^I_J G_{IJ} \]

**Time-gauge**

- Expressions can be simplified by choosing three tetrad components \( \chi_i \) to be zero.
- This choice fixes the boost constraint, since \( \chi_i \approx 0 \) and \( G_{i}^{\text{boost}} \approx 0 \) form a second-class pair.
- In time-gauge, only nonvanishing momenta are \( \hat{E}^a_i = E^a_i \) (densitized triad), once the second-class constraints are implemented.
- Thus, the only canonical pair is \( (A_a^i, E^a_i) \).
SU(2) interpretation

• Final Lagrangian density:

\[ L = E^a_i \partial_t A^i_a - NH - N^a H_a - \frac{1}{2} \epsilon^{ijk} \omega^i_t G^{rot}_i \]

• 9 pairs \((A^i_a, E^a_i)\), 7 first-class constraints \((H, H_a, G^{rot}_i)\) - 2 degrees of freedom per space-time point

• The rotation law \(G^{rot}_i = D_a(A)E^a_i\) acts as the generator of SU(2) symmetry transformations

\[
\begin{align*}
\delta A^i_a &= \left[ \int \theta^k G^{rot}_k, A^i_a \right]_{P.B.} = D_a \theta^i \\
\delta E^a_i &= \left[ \int \theta^k G^{rot}_k, E^a_i \right]_{P.B.} = \epsilon_{ijk} \theta^j E^{ak}
\end{align*}
\]
Essence of all these..

- The new lagrangian leads to a real SU(2) gauge theoretic interpretation for gravity, exactly same as Barbero-Immirzi formulation.

- Provides a topological interpretation of Barbero-Immirzi parameter, and hence clarifies the issue of its origin (unlike Holst).

- A universal prescription for matter coupling, as the topological NY term remains the same with or without matter (unlike Holst) (spin-$\frac{1}{2}$ fermion, Supergravity theories-Kaul and Sengupta PRD, 2010).

- Hence, supercedes the Holst Lagrangian formulation.
Not the end of the story..

There are two other topological densities in 4-D gravity:

**Pontryagin class:**

\[
I_P = \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu IJ}(\omega) R_{\alpha\beta}^{IJ}(\omega)
= 4\partial_\mu \left[ \epsilon^{\mu\nu\alpha\beta} \omega_\nu^{IJ} \left( \partial_\alpha \omega_{\beta IJ} + \frac{2}{3} \omega_{\alpha I} K \omega_{\beta KJ} \right) \right]
\]

**Euler class:**

\[
I_E = \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu IJ}(\omega) \tilde{R}_{\alpha\beta}^{IJ}(\omega)
= 4\partial_\mu \left[ \epsilon^{\mu\nu\alpha\beta} \tilde{\omega}_\nu^{IJ} \left( \partial_\alpha \omega_{\beta IJ} + \frac{2}{3} \omega_{\alpha I} K \omega_{\beta KJ} \right) \right]
\]

In addition to the **Nieh-Yan class:**

\[
I_{NY} = \epsilon^{\mu\nu\alpha\beta} D_\mu e_\nu^I D_\alpha e_{I\beta} + \frac{1}{2} e e_\mu^I e_\nu^J \tilde{R}_{IJ}^{\mu\nu}
= \partial_\mu [\epsilon^{\mu\nu\alpha\beta} e_\nu^I D_\alpha e_{I\beta}]
\]
Thus, the most general Gravity Lagrangian in 4 D-

\[ L(e, \omega) = \frac{1}{2} ee^\mu_I e^\nu_J R_{\mu\nu}^{IJ}(\omega) + \frac{\eta}{2} I_{NY} + \frac{\theta}{4} I_P + \frac{\phi}{4} I_E \]  
(Kaul, Sengupta; PRD, 2012)

This has three independent dimensionless coupling constants in addition to the dimensionful Newton’s constant

Classical EOMs are independent of \( \eta, \theta, \phi \) but the canonical theory gets affected

Quantum theory may see them
• **Parametrisation and redefinition**

Tetrads $e^I_\mu := (E^a_i, \chi_i, N_a, N) \ (16 = 9 + 3 + 3 + 1)$

Spin-connections $\omega^{IJ}_\mu := (A^i_a, K^i_a, \omega^I_t) \ (24 = 9 + 9 + 6)$

• **Time-gauge Lagrangian**

$$
L = \hat{E}^a_i \partial_t A^i_a + \hat{F}^a_i \partial_t K^i_a + t^a_i \partial_t V^i_a - NH - N^a H_a - \frac{1}{2} \epsilon^{ijk} \omega^t_{jk} G^i_{rot}
$$

where,

$$
\hat{E}^a_i = E^a_i - 2 \left( \bar{e}^a_0 + \frac{1}{\eta} e^a_0 \right),
$$

$$
\hat{F}^a_i = 2 \left( \eta + \frac{1}{\eta} \right) \bar{e}^a_0,
$$

$$
e^a_{IJ}(\theta, \phi) = \epsilon^{abc} [(\theta + \eta \phi) R_{bcIJ} + (\phi - \eta \theta) \tilde{R}_{bcIJ}]
$$

(Earlier $\theta = \phi = 0$, $\hat{E}^a_i = E^a_i$, $\hat{F}^a_i = 0$)
Degrees of freedom

\[ L = \hat{E}_i^a \partial_t A_a^i + \hat{F}_i^a \partial_t K_a^i + t_i^a \partial_t V_a^i - NH - N^a H_a - \frac{1}{2} \epsilon^{ijk} \omega_t^{jk} G_{rot}^i \]

- \( N, N_a, \omega_t^{ij} \) are Lagrange multipliers, leading to the secondary constraints
  \( H \approx 0, \ H_a \approx 0, \ G_{rot}^i \approx 0 \)

- There are 9 pairs of second class constraints as \( (\hat{F}_i^a, \ K_a^i) \) are not independent fields

- There are another 9 pairs of second class constraints as \( (t_i^a, \ V_a^i) \) are not independent fields

- Finally in time gauge we have 9 canonical pairs \( (\hat{E}_i^a, \ A_b^j) \) and 7 first class constraints \( H, \ H_a, \ G_{rot}^i \) giving two degrees of freedom per space-time point
\textbf{SU(2) interpretation}

\begin{itemize}
  \item $G^\text{rot}_i = \eta D_a(A)^i \hat{E}_a^i + \epsilon_{ijk} K^j_a \hat{F}_k^a$ generate SU(2) gauge transformations:
    \[
    \left[ G^\text{rot}_i(x), \hat{E}_j^a(y) \right]_D = \epsilon^{ijk} \hat{E}_k^a \delta^{(3)}(x,y),
    \]
    \[
    \left[ G^\text{rot}_i(x), A^j_a(y) \right]_D = -\eta \left( \delta^{ij} \partial_a + \frac{1}{\eta} \epsilon^{ijk} A^k_a \right) \delta^{(3)}(x,y)
    \]
    and also
    \[
    \left[ G^\text{rot}_i(x), \hat{F}_i^a(y) \right]_D = \epsilon^{ijk} \hat{F}_k^a \delta^{(3)}(x,y)
    \]
    \[
    \left[ G^\text{rot}_i(x), K^j_a(y) \right]_D = \epsilon^{ijk} K^k_a \delta^{(3)}(x,y)
    \]
  
  \item Similarly, Dirac brackets of $H_a$ with various fields yield the Lie derivatives of these fields respectively, modulo SU(2) gauge transformations.
\end{itemize}
To summarise..

- We have a real SU(2) formulation of gravity with three topological parameters with the gauge field $A^i_a$

- The gauge-field coupling constant is the Barbero-Immirzi parameter as earlier: $A^i_a = \omega^{0i}_a + \eta \tilde{\omega}^{0i}_a$

- The other two parameters enter the definition of momenta conjugate to $A^i_a$:
  \[ \hat{E}^a_i = E^a_i + \mathcal{E}^a_i(\theta, \phi) \]

- The momenta $\hat{E}^a_i$ is not equal to the densitized triad $E^a_i$, thus the phase space is not the same as in Barbero-Immirzi formulation ($\theta = \phi = 0$)
• The canonical pair \((A, \hat{E})\) have a non-standard (Dirac) bracket, hence quantization using these may not be straightforward.

• However, it is possible to obtain a SU(2) description in terms of a new pair \((\bar{A}, \bar{E})\) which has the standard bracket: \([\bar{A}(x), \bar{E}(y)] = \delta^3(x, y)\).

• \((\bar{A}, \bar{E})\) are related to the Barbero-Immirzi phase space \((A, E)\) through a canonical transformation:

\[
\bar{A} = e^{J_{\theta,\phi}(A,E)} A \ e^{-J_{\theta,\phi}(A,E)} \\
= A + [J_{\theta,\phi}, A] + \frac{1}{2!}[[J_{\theta,\phi}, A], A] + ..
\]

\[
\bar{E} = e^{J_{\theta,\phi}(A,E)} E \ e^{-J_{\theta,\phi}(A,E)} \\
= E + [J_{\theta,\phi}, E] + \frac{1}{2!}[[J_{\theta,\phi}, E], E] + ..
\]

• This is a more natural pair for quantization (Kaul and Sengupta, PRD, 2012)
Summary

• Barbero-Immirzi parameter $\eta$ has a topological origin

• A complete theory of gravity has three independent topological parameters $\eta, \theta, \phi$

• With these, there exist more general real SU(2) formulations, not equivalent to the Barbero-Immirzi formulation (corresp to $\theta = \phi = 0$)

• The generalized SU(2) formulations with $\eta, \theta, \phi$ apply to any arbitrary matter-coupling
To explore further..
Implications for Quantum Geometry?

• In loop gravity framework, the spectrum of geometrical observables like area depend on $\eta$; Does it depend on $\theta, \phi$ also?

• Area operator in terms of the generalized SU(2) variables is not self-adjoint in the spin-network basis; the issue of the emergence of spatially discrete quantum geometry in loop gravity has to be revisited
A nonperturbative vacuum structure in gravity?

- Yang-Mills theory has a non-perturbative $\theta$ vacuum; can be understood through the presence of Pontryagin term $\theta F \tilde{F}$ in the effective Lagrangian; vacuum is characterised by homotopy maps $\Pi_3[SU(2)]$

- Gravity: $\eta I_{NY}$, $\theta I_P$, $\phi I_E$; A rich vacuum structure? (In Euclidean theory, Pontryagin $I_P$ corresponds to homotopy map $\Pi_3[SO(4)] \equiv Z + Z$ and Nieh-Yan $I_{NY}$ corresponds to $\Pi_3[SO(4)] + \Pi_3[SO(5)] \equiv Z + Z + Z$)
• How does Barbero-Immirzi parameter show up in other quantization formulations, e.g. string theory? Axionic moduli?

• Can the (anticipated) nonperturbative effects from $\eta, \theta, \phi$ be understood within nonperturbative formulations like causal dynamical triangulations?

• Nieh-Yan topological charge is \textbf{nontrivial} only for spacetime configurations with nonvanishing \textbf{torsion}: typically associated with degenerate metrics. What is their role in quantum gravity? (Regge, Tseytlin, Horowitz, Zanelli,..)

• Any role of \textbf{quantum torsion}? Such effects cannot be captured in Dirac quantization; one needs alternative methods (eg. Gupta-Bleuler, Coherent state quantization) [Sengupta, CQG, 2010]
What about generic manifolds with boundaries?
Topological densities in manifolds with boundaries

• Possible modifications in the action principle?

• How many independent topological parameters?
ALADS boundary

• At asymptotic boundary, the curvature tensor locally satisfies:

\[ R_{ab}^{IJ} + \frac{1}{l^2}e^{I}_{a}e^{J}_{b} = 0 \]

where, the AdS radius \( l \) is related to the cosmological constant \( \Lambda \) as: \( \Lambda = -\frac{3}{l^2} \)

(This definition of ALADS boundaries in first order gravity is based on the work by Zanelli et al., 2000, PRL, PRD)

• Action principle:

\[
L_{0}(e, \omega) = \frac{1}{8\kappa} \epsilon_{\mu\nu\alpha\beta} \epsilon_{IJKL} \left( e_{I}^{\mu} e_{J}^{\nu} R_{\alpha\beta}^{KL} + \frac{1}{l^2} e_{I}^{\mu} e_{J}^{\nu} e_{K}^{\alpha} e_{L}^{\beta} \right) \\
+ B
\]

• Add the NY density:

\[
L(e, \omega) = L_{0}(e, \omega) + \eta I_{NY}
\]
\[ L(e, \omega) = \frac{1}{8\kappa} \epsilon_{\mu \nu \alpha \beta} \epsilon_{IJKL} \left( \epsilon_{I}^{I} \epsilon_{J}^{J} R_{\alpha \beta}^{KL} + \frac{1}{l^{2}} \epsilon_{I}^{I} \epsilon_{J}^{J} e_{\alpha}^{K} e_{\beta}^{L} \right) + \eta I_{NY} + B \]

- Action has an extremum if
  \[ \delta B = -\epsilon^{abc} \left( \frac{1}{4\kappa} \epsilon_{IJKL} e_{a}^{K} e_{b}^{L} - \eta e_{a}^{I} e_{b}^{J} \right) \delta \omega_{c}^{IJ} \]
  \[ = -\frac{l^{2}}{2} \epsilon^{abc} \left( \frac{1}{4\kappa} \epsilon_{IJKL} R_{ab}^{KL} - \eta R_{abIJ} \right) \delta \omega_{c}^{IJ} \]

- These two terms are precisely the variations of the Chern-Simons densities:
  \[ \delta B = \frac{\eta l^{2}}{2} \delta C_{P} - \frac{l^{2}}{4\kappa} \delta C_{E} \]

where,
\[ C_{E} = \frac{1}{2} \epsilon^{abc} \epsilon_{IJKL} \omega_{a}^{IJ} \left( \partial_{b} \omega_{c}^{KL} + \frac{2}{3} \omega_{b}^{KM} \omega_{cM}^{L} \right) \]
\[ C_{P} = \epsilon^{abc} \omega_{aIJ} \left( \partial_{b} \omega_{c}^{IJ} + \frac{2}{3} \omega_{b}^{IK} \omega_{cK}^{J} \right) \]

- This implies: \[ B = \frac{\eta l^{2}}{2} C_{P} - \frac{l^{2}}{4\kappa} C_{E} \]
**Full Lagrangian**

- Addition of $C_P$ and $C_E$ at the boundary is equivalent to addition of Pontryagin and Euler densities $I_P$ and $I_E$ in the bulk (with fixed coefficients)

- Resulting Lagrangian:

\[
L(e, \omega) = \frac{1}{8\kappa} e^{\mu\nu\alpha\beta} \epsilon_{IJKL} \left( e_I^I e_J^J R_{\alpha\beta}^{KL} + \frac{1}{l^2} e_I^I e_J^J e^K e_L^L \right) \\
+ \frac{l^2}{4\kappa} I_E + \eta I_{NY} - \frac{\eta l^2}{2} I_P
\]

- $L$ contains all three topological densities, but only one independent topological parameter - the BI parameter $\eta$
Emergence of SO(3,2) Pontryagin

- The weights of $I_{NY}$ and $I_P$ are such that these two can be combined into a single topological density, the SO(3,2) Pontryagin density:

$$\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^{AB}(W) F_{\alpha\beta AB}(W) = 4I_P(\omega) - \frac{8}{l^2}I_{NY}(e,\omega)$$

- The corresponding SO(3,2) connection $W^{AB}_\mu$ is built out of the SO(3,1) connection $\omega^{IJ}_\mu$ and tetrad $e^I_\mu$:

$$W^{IJ}_\mu = \omega^{IJ}_\mu; \quad W^{I4}_\mu = \frac{1}{l} e^I_\mu$$
Finally, Lagrangian becomes:

\[ L = \frac{1}{8\kappa} \epsilon^{\mu \nu \alpha \beta} \epsilon_{IJKL} \left( e^I_\mu e^J_\nu R_{\alpha \beta}^K L(\omega) + \frac{1}{l^2} e^I_\mu e^J_\nu e^K e_L \right) + \frac{l^2}{32\kappa} \epsilon^{\mu \nu \alpha \beta} \epsilon_{IJKL} R_{\mu \nu}^{IJ}(\omega) R_{\alpha \beta}^{KL}(\omega) - \frac{\eta l^2}{8} \epsilon^{\mu \nu \alpha \beta} F_{\mu \nu}^{AB}(W) F^{\alpha \beta}_{AB}(W) \]

BI parameter \( \eta \) appears as a coefficient of the SO(3,2) Pontryagin topological density
To summarize..

• Topological densities do affect the Action principle in manifolds with boundaries

• The modifications are gauge invariant

• The ALADS Lagrangian contains all three topological densities, but only one independent topological parameter, the BI parameter $\eta$

(S. Sengupta, PRD, 2013)
Entropy

- Do the Noether charges depend on the topological parameters?

- In Euclidean gravity, the compactified version of $\text{SO}(3,2)$ is $\text{SO}(4,1) \ [\Lambda < 0]$ and $\text{SO}(4,1)$ is $\text{SO}(5) \ [\Lambda > 0]$. For suitably compactified manifolds, these can lead to nontrivial topological numbers through homotopy maps.

- Thus, in the quantum theory of gravity with $\Lambda$, is there a Non-perturbative vacuum structure dictated by any of these larger gauge groups?
THANK YOU