A PROOF OF COMES-KUJAWA’S CONJECTURE

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Abstract. Let $\kappa$ be a commutative ring containing $2^{-1}$. In this paper, we prove the Comes-Kujawa’s conjecture on a $\kappa$-basis of cyclotomic oriented Brauer-Clifford supercategory. As a byproduct, we prove that the cyclotomic walled Brauer-Clifford superalgebra defined by Comes and Kujawa and ours are isomorphic if $\kappa$ is an algebraically closed field with characteristic not two.

1. Introduction

The affine walled Brauer-Clifford supercategory and its cyclotomic quotients are introduced by Comes and Kujawa \cite{8}. These supercategories have closed connections with representations of queer Lie superalgebra $\mathfrak{q}(n)$, and its associated finite $W$-superalgebras, etc. The aim of this paper is to prove the Comes-Kujawa’s conjecture on a basis of cyclotomic oriented Brauer-Clifford supercategory \cite{8, Conjecture 7.1}.

Before we recall Comes-Kujawa’s conjecture, we need some notions in \cite{8} etc. Throughout, we assume that $\kappa$ is an arbitrary commutative ring containing $2^{-1}$.

1.1. The affine walled Brauer-Clifford supercategory. In this paper, we work over the super world. By definition, a supermodule is a module on which there is a $\mathbb{Z}_2$-grading. We are going to freely use the notions of $\kappa$-linear (monoidal) supercategories and superfunctors etc. For more details, we refer a reader to \cite{5, 8} and references therein.

For any two objects $a, b$ in a strict monoidal supercategory, $ab$ represents $a \otimes b$. So $a^k$ represents $a \otimes \ldots \otimes a$, where there are $k$ copies of $a$ in the tensor product. Following \cite{8}, a morphism $g : a \to b$ is drawn as

\begin{center}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}, scale=0.5]
\draw (-1,0) node [below] {$a$} -- (0.5,0) node [above] {$b$};
\end{tikzpicture}
\end{center}

or simply as

\begin{center}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}, scale=0.5]
\draw (-1,0) node [below] {$a$};
\end{tikzpicture}
\end{center}

if there is no confusion for the objects. Note that $a$ is drawn at the bottom while $b$ is at the top. There is a well-defined tensor product of two morphisms such that $g \otimes h$ is given by horizontal stacking:

\begin{center}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}, scale=0.5]
\draw (-1,0) node [below] {$g$} -- (0.5,0) node [above] {$h$};
\end{tikzpicture}
\end{center}

To simplify the notation, the $r$-fold tensor products of $g$ is drawn as

\begin{center}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}, scale=0.5]
\draw (-1,0) node [below] {$g$};
\end{tikzpicture}\quad \overbrace{}^{r}
\end{center}

The composition of two morphisms $g \circ h$ is given by vertical stacking:

\begin{center}
\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}, scale=0.5]
\draw (-1,0) node [below] {$h$} -- (0,0) node [above] {$g$};
\end{tikzpicture}
\end{center}

Following \cite{8}, a diagram involving multiple products is interpreted by first composing horizontally, then composing vertically. The super-interchange law is:

\[(g \otimes h) \circ (k \otimes l) = (-1)^{|[h]|[k]} (g \circ k) \otimes (h \circ l),\]

where $g, h, k, l$ are homogenous elements and $[h]$ is the parity of $h$.

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Definition 1.1. [8, Definition 3.7] The degenerate affine oriented Brauer-Clifford supercategory \( \text{AOBC}_\kappa \) is the \( \kappa \)-linear strict monoidal supercategory generated by two objects \( \uparrow, \downarrow \); four even morphisms \( \uparrow \rightarrow \downarrow, \downarrow \rightarrow \uparrow, \uparrow \uparrow \rightarrow \uparrow \uparrow, \uparrow \rightarrow \uparrow \); and one odd morphism \( \uparrow \rightarrow \uparrow \) subject to the following relations:

\[
\begin{align*}
\begin{array}{c}
\uparrow \downarrow = \downarrow \uparrow, \\
\downarrow \uparrow = \uparrow \downarrow, \\
\uparrow \uparrow = \uparrow \uparrow \\
\end{array}
\end{align*}
\]

(1.1)

\[
\begin{align*}
\begin{array}{c}
\uparrow = \uparrow, \\
\downarrow = \downarrow, \\
\uparrow \uparrow = \uparrow \uparrow \\
\uparrow \rightarrow \uparrow = 0.
\end{array}
\end{align*}
\]

(1.2)

Let \( \langle \uparrow, \downarrow \rangle \) be the set of all words in the alphabets \( \uparrow, \downarrow \) including the empty word \( \emptyset \). Each word \( \mathbf{a}_1 \ldots \mathbf{a}_r \) (resp., empty word \( \emptyset \)) represents \( \mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_r \) (resp., the unit object 1) in \( \text{AOBC}_\kappa \). The objects of previous five morphisms are implicated in the pictures. In fact, they can be read from the consistent orientation of each strand. For example, the objects at both the top and the bottom of \( \uparrow \downarrow \) are \( \uparrow \uparrow \) since the orientations of strands at both the top and the bottom are up-toward. It means that \( \uparrow \downarrow \) is a morphism in \( \text{End}_{\text{AOBC}_\kappa}(\uparrow \downarrow) \). For \( \downarrow \uparrow \), there is no endpoint at the bottom. It means the object at the bottom is the unit object and hence \( \downarrow \uparrow \in \text{Hom}_{\text{AOBC}_\kappa}(1, \uparrow \downarrow) \). Similarly, \( \uparrow \downarrow \in \text{Hom}_{\text{AOBC}_\kappa}(\downarrow \uparrow, 1) \) and \( \uparrow \downarrow \in \text{End}_{\text{AOBC}_\kappa}(\uparrow \downarrow) \). Since any morphism \( g : \mathbf{a} \rightarrow \mathbf{b} \) in \( \text{AOBC}_\kappa \) can be expressed as tensor products and compositions of the five morphisms in Definition 1.1, \( \mathbf{a}, \mathbf{b} \) will be omitted when \( g \) is drawn as a picture. Given an \( \mathbf{a} \in \langle \uparrow, \downarrow \rangle \), following [2], the identity morphism \( 1_\mathbf{a} \in \text{End}_{\text{AOBC}_\kappa}(\mathbf{a}) \) can be drawn by the object \( \mathbf{a} \) itself. For example, \( 1_{\uparrow \downarrow} = \uparrow \downarrow \).

By [8, Definition 3.2], the oriented Brauer-Clifford supercategory \( \text{OBC}_\kappa \) is the subcategory of \( \text{AOBC}_\kappa \) generated by the same objects, and the previous morphisms except \( \uparrow \rightarrow \uparrow \) and \( \downarrow \uparrow \rightarrow \uparrow \). The oriented Brauer category \( \text{OB}_\kappa \) [2, Theorem 1.1] is the supercategory generated by the same objects and the previous morphisms except \( \uparrow \rightarrow \uparrow \) and \( \downarrow \uparrow \rightarrow \uparrow \) subject to the relations (1.1)–(1.2).

1.2. Hom-superspaces of \( \text{AOBC}_\kappa \) and dotted oriented Brauer-Clifford diagrams with bubbles. In order to state bases of Hom-superspaces in \( \text{AOBC}_\kappa \), \( \text{OBC}_\kappa \), and \( \text{OB}_\kappa \), we need to recall the definitions of (dotted) oriented Brauer(-Clifford) diagrams with bubbles in [2, 8].

Definition 1.2. [2] For any two words \( \mathbf{a}, \mathbf{b} \in \langle \uparrow, \downarrow \rangle \), an oriented Brauer diagram of type \( \mathbf{a} \rightarrow \mathbf{b} \) is an oriented diagrammatic representation of a bijection

\[
\{ i \mid a_i = \uparrow \} \sqcup \{ i' \mid b_{i'} = \downarrow \} \sim \{ i \mid b_i = \uparrow \} \sqcup \{ i' \mid a_{i'} = \downarrow \}
\]

(1.5)

obtained by placing \( \mathbf{a} \) below \( \mathbf{b} \), then drawing strands connecting pairs of letters as prescribed by the bijection in (1.5). The consistent orientation to each strand in the diagram is given by the letters of \( \mathbf{a} \) and \( \mathbf{b} \). Two oriented Brauer diagrams are equivalent if they are of the same type and represent the same bijection.

For example, the following is an oriented Brauer diagram of type \( \downarrow \uparrow \rightarrow \downarrow \uparrow \).

\[
\begin{align*}
\begin{array}{c}
\downarrow \uparrow \rightarrow \downarrow \uparrow
\end{array}
\end{align*}
\]

(1.6)

Following [2], \( \Delta_0 := \emptyset \rightarrow \emptyset \) is called a bubble. Two oriented Brauer diagrams with bubbles are equivalent if they have the same number of bubbles and the underlying oriented Brauer diagrams without bubbles are equivalent. It is proven in [2] that two equivalent oriented Brauer diagrams with bubbles of type \( \mathbf{a} \rightarrow \mathbf{b} \) represent the same morphism in \( \text{Hom}_{\text{OBC}_\kappa}(\mathbf{a}, \mathbf{b}) \), and the set of all equivalence classes of oriented Brauer diagrams with bubbles of type \( \mathbf{a} \rightarrow \mathbf{b} \) is a \( \kappa \)-basis of \( \text{Hom}_{\text{OB}_\kappa}(\mathbf{a}, \mathbf{b}) \).
Definition 1.3. \([8, \S 3.3]\) For any two words \(a, b \in \langle \uparrow, \downarrow \rangle\), an oriented Brauer-Clifford diagram (resp., with bubbles) of type \(a \rightarrow b\) is an oriented Brauer diagram (resp., with bubbles) of type \(a \rightarrow b\) such that there are finitely many \(\circ\)‘s on its segments. A dotted oriented Brauer-Clifford diagram (resp., with bubbles) is an oriented Brauer diagram (resp., with bubbles) such that there are finitely many \(\circ\)‘s and \(\bullet\)‘s on its segments.

It is defined in \([8]\) (see also \([2]\) for the second one) that

\[
\begin{align*}
\circ : &= \begin{array}{c}
\displaystyle \bigcirc \bigcirc \\
\end{array} \\
\bullet : &= \begin{array}{c}
\displaystyle \bigcirc \bigcirc \bigcirc \\
\end{array}
\end{align*}
\]

such that any morphism in \(\text{Hom}_{\text{OBC}}(a, b)\) (resp., \(\text{Hom}_{\text{AOBC}}(a, b)\)) can be realized as a \(\kappa\)-linear combination of (resp., dotted) oriented Brauer-Clifford diagrams with bubbles of type \(a \rightarrow b\). In order to give bases of Hom-superspaces in \(\mathcal{OBC}_\kappa\) (resp., \(\mathcal{AOBC}_\kappa\)), Comes and Kujawa introduced the notion of a normally ordered (resp., dotted) oriented Brauer-Clifford diagram.

Definition 1.4. \([8, \S 3.3]\) An oriented Brauer-Clifford diagram is called normally ordered if

\(a\) it has at most one \(\circ\) on each strand and it has no bubble;

\(b\) all \(\circ\)‘s are on outward-pointing boundary segments;

\(c\) all \(\circ\)‘s are positioned at the same height if the segments they occur on have the same orientation.

Definition 1.5. \([8, \text{Definition 3.8}]\) A dotted oriented Brauer-Clifford diagram with bubbles is normally ordered if

\(a\) it is a normally ordered oriented Brauer-Clifford diagram by ignoring all bubbles and all \(\bullet\)‘s;

\(b\) each \(\bullet\) is either on a bubble or on an inward-pointing boundary segment;

\(c\) each bubble has zero \(\circ\)‘s, and an odd number of \(\bullet\)‘s, are crossing-free, counterclockwise, and there are no other strands shielding it from the rightmost edge of the picture;

\(d\) whenever a \(\bullet\) and a \(\circ\) appear on a segment that is both inward and outward-pointing, the \(\bigcirc\) appears ahead of the \(\bullet\) in the direction of the orientation.

For example, the following diagrams represent two morphisms in \(\text{Hom}_{\text{AOBC}}(\downarrow^2 \uparrow^3, \downarrow^2)\). The right one is normally ordered whereas the left one is not.

\[
\begin{align*}
\text{Diagram 1} & \quad \text{Diagram 2}
\end{align*}
\]

Let \(\Delta_k = \bigcirc_k\). Then \(\Delta_k\) is the crossing-free and counterclockwise bubble with \(k\) \(\bullet\)‘s on it. By \([8, \text{Proposition 3.12}]\), \(\Delta_k = 0\) whenever \(k\) is even. This is the reason why Comes-Kujawa require that there are odd numbers of \(\bullet\)‘s on each bubble in Definition 1.5. Later on, only bubbles on which there are odd \(k\) \(\bullet\)‘s will be considered.

Following \([8]\), two normally ordered (resp., dotted) oriented Brauer-Clifford diagrams (resp., with bubbles) are said to be equivalent if their underlying oriented Brauer diagrams (resp., with bubbles) are equivalent and their corresponding strands have the same number of \(\circ\)‘s (resp., \(\bullet\)‘s). It is proven in \([8]\) that two normally ordered oriented Brauer-Clifford diagrams of type \(a \rightarrow b\) represent the same morphism in \(\text{Hom}_{\text{OBC}_\kappa}(a, b)\) and the set of all equivalence classes of normally ordered oriented Brauer-Clifford diagrams of type \(a \rightarrow b\) is a \(\kappa\)-basis of \(\text{Hom}_{\text{OBC}_\kappa}(a, b)\). Unlike the cases for \(\mathcal{OB}_\kappa\) and \(\mathcal{OBC}_\kappa\), two equivalent normally ordered dotted oriented Brauer-Clifford diagrams (with bubbles) may not represent the same morphism in \(\mathcal{AOBC}_\kappa\). The following is the main result of \([8]\).

Theorem 1.6. \([8, \text{Corollary 6.4}]\) For any \(a, b \in \langle \uparrow, \downarrow \rangle\), the set of all equivalence classes of normally ordered dotted oriented Brauer-Clifford diagrams with bubbles of type \(a \rightarrow b\) is a \(\kappa\)-basis of \(\text{Hom}_{\text{AOBC}_\kappa}(a, b)\).

1.3. Cyclotomic quotients and Comes-Kujawa’s conjecture. Fix two nonnegative integers \(a, b\) and \(u = (u_1, \ldots, u_b) \in (\kappa^*)^b\), where \(\kappa^* = \kappa \setminus \{0\}\). Let

\[
f(t) = t^{2a+\epsilon} \prod_{1 \leq i \leq b} (t^2 - u_i),
\]

\([1.9]\)
where \( \varepsilon \in \{0, 1\} \). In [8], Comes and Kujawa defined
\[
\text{OBC}_f^\ell = \text{AOBC}_\kappa / I,
\]
called the cyclotomic quotient of \( \text{AOBC}_\kappa \) or the level \( \ell \) oriented Brauer-Clifford supercategory [8], where \( \ell \) is the degree of \( f(t) \) and \( I \) is the left tensor ideal generated by \( f(1) \).

**Conjecture 1.7.** [8 Conjecture 7.1] Suppose that \( \kappa \) is a field of characteristic not two. Given two words \( a, b \in \langle \uparrow, \downarrow \rangle \), \( \text{Hom}_{\text{OBC}_f^\ell}(a, b) \) has basis given by all equivalence classes of normally ordered dotted oriented Brauer-Clifford diagrams with bubbles of type \( \text{words} \) \( a \rightarrow b \) with fewer than \( \ell \) \( \bullet \)'s on each strand.

Comes and Kujawa proved that Conjecture 1.7 is true when either \( f(t) = t \) or \( f(t) = t^2 - u \) with \( u \neq 0 \). The proof for the second case is inspired by [9, 8]. Let \( \text{AOBC}_\kappa(0) \) (resp., \( \text{OBC}_f^\ell(0) \)) be the supercategory obtained from \( \text{AOBC}_\kappa \) (resp. \( \text{OBC}_f^\ell \)) by imposing the relations \( \Delta_k = 0 \) for all \( k > 0 \). Using certain representations of finite \( W \)-superalgebras associated to queer Lie superalgebras \( q(n) \), Comes and Kujawa are able to prove Conjecture 1.7 for \( \text{OBC}_f^\ell(0) \). In general, as far as we know, their conjecture remains open.

The main result of this paper is that Conjecture 1.7 holds over an arbitrary commutative ring \( \kappa \) containing \( 2^{-1} \). As a by-product, we prove that the cyclotomic walled Brauer-Clifford superalgebra defined by Comes and Kujawa in [8] is isomorphic our cyclotomic walled Brauer-Clifford superalgebra in [9, Definition 3.14] when \( \kappa \) is an algebraically closed field with characteristic not two.

The contents of this paper are organized as follows. In section 2, we consider certain tensor modules in parabolic supercategory \( O \) for \( q(n) \) over \( \mathbb{C} \). In section 3, we prove that Conjecture 1.7 holds over an arbitrary commutative ring \( \kappa \) containing \( 2^{-1} \). In section 4, we prove that the cyclotomic walled Brauer-Clifford superalgebra defined by Comes-Kujawa and ours are isomorphic if \( \kappa \) is an algebraically closed field with characteristic not two.

### 2. Schur-Weyl super-duality

Let \( g \) be the queer Lie superalgebra \( q(n) \) of rank \( n \) over \( \mathbb{C} \). Then \( g \) has a basis \( e_i, j = E_{i, j} + E_{j, i} \) (even element), \( f_i, j = E_{i, j} + E_{j, i} \) (odd element) for \( i, j \in I^+ = \{1, 2, \ldots, n\} \), where \( E_{i, j} \) is the \( 2n \times 2n \) matrix with entry 1 at \( (i, j) \) position and zero elsewhere for \( i, j \in I = I^+ \cup I^-, \) and \( I^- = -I^+ \).

Let \( V = \mathbb{C}^{n|n} = V_{\uparrow} \oplus V_{\downarrow} \) be the natural \( g \)-supermodule (and the natural supermodule of the general linear Lie superalgebra \( gl(n|n) \)) with basis \( \{v_i \mid i \in I\} \). Then \( v_i \) has the parity \( [v_i] = [i] \in \mathbb{Z}_2 \), where \([i] = 0 \) and \([-i] = 1 \) for \( i \in I^* \). Let \( V^* \) be the linear dual of \( V \) with \( \{\overline{v}_i \mid i \in I\} \) being its dual basis. Then \( V^* \) is a left \( g \)-supermodule such that
\[
E_{a, b} \overline{v}_i = -(-1)^{|a|([a]+[b])} \delta_{a,a} \overline{v}_i \quad \text{for} \quad a, b, i \in I.
\]
Let \( b = b_0 \oplus b_\uparrow \) be a Cartan subalgebra of \( g \) with even part \( b_\uparrow = \text{span}\{h_i \mid i \in I^+\} \) and odd part \( b_\uparrow = \text{span}\{h'_i \mid i \in I^+\} \), where \( h_i = e_{i, i} \) and \( h'_i = f_{i, i} \) for all admissible \( i \). Let \( b_\uparrow^* \) be the linear dual of \( b_\uparrow \) with \( \{e_i \mid i \in I^+\} \) being the dual basis of \( \{h_i \mid i \in I^+\} \). Then an element \( \lambda \in b_\uparrow^* \) (called a weight) can be written as
\[
\lambda = \sum_{i \in I^+} \lambda_i e_i.
\]
Let \( b \) be the standard Borel subalgebra of \( g \) with even part \( b_\uparrow = \text{span}\{e_{i, j} \mid i \leq j \in I^+\} \) and odd part \( b_\uparrow = \text{span}\{f_{i, j} \mid i \leq j \in I^+\} \). Let \( O \) be the supercategory of all \( g \)-supermodules \( M \) such that:

- \( a \) \( M \) is finitely generated as a \( g \)-supermodule;
- \( M \) is locally finite-dimensional over \( b \);
- \( M \) is semisimple over \( b_\uparrow \).

For any \( \lambda \in b_\uparrow^* \), let \( I_\lambda \) be the irreducible \( b \)-supermodule. Then the dimension of \( I_\lambda \) is \( 2^{|\ell(\lambda)|} \) (see [7]), where \( \ell(\lambda) \) is the number of non-zero parts of \( \lambda \) and \( [a] \) is the integer part of any nonnegative real number \( a \). Let
\[
M(\lambda) := U(g) \otimes_{U(b)} I_\lambda
\]
be the Verma supermodule with the highest weight \( \lambda \), where in general \( U(f) \) is the universal enveloping algebra of any Lie superalgebra \( f \). Then \( M(\lambda) \) has the simple head denoted by \( L(\lambda) \). It is well known that \( L(\lambda) \) is of finite dimensional if and only if \( \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \) and \( \lambda_i - \lambda_{i+1} = 0 \) implies that \( \lambda_i = 0 \), for \( 1 \leq i < n \).
Fix $\varepsilon \in \{0, 1\}$ and two nonnegative integers $a$ and $b$. We define $n = (n_1, n_2, \ldots, n_{a+b+\varepsilon})$ such that
\[
\sum_{i=1}^{a+b+\varepsilon} n_i, \quad (2.3)
\]
the summation of even positive integers $n_i$ for $1 \leq i \leq a + b + \varepsilon$. Let $p$ be the parabolic super subalgebra of $g$ such that the Levi super subalgebra $l$ is $\oplus_{i=1}^{a+b+\varepsilon} q(n_i)$. Let $\mathcal{O}^p$ be the corresponding parabolic supercategorify $\mathcal{O}$. Then $\mathcal{O}^p$ is the full subcategory of $\mathcal{O}$ consisting of all $g$-supermodules which are locally finite-dimensional over $p$. Throughout, we define
\[
p_0 = 0, \quad p_i = \sum_{j=1}^{i} n_j, \quad p_i = \{p_{i-1} + 1, \ldots, p_i\} \text{ for } 1 \leq i \leq a + b + \varepsilon
\]
and let
\[
\Lambda = \{ \lambda \in h^\vee \mid \lambda_j - \lambda_{j+1} \in \mathbb{Z}_{\geq 0} \text{ and } \lambda_j = 0 \text{ if } \lambda_j = \lambda_{j+1}, \ p_i \leq j < p_{i+1} \}, \quad (2.5)
\]
be the set of $l$-dominant weights. For any $\lambda \in \Lambda$, the irreducible $l$-module $L(\lambda)^0$ with the highest weight $\lambda$ is finite dimensional. The parabolic Verma supermodule $M^p(\lambda)$ with the highest weight $\lambda \in \Lambda$ is
\[
M^p(\lambda) := U(g) \otimes U(p) L(\lambda)^0.
\]
For any $g$-supermodule $M$ and any $r \in \mathbb{Z}_{\geq 0}$, set $M^r = V^\otimes r \otimes M$. For convenience we define the totally ordered set
\[
J = J_1 \cup \{0\} \text{ where } J_1 = \{1, \ldots, r\}, \quad (2.6)
\]
such that $r < r - 1 < \ldots < 1 < 0$. We write $M^r$ as
\[
M^r = \bigotimes_{i \in J} V_i, \quad \text{where } V_0 = M, \ V_i = V \text{ if } i \in J_1.
\]
Hereafter all tensor products will be taken according to the total order $<$ on $J$. Then $M^r$ is a left $U(g)^{\otimes (r+1)}$-supermodule such that the action is given by
\[
\left( \bigotimes_{i \in J} g_i \right) \left( \bigotimes_{i \in J} x_i \right) = (-1)^{\sum_{i \in J} |g_i| \sum_{j < i} |x_j|} \bigotimes_{i \in J} (g_i x_i) \text{ for } g_i \in U(g), \ x_i \in V_i. \quad (2.8)
\]
Via the coproduct of $U(g)$, it is a left $U(g)$-supermodule. In order to define the left action of $AO BC_C$ on $M^r$, we define
\[
\hat{e}_{i,j} = E_{i,j} - E_{i-1,j}, \quad \hat{f}_{i,j} = E_{-i,j} - E_{-i-1,j} \in gl_{n,j},
\]
\[
\Omega_1 = \sum_{i,j \in I^+} \hat{e}_{i,j} \otimes e_{i,j} - \sum_{i,j \in I^+} \hat{f}_{i,j} \otimes f_{j,i} \in gl_{n,n} \otimes g.
\]
Let $c : V \to V$ be the odd linear map such that
\[
c(v_i) = (-1)^{|v_i|} \sqrt{-1} v_i, \quad \text{for all } i \in I. \quad (2.10)
\]
Since $\mathcal{O}$ and $\mathcal{O}^p$ are closed under the functors $V \otimes -$ and $V^* \otimes -$ we can use $\mathcal{O}^p$ (or $\mathcal{O}$) to replace the supercategory $U(g)$-smod of left $U(g)$-supermodules in [4] Theorem 4.4].

**Theorem 2.1.** [4] Theorem 4.4] There is a monoidal superfunctor $\Psi : AO BC_C \to End(\mathcal{O}^p)$ sending the objects $\uparrow, \downarrow$ to the endofunctors $V \otimes -, V^* \otimes -$, respectively, and moreover,
\[
\Psi (\bigotimes) : Id \to V \otimes V^* \otimes -, \quad m \mapsto \sum_{i \in I} v_i \otimes v_i^* \otimes m,
\]
\[
\Psi (\bigotimes) : V^* \otimes V \otimes - \to Id, \quad f \otimes v \otimes m \mapsto f(v)m,
\]
\[
\Psi (\bigotimes) : V \otimes V \otimes - \to V \otimes V \otimes -, \quad u \otimes v \otimes m \mapsto (-1)^{|u||v|} v \otimes u \otimes m,
\]
\[
\Psi (\bigotimes) : V \otimes - \to V \otimes -, \quad v \otimes m \mapsto \Omega_1(v \otimes m),
\]
\[
\Psi (\bigotimes) : V \otimes - \to V \otimes -, \quad v \otimes m \mapsto c(v) \otimes m.
\]

Write $\Psi_M : AO BC_C \to \mathcal{O}$ (resp., $\mathcal{O}^p$) for the composition of $\Psi$ followed by evaluation at $M$ for any highest weight supermodule $M$ in $\mathcal{O}$ (resp., $\mathcal{O}^p$). Given two $h$-supermodule (resp., $g$-supermodule) $M$ and $N$, define $\frac{\partial M}{\partial N}$ to be a supermodule which has a filtration of length two such that the top (resp., bottom) section is isomorphic to $M$ (resp., $N$). Let $\Pi$ be the parity change functor. The following
result can be found in the proof of [11 Lemma 4.37]. Note that $V \cong \bigoplus_{i=1}^{n} I_{i}$, as $h$-supermodules. Moreover, $I_i$ has basis $\{v_i, v_{-i}\}$, for $1 \leq i \leq n$.

**Lemma 2.2.** (cf. [11 Lemma 4.37]) Suppose that $\lambda \in h_{\mathbb{P}}^*$ such that $\ell(\lambda)$ is even. As $h$-supermodules, there is an isomorphism $V \otimes I_{\lambda} \cong \bigoplus_{i=1}^{n} I_{i}$, where

$$I_{i} \cong \begin{cases} I_{(\lambda+\varepsilon_{i})} \oplus \Pi I_{(\lambda+\varepsilon_{i})}, & \text{if } \lambda_i \notin \{0, -1\}; \\ I_{(\lambda+\varepsilon_{i})}, & \text{if } \lambda_i = 0; \\ I_{(\lambda+\varepsilon_{i})}, & \text{if } \lambda_i = -1. \end{cases}$$ (2.11)

Moreover, $I_{i}$ has a basis $\{v_{i} \otimes v, v_{-i} \otimes v \mid v \in S_{\lambda}\}$, where $S_{\lambda}$ is any basis of $I_{\lambda}$.

For the simplification of notation, we use $x$ to denote $\Psi_{M(\lambda)}(\downarrow)$ in the following result.

**Lemma 2.3.** Suppose that $\lambda \in h_{\mathbb{P}}^*$ such that $\ell(\lambda)$ is a nonnegative even integer. Then $V \otimes M(\lambda)$ has an $x$-stable $U(\mathfrak{g})$-filtration

$$0 = M_{0} \subseteq M_{1} \subseteq \ldots \subseteq M_{n} = V \otimes M(\lambda)$$

such that

$$M_{i}/M_{i-1} \cong \begin{cases} M(\lambda + \varepsilon_{i}) \oplus \Pi M(\lambda + \varepsilon_{i}), & \text{if } \lambda_i \notin \{0, -1\}; \\ M(\lambda + \varepsilon_{i}), & \text{if } \lambda_i = 0; \\ M(\lambda+\varepsilon_{i}), & \text{if } \lambda_i = -1. \end{cases}$$ (2.12)

Moreover, $M_{i}/M_{i-1}$ is killed by $x^{2} - \lambda_{i}(\lambda_{i} + 1)$ (resp., $x$) if $\lambda_{i} \neq 0$ (resp., if $\lambda_{i} = 0$).

**Proof.** Recall $M(\lambda) = U(\mathfrak{g}) \otimes U(\mathfrak{b}) I_{\lambda}$. As $U(\mathfrak{g})$-modules,

$$V \otimes M(\lambda) \cong U(\mathfrak{g}) \otimes U(\mathfrak{b}) (V \otimes I_{\lambda}).$$

So the required filtration can be constructed such that $M_{i}/M_{i-1}$ is generated by the images of $\{v_{i} \otimes v, v_{-i} \otimes v \mid v \in S_{\lambda}\}$, where $S_{\lambda}$ is any basis of $I_{\lambda}$. Moreover, (2.12) follows from (2.11). The fact that the filtration is $x$-stable and $M_{i}/M_{i-1}$ is killed by $x^{2} - \lambda_{i}(\lambda_{i} + 1)$ (resp., $x$) follows from arguments in the proof of [3 Lemma 3.2] for $\lambda_{i} \notin \frac{1}{2} \mathbb{Z}$ (see also [3] Lemma 3.5 for $\lambda_{i} \in \frac{1}{2} \mathbb{Z}$). However, their arguments are still available if $\lambda_{i} \in \mathbb{Z}$. The only difference appears when $\lambda_{i} = 0$. We give a sketch of their arguments here.

Let $\{m_{i} \mid 1 \leq i \leq k\}$ be a basis of the even subspace of $I_{\lambda}$. Recall that $h'_{i} = f_{i}$ for all admissible $i$. If $\lambda_{i} \neq 0$, then $\{h'_{i} m_{j} \mid 1 \leq j \leq k\}$ is a basis of the odd subspace of $I_{\lambda}$. Note that $M_{i}/M_{i-1}$ is generated by the images of vectors in $A_{i} = \{v_{i} \otimes m_{j}, v_{-i} \otimes m_{j}, v_{i} \otimes h'_{i} m_{j}, v_{-i} \otimes h'_{i} m_{j} \mid j = 1, \ldots, k\}$. Suppose $\lambda_{i} = 0$. If $\ell(\lambda) > 0$, then we can find a $t$ such that $\lambda_{i} \neq 0$ and $M_{i}/M_{i-1}$ is generated by the images of vectors in $A_{i} = \{v_{i} \otimes m_{j}, v_{-i} \otimes m_{j}, v_{i} \otimes h'_{i} m_{j}, v_{-i} \otimes h'_{i} m_{j} \mid j = 1, \ldots, k\}$. If $\ell(\lambda) = 0$, then $I_{\lambda} = \mathbb{C}$ and $M_{i}/M_{i-1}$ is generated by the images of vectors in $A_{i} = \{v_{i} \otimes 1, v_{-i} \otimes 1\}$ for $1 \leq i \leq n$. For any $v \in A$, we have $(\partial_{r,s} \otimes h_{i} - \tilde{f}_{r,s} \otimes f_{s})v = 0$ unless $r \leq s = i$. If $r < s = i$ then $(\partial_{r,s} \otimes h_{i} - \tilde{f}_{r,s} \otimes f_{s})v = 0$ for any $v \in A$. So, the filtration constructed above is $x$-stable, and $x$ acts on the highest weight space of $M_{i}/M_{i-1}$ via $y_{i} := \partial_{i,i} \otimes h_{i} - \tilde{f}_{i,i} \otimes h'_{i}$. By direct computation (see also the matrix of the endomorphism of $y_{i}$ with respect to the highest weight space of $M_{i}/M_{i-1}$ in the proof of [3 Lemma 3.2]), we have $[y_{i}^{2} - \lambda_{i}(\lambda_{i} + 1)]v = 0$ (resp., $y_{i} v = 0$) for all $v \in A_{i}$ if $\lambda_{i} \neq 0$ (resp., $\lambda_{i} = 0$). Therefore, $M_{i}/M_{i-1}$ is killed by $x^{2} - \lambda_{i}(\lambda_{i} + 1)$ (resp., $x$) if $\lambda_{i} \neq 0$ (resp., $\lambda_{i} = 0$) as required.

Hereafter, we fix a weight $\lambda \in h_{\mathbb{P}}^*$ such that

$$\lambda_{p_{i} + j} = \begin{cases} l_{i} - j + 1, & \text{for } 1 \leq j \leq n_{i+1}, 0 \leq i \leq a + b - 1; \\ 0, & \text{for } 1 \leq j \leq n_{a+b+\varepsilon}, i = a + b + \varepsilon. \end{cases}$$ (2.13)

where $l_{i} = -1$ if $1 \leq i \leq a$ and $l_{i} \notin \mathbb{Z}_{\geq 0} \cup \{-1\}$ if $a + 1 \leq i \leq a + b$. Then $\lambda \in \Lambda$, where $\Lambda$ is the set of $l$-dominant weights defined in (2.5). We identify $\lambda$ with $(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n})$. Define $\lambda^{(i)} = (\lambda_{p_{i} + l_{i}}, \ldots, \lambda_{p_{i}})$ for all $1 \leq i \leq a + b + \varepsilon$. Then $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(a+b)})$. Let $L(\lambda^{(i)})$ be the irreducible $q(n_{i})$-module with the highest weight $\lambda^{(i)}$. Then

$$L(\lambda)^{0} \cong \bigotimes_{i=1}^{a+b+\varepsilon} L(\lambda^{(i)}),$$ (2.14)
where \( L(\lambda^{(a+b+\varepsilon)}) \cong \mathbb{C} \) if \( \varepsilon = 1 \) and \( \lambda^{(i)} \) and \( \lambda^{(i)}+\varepsilon_{p_i-1}+1 \) are typical as weights of \( q(n_i) \) for \( 1 \leq i \leq a+b \) (see \( \Pi \)). Thanks to (2.11) and character considerations (cf. the finite dimensional typical character formula in [11] Theorem 2 or [12] Theorem 4.8), we have \( V \otimes L(\lambda)^0 \cong \bigoplus_{i=1}^{a+b+\varepsilon} L_i \), where

\[
L_i \cong \begin{cases} 
L(\lambda+\varepsilon_{p_i-1}+1)^0, & 1 \leq i \leq a; \\
\Pi L(\lambda+\varepsilon_{p_i-1}+1)^0, & a+1 \leq i \leq a+b; \\
L(\lambda+\varepsilon_{p_i-1}+1)^0, & i = a+b+\varepsilon \text{ and } \varepsilon = 1.
\end{cases}
\]

(2.15)

In order to simplify the notation, similar to the above we still use \( x \) to denote \( \Psi_{M^p(\lambda)} (\bullet) \) (the parabolic version of \( x \)) in the following result. For \( \varepsilon = 0 \) and \( a+b = 1 \), Theorem 2.3(b) can be found in [9] Lemma 4.4(a).

**Theorem 2.4.** For any \( \lambda \) in (2.13), there is an \( x \)-stable \( U(\mathfrak{g}) \)-filtration

\[
0 = M_0 \subset M_1 \subset \ldots \subset M_{a+b+\varepsilon} = V \otimes M^p(\lambda)
\]

such that

\[
M_i/M_{i-1} \cong \begin{cases} 
M^p(\lambda+\varepsilon_{p_i-1}+1) \cong \Pi M^p(\lambda+\varepsilon_{p_i-1}+1), & 1 \leq i \leq a; \\
M^p(\lambda+\varepsilon_{p_i-1}+1) \cong \Pi M^p(\lambda+\varepsilon_{p_i-1}+1), & a+1 \leq i \leq a+b; \\
M^p(\lambda+\varepsilon_{p_i-1}+1), & i = a+b+\varepsilon \text{ and } \varepsilon = 1.
\end{cases}
\]

(2.16)

Moreover,

a) \( M_i/M_{i-1} \) is killed by \( x^2 - l_i(l_i + 1) \) (resp., \( x \)) if \( 1 \leq i \leq a+b \) (resp., if \( i = a+b+\varepsilon \) and \( \varepsilon = 1 \)).

b) \( V \otimes M^p(\lambda) \) is killed by \( f(x) \) where \( f(t) = t^{2a+\varepsilon} \prod_{i=1}^{b}(t^2 - u_i) \), such that \( u_i = l_{a+i}(l_{a+i} + 1) \neq 0 \), for all \( 1 \leq i \leq b \).

c) The superfunctor \( \Psi_{M^p(\lambda)} \) in Theorem 2.3 factors through \( OBC^f \), and thus induces a superfunctor \( \Psi_{M^p(\lambda)}^f : OBC^f \to O^p \).

**Proof.** Recall that \( M^p(\lambda) = U(\mathfrak{g}) \otimes U(b) L(\lambda)^0 \). So, \( V \otimes M^p(\lambda) \cong U(\mathfrak{g}) \otimes U(b) (V \otimes L(\lambda)^0) \). Note that \( L(\lambda)^0 \) is the quotient of \( M(\lambda)_i \), where \( M(\lambda)_i \) is the Verma supermodule of \( I \) with the highest weight \( \lambda \). Let \( \phi : V \otimes M^p(\lambda) \to V \otimes L(\lambda)^0 \) be the epimorphism induced by the canonical epimorphism from \( M(\lambda)_i \) to \( L(\lambda)^0 \). By Lemma 2.3 there is an \( I \)-module filtration of \( V \otimes M(\lambda)_I \)

\[
0 = M_{i,0} \subset M_{i,1} \subset \ldots \subset M_{i,n} = V \otimes M(\lambda)_I
\]

such that \( M_{i,j} \) is generated by the images of \( A_i = \{ v_j \otimes v, v_{j-1} \otimes v \mid v \in S_{\lambda}, j \leq i \} \), and \( M_{i,j}/M_{i,j-1} \) is determined by (2.12) as a filtration of Verma supermodules of \( I \), where \( S_{\lambda} \) is any basis of \( I_{\lambda} \). So, there is an \( I \)-module filtration of \( V \otimes L(\lambda)^0 \)

\[
0 = N_0 \subset N_1 \subset \ldots \subset N_n = V \otimes L(\lambda)^0
\]

(2.17)

such that \( N_i = \phi(M_{i,i}) \). Each \( N_i \) (resp., \( N_i/N_{i-1} \)) is the quotient of \( M_{i,i} \) (resp., \( M_{i,i}/M_{i,i-1} \)) and is generated by the images of \( A_i \) (resp., \( A_i \setminus A_{i-1} \)). Note that \( \lambda + \varepsilon_i \notin \Lambda \) if \( i \neq p_i-1 + 1 \) for any \( 1 \leq j \leq a+b+\varepsilon \) and hence \( L(\lambda + \varepsilon_i)^0 \) is infinite dimensional. Suppose \( i \neq p_i-1 + 1 \). If \( N_i \neq N_{i-1} \), by (2.12), \( N_i/N_{i-1} \) must be infinite dimensional since it is a quotient of \( M_{i,i}/M_{i,i-1} \) and \( L(\lambda + \varepsilon_i)^0 \) is infinite dimensional. This is a contradiction. Thus \( N_i = N_{i-1} \) if \( i \neq p_i-1 + 1 \) for any \( 1 \leq j \leq a+b+\varepsilon \). So, the filtration in (2.16) can be reduced to

\[
0 = N_0 \subset N_1 \subset \ldots \subset N_n = V \otimes L(\lambda)^0
\]

(2.18)

where \( N_{p_i+1}/N_{p_i}+1 \) is generated by images of \( \{ v_{p_i+1} \otimes v, v_{p_i} \otimes v \mid v \in S_{\lambda} \} \). By (2.13) and character consideration, we have

\[
N_{p_i+1}/N_{p_i}+1 \cong L_j, \text{ for } 1 \leq j \leq a+b+\varepsilon - 1, \text{ and } N_1 \cong L_1
\]

(2.19)

where \( L_i \) is given in (2.15). Now define \( M_j := U(\mathfrak{g}) \otimes U(p) N_{p_j+1} \) for \( 1 \leq j \leq a+b+\varepsilon \). Then the required filtration of \( V \otimes M^p(\lambda) \) follows form (2.17)–(2.18) and (a) follows from the proof of Lemma 2.3 Via (a), we immediately have (b). Finally, (c) follows from (b) and Theorem 2.1 \( \square \)

**Remark 2.5.** For any \( 0 \neq u_i \in \mathbb{C} \), there exists \( \ell_{a+i} \notin \mathbb{Z}_{\geq 0} \cup \{-1\} \) such that \( u_i = \ell_{a+i}(\ell_{a+i} + 1) \). This enables us to choose an \( I \)-dominant weight \( \lambda \) in (2.18) such that, for any \( f(t) = t^{2a+\varepsilon} \prod_{i=1}^{b}(t^2 - u_i) \in \mathbb{C}[t] \), there is a superfunctor \( \Psi_{M^p(\lambda)}^f : OBC^f \to O^p \)
For $1 \leq m$ and $1 \leq i, j \leq n$, Sergeev [13] defined $x_{i,j}^\sigma(1) = e_{i,j}$, $x_{i,j}^\tau(1) = f_{i,j}$, and

$$
x_{i,j}^\sigma(m) = \sum_{s=1}^{n}(e_{i,s}x_{s,j}^\sigma(m-1) + (-1)^{m-1}f_{i,s}x_{s,j}^\sigma(m-1)),
$$

$$
x_{i,j}^\tau(m) = \sum_{s=1}^{n}(e_{i,s}x_{s,j}^\tau(m-1) + (-1)^{m-1}f_{i,s}x_{s,j}^\tau(m-1)),
$$

for $m > 1$ and proved the following relations:

$$
[e_{i,j}, x_{s,t}^\sigma(m)] = \delta_{j,s}x_{i,t}^\sigma(m) - \delta_{i,s}x_{s,t}^\sigma(m),
$$

$$
[f_{i,j}, x_{s,t}^\sigma(m)] = (-1)^{m-1}\delta_{j,s}x_{i,t}^\sigma(m) - \delta_{i,s}x_{s,t}^\sigma(m),
$$

$$
[e_{i,j}, x_{s,t}^\tau(m)] = \delta_{j,s}x_{i,t}^\tau(m) - \delta_{i,s}x_{s,t}^\tau(m),
$$

$$
[f_{i,j}, x_{s,t}^\tau(m)] = (-1)^{m-1}\delta_{j,s}x_{i,t}^\tau(m) + \delta_{i,s}x_{s,t}^\tau(m).
$$

Sergeev [13] defined the following central elements in $U$:

$$
S_r := \sum_{i=1}^{n} x_{i,i}^\sigma(2r - 1) \quad \text{for } r \in \mathbb{Z}_{>0}.
$$

By [8] Theorem 4.5, Proposition 4.6, we immediately have the following result.

**Proposition 2.6.** For any positive integer $r$, $\Psi f_{M^p(\lambda)}(\Delta_{2r-1})u = -2\sigma(S_r)u$ for all $u \in M^p(\lambda)$, where $\sigma : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ is the antipode such that $\sigma(g) = -g$, for any $g \in \mathfrak{g}$.

**Definition 2.7.** Given a $\lambda$ in (2.13) and a positive integer $r$, let

$$
z_r(\lambda) = -\sum_{s=1}^{r-1} a_{i,s} \prod_{j=1}^{s} \lambda_{j}^{\lambda_{j}^2 + \lambda_{j}},
$$

where the summation is over all $1 \leq s \leq r, 1 \leq i_1 < i_2 < \ldots < i_s \leq n$, and $a_1, a_2, \ldots, a_s \in \mathbb{N}$ such that $\sum_{s=1}^{r-1} a_{i,s} = r-s$.

The above element $z_r(\lambda)$ can be obtained from the element defined in [6] Lemma 8.4] by replacing $\lambda_i$ with $-\lambda_i$ for all $1 \leq i \leq n$. The following two results and their proofs are essentially the same as [6] Lemma 8.4]. The difference is that we compute the actions of $\sigma(S_r)$ on any parabolic Verma supermodule, while they compute the actions of $S_r$. However, one can not directly get the following two results from [6] Lemma 8.4]. Note that $\sigma$ is a superalgebra anti-involution.

**Lemma 2.8.** Suppose $1 \leq m$ and $1 \leq i < j \leq n$. Then $\sigma(x_{i,j}^\sigma(m)) \equiv \sigma(x_{i,j}^\tau(m)) \equiv 0 \pmod{J}$, where $J$ is the left superideal of $U(\mathfrak{g})$ generated by all $e_{k,l}$ and $f_{k,l}$ such that $k < l$.

**Proof.** Obviously, $\sigma(x_{i,j}^\sigma(1)) \equiv \sigma(x_{i,j}^\tau(1)) \equiv 0 \pmod{J}$. In general, by (2.19), (2.21) and inductive assumption, we have

$$
\sigma(x_{i,j}^\sigma(m)) = \sum_{s=1}^{n} (-\sigma(x_{s,j}^\sigma(m-1)))e_{i,s} + (-1)^{m-1}\sigma(x_{s,j}^\tau(m-1))f_{i,s}
$$

$$
\equiv \sum_{s=1}^{n} (-\sigma(x_{s,j}^\tau(m-1)))e_{i,s} + (-1)^{m-1}\sigma(x_{s,j}^\tau(m-1))f_{i,s} \equiv 0 \pmod{J}.
$$

Similarly, one can verify $\sigma(x_{i,j}^\tau(m)) \equiv 0 \pmod{J}$. \qed

**Proposition 2.9.** Let $u$ be a highest weight vector of $M^p(\lambda)$. Then

1. $\sigma(S_r)u = z_r(\lambda)u$,
2. $\Psi f_{M^p(\lambda)}(\Delta_{2r-1})u = -2z_r(\lambda)u$ for any positive integer $r$.
Proof. Suppose $m$ is an odd positive integer. By Lemma 2.8 and 2.21, we have
\[
\sigma(x_{i,i}(m)) = \sum_{s=1}^{n} (-\sigma(x_{s,i}^{\overline{a}}(m-1)) e_{i,s} + \sigma(x_{s,i}^{\overline{a}}(m-1)) f_{i,s})
\]
\[
\equiv -\sigma(x_{i,i}^{\overline{a}}(m-1)) h_{i} + \sigma(x_{i,i}^{\overline{a}}(m-1)) h_{i}^{\prime} + \sum_{s=1}^{i-1} (-\sigma(x_{s,i}^{\overline{a}}(m-1)) e_{i,s} + \sigma(x_{s,i}^{\overline{a}}(m-1)) f_{i,s}) \pmod{J}
\]
\[
\equiv -\sigma(x_{i,i}^{\overline{a}}(m-1)) h_{i} + \sigma(x_{i,i}^{\overline{a}}(m-1)) h_{i}^{\prime} \pmod{J}.
\]

Similarly, one can verify
\[
\sigma(x_{i,i}^{\overline{a}}(m-1)) \equiv -\sigma(x_{i,i}^{\overline{a}}(m-2)) h_{i} - \sigma(x_{i,i}^{\overline{a}}(m-2)) h_{i}^{\prime} - 2 \sum_{s=1}^{i-1} \sigma(x_{s,i}^{\overline{a}}(m-2)) \pmod{J}
\]
\[
\sigma(x_{i,i}^{\overline{a}}(m-1)) \equiv -\sigma(x_{i,i}^{\overline{a}}(m-2)) h_{i} + \sigma(x_{i,i}^{\overline{a}}(m-2)) h_{i}^{\prime} \pmod{J}.
\]
Combining (2.23) \quad (2.24) yields
\[
\sigma(x_{i,i}^{\overline{a}}(m)) \equiv \sigma(x_{i,i}^{\overline{a}}(m-2))(h_{i}^{2} + h_{i}) + 2 \sum_{s=1}^{i-1} \sigma(x_{s,i}^{\overline{a}}(m-2)) h_{i} \pmod{J}.
\]

By (2.23) and inductive assumption on $r$, we have
\[
\sigma(x_{i,i}^{\overline{a}}(2r-1)) \equiv -\sum 2^{s-1} h_{i_{1}} h_{i_{2}} \cdots h_{i_{s}} y_{i_{1}}^{a_{1}} y_{i_{2}}^{a_{2}} \cdots y_{i_{s}}^{a_{s}} \pmod{J},
\]
where $y_{i} := h_{i}^{2} + h_{i}$ and the summation is over all $1 \leq s \leq r$, $1 \leq i_{1} < i_{2} < \ldots < i_{s} = i$, and $a_{1}, a_{2}, \ldots, a_{s} \in \mathbb{N}$ such that $a_{1} + a_{2} + \ldots + a_{s} = r - s$. Since $xu = 0$ for all $x \in J$, by (2.22) and (2.26), we have $\sigma(S_{r}(u)) = z_{r}(\lambda)u$, proving (1). Finally, (2) follows from (1) and Propositions 2.16. \hfill \Box

3. Proof of Conjecture 1.7

The aim of this section is to give a proof of Conjecture 1.7 over an arbitrary commutative ring $\kappa$ containing $2^{-1}$. First, we assume $\kappa = \mathbb{C}$. Recall that $\lambda$ is a $\ell$-dominant weight given in (2.13). Hereafter, we fix an even highest weight vector $v_{\lambda}$ of $L(\lambda)^{0}$. We always assume $n_{i} \geq 2r_{i}$ for $1 \leq i \leq a+b+\varepsilon$.

Definition 3.1. For $1 \leq i \leq a+b$, let
\[
B_{i} = \{(p_{1},-p_{-},-r-1)^{\delta_{1}}(p_{1},-p_{-},-r-2)^{\delta_{2}} \cdots f_{p_{1},-p_{-},-r+1})^{\delta_{r-1}}(f_{p_{1},-p_{-},-r+2})^{\delta_{r}} \mid \delta_{j} \in \{0,1\}, 1 \leq j \leq r\}.
\]

Lemma 3.2. There is a subset $B$ of the PBW monomial basis, which contains all monomials
\[
\{b_{n_{a+b+n_{a+b+1}} \cdots b_{i}} \mid b_{i} \in B_{i}, 1 \leq i \leq a+b\}
\]
such that $\{b_{w} \mid w \in B\}$ is a basis of $L(\lambda)^{0}$.

Proof. By [9] (4.9), we have the result for $L(\lambda^{(i)})$, $1 \leq i \leq a+b$. Thanks to (2.14), we have the result in general. \hfill \Box

Recall $p_{i}$ in (2.14) for all $1 \leq i \leq a+b+\varepsilon$. Let $u^{l}$ (resp. $u^{l,-}$) be the nilradical (resp., opposite nilradical) of $p_{i}$. Then $B_{u}^{l}$ (resp., $B_{u}^{l,-}$) is a basis of even subspace $u_{\overline{a}}^{l,-}$ (resp., odd subspace $u_{\overline{a}}^{l,-}$) of $u^{l,-}$, where
\[
B_{u}^{l} = \{e_{i,j} \mid i > j, \{i,j\} \not\subseteq p_{k}, 1 \leq k \leq a+b+\varepsilon\},
\]
\[
B_{u}^{l,-} = \{f_{i,j} \mid i > j, \{i,j\} \not\subseteq p_{k}, 1 \leq k \leq a+b+\varepsilon\}.
\]

It is known that the symmetric power $S(u^{l,-}) \equiv S(u_{\overline{a}}^{l,-}) \otimes \wedge u_{\overline{a}}^{l,-}$, where $\wedge u_{\overline{a}}^{l,-}$ is the usual exterior power. Moreover, $S(u^{l,-})$ has basis
\[
B_{u} = \left\{ \prod_{k} (c_{i_{1},i_{k}})^{\delta_{k}} \prod_{m} (f_{m,j_{m}})^{\sigma_{m}} \left| \delta_{k} \in \mathbb{Z}_{\geq 0}, \sigma_{m} \in \{0,1\} \right\} \right\},
\]
where the first product (resp., the second product) is taken over any fixed order (for example, the lexicographic order) on $B_{u}^{l}$ (resp., $B_{u}^{l,-}$).

Corollary 3.3. Let $M$ be the parabolic Verma supermodule $MP(\lambda)$, where $\lambda$ is given in (2.23). Then
(1) $M$ has basis $\{z_{\lambda} \mid z \in B_{M}\}$, where $B_{M} = \{y_{b} \mid y \in B_{u}, b \in B\}$, where $B$ is given in Lemma 2.4.
(2) $V^{\otimes \sigma} \otimes M$ has basis $B_{M,r} = \{v_{i} \otimes u_{\lambda} \mid i \in \Gamma, u \in B_{M}\}$. 

Proof. Thanks to Lemma 3.2 we immediately have (1)-(2).

Let $U(g)^{-}$ be the negative part of $U(g)$. For any $f_{i,j}, e_{i,j} \in U(g)^{-}$, we define $\deg(f_{i,j}) = \deg(e_{i,j}) = 1$. This gives a $\mathbb{Z}$-grading on $U(g)^{-}$. If $x \in U(g)^{-}$ is a PBW monomial, then $\deg(x)$ is equal to the numbers of $f_{i,j}$’s and $e_{i,j}$’s appearing in the product of $x$. For any basis element $ybv_\lambda \in B_{M,r}$, we say $ybv_\lambda$ is of degree $\deg(yb)$.

**Definition 3.4.** Suppose $\beta = (\beta_r, \beta_{r-1}, \ldots, \beta_1) \in \mathcal{L}^r$, where $\mathcal{L} = \{0, 1, \ldots, \ell - 1\}$. Define $v^\beta = v_{i_\ell, \ell} \otimes v_{i_{\ell-1}, \ell-1} \otimes \ldots \otimes v_{i_0, 0}$ and $v^\beta = y_{r, \beta_r} y_{r-1, \beta_{r-1}} \ldots y_{1, \beta_1}$, where

a) $y_{k, \beta_k}$ is the ordered product $(\prod_{j=1}^{k-1} f_{i_{k,j}, i_{k,j+1}}) f_{i_k, 0, i_{k,1}, 1} \leq k \leq r,

b) i_{k,0} = n - k + 1$ and $i_{k,j} = -(i_{k,j-1} - \gamma_j)$, if $1 \leq j \leq \beta_k$ and $1 \leq k \leq r,

c) $\gamma_j = \frac{1-(-1)^{\ell-j}}{2} r + \frac{1+(-1)^{\ell-j}}{2} n_{a+b+e-(\ell-a-1)}$, if $1 \leq j \leq \beta_k$ and $1 \leq k \leq r$.

Write $w^\beta = v^\beta \otimes x^\beta v_\lambda$, such that $x^\beta \in B_M$ can be obtained from $y^\beta$ by changing the order of its factors. We say $w^\beta$ is of degree $\deg(x^\beta)$. Obviously, $\deg(x^\beta) = \deg(y^\beta)$. We define a total order on $\mathcal{L}^r$ such that for any $\beta, \beta' \in \mathcal{L}^r$, $\beta' < \beta$ if either $\sum_i \beta_i' < \sum_i \beta_i$, or $\sum_i \beta_i' = \sum_i \beta_i$ and the leftmost nonzero entry of $\beta - \beta'$ is positive.

Following [8], we count a strand of a dotted oriented Brauer-Clifford diagram from right to left. Let $x_k$ be obtained from the identity diagram by placing a single $\bullet$ on the $k$th strand. For example, $x_2 = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$ (3.4).

If $r = 6$. Let $0 = (0, 0, \ldots, 0) \in \mathcal{L}^r$. We define $c_{\beta, \beta'}$ to be the coefficient of $w^\beta$ when $\Psi_M^f(x_{r}^{\beta'} \cdots x_{1}^{\beta_i})(w^0)$ is written in terms of the basis for $V^\otimes r \otimes M$ in Corollary 3.3.

Recall that $\Omega_1$ is in (2.9). Let $i_{k,j}$ and $i_{k,j+1}$ be given in Definition 3.4.

**Lemma 3.5.** Expressing $\Omega_1(v_{i_{k,j}} \otimes v_\lambda)$ as a linear combination of basis elements in $B_{M,1}$ (see Corollary 3.3), we see that the highest degree of its terms is 1. Moreover,

1) $v_{i_{k,j+1}} \otimes f_{i_{k,j}, i_{k,j+1}} v_\lambda$ is a term of $\Omega_1(v_\lambda)$ if and only if $l = i_{k,j}$.

2) When $l = i_{k,j}$, the coefficient of $v_{i_{k,j+1}} \otimes f_{i_{k,j}, i_{k,j+1}} v_\lambda$ is $\pm 1$.

**Proof.** Straightforward computation. See the definition of $\Omega_1$ in (2.9).

**Lemma 3.6.** Suppose $\beta, \beta' \in \mathcal{L}^r$. We have $c_{\beta, \beta'} = \pm 1$ and $c_{\beta, \beta'} = 0$ if $\beta' < \beta$.

**Proof.** By Theorem 2.4, $x_k$ acts on $V^\otimes r \otimes M$ via $\Psi_M^f(x_k) = 1^\otimes r-k \otimes \Omega_1|_{V^\otimes r-k \otimes M}$. In other words, when we compute $\Psi_M^f(x_k)$, we consider $V$ (resp., $V^\otimes k-1 \otimes M$) as the first (resp., second) tensor factor. Since we are going to consider the terms of $\Psi_M^f(x_{r}^{\beta'} \cdots x_{1}^{\beta_i})(w^0)$ with the highest degree, $x_k$ can be replaced by $\pi_{k,0}(\Omega_1)$, where $\pi_{k,0} : U(g)^\otimes k \to U(g)^\otimes (k+1)$ is the linear map such that $\pi_{k,0}(g_1 \otimes g_2) = g_1 \otimes g_2$, for all $g_1, g_2 \in U(g)$. By Lemma 5.5 (2) and induction on $|\beta| := \sum_i \beta_i$, we have $c_{\beta, \beta'} = \pm 1$.

Since the degree of any term in the expression of $\Psi_M^f(x_{r}^{\beta'} \cdots x_{1}^{\beta_i})(w^0)$ is less than $\sum_i \beta_i'$, we have $c_{\beta, \beta'} = 0$ if $\sum_i \beta_i' < \sum_i \beta_i$. Now, we assume $\sum_i \beta_i' = \sum_i \beta_i$ and $\beta' < \beta$. We prove $c_{\beta, \beta'} = 0$ by induction on $\sum_i \beta_i'$. Obviously, $c_{\beta, \beta'} = 0$ if $\sum_i \beta_i' = 0$. Otherwise, let $k$ be the maximal integer such that $\beta_k > 0$.

Case 1: $\beta_k' = 0$. Suppose that $v_{l} \otimes v_\lambda \in B_{M,r}$, which appears as a term of $\Psi_M^f(x_{r}^{\beta'} \cdots x_{1}^{\beta_i})(w^0)$ with the highest degree. Since we are assuming that $\beta_k = 0$, $v_{i_k}$ (i.e. kth component of $v_\lambda$) must be $v_{n-k+1}$. Further, $v_{i_k}$ is the $k$th component of $v^0$ (see the definition of $v^0$ in Definition 3.4). By the definition of $i_{k,j}$ in Definition 3.3, we have $c_{\beta, \beta'} = 0$.

Case 2: $\beta_k' > 0$. If $c_{\beta, \beta'} \neq 0$, we define $\overline{\beta} = \beta - (0^{r-k}, 1, 0^{k-1})$ and $\overline{\beta'} = \beta' - (0^{r-k}, 1, 0^{k-1})$. By Lemma 3.5 (1), $w^{\overline{\beta}}$ is a term in the expression of $\Psi_M^f(x_{r}^{\beta'} \cdots x_{1}^{\beta_i})(w^0)$ with nonzero coefficient. So, $c_{\overline{\beta}, \overline{\beta'}} \neq 0$. This contradicts our inductive assumption since $\sum_i \overline{\beta_i} = \sum_i \overline{\beta_i'} < \sum_i \beta_i'$ and $\overline{\beta} < \overline{\beta'}$ and $|\beta| = |\beta'|-1$. Thus, $c_{\beta, \beta'} = 0$ as required.

For any dotted oriented Brauer-Clifford diagram $d$ of type $\uparrow^r \rightarrow \uparrow^r$, comes and Kujawa defined the oriented Brauer-Clifford diagram undo($d$), which is obtained from $d$ by removing all $\bullet$s. For example,
Suppose $d$ is a normally ordered dotted oriented Brauer-Clifford diagram without bubbles of type $\uparrow^r \rightarrow \uparrow^r$. Let $\beta_k(d)$ be the number of $\bullet$'s on the $k$th strand of $d$. Then

$$d = \text{undot}(d) \circ x_1^{\beta_1(d)} \circ \cdots \circ x_1^{\beta_1(d)}.$$  \hfill (3.5)

(see \textit{ibid} §5)).

**Proposition 3.7.** Suppose $r$ is a positive integer. Let $\mathcal{OBC}_r^\ell(\delta)$ be the cyclicotomic oriented Brauer-Clifford supercategory over $\mathbb{C}$ with respect to $f(t) = 2^{a+\varepsilon} \prod_{s=1}^b (t^2 - u_i) \in \mathbb{C}[t]$ such that $\delta_{2k}$ is non-zero, where $\delta_k(\lambda)$ is given in Definition \textit{ibid} 2.7 for any positive integer $k$. Let $S$ be the set of all elements undot$(d) \circ x_1^{\beta_1(d)} \circ \cdots \circ x_1^{\beta_1(d)}$, where

a) $d$ ranges over all equivalence classes of normally ordered dotted oriented Brauer-Clifford diagrams without bubbles of type $\uparrow^r \rightarrow \uparrow^r$,

b) $(\beta_r(d), \beta_{r-1}(d), \ldots, \beta_1(d)) \in \ell^k$, where $\ell = \{0, 1, \ldots, \ell - 1\}$.

For a finite subset $A \subset S$, we have $\sum_{d \in A} p_d d = 0$ if and only if $p_d = 0$ for all $d \in A$.

**Proof.** Since undot$(d) : \uparrow^r \rightarrow \uparrow^r$ can be decomposed in terms of $\downarrow^r$'s and $\uparrow^r$'s, we see that undot$(d)$ is invertible. Suppose $\sum_{d \in A} p_d d = 0$. If there is a $d \in A$ such that $p_d \neq 0$, we can find a $d_0 \in A$ such that $p_{d_0} \neq 0$ and $\beta(d_0) \geq \beta(d)$ for all $d \in A$ with $p_d \neq 0$. Thanks to Lemma 3.6 the coefficient of $u^{\beta(d_0)}$ in $\Psi^f_M(\text{undot}(d_0))^{-1} \circ \sum_{d \in A} p_d d (u^0)$ is $p_{d_0} c_{\beta(d_0), \beta(d_0)}$, which is non-zero. So, $\Psi^f_M(\sum_{d \in A} p_d d) \neq 0$, a contradiction.

Now, we consider $z_k(\lambda)$ as a polynomial in variables $n_1, n_2, \ldots, n_{a+b}$ with coefficients in $\mathbb{C}$. Given a weight $\lambda$ in \textit{ibid} 2.13, we consider the morphism

$$\psi : \mathbb{C}^{a+b} \rightarrow \mathbb{C}^{a+b}, \quad (n_1, n_2, \ldots, n_{a+b}) \mapsto (z_1(\lambda), z_2(\lambda), \ldots, z_{a+b}(\lambda)).$$  \hfill (3.7)

**Lemma 3.8.** Let $\psi$ be the morphism in (3.7). Then $\psi$ is dominant over $\mathbb{C}$.

**Proof.** We view $n_1, n_2, \ldots, n_{a+b}$ as variables. So, it is enough to show that the determinant $J_\psi$ is non-zero, where $J_\psi := (\frac{\partial z_k(\lambda)}{\partial n_s})_{1 \leq k, s \leq a+b}$ is the Jacobian matrix.

We claim that the highest degree term of $\frac{\partial z_k(\lambda)}{\partial n_s}$ is $a_k n_s 2^{2k-1}$, for some $a_k \neq 0$ such that $a_k$ is independent of $s$ whenever $s \neq k$. If so, then, up to some non-zero scalar, the term of det $J_\psi$ with the highest degree forms the following determinant

$$\det \begin{pmatrix} n_1 & n_2 & \ldots & n_{a+b} \\ n_1^3 & n_2^3 & \ldots & n_{a+b}^3 \\ \vdots & \vdots & \ddots & \vdots \\ n_1^{2(a+b)-1} & n_2^{2(a+b)-1} & \ldots & n_{a+b}^{2(a+b)-1} \end{pmatrix},$$

which is non-zero. So, it remains to prove the claim. By \textit{ibid} 2.7, the highest degree term of $\frac{\partial z_k(\lambda)}{\partial n_s}$ only appears in some term of $-\frac{\partial}{\partial n_s} \sum_{i=p_{s-1}+1}^{p_s} \lambda_i (\lambda_i^2 + \lambda_s) \lambda_i^{2k-1}$. Therefore, it appears in some term of $-\frac{\partial}{\partial n_s} \sum_{i=p_{s-1}+1}^{p_s} \lambda_i^{2k-1}$. Thanks to \textit{ibid} 2.13, we write

$$g_k, s = \sum_{j=1}^{n_s} (l_s - n_s + j)^{2k-1}.$$  \hfill (2.13)

This shows that the highest degree term of $g_k, s$ is $b_k n_s 2^{2k}$, where $b_k \in \mathbb{C}^\ast$, a non-zero scalar which is independent of $s$. However, when we use \textit{ibid} 2.13 to obtain $g_k, s$, $n_s$ ranges over all big enough even integers. Since $g_k, s$ is a polynomial in variable $n_s$ with coefficients in $\mathbb{C}$, it is available for all $n_s \in \mathbb{C}$. So, the term of $\frac{\partial g_k}{\partial n_s}$ with highest degree is $2k b_k n_s^{2k-1}$. This proves our claim by setting $a_k = -2k b_k$.

**Corollary 3.9.** As $(n_1, n_2, \ldots, n_{a+b})$ ranges over all sequences of even integers such that $n_i \geq 2r$ for $1 \leq i \leq a + b$, the set of points $(z_1(\lambda), z_2(\lambda), \ldots, z_{a+b}(\lambda))$ defined by \textit{ibid} 2.7 is Zariski dense in $\mathbb{C}^{a+b}$.

**Proof.** Thanks to Lemma 3.8 we have the result as required.
For any positive integer $b$, let $Z := Z[\hat{u}_1, \cdots, \hat{u}_b]$ be the ring of polynomials in variables $\hat{u}_1, \cdots, \hat{u}_b$.

**Theorem 3.10.** Suppose $r$ is a positive integer. Let $\mathcal{OBC}_Z^f$ be the cyclotomic oriented Brauer-Clifford supercategory over $Z$ with respect to $f(t) = t^{2a+\tau} \prod_{i=1}^b (t^2 - \hat{u}_i) \in Z[t]$. Let $S$ be the set of all elements

$$p_d(\Delta_1, \Delta_3, \ldots, \Delta_{2(a+b)-1}) \undot(d) \circ x_{\hat{r}}^{\beta_\ell(d)} \circ \cdots \circ x_1^{\beta_1(d)},$$

where

a) $d$ ranges over all equivalence classes of normally ordered dotted oriented Brauer-Clifford diagrams without bubbles of type $\uparrow^r \rightarrow \uparrow^s$,

b) $p_d(t_1, t_3, \ldots, t_{2(a+b)-1})$ ranges over all polynomials in $Z[t_1, \ldots, t_{2(a+b)-1}]$,

c) $(\beta_1(d), \beta_{\ell-1}(d), \ldots, \beta_1(d)) \in \mathcal{L}$, where $\mathcal{L} = \{0, 1, \ldots, \ell - 1\}$.

For a finite subset $A \subset S$,

$$\sum p_d(\Delta_1, \Delta_3, \ldots, \Delta_{2(a+b)-1})d = 0$$

(where the summation is over all elements in $A$) if and only if $p_d(t_1, \ldots, t_{2(a+b)-1}) = 0$ for all $p_d(\Delta_1, \Delta_3, \ldots, \Delta_{2(a+b)-1})d \in A$.

**Proof.** Suppose $\sum p_d(\Delta_1, \Delta_3, \ldots, \Delta_{2(a+b)-1})d = 0$ in $\mathcal{OBC}_Z^f$. There is a ring homomorphism from $Z$ to $\mathbb{C}$, sending $\hat{u}_i$ to $u_i$ for all admissible $i$, where $u_i$'s are given in Theorem 2.4. We have

$$\sum p_d(\Delta_1, \Delta_3, \ldots, \Delta_{2(a+b)-1})d \otimes 1 = 0 \text{ in } \mathcal{OBC}_C^f.$$

Applying the superfunctor $\Psi_M^f(\lambda)$ on the previous equation (see Theorem 2.4) and using Proposition 3.7 we have

$$p_d(\delta_1, \delta_3, \ldots, \delta_{2(a+b)-1}) = 0,$$

where $p_d(\delta_1, \delta_3, \ldots, \delta_{2(a+b)-1})$ is obtained from $p_d(\Delta_1, \Delta_3, \ldots, \Delta_{2(a+b)-1})$ by replacing $u_i$ (resp., $\Delta_{2j-1}$) with $l_{a+i}(l_{a+i} + 1)$ (resp., $\delta_{2j-1}$). Thanks to Corollary 3.9, $p_d(t_1, t_3, \ldots, t_{2(a+b)-1}) = 0$, where $p_d(t_1, t_3, \ldots, t_{2(a+b)-1})$ is obtained from $p_d(t_1, t_3, \ldots, t_{2(a+b)-1})$ by specializing $\hat{u}_i$ at $t_{a+i}(l_{a+i} + 1)$, $1 \leq i \leq b$. By Theorem 2.4 and (2.13), there are infinite choices of $l_{a+1}, l_{a+2}, \ldots, l_{a+b}$ such that $p_d(t_1, t_3, \ldots, t_{2(a+b)-1}) = 0$. Further, the choices of $l_{a+i}$ and $l_{a+j}$ are independent whenever $i \neq j$. Thanks to the fundamental theorem of algebra, we have $p_d(t_1, t_3, \ldots, t_{2(a+b)-1}) = 0$ in $Z[t_1, t_3, \ldots, t_{2(a+b)-1}]$.

**Corollary 3.11.** For any nonnegative $r$ and commutative ring $\kappa$ containing $2^{-1}$, $\text{End}_{\mathcal{OBC}_Z^f}(\uparrow^r)$ has basis given by all equivalence classes of normally ordered darted oriented Brauer-Clifford diagrams with bubbles of type $\uparrow^r \rightarrow \uparrow^s$ with fewer than $\ell \cdot \bullet$'s on each strand.

**Proof.** In [8], Comes and Kujawa have proved that $\text{End}_{\mathcal{OBC}_Z^f}(\uparrow^r)$ is spanned by all equivalence classes of normally ordered darted oriented Brauer-Clifford diagrams with bubbles of type $\uparrow^r \rightarrow \uparrow^s$ with fewer than $\ell \cdot \bullet$'s on each strand. When $\kappa = Z$, the linear independent of such elements immediately follows from Theorem 3.10. This proves the result when $\kappa = Z$. In general, we consider $\kappa$ as the $Z$-module on which $\hat{u}_i$'s act as scalars $u_i$'s for all $1 \leq i \leq b$. By arguments similar to those for $\text{AOBC}_\kappa$ in [8], one can define the $\kappa$-supercategory $\mathcal{OBC}_Z^f \otimes Z \kappa$ with the objects as $\mathcal{OBC}_Z^f$, and the morphisms are

$$\text{Hom}_{\mathcal{OBC}_Z^f \otimes Z \kappa}(a, b) = \text{Hom}_{\mathcal{OBC}_Z^f}(a, b) \otimes Z \kappa.$$

The obvious mutually inverse superfunctors provide an isomorphism of supercategories between $\mathcal{OBC}_Z^f \otimes Z \kappa$ and $\mathcal{OBC}_Z^f$. The result follows from the base change, immediately.

The following result can be proven by arguments similar to those for $\text{AOBC}_\kappa$ in [8] [6]. The difference is that we have to consider $\mathcal{OBC}_Z^f$ whereas they can consider $\text{AOBC}_Z$.

**Theorem 3.12.** Conjecture 2.4 is true over an arbitrary commutative ring $\kappa$ containing $2^{-1}$.

**Proof.** Suppose that $a$ (resp. $b$) consists of $r_1$ (resp., $r'_1$) $\uparrow^r$'s and $r_2$ (resp., $r'_2$ $\downarrow^s$'s). If $r_1 + r'_2 \neq r'_1 + r_2$, then there is no oriented Brauer diagram of type $a \rightarrow b$, forcing $\text{Hom}_{\mathcal{OBC}_Z^f}(a, b) = 0$. When $r_1 + r'_2 = r'_1 + r_2 := r$, and $\kappa$ is a field, there is a $\kappa$-linear isomorphism

$$\text{Hom}_{\text{AOBC}_\kappa}(a, b) \leq k \rightarrow \text{Hom}_{\mathcal{OBC}_Z^f}(\uparrow^r \rightarrow \uparrow^s, \uparrow^r \rightarrow \uparrow^s) \leq k \rightarrow \text{End}_{\mathcal{OBC}_Z^f}(\uparrow^r) \leq k$$

(3.9)
defined in the same way as the top horizontal maps in [8 (4.2)], where \( \text{Hom}_{\mathcal{BC}^j} (a, b) \leq_k \) is the \( \kappa \)-span of all dotted oriented Brauer-Clifford diagrams with bubbles of type \( a \to b \) having at most \( k \) \( \bullet \)'s, and \( 0 \leq k \leq \ell - 1 \). By (8.9),
\[
\dim_\kappa \text{Hom}_{\mathcal{BC}^j} (a, b) = \sum_{k=0}^{\ell-1} \dim_\kappa \text{Hom}_{\mathcal{BC}^k} (a, b) \leq_k,
\]
forcing \( \dim_\kappa \text{Hom}_{\mathcal{BC}^j} (a, b) = \dim_\kappa \text{End}_{\mathcal{BC}^j} (\uparrow^\kappa) \). Comes-Kujawa proved that \( \text{Hom}_{\mathcal{BC}^j} (a, b) \) is spanned by the set of all equivalence classes of normally ordered dotted oriented Brauer-Clifford diagrams with bubbles of type \( a \to b \) with fewer than \( \ell \) \( \bullet \)'s on each strand whenever \( \kappa \) is a commutative ring containing \( 2^{-1} \). Corollary 3.11 immediately implies Theorem 3.12 over the field \( \kappa \). Since the \( \mathcal{Z} \)-linear independent of the set of all equivalence classes of normally ordered dotted Brauer-Clifford diagrams of type \( a \to b \) follows from the corresponding result over the fraction field of \( \mathcal{Z} \), we have Theorem 3.12 over \( \mathcal{Z} \). By (8.8), we have Theorem 3.12 over an arbitrary commutative ring \( \kappa \) containing \( 2^{-1} \).

\[ \square \]

4. Cyclotomic walled Brauer-Clifford superalgebras

The aim of this section is to establish connections between two cyclotomic walled Brauer-Clifford superalgebras defined in [8,9]. The level two cases has been dealt with in [9] under the assumption that \( f(t) = t^2 - u \) with \( u \neq 0 \) over the complex field.

We start by recalling the notion of cyclotomic walled Brauer-Clifford superalgebras in [2]. Let \( \Sigma_r \) be the symmetric group in \( r \) letters. Then \( \Sigma_r \) is generated by \( s_1, \ldots, s_{r-1} \), subject to the relations (for all admissible \( i \) and \( j \)):
\[
s_i^2 = 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_i, \quad s_is_j = s_js_i, \quad \text{if } |i - j| > 1.
\]

The Hecke-Clifford algebra \( HC_r \) is the associative \( \kappa \)-superalgebra generated by even elements \( s_1, \ldots, s_{r-1} \) and odd elements \( c_1, \ldots, c_r \) subject to (4.1) together with the following defining relations (for all admissible \( i \) and \( j \)):
\[
c_i^2 = 1, \quad c_ic_j = -c_jc_i, \quad w^{-1}c_iw = c_{(i)w}, \forall w \in \Sigma_r.
\]

The affine Hecke-Clifford algebra \( HC_r^{\text{aff}} \) is the associative \( \kappa \)-superalgebra generated by even elements \( s_1, \ldots, s_{r-1}, x_1 \) and odd elements \( c_1, \ldots, c_r \) subject to (4.1)–(4.2), together with the following defining relations (for all admissible \( i \) and \( j \)):
\[
x_1x_2 = x_2x_1, \quad x_1c_i = (-1)^{\delta_{i,1}}c_ix_1, \quad s_jx_1 = x_1s_j, \text{if } j \neq 1,
\]

where \( x_2 = s_1x_1s_1 - (1 - c_1c_2)s_1 \).

Let \( \overline{HC}_r \) be the \( \kappa \)-superalgebra generated by the even elements \( \overline{s}_1, \ldots, \overline{s}_{r-1}, \overline{x}_1 \) and odd elements \( \overline{c}_1, \ldots, \overline{c}_r \) subject to the relations (for all admissible \( i \) and \( j \)):
\[
\overline{s}_i^2 = 1, \quad \overline{s}_i\overline{s}_{i+1}\overline{s}_i = \overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1}, \quad \text{and } \overline{s}_i\overline{s}_j = \overline{s}_j\overline{s}_i, \quad \text{if } |i - j| > 1,
\]
\[
\overline{c}_i^2 = 1, \quad \overline{c}_i\overline{c}_j = -\overline{c}_j\overline{c}_i, \quad \text{and } w^{-1}\overline{c}_iw = \overline{c}_{(i)w}, \forall w \in \Sigma_r.
\]

Let \( \overline{HC}_r^{\text{aff}} \) be the \( \kappa \)-superalgebra generated by even elements \( \overline{s}_1, \ldots, \overline{s}_{r-1}, \overline{x}_1 \) and odd elements \( \overline{c}_1, \ldots, \overline{c}_r \) subject to (4.4) together with the following defining relations (for all admissible \( i \) and \( j \)):
\[
\overline{x}_1\overline{x}_2 = \overline{x}_2\overline{x}_1, \quad \overline{x}_1\overline{c}_i = (-1)^{\delta_{i,1}}\overline{c}_i\overline{x}_1, \quad \overline{s}_j\overline{x}_1 = \overline{x}_1\overline{s}_j, \text{if } j \neq 1,
\]

where \( \overline{x}_2 = \overline{s}_1\overline{x}_1\overline{s}_1 - (1 + \overline{c}_1\overline{c}_2)\overline{x}_1 \).

In [10, Theorem 5.1], Jung and Kang defined the walled Brauer-Clifford superalgebra \( BC_{r,t} \). It is the associative \( \kappa \)-superalgebra generated by even generators \( e_1, s_1, \ldots, s_{r-1}, \overline{s}_1, \ldots, \overline{s}_{t-1} \) and odd generators \( c_1, \ldots, c_r, \overline{c}_1, \ldots, \overline{c}_r \) subject to (4.1), (4.2) and (4.3) together with the following defining relations for all admissible \( i,j \):
\[
\begin{align*}
(1) \quad & e_1c_1 = e_1\overline{c}_1, \quad c_1e_1 = \overline{c}_1e_1, \\
(2) \quad & \overline{s}_i\overline{c}_i = c_1\overline{s}_i, \quad s_i\overline{c}_i = \overline{c}_i\overline{s}_i, \\
(3) \quad & \overline{c}_i\overline{s}_i = -\overline{s}_i\overline{c}_i, \quad \overline{s}_i\overline{c}_i = \overline{c}_i\overline{s}_i, \\
(4) \quad & e_1^2 = 0, \\
(5) \quad & e_1s_1e_1 = e_1, \quad e_1\overline{s}_1e_1 = \overline{e}_1, \\
(6) \quad & s_1e_1 = e_1s_1, \quad \overline{s}_1e_1 = \overline{e}_1s_1, \text{if } i \neq 1, \\
(7) \quad & e_1s_1\overline{s}_1e_1 = e_1s_1\overline{s}_1e_1, \\
(8) \quad & s_1e_1s_1\overline{s}_1e_1 = \overline{s}_1e_1s_1\overline{s}_1e_1, \\
(9) \quad & c_1e_1 = e_1c_1 \text{ and } \overline{c}_1e_1 = e_1\overline{c}_1, \text{if } i \neq 1, \\
(10) \quad & e_1c_1e_1 = 0 = e_1\overline{c}_1e_1.
\end{align*}
\]
**Definition 4.1.** [9] Definition 3.1] The affine walled Brauer-Clifford superalgebra $BC_{c}^{aff}$ is the associative $\kappa$-superalgebra generated by odd elements $c_{1}, \ldots, c_{r}, \tau_{1}, \ldots, \tau_{t}$ and even elements $e_{1}, x_{1}, s_{1}, \ldots, s_{r-1}, \bar{x}_{1}, \ldots, \bar{x}_{t-1}$, and two families of even central elements $\omega_{2k+1}, \omega_{2k}, k \in \mathbb{Z} \geq 1$ subject to (4.7) and the above relations (1)–(10) together with the following defining relations for all admissible $i$:

1. $e_{1}(x_{1} + \bar{x}_{1}) = (x_{1} + \bar{x}_{1})e_{1} = 0$,
2. $e_{1}s_{1}s_{1}s_{1} = s_{1}s_{1}s_{1}s_{1}$,
3. $x_{1}(e_{1} + \bar{x}_{1} - \tau_{1}) = (e_{1} - \bar{x}_{1} + \bar{x}_{1})x_{1}$,
4. $e_{1}\bar{x}_{1}s_{1} = \bar{s}_{1}\bar{x}_{1}s_{1}$,
5. $e_{1}x_{1}^{2k+1}e_{1} = \omega_{2k+1}e_{1}$, $\forall k \in \mathbb{N}$,
6. $e_{1}^{2k}e_{1} = 0$, $\forall k \in \mathbb{N}$,
7. $e_{1}\bar{x}_{1}^{k}e_{1} = \omega_{k}e_{1}$, $\forall k \in \mathbb{Z} > 0$,
8. $x_{1}\tau_{1} = \tau_{1}x_{1}$,
9. $x_{1}s_{1} = x_{1}s_{1}$,
10. $x_{1}s_{1} = x_{1}s_{1}$,
11. $x_{1}s_{1} = s_{1}x_{1}$.

By [9] Lemma 3.4, Corollary 3.5], we have to assume that $\omega_{k}$'s satisfy some technical conditions which are defined via $\omega_{2k+1}$ for all $i \in \mathbb{N}$ and moreover, $\omega_{2k}$ is $0$ for $k \in \mathbb{Z} \geq 2$. Otherwise, $e_{1} = 0$ and $BC_{c}^{aff}$ is isomorphic to the outer tensor product of $HC_{r}$ and $HC_{c}^{aff}$ whenever $\kappa$ is a field.

Let $\mathcal{A}BC_{c}(\delta)$ and $\mathcal{O}BC_{c}(\delta)$ be the category obtained from $\mathcal{A}BC_{c}$ and $\mathcal{O}BC_{c}$ by specializing $\Delta_{k}$ at $\delta_{k}$, where $\delta = (\delta_{k})_{k \in \mathbb{Z} \geq 0}$. It is proven in [8,9] that there is a $\kappa$-superalgebra homomorphism

$$\varphi : BC_{c}^{aff} \longrightarrow \text{End}_{\mathcal{A}BC_{c}(\delta)}(\bigotimes_{r,t}). \quad (4.6)$$

Since we are going to use the same notation in [8], we use their homomorphism $\varphi$ in [4,6]. By [8] A.4], $\varphi$ is defined by

\begin{align*}
\delta_{1} & \mapsto \sqrt{-1} \quad s \quad i-1 \quad \uparrow \quad \downarrow \\
\delta_{2} & \mapsto \sqrt{-1} \quad s \quad j-1 \quad \uparrow \quad \downarrow \\
\delta_{i} & \mapsto \sqrt{-1} \quad s \quad r-1 \quad \uparrow \quad \downarrow \\
\omega_{2k+1} & \mapsto \omega_{2k+1} \\
\omega_{2k} & \mapsto \omega_{2k} \quad \delta_{k}' \quad \text{if } k \text{ is odd; } \quad 0 \quad \text{if } k \text{ is even.}
\end{align*}

where $\delta_{k}'$'s $\in \kappa$ are determined by

$$\delta_{k} - \delta_{k} = - \sum_{0 < i < k/2} \delta_{2i-1} \delta_{k-2i}. \quad (4.7)$$

Via the previous homomorphism, we see that [4,7] are the same as those relations in [9] Corollary 3.5]. Let $BC_{c}^{aff}(\delta)$ be the affine walled Brauer-Clifford superalgebra obtained from $BC_{c}^{aff}$ by specializing central generators $\omega_{2k+1}$ and $\omega_{2k+1}$ ($k \in \mathbb{Z} \geq 2$) at $-\delta_{2k+1}$ and $\delta_{k}'$ in $\kappa$. Then $\varphi$ factors through $BC_{c}^{aff}(\delta)$. The basis theorem of $\text{End}_{\mathcal{A}BC_{c}(\delta)}(\bigotimes_{r,t})$ in [8] yields the following $\kappa$-superalgebra isomorphism induced from $\varphi$:

$$\overline{\varphi} : BC_{c}^{aff}(\delta) \longrightarrow \mathcal{B}e_{c}^{aff}, \quad (4.8)$$

where $\mathcal{B}e_{c}^{aff} := \text{End}_{\mathcal{A}BC_{c}(\delta)}(\bigotimes_{r,t})$. In particular, $\overline{\varphi}$ sends each regular monomial of $BC_{c}^{aff}(\delta)$ in [9] Definition 3.1] to a unique equivalence class of normally ordered dotted oriented Brauer-Clifford diagram (without bubbles). The tiny difference between the isomorphisms established in [8,9] is that we number the leftmost strand as the first one in [9] while Comes and Kujawa number the rightmost as the first strand (see, e.g., (3.4]). Moreover, we use right tensor ideal in [9] while they use left tensor ideal for the definition of $\mathcal{O}BC_{c}^{f}$. 
Definition 4.2. Define two linear $\kappa$-linear homomorphisms $\sigma_\uparrow : \text{End}_{\AOBC}(\uparrow) \to \text{End}_{\AOBC}(\downarrow)$ and $\sigma_\downarrow : \text{End}_{\AOBC}(\downarrow) \to \text{End}_{\AOBC}(\uparrow)$ such that, for any $h_1 \in \text{End}_{\AOBC}(\downarrow)$ and $h_2 \in \text{End}_{\AOBC}(\downarrow)$,

$$\sigma_\uparrow(h_1) := (\downarrow \circ \bigstar h_1) \circ (\downarrow \bigstar) = \bigstar,$$

$$\sigma_\downarrow(h_2) := (\uparrow \circ \bigstar h_2) \circ (\uparrow \bigstar) = \bigstar.$$

Lemma 4.3. $\sigma_\uparrow$ and $\sigma_\downarrow$ are mutually inverse to each other.

Proof. The result immediately follows from the first relation in (1.1).

Similarly, we define

$$\sigma_\uparrow(\delta) : \text{End}_{\AOBC}(\delta)(\uparrow) \to \text{End}_{\AOBC}(\delta)(\downarrow),$$

$$\sigma_\downarrow(\delta) : \text{End}_{\AOBC}(\delta)(\downarrow) \to \text{End}_{\AOBC}(\delta)(\uparrow)$$

in an obvious way. Recall $f(t)$ is given in (1.9).

Lemma 4.4. There is a $g(t) \in \kappa[t]$ such that the equation

$$(-1)^s \varphi(e_1 f(x_1)) = \varphi(e_1 g(x_1))$$

(4.9)

holds in $\AOBC(\delta)$.

Proof. Write $f(t) = at^\ell + at_{-1} t^{\ell-1} + \ldots + a_1 t + a_0$, where $a_i \in \kappa$ for all admissible $i$. Define $y_k := \sigma_\uparrow(\delta)^k$ in $\AOBC(\delta)$ for $1 \leq k \leq \ell$. By (3.26), $y_k = \downarrow k + \sum_{i=0}^{k-1} \delta_{k-i-1} \uparrow i$. So,

$$y_k = \downarrow k + \sum_{i=0}^{k-1} \delta_{k-i-1} \uparrow i$$

(4.10)

in $\AOBC(\delta)$. As $\downarrow k = (\delta)^k$, $y_k$ can be considered as a polynomial of with degree $k$. We define

$$g := \sigma_\uparrow(\delta)(f(\delta)) = \bigstar$$

(4.11)

in $\AOBC(\delta)$, where $j = f(\delta)$. So $g = a_\ell y_\ell + a_{\ell-1} y_{\ell-1} + \ldots + a_1 y_1 + a_0$, which can be considered as a polynomial of with degree $\ell$. Let $g(t) \in \kappa[t]$ be obtained from $g$ by replacing $\delta$ by $t$. Then $g(\delta) = g$.

Thanks to Lemma 4.3, $\sigma_\downarrow(\delta)(g(\delta)) = \sigma_\uparrow(\delta) \circ \sigma_\downarrow(\delta)(f(\delta)) = f(\delta)$. Let $h := g(\delta)$. Then

$$f(\delta) = \bigstar$$

(4.12)

Via (1.1–1.2) and the definition of $\varphi$, we have

$$\varphi(x_1)^k = \downarrow^{s-1} r^k \bigstar \quad \text{and hence } \varphi(g(x_1)) = \downarrow^{s-1} r^k.$$

(4.13)

Therefore, we have

$$\varphi(f(x_1)) = (-1)^s f(\downarrow^{s-1} r^{s-1} \bigstar)$$

(4.14)

In order to prove (4.9), by (4.13–4.14), it is enough to prove that

$$\downarrow^{s-1} r^{s-1} \bigstar = \downarrow^{s-1} r^{s-1} \bigstar$$

(4.15)

where $j = f(\delta)$ and $h = g(\delta)$. By (1.1–1.2), we have

$$\downarrow^{s-1} r^{s-1} \bigstar = \downarrow^{s-1} r^{s-1} \bigstar$$

(4.16)
It follows from \cite{8} (3.4) that

\[ \circlearrowleft = \circlearrowright \] \hspace{1cm} (4.17)

Now (4.15) follows from (4.16)–(4.17).

\[ \square \]

**Lemma 4.5.** Suppose \( \kappa \) is an algebraically closed field with characteristic not two. Let \( g(t) \in \kappa[t] \) be defined in the proof of Lemma 4.4. Then there are two non-negative integers \( a_1, b_1 \) and some non-zero scalars \( \pi_i \) in \( \kappa \), \( 1 \leq i \leq b_1 \) such that \( a_1 + 2b_1 = \ell \) and

\[ g(\pi_i) = \pi_i^{a_1} \prod_{j=1}^{b_1} (\pi_i^2 - \pi_j). \] \hspace{1cm} (4.18)

**Proof.** Since we are assuming that \( \kappa \) is an algebraically closed field, we can write \( g(t) = t^{a_1} \prod_{j=1}^{b_1} (t - \pi_j) \) such that \( a_1 + c = \ell \) and \( \pi_j \neq 0 \). This is the place where we have to assume that \( \kappa \) is an algebraically closed field. By Lemma 4.4 and the fact that \( \pi \) is an isomorphism, we have

\[ e_1f(x_1) = (-1)^{\ell}e_1g(x_1) \] \hspace{1cm} (4.19)

in \( BC_{\ell,r,t}(\delta) \). Thanks to \cite{9} Lemma 6.2, we have \( \pi_1g(\pi_1) = (-1)^{a+\varepsilon}g(\pi_1)\pi_1 \) if (4.19) holds, where \( a \) and \( \varepsilon \) can be found in the definition of \( f(t) \). By \cite{9} Theorem 5.15, \( \kappa[\pi_1] \) can be considered as a ring of polynomials in variable \( \pi_1 \). Thus \( \kappa[\pi_1] \) is a unique factorization domain. This means that \( c \) is even and \( \pi_1 - \pi_j, \pi_4 + \pi_j \) appear in \( g(\pi_1) \), simultaneously. Note that \( g(t) \) is a monic polynomial in variable \( t \), we have

\[ g(\pi_i) = \pi_i^{a_1} \prod_{j=1}^{b_1} (\pi_i^2 - \pi_j) \]

for some scalars \( \pi_i \)'s in \( \kappa \), where \( b_1 = c/2 \) such that \( a_1 + 2b_1 = \ell \). \hspace{1cm} \( \square \)

Thanks to (4.9), (4.18)–(4.19), we can define

\[ BC_{\ell,r,t}^f := BC_{\ell,r,t}^{\text{aff}} / I, \] \hspace{1cm} (4.20)

where \( I \) is the two-sided super ideal generated by \( f(x_1) \) and \( g(\pi_1) \). This is the same as the level \( \ell \) walled Brauer-Clifford superalgebra \( BC_{\ell,r,t} \) defined in \cite{9} Definition 3.14.

**Theorem 4.6.** Suppose that \( \kappa \) is an algebraically closed field with characteristic not two. As \( \kappa \)-superalgebras, \( BC_{\ell,r,t}^f \cong \text{End}_{\mathcal{O}BC_{\ell}^f(\delta)}(\ell^{\uparrow \uparrow}) \).

**Proof.** Since \( \mathcal{O}BC_{\kappa}^f(\delta) \) is a quotient supercategory of \( A\mathcal{O}BC_{\kappa}(\delta) \), we have the canonical epimorphism from \( \text{End}_{A\mathcal{O}BC_{\kappa}(\delta)}(\ell^{\uparrow \uparrow}) \) to \( \text{End}_{\mathcal{O}BC_{\kappa}^f(\delta)}(\ell^{\uparrow \uparrow}) \), which is induced by the quotient superfunctor from \( A\mathcal{O}BC_{\kappa}(\delta) \) to \( \mathcal{O}BC_{\kappa}^f(\delta) \) \cite{8}. Composing \( \mathcal{P} \) (see (4.8)) with this epimorphism yields an epimorphism

\[ \mathcal{P}^f : BC_{\ell,r,t}^{\text{aff}}(\delta) \to \text{End}_{\mathcal{O}BC_{\ell}^f(\delta)}(\ell^{\uparrow \uparrow}). \] \hspace{1cm} (4.21)

We have \( f(1) = 0 \in \mathcal{O}BC_{\kappa}^f(\delta) \). So \( \mathcal{P}^f(f(x_1)) = 0 \in \mathcal{O}BC_{\kappa}^f(\delta) \). By (4.11), we have \( g(1) = g = 0 \in \mathcal{O}BC_{\kappa}^f(\delta) \). Hence we have \( \mathcal{P}^f(g(x_1)) = 0 \in \mathcal{O}BC_{\kappa}^f(\delta) \) by (4.13). So, \( \mathcal{P}^f \) factors through \( BC_{\ell,r,t}^f \). It results in an induced surjective superalgebra homomorphism \( \hat{\varphi}^f : BC_{\ell,r,t}^f \to \text{End}_{\mathcal{O}BC_{\ell}^f(\delta)}(\ell^{\uparrow \uparrow}) \). It is easy to check that \( \hat{\varphi}^f \) sends a regular monomial of \( BC_{\ell,r,t}^f \) (see \cite{9} Definition 3.15) to a normally ordered dotted oriented Brauer-Clifford diagram, and moreover, the images of two regular monomials are not equivalent. By \cite{9} Corollary 3.16 and Theorem 3.12 we see that \( \hat{\varphi}^f \) sends a basis of \( BC_{\ell,r,t}^f \) to a basis of \( \text{End}_{\mathcal{O}BC_{\ell}^f(\delta)}(\ell^{\uparrow \uparrow}) \), forcing \( \hat{\varphi}^f \) to be an isomorphism. \hspace{1cm} \( \square \)

As explained in \cite{8}, the proof of Theorem 4.6 does not depend on the result of a basis of \( BC_{\ell,r,t}^f \). Via Theorem 3.12 it can give a proof of the fact that the set of all regular monomials of \( BC_{\ell,r,t}^f \) is a basis of \( BC_{\ell,r,t}^f \) when \( \kappa \) is an algebraically closed field. Finally, one can use Theorem 4.6 to give a presentation of \( \text{End}_{\mathcal{O}BC_{\ell}^f(\delta)}(\ell^{\uparrow \uparrow}) \).
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