Central Extensions of Finite Heisenberg Groups
in Cascading Quiver Gauge Theories

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Abstract

Many conformal quiver gauge theories admit nonconformal generalizations. These generalizations change the rank of some of the gauge groups in a consistent way, inducing a running in the gauge couplings. We find a group of discrete transformation that acts on a large class of these theories. These transformations form a central extension of the Heisenberg group, generalizing the Heisenberg group of the conformal case, when all gauge groups have the same rank. In the AdS/CFT correspondence the nonconformal quiver gauge theory is dual to supergravity backgrounds with both five-form and three-form flux. A direct implication is that operators counting wrapped branes satisfy a central extension of a finite Heisenberg group and therefore do not commute.
1 Introduction

A large class of quiver gauge theories admits the action of certain finite Heisenberg groups [1]. In this paper we continue our investigation of discrete symmetries in quiver gauge theories. In particular, we focus on nonconformal quivers. This is a very interesting class of quiver gauge theories which is anomaly free and nonconformal in the sense that the beta functions corresponding to the gauge couplings are nonzero. This is a very interesting dynamical generalization which lead to interesting QCD-like behavior as exemplified in the case of Klebanov-Strassler model [2] which displays confinement and chiral symmetry breaking.

Let us recall the main result of [1] which is a generalization of [3]. For a large class of quiver gauge theories with gauge group $SU(N)^p$, there is a set of discrete transformations $A, B$ and $C$ satisfying

$$A^q = B^q = C^q = 1, \quad AB = BAC,$$

where $q$ is some integer number which depends on the particular structure of the quiver. These transformations satisfy three important properties: (i) leave the superpotential invariant, (ii) satisfy the anomaly cancelation for all $SU(N)$ gauge groups, and (iii) the above group relations are true up to elements in the center of the gauge group $SU(N)^p$, that is, up to gauge transformations.

The main result of this paper can then be formulated as follows. For generalizations of conformal quivers into nonconformal quivers, that is, we consider gauge theories with gauge group $\prod_{i=1}^p SU(N + \alpha_i M)$ with $\alpha_i$ some positive integers, we find a set of discrete transformations $A, B, C$ and $D$ satisfying

$$A^q = B^q = C^q = D^q = 1, \quad AB = BAC, \quad AC = CAD,$$

here $q$ is the same integer as in the conformal case for general quivers. The conditions are the same as above, that is, invariance of the superpotential, anomaly cancelation and the relations are true up to elements in the center of the gauge group.

An alternative way of describing the above group is as a central extension of the Heisenberg group acting in the conformal case. Let us denote the finite Heisenberg group acting in the conformal case and whose commutation relations are given in (1.1), as $\text{Heis}(\mathbb{Z}_q \times \mathbb{Z}_q)$, then the centrally extended group $H_q$ whose commutations relations are (1.2), is defined via the short exact sequence:
where the $\mathbb{Z}_q$ factor is generated by the central element $D$ in (1.2). Interestingly, the central element in $\text{Heis}(\mathbb{Z}_q \times \mathbb{Z}_q)$ which is denoted by $C$ in (1.1) is no longer central as an element of $H_q$ in (1.2). In section 3 we will explicitly construct all the morphisms involved in the above sequence. Note that when $D$ is the identity one recovers the Heisenberg group of the conformal limit. More precisely, the number $M$ in the gauge groups determines the structure of the element $D$: when $M = 0$ we have that $D = 1$. Thus, the nontriviality of $D$ is directly related to the three-form flux which is proportional to $M$. Alternatively, we can view $M$ as the number of fractional D5 branes in the string theory side.

To a large extent our investigation is motivated by ideas put forward by D. Belov, G. Moore and others suggesting that D-brane charge in string theory with background RR flux is a noncommutative quantity [4, 5]. In this respect our work exploits the AdS/CFT correspondence [6] to learn about fundamental properties of D-branes. We also find the study of discrete symmetries in quiver gauge theories interesting in its own right.

The organization of this note is as follows. In section 2 we review the essential properties of quiver gauge theories that are further used in this paper. In section 3 we explicitly discuss various examples and give some ideas of what a general proof would entail. Section 4 contains some comments on the implications for the string theory description of these cascading quiver gauge theories. In section 5 we conclude with some observations about the limitations of our calculations.

2 Generalities of Quiver Theories

Here we will review some of the general techniques of analysis used in quiver field theories. Our goal is to be self consistent and this section can clearly be skipped by readers who are familiar with the standard properties of quiver gauge theories.

First, we will discuss the role of the adjacency matrix in determining the ranks of the gauge groups. The adjacency matrix component $a_{ij}$ is defined as the number of arrows pointing from the $i^{th}$ node to the $j^{th}$ minus the number pointing from the $j^{th}$ node to the $i^{th}$ node. Thus, even though there is an entry 0 in the adjacency matrix, one may not conclude that there are no arrows between the nodes. For example, the
conifold theory

\[
\begin{align*}
&\begin{array}{c}
U_1^\alpha \\
1 \\
V_2^\alpha \\
2
\end{array}
\end{align*}
\]

has a two by two adjacency matrix with all entries being 0. A general adjacency matrix \( \hat{a} \) is an antisymmetric matrix, and has a certain number of zero eigenvectors that are important for our purposes. The adjacency matrix is a matrix of integers, and so we may always scale its zero eigenvectors to have integer components. Any one of these integer valued zero eigenvectors is a good assignment of gauge groups, assigning the rank of the \( i^{th} \) SU gauge group be the \( i^{th} \) component of that zero eigenvector.

This procedure is simply making sure that the triangle anomaly cancels for any given node (gauge group) of the quiver diagram. Consider a node that denotes a gauge group SU\((N)\). Focusing on that node, the triangle anomaly is proportional to \( \Sigma \text{Tr}(t_i^a t_i^b t_i^c) \), where the sum runs over all other indices (we use the shorthand \( i \)) that label fields, and \( t_i \) are the generators for the representation of this \( i^{th} \) field under the SU\((N)\). This Casimir is zero for the adjoint representation of an SU group, and so only the matter sector contributes. The fundamental and anti-fundamental representations of SU contribute with opposite sign. The sum over other indices includes the gauge indices from the other end of the arrow. Hence, an arrow pointing from a gauge group with gauge group SU\((N')\) to the gauge group in question gets an additional factor of \( N' \) from the sum. Therefore, to count the anomaly, we can simply count the number of arrows from a given gauge group to the group in question and weight this arrow by the rank of the gauge group at the other end, where the sign is given as (+) for arrows pointing away, and (−) for arrows pointing towards the node in question. This is precisely what \( \hat{a} \rightarrow \vec{v} \) measures, the \( i^{th} \) entry being the anomaly at the \( i^{th} \) node, which we of course require to be 0. This argument has nothing to do with supersymmetry, as the anomaly cancels as long as the arrows represent the same type of fields (here, we are considering that they are all chiral superfields).

For illustrative purposes, we will use a toy example. Consider the quiver diagram
The adjacency matrix is

\[
\hat{a} = \begin{pmatrix}
0 & 2 & 0 & -2 \\
-2 & 0 & 2 & 0 \\
0 & -2 & 0 & 2 \\
2 & 0 & -2 & 0
\end{pmatrix}
\]  

(2.1)

There are two linearly independent solutions to the equation \( \hat{a} \vec{v} = 0 \) and they are

\[
\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\]  

(2.2)

Any integer valued zero eigenvector can be expressed as

\[
\vec{v}_0 = N \vec{v}_1 + M \vec{v}_2
\]  

(2.3)

with \( N \) and \( M \) being integers. Thus, the assignment of gauge groups is as follows:

\[
\begin{array}{c}
\text{SU}(N+M) & \text{U}_3^a \\
\text{SU}(N) & \text{SU}(N) \\
\text{SU}(N) & \text{SU}(N+M)
\end{array}
\]

For these to be gauge groups, we require that \( N + M > 1 \), \( N > 1 \) so that the assignments of \( SU(N) \) and \( SU(N + M) \) as the gauge groups makes sense.

Now we go on to consider the \( \beta \) functions for such quivers. For our purposes, we will always consider quivers that have an assignment of gauge group ranks such that there is a conformal case (the need for this assumption will become clear in a moment). These will correspond to vectors of the type \( \vec{v}_1 \) with 1 in every entry.

First, let us consider the \( \beta \) functions associated with the superpotential couplings. As a short hand, we will refer to these couplings as \( h_i \). Non renormalization theorems
simply give that a monomial term in a superpotential with coupling \( h_i \) has a beta function given in terms of the anomalous dimension of the fields that enter the monomial,

\[
\beta_{h_i} = h_i(\mu) \left(-3 + \Sigma \left(1 + \frac{1}{2}\gamma_{ij}\right)\right)
\]

(2.4)

where the anomalous dimension \( \gamma_{ij} \) and sum refer to all fields present in the superpotential monomial. We use the two gauge groups under which the field is charged to label the anomalous dimension. In this discussion, the anomalous dimension is defined through \( \text{dim}(F_{ij}) = 1 + \frac{1}{2}\gamma_{ij} \) so that kinetic terms are scale as \((1 + \gamma_{ij})\) to leading order. Knowing the \( \gamma_{ij} \) is equivalent to knowing the \( R \) charge of the operator, \( \text{dim}(O) = \frac{3}{2}R_O \), in a conformal theory.

The procedure for determining the \( \beta_i \) for the gauge couplings \( g_i \) is much the same as considering the anomaly: they are determined by considering the other other gauge groups in the diagram as flavor symmetries. In the general case for a node with gauge group \( SU(N_i) \) and gauge coupling \( g_i \), the NSVZ beta function is

\[
\beta_i = -\frac{g_i^3}{16\pi^2} \frac{3N_i - \Sigma N_j T_{rij}(1 - \gamma_{ij})}{1 - \frac{2N_i^2}{8\pi^2}}.
\]

(2.5)

\( T_{rij} \) is the Casimir for the field charged under gauge groups \( i \) and \( j \) given as \( \text{Tr}(t^a_{rij} t^b_{rij}) = T_{rij} \delta^{ab} \), and the generators \( t \) are for the \( i^{th} \) gauge group under investigation. \( \gamma_{ij} \) is the anomalous dimension of this field, and \( N_j \) is the rank of the gauge group at the node \( j \).

The anomalous dimensions of the fields are found by taking the conformal case (the \( v_1 \) type vector) and solving \( \beta_i = \beta_{h_i} = 0 \). Of course a maximization is the real principle that allows for solving for the anomalous dimensions, however here we will not concern ourselves with the actual values, and only take that there is a solution that sets all the \( \beta \) functions to zero. In the conformal cases we are considering, all nodes have rank \( N \) gauge groups, and so all terms present are all proportional to \( N \). Now, we assume that the other integer \( M \) multiplying the other zero eigenvector is small. The assignment of \( \gamma_{ij} \) can be seen to depend on \((M/N)^2\) and so if we work to leading order in \( M/N \), the \( \gamma_{ij} \) of the conformal case can still be used. This gives that the \( \beta_{h_i} = 0 \) in this approximation. However, the cancelation of terms in the \( \beta_i \) equations depended crucially on \( N_i \) and \( N_j \) being related. In the new case the leading order is changed because \( N_j = N_i (1 + M\alpha_{ij}/N_i) \) where \( \alpha_{ij} \) is a constant which depends on the particular quiver and node at hand. Therefore, the \( \beta \) functions associated with the
gauge couplings change in general. The leading order must still vanish, and this leaves a term proportional to $M$ left over.

$$\beta_i = \frac{g_i^3}{16\pi^2} \frac{M \Sigma \alpha_{ij} T_{ij} (1 - \gamma_{ij})}{1 - \frac{g_i^2 N_i}{8\pi^2}}. \quad (2.6)$$

Again, let us turn to our example to be more concrete. Let $U_1$ and $U_3$ have the same anomalous dimension $\gamma_1$ and the other fields have the anomalous dimension $\gamma_2$. The superpotential beta functions then give that

$$(-3 + (4 + \gamma_1 + \gamma_2)) = 1 + \gamma_1 + \gamma_2 = 0 \quad (2.7)$$

(the sum is over 4 terms as there is a quartic superpotential), and the gauge couplings give that

$$3N - \Sigma N \frac{1}{2} (1 - \gamma_{ij}) = 0 \rightarrow 1 + \gamma_1 + \gamma_2 = 0 \quad (2.8)$$

(where the sum is over 4 terms, two arrows in and two out of any given node). Note that we have used $T_r (N) = T_r (\bar{N}) = 1/2$. In this case, the beta function equations give the same restriction, and we will find this to be the case in general. More generically, one can eliminate certain anomalous dimensions that are related to others by superpotential terms, and then only discuss the remaining anomalous dimensions.

Perturbing around this fixed point, we find that for nodes 1 and 3 that the new $\beta$ functions are proportional to

$$3N - \Sigma (N + M) \frac{1}{2} (1 - \gamma_{ij}) = 3N - \Sigma N \frac{1}{2} (1 - \gamma_{ij}) - \Sigma M \frac{1}{2} (1 - \gamma_{ij}) = -M (2 - \gamma_1 - \gamma_2) = -3M \quad (2.9)$$

and that for nodes 2 and 4

$$3(N + M) - \Sigma (N) \frac{1}{2} (1 - \gamma_{ij}) = 3M + 3(N) - \Sigma N \frac{1}{2} (1 - \gamma_{ij}) + \Sigma M \frac{1}{2} (1 - \gamma_{ij}) = 3M \quad (2.10)$$

This gives that the new beta functions are non zero, and are proportional to $M$ at leading order.

### 3 Centrally extended finite Heisenberg groups acting on cascading quivers

In this section we discuss various examples of cascading theories and explicitly present a group of discrete transformations acting on these theories. The group in question is a central extension of the finite Heisenberg group acting in the conformal case.
3.1 Orbifolds of $Y^{p,q}$

A very interesting class of gauge theories are the quiver gauge theories obtained as the
gauge theory dual of string theory on $AdS_5 \times Y^{p,q}$ with 5-form flux. A very complete
discussion of $Y^{p,q}$ spaces is presented in [7]. The field theory aspects are presented
in [8–11]. Inclusion of the fractional branes and the subsequent cascade on the field
theory side was studied in [12]. We start with this set of orbifold models because
certain features will be clearer here than in more symmetric cases.

To be concrete, we work out an example. Given a conformal quiver gauge theory,
one can construct a nonconformal phase by appropriately changing the rank of some of
the gauge groups. The precise recipe involves adding multiples of the zero eigenvectors
of the adjacency matrix as reviewed in section 2.

First, let us obtain the zero eigenvectors of interest. The quivers that we will be
dealing with will all respect a “shift” symmetry because there is a fundamental cell
that the quiver is built from. This then allows us to consider only the adjacency matrix
of the sub diagram, identifying the first and last set of arrows. For $Y^{6,3}$ we find the
sub diagram of $Y^{2,1}$ as

\[ \begin{array}{c}
\text{1}
\text{2}
\text{3}
\text{4}
\text{5}
\text{6}
\text{7}
\end{array} \]

with adjacency matrix

\[ \hat{a} = \begin{pmatrix}
0 & 2 & -1 & -1 \\
-2 & 0 & 3 & -1 \\
1 & -3 & 0 & 2 \\
1 & 1 & -2 & 0
\end{pmatrix}. \tag{3.1} \]

The zero eigenvectors of this matrix are

\[ \overrightarrow{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \overrightarrow{v}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}. \tag{3.2} \]

To obtain the zero eigenvectors for the $Y^{6,3}$ case, one simply repeats the entries in the
above vectors 3 times. This process easily generalizes to other quivers with subcells
(except for the cases where \( q = 0 \) or \( q = p \), where the unit cells are the simple \( \tau \) and \( \sigma \) cells), and so one can see that the \( A \) symmetry that shifts these primitive cells into each other is a symmetry in all cases.

We now turn to the explicit construction of some discrete transformations \( A, B, C, \) and \( D \) for the quiver of \( Y^{6,3} \) with the rank of the gauge groups shifted accordingly:

The number in parentheses next to the node denotes the rank of that gauge group.

Next, we will recall the results of [1], where \( M \) is set to 0. In this work, we found a set of transformations \( \tilde{A}, \tilde{B} \) and \( \tilde{C} \):

\[
\tilde{A} : \begin{align*}
(1, 5, 9) & \mapsto (9, 1, 5), \\
(2, 6, 10) & \mapsto (10, 2, 6) \\
(3, 7, 11) & \mapsto (11, 3, 7) \\
(4, 8, 12) & \mapsto (12, 4, 8).
\end{align*}
\]

(3.3)

\[
\tilde{B} : U_i \mapsto u_i U_i
\]

(3.4)

with

\[
\begin{align*}
u_1 &= 1 & u_5 &= \omega^4 & u_9 &= \omega^8 \\
u_2 &= \omega & u_6 &= \omega^3 & u_{10} &= \omega^5 \\
u_3 &= \omega^2 & u_7 &= \omega^6 & u_{11} &= \omega^{10} \\
u_4 &= \omega^{-1} & u_8 &= \omega^{-3} & u_{12} &= \omega^{-35}
\end{align*}
\]

(3.5)

and

\[
\tilde{C} : U_i \mapsto u_i U_i
\]

(3.6)
with
\[ u_1 = \omega^4 \quad u_5 = \omega^4 \quad u_9 = \omega^4 \]
\[ u_2 = \omega^2 \quad u_6 = \omega^2 \quad u_{10} = \omega^2 \]
\[ u_3 = \omega^4 \quad u_7 = \omega^4 \quad u_{11} = \omega^4 \]
\[ u_4 = \omega^{-2} \quad u_8 = \omega^{-2} \quad u_{12} = \omega^{-26} \]

(3.7)

and where \( \omega^{3N} = 1 \). These satisfy a finite Heisenberg group structure

\[ \tilde{A}\tilde{B} = \tilde{B}\tilde{A}, \quad \tilde{A}\tilde{C} = \tilde{C}\tilde{A}, \quad \tilde{A}^3 = \tilde{B}^3 = \tilde{C}^3 = 1 \]

(3.8)

and \( C \) commutes with all generators above. These equations are to be read up to members of the center of the gauge group. The above transformations also satisfy the anomaly cancelation conditions

\[
\begin{align*}
\left(\frac{u_{12}u_1}{u_2}\right)^N &= 1, \quad \left(\frac{u_1u_2}{u_{11}u_{12}u_3}\right)^N = 1, \\
\left(\frac{u_2u_3}{u_1u_4u_5}\right)^N &= 1, \quad \left(\frac{u_3u_4}{u_2}\right)^N = 1, \\
\left(\frac{u_4u_5}{u_6}\right)^N &= 1, \quad \left(\frac{u_5u_6}{u_3u_4u_7}\right)^N = 1, \\
\left(\frac{u_6u_7}{u_5u_8u_9}\right)^N &= 1, \quad \left(\frac{u_7u_8}{u_6}\right)^N = 1, \\
\left(\frac{u_8u_9}{u_{10}}\right)^N &= 1, \quad \left(\frac{u_9u_{10}}{u_7u_8u_{11}}\right)^N = 1, \\
\left(\frac{u_{10}u_{11}}{u_9u_{12}u_1}\right)^N &= 1, \quad \left(\frac{u_{11}u_{12}}{u_{10}}\right)^N = 1.
\end{align*}
\]

(3.9)

We will now generalize these to the non-conformal case. First, we may note that because the \( A \) symmetry shifts gauge groups of the same rank into each other and so this remains a symmetry in the non conformal case. Next, we recall that the \( B \) and \( C \) symmetries were constructed just so that they were \( 3^{rd} \) roots of members of the center of the gauge group. The particular members were labeled by assigning each gauge group a number.
and then fields were rephased as $U_i \rightarrow \omega^{(n_i)} \omega^{(-n_i+1)} U_i$. This then gives that these operators to the $3^{rd}$ power are automatically in the center of the gauge group (as $\omega^3$ is an $N^{th}$ root of unity). Note that $n$ and $n'$ are arbitrary in the conformal case above because each gauge group has the same $\omega$ associated with it, as the gauge groups are the same rank.

We will now set about generalizing this to the non-conformal case. For $M \neq 0$ we define the useful quantity

$$\lambda \equiv \frac{N}{M}. \quad (3.10)$$

We now associate different $\omega$'s to each gauge group. We want that they will eventually be related to the center of the gauge group, and so we define

$$\begin{align*}
\omega_0 &= \omega^{\frac{\lambda+i}{\lambda+3}} \\
\omega_1 &= \omega^{\frac{\lambda+i}{\lambda+1}} \\
\omega_2 &= \omega^{\frac{\lambda+i}{\lambda+2}} \\
\omega_3 &= \omega^{\frac{\lambda+i}{\lambda+3}} = \omega,
\end{align*} \quad (3.11)$$

i.e. we have solved the $(3(\lambda+i)M)^{th}$ roots of unity for $i = 0, 1, 2$ in terms of the $(3(\lambda+3)M)^{th}$ root of unity, which we call $\omega$. Therefore, instead of associating the same $\omega$ with each gauge group, we associate the phase $\omega_i$ with a gauge group with rank $(\lambda+i)M$. Now that different phases are assigned to different gauge groups, the overall numbers $n$ and $n'$ do affect the final result. In our example we will take $n = 0$ and then adjust $n'$ so that $u_5^B = u_5^C$. This will give that the $C$ operation properly “undoes” the affect of shift symmetry $A$. The solution to this restriction is $n' = 0$. 

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Using the above construction, we find

\[ \begin{align*}
(1, 5, 9) & \mapsto (9, 1, 5), \\
(2, 6, 10) & \mapsto (10, 2, 6), \\
(3, 7, 11) & \mapsto (11, 3, 7), \\
(4, 8, 12) & \mapsto (12, 4, 8),
\end{align*} \tag{3.12} \]

and

\[ B : U_i \mapsto u_i U_i, \tag{3.13} \]

with

\[ \begin{align*}
& u_1 = 1, & u_5 = \omega^4 (\lambda + 3)(\lambda - 1) \\
& u_2 = \omega^{\lambda+3} & u_6 = \omega^3 (\lambda + 4) \\
& u_3 = \omega^{2\lambda+1} & u_7 = \omega^{\lambda-2} \\
& u_4 = \omega^{\lambda^2 + \frac{5}{3}} & u_8 = \omega^{3\lambda-12} \\
& u_9 = \omega^{\sqrt{(\lambda + 3)(\lambda + 5)}} & u_{10} = \omega^{\sqrt{(\lambda + 3)(\lambda + 6)}}, \\
& u_{11} = \omega^{10\lambda+4} & u_{12} = \omega^{-35}
\end{align*} \tag{3.14} \]

and

\[ C : U_i \mapsto u_i U_i \tag{3.15} \]

with

\[ \begin{align*}
& u_1 = \omega^4 (\lambda + 3)(\lambda + 3) \\
& u_2 = \omega^{\lambda+3} (\lambda + 1) \\
& u_3 = \omega^{4\lambda+1} \\
& u_4 = \omega^{-2(\lambda - 3)} \\
& u_5 = \omega^4 (\lambda + 3)(\lambda - 1) \\
& u_6 = \omega^2 (\lambda + 5)(\lambda + 5) \\
& u_7 = \omega^{4\lambda+1} \\
& u_8 = \omega^{-2(\lambda - 15)} \\
& u_9 = \omega^4 (\lambda + 3)(\lambda - 5) \\
& u_{10} = \omega^2 (\lambda + 3)(\lambda + 9) \\
& u_{11} = \omega^4 (\lambda + 9) \\
& u_{12} = \omega^{-2(13\lambda + 9)}
\end{align*} \tag{3.16} \]

We find that the above transformations indeed satisfy the new anomaly cancelation conditions

\[ \begin{align*}
\left( \frac{u_1^{\lambda+3} u_3^{\lambda+3}}{u_2^{\lambda+1} u_5^{\lambda+2}} \right)^M &= 1, \\
\left( \frac{u_2^{\lambda+1} u_3^{\lambda+4}}{u_1^{\lambda+2} u_5^{\lambda+2}} \right)^M &= 1, \\
\left( \frac{u_3^{\lambda+3} u_5^{\lambda+3}}{u_1^{\lambda+1} u_5^{\lambda+1}} \right)^M &= 1, \\
\left( \frac{u_6^{\lambda+1} u_7^{\lambda+4}}{u_5^{\lambda+2} u_9^{\lambda+2}} \right)^M &= 1, \\
\left( \frac{u_8^{\lambda+3} u_9^{\lambda+3}}{u_1^{\lambda+2} u_10^{\lambda+1}} \right)^M &= 1, \\
\left( \frac{u_{10}^{\lambda+1} u_{11}^{\lambda+4}}{u_9^{\lambda+2} u_1^{\lambda+2}} \right)^M &= 1.
\end{align*} \tag{3.17} \]
This works in much the same way as the conformal case, where the first and last line impose the condition $\omega^{3(\lambda+3)M} = 1$, and the other equalities are satisfied trivially before raised to the $M^{th}$ power (actually, a 6 could appear instead of 3 as the overall factor in the exponent, however we wish to find operations that when raised to the 3$^{rd}$ power give 1, or a member of the center of the gauge group).

The above relations satisfy

$$AB = BAC \times M \equiv BAC, \quad A^3 = B^3 = C^3 = 1$$

(3.18)

where $M$ is in the center of the gauge group, given by the element

This is in the center because all of the numbers appearing above are divisible by 3.

We now turn to the commutation of $C$ with other generators. It is trivial that $C$ commutes with $B$ because they are both diagonal. However, we now find that

$$AC = CAD$$

(3.19)

with $D$ defined as

$$D : U_i \mapsto u_i U_i$$

(3.20)

with

$$u_1 = \alpha^8 \quad u_5 = \alpha^{-4} \quad u_9 = \alpha^{-4}$$
$$u_2 = \beta^{-8} \quad u_6 = \beta^4 \quad u_{10} = \beta^4$$
$$u_2 = \gamma^8 \quad u_6 = \gamma^{-4} \quad u_{10} = \gamma^{-4}$$
$$u_4 = \omega^{24(\lambda+1)/3} \quad u_8 = \omega^{24/3} \quad u_{12} = \omega^{-24(\lambda+2)/3}$$

(3.21)
where we have defined the useful quantities
\[
\alpha = \omega^{\frac{(\lambda+3)}{(\lambda+1)(\lambda+2)}}, \quad \beta = \omega^{\frac{(\lambda+3)}{(\lambda+2)(\lambda+1)}}, \quad \gamma = \omega^{\frac{1}{(\lambda+1)}}.
\] (3.22)

The powers appearing in the definition of \( \alpha, \beta, \) and \( \gamma \) are related to the rank of the gauge groups that the \((1, 5, 9), (2, 6, 9)\) and \((3, 7, 10)\) fields run between. Note that because the operator \( C \) satisfies the anomaly conditions, and \( A \) simply permutes them, then \( D \) automatically satisfies the anomaly conditions. Also, \( D \) commutes trivially with all generators except \( A \), and so we check that
\[
AD = DA \times M'
\] (3.23)

and we find that \( M' \) is in the center of the gauge group corresponding to the element

We now comment on \( D \). First, one can find that \( D \) is generated by
The crucial difference between the conformal and non conformal cases is now clear: in the conformal case we could raise all of the integer powers by 8 appearing above. Then, only 0 and 24 would appear at each node, both of which are divisible by 3. Another way to say this is that all the numbers appearing above are congruent to 1 mod 3. Thus, $D$ is gauge equivalent to the case where one simply enters 1 for all nodes. This is a trivial operator in the conformal case because the $\omega_i$ associated with each gauge group is the same, however in the nonconformal case the $\omega_i$ associated with each node is different.

Also of interest is the fact that $D$ is simply a $\mathbb{Z}_3$ subgroup of an entire $U(1)$. This is because all of the anomaly cancelation conditions are met trivially (before raising to the $M^\text{th}$ power). Hence, there is no “wrapping” condition that requires the phases appearing in $D$ to be any particular root of unity, and so are arbitrary $U(1)$ phases. Curiously, there is another symmetry that is also a $\mathbb{Z}_3$ subgroup of a $U(1)$:

$$E : U_i \mapsto u_i U_i$$ (3.24)

with

$$
\begin{align*}
  u_1 &= \omega^{\frac{-2(\lambda+3)}{\lambda+2}} \\
  u_2 &= \omega^{\frac{\lambda(\lambda+3)}{(\lambda+1)(\lambda+2)}} \\
  u_4 &= \omega^3 \\
  u_5 &= \omega^{\frac{-2(\lambda+3)}{\lambda+2}} \\
  u_6 &= \omega^{\frac{2\lambda}{\lambda+1}} \\
  u_8 &= \omega^3 \\
  u_9 &= \omega^{\frac{-2(\lambda+3)}{\lambda+2}} \\
  u_{10} &= \omega^{\frac{\lambda(\lambda+3)}{(\lambda+1)(\lambda+2)}} \\
  u_{12} &= \omega^3 \\
\end{align*}
$$ (3.25)

This can be generated from the diagram
One should note that the assignments of $D$ and $E$ correspond exactly to the vectors that determine the rank of the gauge groups: $D$ for the coefficient of $N$ and $E$ for the coefficient of $M$ in the gauge groups. One should also notice that while $D$ is trivial in the conformal case, $E$ is not. The conformal case has an additional symmetry to those found in [1], making the group generators satisfy that of $\tilde{H} \times \mathbb{Z}_3$ using only $E$, or $\tilde{H} \times U(1)$ using the $U(1)$ that $E$ is a subgroup of (denoting the Heisenberg group as $\tilde{H}$). In summary, the algebra that we find is

$$ AB = BAC, \quad AC = CAD, \quad AD = DA, \quad \{B, C, D, E\} \text{ commute} $$
$$ A^3 = B^3 = C^3 = D^3 = E^3 = 1. \quad (3.26) $$

We will refer to the finite Heisenberg group as $\text{Heis}(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and the above group with $E$ removed as $\tilde{H}_3$ (the 3 to denotes that we are really talking about groups modulo 3, and we remove $E$ because it always appears as an uninteresting direct product factor). We note that an arbitrary element of $H_3$ can be written $A^a B^b C^c D^d$ with $a, b, c, d \in \{0, 1, 2\}$. Likewise, an arbitrary element of $\text{Heis}(\mathbb{Z}_3 \times \mathbb{Z}_3)$ may be written as $\tilde{A}^a \tilde{B}^b \tilde{C}^c$ again with $a, b, c \in \{0, 1, 2\}$. One may therefore view $H_3$ as a central extension of the $\text{Heis}(\mathbb{Z}_3 \times \mathbb{Z}_3)$. To be explicit, we take

$$ \mathbb{I} \xrightarrow{f_1} \mathbb{Z}_3(D) \xrightarrow{f_2} H_3 \xrightarrow{f_3} \text{Heis}(\mathbb{Z}_3 \times \mathbb{Z}_3) \xrightarrow{f_4} \mathbb{I} $$

where the $\mathbb{Z}_3$ is the group generated by $D$. Therefore, we take the maps
\[ f_1(\mathbb{I}) = D^0 \]
\[ f_2(D^d) = D^d \]
\[ f_3(A^a B^b C^c D^d) = \tilde{A}^a \tilde{B}^b \tilde{C}^c \]
\[ f_4(\tilde{A}^a \tilde{B}^b \tilde{C}^c) = \mathbb{I} \] (3.27)

and find that

\[
\begin{align*}
\text{Ker}(f_1) &= \mathbb{I} \\
\text{Im}(f_1) &= D^0 \\
\text{Ker}(f_2) &= D^0 \\
\text{Im}(f_2) &= D^d \\
\text{Ker}(f_3) &= D^d \\
\text{Im}(f_3) &= \tilde{A}^a \tilde{B}^b \tilde{C}^c \\
\text{Ker}(f_4) &= \tilde{A}^a \tilde{B}^b \tilde{C}^c \\
\text{Im}(f_4) &= \mathbb{I}.
\end{align*}
\] (3.28) (3.29)

So, the above is an exact sequence of homomorphisms, and further \( D^d \) is in the center of \( H \).

One may worry about the “internal” fields in the diagram. These, however will satisfy the algebra above in the same way. Let us take as an example the \( Z_a \) fields appearing in quiver. The scalings \( z_a \) are always read from the superpotential constraints, and so the \( B \) \( C \) and \( D \) operators are determined directly. Let us take an example. We may show that because \( AB = BAC \) is satisfied for the \( U \) fields that it is also satisfied for the \( Z \) fields. We will refer to the scalings associated with the \( B \) operation acting on the \( U \) fields as \( u^B_a \), and likewise for other fields and operations. We find

\[
\begin{align*}
(AB)Z_a &= (u^B_{a+4} u^B_{a+5})^{-1} Z_{a+4} \\
(BAC)Z_a &= (u^B_a u^B_{a+1} u^C_{a+4} u^C_{a+5})^{-1} Z_{a+4}
\end{align*}
\] (3.30)

Next, because we have already solved the \( U \) problem, we have that

\[ u^B_a u^C_{a+4} \times (G_{a+4,a+5}) = u^B_{a+4} u^B_{a+5} \] (3.31)

where \( G_{a+4,a+5} \) denotes the component of the center of the gauge group that rephases nodes \( a + 4 \) and \( a + 5 \), i.e. those that affect \( U_{a+4} \). From this, we determine that

\[
(AB)Z_a = BAC(G_{a,a+1} G_{a+1,a+2})^{-1} Z_a
\] (3.32)
In the above manipulations one must shift $a$ down by 4 in (3.32) relative to the $a$ that appears in (3.31) so that both sides match after $A$ shifts $a$ up by 4 on the RHS of (3.32). We note that $G_{a,b} = \omega^a_i \omega_j^{-n_b}$ where we use $i$ to denote the rank of the gauge group as $(\lambda + i)M$ at node $a$ and $j$ likewise denotes the rank of the gauge group at node $b$. Thus, $G_{a,b}G_{b,c} = G_{a,c}$. Therefore, we find that

$$(AB)Z_a = BAC(G_{a,a+2})^{-1}Z_a.$$  

(3.33)

as expected. The inverse power shows up precisely because $Z$ is oriented from node $a + 2$ to node $a$ rather than the “forward” direction. Thus, in the $Z$ sector $AB = BAC$ follows from the relation $AB = BAC$ in the $U$ sector. This is likewise true for the fields $Y$ that enter in quartic superpotential terms, only now 3 of the $G$ scalings “telescope” to become the single one needed. The fact that $B^3 = 1$ in the $Z$ sector is also obvious. All equations are of this form and so the $Z$ and $Y$ sectors follow automatically. For this reason we will not treat the internal lines in the remainder of the paper, knowing that the commutation relation follow in a trivial manner given the commutations of the fields on the “outside” of the quiver.

### 3.2 Orbifolds of $S^5 = Y^{p,p}$

A clear set of orbifold examples to explore are those of $S^5$. We will concentrate particularly on orbifolds that correspond to $Y^{p,p}$. These theories have $2p$ gauge groups, and so are $\mathbb{Z}_{2p}$ orbifolds of the sphere. We find it easier to work an example, and then display the generic features. We will concentrate on the $AdS_5 \times Y^{6,6}$ geometry with imaginary self dual three form turned on. The field theory dual to this string background is given by the quiver
We find that the above field theory has the following shift symmetry

\[ A : (1, 3, 5, 7, 9, 11) \mapsto (3, 5, 7, 9, 11, 1), \]
\[ (2, 4, 6, 8, 10, 12) \mapsto (4, 6, 8, 10, 12, 2) \]  \hspace{1cm} (3.34)

and the following rephasing symmetries

\[ B : U_i \mapsto u_i U_i \]  \hspace{1cm} (3.35)

with

\[
\begin{array}{ccccccccc}
  u_1 & = & 1 & & u_3 & = & \omega & & u_5 & = & \omega^{\frac{\lambda-3}{\lambda}} \\
  u_2 & = & \omega & & u_4 & = & \omega^{\frac{\lambda+2}{\lambda}} & & u_6 & = & \omega^{\frac{\lambda+4}{\lambda}} \\
  u_7 & = & \omega^{\frac{\lambda-6}{\lambda}} & & u_9 & = & \omega^{\frac{\lambda-12}{\lambda}} & & u_{11} & = & \omega^{\frac{\lambda-20}{\lambda}} \\
  u_{10} & = & \omega^{\frac{\lambda+14}{\lambda}} & & u_{12} & = & \omega^{\frac{\lambda+18}{\lambda}} ,
\end{array}
\]  \hspace{1cm} (3.36)

and

\[ C : U_i \mapsto u_i U_i \]  \hspace{1cm} (3.37)

with

\[
\begin{array}{ccccccccc}
  u_1 & = & \omega^{\frac{\lambda+2}{\lambda}} & & u_3 & = & \omega & & u_5 & = & \omega^{\frac{\lambda-6}{\lambda}} \\
  u_2 & = & \omega & & u_4 & = & \omega^{\frac{\lambda+2}{\lambda}} & & u_6 & = & \omega^{\frac{\lambda+4}{\lambda}} \\
  u_7 & = & \omega^{\frac{\lambda-4}{\lambda}} & & u_9 & = & \omega^{\frac{\lambda-8}{\lambda}} & & u_{11} & = & \omega^{\frac{\lambda-8}{\lambda}} \\
  u_{10} & = & \omega^{\frac{\lambda+14}{\lambda}} & & u_{12} & = & \omega^{\frac{\lambda+14}{\lambda}} ,
\end{array}
\]  \hspace{1cm} (3.38)

and

\[ D : U_i \mapsto u_i U_i \]  \hspace{1cm} (3.39)

with

\[
\begin{array}{ccccccccc}
  u_1 & = & \omega^{\frac{1}{\lambda}} & & u_3 & = & \omega^{\frac{1}{\lambda}} & & u_5 & = & \omega^{\frac{1}{\lambda}} \\
  u_2 & = & \omega^{-\frac{1}{\lambda}} & & u_4 & = & \omega^{-\frac{1}{\lambda}} & & u_6 & = & \omega^{-\frac{1}{\lambda}} \\
  u_7 & = & \omega^{\frac{1}{\lambda}} & & u_9 & = & \omega^{\frac{1}{\lambda}} & & u_{11} & = & \omega^{\frac{1}{\lambda}} \\
  u_{10} & = & \omega^{-\frac{1}{\lambda}} & & u_{12} & = & \omega^{-\frac{1}{\lambda}} ,
\end{array}
\]  \hspace{1cm} (3.40)
and

$$E : U_i \mapsto u_i U_i$$  \hspace{1cm} (3.41)$$

with

$$u_1 = \omega^{-1} \quad u_3 = \omega^{-1} \quad u_5 = \omega^{-1} \quad u_7 = \omega^{-1} \quad u_9 = \omega^{-1} \quad u_{11} = \omega^{-1}$$

$$u_2 = \omega \quad u_4 = \omega \quad u_6 = \omega \quad u_8 = \omega \quad u_{10} = \omega \quad u_{12} = \omega,$$

and where $\omega^{6(\lambda+1)M} = 1$.

One should note that while we have defined $C$ the operator $C^2$ is also a well-defined symmetry of the system (but not further roots). Likewise, $D$ and $E$ are actually the $\mathbb{Z}_3$ subgroup of two full $U(1)$ symmetries. These symmetries obey the following properties

$$AB = BAC, \quad AC = CAD^{-2}, \quad AD = DA, \quad AE = EA, \quad \{B, C, D, E\} \text{ commute},$$

$$A^6 = B^6 = C^6 = D^6 = E^6 = 1$$  \hspace{1cm} (3.43)$$

where the equal signs are read up to the center of the gauge group. One may read these transformations as coming from the diagrams.
and one should note that again $D$ and $E$ are related to the zero eigenvectors of the adjacency matrix. Finally, we note here that while the conformal case had operators that were order 12, here they are order 6. This change comes about because the new gauge ranks do not respect the original $A$ symmetry. The only candidate $A$ operation would be a map between a quiver where odd nodes have rank $N + M$ and even nodes have rank $N$ to a quiver where odd nodes have rank $N$ and even nodes have rank $N + M$. These, however, are not related by Seiberg duality, as this will shift the ranks of the groups by total factors of $2M$. 
4 String theory interpretation

Various aspects of the physics on this nonconformal quiver gauge theories have been approached from the AdS/CFT point of view, starting with the original idea of adding three-form flux in [13] and including the interpretation of the supergravity background as describing a cascade of Seiberg dualities in [2]. In the concrete case of $Y^{p,q}$ spaces some results were presented in [12]. A further generalization was discussed in [14]. Most of the discussion of the field theory in [12] is based on identifications from [15], where the general role of the wrapped D5 branes was elucidated.

The addition of the possible D5 branes is constrained by chiral anomaly cancelation. As mentioned in section 2, adding D5 branes is correlated with the existence of null eigenvectors of the adjacency matrix. The wrapped D5 branes are described as $G_3$ flux in the supergravity background.

To keep some control of the result near the conformal fixed point we will assume that $M \ll N$. In the supergravity side this limit allows to neglect the backreaction of the three-form flux. When taken into consideration this backreaction generally leads to naked singularities in the supergravity solution [12, 14, 16, 17].

It is worth mentioning that the group of automorphisms of this group is not $SL(2, \mathbb{Z})$. To see this, consider a general automorphism of the type

$$A \mapsto A' = A^{a_1} B^{b_1} C^{c_1} D^{d_1}, \quad B \mapsto B' = A^{a_2} B^{b_2} C^{c_2} D^{d_2},$$

$$C \mapsto C' = A^{a_3} B^{b_3} C^{c_3} D^{d_3}, \quad D \mapsto D' = A^{a_4} B^{b_4} C^{c_4} D^{d_4},$$

which maps $A$, $B$, $C$ and $D$ to arbitrary elements of the group. In order to investigate the primed elements, we first work out the general commutation relation between two group elements. The result is simply

$$(A^{a_i} B^{b_i} C^{c_i} D^{d_i})(A^{a_j} B^{b_j} C^{c_j} D^{d_j}) = (A^{a_i} B^{b_i} C^{c_j} D^{d_j})(A^{a_j} B^{b_j} C^{c_i} D^{d_i}) C^{a_i c_j - a_j c_i} D^{a_i c_j - a_j c_i + b_i a_j (a_j + 1) - b_j a_i (a_i + 1)}.$$

We further denote the order $A$ as $p_a$ (so that $A^{p_a} = 1$), and so on for the other operators.

To show that (4.1) is an automorphism, we need to demonstrate that the primed elements satisfy the same commutation relations as the original elements. We start with the $A'$ and $B'$ relation. Demanding that $A'B' = B'A'C'$ implies that the $C'$ element must be of the form

$$C' \equiv C^{a_1 b_2 - a_2 b_1} D^{a_1 c_2 - a_2 c_1 + b_1 a_2 (a_2 + 1) - b_2 a_1 (a_1 + 1)}.$$

21
Hence in the general form of (4.1) one must require \( a_3 = b_3 = 0 \). Next, by turning to the commutation relation \( A'C' = C'A'D' \), we find that \( D' \) is

\[
D' \equiv D^{a_1 c_3} = D^{a_1(a_1 b_2 - a_2 b_1)},
\]

(4.4)

where we used the relation \( c_3 = a_1 b_2 - a_2 b_1 \) implicit in (4.3). This demonstrates that \( D' \) automatically commutes with everything, and so the rest of its commutation relations are automatically satisfied.

Finally, however, for (4.1) to be an automorphism, the \( B' \) and \( C' \) elements must commute. We now note the problem: \( B' \) contains \( A \) and \( C' \) contains \( C \), so there is an obstruction to their commutation. From (4.3) and (4.4), we find simply

\[
B'C' = C'B'D^{a_2(a_1 b_2 - a_2 b_1)}.
\]

(4.5)

Demanding that the right hand side is equal to \( C'B' \) gives rise to the condition

\[
a_2(a_1 b_2 - a_2 b_1) \equiv 0 \pmod{p_d}.
\]

(4.6)

This is the only condition that needs to be met. All other commutators are trivially satisfied.

The condition (4.6), however, is enough to indicate that the automorphism (4.1) cannot be identified with the \( SL(2, \mathbb{Z}) \) that acts in the conformal case where \( D \equiv 1 \). (For \( D \equiv 1 \), we may formally take \( p_d = 1 \), in which case the above condition becomes trivial.) One way to see this is that the original \( SL(2, \mathbb{Z}) \) acts with \( a_1 b_2 - a_2 b_1 = 1 \), whereupon (4.6) reduces to \( a_2 \equiv 0 \pmod{p_d} \). This is a restriction of \( SL(2, \mathbb{Z}) \) transformations to the shifts \( \tau \to \tau + 1 \) only.

We interpret this breaking of the \( SL(2, \mathbb{Z}) \) symmetry simply as a reflection of the fact that the background breaks the symmetry. Namely, the background contains only D5 branes and no NS5 branes. Hence an \( SL(2, \mathbb{Z}) \) transformation of the form \( \tau \to -1/\tau \) interchanging NS5 with D5 cannot act as an automorphism of the quiver theory. On the other hand, it is interesting to consider the \( SL(2, \mathbb{Z}) \) action on the group elements

\[
A \mapsto A' = A^{a_1} B^{b_1}, \quad B \mapsto B' = A^{a_2} B^{b_2}, \quad C \mapsto C' = C, \quad D \mapsto D' = D,
\]

(4.7)

with \( a_1 b_2 - a_2 b_1 = 1 \). The resulting commutation relations on the primed elements take the form

\[
A'B' = B'A', \quad A'C' = C'A'D'^{a_1}, \quad B'C' = C'B'D'^{a_2}
\]

(4.8)
(with $D'$ a central element). This is suggestive of the $A'C'$ non-commutativity being related to fractional D5 branes, and the $B'C'$ non-commutativity being related to fractional NS5 branes. It would, of course, be interesting to find a more symmetric description of this background. Our guess, implicit in (4.8), is that when NS5 branes wrapping two-cycles are included the group is further extended by an operator that reflects the presences of background NS5 flux.

5 Conclusions

In this paper we have established that a large class of nonconformal quiver gauge theories admits the action of a finite group which is the central extension of the finite Heisenberg group that acts on the conformal case.

The existence of a finite Heisenberg group for a large class of quiver gauge theories was established explicitly in [1]. Here we study a natural generalization to the nonconformal situation. The nonconformal quiver gauge theories have running beta functions and are, therefore more dynamical than their conformal counterparts. The existence of a finite group even in this dynamical case leads us to believe, based on the AdS/CFT correspondence, that in spaces with torsion and RR flux the operators counting the number of wrapped D-branes do not commute and essentially satisfy a centrally extended finite Heisenberg group.

As is well known, the nonconformal quiver gauge theories have running gauge couplings and a natural way to interpret their RG flow is via a cascade of Seiberg dualities [2]. Towards the end of the cascade some nonperturbative superpotential develops, along the lines of the Affleck-Dine-Seiberg superpotential [18]. In this note we have established the existence of a central extension of a finite Heisenberg group acting on the theory only in the ultraviolet regime where $N \gg M$ and thus the theory is still close to the conformal point. As discussed in various articles in the literature, for most of the nonconformal quiver gauge theories that we discussed after a duality cascade the ranks of the gauge groups are changed as $N \to N - M$ [12,15], that is, the quiver is self similar. The self-similarity implies that the same algebraic structure is present with the appropriate redefinitions of the roots $\omega_i$'s of section 3 to incorporate $N \to N - M$. It would be really interesting to follow the action of the group deep into the cascade where nonperturbative effects become important, that is assuming that $M$ is a factor of $N$ as in $M = pN$. After approximately $p$ steps the structure of the theory changes
significantly. We hope to return to this interesting question in the future.

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