ON THE RAYLEIGH-TAYLOR INSTABILITY FOR THE
COMPRESSIBLE NON-ISENTROPIC INVISCID FLUIDS WITH A
FREE INTERFACE

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ABSTRACT. In this paper, we study the Rayleigh-Taylor instability phenomena
for two compressible, immiscible, inviscid, ideal polytropic fluids. Such two
kind of fluids always evolve together with a free interface due to the uniform
gravitation. We construct the steady-state solutions for the denser fluid lying
above the light one. With an assumption on the steady-state temperature
function, we find some growing solutions to the related linearized problem,
which in turn demonstrates the linearized problem is ill-posed in the sense of
Hadamard. By such an ill-posedness result, we can finally prove the solutions to
the original nonlinear problem does not have the property EE(k). Precisely, the
$H^3$ solutions to the original nonlinear problem can not Lipschitz continuously
depend on their initial data.

1. Introduction. In this paper we are concerned with the Rayleigh-Taylor insta-
bility phenomena for two distinct compressible, non-isentropic, inviscid fluids in
an infinite slab, which is denoted by $\Omega := \mathbb{R}^2 \times (-m, l) \subset \mathbb{R}^3$. These two fluids
are separated from each other by a free interface $\Sigma(t)$, which extends to infinity
in any horizontal direction. Consequently, one can define two subsets $\Omega_+(t)$, the
upper part and $\Omega_-(t)$, the lower part. Moreover, $\Omega = \Omega_+(t) \cup \Omega_-(t) \cup \Sigma(t)$ and
$\Sigma(t) = \Omega_+(t) \cap \Omega_-(t)$. Both of these two fluids are governed by the standard full
Euler system with uniform gravitation field terms.

\begin{equation}
\begin{aligned}
\partial_t \rho_\pm + \text{div}(\rho_\pm u_\pm) &= 0, \\
\partial_t (\rho_\pm u_\pm) + \text{div}(\rho_\pm u_\pm \otimes u_\pm + P_\pm \mathbb{I}) &= -g e_3 \rho_\pm, \\
\partial_t (\rho_\pm E_\pm) + \text{div}(\rho_\pm u_\pm E_\pm + P_\pm u_\pm) &= -g e_3 \rho_\pm u_\pm,
\end{aligned}
\end{equation}

here $\rho_\pm, u_\pm, P_\pm, E_\pm$ denote the density, velocity vector, pressure and total energy
respectively. The subscript $+/-$ refers to the upper/lower fluid, which is defined
in $\Omega_+(t)/\Omega_-(t)$. $g > 0$ is the gravitational constant, $e_3 = (0, 0, 1)$ and $\mathbb{I}$ is $3 \times 3$
identity matrix. Here we focus on the ideal polytropic case. Precisely speaking, the
equations of state take the following forms:

\[ P_\pm = R_\pm \rho_\pm \theta_\pm, \quad E_\pm = \frac{1}{2}|u_\pm|^2 + e_\pm, \quad e_\pm = \frac{R_\pm}{\gamma_\pm - 1} \theta_\pm, \]

where \( R_\pm \) are positive constants, and \( e_\pm, \theta_\pm, \gamma_\pm \) denote the inertial energy, the absolute temperature and the adiabatic exponent. Both normal velocity and pressure are assumed to be continuous across the interface ([14]), thus it is natural to impose the following boundary conditions on the free interface \( \Sigma(t) \):

\[
\begin{aligned}
(n \cdot u_+)_{|\Sigma(t)} - (n \cdot u_-)_{|\Sigma(t)} &= 0, \\
P_+_{|\Sigma(t)} - P_-_{|\Sigma(t)} &= 0,
\end{aligned}
\]  

(1.2)

where \( n \) denotes the unit normal vector of the interface \( \Sigma(t) \) and \( f_{|\Sigma} \) is the trace of function \( f \) on \( \Sigma(t) \). And the impenetrable boundary conditions are imposed on the fixed boundaries \( \{ x_3 = -m, l \} \):

\[
\begin{aligned}
u_+(x_1, x_2, l, t) \cdot e_3 &= u_-(x_1, x_2, -m, t) \cdot e_3 = 0, \quad \forall (x_1, x_2) \in \mathbb{R}^2, t \geq 0. \quad (1.3)
\end{aligned}
\]

We first construct the steady-state solutions with \( u_\pm = 0 \) and the interface being located on \( \{ x_3 = 0 \} \) for all \( t \geq 0 \). It is easy to find that \((\rho_\pm, \theta_\pm)\) satisfy

\[
\begin{aligned}
\frac{dR_\pm \rho_\pm \theta_\pm}{dx_3} &= -g \rho_\pm, \\
\rho_\pm' &= -\frac{\rho_\pm}{\theta_\pm}(\theta_\pm' + \frac{\theta_\pm}{R_\pm}).
\end{aligned}
\]  

(1.4)

(1.5)

here \( ' \) denotes the derivative with respect to \( x_3 \) variable.

To demonstrate the Rayleigh-Taylor instability, we always assume the steady density \( \rho_\pm(x_3) \) satisfy \( \rho_+ > \rho_- \) on the steady interface \( \{ x_3 = 0 \} \). Once some suitable temperature function is given, it is not hard to obtain the related density function by solving (1.4) or (1.5). For example, we take \( \theta_\pm = 0 \), which is actually the isothermal case considered in [3]. Of course, it should be guaranteed that the steady-state solutions \( \rho_\pm \) and \( \theta_\pm \) are always positive in the whole interval \((-m, l)\). In order to ensure the Rayleigh-Taylor instability phenomena indeed occur, we impose a condition (2.19) in this paper.

Without loss of generality, we assume that \( m = l = 1 \). To overcome the difficulties caused by the free interface, it is convenient to introduce the Lagrangian coordinates. We define the fixed Lagrangian domains \( \Omega_+ = \mathbb{R}^2 \times (0, 1) \) and \( \Omega_- = \mathbb{R}^2 \times (-1, 0) \) and assume that there exist invertible mappings

\[ \eta_\pm^0 : \Omega_\pm \to \Omega_\pm(0), \]

such that \( \Sigma_0 = \eta_+^0(\{ x_3 = 0 \}), \eta_-^0(\{ x_3 = 1 \}) = \{ x_3 = 1 \} \) and \( \eta_-^0(\{ x_3 = -1 \}) = \{ x_3 = -1 \} \). The Lagrangian transformation can be defined as the solutions to the following problems:

\[
\begin{aligned}
\frac{\partial \eta_\pm}{\partial t} &= u_\pm(\eta_\pm(x, t), t) \\
\eta_\pm(x, 0) &= \eta_\pm^0(x).
\end{aligned}
\]  

(1.6)

Then, we denote the Eulerian coordinates as \((y, t)\) with \( y = \eta(x, t) \). In addition, the upper and lower fluids may slip across each other on the interface, and the slip operator is defined by

\[
S_-(x_1, x_2, t) = \eta_-^{-1}(\eta_+(x_1, x_2, 0, t), t).
\]  

(1.7)
We introduce the unknown functions in Lagrangian coordinates 
\[ q_\pm(x,t) = \rho_\pm(\eta_\pm(x,t),t), \quad v_\pm(x,t) = u_\pm(\eta_\pm(x,t),t), \quad \kappa_\pm(x,t) = \theta_\pm(\eta_\pm(x,t),t), \]
and denote \((\eta_\pm, q_\pm, v_\pm, \kappa_\pm)\) by \((\eta, q, v, \kappa)\) without causing any confusion. Then the equations of \((\eta, q, v, \kappa)\) can be written as
\[
\begin{align*}
\partial_t \eta &= v, \\
\partial_t q + q \text{tr}(ADv) &= 0, \\
\partial_t v + RA \nabla \kappa + R \kappa A \nabla \ln q &= -g A \nabla \eta_3, \\
\partial_t \kappa + (\gamma - 1) \text{tr}(ADv) &= 0,
\end{align*}
\]
where \(A = ((\partial \eta / \partial x)^T)^{-1}\), \(\text{tr}(\cdot)\) denotes matrix trace and \(Dv\) is the Jacobian matrix of \(v\). The jumping conditions in Lagrangian coordinates are in turn
\[
\begin{align*}
(v_+ (x_1, x_2, 0, t) - v_- (S_-(x_1, x_2, t))) \cdot n(x_1, x_2, 0, t) &= 0, \\
R_+ q_+ \kappa_+ (x_1, x_2, 0, t) - R_- q_- \kappa_- (S_-(x_1, x_2, 0, t)) &= 0,
\end{align*}
\]
where the unit normal vector \(n\) is defined by
\[
n := \frac{\partial_x \eta_+ \times \partial x_2 \eta_+}{|\partial_x \eta_+ \times \partial x_2 \eta_+|}.
\]
The fixed boundary conditions are
\[
v_- (x_1, x_2, -1, t) \cdot e_3 = v_+ (x_1, x_2, 1, t) \cdot e_3 = 0.
\]
In the remainder of this section, we shall give a description of the main results and recall some related topics. In the next section, when the steady-state solutions \(\eta = Id, q = \bar{\rho}, \kappa = \bar{\theta}, v = 0\) satisfy the assumptions that \(\rho_+ |_{x_3=0} > \rho_- |_{x_3=0}\) and \(\theta_+ + g R_+ (1 - \frac{1}{\gamma \kappa_+}) \geq 0\), we can prove the ill-posedness theorem of the related linearized problem of (1.8)–(1.10) around \((Id, \bar{\rho}, \bar{\theta})\). In section 3, based on the ill-posedness results for the linearized problems, we show the solutions to the original nonlinear problem (1.8)–(1.10) do not have the property EE(k). Precisely speaking, the \(H^3\)-norms of the solutions to nonlinear problems can not Lipschitz continuously depend on their initial data in any time interval, provided that the solutions indeed exist in the Sobolev space \(H^3\).

The studies on the Rayleigh-Taylor instability began from the pioneering work due to Rayleigh and Taylor in [9, 10, 11]. From then on, many interesting physical phenomena and numerical simulations come from both physical and numerical experiments. We refer to [8] and references therein for general discussion of the physics about Rayleigh-Taylor instability. However, there are only very few analytical results from the mathematical point of view. Only very recently, Guo and Tice established the ill-posedness theory of both linearized problems and nonlinear problems for the compressible isentropic Euler equations with a free interface in [3] by the variational methods. And the dynamical Rayleigh-Taylor instability is considered for the density-dependence Euler equations for an incompressible fluid in a strip by Guo and Hwang in [4], where the background profiles are smooth. When the viscosity or the heat-conductivity is considered, the natural variational structures for the linearized problems are destroyed. Then, Guo and Tice developed a modified variational method in [2], and this method is applied for the MHD in [1, 5, 13] and other models [6, 7, 12] successfully.
Here, we deal with the compressible non-isentropic Euler equations for the ideal polytropic fluids. The effect of temperature field on the motion of fluids should be considered. We give a characterization of temperature function to ensure the Rayleigh-Taylor instability occurs. It should be remarked that the general equations of state replacing the polytropic fluids case can also be studied similarly. When the steady-state temperature is a uniform constant function, then it is reduced into the isothermal case considered in [3]. In principle, we adopt the variational methods developed in [3]. However, we need to overcome the difficulties caused by temperature field. Consequently, we propose a condition on steady temperature function $\bar{\theta}$ to guarantee the Rayleigh-Taylor instability. Of course, it is more interesting to find some stabilization effect of temperature field, which is left for future study.

2. Ill-posedness of linearized problems.

2.1. Construction of growing mode ansatz. Suppose the solution $(\eta, q, v, \theta)$ to (1.8) is the perturbation around the steady-state $(Id, \bar{\rho}, 0, \bar{\theta})$ constructed in last section. Then we obtain the linearized equations

\[
\begin{align*}
\partial_t \eta &= v, \\
\partial_t q + \bar{\rho} \text{div} v &= 0, \\
\bar{\rho} \partial_t v + R \nabla \bar{\rho} \theta + R \nabla q &= -g \bar{\rho} \nabla (e_3 \cdot \eta) - g q e_3, \\
\partial_t \kappa + (\gamma - 1) \bar{\theta} \text{div} v &= 0.
\end{align*}
\]

(2.1)

To construct growing solutions to (2.1), it is assumed that the solution $(\eta, q, v, \kappa)$ to (2.1) takes the following forms

\[
\eta(x, t) = \alpha(x) e^{\lambda t}, \quad q(x, t) = \rho(x) e^{\lambda t}, \quad v(x, t) = u(x) e^{\lambda t}, \quad \kappa(x, t) = \theta(x) e^{\lambda t}.
\]

(2.2)

Plugging (2.2) into (2.1) leads to

\[
\begin{align*}
\lambda \alpha &= u, \\
\lambda \rho + \bar{\rho} \text{div} u &= 0, \\
\lambda \bar{\rho} u + R \nabla \bar{\rho} \theta + R \nabla \bar{\theta} \rho &= -g \bar{\rho} \nabla (e_3 \cdot \alpha) - g \rho e_3, \\
\lambda \theta + (\gamma - 1) \bar{\theta} \text{div} u &= 0.
\end{align*}
\]

(2.3)

From (2.3)\textsubscript{1,2,4}, we have

\[
\begin{align*}
\alpha &= \frac{u}{\lambda}, \quad \rho = -\frac{\bar{\rho} \text{div} u}{\lambda}, \quad \theta = \frac{-(\gamma - 1) \bar{\theta} \text{div} u}{\lambda}.
\end{align*}
\]

(2.4)

Inserting (2.4) into (2.3)\textsubscript{3} gives

\[
\lambda^2 \bar{\rho} u - R \gamma \nabla (\bar{\rho} \text{div} u) = g \bar{\rho} \text{div} u e_3 - g \rho \nabla u_3.
\]

(2.5)

The boundary conditions (1.2) can also be linearized as

\[
[[v \cdot e_3]] = 0, \quad [[P_\rho^g(\bar{\rho}, \bar{\theta}) q + P_\theta^g(\bar{\bar{\rho}}, \bar{\bar{\theta}}) \kappa]] = 0,
\]

here $[.]$ denotes the jump across the interface $\{x_3 = 0\}$. Then due to (2.2) and (2.4), the above jumping conditions can be written as

\[
[[u_3]] = 0, \quad [[(P_\rho^g(\bar{\rho}, \bar{\theta}) \bar{\rho} + P_\theta^g(\bar{\bar{\rho}}, \bar{\bar{\theta}}) (\gamma - 1) \bar{\theta}) \text{div} u]] = 0.
\]

(2.6)

The fixed boundary conditions are

\[
u_3(x_1, x_2, -1) = u_3(x_1, x_2, 1) = 0.
\]

(2.7)
Since the coefficients in the linear equations (2.5) depend only on \( x_3 \) variable, we can adopt the horizontal Fourier transformation to (2.5) to reduce them into ordinary differential equations in term of \( x_3 \) with each spatial frequency as parameters. In the following we will denote the horizontal Fourier transformation of a function \( f \) by \( \hat{f} \) or \( \mathcal{F}(f) \), which is the form
\[
\mathcal{F}(f)(\xi_1, \xi_2) = \int_{\mathbb{R}^2} f(x_1, x_2) e^{-i(x_1\xi_1 + x_2\xi_2)} dx_1 dx_2.
\]
Define
\[
\phi(x_3) = i\tilde{u}_1(\xi_1, \xi_2, x_3), \quad \zeta(x_3) = i\tilde{u}_2(\xi_1, \xi_2, x_3), \quad \psi(x_3) = i\tilde{u}_3(\xi_1, \xi_2, x_3),
\]
and
\[
\mathcal{F}(\text{div} u) = \xi_1 \phi + \xi_2 \zeta + \psi'.
\]
Then we obtain after applying the horizontal Fourier transformation to (2.5) that
\[
\begin{cases}
(\lambda^2 \hat{\phi} + R\gamma \hat{\theta} \xi_1^2)\phi = -\xi_1(R\gamma \hat{\theta} \psi' - g\hat{\psi}) - R\gamma \hat{\theta} \xi_1 \xi_2 \zeta, \\
(\lambda^2 \hat{\psi} + R\gamma \hat{\theta} \xi_2^2)\zeta = -\xi_2(R\gamma \hat{\theta} \psi' - g\hat{\psi}) - R\gamma \hat{\theta} \xi_1 \xi_2 \phi, \\
-(R\gamma \hat{\theta} \psi')' + \lambda^2 \hat{\psi} = \xi_1[(R\gamma \hat{\theta} \psi')' + g\hat{\phi}] + \xi_2[(R\gamma \hat{\theta} \zeta')' + g\hat{\zeta}].
\end{cases}
\]
Assuming \((\phi, \zeta, \psi)\) solves (2.8) for \( \xi_1, \xi_2 \) and \( \lambda \), then for any rotation operator \( R \in SO(2) \) \((\phi, \zeta, \psi) = (R\phi, R\zeta, R\psi)\) solves the same equations for \((\tilde{\xi}_1, \tilde{\xi}_2) = R(\xi_1, \xi_2)\) with \( \psi \) and \( \lambda \) being unchanged. Hence we can choose suitable rotation operator \( R_r \in SO(2) \) so that
\[
\xi_1 = |\xi| > 0, \quad \xi_2 = 0. \tag{2.9}
\]
Therefore, it follows from (2.8) and (2.9) that \( \zeta = 0 \) and
\[
\begin{cases}
-\lambda^2 \hat{\phi} = |\xi|(R\gamma \hat{\theta} \psi' + R\gamma \hat{\theta} |\xi| \phi) - g|\xi| \hat{\psi}, \\
-\lambda^2 \hat{\psi} = -(R\gamma \hat{\theta} \psi' + R\gamma \hat{\theta} |\xi| \phi)' - g|\xi| \hat{\phi}.
\end{cases} \tag{2.10}
\]
Similarly, applying the horizontal Fourier transformation to (2.6) and (2.7) yields
\[
[[\hat{\psi}]] = 0, \quad [[R\gamma \hat{\theta} (\psi' + |\xi| \phi)]] = 0, \quad \psi(-1) = \psi(1) = 0. \tag{2.11}
\]
In what follows, for \( \lambda > 0 \), a variational principle gives a solution \((\phi, \psi)\) to (2.10)-(2.11), then using the inverse operator of the rotational operator \( R_r \), a solution \((\phi, \zeta, \psi)) \) to (2.8) is obtained. Then by the inverse Fourier transformation, the growing solutions to the linearized problem (2.1) along with the boundary conditions (2.6)–(2.7) can be constructed. Moreover, we show that \( \lambda \) depends on the frequency variable \(|\xi|\), and \( \lambda(|\xi|) \) tends to \(+\infty\) as \(|\xi|\) goes to \(+\infty\). Therefore, the linearized equations are proved to be ill-posed in the sense of Hadamard.

### 2.2. Constrained minimization formulation.

To build the variational framework, we multiply \( \hat{\phi}, \hat{\psi} \) to (2.10)\(_1\) and (2.10)\(_2\) respectively to get after integration by parts that
\[
-\frac{\lambda^2}{2} \int_{-1}^1 \hat{\phi}(\phi^2 + \psi^2) dx_3 = \frac{1}{2} \int_{-1}^1 [R\gamma \hat{\theta} (\psi' + |\xi| \phi)^2 - 2g\hat{\phi}|\xi| Re(\hat{\psi} \hat{\phi})] dx_3,
\]
where we have used the boundary conditions (2.11). Note that \(-\lambda^2 \in \mathbb{R}\), then it follows that \( Re(\phi), Re(\psi) \) are also solutions to (2.10)-(2.11). So we restrict ourselves to finding the real solutions to (2.10)-(2.11). For any \(|\xi| > 0\), let
\[
\mu = \mu(|\xi|)
Integration by parts gives that
\[
\int_{-1}^{1} [R\gamma \bar{\rho}(\psi') + |\xi|\phi^2 - 2g\bar{\rho}|\phi\psi|] dx_3 = \int_{-1}^{1} \bar{\rho}(\phi^2 + \psi^2) dx_3 = 1.
\]
\begin{equation}
\text{(2.12)}
\end{equation}

Denote
\[
E(\phi, \psi) = \frac{1}{2} \int_{-1}^{1} [R\gamma \bar{\rho}(\psi') + |\xi|\phi^2 - 2g\bar{\rho}|\phi\psi|] dx_3.
\]
\begin{equation}
\text{(2.13)}
\end{equation}

and
\[
J(\phi, \psi) = \frac{1}{2} \int_{-1}^{1} \bar{\rho}(\phi^2 + \psi^2) dx_3.
\]
\begin{equation}
\text{(2.14)}
\end{equation}

Now we show the infimum value of \(E\) is negative with the constrain that \(J = 1\). Moreover, it achieves its infimum value at some \((\phi, \psi)\). Then we prove the infimum value point \((\phi, \psi)\) is indeed a solution to (2.10)-(2.11). It is convenient to define the following functions space
\[
\mathcal{A} = \{ (\phi, \psi) \in L^2(-1,1) \times H_0^1(-1,1) | J(\phi, \psi) = 1 \}.
\]

**Lemma 2.1.** It holds that \(\inf_{(\phi, \psi) \in \mathcal{A}} E(\phi, \psi) < 0\).

**Proof.** It suffices to show \(\inf_{(\phi, \psi) \in L^2 \times H_0^1} \frac{E(\phi, \psi)}{J(\phi, \psi)} < 0\). We assume that
\[
\phi = -\frac{\psi'}{|\xi|},
\]
for any given \(|\xi| > 0\). It is necessary to construct a function \(\psi \in H_0^1\), so that
\[
\bar{E}(\psi) := E(-\frac{\psi'}{|\xi|}, \psi) = \int_{-1}^{1} g\bar{\rho}\psi' \psi dx_3 < 0.
\]
\begin{equation}
\text{(2.15)}
\end{equation}

Integration by parts gives that
\[
\bar{E}(\psi) = \frac{1}{2} \int_{-1}^{1} g\bar{\rho}(\psi^2)' dx_3
\]
\[
= \frac{1}{2} g\bar{\rho}\psi^2 |_{0}^{1} - \frac{1}{2} \int_{0}^{1} g\bar{\rho}' \psi^2 dx_3 + \frac{1}{2} g\bar{\rho} \psi^2 |_{-1}^{0} - \frac{1}{2} \int_{-1}^{0} g\bar{\rho}' \psi^2 dx_3
\]
\[
= -\frac{g\psi^2(0)}{2} [\bar{\rho}] - g \int_{-1}^{1} \bar{\rho}' \psi^2 dx_3.
\]

Since
\[
\bar{\rho}' = -\frac{\bar{\rho}}{\theta}(\bar{\rho}' + \frac{g}{R}),
\]
then
\[
\bar{E}(\psi) = -\frac{g\psi^2(0)}{2} [\bar{\rho}] + \frac{g}{2} \int_{-1}^{1} \bar{\rho}' (\bar{\rho}' + \frac{g}{R}) \psi^2 dx_3.
\]
\begin{equation}
\text{(2.16)}
\end{equation}

Note that \([\bar{\rho}] = (\rho_+ - \rho_-)|_{x_3 = 0} > 0\), we shall construct a function \(\psi_r \in H_0^1\) satisfying \(\bar{E}(\psi_r) < 0\). Define
\[
\psi_r(x_3) = \begin{cases} (1 - x_3)^{7/2}, & x_3 \in (0, 1), \\ (1 + x_3)^{7/2}, & x_3 \in (-1, 0). \end{cases}
\]
\begin{equation}
\text{(2.17)}
\end{equation}

Then
\[
\left| \int_{-1}^{1} \frac{g}{2} \bar{\rho}' (\bar{\rho}' + \frac{g}{R}) \psi^2 dx_3 \right| \leq \frac{g}{2} \| \bar{\rho}' (\bar{\rho}' + \frac{g}{R}) \|_{L^\infty(-1,1)} \int_{-1}^{1} \psi_r^2 dx_3
\]
For any \( \phi, \psi \in A \) satisfying \( E(\phi, \psi) < 0 \), and
\[
\theta' + \frac{g}{R} (1 - \frac{1}{\gamma}) \geq 0. 
\] (2.19)

Then \( \psi(0) \neq 0 \).

**Proof.** Direct calculation gives that
\[
R \gamma \rho \theta' (\psi' + |\xi| \phi^2 - 2 g \rho \xi |\phi \psi
= \left( \sqrt{R \gamma \rho \theta} (\psi' + |\xi| \phi) - \frac{g \rho \xi}{\sqrt{R \gamma \rho \theta}} \right)^2 + 2 g \rho \psi \psi' - \frac{g^2 \rho}{R \gamma \theta} \psi^2,
\]
and
\[
\int_{-1}^{1} (2 g \rho \psi \psi' - \frac{g^2 \rho}{R \gamma \theta} \psi^2) dx_3
\]
\[
= - g [\rho] |\phi(0)|^2 - \int_{-1}^{1} (g \rho' + \frac{g^2 \rho}{R \gamma \theta}) \psi^2 dx_3
\]
\[
= - g [\rho] |\phi(0)|^2 + \int_{-1}^{1} (\frac{1}{2} \rho' \rho + (1 - 1) \frac{g \rho}{R \theta}) \psi^2 dx_3
\]
\[
= - g [\rho] |\phi(0)|^2 + \int_{-1}^{1} (\theta' - \frac{g}{R} \frac{1}{\gamma} - 1) \frac{g \rho \psi^2}{\theta} dx_3.
\]

Hence
\[
E(\phi, \psi)
\]
\[
= - \frac{1}{2} g [\rho] |\phi(0)|^2
\]
\[
+ \frac{1}{2} \int_{-1}^{1} \left( \sqrt{R \gamma \rho \theta} (\psi' + |\xi| \phi) - \frac{g \rho \xi}{\sqrt{R \gamma \rho \theta}} \right)^2 + (\theta' + \frac{g}{R} (1 - \frac{1}{\gamma}) \frac{g \rho \psi^2}{\theta}) \right] dx_3,
\]
\[
\triangleq - \frac{1}{2} g [\rho] |\phi(0)|^2 + E_1 + E_2,
\]
where \( E_2 \geq 0 \) due to (2.19). Consequently, \( E(\phi, \psi) < 0 \) leads to \( \psi(0) \neq 0. \)

**Proposition 2.1.** \( E \) achieves its infimum in \( A \).

**Proof.** For any \( \phi, \psi \in A \), it holds that
\[
E(\phi, \psi) = \frac{1}{2} \int_{-1}^{1} (R \gamma \rho \theta \psi' + |\xi| \phi^2 + g |\xi| (\rho - \psi)^2) dx_3 - \frac{1}{2} \int_{-1}^{1} g \rho |(\phi^2 + \psi^2)| dx_3
\]
\[
= \frac{1}{2} \int_{-1}^{1} (R \gamma \rho \theta \psi' + |\xi| \phi^2 + g |\xi| (\rho - \psi)^2) dx_3 - g |\xi| \geq -g |\xi|.
\]
Therefore, for any given $|\xi| > 0$, $E$ is bounded below by $-g|\xi|$ in $A$. Let $(\psi_n, \psi_n) \in A$ be a minimizing sequence. Then $\phi_n$ is bounded in $L^2(-1, 1)$, and $\psi_n$ is bounded in $H^1_0(-1, 1)$. Hence, up to the extraction of a subsequence, $\phi_n \to \phi$ weakly in $L^2$, and $\psi_n \to \psi$ weakly in $H^1_0$, moreover, $\psi_n \to \psi$ strongly in $L^2$. Thanks to the weak lower semi-continuity and the strong $L^2$ convergence for $\psi_n$, it holds that

$$E(\phi, \psi) \leq \liminf_{n \to \infty} E(\phi_n, \psi_n) = \inf_A E.$$ 

Now, it remains to show $(\phi, \psi) \in A$. The lower semi-continuity leads that $J(\phi, \psi) \leq 1$. Suppose $J(\phi, \psi) < 1$, then there exits $\nu > 1$ such that $J(\nu \phi, \nu \psi) = 1$. Therefore, $(\nu \phi, \nu \psi) \in A$, and

$$E(\nu \phi, \nu \psi) = \nu^2 E(\phi, \psi) \leq \nu^2 \inf_A E < \inf_A E,$$

which contradicts with the definition of $\inf_A E$ due to $(\nu \phi, \nu \psi) \in A$. Thus $J(\phi, \psi) = 1$, i.e. $(\phi, \psi) \in A$. \hfill \Box

**Remark 2.1.** It is easy to see that $-\lambda^2 = E(\phi, \psi) \geq -g|\xi|$. Then, $\lambda \leq \sqrt{g|\xi|}$.

Next, we give the lower boundedness of $\lambda^2$ for suitably large $|\xi|$. It in turn shows that $\lambda(|\xi|) \to +\infty$ as $|\xi| \to +\infty$.

**Lemma 2.3.** There exist constants $C_0, C_1, C_2$, such that

$$\lambda^2 \geq C_1 |\xi| - C_2, \quad \text{for} \quad |\xi| \geq C_0. \quad (2.20)$$

**Proof.** Let $\psi_\tau(x_3)$ be same as that in (2.17), it follows from (2.18) that

$$\tilde{E}(\psi_\tau) \leq -A_1 + \frac{A_2}{\tau + 1} \quad (2.21)$$

here $A_1$, $A_2$ are independent of $\tau$ and $|\xi|$. Note that

$$J(-\frac{\psi_\tau}{|\xi|}, \psi_\tau) = \frac{1}{2} \int_{-1}^{1} \bar{\rho}(\psi_\tau^2 + \frac{(\psi_\tau')^2}{|\xi|^2})dx_3 \leq \frac{A_3}{\tau + 1} + \frac{1}{|\xi|^2} \frac{A_4}{\tau - 1}, \quad (2.22)$$

where $A_3, A_4$ depend on $\|\bar{\rho}\|_{L^\infty}$. On the other hand,

$$J(-\frac{\psi_\tau}{|\xi|}, \psi_\tau) \geq \frac{A_5}{\tau + 1}, \quad (2.23)$$

where $A_5$ depends on the lower bound of $\bar{\rho}$. Choose $\tau = |\xi| \geq C_0$ sufficiently large, there exist $C_1 > 0$, $C_2 > 0$ such that

$$\frac{E(-\psi_\tau'/|\xi|, \psi_\tau)}{J(-\psi_\tau'/|\xi|, \psi_\tau)} \leq -C_1 |\xi| + C_2. \quad (2.24)$$

Together with $-\lambda^2 = \inf E/J$, (2.20) is proved. \hfill \Box

It follows from Remark 2.1 and Lemma 2.3 that

$$\lambda(|\xi|) \to +\infty, \quad \text{as} \quad |\xi| \to +\infty. \quad (2.25)$$

**Proposition 2.2.** Let $(\phi, \psi) \in A$ be a minimizer of $E$ constructed in Proposition 2.1 and $\mu := E(\phi, \psi) < 0$. Then $(\phi, \psi)$ satisfies

$$\mu \bar{\rho} \phi = |\xi| (R\gamma \bar{\rho} \psi' + R\gamma \bar{\rho} \bar{\psi} \xi |\phi| - g|\xi| \bar{\rho} \psi),$$

$$\mu \bar{\rho} \psi = -(R\gamma \bar{\rho} \psi' + R\gamma \bar{\rho} \bar{\psi} \xi |\phi| - g|\xi| \bar{\rho} \phi), \quad (2.26)$$

along with the jumping conditions

$$[[\psi]] = 0, \quad [[[R\gamma \bar{\rho} \psi' + |\xi| \phi]]] = 0, \quad (2.27)$$
and the fixed boundary conditions
\[ \psi(-1) = \psi(1) = 0. \] (2.28)
Moreover, the solution is smooth on either \((-1, 0)\) or \((0, 1)\).

Proof. For fixed \((\phi_0, \psi_0) \in L^2(-1, 1) \times H_0^1(-1, 1)\), we define
\[ j(t, s) = J(\phi + t\phi_0 + s\phi, \psi + t\psi_0 + s\psi). \] (2.29)
Then \(j(0, 0) = 1\). Moreover, \(j\) is smooth and
\[ \frac{\partial j}{\partial t}(0, 0) = \int_{-1}^{1} \bar{\rho}(\phi\phi_0 + \psi\psi_0)dx_3, \] (2.30)
\[ \frac{\partial j}{\partial s}(0, 0) = \int_{-1}^{1} \bar{\rho}(\phi^2 + \psi^2)dx_3 = 2. \] (2.31)
By the implicit function theorem, there exists a \(C^1\) function \(s = \sigma(t)\) in a neighborhood of 0 so that \(\sigma(0) = 0\) and \(j(t, \sigma(t)) = 1\). Then
\[ \frac{\partial j}{\partial t}(0, 0) + \frac{\partial j}{\partial s}(0, 0)\sigma'(0) = 0, \]
and
\[ \sigma'(0) = -\frac{1}{2} \frac{\partial j}{\partial t}(0, 0) = -\frac{1}{2} \int_{-1}^{1} \bar{\rho}(\phi\phi_0 + \psi\psi_0)dx_3. \] (2.32)
Note that \((\phi, \psi) \in \mathcal{A}\) is a minimizer of \(E\), then it follows
\[
\frac{d}{dt}E(\phi + t\phi_0 + \sigma(t)\phi, \psi + t\psi_0 + \sigma(t)\psi)\bigg|_{t=0} = 0.
\]
Together with (2.32), we have
\[
\int_{-1}^{1} [R\gamma \bar{\rho}(\psi') + |\xi|\phi)(\psi_0' + |\xi|\phi_0) - g|\xi|\bar{\rho}(\psi\psi_0 + \phi\phi_0)]dx_3
= -\sigma'(0) \int_{-1}^{1} [R\gamma \bar{\rho}(\psi') + |\xi|\phi^2 - 2g|\xi|\bar{\rho}\psi]dx_3
= \int_{-1}^{1} [\bar{\rho}(\phi\phi_0 + \psi\psi_0)E(\phi, \psi)]dx_3 = \mu \int_{-1}^{1} [\bar{\rho}(\phi\phi_0 + \psi\psi_0)]dx_3. \] (2.33)
Choosing \(\phi_0, \psi_0\) compactly supported in either \((-1, 0)\) or \((0, 1)\), then it is shown that \((\phi, \psi)\) satisfies (2.26) in the sense of distribution. Hence the standard bootstrapping argument yields that \((\phi, \psi) \in H^k(-1, 0)(\text{resp.} H^k(0, 1))\) for any \(k \geq 0\). Then by choosing arbitrary \(\phi_0, \psi_0 \in C_{\infty}^\infty(-1, 1)\), (2.27) holds. Since \(\psi \in H_0^1(-1, 1) \hookrightarrow C_{c}^{0,1/2}(-1, 1)\), the boundary conditions (2.28) are satisfied trivially. \(\square\)

Let \(\lambda(|\xi|) = \sqrt{-\mu(|\xi|)}\). Then we have the following theorem on the existence of the growing mode solutions to (2.10).

**Theorem 2.1.** For any \(|\xi| > 0\) there exists a solution \((\phi, \psi)\) to (2.10)–(2.11). Moreover, \(\psi(0) \neq 0\) and \((\phi, \psi)\) is smooth restricted on both \((-1, 0)\) and \((0, 1)\).

Now, we give the estimates of \(H^k\)-norms for the solution \((\phi(|\xi|, x_3), \psi(|\xi|, x_3))\).

**Lemma 2.4.** Assume \((\phi(|\xi|, x_3), \psi(|\xi|, x_3))\) is the solution to (2.10) obtained in Theorem 2.1, and \(R_1 > \max\{C_0, 2C_2/C_1\}\) where \(C_0, C_1, C_2\) are the same constants.


as those in Lemma 2.3. Then for $|\xi| > R_1$, and each $k \geq 0$, there exists a constant $A_k > 0$ depending on $\varphi, \theta, g, R_1, \gamma$, so that

$$
\|\phi(|\xi|, \cdot)\|_{H^k(-1, 0)} + \|\psi(|\xi|, \cdot)\|_{H^k(-1, 0)} + \|\phi(|\xi|, \cdot)\|_{H^k(0, 1)} + \|\psi(|\xi|, \cdot)\|_{H^k(0, 1)}
\leq A_k \sum_{j=0}^{k} |\xi|^j. \quad (2.34)
$$

Moreover, there exists a constant $B_0 > 0$ depending on the same parameters, so that

$$
\|\sqrt{\phi^2(|\xi|, \cdot) + \psi^2(|\xi|, \cdot)}\|_{L^2(-1, 1)} \geq B_0, \quad \forall |\xi| > 0. \quad (2.35)
$$

**Proof.** This lemma can be proved by induction arguments, and the proof is similar as that of Lemma 3.8 in [3], so we omit it here for simplicity. \qed

In the rest of this subsection, the growing solution to (2.1) will be obtained from the solution to (2.10) by using the inverse Fourier synthesis.

**Theorem 2.2.** Let $1 < R_1 < R_2 < R_3 < \infty$ with $R_1$ being same as that in Lemma 2.4. Let $f \in C^\infty$ be a real-valued function with $f(|\xi|) = f(|\xi|)$, which satisfies $\text{supp}(f) \subset B(0, R_3) \setminus B(0, R_2)$. For $\xi \in \mathbb{R}^2$, set

$$
\hat{\omega}(\xi, x) = -i\phi(\xi, x_3)e_1 - i\zeta(\xi, x_3)e_2 + \psi(\xi, x_3)e_3,
$$

where $(\phi, \zeta, \psi)$ are solutions to (2.8). Denote $x' \cdot \eta = x_1x_1 + x_2x_2$, we define

$$
\begin{cases}
\eta(x, t) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(\xi)\hat{\omega}(\xi, x_3)e^{\lambda(|\xi|)t}e^{ix' \cdot \xi}d\xi,
\vspace{1em}
v(x, t) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \lambda(|\xi|)f(\xi)\hat{\omega}(\xi, x_3)e^{\lambda(|\xi|)t}e^{ix' \cdot \xi}d\xi,
\vspace{1em}
q(x, t) = -\frac{\rho}{4\pi^2} \int_{\mathbb{R}^2} f(\xi)(\xi_1\phi + \xi_2\zeta + \xi_3\psi')e^{\lambda(|\xi|)t}e^{ix' \cdot \xi}d\xi,
\vspace{1em}
\kappa(x, t) = -\left(\frac{\gamma - 1}{\gamma} - \frac{\varphi}{4\pi^2}\right) \int_{\mathbb{R}^2} f(\xi)(\xi_1\phi + \xi_2\zeta + \xi_3\psi')(x, x_3)e^{\lambda(|\xi|)t}e^{ix' \cdot \xi}d\xi.
\end{cases} \quad (2.36)
$$

Then $(\eta, v, q, \kappa)$ is a solution to (2.1) with the boundary conditions (2.6) and (2.7). For every $k \in \mathbb{N}$, we have the estimates

$$
||\eta(0)||_{H^k} + ||v(0)||_{H^k} + ||q(0)||_{H^k} + ||\kappa(0)||_{H^k}
\leq C_k \int_{\mathbb{R}^2} (1 + |\xi|^{2k+1})|f(\xi)|^2d\xi)^{1/2} < \infty, \quad (2.37)
$$

For every $t > 0$, we have $(\eta(t), v(t), q(t), \kappa(t)) \in H^k(\Omega_k)$ and

$$
\begin{cases}
e^{t\varphi}C_1k - C_2\|\eta(0)\|_{H^k} \leq \|\eta(t)\|_{H^k} \leq e^{t\varphi}C_1k \|\eta(0)\|_{H^k},
\vspace{1em}
e^{t\varphi}C_1k - C_2\|v(0)\|_{H^k} \leq \|v(t)\|_{H^k} \leq e^{t\varphi}C_1k \|v(0)\|_{H^k},
\vspace{1em}
e^{t\varphi}C_1k - C_2\|q(0)\|_{H^k} \leq \|q(t)\|_{H^k} \leq e^{t\varphi}C_1k \|q(0)\|_{H^k},
\vspace{1em}
e^{t\varphi}C_1k - C_2\|\kappa(0)\|_{H^k} \leq \|\kappa(t)\|_{H^k} \leq e^{t\varphi}C_1k \|\kappa(0)\|_{H^k}.
\end{cases} \quad (2.38)
$$

**Proof.** The existence of the solution $(\phi, \zeta, \psi)$ to (2.8) are guaranteed by the solution to (2.10) and a suitable choice of the rotation operator $SO(2)$. One can refer to Corollary 3.9 in [3]. And the proof of Theorem 2.2 is based on the properties of the Fourier transformation and the estimates of the eigenvalue $\lambda(|\xi|)$ in Remark 2.1 and Lemma 2.3, which is similar as that of Theorem 3.10 in [3]. \qed
2.3. Ill-posedness of linearized problem. Suppose \((\eta, q, v, \kappa)\) is the solution to (2.1) constructed in Theorem 2.2. Moreover, we assume that the solution is band-limited at radius \(R > 0\), that is, 
\[
\bigcup_{x_3 \in (-1,1)} \text{supp}(|\tilde{\eta}(\cdot, x_3)| + |\tilde{v}(\cdot, x_3)| + |\tilde{q}(\cdot, x_3)| + |\tilde{\kappa}(\cdot, x_3)|) \subset B(0, R). \tag{2.39}
\]
Denote 
\[
\Lambda(R) := \sup_{0 \leq |\xi| \leq R} \lambda(|\xi|) \leq \sqrt{gR} < \infty. \tag{2.40}
\]
Instead of (2.1), it is convenient to consider 
\[
\tilde{\rho} \partial_t v - R \gamma \nabla (\tilde{\rho} \text{div} v) + g \tilde{\rho} \nabla v_3 - g \tilde{\rho} \text{div} v_3 = 0, \tag{2.41}
\]
with the jumping conditions 
\[
[[\partial_t v]] = 0, \quad [[R \gamma \tilde{\rho} \text{div} v]] = 0, \tag{2.42}
\]
and the fixed boundary conditions 
\[
\partial_t v_3(x_1, x_2, -1, t) = \partial_t v_3(x_1, x_2, 1, t) = 0. \tag{2.43}
\]

**Lemma 2.5.** Suppose \(v\) is a solution to (2.41)-(2.43), then it holds that 
\[
\partial_t \int_\Omega \left( \tilde{\rho} \frac{|\partial_t v|^2}{2} + \frac{R \gamma \tilde{\rho} \theta}{2} |\text{div} v - \frac{g}{R \gamma \theta} v_3|^2 + \frac{\tilde{g} \theta'}{2 \theta} (\tilde{\theta} + \frac{g}{R}(1 - \frac{1}{\gamma})) v_3^2 \right) = \partial_t \int_{\mathbb{R}^2} g[|\tilde{\rho}|] \frac{v_3^2}{2}. \tag{2.44}
\]

**Proof.** Multiplying (2.41) by \(v_t\) and integrating the resulting equation over \(\Omega_+ = \mathbb{R}^2 \times (0, 1)\) give that 
\[
\int_{\Omega_+} v_t \{\tilde{\rho} \partial_t v - R \gamma \nabla (\tilde{\rho} \text{div} v) + g \tilde{\rho} \nabla v_3 - g \tilde{\rho} \text{div} v_3\} = 0.
\]
Direct calculation yields 
\[
\partial_t \int_{\Omega_+} \left( \tilde{\rho} \frac{|v_t|^2}{2} + \frac{R \gamma \tilde{\rho} \theta}{2} |\text{div} v - \frac{g}{R \gamma \theta} v_3|^2 + \frac{\tilde{g} \theta'}{2 \theta} (\tilde{\theta} + \frac{g}{R}(1 - \frac{1}{\gamma})) v_3^2 \right) = - \int_{\mathbb{R}^2} \tilde{R} \gamma \tilde{\rho} \text{div} v_3 t + \partial_t \int_{\mathbb{R}^2} g[|\tilde{\rho}|] \frac{v_3^2}{2}.
\]
In addition, we can obtain the similar equality in \(\Omega_- = \mathbb{R}^2 \times (-1, 0)\) with the opposite sign on the right hand side. Then, summing up the two equalities leads to 
\[
\partial_t \int_{\mathbb{R}^2} \left( \tilde{\rho} \frac{|v_t|^2}{2} + \frac{R \gamma \tilde{\rho} \theta}{2} |\text{div} v - \frac{g}{R \gamma \theta} v_3|^2 + \frac{\tilde{g} \theta'}{2 \theta} (\tilde{\theta} + \frac{g}{R}(1 - \frac{1}{\gamma})) v_3^2 \right)
\]
\[
= - \int_{\mathbb{R}^2} [[R \gamma \tilde{\rho} \text{div} v_3 t]] + \partial_t \int_{\mathbb{R}^2} g[|\tilde{\rho}|] \frac{v_3^2}{2}. \tag{2.45}
\]
The first term on the right hand side of (2.45) disappears due to jumping conditions (2.42). Then, the proof of Lemma 2.5 is completed. \(\square\)

**Lemma 2.6.** Let \(v \in H^1(\Omega)\) be band-limited at radius \(R > 0\) and satisfy the boundary conditions that \(v_3(x_1, x_2, -1, t) = v_3(x_1, x_2, 1, t) = 0\), then it holds that 
\[
\int_{\mathbb{R}^2} g[|\tilde{\rho}|] \frac{v_3^2}{2} - \int_\Omega \left( \frac{R \gamma \tilde{\rho} \theta}{2} |\text{div} v - \frac{g}{R \gamma \theta} v_3|^2 + \frac{\tilde{g} \theta'}{2 \theta} (\tilde{\theta} + \frac{g}{R}(1 - \frac{1}{\gamma})) v_3^2 \right) \leq \frac{\Lambda^2(R)}{2} \int_{\Omega} \tilde{\rho} |v|^2. \tag{2.46}
\]
Proof. By the horizontal Fourier transform, we obtain

\[
4\pi^2 \int_{\mathbb{R}^2} \frac{g(\rho)}{2} \, \hat{v}_3^2 \, d\xi = 4\pi^2 \int_{\Omega} \frac{R^2 g}{R^2} \, v_3^2 \, |\text{div} v - \frac{g}{R} v_3|^2 + \frac{g^2}{2R} \, (\hat{v}^2 + \frac{g}{R}) v_3^2 d\xi dx_3
\]

\[
= \int_{\mathbb{R}^2} \frac{g(\rho)}{2} \, \hat{v}_3^2 \, d\xi - \int_{\Omega} \frac{R^2 g}{R^2} \, |i\xi_1 \hat{v}_1 + i\xi_2 \hat{v}_2 + \partial_x \hat{v}_3 - \frac{g}{R} \hat{v}_3|^2 d\xi dx_3
\]

\[
- \int_{\Omega} \frac{g^2}{2R} \, (\hat{v}^2 + \frac{g}{R}) \, d\xi dx_3
\]

Writing \( \phi(x_3) = i\hat{v}_1(\xi_1, \xi_2, x_3), \psi(x_3) = i\hat{v}_2(\xi_1, \xi_2, x_3), \) we define

\[
\Pi(\phi, \psi, \xi) = - \int_{-1}^{1} \frac{R^2 g}{R^2} \, |\xi_1 \phi + \xi_2 \psi + \psi' - \frac{g}{R} \psi|^2 dx_3
\]

\[
- \int_{-1}^{1} \frac{g^2}{2R} \, (\hat{v}^2 + \frac{g}{R}) \, d\xi dx_3 + g(\rho) \, (|\hat{\psi(0)}|^2 - 2).
\]  

(2.47)

Since \( \Pi \) is invariant under simultaneous rotations of \((\xi_1, \xi_2)\) and \((\phi, \psi)\), we may assume without loss of generality that \((\xi_1, \xi_2) = (|\xi|, 0)\) with \(|\xi| > 0\). Direct calculation gives

\[
\Pi(\phi, \psi, \xi) = -E(\phi, \psi) \leq \frac{\lambda^2(|\xi|)}{2} \int_{-1}^{1} \rho(\phi^2 + \psi^2),
\]

(2.48)

here the form of \( E(\phi, \psi) \) is the same as that in the proof of Lemma 2.2. Then, thanks to Remark 2.1, Lemma 2.3 and Parseval’s equality, we achieve finally the inequality (2.46).

\[
\text{Proposition 2.3.} \quad \text{Assume } v \text{ is a solution to (2.41)–(2.43), and } v \text{ is band-limited at radius } \bar{R} > 0. \text{ Then there exists a positive constant } C, \text{ such that}
\]

\[
\|v(t)\|_{L^2(\Omega)}^2 + \|\partial_t v(t)\|_{L^2(\Omega)}^2
\]

\[
\leq Ce^{\lambda^2 \bar{R}^2}(\|v(0)\|_{L^2(\Omega)}^2 + \|\partial_t v(0)\|_{L^2(\Omega)}^2 + \|\text{div} v(0)\|_{L^2(\Omega)}^2).
\]

(2.49)

\[
\text{Proof. Define}
\]

\[
\|f\|_{\tilde{\rho}} = \int_{\Omega} \tilde{\rho} |f|^2.
\]

(2.50)

By Lemmas 2.5 and 2.6, we have

\[
\frac{1}{2} \left\|\partial_t v\right\|_{\tilde{\rho}}^2 \leq \Lambda^2(R) \, \|v\|_{\tilde{\rho}}^2,
\]

(2.51)

where the positive constant \( A \) depends only on \( \|v(0)\|_{L^2}^2 + \|\partial_t v(0)\|_{L^2}^2 + \|\text{div} v(0)\|_{L^2}^2. \)

Furthermore,

\[
\Lambda(R) \, \|\partial_t v\|_{\tilde{\rho}}^2 = \Lambda(R) \, 2 \, \|\partial_t v, v\| \leq 2\Lambda^2(R) \|v(t)\|_{\tilde{\rho}}^2 + 2 \|\partial_t v(t)\|_{\tilde{\rho}}^2.
\]
and \(\langle \cdot, \cdot \rangle\) denotes the weighted \(L^2\) inner production with wight \(\bar{\rho}\). Combining (2.50) and (2.51) to obtain
\[
\partial_t \| v(t) \|_{\bar{\rho}}^2 \leq \frac{4A}{\Lambda(R)} + 4\Lambda(R)\| v(t) \|_{\bar{\rho}}^2.
\]
Gronwall’s inequality gives that
\[
\| v(t) \|_{\bar{\rho}}^2 \leq e^{4\Lambda(R)t}\| v(0) \|_{\bar{\rho}}^2 + \frac{A}{\Lambda^2(R)}(e^{4\Lambda(R)t} - 1).
\]
Hence
\[
\| \partial_t v(t) \|_{\bar{\rho}}^2 \leq 2A + \Lambda^2(t)(e^{4\Lambda(R)t}\| v(0) \|_{\bar{\rho}}^2) + \frac{A}{\Lambda^2(R)}(e^{4\Lambda(R)t} - 1).
\]
Then, by using the following fact that
\[
A \leq C(\| v(0) \|_{L^2}^2 + \| \partial_t v(0) \|_{L^2}^2 + \| \text{div}v(0) \|_{L^2}^2).
\]
The proof is done.

Next, we show that the solutions constructed in Theorem 2.2 are in fact unique. Let \(\Phi \in C^\infty\) satisfy \(0 \leq \Phi \leq 1\), \(\text{supp}(\Phi) \subset B(0, 1)\) and \(\Phi(x) = 1\) for \(x \in (0, 1/2)\). For \(R > 0\), let \(\Phi_R(x) = \Phi(x/R)\). We define the projection operator \(P_R\) by
\[
P_Rf = F^{-1}(\Phi_RF(f)), \quad f \in L^2(\Omega).
\]
Then the projection operator \(P_R\) has the following properties:
1. \(P_Rf\) is band-limited at radius \(R\);
2. \(P_R\) is a bounded linear operator on \(H^k(\Omega)\) for all \(k \geq 0\);
3. \(P_R\) commutes with partial differential operators and multiplication by functions depending only on \(x_3\);
4. \(P_Rf = 0\) for all \(R > 0\) if and only if \(f = 0\).

Thus, based on Proposition 2.3 and the properties of the projection operator \(P_R\), we can show the solutions to (2.41)-(2.43) are unique. Then by the equations (2.1), we have

**Theorem 2.3.** Solutions to (2.1) are unique.

Now, we come to the position to state the ill-posedness theorem for the linearized problem (2.1).

**Theorem 2.4.** Suppose (2.19) holds, then the linearized problem (2.1) is ill-posed in the sense of Hadamard in \(H^k(\Omega)\) for every \(k\). Precisely speaking, for any given \(k, j \in \mathbb{N}\) with \(j \geq k\), then for any time \(T > 0\) and \(\nu > 0\), there exists a solution sequence \((\eta_n, q_n, v_n, \kappa_n)\) to the linearized problem (2.1), such that
\[
\| \eta_n(0) \|_{H^1} + \| v_n(0) \|_{H^j} + \| q_n(0) \|_{H^j} + \| \kappa_n(0) \|_{H^j} \leq \frac{1}{n},
\]
but
\[
\| v_n \|_{H^k} \geq \| \eta_n \|_{H^k} \geq \nu, \quad \forall t \geq T.
\]

**Proof.** Fix \(j \geq k \geq 0, \nu > 0, T > 0\), and let \(C_j(j = 1, 2)\) be the same constants as in Lemma 2.3, \(R_1, B_0\) be the constants in Lemma 2.4 and \(C_k\) be the same as that in Theorem 2.2. Then, for each \(n \in \mathbb{N}\), choose \(R(n)\) suitably large, such that \(R(n) > R_1, \sqrt{C_1R(n)} - C_2 \geq 1\) and
\[
\frac{\exp(T\sqrt{C_1R(n)} - C_2)}{(1 + (R(n) + 1)^2)^{-1}} \geq \nu^2n^2\frac{C_k^2}{B_0^2}.
\]

(2.54)
In addition, we choose compact supported functions \( f_n \) in the definitions \((2.36)\) satisfying \( f_n \in C_c^\infty(\mathbb{R}^2) \), \( \text{supp}(f_n) \subset B(0, R(n) + 1)/B(0, R(n)) \) and \( f_n \) are real-valued and radial. Moreover, \( 1/C_k^2 n^2 \).

Then, thanks to \((2.36)\), we obtain
\[
\|\eta_n(t)\|_{H^k}^2 \geq \int_{\mathbb{R}^2} (1 + |\xi|^2) |f_n(\xi)|^2 e^{T^\lambda(|\xi|)} \|\hat{\omega}(\xi, \cdot)\|_{L^2((-1,1))}^2 d\xi \\
\geq \exp\left(T \frac{\sqrt{C_1 R(n) - C_2}}{1 + (R(n) + 1)^2} \right) \left[ \int_{\mathbb{R}^2} (1 + |\xi|^2) |f_n(\xi)|^2 \|\hat{\omega}(\xi, \cdot)\|_{L^2((-1,1))}^2 d\xi \right] \\
\geq v^2 n^2 \frac{C_k^2}{B_0} \int_{\mathbb{R}^2} (1 + |\xi|^2) |f_n(\xi)|^2 d\xi = v^2, \quad \text{for } t \geq T,
\]
where we have used \((2.35)\) and the facts that \( \lambda(|\xi|) \geq \sqrt{C_1 R(n) - C_2} \). Since \( \sqrt{C_1 R(n) - C_2} \geq 1 \), then
\[
\|v_n(t)\|_{H^k}^2 \geq \|\eta_n(t)\|_{H^k}^2 \geq \|\eta_n(T)\|_{H^k}^2, \quad \text{for } t \geq T.
\]
Thus we have finished the proof of Theorem 2.4.

3. Ill-posedness for the nonlinear problem. It remains to show the ill-posedness for the original nonlinear problem. First, we recall the nonlinear problem
\[
\begin{aligned}
\partial_t \eta & = v, \\
\partial_t q + q (A_D v) & = 0, \\
q \partial_t v + R \nabla q & = -g A \nabla \eta, \\
\partial_t \kappa + (\gamma - 1) & = 0 (A_D v).
\end{aligned}
\]
(3.1)

Then, we rewrite (3.1) in a perturbation formulation around the steady-state solution. Set
\[
\eta = Id + \tilde{\eta}, \quad \eta^{-1} = Id - \beta, \quad v = 0 + v, \quad q = \tilde{\rho} + \sigma, \quad \kappa = \tilde{\theta} + \pi, \quad A = I - B.
\]
(3.2)

Plugging (3.2) into (3.1) yields
\[
\begin{aligned}
(\partial_t \tilde{\eta} & = v, \\
\partial_t \sigma + (\tilde{\rho} + \sigma) [\text{div} v - \text{tr}(B Dv)] & = 0, \\
(\tilde{\rho} + \sigma) & \partial_t v + R (I - B) \nabla (\tilde{\rho} + \sigma) (\tilde{\theta} + \pi) = -g (\tilde{\rho} + \sigma) (I - B) (c_3 + \nabla \tilde{\eta}), \\
\partial_t \pi + (\gamma - 1) (\tilde{\theta} + \pi) [\text{div} v - \text{tr}(B Dv)] & = 0.
\end{aligned}
\]
(3.3)

Definition 3.1. We say the perturbed problem (3.3) has property EE(k) for \( k \geq 3 \) if there exist \( \delta, t_0, C > 0 \) and a function \( F : [0, t_0) \to \mathbb{R}^+ \) satisfying \( F(z) \leq C z \) for all \( z \in [0, \delta) \), so that the following conditions hold. For any initial data \( (\tilde{\eta}_0, v_0, \sigma_0, \pi_0) \) satisfying
\[
\| (\tilde{\eta}_0, v_0, \sigma_0, \pi_0) \|_{H^k} \leq \delta,
\]
there exists \( (\tilde{\eta}, v, \sigma, \pi) \in L^\infty(0, t_0); H^3(\Omega) \) such that
1. \( (\tilde{\eta}, v, \sigma, \pi)(0) = (\tilde{\eta}_0, v_0, \sigma_0, \pi_0) \); 
2. \( \eta(t) = Id + \tilde{\eta}(t) \) is invertible for \( 0 \leq t < t_0 \); 
3. \( (\tilde{\eta}, v, \sigma, \pi) \) solves the perturbed problem (3.3) on \( \Omega \times (0, t_0) \); 
4. \( \sup_{0 \leq t < t_0} \| (\tilde{\eta}, v, \sigma, \pi)(t) \|_{H^3(\Omega)} \leq F(\| (\tilde{\eta}, v, \sigma, \pi)(0) \|_{H^k(\Omega)}) \).
Theorem 3.1. The perturbed problem (3.3) does NOT have the property EE(k) for any \( k \geq 3 \).

Proof. Suppose the perturbed problem (3.3) has the property EE(k) for some \( k \geq 3 \). Choose \( n \in \mathbb{N} \) so that \( n > k \). Set \( T = t_0/2 \) and \( \nu > 1 \). Let \((\bar{n}, \bar{v}, \bar{\sigma}, \bar{\pi})(t)\) be the solution to the linearized problem (2.1) with the initial data satisfying
\[
\|(\bar{n}, \bar{v}, \bar{\sigma}, \bar{\pi})(0)\|_{H^k} < 1/n,
\]
but due to Theorem 2.4, we have
\[
\|\bar{v}(t)\|_{H^3} \geq \|\bar{n}(t)\|_{H^3} > 1, \quad \text{for } t \geq t_0/2.
\]
For every \( \varepsilon > 0 \), we define
\[
\bar{n}_0 = \varepsilon \bar{n}(0), \quad \bar{v}_0 = \varepsilon \bar{v}(0), \quad \bar{\sigma}_0 = \varepsilon \bar{\sigma}(0), \quad \bar{\pi}_0 = \varepsilon \bar{\pi}(0),
\]
then
\[
\|((\bar{n}_0, \bar{v}_0, \bar{\sigma}_0, \bar{\pi}_0))\|_{H^k} = \|((\bar{n}, \bar{v}, \bar{\sigma}, \bar{\pi})(0))\|_{H^k} < \varepsilon/n.
\]
Choosing \( n \) suitably large so that \( C/n < 1 \), where \( C \) is the positive constant in Definition 3.1, and \( \varepsilon/n < \delta \). Then according to the property EE(k), there exists \((\bar{n}_\varepsilon, \bar{v}_\varepsilon, \bar{\sigma}_\varepsilon, \bar{\pi}_\varepsilon) \in L^\infty((0, t_0); H^3(\Omega))\), which solves the perturbed problem with \((\bar{n}_0, \bar{v}_0, \bar{\sigma}_0, \bar{\pi}_0)\) being the initial data. Moreover,
\[
\sup_{0 \leq t < t_0} \|((\bar{n}_\varepsilon, \bar{v}_\varepsilon, \bar{\sigma}_\varepsilon, \bar{\pi}_\varepsilon))\|_{H^3} \leq F((\bar{n}_0, \bar{v}_0, \bar{\sigma}_0, \bar{\pi}_0))
\]
\[
\leq C \varepsilon \|((\bar{n}, \bar{v}, \bar{\sigma}, \bar{\pi})(0))\|_{H^k} < C \varepsilon/n < \varepsilon.
\]
We define the following re-scaled functions
\[
\bar{n}_\varepsilon = \bar{n}_\varepsilon/\varepsilon, \quad \bar{v}_\varepsilon = \bar{v}_\varepsilon/\varepsilon, \quad \bar{\sigma}_\varepsilon = \bar{\sigma}_\varepsilon/\varepsilon, \quad \bar{\pi}_\varepsilon = \bar{\pi}_\varepsilon/\varepsilon,
\]
then
\[
\sup_{0 \leq t < t_0} \|((\bar{n}_\varepsilon, \bar{v}_\varepsilon, \bar{\sigma}_\varepsilon, \bar{\pi}_\varepsilon))(t)\|_{H^3} < 1,
\]
and \((\bar{n}_\varepsilon, \bar{v}_\varepsilon, \bar{\sigma}_\varepsilon, \bar{\pi}_\varepsilon)(0) = (\bar{n}, \bar{v}, \bar{\sigma}, \bar{\pi})(0)\).

Now, we show that \((\bar{n}_\varepsilon, \bar{v}_\varepsilon, \bar{\sigma}_\varepsilon, \bar{\pi}_\varepsilon) \rightarrow (\bar{n}, \bar{v}, \bar{\sigma}, \bar{\pi})\) in some Sobolev spaces. We make a further assumption that \( \varepsilon \) is sufficiently small, so that
\[
\sup_{0 \leq t < t_0} \|\varepsilon \bar{\sigma}_\varepsilon(t)\|_{L^\infty} < \frac{1}{2} \inf_{-1 \leq x_3 \leq 1} \bar{\rho}(x_3).
\]
Thanks to (3.9), we obtain
\[
\bar{n}_\varepsilon = \varepsilon \bar{n}_\varepsilon, \quad \bar{v}_\varepsilon = \varepsilon \bar{v}_\varepsilon, \quad \bar{\sigma}_\varepsilon = \varepsilon \bar{\sigma}_\varepsilon, \quad \bar{\pi}_\varepsilon = \varepsilon \bar{\pi}_\varepsilon.
\]
Plugging (3.12) into the perturbed system (3.3) gives
\[
\begin{align*}
\varepsilon \partial_t \bar{\sigma}_\varepsilon + \langle \bar{\rho} + \varepsilon \bar{\sigma}_\varepsilon \rangle \varepsilon \nabla \bar{v}_\varepsilon - \varepsilon \text{tr}(\mathbb{B}^\varepsilon D \bar{v}_\varepsilon) &= 0, \\
\langle \bar{\rho} + \varepsilon \bar{\sigma}_\varepsilon \rangle \partial_t (\varepsilon \bar{v}_\varepsilon) + R(I - \mathbb{B}^\varepsilon) \nabla (\bar{\rho} + \varepsilon \bar{\sigma}_\varepsilon) (\bar{\theta} + \varepsilon \bar{\pi}_\varepsilon) &= -g(\bar{\rho} + \varepsilon \bar{\sigma}_\varepsilon) (\bar{\rho} - \varepsilon \bar{\pi}_\varepsilon) (\bar{\rho} - \varepsilon \bar{\pi}_\varepsilon), \\
\varepsilon \partial_t \bar{\pi}_\varepsilon + (\gamma - 1) (\bar{\theta} + \varepsilon \bar{\pi}_\varepsilon) \varepsilon \nabla \bar{v}_\varepsilon - \varepsilon \text{tr}(\mathbb{B}^\varepsilon D \bar{v}_\varepsilon) &= 0,
\end{align*}
\]
where
\[
\mathbb{B}^\varepsilon = I - A^\varepsilon = I - (I + \varepsilon (D\bar{n}_\varepsilon)^T)^{-1} \triangleq \varepsilon \mathbb{B}^\varepsilon,
\]
with $\overline{B}^\varepsilon$ satisfying

$$
\|\overline{B}^\varepsilon\|_{H^2} = \| \sum_{n=1}^{\infty} (-\varepsilon)^{n-1} [(D\overline{\eta}^\varepsilon)^T]^n \|_{H^2} \leq \sum_{n=1}^{\infty} \varepsilon^{n-1} \| (D\overline{\eta}^\varepsilon)^n \|_{H^2}
$$

$$
\leq \sum_{n=1}^{\infty} (\varepsilon K)^{n-1} \| D\overline{\eta}^\varepsilon \|_{H^2} \leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \| \overline{\eta}^\varepsilon \|^{n}_{H^3}
$$

$$
\leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 2,
$$

(3.14)

where $K$ is the optimal Sobolev constant so that $\|FG\|_{H^2} \leq K \|F\|_{H^2} \|G\|_{H^2}$. Here, we further require $\varepsilon$ is suitably small, such that $\varepsilon K < 1/2$.

By (3.13)$_1$, we have

$$
\partial_t \overline{\eta}^\varepsilon = \overline{v}^\varepsilon,
$$

(3.15)

and it follows from (3.13)$_2$ that

$$
\partial_t \bar{\sigma}^\varepsilon + \bar{\nu}\text{div}\bar{v}^\varepsilon + \varepsilon [\bar{\sigma}^\varepsilon \text{div}\bar{v}^\varepsilon - \bar{\rho}\text{tr}(\overline{B}^\varepsilon D\bar{v}^\varepsilon)] - \varepsilon^2 \bar{\sigma}^\varepsilon \text{tr}(\overline{B}^\varepsilon D\bar{v}^\varepsilon)] = 0.
$$

(3.16)

Then we derive from (3.13)$_3$ that

$$
\bar{\rho}\partial_t \bar{v}^\varepsilon + R\nabla \bar{\rho}\bar{v}^\varepsilon + R\nabla \bar{\sigma}^\varepsilon + g\bar{\rho}\nabla \bar{\eta}_5^\varepsilon + g\bar{\sigma}^\varepsilon \cdot e_3
$$

$$
+ \varepsilon [\bar{\sigma}^\varepsilon \partial_t \bar{v}^\varepsilon + R\nabla \bar{\sigma}^\varepsilon \bar{\pi}^\varepsilon - R\bar{\sigma}^\varepsilon \nabla \bar{\rho}\bar{v}^\varepsilon - R\bar{\sigma}^\varepsilon \nabla \bar{\sigma}^\varepsilon + g\bar{\sigma}^\varepsilon \nabla \bar{\eta}_5^\varepsilon - g\bar{\sigma}^\varepsilon \overline{B}^\varepsilon e_3]
$$

$$
+ \varepsilon^2 \bar{\sigma}^\varepsilon \nabla \bar{\sigma}^\varepsilon \bar{\pi}^\varepsilon - g\bar{\sigma}^\varepsilon \nabla \bar{\eta}_5^\varepsilon = 0.
$$

(3.17)

Finally, expanding (3.13)$_4$ yields

$$
\partial_t \bar{\pi}^\varepsilon + (\gamma - 1)\bar{\theta}\text{div}\bar{v}^\varepsilon + \varepsilon [\gamma (\gamma - 1)\bar{\pi}^\varepsilon \text{div}\bar{v}^\varepsilon - (\gamma - 1)\bar{\theta}\text{tr}(\overline{B}^\varepsilon D\bar{v}^\varepsilon)]
$$

$$
- \varepsilon^2 [(\gamma - 1)\bar{\pi}^\varepsilon \text{tr}(\overline{B}^\varepsilon D\bar{v}^\varepsilon)] = 0.
$$

(3.18)

It follows from (3.10) and (3.15) that

$$
\sup_{0 \leq t < t_0} \| \partial_t \bar{\eta}^\varepsilon \|_{H^3} = \sup_{0 \leq t < t_0} \| \bar{v}^\varepsilon \|_{H^3} \leq 1.
$$

(3.19)

By virtue of (3.10) and (3.16), we have

$$
\lim_{\varepsilon \to 0} \sup_{0 \leq t < t_0} \| \partial_t \bar{\sigma}^\varepsilon + \bar{\rho}\text{div}\bar{v}^\varepsilon \|_{H^2} = 0,
$$

(3.20)

and

$$
\sup_{0 \leq t < t_0} \| \partial_t \bar{\sigma}^\varepsilon \|_{H^2} \leq K_1.
$$

(3.21)

Similarly, (3.10) and (3.17) imply that

$$
\lim_{\varepsilon \to 0} \sup_{0 \leq t < t_0} \| \bar{\rho}\partial_t \bar{v}^\varepsilon + R\nabla \bar{\rho}\bar{v}^\varepsilon + R\nabla \bar{\sigma}^\varepsilon + g\bar{\rho}\nabla \bar{\eta}_5^\varepsilon + g\bar{\sigma}^\varepsilon \cdot e_3 \|_{H^2} = 0,
$$

(3.22)

and

$$
\sup_{0 \leq t < t_0} \| \partial_t \bar{v}^\varepsilon \|_{H^2} \leq K_2.
$$

(3.23)

Finally, thanks to (3.10) and (3.18), we obtain

$$
\lim_{\varepsilon \to 0} \sup_{0 \leq t < t_0} \| \partial_t \bar{\pi}^\varepsilon + (\gamma - 1)\bar{\theta}\text{div}\bar{v}^\varepsilon \|_{H^2} = 0,
$$

(3.24)

and

$$
\sup_{0 \leq t < t_0} \| \partial_t \bar{\pi}^\varepsilon \|_{H^2} \leq K_3.
$$

(3.25)
Hence, due to (3.10), we have
\[(\eta\varepsilon, v\varepsilon, \sigma\varepsilon, \pi\varepsilon) \rightharpoonup (\hat{\eta}, \hat{v}, \hat{\sigma}, \hat{\pi}) \text{ weakly in } L^\infty([0, t_0), H^3(\Omega)).\] (3.26)

By the lower semi-continuity of the Sobolev norms \(\| \cdot \|_{H^3}\), the following inequality holds true due to (3.10) and (3.26).
\[
\sup_{0 \leq t < t_0} \| (\hat{\eta}, \hat{v}, \hat{\sigma}, \hat{\pi}) \|_{H^3} \leq 1 \quad (3.27)
\]

Then, based on the uniform estimates (3.19), (3.21), (3.23), (3.25) and (3.26), we obtain
\[(\eta\varepsilon, v\varepsilon, \sigma\varepsilon, \pi\varepsilon) \rightarrow (\hat{\eta}, \hat{v}, \hat{\sigma}, \hat{\pi}) \text{ strongly in } L^\infty([0, t_0), H^{11/4}(\Omega)),\] (3.28)
and then
\[(\partial_t \eta\varepsilon, \partial_t v\varepsilon, \partial_t \sigma\varepsilon, \partial_t \pi\varepsilon) \rightarrow (\partial_t \hat{\eta}, \partial_t \hat{v}, \partial_t \hat{\sigma}, \partial_t \hat{\pi}) \text{ strongly in } L^\infty([0, t_0), H^{7/4}(\Omega)).\] (3.29)

Then it is not hard to find that \((\hat{\eta}, \hat{v}, \hat{\sigma}, \hat{\pi})\) solves the linearized problem (2.1) with initial data \((\eta(0), v(0), \sigma(0), \pi(0))\). By the similar arguments, we can also expand the boundary conditions in terms of the small parameter \(\varepsilon\). Then, owing to the convergence results above, the limit functions \((\hat{\eta}, \hat{v}, \hat{\sigma}, \hat{\pi})\) also satisfy the related jumping conditions and fixed boundary conditions. In addition, the solutions to the linearized problem are unique. Then, we can derive the contradiction from the estimates (3.5) and (3.27). Consequently, the proof of Theorem 3.1 is completed.

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