One-Shot Manipulation of Entanglement for Quantum Channels

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Abstract

We show that the dynamic resource theory of quantum entanglement can be formulated using the superchannel theory. In this formulation, we identify the separable channels and the class of free superchannels that preserve channel separability as free resources, and choose the swap channels as dynamic entanglement golden units. Our first result is that the one-shot dynamic entanglement cost of a bipartite quantum channel under the free superchannels is bounded by the standard log-robustness of channels. The one-shot distillable dynamic entanglement of a bipartite quantum channel under the free superchannels is found to be bounded by a resource monotone that we construct from the hypothesis-testing relative entropy of channels with minimization over separable channels. We also address the one-shot catalytic dynamic entanglement cost of a bipartite quantum channel under a larger class of free superchannels that could generate the dynamic entanglement which is asymptotically negligible; it is bounded by the generalized log-robustness of channels.

I. INTRODUCTION

The emergence of the modern development of quantum information science is tightly linked to a fundamental change of paradigm that characterizes our appreciation of fundamental traits of quantum mechanics. Rather than viewing these merely as counter-intuitive departures from our classical world view, in recent years we have come to recognize fundamental quantum features as resources that enable us to solve technological and information theoretic tasks more efficiently than classical physics would allow. The desire to investigate systematically which aspects of quantum mechanics are responsible for potential operational advantages has led to the development of quantum resource theories [1]. The most basic concept that gives rise to the structure of resource theories is the concept of constraints that are imposed on our ability to operate beyond those that are already enforced by the laws of quantum mechanics. From this emerges by an elegant inevitability the concept of free states and operations. These are those that can be prepared and executed without violation of the constraints. These two main ingredients allow for the formulation of a rigorous theoretical framework in which to analyze resources quantitatively. Perhaps the most fundamental examples are represented by the theory of quantum coherence, which marks the delineation between classical and quantum physics already at the level of individual particles [2]–[4], and, historically having emerged first, the theory of entanglement, which explores the value of quantum correlations as opposed to classical correlations [5], [6]. These were followed by a host of resource theories including that of superposition [7], [8], of reference frames [9], of Gaussianity [10], of quantum optical non-classicality [11]–[13], of indistinguishable particles [14]–[16], and of thermodynamics [17]–[19].

Initially, the focus of attention in entanglement theory was placed squarely on the entanglement content of quantum states, i.e. (i) which states contain entanglement [20], [21], (ii) how the entanglement of states can be transformed under local operations and classical communication [22]–[25], (iii) how entanglement can be verified quantitatively [26]–[29] and (iv) how useful entanglement is in operational tasks, e.g. to enable the realisation of arbitrary non-local quantum operations when only local operations and classical communication is available. For example, maximally entangled states may be employed to achieve general non-local quantum gates between spatially separated parties using only local quantum operations and classical communication [30], [31]. This example is characteristic of the early approaches to entanglement theory in particular and resource theories in general. While task (ii) concerns state-to-state transformations, task (iv) is of a somewhat different nature, as it connects static resources (states) with dynamic ones (quantum operations).

In fact, quantum states may be considered as special cases of quantum channels, and these subsume also quantum measurements and quantum dynamics [32]–[34]. To make this concrete, consider that quantum states and measurements can be thought of as quantum channels with trivial input and classical output, respectively.

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While such approach is legitimate based on the fact that any quantum channel can be simulated with free operations and entangled states, the quantum community is now aiming to encompass all the aspects in a unified manner by studying the properties of quantum channels with modern tools of quantum resource theories. First steps in this direction have been taken with the extension of the entropy of a quantum state to that of quantum channels with its operational meaning given by the channel merging [33]; the entropy of a preparation channel reduces to the usual entanglement of the state it prepares. The resource theory of the coherence of quantum dynamics have been investigated [34]–[37]. The properties of a quantum channel associated with entanglement also has been investigated using the tools of resource theories [38]–[43]. Moreover, there are recent results that built various ways to construct resource monotonies of dynamical resources in general which emphasize similarities and subtle differences from the quantum resources in quantum states [44]. Alongside, operational interpretations of channel resource theories have been identified [34], [35], [45].

In this paper, we will use as basic building blocks specific maximally resourceful operations that play the role that maximally entangled state played in the entanglement theory of states and explore how concatenation and combination with free operations allows us to achieve general quantum channels. As the basic element is itself an operation rather than a static state, this approach is now running under dynamic entanglement theory. As a note of caution though, this should not be equated with an even more general approach in which one indeed considers continuous in time evolution based on some generators. More specifically, we consider the problem of quantum channel manipulation in the one-shot setting, when the allowed set of free superchannels is maximal, in the sense that it comprises all transformations that map separable channels to separable channels. We choose the $K$-swap channels as dynamic entanglement resources, which play the role of $K$-maximally entangled states $\Phi_{A_0B_0}^K$ in the entanglement theory of quantum states. The dynamic entanglement resource is intimately related to the static entanglement resource of quantum states in the sense that the former can generate the maximum static resource under LOCC as well as it requires the maximum static resource to simulate them under LOCC, so it bridges the dynamic entanglement to the maximum static entanglement. In order to treat the dynamic entanglement resource required for conversions between quantum channels, we define the separability-preserving superchannels and use them as the free superchannels. The dynamic entanglement resource of a bipartite quantum channel is investigated in operational ways. To evaluate the one-shot dynamic entanglement cost of a bipartite quantum channel, we ask which amount of dynamic entanglement resource is necessary to simulate the channel by means of free superchannels. We further push the notion of dynamic entanglement cost to its limits, by considering a two-fold variation on the theme: on the one hand, we allow for catalysts, while on the other we define and use a larger set of free superchannels that might generate a small amount of dynamic entanglement. We refer to the resulting modified notion as the one-shot catalytic dynamic entanglement cost of the channel. Finally, in order to get some insight into the asymptotic scenario, we adopt the liberal smoothing to define liberal dynamic entanglement cost and show that it is given by the liberal regularized relative entropy of channels with respect to the free channels.

II. Dynamic Entanglement Resource

We use indexed capital letters such as $A_0$, $B_1$, etc. to denote physical systems, and juxtapose them to indicate physical composites. $\mathcal{B}(\mathcal{H}_{A_0})$ denotes the space of bounded operators acting on a finite dimensional Hilbert space $\mathcal{H}_{A_0}$. The set of linear maps from $\mathcal{B}(\mathcal{H}_{A_0})$ to $\mathcal{B}(\mathcal{H}_{A_1})$ will be denoted with $\mathcal{L}(A) \equiv \mathcal{L}(A_0 \to A_1)$; quantum channels, that is, completely positive trace-preserving linear maps in $\mathcal{L}(A)$, will be collectively denoted with $\text{CPTP}(A) \equiv \text{CPTP}(A_0 \to A_1)$. We use calligraphic letters for quantum channels and abbreviations such as $\mathcal{E}_A \equiv \mathcal{E}_{A_0 \to A_1}$. As an exception, we omit indices if we take the trace map over all the input spaces such as $\text{Tr}(X_{A_0B_0})$. We sometimes omit the identity channel for readability when there’s little chance of confusion. As a distance between two quantum channels, we use the metric induced by the diamond norm denoted as $\| \cdot \|_1$ [46], [47]. $\mathcal{L}(A \to A')$ denotes the set of linear supermaps from $\mathcal{L}(A)$ to $\mathcal{L}(A')$. A Greek letter $\Theta_{A \to B}$ is used to denote supermaps, whose action is expressed as $\Theta_{A_0 \to B_0}[\mathcal{E}_A]$. We write $\Psi_{A_0}$ for the density matrix of the pure state $|\Psi\rangle_{A_0}$, and call $\mathcal{S}(A_0)$ and $\mathcal{D}(A_0)$ the sets of pure and mixed states of system $A_0$, respectively. The set of separable states on $A_0B_0$ is indicated with $\text{SepD}(A_0 : B_0)$, while $\text{SepC}(A : B)$ stands for the set of separable channels from $A_0B_0$ to $A_1B_1$.

A $K$-swap channel $F_{AB}^K$ consists in the application of the $K$-swap gate $F_{AB}^K = \sum_{i,j=0}^{K-1}|ij\rangle\langle ji|_{A_0B_0 \to A_1B_1}$; it is a typical example of a separability-preserving channel that is not actually separable as a map [49]. We use the $K$-swap channel

\footnote{Also called non-entangling map [48].}
\[ \Phi_{A_1A_2}^{K} \quad \Phi_{B_1B_2}^{K} \quad \Phi_{A_1B_2}^{K} \quad \Phi_{B_1A_2}^{K} \]

Fig. 1. Two \( K \)-maximally entangled states generated by the \( K \)-swap channel from locally prepared \( K \)-maximally entangled states.

\[
\begin{array}{c}
\mathcal{F}_{\text{pre}}^K \quad E \quad \mathcal{F}_{\text{post}}^K \\
B_0 \quad A_0 \quad \mathcal{E}_A \quad A_1 \quad B_1
\end{array}
\]

Fig. 2. Structure of a superchannel \( \Theta_{A \rightarrow B} \) acting on a quantum channel \( \mathcal{E}_A \).

\[ \mathcal{F}_{AB}^{K} \text{ as the “golden unit” of resource in the theory of dynamical entanglement [50]; its role is entirely analogous to that of the } \] \( K \)-maximally entangled state \( \Phi_{AB}^{K} \sum_{i=0}^{K-1} |ii\rangle_{A_iB_i} \) in the static entanglement theory. Other reasonable choices for the golden unit are entirely equivalent to ours and therefore lead to the same results. Indeed, consider that a \( K \)-swap channel can generate a pair of \( K \)-maximally entangled states between Alice and Bob under LOCC (Fig. 1), while they also need two \( K \)-maximally entangled states to simulate a \( K \)-swap gate with LOCC.

A superchannel is a linear supermap that sends a quantum channel to another quantum channel [51], [52]. A superchannel \( \Theta_{AB \rightarrow A'B'} \) is called a separability-preserving superchannel (SEPPSC) if

\[ \Theta_{AB \rightarrow A'B'}[\mathcal{M}_{AB}] \in \text{SepC}(A' : B') \quad \forall \mathcal{M}_{AB} \in \text{SepC}(A : B). \quad (1) \]

The set of all separability-preserving superchannels from \( \mathcal{L}(AB) \) to \( \mathcal{L}(A'B') \) is denoted as SEPPSC\( (A : B \rightarrow A' : B') \).

As dynamic entanglement monotones, we use the standard and the generalized robustness of channels. Since having been introduced for states [53]–[55], these measures have found widespread applications to the quantitative analysis of operational tasks, most notably subchannel discrimination [56]–[58]. The standard robustness of a bipartite quantum channel with respect to the separable channels is defined as

\[ R_s(\mathcal{N}_{AB}) := \min \left\{ s \geq 0 : \frac{\mathcal{N}_{AB} + s\mathcal{M}_{AB}}{1 + s} \in \text{SepC}(A : B), \mathcal{M}_{AB} \in \text{SepC}(A : B) \right\}, \quad (2) \]

while the generalized robustness of a bipartite quantum channel with respect to the separable channels is defined as

\[ R(\mathcal{N}_{AB}) := \min \left\{ s \geq 0 : \frac{\mathcal{N}_{AB} + s\mathcal{M}_{AB}}{1 + s} \in \text{SepC}(A : B), \mathcal{M}_{AB} \in \text{CPTP}(AB) \right\}. \quad (3) \]

The standard log-robustness and the generalized log-robustness of channels with respect to the separable channels are given by

\[ LR_s(\mathcal{N}_{AB}) := \log \left\{ 1 + R_s(\mathcal{N}_{AB}) \right\}, \quad LR(\mathcal{N}_{AB}) := \log \left\{ 1 + R(\mathcal{N}_{AB}) \right\}, \quad (4) \]

respectively, where the logarithm uses base 2. For \( \varepsilon \geq 0 \), the smooth versions of the above quantities are defined as

\[ LR_s^\varepsilon(\mathcal{N}_{AB}) = \min_{\mathcal{N}'_{AB} \approx_{\varepsilon} \mathcal{N}_{AB}} LR_s(\mathcal{N}'_{AB}), \quad LR^\varepsilon(\mathcal{N}_{AB}) = \min_{\mathcal{N}'_{AB} \approx_{\varepsilon} \mathcal{N}_{AB}} LR(\mathcal{N}'_{AB}), \quad (5) \]

where \( \mathcal{N}'_{AB} \approx_{\varepsilon} \mathcal{N}_{AB} \) is a shorthand for \( \frac{1}{2} \left\| \mathcal{N}'_{AB} - \mathcal{N}_{AB} \right\|_\varepsilon \leq \varepsilon \). The generalized log-robustness of channels has been found to have an operational meaning in the context of resource erasure [55]; It also can be expressed with the max-relative entropy of channels minimized over the set of the separable channels:

\[ LR^\varepsilon(\mathcal{N}_{AB}) = \min_{\mathcal{N}'_{AB} \approx_{\varepsilon} \mathcal{N}_{AB}} \min_{\mathcal{M}_{AB} \in \text{SepC}(A : B)} D_{\max} \left( \mathcal{N}'_{AB} \parallel \mathcal{M}_{AB} \right). \quad (6) \]
Both robustnesses are monotonically nonincreasing under the free superchannels:

**Lemma II.1.** For $\Theta_{AB \rightarrow A'B'} \in \text{SEPPSC}(AB \rightarrow A'B')$, it holds that

\[
R_s(\Theta_{AB \rightarrow A'B'}[N_{AB}]) \leq R_s(N_{AB}),
\]
\[
R(\Theta_{AB \rightarrow A'B'}[N_{AB}]) \leq R(N_{AB}).
\]

**Proof.** For $r = R_s(N_{AB})$, there exist separable channels $M_{AB}$ and $L_{AB}$ satisfying that

\[
N_{AB} + rM_{AB} = (1 + r)L_{AB}.
\]

For $\Theta_{AB \rightarrow A'B'} \in \text{SEPPSC}(AB \rightarrow A'B')$, we have that

\[
\Theta_{AB \rightarrow A'B'}[N_{AB}] + r\Theta_{AB \rightarrow A'B'}[M_{AB}] = (1 + r)\Theta_{AB \rightarrow A'B'}[L_{AB}],
\]

where $\Theta_{AB \rightarrow A'B'}[M_{AB}]$ and $\Theta_{AB \rightarrow A'B'}[L_{AB}]$ are separable channels. Hence, it follows that $R_s(\Theta_{AB \rightarrow A'B'}[N_{AB}]) \leq R_s(N_{AB})$. The analogous inequality for the generalized robustness follows along the same lines.

In order to calculate the above quantities for the dynamic entanglement resource, i.e., the $K$-swap channel, we review previous results on the robustness of bipartite channels, and especially of unitary bipartite channels:

**Theorem II.2** [59, Theorem 5]. Let $U_{A_0B_0}$ be a unitary bipartite operator whose operator Schmidt decomposition reads

\[
U_{A_0B_0} = \sum_{j} u_j A_j \otimes B_j,
\]

where $A_j A_j^\dagger = |A|_j$, $B_j B_j^\dagger = |B|_j$, and $\text{Tr} A_j^\dagger A_k = \text{Tr} B_j^\dagger B_k = \delta_{jk}$. Then its robustness is given by

\[
R_s(U_{AB}) = R(U_{AB}) = \left(\sum_{j} u_j^2\right) \frac{1}{|A||B|} - 1.
\]

The swap operator $F_{AB}^K$ acting on $K$-dimensional subsystems can be written as

\[
F_{AB}^K = \sum_{i=1}^{K^2} G_i \otimes G_i^\dagger
\]

for any orthonormal operator basis $\{G_i\}_{i=1}^{K^2}$ such that $\text{Tr} G_i^\dagger G_j = \delta_{ij}$ [60]. Using an orthonormal unitary basis, e.g., the discrete Weyl basis [8], the operator Schmidt decomposition of the swap gate is given by

\[
F_{AB}^K = \sum_{i=1}^{K^2} \frac{U_i}{\sqrt{K}} \otimes \frac{U_i^\dagger}{\sqrt{K}}
\]

Hence, the robustness of the $K$-swap channel is given as follows:

\[
R_s(F_{AB}^K) = R(F_{AB}^K) = K^2 - 1.
\]

The following well-known fact concerning separability of the isotropic states will be used afterwards:

**Theorem II.3** [61]. The isotropic state

\[
p\Phi_{A_0B_0}^K + (1 - p)\frac{I_{A_0B_0} - \Phi_{A_0B_0}^K}{K^2 - 1} \quad (0 \leq p \leq 1)
\]

is separable if and only if $p \leq \frac{1}{K}$. 

III. ONE-SHOT DYNAMIC ENTANGLEMENT COST OF A BIPARTITE QUANTUM CHANNEL

The first operational task we investigate consists in simulating a single instance of a known channel \( N_{AB} \) using a \( K \)-swap channel — with \( K \) as small as possible — together with free superchannels, as depicted in Fig. 3. One might call this task dynamic entanglement dilution, in analogy to the entanglement dilution task for quantum states. We can thus give the following formal definition.

**Definition III.1.** Given \( \varepsilon \geq 0 \), the one-shot dynamic entanglement cost of a bipartite quantum channel \( N_{AB} \) under SEPPSC is defined as follows:

\[
E^{(1),\varepsilon}_{C,\text{SEPPSC}}(N_{AB}) := \min \left\{ \log K^2 : \frac{1}{2} \| \Theta_{A'B'\rightarrow AB}[F_{A'B'}^K] - N_{AB} \|_\infty \leq \varepsilon, \right. \\
\left. \Theta_{A'B'\rightarrow AB} \in \text{SEPPSC}(A':B' \rightarrow A:B), \quad K \in \mathbb{N}_0 \right\}.
\]

We now present our first main result. It is a two-fold bound that connects the one-shot dynamic entanglement cost with the smooth standard log-robustness, thus providing an operational meaning of the latter quantity.

**Theorem III.1.** Given \( \varepsilon \geq 0 \), the one-shot dynamic entanglement cost of a bipartite quantum channel \( N_{AB} \) under SEPPSC is bounded as

\[
LR^\varepsilon_s(N_{AB}) \leq E^{(1),\varepsilon}_{C,\text{SEPPSC}}(N_{AB}) \leq LR^\varepsilon_s(N_{AB}) + 2. \tag{15}
\]

**Proof.** We break down the argument into separate proofs of the two bounds.

(i) For the lower bound, let \( \Theta_{A'B'\rightarrow AB} \in \text{SEPPSC}(A':B' \rightarrow A:B) \) be a superchannel that achieves \( E^{(1),\varepsilon}_{C,\text{SEPPSC}}(N_{AB}) \) with \( F_{A'B'}^K \), that is, \( \Theta_{A'B'\rightarrow AB}[F_{A'B'}^K] \approx_{\varepsilon} N_{AB} \). Then we have that

\[
LR^\varepsilon_s(N_{AB}) \leq LR_s(\Theta_{A'B'\rightarrow AB}[F_{A'B'}^K] ) \\
\leq LR_s(F_{A'B'}^K) = \log K^2 = E^{(1),\varepsilon}_{C,\text{SEPPSC}}(N_{AB}).
\]

(ii) For the upper bound, let \( N_{AB}' \) be a channel such that

\[
LR^\varepsilon_s(N_{AB}') = LR_s(N_{AB}') = \log(1 + r), \tag{16}
\]

where \( r = R_s(N_{AB}') \). There exists a separable channel \( M_{AB} \) such that

\[
N_{AB}' + \frac{rM_{AB}}{1 + r} \in \text{SepC}(A:B). \tag{17}
\]

Let \( J^\varepsilon_{AB} := \id_{A_0B_0} \otimes \varepsilon_{AB} \otimes (\Phi^K_{A_0A_0} \otimes \Phi^K_{B_0B_0}) \) be the (normalized) Choi matrix for a quantum channel \( \varepsilon_{AB} \), where \( |\Phi^K\rangle_{A_0B_0} = \frac{1}{\sqrt{K}} \sum_{i=0}^{K-1} |ii\rangle_{A_0B_0} \) is the maximally entangled state. Setting \( K = \lceil \sqrt{1 + r} \rceil \), we construct a SEPPSC \( \Theta_{A'B'\rightarrow AB} \) that simulates \( N_{AB}' \), that is, \( \Theta_{A'B'\rightarrow AB}[F_{A'B'}^K] = N_{AB}' \) as follows:

\[
\Theta_{A'B'\rightarrow AB}[\varepsilon_{A'B'}] = \Tr \left( J^F_{A_0B_0A_0B_0} J^\varepsilon_{A_0B_0A_0B_0} \right) N_{AB}' \\
+ \Tr \left( \left( I_{A_0B_0A_0B_0} - J^F_{A_0B_0A_0B_0} \right) J^\varepsilon_{A_0B_0A_0B_0} \right) M_{AB}.
\]

\^The discrete Weyl basis is composed by the \( d^2 \) unitary operators \( U_{kl} = \sum_{s=0}^{d-1} e^{\frac{2\pi i ls}{d}} |k+s\rangle \langle s| \), where \( k,l = 0,1,\ldots,d-1 \).
While $\Theta_{A'B'\to AB}[F_{A'B'}^{K}] = N_{AB}$ is apparent from the trace terms, we show that $\Theta_{A'B'\to AB}$ is a SEPPSC. Note that the Choi matrix of a separable channel $E_{A'B'}$ is a separable state and $J_{A_{0}B_{0}A_{1}B_{1}}^{E_{A'B'}} = \Phi_{A_{0}B_{0}}^{K} \otimes \Phi_{A_{1}B_{1}}^{K}$, which leads to

$$
\text{Tr} \left( \frac{J_{A_{0}B_{0}A_{1}B_{1}}^{E_{A'B'}} \circ J_{A_{0}B_{0}A_{1}B_{1}}^{E_{A'B'}}}{L} \right) \leq \frac{1}{K^2}
$$

for any $E_{A'B'} \in \text{SepC}(A' : B')$. Therefore, when $E_{A'B'} \in \text{SepC}(A : B)$, we have that

$$
\Theta_{A'B'\to AB}[E_{A'B'}] = \text{Tr} \left( \frac{J_{A_{0}B_{0}A_{1}B_{1}}^{E_{A'B'}} \circ J_{A_{0}B_{0}A_{1}B_{1}}^{E_{A'B'}}}{L} N_{AB} \right) + \text{Tr} \left( \left( I_{A_{0}B_{0}A_{1}B_{1}} - J_{A_{0}B_{0}A_{1}B_{1}}^{E_{A'B'}} \right) J_{A_{0}B_{0}A_{1}B_{1}}^{E_{A'B'}} \right) M_{AB}
$$

$$
= q' N_{AB} + (1 - q') M_{AB}
$$

$$
= q \left( \frac{N_{AB} + r M_{AB}}{1 + r} \right) + (1 - q) M_{AB} \in \text{SepC}(A : B),
$$

where $q = q'(1 + r) \leq 1$ due to $q' \leq \frac{1}{\sqrt{1 + r}}$. We conclude that

$$
E_{C,SEPPS}^{(1),\varepsilon}(N_{AB}) \leq \log K^2
$$

$$
= 2 \log \left( \sqrt{1 + r} \right)
$$

$$
\leq 2 \log (2\sqrt{1 + r})
$$

$$
= \log (1 + r) + 2
$$

$$
\leq LR_{\varepsilon}(N_{AB}) + 2,
$$

where in the third line we observed that $[x] \leq 2x$ for all $x \geq 1$. This concludes the proof.

\[\square\]

IV. ONE-SHOT DISTILLABLE DYNAMIC ENTANGLEMENT OF A BIPARTITE QUANTUM CHANNEL

The converse task to dynamic entanglement dilution is dynamic entanglement distillation. In our setting, this can be thought of as the task of simulating a $K$-swap channel — with $K$ as large as possible — using a noisy channel as a dynamic entanglement resource together with free superchannels. We give a pictorial representation of the process in Fig. 4. We can capture this notion through the following formal definition.

**Definition IV.1.** Given $\varepsilon \geq 0$, the one-shot distillable dynamic entanglement of a bipartite quantum channel $N_{AB}$ under SEPPSC is defined as

$$
E_{D,SEPPS}^{(1),\varepsilon}(N_{AB}) := \max \left\{ \log K^2 : \frac{1}{2} \left\| \Theta_{A'B'\to AB} [N_{AB}] - F_{A'B'}^{K} \right\| \leq \varepsilon, \right. \left. \Theta_{A'B'\to AB} \in \text{SEPPS}(A : B' \to A' : B'), K \in \mathbb{N}_0 \right\} .
$$

We propose to bound the above operational quantity with a measure that is inspired by the one-shot distillable entanglement of a quantum state [62]. It is obtained from the hypothesis-testing relative entropy of channels by means of an additional minimization over the set of separable channels [63], [64]:

\[\text{This follows directly from a well-known result that is reported in the Appendix as Proposition A.2.}\]
Definition IV.2. Given $\varepsilon \geq 0$, we define the hypothesis-testing relative entropy of dynamic entanglement of a bipartite quantum channel $\mathcal{N}_{AB}$ by

$$E^e_H(\mathcal{N}_{AB}) := \max_{\Psi_{A_0}R_0} \sup_{0 \leq Q_{A_1}R_0, N_{AB} \otimes \text{id}_{R_0}(\Psi_{A_0}R_0, R_0)} \text{Tr}\left\{ Q_{A_1}R_0, N_{AB} \otimes \text{id}_{R_0}(\Psi_{A_0}R_0, R_0) \right\} \geq 1 - \varepsilon.$$  

We remark that the above quantity is monotonic in $\varepsilon$ from the definition, implying in particular that $E^e_H(\mathcal{N}_{AB}) \geq E^{e/2}_H(\mathcal{N}_{AB})$. Moreover, the hypothesis-testing relative entropy of dynamic entanglement does not increase under SEPPSC:

**Proposition IV.1.** For a bipartite quantum channel $\mathcal{N}_{AB}$, and $\varepsilon \geq 0$, it holds that

$$E^e_H(\Theta_{AB \to A'B'}[\mathcal{N}_{AB}]) \leq E^e_H(\mathcal{N}_{AB}) \quad \forall \Theta_{AB \to A'B'} \in \text{SEPPSC}(AB \to A'B').$$  

**Proof.** Let $\Psi_{A_0}B_0 \otimes R_0$ and $Q_{A_1}B_1 \otimes R_0$ be optimal arguments of $E^e_H(\Theta_{AB \to A'B'}[\mathcal{N}_{AB}])$, so that

$$E^e_H(\Theta_{AB \to A'B'}[\mathcal{N}_{AB}]) = \min_{\mathcal{M}_{AB} \in \text{SepC}(A'B')} \left\{ -\log \text{Tr} Q_{A_1}B_1 \otimes \mathcal{M}_{AB} \otimes \text{id}_{R_0}(\Psi_{A_0}B_0 \otimes R_0) \right\},$$

where $0 \leq Q_{A_1}B_1 \otimes R_0 \leq I_{A_1}B_1 \otimes R_0$ and

$$\text{Tr}\left\{ Q_{A_1}B_1 \otimes \Theta_{AB \to A'B'}[\mathcal{N}_{AB}] \otimes \text{id}_{R_0}(\Psi_{A_0}B_0 \otimes R_0) \right\} \geq 1 - \varepsilon.$$  

Then using the structure of the superchannel $\Theta_{AB \to A'B'}[\mathcal{E}_{AB}] = U_{A_1}E_0 \rightarrow A_1'B_1 \circ \mathcal{E}_{AB} \circ W_{A_0}B_0 \rightarrow A_0B_0E_0$, with isometries $U_{A_1}B_1 \rightarrow A_1B_1E_0$ and $W_{A_0}B_0 \rightarrow A_0B_0E_0$, we observe that

$$E^e_H(\Theta_{AB \to A'B'}[\mathcal{N}_{AB}]) = \min_{\mathcal{M}_{AB} \in \text{SepC}(A'B')} \left\{ -\log \text{Tr} Q_{A_1}B_1 \otimes \mathcal{M}_{AB} \otimes \text{id}_{E_0R_0}(\Psi_{A_0}B_0 \otimes R_0) \right\} \leq \min_{\mathcal{M}_{AB} \in \text{SepC}(A'B')} \left\{ -\log \text{Tr} Q_{A_1}B_1 \otimes \Theta_{AB \to A'B'}[\mathcal{M}_{AB}] \otimes \text{id}_{E_0R_0}(\Psi_{A_0}B_0 \otimes R_0) \right\} \leq \max_{\Psi_{A_0}R_0} \sup_{0 \leq Q_{A_1}B_1 \otimes R_0 \leq I_{A_1}B_1 \otimes R_0} \min_{\mathcal{M}_{AB} \in \text{SepC}(A'B')} \text{Tr}\left\{ Q_{A_1}B_1 \otimes \mathcal{M}_{AB} \otimes \text{id}_{E_0R_0}(\Psi_{A_0}B_0 \otimes R_0) \right\} \geq 1 - \varepsilon,$$

where $\bar{Q}_{A_1}B_1 \otimes R_0 = U_{A_1}B_1 \rightarrow A_1B_1E_0(\bar{Q}_{A_1}B_1 \otimes R_0)$ and $\bar{\Psi}_{A_0}B_0 \otimes R_0 = W_{A_0}B_0 \rightarrow A_0B_0E_0(\bar{\Psi}_{A_0}B_0 \otimes R_0)$. The last inequality holds since $0 \leq \bar{Q}_{A_1}B_1 \otimes R_0 \leq I_{A_1}B_1 \otimes R_0$ and

$$\text{Tr}\left\{ \bar{Q}_{A_1}B_1 \otimes \mathcal{N}_{AB} \otimes \text{id}_{E_0R_0}(\bar{\Psi}_{A_0}B_0 \otimes R_0) \right\} \geq 1 - \varepsilon.$$  

This completes the proof. \qed

Our second main result connects the two notions identified in Definitions IV.1 and IV.2:

**Theorem IV.2.** Given $\varepsilon \geq 0$ and a bipartite quantum channel $\mathcal{N}_{AB}$, if $|E^e_H(\mathcal{N}_{AB})|$ is even, the one-shot distillable dynamic entanglement from a bipartite quantum channel $\mathcal{N}_{AB}$ under SEPPSC is bounded as

$$|E^e_H(\mathcal{N}_{AB})| \leq E^{(1)}_{D, \text{SEPPSC}}(\mathcal{N}_{AB}) \leq E^{2e}_H(\mathcal{N}_{AB}).$$

(22)
If $|E_H^\varepsilon(N_{AB})|$ is odd, then we have instead that
\[ E_H^\varepsilon(N_{AB}) - 1 \leq E_{D,SEPPSC}^{(1),\varepsilon}(N_{AB}) \leq E_H^{2\varepsilon}(N_{AB}). \]

**Proof.** We break down the argument into separate proofs of the two inequalities.

(i) For the upper bound, let $\Theta_{AB \rightarrow A'B'}$ be an optimal SEPPSC such that $\Theta_{AB \rightarrow A'B'}[N_{AB}] \approx \varepsilon \mathcal{F}_{A'B'}^K$, with $E_{D,SEPPSC}^{(1),\varepsilon}(N_{AB}) = \log K^2$. From the above two propositions we have that
\[
E_H^{2\varepsilon}(N_{AB}) \geq E_H^{2\varepsilon}(\Theta_{AB \rightarrow A'B'}[N_{AB}])
\]
\[
= \max_{\Psi_{A_0'B_0'}R_0} \sup_{0 \leq \mathcal{M}_{A_0'B_0'} \in \text{SepC}(A'B')} \left\{ - \log \text{Tr} \left( \mathcal{M}_{A_0'B_0'} \otimes \text{id}_{R_0}(\Psi_{A_0'B_0'}R_0) \right) \right\}
\]
\[
> \min_{\mathcal{M}_{A_0'B_0'} \in \text{SepC}(A'B')} \left\{ - \log \text{Tr} \left( \mathcal{F}_{A'B'}^K(\Psi_{A_0'B_0'}R_0) \right) \right\}
\]
\[
= \log K^2
\]
\[
= E_{D,SEPPSC}^{(1),\varepsilon}(N_{AB}),
\]
where the second inequality has been derived by making the ansatz $\Psi_{A_0'B_0'}R_0 = \Phi_{A_0}^{K} \otimes \Phi_{B_0}^{K}$, and the fourth line follows from Proposition A.2. That this is a valid choice is confirmed by the fact that
\[
\frac{1}{2} \left\| \Theta_{AB \rightarrow A'B'}[N_{AB}](\Psi_{A_0'B_0'}R_0) - \mathcal{F}_{A'B'}^K(\Psi_{A_0'B_0'}R_0) \right\|_1 \leq \frac{1}{2} \left\| \Theta_{AB \rightarrow A'B'}[N_{AB}] - \mathcal{F}_{A'B'}^K \right\|_1 \leq \varepsilon
\]
for any (pure) state $\Psi_{A_0'B_0'}R_0$, in turn implying that\[ F \left( \Theta_{AB \rightarrow A'B'}[N_{AB}](\Psi_{A_0'B_0'}R_0), \mathcal{F}_{A'B'}^K(\Psi_{A_0'B_0'}R_0) \right) \geq \left( 1 - \frac{1}{2} \left\| \Theta_{AB \rightarrow A'B'}[N_{AB}](\Psi_{A_0'B_0'}R_0) - \mathcal{F}_{A'B'}^K(\Psi_{A_0'B_0'}R_0) \right\|_1 \right)^2 \geq (1 - \varepsilon)^2 \geq 1 - 2\varepsilon.
\]

(ii) For the lower bound, let $\Psi_{A_0B_0}R_0$ and $Q_{A_1B_1}^*$ be optimal arguments of $E_H^\varepsilon(N_{AB})$, which satisfy that $E_H^\varepsilon(N_{AB}) = \max_{\mathcal{M}_{A_0B_0} \in \text{SepC}(A:B)} \left\{ \text{Tr} \left( Q_{A_1B_1}^* \cdot \mathcal{M}_{A_0B_0} \otimes \text{id}_{R_0}(\Psi_{A_0B_0}R_0) \right) \right\}$. Setting $K = 2^{\frac{1}{2}E_H^\varepsilon(N_{AB})}$ for $E_H^\varepsilon(N_{AB})$ even, and $K = 2^{\frac{1}{4}E_H^\varepsilon(N_{AB})}$ otherwise, we can construct a SEPPSC $\Theta_{AB \rightarrow A'B'}$ as follows:
\[
\Theta_{AB \rightarrow A'B'}[E_{AB}] := \text{Tr} \left\{ Q_{A_1B_1}^* \cdot \mathcal{E}_{AB}(\Psi_{A_0B_0}R_0) \right\} \mathcal{F}_{A'B'}^K + \text{Tr} \left\{ (I_{A_1B_1} - Q_{A_1B_1}^*) \mathcal{E}_{AB}(\Psi_{A_0B_0}R_0) \right\} \mathcal{G}_{A'B'}^K,
\]
where $\mathcal{G}_{A'B'}^K$ is the quantum channel corresponding to the following (normalized) Choi operator:
\[
\mathcal{G}_{A'B'}^K = \frac{I_{A_0B_0A_1B_1} - \mathcal{F}_{A_0B_0A_1B_1}}{K^4 - 1} - \frac{I_{A_0B_0A_1B_1} - \Phi_{A_0B_0A_1B_1}^K \otimes \Phi_{A_0B_0A_1B_1}^K}{K^4 - 1} \in \text{SepD}(A_0' : B_0'),
\]

*Here we are making use of the Fuchs–van de Graaf inequalities [63]. They establish the relations $1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \left\| \rho - \sigma \right\|_1 \leq \sqrt{1 - F(\rho, \sigma)}$ between trace distance and quantum fidelity.*
which implies that \( G_{AB}^{K} \in \text{SepC}(A' : B') \). For \( M_{AB} \in \text{SepC}(A : B) \), we observe that

\[
\Theta_{AB\rightarrow A'B'}[M_{AB}] := \text{Tr} \left\{ Q_{A'B'}^{*}M_{AB}(\Psi_{A'B'}^{*}) \right\} F_{AB}^{K} + \text{Tr} \left\{ (I_{A'B'} - Q_{A'B'}^{*}) M_{AB}(\Psi_{A'B'}^{*}) \right\} G_{A'B'}^{K} = qF_{A'B'}^{K} + (1 - q)G_{A'B'}^{K}
\]

because of \( q = \text{Tr} \left\{ Q_{A'B'}^{*}M_{AB}(\Psi_{A'B'}^{*}) \right\} \leq \frac{1}{K} \) and Theorem 17 regarding the Choi matrix. Denoting

\[
q^{*} = \text{Tr} \left\{ Q_{A'B'}^{*}M_{AB}(\Psi_{A'B'}^{*}) \right\} \geq 1 - \varepsilon,
\]

we have that

\[
\frac{1}{2} \|\Theta_{AB\rightarrow A'B'}[\mathcal{N}_{AB}] - F_{A'B'}^{K}\|_{o} = \frac{1}{2} \|q^{*}F_{A'B'}^{K} + (1 - q^{*})G_{A'B'}^{K} - F_{A'B'}^{K}\|_{o} \\
\leq \frac{1}{2} \|(1 - q^{*})F_{A'B'}^{K}\|_{o} + \frac{1}{2} \|(1 - q^{*})G_{A'B'}^{K}\|_{o} = 1 - q^{*} \leq \varepsilon,
\]

where we used that \( \|E\|_{o} = 1 \) for \( E \in \text{CPTP}(A'B') \). Therefore, we conclude that, for \( [E_{H}^{\varepsilon}(\mathcal{N}_{AB})] \) even,

\[
E_{D,\text{SEPPSC}}^{(1),\varepsilon}(\mathcal{N}_{AB}) \geq \log K^{2} = [E_{H}^{\varepsilon}(\mathcal{N}_{AB})].
\]

When \( [E_{H}^{\varepsilon}(\mathcal{N}_{AB})] \) is odd, noticing that it holds that \( \left\lfloor \frac{q}{p} \right\rfloor \geq \frac{q + 1}{p} - 1 \) for integers \( q, p \in \mathbb{N} \), we obtain that

\[
E_{D,\text{SEPPSC}}^{(1),\varepsilon}(\mathcal{N}_{AB}) \geq \log K^{2} = 2 \left\lfloor \frac{1}{2} E_{H}^{\varepsilon}(\mathcal{N}_{AB}) \right\rfloor \\
\geq 2 \left( \frac{E_{H}^{\varepsilon}(\mathcal{N}_{AB}) + 1}{2} - 1 \right) = E_{H}^{\varepsilon}(\mathcal{N}_{AB}) - 1.
\]

This concludes the proof.

\[\square\]

V. ONE-SHOT CATALYTIC DYNAMIC ENTANGLEMENT COST OF A BIPARTITE QUANTUM CHANNEL

The third operational task we consider is a variation on the theme of dynamic entanglement cost. We push this notion further by introducing two tweaks: (i) we allow an additional dynamic entanglement resource that could be used as a catalyst while simulating a bipartite channel, with the stipulation that the catalyst channel be returned intact after the task; and (ii) we introduce a class of superchannels that might generate a small amount of dynamic entanglement when acting on separable channels.

**Definition V.1.** For \( \delta \geq 0 \), a superchannel \( \Theta_{AB\rightarrow A'B'} \) is called \( \delta \)-separability-preserving superchannel (\( \delta \)-SEPPSC) if

\[
R(\Theta_{AB\rightarrow A'B'}[M_{AB}]) \leq \delta \quad \forall M_{AB} \in \text{SepC}(A : B),
\]

where \( R(\mathcal{N}_{AB}) = \min \{ s \geq 0 : \mathcal{N}_{AB} \leq (1 + s)M_{AB}, M_{AB} \in \text{SepC}(A : B) \} \) is the generalized robustness with respect to the separable channels.

The choice of the generalized robustness to quantify the maximum amount of entanglement generation allowed in the above definition may seem rather arbitrary. A compelling reason why this is in fact a reasonable and natural choice comes from the study of entanglement theory for states. Indeed, it is known that a condition analogous to (26) leads to a universally reversible theory of entanglement manipulation [48]. The role of the generalized robustness in this context is quite unique, in the sense that using alternative measures — such as the trace norm distance from the set of separable states — is known to trivialize the problem [48, Section V]. In light of this, Definition V.1 identifies a good candidate for a useful enlargement of the set of free superchannels.
As expected, the generalized log-robustness and its smooth version might increase under a \( \delta \)-separability-preserving superchannel as the following results show:

**Proposition V.1.** Let \( \Theta_{AB \to A'B'} \) be a \( \delta \)-SEPPSC. For any bipartite quantum channel \( \mathcal{N}_{AB} \), the following holds:

\[
LR(\Theta_{AB \to A'B'}[\mathcal{N}_{AB}]) \leq LR(\mathcal{N}_{AB}) + \log(1 + \delta).
\]

**Proof.** Let \( r \equiv R(\mathcal{N}_{AB}) \) such that

\[
\mathcal{N}_{AB} + r\mathcal{E}_{AB} = (1 + r)\mathcal{M}_{AB},
\]

where \( \mathcal{M}_{AB} \in \text{SepC}(A : B) \). It follows that

\[
\Theta_{AB \to A'B'}[\mathcal{N}_{AB}] + r\Theta_{AB \to A'B'}[\mathcal{E}_{AB}] = (1 + r)\Theta_{AB \to A'B'}[\mathcal{M}_{AB}].
\]

Also, we have that

\[
\Theta_{AB \to A'B'}[\mathcal{M}_{AB}] + r'\mathcal{G}_{A'B'} = (1 + r')\mathcal{M}_{A'B'},
\]

where \( r' \equiv R(\Theta_{AB \to A'B'}[\mathcal{M}_{AB}]) \leq \delta \), and \( \mathcal{M}_{A'B'} \in \text{SepC}(A' : B') \). From these two equations, it follows that

\[
\Theta_{AB \to A'B'}[\mathcal{N}_{AB}] + r\Theta_{AB \to A'B'}[\mathcal{E}_{AB}] + (1 + r)r'\mathcal{G}_{A'B'} = (1 + r)(1 + r')\mathcal{M}_{A'B'},
\]

which implies that \( 1 + R(\Theta_{AB \to A'B'}[\mathcal{N}_{AB}]) \leq (1 + r)(1 + r') \).

**Lemma V.2.** For \( \Theta_{AB \to A'B'} \in \delta \text{-SEPPSC}(A : B \to A' : B') \), we have that

\[
LR^e(\Theta_{AB \to A'B'}[\mathcal{N}_{AB}]) \leq LR^e(\mathcal{N}_{AB}) + \log(1 + \delta).
\]

**Proof.** Let \( \mathcal{N}_{AB}^\gamma \) be a quantum channel satisfying that \( LR^e(\mathcal{N}_{AB}^\gamma) = LR(\mathcal{N}_{AB}^\gamma) \). We have that

\[
LR^e(\Theta_{AB \to A'B'}[\mathcal{N}_{AB}]) \leq LR(\Theta_{AB \to A'B'}[\mathcal{N}_{AB}^\gamma]) \\
\leq LR(\mathcal{N}_{AB}^\gamma) + \log(1 + \delta) \\
= LR^e(\mathcal{N}_{AB}) + \log(1 + \delta),
\]

concluding the proof.

With these tools, we give the formal definition of the one-shot catalytic dynamic entanglement cost of a bipartite channel as follows:

**Definition V.2.** Given \( \delta > 0 \) and \( \varepsilon \geq 0 \), the one-shot catalytic dynamic entanglement cost of a bipartite quantum channel \( \mathcal{N}_{AB} \) under \( \delta \)-SEPPSC is defined as

\[
\bar{E}^{(1),\varepsilon}_{C,\delta \text{-SEPPSC}}(\mathcal{N}_{AB}) := \min \left\{ \log K^2 : \Theta_{A'B'CD \to ABCD}[\mathcal{F}_{A'B'}^K \otimes \mathcal{F}_{CD}^L] = \mathcal{N}_{AB}^{'\gamma} \otimes \mathcal{F}_{CD}^L, \right.
\]

\[
\Theta_{A'B'CD \to ABCD} \in \delta \text{-SEPPSC}(A' : B'D \to A'C : BD), \\
\left. \frac{1}{2} \left\| \mathcal{N}_{AB}^{'\gamma} - \mathcal{N}_{AB} \right\|_{\infty} \leq \varepsilon, K, L \in \mathbb{N}_0 \right\}.
\]

In order to bound the one-shot catalytic dynamic entanglement cost of a bipartite channel, the following lemma uses a twisted twirling superchannel that separates the \( K \)-swap channel from the others.
Lemma V.3. For a bipartite quantum channel $\mathcal{N}_{AB}$ and $\varepsilon \geq 0$, there is a quantum channel $\mathcal{M}_{ABCD}^\varepsilon$ given by
\begin{equation}
\mathcal{M}_{ABCD}^\varepsilon = p\mathcal{N}_{AB}^\varepsilon \otimes \mathcal{F}_{CD}^L + (1 - p)\mathcal{L}_{ABCD},
\end{equation}
where $\mathcal{L}_{ABCD}$ is a quantum channel, $\frac{1}{2}\|\mathcal{N}_{AB}^\varepsilon - \mathcal{N}_{AB}\|_\diamond \leq |A_0| |B_0| \sqrt{2\varepsilon}$, and $p \geq 1 - 2\varepsilon$. It also satisfies that
\begin{equation}
LR(\mathcal{M}_{ABCD}^\varepsilon) \leq LR^\varepsilon(\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L).
\end{equation}

Proof. Let $\widetilde{\mathcal{M}}_{ABCD}^\varepsilon$ be a quantum channel satisfying
\begin{equation}
LR^\varepsilon(\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L) = LR(\widetilde{\mathcal{M}}_{ABCD}^\varepsilon) \equiv I,
\end{equation}
which implies the existence of a separable channel $\Sigma_{ABCD} \in \text{SepC}(AC : BD)$ such that
\begin{equation}
\widetilde{\mathcal{M}}_{ABCD}^\varepsilon \leq 2^\varepsilon \Sigma_{ABCD}.
\end{equation}
Since $\widetilde{\mathcal{M}}_{ABCD}^\varepsilon$ is $\varepsilon$-close to $\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L$ by definition, we expect to have $\mathcal{M}_{ABCD}^\varepsilon$ by properly pinching it. We try the following twisted twirling superchannel, which can be performed via LOCC:
\begin{equation*}
\Omega_{AB}[\mathcal{E}_{AB}] := \int \mathcal{U}_{A_0} \otimes \mathcal{V}_{B_1} \circ \mathcal{E}_{AB} \circ \mathcal{V}_{B_1}^\dagger \otimes \mathcal{U}_{A_0}^\dagger.
\end{equation*}

For a quantum channel $\mathcal{E}_{AB}$, the twisted twirling superchannel turns its (normalized) Choi matrix into a structured form:
\begin{equation}
J_{\mathcal{E}_{AB}}^{[\mathcal{E}_{AB}]} = \int \mathcal{V}_{A_0} \otimes \mathcal{U}_{B_0} \otimes \mathcal{U}_{A_1} \otimes \mathcal{V}_{B_1} \left( J_{\mathcal{E}_{AB}}^{[\mathcal{E}_{AB}]} \right)
\end{equation}
\begin{equation*}
= p_0 \Phi^K_{A_0 B_1} \otimes \Phi^K_{A_1 B_1} + p_1 \Phi^K_{A_0 B_1} \otimes \frac{I - \phi^K_{A_0 B_1}}{K^2 - 1}
\end{equation*}
\begin{equation*}
+ p_2 \frac{I - \phi^K_{A_0 B_1}}{K^2 - 1} \otimes \Phi^K_{A_1 B_0} + p_3 \Phi^K_{A_0 B_0} \otimes \frac{I - \phi^K_{A_1 B_1}}{K^2 - 1}
\end{equation*}
\begin{equation*}
= p_0 J_{\mathcal{E}_{AB}}^{[\mathcal{E}_{AB}]} + (1 - p_0) J_{\mathcal{E}_{AB}}^{[\mathcal{E}_{AB}]}.
\end{equation*}

Note that $\text{Tr}(J_{\mathcal{E}_{AB}}^{[\mathcal{E}_{AB}]} - J_{\mathcal{E}_{AB}}^{[\mathcal{E}_{AB}]}) = 0$. Applying the twisted twirling superchannel $\Omega_{CD}$ on $\widetilde{\mathcal{M}}_{ABCD}^\varepsilon$, we devise $\mathcal{M}_{ABCD}^\varepsilon$ by as follows:
\begin{equation}
\mathcal{M}_{ABCD}^\varepsilon = \Omega_{CD} \left[ \widetilde{\mathcal{M}}_{ABCD}^\varepsilon \right]
\end{equation}
\begin{equation*}
= p\mathcal{N}_{AB}^\varepsilon \otimes \mathcal{F}_{CD}^L + (1 - p)\mathcal{L}_{ABCD}.
\end{equation*}

We show that $\mathcal{M}_{ABCD}^\varepsilon$ satisfies the insisted properties. Firstly, by construction,
\begin{equation}
\mathcal{M}_{ABCD}^\varepsilon = \Omega_{CD} \left[ \widetilde{\mathcal{M}}_{ABCD}^\varepsilon \right] \leq 2^\varepsilon \Omega_{CD} [\Sigma_{ABCD}].
\end{equation}

Since $\Omega_{CD}$ can be done by LOCC, we have $\Omega_{CD} [\Sigma_{ABCD}] \in \text{SepC}(AC : BD)$. Therefore, we have
\begin{equation}
LR(\mathcal{M}_{ABCD}^\varepsilon) \leq LR^\varepsilon(\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L).
\end{equation}

From the contractivity of the diamond distance under a superchannel, it follows that
\begin{equation}
\varepsilon \geq \frac{1}{2}\|\mathcal{M}_{ABCD}^\varepsilon - \mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L\|_\diamond
\end{equation}
\begin{equation*}
\geq \frac{1}{2}\|\Omega_{CD} \left[ \widetilde{\mathcal{M}}_{ABCD}^\varepsilon \right] - \mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L\|_\diamond,
\end{equation*}
where we used $\Omega_{CD}[\mathcal{F}_{CD}^L] = \mathcal{F}_{CD}^L$. Using Theorem [A.4] we get to
\begin{equation}
1 - 2\varepsilon \leq F \left( J_{\Omega_{CD} \left[ \widetilde{\mathcal{M}}_{ABCD}^\varepsilon \right]}, J_{\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L} \right)
\end{equation}
\begin{equation*}
= F \left( pJ_{\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L} + (1 - p)J_{\mathcal{L}_{ABCD}}, J_{\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L} \right)
\end{equation*}
\begin{equation*}
\leq pF \left( J_{\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L}, J_{\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L} \right) + (1 - p)F \left( J_{\mathcal{L}_{ABCD}}, J_{\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L} \right)
\end{equation*}
\begin{equation*}
\leq pF \left( J_{\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L}, J_{\mathcal{N}_{AB} \otimes \mathcal{F}_{CD}^L} \right),
\end{equation*}
\begin{equation*}
= pF \left( J_{\mathcal{N}_{AB}}, J_{\mathcal{N}_{AB}} \right),
\end{equation*}
where the second inequality follows from the joint concavity of the fidelity, and the third from the orthogonality of the Choi matrices. From the above, we read that $p \geq 1 - 2\varepsilon$ and $F \left( J^{\epsilon_{AB}}_{\delta}, J^{\epsilon_{AB}}_{\delta} \right) \geq 1 - 2\varepsilon$ due to $p \leq 1$ and $F \left( J^{\epsilon_{AB}}_{\delta}, J^{\epsilon_{AB}}_{\delta} \right) \leq 1$. Furthermore, $F \left( J^{\epsilon_{AB}}_{\delta}, J^{\epsilon_{AB}}_{\delta} \right) \geq 1 - 2\varepsilon$ together with Theorem A.3 and the Fuchs-van der Graaf inequality implies the following:

$$
\frac{1}{2} \| N_{AB}^{\epsilon} - N_{AB} \|_\infty \leq |A_0| \| B_0 | \frac{1}{2} \| J^{\epsilon_{AB}}_{\delta} - J^{\epsilon_{AB}}_{\delta} \|_1 \\
\leq |A_0| \| B_0 | \sqrt{1 - F \left( J^{\epsilon_{AB}}_{\delta}, J^{\epsilon_{AB}}_{\delta} \right)} \\
\leq |A_0| \| B_0 | \sqrt{2\varepsilon}.
$$

This completes the proof. \square

We bound the one-shot catalytic dynamic entanglement cost of a bipartite channel as follows:

**Theorem V.4.** Given $\delta > 0$, $\varepsilon \geq 0$, there exists $L \in \mathbb{N}$ such that $L^2 \geq 1 + \frac{1}{\delta}$, and the one-shot catalytic dynamic entanglement cost for any bipartite quantum channel $N_{AB}$ is bounded as

$$LR^\varepsilon (N_{AB} \otimes F^L_{CD}) - \log L^2 - \log (1 + \delta)
\leq F^{(1)\varepsilon}_{\delta, SEPPSC}(N_{AB})
\leq LR^\varepsilon (N_{AB} \otimes F^L_{CD}) - \log L^2 - \log (1 - 2\varepsilon') + 2,$$

where $\varepsilon' = \varepsilon^2 / \left(2 \| A_0 \|^2 \| B_0 \|^2 \right)$. From the first and the second equation above, it follows that

$$LR^\varepsilon (M_{ABCD}^\varepsilon) \leq LR^\varepsilon (N_{AB} \otimes F^L_{CD}),
M_{ABCD}^\varepsilon = pN_{AB}^{\varepsilon} \otimes F^L_{CD} + (1 - p)\mathcal{L}_{ABCD},$$

$$\frac{1}{2} \| N_{AB}^{\varepsilon} - N_{AB} \|_\infty \leq \varepsilon,
\text{ where } \varepsilon' \geq 1 - 2\varepsilon',$$

where $\varepsilon' = \varepsilon^2 / \left(2 \| A_0 \|^2 \| B_0 \|^2 \right)$. From the first and the second equation above, it follows that

$$\mathcal{M}_{ABCD}^{\varepsilon'} = pN_{AB}^{\varepsilon} \otimes F^L_{CD} + (1 - p)\mathcal{L}_{ABCD} \leq 2LR^\varepsilon (N_{AB} \otimes F^L_{CD}) \Sigma_{ABCD},$$

where $\Sigma_{ABCD} \in \text{SepC}(A'C : BD)$. Since $\mathcal{L}_{ABCD} \geq 0$, we have that

$$N_{AB}^{\varepsilon} \otimes F^L_{CD} \leq 2LR^\varepsilon (N_{AB} \otimes F^L_{CD}) - \log p \Sigma_{ABCD}
\leq 2LR^\varepsilon (N_{AB} \otimes F^L_{CD}) - \log (1 - 2\varepsilon') \Sigma_{ABCD},$$

which leads to the existence of a quantum channel $R_{ABCD}$ such that

$$N_{AB}^{\varepsilon} \otimes F^L_{CD} + \left\{ 2LR^\varepsilon (N_{AB} \otimes F^L_{CD}) - \log (1 - 2\varepsilon') \right\} R_{ABCD}$$

$$= N_{AB}^{\varepsilon} \otimes F^L_{CD} + (r - 1)R_{ABCD}
\propto \text{SepC}(A'C : BD),$$

where we denote $r = 2LR^\varepsilon (N_{AB} \otimes F^L_{CD}) - \log (1 - 2\varepsilon')$. With the insight gained above, we construct a superchannel

$$\Theta_{A'B'CD \rightarrow ABCD} \in \delta\text{-SEPPSC}(A'C : B'D \rightarrow AC : BD)$$

as follows:

$$\Theta_{A'B'CD \rightarrow ABCD}[E_{A'B'CD}] := \text{Tr} \left( J^{F^K_{A'B'} \otimes F^L_{CD}} J^{E_{A'B'CD}} N_{AB}^{\varepsilon} \otimes F^L_{CD} \right) + \text{Tr} \left\{ \left( I - J^{F^K_{A'B'} \otimes F^L_{CD}} \right) J^{E_{A'B'CD}} \right\} R_{ABCD},$$

where the second inequality follows from the joint concavity of the fidelity, and the third from the orthogonality of the Choi matrices. From the above, we read that $p \geq 1 - 2\varepsilon$ and $F \left( J^{\epsilon_{AB}}_{\delta}, J^{\epsilon_{AB}}_{\delta} \right) \geq 1 - 2\varepsilon$ due to $p \leq 1$ and $F \left( J^{\epsilon_{AB}}_{\delta}, J^{\epsilon_{AB}}_{\delta} \right) \leq 1$. Furthermore, $F \left( J^{\epsilon_{AB}}_{\delta}, J^{\epsilon_{AB}}_{\delta} \right) \geq 1 - 2\varepsilon$ together with Theorem A.3 and the Fuchs-van der Graaf inequality implies the following:
Finally, regarding (i) and (ii), we can choose 
\[ L = \max \left\{ \tilde{L}_0, \sqrt{1 + \frac{1}{\delta}} \right\} \]
which provides both the upper and the lower bound on \( \tilde{E}_{C,\delta,\text{SEPPSC}}(N_{AB}) \) for any bipartite quantum channel \( N_{AB} \). This completes the proof. \( \square \)

One can feed any product state \( \rho_A \otimes \rho_B \) into the quantum channel and subsequently trace away some subsystems at the output.
VI. CONCLUSION

We found that entanglement of quantum channels can be naturally understood adopting the superchannel framework. Taking the separable channels as our free resource, we defined the separability-preserving superchannels as the resource non-generating superchannels. The $K$-swap channel $F_{AB}^{K}$ is chosen as the dynamic entanglement resource, mimicking the role of the $K$-maximally entangled state in the resource theory of static entanglement. In fact, these two objects are totally interchangeable, because a $K$-swap channel can be transformed into a pair of $K$-maximally entangled states under LOCC, and vice versa two $K$-maximally entangled states can be used to implement a $K$-swap channel with LOCC — more precisely, by performing two times a teleportation protocol. Our results provide an operational meaning to the standard and the generalized log-robustness of channels as well as the hypothesis-testing relative entropy of dynamic entanglement that we constructed from the hypothesis-testing relative entropy of channels with minimization over the set of separable channels: The one-shot dynamic entanglement cost can be bounded by the standard log-robustness of channels with respect to the separable channels. The one-shot distillable dynamic entanglement is bounded by the hypothesis-testing relative entropy of dynamic entanglement. When it comes to the catalytic scenario where additional dynamic entanglement resource is supplied and returned back after the free superchannel, we find that the one-shot catalytic dynamic entanglement cost can be bounded by the liberal dynamic entanglement cost of bipartite quantum channels: The one-shot dynamic entanglement cost can be bounded by the generalized log-robustness of channels as well as the hypothesis-testing relative entropy of dynamic entanglement — more precisely, by performing two times a teleportation protocol. Our results provide an operational meaning to the standard and the generalized log-robustness of channels with respect to the set of separable channels. Finally, in the appendices, we investigate the asymptotic scenario, using the liberal dynamic entanglement cost of a bipartite quantum channel, which features the liberal smoothing instead of the diamond norm smoothing. It is shown that the quantity is equal to the liberal regularized relative entropy of channels minimized over the separable channels.

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NOTE ADDED

During the completion of this manuscript, we became aware of two independent works on dynamic resource theories: B. Regula and R. Takagi [68] formulated one-shot manipulation of dynamic resources in a general setting, while X. Yuan, et al. [69] also investigated one-shot distillation and dilution of dynamic resources in a general setting.

APPENDIX

A. Liberal Dynamic Entanglement Cost of a Bipartite Channel

There are several alternative ways of smoothing in channel resource theories utilized in the study of the asymptotic equipartition properties [44]. For a quantum channel $N_{A}$ and a quantum state $\varphi_{A_{0}R_{0}}$, we denote the $\varepsilon$-diamond ball and the $\varepsilon$-liberal ball as

$$B_{\varepsilon}(N_{A}) := \left\{ N'_{A} \in \text{CPTP}(A) : \tfrac{1}{2} \left\| N'_{A} - N_{A} \right\|_{1} \leq \varepsilon \right\},$$

$$B_{\varepsilon}^{\varphi}(N_{A}) := \left\{ N'_{A} \in \text{CPTP}(A) : \tfrac{1}{2} \left\| N'_{A}(\varphi_{A_{0}R_{0}}) - N_{A}(\varphi_{A_{0}R_{0}}) \right\|_{1} \leq \varepsilon \right\}.$$ 

Observe that $B_{\varepsilon}(N_{A}) \subset \cap_{\varphi_{A_{0}R_{0}}} B_{\varepsilon}^{\varphi}(N_{A})$. For a set of free resource $\mathcal{F}$, the relevant liberal quantities are defined as follows:

$$LR_{\mathcal{F}}^{\varepsilon}(N_{A}) := \min_{N'_{A} \in B_{\varepsilon}^{\varphi}(N_{A})} LR_{\mathcal{F}}(N'_{A}),$$

$$LR_{\mathcal{F}}(N_{A}) := \max_{\varphi_{A_{0}R_{0}}} \min_{N'_{A} \in B_{\varepsilon}^{\varphi}(N_{A})} LR_{\mathcal{F}}(N'_{A}),$$

$$LR_{\mathcal{F}}^{\varepsilon,n}(N_{A}) := \frac{1}{n} \max_{\varphi_{A_{0}R_{0}}} LR_{\mathcal{F}}^{\varepsilon,\otimes n}(N_{A}^{\otimes n}),$$

$$LR_{\mathcal{F}}(\infty)(N_{A}) := \lim_{\varepsilon \to 0^{+}} \lim_{n \to \infty} LR_{\mathcal{F}}^{\varepsilon,n}(N_{A}).$$
The liberal regularized relative entropy of a channel $\mathcal{N}_A$ with respect to a free resource $\mathcal{F}$ is defined as

$$D_{LR}^{(\infty)}(\mathcal{N}_A) := \lim_{n \to \infty} \frac{1}{n} \max_{\varphi_{A_n,0_n}} D\left(\mathcal{N}_A^{\otimes n}(\varphi_{A_n,0_n}) \parallel \mathcal{M}_A^{\otimes n}(\varphi_{A_n,0_n})\right).$$

(46)

It is shown in [44] that the asymptotic equipartition property holds as

$$L R_{LR}^{(\infty)}(\mathcal{N}_A) = D_{LR}^{(\infty)}(\mathcal{N}_A).$$

(47)

**Definition A.1.** Given $\varepsilon \geq 0$, the $\varepsilon$-liberal one-shot dynamic entanglement cost of a bipartite quantum channel $\mathcal{N}_{AB}$ under SEPPSC is defined as

$$E_{C,SEPPSC}^{(1),\varepsilon}(\mathcal{N}_{AB}) := \max_{\varphi_{A_n,0_n}} E_{C,SEPPSC}^{(1),\varepsilon,\varphi}(\mathcal{N}_{AB}),$$

(48)

where

$$E_{C,SEPPSC}^{(1),\varepsilon,\varphi}(\mathcal{N}_{AB}) := \min_{N_{AB} \in B_r(\mathcal{N}_{AB})} E_{C,SEPPSC}^{(1),0}(N_{AB}).$$

(49)

The liberal (asymptotic) dynamic entanglement cost of a bipartite quantum channel $\mathcal{N}_{AB}$ under SEPPSC is defined as

$$E_{C,SEPPSC}^{(1)}(\mathcal{N}_{AB}) := \lim_{n \to \infty} \liminf_{\varepsilon \to 0^+} \frac{1}{n} \max_{\varphi_{A_n,0_n}} E_{C,SEPPSC}^{(1),\varepsilon,\varphi}(\mathcal{N}_{AB}).$$

(50)

While an operational meaning of the above quantity is missing yet, it is given by the liberal regularized relative entropy:

**Theorem A.1.** The liberal dynamic entanglement cost of a bipartite quantum channel $\mathcal{N}_{AB}$ is given by

$$E_{C,SEPPSC}^{(1)}(\mathcal{N}_{AB}) = D_{LR}^{(\infty)}(\mathcal{N}_{AB}).$$

(51)

**Proof.** From Theorem [III.1] for any $\varepsilon \geq 0$ and $\varphi$ it holds that

$$L R_{LR}^{(1),\varphi}(\mathcal{N}_{AB}) \leq E_{C,SEPPSC}^{(1),\varepsilon,\varphi}(\mathcal{N}_{AB}) \leq L R_{LR}^{(1),\varphi}(\mathcal{N}_{AB}) + 2.$$

(52)

The asymptotic equipartition property leads to the conclusion.

B. A Few Technical Results

**Proposition A.2.** Let $|\Phi^K_{A_0,B_0}\rangle = \frac{1}{\sqrt{K}} \sum_{i=0}^{K-1} |ii\rangle_{A_0,B_0}$ be a $K$-maximally entangled state where $|A_0| \equiv \text{dim} A_0 \geq K$ and $|B_0| \equiv \text{dim} B_0 \geq K$. We have that

$$\max_{\sigma_{A_0,B_0} \in \text{SepD}(A_0;B_0)} \text{Tr} \Phi^K_{A_0,B_0} \sigma_{A_0,B_0} = \frac{1}{K}.$$  

(53)

**Proof.** A separable state $\sigma_{A_0,B_0}$ can be written as a convex sum of pure product states $\sigma_{A_0,B_0} = \sum_{\alpha} p_{\alpha} \phi_{\alpha} \otimes \psi_{\alpha}$:

$$\text{Tr} \Phi^K_{A_0,B_0} \sigma_{A_0,B_0} = \frac{1}{K} \sum_{\alpha} p_{\alpha} \sum_{i,j=0}^{K-1} \langle i|\phi_{\alpha}\rangle \langle i|\psi_{\alpha}\rangle \langle \phi_{\alpha}|j\rangle \langle \psi_{\alpha}|j\rangle$$

$$= \frac{1}{K} \sum_{\alpha} p_{\alpha} \left\{ \sum_{i=0}^{K-1} \langle i|\phi_{\alpha}\rangle \langle i|\psi_{\alpha}\rangle \right\}^2$$

$$\leq \frac{1}{K} \sum_{\alpha} p_{\alpha} \left\{ \sum_{i=0}^{K-1} |\langle i|\phi_{\alpha}\rangle|^2 \right\} \left\{ \sum_{i=0}^{K-1} |\langle i|\psi_{\alpha}\rangle|^2 \right\}$$

$$\leq \frac{1}{K}$$

where the Cauchy-Schwarz inequality is used for the first inequality.

**Theorem A.3.** Let $\mathcal{N}_A$ and $\mathcal{M}_A$ be quantum channels, and $J^{\mathcal{N}_A}$ and $J^{\mathcal{M}_A}$ be their (normalized) Choi matrices, respectively. It holds that

$$\frac{1}{|A|} \|\mathcal{N}_A - \mathcal{M}_A\|_\infty \leq \|J^{\mathcal{N}_A} - J^{\mathcal{M}_A}\|_1 \leq \|\mathcal{N}_A - \mathcal{M}_A\|_\infty.$$

(54)
Proof. The second inequality follows from the definition of the diamond norm. For the first inequality, let $\Psi_{A_0 R_0}$ be the optimal pure state for the diamond distance as

$$
\|N_A - M_A\|_\diamond = \|N_A \otimes \text{id}_{R_0} (\Psi_{A_0 R_0}) - M_A \otimes \text{id}_{R_0} (\Psi_{A_0 R_0})\|_1,
$$

where $|A_0| = |R_0|$. One can denote $|\Psi\rangle_{A_0 R_0} = I_{A_0} \otimes X_{R_0} (\phi^+_{A_0 R_0})$, where $|\phi^+_{A_0 R_0}\rangle = \sum_{i=0}^{|A_0|-1} i |i\rangle_{A_0 R_0}$ and the operator $X_{R_0}$ satisfies $\text{Tr}_{A_0 R_0} \Psi_{A_0 R_0} = 1 = \text{Tr}_{R_0} X_{R_0}^\dagger X_{R_0} = \|X_{R_0}\|_2^2$. With $\Phi^+_{A_0 R_0} = \frac{\phi^+_{A_0 R_0}}{|\phi^+_{A_0 R_0}|}$, we have

$$
\|N_A - M_A\|_\diamond = \|N_A \otimes \text{id}_{R_0} (\Psi_{A_0 R_0}) - M_A \otimes \text{id}_{R_0} (\Psi_{A_0 R_0})\|_1
= \|\left( N_A - M_A \right) \otimes \text{id}_{R_0} \left( I_{A_0} \otimes X_{R_0} \cdot |A_0\rangle \Phi^+_{A_0 R_0} \cdot I_{A_0} \otimes X_{R_0}^\dagger \right)\|_1
= |A_0| \|I_{A_0} \otimes X_{R_0} \cdot (J_{N_A} - J_{M_A}) \cdot I_{A_0} \otimes X_{R_0}^\dagger\|_1
\leq |A_0| \|X_{R_0}\|_\infty \|X_{R_0}^\dagger\|_\infty \|J_{N_A} - J_{M_A}\|_1
\leq |A_0| \|X_{R_0}\|_2 \|X_{R_0}^\dagger\|_2 \|J_{N_A} - J_{M_A}\|_1
\leq |A_0| \|J_{N_A} - J_{M_A}\|_1,
$$

where we used the Hölder inequality for the first inequality.

**Theorem A.4.** Let $N_A$ and $M_A$ be quantum channels. Given $\varepsilon \geq 0$, we have

$$
\frac{1}{2} \|N_A - M_A\|_\diamond \leq \varepsilon \iff \min_{\Psi_{A_0 R_0}} F(N_A(\Psi_{A_0 R_0}), M_A(\Psi_{A_0 R_0})) \geq (1 - \varepsilon)^2 \geq 1 - 2\varepsilon. \tag{56}
$$

Conversely, it follows that

$$
\min_{\Psi_{A_0 R_0}} F(N_A(\Psi_{A_0 R_0}), M_A(\Psi_{A_0 R_0})) \geq 1 - \varepsilon \iff \frac{1}{2} \|N_A - M_A\|_\diamond \leq \sqrt{\varepsilon}. \tag{57}
$$

**Proof.** Both follow from the Fuchs-van de Graaf inequality, while $(1 - \varepsilon)^2 = 1 - 2\varepsilon + \varepsilon^2 \geq 1 - 2\varepsilon$ for $\varepsilon \geq 0$.

**Proposition A.5.** Let $N_A$ and $M_A$ be quantum channels, and $J_{N_A}$ and $J_{M_A}$ be their (normalized) Choi matrices, respectively. If $F(J_{N_A}, J_{M_A}) \geq 1 - \varepsilon$, then $\frac{1}{2} \|N_A - M_A\|_\diamond \leq n \sqrt{\varepsilon}$.

**Proof.** From Theorem A.3 and the Fuchs-van der Graaf inequality, it follows that

$$
\frac{1}{2} \|N_A - M_A\|_\diamond \leq n \frac{1}{2} \|J_{N_A} - J_{M_A}\|_1
\leq n \sqrt{1 - F(J_{N_A}, J_{M_A})}
\leq n \sqrt{\varepsilon}.
$$

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