Equation of State of Quantum Gases Beyond the Van der Waals Approximation

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A recently suggested equation of state with the induced surface tension is generalized to the case of quantum gases with mean-field interaction. The self-consistency conditions and the necessary one to obey the Third Law of thermodynamics are found. The quantum virial expansion of the Van der Waals models of such type is analyzed and its virial coefficients are given. In contrast to traditional beliefs, it is shown that an inclusion of the third and higher virial coefficients of the gas of hard spheres into the interaction pressure of the Van der Waals models either breaks down the Third Law of thermodynamics or does not allow one to go beyond the Van der Waals approximation at low temperatures. It is demonstrated that the generalized equation of state with the induced surface tension allows one to avoid such problems and to safely go beyond the Van der Waals approximation. Besides, the effective virial expansion for quantum version of the induced surface tension equation of state is established and all corresponding virial coefficients are found exactly. The explicit expressions for the true quantum virial coefficients of an arbitrary order of this equation of state are given in the low density approximation. A few basic constraints on such models which are necessary to describe the nuclear and hadronic matter properties are discussed.

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I. INTRODUCTION

Investigation of equation of state (EoS) of strongly interacting particles at low temperatures is important for studies of the nuclear liquid-gas phase transition and for properties of neutron stars \cite{18}. To have a realistic EoS one has to simultaneously account for a short range repulsive interaction, a medium range attraction and the quantum properties of particles. Unfortunately, it is not much known about the in-medium quantum distribution functions of particles which experience a strong interaction. Therefore, a working compromise to account for all these features is to introduce the quasi-particles with quantum properties which interact via the mean-field. One of the first successful models of such type was a Walecka model \cite{4}. However, the strong demands to consider more realistic interaction which is not restricted to some kind of effective Lagrangian led to formulating a few phenomenological generalizations of relativistic mean-field model \cite{5–7}. Although a true breakthrough among them was made in work \cite{7} in which the hard-core repulsion was suggested for fermions, an introduction of phenomenological attraction in the spirit of Skyrme-Hartree-Fock approach \cite{8} which depends not on the scalar field, but on the baryonic charge density, was also important, since such a dependence of attractive mean-field is typical for the EoS of real gases \cite{9}.

However, in addition to the usual defect of the relativistic mean-field models, that they break down the first and the second Van Hove axioms of statistical mechanics \cite{10,11}, the usage of non-native variable, namely a particle number density, in the grand canonical ensemble led to formulation of the self-consistency conditions \cite{5,6}. In contrast to Walecka model \cite{4} and its followers for which the structure of Lagrangian and the extremum condition of the system pressure with respect to each mean-field automatically provide the fulfillment of the thermodynamic identities, the phenomenological mean-field EoS of hadronic matter had to be supplemented by the self-consistency conditions \cite{5,6,12}. The latter allows one to, formally, recover the first axiom of statistical mechanics \cite{10,11} (for more recent discussion of the self-consistency conditions see \cite{13–15}). An exception is given by the Van der Waals (VdW) hard-core repulsion \cite{7}, since in the grand canonical ensemble such an interaction depends on the system pressure which is the native variable for it.

Due to its simplicity the VdW repulsion is very popular in various branches of modern physics, but even in case of Boltzmann statistics it is valid only at low particle densities for which an inclusion of the second virial coefficient is sufficient. For the classical gases the realistic EoS which are able to account for several virial coefficients are well-known \cite{9,16}, while a complete quantum mechanical treatment of the third and higher virial coefficients is rather hard \cite{17}. Hence, the quantum EoS with realistic interaction allowing one to go beyond the second virial coefficient are of great interest not only for the dense hadronic and nuclear/neutron systems, but also for quantum and classical liquids. It is widely believed that one possible way to go beyond the VdW approximation, i.e. beyond the second virial coefficient, is to include a sophisticated interaction known from the classical models \cite{9,16} into the relativistic mean-field models with the quantum distribution functions for quasi-particles \cite{13,14}.

On the other hand, a great success in getting a high
A quality description of experimental hadronic multiplicities measured in the central nuclear collisions from AGS (BNL) to LHC (CERN) energies is achieved recently with the hadron resonance gas model which employs both the traditional VdW repulsion \[18\]–\[23\] and the induced surface tension (IST) concept for the hard-core repulsion \[24\]–\[26\] motivate us to formulate and thoroughly inspect the quantum version of this novel class of IST EoS in order to apply it in the future to the description of the properties of dense hadronic, nuclear, neutron matter and dense quantum liquids on the same footing. This is a natural choice, since the Boltzmann version of IST EoS \[24\]–\[25\] for a single sort of particles simultaneously accounts for the second, third and fourth virial coefficients of the classical gas of hard spheres and, thus, it allows one to go beyond the VdW approximation, whereas the multicomponent formulation of such EoS applied to the mixture of nuclear fragments of all possible sizes \[27\] not only allows one to introduce a compressibility of atomic nuclei into an exactly solvable version \[28\] of the statistical nuclear multifragmentation model \[29\], but also it sheds light on the reason of why this model employing the proper volume approximation for the hard-core repulsion is able to correctly reproduce the low density virial expansion for all atomic nuclei.

Therefore, the present work has two aims. First, we would like to analyze the popular quantum VdW models \[13\]–\[15\] at high and low temperatures in order to verify whether a tuning of interaction allows one to go beyond the VdW treatment. In addition, we calculate all virial coefficients for the pressure of point-like particles of the quantum VdW EoS. Second, we generalize the recently suggested IST EoS \[24\]–\[25\] to the quantum case, obtain its effective virial expansion and calculate all quantum virial coefficients, including the true virial coefficients for the low density limit. Using these results, we discuss a few basic constraints on the quantum EoS which are necessary to model the properties of nuclear/neutron and hadronic matter.

The work is organized as follows. In Sect. 2 we analyze the quantum VdW EoS and its virial expansion, and discuss the pitfalls of this EoS. The quantum version of the IST EoS is suggested and analyzed in Sect. 3. In Sect. 4 we obtain several virial expansions of this model and discuss the Third Law of thermodynamics for the IST EoS. Some simplest applications to nuclear and hadronic matter EoS are discussed in Sect. 5, while our conclusions are formulated in Sect. 6.

II. QUANTUM VIRIAL EXPANSION FOR THE VDW QUASI-PARTICLES

Similarly to the ordinary gases, in the hadronic or nuclear systems the source of hard-core repulsion is connected to the Pauli blocking effect between the interacting fermionic constituents existing interior the composite particles (see, for instance, \[3\]). This effect appears due to the requirement of antisymmetrization of the wave function of all fermionic constituents existing in the system and at very high densities it may lead to the Mott effect, i.e. to a dissociation of composite particles or even the clusters of particles into their constituents \[2\]. Therefore, it is evident that at sufficiently high densities one cannot ignore the hard-core repulsion or the finite (effective) size or composite particles and the success of traditional EoS used in the theory of real gases \[9\] based on the hard-core repulsion approach tells us that this is a fruitful framework also for quantum systems. Hence we start from the simplest case, i.e. the quantum VdW EoS \[14\]–\[15\]. The typical form of EoS for quantum quasi-particles of mass $m_p$ and degeneracy factor $d_p$ is as follows

$$p(T, \mu, n_{id}) = p_{id}(T, \nu(\mu, n_{id})) - P_{int}(T, n_{id})$$ \hspace{0.5cm} (1)

$$p_{id}(T, \nu) = d_p \int \frac{dk}{(2\pi)^3} \frac{k^2}{3 E(k)} \left( \frac{e^{E(k)/T}}{1 + e^{E(k)/T}} + \zeta \right)$$ \hspace{0.5cm} (2)

$$\nu(\mu, n_{id}) = \mu - b_p + U(T, n_{id})$$ \hspace{0.5cm} (3)

where the constant $b \equiv 4 V_0 = \frac{16 \pi}{3} R_0^3$ is the excluded volume of particles with the hard-core radius $R_0$ (here $V_0$ is their proper volume), the relativistic energy of particle of momentum $k$ is $E(k) \equiv \sqrt{k^2 + m_p^2}$ and the density of point-like particles is defined as $n_{id}(T, \nu) \equiv \frac{\partial p_{id}(T, \nu)}{\partial \nu}$. The parameter $\zeta$ switches between the Fermi ($\zeta = 1$), the Bose ($\zeta = -1$) and the Boltzmann ($\zeta = 0$) statistics. The interaction part of pressure $P_{int}(T, n_{id})$ and the mean-field function $U(T, n_{id})$ will be specified later.

Note that similarly to the Skyrme-like EoS and the EoS of real gases it is assumed that the interaction between quasi-particles described by the system \[1\]–\[3\] is completely accounted by the excluded volume (hard-core repulsion), by the mean-field potential $U(T, n_{id})$ and by the pressure $P_{int}(T, n_{id})$. This is in contrast to the relativistic mean-field models of Walecka type in which the mass shift of quasi-particles is taken into account. Since such an effect may be important for the modeling the chiral symmetry restoration in hadronic matter the strongest arguments of whose existence are recently given in \[29\], we leave it for a future exploration and concentrate here on a simpler EoS defined by Eqs. \[1\]–\[3\].

The functions $U(T, n_{id})$ and $P_{int}(T, n_{id})$ are not independent, due to the thermodynamic identity $n(T, \nu(T, n_{id})) \equiv \frac{\partial p(T, \nu(T, n_{id}))}{\partial \nu(T)}$. Therefore, the mean-field terms $U$ and $P_{int}$ should obey the self-consistency condition

$$n_{id}\frac{\partial U(T, n_{id})}{\partial n_{id}} = \frac{\partial P_{int}(T, n_{id})}{\partial n_{id}} \Rightarrow$$ \hspace{0.5cm} (4)

$$P_{int}(T, n_{id}) = n_{id} U(T, n_{id}) - \int_0^{n_{id}} dn U(T, n)$$ \hspace{0.5cm} (5)

After integrating by parts Eq. \[1\], we used in \[5\] an obvious condition $U(T, 0) < \infty$. If the condition \[5\] is obeyed, then the direct calculation of the $\mu$-derivative of
the pressure \( n \) gives one the usual expression for particle number density in terms of the density of point-like particles

\[
n = \frac{n_{id}}{1 + b n_{id}}, \quad (6)
\]

\[
n_{id}(T, \nu) = d_p \int \frac{dk}{(2\pi)^3} \frac{1}{e^{\frac{E(k)}{T}} + 1}, \quad (7)
\]

From these equations one finds that \( n \to b^{-1} \) for \( n_{id} \to \infty \). The limit \( n_{id} \to \infty \) is provided by the conditions \( \nu \to \infty \) or \( T \to \infty \) for \( \zeta = (0, 1) \), while for \( \zeta = -1 \) it is provided by the conditions \( \nu \to m_p - 0 \) or \( T \to \infty \).

Note that in contrast to other works discussing Eqs. (15) and (16) through this paper we will use the density of point-like particles \( n_{id} \) as an argument of the functions \( U(T, n_{id}) \) and \( P_{int}(T, n_{id}) \) instead of the physical density of particles \( n \) because for more sophisticated EOS their relation will be more complicated than \( [4] \). Also such a representation is convenient for a subsequent analysis because in terms of \( n_{id}(T, \nu) \) the virial expansion of \( p_{id}(T, \nu) \) looks extremely simple \([17] \)

\[
p_{id}(T, \nu) = T \sum_{l=1}^{\infty} a^{(l)} \left[ n_{id}(T, \nu) \right] \nu^l, \quad (8)
\]

\[
a^{(0)} = 1, \quad a^{(2)} = -b^{(2)}, \quad (9)
\]

\[
a^{(3)} = 4 \left[ b^{(1)} \right]^2 - 2 b^{(3)}, \quad (10)
\]

\[
a^{(4)} = -20 \left[ b^{(2)} \right]^3 + 18 b^{(2)} b^{(3)} - 3 b^{(4)}, \quad (11)
\]

\[
......... \quad (12)
\]

\[
......... \quad (13)
\]

Here the first few virial coefficients \( a^{(l)} \) of an ideal quantum gas are expressed in terms of the corresponding cluster integrals \( b^{(l)} \) which depend only on temperature. The latter are expressed via the thermal density of the auxiliary Boltzmann system \( n_{id}^{(0)}(T, \nu) \equiv n_{id}(T, \nu)|_{\zeta=0} \) of Eq. (7) \([17, 30]\)

\[
b^{(l)} = \frac{(\mp 1)^{l+1} T}{l^2 \pi} n_{id}^{(0)}(T, \nu) \left[ n_{id}^{(0)}(T, \nu) \right]^{-l}, \quad (14)
\]

where the upper (lower) sign corresponds to Fermi (Bose) statistics. For the non-relativistic case the expression \([14] \) can be further simplified \([17] \) and for an arbitrary degeneracy factor \( d_p \) it acquires the form \([30]\)

\[
b^{(l)} \bigg|_{nonrel} \simeq \frac{(\mp 1)^{l+1}}{l^2} \left( \frac{1}{d_p} \right) \left[ \frac{2 \pi}{T m_p} \right]^{\frac{2}{l-1}}, \quad (15)
\]

For high temperatures one can write an ultra-relativistic analog of Eq. \([15] \) for a few values of \( l = 2, 3, \ldots \ll T/m_p \)

\[
b^{(l)} \bigg|_{urel} \simeq \frac{(\mp 1)^{l+1}}{l^4} \left[ \left( \frac{\pi^2}{d_p T^3} \right)^{\frac{2}{l-1}}, \quad (16)
\]

Suppose that the coefficients \( a^{(0)} \) from Eq. \([8] \) are known and that the virial expansion is convergent for the considered \( T \). Then using Eq. \([6] \) one finds \( n_{id} = n/(1-bn) \) and, hence, one can rewrite Eq. \([8] \) as

\[
p_{id}(T, \nu) = \frac{1}{1 - bn} + \sum_{l=2}^{\infty} a^{(l)} \left[ n \right]^{l-1} \left[ 1 - bn \right]^{l-1}. \quad (17)
\]

As one can see from Eqs. \([15] \) and \([16] \) at high temperatures all cluster integrals and virial coefficients of ideal quantum gas strongly decrease with the temperature \( T \) and, hence, at high temperatures the virial expansion of \( p_{id}(T, \nu) \) is defined by the first (classical) term on the right hand side of \([17] \), i.e. in this case one gets

\[
p_{id}(T, \nu) \simeq 1 + 4V_0 n + (4V_0 n)^2 + (4V_0 n)^3 + \ldots. \quad (18)
\]

where after expanding the first term on the right hand side of \([17] \) we used the relation between \( b \) and \( V_0 \). From this equation one sees that only the second virial coefficient, \( 4V_0 \), coincides with the one for the gas of hard spheres, while the third, \( 16V_0^2 \), and the fourth, \( 64V_0^3 \), virial coefficients are essentially larger than their counterparts \( B_3 = 10V_0^2 \) and \( B_4 = 18.36V_0^3 \) of the gas of hard spheres. Also Eq. \([17] \) can naturally explain why in the work \([13] \) the authors insisted on the interaction pressure \( P_{int} \) to be a linear function of \( T \) (see a statement after Eq. (62) in \([13] \)): if one chooses the interaction pressure in the form

\[
P_{int}(T, n(n_{id})) = T n \left( (b^2 - B_3)n^2 + (b^3 - B_3)n^3 + (b^4 - B_5)n^4 + \ldots \right), \quad (19)
\]

then at high temperatures the quantum corrections are negligible and, hence, for such a choice of \( P_{int}(T, n(n_{id})) \) with the corresponding value for the mean-field potential \( U(T, n(n_{id})) \) obeying the self-consistency condition \([4] \), one can improve the total pressure of mean-field model by matching its repulsive part to the pressure of hard spheres.

The problem, however, arises at low temperatures, while calculating the entropy density for the model with \( P_{int}(T, n(n_{id})) \) \([19] \). Indeed, for the choice \( U(T, n(n_{id})) = g(T) f(n(n_{id})) \) from the thermodynamic identities \( s = \frac{\partial n(T, \mu)}{\partial T} \) and \( s_{id} = \frac{\partial P_{id}(T, \nu)}{\partial T} \) one finds \([13] \)

\[
s(T, \mu) = \left[ s_{id} + \left[ n_{id} \frac{\partial U}{\partial T} - \frac{\partial P_{id}}{\partial T} \right] \right] \left[ 1 + bn_{id} \right]^{-1} = \left[ s_{id} + \frac{dg(T)}{dT} \int_{n_{id}}^{n} \frac{d\tilde{n}}{T} f(\tilde{n}) \right] \left[ 1 + bn_{id} \right]^{-1}, \quad (20)
\]

where in deriving Eq. \([21] \) we used an explicit form of \( P_{int} \) \([19] \) and Eq. \([5] \). As one can see from \([21] \) the mean-field model with linear \( T \) dependence of \( U \) or, equivalently, of \( P_{int} \), i.e. \( g(T) = T \Rightarrow \frac{dg(T)}{dT} = 1 \), breaks down the Third Law of thermodynamics, since at \( T = 0 \) one finds \( s_{id}(T = 0, \nu) = 0 \) by construction, whereas \( s(T = 0, \mu) = [1 + bn_{id}^{-1}] \int_{0}^{n_{id}} d\tilde{n} f(\tilde{n}) \neq 0 \), unless
\( f \equiv 0 \). Hence, the mean-field model with the linear \( T \) dependence of \( P_{\text{id}} \) may be very good at high temperatures, for which the Boltzmann statistics is valid, but it is unphysical at \( T = 0 \). Of course, one can repair this defect by choosing more complicated function \( g(T) \), which \( g(T) \sim T \) at high \( T \), but its derivative \( g'(T) \) vanishes at \( T = 0 \) providing the fulfillment of the Third Law of thermodynamics. However, in this case the whole idea to compensate the defects of the VdW EoS by tuning the interacting part of pressure does not work at low \( T \), since in this case \( P_{\text{id}} = g(T)F(n_{id}) \) would vanish faster than the term \( Tn_{id} \) on the right hand side of Eq. (17). Thus, we explicitly showed here that at low \( T \) the mean-field models defined by Eqs. (7)-(9) either are unphysical, or \( P_{\text{id}} = TF(n_{id}) \), or they cannot go beyond the VdW approximation by adjusting their interaction pressure \( P_{\text{id}} \).

Such a conclusion can be also applied to the one of two ways to introduce the excluded volume correction into the quantum second virial coefficients discussed in Ref. [31]. Although the model of Ref. [31] contains the scalar mean-fields which modify the masses of particles, the effective potential approach to treat the excluded volume correction of Ref. [31] with the linear \( T \) dependence of the repulsive effective potential \( W_i \) (equivalent to the mean-field potential \(-U\) in our notations) of the \( i\)-th particle sort (see Eqs. (20) and (46) and (47) in [31]) should unavoidably lead to a break down of the Third Law of thermodynamics. Therefore, we conclude that such a way to introduce the excluded volume correction into the quantum second virial coefficients discussed in [31] is unphysical. Thus, despite the claims of author of Ref. [31] there is only a single physical way to include the hard-core repulsion in quantum systems.

To end this section we express the traditional virial coefficients \( a_k^Q \) of the quantum VdW gas of Eq. (17) in terms of the classical excluded volume \( b \) and the quantum virial coefficients of point-like particles \( a_k^{(0)} \). Expanding each denominator in Eq. (17) into a series of powers of \( n \), one can easily find

\[
p_{id}(T, \nu) = T \left[ n + \sum_{k=2}^{\infty} a_k^Q n^k \right], \quad \text{where} \quad (22)
\]

\[
a_2^Q = b + a_2^{(0)}, \quad (23)
\]

\[
a_3^Q = b^2 + 2b a_2^{(0)} + a_3^{(0)}, \quad (24)
\]

\[
a_4^Q = b^3 + 3b^2 a_2^{(0)} + 3b a_3^{(0)} + a_4^{(0)}, \quad (25)
\]

\[
a_k^Q = b^{k-1} + \sum_{l=2}^{k} \frac{(k-1)!}{(l-1)!(k-l)!} b^{k-l} a_l^{(0)}. \quad (26)
\]

If the interaction pressure \( P_{\text{id}}(T, n_{id}(n)) \) of the model [1] can be expanded into the Taylor series of particle number density \( n \) at \( n = 0 \), then one can obtain the full quantum virial expansion of this EoS. Note that the coefficients \( a_k^{(0)} \) for the model [1] depend on temperature only, while specific features of the EoS are stored in \( b \) and in \( P_{\text{id}}(T, n_{id}(n)) \). For example, using the coefficients \( b = 3.42 \text{ fm}^3 \) and \( P_{\text{id}}(T, n) = a_{\text{attr}} R_p^2 \) \( (a_{\text{attr}} = 329 \text{ MeV-fm}^3) \) found in [14] for the quantum VdW EoS of nuclear matter, one can calculate the full quantum second virial coefficient of the model as

\[
a_2^Q = b + a_2^{(0)} - \frac{a_{\text{attr}}}{T} \simeq b + \frac{1}{2} \frac{2\pi}{T m_p} \left( \frac{2}{3} - \frac{a_{\text{attr}}}{T} \right), \quad (27)
\]

where in the second step of derivation we used the non-relativistic expression for the cluster integral \( b_2^{(0)} \) [15]. Taking results from [14], one can find that for nucleons \( (d_p = 4, m_p = 939 \text{ MeV}) \) the coefficient \( a_2^Q(T) \) is zero at \( T \simeq 5.76 \text{ MeV} \) and \( T \simeq 70.4 \text{ MeV} \), is negative between these temperatures and then above \( T \simeq 70.4 \text{ MeV} \) it grows almost linearly with \( T \) to \( a_2^Q(T = 150 \text{ MeV}) \simeq (3.42 + 0.403 - 2.199) \text{ fm}^3 \simeq 1.63 \text{ fm}^3 \) which corresponds to the equivalent hard-core radius \( R_{\text{eq}} \simeq 0.46 \text{ fm} \) at \( T = 150 \text{ MeV} \). From this estimate it is evident that the large value of the equivalent hard-core radius \( R_{\text{eq}} \) for the model [13] is a consequence of the unrealistically large hard-core radius of nucleons \( R_n \simeq 0.59 \text{ fm} \) obtained in [13] (also, see a discussion later), whereas in the most advanced version of the hadron resonance gas model the hard-core radius of nucleons is 0.365 fm [24-26] and in the IST EoS of the nuclear matter this radius is below 0.4 fm [22]. It is obvious that more realistic attraction than the one used in [13] would decrease the values of \( R_{\text{eq}} \) and \( R_n \) to physically more adequate ones. Although the explicit quantum virial expansion (22)-(27) can be used to find the appropriate attraction in order to cure the problems of the VdW EoS and extend it to higher particle number densities and high/low \( T \) values, the true solution of this problem is suggested below.

### III. EOS WITH INDUCED SURFACE TENSION

In order to overcome the difficulties of the quantum VdW EoS at high particle number densities we suggest the following EoS

\[
p = p_{id}(T, \nu_1) - P_{\text{id}}(T, n_{id}) , \quad (28)
\]

\[
\Sigma = R_p [p_{id}(T, \nu_2) - P_{\text{id}}(T, n_{id})], \quad (29)
\]

\[
\nu_1 = \mu - V_0 p - S_0 \Sigma + U_1(T, n_{id}) , \quad (30)
\]

\[
\nu_2 = \mu - V_0 p - \alpha S_0 \Sigma + U_2(T, n_{id}), \quad (31)
\]

where \( n_{id,A} \equiv \frac{d\rho_{id}(T, \nu_A)}{d\nu_A} \) with \( A = \{1; 2\} \), \( S_0 = 4\pi R_p^2 \) denotes the proper surface of the hard-core volume \( V_0 \), Eq. (25) is an analog of Eq. (1), while the equation for the induced surface tension coefficient \( \Sigma \) (29) was first introduced for the Boltzmann statistics in [22]. The system (25)-(31) is a quantum generalization of the Boltzmann EoS in the spirit of work [7]. As it was argued above the alternative way to introduce the hard-core repulsion in quantum systems mentioned in [31] is unphysical at
low temperatures, at the moment it seems that such a generalization of the Boltzmann version is a unique one.

The quantity $\Sigma$ defined by \ref{30} is the surface part of the hard-core repulsion \ref{25}. As it will be shown later, representing the hard-core repulsion in pressure \ref{28} in two terms, namely via $-V_0 \rho$ and $-S_0 \Sigma$, instead of a single term $-4V_0 \rho$ as it is done in the quantum VdW EoS, has great advantages and allows one to go beyond the VdW approximation.

Evidently, the self-consistency conditions for the IST EoS are similar to Eqs. \ref{3} and \ref{4} ($A = \{1; 2\}$)

$$n_{id} \frac{\partial U_A(T, n_{id} A)}{\partial n_{id} A} = \frac{\partial P_{\text{int}}(T, n_{id} A)}{\partial n_{id} A}, \quad (32)$$

The model parameter $\alpha > 1$ is a switch between the excluded and proper volume regimes. To demonstrate this property let us consider the quantum distribution function

$$\phi_{id}(k, T, \nu_2) \equiv \frac{1}{e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} + \zeta} = e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} + \zeta \left[ 1 - e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \right]^{-1},$$

where in the last step of the derivation we have expanded the longest denominator above into a series of $\phi_{id}(k, T, \nu_2) \zeta \left(1 - e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}}\right)$ powers. Consider two limits of \ref{33}, namely $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \rightarrow 1$ and $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \rightarrow 0$ for $\zeta \neq 0$. Then the distribution function \ref{33} can be cast as:

$$\phi_{id}(k, T, \nu_2) \rightarrow \phi_{id}(k, T, \nu_1) e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \begin{cases} \text{for } \zeta \neq 0, & \text{if } e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \sim 1, \\ \text{for } \forall \zeta, & \text{if } e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \rightarrow 0. \end{cases} \quad (34)$$

Further on we assume that the inequality

$$(\alpha - 1)S_0 \Sigma/n_{id 2} \gg (U_2 - U_1)/n_{id 2}, \quad (35)$$

holds in either of the considered limits for $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}}$. Note that for the case $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \sim 1$ the condition \ref{35} is a natural one because at low particle densities it means that the difference of two mean-field potentials $(U_2 - U_1)$ is weaker than the hard-core repulsion term $(\alpha - 1)S_0 \Sigma$; whereas for $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \rightarrow 0$ it means that such a difference is simply restricted from above for large values of $\Sigma$, i.e. \max\{U_1; U_2\} < Const < \infty. Evidently, in this limit the mean-field pressures should be also finite, i.e. $|P_{\text{int}}| < \infty$.

For the case $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \sim 1$ one immediately recovers the following relation $p_{id}(T, \nu_2) \sim e^{\frac{\varepsilon_{id}(k, \nu_2)}{T} - (\alpha - 1)S_0 \Sigma} p_{id}(T, \nu_1)$ for $\zeta \neq 0$, which exactly corresponds to the Boltzmann statistics version \ref{25} of the system \ref{28}-\ref{31} and, hence, one recovers the virial expansion of $p_{id}(T, \nu_1)$ \ref{25} in terms of the density of particle number $n_1 = \frac{\partial p_{id}(T, \nu_1)}{\partial \mu} |_{\nu_1}$ which is calculated under the condition $U_1 = const$

$$p_{id}(T, \nu_1) \sim 1 + 4V_0 n_1 + [16 - 18(\alpha - 1)] V_0^2 n_1^2 + \left[64 + \frac{243}{2}(\alpha - 1)^2 - 216(\alpha - 1)\right] V_0^3 n_1^3 + \ldots \quad (36)$$

Note that due to the self-consistency condition \ref{32} one finds $\frac{\partial p_{id}(T, \nu_1)}{\partial \mu} |_{\nu_1} = \frac{\partial p_{id}(T, \nu_1)}{\partial \mu} |_{\nu_1}$, and, therefore, $n_1$ is the physical particle number density.

As it was explained in \ref{25} for $\alpha = \alpha_B \equiv 1.245$ one can reproduce the fourth virial coefficient of the gas of hard spheres exactly, while the value of the third virial coefficient of such a gas is recovered with the relative error about 16% only. Therefore, for low densities, i.e. for $V_0 n_1 \ll 1$, the IST EoS \ref{28}-\ref{31} reproduces the results obtained for $\zeta = 0$, if the condition \ref{30} is fulfilled.

On the other hand, from Eqs. \ref{33} and \ref{34} one sees that in the limit $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \rightarrow 0$ the distribution function $\phi_{id}(k, T, \nu_2)$ with $\zeta \neq 0$ acquires the Boltzmann form. In this limit we find $p_{id}(T, \nu_2) \sim p_{id}(T, \nu_1) e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}}$ and $n_{id 2} \sim n_{id 1} e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}}$. Using these results and Eq. \ref{35} we can rewrite \ref{29} as

$$\Sigma \sim R_p \left[ p_{id}(T, \nu_1) e^{\frac{(1-\alpha)S_0 \Sigma}{\rho}} - P_{\text{int}}(T, n_{id 2}^{(0)}) \right]. \quad (37)$$

Here we use the same notation as in previous section (see a paragraph before Eq. \ref{14}). From Eq. \ref{37} one can see that for $\frac{V_0 p_{id}(T, \nu_1)}{T} \gg 1$ the surface tension coefficient $\Sigma$ is strongly suppressed compared to $R_p p_{id}(T, \nu_1)$, i.e. $\Sigma \sim \frac{T}{S_0(\alpha - 1)} \ln \left[ \frac{R_p p_{id}(T, \nu_1)}{\Sigma} \right] \ll R_p p_{id}(T, \nu_1)$. Note that for $\alpha > 1$ the condition $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \rightarrow 0$ can be provided by $S_0 \Sigma/T \gg 1$ only. Thus, the second term on the right hand side of Eq. \ref{37} cannot dominate, since it is finite. It is evident that the inequality $\frac{V_0 p_{id}(T, \nu_1)}{T} \gg 1$ also means that $n_{id 1} V_0 \gg 1$. Therefore, in this limit the effective chemical potential \ref{30} can be approximated as

$$\nu_1 \simeq \mu - V_0 p + U_1(T, n_{id 1}^{(0)}), \quad (38)$$

i.e. the contribution of the induced surface tension is negligible compared to the pressure. This result means that for $n_{id 1}^{(0)} V_0 \gg 1$, i.e. at high particle densities or for $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \rightarrow 0$, the IST EoS corresponds to the proper volume approximation.

On the other hand, Eq. \ref{36} exhibits that at low densities, i.e. for $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \simeq 1$, the IST EoS recovers the virial expansion of the gas of hard-spheres up to fourth power of particle density $n_1$. Therefore, it is natural to expect that for intermediate values of the parameter $e^{\frac{\varepsilon_{id}(k, \nu_2)}{T}} \in [0; 1]$ the IST EoS will gradually evolve from the low density approximation to the high density one, if the condition
is obeyed. This is a generalization of the previously obtained result \[33\] onto the quantum statistics case.

Already from the virial expansion [36] one can see that the case \(\alpha = 1\) recovers the VdW EoS with the hard-core repulsion. If, in addition, the mean-field potentials are the same, i.e., \(U_2 = U_1\) and, consequently, \(P_{\text{int}2} = P_{\text{int}1}\), then one finds that \(\nu_2 = \nu_1\) and \(\Sigma = R_p p(T, \nu_1)\). In this case the term \(V_0 p + S_0 \Sigma = 4V_0 p\) exactly corresponds to the VdW hard-core repulsion. If, however, \(U_2 \neq U_1\), but both mean-field potentials are restricted from above, then the model can deviate from the VdW EoS at low temperatures only, while at high temperatures it again corresponds to the VdW EoS. For the case \(U_2 < U_1\) this can be easily seen from Eqs. \[33\] and \[34\] for the case \(e^{\frac{\nu_2}{\nu_1}} \approx 0\), if one sets \(\alpha = 1\). Then using the same logic as in deriving Eq. \[37\], one can find that \(\Sigma \ll R_p p_{\text{int}}(T, \nu_1)\) and, hence, the effective chemical potential \(\nu_1\) acquires the form \[38\]. In other words, at low \(T\) the surface tension effect becomes negligible and the IST EoS corresponds to the proper volume approximation, if \(e^{\frac{\nu_2}{\nu_1}} \approx 0\).

Finally, if the inequality \(U_2 > U_1\) is valid, then at low \(T\) an expansion \[33\] has to be applied to the distribution function \(\phi_{id}(k, T, \nu_1)\) instead of \(\phi_{id}(k, T, \nu_2)\) and then one arrives at the unrealistic case, since \(\Sigma \gg R_p p_{\text{int}}(T, \nu_1)\). In this case the hard-core repulsion would be completely dominated by the induced surface tension term and, hence, even the second virial coefficient would not correspond to the excluded volume of particles.

IV. GOING BEYOND VDW APPROXIMATION

Let us closely inspect the IST EoS and show explicitly its major differences from the VdW one. For such a purpose in this section we analyze its effective and true virial expansions and discuss somewhat unusual properties of the entropy density.

A. Effective virial expansion

First we analyze the particle densities \(n_1(T, \nu_1) = \frac{\partial p(T, \nu_1)}{\partial \mu}\) and \(n_2(T, \nu_2) = \frac{\partial^2 p(T, \nu_2)}{\partial \mu^2}\) for this purpose we differentiate Eqs. \[28\] and \[29\] with respect to \(\mu\) and apply the self-consistency conditions \[32\]

\[
n_1 = n_{id1} \left[ 1 - V_0 n_1 - S_0 \frac{\partial \Sigma}{\partial \mu} \right], \tag{39}
\]

\[
\frac{\partial \Sigma}{\partial \mu} = R_p n_{id2} \left[ 1 - V_0 n_1 - \alpha S_0 \frac{\partial \Sigma}{\partial \mu} \right]. \tag{40}
\]

Expressing \(\frac{\partial \Sigma}{\partial \mu}\) from Eq. \[40\] and substituting it into \[39\], one finds the densities of particle number \(\bar{n}_2(T, \nu_2) \equiv n_2(1 - V_0 n_1)\)

\[
n_1 = \frac{n_{id1} (1 - 3V_0 n_2)}{1 + V_0 n_{id1} (1 - 3V_0 n_2)}, \tag{41}
\]

\[
n_2 = \frac{n_{id2}}{1 + \alpha 3V_0 n_{id2}}, \tag{42}
\]

where we used the relation \(R_p S_0 = 3V_0\) for hard spheres.

From Eq. \[42\] for \(n_2\) one finds that for \(\alpha > 1\) the term \((1 - 3V_0 n_2)\) staying above is always positive, since, taking the limit \(n_{id2} \rightarrow \infty\) in Eq. \[42\] one finds the limiting density of \(\max\{n_2\} = |3aV_0|^{-1}\). Therefore, irrespective of the value of \(n_{id2} \geq 0\) in the limit \(n_{id1}V_0 \gg 1\) one finds that \(\max\{n_1\} = V_0^{-1}\). This is another way to prove that the limiting density of the IST EoS corresponds to the proper volume limit, since at high densities it is four times higher than the one of the VdW EoS. Writing the particle number density \(n_{id1}\) from Eq. \[41\] as

\[
n_{id1} = \frac{n_1}{(1 - V_0 n_1)(1 - 3V_0 n_2)}, \tag{43}
\]

one can get the formal virial-like expansion for the IST pressure \(p_{id}(T, \nu_1) \tag{28}\)

\[
p_{id}(T, \nu_1) \equiv \frac{\sum_{k=1}^{\infty} a_k^{(0)} \frac{n_1^k}{[1 - V_0 n_1]^k}}{T}, \tag{44}
\]

with the coefficients \(a_k^{(0)}\) which depend not only on \(T\), but also on \(n_2\). Similarly to deriving Eq. \[29\], from \[45\] one can get the quantum virial expansion for IST pressure \(p_{id}(T, \nu_1) \tag{46}\)

\[
p_{id}(T, \nu_1) = T \sum_{k=1}^{\infty} a_k^{Q,IST} n_1^k, \tag{46}
\]

\[
a_k^{Q,IST} = \sum_{l=1}^{k} C_l^{(k)} \frac{n_2^{l-1} a_l^{(0)}}{(l-1)!}, \tag{47}
\]

\[
C_l^{(k)} = \frac{(k-1)!}{(l-1)![(k-l)]} V_0^{k-l} a_l^{(0)}, \tag{48}
\]

with the coefficients \(a_k^{Q,IST}\) which are \(T\) and \(n_2\) dependent. For the interaction pressure \(P_{\text{int}}(T, n_{id1})\) which is expandable in terms of the density \(n_1\), Eq. \[47\] can be used to estimate the full quantum virial coefficients of higher orders. Of course, Eq. \[46\] is not the traditional virial expansion, but the fact that it can be exactly obtained from the grand canonical ensemble formulation of the quantum version of the IST EoS for the third, the fourth and higher order virial coefficients is still remarkable.

B. True quantum virial coefficients

Now we consider an example on how to employ the results \[46\]–\[48\] to estimate the true virial coefficients at low densities and at sufficiently high temperature.
which provide the convergence of virial expansion (46). Apparently, in this case one can expand the density \( n_2 \simeq B_1 n_1 (1 + B_2 n_1) \) in powers of the density \( n_1 \). From our above treatment of the low density limit \( e^{\frac{\nu_0 n_1}{n_2}} \approx 1 \) it is clear that \( B_1 = 1 \). Substituting this expansion for \( n_2 \) into Eqs. (46) and (47) and keeping only the terms up to \( n_2^2 \) one can get the true quantum virial coefficients \( a_k^{Q,\text{tot}} \) as

\[
a_2^{Q,\text{tot}} = V_0 + a_{2}^{(0)} + 3V_0B_1 = 4V_0 + a_{2}^{(0)},
\]

\[
a_3^{Q,\text{tot}} \approx 13V_0^2 + 3V_0B_2 + 5V_0a_{2}^{(0)} + a_{3}^{(0)},
\]

\[
a_{k \geq 3}^{Q,\text{tot}} \approx \sum_{l=1}^{k} C_{l}^{(k)} + 3V_0B_1 \sum_{l=1}^{k-1} C_{l}^{(k-1)} l
+ 3V_0B_1 \sum_{l=1}^{k-2} C_{l}^{(k-2)} \left[ \frac{3}{2} l(l+1)V_0B_1 + B_2 \right].
\]

and replace the coefficients \( a_k^{Q,\text{IST}} \) in Eq. (46) with the true quantum virial coefficients \( a_k^{Q,\text{tot}} \) which depend on \( T \) only. Note that an expression for the second virial coefficient \( a_2^{Q,\text{tot}} \) is exact, while the expressions for the higher order virial coefficients are the approximated ones, which, nevertheless, at high values of temperature are rather accurate. Considering the limit of high temperatures which allows one to ignore the quantum corrections in Eqs. (49) and (50), one can find the coefficients \( B_1 = 1 \) exactly and \( B_2 \approx [7 - 6\alpha]V_0 \) approximately by comparing the expressions (49) and (50) with the corresponding virial coefficients of Boltzmann gas in Eq. (36). Substituting the obtained expressions for \( B_1 \) and \( B_2 \) coefficients into Eq. (51) one gets the approximate formula for higher order virial coefficients \( a_k^{Q,\text{tot}} \):

\[
a_{k \geq 3}^{Q,\text{tot}} \approx \sum_{l=1}^{k} C_{l}^{(k)} + 3V_0 \sum_{l=1}^{k-1} C_{l}^{(k-1)} l
+ 3V_0 \sum_{l=1}^{k-2} C_{l}^{(k-2)} \left[ \frac{3}{2} l(l+1) + (7 - 6\alpha) \right].
\]

Comparing Eq. (52) for the IST EoS and Eq. (26) for the VdW EoS one can see that the first sum on the right hand side of (52) is identical to the expression for the VdW quantum virial coefficients with the excluded volume \( b = 4V_0 \) replaced by the proper volume \( V_0 \). Apparently, the other two sums on the right hand side of (52) are the corrections due to induced surface tension coefficient.

Note that it is not difficult to get the exact expressions for the third or the fourth virial coefficients \( a_k^{Q,\text{tot}} \) by inserting the higher order terms of the expansion \( n_2(n_1) \) in power of density \( n_1 \) into Eqs. (46) and (47), although comparing the coefficients in front of \( B_1 \) and \( B_2 \) in the last sum of Eq. (51), one can see that even for \( l = 1 \) the coefficient staying before \( B_1 \) is essentially larger than the one staying before \( B_2 \). This means that at low densities the role of \( B_2 \) is an auxiliary one, if \( \alpha \) is between 1 and 1.5.

C. Virial expansion for compressible spheres

It is interesting that the \( k \)-th term \( \frac{(-1)^k V_0^{k-1}}{(1 - V_0n_1)^k} \) in the sum (44) allows for a non-trivial interpretation. Comparing Eq. (17) and Eq. (44) and recalling the fact that the particle number density \( n_1 \) is proportional to the number of spin-isospin configurations \( d_p \), one can introduce an effective number of such configurations as \( d_p^{eff} = \frac{d_p}{1 - V_0n_2} \) with simultaneous replacement of \( V_0 \) by the effective proper volume \( V_0^{eff} = V_0 (1 - 3V_0n_2) \) in the term \( [1 - V_0n_1] \) on the right hand side of (44). Then at high densities the effective number of spin-isospin configurations \( d_p^{eff} \leq \frac{d_p}{\frac{\alpha}{2}} \) can be sizably larger than \( d_p \), while the effective proper volume \( V_0^{eff} \) can be essentially smaller than \( V_0 \) (i.e. such effective particles are compressible), if the coefficient \( \alpha > 1 \) is close to 1. Moreover, one can also establish an equivalent virial expansion of pressure (44) in terms of \( \frac{1}{(1 - 3V_0n_2)} \) powers. Then instead of the coefficients \( a_k^{Q,\text{IST}} \) one would get

\[
a_k^{Q,\text{IST}} = \sum_{l=1}^{k} \frac{(k - 1)!}{(l - 1)! (k - l)!} \left[ V_0^{eff} \right]^{k-l} a_l^{(0)},
\]

which shows that at high densities the contributions of low order virial coefficients \( a_l^{(0)} \) into the coefficient \( a_k^{Q,\text{IST}} \) are suppressed due to decrease of \( V_0^{eff} \). Eq. (53) quantifies the source of softness of the IST EoS compared to VdW one at high densities. It is also interesting that the monotonic decrease of \( V_0^{eff} \) at high densities is qualitatively similar to the effect of Lorentz contraction of proper volume for relativistic particles (33).

Although the present model does not know anything about the internal structure of considered particles, but the fact that \( d_p^{eff} \) increases with the particle number density \( n_2 \) can be an illustration of the in-medium effect that the IST hard-core interaction ‘produces’ the additional (or ‘enhances’ the number of existing) spin-isospin states which are well known in quantum physics as excited states, but with an excitation energy being essentially smaller than the mean value of particle free energy. In this way one can see that at high densities the IST effectively increases the degeneracy factor of particles. This finding is a good illustration that the claim of Ref. (31) that accounting for the excluded volume correction in quantum case via the effective degeneracy leads to the reduction of latter (see a discussion of Eqs.(18) and (19) in (31)) is not a general one.

It is apparent that for \( \alpha \gg 1 \) the quantities \( V_0^{eff} \) and \( d_p^{eff} \) are practically independent of \( n_2 \), i.e. in this case the coefficients \( a_k^{Q,\text{IST}} \) and \( a_k^{Q,\text{IST}} \) are the true quantum virial coefficients of the VdW EoS, but with the excluded volume \( b = 4V_0 \) replaced by \( V_0 \).

D. Properties of entropy density

Next we study the entropy density of the IST EoS. Similarly to finding the derivatives of Eqs. (28) and (29)
with respect to \( \mu \), one has to find their derivatives with respect to \( T \) in order to get the entropy per particle

\[
\frac{s_1}{n_1} = \frac{\tilde{s}_{idA} - 3V_0 n_2 \cdot \tilde{s}_{id2}}{[1 - 3V_0 n_2]},
\]

where the entropy density of point-like particles is defined as \( s_{idA} \equiv \frac{\partial U_A}{\partial T} + \frac{\partial P_{intA}}{\partial T} \), and

\[
\tilde{s}_{idA} \equiv s_{idA} + n_{idA} \left( \frac{\partial U_A}{\partial T} - \frac{\partial P_{intA}}{\partial T} \right).
\]

V. APPLICATION TO NUCLEAR AND HADRONIC MATTER

As a pedagogical example to our discussion we consider the IST EoS for the nuclear matter and compare it with the VdW EoS having the following interaction

\[
P^{VdW}_{int}(T, n_{id}) = a \left[ \frac{n_{id}}{1 + b n_{id}} \right]^4 + T n_{id} - g(T) \frac{n_{id}}{1 + b n_{id}} - g(T) \frac{B_3 n_{id}^3}{1 + b n_{id}} \frac{B_4 n_{id}^4}{1 + b n_{id}} = 0.
\]

Another important case corresponds to the choice \( U_1 > 0 \) and \( U_2 < 0 \), i.e. the mean-field \( U_1 \) describes an attraction, while \( U_2 \) represents a repulsion. Clearly, the condition (35) in this case is also fulfilled for any choice of parameters. Using the self-consistency relations (52), or its more convenient form (5), one can find that the term describing the entropy of mean-field in \( \tilde{s}_{id2} \) can be negative, i.e. \( n_{id2} \frac{\partial U_{id2}}{\partial T} - \frac{\partial P_{int}}{\partial T} = 0 \)\( \int_0^b f_2^2(n) < 0 \), if \( g(T) > 0, \frac{\partial g(T)}{\partial T} > 0 \) and \( U_2 < 0 \) for \( T \geq 0 \). Such a choice of interaction allows one to decrease the effective entropy density \( \tilde{s}_{id2} \) or even to make it negative by tuning the mean-field \( U_2 \) related to the IST coefficient. As a result this would increase the physical entropy density \( s_1 \). Note that for the VdW EoS this is impossible.
where \( n \) \( \exp \) (for the Carnahan-Starling (CS) EoS \[16\] is \( f \) which for the VdW case is \( \eta \) pressure can be written as \[34\] (VdW EoS (GVdW hereafter) was suggested in \[34\]. This concept \[32\].

Results obtained for nuclear matter EoS within the IST mal nuclear matter. This conclusion is supported by the so strong repulsion to reproduce the properties of nor-

The presence of the function \( w(\eta) \) in front of the ideal gas pressure in \[57\] allows one to reproduce the famous CS EoS \[16\] at high temperatures, while it creates the problems with formulating the GVdW model for several hard-core radii, since the pressures of point-like particles of kinds 1 and 2 cannot be added to each other, if their functions \( w(\eta_1) \) and \( w(\eta_2) \) are not the same.

Using the quantum virial expansion \[8\] and the particle number density expression \( n = f(\eta) n_{id}(T, \nu_G) \[34\], for \( P_{IG} \equiv w(\eta) p_{id}(T, \nu_G) \) one obtains

\[
P_{IG} = w(\eta) T \left[ \frac{n}{f(\eta)} + \sum_{i=2}^{\infty} a_i(0) \left( \frac{n}{f(\eta)} \right)^i \right],
\]

(59)

where \( n \) is the particle density, and the multiplier \( w(\eta) \equiv (f(\eta) - \eta f'(\eta)) \) is given in terms of the function \( f(\eta) \) which for the VdW case is \( f^{\text{VdW}}(\eta) = 1 - 4\eta \) and for the Carnahan-Starling (CS) EoS \[16\] is \( f^{\text{CS}}(\eta) = \exp \left[ \frac{(4 - 3\eta)\eta}{1 - \eta^2} \right] \). The interaction terms of the GVdW EoS are given in terms of a function \( u(n) \): \( U_G = u(n) + nu'(n) \) and \( P_{int \ G} = -n^2 u''(n) \). This choice automatically provides the self-consistency condition fulfillment. Since the potentials \( U_G \) and \( P_{int \ G} \) are temperature independent, the Third Law of thermodynamics is obeyed.

Although this effective expansion can be used to derive the true virial expansion for the CS parameterization of the GVdW EoS (for the VdW one it is given above), the result is cumbersome. Nevertheless, these equations show that due to the multiplier \( w(\eta) \) the first term of the quantum virial expansion in Eqs. \[59\], \[8\], \[17\] and \[16\], i.e. the classical term, exactly reproduces the pressure of corresponding classical EoS. Hence, all other terms in Eqs. \[59\], \[17\], \[16\] and \[59\] are the quantum ones. A direct comparison of the IST with \( \alpha = 1.245 \) and CS EoS for classical gases shows that for packing fractions \( \eta > 0.22 \) the IST EoS is softer than the CS one \[24\] [25].

From Fig. 2 one can see that the quantum compressibility factors \( \Delta Z_Q^{\text{CS}}(\eta) = \frac{P_{IG} - w(\eta) n_{id}(T, \nu_G)}{n T} \) for the CS EoS of the GVdW model and the one for the IST EoS \( \Delta Z_Q^{\text{IST}}(\eta) = \frac{P_{IG} - w(\eta) n_{id}(T, \nu_G)}{n T} \) taken both for the same parameters \( b = 3.42 \text{ fm}^3 \), \( P_{int \ G}(T, n) = a_{\text{attr}} n^2 \) with \( a_{\text{attr}} = 329 \text{ MeV-fm}^3 \) (see \[34\] for more details) differ for

**FIG. 1:** Behavior of pressure as a function of particle number density for isotherms of nuclear matter (see text for details).

**FIG. 2:** Packing fraction dependence of the quantum compressibility factors \( \Delta Z_Q \) of the GVdW EoS and IST EoS (see text).
\( \eta \geq 0.05 \). Therefore, for \( \eta \geq 0.1 \) both the classical and the quantum parts of the IST pressure with \( \alpha = 1.245 \) \cite{24} are essentially softer than the corresponding terms of the CS version of GVdW model of Ref. \cite{34}. One can easily understand such a conclusion comparing the expansions \( \left( \ref{59} \right) \) and \( \left( \ref{44} \right) \). Since for the same packing fraction \( \eta \geq 0.1 \) the function \( f^{CS}\left( \eta \right) \) of the CS version of GVdW EoS vanishes essentially faster than the term \( \left[ 1 - 3V_{\text{HR}} \right] \left[ 1 - V_{\text{HR}} \right] \) of the IST EoS, then each term proportional to \( n^5 \) in \( \left( \ref{59} \right) \) with \( k > 1 \) is larger than the corresponding term proportional to \( n^5 \) in \( \left( \ref{44} \right) \). It is necessary to note that such a property is very important because the softer EoS provides a wider range of thermodynamic parameters for which the EoS is causal, i.e. its speed of sound is smaller than the speed of light.

Let’s now discuss the most important constraints on the considered mean-field models which are necessary to describe the strongly interacting matter properties. According to Eqs. \( \left( \ref{17} \right), \left( \ref{46} \right) \) and \( \left( \ref{59} \right) \) the fermionic pressure of considered EoS consists of three contributions: the classical pressure (the first term on the right hand side of \( \left( \ref{17} \right), \left( \ref{46} \right) \) and \( \left( \ref{59} \right) \), the quantum part of pressure and the mean-field \( P_{\text{int}} \). At temperatures below 1 MeV the classical part is negligible, but the usage of virial expansions discussed above is troublesome due to convergence problem. Since the exact parameterization of the function \( P_{\text{int}} \) on the particle number density of nucleons is not known, it is evident that all considered models are effective by construction. To fix their parameters one has to reproduce the usual properties of normal nuclear matter, i.e. to get a zero value for the total pressure at normal nuclear density \( n_0 \approx 0.16 \text{ fm}^{-3} \) and the binding energy \( W = -16 \text{ MeV} \) at this density \cite{1}. Similarly to high temperature case discussed at the end of Section 2 there is exist a freedom of parametrizing the hard-core radius of nucleons, since the attraction pressure can be always adjusted to reproduce the properties of normal nuclear matter and, therefore, all the model parameters are effective by construction.

Fortunately, there is the so called flow constraint at nuclear densities \( n = (2 - 5)n_0 \) \cite{35}, which sets strong restrictions on the model pressure dependence on nuclear particle density and requires rather soft EoS at \( n = (2 - 5)n_0 \). Hence, it can be used to determine the parameters of realistic EoS at high nuclear densities and \( T = 0 \). Traditionally, such a constraint creates troubles for the relativistic mean-field EoS based on Walecka model \cite{21, 36, 37}. Indeed, as one can see from Ref. \cite{36} only 104 of such EoS out of 263 analyzed in \cite{36} are able to obey the flow constraint despite the fact that they have 10 or even more adjustable parameters. At the same time as one can see from the simplest realization of the IST EoS suggested in Ref. \cite{32}, the 4-parametric EoS is able to simultaneously reproduce all properties of normal nuclear matter and the flow constraint. Furthermore, the IST EoS is able not only to reproduce the flow constraint, but simultaneously it is able to successfully describe the neutron star properties with the masses more than two Solar ones \cite{38}, which set another strong constraint on the stiffness of the realistic EoS at high particle densities and zero temperature. On the other hand, Fig. 2 shows that the existing CS version of GVdW EoS of Ref. \cite{34} is very stiff and, hence, it will also have troubles to obey the flow constraint \cite{35}.

From the virial expansions of all models discussed here one sees that the EoS calibration on the properties of nuclear matter at low \( T \) and at high densities involves mainly the quantum and the mean-field pressures, but, unfortunately, it also fixes the parameters of the classical pressure at higher temperatures. It is, however, clear that the one component mean-field models of nuclear matter cannot be applied at temperatures above 50 MeV, since one has to include the mesons, other baryons and their resonances \cite{30, 39}. Moreover, in this case the mean-fields and the parameters of interaction should be recalibrated because the very fact of resonance existence already corresponds to partial accounting of the interaction \cite{39}. For many years it is well known that for temperatures below 170 MeV and densities below \( n_0 \) the mixture of stable hadrons and their resonances whose interaction is taken into account by the quantum second virial coefficients behaves as the mixture of nearly ideal gases of stable particles which, in this case, includes both the hadrons and the resonances, but taken with their averaged masses \cite{39}. The main reason for such a behavior is rooted in a nearly complete cancellation between the attraction and repulsion contributions. The resulting deviation from the ideal gas (a weak repulsion) is usually described in the hadron resonance gas model (HRGM) \cite{18, 20} by the classical second virial coefficients. Nevertheless, such a repulsion is of principal importance for the HRGM, otherwise, if one considers the mixture of ideal gases of all known hadrons and their resonances, then at high temperatures the pressure of such a system will exceed the one of the ideal gas of massless quarks and gluons \cite{40}. Since such a behavior contradicts to the lattice version of quantum chromodynamics, the (weak) hard-core repulsion in the HRGM is absolutely necessary. Moreover, to our best knowledge there is no other approach which is able to include all known hadronic states into consideration and to be consistent with the thermodynamics of lattice quantum chromodynamics at low energy densities and which, simultaneously, would not contradict it at the higher ones. Therefore, it seems that the necessity of weak repulsion between the hadrons is naturally encoded in the smaller values of their hard-core radii \( R_p < 0.4 \text{ fm} \) obtained within the HRGM compared to the larger hard-core radius of nucleons in nuclear matter \( R_n \geq 0.52 \text{ fm} \) found in \cite{33}. This conclusion is well supported by the recent simulations of the neutron star properties with masses more than two Solar ones \cite{38}, which also favor the nucleon hard-core radii below than 0.52 fm. Furthermore, the small values of the hard-core radii provide the fulfillment of the causality condition in hadronic phase \cite{24, 25, 38, 53}, while a possible break of causality occurs in the region where the
hadronic degrees of freedom are not relevant [13]. Hence, in contrast to Ref. [34], we do not see any reason to believe that the mean-field model describing the nuclear matter properties may set any strict conditions on the hadronic hard-core radii of the HRGM.

Moreover, we would like to point out that a great success achieved recently by the HRGM [18–26] sets a strong restriction on any model of hadronic phase which is claimed to be realistic. The point is that at the chemical freeze-out curve \( \mu = \mu_{\text{CFO}}(T) \) the mean-field interaction term of pressure (1) or (28) must vanish, otherwise one would need a special procedure to transform the mean-field potential energy into the masses and kinetic energy of non-interacting hadrons (the kinetic freeze-out problem [41])

Therefore, the HRGM plays a special role in the phenomenology of heavy ion collisions and, hence, any realistic hadronic EoS should be able to reproduce the pressure, entropy and all charge densities obtained by the HRGM at the chemical freeze-out curve \( \mu = \mu_{\text{CFO}}(T) \). In particular, for the mean-field models discussed here it means that they should be extended in order to include all other hadrons and that at the curve \( \mu = \mu_{\text{CFO}}(T) \) the total interaction pressure must vanish, i.e. \( P_{\text{int}} = 0 \), since it does not exist in the HRGM. In other words, if at the chemical freeze-out curve such a model has non-vanishing interaction, then it must have an additional repulsion to provide \( P_{\text{int}} = 0 \). Only this condition will help to avoid a hard mathematical problem of kinetic freeze-out of converting the interacting particles into a gas of free streaming particles [41] [42], since the HRGM with the hard-core repulsion and with vanishing mean-field interaction has the same energy per particle as an ideal gas (see Appendix for details). Also the condition \( P_{\text{int}} = 0 \) at chemical freeze-out curve will make a direct connection between the realistic EoS and the hadron multiplicities measured in heavy ion collisions. It is clear, that without \( T \)-dependent mean-field interaction \( P_{\text{int}} \) such a condition cannot be fulfilled.

Despite many valuable results obtained with the HRGM the hard-core radii are presently well established for the most abundant hadrons only, namely for pions \( (\pi \sim 0.15 \text{ fm}) \), the lightest \( K^- \)-mesons \( (K \sim 0.395 \text{ fm}) \), nucleons \( (p \sim 0.365 \text{ fm}) \) and the lightest \( \Lambda \) hyperons \( (\Lambda \sim 0.085 \text{ fm}) \) [24, 25]. Nevertheless, we hope that the new data of high quality obtained during the Beam Energy Scan II at RHIC BNL 

and FAIR, and the ones which are expected to be measured during the Beam Energy Scan II at RHIC BNL (Brookhaven) [43], and at the accelerators of new generations, i.e. at NICA JINR (Dubna) [46, 47] and FAIR GSI (Darmstadt) [18, 49] will help us to determine their hard-core radii with high accuracy. We have to add only that the IST EoS for quantum gases is well suited for such a task due to additive pressure \( p_{\text{id}}(T, \nu_1, 2) \), whereas the generalization of the CS EoS of Ref. [34] to a multicomponent case looks rather problematic, since the CS EoS [10] is the one component EoS by construction.

VI. CONCLUSIONS

The self-consistent generalization of the IST EoS for quantum gases is worked out. It is shown that with this EoS one can go beyond the VdW approximation at any temperature. The restrictions on the temperature dependence of the mean-field potentials are discussed. It is found that at low temperatures these potentials either should be \( T \)-independent or should vanish faster than the first power of temperature providing the fulfillment of the Third Law of thermodynamics. The same is true for the quantum VdW EoS. Hence, the idea to improve the quantum VdW EoS by tuning the interaction part of pressure [13, 44] is disproved for low temperature \( T \): if this part of pressure is linear in \( T \), then the VdW EoS breaks down the Third Law of thermodynamics; if it vanishes faster than the first power of \( T \), then the interaction part of pressure is useless, since it vanishes faster than the first term of the quantum virial expansion. An alternative EoS [34] allowing one to abandon the VdW approximation for nuclear matter is analyzed here and it is shown that for the same parameters at low temperatures the IST EoS is softer at packing fractions \( \eta \geq 0.05 \).

The virial expansions for quantum VdW and IST EoS are established and the explicit expressions for all quantum virial coefficients, exact for VdW and approximative ones for IST EoS, are given. Therefore, for the first time the analytical expressions for the third and fourth quantum virial coefficients are derived for the EoS which is more realistic than the VdW one. The source of softness of the IST EoS is demonstrated using the effective virial expansion for the effective proper volume which turns out to be compressible. The generalization of the traditional virial expansions for the mixtures of particles with different hard-core radii is straightforward.

The general constraints on realistic EoS for nuclear and hadronic matter are discussed. We hope that using the revealed properties of the IST EoS for quantum gases it will be possible to go far beyond the traditional VdW approximation and that due to its advantages this EoS will become a useful tool for heavy ion physics and for nuclear astrophysics. Furthermore, we hope that the developed EoS will help us to determine the hard-core radii of the hadrons from the new high quality ALICE data and the ones which will be measured at RHIC, NICA and FAIR.

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energy per particle for the gas of point-like particles with
\[ n \]
where Eq. (61) was also used. Approximating the partici-
lation (34) for the distribution functions
\[ s \]
result follows from the condition (35). Then in the low
density limit, i.e. for \( e^{\frac{\pi^2 v}{2}} \simeq 1 \), one gets the re-
lation (34) for the distribution functions \( \phi(k) \), which can be approximated further on as \( \phi(k, T, \nu_v) \simeq \phi(k, T) \) and, therefore, one
finds \( p_{id}(T, \nu_v) \simeq p_{id}(T, \nu_1), \) \( n_{id}(T, \nu_v) \simeq n_{id}(T, \nu_1) \) and
\[ s \]
the energy per particle for the EoS (28) can be found from
the thermodynamic identity
\[ \frac{\epsilon}{n} = T \frac{s}{n} + \frac{p}{n} \cdot (\mu - \frac{n}{1}) \cdot (p_{id} - P_{int \#}) \., \]
Expressing the chemical potential \( \mu \) via an effective one
\[ \nu_1 \] from Eq. (40) one can write \( \mu = \nu_1 + V_0 \rho_{id} - V_0 P_{int \#} + 3V_0 \rho_{id} - 3V_0 P_{int \#} - U_1 \). Substituting this result into Eq. (62), one finds
\[ \frac{\epsilon}{n} \simeq T \frac{s_{id}}{n_{id}} + \nu_1 - U_1 + \left[ V_0 - \frac{1}{n_1} \right] \left( p_{id} - P_{int \#} \right) + 3V_0 \left( p_{id} - P_{int \#} \right) \, \]
where Eq. (61) was also used. Approximating the particle
number density \( n_1 \) in Eq. (41) as
\[ n_1 \simeq \frac{n_{id}}{1 + V_0 n_{id} + 3V_0 n_2} \, \]
and substituting it into Eq. (63), one obtains
\[ \frac{\epsilon}{n} \simeq \frac{\epsilon_{id}}{n_{id}} + 3V_0 \frac{n_2}{n_1} \left( \frac{p_{id}}{n_2} - \frac{p_{id}}{n_{id}} \right) - U_1 - \left[ V_0 - \frac{1}{n_1} \right] P_{int \#} - 3V_0 P_{int \#} \, \]
where we applied the thermodynamic identity (62) to the
the chemical potential \( \nu_1 \). To simplify the evaluation for
the moment we assume that all mean-field interaction
terms obey the following equality
\[ \frac{1 - V_0 \nu_1}{n_1} P_{int \#} / (n_{id}) - 3V_0 P_{int \#} (n_{id}) = U_1 (n_{id}) \, \]
Using in Eq. (65) the first two terms of virial expansion
(8) for the pressures \( p_{id} \) and \( P_{int \#} \) for \( n_2 \) one finds
\[ \frac{p_{id}}{n_2} - \frac{p_{id}}{n_{id}} \simeq T \left[ (1 + a_1 \nu_1) (1 + 3 \alpha \nu_0 n_{id}) \right] - (1 + a_2 \nu_1) \, \]
where in the last step of derivation we used the low den-
sity approximation \( n_{id} \simeq n_{id} \). Finally, under the con-
dition (66) Eq. (65) acquires the form
\[ \frac{\epsilon}{n} \simeq \frac{\epsilon_{id}}{n_{id}} + 9 \alpha V_0^2 n_2 / T (1 + a_2 \nu_1) \, \]
Since the typical packing fractions \( \eta = V_0 n_1 \simeq V_0 n_2 \simeq V_0 n_{id} \) of the hadron resonance gas model at chemical freeze-out do not exceed the value 0.05 [24], then the second term on the right hand side of Eq. (68) is not larger than
\[ 0.025 \alpha T (1 + a_2 \nu_1) \, \]
Comparing this estimate with the energy per particle for
the lightest hadrons, i.e. for pions, in the non-relativistic
limit \( \frac{\epsilon_{id}}{n_{id}} \) \( m_\pi + \frac{3}{2} T \) (here \( m_\pi \simeq 140 \text{ MeV} \)), one can be sure that for temperatures at which the hadron gas exists, i.e. for \( T < 160 \text{ MeV} \), the term (69) is negligible and, hence, with high accuracy one finds \( \frac{\epsilon}{n} \simeq \frac{\epsilon_{id}}{n_{id}} \).

Now let’s discuss the condition (66). It is apparent that in
the general case it can hold, if the mean-field interaction
is absent, i.e. \( U_1 = U_2 = 0 \) and \( P_{int \#} = P_{int \#} = 0 \). This is exactly the case of the hadron resonance gas model. However, one might think that there exist a special case for which Eq. (66) is the simple differential equation for two independent variables \( n_{id} \) and \( n_{id} \). Let’s show that this is impossible. First, with the help of Eq. (44) we rewrite the term \( \frac{1 - V_0 \nu_1}{n_1} = [n_{id} (1 - 3V_0 n_2)]^{-1} \). Then Eq. (66) can be cast as
\[ P_{int \#} (n_{id}) / (n_{id}) - 3V_0 P_{int \#} (n_{id}) = U_1 (n_{id}) \, \]
From this equation one sees that the only possibility to
decouple the dependencies on \( n_{id} \) and \( n_2 \) in the first
above is to assume that \( P_{int \#} = C n_{id} \) where \( C \) is
some constant. However, in this case one finds that the
\( n_{id} \)-dependence of the right hand side of Eq. (70)
remains, since \( U_1 = C \ln(n_{id}) \). Therefore, there is a
single possibility to decouple the functional dependence
of \( n_{id} \) from \( n_2 \), namely that \( C = 0 \) which means that
\( P_{int \#} = 0 \).
One can, however, consider Eq. (70) under the low density approximation assuming that $n_{id2} = n_{id1}$. In this case Eq. (70) defines the functional dependence of $P_{tot2}(n_{id1})$ for any reasonable choice of the potential $U_1(n_{id1})$. Note that in this case the function $P_{tot2}(n_{id1})$ can be rather complicated even for the simplest choice of $U_1(n_{id1})$ and, hence, the practical realization of the dependence (70) seems to be problematic. Therefore, the most direct way to avoid the problem to convert the interacting particles into the free streaming ones \[\text{[41, 42]},\] is to use only the hard-core repulsion between hadrons and set to zero all other interactions at chemical freeze-out.

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