Jacobson generators of the quantum superalgebra 
$U_q[sl(n + 1|m)]$ and Fock representations

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Abstract

As an alternative to Chevalley generators, we introduce Jacobson generators for the quantum superalgebra $U_q[sl(n + 1|m)]$. The expressions of all Cartan-Weyl elements of $U_q[sl(n + 1|m)]$ in terms of these Jacobson generators become very simple. We determine and prove certain triple relations between the Jacobson generators, necessary for a complete set of supercommutation relations between the Cartan-Weyl elements. Fock representations are defined, and a substantial part of this paper is devoted to the computation of the action of Jacobson generators on basis vectors of these Fock spaces. It is also determined when these Fock representations are unitary. Finally, Dyson and Holstein-Primakoff realizations are given, not only for the Jacobson generators, but for all Cartan-Weyl elements of $U_q[sl(n + 1|m)]$.

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I Introduction

The quantization of simple Lie algebras [1, 2] or Lie superalgebras [3, 4, 5, 6, 7] as quasitriangular Hopf (super)algebras has been carried out more than a decade ago. Since then, these structures have received much attention both in the mathematical and physical literature. In a physical context, one is mostly dealing with representations or realizations of these quantized algebras. This is in fact the main topic of the present paper: certain special representations (Fock representations) and related realizations (Dyson and Holstein-Primakoff) of the quantum superalgebra $U_q[sl(n + 1|m)]$ are presented.

The Lie superalgebra $sl(n + 1|m)$ is one of the basic classical simple Lie superalgebras in Kac's classification [8]. It can be considered as the superanalogue of the special linear Lie algebra $sl(n + 1)$. The quantum superalgebra $U_q[sl(n + 1|m)]$ is a Hopf superalgebra deformation of the associative superalgebra $U[sl(n + 1|m)]$, the universal enveloping superalgebra of $sl(n + 1|m)$. At this point, it is already worth observing that the more familiar case of $sl(n + 1)$ and $U_q[sl(n + 1)]$ just follows by putting $m = 0$. The readers who are interested in this case only can still use all formulas presented in this paper, simply taking $m$ equal to 0.

For a definition of the quantum superalgebra $U_q[sl(n + 1|m)]$, we refer to [3, 4, 5, 6, 7]. Usually, $U_q[sl(n + 1|m)]$ is defined by its Chevalley generators (often denoted by $e_i$, $f_i$ and $h_i$, with $i = 1, \ldots, n + m$), subject to the Cartan-Kac relations and the Serre relations [3, 4, 5, 6, 7]. Besides these defining relations, also the other Hopf superalgebra maps (comultiplication, co-unit and antipode) are part of the definition. In this paper, however, we do not use these other Hopf superalgebra maps; so we shall concentrate on $U_q[sl(n + 1|m)]$ as an associative superalgebra.

The definition in terms of Chevalley generators has the advantage that the comultiplication, co-unit and antipode are easy to give. Furthermore, certain representations can be constructed explicitly (e.g. for the essentially typical representations a Gelfand-Zetlin basis exist for which the action of the Chevalley generators is known [3]). Having certain physical applications in mind, however, it is sometimes more useful to work with a different set of generators for $U_q[sl(n + 1|m)]$. The different set of generators for $U_q[sl(n + 1|m)]$ given here are the Jacobson generators (denoted by $a_i^+$, $a_i^-$ and $H_i$, with $i = 1, \ldots, n + m$). For the case of $sl(n + 1)$, such generators
were originally introduced by Jacobson [10, 11]. The use of Jacobson generators has a number of advantages.

First of all, in certain applications it is necessary to have a complete basis of $U_q[sl(n + 1|m)]$ (following from the Poincaré-Birkhoff-Witt theorem). Such a basis is given in terms of the Cartan-Weyl elements. Although it is possible to express all Cartan-Weyl elements in terms of the Chevalley generators, such expressions soon become rather unmanageable. In terms of the Jacobson generators, the description of all Cartan-Weyl elements is very easy.

Secondly, the Jacobson generators allow for the construction of a class of irreducible $U_q[sl(n + 1|m)]$ modules $W_p$, $p \in \mathbb{C}$, called Fock representations. The Fock representations corresponding to different $p$ are inequivalent. For $p$ a positive integer they provide an explicit construction (basis and transformation of the basis under the action of the generators) of (deformations of) atypical representations of $U_q[sl(n + 1|m)]$. This is an interesting mathematical result, since even in the nondeformed case all atypical representations of $sl(n + 1|m)$ were not explicitly constructed so far (e.g. even a dimension formula is unknown).

A disadvantage of the Jacobson generators compared to the Chevalley generators is that the explicit expressions for the other Hopf (super)algebra maps (comultiplication, co-unit and antipode) become very lengthy and complicated.

The results of the present paper provide a mathematical background for further studies of noncanonical quantum statistics initiated in [12] (see also [11] and [13] for further references). The approach is based on the concept of creation and annihilation operators (CAO’s) of a simple Lie (super)algebra $\mathcal{A}$ and its Fock representations [14]. The CAO’s of $\mathcal{A}$ provide a description of $\mathcal{A}$ in terms of generators and relations, which are different from the Chevalley generators. In this terminology any $n$ pairs of para-Fermi operators [15] are CAO’s of $so(2n + 1)$ [16] and any $n$ pairs of para-Bose operators [15] are CAO’s of the orthosymplectic Lie superalgebra $osp(1|2n)$ [17]. The CAO’s of $sl(n + 1)$ [12] and of $sl(1|n)$ [14] lead to new quantum statistics. Generalizing the results of Jacobson on Lie triple systems [10], Okubo has reformulated all above statistics in terms of Lie supertriple systems [18]. In this setting the CAO’s of the Lie (super)algebras mentioned above are generators of the related (super)triple systems. This is another reason to call them Jacobson generators (JG’s). The link between the JG’s and the simple Lie superalgebras
provides a natural background for their $q$-deformations (we refer to [19] for more discussion in this respect).

The representations of (quantum) superalgebras have certainly wider applications. These algebras (and in particular $U_q[gl(n + 1|m)]$ [20]) play a role for finding new solutions of the quantum Yang-Baxter equations and for the construction of solvable models. As examples we mention the supersymmetric solvable $t - J$ models of correlated electrons [21] and their quantum analogue [22]. Some other potential physical applications are mentioned in the last section.

In section II we define the Jacobson generators of $U_q[sl(n + 1|m)]$, as a special subset of the Cartan-Weyl elements. The description of all Cartan-Weyl elements in terms of the Jacobson generators becomes very simple (Theorem 1). However, in order to apply these results (e.g. in representations) one must have a list of all (super)commutation relations between these Cartan-Weyl elements; in terms of Jacobson generators, this means one has to determine certain triple relations. These are also given in Theorem 1, together with their proof.

In section III we define Fock representations for $U_q[sl(n + 1|m)]$, related to the earlier defined Jacobson generators. The main part of this section is devoted to the proof of Theorem 2, describing the action of the Jacobson generators on a basis of the Fock representation. This proof is rather technical and lengthy, and has been divided in a number of lemmas. The essential result is that these Fock representations are labelled by a number $p$; when $p$ is a nonnegative integer, the Fock representation is finite-dimensional.

The Fock representations determined in section III are further analysed in section IV. In particular, following conditions required in a physical context, it is determined when these Fock representations are unitary (or unitarizable, or Hermitian), see Theorem 6. In that case, an orthonormal basis of the Fock space is given, together with the action of the Jacobson generators on these basis elements.

Inspired by the Fock representations, we can give new expressions for the Dyson and Holstein-Primakoff realizations of $U_q[sl(n + 1|m)]$ (section V). In [23], the Dyson and Holstein-Primakoff realizations for the Chevalley generators of $U_q[gl(n|m)]$ was already given. Here, we give Dyson and Holstein-Primakoff realizations for the Jacobson generators of $U_q[sl(n + 1|m)]$ (Theorems 7 and 8); from these, the corresponding realization for all Cartan-Weyl elements are deduced. All these realizations are in terms of $n$ pairs of
Bose and $m$ pairs of Fermi creation and annihilation operators. The Holstein-Primakoff realization becomes particularly simple when expressed in terms of $q$-deformed Bose and Fermi creation and annihilation operators.

Unless otherwise stated, we consider in this paper $U_q[sl(n + 1|m)]$ as a module over the algebra $\mathbb{C}[[h]]$ (with $q = e^h$) of formal power series over an indeterminate $h$. It is important to note however that all considerations remain true if one replaces $h$ by a complex number such that $q = e^h$ is not a root of unity. In fact, most of our results hold also for $q$ being a root of 1, including the unitary Fock representations and the Dyson and Holstein-Primakoff realizations.

Throughout the paper we use the notation: JGs for Jacobson generators; $\mathbb{Z}$ (resp. $\mathbb{Z}_+$) for the set of all integers (resp. of all nonnegative integers); $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ for the ring of all integers modulo 2; $\mathbb{C}$ for all complex numbers. Furthermore:

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \text{ when } x \in \mathbb{C}, \quad (1.1)$$

$$[r; s] = \{r, r + 1, r + 2, \ldots, s - 1, s\}, \text{ for } r \leq s \in \mathbb{Z}; \quad (1.2)$$

$$\theta_i = \begin{cases} \bar{0} & \text{if } i \in [0; n] \\ \bar{1} & \text{if } i \in [n + 1; n + m] \end{cases}; \quad \theta_{ij} = \theta_i + \theta_j; \quad (1.3)$$

$$[a, b] = ab - ba, \quad \{a, b\} = ab + ba, \quad [a, b] = ab - (-1)^{\deg(a) \deg(b)} ba; \quad (1.4)$$

$$[a, b]_x = ab - xba, \quad \{a, b\}_x = ab + xba, \quad [a, b]_x = ab - (-1)^{\deg(a) \deg(b)} xba, \quad (1.5)$$

where $\deg(a) \in \mathbb{Z}_2$ refers to the degree (or grading) of $a$ when $a$ is a homogeneous element of a superalgebra.

II Jacobson generators of $U_q[sl(n + 1|m)]$

Although the quantization ($q$-deformation) of simple Lie algebras and basic Lie superalgebras is usually carried out in terms of their Chevalley generators, there exist alternative descriptions in terms of so-called deformed creation and annihilation operators for the $q$-deformation of $osp(1|2n)$ $[24]$, $so(2n + 1)$ $[23]$, $osp(2n + 1|m)$ $[13]$, $sl(n + 1)$ $[22]$ and $sl(n + 1|m)$ $[27]$. These alternative generators have the advantage that in some natural interpretation they have a direct physical significance; furthermore, they allow the definition and construction of a mathematically interesting and physically important class of irreducible representations, the Fock representations.
The Hopf superalgebra $U_q[sl(n+1|m)]$ is defined in the sense of Drinfeld [1], as a topologically free $\mathbb{C}[[h]]$ module. As a superalgebra, $U_q[sl(n+1|m)]$ is usually defined by means of its Chevalley generators, subject to the Cartan-Kac relations and the Serre relations [3, 4, 7]. Unlike the Lie algebra case, there is an “extra Serre relation” involving the generator associated with an odd simple root [3, 4, 7, 28]. This property was investigated further by Yamane [29, 30]. Indeed, for the basic classical Lie superalgebras there exist many non-isomorphic simple root systems [8]; one of these, having only one odd simple root, is known as the distinguished simple root system [8]. The classical description of $U_q[sl(n+1|m)]$ is in terms of relations and generators associated with this distinguished simple root system. Yamane [29, 30] studied Hopf superalgebras in terms of relations and generators associated with other simple root systems. Apparently, this gives rise to more involved extra Serre relations. Moreover, the structure of the Hopf superalgebra seems to depend on the choice of simple root system [29, 30]. In this paper, $U_q[sl(n+1|m)]$ stands for the usual Hopf superalgebra associated with the distinguished simple root system. But we shall be dealing with an alternative set of generators (and relations) for $U_q[sl(n+1|m)]$.

In this section we shall recall the definition of deformed creation and annihilation operators of $U_q[sl(n+1|m)]$, and refer to them as Jacobson generators (JGs) since they are closely related to generators in the sense of a Lie supertriple system [18] (and for Lie triple systems, such generators were originally introduced by Jacobson [10, 11]). The definition of JGs can be best presented in the framework of a set of Cartan-Weyl elements of $U_q[sl(n+1|m)]$. Furthermore, in order to construct the Fock representations explicitly, it is necessary to have a complete list of so-called triple relations between the JGs. Such relations can be deduced from the supercommutation relations between all Cartan-Weyl elements. So we begin this section by recalling some properties of Cartan-Weyl elements of $U_q[gl(n+1|m)]$, deduced in [31], which are then easily restricted to the case of $U_q[sl(n+1|m)]$.

Although a set of generators, such as the Chevalley generators, is sufficient for the definition of $U_q[gl(n+1|m)]$ as an associative algebra, it is not sufficient for describing a basis of $U_q[gl(n+1|m)]$. For this purpose, the construction of a set of Cartan-Weyl elements is necessary. For $U_q[gl(n+1|m)]$, a set of Cartan-Weyl elements is given by elements $e_{ij}$, with $i, j \in [0; n + m]$; for an explicit expression of these elements $e_{ij}$ in terms of the standard Chevalley generators, see [31]. Finding a set of Cartan-Weyl elements, and their (super)commutation relations, is necessary for the construction of a Poincaré-Birkhoff-
Witt basis of $U_q[gl(n + 1|m)]$. The elements $e_{ij}$ are the $q$-analogues of the defining basis of $gl(n + 1|m)$; their grading is given by $\deg(e_{ij}) = \theta_{ij}$. We shall refer to $e_{ij}$ as a positive root vector (resp. negative root vector) if $i < j$ (resp. $i > j$). For the formulation of the Poincaré-Birkhoff-Witt theorem, it is necessary to fix a total order for the set of elements $e_{ij}$. Among the positive root vectors, this order is given by

$$e_{ij} < e_{kl}, \text{ if } i < k \text{ or } i = k \text{ and } j < l; \quad (2.1)$$

for the negative root vectors $e_{ij}$ one takes the same rule (2.1), and total order is fixed by choosing

positive root vectors < negative root vectors < $e_{ii}$.

The order among the elements $e_{ii}$ is of no real importance since they commute. A complete set of relations between the Cartan-Weyl elements $e_{ij}$ is given by (2.2)-(2.7) (see eqs. (3.10)-(3.15) of [31]):

$$[e_{ii}, e_{jj}] = 0; \quad (2.2)$$

$$[e_{ii}, e_{jk}] = (\delta_{ij} - \delta_{ik})e_{jk}; \quad (2.3)$$

$$(e_{ij})^2 = 0 \text{ if } \theta_{ij} = 1; \quad (2.4)$$

for two positive root vectors $e_{ij} < e_{kl}$:

$$[e_{ij}, e_{kl}]_{q^{(-1)^{\theta_{ij}}\delta_{ij}(-1)^{\theta_{jk}}\delta_{jk}+(-1)^{\theta_{ik}}\delta_{ik}}} = \delta_{jk}e_{il} + (q - q^{-1})(-1)^{\theta_k}\theta(l > j > k > i)e_{kj}e_{il}; \quad (2.5)$$

for two negative root vectors $e_{ij} > e_{kl}$:

$$[e_{ij}, e_{kl}]_{q^{(-1)^{\theta_{ij}}\delta_{ij}(-1)^{\theta_{jk}}\delta_{jk}+(-1)^{\theta_{ik}}\delta_{ik}}} = \delta_{jk}e_{il} - (q - q^{-1})(-1)^{\theta_k}\theta(i > k > j > l)e_{kj}e_{il}; \quad (2.6)$$

and finally for a positive root vector $e_{ij}$ and a negative root vector $e_{kl}$:

$$[e_{ij}, e_{kl}] = \frac{\delta_{il}\delta_{jk}}{q - q^{-1}} \left( q^{e_{ii} - (-1)^{\theta_{ij}}e_{ij}} - q^{-e_{ii} - (-1)^{\theta_{ij}}e_{ij}} \right)$$

$$+ \left( (q - q^{-1})\theta(j > k > i > l)(-1)^{\theta_k}\theta(j > k)(-1)^{\theta_{kl}}e_{kj} + \delta_{jk}\theta(i > l)e_{il} \right)$$

$$\times q^{(-1)^{\theta_k}e_{kk} - (-1)^{\theta_{ij}}e_{ij}} + q^{(-1)^{\theta_i}e_{ii} - (-1)^{\theta_{ij}}e_{ij}} \left( - (q - q^{-1})\theta(k > j > l > i)(-1)^{\theta_k}e_{il}e_{kj} - \delta_{kl}\theta(k > j)(-1)^{\theta_{ij}}e_{kj} + \delta_{jk}\theta(l > i)e_{il} \right). \quad (2.7)$$

Herein,

$$\theta(i_1 > i_2 > \ldots > i_r) = \begin{cases} 
1, & \text{if } i_1 > i_2 > \ldots > i_r, \\
0, & \text{otherwise}. \end{cases} \quad (2.8)$$
The difference between $U_q[sl(n+1|m)]$ and $U_q[gl(n+1|m)]$ lies in the elements of the Cartan subalgebra. For $U_q[gl(n+1|m)]$ the Cartan subalgebra is generated by $e_{ii}$ ($i \in [0; n+m]$). For $U_q[sl(n+1|m)]$ the Cartan subalgebra is generated by the elements $H_i$, with

$$H_i = e_{00} - (-1)^{\theta_i} e_{ii}, \quad i \in [1; n+m].$$  \hspace{1cm} (2.9)

Sometimes, it will be useful to work with the elements $L_i$ and $\bar{L}_i$, where

$$L_i = q^{H_i}, \quad \bar{L}_i = q^{-H_i}, \quad i \in [1; n+m].$$  \hspace{1cm} (2.10)

The Cartan-Weyl elements of $U_q[sl(n+1|m)]$ are now given by

$$\{H_i; i \in [1; n+m]\} \cup \{e_{ij}; i \neq j \in [0; n+m]\}.$$  \hspace{1cm} (2.11)

The complete set of supercommutation relations between these Cartan-Weyl elements is given by

$$[H_i, H_j] = 0;$$  \hspace{1cm} (2.11)

$$[H_i, e_{jk}] = (\delta_{0j} - \delta_{0k} - (-1)^{\theta_i}(\delta_{ij} - \delta_{ik})) e_{jk};$$  \hspace{1cm} (2.12)

and finally the relation between a positive root vector $e_{ij}$ and a negative root vector $e_{kl}$:

$$[e_{ij}, e_{kl}] = \frac{\delta_{il} \delta_{jk}}{q - q^{-1}} (L_j (-1)^{\theta_j} \bar{L}_i (-1)^{\theta_i} - \bar{L}_j (-1)^{\theta_i} L_i (-1)^{\theta_i})$$  \hspace{1cm} (2.13)

$$+ \left((q - q^{-1})\theta(j > k > i > l)(-1)^{\theta_k} e_{kl} e_{il} - \delta_{il} \theta(j > k)(-1)^{\theta_k} e_{kj} + \delta_{jk} \theta(i > l) e_{il}\right) L_i \bar{L}_k$$

$$+ L_j \bar{L}_l \left(-(q - q^{-1})\theta(k > j > l > i)(-1)^{\theta_i} e_{il} e_{kj} - \delta_{il} \theta(k > j)(-1)^{\theta_k} e_{kj} + \delta_{jk} \theta(l > i) e_{il}\right).$$

The Jacobson generators of $U_q[sl(n+1|m)]$ are now defined as the Cartan elements $H_i$ ($i \in [1; n+m]$) together with the elements

$$a^-_i = e_{0i}, \quad a^+_i = e_{i0}, \quad i \in [1; n+m].$$  \hspace{1cm} (2.14)

From (2.13) it is easy to deduce that

$$[a^-_i, a^+_j] = -(-1)^{\theta_i} L_i e_{ji}, \quad (i < j); \quad [a^-_i, a^-_j] = (-1)^{\theta_i} e_{ji} \bar{L}_j, \quad (i > j).$$  \hspace{1cm} (2.15)

However, these relations are not complete in order to reshuffle all Cartan-Weyl elements in an arbitrary expression in the right order. For this purpose, we have the following result:
A set of Cartan-Weyl elements of \( U_q(sl(n+1|m)) \) is given by \( H_i, a_i^\pm, [a_i^+, a_j^-] \) \((i \neq j \in [1; n+m]). \) A complete set of supercommutation relations between these elements is given by :

\[
\begin{align*}
[H_i, H_j] &= 0; \\
[H_i, a_j^\pm] &= \mp(1 + (-1)^{\delta_{ij}})a_j^\pm; \\
[a_i^-, a_j^+] &= \frac{L_i - L_j}{q - q^{-1}}; \\
[a_i^\eta, a_j^\eta]_q &= 0 \quad (i < j); \\
(a_j^\pm)^2 &= 0 \quad (i \in [n + 1; n + m]); \\
[[a_i^\eta, a_j^\eta], a_k^\eta]_{q^{\xi(l+1)\delta_{ijk}}} &= \eta^{\delta_{ij}}\delta_{jk}L_k^\xi a_i^\eta + (-1)^{\theta_k}\epsilon(j,k,i)(q - \bar{q})[a_k^\eta, a_j^-]a_i^\eta \\
&= \eta^{\delta_{ij}}\delta_{jk}L_k^\xi a_i^\eta + (-1)^{\theta_k}\epsilon(j,k,i)q^\xi(q - \bar{q})a_i^\eta [a_k^\eta, a_j^-]a_i^\eta, \\
where (j - i)\xi &> 0, \quad \xi, \eta = \pm \\
and \epsilon(j,k,i) &= \begin{cases} 
1, & \text{if } j > k > i; \\
-1, & \text{if } j < k < i; \\
0, & \text{otherwise,}
\end{cases}
\end{align*}
\]

and we have used the notation \( \bar{q} = q^{-1}. \)

**Proof.** The first part of the statement is obvious. Relation (2.16) follows from (2.11) and (2.12); (2.17) follows from (2.13) with \( l = i < j = k; \) the first relation in (2.18) follows from (2.3) with \( i = k < j < l \) and from (2.6) with \( l = j < k < i, \) whereas the second relation in (2.18) comes from (2.4). Finally, it remains to prove (2.19). There are four similar cases to consider, according to \( \eta = \pm \) and \( \xi = \pm. \) For \( \eta = - \) and \( \xi = +, \) we use the first relation in (2.15) and find :

\[
[[a_i^-, a_j^+], a_k^\eta]_{q^{\xi(l+1)\delta_{ijk}}} = \mp(-1)^{\theta_k}L_i[e_{ji}, a_k^\eta]_{q^{\xi(l+1)\delta_{ijk}}} = \mp(-1)^{\theta_k}L_i[e_{ji}, a_k^\eta] = \mp(-1)^{\theta_k+\theta_i+\theta_j}L_i[e_{0k}, e_{ji}].
\]

Herein, we have used the last equation of (2.16) to change the order of \( L_i \) and \( a_k^- \). For the last supercommutator, we use (2.13) :

\[
[[a_i^-, a_j^+], a_k^-]_{q^{\xi(l+1)\delta_{ijk}}} = (-1)^{\theta_i+\theta_j+\theta_k}L_iL_k\bar{L}_i (-(q - \bar{q})\theta(j > k > i)(-1)^{\theta_k}e_{0k}e_{jk} + \delta_{kj}e_{0i}).
\]

Using trivial properties of the \( \theta_i \)-symbols, the second term in the rhs of this expression becomes \((-1)^{\theta_j}\delta_{jk}L_k a_i^-; \) for the first term we use similar properties and replace according to (2.13) \( e_{jk} \) by \((-1)^{\theta_k}\bar{L}_k[a_k^-, a_j^+]\), so there comes

\[
(-1)^{\theta_k}q(q - \bar{q})\theta(j > k > i)a_i^- [a_k^-, a_j^+].
\]
This coincides with the second expression in (2.19). Exchanging indices $i$ and $k$, and using the relation just obtained, one shows that also the first expression in (2.19) is valid.

For the remaining choices of $\eta$ and $\xi$, the proof is similar. \(\square\)

Finally, we wish to remark that in order to construct $U_q[sl(n+1|m)]$ by means of the JGs subject to a set of relations, not all relations of Theorem 1 are needed. Such a minimal set of relations was determined in [27].

### III Fock representations

In this section we shall construct so-called Fock representations of $U_q[sl(n+1|m)]$. The representations considered here are diagonal with respect to the Cartan elements $H_i$. So it will be convenient to fix $q$ (or $h$) as a complex number in this and in the following section.

The Fock representations, or modules, can be defined by means of an induced module construction. First observe that $G = U_q[sl(n+1|m)]$, with Cartan-Weyl elements $H_i$, $a_i^\pm$ and $[a_i^+, a_j^-]$ ($i \neq j \in [1; n+m]$), has a subalgebra $H = U_q[gl(n|m)]$ with Cartan-Weyl elements $H_i$ and $[a_i^+, a_j^-]$ ($i \neq j \in [1; n+m]$). A trivial one-dimensional $H$ module is defined as follows:

$$[a_i^-, a_j^+]|0\rangle = 0, \quad (i \neq j \in [1; n+m]) \quad (3.1)$$

$$H_i|0\rangle = p|0\rangle, \quad (3.2)$$

where $p$ is any complex number. Let $P$ be the (associative) subalgebra of $G = U_q[sl(n+1|m)]$ generated by the elements of $H$ and $\{a_i^-; i \in [1; n+m]\}$. The one-dimensional module $C|0\rangle$ can be extended to a one-dimensional module of $P$ by requiring:

$$a_i^-|0\rangle = 0, \quad i \in [1; n+m]. \quad (3.3)$$

Now the $G$ module $\bar{W}_p$ is defined as

$$\bar{W}_p = \text{Ind}_P^G C|0\rangle.$$

By construction, this means that $\bar{W}_p$ is freely generated by the generators $a_i^+$ ($i \in [1; n+m]$) acting on $|0\rangle$. In other words, a basis for $\bar{W}_p$ is given by

$$|p; r_1, r_2, \ldots, r_{n+m}\rangle \equiv (a_1^+)^{r_1}(a_2^+)^{r_2}\ldots(a_n^+)^{r_n}(a_{n+1}^+)^{r_{n+1}}(a_{n+2}^+)^{r_{n+2}}\ldots(a_{n+m}^+)^{r_{n+m}}|0\rangle$$

where $r_i \in \mathbb{Z}_+$ for $i \in [1; n]$ and $r_i \in \{0, 1\}$ for $i \in [n+1; m]$. \(3.4\)
So $\tilde{W}_p$ is an infinite-dimensional $G$ module. The main part of this section is devoted to the computation of the action of the JGs on the basis vectors (3.4) of $\tilde{W}_p$. This, of course, completely determines the action of $U_q[sl(n+1|m)]$ on $\tilde{W}_p$.

**Theorem 2** The transformation of the basis (3.4) of $\tilde{W}_p$ under the action of the JGs reads:

$$H_i |p; r_1, r_2, \ldots, r_{n+m}\rangle = \left(p - (-1)^{q_i} r_i - \sum_{j=1}^{n+m} r_j\right) |p; r_1, r_2, \ldots, r_{n+m}\rangle,$$

(3.5)

$$a_i^- |p; r_1, r_2, \ldots, r_{n+m}\rangle = (-1)^{q_i} r_i \left[ \frac{q^{2r_i} - 1}{q^2 - 1} \right]^{-1} [a_i^-, a_j^+] \quad \text{when } i < j,$$

(3.6)

$$a_i^+ |p; r_1, r_2, \ldots, r_{n+m}\rangle = (-1)^{q_i} r_i \left[ \frac{q^{2r_i} - 1}{q^2 - 1} \right]^{-1} [a_i^-, a_j^+] \quad \text{when } i > j,$$

(3.7)

where $i \in [1; n + m]$.

**Proof.** Equation (3.5) is an immediate consequence of $[H_i, a_j^+] = -(1 + (-1)^{q_i} \delta_{ij}) a_j^+$, which is one of the last relations in (2.16). Also the action of $a_i^+$ on the basis vectors is easy: (3.7) follows directly from (2.18). The hard work lies in the proof of (3.6). For this purpose, we shall use a number of technical lemmas.

**Lemma 3** The following relations hold:

- $[A, B_1 B_2 \ldots B_{i-1} B_i B_{i+1} \ldots B_j]_{q^{b_1+b_2+\ldots+b_j}}$

$$= \sum_{i=1}^{j} q^{b_1+b_2+\ldots+b_i-1} (-1)^{\alpha(\beta_1+\ldots+\beta_{i-1})} B_1 B_2 \ldots B_{i-1} [A, B_i]_{q^{b_i}} B_{i+1} \ldots B_j,$$

where $\alpha = \deg(A)$ and $\beta_i = \deg(B_i);$ (3.8)

- $[a_i^-, (a_j^+)^r] = \begin{cases} \frac{q^{2r_i} - 1}{q^2 - 1} (a_j^+) r^{-1} [a_i^-, a_j^+] & \text{when } i < j, \\ \frac{q^{2r_i} - 1}{q^2 - 1} (a_j^+) r^{-1} [a_i^-, a_j^+] & \text{when } i > j; \end{cases}$

(3.9)

- $[a_i^-, (a_i^+)^r] = \left( \frac{q^{2r_i} - 1}{q^2 - 1} L_i - \frac{q^{2r_i} - 1}{q^2 - 1} \right);$

(3.10)

- $[[a_i^-, a_j^+], (a_k^+)^r]_{q^r} = (-1)^{q_r} \frac{q^{2r} - 1}{q^2 - 1} L_i a_j^+(a_k^+) r^{-1}, \quad i > j,$

(3.11)

- $[[a_i^-, a_j^+], (a_k^+)^r]_{q^r} = (-1)^{q_r} (q^{2r_i} - 1) a_j^+(a_k^+) r^{-1} [a_i^-, a_j^+], \quad i > k > j.$

(3.12)
Proof. Equation (3.8) follows by direct calculation. We need to prove equation (3.9) only when \( r > 1 \), i.e. only when \( \theta_j = 0 \). Then one writes, using (3.8),

\[
[a_i^-, (a_j^+)^r] = [a_i^-, (a_j^+)^{r-1}a_j^+] = [a_i^-, (a_j^+)^{r-1}]a_j^+ + (a_j^+)^{r-1}[a_i^-, a_j^+].
\]

Now the result follows using induction on \( r \) and using the triple relation (2.19) with \( k = j \) and \( \eta = - \). The proof of (3.10) is similar, using (3.8), induction on \( r \), and (2.17). Also the proof of (3.11) goes along the same line: first one writes \((a_j^+)^r\) as \((a_i^+)^{r-1}\) for \( r > 1 \); using (3.8) this yields two terms: on the first term one applies (2.19), and on the second term one applies (3.11) by induction; then the result follows. The proof of (3.12) is essentially the same.

\[\square\]

Lemma 4 For \( i > 1 \) the following relation holds:

\[
[a_i^-, a_1^+] (a_2^+)^{r_2} \ldots (a_n^+)^{r_n+m} |0\rangle = -(-1)^{\theta_1+\theta_2+r_2+\cdots+\theta_{i-1}+r_{i-1}} q^{2r_2+\cdots+2r_{i-1}+r_i+\cdots+r_{n+m}-p[r_i]} \times a_1^+ (a_2^+)^{r_2} \ldots (a_{i-1}^+)^{r_{i-1}} (a_i^+)^{r_i-1} (a_{i+1}^+)^{r_{i+1}} \ldots (a_{n+m}^+)^{r_{n+m}} |0\rangle. \tag{3.13}
\]

Proof. Consider first \( i = 2 \). Using (3.8), one finds

\[
[a_2^-, a_1^+] (a_2^+)^{r_2} \ldots (a_n^+)^{r_n+m} |0\rangle = [[a_2^-, a_1^+], (a_2^+)^{r_2} \ldots (a_n^+)^{r_n+m}] q^{2r_2+\cdots+r_{n+m}} |0\rangle = \left(-(-1)^{\theta_1+\theta_2+r_2} q^{2r_2} (a_3^+)^{r_3} \ldots (a_n^+)^{r_n+m} |0\rangle \right) + \left((-1)^{\theta_1+\theta_2+r_2} q^{2r_2} [a_2^-, a_1^+], (a_3^+)^{r_3} \ldots (a_n^+)^{r_n+m} |0\rangle \right). \tag{3.14}
\]

From (3.8) and (2.19) it follows that the second term in the rhs of (3.14) is zero. For the first term, apply (3.11) and use the action of \( \tilde{L}_2 \) as given by (3.7) and (2.10). Then the result follows.

Next we shall use induction on \( i \) to prove (3.13) in general. So suppose (3.13) holds for all \( j = 2, 3, \ldots, i-1 \), i.e.

\[
[a_j^-, a_1^+] (a_2^+)^{r_2} \ldots (a_{n+m}^+)^{r_{n+m}} |0\rangle = -(-1)^{\theta_1+\theta_2+r_2+\cdots+\theta_{j-1}+r_{j-1}} q^{2r_2+\cdots+2r_{j-1}+r_j+\cdots+r_{n+m}-p[r_j]} \times a_1^+ (a_2^+)^{r_2} \ldots (a_{j-1}^+)^{r_{j-1}} (a_j^+)^{r_j-1} (a_{j+1}^+)^{r_{j+1}} \ldots (a_{n+m}^+)^{r_{n+m}} |0\rangle. \tag{3.15}
\]
Making a shift of indices in (3.15) (thereby putting the last \( r_k \)-values equal to zero), leads to the following equivalent equation:

\[
\begin{align*}
[a_i^- , a_j^+ ](a_{j+1}^-)^{r_{j+1}} \cdots (a_{n+m}^+)^{r_{n+m}} |0 \rangle &= -(-1)^{\theta_j + \theta_{j+1} + \theta_{j+2} + \cdots + \theta_{i-1} + \theta_{i} + \theta_{i+1} + \cdots + \theta_{n+m} - p} [r_i] \\
\times a_j^+(a_{j+1}^+)^{r_{j+1}} \cdots (a_{i-1}^+)^{r_{i-1}}(a_{i}^+)^{r_i} \cdots (a_{n+m}^+)^{r_{n+m}} |0 \rangle, \quad (j < i).
\end{align*}
\]

(3.16)

Now consider the lhs of (3.13) and apply (3.8):

\[
\begin{align*}
\sum_{k=2}^{n+m} q^{r_{2} + \cdots + r_{k-1}}(-1)^{\theta_i + \theta_{i+1} + \theta_{i+2} + \cdots + \theta_{k-1} - r_{k-1}} \\
\times (a_2^+)^{r_{2}} \cdots (a_{k-1}^+)^{r_{k-1}}[a_i^-, a_k^+], (a_k^+)^{r_{k}} q^{r_k} (a_{k+1}^+)^{r_{k+1}} \cdots (a_{n+m}^+)^{r_{n+m}} |0 \rangle.
\end{align*}
\]

(3.17)

In this last sum, all terms with \( k > i \) are easily seen to vanish. For the terms with \( k < i \), we apply (3.12), and for the term with \( k = i \), we apply (3.11). Then there comes:

\[
\begin{align*}
\sum_{k=2}^{i-1} (-1)^{\theta_1 + \theta_{2} + \theta_{3} + \cdots + \theta_{k-1} - r_{k-1}} (q^{2r_{k}} - 1) \\
\times a_1^+(a_2^+)^{r_2 \cdots (a_{k-1}^+)^{r_{k-1}}(a_k^+)^{r_{k}} [a_i^-, a_k^+],(a_k^+)^{r_{k+1}} \cdots (a_{n+m}^+)^{r_{n+m}} |0 \rangle \\
- (-1)^{\theta_1 + \theta_{2} + \theta_{3} + \cdots + \theta_{i-1} - r_{i-1}} q^{-p + \sum_{l=i+1}^{n+m} r_l} [r_i] \\
\times a_1^+(a_2^+)^{r_2 \cdots (a_{i-1}^+)^{r_{i-1}}(a_i^+)^{r_i} \cdots (a_{n+m}^+)^{r_{n+m}} |0 \rangle.
\end{align*}
\]

(3.18)

For the terms in (3.18) with \( k < i \), we can apply (3.16). Then it is a matter of appropriately summing all contributions, which leads finally to the rhs of (3.13). \( \square \)

**Proof of Theorem 2** There remains to prove equation (3.6). First, assume that \( i = 1 \) in (3.6); then we have according to (3.8)

\[
\begin{align*}
[a_1^- , a_j^+ ](a_{j+1}^+)^{r_{j+1}} \cdots (a_{n+m}^+)^{r_{n+m}} |0 \rangle &= [[a_1^-, a_1^+], (a_2^+)^{r_2} \cdots (a_{n+m}^+)^{r_{n+m}} |0 \rangle \\
= \sum_{j=2}^{n+m} \sum_{k=2}^{i-1} (-1)^{\theta_1 + \theta_{2} + \theta_{3} + \cdots + \theta_{j-1} - r_{j-1}} \\
\times (a_1^+)^{r_1} \cdots (a_{j-1}^+)^{r_{j-1}} [a_1^-, (a_j^+)^{r_j}], (a_{j+1}^+)^{r_{j+1}} \cdots (a_{n+m}^+)^{r_{n+m}} |0 \rangle.
\end{align*}
\]

(3.19)

The terms with \( j \geq 2 \) in the rhs of (3.19) are found to be zero using (3.3) and (2.19). So only the first term in the rhs of (3.19) gives a contribution; using (3.10) this is

\[
[r_1]|p - \sum_{j=1}^{n+m} r_j + 1||p; r_1 - 1, r_2, \ldots, r_{n+m}\rangle.
\]

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so the case $i = 1$ is proved. Now we use again induction on $i$. So the following equation holds for $j < i$:

$$a_j^- (a_1^+)^{r_1} \cdots (a_{n+m}^+)^{r_{n+m}} |0\rangle = (-1)^{\theta_1 r_1 + \theta_2 r_2 + \cdots + \theta_{j-1} r_{j-1}} q^{r_1 + \cdots + r_{j-1}}$$

$$\times [r_j]p - \sum_{l=1}^{n+m} r_l + 1 |p; r_1, r_2, \ldots, r_{j-1}, r_j - 1, r_{j+1}, \ldots, r_{n+m}\rangle. \quad (3.20)$$

In this equation, put $r_{n+m} = 0$ and raise all indices by 1. Then the following (equivalent) equation holds:

$$a_i^- (a_2^+)^{r_2} \cdots (a_{n+m}^+)^{r_{n+m}} |0\rangle = (-1)^{\theta_2 r_2 + \theta_3 r_3 + \cdots + \theta_{i-1} r_{i-1}} q^{r_2 + \cdots + r_{i-1}}$$

$$\times [r_i]p - \sum_{l=2}^{n+m} r_l + 1 (a_2^+)^{r_2} \cdots (a_{i-1}^+)^{r_{i-1}} (a_i^+)^{r_i-1} (a_{i+1}^+)^{r_{i+1}} \cdots (a_{n+m}^+)^{r_{n+m}} |0\rangle. \quad (3.21)$$

Now consider

$$a_i^- |p; r_1, r_2, \ldots, r_{n+m}\rangle = [a_i^- , (a_1^+)^{r_1} \cdots (a_{n+m}^+)^{r_{n+m}}] |0\rangle$$

$$= [a_i^-, (a_1^+)^{r_1} (a_2^+)^{r_2} \cdots (a_{n+m}^+)^{r_{n+m}}] |0\rangle + (-1)^{\theta_i r_i} [a_i^-, (a_2^+)^{r_2} \cdots (a_{n+m}^+)^{r_{n+m}}] [a_i^- , (a_1^+)^{r_1} \cdots (a_{n+m}^+)^{r_{n+m}}] |0\rangle$$

$$= q^{2r_i-1} [a_1^-, a_i^-] [a_2^+)^{r_2} \cdots (a_{n+m}^+)^{r_{n+m}}] |0\rangle$$

$$+ (-1)^{\theta_i r_i} [a_i^-, (a_2^+)^{r_2} \cdots (a_{n+m}^+)^{r_{n+m}}] |0\rangle. \quad (3.22)$$

This was obtained by applying (1.9) on the first term. The rhs of (3.22) can now be determined as follows: for the first term we use (3.13), and for the second term, we use (3.21) in which both sides have been multiplied (on the left) by $(a_1^+)^{r_1}$. Adding both contributions leads to the desired result.

The action of the elements $H_i$ and $a_i^\pm$ ($i \in [1; n + m]$) on the basis vectors of $\tilde{W}_p$, determined in Theorem 3, clearly imply that $\tilde{W}_p$ has an invariant submodule when $p$ is a nonnegative integer. From now on we shall assume that $p \in \mathbb{Z}_+$. Then we have

**Corollary 5** The $U_q[sl(n + 1|m)]$ module $\tilde{W}_p$ has an invariant submodule $V_p$ with basis vectors

$$|p; r_1, r_2, \ldots, r_{n+m}\rangle, \text{ with } \sum_{i=1}^{n+m} r_i > p.$$ 

The quotient module $W_p = \tilde{W}_p/V_p$ is an irreducible representation for $U_q[sl(n + 1|m)]$. The basis vectors of $W_p$ are given by (the representatives of)

$$|p; r_1, r_2, \ldots, r_{n+m}\rangle, \text{ with } \sum_{i=1}^{n+m} r_i \leq p. \quad (3.23)$$
These finite-dimensional irreducible $U_q[sl(n+1|m)]$ modules $W_p$ are referred to as the Fock modules or Fock representations of $U_q[sl(n+1|m)]$. Also in the Fock modules, the action of the elements $H_i$ and $a_i^{\pm} (i \in [1; n+m])$ on the basis vectors (3.23) is essentially given by the equations of Theorem 2.

One can verify that the irreducible Fock representations $W_p$ are so-called atypical representations of $U_q[sl(n+1|m)]$. Atypicality is usually defined for highest weight representations of simple Lie superalgebras [32], but it can be extended to highest weight representations of the corresponding Hopf superalgebras [33]. In the standard basis, the Dynkin labels of $W_p$ (or of its highest weight) are given by $(p,0,\ldots,0)$. This means that in general the representation $W_p$ is multiply atypical [32, 34]. More precisely, if $n \geq m$, then $W_p$ is $m$-fold atypical; if $n < m$, then $W_p$ is $(n+1)$-fold atypical for $p < m - n$ and $n$-fold atypical for $p \geq m - n$. Observe that in this way we have obtained the action of a set of generators of $U_q[sl(n+1|m)]$ on a class of atypical irreducible representations, i.e. the Fock modules. In general, an explicit basis for atypical representations is not known, not even in the case of $sl(n+1|m)$. For typical representations of $U_q[sl(n+1|m)]$, it is easier to construct a basis. For a subclass of these, the so-called essentially typical representations, a (Gelfand-Zetlin) basis has been constructed together with the action of the Chevalley generators [9].

IV Unitary Fock representations

In this section we select a class of Fock modules important for physical applications. These are the ones for which the standard Fock metric is positive definite, and for which the representatives of $a_i^{\pm}$ and $H_i$ ($i \in [1; n+m]$) satisfy the Hermiticity conditions:

$$(a_i^{+})^\dagger = a_i^{-}, \quad (a_i^{-})^\dagger = a_i^{+}, \quad (H_i)^\dagger = H_i. \quad (4.1)$$

In quantum mechanics, including its generalization to the noncommutative case (see, for instance [35, 36], (4.1) follows from the relations $a_k^{\pm} = \text{const}(x_k \mp ip_k)$ and the requirement that the position operators $x_k$ and the momentum operators $p_k$ should be selfadjoint operators. By definition, representations for which (4.1) holds are said to be unitary (with respect to the anti-involution in $U_q[sl(n+1|m)]$ defined by (4.1), and the Fock space scalar product).
For the Fock representation $W_p$, we can define a Hermitian form $(\ , \ )$ by requiring

$$(0, 0) = \langle 0|0 \rangle = 1,$$  \hspace{1cm} (4.2)$$

and by postulating that the Hermiticity conditions (4.1) should be satisfied, i.e.

$$(a_i^v, w) = (v, a_i^w), \quad \forall v, w \in W_p.$$  \hspace{1cm} (4.3)$$

It is now easy to determine that any two vectors $|p; r_1, r_2, \ldots, r_{n+m}\rangle$ and $|p; r_1', r_2', \ldots, r_{n+m}'\rangle$ with $(r_1, r_2, \ldots, r_{n+m}) \neq (r_1', r_2', \ldots, r_{n+m}')$ are orthogonal. Furthermore, one can compute:

$$(|p; r_1, r_2, \ldots, r_{n+m}\rangle, |p; r_1', r_2', \ldots, r_{n+m}'\rangle) = \frac{[p]!}{p-R!} \prod_{i=1}^{n+m} [r_i]! = \frac{[p]!}{p-R!} \prod_{i=1}^{n} [r_i]!, \quad (4.4)$$

where $R = r_1 + r_2 + \ldots + r_{n+m}$. Clearly, it holds for $R = 0$; then use induction on $R$ together with (3.9)-(3.11).

Assume now that $1 \leq i < j \leq n + m$. According to (4.4) we have

$$(a_i^v a_j^+|0\rangle, a_i^+ a_j^v|0\rangle) = [p][p - 1].$$  \hspace{1cm} (4.5)$$

From (2.18) we have $a_i^+ a_j^+ = (-1)^{\theta_i \theta_j} q a_j^+ a_i^+$ (since $(-1)^{\theta_i \theta_j} = (-1)^{\theta_i}$ for $i < j$); thus we find

$$(a_j^+ a_i^+|0\rangle, a_i^+ a_j^+|0\rangle) = ((-1)^{\theta_i} \bar{q} a_j^+ a_i^+|0\rangle, a_i^+ a_j^+|0\rangle) = (-1)^{\theta_i} \bar{q}^* [p][p - 1], \quad (4.6)$$

where $\bar{q}^*$ is the complex conjugate of $\bar{q} = q^{-1}$. On the other hand, using (3.6),

$$(a_j^+ a_i^+|0\rangle, a_i^+ a_j^+|0\rangle) = (a_i^+|0\rangle, a_j^+|p; 0, \ldots, 0, 1_i, 0, \ldots, 0, 1_j, 0, \ldots, 0)$$

$$= (a_i^+|0\rangle, (-1)^{\theta_i} \bar{q} [p - 1]|p; 0, \ldots, 0, 1_i, 0, \ldots, 0)$$

$$= (-1)^{\theta_i} \bar{q} [p - 1](a_i^+|0\rangle, a_i^+|0\rangle) = (-1)^{\theta_i} \bar{q} [p][p - 1]. \quad (4.7)$$

Herein, $1_i$ stands for a number 1 at the position $i$. When $p \geq 2$, the comparison of (4.6) and (4.7) yields $|q|^2 = 1$. Hence a necessary condition for the Fock space to be unitary is that $q$ must be a phase, i.e.

$$q = e^{i\phi}, \quad (-\pi < \phi < \pi). \quad (4.8)$$
Let us now further investigate when the Fock module is unitary, i.e. when the Hermitian form $(\cdot, \cdot)$ is an inner product. This means that for every $(r_1, \ldots, r_{n+m})$ with $0 \leq R \leq p$, the value in (4.4) should be positive. In particular, this implies that all the numbers $[p], [p - 1], [p - 2], \ldots, [2], [1]$ should be positive. However, since $q = e^{i\phi}$ is a phase, we have

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{\sin(k\phi)}{\sin(\phi)}.$$ 

So we are left with the following question: let $p > 1$, find the values of $\phi (-\pi < \phi < \pi)$ where all of the following functions

$$\frac{\sin(2\phi)}{\sin(\phi)}, \frac{\sin(3\phi)}{\sin(\phi)}, \ldots, \frac{\sin(p\phi)}{\sin(\phi)}$$

are positive. For each of these functions $\frac{\sin(k\phi)}{\sin(\phi)}$, the zeros and hence the signs are easy to determine. So the common domain where all of these functions are positive is given by

$$\frac{-\pi}{p} < \phi < \frac{\pi}{p}.$$ 

Thus we have

**Theorem 6** The irreducible Fock module $W_p$ ($p \geq 2$) is unitary if and only if $q$ is a phase, i.e. $q = e^{i\phi}$, with $\frac{-\pi}{p} < \phi < \frac{\pi}{p}$.

Observe that whether $q$ is a root of unity or not does not have any effect on the irreducibility or unitarity of the Fock module $W_p$, as long as the conditions of Theorem 6 are satisfied. Indeed, suppose that $q = e^{i\phi}$ is a root of unity with $\phi$ a rational multiple of $\pi$ and $\frac{-\pi}{p} < \phi < \frac{\pi}{p}$. Then the smallest integer $N$ for which $q^N = -1$ is greater than $p$. As a consequence, the rhs in (4.4) is never zero. This implies that there are no singular vectors among the weight vectors $[p; r_1, \ldots, r_{n+m}]$, and thus irreducibility holds.

Under the conditions of Theorem 6, we can define an orthonormal basis of $W_p$:

$$[p; r_1, r_2, \ldots, r_{n+m}] = \sqrt{\frac{[p - \sum_{l=1}^{n+m} r_l]!}{[p]! [r_1]! \ldots [r_{n+m}]!}} [p; r_1, r_2, \ldots, r_{n+m}], \quad (4.9)$$
where \(0 \leq \sum_{i=1}^{n+m} r_i \leq p\). In the new basis \(\text{(1.3)}\) the transformation formulas \(\text{(3.5)}-\text{(3.7)}\) read \((i \in [1; n + m])\):

\[
H_i|p; r_1, r_2, \ldots, r_{n+m}\rangle = (p - (-1)^{\theta_i}r_i - \sum_{j=1}^{n+m} r_j) |p; r_1, r_2, \ldots, r_{n+m}\rangle, \quad (4.10)
\]

\[
a_i^-|p; r_1, \ldots, r_{n+m}\rangle = (-1)^{\theta_i(r_i+1)} \times q^{r_i+\ldots+1} \sqrt{|r_i|} |p; r_1, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{n+m}\rangle, \quad (4.11)
\]

\[
a_i^+|p; r_1, \ldots, r_{n+m}\rangle = (-1)^{\theta_i(r_i+1)} \times q^{r_i+\ldots+1} (1 - \theta_i r_i) \sqrt{|r_i|} |p; r_1, \ldots, r_{i-1}, r_i + 1, \ldots, r_{n+m}\rangle. \quad (4.12)
\]

From \(\text{(2.15)}\) it is now easy to determine the action of the remaining Cartan-Weyl generators \(e_{ji}\) on the basis elements of \(W_p\):

\[
e_{ji}|p; r_1, \ldots, r_{n+m}\rangle = (-1)^{\theta_i(r_i+1)+\theta_i+1} q^{r_i+\ldots+1} \times \sqrt{|r_i|} |p; r_1, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{n+m}\rangle, \quad (i < j), \quad (4.13)
\]

\[
e_{ji}|p; r_1, \ldots, r_{n+m}\rangle = (-1)^{\theta_j(r_j+1)+\theta_i} q^{r_j+\ldots+1} \times \sqrt{|r_i|} |p; r_1, \ldots, r_{j-1}, r_j + 1, \ldots, r_{n+m}\rangle, \quad (i > j). \quad (4.14)
\]

In particular, it is possible to extend \(W_p\) to a \(U_q[gl(n + 1|m)]\) module, the actions being given by \(\text{(4.11)}, \text{(4.12)}, \text{(4.13)}, \text{(4.14)}\) and

\[
e_{00}|p; r_1, \ldots, r_{n+m}\rangle = (p - \sum_{l=1}^{n+m} r_l)|p; r_1, \ldots, r_{n+m}\rangle, \quad (4.15)
\]

\[
e_{ii}|p; r_1, \ldots, r_{n+m}\rangle = r_i|p; r_1, \ldots, r_{n+m}\rangle, \quad i \in [1; n + m]. \quad (4.16)
\]

V

**Dyson and Holstein-Primakoff realizations of** \(U_q[sl(n + 1|m)]\)

Consider \((n + m)\) \(\mathbb{Z}_2\)-graded indeterminates \(c_i^\pm (i \in [1; n + m])\) with

\[
\text{deg}(c_i^\pm) = \theta_i. \quad (5.1)
\]
Denote by $W(n|m)$ the free $\mathbb{C}[[h]]$ module (completed in the $h$-adic topology) generated by the elements $c_i^\pm$ subject to the relations
\[ [c_i^-, c_j^+] = \delta_{ij}, \quad [c_i^+, c_j^+] = [c_i^-, c_j^-] = 0. \tag{5.2} \]

As usual, let
\[ N_i = c_i^+ c_i^-, \quad N = \sum_{j=1}^{n+m} N_j. \tag{5.3} \]

The algebra $W(n|m)$ of $n$ pairs of Bose and $m$ pairs of Fermi CAO’s has a natural action in the Fock space $\mathcal{F}(n|m)$, defined as follows. Let $\mathcal{F}(n|m)$ be the free $W(n|m)$ module generated by a vector $|0\rangle$ subject to the relations
\[ c_i^- |0\rangle = 0, \text{ for all } i \in [1; n+m]. \]

Then it follows easily that a basis of $\mathcal{F}(n|m)$ is given by
\[ (c_1^+)^{l_1}(c_2^+)^{l_2} \cdots (c_{n+m}^+)^{l_{n+m}} |0\rangle \equiv |l_1, l_2, \ldots, l_{n+m}\rangle, \tag{5.4} \]
where
\[ l_i \in \mathbb{Z}_+ \text{ for } i \in [1; n] \text{ and } l_i \in \{0, 1\} \text{ for } i \in [n+1; n+m]. \]

The Dyson \cite{37} and Holstein-Primakoff \cite{38} realizations of $U_q[sl(n+1|m)]$ are two different algebra homomorphisms of $U_q[sl(n+1|m)]$ into $W(n|m) \cite{23}$. Since $W(n|m)$ has the natural Fock representation $\mathcal{F}(n|m)$, these realizations will provide representations of $U_q[sl(n+1|m)]$ in $\mathcal{F}(n|m)$.

**Theorem 7 (Dyson realization)** Let $p$ be any complex number. The linear map $\rho : U_q[sl(n+1|m)] \to W(n|m)$, defined on the Jacobson generators by
\[
\begin{align*}
\rho(H_i) &= p - (-1)^\theta c_i^+ c_i^- - \sum_{j=1}^{n+m} c_j^+ c_j^- = p - (-1)^\theta N_i - N, \\
\rho(a_i^-) &= q^{N_i+N_{i-1}+\ldots+N_{i-1}} \frac{[N_i+1]}{N_i+1} [p-N]c_i^-,
\rho(a_i^+) &= \bar{q}^{N_i+N_{i-1}+\ldots+N_{i-1}} c_i^+,
\end{align*}
\tag{5.5} \]
is a (associative algebra) homomorphism of $U_q[sl(n+1|m)]$ into $W(n|m)$.
The inspiration of this mapping comes from Theorem 2. The actual proof of Theorem 3 is straightforward but tedious: one has to verify that all relations in Theorem 1 are satisfied under the substitution of $H_i$ and $a_i^\pm$ by $\rho(H_i)$ and $\rho(a_i^\pm)$. These computations are lengthy and based upon easy relations such as

$$f(N_i)e_j^\pm = e_j^\pm f(N_i \pm \delta_{ij}), \quad i, j \in [1;n+m]; \quad q^{N_i} = 1 - N_i + qN_i \text{ for } i > n,$$

or simple $q$-identities such as $[x + 1][y] - [x][y + 1] = [y - x]$.

The Dyson realization of the JGs of $U_q[sl(n + 1|m)]$ leads to an explicit realization of all Cartan-Weyl elements of $U_q[sl(n + 1|m)]$ in terms of the Bose and Fermi CAO’s. Indeed, using (2.15) and (5.5) one obtains:

$$\rho(e_{ji}) = q^{2\theta_j(N_j-1)+N_j+1+N_{j+2}+\ldots+N_{i-1}}\frac{[N_i+1]}{N_i+1}e_j^+e_i^-, \quad (j < i) \tag{5.6}$$

$$\rho(e_{ji}) = q^{2\theta_iN_i+N_i+1+N_{i+2}+\ldots+N_{j-1}}\frac{[N_j+1]}{N_j+1}e_j^+e_i^-, \quad (j > i). \tag{5.7}$$

In (5.6), the convention is that the summation (in the power of $q$) is 0 when $j = i - 1$ (and similarly for (5.7)). Since $F(n|m)$ is a $W(n|m)$ module, the Dyson realization provides a representation of $U_q[sl(n + 1|m)]$ into $F(n|m)$. It is easy to see that the action of every $\rho(H_i)$ and $\rho(a_i^\pm)$ upon $|l_1, \ldots, l_{n+m}\rangle$ is the same as the action of $H_i$ and $a_i^\pm$ in the representation on $\tilde{W}_p$, given by Theorem 3, under the identification

$$|l_1, \ldots, l_{n+m}\rangle \equiv |p; l_1, \ldots, l_{n+m}\rangle.$$

Therefore, it follows that the representation $\rho$ of $U_q[sl(n + 1|m)]$ into $F(n|m)$ (under the Dyson realization) is irreducible when $p \notin \mathbb{Z}_+$. When $p \in \mathbb{Z}_+$, the representation $\rho$ is indecomposable. The subspace $F_1(n|m)$, spanned on the vectors

$$|l_1, \ldots, l_{n+m}\rangle \text{ with } l_1 + \cdots + l_{n+m} > p$$

is clearly invariant under the action of $U_q[sl(n + 1|m)]$. We denote the (finite dimensional) quotient module by $F_0(n|m) = F(n|m)/F_1(n|m)$, and (by abuse of notation) its vectors are denoted by

$$|l_1, \ldots, l_{n+m}\rangle \text{ with } l_1 + \cdots + l_{n+m} \leq p.$$

For $h$ an indeterminate ($q = e^h$), the representation of $U_q[sl(n + 1|m)]$ into $F_0(n|m)$ is irreducible. It is obvious how to identify $F_0(n|m)$ with $W_p$. 

20
In order to turn $F_0(n|m)$ into a unitary $U_q[sl(n+1|m)]$ module, we introduce the Holstein-Primakoff realization.

**Theorem 8 (Holstein-Primakoff realization)** Let $p \in \mathbb{C}$. The linear map $\rho : U_q[sl(n+1|m)] \rightarrow W(n|m)$, defined on the Jacobson generators by

\[
\begin{align*}
\rho(H_i) &= p - (-1)^\theta c_i^+ c_i^- - \sum_{j=1}^{n+m} c_j^+ c_j^- = p - (-1)^\theta N_i - N, \\
\rho(a_i^-) &= q^{N_1+\ldots+N_{i-1}} \sqrt{\frac{[N_i+1]}{N_i+1}} [p - N] c_i^-,
\end{align*}
\]

\[
\rho(a_i^+) = q^{N_1+\ldots+N_{i-1}} \sqrt{\frac{[N_i]}{N_i}} [p - N + 1] c_i^+,
\]

is a homomorphism of $U_q[sl(n+1|m)]$ into $W(n|m)$.

Let us now also consider the special case that $p$ is a positive integer. Just as in the previous case, the subspace $F_1(n|m)$ is invariant for the action of $U_q[sl(n+1|m)]$ under $\rho$ when $p \in \mathbb{Z}_+$. It is clearly invariant under the action of $U_q[sl(n+1|m)]$. Let us consider the following basis of the (finite dimensional) quotient module $F_0(n|m)$:

\[
\begin{align*}
\frac{(c_1^+)^{l_1}(c_2^+)^{l_2}\ldots(c_{n+m}^+)^{l_{n+m}}}{\sqrt{l_1! l_2! \ldots l_{n+m}!}} |0\rangle &\equiv |l_1, l_2, \ldots, l_{n+m}\rangle, \quad l_1 + \ldots + l_{n+m} \leq p. \quad (5.9)
\end{align*}
\]

It is easy to verify that the action of every $\rho(H_i)$ and $\rho(a_i^\pm)$ upon $|l_1,\ldots,l_{n+m}\rangle$ is the same as the action of $H_i$ and $a_i^\pm$ in the representation on $W_p$ given by (4.11)-(4.12), under the identification

\[
|l_1,\ldots,l_{n+m}\rangle \equiv |p;l_1,\ldots,l_{n+m}\rangle.
\]

Therefore, it follows that the representation $\rho$ of $U_q[sl(n+1|m)]$ into $F_0(n|m)$ (under the Holstein-Primakoff realization with $p \in \mathbb{Z}_+$) is an irreducible unitary module when

\[q = e^{i\phi} \text{ with } -\frac{\pi}{p} < \phi < \frac{\pi}{p}.\]

From (2.13) and (5.8), one obtains the Holstein-Primakoff realization of the remaining Cartan-Weyl elements of $U_q[sl(n+1|m)]$:

\[
\begin{align*}
\rho(e_{ji}) &= q^{2\delta_j(N_j-1)+N_{j+1}+N_{j+2}+\ldots+N_{i-1}} \sqrt{\frac{[N_j][N_i+1]}{N_j(N_i+1)}} c_j^+ c_i^- \quad (j < i), \\
\rho(e_{ji}) &= q^{2\delta_i(N_i-1)+N_{i+1}+N_{i+2}+\ldots+N_{j-1}} \sqrt{\frac{[N_j][N_i+1]}{N_j(N_i+1)}} c_j^+ c_i^- \quad (j > i). \quad (5.10)
\end{align*}
\]
Observe that there is an alternative description of the Holstein-Primakoff realization, in terms of deformed Bose \cite{39, 40, 41, 42} and Fermi \cite{43} CAO's \( \tilde{c}^\pm_i \) defined as

\[
\tilde{c}^-_i = \sqrt{\frac{N_i + 1}{N_i + 1}} c^-_i, \quad \tilde{c}^+_i = \frac{N_i}{N_i} c^+_i, \quad \tilde{N}_i = N_i, \quad i \in [1; n + m]. \tag{5.11}
\]

These elements of \( W(n|m) \) satisfy the relations

\[
[\tilde{c}^-_i, \tilde{c}^+_j] = \delta_{ij} \tilde{q}^{(-1)^{\delta_{ij}} \tilde{N}_i}, \quad [\tilde{N}_i, \tilde{c}^\pm_j] = \pm \delta_{ij} \tilde{c}^\pm_j, \quad [\tilde{c}^\pm_i, \tilde{c}^\mp_j] = [\tilde{N}_i, \tilde{N}_j] = 0. \tag{5.12}
\]

The Holstein-Primakoff realization can be rewritten in terms of these deformed Bose and Fermi operators \( \tilde{c}^\pm_i \). We give it here for all Cartan-Weyl elements:

\[
\varrho(H_i) = p - (-1)^{\delta_{0i}} \tilde{N}_i - \tilde{N}, \quad \varrho(a^-_i) = \tilde{q}^{\tilde{N}_1 + \cdots + \tilde{N}_{i-1}} \sqrt{[p - \tilde{N}] \tilde{c}^-_i}, \\
\varrho(a^+_i) = \tilde{q}^{\tilde{N}_1 + \cdots + \tilde{N}_{i-1}} \sqrt{[p - \tilde{N} + 1] \tilde{c}^+_i}, \\
\varrho(e_{ji}) = q^{2\theta_i (\tilde{N}_j - 1)} \tilde{N}_{j+1} + \tilde{N}_{j+2} + \cdots + \tilde{N}_{i-1} \tilde{c}^+_j \tilde{c}^-_i, \quad (j < i), \\
\varrho(e_{ji}) = q^{2\theta_j (\tilde{N}_i - 1)} \tilde{N}_{i+1} + \tilde{N}_{i+2} + \cdots + \tilde{N}_{j-1} \tilde{c}^+_j \tilde{c}^-_i, \quad (j > i). \tag{5.13}
\]

Furthermore, this is easy to extend to a Holstein-Primakoff realization of \( U_q[gl(n+1|m)] \) by

\[
\varrho(e_{00}) = p - \tilde{N}, \quad \varrho(e_{ii}) = \tilde{N}_i. \tag{5.14}
\]

The Holstein-Primakoff realization given us a realization in terms of oscillators (in (5.8) and (5.10)) or \( q \)-oscillators (in (5.13)). Observe that this oscillator realization is different from the one given by Floreanini et al \cite{43} : in \cite{43} only the Chevalley generators are realized in terms of oscillators or \( q \)-oscillators. Furthermore all generators are bilinear expressions in the oscillators, whereas here the JGs are linear expressions in the oscillators.

**VI Conclusions**

We have constructed a class of representations of the quantum superalgebra \( U_q[sl(n+1|m)] \), which was also extended to \( U_q[gl(n+1|m)] \). Our approach is entirely along the lines of Fock representations of parastatistics of order \( p \), for which the defining relations are given by (3.1)-(3.3). The analogy with parastatistics goes further : within the Fock representations, the JGs \( a^\pm_i \) can be interpreted as operators creating or annihilating (quasi)particles, or excitations of a new kind of quantum statistics.
In order to be more concrete consider a Hamiltonian \( H = \sum_{i=1}^{n+m} \varepsilon_i e_{ii} \). Then, see (4.16),

\[
H|p; r_1, \ldots, r_{n+m}) = \sum_{i=1}^{n+m} \varepsilon_i r_i |p; r_1, \ldots, r_{n+m})
\]

Therefore the vector \( |p; r_1, \ldots, r_{n+m}) \) can be interpreted as a state consisting of \( r_1 \) particles with energy \( \varepsilon_1 \), \( r_2 \) particles with energy \( \varepsilon_2 \), and so on, \( r_{n+m} \) particles with energy \( \varepsilon_{n+m} \). Moreover, according to (4.11)-(4.12) any operator \( a_i^+ \) (resp. \( a_i^- \)) creates (resp. annihilates) a particle on the orbital \( i \). Since \( r_i \in \mathbb{Z}_+ \) for \( i \in [1; n] \) and \( r_i \in \{0, 1\} \) for \( i \in [n+1; m] \) the particles on the first \( n \) orbitals behave like bosons, and the particles on the next orbitals like fermions. This is however not quite the case if \( p < n + m \), since \( \sum_{i=1}^{n+m} r_i \leq p \). In other words the system cannot accommodate more than \( p \) particles. Therefore the statistics falls in the group of exclusion statistics in the broad sense [44]: the number of particles to be accommodated on a certain orbital depends on the number of particles already accommodated in the system. What are the properties of the underlying statistics is a question still to be answered.

Another property worth to be studied is to analyze what happens if \( p \to \infty \) and \( q \to 1 \). Having in mind the results from [11] we expect that in this limit the operators \( A(p, q)_i^\pm = a_i^\pm / \sqrt{p} \) become genuine Bose CAO’s for \( i \in [1; n] \) and genuine Fermi CAO’s for \( i \in [n+1; n+m] \). If so, then for large \( p \) and values of \( q \) close to 1 the operators \( A(p, q)_i^\pm \) describe small deviations from the canonical quantum statistics. Moreover these CAO’s are defined in a state space with positive definite scalar product. Among the various noncanonical statistics (Gentile intermediate statistics [15], parastatistics [15], infinite statistics [46], parons [47], quons [48]) only the quons have the same property. Therefore parallel to quons the \( A(p, q)_i^\pm \) operators may appear as another candidate to describe eventual small violations of canonical quantum statistics (see [17] where also experiments for detecting small violations of statistics are discussed).

We believe also (having in mind again the results in [11]) that the CAO’s of \( U_q[sl(n + 1|m)] \) and their Fock representations will be natural “building blocks” for any multicomponent \( t – J \) supersymmetric lattice model. To this end we note that at each site \( i \) the Hubbard operators \( X^0_k \) and \( X^{k0} \) (we suppress the site index) are nothing but nondeformed Jacobson generators \( a_k^- \) and \( a_k^+ \), respectively. Then the representations with \( p = 1 \) satisfy the hard-core condition forbidding configurations with two or more particles to be accommodated simultaneously on each lattice site.
Some of the results related to this quantum statistics were already published in an earlier paper [50]. Let us underline the new contributions in the present paper. Theorem 1 (section II) was already stated without proof in [50], since it is the main ingredient to describe the quantum statistics; here we have given its relevant background and a complete proof. Sections III and IV contain our key results; all of them are original. We have constructed a class of representations of \( U_q[sl(n+1|m)] \) labelled by an arbitrary number \( p \). When \( p \) is a positive integer, this representation is indecomposable and the corresponding quotient module is finite dimensional. The derivation of the action of the JGs on basis elements of these representations is highly nontrivial. In section IV we have selected the unitary representations, with respect to the (in physics) natural Hermiticity condition (4.1) considered as an anti-involution, and the requirement that the usual Fock space metric should be positive definite. It is interesting to note that the selected representations remain irreducible when \( q \) is a root of unity.

The Dyson and Holstein-Primakoff realizations of \( U_q[sl(n+1|m)] \) were given in an earlier paper by one of us [23], but only for the Chevalley generators. Here, in section V, we give the realization for all Cartan-Weyl elements of \( U_q[sl(n+1|m)] \). Such realizations are relevant since also in the classical case \((q = 1)\) the realization of all Cartan-Weyl generators (Bargmann-Schwinger realizations, ladder representations) are of physical importance. Observe that it is far from trivial to deduce the realization for all Cartan-Weyl elements from those of the realization for the Chevalley generators. This would be rather hard because the expressions of all Cartan-Weyl elements in terms of the Chevalley generators are very involved and difficult to manage, see e.g. [31]. In the present case, the problem was overcome because we were able to give the Dyson and Holstein-Primakoff realizations of the Jacobson generators of \( U_q[sl(n+1|m)] \). Since the expressions of the remaining Cartan-Weyl elements in terms of the Jacobson generators is simple, the Dyson and Holstein-Primakoff realizations of all Cartan-Weyl elements followed without too much trouble.

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