Global existence, boundedness and asymptotic behavior to a logistic chemotaxis model with density-signal governed sensitivity and signal absorption*

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Abstract

In present paper, we consider a chemotaxis consumption system with density-signal governed sensitivity and logistic source:

\[ u_t = \Delta u - \nabla \cdot \left( \frac{S(u)}{v} \nabla v \right) + ru - \mu u^2, \quad v_t = \Delta v - uv, \]

in a smooth bounded domain \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)), where parameters \( r, \mu > 0 \) and density governed sensitivity fulfills \( S(u) \simeq u(u+1)^{\beta-1} \) for all \( u \geq 0 \) with \( \beta \in \mathbb{R} \). It is proved that for any \( r, \mu > 0 \), there exists a global classical solution if \( \beta < 1 \) and \( n \geq 2 \). Moreover, the global boundedness and the asymptotic behavior of the classical solution are determined for the case \( \beta \in [0,1) \) in two dimensional setting, that is, the global solution \((u, v)\) is uniformly bounded in time and \((u, v, |\nabla v|) \to \left( \frac{r}{\mu}, 0, 0 \right) \) in \( L^\infty(\Omega) \) as \( t \to \infty \), provided \( \mu \) sufficiently large.

2010MSC: 35B35, 35B40, 35K55, 92C17

Keywords: Chemotaxis; Global boundedness; Signal-dependent sensitivity; Logistic source

1 Introduction

In this paper, we consider the following chemotaxis consumption system with general sensitivity and logistic source:

\[
\begin{aligned}
& u_t = \Delta u - \nabla \cdot \left( \frac{S(u)}{v} \nabla v \right) + ru - \mu u^2, \quad x \in \Omega, \quad t > 0, \\
& v_t = \Delta v - uv, \quad x \in \Omega, \quad t > 0, \\
& \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
& (u(x,0), v(x,0)) = (u_0(x), v_0(x)), \quad x \in \Omega,
\end{aligned}
\]

(1.1)

*Supported by the National Natural Science Foundation of China (11571020, 11671021).
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in a bounded and smooth domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\), where \( \partial/\partial n \) denotes the derivative with respect to the outer normal of \( \partial \Omega \), and the initial data \((u_0, v_0)\) satisfies

\[
\begin{aligned}
&u_0(x) \in C^0(\overline{\Omega}), \ u_0(x) \geq 0 \text{ with } u_0(x) \not\equiv 0, \ x \in \overline{\Omega}, \\
v_0(x) \in W^{2,\infty}(\Omega), \ v_0(x) > 0, \ x \in \overline{\Omega}, \ \text{and } \frac{\partial v_0(x)}{\partial n} = 0, \ x \in \partial \Omega.
\end{aligned}
\] (1.2)

In the model (1.1), the bacteria (with density \(u\)) move towards the location with higher concentration gradient of oxygen \(v\) (with concentration \(v\)), which involves general chemotactic cross-diffusion mechanisms with the density-dependent sensitivity \(S(u)\) and signal-dependent sensitivity \(\varphi(v) = \frac{1}{v}\) \(v\) as a nutriment is consumed by \(u\) through contacting. Moreover, (1.1) also characterizes the cells-kinetics mechanism, which is exhibited by logistic source \(f(u) = ru - \mu u^2\) with \(r, \mu > 0\). This model is a variant of a phenomenological system introduced by Keller and Segel [12] as follows:

\[
\begin{aligned}
&u_t = \Delta u - \nabla \cdot \left( \frac{\mu}{v} \nabla v \right), \ x \in \Omega, \ t > 0, \\
v_t = \Delta v - uv, \quad x \in \Omega, \ t > 0,
\end{aligned}
\] (1.3)

which captures the experimental works about motion of bacteria placed in one end of a capillary tube containing oxygen [1, 2]. It is important to note that there are mathematical difficulties in treating system (1.3), caused by singular chemotaxis sensitivity with absorption of \(v\). See more detailed arguments in [13, 16].

For some related Keller-Segel models (cf. [10, 11]), \(v\) does not stand for a nutrient to be consumed but a chemical signal actively secreted by bacteria (or cells) themselves, i.e. the evolution is governed by

\[
\begin{aligned}
&u_t = \Delta u - \chi \nabla \cdot \left( \frac{\mu}{v} \nabla v \right) + f(u), \ x \in \Omega, \ t > 0, \\
v_t = \Delta v - v + u, \quad x \in \Omega, \ t > 0,
\end{aligned}
\] (1.4)

where logistic function \(f(u) \in C^0[0, \infty)\) with \(f(0) \geq 0\). For the case of \(f(u) \equiv 0\), [6] gave uniform-in-time boundedness of solutions to (1.4) if \(\chi < \sqrt{\frac{2}{n}}\). Lankeit established global existence and boundedness of solutions in a convex two-dimensional domain for \(\chi \in (0, \chi_0)\) with some \(\chi_0 > 1\) [15]. Moreover, [25] showed the existence of weak solutions to (1.4) as long as \(\chi < \sqrt{\frac{n+2}{3n-4}}\). In [20], a generalized solution was constructed under radially symmetric setting, and certain global bounded solution was obtained regardless of the size of \(\chi > 0\). For the case of \(f(u) = ru - \mu u^2\), it has been proved that the system (1.4) with \(n = 2\) possesses a global classical solution for any \(r \in \mathbb{R}, \chi, \mu > 0\), and the global solution is bounded if \(r > \frac{\chi^2}{2}\) for \(0 < \chi \leq 2\), or \(r > \chi - 1\) for \(\chi > 2\) [30]. Also see, e.g., [18, 8, 7] for results to the corresponding parabolic-elliptic models with signal-dependent sensitivity or logistic source.

Now, turn back to a chemotaxis consumption system as follow:

\[
\begin{aligned}
&u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), \ x \in \Omega, \ t > 0, \\
v_t = \Delta v - uv, \quad x \in \Omega, \ t > 0.
\end{aligned}
\] (1.5)
where \( \chi > 0 \) and \( f(s) \in C^0[0, \infty) \) with \( f(0) \geq 0 \). When \( \chi = 1 \) and \( f(u) \equiv 0 \), the global existence of classical solutions has been established for large-data with \( n = 2 \) [22] or small initial data with \( n \geq 3 \) [21]. Moreover, Tao and Winkler presented the problem (1.5) under convexity hypothesis admits at least one global weak solution for \( n = 3 \), and eventual smoothness as well as stabilization of the weak solutions has also been discussed in [22]. When \( f(u) = ru - \mu u^2 \) with \( r \in \mathbb{R} \) and \( \mu > 0 \), [14] emphasized the effects of logistic source and gave the global boundedness result of classical solutions to (1.5) provided \( \mu \) suitably large. Also, [14] proved that there exists a global weak solution to (1.5) for any \( \mu > 0 \). In addition, the chemotaxis consumption model with nonlinear diffusion and density-signal dependent sensitivity, i.e.

\[
\begin{aligned}
\begin{cases}
  u_t = \nabla \cdot (D(u) \nabla u - \frac{S(u)}{\chi} \nabla v), & x \in \Omega, \ t > 0, \\
  v_t = \Delta v - uv, & x \in \Omega, \ t > 0,
\end{cases}
\end{aligned}
\tag{1.6}
\]

has also been studied recently, where \( D(s) \in C^1([0, \infty)) \) and \( S \in C^2([0, \infty)) \) with \( S(0) = 0 \). For the case of \( D(u) \equiv 1 \) and \( S(u) = \chi u (\chi > 0) \), if \( n = 2 \), Winkler gave the global existence of a generalized solution to (1.5) with \( v \to 0 \) in \( L^p(\Omega) \) as \( t \to \infty \) [26], and the solution becomes eventually smooth and converges to the homogeneous steady state as long as the initial mass \( \int_{\Omega} u_0 dx \) is small enough [27]. In particular, under an explicit smallness condition on \( u_0 \ln u_0 \in L^1(\Omega) \) and \( \nabla \ln v_0 \in L^2(\Omega) \), the system (1.5) possesses a global classical solution [27]. If \( \Omega \subset \mathbb{R}^n \) with \( n \geq 2 \) is a ball, [28] constructed a global renormalized solution, and moreover this established that the renormalized solution solves (1.5) classically in \((\Omega \setminus \{0\}) \times [0, \infty)\). For the case of \( D(u) \geq \delta u^\alpha \) (\( \delta > 0, \alpha \geq 1 \)) and \( S(u) = u \), Lankeit [13] proved for \( \alpha > 1 + \frac{n}{4} \) with \( n \geq 2 \) there is a global classical solution to (1.5) under strict positivity of nonlinear diffusion \( D \), or a global weak solution for degenerate case \( D(0) = 0 \). For the case of \( D(u) \equiv 1 \) and \( 0 < S(u) \leq \chi(u + 1)^\beta \) (\( \chi, \beta > 0 \)), [17] presented that global classical solution exists for (1.5) when either \( n = 1 \) with \( \beta < 2 \) or \( n \geq 2 \) with \( \beta < 1 - \frac{4}{n} \).

The aim of this paper is to give the global existence of classical solutions to (1.1) and determine the asymptotic behavior of the solutions for \( n = 2 \). In present work, we assume density-dependent sensitivity \( S \in C^2([0, \infty)) \) with \( S(0) = 0 \) satisfies

\[
b_0 u(u + 1)^{\beta - 1} \leq S(u) \leq b_1 u(u + 1)^{\beta - 1} \quad \text{for all } u > 0,
\tag{1.7}
\]

where parameters \( b_0, b_1 > 0 \) with \( b_1 \geq b_0 \) and \( \beta \in \mathbb{R} \).

Under these hypotheses, we state the following theorem to demonstrate the global existence of solutions to (1.1).

**Theorem 1.** Let \( \Omega \subset \mathbb{R}^n \ (n \geq 2) \) be a bounded domain with smooth boundary and \( r, \mu > 0 \). Assume that \( S \) satisfies (1.7) with \( \beta < 1 \). Then for any initial data \( (u_0, v_0) \) as in (1.2), the problem (1.1) possesses a global classical solution \((u, v)\).

**Remark 1.** Focused on the problem (1.1) without logistic source, Liu presented the global existence of classical solution under the hypothesis of \( \beta < 1 - \frac{4}{n} \) [17]. Here, thanks to
the effects of the logistic source on properties of solutions, we can reduce the requirement of parameter $\beta$ to the condition $\beta < 1$ for the desired conclusion. Besides, in comparison to the problem (1.1) with linear density-dependent sensitivity (namely $S(u) \equiv \chi_0 u$), [16] gave a global existence result provided $\chi_0 < \sqrt{\frac{2}{n}}$ and $\mu > \frac{n-2}{n}$, whereas for our situation $S(u) \simeq \chi_0 u^\beta$ ($u \geq 1$) with $\beta < 1$, the global existence conclusion still holds regardless of the size of $\mu, \chi_0 > 0$, since the density-dependent sensitivity has a sublinear growth for $u > 1$.

For dimension $n = 2$ and $0 < \beta < 1$, we can give the global boundedness result, that is,

**Theorem 2.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that $r > 0$ and $S$ satisfies (1.7) with $0 < \beta < 1$, then there exists $\mu_* > 0$ with property: the condition $\mu > \mu_*$ ensures the solution $(u, v)$ of (1.1) is globally bounded.

**Remark 2.** For the Keller-Segel system (1.1) with a general logistic source $f(u)$ satisfying $f(u) \leq ru - \mu u^k$, the results obtained in Theorem 1&2 are still valid if $k > 2$.

Moreover, with even large $\mu$, the asymptotic stability of solutions also can be obtained as below:

**Theorem 3.** Under the conditions of Theorem 2. Then there exists $\mu^* > \mu_*$ with property: if $\mu > \mu^*$, the global bounded solutions presented in Theorem 2 satisfy

$$(u, v, \frac{\nabla u}{v}) \longrightarrow \left(\frac{r}{\mu}, 0, 0\right) \text{ in } L^\infty(\Omega) \text{ as } t \to \infty.$$

**Ideas of proof for Theorem 3.** In order to research the asymptotic behavior of global solutions (ensured by Theorem 1) to system (1.1), we first utilize logistic source to find some $t_1 > 0$ such that estimates $\int_{t_1}^t u(\cdot, t) \simeq \frac{1}{t}$ and $\int_{t_1}^t \int_{t_1}^t u^2 \simeq \frac{1}{t^\beta}$ are valid for all $t > t_1$. It should be mentioned that the aforementioned estimates are revelent to $\mu$: if we set $\mu$ large, then this can make it small for the terms $\int_{t_1}^t \int_{\Omega} u^2$ and $\int_{\Omega} u(\cdot, t)$.

Next, we introduce an energy functional for problem (2.2) with the form

$$F(u, w)(t) := \int_{\Omega} G(u(x, t))dx + \frac{1}{2} \int_{\Omega} |\nabla w(x, t)|^2 dx,$$

$$G(s) = \int_{s^\mu}^{s} \int_{T_\sigma}^{s} \frac{1}{S(\sigma)} d\sigma d\rho$$

here, $w$ is defined as (2.1). By taking $\mu$ properly large (namely, taking $\int_{t_1}^t \int_{\Omega} u^2$ and $\int_{\Omega} u(\cdot, t)$ small), it can be obtain that $F(u, w)(t)$ decreases from some point $t_*$ in time after $t_1$, which is exhibited in Lemma 4.1. As a direct result of the monotonicity and structure for $F(u, w)$, we can claim that $\int_{\Omega} |\nabla w(\cdot, t)|^2 \simeq \left(\frac{1}{\mu}\right)^2$ with $t > t_*$. Starting with the estimates presented above and afresh enlarging parameter $\mu$, the $L^2$-boundedness of $u$ is established together with the estimate to the integral $\int_{\Omega} |\nabla w(\cdot, t)|^4$ via doing energy estimates. Followed by these, we can get a bound of $u$ and $\nabla w$ in $L^\infty(\Omega)$ by means of semigroup estimates, which presents the global boundedness of the classical solutions to (1.1).
Then, we certainly need to pursue in some estimate like \( \sup_{s \in (t, \infty)} \mu^p \|u(\cdot, t) - \bar{u}(t)\|_{L^p(\Omega)} \to 0 \) as \( \mu \to 0 \) for some large \( t > t_* \) and \( p, \gamma > 1 \) (the condition of \( \gamma > 1 \) is necessary in our arguments). To this end, we give a lower bound of \( \lim \inf \) and a pair of functions \( (u, v) \) in the form of \( \|u(\cdot, t) - \bar{u}(t)\|_{L^\infty(\Omega)} \) and a lower bound of \( \bar{u}(t) \). Subsequently, setting \( U(x, t) := u(x, t) - \bar{u}(t) \) and doing energy estimates via the evolution equation of \( U \) lead to an estimate of \( \|U(\cdot, t)\|_{L^2(\Omega)} \simeq \left( \frac{\ln \mu}{\mu} \right)^{\frac{3}{2}} \) (see Lemma 5.4).

Finally, we turn the bound on \( \int_\Omega U^2(\cdot, t) \) into the asymptotic stability of \( U(\cdot, t) \) in \( L^\infty(\Omega) \) by applying similar arguments as [29, Lemma 7.1]. It’s noteworthy that the system of \((U, v)\) in present paper is distinct from that of \((U, v)\) in [29]. Hence, we introduce a definition of \( T \) involving both \( \|\nabla w\|_{L^\infty(\Omega)} \) and \( \|U\|_{L^\infty(\Omega)} \) (see (5.22)), to control the natural growth term \( |\nabla w|^2 \) in (5.11). Find more details in the proof of Theorem 3.

This paper is organized as follows. Sections 2 gives the local existence of solutions to (1.1), some fundamental estimates of \((u, v)\) and semigroup estimates as preliminaries. Then we establish a crucial estimate of the integral \( \int_\Omega u^p v^{-q} \) with properly large \( p \) in Section 3, which leads to the global existence of classical solutions to (1.1) (Theorem 1). Finally, under the conditions of dimension \( n = 2 \) and parameter \( 0 \leq \beta < 1 \), Section 4&5 are devoted to discuss the global boundedness and the asymptotic stability of the solutions (Theorem 2&3).

## 2 Local existence and some properties

We begin with the local existence of classical solutions to (1.1), the proof of which is standard. Refer to, e.g., [9, 15] for details.

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) be a bounded domain with smooth boundary. Assume that (1.7) is valid for \( S \) and initial data \((u_0, v_0)\) fulfills (1.2). Then there exist \( T_{\max} \in (0, \infty) \) and a pair of functions \((u, v)\) from \( C^{2, 1}(\bar{\Omega} \times (0, T_{\max})) \cap C(\bar{\Omega} \times [0, T_{\max}]) \) satisfying (1.1) classically. Here, either \( T_{\max} = \infty \), or \( \lim_{t \to T_{\max}} u(\cdot, t) \|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{H^1, 0(\Omega)} = \infty \), or \( \liminf_{t \to T_{\max}} \inf_{x \in \Omega} v(x, t) = 0 \). Moreover, we have \( u, v > 0 \) in \( \Omega \times (0, T_{\max}) \).

In order to get some essential estimates, we do a transformation of \( v \) ensured by Lemma 2.1 as [13, 31]. Denote

\[
\begin{align*}
w(x, t) := & - \ln \frac{v(x, t)}{\|v_0(x)\|_{L^\infty(\Omega)}}. \\
\end{align*}
\]  

(2.1)

Apparently (1.1) with (2.1) yields that \( w_t = \Delta w - |\nabla w|^2 + u \) on \( \Omega \times (0, T_{\max}) \), and then we have that the pair \((u, w)\) solves the following system

\[
\begin{cases}
u_t = \Delta u + \nabla \cdot (S(u) \nabla w) + ru - \mu u^2, & x \in \Omega, \quad t > 0, \\
w_t = \Delta w - |\nabla w|^2 + u, & x \in \Omega, \quad t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, \quad t > 0, \\
(u(x, 0), w(x, 0)) = (u_0(x), - \ln \frac{v_0(x)}{\|v_0(x)\|_{L^\infty(\Omega)}}), & x \in \Omega.
\end{cases}
\]  

(2.2)
Now we introduce some basic estimates of $u, v$ and $w$.

**Lemma 2.2.** Let $(u, v)$ be a solution of (1.1) and $w$ be defined as (2.1). Then the following estimate

$$
\int_{\Omega} u(x,t) dx \leq C, \ \forall \ t \in (0, T_{\text{max}})
$$

holds with $C = C(r, \mu) > 0$. In addition, we have

$$
0 < v(x,t) \leq \|v_0\|_{L^\infty(\Omega)}, \ w(x,t) \geq 0, \ \forall \ (x,t) \in \Omega \times (0, T_{\text{max}}).
$$

**Proof.** It is obvious from (1.1), the H"older inequality and Young’s inequality that

$$
\frac{d}{dt} \int_{\Omega} u(x,t) dx = r \int_{\Omega} u(x,t) dx - \mu \int_{\Omega} u^2(x,t) dx,
$$

$$
\leq r \int_{\Omega} u(x,t) dx - \frac{\mu}{|\Omega|} \left( \int_{\Omega} u(x,t) dx \right)^2
$$

$$
\leq -\frac{\mu}{2|\Omega|} \left( \int_{\Omega} u(x,t) dx \right)^2 + \frac{r^2|\Omega|}{2\mu}, \ \forall \ t \in (0, T_{\text{max}}).
$$

This along with an argument of ODI entails (2.3), and see (2.4) in [16, Lemma 4.1&3.6].

**Lemma 2.3.** ([24, Lemma 1.3] and [3, Lemma 2.1]) Let $n \geq 2$, $0 < T \leq \infty$, $\{e^{t\Delta}\}_{t \geq 0}$ be the Neumann heat semigroup in $\Omega$, and $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in $\Omega \subset \mathbb{R}^n$ under the Neumann boundary condition. Then there exist $K_1, \ldots, K_4 > 0$ depending on $\Omega$ only such that the following estimates hold.

(i) If $1 \leq q \leq p \leq \infty$, then

$$
\|e^{t\Delta} w\|_{L^p(\Omega)} \leq K_1 (1 + t^{-\frac{n}{2} \left(1 - \frac{1}{p}\right)}) e^{-\lambda_1 t \|w\|_{L^q(\Omega)}}, \ \forall \ t \in (0, T)
$$

is true for all $w \in L^q(\Omega)$ satisfying $\int_{\Omega} w = 0$.

(ii) If $1 \leq q \leq p < \infty$, then

$$
\|
abla e^{t\Delta} w\|_{L^p(\Omega)} \leq K_2 (1 + t^{-\frac{n}{2} \left(1 - \frac{1}{p}\right)} - \frac{1}{p}) e^{-\lambda_1 t \|\nabla w\|_{L^q(\Omega)}}, \ \forall \ t \in (0, T)
$$

holds for each $w \in L^q(\Omega)$.

(iii) If $2 \leq q \leq p \leq \infty$, then

$$
\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \leq K_3 (1 + t^{-\frac{n}{2} \left(1 - \frac{1}{q}\right)}) e^{-\lambda_1 t \|\nabla w\|_{L^q(\Omega)}}, \ \forall \ t \in (0, T)
$$

is valid for all $w \in W^{1,q}(\Omega)$. 

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(iv) If \(1 < q \leq p < \infty\) or \(1 < q < \infty\) and \(p = \infty\), then
\[
\|e^{t\Delta} \nabla \cdot w\|_{L^p(\Omega)} \leq K_4(1 + t^{-\frac{1}{2}} \eta \frac{1}{2}(1 - \frac{1}{p}))e^{-\lambda t}\|w\|_{L^q(\Omega)}, \quad \forall \ t \in (0, T)
\]
holds for all \(w \in (L^q(\Omega))^n\).

We give some properties of solutions for an differential inequality as a lemma here, which is important to obtain the boundedness result.

**Lemma 2.4.** Let \(0 < t_0 < T \leq \infty\) and \(\eta, \chi > 0\). Suppose that \(y(t), h(t), g(t)\) are nonnegative integrable functions defined on \([t_0, T]\) and \(y \in C^0[t_0, T) \cap C^1(t_0, T)\) fulfills
\[
y'(t) + (\chi - \eta y(t))h(t) + g(t) \leq 0, \quad \forall \ t \in (t_0, T). \tag{2.6}
\]
If \(y(t_0) < \frac{\chi}{2\eta}\), then we have \(y' \leq 0\) on \([t_0, T]\). Moreover, the following estimate
\[
y(t) + \frac{1}{2} \int_{t_0}^t h(\tau)d\tau + \int_{t_0}^t g(\tau)d\tau < y(t_0) \tag{2.7}
\]
is true for any \(t \in [t_0, T]\).

**proof.** We claim that for any \(t \in [t_0, T)\), the estimate
\[
y(t) < \frac{\chi}{2\eta} \tag{2.8}
\]
holds. If (2.8) were false, there would be \(T^* \in (t_0, T)\) satisfying \(y < \frac{\chi}{2\eta}\) on \([t_0, T^*)\) and \(y(T^*) = \frac{\chi}{2\eta}(T^* = \sup\{t \in (t_0, T) \mid y(s) < \frac{\chi}{2\eta}, \ \forall \ s \in [t_0, t]\}\) is well defined). Thus, it could be derived from (2.6) that
\[
y'(t) + g(t) \leq 0, \quad \forall \ t \in (t_0, T^*]. \tag{2.9}
\]
(2.9) along with nonnegativity of \(g\) would lead to \(y(T^*) \leq y(t_0) < \frac{\chi}{2\eta}\), which produces a contraction with \(y(T^*) = \frac{\chi}{2\eta}\). Hence, \(y < \frac{\chi}{2\eta}\) and \(y' \leq 0\) are true for all \(t \in [t_0, T]\). This combined with an integration of (2.6) infers (2.7) readily. \(\square\)

In present work, we will use extended versions of the Gagliardo-Nirenberg inequality.

**Lemma 2.5.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary,
(i) If \(0 < q < \infty, \ s > 0\) and \(\gamma > 0\). Assume \(p \in [q, \infty)\) fulfilling
\[
a = \frac{1}{q} - \frac{1}{s} - \frac{1}{2} \in (0, 1).
\]
Then there is \(C_{GN} = C_{GN}(p, q, s, \Omega) > 0\) such that
\[
\|\varphi\|_{L^p(\Omega)}^\gamma \leq C_{GN}\|\nabla \varphi\|_{L^2(\Omega)}^{a\gamma} \|\varphi\|_{L^2(\Omega)}^{(1-a)\gamma} + C_{GN}\|\varphi\|_{L^s(\Omega)}^\gamma
\]
holds for all \(\varphi \in W^{1,2}(\Omega) \cap L^2(\Omega) \cap L^s(\Omega)\).
(ii) There exists \( L_1 = L_1(Ω) > 0 \) fulfilling
\[
\|\nabla ϕ\|_{L^4(Ω)}^4 \leq \frac{L_1}{2} \|Δ ϕ\|_{L^2(Ω)}^2 \|\nabla ϕ\|_{L^2(Ω)}^2
\]
for all \( ϕ \in W^{2,2}(Ω) \) with \( \frac{∂ϕ}{∂n} = 0 \) on \( ∂Ω \).

(iii) There is \( L_2 = L_2(Ω) > 0 \) such that
\[
\|ϕ\|_{L^3(Ω)}^3 \leq L_2 \|ϕ\|_{W^{1,2}(Ω)}^2 \|ϕ\|_{L^1(Ω)} + L_2 \|ϕ\|_{L^1(Ω)}^3
\]
is valid for any \( ϕ \in W^{1,2}(Ω) \).

**Proof.** See [13, Lemma 3.4], [28] for the items (i) and (ii). As for (iii), the Gagliardo-Nirenberg inequality yields that
\[
\|ϕ\|_{L^3(Ω)}^3 \leq C_1 \|\nabla ϕ\|_{L^2(Ω)}^2 \|ϕ\|_{L^1(Ω)} + C_1 \|ϕ\|_{L^2(Ω)}^2 \|ϕ\|_{L^1(Ω)}
\]
with \( C_1 = C_1(Ω) > 0 \). It follows from the Hölder inequality that
\[
\|ϕ\|_{L^2(Ω)}^2 \leq \|ϕ\|_{L^3(Ω)}^\frac{2}{3} \|ϕ\|_{L^1(Ω)}^{\frac{1}{3}}.
\]
Substituting this into (2.12) shows
\[
\frac{1}{2} \|ϕ\|_{L^3(Ω)}^3 \leq C_1 \|\nabla ϕ\|_{L^2(Ω)}^2 \|ϕ\|_{L^1(Ω)} + \frac{C_1^2}{2} \|ϕ\|_{L^1(Ω)}^3
\]
by Cauchy’s inequality. Taking \( L_2 = \max\{2C_1, C_1^2\} \) ends our proof.

3 Global existence of solutions

This section is devoted to give the global existence of solutions to (1.1). Let \((u, v)\) be the local classical solution ensured by Lemma 2.1, then for any \( T \in (0, T_{max}] \) satisfying \( T < \infty \), we shall develop a crucial estimate of \( \intΩ u^p v^{-q} \) with proper \( p, q > 0 \), which is resolved in following three steps.

**Lemma 3.1.** For each \( q > 0 \), we have
\[
\frac{d}{dt} \intΩ v^{-q} dx = -(q + 1)q \intΩ \frac{|∇ v|^2}{v^{q+2}} dx + q \intΩ \frac{u}{v^q} dx, \quad ∀ t \in (0, T).
\]

**Proof.** Multiplying the second equation in (1.1) by \( -\frac{q}{v^{q+1}} \) and integrating by parts over \( Ω \), we obtain (3.1) easily.

**Lemma 3.2.** Assume that (1.7) is valid for \( S \) with \( β < 1 \), and \( r, µ > 0 \). Then for any \( p, q > 0 \), we have following estimate
\[
\frac{d}{dt} \intΩ u^p v^{-q} dx = -(p - 1)p \intΩ u^{p-2} v^{-q} |∇ u|^2 dx - (q + 1)q \intΩ u^p v^{-q-2} |∇ v|^2 dx
\]
+ 2pq \int_\Omega u^{p-1}v^{-q-1}|\nabla u||\nabla v|dx + p(p - 1)b_1 \int_\Omega u^{p-2+\beta}v^{-q-1}|\nabla u||\nabla v|dx \\
+ rp \int_\Omega u^pv^{-q}dx + (q - \mu p) \int_\Omega u^{p+1}v^{-q}dx, \quad \forall t \in (0, T). \quad (3.2)

**Proof.** Differentiate \( \int_\Omega u^pv^{-q}dx \) with respect to \( t \), we have from (1.1) that

\[
\frac{d}{dt} \int_\Omega u^pv^{-q}dx = p \int_\Omega u^{p-1}v^{-q}u_t dx - q \int_\Omega u^pv^{-q-1}v_t dx \\
= p \int_\Omega u^{p-1}v^{-q}(\Delta u - \nabla \cdot (\frac{S(u)}{v}\nabla v) + ru - \mu u^2)dx \\
- q \int_\Omega u^pv^{-q-1}(\Delta v - vu)dx. \quad (3.3)
\]

Due to (1.7) with \( \beta < 1 \), it is easy to check that \( 0 \leq S(u) \leq b_1 u^\beta \). Integrating the terms on the right side of (3.3) by parts yields

\[
\int_\Omega u^{p-1}v^{-q}(\Delta u - \nabla \cdot (\frac{S(u)}{v}\nabla v) + ru - \mu u^2)dx \\
= -(p - 1) \int_\Omega u^{p-2}v^{-q} |\nabla u|^2 dx + q \int_\Omega u^{p-1}v^{-q-1} \nabla u \cdot \nabla v dx \\
- q \int_\Omega u^{p-1}v^{-q-2} S(u) |\nabla v|^2 dx + (p - 1) b_1 \int_\Omega u^{p+\beta-2}v^{-q-1} |\nabla u||\nabla v| dx + r \int_\Omega u^{p-1}v^{-q}dx - \mu \int_\Omega u^{p+1}v^{-q}dx \quad (3.4)
\]

and

\[
- \int_\Omega u^pv^{-q-1}(\Delta v - vu)dx = -(q + 1) \int_\Omega u^{p-2}v^{-q-2} |\nabla v|^2 dx \\
+ p \int_\Omega u^{p-1}v^{-q-1} \nabla u \cdot \nabla v dx - \int_\Omega u^{p+1}v^{-q}dx \\
\leq -(q + 1) \int_\Omega u^{p-2}v^{-q-2} |\nabla v|^2 dx \\
+ p \int_\Omega u^{p-1}v^{-q-1} |\nabla u||\nabla v| dx - \int_\Omega u^{p+1}v^{-q}dx. \quad (3.5)
\]

Finally the assertion (3.2) follows by combing (3.3)-(3.5).

**Lemma 3.3.** Assume that (1.7) is valid for \( S \) with \( \beta < 1 \) and \( r, \mu > 0 \). Then for any \( p > 1 \), we have

\[
\sup_{t \in [0, T]} \|u(\cdot, t)\|_{L^p(\Omega)} < \infty.
\]
**Proof.** For any fixed $p > 1$, we pick some $0 < q < \min\{\mu p, p - 1\}$. It is easy to find out $\frac{p^2 q^2}{(p-1)p} < q(q+1)$ due to $q < p - 1$, hence there is $p^* \in (1, p)$ fulfilling $\frac{p^2 q^2}{(p^* - 1)p^*} < q(q+1)$. An application of Cauchy's inequality implies that

$$2pq \int_\Omega u^{p-1}v^{-q-1}|\nabla u||\nabla v|dx \leq (p^* - 1)p^* \int_\Omega w^{p-2}v^{-q}|\nabla u|^2dx + \frac{p^2q^2}{(p^* - 1)p^*} \int_\Omega w^{p-2}v^{-q}|\nabla v|^2dx. \quad (3.6)$$

By using Cauchy's inequality again and recalling the fact $\beta < 1$, it can be obtained that for any $\varepsilon_1, \varepsilon_2 > 0$

$$p(p - 1)b_1 \int_\Omega u^{p-2+\beta}v^{-q-1}|\nabla u||\nabla v|dx$$

$$\leq \varepsilon_1 \int_\Omega u^{p-2}v^{-q}|\nabla u|^2dx + \frac{4p^2(p - 1)b_1^2}{\varepsilon_1} \int_\Omega u^{p+2\beta}v^{-q-2}|\nabla v|^2dx$$

$$\leq \varepsilon_1 \int_\Omega u^{p-2}v^{-q}|\nabla u|^2dx + \int_\Omega u^p v^{-q-2}|\nabla v|^2dx + C_1 \int_\Omega v^{-q-2}|\nabla v|^2dx, \quad (3.7)$$

here $C_1 = C_1(\varepsilon_1, \varepsilon_2, p, q, \beta, b_1) > 0$. With taking $\varepsilon_1 = (p-1)p - (p^* - 1)p^*$ and $\varepsilon_2 = (q + 1)q - \frac{p^2q^2}{(p^* - 1)p^*}$, (3.6)-(3.7) results in

$$2pq \int_\Omega u^{p-1}v^{-q-1}|\nabla u||\nabla v|dx + p(p - 1)b_1 \int_\Omega u^{p-2+\beta}v^{-q-1}|\nabla u||\nabla v|dx$$

$$\leq (p - 1)p \int_\Omega u^{p-2}v^{-q}|\nabla u|^2dx + (q + 1)q \int_\Omega u^p v^{-q-2}|\nabla v|^2dx + C_2 \int_\Omega v^{-q-2}|\nabla v|^2dx \quad (3.8)$$

with $C_2 = C_2(p, q, \beta, b_1) > 0$. In light of (3.1), (3.2) and (3.8), we have

$$\frac{d}{dt} \left( \int_\Omega u^p v^{-q}dx + \frac{C_2}{(q+1)q} \int_\Omega v^{-q}dx \right) \leq rp \int_\Omega u^p v^{-q}dx + \frac{C_2}{q+1} \int_\Omega uv^{-q}dx + (q - \mu p) \int_\Omega u^{p+1}v^{-q}dx$$

$$\leq (rp + \frac{C_2}{q+1}) \int_\Omega u^p v^{-q}dx + \frac{C_2}{q+1} \int_\Omega v^{-q}dx \quad (3.9)$$

due to $q < \mu p$. By an argument of ODI, we can find $C_3(T) > 0$ relying on $T, p, q, \beta, b_1, r$ such that

$$\int_\Omega u^p v^{-q}dx < C_3(T), \quad \forall t \in (0, T).$$
This in conjunction with (2.4) leads to
\[ \int_{\Omega} u^p dx \leq \| v_0 \|_{L^\infty(\Omega)}^q \int_{\Omega} u^{p-q} dx < C_4(T), \quad \forall \ t \in (0, T) \]
with \( C_4(T) = C_4(T, p, q, \beta, b_1, r) > 0 \). The proof is finished. \( \Box \)

Thus, we get the boundedness of \( \int_{\Omega} u^p \) for any \( p > 1 \) (apparently valid for each \( p > n+1 \)). According to positivity of \( w \) and (2.2), we have for any \( t > 0 \)
\[ \| w(\cdot, t) \|_{L^\infty(\Omega)} \leq \| e^{t\Delta} w_0 \|_{L^\infty(\Omega)} + \int_0^t \| e^{(t-s)\Delta} u \|_{L^\infty(\Omega)} ds \]
\[ \leq K_1 \| w_0 \|_{L^\infty(\Omega)} + K_1 \int_0^t \left( 1 + (t-s)^{-\frac{n}{2}} \right) e^{-\lambda_1(t-s)} \| u \|_{L^n(\Omega)} ds \]
which entails the \( L^\infty \)-boundedness of \( w \) (see details in [16, Lemma 4.2]). Then, it follows from the definition of \( w \) that \( v(x, t) > C_5(T) \) on \( \Omega \times (0, T) \) with some \( C_5(T) > 0 \) (see details in [16, Lemma 4.3]). Meanwhile, (1.1) tells that for any \( t > 0 \)
\[ \| \nabla v(\cdot, t) \|_{L^\infty(\Omega)} \leq \| e^{t\Delta} v_0 \|_{L^\infty(\Omega)} + \int_0^t \| e^{(t-s)\Delta} u \|_{L^\infty(\Omega)} ds \]
\[ \leq K_3 \| \nabla v_0 \|_{L^\infty(\Omega)} + K_2 \sup_{s \in (0,t)} \| v(\cdot, s) \|_{L^\infty(\Omega)} \]
\[ \times \int_0^t \left( 1 + (t-s)^{-\frac{n}{2(n+1)}} \right) e^{-\lambda_1(t-s)} \| u \|_{L^{n+1}(\Omega)} ds, \quad \forall \ t \in (0, T). \]
This combined with (2.4) and the bound of \( \int_{\Omega} u^{n+1} dx \) implies \( \sup_{t \in (0,T)} \| \nabla v(\cdot, t) \|_{L^\infty(\Omega)} < C_6(T) \) with some \( C_6(T) > 0 \).

Because of the results above, we can give the proof of Theorem 1.

Proof of Theorem 1. Choosing \( p \) large enough and invoking Lemma 3.3, we can find some \( C(T) = C(T, \beta, b_1, r, \mu) > 0 \) fulfilling \( \int_{\Omega} u^p dx < C(T) \). This combined with the lower bound of \( v \) and the \( L^\infty \)-bound of \( \nabla v \) in enables us to use the well-known Moser-Alikakos iteration technique to (1.1) (see a survey in [23, Appendix]) and find \( C'(T) = C'(T, \beta, b_1, r, \mu, \Omega) > 0 \) satisfying
\[ \sup_{t \in [0,T]} \| u(\cdot, t) \|_{L^\infty(\Omega)} < C'(T), \]
which along with the extensibility criterion provided by Lemma 2.1 guarantees the global existence of classical solutions to (1.1). \( \Box \)
4 Global boundedness of solutions

In this section, we research the global boundedness of the classical solution \((u, v)\) presented in Theorem 1, under conditions of \(n = 2\) and \(0 \leq \beta < 1\). At first, we give some foundational estimates of the solution \((u, w)\) to the problem (2.2), which is demonstrated in following lemma and these boundedness results are ensured by logistic source.

**Lemma 4.1.** For any \(\mu > 0\), there exists \(t_1 > 0\) such that
\[
\int_{\Omega} u(x, t)dx \leq \frac{2|\Omega|r}{\mu}, \quad \forall \ t > t_1.
\]
Moreover,
\[
\int_{t_1}^{t} \int_{\Omega} u^2dxds \leq \frac{2|\Omega|r^2}{\mu^2} (t - t_1) + \frac{2|\Omega|r}{\mu^2}, \quad \forall \ t > t_1,
\]
\[
\int_{\Omega} w(x, t)dx + \int_{t_1}^{t} \int_{\Omega} |\nabla w|^2dxds \leq \frac{2|\Omega|r}{\mu} (t - t_1) + \int_{\Omega} w(x, t_1)dx, \quad \forall \ t > t_1.
\]

**Proof.** According to (2.5), we have
\[
\frac{d}{dt} \int_{\Omega} udx = r \int_{\Omega} udx - \mu \int_{\Omega} u^2dx \leq r \int_{\Omega} udx - \frac{\mu}{|\Omega|} \left( \int_{\Omega} udx \right)^2, \quad \forall \ t > 0,
\]
which along with an application of the Bernoulli inequality [4, Lemma 1.2.4] leads to
\[
\limsup_{t \to \infty} \int_{\Omega} u(x, t)dx \leq \frac{|\Omega|r}{\mu}.
\]
Hence, we can find \(t_1 > 0\) satisfying \(\int_{\Omega} u(x, t)dx \leq \frac{2|\Omega|r}{\mu}\) for any \(t > t_1\). And this together with (4.4) infers (4.2) by an integration of \(t\). In addition, we have from (2.2) and (4.1) that
\[
\frac{d}{dt} \int_{\Omega} wdx = - \int_{\Omega} |\nabla w|^2dx + \int_{\Omega} udx \leq - \int_{\Omega} |\nabla w|^2dx + \frac{2|\Omega|r}{\mu}, \quad \forall \ t > t_1.
\]
This yields (4.3) readily. \(\square\)

On account of estimates (4.1)-(4.3) provided by logistic source, we introduce an energy functional concerning \((u, w)\) and assert the boundedness of this functional for \(t\) properly large. Denote
\[
G(s) := \int_{\mu}^{s} \int_{\rho}^{S(\sigma)} \frac{1}{S(\sigma)} d\sigma d\rho, \quad \forall \ s > 0
\]
and
\[
\mathcal{F}(u, w)(t) := \int_{\Omega} G(u(x, t))dx + \frac{1}{2} \int_{\Omega} |\nabla w(x, t)|^2dx, \quad \forall \ t > 0.
\]

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Based on (1.7) with $\beta < 1$, it can be deduced by integration by parts that
\[
G(s) \leq \int_{\frac{s}{\mu}}^{s} \int_{\frac{\sigma}{\mu}}^{\frac{\sigma}{\mu}+1-\beta} \frac{d\sigma d\rho}{b_{0}\sigma} \leq \frac{2}{b_{0}} \int_{\frac{s}{\mu}}^{s} \int_{\frac{\sigma}{\mu}}^{\frac{\sigma}{\mu}+1-\beta} \frac{d\sigma d\rho}{\sigma}
\]
\[
\leq \frac{2}{b_{0}} \int_{\frac{s}{\mu}}^{s} \left( \ln \rho - \ln \frac{r}{\mu} + \frac{1}{1-\beta} \frac{1}{\rho^{1-\beta}} - \frac{1}{1-\beta} \left( \frac{r}{\mu} \right)^{1-\beta} \right) d\rho
\]
\[
\leq \frac{2}{b_{0}} \left( 2 \ln s + \frac{2s^{2-\beta}}{b_{0}(1-\beta)(2-\beta)} + \frac{2r}{b_{0}\mu} + \frac{2r^{2-\beta}}{b_{0}(2-\beta)\mu^{2-\beta}} \right), \quad \forall \ s > 0.
\]

Similarly, we also have
\[
G(s) \geq \int_{\frac{s}{\mu}}^{s} \int_{\frac{\sigma}{\mu}}^{\frac{\sigma}{\mu}+1-\beta} \frac{d\sigma d\rho}{b_{1}(\sigma+1-\beta)\sigma^{1-\beta}} \geq \frac{1}{b_{1}} \int_{\frac{s}{\mu}}^{s} \int_{\frac{\sigma}{\mu}}^{\frac{\sigma}{\mu}+1-\beta} \frac{d\sigma d\rho}{\sigma^{1-\beta}}
\]
\[
\geq \frac{1}{b_{1}(1-\beta)} \int_{\frac{s}{\mu}}^{s} \left( \rho^{1-\beta} - \left( \frac{r}{\mu} \right)^{1-\beta} \right) d\rho \quad (4.7)
\]
\[
\geq \frac{s^{2-\beta}}{b_{1}(1-\beta)(2-\beta)} - \frac{sr^{1-\beta}}{b_{1}(1-\beta)\mu^{1-\beta}} + \frac{r^{2-\beta}}{b_{1}(2-\beta)\mu^{2-\beta}}, \quad \forall \ s > 0 \quad (4.8)
\]
and (4.7) infers that $G(s) > 0$ for any $s > 0$. Thus,
\[
\int_{\Omega} |\nabla w(x,t)|^{2} dx \leq 2F(u,w)(t), \quad \forall \ t > 0.
\]

We will show for dimension $n = 2$ that $F(u,w)(t)$ is decreasing after some point in time if $\mu$ suitably large (without loss of generality, we assume $\mu > e$ in our proofs). First, we give following estimate of $F(u,w)(t)$ for any $t > 0$.

**Lemma 4.2.** Let $n = 2$, and (1.7) be valid for $S$ with $\beta < 1$. Then we have following estimate
\[
\frac{d}{dt} F(u,w)(t) + \left( 1 - \frac{L_{1}}{2} \right) \int_{\Omega} |\nabla w|^{2} dx \int_{\Omega} |\Delta w|^{2} dx + \int_{\Omega} \frac{|\nabla u|^{2}}{S(u)} dx \leq 0, \quad \forall \ t > 0. \quad (4.10)
\]

**Proof.** It follows from (2.2) that
\[
\frac{d}{dt} \int_{\Omega} G(u) dx = - \int_{\Omega} \frac{|\nabla u|^{2}}{S(u)} dx - \int_{\Omega} \nabla u \cdot \nabla w dx - \int_{\Omega} \mu u(u - \frac{r}{\mu})(\int_{\frac{\rho}{\sigma}}^{\frac{\rho}{\rho}-1} \frac{1}{S(\sigma)} d\sigma) dx
\]
\[
\leq - \int_{\Omega} \frac{|\nabla u|^{2}}{S(u)} dx - \int_{\Omega} \nabla u \cdot \nabla w dx. \quad (4.11)
\]
By differentiating the integral $\int_{\Omega} |\nabla w|^{2} dx$ with respect to $t$ and utilizing Young's inequality, we have from (2.2) that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla w|^{2} dx = \int_{\Omega} \nabla w \cdot \Delta w dx - \int_{\Omega} \nabla w \cdot \nabla |\nabla w|^{2} dx + \int_{\Omega} \nabla u \cdot \nabla w dx
\]
\[
= - \int_{\Omega} |\Delta w|^{2} dx - \int_{\Omega} \Delta |w|^{2} dx + \int_{\Omega} \nabla u \cdot \nabla w dx
\]
\begin{equation}
\leq -\frac{1}{2} \int_\Omega |\Delta w|^2 dx + \int_\Omega \nabla u \cdot \nabla w dx + \frac{1}{2} \int_\Omega |\nabla w|^4 dx \tag{4.12}
\end{equation}
where the last integral can be estimated
\begin{equation}
\int_\Omega |\nabla w|^4 dx = \|\nabla w\|_{L^4(\Omega)}^4 \leq \frac{L_1}{2} \left( \int_\Omega |\Delta w|^2 dx \right) \left( \int_\Omega |\nabla w|^2 dx \right) \tag{4.13}
\end{equation}
due to (2.10). By virtue of (4.11)-(4.13), we can see
\begin{equation}
\frac{d}{dt} \mathcal{F}(u, w)(t) + \int_\Omega \frac{|\nabla u|^2}{S(u)} dx + \frac{1}{2} \left( 1 - \frac{L_1}{2} \int_\Omega |\nabla w|^2 dx \right) \int_\Omega |\Delta w|^2 dx \leq 0, \quad \forall \ t > 0 \tag{4.14}
\end{equation}
as desired.

Let \( t_1 \) be specific in Lemma 4.1, then we have following lemma.

\textbf{Lemma 4.3.} \textit{Under conditions of Lemma 4.2. There exists \( \mu_1 > 0 \) with property: if \( \mu > \mu_1 \), then there is \( t_\ast > t_1 \) such that}
\begin{equation}
\frac{d}{dt} \mathcal{F}(u, w)(t) \leq 0, \quad \forall \ t > t_\ast. \tag{4.15}
\end{equation}

\textbf{Proof.} Due to (4.2), we have
\begin{equation}
\int_{t_1}^{t} \int_\Omega u^2 dxds \leq \frac{2|\Omega|r^2}{\mu^2}(t - t_1) + \frac{2|\Omega|r}{\mu^2} \leq \frac{4|\Omega|r^2}{\mu^2}(t - t_1), \quad t \geq t_1 + \frac{1}{r}. \tag{4.16}
\end{equation}
Since \( \int_\Omega w(x, t_1) dx < \infty \) ensured by Lemma 2.2, we can pick \( t_2 > t_1 + \frac{1}{r} \) so large that
\begin{equation}
\int_\Omega w(x, t_1) dx \leq \frac{2|\Omega|r}{\mu}(t_2 - t_1). \tag{4.17}
\end{equation}
Now, denote
\begin{align*}
S_1 & := \left\{ t \in (t_1 + \frac{1}{r}, t_2) \left| \int_\Omega u^2 dx > \frac{16|\Omega|r^2}{\mu^2} \right. \right\}, \\
S_2 & := \left\{ t \in (t_1 + \frac{1}{r}, t_2) \left| \int_\Omega |\nabla w|^2 dx > \frac{16|\Omega|r}{\mu} \right. \right\}.
\end{align*}
Then (4.16) indicates that
\begin{align*}
|S_1| & < \frac{\mu^2}{16|\Omega|r^2} \int_{t_1 + \frac{1}{r}}^{t_2} \int_\Omega u^2 dxds \leq \frac{\mu^2}{16|\Omega|r^2} \int_{t_1}^{t_2} \int_\Omega u^2 dxds \leq \frac{t_2 - t_1}{4}, \\
|S_2| & < \frac{\mu}{16|\Omega|r} \int_{t_1 + \frac{1}{r}}^{t_2} \int_\Omega |\nabla w|^2 dxds \leq \frac{\mu}{16|\Omega|r} \left( \frac{2|\Omega|r}{\mu} (t_2 - t_1) + \int_\Omega w(x, t_1) dx \right) \leq \frac{t_2 - t_1}{4},
\end{align*}
Moreover, in view of (4.3) and (4.17), we can see
Hence set \((t_1 + \frac{1}{r}, t_2)\)\((S_1 \cup S_2)\) is nonempty with taking \(t_2 > t_1 + \frac{2}{7}\). This allows us to pick \(t_* \in (t_1 + \frac{1}{r}, t_2)\) satisfying
\[
\int_{\Omega} u^2(x, t_*) dx \leq \frac{16|\Omega| r^2}{\mu^2}, \tag{4.18}
\]
\[
\int_{\Omega} |\nabla w(x, t_*)|^2 dx \leq \frac{16|\Omega| r}{\mu}. \tag{4.19}
\]
Because of \(\beta \geq 0\), we know by (4.6) that
\[
\int_{\Omega} G(u(x, t_*)) dx \leq \frac{2}{b_0} \int_{\Omega} u(x, t_*) \ln u(x, t_*) dx + \frac{2}{b_0 (1 - \beta)(2 - \beta)} \int_{\Omega} u^{2-\beta}(x, t_*) dx
\]
\[
+ \frac{2}{b_0} \ln \frac{\mu}{r} \int_{\Omega} u(x, t_*) dx
\]
\[
+ \left( \frac{2}{b_0 (1 - \beta)(2 - \beta)} + \frac{2}{b_0} \right) \int_{\Omega} u^2(x, t_*) dx
\]
\[
+ \left( \frac{2}{b_0 (1 - \beta)(2 - \beta)} + \frac{2}{b_0} \ln \frac{\mu}{r} \right) \int_{\Omega} u(x, t_*) dx
\]
\[
+ \left( \frac{r^2}{\mu} \right)^{2-\beta} \frac{2|\Omega|}{b_0 (2 - \beta)} + \frac{2|\Omega| r}{b_0} \frac{\mu}{r}. \tag{4.20}
\]
This combined with (4.1), (4.5), (4.18) and (4.19) tells that
\[
\mathcal{F}(u, w)(t_*) \leq \int_{\Omega} G(u(x, t_*)) dx + \frac{1}{2} \int_{\Omega} |\nabla w(x, t_*)|^2 dx
\]
\[
\leq \left( \frac{32|\Omega|}{b_0 (1 - \beta)(2 - \beta)} + \frac{32|\Omega|}{b_0} \right) \left( \frac{r}{\mu} \right)^2 + \left( \frac{4|\Omega|}{b_0 (1 - \beta)(2 - \beta)} + \frac{4|\Omega|}{b_0} \ln \frac{\mu}{r} \right) \frac{r}{\mu}
\]
\[
+ \left( \frac{r^2}{\mu} \right)^{2-\beta} \frac{2|\Omega|}{b_0 (2 - \beta)} + \frac{2|\Omega| r}{b_0} \frac{\mu}{r} + \frac{8|\Omega| r}{\mu}. \tag{4.21}
\]
which ensures the existence of \(\mu_1\) such that whenever \(\mu > \mu_1\),
\[
\mathcal{F}(u, w)(t_*) < \frac{1}{2L_1}. \tag{4.22}
\]
By invoking (4.9) and (4.10), we also have
\[
\frac{d}{dt} \mathcal{F}(u, w)(t) + \frac{1}{2} \left( 1 - L_1 \mathcal{F}(u, w)(t) \right) \int_{\Omega} |\Delta w|^2 dx + \int_{\Omega} |\nabla u|^2 dx \leq 0, \; \forall t > t_* \tag{4.23}
\]
Hence, (4.15) is an immediate consequence of Lemma 2.4 with taking \(y(t) = \mathcal{F}(u, w)(t)\),
\(h(t) = \frac{1}{2} \int_{\Omega} |\Delta w|^2 dx\) and \(g(t) = \int_{\Omega} |\nabla u|^2 dx\). This concludes our proof.

As an evident corollary of Lemma 4.3, we have following result.

**Corollary 4.1.** Under the conditions of Lemma 4.2, we have for \(\mu > \mu_1\) that
\[
\mathcal{F}(u, w)(t) + \int_{t_*}^t \int_{\Omega} |\nabla u|^2 S(u) dx ds + \frac{1}{4} \int_{t_*}^t \int_{\Omega} |\Delta w|^2 dx ds \leq \frac{C \ln \mu}{\mu}, \; \forall t > t_* \tag{4.24}
\]
\[
\int_\Omega |\nabla w|^2(x,t)dx \leq \frac{C \ln \mu}{\mu}, \quad \forall \ t > t_*, \tag{4.25}
\]

with some \( C = C(\beta, b_0, r, \Omega) > 0 \).

**Proof.** According to (4.22), (4.23) and (2.7) of Lemma 2.4, we arrive at
\[
\mathcal{F}(u,w)(t) + \int_{t_*}^{t} \int_\Omega \frac{|\nabla u|^2}{S(u)} \, dx \, ds + \frac{1}{4} \int_{t_*}^{t} \int_\Omega |\Delta w|^2 \, dx \, ds \leq \mathcal{F}(u,w)(t_*), \quad \forall \ t > t_*,
\]
which combined with (4.21) leads to (4.24). Then (4.25) follows from (4.9) easily. \( \square \)

Thanks to the estimate (4.25) (which gives an appropriate smallness on \( \int_\Omega |\nabla w(\cdot, t)|^2 \) under setting \( \mu \) big enough), we can establish the bound for \( \int_\Omega u^2(\cdot, t) \) and \( \int_\Omega |\nabla w(\cdot, t)|^4 \).

**Lemma 4.4.** Under the conditions of Lemma 4.2. There exists \( \mu_3 > \mu_1 \) having property: if \( \mu > \mu_3 \), there is \( \delta_0 > 0 \) such that following estimate
\[
\int_\Omega u^2(x,t)dx + \int_\Omega |\nabla w(x,t)|^4 dx \leq C \left( \frac{\ln \mu}{\mu} \right)^2, \quad \forall \ t > t_* + \delta_0 \tag{4.26}
\]
holds with some \( C = C(\beta, b_0, r, \Omega) > 0 \).

**Proof.** Multiply the first equation in (1.1) by \( u \), integrate by parts over \( \Omega \) and use Young’s inequality to get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx = -\int_\Omega |\nabla u|^2 dx - \int_\Omega S(u) \nabla u \cdot \nabla w dx + r \int_\Omega u^2 dx - \mu \int_\Omega u^3 dx \leq -\frac{1}{2} \int_\Omega u^2 dx + b_1^2 \int_\Omega u^2 |\nabla w|^2 dx \\
+ (r + \frac{1}{2}) \int_\Omega u^2 dx - \mu \int_\Omega u^3 dx, \quad \forall \ t > t_*, \tag{4.27}
\]
here, we use the fact \( S(u) \leq b_1 u \) due to \( \beta < 1 \). Since \( (r + \frac{1}{2})u^2 < \frac{\mu}{2} u^3 \) for \( u > \frac{2r+1}{\mu} \), and \( (r + \frac{1}{2})u^2 \leq \frac{(2r+1)^3}{2\mu^2} \) for \( u \leq \frac{2r+1}{\mu} \), it can be obtained that
\[
(r + \frac{1}{2}) \int_\Omega u^2 dx \leq \frac{\mu}{2} \int_\Omega u^3 dx + \frac{(2r + 1)^3}{2\mu^2} |\Omega|. \tag{4.28}
\]
Applying Young’s inequality to the term \( b_1^2 \int_\Omega u^2 |\nabla w|^2 dx \) in (4.27) and combing with (4.28), we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 dx + \frac{1}{2} \int_\Omega u^2 dx \leq C_1 \int_\Omega |\nabla w|^6 dx + C_1 \int_\Omega u^3 dx + \frac{\mu}{2} \int_\Omega u^3 dx \\
+ \frac{(2r + 1)^3}{2\mu^2} |\Omega| - \mu \int_\Omega u^3 dx
\]

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\[
\begin{align*}
&= C_1 \int_\Omega |\nabla w|^6 dx + (C_1 - \frac{\mu}{2}) \int_\Omega u^3 dx \\
&\quad + \frac{(2r + 1)^3}{2\mu^2} |\Omega|, \quad \forall t > t_*
\end{align*}
\] (4.29)

with \( C_1 = C_1(b_1) > 0 \). Moreover, it can be derived by (2.2)\(_2\) that

\[
\nabla w_t = \nabla \Delta w - \nabla |\nabla w|^2 + \nabla u, \quad \forall t > t_*.
\] (4.30)

Testing (4.30) by \( 4|\nabla w|^2 \nabla w \) and recalling the identity \( \nabla w \cdot \nabla \Delta w = \frac{1}{2} \Delta |\nabla w|^2 - |\nabla^2 w|^2 \),

\[
\begin{align*}
\frac{d}{dt} \int_\Omega |\nabla w|^4 dx &\leq 2 \int_\Omega \Delta |\nabla w|^2 |\nabla w|^2 dx - 4 \int_\Omega |D^2 w|^2 |\nabla w|^2 dx \\
&\quad - 4 \int_\Omega |\nabla w|^2 \nabla w \cdot \nabla |\nabla w|^2 dx + 4 \int_\Omega |\nabla w|^2 \nabla w \cdot \nabla w dx, \quad \forall t > t_*.
\end{align*}
\] (4.31)

Since \( \frac{\partial w}{\partial \nu} = \text{on } \partial \Omega \), we can see by integrating by parts that

\[
\begin{align*}
\frac{d}{dt} \int_\Omega |\nabla w|^4 dx &\leq 2 \int_\Omega |\nabla |\nabla w|^2|^2 dx + 4 \int_\Omega |D^2 w|^2 |\nabla w|^2 dx \\
&\quad - 4 \int_\partial \Omega \frac{\partial |\nabla w|^2}{\partial \nu} |\nabla w|^2 d\sigma - 4 \int_\Omega |\nabla w|^2 \nabla w \cdot |\nabla w|^2 dx \\
&\quad - 4 \int_\Omega u \nabla w \cdot |\nabla w|^2 dx - 4 \int_\Omega u |\nabla w|^2 \Delta w dx, \quad \forall t > t_*.
\end{align*}
\] (4.32)

It follows from [19, Lemma 4.2] that

\[
\frac{\partial |\nabla w|^2}{\partial \nu} \leq 2k|\nabla w|^2,
\]

where \( k = k(\Omega) > 0 \) is an upper bound of the curvature of \( \partial \Omega \), then the trace inequality tells that

\[
2 \int_\partial \Omega |\nabla w|^2 \frac{\partial |\nabla w|^2}{\partial \nu} d\sigma \leq 4k \int_\partial \Omega |\nabla w|^4 d\sigma \\
\leq \frac{1}{4} \int_\Omega |\nabla |\nabla w|^2|^2 dx + C_2 \int_\Omega |\nabla w|^4 dx
\] (4.33)

with \( C_2 = C_2(\Omega) > 0 \). Utilizing Young’s inequality to the terms on the right side of (4.32) leads to

\[
\begin{align*}
-4 \int_\Omega |\nabla w|^2 \nabla w \cdot |\nabla w|^2 dx &\leq \frac{1}{4} \int_\Omega |\nabla |\nabla w|^2|^2 dx + 16 \int_\Omega |\nabla w|^6 dx \\
-4 \int_\Omega u \nabla w \cdot |\nabla w|^2 dx &\leq \frac{1}{4} \int_\Omega |\nabla |\nabla w|^2|^2 dx + 16 \int_\Omega u^2 |\nabla w|^2 dx
\end{align*}
\] (4.34) (4.35)
as well as

$$\begin{align*}
-4 \int_{\Omega} u|\nabla w|^2 \Delta w dx & \leq 2 \int_{\Omega} |\nabla w|^2 |\Delta w|^2 dx + 2 \int_{\Omega} u^2 |\nabla w|^2 dx \\
& \leq \int_{\Omega} |\nabla w|^2 |D^2 w|^2 dx + 2 \int_{\Omega} u^2 |\nabla w|^2 dx
\end{align*}$$

(4.36)

by pointwise estimate $2|\Delta w|^2 \leq |D^2 w|^2$. Hence (4.32)-(4.36) results in

$$\begin{align*}
\frac{d}{dt} \int_{\Omega} |\nabla w|^4 dx + \int_{\Omega} |\nabla |\nabla w|^2|^2 dx \\
& \leq 16 \int_{\Omega} |\nabla w|^6 dx + C_2 \int_{\Omega} |\nabla w|^4 dx + 18 \int_{\Omega} u^2 |\nabla w|^2 dx \\
& \leq C_3 \int_{\Omega} |\nabla w|^6 dx + C_3 \int_{\Omega} |\nabla w|^4 dx + C_3 \int_{\Omega} u^3 dx
\end{align*}$$

(4.37)

with $C_3 = C_3(\Omega) > 0$. With taking $\mu_2 > 2C_1 + 2C_3$, we have from (4.29) and (4.37) that

$$\begin{align*}
\frac{d}{dt}\left( \frac{1}{2} \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla w|^4 dx \right) + \frac{1}{2} \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla |\nabla w|^2|^2 dx \\
& \leq (C_1 + C_3) \int_{\Omega} |\nabla w|^6 dx + C_3 \int_{\Omega} |\nabla w|^4 dx + \frac{(2r + 1)^3}{2\mu^2} |\Omega| \\
& \quad + (C_1 + C_3 - \frac{\mu}{2}) \int_{\Omega} u^3 dx \\
& \leq (C_1 + C_3) \int_{\Omega} |\nabla w|^6 dx + C_3 \int_{\Omega} |\nabla w|^4 dx + \frac{(2r + 1)^3}{2\mu^2} |\Omega|, \quad \forall t > t_*
\end{align*}$$

(4.38)

as long as $\mu > \mu_2$. An application of the Gagliardo-Nirenberg inequality and Young’s inequality infers that

$$\begin{align*}
\int_{\Omega} |\nabla w|^4 dx &= |||\nabla w|^2||^2_{L^2(\Omega)} \\
& \leq C_4 |||\nabla |\nabla w|^2||_{L^2(\Omega)} |||\nabla w|^2||_{H^1(\Omega)} + C_4 |||\nabla w|^2||^2_{L^1(\Omega)} \\
& \leq \frac{1}{2(C_3 + 1)} \int_{\Omega} |\nabla |\nabla w|^2|^2 dx + C_5 \left( \int_{\Omega} |\nabla w|^2 dx \right)^2
\end{align*}$$

(4.39)

with $C_4 = C_4(\Omega) > 0$, $C_5 = C_5(\Omega) > 0$. Moreover, in view of (2.11),

$$\begin{align*}
\int_{\Omega} |\nabla w|^6 dx &= |||\nabla w|^2||^3_{L^3(\Omega)} \\
& \leq L_2 \left( \int_{\Omega} |\nabla w|^2 dx \right)^2 \left( \int_{\Omega} |\nabla w|^2 dx \right) + L_2 \left( \int_{\Omega} |\nabla w|^2 dx \right)^3.
\end{align*}$$

(4.40)

Substituting (4.40) and (4.39) into (4.38) yields that

$$\begin{align*}
\frac{d}{dt}\left( \frac{1}{2} \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla w|^4 dx \right) + \frac{1}{2} \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla w|^4 dx + \int_{\Omega} |\nabla |\nabla w|^2|^2 dx
\end{align*}$$
\[ \leq (C_1 + C_3) \int_{\Omega} |\nabla w|^6 dx + (C_3 + 1) \int_{\Omega} |\nabla w|^4 dx + \frac{(2r + 1)^3}{2\mu^2} |\Omega| \]
\[ \leq L_2(C_1 + C_3) \left( \int_{\Omega} |\nabla w|^2 |dx| \right) \left( \int_{\Omega} |\nabla w|^2 dx \right) + \frac{1}{2} \int_{\Omega} |\nabla |\nabla w|^2 |dx| \]
\[ + L_2(C_1 + C_3) \left( \int_{\Omega} |\nabla w|^2 dx \right)^3 + C_5(C_3 + 1) \left( \int_{\Omega} |\nabla w|^2 dx \right)^2 \]
\[ + \frac{(2r + 1)^3}{2\mu^2} |\Omega|, \quad \forall \ t > t_* . \quad (4.41) \]

Due to (4.25), there exists some \( \mu_3 \geq \mu_2 \) with property: if \( \mu > \mu_3 \), then
\[ L_2(C_1 + C_3) \int_{\Omega} |\nabla w|^2 dx \leq \frac{1}{2} \quad (4.42) \]
is true for all \( t > t_* \). Therefore, by choosing \( \mu > \mu_3 \), (4.41)-(4.42) allows us to find positive constants \( C_6 = C_6(\beta, b_0, r, \Omega) \) and \( C_7 = C_7(\beta, b_0, r, \Omega) \) fulfilling
\[ \frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla w|^4 dx \right) + \frac{1}{2} \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla w|^4 dx \]
\[ \leq C_6 \left( \frac{\ln \mu}{\mu} \right)^3 + C_6 \left( \frac{\ln \mu}{\mu} \right)^2 + \frac{(2r + 1)^3}{2\mu^2} |\Omega| \leq C_7 \left( \frac{\ln \mu}{\mu} \right)^2, \quad \forall \ t > t_* . \quad (4.43) \]

Letting \( y(t) = \frac{1}{2} \int_{\Omega} u^2(\cdot, t) dx + \int_{\Omega} |\nabla w(\cdot, t)|^4 dx \) on \( t \in [t_*, \infty) \). From (4.43), we see that \( y(t) \) satisfies
\[ y'(t) + y(t) \leq C_7 \left( \frac{\ln \mu}{\mu} \right)^2, \quad \forall \ t > t_* . \quad (4.44) \]

Finally invoking the Bernoulli inequality implies
\[ \lim_{t \to \infty} y(t) \leq C_7 \left( \frac{\ln \mu}{\mu} \right)^2 \]
for \( \mu > \mu_3 \). By the continuity of \( y(t) \), we arrive at (4.26) readily with some \( \delta_0 \) relying on \( y(t_*) \) and \( \mu \). \( \square \)

Based on the bound of \( \int_{\Omega} u^2(\cdot, t) \) and \( \int_{\Omega} |\nabla w(\cdot, t)|^4 \), which are shown in Lemma 4.4, we can get the \( L^\infty \)-boundedness of \( u(x, t) \) by applying semigroup estimates. Denote \( \tilde{t} := t_* + \delta_0 \) for convenience, then we present the following lemma:

**Lemma 4.5.** Under the conditions of Lemma 4.4, we have for \( \mu > \mu_3 \) that
\[ \| \nabla w(\cdot, t) \|^2_{L^\infty(\Omega)} + \| u(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{C \ln \mu}{\mu}, \quad \forall \ t > \tilde{t} + 2 \quad (4.45) \]
with some \( C = C(\beta, b_0, b_1, r, \Omega) > 0 \).
Proof. According to (2.2)\textsubscript{2}, we represent \( \nabla w \) as

\[
\nabla w(\cdot, t) = \nabla e^{(t-\bar{t})\Delta}w(\cdot, \bar{t}) - \int_{\bar{t}}^{t} \nabla e^{(t-s)\Delta} |\nabla w(\cdot, s)|^2 ds \\
+ \int_{\bar{t}}^{t} \nabla e^{(t-s)\Delta} u(\cdot, s) ds, \quad \forall \ t > \bar{t}.
\]

(4.46)

For any \( q > 2 \), we have by Lemma 4.4 that

\[
\| \nabla w(\cdot, t) \|_{L^q(\Omega)} \leq \| \nabla e^{(t-\bar{t})\Delta}w(\cdot, \bar{t}) \|_{L^q(\Omega)} + \int_{\bar{t}}^{t} \| \nabla e^{(t-s)\Delta} |\nabla w(\cdot, s)|^2 \|_{L^q(\Omega)} ds \\
+ \int_{\bar{t}}^{t} \| \nabla e^{(t-s)\Delta} u(\cdot, s) \|_{L^q(\Omega)} ds \\
\leq K_3 \left( 1 + (t - \bar{t})^{-\frac{1}{2} + \frac{q}{4}} \right) e^{-\lambda_1 (t-\bar{t})} \| \nabla w(\cdot, \bar{t}) \|_{L^q(\Omega)} \\
+ K_2 \int_{\bar{t}}^{t} \left( 1 + (t - s)^{-\frac{1}{4} + \frac{q}{4}} \right) e^{-\lambda_1 (t-s)} \| \nabla w(\cdot, s) \|^2 \|_{L^q(\Omega)} ds \\
+ K_2 \int_{\bar{t}}^{t} \left( 1 + (t - s)^{-\frac{1}{4} + \frac{q}{4}} \right) e^{-\lambda_1 (t-s)} \| u(\cdot, s) \|_{L^q(\Omega)} ds \\
\leq C_1 \left( \ln \frac{\mu}{\mu} \right)^{\frac{1}{2}} + C_1 \frac{\ln \mu}{\mu} \leq 2C_1 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}}, \quad \forall \ t > \bar{t} + 1
\]

(4.47)

with \( C_1 = C_1(\beta, b_1, r, \Omega) > 0 \). Due to the positivity of \( u \) and (2.2)\textsubscript{1}, we can see

\[
u(\cdot, t) \leq e^{(t-\bar{t})(\Delta-1)}u(\cdot, \bar{t}) + \int_{\bar{t}}^{t} e^{(t-s)(\Delta-1)} \nabla \cdot (S(\cdot, s)) \nabla w(\cdot, s) ds \\
+ (r + 1) \int_{\bar{t}}^{t} e^{(t-s)(\Delta-1)} u(\cdot, s) ds, \quad \forall \ t > \bar{t}.
\]

Then for any \( p > 2 \), we have

\[
\| u(\cdot, t) \|_{L^p(\Omega)} \leq \| e^{(t-\bar{t})(\Delta-1)} u(\cdot, \bar{t}) \|_{L^p(\Omega)} + \int_{\bar{t}}^{t} \| e^{(t-s)(\Delta-1)} \nabla \cdot (S(\cdot, s)) \nabla w(\cdot, s) \|_{L^p(\Omega)} ds \\
+ (r + 1) \int_{\bar{t}}^{t} \| e^{(t-s)(\Delta-1)} u(\cdot, s) \|_{L^p(\Omega)} ds, \quad \forall \ t > \bar{t} + 1.
\]

(4.48)

It follows by (4.1) that

\[
\| e^{(t-\bar{t})(\Delta-1)} u(\cdot, \bar{t}) \|_{L^p(\Omega)} \leq \| e^{(t-\bar{t})(\Delta-1)} (u - \bar{u})(\cdot, \bar{t}) \|_{L^p(\Omega)} + \| e^{(t-\bar{t})(\Delta-1)} \bar{u}(\cdot, \bar{t}) \|_{L^p(\Omega)} \\
\leq K_1 \left( 1 + (t - \bar{t})^{-1 + \frac{p}{2}} \right) e^{-\lambda_1 (t-\bar{t})} \| u(x, \bar{t}) \|_{L^1(\Omega)} + \| \bar{u}(x, \bar{t}) \|_{L^p(\Omega)} \\
\leq \frac{2K_1 |\Omega|^r}{\mu} + \frac{2|\Omega|^\frac{p}{2} r}{\mu}, \quad \forall \ t > \bar{t} + 1.
\]

(4.49)

By using the Hölder inequality for the case \( \beta \neq 0 \), we obtain from (4.26) and (4.47) that

\[
\int_{\bar{t}}^{t} \| e^{(t-s)\Delta} \nabla \cdot (S(\cdot, s)) \nabla w(\cdot, s) \|_{L^p(\Omega)} ds
\]
\[ \leq b_1 K_4 \int_{\bar{t}}^{t} \left(1 + (t - s)^{-1 + \frac{1}{p}} \right) e^{-(\lambda_1 + 1)(t - s)} \|u^\beta(s, s)\|_{L^2(\Omega)} ds \]
\[ \leq b_1 K_4 \int_{\bar{t}}^{t} \left(1 + (t - s)^{-1 + \frac{1}{p}} \right) e^{-(\lambda_1 + 1)(t - s)} \|\nabla w(s, s)\|_{L^2(\Omega)} \frac{\|\nabla w(s, s)\|_{L^{1/\beta}(\Omega)}}{\|\nabla w(s, s)\|_{L^{1/\beta}(\Omega)}} ds \]
\[ \leq C_2 \left( \frac{\ln \mu}{\mu} \right)^{\frac{\beta}{2} + \frac{1}{\beta}}, \quad \forall \ t > \bar{t} + 1 \quad (4.50) \]

with \( C_2 = C_2(p, \beta, b_0, b_1, \Omega) > 0 \). Moreover, noticing that \( S(u) \leq b_1 \) for all \( u \geq 0 \) if \( \beta = 0 \), hence we can conclude that (4.50) is indeed valid for any \( \beta \in [0, 1) \). It can be deduced from (4.1) that

\[
\int_{\bar{t}}^{t} \left\| e^{(t-s)(\Delta-1)} u(\cdot, s) \right\|_{L^p(\Omega)} ds \\
\leq \int_{\bar{t}}^{t} \left\| e^{(t-s)(\Delta-1)} (u - \bar{u})(\cdot, s) \right\|_{L^p(\Omega)} ds + \int_{\bar{t}}^{t} \left\| e^{(t-s)(\Delta-1)} \bar{u}(\cdot, s) \right\|_{L^p(\Omega)} ds \\
\leq K_1 \int_{\bar{t}}^{t} \left(1 + (t - s)^{-1 + \frac{1}{p}} \right) e^{-(\lambda_1 + 1)(t - s)} \|u(\cdot, s)\|_{L^1(\Omega)} ds \\
+ \int_{\bar{t}}^{t} e^{-(t-s)} \|e^{(t-s)\Delta} \bar{u}(\cdot, s)\|_{L^p(\Omega)} ds \\
\leq \frac{C_3}{\mu}, \quad \forall \ t > \bar{t} + 1 \quad (4.51) \]

with \( C_3 = C_3(p, r, \Omega) > 0 \). In view of (4.48)-(4.51), we can find \( C_4 = C_4(p, \beta, b_1, r, \Omega) > 0 \) fulfilling

\[
\|u(\cdot, t)\|_{L^p(\Omega)} \leq C_4 \left( \frac{\ln \mu}{\mu} \right)^{\frac{\beta}{2} + \frac{1}{\beta}}, \quad \forall \ t > \bar{t} + 1. \quad (4.52) \]

Next, we use (4.47) and (4.52) to estimate and get

\[
\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \leq K_3 \left(1 + (t - (\bar{t} + 1))^{-\frac{1}{2}} \right) e^{-\lambda_1(t - (\bar{t} + 1))} \|\nabla w(\cdot, \bar{t} + 1)\|_{L^4(\Omega)} \\
+ K_2 \int_{\bar{t}+1}^{t} \left(1 + (t - s)^{-\frac{1}{4}} \right) e^{-\lambda_1(t - s)} \|\nabla w(\cdot, s)\|_{L^4(\Omega)} ds \\
+ K_3 \int_{\bar{t}+1}^{t} \left(1 + (t - s)^{-\frac{1}{4}} \right) e^{-\lambda_1(t - s)} \|u(\cdot, s)\|_{L^4(\Omega)} ds \\
\leq C_5 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} + C_5 \left( \frac{\ln \mu}{\mu} \right)^{\frac{\beta}{2} + \frac{1}{\beta}} + C_5 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} \leq 3C_5 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}}, \quad \forall \ t > \bar{t} + 2 \quad (4.53) \]

with \( C_5 = C_5(p, \beta, b_1, r, \Omega) > 0 \). Lastly, we do the term \( \|u(\cdot, t)\|_{L^\infty(\Omega)} \) by similar arguments as above:

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \|e^{(t-(\bar{t}+1))(\Delta-1)} u(\cdot, \bar{t} + 1)\|_{L^\infty(\Omega)} \\
+ \int_{\bar{t}+1}^{t} \|e^{(t-s)(\Delta-1)} \nabla \cdot (S(u(\cdot, s)) \nabla w(\cdot, s))\|_{L^p(\Omega)} ds 
\]

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entails the existence of \( \tilde{t} \). Therefore, \( \frac{4K_1|\Omega|}{\mu} + \frac{2r}{\mu}, \forall t > \tilde{t} + 2. \) \hfill (4.55)

It can be derived from (4.52) and (4.53) that
\[
\int_{\tilde{t}+1}^{t} \|e^{(t-s)(\Delta-1)} u(\cdot, s)\|_{L^\infty(\Omega)} ds \\
\leq b_1 K_4 \int_{\tilde{t}}^{t} \left( 1 + (t-s)^{\frac{3}{2}} \right) e^{-\left(\lambda_1+1\right)(t-s)} \|u(\cdot, s)\|_{L^4(\Omega)} ds \\
\leq b_1 K_4 \int_{\tilde{t}}^{t} \left( 1 + (t-s)^{\frac{3}{2}} \right) e^{-\left(\lambda_1+1\right)(t-s)} \|u(\cdot, s)\|_{L^4(\Omega)} \|\nabla w(\cdot, s)\|_{L^\infty(\Omega)} ds \\
\leq C_6 \left( \frac{\ln \mu}{\mu} \right)^{\frac{2}{3}+1}, \forall t > \tilde{t} + 2 \hfill (4.56)
\]

by the Hölder inequality, here \( C_6 = C_6(\beta, b_1, r, \Omega) > 0 \). In addition, (4.1) and (4.26) indicate
\[
\int_{\tilde{t}+1}^{t} \|e^{(t-s)(\Delta-1)} u(\cdot, s)\|_{L^\infty(\Omega)} ds \\
\leq \int_{\tilde{t}+1}^{t} \|e^{(t-s)(\Delta-1)} (u - \tilde{u})(\cdot, s)\|_{L^\infty(\Omega)} ds + \int_{\tilde{t}+1}^{t} \|e^{(t-s)(\Delta-1)} \tilde{u}(\cdot, s)\|_{L^\infty(\Omega)} ds \\
\leq K_1 \int_{\tilde{t}+1}^{t} \left( 1 + (t-s)^{\frac{3}{2}} \right) e^{-\left(\lambda_1+1\right)(t-s)} \|u(\cdot, s)\|_{L^2(\Omega)} ds \\
+ \int_{\tilde{t}+1}^{t} e^{-\left(\lambda_1+1\right)(t-s)} \|\Delta \tilde{u}(\cdot, s)\|_{L^\infty(\Omega)} ds \\
\leq \frac{C_7 \ln \mu}{\mu} + \frac{C_7 \ln \mu}{\mu \mu} \leq \frac{2C_7 \ln \mu}{\mu}, \forall t > \tilde{t} + 2 \hfill (4.57)
\]

with \( C_7 = C_7(\beta, b_0, b_1, r, \Omega) > 0 \). As a consequence of (4.54)-(4.57), there exists \( C_8 = C_8(\beta, b_0, b_1, r, \Omega) > 0 \) fulfilling
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{C_8 \ln \mu}{\mu}, \forall t > \tilde{t} + 2. \hfill (4.58)
\]

Therefore, (4.53) and (4.58) conclude our desired result. \( \square \)

**Proof of Theorem 2.** Let \( \mu_* = \mu_3 \) provided as Lemma 4.4, then for any \( \mu > \mu_* \), Lemma 4.5 entails the existence of \( \tilde{t} > 0 \) such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1, \forall t > \tilde{t} + 2 \hfill (4.59)
\]
with \( C_1 = C_1(\beta, b_0, b_1, r) > 0 \). Moreover, according to Theorem 1, we can find \( C_2 = C_2(\beta, b_0, b_1, r, \mu) > 0 \) ensuring
\[
\sup_{t \in [0, t+2]} \|u(\cdot, t)\|_{L^\infty(\Omega)} < C_2.
\] 
(4.60)

Hence, the global boundedness statement follows from (4.59) and (4.60) directly.

\[ \square \]

5 Asymptotic behavior of solutions

This section is devoted to establish the asymptotic behavior of solutions. We first give following lemma on the H"older continuity of \( u \) without proof, find details in [27, Lemma 4.5].

**Lemma 5.1.** Let \( n = 2, p > 2, m_0 > 0, M > 0 \) and \( \delta > 0 \). Then there exist \( \theta = \theta(p) \in (0,1) \) and \( C(p, m_0, M, \delta) > 0 \) with the property that whenever \((u, w) \in (C^{2,1}(\Omega \times (t_0, \infty)))^2 \) is a classical solution of the boundary value problem in (2.2) in \( \Omega \times (t_0, \infty) \) for some \( t_0 \geq 0 \), satisfying \( u \geq 0 \) in \( \Omega \times (t_0, \infty) \) and
\[
\int_\Omega u(x,t)dx \leq m_0, \ \forall \ t \geq t_0,
\]
as well as
\[
\int_\Omega |\nabla w(x,t)|^4 dx \leq M, \ \forall \ t \geq t_0,
\]
then
\[
\|u\|_{C^{\theta, \theta/2}(\Omega \times [t, t+1])} \leq C(r, \mu, m_0, M, \delta, \Omega), \ \forall \ t \geq t_0 + \delta.
\]

Let \( \mu_3 \) be given as Lemma 4.4, then for any global solution \((u, w)\) to the system (2.2) with \( \mu > \mu_3 \), we show the \( L^\infty \)-norm decaying of \((u - \bar{u})(t)\) as below.

**Lemma 5.2.** Let \( n = 2 \) and (1.7) be valid for \( S \) with \( \beta < 1 \). If \( \mu > \mu_3 \), then we have
\[
\|u(\cdot, t) - \bar{u}(t)\|_{L^\infty(\Omega)} \to 0 \ \text{as} \ \ t \to \infty.
\] 
(5.1)

**Proof.** According to Corollary 4.1, Lemma 4.5 and Lemma 5.1, there are \( \bar{t} > 0, C_1 = C_1(\beta, b_0, b_1, r, \Omega) > 0 \) and \( C_2 = C_2(\beta, b_0, b_1, r, \mu, \Omega) > 0 \) such that
\[
\int_\bar{t}^t \int_\Omega \frac{|\nabla u|^2}{S(u)} dx ds + \|\nabla w(\cdot, t)\|_{L^\infty(\Omega)}^2 + \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{C_1 \ln \mu}{\mu}, \ \forall \ t > \bar{t}
\] 
(5.2)
as well as
\[
\|u\|_{C^{\theta, \theta/2}(\Omega \times [t, t+1])} \leq C_2, \ \forall \ t > \bar{t}.
\] 
(5.3)
with some $\theta \in (0,1)$. Moreover by the Sobolev-Poincaré inequality, we can see
\[
\|u(\cdot, t) - \bar{u}(t)\|_{L^2(\Omega)} \leq C_3\|\nabla u(\cdot, t)\|_{L^1(\Omega)}
\]  
with $C_3 = C_3(\Omega) > 0$. The Hölder inequality and (4.1) lead to
\[
\left(\int_\Omega |\nabla u| dx\right)^2 \leq \int_\Omega \frac{|\nabla u|^2}{S(u)} dx \int_\Omega S(u) dx
\leq b_1 \left(\int_\Omega \frac{|\nabla u|^2}{S(u)} dx\right) \int_\Omega u dx \leq C_4 \int_\Omega \frac{|\nabla u|^2}{S(u)} dx, \quad \forall \; t > \tilde{t}
\]  
with $C_4 = C_4(b_1, r, \Omega) > 0$. The Hölder inequality and (4.1) lead to
\[
\left(\int_\Omega \frac{|\nabla u|^2}{S(u)} dx\right) \leq b_1 \left(\int_\Omega \frac{|\nabla u|^2}{S(u)} dx\right) \int_\Omega u dx \leq C_4 \int_\Omega \frac{|\nabla u|^2}{S(u)} dx, \quad \forall \; t > \tilde{t}
\]  
with $C_5 = C_5(\beta, b_0, b_1, r, \Omega) > 0$. Now if (5.1) was false, there would exist $(\tilde{t}_k)_{k \in \mathbb{N}} \in (\tilde{t}, \infty)$ and $C_6 > 0$ such that $\tilde{t}_k \rightarrow \infty$ as $k \rightarrow 0$,
\[
\|u(\cdot, \tilde{t}_k) - \bar{u}(\tilde{t}_k)\|_{L^\infty(\Omega)} \geq C_6, \quad \forall \; k \in \mathbb{N},
\]
which along with the uniform continuity of $u$ in $\overline{\Omega} \times (\tilde{t}, \infty)$ would allow us to find $(x_k)_{k \in \mathbb{N}} \in \Omega$, $r > 0$ and $\tau > 0$ with property $B_r(x_k) \subset \Omega$ for all $k \in \mathbb{N}$ and
\[
|u(x, t) - \bar{u}(t)| \geq \frac{C_6}{2}, \quad \forall \; x \in B_r(x_k) \quad \text{and} \quad t \in (\tilde{t}_k, \tilde{t}_k + \tau).
\]
This would imply that
\[
\int_{\tilde{t}_k}^{\tilde{t}_k + \tau} \|u(\cdot, t) - \bar{u}(t)\|^2_{L^2(\Omega)} dt \geq \tau \frac{C_6}{4} \pi r^2, \quad \forall \; k \in \mathbb{N},
\]
which contradicts (5.6). Hence (5.1) is valid.

\textbf{Lemma 5.3.} Under the conditions of Lemma 5.2, for any $\mu > \mu_3$, there exists $\delta_2 > 0$ such that
\[
\int_\Omega u(x, t) dx \geq \frac{r}{2\mu} |\Omega|, \quad \forall \; t > \tilde{t} + \delta_2.
\]  
Moreover, we have
\[
u(x, t) \geq \frac{r}{4\mu}, \quad \forall \; t > \tilde{t} + \delta_2.
\]

\textbf{Proof.} Since $\|u(\cdot, t) - \bar{u}(t)\|_{L^\infty(\Omega)} \rightarrow 0$ as $t \rightarrow \infty$, we can find $\delta_1 > 0$ such that
\[
\|u(\cdot, t) - \bar{u}(t)\|_{L^\infty(\Omega)} \leq \frac{r}{4\mu}, \quad \forall \; t > \tilde{t} + \delta_1
\]  
(5.9)
which ensures $\bar{u}(t) - \frac{r}{4\mu} \leq u(x, t) \leq \bar{u}(t) + \frac{r}{4\mu}$ on $\Omega \times (\bar{t} + \delta_1, \infty)$. Integrating (2.2)$_1$ over $\Omega$, we have

$$\frac{d}{dt} \int_\Omega u dx = r \int_\Omega u dx - \mu \int_\Omega u^2 dx \geq r \int_\Omega u dx - \mu(\bar{u}(t) + \frac{r}{4\mu}) \int_\Omega u dx = \frac{3r}{4} \int_\Omega u dx - \frac{\mu}{\Omega}(\int_\Omega u dx)^2, \ \forall \ t > \bar{t} + \delta_1. \quad (5.10)$$

Applying the Bernoulli inequality to (5.10) tells that

$$\liminf_{t \to \infty} \int_\Omega u(x, t) dx \geq \frac{3r}{4\mu} |\Omega|.$$ 

This implies the existence of $\delta_2 > \delta_1$ fulfilling

$$\int_\Omega u(x, t) dx \geq \frac{r}{2\mu} |\Omega|, \ \forall \ t > \bar{t} + \delta_2.$$ 

Because of $u(x, t) \geq \bar{u}(t) - \frac{r}{4\mu}$ on $\Omega \times (\bar{t} + \delta_2, \infty)$, we have $u(x, t) \geq \frac{r}{4\mu}$ for any $x \in \Omega$ and $t \in (\bar{t} + \delta_2, \infty)$. This proof is complete. \[\square\]

Next, we are going to present an estimate for the integral $\int_\Omega (u - \frac{r}{\mu})^2(x, t)$. Before this, denote $U(x, t) := u(x, t) - \frac{r}{\mu}$ for convenience, then we have from (2.2) that

$$\begin{cases}
U_t = \Delta U + \nabla \cdot (S(u)\nabla w) - rU - \mu U^2, & x \in \Omega, \ t > 0, \\
w_t = \Delta w - |\nabla w|^2 + U + \frac{r}{\mu}, & x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
(U(x, 0), w(x, 0)) = (u_0(x) - \frac{r}{\mu}, -\ln \frac{v_0(x)}{\|v_0(x)\|_{L^\infty(\Omega)})}, & x \in \Omega.
\end{cases} \quad (5.11)$$

**Lemma 5.4.** Under the conditions of Lemma 5.2. If $\mu > \mu_3$, then for $U = u - \frac{r}{\mu}$, there is $\delta_3 > 0$ such that the following estimate

$$\|U\|_{L^2(\Omega)} < C(\frac{\ln \mu}{\mu})^\frac{3}{2}, \ \forall \ t > \bar{t} + \delta_3 \quad (5.12)$$

holds with some $C = C(\beta, b_0, b_1, r, \Omega) > 0$.

**Proof.** Multiplying (5.11)$_1$ by $U$, integrating by parts and using Young’s inequality, we have from (1.7) that

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega U^2 dx &= -\int_\Omega |\nabla U|^2 dx - \int_\Omega S(u)\nabla w \cdot \nabla U dx - r \int_\Omega U^2 dx - \mu \int_\Omega U^3 dx \\
&\leq -\frac{1}{2} \int_\Omega |\nabla U|^2 dx + \int_\Omega S^2(u)|\nabla w|^2 dx - r \int_\Omega U^2 dx - \mu \int_\Omega U^3 dx
\end{align*}$$
\[
\leq b_1^2 \int_{\Omega} u^2 |\nabla w|^2 dx - r \int_{\Omega} U^2 dx - \mu \int_{\Omega} U^3 dx.
\] (5.13)

Since \( \mu > \mu_3 \), (4.45) entails that the term \( b_1^2 \int_{\Omega} u^2 |\nabla w|^2 dx \) can be estimated as

\[
b_1^2 \int_{\Omega} u^2 |\nabla w|^2 dx \leq C_1 \left( \frac{\ln \mu}{\mu} \right)^3, \quad \forall \ t > \tilde{t} + \delta_2
\] (5.14)

with \( C_1 = C_1(\beta, b_0, b_1, r, \Omega) > 0 \). Moreover, by simple calculations, we can check that

\[-\mu U^3 \leq 0 \text{ with } u \geq \frac{r}{\mu} \quad \text{and} \quad -\mu U^3 \leq \frac{3r}{4} U^2 \text{ with } \frac{r}{4\mu} \leq u < \frac{r}{\mu}.
\]

Thus, we have from (5.8) that

\[-\mu \int_{\Omega} U^3 dx \leq \frac{3r}{4} \int_{\Omega} U^2 dx.
\] (5.15)

This together with (5.13) and (5.14) indicates that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} U^2 dx \leq -\frac{r}{4} \int_{\Omega} U^2 dx + C_1 \left( \frac{\ln \mu}{\mu} \right)^3, \quad \forall \ t > \tilde{t} + \delta_2.
\] (5.16)

Setting \( \bar{y}(t) = \int_{\Omega} U^2(\cdot, t)dx \) for \( t \in [\tilde{t} + \delta_2, \infty) \), it can be deduced by (5.16) that

\[
\bar{y}'(t) + \frac{r}{2} \bar{y}(t) \leq 2C_1 \left( \frac{\ln \mu}{\mu} \right)^3, \quad \forall \ t > \tilde{t} + \delta_2
\] (5.17)

This combined with the Bernoulli inequality results in

\[
\limsup_{t \to \infty} \bar{y}(t) \leq \frac{4C_1}{r} \left( \frac{\ln \mu}{\mu} \right)^3
\] (5.18)

Hence, (5.12) is an immediate consequence of (5.18) by choosing \( \delta_3 \) large enough.

Now, let \( \tilde{t} \) and \( \delta_3 \) provided as above, based on the results of Lemma 4.5&5.4, we can claim for \( \mu > \mu_3 \) that

\[
\|\nabla w\|_{L^2(\Omega)}^2 + \|u\|_{L^\infty(\Omega)} + \|u\|_{L^4(\Omega)} + \|U\|_{L^\infty(\Omega)} \leq C_0 \frac{\ln \mu}{\mu}, \quad \forall \ t > \tilde{t} + \delta_3
\] (5.19)

as well as

\[
\|U\|_{L^2(\Omega)} < \tilde{C}_0 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}}, \quad \forall \ t > \tilde{t} + \delta_3
\] (5.20)

with \( C_0, \tilde{C}_0 \) independent of \( \mu \). Because of estimates (5.19) and (5.20), we can discuss the asymptotic stability of solutions to (1.1). Before this, we introduce some nations for convenience: \( \hat{\lambda} \) is some fixed constant less than \( \frac{\lambda_2}{2} \); \( \hat{c} := \int_0^\infty (1 + \sigma^{-\frac{1}{2}}) e^{-(\lambda_1 - \hat{\lambda})\sigma} d\sigma < \infty; \)

\( \tilde{c} := \int_0^\infty (1 + \sigma^{-\frac{1}{2}}) e^{-(\lambda_1 + r - \hat{\lambda})\sigma} d\sigma < \infty; \tilde{c} := \int_0^\infty (1 + \sigma^{-\frac{1}{2}}) e^{-(\lambda_1 + r - \hat{\lambda})\sigma} d\sigma < \infty; t^* = \tilde{t} + \delta_3. \)

Now, we give our proof of Theorem 3:
Proof of Theorem 3. Define
\[ T := \sup \left\{ \tilde{T} \in (0, \infty) \left| \left\| U \right\|_{L^\infty(\Omega)} \leq C_1 e^{-\tilde{\lambda}(t-t^*)} \text{ for all } t \in (t^*, \tilde{T}) \right. \right\} \] (5.21)
and \[ \left\| \nabla w \right\|_{L^\infty(\Omega)} \leq \tilde{C}_1 e^{-\tilde{\lambda}(t-t^*)} \text{ for all } t \in (t^*, \tilde{T}) \] (5.22)
with \( C_1 = 2C_0 \) and \( \tilde{C}_1 = (2\tilde{c}K^2 + 2)(C_1 + \frac{1}{2}) \). The definitions of \( C_1, \tilde{C}_1, \) and (5.19) ensure that \( T \) is well defined. Now, we will claim \( T = \infty \) if \( \mu > \mu_3 \) is big enough. (5.11)_2 tells that
\[
\nabla w(\cdot, t) = \nabla e^{(t-t^*)}\Delta w(\cdot, t^*) - \int_{t^*}^t \nabla e^{(t-s)}\Delta |\nabla w|^2(\cdot, s) ds
+ \int_{t^*}^t \nabla e^{(t-s)}\Delta U(\cdot, s) ds + \int_{t^*}^t \nabla \left( e^{(t-s)}\Delta \frac{\sigma}{\mu} \right) ds, \ \forall \ t > t^*.
\] (5.23)
Since \( e^{t\Delta \frac{\sigma}{\mu}} = \frac{t}{\mu} \) on \( \Omega \times [0, \infty) \) yields that \( \nabla e^{(t-s)}\Delta \frac{\sigma}{\mu} = 0 \), it follows from (5.23) that for any \( t > t^* \)
\[
\left\| \nabla w(\cdot, t) \right\|_{L^\infty(\Omega)} \leq \left\| \nabla e^{(t-t^*)}\Delta w(\cdot, t^*) \right\|_{L^\infty(\Omega)} + \int_{t^*}^t \left\| \nabla e^{(t-s)}\Delta |\nabla w|^2(\cdot, s) \right\|_{L^\infty(\Omega)} ds
+ \int_{t^*}^t \left\| \nabla e^{(t-s)}\Delta U(\cdot, s) \right\|_{L^\infty(\Omega)} ds
=: I_1 + I_2 + I_3.
\] (5.24)
We estimate the terms \( I_1, I_2 \) and \( I_3 \) respectively. Applying (5.19) to \( I_1 \) yields that
\[
I_1 = \left\| \nabla e^{(t-t^*)}\Delta w(\cdot, t^*) \right\|_{L^\infty(\Omega)}
\leq K_3 e^{-\lambda_1(t-t^*)} \left\| \nabla w(\cdot, t^*) \right\|_{L^\infty(\Omega)}
\leq K_3 C_0^2 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} e^{-\tilde{\lambda}(t-t^*)}, \ \forall \ t > t^*.
\] (5.25)
As a consequence of (5.19) and (5.22), we obtain
\[
I_2 = \int_{t^*}^t \left\| \nabla e^{(t-s)}\Delta |\nabla w|^2(\cdot, s) \right\|_{L^\infty(\Omega)} ds
\leq K_2 \int_{t^*}^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} \left\| \nabla w|^2(\cdot, s) \right\|_{L^\infty(\Omega)} ds
\leq K_2 C_0^2 \tilde{C}_1 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} \int_{t^*}^t (1 + (t-s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} e^{-(\tilde{\lambda}-\tilde{\lambda})(s-t^*)} ds
\leq K_2 C_0^2 \tilde{C}_1 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} \left( \int_{0}^{t-t^*} (1 + \sigma^{-\frac{1}{2}}) e^{-(\lambda_1-\tilde{\lambda})\sigma} d\sigma \right) e^{-\tilde{\lambda}(t-t^*)}
= cK_2 C_0^2 \tilde{C}_1 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} e^{-\tilde{\lambda}(t-t^*)}, \ \forall \ t^* < t < T.
\] (5.26)
We use (5.21) to estimate \( I_3 \) and get
\[
I_3 = \int_{t^*}^t \left\| \nabla e^{(t-s)}\Delta U(\cdot, s) \right\|_{L^\infty(\Omega)} ds
\]
\[
\begin{align*}
&\leq K_2 \int_{t^*}^{t} (1 + (t - s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} \|U(\cdot, s)\|_{L^\infty(\Omega)} ds \\
&\leq K_2 C_1 \int_{t^*}^{t} (1 + (t - s)^{-\frac{1}{2}}) e^{-\lambda_1(t-s)} e^{-\tilde{\lambda}(s-t^*)} ds \\
&\leq \left( K_2 C_1 \int_{0}^{t-t^*} (1 + \sigma^{-\frac{1}{2}}) e^{-(\lambda_1-\tilde{\lambda})\sigma} \right)e^{-\lambda(t-t^*)} \\
&\leq \hat{c} K_2 C_1 e^{-\tilde{\lambda}(t-t^*)} \\
&\leq \frac{\tilde{C}_1}{2} e^{-\tilde{\lambda}(t-t^*)}, \quad \forall \ t^* < t < T.
\end{align*}
\]

In view of (5.24)-(5.27), we obtain that for any \( t \in (t^*, T) \)

\[
\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \leq \left( K_3 C_0 \left( \ln \frac{\mu}{\mu} \right)^\frac{1}{2} + \hat{c} K_2 C_0 \tilde{C}_1 \left( \ln \frac{\mu}{\mu} \right)^\frac{1}{2} + \frac{\tilde{C}_1}{2} \right)e^{-\tilde{\lambda}(t-t^*)}.
\]

Moreover, it follows from (5.11)_1 that

\[
U(x, t) = e^{(t-t^*)(\Delta-r)}U(x, t^*) + \int_{t^*}^{t} e^{(t-s)(\Delta-r)} \nabla \cdot (S(u(\cdot, s)) \nabla w(\cdot, s)) ds \\
- \mu \int_{t^*}^{t} e^{(t-s)(\Delta-r)} U^2(\cdot, s) ds, \quad \forall \ t > t^*.
\]

Then, we can see

\[
\|U(\cdot, t)\|_{L^\infty(\Omega)} \leq \|e^{(t-t^*)(\Delta-r)}U(\cdot, t^*)\|_{L^\infty(\Omega)} \\
+ \int_{t^*}^{t} \|e^{(t-s)(\Delta-r)} \nabla \cdot (S(u(\cdot, s)) \nabla w(\cdot, s))\|_{L^\infty(\Omega)} ds \\
+ \mu \int_{t^*}^{t} \|e^{(t-s)(\Delta-r)} U^2(\cdot, s)\|_{L^\infty(\Omega)} ds \\
=: II_1 + II_2 + II_3.
\]

The term \( II_1 \) can be estimated by (5.19) as

\[
\begin{align*}
II_1 &= \|e^{(t-t^*)(\Delta-r)}U(x, t^*)\|_{L^\infty(\Omega)} \\
&\leq K_1 e^{-\lambda_1(t-t^*)} \|U(\cdot, t^*)\|_{L^\infty(\Omega)} \\
&\leq K_1 C_0 \ln \frac{\mu}{\mu} e^{-\tilde{\lambda}(t-t^*)}, \quad \forall \ t > t^*.
\end{align*}
\]

As to the term \( II_3 \), we apply (5.19) and (5.22) to obtain that

\[
II_2 = \int_{t^*}^{t} \|e^{(t-s)(\Delta-r)} \nabla \cdot (S(u(\cdot, s)) \nabla w(\cdot, s))\|_{L^\infty(\Omega)} ds \\
\leq K_4 \int_{t^*}^{t} (1 + (t-s)^{-\frac{3}{2}}) e^{-(\lambda_1+r)(t-s)} \|S(u(\cdot, s)) \nabla w(\cdot, s)\|_{L^4(\Omega)} ds
\]

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\[
\begin{align*}
\leq K_4 b_1 \int_{t^*}^{t} (1 + (t - s)^{-\frac{3}{2}}) e^{-(\lambda_1 + r)(t-s)} \|u(\cdot, s)\|_{L^4(\Omega)} \|\nabla w(\cdot, s)\|_{L^\infty(\Omega)} ds \\
\leq K_4 b_1 C_0 \frac{\ln \mu}{\mu} \int_{t^*}^{t} (1 + (t - s)^{-\frac{3}{2}}) e^{-(\lambda_1 + r)(t-s)} \|\nabla w(\cdot, s)\|_{L^\infty(\Omega)} ds \\
\leq K_4 b_1 C_0 \bar{C}_1 \frac{\ln \mu}{\mu} \int_{t^*}^{t} (1 + (t - s)^{-\frac{3}{2}}) e^{-(\lambda_1 + r)(t-s)} e^{-\lambda(s-t^*)} ds \\
\leq K_4 b_1 C_0 \bar{C}_1 \frac{\ln \mu}{\mu} \left( \int_{0}^{t-t^*} (1 + \sigma^{-\frac{3}{2}}) e^{-(\lambda_1 + r - \tilde{\lambda})\sigma} d\sigma \right) e^{-\tilde{\lambda}(t-t^*)} \\
\leq \tilde{c} K_4 b_1 C_0 \bar{C}_1 \frac{\ln \mu}{\mu} e^{-\tilde{\lambda}(t-t^*)}, \quad \forall \, t^* < t < T. \quad (5.31)
\end{align*}
\]

Lastly, substituting (5.20) and (5.21) to $I_3$ results in
\[
I_3 = \mu \int_{t^*}^{t} \|e^{(t-s)(\Delta - r)} U^2(\cdot, s)\|_{L^\infty(\Omega)} ds \\
\leq K_1 \mu \int_{t^*}^{t} (1 + (t - s)^{-\frac{3}{2}}) e^{-(\lambda_1 + r)(t-s)} \|U(\cdot, s)\|_{L^\infty(\Omega)} \|U(\cdot, s)\|_{L^2(\Omega)} ds \\
\leq K_1 \bar{C}_0 C_1 \frac{\ln \mu}{\mu} \frac{\tilde{\lambda}}{2} \int_{t^*}^{t} (1 + (t - s)^{-\frac{3}{2}}) e^{-(\lambda_1 + r)(t-s)} e^{-\tilde{\lambda}(s-t^*)} ds \\
\leq K_1 \bar{C}_0 C_1 \frac{\ln \mu}{\mu} \frac{\tilde{\lambda}}{2} \left( \int_{0}^{t-t^*} (1 + \sigma^{-\frac{3}{2}}) e^{-(\lambda_1 + r - \tilde{\lambda})\sigma} d\sigma \right) e^{-\tilde{\lambda}(t-t^*)} \\
\leq \tilde{c} K_1 \bar{C}_0 C_1 \frac{\ln \mu}{\mu} \frac{\tilde{\lambda}}{2} e^{-\tilde{\lambda}(t-t^*)}, \quad \forall \, t^* < t < T. \quad (5.32)
\]

In conjunction with (5.29)-(5.31), this yields for any $t \in (t^*, T)$
\[
\|U(\cdot, t)\|_{L^\infty(\Omega)} \leq \left( K_1 C_0 \frac{\ln \mu}{\mu} + \tilde{c} K_4 b_1 C_0 \bar{C}_1 \frac{\ln \mu}{\mu} + \tilde{c} K_1 \bar{C}_0 C_1 \frac{(\ln \mu)^{\frac{3}{2}}}{\mu^2} \right) e^{-\tilde{\lambda}(t-t^*)}. \quad (5.33)
\]

It is apparent that $K_3 C_0^{\frac{1}{2}} \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} + \tilde{c} K_2 C_0^{\frac{1}{2}} \bar{C}_1 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} + \frac{\tilde{C}_1}{2} \rightarrow \frac{\tilde{C}_1}{2}$ as $\mu \nearrow \infty$. Similarly, we also have $K_1 C_0 \frac{\ln \mu}{\mu} + \tilde{c} K_4 b_1 C_0 \bar{C}_1 \frac{\ln \mu}{\mu} + \tilde{c} K_1 \bar{C}_0 C_1 \frac{(\ln \mu)^{\frac{3}{2}}}{\mu^2} \rightarrow 0$ as $\mu \nearrow \infty$. Hence there is $\mu^* > \mu_3$ with property: the condition $\mu > \mu^*$ ensures
\[
K_3 C_0^{\frac{1}{2}} \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} + \tilde{c} K_2 C_0^{\frac{1}{2}} \bar{C}_1 \left( \frac{\ln \mu}{\mu} \right)^{\frac{1}{2}} + \frac{\tilde{C}_1}{2} \leq \frac{3 \tilde{C}_1}{4}
\]
as well as
\[
K_1 C_0 \frac{\ln \mu}{\mu} + \tilde{c} K_4 b_1 C_0 \bar{C}_1 \frac{\ln \mu}{\mu} + \tilde{c} K_1 \bar{C}_0 C_1 \frac{(\ln \mu)^{\frac{3}{2}}}{\mu^2} \leq \frac{3 C_1}{4}.
\]

Then this along with (5.29) and (5.33) entails that
\[
\|U(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{3 C_1}{4} e^{-\tilde{\lambda}(t-t^*)}. \quad (5.34)
\]
and
\[
\|\nabla w(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{3\tilde{C}_1}{4} e^{-\hat{\lambda}(t-t^*)}
\] (5.35)

are valid for all \( t \in [t^*, T) \), which by continuity of \( U \) leads to that \( T \) cannot be finite as long as \( \mu > \mu^* \). Consequently, for any global solution to the problem (2.2) with \( \mu > \mu^* \), we have
\[
\|u - \frac{r}{\mu}\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.
\] (5.36)

Due to the asymptotic stability of \( u \), it is easy to find \( \bar{t} > t^* \) satisfying
\[
u(x, t) \geq \frac{r}{2\mu} \quad \text{for all } x \in \Omega \text{ and } t > \bar{t}.
\]

Utilizing the comparison principle to (2.2) and combing with the nonnegativity of \( w \) concludes
\[
w(x, t) \geq \frac{r}{2\mu}(t - \bar{t}) \quad \text{for all } x \in \Omega \text{ and } t > \bar{t}.
\]

This together with (2.1) guarantees that \( \|v\|_{L^\infty(\Omega)} \to 0 \) as \( t \to \infty \). \( \square \)

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