GEODESICS IN LEWIS SPACETIME

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Abstract

The geodesic equations are integrated for the Lewis metric and the effects of the different parameters appearing in the Weyl class on the motion of test particles are brought out. Particular attention deserves the appearance of a force parallel to the axial axis and without Newtonian analogue.

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1 Introduction

Lewis metric [1] describes the spacetime outside an axially symmetric unbounded, along the axial axis, source, endowed with angular momentum. Usually this metric is presented with four parameters [2] which may be real (Weyl class) or complex (Lewis class). In recent papers [3, 4], the physical meaning of these parameters have been discussed for both classes. Thus for the Weyl class, it appears that one of the parameters is proportional to the energy per unit length (at least in the Newtonian limit), a second parameter is the arbitrary constant potential which is always present in the Newtonian solution, and the remaining two parameters are responsible for the non staticity of the metric, although affecting staticity in different ways.

It is the purpose of this work to delve more deeply into the physical meaning of the aforesaid parameters. Accordingly we shall study the motion of test particles in the Lewis spacetime. In the next section we shall present the Lewis metric, the geodesic equations are given in section 3 and the specific case of circular geodesics is considered in section 4. The two next sections are devoted to the Weyl class. Two specific examples for circular geodesics are studied in section 5 exhibiting the difference in the role played by the two parameters responsible for the non staticity. Radial geodesics are considered for several different cases in the last section. Furthermore, in this last section, the geodesics along the axial axis shows the appearance of a force parallel to the axis, without Newtonian analogue. This force tends to damp the motion along the $z$ axis, when the particle approaches this axis, whereas this tendency reverses when the particle moves away from it.

As it will be seen below, besides the usual dragging effect, to be expected whenever a rotating source is involved, another, mixed, effect appears, which we call topological frame dragging, related simultaneously to the rotation of the source and to the topological defect associated with some metric parameters.

2 Lewis spacetime

The general line element for a cylindrically symmetric stationary spacetime is given by

$$ds^2 = -f dt^2 + 2k dt d\phi + e^\mu (dr^2 + dz^2) + l d\phi^2,$$  
(1)
where \( f, k, \mu \) and \( l \) are functions only of \( r \), and the ranges of the coordinates \( t, z \) and \( \phi \) are

\[
-\infty < t < \infty, \quad -\infty < z < \infty, \quad 0 \leq \phi \leq 2\pi,
\]

the hypersurfaces \( \phi = 0 \) and \( \phi = 2\pi \) being identified. The coordinates are numbered

\[
x^0 = t, \quad x^1 = r, \quad x^2 = z, \quad x^3 = \phi.
\]

Einstein’s field equations for vacuum are

\[
R_{\mu\nu} = 0,
\]

and with metric (1) the non identically null components become in van Stockum’s notation,

\[
2\epsilon^\mu DR^0_0 = \left( \frac{lf' + kk'}{D} \right)',
\]

\[
2\epsilon^\mu DR^2_0 = \left( \frac{f k' - k f'}{D} \right)',
\]

\[
2\epsilon^\mu DR^0_3 = \left( \frac{k l' - l k'}{D} \right)',
\]

\[
2\epsilon^\mu DR^3_3 = \left( \frac{f l' + k k'}{D} \right)',
\]

\[
2R_1 = -\mu'' + \mu \frac{D'}{D} - 2 \frac{D''}{D} + \frac{k^2 + f' l'}{D^2},
\]

\[
2R_{22} = -\mu'' - \mu \frac{D'}{D},
\]

where the primes stand for differentiation with respect to \( r \) and

\[
D^2 = k^2 + f l.
\]

The four equations (5,6,7,8) are not all independent; any one of them can be expressed in terms of the remaining three. The general solution of (4) for (1) is the stationary Lewis metric \( f \) given in the notation of \( g \)

\[
f = ar^{-n+1} - \frac{c^2}{n^2a} r^{n+1},
\]

3
\[ k = -Af, \quad \text{(13)} \]
\[ e^\mu = r^{(n^2-1)/2}, \quad \text{(14)} \]
\[ l = \frac{r^2}{f} - A^2 f, \quad \text{(15)} \]

with
\[ A = \frac{c r^{n+1}}{n a f} + b. \quad \text{(16)} \]

The constants \( n, a, b \) and \( c \) can be either real or complex, and the corresponding solutions belong to the Weyl or Lewis classes. For the Lewis class these constants are given by
\[ n = im, \quad \text{(17)} \]
\[ a = \frac{1}{2}(a_1^2 - b_1^2) + ia_1 b_1, \quad \text{(18)} \]
\[ b = \frac{a_1 a_2 + b_1 b_2}{a_1^2 + b_1^2} + \frac{i}{a_1^2 + b_1^2}, \quad \text{(19)} \]
\[ c = \frac{1}{2} m (a_1^2 + b_1^2), \quad \text{(20)} \]

where \( m, a_1, b_1, a_2 \) and \( b_2 \) are real constants and satisfy
\[ a_1 b_2 - a_2 b_1 = 1. \quad \text{(21)} \]

The equations (17,18,19,20,21) reveal that if the values of the parameters \( n \) and \( a \), or \( n \) and \( b \) are known, we can obtain the parameter \( c \). However, knowing \( n \) and \( c \), we cannot obtain \( a \) and \( b \). The metric coefficients (12,13,14,15) with (17,18,19,20) become
\[ f = (a_1^2 - b_1^2) r \cos(m \ln r) + 2a_1 b_1 r \sin(m \ln r), \quad \text{(22)} \]
\[ k = -(a_1 a_2 - b_1 b_2) r \cos(m \ln r) - (a_1 b_2 + a_2 b_1) r \sin(m \ln r), \quad \text{(23)} \]
\[ e^\mu = r^{-(m^2+1)/2}, \quad \text{(24)} \]
\[ l = -(a_2^2 - b_2^2) r \cos(m \ln r) - 2a_2 b_2 r \sin(m \ln r). \quad \text{(25)} \]

3 Geodesics

The equations governing geodesics can be derived from the Lagrangian
\[ 2\mathcal{L} = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}. \quad \text{(26)} \]
where $\lambda$ is an affine parameter along the geodesics. For timelike geodesics $\lambda$ is the proper time. From the extremal problem it emerges the Euler-Lagrange equations

$$\frac{d}{d\lambda} \left( \frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0,$$

(27)

and from them follow the geodesics given by

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0,$$

(28)

where the overdot stands for differentiation with respect to $\lambda$. For spacetime (1) the Lagrangian (26) is

$$2\mathcal{L} = -ft^2 + 2k\dot{t}\dot{\phi} + c^\mu (\dot{r}^2 + \dot{z}^2) + l\dot{\phi}^2,$$

(29)

and the geodesic equations (28) are

$$D\ddot{t} + \frac{lf' + kk'}{D} \dot{t} + \frac{kl' - ll'}{D} \dot{\phi} = 0,$$

(30)

$$2\ddot{r} + e^{-\mu}(f't^2 - 2k'i\dot{\phi} - l't^2) + \mu'(\dot{r}^2 - \dot{z}^2) = 0,$$

(31)

$$\ddot{z} + \mu' \dot{r} \dot{z} = 0,$$

(32)

$$D\ddot{\phi} + \frac{ff' - kk'}{D} \dot{t} + \frac{fl' + kl'}{D} \dot{r} = 0.$$

(33)

The corresponding canonical momenta to (29) are

$$p_t = -\frac{\partial L}{\partial \dot{t}} = ft - k\dot{\phi},$$

(34)

$$p_r = \frac{\partial L}{\partial \dot{r}} = e^\mu \dot{r},$$

(35)

$$p_z = \frac{\partial L}{\partial \dot{z}} = e^\mu \dot{z},$$

(36)

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = kl + l\dot{\phi}.$$

(37)

The integrals of motion follow from the Euler-Lagrange equations (27)

$$\frac{dp_t}{d\lambda} = \frac{\partial L}{\partial t} = 0,$$

(38)

$$\frac{dp_r}{d\lambda} = \frac{\partial L}{\partial r} = 0,$$

(39)

$$\frac{dp_z}{d\lambda} = \frac{\partial L}{\partial z} = 0,$$

$$\frac{dp_\phi}{d\lambda} = \frac{\partial L}{\partial \phi} = 0,$$

(40)
and the conserved quantities are

\[ p_t = E, \quad p_z = P, \quad p_\phi = L, \quad (41) \]

where the constants \( E, P \) and \( L \) represent respectively the total energy of the test particle, its momentum along the \( z \) axis and its angular momentum about the \( z \) axis.

\[ \dot{t} = \frac{1}{D^2}(Lk + El), \quad (42) \]

\[ \dot{z} = Pe^{-\mu}, \quad (43) \]

\[ \dot{\phi} = \frac{1}{D^2}(Lf - Ek), \quad (44) \]

and instead of integrating (31) we can use the line element (1) to obtain

\[ -\epsilon = -f\dot{t}^2 + 2kt\dot{\phi} + e^{\mu}(\dot{r}^2 + \dot{z}^2) + l\dot{\phi}^2, \quad (45) \]

where \( \epsilon = 0, 1 \) or \(-1\) if the geodesics are respectively null, timelike or spacelike.

### 4 Circular geodesics

Now we restrict ourselves to the study of circular geodesics assuming

\[ \dot{r} = \dot{z} = 0, \quad \dot{\phi} = 0, \quad (46) \]

then (31) becomes

\[ f't'^2 - 2k't'\dot{\phi} - l'\dot{\phi}^2 = 0. \quad (47) \]

The angular velocity of the test particle is given by

\[ \omega = \frac{\dot{\phi}}{t}, \quad (48) \]

which becomes using (47)

\[ \omega = \frac{1}{t'} \left[ -k' \pm \left( k'^2 + f't'^2 \right)^{1/2} \right]. \quad (49) \]
For a stationary spacetime the normal velocity $W^\mu$ of the particle defined as the change in the displacement normal to $\tau^\mu = (1, 0, 0, 0)$ relative to its displacement parallel to $\tau^\mu$, where $\tau^\mu$ is a timelike Killing vector, is

$$W^\mu = \left[(-g_{00})^{1/2}\left(dx^0 + \frac{g_{0a}}{g_{00}}dx^a\right)\right]^{-1}V^\mu,$$  \hspace{1cm} (50)

where

$$V^\mu = \left(-\frac{g_{0a}}{g_{00}}dx^a, dx^1, dx^2, dx^3\right),$$  \hspace{1cm} (51)

and Latin indexes range from 1 to 3. Considering the metric (1) and the expression for $\omega$, (48), then (50) becomes

$$W^\mu = \frac{(k\omega, 0, 0, f\omega)}{f^{1/2}(f - k\omega)}.$$  \hspace{1cm} (52)

The modulus of $W^\mu$, defined by $W^2 = W^\mu W_\mu$, and using (52) is

$$W = \frac{D\omega}{f - k\omega}.$$  \hspace{1cm} (53)

We can write (45) by considering (48)

$$\frac{\epsilon}{t^2} = f - 2k\omega - l\omega^2.$$  \hspace{1cm} (54)

Substituting (53) into (54) we obtain

$$\frac{\epsilon}{t^2} = \frac{D^2f}{(D+ kW)^2}(1 - W^2),$$  \hspace{1cm} (55)

which shows that circular geodesics are timelike, null, or spacelike for, respectively, $W < 1, W = 1, $ or $W > 1$.

## 5 Circular geodesics in the Weyl class

In this section we study circular geodesics in the Lewis spacetime for the Weyl class, which means that the parameters $n, a, b$ and $c$ appearing in (12,13,14,15) are real.
The parameter $n$ is associated to the Newtonian mass per unit length $\sigma$,

$$\sigma = \frac{1}{4}(1 - n), \quad (56)$$

of an uniform line mass in the low density regime. The parameter $a$ is connected to the constant arbitrary potential that exists in the corresponding Newtonian solution. In the static and locally flat limit produces angular deficit $\delta$, given by

$$\delta = 2\pi \left(1 - \frac{1}{a^{1/2}}\right). \quad (57)$$

The parameter $b$ is associated, in the locally flat limit, with the angular momentum of a spinning string. Finally, the parameter $c$ is related to the vorticity of the source when it is represented by a stationary completely anisotropic fluid. This parameter, together with the parameter $b$, is responsible for the nonstaticity of the metric. All these interpretations are restricted to the Weyl class only. For further details see reference [3].

We shall now discuss the expression for the angular velocity (49) and the tangential velocity (53). In order to exhibit more clearly the role played by different parameters, we shall consider two different cases, when $b = 0, c \neq 0$ and $b \neq 0, c = 0$.

### 5.1 Case $b = 0, c \neq 0$

For this case substituting metric (12,13,14,15) into (49) we obtain

$$\omega = \frac{c}{n} \pm \omega_0 = \frac{c}{1 - 4\sigma} \pm \omega_0, \quad (58)$$

where $\omega_0$ is the angular velocity when the spacetime is static given by Levi Civita’s metric,

$$\omega_0^2 = \frac{1 - n}{1 + n} a^2 r^{-2n} = \frac{2\sigma}{1 - 2\sigma} a^2 r^{2(4\sigma - 1)}, \quad (59)$$

and (56) has been used.

The term $c/(1 - 4\sigma)$ represents an inertial frame dragging correction to the static case, similar to the one appearing in the case of the Kerr metric [7] or in the field of a massive charged magnetic dipole [8]. The presence of $c$ in
that term, becomes intelligible when we recall that this parameter measures
the vorticity of the source when described by a rigidly rotating anisotropic
cylinder [3]. It increases or diminishes the modulus of ω if the vorticity is in
the same or oposite direction, respectively, of the rotation of the test particle.
Furthermore, the dragging term is increased by the Newtonian mass per unit
length σ.

Calculating the tangential velocity of the particle (53) with (12,13,14,15)
and (56) we obtain

\[ W = \left( \frac{c}{n} \pm \omega_0 \right) \frac{r^n}{a} \left( 1 \pm \omega_0 \frac{cr}{na^2} \right)^{-1} \]

(60)

\[ = \left[ \frac{cr^{1-4\sigma}}{a(1-4\sigma)} \pm W_0 \right] \left[ 1 \pm W_0 \frac{cr^{1-4\sigma}}{a(1-4\sigma)} \right]^{-1}, \]

(61)

where \( W_0 \) is the velocity in the static Levi-Civita spacetime

\[ W_0^2 = \frac{1-n}{1+n} = \frac{2\sigma}{1-2\sigma}, \]

(62)

which can also be rewritten with (59)

\[ W_0 = \frac{r^n}{a} \omega_0, \]

(63)

showing clearly the Newtonian limit \( W_0 = r\omega_0 \). We see that the dragging
term \((cr^{1-4\sigma})/[a(1-4\sigma)]\) for low energy densities and \( a = 1 \) is proportional
to the vorticity times the radius. However it is worth noticing that this latter
term decreases with \( a \), and furthermore since \( W_0 < 1 \), \( W \) also decreases with
\( a \). Thus we have a dragging effect corrected by a topological defect. In
order to understand why \( W \) decreases with \( a \), unlike the angular velocity
\( \omega \), which grows with \( a \), it should be observed that the arc length \( \Lambda \) along a
circular path in this case with \( n = 1 \) becomes \( d\Lambda = (r/a^{1/2})d\phi \). Therefore
for a given \( d\phi \), \( d\Lambda \) decreases with \( a \), explaining thereby the decreasing of \( W \).
On the other hand, for a given \( \Lambda \), \( d\phi \) increases with \( a \), which explains why
the angular velocity also grows with \( a \). This argument is not without its
problems. Indeed if \( c = 0 \) the spacetime reduces to the static Levi-Civita
spacetime but the tangential velocity \( W_0 \) in unaffected by \( a \) contrary to the
argument above. The same is true for the next case below, with respect to
\( b \).
We notice here too, in this case, that $c$ plays no role as a topological defect, i.e., does not appear in the metric term $l$, hence it is restricted to its frame dragging effect.

Considering (54) and (12,13,14,15) we obtain,

$$\epsilon \frac{t^2}{l^2} = \frac{r^{n+1}}{a} \left[ a^2 r^{-2n} - \left( \omega - \frac{c}{n} \right)^2 \right], \tag{64}$$

and using (56,58,59) it becomes

$$\epsilon \frac{t^2}{l^2} = \frac{r^{n+1}}{a} \left( a^2 r^{-2n} - \omega_0^2 \right) = \frac{1 - 4\sigma}{1 - 2\sigma} a r^{4\sigma}. \tag{65}$$

We see from (65) that for circular geodesics the conditions for timelike, null and spacelike orbits, correspondingly $0 \leq \sigma < 1/4$, $\sigma = 1/4$ and $1/4 < \sigma \leq 1/2$ are unaffected by the vorticity of the source $c$.

### 5.2 Case $b \neq 0$, $c = 0$

For this case substituting the metric (12,13,14,15) into (49) we obtain

$$\omega = \frac{\omega_0(b\omega_0 \pm 1)}{1 - b^2 \omega_0^2} \tag{66}$$

Now we have two different kinds of terms appearing in (66). On one hand the angular velocity corresponding to the Levi-Civita spacetime corrected by the factor $1 - b^2 \omega_0^2$, and on the other the topological frame dragging term associated with $b$. We call it topological frame dragging since $b$ produces also a topological defect and to distinguish from the frame dragging engendered by $c$ that does not produce topological defect in the previous case. The increasing of $\omega$ with $b$ may be interpreted as follows. In the $n = 1$ case the line element (1) takes the form

$$ds^2 = -dt^2 - 2ba^{1/2}dtd\phi + dr^2 + dz^2 + \left( \frac{r^2}{a} - b^2a \right) d\phi^2. \tag{67}$$

Therefore, from (67), the arc length $\Lambda$ along the circular path is given by

$$d\Lambda = \left( \frac{r^2}{a} - b^2a \right)^{1/2} d\phi. \tag{68}$$
Thus, for a given \( d\Lambda \), larger \( b \)'s lead to larger \( d\phi \)'s and thereby to larger angular velocities.

Considering (54) and (12,13,14,15) we obtain

\[
\frac{\epsilon}{t^2} = ar^{-n+1}(1 + b\omega)^2 - \frac{r^2}{f}\omega^2, \tag{69}
\]

and using (56,59,66) it becomes

\[
\frac{\epsilon}{t^2} = \frac{1}{(1 \mp b\omega_0)^2} \frac{r^{n+1}}{a} (a^2r^{-2n} - \omega_0^2) = \frac{1}{(1 \mp b\omega_0)^2} \frac{1 - 4\sigma}{1 - 2\sigma} ar^{4\sigma}. \tag{70}
\]

We see from (70) that for circular geodesics the condition for timelike, null and spacelike orbits are the same as in case \( b = 0 \) and \( c \neq 0 \).

Calculating the tangential velocity of the particle (53) with (12,13,14,15) we obtain

\[
W = \frac{r^n}{a} \omega_0 = W_0. \tag{71}
\]

In the general case, \( b \neq 0 \) and \( c \neq 0 \), both parameters contribute in a cumbersome way to frame dragging and topological effects.

### 6 Geodesics in the Weyl class

Substituting (42,43,44) into (45) we have an expression for the radial speed \( \dot{r}^2 \) of the test particle,

\[
\dot{r}^2 = e^{-\mu} \left( E^2 \frac{l^2}{D^2} + 2EL \frac{k}{D^2} - \epsilon - L^2 \frac{f}{D^2} - P^2 e^{-\mu} \right). \tag{72}
\]

We restrict the study of geodesics to \( 0 < n < 1 \), which is the condition for circular timelike geodesics as expected in the Newtonian analog [3].

\[
\dot{z} = Pr^{(1-n^2)/2}, \tag{73}
\]

which means that, if \( P \neq 0 \), particles increase their speed along \( z \) when distancing radially from de axis, while diminish their axial speed when moving radially towards the axis. This result indicates that a force parallel to the \( z \) axis appears. In the flat case \( n = 1 \) such effect vanishes, bringing out its non Newtonian nature.
\[ \ddot{z} = \frac{1 - n^2}{2} \frac{\dot{r} \dot{z}}{r}, \quad (74) \]

therefore, this force tends to damp any motion along the \( z \) axis whenever the particle approaches that axis, and reverses this tendency, in the opposite case. A similar result is obtained for the van Stockum spacetime [4], although in that case the effect is related to the vorticity of the source, whereas in our case, neither \( b \) nor \( c \) appear involved in (73). It is also worth noticing that non Newtonian forces, parallel to the \( z \) axis, also appear in the field of axially symmetric rotating bodies [10]. However, as in the example of reference [9], the force parallel to \( z \), in [10], unlike our case, is directly related to the spin of the source. In a stationary rotating universe it has been observed too a repulsive potential along the axis of rotation [11].

Now we study (72) for the two separate cases considered for circular geodesics and concentrate for null and timelike particles, i.e. \( \epsilon = 0 \) and 1.

### 6.1 Case \( b = 0 \), \( c \neq 0 \)

For this case substituting (12,13,14,15) into (72) we obtain

\[ \dot{r}^2 = r^{-(1-n)^2/2} [V_0 - V(r)], \quad (75) \]

where

\[ V_0 = \frac{1}{a} \left( E - \frac{Lc}{n} \right)^2 > 0, \quad (76) \]

\[ V(r) = \epsilon r^{1-n} + P^2 r^{(1-n)(3+n)/2} + aL^2 r^{-2n} > 0. \quad (77) \]

The radial acceleration of the particle is from (75)

\[ \ddot{r} = \frac{1}{4} r^{-(1-n)^2/2} \left[ -(1-n)^2 V_0 - (1-n)(1+n)\epsilon r^{1-n} 
+ (1+n)^2 aL^2 r^{-2n} - (1-n)(1+n) P^2 r^{(1-n)(3+n)/2} \right]. \quad (78) \]

\[ V'(r) = (1-n) \epsilon r^{-n} - 2naL^2 r^{-1-2n} + \frac{1}{2} (1-n)(3+n) P^2 r^{-1+(1-n)(3+n)/2}. \quad (79) \]

The equation \( V'(r) = 0 \) has a solution, say \( r = r_c \), which satisfies

\[ 2naL^2 r_c^{-1-2n} = (1-n) \epsilon r_c^{-n} + \frac{1}{2} (1-n)(3+n) P^2 r_c^{-1+(1-n)(3+n)/2}. \quad (80) \]
Since
\[ V''(r) = \frac{1}{r} \left\{ -n(1-n)\epsilon r^{-n} + 2n(2n+1)aL^2r^{-1-2n} ight. \\
+ \left. \frac{1}{2}(1-n)(3+n) \left[ \frac{1}{2}(1-n)(3+n) - 1 \right] P^{2r^{-1+(1-n)(3+n)/2}} \right\} , \tag{81} \]
and with (80) we have
\[ V''(r_c) = \frac{1}{r_c} \left\{ (1-n)(1+n)\epsilon r^{-n} ight. \\
+ \left. \frac{1}{2}(1-n)(3+n) \left[ 2n + \frac{1}{2}(1-n)(3+n) \right] P^{2r_c^{-1+(1-n)(3+n)/2}} \right\} . \tag{82} \]
On the other hand, from (77), if \( \epsilon \neq 0 \) and/or \( P \neq 0 \), we have when \( r \to 0 \) or \( \infty \), \( V(r) \to \infty \). Furthermore, from (82) we have \( V''(r_c) > 0 \), hence the equation \( V'(r) = 0 \) has only one solution \( r = r_c \) and is a minimum.

We see that in this case the parameter \( c \) only affects \( V_0, (76) \), by modifying the energy of the test particle, and leaving otherwise the geodesics indistinguishable from the static Levi-Civita spacetime.

### 6.1.1 Case \( \epsilon = 0, P = 0 \)

\[ V(r) = aL^2r^{-2n}, \tag{83} \]
\[ \ddot{r} = \frac{1}{4}r^{-1-(1-n)^2/2}[-(1-n)^2V_0 + (1+n)^2aL^2r^{-2n}]. \tag{84} \]

A null particle with large \( r \) approaches \( z \) with decreasing negative acceleration, \( \ddot{r} < 0 \), and increasing speed \( \dot{r} \). Its speed attains a maximum at \( \ddot{r} = 0 \) and then diminishes since the acceleration becomes positive, \( \ddot{r} > 0 \). The null particle arrives to a minimum distance \( r = r_{min} \) when \( \dot{r} = 0 \) and \( V_0 = V(r) \).

At \( r = r_{min} \) the null particle is reflected to infinity, \( r \to \infty \), where \( \dot{r} \to 0 \).

### 6.1.2 Case \( \epsilon \neq 0, P = 0 \)

\[ V(r) = \epsilon r^{1-n} + aL^2r^{-2n}, \tag{85} \]
\[ \ddot{r} = \frac{1}{4}r^{-1-(1-n)^2/2}[-(1-n)^2V_0 \\
- (1-n)(1+n)\epsilon r^{1-n} + (1+n)^2aL^2r^{-2n}]. \tag{86} \]
For this case \( V_0 = V(r) \) has two real roots, say \( r_{\text{min}} \) and \( r_{\text{max}} \). The timelike particle approaching \( z \) is reflected at \( r = r_{\text{min}} \) and moves outwards till it attains \( \dot{r} = 0 \) at \( r = r_{\text{max}} \) repeating endlessly this trajectory. This kind of confinement of the test particle has also been found in the van Stockum spacetime \([9]\). In this case if the motion is circular, \( \dot{r} = 0 \), there are stable orbits, since \( V(r) \) has a minimum.

### 6.1.3 Case \( P \neq 0 \)

For this case while the particles move along the \( z \) axis, as described by (73), there is a confinement similar to the previous case in the \( z = \text{constant} \) plane. The effect of \( P \), if \( \epsilon \neq 0 \), is to diminish the \( r_{\text{min}} \) and \( r_{\text{max}} \) attained by the particle. It is interesting to observe that even if \( \epsilon = 0 \) the null particle has a similar trajectory in the \( z = \text{constant} \) plane which is not allowed if \( P = 0 \), as described previously.

### 6.2 Case \( b \neq 0, c = 0 \)

For this case substituting (12,13,14,15) into (72) we obtain

\[
\dot{r}^2 = r^{-(1-n)^2/2}[V_0 - V(r)],
\]

where

\[
V_0 = \frac{1}{a}E^2 > 0, \quad (88)
\]

\[
V(r) = \epsilon r^{1-n} + P^2 r^{(1-n)(3+n)/2} + a(L + bE)^2 r^{-2n} > 0. \quad (89)
\]

We see that contrary to \( c \), \( b \) affects \( V(r) \) and not \( V_0 \), by modifying the angular momentum of the test particle. From (89) we see that even if \( L = 0 \) there is an angular momentum term acting on the particle. This difference between the parameters \( b \) and \( c \) is expected from our analysis for the circular geodesics. For \( L \neq 0 \) and \( 0 \) and \( \epsilon = 0 \), \( P = 0 \); \( \epsilon \neq 0 \), \( P = 0 \); and \( P \neq 0 \) the behaviour of the geodesics is qualitatively similar to the case \( b = 0, c \neq 0 \) with \( L \neq 0 \).

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