Markovian and non-Markovian dynamics from probability amplitudes perspective

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One-spin and two-spin in a thermodynamic limit spin-1/2 chain as open quantum systems are considered. The dynamics of system is generated by XX Heisenberg interaction and three-spin interaction (TSI) among all the nearest three spins in the chain. Two variety Hamiltonians of the three-spin interaction are considered. Using the fermionization technique and calculating the trace distance, non-Markovianity as a function of the TSI is evaluated, and the results for the two different open quantum systems are compared. In addition, the time behavior of the probability amplitudes are studied. The results show that if all probability amplitudes except one of them will almost vanish after some time, the dynamics of the open quantum system will be Markovian. In the non-Markovian dynamics, more than one of the probability amplitudes fluctuate in time and will never reach zero value.

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I. INTRODUCTION

The dynamics of open quantum systems has been paid attention lately. Interesting study, both theoretical and experimental, have been done on the dynamical behavior of open quantum systems. Using the theory of open quantum systems, the dynamical behavior of the system that is in interaction with its environment is classified into two categories: Markovian and non-Markovian. The Markovian process involves the flow of information from the system to the environment. Note that in Markovian dynamics, the concurrence decreases exponentially. A common example used to introduce Markov chains is the weather: the chance of sunny, cloudy, or rainy day of tomorrow depends only on what the weather is like today, and is independent of past weather conditions. On the other hand, the process by which part of the information is returned from the environment to the system is called non-Markovian dynamics. Dynamics of open quantum system investigation had been initially allocated to the Markovian behavior, recently have been expanded to the non-Markovian behavior. Systems which are in contact with different non-Markovian environments show the revival of entanglement. It is essential to show the process of the revival of entanglement to introduce a general measurement of non-Markovianity degree in open quantum systems.

Theoretically, in a recent work, the dynamics of a qubit coupled to a spin chain environment has been studied. The environment is described by a XY model in the transverse magnetic field. The dynamics of the qubit is Markovian at a special point. Two regions which are separated by this point are non-Markovian and leads to two completely different dynamical behaviors. It has been shown that the contribution of energy density is responsible for the non-Markovian effects. This result for an infinite environment is also correct. From experimental point of view, a non-Markovianity-assisted high-fidelity refined Deutsh-Jozsa algorithm is implemented with a solid spin in a diamond. Specially, a non-Markovian quantum process is observed by measuring the non-Markovianity of the spin system. The control of the degree of non-Markovianity in the dynamics of an nitrogen-vacancy center electron spin is also demonstrated experimentally. They have showed that, by changing the population of the nitrogen spin, the non-Markovianity of the electron spins dynamics is tuneable. In addition, using a randomized set of central radio-frequency fields, a non-Markovian environment for a single nuclear magnetic resonance qubit is effectively realized.

The task of quantum correlations of the environment has also been studied in the evolution of spin chains. Recently, systems involving multiple interactions, such as three-spin interaction, four-spin interaction, etc., are significant by various parts of physics. A wide range of spin-1/2 Hamiltonians can be created in different configurations of an optical lattice. One type of multiple-spin interaction is a three-spin interaction (TSI) which can be represented by a triangular configuration. TSI has been proposed with various Hamiltonians, including the following:

\[ TSI_1 = J' \sum_{j=1}^{N} (S_j^z S_{j+1}^z + S_j^y S_{j+1}^y - S_j^y S_{j+1}^z S_{j+2}^z), \]

\[ TSI_2 = J'' \sum_{j=1}^{N} (S_j^z S_{j+1}^z S_{j+2}^z + S_j^y S_{j+1}^y S_{j+2}^y). \] (1)

In this paper, for the purpose of simplicity, one of them is called \( TSI_1 \) and the other \( TSI_2 \). Thermodynamic properties and quantum phase transition for both interactions with the Heisenberg spin-1/2 XX model have been investigated. There is a significant difference between the results of these two models. Some of the differences express: The model that containing \( TSI_1 \) shows that it has spontaneous magnetization in its ground state, while the second model has zero magnetization in its ground state. In the study of the specific heat as a function of \( T/J \), it is observed that the first model has a
two-peak structure, while the second model has a single-peak structure.

The dynamics of entanglement and dynamical phase transition from the Markovian to the non-Markovian regime for a one-dimensional spin-1/2 XX model with TSI1 has been studied. In this work, we consider the dynamical phase transition for the system involving TSI2 to determine whether different three-spin interactions cause different dynamical behaviors or not. In addition, we are interested to consider the dynamical behavior of a single-spin system with two different three-spin interaction models, TSI1 and TSI2, and realizing the difference in the results. Finally, this work focuses on the time behavior of probability amplitudes of the quantum state of single-spin and two-spin open quantum systems. We have shown that in the Markovian regime, all probability amplitudes, except one, are being close to zero eventually.

The paper is structured as follows. In section II, first the model is introduced, then using the fermionization technique, the Hamiltonian is diagonalized and analytical results are given. In section III, dynamics of single-spin and two-spin open quantum systems are presented. In section IV, the probability amplitudes are calculated and results are discussed. Finally, we conclude all of our results in section V.

II. THE MODEL

The Hamiltonian of a spin-1/2 XX model with TSI1 is considered as

\[ H = -J \sum_{j=1}^{N} (S_{j}^{x} S_{j+1}^{x} + S_{j}^{y} S_{j+1}^{y}) - J' \sum_{j=1}^{N} (S_{j}^{x} S_{j+1}^{x} S_{j+2}^{x} + S_{j}^{y} S_{j+1}^{y} S_{j+2}^{y}), \]

where \( S_{j} \) is the spin-1/2 operator in the \( j \)-th site, \( J \) and \( J' \) are respectively the exchange coupling between the spins on the nearest neighbors sites and on the three nearest neighbors sites. \( J = 0 \) represents the isotropic XX model. To solve the model, the Hamiltonian should be diagonalized. In the first step, the Jordan-Wigner transformation is used as

\[ S_{j}^{+} = a_{j}^{\dagger} \exp(i \sum_{i<j} a_{i}^{\dagger} a_{i}), \]

\[ S_{j}^{-} = a_{j} \exp(-i \sum_{i<j} a_{i}^{\dagger} a_{i}), \]

\[ S_{j}^{z} = a_{j}^{\dagger} a_{j} - \frac{1}{2}. \]

Using this transformation, fermionic Hamiltonian that contains spinless fermions is obtained,

\[ H_{f} = \frac{J}{2} \sum_{j=1}^{N} (a_{j}^{\dagger} a_{j+1} + a_{j+1}^{\dagger} a_{j}) + \frac{J'}{4} \sum_{j=1}^{N} (a_{j}^{\dagger} a_{j+2} + a_{j+2}^{\dagger} a_{j}). \]

In the next step, using Fourier transformation and transferring to the momentum space as \( a_{k}^{\dagger} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i k j} a_{j}^{\dagger} \), consequently the diagonalized Hamiltonian becomes

\[ H_{f} = \sum_{k} \varepsilon(k) a_{k}^{\dagger} a_{k}, \]

where the dispersion relation is given by

\[ \varepsilon(k) = -J(\cos(k) - \frac{\alpha}{2} \cos(2k)), \]

with \( \alpha = \frac{J'}{J} \).

Similarly, the Hamiltonian of a spin-1/2 XX model with TSI2 is

\[ H = -J \sum_{j=1}^{N} (S_{j}^{x} S_{j+1}^{x} + S_{j}^{y} S_{j+1}^{y}) - J'' \sum_{j=1}^{N} (S_{j}^{x} S_{j+1}^{x} S_{j+2}^{x} - S_{j}^{y} S_{j+1}^{y} S_{j+2}^{y}), \]

where \( J \) and \( J'' \) are respectively the exchange coupling between the spins on the nearest neighbors sites and on the three nearest neighbors sites. \( J'' = 0 \) displays the isotropic XX model. All of the above steps are done again and the dispersion relation is determined by

\[ \varepsilon(k) = -J(\cos(k) - \frac{\alpha}{2} \sin(2k)), \]

where \( \alpha = \frac{J''}{J} \).

The second kind of the three spin interaction breaks \( \pi/2 \)-rotation along z-axis symmetry. Since the dynamics of system is governed by Hamiltonian, so we expect remarkable changes of the dynamics will occur.

We are interested to study the dynamical behavior of open quantum systems and understanding how the different Hamiltonian model for three-spin interaction changes the dynamical behavior of the systems. To study the dynamics and measuring the degree of non-Markovianity, we use trace distance, hence we need to calculate the reduced density matrix. In the first step, it is necessary to determine the initial state of the system at \( t = 0 \). Then by using the time evolution operator as

\[ U(t) = e^{-it \sum \varepsilon(k) a_{k}^{\dagger} a_{k} / \hbar}, \]
the physical state of the system at any time $t$ is obtained. The reduced density matrix between two spins at sites $i$ and $j$ is now computable as
\[
\rho_{i,j} = \begin{pmatrix}
< P_i^+ P_j^+ > & < P_i^+ S_j^- > & < S_i^- P_j^+ > & < S_i^- S_j^- > \\
< P_i^+ S_j^- > & < S_i^- P_j^+ > & < P_i^+ P_j^+ > & < P_i^+ S_j^- > \\
< S_i^- P_j^+ > & < S_i^- S_j^- > & < P_i^+ P_j^+ > & < P_i^+ S_j^- > \\
< S_i^- S_j^- > & < S_i^- S_j^- > & < P_i^+ P_j^+ > & < P_i^+ S_j^- >
\end{pmatrix},
\]
where $P_i^\dagger = \frac{1}{2} + S_i^z, P_i^\dagger = \frac{1}{2} - S_i^z$ and $S_i^z = S_i^x \pm iS_i^y$.

The trace distance which gives a measure of the distinguishability between two initial quantum states, scilicet $\rho_1$ and $\rho_2$ can be expressed as
\[
D(\rho_1, \rho_2) = \frac{1}{2} \text{tr} |\rho_1 - \rho_2|,
\]
where $|\rho_1 - \rho_2| = \sqrt{(\rho_1 - \rho_2)(\rho_1 - \rho_2)}$. At any time, if the information is returned from the environment to the system, the trace distance will increase; therefore, it is obviously related to existence of the non-Markovianity dynamics. The witness of the non-Markovianity $N$ is defined as
\[
N = \max \int_{\sigma > 0} \sigma(t, \rho_{i,2}(0)) dt,
\]
where
\[
\sigma(t, \rho_{i,2}(0)) = \frac{d}{dt} D(\rho_1(t), \rho_2(t)),
\]
is the rate of change of the trace distance at time $t$.

**III. MEASURE OF NON-MARKOVIANITY**

In order to calculate the witness of non-Markovianity, $\rho_1$ and $\rho_2$ must be determined. Hence, we have to define two initial states. Using the method which was explained in the previous section, we determine the dynamics of cluster spin-1/2 XX chain in the thermodynamic limit. We will select single-spin and two nearest neighbor spins as an open quantum system and investigate dynamical behavior of the two systems. Finally, the results will be compared.

**A. Two-spin system**

In this part, a spin-1/2 chain will be considered. We will select two nearest neighbor spins as an open quantum system, while the rest of the chain plays the role of the environment. A symbolic form of the system is shown in Fig. 1 (a). Two initial states are defined as
\[
|\psi_1(t = 0)\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) S \otimes (|\downarrow\downarrow\rangle \ldots \downarrow\rangle) E,
\]
\[
|\psi_2(t = 0)\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + e^{i\phi} |\downarrow\uparrow\rangle) S \otimes (|\downarrow\downarrow\rangle \ldots \downarrow\rangle) E.
\]

The system contains two spins at sites $m$ and $m + 1$. It should be pointed out the two mentioned initial states are extracted from the following equation
\[
|\psi(t = 0)\rangle = \frac{1}{\sqrt{2}} (a_m^\dagger |0\rangle + e^{i\phi} a_{m+1}^\dagger |0\rangle),
\]
where $|0\rangle$ is the vacuum state of the original fermions, i.e. $a_m|0\rangle = 0$ and $\phi$ is a phase factor. Using the defined initial states, the density matrix is calculated. Ultimately, the trace distance is achievable. The trace distance is bounded as $0 < D(\rho_1, \rho_2) < 1$. If the pair states are the same, $D(\rho_1, \rho_2) = 0$; and if the pair states are orthogonal, $D(\rho_1, \rho_2) = 1$. It should be note that for calculating Eq. (12), we examined different quantities of $\phi$ and found the maximum value for $N$.

It is important to determine Markovian or non-Markovian behavior of the system, so we calculated witness of non-Markovianity $N$ as a function of the three-spin interactions strength. The results are shown in Fig. 2 (a), (b).

As it can be seen in Fig. 2 (a), the witness of non-Markovianity $N$ is equal to zero in the absence of $TSI_2$, namely $\alpha = 0$. This means that Markovian behavior in this system is detectable when we just consider nearest neighbors interaction. By applying $TSI_2$, $N$ will be non-zero; thus, the non-Markovian behavior will be observed. In consequence, a dynamical transition from the Markovian to the non-Markovian regime occurs by applying the second kind of Hamiltonian of three-spin interaction. Furthermore, the dynamical transition from Markovian to the non-Markovian for $TSI_1$ had been specified (10) (Fig. 2(b)). It was observed that if the $TSI_1$ did not exist, Markovian behavior would be observed in the system.
By exerting the TSI$_1$, for $\alpha \lesssim 0.5$, we still see the Markovian behavior. For $\alpha \gtrsim 0.5$, the system has the non-Marvokian behavior, hence the dynamical phase transition is occurred. We have to mention that the quantitative deviation of the results presented in Fig. 2(b) with Ref. [x] in the region $\alpha > 0.5$ is related to the better accuracy of our code in solving numerically Eq. (12). Comparing the results, it is clearly seen that the second kind of the three spin interaction which breaks $\pi/2$-rotation along $z$-axis symmetry, makes remarkable changes of the dynamics. It is comparability of the result of the study of the thermodynamic properties for the two types of the cluster interaction [23,24].

**B. Single-spin system**

In this section, we will select a single spin in a spin-1/2 chain as an open quantum system, while the rest of the chain plays the role of the environment. The symbolic form of the system is displayed in the Fig. 1(b). For this system, the two initial states are defined as

\[ |\psi_1(t = 0)\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle + |\uparrow\rangle)_S \otimes (|\downarrow\downarrow\ldots\downarrow\rangle)_E, \]

\[ |\psi_2(t = 0)\rangle = (\sin(\phi)|\downarrow\rangle + \cos(\phi)|\uparrow\rangle)_S \otimes (|\downarrow\downarrow\ldots\downarrow\rangle)_E. \]  

(16)

Similar to the two-spin system, we want to determine whether the dynamical behavior of the system is Markovian or non-Markovian. Witness of non-Markovianity $N$ as a function of the three-spin interactions strength are shown in Fig. 2(a), (b). It can be seen in Fig. 2(c) that in the absence of TSI$_2$, namely $\alpha = 0$, the dynamics of the system is non-Markovian. By applying the TSI$_2$, the non-Markovian behavior is still observed. Therefore, the dynamical transition does not occur since Markovian behavior is not seen.

To compare the behavior of the two types of the three-spin interaction models, we seek the dynamical behavior of single-spin system by applying TSI$_1$ too. The result is presented in Fig. 2(d). As it can be seen, again in the absence and presence of the cluster interaction, non-Markovian behavior is observed. Consequently, no dynamical phase transition is occurred when we have single-spin open quantum system; while in the two-spin open quantum system the dynamical phase transition from the Markovian to the non-Markovian regime was happened if the three-spin interaction took into account. The dynamical behavior of system again notably changes when we consider the Hamiltonian which breaks $\pi/2$-rotation along $z$-axis symmetry. Comparing two diagrams in Fig. 2(c), (d), we can see at the range of $0 \leq \alpha \lesssim 0.7$ the value of witness of non-Markovianity decreases for the case of TSI$_2$, while the value increases for the case of TSI$_1$. Increasing values of $\alpha$, witness of non-Markovianity respectively increases for the case of TSI$_2$, and decreases for the case of TSI$_1$.

**IV. PROBABILITY AMPLITUDE**

In the following, we will inspect the Markovian and the non-Markovian behavior of the open quantum system as probability amplitude perspective. The time evolution of the initial non-stationary state of the system can be observed. For an open quantum system which consists of $m$ spins, the dimension of the Hilbert space is $2^m$ and the base kets are mostly selected as the eigenstates of $S^z_{tot} = \sum_{i=1}^{m} S^z_i$. The most general form of the initial state of the open quantum system is written as

\[ |\psi(t = 0)\rangle = c_1 |\uparrow\uparrow\ldots\rangle + c_2 |\uparrow\ldots\downarrow\rangle + \ldots + c_{2^m} |\downarrow\downarrow\ldots\rangle, \]  

(17)

where $|c_1|^2, |c_2|^2, \ldots, |c_{2^m}|^2$ are known as the probability amplitudes with $\sum_{i=1}^{2^m} |c_i|^2 = 1$. By choosing the initial states of the single-spin and two-spin open quantum system respectively as

\[ |\psi(t = 0)\rangle = \frac{1}{\sqrt{2}} (|\downarrow\rangle + |\uparrow\rangle)_S \otimes (|\downarrow\downarrow\ldots\downarrow\rangle)_E, \]

\[ |\psi(t = 0)\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)_S \otimes (|\downarrow\downarrow\ldots\downarrow\rangle)_E, \]  

(18)
we calculated the time-dependent probability amplitudes, $|c_i(t)|^2$. The Hamiltonian is considered as the spin-1/2 XX model with the first kind of three-spin interaction which preserves all symmetries. Probability amplitudes of single-spin open quantum system, namely $|c_1|^2$ and $|c_2|^2$, are plotted as a function of time for two different values of $\alpha = 0$ and $\alpha = 0.7$, Fig. 3(a), (b). The results prove that the time-dependent quantum state of the open single-spin system remains a superposition of two eigenstates; therefore, the dynamics of the system is non-Markovian which is in consistence with the consequence of the last section which was shown that the dynamics of the single-spin open quantum system is non-Markovian independent of the value of $\alpha$.

Results on the time behavior of the probability amplitudes of two-spin open quantum system for two different values of $\alpha = 0$ and $\alpha = 0.7$ are presented in Fig. 3(c), (d). In the last section we showed that the dynamics of system was non-Markovian only in the region $\alpha \geq \alpha_c \sim 0.5$. It can be seen in Fig. 3(c) that in the case of $\alpha = 0$, one of the probability amplitudes increases by time and will be close to one; while other probability amplitudes reach zero eventually. In fact, the system behaves as a one-level system and no information can be reserved which agrees that the system is in the Markovian region. In the case of $\alpha = 0.7$, Fig. 3(d) shows that more than one probability amplitude fluctuate with time and system behaves as a three-level system with a potential for reserving the information, which agrees that the system is in the non-Markovian region.

V. CONCLUSION

In this paper the dynamical behavior of single-spin and two-spin open quantum systems in a one-dimensional spin-1/2 isotropic XX Heisenberg model with three-spin interaction have been considered. Two kinds of Hamiltonian of the three-spin interaction were evaluated; one is symmetric under parity, time and rotation while the other breaks $\pi/2$-rotation along $z$-axis symmetry. We have shown that notable changes of the dynamics of the systems have been occurred since the dynamics of system is governed by Hamiltonian. Calculating witness of non-Markovianity for single-spin system with the absence and presence of the two kinds of Hamiltonian, we have shown that the open quantum single-spin system experiences no dynamical transition, since one-way flow of information from the system to the environment were never observed. However; we have seen that the system has treated differently when we compare the diagrams of non-Markovianity as a function of the cluster interactions.

In the two-spin system, we have shown that in the absence and presence of the three-spin interaction, the system is respectively Markovian and non-Markovian, which means that by applying the cluster interaction, the system experiences a dynamical phase transition from the Markovian to the non-Markovian regime. Moreover, the effect of two types of the three-spin interaction ($TSI_1$ and $TSI_2$) on the dynamical behavior was compared.

In the case of applying $TSI_1$ the system is Markovian for $\alpha \lesssim 0.5$. By increasing the value of $\alpha$, the system displays non-Markovian behavior. While, in the case of having $TSI_2$, the system is Markovian for $\alpha = 0$.

In addition, the probability amplitudes as a function of time for single-spin and two-spin open quantum systems were determined. Results showed that in the Markovian regime, all the probability amplitudes, except one, will almost vanish after some time. But, more than one probability amplitudes fluctuate in time, when the dynamics of the open quantum systems is non-Markovian.

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