The study of formal arcs was initiated by Nash in a 1967 preprint; published much later as [Nas95]. Arc spaces of smooth varieties have a rather transparent structure but difficult problems arise for arcs passing through singularities. The Nash conjecture on the irreducible components of such arc spaces was proved for surfaces [FdBP12b] and for toric singularities [IK03], but counter examples were found in higher dimensions [IK03, dF12, Kol12].

Here we start the study of holomorphic arcs; these are holomorphic maps \( \phi : \overline{D} \to X \) of the closed unit disk to a complex analytic space \( X \). As one expects, there is not much conceptual difference between the set of formal arcs and the set of holomorphic arcs since every formal arc can be approximated by holomorphic arcs. However, a formal deformation of an arc is a much more local object than a holomorphic deformation. (See Remark 3 and Example 4 for more details.) Thus, in many cases, the space of holomorphic arcs has infinitely many connected components while the space of formal arcs always has only finitely many.

For a complex analytic space \( X \) we define two variants – the space of \textit{arcs}, denoted by \( \text{Arc}(X) \) and the space of \textit{short arcs}, denoted by \( \text{ShArc}(X) \) – and we study their connected components. For short arcs we obtain complete answers for surface singularities and for isolated quotient singularities in all dimensions.

1. ARCS ON ANALYTIC SPACES

\textbf{Definition 1 (Arcs).} Let \( X \) be a complex analytic space. Let \( D \subset \mathbb{C} \) denote the open unit disk and \( \overline{D} \subset \mathbb{C} \) its closure. A complex analytic arc in \( X \) is a holomorphic morphism \( \phi : \overline{D} \to X \). (That is, \( \phi \) is defined and holomorphic in some neighborhood of \( \overline{D} \).) The center of \( \phi \) is the point \( \phi(0) \in X \).

We think of an arc as describing the local behavior of a morphism of a Riemann surface to \( X \). The interesting aspects happen if the image passes through a singular point of \( X \). We thus localize at such a point and get a complex analytic morphism \( \phi : \overline{D} \to X \) such that \( \text{Supp} \, \phi^{-1}(\text{Sing} \, X) = \{0\} \). This will be called a short complex analytic arc or a short arc.

More generally, if \( Z \subset X \) is a subset then a short arc on \( (Z \subset X) \) is a complex analytic morphism \( \phi : \overline{D} \to X \) such that \( \text{Supp} \, \phi^{-1}(Z) = \{0\} \). Thus a short arc on \( X \) is the same as a short arc on \( (\text{Sing} \, X \subset X) \).

For many purposes it is easier to work with formal arcs. These are morphisms \( \phi : \text{Spec} \, \mathbb{C}[t] \to X \). Most earlier works on arc spaces considered formal arcs; see [Nas95, Mus02, IK03, Ish04, EM09, FdBP12b] and the references there.

\textbf{Definition 2 (Spaces of arcs).} Let \( X \) be a complex space and \( Z \subset X \) a subset. We consider various spaces of arcs. First we define these only as sets, then we endow them with a natural topology.

(1) \( \text{Arc}(X) \) is the set of all arcs \( \phi : \overline{D} \to X \).
(2) $\Arc^c(Z \subset X)$ is the set of those arcs $\phi$ for which $\phi(\partial D) \subset X \setminus Z$.

(3) $\ShArc(Z \subset X)$ is the set of short arcs on $Z \subset X$.

If $Z = \Sing X$ then we also use the notation

(4) $\Arc^c(X) := \Arc^c(\Sing X \subset X)$ and

(5) $\ShArc(X) := \ShArc(\Sing X \subset X)$.

Fixing a continuous metric $d(\ , \ )$ on $X$, we get a metric on $\Arc(X)$ by setting

$$d_\alpha(\phi, \psi) := \sup\{d(\phi(t), \psi(t)) : t \in D\}.$$ 

It is clear that the topology induced by this metric does not depend on the choice of the metric $d(\ , \ )$. The other arc spaces inherit their topology from $\Arc(X)$.

It is quite likely that these spaces also have a natural structure as an infinite dimensional complex space, but we do not establish this. (See Conjecture 73 for a more precise version.) In all our proofs, we essentially write down finite dimensional complex subspaces of $\Arc(X)$ and work with them.

**Remark 3** (Comparison of the arc spaces). The arc spaces $\Arc(X)$, $\ShArc(X)$ and the space of formal arcs, which we denote by $\hat{\Arc}(X)$, are different as sets, but the main distinguishing feature comes from the deformations that we allow.

Consider for instance a short arc $\phi_0 : (0 \in D) \to (Z \subset X)$ and a deformation of it as an arc $\{\phi_s : s \in [0, 1]\}$. In general, $\phi_s^{-1}(Z)$ breaks up into several points; write these as $p_1(s), \ldots, p_m(s) \in D$.

If $m > 1$ then, from our point of view, $\phi_s$ is a global object. Picking any of the $p_i(s)$ and ignoring the others corresponds to a family of formal arcs. Note that for each $s \in [0, 1]$ we can switch to a smaller disk $D_s := D(\epsilon_s) \supset p_i(s)$ of radius $\epsilon_s$ such that $\phi_{s|D_s}$ becomes a short arc. However, $\lim_{s \to 0} \epsilon_s = 0$, thus we never get a family of short arcs if $m > 1$.

If $m = 1$ then, after a translation and rescaling, we do get a family of short arcs. Thus, working with short arcs is essentially an equisingularity condition on families.

Spaces of formal arcs always have only finitely many (connected or irreducible) components. Since an arc can be through of as a collection of many formal arcs, it is not surprising that $\Arc^c(X)$ usually has infinitely many connected components. By contrast, $\ShArc(X)$ is essentially a dense subset of $\hat{\Arc}(X)$. However, the equisingularity condition turns out to be quite restrictive, and frequently we again get infinitely many connected components.

**Example 4.** Consider the surface singularity

$$S := (xyz = x^4 + y^4 + z^4) \subset \mathbb{C}^3.$$ (1)

(This is one of the simplest cusp singularities; we study the general case in Sections 9.10.) Any arc $(x(t), y(t), z(t))$ on $S$ can be written uniquely as

$$(x(t), y(t), z(t)) = (t^m u(t), t^n v(t), t^m w(t))$$

where $m \in \mathbb{N}$ and $(u(0), v(0), w(0)) \neq (0, 0, 0)$. If $u(t)^4 + v(t)^4 + w(t)^4$ is nowhere zero on $D$ then we can write this arc as

$$ (x(t), y(t), z(t)) = \left(u \cdot \frac{uvw}{u^4 + v^4 + w^4}, v \cdot \frac{uvw}{u^4 + v^4 + w^4}, w \cdot \frac{uvw}{u^4 + v^4 + w^4}\right).$$ (2)

Conversely (and this is a very special occurrence) for any $(u(t), v(t), w(t))$ such that

$$u(t)^4 + v(t)^4 + w(t)^4 \neq 0 \quad \text{for} \quad t \in D,$$

(3)
the formula \( (1) \) defines an arc on \( S \). (The condition \( (1) \) is open on \( \text{Arc}(S) \) and it always holds on a smaller disc. It turns out that, as far as local deformations are concerned, the arcs satisfying \( (1) \) are typical among all arcs.) The center of this arc is the origin iff

\[
u(0) \cdot v(0) \cdot w(0) = 0.\] (1.4)

Under the traditional, Nash version of the space of arcs on \( (0 \in S) \), we can perturb \( (u(t), v(t), w(t)) \) such that one of the functions vanishes at the origin with multiplicity 1 and the other two do not vanish. This shows that \( \widetilde{\text{Arc}}(0 \in S) \) has 3 irreducible components, corresponding to which of the three values \( u(0), v(0), w(0) \) vanishes.

By contrast, \( \text{ShArc}(0 \in S) \) has infinitely many connected components. Indeed, if one of the functions \( u(t), v(t), w(t) \) vanishes at a point \( t_0 \in \mathbb{D} \) then the arc \( (1) \) sends \( t_0 \) to the origin. Thus if the arc \( (1) \) is short then the functions \( u(t), v(t), w(t) \) vanish only at the origin. This implies that the multiplicity of their zero at the origin is a locally constant function on \( \text{ShArc}(0 \in S) \). (Strictly speaking, so far we have established this only for the open set where \( (1) \) holds, but it turns out to be true in general.) Thus we get 3 doubly infinite families of connected components given by the possibilities

\[
\text{mult}_0(u(t), v(t), w(t)) = (n_1, n_2, n_3) \in \mathbb{N}^3 \quad \text{where} \quad n_1n_2n_3 = 0.
\]

5 (Connected components of \( \text{ShArc} \)). Let \( X \) be a complex space and \( Z \subset X \) a closed subset such that \( X \setminus Z \) is connected. Given any arc \( \phi \in \text{Arc}^0(Z \subset X) \) we can restrict it to the boundary to get

\[
\phi|_{\partial X} : \partial \overline{X} \cong S^1 \to X \setminus Z.
\]

There is no natural basepoint, thus this map defines an element of

\[
\pi_1(X \setminus Z) / \text{(conjugation)}.
\]

(We fix the counterclockwise orientation on \( \partial \overline{X} \).) The map is clearly locally constant on \( \text{Arc}^0(Z \subset X) \), thus it descends to

\[
\pi_0(\text{Arc}^0(Z \subset X)) \to \pi_1(X \setminus Z) / \text{(conjugation)}.\] (5.1)

Since \( \text{ShArc}(Z \subset X) \subset \text{Arc}^0(Z \subset X) \), we get similar maps for \( \text{ShArc}(Z \subset X) \).

Note further that the semigroup \((\mathbb{R}_{>0}, +)\) acts freely on \( \text{ShArc}(Z \subset X) \) by

\[
\sigma_s : \phi(t) \mapsto \phi(e^{-s} \cdot t)
\]

and we can view \( \text{ShArc}(Z \subset X) \) as an interval bundle over the orbit space of this action. In particular, if \( Z \subset U \subset X \) is any open subset then \( \text{ShArc}(Z \subset X) \) and \( \text{ShArc}(Z \subset U) \) are homeomorphic, so understanding the topology of \( \text{ShArc}(Z \subset X) \) is a local problem near \( Z \). (By contrast, \( \text{Arc}^0(Z \subset U) \) does depend on the choice of \( Z \subset U \subset X \); see, however, Conjecture (76))

Thus, for an isolated singularity \((0 \in X)\), the topology of \( \text{ShArc}(0 \in X) \) is independent of the particular representative that we choose. If \( X \) is contractible then \( X \setminus \{0\} \) is homotopy equivalent to \( \text{link}(0 \in X) \), thus we get a natural map

\[
w : \pi_0(\text{ShArc}(0 \in X)) \to \pi_1(\text{link}(0 \in X)) / \text{(conj)}.\] (5.2)

More generally, if \( Z \subset X \) is compact, and \( \text{link}(Z \subset X) \) denotes the boundary of a regular neighborhood of \( Z \subset X \) then we have a natural map

\[
w : \pi_0(\text{ShArc}(Z \subset X)) \to \pi_1(\text{link}(Z \subset X)) / \text{(conj)}.\] (5.3)
We call these the winding number maps.

Our main result says that for surfaces these maps are injective.

**Theorem 6.** Let \( (0 \in X) \) be a normal surface singularity. Then the winding number map \( (5.2) \)

\[
\pi_0(\text{ShArc}(0 \in X)) \to \pi_1(\text{link}(0 \in X))/(\text{conj}) \tag{6.1}
\]
is an injection.

The map \( (6.1) \) is rarely surjective since its image is clearly contained in \[
\ker[\pi_1(\text{link}(0 \in X)) \to \pi_1(E)]/(\text{conjugation by } \pi_1(\text{link}(0 \in X))) \tag{6.2}
\]
where \( E \) is the exceptional curve of the minimal resolution of \( (0 \in X) \). The image of the map \( (6.1) \) can be explicitly given in terms of the minimal resolution, see Theorem 37. The description shows that, with a few exceptions, the image of the winding number map is an infinite but small subset of \( (6.2) \).

Since the topology of the minimal resolution is determined by the oriented homeomorphism type of the link \([\text{[Neu81]}]\), we obtain the following consequence.

**Corollary 7.** For a normal surface singularity \( (0 \in X) \) the image of the winding number map \( (6.1) \) — and hence \( \pi_0(\text{ShArc}(0 \in X)) \) — is determined by the oriented homeomorphism type of \( \text{link}(0 \in X) \). \( \square \)

For some interesting classes of surface singularities there are even better descriptions. For quotient singularities, this works in all dimensions.

**Theorem 8.** Let \( (0 \in X) \cong (0 \in \mathbb{C}^n)/G \) be an isolated quotient singularity. Then there are natural identifications

\[
\pi_0(\text{ShArc}(0 \in X)) = \pi_1(\text{link}(0 \in X))/(\text{conj}) = G/(\text{conj}).
\]

For \( G \subset \text{SL}(2, \mathbb{C}) \), this gives a concrete realization of the McKay correspondence between nontrivial conjugacy classes of \( G \) and exceptional curves of the minimal resolution of \( \mathbb{C}^2/G \). We also get a McKay-type correspondence in higher dimensions, see Paragraph 30.

It turns out that \( \pi_0(\text{ShArc}(0 \in S)) \) is infinite for every other surface singularity; see Theorem 37. We have especially clear descriptions in two further cases.

Let \( (0 \in S) \) be a surface cusp singularity. (See \([\text{Hir73]}\) or Section 9 for the definition and basic properties.) The fundamental group of the link of a cusp is given by an extension

\[
0 \to H \cong \mathbb{Z}^2 \to \pi_1(\text{link}(0 \in S)) \to \mathbb{Z} \to 0.
\]

Under conjugation, \( H \) acts trivially on itself, thus conjugation by \( \pi_1(\text{link}(0 \in S)) \) on \( H \) can be described by a single matrix \( M \in \text{SL}(2, \mathbb{Z}) \) which can be easily computed from the minimal resolution. The conjugacy classes are the orbits of the action generated by \( M \). The dual cusp gives the same link but with a different orientation; see Paragraph 64.

**Theorem 9.** Let \( (0 \in S) \) be a surface cusp singularity. Then

1. The image of the winding number map \( (5.2) \) is contained in the subgroup \( H \) modulo conjugation by \( \pi_1(\text{link}(0 \in S)) \).
Every nontrivial conjugacy class of $H$ is obtained uniquely from a connected component of the space of short arcs in either the cusp or the dual cusp for a suitable orientation of $\partial \overline{D}$.

We get a different type of answer for normal surface singularities $(0 \in S)$ with a good $\mathbb{C}^*$-action given by $(\lambda, s) \mapsto \rho(\lambda, s)$. We say that an arc $\phi : D \to (0 \in S)$ is equivariant if there are $a, b \in \mathbb{N}$ such that

$$\phi(\lambda^a t) = \rho(\lambda^a, \phi(t)) \quad \text{for} \quad t, \lambda \in \overline{D}.$$

We can assume that $(a, b) = 1$. Note that $b$ divides the order of the stabilizer subgroup of the $\mathbb{C}^*$-action along the image of $\phi$, thus $b = 1$ for most equivariant arcs.

**Proposition 10.** Let $(0 \in X)$ be a normal surface singularity with a good $\mathbb{C}^*$-action.

1. Every connected component of $\text{ShArc}(0 \in X)$ contains an equivariant arc.
2. This equivariant arc is unique, except when
   (a) either $(0 \in X)$ is a quotient singularity
   (b) or $b = 1$ in the notation above, in which case there is an irreducible 1-parameter family.

**11 (The method of the proof of Theorem 6).** Let $(0 \in X)$ be a normal surface singularity and $f : Y \to X$ a proper bimeromorphic morphism that is an isomorphism outside 0. Set $Z := \text{Supp} f^{-1}(0)$. Composing with $f$ gives continuous bijections

$$f_\ast : \text{Arc}(Z \subset Y) \to \text{Arc}(0 \in X) \quad \text{and} \quad f_\ast : \text{ShArc}(Z \subset Y) \to \text{ShArc}(0 \in X).$$

However, these maps are usually not homeomorphisms and not even the number of connected components is preserved.

In general, there is a tension between two requirements.

- The local structure of $(Z \subset Y)$ should be simple in order to be able to describe $\text{ShArc}(Z \subset Y)$.
- The map $f$ should be simple so that $f_\ast$ be a homeomorphism, or at least that $f_\ast : \pi_0(\text{ShArc}(Z \subset Y)) \to \pi_0(\text{ShArc}(0 \in X))$ be a bijection.

Following the Nash conjecture [Nas95], a first attempt could be to take $Y$ to be the minimal resolution (or the minimal log resolution) of $(0 \in X)$. Both of these fail the second requirement, even in very simple cases like $X = (xy = z^n)$ for any $n \geq 2$.

Our results rest on the observation that the minimal dlt modification (to be defined in Section 7) has both of the above good properties. The proof uses the tight connection between the minimal dlt modification and the plumbing construction of links [Neu81].

From the point of view of the plumbing construction, Theorems 8–9 correspond to the two exceptional cases and Proposition 10 to the simplest case.

**12 (Description of the sections).** We start our discussions by describing the various arc spaces of the unit disk $(0 \in D)$ in Section 3. This is then used in Section 5 to determine the connected components of the spaces of short arcs on simple normal crossing pairs. We also study equivariant versions. Arcs on quotient singularities are related to equivariant arcs on simple normal crossing pairs. This leads to the proof of Theorem 8 in Section 6.
The Nash conjecture \cite{Nas95}, proved in \cite{FdBP12b} for surfaces, describes the irreducible components of the space of formal arcs on a surface $S$ in terms of the minimal resolution. Similarly, in Section 7 we use the minimal dlt modification to understand the irreducible components of $\text{ShArc}(S)$. This gives an explicit description of the image of the map (6.1).

As with the Nash conjecture, it is easy to come up with a list of candidates for the irreducible components of an arc space; the hard part is to prove that none of them is contained in the closure of another. In Section 8 we study the conjugacy problem for the fundamental group of links of surface singularities to prove that the map (6.1) is injective and to give an explicit description of its image. The particular cases of Seifert manifolds is discussed in \cite{56,57}. At the end we also show Proposition 10.

A topological proof of Theorem 9 is given in Section 9. A cusp and its dual can be seen very clearly on hyperbolic Inoue surfaces. This leads to another proof in Section 10.

Comments and conjectures on long arcs, higher dimensions and real arcs are in Section 11.

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2. Families of arcs

Definition 13 (Families of arcs). Let $X$ be a complex space and $V$ a topological space. A continuous family of arcs parametrized by $V$ is a continuous function $F(\ , ) : V \times \overline{D} \to X$ such that $F(v, \ ) : \overline{D} \to X$ is holomorphic \eqref{103} for every $v \in V$. If $V$ is a complex space and $F$ is holomorphic, we have a holomorphic family of arcs.

A family of arcs is essentially the same as the corresponding classifying map $F_c : V \to \text{Arc}(X)$.

We have not defined a complex structure on $\text{Arc}(X)$, but the following approximation allows us to think about holomorphic classifying maps.

Let $\mathcal{H}(\overline{D})$ denote the Banach space of holomorphic functions of $\overline{D}$, that is, functions that are defined and holomorphic in some neighborhood of $\overline{D}$. An arc in $\mathbb{C}^N$ is the same as $N$ holomorphic functions

$\phi_1, \ldots, \phi_N \in \mathcal{H}(\overline{D})$.

Thus we can identify $\text{Arc}(\mathbb{C}^N)$ with the complex Banach space $\mathcal{H}(\overline{D})^N$. A norm is given by

$d(\phi, \psi) : \sum_i \sup\{|\phi_i(t) - \psi_i(t)| : t \in \overline{D}\}$.

If $U \subset \mathbb{C}^N$ is an open set then $\text{Arc}(U)$ is naturally an open subset of $\text{Arc}(\mathbb{C}^N)$ and if $X \subset U$ is a closed subset defined by the equations $f_i(x_1, \ldots, x_N) = 0$ then $\text{Arc}(X)$ is a closed subset of $\text{Arc}(U)$ defined by the equations

$f_i(\phi_1(t), \ldots, \phi_N(t)) \equiv 0 \ \forall \ i$. 

Note that here we think of $f_i$ as a map $f_i : \text{Arc}(U) \to \mathcal{H}(\overline{D})$, thus $\text{Arc}(X) \subset \text{Arc}(U)$ has infinite codimension.

If $X$ is smooth of dimension $n$, then this construction establishes a complex Banach manifold structure on $\text{Arc}(X)$, locally modeled on $\mathcal{H}(\overline{D})^n$.

Note also that $\text{Arc}(\mathbb{C}^N)$ is a complex vector space, thus it can be written as a union of finite dimensional complex vector spaces. However, it seems that a typical $\text{Arc}(X) \subset \text{Arc}(\mathbb{C}^N)$ intersects all finite dimensional vector subspaces of $\text{Arc}(\mathbb{C}^N)$ in a finite set of points, thus we do not get interesting analytic subspaces of $\text{Arc}(X)$ this way.

**Definition 14** (Irreducibility). Let $X$ be a complex space and $A \subset \text{Arc}(X)$ a subset. We say that $A$ is strongly irreducible if for every finite subset $P \subset A$ there is a (finite dimensional) irreducible complex space $V_P$ and a holomorphic family of arcs $F_P : V_P \times \overline{D} \to X$ such that the image of the classifying map $F_P : V_P \to \text{Arc}(X)$ lies in $A$ and contains $P$.

If $A$ is a (finite dimensional) complex space then $A$ is strongly irreducible iff it is irreducible.

There are two simple cases when we prove that some infinite dimensional arc space $A$ is strongly irreducible.

First, assume that we can write $A$ as a convex open subset in some complex Banach space $\mathcal{H}$. If $V \subset \mathcal{H}$ is any finite dimensional vector subspace then $V \cap A$ is a convex open subset of $V \cong \mathbb{C}^m$, hence an irreducible complex manifold.

Second, assume that we can write $A$ as an open subset of $M \times \mathcal{H}$ where $M$ is a (finite dimensional) irreducible complex space and $\mathcal{H}$ is a complex Banach space. Assume further that $A$ contains a constant section $M \times \{v\}$ for some $v \in \mathcal{H}$ and $(\{m\} \times \mathcal{H}) \cap A$ is convex for every $m \in M$. If $V \subset \mathcal{H}$ is any finite dimensional vector subspace containing $v$ then $A \cap (M \times V)$ is a finite dimensional complex manifold. The fibers of its projection to $M$ are convex, open in $V \cong \mathbb{C}^m$ and $M \times \{v\}$ gives a section. Thus $A \cap (M \times V)$ is irreducible.

Our definition is dictated by what we could prove in some cases and it would be useful to develop a notion of irreducibility of arc spaces that is more in line with the usual finite dimensional concept. See Conjecture 75 for further discussion of this topic.

### 3. Arc spaces of the disk

In this section we describe the connected components of the various arc spaces of the disk $\mathbb{D}$. We show that they are all complex Banach manifolds, have finite dimensional approximations and irreducible in the strong sense of (14). In subsequent sections we build up arc spaces of higher dimensional complex spaces from products of arc spaces of the disk.

**15 (Arc spaces of the disk).** One can think of $\text{Arc}(\mathbb{D})$ as the space of holomorphic functions $\phi : \overline{D} \to \mathbb{D}$. Thus $\text{Arc}(\mathbb{D}) = B_{<1}(\mathcal{H}(\overline{D}))$, the open unit ball in $\mathcal{H}(\overline{D})$, hence $\text{Arc}(\mathbb{D})$ is strongly irreducible by (14).

$\text{ShArc}(0 \in \mathbb{D})$ is the space of holomorphic functions $\phi : \overline{D} \to \mathbb{D}$ whose only zero is at the origin. The multiplicity of the zero is locally constant, giving the connected components $\{\text{ShArc}(0 \in \mathbb{D})_m : m = 1, 2, \ldots \}$. It is convenient to set $\text{ShArc}(0 \in \mathbb{D})_0 := \text{Arc}(\mathbb{D} \setminus \{0\})$, the set of arcs that do not pass through 0.
For $m = 0$, the function $\phi$ has no zeros, thus it has a logarithm
\[ \log \phi : \overline{D} \to \{ h \in \mathcal{H}(\overline{D}) : \Re(h) < 0 \}. \]
The space of functions whose real part is everywhere negative is a convex open subset of $\mathcal{H}(\overline{D})$, thus has the same type of finite dimensional approximations as the unit ball. The actual arc space is the quotient
\[ \text{ShArc}(0 \in D)_0 \equiv \{ h \in \mathcal{H}(\overline{D}) : \Re(h) < 0 \}/2\pi i \mathbb{Z}. \]

If $\phi$ has an $m$-fold zero at the origin then $t^{-m}\phi$ has no zeros. On the boundary of $\overline{D}$ it is still strictly less than 1 in absolute value, hence, by the maximum principle, $t^{-m}\phi \in \text{ShArc}(0 \in D)_0$. Thus multiplication by $t^m$ gives an isomorphism
\[ t^m : \text{ShArc}(0 \in D)_0 \cong \text{ShArc}(0 \in D)_m. \]

Arc$^\circ(0 \in \mathbb{D})$ is the space of holomorphic functions $\phi : \overline{D} \to \mathbb{D}$ that have no zero on the boundary. The number of zeros in $\mathbb{D}$ is locally constant, giving the connected components $\{ \text{Arc}^\circ(0 \in D)_m : m \in \mathbb{N} \}$. Note that $\text{Arc}^\circ(0 \in \mathbb{D})_0 = \text{ShArc}(0 \in D)_0$.

For $m > 0$ we can use Blaschke products to write any $\phi \in \text{Arc}^\circ(0 \in \mathbb{D})_m$ uniquely in the form
\[ \phi(t) = g(t) \cdot \prod_{i=1}^m \frac{t - a_i}{1 - a_i t} \]
where $g \in \text{Arc}^\circ(0 \in D)_0$ and $a_i$ are the zeros of $\phi$. This gives a surjective real analytic map
\[ \mathbb{D}^m \times \{ h \in \mathcal{H}(\overline{D}) : \Re(h) < 0 \} \to \text{Arc}^\circ(0 \in \mathbb{D})_m \]
which is equivariant with respect to the $2\pi i \mathbb{Z}$ action on the second factor and the symmetric group action permuting the coordinates on the first factor. This shows that $\text{Arc}^\circ(0 \in \mathbb{D})_m$ is connected, but it does not give enough complex analytic subspaces.

In order to get the complex structure right, we write
\[ \phi(t) = \psi(t) \cdot \prod_{i=1}^m (t - a_i) \]
where $a_i$ are the zeros of $\phi$. The condition on $\psi(t)$ is more complicated than before; we need that
\[ 0 < |\psi(t)| < \left| \prod_{i=1}^m \frac{1}{1 - a_i} \right| \quad \text{for every} \quad t \in \mathbb{D}. \]
As before, by passing to $\log \psi$ this becomes a convex condition
\[ \Re(\log \psi(t)) < \log \left| \prod_{i=1}^m \frac{1}{1 - a_i} \right| \quad \text{for every} \quad t \in \mathbb{D}. \]
Thus we get a holomorphic parametrization of $\text{Arc}^\circ(0 \in \mathbb{D})_m$ by an open subset
\[ U_m \subset \mathbb{D}^m \times \mathcal{H}(\overline{D}) \]
whose intersection with each $\{(a_1, \ldots, a_m) \times \mathcal{H}(\overline{D})\}$ is convex. Note that $U_m$ does not contain any constant section $\mathbb{D}^m \times \{c\}$, but if $\epsilon > 0$ and $\Re(c)$ is very negative then $\mathbb{D}(1 - \epsilon)^m \times \{c\} \subset U_m$. This is enough to show that $U_m$ is strongly irreducible by [14].

In Section 16 we will also need equivariant versions of the above arc spaces.

16 (Equivariant arc spaces of the disk). Let $G \subset \text{Aut}(0 \in \overline{D})$ be a finite subgroup and $\rho : G \to \text{Aut}(Z \subset X)$ a representation. An arc $\phi$ is called $\rho$-equivariant if
\[ \phi(g(t)) = \rho(g)(\phi(t)). \]
The set of $\rho$-equivariant short arcs is denoted by $\text{ShArc}(Z \subset X)^\rho$. 
Note that $\text{Aut}(0 \in \mathbb{D})$ is just the group of rotations, thus every finite subgroup is cyclic and generated by $\epsilon = e^{2\pi i / m}$ for some $m \in \mathbb{N}$.

For arc spaces of a disk there are even fewer possibilities. Fix natural numbers $m > 0$, $a \geq 0$ and a primitive $m$th root of unity $\epsilon$.

Let $\mathbb{D}(t)$ be a closed disk with coordinate $t$ and $\mathbb{Z}/m$-action $t \mapsto \epsilon t$ and $\mathbb{D}(z)$ be an open disk with coordinate $z$ and $\mathbb{Z}/m$-action $z \mapsto \epsilon^a z$. The representation $\rho = \rho(a)$ is determined by $a$ modulo $m$.

We study the spaces of $\rho(a)$-equivariant arcs. These can be thought of as functions $z = \phi(t)$ such that

$$\phi(\epsilon t) = \epsilon^a \phi(t).$$  \hfill (15.1)

Every such $\phi$ can be uniquely written as $\phi(t) = t^a \psi(t^m)$ for some $\psi$ in the unit ball of $\mathcal{H}(\mathbb{D}(s))$ where $s = t^m$. Thus the space of $\rho(a)$-equivariant arcs is connected and strongly irreducible.

The space $\text{ShArc}(0 \in \mathbb{D}(z))$, contains a $\rho(a)$-equivariant arc only if $r \equiv a \mod m$ and then these can be written as $\phi(t) = t^a \psi(t^m)$ for some invertible $\psi$ in the unit ball of $\mathcal{H}(\mathbb{D}(s))$ where $s = t^m$. Thus multiplication by $z^r$ identifies $\text{ShArc}(0 \in \mathbb{D}) = \text{Arc}(\mathbb{D} \setminus \{0\})$ (where we think of $\mathbb{D}$ as the unit disk with coordinate $s = t^m$) with the $\rho(a)$-equivariant arcs in $\text{ShArc}(0 \in \mathbb{D})$.

4. Arc spaces of the polydisk

It is clear from the definition that

$$\text{Arc}(\prod_{i \in I} X_i) = \prod_{i \in I} \text{Arc}(X_i).$$

In particular, $\text{Arc}(\mathbb{D}^n) \cong \text{Arc}(\mathbb{D})^n$ is a convex open set in the Banach space $\mathcal{H}(\mathbb{D})^n$.

By contrast, the spaces of short arcs have a more complicated behavior. To start with, there are several ways to define the product of pairs $(Z_i \subset X_i)$.

**Definition 17** (Products of pairs). For simple normal crossing pairs the natural notion is the product (or maximal product) of pairs $(Z_i \subset X_i)$, defined as

$$\prod_{i \in I} (Z_i \subset X_i) := \left( \bigcup_{i \in I} Z_i \times \prod_{j \neq i} X_j \subset \prod_{i \in I} X_i \right).$$  \hfill (17.1)

The minimal product is

$$\left( \prod_{i \in I} Z_i \subset \prod_{i \in I} X_i \right).$$  \hfill (17.2)

18 (Short arcs on maximal products). In order to write the space of short arcs on a product it is convenient to set

$$\text{ShArc}^+(Z \subset X) := \text{ShArc}(Z \subset X) \prod_i \text{Arc}(X \setminus Z).$$  \hfill (18.1)

Then

$$\text{ShArc}^+ \left( \prod_{i \in I} (Z_i \subset X_i) \right) = \prod_{i \in I} \text{ShArc}^+(Z_i \subset X_i).$$  \hfill (18.2)

In particular,

$$\text{ShArc}^+ \left( (0 \in \mathbb{D})^r \times \mathbb{D}^s \right) = \left( \text{ShArc}^+(0 \in \mathbb{D}) \right)^r \times \left( \text{Arc}(\mathbb{D}) \right)^s.$$  \hfill (18.3)

The latter has an obvious equivariant version. If $\mathbb{Z}/m$ acts on $\mathbb{D}$ by $t \mapsto \epsilon t$ and on the target by

$$\rho : (x_1, \ldots, x_r, y_1, \ldots, y_s) \mapsto (\epsilon^{a_1} x_1, \ldots, \epsilon^{a_r} x_r, \epsilon^{b_1} y_1, \ldots, \epsilon^{b_s} y_s)$$

then

$$\text{ShArc}^+ \left( (0 \in \mathbb{D})^r \times \mathbb{D}^s \right)^\rho = \prod_{i=1}^r \text{ShArc}^+ (0 \in \mathbb{D})^{\rho(a_i)} \times \prod_{i=1}^s \text{Arc}(\mathbb{D})^{\rho(b_i)}.$$  \hfill (18.4)
It is harder to study short arcs on a minimal product since the product of two non-short arcs is frequently short. Especially for singular spaces, the precise answer seems complicated.

In our applications the ambient space is the polydisc but the subvariety sits between the minimal and the maximal product.

**Definition 19.** Let \( \text{Arc}(0 \in \mathbb{D}) \subset \text{Arc}(\mathbb{D}) \) denote the set of those arcs \( \phi \) for which \( \phi(0) = 0 \). Let \( Z_2 \subset \mathbb{D}^n \) be the union of the codimension 2 coordinate hyperplanes \( \bigcup_{i \neq j} (x_i = x_j = 0) \). Set
\[
\text{ShArc}^*(0 \in \mathbb{D}^n) := \{ \phi \in \text{Arc}(0 \in \mathbb{D})^n : \text{Supp} \phi^{-1}(Z_2) = \{0\} \}.
\]
It is clear that \( \text{ShArc}^*(0 \in \mathbb{D}^n) \subset \text{ShArc}(0 \in \mathbb{D}^n) \subset \text{Arc}(0 \in \mathbb{D})^n \).

More generally, given representations \( \rho_1 : \mathbb{Z}/m \to \mathbb{C}^* \) and their product \( \rho : \mathbb{Z}/m \to (\mathbb{C}^*)^n \) set
\[
\text{ShArc}^*(0 \in \mathbb{D}^n)^\rho := \{ \phi \in \text{ShArc}(0 \in \mathbb{D}^n)^\rho : \text{Supp} \phi^{-1}(Z_2) = \{0\} \}.
\]

**Proposition 20.** For \( n \geq 2 \) the arc space \( \text{ShArc}^*(0 \in \mathbb{D}^n) \) is connected, strongly irreducible and dense in \( \text{Arc}(0 \in \mathbb{D})^n \). Thus \( \text{ShArc}(0 \in \mathbb{D}^n) \) is also connected.

The same holds for \( \text{ShArc}^*(0 \in \mathbb{D}^n)^\rho \) and \( \text{ShArc}(0 \in \mathbb{D}^n)^\rho \).

Proof. For \( i \neq j \), let \( W_{ij} \subset \text{Arc}(0 \in \mathbb{D})^n \) be the set of those functions \( (f_1, \ldots, f_n) \) such that \( f_i \) and \( f_j \) have a common zero in \( \mathbb{D} \setminus \{0\} \). These exactly correspond to arc that intersect the codimension 2 coordinate hyperplane \( (x_i = x_j = 0) \) outside the origin. Thus
\[
\text{ShArc}^*(0 \in \mathbb{D}^n) = \text{Arc}(0 \in \mathbb{D})^n \setminus \bigcup_{i \neq j} W_{ij}.
\]
The structure of the subsets \( W_{ij} \) is somewhat complicated (they are only real subanalytic), but we show in 21 that they are locally contained in a complex hypersurface.

By Lemma 23, this implies that \( \text{ShArc}^*(0 \in \mathbb{D}^n) \) is connected, strongly irreducible and dense in \( \text{ShArc}(0 \in \mathbb{D}^n) \). Thus any subspace in between these two is also connected, hence \( \text{ShArc}(0 \in \mathbb{D}^n) \) is connected.

The argument is the same for the equivariant versions. \( \square \)

More generally, we study the locus where a collection of holomorphic functions has an unexpected common zero.

21 (Common zeros of holomorphic functions). Let \( V_1, \ldots, V_k \) be irreducible complex spaces parametrizing holomorphic functions \( F_i : V_i \times \mathbb{D} \to \mathbb{D} \). For \( f_1, \ldots, f_k \in \mathcal{H}(\mathbb{D}) \) let \( Z(f_1, \ldots, f_k) \subset \mathbb{D} \) denote their common zero set (with multiplicity). The common zero set of the families \( F_1, \ldots, F_k \) is then
\[
Z(F_1, \ldots, F_k) = \bigcap \{ Z(f_1, \ldots, f_k) : (f_1, \ldots, f_k) \in V_1 \times \cdots \times V_k \}
\]
where we again keep track of the multiplicities. Let
\[
W^{\text{exc}} = W^{\text{exc}}(F_1, \ldots, F_k) \subset V_1 \times \cdots \times V_k
\]
denote the set of those \( (f_1, \ldots, f_k) \) that have extra zeros, that is for which
\[
Z(f_1, \ldots, f_k) \supset Z(V_1, \ldots, V_k).
\]
\(W^{\text{exc}}\) is a subset of \(V_1 \times \cdots \times V_k\), which can be understood as follows.

Pick any function \(q \in \mathcal{H}(\mathbb{D})\) whose zero set equals \(Z(V_1, \ldots, V_k)\). Replacing each \(F_i\) by \(F_i / g\), we get new families of holomorphic functions on \(\overline{\mathbb{D}}\) such that
\[
Z(F_1/g, \ldots, F_k/g) = \emptyset.
\]
Furthermore,
\[
W^{\text{exc}}(F_1, \ldots, F_k) = W^{\text{exc}}(F_1/g, \ldots, F_k/g).
\]
Thus, it is sufficient to understand \(W^{\text{exc}}\) in the special case when \(Z(F_1, \ldots, F_k) = \emptyset\). We assume the latter from now on.

Let \(H_i \subset V_1 \times \cdots \times V_k \times \overline{\mathbb{D}}\) be the pull-back of the zero set of \(F_i\) by the \(i\)th coordinate projection. Then \(W^{\text{exc}}\) is the image of \(H_1 \cap \cdots \cap H_k\) under the projection
\[
\Pi : V_1 \times \cdots \times V_k \times \overline{\mathbb{D}} \to V_1 \times \cdots \times V_k.
\]
\(\Pi\) is proper since \(\overline{\mathbb{D}}\) is compact, so \(W^{\text{exc}}\) is closed in \(V_1 \times \cdots \times V_k\).

To understand its local structure, pick a point \((f_1, \ldots, f_k) \in V_1 \times \cdots \times V_k\) such that not all the \(f_i\) are identically 0. We can extend the functions \(F_i\) to a larger open disk \(\mathbb{D}(1 + \epsilon)\) where some of the \(f_i\) has only finitely many zeros. Then \(\Pi\) restricts to a finite morphism \(\Pi(\epsilon)\) on
\[
H_1(\epsilon) \cap \cdots \cap H_k(\epsilon) \subset V_1 \times \cdots \times V_k \times \overline{\mathbb{D}}(1 + \epsilon).
\]
The image of \(\Pi(\epsilon)\) describes those \((f_1, \ldots, f_k)\) that have a common zero in \(\mathbb{D}(1 + \epsilon)\). Thus \(W^{\text{exc}}\) is a subset of this.

Of course this is useful information only if \(\Pi(\epsilon)\) is not dominant near \((f_1, \ldots, f_k)\), that is when \(H_1(\epsilon) \cap \cdots \cap H_k(\epsilon)\) has codimension \(\geq 2\) near \(\{(f_1, \ldots, f_k)\} \times \overline{\mathbb{D}}(1 + \epsilon)\).

We claim that this holds if at least 2 of the \(f_i\) are not identically 0.

To see this, pick a point \(z_0 \in Z(f_1, \ldots, f_k)\). Since \(Z(F_1, \ldots, F_k) = \emptyset\), there is at least one index, say \(i = 1\), and a 1-parameter family \(f_i(s) \subset V_1\) such that \(f_1(s)(z_0) \neq 0\) for \(s \neq 0\). Since at least 2 of the \(f_i\) are not identically 0, up to re-indexing we may assume that \(f_2\) is not identically 0. Near \(z_0\) the only zero of \(f_2\) is \(z_0\) which is not a zero of \(f_1(s)\) for \(s \neq 0\). Thus the restriction of \(\Pi\) to \(H_1 \cap H_2\) is not dominant near \(\{(f_1, \ldots, f_k)\} \times \{z_0\}\), hence \(H_1 \cap H_2\) has codimension \(\geq 2\) near \(\{(f_1, \ldots, f_k)\} \times \{z_0\}\).

**Definition 22.** Let \(V\) be a complex space and \(W \subset V\) a closed subset. We say that \(W\) has **complex codimension** \(\geq k\) if each point \(w \in W\) has an open neighborhood \(w \in U_w \subset V\) and a closed, complex subspace \(Z_w \subset U_w\) of codimension \(\geq k\) such that \(W \cap U_w \subset Z_w\). We are mostly interested in the case \(k = 1\).

A finite union of closed subsets of complex codimension \(\geq 1\) also has complex codimension \(\geq 1\).

**Lemma 23.** Let \(V\) be a normal, irreducible complex space and \(W \subset V\) a closed subset of complex codimension \(\geq 1\). Then \(V \setminus W\) is also an irreducible complex space.

Proof. Let \(Y \subset V \setminus W\) be a irreducible component. Let \(w \in W\) be any point. Then \(Y \cap (U_w \setminus Z_w)\) is an irreducible component of \(U_w \setminus Z_w\). Since \(U_w\) is normal, the latter is irreducible. Thus if \(Y \cap (U_w \setminus Z_w) \neq \emptyset\) then the closure \(\overline{Y} \subset V\) contain \(U_w\). Thus \(\overline{Y}\) is a closed analytic subset of \(V\), hence an irreducible component. Thus \(\overline{Y} = V\) and so \(Y = V \setminus W\). \(\square\)
By applying the above arguments to suitable projections we obtain the following result, which implies that we can move arcs away from codimension 2 subsets of the smooth locus without changing the connected components of the arc spaces.

**Proposition 24.** Let $X$ be a complex space and $Z \subset Z_2 \subset X$ closed subsets. Assume that $X$ is smooth along $Z_2 \setminus Z$ and $\text{codim}_X(Z_2 \setminus Z) \geq 2$. Then

\[ \{ \phi \in \text{ShArc}(Z \subset X) : \phi^{-1}(Z_2 \setminus Z) \neq \emptyset \} \]

has complex codimension $\geq 1$ in $\text{ShArc}(Z \subset X)$.

\[ \square \]

5. **Simple normal crossing pairs**

25 (Intersection number of arcs and divisors). Let $Y$ be a complex manifold and $D \subset Y$ a divisor. For some $Z \subset D$, let $\phi \in \text{Arc}^\circ(Z \subset X)$ be an arc. We claim that the intersection number $(\phi \cdot D)$ is defined and that it is a locally constant function on $\text{Arc}^\circ(Z \subset X)$.

The intersection number will be a sum of terms for each point $p \in \text{Supp} \phi^{-1}(D)$. To define the local contribution at $p$, we restrict $\phi$ to a smaller disk $\mathbb{D}(p, \epsilon)$ around $p$. Let $\phi(p) \in U_p \subset X$ be an open neighborhood such that $D \cap U_p = (F_p) = 0$ for some holomorphic function $F_p$ on $U_p$. By choosing $\epsilon$ small enough, we may assume that $\phi(\mathbb{B}(p, \epsilon)) \subset U_p$. Then $F_p \circ \phi$ is a holomorphic function on $\mathbb{D}(p, \epsilon)$ and the local contribution is the multiplicity of its zero at $p$.

If we perturb $\phi$, these numbers stay constant as long as the image of $\partial \mathbb{D}(p, \epsilon)$ stays disjoint from $D$. Thus $\phi \mapsto (\phi \cdot D)$ is a locally constant function on $\text{Arc}^\circ(Z \subset X)$.

Note also that if $\phi \in \text{ShArc}(Z \subset Y)$, then $(\phi \cdot D) > 0$ iff $\phi(0) \in D$.

Let now $D := \sum_{i \in I} D_i \subset Y$ be a reducible divisor. Fix natural numbers $\{m_i : i \in I\}$ and set

\[ A(m_i : i \in I) := \{ \phi : (\phi \cdot D_i) = m_i : i \in I \} \subset \text{Arc}^\circ(D \subset Y). \]

Since each $(\phi \cdot D_i)$ is locally constant on $\text{Arc}^\circ(D \subset Y)$, every $A(m_i : i \in I)$ is a union of some connected components of $\text{Arc}^\circ(D \subset Y)$ and $\text{Arc}^\circ(D \subset Y)$ is the disjoint union of all the $A(m_i : i \in I)$ for all $(m_1, \ldots) \in \mathbb{N}^{\vert I \vert}$.

Furthermore, if $\phi \in \text{ShArc}(D \subset Y)$, then

\[ \phi \in A(m_i : i \in I) \quad \text{iff} \quad \phi(0) \in \bigcap_{i : m_i > 0} D_i. \]

Let $W \subset \bigcap_{i : m_i > 0} D_i$ be a connected component and set

\[ \text{SA}(W, m_i : i \in I) := \{ \phi : \phi(0) \in W \text{ and } (\phi \cdot D_i) = m_i : i \in I \} \subset \text{ShArc}(D \subset Y). \]

By the above considerations, $\text{SA}(W, m_i : i \in I)$ is a union of connected components of $\text{ShArc}(D \subset Y)$.

Usually the $\text{SA}(W, m_i : i \in I)$ are disconnected, but they are connected and irreducible in the following important special case which gives our basic supply of connected arc spaces.

**Proposition 26.** Let $Y$ be a complex manifold and $D := \sum_{i \in I} D_i \subset Y$ a simple normal crossing divisor. Then

1. the above $\text{SA}(W, m_i : i \in I)$ are connected and
2. they give all the connected components of $\text{ShArc}(D \subset Y)$. 

Proof. Sending an arc to its center gives a continuous map $SA(W, m_i : i \in I) \to W$. Since $W$ is connected, it remains to show that each point $p \in W$ has an open neighborhood whose preimage in $SA(W, m_i : i \in I)$ is connected. As we noted in [15], this is a local question near $W$, hence it is enough to prove the lemma when $Y = \mathbb{D}^n$, $D_i = (x_i = 0)$ and $W = \cap_{i,m_i>0}(x_i = 0)$. This follows from the product formula (18.4).

Remark 27. If $X$ is an algebraic variety then each $SA(W, m_i : i \in I)$ contains a strongly irreducible dense open subset. This can be seen as follows. After replacing $X$ by a dense, open subset we may assume that there is a finite morphism $\pi : X \to \mathbb{C}^n$ such that $D_i = \pi^{-1}(x_i = 0)$. It is easy to see that our arc spaces are irreducible if $X = \mathbb{C}^n$ and the $D_i$ are hyperplanes. The preimages of the resulting finite dimensional complex subspaces give the necessary finite dimensional complex subspaces of $SA(W, m_i : i \in I)$.

6. Quotient singularities

Let $G \subset \text{GL}(n, \mathbb{C})$ be a finite subgroup. Then $G$ acts on $\mathbb{C}^n$ and $X = \mathbb{C}^n/G$ is called a quotient singularity. The quotient $X$ does not determine $G$ uniquely but there is a smallest choice given by $G = \pi_1(X \setminus \text{Sing} X)$. This $G$ is characterized by the property that it contains no pseudo-reflections, that is, elements whose fixed point set is a hyperplane.

We also need to work with quotients of simple normal crossing pairs, thus we consider groups $G$ that also stabilize some of the coordinate hyperplanes. This forces part of $G$ to be diagonal.

28 (Quotient singularities). Write

$$\mathbb{C}^n = \mathbb{C}^*_{x_1,...,x_r} \times \mathbb{C}^*_y, ...., y_s.$$ 

Set $D_i = (x_i = 0) \subset \mathbb{C}^n$ and $D = D_1 + \cdots + D_r$. Let $G \subset (\mathbb{C}^*)^r \times \text{GL}(s, \mathbb{C})$ be a finite subgroup without pseudo-reflections; then each of the $D_i$ is $G$-invariant. Let $F \subset \mathbb{C}^n$ be the union of the fixed point sets of the non-identity elements. Thus $F$ is a union of linear subspaces of codimension $\geq 2$ and $G$ acts freely on $\mathbb{C}^n \setminus F$.

$(0 \in X) := (0 \in \mathbb{C}^n)/G$ is a quotient singularity; let $\pi : \mathbb{C}^n \to X$ be the quotient map. Note that $\text{Sing} X = \pi(F)$. Set $E_i := \pi(D_i)$ and $E = E_1 + \cdots + E_r$. For uniformity of notation, we set $E := \{0\}$ if $r = 0$.

We have an exact sequence

$$0 \to \mathbb{Z}^r \to \pi_1(X \setminus (E \cup \text{Sing} X)) \to G \to 1$$

and the $G$-action on $\mathbb{Z}^r$ is trivial. Thus, for every $g \in G$, the preimage $\tau^{-1}(g)$ is an Abelian group, in fact isomorphic to $\mathbb{Z}^r$. This shows that if $\gamma_1, \gamma_2 \in \pi_1(X \setminus (E \cup \text{Sing} X))$ are conjugate then $\tau(\gamma_1), \tau(\gamma_2) \in G$ are conjugate and if $\tau(\gamma_1) = \tau(\gamma_2)$ then $\gamma_1, \gamma_2$ are conjugate iff $\gamma_1 = \gamma_2$.

With the above notation, let

$$\text{ShArc}^*(E \subset X) \subset \text{ShArc}(E \subset X)$$

denote the set of those arcs $\phi : \mathbb{D} \to X$ such that $\text{Supp} \phi^{-1}(\text{Sing} X) = \{0\}$. If $(0 \in X)$ is an isolated singularity, which is the main case that we are interested in, then $\text{ShArc}^*(E \subset X) = \text{ShArc}(E \subset X)$.
Theorem 29. Let \((0 \in (X,E)) = (0 \in (\mathbb{C}^n,D))/G\) be a quotient singularity as above. Then the irreducible components of \(\text{ShArc}^*(E \subset X)\) are in one-to-one correspondence with the conjugacy classes of \(\pi_1(X \setminus (E \cup \text{Sing} X))\).

It is interesting to connect the above result with the McKay correspondence; see [Rei02] for a survey and for further references.

30 (Short arcs and the McKay correspondence). Let \((0 \in \mathbb{C}^n/G)\) be an isolated quotient singularity and \(p : Y \to (0 \in \mathbb{C}^n/G)\) a resolution of singularities. For any short arc \(\phi : \overline{D} \to (0 \in \mathbb{C}^n/G)\), let \(p^{-1} \circ \phi : \overline{D} \to Y\) be its lift. As \(\phi\) varies in a dense open subset of an irreducible component of \(\text{ShArc}(0 \in \mathbb{C}^n/G)\), the centers of the lifts \(p^{-1} \circ \phi\) sweep out a dense subset of an irreducible subvariety of \(Y\). By Theorem 29, the irreducible components of \(\text{ShArc}(0 \subset \mathbb{C}^n/G)\) correspond to the conjugacy classes of \(G\), hence we get a natural map

\[
\{\text{conjugacy classes of } G\} \to \{\text{subvarieties of } Y\}.
\]

(The trivial conjugacy class corresponds to arcs that can be deformed away from the origin. The centers of such deformations sweep out a dense subset of \(Y\). Thus it is natural to say that \(Y\) itself corresponds to the trivial conjugacy class.)

Assume next that \(G \subset \text{SL}(2, \mathbb{C})\) and let \(p : Y \to (0 \in \mathbb{C}^2/G)\) be the minimal resolution. Since the Nash conjecture holds for surfaces, each irreducible exceptional curve of \(p\) corresponds to an irreducible component of the space of formal arcs \(\text{Arc}(0 \in \mathbb{C}^2/G)\) and each irreducible component of the space of formal arcs also gives an irreducible component of the space of short arcs.

Thus short arcs give a concrete realization of the McKay correspondence between nontrivial conjugacy classes of \(G\) and exceptional curves of the minimal resolution.

The McKay correspondence is not fully established in higher dimensions. Our results suggest a connection between the McKay correspondence and the Nash conjecture for higher dimensional quotient singularities. This topic should be explored further.

Our proof of Theorem 29 is not very illuminating. We write down a complete list of the irreducible components of \(\text{ShArc}^*(E \subset X)\) and then observe the one-to-one correspondence with the conjugacy classes.

31 (Arcs on quotient singularities). Let \(\phi : \overline{D} \to X\) be an arc in \(\text{ShArc}^*(E \subset X)\). Set \(D(\phi) = \overline{D} \times_X \mathbb{C}^n\) with normalization \(\overline{D}(\phi)\). Then \(\overline{D}(\phi)\) is a disjoint union of several disks \(\coprod_i \overline{D}(\phi)_i\) and \(\phi\) lifts to \(\phi_i : \overline{D}(\phi)_i \to \mathbb{C}^n\).

The restriction of \(\pi\) gives \(\pi_i : \overline{D}(\phi)_i \to \overline{D}\); these are Galois covers with cyclic Galois group \(C_i \subset G\). The different \(C_i\) are conjugates of each other. Set \(m := |C_i|\).

Fixing a coordinate \(s_i\) on \(\overline{D}(\phi)_i\) such that \(s_i^m = \pi_i^* t\) determines a generator \(g_i \in C_i\).

Thinking of \(g_i\) as a representation \(\rho_i : \mathbb{Z}/m \to \text{GL}(r+s, \mathbb{C})\), \(\phi_i : \overline{D}(\phi)_i \to \mathbb{C}^n\) is \(\rho_i\)-equivariant.

Let \(\text{ShArc}^*(D \subset \mathbb{C}^n) \subset \text{ShArc}(D \subset \mathbb{C}^n)\) denote the set of those arcs \(\phi : \overline{D} \to \mathbb{C}^n\) such that \(\text{Supp} \phi^{-1}(E) = \{0\}\). Then \(\phi_i \in \text{ShArc}^*(D \subset \mathbb{C}^n)\).

The construction can be carried out in any connected family of arcs, except that we may get a monodromy action of the fundamental group of the base on the set of the disks \(\{\overline{D}(\phi)_i\}\). This can be eliminated after a finite covering.

Conversely, given \(g \in G\), let \(m\) be the order of \(g\) and \(\text{ShArc}^*(D \subset \mathbb{C}^n)^{\rho(g)} \subset \text{ShArc}^*(D \subset \mathbb{C}^n)\) the set of \(\rho(g)\)-equivariant arcs. Thus we have established the following description of \(\text{ShArc}^*(E \subset X)\).
(1) Pick $g \in G$. Set $m = \text{ord}(g)$, write $g$ as 

$$g = (e^{a_1}, \ldots, e^{a_r}, g_s) \quad \text{where} \quad g_s \in \text{GL}(s, \mathbb{C})$$

and let $\rho(g) : \mathbb{Z}/m \to \text{GL}(r+s, \mathbb{C})$ denote the corresponding representation. Note that $(a_1, \ldots, a_r)$ are unchanged by conjugation.

(2) Pick $m_1, \ldots, m_r \in \mathbb{N}$, not all zero, such that $m_i \equiv a_i \mod m$.

(3) By (20) the $m_i$ determine $SA(m_1, \ldots, m_r) \subset \text{ShArc}(D \subset \mathbb{C}^n)$ which is a connected and irreducible component.

(4) Let $SA(m_1, \ldots, m_r)^{(g)} \subset SA(m_1, \ldots, m_r)$ denote the subset of $\rho(g)$-equivariant arcs. By (18)3 and (16), $SA(m_1, \ldots, m_r)^{(g)}$ is connected and irreducible.

(5) We saw in (20) that $SA^*(m_1, \ldots, m_r)^{(g)} := SA(m_1, \ldots, m_r)^{(g)} \cap \text{ShArc}^*(D \subset \mathbb{C}^n)$ is also connected and irreducible.

(6) These arcs descend to arcs on $X$, giving $SA^*(m_1, \ldots, m_r)^{(g)} \to \text{ShArc}^*(E \subset X)$.

(7) The image is a connected component of $\text{ShArc}^*(E \subset X)$.

(8) Conjugate elements give the same connected component and every connected component of $\text{ShArc}^*(E \subset X)$ is obtained this way.

Next we make this more explicit in two cases that are of special interest.

Assume first that $(0 \in X) = (0 \in \mathbb{C}^n)/G$ is an isolated quotient singularity and there are no divisors. Thus $r = 0$. Given $g \in G$ of order $m$, $A^{(g)} \subset \text{ShArc}(0 \in \mathbb{C}^n)$ is the set of $\rho(g)$-equivariant arcs. This $A^{(g)}$ is connected and irreducible by (20).

Thus we obtain a proof of the following restatement of Theorem 8.

**Corollary 32.** Let $(0 \in X) = (0 \in \mathbb{C}^n)/G$ be an isolated quotient singularity where $G \subset \text{GL}(n, \mathbb{C})$ acts freely outside the origin. Then the irreducible components of $\text{ShArc}(0 \in X)$ are in one-to-one correspondence with the conjugacy classes of $G$. □

We also need to understand cyclic quotients of $((x = 0) \subset \mathbb{C}^2)$. Consider the pair

$$((x = 0) \subset \mathbb{C}^2)/\mathbb{Z}_m(q, 1).$$

(The notation represents the quotient of $((x = 0) \subset \mathbb{C}^2)$ by the group action $(x, y) \mapsto (e^c x, e^c y)$ where $c = e^{2\pi i/m}$.) Then $\pi_1(X \setminus E)$ is an extension of $\mathbb{Z}/m$ by $\mathbb{Z} \cong \pi_1(\mathbb{C}^2 \setminus D)$ where $D = (x = 0)$. The universal cover of $\mathbb{C}^2 \setminus D$ is

$$\rho : \mathbb{C}_uv^2 \to \mathbb{C}_xy^2 \setminus D \quad \text{where} \quad \rho(u, v) = (u, e^{2\pi iv}).$$

The group of deck transformations is generated by $(u, v) \mapsto (u, v + 1)$. We see that $\mathbb{C}_uv^2$ is also the universal cover of $X \setminus E$ and the group of deck transformations is generated by

$$(u, v) \mapsto (e^{2\pi iv/m}u, v + \frac{1}{m}).$$

Given $a \in \{0, \ldots, m - 1\}$ (which we think of as an element of $\mathbb{Z}/m$) let $m_1$ be a positive integer congruent to $a \mod m$ and $c$ the smallest nonnegative integer such that $a \equiv cq \mod m$. Then the arc

$$\phi(m_1) : \mathbb{C} \to \mathbb{C}^2 \quad \text{given by} \quad t \mapsto (t^{m_1}, t^c)$$

is a typical element of $SA(m_1)$ in (31.3). The intersection number of $\phi(m_1)$ with $D = (x = 0)$ is $m_1$. Thus $\phi(m_1)$ descends to an arc in $X$ whose intersection number with $E$ is $m_1/m$. Thus we obtain the following.
Corollary 33. Let \((E \subset X) := ((x = 0) \subset \mathbb{C}^2)/\frac{1}{m}(q,1)\). The intersection number establishes a bijection \(\pi_0(\text{ShArc}(E \subset X)) \leftrightarrow \frac{1}{m}\mathbb{Z}_{>0}\).

If \(a \in \frac{1}{m}\mathbb{Z}_{>0}\) is not an integer, then the corresponding arcs are all centered at the origin. If \(a \in \mathbb{Z}_{>0}\) then a corresponding general arc has center on \(E \setminus \{0\}\). □

7. Minimal dlt modifications

As outlined in Paragraph 11, the first step toward proving Theorem 8 is the construction and study of minimal dlt modifications of surfaces.

Definition 34 (Minimal dlt modification). Let \(Y\) be a normal surface and \(E \subset Y\) a reduced curve. The pair \((Y,E)\) is divisorial log terminal (abbreviated as dlt) if everywhere it has one of the following normal forms (analytically locally).

1. (Normal crossing points) \(((xy = 0) \subset \mathbb{C}^2)\text{ or }((x = 0) \subset \mathbb{C}^2)\text{ or }\emptyset \subset \mathbb{C}^2\).

2. (Cyclic quotients) \(((x = 0) \subset \mathbb{C}^2)/\frac{1}{m}(q,1)\text{ where } (q,m) = 1\).

3. (Quotients) \(\emptyset \subset \mathbb{C}^2)/G\text{ where } G \subset \text{GL}(2,\mathbb{C})\text{ is a finite subgroup acting freely outside the origin.} (\text{For a more conceptual definition, see } \text{[KM98, 2.37].})

Let \((0 \in X)\) be a normal surface singularity and \(f: Y \to X\) a proper birational morphism that is an isomorphism outside 0. Set \(E := \text{Supp} f^{-1}(0)\). We say that \(f: (E \subset Y) \to (0 \in X)\) is a dlt modification if

4. \((Y,E)\) is dlt and

5. \(((K_Y + E) \cdot E_i) \geq 0\text{ for every irreducible curve } E_i \subset E.\)

A dlt modification is called minimal if, in addition,

6. there are no exceptional curves \(E_i \subset E\) such that \(Y\) is smooth along \(E_i\), \(\mathbb{P}^1 \cong E_i\), \(E_i^2 = -1\), \(((K_Y + E) \cdot E_i) = 0\) and \(E_i\) intersects the rest of \(E\) in 2 points.

35 (Construction of minimal dlt modifications). Let \((0 \in X)\) be a normal surface singularity.

If \(X\) is a quotient singularity, then it is its own minimal dlt modification.

If \(X\) is not a quotient singularity, then its minimal dlt modification can be constructed as follows.

Let \(g: (E \subset Y) \to (0 \in X)\) be a log resolution, that is, \(Y\) is smooth and \(E := \text{Supp} g^{-1}(0)\) is a curve with nodes only. If \(\mathbb{P}^1 \cong E_i \subset E\) is a \(-1\)-curve that intersects the rest of \(E\) in at most 2 distinct points, then we can contract \(E_i\) to get another log resolution. After all such contractions we get the minimal log resolution \(g^m: (E^m \subset Y^m) \to (0 \in X)\).

Next we contract maximal tails of rational chains. A sequence of curves \(E_1,\ldots,E_m\) is called a rational chain if these are smooth rational curves and

\[(E_i \cdot E_j) = \begin{cases} 0 & \text{if } |i-j| > 1 \text{ and} \\ 1 & \text{if } |i-j| = 1. \end{cases}\]

The sequence is called a tail if it intersects the rest of \(E\) in a single point of \(E^m\). The maximal tails of rational chains are disjoint from each other. If \((0 \in X)\) is not
a quotient singularity then contracting each of the maximal tails of rational chains gives the minimal dlt modification
\[ g^\text{dlt} : (E^\text{dlt} \subset X^\text{dlt}) \to (0 \in X). \]

Note that if \((0 \in X)\) is a cyclic quotient singularity then the latter recipe gives the correct answer: \(X^\text{dlt} = X\). If \((0 \in X)\) is a non-cyclic quotient singularity the above recipe gives \(X' \to X\) with a unique exceptional curve \(E'\) and 3 singular points on \(X'\). This is not a dlt modification since \( (K_{X'} + E') \cdot E' < 0 \). The correct minimal dlt modification is again \(X^\text{dlt} = X\).

Following the method of (26), we get the following description of the arc space of dlt pairs. The only difference is that we have to understand arcs centered at the singular points; these were described in Corollary 33.

**36 (Short arcs on dlt pairs).** Let \((E \subset Y)\) be a dlt pair. Then the following is a list of the connected components of \(\text{ShArc}(E \subset Y)\).

1. Let \(E_i\) be an irreducible component of \(E\) and \(m_i \in \mathbb{Z}_{>0}\). We get \(\text{SA}(E_i, m_i)\), consisting of arcs whose intersection number with \(E_i\) is \(m_i\) (and with every other \(E_j\) is 0).
2. Let \(E'_p, E''_p\) be the two local branches of \(E\) at a singular point \(p\) and \(m'_p, m''_p \in \mathbb{Z}_{>0}\). We get \(\text{SA}(p, m'_p, m''_p)\), consisting of arcs whose center is at \(p\) and whose intersection numbers with the two local branches are \(m'_p, m''_p\).
3. Let \(q\) be a singular point of \(X\) of the form
   \[ \left((x = 0) \subset \mathbb{C}^2\right)/\mathbb{Z} \] and \(m_q \in \mathbb{Z}_{>0} \setminus \mathbb{Z}_{>0}\).

We get \(\text{SA}(q, m_q)\), consisting of arcs whose center is at \(q\) and whose intersection number with \(E\) is \(m_q\).

In the above cases it is easy to work out the homotopy types of the different connected components of \(\text{ShArc}(E \subset Y)\).

(1') Let \(E^0_i \subset E_i\) be the complement of the set of singular points of \(E\). Then \(\text{SA}(E_i, m_i)\) is homotopy equivalent to an \(S^1\)-bundle over \(E^0_i\). If \(E_i\) has genus \(g\) and contains \(r > 0\) singular points then \(\text{SA}(E_i, m_i)\) is homotopy equivalent to \(S^1 \times \mathbb{C}^{2g+r-1} \times S^1\). The exceptional case is when \(E = E_i\) consists of a single smooth curve, thus \(r = 0\). Then we get an \(S^1\)-bundle over a compact Riemann surface (without boundary) whose Chern class is \(m_i (-E^2_i)\).

(2’) Taking the angle component of the leading coefficients of the Taylor series of an arc shows that \(\text{SA}(p, m'_p, m''_p)\) is homotopy equivalent to \(S^1 \times S^1\).

(3’) Working with equivariant Taylor series on the universal cover shows that \(\text{SA}(q, m_q)\) is homotopy equivalent to \(S^1\).

These cases are topologically distinct, except that \(\text{SA}(E_i, m_i)\) is homotopy equivalent to \(S^1 \times S^1\) if \(g = 0\) and \(r = 2\).

Our main result says that (36 1–3) gives a complete list of the connected components of \(\text{ShArc}(0 \in X)\) for normal surface singularities.

**Theorem 37.** Let \((0 \in X)\) be a normal surface singularity with minimal dlt modification \(g^\text{dlt} : (E^\text{dlt} \subset X^\text{dlt}) \to (0 \in X)\). Then composing an arc with \(g^\text{dlt}\) gives a bijection
\[ g^\text{dlt}_* : \pi_0\left(\text{ShArc}(E^\text{dlt} \subset X^\text{dlt})\right) \leftrightarrow \pi_0\left(\text{ShArc}(0 \in X)\right). \]
Proof. If \((0 \in X)\) is a quotient singularity then \(X^{\text{dlt}} = X\) and there is nothing to prove.

Since \(f_*\) gives a continuous bijection
\[
f_* : \text{ShArc}(E^{\text{dlt}} \subset Y^{\text{dlt}}) \to \text{ShArc}(0 \in X),
\]
the induced map on \(\pi_0\) is surjective and it remains to prove that it is injective. Using the winding number maps (5.2–3), (37.1) sits in a diagram
\[
\pi_0 \left( \text{ShArc}(E^{\text{dlt}} \subset X^{\text{dlt}}) \right) \xrightarrow{w} \pi_1 \left( \text{link}(E^{\text{dlt}} \subset X^{\text{dlt}}) \right)/(\text{conj})
\]
\[
g_*^{\text{dlt}} \downarrow \quad \cong \quad \downarrow
\]
\[
\pi_0 \left( \text{ShArc}(0 \in X) \right) \xrightarrow{w} \pi_1 \left( \text{link}(0 \in X) \right)/(\text{conj}).
\]

Thus it is sufficient to prove that the winding number map
\[
w : \pi_0 \left( \text{ShArc}(E^{\text{dlt}} \subset X^{\text{dlt}}) \right) \to \pi_1 \left( \text{link}(E^{\text{dlt}} \subset X^{\text{dlt}}) \right)/(\text{conj})
\]
is an injection. The latter is proved in Section 8.

Remark 38. It is possible that the map
\[
g_*^{\text{dlt}} : \text{ShArc}(E^{\text{dlt}} \subset Y^{\text{dlt}}) \to \text{ShArc}(0 \in X)
\]
is a homeomorphism but we do not know how to prove this.

8. Fundamental groups of normal surface singularity links

For general introductions to the topics of 3–manifold topology that we use, see [Sei32, Hem76, ST80, Jac80, Sco83]. Connections with links are treated in detail in [Neu81].

In this section we denote by \(L\) the oriented link of a normal surface singularity \((0 \in X)\), and by \(\pi_1\) its fundamental group. Recall that \(L\) is connected.

Theorem 39. The following properties hold.

1. Singularity links are irreducible 3–manifolds (that is, every embedded \(S^2\) in \(L\) bounds a 3–ball).
2. \(\pi_1\) is finite iff \(X\) is a quotient singularity.
3. If \(\pi_1\) is infinite then \(L\) is the Eilenberg–Mac Lane space of \(\pi_1\), that is \(L = K(\pi_1, 1)\). Moreover, in this case, \(\pi_1\) is torsion free.

Proof. Part (1) follows e.g. from [Neu81, Thm.1]. If \(\pi_1\) is finite then the universal cover of \(L\) is \(S^3\), the link of \(C^2\) at the origin, and the action of the finite \(\pi_1\) can be linearized. For (3) see e.g. [AFW12, p.32]. The argument is the following. By the Sphere Theorem, if \(N\) is an orientable 3–manifold with \(\pi_2(N) \neq 0\), then \(N\) contains an embedded 2–sphere which is homotopically non–trivial. Hence, since \(L\) is irreducible, we must have \(\pi_2(L) = 0\). Since \(\pi_1\) in infinite, the 3–manifold \(\bar{L}\), the universal cover of \(L\), has \(H_3(\bar{L}) = 0\). Hence, by the Hurewicz theorem \(\pi_i(L) = \pi_i(\bar{L}) = H_i(\bar{L}) = 0\) for any \(i > 1\). Next, any finite dimensional Eilenberg–Mac Lane space \(L = K(\pi_1, 1)\) has a torsion free fundamental group.

Part (3) can also be proved using the JSJ (=Jaco–Shalen–Johannson) decomposition of \(L\) (see below), and using Bass–Serre theory (for the \(K(\pi_1, 1)\) property) and the Torsion Theorem for free products with amalgamations and HNN (=Higman–Neumann–Neumann) extensions (for the torsion free property). Similar arguments will be used in the next paragraphs for conjugacy properties.
The case of finite $\pi_1$ is completely classified \[\text{[Bri68, DV64]}, \text{they are the quotient singularities. The relevant statement regarding ShArc is given in Section 6.}\]

In the sequel we will assume that $\pi_1$ is an infinite group.

The fundamental group $\pi_1$ has a presentation in terms of the plumbing realization of the link via a dual resolution graph. This presentation usually is rather involved due to the cycles in the graph. Hence, for the first conceptual conjugacy statements, we will use the structural decomposition of $\pi_1$ as the fundamental group of a graph of groups induced by the JSJ decomposition. Then, for each independent component of the JSJ decomposition (whose graphs have no cycles anymore) we will consider the corresponding concrete plumbing presentation.

40 (Plumbing graphs). Let us fix the minimal log resolution of $(0 \in X)$ as in (35). Then the link appears as the boundary of a small tubular neighbourhood of the exceptional curve, or, as a plumbed 3–manifold associated with the dual resolution graph $\Gamma$ of the irreducible exceptional components. The set of vertices of this graph will be denoted by $V$, the set of edges by $E$. The nodes are those vertices $v$ which either have genus $g_v > 0$ or valency $\geq 3$.

The plumbing description of the link provides a presentation of $\pi_1$. For rational homology sphere links (when the exceptional curve has trivial fundamental group, and $\pi_1$ is generated by the loops around the irreducible exceptional components) this was determined by Mumford [Mum61], for the general case see [Cat06, Aus00]. We will use this concrete presentation only for the Seifert pieces, cf. (47).

Each edge of the plumbing provides naturally an embedded torus in $L$. The tori of the minimal JSJ decomposition are such tori: from each maximal chain with node ends (which can be the very same node in the case of a ‘loop’) we choose one edge, their collection serves as the JSJ–tori for the minimal decomposition. (For a fixed such chain, the choice of the edge will not alter the torus, but only a canonical basis in its homology $\mathbb{Z}^2$.) In particular, the pieces $\{L_j\}_j$ of the decomposition are indexed by nodes, and each piece has a Seifert structure whose central vertex is the corresponding node.

In fact, there is an exception to this description, when $\Gamma$ is a cyclic graph with all genus decorations zero (the case of cusps): one has no nodes, exactly one JSJ torus provided by one of the edges arbitrarily chosen, and one Seifert piece.

In the plumbing representation of the pieces $L_j$ we extend the above notation of the plumbing graphs: the boundary components of $L_j$ will be denoted by arrowhead vertices and their set is denoted by $A$. The set of the other, non–arrowhead vertices, is denoted by $W$. Hence $V = W \cup A$. If an arrow $a$ is supported by the non–arrowhead $v$, then the corresponding boundary component is the boundary of the tubular neighborhood of a generic $S^1$–fiver $S_a$ over the surface (irreducible exceptional curve) indexed by $v$ in the plumbing construction. The plumbing graph of the collection of $\{L_j\}_j$ is obtained from the graph of $L$ by replacing each chosen edge of the chains by two arrowheads (supported by the two endpoints of the edge).

In the remaining part of this section we assume that $\Gamma$ has at least one node. The case of cyclic graphs without nodes will be treated in Section 9.

41 (The non–Seifert case. The presentation of $\pi_1$ via graph of groups). Assume first that $L$ is not a Seifert 3–manifold and $\Gamma$ has at least one node. Then, being irreducible, and having at least one incompressible JSJ torus, it is Haken. The JSJ decomposition of $L$ provides a splitting of $\pi_1$ as a fundamental group of a graph of groups [Ser03, Pre06]. More precisely, the JSJ tori form a collection of
disjointly embedded incompressible tori \( \{T_j\}_T \) (where incompressibility means that \( \pi_1(T_j) \to \pi_1(L) \) is injective), such that the components \( \{L_j\}_T \) of \( L \setminus \cup_T T \) are Seifert fibred. Each \( T \) is two-sided in \( L \), let \( T^+ \) and \( T^- \) be the two boundaries of a small tubular neighbourhood of \( T \) in \( L \).

The underlying graph \( \mathfrak{G} \) of the graph of groups has one vertex \( v_j \) for each piece \( L_j \), we will also write \( L(v_j) = L_j \). To each \( T \) correspond two edges of \( \mathfrak{G} \), \( e_\tau := e(T) \) and \( \overline{e(T)} := e(T)^{-1} \), inverse to each other. If \( L_k \) and \( L_l \) are such that \( T^- \subset L_k \) and \( T^+ \subset L_l \), then we say that \( e(T) \) has origin \( v(L_k) \) and extremity \( v(L_l) \).

The graph of groups is obtained by assigning to each vertex \( v_j \) the vertex group \( G(v_j) = \pi_1(L(v_j)) \), and to each edge \( e_\tau \) the edge group \( G(e_\tau) = \pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z} \). For each edge \( e_\tau \), with origin \( v_0 \) and extremity \( v_1 \), one defines the monomorphisms \( \phi^-_\tau : G(e_\tau) \to G(v_0) \) and \( \phi^+_\tau : G(e_\tau) \to G(v_1) \). (Strictly speaking, this depends on the choice of base points \( *_0 \) in \( L_j, *_1 \in T^\pm \), and paths from \( *_- \) to \( *_0 \) and from \( *^+ \) to \( *_1 \) respectively.) Their images in the corresponding vertex groups will be denoted by \( G(e_\tau)^- \) and \( G(e_\tau)^+ \), and we set \( \phi_{e_\tau} = \phi^+_\tau \circ (\phi^-_\tau)^{-1} : G(e_\tau)^- \to G(e_\tau)^+ \). For the inverse edges \( G(e)^- = G(e), G(e)^\pm = G(e)^\mp, \phi_e = \phi_e^- \).

These data determine \( \pi_1 \) as the fundamental group of the graph of groups, once \( \mathfrak{G} \) is chosen. An edge of \( \mathfrak{G} \) is called \( \mathfrak{T} \)-separating if it belongs to \( \mathfrak{T} \), and \( \mathfrak{T} \)-non–separating otherwise. Then, if for each vertex \( v \), \( \langle S_c | R_v \rangle \) is the presentations of \( G(v) \), then the presentation of \( \pi_1 \) is the following (cf. [Ser03, 5.1] or [Pré06 page 179]):

Generators: \( \bigcup_{j=1}^n S_{v_j} \cup \{t_e \mid e \text{ edge of } \mathfrak{G}\} \).

Relations:

\[
\bigcup_{j=1}^n R_{v_j} \cup \{ \text{for all edge } e \text{ of } \mathfrak{G} \text{ and generator } c \text{ of } G(e)^- : t_e \phi_e(c) t_e^{-1} = c \}
\]

\[
\cup \{ \text{for all edge } e \text{ of } \mathfrak{G} : t_e = t_e^{-1} \}
\]

\[
\cup \{ \text{for all } \mathfrak{T} \text{-separating edge } e \text{ of } \mathfrak{G} : t_e = 1 \}.
\]

The presentation \( \langle S_c | R_v \rangle \) of each contributing group \( G(v) \) will be given in (47).

42 (Reduced and cyclically reduced forms). [Ser03] [Pré06] A path in the graph \( \mathfrak{G} \), of length \( m \geq 0 \), is a sequence \( (v_{j_0}, c_{k_0}, v_{j_1}, \ldots, c_{k_{m-1}}, v_{j_m}) \), where the origin and the extremity of \( e_c \) are \( v_{j_0} \) and \( v_{j_m+1} \) respectively. A path is called a loop if \( j_0 = j_m \).

A word of type \( C \) is a couple \( (C, \mu) \), where \( C \) is a loop in \( \mathfrak{G} \), and \( \mu \) is a sequence \( \mu = (\mu_0, \ldots, \mu_m) \), such that \( \mu_i \in G(v_{j_i}) \) for all \( i \).

Once a base point (vertex) \( v_* \) in \( \mathfrak{G} \) is fixed, we might consider loops with \( v_{j_0} = v_{j_0} = v_* \). Then any such word \( (C, \mu) \) gives an element of the fundamental group of the graph of groups, denoted by \( [C, \mu] \). In the presentation (41) it is the word \( \mu_{j_0} t_{k_0} \mu_{j_1} \cdots t_{k_{m-1}} \mu_{j_m} \). We say that \( [C, \mu] \) is the label of \( (C, \mu) \).

A word \( (C, \mu) \) is called reduced if either \( m = 0 \) and \( \mu_{j_0} \neq 1 \), or \( m > 0 \) and any time when \( c_{k_{r-1}} = \overline{e_{k_r}} \) one has \( \mu_i \in G(v_{j_i}) \setminus G(e_{k_r})^- \). Given any non-trivial element of the fundamental group, there exists a reduced form associated with it.

A cyclic conjugate of the word \( (C, \mu) \) in \( (C', \mu') \), where

\[
C' = (v_{j_1}, c_{k_1}, v_{j_1+1}, \ldots, c_{k_{m-1}}, v_{j_m}, c_{k_0}, \ldots, c_{k_{r-1}}, v_{j_1})
\]

and \( \mu' = (\mu_i, \mu_{i+1}, \ldots, \mu_{m-1}, \mu_m, \mu_0, \ldots, \mu_{r-1}, 1) \), for some \( 1 \leq i \leq m \).

A word is called cyclically reduced if all its cyclic conjugates are reduced, and if \( \mu_m = 1 \) whenever \( m > 0 \). Any non-trivial conjugacy class of the fundamental group can be represented by a cyclically reduced form whose label is an element of the class.
We say that a word \((\mathcal{C}, \mu)\) is 0-reduced, if \(\mu_i = 1\) for all indices \(i\), except one. Note that a 0-reduced word has a cyclically reduced form of length 0.

**43. The Conjugacy Theorem in the group of graph of groups** characterizes (cyclically reduced) words whose labels are conjugate in the group; see e.g. \([\text{Pr}06, \text{Theorem 3.1}]\). Since in our application we need the conjugacy properties of special 0-reduced words, we state the theorem only for them. Below \(\sim\) denotes conjugacy.

**Theorem 44.** \([\text{Pr}06]\) Suppose that \((\mathcal{C}, \mu) = (v_0, \mu_0)\) and \((\mathcal{C}', \mu') = (v_0', \mu_0')\) are two cyclically reduced forms, both of length zero, whose labels \(\mu_0\) and \(\mu_0'\) are conjugate in \(\pi_1\). Then

1. either \(v_0 = v_0'\), and the elements \(\mu_0\) and \(\mu_0'\) are conjugate in \(G(v_0)\),
2. or there exists a path \((v_{j_0}, e_{j_0}, v_{j_1}, \ldots, e_{j_m}, v_{j_m})\) of length \(m > 0\), with \(v_{j_0} = v_0\) and \(v_{j_m} = v_0'\), and a sequence of elements \((c_0, \ldots, c_{m-1})\) with \(c_i \in G(e_{j_i})\), such that:

\[
\mu_0 \sim \phi_{e_{j_0}}^+(c_0) \text{ in } G(v_{j_0}),
\]

\[
\phi_{e_{j_{i-1}}}^-(c_{i-1}) \sim \phi_{e_{j_i}}^+(c_i) \text{ in } G(v_{j_i}), \quad 1 \leq i \leq m - 1,
\]

\[
\phi_{e_{j_{m-1}}}^-(c_{m-1}) \sim \mu_0' \text{ in } G(v_{j_m}).
\]

**45 (Arc–generators and m-arc-generators).** Consider again the plumbing representation of \(L\), cf. \([10]\). For a vertex \(v \in V\) of the plumbing graph \(\Gamma\) let \(\gamma_v\) be the oriented \(S^1\)-fiber associated with \(v\) in the plumbing construction. Note that \(\gamma_v\) is well defined up only to conjugacy; nevertheless, if we fix a base point \(\ast\) of \(L\) and a connecting path from \(\ast\) to a point of this loop, we get an element of \(\pi_1\). If \(u, v\) are connected by an edge in \(\Gamma\), then we can consider for both of them the same connecting path, in which case \(\gamma_u\) and \(\gamma_v\) commute in \(\pi_1\). (Similarly one can be achieve that all the \(\gamma\)-generators along a chain commute with each other; see also the relations between them in the second paragraph of \([17]\).)

Similarly, assume that the JSJ decomposition is fixed (that is, the separating edges of the plumbing graph are fixed). Then \(\gamma_v\) for each \(v \in V(L_j)\) determines a well-defined conjugacy class of \(\pi_1(L_j)\). Moreover, for each arrowhead \(a \in A(L_j)\), inherited from the edge \(e_r\) such that \(L_j \supset T^*_r\), \(o = +\) or \(-\), we can consider \(\gamma_a\), the oriented loop around \(S_o\). If \(v(a)\) is the supporting vertex of \(a\) in the graph of \(L_j\), then by taking for the connecting path of \(\gamma_a\) and \(\gamma_{v(a)}\) the path connecting the basepoints \(\ast_j\) and \(\ast^+_r\) already considered in \([11]\), we obtain that \(\gamma_a\) and \(\gamma_{v(a)}\) generate \(G(e_r)^o \cong \mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(L_j)\).

In the sequel we abbreviate \(\pi_1(L_j)\) by \(\pi_{1,j}\) and \(G(e_r)^o\) by \(H_r \subset \pi_{1,j}\).

We consider the following conjugacy classes of \(\pi_1\): for each \(v \in V\) we take the class of \(\gamma_v^m\) for any fixed \(m_v > 0\), and for each pair \(v, u \in V, (u, v) \in E\), the class \(\gamma_u^{m_u} \gamma_v^{m_v}\) for \(m_u, m_v > 0\) (with common connecting paths for \(u, v\)). We call these classes arc-generators. Their definition is motivated by the map \(\text{ShArc} \to \pi_1/(\text{conj})\), see \([32]\) and \([46]\). It is convenient to extend this family and include also those powers which have negative exponents: we call these classes \(m\)-arc-generators (‘\(m\)’ from ‘meromorphic’). That is, they consists of classes of type \(\gamma_u^{m_u}\), with \(m_u \in \mathbb{Z}\) and \(\gamma_u^{m_u} \gamma_v^{m_v}\) with \(m_u, m_v \in \mathbb{Z}\) for \((u, v) \in E\). Thus we conclude the following.

**Lemma 46.** The conjugacy classes of \(m\)-arc-generators of \(\pi_1\) can be represented by 0-reduced words, hence their cyclically reduced forms have length zero. \(\Box\)
In particular, for conjugate arc-generators Theorem 44 can be applied. Moreover, we will show that for arc-generators necessarily case (1) happens, hence we have to understand conjugate arc-generators of some fixed \( \pi_{1,j} \) only. For this we need the precise presentation of \( \pi_{1,j} \).

47 (The presentation of \( \pi_{1,j} \)). The plumbing graph of \( L_j \) is star–shaped. Let the number of chains without arrows at the end be \( n \geq 0 \) and the number of chains with arrow at the end be \( k \geq 1 \).

Each chain without arrowhead at the end has a Seifert invariant \((\alpha_i, \omega_i), 0 < \omega_i < \alpha_i, \gcd(\omega_i, \alpha_i) = 1\). If \([b_1, \ldots, b_s]\) is the Hirzebruch continued fraction of \( \alpha_i/\omega_i \), then the chain has \( s \) vertices, all of them of genus \( 0 \), their Euler numbers are \(-b_1, \ldots, -b_s\) such that the vertex decorated with \(-b_1\) is connected with the node. If \( \gamma_1, \ldots, \gamma_s \) are the corresponding elements in the fundamental group, then one has the relations \( \gamma_s^{b_1} = \gamma_{s-1}, \ldots, \gamma_s^{b_i} = \gamma_{i-1} \gamma_{i+1} \) (see e.g. [Mum61]). In particular, all the generators \( \gamma_i \) \((1 < i \leq s)\) can be eliminated: they appear as certain powers of \( \gamma_s \). Indeed, let \( \det[b_k, \ldots, b_l] \) be the numerator of the corresponding continued fraction (or the determinant of the corresponding subgraph), then \( \gamma_i = \gamma_s \det[b_k, \ldots, b_l] \).

Similar elimination happens for a chain with arrow at the end, but now the generator of that vertex survives which is connected with the node (independently whether it is arrowhead or non–arrowhead), see also the proof of Lemma 49. (Note also that the Euler numbers along these arrowed chains become irrelevant in the topology of \( L_j \) — with non-framed boundary —, and in the presentation of \( \pi_{1,j} \) as well.)

We rename the non-eliminated generators: \( h \) is the generator of the node, \( \{g_i\}_{i=1}^n \) are the generators of the end vertices of chains without arrow, and \( \{f_r\}_{r=1}^k \) are the generators of vertices next to the node on the chains with arrows. Let \(-b \) and \( g \) be the Euler number and genus of the node. Then, if \( g \neq 0 \) we will have \( 2g \) more generators, \( \{a_m, b_m\}_{m=1}^g \) and \( \pi_{1,j} \) has the following presentation [Mum61] [Pré06].

\[
\langle h, \{g_i\}, \{f_r\}_r, \{a_m, b_m\}_m \mid h \text{ central}, g_i^{a_i} = h, \forall i, h^b = \prod_m [a_m, b_m] \prod_i g_i^{a_i} \prod_r f_r \rangle.
\]

Note that this group is independent of \( b \) since we can eliminate one of the generators \( f_{r_0} \) and the last equation. Then the generators \( \{f_r\}_{r \neq r_0} \) and \( \{a_m, b_m\}_m \) become free generators, and \( \pi_{1,j} \) is a free product \( G \ast F_{2g+k-1} \), where \( F_r \) is the free group with \( r \) letters, and

\[
G := \langle h, \{g_i\}_{i=1}^n \mid g_i^{a_i} = h, \forall i \rangle. \tag{47.1}
\]

Since the central element \( h \) has infinite order, cf. (39), it generates a normal subgroup \( \langle h \rangle \cong \mathbb{Z} \). Moreover,

\[
\overline{\pi_{1,j}} := \pi_{1,j}/\langle h \rangle \cong \mathbb{Z}_{a_1} \ast \cdots \ast \mathbb{Z}_{a_n} \ast F_{2g+k-1}. \tag{47.2}
\]

Notation 48. In the sequel for any \( x \in \pi_{1,j} \) we denote by \( \overline{x} \) its image in \( \overline{\pi_{1,j}} \), and \( \ell := \prod_m [a_m, b_m] \cdot \prod_i g_i^{a_i} \).

Lemma 49. Fix an arrow (or boundary torus) indexed by \( \tau \) \((1 \leq \tau \leq k)\). Then \( h \) and \( f_{\tau} \) generate \( H_{\tau} \cong \mathbb{Z} \ast \mathbb{Z} \subset \pi_{1,j} \). (Recall \( H_{\tau} = G(e_{\tau})^{\circ} \), cf. (43).)

In particular, the \( m \)-arc-generators of \( L_j \) in \( \pi_{1,j} \) are the following: either elements of \( H_{\tau} \) \((1 \leq \tau \leq k)\), or integral powers of \( g_i \) \((1 \leq i \leq n)\).

Proof. Let \(-b_1, \ldots, -b_s\) be the Euler numbers of the vertices of the corresponding chain, where the \( s \)-th vertex supports the arrow. Via similar identities as in (47),
any $\gamma_a$ along this chain equals $f_x^a h^b$, where the exponents are certain determinants; in particular, with the notation of (45):
\[
\gamma_a = f_x^{\det[b_1, \ldots, b_n]} h^{-\det[b_2, \ldots, b_n]}, \quad \gamma_v(a) = f_x^{\det[b_2, \ldots, b_n]} h^{-\det[b_2, \ldots, b_n-1]}.
\]
Since this transformation from $(f_x, h)$ to $(\gamma_a, \gamma_v(a))$ has unimodular matrix, and $H_\tau$ is generated by $\gamma_a$ and $\gamma_v(a)$, the lemma follows.

**Definition 50.** We say that two m-arc-generators of $L_j$ have the ‘same type’ if both belong to the same $H_\tau$, or both of them are powers of the same $g_1$ (cf. Lemma 49). An m-arc-generator of a certain type is called ‘pure’ if it is not equal to another m-arc-generator of different type.

**Theorem 51.** Consider the presentation of $\pi_{1,j}$ from (47) and the m-arc-generators of $L_j$, represented in $\pi_{1,j}$ as in Lemma 24. Then if two m-arc-generators are conjugate in $\pi_{1,j}$ then they are equal in $\pi_{1,j}$.

A non-pure m-arc-generator of $\pi_{1,j}$ is a power of $h$.

Proof. We analyze several cases according to the types of the m-arc-generators.

**Case 1.** Assume that $g_i^r \sim g_s^r (1 \leq i \leq n; \ r, s \in \mathbb{Z})$.

Then $g_i^r \sim g_s^r$ and $h^r \sim h^s$. Since $h$ is central and of infinite order, $r = s$.

**Case 2.** Assume that $A \sim A'$, where $A, A' \in H_\tau$.

Let us write $A = h^r f_x^a$ and $A' = h^s f_x^{a'}$. Since $h$ is central we can assume $s = 0$.

Set $y := \prod_{r \geq t} f_x \cdot \ell \cdot \prod_{s < r} f_x$. Then $h^b = \ell \prod_{s < r} f_x \cdot f_x \cdot \prod_{r > s} f_x = y f_x$, hence $A = y^{-c h^{b+r}}$ and $A' = y^{-d h^{b+d}}$. Projected into $\pi_{1,j}$, we get $\tilde{y}^r \sim \tilde{y}^d$. By the ‘Conjugacy theorem of free products’ (LS77 Theorem 1.4, Ch IV), $c = d$. Next, write $A'$ as $x^{-1} A x$. Then $h^r = [x, f_x^a]$, hence $h^r$ projected into $H_1(L_j) = \pi_{1,j}/[\pi_{1,j}, \pi_{1,j}]$ is trivial. But the image of $h$ in $H_1(L_j)$ has infinite order (use the abelianized $G$ from (47)1), hence $r = 0$ too.

**Case 3.** Assume that $g_i^r \sim g_j^r (1 \leq i, j \leq n, \ i \neq j; r, s \in \mathbb{Z})$, or $g_i^r \sim A (1 \leq i \leq n; r \in \mathbb{Z})$, where $A \in H_\tau (1 \leq \tau \leq k)$.

Consider the projection of the two elements in $\pi_{1,j}$: $g_i^r$ projects into the factor $Z_{n_{\tau}}$, while $H_\tau$ in $F_{2g+k-1}$. (Hence, either they belong to different factors, or, if $2g+k = 1$, then $A$ projects into a power of $\ell$ in $\pi_{1,j}$.) In any case, the projections can be conjugate in $\pi_{1,j}$ only if they are both trivial, cf. [LS77] Theorem 1.4, Ch IV. That means that the two original elements belong to the central subgroup ($h$). Since $h$ is central, if they are conjugate they should be equal.

**Case 4.** Assume that $A_{\tau} \sim A_{\sigma}$, where $A_{\tau} \in H_\tau$, $A_{\sigma} \in H_\sigma$, $\tau \neq \sigma$ ($k > 2$).

Let us write $A_{\tau} = h^r f_x^a$ and $A_{\sigma} = h^s f_x^{a'}$. Their projections $\tilde{f}_x^a$ and $\tilde{f}_x^{a'}$ belong to the factor $F_{2g+k-1}$ of $\tilde{\pi}_{1,j}$. Assume $2g+k > 2$. When we define the factor $F_{2g+k-1}$ we can eliminate a variable different from $f_x$ or $f_x$, hence we can write $F_{2g+k-1}$ as a free product such that $\tilde{f}_x^a$ and $\tilde{f}_x^{a'}$ belong to two different non-trivial factors. Therefore $\tilde{f}_x^a = \tilde{f}_x^{a'} = 1$. If $F_{2g+k-1} = \mathbb{Z}$, we can assume that its generator is $\tilde{f}_x$, and $\tilde{f}_x = (\ell f_x)^{-1}$. Hence $\tilde{f}_x^a \sim (\ell f_x)^{-a}$. Then, again by [LS77] Theorem 1.4, Ch IV, $c = d = 0$. Hence, in any case, $A_{\tau}$ and $A_{\sigma}$ belong to ($h$), and they are equal. □

**52.** Now, using (49) and Theorem 51 we are ready to apply Theorem 44 for m-arc and arc-generators. For $\pi_1$ we use the notations of (11–15).

**Theorem 55.** Suppose that $(C, \mu) = (v_0, \mu_0)$ and $(C', \mu') = (v'_0, \mu'_0)$ are two cyclically reduced forms of length zero, whose labels $\mu_0$ and $\mu'_0$ are conjugate in $\pi_1$. 

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**HOLOMORPHIC ARCS ON ANALYTIC SPACES**

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(1) If \( \mu_0 \) and \( \mu'_0 \) are m-arc-generators, then the following cases can happen:
(a) \((m=0)\) \( v_0 = v'_0 \) and \( \mu_0 = \mu'_0 \) \( \in G(v_0) \).
(b) \((m=1)\) \( v_0 \) and \( v'_0 \) are connected by the edge \( e_{\tau_0} \), \( \mu_0 = \phi_{e_{\tau_0}}(\mu_0) \), and at least one of the generators \( \mu_0 \in G(e_{\tau_0})^- \) and \( \mu'_0 \in G(e_{\tau_0})^+ \) is pure.
(c) \((m=2)\) there exists a path \((v_0, e_{\tau_0}, v_1, e_{\tau_1}, v'_0) \) as in \((\text{4.4})(2)\), with \( e_{\tau_1} \neq e_{\tau_0} \) and an element \( \mu_1 \in G(v_{j_1}) \) such that \( \mu_0 = \phi_{e_{\tau_0}}^{-1}(\mu_1) \), \( \mu'_0 = \phi_{e_{\tau_1}}(\mu_1) \), and both \( \mu_0 \in G(e_{\tau_0})^- \) and \( \mu'_0 \in G(e_{\tau_1})^+ \) are pure.

(2) If \( \mu_0 \) and \( \mu'_0 \) are arc-generators, then the cases \((m=1)\) and \((m=2)\) from \((1)\) can not happen. In the \((m=0)\) case the coincidence \( \mu_0 = \mu'_0 \) of the labels in \( \pi_1 \) associated with two different arc-generators can happen only if both are supported by the same chain without arrow (including the generators of type \( h'' \)): all of them are positive powers of the corresponding \( g_j \).

Proof. (1) We apply Theorem \( \text{4.4} \) By Lemma \( \text{4.9} \) all elements of type \( \phi_i^+ (c_i) \) are m-arc-generators in the corresponding groups \( G(v_{j_i}) \). Hence, by Theorem \( \text{4.1} \) all conjugacies in Theorem \( \text{4.4}(1)-(2) \) are equalities. Let us write \( \mu_i := \phi_i^+ (c_i-1) = \phi_i^- (c_i) \). Then \( \phi_{e_{\tau_i}}(\mu_i) = \mu_{i+1} \) for \( 0 \leq i \leq m-1 \) (where \( \mu_m := \mu'_0 \)). If \( e_{\tau_i} = e_{\tau_{i+1}} \) is a ‘backtracking’, then \( \mu_{i+1} = \mu_{i-1} \), and the segment of path \((v_{j_{i-1}}, e_{\tau_{i-1}}, v_{j_i}, e_{\tau_i}, v'_{j_{i+1}}) \) can be shortened to \( v_{j_{i-1}} \) (although, in such a case, \( \mu_{i+1} \) is special, it is in the image of \( \phi_{e_{\tau_i}} \), and this information disappears after we shorten the path).

Hence, we can assume that we have no backtracking in the path.

Assume that \( m \geq 3 \), and we consider the segment of path \((e_{\tau_0}, v_{j_{1}}, e_{\tau_{1}}, v_{j_2}, e_{\tau_2}) \). Then the very same element \( \mu_1 := \phi_{e_{\tau_1}}(c_0) = \phi_{e_{\tau_1}} (c_1) \) is represented in two different ways as m-arc-generator, of different types (since \( \tau_1 \neq \tau_0 \)). Hence, by Theorem \( \text{4.1} \) \( \mu_1 \) is a power \( h_{1i} \) of the central element \( h_1 \) of the Seifert piece \( G(v_{j_1}) \). The same fact is true in \( G(v_{j_2}) \), one has \( \mu_2 = h_{2i} \). But, we claim that an identity of type \((\dagger)\) \( \phi_{e_{\tau_1}} (h_{1i}) = h_{2i} \) cannot happen unless \( n_1 = n_2 = 0 \). This basically follows from the fact that \( G(e_{\tau_1}) = \mathbb{Z}^2 \) injects into \( \pi_1 \), and \((\dagger)\) would give a relation in this \( \mathbb{Z}^2 \). The proof imitates the proof of Lemma \( \text{4.9} \). Consider the chain connecting \( v_{j_{1}} \) and \( v_{j_2} \), and assume that the Euler numbers are \( -b_1, \ldots, -b_s \). Let \( \gamma_i \) be the arc-generator associated with the vertex adjacent to \( v_{j_i} \). Then \( h_2 = \phi_{e_{\tau_1}}(\gamma_i^0) = h_{2i} \). Moreover, \( h_1 \) and \( \gamma_1 \) generate \( \mathbb{Z}^2 \) in \( \pi_1 \). But \( \det [b_{i_1}, \ldots, b_{i_s}] \neq 0 \) (since the graph is negative definite), hence \((\dagger)\) can not happen.

All the other restrictions in part \((1)\) follow similarly.

(2) Let us start with case \((m=1)\), and we assume that \( \mu_0 \notin \langle h \rangle \). Connect \( v_0 \) with \( v'_0 \) by a chain with decorations \( -b_1, \ldots, -b_s \). Let \( h = \gamma_0 \cdot \gamma_1 \cdot \ldots \cdot \gamma_s \cdot h' = \gamma_{s+1} \) be the corresponding arc-generators. Fix two integers \( 0 \leq i < j \leq s+1 \) and write \( \mu_0 = \gamma_i^m \gamma_j^{m+1} \), where \( n_i \geq 0 \) and \( n_{i+1} > 0 \), and \( \mu'_0 = \gamma_j^{m'} \gamma_j^{m+1} \). The case \( \mu'_0 \notin \langle h' \rangle \) (when both generators are associated with the interior of the chain) corresponds to \( m_j \geq 0 \) and \( m_{j+1} > 0 \), while \( \mu'_0 \in \langle h' \rangle \) is covered by \( m_j = 0 \), \( m_{j+1} \in \mathbb{Z} \) for \( j = s \).

The last condition covers the case when \( \mu'_0 \) is not pure too.) Then
\[
\begin{pmatrix}
[b_{i+1}, \ldots, b_j] & [b_{i+1}, \ldots, b_{j-1}] \\
-[b_{i+2}, \ldots, b_j] & -[b_{i+2}, \ldots, b_{j-1}]
\end{pmatrix} \begin{pmatrix}
m_{j+1} \\
n_i
\end{pmatrix} = \begin{pmatrix}
m_j \\
n_{i+1}
\end{pmatrix}.
\]

But this system has no solution with the above (positivity) restrictions.

The cases \((m=2)\), or \((m=0)\) of different types, are eliminated similarly as the \((m=1)\) \( \mu'_0 \in \langle h' \rangle \) case. The case \((m=0)\) of the same \( H_\tau \) is eliminated as \((m=1)\) \( \text{‘interior chain’} \) case. \( \square \)
Corollary 54. Theorem 2 is true if L is not a Seifert manifold or a cusp link.

Proof. This follows from the list of components from Paragraph 30 and from Theorem 53. The coincidences from Theorem 53(2) are eliminated by considering Proposition 57, the following arc-generators in $\pi_1$ can be realized by equivariant arcs (parametrizations of the generic and special curves). Theorem 6 is true if Corollary 58.

55. The above proof can be adapted to show Theorem 6 for cusp singularities as well. A more precise description is given in Section 6.

56 (The Seifert case). We run the same strategy as above. The plumbing graph has no arrowheads, and $\pi_1 = \pi_1(L)$ and $\pi_i := \pi_1/\langle h \rangle$ are the following:

$$\pi_1 = \langle h, \{g_i\}_{i=1}^n, \{a_m, b_m\}_{m=1}^g \mid h \text{ central, } g_i^{a_i} = h, \forall i, \ h^b = \prod_m [a_m, b_m] \cdot \prod_i g_i^{a_i} \rangle.$$

$$\pi_i = \langle \{g_i\}_{i=1}^n, \{a_m, b_m\}_{m=1}^g \mid \bar{g}_i^{a_i} = 1, \forall i, \ \prod_m [a_m, b_m] \cdot \prod_i g_i^{a_i} = 1 \rangle.$$

By $x_i := \bar{g}_i^{a_i}, \bar{g}_i := x_i^{a_i}$, where $\omega a_i \equiv 1$ mod $\alpha_i$, $\pi_i$ transforms into the crystallographic group (if $g = 0$, it is also named Dyck-Schwartz polygonal group)

$$\{\{x_i\}_{i=1}^n, \{a_m, b_m\}_{m=1}^g \mid x_i^{a_i} \cdots x_n^{a_n} = \prod_m [a_m, b_m] \cdot \prod_i x_i = 1 \}.$$

Then $\pi_1$ is infinite iff $\pi_i$ is infinite. Moreover, infinite crystallographic groups have the following property, see e.g. [JNS, page 62].

Proposition 57. If $\bar{g}_i ^{a_i} \sim \bar{g}_j ^{a_j}$ in $\pi_i$ ($1 \leq i \neq j \leq n, r, s \in \mathbb{Z}$), then $\bar{g}_i ^{a_i} = \bar{g}_j ^{a_j} = 1$. □

Corollary 58. Theorem 2 is true if L is a Seifert manifold.

Proof. Quotient singularities were already treated in Section 9. Otherwise, by Proposition 57 the following arc-generators in $\pi_1$ belong to different conjugacy classes: $\{h^m\}_{m \in \mathbb{Z}^\geq 0}, \{g_i^m\}_{m \in \mathbb{Z}^\geq 0 \setminus \{a_i, 2\}}$, $1 \leq i \leq n$. Considering the minimal dlt modification we hit exactly the components from Paragraph 30. □

59 (Proof of Proposition 10). (1) follows from the fact that the classes $h$ and $g_i$ can be realized by equivariant arcs (parametrizations of the generic and special orbits). (2) follows from Corollary 58 (or its proof), saying that the conjugacy classes $\{h^m\}_{m \in \mathbb{Z}^\geq 0}, \{g_i^m\}_{m \in \mathbb{Z}^\geq 0 \setminus \{a_i, 2\}}$ are all different. The class $h^m$ can be realized by a family of equivariant arcs parametrized by the smooth part of the central curve.

9. Cusp singularities

Definition 60 (Cusps and their canonical representation). Cusp normal surface singularities are characterized by the fact that the dual graph of their minimal log resolution is a cyclic graph with all genus decorations zero. If $E$ is the exceptional curve, then $\pi_1(E) = \mathbb{Z}$, and fixing the generator of $\pi_1(E)$ is equivalent with fixing the cyclic order of the vertices of the graph. Let us fix such an ordering. The corresponding Euler numbers of the ordered vertices will be denoted by $-b_1, \ldots, -b_k$ (up to cyclic permutation), where each $b_i \geq 2$ and $\sum_i (b_i - 2) > 0$.

The dual graph $\Gamma$ has no nodes, nevertheless it has exactly one JSJ torus $T$, which can be chosen as the torus corresponding to one of the edges of the graph. We rename the vertices in such a way that the cutting edge is $(k, 1)$, hence if we cut the link, or the graph $\Gamma$, along this torus, we get
Introduce the column vectors \( \gamma_i \) denoted by paths we may assume that they commute with each other and satisfy the relations
\[
\gamma_i^b = \gamma_i \gamma_{i+1}, \quad \text{for } 1 \leq i \leq k.
\]
In particular, \( \gamma_0 \) and \( \gamma_1 \) generate freely \( \mathbb{Z}^2 \cong \pi_1(T) \), and all \( \gamma_i \) can be expressed in terms of them. It is convenient to organize this as follows.

For any collection of integers \( a_1, \ldots, a_n \) we define the matrix
\[
M(a_1, \ldots, a_n) := \begin{pmatrix} a_1 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ -1 & 0 \end{pmatrix} \in \operatorname{SL}(2, \mathbb{Z}).
\]
Introduce the column vectors \( \{v_i\}_{i=1}^{k+1} \) of \( \mathbb{Z}^2 \) by \( v_0 = (0) \) and \( M(b_1, \ldots, b_k) = (v_{i+1}, v_i) \) for \( 1 \leq i \leq k \). E.g., \( v_1 = \left( \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \right) \), \( v_2 = \left( \begin{smallmatrix} b_1 \\ -1 \end{smallmatrix} \right) \). For any \( v = (\lambda) \) we set
\[
\gamma^v := \gamma_1^\lambda \in \pi_1(T). \quad \text{Then, by induction, } \gamma_i = \gamma^v_i \text{ for any } 0 \leq i \leq k + 1.
\]
The matrix \( M = M(b_1, \ldots, b_k) \in \operatorname{SL}(2, \mathbb{Z}) \) is called the monodromy operator associated with the choice of the torus \( T \) and the bases \( \{v_0, v_1\} \) of \( \mathbb{Z}^2 \), and the representation \( \gamma_0 \mapsto v_0, \; \gamma_1 \mapsto v_1 \) of \( \pi_1(T) \to \mathbb{Z}^2 \). Note that \( Mv_0 = v_k \) and \( Mv_1 = v_{k+1} \). The sequence of vectors \( v_i \) can be extended to a sequence \( \{v_i\}_{i \in \mathbb{Z}} \) of \( \mathbb{Z}^2 \) by \( v_{i+1} := Mv_i \) (\( i \in \mathbb{Z} \)), which evidently correspond to arc generators of the infinite string \( \ldots, b_1, \ldots, b_k, b_1, \ldots, b_k, \ldots \). Then, the link is a torus bundle over \( S^1 \), (where the fiber is identified with \( T \)), hence one has the (HNN) extension
\[
1 \to \mathbb{Z}^2 = \pi_1(T) \to \pi_1 \to \mathbb{Z} = \pi_1(E) \to 1,
\]
where \( \pi_1(E) \) acts on \( \pi_1(T) \) via the monodromy \( M \). In other words,
\[
\pi_1 = \langle v, t \mid v \in \mathbb{Z}^2, \; tvt^{-1} = Mv \rangle.
\]
This identification of \( \pi_1(T) \) with \( \mathbb{Z}^2 \) will be called its ‘canonical representation’.

**61** (The monodromy operator \( M \)). The monodromy operator \( M \in \operatorname{SL}(2, \mathbb{Z}) \) satisfies \( \tau := \text{trace}(M) \geq 3 \). This shows that the characteristic polynomial \( \lambda^2 - \tau \lambda + 1 \) has two positive (non-rational) roots. We denote them \( \lambda_1 > 1 > \lambda_2 > 0 \). Let \( V_1 \) and \( V_2 \) in \( \mathbb{R}^2 \) be the corresponding eigenvectors chosen such that \( V_1 \) is in the fourth quadrant and \( V_2 \) in the second one. Then one has the following limits of halflines
\[
\lim_{i \to \infty} \mathbb{R}_{>0}v_i = \mathbb{R}_{>0}V_1, \quad \lim_{i \to -\infty} \mathbb{R}_{>0}v_i = \mathbb{R}_{>0}V_2.
\]
Note that \( (\mathbb{R}_{>0}V_1 \cup \mathbb{R}_{>0}V_2) \cap \mathbb{Z}^2 = \emptyset \).

**Proposition 62.** Let \( \text{Cone}(\Gamma) \) be the real (open) positive cone \( \mathbb{R}_{>0}(V_1, V_2) \) generated by \( V_1 \) and \( V_2 \), and set \( \text{Cone}_{\mathbb{Z}}(\Gamma) := \text{Cone}(\Gamma) \cap \mathbb{Z}^2 \). Consider also the action of \( \mathbb{Z} \) on \( \text{Cone}_{\mathbb{Z}}(\Gamma) \) given by \( \mathbb{Z} \times \mathbb{Z}^2 \ni (\ell, v) \mapsto M^\ell v \). Then the image of \( w : \pi_0(\text{ShArc}(0 \in X)) \to \pi_1(\text{link}(0 \in X)) / (\text{conj}) \) is in \( \pi_1(T) / (\text{conj}) \), and under the above ‘canonical representation’ it is identified with \( \text{Cone}_{\mathbb{Z}}(\Gamma) / \mathbb{Z} = \text{Cone}_{\mathbb{Z}}(\Gamma) / M \).

Proof. Arc generators of type \( \gamma_i^m \) (\( m \in \mathbb{Z}_{>0} \)) are represented on the lattice points on the ray \( \mathbb{R}_{>0}v_i \), and those of type \( \gamma_i^m \gamma_{i+1}^{m+1} \) (\( m_i, m_{i+1} \in \mathbb{Z}_{>0} \)) by the lattice points in the open cone \( \mathbb{R}_{>0}(v_i, v_{i+1}) \), modulo the monodromy action. \( \square \)
63 (Dependencies). $M$ depends on the above choice of ‘canonical representation’ (of $\pi_1 \to \mathbb{Z}^2$, $\gamma_0 \mapsto v_0$, $\gamma_1 \mapsto v_1$). If the JSJ torus is given by the edge $(1, 2)$, then the new monodromy operator is $M_{(1, 2)} = M(b_2, \ldots, b_k, b_1) = M(b_1)^{-1} \cdot M \cdot M(b_1)$, hence $M_{(1, 2)}$ and $M$ are conjugate. The same is true for any other choice of $T$.

In fact, the correspondence $(b_1, \ldots, b_k) \mapsto M$ from oriented cycles to the set of conjugacy classes of $A \in \text{SL}(2, \mathbb{Z})$ with $\text{trace}(A) \geq 3$ is a bijection (cf. [Neu81, 6.3]). (One recovers $(b_1, \ldots, b_k)$ from $M$ via an infinite periodic continued fraction associated with $\text{Cone}_2(\Gamma)$.)

If we change the orientation in the cycle of the graph (or, the generator of $\pi_1(T)$), then the new monodromy operator will be $\overline{M} = M(b_k, \ldots, b_1)$, and

$$\overline{M} = SM^{-1}S^{-1}, \quad \text{where} \quad S = S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

If $T(A)$ denotes the torus bundle over $S^1$ with monodromy $A$ (where $\text{trace}(A) \geq 3$ always), then $T(A)$ is orientation preserving diffeomorphic with $T(B)$ iff $A$ is conjugate in $\text{SL}(2, \mathbb{Z})$ to either $B$ or $BS^{-1}S^{-1}$ [Neu81, 6.2]. Hence $T(M) \simeq T(\overline{M})$, and their fundamental groups are isomorphic via $t \leftrightarrow t^{-1}$, $v \leftrightarrow Sv$ and $M \leftrightarrow M^{-1}$.

Changing the orientation of $\partial M$, the image of $w$ is the set of lattice points of $-\text{Cone}_2(\Gamma)$ up to the monodromy action of $M$. Our next goal is to identify the lattice points of the complementary cone of $\text{Cone}(\Gamma)$ (up to the action of $M$) with $\pi_0(\text{ShArc}(0 \in X^*)))$, where $X^*$ is the dual cusp.

64 (The dual cusp). We wish to identify $\pi_1(T)$ for the cusp $X$ and its dual $X^*$ in a canonical way, such that both monodromy actions will act on the same $\mathbb{Z}^2$.

A possible plumbing graph of the link $L^*$ of the dual cusp $X^*$ is $-\Gamma$. This means that we replace each decoration $-b_i$ of $\Gamma$ by $b_i$, and each edge decoration (which was $+$) by $-$. The effect of this is that $L^*$ is $-L$, $L$ with opposite orientation.

By (oriented) plumbing calculus (see [Neu81]) $-\Gamma$ can be replaced by another graph, $\Gamma^*$, which has all Euler decorations $\leq -2$ and edge decorations $+$, and it is the dual graph of the minimal log resolution of $X^*$. Before we describe it, we fix/identify the cutting JSJ tori in both 3-manifolds. Let us fix a cyclic ordering and the cutting edge in $\Gamma$ as in [60]; and mark the corresponding Euler numbers by $-b_1, \ldots, -b_k$. Since at least one of them is $\leq -3$, we can assume that $b_k \geq 3$.

We choose in $-\Gamma$ the very same edge as cutting edge and the same orientation of the cycle (generator of $\pi_1(E)$). Then, after running the plumbing calculus (in such a way that we never blow up/down the cutting edge) we get the vertices with decorations $-b_1^\star, \ldots, -b_k^\star$. One checks (see e.g. [Neu81]) that if the $b$-sequence is

$$2^{k_1^\star} - 1, k_1 + 2, 2^{k_2^\star} - 1, k_2 + 2, \ldots, k_g + 2,$$

then the $b^\star$ sequence in $\Gamma^*$ is

$$k_1^\star + 2, 2^{k_1^\star} - 1, k_2^\star + 2, 2^{k_2^\star} - 1, \ldots, 2^{k_g^\star} - 1,$$

where $2^k$ represents $k$ copies of $2$.

The goal is to identify the two tori-fibers, that is, to see both tori in the same graph. This seems to appear only implicitly in the literature [Neu81, PP07]. Consider the following plumbing graph:
Then, we express in the above graph the elements $\gamma$ the arc generators associated with the vertices compatibly to the notation of (60).

By plumbing calculus (0-chain and oriented handle absorptions), this graph is equivalent to a single vertex with genus 1 and Euler number 0, hence representing the trivial torus bundle over $S^1$. Since the monodromy acting on the torus is trivial, the tori associated with the edges can be identified by canonical isomorphisms.

In this graph we can proceed on the ‘top subchain’ the calculus which provided (60) for it), we would send $\gamma$ to $0$ (just repeating the steps which provides the normal form of $-\Gamma$). Then we get the graph (here we use the fact that $b_k \geq 3$):

Only one edge has decoration $-$, the others have $+$. We inserted also some of the arc generators associated with the vertices compatibly to the notation of (60).

Next, we fix $\pi_1(T) = \mathbb{Z}^2$ under the canonical identification of $\Gamma$ as in subsection (60) with base elements $v_0 = (0)_{(1)}$ and $v_1 = (1)_{(0)}$, and we represent $\gamma_i = \gamma_i^v$ for $i = 1, 2$. Then, we express in the above graph the elements $\gamma_i^v$ ($i = 1, 2$) in terms of $\gamma_0$ and $\gamma_1$. Namely, using the relations $\gamma_0^v = \gamma_1 \gamma_0^v$ and $\gamma_1^v = \gamma_0 \gamma_1^v$, we get $\gamma_0^v = \gamma_0^v \gamma_1^v$ and $\gamma_0 \gamma_1^v = 1$. Hence, although in the canonical representation of $\Gamma$ (considered independently from $\Gamma$), and repeating (60) for it), we would send $\gamma_0^v$ to $0$ and $\gamma_1^v$ to $0$, in this representation compatible with the canonical representation of $\Gamma$ (when we send $\gamma_0$ to $0$ and $\gamma_1$ to $0$) we have to send $\gamma_0^v$ to $u_0 := (-1)$ and $\gamma_1^v$ to $u_1 := (-1)$.

Let $\text{Cone}^v(\Gamma)$ be the complementary cone of $\text{Cone}(\Gamma)$, namely $\mathbb{R}_{>0}(-V_1, V_2)$, and set also $\text{Cone}^v(\Gamma) = \text{Cone}^v(\Gamma) \cap \mathbb{Z}^2$. Then using the properties of $M$ one verifies that $u_0, u_1 \in \text{Cone}^v(\Gamma)$. Moreover, one has the following identity: cf. [Nak80], 7.6.

**Proposition 65.** Set $T := \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$ and consider the monodromy operators $M = M(b_1, \ldots, b_k)$ and $M^* = M(b_1^*, \ldots, b_k^*)$. Then $MT = TM^*$, hence the eigenvalues of $M$ and $M^*$ are the same.

**Proof.** In the above graph the monodromy operator of the trivial torus-fibration is the identity. This reads as (starting from the left $-1$-vertex)

$$M(1, 2) \cdot M \cdot M(1, 2) \cdot M(b_k^*, \ldots, b_1^*) = -I,$$

where the sign is given by the unique negative edge. Since $M(1, 2) = ST^{-1} = -TS$ and by (60) one also has $M(b_k^*, \ldots, b_1^*) = S(M^*)^{-1}S$, we get $MT = TM^*$. □

Note that $\det(T) = -1$, hence the proposition does not say that $M$ and $M^*$ are necessarily conjugate in $\text{SL}(2, \mathbb{Z})$ (although, in special cases they can be conjugate or even equal, see e.g the case $(3, 3, 3)$ appearing in Example [4] which is auto-dual).
Corollary 66. Let $V^*_1$ and $V^*_2$ be the eigenvectors of $M^*$ defined via the canonical recipe of (61). Then

1. $TV^*_1 \in -\mathbb{R}_{>0}V_1$ and $TV^*_2 \in \mathbb{R}_{>0}V_2$. In particular, $T$ sends Cone$_2(\Gamma^*)$ isomorphically onto Cone$_2(\Gamma)$ identifying the action of $M^*$ on the first one with the action of $M$ on the second one.

2. Cone$_2(\Gamma)/M$ is identified via $T^{-1}$ with Cone$_2(\Gamma^*)/M^*$ and Proposition 62 identifies the latter with $\pi_0(\text{ShArc}(0 \in X^*))$. $\square$

Remark 67 (The four cones and orientations). Let us fix a generator of $\pi_1(E)$ as above. There are two sources of orientations on the torus $T$.

First, regarding $L$ as a torus bundle over $S^1$, $\pi_1(S^1)$ and $\pi_1(E)$ are canonically identified, hence an orientation of $L$ is equivalent to an orientation of $T$.

The plumbing representation of $L$ gives another orientation as we glue several $S^1$-bundles over annuli. Let us call the circle in the annulus $[0, 1] \times S^1$ the ‘angle circle’ and the $S^1$-fiber of the bundles the ‘fiber circle’. Then $T$ consists of their product, and the orientation of $T$ is the product of angle and fiber circle orientations. Switching both circle orientations does not alter orientation of the torus or the link.

The realization of the link as a tubular neighborhood of holomorphic curves fixes an orientation of the fiber circle. The latter is not a topological invariant of the oriented link.

10. INOUE SURFACES AND SHORT ARCS ON CUSPS

Short arcs on a cusp and on its dual can be seen together very nicely on hyperbolic Inoue surfaces [Ino75, Ino77]. We need to understand mostly their cusps, treated in detail in [Hir73, §2].

Definition 68 (Hyperbolic Inoue surfaces). Let $K = \mathbb{Q}(\sqrt{d}) \subset \mathbb{R}$ be a real quadratic field. Conjugation is denoted by $w \mapsto w'$.

Let $H \subset K$ be a free $\mathbb{Z}$-module of rank 2. Let $u \in K$ be a nontrivial element such that $u > 0$, $u' > 0$ and $uH = H$. Let $V = \langle u \rangle$ be the group it generates. We may assume that $H$ is an ideal in $\mathcal{O}_K$; in this case $u$ is a unit in $\mathcal{O}_K$ and this makes some formulas simpler.

Let $G = G(K, H, u) \subset \text{GL}(2, \mathbb{R})$ be the group

$$G := \left\{ \begin{pmatrix} v & m \\ 0 & 1 \end{pmatrix} : v \in V, m \in H \right\}.$$ 

There is a natural group extension

$$0 \to H \to G \to V \to 0.$$ 

The group $G$ acts on $\mathbb{C}^2$ by the rule

$$\rho(v, m) : (z_1, z_2) \mapsto (vz_1 + m, vz_2 + m').$$

The action is properly discontinuous on $\mathbb{H} \times \mathbb{C}$ where $\mathbb{H} \subset \mathbb{C}$ is the upper half plane. It is proved in [Hir73, §2] that the quotient $(\mathbb{H} \times \mathbb{C})/G$ can be compactified by adding 2 points; resulting in a singular complex surface with 2 cusps. The sets

$$\{(z_1, z_2) : \Im z_1 \cdot \Im z_2 > 1/\epsilon \} \quad \text{resp.} \quad \{(z_1, z_2) : \Im z_1 \cdot \Im z_2 < -1/\epsilon \}$$

give open neighborhoods of the $+\infty$-cusp (resp. $-\infty$-cusp).

The minimal resolution of this singular surface is called a hyperbolic Inoue surface.
Similarly, if \( m > p \) the punctured unit disc \( D \).

Note that \( \tilde{\phi} \) and the image of \( R \) descends to an holomorphic map \( \phi \) by

\[
\rho(v) : (y_1, y_2) \mapsto (vy_1, v^{-1}y_2).
\]

Thus \( (\mathbb{H} \times C)/H \) is a torus bundle over \( R^{>0} \times R \). It is convenient to extend the \( G \)-action to \( \mathbb{H} \times C \). The \( H \)-action is still properly discontinuous and \( (\mathbb{H} \times C)/H \) is a torus bundle over \( R^{\geq 0} \times R \). The advantage is that the fiber over \( (0, 0) \in R^{\geq 0} \times R \) is canonically identified with \( R \otimes Q K/H \), giving a canonical isomorphism

\[
H_1(\mathbb{S}^1(0,0)/H, \mathbb{Z}) \cong H.
\]

(For other fibers, this identification is possible only up to the \( V \)-action on \( H \).)

We see that the real hypersurface

\[
T := (\mathbb{S}^2 = 0) \subset \mathbb{H} \times C
\]

is \( \rho \)-invariant. The quotient \( T/G \) is a torus bundle over \( \mathbb{S}^1 \).

The complement of \( T \) is decomposed into 2 pieces

\[
W^+ := \mathbb{H} \times \mathbb{H} \quad \text{and} \quad W^- := \mathbb{H} \times (-\mathbb{H}).
\]

The \( G \)-equivariant map \((z_1, z_2) \mapsto (z_1, \mathbb{R}z_2)\) shows that

\[
W^+/G \sim (T/G) \times R^{>0} \quad \text{and} \quad W^-/G \sim (T/G) \times R^{<0}.
\]

Thus \( W^+/G \cup \{\infty\} \) and \( W^-/G \cup \{-\infty\} \) are both homeomorphic to cones over \( T/G \).

70 (Short arcs on hyperbolic Inoue surfaces). One can construct representatives of the spaces of short arcs of the two cusps as follows.

First let \( m \in H \) be an element such that \( m > 0, m' > 0 \). Then

\[
\tilde{\phi}_m : \mathbb{H} \rightarrow W^+ \quad \text{given by} \quad w \mapsto (mw, m'w)
\]

descends to an holomorphic map \( \phi_m^* : \mathbb{D}^* \cong \mathbb{H}/Z[1] \rightarrow W^+/G \). Here \( Z[1] \) denotes the translation action \( w \mapsto w + 1 \) and the quotient \( \mathbb{H}/Z[1] \) is identified with the punctured unit disc \( \mathbb{D}^* \). Next \( \phi_m^* \) extends to a short arc

\[
\phi_m : \mathbb{D} \rightarrow W^+/G \cup \{\infty\}.
\]

Similarly, if \( m > 0 \) but \( m' < 0 \) then we get a short arc

\[
\phi_m : \mathbb{D} \rightarrow W^-/G \cup \{\infty\}.
\]

Note that \( \tilde{\phi}_m \) extends to

\[
\tilde{\phi}_m : \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}
\]

and the image of \( \mathbb{R} = \partial \mathbb{H} \) gives a homology class in \( \mathbb{S}^1(0,0)/H \). Under the identification \( (69.1) \), this homology class is exactly \( m \in H \).

More generally, any short arc through the \( +\infty \) cusp lifts to a holomorphic map \( \phi : \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H} \) that satisfies the condition

\[
\phi(w + 1) = \phi(w) + (m, m') \quad \text{for some} \quad m \in H.
\]

Then \( \psi(w) := \phi(w) - (mw, m'w) \) satisfies \( \psi(w + 1) = \psi(w) \). Thus \( \psi \) can be expanded into a Fourier series

\[
\psi(w) = \left( \sum_{n \in \mathbb{Z}} a_n e^{2\pi i nw}, \sum_{n \in \mathbb{Z}} b_n e^{2\pi i nw} \right).
\]
We need to extend this across the cusp, hence $\psi$, as a function on $D^* = \mathbb{H}/\mathbb{Z}[1]$, can not have essential singularities. This gives the general solution

$$\phi(w) = (mw + \sum_{n \geq 0} a_n e^{2\pi inw}, m'w + \sum_{n \geq 0} b_n e^{2\pi inw}). \quad (70.1)$$

Note finally that the monodromy action of Definition 60 corresponds to multiplication by $u$ on $H$. The 4 cones in Paragraph 61 are given by the conditions

$$(m > 0, m' > 0), \ (m > 0, m' < 0), \ (m < 0, m' > 0), \ (m < 0, m' < 0).$$

We see that short arcs on the $+\infty$ cusp give elements in the first cone, and short arcs on the $-\infty$ cusp give elements in the second cone. Thus our computations give the same results as the topological considerations of Section 9.

Remark 71. It is interesting that writing families of short arcs as in (70.1) treats all of them the same way. By contrast, our original approach suggests that arc families arising from irreducible components of the exceptional set of the minimal resolution and arc families arising from the singularities of the exceptional set do behave differently.

By Paragraph 66 these two types of families can be distinguished topologically as long as the irreducible component has either genus $\geq 1$ or has at least 3 nodal points. For cusps, every irreducible component of the exceptional set is a rational curve with 2 nodal points.

11. Open problems

There are many open questions about arc spaces. Here we collect some of them that seem most interesting to us.

Local structure of arc spaces.

Definition 72. Let $X$ be an analytic space. A finite type family of arcs on $X$ is given by

1. an analytic space $V \subset \mathbb{C}^r_x$ and
2. a holomorphic vector function

$$F(x_1, \ldots, x_r; y_1, \ldots, y_s, t) : V \times \mathbb{D}_x^s \times \mathbb{D}_t^t \to X$$

such that $t \mapsto F(c_1, \ldots, c_r; u_1(t), \ldots, u_s(t), t)$ is an arc in $X$ for all $(c_1, \ldots, c_r) \in V$ and $u_j(t) \in B_{<1} \mathcal{H}(\mathcal{O}_X)$ for $j = 1, \ldots, s$.

Thus $F$ gives a family of arcs parametrized by $V \times B_{<1} \mathcal{H}(\mathcal{O}_X)^s$.

Note that a finite type family of arcs is much more restrictive than a family of arcs parametrized by $V \times B_{<1} \mathcal{H}(\mathcal{O}_X)^r$.

We have not checked the details but the methods of [HM01, GK00, Dr02] should imply that if $\phi : \mathbb{D} \to X$ is an arc whose image is not contained in $\text{Sing} \ X$ then $[\phi] \in \text{Arc}(X)$ has an open neighborhood $[\phi] \in U(\phi) \subset \text{Arc}(X)$ that is obtained from a finite type family of arcs.

Conjecture 73. Let $X$ be a complex space and $\text{Arc}^*(X) \subset \text{Arc}(X)$ the space of arcs that are not contained in $\text{Sing} \ X$.

Then $\text{Arc}^*(X)$ has a natural atlas consisting of finite type families.

Remark 74. We are intentionally vague about what types of coordinate changes we allow in such a “natural atlas.” Our hope is that the following three types of operations suffice.
(1) (Shrinking $V$) Replacing $V$ with an open subset of it.

(2) (Changing coordinates) Biholomorphisms $V_1 \cong V_2$ and $V_1 \times \mathbb{D}^s \cong V_2 \times \mathbb{D}^s$ (the latter defined in neighborhoods of $V_1 \times \{0\}$) that are compatible with the projections

$$V_1 \times \mathbb{D}^s \cong V_2 \times \mathbb{D}^s$$

$$\downarrow \quad \downarrow$$

$$V_1 \cong V_2$$

(3) (Splitting off the leading coefficient) A finite type family $(F : V \times \mathbb{D}^s \times \mathbb{D}_t \to X)$ is replaced by $(F^s : (V \times \mathbb{D}) \times \mathbb{D}^s \times \mathbb{D}_t \to X)$ using the rule

$$(c_1, \ldots, c_r; u_1(t), \ldots, u_s(t)) \mapsto (c_1, \ldots, c_r, u_1(0); \frac{u_1(t) - u_1(0)}{t}, u_2(t), \ldots, u_s(t)).$$

The methods and results of [HM94, BH10] seem quite close to a solution along the above lines.

Even weaker versions of this would be quite important. For instance, without a complex structure on $\text{Arc}^\ast(X)$, it is not even clear how to define the notion of irreducible components. Our Definition 14 seems to work, but it leaves several basic questions unanswered. The most important is the following.

**Curve selection conjecture.**

Let $X$ be an irreducible algebraic variety and $p_1, \ldots, p_m \in X$ a collection of points. Then there is always an irreducible curve $C \subset X$ that contains all these points. This implies that there is an arc $\phi : \mathbb{D} \to X$ whose image contains the points $p_1, \ldots, p_m$. For irreducible complex spaces much stronger results are proved in [Win05, FW05].

A similar assertion is much harder for infinite dimensional complex spaces, in particular for the arc spaces $\text{Arc}^\circ(X)$ and $\text{ShArc}(X)$.

**Conjecture 75.** Let $X$ be a complex space, $W$ an irreducible component of $\text{Arc}^\circ(X)$ or $\text{ShArc}(X)$ and $[\phi_1], \ldots, [\phi_m]$ arcs in $W$. Then there is a holomorphic family of arcs $F : \mathbb{D} \times \mathbb{D}_t \to X$ and $p_1, \ldots, p_m \in \mathbb{D}$ such that $\phi_i(t) = F(p_i, t)$ for $i = 1, \ldots, m$.

The notion of a strongly irreducible space was introduced in Definition 14 to go around this conjecture. We do not know how to prove that a connected component of $\text{Arc}(X)$ is strongly irreducible iff it contains a strongly irreducible dense open subset. These types of questions are quite difficult; see for instance [FdBP12a].

**Is $\text{Arc}^\circ(0 \in X)$ well defined?**

One usually thinks of a singularity as an equivalence class of pointed singular spaces $(0 \in X)$. The set of short arcs $\text{ShArc}(X)$ does depend on the choice of the representative $(0 \in X)$ but we saw in Paragraph 5 that the resulting arc spaces $\text{ShArc}(X)$ are naturally homeomorphic to each other.

If several points of an arc pass through the singularities then the rescaling trick used in Paragraph 5 does not work and the homeomorphism type of $\text{Arc}^\circ(X)$ does depend on the choice of the representative $(0 \in X)$. For instance,

$$\text{Arc}^\circ(0 \in \mathbb{D}) \quad \text{and} \quad \text{Arc}^\circ(0 \in \mathbb{D} \setminus \{\frac{1}{2}\})$$

have different connected components.

A positive answer to the following would be a suitable replacement.
**Conjecture 76.** Let \( (0 \in X) \) be an isolated singularity and \( 0 \in U \subset X \) a contractible, Stein neighborhood. Then the homeomorphism types of \( \text{Arc}(U) \) and of \( \text{Arc}^c(U) \) are independent of \( U \).

For singularities with a good \( \mathbb{C}^* \)-action, sufficiently convex neighborhoods \( U \) do give homeomorphic arc spaces, but we have very little other evidence to support the above conjecture.

**Computing \( \text{Arc}^c(X) \) for surface singularities.**

Let \( (0 \in S) \) be a contractible, Stein, normal surface singularity. Assuming that at least \( \pi_0(\text{Arc}^c(S)) \) is independent of the choice of the representative \( S \), the natural problem is the following variant of Theorem 6.

**Question 77.** Let \( (0 \in S) \) be a contractible, Stein, normal surface singularity. Is the winding number map (5.1)

\[
\pi_0(\text{Arc}^c(S)) \to \pi_1(\text{link}(0 \in S))/(\text{conj})
\]

an injection?

We have not been able to compute any examples. It is also possible that a connected component of \( \text{Arc}^c(0 \in S) \) has several irreducible components and determining them may well be the more important question.

**Achs on complex manifolds.**

It would also be interesting to study arcs on complex manifolds, even in the normal crossing cases.

Let \( X \) be a smooth, affine variety (or a Stein manifold) and \( Z \subset X \) a normal crossing divisor. As before, let \( \text{Arc}^c(Z \subset X) \) denote the space of those arcs \( \phi : D \to X \) for which \( \phi(\partial D) \subset X \setminus Z \). We get a winding number map

\[
w_{X,Z} : \pi_0(\text{Arc}^c(Z \subset X)) \to \ker[\pi_1(X \setminus Z) \to \pi_1(X)]/(\text{conj}).
\]

Note that the kernel on the right hand side may not even be finitely generated.

There are some cases when \( Z \neq 0 \) yet \( \pi_1(X \setminus Z) = 1 \). This happens for instance for the affine quadric

\[
X = (x^2 + y^2 = z^2 + 1) \subset \mathbb{C}^3 \quad \text{and} \quad Z = (x - z = y - 1 = 0).
\]

There are, however, many cases when every irreducible component of \( Z \) adds a new generator to \( \pi_1(X \setminus Z) \). This happens if \( X = \mathbb{C}^n \) or, more generally, if \( H^2(X, \mathbb{Z}) = 0 \).

**Problem 78.** Study the above winding number map, or its restriction to short arcs, in some interesting global situations.

A starting case could be when \( X = \mathbb{C}^n \) and \( Z \) is a union of hyperplanes intersecting transversally.

**Short arcs in higher dimensions.**

As the examples in [IK03, dF12, Kol12] show, the original Nash problem needs to be reformulated in higher dimensions. A variant was proposed in [Kol12] and it is not hard to develop a version that applies to \( \text{ShArc}(0 \in X) \) in all dimensions.

However, we feel that both the questions in [Kol12] and their ShArc versions need to be tested in some nontrivial cases. For \( cA \)-type singularities, the irreducible components of \( \text{ShArc}(0 \in X) \) are studied in [JK13].
Real arcs.

**Definition 79** (Real arcs). Let $X$ be a real analytic space. A **real analytic arc** in $X$ is a real analytic morphism $\phi : [-1, 1] \to X$. (As before, $\phi$ is defined and analytic in some neighborhood of $[-1, 1]$.) If, in addition, $\text{Supp} \phi^{-1}(\text{Sing} X) = \{0\}$ then $\phi$ is called a **short real analytic arc** or **short arc**.

For real analytic arcs one should use the topology defined by convergence in all derivatives. (In the real case it makes sense to talk about $C^m$ arcs where $m \in \{0, 1, \ldots, \infty\}$. These behave quite differently from real analytic arcs, even for $m = \infty$.)

The basic observation is the following finiteness result.

**Proposition 80.** The space of short real arcs $\text{ShArc}_R(X)$ has only finitely many irreducible components.

Proof. Let $f : Y \to X$ be a log resolution with exceptional divisor $E$. As in Section 7, it is enough to prove that $\text{ShArc}_R(E \subset Y)$ has only finitely many irreducible components.

We follow the arguments in Paragraphs 18–26. In the complex case, the irreducible components of $\text{ShArc}(E \subset Y)$ were described by the strata of $E$ and by the intersection numbers with the divisors $E_i \subset E$.

In the real case only the parity of the intersection numbers matters as shown by the deformations

$$(t, s) \mapsto t^{2r+1} + s^2 t \quad \text{and} \quad (t, s) \mapsto t^{2r} + s^2 t^2.$$ 

We also need some new information. First, by strata we mean not the irreducible components of the intersections but their connected components. Furthermore, if $E_j$ is 2-sided in $X$ then an extra datum that appears is the side that contains the image of $(0, 1]$.

All together we get finitely many irreducible components for $\text{ShArc}_R(E \subset Y)$ and so finitely many irreducible components for $\text{ShArc}_R(X)$. □

We know very little about the connected or irreducible components for $\text{ShArc}_R(X)$ in higher dimensions, but the surface case should be easier to handle.

**Example 81** (A-type singularities). There are two real forms of A-type singularities with enough real points.

\begin{enumerate}
  \item \((x^2 + y^2 = z^m)\) These are quotient singularities. If $m$ is even, one of the exceptional curves of the minimal resolutions is real, if $m$ is odd then none of them are real. This shows the following.

  - $\text{ShArc}_R(x^2 + y^2 = 2z^{2m+1})$ has one connected component. A typical general arc is \((t^{2m+1}, t^{2m+1}, t^2)\).
  - $\text{ShArc}_R(x^2 + y^2 = 2z^{2m})$ has two connected components. Typical general arcs are \((t^m, t^m, \pm t)\).

  \item \((xy = z^m)\) These are not quotient singularities. All $m - 1$ exceptional curves of the minimal resolution are real. Following the arguments of Proposition 80 we expect 2 types of connected components in $\text{ShArc}_R(xy = z^m)$.

  \begin{enumerate}
    \item Typical arcs are \((t^i, t^{m-i}, t)\) for $0 < i < m$. These are the arcs whose lift to the minimal resolution intersects one exceptional curve with multiplicity 1.
  \end{enumerate}
\end{enumerate}
Adding signs to two of these terms we get $4m - 8$ connected components of the space of short real arcs. (The curves at the end give only 2 components each.)

81.2.b. Typical arcs are $(t_i, t^{2m-i}, t^2)$ for $0 < i < 2m$. These are the arcs whose lift to the minimal resolution intersects either one exceptional curve with multiplicity 2 (for $i$ even) or two exceptional curves (for $i$ odd).

Most of the time, these do not give new components. For instance, if $i < m$ then we have the deformation

$$\left(t^i, t^{m-i}(t - \epsilon)^m, t(t - \epsilon)\right),$$

and similarly if $i > m$. However, for $i = m$ the similar deformation

$$\left(t^m, (t - \epsilon)^m, t(t - \epsilon)\right),$$

gives arcs that do not pass through the origin.

These suggest that $\text{ShArc}_R(xy = z^m)$ should have $4m - 6$ connected components.

The above example shows that the method of Proposition 80 can give too many candidates for connected components, when applied to the minimal resolution. One can, however, hope that the minimal dlt modification is again the right choice.

**Problem 82.** Let $X$ be a normal, real algebraic surface with minimal dlt modification $g^{dlt} : (E^{dlt} \subset X^{dlt}) \rightarrow X$. Does composing with $g^{dlt}$ induce a bijection between the (connected or irreducible) components of $\text{ShArc}_R(E^{dlt} \subset X^{dlt})$ and $\text{ShArc}_R(0 \in X)$?

The proof of Theorem 37 relies on the fundamental group of the complex link of a singularity; we know no analogs of it in the real case.

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