Accuracy of lumped-parameter representations for heat conduction modeling in multilayer slabs

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Abstract. Heat conduction in homogeneous solids can be studied by resorting to one-dimensional schemes, as is often done, e.g., for building construction elements. In such situations, a simple model often employed makes use of an electrical analogy between temperature and heat flux, on one side, and voltage and electrical current on the other side. Within this framework, a few lumped-parameter representations have been described in literature to describe the thermal behavior of a single homogeneous slab or of multilayer slabs. Such models have the advantage of providing some physical insight into the phenomenon of one-dimensional heat conduction, by conveying the concepts of thermal resistance and thermal capacitance, the latter related to heat storage ability. There is, however, a certain degree of approximation in such models. The simplifying assumptions and approximations underlying these approaches will be reviewed and discussed in this contribution.

The accuracy of some lumped-parameter model will be analyzed in order to show under which circumstances the approximate solutions can be satisfactorily employed. In particular, the focus will be on the comparison of the predictions that approximate and accurate methods provide when studying the influence of layer order and distribution on the thermal performance of multilayer structures.

1. Introduction

The modeling of heat conduction in multilayer structures is an essential part for the evaluation of thermal loads in buildings. The accuracy of such modeling reflects in the proper sizing of the HVAC systems and as such can be the source of cost-effective energy use [1, 2, 3, 4, 5]. Equivalent electric circuits have been widely employed to model heat conduction [6, 7]. Their use has a double value. First, they offer a simpler mathematical treatment to be used as compared to exact solutions. Second, they have a considerable appeal because they trace thermal problems back to familiar circuit behavior using equivalent electric circuits essentially based on (single or chains of) simple RC cells. Such equivalence has been used extensively both for time sinusoidal variation regime and for transient phenomena [8]. The validity of lumped representations has been investigated by several authors [6, 7, 9]. Assessments lead to markedly different results for the two regimes cited above (steady state sinusoidal and transient). Even for single slabs, transient regime is considerably more demanding than sinusoidal regime. It has been shown [9] that good approximations for short times (0.1 RC) may require up to 50 lumped elements. Sinusoidal cases can be treated with simpler circuitry, the basic reason for this being that the
fundamental frequency to be taken into account for daily variation phenomena is about 12 μHz
while the spectrum required to account for functions affected by discontinuities may contain
much higher frequencies.

In recent times, multilayer walls came into play, and analyses of their performances in
sinusoidal regime were investigated, mainly by using approximate methods. In particular, a
resurgence of electric analogs took place [10, 11, 12, 13, 14, 15]. Available results show some
discrepancies, and the need for an analysis of RC model reliability in the realm of layered
structures is evident.

In this context, an issue that has been frequently investigated in the design of multilayer
walls is that of the influence of layers order and distribution on the wall’s thermal performances
[13, 16, 17, 18, 19, 20]. This subject will be studied in this work using the full analytical
model of a multilayer structure in sinusoidal regime and certain relevant results will be derived.
After that, the comparison with the predictions provided by the approximate lumped-parameter
models will be presented.

The analysis will concern summer weather conditions, as the most demanding in typical
Mediterranean climates, and on the decrease effect that the building envelope produces on
the inner temperature variations. Of course, similar evaluations can be similarly carried out
when the issue is to limit as much as possible the dispersion of internal heat toward the outer
environment in winter conditions.

2. Harmonic temperature variation on a multilayer wall

We recall here some basic equations that are used to describe, in a one-dimensional model, the
behavior of a multilayer wall subject to harmonic temperature variation. We focus in particular
on the exact model of the system and on lumped-parameter models based on electrical analogy.

2.1. Exact model

The one-dimensional heat conduction problem in a homogeneous wall of thickness \( L \), thermal
conductivity \( \lambda \) and thermal diffusivity \( D \), subject to harmonic temperature variation on its
boundary faces can be analytically described [21] by linking temperatures and heat fluxes on its
faces by a transfer matrix in the following way:

\[
\begin{bmatrix}
T_0 \\
q_0
\end{bmatrix} = \begin{bmatrix}
\cosh(KL) & \sinh(KL)/(\lambda K) \\
\lambda K \sinh(KL) & \cosh(KL)
\end{bmatrix}\begin{bmatrix}
T_L \\
q_L
\end{bmatrix} = \begin{bmatrix}
C & \frac{S}{V} \\
YS & C
\end{bmatrix}\begin{bmatrix}
T_L \\
q_L
\end{bmatrix},
\]

where \( K = (1 + j)\sqrt{\omega/(2D)} \) and \( \omega = 2\pi f \) is the angular frequency (typically considered to
be that of the daily temperature variation: \( \omega = 2\pi/86400 = 7.27 \cdot 10^{-5} \text{ rad/s} \)). A shorthand
notation to be used in the following has also been introduced, with \( Y = \lambda K \) denoting a thermal
admittance. It is worth noting that the transfer matrix in equation (1) has unity determinant,
a property that reduces to three the independent matrix elements. This of course is still valid
for the matrix product that is employed for the description of a multilayer wall.

It is possible to introduce a temperature decrement factor as the ratio \( F_{\text{dec}} = T_0/T_L \) that
quantifies the effect that the massive wall has on the inner temperature variations. If an adiabatic
boundary condition is selected \((q_L = 0), F_{\text{dec}} = \cosh(KL)\). The decrement factor is a complex
quantity that takes into account both an amplitude reduction and a phase lag (illustrated in the
following by a corresponding time delay between outer and inner temperatures). Similarly, a
heat flux decrement factor \( F_{\text{dec}}^q = q_0/q_L = \cosh(KL) \) can be defined if an isothermal boundary
condition is selected at the inner face \((T_L = 0)\).

For small values of \(|KL|\), a second-order Taylor expansion of the \( \cosh(\cdot) \) function gives in the
adiabatic case \( F_{\text{dec}} \approx 1 + K^2L^2/2 = 1 + j\omega RC/2 \), where \( R = L/\lambda \) and \( C = \rho c_p L \) have been
introduced, with $\rho$ the volume density and $c_p$ the specific heat of the homogeneous wall. At this level, $R$ and $C$ just define a thermal resistance and a thermal capacitance (per unit surface in both cases) of the wall but they still do not imply a specific lumped-parameter scheme that will be introduced instead in the following. In the general case (no restriction to adiabatic boundary condition), by approximating both terms in the expression of $T_0$ through their leading terms in the Taylor expansion (using $\sinh(x) \simeq x + x^3/6$), one obtains

$$T_0 = T_L \left(1 + j\frac{\omega RC}{2}\right) + q_L \left(1 + j\frac{\omega R^2 C}{6}\right)$$

(2)

For multilayer walls subject to sinusoidal temperature variations, assuming continuity of temperature and heat flux at the interfaces, the description is simply based on the sequential application of transfer matrices of the type given in equation (1), each one depending on the thickness and thermophysical properties of the pertaining layer.

2.2. Two simple lumped electrical models

Two widely used models [6] of a homogeneous wall based on the electro-thermal analogy are recalled in the following. The 2R1C or T model, is a first-order model with two equal resistances and a capacitance connected as sketched in figure 1a. The lumped parameters are the thermal resistance per unit surface $R_L = L/\lambda$ and the thermal capacitance per unit surface $C_L = \rho c_p L$. In this case temperatures and heat fluxes on the two boundary faces can still be related by a matrix that reads

$$\begin{bmatrix} T_0 \\ q_0 \end{bmatrix} = \begin{bmatrix} 1 + Z_R/Z_C & 2Z_R + Z_R^2/Z_C \\ 1/Z_C & 1 + Z_R/Z_C \end{bmatrix} \begin{bmatrix} T_L \\ q_L \end{bmatrix},$$

(3)

with $Z_R = R_L/2$ and $Z_C = 1/(j\omega C_L)$.

By explicitly writing the frequency dependence in equation (3), it is easily seen that, in the adiabatic case, the lumped-parameter value of $F_{\text{dec}}$ coincides with the small-value expression found in section 2.1, whereas a small difference occurs in the general case where one obtains

$$T_0 = T_L \left(1 + j\frac{\omega R_L C_L}{2}\right) + q_L \left(R_L + j\frac{\omega R_L^2 C_L}{4}\right).$$

(4)

This shows that the lumped-parameter T representation of a wall gives very similar but not exactly identical results to a small-value approximated expression of the full analytical representation.
Another employed model is a 1R2C model, or Π model, that is a second-order model including two equal capacitances and a resistance connected as sketched in figure 1b.

\[
\begin{bmatrix}
    T_0 \\
    q_0
\end{bmatrix} = \begin{bmatrix}
    1 + Z_R/Z_C & Z_R \\
    2/Z_C + Z_R/Z_C^2 & 1 + Z_R/Z_C
\end{bmatrix} \begin{bmatrix}
    T_L \\
    q_L
\end{bmatrix},
\]

where now \( Z_R = R_L \) and \( Z_C = 2/(j\omega C_L) \). As in the exact case, both the determinant of the transfer matrix in equation (3) and in equation (5) are one. Also for the Π model the value of the decrement factor in the adiabatic case is \( F_{\text{dec}} = 1 + j\omega RC/2 \).

### 2.3. Higher-order lumped electrical models

Higher-order Taylor expansions of the hyperbolic functions appearing in the transfer matrix in equation (1) give rise to more accurate lumped-parameter approximations of the behavior of the multilayer wall. Correspondingly, more complex electrical circuits can be devised to model one-dimensional heat conduction (see, e.g., refs. [10, 22]).

### 3. Comparison of model predictions

The accuracy of the approximated models introduced in the previous section will be now analyzed by comparing their predictions with the ones of the full matrix treatment. As anticipated, the comparison will deal with the effect that the order of layers and their distribution have on the overall wall performance. At first, a few significant results will be derived by means of the exact analytical model. Then the issue will be addressed within the lumped-parameter schemes.

#### 3.1. Exact model

##### 3.1.1. Layer order

Let us consider a double-layer wall composed by a conducting layer (identified by the subscript \( c \) in the following) and an insulating layer (subscript \( i \)). The matrix treatment of the problem immediately shows that changing the order of the layers affects the resulting inner temperature for a given outer temperature, being the matrix product non commutative. Indeed, for the two alternate arrangements, using equation (1) it is possible to write

\[
\begin{bmatrix}
    T_0 \\
    q_0
\end{bmatrix} = \begin{bmatrix}
    C_c C_i + \frac{Y_c}{Y} S_i S_c & \frac{C_c S_i}{Y_c} + \frac{C_i S_c}{Y_i} \\
    C_s Y_i + C_s Y_c & C_c C_i + \frac{Y_c}{Y} S_i S_c
\end{bmatrix} \begin{bmatrix}
    T_L \\
    q_L
\end{bmatrix},
\]

or

\[
\begin{bmatrix}
    T_0 \\
    q_0
\end{bmatrix} = \begin{bmatrix}
    C_c C_i + \frac{Y_c}{Y} S_i S_c & \frac{C_c S_i}{Y_c} + \frac{C_i S_c}{Y_i} \\
    C_s Y_i + C_s Y_c & C_c C_i + \frac{Y_c}{Y} S_i S_c
\end{bmatrix} \begin{bmatrix}
    T_L \\
    q_L
\end{bmatrix}.
\]

It can be observed that the layer arrangement affects only the diagonal elements. If one is interested in the temperature decrement factor, under adiabatic conditions \( F_{\text{dec}} = C_c C_i + \frac{Y_c}{Y} S_i S_c \) and therefore the different complex ratios \( Y_c/Y_i \) and \( Y_i/Y_c \) are at the origin of the different wall behaviors in the two situations. A few examples are shown in the following. Let us consider typical values of the thermophysical parameters of concrete (\( \lambda_c = 0.81 \text{W/(m·K)} \), \( D_c = 5.96 \cdot 10^{-7} \text{m}^2/\text{s} \)) and polystyrene, (\( \lambda_i = 0.034 \text{W/(m·K)} \), \( D_i = 11.55 \cdot 10^{-7} \text{m}^2/\text{s} \)). Figure 2a shows the ratio \( |T_L/T_0| = |F_{\text{dec}}|^{-1} \) for the two alternate arrangements of an insulating and of a conducting layer (CI, conductor in the outer layer, or IC, insulator in the outer layer) when they have equal thickness (\( \alpha = 0.5 \)) or when the conducting layer thickness is 90% (\( \alpha = 0.9 \)) of the total wall extension \( L \). This comparison shows that data reported in literature on the layer order influence [13, 17] are easily explained within our
model that indicates how arranging an insulating layer closer to the heat source (red curves) is always more effective than starting with a conducting layer (black curves) to reduce temperature oscillations. It is also apparent that order is more important than thickness and that using a thin insulating layer in IC configuration (red dashed line in figure 2a) is more effective than using a five times larger insulating layer in CI configuration (black full line in figure 2a). It is possible
to calculate the time delay introduced by the bilayer wall in the same conditions: this is shown in figure 2b suggesting that time delay is also significantly affected by layer order. In this case a more critical role of the insulation thickness is observed. For smaller values of $L_c$, however, IC configuration provide increased time delays in addition to larger temperature attenuation.

3.1.2. Layer splitting  Recent works [13, 17] have shown the possibility that alternating insulator and conducting layers by reducing their thicknesses, while keeping their total amount constant, could improve the insulation capabilities of a wall. We investigate here this issue. Figure 3a shows that halving the thickness of individual layers in an IC configuration, employing an ICIC arrangement, can be effective to significantly increase the wall attenuation. However this is true only above a given critical thickness of the conducting layer, below which halving layer thickness results instead in a worsening of the wall insulating properties. For the example shown in figure 3a (where $L_i = 0.2L_c$) this critical value is $L_c=13.4$ cm.

Also the time delay between $T_L$ and $T_0$ is affected by the layer distribution, as shown in figure 3b, mainly above $L_c=6.0$ cm for the example shown here. The critical thickness that makes layer halving detrimental for insulation is plotted as a function of the ratio $L_i/L_c$ in figure 4 that shows how this value becomes larger as long as smaller percentages of insulation are employed.

3.2. $T$ and $Π$ models
The effect that layer order and distribution have on temperature reduction and delay will be investigated in this paragraph in the cases in which the approximate $T$ and $Π$ models are employed. The adiabatic hypothesis is again applied.

It is generally found that the qualitative behavior is reproduced with the lumped-parameter models, but that there are quantitative discrepancies that can be large. For this reason, in
Figure 3. (a) Modulus of the inverse of the decrement factor versus the conducting layer thickness for IC or ICIC configuration, where the second configuration refers to layers with halved thickness; (b) Time delay in hours versus the conducting layer thickness for IC or ICIC configuration where the second configuration refers to layers with halved thickness.

Figure 4. Value of the minimum thickness of the conducting layer below which halving it worsens the wall insulation capability. The plot is made as a function of the relative amount of insulating and conducting layer. IC and ICIC configurations are compared.

the following only error plots will be presented, omitting graphs similar to the ones shown in figures 2-4, as approximate models provide trends similar to those indicated by the exact model.

3.2.1. Layer order Figures 5a and b display the percent error on the inverse modulus of the decrement factor arising from the use of the lumped-parameter models: it can be observed that, while both models provide very similar results for CI configurations, where the attenuating behavior of the wall is less effective, they largely differ in the case where the decrement factor is more significant, i.e. in IC configurations. In such cases, the T model can be satisfactory up to \( L_c = 0.2 - 0.3 \) m (depending on the relative ratio \( L_i/L_c \)), while the II model validity ceases for thinner walls. If also the delay estimation is considered, however (see figures 6a and 6b) the indication is that the use of approximate models can be acceptable if the layers are even thinner.
Figure 5. Percent error on the prediction of the inverse modulus of the decrement factor obtained with T and Π models for (a) \( L_c/L = 0.5 \) and (b) \( L_c/L = 0.9 \). In (b) black dashed and red dashed curves are almost completely superimposed.

Figure 6. (a) Percent error on the prediction of the delay obtained with T and Π models for (a) \( L_c/L = 0.5 \) and (b) \( L_c/L = 0.9 \). (e.g. approximately smaller than 0.10-0.15 m if the accepted error threshold is set at 10%).

3.2.2. Layer splitting A relevant result derived in section 3.1.2 is the determination of a critical conductor thickness below which the performances of a double-layer wall (of IC type) worsen if both of them are split in two equal parts and alternated. It is shown in figures 7a and 7b that, using lumped-parameter models, such critical value is determined with percent errors that can be large, especially when using the Π model. In both cases the largest errors arise for small values of the ratio \( L_i/L_c \) that imply, as shown in figure 3, larger values of the critical thickness \( L_c \). In this latter cases, as expected, less accurate descriptions of the wall behavior are obtained in the lumped-parameter schemes.

4. Conclusions
This work analyzed the behavior of multilayer walls in sinusoidal regime as far as layer order and distribution are concerned. Both exact and approximate equivalent electrical models were
Figure 7. (a) Percent error on the prediction of the T model for the estimation of the minimum thickness of the conducting layer below which halving the thickness worsens the wall insulation capability; (b) The same as in (a) but for the Π model.

employed and their predictions were compared. The conclusion turns out to be that approximate models are usually sufficient to capture the trends of the different behaviors of multilayer walls. Concerning quantitative comparisons, however, they can largely fail especially for larger thicknesses. In such cases more than one electrical equivalent network can be necessary to model a single layer, and the question can be how useful lumped parameter models still prove being the exact treatment not only more accurate but, paradoxically enough, even simpler than approximate electrical models.

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