ON LINKS WITH KHOVANOV HOMOLOGY OF SMALL RANKS

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Abstract. We classify all links whose Khovanov homology have ranks no greater than 8, and all three-component links whose Khovanov homology have ranks no greater than 12, where the coefficient ring is \( \mathbb{Z}/2 \). The classification is based on the previous results of Kronheimer-Mrowka [KM11], Batson-Seed [BS15], Baldwin-Sivek [BS18], and the authors [XZ19a].

1. Introduction

Khovanov homology [Kho00] is a combinatorially defined invariant for oriented links in \( S^3 \). For a commutative ring \( R \) and an oriented link \( L \), the Khovanov homology assigns a bi-graded \( R \)-module \( \text{Kh}(L; R) \).

The detection properties of Khovanov homology have been studied intensively in the past decade. In 2011, Kronheimer and Mrowka [KM11] proved that Khovanov homology detects the unknot. Since then, many other detection results of Khovanov homology have been obtained. It is now known that Khovanov homology detects the unlink [BS15, HN13], the trefoil [BS18], the Hopf link [BSX18], the forest of unknots [XZ19a], the splitting of links [LS19], and the torus link \( T(2, 6) \) [Mar20].

In this paper, we classify all the links \( L \) such that rank \( \mathbb{Z}/2 \text{Kh}(L; \mathbb{Z}/2) \leq 8 \), and all the 3-component links \( L \) such that rank \( \mathbb{Z}/2 \text{Kh}(L; \mathbb{Z}/2) \leq 12 \). Since the rank of the Khovanov homology does not depend on the orientation, it makes sense to refer to rank \( \mathbb{Z}/2 \text{Kh}(L; \mathbb{Z}/2) \) without orienting \( L \).

Let \( \text{Khr}(L; R) \) be the reduced Khovanov homology of \( L \) with coefficient ring \( R \). By [Shu14, Corollary 3.2.C], we have

\[
\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 2 \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2).
\]

Moreover, the parity of \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) \) is invariant under crossing changes of \( L \). Therefore, \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) \) has the form \( 4k + 2 \) \( (k \in \mathbb{Z}) \) if \( L \) is a knot, and is a multiple of 4 if \( L \) is a link with at least two components.

On the other hand, it is well-known that if \( L \) is a link with \( n \) components, then \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) \geq 2^n \) (see, for example, [XZ19a, Equation (1)]). As a consequence, if \( L \) is a knot such that \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) \leq 8 \), then \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) = 2 \) or 6, hence \( L \) is an unknot or a trefoil by [KM11, BS18]. If \( L \) is a 2-component link such that \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) \leq 8 \), then \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) = 4 \) or 8. If \( L \) is a 3-component link such that \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) \leq 12 \), then \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) = 8 \) or 12. If \( L \) has at least 4 components, then \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) \geq 16 \). In [XZ19a], the authors have classified all the \( n \)-component links \( L \) with \( \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) = 2^n \).

Therefore, the new content of this paper is given by the following two results:

**Theorem 1.1.** Suppose \( L \) is a 2-component link in \( S^3 \), then

\[
\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 8
\]
Figure 1. The link L4a1.  

Figure 2. The link L6n1.

if and only if \( L \) is isotopic to the link L4a1 in the Thistlethwaite link table, which is the link given by Figure 1, or its mirror image.

**Remark 1.2.** In [XZ19a, Corollary 1.4], the authors proved that Khovanov homology (together with the bi-grading) distinguishes an oriented link whose underlying unoriented link is isotopic to L4a1. Theorem 1.1 is a stronger version of that result.

**Theorem 1.3.** Suppose \( L \) is a three-component link in \( S^3 \), then

\[
\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 12
\]

if and only if \( L \) is isotopic to the link L6n1 in the Thistlethwaite link table, which is the link given by Figure 2, or its mirror image.

Combining Theorem 1.1, Theorem 1.3, and the results in [KM11,BS18,XZ19a], we have the following two corollaries.

**Corollary 1.4.** Suppose \( L \subset S^3 \) is a link such that \( \text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) \leq 8 \), then \( L \) is isotopic to one of the following:

- the unlink with at most 3 components;
- the left-handed or right-handed trefoil;
- the Hopf link;
- the connected sum of two Hopf links;
- the disjoint union of a Hopf link and an unknot;
- the link L4a1 or its mirror image.

**Corollary 1.5.** Suppose \( L \subset S^3 \) is a link with three components such that \( \text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) \leq 12 \), then \( L \) is isotopic to one of the following:

- the unlink with 3 components;
- the connected sum of two Hopf links;
- the disjoint union of a Hopf link and an unknot;
- the link L6n1 or its mirror image.

2. Preliminaries

Let \( L \) be a link \( L \) in the (framed) solid torus \( S^1 \times D^2 \), its annular instanton Floer homology \( \text{AHII}(L) \) is defined in [Xie18], and the theory is further developed by [XZ19b,XZ19a]. This section reviews several results from [Xie18,XZ19b,XZ19a] that will be used later.
The annular instanton Floer homology is a \( \mathbb{Z} \)-graded complex vector space, and the grading is called the f-grading. We use \( \text{AHI}(L, i) \) to denote the component of \( \text{AHI}(L) \) with f-degree \( i \). For each \( i \in \mathbb{Z} \), we have
\[
\text{AHI}(L, i) \cong \text{AHI}(L, -i). \tag{2.1}
\]
We recall the following definition from [XZ19b, Definition 1.5].

**Definition 2.1.** A properly embedded, connected surface \( S \subset S^1 \times D^2 \) is called a meridional surface if \( \partial S \) is a meridian of \( S^1 \times D^2 \).

We have the following two results.

**Theorem 2.2** ([XZ19b, Theorem 8.2]). Given a link \( L \) in \( S^1 \times D^2 \), let \( S \) be a meridional surface that intersects \( L \) transversely. Let \( g \) be the genus of \( S \), and let \( n = |S \cap L| \). Suppose \( S \) minimizes the value of \( (2g + n) \) among meridional surfaces that intersect \( L \) transversely, then we have
\[
\text{AHI}(L, \pm (2g + n)) \neq 0,
\]
and
\[
\text{AHI}(L, i) = 0
\]
for all \( |i| > 2g + n \).

**Proposition 2.3** ([XZ19b, Corollary 8.4]). Let \( L \) be a link in \( S^1 \times D^2 \), let \( n \) be a positive integer. Then \( L \) is isotopic to the closure of a braid with \( n \) strands if and only if the top f-grading of \( \text{AHI}(L) \) is \( n \) and \( \text{AHI}(L, n) \cong \mathbb{C} \).

If \( K \subset S^3 \) is a knot, we will use \( N(K) \) to denote the open tubular neighborhood of \( K \). Suppose \( K \) is an unknot, then \( S^3 - N(K) \) is a solid torus. Choose the framing of \( S^3 - N(K) \) such that the preferred longitude of \( S^3 - N(K) \) is a meridian of \( K \). Then for every link \( L \) that is disjoint from \( N(K) \), we can take the annular instanton Floer homology \( \text{AHI}(L) \) by viewing \( L \) as a link in \( S^3 - N(K) \). Notice that in this case, a meridional surface of \( S^3 - N(K) \) induces a Seifert surfaces of \( K \) and vice versa.

The following proposition establishes a relation between the annular instanton Floer homology and the reduced Khovanov homology.

**Proposition 2.4.** Suppose \( L \subset S^3 \) is a link, \( U \subset L \) is a component of \( L \) that is an unknot, and let \( p \in U \) be a base point on \( U \). Let \( L_0 = L - U \), then \( L_0 \) is a link in the solid torus \( S^3 - N(U) \). We have
\[
\text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2) \geq \dim_{\mathbb{C}} \text{AHI}(L_0).
\]

**Proof.** By Kronheimer-Mrowka’s spectral sequence [KM11, Theorem 8.2], we have
\[
\dim_{\mathbb{C}} \hat{F}(L, p; \mathbb{C}) \leq \dim_{\mathbb{C}} \text{Khr}(\bar{L}, \bar{p}; \mathbb{C}),
\]
where \( (\bar{L}, \bar{p}) \) is the mirror image of \( (L, p) \), and \( \hat{F} \) is the reduced singular instanton Floer homology introduced in [KM11]. By [Kho00, Corollary 11], we have
\[
\dim_{\mathbb{C}} \text{Khr}(\bar{L}, \bar{p}; \mathbb{C}) = \dim_{\mathbb{C}} \text{Khr}(L, p; \mathbb{C}).
\]
By the universal coefficient theorem,
\[
\dim_{\mathbb{C}} \text{Khr}(L; \mathbb{C}) \leq \text{rank}_{\mathbb{Z}/2} \text{Khr}(L; \mathbb{Z}/2).
\]
By [XZ19a, Proposition 2.6], \( \hat{F}(L, p; \mathbb{C}) \cong \text{AHI}(L_0) \). Therefore the proposition is proved. \( \square \)
Let $U_k \subset S^1 \times D^2$ be the unlink with $k$ components, and let $K_i \subset S^1 \times D^2$ be
the link given by $S^1 \times \{p_1, \ldots, p_i\}$. Let $U_k \sqcup K_i$ be the disjoint union of $U_k$ and
$K_i$ such that $U_k$ is included in a 3-ball disjoint from $K_i$. By [Xie18, Example 4.2,
Proposition 4.3],
\[
\text{AHI}(U_k \sqcup K_i) \cong \mathbb{C}^{g_k}_{(0)} \otimes (\mathbb{C}_{(1)} \oplus \mathbb{C}_{(-1)}) \otimes \mathbb{C},
\]
where the subscripts represent the f-gradings.

**Proposition 2.5** ([XZ19a, Proposition 4.3]). Suppose $L \subset S^1 \times D^2$ is an oriented
link such that every component of $L$ has winding number 0 or $\pm 1$. Assume there
are $k$ components with winding number 0 and $l$ components with winding number
$\pm 1$, then we have
\[
\dim \text{AHI}(L, i) \geq \dim \text{AHI}(U_k \sqcup K_i, i)
\]
for all $i \in \mathbb{Z}$.

**Proposition 2.6** ([Xie18, Section 4.4]). Suppose $L_1, L_2$ are two links in $S^1 \times D^2$.
If $L_1$ and $L_2$ are homotopic to each other in $S^1 \times D^2$, then
\[
\dim \text{AHI}(L_1, i) \equiv \dim \text{AHI}(L_2, i) \mod 2
\]
for all $i \in \mathbb{Z}$.

## 3. Proof of Theorem 1.1

Let $l \geq 2$ be an integer, recall that the $l$–strand braid group $B_l$ has the following
presentation:
\[
B_l = \langle \sigma_1, \ldots, \sigma_{l-1} | \sigma_l \sigma_{l+1} \sigma_i = \sigma_{l+1} \sigma_l \sigma_i, \sigma_l \sigma_j = \sigma_j \sigma_l \ (j - i \geq 2) \rangle
\]
The reduced Burau representation (see [Bir74]) is a group homomorphism
\[
\rho : B_l \to GL(l - 1, \mathbb{Z}[t, t^{-1}])
\]
which maps $\sigma_i$ to
\[
\begin{pmatrix}
I_{l-2} & 1 & 0 & 0 \\
t & -t & 1 \\
0 & 0 & 1 & I_{l-1-2}
\end{pmatrix},
\]
where the matrix is truncated appropriately when $i = 1$ or $l - 1$. Notice that
\[
\det(\rho(\sigma_i)) = -t \text{ for all } i,
\]
hence $\det(\rho(\beta)) = \pm t^a$ for all $\beta \in B_l$.

**Definition 3.1.** Suppose $\beta \in B_l$ is a braid, let $U \subset S^3$ be an unknot, let $\hat{\beta} \subset
S^1 \times D^2 \cong S^3 - N(U)$ be the braid closure of $\beta$. Define $U \cup \hat{\beta}$ to be the union of $U$
and $\hat{\beta}$ under the standard framing of $S^3 - N(U)$.

**Remark 3.2.** The link L4a1 is isotopic to $U \cup \hat{\sigma}_1$, where $\sigma_1$ is a generator of $B_2$.

**Theorem 3.3** ([Mor85, Theorem 3]). Let $\beta \in B_l$, and let $L = U \cup \hat{\beta}$. Suppose $\hat{\beta}$ is
connected, then the multi-variable Alexander polynomial $\Delta_L(x, t)$ of $L$ is given by
\[
\Delta(x, t) = \det(xI - \rho(\beta)(t)),
\]
where $x$ and $t$ are the variables corresponding to $U$ and $\hat{\beta}$ respectively, and the sign
"\(\equiv\)" means that the two sides are equal up to a multiplication by $\pm x^a t^b$. 

Remark 3.4. The ambiguity in the notation \( \pm \) is necessary because the multi-variable Alexander polynomial (before normalization) is only well-defined up to a multiplication by \( \pm x^a t^b \).

We also need the following result:

**Theorem 3.5 ([Tor53]).** Suppose \( L = K_1 \cup K_2 \) is a 2-component link, and let \( \Delta_L(x, y) \) be the multi-variable Alexander polynomial of \( L \) where \( x \) and \( y \) are the variables corresponding to \( K_1 \) and \( K_2 \) respectively. Then we have

\[
\Delta_L(x, 1) = \frac{1 - x^l}{1 - x} \Delta_{K_1}(x),
\]

where \( \Delta_{K_1}(x) \) is the Alexander polynomial of \( K_1 \), and \( l = |\text{lk}(K_1, K_2)| \) is the absolute value of the linking number of \( K_1 \) and \( K_2 \).

The next lemma is an immediate corollary of the results in [Dow18, OS08], and is essentially contained in the proof of [XZ19a, Lemma 6.1]. We state it here as a separate lemma for future reference.

**Lemma 3.6.** Suppose \( L \) is a link with \( n \) components, let \( \Delta_L(x_1, \cdots , x_n) \) be the (multi-variable) Alexander polynomial of \( L \). Let \( p \in L \) be a base point.

1. If \( n = 1 \), then the sum of the absolute values of the coefficients of \( \Delta_L(x_1) \) is less than or equal to \( \text{rank}_Q \text{Khr}(L, p; \mathbb{Q}) \).

2. If \( n \geq 2 \), then the sum of the absolute values of the coefficients of \( (x_1 - 1) \cdots (x_n - 1) \Delta_L(x_1, \cdots , x_n) \) is less than or equal to \( 2^{n-1} \text{rank}_Q \text{Khr}(L, p; \mathbb{Q}) \).

**Proof.** We use \( \widehat{\text{HFK}} \) and \( \widehat{\text{HFL}} \) to denote the Heegaard knot Floer homology [OS04, Ras03] and the link Floer homology [OS08] respectively. The link Floer homology was originally defined for \( \mathbb{Z}/2 \)-coefficients, and was generalized to \( \mathbb{Z} \)-coefficients in [Sar11]. It is known that

\[
\text{rank}_Q \widehat{\text{HFK}}(L; \mathbb{Q}) = \text{rank}_Q \widehat{\text{HFL}}(L; \mathbb{Q}),
\]

but \( \widehat{\text{HFL}}(L; \mathbb{Q}) \) carries more refined gradings.

By [Dow18, Corollary 1.7], we have

\[
\text{rank}_Q \widehat{\text{HFK}}(L; \mathbb{Q}) \leq 2^{n-1} \text{rank}_Q \text{Khr}(L; \mathbb{Q}).
\]

By [OS08, Equation (1)], the multi-graded Euler characteristics of \( \widehat{\text{HFL}}(L; \mathbb{Q}) \) satisfy

\[
\chi(\widehat{\text{HFL}}(L; \mathbb{Q})) = \begin{cases} \Delta_L(x_1) & \text{if } n = 1, \\ (x_1 - 1) \cdots (x_n - 1) \Delta_L(x_1, \cdots , x_n) & \text{if } n \geq 2, \end{cases}
\]

hence the result is proved. \( \square \)

Now let \( l \geq 2 \) be an integer, let \( \beta \in B_i, L = U \cup \beta \), and let \( \Delta_L(x, y) \) be the multi-variable Alexander polynomial of \( L \) such that \( x \) and \( y \) are the variables corresponding to \( U \) and \( \beta \) respectively. By (3.1), we have

\[
\Delta_L(x, y) \doteq (-1)^{l-1} \det(\rho(\beta_2)(y)) + f_1(y)x + \cdots + f_{l-2}(y)x^{l-2} + x^{l-1} = \pm y^a + f_1(y)x + \cdots + f_{l-2}(y)x^{l-2} + x^{l-1}
\]

(3.5)
for some \(a \in \mathbb{Z}, f_i \in \mathbb{Z}[y, y^{-1}].\) By Theorem 3.5,
\[
\Delta_L(x, 1) \triangleq (1 + x + x^2 + \cdots + x^{l-1})\Delta_U(x) = 1 + x + x^2 + \cdots + x^{l-1}.
\]
Therefore in Equation (3.5), we have \(f_i(1) = 1\) for all \(i,\) and the sign in front of the term \(y^a\) is positive.

A 2-component link \(K_1 \cup K_2\) is called exchangeably braided, if both \(K_1, K_2\) are unknots, and for each \((i, j) \in \{(1, 2), (2, 1)\},\) the knot \(K_i\) is a braid closure with axis \(K_j.\) The concept of exchangeably braided links was introduced and studied by Morton in [Mor85]. If we further assume that \(L\) is exchangeably braided, then by symmetry and (3.5), we have
\[
\Delta_L(x, y) = x^b + g_1(x)y + \cdots + g_{l-2}(x)y^{l-2} + y^{l-1}
\]
for some \(b \in \mathbb{Z}\) and \(g_i(x) \in \mathbb{Z}[x, x^{-1}]\) with \(g_i(1) = 1.\)

**Lemma 3.7.** Let \(L\) be a mutually braided link with linking number \(l \geq 3,\) let \(\Delta_L(x, y)\) be the multi-variable Alexander polynomial of \(L.\) Then the sum of the absolute values of the coefficients of \((x - 1)(y - 1)\Delta_L(x, y)\) is at least 12.

**Proof.** Let \(f_i(y)\) be as in (3.5), and set \(f_0(y) = y^a\) and \(f_{l-1}(y) = 1.\) Then we have
\[
(x - 1)(y - 1)\Delta_L(x, y)
\]
\[
\triangleq (y - 1)\left(- y^a + (y^a - f_1(y))x + \cdots + (f_{i-3}(y) - f_{i-2}(y))x^{l-2}
\right)
\]
\[
+ (f_{i-2}(y) - 1)x^{l-1} + x^l
\]
\[
= - (y - 1)y^a + (y - 1)x^l + \sum_{i=1}^{l-1}(y - 1)(f_{i-1}(y) - f_i(y))x^i.
\]

We discuss three cases. If \(f_{i-1} = f_i\) for all \(i \in \{1, \cdots , l - 1\},\) then \(a = 0,\) and (3.5) gives
\[
\Delta_L(x, y) \triangleq 1 + x + \cdots + x^{l-1},
\]
which contradicts (3.6) and the assumption that \(l \geq 3.\)

If there is exactly one element \(i \in \{1, \cdots , l - 1\}\) such that \(f_{i-1}(y) \neq f_i(y),\) then by (3.5), we have
\[
\Delta_L(x, y) \triangleq y^a + y^a x + \cdots + y^a x^{l-1} + x^i + \cdots + x^{l-1},
\]
which also contradicts (3.6) and the assumption that \(l \geq 3.\)

If there exist at least two elements \(i \in \{1, \cdots , l - 1\}\) such that \(f_{i-1}(y) \neq f_i(y),\) notice that \(f_{i-1}(1) - f_i(1) = 1 - 1 = 0,\) hence \((y - 1)\) is a factor of \((f_{i-1}(y) - f_i(y)).\) Therefore for such an \(i,\) the sum of the absolute values of the coefficients of
\[
(y - 1)(f_{i-1}(y) - f_i(y))
\]
is even and strictly greater than 2, hence it is at least 4. The desired result then follows from (3.7).

The following lemma refines the proof above and obtains a necessary condition for attaining the lower bound in Lemma 3.7. This result will not be used in the proof of Theorem 1.1.
Lemma 3.8. Suppose $L$ is an exchangeably braided link with linking number $l \geq 3$, let $\Delta_L(x,y)$ be the multi-variable Alexander polynomial of $L$. If the sum of the absolute values of the coefficients of $(x-1)(y-1)\Delta_L(x,y)$ is equal to 12, then $l = 3$.

Proof. We use the same notation as in the proof of Lemma 3.7. If the sum of the absolute values of the coefficients of $(x-1)(y-1)\Delta_L(x,y)$ is equal to 12, then the proof of Lemma 3.7 indicates that there are exactly two elements $i \in \{1, \ldots, l-1\}$ such that $f_i - 1(y) \neq f_i(y)$. Therefore by (3.5), there exists $f(y) \in \mathbb{Z}[y, y^{-1}]$ and $0 \leq k_1 < k_2 \leq l-2$, such that $f(y) \neq 1$, $f(y) \neq y^a$, and

$$\Delta_L(x,y) \equiv y^a(1 + \cdots + x^{k_1}) + f(y)(x^{k_1+1} + \cdots + x^{k_2}) + x^{k_2+1} + \cdots + x^{l-1},$$

hence

$$\Delta_L(1,y) \equiv (1 + k_1)y^a + (k_2 - k_1)f(y) + (l - 1 - k_2).$$

On the other hand, by Theorem 3.5, we have

$$\Delta_L(1,y) \equiv 1 + y + \cdots + y^{1-1}$$

Since $l \geq 3$, comparing the two equations above gives $k_2 - k_1 = 1$. Hence

$$\Delta_L(x,y) \equiv y^a(1 + \cdots + x^{k_1}) + f(y)x^{k_1+1} + x^{k_1+2} + \cdots + x^{l-1}.$$ 

Since $\Delta_L(x,y) \equiv \Delta_L(x^{-1},y^{-1})$, and recall that $f(y) \neq 1$, $f(y) \neq y^a$, we have $l = 2m + 1$ for $m \in \mathbb{Z}$, and

$$\Delta_L(x,y) \equiv y^a(1 + \cdots + x^{m-1}) + f(y)x^m + x^{m+1} + \cdots + x^{2m}.$$ 

View $\Delta_L(x,y)$ as a Laurent polynomial of $x$ with coefficients in $\mathbb{Z}[y, y^{-1}]$, the equation above shows that there is at most one power of $x$ whose coefficient is not a monomial of $y$. Switching the roles of $x$ and $y$ and repeating the same argument, we conclude that there is at most one power of $y$ in $\Delta_L(x,y)$ whose coefficient is not a monomial of $x$. Therefore we must have $m = 1$ and $l = 3$. □

Combining (3.2), (3.4), Lemma 3.7, and Lemma 3.8, we obtain the following corollary, which may be of independent interest.

Corollary 3.9. Suppose $L$ is an exchangeably braided link with linking number $l \geq 3$, then $\text{rank}_\mathbb{Q} \text{HFK}(L; \mathbb{Q}) \geq 12$. Moreover, if $\text{rank}_\mathbb{Q} \text{HFK}(L; \mathbb{Q}) = 12$, then $l = 3$. □

Proof of Theorem 1.1. The “if” part of the theorem follows from a straightforward computation. Now suppose $L$ is a 2-component link such that

$$\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 8,$$ 

we prove that $L$ is isotopic to L4a1 or its mirror image.

Denote the two components of $L$ by $K_1$ and $K_2$. Batson-Seed’s inequality [BS15, Corollary 1.6] gives

$$\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) \geq \text{rank}_{\mathbb{Z}/2} \text{Kh}(K_1; \mathbb{Z}/2) \cdot \text{rank}_{\mathbb{Z}/2} \text{Kh}(K_2; \mathbb{Z}/2).$$

Since $\text{rank}_{\mathbb{Z}/2} \text{Kh}(K_1; \mathbb{Z}/2) \geq 2$, we have

$$\text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i; \mathbb{Z}/2) \leq 4.$$ 

By [Shu14, Corollary 3.2.C],

$$\text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i; \mathbb{Z}/2) = 1/2 \text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i; \mathbb{Z}/2) \leq 2.$$
The parity of $\text{rank}_{\mathbb{Z}/2} \text{Khr}(K; \mathbb{Z}/2)$ is invariant under crossing changes, therefore $\text{rank}_{\mathbb{Z}/2} \text{Khr}(K; \mathbb{Z}/2)$ is odd, hence it has to be 1. Kronheimer-Mrowka’s unknot detection theorem [KM11] then implies that both $K_1$ and $K_2$ are unknotted.

Pick a base point $p \in K_2$. We have
\[
\text{rank}_{\mathbb{Z}/2} \text{Khr}(L, p; \mathbb{Z}/2) = \frac{1}{2} \text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 4.
\]
By Proposition 2.4,
\[
\dim_{\mathbb{C}} \text{AHI}(K_1) \leq \text{rank}_{\mathbb{Z}/2} \text{Khr}(L, p; \mathbb{Z}/2) = 4,
\]
where $K_1$ is viewed as a knot in the solid torus $S^3 - N(K_2)$.

If $\text{lk}(K_1, K_2) = 0$, then $K_1$ is homotopic to the unknot in the solid torus. By Proposition 2.5 and Proposition 2.6, we have
\[
\dim_{\mathbb{C}} \text{AHI}(K_1; 0) \geq \dim_{\mathbb{C}} \text{AHI}(U_1; 0) = 2,
\]
and
\[
\dim_{\mathbb{C}} \text{AHI}(K_1; i) \equiv \dim_{\mathbb{C}} \text{AHI}(U_1; i) \equiv 0 \pmod{2}, \quad \text{for all } i.
\]
Therefore, by (2.1) and (3.9), $\text{AHI}(K_1)$ must be supported at $t$-degree 0. By Theorem 2.2, this implies $K_2$ bounds a disk that is disjoint from $K_1$, hence $L$ is the disjoint union of $K_1$ and $K_2$, which implies $L$ is the unlink. However, the unlink does not satisfy the assumption (3.8), which is a contradiction.

Therefore, we have $l = |\text{lk}(K_1, K_2)| > 0$, hence $K_1$ is homotopic to the closure $\hat{\beta}$ of an $l$-braid $\beta$ in the solid torus $S^3 - N(K_2)$. By Proposition 2.6, we have
\[
\dim_{\mathbb{C}} \text{AHI}(K_1; i) \equiv \dim_{\mathbb{C}} \text{AHI}(\hat{\beta}; i) \pmod{2},
\]
and by (the easy direction of) Proposition 2.3, we have
\[
\dim_{\mathbb{C}} \text{AHI}(\hat{\beta}; \pm l) = 1,
\]
\[
\dim_{\mathbb{C}} \text{AHI}(\hat{\beta}; \pm i) = 0 \text{ for all } i > l.
\]
Therefore (2.1) and (3.9) yield that
\[
\dim \text{AHI}(K_1; \pm l) = 1
\]
and
\[
\dim \text{AHI}(K_1; \pm i) = 0 \text{ for all } i > l.
\]
By Proposition 2.3, this implies $K_1$ is the closure of an $l$-braid in $S^3 - N(K_2)$. A similar argument shows that $K_2$ is the closure of an $l$-braid in $S^3 - N(K_1)$. Therefore $L$ is exchangeably braided.

By the universal coefficient theorem,
\[
\text{rank}_{\mathbb{Q}} \text{Khr}(L, p; \mathbb{Q}) \leq \text{rank}_{\mathbb{Z}/2} \text{Khr}(L, p; \mathbb{Z}/2) = \frac{1}{2} \text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 4.
\]
Therefore by Lemma 3.6 and Lemma 3.7, we have $l \leq 2$.

If $l = 1$, then $L$ is the Hopf link, which does not satisfy (3.8).

If $l = 2$, then $L = U \cup \hat{\sigma}_1^m$, where $\sigma_1 \in B_2$ is a generator of the braid group with 2 strands and $m \in \mathbb{Z}$. Since both components of $L$ are unknotted, we have $m = \pm 1$, hence $L$ is isotopic to L4a1 or its mirror image. □
4. Proof of Theorem 1.3

The “if” part of the theorem follows from a straightforward calculation. Now suppose \( L \) is a 3-component link with \( \text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 12 \), we prove that \( L \) isotopic to \( L6n1 \) or its mirror image.

Denote the three components of \( L \) by \( K_1, K_2, K_3 \). By Batson-Seed’s inequality [BS15, Corollary 1.6], we have

\[
\text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i \cup K_j; \mathbb{Z}/2) \leq \text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 12
\]

for all triples \((i, j, k)\) with \( \{i, j, k\} = \{1, 2, 3\} \). Since \( \text{rank}_{\mathbb{Z}/2} \text{Kh}(K_k; \mathbb{Z}/2) \geq 2 \), we have

\[
\text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i \cup K_j; \mathbb{Z}/2) \leq 6.
\]

We apply the same parity argument as before. By [Shu14, Corollary 3.2.C], we have

\[
\text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i \cup K_j; \mathbb{Z}/2) = \frac{1}{2} \text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i \cup K_j; \mathbb{Z}/2) \leq 3.
\]

Crossing changes do not change the parity of \( \text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i \cup K_j; \mathbb{Z}/2) \), hence \( \text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i \cup K_j; \mathbb{Z}/2) \) is even, therefore

\[
\text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i \cup K_j; \mathbb{Z}/2) \leq 2,
\]

and we have

\[
\text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i \cup K_j; \mathbb{Z}/2) = 2 \text{rank}_{\mathbb{Z}/2} \text{Kh}(K_i \cup K_j; \mathbb{Z}/2) \leq 4.
\]

By [XZ19a, Theorem 1.2], \( K_i \cup K_j \) is either a Hopf link or an unlink. In particular, \( |\text{lk}(K_i, K_j)| = 0 \) or 1. Hence after permuting the labels of the components, we may assume that \( |\text{lk}(K_3, K_1)| = |\text{lk}(K_3, K_2)| \).

Pick a base point \( p \in K_3 \). By [Shu14, Corollary 3.2.C], we have

\[
\text{rank}_{\mathbb{Z}/2} \text{Kh}(L, p; \mathbb{Z}/2) = \frac{1}{2} \text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 6.
\]

View \( K_1 \cup K_2 \) as a link in the solid torus \( S^3 - N(K_3) \), Proposition 2.4 gives

\[
\dim_{\mathbb{C}} \text{AHI}(K_1 \cup K_2) \leq \text{rank}_{\mathbb{Z}/2} \text{Kh}(L, p; \mathbb{Z}/2) = 6. \quad (4.1)
\]

We discuss two cases.

**Case 1.** \( |\text{lk}(K_3, K_1)| = |\text{lk}(K_3, K_2)| = 0 \). Then \( K_1 \cup K_2 \) is homotopic to the unlink in the solid torus \( S^3 - N(K_3) \). Hence by Proposition 2.5 and Proposition 2.6, we have

\[
\dim_{\mathbb{C}} \text{AHI}(K_1 \cup K_2, 0) \geq \dim_{\mathbb{C}} \text{AHI}(\mathcal{U}_2, 0) = 4,
\]

and

\[
\dim_{\mathbb{C}} \text{AHI}(K_1 \cup K_2, i) \equiv \dim_{\mathbb{C}} \text{AHI}(\mathcal{U}_2, i) \equiv 0 \mod 2, \text{ for all } i.
\]

By (2.1) and (4.1), \( \text{AHI}(K_1 \cup K_2) \) must be supported at f-degree 0. By Theorem 2.2, this implies \( K_1 \cup K_2 \) is spilt from \( K_3 \), so \( L \) is either the unlink or the disjoint union of a Hopf link and an unknot. In both cases we have \( \text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 8 \), which contradicts the assumption.

**Case 2.** \( |\text{lk}(K_3, K_1)| = |\text{lk}(K_3, K_2)| = 1 \). Recall that the link \( K_2 \subset S^1 \times D^2 \) is defined by \( K_2 = S^1 \times \{p_1, p_2\} \), and it can be viewed as a link in \( S^3 - N(K_3) \).
The assumption above implies that $K_1 \cup K_2$ is homotopic to $K_2$. By Proposition 2.5 and Proposition 2.6, we have
\[
\dim \text{AHI}(K_1 \cup K_2; i) \geq \dim \text{AHI}(K_2; i) \tag{4.2}
\]
and
\[
\dim \text{AHI}(K_1 \cup K_2; i) \equiv \dim \text{AHI}(K_2; i) \pmod{2} \tag{4.3}
\]
for all $i$. Recall that $\text{AHI}(K_2) \cong C_2 \oplus C_{-2} \oplus C_0^2$, \hspace{1cm} (4.4)
where the subscripts represent the f-gradings. It then follows from (2.1), (4.1), (4.3), (4.4) that
\[
\text{AHI}(K_1 \cup K_2) \cong C_2 \oplus C_{-2} \oplus C_4^2.
\]

By Proposition 2.3, this implies $K_1 \cup K_2$ is the closure of a 2-braid $\beta \in B_2$ in the solid torus $S^3 - N(K_3)$. Since $K_1 \cup K_2$ is either the unlink or the Hopf link, $\beta$ is either trivial or $\sigma_1^{\pm 2}$, where $\sigma_1$ is a generator of $B_2$. If $\beta$ is trivial, then $L$ is the connected sum of two Hopf links, hence $\text{rank}_{\mathbb{Z}/2} \text{Kh}(L; \mathbb{Z}/2) = 8$, which contradicts the assumption. Therefore $\beta = \sigma_1^{\pm 2}$, hence $L$ is isotopic to $L_6n1$ or its mirror image, and the result is proved.

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