DC conductivity with external magnetic field in hyperscaling violating geometry

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We investigate the holographic DC conductivity of (2+1) dimensional systems while considering hyperscaling violating geometry in bulk. We consider Einstein-Maxwell-Dilaton system with two gauge fields and Liouville type potential for dilaton. We also consider axionic fields in bulk to introduce momentum relaxation in the system. We apply an external magnetic field to study the response of the system and obtain analytic expressions for DC conductivity, Hall angle and (thermo)electric conductivity.

I. INTRODUCTION

String theory provides us a valuable tool to investigate strongly coupled gauge theories using AdS/CFT duality (holography) [1–3]. The duality establishes a connection between strongly coupled gauge theory in d-dimensions on the boundary and its weakly coupled gravity dual in (d+1)-dimensional bulk spacetime. Many important phenomena of nuclear physics, condensed matter physics and high $T_c$ superconductors are being explored using this duality [4–9].

Recently, considerable interest is seen to study realistic condensed matter systems using holography. The investigation of different phases of strongly coupled systems requires new holographic models. Significant results have been obtained in this area after including momentum dissipation term in holographic models. Realistic examples of strongly coupled systems have finite DC conductivity either due to the presence of impurity or as a result of broken translational invariance. One can introduce momentum relaxation in holographic systems through various ways; by introducing impurity in holographic set-up [10, 11] or by introducing spatial source field which breaks translational symmetry [12, 13]. Also the breaking of diffeomorphism invariance using massive gravity term in the bulk theory, results in finite DC conductivity [14–18].

Further, efforts are being made to study strongly coupled condensed matter systems near quantum critical points using holography. These critical points are realized in the boundary system by opting for anisotropic scaling between temporal and spatial directions in the gravity set-up. Although the introduction of anisotropy results in breaking of Lorentz invariance, the metric remains scale invariant. The system with this anisotropy while possessing scaling symmetry is known as Lifshitz-like geometry,

$$\bar{x} \rightarrow \lambda \bar{x} \quad t = \lambda^z t. \quad (1)$$

where $\bar{x}$ is spatial coordinate, $t$ is temporal coordinate and $z > 1$ is known as the dynamical critical exponent.

Several efforts are being made to realize this type of geometry (Lifshitz-like) in gravity set-up. The most common way is working with Einstein-Maxwell-Dilaton (EMD) theories in the gravity system. This geometry was first introduced by [19] and found wide range of application in analyzing thermodynamical and hydrodynamical aspects of strongly coupled systems [20–28]. Generalized system with warped metric is also used for the detailed studies of realistic condensed matter system [29–36].

$$ds_{d+2}^2 = r^{2\theta} \left( -r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 \sum_{i=1}^{d} dx_i^2 \right), \quad (2)$$

where $\theta$ is known as hyperscaling violating parameter in d-dimensions.

In this work, we investigate the DC conductivity and (thermo)electric conductivity of (2+1) dimensional systems with the hyperscaling violating term for Lifshitz-like geometry. Hence, we consider two different gauge fields in gravity set-up, one field will introduce Lifshitz-like geometry while other introduces finite charge density. Linear axionic fields have been introduced in the system to break translational invariance and obtain finite DC conductivity. Introducing

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The metric ansatz for the above action (3) is given by Lifshitz-like, hyperscaling violating black-brane solution as in [37,40]. Using the gravity solution, the parameters of given model are related by, \[29\]. We have studied these RG flow equations and extract the transport coefficients of the boundary theory.

The external magnetic field is introduced in the set-up in the given form,\[43, 46, 49\]. We have studied these coupled differential equations reduces to first order ordinary differential equations \[43, 46, 49\]. We have used Wilsonian RG (renormalization group) flow approach \[57\]. The advantage of this approach is that second order correlation function, one can use various approaches \[41–56\]. In this work, we have simplified the calculation while presence of external magnetic field and multiple gauge fields. To read transport coefficients for boundary theory using Wilsonian RG flow approach \[57\]. The main motivation of our present work. We discuss the temperature dependence of various conductivities is also investigated. The concluding section contains the detailed discussion and summary of the work done.

II. EMD SYSTEM

Let us consider the EMD system with two gauge fields and axionic fields. The gravity action is given by,

\[ S = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 + V(\phi) - \frac{1}{4} \sum_{i=1}^{2} e^{\lambda_i \phi} F_i^2 - e^{\lambda_0 \phi} \sum_{i=1}^{2} \partial_i^2 \right), \]  

(3)

where \( F_1 \) and \( F_2 \) are two \( U(1) \) gauge fields, the role of first gauge field is to break Lorentz-invariance and introduce Lifshitz-like geometry while second gauge field introduced the charge in the gravitational set-up. \( V(\phi) \) is dilaton potential and \( \chi_i \) are the axionic fields. In this work, we take the potential of form, \( V(\phi) = -2\Lambda e^{\lambda_0 \phi} \) for further calculation.

The Einstein equation from the above action is given by,

\[ R_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \Lambda e^{\lambda_0 \phi} g_{\mu\nu} + \frac{1}{2} e^{\lambda_i \phi} (\partial_\mu \chi_i \partial_\nu \chi_i) + \sum_{i=1}^{2} \frac{1}{2} e^{\lambda_i \phi} (F_i \mu \nu F_i^\mu F_i^\nu - \frac{1}{4} F_i^2 g_{\mu\nu}). \]  

(4)

The matter fields equations of motion are obtained as,

\[ \Box \phi = \frac{1}{2} \lambda_3 \sum_{i=1}^{2} (\partial \chi_i) + \frac{1}{4} \sum_{i=1}^{2} \lambda_i e^{\lambda_i \phi} F_i^2 + 2\lambda_0 \Lambda e^{\lambda_0 \phi}, \]  

(5)

\[ 0 = \nabla_\mu \left( e^{\lambda_1 \phi} F_1^{\mu \nu} \right), \quad 0 = \nabla_\mu \left( e^{\lambda_2 \phi} F_2^{\mu \nu} \right), \]  

(6)

\[ 0 = \nabla_\mu \left( e^{\lambda_3 \phi} \nabla_\nu \chi_i \right). \]  

(7)

The metric ansatz for the above action \[3\] is given by Lifshitz-like, hyperscaling violating black-brane solution as in \[27, 29, 30\],

\[ ds^2 = r^\theta \left( -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 (dx^2 + dy^2) \right). \]  

(8)

The external magnetic field is introduced in the set-up in the given form,

\[ A_2 = A_2(r) dt + Bxdy. \]  

(9)

The axion fields are linear in spatial direction given by, \( \chi_1 = \alpha x \) and \( \chi_2 = \alpha y \) where \( \alpha \) is considered as the strength of the momentum dissipation.

Using the gravity solution, the parameters of given model are related by, \[29\]

\[ \gamma = \sqrt{(\theta + 2)(\theta + 2z - 2)}, \quad \lambda_0 = \frac{-\theta}{\gamma}, \quad \lambda_1 = \frac{-4 + \theta}{\gamma}, \]  

\[ \lambda_2 = \frac{\theta + 2z - 2}{\gamma}, \quad \lambda_3 = \frac{\gamma}{\theta + 2}, \quad q_1 = \sqrt{2(z - 1)(\theta + z + 2)}, \]  

(10)

(11)
with \( \Lambda = \frac{-1}{2}(\theta + z + 1)(\theta + z + 2) \) and \( \phi = \gamma \log r \).

From the temporal component of the gauge field equation, we obtain,
\[
J^0_\ell = q_\ell = r^{-\ell+3}e^{\lambda_\gamma}A^\ell_\ell'_t,
\]
(12)

considering \( q_\ell \) as the charges of two gauge fields. Here the role of \( q_1 \) is the used to introduce Lifshitz-scaling whereas \( q_2 \) is interpreted as the black hole charge.

The black hole factor with an external magnetic field (B), mass (m) and charge (q2) along with axionic strength(\( \alpha \)) is given by,
\[
f(r) = 1 - \frac{m}{r^{\theta+\ell+2}} + \frac{q_2^2 + B^2}{2(\theta + 2)(\theta + z)}r^{2(\theta + z + 1)} + \frac{\alpha^2}{2(\theta + 2)(\theta + z - 2)r^{\theta+2z}}.
\]
(13)

The Hawking temperature of black hole is obtained from the expression given below,
\[
T = \frac{r^\ell h'}{4\pi f(r)},
\]
(14)

Thus,
\[
T = \frac{z + 2 + \theta}{4\pi}r^\ell h_\ell - \frac{q_2^2}{8\pi(2 + \theta)}r^\ell_{\ell+2+2\gamma} - \frac{\alpha^2}{4\pi(2 + \theta)}r^\ell_{\ell+\theta}.
\]
(15)

The constraint from the gravity solution is that every point in space-time follows the null energy condition (NEC) i.e., \( T_{\mu\nu}V^\mu V^\nu \geq 0 \), where \( V^\mu \) is the light like vector. Thus, the allowed values of ‘\( z \)’ and ‘\( \theta \)’ consistent with the gravity solution are [29],
\[
(2 + \theta)(2z - 2 + \theta) \geq 0,
\]
(16)
\[
(z - 1)(2 + z + \theta) \geq 0.
\]
(17)

Later, we shall see that the consistency of coupled equations demand that \( \theta = z - 1 \) and both the conditions require \( z \geq 1 \). Further at \( z = 2 \), the solution of black hole factor [13] is not valid as the last term changes sign. So we shall consider the range \( 1 \leq z < 2 \), which corresponds to \( 0 < \theta < 1 \).

To study the response of system, we introduce the following perturbations,
\[
\delta A_{1i}(t,r) = \int_\infty^{\infty} \frac{d\omega}{2\pi} a_{1i}(r)e^{-i\omega t},
\]
(18)
\[
\delta A_{2i}(t,r) = \int_\infty^{\infty} \frac{d\omega}{2\pi} a_{2i}(r)e^{-i\omega t},
\]
(19)
\[
\delta \chi_i(t,r) = \int_\infty^{\infty} \frac{d\omega}{2\pi} b_i(r)e^{-i\omega t},
\]
(20)
\[
\delta g_{1i}(t,r) = \int_\infty^{\infty} \frac{d\omega}{2\pi} r^{\theta+2}\chi_i(t,r)e^{-i\omega t}.
\]
(21)

The coupled linearized equations of motion for the fields are obtained as,
\[
0 = (r^{z-3-\theta}fa'_{1i})' + \frac{\omega^2a_{1i}}{r^{3+z+\theta}} + \omega q_1h'_{1i},
\]
\[
0 = (r^{z-1-\theta}fa'_{2i})' + \frac{\omega^2a_{2i}}{r^{3-\theta}} + q_2h'_{1i} + \epsilon_{ij}i\frac{\omega Bh_{ij}}{f^{3-z}},
\]
\[
0 = (r^{z-2}fb'_{ij})' + \frac{\omega^2b_{ij}}{f^{3-z}} - \frac{i\omega \epsilon_{ij}}{f^{3-z}}
\]
(22)

where \( \epsilon_{ij} \) is the Levi-Civita tensor. The constraint equation of the set-up is given by,
\[
0 = \omega r^{5-z+\theta}h'_{1i} + \omega q_1a_{1i} + \omega q_2a_{2i} + i\alpha r^{5-z}fb'_{ij} - q_2Bh_{1i} - Bfa'_{2i}r^{3z-1+\theta}.
\]
(23)

whereas metric perturbation equation is obtained in the given form,
\[
0 = (r^{z+3+\theta}h'_{1i})' - q_1r^{z-1-\theta}a'_{1i} - q_2r^{z-1-\theta}a'_{2i} + \frac{(\alpha^2r^{-\theta-2z+2} + B^2r^{2z-4})h_{1i}}{f} + \frac{i\omega r^{-\theta-2z+2}h_{1i}}{f} + \epsilon_{ij}i\frac{\omega Br^{2z-4}a_{2i}}{f}.
\]
(24)
III. HOLOGRAPHIC APPROACH

To study the transport properties of (2+1) dimensional boundary system, one has to solve coupled equations of motion (22), (23) and (24) using the standard procedure as mentioned in [39, 52]. However in this work we follow the approach introduced by [57] and developed through various studies [49, 53, 57]. Here, DC conductivities are obtained from first order RG flow equations in the near horizon limit. To study the conductivity of a system we simply apply Ohm’s Law as given below.

\[ J = \sigma E. \]  

(25)

Applying holographic techniques to obtain transport coefficients we use the Onsager relation \((J = \tau X)\) in the matrix form as shown,

\[
\left( \begin{array}{cc} J_{1i} & J_{1j} \\ J_{2i} & J_{2j} \end{array} \right) = \tau \left( \begin{array}{cc} X_{1i} & X_{1j} \\ X_{2i} & X_{2j} \end{array} \right),
\]

(26)

where ‘X_i’ are the linear independent sources and ‘J_i’ are the responses of system. \(\tau\) matrix are the coefficients evaluated in the near horizon limit. The detailed discussion and application of the formalism is presented in [49, 53, 57]. Also, we could use the following notation

\[
\| J_1 \| = \| \tau \| \| X_1 \|
\]

(27)

And \(\tau = JX^{-1}\) can be expressed as,

\[
\tau = \left( \| J_1 \| \| X_1 \| \right)^{-1}
\]

(28)

The linearized equations of motion (22) and (24) can be put in the matrix form as,

\[
\tau = \begin{bmatrix}
-\tau_{z-3-\theta} f a_{1i}' & -\tau_{z+\theta} f a_{2i}' \\
-\tau_{z-3+\theta} f a_{2i}' & -\tau_{z+\theta} f a_{1i}' \\
-\tau_{z-5-\theta} f b_{1i}' & -\tau_{z+\theta} f b_{1i}' \\
-\tau_{z-5+\theta} f b_{1i}' & -\tau_{z+\theta} f b_{1i}'
\end{bmatrix}^{-1}
\]

(29)

Now, taking the radial derivative and substituting the equations of motion we get,

\[
\tau' = \begin{bmatrix}
\frac{\omega^2 a_{2i}}{\tau_{z+\theta} f} + \frac{\omega^2 h_{1i}}{\tau_{z-\theta} f} + q_1 h_{1i}' & q_1 r^z-\theta a_{2i}' + \frac{\omega B r^z-\theta a_{2j}'}{\tau_{z-\theta} f} - \frac{\omega \omega r^z-2z+2 b_{1i}}{\tau_{z-\theta} f} \\
-\frac{\omega}{\tau_{z+\theta} f} & 0 & 0 & 0 & 0 \\
0 & -\frac{\omega}{\tau_{z+\theta} f} & 0 & 0 & 0 \\
0 & 0 & -\frac{\omega}{\tau_{z+\theta} f} & 0 & 0 \\
0 & 0 & 0 & -\frac{\omega}{\tau_{z+\theta} f} & 0
\end{bmatrix} \tau
\]

Further simplifying we get,

\[
\tau' = \begin{bmatrix}
\frac{-i\omega}{\tau_{z+\theta} f} & 0 & 0 & 0 & 0 \\
0 & \frac{-i\omega}{\tau_{z+\theta} f} & 0 & 0 & 0 \\
0 & 0 & \frac{-i\omega}{\tau_{z+\theta} f} & 0 & 0 \\
0 & 0 & 0 & \frac{-i\omega}{\tau_{z+\theta} f} & 0 \\
0 & 0 & 0 & 0 & \frac{-i\omega}{\tau_{z+\theta} f}
\end{bmatrix} \tau - \tau
\]

(30)
Multiplying the above metric by black hole factor \( f(r) \) and taking its near horizon limit, where \( f(r_h) = 0 \) we obtain \( \tau_h \).

Considering the constraint equation (23) in the matrix form as,

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
iB & \alpha & -i\omega & \tau_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_2 \\
-\frac{q_2\omega}{\omega} \\
0
\end{pmatrix}
\]

we obtain,

\[
\begin{align*}
iB\tau_{11} + \alpha\tau_{21} - i\omega\tau_{31} &= -q_1, \\
iB\tau_{12} + \alpha\tau_{22} - i\omega\tau_{32} &= -q_2, \\
iB\tau_{13} + \alpha\tau_{23} - i\omega\tau_{33} &= 0, \\
iB\tau_{14} + \alpha\tau_{24} - i\omega\tau_{34} &= -\frac{q_2B}{\omega}.
\end{align*}
\]

To maintain the consistency of the equations, we fixed \( \theta = z - 1 \) and the \( \tau_h \) matrix is given as,

\[
\tau_h = \begin{pmatrix}
r_h^{-3z} & 0 & 0 \\
0 & r_h^{-3+3z} & 0 \\
0 & 0 & \frac{r_h^{-2z+4}}{i\omega}
\end{pmatrix}
\]

By substituting \( \tau_h \) in equation (29), we find the flow equation in the near horizon limit,

\[
\begin{align*}
(-r_h^{-2} f a_{1i})' &= r_h^{-3z} i\omega a_{1i}, \\
(-r_h^{4z-2} f a_{2i})' &= r_h^{-3+3z} i\omega a_{2i} + \epsilon_{ij} iB r_h^{-3+3z} h_{ij}, \\
(-r_h^{5z-2} f b_i) &= r_h^{-2z+4} i\omega b_i + \alpha r_h^{-2z+4} h_{ii}, \\
(-r_h^{4z} h_{ti}) &= (iB r_h^{-3z} + q_2) a_{2i} + r_h^{4z-2z} \alpha b_i + q_1 a_{1i} - \frac{(iB r_h^{-3z} + \alpha r_h^{4z-2z} + q_2 B) h_{ij}}{\omega}.
\end{align*}
\]

Near horizon limit of equation (24) is given by,

\[
(B^2 r_h^{2z-4} + \alpha^2 r_h^{-3z+4}) h_{ti} = q_1 a_{1i} + q_2 a_{2i} + \epsilon_{ij} i\omega B a_{2j} r_h^{2z-4} - i\alpha \omega r_h^{-3z+3} b_i.
\]

Simplifying the above expression using the flow equations (37) we obtain the expression for the metric perturbation as,

\[
\left. h_{ti}\right|_{r=r_h} = -i\omega \frac{(q_1 r_h^{-3z-1}) a_{1i} + \epsilon_{ij} (q_2 r_h^{-3z-1} + iB r_h^{2z-4}) a_{2j} + i\alpha \omega r_h^{-3z+3} b_i}{(B^2 r_h^{2z-4} + \alpha^2 r_h^{-3z+3}) - iq_2 B r_h^{-2z-4} - iq_2 B r_h^{-3z+4}}.
\]

### A. DC conductivity

Using gauge field equation (22), we get the expression for conserved currents for first gauge field as,

\[
J_{1i} = -r_h^{-3z-\theta} f a_{1i}' - q_1 h_{ti}.
\]

On substituting the equation (37) the expression takes the form,

\[
J_{1i} = r_h^{-3z-\theta} i\omega a_{1i} - q_1 h_{ti}.
\]

Now from equation (39) neglecting the second term (i.e. axion perturbation part) we obtain the DC conductivity using,

\[
\sigma_{ij} = \frac{\partial J_i}{\partial E_j}, \quad \text{where} \quad E_j = i\omega a_j.
\]
Thus we obtain,

\[ \sigma_{xx}^{11} = \sigma_{yy}^{11} = r_h^{-3-z} + \frac{q_1^2 (B^2 r_h^{-3+\frac{z}{2}} + \alpha^2 r_h^{-\frac{3z}{2} + 1})}{(B^2 r_h^{z-4} + \alpha^2 r_h^{-3+z+1})^2 + B^2 q_2^2 r_h^{2z-2}}, \] (43)

Also, we have some mixed terms for DC conductivity, where charges of both the fields effect the conductivity as shown,

\[ \sigma_{xx}^{12} = \sigma_{yy}^{12} = \frac{q_1 q_2 \alpha^2 r_h^{-\frac{3z}{2} + 2}}{(B^2 r_h^{-z+1} + \alpha^2 r_h^{-3+z+1})^2 + B^2 q_2^2 r_h^{2z-2}}, \] (44)

\[ \sigma_{xy}^{11} = -\sigma_{yx}^{11} = \frac{q_1^2 q_2 B r_h^{-2}}{(B^2 r_h^{z-3} + \alpha^2 r_h^{-4+3})^2 + B^2 q_2^2 r_h^{2z-2}}, \] (45)

\[ \sigma_{xy}^{12} = -\sigma_{yx}^{12} = q_1 B \frac{B^2 r_h^{2z-8} + \alpha^2 r_h^{-z-1} + q_2^2 r_h^{-2-2z}}{(B^2 r_h^{z-3} + \alpha^2 r_h^{-3+z+1})^2 + B^2 q_2^2 r_h^{2z-2}}. \] (46)

Since these expressions are quite complex to analyse we numerically study the dependence of conductivities on magnetic field and momentum relaxation term for two different values of the dynamical exponent, \( z = 1 \) and \( z = 4/3 \) in Fig.1 to Fig.4.

FIG. 1: Variation of \( \sigma_{xx}^{11} \) with \( B \) and \( \alpha \) for \( z = 1 \)(left) and \( z = 4/3 \)(right)

FIG. 2: Variation of \( \sigma_{xy}^{11} \) with \( B \) and \( \alpha \) for \( z = 1 \)(left) and \( z = 4/3 \)(right)

Keeping \( z = 1 \) reduces the geometry to RN-AdS and we observe trivial DC conductivity flow. However, for \( z \neq 1 \) non-trivial dependence of DC conductivity on magnetic field and momentum relaxation strength is seen. We also observe a discontinuity while considering \( z = 1 \) (\( \theta = 0 \)) case, as \( \sigma_{xx}^{11} = r_h^{-4} \). On the other hand for RN-AdS black hole the conductivity is given is a constant term given by, \( \sigma_{xx}^{11} = 1 \). On the other hand for RN-AdS black hole the conductivity is given is a constant term given by, \( \sigma_{xx}^{11} = 1 \).

Similarly, for the second gauge field we can evaluate the conductivity accordingly. The conserved current for the boundary theory is obtained using equation (22),

\[ J_{2i} = -r_h^{3z-1+\theta} f d_{2i} - q_2 h_{ti}, \] (47)
On substituting equation (37) we obtain the modified expression for $J_{2i}$ as,

$$J_{2i} = r_h^{-3+3z}i\omega a_{2i} + \epsilon_{ij}Br_h^{-3+3z}h_{tj} - q_2hq_{ti}. \quad (48)$$

Then the DC conductivity is evaluated concerning the second gauge field.

$$\sigma_{22}^{xx} = \sigma_{22}^{yy} = \frac{\alpha^2 r_h^{2z-4} [B^2 + r_h^{-5z+7} (q_2^2 h_r^{-z-1} + \alpha^2)]}{B^2 q_2^2 r_h^{2z-2} + (B^2 r_h^{2z-4} + \alpha^2 r_h^{-3z+3})^2}, \quad (49)$$

$$\sigma_{xy}^{22} = -\sigma_{yx}^{22} = \frac{q_2 Br_h^{4z-8} [B^2 + r_h^{-5z+7} (q_2^2 r_h^{-z-1} + 2\alpha^2)]}{B^2 q_2^2 r_h^{2z-2} + (B^2 r_h^{2z-4} + \alpha^2 r_h^{-3z+3})^2}, \quad (50)$$

The above expressions are complicated to interpret. So we study the dependence of these conductivities on magnetic field and $\alpha$ using plots as shown in Fig.5 to Fig.6.

Also from the plots of different DC conductivities (for both the fields) we observe at fixed momentum relaxation strength conductivity $\sigma_{22}^{xx}$ shows a monotonic dependence on the external magnetic field whereas in $\sigma_{xy}^{22}$ this behavior is not seen.

Let us consider the limit $B \rightarrow 0$.

$$\sigma_{xx}^{22} = r_h^{3z-3} + r_h^{2z-4} \frac{q_2^2}{\alpha^2}, \quad \sigma_{xy}^{22} = 0 \quad \text{ (51)}$$

From the above expression it is noted that electric conductivity obeys inverse Matthiessen’s rule given by,

$$\sigma_{DC} = \sigma_Q + \sigma_D \quad \text{ (52)}$$

where $\sigma_Q$ is the charge conjugation symmetric part and $\sigma_D$ is the momentum dissipation part.

The temperature dependence of conductivity is governed by equation (15) and $T \sim r_h^{-3}$. Here we observe the following scaling in the DC conductivity,
FIG. 5: Variation of $\sigma_{xx}^{22}$ with B and $\alpha$ for $z = 1$(left) and $z = 4/3$(right)

FIG. 6: Variation of $\sigma_{xy}^{22}$ with B and $\alpha$ for $z = 1$(left) and $z = 4/3$(right)

i For $z = 1$, $\sigma_{xx}^{22} \sim 1 + \frac{q^2}{\alpha^2 T^2}$

ii For $z = 4/3$, $\sigma_{xx}^{22} \sim T^{3/4} + \frac{q^2}{T \alpha^2}$

iii For $z \to 2$, $\sigma_{xx}^{22} \sim T^{3/2} + \frac{q^2}{\alpha^2}$

In our holographic model, we are able to capture the low temperature behavior of the DC conductivity obeying Fermi-Liquid law ($\sigma_{DC} \sim \frac{1}{T^2}$) for $z = 1$ along with a constant term. This behavior changes to unconventional metallic behavior ($\sigma_{DC} \sim \frac{1}{T}$) as we increase the Lifshitz scaling to $z = 4/3$ and becomes constant in the limiting case $z \to 2$. Thus, there is non-trivial dependence of conductivity on temperature for hyperscaling range ($1 < z < 2$).

B. Halls Angle

We can obtain the expression for Hall angle using equations (49) and (50). Thus,

$$\tan \theta_H = \frac{\sigma_{xy}^{22}}{\sigma_{xx}^{22}}$$

$$\theta_H = \frac{B q_2 r_h^{2z-4} [B^2 + r_h^{7-5z} (2\alpha^2 + q_2^2 r_h^{-z-1})]}{\alpha^2 [B^2 + r_h^{7-5z} (\alpha^2 + q_2^2 r_h^{-z-1})]}$$

From the above expression it is observed that, $\theta_H \propto \frac{B q_2 r_h^{2z-4}}{\alpha^2}$ as the terms in the bracket is consider as a geometric quantity [52].

Comparing the result with that of DC conductivity Hall angle consists of only dissipation part ($\sigma_D$), unlike DC conductivity which is the combination of two different terms (shown in equation [51]). This is responsible for the
FIG. 7: Variation of $\theta_H$ with B and $\alpha$ for $z = 1$(left) and $z = 4/3$(right)

presence of different scaling in strange metals\cite{58}. We plot the dependence of Hall angle on the magnetic field applied and the strength of momentum relaxation in Fig.7 for fixed $q_2$. Let us consider the temperature dependence of Hall angle,

i For $z = 1$, $\theta_H \sim 1/T^2$

ii For $z = 4/3$, $\theta_H \sim 1/T$

iii For $z \to 2$, $\theta_H \sim 1/T^0$

For $z = 1$, we observe the temperature dependence is same as measured in cuprates \cite{59}. However the behavior changes to $\theta_H \sim 1/T$ with the non-trivial scaling $z \neq 4/3$. Further $\theta_H$ reduces to a constant for $z \to 2$. We show the temperature and magnetic field dependence on the Hall angle from plots given in Fig.8 and Fig.9.

FIG. 8: $\theta$ vs T at $\alpha = 0.1$(blue),0.5(red),1(green) for $z = 1$(left) and $z = 4/3$(right)

FIG. 9: $\theta$ vs B at $T = 0.5$(blue),1(red),1.5(green) for $z = 1$(left) and $z = 4/3$(right)
C. Thermoelectric conductivity

Next, we evaluate thermoelectric conductivity using heat current expression.

\[ Q_i = -4\pi T g_{xx} \delta_{ti}, \quad \text{and} \quad \alpha_{ij} = \frac{\partial Q_i}{\partial E_j} \]  

(55)

We get the following expression for thermoelectric conductivity depending on external magnetic field and dynamical exponent for our model as shown in Fig. 10 and Fig. 11.

\[ \alpha_{xx}^{22} = \alpha_{yy}^{22} = \frac{4\pi r^{-3z+3} q z \alpha^2}{(B^2 r^{-2z-4} + \alpha^2 r^{-3z+3})^2 + B^2 q^2 r^{-2z-2}}, \]  

(56)

\[ \alpha_{xy}^{22} = -\alpha_{yx}^{22} = \frac{4\pi B r^{-1+1} (B^2 r^{-4z-8} + \alpha^2 r^{-2z-1} + q^2 r^{-2z-2})}{(B^2 r^{-2z-4} + \alpha^2 r^{-3z+3})^2 + B^2 q^2 r^{-2z-2}}, \]  

(57)

In the limiting case \( B \to 0 \), we obtain,

\[ \alpha_{xx}^{22} = \frac{4\pi q z \alpha^3}{\alpha^2}, \quad \alpha_{xy}^{22} = 0 \]  

(58)

Thus, the thermoelectric conductivity also shows non-trivial dependence on hyperscaling and unconventional temperature dependence.

FIG. 10: Variation of \( \alpha_{xx}^{22} \) with \( B \) and \( \alpha \) for \( z = 1 \)(left) and \( z = 4/3 \)(right)

FIG. 11: Variation of \( \alpha_{xy}^{22} \) with \( B \) and \( \alpha \) for \( z = 1 \)(left) and \( z = 4/3 \)(right)
D. Seebeck Coefficient

The generation of transverse electric field in the system is given by the thermoelectric power (Seebeck coefficient). Using the results of the conductivity we obtain,

$$ S = \frac{\sigma_{2x}^{zz}}{\sigma_{zz}^{xx}} = \frac{4\pi r_h^{-5z+7} q_2}{[B^2 + r_h^{-5z+7}(q_2^2 r_h^{-z-1} + \alpha^2)]}. \quad (59) $$

FIG. 12: Variation of $S$ with $B$ and $\alpha$ for $z = 1$ (left) and $z = 4/3$ (right)

FIG. 13: $S$ vs $T$ at $\alpha = 0.5$ (blue), 1 (red), 1.5 (green) for $z = 1$ (left) and $z = 4/3$ (right)

FIG. 14: $S$ vs $B$ at $T = 0.5$ (blue), 1 (red), 1.5 (green) for $z = 1$ (left) and $z = 4/3$ (right)

The variation of Seebeck coefficient with applied magnetic field and momentum relaxation strength is shown in Fig. 12. The result from experiments suggest that at high temperature Seebeck coefficient remains constant. The dependence of Seebeck coefficient on model parameters is shown in Fig.13 and Fig.14. We observe at different temperature the behavior of Seebeck coefficients does not change appreciably for non-trivial scaling. The temperature scaling of the coefficient is still unclear from experimental results[60].
E. Thermal conductivity

Using the results for thermoelectric and DC conductivity, we can also obtain the thermal conductivity \[^{[49]}\]. The thermal conductivity for non-zero magnetic field can be obtained using the relation given below,

\[
\begin{pmatrix}
< J_i > \\
< Q_i > 
\end{pmatrix} = \begin{pmatrix}
\sigma_{ij} \\
\alpha_{ij}T
\end{pmatrix} \begin{pmatrix}
E_j \\
(\nabla_j T)/T
\end{pmatrix}
\]

(60)

Considering the thermal current \(Q_x = 0\) and \(E_y = 0\) we obtain the expression for the thermal conductivity using,

\[
\bar{\kappa}_{xx}^{22} = \frac{T(\sigma_{xx}^{22})^2}{\sigma_{xx}^{22} - \sigma_{xx}^{22}(0)}, \quad \text{and} \quad \bar{\kappa}_{xy}^{22} = \frac{T\alpha_{xy}^{22}\sigma_{xx}^{22}}{\sigma_{xx}^{22}}
\]

(61)

where \(\sigma_{xx}^{22}(0)\) is the electric conductivity for \(Q_x = 0\) (vanishing heat currents). Thus, we obtain,

\[
\bar{\kappa}_{xx}^{22} = \frac{16\pi^2 r_h^{5+4z} T(B^2 r_h^{5z} + r_h^7)}{B^4 r_h^{10z} + r_h^{14} + B^2 r_h^{6+4z}(q_2^2 + 2r_h^{1+z} \alpha^2)}
\]

(62)

\[
\bar{\kappa}_{xy}^{22} = -\bar{\kappa}_{yx}^{22} = \frac{16\pi^2 TB^2 r_h^{8+6z}}{B^4 r_h^{10z} + r_h^{14} + B^2 r_h^{6+4z}(q_2^2 + 2r_h^{1+z} \alpha^2)}
\]

(63)

Variation of thermal conductivity with different model parameters is shown in Fig.15 and Fig.16 which also shows non-trivial dependence on hyperscaling.

Let us discuss the limiting case \(B \to 0\),

\[
\bar{\kappa}_{xx}^{22} = \frac{16\pi^2 T r_h^{4z-2}}{\alpha^2}, \quad \bar{\kappa}_{xy}^{22} = 0,
\]

(64)

F. Lorenz ratio

To complete the discussion, we check the expression for Lorenz ratio. First we obtain the Hall Lorentz ratio using \[^{[53]}\],

\[
L = \frac{\bar{\kappa}_{xy}}{T\sigma_{xy}} = \frac{16\pi^2 r_h^{2z+8}}{B^2 r_h^{6z} + q_2^2 r_h^{6} + 2\alpha^2 r_h^{2z+7}}.
\]

(65)

The expression for the Lorenz ratio is,

\[
\bar{L} = \frac{\bar{\kappa}_{xx}}{T\sigma_{xx}} = \frac{16\pi^2 r_h^{2z+1} \left( B^2 r_h^{5z} + \alpha^2 r_h^{7} \right)}{\alpha^2 \left( B^2 r_h^{6z} + q_2^2 r_h^{6} + \alpha^2 r_h^{2z+7} \right)}.
\]

(66)
FIG. 16: Variation of $\kappa_{xy}^{22}$ with $B$ and $\alpha$ for $z = 1$ (left) and $z = 4/3$ (right).

FIG. 17: Variation of $L$ with $B$ and $\alpha$ for $z = 1$ (left) and $z = 4/3$ (right).

The explicit dependence of Lorenz ratio on the dynamical scaling and momentum relaxation strength is shown in Fig. 17.

For $B \to 0$,

$$\bar{L} = \frac{16\pi^2 r_{h}^{2z+2}}{q_{h}^2 + \alpha^2 r_{h}^{2 + 1}}.$$  \hfill (67)

We obtain the Lorenz ratio ratio at zero temperature for vanishing magnetic field keeping $z = 1$ as,

$$\bar{L} = \frac{r_{xx}}{T \sigma_{xx}} |_{T, B \to 0} = \frac{4 \pi^2}{3} \left( 1 + \frac{\alpha^2}{\sqrt{\alpha^4 + 12q_{h}^2}} \right).$$  \hfill (68)

The given expression indicates, at $B = 0$ the WF law is valid and we obtain Fermi-liquid type ground state for $z = 1$.

The temperature dependence of Lorenz ratio is shown in Fig. 18 for different momentum relaxation strength. According to Wiedemann-Franz (WF) law the Lorenz ratio is constant for normal metals. Since our results show explicit dependence on temperature and also on Lifshitz scaling, the WF Law is violated in this model. The temperature dependence can not be extracted in a simple manner because of the interplay of different parameters in the system. We also plot the variation of the Lorenz ratio with the magnetic field in Fig. 19.

IV. CONCLUSIONS AND SUMMARY

In this paper, we have investigated the DC transport of holographic systems with Lifshitz-like geometry and hyperscaling violation. The geometry is dual to non-relativistic ($z \neq 1$) condensed matter systems under the applied external magnetic field. We considered the near horizon limit of linearized equations of motion and calculated DC conductivity, thermoelectric and thermal conductivity analytically.

We introduced the perturbations in both the gauge fields and obtained the expressions for the transport coefficients. The behavior of transport coefficients is depicted by numerical plots for different values, $z = 1$ and $z = 4/3$, of the
dynamical exponent. While we have non-trivial scaling for \( z = 4/3 \), the geometry reduces to RN-AdS black hole for \( z = 1 \). Dependence of DC transport on magnetic field and momentum relaxation strength is also studied while considering different Lifshitz scaling. The pattern of different type of conductivities (\( \sigma^{ij}_{xx}, \alpha^{ij}_{xx} \) and \( \tilde{\kappa}^{ij}_{xx} \)) are quite similar showing monotonic dependence on the magnetic field but this behavior is absent in \( \sigma^{ij}_{xy} \) etc.

We also obtained the Hall angle, Seebeck coefficient and Lorenz ratio for the system and plotted them as a function of temperature and magnetic field. Hall angle shows \( 1/T^2 \) in temperature behavior as given in equation (54) for \( z = 1 \) but it changes to \( 1/T \) for \( z = 4/3 \). The Wiedemann-Franz law is violated for our holographic model depicting unconventional metallic behavior. However, at zero temperature and \( B \to 0 \) the Fermi-liquid behavior is obtained for \( z = 1 \). Seebeck coefficient and Lorenz ratio also showed non-trivial dependence on the hyperscaling parameter. This unconventional dependence of transport coefficient on temperature can be useful to study the strange metal phenomenon [52, 56].

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