Polynomial-time algorithms for coding across multiple unicasts

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Abstract

We consider the problem of network coding across multiple unicasts. We give, for wired and wireless networks, efficient polynomial time algorithms for finding optimal network codes within the class of network codes restricted to XOR coding between pairs of flows.

I. INTRODUCTION

In this paper we consider network coding across multiple unicasts, using the class of pairwise XOR codes introduced in [4] for wired networks. This class of codes includes “reverse carpooling” and two-flow “star coding” for wireless networks [3]. We give efficient polynomial time algorithms for finding optimal network codes within this class on wired and wireless networks.

II. PROBLEM SETUP

A. Multiple unicasts problem on wired network

$K$ unicast sessions are transmitted over a network represented as a directed graph $G = (N, L)$ of $N = |N|$ nodes and $M = |L|$ links. Network coding is limited to XOR coding between pairs of uncoded flows. Each session $c = 1, \ldots, K$, demands a communication rate $r_c$. Each link $(a, b) \in L$ has a capacity denoted $C_{ab}$, which, if greater than $\max_c r_c$, is set to $\max_c r_c$. The total incoming capacity and total outgoing capacity of each node is upper bounded by $\bar{C}$. A solution for a given set of demanded rates $\{r_c\}$ is an assignment of values to variables

$$\{\nu_{ab}^c, \nu_{va}^c, \pi_{ib}^{(c,v')}, \rho_{ab}^{(c,v')}, \gamma_i^{(c,v,v')}, \sigma_i^{(c,v')}, \eta_i^{(c,v')}\}$$

satisfying:

$$\sum_b (\nu_{ib}^v + \nu_{ib}^{cv'}) + \sum_{c' \neq c, v'} \gamma_i^{(cv,v')} = \sum_a \nu_{ai}^c + \sum_{v'} \nu_{va}^{cv'} + \sum_{c'} \gamma_i^{cv'} + \left\{ \begin{array}{ll} r_c & i = v = s_c \\ 0 & v \neq s_c, d_c, i \neq d_c \end{array} \right. \quad (1)$$

$$\sum_b \pi_{ib}^{(c,v')j} + \sigma_i^{(c,v')j} = \sum_a \pi_{ai}^{(c,v')j} + \sum_{v,v'} \gamma_i^{(c,v,v')} \quad (2)$$

$$\sum_b \rho_{ib}^{(c,v')j} + \eta_i^{(c,v')j} = \sum_a \rho_{ai}^{(c,v')j} + \sigma_i^{(c,v')j} \quad (3)$$

$$\sum_b (\nu_{ab}^c + \nu_{va}^c) + \sum_{c,c',v} (\rho_{ab}^{(c,v')j} + \pi_{ib}^{(c,v')j}) + \sum_{c,c'} \pi_{ab}^{(c,v')j} \leq C_{ab} \quad (4)$$
We show how to find a solution for the problem with rates \( \{r_c\} \) if there exists a solution for the problem with slightly higher rates \( \{(1 + 2\epsilon)r_c\} \) for any \( \epsilon > 0 \).

We define a number of queues at each node \( i \) which can be interpreted as follows:

- \( U^{cv}_i \): uncoded session \( c \) data previously at node \( v \)
- \( P_a^{(c,e,j)} \): data from sessions \( c, c' \) coded at node \( j \) meant for both sinks
- \( P_a^{(c,e,j)} \): data from sessions \( c, c' \) coded at node \( j \) meant for sink \( d_c' \)
- \( R_a^{(c,e,j)} \): remedy for session \( c \) data that has been coded with \( c' \) data at node \( j \)

Data can be transmitted on a link \( (a, b) \in \mathcal{L} \) from queue \( U^{cv}_a \) to \( U^{cv}_b \), from \( U^{ca}_a \) to \( U^{ca}_b \), from \( R_a^{(c,e,j)} \) to \( R_b^{(c,e,j)} \), from \( P_a^{(c,e,j)} \) to \( P_b^{(c,e,j)} \), or from \( P_a^{(c,e,j)} \) to \( P_b^{(c,e,j)} \). A coding operation at a node \( a \) transforms \( f \) units from each of a pair of queues \( (U^{cv}_a, U^{cv}_b) \) into \( f \) units in each of the queues \( R_a^{(c,e,j)} \) and \( R_b^{(c,e,j)} \). A decoding operation at \( a \) transforms \( f \) units from each of a pair of queues \( (R_a^{(c,e,j)}, R_b^{(c,e,j)}) \) into \( f \) units in \( U^{(c,e,j)}_a \). A branching operation at \( a \) transforms \( f \) units from a queue \( P_a^{(c,e,j)} \) into \( f \) units in each of the queues \( P_a^{(c,e,j)} \) and \( P_b^{(c,e,j)} \).

Then the solution variables can be interpreted as follows:

- \( \nu_{ab}^{cv} \): average flow rate from \( U^{cv}_a \) to \( U^{cv}_b \)
- \( \nu_{ab}^{ca} \): average flow rate from \( U^{ca}_a \) to \( U^{ca}_b \)
- \( \pi_{ab}^{(c,e,j)} \): average flow rate from \( P_a^{(c,e,j)} \) to \( P_b^{(c,e,j)} \)
- \( \pi_{ab}^{(c,e,j)} \): average flow rate from \( P_a^{(c,e,j)} \) to \( P_b^{(c,e,j)} \)
- \( \rho_{ab}^{(c,e,j)} \): average flow rate from \( P_a^{(c,e,j)} \) to \( P_b^{(c,e,j)} \)
- \( \gamma_a^{(c,e',v')} \): average rate of coding transformation from \( (U^{cv}_a, U^{cv}_b) \) to \( (R_a^{(c,e,j)}, R_b^{(c,e,j)}) \)
- \( \eta_a^{(c,e,j)} \): average rate of decoding from \( (R_a^{(c,e,j)}, R_b^{(c,e,j)}) \) to \( U^{(c,e,j)}_a \)
- \( \sigma_a^{(c,e,j)} \): average rate of branching transformation from \( P_a^{(c,e,j)} \) to \( P_a^{(c,e,j)}, P_b^{(c,e,j)} \)

### B. Modified problem

For a given problem instance, we consider a modified problem any solution of which is equivalent to a solution of the original problem and vice versa. The modified problem reverses the direction of the poison flows and imposes some additional constraints. Specifically, it is defined as follows:

- Each source node \( s_c \) has a source queue \( U^c \), an overflow queue \( \bar{U}^c \), and a virtual source link of capacity \( C \) from \( U^c \) to queue \( U^c_{s_c} \).
- Each node \( a \) has a virtual coding link, a virtual decoding link and a virtual branching link, each of capacity \( \bar{C}/2 \).
- Each real and virtual link \( e \) is associated with a set \( \mathcal{P}_e \) of pairs \( (\mathcal{O}, \mathcal{D}) \) such that data units from each queue in the set \( \mathcal{O} \) are transformed into an equivalent number of units in each queue in the set \( \mathcal{D} \) via \( e \):
  - if \( e \) is the virtual source link for session \( c \), \( \mathcal{P}_e = (U^c, U^c_{s_c}) \)
  - if \( e \) is a real link \( (a, b) \in \mathcal{L} \),
    \[
    \mathcal{P}_e = \{(U_a^{cv}, U_b^{cv}), (U_a^{ca}, U_b^{ca}), (P_b^{(c,e,j)}, P_a^{(c,e,j)}), (P_b^{(c,e,j)}, P_a^{(c,e,j)}), (P_b^{(c,e,j)}, R_b^{(c,e,j)})\}
    \]
    - if \( e \) is the virtual coding link at node \( a \), \( \mathcal{P}_e = \{(U_a^{cv}, U_a^{cv}), (R_a^{(c,e,j)}, R_b^{(c,e,j)}) \}: c \neq c'; a \neq v, v' \}
    - if \( e \) is the virtual decoding link at node \( a \), \( \mathcal{P}_e = \{(R_a^{(c,e,j)}, U_a^{(c,e,j)}) \}: c \neq c'; a \neq j \}

- if \( e \) is the virtual branching link at \( a \), \( \mathcal{P}_e = \left\{ \left( \left( P^{e,c_j}_a, P^{e,c_j}_a \right), P^{(e,c')}_{j} \right) : c \neq c'; a \neq j \right\} \)
- Data is removed from queues \( U^{cv}_{a,l}, P^{e,c_j}_j \), and \( P^{(c,e')}_{j} \).

C. Wireless case

We also consider the multiple unicasts problem and corresponding modified problem in a wireless setting, using the following wireless network model. We model wireless transmissions by generalized links, denoted by \((a,Z)\), where \( a \) is the originating node and \( Z \) is the set of destination nodes. The network connectivity and link transmission rates depend on the transmitted signal and interference powers according to some underlying physical layer model. For example, the transmission rate per unit bandwidth \( \mu_{ij} \) from node \( i \) to node \( j \), with other nodes \( n \in \mathcal{N} \) transmitting independent information simultaneously, may be given by the Shannon formula [2]

\[
\mu_{ij}(P, S) = \log \left( 1 + \frac{P_i S_{ij}}{N_0 + \sum_{n \in \mathcal{N}} P_n S_{nj}} \right)
\]

where \( P_i \) is the power transmitted by node \( l \), \( S_{ij} \) is the channel gain from node \( l \) to node \( j \) and \( N_0 \) is additive white Gaussian noise power over the signaling bandwidth. For simplicity, we consider a finite set \( \mathcal{L} \) of links and a finite set \( \mathcal{U} \) of sets of simultaneously achievable link rates. We denote by \( C_{(a,Z),u} \) the capacity of link \((a,Z)\) in set \( u \in \mathcal{U} \). A solution to the multiple unicasts problem consists of a convex combination of sets in \( \mathcal{U} \), which gives a set of average link capacities achievable by timesharing, and a network code that operates over the network with these average link capacities.

In the wireless case, each node has a virtual source link, coding link and decoding link but no virtual branching link; a branching operation occurs over a real wireless link. For virtual source, coding and decoding links \( e \), the set \( \mathcal{P}_e \) is defined exactly as in the wired case. For real wireless links \((a,Z)\),

\[
\mathcal{P}_{(a,Z)} = \left\{ \left( U^{cv}_{a}, U^{cv}_{b} \right), (U^{cv}_{a}, U^{cv}_{b}), \left( P^{(e,c')}_{b}, P^{(e,c')}_{b} \right), \left( P^{e,j}_{b}, P^{e,j}_{b} \right), \left( R^{e,j}_{b}, R^{e,j}_{b} \right), \left( \left( U^{cv}_{a}, R^{e,c_j}_{b} \right), \left( U^{cv}_{b}, R^{e,c_j}_{b} \right) \right), \left( \left( P^{e,j}_{b}, P^{e,j}_{b} \right), \left( P^{(e,c')}_{j}, P^{(e,c')}_{j} \right) \right) : c \neq c'; b, b' \in \mathcal{Z} \right\}
\]

III. BACK-PRESSURE APPROXIMATION ALGORITHM

Queues \( U^{cv}_{a}, U^{c}, R^{cv}_{a}, R^{cv}_{a} \) are associated with session \( c \). We consider each joint poison queue \( P^{(e,c')}_{j} \) as a pair of queues, one associated with each session \( c \) and \( c' \), such that equal amounts are always added or removed from each of the pair of queues.

The overflow queue for each session \( c \) has a potential \( \alpha_c l e^{\alpha_B r} \) that is a function of its length \( l \), where

\[
\alpha_c = \frac{e}{24F r_c},
\]

and \( B \) is a constant to be determined. We divide each other queue into subqueues, one for each link for which it is an origin or destination. Each subqueue \( Q \) has a potential \( \phi_{c(Q)}(l_Q) = e^{\alpha_c(l_Q)} \) that depends on its length \( l_Q \) and its session \( c(Q) \). For notational simplicity, we will abbreviate subscripts \( c(Q) \) as subscripts \( Q \).

Flow entering or leaving subqueues associated with session \( c \) is partitioned into packets of size \( p_c = (1 + e)r_c \). Besides its true length \( l_Q \), each subqueue \( Q \) has an approximate length \( \hat{l}_Q \) that is an integer multiple of the packet size. The approximate length of a subqueue is updated only when its true length

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This text is a continuation of a discussion about network coding in wireless and wired networks, focusing on the capacity of links and the design of efficient algorithms for data transmission. The wireless case introduces additional complexities due to the physical layer characteristics, such as channel gains and noise power, which must be accounted for in the network coding design.

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The equations and notations used in this text are typical in network coding literature, where the goal is to maximize the throughput of data transmission in a network with multiple links and sessions, considering both wired and wireless settings.

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The last part of the text mentions the Back-pressure approximation algorithm, which is a fundamental concept in network flow and queueing theory, used for efficient data transmission in networks. The algorithm balances the trade-off between maintaining a high throughput and avoiding congestion by dynamically adjusting the flow rates based on the current state of the queues.
has changed by at least one packet since the last update of its approximate length, as follows: \( \hat{l}_Q \) is set to \( kpQ \) where \( k = \left\lfloor \frac{l_Q - 1}{p_Q} \right\rfloor \) if \( Q \) is an origin subqueue or \( k = \left\lceil \frac{l_Q + 1}{p_Q} \right\rceil \) if \( Q \) is a destination subqueue. Between updates, \( l_Q \) and \( \hat{l}_Q \) satisfy:

- For an origin subqueue, \( l_Q - 3p_Q \leq \hat{l}_Q \leq l_Q \) for \( Q \) is an origin subqueue, or \( l_Q \leq \hat{l}_Q \leq l_Q + 3p_Q \) for a destination subqueue.

We denote by \( \bar{P}_e \) the subset of \( P_e \) consisting of pairs \( (O, D) \in P_e \) satisfying:

\[
\min_{Q \in O} \hat{l}_Q > 0 \\
\max_{Q \in D} \hat{l}_Q < BrQ + \ln((L + 1)\rho)/\alpha_Q + 3p_Q,
\]

where

\[ \rho = \frac{\max_c \tau_c}{\min_c \tau_c}. \]

### A. Wired case

In each round \( t \), the algorithm carries out the following:

1) Add \( (1 + \epsilon)\tau_c \) units to the overflow queue \( U^c \) of each session \( c \), then transfer as much as possible to \( U^c \) subject to a maximum length constraint of \( Br_c \) for \( U^c \).

2) For each real and virtual link \( e \), flow is pushed for zero or more origin-destination pairs \( (O, D) \in \bar{P}_e \) such that the total amount pushed is at most the link capacity \( C_e \). Specifically, initialize \( C \) to \( C_e \) and repeat:

   - Choose the pair \( (O, D) \in \bar{P}_e \) that maximizes
     
     \[
     w_{(O,D)} = \sum_{Q \in O} \phi'_c(\hat{l}_Q) - \sum_{Q \in D} \phi'_c(\hat{l}_Q).
     \]
     
     Let
     
     \[ C' = \min \left( C, \min_{Q \in (O \cup D)} (p_Q - |\delta_Q|) \right) \]
     
     where \( \delta_Q \) is the change in \( l_Q \) since the last update of \( \hat{l}_Q \). Subtract \( C' \) units from \( C \), subtract \( C' \) units from \( l_Q \) for each \( Q \in O \) and add units \( C' \) to \( l_Q \) for each \( Q \in D \). For \( Q \in (O \cup D) \), if \( C' = p_Q - |\delta_Q| \), update \( \hat{l}_Q \).

   - If \( C = 0 \) then end.

3) Zero out all subqueues \( U^c_{ij}, P^{c,e}_j \) and \( P^{(c,c')}_{ij} \).

4) For each queue that has at least one subqueue whose actual length has changed during the round, reallocate data units to equalize the actual lengths of all its subqueues. If the actual length of any subqueue has changed by at least one packet since the last update of its approximate length, update its approximate length.

When the amount of flow remaining in the network queues is an \( \epsilon \)-fraction of the total amount that has entered the network, the flow values for each link are averaged over all rounds to give the solution.
B. Wireless case

The algorithm is the same as for the wired case, except for Phase 2, which is as follows. For each virtual link \( e \), flow is pushed for zero or more origin-destination pairs \( (O, D) \in \bar{P}_e \) exactly as in the wired case.

For each real link \( e = (a, Z) \), flow is pushed for zero or more origin-destination pairs \( (O, D) \in \bar{P}_e \) such that the total amount pushed is at most the average link capacity \( \lambda_u C_{e,u} \) for some \( \lambda_u \) satisfying \( \sum_{u \in U} \lambda_u \leq 1 \).

Specifically, initialize \( T \) to 1 and repeat

- For each real link \( e = (a, Z) \), let
  \[
  T' = \min \left( T, \min_{e: C_{e,u} > 0} Q \in (O_e \cup D_e) \left( \frac{p_Q - |\delta_Q|}{C_{e,u}} \right) \right)
  \]

  where \( \delta_Q \) is the change in \( l_Q \) since the last update of \( \tilde{l}_Q \). Subtract \( T' \) units from \( T \), and for each \( e \), subtract \( T'C_{e,u} \) units from \( l_Q \) for each \( Q \in O_e \) and add the same amount to \( l_Q \) for each \( Q \in D_e \), updating \( \tilde{l}_Q \) if \( T'C_{e,u} = p_Q - |\delta_Q| \).

- If \( T = 0 \) then end.

IV. POTENTIAL ANALYSIS

Our analysis is partly based on the approach in [1]. We lower bound the decrease in potential over a round \( t \). We denote by \( Q(t) \) the actual length of a subqueue \( Q \) at the end of Phase 1 of round \( t \). From [1], the increase in potential during Phase 1 is upper bounded by

\[
\sum_c p_c \phi_c'(U^c(t)).
\]

A. Flow solution-based algorithm

We lower bound the decrease in potential during Phases 2 and 3 by comparison with the potential decrease resulting from pushing flow based on a flow solution for rates \( f_c = (1 + 2\epsilon)r_c \). This algorithm differs from the back pressure algorithm only in the specific portion of Phase 2 determining the amount of flow pushed for each pair \( (O, D) \in \bar{P}_e \) of each link \( e \).

Consider a flow solution for rates \( f_c \). Partition the flow for each session \( c \) into elementary flows \( F^c_n \) such that all data in an elementary flow undergoes the same routing and coding operations. Each elementary flow \( F^c_n \) has a size denoted \( f^c_n \), and consists of a set of links with associated origin and destination subqueues, comprising a primary path from \( s_c \) to \( d_c \) and a remedy path associated with each coding node. Note that each flow \( F^c_n \) starts from queue \( U^c \), which consists of a single subqueue, and that
\( f_c = \sum_n f_n^c \). Let \( L \) be the length of the longest primary path and \( F \) the maximum number of links in an elementary flow. A subqueue in \( \mathcal{F}_n^c \) is considered upstream or downstream of another according to the direction of flow in the modified problem defined in Section II-B.

Phase 2 of the flow solution-based algorithm consists of a preprocessing part and a flow pushing part.

1) **Preprocessing procedure:**

- **Initialization:**
  - Remove from each \( \mathcal{F}_n^c \) any portions of each branch that are downstream of a subqueue \( Q \) in \( \mathcal{F}_n^c \) for which \( l_Q \leq 3p_c \). Note that all subqueues of each queue have been equalized in Phase 4 of the previous round. Let \( \mathcal{G} \) be the set of flows \( \mathcal{F}_n^c \) containing some subqueue of length at least \( B_{rc} + \ln((L + 1)\rho)/\alpha_c \). For flows \( \mathcal{F}_n^c \in \mathcal{G} \), let \( \bar{Q}_n^c \) be the furthest downstream origin subqueue in \( \mathcal{F}_n^c \) of length at least \( B_{rc} + \ln((L + 1)\rho)/\alpha_c \); for flows \( \mathcal{F}_n^c \notin \mathcal{G} \), let \( Q_n^c \) be the longest origin subqueue in \( \mathcal{F}_n^c \) (if there is a tie, choose the furthest downstream).
  - Initialize \( \mathcal{H} \) as the set \( \mathcal{H}_0 \) of all flows \( \mathcal{F}_n^{sc} \), initialize \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) as empty sets, and for each \( c, n \), initialize \( Q_n^{sc} \) as \( Q_n^c \).

- **Phase A:** Repeat
  - Choose some flow \( \mathcal{F}_n^c \in \mathcal{G} \cap \mathcal{H} \) and remove from \( \mathcal{H} \) all flows \( \mathcal{F}_n^{c'} \) such that \( \mathcal{F}_n^{c'} \) shares any portion of a common joint poison path segment, or the coding and branching links at either end, with \( \mathcal{F}_n^c \). Remove \( \mathcal{F}_n^c \) from \( \mathcal{H} \) and add it to \( \mathcal{H}_1 \).
  - If there is no such flow remaining, end Phase A.

- **Phase B:** Associate with each flow \( \mathcal{F}_n^c \in \mathcal{H} \) a weight \( w_n^c \), initially set to \( \phi_c(U^c(t)) \). For each flow \( \mathcal{F}_n^{c'} \in \mathcal{H} \), set \( \mathcal{I}_n^{c'} \) to be the set of subqueues \( \bar{Q}_n^{c'} \) such that \( \mathcal{F}_n^{c'} \) is the shared flow corresponding to \( \bar{Q}_n^{c'} \), and either
  - \( \bar{Q}_n^{c'} \) is a remedy or individual poison subqueue whose corresponding branching link is in \( \mathcal{F}_n^c \) and whose corresponding coding link is in \( \mathcal{F}_n^{c'} \), or
  - \( \bar{Q}_n^{c'} \) is a joint poison subqueue whose corresponding branching link is in \( \mathcal{F}_n^{c'} \).

Repeat
  - Choose some flow \( \mathcal{F}_n^{c'} \in \mathcal{H} \) such that
    \[
    w_n^{c'} \leq \sum_{\bar{Q}_n^{c'} \in \mathcal{I}_n^{c'}} \left( \phi'_c(\bar{Q}_n^{c'}(t)) - w_n^c \right),
    \]
    * remove \( \mathcal{F}_n^{c'} \) from \( \mathcal{H} \),
    * remove any subqueues in \( \mathcal{F}_n^{c'} \) from \( \mathcal{I}_n^c \) \( \forall c, n \),
    * set the weight \( w_n^{c'} \) of each flow \( \mathcal{F}_n^{c'} \) in \( \mathcal{I}_n^{c'} \) to \( \phi'_c(\bar{Q}_n^{c'}(t)) \).
  - If there is no such flow, for each \( \mathcal{I}_n^{c'} \) and each \( Q_n^{sc} \in \mathcal{I}_n^{c'} \) set \( Q_n^{sc} \) to \( U^c \), and end Phase B.

- **Phase C:** Repeat
  - Choose some flow \( \mathcal{F}_n^c \in \mathcal{H} \) such that the longest individual poison subqueue along one of its poison branches, \( Q \), is longer than \( Q_n^{sc} \), and the entire joint portion of that branch together with the branching link are not in the corresponding shared flow. Set \( Q_n^{sc} \) to \( Q \) and remove from \( \mathcal{F}_n^c \) all but the portion downstream of \( Q \).
  - If there is no such flow remaining, end Phase C.
Observe that:

- At most $L$ flows are removed by each flow in $G$, and
  \[
  \phi_c'(B_{rc} + \ln((L+1)\rho)/\alpha_c) \geq (L+1) \max_c \phi_c'(B_{rc}) \geq (L+1) \max_c \phi_c'(U^c(t))
  \]
  \[\text{(6)}\]

- $f_n^c = f_n'I$ for all $F_n^c$ and $F_n'^c$ that share a common joint poison path segment.

- At the end of the preprocessing procedure,
  \[
  2 \sum_{F_n \in \mathcal{H}} \phi_c'(Q_n^c(t)) f_n^c \geq \sum_{F_n \in \mathcal{H}} \phi_c'(\bar{Q}_n^c(t)) f_n^c.
  \]
  \[\text{(7)}\]

To see this, note that at the end of Phase B there is a one-to-one correspondence between flows $F_n^c \in \mathcal{H}$ for which $Q_n^c \neq \bar{Q}_n^c$ and elements $\bar{Q}_n^c$ in the sets $\mathcal{I}_n'$ of flows $F_n'^c \in \mathcal{H}$. For each $F_n'^c \in \mathcal{H}$, by (5),
\[
\phi_c'(Q_n^c(t)) - w_n^c = \sum_{\bar{Q}_n^c \in \mathcal{I}_n'} \left( \phi_c'(\bar{Q}_n^c(t)) - w_n^c \right).
\]

Multiplying by $f_n^c$ and summing over all $F_n^c \in \mathcal{H}$, and noting that at the end of the preprocessing procedure
\[
\phi_c'(Q_n^c(t)) \geq w_n^c
\]
gives (7).

- Phase B and C maintain the invariant
  \[
  \sum_{F_n \in \mathcal{H}} w_n^c f_n^c \geq \sum_{F_n \in \mathcal{H}_2} \phi_c'(U^c(t)) f_n^c
  \]
  \[\text{(9)}\]

where $\mathcal{H}_2$ is the value of $\mathcal{H}$ at the start of Phase B. The invariant holds since both sides are equal at the start of Phase B, and the left-hand side is monotonically non-decreasing. From (6), (8) and (9), at the end of the preprocessing procedure we have
\[
\sum_{F_n \in \mathcal{H}} \phi_c'(Q_n^c(t)) f_n^c \geq \sum_{F_n \in \mathcal{H}_0} \phi_c'(U^c(t)) f_n^c.
\]
\[\text{(10)}\]

2) Flow pushing procedure: For each flow $F_n^c \in \mathcal{H}_1 \cup \mathcal{H}$, let $F_n'^c$ be the portion of $F_n^c$ downstream of $Q_n^c$. Partition its links into a set $\mathcal{S}_n^c$ of subsets. Each of these subsets $S \in \mathcal{S}_n^c$ may be
\begin{itemize}
  \item a path from $Q_n^c$, if it is a poison subqueue, to its associated coding node $a(S)$
  \item a path from $Q_n^c$, if it is a remedy subqueue, up to and including its associated decoding link at node $w(S)$, and the associated poison path from $w(S)$ to its associated coding node $a(S)$ via the branching link at node $b(S)$,
  \item a path associated with uncoded flow ending in sink node $d_c$, or
  \item a path associated with uncoded flow ending in a coding link at a node $a(S)$, together with the associated remedy path up to and including the decoding link at a node $w(S)$, and the associated poison path from $w$ to $a$ via the branching link at a node $b$.
\end{itemize}

Note that each subset starts either at $Q_n^c$ or at an uncoded subqueue.
The flow solution-based algorithm pushes flow as follows. First, for each pair \((S_c, S_{c'}) \in G_n^c \times G_n^{c'}\) that shares any portion of a common joint poison path segment, note that \(f_n^c = f_n^{c'}\) and that Phase B of the preprocessing procedure ensures that both subsets or neither contains the coding link; in the latter case both or neither contains the branching link. Thus, the following are the only two cases:

- Case 1: One of the subsets, say \(S_c\), contains both the coding link at \(a(S_c)\) and the branching link at \(b(S_c)\), while \(S_{c'}\) contains the coding link but not the branching link. Phase C of the preprocessing procedure ensures that all session \(c\) individual poison subqueues along the joint poison path segment are shorter than \(Q_n^{c}\). Push \(f_n^c\) units through \(S_c \cup S_{c'}\), pushing session \(c\) individual poison units through the joint poison path segment.

- Case 2: Both subsets contain the same portion of the joint poison path segment. Push \(f_n^c\) units through \(S_c \cup S_{c'}\).

Next, for each subset \(S\) of some \(F_n^{c}\) that does not have any coded segments in common with \(F_n^{c'}\) for all \(c' \neq c, n'\), we have the following cases:

- Case 1: \(Q_n^{c}\) is an individual poison subqueue in \(S\). Phase B of the preprocessing procedure ensures that \(Q_n^{c}\) is the longest individual poison subqueue in \(F_n^{c'}\). Push \(f_n^c\) individual poison units through \(S\).

- Case 2: \(Q_n^{c}\) is a joint poison subqueue in \(S\). Push \(f_n^c\) joint poison units along the path in \(S\) downstream of \(Q_n^{c}\).

- Case 3: \(Q_n^{c}\) is a remedy subqueue in \(S\). Phase C of the preprocessing procedure ensures that \(Q_n^{c}\) is longer than all session \(c\) individual poison subqueues along the primary path of \(S\). Push \(f_n^c\) remedy units along \(S\) through the decoding link at \(w(S)\), and \(f_n^c\) individual poison units along the primary path of \(S\).

- Case 4: \(Q_n^{c}\) is an uncoded subqueue or is not in \(S\). Push \(f_n^c\) uncoded units along the primary path of \(S\) starting from its longest session \(c\) uncoded subqueue.

Note that flow is pushed only from origin subqueues \(Q\) for which

\[ l_Q > 3p_Q \Rightarrow \tilde{l}_Q > 0 \]

and only to destination subqueues \(Q\) for which

\[ l_Q < Br_Q + \ln((L + 1)\rho)/\alpha_Q \Rightarrow \tilde{l}_Q < Br_Q + \ln((L + 1)\rho)/\alpha_Q + 3p_Q. \]

**B. Potential decrease in Phases 2 and 3**

The decrease in potential from pushing \(f\) units across a link from a set \(O\) of origin to a set \(D\) of destination subqueues is at least

\[ \sum_{Q \in O} (f \varphi'_Q(l_Q) - f^2 \varphi''_Q(l_Q)) - \sum_{Q \in D} (f \varphi'_Q(l_Q) + f^2 \varphi''_Q(l_Q + f)) \]

where \(l_Q\) denotes the initial length of each subqueue \(Q\).
1) Flow solution-based algorithm: We denote by \( O(\mathcal{F}_n^{oc}) \) and \( D(\mathcal{F}_n^{oc}) \) the sets of origin and destination subqueues of a flow \( \mathcal{F}_n^{oc} \), and by \( O_{c,e} \) and \( D_{c,e} \) the sets of session \( c \) origin and destination subqueues of a link \( e \). For each subqueue \( Q \), denote by \( f_Q \) the total flow out of \( Q \) (if \( Q \) is an origin subqueue) or into \( Q \) (if \( Q \) is a destination subqueue) in the flow pushing procedure of the flow-solution-based algorithm. The potential drop over Phases 2 and 3 in the flow solution-based algorithm is at least

\[
\sum_{c,e} \left( \sum_{Q \in O_{c,e}} (f_Q \phi'_c(l_Q) - f_Q \phi''_c(l_Q + f_Q)) - \sum_{Q \in D_{c,e}} f_Q (\phi'_c(l_Q) + f_Q \phi''_c(l_Q + f_Q)) \right)
\]

\[
\geq \sum_{c,e} \left( \sum_{Q \in O_{c,e}} f_Q (\phi'_c(l_Q) - f_c \phi''_c(l_Q + f_c)) - \sum_{Q \in D_{c,e}} f_Q (\phi'_c(l_Q) + f_c \phi''_c(l_Q + f_c)) \right)
\]

\[
\geq \sum_{\mathcal{F}_n^{oc}} \left( \sum_{Q \in \mathcal{O}(\mathcal{F}_n^{oc})} f_n^c (\phi'_c(l_Q) - f_c \phi''_c(l_Q + f_c)) - \sum_{Q \in D(\mathcal{F}_n^{oc})} f_n^c (\phi'_c(l_Q) + f_c \phi''_c(l_Q + f_c)) \right)
\]

Since all subqueues of each queue have been equalized in Phase 4 of the previous round, and since each flow \( \mathcal{F}_n^{oc} \) has one origin subqueue of length \( Q_n^{oc}(t) \), at most \( L + 1 \) destination subqueues each of length less than \( 3p_c \), and a total of at most \( F \) links, this potential drop is lower-bounded by

\[
\sum_{\mathcal{F}_n^{oc}} f_n^c (\phi'_c(Q_n^{oc}(t)) - (L + 1) \phi'_c(3p_c)) - \sum_{\mathcal{F}_n^{oc}} 3F f_n^c f_c \phi''_c(Q_n^{oc}(t) + f_c) - \sum_{\mathcal{F}_n^{oc}} 3F f_n^c f_c \phi''_c(Q_n^{oc}(t) + f_c)
\]

From (11), we have

\[
3F f_c \phi''_c(l + f_c) \leq 3F(1 + 2\epsilon) r_c \phi''_c(l + (1 + 2\epsilon) r_c)
\]

\[
= 3F(1 + 2\epsilon) r_c \alpha^2 e^{\alpha}(1 + 2\epsilon) r_c
\]

\[
= 3F(1 + 2\epsilon) \alpha e^{\alpha(1 + 2\epsilon) r_c} \phi'_c(l)
\]

\[
= \frac{e(1 + 2\epsilon)}{8} e^{\frac{(1 + 2\epsilon)}{4\epsilon}} \phi'_c(l)
\]

\[
\leq \phi'_c(l)/4.
\]

This yields, using (11), the following lower bound on the potential drop:

\[
\sum_{\mathcal{F}_n^{oc} \in (\mathcal{H}_1 \cup \mathcal{H})} f_n^c (\phi'_c(Q_n^{oc}(t)) - (L + 1) \phi'_c(3p_c)) - \sum_{\mathcal{F}_n^{oc} \in \mathcal{H}_1} f_n^c e \phi'_c(Q_n^{oc}(t))/4 - \sum_{\mathcal{F}_n^{oc} \in \mathcal{H}} f_n^c \phi''_c(Q_n^{oc}(t))/4
\]

\[
\geq \sum_{\mathcal{F}_n^{oc} \in (\mathcal{H}_1 \cup \mathcal{H})} f_n^c \left( \left( 1 - \frac{\epsilon}{2} \right) \phi'_c(Q_n^{oc}(t)) - (L + 1) \phi'_c(3p_c) \right)
\]

\[
\geq \sum_{c} (1 + 2\epsilon) r_c \left( \left( 1 - \frac{\epsilon}{2} \right) \phi'_c(U^c(t)) - (L + 1) \phi'_c(3p_c) \right)
\]

\[
\geq \sum_{c} r_c \left( \left( 1 + \frac{3\epsilon}{2} - \epsilon^2 \right) \phi'_c(U^c(t)) - (L + 1)(1 + 2\epsilon) \phi'_c(3p_c) \right)
\]

(12)
2) Back pressure algorithm: If the packet size $p_c$ were infinitesimally small rather than $(1 + \epsilon)r_c$, then $l_Q = \tilde{l}_Q$ and the procedure in Phase 2 of the back pressure algorithm would give, for each link $e$, the maximum possible potential decrease from pushing flow for zero or more origin-destination pairs $(O, D) \in \mathcal{P}_e$ such that the total amount pushed is at most the link capacity $C_e$. Since $p_c = \Theta(r_c)$ and

$$
\phi'_e(l + \Theta(r_c)) - \phi''_e(l) \leq \Theta(r_c)\phi''_e(l + \Theta(r_c)),
$$

the back pressure algorithm achieves, for each link $e$, a potential decrease of at least that achieved by the flow solution-based algorithm minus an error term

$$
\sum_{Q \in (O_e \cup D_e)} \Theta(f_Q r_Q)\phi''_Q(l_Q + \Theta(r_Q))
$$

where $f_Q$ is the amount of flow added or removed from $Q$ in Phase 2 of the flow solution-based algorithm. Since $f_Q = \Theta(r_Q)$, decreasing each $\alpha_c$ by some constant factor is sufficient to ensure that (12) applies to the back pressure algorithm.

C. Overall potential change and number of rounds

The potential does not increase during Phase 4. Thus, from (4) and (12), the overall potential decrease during the round is lower bounded by

$$
\sum c_r \left( \frac{\epsilon}{2} - \epsilon^2 \right) \phi'_c(U^c(t)) - (L + 1) (1 + \epsilon)\phi'_c(3p_c)
$$

$$
= \sum c_r \left( \frac{\epsilon}{2} - \epsilon^2 \right) \phi'_c(U^c(t)) - (L + 1) (1 + \epsilon)\frac{\epsilon}{24F}e^{\frac{\epsilon(1+\epsilon)}{2F}}
$$

If $U^c(t) = Br_c$ for some $c$, then the decrease in potential is at least

$$
\phi'_c(Br_c) - K (L + 1) (1 + \epsilon)\frac{\epsilon}{24F}e^{\frac{\epsilon(1+\epsilon)}{2F}}
$$

$$
= \alpha_c \frac{\epsilon}{2} - \epsilon^2 \phi'_c(Br_c) - K (L + 1) (1 + \epsilon)\frac{\epsilon}{24F}e^{\frac{\epsilon(1+\epsilon)}{2F}}
$$

which is non-negative if

$$
Br_c = \frac{1}{\alpha_c} \ln \left( \frac{K (L + 1) (1 + \epsilon)}{\epsilon(1 - 2\epsilon)} \right) + 3p_c
$$

$$
= \Theta \left( \frac{1}{\alpha_c} \ln \left( \frac{KL}{\epsilon} \right) \right)
$$

If $U^c(t) < Br_c$ for all $c$, the overflow queues are empty and, since there are $\Theta(NMK)$ session-$c$ subqueues each of potential less than

$$
\phi_c(Br_c + \ln((L + 1)\rho)/\alpha_c + 4p_c) = \Theta \left( Lpe^{\frac{\epsilon l}{24F}} \right),
$$

the overall potential in the system at the end of the round is at most

$$
\Theta \left( NMK^2 Lpe^{\frac{\epsilon l}{24F}} \right)
$$
By induction, this is also an upper bound on the total potential at the end of every round. The length of the overflow queue for session $c$ is thus never more than

$$O\left(\frac{N M K^2 L \rho e^{\frac{KL\rho}{1 - \epsilon}}}{\alpha_c e^{\frac{KL\rho}{1 - \epsilon}}}\right) = O\left(\frac{N M K^2 L \rho}{\alpha_c}\right)$$

and the total units for session $c$ is at most

$$O\left(\frac{N M K^2 L \rho}{\alpha_c} + N M K (Br_c + \ln(L\rho)/\alpha_c + 4p_c)\right) = O\left(\frac{N M K}{\alpha_c} \left(KL\rho + \ln\left(\frac{KL\rho}{\epsilon}\right)\right)\right)$$

Thus, at most

$$O\left(\frac{N M K F}{\epsilon} \left(\frac{KL\rho}{\epsilon}\right)\right)$$

rounds of input flow for each session remain in the network at any time. For

$$t = O\left(\frac{N M K F}{\epsilon^2} \left(\frac{KL\rho}{\epsilon}\right)\right)$$

amount remaining in the network is at most a fraction $\epsilon$ of the total amount that has entered the network up to round $t$.

**D. Number of operations**

In each round, at each node, pushing flow across links results in a total decrease of at most $O(\hat{C})$ in the actual lengths of origin subqueues and a total increase of at most $O(\hat{C})$ in the actual lengths of destination subqueues. The total change in subqueue lengths from rebalancing is not more than the total change resulting from pushing flow across links. Thus, at most $O(ND)$ approximate subqueue lengths in the network are updated in each round, where $D = \max_c \frac{\hat{C}}{r_c}$. Assuming the subqueue differences computed by the algorithm can be stored, only those differences involving subqueues whose approximate lengths have changed are recomputed.

For the coding link at each node $i$, for each pair of sessions $\{c, c'\}$, the values of $w(\hat{O}, \hat{D})$, defined in 2, for pairs $(\hat{O}, \hat{D}) \in \bar{P}_e$ of the form $((U^{c'}_i, U^c_i), (R^{c'}_{v'i}, R^c_{vi}))$ are stored in a sorted list of length $O(N^2)$, and the list indexes are ranked according to the maximum value in each list. If $U^{c'}_i$ or $U^c_i$ changes, $N$ differences are updated in each of $K$ lists, requiring $O(NK \log N)$ operations, and the rank of these $K$ lists are updated using $O(K \log K)$ operations. If $R^{c'}_{v'i}$ or $R^c_{vi}$ changes, $N$ differences are updated in one of the lists and fewer operations are required.

For the decoding link $e$ at each node $i$, the values of $w(\hat{O}, \hat{D})$ for pairs $(\hat{O}, \hat{D}) \in \bar{P}_e$ are similarly stored in a sorted list of length $O(NK^2)$. A change to any of the approximate subqueue lengths requires updating of at most $K$ differences, for which $O(K \log(NK))$ operations suffices.

1) **Wired case:** For each real and virtual branching link, the values of $w(\hat{O}, \hat{D})$ for pairs $(\hat{O}, \hat{D}) \in \bar{P}_e$ are similarly stored in a sorted list of length $O(NK^2)$. A change to any of the approximate subqueue lengths requires updating of $O(1)$ differences, for which $O(\log(NK))$ operations suffices.

Thus, each round has complexity $O(N^2 KD \log(NK))$, and the algorithm has complexity

$$O\left(\frac{N^3 M K^2 F D \log(NK)}{\epsilon^2} \left(\frac{KL\rho}{\epsilon}\right)\right)$$.
2) Wireless case: We assume that for each wireless link \((a, Z)\), \(|Z| = O(1)\). For a wireless link \(e = (a, Z)\), for each pair of sessions \(\{c, c'\}\), the values of \(w_{(O,D)}\) for pairs \((O,D) \in \bar{P}_e\) of the form \(\left((U^{c_{cv}}_a, R^{c_{cv}}_a), (U^{c_{ca}}_b, R^{c_{ca}}_b)\right)\) are stored in a sorted list of length \(O(N^2)\), and the values of \(w_{(O,D)}\) for all other pairs \((O,D) \in \bar{P}_e\) are stored in a sorted list of length \(O(NK^2)\). The values of \(y_u\), defined in (3), for each set \(u \in U\) are also stored.

Of all the subqueues of \((a, Z)\), a change in the approximate length of some subqueue \(U^{c_{cv}}_a\) or \(U^{c_{ca}}_b\) requires the most operations: \(O(N)\) differences are updated in each of \(O(K)\) length-\(O(N^2)\) lists, \(O(1)\) differences are updated in the length-\(O(NK^2)\) list and \(O(K)\) of the \(O(K^2)\) list indexes are repositioned, requiring a total of \(O(NK \log N) + O(\log(NK)) + O(K \log K) = O(NK \log(NK))\) operations. A further \(O(|U|)\) operations suffices to update the values of \(y_u\) and find the maximum among them.

Thus, each round has complexity \(O(ND(NK \log(NK) + |U|))\), and the algorithm has complexity

\[
O\left(\frac{N^2 MKFD(NK \log(NK) + |U|)}{\epsilon^2} \left(KL\rho + \ln\left(\frac{KL\rho}{\epsilon}\right)\right)\right).
\]

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