Anytime Minibatch with Delayed Gradients

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Abstract—Distributed optimization is widely deployed in practice to solve a broad range of problems. In a typical asynchronous scheme, workers calculate gradients with respect to out-of-date optimization parameters while the master uses stale (i.e., delayed) gradients to update the parameters. While using stale gradients can slow the convergence, asynchronous methods speed up the overall optimization with respect to wall clock time by allowing more frequent updates and reducing idling times. In this paper, we present a variable per-epoch minibatch scheme called Anytime Minibatch with Delayed Gradients (AMB-DG). In AMB-DG, workers compute gradients in epochs of a fixed time while the master uses stale gradients to update the optimization parameters. We analyze AMB-DG in terms of its regret bound and convergence rate. We prove that for convex smooth objective functions, AMB-DG achieves the optimal regret bound and convergence rate. We compare the performance of AMB-DG with that of Anytime Minibatch (AMB) which is similar to AMB-DG but does not use stale gradients. In AMB, workers stay idle after each gradient transmission to the master until they receive the updated parameters from the master while in AMB-DG workers never idle. We also extend AMB-DG to the fully distributed setting. We compare AMB-DG with AMB when the communication delay is long and observe that AMB-DG converges faster than AMB in wall clock time. We also compare the performance of AMB-DG with the state-of-the-art fixed minibatch approach that uses delayed gradients. We run our experiments on a real distributed system and observe that AMB-DG converges more than two times.

I. INTRODUCTION

Distributed stochastic optimization has become an important tool in solving large-scale problems such as those found in modern machine learning applications. Traditionally, such problems have been solved using gradient-based methods in a serial manner on a single compute node. However, training deep neural networks using large datasets with millions of data points in a reasonable amount of time often requires parallelizing across multiple machines [3], [4], [5], [6]. Furthermore, in many online learning systems, such as search engines, data (e.g., queries) arrive at rates that a single node cannot handle or process in real time [7]. Although training deep networks using a single GPU is possible, in many large-scale training and online learning problems, multiple nodes are deployed in parallel to cooperatively execute the desired tasks.

In the past decade, a large body of work has proposed and analyzed various schemes to parallelize stochastic optimization and online learning problems [3], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]. A common approach is to use a network of nodes arranged according to a hub-and-spoke topology. In this topology, a central node, known as the master, keeps track of and updates the optimization parameters while the other nodes, known as workers, calculate the gradients of the objective function with respect to individual data points and submit these gradients to the master.

Distributed optimization methods can be categorized as synchronous or asynchronous. In synchronous methods, the master waits for all workers to submit their results before updating the optimization parameters. As a result, the speed of the computation in synchronous methods is limited by the speed of the slowest workers known as “stragglers” [20], [21], [22]. To alleviate the straggler effect, asynchronous methods have been proposed in which the master does not wait for all workers. Instead, the master updates the optimization parameters using “stale” (i.e., delayed) gradients.

Recently, we introduced a novel synchronous technique called Anytime Minibatch (AMB) in [13]. Our main idea was to fix the per-epoch compute time across all workers rather than fixing the job size as is typical. Since workers progress at different rates, fixing the compute time results in a variable amount of work being completed by each worker in each epoch. A drawback of AMB is that when workers submit their results to the master, they idle while waiting to receive back the updated optimization parameters. When this wait time is long, the wall time convergence can be greatly prolonged. To alleviate this deficiency, in this paper we propose Anytime Minibatch with Delayed Gradients (AMB-DG). In AMB-DG, workers keep processing data points and calculate gradients at all times while the master updates the parameters using delayed gradients. We analyze the performance of AMB-DG for a convex smooth objective function under the assumption of fixed gradient staleness. Under these assumptions, we prove that the expected regret bound attained is $O(\sqrt{m})$ where $m$ is the number of samples observed across all nodes. This bound is well-known to be the optimal regret bound achievable by gradient-based methods on arbitrary convex objective functions [7]. We also show that under the same assumptions the expected convergence rate (i.e., optimality gap) achieved is $O(1/\sqrt{m})$. Our theoretical derivations show that asymptotically in the number of data points observed, the impact of the fixed delay on convergence is negligible for smooth convex loss functions. We compare the performance of AMB-DG with that of AMB and observe that under long communication delays AMB-DG converges faster than AMB in terms of wall clock time. We also implement AMB-DG on
the performance of AMB-DG with existing asynchronous delayed-gradient-based methods. AMB-DG converges almost 2 times faster in our examples.

The remainder of the paper is organized as follows. In Sec. II, we review the literature. In Sec. III, we present the system model and the algorithmic description. The convergence analysis and theorems are presented in Sec. IV. Detailed proofs are deferred to Appendix A. In Sec. VI, we present numerical results illustrating the performance of AMB-DG. Finally, Sec. VII summarizes and concludes the paper.

II. RELATED WORK

Asynchronous distributed optimization has been studied at least since the seminal work of Tsitsiklis et al. [23]. In recent years, asynchronous schemes have played a key role in solving large-scale machine learning and online optimization problems. For instance, Google’s distbelif [10] was one of the earliest implementation of asynchronous distributed optimization for large-scale training in machine learning. In general, the literature pertaining to asynchronous distributed optimization and online learning focuses on two aspects: algorithmic development and convergence analysis.

Developing schemes for asynchronous distributed optimization has been an active area of research in recent years. Examples of such algorithms include [6], [12], [15], [24]. In [24], the authors study parallel optimization with sparse data. They propose an asynchronous dual averaging approach termed AsyncDA. In AsyncDA, the master keeps and updates the dual variable while each worker keeps its own primal optimization variable. Asynchronous methods use stale gradients, as was mentioned earlier. While asynchronous schemes can tolerate gradient staleness, large staleness can slow convergence. This is known as the stale gradient problem. For instance, in [15], gradient staleness scales linearly in the number of workers. Thus using larger network may have an undesirable impact on convergence. To limit the negative impact of staleness, the authors in [25] propose a stale-gradient-based scheme and control the staleness in a spoke-and-hub setting by forcing fast workers to wait for slow ones. One of the challenges encountered in distributed optimization is dealing with the communication load. When the number of workers increases, worker-master communication may become a bottleneck. To reduce the communication load and alleviate the bottleneck, techniques that compress the information before transmission have been proposed and analyzed such as gradient quantization [26], [27], [28] and gradient sparsification (i.e., randomly dropping out coordinates of the gradient and amplifying the rest) [29].

In another thread of works, convergence analyses of existing asynchronous methods have been developed. In [8], the authors analyze the convergence of distributed optimization of smooth convex loss functions using delayed gradients under the assumption of bounded gradient delay. They prove that the effect of gradient delay is negligible asymptotically in the number of processed data points and that the optimal rate is achievable. In [30], the convergence analysis is extended to general regularized convex cost functions. In [17], the authors study the convergence of stochastic gradient methods for nonconvex objective functions and establish an ergodic convergence rate of $O(1/\sqrt{T})$, where $T$ is the number of iterations.

III. AMB-DG SCHEME

In this section, we introduce AMB-DG. We first introduce the system model and then describe the algorithm. The pseudocode of the scheme is found in Appendix C.

A. System Model

In this paper we study a synchronized network consisting of a master and $n$ workers. Processing proceeds in epochs, each of duration $T_p$ seconds. Epochs are indexed by $t \in \mathbb{Z}_+$. The $t$-th epoch therefore ends at time $tT_p$. At the end of epoch $t$ workers synchronously and in parallel transmit messages to the master. Worker $i$ transmits message $m_i(t)$. All worker-to-master transmissions take $\frac{1}{2}T_c$ second. Based on its current state and the $n$ messages received, the master updates its estimate of the optimization parameters. The $t$-th update of the optimization parameter is denoted by $w(t + 1)$. We assume the master’s update occurs instantaneously. Therefore the master’s $t$-th update occurs at time $tT_p + \frac{1}{2}T_c$. The master then broadcasts its $t$-th parameter update to all workers. This is done in parallel and all messages take $\frac{1}{2}T_c$ second to reach each worker. Therefore, each worker receives $w(t + 1)$ at time $tT_p + T_c$. Naturally, depending on the relative durations of $T_p$ and $T_c$ the workers will be basing their calculation on more or less stale information. The ratio $\tau = \frac{T_c}{T_p} \in \mathbb{Z}_+$ plays an important role in the ensuing derivation. We term this ratio the “staleness parameter”. Note that $T_p$ and $T_c$ are both deterministic: hence $\tau$ is deterministic, too. The staleness parameter, $\tau$, represents the number of updates the master makes to the optimization parameter from the time when it calculates and broadcasts $w(t)$ until it receives gradients with respect to the same $w(t)$.

The above description is diagrammed in Fig. 1. At the core of our interest in this system is the fact that workers may be operated at different computing speeds and therefore may process different amounts of data in each processing epoch. In epoch $t$ worker $i$ processes $b_i(t)$ data points. These data are indexed as $x_i(t, s)$ where $s \in [b_i(t)]$. The total data processed in epoch $t$ is denoted $b(t) = \sum_{i=1}^n b_i(t)$. In Fig. 1 the variable-sized mini-batches are indicated by the different heights of the bar plots associated with each worker. Fig. 1 illustrates a system with three workers.

B. Algorithmic Description

Our objective is to solve the following optimization problem

$$\min_{w \in \mathbb{W}} F(w) \text{ where } F(w) := \mathbb{E}_P [f(w, x)]. \quad (1)$$

In (1) $f : \mathcal{W} \times \mathcal{X} \rightarrow \mathbb{R}$ and the expectation is taken with respect to some (unknown) distribution $P$ over the set $\mathcal{X} \subseteq \mathbb{R}^d$. In particular, we are interested in solving for $w^* = \arg\min_{w \in \mathbb{W}} F(w)$.
The algorithm we study in the context of the above system model is dual averaging [31], [32]. We first describe the distributed dual averaging without stale gradients (i.e., when $T_c = 0$). Then, we explain how to extend the algorithm to cater for stale gradients.

**Gradients computation:** In each epoch $t$, worker $i$ calculates $b_i(t)$ gradients of $f(w,x)$ with respect to the optimization parameters $w(t)$ and data points $\{x_i(t,s)\}, s \in [b_i(t)]$. The sum of gradients calculated by worker $i$ at the end of epoch $t$ is

$$g_i(t) = \sum_{s=1}^{b_i(t)} \nabla_w f(w(t), x_i(t,s)),$$

where the $x_i(t,s)$ are sampled by worker $i$ in epoch $t$ in an independent and identically distributed (i.i.d.) manner from $P$. At the end of each gradient computation epoch, each worker $i$ sends message $m_i(t) := (g_i(t), b_i(t))$ to the master and proceeds immediately to the next epoch.

**Parameters update:** The master calculates the average of the received gradients $g(t) = \frac{1}{b(t)} \sum_{i=1}^{n} g_i(t)$, and updates $z(t)$ and $w(t)$ using the following dual averaging update rules

$$z(t+1) = z(t) + g(t),$$

$$w(t+1) = \arg \min_{w \in \mathcal{W}} \left\{ \langle z(t+1), w \rangle + \frac{1}{\alpha(t+1)} \psi(w) \right\},$$

where $\alpha(t)$ is a nonincreasing sequence of step sizes. Since we work in the Euclidean space, a typical choice of $\psi(w)$ is $\|w\|^2$. The master broadcasts $w(t+1)$ to all workers immediately. The workers use the updated optimization parameter to calculate the next set of gradients.

**Stale gradients:** Next, we consider the case when $T_c \neq 0$. To simplify our analysis, we assume that $T_c$ is an integer multiple of $T_p$, so that $\tau = T_c/T_p$. This is possible by choosing the appropriate $T_p$. Note that for short communication delay or when $T_c$ is smaller than $T_p$, $\tau \leq 1$. In this case, we can use AMB instead of AMB-DG as workers’ idle time between two consecutive epochs is relatively small and the benefits of using non-stale gradients can result in faster convergence despite the short intermittent idle times. In our analysis, we are interested in operating in the large communication delay regimes. Large communication delay has been observed to be a major bottleneck in large scale distributed computing. In this situation, slow epoch-wise convergence due to delayed gradients is compensated by a faster rate of parameters update. Furthermore, since $[T_c/T_p] \geq T_c/T_p$ with equality if and only if $T_c$ is an integer multiple of $T_p$, setting $T_p$ such $T_c$ is an integer multiple of $T_p$ ensures that workers will immediately use the new information; i.e., a new compute epoch starts as soon as updated parameters arrive. Also, since we are interested in a large communication delay regime, such a choice of $T_p$ is not very limiting.

We assume the use of a non-blocking communication protocol. That is, after each gradient computation epoch ends, workers send their messages to the master and immediately start a new round of computation while communication proceeds in the background. Note that workers receive the first update from the master at time $T_p + T_c = (\tau+1)T_p$. Hence, worker $i$ uses $w(t)$ to calculate gradients $g_i(t)$ for epochs $1 \leq t \leq \tau + 1$. In other words,

$$g_i(t) = \sum_{s=1}^{b_i(t)} \nabla_w f(w(1), x_i(t,s)).$$

On the other hand, since workers receive $w(t+1)$ at time $tT_p + T_c = (t + \tau)T_p$, then for epochs $t \geq \tau + 2$, each worker $i$ calculates $g_i(t)$ as

$$g_i(t) = \sum_{s=1}^{b_i(t)} \nabla_w f(w(t-\tau), x_i(t,s)).$$

Therefore, the master calculates $g(t)$ as

$$g(t) = \sum_{i=1}^{n} \sum_{s=1}^{b_i(t)} \nabla_w f(w(t-\tau), x_i(t,s)),$$

where for $t - \tau \leq 0$, $w(t-\tau) := w(1)$, the initial value.

Figure 1 illustrates the delayed gradient model when the communication time $T_c = 3T_p$. Workers calculate gradients with respect to $w(1)$ in the first four epochs and send their messages to the master at times $tT_p$ for $t = 1, 2, 3, 4$. The master uses $z(2) = g(1) = \sum_{s=1}^{b_i(t)} g_i(1)$ to calculate $w(2)$ and broadcasts it to all workers while it uses delayed gradients $g(2)$, which are calculated with respect to $w(1)$, to update

![Fig. 1: AMB-DG system model](image-url)
z(3) before calculating w(3). Worker i receives w(2) at time $4T_p$ and uses it to calculate $g_i(t)$. The master uses $g_i(t)$ in turn to calculate $z(6)$ and hence $w(6)$ at time $5T_p + \frac{1}{4}T_c$. Since $T_c = 3T_p$, the staleness parameter in this example is $\tau = 3$. Observe that the master calculates $w(6)$ using gradients calculated with respect to $w(2)$ instead of $w(5)$, so gradient staleness $5-2 = 3 = \tau$. Except for the first $\tau + 1 = 4$ epochs, the master calculates $w(t)$ for $t \geq \tau + 2 = 5$ using gradients calculated with respect to $w(t - \tau) = w(t - 3)$.

We note that although we use dual averaging as an algorithmic workhorse, AMB-DG can be implemented using other gradient-based algorithms as well. The motivation behind using dual averaging stems from the following. Dual averaging is asymptotically optimal. Hence, we anticipate that using dual averaging yields the desired convergence rate.

**IV. CONVERGENCE RESULTS**

In this section, we develop convergence results for AMB-DG in terms of expected regret and expected convergence rate. The latter is also known as the optimality gap. Regret is the standard metric of performance measurement in the online learning literature while optimality gap is the standard metric in stochastic optimization. Let $R(T)$ and $G(T)$, respectively, denote the regret and the optimality gap after $T$ epochs (i.e., parameters updates). Then,

$$R(T) = \sum_{t=1}^{T} \left[ f(w(t + 1), x(t + 1)) - f(w^*, x(t + 1)) \right],$$

while

$$G(T) = F(\bar{w}(T)) - F(w^*),$$

where $w^* = \text{argmin}_{w \in W} F(w)$ and $\bar{w}(T) = \frac{1}{T} \sum_{t=1}^{T} w(t + 1)$ is the time-average of the optimization variable over the $T$ epochs.

Before turning to our performance analysis, we first state the main assumptions used in our derivations.

**A. Preliminaries**

As was stated in Sec. III, we are interested in solving the online stochastic optimization problem of (1). We assume that the feasible set $W$ is closed and convex and that an optimal solution, $w^* \in W$, exists. We assume that $f(w, x)$ is differentiable and convex in $w$ for every $x$. Hence, $F(w)$ is also convex and differentiable and the gradient of the objective function is $\nabla F(w) := \mathbb{E}_x[\nabla f(w, x)]$. We assume that $F(w)$ and $\nabla f(w, x)$ are both Lipschitz continuous with parameters $J$ and $L$, respectively. In other words,

$$|F(w_1) - F(w_2)| \leq J \|w_1 - w_2\| \quad \forall w_1, w_2 \in W,$$

and for all $w_1, w_2 \in W$ and for all $x \in X$

$$\|\nabla f(w_1, x) - \nabla f(w_2, x)\| \leq L \|w_1 - w_2\|$$

where $\|\cdot\|$ is the $l_2$ norm. The smoothness property in (9) implies that

$$f(w_2, x) \leq f(w_1, x) + (\nabla f(w_1, x), w_2 - w_1) + \frac{L}{2} \|w_1 - w_2\|^2.$$

We also assume the following variance bound, for all $w \in W, \mathbb{E}_x[\|\nabla f(w, x) - \nabla F(w)\|^2] \leq \sigma^2$, for some $\sigma \geq 0$. Furthermore, the proximal function, $\psi(w)$, in the primal variable update rule (4) is $1$-strongly convex. This means that $\psi$ satisfies

$$\psi(w_2) \geq \psi(w_1) + (g, w_2 - w_1) + \frac{1}{2} \|w_2 - w_1\|^2$$

for all $w_1, w_2 \in W$ and for all $g \in \partial \psi(w)$ where $\partial \psi(w)$ is the set of sub-gradients defined as $\partial \psi(w) := \{g \in \mathbb{R}^d | \psi(w_2) \geq \psi(w_1) + (g, w_2 - w_1), \forall w_2 \in \text{dom}(\psi)\}$.

Furthermore, for the optimal solution, $w^*$, we assume that $\psi(w^*) \leq C^2/2$ and that for all $w \in W, D_\psi(w^*, w) \leq C^2$, for some $C \in \mathbb{R}$. In these definitions, $D_\psi(w^*, w)$ is the Bregman divergence between $w^*$ and $w$ defined as

$$D_\psi(w^*, w) := \psi(w^*) - \psi(w) - \langle \nabla \psi(w), w^* - w \rangle.$$

**B. Regret and Convergence Analysis**

We analyze the performance of AMB-DG in terms of the regret achieved after $T$ epochs. $b_i(t)$ are modeled as i.i.d. random variables. Let $\bar{b}$ be the expected minibatch per epoch. That is, $\mathbb{E}[b(t)] = \bar{b}$. Then $\mathbb{E}[b(t)] = \mathbb{E}\left[ \sum_{i=1}^{n} b_i(t) \right] = n\bar{b}$. Therefore, $\mathbb{E}[b_i(t)] = \bar{b}/n$ for all $i \in [n]$ and for all $t \in [T]$. Furthermore, assume that $b(t) \geq \bar{b}$. Let $m$ be the expected total number of samples observed from epoch 1 to $T$, i.e.,

$$m = \mathbb{E}\left[ \sum_{t=1}^{T} b(t) \right] = \sum_{t=1}^{T} \sum_{i=1}^{n} \mathbb{E}[b_i(t)] = T\bar{b}.$$

We assume that all gradients received are delayed by $\tau$ steps where $\tau$ is a nonnegative integer.

$$\mathbb{E}[R(T)] := \mathbb{E}\left[ \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{s=1}^{b_i(t)} \left[ f(w(t + 1), x_i(t + 1, s)) - f(w^*, x_i(t + 1, s)) \right] \right].$$

$$\text{(14)}$$

We next state the main theoretical results of our work. The theorem bounds the expected regret and the corollary bounds the optimality gap.
Theorem IV.1. Assume that the master receives a set of gradients over $T$ epochs during which the expected number of data points workers sample from $P$ in an i.i.d. manner is $m$. Let $\hat{b} \neq 0$ and $\hat{b}$ be as defined above. Choosing $\alpha(t)^{-1} = L + \sqrt{(t + \tau)/\hat{b}}$, the expected regret for AMB-DG with deterministic common gradient delay $\tau$ across all master-worker links in each epoch is

$$\mathbb{E}[R(T)] \leq \frac{C^2}{2}(L + \sqrt{(T + 1 + \tau)/\hat{b}}) + 2\tau JC\hat{b}$$

$$+ 2LJ^2(\tau + 1)^2\hat{b}^2(1 + \log T) + \frac{\hat{b}}{b}\sigma^2\sqrt{m}. \quad (15)$$

In both Theorem IV.1 and Corollary IV.2, next, $J$ and $L$ are, respectively, the Lipschitz constants of $F(w)$ and $\nabla f(w,x)$, $\sigma^2$ is the bound on the variance of $\nabla f(w,x)$ and $C$ is a constant that satisfies that for all $w \in \mathcal{W}$, the Bregman divergence $B_w(w^*, w) \leq C^2$.

Let $\tilde{w}(T)$ be the average of the optimization variable over $T$ epochs, i.e.,

$$\tilde{w}(T) = \frac{1}{T} \sum_{t=1}^{T} w(t + 1). \quad (16)$$

Using $\tilde{w}(T)$ the expected optimality gap is

$$\mathbb{E}[G(T)] := \mathbb{E}\left[F\left(\frac{1}{T} \sum_{t=1}^{T} w(t + 1)\right) - F(w^*)\right]. \quad (17)$$

Using the results from Theorem IV.1 and the definition of the conditional expectation of the optimality gap in (17), we develop the following corollary on the upper bound of the expected optimality gap achieved by AMB-DG.

Corollary IV.2. The expected optimality gap for the AMB-DG scheme is

$$\mathbb{E}[G(T)] \leq \hat{b}\left(\frac{C^2}{2m}(L + \sqrt{(T + 1 + \tau)/\hat{b}}) + \frac{2\tau JC}{m}\right)$$

$$+ 2LJ^2(\tau + 1)^2\hat{b}^2(1 + \log T) + \frac{\sigma^2}{b\sqrt{m}}. \quad (18)$$

The proofs for Theorem IV.1 and Corollary IV.2 are provided in Appendix A. The proof techniques exploit the convexity of $F$ and the Lipschitz continuity of $\nabla F$ to bound $F(w(t + 1)) - F(w^*)$. We define the error term $e(t) = \nabla F(w(t)) - g(t)$ and use dual averaging and some results from [8] to further bound $F(w(t + 1)) - F(w^*)$. The choice of a nonincreasing sequence of the learning rates $\{\alpha(t)\}$ for $t \in [T]$ helps achieve the optimal regret and optimality gap bounds.

C. Discussion

Our results in Theorem IV.1 and Corollary IV.2 can be interpreted as follows. As long as gradient staleness $\tau \leq O(m^{-1/4})$, then the expected regret $\mathbb{E}[R(T)] \leq O(\sqrt{m})$ and the convergence rate $\mathbb{E}[G(T)] \leq O(1/\sqrt{m})$. These are the optimal achievable bounds for each metric. We highlight that in AMB-DG, we can control the gradient staleness by choosing the appropriate fixed computation time, $T_p$. Since in AMB-DG $\tau = [T_c/T_p]$, if we choose $T_p$ such that $[T_c/T_p] \leq O(m^{1/4})$, we can guarantee $\tau \leq O(m^{1/4})$. Unlike other existing schemes such as those in [15], [12], the gradient staleness does not depend on the number of workers. It only depends on the ratio of $T_c$ to $T_p$. Hence, AMB-DG can scale to larger cluster sizes without experiencing the performance degradation that can result from larger staleness due to larger cluster.

The regret bound in (15) and the optimality gap bound in (18) depend on $\hat{b}$, $\hat{b}$ and the ratio $b/b$. In a real system, these parameters would be determined by the characteristics of the workers and their computing speeds. They further depend on the fixed processing time $T_p$. As might be expected, in our experiments on a real distributed system we observe that $\hat{b}$ and $\hat{b}$ scale linearly with $T_p$. In addition, we observe that the ratio $b/b$ is typically bounded by some small constant. Therefore, both bounds in (15) and (18) are asymptotically dominated by $m$.

Since $\hat{b} \leq \hat{b}$ with equality if and only if $b(t) = \hat{b}$ for all $t \in [T]$, the regret bound is minimized when the system observes the same number of data points in each epoch. This is achieved if there are no stragglers. Hence, the analysis reveals the penalty incurred due to stragglers. However, this penalty becomes negligible asymptotically in $m$. Similarly, the dominant term in the convergence rate bound in (18) is scaled by $b/b \geq 1$. When there are no stragglers, that dominant term is minimized. Hence, the ratio $b/b$ reflects the impact the stragglers can have on the convergence rate. This impact diminishes as $m$ gets larger.

V. EXTENSION TO THE FULLY DECENTRALIZED SETTING

The previous analysis assume that the time required by the master to aggregate gradients and update parameters is negligible. In practice, the time spent by the master to aggregate gradients increases with the number of the workers. The added delay can be accounted for an increase in the effective communication time that each worker node experiences. For this reason it can be difficult to scale the hub-and-spoke model to large networks of nodes.

In contrast, the fully decentralized setting, in which no master node exists, scales well to large network size avoiding significant increases in communication time. In this setting, each node communicates with its neighboring nodes using distributed averaging protocols [33]. During communication, workers exchange information (e.g., primal and dual variables) with their neighbors a few times and calculate a weighted-average of the received information to achieve consensus. If perfect consensus is achieved, then each worker ends up with the same value which corresponds to the average of the information prior to communication across all workers. In theory, achieving perfect consensus may require infinite number of information exchange. Since this is not possible in practice as workers exchange information a few times, workers often end up with imperfect consensus.

The model we analyze to investigate the decentralized realization of AMB-DG consists of two-way noiseless communication channel between certain pairs of workers. We
represent these connections as a graph $G(E, V)$ in which each worker $i \in V$ is a vertex in the graph. Each edge represented two neighbour nodes, i.e., if workers $i$ and $j$ can communicate with each other then $(i, j) \in E$. Furthermore, the graph is undirected since if worker $i$ can send to worker $j$, then also worker $j$ can send to worker $i$. This means that edge $(i, j)$ is the same as the edge $(j, i)$. We assume that communication time between all connected pairs is equal. Let $Q$ be a matrix that is positive semi-definite and doubly stochastic (i.e., all columns and rows sum to 1) such that $Q_{ij} > 0$ if $i = j$ or $(i, j) \in E$, otherwise $Q_{ij} = 0$. The entries of $Q$ are the weights that workers use to scale the messages received from their neighbours so that nodes can eventually calculate the average of their messages. $Q$ is sometimes referred to as communication matrix.

A. AMB-DG Algorithm in the Masterless Setting

The version of the decentralized AMB-DG we analyze is similar to the one explained in Sec. III. The main difference is that instead of master-worker communication, we have inter-node communication using a distributed averaging protocol. This is similar to the model we analyzed for the AMB scheme in [13].

Gradient Computation: This step is the same as in the hub-and-spoke setting. Each worker computes gradients for $T_p$ seconds. Due to imperfect consensus, in each epoch, workers calculate gradients with respect to different optimization parameters. Let $w_i(t)$ be the optimization parameter at worker $i$ in the $t$-th epoch. The resulting sum of the gradients computed by worker $i$ in epoch $t$ is

$$g_i(t) = \frac{1}{b_i(t)} \sum_{s=1}^{b_i(t)} \nabla_{w} f(w_i(t), x_i(t, s)).$$

(19)

Consensus Phase: The consensus phase starts immediately after the gradient computation epoch. At the end of this phase, each worker updates its own optimization parameter. In each consensus phase, workers spend $T_c$ seconds to exchange information over $r$ rounds of consensus. In the $t$-th phase, worker $i$ begins by sending message $m_i^{(0)}(t) = nb_i(t)(z_i(t) + g_i(t))$. After $k$ rounds of consensus, where $k \in [r]$, worker $i$ has the message

$$m_i^{(k)}(t) = \sum_{j=1}^{n} Q_{ij}m_j^{(k-1)}(t) = \sum_{j=1}^{n} (Q_{ij})^k m_i^{(0)}(t).$$

(20)

For a large enough number of iterations, $r$, and if the graph is connected and the second largest eigenvalue of $Q$ is strictly less than one, then the consensus rounds converge to $m_i^{(r)}(t) = b(t)(z(t) + g(t))$, where $z(t) := \frac{1}{b(t)} \sum_{s=1}^{b(t)} b_i(t) z_i(t), \quad g(t) = \frac{1}{b(t)} \sum_{s=1}^{b(t)} g_i(t).$ We note that if the number of iteration per consensus round increases, then the duration of each consensus phase becomes larger. In other words, as $r$ increases, $T_c$ increases. This means that if $r$ is too large, then gradient staleness, $\tau$, also increases since $\tau$ increases with $T_c$. We highlight that in the master-worker setup the communication phases can overlap as can be seen in Fig. 1. In the fully distributed setting, we also allow such an overlap. To manage multiple messages, each worker $i$ sends the index of the epoch, $t$, to its neighbours when sending message $m_i^{(r)}(t)$.

Parameters Update: Worker $i$ updates its dual variable after $r$ rounds of consensus according to

$$z_i(t + 1) = \frac{1}{b(t)} m_i^{(r)}(t) = \bar{z}_i(t) + g(t).$$

(21)

However, often having a large number of consensus rounds may not be practical and workers complete a small number of rounds. Therefore, the actual dual variable update contains an additive error term and can be expressed as

$$z_i^{(r)}(t + 1) = \bar{z}_i(t) + g(t) + \epsilon_i(t)$$

(22)

where the superscript $r$ in $z_i^{(r)}$ denotes the number of consensus rounds and $\epsilon_i(t)$ is the consensus error at worker $i$. The optimization parameter $w_i(t)$ is updated according to (4). Let $z(t)$ denote the dual variable with perfect consensus. According to Lemma 1 from our previous work [13], let $\delta \geq 0$ and $\lambda_2(Q)$ be the second largest eigenvalue of $Q$, then

$$\|z_i^{(r)}(t + 1) - z(t + 1)\| \leq \delta$$

(23)

if the number of consensus rounds per worker in each consensus phase satisfies

$$r \geq \left\lceil \frac{\log(2\sqrt{n}(1 + 2J/\delta))}{1 - \lambda_2(Q)} \right\rceil$$

(24)

where $J$ is the Lipschitz parameter of the objective function $F(w)$.

B. Convergence Results

The convergence analysis for the fully distributed case is similar to the one we developed in Sec. IV for the hub-and-spoke. The only exception is that we must account for the consensus error. In addition, the optimization parameter $w(t)$ in the regret in (14) is replaced with $w_i(t)$.

Theorem V.1. Assume that the workers process gradients over $T$ epochs during which the expected number of data points workers sample from $P$ in an i.i.d. manner is $m$. Let $b \neq 0$ and $\tilde{b}$ be as defined in Sec. IV. Choosing $\alpha(t)^{-1} = L + \sqrt{(t + \tau)/b}$, the expected regret for AMB-DG in the fully decentralized setting with deterministic common gradient delay $\tau$ for all links in each epoch is

$$\mathbb{E}[R(T)] \leq \frac{\tilde{b}}{\alpha(T + 1)} \psi(w^*) + 2rJC\tilde{b} + 2LJ^2(\tau + 1)^2(1 + \log T)\tilde{b}^2 + \sigma^2 \frac{\tilde{b}}{b} \sqrt{m} + 2J\delta\tilde{b}^{3/2}\sqrt{m}. \quad \text{(25)}$$

The proof of Theorem V.1 is provided in Appendix B. The regret bound in (25) is similar to the one obtained in (15). The only difference is that (25) has an extra term related to the consensus error (i.e., the term with $\delta$). If workers achieve
perfect consensus, then $\delta = 0$ and thus (25) reduces to (15). In contrast, if the communication time per consensus iteration is long and workers are forced to keep $r$ small, then $\delta$ will be large and hence the regret bound is larger.

VI. Numerical Results

In this section we demonstrate the performance of AMB-DG and compare it to other schemes. We evaluate the performance in two ways. First, we solve a linear regression problem using a synthetic dataset. We model the per-node compute time using the shifted exponential distribution. We compare the performance of AMB-DG with that of AMB under the assumption of long communication delays. Furthermore, we compare the performance of AMB-DG with a fixed minibatch delayed-gradient-based scheme called $K$-batch async [34], [17]. $K$-batch async is a suitable baseline to compare AMB-DG with since both schemes ensure that workers never idle. Hence, they both use stale gradients. Moreover, both schemes exploit all jobs completed by all workers. In $K$-batch async, in each epoch each worker calculates $b/K$ gradients and submits their sum to the master. Here $b$ is the per-epoch fixed minibatch size and $K$ is an integer that divides $b$. The master updates the parameter $w$ after receiving $K$ messages from the workers. These $K$ messages need not be from distinct workers. Faster workers can contribute more messages. Second, we compare the performance of AMB-DG with that of $K$-batch async when training a neural network to classify images from the CIFAR-10 dataset [35]. We run these neural network experiments on SciNet which is an academic high performance computing cluster. All experiments are repeated 10 times and the average performance is reported. We note that the purpose of these experiments is to compare the performance of our approach that is based on fixing the compute time with the that of the state-of-the-art approach that uses a fixed minibatch per epoch. Methods that speed up convergence when using delayed gradients such as those found in [36] and [37] can also be applied to AMB-DG.

A. Linear regression

1) Dataset: For the linear regression problem we generate the global optimizer vector $w^* \in \mathbb{R}^d$ from the multivariate Gaussian $\mathcal{N}(0, I)$. In each compute epoch, $t$, worker $i$ receives a stream of a sequence of inputs of the form $(\xi(t, s), y_i(t, s))$. If the $i$-th node observes $b_i(t)$ data points in epoch $t$ then $1 \leq s \leq b_i(t)$. The data points $\xi_i(t, s), y_i(t, s)$ are generated as following. First, we generate $\xi_i(t, s) \in \mathbb{R}^d$ in an i.i.d. manner according to Gaussian $\mathcal{N}(0, I)$. Then the labels $y_i(t, s)$ are defined as $y_i(t, s) = \xi_i(t, s)^T w^* + \epsilon_i(t, s)$, where $T$ denotes the transpose operation and the $\epsilon_i(t, s)$ are i.i.d. zero-mean Gaussian noise samples with variance $\sigma^2$. In our simulations, we choose $d = 10^4$ and $\sigma^2 = 10^{-3}$.

2) Objective function: The objective function the system is designed to minimize is

$$F(w) = \sum_t \sum_{i=1}^n b_i(t) \sum_{s=1}^{b_i(t)} [\xi_i(t, s)^T w - y_i(t, s)]^2. \quad (26)$$

We assume that the system initializes the optimization variable to the all-zero vector $w(1) = 0$. In each worker’s compute epoch $t$, the worker calculates gradients $g_i(t)$ according to

$$g_i(t) = \sum_{s=1}^{b_i(t)} [\xi_i(t, s)^T w(t) - y_i(t, s)] \xi_i(t, s). \quad (27)$$

The error rate achieved in the $t$-th epoch is calculated as

$$\text{Err}(t) = \frac{\|A[w(t) - w^*]\|^2}{\|A w^*\|^2} \quad (28)$$

where $A$ is an $N \times d$ matrix whose rows are i.i.d. $\mathcal{N}(0, I)$ and $\|\cdot\|$ is the $l_2$ norm. We choose $N = 250000$. Since $w(1) = 0$, the initial error rate is always $\text{Err}(1) = 1$.

3) Shifted exponential model: To evaluate performance versus wall clock time, we model the time that each node takes to calculate a fixed local minibatch of size $b$ as random variable that follows a shifted exponential distribution. Exponential and shifted exponential distributions are widely used in the literature to model variable compute times [34], [16]. We use $T_i(t)$ to denote the time taken by worker $i$ to calculate $b$ gradients in the $t$-th epoch. The probability density function of $T_i(t)$ is

$$f_{T_i}(\tau) = \lambda e^{-\lambda(\tau - \xi)} \quad \tau \geq \xi \geq 0 \quad (29)$$

where $\xi$ is the minimum duration a single worker takes to calculates $b$ gradients while $\lambda^{-1} + \xi$ is the expected time to calculate $b$ gradients.

Conditioned on $T_i(t)$, the worker is assumed to make linear progress through the dataset, i.e., it takes $kT_i(t)/b$ to calculate $k$ gradients (note that $k \geq b$ is allowed). In our simulations, we choose $\lambda = 2/3$ and $\xi = 1$. Per our model, in both AMB and AMB-DG, node $i$ is given a fixed time $T_p$ to compute $b_i(t)$ gradients in epoch $t$. Per our assumptions above, $b_i(t) = bT_p/T_i(t)$, for some fixed $b \leq \mathbb{E}[b_i(t)]$. In our simulations, we assume a network of $n = 10$ workers and that $T_p = 2.5$. We choose $b = 60$ to ensure that the expected minibatch per epoch is $\mathbb{E}[b(t)] \geq nb = 600$. This lower bound is according to Theorem 7 of [13]. Moreover, in our examples, we set $T_c = 10$.

4) Performance evaluation of AMB-DG vs. AMB: Figure 2 compares the performance of AMB-DG with that of AMB. The two plots in Fig. 2 compare the performance of each scheme with respect to the number of epochs as well as wall clock time.

Figure 2a shows that the per-epoch performance of AMB is better than that of AMB-DG. This is expected since AMB uses “fresh” gradients to update the optimization parameter in each epoch whereas AMB-DG uses stale ones. However, when comparing the performance against the wall clock time, we observe in Fig. 2b that AMB-DG is almost three times faster than AMB. In particular, AMB-DG achieves an error rate of 0.35 at 55 s whereas AMB takes about 182 s to achieve the same error rate. We highlight that (akin to AMB) the gradients in the first epoch of AMB-DG do not suffer from staleness since they are calculated with respected $w(1)$. Therefore, both schemes achieve similar error rates after the first epoch.

In contrast, the communication time per consensus iteration is long and workers are forced to keep $r$ small, then $\delta$ will be large and hence the regret bound is larger.
AMB-DG, gradient staleness increases gradually until the fifth epoch from which point $\tau = 4$. This higher staleness explains why, as can be observed in Fig.2a, AMB-DG achieves higher per-epoch error rates especially from the fifth epoch onward.

The comparison above highlights one of the shortcomings of AMB. During communications, AMB workers stay idle. This can slow convergence when the communication time is long. On the other hand, AMB-DG exploits the communication time to process additional gradients which, albeit being stale, help improve convergence speed. The improvement becomes especially pronounced when communication delays are long. Note that in AMB, except for the first update, the master updates the optimization parameters every $T_p + T_c = 12.5$ s whereas in AMB-DG the parameters are updated every $T_p = 2.5$ s. The first update in both schemes takes place after $T_p + \frac{3}{2}T_c = 7.5$ s. Hence, the epoch duration in AMB is larger than that in AMB-DG. As the communication time is increased, this difference in epoch durations increases. Conversely, as the communication time approaches zero, the epoch duration in both schemes become identical. In this limiting case, the gradient delay in AMB-DG goes to $\tau = 0$ which means that AMB-DG reduces to AMB.

5) Performance comparison of AMB-DG with $K$-batch async: In this experiment we again model the time that takes worker $i$ to calculate $b = 60$ gradients in the $t$-th epoch according to the same shifted exponential distribution in (29). We set $\lambda = 2/3$ and $\xi = 1$ as before. We run the simulations for $n = 10$ workers and let $T_c = 10$. For AMB-DG, we set $T_p = 2.5$. Thus, the expected minibatch per epoch in AMB-DG is roughly 600. For the $K$-batch async, each worker computes 60 gradients per epoch and we set $K = 10$. Hence, the master updates the optimization parameters whenever it receives $\tilde{b} = 600$ gradients. Fig. 3 illustrates that AMB-DG achieves faster convergence when compared to $K$-batch async. For instance, the error rate achieved by AMB-DG after almost 30 s is achieved by $K$-Batch Async at around 47 s meaning that AMB-DG is over 1.5 times faster than $K$-Batch Async. Observe that the first update takes a long time due to the long communication delay. This impacts both schemes. If we ignore this initial shared delay and re-scale time, AMB-DG becomes about 1.7 times faster than $K$-Batch Async.

To gain further insights into why AMB-DG outperforms $K$-batch async, we look at the distribution of gradient staleness in both cases. As explained in Sec. III, the gradient staleness suffered by AMB-DG after the first few epochs is $\tau = T_c/T_p = 4$. On the other hand, the distribution of gradient staleness experienced by $K$-batch async as depicted in Figure 4 shows that almost 80% of the gradients received by the master are delayed by five or more steps. Therefore, although the average per-epoch minibatch size and the average per-update time in both schemes are similar, AMB-DG suffers less staleness when compared to $K$-batch async and thus converges faster.

B. CIFAR-10 Image Classification

Although our analysis assumed convex smooth objective functions, we complement our work by presenting results for a nonconvex image classification problem. We train a neural network to classify CIFAR-10 images. We work with the neural network architecture of [38] that consists of 14 layers: 9 convolutional layers followed by 5 fully connected layers. The loss function that we optimize is the cross-entropy
function. We run our experiments on the SciNet using \( n = 4 \) workers and a master node. Each node in the SciNet cluster is equipped with a total of 40 Intel Skylake CPU cores and 202 GB of memory. Since we use a high performance computing platform, the inter-node communication time is much smaller than the per-epoch compute time. Therefore, we artificially delay communication to mimic operating environments in which communication time is long. We induce \( T_c = 10 \text{ s} \) while we choose \( T_p = 10 \text{ s} \) for AMB-DG. For \( K \)-batch async, we set \( b = 60 \) and \( K = 4 \). The per-epoch minibatch size is in \( K \)-batch async is 240 which is approximately the same as the average minibatch per epoch in AMB-DG. Figure 5 compares the performance of both schemes when \( T_c = 10 \). It illustrates that AMB-DG achieves a lower training loss and higher testing accuracy than that achieved by \( K \)-batch async when the neural network is trained for the same duration. In particular, the training loss achieved by AMB-DG after 4 hours is achieved by \( K \)-batch async after almost 7.5 hours showing that AMB-DG is about 1.9 times faster than \( K \)-batch async. Similarly, the testing accuracy achieved by AMB-DG after about 7.5 hours is already achieved by AMB-DG after only 4 hours. We remind the reader that the reported performance is an average of 10 experiments and that the purpose of these experiments is to compare the wall-clock performance of our approach to the existing fixed minibatch approach. The aim is not to build a deep learning model that achieves higher training error or testing accuracy than the existing best in class so, for example, we did not implement dropout or data augmentation in our simulations.

C. Variable Minibatches of AMB-DG in a Real Distributed System

In our analysis in Sec. IV, we observe that the regret depends on several parameters that are system specific. These are the minimum per-epoch minibatch \( \hat{b} \) and the average minibatch \( \bar{b} \). To understand how these parameters scale in real distributed systems, we ran our algorithm on SciNet using the MNIST dataset [39] for several values of the processing time, \( T_p \), ranging from 200 \( \mu \text{s} \) to 2000 \( \mu \text{s} \). For each choice of \( T_p \), we ran the algorithm for 200 epochs and recorded the minimum \( \hat{b} \) and the average \( \bar{b} \) minibatches across these 200 epochs. Fig. 6 shows that these parameters scale almost linearly with \( T_p \). Hence, for a given \( T_p \), these parameters are bounded and thus the regret bound in (15) and the convergence rate in (18) are dominated by \( m \) asymptotically as discussed in Sec. IV-C. Furthermore, the ratio \( \bar{b}/\hat{b} \) is observed to be bounded above by \( \gamma < 1.1 \) as can be read from the right y-axis in Fig. 6.

VII. CONCLUSION

In this paper, we present the Anytime Minibatch with Delayed Gradients (AMB-DG) algorithm. AMB-DG is a variable-minibatch scheme for online distributed stochastic optimization. In AMB-DG, workers are given a fixed compute time to calculate as many gradients as they can before submitting the sum of the computed gradients, along with their local minibatch size, to the master. Following each submission, workers continue to calculate gradients using the existing optimization parameters, without waiting for updates from the master. As a result, the master uses stale gradients to update the optimization parameters. Our theoretical analysis shows that for smooth convex objective functions, AMB-DG
achieves the optimal regret bound of $O(\sqrt{m})$ and the optimal convergence rate of $O(1/\sqrt{m})$, where $m$ is the total number of samples observed by the entire system and that the impact of gradient delay is negligible asymptotically in the number of samples observed, $m$. We compare AMB-DG with AMB under long communication delay setting when solving a linear regression problem and observe that AMB-DG outperforms AMB in terms of wall clock time convergence. We also compare the performance of AMB-DG with that of $K$-batch async algorithm for the same linear regression problem and observe that AMB-DG achieves lower staleness and faster convergence in wall clock time. We also implement AMB-DG and $K$-batch async on a real distributed system to train a neural network using the CIFAR-10 dataset. We observe that AMB-DG is almost 2 times faster.

In our analysis we assumed fixed communication delay. As a future extension of this work, one can consider a more interesting regime of operation in which the communication time is random. Furthermore, this paper analyzed AMB-DG for the master-worker distributed setting. AMB-DG can be extended to a fully decentralized setup and its performance and convergence can be analyzed as in [13].

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Lemma A.3. Let the assumptions of Sec. IV-A hold. Let \( \alpha(t) = L + \eta(t) \) for \( \eta(t) > 0 \) and \( L \geq 0 \). Then
\[
\mathbb{E}\left[ \|w(t) - w(t + 1)\|^2 \right] \leq \frac{4L^2}{\alpha(t)}.
\]

The convergence of our scheme is impacted by the following two factors. First, gradients are calculated with respect to \( f(w, x) \) rather than \( F(w) \). Furthermore, in each epoch \( t \), the master receives the gradients \( g(t) \) which are delayed by \( \tau \) steps. We define the error term \( e(t) \).
\[
e(t) := \nabla F(w(t)) - g(t) \tag{30}
\]
First, we look at \( F(w(t)) - F(w^*) \) and develop some bounds.
\[
F(w(t)) - F(w^*) \leq \langle \nabla F(w(t)), w(t) - w^* \rangle = \langle \nabla F(w(t)), w(t + 1) - w^* \rangle + \langle \nabla F(w(t)), w(t) - w(t + 1) \rangle \tag{31}
\]
\[
\leq \langle \nabla F(w(t)), w(t + 1) - w^* \rangle + F(w(t)) - F(w(t + 1)) + \frac{L}{2} \|w(t) - w(t + 1)\|^2, \tag{32}
\]
where (31) follows from the convexity of \( F \) and (33) from the L-Lipschitz continuity of \( \nabla F \), cf., (10).

From (30), \( \nabla F(w(t)) = e(t) + g(t) \). Substituting this in (33) and rearranging, we get a bound for \( F(w(t + 1)) - F(w^*) \).
\[
F(w(t + 1)) - F(w^*) \leq \langle e(t) + g(t), w(t + 1) - w^* \rangle + \frac{L}{2} \|w(t) - w(t + 1)\|^2 \tag{34}
\]
\[
= \langle e(t), w(t + 1) - w^* \rangle + \langle g(t), w(t + 1) - w^* \rangle + \frac{L}{2} \|w(t) - w(t + 1)\|^2 \tag{35}
\]
\[
= \langle z(t + 1) - z(t), w(t + 1) - w^* \rangle + \langle e(t), w(t + 1) - w^* \rangle + \frac{L}{2} \|w(t) - w(t + 1)\|^2 \tag{36}
\]
\[
= \langle z(t + 1), w(t + 1) - w^* \rangle - \langle z(t), w(t + 1) - w^* \rangle + \langle e(t), w(t + 1) - w^* \rangle + \frac{L}{2} \|w(t) - w(t + 1)\|^2, \tag{37}
\]
where (36) follows by observing that \( g(t) = z(t + 1) - z(t) \) from (3).

Recall that \( w(t) = \arg \min_{w \in W} \{z(t), w(t) + \alpha(t)^{-1} \psi(w)\} \) according to (4). We apply Lemma A.1 to (37) where we replace \( x^+, x, z \) and, \( A \) from Lemma A.1 with \( w(t) \), \( w(t + 1) \), \( z(t) \), and \( \alpha(t)^{-1} \), respectively, and re-arrange to get
\[
-\langle z(t), w(t + 1) - w^* \rangle \leq -\langle z(t), w(t) - w^* \rangle + \frac{1}{\alpha(t)} [\psi(w(t + 1)) - \psi(w(t))] - \frac{1}{\alpha(t)} D_\psi(w(t+1), w(t)). \tag{38}
\]
Substitute (38) in (37) yields
\[F(w(t+1)) - F(w^*) \leq \langle z(t+1), w(t+1) - w^* \rangle \]
\[-\langle z(t), w(t) - w^* \rangle + \frac{1}{\alpha(t)} \left[ \psi(w(t+1)) - \psi(w(t)) \right] \]
\[+ \langle e(t), w(t+1) - w^* \rangle - \frac{1}{\alpha(t)} D_\psi(w(t+1), w(t)) \]
\[+ \frac{L}{2} \| w(t) - w(t+1) \|^2. \] (39)

Since \( \alpha(t) \) is a sequence of positive nonincreasing step sizes, define a positive nondecreasing sequence \( \eta(t) \) such that \( \alpha(t)^{-1} = L + \eta(t) \). As will be seen later, \( \eta(t) \) will subsequently be optimized to achieve the desired regret bound. Typically, \( \eta(t) \) is set proportional to \( \sqrt{t+1} \). Substituting \( L + \eta(t) \) for \( \alpha(t)^{-1} \) and observing that \( \frac{1}{2} \| w(t) - w(t+1) \|^2 \leq D_\psi \left( w(t+1), w(t) \right) \) since \( \psi(x) \) is strongly convex, and then noticing that the \( LD_\psi(w(t+1), w(t)) \) term cancels. Equation (39) then becomes

\[F(w(t+1)) - F(w^*) \leq \]
\[-\langle z(t+1), w(t+1) - w^* \rangle - \langle z(t), w(t) - w^* \rangle + \frac{1}{\alpha(t)} \left[ \psi(w(t+1)) - \psi(w(t)) \right] \]
\[+ \langle e(t), w(t+1) - w^* \rangle - \eta(t) D_\psi(w(t+1), w(t)). \] (40)

Now we sum both sides of (40) from \( t = 1 \) to \( t = T \) and note that the term \( \langle z(1), w(1) - w^* \rangle = 0 \) since \( z(1) = 0 \) to get

\[\sum_{t=1}^{T} F(w(t+1)) - F(w^*) \leq \]
\[-\langle z(T+1), w(T+1) - w^* \rangle + \frac{\psi(w(T+1))}{\alpha(T)} - \]
\[\frac{\psi(w(1))}{\alpha(1)} + \sum_{t=2}^{T} \frac{\psi(w(t))}{\alpha(t-1)} - \frac{1}{\alpha(t)} \]
\[+ \sum_{t=1}^{T} \eta(t) D_\psi(w(t+1), w(t)) + \sum_{t=1}^{T} \langle e(t), w(t+1) - w^* \rangle \]
\[\leq \langle z(T+1), w(T+1) - w^* \rangle + \frac{\psi(w(T+1))}{\alpha(T+1)} \]
\[-\sum_{t=1}^{T} \eta(t) D_\psi(w(t+1), w(t)) + \sum_{t=1}^{T} \langle e(t), w(t+1) - w^* \rangle \]
\[\leq \frac{\psi(w^*)}{\alpha(T+1)} - \sum_{t=1}^{T} \eta(t) D_\psi(w(t+1), w(t)) \]
\[+ \sum_{t=1}^{T} \langle e(t), w(t+1) - w^* \rangle \] (43)

where (42) follows from the facts that \( \frac{1}{\alpha(t)} - \frac{1}{\alpha(t+1)} \leq 0 \) (and hence \( \alpha(T) \geq \alpha(T+1) \)) and \( \psi(w) \geq 0 \) so we can drop the negative term \( -\frac{\psi(w(1))}{\alpha(1)} \). Equation (43) follows from that \( \langle z(T+1), w(T+1) - w^* \rangle + \frac{\psi(w(T+1))}{\alpha(T+1)} \leq \frac{\psi(w^*)}{\alpha(T+1)} \). To see this, observe that according to dual average rule \( w(T+1) = \arg \min_{w \in \mathcal{W}} \left\{ \langle z(T+1), w \rangle + \frac{1}{\alpha(T+1)} \psi(w) \right\} \). So, if we substitute \( w^* \) for \( w(T+1) \) in the first two term on the right hand side of (42), the expression will be larger. This substitution eliminates the inner product term and we get (43).

Next, we look at the last term in (43); i.e., \( \sum_{t=1}^{T} \langle e(t), w(t+1) - w^* \rangle \).

\[\sum_{t=1}^{T} \langle e(t), w(t+1) - w^* \rangle = \]
\[\sum_{t=1}^{T} (\nabla F\left(w(t+1)\right) - \nabla F\left(w(t)\right), w(t+1) - w^*) \]
\[+ \sum_{t=1}^{T} \langle \nabla F\left(w(t)\right) - g(t), w(t+1) - w^* \rangle. \] (44)

Using Lemma A.2, and noting that from Lemma A.3 that \( \mathbb{E} \left[ \| w(t) - w(t+1) \|^2 \right] \leq \frac{4LJ^2}{\eta(t)} \), we can bound the expected value of the first term on the right hand side of (44) as

\[\mathbb{E} \left[ \sum_{t=1}^{T} \langle \nabla F\left(w(t)\right) - \nabla F\left(w(t)\right), w(t+1) - w^* \rangle \right] \]
\[\leq 2\tau JC + 2LJ^2(\tau + 1)^2 \sum_{t=1}^{T} \frac{1}{\eta(t-t)^2}, \] (45)

where the nonincreasing sequence \( \kappa(t) \) in Lemma A.2 is replaced by \( \frac{1}{\kappa(t)} \) since \( \eta(t) \) is nondecreasing.

Now, we look at the second term in (44)

\[\langle \nabla F\left(w(t)\right) - g(t), w(t+1) - w^* \rangle \]
\[= \langle \nabla F\left(w(t)\right) - g(t), w(t) - w^* \rangle \]
\[+ \langle \nabla F\left(w(t)\right) - g(t), w(t) - w(t) \rangle \]
\[\leq \langle \nabla F\left(w(t)\right) - g(t), w(t) - w^* \rangle \]
\[+ \frac{1}{2\eta(t)} \left\| \nabla F\left(w(t)\right) - g(t) \right\|^2 + \frac{\eta(t)}{2} \left\| w(t) - w(t) \right\|^2. \] (46)

(47)

The last inequality follows from the Fenchel-Young inequality which states that for every \( u, v \in \mathbb{R}^n \) given a real valued convex function \( h(u) \) and its convex conjugate \( h^*(u) \), the inner product \( \langle u, v \rangle \leq h(u) + h^*(v) \). In (47), we let \( u = \nabla F\left(w(t)\right) - g(t), v = w(t) - w(t) \), \( h(u) = \frac{\eta(t)}{2} \left\| u \right\|^2 \) and \( h^*(v) = \frac{\eta(t)}{2} \left\| v \right\|^2 \). Note that conditioned on \( g(1), g(2), ..., g(t-1) \), \( w(t) \) is not random and hence \( w(t) \) does not depend on \( \nabla F\left(w(t)\right) - g(t) \). Furthermore, since by definition \( \nabla F(w) = \mathbb{E}\left[ \nabla f(w, x) \right] \), then \( \mathbb{E}\left[ \nabla F\left(w(t)\right) - g(t) \right] = 0 \). Hence, the first term on the right hand side of (47) has zero expectation. Due to the bounded variance assumption, and assuming that the master receives \( b(t) \) gradients in epoch \( t \),
Let $\mathbb{E}[\|\nabla F(w(t) - \tau)) - g(t)\|^2 | b(t) | = \mathbb{E}[\|\nabla F(w(t) - \tau)) - g(t)\|^2 | b(t) |] - \frac{1}{t\eta(t)} \sum_{i=1}^{n} b_i(t) \sum_{s=1}^{F(s)} [f(w(t) - \tau), x_i(t, s)] ||^2 | b(t) | \leq \frac{s^2}{b(t)}.

The bound above follows from the fact that $x_i(t, s)$ are sampled i.i.d, so the cross terms have zero expectation. Taking the expected value of (47) conditioned on $b(t)$

$\mathbb{E}\left[(\nabla F(w(t) - \tau)) - g(t), w(t+1) - w^*\right] | b(t) \leq \frac{1}{2} \mathbb{E}\left[||w(t+1) - w(t)||^2\right]$.

Taking the expectation on both sides in (44) conditioned on $b(t) \in [T]$ and substituting (45) and (48),

$\sum_{t=1}^{T} \mathbb{E}[\langle e(t), w(t+1) - w^*\rangle | b(t) | \leq 2\tau JC + 2LJ^2(\tau + 1)^2 \sum_{t=1}^{T} \frac{1}{t^2} + \frac{s^2}{2} \sum_{t=1}^{T} \eta(t) b(t) + \frac{\eta(t)}{2} \mathbb{E}\left[||w(t+1) - w(t)||^2\right]$.

Substituting this into (43) after taking the conditional expectation given $B_{tot}$ and observing that the term $-\eta(t) D(w(t+1), w(t))$ in (43) is upper bounded by $-\frac{c^2}{2} \sum_{i=1}^{n} b_i(t) \left[\sum_{i=1}^{n} b_i(t) \mathbb{E}\left[F(w(t+1) - F(w^*)\right]\right.$

$\leq \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{b_i(t)}{n} \mathbb{E}[F(w(t+1) - F(w^*)$]

where equation (54) follows from the fact that $\mathbb{E}[f(w(t), x(t))] = \mathbb{E}[\mathbb{E}[f(w(t), x(t)|w(t))] = \mathbb{E}[f(w(t))$. Substituting (52), (53), and (51) in (56), we get

$\mathbb{E}[R(T)] \leq \frac{\bar{b}}{\alpha(T + 1)} \mathbb{E}[\psi(w^*)] + 2\tau JC\bar{b} + 2LJ^2(\tau + 1)^2 \sum_{t=1}^{T} \frac{1}{t^2} + \frac{s^2}{2} \sum_{t=1}^{T} \eta(t) b(t)$.

Finally, the bound in (15) of Theorem IV.1 is obtained from (57) by observing that $\psi(w^*) \leq \frac{C^2}{2}$ and $\alpha(t)^{-1} = L + \sqrt{t + \tau}/\bar{b}$. Hence, $1/\alpha(T+1) = L + \sqrt{t + \tau}/\bar{b}$ since $\bar{b} = m/T$.

B. Proof of Corollary IV.2

Let $\hat{w}(T) = \frac{1}{T} \sum_{t=1}^{T} w(t+1)$, then the expected optimality gap can be bounded by
\[ \mathbb{E}[G(T)] = \mathbb{E}[F(\tilde{w}(T))] - F(w) \]  
(58)

\[ \leq \mathbb{E}\left[ \frac{1}{T} \sum_{t=1}^{T} F(w(t+1)) - F(w^*) \right] \]  
(59)

\[ \leq \mathbb{E}\left[ \frac{1}{T} \sum_{t=1}^{T} F(w(t+1)) - F(w^*) \right] \]  
(60)

\[ = \mathbb{E}\left[ \frac{\hat{b}}{m} \sum_{i=1}^{T} F(w(t+1)) - F(w^*) \right] \]  
(61)

\[ = \frac{\hat{b}}{m} \mathbb{E}\left[ \sum_{t=1}^{T} F(w(t+1)) - F(w^*) \right] \]  
(62)

\[ \leq \frac{\hat{b}}{m} \left( \frac{C^2}{2m} L + \frac{\sqrt{(T+1+\tau)}/b}{m} + \frac{2LJ^2(\tau+1)^2\hat{b}(1+\log T)}{m} + \frac{\sigma^2}{b\sqrt{m}} \right). \]  
(63)

where (60) follows from the convexity of \( F(w) \) and (63) follows from Theorem IV.1.

### Appendix B

**Proof of Theorem V.1**

Define the sample path \( B_{\text{tot}} = \{b_i(t), i \in [n], t \in [T]\} \).

Then conditioned on this sample path, the expected regret is given by

\[ \mathbb{E}[R(T) \mid B_{\text{tot}}] = \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ F(w_i(t+1), x_i(t+1, s)) - F(w^*, x_i(t+1, s)) \right] \]  
(64)

\[ = \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ \sum_{s=1}^{b_i(t)} F(w_i(t+1)) - F(w^*) \right] \]  
(65)

\[ \leq \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ \sum_{s=1}^{b_i(t)} F(w_i(t+1)) - F(w^*) \right] + \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ F(w_i(t+1)) - F(w(t+1)) \right] \]  
(66)

\[ = \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ \sum_{s=1}^{b_i(t)} F(w(t+1)) - F(w^*) \right] + \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ F(w_i(t+1)) - F(w(t+1)) \right] \]  
(67)

\[ \leq \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ \sum_{s=1}^{b_i(t)} F(w(t+1)) - F(w^*) \right] + \sum_{i=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{b_i(t)} \mathbb{E}[\|w_i(t+1) - w(t+1)\|] \]  
(68)

where (68) follows from the Lipschitz continuity assumption (8). To bound the the term \( \|w_i(t+1) - w(t+1)\| \) above we use Lemma 2 from [19] which states that for all \( i \in [n] \) and all \( t \in [T] \),

\[ \|w_i(t+1) - w(t+1)\| \leq \alpha(t+1) \| z_i^{(r)}(t+1) - z(t+1) \|. \]  
(69)

Substituting (23) and (69) in (68), we get

\[ \mathbb{E}[R(T) \mid B_{\text{tot}}] \leq \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ \sum_{s=1}^{b_i(t)} F(w(t+1)) - F(w^*) \right] \]  
(70)

\[ = \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ \sum_{s=1}^{b_i(t)} F(w(t+1)) - F(w^*) \right] + \sum_{i=1}^{n} \sum_{t=1}^{T} J\delta \alpha(t+1) \]  
(71)

Since \( \mathbb{E}[R(T)] = \mathbb{E}[\mathbb{E}[R(T) \mid B_{\text{tot}}]] \), then the expected regret is

\[ \mathbb{E}[R(T)] \leq \sum_{i=1}^{n} \sum_{t=1}^{T} \mathbb{E}\left[ \sum_{s=1}^{b_i(t)} F(w(t+1)) - F(w^*) \right] \]  
(72)

\[ + J\delta \sum_{i=1}^{T} \alpha(t+1). \]  
(73)

We look at the second term \( \sum_{i=1}^{T} J\delta \alpha(t+1) \) and observe that \( \alpha(t+1) \leq 1/\sqrt{(t+\tau)/b} \). Thus,

\[ J\delta \sum_{i=1}^{T} \alpha(t+1) \leq J\delta \sum_{i=1}^{T} \frac{1}{\sqrt{(t+\tau)/b}} \]  
(74)

\[ = J\delta \sum_{i=1}^{T} \frac{1}{\sqrt{(t+\tau)}} \]  
(75)

The first term in (72) is the expected regret for the hub-and-spoke case so we can substitute its bound from (57). Thus, we have

\[ \mathbb{E}[R(T)] \leq \frac{\hat{b}}{\alpha(T+1)} \psi(w^*) + 2\tau JC\hat{b} + 
2LJ^2(\tau+1)^2(1+\log T)\hat{b}^2 + \sigma^2 \frac{\hat{b}}{b}\sqrt{m} + 2J\delta \hat{b}^{3/2}\sqrt{m}. \]  
(76)

### Appendix C

**AMB-DG Pseudocode**

In this section, we present pseudocode for AMB-DG. Algorithm 1 details the process at the workers while Algorithm 2 shows the AMB-DG steps at the master. In Algorithm 1, line 2 is about parameter initialization at the workers, Lines 4-11 correspond to the gradient compute phase. Line 12 is related to worker-to-master communication while in lines 8-10 and
13-15, workers receive the update from the master and update their local versions of the optimization parameters. On the other hand, in Algorithm 2, lines 3-7 are related to worker-to-master communication in which the master receives messages from the workers. Lines 8-10, correspond to the parameter update phase while line 12 represent the communication from the master to all workers.

Algorithm 1 AMB-DG Algorithm (Worker Node)
1: for all $t = 1, 2, \ldots$ do
2: initialize $g_i(t) = 0, b_i(t) = 0, s = 0$
3: $T_0 = \text{current\_time}$
4: while $\text{current\_time} \leq T_p - T_0$ do
5: sample i.i.d input data $x_i(t, s)$ from $P$
6: calculate $g_i(t) = g_i(t) + \nabla f(w(t), x_i(t, s))$
7: $b_i(t)++$, $s+s$
8: end while
9: send $g_i(t), b_i(t)$ to the master
10: if received updated parameter $\tilde{w}$ from master then
11: set $w(t) = \tilde{w}$
12: end if
13: end for

Algorithm 2 AMB-DG Algorithm (Master Node)
1: for all $t = 1, 2, \ldots$ do
2: initialize $b(t) = 0, g(t) = 0$
3: for all $i = 1, 2, \ldots, n$ do
4: receive $b_i(t), g_i(t)$ from worker $i$
5: $b(t) += b_i(t)$
6: $g(t) += g_i(t)$
7: end for
8: $z(t+1) = z(t) + \frac{1}{b(t)} g(t)$
9: $w(t+1) = \text{arg min}_{w \in W} \{ \langle w, z(t+1) \rangle + \frac{1}{\alpha(t+1)} \psi(w) \}$
10: send $w(t+1)$ to all workers
11: end for

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