Theory of a dilute low-temperature trapped Bose condensate

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This set of four lectures reviews various aspects of the theory of a dilute low-temperature trapped Bose gas, starting with (I) a review of the Bogoliubov description of the elementary excitations in a uniform system. The treatment is then generalized (II) to include the new physical effects of a confining harmonic trap potential on the condensate and its normal modes. An equivalent hydrodynamic description (III) focuses directly on the density and velocity fluctuations. The physics of vortices (IV) in an incompressible fluid is summarized and extended to the case of a trapped Bose condensate.

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I. UNIFORM DILUTE BOSE GAS

Bogoliubov’s seminal 1947 paper originated the modern theoretical description of a dilute interacting low-temperature Bose gas. He considered how a weak repulsive interparticle potential affects the ground state and low-lying excited states of a uniform condensed Bose gas, showing that the interactions qualitatively alter the dispersion relation at long wavelengths from the familiar quadratic free-particle form $p^2/2M$ to a linear “phonon”-like structure $sp$ with speed of sound $s$.

A. Brief review of a uniform ideal Bose gas

The standard textbook example of an ideal Bose gas treats a uniform system of $N$ particles in a cubical box of volume $V = L^3$ with periodic boundary conditions and number density $n = N/V$. The relevant single-particle states are plane waves $V^{-1/2} \exp(ik \cdot r)$, with $k = (2\pi/L)(n_1, n_2, n_3)$, where $n_i$ is any integer; the energy of such a state is $\epsilon_k^0 = \hbar^2 k^2/2M$, where $M$ is the particle’s mass. In the classical limit, the thermal de Broglie wavelength $\Lambda_T = (2\pi\hbar^2/Mk_BT)^{1/2}$ is much smaller than the interparticle spacing $l \approx n^{-1/3}$, so that diffraction effects are negligible. The inequality $\Lambda_T \ll l$ necessarily fails as the temperature is reduced (or the number density increases), and quantum diffraction becomes important at a temperature $T_c$ determined by the approximate condition $\Lambda_T \approx l$ (or, equivalently, $n\Lambda_T^3 \approx 1$). A detailed calculation yields the exact expression

$$n\Lambda_T^3 = \zeta(\frac{3}{2}) \approx 2.612,$$

where $\zeta(\frac{3}{2})$ is a Riemann zeta function. The transition at $T_c$ represents the onset of the special form of quantum degeneracy known as Bose-Einstein condensation. Alternatively, the transition temperature $T_c$ is given by the approximate relation

$$k_B T_c \approx \frac{\hbar^2 n^{2/3}}{M}$$

that characterizes the onset of degeneracy in any ideal quantum gas, providing a qualitative description of electrons in metals and white dwarf stars, nucleons in nuclear matter and neutron stars, and the fermionic isotope $^3$He, as well as the bosonic isotope $^4$He.

For $T < T_c$, a uniform ideal Bose gas has a macroscopic number of particles $N_0(T)$ in the lowest single-particle state (here, that with $k = 0$), given by

$$\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2} \quad \text{for } T \leq T_c.$$

Thus a finite fraction of all the particles in a condensed Bose gas occupies a single quantum-mechanical state. In the limit $T = 0$, this fraction is unity; all the particles in a uniform ideal Bose gas at zero temperature and fixed volume $V$ have zero momentum (and therefore exert no pressure on the confining walls).
B. Effect of weak repulsive interactions

Bogoliubov \[1\] introduced the microscopic treatment of a “weakly interacting” or “nearly ideal” Bose gas, starting from the second-quantized Hamiltonian

\[
\hat{H} = \sum_k \epsilon_k \hat{a}_k \hat{a}_k^\dagger + \frac{1}{2V} \sum_{k_1,k_2,k_3,k_4} \hat{V}_{k_1-k_3} \hat{a}_{k_1}^\dagger \hat{a}_{k_2} \hat{a}_{k_3} \hat{a}_{k_4} \delta_{k_1+k_2,k_3+k_4},
\]

(1.4)

where the Kronecker \(\delta\) ensures momentum conservation, and \(\hat{V}_k = \int d^3r \exp(-ik \cdot r)V(r)\) is the Fourier transform of the interparticle potential. The operators \(\hat{a}_k^\dagger\) and \(\hat{a}_k\) obey bosonic commutation relations

\[
[a_{k_1}, a_{k_2}^\dagger] = \delta_{k_1,k_2}, \quad [a_{k_1}, a_{k_2}] = [a_{k_1}^\dagger, a_{k_2}^\dagger] = 0.
\]

(1.5)

Bogoliubov’s basic idea is very simple and elegant. If the interacting Bose gas is sufficiently dilute, it should differ only slightly from an ideal gas, so that most of the particles in the true interacting ground state should have zero momentum. Furthermore, only two-body collisions with small momentum transfer are significant, and these may be characterized by a single parameter known as the s-wave scattering length \(a\) (here generally taken as positive, corresponding to a repulsive interaction). For a strong repulsive potential, \(a\) is simply the range, and the more general case is treated below. In the dilute limit \(a \ll l\) (or, equivalently, \(na^3 \ll 1\)), the interparticle potential may be approximated by a “pseudopotential” \(V(r) \approx g\delta^{(3)}(r)\), with a constant Fourier transform \(\hat{V}_k = g\), and the original Hamiltonian in eq. (1.4) becomes

\[
\hat{H} = \sum_k \epsilon_k \hat{a}_k \hat{a}_k^\dagger + \frac{g}{2V} \sum_{k_1,k_2,k_3,k_4} \hat{a}_{k_1}^\dagger \hat{a}_{k_2} \hat{a}_{k_3} \hat{a}_{k_4} \delta_{k_1+k_2,k_3+k_4}.
\]

(1.6)

C. Review of scattering theory

The model interaction strength \(g\) can be related to the more physical parameter \(a\) by requiring that the many-body Hamiltonian (1.6) reproduce the correct two-body scattering in vacuum \[3\]. The Schrödinger equation for the scattering wave function \(\psi_k(r)\) of two particles with mass \(M\) and initial relative wave vector \(k\) can be rewritten in terms of the relative separation \(r = r_1 - r_2\) and the reduced mass \(\frac{1}{2}M\)

\[
-\frac{\hbar^2}{M} \nabla^2 \psi_k(r) + V(r)\psi_k(r) = E\psi_k(r),
\]

(1.7)

where \(E = 2\epsilon_k = \hbar^2 k^2 / M\). With the outgoing-wave Green’s function of the Helmholtz equation [1], this partial differential equation can be recast as an integral equation

\[
\psi_k(r) = e^{ik \cdot r} - \frac{M}{4\pi\hbar^2} \int d^3r' \frac{e^{ik \cdot (r - r')}}{|r - r'|} V(r') \psi_k(r').
\]

(1.8)

For simplicity, assume that the potential has a finite range (and no bound states), so that eq. (1.8) has the asymptotic form for \(|r| = r \to \infty\)

\[
\psi_k(r) \sim e^{ik \cdot r} + f(k', k) \frac{e^{ikr}}{r}.
\]

(1.9)

This expression is the sum of an incident plane wave \(\exp(ik \cdot r)\) with wave vector \(k\), and an outgoing scattered spherical wave \(r^{-1} \exp(ikr)\) with a proportionality factor

\[
f(k', k) \equiv -\frac{M}{4\pi\hbar^2} \int d^3r' e^{-ik' \cdot r'} V(r') \psi_k(r').
\]

(1.10)

known as the scattering amplitude for a transition from the initial relative wave vector \(k\) to a final relative wave vector \(k'\). Note that this definition remains valid even for a singular potential because it involves the true wave function \(\psi_k\) that vanishes wherever \(V(r)\) diverges. In contrast, the Born approximation takes \(\psi_k(r) \approx \psi_k^B(r) = \exp(ik \cdot r)\), and the resulting Born approximation to the scattering amplitude
\[ f_B(k', k) = -\frac{M}{4\pi\hbar^2} \tilde{V}_{k' - k} \quad (1.11) \]

clearly requires that \( V(r) \) be integrable and hence non-singular.

For a spherically symmetric potential with \( k' = k \) (so that energy is conserved), the scattering amplitude depends on \( k \) and the scattering angle \( \theta \) (defined by \( \cos \theta = \hat{k} \cdot \hat{k}' \))

\[ f(k, \theta) = \sum_{l=0}^{\infty} \frac{2l + 1}{k} e^{i\delta_l} \sin \delta_l P_l(\cos \theta), \quad (1.12) \]

where \( \delta_l \) is the energy-dependent phase shift for the \( l \)th partial wave. In the low-energy limit \( (k \to 0) \), the \( s \)-wave phase shift reduces to \( \delta_0 = -ka \), which defines the \( s \)-wave scattering length \( a \), and higher-order phase shifts are of order \( k^{2l+1} \) for \( l \geq 1 \). Thus, only the \( s \)-wave term \( (l = 0) \) contributes as \( k \to 0 \), and the scattering amplitude takes the simple form

\[ \lim_{|k| = |k'| \to 0} f(k', k) \approx -a. \quad (1.13) \]

For many purposes, it is convenient to introduce a Fourier decomposition of the scattering wave function, with

\[ \psi_k(r) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot r} \tilde{\psi}_k(p), \quad (1.14) \]

and the exact scattering amplitude in eq. (1.10) can be rewritten as a convolution of the Fourier transforms of the interaction potential and the wave function

\[ f(k', k) = -\frac{M}{4\pi\hbar^2} \int \frac{d^3p}{(2\pi)^3} \tilde{V}_{k' - p} \tilde{\psi}_k(p). \quad (1.15) \]

In addition, the outgoing-wave Green’s function has the Fourier representation \( \tilde{\psi}_k(p) \)

\[ -\frac{e^{i|k||r - r'|}}{|r - r'|} = \int \frac{d^3q}{(2\pi)^3} e^{iq \cdot (r - r')} \frac{\delta_0}{k^2 - q^2 + i\eta}, \quad (1.16) \]

where \( \eta \to 0^+ \) defines the appropriate contour in the complex \( q \) plane. A combination with the integral equation (1.8) then yields an exact expression for the Fourier transform \( \tilde{\psi}_k(p) \)

\[ \tilde{\psi}_k(p) = (2\pi)^3 \delta^{(3)}(k - p) - \frac{4\pi f(p, k)}{k^2 - p^2 + i\eta}, \quad (1.17) \]

again expressed as the sum of an incident wave and an outgoing scattered wave.

As a final step, a combination of eqs. (1.15) and (1.17) gives a corresponding integral equation for the scattering amplitude

\[ -\frac{4\pi\hbar^2}{M} f(k', k) = \tilde{V}_{k' - k} - 4\pi \int \frac{d^3p}{(2\pi)^3} \tilde{V}_{k' - p} f(p, k), \quad (1.18) \]

which requires knowledge of the scattering amplitude for \( p^2 \neq k^2 \) (known as “off-the-energy-shell”). Note that the leading term of eq. (1.18) is simply the Born approximation \( f_B \), and an iteration gives the first correction

\[ -\frac{4\pi\hbar^2}{M} f(k', k) \approx \tilde{V}_{k' - k} + \frac{M}{\hbar^2} \int \frac{d^3p}{(2\pi)^3} \tilde{V}_{k' - p} \tilde{V}_{p - k} + \cdots. \quad (1.19) \]

In the limit \( k'^2 = k^2 \to 0 \), this equation becomes a perturbation expansion for the scattering length

\[ \frac{4\pi\hbar^2 a}{M} \approx \tilde{V}_0 - \frac{M}{\hbar^2} \int \frac{d^3p}{(2\pi)^3} |\tilde{V}_p|^2 \frac{1}{p^2} + \cdots, \quad (1.20) \]

and the first correction is finite if \( \tilde{V}_p \) vanishes sufficiently rapidly as \( |p| \to \infty \).
The situation is different in the singular case of the pseudopotential \( \tilde{V}_p = g \), for eq. (1.20) reduces to

\[
\frac{4\pi \hbar^2 a}{M} \approx g - \frac{M g^2}{\hbar^2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 + \cdots}, \tag{1.21}
\]

To leading order, this equation relates the pseudopotential \( g \) to the measurable scattering amplitude

\[
g \approx \frac{4\pi \hbar^2 a}{M}. \tag{1.22}
\]

It typifies the general prescription that a first-order perturbation calculation can usually be extended to treat a dilute hard-core gas by substituting the true \( s \)-wave scattering length \( a \) for the first Born approximation \( a_B = MV_0/4\pi\hbar^2 \); in the present context of a dilute Bose gas, this procedure was suggested by Landau (see last footnote of ref. [1]). In contrast, the second-order correction in eq. (1.21) diverges for large values of \( |p| \). As seen below, the Bogoliubov theory yields a similar divergence for the ground-state energy that disappears when re-expressed in terms of the physical quantity \( a \) (for an alternative treatment of the divergence arising from the use of the pseudopotential, see ref. [1]).

There is one remaining subtlety, for the \( s \)-wave scattering length has been defined as if the particles were distinguishable, whereas the true overall wave function is symmetric. Consequently, the actual differential cross section is obtained from a symmetrized scattering amplitude

\[
\frac{d\sigma}{d\Omega} = |f(k, \theta) + f(k, \pi - \theta)|^2, \tag{1.23}
\]

which reduces to \(|2a|^2 = 4a^2\) in the low-energy limit, four times that for distinguishable particles. The corresponding total cross section \( \sigma_T = 8\pi a^2 \) is obtained by integrating over one hemisphere, because the particles are indistinguishable.

### D. Bogoliubov quasiparticles

Given the approximate relation in eq. (1.22) between the pseudopotential \( g \) and the \( s \)-wave scattering length \( a \), it is now feasible to proceed with Bogoliubov’s [1] analysis of the model hamiltonian in eq. (1.6), following the treatment of ref. [1], secs. 35 and 55. The basic observation is that an imperfect Bose gas at low temperature should resemble an ideal Bose gas to the extent that a macroscopic number \( N_0 \) of particles occupies one single-particle state. In the simplest case of a stationary uniform system, this condensation occurs in the mode with \( k = 0 \), but many other possibilities also can occur (for example, a condensate moving with velocity \( v \)). The effect of the repulsive interactions is to scatter particles from the ideal-gas ground state \( |0\rangle \) to pairs of higher-momentum states with \( \pm k \neq 0 \). This depletion of the zero-momentum condensate implies that \( N_0 \) is definitely less than \( N \), even at \( T = 0 \), but the “condensate fraction” \( N_0/N \) is assumed to remain finite in the thermodynamic limit \( (N \to \infty, V \to \infty) \), with \( N/V \) fixed; this assumption must naturally be verified at the end of the calculation.

Recall that the ground state of an ideal uniform Bose gas \( |\Phi_0(N)\rangle = |N, 0, 0, \cdots\rangle = (a_0)\sqrt{N} (N!)^{-1/2} |0, 0, 0, \cdots\rangle \) has all \( N \) particles in the single-particle mode with \( k = 0 \). The associated annihilation and creation operators then yield macroscopic coefficients when applied to this ground state

\[
a_0|\Phi_0(N)\rangle = \sqrt{N} |\Phi_0(N-1)\rangle, \quad a_0^\dagger|\Phi_0(N)\rangle = \sqrt{N+1} |\Phi_0(N+1)\rangle, \tag{1.24}
\]

whereas their commutator

\[
[a_0, a_0^\dagger] |\Phi_0(N)\rangle = |\Phi_0(N)\rangle \tag{1.25}
\]

gives a coefficient unity. To the extent that the thermodynamic properties of an ideal Bose gas are independent of the addition or subtraction of one particle, it is natural to treat the condensed mode differently from all the others, replacing the operators \( a_0 \) and \( a_0^\dagger \) by pure numbers, because their commutator is of order \( N^{-1/2} \) relative to their individual matrix elements.

The same situation is assumed to describe the ground state \( |\Phi\rangle \) of an imperfect uniform Bose gas, as long as the condensate density

\[
\frac{\langle \Phi| a_0^\dagger a_0 |\Phi\rangle}{V} = \frac{N_0}{V} = n_0 \tag{1.26}
\]
remains finite in the thermodynamic limit. Consequently, Bogoliubov proposed the now famous prescription
\[ a_0, a_0^\dagger \rightarrow \sqrt{N_0}, \] (1.27)
with the operators \( a_0 \) and \( a_0^\dagger \) reinterpreted as pure numbers that commute. For simplicity, he also imposed the additional stronger condition of small total depletion \( N - N_0 \ll N \), which limits the strength of the interparticle potential and requires that the temperature \( T \) be small compared to the critical temperature \( T_c \) for the onset of Bose condensation. It is important to realize that this second assumption is distinct from the Bogoliubov prescription in eq. (1.27), which merely requires the existence of a finite condensate density \( n_0 \) in the thermodynamic limit (the more general case at finite temperature is described, for example, in refs. [1] as well as in Griffin’s lectures in this volume).

Substitute the Bogoliubov prescription (1.27) into the model hamiltonian (1.6) and keep only the leading terms of order \( N_0^2 \) and \( N_0^3 \) (for a uniform gas, the terms of order \( N_0^{3/2} \) vanish identically because of momentum conservation, but a similar situation holds in general). The resulting interaction part of the hamiltonian becomes
\[ \hat{H}_{\text{int}} \approx \frac{g}{2V} \left[ N_0^2 + 2N_0 \sum_{\mathbf{k} \neq 0} \left( a_k^\dagger a_k + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} \right) \right. \] (1.28)
\[ + \left. N_0 \sum_{\mathbf{k} \neq 0} \left( a_k^\dagger a_{-\mathbf{k}} + a_k a_{-\mathbf{k}} \right) \right]. \]

In the present context, it is convenient to consider only states with a fixed number \( N \) of particles (see sec. II for a more general approach using the grand canonical ensemble). The number operator
\[ \hat{N} = \sum_k a_k^\dagger a_k \approx N_0 + \frac{1}{2} \sum_{\mathbf{k} \neq 0} (a_k^\dagger a_k + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}}) \] (1.29)
can then be replaced by its eigenvalue \( N \), and eq. (1.29) thus serves to eliminate \( N_0 \) in favor of \( N \) through terms of order \( N^2 \) and \( N \). As a result, the model hamiltonian (1.6) becomes
\[ \hat{H} \approx \frac{gN^2}{2V} + \frac{1}{2} \sum_{\mathbf{k} \neq 0} \left[ (\epsilon_k^0 + ng) \left( a_k^\dagger a_k + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} \right) + ng \left( a_k^\dagger a_{-\mathbf{k}}^\dagger + a_k a_{-\mathbf{k}} \right) \right]. \] (1.30)

This approximation evidently neglects terms like \( \sum_{\mathbf{k} \neq 0} a_k^\dagger a_k a_k^\dagger a_k^\dagger \), which are small for \( N - N_0 \ll N \). The Bogoliubov hamiltonian (1.30) is a quadratic form in the creation and annihilation operators and can be diagonalized with a canonical transformation.

To demonstrate this remarkable feature, Bogoliubov introduced a new set of “quasiparticle” operators \( \alpha_k \) and \( \alpha_k^\dagger \), defined by the linear transformation
\[ a_k = u_k \alpha_k - v_k \alpha_k^\dagger, \] (1.31a)
\[ a_{-\mathbf{k}}^\dagger = u_k \alpha_{-\mathbf{k}} - v_k \alpha_{-\mathbf{k}}^\dagger, \] (1.31b)
with real isotropic coefficients \( u_k \) and \( v_k \). Note that both operators \( a_k \) and \( a_{-\mathbf{k}}^\dagger \) reduce the total momentum by \( \hbar k \). This transformation is canonical if the new operators also obey bosonic commutation relations, with \([\alpha_k, \alpha_{k'}^\dagger] = \delta_{k,k'} \) and \([\alpha_k, \alpha_k'] = [\alpha_k^\dagger, \alpha_k^\dagger] = 0 \). It is easy to verify that this condition requires
\[ u_k^2 - v_k^2 = 1 \quad \text{for all } k \neq 0. \] (1.32)

Direct substitution of eq. (1.31) into the Bogoliubov hamiltonian (1.30) yields
\[ \hat{H} = \frac{1}{2} g n^2 V + \sum_{\mathbf{k} \neq 0} \left[ (\epsilon_k^0 + ng) v_k^2 - ng u_k v_k \right] \]
\[ + \frac{1}{2} \sum_{\mathbf{k} \neq 0} \left[ (\epsilon_k^0 + ng) (u_k^2 + v_k^2) - 2ng u_k v_k \right] \left( \alpha_k^\dagger \alpha_k + \alpha_{-\mathbf{k}}^\dagger \alpha_{-\mathbf{k}} \right) \]
\[ + \frac{1}{2} \sum_{\mathbf{k} \neq 0} \left[ ng (u_k^2 + v_k^2) - 2 (\epsilon_k^0 + ng) u_k v_k \right] \left( \alpha_k^\dagger \alpha_{-\mathbf{k}}^\dagger + \alpha_k \alpha_{-\mathbf{k}} \right). \] (1.33)
Here, the first line is a pure number with no operator character, and the second line is diagonal in the quasiparticle number operator \( \alpha_k^\dagger \alpha_k \). In contrast, the third is off-diagonal in the quasiparticle number, but it can be eliminated entirely by choosing the coefficients \( u_k \) and \( v_k \) to satisfy

\[
ng \left( u_k^2 + v_k^2 \right) = 2 \left( \epsilon_k^0 + ng \right) u_k v_k.
\]

The subsidiary condition \( u_k^2 - v_k^2 = 1 \) in eq. (1.32) can be incorporated automatically by writing

\[
u_k = \cosh \theta_k \quad \text{and} \quad v_k = \sinh \theta_k,
\]

and a combination with eq. (1.34) readily yields

\[
tanh 2\theta_k = \frac{ng}{\epsilon_k^0 + ng}.
\]

If \( g \) is negative, the right-hand side of this equation will diverge for some \( k \), implying unphysical values for a range of the \( \theta_k \)'s. The present case of a uniform dilute Bose condensate therefore requires repulsive interactions, with both \( g \) and \( a \) positive.

The resulting equations for \( u_k^2 \) and \( v_k^2 \) are easily solved to yield

\[
v_k^2 = u_k^2 - 1 = \frac{1}{2} \left( \frac{\epsilon_k^0 + ng}{E_k} - 1 \right),
\]

where

\[
E_k = \sqrt{(\epsilon_k^0 + ng)^2 - (ng)^2} = \sqrt{(\epsilon_k^0)^2 + 2nge_k^0}
\]

is defined as the positive square root. Substitution into eq. (1.33) gives the final and simple quasiparticle hamiltonian

\[
\hat{H} = \frac{1}{2} gn^2 V - \frac{1}{2} \sum_{k \neq 0} \left( \epsilon_k^0 + ng - E_k \right) + \frac{1}{2} \sum_{k \neq 0} E_k \left( \alpha_k^\dagger \alpha_k + \alpha_k^\dagger \alpha_{-k} \right),
\]

which involves only the quasiparticle number operator for each normal mode \( k \). Since \( \alpha_k^\dagger \alpha_k \) has the eigenvalues 0, 1, 2, \( \cdots \), it follows immediately that the ground state \( | \Phi \rangle \) of this model hamiltonian is simply the quasiparticle vacuum, defined by

\[
\alpha_k | \Phi \rangle = 0 \quad \text{for all} \ k \neq 0;
\]

evidently, \( | \Phi \rangle \) is a very complicated combination of unperturbed particle states, for neither \( a_k \) nor \( a_k^\dagger \) annihilates it. In addition, the ground-state energy is given by

\[
E_g = \langle \Phi | \hat{H} | \Phi \rangle = \frac{1}{2} gn^2 V - \frac{1}{2} \sum_{k \neq 0} \left( \epsilon_k^0 + ng - E_k \right) = \frac{1}{2} gn^2 V - \sum_{k \neq 0} E_k v_k^2.
\]

The physics of the quasiparticle hamiltonian is very transparent, for all the excited states correspond to various numbers of non-interacting bosonic quasiparticles, each with a wave vector \( k \) and an excitation energy

\[
E_k = \sqrt{\frac{ng \hbar^2 k^2}{M} + \left( \frac{\hbar^2 k^2}{2M} \right)^2}
\]

\[
\approx \begin{cases} 
\sqrt{\frac{ng}{M} \hbar k} = \sqrt{\frac{4\pi \hbar^2 an}{M^2}} \hbar k, & \text{for} \ k \to 0, \\
\epsilon_k^0 + \frac{4\pi \hbar^2 an}{M}, & \text{for} \ k \to \infty.
\end{cases}
\]

In the long-wavelength limit (\( k \to 0 \)), the elementary excitations represent sound waves with the propagation speed

\[
s = \sqrt{\frac{ng}{M}} = \frac{\hbar}{M} \sqrt{4\pi an},
\]

\[
6
\]
which again shows that $g$ and $a$ must both be positive. For short wavelengths, in contrast, the spectrum is that of a free particle, but shifted upward by a constant “optical potential” $gn = 4\pi\hbar^2 an/M$ arising from the Hartree interaction with the remaining particles (equivalently, this shift reflects the forward scattering from the background medium, producing an effective index of refraction).

The cross-over between the linear and quadratic regimes occurs when $\epsilon_k^0/ng \approx 1$. This ratio can be rewritten as $k^2/8\pi na \approx 1$, suggesting the introduction of a characteristic length

$$\xi = \frac{1}{\sqrt{8\pi na}}$$

(1.45)

that is frequently known as the “correlation length” [3] or the “coherence length;” as shown in sec. [4], however, the term “healing length” is usually preferable in the context of a non-uniform dilute trapped Bose gas. Note that $\xi \to \infty$ for a non-interacting gas ($a \to 0$); in this limit, $E_k$ reduces to the quadratic free-particle spectrum for all $k$, and the linear phonon-like region then vanishes entirely. The energy spectrum in eq. (1.42) can be rewritten as $E_k/ng = E_k M/(4\pi\hbar^2 an) = k\xi \sqrt{2 + k^2\xi^2}$, shown in fig. 1 as a function of the dimensionless combination $k\xi$.

The Bogoliubov transformation (1.31) can be inverted to express the quasiparticle operators in terms of the original particle operators

$$\alpha_k = u_k a_k + v_k a_{-k}^\dagger,$$  \hspace{1cm} (1.46a)

$$\alpha_{-k}^\dagger = u_k a_{-k}^\dagger + v_k a_k.$$  \hspace{1cm} (1.46b)

The quasiparticle operators are coherent linear superpositions of particle and hole operators, with $u_k$ and $v_k$ as the weight factors. In the phonon regime ($k\xi \ll 1$), these coefficients are both large, with $u_k^2 \approx v_k^2 \approx (\sqrt{8k}\xi)^{-1} \gg 1$, and the long-wavelength quasiparticle operators therefore represent a coherent nearly equal admixture of a particle and a hole. In the free-particle regime ($k\xi \gg 1$), however, the behavior changes to $u_k^2 \approx 1$ and $v_k^2 \approx (4k^4\xi^4)^{-1} \ll 1$, so that the short-wavelength quasiparticle creation operator is effectively a pure particle creation operator. Figure 2 shows the quantities $u_k^2$ and $v_k^2$ as a function of the dimensionless wavenumber $k\xi$, illustrating clearly the qualitative effect of the repulsive interactions on the long-wavelength properties of a uniform dilute Bose gas.

E. Macroscopic properties

Within the Bogoliubov approximation, the low-temperature behavior can be described with the quasiparticle Hamiltonian $\hat{H}$ in eq. (1.31) and an associated Gibbs ensemble density operator

$$\hat{\rho} = \frac{\exp(-\beta \hat{H})}{\text{Tr} \left[ \exp(-\beta \hat{H}) \right]},$$

(1.47)

where $\beta^{-1} = k_BT$ and the denominator is simply the partition function (the notation $\text{Tr}$ denotes a sum over all states). Manipulations with this density operator are straightforward, for $\hat{H}$ is simply a sum of non-interacting bosonic contributions, yielding a thermal average over familiar harmonic-oscillator states. For example, consider the total particle number at low temperature

$$N = N_0 + \frac{1}{2} \sum_{k \neq 0} \text{Tr} \left[ \hat{\rho} \left( a_{-k} a_k + a_k^\dagger a_{-k}^\dagger \right) \right].$$

(1.48)

The trace is invariant under a canonical transformation, and it is most readily evaluated in the quasiparticle basis where $\hat{H}$ is diagonal. Use of the Bogoliubov transformations from eq. (1.31) leads to various products of two quasiparticle and quasihole operators. Those involving off-diagonal quantities like $\text{Tr} \left( \hat{\rho} \alpha_{-k}^\dagger \alpha_k \right)$ and $\text{Tr} \left( \hat{\rho} \alpha_k \alpha_{-k} \right)$ vanish identically, and the diagonal quantities are just the finite-temperature harmonic-oscillator occupation numbers

$$\text{Tr} \left( \hat{\rho} \alpha_k^\dagger \alpha_k \right) = f(E_k) \equiv \frac{1}{\exp(\beta E_k) - 1} \quad \text{and} \quad \text{Tr} \left( \hat{\rho} \alpha_k \alpha_k^\dagger \right) = 1 + f(E_k),$$

(1.49)

involving the familiar Planck (or Bose-Einstein) distribution function $f(E_k)$ for a quasiparticle with energy $E_k$ in thermal equilibrium at temperature $T$ [note that $f(E_k)$ has no chemical potential because quasiparticles are not conserved].
A combination of these results with eq. (1.48) immediately yields the relation

$$N = N_0 + \sum_{k \neq 0} \left[ v_k^2 + \frac{u_k^2 + v_k^2}{\exp(\beta E_k) - 1} \right] = N_0 + \sum_{k \neq 0} \left[ v_k^2 + \left( u_k^2 + v_k^2 \right) f(E_k) \right].$$  \hspace{1cm} (1.50)

Since $N$ is fixed, this equation determines implicitly the temperature-dependent condensate number $N_0(T)$. In a similar way, the low-temperature internal energy $E(T)$ is simply the finite-temperature average of the Bogoliubov quasiparticle hamiltonian in eq. (1.33)

$$E(T) = \text{Tr} \left( \rho H \right) = E_g + \sum_{k \neq 0} \frac{E_k}{\exp(\beta E_k) - 1} = E_g + \sum_{k \neq 0} E_k f(E_k);$$  \hspace{1cm} (1.51)

expressed as the zero-temperature ground-state energy $E_g$ plus the sum over all thermally weighted excited states.

For simplicity, consider the total number of non-condensate particles $N' \equiv N - N_0$ (the depletion of the condensate arising from the repulsive interactions). At zero temperature, the thermal factors vanish, yielding

$$N'(T = 0) = N - N_0(T = 0) = \sum_{k \neq 0} v_k^2.$$  \hspace{1cm} (1.52)

This equation shows that $v_k^2$ can be interpreted as the number $N'_k$ of non-condensate particles with wave number $k$ in the Bogoliubov ground state $|0\rangle$; since $v_k^2 \propto (k\xi)^{-1}$ is large for $k\xi \ll 1$, the low-lying phonon-like modes have large occupation numbers (this behavior reflects the Bose condensation in the mode with $k = 0$). Nevertheless, the total noncondensate number remains finite because $v_k^2$ vanishes rapidly for $k\xi \gg 1$.

The sum over the excited plane-wave states $k$ in eq. (1.52) can be approximated as an integral $\sum_{k \neq 0} \cdots \approx V(2\pi)^{-3} \int d^3k \cdots$, and the total zero-temperature fractional depletion becomes

$$\frac{N'(T = 0)}{N} = \frac{N - N_0(T = 0)}{N} \approx \frac{1}{n} \int \frac{d^3k}{(2\pi)^3} v_k^2$$

$$= 4 \left( \frac{2na^3}{\pi} \right)^{1/2} \int_0^\infty y^2 dy \left[ \frac{y^2 + 1}{(y^4 + 2y^2)^{1/2}} - 1 \right]$$

$$= \frac{8}{3} \left( \frac{na^3}{\pi} \right)^{1/2}.$$  \hspace{1cm} (1.53)

This result exhibits $\sqrt{na^3}$ as the relevant expansion parameter, and the assumption of small total depletion requires that $\sqrt{na^3} \ll 1$. Note that the fractional depletion is non-analytic in the interaction strength $a$ (or $g$), for the $\frac{3}{4}$ power requires a branch cut in the complex $a$ plane. As a result, this and other corrections to the physical properties of an ideal Bose gas are non-perturbative; they cannot be obtained with conventional perturbation theory. Such a conclusion here presents no conceptual difficulty, however, for it relies on (a non-perturbative) canonical transformation. It is not difficult to obtain the first low-temperature contribution to the depletion

$$\frac{N'(T)}{N} = \frac{N - N_0(T)}{N} \approx \frac{8}{3} \left( \frac{na^3}{\pi} \right)^{1/2} + \frac{M(k_BT)^2}{12\hbar^38n},$$ \hspace{1cm} (1.54)

It is also interesting to evaluate the ground-state energy from eq. (1.41), which has the form

$$E_g = \frac{1}{2}gn^2V - \frac{g}{2} \sum_{k \neq 0} \left( \epsilon_k^0 + n_g - E_k \right).$$ \hspace{1cm} (1.55)

An expansion of $E_k$ for large $k\xi$ shows that the sum diverges like $g^2 \sum_{k \neq 0} k^{-2}$, which reflects a failure of second-order perturbation theory for a dilute Bose gas. Here, it arises from the assumption that the Fourier transform $g$ of the interparticle potential remains constant for large wave numbers, and the same divergence appeared in eq. (1.21) for the scattering length to order $g^2$. Equation (1.55) can be rewritten by adding and subtracting the divergent second-order contribution

$$E_g = \frac{1}{2}gn^2V - \frac{g}{2}n^2 \sum_{k \neq 0} \frac{M}{\hbar^2k^2} + \frac{1}{2} \sum_{k \neq 0} \left( E_k - \epsilon_k^0 - n_g + \frac{Mg^2n^2}{\hbar^2k^2} \right),$$  \hspace{1cm} (1.56)
and it is easy to verify that the last sum converges. In addition, the first two terms on the right-hand side are precisely $\frac{1}{4}n^2V$ times those in eq. (21) for $4\pi a^3/M$, where $a$ is the physical scattering length. A detailed evaluation yields the ground-state energy per particle \[ \frac{E_g}{N} = \frac{1}{2}n \left( g - \frac{Mg^2}{\hbar^2} \right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} + \frac{1}{2}g \int \frac{d^3k}{(2\pi)^2} \left( \frac{E_k - \frac{\hbar^2}{2M}}{\hbar^2} - 1 + \frac{nq}{2E_k} \right) 
 \]
\[ = \frac{2\pi \hbar^2 an}{M} \left[ 1 + 8 \left( \frac{na^3}{\pi} \right)^{1/2} \right] \int_0^\infty y^2 dy \left[ (y^4 + 2y^2)^{1/2} - y^2 - 1 + \frac{1}{2y^2} \right] \] \[ = \frac{2\pi \hbar^2 an}{M} \left[ 1 + 128 \left( \frac{na^3}{\pi} \right)^{1/2} \right], \quad (1.57) \]

which again exhibits the same non-analytic dependence on $\sqrt{na^3}$.

Equation (1.57) expresses the ground-state energy as an explicit function of the number $N$ and volume $V$, and standard thermodynamics provides many other important ground-state properties. For example, the chemical potential $\mu$ and pressure $p$ follow from

\[ \mu = \left( \frac{\partial E_g}{\partial N} \right)_V = \frac{4\pi \hbar^2 an}{M} \left[ 1 + \frac{32}{3} \left( \frac{na^3}{\pi} \right)^{1/2} \right], \quad (1.58) \]

and

\[ p = - \left( \frac{\partial E_g}{\partial V} \right)_N = \frac{2\pi \hbar^2 an^2}{M} \left[ 1 + \frac{64}{5} \left( \frac{na^3}{\pi} \right)^{1/2} \right]. \quad (1.59) \]

In addition, the speed of sound $s$ follows from the compressibility through the general formula $s^2 = M^{-1} (\partial p/\partial n)$, giving

\[ s = \frac{\hbar}{M} \sqrt{4\pi an} \left[ 1 + 8 \left( \frac{na^3}{\pi} \right)^{1/2} \right]. \quad (1.60) \]

It is striking that the speed of sound obtained here with macroscopic thermodynamics agrees to leading order with the slope of the quasiparticle spectrum given in eq. (1.44); although this zero-temperature relation is not entirely obvious, it has, in fact, been verified to all orders in perturbation theory that the slope of the quasiparticle spectrum agrees precisely with the thermodynamic speed of sound.

F. Moving condensate

It is not difficult to generalize this analysis to the case of Bose condensation in a mode with non-zero momentum $\hbar q$, which implies that the condensate moves with velocity $v = \hbar q/M$. The quasiparticle energy spectrum $E_k$ for the state with the proper positive normalization $|u|^2 - |v|^2 = 1$ is simply related to the energy $E_k^0$ for a stationary condensate:

\[ E_k = \hbar k \cdot v + E_k^0. \quad (1.61) \]

This expression can be considered a Doppler shift in the frequency; its form agrees precisely with Landau’s general result in his celebrated 1941 paper on superfluid $^4$He. In the long-wavelength limit, the excitation energy reduces to $E_k \approx \hbar k (v \cos \theta + s)$, where $s$ is the Bogoliubov speed of sound and $\theta$ is the angle between $k$ and the flow velocity $v$. For $v \leq s$, the quasiparticle energy is positive for all physical angles $\theta$; if $v$ exceeds $s$, however, the quasiparticle energy becomes negative in a cone around the backward direction where $\cos \theta \approx -1$. Consequently, the quasiparticle hamiltonian in eq. (1.39) becomes unstable, and the system can lower its energy arbitrarily by creating quasiparticles with negative energy. This behavior corresponds to the well-known Landau critical velocity $v_c$ for the onset of dissipation in a superfluid. In the Bogoliubov model, $v_c$ is equal to the speed of sound $s$. Physically, the finite $s$ reflects the presence of repulsive interactions [see eq. (1.44)] and therefore $s$ vanishes for an ideal Bose gas. As a corollary, an ideal Bose gas also has $v_c = 0$ and hence cannot sustain superfluid flow.
At $T = 0$, it is straightforward to prove that the total momentum $P$ carried by both the condensed and non-condensed particles is $P = NMv$, where $N$ is the total number of particles, not the condensate number $N_0$. This result implies that the Bose condensation induces a coherence or rigidity, ensuring that, in effect, all the particles participate in the motion. The same conclusion also follows with a Galilean transformation from the rest frame of the condensate to a frame moving with velocity $-v$.

At finite temperature $T \neq 0$, the corresponding total momentum decreases, according to the expression $P = (\rho_s/\rho) NMv$, where the reduction factor $\rho_s(T)/\rho \leq 1$ is the temperature-dependent superfluid fraction first calculated by Landau [11,12]. In his model for superfluidity, the total density $\rho$ consists of two separate parts, the superfluid density $\rho_s$ and the normal fluid density $\rho_n$, with $\rho_s + \rho_n = \rho$. The normal fluid density vanishes at $T = 0$ and depends solely on the excitation spectrum in the rest frame of the condensate. For the Bogoliubov energy $E_k$, the normal fluid fraction $\rho_n/\rho$ is proportional to $T^4$ at low temperatures, similar to the phonon contribution in superfluid $^4$He [13].

II. DILUTE BOSE GAS IN A HARMONIC TRAP

The basic physics of a uniform dilute Bose gas has been well understood for over 30 years [6,10]. The recent intense renewed interest in such systems arises from the remarkable experimental demonstration of Bose-Einstein condensation of dilute ultra-cold alkali atoms in harmonic traps [14–16]. The presence of a trap alters the physics in several important ways, and it is essential to reformulate the theory to incorporate the many qualitatively new features [17].

A. Ideal Bose gas in a harmonic trap

In the usual case, the trap is an axisymmetric harmonic-oscillator potential

$$V_{tr}(r) = V_{tr}(r_\perp, z) = \frac{1}{2} M (\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2),$$

(2.1)

where the coordinate vector $r$ is expressed in cylindrical polar coordinates $(r_\perp, \phi, z)$. Here, $\omega_\perp$ and $\omega_z$ are the radial and axial angular frequencies, and it is common to introduce the “anisotropy parameter”

$$\lambda \equiv \frac{\omega_z}{\omega_\perp}.$$  

(2.2)

A particle with mass $M$ in such a potential has a Gaussian ground-state wave function

$$\psi_g(r_\perp, z) \propto \exp \left[ - \frac{1}{2} \left( \frac{r_\perp^2}{d_\perp^2} + \frac{z^2}{d_z^2} \right) \right],$$

(2.3)

with radial and axial dimensions

$$d_\perp = \sqrt{\frac{\hbar}{M \omega_\perp}} \quad \text{and} \quad d_z = \sqrt{\frac{\hbar}{M \omega_z}} = \frac{d_\perp}{\lambda^{1/2}},$$

(2.4)

which are the relevant oscillator lengths (the mere presence of the trap introduces a new characteristic length scale that will play a significant role in the physics of these systems). The anisotropy in the ideal-gas density profile is simply

$$\frac{d_z^2}{d_\perp^2} = \frac{1}{\lambda} \quad \text{for a non-interacting gas},$$

(2.5)

where $\lambda \gg 1$ represents a “pancake” and $\lambda \ll 1$ represents a “cigar.”

What is the transition temperature $T_c$ for the onset of Bose-Einstein condensation in an ideal gas in such a trap? For many qualitative discussions, it is convenient to define a mean frequency $\omega_0 = (\omega_\perp^2 \omega_z)^{1/3}$ and a mean oscillator length $d_0 = (d_\perp^2 d_z^2)^{1/3}$. Consider the classical limit, with $k_B T \gg \hbar \omega_0$. Instead of the ground-state density $|\psi_g(r)|^2$, the classical density profile has the Maxwell-Boltzmann form

$$n(r) \propto \exp \left[ - \beta V_{tr}(r) \right] = \exp \left[ - \frac{V_{tr}(r)}{k_B T} \right],$$

(2.6)
where $\beta^{-1} = k_B T$. For simplicity, consider a spherical trap ($\lambda = 1$), when the density

$$n(r) \propto \exp \left(-\frac{r^2}{R_T^2}\right), \quad (2.7)$$

is isotropic with a mean thermal radius $R_T$ given by

$$R_T^2 = \frac{2k_B T}{M\omega_0^2} = d_0^2 \frac{2k_B T}{\hbar\omega_0} \gg d_0^2. \quad (2.8)$$

In this classical limit, the density is again Gaussian, but its mean dimension $R_T$ necessarily exceeds the mean ground-state radius $d_0$. Correspondingly, the mean number density is of order $n \sim N/R_T^3$.

Recall the estimate $k_B T_c \sim \hbar^2 n^{2/3}/M$ in eq. (1.2) for the onset of quantum degeneracy in an ideal Bose gas. A combination of these expressions readily shows that

$$k_B T_c \sim N^{1/3} \hbar \omega_0, \quad (2.9)$$

and a detailed calculation gives the more accurate value $k_B T_c = [\zeta(3)]^{-1/3} N^{1/3} \hbar \omega_0 \approx 0.941 N^{1/3} \hbar \omega_0$. Since $N$ typically is large in most cases of interest, the Bose-Einstein transition indeed occurs in the classical regime ($k_B T_c \gg \hbar \omega_0$). For $T < T_c$, a Bose condensate forms in the lowest single-particle state of the harmonic oscillator potential, whose width is the oscillator length $d_0$, much smaller than the thermal width $R_T$ of the non-condensate for $T \lesssim T_c$. The appearance of this sharp narrow spike in the particle density served as a dramatic signal of the first Bose condensation in a dilute alkali gas [14].

In a typical situation (see, for example, ref. [18]), the trap has a frequency $\omega_0/2\pi \sim 100$ Hz and an oscillator length $d_0 \sim 2 \mu m$, containing $N \sim 10^6$ particles. The ground-state energy corresponds to a temperature $\hbar \omega_0/k_B \sim 5 \times 10^{-9}$ K. In this case, $N^{1/3} \sim 100$, so that $T_c \sim 0.5 \times 10^{-6}$ K; at this temperature, the ratio $R_T/d_0 \sim N^{1/6}$ is of order 10.

### B. Effect of interactions on a trapped Bose condensate

As in the case of a uniform condensate, the Bose gas is assumed to be dilute, in the sense that the interparticle spacing $l \sim n^{-1/3}$ is large compared to the $s$-wave scattering length $a$ and the range of the interactions. Thus the interparticle potential is again approximated as a pseudopotential $V(r) \approx g \delta^{(3)}(r)$, with $g \approx 4\pi \hbar^2 a/M$, as in eq. (1.22). Typically this low-density condition is well satisfied, with $a \sim 4$ nm and mean number density $n \sim 10^{30}$ m$^{-3}$, implying $l \sim 4 \times 10^{-7}$ m and $a \ll l$. In addition, the condition $a \ll d_0$ holds whenever $na^3 \ll 1$, since the existence of a macroscopic Bose condensate implies that $l \ll d_0$.

The first question is how to generalize the Bogoliubov theory to describe a non-uniform Bose gas in a trap. Although various approaches lead to essentially the same final results, the treatment here [19] emphasizes the field-theory aspects of the many-body Hamiltonian

$$\hat{H} = \int d^3 r \left[ \hat{\psi}^\dagger (T + V_{tr}) \hat{\psi} + \frac{i}{\hbar} \hat{\psi}^\dagger \hat{\rho}^\dagger \hat{\psi} \right], \quad (2.10)$$

where

$$T = -\frac{\hbar^2 \nabla^2}{2M} \quad (2.11)$$

is the kinetic-energy operator, $V_{tr}$ is the harmonic potential given in eq. (2.1), and the two-body potential has been replaced by a short-range pseudopotential $V(r) \approx g \delta^{(3)}(r)$. Here, $\hat{\psi}$ and $\hat{\psi}^\dagger$ are field operators that obey bosonic commutation relations

$$\left[ \hat{\psi}(r), \hat{\psi}^\dagger (r') \right] = \delta^{(3)}(r - r'), \quad \left[ \hat{\psi}(r), \hat{\psi}(r') \right] = \left[ \hat{\psi}^\dagger (r), \hat{\psi}^\dagger (r') \right] = 0. \quad (2.12)$$

As an aside, the field operator for the uniform system has the simple form

$$\hat{\psi}(r) = \sum_k \frac{1}{\sqrt{V}} e^{ikr} a_k, \quad (2.13)$$
using the normalized plane-wave single-particle wave functions. More generally, given any complete set of normalized single-particle wave functions \( \{ \chi_j(r) \} \), the corresponding field operator is a sum over all normal modes

\[
\hat{\psi}(r) = \sum_j \chi_j(r) a_j, 
\]

(2.14)

where \( a_j \) is a corresponding bosonic annihilation operator for the single-particle mode \( j \).

In the Bogoliubov approximation for a non-uniform system, one single mode is assumed to be macroscopically occupied. The exact field operator is therefore separated into two terms,

\[
\hat{\psi}(r) \approx \Psi(r) + \hat{\phi}(r), 
\]

(2.15)

where \( \Psi \) is the (large) condensate wave function [analogous to the Bogoliubov prescription in eq. (1.27)] and \( \hat{\phi} \) is the (small) operator characterizing the remaining non-condensate. Formally, the condensate wave function \( \Psi = \langle \hat{\psi} \rangle \) is an ensemble average of the full field operator in a particular ensemble that breaks the symmetry with respect to the phase \( \theta \), and the remaining fluctuation part \( \hat{\phi} \) can be defined by the condition \( \langle \hat{\phi} \rangle = 0 \). In particular, the condensate density \( n_0(r) \) is given by

\[
n_0(r) = |\Psi(r)|^2, 
\]

(2.16)

and the total number of condensate particles is

\[
N_0 = \int d^3r |\Psi|^2. 
\]

(2.17)

It is most convenient to introduce the grand canonical ensemble at temperature \( T \) and chemical potential \( \mu \), studying the modified hamiltonian

\[
\hat{K} = \hat{H} - \mu \hat{N}, 
\]

(2.18)

where \( \hat{N} = \int d^3r \hat{\psi}^\dagger \hat{\psi} \) is the number operator. For a uniform system, the condensate number density \( n_0 \) is simply a spatial constant. For a non-uniform system, however, the determination of \( \Psi \) and \( n_0 \) is in general quite complicated. In analogy with the leading term of the Bogoliubov hamiltonian in eq. (1.30), where all the creation and annihilation operators were replaced by the condensate contributions \( \sqrt{N_0} \), the condensate part of the modified hamiltonian

\[
K_0 = H_0 - \mu N_0 = \int d^3r \left[ \Psi^* (T + V_{tr} - \mu) \Psi + \frac{1}{2} g |\Psi|^2 \Psi \Psi^* \right] 
\]

(2.19)

is assumed to dominate the physics at low temperature.

In the grand canonical ensemble at zero temperature, equilibrium corresponds to the minimum value of the thermodynamic potential, which is the ensemble average of the modified hamiltonian \( \langle \hat{K} \rangle \). For the present situation of a macroscopic non-uniform condensate, this quantity is approximately \( K_0 \) in eq. (2.19), and the only available parameter to vary is the condensate wave function itself. Thus, the value of the integral \( K_0 \) must be stationary under the variation \( \Psi^* \to \Psi^* + \delta \Psi^* \). The corresponding Euler-Lagrange equation

\[
(T + V_{tr} - \mu) \Psi + g |\Psi|^2 \Psi = 0 
\]

(2.20)

is known as the Gross-Pitaevskii (GP) equation [20,21]. This non-linear Schrödinger equation for the condensate wave function is expected to hold at low temperature; it is similar in form (but not in spirit) to the non-linear Ginzburg-Landau equation for the superconducting order parameter that applies near the superconducting transition temperature (see, for example, ref. [12], sec. 45).

It is convenient to introduce the “Hartree” potential energy \( V_H(r) \), which is the interaction potential of a particle at \( r \) with all the other condensed particles

\[
V_H(r) \approx \int d^3r' V(r - r') n_0(r') \approx g n_0(r) = g |\Psi(r)|^2, 
\]

(2.21)

so that the GP equation assumes the equivalent form

\[
(T + V_{tr} + V_H - \mu) \Psi = 0. 
\]

(2.22)
The interaction energy of the condensate is the last term of eq. (2.19)

\[ E_{\text{int}} = \langle V_{\text{int}} \rangle_0 = \int d^3r \frac{1}{2} g |\Psi|^4 = \frac{1}{2} \int d^3r V_H(r) |\Psi(r)|^2 = \frac{1}{2} \langle V_H(0) \rangle, \]

(2.23)

where the angular bracket here denotes an integral \( \langle \cdots \rangle_0 = \int d^3r \Psi^* \cdots \Psi \) over the ground-state condensate wave function; note that \( E_{\text{int}} \) is half the mean Hartree energy of the condensate.

In time-dependent form, the GP equation becomes

\[ i\hbar \frac{\partial \Psi}{\partial t} = (T + V_t + V_H) \Psi, \]

(2.24)

where the condensate wave function is interpreted as an off-diagonal matrix element of the destruction operator \( \Psi(r, t) = \langle N - 1|\psi(r, t)|N \rangle \). The time-dependence of the operator is given by the usual Heisenberg picture \( \psi(r, t) = \exp(i\hat{H}t/\hbar) \hat{\psi}(r) \exp(-i\hat{H}t/\hbar) \). The time-dependent phase factor in the matrix element thus involves the change in the ground-state energy \( E_g(N - 1) - E_g(N) \) when a particle is removed from the system, which immediately reproduces the original GP eq. (2.20) because \( \mu \approx \partial E_g/\partial N \) at \( T = 0 \).

The ground-state solution of the GP equation satisfies several important relations. The first is the ground-state energy \( E_g \), which is the expectation value of the condensate hamiltonian in eq. (2.19)

\[ E_g = \langle H_0 \rangle_0 = \langle T \rangle_0 + \langle V_t \rangle_0 + \langle V_{\text{int}} \rangle_0 = \langle T \rangle_0 + \langle V_t \rangle_0 + \frac{1}{2} \langle V_H \rangle_0, \]

(2.25)

Second, the expectation value of the GP equation itself yields an expression for the chemical potential

\[ N_0 \mu = \langle T \rangle_0 + \langle V_t \rangle_0 + \langle V_H \rangle_0 = \langle T \rangle_0 + \langle V_t \rangle_0 + 2\langle V_{\text{int}} \rangle_0. \]

(2.26)

Finally, consider a dimensionless scale transformation on the coordinates \( r \rightarrow \zeta r \). The scaling of the various terms in \( H_0 \) follows by inspection to give

\[ E_g \rightarrow \zeta^{-2} \langle T \rangle_0 + \zeta^2 \langle V_t \rangle_0 + \zeta^{-3} \langle V_{\text{int}} \rangle_0; \]

(2.27)

since this scale transformation cannot change the ground-state energy, it follows that \( (\partial E_g/\partial \zeta) |_{\zeta=1} = 0 \), which readily yields the virial theorem \[22\]

\[ 2\langle V_t \rangle_0 = 2\langle T \rangle_0 + 3\langle V_{\text{int}} \rangle_0 = 2\langle T \rangle_0 + \frac{5}{3} \langle V_H \rangle_0. \]

(2.28)

A combination of this identity with Eqs. (2.27) and (2.28) provides the simpler expressions

\[ E_g = \frac{1}{3} \langle T \rangle_0 + \frac{5}{3} \langle V_t \rangle_0, \]

(2.29a)

\[ N_0 \mu = -\frac{1}{3} \langle T \rangle_0 + \frac{7}{3} \langle V_t \rangle_0 \]

(2.29b)

that do not involve the Hartree or interaction energy.

### C. Basic physics of the Gross-Pitaevskii equation

The presence of the trap introduces a new characteristic energy \( \hbar \omega_0 \) and a new length scale \( d_0 = \sqrt{\hbar/2M\omega_0} \). The repulsive interparticle interactions tend to expand the condensate, so that the true mean size \( R_0 \) generally exceeds the size \( d_0 \) of the ideal Bose gas in the Gaussian ground state. To estimate the dimensionless expansion ratio \( R \equiv R_0/d_0 \), consider the order-of-magnitude of the various terms in the ground-state energy. Since the ground-state wave function is nodeless, the kinetic energy per particle has the order-of-magnitude

\[ \frac{\langle T \rangle_0}{N_0} \sim \frac{\hbar^2}{MR_0^2} \sim \frac{\hbar \omega_0}{R^2}. \]

(2.30a)

Similarly,

\[ \frac{\langle V_t \rangle_0}{N_0} \sim \hbar \omega_0 R^2. \]

(2.30b)
and

\[
\frac{\langle V_H \rangle_0}{N_0} \sim gn \sim \frac{\hbar^2 a_n}{M} \sim \frac{\hbar^2 a N}{MR_0^3} \sim \frac{\hbar \omega_0}{R^3} \frac{Na}{d_0},
\]

(2.30c)

with the new dimensionless parameter \(Na/d_0\) that specifically reflects the presence of the trap. Although the ratio \(a/d_0\) is generally small, of order \(\sim 10^{-3}\), \(N\) is often of order \(10^6\)–\(10^7\), so that the product \(Na/d_0\) can itself be large, of order \(10^3\)–\(10^4\).

The total energy per particle is the sum of these three contributions

\[
\frac{E_a}{N_0} \sim \hbar \omega_0 \left( \frac{1}{R^2} + R^2 + \frac{Na}{d_0} \frac{1}{R^3} \right),
\]

(2.31)

with the dimensionless expansion ratio \(R\) as a variational parameter. Two limits are easy to analyze.

1. If \(Na/d_0 \ll 1\) (a nearly ideal Bose gas), the last term of eq. (2.31) can be omitted, and the minimum ground-state energy occurs at \(R = 1\), which is just the familiar variational solution for the simple harmonic oscillator with \(\langle T \rangle_0 = \langle V_{tr} \rangle_0\).

2. In the opposite limit that \(Na/d_0 \gg 1\), the repulsive interactions are crucial (even though the gas is assumed to remain dilute with \(na^3 \ll 1\)). In this case, the kinetic energy \((\propto R^{-1})\) can be neglected because the condensate is significantly larger than its ideal-gas dimension, and the minimum ground-state energy occurs when the last two terms in eq. (2.31) are comparable, namely when

\[
R^5 = R_0^5/d_0^5 \sim Na/d_0 \gg 1.
\]

(2.32)

Before proceeding with an analysis of this very important scaling relation that holds in what has become known as the Thomas-Fermi (TF) limit, it is helpful to recall the healing (or coherence) length \(\xi_{\text{coherence}}\) as the Thomas-Fermi (TF) limit, it is helpful to recall the healing (or coherence) length \(\xi = (8\pi na)^{-1/2}\) defined in eq. (1.45) for a uniform unbounded condensate. Physically, \(\xi\) arises from the balance between the kinetic energy \(\hbar^2/2M\xi^2\) and the Hartree energy \(gn = 4\pi \hbar^2 a/M\). When the bulk condensate wave function \(\Psi\) is perturbed locally (for example, an otherwise uniform Bose gas in a half space bounded by an external plane), the length \(\xi\) characterizes the distance required for \(\Psi\) to heal back to the previous background value, justifying the name “healing length.”

The presence of a closed boundary (with an additional associated length scale) complicates the situation, even in the absence of an external confining potential like \(V_{tr}\). One simple and exactly soluble example is a three-dimensional dilute Bose gas with uniform areal density \(N/A\) in the \(yz\) plane and confined to a slab \(|x| \leq \frac{1}{2}L\) by rigid transverse walls \([23]\).

Here the real condensate wave function \(\Psi(x)\) obeys a one-dimensional GP equation \(-\frac{\hbar^2}{2M} \Psi'' - \mu \Psi + g |\Psi|^2 \Psi = 0\) and vanishes at \(x = \pm \frac{1}{2}L\). The nearly ideal limit can be characterized by the condition \(\xi^2 \sim AL/Na \gg L^2\), when the condensate density \(n_0(x) = |\Psi(x)|^2\) is everywhere small and proportional to \(\cos^2(\pi x/L)\). In this case, the length scale \(L\) characterizes the spatial variation of \(n_0\), so that \(L\) itself plays the role of the healing length in the nearly ideal limit. As \(N\) increases (and \(\xi\) decreases), however, the repulsive interactions act to flatten and spread the central density maximum. In the opposite limit \(\xi^2 \ll L^2\), the detailed solution shows that the condensate density \(n_0\) forms a plateau with the density falling to zero at the walls in a skin thickness of order \(\xi\). This simple model shows that the physical healing length in a confined Bose condensate should be taken as the smaller of the two lengths \(L\) and \(\xi\).

A similar situation holds for a dilute three-dimensional Bose gas in a harmonic trap, apart from the additional complication that the condensate’s volume \(\sim R_0^3\) can exceed the ideal-gas volume \(\sim d_0^3\) when the interactions are sufficiently strong. The squared coherence length \(\xi^2 \sim 1/na \sim R_0^3/Na\) provides an instructive way to think about the various limits discussed above in connection with the variational energy in eq. (2.31). In the nearly ideal limit \((Na/d_0 \ll 1\) and \(R_0 \approx d_0\)), it is easy to see that \(\xi^2 \gg d_0^2\); the ground-state Gaussian condensate density varies on the length scale \(d_0\), which thus acts as the healing length. As \(N\) increases, however, the coherence length \(\xi\) shrinks, becoming comparable with the ideal-gas condensate dimension \(d_0\) when \(Na/d_0\) is of order 1.

In the opposite (TF) limit \(Na/d_0 \gg 1\), the coherence length \(\xi\) falls significantly below \(d_0\), and \(\xi\) itself now plays the role of the healing length (for example, near a hole produced by a localized blue-detuned laser beam). In addition, the characteristic condensate dimension \(R_0\) expands beyond \(d_0\). To analyze this last result in detail, note that the definition of the coherence length implies that \(\xi^2 R_0^5 \sim R_0^5/Na\), and use of the preceding qualitative TF relation \(R_0^5/d_0^5 \sim Na/d_0\) from eq. (2.32) yields

\[
\xi^2 R_0^5 \sim d_0^5 \frac{R_0^5}{d_0^5} \frac{d_0}{Na} \sim d_0^4, \quad \text{or, equivalently,} \quad \frac{\xi}{d_0} \sim \frac{d_0}{R_0} \ll 1.
\]

(2.33)

Consequently, in the TF limit, the oscillator length \(d_0\) is approximately the geometric mean of the small healing length \(\xi\) and the large condensate radius \(R_0\).
If the gas remains dilute with \( na^3 \ll 1 \), the healing length is considerably larger than the interparticle spacing. The situation changes in the extreme large-\( N \) (high-density) limit with \( na^3 \sim 1 \), for the healing length \( \xi \) then becomes comparable with the interparticle spacing \( l \sim n^{-1/3} \). Although this dense regime is probably unattainable for the trapped alkali gases because of three-body recombination effects, it is worth mentioning that bulk superfluid \(^4\)He does have \( na^3 \sim 1 \), for its healing length \( \xi \) is comparable with an atomic dimension (and thus with the interparticle spacing).

The original Bogoliubov theory of a dilute uniform Bose gas with \( na^3 \ll 1 \) has no independent length scale for the dimension of the condensate. In contrast, the new dimension \( d^0 \) for the trapped Bose gas correspondingly yields two distinct physical regimes (although both are dilute with \( na^3 \ll 1 \)):

1. the nearly ideal regime with \( \xi \gg d^0 \) and \( Na/d^0 \ll 1 \);
2. the interacting regime with \( \xi \ll d^0 \) and \( Na/d^0 \gg 1 \).

This comparison of the uniform and trapped dilute Bose gases emphasizes the qualitatively new physics associated with the presence of the trap.

### D. The behavior of the condensate for large \( N \)

In the limit \( Na/d^0 \gg 1 \), the GP equation (2.20) can be simplified by omitting the kinetic energy \(^{[24,25]} \), yielding a simple algebraic equation

\[
(V_{tr} + V_H - \mu)\Psi = 0.
\]

Either \( \Psi = 0 \) or the Hartree potential and condensate density are given by \( V_H = g|\Psi|^2 = \mu - V_{tr} \). In the present case of a harmonic potential, the approximate condensate density has a parabolic profile

\[
|\Psi(r)|^2 = n_0(r) = n(0) \left( 1 - \frac{r^2}{R_1^2} - \frac{z^2}{R_z^2} \right) \Theta \left( 1 - \frac{r^2}{R_1^2} - \frac{z^2}{R_z^2} \right),
\]

where \( \Theta \) denotes the unit positive step function. Here,

\[
n(0) = \frac{\mu}{g} = \frac{\mu M}{4\pi\hbar^2 a}
\]

is the central density in the trap and

\[
R_1^2 = \frac{2\mu}{M\omega_1^2} \quad \text{and} \quad R_z^2 = \frac{2\mu}{M\omega_z^2}
\]

characterize the radial and axial semiaxes of the ellipsoidal condensate density. In this large-\( N \) limit, the slowly varying trap potential determines the local density [often called the Thomas-Fermi (TF) limit in analogy with the corresponding treatment of the electron density in large atoms]. Note that the anisotropy of the condensate in this TF limit is now given by

\[
\frac{R_z^2}{R_1^2} = \frac{\omega_z^2}{\omega_1^2} = \frac{1}{\lambda^2} \quad \text{in TF limit},
\]

differing from the anisotropy \( 1/\lambda \) in eq. (2.5) for the near-ideal condensate.

Typically, the fractional depletion \( N'/N = 1 - N_0/N \) is small, and it is sufficient to set \( N_0 \approx N \) and hence \( n_0 \approx n \). In this case, the normalization of the condensate wave function requires

\[
\int d^3r \, n(r) \approx \int d^3r \, |\Psi(r)|^2 \approx N.
\]

A straightforward calculation \(^{[24]} \) with the TF density from eq. (2.35) yields

\[
N = \frac{1}{15} \frac{R_0^3}{ad_0^3},
\]

where \( R_0^3 = R_1^2 R_z \). This result can be rewritten in dimensionless form as

\[
N = \frac{1}{15} \frac{R_0^3}{ad_0^3}.
\]
which has many important implications for the behavior of the condensate in the large-$N$ (or TF) limit.

1. The dimensionless expansion factor $R$ grows relatively slowly, like $N^{1/5}$, which thus characterizes the expansion of the mean condensate radius $R_0$. Equivalently, for a given radius $R_0$, the total number increases like $N \propto R_0^5$.

2. The central density $n(0)$ is of order $N/R_0^3$, and the preceding dependence implies that $n(0) \propto R_0^2$.

3. The chemical potential is $\mu \approx \frac{1}{2}\hbar\omega_0 R^2$, with $R \gg 1$, so that $\mu$ far exceeds the ground-state energy of a single particle in the trap.

4. The ground-state condensate energy $E_g \approx \frac{5}{3} \hbar^2 \omega_0 R^2$, from eq. (2.29), and direct evaluation of $\langle r^2 \rangle_0 = \frac{5}{3} \hbar \omega_0 R^2 N \propto N^{7/5}$ follows from the virial theorem in eq. (2.29) and item 6 above shows that the central density $n(0)$ is negligible in the TF limit.

5. The TF chemical potential given above in item 3 is consistent with the general thermodynamic identity $\mu = \langle \partial E_g/\partial N \rangle \approx \frac{5}{3} E_g/N \propto N^{2/5}$, as well as the virial theorem in eq. (2.29).

6. It is natural to define the healing length for the trapped condensate in terms of the central density, with $\xi = [8\pi n(0)a]^{-1/2}$. In the TF limit, this choice implies that

$$\xi R_0 = d_0^2, \quad \text{or, equivalently,} \quad \frac{\xi}{d_0} = \frac{d_0}{R_0} \ll 1,$$

quantifying the previous qualitative TF conclusion from eq. (2.33) that the bare trap size $d_0$ is the geometric mean of the healing length $\xi$ and the mean condensate radius $R_0$.

7. This whole TF analysis is easily generalized to an arbitrary anisotropic (triaxial) trap with $\omega_x \neq \omega_y = \omega_z$ by interpreting the mean frequency as $\omega_0 = \omega_x \omega_y \omega_z$, with similar expressions for $d_0^3 = dx dy dz$ and $R_0^3 = Rx Ry Rz$.

Recall that a uniform Bose gas has the speed of sound $s = \sqrt{gn/M}$ from eq. (1.44). In a trap, the central density $n(0)$ provides the corresponding estimate $s \approx \sqrt{gn(0)/M}$, and item 2 above shows that the central density $n(0) \propto R_0^2$ scales with the square of the condensate radius $R_0$ in the TF limit. Thus, the TF speed of sound in a trapped condensate increases linearly with $R_0$. With item 6 above, it is easy to find the quantitative expression

$$s = \frac{\hbar}{\sqrt{2M}} \frac{R_0}{d_0} = \frac{\hbar}{\sqrt{2M}\xi}.$$  

(2.43)

This linear dependence on the condensate radius has the following remarkable implication for the lowest-lying compressional modes of the condensate. The fundamental mode has a wavelength of order $R_0$, and the lowest frequency will therefore be of order $\omega \sim s/R_0 \sim \hbar/M d_0^2 \sim \omega_0$, namely the mean trap frequency $\omega_0$. Consequently, the lowest normal-mode frequencies of the condensate should be independent of the condensate radius $R_0$ (and hence the condensate number $N$) in the TF limit of a large condensate. Several experimental studies [26, 27] have verified this prediction (see sec. 11 below contains an account of the theoretical analysis [28]).

It is also instructive to consider the Landau critical velocity $v_c$ for the onset of dissipation in the condensate, which is simply the speed of sound in the Bogoliubov description of a uniform dilute Bose gas. In the present case of a trapped condensate, it is not obvious how to send an impurity rapidly through the sample, but a rigid rotation with angular velocity $\Omega$ can accomplish much the same effect. The equatorial speed is $\sim \Omega R_0$, and the onset of dissipation is expected when this speed equals $v_c \approx s$. Thus the critical rotation speed should be of order $\Omega_c \sim v_c/R_0 \approx s/R_0 \sim \omega_0$ as shown above. This result means that the condensate should become unstable if the rotation speed exceeds any of the trap frequencies (a similar instability occurs for a particle in a harmonic potential).

E. Effect of an attractive interaction

Bose-Einstein condensation has been observed [14] in $^7$Li, which is known to have a negative scattering length $a$, corresponding to an attractive interaction. For a uniform dilute Bose gas, the corresponding energy spectrum in eq. (2.42) becomes imaginary at long wavelengths (see fig. 3)

$$E_k^2 = \left( \frac{\hbar^2}{2M} \right)^2 k^2 \left( k^2 - 16\pi n|a| \right),$$  

(2.44)

implying an instability for those modes with $k^2 \leq 16\pi n|a|$. Equivalently, the speed of sound becomes imaginary. For a trap, however, the wavenumber cannot be arbitrarily small, and the minimum value is of order $k_{\text{min}} \approx \pi/R_0$. Hence
the system can remain stable if $\pi^2/R_0^2 \gtrsim 16\pi n|a|$. Since the density is of order $n \sim N/R_0^3$, this analysis predicts the critical number

$$N_c \approx \frac{\pi R_0}{16|a|} \approx \frac{\pi d_0}{16|a|}, \quad (2.45)$$

because $R_0 \approx d_0$ for such small values of the ratio $N_c|a|/d_0 \approx \pi/16$. For $N \lesssim N_c$, the trapped condensate with attractive interactions should remain stable because of the positive kinetic energy arising from the confinement; for $N \gtrsim N_c$, however, the attractive interaction energy predominates, and the condensate should collapse.

It is not difficult to make a variational estimate of the critical condensate number $N_c$ for a spherical trap with attractive interactions. As a simple model, take a Gaussian trial function [a similar variational estimate was used by Baym and Pethick [25] in their initial study of the GP equation for repulsive interactions; compare also the qualitative estimate in eq. (2.31)],

$$\Psi(r) \propto \exp\left(-\frac{r^2}{2\beta^2 d_0^2}\right) \quad (2.46)$$

with $\beta$ as the variational parameter; the condensate radius is $\approx \beta d_0$ with $\beta < 1$ for attractive interactions. The ground-state energy is readily evaluated to be [28, 31]

$$E_g(\beta) = \frac{1}{2} N\hbar\omega_0 \left[\frac{3}{2} \left(\frac{1}{\beta^2} + \beta^2\right) - \sqrt{\frac{2}{\pi} N|a| d_0^{-1} \beta^3}\right], \quad (2.47)$$

and fig. 4 shows this function of $\beta$ for several values of the interaction parameter $N|a|/d_0$. Evidently, $E_g(\beta)$ has no global minimum, for it becomes arbitrarily negative as $\beta \to 0$. Nevertheless, the energy does have a local minimum if $N < N_c$. A straightforward analysis shows that the minimum disappears at the critical value $N_c|a|/d_0 \approx 0.671$, and that the critical condensate radius is reduced by a factor $\beta_c \approx 5^{-1/4} = 0.669$. For comparison, a numerical study of the GP equation [32] yields the value $N_c|a|/d_0 \approx 0.575$, which differs from the variational estimate by $\approx 17\%$.

### F. Surface region of Thomas-Fermi condensate

For a spherical trap, the slope of the TF condensate wave function

$$\Psi_{TF}(r) \propto \sqrt{1 - \frac{r^2}{R_0^2}} \Theta(R_0 - r) \quad (2.48)$$

becomes infinite at the surface ($r \to R_0$), leading to logarithmic singularity in the kinetic energy. Consequently, it is essential to introduce a boundary layer near the surface, where the true wave function is modified from its approximate TF form [33]. To study this surface region, it is convenient to focus on the region $|r - R_0| \ll R_0$, writing $r = R_0(1 + X\delta)$, where $X$ is a new scaled dimensionless variable measured from the TF surface, and $\delta \ll 1$ is the dimensionless thickness of the boundary layer, measured in units of $R_0$. Equivalently, $X = (r - R_0)/(R_0\delta)$. With this change of variable, the GP equation for a spherical trap

$$-\frac{\hbar^2}{2M} \frac{d^2}{dr^2} \Psi - \frac{\hbar^2}{Mr} \frac{d}{dr} \Psi + \left(\frac{1}{2} M\omega_0^2 r^2 - \mu\right) \Psi + \frac{4\pi\hbar^2 a}{M}|\Psi|^2 \Psi = 0 \quad (2.49)$$

takes the approximate form

$$-\frac{\hbar^2}{2MR_0^2} \frac{1}{\delta^2} \frac{d^2}{dX^2} \Psi + M\omega_0^2 R_0^2 \delta X \Psi + \frac{4\pi\hbar^2 a}{M}|\Psi|^2 \Psi = 0. \quad (2.50)$$

The balance between the first two terms gives the relation

$$2\delta^3 = \frac{\hbar^2}{M^2 \omega_0^2 R_0^4} \frac{a^4}{R_0^4} = R^{-4} \propto N^{-4/5}, \quad (2.51)$$

verifying that the boundary layer is indeed thin and showing that $\delta \propto R^{-4/3}$ scales with the condensate number like $N^{-4/15}$. In contrast, the balance between the last two terms shows that the condensate density $|\Psi|^2$ near the
surface is small (of order \( \delta \)). A detailed study of the condensate wave function in the surface region involves a non-linear differential equation. Its solution matches the TF behavior in the interior of the condensate for \( X \ll -1 \) and vanishes exponentially for \( X \gg 1 \), thus providing a uniform extension across the boundary layer. The resulting kinetic energy per particle

\[
\frac{(T)}{N} \approx \frac{5}{2} \frac{\hbar^2}{M R_0^3} \ln \left( \frac{R_0}{1.3d_0} \right)
\]

(2.52)

exhibits the role of the boundary layer in cutting off the logarithmic singularity. This contribution to the total energy is much smaller than the other dominant contributions \( \langle V_{tr}\rangle/N \approx \frac{3}{14} M\omega_0^2 R_0^2 \) and \( \langle V_{int}\rangle/N = \frac{1}{2} \langle V_H\rangle/N \approx \frac{1}{7} M\omega_0^2 R_0^2 \), whose sum gives the previous expression \( E_g/N \approx \frac{14}{14} M\omega_0^2 R_0^2 \).

G. Excited states of a trapped Bose condensate

In the case of the uniform stationary dilute Bose gas, the condensate wave function is a spatial constant. This simple structure allows a complete explicit solution for the elementary excitations, which are the Bogoliubov quasiparticles. For a non-uniform trapped Bose condensate, however, the situation is more complicated because the determination of the condensate wave function \( \Psi(\mathbf{r}) \) already constitutes a significant problem. Nevertheless, given \( \Psi \), a well-defined procedure leads to the corresponding equations for the small-amplitude normal modes of the condensate. These equations were first studied by Pitaevskii in the context of a long straight vortex line in an unbounded condensate, and they have subsequently been generalized as operator equations in ref. \[19\].

Recall that eq. (2.13) separated the field operator into a (large) condensate part and a (small) fluctuation operator

\[
\hat{\psi} = \Psi + \hat{\phi},
\]

(2.53)

where \( \langle \hat{\psi} \rangle \) and \( \langle \hat{\phi} \rangle = 0 \), and the angular brackets denote an ensemble average in a restricted grand-canonical ensemble that breaks the symmetry with respect to the phase \[10\]. The derivation of the GP equation for the condensate wave function retained only the dominant terms involving \( \Psi \), giving the leading condensate approximation \( K_0 \) to the thermodynamic potential in eq. (2.19), thereby ignoring the contribution of the fluctuation operators. To include the first corrections, it is necessary to retain the fluctuation contributions \( \hat{\phi} \) and \( \hat{\phi}^\dagger \) to the original field operators; they obey the approximate bosonic commutation relations \[19\]

\[
\left[ \hat{\phi}(\mathbf{r}), \hat{\phi}^\dagger(\mathbf{r}') \right] \approx \delta^{(3)}(\mathbf{r} - \mathbf{r}'), \quad \left[ \hat{\phi}(\mathbf{r}), \hat{\phi}(\mathbf{r}') \right] = \left[ \hat{\phi}^\dagger(\mathbf{r}), \hat{\phi}^\dagger(\mathbf{r}') \right] = 0.
\]

(2.54)

Substitution into the full modified hamiltonian \( \hat{K} = \hat{H} - \mu \hat{N} \) from eqs. (2.10) and (2.18) gives the previous \( K_0 \) plus a quadratic correction

\[
\hat{K}' = \int d^3r \hat{\phi}^\dagger (T + V_{tr} - \mu) \hat{\phi} + \frac{2\pi\hbar^2a}{M} \int d^3r \left[ 4|\Psi|^2 \hat{\phi}^\dagger \hat{\phi} + \Psi^2 \hat{\phi}^\dagger \hat{\phi}^\dagger + (\Psi^*)^2 \hat{\phi}^\dagger \hat{\phi} \right].
\]

(2.55)

The linear contribution to \( \hat{K} \) vanishes identically because \( \hat{K} \) is stationary with respect to small changes around the GP solution \( \Psi \); this condition merely requires that \( \Psi \) obey the appropriate Euler-Lagrange equation.

The mean number density in the ground state is given by \( n(\mathbf{r}) = \langle \hat{\phi}^\dagger(\mathbf{r})\hat{\phi}(\mathbf{r}) \rangle \). The expansion in eq. (2.53) shows that \( n(\mathbf{r}) = |\Psi(\mathbf{r})|^2 + \langle \hat{\phi}^\dagger(\mathbf{r})\hat{\phi}(\mathbf{r}) \rangle \), because, by assumption, the linear contributions have zero ground-state ensemble average. Consequently, the mean total number of particles is a sum of condensate and noncondensate contributions \( N = N_0 + N' \), where \( N_0 = \int d^3r n_0(\mathbf{r}) \) and

\[
N' = \int d^3r \langle \hat{\phi}^\dagger(\mathbf{r})\hat{\phi}(\mathbf{r}) \rangle.
\]

(2.56)

The dynamical equations for the fluctuation operators follow immediately from the commutators with \( \hat{K}' \):

\[
\frac{\hbar}{i} \frac{\partial \hat{\phi}}{\partial t} = \left[ \hat{\phi}, \hat{K}' \right] \quad \text{and} \quad \frac{\hbar}{i} \frac{\partial \hat{\phi}^\dagger}{\partial t} = \left[ \hat{\phi}^\dagger, \hat{K}' \right].
\]

(2.57)

A straightforward calculation yields the coupled operator equations
The primed sum runs over all the excited states. Here, the quasiparticle operators have harmonic time dependence \( \alpha_j(t) = \alpha_j \exp(-i E_j t/h) \) and \( \alpha^\dagger_j(t) = \alpha_j^\dagger \exp(i E_j t/h) \), obeying bosonic commutation relations at equal times. Substitution into eqs. (2.58) leads to the coupled “Bogoliubov equations” \( [30] \):

\[
\begin{align*}
(T + V_{tr} - \mu + \frac{8\pi \hbar^2 a}{M} |\Psi|^2) & u_j - \frac{4\pi \hbar^2 a}{M} \Psi^2 v_j = E_j u_j, \\
(T + V_{tr} - \mu + \frac{8\pi \hbar^2 a}{M} |\Psi|^2) & v_j - \frac{4\pi \hbar^2 a}{M} (\Psi^*)^2 u_j = -E_j v_j
\end{align*}
\]

(2.60a)

(2.60b)

that determine the pair of quantum eigenamplitudes (in effect, they are wave functions) \( u_j(r) \) and \( v_j(r) \) along with the associated energy eigenvalue \( E_j \) for the \( j \)th eigenstate. In addition, direct substitution of eq. (2.60) gives the quasiparticle modified hamiltonian \( \hat{K}' \) and use of eq. (2.61) gives the simple and intuitive expression

\[
\hat{K}' = -\sum_j E_j N^+_j + \sum_j E_j \alpha^\dagger_j \alpha_j,
\]

(2.62)
which is a direct generalization of the Bogoliubov quasiparticle hamiltonian \([1.39]\) for a uniform system. The first term is the ground-state thermodynamic potential \(K_g = \langle \hat{\mathcal{H}} \rangle = -\sum_j E_j N_j',\) and the second shows that the excited states consist of various numbers of non-interacting bosons with energies \(E_j\) and occupation-number operators \(\alpha_j^\dagger \alpha_j\). Since these operators have non-negative integers as their eigenvalues, stability of the ground state requires that \(E_j \geq 0\) for all eigenstates \(j\) that obey the proper normalization in eq. \((2.63)\).

At low but finite temperature in the grand canonical ensemble, the appropriate (unnormalized) weight factor for the excited states is \(\exp(-\beta \hat{\mathcal{H}}')\); the average of any operator \(\hat{O}\) is simply the trace \(\langle \hat{O} \rangle = \text{Tr}[\hat{O} \exp(-\beta \hat{\mathcal{H}}')]/\text{Tr}[\exp(-\beta \hat{\mathcal{H}}')]\) with the same weight factor (normalized by the grand partition function \(Z = \text{Tr}[\exp(-\beta \hat{\mathcal{H}}')]\)). For example, the non-condensate density is

\[
n'(\mathbf{r}) = \sum_j \left\{ f(E_j)|u_j(\mathbf{r})|^2 + [1 + f(E_j)]|v_j(\mathbf{r})|^2 \right\},
\]

where \(f(E_j) = [\exp(\beta E_j) - 1]^{-1}\) is the Bose-Einstein distribution function for the \(j\)th excited state. At zero temperature, the spatial integral gives the preceding result for the total non-condensate number.

It is instructive to specialize the present Bogoliubov quasiparticles to a uniform system. In sec. \(\text{I}\), the condensate number \(N_0\) was eliminated in favor of the total number \(N\), leading to a number-conserving description (with no chemical potential \(\mu\) in the hamiltonian); in contrast, the theory here relies on the grand canonical ensemble at fixed temperature \(T\) and chemical potential \(\mu\), leading to a modified hamiltonian \(\hat{\mathcal{H}} = \hat{H} - \mu \hat{N}\). In this latter formalism, the ground-state thermodynamic potential per unit volume for the uniform condensate is \(K_0/V = \frac{1}{2}gn_0^2 - \mu n_0\), where \(n_0 = N_0/V\) is the condensate density (the kinetic energy vanishes for a uniform condensate). In equilibrium at fixed chemical potential and temperature (here, \(T = 0\)), the system adjusts all free parameters to minimize the thermodynamic potential. In the present case where the non-condensate is neglected, only the condensate density \(n_0\) remains undetermined, and the condition \((\partial K_0/\partial n_0)\mu = 0\) immediately gives the expected result \(\mu = gn_0\), which is the Hartree energy for adding one particle. The corresponding solutions of the Bogoliubov equations are plane waves \(u_k(\mathbf{r}) = u_k \exp(i \mathbf{k} \cdot \mathbf{r})\) and \(v_k(\mathbf{r}) = v_k \exp(i \mathbf{k} \cdot \mathbf{r})\). Use of the preceding expression for \(\mu\) in the Bogoliubov equations \((2.66)\) readily reproduces all of previous Bogoliubov results, for example, \(E_k = \sqrt{2gn_0\epsilon_k^0 + (\epsilon_k^0)^2}\).

More generally, the Bogoliubov equations provide a flexible formalism to study the effect of a non-uniform condensate wave function. A particularly interesting example is a uniform system in which the condensate moves with velocity \(\mathbf{v} = \hbar \mathbf{q}/M\) (discussed briefly at the end of sec. \(\text{I}\)). The condensate wave function has the form \(\Psi \propto e^{i\mathbf{q} \cdot \mathbf{r}}\), and the GP equation then shows that the chemical potential becomes \(\mu = \epsilon_q^0 + gn_0 = \frac{1}{2}Mv^2 + gn_0\). The structure of the two Bogoliubov eqs. \((2.66)\) implies that the pair of amplitudes has the form

\[
\begin{pmatrix}
u(\mathbf{r}) \\
u(\mathbf{r})
\end{pmatrix} = \begin{pmatrix} u_k e^{iq \cdot r} e^{ik \cdot r} \\
v_k e^{-iq \cdot r} e^{ik \cdot r}
\end{pmatrix},
\]

and the resulting eigenvalue is given by \(E_k = \hbar \mathbf{k} \cdot \mathbf{v} \pm \epsilon_k^0\), where \(\epsilon_k^0\) is the Bogoliubov energy for the stationary condensate. Substitution back into the Bogoliubov equations shows that the \(\pm\) sign here corresponds to the normalization \(u_k^2 - v_k^2 = \pm 1\), so that the bosonic commutation relations require the choice \(E_k = \hbar \mathbf{k} \cdot \mathbf{v} + \epsilon_k^0\) (as also follows by requiring continuity with the small-\(v\) limit). For a detailed discussion of these solutions, along with the associated mass current and superfluid density (as well as the generalization to a mixture of two distinct interacting species), see, for example, refs. \([11][12]\).

Consider an anisotropic harmonic trap with arbitrary frequencies \(\omega_\alpha (\alpha = x, y, z)\), and let \(\Psi\) denote any exact solution of the GP equation. In general, the Bogoliubov equations are difficult to solve, but they do have three exact solutions (the “dipole” modes) in which this general condensate oscillates rigidly in each orthogonal direction with frequency \(\omega_\alpha\). To construct these explicit solutions \([11]\), define the familiar raising and lowering operators

\[
a^\dagger_\alpha = \frac{1}{\sqrt{2}} \left( \frac{x_\alpha}{d_\alpha} - id_\alpha \frac{\partial}{\partial x_\alpha} \right),
\]

\[
a_\alpha = \frac{1}{\sqrt{2}} \left( \frac{x_\alpha}{d_\alpha} + id_\alpha \frac{\partial}{\partial x_\alpha} \right).
\]

It is not difficult to prove that the corresponding dipole solutions have the explicit form

\[
\begin{pmatrix} u_\alpha(\mathbf{r}) \\
u_\alpha(\mathbf{r})
\end{pmatrix} = \begin{pmatrix} a_\alpha^\dagger \Psi(\mathbf{r}) \\
a_\alpha \Psi^*(\mathbf{r})
\end{pmatrix}.
\]

20
For the non-interacting Bose gas, the solution of the GP equation is just the Gaussian ground state $\Psi_g = \sqrt{N}\psi_g$ of the trap given in eq. (2.3); the associated dipole excitations are then characterized by $u_\alpha \propto a_\alpha^\dagger \psi_g$ (namely, the first excited states of the oscillator) and $v_\alpha = 0$ (because $a_\alpha$ annihilates the ground state $\psi_g$).

It is worth noting that the Bogoliubov equations (2.60) have an equivalent derivation [42–44] based directly on the linear response of the time-dependent GP equation (although this approach leads more directly to the final equations, it relies only on wave functions and does not treat the field operators explicitly). Add a weak sinusoidal perturbation to the GP equation

$$i\hbar \frac{\partial \Psi}{\partial t} = (T + V_{tr} + g|\Psi|^2) \Psi + (f_+ e^{-i\omega t} + f_- e^{i\omega t}) \Psi,$$

and seek a solution in the form

$$\Psi(r, t) = e^{-i\mu t/\hbar} [\Psi(r) + u(r) e^{-i\omega t} + v(r) e^{i\omega t}].$$

Linearizing in the small amplitudes $u$ and $v$ yields an inhomogeneous form of the Bogoliubov equations with driving terms proportional to $f_{\pm}(r)\Psi$. A normal-mode expansion in terms of eigenfunctions $u_j$ and $v_j$ gives precisely the same coupled equations as in eq. (2.60), identifying the frequencies $\omega_j = E_j/\hbar$ as resonances in the linear response.

III. HYDRODYNAMIC DESCRIPTION OF A DILUTE TRAPPED BOSE GAS

Although the Bogoliubov equations, in principle, characterize all the excited states of the condensate, they are difficult to solve and their physical meaning is not always transparent. Thus, it is important to consider alternative formulations, and the present section concentrates on the hydrodynamic approach that emphasizes the familiar concepts of density and current fluctuations.

A. Uniform dilute Bose gas

It is instructive to review the theory of density fluctuations in a uniform Bose gas, where the field operator has the familiar form

$$\hat{\psi}(r) = \sum_k \frac{1}{\sqrt{V}} e^{ikr} a_k.$$

The corresponding density operator $\hat{n}(r) = \hat{\psi}^\dagger(r)\hat{\psi}(r)$ has a Fourier expansion

$$\hat{n}(r) = \sum_q e^{iqr} \hat{n}_q,$$

where

$$\hat{n}_q = \sum_k a_k^\dagger a_{k+q}$$

is an operator that removes one quantum of density fluctuation with wave number $q$. It is easy to see that the associated creation operator is given by $\hat{n}_q^\dagger = \hat{n}_{-q}$. For an ideal Fermi gas at $T = 0$, the operator $\hat{n}_q$ in eq. (3.3) removes a particle with wave number $k$ (which must lie inside the Fermi sea with $|k| \leq k_F$) and simultaneously creates a particle with wave number $k - q$ (which must lie outside the Fermi sea with $|k - q| \geq k_F$); the final operator $\hat{n}_q$ is a sum over all possible $k$ consistent with the above restrictions. Thus the density-fluctuation operator $\hat{n}_q$ is really a particle-hole operator that conserves particle number, and a similar characterization applies to both a general interacting Fermi gas and a non-condensed Bose gas. Typically, density fluctuations in a Fermi gas have a well-defined dispersion relation $\omega_k$, and they are often called “collective modes” to distinguish them from the “quasiparticles” that are associated with single-particle excitations (those resulting from the removal or addition of one particle).

The situation is qualitatively different for a condensed Bose gas, because the condensate itself plays a special role. For simplicity, consider a dilute uniform Bose gas with stationary condensate at low temperature. In this case, the two terms arising from the Bogoliubov prescription dominate the general sum in eq. (3.3), leading to the simple density-fluctuation creation operator
that is a coherent linear combination of a quasiparticle and a quasihole. When this operator acts on the Bogoliubov ground state \( |\Phi\rangle \), it gives

\[
\hat{n}_q^\dagger\hat{n}_q = \sqrt{N_0} (a_q^\dagger + a_{-q}^\dagger) \approx \sqrt{N_0} (u_q - v_q) (a_q^\dagger + a_{-q}^\dagger)
\]

(3.4)

because, by definition, \( \alpha_{-q}^\dagger |\Phi\rangle \) vanishes. Apart from an overall factor, the action of the density-fluctuation operator \( \hat{n}_q^\dagger \) on the Bogoliubov ground state is the same as that of the quasiparticle operator \( \alpha_q^\dagger \). Consequently, the excitation spectrum associated with a density fluctuation in a dilute Bose gas is necessarily identical with that for a quasiparticle; as discussed above, this situation is completely different from that for a dilute Fermi gas.

The current-density operator has a similar expansion

\[
\hat{j}(r) = \frac{\hbar}{2M} \left[ \hat{\psi}^\dagger(r) \nabla \hat{\psi}(r) - \left( \nabla \hat{\psi}^\dagger(r) \right) \hat{\psi}(r) \right] = \sum_q \hat{J}_q e^{i\mathbf{q} \cdot \mathbf{r}},
\]

(3.6)

with

\[
\hat{J}_q = \frac{\hbar}{M} \sum_k \left( k - \frac{1}{2}\mathbf{q} \right) a_{k-q}^\dagger a_k.
\]

(3.7)

In the Bogoliubov approximation, two terms again predominate with

\[
\hat{J}_q \approx \frac{\hbar \mathbf{q}}{2M} \sqrt{N_0} \left( a_q - a_{-q}^\dagger \right) \approx \frac{\hbar \mathbf{q}}{2M} \sqrt{N_0} (u_q + v_q) \left( a_q - a_{-q}^\dagger \right);
\]

(3.8)

as expected for a dilute gas, this operator is manifestly longitudinal (namely, along \( \mathbf{q} \)).

For time-dependent (Heisenberg) operators, conservation of particle number requires that

\[
\frac{\partial \hat{n}_q(t)}{\partial t} + i\mathbf{q} \cdot \hat{J}_q(t) = 0.
\]

(3.9)

It is clear from eq. (3.4) that the density-fluctuation operator \( \hat{n}_q(t) \) in a dilute Bose gas oscillates at the Bogoliubov frequency \( \omega_q = E_q/\hbar \).

In a bulk uniform system, the ensemble average of \( \hat{n}_q^\dagger \hat{n}_q \) is called the “static structure function,” with the precise definition

\[
S(q) = N^{-1} \langle \hat{n}_q^\dagger \hat{n}_q \rangle.
\]

For a dilute Bose gas at low temperature, this quantity is readily evaluated with eq. (3.4) to give

\[
S(q) \approx (u_q - v_q)^2 \langle \alpha_q^\dagger \alpha_q^\dagger + \alpha_q^\dagger \alpha_{-q}^\dagger \rangle = \frac{e^0}{E_q} \coth \left( \frac{E_q}{2k_B T} \right), \quad \text{in the Bogoliubov approximation},
\]

(3.10)

since off-diagonal ensemble averages like \( \langle \alpha_q^\dagger \alpha_{-q} \rangle \) vanish. At \( T = 0 \), the thermal factor is simply 1, giving the zero-temperature limits

\[
S(q) = \frac{e^0}{E_q} \approx \begin{cases} \frac{\hbar q}{2Ms} & \text{for } q\xi \ll 1, \\ 1 & \text{for } q\xi \gg 1, \end{cases}
\]

(3.11)

where \( s \) is the speed of sound. More generally, the Bogoliubov approximation at \( T = 0 \) reproduces the form of Feynman’s variational approximation for the zero-temperature density excitation spectrum of superfluid \(^4\text{He} \).

\[
E_q \approx \frac{\hbar^2 q^2}{2Ms^2} \cdot S(q).
\]

(3.12)

with \( S(q) \) the measured structure function. For low but finite temperature, in contrast, the static structure function for a dilute Bose gas has the limiting long-wavelength form

\[
S(q) \approx \frac{k_B T}{Ms^2} \quad \text{as } q \to 0.
\]

(3.13)
B. Sum rules for a uniform dilute Bose gas

For both Fermi and Bose systems, sum rules provide a powerful and exact approach to the description of collective modes \[46,47\]. Consider the response of a uniform system to a weak scalar field $\Phi(r, t)$ that couples to the density with a perturbation Hamiltonian (here expressed in Fourier components)

$$\hat{H}_{\text{int}}(t) = \sum_q \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{n}_q^\dagger \Phi(q, \omega) e^{-i\omega t}$$

(3.14)

The general theory of linear response shows that the induced change in the density occurs at the same wave vector $q$ and frequency $\omega$, with the explicit form

$$\delta \langle \hat{n}(q, \omega) \rangle = -\chi(q, \omega) \Phi(q, \omega).$$

(3.15)

Here, the coefficient is known as the “dynamic susceptibility,” and the explicit minus sign is added here for later convenience; $\chi$ is really the Fourier transform of a density-density correlation function. Linear-response theory \[46\] yields the explicit zero-temperature form

$$\chi(q, \omega) = -\frac{1}{N} \sum_{j \neq 0} |\langle j | \hat{n}_q^\dagger | 0 \rangle|^2 \left[ \frac{1}{\omega - \omega_j + i\eta} - \frac{1}{\omega + \omega_j + i\eta} \right],$$

(3.16)

where the limit $\eta \to 0^+$ is taken at the end of the analysis. Here, $|j\rangle$ is an exact eigenstate of the interacting but unperturbed many-particle system, and $\omega_j = (E_j - E_0)/\hbar$ is the exact excitation frequency of this state relative to the ground state.

It is clear by inspection that $\chi(q, \omega)$ has poles in the complex $\omega$ plane at the points $\omega = \pm \omega_j - i\eta$. Use of the formal identity

$$\lim_{\eta \to 0^+} \frac{1}{x - a + i\eta} = \mathcal{P} \frac{1}{x - a} - i\pi \delta(x - a),$$

(3.17)

where $\mathcal{P}$ denotes the Cauchy principal value of an integral, yields the explicit expression for the imaginary part

$$\text{Im} \chi(q, \omega) = \frac{\pi}{N} \sum_{j \neq 0} |\langle j | \hat{n}_q^\dagger | 0 \rangle|^2 \left[ \delta(\omega - \omega_j) - \delta(\omega + \omega_j) \right].$$

(3.18)

Note that $\text{Im} \chi(q, \omega)$ is an odd function of $\omega$ and therefore vanishes at $\omega = 0$. Furthermore, a combination of eqs. (3.16) and (3.18) yields the spectral representation for the susceptibility (the response function) at wave vector $q$ and frequency $\omega$

$$\chi(q, \omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\text{Im} \chi(q, \omega')}{\omega' - \omega - i\eta}.$$

(3.19)

In particular, the limit $\omega \to 0$ is simply the static susceptibility

$$\chi_q \equiv \chi(q, 0) = \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\text{Im} \chi(q, \omega')}{\omega'}$$

(3.20)

that characterizes the response to a static plane-wave distortion with wave vector $q$ [the convergence factor $i\eta$ is now unnecessary because $\text{Im} \chi(q, 0) = 0$].

This result can be generalized by defining the $i$th moment of $\text{Im} \chi$

$$\mathcal{M}_i \equiv \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \omega^i \text{Im} \chi(q, \omega) = \frac{1}{N} \sum_{j \neq 0} |\langle j | \hat{n}_q^\dagger | 0 \rangle|^2 (\omega_j)^i \left[ 1 - (-1)^i \right];$$

(3.21)

\[From this perspective, the preceding eq. (3.20) is the $i = -1$ moment

$$\mathcal{M}_{-1} = \frac{2}{\pi} \int_0^{\infty} \frac{d\omega}{\omega} \text{Im} \chi(q, \omega) = \chi_q.$$
In the limit $\mathbf{q} \to 0$, this quantity is related to the bulk thermodynamic compressibility because $\chi_\mathbf{q}$ is the response to a static plane wave $\propto e^{i \mathbf{q} \cdot \mathbf{r}}$; a detailed analysis \cite{46} shows that

$$\lim_{\mathbf{q} \to 0} \chi_\mathbf{q} = \frac{\hbar}{M s^2},$$  \hspace{1cm} (3.23)

giving the limit of $\mathcal{M}_{-1}$ as $q \to 0$ (this result is known as the “compressibility sum rule”).

It is clear that the even moments vanish. The next odd moment

$$\mathcal{M}_1 = \int_0^\infty d\omega \omega \text{Im} \chi(\mathbf{q}, \omega)$$

$$= \frac{2}{N} \sum_{j \neq 0} |\langle j | \hat{n}_\mathbf{q}^\dagger | 0 \rangle|^2 \omega_j$$  \hspace{1cm} (3.24)

can be related \cite{46} to the ground-state average of a double commutator $\langle 0 \mid [\hat{n}_\mathbf{q}, [\hat{H}, \hat{n}_\mathbf{q}^\dagger]] \mid 0 \rangle$. The inner commutator follows from the dynamical eq. (3.9) that expresses the conservation of particles since $[\hat{H}, \hat{n}_\mathbf{q}^\dagger] = i \hbar \partial \hat{n}_\mathbf{q}^\dagger / \partial t = -\mathbf{q} \cdot \hat{\mathbf{j}}_\mathbf{q}^\dagger$, leading to the final result

$$\mathcal{M}_1 = \frac{\hbar q^2}{M},$$  \hspace{1cm} (3.25)

which is known as the “$f$-sum rule” \cite{46}.

It is also valuable to consider the third moment \cite{48, 50}

$$\mathcal{M}_3 = \frac{2}{\pi} \int_0^\infty d\omega \omega^3 \text{Im} \chi(\mathbf{q}, \omega)$$

$$= \frac{2}{N} \sum_{j \neq 0} |\langle j | \hat{n}_\mathbf{q}^\dagger | 0 \rangle|^2 (\omega_j)^3$$  \hspace{1cm} (3.26)

which can be related to the more complicated matrix element

$$\langle 0 \mid [\hat{n}_\mathbf{q}, [\hat{H}, \hat{n}_\mathbf{q}^\dagger]] \mid 0 \rangle \propto \langle 0 \mid [\mathbf{q} \cdot \hat{\mathbf{j}}_\mathbf{q}^\dagger, [\hat{H}, \mathbf{q} \cdot \hat{\mathbf{j}}_\mathbf{q}^\dagger]] \mid 0 \rangle.$$  

For a dilute uniform Bose gas, a detailed analysis \cite{50} that neglects a small kinetic-energy part of order $\sqrt{n a^3}$ yields

$$\mathcal{M}_3 = \frac{q^2}{\hbar M} \epsilon_0^q (\epsilon_0^q + 2ng) = \frac{q^2}{\hbar M} E_q^2,$$  \hspace{1cm} (3.27)

where $E_q$ is seen to be the Bogoliubov energy spectrum in eq. (1.42).

These various sum rules can serve to characterize the detailed structure of $\text{Im} \chi(\mathbf{q}, \omega)$ \cite{48, 50} for a dilute uniform Bose gas. Assume that an excitation with wave vector $\mathbf{q}$ constitutes a single long-lived collective mode with frequency $\omega_\mathbf{q}$. In this case, the spectral weight has only one $\delta$-function, with $\text{Im} \chi(\mathbf{q}, \omega) = A_\mathbf{q} \delta(\omega - \omega_\mathbf{q})$ for positive $\omega$. Direct substitution shows that

$$\mathcal{M}_1 = \frac{\hbar q^2}{M} = \frac{2}{\pi} A_\mathbf{q} \omega_\mathbf{q},$$  \hspace{1cm} (3.28a)

$$\mathcal{M}_3 = \frac{q^2}{\hbar M} E_q^2 = \frac{2}{\pi} A_\mathbf{q} \omega_\mathbf{q}^3.$$  \hspace{1cm} (3.28b)

The ratio of these results shows that $\omega_\mathbf{q}$ is precisely the Bogoliubov frequency $E_q/\hbar$, with the weight factor $A_\mathbf{q} = \pi \epsilon_0^q / E_q$. In addition, substitution of the single-mode approximation into the compressibility sum rule (3.20) gives the static susceptibility

$$\chi_\mathbf{q} = \frac{\hbar q^2}{M \omega_\mathbf{q}^2} \to \frac{\hbar}{M s^2} \text{ as } q \to 0,$$  \hspace{1cm} (3.29)

in agreement with the general result in eq. (3.23).
Finite temperature requires an ensemble average, weighted with the Gibbs factor $e^{-\beta E_i}$ for the general initial state $|i\rangle$, along with the partition function $Z \equiv \sum_j e^{-\beta E_j}$. The “dynamical structure factor” $S(q, \omega)$ is defined as

$$S(q, \omega) = \frac{1}{NZ} \sum_{f_i} |\langle f | \hat{n}_q^{\dagger} | i \rangle|^2 \delta(\omega - \omega_{f_i}),$$

where $\omega_{f_i} = (E_f - E_i)/\hbar$ can now be both positive and negative. For liquid $^4$He, neutron scattering can measure $S(q, \omega)$ directly. In the Bogoliubov approximation for a dilute uniform Bose gas, it is not difficult to show that

$$S(q, \omega) \approx \frac{e_0}{E_q} \left\{ 1 + f(E_q) \right\} \delta(\omega - \frac{E_q}{\hbar}) + f(E_q) \delta(\omega + \frac{E_q}{\hbar}) \right\},$$

where $f(E_q)$ is the Bose-Einstein distribution function. The first term (representing emission of a quasiparticle with energy $E_q$) is known as a “Stokes” process; it persists even at zero temperature. The second term (representing the absorption of a quasiparticle with energy $E_q$) is an “anti-Stokes process” that requires the presence of a previously excited quasiparticle and thus vanishes as $T \to 0$. On general grounds, the dynamical structure factor satisfies several important relations; for example, its zeroth moment is the static structure function $S(q)$, which here gives

$$S(q) = \int_{-\infty}^{\infty} d\omega S(q, \omega) = \frac{e_0}{E_q} \coth\left(\frac{E_q}{2k_B T}\right),$$

in agreement with the previous eq. (3.10); furthermore, the first moment is just the $f$-sum rule at finite temperature

$$\int_{-\infty}^{\infty} d\omega \omega S(q, \omega) = \frac{\hbar q^2}{2M},$$

finally, $S(q, \omega)$ obeys a detailed-balance condition $S(q, -\omega) = e^{-\beta h \omega} S(q, \omega)$. These relations provide non-trivial checks on the Bogoliubov approximation to $S(q, \omega)$ in eq. (3.31).

C. Hydrodynamic description of non-uniform dilute Bose gas

The condensate wave function $\Psi$ can be rewritten as

$$\Psi = |\Psi|e^{iS},$$

where the absolute value is related to the condensate density $|\Psi(r)| = \sqrt{n_0(r)}$. Furthermore, the condensate particle current

$$j_0 = \frac{\hbar}{2M} [\Psi^* \nabla \Psi - (\nabla \Psi^*) \Psi] = \frac{\hbar}{M} n_0 \nabla S,$$

identifies the condensate velocity as

$$v_0(r) = \frac{\hbar}{M} \nabla S(r).$$

The local superfluid velocity $v_0$ is proportional to the gradient of the phase of the condensate wave function, implying that $\nabla \times v_0 = 0$. Hence the condensate flow is longitudinal (or irrotational), as proposed by Landau [1] for the superfluid velocity $v_s$ in $^4$He.

Substitution of eq. (3.34) into the GP equation (2.20) provides an intuitive hydrodynamic picture of the condensate. The real part gives a generalized Bernoulli’s equation

$$\mu = V_t + \frac{1}{2} M v_0^2 + \frac{4\pi \hbar^2 a}{M} n_0 - \frac{\hbar^2}{2M \sqrt{n_0}} \nabla^2 \sqrt{n_0},$$

where the last term is a quantum pressure associated with spatial variations in the condensate amplitude $|\Psi|$ (and thus the condensate density); the imaginary part is the static continuity equation for the condensate

$$\nabla \cdot (n_0 v_0) = \nabla \cdot j_0 = 0.$$
What determines the dynamics of small time-dependent perturbations about the static solution $\Psi(r)$? Two equivalent approaches yield the same final results.

1. Write the time-dependent condensate wave function as $\Psi(r, t) = \sqrt{n(r, t)} \exp [iS(r,t)]$ and substitute into the time-dependent GP equation (2.24). The real and imaginary parts yield time-dependent generalizations of eqs. (3.37) and (3.38), which can then be linearized around the static condensate contributions [28].

2. Retain the operators $\hat{\phi}$ and $\hat{\phi}^\dagger$ in the expansion of the field operator $\hat{\psi} = \Psi + \phi$ from eq. (2.53). This latter method allows a transparent treatment of the operator aspects of the hydrodynamic variables [51] and will be used here.

In a dilute Bose gas at low temperature, the total number-density operator

\[ \hat{n} = \hat{\psi}^\dagger \hat{\psi} \approx |\Psi|^2 + \Psi^\ast \hat{\phi} + \Psi \hat{\phi}^\dagger \]  

(3.39)

naturally separates into a large condensate part $n_0 = |\Psi|^2$ and a small non-condensate part

\[ \hat{n}' \approx \Psi^\ast \hat{\phi} + \Psi \hat{\phi}^\dagger \]  

(3.40)

that characterizes the local time-dependent particle-density fluctuations. Similarly, the local time-dependent fluctuations in the current-density operator have the form

\[ \hat{j}' = \hat{n}' \mathbf{v}_0 + n_0 \nabla \hat{\Phi}', \]  

(3.41)

where

\[ \hat{\Phi}' = \frac{\hbar}{2M|\Psi|^2} \left( \Psi^\ast \hat{\phi} - \Psi \hat{\phi}^\dagger \right) \]  

(3.42)

is the perturbation in the velocity potential operator (this quantity is effectively the perturbation in the phase operator). Since both $\hat{n}'$ and $\hat{\Phi}'$ are linear functions of $\hat{\phi}$ and $\hat{\phi}^\dagger$, it follows that $\langle \hat{n}' \rangle$ and $\langle \hat{\Phi}' \rangle$ vanish in the ensemble discussed below eq. (2.53), although correlation functions like $\langle \hat{n}'(r, t) \hat{n}'(\mathbf{r}', t') \rangle$ are finite. Note that $\hat{n}'$ is distinct from the operator $\hat{\phi}^\dagger \hat{\phi}$ whose ensemble average is the non-condensate density in eq. (2.56).

It is straightforward to verify from the equations of motion (2.58) for $\hat{\phi}$ and $\hat{\phi}^\dagger$ that these hydrodynamic operators $\hat{n}'$ and $\hat{\Phi}'$ obey the linearized conservation law

\[ \frac{\partial \hat{n}'}{\partial t} + \nabla \cdot \hat{j}' = \frac{\partial \hat{n}'}{\partial t} + \nabla \cdot \left( \hat{n}' \mathbf{v}_0 + n_0 \nabla \hat{\Phi}' \right) = 0 \]  

(3.43)

and the linearized Bernoulli’s equation

\[ \frac{\partial \hat{\Phi}'}{\partial t} + \mathbf{v}_0 \cdot \nabla \hat{\Phi}' + \frac{4\pi \hbar^2 a}{M^2} \hat{n}' - \frac{\hbar^2}{M^2 n_0} \nabla \cdot \left[ n_0 \nabla \left( \frac{n'}{n_0} \right) \right] = 0 \]  

(3.44)

that is a direct quantum generalization of Bernoulli’s equation for irrotational compressible flow [51] at constant entropy. Here, the first two terms constitute the total (or hydrodynamic) derivative (see, for example, ref. [4], sec. 48), and the last term is again the quantum (kinetic-energy) contribution. This eq. (3.44) has one significant advantage over the original field equations (2.58), for the chemical potential no longer appears explicitly ($\mu$ does, of course, affect the condensate wave function and condensate density).

These coupled operator equations can be rewritten with the normal-mode expansions from eq. (2.59). In this way, the hydrodynamic operators themselves have normal-mode expansions

\[ \hat{n}' = \sum_j \left( n'_j \alpha_j + n'_j^\ast \alpha_j^\dagger \right), \]  

(3.45a)

\[ \hat{\Phi}' = \sum_j \left( \Phi'_j \alpha_j + \Phi'_j^\ast \alpha_j^\dagger \right), \]  

(3.45b)

where the amplitudes $n'_j$ and $\Phi'_j$ obey the same coupled eqs. (3.43) and (3.44) as the operators. Comparison with the Bogoliubov eqs. (2.60) shows that

\[ n'_j = \Psi^\ast u_j - \Psi v_j, \]  

(3.46a)
\[ \Phi_j' = \frac{\hbar}{2M|\Psi|^2} (\Psi^* u_j + \Psi v_j), \] (3.46b)

showing that the Bogoliubov amplitudes \( u_j \) and \( v_j \) are linearly related to the hydrodynamic amplitudes \( n_j' \) and \( \Phi_j' \). Thus, any solution for \( n_j' \) and \( \Phi_j' \) gives a corresponding solution \( u_j \) and \( v_j \), and vice versa.

For example, eq. (2.66) shows that the Bogoliubov equations have exact dipole-mode solutions \( u_\alpha = a_\alpha^\dagger \Psi \) and \( v_\alpha = a_\alpha \Psi^* \), in which the condensate oscillates rigidly with frequency \( \omega_\alpha \), along the three orthogonal directions (this holds for any solution \( \Psi \) of the GP equation). Use of the explicit form of the raising and lowering operators \( a_\alpha^\dagger \) and \( a_\alpha \) in eqs. (2.63) and the linear relations in eqs. (3.46) readily yields the corresponding density perturbation

\[ n'_\alpha = -\frac{d_\alpha}{\sqrt{2}} \frac{\partial |\Psi|^2}{\partial x_\alpha}, \] (3.47)

which is a rigid shift of the condensate density \( |\Psi|^2 = n_0 \), and

\[ \Phi'_\alpha = \frac{\hbar}{\sqrt{2M} i} \left(x_\alpha - id_\alpha \frac{\partial S}{\partial x_\alpha}\right). \] (3.48)

The last term involving the phase \( S \) is absent if \( \Psi \) is real, and the velocity perturbation is then a spatial constant, moving 90° out of phase with the density oscillation. More generally, the last term reflects the change in the local velocity arising from the rigid motion of the condensate velocity.

For a nearly ideal Bose gas, the Bogoliubov equations for \( u_j \) and \( v_j \) often provide a simpler description than the hydrodynamic variables; in contrast, the hydrodynamic amplitudes \( n_j' \) and \( \Phi_j' \) are usually preferable in the large-condensate (TF) limit, where Stringari (see below) has constructed explicit solutions for the low-lying normal modes [28]. As an application of the linear relation between the two descriptions, these TF hydrodynamic solutions immediately give the corresponding Bogoliubov amplitudes \( u_j \) and \( v_j \) in the TF limit. The spatial integral of \( |v_j|^2 \) is the zero-temperature occupation number \( N_j'(0) \) of the \( j \)th mode, and a detailed evaluation for a large spherical condensate shows that they are large (of order \( R_0^2/d_0^2 \)) [1].

D. Hydrodynamics in the Thomas-Fermi limit

Assume a large stationary condensate with \( v_0 = 0 \) and condensate density \( n_0 = (M/4\pi\hbar^2a)(\mu - V_\text{tr})\Theta(\mu - V_\text{tr}) \). In this TF limit, it is consistent to omit the quantum contribution in Bernoulli’s equation, in which case the hydrodynamic amplitudes \( n_j'(r)e^{-i\omega_j't} \) and \( \Phi_j'(r)e^{-i\omega_j't} \) obey two coupled linear equations

\[ -i\omega_j n_j' + \nabla \cdot (n_0 \nabla \Phi_j') = 0, \] (3.49a)

\[ -i\omega \Phi_j' + \frac{4\pi\hbar^2a}{M^2} n_j' = 0. \] (3.49b)

They can be combined into a single second-order equation for the density perturbation [28]

\[ \omega_j^2 n_j' + \frac{1}{M} \nabla \cdot \left[(\mu - V_\text{tr})\nabla n_j'\right] = 0 \] (3.50)

that has numerous explicit solutions.

1. spherical trap

In the TF limit, the chemical potential becomes \( \mu \approx \frac{1}{2}\hbar\omega_0 R_0^2/d_0^2 = \frac{1}{2}M\omega_0^2 R_0^2 \), so that eq. (3.50) becomes

\[ \omega_j^2 n_j' + \frac{1}{2M} \omega_0^2 \nabla \cdot [(R_0^2 - r^2) \nabla n_j'] = 0. \] (3.51)

The spherical symmetry allows solutions of the form \( n_j''(r)Y_{ml}(\theta, \phi) \), where \( Y_{ml} \) is a spherical harmonic and \( n_j'' \) is the corresponding \( n \)th radial eigenfunction. Solutions with different \( m \) are degenerate, and the TF frequencies [28]
\[ \omega_{nl}^2 = \omega_0^2 \left[ l + n (2n + 2l + 3) \right], \]  
\[ (3.52) \]
can be compared with the frequencies \( \omega_{nl} = \omega_0 (2n + l) \) of an ideal gas in an isotropic harmonic potential. For the lowest radial modes with \( n = 0 \), the TF eigenfrequencies \( \omega_0 \sqrt{I} \) differ from those for the harmonic oscillator \( \omega_0 l \) except for the lowest dipole mode \( (l = 1) \).

The associated eigenfunctions are polynomials of the form \( n_{nl}(r) = r^l P_{nl}(r^2/R_0^2) \), where \( P_{nl}(u) \) is a polynomial of order \( n \) (a type of Jacobi polynomial), for example, \( P_{0l}(u) = 1 \) and \( P_{1l}(u) = 1 - (2l + 5)u/(2l + 3) \). The radial eigenfunctions are bounded at both the origin and the TF surface \( r = R_0 \).

2. anisotropic axisymmetric trap

The preceding treatment can be generalized to an axisymmetric but anisotropic trap with \( V_T = \frac{1}{2} M \omega_\perp^2 (r_\perp^2 + \lambda^2 z^2) \), where \( \lambda = \omega_z/\omega_{\perp} \) is the asymmetry parameter. The condensate density \( n_0 \) is proportional to \( 1 - r_\perp^2/R_0^2 - z^2/R_0^2 \), where \( R_\perp \) and \( R_z \) are the radial and axial semiaxes of the ellipsoidal condensate with anisotropy ratio \( R_z/R_\perp = 1/\lambda \).

Equation (3.50) now has eigenfunctions \( n' \) proportional to \( e^{im\phi} \) and eigenvalues that depend explicitly on the azimuthal quantum number \( m \). They fall into two distinct classes \[ 28, 3, 54 \]
a. \( n' \propto r_\perp^{|m|} e^{im\phi} \times \) a polynomial in \( (r_\perp^2, z^2) \) (these modes are even in \( z \)); 
b. \( n' \propto z r_\perp^{|m|} e^{im\phi} \times \) a polynomial in \( (r_\perp^2, z^2) \) (these modes are odd in \( z \)).

Numerous special cases are of interest, and it is simplest to start with the choice that the polynomial equals one.

a. The solutions of type a take the form \( n' \propto r^l Y_{lm} \) with \( m = \pm l \), and the associated frequency is \( \omega_{m=\pm l}^2 = \lambda^2 \omega_{\perp}^2 \), independent of the asymmetry parameter \( \lambda \). This mode is a generalization of that with frequency \( \omega_{0l}^2 = \omega_0^2 \) in a spherical trap. For \( l = 1 \), these two modes are the circularly polarized radial dipole oscillations with frequency \( \omega_\perp \) and \( m = \pm 1 \).

b. The solutions of type b take the form \( n' \propto r^l Y_{lm} \) with \( m = \pm (l-1) \), and the associated frequency is \( \omega_{m=\pm l-1}^2 = (l-1) \omega_\perp^2 + \lambda^2 \omega_z^2 \). For \( l = 1 \), this mode is the axial dipole oscillation with \( m = 0 \) and frequency \( \omega_z \).

The next simplest class takes a polynomial of the form \( A + Br_\perp^2 + Cz^2 \). For \( m = 0 \), those of type a have the two frequencies

\[ \frac{\omega_{m=0}^2}{\omega_{\perp}^2} = 2 + \frac{3}{2} \lambda^2 \pm \sqrt{(2 - \frac{3}{2} \lambda^2)^2 + 2 \lambda^4}. \]
\[ (3.53) \]
The two eigenstates involve a coupled motion of a monopole mode with \( n = 1, l = 0 \) and a quadrupole mode with \( n = 0, l = 2, m = 0 \). For a spherical trap \( (\lambda = 1) \), the frequencies reproduce both the results given in eq. \[ 3.52 \]. For a disk-shape trap with \( \lambda = \sqrt{5} \), this expression gives a good fit to the observations on \(^{87}\text{Rb} \[ 27 \]. For a cigar-shape trap with \( \lambda \ll 1 \), the two frequencies reduce to \( \sqrt{2} \omega_\perp \) and \( 2\omega_\perp \); the first fits the observations on \(^{23}\text{Na} \[ 27 \] within 1\%. More generally, theoretical eigenvalues and eigenfunctions are readily determined for larger values of \( |m| \) and for the solutions of class b.

E. Sum-rules for a non-uniform Bose condensate

Stringari \[ 28 \] has modified the sum rules for a uniform medium to obtain accurate estimates of the first few lowest frequencies of the collective modes in a trap for intermediate values of the interaction parameter \( Na/d \), where neither the nearly ideal gas nor the TF approximation is accurate. Consider a general operator \( \hat{F} \) that excites the system from its ground state \( |0\rangle \) to an excited state \( |j\rangle \). Here, \( \hat{H}|j\rangle = E_j |j\rangle \) and \( \hat{H}|0\rangle = E_0 |0\rangle \), where \( \hat{H} \) is the interacting many-particle Hamiltonian and \( E_j \) and \( E_0 \) are the corresponding exact energies.

In analogy with eq. \( 3.18 \), define a “spectral-density function”

\[ S(\omega) = \sum_{j \neq 0} \frac{|\langle j|\hat{F}|0\rangle|^2}{\omega_j - \omega} \delta (\omega - \omega_j), \]
\[ (3.54) \]
where \( \omega_j = (E_j - E_0)/\hbar \). In the following analysis, the matrix elements of \( \hat{F} \) are assumed to have the following time-reversal behavior

\[ |\langle j|\hat{F}|0\rangle|^2 = |\langle j|\hat{F}^\dagger|0\rangle|^2 = |\langle 0|\hat{F}|j\rangle|^2, \]
\[ (3.55) \]
which restricts the allowed class of excitation operators. The various moments of the spectral-density function are

\[
\mathcal{M}_i = \int_0^{\infty} d\omega \omega^i S(\omega).
\] (3.56)

For example, the first moment is

\[
\mathcal{M}_1 = \sum_{j \neq 0} \left| \langle j | \hat{F} | 0 \rangle \right|^2 \omega_{j0}
= \frac{1}{2} \sum_{j \neq 0} \left( \langle 0 | \hat{F}^\dagger | j \rangle \omega_{j0} \langle j | \hat{F} | 0 \rangle + \langle 0 | \hat{F} | j \rangle \omega_{j0} \langle j | \hat{F}^\dagger | 0 \rangle \right)
= \frac{1}{2\hbar} \sum_{j \neq 0} \left( \langle 0 | \hat{F}^\dagger | j \rangle \left[ \hat{H}, \hat{F} \right] | 0 \rangle - \langle 0 | \hat{F} | j \rangle \left[ \hat{H}, \hat{F}^\dagger \right] | j \rangle \right). \] (3.57)

The sum over the excited states can now be performed with the completeness relation [this is essentially the method used to prove the \( f \)-sum rule in eq. (3.25)], leading to the expression

\[
\mathcal{M}_1 = \frac{1}{2\hbar} \langle 0 | \left[ \hat{F}^\dagger, \left[ \hat{H}, \hat{F} \right] \right] | 0 \rangle, \] (3.58)

expressed in terms of a ground-state expectation value of a double commutator. Similarly,

\[
\mathcal{M}_3 = \frac{1}{2\hbar^3} \langle 0 | \left[ \left[ \hat{F}^\dagger, \hat{H} \right], \left[ \hat{H}, \left[ \hat{H}, \hat{F} \right] \right] \right] | 0 \rangle. \] (3.59)

1. surface modes

It is simplest to treat a spherical trap with a spherical condensate density. Consider the excitation operator

\[
F = \sum_{i=1}^N r_i Y_{lm}(\theta_i, \phi_i) \quad \text{in first-quantized notation, or}
\]

\[
\hat{F} = \int d^3r \hat{\psi}^\dagger(r) r^l Y_{lm}(\theta, \phi) \hat{\psi}(r) \quad \text{in second-quantized notation.} \] (3.60a)

This operator has no radial node and thus predominantly excites modes with zero radial quantum number \( n = 0 \) (called “surface modes” 28). A direct evaluation of the two moments for the ground-state condensate wave function (the solution of the GP equation) yields the ratio \( \mathcal{M}_3/\mathcal{M}_1 \), which is an upper bound for the squared frequency of the transition from the ground state to the first state excited by the operator \( \hat{F} \). In fact, this ratio should provide a good approximation to the exact lowest squared frequency since these surface modes are orthogonal to the higher radial modes.

For the special case of the quadrupole surface mode \((n = 0 \text{ and } l = 2)\), this ratio gives the expression

\[
\omega_0^2 = \frac{2\omega_0^2}{1 + \left( \frac{T_0}{V_{tr0}} \right)}.
\] (3.61)

where the angular brackets are the ground-state expectation values of the kinetic and trap potential energies evaluated with the condensate wave function \( \Psi \). This expression holds for all values of the interaction parameter \( Na/d_0 \), providing a uniform interpolation between the ideal gas and the TF limit.

a. For an ideal gas, the familiar properties of the harmonic oscillator show that the kinetic energy and trap potential energy are equal, reproducing the expected result that \( \omega_0^2 = 2\omega_0 \) for the excitation to the lowest quadrupole state of the oscillator.

b. In contrast, the kinetic energy is negligible in the TF limit, reproducing the previous TF result from eq. (3.52) that \( \omega_0^2 = \sqrt{2}\omega_0 \).
2. lowest compressional mode

The operator

\[ F = \sum_{i=1}^{N} r_i^2, \quad \text{or, equivalently,} \]

\[ \hat{F} = \int d^3r \, \psi^\dagger(r) r^2 \psi(r) \]

excites the lowest monopole mode with \( n = 1 \) and \( l = 0 \). Direct evaluation of the moments \( M_1 \) and \( M_3 \), along with the virial theorem from eq. (2.28) gives the result

\[ \omega_{10}^2 = \omega_0^2 \left( 5 - \frac{\langle T \rangle_0}{\langle V_{tr} \rangle_0} \right). \]

(a) In the ideal-gas limit, the last ratio is 1, yielding the expected expression \( \omega_{10} = 2\omega_0 \) [from the general expression \( \omega_{nl} = \omega_0(2n + l) \) for an isotropic oscillator].

(b) In the TF limit, the last ratio is negligible, yielding \( \omega_{10} = \sqrt{5} \omega_0 \), in agreement with the previous TF result from eq. (3.52).

IV. VORTICES IN A DILUTE TRAPPED BOSE GAS

The prediction and detection of quantized vortices in superfluid \( ^4\text{He} \) were early milestones in understanding the role of the condensate and the associated macroscopic wave function (see, for example, refs. [55,56]). Thus it is natural to consider similar vortex structures in the dilute trapped Bose gases, but the analysis involves several new and subtle features.

A. Review of classical vortices

Consider an incompressible nonviscous fluid with uniform number density \( \bar{n} \). A long straight vortex line at the origin has the velocity field

\[ \mathbf{v}(r) = \frac{\kappa}{2\pi r_\perp} \hat{\phi}, \]

where \( \hat{\phi} \) is a unit tangential vector in cylindrical polar coordinates. The flow velocity consists of circular streamlines, and its magnitude diverges near the symmetry axis (as \( r_\perp \to 0 \)). It is obvious from the flow pattern that \( \nabla \cdot \mathbf{v} = 0 \), so that the flow is transverse (or solenoidal). In addition, it is not difficult to verify that \( \nabla \times \mathbf{v} = 0 \) except on the axis, and use of Stokes’s theorem shows that

\[ \nabla \times \mathbf{v} = \kappa \hat{z} \delta^{(2)}(r_\perp). \]

Thus the flow is longitudinal (or irrotational) \emph{nearly everywhere}, but it is singular on the axis (analogous to the magnetic field of a long straight current-carrying filament). Finally, the line integral of \( \mathbf{v} \) is known as the “circulation;” for any contour \( C \) that encircles the vortex once, the circulation

\[ \oint_C d\mathbf{l} \cdot \mathbf{v} = \kappa, \]

is independent of \( C \), with the constant \( \kappa \) in eq. (4.1) fixing the strength of the vortex.

The energy is purely kinetic in an incompressible fluid. In the present case of a long vortex filament, it has the value (per unit length)

\[ E_v = \frac{1}{2} \bar{M} \bar{n} \int d^2r_\perp v^2 \]

\[ = \frac{\bar{M} \bar{n} \kappa^2}{4\pi} \int \frac{dr_\perp}{r_\perp} \approx \frac{\bar{M} \bar{n} \kappa^2}{4\pi} \ln \left( \frac{R_0}{\xi} \right). \]

Here, the integral must be cut off at an upper limit \( R_0 \), which can be interpreted as either the size of the container or the intervortex separation, and a lower limit \( \xi \), which can be interpreted as the size of the vortex core.
1. quantized circulation

The idea of a quantized vortex in superfluid $^4$He can be motivated by recalling the Bohr-Sommerfeld quantization condition $\oint d\mathbf{r} \cdot \mathbf{p} = \hbar$ for the periodic motion of a particle in phase space. Since $\mathbf{p} = M\mathbf{v}$ for a particle in the vortex, this result immediately suggests that the circulation in superfluid $^4$He should be quantized in units of $\hbar/M$. Note that $\kappa$ has the dimensions of length/time (like a diffusion constant or a thermal conductivity or a kinematic viscosity). Onsager [57] first publicly proposed the concept of quantized circulation in 1949 (sec. 2.3 of ref. [56] contains a brief history of Onsager’s unpublished work). London [58] documented one of Onsager’s earlier remarks and made a similar proposal [59] concerning quantized magnetic flux in superconductors in 1950. Independently, Feynman developed a more complete description of quantized vortex lines in superfluid $^4$He [60], based on the identification of the velocity with the gradient of the phase of the appropriate many-body wave function [compare eq. (3.36) for the condensate 4].

A classical vortex in an incompressible fluid requires a specific model for the core radius $\xi$. In fact, all real fluids have a non-zero compressibility, implying a finite speed of sound $s$. If the hydrodynamic flow speed $|v|$ is small compared to $s$, then the fluid is essentially incompressible. This picture necessarily fails near the vortex core, where eq. (1.3) shows the flow speed diverges; hence a singularity develops when the magnitude of the circulating velocity becomes comparable with the speed of sound. Setting $v(\xi) = \kappa/2\pi\xi \approx s$ gives the resulting core radius

$$\xi \approx \frac{\kappa}{2\pi s}. \quad (4.5)$$

If the vortex is quantized with circulation $\hbar/M$, this result implies that the vortex core has a radius

$$\xi \approx \frac{\hbar}{Ms}. \quad (4.6)$$

As seen in eq. (2.43), the same qualitative expression holds for the healing length $\xi = \hbar/(\sqrt{2}Ms)$ in a dilute trapped Bose gas in the TF limit, suggesting that $\xi$ also characterizes the vortex core radius in such a condensate.

2. classical vortex dynamics

The dynamics of a vortex in an incompressible fluid follows from the celebrated “circulation” theorem of classical hydrodynamics (see, for example, sec. 48 of ref. [4]); it states that the circulation around any contour that moves with the fluid is a strict constant of the motion. If the contour initially contains a vortex with circulation $\kappa$, the vortex can never escape the moving contour. In the particular case of a single long straight vortex line in otherwise stationary fluid, this result implies that the vortex itself remains at rest. Similarly, in the presence of a uniform external flow $\mathbf{v}_{\text{ex}}$, the vortex line moves rigidly with the same velocity (this conclusion also follows with a Galilean transformation to a moving coordinate system).

A more interesting case is two long straight parallel vortices a distance $d$ apart. Since neither vortex moves under its own influence, the motion arises solely from its neighbor. Assume first that they have the same sense of circulation, as shown in fig. 5a. In this case, it is easy to see from the circulating velocity fields that they each execute a circular trajectory at fixed separation $d$. Since the interaction energy depends only on $d$ (and the individual circulation $\kappa$), the constancy of the separation $d$ follows from the conservation of energy in a nonviscous fluid. In the presence of weak dissipation, the system will act to lower the energy; here, the two vortices will slowly spiral apart because the total energy of two identical vortex lines is lowest at infinite separation (for $d \to 0$, the vortices fuse into a single vortex with circulation $2\kappa$ and four times the energy of a single vortex line, whereas, for $d \to \infty$, the energy reduces to two times the energy of a single vortex line).

If the parallel vortices have opposite sense of circulation, they are known as a “vortex pair” (shown in fig. 5b). The mutual influence of the circulating velocity fields means that they move perpendicular to the line joining them in the same sense as the local fluid between them, with fixed separation $d$ (again because of conservation of energy). This configuration is the two-dimensional analog of a vortex ring, which moves perpendicular to its plane maintaining a fixed radius. In the presence of weak dissipation, the two vortices slowly drift together and annihilate when the separation $d$ is comparable with the vortex core size $\xi$.

3. effect of rigid boundaries in classical hydrodynamics

As a simple example, consider a long circular cylinder with rigid walls of radius $R_0$. If a vortex line is located on the symmetry axis, its circulating flow velocity is everywhere parallel to the boundary, so that the flow field is unperturbed
One of the original motivations for the introduction of quantized vortices was the observed behavior of a rotating superfluid. For example, a bucket of $^4$He rotating at an angular velocity $\Omega$ was known to retain its classical parabolic meniscus $z = \frac{1}{2}\Omega^2 r^2/g$ far below the superfluid transition temperature $T_\lambda$. This observation appeared to contradict Landau’s assertion that the superfluid velocity is irrotational with $\nabla \times v = 0$ (implying that it could not rotate) and that $\rho_s/\rho \approx 1$ (so that the normal fluid played a negligible role). To explain this apparent paradox, Feynman relaxed the irrotational condition of strictly zero vorticity, allowing the vorticity to be singular at the cores of the quantized vortices [compare eq. (4.2)]; in particular, he suggested that a rotating superfluid contains an array of quantized vortex lines with areal density $n_v = 2\Omega/\kappa = M\Omega/(\pi\hbar)$ and mean vorticity $n_{\kappa,\kappa} = 2\Omega$, thereby mimicking the solid-body rotation required to explain the curved meniscus (see sec. 2.4 of ref. [50]).

From this perspective, it becomes interesting to consider the effect of rotating a physical container at an angular velocity $\Omega = \Omega z$. In the laboratory (non-rotating) frame, the moving walls represent time-dependent potentials that do work on the system, precluding a description in terms of thermodynamic equilibrium. In the rotating frame, however, the walls are stationary, so that the hamiltonian becomes time-independent, and the usual thermal Gibbs distribution remains valid when expressed in terms of the energy levels appropriate for the rotating frame. In the rotating frame, the new hamiltonian is known to be $H - \Omega \cdot L = H - \Omega L_z$ (see, for example, sec. 5 of ref. [11] and sec. 34 of ref. [2]).

At zero temperature, a system rotating at an angular velocity $\Omega$ will adjust any free parameters to minimize the quantity $F = \langle H - \Omega L_z \rangle = E - \Omega L_z$, which can be considered a “free energy” that depends on the parameter $\Omega$. As a simple example, consider a long cylinder of radius $R_0$. If the fluid does not contain a vortex, the velocity is everywhere zero, so that the energy $E_0$ and angular momentum $L_0$ per unit length both vanish, with $F_0 = E_0 - \Omega L_0 = 0$. When a vortex is placed at the center of the cylinder, however, the energy per unit length is

$$E_v = \frac{M\bar{\kappa}^2}{4\pi} \ln\left(\frac{R_0}{\xi}\right)$$  

(4.7)

from eq. (4.4), and an elementary calculation shows that the corresponding angular momentum per unit length is $L_v = \frac{1}{2}M\bar{\kappa}R_0^2$. Thus the free energy per unit length of the fluid containing one vortex is given by

$$F_v = E_v - \Omega L_v = \frac{M\bar{\kappa}^2}{4\pi} \ln\left(\frac{R_0}{\xi}\right) - \frac{1}{2}M\bar{\kappa}R_0^2.$$  

(4.8)

This quantity is positive for sufficiently small $\Omega$, but it decreases linearly with increasing angular velocity and becomes negative (and hence less than $F_0$ for the no-vortex state) at a critical angular velocity $\Omega_{c1} = (\kappa/2\pi R_0^2)\ln(R_0/\xi)$ for the creation of a single vortex line in a long cylinder of radius $R_0$. Use of the quantized circulation $\kappa = \hbar/M$ yields the expression (see, for example, sec. 2.4 of ref. [50])

$$\Omega_{c1} \approx \frac{\hbar}{M R_0^2} \ln\left(\frac{R_0}{\xi}\right)$$  

for a quantized vortex in uniform fluid. (4.9)

This elementary analysis can be generalized to the case of a long straight vortex a distance $r_0$ from the center of the cylinder. The free energy per unit length remains zero if there is no vortex, and a detailed calculation gives the free energy per unit length of a vortex displaced a fractional distance $x_0 = r_0/R_0$ from the center

$$F_v(x_0) = \frac{M\bar{\kappa}^2}{4\pi} \left[ \ln\left(\frac{R_0}{\xi}\right) + \ln(1 - x_0^2) - \frac{\Omega}{\Omega_0}(1 - x_0^2) \right].$$  

(4.10)

by the presence of the cylinder. The situation is more complicated, however, if the vortex is shifted rigidly a distance $r_0 < R_0$ off-axis. In this case, the flow of the vortex by itself no longer matches the boundary condition that the normal component of the flow velocity must vanish. This boundary-value problem has an elementary solution consisting of a single opposite image located along the same ray at a distance $R_0^2/r_0$ outside the cylinder (see fig. 6). Comparison with fig. 5b shows that the vortex and its image constitute a vortex pair; in the absence of dissipation, the original vortex will therefore execute a circular orbit at fixed $r_0$ under the influence of its image (which moves synchronously with the vortex). If the system is weakly dissipative, the vortex will slowly spiral outward and eventually annihilate with the image vortex when $r_0 \approx R_0 - \xi$.

4. effect of rotation
where \( \Omega_0 = \kappa/(2\pi R_0^2) = \hbar/(MR_0^2) \) is a characteristic angular velocity. Figure 7 shows this quantity as a function of \( x_0 \) for various fixed values of \( \Omega \). If \( \Omega = 0 \), the free energy decreases monotonically with increasing \( x_0 \), indicating that the vortex is unstable in the presence of a weak dissipation. This behavior persists to the value \( \Omega = \Omega_0 \), when the curvature at the origin \( x_0 = 0 \) vanishes. For \( \Omega_0 < \Omega < \Omega_0 \), the free energy develops a local minimum at \( x_0 = 0 \), although \( F_0(0) \geq F_0(1) \), with an intermediate barrier that hinders the vortex from reaching the outer wall at \( x_0 = 1 \). This behavior suggests that \( \Omega_m = \Omega_0 \) represents the onset of metastability in the presence of weak dissipation, which persists up to the critical angular velocity \( \Omega_0 \), when the free energy \( F_0(0) \) for a vortex at the origin becomes equal to that \( F_0(1) \) for a vortex at the wall. For \( \Omega > \Omega_0 \), the vortex becomes stable relative to the no-vortex state. Experiments on rotating superfluid \(^4\)He \(^2\) have verified these predictions in considerable detail.

### B. Vortices in a dilute uniform Bose gas

The original application of the GP equation \(^2\) was the description of a long straight vortex in an otherwise uniform dilute Bose gas. Recall eq. \(^2\) for the condensate velocity \( v_0 = (\hbar/M)\nabla S \), where \( S \) is the phase of the condensate wave function, and consider the circulation \( \Gamma = \oint_C \mathbf{r} \cdot \mathbf{v}_0 \) around an arbitrary closed contour \( C \). A combination of these expressions gives

\[
\Gamma = \hbar \oint_C \mathbf{r} \cdot \mathbf{v}_0 = \frac{\hbar}{M} \Delta S|_C,
\]

(4.11)

where \( \Delta S|_C \) is the change in the phase on once going around \( C \). If the condensate wave function is single valued, this change must be an integer times \( 2\pi \), implying that the circulation in any condensate is quantized in units of \( \hbar/M \).

In general, the GP equation for a bulk dilute Bose gas is

\[
(T + g|\Psi|^2 - \mu) \Psi = 0,
\]

(4.12)

since \( V_\perp \) is absent. Assume that the condensate contains a singly quantized vortex line at \( r_\perp = 0 \), oriented along \( \hat{z} \), with bulk condensate density \( n_0 \) far from the vortex \( r_\perp \to \infty \). The condensate wave function has the form \(^2\)

\[
\Psi = \sqrt{n_0} e^{i\phi} f(r_\perp),
\]

(4.13)

where \( f \) is real and approaches 1 as \( r_\perp \to \infty \). Here, the phase is given by \( S = \phi \), so that the condensate velocity \( \mathbf{v}_0 = (\hbar/M)\nabla \phi = (\hbar/Mr_\perp)\phi \) is precisely that for a vortex with circulation \( \hbar/M \). The condensate density \( |\Psi|^2 = n_0(r_\perp) = n_0 f(r_\perp)^2 \) is cylindrically symmetric. The radial amplitude obeys the nonlinear GP equation

\[
\left( -\frac{\hbar^2}{2M} \frac{1}{r_\perp} \frac{d}{dr_\perp} r_\perp \frac{d}{dr_\perp} + \frac{\hbar^2}{2Mr_\perp^2} + gn_0 f^2 - \mu \right) f = 0,
\]

(4.14)

where the asymptotic behavior of \( f \) fixes the chemical potential \( \mu = gn_0 \) at the value for the uniform system.

In a dilute uniform Bose gas, the healing length is defined by \( \xi^2 = \hbar^2/(2Mgn_0) = 1/(8\pi n_0) \), balancing the kinetic energy and the interaction energy. With the dimensionless variable \( x = r_\perp/\xi \), the radial GP equation becomes

\[
\left( - \frac{1}{x} \frac{d}{dx} x \frac{d}{dx} + \frac{2}{x^2} + f^2 - 1 \right) f = 0,
\]

(4.15)

where the term \( 1/x^2 \) is the centrifugal barrier associated with the circulating velocity field. For small \( x \ll 1 \), this barrier forces the amplitude to vanish linearly inside the vortex core \( x \lesssim 1 \), and a detailed analysis shows that

\[
f(x) \approx \begin{cases} 0.583 x, & \text{for } x \ll 1, \\ 1 - (2x^2)^{-1}, & \text{for } x \gg 1. \end{cases}
\]

(4.16)

Figure 8 shows the form of this function, indicating that this quantum vortex automatically forms its own core with radius \( \approx \xi = \hbar/(\sqrt{2M}s) \), where \( s \) is the Bogoliubov speed of sound [compare eq. \(^4\)]. The condensate density \( n_0(r_\perp) \) varies like \( r_\perp^2 \) near the origin, and the corresponding condensate current \( j_0(r_\perp) = n_0(r_\perp)\mathbf{v}_0(r_\perp) \) reaches a maximum near the core and then vanishes linearly as \( r_\perp \to 0 \), even though the velocity itself diverges. Numerical analysis yields the GP vortex energy per unit length \(^4\)

\[
E_v \approx \frac{\pi \hbar^2 n_0}{M} \ln \left( \frac{1.46 R_0}{\xi} \right) \approx \frac{Mn_0\kappa^2}{4\pi} \ln \left( \frac{1.46 R_0}{\xi} \right),
\]

(4.17)
The GP description of a vortex in a bulk dilute Bose gas can be generalized to study the dynamics of a set of widely separated parallel vortex lines \([24]\) located at \([\mathbf{r}_{\perp j}]\), assuming that \(|\mathbf{r}_{\perp i} - \mathbf{r}_{\perp j}| \gg \xi\) for all pairs. An approximate GP condensate wave function can be constructed as a product of the individual radial functions \(f(|\mathbf{r}_{\perp} - \mathbf{r}_{\perp j}|)\), each of which approaches 1 far from the vortex cores; the phase is given as the sum of the azimuthal angles \(S_j\) measured relative to \(\mathbf{r}_{\perp j}\) as origin, where \(\tan S_j = (y_j - y_j)/(x_i - x_j)\), so that

\[
\Psi(\mathbf{r}_{\perp}, t) = \sqrt{N_0} e^{-i\mu t/\hbar} \prod_j [e^{i S_j} f(|\mathbf{r}_{\perp} - \mathbf{r}_{\perp j}|)].
\]

(4.18)

This state changes with time both because of the overall chemical potential and because each vortex follows its own dynamical trajectory. Substitution of this condensate wave function into the time-dependent GP eq. (2.24) determines the subsequent motion of the individual vortices; they obey the classical hydrodynamic prescription that each vortex moves with the local velocity induced at its position by all the other ones. This result should not be very surprising, because the GP equation itself can be recast in a hydrodynamic form.

C. A vortex in a dilute trapped Bose condensate

Assume a singly quantized vortex in an axisymmetric trap, where the condensate wave function takes the form

\[
\Psi(\mathbf{r}) = \sqrt{N_0} e^{i\phi} f(r_{\perp}, z).
\]

(4.19)

Like the case of a vortex in a uniform condensate, the condensate velocity field here is just \(\mathbf{v}_0 = (\hbar/Mr_{\perp}) \dot{\phi}\), representing a vortex with circulation \(\kappa = \hbar/M\). The amplitude function \(f\) satisfies the GP equation and can be determined numerically \([22]\). This function vanishes linearly as \(r_{\perp} \to 0\), giving a node on the symmetry axis. Consequently, the condensate density for a vortex in a trapped condensate is toroidal (in contrast to the non-vortex condensate, where the density decreases monotonically away from the center).

1. critical angular velocity \(\Omega_{c1}\)

Given the condensate wave function for the vortex state and that for no vortex, it is straightforward to evaluate the additional energy \(\Delta E\) associated with the formation of the vortex, along with the angular momentum \(L_z\). If the system rotates with angular velocity \(\Omega\), a vortex line on the symmetry axis becomes stable when \(\Delta F = \Delta E - \Omega L_z\) becomes negative, so that the critical angular velocity is \(\Omega_{c1} = \Delta E/L_z\).

This quantity can be evaluated analytically in two limits. If the system is nearly ideal, with \(N_0a/d_0 \ll 1\), then the condensate wave function for the vortex is one with all \(N_0\) particles in the single-particle state

\[
\chi_{10}(\mathbf{r}_{\perp})\psi_0(z) \propto r_{\perp} e^{i\phi} \exp \left(-\frac{r_{\perp}^2}{2d_{\perp}^2} - \frac{z^2}{2d_z^2}\right),
\]

(4.20)

where \(\chi_{n_+,n_-}\) is a two-dimensional oscillator state with \(n_+\) and \(n_-\) right- and left-circular quanta and \(\psi_n\) is the usual one-dimensional oscillator wave function \([35]\). The increased energy is essentially \(N_0\) times the increased single-particle energy \(\hbar \omega_{\perp}\) associated with the transition from the two-dimensional ground state \(\chi_{00}\) to the vortex state \(\chi_{10}\), and the associated angular momentum is just \(N_0\hbar\) because every particle has unit angular momentum. Thus \(\Omega_{c1} \to \omega_{\perp}\) for an ideal gas in an axisymmetric trap. In this ideal-gas limit, however, the transition is hugely degenerate, for the same \(\Omega_{c1}\) also characterizes a condensate with multiple quanta of circulation. With the GP equation, it is not difficult to evaluate the first correction to this value for a singly quantized vortex in a nearly ideal Bose gas

\[
\frac{\Omega_{c1}}{\omega_{\perp}} \approx 1 - \frac{1}{\sqrt{8\pi}} \frac{N_0a}{d_z} + \cdots,
\]

(4.21)

showing that the critical angular velocity decreases linearly as the interaction parameter \(N_0a/d_z = \sqrt{\lambda} N_0a/d_\perp\) initially increases from 0 (here, \(\lambda = \omega_z/\omega_{\perp}\) is the asymmetry parameter for the trap).

In the opposite limit of a large interaction parameter \(N_0a/d_0 \gg 1\), initial estimates \([20]\) relied on classical hydrodynamics, using eq. (4.19) to write
\[ \frac{\Omega_{c1}}{\omega_\perp} \approx \frac{\hbar}{M \omega_\perp R_\perp^2} \ln \left( \frac{R_\perp}{\xi} \right) = \frac{d_1^2}{R_\perp^2} \ln \left( \frac{R_\perp}{\xi} \right), \]  
(4.22)

since \( d_1^2 = \hbar/M \omega_\perp \). Note that this ratio is small in the TF limit where \( d_1^2/R_\perp^2 \sim \xi/R_\perp \ll 1 \); the explicit TF relation

\[ \frac{d_1^2}{R_\perp^2} = \left( \frac{d_1}{15 N \alpha \lambda} \right)^{2/5} \]
(4.23)

shows precisely how this small parameter scales with \( N \) and \( \lambda \).

An improved TF estimate follows by writing the condensate wave function as \( \Psi = e^{i\phi} |\Psi| \), and ignoring the gradients of \(|\Psi|\). The new feature is that the TF density profile now has a centrifugal barrier [66,67]

\[ g_{n1} \approx \left( \mu_1 - \frac{1}{2} M \omega_\perp^2 r_\perp^2 - \frac{\hbar^2}{2 M R_\perp^2} \right) \Theta \left( \mu_1 - \frac{1}{2} M \omega_\perp^2 r_\perp^2 - \frac{\hbar^2}{2 M R_\perp^2} \right), \]
(4.24)

where \( n_1 \) and \( \mu_1 \) are the density and chemical potential for the state with one quantum of circulation. This density can be compared with that for the no-vortex state in eq. (2.35)

\[ g_{n0} \approx (\mu_0 - \frac{1}{2} M \omega_\perp^2 r_\perp^2 - \frac{\hbar^2}{2 M R_\perp^2} \Theta \left( \mu_0 - \frac{1}{2} M \omega_\perp^2 r_\perp^2 - \frac{\hbar^2}{2 M R_\perp^2} \right). \]
(4.25)

The fractional change in the chemical potential caused by the vortex \((\mu_1 - \mu_0)/\mu_0\) can be shown [68] to be small, of order \((d_1^2/R_\perp^2) \ln (R_\perp^2/d_1^2)\). As a result, the TF density profile for the condensate with a singly quantized vortex has the simple form

\[ n_1(r_\perp, z) \approx n_0(0) \left( 1 - \frac{r_\perp^2}{R_\perp^2} - \frac{z^2}{R_\perp^2} - \frac{\xi^2}{R_\perp^2} \right) \Theta \left( 1 - \frac{r_\perp^2}{R_\perp^2} - \frac{z^2}{R_\perp^2} - \frac{\xi^2}{R_\perp^2} \right), \]
(4.26)

where \( n_0(0) \) is the central density for the no-vortex condensate. The important new qualitative feature of a vortex in the TF limit is the appearance of a small flared hole of radius \( \sim \xi \) [34,69,67], but the remainder of the condensate density is essentially unchanged; in practice, it is usually sufficient to retain the no-vortex density and simply cut off any divergent radial integrals at the core size \( \xi \).

A detailed solution of the GP equation for an axisymmetric trap in the TF limit [4,58] gives the more careful estimate

\[ \Omega_{c1} \approx \frac{5}{2} \frac{\hbar}{M R_\perp^2} \ln \left( \frac{R_\perp}{\xi} \right) \quad \text{or, equivalently,} \quad \frac{\Omega_{c1}}{\omega_\perp} \approx \frac{5}{2} \frac{d_1^2}{R_\perp^2} \ln \left( \frac{R_\perp}{\xi} \right), \]
(4.27)

which holds with logarithmic accuracy: it is larger than the previous naive estimate in eq. (4.24) by a factor \( \frac{5}{2} \) that arises from the reduced angular momentum for the non-uniform density relative to that for a uniform density. Reference [24] shows that this asymptotic estimate agrees well with the numerical work of ref. [22] for the particular asymmetry \( \lambda = \sqrt{8} \), appropriate for the original experiments [8] on \(^{87}\)Rb.

The TF description has also been extended [69] to the case of a rotating axisymmetric trap containing a straight vortex that is displaced laterally a distance \( r_0 \) from the center. The dependence of the free energy on the dimensionless parameter \( x_0 = r_0/R_\perp \) for fixed \( \Omega \) is qualitatively similar to that in fig. 7, but the details differ considerably. In particular, the vortex at the center is unstable for \( \Omega \leq \Omega_m = \frac{1}{2} \Omega_{c1} \), where \( \Omega_{c1} \) is given above in eq. (4.27), and metastable for \( \Omega_m \leq \Omega \leq \Omega_{c1} \). Thus the presence of the trap reduces (but does not eliminate) the regime of metastability. A similar result also holds for a simpler classical model where the uniform density \( \bar{n} \) is replaced by a parabolic radial density profile \( \bar{n}(r) \). In both these cases, the reduced outer density in a trap lowers the free energy \( F_0(x_0) \) for \( x_0 \gtrsim \frac{1}{4} \) compared to that for uniform density, raising the threshold for metastability relative to that in fig. 7.

2. possible scenario for creation of a vortex

A basic question is: how can the condensate be made to rotate? For \(^4\)He in a rotating circular container, the microscopically rough walls can spin up the normal (viscous) fluid when the temperature is above the superfluid transition. If the fluid is then cooled below the transition temperature, the superfluid is created in a state of rotation for sufficiently large rotation speeds. This method fails for a condensate in an axisymmetric trap, because the trap is simply a smooth trap. Consequently, only a non-axisymmetric trap can spin up the condensate, for the
asymmetry in the trap potential exerts a torque on the condensate, similar to the effect of moving walls in classical hydrodynamics.

It is helpful first to consider the classical example of a uniform fluid in a long rotating elliptical cylinder with semi-major and semi-minor axes \(a\) and \(b\). If \(a > b\), the rotating walls push the fluid, creating a classical velocity potential \(\Phi_{cl} = \Omega xy (a^2 - b^2)/(a^2 + b^2)\) \([71]\), even in the absence of a vortex. The resulting angular momentum \(L_z = I_0 \Omega\) and kinetic energy \(E_0 = \frac{1}{2} I_0 \Omega^2\) have the expected dependence on the rotation speed \(\Omega\), with an effective moment of inertia per unit length \(I_0\) that is less than the classical solid-body value \(I_{sb}\) by the ratio

\[
\frac{I_0}{I_{sb}} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2.
\]

This quantity vanishes for a circular cylinder \((a = b)\), but it can approach 1 from below for extreme asymmetry \((b \ll a)\). Unlike the case of a circular cylinder, the free energy \(F_0 = E_0 - \Omega L_{z,0} = -\frac{1}{2} I_0 \Omega^2\) for a vortex-free state is now negative. Note that the classical velocity \(v_{cl} = \nabla \Phi_{cl}\) is everywhere irrotational with zero vorticity \(\nabla \times v_{cl} = 0\). Nevertheless, the moment of inertia \(I_0\) is non-zero; indeed, for \(b \ll a\), it can be comparable with that for solid-body rotation \(v_{sb} = \Omega \times r\), whose vorticity is uniform with \(\nabla \times v_{sb} = 2\Omega\).

The critical angular velocity \(\Omega_{c1}\) for creating a vortex at the center depends on the asymmetry ratio \(b/a\) \([72]\), and experiments confirm the theoretical predictions in considerable detail \([73]\). In the extreme asymmetric limit \(b \ll a\), a detailed calculation yields

\[
\Omega_{c1} \approx \frac{\hbar}{2 M b^2} \ln \left(\frac{b}{\xi}\right),
\]

which holds with logarithmic accuracy (here, the circulation is taken as \(h/M\)). Qualitatively, this value can be interpreted as that for a circular cylinder with the radius equal to the largest inscribed circle that fits inside the elliptical cross section.

An approximate TF solution of the GP equation in a rotating triaxial (totally asymmetric) trap with \(\omega_z \neq \omega_x \neq \omega_y\) bears out these qualitative conclusions \([63]\). As usual in the TF limit, the chemical potential \(\mu\) determines the condensate radii \(R_x^2 = 2 \mu/M \omega_x^2\). Even in the absence of a vortex, the rotating walls push the condensate, inducing a phase for the condensate wave function that is linear in the angular velocity \(\Omega = \Omega \hat{z}\)

\[
S_0(x, y) \approx \frac{M \Omega}{\hbar} \left(\frac{\omega_x^2 - \omega_y^2}{\omega_x^2 + \omega_y^2}\right) xy = \frac{M \Omega}{\hbar} \left(\frac{R_x^2 - R_y^2}{R_x^2 + R_y^2}\right) xy.
\]

This expression has the correct angular symmetry \(\propto xy = \frac{1}{2} r^2 \sin 2\alpha\) and vanishes for an axisymmetric trap; it is the quantum analog of the classical velocity potential for a rotating elliptic cylinder \([71]\).

For a non-axisymmetric trap rotating about the \(\hat{z}\) axis, it is convenient to define the mean radius

\[
R_\perp^2 = \frac{2 R_x^2 R_y^2}{R_x^2 + R_y^2}, \quad \text{or, equivalently,} \quad \frac{2}{R_\perp^2} = \frac{1}{R_x^2} + \frac{1}{R_y^2},
\]

so that \(R_\perp \to R_\perp = R_z = R_y\) for an axisymmetric trap. A calculation of the energy and angular momentum of a vortex on the axis of the trap yields the critical angular velocity \([63]\)

\[
\Omega_{c1} \approx \frac{5}{2} \frac{\hbar}{M R_\perp^2} \ln \left(\frac{R_\perp}{\xi}\right)
\]

for the creation of a vortex in an arbitrary non-axisymmetric trap in the TF limit. For an axisymmetric trap \((R_\perp \to R_\perp)\), the resulting expression for \(\Omega_{c1}\) reproduces the previous result in eq. \((4.27)\); in contrast, it becomes

\[
\Omega_{c1} \approx \frac{5}{4} \frac{\hbar}{M R_y^2} \ln \left(\frac{R_y}{\xi}\right)
\]

in the extreme anisotropic limit \(R_y \ll R_x\), very similar to the classical expression in eq. \((1.29)\).

One possible experimental scenario for creating a vortex in a trap follows from the analogy with superfluid \(^4\)He \([62]\). Cool a rotating trap with significant anisotropy in the \(xy\) plane from the normal state through \(T_c\) to some low temperature \(T \ll T_c\). If \(\Omega\) exceeds the critical angular velocity \(\Omega_{c1}\), the resulting Bose condensate should have a vortex. Note the crucial order of operations, with the condensate created in a rotating state. Although reversing the order (cooling and then rotating) does often create a vortex in superfluid \(^4\)He, such a procedure is hysteretic and displays considerable metastability \([74, 75]\). Type-II superconductors exhibit similar metastability when a magnetic field is applied after the sample has been cooled in zero field.
In principle, it may be possible to observe directly the reduced density in the vortex core (somewhat like the optical detection of localized trapped charge in the core of vortex lines in superfluid $^4$He \cite{4,72}), but the small healing length $\xi$ in the TF limit may well suppress the effect. An alternative and promising approach considers the effect of a vortex on the collective modes of the condensate. The basic physical idea is that a uniform rotation splits the time-reversal degeneracy of the two modes with $\pm |m|$ that are originally degenerate (like the Zeeman effect of a uniform magnetic field on the states with different azimuthal components of magnetic moment).

Two distinct approaches have been developed (they both yield the same results in the TF limit). One uses sum rules \cite{33} to generalize the earlier study of the low-lying collective modes for a vortex-free condensate \cite{28}, discussed in the previous section. The other uses the hydrodynamic equations in the TF limit \cite{68}, where the vortex provides a weak perturbation (ref. \cite{68} also obtained some, but not all, of the same results).

To estimate the magnitude of the effect, recall that a static velocity field $v_0$ alters the usual time derivative $\partial / \partial t$ to the hydrodynamic derivative $\partial / \partial t + v_0 \cdot \nabla$. Thus the original frequency $\omega$ of a wave is shifted to $\sim \omega \pm v_0 / R_0$, where $R_0$ is the typical dimension of the condensate. For a vortex, the circulating velocity is of order $v_0 \sim \hbar / (MR_0)$, and the fractional change in the frequency is $\delta \omega / \omega \sim \hbar / (MR_0^2 \omega_\perp) \sim d_\perp^2 / R_\perp^2 \ll 1$ in the TF limit. Alternatively, the fractional change in the frequency is comparable to the ratio $v_0 / s$, where $s$ is the speed of sound. In a dilute trapped Bose gas, this speed is of order $s \sim \hbar / (M \xi)$, which becomes $s \sim \hbar R_0 / (M d_\perp^2)$ in the TF limit and reproduces the previous estimate for $\delta \omega / \omega$. Similar ideas (see, for example, pp. 144-145 of ref. \cite{36}) have served to study the effect of a quantized vortex line in a type-II superconductor on the BCS quasiparticles.

Consider a large (TF) vortex-free condensate in an axisymmetric trap, and focus on a particular collective mode with degenerate frequency $\omega^0_{|m|}$. In the presence of a vortex, the fractional splitting of the modes with $\pm |m|$ can be characterized by the ratio

$$\frac{\omega_{|m|} - \omega_{-|m|}}{\omega^0_{|m|}} = \Delta_{|m|} \left( \frac{d_\perp}{R_\perp} \right)^2 \approx \Delta_{|m|} \left( \frac{d_\perp}{15 \eta a \lambda} \right)^2,$$

where $\Delta_{|m|}$ is a dimensionless number that depends on the specific mode in question. For $|m| > 1$, it is proportional to the matrix element of $1 / r_\perp^2$, evaluated with the normalized unperturbed (vortex-free) eigenfunctions, but the case of $|m| = 1$ requires a more careful analysis \cite{68}.

- a. For the lowest quadrupole mode with $|m| = 2$, the unperturbed frequency is $\omega^0_2 = \sqrt{2} \omega_\perp$, and the dimensionless factor is $\Delta_2 = 7 / \sqrt{2}$.
- b. For the lowest quadrupole mode with $|m| = 1$, the unperturbed frequency is $\omega^0_1 = \sqrt{1 + \lambda^2} \omega_\perp = \sqrt{\omega_\perp^2 + \omega_\parallel^2}$, and the dimensionless factor is $\Delta_1 = 7 \lambda^2 / (1 + \lambda^2)^{3/2}$.

Typically, the predicted fractional splitting is of order 10% for the low-lying modes. Current measurements of frequencies for non-rotating condensates are accurate to $\lesssim 1\%$, so that this vortex-induced splitting should be readily detectable.

### 4. Stability of a Vortex in a Trapped Bose Condensate

If the condensate rotates with an angular velocity $\Omega \gtrsim \Omega_{\perp}$, a singly quantized vortex is believed to be the stable equilibrium configuration. If, however, the angular velocity is reduced so that $\Omega \ll \Omega_{\perp}$, does the vortex remain stable (especially if $\Omega / \Omega_{\perp} \ll 1$)? In the present case of a trapped condensate, the question can become even more intricate, depending on whether the trap remains non-axisymmetric (so that the reduced rotation exerts a braking torque on the condensate) or is first adiabatically transformed to axisymmetric (in which case, the reduced angular velocity of the trap potential cannot affect the rotation of the condensate).

Rokhsar \cite{67} has given a qualitative argument that a vortex in a non-rotating axisymmetric nearly ideal condensate should be unstable. In this limit, the single-particle vortex wave function has the form $\psi_v \approx e^{i \theta} \psi_\perp$, with the radius of the vortex core comparable to the oscillator length and condensate radius $d_0$ (note that the coherence length $\xi$ here is much larger than $d_0$ and therefore does not characterize the vortex core radius in the nearly ideal limit). This large depleted region of reduced repulsive Hartree potential energy favors the formation of a localized bound core state, with a real single-particle wave function $\psi_\perp$. The actual condensate wave function becomes a linear combination $\Psi \approx \sqrt{N_v} \psi_\perp + \sqrt{N_h} \psi_h$, where $N_v$ is the number of particles in the vortex condensate and $N_h = N - N_v$ is that in the bound core state. For small $N_v$, this complex wave function $\Psi \approx \sqrt{N_v} e^{i \theta} \psi_\perp + \sqrt{N_h} \psi_h$ has a node away from the axis of symmetry, so that the vortex core shifts radially outward from the origin. This motion can occur only if
there is some mechanism for the condensate to lose energy and angular momentum; it is a quantum analog of the hydrodynamic instability below the threshold for metastability, discussed in connection with fig. 7. As in that case, the time for a vortex to move outward depends on the details of the dissipation. In addition, the vortex core itself becomes small in the TF limit and cannot support a bound state for $Na/d_0 \gtrsim 1 \[77\]$, so that this argument is not applicable in the TF limit.

As an alternative physical picture in the non-interacting limit, note that the vortex has all the particles in the single-particle state $\chi_0(r, z)\chi_0(z)$ given in eq. (4.21), with excitation energy $\hbar\omega_\perp$ and angular momentum $\hbar$. Clearly, the system can lower its energy by transferring one particle from this (excited) condensate to the original ground state $\chi_0(r, z)\chi_0(z)$; the same process can be repeated many times, which is the physical source of the instability.

This argument can be sharpened somewhat by considering the Bogoliubov equations for a singly quantized vortex in a nearly ideal condensate with $Na/d_2 \ll 1 \[77\]$. Among the many normal modes with positive norm $\int d^3r (|u|^2 - |v|^2) = 1$, only one is anomalous and has a negative frequency

$$\frac{\omega_a}{\omega_\perp} \approx -1 + \frac{1}{\sqrt{8\pi}} \frac{N_0a}{d_2} + \cdots \quad \text{(4.35)}$$

This mode consists of a coherent linear combination of both $u$ and $v$ amplitudes with an angular dependence $m_a = -1$ relative to the vortex. When re-expressed in terms of the particle density, it represents a displacement of the vortex core relative to the center of mass of the condensate.

More generally, Dodd et al. \[77\] have studied the normal modes of a singly quantized vortex numerically for small and intermediate values of the dimensionless coupling parameter $Na/d_0$ (note that ref. \[77\] uses $m$ to denote the quantum number of the static vortex condensate, whereas $m$ here denotes the angular dependence $\propto e^{im\phi}$ of a density perturbation \[58\], in direct analogy with the same $m$ dependence of a spherical harmonic $Y_{lm}$). For the lowest-lying dipole modes with $|m| = 1$ relative to the vortex, Dodd et al. \[77\] find two dipole-sloshing modes with frequency $\omega_\perp$ in which the condensate oscillates rigidly with right- and left-circular helicity, independent of the interaction parameter (as expected). In addition, they exhibit an additional (anomalous) dipole mode with $m = -1$ that has the frequency $\omega_\perp$ for the ideal gas and decreases toward zero with increasing coupling strength. Rokhsar \[77\] argues that this mode actually describes an antiparticle; it is the opposite-frequency and negative-norm partner of a physical mode with negative frequency and positive norm. With this reinterpretation, the frequency of the anomalous mode reduces to the expected value $-\omega_\perp$ for an ideal gas and grows toward zero from below with increasing $Na/d_2$. In the context of the preceding discussion, this anomalous mode is precisely the unstable (negative-frequency) mode found analytically in eq. (4.33) from the Bogoliubov equations in the weak-coupling limit. At present, little is known of its behavior in the TF limit, when $Na/d_2 \gg 1$, although the trend of the numerical data \[77\] suggests that the frequency remains negative.

As noted several times, an applied rotation can stabilize a vortex. Specifically, the frequency $\omega_j$ of a given normal mode in a non-rotating condensate is shifted to $\omega_j - \Omega_{m_j}$ when the system rotates with angular velocity $\Omega$. In particular, the anomalous mode with negative frequency $\omega_a$ has $m_a = -1$; it therefore has an apparent frequency $\omega_a + \Omega$ when observed in the rotating frame, indicating that a positive rotation raises the (negative) frequency, tending to stabilize the mode. Recall that eq. (4.21) gives the thermodynamic critical angular velocity $\Omega_{m}$ for the creation of a singly quantized vortex in the near-ideal limit, and comparison with the independently computed negative frequency $\omega_a$ in eq. (4.35) shows that $\omega_a + \Omega_{m}$ vanishes identically for small $N_0a/d_2$. Hence, a rotation at $\Omega_{m}$ just suffices to stabilize this unstable anomalous mode in the weak-coupling limit. The corresponding situation in the TF limit remains unknown, where the metastable frequency $\Omega_m < \Omega_{m}$ discussed in connection with fig. 7 may well be relevant.

In principle, various other methods can stabilize a vortex in a dilute trapped Bose condensate; in the present context, the most relevant proposal is to force the non-rotating condensate to assume a toroidal form \[78,79\] (for example, piercing it with a blue-detuned narrowly focused laser beam that acts as a repulsive localized force). In this case, the externally imposed hole in the stationary condensate renders it multiply connected, even in the absence of a vortex. Strictly speaking, the excited states of interest here are those with irrotational flow and quantized circulation around the circumference of the torus, but they are frequently considered to represent the corresponding quantized vortex with its core “pinned” in the central void. Such states have been studied in detail for superfluid $^4$He in a rotating annulus (see, for example, secs. 2.6 and 5.3 of ref. \[69\]), where a row of physical vortices eventually appears in the gap between the walls at sufficiently high angular velocity.

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FIG. 1. Dimensionless excitation-energy spectrum $E/\hbar\omega_0$ as a function of the dimensionless wavenumber $k\xi$: (a) Bogoliubov spectrum in eq. (1.42); (b) free-particle spectrum.

FIG. 2. Bogoliubov coherence factors (a) $\nu^2$ and (b) $v^2$ from eq. (1.37) as functions of the dimensionless wavenumber $k\xi$.

FIG. 3. Dimensionless squared Bogoliubov energy $E^2/n^2g^2$ in eq. (2.44) for a uniform system subject to an attractive interaction with $s$-wave scattering length $-|a|$, as a function of the dimensionless wavenumber $k\xi$, where $8\pi n|a|\xi^2 = 1$. In these units, the unstable range is $0 \leq k\xi \leq \sqrt{2}$.

FIG. 4. Dimensionless variational energy $2E/(N\hbar\omega_0)$ in eq. (2.47) for a trapped condensate, as a function of the dimensionless parameter $\beta$ that characterizes the actual radius $\beta d_0$ for various values of the interaction parameter: (a) $Na/d_0 = 0.5$ (locally stable), (b) $Na/d_0 = 0.67$ (onset of local instability), and (c) $Na/d_0 = 0.84$ (locally unstable).
FIG. 5. Induced local velocity for two parallel vortex lines a distance \(d\) apart in unbounded incompressible dissipationless fluid: (a) Two parallel vortex lines with \textit{same} sense of circulation; induced motion as shown yields circular orbits at fixed separation \(d\); (b) two antiparallel vortex lines with \textit{opposite} sense of circulation; induced motion as shown yields uniform motion perpendicular to line joining centers in direction of fluid at center, maintaining fixed separation \(d\).

FIG. 6. Cylinder of radius \(R_0\) containing a long straight vortex line displaced from the center by a distance \(r_0 < R_0\). One image vortex with opposite circulation on same ray at the exterior point with radius \(R_2^2/r_0\) suffices to satisfy the boundary condition that the normal component of velocity vanish at the boundary.

FIG. 7. Dimensionless free energy per unit length \(MF_c/(\pi \bar{n} h^2)\) from eq. (4.10) for a vortex in a cylinder of radius \(R_0\); the vortex is displaced radially a fractional distance \(x_0 = r_0/R_0\) from the axis of symmetry, and the different curves correspond to various fixed values of the angular velocity \(\Omega\): (a) \(\Omega = 0\) (unstable); (b) \(\Omega = \Omega_m = \Omega_0 \equiv h/(MR_0^2)\) (marginally metastable at origin); (c) \(\Omega = 0.5 \Omega_{c1}\) (metastable at origin); (d) \(\Omega = \Omega_{c1}\) (stable at origin), where \(\Omega_{c1} = \Omega_0 \ln(R_0/\xi) \equiv (h/MR_0^2) \ln(R_0/\xi)\) is evaluated for \(R_0/\xi \approx 100\).

FIG. 8. Radial wave function \(f(r_\perp/\xi)\) obtained by numerical solution of eq. (4.15).
\[ \frac{E^2}{n^2 g^2} \]
image vortex at $R_0^2/r_0$
$f(r_{\perp}/\xi)$

$\frac{r_{\perp}}{\xi}$