Abstract. We consider the classical geometric problem of prescribing the scalar and boundary mean curvatures via conformal deformation of the metric on a \( n \)-dimensional compact Riemannian manifold. We deal with the case of negative scalar curvature and positive boundary mean curvature. It is known that if \( n = 3 \) all the blow-up points are isolated and simple. In this work we prove that, for a linear perturbation, this is not true anymore in low dimensions \( 4 \leq n \leq 7 \). In particular, we construct a solution with a clustering blow-up boundary point (i.e. non-isolated), which is non-umbilic and is a local minimizer of the norm of the trace-free second fundamental form of the boundary.

1. Introduction

Given a compact Riemannian manifold \((M, g)\) of dimension \( n \geq 3 \) with boundary \( \partial M \), a widely studied geometric problem is the following one: given two smooth functions \( K \) and \( H \) find a metric conformal to \( g \) whose scalar curvature is \( K \) and boundary mean curvature is \( H \).

As it is well known, the geometric problem can be rephrased into the following one: given two smooth functions \( K \) and \( H \) find a positive solution to the PDE

\[
\begin{align*}
- \frac{4(n-1)}{n-2} \Delta_g u + S_g u &= Ku^{\frac{n+2}{n-2}} \quad \text{in } M \\
\frac{2}{n-2} \frac{\partial u}{\partial \nu} + h_g u &= Hu^{\frac{n+2}{n-2}} \quad \text{on } \partial M.
\end{align*}
\]

Here \( \Delta_g \) is the Laplace-Beltrami operator, \( S_g \) is the scalar curvature and \( h_g \) the boundary mean curvature associated to the metric \( g \) and \( \nu \) is the outward unit normal vector to \( \partial M \). The metric \( \tilde{g} = u^{\frac{4}{n-2}} g \) is conformal to \( g \) and its scalar and boundary mean curvatures are nothing but \( K \) and \( H \), respectively.

The study began with the work of Cherrier [13] who gave a first criterion for the existence and regularity of solution of (1.1). Successively, Escaur in a series of papers [19, 21, 20] found a solution to (1.1) when either \( K = 0 \) (i.e. scalar flat metric) and \( H \) constant or \( H = 0 \) (i.e. minimal boundary) and \( K \) is constant. The proof strongly relies on the dimensions of the manifold, on the properties of the boundary (e.g. being or not umbilic) and on vanishing properties of the Weyl tensor (e.g. being identically zero or not on the boundary or on the whole manifold). Important contributions in this framework are due to the work of Marques in [32, 31], Almaraz [3], Brendle & Chen [9] and Mayer & Ndiaye [33]. The case when \( K > 0 \) and \( H \) is an arbitrary constant, has been successfully treated by Han & Li in [26, 25] and Chen, Ruan & Sun [12].

There are a few results concerning the general case in which \( K \) and \( H \) are functions (not necessarily constants) and all of them have been obtained for special manifolds (e.g. typically the unit ball or the half sphere). In particular, we refer to the works of Ben Ayed, El Melli & Ould Ahmedou [7, 8] and Li [27] when \( H = 0 \) and Abdelhedi, Chtioui & Ould Ahmedou [1], Chang, Xu & Yang [10], Djadli,
Recently, Cruz-Blázquez, Malchiodi and Ruiz [15] considered a manifold whose scalar curvature $S_g \leq 0$ and the case $K$ negative and $H$ of arbitrary sign. They introduce the scaling invariant quantity

$$D_n(p) = \frac{1}{n(n-1)} \frac{H(p)}{\sqrt{|K(p)|}} , \quad p \in \partial M$$

(1.2)

and established the existence of a solution to (1.1) whenever $D_n < 1$ along the whole boundary. On the other hand, if $D_n > 1$ at some boundary points they got a solution only in a three dimensional manifold, for a generic choice of $K$ and $H$. Let us describe more carefully their result. First of all, via the conformal change of metric due to Escobar [21], one can assume that the mean curvature $h_g = 0$ and $S_g$ has constant sign (this will be also assumed understood in the rest of our paper), so problem (1.1) reads as

$$\begin{cases}
-\frac{4(n-1)}{n-2} \Delta_g u + S_g u = Ku^{\frac{n+2}{n-2}} & \text{in } M \\
\frac{2}{n-2} \frac{\partial u}{\partial \nu} = Hu^{\frac{n}{n-2}} & \text{on } \partial M.
\end{cases}$$

(1.3)

Problem (1.3) is variational in nature, i.e. the solutions of (1.3) are critical points of the energy functional defined on $H^1(M)$

$$J(u) = \frac{2(n-1)}{n-2} \int_M |\nabla u|^2 - \frac{1}{2} \int_M S_g u^2 - \frac{1}{2^*} \int_M K(u^+)^{2^*} - (n-2) \int_{\partial M} H(u^+)^{2^*}$$

where $2^* = \frac{2n}{n-2}$ and $2^* = \frac{2(n-1)}{n-2}$ are the critical Sobolev exponent for $M$ and the critical trace embedding exponent for $\partial M$, respectively. In [15] the authors show that if $S_g \leq 0$ and $D_n < 1$ along the whole boundary, the functional becomes coercive and they found a global minimizer. On the other hand, if there exists $p \in \partial M$ such that $D_n(p) > 1$, they construct a sequence of functions $u_i$ such that the energy $J(u_i) \to -\infty$ and the minimum point does not exist anymore. However, on a $3-$dimensional manifold they recover the existence of a positive solution by using a mountain pass type argument. Their proof relies on a careful blow-up analysis: first they show that the blow-up phenomena occurs at boundary points $p$ with $D_n(p) \geq 1$, with different behaviours depending on whether $D_n(p) = 1$ or $D_n(p) > 1$. To deal with the loss of compactness at points with $D_n(p) > 1$, where bubbling of solutions occurs, it is shown that in dimension three all the blow-up points are isolated and simple (see also [16]). As a consequence, the number of blow-up points is finite and the blow-up is excluded via integral estimates that hold true when $S_g \leq 0$. In that regard, $n = 3$ is the maximal dimension for which one can prove that the blow-up points with $D_n > 1$ are isolated and simple for generic choices of $K$ and $H$. In the closed case such a property is assured up to dimension four (see [28]) but, as observed in [16], the presence of the boundary produces a stronger interaction of the bubbling solutions with the function $K$.

Therefore, a natural question arises.

(Q) In higher dimensions $n \geq 4$, are the blow-up points still isolated and simple?

In the present paper we give a partial negative answer.

Let us consider the linearly perturbed problem

$$\begin{cases}
-\frac{4(n-1)}{n-2} \Delta_g u + S_g u = Ku^{\frac{n+2}{n-2}} & \text{in } M \\
\frac{2}{n-2} \frac{\partial u}{\partial \nu} + \varepsilon u = Hu^{\frac{n}{n-2}} & \text{on } \partial M
\end{cases}$$

(1.4)
where $\varepsilon$ is a small and positive parameter. Let $\pi$ be the second fundamental form of $\partial M$. Our main result reads as follows

**Theorem 1.1.** Assume

(i) $4 \leq n \leq 7$ and $S_g > 0$,

(ii) $H > 0$ and $K < 0$ are constant functions such that $\mathcal{D}_n > 1$,

(iii) $p \in \partial M$ is non-umbilic (i.e. $\pi(p) \neq 0$) and non-degenerate minimum point of $\|\pi(\cdot)\|^2$.

Then $p$ is a “clustering” blow-up point, i.e. for any $k \in \mathbb{N}$, there exist $p_k^j \in \partial M$ for $j = 1, \ldots, k$ and $\varepsilon_k > 0$ such that for all $\varepsilon \in (0, \varepsilon_k)$ the problem (1.4) has a solution $u_\varepsilon$ with $k$ positive peaks at $p_k^j$ and $p_k^j \to p$ as $\varepsilon \to 0$.

**Remark 1.2.** We remind that a point $p \in \partial M$ is non-umbilic if the trace-free part of the second fundamental form of $\partial M$ does not vanish at $p$. Since $h_g = 0$, the tensor $T_{ij} = h_{ij} - h_g g_{ij}$ reduces to the second fundamental form $\pi$ whose components are $h_{ij}$ and so $p$ is non-umbilic if $\|\pi(p)\| > 0$.

We believe that the non-degeneracy assumption is satisfied for generic Riemannian metrics (this could be proved using transversality tools as in [23, 35, 34]).

The main ingredients of our construction are the so-called bubbles, i.e. the solutions of the problem

$$
\begin{cases}
-c_n \Delta u = Ku^{n+2} & \text{in } \mathbb{R}_+^n \\
\frac{2}{n-2} \frac{\partial u}{\partial \nu} = Hu^{\frac{n}{n-2}} & \text{on } \partial \mathbb{R}_+^n
\end{cases}
$$

where $c_n := \frac{4(n-1)}{n-2}$ and $\mathcal{D}_n := \sqrt{n(n-1)H/|K|} > 1$ ($\nu$ is the exterior normal vector to $\partial \mathbb{R}_+^n$). Solutions to (1.5) are completely classified in [14] (see also [29]). These are given by

$$
U_{\delta, y}(x) := \frac{1}{\delta^\frac{n-2}{2}} U \left( \frac{x - y}{\delta} \right), \quad U(x) := \frac{\alpha_n}{|K|^{\frac{n-2}{2}}} \frac{1}{(|\hat{x}|^2 + (x_n + \mathcal{D}_n)^2 - 1)^\frac{n-2}{2}}
$$

where $\alpha_n := (4n(n-1))^{-\frac{n-2}{2}}$, $x = (\hat{x}, x_n)$, $y = (\hat{y}, 0)$ and $\delta > 0$. The solutions we are looking for are the sum of $k$ positive bubbles which concentrate at the same boundary point $p$ with the same speeds, i.e. in local coordinates (see (3.1) and (4.4)) around $p$

$$
u_\varepsilon(x) \sim \sum_{j=1}^k \frac{1}{\delta_j^\frac{n-2}{2}} U \left( \frac{x - \eta_j}{\delta_j} \right)
$$

where the all concentration parameters $\delta_j$ have the same speed with respect to $\varepsilon$ and all the concentration points $\eta_j$ collapse to 0 as $\varepsilon \to 0$ (see (4.5), (4.6) and (4.7)).

Unfortunately this first approximation is not as good as one can expect. We need to refine it adding some extra terms which solve the linear problem (3.2). To find this extra terms, it is crucial the study of the linear theory developed in Section 2. The novelty is Theorem 2.1 which states the non-degeneracy of the bubble (1.6), i.e. all the solution of the linearized problem

$$
\begin{cases}
-c_n \Delta v - \frac{n+2}{n-2} Ku^{\frac{n}{n-2}} v = 0 \text{ in } \mathbb{R}_+^n, \\
\frac{2}{n-2} \frac{\partial v}{\partial \nu} - \frac{n}{n-2} Hu^{\frac{n}{n-2}} v = 0 \text{ on } \partial \mathbb{R}_+^n,
\end{cases}
$$

are a linear combination of the functions

$$
\mathcal{J}_i(x) := \frac{\partial U}{\partial x_i}(x), \quad i = 1, \ldots, n-1, \quad \text{and} \quad \mathcal{J}_n(x) := \left( \frac{2-n}{2} U(x) - \nabla U(x) \cdot (x + \mathcal{D}_n c_n) + \mathcal{D}_n \frac{\partial U}{\partial x_n} \right).
$$

The proof relies on some new ideas which allow a comparison among solutions to the linear problem (1.8) and the eigenfunctions of the Neumann problem on the ball equipped with the hyperbolic metric.
(see Lemma 2.3). It is worthwhile to point out that Han & Lin in [25] and Almaraz in [2] related similar linear problems in the case $K \geq 0$ with some eigenvalue problems on spherical caps with standard metric.

Once the refinement of the ansatz is made, we argue using a Ljapunov-Schmidt procedure. As it is usual, the last step consists in finding a critical point of the so-called reduced energy and to achieve this goal it is necessary to know the energy of each bubble together with its correction. The contribution of the correction to the energy is relevant and to capture it is necessary to know the exact expression of the correction itself. This part is new and requires a lot of work. This is done in Section 3. Finally, we can write the main terms of the reduced energy which come from the contribution of each peak $\eta_j$, the interaction between different peaks $\eta_j$ and $\eta_i$ and the linear perturbation $\epsilon-$term. For example, in dimension $n \geq 5$ (up to some constants) it looks like

$$
\sum_{j=1}^{k} \left[ \delta_j^2 \left( \|\pi(p)\|^2 + \Omega(p)(\eta_i, \eta_j) \right) + \sum_{i \neq j} \frac{(\delta_i \delta_j)^{n-2}}{|\eta_i - \eta_j|^{n-2}} - \epsilon \delta_j \right] + \text{h.o.t.} \tag{1.9}
$$

here $\Omega(p)$ is the quadratic form associated with the second derivative of $\|\pi(\cdot)\|^2$ at the point $p$ which is supposed to be positively definite (remind that $p$ is a minimum point of $\pi$). Now, if we choose

$$
\delta_j \sim \epsilon \text{ and } |\eta_j| \sim \eta \text{ with } \epsilon^2 \eta^2 \sim \frac{\epsilon^{n-2}}{\eta^{n-2}}
$$

we can minimize the leading term in (1.9) as soon as the term “h.o.t.” is really an higher order term and this is true only in low dimensions $4 \leq n \leq 7$. We believe that this is not merely a technical issue. It would be extremely interesting to understand if in higher dimensions the clustering phenomena appears if the blow-up point is umbilic, i.e. $\pi(p) = 0$. It is clear that in this case building the clustering configuration is even more difficult than in the non-umbilic case, because the ansatz must be refined at an higher order.

Even if our result holds true in low dimensions we decide to write all the steps of the Ljapunov-Schmidt procedure in any dimensions because it would be useful in studying some related problems. In particular, our argument allows to prove that if $n \geq 4$ the problem (1.4) has always a solution with one blow-up boundary point $p$ which is non-umbilic and minimizes $\|\pi(\cdot)\|$. In fact, if $k = 1$ the expansion of the reduced energy in (1.9) holds true in any dimensions. The existence of solutions with a single blow-up point was studied by Ghimenti, Micheletti & Pistoia in [24, 22] when $K = 0$ and $H = 1$ in presence of a linear non-autonomous perturbation $\epsilon \gamma u$. being $\gamma \in C^2(\partial M)$.

We remark that very recently Ben Ayed & Ould Ahmedou [6] found solutions with clustering blow-up points on half spheres of dimension greater than five for a subcritical approximation of the geometric problem (1.1), with a nonconstant function $K > 0$ and $H = 0$. As far as we know, our result is a pioneering work in the construction of solutions with clustering blow-up points for the problem (1.4) with $K$ and $H$ not identically zero. In particular, it is the first time that this argument is carried out with $K < 0$ and $H > 0$, which has been proved to be especially challenging due to the existing competition between the critical terms of the energy functional.

Finally, we point out that the clustering and towerin phenomena for Yamabe-type equations have been largely studied in the literature, although most of the results available concern the problem on closed compact manifolds. Consider for instance the linear perturbation of the classical Yamabe equation,

$$
-\Delta_g u + S_g u + \varepsilon u = u^{n+2} \text{ in } M. \tag{1.10}
$$

It is known that in 3—dimensional manifolds all the solutions to (1.10) have isolated and simple blow-up points (see Li and Zhu [30]). However, this property is lost in higher dimensions. If $n \geq 7$, Pistoia & Vaira [37] build a solution to (1.10) with a clustering (i.e. non-isolated) blow-up point at a non-degenerate and non-vanishing minimum point of the Weyl’s tensor. In any dimensions
\( n \geq 4 \) the clustering phenomena appears if the linear perturbation term \( \epsilon u \) is replaced with a function \( h_\epsilon \) converging to a suitable function \( h_0 \) as showed by Druet & Hebey [18] and Robert & Vétois [40] if \( n \geq 6 \) and by Thizy & Vétois [41] if \( n = 4, 5 \).

The existence of solutions to (1.10) with a towering (i.e. isolated but non-simple) blow-up point has been proved in dimensions \( n \geq 7 \), by Morabito, Pistoia & Vaira [36] on symmetric nonlocally conformally flat manifolds and by Premoselli [38] in the locally flat case.

In the spirit of [18, 40] it would be interesting to replace the linear perturbation term in (1.4) with some functions \( h_\epsilon \) in order to build a solution with a clustering blow-up point in any dimensions \( n \geq 4 \). Moreover, inspired by the above results we strongly believe that it would be possible to build solutions to problem (1.4) with a towering blow-up point in any dimensions \( n \geq 4 \). This will be the topic (at least in a symmetric setting) of a forthcoming paper.

The paper is organized as follows. In Section 2 we study the linear problem (1.8). In Section 3 we find out the correction term. In Section 4 we sketch the main steps of the proof, which relies on standard arguments typical of the Ljapunov-Schmidt procedure. However, since it involves a lot of new delicate and quite technical estimates, in order to streamline the reading of the work, we have decided to postpone them in the appendices.

In what follows we agree that \( f \lesssim g \) means \( |f| \leq c|g| \) for some positive constant \( c \) which is independent on \( f \) and \( g \) and \( f \sim g \) means \( f = g(1 + o(1)) \).

2. THE KEY LINEAR PROBLEM

First of all, it is necessary to study the set of the solutions for the linearized problem:

\[
\begin{aligned}
-\frac{4(n-1)}{n-2} \Delta v + \frac{n+2}{n-2} |K| U^{n-2} v &= 0 \quad \text{in } \mathbb{R}^n_+, \\
\frac{2}{n-2} \frac{\partial v}{\partial \nu} - \frac{n}{n-2} H U^{n-2} v &= 0 \quad \text{on } \partial \mathbb{R}^n_+, \\
\end{aligned}
\]

where \( \nu = -e_n \) is the exterior normal vector to \( \partial \mathbb{R}^n_+ \) and

\[
U(x) = U_{1,x_0(1)}(\tilde{x}, x_n) = \frac{\alpha_n}{|K|^{\frac{n-2}{2}} (|\tilde{x}|^2 + (x_n + D_n)^2 - 1)^{\frac{n-2}{2}}} \tag{2.2}
\]

where \( \tilde{x} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) and \( x_n \geq 0 \), stands for the simplest solution to the boundary Yamabe problem defined in (1.5) when \( D_n(p) > 1 \).

**Theorem 2.1.** Let \( v \in H^1(\mathbb{R}^n_+) \) be a solution of (2.1). Then \( v \) is a linear combination of the functions

\[
\hat{\mathfrak{g}}_i(x) := \frac{\partial U}{\partial x_i}(x) = \frac{\alpha_n}{|K|^{\frac{n-2}{2}} (|\tilde{x}|^2 + (x_n + D_n)^2 - 1)^{\frac{n-2}{2}}} (2 - n)x_i, \quad i = 1, \ldots, n - 1 
\]

and

\[
\hat{\mathfrak{g}}_n(x) := \left( \frac{2 - n}{2} U(x) - \nabla U(x) \cdot (x + D_n e_n) + D_n \frac{\partial U}{\partial x_n} \right) = \frac{\alpha_n}{|K|^{\frac{n-2}{2}} 2} \frac{n-2}{(|\tilde{x}|^2 + (x_n + D_n)^2 - 1)^{\frac{n-2}{2}}} \tag{2.4}
\]

The proof of Theorem 2.1 will require some preliminary results. In particular it is useful to recall the properties of the conformal Laplacian and boundary operator. For a given metric \( g \), they are defined as

\[
L_g v = \frac{4(n-1)}{n-2} \Delta_g v + S_g v \quad \text{and}, \quad B_g v = \frac{2}{n-2} \frac{\partial v}{\partial \nu} + h_g v,
\]
being $S_g$ and $h_g$ the scalar and boundary mean curvatures. If we choose a conformal metric of the form $\rho^{\frac{4}{n+2}}g$, then $L_g$ and $B_g$ are conformally invariant in the following sense:

\[
L_g v = \rho^{\frac{n+2}{4}} L \left( \frac{n+2}{4} \rho^{-1} v \right) \quad \text{and} \quad B_g v = \rho^{\frac{n+2}{4}} B \left( \frac{n+2}{4} \rho^{-1} v \right).
\]  

(2.5)

**Lemma 2.2.** For every $i = 0, 1, \ldots$, let us consider the following boundary eigenvalue problem:

\[
\begin{aligned}
\gamma''_i + (n-1) \coth \gamma'_i - \left( \frac{i(i+n-2)}{\sinh^2 t} + n \right) \gamma_i &= 0, & 0 < t < T, \\
\gamma_i(T) - \gamma_i(T) &= 0, \\
\end{aligned}
\]

(2.6)

with $\mu \in \mathbb{R}$. Then the following hold true:

(i) If $i = 0$, the only bounded solutions are of the form $\gamma_0(t) = c_1 \cosh t$ for $c_1 \in \mathbb{R}$, and satisfy (2.6) with $\mu = \mu_0 := \tanh T$.

(ii) If $i = 1$, the only bounded solutions can be written in the form $\gamma_1(t) = c_2 \sinh t$ with $c_2 \in \mathbb{R}$, and solve (2.6) with $\mu = \mu_1 := \left( \tanh T \right)^{-1}$.

(iii) If $i \geq 2$ and $\mu \leq \mu_1$, (2.6) does not admit bounded solutions.

**Proof.** The proofs for (i) and (ii) use the exact same argument, so for the sake of brevity we will only show the proof for (ii)

Firstly, observe that $\sinh t$ solves the first equation of (2.6) with $i = 1$, and it is positive and bounded in $[0, T]$. Therefore, by linear ODE theory, we can write any solution to the equation in the form $\gamma_1(t) = c(t) \sinh t$ for some function $c(t)$. Straightforward computations show that $c(t)$ must solve the following relation:

\[
c''(t) \sinh t + (2 \cosh t + (n-1) \coth t \sinh t) c'(t) = 0.
\]

(2.7)

If $c(t)$ is nonconstant, (2.7) can be integrated and its solutions can be calculated explicitly. For $t$ small enough, they present the asymptotic behaviour

\[
c(t) = c_3 \left( \frac{1}{t} - (n-1) \ln t + O(t) \right), \quad \text{with} \quad c_3 \neq 0.
\]

Thus, $c(t)$ must be constant. The second part of (ii) can be proved by direct computation.

Finally, let us prove (iii). We will consider the unique solution to (2.6) with $\gamma_i(T) = 1$, so we study the following situation:

\[
\begin{aligned}
\gamma''_i + (n-1) \coth \gamma'_i - \left( \frac{i(i+n-2)}{\sinh^2 t} + n \right) \gamma_i &= 0, & 0 < t < T, \\
\gamma_i(T) &= \mu, \\
\gamma_i(T) &= 1.
\end{aligned}
\]

Let $i \geq 2$ and define $u_i = \gamma_i - \gamma_1$, with $\gamma_1$ denoting the unique solution to (2.6) with $i = 1$, $\mu = \mu_1$, and $\gamma_1(T) = 1$. Then $u_i$ satisfies

\[
\begin{aligned}
u''_i + (n-1) \coth t u'_i - \left( \frac{i(i+n-2)}{\sinh^2 t} + n \right) u_i &= \frac{(i-1)(i+n-1)}{\sinh^2 t} \gamma_1, \\
u'_i(T) &= \mu - \mu_1, \\
u_i(T) &= 0.
\end{aligned}
\]

(2.8)

Firstly, we will show that $u'_i(T) \geq 0$, proving that $\mu \geq \mu_1$. Assume by contradiction that $u'_i(T) < 0$. Then since $u(T) = 0$, there exists a small interval $(t_0, T)$ where $u(t) > 0$. By the first equation of (2.8), since $i \geq 2$,

\[
u''_i(t) + (n-1) \frac{\cosh t}{\sinh t} u'_i(t) \geq 0, \quad \text{for} \quad t_0 < t < T.
\]

(2.9)
Inequality (2.9) can be written in the more convenient way
\[
((\sinh t)^{n-1}u'_t)^2 \geq 0, \quad \text{for } t_0 < t < T.
\]
Consequently, \((\sinh T)^{n-1}u'_T(T) \geq (\sinh t_0)^{n-1}u'_T(t_0)\). In view of this, if \(t_0 = 0\), then \(u'_T(T) \geq 0\), a contradiction. However, if \(t_0 > 0\), then \(u_t(t_0) = u_t(T) = 0\) so there exists \(t_1 \in (t_0, T)\) with \(u'_T(t_1) = 0\), again a contradiction.

To see that the inequality is strict we only need to show that there are no solutions for \(i \geq 2\) and \(\mu = \mu_1\). Let us define the sequence of linear operators
\[
A_i(\phi)(t) = -\phi''(t) - (n - 1) \coth t \phi'(t) + \left(\frac{i(i + n - 2)}{\sinh^2 t}\right) \phi(t),
\]
subject to the boundary conditions \(\phi(T) = 1\) and \(\phi'(T) = \mu_1\). \((i)\) implies that \(A_0\) admits no solution, while \((ii)\) gives us a positive function \(\phi_1\) satisfying \(A_1(\phi_1) = 0\). Therefore, \(A_1\) is a nonnegative operator. Now, notice that the following relation holds:
\[
A_i = A_1 + (i - 1)(i + n - 1).
\]
Consequently, \(A_i\) is a positive operator if \(i \geq 2\) and \(A_1(\phi) = 0\) only admits the trivial solution. \(\Box\)

Lemma 2.3. Let \(n \geq 3\). Denote by \(B_R\) the ball of radius \(0 < R < 1\) centered at the origin of \(\mathbb{R}^n\), equipped with the hyperbolic metric
\[
g_{\mathbb{H}} = \frac{4|dx|^2}{(1 - |x|^2)^2}.
\]
The first eigenvalue of the Neumann boundary problem
\[
\begin{cases}
\Delta_{\mathbb{H}} \phi - n \phi &= 0 \quad \text{in } B_R, \\
\frac{\partial \phi}{\partial n} &= \mu \phi \quad \text{on } \partial B_R.
\end{cases}
\]
is \(\mu_0 = \frac{2R}{1 + R^2}\), with corresponding eigenfunction given by \(\phi_0(x) = \frac{|x|^2}{1 - |x|^2}\). The second eigenvalue is \(\mu_1 = \frac{1 + R^2}{2R}\) and the corresponding eigenspace is \(n\)-dimensional and generated by the family of eigenfunctions
\[
\left\{ \phi_i^n(x) = \frac{|x|^n x_i}{1 - |x|^2}; \quad i = 1, \ldots, n \right\}.
\]

Proof. Let \(d_{\mathbb{H}}\) denote the geodesic distance from the origin, given by \(d_{\mathbb{H}}(x) = \ln \frac{1 + |x|}{1 - |x|}\), and let \((t, \theta)\) be the geodesic polar coordinates of a point in \(B_R \setminus \{0\}\), where \(0 < t < T = \ln \frac{1 + R}{1 - R}\) and \(\theta \in \mathbb{S}^{n-1}\). In these coordinates, the hyperbolic metric takes the form
\[
g_{\mathbb{H}} = dt^2 + \sinh^2 t g_{\mathbb{S}^{n-1}},
\]
where \(g_{\mathbb{S}^{n-1}}\) is the standard metric on \(\mathbb{S}^{n-1}\), and (2.10) is equivalent to the following problem:
\[
\begin{cases}
\frac{\partial^2 \phi}{\partial t^2} + (n - 1) \coth t \frac{\partial \phi}{\partial t} + \frac{\Delta_{\mathbb{S}^{n-1}} \phi}{\sinh^2 t} + n \phi &= 0 \quad \text{in } B_T, \\
\frac{\partial \phi}{\partial t} &= \mu \phi \quad \text{on } \partial B_T.
\end{cases}
\]
See [39] for more details. Using the fact that spherical harmonics generate \(L^2(\mathbb{S}^{n-1})\), we write \(\phi(t, \theta) = \sum_i \gamma_i(t) \xi_i(\theta)\), with \(\xi_i\) satisfying the equation
\[
-\Delta_{\mathbb{S}^{n-1}} \xi_i = i(i + n - 2) \xi_i, \quad i = 0, 1, \ldots
\]
Therefore, separating variables, we can rewrite (2.11) in the following form:
\[
\begin{cases}
\sum_i \left( \gamma''_i + (n - 1) \coth t \gamma'_i - \left(\frac{i(i + n - 2)}{\sinh^2 t} + n\right) \gamma_i \right) \xi_i = 0, \\
\sum_i \left( \gamma_i(T) - \mu \gamma_i(T) \right) \xi_i = 0.
\end{cases}
\]
Since the functions $\xi_i$ are orthogonal, the consequence is that each $\gamma_i$ is a solution of (2.6). By Lemma 2.2, if $\mu = \mu_0 = \tanh T = \frac{2R}{1+R^2}$, there exists a solution for (2.6) associated to $i = 0$, and consequently a solution for (2.11):

$$\phi_0(t, \theta) = \cosh t.$$  

$\phi_0$ is nonnegative in $[0, T]$, so $\mu_0$ must be the first eigenvalue of (2.10). Again by Lemma 2.2, for $\mu = \mu_1 = (\tanh T)^{-1} = \frac{1+R^2}{2R}$ there exists a solution for (2.6) associated to $i = 1$, which produces the family of solutions for (2.11):

$$\{ \phi_i(t, \theta) = \xi_i(\theta) \sinh t : i = 1, \ldots, n \}.$$  

The same result guarantees that any other solution of (2.6) must have $\mu > \mu_1$, finishing the proof. $\square$

Finally, we are in position to prove Theorem 2.1.

**Proof of Theorem 2.1.** This proof follows the ideas of [2, Lemma 2.2], with the fundamental difference that our problem is equivalent to one on a geodesic ball in the Hyperbolic space and not in the Euclidean sphere.

Let us denote $g_* = |K| U^{\frac{4}{n-2}} g_0$. The scalar and boundary mean curvatures of $\mathbb{R}^n_+$ with respect to $g_*$ are given by (1.1):

$$S_* = -1, \quad h_* = \frac{\mathcal{D}_n(p)}{\sqrt{n(n-1)}}.$$  

By means of (2.5), it is possible to rewrite (2.1) as follows:

$$\begin{cases}
\Delta_+ \bar{v} - \frac{1}{n-1} \bar{v} = 0 & \text{in } \mathbb{R}^n_+,
\frac{\partial \bar{v}}{\partial \nu} - \frac{\mathcal{D}_n(p)}{\sqrt{n(n-1)}} \bar{v} = 0 & \text{on } \partial \mathbb{R}^n_+.
\end{cases}$$

with $\bar{v} = |K|^{-\frac{n-2}{4}} U^{-1} v$. The differential operators are explicit and their expressions are given by:

$$\Delta_+ \bar{v} = \left(1 - |\bar{x}|^2 - (x_n + \mathcal{D}_n(p))^2 \right)^2 \Delta \bar{v} + \frac{n-2}{2n(n-1)} \left(1 - |\bar{x}|^2 - (x_n + \mathcal{D}_n(p))^2 \right) \nabla \bar{v} \cdot (x + \mathcal{D}_n(p)e_n),$$

$$(2.12)$$

$$\frac{\partial \bar{v}}{\partial \nu} = \frac{1 - |\bar{x}|^2 - (x_n + \mathcal{D}_n(p))^2}{2\sqrt{n(n-1)}} \frac{\partial \bar{v}}{\partial \eta}.$$  

(2.13)

Now let us denote by $\Phi$ the map given by

$$\Phi = K^{-1} \circ \tau_{\mathcal{D}_n(p)} : \mathbb{R}^n_+ \rightarrow B_1(0) \subset \mathbb{R}^n,$$  

(2.14)

where $\tau_{\mathcal{D}_n(p)}$ is the translation $x \rightarrow x + \mathcal{D}_n(p)e_n$ and $K$ is the Cayley transform, which maps conformally the ball of radius 1 centered at the origin of $\mathbb{R}^n$ to the half-space $\mathbb{R}^n_+$. It can be proved that, up to composing with a certain isometry of $\mathbb{H}^n$, $\text{Im}(\Phi) = B_R(0)$ with $R = \mathcal{D}_n(p) - \sqrt{\mathcal{D}_n(p)^2 - 1}$. Moreover, $\Phi$ is a conformal map and satisfies

$$\Phi^* g_\mathbb{H} = \frac{|K|}{n(n-1)} U^{\frac{4}{n-2}} g_0, \quad \text{where } g_\mathbb{H} = \frac{4 \left|dx\right|^2}{(1 - |x|^2)^2} \text{ on } B_R.$$  

(2.15)

Multiplying (2.12) by $n(n-1)$ and (2.13) by $\sqrt{n(n-1)}$ and applying (2.15), one can see that $\hat{v} = (U^{-1} v) \circ \Phi^{-1}$ is in $H^1(B_R)$ (see [24, Lemma 6]) and satisfies the following problem:

$$\begin{cases}
\Delta_+ \hat{v} - n \hat{v} = 0 & \text{in } B_R,
\frac{\partial \hat{v}}{\partial \nu} = \mathcal{D}_n(p) \hat{v} & \text{on } \partial B_R.
\end{cases}$$  

(2.16)
being
\[ \Delta_v \hat{\nu} = \frac{(1 - |x|^2)^2}{4} \Delta \hat{\nu} + \frac{n-2}{2} \nabla \hat{\nu} \cdot x, \]
and
\[ \frac{\partial \hat{\nu}}{\partial \nu_{\hat{M}}} = 1 - |x|^2 \frac{\partial \hat{\nu}}{\partial \hat{\eta}} \]
the Laplace-Beltrami operator and normal derivative on \( B_R \) considered with respect to the hyperbolic metric \( g_{\hat{M}} \). Theorem 2.1 follows from Lemma 2.3, taking into account that \( D_n(p) = \frac{1+R^2}{2R} \) and
\[ \hat{j}_i = c_i \phi^i_1 \quad \text{for every } i = 1, \ldots, n. \] (2.16)

\[ \square \]

3. THE BUILDING BLOCK

Let \( p \in \partial M \). The main ingredient to cook up our solutions are the bubbles defined in (1.6) together with the correction found out in Proposition 3.1, i.e. the building block of the solutions we are looking for is
\[ \mathcal{W}_p(\xi) := \chi \left( \left( \psi_p^0 \right)^{-1} (\xi) \right) \left[ \frac{1}{\delta^{\frac{n-2}{2}}} U \left( \frac{\left( \psi_p^0 \right)^{-1}(\xi)}{\delta} \right) + \frac{1}{\delta^{\frac{n-2}{2}}} V_p \left( \frac{\left( \psi_p^0 \right)^{-1}(\xi)}{\delta} \right) \right] \]
where \( \psi_p^0 : \mathbb{R}^n_+ \to M \) are the Fermi coordinates in a neighborhood of \( p \) and \( \chi \) is a radial cut-off function, with support in a ball of radius \( R \). Here \( U \) is the bubble defined in (2.2) and \( V_p \) solves (3.2).

3.1. The correction of the bubble. Let us introduce the correction term as the function \( V_p : \mathbb{R}^n_+ \to \mathbb{R} \) which is defined below.

**Proposition 3.1.** Let \( U \) be as in (2.2) and set
\[ E_p(x) = \sum_{i,j=1}^{n-1} \frac{8(n-1)}{n-2} \frac{h^{ij}(p)}{\partial x_i \partial x_j} x_n, \quad x \in \mathbb{R}^n_+ \]
where \( h^{ij}(p) \) are the coefficients of the second fundamental form of \( M \) at the point \( p \in \partial M \). Then the problem
\[ \begin{cases} -\frac{4(n-1)}{n-2} \Delta V + \frac{n+2}{n-2} |K| U \frac{n-2}{n} V = E_p & \text{in } \mathbb{R}^n_+, \\ \frac{2}{n-2} \frac{\partial V}{\partial \nu} - \frac{n-2}{n-2} H U \frac{n-2}{n} V = 0 & \text{on } \partial \mathbb{R}^n_+ \end{cases} \]
admits a solution \( V_p \) satisfying the following properties:

(i) \( \int_{\mathbb{R}^n_+} V_p(x) \hat{j}_i(x) dx = 0 \) for any \( i = 1, \ldots, n \) (see (2.3) and (2.4))

(ii) \( |\nabla^\alpha V_p| (x) \lesssim \frac{1}{(1+|x|)^{n-\alpha}} \) for any \( x \in \mathbb{R}^n_+ \) and \( \alpha = 0, 1, 2 \)

(iii) \[ |K| \int_{\mathbb{R}^n_+} U \frac{n+2}{n-2} V_p dx = (n-1) H \int_{\partial \mathbb{R}^n_+} U \frac{n-2}{n} V_p d\hat{x}. \]

(iv) if \( n \geq 5 \)
\[ \int_{\mathbb{R}^n_+} \left( -\frac{4(n-1)}{n-2} \Delta V_p + \frac{n+2}{n-2} |K| U \frac{n-2}{n} V_p \right) V_p \geq 0, \]

(v) the map \( p \mapsto V_p \) is \( C^2(\partial M) \).
Choosing \((\psi)\) by the area formula and (2.1), we will introduce some notation to reduce ourselves to the study of a problem similar to (2.1). Let \(\bar{U} = |K|^{\frac{1}{n-2}} U\), then we can rewrite (3.2) as:

\[
\begin{align*}
-\frac{4(n-1)}{n-2} \Delta v + \frac{n+1}{n-2} \bar{U} \frac{1}{n-2} v &= f \quad \text{in } \mathbb{R}^n, \\
\frac{2}{n-2} \frac{\partial v}{\partial r} - \frac{n}{n-2} \frac{\mathcal{D}_n(p)}{\sqrt{n(n-1)}} \bar{U} \frac{2}{n-2} v &= 0 \quad \text{on } \partial \mathbb{R}^n.
\end{align*}
\]

Let \(\Phi\) be as in (2.14). We set

\[
\hat{f}(\Phi^{-1}(x)) = \frac{n(n-2)}{4} f(x) \bar{U}(x)^{-\frac{n+2}{2}}
\]

Arguing as in the proof of Theorem 2.1, we see that it is enough to consider the following problem for \(\hat{v} = (\bar{U}^{-1} v) \circ \Phi^{-1}\):

\[
\begin{align*}
\Delta_H \hat{v} - n \hat{v} &= \hat{f} \quad \text{in } B_R, \\
\frac{\partial \hat{v}}{\partial \nu_H} &= \mathcal{D}_n(p) \hat{v} \quad \text{on } \partial B_R,
\end{align*}
\]

By the area formula and (2.16):

\[
\int_{B_r} \phi_k^1(z) \hat{f}(z) d\mu_{2n} = c_n \int_{\mathbb{R}^n_+} \phi_k^1(\Phi^{-1}(x)) h^{ij}(p) \frac{\partial^2 U(x)}{\partial x_i \partial x_j} x_n U^{-\frac{n+2}{2}} |\text{Jac } \Phi^{-1}| \, dx
\]

\[
= c_n \int_{\mathbb{R}^n_+} \tilde{j}_k(x) h^{ij}(p) \frac{\partial^2 U(x)}{\partial x_i \partial x_j} x_n \, dx
\]

\[
= c_n \sum_{i,j=1}^{n-1} \int_0^{+\infty} \int_{\mathbb{R}^{n-1}} \frac{x_i x_j p_k(\tilde{x}, x_n)}{|\tilde{x}|^2 - 1} \, d\tilde{x} \, dx_n,
\]

being \(p_k\) a polynomial in \(x\) with \(\deg p_k = 1\) if \(k = 1, \ldots, n-1\), and \(\deg p_n = 2\). To get the last identity we have also used definitions (2.2), (2.3), (2.4) and the condition \(\sum_i h^{ii}(p) = 0\). Now, if we take polar coordinates in \(\mathbb{R}^{n-1}\) and use the fact that

\[
\int_{S^{n-2}} p^\gamma = \frac{r^2}{\gamma(\gamma + n - 3)} \int_{S^{n-2}} \Delta p^\gamma
\]

for every homogeneous polynomial \(p^\gamma\) of degree \(\gamma\), we can check that

\[
\int_{B_r} \phi_k^1(z) \hat{f}(z) d\mu_{2n} = 0 \quad \text{for all } k = 1, \ldots, n.
\]

By elliptic linear theory, there exists a solution \(\hat{v}\) to (3.3) which is orthogonal to \(\{\phi_k^1\}_{k=1}^n\). Consequently, \(v = \bar{U}(\hat{v} \circ \Phi)\) is a solution of (3.2) orthogonal to \(\{\tilde{j}_k\}_{k=1}^n\).

Given \(z \in B_R\), let \(G_{z_0}\) denote the Green’s function solving the problem

\[
\begin{align*}
\Delta_H G_z - n G_z &= \delta_z - \sum_{k=1}^n \frac{\phi_k^1(z) \phi_k^1}{\|\phi_k^1\|^2_{L^2}} \quad \text{in } B_R, \\
\frac{\partial G_z}{\partial \nu_H} - \mathcal{D}_n(p) G_z &= 0 \quad \text{on } \partial B_R.
\end{align*}
\]

Then, by Green’s representation formula

\[
\psi(z) = \sum_{k=1}^n \int_{B_R} \frac{\phi_k^1(z) \phi_k^1(w)}{\|\phi_k^1\|^2_{L^2}} \psi(w) d\mu_{2n}(w) - \int_{B_R} G_z(w) \Delta_H \psi(w) d\mu_{2n}(w)
\]

\[
- \int_{\partial B_R} G_z(w) \left( \frac{\partial}{\partial \nu_H} - \mathcal{D}_n(p) \right) \psi(w) d\mu_{2n}(w).
\]

Choosing \(\psi = \hat{v}\) in (3.4),

\[
\hat{v}(z) = - \int_{B_R} G_z(w) \hat{f}(w) d\mu_{2n}(w).
\]
then
\[ |\hat{v}(z)| \leq c_n |h^{ij}(p)| \int_{B_R} |w - z|^{2-n} |w + \mathcal{D}_n(p) e_n|^{-3} d\mu_{gn}(w). \]

By [4, Proposition 4.12] with \( \alpha = 2 \) and \( \alpha = n - 3 \),
\[ \hat{v}(z) \leq c_n |h^{ij}(p)| |z + \mathcal{D}_n(p) e_n|^{-1}. \]

Hence, \( u = \hat{U}(\hat{u} \circ \Phi) \) satisfies the estimate (ii). To prove (iii), integrate by parts (3.3) to obtain:
\[ n \int_{B_R} \hat{v} d\mu_{gn} - \int_{B_R} \hat{f} d\mu_{gn} = \mathcal{D}_n(p) \int_{\partial B_R} \hat{v} ds_{gn}. \] (3.5)

By (2.15),
\[ d\mu_{gn} = (n(n - 1))^{\frac{1}{2}} |K|^{\frac{1}{2}} U^{\frac{2n}{n - 2}} |dx|^2, \] (3.6)
\[ ds_{gn} = (n(n - 1))^{-\frac{1}{2}} |K|^{-\frac{1}{2}} U^{-\frac{2n}{n - 2}} |d\vec{x}|^2. \] (3.7)

Therefore, by the area formula:
\[ \int_{B_R} \hat{f} d\mu_{gn} = c_n \int_{\mathbb{R}^n} h^{ij}(p) \frac{\partial^2 U(x)}{\partial x_i \partial x_j} x_n U(x) dx = 0. \] (3.8)

Combining (3.5) and (3.8) with the relations (3.6) and (3.7), we get the desired equality.

Finally, integrating by parts we obtain
\[ -\int_{B_R} (\Delta H \hat{v}) \hat{v} d\mu_{gn} = \int_{B_R} |\nabla H \hat{v}|^2 d\mu_{gn} + \mathcal{D}_n(p) \int_{\partial B_R} \hat{v}^2 ds_{gn}. \] (3.9)

By Lemma 2.3, we know that
\[ \inf \left\{ \int_{B_R} \left( |\nabla H \tilde{v}|^2 + n \psi^2 \right) d\mu_{gn} : \int_{\partial B_R} \psi_0 ds_{gn} = 0 \right\} = \mathcal{D}_n(p). \]

If we showed that \( \hat{v} \) is orthogonal to \( \phi_0 \) in \( L^2(\partial B_R) \), we would get:
\[ \int_{B_R} |\nabla H \hat{v}|^2 d\mu_{gn} + n \int_{B_R} \hat{v}^2 d\mu_{gn} \geq \mathcal{D}_n(p) \int_{\partial B_R} \hat{v}^2 ds_{gn}. \] (3.10)

Then, combining (3.9) and (3.10), we obtain
\[ -\int_{B_R} (\Delta H \hat{v}) \hat{v} d\mu_{gn} + n \int_{B_R} \hat{v}^2 d\mu_{gn} \geq 0. \] (3.11)

By the properties of the conformal Laplacian, we know that
\[ L_{\mathbb{H} \Pi} \psi = \frac{4(n - 1)}{n - 2} \Delta_{\mathbb{H} \Pi} \psi - n(n - 1) \psi = n(n - 1) L_\star \phi, \]
with \( \psi \circ \Phi^{-1} = \phi \). Thus, multiplying (3.11) by \( \frac{4(n - 1)}{n - 2} \), we obtain
\[
0 \leq n(n - 1) \int_{\mathbb{R}^n_+} L_\star \left( U^{-1} v \right) U^{-1} v d\mu_\star + n(n - 1) \left( 1 + \frac{4}{n - 2} \right) \int_{\mathbb{R}^n_+} U^{\frac{4}{n - 2}} v^2 dx \\
n = n(n - 1) \left( \frac{4(n - 1)}{n - 2} \int_{\mathbb{R}^n_+} (\Delta v) v dx + \frac{n + 2}{n - 2} \int_{\mathbb{R}^n_+} |K| U^{\frac{4}{n - 2}} v^2 dx \right). 
\]
We conclude the proof by showing that \( \int_{\partial B_R} \hat{\psi}_0 \, ds_{g_R} = 0 \). We will use the fact that \( \phi_0 \) solves (2.10) for \( \mu = \mathcal{D}_n(p)^{-1} \) and that \( \hat{v} \) is a solution of (3.3). Integrating by parts:

\[
0 = \int_{B_R} \hat{f} \, \phi_0 \, d\mu_{g_R} = \int_{B_R} (\phi_0 \Delta \hat{v} - \hat{v} \Delta \phi_0) \, d\mu_{g_R} = \int_{\partial B_R} \left( \frac{\partial \hat{v}}{\partial \nu_{g_R}} \phi_0 - \frac{\partial \phi_0}{\partial \nu_{g_R}} \hat{v} \right) \, ds_{g_R} = \left( 4 \mathcal{D}_n(p) - \frac{1}{\mathcal{D}_n(p)} \right) \int_{\partial B_R} \hat{\psi}_0 \, ds_{g_R},
\]

where the first identity can be proved using the same argument as in (i).

For the proof of (v) we can reason as in Proposition 7 of [24]. \( \square \)

We end this section by giving a more careful description of the function \( V_p \). In particular, we need to detect the leading part of \( V_p \) and since its decay changes as \( n = 4 \) or \( n \geq 5 \) we have to distinguish the two cases.

**Case** \( n = 4 \). We decompose \( V_p \) into three parts: the main part \( \bar{w}_p \) is almost a rational function, the second part \( \zeta_p \) is a harmonic function with prescribed boundary condition and the third one \( \psi_p \) is an higher order term. More precisely, let

\[
V_p = \bar{w}_p + \zeta_p + \psi_p
\]

where \( \bar{w}_p, \zeta_p \) and \( \psi_p \) solve respectively the following problems

\[
-6 \Delta \bar{w}_p = E_p(x), \quad \text{in } \mathbb{R}^4_+
\]

\[
\begin{aligned}
\begin{cases}
-6 \Delta \zeta_p = 0 & \text{in } \mathbb{R}^4_+ \\
\frac{\partial \zeta_p}{\partial \nu} = 2HU \zeta_p + \left( 2HU \bar{w}_p - \frac{\partial \bar{w}_p}{\partial \nu} \right) & \text{on } \partial \mathbb{R}^4_+
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
\begin{cases}
-6 \Delta \psi_p + 3|K|U^2 \psi_p = -3|K|U^2 (\bar{w}_p + \zeta_p) & \text{in } \mathbb{R}^4_+ \\
\frac{\partial \psi_p}{\partial \nu} = 2HU \psi_p & \text{on } \partial \mathbb{R}^4_+
\end{cases}
\end{aligned}
\]

The following holds:

**Lemma 3.2.** Set

\[
\bar{w}_p^0(x) := \sum_{i,j=1}^{3} M_{ij}(p) \frac{x_i x_j}{|x|^2 + (x_4 + \mathcal{D}_4)^2 - 1}, \quad \text{with } M_{ij}(p) = \frac{2h_{ij}(p)\alpha_4}{|K|^2}.
\]

Then

\[
\bar{w}_p(x) - \bar{w}_p^0(x) = \mathcal{O} \left( \frac{1}{1 + |x|^2} \right) \quad \text{and} \quad |\nabla \bar{w}_p(x) - \nabla \bar{w}_p^0(x)| = \mathcal{O} \left( \frac{1}{1 + |x|^2} \right),
\]

\[
|\zeta_p| \lesssim \frac{1}{1 + |x|} \quad \text{and} \quad |\nabla \zeta_p| \lesssim \frac{1}{1 + |x|^2}
\]

\[
|\psi_p| \lesssim \frac{1}{1 + |x|^3} \quad \text{and} \quad |\nabla \psi_p| \lesssim \frac{1}{1 + |x|^4}.
\]

**Proof.** First we observe that the estimates (3.17) and (3.18) follows by using the same arguments of Proposition 3.1 applied to problems (3.14) and (3.15).

Now it remains to show (3.16).

We remark that we can write

\[
\bar{w}_p = 2h_{ij}(p) \bar{z}_i \bar{z}_j, \quad i, j = 1, \ldots, 3 \quad i \neq j
\]

where \( \bar{z}_p \) solves the problem

\[
- \Delta \bar{z}_p = U(x)x_4 = \frac{\alpha_4}{|K|^2} \frac{x_4}{|x|^2 + (x_4 + \mathcal{D}_4)^2 - 1} \quad \text{in } \mathbb{R}^4_+.
\]
The aim is then to understand the main term of the solution $z_p$ of (3.19).
It holds that
\[
\frac{x_4 + D_4}{|x|^2 + (x_4 + D_4)^2 - 1} = \frac{1}{2} \ln(|x|^2 + (x_4 + D_4)^2 - 1).
\]
Thus if we take $\Phi_0$ a solution of
\[
- \Delta \Phi_0 = \ln \left( \frac{|x|^2 + (x_4 + D_4)^2 - 1}{|x|^2 + (x_4 + D_4)^2} \right), \quad \text{in } \mathbb{R}^4_+	ag{3.20}
\]
and $\Phi_1$ a solution of
\[
- \Delta \Phi_1 = \frac{D_4}{|x|^2 + (x_4 + D_4)^2 - 1} \quad \text{in } \mathbb{R}^4_+, \tag{3.21}
\]
then,
\[
z_p = \frac{\alpha_4}{|K|^4} \left( \frac{1}{2} \frac{\partial \Phi_0}{\partial x_4} - \Phi_1 \right)
\]
solves (3.19).

The advantage of (3.20) and (3.21) over (3.19) is that, under a change of variables
\[
- \Delta [\Phi_0(\tilde{x}, x_4 - D_4)] = \ln(|\tilde{x}|^2 + x_4^2 - 1) = \ln(|x|^2 - 1) \quad \text{in } \mathbb{R}^4_+
\]
and
\[
- \Delta [\Phi_1(\tilde{x}, x_4 - D_4)] = \frac{D_4}{|\tilde{x}|^2 + x_4^2 - 1} = \frac{D_4}{|x|^2 - 1} \quad \text{in } \mathbb{R}^4_+.
\]
If we assume that $\Phi_0(\tilde{x}, x_4 - D_4)$ and $\Phi_1(\tilde{x}, x_4 - D_4)$ are radially symmetric, i.e. $\hat{\Phi}_0(|x|) = \Phi_0(\tilde{x}, x_4 - D_4)$ and $\hat{\Phi}_1(|x|) = \Phi_1(\tilde{x}, x_4 - D_4)$ then it is reduced to solve the equations
\[
- \hat{\Phi}_0'' - \frac{N - 1}{r} \hat{\Phi}_0' = \ln(r^2 - 1) \quad \text{in } (1, +\infty)
\]
and
\[
- \hat{\Phi}_1'' - \frac{N - 1}{r} \hat{\Phi}_1' = \frac{D_4}{r^2 - 1} \quad \text{in } (1, +\infty).
\]
The general solutions are expressed as
\[
\hat{\Phi}_0(r) = \frac{c_1 + 3r^4 - 2(r^2 - 1)^2 \ln(r^2 - 1)}{16r^2},
\]
\[
\hat{\Phi}_1(r) = \frac{c_2}{r^2} + \frac{D_4 \ln(r^2 - 1)}{4r^2} - \frac{D_4 \ln(r^2 - 1)}{4},
\]
with $c_1, c_2 \in \mathbb{R}$. Using the symmetries of the coefficients $h^{ij}$ (with the aid of computer assisted proof), we get
\[
w_p(x) = \sum_{\substack{i,j=1 \atop i<j}}^3 M_{ij} x_i x_j \left( \frac{1}{|x|^2 + (x_4 + D_4)^2 - 1} \right)^2 + O \left( \frac{1}{(1 + |x|)^2} \right) \quad \text{in a } C^1\text{-sense.}
\]
That concludes the proof.

\[\square\]

**Case** $n \geq 5$. We can decompose $V_p = w_p + \psi_p$ where $w_p$ solves
\[
- c_n \Delta w_p + c_n \frac{n(n + 2)}{(|x|^2 + (x_n + D_n)^2 - 1)^2} w_p = E_p(x) \quad \text{in } \mathbb{R}^n_+, \tag{3.22}
\]
and $\psi_p$ solves...
\[
\begin{cases}
-c_n \Delta \psi_p + c_n \frac{n(n+2)}{|x|^2 + (x + \mathbb{D}_n)^2 - 1} \psi_p = 0, \\
\frac{\partial \psi_p}{\partial \nu} = \frac{n \mathbb{D}_n}{(|x|^2 + \mathbb{D}_n^2 - 1)} \psi_p + \left( \frac{n \mathbb{D}_n}{(|x|^2 + \mathbb{D}_n^2 - 1)} w_p - \frac{\partial w_p}{\partial \nu} \right) \quad \text{on } \partial \mathbb{R}_+^n
\end{cases}
\] (3.23)

We claim that

\[w_p(x) = \beta_n \sum_{n \neq j}^{n-1} h^{ij}(p)x_i x_j (x_n - \mathbb{D}_n)\]

(3.24)

Indeed, we look for a solution of (3.22) of the form

\[w_p(x) = \frac{q(x)}{|x|^2 + (x + \mathbb{D}_n)^2 - 1}^\frac{n}{2},\]

with \(q(x)\) a polynomial function. Straightforward computations show that \(q(x)\) has to verify the equation

\[L(q(x)) = \beta_n x_n \sum_{n \neq j}^{n-1} h^{ij}(p)x_i x_j,\]

being

\[L(q) = -(|x|^2 + (x + \mathbb{D}_n)^2 - 1) \Delta q + 2n \nabla q \cdot (x + \mathbb{D}_n c_n) - 2nq\]

with \(\beta_n := \frac{2n(n-2)\alpha_n}{|K|^{-\frac{n-2}{2}}}\).

Observe that it is possible to write

\[q(x) = \beta_n \sum_{n \neq j}^{n-1} h^{ij}(p)q^{ij}(x),\]

where every \(q^{ij}\) is a polynomial solving \(L(q^{ij}) = x_i x_j x_n\). We note that \(L(x_i x_j x_n) = 4n x_i x_j x_n + 2n x_i x_j \mathbb{D}_n\) and \(L(x_i x_j) = 2n x_i x_j\), so

\[L \left( \frac{1}{4n} x_i x_j (x_n - \mathbb{D}_n) \right) = x_i x_j x_n.\]

Therefore, (3.24) follows.

### 3.2. The energy of the building block.

Let us define the energy \(J_e : H^1(M) \to \mathbb{R}\)

\[J_e(u) := \int_M \left( \frac{c_n}{2} |\nabla_g u|^2 + \frac{1}{2} S_g u^2 - K \mathfrak{G}(u) \right) dv_g - c_n \frac{n - 2}{2} \int_{\partial M} H \mathfrak{F}(u) d\sigma_g\]

(3.25)

where

\[\mathfrak{G}(s) := \int_0^s \varrho(t) dt, \quad \varrho(t) := (t^+)^{\frac{n-2}{2}} \quad \text{and} \quad \mathfrak{F}(s) := \int_0^s \varphi(t) dt, \quad \varphi(t) := (t^+)^{\frac{n}{2}}.\]

It is useful to introduce the integral quantities whose properties are listed in Appendix A:

\[R_m^\alpha := \int_0^{+\infty} \frac{\varrho^\alpha}{(1 + \varrho^2)^m} d\rho\]

(3.26)

and if \(p \in \partial M\)

\[\varphi_m(p) := \int_{\mathbb{D}_n}^{+\infty} \frac{1}{(t^2 - 1)^m} dt \quad \text{and} \quad \hat{\varphi}_m(p) := \int_{\mathbb{D}_n}^{+\infty} \frac{(t - \mathbb{D}_n)^2}{(t^2 - 1)^m} dt.\]

(3.27)
We will assume that $H$ and $K$ are constant functions. We remark that $\mathcal{D}_n$, $\varphi_m$ and $\hat{\varphi}_m$ are also constant functions, so we will omit the dependence on $p$.

In the following result we compute the energy of the building block (3.1) (the proof is quite technical and is postponed in Appendix B).

**Proposition 3.3.** It holds true that

$$J_c(\mathcal{W}_p) = \mathcal{E} - \zeta_n(\delta) \left[ b_n \| \pi(p) \|^2 + o'_n(1) \right] + \varepsilon \delta \left[ c_n + o''_n(1) \right]$$

where (see (3.26) and (3.27))

$$\mathcal{E} := \frac{a_n}{|K|^\frac{2}{n-2}} \left[ -(n-1)\varphi_{n+1} + \frac{\mathcal{D}_n}{(\mathcal{D}_n^2 - 1)^{\frac{n}{2}}} \right], \quad a_n := \alpha_n^2 \omega_{n-1} I_{n-1}^n \frac{n-3}{(n-1)\sqrt{n(n-1)}},$$

and

$$b_n := \frac{1}{2} b_n + \frac{n-2}{n-1} \alpha_n^2 \omega_{n-1} I_{n-1}^n \frac{1}{|K|^\frac{2}{n-2}} \left( 4(n-3)\varphi_{n-1} + \varphi_{n-3} \right), \quad n \geq 5$$

$$b_4 := \frac{192\pi^2}{|K|^\frac{2}{n-2}} \left( \frac{\alpha_n^2 \omega_n I_3^4}{(\mathcal{D}_n^2 - 1)^{\frac{n}{2}}} \right) I_{n-1}^n -$$

Moreover (see Proposition B.2 for the definition of $f_n$)

$$c_n := 2(n-2)\omega_{n-1} \alpha_n^2 \frac{1}{|K|^\frac{2}{n-2}} \left( \frac{\alpha_n^2 \omega_n I_3^4}{(\mathcal{D}_n^2 - 1)^{\frac{n}{2}}} \right) I_{n-1}^n -$$

Moreover

$$\zeta_4(\delta) := \delta^2 |\ln \delta| \quad \text{and} \quad \zeta_n(\delta) := \delta^2 \quad \text{if } n \geq 5.$$

and

$$o'_n(1) = \begin{cases} O(\delta) & \text{if } n \geq 6, \\ O(\delta |\ln \delta|) & \text{if } n = 5, \\ O(|\ln \delta|^{-1}) & \text{if } n = 4 \end{cases} \quad \text{and} \quad o''_n(1) = \begin{cases} O(\delta) & \text{if } n \geq 5, \\ O(\delta |\ln \delta|) & \text{if } n = 4. \end{cases}$$

4. PROOF OF THEOREM 1.1

4.1. Preliminaries. Since $(M,g)$ belongs to the positive Escobar class (i.e. the quadratic part of the Euler functional associated to the problem is positive definite), we can provide the Sobolev space $H^1(M)$ with the scalar product

$$\langle u, v \rangle := \int_M \left( c_n \nabla_g u \nabla_g v + \mathcal{S}_g uv \right) \, dv_g$$

where $dv_g$ is the volume element of the manifold. We let $\| \cdot \|$ be the norm induced by $\langle \cdot, \cdot \rangle$.

Moreover, for any $u \in L^q(M)$ (or $u \in L^q(\partial M)$) we denote the $L^q$-norm of $u$ by $\|u\|_{L^q(M)} := (\int_M |u|^q \, dv_g)^\frac{1}{q}$ (respectively $\|u\|_{L^q(\partial M)} := (\int_{\partial M} |u|^q \, d\sigma_g)^\frac{1}{q}$ where $d\sigma_g$ is the volume element of $\partial M$.) We have the well-known embedding continuous maps

$$i_{\partial M} : H^1(M) \rightarrow L^1(\partial M) \quad i_M : H^1(M) \rightarrow L^{\frac{2n}{n-2}}(M)$$

$$i_{\partial M}^t : L^t(\partial M) \rightarrow H^1(M) \quad i_M^t : L^{\frac{2n}{n-2}}(M) \rightarrow H^1(M)$$

for $1 \leq t \leq \frac{2(n-1)}{n-2}$.

Now given $f \in L^{\frac{2n}{n-2}}(\partial M)$ the function $w_1 = i_{\partial M}^t(f)$ in $H^1(M)$ is the unique solution of the equation

$$\begin{cases} -c_n \Delta_g w_1 + \mathcal{S}_g w_1 = 0 & \text{in } M \\ \frac{\partial w_1}{\partial \nu} = f & \text{on } \partial M. \end{cases} \quad (4.1)$$
Moreover, if we let \( g \in L^{\frac{2n}{n-2}}(M) \), the function \( w_2 = i^*_M(g) \) is the unique solution of the equation
\[
\begin{cases}
-\kappa \Delta_g w_2 + S_g w_2 = g & \text{in } M \\
\frac{\partial w_2}{\partial \nu} = 0 & \text{on } \partial M.
\end{cases}
\tag{4.2}
\]

By continuity of \( i_M, i_{\partial M} \) we get
\[
\| i_{\partial M}^*(f) \| \leq C_1 \| f \|_{L^{\frac{2(n-1)}{n-2}}(\partial M)} \quad \| i^*_M(g) \| \leq C_2 \| g \|_{L^{\frac{2n}{n-2}}(M)}
\]
for some \( C_1 > 0 \) and independent of \( \delta \) and some \( C_2 > 0 \) and independent of \( g \).

Then we rewrite the problem (1.4) as
\[
u = i^*_M(K \chi(u)) + i^*_{\partial M} \left( \frac{n-2}{2} (HF(u) - \varepsilon u) \right), \quad \text{with } g(u) := (u^+)\frac{n+2}{n-2} \text{ and } f(u) = (u^+)\frac{n}{n-2}
\]

\section*{4.2. The ansatz.}

Having in mind Proposition 3.3, we fix a non-umbilic and non-degenerate minimum point \( p \in \partial M \) of the function \( \| \pi(\cdot) \| \) with and we choose
\[
d_0 := \frac{c_n}{2b_n \| \pi(p) \|^2}
\tag{4.3}
\]
where \( b_n \) and \( c_n \) are positive constants defined in (3.29), (3.30) and (3.31). For any integer \( k \geq 1 \), we look for solutions of (1.4) of the form
\[
u_\varepsilon(\xi) := \sum_{j=1}^{k} W_j(\xi) + \Phi_\varepsilon(\xi) \quad \xi \in M
\tag{4.4}
\]
where
\[
W_j(\xi) = \chi \left( \left( \psi^\varepsilon_p \right)^{-1}(\xi) \right) W_j(\xi)
\]
and
\[
W_j(\xi) := \frac{1}{\delta_j^{\frac{n-2}{2}}} U \left( \frac{\left( \psi^\varepsilon_p \right)^{-1}(\xi) - \eta(\varepsilon)\tau_j}{\delta_j} \right) + \delta_j \frac{1}{\delta_j^{\frac{n-2}{2}}} V_p \left( \frac{\left( \psi^\varepsilon_p \right)^{-1}(\xi) - \eta(\varepsilon)\tau_j}{\delta_j} \right).
\]

Here \( \chi \) is a radial cut-off function with support in a ball of radius \( R \), the bubble \( U \) is defined in (2.2) and \( V_p \) solves (3.2). Moreover,
\[
\tau_j \in \mathcal{C} := \{ (\tau_1, \ldots, \tau_k) \in \mathbb{R}^{(n-1)k} : \tau_i \neq \tau_j \text{ if } i \neq j \}
\tag{4.5}
\]
and given \( d_0 \) as in (4.3) the concentration parameters \( \delta_j \) and the rate of the concentration points \( \eta(\varepsilon) \) are chosen as follows:
\[
\delta_j := \varepsilon (d_0 + \eta(\varepsilon)d_j), \quad d_j \in [0, +\infty) \text{ and } \eta(\varepsilon) := \varepsilon^\alpha \text{ with } \alpha := \frac{n-4}{n} \text{ if } n \geq 5
\tag{4.6}
\]
or
\[
\delta_j := \rho(\varepsilon) (d_0 + \eta(\varepsilon)d_j) \quad d_j \in [0, +\infty) \text{ and } \eta(\varepsilon) := \frac{1}{|\ln \rho(\varepsilon)|^+} \text{ if } n = 4
\tag{4.7}
\]
where \( \rho \) is the inverse function of \( \ell : (0, e^{-\frac{1}{2}}) \to \left( 0, \varepsilon^{-1}\frac{1}{2} \right) \) defined by \( \ell(s) = -s \ln s \). We remark that \( \rho(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

Finally, the remainder term \( \Phi_\varepsilon(\xi) \) belongs to \( \mathcal{K}^2 \) defined as follows.

Let us define for \( i = 1, \ldots, n, \text{ and } j = 1, \ldots, k \)
\[
Z_{j,i}(\xi) = \chi \left( \left( \psi^\varepsilon_p \right)^{-1}(\xi) \right) Z_{j,i}(\xi), \text{ with } Z_{j,i}(\xi) := \frac{1}{\delta_j^{\frac{n-2}{2}}} \left( \frac{\left( \psi^\varepsilon_p \right)^{-1}(\xi) - \eta(\varepsilon)\tau_j}{\delta_j} \right)
\]
where \( y_i \) are given in (2.3) and (2.4).

We decompose \( H^1(M) \) in the direct sum of the following two subspaces

\[
\mathcal{K} = \text{span} \{ Z_{j,i} : i = 1, \ldots, n, \ j = 1, \ldots, k \}
\]

and

\[
\mathcal{K}^\perp := \{ \psi \in H^1(M) : \langle \psi, Z_{j,i} \rangle = 0, \ i = 1, \ldots, n, \ j = 1, \ldots, k \}.
\]

4.3. The reduction process. We define the projections

\[
\Pi : H^1(M) \to \mathcal{K} \quad \Pi^\perp : H^1(M) \to \mathcal{K}^\perp.
\]

Therefore solving (4.1) is equivalent to solve the couple of equations

\[
\Pi^\perp \left\{ u_\varepsilon - i^*_M (K g (u_\varepsilon)) - i^*_M \left( \frac{n-2}{2} \left( H f (u_\varepsilon) - \varepsilon u_\varepsilon \right) \right) \right\} = 0 \tag{4.8}
\]

\[
\Pi \left\{ u_\varepsilon - i^*_M (K g (u_\varepsilon)) - i^*_M \left( \frac{n-2}{2} \left( H f (u_\varepsilon) - \varepsilon u_\varepsilon \right) \right) \right\} = 0 \tag{4.9}
\]

where \( u_\varepsilon \) is defined in (4.4).

4.4. Solving the equation (4.8): the remainder term. We shall find the remainder term \( \Phi_\varepsilon \in \mathcal{K}^\perp \) in (4.4). Let us rewrite the equation (4.8) as

\[
\mathcal{E} + \mathcal{L}(\Phi_\varepsilon) + \mathcal{N}(\Phi_\varepsilon) = 0 \tag{4.10}
\]

where the error term \( \mathcal{E} \) is

\[
\mathcal{E} := \Pi^\perp \left\{ W - i^*_M (K g (W)) - i^*_M \left( \frac{n-2}{2} \left( H f (W) - \varepsilon W \right) \right) \right\},
\]

the linear operator \( \mathcal{L} \) is

\[
\mathcal{L}(\Phi_\varepsilon) := \Pi^\perp \left\{ \Phi_\varepsilon - i^*_M (K g' (W) \Phi_\varepsilon) - i^*_M \left( \frac{n-2}{2} \left( H f' (W) \Phi_\varepsilon - \varepsilon \Phi_\varepsilon \right) \right) \right\}
\]

and the quadratic term \( \mathcal{N}(\Phi_\varepsilon) \) is

\[
\mathcal{N}(\Phi_\varepsilon) := \Pi^\perp \left\{ - i^*_M \left[ K (g (W + \Phi_\varepsilon) - g (W) - g' (W) \Phi_\varepsilon) \right] - i^*_M \left[ \frac{n-2}{2} H (f (W + \Phi_\varepsilon) - f (W) - f' (W) \Phi_\varepsilon) \right] \right\}.
\]

The following result holds true.

**Proposition 4.1.** For any compact subset \( \mathcal{A} \subset (0, +\infty)^k \times \mathcal{C} \) (see (4.5)) there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and for any \( (d_1, \ldots, d_k, \tau_1, \ldots, \tau_k) \in \mathcal{A} \) there exists a unique function \( \Phi_\varepsilon \in \mathcal{K}^\perp \) which solves equation (4.10). Moreover, the map \( (d_1, \ldots, d_k, \tau_1, \ldots, \tau_k) \mapsto \Phi_\varepsilon (d_1, \ldots, d_k, \tau_1, \ldots, \tau_k) \) is of class \( C^1 \) and

\[
\| \Phi_\varepsilon \| \lesssim \begin{cases} \\
\varepsilon^2 & \text{if } n \geq 7 \\
\varepsilon^2 \ln \varepsilon^{1/2} & \text{if } n = 6 \\
\varepsilon^{2} & \text{if } n = 5 \\
\rho(\varepsilon) & \text{if } n = 4.
\end{cases}
\]

We omit the proof because it is standard and relies on the following two key results whose proof is given in Appendix C and Appendix D respectively. First, we estimate the size of the error term \( \mathcal{E} \).
Lemma 4.2. Let $n \geq 4$. For any compact subset $A \subset (0, +\infty)^k \times C$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and for any $(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k) \in A$ it holds

$$
\|E\| \lesssim \left\{ \begin{array}{ll}
\epsilon^2 & \text{if } n \geq 7 \\
\epsilon^2 |\ln \epsilon|^\frac{n}{4} & \text{if } n = 6 \\
\epsilon^\frac{n}{2} & \text{if } n = 5 \\
\rho(\epsilon) & \text{if } n = 4.
\end{array} \right.
$$

Next, we study the invertibility of the linear operator $L$.

Lemma 4.3. Let $n \geq 4$. For any compact subset $A \subset (0, +\infty)^k \times C$ there exist a positive constant $C > 0$ and $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and any $(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k) \in A$ it holds

$$
\|L(\phi)\| \geq C\|\phi\|.
$$

Next, we have to

4.5. Solving equation (4.9): the reduced problem. We know that solutions to problem (1.4) are critical points of the energy functional $J_\epsilon$ defined in (3.25). Let us introduce the so-called reduced energy

$$
\tilde{J}_\epsilon(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k) := J_\epsilon(W + \Phi_\epsilon)
$$

where the remainder term $\Phi_\epsilon$ has been found in Proposition 4.1. We shall prove that a critical point of the reduced energy provides a solution to our problem.

Proposition 4.4. It holds true that

1. If $(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k) \in [0, +\infty)^k \times (\mathbb{R}^n)^k$ is a critical point of the reduced energy (4.11), then $W + \Phi_\epsilon$ is a critical point of $J_\epsilon$ and so it solves (1.4).

2. The following expansion holds true

$$
\tilde{J}_\epsilon(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k) = k\mathcal{E} + k\theta_n(\epsilon) \left( c_n d_0 - b_n \|\pi(p)\|^2 d_0^2 \right)
$$

$$
+ \Theta_n(\epsilon) \left( -b_n \sum_{i=1}^k \Omega(p)(\tau_i, \tau_i) - b_n \|\pi(p)\|^2 \sum_{i=1}^k d_i^2 - \frac{d_n}{|K|^{\frac{n-2}{2}}} \sum_{i<j} \frac{d_{i,j}^{n-2}}{1 + \tau_i - \tau_j} \right)
$$

$$
=: \mathcal{F}_\epsilon(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k)
$$

$C^0$– uniformly with respect to $(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k)$ in a compact set of $(0, +\infty)^k \times C$. Here $\Omega(p)$ is the quadratic form associated with the second derivative of $p \rightarrow \|\pi(p)\|^2$ (being zero the first derivative), $\mathcal{E}$, $c_n$ are constants defined in (3.28) and (3.31), respectively, and

$$
d_n := \alpha_n \omega_{n-1} l_{n-2}^\frac{n-2}{2}.
$$

Moreover

$$
\theta_4(\epsilon) = \Theta_4(\epsilon) := \rho(\epsilon) |\ln \rho(\epsilon)| \quad \text{if } n = 4 \quad \text{and} \quad \theta_n(\epsilon) = \epsilon^2, \quad \Theta_n(\epsilon) := \epsilon \frac{4(n-2)}{n} \quad \text{if } n \geq 5.
$$

As we claimed above, we shall postpone in Appendix E the proof because even if it relies on standard arguments, it requires a lot of new elaborated and technical estimates.

4.6. Proof of Theorem 1.1: completed. The claim immediately follows by Proposition 4.4 taking into account that the function $\mathcal{F}_\epsilon(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k)$ has a maximum point which is stable under $C^0$– perturbations.
Appendix A. Auxiliary results

We have (see [2] Lemmas 9.4 and 9.5) the following results:

\[ I_m^n := \int_0^{+\infty} \frac{\rho^\alpha}{(1 + \rho^2)^m} d\rho = \frac{2m}{\alpha + 1} I_{m+1}^{n+2}, \quad \text{for } \alpha + 1 < 2m \]
\[ I_m^n = \frac{2m}{2m - \alpha - 1} I_{m+1}^\alpha, \quad \text{for } \alpha + 1 < 2m + 2 \]
\[ I_m^n = \frac{2m - \alpha - 3}{\alpha + 1} I_m^{n+2}, \quad \text{for } \alpha + 3 < 2m. \]

In particular, if \( n \geq 4 \)

\[ I_n^n = I_{n-2}^n = \frac{n - 3}{(n - 1)} I_{n-1}^n, \quad I_{n-2}^n = \frac{n - 3}{n - 1} I_{n-1}^n, \quad I_{n-2}^n = \frac{2(n - 2)}{n - 1} I_{n-1}^n. \quad (A.1) \]

**Lemma A.1.** It holds:

\[ \int_{\mathbb{R}^n} \frac{|\tilde{x}|^\alpha}{(|\tilde{x}|^2 + (x + \mathcal{D}_n)^2 - 1)^m} \ dx = \omega_{n-1} I_m^{n-2+\alpha} \varphi_2^{m-n-a+1}, \quad \text{for } n + \alpha < 2m \quad (A.2) \]

\[ \int_{\mathbb{R}^{n-1}} \frac{|\tilde{x}|^\alpha}{(|\tilde{x}|^2 + \mathcal{D}_n^2 - 1)^m} \ dx = \omega_{n-1} (\mathcal{D}_n^2 - 1)^{n+a-1-2m} I_m^{n-2+\alpha}, \quad \text{for } n - 1 + \alpha < 2m \quad (A.3) \]

\[ \int_{\mathbb{R}^n} \frac{x_1^2 |\tilde{x}|^\alpha}{(|\tilde{x}|^2 + (x + \mathcal{D}_n)^2 - 1)^m} \ dx = \omega_{n-1} I_m^{n-2+\alpha} \varphi_2^{m-n-a+1}, \quad \text{for } n + 2 + \alpha < 2m \quad (A.4) \]

**Proof.** Let us show (A.2).

\[
\int_{\mathbb{R}^n} \frac{|\tilde{x}|^\alpha}{(|\tilde{x}|^2 + (x + \mathcal{D}_n)^2 - 1)^m} \ dx = \int_0^{+\infty} \int_{\mathbb{R}^{n-1}} \frac{|\tilde{x}|^\alpha}{(|\tilde{x}|^2 + (x + \mathcal{D}_n)^2 - 1)^m} \ dx
\]
\[
= \int_0^{+\infty} \frac{\lambda}{\lambda^2 + 1} \int_{\mathbb{R}^{n-1}} \frac{|\tilde{x}|^\alpha}{(|\tilde{x}|^2 + \mathcal{D}_n^2 - 1)^m} \ dx
\]
\[
= \omega_{n-1} \int_0^{+\infty} \frac{\lambda^{n+a-2m}}{\lambda^2 + 1} \int_0^{+\infty} \frac{r^{n-2+\alpha}}{(r^2 + 1)^m} \ dx
\]
\[
= \omega_{n-1} I_{m-2+\alpha} \int_{\mathcal{D}_n} \frac{1}{n - 1} 2m - n - a + 1 dt
\]
\[
= \omega_{n-1} I_{m-2+\alpha} \varphi_2^{m-n-a+1}. \]

For what concerning (A.3) we have

\[
\int_{\mathbb{R}^{n-1}} \frac{|\tilde{x}|^\alpha}{(|\tilde{x}|^2 + \mathcal{D}_n^2 - 1)^m} \ dx = \omega_{n-1} \int_0^{+\infty} \frac{r^{n-2+\alpha}}{(r^2 + \mathcal{D}_n^2 - 1)^m} \ dx
\]
\[
= \omega_{n-1} (\mathcal{D}_n^2 - 1)^{n+a-1-2m} \int_0^{+\infty} \frac{r^{n-2+\alpha}}{(r^2 + 1)^m} \ dx
\]
\[
= \omega_{n-1} I_{m-2+\alpha} (\mathcal{D}_n^2 - 1)^{n+a-1-2m}. \]
Finally
\[ \int_{\mathbb{R}^n_+} \frac{x_n^2 |\tilde{x}|^\alpha}{(x_n^2 + (x_n + \mathcal{D}_n)^2)^m} dx = \int_0^{+\infty} \frac{\lambda}{(\sqrt{\lambda^2 + 1} - \mathcal{D}_n)^2} \frac{|\tilde{x}|^\alpha}{\sqrt{\lambda^2 + 1}} d\lambda \]
\[ = \omega_n - 1 R_m^{n-2+\alpha} \int_0^{+\infty} \frac{\lambda^{n+\alpha-2m}(\sqrt{\lambda^2 + 1} - \mathcal{D}_n)^2}{\lambda^2 + 1} d\lambda \]
\[ = \omega_n - 1 R_m^{n-2+\alpha} \int_0^{+\infty} \frac{(t - \mathcal{D}_n)^2}{(t^2 - 1)^{2m-n-\alpha+1}} dt \]
\[ = \omega_n - 1 R_m^{n-2+\alpha} \int_0^{+\infty} \frac{(t^2 - 1)^{2m-n-\alpha+1}}{t} \]

Lemma A.2. It holds:
\[ \varphi_{n+1}^n = \frac{1}{n-1} \mathcal{D}_n - \frac{n-2}{n-1} \varphi_{n-1}^n \]
and
\[ \varphi_{n+2}^n = \frac{1}{n-3} \mathcal{D}_n - \frac{n-4}{n-3} \varphi_{n-2}^n \]
Moreover
\[ \hat{\varphi}_m := \varphi_{m-1} + (\mathcal{D}_n^2 + 1) \varphi_m - \frac{1}{m-1} \mathcal{D}_n \]

Proof. It is enough to integrate by parts. \( \square \)

It is also useful to remind the expansion of the metric given in [19].

Lemma A.3. Let \((M, g)\) be a compact Riemannian manifold with boundary. If \(x = (\tilde{x}, x_n) = (x_1, \ldots, x_n)\) are the Fermi coordinates centered at a point \(p \in \partial M\), then the following expansion holds:

- \(\sqrt{|g(x)|} = 1 - \frac{1}{2} \left( \|\pi(p)\|^2 + \text{Ric}_\nu(p) \right) x_n^2 - \frac{1}{6} \tilde{R}_{ij}(p)x_i x_j + O(|x|^3)\),
- \(g^{ij}(x) = \delta_{ij} + 2h^{ij}(p)x_n + \frac{1}{3} \tilde{R}_{ijkl}(p)x_k x_l + \frac{2h_{ij}^{\nu}(p)}{\partial x_k} x_k x_n + (\text{Ric}_\nu(p) + 3h_{ik}(p)h_{jk}(p)) x_n^2 + O(|x|^3)\),
- \(g^{on}(x) = \delta_{on}\)
- \(\Gamma^b_{ij}(x) = O(|x|)\)

where \(\pi(p)\) is the second fundamental form at \(p\), \(h^{ij}(p)\) are its coefficients, \(\tilde{R}_{ijkl}(p)\) and \(R_{abcd}(p)\) are the curvature tensor of the boundary \(\partial M\) and \(M\), respectively, \(\tilde{R}_{ij}(p) = \tilde{R}_{ikj}(p)\) are the coefficients of the Ricci tensor, and \(\text{Ric}_\nu(p) = R_{\min}(p) = R_{\max}(p)\). Here the indices \(i, j, k = 1, \ldots, n - 1\) and \(a, b = 1, \ldots, n\).

Finally, we also remind the useful estimate:
\[ ||s + t||^q - s^q \lesssim \begin{cases} \min\{s^{q-1}|t|, |t|^q\} & \text{if } 0 < q \leq 1 \\ s^{q-1}|t| + |t|^q & \text{if } q > 1 \end{cases} \quad \text{for any } s > 0 \text{ and } t \in \mathbb{R}. \quad \text{(A.5)} \]

Appendix B. Proof of Proposition 3.3

First we need two technical propositions in which we compute the contribution of correction term \(V_p\) to the energy.

Proposition B.1. Let \(n = 4\) and \(\bar{w}_p\) the solution of (3.13) (the first term of the expansion of \(V_p\) in (3.12)). Then
\[ \int_{B^+_{\pi \delta}} |\nabla \bar{w}_p|^2 = \frac{64\pi^2}{|\pi|} ||\pi(p)||^2 \ln \delta + O(1). \quad \text{(B.1)} \]
Proof. We first reduce the integral into \((B.1)\) to one in a simpler domain. Let \(Q^+_r\) denote the upper half of the ball of radius \(r\) in the \(\|\cdot\|_\infty\) of \(\mathbb{R}^4\), that is,

\[
Q^+_r = \{ x \in \mathbb{R}^4 : x_4 \geq 0 \text{ and } -r \leq x_i \leq r, \ i = 1, 2, 3. \}
\]

and let \(A^+(\frac{R}{3}, \frac{R}{3})\) the upper half of the annulus with radii \(r_1 = \frac{R}{3}\) and \(r_2 = \frac{R}{3}\).

Then, we can write

\[
B^+_{\frac{R}{3}} = Q^+_{\frac{R}{3}} \cup \Omega_\delta,
\]

with \(\Omega_\delta := B^+_{\frac{R}{3}} \setminus Q^+_{\frac{R}{3}}\). Notice that \(\Omega_\delta\) satisfies \(\Omega_\delta \subset A^+(\frac{R}{3}, \frac{R}{3})\). Then, by using also Lemma 3.2, we get

\[
\int_{B^+_{\frac{R}{3}}} |\nabla \tilde{w}_p|^2 = \int_{Q^+_{\frac{R}{3}}} |\nabla \tilde{w}_p|^2 + \int_{\Omega_\delta} |\nabla \tilde{w}_p|^2 = \int_{Q^+_{\frac{R}{3}}} |\nabla \tilde{w}_p^0|^2 + \int_{\Omega_\delta} |\nabla \tilde{w}_p|^2 + \mathcal{O}(1),
\]

and

\[
\int_{\Omega_\delta} |\nabla \tilde{w}_p|^2 \leq \int_{A^+(\frac{R}{3}, \frac{R}{3})} |\nabla \tilde{w}_p|^2 \leq C \int_{B^+_{\frac{R}{3}}} (1 + r)^{-4} r^3 dr = \mathcal{O}(1).
\]

Then

\[
\int_{B^+_{\frac{R}{3}}} |\nabla \tilde{w}_p|^2 = \int_{Q^+_{\frac{R}{3}}} |\nabla \tilde{w}_p^0|^2 + \mathcal{O}(1).
\]

The latter integral can be calculated explicitly with the help of mathematical software. Firstly, we compute

\[
\int_0^{\frac{R}{3}} \int_0^{\frac{R}{3}} \int_0^{\frac{R}{3}} \int_0^{\frac{R}{3}} \left| \frac{\partial \tilde{w}_p}{\partial x_i} \right|^2 dx_1 dx_2 dx_3 dx_4 = \frac{\pi^2}{30} \left( \sum_{k > j} \sum_{j = 1}^3 \left( 3M^2_{ij} + M^2_{jk} \right) \right) |\ln \delta| + \mathcal{O}(1),
\]

for \(i = 1, 2, 3\). Similarly,

\[
\int_0^{\frac{R}{3}} \int_0^{\frac{R}{3}} \int_0^{\frac{R}{3}} \int_0^{\frac{R}{3}} \left| \frac{\partial \tilde{w}_p^0}{\partial x_4} \right|^2 dx_1 dx_2 dx_3 dx_4 = \frac{\pi^2}{30} \left( \sum_{i = 1}^3 3M^2_{ij} \right) |\ln \delta| + \mathcal{O}(1).
\]

Hence, by the definition of \(M_{ij}(p) = \frac{8\sqrt{3}}{|K|} h^{ij}(p)\):

\[
\int_{Q^+_{\frac{R}{3}}} |\nabla \tilde{w}_p^0|^2 = \frac{64\pi^2}{|K|} \|\pi(p)\|^2 |\ln \delta| + \mathcal{O}(1),
\]

being \(\|\pi(p)\|^2 = h^2_{12}(p) + h^2_{13}(p) + h^2_{23}(p)\).

\[\square\]

Proposition B.2. Let \(n \geq 5\). Let \(V_p\) a solution of (3.2), then there exists a nonnegative constant \(f_n\) depending only on \(n\) and \(\mathcal{D}_n\) such that

\[
\int_{\mathbb{R}^n} \left( -c_n \Delta V_p + \frac{n + 2}{n - 2} |K| U^\frac{4}{n-2} V_p \right) V_p = f_n \|\pi(p)\|^2.
\]

Proof. Let decompose \(V_p = w_p + \psi_p\) where \(w_p\) solves (3.22) and \(\psi_p\) solves (3.23).

For sake of convenience, let us define

\[
b(\tilde{x}, 0) = \frac{n \mathcal{D}_n}{(|\tilde{x}|^2 + \mathcal{D}_n^2 - 1)} w_p(\tilde{x}, 0) + \left| \frac{\partial w_p}{\partial x_n} \right|_{x_n = 0}.
\]
Let us evaluate separately $I_{w_p}$ and $I_{\psi_p}$.

$$I_{w_p} = \beta_n h^{ij}(p) \int_{\mathbb{R}^n_+} \frac{x_i x_j x_n}{(|x|^2 + (x_n + D_n)^2 - 1)^{n+1}} w_p \, dx$$

$$= \frac{\beta_n^2}{4n} \sum_{i,j=1}^{n-1} h^{ij}(p) h^{k\ell}(p) I_{ijk\ell}$$

Notice that by symmetry reasons and the fact that $h^{ii}(p) = 0$ for every $i = 1, \ldots, n - 1$, we can write

$$\frac{\beta_n^2}{4n} \sum_{i,j=1}^{n-1} h^{ij}(p) h^{k\ell}(p) I_{ijk\ell} = \frac{\beta_n^2}{n} \sum_{i,j=1}^{n-1} h^{ij}(p) h^{k\ell}(p) I_{ijk\ell}.$$  \hfill (B.3)

In view of (B.3), if $(i,j) \neq (k,\ell)$, there exists an index, let say $i$, such that $i \notin \{j,k,\ell\}$. In that case, it is easy to see that

$$h^{ij}(p) h^{k\ell}(p) I_{ijk\ell} = \sum_{i,j=1}^{n-1} h^{ij}(p)^2 \int_{\mathbb{R}^n_+} \frac{x_i x_j x_n (x_n - D_n)}{(|x|^2 + (x_n + D_n)^2 - 1)^{n+1}} \, dx$$

Consequently,

$$h^{ij}(p) h^{k\ell}(p) I_{ijk\ell} = \sum_{i,j=1}^{n-1} h^{ij}(p)^2 \int_{\mathbb{R}^n_+} \frac{x_i x_j x_n (x_n - D_n)}{(|x|^2 + (x_n + D_n)^2 - 1)^{n+1}} \, dx.$$  \hfill (B.4)

For every $i \neq j$, using polar coordinates in $\mathbb{R}^{n-3}$, we can see that

$$\int_{\mathbb{R}^n_+} \frac{x_i x_j x_n (x_n - D_n)}{(|x|^2 + (x_n + D_n)^2 - 1)^{n+1}} \, dx = \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i x_j x_n (x_n - D_n) \omega_{n-4} r^{n-4} dr$$

$$\times \int_0^{+\infty} \frac{\omega_{n-4} r^{n-4}}{(r^2 + x_i^2 + x_j^2 + (x_n + D_n)^2 - 1)^{n+1}} dx_i dx_j dx_n$$

$$= \omega_{n-4} \frac{\Gamma \left( \frac{a+3}{2} \right) \Gamma \left( \frac{a+5}{2} \right)}{2n!} \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_i x_j x_n (x_n - D_n) \frac{\omega_{n-4} r^{n-4}}{(r^2 + x_i^2 + x_j^2 + (x_n + D_n)^2 - 1)^{n+1}} dx_i dx_j dx_n$$

$$= \omega_{n-4} \frac{\Gamma \left( \frac{a+3}{2} \right) \Gamma \left( \frac{a+5}{2} \right)}{n!(n-1)(n+1)(n+3)} \int_0^{+\infty} \frac{x_n (x_n - D_n)}{((x_n + D_n)^2 - 1)^{n+1}}.$$

Substituting in (B.2):

$$I_{w_p} = \frac{\omega_{n-4} \pi \beta_n^2 \Gamma \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n+5}{2} \right)}{4n(n-1)(n+3)(n+1)!} f_1(n, D_n) \| \pi(p) \|^2.$$
Let us now study the term $I_{\psi_p}$. Multiplying (3.2) by $\psi_p$ and integrating by parts we obtain:

$$b(\bar{x}, 0) = \frac{\beta_n}{4n} \frac{\sum_{i,j=1}^{n-1} h^{ij}(p) x_i x_j}{(\bar{x}^2 + \mathcal{D}_n^2 - 1)^{\frac{n}{2}}}$$

and arguing as before:

$$\int_{\mathcal{R}_+^n} b(\bar{x}, 0) w_p(\bar{x}, 0) d\bar{x} = -\frac{\mathcal{D}_n \beta_n^2}{16n^2} \sum_{i,j=1}^{n-1} h^{ij}(p)^2 \int_{\mathcal{R}^{n-1}} \frac{x_i x_j^2}{(\bar{x}^2 + \mathcal{D}_n^2 - 1)^{n}} d\bar{x}.$$

Now, for $i \neq j$ fixed,

$$\int_{\mathcal{R}^{n-1}} \frac{x_i x_j^2}{(\bar{x}^2 + \mathcal{D}_n^2 - 1)^{n}} d\bar{x} = \frac{\omega_{n-4} \Gamma \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n+3}{2} \right)}{2(n-1)!} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \frac{\omega_{n-4} r^{n-4} dr}{(r^2 + x_i^2 + x_j^2 + \mathcal{D}_n^2 - 1)} dx_i dx_j$$

$$= \frac{\omega_{n-4} \pi \Gamma \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n+3}{2} \right)}{(n-1)!} \frac{\omega_{n-4} \pi \Gamma \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n+3}{2} \right) (\mathcal{D}_n^2 - 1)^{\frac{3-n}{2}}}{(n-3)(n-1)(n+1)!}.$$

Finally,

$$\int_{\mathcal{R}_+^n} b(\bar{x}, 0) \psi_p(\bar{x}) d\bar{x} = -\frac{\mathcal{D}_n \beta_n^2 \omega_{n-4} \pi \Gamma \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n+3}{2} \right) (\mathcal{D}_n^2 - 1)^{\frac{3-n}{2}}}{16n(n-1)(n-3)(n+1)!} ||\pi(p)||^2.$$

Finally we address the terms with $\psi_p$. 

Since \( \psi_p \) solves (3.23) we can write
\[
\psi_p = \frac{\beta_n}{4n} \sum_{i,j \neq j} h^{ij}(p) \psi_{ij}
\]
where \( \psi_{ij} \) solves
\[
\begin{cases}
- \Delta \psi_{ij} + \frac{n(n+2)}{2} (|\bar{x}|^2 + (x_n + D_n)^2 - 1)^2 \psi_{ij} = 0, & \text{in } \mathbb{R}^n_+ \\
\frac{\partial \psi_{ij}}{\partial n} - \frac{n D_n}{(|\bar{x}|^2 + D_n^2 - 1)^2} \psi_{ij} = \frac{x_i x_j}{(|\bar{x}|^2 + D_n^2 - 1)^2} & \text{on } \partial \mathbb{R}^n_+
\end{cases}
\]
It is not difficult to check that \( \psi_{ij} \) is odd in \( x_i \) and \( x_j \) and even in all the other variables \( x_\ell, \ell = 1, \ldots, n-1, \) and so
\[
\int_{\mathbb{R}^{n-1}} \frac{x_i x_j}{(|\bar{x}|^2 + D_n^2 - 1)^2} \psi_{\kappa}(\bar{x},0) d\bar{x} = 0 \text{ if } (i,j) \neq (\ell,\kappa). \tag{B.4}
\]
Moreover it holds that \( \psi_{ij} = \psi_{ji} \) and \( \psi_{ij} = \psi_{12}(\sigma_{ij} x) \), where \( \sigma_{ij} \) permutes the \( x_i \) and \( x_j \) variables, i.e.
\[
\sigma_{ij}(x_1, \ldots, x_i, \ldots, x_j, \ldots x_n) = (x_1, \ldots, x_j, \ldots, x_i, \ldots x_n).
\]
Multiplying by \( \psi_p \) in (3.23), integrating by parts and using (B.4) we immediately see that
\[
\begin{align*}
\int_{\mathbb{R}^n_+} |\nabla \psi_p|^2 + \int_{\mathbb{R}^n_+} &\frac{n(n+2)}{2} (|\bar{x}|^2 + (x_n + D_n)^2 - 1)^2 \psi_p^2 - \int_{\partial \mathbb{R}^n_+} \frac{n D_n}{|\bar{x}|^2 + (x_n + D_n)^2 - 1} \psi_p^2 \\
= \beta_n &\int_{\mathbb{R}^n_+} h^{ij}(p) x_i x_j \\
= \frac{\beta_n}{4n} &\sum_{i,j=1}^{n-1} h^{ij}(p) \sum_{\ell,\kappa=1}^{n} h^{\kappa \ell}(p) \int_{\mathbb{R}^{n-1}} \frac{x_i x_j}{(|\bar{x}|^2 + D_n^2 - 1)^2} \psi_{\kappa}(\bar{x},0) d\bar{x} \\
= \sum_{i,j=1}^{n-1} &h^{ij}(p)^2 \int_{\mathbb{R}^{n-1}} \frac{x_i x_j}{(|\bar{x}|^2 + D_n^2 - 1)^2} \psi_{ij}(\bar{x},0) d\bar{x} \\
= \sum_{i,j=1}^{n-1} &h^{ij}(p)^2 \left( \int_{\mathbb{R}^{n-1}} \frac{x_1 x_2}{(|\bar{x}|^2 + D_n^2 - 1)^2} \psi_{12}(\bar{x},0) d\bar{x} \right) \\
= f_2(n, D_n) ||\pi(p)||^2,
\end{align*}
\]
where \( f_2 \) only depends on \( n \) and \( D_n \) because the functions \( \psi_{ij} \) do not depend on the point \( p \).

Collecting all the previous estimates, we get a constant \( f_n \), which only depends on \( n \) and \( D_n \), such that
\[
\int_{\mathbb{R}^n_+} \left( -c_n \Delta V_p + \frac{n+2}{n-2} |K| U^{\frac{4}{n-2}} V_p \right) V_p = f_n ||\pi(p)||^2.
\]
Notice that \( f_n \) needs to be nonnegative because \( V_p \) satisfies Proposition 3.1-(iv). \qedhere

**Proof of Proposition 3.3.** We write \( W_p = \chi \left( (\psi_p^\delta)^{-1}(\xi) \right) W \) with
\[
W(\xi) := \frac{1}{\delta} U \left( \frac{(\psi_p^\delta)^{-1}(\xi)}{\delta} \right) + \delta \frac{1}{\delta^{n-2}} V_p \left( \frac{(\psi_p^\delta)^{-1}(\xi)}{\delta} \right),
\]
and also $\mathcal{W}_p = \mathcal{W} = \mathcal{U} + \delta \mathcal{V}$ with

$$\mathcal{U}(\xi) = \chi \left( \left( \psi_p^0 \right)^{-1}(\xi) \right) \mathcal{U}(\xi) \quad \text{and} \quad \mathcal{V}(\xi) = \chi \left( \left( \psi_p^0 \right)^{-1}(\xi) \right) \mathcal{V}(\xi).$$

We have

$$J_\varepsilon(\mathcal{W}) = \frac{C_0}{2} \int_M |\nabla g(\mathcal{U} + \delta \mathcal{V})|^2 \, d\mathcal{I}_1 + \frac{1}{2} \int_M S_g(\mathcal{U} + \delta \mathcal{V})^2 + (n - 1)\varepsilon \int_{\partial M} (\mathcal{U} + \delta \mathcal{V})^2 \, d\mathcal{I}_2 - (n - 2) \int_{\partial M} H \left[ \left( \left( \mathcal{U} + \delta \mathcal{V} \right)^{\frac{2(n-1)}{n-2}} \mathcal{U} + \delta \mathcal{V} \right) \left( \left( \mathcal{U} + \delta \mathcal{V} \right)^{\frac{4n}{n-2}} - \mathcal{U}^{\frac{4n}{n-2}} \right) \right] \, d\mathcal{I}_3 - (n - 2) \int_{\partial M} H \mathcal{U}^{\frac{2(n-1)}{n-2}} \, d\mathcal{I}_4 - \frac{n - 2}{2n} \int_M K \left[ \left( \left( \mathcal{U} + \delta \mathcal{V} \right)^{\frac{4n}{n-2}} - \mathcal{U}^{\frac{4n}{n-2}} \right) \right] - \frac{n - 2}{2n} \int_M K \mathcal{U}^{\frac{2n}{n-2}} \, d\mathcal{I}_5$$

Estimate of $I_2$ By (A.2) (with $\alpha = 0$ and $m = n - 2$) and (A.1), if $n \geq 5$

$$I_2 := \frac{1}{2} \delta^2 \int_{\mathbb{R}^n_+} S_g(\delta x) (U(x)\chi(\delta x) + \delta V_p(x)\chi(\delta x))^2 |g(\delta x)|^{\frac{1}{2}} \, dx$$

$$= \frac{1}{2} \delta^2 S_g(\delta x) \int_{\mathbb{R}^n_+} U^2(x) \, dx + \begin{cases} \mathcal{O}(\delta^3) & \text{if } n \geq 6 \\ \mathcal{O}(\delta^3 |\ln \delta|) & \text{if } n = 5 \end{cases}$$

$$= \frac{1}{2} \delta^2 S_g(\delta x) \int_{\mathbb{R}^n_+} \frac{\alpha_n^2}{|\nabla g(\delta x)|^2} \, dx + \begin{cases} \mathcal{O}(\delta^3) & \text{if } n \geq 6 \\ \mathcal{O}(\delta^3 |\ln \delta|) & \text{if } n = 5 \end{cases}$$

$$= \frac{1}{2} \delta^2 \alpha_2^2 \omega_{n-1} \frac{2(n-2)}{n-1} \int_{\mathbb{R}^n} S_g(\delta x) \frac{\alpha_n^2}{|\nabla g(\delta x)|^2} \varphi^{n-3} + \begin{cases} \mathcal{O}(\delta^3) & \text{if } n \geq 6 \\ \mathcal{O}(\delta^3 |\ln \delta|) & \text{if } n = 5 \end{cases}$$

and if $n = 4$

$$I_2 = \frac{\alpha_3^2}{2} \delta^2 S_g(\delta x) \int_{\mathbb{R}^n} \frac{1}{(\delta x)^2 + (\delta x + \mathcal{D}_4(\delta x))^2 - 1} \, dx + \mathcal{O}(\delta^2)$$

$$= \frac{\alpha_3^2}{2} \delta^2 S_g(\delta x) \int_{\mathbb{R}^n} \frac{1}{(\delta x + \mathcal{D}_4(\delta x))^2} \, dx + \mathcal{O}(\delta^2)$$

$$= - \frac{2\alpha_3^2 \omega_3}{3} \int_{\mathbb{R}^n} I_3 \delta^2 \ln \delta + \mathcal{O}(\delta^2)$$

Estimate of $I_3$ By (A.3) (with $\alpha = 0$ and $m = n - 2$) and (A.1), if $n \geq 4$ that

$$I_3 := (n - 1)\varepsilon \int_{\mathbb{R}^n} (U(\delta \tilde{x}, 0)\chi(\delta \tilde{x}, 0) + \delta V_p(\delta \tilde{x}, 0)\chi(\delta \tilde{x}, 0))^2 |g(\delta \tilde{x}, 0)|^{\frac{1}{2}} \, d\tilde{x}$$

$$= (n - 1)\varepsilon \int_{\mathbb{R}^n} U^2(\delta \tilde{x}, 0) \, d\tilde{x} + \begin{cases} \mathcal{O}(\varepsilon^2 |\ln \delta|) & \text{if } n \geq 5 \\ \mathcal{O}(\varepsilon^2 |\ln \delta|) & \text{if } n = 4 \end{cases}$$

$$= (n - 1)\varepsilon \int_{\mathbb{R}^n} U^2(\delta \tilde{x}, 0) \, d\tilde{x} + \begin{cases} \mathcal{O}(\varepsilon^2 |\ln \delta|) & \text{if } n \geq 5 \\ \mathcal{O}(\varepsilon^2 |\ln \delta|) & \text{if } n = 4 \end{cases}$$

$$= \varepsilon \int_{\mathbb{R}^n} \frac{1}{(\delta \tilde{x})^2 + (\delta \tilde{x} + \mathcal{D}_n(\delta \tilde{x}))^2 - 1} \, d\tilde{x} + \begin{cases} \mathcal{O}(\varepsilon^2 |\ln \delta|) & \text{if } n \geq 5 \\ \mathcal{O}(\varepsilon^2 |\ln \delta|) & \text{if } n = 4 \end{cases}$$

$$:= \varepsilon \int_{\mathbb{R}^n} \frac{1}{(\delta \tilde{x})^2 + (\delta \tilde{x} + \mathcal{D}_n(\delta \tilde{x}))^2 - 1} \, d\tilde{x} + \begin{cases} \mathcal{O}(\varepsilon^2 |\ln \delta|) & \text{if } n \geq 5 \\ \mathcal{O}(\varepsilon^2 |\ln \delta|) & \text{if } n = 4 \end{cases}$$
Estimate of $I_5$ By (A.3) (with $\alpha = 2$ and $m = n - 1$) and Lemma A.3, if $n \geq 4$

$$I_5 := -(n - 2) \int_{\mathbb{R}^{n-1}} H \left( U(\bar{x}, 0) \chi(\delta \bar{x}, 0) \right)^2 |g(\delta \bar{x}, 0)|^{\frac{n}{2}} \, d\bar{x}$$

$$= -(n - 2) H \int_{\mathbb{R}^n+} U^{2^*}(\bar{x}, 0) \, d\bar{x} + \frac{n - 2}{6} \delta^2 R_{ij}(p) H \int_{\mathbb{R}^{n-1}} U^{2^*}(\bar{x}, 0) \bar{x}_i \bar{x}_j \, d\bar{x} + \mathcal{O}(\delta^3)$$

$$= -(n - 2) H \int_{\mathbb{R}^n+} U^{2^*}(\bar{x}, 0) \, d\bar{x} + \frac{\delta^2 \alpha_n^2 (n - 2)}{6(n - 1)} \int_{\mathbb{R}^{n-1}} \frac{|\bar{x}|^2}{|K|^{\frac{n}{2}}} \, d\bar{x} + \mathcal{O}(\delta^3)$$

Estimate of $I_7$ By (A.2) (with $\alpha = 2$ and $m = n$), (A.4) (with $\alpha = 0$ and $m = n$), Lemma A.3 and (A.1), if $n \geq 4$

$$I_7 := -\frac{n - 2}{2n} \int_{\mathbb{R}^n_+} K \left( U(x) \chi(\delta x) \right)^2 |g(\delta x)|^{\frac{n}{2}} \, dx$$

$$= -\frac{n - 2}{2n} |K| \int_{\mathbb{R}^n_+} U^{2^*}(x) \, dx - \frac{n - 2}{4n} \left( \|\pi(p)\|^2 + \text{Ric}(p) \right) \delta^2 |K| \int_{\mathbb{R}^n_+} U^{2^*} x_n^2 \, dx$$

$$= -\frac{n - 2}{2n} |K| \int_{\mathbb{R}^n_+} U^{2^*}(x) \, dx$$

$$- \frac{\delta^2 \alpha_n^2 (n - 2)}{4n} \left( \|\pi(p)\|^2 + \text{Ric}(p) \right) \frac{|K|}{|K|^\frac{n}{2}} \int_{\mathbb{R}^n_+} x_n^2 \, dx$$

$$= -\frac{n - 2}{2n} \delta^2 \alpha_n^2 \left( \|\pi(p)\|^2 + \text{Ric}(p) \right) \frac{|K|}{|K|^\frac{n}{2}} \int_{\mathbb{R}^n_+} \frac{|\bar{x}|^2}{|\bar{x}|^2 + (x_n + \mathcal{D}^2)^2} \, dx + \mathcal{O}(\delta^3)$$

$$= -\frac{n - 2}{2n} \delta^2 \alpha_n^2 \left( \|\pi(p)\|^2 + \text{Ric}(p) \right) \frac{|K|}{|K|^\frac{n}{2}} \int_{\mathbb{R}^n_+} \frac{|\bar{x}|^2}{|\bar{x}|^2 + (x_n + \mathcal{D}^2)^2} \, dx + \mathcal{O}(\delta^3)$$

Estimate of $I_4$ and $I_6$ By Lemma A.3-(i), if $n \geq 4$

$$I_4 = -(n - 2) \int_{\mathbb{R}^n_+} H \left( (U + \delta V_p)^+ \right)^{\frac{2^{(n-1)}}{n^2}} \left( \bar{x}, 0 \right) |g(\delta \bar{x}, 0)|^{\frac{1}{2}} \, d\bar{x}$$

$$= -2(n - 1) \delta H \int_{\mathbb{R}^n_+} U^{\frac{n-7}{2}} V_p \, d\bar{x} - \frac{n(n - 1)}{n - 2} \delta^2 H \int_{\mathbb{R}^n_+} U^{\frac{n-5}{2}} V_p^2 \, d\bar{x} + \mathcal{O}(\delta^3)$$

and similarly

$$I_6 = |K| \delta \int_{\mathbb{R}^n_+} U^{\frac{n+4}{2}} V_p \, d\bar{x} + \frac{n + 2}{2(n - 2)} \delta^2 |K| \int_{\mathbb{R}^n_+} U^{\frac{n+2}{2}} V_p^2 + \mathcal{O}(\delta^3).$$
Estimate of $I_1$ First we have
\begin{align*}
I_1 := \frac{c_n}{2} \int_M |\nabla g \delta_\beta|^2 dv_g + c_n \delta \int_M \nabla g \delta_\beta \nabla g \delta_\beta dv_g + \frac{c_n \delta^2}{2} \int_M |\nabla g \delta_\beta|^2 dv_g \\
:= I_{1_1}^1 + I_{1_2}^1 + I_{1_3}^1
\end{align*}
and we separately estimate the terms $I_{1_i}^1$ with $i = 1, 2, 3$. Set $B_\delta := \{ x \in \mathbb{R}^n_+ : |\delta x| \leq R \}$.

Estimate of $I_{1_1}^1$ By Lemma A.3-(ii)-(iii) we get
\begin{align*}
I_{1_1}^1 &= \frac{c_n}{2} \int_{\mathbb{R}^n_+} g^{ab} (\delta x) \frac{\partial}{\partial x_a} (U(x) \chi (\delta x)) \frac{\partial}{\partial x_b} (U(x) \chi (\delta x)) |g(\delta x)|^{\frac{1}{2}} dx \\
&= c_n \int_{B_\delta} \left[ \frac{|\nabla U|^2}{2} + \left( \delta_h x_n + \frac{\delta}{6} R_{ijk} x_k x_i + \delta^2 \frac{\partial h_j}{\partial x_k} x_k x_i + \frac{\delta^3}{6} (R_{ijmn} + 3 h_{ik} h_{kj}) x_n^2 \right) \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \right] \\
&\quad \times \left( 1 - \frac{\delta^2}{2} \left( \|\nabla U\|^2 + \text{Ric}_\nu \right) x_n^2 - \frac{\delta^2}{6} \nabla_x \delta x_m \right) dx + O(\delta^3) \\
&= \int_{B_\delta} \left( c_n \frac{2}{\delta} |\nabla U|^2 + \frac{\delta^2 c_n}{2} \left( \|\nabla U\|^2 + \text{Ric}_\nu \right) x_n^2 \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \right) \\
&\quad \times \left( 1 - \frac{\delta^2}{2} \left( \|\nabla U\|^2 + \text{Ric}_\nu \right) x_n^2 - \frac{\delta^2}{6} \nabla_x \delta x_m \right) dx + O(\delta^3).
\end{align*}
Moreover, by (A.4) (with $\alpha = 0$ or $\alpha = 2$ and $m = n - 1$ and with $\alpha = 0$ and $m = n$ secondly), (A.2) (with $\alpha = 2$ and $m = n - 1$ first and with $\alpha = 2$ and $m = n$ secondly) and (A.1), if $n \geq 5$
\begin{align*}
I_{1_1}^1 &= \frac{c_n}{2} \int_{\mathbb{R}^n_+} |\nabla U|^2 - \frac{c_n}{4} \delta^2 (\|\nabla U\|^2 + \text{Ric}_\nu (p)) \int_{\mathbb{R}^n_+} x_n^2 \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \\
&\quad + \frac{c_n^2}{2} \frac{(n - 2)^2}{n - 1} \delta^2 \left( \|\nabla U\|^2 + \text{Ric}_\nu (p) \right) \int_{\mathbb{R}^n_+} x_n^2 \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} + O(\delta^3) \\
&= \frac{c_n}{2} \int_{\mathbb{R}^n_+} |\nabla U|^2 - \frac{c_n}{4} \delta^2 \left( \|\nabla U\|^2 + \text{Ric}_\nu (p) \right) \int_{\mathbb{R}^n_+} x_n^2 \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \\
&\quad - \frac{c_n^2}{2} \frac{(n - 2)^2}{n - 1} \delta^2 \left( \|\nabla U\|^2 + \text{Ric}_\nu (p) \right) \int_{\mathbb{R}^n_+} x_n^2 \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} + O(\delta^3)
\end{align*}
and if \( n = 4 \)

\[
I_1^1 = \frac{c_4}{2} \int_{\mathbb{R}^4_+} |\nabla U|^2 - c_4 \alpha_4^2 \left( (\|\pi(p)\|^2 + \text{Ric}_\nu(p)) \right) \delta^2 \int_{B_3} \frac{x_4^2}{(|x|^2 + (x_4 + \mathcal{D}_4)^2 - 1)^3} \, dx \\
- \frac{1}{9} c_4 \alpha_4^2 \frac{R_{\ell(p)}(p)}{|K|} \delta^2 \int_{B_3} \frac{\bar{x}^2}{(|x|^2 + (x_4 + \mathcal{D}_4)^2 - 1)^3} \, dx \\
+ \frac{2}{3} c_4 \alpha_4^2 \left( 3\|\pi(p)\|^2 + \text{Ric}_\nu(p) \right) \delta^2 \int_{B_3} \frac{x_4^2 |\bar{x}|^2}{(|x|^2 + (x_4 + \mathcal{D}_4)^2 - 1)^4} \, dx + \mathcal{O}(\delta^2) \\
= \frac{c_4}{2} \int_{\mathbb{R}^4_+} |\nabla U|^2 + \frac{1}{3} c_4 \omega_3 \alpha_4^2 I_3 \delta^2 \int_{\mathbb{R}^4_+} (\|\pi(p)\|^2 + \text{Ric}_\nu(p)) \delta^2 \ln \delta + \frac{1}{9} c_4 \omega_3 \alpha_4^2 I_3 \frac{R_{\ell(p)}(p)}{|K|} \delta^2 \ln \delta \\
- \frac{1}{54} c_4 \omega_3 \alpha_4^2 \frac{(3\|\pi(p)\|^2 + \text{Ric}_\nu(p))}{|K|} I_3 \delta^2 \ln \delta + \mathcal{O}(\delta^2).
\]

\text{Estimate of } I_1^1 \text{ and } I_1^3 \text{ if } n \geq 5

We have

\[
I_1^3 = \delta c_n \int_{\mathbb{R}^n_+} g^{0\delta}(\delta x) \frac{\partial}{\partial x_\alpha} (U(x) \chi(\delta x)) \frac{\partial}{\partial x_\beta} (V_p(x) \chi(\delta x)) |g(\delta x)| \frac{\partial}{\partial x_\gamma} dx
\]

\[
= \delta c_n \int_{\mathbb{R}^n_+} \nabla U \nabla V_p \, dx + \delta^2 2c_n h^{ij}(p) \int_{\mathbb{R}^n_+} x_n \frac{\partial U}{\partial x_i} \frac{\partial V_p}{\partial x_j} \, dx + \mathcal{O}(\delta^3)
\]

\[
= -\frac{n + 2}{n - 2} |K| \int_{\mathbb{R}^n_+} U^{n+2} V_p \, dx + c_n \frac{n}{2} H \int_{\partial \mathbb{R}^n_+} U^{n+2} V_p \\
- c_n \delta^2 \int_{\mathbb{R}^n_+} \left| \nabla V_p \right|^2 \, dx + c_n \frac{n}{2} H \int_{\partial \mathbb{R}^n_+} U^{n+2} V_p \, dx - \frac{n + 2}{n - 2} |K| \delta^2 \int_{\mathbb{R}^n_+} U^{n+2} V_p \, dx + \mathcal{O}(\delta^3)
\]

since

\[
c_n \int_{\mathbb{R}^n_+} \nabla U \nabla V_p = -c_n \int_{\mathbb{R}^n_+} U \Delta V_p + c_n \int_{\mathbb{R}^n_+} U(\bar{x}, 0) \frac{\partial V_p}{\partial \nu} \\
= -\frac{n + 2}{n - 2} |K| \int_{\mathbb{R}^n_+} U^{n+2} V_p \, dx + c_n \frac{n}{2} H \int_{\partial \mathbb{R}^n_+} U^{n+2} (\bar{x}, 0)V_p \, d\bar{x} \\
+ \frac{8}{n - 2} \int_{\mathbb{R}^n_+} h^{ij}(p) \frac{\partial^2 U}{\partial x_i \partial x_j} x_n U \]

and

\[
2c_n h^{ij}(p) \int_{\mathbb{R}^n_+} x_n \frac{\partial U}{\partial x_i} \frac{\partial V_p}{\partial x_j} \, dx = -2c_n h^{ij}(p) \int_{\mathbb{R}^n_+} x_n \frac{\partial^2 U}{\partial x_i \partial x_j} V_p \, dx
\]

\[
= - \int_{\mathbb{R}^n_+} \left( -c_n \Delta V_p + \frac{n + 2}{n - 2} |K| U^{n+2} \right) V_p \, dx
\]

\[
= -c_n \int_{\mathbb{R}^n_+} |\nabla V_p|^2 + c_n \int_{\partial \mathbb{R}^n_+} V_p \frac{\partial V_p}{\partial \nu} - \frac{n + 2}{n - 2} |K| \int_{\mathbb{R}^n_+} U^{n+2} V_p^2 \\
= -c_n \int_{\mathbb{R}^n_+} |\nabla V_p|^2 + c_n \frac{n}{2} H \int_{\partial \mathbb{R}^n_+} U^{n+2} V_p^2 - \frac{n + 2}{n - 2} |K| \int_{\mathbb{R}^n_+} U^{n+2} V_p^2
\]

and also

\[
I_1^3 := \frac{c_n}{2} \delta^2 \int_{\mathbb{R}^n_+} |\nabla V_p|^2 + \mathcal{O}(\delta^3).
\]
Estimate of $I_1^2$ and $I_3^2$ if $n = 4$
Let $V_p = \bar{w}_p + \zeta_p + \psi_p$ as in (3.12) and set $w_p = \bar{w}_p + \zeta_p$ so that $w_p$ solves the problem
\[
\begin{align*}
&\begin{cases}
-6\Delta w_p = E_p(x) & \text{in } \mathbb{R}^4_+ \\
\frac{\partial w_p}{\partial \nu} = 2HU w_p, & \text{on } \partial \mathbb{R}^4_+
\end{cases}
\end{align*}
\]
(B.5)
Then
\[
I_1^2 = \delta c_4 \int_{\mathbb{R}^4_+} g^{ab}(\delta x) \frac{\partial}{\partial x_a} (U(x) \chi(\delta x)) \frac{\partial}{\partial x_b} (V_p(x) \chi(\delta x)) |g(\delta x)|^{\frac{1}{2}} dx
\]
\[
= -\delta \int_{\mathbb{R}^4_+} U^3 V_p dx + 2c_4 \delta H \int_{\partial \mathbb{R}^4_+} U^2 V_p
\]
\[
+ \delta^2 12h^{ij}(p) \int_{B^+} x_4 \frac{\partial U}{\partial x_i} \frac{\partial V_p}{\partial x_j} dx + \mathcal{O}(\delta^2)
\]
(B.6)
Here we have used the fact that
\[
\int_{B^+} x_4 \frac{\partial U}{\partial x_i} \frac{\partial \psi_p}{\partial x_j} dx = \mathcal{O}(1).
\]
Let us study the last integral term of (B.6). Integrating by parts in $x_j$ and using the equation (B.5):
\[
12h^{ij}(p) \int_{B^+} x_4 \frac{\partial U}{\partial x_i} \frac{\partial w_p}{\partial x_j} dx = 12h^{ij}(p) \int_{\partial + B^+} x_4 \frac{\partial U}{\partial x_i} \frac{x_j}{|x|} w_p - \int_{B^+} E_p w_p
\]
\[
= 12h^{ij}(p) \int_{\partial + B^+} x_4 \frac{\partial U}{\partial x_i} \frac{x_j}{|x|} w_p - 6 \int_{B^+} |\nabla w_p|^2
\]
\[
+ 12 \int_{\partial + B^+} HU w_p^2 + 6 \int_{\partial + B^+} \nabla w_p \cdot \frac{x}{|x|} w_p
\]
\[
= -6 \int_{B^+} |\nabla w_p|^2 + \mathcal{O}(1),
\]
(B.7)
since
\[
\int_{\partial + B^+} HU w_p^2 = \mathcal{O}(1),
\]
\[
\int_{\partial + B^+} x_4 \frac{\partial U}{\partial x_i} \frac{x_j}{|x|} w_p \leq R \left( 1 + \frac{1}{\delta} \right)^{-3} \left( 1 + \frac{1}{\delta} \right)^{-1} \omega_{n-1} \frac{R^3}{2 \delta^3} = \mathcal{O}(1),
\]
\[
\int_{\partial + B^+} \nabla w_p \cdot \frac{x}{|x|} w_p \leq \left( 1 + \frac{R}{\delta} \right)^{-2} \left( 1 + \frac{R}{\delta} \right)^{-1} \omega_{n-1} \frac{R^3}{2 \delta^3} = \mathcal{O}(1).
\]
By (B.6) and (B.7),
\[
I_1^2 = -6\delta^2 \int_{B^+} |\nabla w_p|^2 + \mathcal{O}(\delta^2).
\]
(B.8)
Analogously,
\[ I_1^3 := \frac{c_1}{2} \delta^2 \int_{B_{\frac{1}{4}}} |\nabla V_p|^2 + O(\delta^2) = 3 \delta^2 \int_{B_{\frac{1}{4}}} |\nabla w_p|^2 + O(\delta^2). \] (B.9)

Finally, combining (B.8) and (B.9), we obtain
\[ I_1^1 + I_1^3 = -3 \delta^2 \int_{B_{\frac{1}{4}}} |\nabla w_p|^2 + O(\delta^2). \] (B.10)

Now, using the fact that \( w_p = \bar{w}_p + \zeta_p \) and the decay estimate (3.17) we get
\[
\int_{B_{\frac{1}{4}}^+} |\nabla w_p|^2 = \int_{B_{\frac{1}{4}}^+} |\nabla \bar{w}_p|^2 + \int_{B_{\frac{1}{4}}^+} |\nabla \zeta_p|^2 + \int_{B_{\frac{1}{4}}^+} \nabla \bar{w}_p \nabla \zeta_p
\]
\[
= \int_{B_{\frac{1}{4}}^+} |\nabla \bar{w}_p|^2 + \int_{B_{\frac{1}{4}}^+} |\nabla \zeta_p|^2 + \int_{\partial B_{\frac{1}{4}}^+} \nabla \zeta_p \cdot \frac{x}{|x|} \bar{w}_p
\]
\[
+ \int_{\partial B_{\frac{1}{4}}^+} \left( 2HU \zeta_p + \left(2HU \bar{w}_p - \frac{\partial \bar{w}_p}{\partial \nu} \right) \right) \bar{w}_p
\]
\[
= \int_{B_{\frac{1}{4}}^+} |\nabla \bar{w}_p|^2 + O(1)
\]

since again, by using the decay properties, we get
\[
\int_{\partial B_{\frac{1}{4}}^+} \left( 2HU \zeta_p + \left(2HU \bar{w}_p - \frac{\partial \bar{w}_p}{\partial \nu} \right) \right) \bar{w}_p = O(1)
\]

and
\[
\int_{\partial B_{\frac{1}{4}}^+} \nabla \zeta_p \cdot \frac{x}{|x|} \bar{w}_p = O(1)
\]

and by using the problem solved by \( \zeta_p \), i.e. (3.14), we get
\[
0 = \int_{B_{\frac{1}{4}}^+} |\nabla \zeta_p|^2 - \int_{\partial B_{\frac{1}{4}}^+} \nabla \zeta_p \cdot \frac{x}{|x|} \zeta_p + \int_{B_{\frac{1}{4}}^+} \left( 2HU \zeta_p + \left(2HU \bar{w}_p - \frac{\partial \bar{w}_p}{\partial \nu} \right) \right) \zeta_p
\]
\[
= \int_{B_{\frac{1}{4}}^+} |\nabla \zeta_p|^2 + O(1)
\]

from which it follows that
\[
\int_{B_{\frac{1}{4}}^+} |\nabla \zeta_p|^2 = O(1).
\]

By Proposition B.1 we get
\[
\int_{B_{\frac{1}{4}}^+} |\nabla \bar{w}_p|^2 = \frac{64 \pi^2}{|K|} \|\pi(p)\|^2 \ln \delta + O(1)
\]

Then (B.10) reduces to
\[
I_1^2 + I_1^3 = -3 \delta^2 \int_{Q_{\frac{1}{4}}} |\nabla \bar{w}_p|^2 + O(\delta^2) = -3 \delta^2 \int_{Q_{\frac{1}{4}}} |\nabla w_0|^2 + O(\delta^2) = -3 \frac{64 \pi^2}{|K|} \|\pi(p)\|^2 \delta^2 \ln \delta + O(\delta^2).
\]

Conclusion.

We collect all the previous estimates and we take into account that
- the terms of order \( \delta \) cancel because of Proposition 3.1-(iii)
- the higher order terms which contain Ric$\nu(p)$ and $R_{\ell\ell}(p)$ (di order $\delta^2$ if $n \geq 5$ and $\delta^2 |\ln \delta|$ if $n = 4$) cancel, because by Lemma A.2 and the fact that $\mathcal{S}_\nu(p) = 2\text{Ric}_\nu(p) + R_{\ell\ell}(p) + \|\pi(p)\|^2$

$$\delta^2 \alpha_n^2 \omega_{n-1} I_{n-1}^n \frac{n-2}{n-1} \text{Ric}_\nu(p) \left( 2 \rho_{n-3} - (n-3)(n-1) \frac{\varphi}{n+1} \right) = 0,$$

and

$$\delta^2 \alpha_n^2 \omega_{n-1} I_{n-1}^n \frac{n-2}{3(n-1)} \text{Ric}_\nu(p) \left( - (n-4) \varphi_{n-3} - (n-3) \frac{\varphi}{n+1} + \frac{\mathcal{D}_n}{(\mathcal{D}_n - 1)^{n/2}} \right) = 0.$$

Finally, we have if $n = 4$

$$J_\varepsilon(W) = \mathcal{E} - \delta^2 |\ln \delta| \left( \frac{192\pi^2}{\mathcal{K}} + \alpha_n^4 \omega_3 I_3^4 \left( \frac{1}{\mathcal{K}} \right) \right) \|\pi(p)\|^2 + \varepsilon \delta c_4 + O(\delta^2).$$

and if $n \geq 5$

$$J_\varepsilon(W) = \mathcal{E} - \delta^2 \left( \frac{1}{2} f_n + f_1^n \right) \|\pi(p)\|^2 + \varepsilon \delta c_n + \begin{cases} O(\delta^3) & \text{if } n \geq 6 \\ O(\delta^3 |\ln \delta|) & \text{if } n = 5 \end{cases}$$

because the higher order terms which contain $\|\pi(p)\|^2$ reduces to

$$\delta^2 \alpha_n^2 \frac{n-2}{n-1} \omega_{n-1} I_{n-1}^n \left( \frac{\varphi}{n+1} - (n-1)(n-3) \frac{\varphi}{n+1} - (n-1)(n-3) \frac{\varphi}{n+1} + 3(n-3) \frac{\varphi}{n+1} \right)$$

$$= - \delta^2 \alpha_n^2 \frac{n-2}{n-1} \omega_{n-1} I_{n-1}^n \left( \frac{1}{\mathcal{K}} \right) \left( 4(n-3) \frac{\varphi}{n+1} + \frac{\varphi}{n+1} \right) \|\pi(p)\|^2$$

and, by Proposition B.2,

$$\frac{1}{2} \int_{\mathbb{R}_+^n} \left( - c_n \Delta V + \frac{n+2}{n-2} \|\mathcal{K} U \|^2 \right) V_p = \frac{1}{2} f_n \|\pi(p)\|^2$$

Here the energy of the bubble $\mathcal{E}$ is constant and is computed in the remark below. \hfill \Box

**Remark B.3.** The energy of the bubble is given by

$$\mathcal{E} = \frac{c_n}{2} \int_{\mathbb{R}_+^n} |\nabla U|^2 - \frac{n-2}{2n} K \int_{\mathbb{R}_+^n} U^{2^*} - (n-2) H \int_{\partial \mathbb{R}_+^n} U^{2^*}$$

where $c_n := \frac{4(n-1)}{n-2}$ and

$$U(\tilde{x}, x_n) := \frac{\alpha_n}{\left| \mathcal{K} \right|^{n/4}} \frac{1}{\left( |\tilde{x}|^2 + (x_n + \mathcal{D}_n)^2 - 1 \right)^{n/4}}$$

and $\alpha_n := (4n(n-1))^{n-2}$ and $\mathcal{D}_n := \sqrt{n(n-1)} \frac{H}{\sqrt{|\mathcal{K}|}}$.

We recall that $U$ satisfies (1.5). Hence

$$\frac{c_n}{2} \int_{\mathbb{R}_+^n} |\nabla U|^2 = \frac{c_n(n-2)}{4} H \int_{\partial \mathbb{R}_+^n} U^{2^*} - \frac{1}{2} \|\mathcal{K} \| \int_{\mathbb{R}_+^n} U^{2^*}$$
Then
\[
\mathcal{E} = -\frac{1}{n} |K| \int_{\mathbb{R}^n_+} U^{2^\ast} + H \int_{\partial \mathbb{R}^n_+} U^{2^\ast} \\
= -\frac{1}{n} |K| \frac{\alpha_n^2}{|K|^\frac{n}{2}} \int_{\mathbb{R}^n_+} \frac{1}{(|\tilde{x}|^2 + (x_n + D_n)^2 - 1)^n} \, d\tilde{x} \, dx_n \\
+ H \frac{\alpha_n^2}{|K|^\frac{n}{2}} \int_{\partial \mathbb{R}^n_+} \frac{1}{(|\tilde{x}|^2 + D_n^2 - 1)^{n-1}} \, d\tilde{x}.
\]

Now, by using (A.3) with \( \alpha = 0 \) and \( m = n - 1 \) we get for \( n \geq 4 \)
\[
\int_{\partial \mathbb{R}^n_+} \frac{1}{(|\tilde{x}|^2 + D_n^2 - 1)^{n-1}} \, d\tilde{x} = \omega_{n-1}^{-1} \frac{n - 3}{n - 1} \frac{I_{n-1}^n}{(D_n^2 - 1)^{\frac{n}{2}}}. \]

Instead, by using (A.2) with \( \alpha = 0 \) and \( m = n \) we get
\[
\int_{\mathbb{R}^n_+} \frac{1}{(|\tilde{x}|^2 + (x_n + D_n)^2 - 1)^n} \, d\tilde{x} \, dx_n = \omega_{n-1}^{-1} \frac{n - 3}{2(n - 1)} \frac{I_{n-1}^n}{|\psi|^\frac{n}{2}}.
\]

Collecting all the previous terms, by Lemma A.2
\[
\mathcal{E} = \frac{\alpha_n^2 (n - 3)}{2n(n - 1)} \omega_{n-1}^{-1} I_{n-1}^n \frac{\varphi_{\frac{n}{2}}}{|K|^\frac{n}{2}} + \frac{\alpha_n^2 (n - 3)}{n - 1} \omega_{n-1}^{-1} I_{n-1}^n \frac{H}{|K|^\frac{n}{2}} (D_n^2 - 1)^{\frac{n}{2}} \\
= \alpha_n^2 \omega_{n-1}^{-1} I_{n-1}^n \frac{n - 3}{n - 1} \frac{1}{|K|^\frac{n}{2}} \left[ -\frac{\alpha_n^2 - 2^\ast}{2n} \frac{\varphi_{\frac{n}{2}}}{|K|^\frac{n}{2}} + \frac{H}{|K|^\frac{n}{2}} (D_n^2 - 1)^{\frac{n}{2}} \right] \\
= \alpha_n^2 \omega_{n-1}^{-1} I_{n-1}^n \frac{n - 3}{(n - 1)\sqrt{n(n - 1)}} \frac{1}{|K|^\frac{n}{2}} \left[ -(n - 1)\varphi_{\frac{n}{2}} + \frac{D_n}{(D_n^2 - 1)^{\frac{n}{2}}} \right] := a_n.
\]

**APPENDIX C. PROOF OF LEMMA 4.2**

In the following we use the following notation
\[
W_j(\xi) := \frac{1}{\delta_j^{\frac{n}{2}}} U \left( \frac{\psi_p^\delta (-1)(\xi) - \eta(\varepsilon)\tau_j}{\delta_j} \right) + \delta_j \frac{1}{\delta_j^{\frac{n}{2}}} V_p \left( \frac{\psi_p^\delta (-1)(\xi) - \eta(\varepsilon)\tau_j}{\delta_j} \right) := U_j^\delta + V_j^\delta,
\]
and
\[
U_j(\xi) = \chi \left( \frac{\psi_p^\delta (-1)(\xi)}{\delta_j} \right) U_j(\xi), \quad V_j(\xi) = \chi \left( \frac{\psi_p^\delta (-1)(\xi)}{\delta_j} \right) V_j(\xi).
\]

**Proof.** Let
\[
\gamma_M := i_M^\ast (K g(W)) \quad \text{and} \quad \gamma_{\partial M} := i_{\partial M}^\ast \left( \frac{n - 2}{2} (H f(W) - \varepsilon W) \right).
\]

By using the equations that \( \gamma_M \) and \( \gamma_{\partial M} \) satisfy (see (4.1), (4.2)) we get
\[
\|\mathcal{E}\|^2 = c_n \int_M |\nabla_g (W - \gamma_M - \gamma_{\partial M})|^2 \, d\nu_g + \int_M S_g (W - \gamma_M - \gamma_{\partial M})^2 \, d\nu_g
\]
\[
= -c_n \int_M [\Delta_g (W - \gamma_M - \gamma_{\partial M}) (W - \gamma_M - \gamma_{\partial M})] \, d\nu_g + \int_M S_g (W - \gamma_M - \gamma_{\partial M})^2 \, d\nu_g
\]
\[
+ c_n \int_{\partial M} \frac{\partial}{\partial \nu} (W - \gamma_M - \gamma_{\partial M}) (W - \gamma_M - \gamma_{\partial M}) \, d\sigma_g
\]
\[
= \sum_{j=1}^k \left[ -c_n \Delta_g W_j + S_g W_j - K g(W_j) \right] \mathcal{E} \, d\nu_g
\]
\[
+ c_n \int_{\partial M} \left( \frac{\partial}{\partial \nu} W_j - \frac{n-2}{2} H f(W_j) + \frac{n-2}{2} \epsilon W_j \right) \mathcal{E} \, d\nu_g
\]
\[
+ \int_M K \left( \sum_{j=1}^k g(W_j) - g \left( \sum_{j=1}^k W_j \right) \right) \mathcal{E} \, d\nu_g
\]
\[
+ \frac{n-2}{2} c_n \int_{\partial M} H \left( \sum_{j=1}^k f(W_j) - f \left( \sum_{j=1}^k W_j \right) \right) \mathcal{E} \, d\nu_g
\]

Let us estimate \((I)\), which is the sum of the contribution of each peak. We estimate each term in the sum and for the sake of simplicity, we replace \(W_j\) by \(W\). Each term looks like

\[
\int_M \left( -c_n \Delta_g W - K U^{\frac{n+2}{n}} \right) \mathcal{E}
\]

\[
+ \int_{\partial M} \left( c_n \frac{\partial U}{\partial \nu} - 2(n-1) H U^{\frac{n}{n-2}} \right) \mathcal{E}
\]

\[
+ 2(n-1) \int_{\partial M} \left( H (f(U) - f(U + \delta V)) + \frac{2 \delta}{n-2} \frac{\partial V}{\partial \nu} \right) \mathcal{E}
\]

\[
+ \int_M S_g W \mathcal{E}
\]

\[
+ 2(n-1) \epsilon \int_{\partial M} W \mathcal{E}.
\]

Estimate of \((I_1)\). We have

\[
|I_1| \lesssim \|\mathcal{E}\|_{H^1(M)} \|A_{\delta, n} \|_{L^{n+2}(M)}
\]
with
\[ A_\delta = -c_n \Delta_y (U + \delta V) - K U \frac{n+2}{n-2} + K (g(U) - g(U + \delta V)). \]

Now, in local coordinates, the Laplace-Beltrami operator reads as:
\[ \Delta_y \phi = \Delta \phi + (g^{ij} - \delta^{ij}) \partial_i \partial_j \phi - g^{ij} \Gamma^k_{ij} \partial_k \phi. \]  
(C.1)

and so by the decay of \( U \) and \( V_p \) (see Proposition 3.1) and by Lemma A.3 in variables \( x = \delta y \) with \( |\delta y| \leq R \)

\[
A_\delta(y) = -\delta^{-\frac{n+2}{2}} c_n \Delta U(y) \chi(\delta y) - \frac{8(n-1)}{n-2} \delta^{-\frac{n+2}{2}} h^{ij}(y) \partial^2_{ij} U(y) \chi(\delta y) y_n - \delta^{-\frac{n+2}{2}} K \chi^{\frac{n+2}{2}} (\delta y) U \frac{n+2}{n-2}(y) \\
+ \delta^{-\frac{n+2}{2}} K \chi^{\frac{n+2}{2}} (\delta y) (g(U) - g(U + \delta V_p)) - \delta^{-\frac{2}{2}} c_n \Delta V_p + \delta^{-\frac{n+2}{2}} \Lambda (y) \\
= \delta^{-\frac{n+2}{2}} K \left( \chi(\delta y) - \chi^{\frac{n+2}{2}} (\delta y) \right) U \frac{n+2}{n-2}(y) \\
+ \delta^{-\frac{n+2}{2}} K \chi^{\frac{n+2}{2}} (\delta y) (g(U) - g(U + \delta V_p)) + \delta^{-\frac{2}{2}} K \chi(\delta y) g'(U) V_p(y) + \delta^{-\frac{n+2}{2}} \Lambda (y)
\]

where
\[ |\Lambda(y)| \lesssim \frac{1}{1 + |y|^{n-2}} \text{ if } |\delta y| \leq R. \]

Finally,

\[
\|A_\delta\|_{L^{\frac{2n}{n+2}}(M)} \lesssim \left\| K \left( \chi(\delta y) - \chi^{\frac{n+2}{2}} (\delta y) \right) U \frac{n+2}{n-2}(y) \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\
+ \delta \left\| K \left( \chi(\delta y) - \chi^{\frac{n+2}{2}} (\delta y) \right) U \frac{n+2}{n-2}(y) V_p(y) \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\
+ \left\| K \chi^{\frac{n+2}{2}} (\delta y) (g(U + \delta V_p) - g(U) - \delta g'(U) V_p(y)) \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\
+ \delta^2 \|\Lambda\|_{L^{\frac{2n}{n+2}}(B(0,R/\delta))} \\
\lesssim \left\{ \begin{array}{ll}
\delta^2 & \text{if } n \geq 7 \\
\delta^2 |\ln \delta|^2 & \text{if } n = 6 \\
\delta^{\frac{n-2}{2}} & \text{if } n = 4, 5,
\end{array} \right.
\]

because by (A.5)

\[
\left| (g(U + \delta V_p) - g(U) - \delta g'(U) V_p(y)) \right| \lesssim \left\{ \begin{array}{ll}
U \frac{n-2}{n-2} (\delta V_p)^2 & \text{if } n \geq 6 \\
U \frac{n-2}{n-2} (\delta V_p)^2 + (\delta V_p) \frac{n+2}{n-2} & \text{if } n = 4, 5
\end{array} \right.
\]

which implies

\[
\left\| (g(U + \delta V_p) - g(U) - \delta g'(U) V_p(y)) \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \lesssim \left\{ \begin{array}{ll}
\delta^{\frac{n+2}{n-2}} & \text{if } n \geq 6 \\
\delta^2 & \text{if } n = 4, 5.
\end{array} \right.
\]

and also

\[
\delta^2 \|\Lambda\|_{L^{\frac{2n}{n+2}}(B(0,R/\delta))} \lesssim \left\{ \begin{array}{ll}
\delta^2 & \text{if } n \geq 7 \\
\delta^2 |\ln \delta|^2 & \text{if } n = 6 \\
\delta^{\frac{n-2}{2}} & \text{if } n = 4, 5.
\end{array} \right.
\]
Estimate of \((I_2)\). We have

\[
(I_2) = 2(n-1) \int_{\partial M} \left( \frac{2}{n-2} \frac{\partial U}{\partial \nu} - H U^{\frac{n}{n-2}} \right) E
\]

and

\[
\left\| \frac{2}{n-2} \frac{\partial U}{\partial \nu} - H U^{\frac{n}{n-2}} \right\|_{L^2(\partial M)}^{2(n-1)}
\]

\[
\lesssim \left( \int_{\partial M} \left( \frac{2}{n-2} \frac{\partial U}{\partial \nu} - H U^{\frac{n}{n-2}} \right)^{2(n-1)} \right)^{\frac{n}{2(n-1)}}
\]

\[
\lesssim \left( \int_{\partial \mathbb{R}^n_+} H^{2(n-1)} (\tilde{y}) \left( \chi(\delta \tilde{y}) - \chi^{\frac{n}{n-2}}(\delta \tilde{y}) \right) \right)^{\frac{n}{2(n-1)}}
\]

\[
\lesssim \delta^2
\]

Estimate of \((I_3)\). We have

\[
|I_3| \lesssim \|E\|_{H^1(M)} \left\| H (\tilde{f}(U) - \tilde{f}(U + \delta V)) + \frac{2\delta}{n-2} \frac{\partial V}{\partial \nu} \right\|_{L^2(\partial M)}^{2(n-1)}
\]

and by the decay of \(V_p\) and \((A.5)\)

\[
\left\| H (\tilde{f}(U) - \tilde{f}(U + \delta V)) + \frac{2\delta}{n-2} \frac{\partial V_p}{\partial \nu} \right\|_{L^2(\partial M)}^{2(n-1)}
\]

\[
\lesssim \left\| H (\tilde{f}(U) - \tilde{f}(U + \delta V_p)) \chi^{\frac{n}{n-2}}(\delta \tilde{y}) + \frac{2\delta}{n-2} \frac{\partial V_p}{\partial \nu} \chi(\delta \tilde{y}) \right\|_{L^2(\partial M)}^{2(n-1)}
\]

\[
\lesssim \delta \left\| H \tilde{f}'(U + \delta V_p) V_p \chi^{\frac{n}{n-2}}(\delta \tilde{y}) - \frac{n}{n-2} H U^{\frac{n}{n-2}} V_p \chi(\delta \tilde{y}) \right\|_{L^2(\partial M)}^{2(n-1)}
\]

\[
\lesssim \delta \left\| H \left( \chi^{\frac{n}{n-2}}(\delta \tilde{y}) - \chi(\delta \tilde{y}) \right) \tilde{f}'(U + \delta V_p) V_p \right\|_{L^2(\partial M)}^{2(n-1)}
\]

\[
\lesssim \delta \left\| H \chi(\delta \tilde{y}) \left( \tilde{f}'(U + \delta V_p) - \tilde{f}'(U) \right) V_p \right\|_{L^2(\partial M)}^{2(n-1)}
\]

\[
\lesssim \delta \frac{n \ln \delta}{\delta} \text{ if } n \geq 5
\]

\[
\lesssim \delta \frac{n \ln \delta}{\delta} \text{ if } n = 4.
\]

because by \((A.5)\)

\[
|\left( \tilde{f}'(U + \delta V_p) - \tilde{f}'(U) \right) V_p| \lesssim \delta U^{\frac{4-n}{n-2}} V_p^2.
\]

Estimate of \((I_4)\). By Hölder’s inequality

\[
|I_4| \lesssim \int_M |\tilde{U}| |\tilde{E}| \, d\nu_g + \delta \int_M |\tilde{V}| |\tilde{E}| \, d\nu_g \lesssim \|\tilde{E}\| \left( \|\tilde{U}\|_{L^\infty(M)} + \delta \|\tilde{V}\|_{L^2(M)} \right),
\]
with
\[ \|U\|_{L^{2n/(n+2)}(M)} \lesssim \begin{cases} \delta^2 & \text{if } n \geq 7 \\ \delta^2 |\ln \delta|^{1/2} & \text{if } n = 6 \\ \delta^{n-2} & \text{if } n = 4, 5 \end{cases} \]

and
\[ \|V\|_{L^2(M)} \lesssim \begin{cases} \delta^2 & \text{if } n \geq 7 \\ \delta^2 |\ln \delta|^{1/2} & \text{if } n = 6 \\ \delta^{n-2} & \text{if } n = 4, 5 \end{cases} \]

**Estimate of \((I_5)\).** By Hölder’s inequality
\[ |(I_5)| \lesssim \varepsilon \left( \|U\|_{L^{2(n-1)/(n+1)}(\partial M)} + \delta \|V\|_{L^2(\partial M)} \right) \|\mathcal{E}\|, \]

with
\[ \|U\|_{L^{2(n-1)/(n+1)}(\partial M)} \lesssim \begin{cases} \delta & \text{if } n \geq 5 \\ |\ln \delta|^{1/2} & \text{if } n = 4 \end{cases} \]

and
\[ \|V\|_{L^2(\partial M)} \lesssim \begin{cases} \delta^{1/2} & \text{if } n \geq 6 \\ |\ln \delta|^{1/2} & \text{if } n = 5 \\ \delta & \text{if } n = 4 \end{cases} \]

Finally, collecting all the previous estimates, by the choice of \(\delta_j\) in (4.6) and (4.7)
\[ |(I)| \lesssim \begin{cases} \varepsilon^2 & \text{if } n \geq 7 \\ \varepsilon^2 |\ln \varepsilon|^{1/2} & \text{if } n \geq 6 \\ \varepsilon^{1/2} & \text{if } n = 5 \\ \rho(\varepsilon) & \text{if } n = 4 \end{cases} \]

• Let us estimate the interaction terms \((II)\) and \((III)\).

Set for any \(h = 1, \ldots, k\) \(B_h^+ = B^+(\eta(\varepsilon)\tau_h, \eta(\varepsilon)\sigma/2)\) where \(\sigma > 0\) is small enough and \(\partial' B_h^+ = B_h^+ \cap \partial \mathbb{R}^{n+1}_+\). Since \(\sigma\) is small then \(B_h^+ \subset B_R^+\) and \(\partial' B_h^+ \subset \partial' B_R^+\) and they are disjoint.

We remark that in \(B_{2R}^+(0)\) we get
\[ W_i(x) \lesssim \frac{\delta_i^{n-2}}{|x - \eta(\varepsilon)\tau_i|^{n-2}}. \]

Then
\[ \left| \int_M K \left( \sum_j g(W_j) - g \left( \sum_j W_j \right) \right) \mathcal{E} \right| \lesssim \left\| \sum_j g(W_j) \right\|_{L^{\frac{2n}{n+2}}(M)} \|\mathcal{E}\|. \]
Hence

\[
\| \cdots \|_{L^{2n+2}}(M) \lesssim \left[ \int_{B^+_{2R} \setminus B^+_{R}} | \cdots |^{2n+2} |g(x)|^{\frac{1}{2}} \ dx \right]^\frac{n+2}{2n} \\
+ \left[ \int_{B^+_{2R} \setminus \bigcup_{h} B^+_{R}} | \cdots |^{2n+2} |g(x)|^{\frac{1}{2}} \ dx \right]^\frac{n+2}{2n}
\]

\[
\lesssim \sum_{i=1}^{k} \left[ \int_{B^+_{R} \setminus \bigcup_{h} B^+_{R}} (1 - \chi^{2^*}(|x|)) |W_i|^{2^*} |g(x)|^{\frac{1}{2}} \ dx \right]^\frac{n+2}{2n} \lesssim \sum_{i=1}^{k} \left[ \int_{B^+_{R} \setminus \bigcup_{h} B^+_{R}} \frac{\delta_i^{n+2}}{|x - \eta(\varepsilon) \tau_i|^{2n}} \ dx \right]^\frac{n+2}{2n} \\
\lesssim \sum_{i=1}^{k} \frac{\delta_i^{n+2}}{\eta(\varepsilon)^{\frac{n+2}{2}}} \left[ \int_{B^+_{R} \setminus \bigcup_{h} B^+_{R}} \frac{1}{|y - \tau_i|^{2n}} \ dy \right]^\frac{n+2}{2n} \lesssim \varepsilon^{(1+\alpha)\frac{n+2}{2}} (\rho(\varepsilon))^3 |\ln \rho(\varepsilon)|^2 \quad \text{if } n \geq 5
\]

Now for \( n \geq 7 \)

\[
\sum_{h=1}^{k} \left[ \int_{B^+_{h}} |W_i|^{2^*} \sum_{i \neq h} W_i |g(x)|^{\frac{1}{2}} \ dx \right]^\frac{n+2}{2n} \lesssim \sum_{h=1}^{k} \sum_{i \neq h} \left[ \int_{B^+_{h}} \frac{\delta_i^{n+2}}{|x - \varepsilon^0 \tau_i|^{\frac{n+2}{2}}} \ dx \right]^\frac{n+2}{2n} \lesssim \sum_{h=1}^{k} \sum_{i \neq h} \delta_i^2 \delta_i^{n-2} \left( \int_{B^+_{h}} \frac{1}{|y|^{n+2}} \ dy \right) \lesssim \varepsilon^{\frac{n+2}{2} + \alpha \frac{n+6}{2n}}
\]
while for \( n = 4, 5, 6 \)

\[
\sum_{h=1}^{k} \left[ \int_{B_h^+} |W_h|^{2^* - 2} \sum_{i \neq h} W_i \right]^{\frac{2n}{n+2}} \left[ \left( g(x) \right)^{\frac{1}{2}} dx \right]^{\frac{n+2}{2n}} \\
\lesssim \sum_{h=1}^{k} \sum_{i \neq h} \frac{\delta_h^2 \delta_{i-1} \eta(\varepsilon)}{n+2} \left( \int_{B_h^+} \frac{1}{\left( 1+\|y\|^2 + (y_n + D_h \eta(\varepsilon)^{-1})^2 - \delta_h^2 \eta(\varepsilon)^{-2} \right)^{\frac{3n}{n+2}}} \right)^{\frac{n+2}{2n}} \\
\lesssim \sum_{h=1}^{k} \sum_{i \neq h} \frac{\delta_h^2 \delta_{i-1} \eta(\varepsilon)}{n+2} \left( \int_{B_h^+} \frac{1}{\left( 1+\|y\|^2 + (\delta_h \eta(\varepsilon)^{-1})^2 (D_h^2(p) - 1) \right)^{\frac{3n}{n+2}}} \right)^{\frac{n+2}{2n}} \\
\lesssim \begin{cases} 
\varepsilon^4 |\ln \varepsilon|^{\frac{2}{3}} & \text{if } n = 6 \\
\varepsilon^3 & \text{if } n = 5 \\
(\rho(\varepsilon))^3 |\ln \rho(\varepsilon)| & \text{if } n = 4.
\end{cases}
\]

At the end

\[
\sum_{h=1}^{k} \left[ \int_{B_h^+} \sum_{i \neq h} W_i |g(x)|^{\frac{1}{2}} \right]^{\frac{n+2}{2n}} \lesssim \sum_{h=1}^{k} \sum_{i \neq h} \left[ \int_{B_h^+} \frac{\delta_h^n}{|x - \eta(\varepsilon)\tau|^2} \right]^{\frac{n+2}{2n}} \\
\lesssim \begin{cases} 
\varepsilon^{(1-n)\frac{n+2}{2n}} & \text{if } n \geq 5 \\
(\rho(\varepsilon))^3 |\ln \rho(\varepsilon)|^{\frac{2}{3}} & \text{if } n = 4.
\end{cases}
\]

For the term \((III)\) we get

\[ \| (III) \| \lesssim \left\| \sum_{j} f(W_j) - f \left( \sum_{j} W_j \right) \right\|_{L^{2(n-1)}(\partial M)} \| \xi \| \]

Hence

\[
\| \cdots \|_{L^{2(n-1)}(\partial M)} \lesssim \int_{\partial \Omega_{B_2}} \left( 1 - \chi_{\Omega_{B_2}}(\tilde{x}, 0) \right) |g(\tilde{x}, 0)|^{\frac{1}{2}} \left( \sum_{j} W_j \right) d\tilde{x} \]

\[
+ \int_{\partial \Omega_{B_2}} \left( 1 - \chi_{\Omega_{B_2}}(\tilde{x}, 0) \right) |g(\tilde{x}, 0)|^{\frac{1}{2}} \left( \sum_{j} W_j \right) d\tilde{x} \]

\[
\leq \sum_{i=1}^{k} \int_{\partial \Omega_{B_2}} \left( 1 - \chi_{\Omega_{B_2}}(\tilde{x}, 0) \right) |W_i|^{2^*} |g(\tilde{x}, 0)|^{\frac{1}{2}} d\tilde{x} \]

\[
+ \sum_{h=1}^{k} \int_{\partial \Omega_{B_2}} \left( 1 - \chi_{\Omega_{B_2}}(\tilde{x}, 0) \right) |W_h|^{2^*} |g(\tilde{x}, 0)|^{\frac{1}{2}} d\tilde{x} \]

\[
+ \sum_{h=1}^{k} \int_{\partial \Omega_{B_2}} \left( 1 - \chi_{\Omega_{B_2}}(\tilde{x}, 0) \right) \left( \sum_{i \neq h} W_i \right)^{2^*} |g(\tilde{x}, 0)|^{\frac{1}{2}} d\tilde{x} \]

Now
\[
\sum_{i=1}^{k} \left[ \int_{\partial B_{2n}^{+}} |W_i|^{2} |g(\tilde{x}, 0)|^{\frac{1}{2}} d\tilde{x} \right]^{\frac{n}{2(n-1)}} \leq \sum_{i=1}^{k} \left[ \int_{\mathbb{R}^{n-1} \setminus \partial B_{\tilde{R}}^+} \frac{\delta_i^{n-1}}{|\tilde{x} - \eta(\varepsilon) \tilde{r}_i|^{2(n-1)-2}} d\tilde{x} \right]^{\frac{n}{2(n-1)}}
\]
\[
\lesssim \sum_{i=1}^{k} \left[ \int_{\mathbb{R}^{n-1} \setminus \partial B_{\tilde{R}}^+} \frac{1}{|\tilde{y} - \tilde{r}_i|^{2(n-1)}} d\tilde{x} \right]^\frac{n}{2(n-1)}
\]
\[
\lesssim \begin{cases} 
\varepsilon^{(1-\alpha)\frac{n}{2}} & \text{if } n \geq 5 \\
(\rho(\varepsilon))^2 |\ln \rho(\varepsilon)|^\frac{2}{\alpha} & \text{if } n = 4 
\end{cases}
\]

Similarly
\[
\left[ \int_{\partial B_{2n}^{+} \setminus \bigcup_{h} \partial B_{\tilde{R}}^+} |W_i|^{2} |g(\tilde{x}, 0)|^{\frac{1}{2}} d\tilde{x} \right]^{\frac{n}{2(n-1)}} \lesssim \begin{cases} 
\varepsilon^{(1-\alpha)\frac{n}{2}} & \text{if } n \geq 5 \\
(\rho(\varepsilon))^2 |\ln \rho(\varepsilon)|^\frac{2}{\alpha} & \text{if } n = 4 
\end{cases}
\]

Now for \( n \geq 5 \)
\[
\sum_{h=1}^{k} \left[ \int_{\partial B_{\tilde{R}}^+} \left| W_h \right|^{2} \sum_{i \neq h} W_i \right]^{2} \left| g(\tilde{x}, 0) \right|^{\frac{1}{2}} d\tilde{x} \right]^{\frac{n}{2(n-1)}}
\]
\[
\lesssim \sum_{h=1}^{k} \sum_{i \neq h} \delta_h \delta_i^{n-2} \left[ \int_{\partial B_{\tilde{R}}^+} \frac{1}{|\tilde{x} - \eta(\varepsilon) \tilde{r}_i|^{2(n-1)}} d\tilde{x} \right]^{\frac{n}{2(n-1)}}
\]
\[
\lesssim \sum_{h=1}^{k} \sum_{i \neq h} \delta_h \delta_i^{n-2} \left[ \int_{\partial B_{\tilde{R}(\varepsilon)^{\frac{1}{2}}}^{+}} \frac{1}{|\tilde{y}|^{2(n-1)}} d\tilde{x} \right]^{\frac{n}{2(n-1)}} \lesssim \varepsilon^{\frac{n}{2} + \frac{\alpha - 4}{2}}
\]

and for \( n = 4 \)
\[
\sum_{h=1}^{k} \left[ \int_{\partial B_{\tilde{R}}^+} \left| W_h \right| \sum_{i \neq h} W_i \right] \left| g(\tilde{x}, 0) \right|^{\frac{1}{2}} d\tilde{x} \right]^{\frac{4}{3}} \lesssim \rho(\varepsilon).
\]

Finally
\[
\sum_{h=1}^{k} \left[ \int_{\partial B_{\tilde{R}}^+} \sum_{i \neq h} W_i \right]^{2} \left| g(\tilde{x}, 0) \right|^{\frac{1}{2}} d\tilde{x} \right]^{\frac{2}{n}} \lesssim \begin{cases} 
\varepsilon^{(1-\alpha)\frac{n}{2}} & \text{if } n \geq 5 \\
(\rho(\varepsilon))^2 |\ln \rho(\varepsilon)|^\frac{2}{\alpha} & \text{if } n = 4 
\end{cases}
\]

We collect all the above estimates and the claim follows. \( \square \)

**APPENDIX D. PROOF OF LEMMA 4.3**

**Proof.** We argue by contradiction. Assume there exist sequences \( \varepsilon_m \to 0 \), \( p_m \to p \in \partial M \), \( \tau_m \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n \), \( d_m \in \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \), \( \phi_m, \psi_m \in \mathcal{K} \) such that \( \tau_m \to \tau, d_m \to d \), \( \tau_{mj} \to \tau_j, d_{mj} \to d_j \) for every \( j = 1, \ldots, k \) and
\[
\mathcal{L}(\phi_m) = \psi_m, \quad \|\phi_m\|_{HY^1} = 1, \quad \|\psi_m\|_{HY^1} \to 0.
\]

We will write \( W_m := \mathcal{W}(d_m, \tau_m, \varepsilon_m) \), and
\[
\phi_m l(\xi) := \delta_m^{n-2} \phi_m (\psi_m^{0} (\delta_m l y + \eta(\varepsilon) \tau_m l) \chi(\delta_m l y + \eta(\varepsilon) \tau_m l)).
\]
Since \( \|\phi_m\|_{H^1} = 1 \), \( \phi_{ml} \) is bounded in \( \mathcal{D}^{1,2}(\mathbb{R}^n_+) \), there exists \( \phi_l \) such that
\[
\phi_{ml} \rightharpoonup \phi_l \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^n_+),
\phi_{ml} \to \phi_l \quad \text{weakly in } L^{\frac{2n}{{n-2}}} (\mathbb{R}^n_+),
\phi_{ml} \to \phi_l \quad \text{strongly in } L^2_{loc} (\partial \mathbb{R}^n_+),
\] (D.1)

Firstly, notice that by definition of \( \mathcal{L} \),
\[
\psi_m - \phi_m - i_M^* (Kg'(W_m) \phi_m) - i_M^* \left( \frac{n-2}{2} H^l(W_m) \phi_m - \varepsilon_m \phi_m \right) = \sum_{j=1, \ldots, k} C_{mj}^i Z_{j,i}.
\] (D.2)

We will show that \( C_{mj}^i = o_m(1) \) for every \( j, i \). Multiply equation (D.2) by \( Z_{i,q} \) and integrate by parts to obtain
\[
C_{mj}^i \langle Z_{j,i}, Z_{l,q} \rangle = \left( \langle \psi_m, Z_{l,q} \rangle - \langle \phi_m, Z_{l,q} \rangle - \langle i_M^* (Kg'(W) \phi_m), Z_{l,q} \rangle \right) = 0
\]
\[
- \left( i_M^* \left( \frac{n-2}{2} H^l(W_m) \phi_m, Z_{l,q} \right) \right) + \varepsilon_m \left( \langle i_M^* (\phi_m), Z_{l,q} \rangle \right)
= \frac{n+2}{n-2} \int M |K| W_m^\frac{2}{n-2} \phi_m Z_{l,q} - c_n \frac{n}{2} \int \partial M H W_m^\frac{2}{n-2} \phi_m Z_{l,q}
+ \varepsilon_m \int \partial M \phi_m Z_{l,q}.
\]

Observe that
\[
\varepsilon_m \int \partial M \phi_m Z_{l,q} = \varepsilon_m \int \partial B^+_s(\eta(\varepsilon_m) r_m) \phi_m \frac{1}{\delta_m^2} \delta_q \left( \frac{x - \eta(\varepsilon_m) r_m}{\delta_m} \right) = \varepsilon_m \delta_m^2 \int \partial B^+_s(\eta(\varepsilon_m) r_m) \phi_m \delta_q
= \varepsilon_m \times \left\{ \begin{array}{ll}
\delta_m^2 \ln \delta_m & \text{if } 1 \leq q < n,
\delta_m & \text{if } q = n.
\end{array} \right.
\]

Then, by (D.6), (D.8) and (D.9):
\[
C_{mj}^i o_m(1) = \frac{n+2}{n-2} \int B^+_s(\eta(\varepsilon_m) r_m) |K| U^\frac{4}{n-2} \phi_m \delta_q - c_n \frac{n}{2} \int \partial B^+_s(\eta(\varepsilon_m) r_m) H U^\frac{2}{n-2} \phi_m \delta_q + o_m(1),
\] (D.3)

On the other hand, since \( \delta_q \) satisfies (2.1) we get
\[
\begin{align*}
0 &= \langle \phi_m, Z_{l,q} \rangle = c_n \int M \nabla g \phi_m \cdot \nabla g Z_{l,q} + \int M S_g \phi_m Z_{l,q}
= -c_n \int M \phi_m \Delta g Z_{l,q} + c_n \int M \frac{\partial \phi_m}{\partial \nu} \frac{\partial Z_{l,q}}{\partial \nu} + \int M S_g \phi_m Z_{l,q}
= -c_n \int B^+_s(\eta(\varepsilon_m) r_m) \phi_m \frac{1}{\delta_m^2} \Delta \delta_q \left( \frac{x - \eta(\varepsilon_m) r_m}{\delta_m} \right)
+ c_n \int B^+_s(\eta(\varepsilon_m) r_m) \phi_m \frac{1}{\delta_m^2} \frac{\partial \delta_q}{\partial \nu} \left( \frac{x - \eta(\varepsilon_m) r_m}{\delta_m} \right)
+ c_n \int B^+_s(\eta(\varepsilon_m) r_m) \phi_m \frac{1}{\delta_m^2} \delta_q \left( \frac{x - \eta(\varepsilon_m) r_m}{\delta_m} \right) + o_m(1)
= -\frac{n+2}{n-2} \int B^+_s(\eta(\varepsilon_m) r_m) |K| U^\frac{4}{n-2} \phi_m \delta_q + c_n \frac{n}{2} \int \partial B^+_s(\eta(\varepsilon_m) r_m) H U^\frac{2}{n-2} \phi_m \delta_q + o_m(1).
\end{align*}
\]
Hence,
\[
\frac{n+2}{n-2} \int_{B^+_{\frac{1}{\sigma_m}}} |K| \frac{\partial}{\partial x} \phi_{mj} \delta q - c_n \frac{n}{2} \int_{\partial B^+_{\frac{1}{\sigma_m}}} H U \frac{\partial^2}{\partial x^2} \phi_{mj} \delta q = o_m(1). \tag{D.4}
\]

The claim follows from (D.3) and (D.4). Now, take any \( \varphi \in C^2(\mathbb{R}^n_+) \) with compact support, and define
\[
\varphi_{mj}(\xi) = \frac{1}{\delta_{mj}} \varphi \left( \left( \psi_{p_m}^j \right)^{-1}(\xi) - \eta(\varepsilon_m) \tau_{mj} \right) \chi \left( \left( \psi_{p_m}^j \right)^{-1}(\xi) \right).
\]

By (D.2), arguing as in the proof of Lemma D.2:
\[
\langle \phi_m, \varphi_{mj} \rangle = C_{mj}^i \langle Z_{j,i}, \varphi_{mj} \rangle + \langle i_M^* (K g'(W) \phi_m), \varphi_{mj} \rangle + \langle i_M^* (H f(W) \phi_m), \varphi_{mj} \rangle - \varepsilon_m \langle i_M^* (\varphi_m), \varphi_{mj} \rangle + \langle \psi_m, \varphi_{mj} \rangle
\]
\[
= - \frac{n+2}{n-2} K \int_{B^+_{\frac{1}{\sigma_m}}} U \frac{\partial}{\partial x} \phi_{mj} \varphi + c_n \frac{n}{2} \int_{\partial B^+_{\frac{1}{\sigma_m}}} H U \frac{\partial^2}{\partial x^2} \phi_{mj} \varphi + o_m(1).
\]

Moreover,
\[
\langle \phi_m, \varphi_{mj} \rangle = c_n \int_M \nabla_g \phi_m \cdot \nabla_g \varphi_{mj} + \int_M S_g \phi_m \varphi_{mj}
\]
\[
= c_n \int_{B^+_{\frac{1}{\sigma_m}}} \nabla \phi_{mj} \cdot \nabla \varphi + o_m(1).
\]

Therefore
\[
c_n \int_{B^+_{\frac{1}{\sigma_m}}} \nabla \phi_{mj} \cdot \nabla \varphi + \frac{n+2}{n-2} K \int_{B^+_{\frac{1}{\sigma_m}}} U \frac{\partial}{\partial x} \phi_{mj} \varphi
\]
\[
- c_n \frac{n}{2} \int_{\partial B^+_{\frac{1}{\sigma_m}}} H U \frac{\partial^2}{\partial x^2} \phi_{mj} \varphi = o_m(1). \tag{D.5}
\]

If we take limits in (D.5) (possible thanks to (D.1)), we see that \( \phi_j \) is a weak solution to (2.1), and then by Theorem 2.1 \( \phi_j \in \text{Span}(\{Z_{\gamma} : \gamma = 1, \ldots, n\}) \). Now, by the orthogonality condition
\[
0 = \langle \phi_m, Z_{j,i} \rangle = \langle \phi_m, Z_{l,q} \rangle + o_m(1), \quad \text{for every } i = 1, \ldots, n,
\]
which implies \( \phi_j = 0 \) for every \( j = 1, \ldots, k \). However, again by (D.2):
\[
\| \phi_m \|^2_{H^1} = - \frac{n+2}{n-2} K \int_{B^+_{\frac{1}{\sigma_m}}} U \frac{\partial}{\partial x} \phi_{mj} + c_n \frac{n}{2} \int_{\partial B^+_{\frac{1}{\sigma_m}}} H U \frac{\partial^2}{\partial x^2} \phi_{mj} + o_m(1)
\]
\[
= o_m(1).
\]

This contradicts \( \| \phi_m \|^2_{H^1} = 1 \) and finishes the proof. \( \Box \)

**Lemma D.1.** It holds true that
\[
\langle Z_{j,i}, Z_{l,q} \rangle = \begin{cases} 
\delta^{ij} \| Z_{l,q} \|^2_{H^1(M)} = \delta^{ij} O(1) & \text{if } j = l \\
O \left( \frac{\delta^{i} \delta^{j}}{\delta^{l} \delta^{q}} \right) & \text{if } j \neq l.
\end{cases} \tag{D.6}
\]

**Proof.** The first statement is a consequence of the orthogonality of \( Z_i \) and \( Z_q \) in \( H^1(\mathbb{R}^n_+) \), so let us assume \( j \neq l \). The following decay property will be used throughout this proof:
\[
|\nabla^\alpha Z_{j,i}| \leq \begin{cases} 
\frac{\delta^j}{\delta^i} |x - \eta(\varepsilon) \tau_i|^{1-\alpha-n} & \text{if } i = 1, \ldots, n-1, \\
\frac{\delta^j}{\delta^i} |x - \eta(\varepsilon) \tau_j|^{2-\alpha-n} & \text{if } i = n,
\end{cases} \tag{D.7}
\]
for $\alpha = 0,1$. Take a small $\sigma > 0$ such that $B^+_\sigma(\eta(\varepsilon)\tau_j) \cap B^+_\sigma(\eta(\varepsilon)\tau_i) = \emptyset$, and let $\Omega_\sigma = B^+_R \setminus (B^+_\sigma(\eta(\varepsilon)\tau_j) \cup B^+_\sigma(\eta(\varepsilon)\tau_i))$. First, observe that

$$\int_M S_g Z_{j,i} Z_{l,q} = \int_{\Omega_\sigma} S_g Z_{j,i} Z_{l,q} + \int_{B^+_\sigma(\eta(\varepsilon)\tau_j)} S_g \frac{1}{\delta^2_j} \left( \frac{x - \eta(\varepsilon)\tau_j}{\delta_j} \right) Z_{l,q}$$

$$+ \int_{B^+_\sigma(\eta(\varepsilon)\tau_i)} S_g \frac{1}{\delta^2_j} \left( \frac{x - \eta(\varepsilon)\tau_i}{\delta_i} \right) Z_{j,i}$$

$$= \int_{\Omega_\sigma} S_g Z_{j,i} Z_{l,q} + \delta^{\frac{n-2}{2}} \int_{B^+_\sigma} S_g (\delta_j y + \eta(\varepsilon)\tau_j) Z_{l,q}(\delta_j y + \eta(\varepsilon)\tau_j) dy$$

$$+ \delta^{\frac{n-2}{2}} \int_{B^+_\sigma} S_g (\delta_i y + \eta(\varepsilon)\tau_i) Z_{j,i}(\delta_i y + \eta(\varepsilon)\tau_i) dy$$

where $\phi \in H^4(M)$ it holds true

$$\int_M |K| |W|^{\frac{n-2}{n}} \phi Z_{l,q} = |K| \int_{B^+_\sigma} U^{\frac{n-2}{2}} \phi \delta_l \eta_q + o(1), \quad (D.8)$$

$$\int_{\partial M} H |W|^{\frac{n-2}{n}} \phi Z_{l,q} = H \int_{\partial B^+_\sigma} U^{\frac{n-2}{2}} \phi \delta_l \eta_q + o(1). \quad (D.9)$$

where $\phi_l(y) = \frac{n-2}{2} \phi(\delta_j y + \eta(\varepsilon)\tau_i)$ and $\sigma > 0$ is small enough.
Proof. For the sake of brevity we will only prove (D.8), as the proof of (D.9) follows the same argument. Take \( \sigma > 0 \) small enough such that

\[
\{ B_\sigma^+ (\eta(\varepsilon) \tau_\gamma) : \gamma = 1, \ldots, k. \}
\]

is a disjoint family, and denote by \( \Omega_\sigma = B_\sigma^+ \setminus \bigcup_{\gamma=1}^k B_\sigma^+ (\eta(\varepsilon) \tau_\gamma) \). Then,

\[
\int_M |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q} = \int_{B_\sigma^+} |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q}
\]

\[
= \int_{\Omega_\sigma} |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q} + \int_{B_\sigma^+ (\eta(\varepsilon) \tau_\gamma)} |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q} + \sum_{\gamma=1}^k \int_{B_\sigma^+ (\eta(\varepsilon) \tau_\gamma)} |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q}
\]

\[
\simeq \int_{\Omega_\sigma} |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q} + \int_{B_\sigma^+ (\eta(\varepsilon) \tau_\gamma)} |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q} + \sum_{\alpha=1}^k \int_{B_\sigma^+ (\eta(\varepsilon) \tau_\gamma)} |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q}
\]

We proceed to estimate every term in the right-hand side. To that aim, we will use the bound (D.7) together with the fact that

\[
|W_\alpha(x)| \leq C \delta_\alpha^{\frac{q}{2}} \left( \frac{1}{|x - \eta(\varepsilon) \tau_\gamma|^{n-2}} + \frac{1}{|x - \eta(\varepsilon) \tau_\gamma|^{n-3}} \right). \tag{D.10}
\]

First of all, by (D.7) and (D.10), it is easy to see that

\[
\int_{\Omega_\sigma} |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q} = \begin{cases} \mathcal{O} \left( \sum_{\alpha=1}^k \delta_\alpha^2 \delta_l^{\frac{q}{2}} \right) & \text{if } 1 \leq q < n, \\ \mathcal{O} \left( \sum_{\alpha=1}^k \delta_\alpha^2 \delta_l^{\frac{n+2}{2}} \right) & \text{if } q = n. \end{cases}
\]

The second addend gives us the main term of (D.8):

\[
\int_{B_\sigma^+ (\eta(\varepsilon) \tau_\gamma)} |K| |W|^{\frac{4}{n-2}} \phi Z_{l,q}
\]

\[
= \int_{B_\sigma^+ (\eta(\varepsilon) \tau_\gamma)} |K| \phi \frac{1}{\delta_l^{\frac{4}{2}}} \left( \frac{x - \eta(\varepsilon) \tau_\gamma}{\delta_l} \right) \left( U + \delta_l V_p \right) \left( \frac{x - \eta(\varepsilon) \tau_\gamma}{\delta_l} \right)^{\frac{4}{n-2}} dx
\]

\[
= \int_{B_\sigma^+ (\eta(\varepsilon) \tau_\gamma)} |K| \phi(y) \beta_q(y) (U + \delta_l V_p)(y)^{\frac{4}{n-2}} dy
\]

\[
= \int_{B_\sigma^+ (\eta(\varepsilon) \tau_\gamma)} |K| \phi(y) \beta_q(y) U(y)^{\frac{4}{n-2}} dy + o(1).
\]
The rest of the terms go to zero as $\delta_l \to 0$:

$$
\sum_{\alpha=1}^{k} \int_{B^+_{\gamma}(\eta(\varepsilon)\tau_\gamma)} |K| |W_\alpha|^{\frac{4}{n-2}} \phi Z_{l,q}
\leq \sum_{\alpha=1}^{k} \frac{\delta_\alpha^2 \delta_{\gamma}^{\frac{n+2}{2}}}{\alpha \neq \gamma} \int_{B^+_{\gamma}(\eta(\varepsilon)\tau_\gamma)} |K| \phi(\delta_l y + \eta(\varepsilon)\tau_\gamma) dy
$$

\[
= \begin{cases} 
O\left(\sum_{\alpha=1}^{k} \frac{\delta_\alpha^2 \delta_{\gamma}^{\frac{n+2}{2}}}{\alpha \neq \gamma}\right) & \text{if } 1 \leq q < n, \\
O\left(\sum_{\alpha=1}^{k} \frac{\delta_\alpha^2 \delta_{\gamma}^{\frac{n+2}{2}}}{\alpha \neq \gamma}\right) & \text{if } q = n.
\end{cases}
\]

Analogously,

$$
\sum_{\gamma=1}^{k} \int_{B^+_{\gamma}(\eta(\varepsilon)\tau_\gamma)} |K| |W_\gamma|^{\frac{4}{n-2}} \phi Z_{l,q}
\leq \sum_{\gamma=1}^{k} \frac{\delta_{\gamma}^{n-2}}{\gamma \neq l} \int_{B^+_{\gamma}(\eta(\varepsilon)\tau_\gamma)} |K| \left|U + \delta_{\gamma} V_p\right|^{\frac{4}{n-2}} \phi(\delta_{\gamma} y + \eta(\varepsilon)\tau_\gamma) Z_{l,q}
$$

\[
= \begin{cases} 
O\left(p_n(\delta_\gamma) \delta_{\gamma}^{\frac{n}{2}}\right) & \text{if } 1 \leq q < n, \\
O\left(p_n(\delta_\gamma) \delta_{\gamma}^{\frac{n+2}{2}}\right) & \text{if } q = n,
\end{cases}
\]

being

$$
p_n(\delta_\gamma) = \begin{cases} 
\delta_\gamma & \text{if } n = 3, \\
\delta_{\gamma}^{2} |\ln \delta_\gamma| & \text{if } n = 4, \\
\delta_{\gamma}^{2} & \text{if } n \geq 5.
\end{cases}
$$

Finally,

\[
\sum_{\gamma=1}^{k} \int_{B^+_{\gamma}(\eta(\varepsilon)\tau_\gamma)} |K| |W_\gamma|^{\frac{4}{n-2}} \phi Z_{l,q} = \begin{cases} 
O\left(\sum_{\alpha=1}^{k} \frac{\delta_\alpha^2 \delta_{\gamma}^{\frac{n}{2}}}{\alpha \neq \gamma}\right) & \text{if } 1 \leq q < n, \\
O\left(\sum_{\alpha=1}^{k} \frac{\delta_\alpha^2 \delta_{\gamma}^{\frac{n+2}{2}}}{\alpha \neq \gamma}\right) & \text{if } q = n.
\end{cases}
\]

\[\square\]

**Appendix E. Proof of Proposition 4.4**

**Proof of (1).** Take $\lambda = 1, \ldots, k$ and $s = 1, \ldots, n-1$. Since $(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k)$ is a critical point for $J_\varepsilon$, then

$$
0 = \frac{\partial}{\partial \tau_\lambda} J_\varepsilon(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k) = \left\langle W + \Phi_\varepsilon - i_M^* (Kg(W + \Phi_\varepsilon)) \right. \\
- i_M^* \left(\frac{n-2}{2} (Hf(W + \Phi_\varepsilon) - \varepsilon(W + \Phi_\varepsilon))\right), \frac{\partial(W + \Phi_\varepsilon)}{\partial \tau_\lambda}\right\rangle.
\]

Using equation (4.8), we can write

$$
W + \Phi_\varepsilon - i_M^* (Kg(W + \Phi_\varepsilon)) - i_M^* \left(\frac{n-2}{2} (Hf(W + \Phi_\varepsilon) - \varepsilon(W + \Phi_\varepsilon))\right) = \sum_{j=1}^{k} \sum_{i=1}^{n} c_{ji} Z_{j,i}.
$$
The proof concludes if we show that $c_{j,i} \to 0$ as $\varepsilon \to 0$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Taking this into account, (E.1) becomes

$$0 = \sum_{i,j} c_{j,i} \left< \nabla_{j,i}, \frac{\partial W}{\partial \tau_{\lambda}} + \Phi_{\varepsilon} \right> = \sum_{i,j} c_{j,i} \left< \nabla_{j,i}, \frac{\partial W}{\partial \tau_{\lambda}} \right> - \left< \frac{\partial Z_{j,i}}{\partial \tau_{\lambda}}, \Phi_{\varepsilon} \right>,$$

where for the last identity we have used that $\Phi_{\varepsilon} \in K_{\lambda}$, so

$$0 = \frac{\partial}{\partial \tau_{\lambda}} \left< \nabla_{j,i}, \Phi_{\varepsilon} \right> = \left< \frac{\partial \Phi_{\varepsilon}}{\partial \tau_{\lambda}}, \frac{\partial Z_{j,i}}{\partial \tau_{\lambda}} \right> + \left< \frac{\partial Z_{j,i}}{\partial \tau_{\lambda}}, \Phi_{\varepsilon} \right>.$$

By Lemma E.1,

$$0 = c_{n,\lambda} \frac{\partial}{\partial \lambda} \left\| \nabla_{j,i} \right\|_{L^2(R^n_+)}^2 + O \left( \eta(\varepsilon)^2 \right).$$

This proves $c_{n,\lambda} \to 0$ as $\varepsilon \to 0$ for every $\lambda = 1, \ldots, k$ and $s = 1, \ldots, n - 1$. By taking derivatives with respect to $d_{j,i}$ and arguing as above, we can prove that $c_{n,\eta} \to 0$ as $\varepsilon \to 0$ and claim follows. □

**Lemma E.1.** For any $\lambda = 1, \ldots, k$ and $s = 1, \ldots, n - 1$, it holds true that

$$\left< \nabla_{j,i}, \frac{\partial W}{\partial \tau_{\lambda}} \right> = \frac{\partial}{\partial \lambda} \sum_{i,j} c_{n,\lambda} \frac{\partial}{\partial \lambda} \left\| \nabla_{j,i} \right\|_{L^2(R^n_+)}^2 + O \left( \eta(\varepsilon)^2 \right),$$

$$\left\| \nabla_{j,i} \right\|^2_{L^2(R^n_+)} = \frac{\partial}{\partial \lambda} \sum_{i,j} c_{n,\lambda} \frac{\partial}{\partial \lambda} \left\| \nabla_{j,i} \right\|_{L^2(R^n_+)}^2 + O \left( \eta(\varepsilon)^2 \right).$$

**Proof.** First of all, observe that the following estimates hold:

$$\left| \nabla_{j,i} \frac{\partial Z_{j,i}}{\partial \tau_{\lambda}} \right| \leq \delta_{j,i} \times \left\{ \begin{array}{ll}
\delta_{j,i} \eta(\varepsilon) |x - \eta \varepsilon \varepsilon|^{1-n-\alpha} & \text{if } i = 1, \ldots, n - 1, \\
\delta_{j,i} \eta(\varepsilon) |x - \eta \varepsilon \varepsilon|^{1-n-\alpha} & \text{if } i = n.
\end{array} \right.$$  

(E.2)

Assume $j \neq \lambda$, take $\sigma > 0$ small enough such that $B^+_{\sigma}(\eta(\varepsilon) \varepsilon) \cap B_{\sigma}(\eta(\varepsilon) \varepsilon) = \emptyset$, and call $\Omega_{\sigma} = M \setminus B^+_{\sigma}(\eta(\varepsilon) \varepsilon) \cup B_{\sigma}(\eta(\varepsilon) \varepsilon)$. Then, by (D.7) and (E.2),

$$\left< \nabla_{j,i}, \frac{\partial W}{\partial \tau_{\lambda}} \right> = \left< \nabla_{j,i}, \frac{\partial W_{\lambda}}{\partial \tau_{\lambda}} \right> = \int_M c_n \nabla_g Z_{j,i} \cdot \nabla_g \frac{\partial W_{\lambda}}{\partial \tau_{\lambda}} + S_g Z_{j,i} \frac{\partial W_{\lambda}}{\partial \tau_{\lambda}}$$

$$= \int_{\Omega_{\sigma}} c_n \nabla_g Z_{j,i} \cdot \nabla_g \frac{\partial W_{\lambda}}{\partial \tau_{\lambda}} + S_g Z_{j,i} \frac{\partial W_{\lambda}}{\partial \tau_{\lambda}} + \int_{B^+_{\sigma}(\eta(\varepsilon) \varepsilon)} c_n \nabla_g Z_{j,i} \cdot \nabla_g \frac{\partial W_{\lambda}}{\partial \tau_{\lambda}}$$

$$+ \int_{B^+_{\sigma}(\eta(\varepsilon) \varepsilon)} S_g Z_{j,i} \frac{\partial W_{\lambda}}{\partial \tau_{\lambda}} + \int_{B_{\sigma}(\eta(\varepsilon) \varepsilon)} c_n \nabla_g Z_{j,i} \cdot \nabla_g \frac{\partial W_{\lambda}}{\partial \tau_{\lambda}} + \int_{B_{\sigma}(\eta(\varepsilon) \varepsilon)} S_g Z_{j,i} \frac{\partial W_{\lambda}}{\partial \tau_{\lambda}}$$

$$\leq O \left( \eta(\varepsilon) \delta_{j,i}^{\frac{n-2}{2\lambda}} \right) + c_n \delta_{j,i}^{\frac{n-2}{2\lambda}} \eta(\varepsilon) \int_{B^+_{\sigma}(\eta(\varepsilon) \varepsilon)} \delta_{j,i}^{\frac{n-2}{2\lambda}} |\nabla_{j,i}| + \delta_{j,i}^{\frac{n-2}{2\lambda}} \eta(\varepsilon) \int_{B_{\sigma}(\eta(\varepsilon) \varepsilon)} \delta_{j,i}^{\frac{n-2}{2\lambda}} |\nabla_{j,i}|$$

$$+ c_n \delta_{j,i}^{\frac{n-2}{2\lambda}} \eta(\varepsilon) \int_{B^+_{\sigma}(\eta(\varepsilon) \varepsilon)} \delta_{j,i}^{\frac{n-2}{2\lambda}} |\nabla_{j,i}|$$

$$+ c_n \delta_{j,i}^{\frac{n-2}{2\lambda}} \eta(\varepsilon) \int_{B^+_{\sigma}(\eta(\varepsilon) \varepsilon)} \frac{\partial}{\partial x_{\lambda}} \left( U + \delta_{\lambda} V_{\lambda} \right) + \delta_{j,i}^{\frac{n-2}{2\lambda}} \eta(\varepsilon) \int_{B^+_{\sigma}(\eta(\varepsilon) \varepsilon)} \frac{\partial}{\partial x_{\lambda}} \left( \frac{\partial}{\partial x_{\lambda}} \frac{\partial}{\partial \tau_{\lambda}} \right)$$

$$= O \left( \eta(\varepsilon) \delta_{j,i}^{\frac{n-2}{2\lambda}} \delta_{\lambda}^{\frac{n-2}{2\lambda}} |\ln \delta_{\lambda}| \right).$$
On the other hand, if $\lambda = j$,
\[
\left\langle Z_{j,i}^\lambda, \frac{\partial W}{\partial \tau_j^x} \right\rangle = \left\langle Z_{j,i}^\lambda, \frac{\partial W_j}{\partial \tau_j^x} \right\rangle = \int_M c_n \nabla_g Z_{j,i} \cdot \nabla_g \frac{\partial W_j}{\partial \tau_j^x} + S_g Z_{j,i} \frac{\partial W_j}{\partial \tau_j^x} \\
= \mathcal{O} \left( \eta(\varepsilon) \delta_j^{n-2} \right) + c_n \frac{\eta(\varepsilon)}{\delta_j} \int_{B^+_{\eta^2}} \nabla \delta_i \cdot \nabla \delta_s + \delta_j \eta(\varepsilon) \int_{B^+_{\eta^2}} S_g(0) \delta_i \delta_s \\
+ \delta_j^2 \eta(\varepsilon) \int_{B^+_{\eta^2}} S_g(0) \delta_i \frac{\partial V}{\partial x_s} = \delta^s c_n \frac{\eta(\varepsilon)}{\delta_j} \left\| \nabla \delta_i \right\|_{L^2(\mathbb{R}_+^n)}^2 + \mathcal{O}(\eta(\varepsilon))
\]
This proves the first part of the proposition. Now, reasoning as before,
\[
\left\| \frac{\partial Z_{j,i}}{\partial \tau_j} \right\|^2 = \delta_j^\lambda \left\langle Z_{j,i}^\lambda, \frac{\partial Z_{j,i}}{\partial \tau_j} \right\rangle = \delta_j^\lambda \left( \mathcal{O}(\eta(\varepsilon)^{n-2}) + \frac{\eta(\varepsilon)^2}{\delta_j} c_n \int_{B^+_{\eta^2}} \left\| \nabla \delta_i \right\|^2 + \eta(\varepsilon)^2 \int_{B^+_{\eta^2}} S_g(0) \left\| \frac{\partial j_i}{\partial x_s} \right\|^2 \right) \\
= \delta_j^\lambda \frac{\eta(\varepsilon)^2}{\delta_j^2} c_n \left\| \nabla \delta_i \right\|^2_{L^2(\mathbb{R}_+^n)} + \mathcal{O}(\eta(\varepsilon)^2).
\]

\[\blacksquare\]

**Proof of (2).**

(i) First we prove that
\[
J_\varepsilon(W + \Phi_\varepsilon) = J_\varepsilon(W) + \mathcal{O}(\|\Phi_\varepsilon\|^2) \quad \text{(E.3)}
\]
$C^0$—uniformly with respect to $(d_1, \ldots, d_k, \tau_1, \ldots, \tau_k)$ in a compact subset of $[0, +\infty)^k \times \mathcal{C}$. We will follow the arguments of the proof of Proposition 4.2, so many details will be skipped for the sake of brevity. By Taylor expansion, there exists $\sigma \in (0, 1)$ such that:
\[
J_\varepsilon(W + \Phi_\varepsilon) - J_\varepsilon(W) = \frac{1}{2} J''_\varepsilon(W + \sigma \Phi_\varepsilon)[\Phi_\varepsilon, \Phi_\varepsilon] \\
= \frac{1}{2} \int_{\partial M} H(W) \Phi_\varepsilon^2 + 2\varepsilon(n-1) \int_{\partial M} W \Phi_\varepsilon + \frac{1}{2} \|\Phi_\varepsilon\|^2 + \varepsilon(n-1) \int_{\partial M} \Phi_\varepsilon^2 \\
+ \frac{1}{2} \int_{\partial M} |K| g'(W + \sigma \Phi_\varepsilon) \Phi_\varepsilon^2 - c_n \frac{n-2}{4} \int_{\partial M} H \Phi_\varepsilon^2.
\]
Immediately, by Sobolev embedding:
\[
\varepsilon(n-1) \int_{\partial M} \Phi_\varepsilon^2 \lesssim \varepsilon \|\Phi_\varepsilon\|^2,
\]
Instead,
\[
\int_M S_g W \Phi_\varepsilon = \sum_{j=1}^k \int_M S_g \mu_j \Phi_\varepsilon + \delta_j \sum_{j=1}^k \int_M S_g \nu_j \Phi_\varepsilon
\]
and

\[
\int_M S_g U_j \Phi_\varepsilon \lesssim \|\Phi_\varepsilon\| \|U_j\| \frac{2(n-1)}{n} (M) \lesssim \|\Phi_\varepsilon\| \times \begin{cases} O(\delta_j^2) & \text{if } n \geq 7, \\
O(\delta_j^2 \ln(\delta_j))^\frac{2}{7} & \text{if } n = 6 \\
O(\delta_j^2) & \text{if } n = 4, 5. \end{cases}
\]

\[
\delta_j \int_M S_g V_j \Phi_\varepsilon \lesssim \delta \|\Phi_\varepsilon\| \|V_j\|_{L^2(M)} \lesssim \|\Phi_\varepsilon\| \times \begin{cases} O(\delta_j^2) & \text{if } n \geq 7, \\
O(\delta_j^2 \ln(\delta_j))^\frac{2}{7} & \text{if } n = 6 \\
O(\delta_j^2) & \text{if } n = 4, 5. \end{cases}
\]

Moreover

\[
\varepsilon \int_{\partial M} U_j \Phi_\varepsilon \lesssim \varepsilon \|\Phi_\varepsilon\| \|U_j\| \frac{2(n-1)}{n} (\partial M) \lesssim \varepsilon \|\Phi_\varepsilon\|_{H^1(M)} \times \begin{cases} O(\delta_j) & \text{if } n \geq 5, \\
O(\delta_j \ln(\delta_j))^\frac{2}{7} & \text{if } n = 4. \end{cases}
\]

\[
\varepsilon \int_{\partial M} V_j \Phi_\varepsilon \lesssim \varepsilon \|\Phi_\varepsilon\| \|V_j\|_{L^2(\partial M)} \lesssim \varepsilon \|\Phi_\varepsilon\| \times \begin{cases} O(\delta_j^2) & \text{if } n \geq 6, \\
O(\delta_j^2 \ln(\delta_j))^\frac{1}{2} & \text{if } n = 5 \\
O(\delta_j^2) & \text{if } n = 4. \end{cases}
\]

Integrating by parts,

\[
c_n \int_M \nabla g \nabla g \Phi_\varepsilon + \int_M |K| g(W) \Phi_\varepsilon - 2(n-1) \int_{\partial M} H f(W) \Phi_\varepsilon
= \sum_{j=1}^k \left( \int_M \left( -c_n \Delta g W_j \Phi_\varepsilon + |K| g(W_j) \Phi_\varepsilon - 2(n-1) \int_{\partial M} H f(W_j) \Phi_\varepsilon + \int_{\partial M} c_n \frac{\partial W_j}{\partial \nu} \Phi_\varepsilon \right) \right)
+ \int_M |K| \left( g(W) - g \left( \sum_j W_j \right) \right) \Phi_\varepsilon - 2(n-1) \int_{\partial M} H \left( f(W) - f \left( \sum_j W_j \right) \right) \Phi_\varepsilon
\lesssim \|\Phi_\varepsilon\| \times \begin{cases} O(\delta_j^2) & \text{if } n \geq 5, \\
O(\delta_j^2 \ln(\delta_j))^\frac{2}{7} & \text{if } n = 4 \end{cases}
+ \begin{cases} O((\varepsilon^{1-\alpha})^\frac{2}{7}) & \text{if } n \geq 5, \\
O\left( (\rho(\varepsilon))^3 \ln(\rho(\varepsilon))^\frac{2}{7} \right) & \text{if } n = 4. \end{cases}
\]

Finally,

\[
\int_M |K| g'(W + \sigma \Phi_\varepsilon) \Phi_\varepsilon^2 \lesssim \|\Phi_\varepsilon\|^2 \|W + \sigma \Phi_\varepsilon\| \frac{n-2}{L_{\sigma}(M)} \lesssim \|\Phi_\varepsilon\|^2
\]

and similarly

\[
\int_{\partial M} |K| g'(W + \sigma \Phi_\varepsilon) \Phi_\varepsilon^2 \lesssim \|\Phi_\varepsilon\|^2 \|W + \sigma \Phi_\varepsilon\| \frac{n-2}{L_{\sigma}(\partial M)} \lesssim \|\Phi_\varepsilon\|^2.
\]

Collecting all the above estimates (E.3) follows.

(ii) Next we estimate the leading term $J_\varepsilon(W)$. 

We claim that

\[ J_\varepsilon(W) := \sum_{i=1}^{k} J_\varepsilon(W_i) - \sum_{j<i} \int_M K g(W_i) W_j \, dv_g - \sum_{j<i} c_n \frac{n-2}{2} \int_{\partial M} H \overline{f}(W_i) W_j \, d\sigma_g \]

\[ + \sum_{i<j} \int_M (c_n \nabla g W_i \nabla g W_j + S_g W_i W_j - K g(W_i) W_j) \, dv_g \]

\[ - c_n \frac{n-2}{2} \int_{\partial M} H \left( \overline{\delta} \left( \sum_{i=1}^{k} W_i \right) - \sum_{i=1}^{k} \overline{\delta}(W_i) - \overline{f}(W_i) W_j \right) \, d\sigma_g \]

\[ - \int_M K \left( \Theta \left( \sum_{i=1}^{k} W_i \right) - \sum_{i=1}^{k} \Theta(W_i) - g(W_i) W_j \right) \, dv_g + (n-1)\varepsilon \sum_{i \neq j} \int_{\partial M} W_i W_j \, d\sigma_g \]

\[ = k \varepsilon - \sum_{i=1}^{k} \zeta_n (\delta_i) \left[ b_n \| \pi(p_i) \|^2 + o'_n(1) \right] - \varepsilon \sum_{i=1}^{k} \delta_i (c_n + o''_n(1)) \]

\[- \sum_{j<i} \frac{1}{|K|^{n-2}} \frac{\delta_i^{n-2}}{\delta_j^{n-2}} \eta(\varepsilon)^n \frac{1}{|\tau_i - \tau_j|^{n-2}} + o \left( \frac{\delta_i^{n-2}}{\delta_j^{n-2}} \frac{\eta(\varepsilon)^n}{\eta(\varepsilon)^n} \right). \]

The contribution of each single bubble is encoded in the first term (I) whose expansion is given in Proposition 3.3. All the other terms come from the interaction among different bubble. First we estimate the leading term \((II) + (III)\).

For any \( h = 1, \ldots, k \) let \( B^+_h := B^+ (\eta(\varepsilon) \tau_h, \eta(\varepsilon) \tau_h^*) \subset B^+_R \) provide \( \sigma \) is small enough and moreover \( B^+_h \) are disjoint each other and \( \partial' B^+_h = B^+_h \cap \partial \mathbb{R}^n_+ \).

\[(II) = \int_{B^+_h} K g(W_i(x)) W_j(x)|g(x)|^{\frac{1}{2}} \, dx + \int_{B^+_R \setminus B^+_h} K g(W_i(x)) W_j(x)|g(x)|^{\frac{1}{2}} \, dx \]

\[ + \int_{B^+_R} (1 - \chi^{2*}(|x|)) K g(W_i(x)) W_j(x)|g(x)|^{\frac{1}{2}} \, dx \]
Now, the main term in (II) is given by

\[
\int_{B^+_i} K g (\mathcal{W}_i(x)) \mathcal{W}_j(x) |g(x)|^{\frac{1}{2}} dx = \\
\int_{B^+_i} K g \left( \delta_i^{n-2} U \left( \frac{x - \eta(\varepsilon) \tau_i}{\delta_i} \right) + \delta_i \delta_i^{n-2} V_{\eta} \left( \frac{x - \eta(\varepsilon) \tau_i}{\delta_i} \right) \right) \times \\
\left( \delta_j^{n-2} U \left( \frac{x - \eta(\varepsilon) \tau_j}{\delta_j} \right) + \delta_j \delta_j^{n-2} V_{\eta} \left( \frac{x - \eta(\varepsilon) \tau_j}{\delta_j} \right) \right) |g(x)|^{\frac{1}{2}} dx \\
= \delta_i^{n-2} \delta_j^{n-2} \frac{\alpha_n}{|K|^{\frac{n+2}{n}}} \int_{B^+_i} K g (U + \delta_i V_{\eta}) \times \\
\left( \frac{1}{(\delta_i y + \eta(\varepsilon) \tau_i)^2 + (\delta_i y + \delta_j D_n)^2 - \delta_j^{n-2}} \right) |g(\delta_i y + \eta(\varepsilon) \tau_i)|^{\frac{1}{2}} dy \\
+ O \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(\varepsilon)^{n-3}} \int_{B^+_i} g (U + \delta_i V_{\eta}) dy \right) \\
= \frac{\alpha_n}{|K|^{\frac{n+2}{n}}} \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(\varepsilon)^{n-2}} K \int_{\mathbb{R}^n_+} U^{2^*-1} dy + o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(\varepsilon)^{n-2}} \right) \\
= -\frac{\alpha_n}{|K|^{\frac{n+2}{n}}} \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(\varepsilon)^{n-2}} \int_{\mathbb{R}^n_+} \frac{1}{(y^2 + (y_n + D_n)^2 - 1)^{\frac{n-2}{2}}} dy + o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(\varepsilon)^{n-2}} \right) \\
= -\alpha_n^{2^n} \omega_{n-1} \int_0^{+\infty} \frac{r^{n-2}}{(1 + r^2)^{\frac{n+2}{2}}} dr \phi \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(\varepsilon)^{n-2}} \frac{1}{|K|^{\frac{n+2}{n}}} + o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(\varepsilon)^{n-2}} \right) \\
= -\delta_n \left( \frac{D_n}{(D_n^2 - 1)^2} - 1 \right) \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(\varepsilon)^{n-2}} \frac{1}{|K|^{\frac{n+2}{n}}} + o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(\varepsilon)^{n-2}} \right) .
\]

Now

\[
\left| \int_{B^+_i \setminus B^+_i} K g (\mathcal{W}_i(x)) \mathcal{W}_j(x) |g(x)|^{\frac{1}{2}} dx \right| \\
\leq \int_{\mathbb{R}^n_+ \setminus (\eta(\varepsilon) \tau_i, \eta(\varepsilon) \tau_j)} \frac{\delta_i^{n+2}}{|x - \eta(\varepsilon) \tau_i|^{n+2}} \frac{\delta_j^{n+2}}{|x - \eta(\varepsilon) \tau_j|^{n+2}} dx = (setting \ x = \eta(\varepsilon) y) \\
= c \frac{\delta_i^{n+2} \delta_j^{n+2}}{\eta(\varepsilon)^n} \int_{\mathbb{R}_+^2 \setminus (\tau_i, \tau_j)} \frac{1}{|y - \tau_i|^{n+2}} \frac{1}{|y - \tau_j|^{n+2}} dx \\
= o \left( \frac{\delta_i^{n+2} \delta_j^{n+2}}{\eta(\varepsilon)^n} \right)
\]
and similarly
\[ \left| \int_{B^*_R} (1 - \chi^2(x)) K g(W_i(x)) W_j(x) \right| dx = o \left( \frac{n}{(\eta(x)^{n-2})^2} \right). \]

For the interaction terms (III), we argue as before, obtaining that
\[ 2(n-1) \int_{\partial M} H f(W_i) W_j \]
\[ = 2(n-1) \alpha_n \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^{n-2}} \frac{H}{|K|^{1/2}} \int_{\mathbb{R}^n-1} U^{2n-1} d\tilde{x} (1 + o(1)) \]
\[ = 2(n-1) \alpha_n \frac{\omega_{n-1}}{\sqrt{n(n-1)}} \int_0^{+\infty} \frac{r^{n-2}}{(1 + r^2)^{1/2}} dr \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^{n-2}} \frac{\mathcal{D}_n}{|K|^{1/2}} (\mathcal{D}_n - 1)^{1/2} (1 + o(1)). \]

Then,
\[ (II) + (III) = \sum_{j < i} d_n \frac{1}{|K|^{n-2}} \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^{n-2}} \frac{1}{|\tau_i - \tau_j|^{n-2}} + o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^{n-2}} \right) \]
since a simple computation shows that \( d_n - b_n = 0 \).

Now we evaluate the remaining terms.

For \( i \neq j \)
\[ \left| \epsilon \int_{\partial M} W_i W_j d\sigma_g \right| \lesssim \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^{n-2}} \frac{1}{|\tau_i - \tau_j|^{n-2}} \frac{1}{|\tilde{y} - \tilde{\tau_i}|^{n-2}} \frac{1}{|\tilde{y} - \tilde{\tau_j}|^{n-2}} d\tilde{y} = o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^{n-2}} \right) \]

Now
\[ \sum_{i < j} \left[ \int_M \nabla_g W_i \nabla_g W_j + S_g W_i W_j - K g(W_i) W_j \right] d\sigma_g - 2(n-1) \sum_{i < j} \int_{\partial M} H f(W_i) W_j d\sigma_g \]
\[ = \sum_{i < j} \left[ \int_M \left( -c_n \Delta_g W_i + S_g W_i - K g(W_i) \right) W_j d\sigma_g \right] + \sum_{i < j} c_n \int_{\partial M} \left( \frac{\partial W_i}{\partial \nu} - \frac{n-2}{2} H f(W_i) \right) W_j d\sigma_g \]

and by using (C.1) we get that
\[ \left| -c_n \Delta_g W_i + S_g W_i - K g(W_i) \right| \lesssim \frac{\delta_i^{n-2} \delta_j^{n-2}}{(|\tilde{x} - \eta(x)\tilde{\tau_i}|^2 + (x_n - \eta(x)\tau_{i,n} + \mathcal{D}_n d_i)^2 - \delta_i^{n-2}} \]

Hence
\[ \sum_{i < j} \left[ \int_M \left( -c_n \Delta_g W_i + S_g W_i - K g(W_i) \right) W_j d\sigma_g \right] \lesssim \sum_{i < j} \int_{B^*_R} \frac{\delta_i^{n-2} \delta_j^{n-2}}{x - \eta(x)\tau_i^{n-2}} |x - \eta(x)\tau_j|^{n-2} \]
\[ \lesssim \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^{2(n-2)}} \eta(x)^n \int_{\mathbb{R}^n} \frac{1}{|y - \tau_i|^{n-2}} \frac{1}{|y - \tau_j|^{n-2}} = o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^{n-2}} \right) \]

and similarly
\[ \sum_{i < j} c_n \int_{\partial M} \left( \frac{\partial W_i}{\partial \nu} - \frac{n-2}{2} H f(W_i) \right) W_j d\sigma_g = o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^{n-2}} \right) \]
Now
\[
\int_M K \left[ \sum_{j=1}^k \mathcal{W}_j - \sum_{j=1}^k \mathcal{G}(W_j) - \sum_{i \neq j} g(W_i)W_j \right] dx
\]
\[
= \sum_{h=1}^k \int_{B_h^+} K \left[ \sum_{j=1}^k W_j - \sum_{j=1}^k \mathcal{G}(W_j) - \sum_{i \neq j} g(W_i)W_j \right] |g(x)|^{\frac{1}{2}} dx
\]
\[
+ \int_{B_R^+ \setminus \bigcup_h B_h^+} K \left[ \sum_{j=1}^k W_j - \sum_{j=1}^k \mathcal{G}(W_j) - \sum_{i \neq j} g(W_i)W_j \right] |g(x)|^{\frac{1}{2}} dx
\]
\[
+ \int_{B_R^+} (1 - \chi^2(|x|)) K \left[ \sum_{j=1}^k W_j - \sum_{j=1}^k \mathcal{G}(W_j) - \sum_{i \neq j} g(W_i)W_j \right] |g(x)|^{\frac{1}{2}} dx
\]
It is immediate that
\[
\left| \int_{B_R^+} (1 - \chi^2(|x|)) K \left[ \sum_{j=1}^k W_j - \sum_{j=1}^k \mathcal{G}(W_j) - \frac{2n}{n-2} \sum_{i \neq j} g(W_i)W_j \right] |g(x)|^{\frac{1}{2}} dx \right|
\]
\[
= \mathcal{O} \left( \delta_j^n + \delta_i^2 \frac{\eta j}{\eta \epsilon |\epsilon| - 2} \right).
\]
Now, outside the \(k\)-balls
\[
\left| \int_{B_R^+ \setminus \bigcup_h B_h^+} K \left[ \sum_{j=1}^k W_j - \sum_{j=1}^k \mathcal{G}(W_j) - \sum_{i \neq j} g(W_i)W_j \right] |g(x)|^{\frac{1}{2}} dx \right|
\]
\[
\leq \sum_{i \neq j} \int_{B_R^+ \setminus \bigcup_h B_h^+} \left| W_i |^{2^*-2} W_j^2 + | W_j |^{2^*-2} W_i^2 \right| dx = o \left( \frac{\delta_j^{n-2}}{\eta (\epsilon)^{n-2}} \right)
\]
because if \(i \neq j\)
\[
\int_{B_R^+ \setminus \bigcup_h B_h^+} | W_i |^{2^*-2} W_j^2 \leq \int_{B_R^+ \setminus \bigcup_h B_h^+} \frac{\delta_i^2}{|x - \eta \epsilon \tau_j|^4} \frac{\delta_j^{n-2}}{|x - \eta \epsilon \tau_j|^{2(n-2)}} dx
\]
\[
\leq \frac{\delta_i^{n-2}}{\eta \epsilon |\epsilon| - 2} \left( \frac{\eta j}{\eta \epsilon |\epsilon| - 2} \right).
\]
On each ball \(B_h^+\) we also have
\[
\int_{B_h^+} K \left[ \sum_{j=1}^k W_j - \sum_{j=1}^k \mathcal{G}(W_j) - \sum_{i \neq j} g(W_i)W_j \right] |g(x)|^{\frac{1}{2}} dx
\]
\[
\leq \int_{B_h^+} \left| \mathcal{G}(W_h + \sum_{i \neq h} W_i) - \mathcal{G}(W_h) - \sum_{j \neq h} g(W_h)W_j \right| dx + \sum_{i \neq h} \int_{B_h^+} |\mathcal{G}(W_i)| dx
\]
\[
+ \sum_{i \neq j} \int_{B_h^+} |g(W_h)W_j| dx \leq \sum_{i \neq h} \int_{B_h^+} W_i^{2^*-2} W_i^2 + c \sum_{i \neq h} \int_{B_h^+} W_i^2 dx
\]
\[
+ c \sum_{i \neq h} \int_{B_h^+} W_i^{2^*-1} W_j dx
\]
Now if \( i \neq h \) and \( n \geq 5 \) then
\[
\int_{B_h^+} W_h^{2(n-2)} W_i^2 \lesssim \int_{B_h^+} \frac{\delta_h^{n-2}}{|x - \eta(x)\tau_i|^4} \frac{\delta_i^{n-2}}{|x - \eta(x)\tau_i|^{2(n-2)}} \, dx
\]
\[
\lesssim \frac{\delta_h^{n-2}}{\eta(x)^n} \int_{B_h^+(\eta(x)/2)} \frac{1}{|y - \tau_i|^4} \frac{1}{|y - \tau_i|^{2(n-2)}} = o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^n} \right).
\]

If, instead, \( n = 4 \) then
\[
\int_{B_h^+} W_h^{2(n-2)} W_i^2 \lesssim \frac{\delta_h^{n-2} \delta_i^{n-2}}{\eta(x)^n} \int_{B_h^+(\eta(x)/2)} \frac{1}{(|x|^2 + 1)^2} \, dx
\]
\[
\lesssim \frac{\delta_h^{n-2} \delta_i^{n-2}}{\eta(x)^n} \left| \ln \frac{\eta(x)}{\delta_h} \right| = o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^n} \right).
\]

If \( i, j \neq h \) then
\[
\int_{B_h^+} W_i^{2(n-1)} W_j \leq \int_{B_h^+} \frac{\delta_i^{n-2}}{|x - \eta(x)\tau_i|^{n+2}} \frac{\delta_j^{n-2}}{|x - \eta(x)\tau_j|^{n-2}}
\]
\[
\leq \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^n} = o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^n} \right).
\]

If \( i \neq h \) then
\[
\int_{B_h^+} W_i^{2(n-1)} W_h \leq \int_{B_h^+} \frac{\delta_i^{n-2}}{|x - \eta(x)\tau_i|^{n+2}} \frac{\delta_h^{n-2}}{|x - \eta(x)\tau_h|^{n-2}}
\]
\[
\leq \frac{\delta_i^{n-2} \delta_h^{n-2}}{\eta(x)^n} = o \left( \frac{\delta_i^{n-2} \delta_h^{n-2}}{\eta(x)^n} \right)
\]

Finally, if \( i \neq h \)
\[
\int_{B_h^+} W_i^{2(n-1)} \leq \int_{B_h^+} \frac{\delta_i^n}{|x - \eta(x)\tau_i|^n} \leq \frac{\delta_i^n}{\eta(x)^n} = o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^n} \right)
\]

In a similar way
\[
\int_{\partial M} \left[ \mathfrak{B} \left( \sum_i W_i - \sum_i \mathfrak{B}(W_i) - \sum_{i \neq j} (W_i)W_j \right) \right] \, d\nu_g
\]
\[
= \sum_{h=1}^k \int_{\partial' B_h^+} \left[ \ldots \right] |g(x)|^{\frac{2}{n-2}} \, dx + \int_{\partial' B_h^+ \setminus \bigcup_h \partial' B_h^+} \left[ \ldots \right] |g(x)|^{\frac{2}{n-2}} \, dx
\]
\[
+ \int_{\partial' B_h^+} (1 - \chi^{2n}) \left[ \ldots \right] |g(x)|^{\frac{2}{n-2}}
\]
\[
= o \left( \frac{\delta_i^{n-2} \delta_j^{n-2}}{\eta(x)^n} \right).
\]
Let us look at the main term of (E.4). Let $\mathcal{Q}(p)$ be the quadratic form associated with the second derivative of $p \to \|\pi(p)\|^2$ (being zero the first derivative) If $n = 4$ by (4.7)

$$
\sum_{i=1}^{k} (\delta_{i}^2 \ln \delta_i) \left[ b_i \|\pi(p_i)\|^2 + o'_{n}(1) \right] - \varepsilon \sum_{i=1}^{k} \delta_i (c_n + o''_{n}(1))
$$

$$
- \sum_{j<i} \partial_{n} \frac{1}{|K|^{n-2}} \frac{\delta_{i}^{-n-2} \delta_{j}^{-n-2}}{n_{i}-n_{j}} + o \left( \frac{\delta_{i}^{-n-2} \delta_{j}^{-n-2}}{n_{i}-n_{j}} \right)
$$

$$
= \sum_{i=1}^{k} \rho_{i}^2 (d_{0} + \eta d_{j})^2 (|\ln \rho| + \mathcal{O}(1)) b_i \left[ \|\pi(p)\|^2 + \frac{1}{2} \eta^2 \mathcal{Q}(\tau_i, \tau_i) + \mathcal{O}(\eta^3) + o'_{n}(1) \right]
$$

$$
- \varepsilon \sum_{i=1}^{k} \delta_i (c_n + o''_{n}(1))
$$

$$
- \frac{\rho_{i}^2}{\eta^2} \sum_{j<i} \partial_{n} \frac{1}{|K|^{n-2}} \frac{d_{0}^2}{n_{i}-n_{j}} + o \left( \frac{\rho_{i}^2}{\eta^2} \right)
$$

and the claim follows because of the choice of $d_0$ in (4.3) and the fact that since $\eta = |\ln \rho|^{-\frac{1}{4}}$ (see (3.32))

$$
o'_{n}(1) = \mathcal{O}(|\ln \rho|) = o(\eta^2).
$$

If $5 \leq n \leq 7$ by (4.6)

$$
\sum_{i=1}^{k} \delta_{i}^2 \left[ b_i \|\pi(p_i)\|^2 + o'_{n}(1) \right] - \varepsilon \sum_{i=1}^{k} \delta_i (c_n + o''_{n}(1))
$$

$$
- \sum_{j<i} \partial_{n} \frac{1}{|K|^{n-2}} \frac{\delta_{i}^{-n-2} \delta_{j}^{-n-2}}{n_{i}-n_{j}} + o \left( \frac{\delta_{i}^{-n-2} \delta_{j}^{-n-2}}{n_{i}-n_{j}} \right)
$$

$$
= \sum_{i=1}^{k} \varepsilon^2 (d_{0} + \eta d_{j})^2 b_n \left[ \|\pi(p)\|^2 + \frac{1}{2} \eta^2 \mathcal{Q}(\tau_i, \tau_i) + \mathcal{O}(\eta^3) + o'_{n}(1) \right]
$$

$$
- \varepsilon \sum_{i=1}^{k} \delta_i (c_n + o''_{n}(1))
$$

$$
- \frac{\varepsilon^{n-2}}{\eta^{n-2}} \sum_{j<i} \partial_{n} \frac{1}{|K|^{n-2}} \frac{d_{0}^{n-2}}{n_{i}-n_{j}^{n-2}} + o \left( \frac{\varepsilon^{n-2}}{\eta^{n-2}} \right)
$$

and the claim follows because of the choice of $d_0$ in (4.3) and the fact that since $\eta = \varepsilon^{-\frac{1}{4}}$ (see (3.32))

$$
o'_{n}(1) = \mathcal{O}(\varepsilon) = o(\eta^2) \text{ if } n = 6, 7 \text{ and } o'_{n}(1) = \mathcal{O}(\varepsilon |\ln \varepsilon|) = o(\eta^2) \text{ if } n = 5.
$$

We point out that in higher dimensions $n \geq 8$ this is not true anymore.

\[ \square \]

References

[1] W. Abdelhedi, H. Chtioui, M.O. Ahmedou, A Morse theoretical approach for the boundary mean curvature problem on $B^4$, J. Funct. Anal. 254 (2008), no. 5, 1307–1341.
[2] S.M. Almaraz, A compactness theorem for scalar-flat metrics on manifolds with boundary. Calc. Var. Part. Differ. Equ. 41 (2011), no. 3–4, 341–386.
[3] S.M. Almaraz, An existence theorem of conformal scalar flat metrics on manifolds with boundary, Pacific J. Math. 248 (2010), no. 1, 1–22.
[4] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, Springer Monographs in Mathematics. Springer, Berlin (1998).
[5] A. Ambrosetti, Y.Y. Li, A. Malchiodi, *On the Yamabe problem and the scalar curvature problems under boundary conditions*, Math. Ann. **322** (2002), no. 4, 667–699.

[6] M. Ben Ayed, M.O. Ahmedou, *Non simple blow ups for the Nirenberg problem on half spheres*, https://arxiv.org/abs/2108.08608

[7] M. Ben Ayed, K. El Mehdi, M.O. Ahmedou, *Prescribing the scalar curvature under minimal boundary conditions on the half sphere*, Adv. Nonlinear Stud. **2** (2002), no. 2, 93–116.

[8] M. Ben Ayed, K. El Mehdi, M.O. Ahmedou, *The scalar curvature problem on the four dimensional half sphere*, Calc. Var. Part. Differ. Equ. **22** (2005), no. 4, 465–482.

[9] S. Brendle, S.Y.S. Chen, *An existence theorem for the Yamabe problem on manifolds with boundary*, J. Eur. Math. Soc. **16** (2014), no. 5, 991–1016.

[10] A. Chang, X. Xu, P. Yang, *A perturbation result for prescribing mean curvature*, Math. Ann. **310** (1998), no. 3, 473–496.

[11] X. Chen, P.T. Ho, L. Sun, *Liming Prescribed scalar curvature plus mean curvature flows in compact manifolds with boundary of negative conformal invariant*, Ann. Global Anal. Geom. **53** (2018), no. 1, 121–150.

[12] X. Chen, Y. Ruan, L. Sun, *The Han-Li conjecture in constant scalar curvature and constant boundary mean curvature problem on compact manifolds*, Adv. Math. **358** (2019), 56 pp.

[13] P. Cherrier, *Problèmes de Neumann non linéaires sur les varietes riemanniennes*, J. of Funct. Anal. **57** (1984), 154–206.

[14] M. Chipot, M. Fila, I. Shafrir, *On the Solutions to some Elliptic Equations with Nonlinear Neumann Boundary Conditions*. Adv. in Diff. Eq., **1**, 1 (1996), 91–110.

[15] S. Cruz-Blázquez, A. Malchiodi, D. Ruiz, *Conformal metrics with prescribed scalar and mean curvature, to appear on J. für die Reine und Ang. Math.*

[16] Z. Djadli, A. Malchiodi, M.O. Ahmedou, *Prescribing Scalar and Boundary Mean Curvature on the Three Dimensional Half Sphere*, The Journal of Geom. Anal. **13** (2003), no. 2, 255–289.

[17] Z. Djadli, A. Malchiodi, M.O. Ahmedou, *The prescribed boundary mean curvature problem on $B^3$*, J. Diff. Eqs. **206** (2004), no. 2, 373–398.

[18] O. Druet, E. Hebey, *Blow-up examples for second order elliptic PDEs of critical Sobolev growth*, Trans. Amer. Math. Soc. **357** (5) (2005) 1915–1929.

[19] J. Escobar, *Conformal deformation of a Riemannian metric to a scalar at metric with constant mean curvature on the boundary*, Ann. Math. **136** (1992), 1–50.

[20] J. Escobar, *Conformal metrics with prescribed mean curvature on the boundary*, Calc. Var. **4** (1996), 559–592.

[21] J. Escobar, *The Yamabe problem on manifolds with boundary*, J. Differ. Geom., **35** (1992), 21–84.

[22] M. Ghimenti, A.M. Micheletti, A. Pistoia, *Blow-up phenomena for linearly perturbed Yamabe problem on manifolds with umbilic boundary*, J. Differential Equations **267** (2019), no. 1, 587–618.

[23] M. Ghimenti, A.M. Micheletti, A. Pistoia, *Blow-up solutions concentrated along minimal submanifolds for some supercritical elliptic problems on Riemannian manifolds*, J. Fixed Point Theory Appl. **14** (2013), no. 2, 503–525.

[24] M. Ghimenti, A.M. Micheletti, A. Pistoia, *Linear Perturbation of the Yamabe Problem on Manifolds with Boundary*, J. Geom. Anal. **28** (2018) 1315–1340.

[25] Z.C. Han, Y.Y. Li, *The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature*, Comm. Anal. Geom. **8** (2000), 809–869.

[26] Z.C. Han, Y.Y. Li, *The Yamabe problem on manifolds with boundaries: existence and compactness results*, Duke Math. J. **99** (1999) 489–542.

[27] Y.Y. Li, *The Nirenberg problem in a domain with boundary*, Topol. Meth. Nonlinear Anal. **6** (1995), no. 2, 309–329.

[28] Y.Y. Li, *Prescribing scalar curvature on $S^n$ and related problems. II. Existence and compactness*, Comm. Pure Appl. Math. **49** (1996), no. 6, 541–597.

[29] Y.Y. Li, M. Zhu, *Uniqueness theorems through the method of moving spheres*, Duke Math. J. **80** (1995), no. 2, 383–417.

[30] Y.Y. Li, M. Zhu, *Yamabe type equations on three-dimensional Riemannian manifolds*, Commun. Contemp. Math. **1** (1) (1999) 1–50.

[31] F.C. Marques, *Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary*, Comm. Anal. Geom. **15** (2007), no. 2, 381–405.

[32] F.C. Marques, *Existence results for the Yamabe problem on manifolds with boundary*, Indiana Univ. Math. J. (2005), 1599–1620.

[33] M. Mayer, C.B. Ndiaye, *Barycenter technique and the Riemann mapping problem of Cherrier-Escobar*, J. Differential Geom. **107** (2017), no. 3, 519–560.

[34] A.M. Micheletti, A. Pistoia, *A generic result on Weyl tensor*, Top. Meth. Nonlin. Anal. **53** (2019), no. 1, 257–269.

[35] A.M. Micheletti, A. Pistoia, *Generic properties of critical points of the Weyl tensor*, Adv. Nonlinear Stud. **17** (2017), no. 1, 99–109.

[36] F. Morabito, A. Pistoia, G. Vaira, *Towering phenomena for the Yamabe equation on symmetric manifolds*, Potential Anal. **47** (1) (2017), 53–102.
[37] A. Pistoia, G. Vaira, *Clustering phenomena for linear perturbation of the Yamabe equation*, Partial Differential Equations Arising from Physics and Geometry, London Mathematical Society Lecture Note Series 450 (2019), 311–331. Cambridge University Press, Cambridge.

[38] B. Premoselli, *Towers of bubbles for Yamabe-type equations and for the Brézis-Nirenberg problem in dimensions n ≥ 7*, J. Geom. Anal. 32 (2022), no. 3, paper no. 73, 65 pp.

[39] F. Punzo, *On well-posedness of semilinear parabolic and elliptic problems in the hyperbolic space*, J. Differential Equations 251 (2011), no. 7, 1972–1989.

[40] F. Robert, J. Vétois, *Examples of non-isolated blow-up for perturbations of the scalar curvature equation on non-locally conformally flat manifolds*, J. Differential Geom. 98 (2) (2014) 349–356.

[41] P.D. Thizy, J. Vétois, *Positive clusters for smooth perturbations of a critical elliptic equation in dimensions four and five*, J. Funct. Anal. 275 (2018), no. 1, 170–195.

[42] X. Xu, H. Zhang, *Conformal metrics on the unit ball with prescribed mean curvature*, Math. Ann. 365 (2016), no. 1–2, 497–557.

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