Number-resolving photon detectors with quantum emitters coupled to waveguides

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Number-resolving single-photon detectors represent a key technology for a host of quantum optics protocols, comprising quantum state preparation, quantum metrology, entanglement distribution, and quantum computing. Despite significant advances over the last few decades, state-of-the-art technology can distinguish only very low photon numbers with high fidelity. We show that (artificial) atoms coupled to photonic waveguides constitute an ideal platform to entirely absorb incident photonic wavepackets, even in the presence of disorder and finite Purcell factors, with a fidelity increasing with the number of atoms. Absorption is achieved through engineering a decay channel to a metastable state, such that readout of the atomic state after absorption allows number-resolving photon detection.

**Introduction.**—Single-photon detectors have a long history [1], with a plethora of technologies available [2]. Applications in quantum optics, such as quantum state preparation, quantum metrology [3], entanglement distribution [4], and quantum computing [5, 6] have placed a renewed focus on single-photon detectors capable of resolving the number of incoming photons. Perhaps the most promising technology in this regard are superconducting transition-edge sensors, which have been demonstrated to achieve a (per-photon) detection efficiency (percentage of detected photons) of \( \eta \approx 95\% \) [7] and to distinguish up to seven photons, with a negligible dark count rate (clicks in the absence of incoming photons). They are based on the principle that near the critical temperature, the resistance of a superconductor is very sensitive to temperature changes, down to the level of single-photon energies. While very impressive, this is limited to optical photons, and implies that for a pulse containing seven photons, there is a 30\% chance of losing one.

Another strategy, which does not rely on direct or indirect energy measurements, can generically be described as a setup in which incoming photons interact with a local degree of freedom that can be measured with a strong measurement, such as the state of a few-level system. This has been used to detect single microwave photons, for example with Josephson photomultipliers [8–11] or impedance-matched \( \Lambda \)-systems [12–14], and may enable a quantum non-demolition (QND) photon number measurement [15–17].

In recent years, quantum emitters coupled to waveguides have emerged as a powerful experimental paradigm [18, 19]. Examples of such systems include cold atoms levitated near optical fibres [20, 21] or photonic crystal waveguides [22], but also solid-state realizations such as quantum dots [23, 24], superconducting qubits [25, 26], or nitrogen-vacancy centres [27–29]. As a result of the strong confinement of light, even a single emitter can have a profound effect on light propagation, and many emitters show remarkable collective effects [30]. At the same time, impressive experimental breakthroughs have enabled the control [31] and, in particular, readout of atoms [32], superconducting qubits [33], and trapped ions [34] with essentially unity fidelity (99.94\% in arrays of 160 atoms in recent experiments [32]). Thus, quantum emitters appear to be ideally suited for number-resolving photon detectors.

In the absence of additional dissipation, a photon absorbed by an atom is re-emitted. Irreversible absorption requires a second decay channel, allowing the atom decay to a third, metastable state \(|s\rangle\). This way, the atom keeps a memory of the absorbed photon [12–14]. Generalizing to arrays of quantum emitters may enable number-resolving photon detection, as has been first proposed in the context of circuit quantum electrodynamics [35–37].

Building on this idea, we use input-output theory to show that full absorption can generically be attained by engineering an additional decay channel. We elucidate the underlying mechanism by showing that full absorption corresponds to moving a pole of the inverse scattering matrix for the system onto the real line. This allows us to draw conclusions about detector properties from only the eigenvalues of the system. In view of experimental realizations, we provide recipes how to engineer an additional decay channel in atomic arrays, and pay particular attention to positional disorder of the atoms, as the latter fundamentally modifies the eigenmodes of the system. Surprisingly, it is possible to engineer dissipation to achieve full absorption independent of disorder. Even with moderate Purcell factor (ratio of waveguide to free-space decay rate), we find that detection efficiencies \( \eta > 99\% \) can be reached for...
intermediate numbers of atoms ($N > 20$), as a consequence of collective enhancement in the atom-waveguide coupling. Both detection efficiency and bandwidth grow with atom number. The scheme is scalable, as the number of atoms required grows polynomially with the number of photons. Finally, we also considering emitters coupled to chiral waveguides, which have distinct advantages due to the natural suppression of backscattering.

Absorption in atomic arrays.—We first illustrate the general principle of photon absorption in an atomic array coupled to a one-dimensional waveguide. Below, we consider both infinite waveguides (Fig. 1b) and semi-infinite waveguides terminated by a mirror (Fig. 1a), but the principle remains the same. A key ingredient in our proposal is tunable dissipation, engineered for example by driving a forbidden transition from |e⟩ to some auxiliary state |f⟩ that decays to a fourth level |s⟩, but neither to |g⟩ nor to |e⟩, as sketched in Fig. 1c. We discuss experimental details further below and in the Supplementary Information (SI).

A generic Hamiltonian describing the system-waveguide interaction is given through

$$H = \int_0^\infty \frac{dk}{2\pi} \sum_\nu (\omega_k - \omega_0) a_{\nu, k}^\dagger a_{\nu, k} - \sum_{n, \nu} g_{\nu, n} \beta(n) a_{\nu, k} + \text{H.c.}$$

In Eq. (1), the label $\nu$ runs over different sets of waveguide modes, and the individual modes are labelled by their wavevector $k$. In an infinite waveguide (Fig. 1b), $g_{\nu, n} = \sqrt{2\Gamma_{id}} \exp(ikx_n)$ and $\nu \in \pm$, corresponding to left- and right-moving modes, whereas for the semi-infinite waveguide (Fig. 1a), there is only one set of waveguide modes with coupling $g_k = \sqrt{\Gamma_{id}} \sin(kx_n)$. In these expressions, $\Gamma_{id}$ is the decay rate of an individual atom into the waveguide, and $\omega_k = ck$. Integrating out the bath modes yields quantum Langevin equations for the spin operators (see SI), which we linearize using a Holstein-Primakoff transformation, valid for small numbers of excitations in the atomic chain [18, 38-40]

$$\dot{b} = (-i\Gamma_{eff} - \Gamma'/2) b + b_{in}.$$

In Eq. (2), the vector $b$ contains lowering operators for the (bosonized) $N$ atoms, the non-Hermitian Hamiltonian $H_{eff}$ describes both coherent interaction and decay into the waveguide, and the coupling of each mode to the input operator $a_{in}$ is given by the generically non-square matrix $L$. In the infinite waveguide, $a_{in} = (a_{in,+}, a_{in,-}, \ldots)$, such that $L$ is a $N \times 2$ matrix, whereas in the semi-infinite waveguide, there is only input mode, such that $L$ is a vector. In Eq. (2) we have already explicitly included the additional decay channels, which consists of engineered and free-space decay at total rate $\Gamma' = \Gamma_{eng} + \Gamma_{free}$. The derivation below works for generic, Hermitian $\Gamma'$, but here we only consider uniform dissipation for simplicity. Strictly speaking, decay to |s⟩ eliminates the atom from the dynamics rather than de-exciting it, but this effect is neglected in the linearization. The Langevin equations are accompanied by input-output equations

$$a_{out}(\omega) = \left\{ 1 - L^\dagger [(\Gamma'/2 - i\omega) + iH_{eff}]^{-1} L \right\} a_{in}(\omega)$$

$$\equiv S(\omega) a_{in}(\omega).$$

(3)

A useful detector will count the number of photons in a specific input port (photons impinging on the system from the left, say). This is captured by our key figure of merit, the detection efficiency $\eta$, which is the product of the probability that a (right-moving, say) photon in the waveguide is absorbed $p_{abs} = 1 - \sum_{\nu=\pm} |S_{\nu}(\omega)|^2$, and the probability that it is dissipated via the engineered channel rather than into free space. $\eta = p_{abs} \Gamma_{eng}/(\Gamma_{end} + \Gamma_{free})$. Thus, we require both $\Gamma_{eng} \gg \Gamma_{free}$ and $p_{abs} \approx 1$.

Near unity absorption is attained if one of the eigenvalues of the scattering matrix $S(\omega)$ is zero. This corresponds to a pole of the inverse scattering matrix $S^{-1} = [1 + L^\dagger (-i\omega + iH_{eff} + \Gamma'/2)^{-1} L]$ (derived in SI). A pole of $S^{-1}$ arises whenever $\omega$ coincides with an eigenvalue of $H_{eff}^2 - \Gamma'/2$. We can immediately conclude that the scattering matrix $S$ has a zero if $\omega = -i\Gamma'/2$ coincides with an eigenvalue of $H_{eff}$. Absorption based on this principle has been observed in a variety of systems [41-43], and later been termed coherent perfect absorption (CPA) [44]. The key point here is that to reach this conclusion we did not have to assume anything about the form of $H_{eff}$ (apart from linearity), which is the reason it works for arbitrary disordered systems. If these conditions are fulfilled, there exists an eigenvector $e_0$, such that $S(\omega_0)e_0 = 0$. Generically, $e_0$ is a linear combination of various input modes of the system, which implies that it is only a sufficient condition for full absorption in the mirror geometry, but not in the infinite waveguide. Nevertheless, we find that efficient detection is still possible in both setups, in the limit of large atom numbers.

Mirror geometry.—In the mirror geometry, the effective Hamiltonian reads

$$H_{eff, mn} = \Gamma_{id} \frac{1}{4} \left[ e^{ik_0|x_m-x_n|} - e^{ik_0(x_m+x_n)} \right].$$

(4)

where, as before, $\Gamma_{id}$ is single-atom decay rate into the waveguide, $x_n$ the position of the $n$th atom, and $k_0$ is the wavevector of the emitted light (wavelength $\lambda = 2\pi/k_0$). The coupling of the atoms via the waveguide contain both a term due to photons travelling directly in between them, accumulating a phase $k_0|x_m-x_n|$, and one mediated by photons being reflected from the mirror, which invercs a minus sign and a phase $k_0|x_m+x_n|$. Since there is only one input and output field [cf. Fig. 1a], the matrix $L$ is now a vector $L_n = \sqrt{\Gamma_{id}} \sin(k_0x_n)$.

It is instructive to see an example of how perfect absorption manifests in this setup. Placing the atoms in the atomic mirror configuration at positions $x_n = (1/4+n)\lambda$ (due to the infinite range interactions, the lattice need not have unity filling), the photonic field only couples to the symmetric collective atomic excitation $B = \sum b_n$, which at the same time is an eigenmode of the atomic array. All other modes are dark and do not participate in the dynamics. In terms of this collective mode,
the governing equations reduce to the input-output equations for a one-sided cavity [45] with internal dissipation

\[ \dot{B}(t) = -\frac{\Gamma_{\text{tot}}}{2} B(t) + \sqrt{N\Gamma_{1d}} a_{\text{in}}(t), \]

(5a)

\[ a_{\text{out}}(t) = a_{\text{in}}(t) - \sqrt{N\Gamma_{1d}} B(t), \]

(5b)

where we have introduced the total decay rate \( \Gamma_{\text{tot}} = N\Gamma_{1d} + \Gamma' \). Solving Eqs (5) in frequency space, we find the number of photons in the output field

\[ \langle a_{\text{out}}^{\text{†}}(\omega)a_{\text{out}}(\omega) \rangle = \left| 1 - \frac{N\Gamma_{1d}}{\Gamma_{\text{tot}}/2 - i\omega} \right|^2 \langle a_{\text{in}}^{\text{†}}(\omega)a_{\text{in}}(\omega) \rangle. \]

(6)

If the engineered decay is tuned such that \( \Gamma_{\text{tot}} = 2N\Gamma_{1d} \), there is perfect absorption on resonance (\( p_{\text{abs}} = 1 \)), with a bandwidth of \( 2N\Gamma_{1d} \). Since \( \Gamma_{\text{eng}} \propto N \), as the atom number is increased, the detection efficiency \( \eta = p_{\text{abs}}\Gamma_{\text{eng}}/(\Gamma_{\text{eng}} + \Gamma_{\text{free}}) \) can become arbitrarily close to 1, illustrating that collective decay can be used to overcome a finite Purcell factor.

In Fig. 2a we include spatial disorder and show how the photon loss \( p_{\text{loss}} = 1 - \eta \) scales with atom number. Clearly, while it works very well for low spatial disorder (\( \sigma / \lambda < 1\% \)), the discussed setup suffers significantly from disorder. In the following we show how this is mitigated.

Disorder shifts the energies and decay rates of the eigenmodes of the atomic array. As a result, many collective atomic modes couple to the input field. Yet, as we have shown following Eq. (3), full absorption can be attained generically, independent of disorder. Since this relies on tuning dissipation to a specific eigenmode, which are not known \textit{a priori}, this appears infeasible. Surprisingly, one can still vastly improve over the results of the atomic mirror configuration, if the standard deviation \( \sigma \equiv \sqrt{\langle \delta x^2 \rangle} \) of atomic positions is known. For a given \( N, \sigma \), one can calculate the average largest eigenvalue \( \langle \mu \rangle_{N,\sigma} \) and tune the engineered dissipation to its imaginary part \( \Gamma_{\text{eng}} = \text{Im}[\langle \mu \rangle_{\sigma,N}] \). This restores the favourable scaling of detection efficiency with \( N \), as illustrated by the blue (\( \sigma = .1\lambda \)) and green (\( \sigma = \lambda \)) curves in Fig. 2b. Most strikingly, this works even in the presence of disorder on the scale of the lattice spacing \( \sigma = a \), essentially equivalent to a fully random configuration. The reason it works lies in the fact that the absorption bandwidth grows with the imaginary part of the average eigenvalue, whereas the fluctuations (the detuning from the CPA point) only grow with its square root.

If the disorder is fixed as a result of fabrication, such as in solid-state implementations, one can further improve the scaling by first characterizing the system, such that the largest eigenvalue is known. This situation is illustrated by the red curve in Fig. 2, calculated for completely random configurations, but with \( \Gamma_{\text{eng}} \) matched to the largest eigenvalue in each disorder realization.

So far, we have just discussed absorption and detection on resonance. Equally important is the detection bandwidth, given by the engineered decay rate. Since also the maximum detection efficiency \( \eta \) depends on the ratio between engineered dissipation and total decay rate, best detection is achieved when tuning to the most dissipative eigenmode (cf. Fig. 2c). This is our choice for all other plots.

\textit{Infinite waveguide.}—Let us now turn to an atomic array coupled to an infinite waveguide, which has the simpler effective Hamiltonian

\[ H_{\text{eff},mn} = \Gamma_{1d} \exp(ik_0|x_m - x_n|), \]

since there is only one path for a photon to travel from one atom to the next. As illustrated in Fig. 1b, there are now two input and two output modes, a right-moving one (†) and a left-moving one (−). [cf. Eq. (2)]. The atomic lowering operators couple to the input operators via the \( N \times 2 \) matrix \( L_{\nu} = \sqrt{\Gamma_{1d}} \exp(ik_0\nu x_n) \), where \( \nu \in \{ \pm1 \} \). The scattering matrix may be written

\[ S(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} L^\dagger M^{-1}(\omega)L & L^\dagger M^{-1}(\omega)L^\dagger \\ LM^{-1}(\omega)L & LM^{-1}(\omega)L^\dagger \end{pmatrix}, \]

(7)
where have defined $M \equiv iH_{\text{eff}} + i\omega + \Gamma'/2$ and the vector $L \equiv L_+$ for brevity (note $L_+ = L_1^+$. Since $M$ is symmetric, transmission of right- and left-moving waves (the diagonal elements of $S$) are equal.

In this geometry, CPA does not imply $p_{\text{abs}} = 1$. For example, in the atomic mirror configuration (lattice spacing $a = \lambda$), which is equivalent to all atoms placed at the same point, tuning to CPA ($\Gamma_{\text{eng}} = N\Gamma_{1d}$) only gives perfect absorption for symmetric wavepackets. Absorption of wavepackets incident from one direction is limited to 50% [35]. Instead, any other lattice spacing will do, as has been realized in Ref. [35]. Otherwise the analysis of the infinite waveguide case is very similar to the mirror geometry, and can be found in the SI. The upshot is that arbitrary detection efficiencies can again be attained by increasing atom number, independent of disorder.

Chiral waveguide.—Interestingly, in the recently demonstrated platforms for chiral atom-waveguide coupling [46] are another architecture in which robust photon detection may be achieved. This situation describes a range of situation, for example when the light field is strongly confined [47–49], when giant atoms are tuned to give a chiral coupling [50], or in topological systems [51, 52].

By design, (almost) no backscattering occurs in these systems, there are no collective effects, and the spacing of the atoms is immaterial, making the analysis straightforward. Clearly, the probability for an incoming photon to pass $N$ quantum emitters decreases exponentially with $N$, $|T|^2 = |t|^2N$. In the limit of large $N$ and in the almost directional regime, $\gamma_+ \gg \gamma_-$, where $\gamma_{\pm}$ are the coupling rates to right- and left-moving photons, the detection efficiency $\eta$ is given to first order in $\gamma_-/\gamma_+$ by (derivation in SI)

$$\eta_{\text{chiral}} = \frac{\Gamma_{\text{eng}}}{\Gamma'} \left(1 - \frac{\gamma_-}{\gamma_+ + \Gamma'}\right).$$

Note that even for moderate $\gamma_-/\gamma_+$, the second term can be reduced arbitrarily by increasing $\Gamma_{\text{eng}}$, with the caveat that a larger number of emitters is needed before complete extinction is attained. In the absence of backscattering, this scheme is intrinsically robust against disorder. On top of that, the detection bandwidth depends only on the analysis straightforward and thus is—at least in principle—indeed independent of $\Gamma_{\text{eng}}$. This comes again with the caveat that photons far detuned from resonance on a scale of $\Gamma_{\text{eng}}$ can only be absorbed with a large number of emitters.

Experimental considerations.—A tacit assumption in our discussion so far has been that the engineered decay $\Gamma_{\text{eng}}$ can be arbitrarily tuned, even though ultimately it is limited by the decay rate to the metastable state $\Gamma'$. In circuit quantum electrodynamics, these parameters can be tuned arbitrarily [36], but in other systems generically $\Gamma_{\text{eng}} \sim \Gamma_{\text{free}}$, severely limiting the detection efficiency $\eta$. In solid-state systems, this may be overcome by designing a bandstructure for light with a high density of states at the transition frequency from $|f\rangle$ to $|s\rangle$, whereas in atomic systems it may be possible to choose $g \leftrightarrow e$ to be a narrow transition.

A more flexible solution is shown in Fig. 3 and consists of replacing the direct transition from $|g\rangle$ to $|e\rangle$ by another Raman transition, which yields a tunable waveguide decay rate $\Gamma_{1d}(\Omega_1)$. If $|e\rangle$ is chosen to be a hyperfine state that does not decay to $|g\rangle$, the Purcell factor $\Gamma_{1d}/\Gamma_{\text{free}}$ is fixed, but $\Gamma_{1d}$ and $\Gamma_{\text{free}}$ can be tuned independently from $\Gamma_{\text{eng}}$. Ultimately, $\Gamma_{\text{eng}}$ still limits detection bandwidth, but not detection efficiency. Analyzing the effect of decays from $|f_1\rangle$ back to $|e\rangle$ (see SI), we find that only $|f_2\rangle \rightarrow |e\rangle$ is relevant, which should therefore be minimized by choosing a forbidden transition.

We have neglected inhomogeneous broadening and atomic motion, which could be taken into account in the same way as positional disorder and do not modify our conclusions. Furthermore, the relative effect of disorder decreases as the number of atoms increases [36]. We have also neglected the effect of non-linearities in our analysis, on the basis that they scale as $1/N$. Their effect can be estimated as a reduction in the collective decay rate with each absorbed photon. Using Eq. (6) to estimate the effect (the reduction in decay rate is most prominent in the atomic mirror configuration), we find that absorbing $m$ photons reduces $p_{\text{abs}}$ by $m^2/4N^2$. Thus, assuming we start from $p_{\text{abs}} = 1$, the probability that $m$ photons are absorbed is $1 - (2m^3 - 3m^2 + m)/24N^2$ to third order in $1/N$.

Conclusion.—We have explored the use of arrays of quantum emitters coupled to waveguides for number-resolving photon detection. Paying particular heed to experimental limitations such as disorder and free-space decay, we have found that both can be overcome, leaving no fundamental limitation to the achievable detection efficiency. In a nutshell, our proposal builds on four facts that together enable highly efficient detectors: (1), few-level systems allow for strong, projective measurements of their state due to their intrinsic nonlinearity, (2), nevertheless, sufficiently large ensembles of atoms are linear, (3), collective decay mitigates errors due to non-idealities, and (4), in linear systems one can always engineer dissipation to obtain complete absorption. We hope that the ideas outlined here will mark a step towards high-fidelity number-resolving photon detectors.

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[53] J. D. Hood, A. Goban, A. Asenjo-Garcia, M. Lu, S.-P. Yu, D. E. Chang, and H. J. Kimble, Proceedings of the National Academy of Sciences 113, 10507 (2016).

[54] D. Porras and J. I. Cirac, Physical Review A 78, 053816 (2008).
Supplementary Information:
Number-resolving photon detectors with quantum emitters coupled to waveguides

Derivation of Langevin equations

We study quantum emitters coupled to a semi-infinite waveguide terminated at \( x = 0 \) by a mirror. First assuming that the waveguide is also terminated on the other side after a length \( L \), the bath eigenmodes have wavefunction \( \phi_n(x) = \sin(k_n x) \sqrt{2/L} \), where \( k_n = \pi n / L \) for all natural \( n \). Taking the length of the waveguide to infinity, we recover the Hamiltonian given in the main text [Eq. (1)]. Within the Holstein-Primakoff the Hamiltonian reads

\[
H = \int_0^\infty \frac{dk}{2\pi} \left\{ (\omega_k - \omega_0) a_k^\dagger a_k - g \sum_n \sin(k x_n) \left( a_k^\dagger b_n + \text{H.c.} \right) \right\}
\]

(9)

Defining the sine transform and its inverse through

\[
\tilde{f}(\nu) = 2 \int_0^\infty f(t) \sin(\nu t) \, dt, \quad f(t) = \frac{1}{\pi} \int_0^\infty \tilde{f}(\nu) \sin(\nu t) \, d\nu,
\]

(10)

we can write the field in the waveguide as

\[
\phi(x, t) = \int_0^\infty \frac{dk}{\pi} \sin(k x) \left( a_k(t) + a_{-k}^\dagger(t) \right).
\]

(11)

Defined this way, the commutation relation \([\phi(x, t), \phi(x', t)] = \delta(x - x')\) (for positive \( x \) only) implies canonical commutation relations \([a_k(t), a_{-q}^\dagger(t)] = 2\pi \delta(k - q)\). In terms of complex amplitudes, we have

\[
a(x, t) = \int_0^\infty \frac{dk}{\pi} \sin(k x) a_k(t), \quad a_k(t) = 2 \int_0^\infty dx \sin(k x) a(x, t).
\]

(12)

Solving the operator equations of motion

\[
\dot{a}_k = -i(\omega_k - \omega_0) a_k + ig \sum_n \sin(k x_n) b_n, \quad \dot{b}_n = ig \int \frac{dk}{2\pi} \sin(k x_n) a_k.
\]

(13)

yields

\[
a_k(t) = e^{-i(\omega_k - \omega_0)t} a_k(0) + ig \int_0^t dt e^{-i(\omega_k - \omega_0)(t-\tau)} \sum_m \sin(k x_m) b_m(\tau).
\]

(14)

This solution can be plugged into the equation of motion for \( b_n \). We need the following integral

\[
\int_0^\infty \frac{d\omega}{\pi} \sin(\omega x_n/c) \sin(\omega x/c) e^{-i(\omega - \omega_0)t}
\]

\[
e^{-i\omega_0 t} \left[ \delta \left( \frac{x_n + x}{c} - t \right) + \delta \left( \frac{x_n + x}{c} + t \right) - \delta \left( \frac{x_n - x}{c} - t \right) - \delta \left( \frac{x_n - x}{c} + t \right) \right]
\]

(15)

\[
e^{-i\omega_0 t} \left[ \delta \left( \left| \frac{x_n - x}{c} \right| - t \right) - \delta \left( \frac{x_n + x}{c} - t \right) - \delta \left( \frac{x_n + x}{c} + t \right) \right], \quad \text{if } x, x_n, t > 0.
\]

The equation of motion for \( b_n \) becomes

\[
\dot{b}_n(t) = \frac{ig e^{-i\omega_0 t}}{4} \left[ a(x_n + ct, 0) - a(ct - x_n, 0) \right]
\]

\[
+ \frac{g^2}{8c} \sum_m \left[ e^{ik_0(x_m + x_n)} b_m \left( t - \frac{x_m + x_n}{c} \right) - e^{ik_0|x_m - x_n|} b_m \left( t - \frac{|x_m - x_n|}{c} \right) \right].
\]

(16)
We define the input field $a_{in}(t)$ as the portion of the waveguide field that was at a position $x = ct$ at time $t = 0$ and has since travelled all the way to the atoms. Thus, $a_{in}(t) = -\sqrt{\frac{2}{\hbar c}} e^{i\omega_0 t} \phi(ct,0)$, where the pre-factor is fixed by the commutation relations of $a_{in}$, up to an arbitrary phase. This yields the Langevin equation

$$\dot{b}_n(t) = \frac{g}{2\sqrt{c}} \left\{ e^{ik_0 x_n} a_{in}(t - x_n/c) + e^{-ik_0 x_n} a_{in}(t + x_n/c) \right\}$$

$$+ \frac{g^2}{8c} \sum_m \left[ e^{ik_0 (x_m + x_n)} b_m \left( t - \frac{x_m + x_n}{c} \right) - e^{ik_0 |x_n - x_m|} b_m \left( t - \frac{|x_m - x_n|}{c} \right) \right].$$

(17)

As defined, $a_{in}(t)$ is a slow variable, so if the dynamics of the system and the bandwidth of the input state around $\omega_0$ are slow compared to the time it takes for light to travel a distance $2x_n/c$, we can neglect the retardation, rendering our description Markovian. The same applies to the atomic lowering operators.

$$\dot{b}_n(t) = \frac{g}{\sqrt{2c}} \sin(k_0 x_n) a_{in}(t) + \frac{g^2}{8c} \sum_m \left[ e^{ik_0 (x_m + x_n)} - e^{ik_0 |x_n - x_m|} \right] b_m(t).$$

(18)

In order to calculate the output field, we take the integrated equation of motion for the light field and apply a sine transform. This is essentially the same as the right-hand side of the equation of motion for $b_n$, except evaluated at a different point in space. Choosing this point to be $x_R + \varepsilon$, i.e., a small distance to the right of the rightmost atom, and again neglecting retardation, we find

$$\int \frac{dk}{\pi} \sin(k x) a_k(t) = -i \sqrt{\frac{2}{\hbar c}} \sin(k_0 (x_R + \varepsilon)) a_{in}(t) - i \frac{g}{4c} \sum_m e^{ik_0 (x_R + \varepsilon)} 2i \sin(k_0 x_m) b_m(t).$$

(19)

Further choosing $\varepsilon$ such that $\sin(k_0 (x_R + \varepsilon)) = 1$, and defining $a_{out}(t) = -\sqrt{\frac{2}{\hbar c}} e^{i\omega_0 t} a(x_R + \varepsilon, t)$, we have

$$a_{out}(t) = a_{in}(t) - \frac{g}{\sqrt{2c}} \sum \sin(k_0 x_m) b_m(t).$$

(20)

Finally, let us define the decay rate $\Gamma_{1d} = \frac{g^2}{2c}$.

$$\dot{b}_n(t) = \sqrt{\Gamma_{1d}} \sin(k_0 x_n) a_{in}(t) - \frac{\Gamma_{1d}}{4} \sum_m \left[ e^{ik_0 |x_m - x_n|} - e^{ik_0 (x_n + x_m)} \right] b_m(t),$$

(21a)

$$a_{out}(t) = a_{in}(t) - \sqrt{\Gamma_{1d}} \sum \sin(k_0 x_m) b_m(t).$$

(21b)

The derivation for the infinite waveguide proceeds in much the same way and can be found elsewhere [40].

**dissipation engineering**

As is illustrated in Fig. 1, the metastable level $|s\rangle$ forms a $\Lambda$-scheme together with the auxiliary state $|f\rangle$ and the excited atomic state $|e\rangle$. Adiabatic elimination of $|f\rangle$, valid if the detuning $\Delta$ is large compared to the Rabi frequency, yields an artificial decay channel from $|e\rangle$ to $|s\rangle$. This can be modelled through the Hamiltonian

$$H = \sum_k (\omega_k - \omega_0) c^\dagger_k c_k - \frac{g_{\text{diss}}}{\hbar} \sum_n \sigma_{es}^n c_k + \text{H.c.} + H_{\text{atom-waveguide}},$$

(22)

where $H_{\text{atom-waveguide}}$ is the Hamiltonian given in the main text [Eq. (1)], and $\sigma_{es}^n \equiv |e\rangle_n \langle s|$. For our purposes it does not matter if $\{c_k\}$ are waveguide modes (in a different frequency range from the bandwidth of the detector), free-space modes, or some other guided modes. While it is well-known that the decay of atoms into free space does not lead to collective effects in 1D arrays of atoms in optical lattices (i.e., spaced by at least the wavelength of light), interestingly even if the transition $e \leftrightarrow s$ is coupled to a 1D waveguide, there are no collective effects that affect the detector, as we will show in the following.

A Hamiltonian that combines both bath couplings reads

$$H = \sum_{\nu = \pm} \int \frac{dk}{2\pi} \sum_{\alpha} (\omega_k - \omega_0) a^\dagger_{k,\nu,\alpha} a_{k,\nu,\alpha} - \sum_n \left[ \sqrt{\Gamma_1} e^{i(\nu k - k_L)} x_n \sigma_{eg}^n a_{k,\nu,1} + \sqrt{\Gamma_2} e^{i(\nu k - k_L)} x_n \sigma_{es}^n a_{k,\nu,2} + \text{H.c.} \right].$$

(23)
There are two waveguide fields, \( \alpha = 1, 2 \), distinguished either in frequency, polarization, or by being in a different waveguide. From Eq. (23), we can derive the bath equations of motion, integrate them up and Fourier transform them [40]

\[
a_{\nu,1}(x,t) = e^{i\omega t}a_{\nu,1}(x-ct,0) + ig_t\Theta[(x-\nu x_n)/c]e^{ik_1(x-\nu x_n)+ik_L,1x_n}\sigma_{ge}^n[t-(x-\nu x_n)/c].
\]

(24)

For the other field, we have to exchange \( 1 \leftrightarrow 2 \) and \( g \leftrightarrow s \) As above, we next derive the atomic equations of motion

\[
\dot{\sigma}_{ge}^n = \sum_{\nu=\pm} \int \frac{dk}{2\pi} e^{i(\nu k-L,1)x_n}a_{\nu,1}(\sigma_{gg}^n - \sigma_{ee}^n) + \Gamma_1 \sum_{m} e^{i k_1|x_m-x_n|}a_{\nu,1}(\sigma_{gg}^n - \sigma_{ee}^n)\sigma_{ge}^m,
\]

(25a)

\[
\dot{\sigma}_{sc}^n = \sum_{\nu=\pm} \int \frac{dk}{2\pi} e^{i(\nu k-L,2)x_n}a_{\nu,2}(\sigma_{ee}^n - \sigma_{ss}^n) + \Gamma_1 \sum_{m} e^{i k_1|x_m-x_n|}a_{\nu,2}(\sigma_{ee}^n - \sigma_{ss}^n)\sigma_{se}^m,
\]

(25b)

\[
\dot{\sigma}_{gs}^n = \sum_{\nu=\pm} \int \frac{dk}{2\pi} e^{i(\nu k-L,1)x_n}a_{\nu,1}(\sigma_{gs}^n + \sigma_{ge}^n) + \Gamma_1 \sum_{m} e^{i k_1|x_m-x_n|}a_{\nu,1}^+\sigma_{ge}^m.
\]

(25c)

and replace the photon field [Eq. (24)]

\[
\dot{\sigma}_{ge}^n = \sqrt{\Gamma_1} \sum_{\nu=\pm} e^{i(k_1-\nu k-L,1)x_n}(\sigma_{gg}^n - \sigma_{ee}^n)a_{\nu,1} - \Gamma_1 \sum_{m} e^{i k_1|x_m-x_n|}a_{\nu,1}(\sigma_{gg}^n - \sigma_{ee}^n)\sigma_{ge}^m
\]

\[
+ \sqrt{\Gamma_2} \sum_{\nu=\pm} e^{i(k_2-\nu k-L,2)x_n}\sigma_{gs}^n a_{\nu,2} - \Gamma_2 \sum_{m} e^{i k_2|x_m-x_n|}a_{\nu,2}(\sigma_{gs}^n - \sigma_{ee}^n)\sigma_{ge}^m,
\]

(26a)

\[
\dot{\sigma}_{sc}^n = \sqrt{\Gamma_2} \sum_{\nu=\pm} e^{i(k_2-\nu k-L,2)x_n}(\sigma_{ss}^n - \sigma_{ee}^n)a_{\nu,2} - \Gamma_2 \sum_{m} e^{i k_2|x_m-x_n|}a_{\nu,2}(\sigma_{ss}^n - \sigma_{ee}^n)\sigma_{se}^m
\]

\[
+ \sqrt{\Gamma_1} \sum_{\nu=\pm} e^{i(k_1-\nu k-L,1)x_n}\sigma_{gs}^n a_{\nu,1} - \Gamma_1 \sum_{m} e^{i k_1|x_m-x_n|}a_{\nu,1}(\sigma_{gs}^n - \sigma_{ee}^n)\sigma_{se}^m
\]

(26b)

\[
\dot{\sigma}_{gs}^n = \sqrt{\Gamma_1} \sum_{\nu=\pm} e^{i(k_1-\nu k-L,1)x_n}(-\sigma_{ex}^n)a_{\nu,1} - \Gamma_1 \sum_{m} e^{i k_1|x_m-x_n|}a_{\nu,1}(\sigma_{gs}^n - \sigma_{ee}^n)\sigma_{ge}^m
\]

\[
+ \sqrt{\Gamma_2} \sum_{\nu=\pm} e^{i(k_2-\nu k-L,2)x_n}\sigma_{gs}^n a_{\nu,2} - \Gamma_2 \sum_{m} e^{i k_2|x_m-x_n|}a_{\nu,2}(\sigma_{gs}^n - \sigma_{ee}^n)\sigma_{ge}^m
\]

(26c)

\[
a_{\nu,1} = a_{\nu,1} - \sqrt{\Gamma_1} \sum_{n} e^{i(k_1-L,1)x_n}\sigma_{ge}^n,
\]

(26d)

\[
a_{\nu,2} = a_{\nu,2} - \sqrt{\Gamma_2} \sum_{n} e^{i(k_2-L,2)x_n}\sigma_{sc}^n.
\]

(26e)

Linearizing, we’ll replace \( \sigma_{ge} \rightarrow b, \sigma_{gs} \rightarrow c, \sigma_{gg} \rightarrow 1, \sigma_{ee} \rightarrow 0 \) (and \( c \) being bosonic annihilation operators), and derive the equations of motion in the same vein as above, setting \( k_{L,j} = 0 \) for convenience

\[
\dot{b}_n = \sqrt{\Gamma_1} \sum_{\nu=\pm} e^{i k_1|x_n-x_m|}a_{\nu,1} - \Gamma_1 \sum_{\nu=\pm} e^{i k_1|x_n-x_m|}b_{m}
\]

\[
+ \sqrt{\Gamma_2} \sum_{\nu=\pm} e^{i k_2|x_n-x_m|}c_{n} - \Gamma_2 \sum_{\nu=\pm} e^{i k_2|x_n-x_m|}c_{m} b_{n},
\]

(27a)

\[
\dot{c}_n = \sqrt{\Gamma_1} \sum_{\nu=\pm} e^{i k_1|x_n-x_m|}a_{\nu,2} - \Gamma_1 \sum_{\nu=\pm} e^{i k_1|x_n-x_m|}(-b_{n}^+c_{n})b_{m}
\]

\[
+ \sqrt{\Gamma_2} \sum_{\nu=\pm} e^{-i k_2|x_n-x_m|}c_{n} a_{\nu,2} - \Gamma_2 \sum_{\nu=\pm} e^{-i k_2|x_n-x_m|}b_{n}^+b_{m} c_{n},
\]

(27b)

\[
a_{\nu,1} = a_{\nu,1} - \sqrt{\Gamma} \sum_{n} e^{-i k_1 x_n}b_{n},
\]

(27c)

\[
a_{\nu,2} = a_{\nu,2} - \sqrt{\Gamma_2} \sum_{n} e^{-i k_2 x_n}b_{n}^+c_{n}.
\]

(27d)

The bosonization and indeed everything we do here is in the approximation of low excitation numbers. Thus we normal-order the non-linear terms (which picks up a commutator), and then throw away terms like \( c_{n}^+ b_{m} \), as they are small, as well as those
containing $a_{in,n,2}$ on the basis that this field is in vacuum. Thus, we find

$$\dot{b}_n = \sqrt{\Gamma_1} \sum_{\nu=\pm} e^{i k_1 \nu x_n} a_{in,\nu,1} - \sum_m \Gamma_1 e^{ik_2 x_m - x_n} b_m - \Gamma_2 b_n \quad \text{(28a)}$$

$$\dot{c}_n = \sqrt{\Gamma_2} \sum_{\nu=\pm} e^{-ik_2 \nu x_n} b_n a_{in,\nu,2}^\dagger - \Gamma_2 c_n, \quad \text{(28b)}$$

$$a_{out,\nu,1} = a_{in,\nu,1} - \sqrt{\Gamma_1} \sum_n e^{-i\nu k_1 x_n} b_n, \quad \text{(28c)}$$

$$a_{out,\nu,2} = a_{in,\nu,2} - \sqrt{\Gamma_2} \sum_n e^{-i\nu k_2 x_n} b_n c_n. \quad \text{(28d)}$$

In the linearized regime, the decay to the metastable state is therefore simply characterized by an additional decay rate. An important point is that the atoms in a metastable state cease to participate in the dynamics such that the collective decay rate decreases accordingly, which is discussed in the main text.

Non-idealities in double $\Lambda$-system

![Double Lambda Diagram](image)

FIG. 4. The double-$\Lambda$ scheme with additional decay channels from $|f_1\rangle$ and $|f_2\rangle$ back to $|e\rangle$.

In order to evaluate the effect of additional decays from $|f_1\rangle$ and $|f_2\rangle$ to $|e\rangle$, we derive the effective quantum master equation of the double-$\Lambda$ shown in Fig. 4. Neglecting the energy shifts due to the pumps, the dynamics are purely dissipative, given by the jump operators

$$\hat{L}_{g,\text{eff}} = \frac{\sqrt{\Gamma_2 \Omega_1}}{2\Delta_1 - i(\Gamma_g + \Gamma_{1,e})} |g\rangle\langle e|, \quad \hat{L}_{s,\text{eff}} = \frac{\sqrt{\Gamma_s \Omega_2}}{2\Delta_2 - i(\Gamma_s + \Gamma_{2,e})} \langle s|\langle e|, \quad \text{(29a)}$$

$$\hat{L}_{ee,\text{eff}} = \left( \frac{\sqrt{\Gamma_{1,e} \Omega_1}}{2\Delta_1 - i(\Gamma_g + \Gamma_{1,e})} + \frac{\sqrt{\Gamma_{2,e} \Omega_2}}{2\Delta_2 - i(\Gamma_s + \Gamma_{2,e})} \right) \langle e|\langle e|. \quad \text{(29b)}$$

We will refer to the rates corresponding to these jump operators as $\Gamma_{id} = \Gamma_{g,\text{eff}}, \Gamma_{eng} = \Gamma_{s,\text{eff}},$ and $\Gamma_{ee,\text{eff}}$.

In a regime in which the decay from $|e\rangle$ to $|g\rangle$ is collectively enhanced, $\Gamma_{id} \ll \Gamma_{eng}$ to achieve CPA conditions. Thus, the dephasing due to the decay $|f_1\rangle \to |e\rangle$ (first term in $\Gamma_{ee,\text{eff}}$) is negligible. On the other hand, the dephasing due to the decay $\Gamma_{2,e}$ has the same dependence on driving parameters ($\Omega_2, \Delta_2$) as the engineered dissipation $\Gamma_{eng}$. In order to be negligible, we require $\Gamma_{2,e} \ll \Gamma_s$, which is for example realized if $|f_2\rangle \to |e\rangle$ is a forbidden decay, as we have proposed in the main text.

Inverse scattering matrix

It can be checked explicitly that $S^{-1} = [1 + L^\dagger(-i\omega + iH_\text{eff}^\dagger + \Gamma/2)^{-1}L]$ is the inverse of $S$. Note that now we are taking $\Gamma'$ to be a matrix, for full generality. Multiplying both, we find

$$S^{-1}S = 1 - L^\dagger (-i\omega + iH_\text{eff}^\dagger + \Gamma/2)^{-1} \left[-i(H_\text{eff} - H_\text{eff}^\dagger) + LL^\dagger \right] (-i\omega + iH_\text{eff}^\dagger + \Gamma/2)^{-1}L = 1. \quad \text{(30)}$$

The square brackets in the above expression vanish, which can be checked explicitly for the two examples in the main text. Mathematically, one can show this holds generically, since if $\Gamma' = 0$, the scattering matrix is unitary $S^{-1} = S^\dagger$ (provable, e.g.,
FIG. 5. a Percentage of undetected photons on resonance for an atomic array coupled to an infinite waveguide as a function of atom number. The blue line denotes the limit of a perfectly-ordered array with spacing $a = \lambda/4$ (or $5\lambda/4$ etc). Purcell factor is $P = 10$, the average for the other cases was performed over 150 disorder realization (standard deviation shown as lightly coloured area). b Detection probability for a disorder-free array of $N$ atoms with different spacings. c Scaling of largest eigenvalue in ordered arrays with varying spacing compared to a fully random one (red). While the atomic mirror configuration ($a = \lambda$) is clearly different, it appears to be a fine-tuned exception, with all other generic arrays (ordered or disordered) behaving remarkably similar. The robustness of our scheme relies to a large degree on this universal eigenvalue scaling.

from the canonical commutation relations of output and input operators), which is only true if the term in square brackets vanishes (as $\omega$ is arbitrary). Physically, this is the fluctuation-dissipation theorem, as the anti-Hermitian part of $H_{\text{eff}}$ specifies the damping, whereas $L$ captures how strongly the modes are coupled to the input noise operator.

**Full Langevin equations**

For completeness, we state here the Langevin equations without dropping the additional quantum noise terms. For the mirror geometry, they read

$$
\dot{b}_n(t) = \sqrt{\Gamma_{1d}} \sin(k_0 x_n) a_{in}(t) - \frac{\Gamma_{1d}}{4} \sum_m \left[ e^{ik_0 |x_m-x_n|} - e^{ik_0 (x_m+x_n)} \right] b_m(t) \\
- \frac{\Gamma_{\text{eng}} + \Gamma_{\text{free}}}{2} b_n(t) + \sqrt{\Gamma_{\text{eng}}} c_{in}(t) + \sqrt{\Gamma_{\text{free}}} d_{in}(t),
$$

(31a)

$$
a_{out}(t) = a_{in}(t) - \sqrt{\Gamma_{1d}} \sum_n \sin(k_0 x_n) b_n(t),
$$

(31b)

whereas for the infinite waveguide we have

$$
\dot{b}_n(t) = \sqrt{\Gamma_{1d}} \sum_{\nu=\pm} e^{ik_0 \nu x_n} a_{in,\nu}(t) - \Gamma_{1d} \sum_m e^{ik_0 |x_m-x_n|} b_m(t) - \frac{\Gamma_{\text{eng}} + \Gamma_{\text{free}}}{2} b_n(t) + \sqrt{\Gamma_{\text{eng}}} c_{in}(t) + \sqrt{\Gamma_{\text{free}}} d_{in}(t),
$$

(32a)

$$
a_{out,\nu}(t) = a_{in,\nu}(t) - \sqrt{\Gamma_{1d}} \sum_n e^{-ik_0 \nu x_n} b_n(t).
$$

(32b)

If the modes corresponding to $c_{in}, d_{in}$ are in vacuum, which we assume, normal-ordered expectation values involving these inputs vanish.

**Infinite waveguide analysis**

In this geometry, CPA does not imply $p_{\text{abs}} = 1$. For example, in the atomic mirror configuration (lattice spacing $a = \lambda$), tuning to CPA ($\Gamma_{\text{eng}} = N\Gamma_{1d}$) yields an amplitude scattering matrix on resonance

$$
S_{\text{AMC}}(\omega = 0) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
$$

(33)

This gives perfect absorption for symmetric wavepackets, but not for wavepackets incident from one direction. For this reason, the atomic mirror configuration is not suited to build a detector. Instead, any other lattice spacing will do, which is shown explicitly in...
As before, we can calculate the off-diagonal coupling. The detection probability per atom is then given through

$$\eta_{\text{detect}} = \frac{p_{\text{detect}}}{p_{\text{detect}} + p_{\text{loss}}} = \left(1 - \frac{\gamma_-}{\gamma_+ + \Gamma'}\right) \frac{\Gamma_{\text{eng}}}{\Gamma'}. \quad (37)$$

The upshot is that arbitrary detection efficiencies can again be attained by increasing atom number, independent of disorder.

If \(m\) captured by the random variable \(y\) distribution of atomic positions. Assuming a Gaussian distributions of positions around the atomic mirror configuration, if fast, some of the effect of disorder will be averaged out. This can be modelled by instead averaging the atomic coupling over a fully random array. This is explicitly demonstrated in Fig. 5b,c. In the main text we take

$$\bar{\Gamma}_{1d} = \exp(i\phi) J, \quad J = \left(\begin{array}{cc} A & B \\ B^* & A^* \end{array}\right),$$

with eigenvalues \(\mu = A \pm B \exp(i\phi)\). In this system, coherent perfect absorption (one eigenvalue is zero) is equivalent to

$$B = \exp(i\theta)A \quad \text{and} \quad \exp(i\phi + i\theta) = \pm 1.$$\n
In the end, full absorption of a uni-directional wavepacket may only be attained if \(S = 0\).

Interestingly, this happens for all arrays except \(B = \exp(i\theta)A\). This allows us to calculate the absorption probability (for the + mode)

$$\eta_{\text{chiral}} = \frac{p_{\text{detect}}}{p_{\text{detect}} + p_{\text{loss}}} = \left(1 - \frac{\gamma_-}{\gamma_+ + \Gamma'}\right) \frac{\Gamma_{\text{eng}}}{\Gamma'}. \quad (37)$$

Fast thermal motion of atoms

In the main text we have only considered static disorder, which is valid for slowly moving atoms. If the thermal motion of atoms is fast, some of the effect of disorder will be averaged out. This can be modelled by instead averaging the atomic coupling over a distribution of atomic positions. Assuming a Gaussian distributions of positions around the atomic mirror configuration, captured by the random variable \(y_m\)

$$x_m = \frac{\pi}{k_0} \left(\frac{1}{2} + 2m\right) + y_m, \quad (38)$$

we can calculate the off-diagonal coupling

$$g_{mn} = -\frac{\Gamma_{1d}}{8\pi \sigma^2} \int dy_m dy_n e^{-(y_m^2 + y_n^2)/2\sigma^2} \left[ e^{ik_0|y_m - y_n|} + e^{-ik_0(y_m + y_n)} \right]. \quad (39)$$

If \(m = n\), there should only be one integral, giving \(g_{nn} = -[1 + \exp(-k_0^2\sigma^2)]\Gamma_{1d}/(4\phi)\). In the case \(m \neq n\), we can straightforwardly evaluate the second term, which yields \(-\Gamma_{1d} e^{-k_0^2\sigma^2}/4\) overall. For the first term, we first shift \(y_m \rightarrow y_m + y_n\).
in which case the \( y_m \) integral becomes a straightforward Gaussian giving a factor of \( \sqrt{\pi \sigma^2 e^{y_m^2/4\sigma^2}} \). The leftover integral reads

\[
- \frac{\Gamma_1 d}{8\sqrt{\pi} \sigma^2} \int dy_m e^{-y_m^2/4\sigma^2} e^{ik_0 |y_m|} = -\frac{\Gamma_1 d}{4} e^{-k_0^2 \sigma^2} [1 + i\text{erfi}(k_0 \sigma)].
\]

(40)

Taken together, we get

\[
\bar{g}_{mn} = -\frac{\Gamma_1 d}{2} \left[ e^{-k_0^2 \sigma^2} + \frac{i}{\sqrt{\pi}} F(k_0 \sigma) \right],
\]

(41)

where \( F(x) \) is the purely real Dawson integral. It is peaked at \( x = 1 \), is odd, and obeys \( |F(x)| < 0.6 \), and \( F(x) \to 0^\pm \) as \( x \to \pm \infty \). Equation (41) predicts that the coupling decreases exponentially in \( (k_0 \sigma)^2 \). For low to moderate \( k_0 \sigma \), the effect of fast thermal motion can simply be accounted for by rescaling the couplings, without affecting the conclusions in the main text.