Comparing metric and Palatini approaches to vector Horndeski theory

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Abstract

We compare cosmologic and spherically symmetric solutions to metric and Palatini versions of vector Horndeski theory. It appears that Palatini formulation of the theory admits more degrees of freedom. Specifically, homogeneous isotropic configuration is effectively bimetric, and static spherically symmetric configuration contains non-metric connection. In general, the exact solution in metric case coincides with the approximative solution in Palatini case. The Palatini version of the theory appears to be more complicated, but the resulting non-linearity may be useful: we demonstrate that it allows the specific cosmological solution to pass through singularity, which is not possible in metric approach.

1 Introduction

The stages of accelerated expansion of our universe were not predicted by Einstein gravity. This strongly motivates searching for new gravity theories. However, when we specify the action of new theory, we immediately face the question: whether the compatibility of connection with metric is an accidental feature of Einstein gravity or a fundamental property of our universe? Indeed, there are two main approaches to variation procedure of the gravitational action, which are inequivalent. In Palatini/first-order formalism the affine connection $\Gamma_{\sigma}^{\alpha}$ is treated as an independent variable, while in metric/second-order formalism the connection is restricted to be Levi-Civita connection of the metric $g_{\rho\sigma}$:

$$\Gamma_{\rho\sigma}^{\alpha} = \{^\alpha_{\rho\sigma} \} = \frac{1}{2} (\partial_\rho g_{\lambda\sigma} + \partial_\sigma g_{\lambda\rho} - \partial_\lambda g_{\rho\sigma}).$$

Let us denote, for brevity, the two formalisms as F1 and F2, correspondingly.

Metric theory of gravity is present in any textbook on gravitation. A detailed review of Palatini formulation of modified gravity theories can be found in [1]. Worth to mention, there exist a plethora of other options. One can consider bimetric theory, in which connection is compatible with independent second metric [2], biconnection theory with two affine connections [3], the action may depend both on metric and affine geometrical objects [4] etc. But those approaches are beyond the scope of our consideration.

Einstein gravity appears to be physically equivalent in both formalisms: even if connection is regarded as an independent variable in Einstein-Hilbert action, the solution to corresponding
equation of motion is a connection, which admits the same geodesics and Einstein equations as the Levi-Civita connection \[5\,6\]. There are known few more theories which exhibit such equivalence \[7, 8\], but for an arbitrary gravity theory the equivalence is generally broken. Identically looking actions provide different equations of motion in F1 and F2 approaches \[9\]. For instance, \( f(R) \) gravity contains an additional scalar degree of freedom, compared to Einstein gravity. In F2 case, this degree of freedom is dynamical and allows describing quint-essence and inflation. In F1 this would be a non-dynamical degree of freedom \[10\], which is similar to an effective cosmological constant \[11, 12\].

Consideration of independent connection may lead to ambiguities in definition of geodesics, it admits traveling faster than light and other potentially unpleasant phenomena, yet admissible at high energies. On other hand, the artificial restriction on connection may appear unphysical. Therefore both approaches to variation of gravitational action are justified, and it is important to establish an actual difference between F1 and F2 versions of gravity theories actively studied nowadays.

In current research we would like to compare metric and Palatini formulations of vector Horndeski theory. Horndeski models, both in scalar \[13\] and vector \[14\] case have attracted a lot of interest in recent years, because the issues of inflation and/or dark energy can be resolved in a very natural way within their framework \[15, 16, 17\]. Although some of the models may be plagued by ghosts and instabilities \[18\], their elegant mathematical structure \[19, 20\] motivates the detailed study. The particular case of Horndeski prescription in Palatini approach remains barely covered by the investigations, though the theories with non-minimally coupled scalar field were extensively explored recently in both metric and Palatini formulations \[21, 22, 23, 24\]. However, in Horndeski case the model with scalar field is quite complicated, because it contains several distinct types of coupling between matter and geometry. On the contrary, the vector Horndeski model is unique in its simplicity, so we chose it as the starting point for studying.

One can find several studies of cosmological \[25, 26, 17\] and spherically symmetric \[27, 28\] solutions to vector Horndeski theory in metric approach. But the Palatini version of the theory remains unexplored, to the best of our knowledge. Modified theories of gravitation with electromagnetic fields were pretty well studied in the Palatini approach \[29, 30, 31\], but in all the models examined, the interaction between matter and gravity was realized at the level of effective scalars with the lagrangians of the form: \( L = L(R, R_{\rho\sigma} R^\rho^\sigma, F_{\rho\sigma} F^{\rho\sigma}, F_{\rho\sigma} \tilde{F}^{\rho\sigma}) \). And the vector Horndeski model exhibits much closer ties between matter and geometry, implying contractions of curvature tensor with the field tensors:

\[
L = \tilde{R}^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} = R_{\alpha\beta\mu\nu} \tilde{F}^{\alpha\beta} \tilde{F}^{\mu\nu}.
\]

(1)

Here \( R_{\alpha\beta\mu\nu} \) is curvature tensor, \( F_{\mu\nu} \) — field tensor, and their duals \( \tilde{R}^{\alpha\beta\mu\nu}, \tilde{F}^{\alpha\beta} \) are obtained by contractions with Levi-Civita tensor:

\[
\tilde{R}^{\alpha\beta\gamma\delta} = \frac{1}{4} \epsilon^{\alpha\beta\mu\nu} R_{\rho\mu\nu\sigma} \epsilon^{\rho\sigma\gamma\delta}, \quad \tilde{F}^{\alpha\beta} = -\frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu}.
\]

(2)

Mention that compared to above, in literature the double dual of Riemann tensor is often defined with minus sign.

## 2 Non-minimal gravity in F1 and F2

Despite focusing on vector Horndeski model, we start our research from study of a gravity theory with non-minimally coupled matter in general case. The reason is simple: we need to distinguish peculiarities of Horndeski model and general features of a theory with non-minimal coupling between matter and gravity. During this general consideration we choose the coupling matter in a
form of scalar field, for simplicity. We will explain later that the results will be also relevant to vector Horndeski theory.

So we start with the action

$$S = \frac{1}{2} \int \left[ R - \lambda L_c(R_{\rho\sigma\nu}^\alpha, g_{\rho\sigma}, \phi, \nabla_\mu \phi, \nabla_\rho \nabla_\sigma \phi) \right] \sqrt{-g} \; d^4x + \int L_m \sqrt{-g} \; d^4x. \tag{3}$$

The minimally coupled term $L_m$ represents some standard matter lagrangian without curvature, connection and second-order derivatives. One can naturally assume that at relatively low energies the non-minimal coupling effects can be observed as small corrections to Einstein gravity. The parameter of non-minimal coupling, $\lambda$, may be used to establish energy scale at which the coupling between matter and geometry becomes significant. Although a particular physical interpretation of $\lambda$ depends on specific lagrangian, we may formally assume that for a certain class of theories the solutions can be expanded in powers of $\lambda$, and the approximative solutions method can be applied. Exact solutions in a theories with non-minimal coupling are rare, so many results can be obtained with the aid of approximative solutions.

2.1 F1 case

Let us consider the Palatini approach, first. Affine connection $\Gamma$ is then an independent variable, and its difference from metric connection $\{\}$ is given by distortion tensor:

$$C_{\alpha \rho \sigma} = \{\rho \sigma\} - \Gamma_{\alpha \rho \sigma}^\beta. \tag{4}$$

Throughout the article we will deal with symmetric connection only: $\Gamma_{\rho \sigma}^\alpha = \Gamma_{\sigma \rho}^\alpha$. Though in $f(R, R_{\rho\sigma} R^\rho_{\sigma})$ theories torsion does not affect Einstein and field equations due to projective invariance of scalar curvature $\{12\}$, in our case there is a curvature tensor in the action. Hence setting torsion to zero is a subtle question $\{32\}$. Nonetheless, we let it vanish a priori, in order to simplify the calculations.

Covariant derivative with respect to metric connection will be denoted as $\nabla$, and covariant derivative with respect $\Gamma$ will be denoted as $\nabla^\Gamma$. The former preserves metric: $\nabla_\alpha g_{\rho\sigma} = 0$. The latter, when acting on metric, generates non-metricity tensor by the formula:

$$Q_{\alpha \rho \sigma} = -\nabla^\Gamma_\alpha g_{\rho \sigma}; \quad Q_\alpha = \frac{1}{4} Q_{\alpha \rho \sigma}^\rho; \quad \bar{Q}_\sigma = Q_{\sigma \rho \sigma}^\rho.$$

For convenience of further use we’ve introduced also the traces of non-metricity. The first of them, $Q_\alpha$, is often called Weyl vector, and the factor $1/4$ is standard in its definition. Mention that affine derivative of contravariant metric tensor is equal to non-metricity with raised indices taken with positive sign: $\nabla^\Gamma_\alpha g^{\rho \sigma} = Q_{\rho \sigma}^\alpha$. The relation between distortion and non-metricity is following:

$$C_{\alpha \rho \sigma} = -\frac{1}{2} (Q_{\rho \sigma \alpha} + Q_{\sigma \alpha \rho} - Q_{\alpha \rho \sigma}). \tag{6}$$

The action $\{3\}$ now contains covariant derivatives with respect to $\Gamma$, and Riemann tensor $R_{\rho\sigma\nu}^\alpha$ depends solely on $\Gamma$:

$$S = \frac{1}{2} \int \left[ R(\Gamma, g) - \lambda L_c(R_{\rho\sigma\nu}^\alpha(\Gamma), g_{\rho\sigma}, \phi, \nabla^\Gamma_\mu \phi, \nabla^\Gamma_\rho \nabla^\Gamma_\sigma \phi) \right] \sqrt{-g} \; d^4x + \int L_m \sqrt{-g} \; d^4x. \tag{7}$$

The derivation of equations of motion usually requires a permutation of the derivatives $\nabla^\Gamma$. During this procedure, there appears a term with Weyl vector due to presence of the factor $\sqrt{-g}$. For example, in case of two arbitrary tensors $T$ and $K$ combined into scalar density one has

$$\sqrt{-g} T^{\mu \alpha (\alpha \beta \rho)} = \partial_\mu \left( \sqrt{-g} T_{\beta (k)}^{\mu \alpha (\alpha \beta \rho)} K_{\alpha (\alpha \beta \rho)}^{(\beta (k))} \right) - \sqrt{-g} K^{\beta (k) \alpha (\alpha \beta \rho)} (\nabla^\Gamma_\mu - 2Q_\mu) T_{\beta (k)}^{\mu \alpha (\alpha \beta \rho)}. \tag{8}$$
Thus it is useful to define the extended affine covariant derivative:
\[
\bar{\nabla}_\nu^\Gamma \equiv \nabla_\nu^\Gamma - 2Q_\nu.
\] (9)

The corresponding covariant divergence of a vector or antisymmetric tensor coincides with metric covariant divergence:
\[
\bar{\nabla}_\nu^\Gamma V^\nu = \nabla_\nu V^\nu, \quad \bar{\nabla}_\nu^\Gamma V[^{\nu[\mu_1}..\mu_n]] = \nabla_\nu V[^{\nu[\mu_1}..\mu_n]].
\] (10)

Last property comes from the fact that both connections \( \Gamma_{\beta\mu}^\alpha \) and \{\( \alpha_{\beta\mu} \)\} are symmetric and their contraction in two indices with antisymmetric tensor identically vanishes. Finally, let us introduce the two tensors in order to make the formulas more compact:
\[
M_{\cdot \rho\sigma\nu}^\alpha \equiv \frac{\partial L}{\partial R_{\rho\sigma\nu}^\alpha}, \quad N_{\rho\sigma}^\alpha \equiv \frac{\partial L}{\partial \bar{\nabla}_\rho^\Gamma \bar{\nabla}_\sigma^\Gamma \phi}.
\] (11)

The first one is antisymmetric in \( \mu, \nu \), and the second one is symmetric.

Equation of motion for scalar field \( \phi \) can be easily written as
\[
-\lambda \bar{\nabla}_\rho^\Gamma \bar{\nabla}_\sigma^\Gamma N_{\rho\sigma}^\alpha + \left[ \nabla_\mu \frac{\partial}{\partial \mu_\phi} - \frac{\partial}{\partial \phi} \right] (2L_m - \lambda L_c) = 0.
\] (12)

Mention that affine and metric covariant divergences of the vector \( \partial L/\partial \mu_\phi \) coincide, as it follows from Eq. (10).

Einstein equations in F1 look also quite simply:
\[
G_{\rho\sigma}^{(\Gamma, g)} = T_{\rho\sigma}^{(1)}(\Gamma, g), \quad \text{where} \quad T_{\rho\sigma}^{(1)} \equiv -\frac{\partial}{\sqrt{-g} \partial g^{\rho\sigma}} [ (2L_m - \lambda L_c) \sqrt{-g} ].
\] (13)

Einstein tensor \( G_{\rho\sigma}(\Gamma, g) \) is not symmetric for arbitrary affine connection \( \Gamma \), and only its symmetric part joins the equations.

In order to derive the connection equation one should vary the lagrangian density from (7) in \( \Gamma \):
\[
\frac{\delta (\sqrt{-g} L)}{\sqrt{-g}} = \frac{1}{2} (\delta^\sigma_{(a} g^{\rho\nu)} - \lambda M_{(a}^{\rho\sigma\nu}) ) \delta \Gamma_{\rho\sigma\nu}^\alpha (\Gamma) - \lambda N_{\rho\sigma}^\alpha \delta \nabla_\rho^\Gamma \nabla_\sigma^\Gamma \phi.
\] (14)

The variation of curvature tensor with respect to connection reads as
\[
\delta R_{\rho\sigma\nu}^\alpha = \nabla_\sigma^\Gamma \delta \Gamma_{\rho\nu}^\alpha - \nabla_\nu^\Gamma \delta \Gamma_{\rho\sigma}^\alpha,
\] (15)

while for second-order covariant derivative one has
\[
\delta \Gamma_{\rho\sigma}^\Gamma \nabla_\sigma^\Gamma \phi = -\nabla_\alpha^\Gamma \phi \delta \Gamma_{\rho\sigma}^\alpha.
\] (16)

Then the variation takes the form
\[
\frac{\delta (\sqrt{-g} L)}{\sqrt{-g}} = - \left( \delta^{[\nu}_{(a} g^{\sigma]} - \lambda M_{(a}^{\rho\sigma\nu}) \right) \nabla_\nu^\Gamma \delta \Gamma_{\rho\sigma}^\alpha + \frac{\lambda}{2} N_{\rho\sigma}^\alpha \nabla_{\alpha\phi}^\Gamma \delta \Gamma_{\rho\sigma}^\alpha
\]
\[
= \left[ \nabla_\nu^\Gamma \left( \delta^{[\nu}_{(a} g^{\sigma]} - \lambda M_{(a}^{\rho\sigma\nu}) \right) + \frac{\lambda}{2} N_{\rho\sigma}^\alpha \nabla_{\alpha\phi}^\Gamma \right] \delta \Gamma_{\rho\sigma}^\alpha + \text{tot. der.}
\] (17)

As a result, the equation on symmetric affine connection reads as
\[
\nabla_\nu^\Gamma \left( \delta^{[\nu}_{(a} g^{\sigma]} - \lambda M_{(a}^{\rho\sigma\nu}) \right) + \frac{\lambda}{2} N_{\rho\sigma}^\alpha \nabla_{\alpha\phi}^\Gamma + (\rho \leftrightarrow \sigma) = 0.
\] (18)
Let us introduce hyperstress tensor:

$$\Theta_{\alpha}^{\rho \sigma} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_{c}}{\delta \Gamma_{\rho \sigma}^{\alpha}} = -2\lambda \nabla_{\nu} M_{\alpha}^{(\rho \sigma) \nu} + \lambda N_{\rho \sigma} \nabla_{\alpha} \phi .$$  

(19)

It is the variational derivative of non-minimal term in the action (7) with respect to connection. Mention that the variation by connection of the entire action is often called the connection tensor, and variation of matter part of the action (coupled to connection via covariant derivatives only) is called hypermomentum [33]. Here we deal with the action which is explicitly split onto Einstein-Hilbert and coupling terms. The variation of EH term is trivial, while the variation of coupling term is a matter of interest. However it is more than just a material current coupled to connection, so it deserves an individual name. By construction, hyperstress is symmetric in last two indices, $\rho$ and $\sigma$.

The derivative $\nabla_{\Gamma}$ acting on metric tensor spawns non-metricity tensor and its traces. Therefore connection equation can be rewritten as following:

$$Q_{\alpha}^{\rho \sigma} - 2g^{\rho \sigma} Q_{\alpha} + 2\delta^{(\rho}_{\alpha} Q^{\sigma)} - \delta^{(\rho}_{\alpha} \bar{Q}^{\sigma)} = -\Theta_{\alpha}^{\rho \sigma} .$$  

(20)

This is the differential equation for connection, but if $L_{c}$ is linear in curvature, it turns to algebraic one. It hardly can be resolved in general, but one can use successive approximation method, expanding the equation in powers of $\lambda$. For this, let us expand the r.h.s. of connection equation (20) in powers of $\lambda$ up to $O(\lambda)$ term. The result is the hyperstress calculated on solutions to minimally coupled theory, i.e. with $\lambda = 0$, because the difference between solutions with vanishing and non-vanishing $\lambda$ has $O(\lambda)$ order, and hyperstress already contains $\lambda$ as a factor.

The approximative solution for connection, valid in the order $O(\lambda)$, should therefore satisfy the equation

$$Q_{\alpha}^{\rho \sigma} - 2g^{\rho \sigma} Q_{\alpha} + 2\delta^{(\rho}_{\alpha} Q^{\sigma)} - \delta^{(\rho}_{\alpha} \bar{Q}^{\sigma)} = -\Theta_{\alpha}^{\rho \sigma} ,$$  

(21)

in which the r.h.s. now is independent of affine connection and is known function derived from solution to minimally coupled theory. It is convenient also introduce the traces of hyperstress:

$$\Theta_{\alpha} \equiv \Theta_{\alpha}^{\rho} \cdot \rho , \quad \bar{\Theta}_{\sigma} \equiv \Theta_{\alpha}^{\cdot \cdot \cdot \bar{\sigma}} .$$  

(22)

Then, taking traces of the Eq. (21) and substituting it back in the equation, after simple algebra one finds the expression for non-metricity:

$$Q_{\alpha \rho \sigma} = \Theta_{\alpha \rho \sigma} - \frac{2}{3} g_{\alpha (\rho} \bar{\Theta}_{\sigma)} - \left( \frac{1}{2} \Theta_{\alpha} - \frac{1}{3} \Theta_{\bar{\alpha}} \right) g_{\rho \sigma} .$$  

(23)

The distortion tensor can be obtained from (23) with known non-metricity:

$$C_{\alpha \rho \sigma} = -\frac{1}{2} \left[ \Theta_{\rho \sigma \alpha} + \Theta_{\sigma \alpha \rho} - \Theta_{\alpha \rho \sigma} \right] - \frac{1}{3} g_{\alpha (\rho} \bar{\Theta}_{\sigma)} + \frac{1}{2} g_{\alpha (\rho} \Theta_{\sigma)} - \frac{g_{\rho \sigma}}{4} \left( \Theta_{\alpha} - 2 \bar{\Theta}_{\alpha} \right) .$$  

(24)

We see that affine connection $\Gamma$ appears to be incompatible with metric $g_{\rho \sigma}$, unless hyperstress is vanishing.

### 2.2 F2 case

Now turn to metric formulation with the action

$$S = \frac{1}{2} \int \left[ R(g) - \lambda L_{c}(R_{\rho \sigma \nu}^{\alpha}(g), g_{\rho \sigma}, \phi, \nabla_{\mu} \phi, \nabla_{\rho} \nabla_{\sigma} \phi) \right] \sqrt{-g} \ d^{4}x + \int L_{m} \sqrt{-g} \ d^{4}x .$$  

(25)
The field equation differs from F1 case (12) by replacement of the covariant derivatives:

\[- \lambda \nabla_\rho \nabla_\sigma N^{\rho\sigma} + \left[ \nabla_\mu \frac{\partial}{\partial \phi} - \frac{\partial}{\partial \phi} \right] (2L_m - \lambda L_c) = 0. \tag{26}\]

The terms \(N^{\mu\nu}\) and \(L_c\) are also different in F1 and F2 cases, because they depend on different connections. However, the difference has the order \(O(\lambda)\), and \(N^{\mu\nu}\) and \(L_c\) enter the equation already with the factor \(\lambda\). Therefore the equations for scalar field in F1 and F2 formulations are identical in \(O(\lambda)\) order.

The derivation of Einstein equation in F2 case is a bit more complicated, than in F1. Now the variation of connection and curvature tensor should be expressed via metric variation:

\[
\delta \Gamma^\alpha_{\rho\sigma}(g) = g^{\alpha\lambda} \left( \nabla_\rho \delta g_{\lambda\sigma} + \nabla_\sigma \delta g_{\lambda\rho} - \nabla_\lambda \delta g_{\rho\sigma} \right). \tag{27}\]

As a result, in F2 version of Einstein equations a new term, \(T^{(2)}_{\rho\sigma}\), appears. In order to calculate it, one should vary the coupling term with respect to connection, and then take into account the dependence of \(\delta \Gamma\) on \(\delta g\). So, on the first step we will get nothing more than hyperstress \((19)\), calculated on the metric connection. After that, we should swap covariant derivatives and apply the permutation of indices reflecting the structure of the Eq. (27). Combining all terms together we obtain:

\[ G_{\rho\sigma}(g) = T^{(1)}_{\rho\sigma}(g) + T^{(2)}_{\rho\sigma}(g), \quad \text{where} \quad T^{(2)}_{\rho\sigma}(g) = -\frac{1}{2} \nabla^\alpha \left( \Theta_{\rho\alpha\sigma}(g) + \Theta_{\sigma\alpha\rho}(g) - \Theta_{\alpha\rho\sigma}(g) \right). \tag{28}\]

The shape of term \(T^{(1)}_{\rho\sigma}\) coincides with the one derived in F1 case (13), but now it depends on metric connection. Hyperstress in the expression for \(T^{(2)}_{\rho\sigma}\) also depends on metric connection \(\{^\alpha_{\rho\sigma}\}\).

The fact that F1 and F2 versions of Einstein equation differ by \(T^{(2)}_{\rho\sigma}\) term was already established in literature [8, 34]. And now we will calculate the actual difference at the level of solutions, expanding the equations in powers of \(\lambda\).

### 2.3 Comparing F1 and F2

We have started with two actions (7), (25), which just look similar but are actually different, and have applied two different variational procedures. So no wonder that we eventually obtained two different versions of Einstein equations. Mathematically, F1 and F2 formulations are distinct, and that is a well-known fact. Here we would like to accentuate that from physical point of view the two formulations may appear less distinguishable. When the modification to Einstein gravity is considered as a correction at high energies, the corresponding Observable phenomena should be barely visible at low energy scale accessible to us. Those phenomena can be described by approximative solutions. But what is the difference between approximative solutions in F1 and F2?

Let us expand Einstein equations (13, 28) in \(O(\lambda)\) order. All connection-dependent terms in right hand sides already contain the factor \(\lambda\), therefore the difference between F1 and F2 connections will contribute in the r.h.s. only into terms of the order \(O(\lambda^2)\). Einstein tensor in l.h.s. does not contain the factor \(\lambda\), so the difference between F1 and F2 versions of Einstein tensors,

\[
\delta G_{\rho\sigma} = G_{(\rho\sigma)}(\Gamma, g) - G_{\rho\sigma}(g), \tag{29}\]

emerges already at \(O(\lambda)\) terms. In order to calculate \(\delta G_{\rho\sigma}\) in \(O(\lambda)\) order let us use the Palatini identity

\[
\delta R_{(\mu\nu)} = \nabla_\lambda \delta \Gamma^\lambda_{\mu\nu} - \nabla_{(\mu} \delta \Gamma^\lambda_{\nu)} , \tag{30}\]

where \(\Gamma^\alpha_{\rho\sigma}\) are the Christoffel symbols associated with the connection \(\Gamma\).
with difference between connections given by distortion tensor \( \delta \Gamma_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda - \{ \beta \}_{\mu\nu} = -C_{\mu\nu}^\lambda \).

After short calculations one can see that

\[
\delta G_{\rho\sigma} = \frac{1}{2} \nabla^\alpha (\Theta_{\rho\sigma\alpha} + \Theta_{\sigma\alpha\rho} - \Theta_{\alpha\rho\sigma}) + O(\lambda^2).
\]

Shortly speaking, Einstein equations (13) and (28) in F1 and F2 cases look like

\[
F_1 : \quad G_{\rho\sigma}(g) - T_{\rho\sigma}^{(2)}(g) = T_{\rho\sigma}^{(1)}(g) + O(\lambda^2),
\]

\[
F_2 : \quad G_{\rho\sigma}(g) = T_{\rho\sigma}^{(1)}(g) + T_{\rho\sigma}^{(2)}(g),
\]

where all terms are evaluated for metric connection \( \{ \alpha \}_{\beta\mu} \).

The apparent equivalence of F1 and F2 versions of Einstein equations in \( O(\lambda) \) order is not accidental, of cause. Both terms \( \delta G_{\rho\sigma} \) and \( T_{\rho\sigma}^{(2)} \) reflect the structure of solutions to connection equation. In F2 case the equation on Levi-Civita connection is linear, while in F1 case the equation on distortion tensor is linear only in \( O(\lambda) \) order. So it appears that exact solutions to metric version of gravity theory \( (\lambda) \) coincide with approximative \( O(\lambda) \) solutions obtained in Palatini formulation of the theory.

### 3 Vector Horndeski theory

Let us now turn from general consideration to investigation of particular theory. The covariant derivatives of vector and scalar fields are different, so in general the results of previous section can not be directly applied to theories with non-minimally coupled vector field. However the lagrangian \( (\mathbb{I}) \) does not contain covariant derivatives of vector field due to our choice of vanishing torsion. It contains only the first-order partial derivatives, which are the same for vector and scalar fields. Consequently, all calculations from previous section remain valid.

The lagrangian \( (\mathbb{I}) \) for vector Horndeski model is given in metric version. We suggest that it can be slightly modified to suit better the F1 case. Gauge field comes as tensor \( \tilde{F}^{\alpha\beta} \tilde{F}_{\mu\nu} \) which is antisymmetric in first and second pairs of indices and symmetric with respect to pair exchange \( (\alpha\beta) \leftrightarrow (\mu\nu) \). In metric case the curvature tensor shares this symmetry, but not in Palatini case. In latter case gauge field actually is coupled to symmetrized part of curvature tensor (or its dual):

\[
S_{\alpha\beta\mu\nu} = \frac{1}{2} \left( g_{\lambda[\alpha} R_{\beta]\mu\nu \lambda] + g_{\lambda[\mu} R_{\nu]\alpha\beta \lambda]} \right), \quad \tilde{S}_{\alpha\beta\gamma\delta} = \frac{1}{4} \epsilon_{\alpha\beta\mu\nu} S_{\mu\nu\rho\sigma} \epsilon_{\rho\sigma\gamma\delta}.
\]

The remaining components of curvature tensor do not interact with \( \tilde{F}^{\alpha\beta} \tilde{F}_{\mu\nu} \), because their contraction identically vanishes. For metric curvature tensor one has \( S_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} \), so the coupling in the form \( S \tilde{F} = \tilde{S} FF \) correctly describes both F1 and F2 version of vector Horndeski model.

Finally, we would like to consider the action containing both gravity-coupled and standard terms:

\[
S = \frac{1}{2} \int \left[ R - Tr(\Phi_{\mu\nu} F_{\mu\nu}) \right] \sqrt{-g} d^4x.
\]

Here the “induction” tensor is introduced for convenience, it absorbs minimally and non-minimally coupled field tensors:

\[
\Phi_{\mu\nu} = F_{\mu\nu} + \frac{\lambda}{2} \tilde{S}_{\mu\nu\lambda\tau} F_{\lambda\tau}.
\]

The trace operators appear because the vector field can be non-Abelian with SU(2) gauge group. Then

\[
A_\mu = A_\mu^a T_a, \quad F_{\mu\nu} = F_{\mu\nu}^a T_a = 2 \nabla_{[\mu} A_{\nu]} - i [A_\mu, A_\nu] ,
\]
where SU(2) gauge group generators are

\[ [T_a, T_b] = i \varepsilon_{ab}^c T_c, \quad \text{Tr}(T_a T_b) = \frac{1}{2} \delta_{ab}. \]  (37)

In Maxwell case trace operators should be replaced by factors 1/2 in order to get correct factors everywhere.

The field equation is the conservation law:

\[
F_1 \text{ and } F_2: \quad D_\nu \Phi^{\mu \nu} = 0, \quad \text{where } D_\nu \equiv \nabla_\nu + [A_\nu, \cdot]. \]  (38)

Covariant divergence here can be calculated with metric connection in both F2 and F1 approaches, because $\Phi^{\mu \nu}$ is antisymmetric, see Eq. (10). Einstein equations look quite different in two formulations:

\[
F_1: \quad G_{(\rho \sigma)} = \text{Tr} \left( F_{(\rho \alpha} \Phi^{\alpha \sigma)} - \frac{1}{2} g_{\rho \sigma} F^{\alpha \beta} F_{\alpha \beta} \right), \]

\[
F_2: \quad G_{(\rho \sigma)} = 2 \text{Tr} \left( F_{(\rho \alpha} \Phi^{\alpha \sigma)} - \frac{1}{4} g_{\rho \sigma} F^{\alpha \beta} \Phi^{\alpha \beta} \right) + \lambda \text{Tr} \left( R^{\alpha \beta} \tilde{F}_{\alpha \rho} \tilde{F}_{\beta \sigma} + D^\alpha \tilde{F}_{\beta \rho} D^\beta \tilde{F}_{\alpha \sigma} + F^{\alpha \beta} \left[ \tilde{F}_{\alpha \rho}, \tilde{F}_{\beta \sigma} \right] \right). \]  (40)

The r.h.s. of connection equation (20) now contains the hyperstress

\[
\Theta_{\rho}^{\sigma} \rho^\sigma = -\lambda \nabla_\nu^\Gamma \text{Tr} \left( \tilde{F}_{\gamma}^{(\rho} \tilde{F}_{\sigma)}^{\nu} \right). \]  (41)

The derivatives of hyperstress tensor arising in F2 version of Einstein equations generally contain higher order derivatives of matter field. However, for Horndeski prescription the terms with higher order derivatives are totally annihilated by virtue of Bianchi identities for gauge field and curvature tensor. The arbitrary couplings of the form $R F_{\mu \nu} F^{\mu \nu}, R_{\mu \nu} F^{\mu \lambda} F_{\lambda \nu}$, e.t.c. don’t share this property.

One can also check that in $O(\lambda)$ order Einstein equations in F1 and F2 cases do coincide. For instance, the difference between affine and metric Einstein tensors can be found from the Eq. (32) with hyperstress given above:

\[
\delta G_{(\rho \sigma)} = \text{Tr} \left( \frac{1}{2} g_{\rho \sigma} F^{\alpha \beta} \left[ \Phi^{\alpha \beta} - F^{\alpha \beta} \right] - F_{(\rho \alpha} \Phi^{\alpha \sigma)} \right) - \lambda \text{Tr} \left( R^{\alpha \beta} \tilde{F}_{\alpha \rho} \tilde{F}_{\beta \sigma} + D^\alpha \tilde{F}_{\beta \rho} D^\beta \tilde{F}_{\alpha \sigma} + F^{\alpha \beta} \left[ \tilde{F}_{\alpha \rho}, \tilde{F}_{\beta \sigma} \right] \right). \]  (42)

Combining it with F1 version of Einstein equation we obtain precisely the F2 version.

Now let us compare the solutions to F1 and F2 versions of vector Horndeski model beyond $O(\lambda)$ order. The equations of motion are quite complicated, so we will use suitable models mostly for illustrative purposes. The detailed investigation of solutions in each case is not the subject of the current consideration.

3.1 Homogeneous isotropic model

Unlike Maxwell case, the vector Horndeski model with SU(2) Yang–Mills field admits homogeneous and isotropic cosmological solutions, including inflationary ones [17]. So here we consider the non-Abelian version of the action (34). The ansatz for metric in proper time gauge is standard:

\[
ds^2 = -dt^2 + a^2 \left[ d\chi^2 + \Sigma_k^2(\chi)(d\theta^2 + \sin^2 \theta d\varphi^2) \right], \]  (43)
where $\Sigma_k(\chi) = \{\sin \chi, \chi, \sinh \chi\}$ for the closed, flat and open universe, labeled by $k = 1, 0, -1$, correspondingly. For SU(2) Yang–Mills field the most general cosmological ansatz preserving the isotropy and homogeneity of the metrics can be written in terms of a single function $f(t)$ in all three cases $k = 0, \pm 1$ [35]:

$$A = f(t)T_0d\chi + \left[f(t)\Sigma_kT_0 + (\Sigma_k' - 1)T_\varphi\right]d\theta + \left[f(t)\Sigma_kT_\varphi - (\Sigma_k' - 1)T_0\right] \sin \theta d\varphi.$$  \hspace{1cm} (44)

Here the group generators, $T_\alpha$, are the Pauli matrices $\tau_\beta/(2i)$ contracted with spherical unit vectors $n^b(\chi, \theta, \varphi)$. Then the field tensor takes the form [36]:

$$F = \dot{f} (T_\chi dt \wedge d\chi + T_0 \Sigma_k dt \wedge d\theta + T_\varphi \Sigma_k \sin \theta dt \wedge d\varphi) + \Sigma_k (f^2 - k)(T_\varphi d\chi \wedge d\theta - T_\theta \sin \theta d\chi \wedge d\varphi + T_\chi \Sigma_k \sin \theta d\theta \wedge d\varphi).$$  \hspace{1cm} (45)

Dot over the letter represents the time-derivative. Such configuration (with $k = 0$) in metric formulation was investigated in [17, 18], so here we may focus on Palatini case.

It is easy to calculate hyperstress (41) in $O(\lambda)$ order, since then we may use Levi-Civita connection of the metric (43) instead of unknown affine connection. It appears that all components of hyperstress in $O(\lambda)$ order can be expressed in terms of its trace:

$$\Theta_{\alpha \rho \sigma} = \frac{1}{3} (\Theta_{\alpha g_{\rho \sigma}} - \Theta_{(\rho g_{\sigma})\alpha}) ,$$  \hspace{1cm} (46)

and only the temporal component of the trace is non-vanishing:

$$\Theta_\alpha = 3\lambda \left[ \left( \frac{(f^2 - k)^2}{a^2} - \frac{j^2}{a^2} \right) H - \frac{1}{a^4} \frac{d}{dt} (f^2 - k)^2 \right] \delta_{\alpha t} .$$  \hspace{1cm} (47)

Here $H = \dot{a}/a$ is Hubble parameter, as usual. Then the non-metricity tensor in $O(\lambda)$ order can be found from the Eq. (23), and is equal to

$$Q_{\alpha \rho \sigma} = -\frac{1}{3} \Theta_{\alpha g_{\rho \sigma}} .$$  \hspace{1cm} (48)

It is traceless, which happens only when covariant derivative is compatible the conformally transformed metric $h_{\rho \sigma} = e^{2\omega(t)} g_{\rho \sigma}$, where $\omega(t)$ is some unknown function.

One can check that the ansatz for connection taken in the form of Levi-Civita connection compatible with metric $h_{\rho \sigma}$,

$$\Gamma^\alpha_{\rho \sigma} = \frac{(h^{-1})^\alpha}{2} (\partial_\rho h_{\lambda \sigma} + \partial_\sigma h_{\lambda \rho} - \partial_\lambda h_{\rho \sigma}) ,$$  \hspace{1cm} (49)

passes through the full equations of motion (not restricted by $O(\lambda)$ order), if $\dot{\omega} = -\Theta_t/6$. So, homogeneous isotropic vector Horndeski model in Palatini approach is effectively a bimetric theory. However the two metrics differ only by a conformal factor, hence there is only one additional degree of freedom compared to metric case.

The equations of motion are non-integrable and extremely complicated. So we would like to simplify the model in order to get the exact solutions, which can be easily analyzed. For this, let us now turn to a so-called cosmological sphaleron solution. The classical stress-energy tensor of minimally coupled theory reads as:

$$T^t_t = \rho , \hspace{1cm} T^k_i = \frac{\rho}{3} \delta^k_i , \hspace{1cm} \text{where} \hspace{1cm} \rho = \frac{3}{2} \left( \frac{j^2}{a^2} + \frac{(f^2 - k)^2}{a^4} \right) .$$  \hspace{1cm} (50)
In closed universe the gauge field potential \((f^2 - 1)^2/a^4\) acquires a double-valley form. The unstable static solution to field equation, \(f = 0\), corresponds to the top of a barrier which separates two distinct vacua states with \(f = \pm 1\). It has non-zero energy density \(\rho = 3/(2a^4)\) in Einstein gravity theory. Hence one obtains the model with non-dynamical gauge field and non-trivial stress-energy tensor which is very suitable for investigation of complicated theory.

The equation \(\dot{\omega} = -\Theta_t/6\) can be easily integrated, since now \(\Theta_t = 3\lambda H/a^4\). Integrating it with natural initial condition \(\Gamma^\alpha_{\rho\sigma}|_{\lambda=0} = \{\rho\}_\lambda\) we find the exact solution to connection for cosmological sphaleron model in F1 approach. It is the Levi-Civita connection of conformally transformed metric

\[
h_{\rho\sigma} = \exp \left( \frac{\lambda}{4a^4} \right) g_{\rho\sigma}.
\]

The factor \(1/a^4\) in above formula is proportional to energy density of Yang-Mills field. So when the field energy density goes beyond the scale at which non-minimal coupling joins the game, the difference between two metrics \(g\) and \(h\) starts growing exponentially fast.

Now the Einstein equations (39–40) are greatly simplified and become integrable. We would like to present them in a form of Friedmann equation:

\[
\begin{align*}
F1 : \quad H^2 &= \left( \frac{1}{2a^4} - \frac{1}{a^2} \right) \left( 1 - \frac{\lambda}{2a^4} \right)^{-2}, \\
F2 : \quad H^2 &= \left( \frac{1}{2a^4} - \frac{1}{a^2} \right) \left( 1 - \frac{\lambda}{a^4} \right)^{-1}.
\end{align*}
\]

Compared to Einstein gravity, the r.h.s. of the Friedmann equations acquire additional factors, which differ starting from \(O(\lambda^2)\) order.

Let us briefly spell out the difference between Palatini and metric versions of Horndeski theory in case of particular cosmological sphaleron configuration. Non-minimal coupling in both cases produces additional singularity when the Hubble parameter diverges. In F1 case the singularity takes place at the point \(a^4 = \lambda/2\), while in F2 — at another point \(a^4 = \lambda\). The behavior of solutions in vicinity of those points is distinct. The r.h.s. of the Friedman equation (53) changes sign at singularity, which is prohibited for l.h.s. Therefore the F2 solution just stops there, which implies a so-called “Big Freeze” singularity. In F1 case, the r.h.s. of the Friedmann equation (52) remains positive after crossing the critical point, and solutions can be analytically continued through it. This picture resembles a phase transition in the universe.

The behavior of F1 and F2 solutions in vicinity of initial singularity, \(a \to 0\), is also different. In metric case the Hubble parameter approaches constant, \(H^2 \to -1/(2\lambda)\). Such evolution of Hubble parameter takes place when the effective equation of state corresponds to inflaton, \(p = -\rho\). The Palatini approach provides another asymptotic: \(a \propto t^{-1/2}\), which corresponds to phantom equation of state with \(p = -\frac{7}{4}\rho\).

### 3.2 Static spherically-symmetric model

We have found that for homogeneous isotropic configurations the connection is compatible with some effective metric. However this is not an inherent property of vector Horndeski model. Let us see what happens in case of static spherically symmetric configuration. The metric now reads as

\[
ds^2 = -w(r)dt^2 + w(r)^{-1}dr^2 + \rho(r) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right),
\]

and the ansatz for vector field is

\[
A = f(r) \ dt + p \cos \theta \, d\varphi,
\]
which is often called the scalar electrodynamics. There is no need in non-Abelian configuration here, so we better consider the Maxwell field for simplicity. The solutions to metric version of such model can be found in literature \cite{27,28}, so we proceed with investigation of Palatini case.

The expression for hyperstress in \( O(\lambda) \) order is not so simple, and does not allow to guess the ansatz for connection. However the connection equation (20) is linear in connection, because the action of Horndeski theory \cite{31} is linear in curvature. The system of linear algebraic equations can be easily resolved with a computer, so we can only present the results. In purely electric case \((p = 0)\) one has the following non-vanishing components of connection:

\[
\Gamma_{tr}^t = \frac{w'}{2w} + \frac{\lambda f'^2}{2\rho^2} \left( \frac{2}{3} + \lambda \frac{f'^2}{24} \right), \quad \Gamma_{tt}^t = w^2 \left[ \frac{w'}{2w} + \frac{\lambda f'^2}{2\rho^2} \left( 1 + \lambda \frac{f'^2}{24} \right) \right], \\
\Gamma_{rr}^t = -\frac{w'}{2w} + \frac{\lambda f'^2}{2\rho} \left( \frac{1}{3} + \lambda \frac{f'^2}{24} \right), \quad \Gamma_{r\theta}^\theta = \frac{\rho'}{\rho} \left( 1 + \lambda \frac{f'^2}{12} \right), \\
\Gamma_{\theta\phi}^\phi = -w \rho p', \quad \Gamma_{\phi\phi}^\phi = \Gamma_{\theta\theta}^\phi \sin^2 \theta, \quad \Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta, \quad \Gamma_{\phi\phi}^{\phi\phi} = \frac{\cos \theta}{\sin \theta}.
\]

The solution to magnetic case \((q = 0)\) reads as

\[
\Gamma_{tr}^t = -\Gamma_{rr}^t = \frac{w'}{2w} + \frac{\lambda \rho f'^2}{6\rho^2} \left( 1 + \lambda \frac{f'^2}{6\rho^2} \right)^{-1}, \quad \Gamma_{tt}^t = w^2 \left[ \frac{w'}{2w} + \frac{\lambda \rho f'^2}{2\rho^2} \left( 1 + \lambda \frac{f'^2}{6\rho^2} \right)^{-1} \right], \\
\Gamma_{r\theta}^\theta = \Gamma_{\theta\phi}^\phi = \frac{\rho'}{\rho} \left( 1 - \lambda \frac{f'^2}{6\rho^2} \right) \left( 1 + \lambda \frac{f'^2}{6\rho^2} \right)^{-1}, \quad \Gamma_{\theta\theta}^\theta = -w \rho p' \left( 1 - \lambda \frac{f'^2}{2\rho^2} \right), \\
\Gamma_{\phi\phi}^\phi = \Gamma_{\theta\theta}^\phi \sin^2 \theta, \quad \Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta, \quad \Gamma_{\phi\phi}^{\phi\phi} = \frac{\cos \theta}{\sin \theta}.
\]

The solution which incorporates both \( q, p \) non-zero is a quite complicated combination of the above solutions, and there is no need presenting it.

In electric case the connection is parameterized by five nontrivial functions, while in magnetic case there are only four. However, neither of these connections is Levi-Civita one. One can easily show this considering just \( O(\lambda) \) order. If connection is non-metric in that order, it is non-metric in general.

Suppose that connection is compatible with some metric \( h_{\rho\sigma} \), so that \( \nabla_{\alpha} h_{\rho\sigma} = 0 \). Let us then calculate the difference between the two metrics, \( \delta h_{\rho\sigma} \equiv (h_{\rho\sigma} - g_{\rho\sigma}) \) up to \( O(\lambda) \) order. By definition,

\[
\nabla_{\alpha} \delta h_{\rho\sigma} = -\nabla_{\alpha} g_{\rho\sigma} = Q_{\alpha\rho\sigma}.
\]

From other hand, \( \nabla_{\alpha} \delta h_{\rho\sigma} = \nabla_{\alpha} \delta h_{\rho\sigma} + O(\lambda^2) \). Consequently, in \( O(\lambda) \) order one has

\[
\nabla_{\alpha} \delta h_{\rho\sigma} = Q_{\alpha\rho\sigma},
\]

where non-metricity tensor is given by the Eq. (23). The commutator of two covariant derivatives, \([\nabla_{\nu}, \nabla_{\alpha}]\), generates algebraic equations on \( \delta h_{\rho\sigma} \):

\[
\delta h_{(\lambda)\rho\sigma} = \nabla_{\[\nu} Q_{\alpha\]\rho\sigma}.
\]

Here the Riemann tensor and covariant derivatives of non-metricity can be calculated on solutions to non-coupled theory (with \( \lambda = 0 \)). In our case the corresponding solution is Reissner-Nordström metric:

\[
f = \frac{q}{r}, \quad \rho = r, \quad w = 1 - \frac{M}{r} + \frac{p^2 + q^2}{4r^2}.
\]

Then it is not difficult to find the Riemann tensor and hyperstress (19). Non-metricity will be given by the Eq. (23), and all covariant derivatives should be taken with metric connection.
The number of equations in (60) exceeds the number of independent variables $\delta h_{\rho\sigma}$, so the existence of solution is not guaranteed, in general. The system of linear algebraic equations on ten functions $\delta h_{\rho\sigma}$ with known coefficients can be easily investigated. In case of considered spherically-symmetric configuration there are no solutions to the Eq. (60). The Horndeski prescription ruins the metricity of connection even in a case of spherically-symmetric configuration. Mention that in Palatini version of modified gravities with minimally coupled Maxwell field the connection remains metric-compatible $^{[29,37]}$.

Unfortunately, now there are no such simple exact solutions like those obtained in cosmological case. The equations on metric functions are very complicated. Since our goal was to establish the fundamental difference between F1 and F2 approaches, we will not go into detailed investigation of the solutions. The main result is that connection is non-metric in Palatini formulation.

4 Conclusion

It is a well-known fact that metric and Palatini versions of modified gravity theories are mathematically distinct, but the actual difference for specific theories remains unknown. Here we have compared the most common solutions (homogeneous isotropic and static with spherical symmetry) to vector Horndeski model, which were derived in two formalisms. Though that is not a general consideration, it provides good practical insight on the difference between metric and Palatini versions of the vector Horndeski theory.

It appears that in Palatini case there are more degrees of freedom, because independent connection is not compatible with the metric. The connection may be compatible with another metric, or, most probably, it will be non-metric. The connection equation for particular Horndeski prescription is linear and algebraic. One may solve it and substitute the solution into Einstein equations. After carrying this procedure, one can find that Palatini version of Einstein equations incorporates metric Einstein equations as $O(\lambda)$ term. However there will be also the terms with higher orders in $\lambda$, which makes the equations much more complicated. But it does not make them worse. For example, in cosmological sphaleron model the non-linearity allows to analytically continue the solution through the certain singularity, which is not possible in metric case. Thus, both approaches have their advantages and disadvantages and deserve equal consideration.

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