ON THE RATIONALITY OF THE MODULI SPACE OF INSTANTON BUNDLES ON THE PROJECTIVE 3-SPACE

M. HALIC, R. TAJAROD

Abstract. We describe the geometry of the moduli space of arbitrary rank instanton-like vector bundles on $\mathbb{P}^3$. In particular, we address the rationality of the moduli spaces of rank-2 mathematical instanton bundles. The results are related to a topic to be presented at a satellite conference of the ICM 2018.

1. Introduction

The interest in the so-called rank-2 ‘mathematical’ instanton bundles on the projective space has its origins in the works of Atiyah-Ward [2], Atiyah et al. [1], and Barth-Hulek [3, 4]. During the past decade, the irreducibility and rationality of their moduli spaces has been intensively investigated by Tikhomirov et al. in an extensive series of papers (cf. [6, 7, 8], references therein) and in recent preprints.

The first author [5] described the geometry of a certain component of the moduli space of semi-stable vector bundles of arbitrary rank on $\mathbb{P}^2$-bundles over $\mathbb{P}^1$ and proved its rationality. The goal of this note is to specialize those general results to the projective 3-space and to address the rationality issue which is mentioned above. Although this application is clearly pointed out in op. cit. [Introduction, p. 3], we believe that this short completion has its interest, taking into consideration the prospective talk [9].

Theorem. The moduli space of special-instanton-like vector bundles on $\mathbb{P}^3$ (of rank $r$, with Chern classes $0, n, 0$) is non-empty, irreducible of dimension $4rn - r^2 + 1$, and it is rational. In particular, the statement holds for the moduli space of rank-2 mathematical instanton bundles on $\mathbb{P}^3$.

Moreover, we give an explicit description of the general special-instanton-like vector bundle (cf. Theorem 2.8). Note that, in spite of the extensive literature on the rank-2 case, there are few articles dealing with higher rank instantons on $\mathbb{P}^3$. The relevant definitions are given below. Throughout the article, we work over an algebraically closed field of characteristic zero.

2. The proofs

Henceforth, we agree that vector bundles on $\mathbb{P}^3$ are always overlined, while those on its blow-up (to be denoted by $Y$) are not.

Definition 2.1 (vector bundles on the projective space) An instanton-like vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ of charge $n$ and rank $r$, with $n \geq r$, is defined by the following properties:

(i) $c_1(\mathcal{F}) = 0, c_2(\mathcal{F}) = n, c_3(\mathcal{F}) = 0$.

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(ii) it is slope semi-stable, for the slope \( \mu_{P^3}^{O_{P^3}}(1) \) obtained by pairing with \([O_{P^3}(1)]^2\);

(iii) its restriction to some (general) line \( \ell_{\text{gen}} \subset P^3 \) is trivializable;

(iv) \( H^1(\mathcal{F}(-2)) = H^2(\mathcal{F}(-2)) = 0 \).

We say that \( \mathcal{F} \) is special-instanton-like if, in addition, it satisfies:

(v) \( H^2(\text{End}(\mathcal{F})(-2)) = 0 \).

**Remark 2.2** Barth-Hulek [3, §7] showed that vector bundles satisfying (i)-(iv) above can be written as the cohomology of a linear monad

\[
O_{P^3}(-1)^\oplus n \to O_{P^3}^{\oplus r + 2n} \to O_{P^3}(1)^\oplus n.
\]

(2.1)

As mentioned in the Introduction, the case \( r = 2 \), with the conditions (i)-(ii), corresponds to the mathematical (‘t Hooft) instanton bundles and it has been intensely investigated. Note that, in this case, the Grauert-Müllich theorem automatically yields the property (iii).

**Notation 2.3** Let \( \sigma : Y \to P^3 \) be the blow-up along a line \( \ell \subset P^3 \); the exceptional divisor is \( E \cong P^1 \times P^1 \). Note that \( Y \cong P^1(O_{P^1} \oplus O_{P^1} \oplus O_{P^1}(-1)) \) admits a \( P^2 \)-fibre bundle structure over \( P^1 \); let \( \pi : Y \to P^1 \) be the projection and \( P \cong P^2 \) be its general fibre. The relatively ample line bundle has the property \( O_\sigma(1) = \sigma^* O_{P^3}(1) \).

For a vector bundle \( \mathcal{F} \) on \( Y \) and \( k, l \in \mathbb{Z} \), we denote \( \mathcal{F}(k,l) := \mathcal{F} \otimes O_\pi(k) \otimes \pi^* O_{P^1}(l) \). A straightforward computation yields \( O_Y(E) = O(1,-1) \) and that the canonical bundle is \( \kappa_Y = O(-3,-1) \).

**Definition 2.4 (vector bundles on the blow-up)** Let \( L_c := O(1,c) \), with \( c > r(r-1)n \).

A vector bundle \( \mathcal{F} \) on \( Y \) is called a \( \pi \)-instanton if it satisfies the following properties:

(i) \( c_1(\mathcal{F}) = 0, c_2(\mathcal{F}) = n \cdot [O_\pi(1)]^2, c_3(\mathcal{F}) = 0 \);

(ii) It is \( L_c \)-semi-stable (cf. [5, Definition 2.1]), for the slope \( \mu_{Y}^{L_c} \) obtained by pairing with \([O_\pi(1)]^2 + c \cdot [O_\pi(1)] \cdot [O_{P^1}(1)]\). The (semi-)stability condition is independent of \( c \) and the restriction of \( \mathcal{F} \) to \( P \) is semi-stable (cf. [5, Theorem 2.8]).

(iii) The restrictions of \( \mathcal{F} \) to \( \Lambda := P(O_{P^1} \oplus O(0)) \subset Y \) and to some line in \( P \) are trivializable.

(iv) (a) \( R^2 \pi_Y \mathcal{F}(-2,0) = R^2 \pi_Y \mathcal{F}(-2,0) = 0 \);

(b) \( H^1(Y,\mathcal{F}(-1,-1)) = H^1(Y,\mathcal{F}(-2,0)) = 0 \).

We say that \( \mathcal{F} \) is a special-\( \pi \)-instanton if, in addition, it holds:

(v) \( H^1(\text{End}(\mathcal{F})(-1,-1)) = 0 \).

**Remark 2.5** Recall [5, Theorem 4.1] that \( \pi \)-instantons can be written as the cohomology of a monad

\[
O_\pi(-1)^\oplus n \to O_Y^{\oplus r + 2n} \to O_\pi(1)^\oplus n.
\]

(2.2)

In particular, since the restriction of \( O_\pi(1) \) to the exceptional divisor \( E \) is trivial, the same holds for \( \mathcal{F} \). So \( \mathcal{F} := \sigma^* \mathcal{F} \) is a vector bundle on \( P^3 \), trivializable along \( \ell \), and \( \mathcal{F} = \sigma^* \mathcal{F} \).

The restrictions of \( \mathcal{F} \) to both \( P \) and \( E \)—they intersect along a line—are semi-stable. Moreover, the general divisor \( D \in |O_\pi(1)| \)—isomorphic to the blow-up of \( P^2 \) at a point—can be deformed into the union \( P \cup E \). The semi-stability is an open condition, hence the restriction \( \mathcal{F}_D \) is semi-stable, too.

**Notation 2.6** We consider the following quasi-projective varieties:
(i) $\text{SI}_{\mathbb{P}^3}(r; n)$, the moduli space of special instanton-like vector bundles, as in 2.1. It is an open subset of the moduli space of semi-stable sheaves on $\mathbb{P}^3$.

(ii) $\text{SI}_{\mathbb{P}^3}(r; n)_\ell$, the open subspace corresponding to bundles which are trivializable along the line $\ell \subset \mathbb{P}^3$.

(iii) $\text{f-SI}_{\mathbb{P}^3}(r; n)_\ell$, the moduli space of pairs consisting of an instanton-like vector bundle $\tilde{F}$ and a framing $\tilde{F}_\ell \cong \mathcal{O}_\ell$. It is a principal $\text{PGL}(r)$ bundle over $\text{SI}_{\mathbb{P}^3}(r; n)_\ell$ and $\dim \text{f-SI}_{\mathbb{P}^3}(r; n)_\ell = 4rn$ (see below).

(iv) Similarly, we define $\text{SI}_Y(r; n)_E, \text{f-SI}_Y(r; n)_E$.

(v) Let $M_{\mathbb{P}^2}(r; n)$ (resp. $M_{\mathbb{P}^2}(r; n)_{\text{line}}$) be the moduli space of rank-$r$, slope semi-stable (resp. framed) vector bundles on $\mathbb{P}^2$, with $c_1 = 0, c_2 = n$. They are $(2rn - r^2 + 1)$- (resp. $2rn$)-dimensional, cf. [5, §3]. For simplicity, we call such vector bundles $\mathbb{P}^2$-instantons.

Now we transfer the problem from $\mathbb{P}^3$ to $Y$.

**Lemma 2.7**

(i) The push-forward by $Y \stackrel{\sigma}{\rightarrow} \mathbb{P}^3$ induces a birational morphism

$$\sigma_* : \text{SI}_Y(r; n)_E \rightarrow \text{SI}_{\mathbb{P}^3}(r; n).$$

Its inverse is induced by the pull-back and is well-defined on $\text{SI}_{\mathbb{P}^3}(r; n)_\ell$.

(ii) In both moduli spaces, the loci corresponding to stable bundles are dense. Thus they are pure dimensional, of dimension $4rn - r^2 + 1$.

**Proof.** (i) from $Y$ to $\mathbb{P}^3$ By 2.5, any $\pi$-instanton $\mathcal{F}$ is the pull-back of a vector bundle $\tilde{F}$, trivializable along $\ell$. Its Chern classes are indeed as required; the isomorphism $H^1(\mathcal{F}(-2)) \cong H^1(\mathcal{F}_\ell(-2), 0)$ implies the condition 2.1(iv).

from $\mathbb{P}^3$ to $Y$ Clearly, $\text{PGL}(4)$ acts transitively on the lines in $\mathbb{P}^3$. It also acts on $\text{SI}_{\mathbb{P}^3}(r; n)$ and the components of the latter are invariant under the action. Therefore, the irreducible components of $\text{SI}_{\mathbb{P}^3}(r; n)$ correspond to those of $\text{SI}_{\mathbb{P}^3}(r; n)_\ell$. Hence it suffices to study the pull-back of an instanton-like bundle $\mathcal{F}$ which is trivializable along $\ell$; let $\mathcal{F} := \sigma^* \mathcal{F}$.

The Chern classes of $\mathcal{F}$ are as required. For any small deformation $\ell'$ of $\ell$, the restriction $\mathcal{F}|_{\ell'}$ is still trivial; in particular, this holds for (some) $\ell'$ intersecting $\ell$, which yields a line in the general fibre of $\pi$. Let $H \subset \mathbb{P}^3$ be an arbitrary 2-plane. The exact sequence $0 \rightarrow \mathcal{F}(1) \rightarrow \mathcal{F} \rightarrow \mathcal{F}_H \rightarrow 0$ and the cohomological properties of $\mathcal{F}$ imply $H^2(\mathcal{F}_H(-2)) = 0$. In particular, this holds for the planes containing $\ell$, so $\text{PGL}(4)$ acts transitively on $\dim \text{SI}_{\mathbb{P}^3}(r; n)_\ell$.

Concerning the ‘special’ conditions, they are equivalent, by Serre duality:

$$H^1(Y, \mathcal{E}nd(\mathcal{F})(-1, -1)) \cong H^2(Y, \sigma^*(\mathcal{E}nd(\mathcal{F})(-2))) \cong H^2(\mathbb{P}^3, \mathcal{E}nd(\mathcal{F})(-2)).$$

So far, we ignored in both directions the (semi-)stability issue. For the push-forward, one has the identity $\mu^L_Y(\sigma^* \mathcal{G}) = (1 + c)\mu^L_{\mathbb{P}^3}(\mathcal{G})$, for any reflexive sheaf $\mathcal{G}$ on $\mathbb{P}^3$, which yields the implication:

$$\mathcal{F} \text{ is (semi-)stable} \quad \Rightarrow \quad \tilde{\mathcal{F}} \text{ is (semi-)stable.}$$

(The semi-stability of $\mathcal{F}$ follows also from the fact that $\tilde{\mathcal{F}}$ is trivializable.) Thus the push-forward induces a well-defined morphism.
Conversely, for $\mathcal{F} \in \text{SI}_{\mathbb{P}^3}(r; n)_\ell$, the restrictions to the general line in $\mathbb{P}^3$ and to the general line intersecting $\ell$ are trivializable. These represent, respectively, the Poincaré duals of $[\mathcal{O}_\mathbb{P}(1)]^2$ and $[\mathcal{O}_\mathbb{P}(1) \cdot \pi^*\mathcal{O}_{\mathbb{P}^2}(1)]$. We deduce that $\mathcal{F}$ is $L_c$-semi-stable. We remark that the implication about the stability is not clear.

(ii) For $\text{SI}_Y(r; n)_E$, the statement is [5, Theorem 4.7]. Second, $\text{SI}_{\mathbb{P}^3}(r; n)_\ell \subset \text{SI}_{\mathbb{P}^3}(r; n)$ is dense, so it suffices to prove the claim for the former. Let $\mathcal{F} \in \text{SI}_{\mathbb{P}^3}(r; n)_\ell$ be arbitrary; by density, there are $L_c$-stable deformations $\mathcal{F}' \in \text{SI}_Y(r; n)_E$ of $\mathcal{F} = \sigma^*\mathcal{F}$. Then $\mathcal{F}' := \sigma_\ast \mathcal{F}'$ is a stable deformation of $\mathcal{F} = \sigma_\ast \mathcal{F}$; hence the stable locus is dense in $\text{SI}_{\mathbb{P}^3}(r; n)_\ell$.

At a stable point, the deformation theory is unobstructed (the relevant $H^2$-cohomology group vanishes), and the dimension follows from the Riemann-Roch formula. □

Now we are in position to prove the main result, stated in the Introduction.

**Proof.** By the previous lemma, $\text{SI}_Y(r; n)_E \simeq \text{SI}_{\mathbb{P}^3}(r; n)$ is birational. The hypotheses of [5, Theorem 4.7] are satisfied, so the right-hand side is non-empty, irreducible, and rational. □

The construction of special-instanton-like vector bundles of arbitrary rank, thus proving the non-emptiness of $\text{SI}_Y(r; n)_E$, is not trivial. This is achieved in [5, §4.3.2].

Finally, we remark that one obtains an explicit description of the general special-instanton-like bundle on the projective space.

**Theorem 2.8** The assignment $\mathcal{F} \mapsto (\mathcal{F}_D, \mathcal{F}_P)$ yields a birational morphism

$$\ell-\text{SI}_Y(r; n)_E \to M_{\mathbb{P}^2}(r; n)_{\text{line}} \times M_{\mathbb{P}^2}(r; n)_{\text{line}},$$

which, in turn, induces the birational morphism $\text{SI}_{\mathbb{P}^3}(r; n)_\ell \to M_{\mathbb{P}^2}(r; n) \times M_{\mathbb{P}^2}(r; n)_{\text{line}}$.

Consequently, a general special-instanton-like vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ is uniquely determined by its restrictions $(\mathcal{F}_D, \mathcal{F}_P)$ to a pair (wedge) of 2-planes $D, P \subset \mathbb{P}^3$, intersecting along the line $\ell$, and the gluing data $\mathcal{F}_\ell \cong \mathcal{O}_\ell \cong \mathcal{F}_\ell$ (trivializations, up to simultaneous $\text{PGL}(r)$-action).

At infinitesimal, deformation-theoretic level, this fact is reflected in the isomorphism:

$$H^1(\mathbb{P}^3, \text{End}(\mathcal{F})) \cong H^1(D \cup P, \text{End}(\mathcal{F})).$$

**Proof.** See op. cit. □

**Remark 2.9** (i) Barth-Hulek [3, §6] gave a (linear) monad-theoretic description of the $\mathbb{P}^2$-instantons, analogous to (2.1). The results obtained in [5, Theorem 3.6] yield a detailed description of $M_{\mathbb{P}^2}(r; n)$, too: it is an irreducible, $(2nr - r^2 + 1)$-dimensional, rational variety. Its general element is the kernel of a surjective homomorphism,

$$\mathcal{F}_p^a(a)^{\otimes r - \rho} \oplus \mathcal{O}_{\mathbb{P}^2}(a + 1)^{\otimes \rho} \to \bigoplus_{j=1}^n \mathcal{O}_{l_j}(1),$$

where $n = ar + \rho$, $0 \leq \rho < r$, and $l_1, \ldots, l_n \subset \mathbb{P}^2$ is a bouquet of distinct lines passing through the point $p \in \mathbb{P}^2$.

(ii) In the case of mathematical instantons—that is, $r = 2$—Tikhomirov [7, 8] proved that their moduli spaces are irreducible, so the ‘special’ condition 2.1(\(\gamma\)) can be dropped. Prior to this, he characterized [6] the ‘physical’ instantons by the cohomological property

$$H^2(\text{End}(\mathcal{F})(-1)) = H^2(\text{End}(\mathcal{F})) = 0.$$
For instanton-like vector bundles of arbitrary rank, one may ask for the relationship between the property above and the ‘special’ condition. Let \( \ell \) be a trivializing line for \( \mathcal{F} \) and \( H \) a plane containing \( \ell \) (so \( \mathcal{F}_H \) is semi-stable). The exact sequences

\[
0 \to \mathcal{O}_{P^3}(-2) \to \mathcal{O}_{P^3}(-1) \to \mathcal{O}_H(-1) \to 0 \\
0 \to \mathcal{O}_{P^3}(-2) \to \mathcal{O}_{P^3}(-1)^{\oplus 2} \to J_\ell \to 0
\]

yield the implication:

\[
H^2(\text{End}(\mathcal{F})(-2)) = 0 \implies H^2(\text{End}(\mathcal{F})(-1)) = H^2(\text{End}(\mathcal{F})) = 0.
\]

The converse is not clear, although the second exact sequence suggests that it is likely to hold generically.

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E-mail address: mihai.halic@gmail.com, roshan.tajarod@gmail.com

CRM, UMI 3457, Montréal H3T 1J4, Canada