FUZZY COVERING SPACES AND ITS PROPERTIES

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Abstract. The properties of fuzzy path covering space, fuzzy lifting from an arbitrary fuzzy map to a fuzzy path covering map are discussed in this paper. Also the properties of fuzzy path covering space and fuzzy pointed topological space in fuzzy Peano space are discussed.

1. Introduction
The concepts of fuzzy set theory and fuzzy logic was given in [7]. The idea of fuzzy topological space was proposed in [2]. The concept of covering space was given by Massey[6] and the definition of Fuzzy Peano space was given by Sugapriya and Amudhambigai[4]. Throughout this paper the properties of fuzzy path covering space are studied. Also, the concept of fuzzy lifting from an arbitrary fuzzy map to a fuzzy path covering map is discussed. Some of the properties of fuzzy lifting are explained and they are used to frame the Necessary and Sufficient for the existence of fuzzy lifting. Finally, a necessary condition for the given fuzzy path covering space to be a fuzzy path cover for another fuzzy path covering space is studied.

Preliminaries
The basic preliminaries required for the study of this paper are given below:

Definition 1.1. [2] A fuzzy topology is a family T of fuzzy sets in X which satisfies the following conditions:
(i) 0X, 1X ∈ T
(ii) If λ, µ ∈ T then λ ∧ µ ∈ T,
(iii) If λi ∈ T for each i ∈ J then ∨λi ∈ T. T is called a fuzzy topology for X, and the pair (X, T) is a fuzzy topological space, or fts for short.

Definition 1.2. [6] A fuzzy Point P in X is a special fuzzy set with membership function defined by

\[ P(x) = \begin{cases} 
\lambda & \text{if } x = y \\
0 & \text{if } x \neq y 
\end{cases} \]

where 0 < λ ≤ 1. P is said to have support y, value λ and is denoted by \( P^\lambda_y \) or P(y, λ).

Definition 1.3. [5] Let (X, T) be a fuzzy topological space. Then

\[ \tilde{T} = \{ A \in FP(X) \mid \text{Supp}\ A \in T \} \]
is a fuzzy topology on X, called the fuzzy topology on X introduced by T and (X, ˜T) is called the fuzzy topological space introduced by (X, T).

**Definition 1.4.** [3] Let (X, T) and (Y, S) be any two fuzzy topological spaces. A function f : (X, T) → (Y, S) is said to be fuzzy continuous if the inverse image of every fuzzy open set in IY is fuzzy open in IX.

**Definition 1.5.** [2] Let (X, T) be a fuzzy topological space. A continuous function f : I → X is said to be a fuzzy path in (X, T) if for any two fuzzy points p, q ∈ X such that f(0) = p, f(1) = q.

**Definition 1.6.** [2] Any fuzzy topological space (X, T) is said to be a fuzzy path connected space of (X, T), C is fuzzy open in X, called the fuzzy topology on X introduced by T and (X, T) is the maximal fuzzy path connected set in (X, T) that contains x1 and denoted by C.

**Notation 1.1.** [5] The collection of all fuzzy Path connected points in (X, T) is denoted by FPCP(X).

**Notation 1.2.** [5] Let (X, T) be a fuzzy topological space. Let μ ∈ IX be a fuzzy open set in (X, T) and x1 ∈ FPCP(X) be a fuzzy path connected point with x1 ≤ μ, then the fuzzy path component of x1 ∈ FPCP(X) is denoted by C(x1, μ).

**Definition 1.7.** [2] A fuzzy path component of a fuzzy point x1 in a fuzzy topological space (X, T) is the maximal fuzzy path connected set in (X, T) that contains x1 and denoted by C.

**Definition 1.8.** [4] A fuzzy topological space (X, T) is said to be a fuzzy Peano space if it is a fuzzy locally path connected space and a fuzzy connected space.

### 2. FUZZY PATH COVERING PROPERTIES

**Definition 2.1.** Let the two fuzzy path connected spaces be (X, τ) and (X, τ̃). We call (X, τ̃) is a fuzzy path covering space of (X, τ) if every fuzzy path connected Point x1 ∈ FPCP(X) and any fuzzy continuous map f : (X, τ̃) → (X, τ) there exists a λ ∈ IX which is both fuzzy path connected and fuzzy open neighbourhood such that each fuzzy path component of f−1(λ) is mapped fuzzy homeomorphically onto λ by f.

**Definition 2.2.** For a non-empty set, a fuzzy path p : [0, 1] → FP(X) is said to be fuzzy loop, for x1 ∈ FP(X), p(0) = p(1) = x1.

**Definition 2.3.** Let the two fuzzy path connected spaces be (X, τ) and (X, τ̃). A fuzzy continuous function f : FPCP(X) → FPCP(X) is called a fuzzy path if x̃, ỹ ∈ FPCP(X) and x1, y1 ∈ FPCP(X) f(x̃) = x1 and f(ỹ) = y1.

**Proposition 2.4.** Let the two fuzzy path connected spaces be (X, τ) and (X, τ̃) and (X, τ̃) be a fuzzy path covering space of (X, τ, p). Let f : (X, τ̃) → (X, τ) be a fuzzy continuous map and p : FPCP(X) → FPCP(X) be a fuzzy path such that p(x̃) = x1, where x̃ ∈ FPCP(X) and x1 ∈ FPCP(X). Then for any fuzzy paths p1 : [0, 1] → FPCP(X) and p2 : [0, 1] → FPCP(X) with p1(0) = x1 and p2(0) = (x̃i) such that p ◦ p2 = p1.

**Proof.** Let λ ∈ IX be a fuzzy neighbourhood of x1 ∈ FPCP(X) such that x1 ≤ λ. Let C(p−1(λ), X) be the fuzzy path component of p−1(λ). Since (X, τ̃) be a fuzzy path covering space of (X, T), p : FPCP(X) → FPCP(X) is fuzzy homeomorphism. Therefore p maps C(p−1(λ), X) fuzzy homeomorphically into λ and there exists a unique fuzzy path p2 : [0, 1] → FPCP(X) with p2(0) = (x̃i) such that p ◦ p2 = p1.

The uniqueness of the fuzzy path is given in the following Proposition 3.2
Proposition 2.5. Let the two fuzzy path connected spaces be \((X, \tau)\) and \((\hat{X}, \hat{\tau})\) be a fuzzy path covering space of \((X, \tau)\). Given any two fuzzy continuous functions \(g_0, g_1 : [0, 1] \to \mathcal{FPCP}(\hat{X})\) such that \(f \circ g_0 = f \circ g_1\), then \(\{y_0 \in \mathcal{FP}(X) : g_0(y_0) = g_1(y_0)\} = \{0_X\} \cup \{1_X\} \). 

Proof. In a fuzzy connected space, the fuzzy set that is fuzzy open as well as fuzzy closed is either \(0_X\) or \(1_X\). So we will prove that the fuzzy set \(\lambda = \{y_0 \in \mathcal{FP}(X) : g_0(y_0) = g_1(y_0)\}\) is fuzzy closed. Let \(y_t \in \text{cl}(\lambda)\) and \(x_t = f(g_0(y_t)) = f(g_1(y_t))\). Assume \(g_0(y_t) \neq g_1(y_t)\). This will leads to a contradiction. Let \(\mu \in I^X\) be a fuzzy neighbourhood of \(x_t\) and \(C_1(p^{-1}(\lambda), X)\) and \(C_2(p^{-1}(\lambda), \hat{X})\) be the fuzzy path components of \(p^{-1}(\lambda)\). Since \(g_0\) and \(g_1\) are both fuzzy continuous, we can find a fuzzy neighbourhood \(\gamma \in I^Y\) of \(y_t\) such that \(g_0(\gamma) \subseteq C_1(p^{-1}(\lambda), \hat{X})\) and \(g_1(\gamma) \subseteq C_2(p^{-1}(\lambda), \hat{X})\). This is a contradiction to \(y_t\) and \(\lambda\). Thus \(\lambda\) contains all its fuzzy closure and it therefore fuzzy closed. Similarly we can prove every fuzzy point in \(\lambda\) is an fuzzy interior and therefore the \(\lambda\) is fuzzy open. Hence \(\{y_0 \in \mathcal{FP}(X) : g_0(y_0) = g_1(y_0)\} = \{0_X\} \cup \{1_X\}\). \(\square\)

Definition 2.6. Let \((I, \varepsilon^I_1)\) and \((I, \varepsilon^I_2)\) be any two fuzzy path connected spaces introduced by the Euclidean subspace fuzzy topologies \(\varepsilon^I_1\) and \(\varepsilon^I_2\) respectively. Let the two fuzzy path connected spaces \((X, \tau)\) and \((Y, \sigma)\). Any two fuzzy continuous maps \(f, g : \mathcal{FPCP}(X) \to \mathcal{FPCP}(Y)\) are said to be fuzzy Path Connected Homotopic (in short \(f \cong_{fp} g\) or \(f \cong_{fp}\)), if there is a fuzzy continuous map \(P : (I, \varepsilon^I_1) \times (I, \varepsilon^I_2) \to \mathcal{FPCP}(X)\) such that \(P(0, \gamma) = x_t, P(1, \gamma) = y_t\) for all \(\gamma \in I^Y\). Thus \(\mathcal{FPCP}(X)\) and \(\mathcal{FPCP}(Y)\) is fuzzy continuous. Let \(\varepsilon^I_1 \in \mathcal{FPCP}(\varepsilon^I_1)\) and \(\varepsilon^I_2 \in \mathcal{FPCP}(\varepsilon^I_2)\) and \(f(\varepsilon_1, 1) = g(y_t)\) for all \(\varepsilon_1 \in \mathcal{FPCP}(\varepsilon^I_1)\). 

Definition 2.7. Let the two fuzzy path connected spaces be \((X, \tau)\) and \((Y, \sigma)\) and let \(f, g : (X, \tau) \to (Y, \sigma)\) be any two fuzzy continuous maps. Then the fuzzy path homotopy class of a fuzzy continuous map \(f\) is denoted by \([f]\) and is defined by \([f] = \{g : f \cong_{fp} g\}\). 

Notation 2.1. The collection of all fuzzy path homotopy classes of fuzzy loops is denoted by \(\pi(X, x_t)\). 

Proposition 2.8. Let the two fuzzy path connected spaces be \((X, \tau)\) and \((\hat{X}, \hat{\tau})\) be fuzzy path covering space of \((X, \tau)\) and \(f : (\hat{X}, \hat{\tau}) \to (X, \tau)\) be a fuzzy continuous map and \(p : \mathcal{FPCP}(\hat{X}) \to \mathcal{FPCP}(X)\) be a fuzzy path such that for each \(x_t \in \mathcal{FPCP}(\hat{X})\) and \(x_t \in \mathcal{FPCP}(X)\), \(x_t = p(\hat{x}_t)\). Then the fuzzy homomorphism \(f_\ast : \pi(\hat{X}, \hat{x}_t) \to \pi(X, x_t)\) is fuzzy injective. 

Proof. Let \(f : (\hat{X}, \hat{\tau}) \to (X, \tau)\) be a fuzzy continuous map then by Definition 3.1, \(f_\ast : \pi(\hat{X}, \hat{x}_t) \to \pi(X, x_t)\) is the induced fuzzy homomorphism. For any two fuzzy path homotopy classes in \((\hat{X}, \hat{T})\) be \([\gamma]\) and \([\delta]\). Suppose \(g_\gamma\) and \(g_\delta\) are two fuzzy paths in \([\gamma]\) and \([\delta]\) respectively. Let \(f_\ast[\gamma] = f_\ast[\delta]\). This implies that \(f_{g_\gamma} \cong_{fp} f_{g_\delta}\). It follows that from Proposition 3.2, \(g_\gamma \cong_{fp} g_\delta\) in \((\hat{X}, \hat{\tau})\). So \([\gamma] = [\delta]\). Thus the fuzzy homomorphism \(f_\ast : \pi(\hat{X}, \hat{x}_t) \to \pi(X, x_t)\) is fuzzy injective. \(\square\)

Definition 2.9. Let \(X\) be a non-empty set and let \(p : [0, 1] \to \mathcal{FP}(X)\) be a fuzzy path with \(p(0) = p(1) = x_t, x_t \in \mathcal{FP}(X)\). Then \(x_t\) is said to be a base fuzzy point. 

Definition 2.10. Let the fuzzy topological space be \((X, \tau)\). A fuzzy pointed topological space is a fuzzy topological space with a fuzzy base point \(x_t \in \mathcal{FP}(X)\). A fuzzy pointed topological space is denoted by \((X, x_t)\).

Definition 2.11. Let the fuzzy topological space be \((X, \tau)\) and \((X, x_t)\) be a fuzzy pointed topological space with a fuzzy base point \(x_t\). Any fuzzy map \(f : (X, x_t) \to (Y, y_t)\) is said to be a fuzzy based map if \(f\) is fuzzy continuous with respect to \((X, \tau)\) and \((Y, \sigma)\), and if \(f(x_t) = y_t\) where \(x_t \in \mathcal{FP}(X)\) and \(y_t \in \mathcal{FP}(Y)\).
Proposition 2.12. Let the three fuzzy path connected spaces be \((X, \tau), (Y, \sigma)\) and \((\tilde{X}, \tilde{\tau})\) be a fuzzy path covering space of \((X, \tau)\) and let \((Y, \sigma)\) be a fuzzy Peano space. Let \(f: (\tilde{X}, \tilde{\tau}) \to (X, \tau)\) a fuzzy continuous map and there is a fuzzy path \(p: FPCP(\tilde{X}) \to FPCP(X)\) such that for each \(\tilde{x}_t \in FPCP(\tilde{X})\) and \(x_t \in FPCP(X)\) with \(x_t = p(\tilde{x}_t)\). Given any fuzzy continuous map \(\phi: (Y, y_t) \to (X, x_t)\), there is a fuzzy lifting \(\tilde{\phi}: (\tilde{Y}, \tilde{y}_t) \to \tilde{FPCP}(\tilde{X})\) such that for each \(\tilde{x}_t \in \tilde{FPCP}(\tilde{X})\) and \(x_t \in FPCP(X)\) with \(x_t = p(\tilde{x}_t)\). Assume \(\phi(\tilde{y}_t) \in \tilde{FPCP}(\tilde{X})\) and constructive suggestions which have improved this paper.

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Proposition 2.13. Let the three fuzzy path connected spaces be \((X, \tau), (\tilde{X}_1, \tilde{\tau}_1)\) and \((\tilde{X}_2, \tilde{\tau}_2)\) be a fuzzy path covering space of \((X, \tau)\) and \(\phi: (\tilde{X}_1, \tilde{\tau}_1) \to (\tilde{X}_2, \tilde{\tau}_2)\) be a fuzzy homomorphism. Then \((\tilde{X}_1, \tilde{\tau}_1)\) is a fuzzy path covering space of \((\tilde{X}_2, \tilde{\tau}_2)\).

Proof. Let \(p_1, p_2: FPCP(\tilde{X}_1) \to FPCP(\tilde{X}_2)\) be any two fuzzy paths. Let \(x_t \in FPCP(X)\) be an fuzzy open neighbourhood \(\lambda \in I^X\) such that \(x_t \leq \lambda\) is the fuzzy open neighbourhood of \((\tilde{X}_1, \tilde{\tau}_1)\) and \((\tilde{X}_2, \tilde{\tau}_2)\) respectively. Let \(X_2, X_1 \in I^X\) and \(x_0, x_1 \in X_2\) be any two fuzzy neighbourhoods of \((X_1, \tau_1)\) and \((X_2, \tau_2)\) respectively. Take \(\lambda = \lambda_1 \wedge \lambda_2\). Next we have prove that \(\phi: (\tilde{X}_1, \tilde{\tau}_1) \to (\tilde{X}_2, \tilde{\tau}_2)\) is onto. Let \(y_t \in FPCP(\tilde{X}_2)\). We need to show that there is a fuzzy point \(x_t \in FPCP(\tilde{X}_1)\) such that \(\phi(x_t) = y_t\). Consider a base fuzzy point \(x_t \in FPCP(\tilde{X}_1)\) and \(x_2 = \phi(x_1), x_0 = \phi(x_1)\) and \(x_0 = p_1(x_1) = p_2(x_2)\). Choose a fuzzy path \(p_3: [0, 1] \to FPCP(\tilde{X}_2)\) with \(p_3(0) = x_2\) and \(p_3(1) = x_3\). Let \(p_4 = p_2p_3\). From Proposition 3.2, there is a unique fuzzy path \(p_5: [0, 1] \to FPCP(\tilde{X}_1)\) with \(p_5(0) = x_1\) and \(p_5(1) = x_1\) such that \(p_1p_5 = p_4\). Since \(\phi(x_1) = p_5(0) = p_3(0) = x_2, p_2, p_5 = p_4\) and \(p_2\phi(p_5 = p_4 = p_2p_3\). Therefore by Proposition 3.2, \(\phi(x_t) = y_t\).

3. Conclusion

In this paper the properties of fuzzy path covering space are studied. Also, the concept of fuzzy lifting from an arbitrary fuzzy map to a fuzzy path covering map is discussed. Some of the properties of fuzzy lifting are explained and necessary condition of the fuzzy path covering space is studied.

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