Local Distance Antimagic Vertex Coloring of Graphs

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Abstract

A bijective function \( f : V \to \{1, 2, 3, ..., |V|\} \) is said to be a local distance antimagic labeling of a graph \( G = (V, E) \), if \( w(u) \neq w(v) \) for any two adjacent vertices \( u, v \) where the weight \( w(v) = \sum_{z \in N(v)} f(z) \). The local distance antimagic labeling of \( G \) induces a proper coloring in \( G \), called local distance antimagic chromatic number denoted by \( \chi_{ld}(G) \). In this article, we introduce the parameter \( \chi_{ld}(G) \) and compute the local distance antimagic chromatic number of graphs.

Keywords: Distance antimagic labeling, Local distance antimagic labeling, Local distance antimagic chromatic number.

Mathematics Subject Classification : 05C78, 05C15

1 Background and Introduction

Let \( G = (V, E) \) be a simple, connected, undirected graph with \( |V| = n \) and \( |E| = m \). For graph theoretic terminology we refer to Chartrand and Lesniak [4]. A graph \( G \) is antimagic if \( m \) edges are distinctly labeled by positive integers \( 1, 2, ..., |E| \) such that weight of each vertex (sum of the label of edges incident to that vertex) is distinct. The concept of antimagic labeling was introduced by Hartsfield and Ringel [8]. They conjectured that every connected graph other than \( K_2 \) is antimagic and every tree other than \( K_2 \) is antimagic. These conjectures gained more attention and results were established for several classes of graphs. In 2012, Arumugam and Kamatchi [1] introduced the term \( (a, d) \)-distance antimagic labeling of graphs which is defined as follows: A bijective function \( f : V \to \{1, 2, 3, ..., n\} \) is said to be \( (a, d) \)-distance antimagic labeling
if \( w(u) \neq w(v) \) for every vertex \( u \) and \( v \), \( w(v) = \sum_{u \in N(v)} f(u) \) where each vertex weight forms an arithmetic progression \( \{a, a + d, a + 2d, \ldots, a + (n - 1)d\} \) with common weight difference \( d \) and minimum vertex weight \( a \). A graph \( G \) is \((a, d)\)-distance antimagic if \( G \) admits \((a, d)\)-distance antimagic labeling. The \((a, d)\)-distance antimagic labeling is distance antimagic if \( d \geq 0 \). A bijective function \( f : V \to \{1, 2, 3, \ldots, n\} \) be a distance antimagic labeling, if \( w(x) \neq w(y) \) for every two vertices \( x \) and \( y \) in \( V \). A graph \( G \) is distance antimagic if \( G \) admits a distance antimagic labeling. In 2013, Kamatchi and Arumugam [12] proved that the cycles \( \{C_n, n \neq 4\} \), the wheels \( \{W_n, n \neq 4\} \), the paths \( \{P_n\} \) are distance antimagic graphs. Further they put forth the following conjectures that are yet open.

**Conjecture 1.1.** A graph \( G \) is distance antimagic if and only if \( N(u) \neq N(v) \) for any two distinct vertices \( u, v \) in \( V(G) \).

**Conjecture 1.2.** A tree \( T \) is distance antimagic if and only if every support vertex \( v \) has precisely one leaf adjacent to \( v \).

Kamatchi et al. [13] proved hypercube and several classes of disconnected graphs are distance antimagic. Rinovia Simanjuntak, Kristiana Wijaya proved that sun graphs, prism graphs, complete graphs, wheel graphs, fans and friendship graphs are distance antimagic [16].

S. Arumugam et al. [2] introduced the local antimagic graphs induced by local antimagic labeling which is defined as follows: Let \( G = (V, E) \) be a graph. A bijective function \( f : E \to \{1, 2, 3, \ldots, m\} \) is a local antimagic labeling, if for all \( uv \in E \), \( w(u) \neq w(v) \) where \( w(u) = \sum_{e \in E(u)} f(u) \). If \( G \) is local antimagic, then \( G \) has local antimagic labeling. The local antimagic chromatic number \( \chi_{la}(G) \) is the proper coloring of \( G \) induced by local antimagic labeling. They investigated the local antimagic chromatic number of tree \( \chi_{la}(T) \geq l + 1 \) and conjectured that every connected graph other than \( K_2 \) is local antimagic and every tree other than \( K_2 \) is local antimagic are still open. Further, they provided a local chromatic number of cycle graphs, complete bipartite graphs, friendship graphs, wheel graphs. John Haslegrave [10] proved the conjecture raised in [2], every connected graph other than \( K_2 \) is local antimagic using the probabilistic method. Julien Bensmail et al. [11] positively proved the conjecture, every tree other than \( K_2 \) is local antimagic raised in [2].

The corona product of graphs was introduced by Frucht and Harary in 1970 [7]. The corona product of two graphs \( G_1 \) and \( G_2 \) denoted by \( G_1 \circ G_2 \) is constructed by taking one copy of \( G_1 \) along with \( |V(G_1)| \) copies of \( G_2 \) and joining the \( i \)-th vertex of \( G_1 \) to every vertex of \( i \)-th copy of \( G_2 \), for \( 1 \leq i \leq |V(G_1)| \). Note that \( K_m \) denote the complement of \( K_m \), also called the null graph on \( m \) vertices.
Arunugam et al. [3] provided the local antimagic chromatic number for corona product of graphs $G$ with $\overline{K_m}$ for $m \geq 1$. They proved that $\chi_{la}(P_n \circ \overline{K_m})$ is $mn + 2$. Further they provided the local antimagic chromatic number for corona product of both graphs, the odd cycle $C_n$, $n \geq 5$ and the complete $K_n$, $n \geq 3$ with the null graph $\overline{K_m}$ for $m \geq 2$.

Motivated by these observations, we introduce a concept named a local distance antimagic labeling of graphs and a new parameter, local distance antimagic chromatic number its denotes $\chi_{ld}(G)$.

**Definition 1.3.** A bijective function $f : V \rightarrow \{1, 2, 3, ..., |V| = n\}$ is local distance antimagic labeling for a graph $G = (V, E)$, if $w(u) \neq w(v)$ for any two adjacent vertices $u$ and $v$, where $w(v) = \sum_{z \in N(v)} f(z)$, $N(v)$ is the open neighborhood of $v$. A graph $G$ is local distance antimagic if $G$ admits a local distance antimagic labeling. This induces a proper color where the vertex $v$ is assigned the color $w(v)$. This leads to the following concept.

**Definition 1.4.** Local distance antimagic chromatic number $\chi_{ld}(G)$ is defined as the minimum number of colors required to proper color the graph induced by local distance antimagic labeling of $G$. If $G$ is distance antimagic, then $G$ is local distance antimagic.

![Figure 1: Local distance antimagic chromatic number for $K_{4,4}$](image)

If $G$ is distance antimagic, then $G$ is local distance antimagic, but the converse is not true. We observe that the local distance antimagic chromatic number of the graph $G$ shown in Figure 1 is $\chi_{ld}(G) = 2 = \chi(G)$.

The graph $K_2$ is neither antimagic nor local antimagic. Many researchers tried hard to solve the conjecture in [8], [2], every connected graph other than $K_2$ is antimagic and local antimagic, respectively. But $K_2$, a magic graph. We observe that $K_2$ is distance antimagic. All distance antimagic graphs are local distance antimagic, but the converse is not valid. Hence on achieving the local distance antimagic labeling on $K_2$, we receive the minimum labels $(v_1, v_2)$ with $(1, 2)$ respectively, thus obtaining unique vertex colors. Therefore $K_2$ is a local distance antimagic graph. Hence we start with the following observation.
Observation 1.5. Every connected graph is local distance antimagic.

Remark 1.6. $\chi_{ld}(G) \geq \chi(G)$. Then $\chi_{ld}(G) - \chi(G)$ is arbitrarily large is shown in the following theorems.

We provide a new concept to compute the local distance antimagic chromatic number of graphs in the following sections.

2 Main Results

2.1 Local Distance Antimagic Chromatic Number of Star Related Graphs

This section observed that the local distance antimagic graphs naturally provide the proper color induced by local distance antimagic labeling. Here we compute the local distance antimagic chromatic number of star-related graphs. We know that the chromatic number of star graph $\chi(S_n)$ is 2. The following Theorem provides the local distance antimagic chromatic number of star graph $\chi_{ld}(S_n)$ is 2. Thus, we infer that $\chi_{ld}(S_n) = \chi(S_n)$.

**Theorem 2.1.** For a star graph $S_n$ on $n \geq 2$, we have $\chi_{ld}(S_n) = 2$.

**Proof.** Let $G$ be a star graph $S_n$ with $n+1$ vertices. Let $V(G) = \{c \cup v_i, f o r 1 \leq i \leq n\}$ and $E(G) = \{c v_i, f o r 1 \leq i \leq n\}$ where $c$ is the internal vertex and $v_i$ are pendant vertices. We have $|V(G)| = n+1$ and $|E(G)| = n$. We define a bijective function $f : V \rightarrow \{1, 2, 3, ..., n+1\}$ by

\[
f(c) = n + 1 \\
f(v_i) = i, \ for \ 1 \leq i \leq n
\]

Then the vertex weights are as follows

\[
w(v_i) = n + 1 \\
w(c) = \frac{n(n + 1)}{2}
\]

Thus $\chi_{ld}(S_n) = 2$. We know that, $\chi(S_n) = 2$. Hence $\chi_{ld}(S_n) = 2$.

The Join of two graphs $G = G_1 + G_2$ is structured by adjoining every vertex at $G_1$ to every vertex at $G_2$. As an immediate consequences of Theorem 2.1, we proved the following Corollary.

**Corollary 2.2.** For the graph $G = K_1 + \overline{K_n}$, with $n \geq 1$, we have $\chi_{ld}(G) = 2$. 


**Theorem 2.3.** For a subdivision of star graph $S^m_n$ with $n \geq 3$, $m = 1$, we have $\chi_{ld}(S^m_n) = n+1$.

**Proof.** Let $V(S^1_n) = \{c \cup v^i_j, \text{for } 1 \leq i \leq n, 1 \leq j \leq 2\}$ and $E(S^1_n) = \{v^i_j \cup v^i_j v^{i+1}_j, \text{for } 1 \leq i \leq n, j = 1\}$. Then $|V(S^1_n)| = 2n+1$ and $|E(S^1_n)| = 2n$. Let us define a function $f : V \to \{1, 2, 3, ..., 2n+1\}$ by

$$f(v^i_j) = \begin{cases} 
2n+1, & \text{for } i = 1, j = 1 \\
n + i, & \text{for } 2 \leq i \leq n, j = 1 \\
i, & \text{for } 1 \leq i \leq n, j = 2 
\end{cases}$$

$$f(c) = n + 1$$

Then the vertex weights are

$$w(v^i_j) = \begin{cases} 
n + 1 + i, & \text{for } 1 \leq i \leq n, j = 1 \\
2n+1, & \text{for } i = 1, j = 2 \\
n + i, & \text{for } 2 \leq i \leq n, j = 2 
\end{cases}$$

$$w(c) = 2n+1 + \sum_{i=2}^{n} (n + i)$$

Thus, $\chi_{ld}(S^1_n) \leq n+1$. Suppose $n = 3$, the minimum possible vertex weight $w(c) = \frac{n(n+1)}{2} = 6$, such that $f(v^1_1) = 1, 2, ..., n$. Thus vertex $c$ receives a first color. $v^2_i$ are pendant vertices. Therefore, $w(v^2_1) = 1, 2, ..., n$. Thus all pendant vertices receive $n$ colors. Hence, $S^1_n$ receives $n+1$ colors. Similarly we can prove for $n \geq 4$, the minimum possible weight of central vertex $w(c) = \frac{n(n+1)}{2}$, but $|V(S^1_n)| = 2n+1$. Thus $w(c) > |V(S^1_n)|$. Note that $w(c)$ receives a new color. Therefore, $\chi_{ld}(S^1_n) \geq n+1$. Hence for $m = 1$, $\chi_{ld}(S^m_n) = n+1$. 

The bistar $B_{n,n}$ is a graph constructed by joining $n$ pendant edges to endpoints of $K_2$. Any bistar graphs are local distance antimagic, shown in the following Theorem.

**Theorem 2.4.** For any bistar graph $B_{n,n}$ with $n \geq 2$, we have $\chi_{ld}(B_{n,n}) = 4$.

**Proof.** Let $V(B_{n,n}) = \{u, v, u_i, v_i, \text{for } 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv \cup uu_i \cup vv_i, \text{for } 1 \leq i \leq n\}$. We have $|V(B_{n,n})| = 2(n+1)$ and $|E(B_{n,n})| = 2n + 1$. Let $f : V \to \{1, 2, ..., 2(n+1)\}$ be a bijective function defined by

$$f(u) = 2n + 1$$

$$f(v) = 2n + 2$$

$$f(u_i) = i, \text{for } 1 \leq i \leq n$$

$$f(v_i) = n + i, \text{for } 1 \leq i \leq n$$
The weight of the vertices are

\[ w(u) = \frac{n(n+5) + 4}{2} \]
\[ w(v) = \frac{3n^2 + 5n + 2}{2} \]
\[ w(u_i) = f(u) \]
\[ w(v_i) = f(v) \]

Thus \( \chi_{ld}(B_{n,n}) \leq 4 \). The chromatic number of a bistar is \( \chi(B_{n,n}) = 2 \). Suppose \( \chi_{ld}(B_{n,n}) = 2 \). The weight of the pendant vertex receives its adjacent vertex label. Therefore the graph receives \( w(u_i) = f(u) \) and \( w(v_i) = f(v) \). Thus there exists 2 colors. Then \( w(u) = f(u_i) + f(v) \), meant that the weight of the vertex \( u \) is the sum of the labels of \( u_i \) pendant vertices and \( f(v) \). Also \( w(v) = f(v_i) + f(u) \). Therefore \( w(u) \) and \( w(v) \) receives two new colors, because \( u, v \) are adjacent vertices. Therefore \( \chi_{ld}(B_{n,n}) \geq 4 \). Hence \( \chi_{ld}(B_{n,n}) = 4 \).  

The Doublestar graph is obtained by joining the \( m \) pendant edges to one end of \( K_2 \) and \( n \) pendant edges to the other end of \( K_2 \), denoted by \( B_{m,n} \). As an immediate consequence of Theorem 2.4, we proved the following Corollary.

**Corollary 2.5.** For any doublestar graph \( B_{m,n} \), with \( m, n \geq 2 \), we have \( \chi_{ld}(B_{m,n}) = 4 \).

**Theorem 2.6.** The subdivision of a bistar graph admits a local distance antimagic labeling, thus we have \( \chi_{ld}(BS(n,n)) = 4 \), for \( n \geq 4 \).

**Proof.** For \( 1 \leq i \leq n \), consider \( V(BS(n,n)) = \{c, u, v, u_i, v_i\} \) and \( E(BS(n,n)) = \{cu \cup cv \cup uu_i \cup vv_i\} \). Thus \( |V(BS(n,n))| = 2n+3 \) and \( |E(BS(n,n))| = 2(n+1) \). Let us define a bijective function \( f : V(BS(n,n)) \rightarrow \{1, 2, 3, ..., 2n+3\} \) by

Case 1: When \( n = 3 \)

\[ f(c) = 2n + 3 \]
\[ f(u) = 2n \]
\[ f(v) = 2(n + 1) \]
\[ f(u_i) = \begin{cases} 2i - 1, & \text{for } i = 1, 2 \\ 2i + 1, & \text{for } i = 3 \end{cases} \]
\[ f(v_i) = \begin{cases} 2i, & \text{for } i = 1, 2 \\ 2i - 1, & \text{for } i = 3 \end{cases} \]
Then the vertex weights are

\[
\begin{align*}
  w(c) &= 2(2n + 1) \\
  w(u_i) &= 2n \\
  w(v_i) &= 2(n + 1) \\
  w(u) &= w(v) = 5(n + 1)
\end{align*}
\]

Case 2: When \( n \geq 5 \) and \( n \) is odd

\[
\begin{align*}
  f(c) &= 6 \\
  f(u) &= 2(n + 1) \\
  f(v) &= 2n + 3 \\
  f(u_i) &= \begin{cases} 
  2i - 1, & \text{for } i = 1, 2 \\
  2i + 1, & \text{for } 3 \leq i \leq n, \text{ where } i \text{ is odd} \\
  2i, & \text{for } 4 \leq i \leq n, \text{ where } i \text{ is even}
\end{cases} \\
  f(v_i) &= \begin{cases} 
  2i, & \text{for } i = 1, 2 \text{ and } 5 \leq i \leq n, \text{ where } i \text{ is odd} \\
  2i - 1, & \text{for } 4 \leq i \leq n, \text{ where } i \text{ is even}
\end{cases}
\end{align*}
\]

Then the vertex weights are

\[
\begin{align*}
  w(c) &= 4n + 5 \\
  w(u_i) &= 2(n + 1) \\
  w(v_i) &= 2n + 3 \\
  w(u) &= w(v) = \frac{n(2n + 3) + 7}{2}
\end{align*}
\]

Case 3: When \( n \geq 4 \) and \( n \) is even

\[
\begin{align*}
  f(c) &= 2n + 1 \\
  f(u) &= 2(n + 1) \\
  f(v) &= 2n + 3 \\
  f(u_i) &= \begin{cases} 
  2i - 1, & \text{for } 1 \leq i \leq n, \text{ where } i \text{ is odd} \\
  2i, & \text{for } 1 \leq i \leq n, \text{ where } i \text{ is even}
\end{cases} \\
  f(v_i) &= \begin{cases} 
  2i, & \text{for } 1 \leq i \leq n, \text{ where } i \text{ is odd} \\
  2i - 1, & \text{for } 1 \leq i \leq n, \text{ where } i \text{ is even}
\end{cases}
\end{align*}
\]
We have

\[ w(c) = 4n + 5 \]
\[ w(u_i) = 2(n + 1) \]
\[ w(v_i) = 2n + 3 \]
\[ w(u) = w(v) = \frac{(4n + 2)(n + 2)}{4} \]

Thus \( \chi_{ld}(BS(n, n)) \leq 4 \). We know that \( \chi(BS(n, n)) = 2 \). If \( \chi_{ld}(BS(n, n)) = 2 \), the weight of the pendant vertex receives the adjacent vertex label. Thus the graph receives \([w(u_i) = f(u) \text{ and } w(v_i) = f(v)]\), 2-colors. The minimum possible vertex weight \( w(u) = \frac{(n+1)(n+2)}{2} \) is greater than \([V(BS(n, n))])\). Similar condition is applied for \( w(v) \). Then \( w(u) = w(v) \) receives one new color. The weight \( w(c) = w(u_i) \text{ or } w(v_i) \) may occur, but the vertex weight \( w(c) = f(u) + f(v) \). We know the value of \( w(u_i) \) and \( w(v_i) \). Further \( w(c) \neq w(u_i) \text{ or } w(v_i) \), vertex \( c \) receives another new color. Therefore \( \chi_{ld}(BS(n, n)) \geq 4 \). Hence \( \chi_{ld}(BS(n, n)) = 4 \). \( \square \)

The shadow graph \( D_2[G] \) is constructed from two graphs \( G_1 \) and \( G_2 \) by joining the vertex \( v \) in \( G_1 \) to the neighbors of corresponding vertex in \( G_2 \). The following result provides the local distance antimagic chromatic number of the shadow graph of a bistar.

**Theorem 2.7.** \( \chi_{ld}(D_2[B_{n,n}]) = 5 \), where \( D_2[B_{n,n}] \), \( n \geq 2 \) is the shadow graph of a bistar graph.

**Proof.** Let \( V(D_2[B_{n,n}]) = \{u, v, u_\alpha, v_\alpha, u_i, v_i, u_\alpha^i, v_\alpha^i, \text{ for } 1 \leq i \leq n \} \) and \( E(D_2[B_{n,n}]) = \{u v \cup u u_i \cup v v_i \cup u_\alpha u_\alpha^i \cup v_\alpha v_\alpha^i \cup u u_\alpha \cup u_\alpha^i \cup v v_\alpha \cup v_\alpha^i, \text{ for } 1 \leq i \leq n \} \). We have \([V(D_2[B_{n,n}])]| = 4(n + 1) \) and \([E(D_2[B_{n,n}])]| = 2(4n + 1) \). Define \( f : V(D_2[B_{n,n}]) \rightarrow \{1, 2, 3, ..., 4(n + 1)\} \) by

\[
\begin{align*}
  f(u_i) & = i, \text{ for } 1 \leq i \leq n \\
f(v_i) & = 2n + i, \text{ for } 1 \leq i \leq n \\
f(u_\alpha) & = n + i, \text{ for } 1 \leq i \leq n \\
f(v_\alpha) & = 3n + i, \text{ for } 1 \leq i \leq n \\
f(u) & = 4n + 1 \\
f(v) & = 2(2n + 1) \\
f(u_\alpha) & = 4(n + 1) \\
f(v_\alpha) & = 4n + 3
\end{align*}
\]
Then the weight of the vertices are
\[
\begin{align*}
    w(u) &= 2n^2 + 5n + 2 \\
    w(v) &= 6n^2 + 5n + 1 \\
    w(u_\alpha) &= 2n^2 + 5n + 3 \\
    w(v_\alpha) &= 6n^2 + 5n + 4
\end{align*}
\]

Also \(w(u_i) = w(v_i) = w(u_\alpha^i) = w(v_\alpha^i) = 8n + 5\). Thus \(\chi_{ld}(D_2[B_{n,n}]) \leq 5\). Suppose \(w(u) = w(v_\alpha)\), where \(u\) and \(v_\alpha\) are non-adjacent vertices. The neighbours of the vertex \(u\) and \(v_\alpha\) must receive same labels. This is not possible. There is a contradiction. Thus \(w(u) \neq w(v_\alpha)\). Thus there exists two distinct colors. Similar argument satisfy for \(w(v) \neq w(u_\alpha)\). All the 2- degree vertices receives a new color. Thus \(\chi_{ld}(D_2[B_{n,n}]) \geq 5\). Hence \(\chi_{ld}(D_2[B_{n,n}]) = 5\). Thus the shadow graph of any bistar can be induced properly by local distance antimagic exactly by five colors. Figure 2 shows the local distance antimagic chromatic number of \(D_2[B_{5,5}]\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Local distance antimagic chromatic number of \(D_2[B_{5,5}]\)}
\end{figure}

### 2.2 Local Distance Antimagic Chromatic Number of Complete \(k\)-partite Graphs

Rinovia Simanjuntak, Kristiana Wijaya [16] proved that complete graphs are distance antimagic. They proved that complete graphs admits \((a,d)\)-distance antimagic if and only if \(d = 1\). Further, in addition to that, they proved all complete multipartite graphs are not distance antimagic. Here in this section,
we proved that all complete $k$-partite graphs are local distance antimagic. We observe that $\chi_{ld}(G) = \chi(G)$, where $G$ is a complete $k$-partite graphs.

**Theorem 2.8.** $\chi_{ld}(K_n) = n$, where $K_n$ is a complete graph on $n \geq 3$ vertices.

**Proof.** Let $V(K_n) = \{v_i, \text{for } 1 \leq i \leq n\}$ and $E(K_n) = \{v_i, v_{i+j} \cup v_1, v_n, \text{for } 1 \leq i \leq n, 1 \leq j \leq n - 2\}$. Thus we have $|V(K_n)| = n$ and $|E(K_n)| = \frac{n(n-1)}{2}$. Let us define a function $f : V \to \{1, 2, 3, ..., n\}$, for $i = 1, 2, ..., n$ such that $f(v_i) = i$, for $1 \leq i \leq n$. Then the vertex weights are $w(v_i) = \frac{n(n+1) - 2i}{2}$. Thus, $\chi_{ld}(K_n) \leq n$. We know that $\chi(K_n) = n$. Hence, $\chi_{ld}(K_n) = n$. \□

Note that the line graph of a star graph is a complete graph. As a immediate consequences of Theorem 2.8, the following Corollary is proved.

**Corollary 2.9.** $\chi_{ld}(L(S_n)) = n$, where $S_n$ is a star graph on $n \geq 2$.

In [2], S. Arumugam et al. proved that complete bipartite graphs $K_{2,n}$ is local antimagic.

**Theorem 2.10.** [2] For any two distinct integers $m, n \geq 2$, $\chi_{ld}(K_{m,n}) = 2$ if and only if $m \equiv n (\text{mod} 2)$.

**Theorem 2.11.** [2] For any complete bipartite graph $K_{2,n}$, we have

\[
\chi_{ld}(K_{2,n}) = \begin{cases} 
2, & \text{for } n \text{ is even and } n \geq 4 \\
3, & \text{for } n \text{ is odd or } n = 2
\end{cases}
\]

We proved that any complete bipartite graphs $K_{m,n}$, for $m, n \geq 2$ are local distance antimagic is shown in Theorem 2.12.

**Theorem 2.12.** For any complete bipartite graph $K_{m,n}$, we have $\chi_{ld}(K_{m,n}) = 2$, for $m, n \geq 2$.

**Proof.** Let $V(K_{m,n}) = \{v_i^1 \cup v_j^2, \text{for } 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(K_{m,n}) = \{v_i^1, v_j^2, \text{for } 1 \leq i \leq m, 1 \leq j \leq n\}$. Then $|V(K_{m,n})| = m + n$ and $|E(K_{m,n})| = mn$. Let us define the bijective function $f : V \to \{1, 2, 3, ..., m + n\}$ by

\[
f(v_i^1) = i, \text{for } 1 \leq i \leq m \\
f(v_j^2) = m + j, \text{for } 1 \leq j \leq n
\]

Then the vertex weights are

\[
w(v_i^1) = \sum_{j=1}^{n} m + j \\
w(v_j^2) = \sum_{i=1}^{m} i
\]

Thus $\chi_{ld}(K_{m,n}) \leq 2$. We know that, $\chi(K_{m,n}) = 2$. Therefore $\chi_{ld}(K_{m,n}) = 2$. \□
Theorem 2.13. $\chi_{ld}(K_{x,y,z}) = 3$, where $K_{x,y,z}$, $x, y, z \geq 3$ is a complete tripartite graph on $n$ vertices and $m$ edges.

Proof. Let $V(K_{x,y,z}) = \{v_1^i \cup v_j^2 \cup v_k^3, \text{for } 1 \leq i \leq x, 1 \leq j \leq y, 1 \leq k \leq z\}$ and $E(K_{x,y,z}) = \{v_1^i v_2^j \cup v_1^i v_k^3 v_3^j, v_2^j v_3^k, \text{for } 1 \leq i \leq x, 1 \leq j \leq y, 1 \leq k \leq z\}$. Then $|V(K_{x,y,z})| = x + y + z$ and $|E(K_{x,y,z})| = x(y + z) + yz$. Define $f: V \rightarrow \{1, 2, 3, ..., x+y+z\}$ by

$$
\begin{align*}
&f(v_1^i) = i, \text{ for } 1 \leq i \leq x \\
&f(v_2^j) = x + j, \text{ for } 1 \leq j \leq y \\
&f(v_3^k) = y + k, \text{ for } 1 \leq k \leq z
\end{align*}
$$

Then the vertex weights are

$$
\begin{align*}
w(v_1^i) &= \frac{n(n+1)}{2} - \frac{x(x+1)}{2} \\
w(v_2^j) &= \frac{n(n+1)}{2} - \sum_{j=1}^{y} f(v_2^j) \\
w(v_3^k) &= \frac{(x+y)(x+y+1)}{2}
\end{align*}
$$

Thus $\chi_{ld}(K_{x,y,z}) \leq 3$. We know that, $\chi(K_{x,y,z}) = 3$. Therefore $\chi_{ld}(K_{x,y,z}) = 3$. \hfill \Box

In general, from Theorem 2.12 and 2.13, we observed that the local distance antimagic chromatic number of $k$-partite graph is $k$.

In [14], Martin Bac˘a, Mirka Miller, Oudone Phanalasy, Andrea Semaničová-Fečňovčíková proved that regular-complete multipartite graphs are antimagic. Rinovia Simanjuntak, Kristiana Wijaya [16], proved all complete multipartite graphs are not distance antimagic. A complete multipartite graph is complete $k$-partite graphs. We proved that all complete multipartite graphs are local distance antimagic in Theorem 2.14.

Theorem 2.14. $\chi_{ld}(K_{p_1,p_2,...p_k}) = k$, where $K_{p_1,p_2,...p_k}$ is a complete multipartite graph with $|p_1| = |p_2| = ... = |p_k|$ with $k \geq 4$, $n \geq 2$.

Proof. Let $V(K_{p_1,p_2,...p_k}) = \{v_i^j, \text{ for } 1 \leq i \leq n, 1 \leq j \leq k\}$. Then $|V(K_{p_1,p_2,...p_k})| = np$. Let us define the function $f: V \rightarrow \{1, 2, 3, ..., np\}$ by

$$
\begin{align*}
f(v_i^j) &= \begin{cases} 
i, & \text{for } j = 1 \\
(nj - 1) + i, & \text{for } j \geq 2
\end{cases}
\end{align*}
$$

Then the vertex weight is given by

11
\[ w(v_i^j) = \left\lfloor \frac{nk \times n(k+1)}{2} \right\rfloor - \sum_{j=1}^{k} i + n(j-1), \text{ for } 1 \leq i \leq n. \]

The chromatic number of the complete multipartite graph is equal to the number of partitions in the graph. The set of vertices in each independent set receives a distinct color. Thus \( \chi(K_{p_1,p_2,...,p_k}) = k = \chi_{ld}(K_{p_1,p_2,...,p_k}) \). Hence \( \chi_{ld}(K_{p_1,p_2,...,p_k}) = k. \)

Figure 3 shows the local distance antimagic chromatic number of the turan graph. It is a special case of the complete multipartite graph in which the size of two independent sets differs by at most one vertex. The chromatic number of Turan graph \( \chi(T_{n,k}) = k \), where \( k \) is the number of partitions and \( |V| = n \). We have \( \chi(T_{13,4}) = 4 = \chi_{ld}(T_{13,4}). \)

### 2.3 Local Distance Antimagic Chromatic Number of Graphs

Rinovia Simanjuntak, Kristiana Wijaya \[16\], investigated the distance antimagic labeling of friendship graphs. Nalliah, M \[15\] investigated the \((a,1)\)-distance antimagic labeling of friendship graph \( F_n \) for \( n = 1 \) or \( 2 \) and there is no \((a,d)\)-distance antimagic labeling of friendship graph if \( d \geq 2 \). Handa, A.K, Aloysius Godinho, Singh. T \[9\] provided the distance antimagic labeling of ladder graphs. D. Froncek \[5,6\] proved the cartesian product of two disjoint copies of complete graphs and its complement are \((a,2)\)- distance antimagic and \((a,1)\)-distance an-
timagic respectively.

In this section, we prove that friendship graphs and ladder graphs are local distance antimagic. A friendship graph $F_n$, where $|V| = p$ is the graph constructed by joining the $n$ copies of cycle $C_3$ to a common vertex.

**Theorem 2.15.** $\chi_{ld}(F_n) = 2n + 1$, for $n \geq 2$.

*Proof.* Let $V(F_n) = \{c \cup v_i \cup u_i, \text{for } 1 \leq i \leq n\}$ and $E(F_n) = \{c v_i \cup c u_i \cup u_i v_i, \text{for } 1 \leq i \leq n\}$. We have $|V(F_n)| = 2n + 1$, $|E(F_n)| = 3n$. Define a bijective function $f : V \to \{1, 2, 3, ..., 2n + 1\}$ by

$$f(c) = 2n + 1$$
$$f(v_i) = 2i - 1, \text{for } 1 \leq i \leq n$$
$$f(u_i) = 2i, \text{for } 1 \leq i \leq n$$

Then the vertex weights are

$$w(c) = n(2n + 1)$$
$$w(v_i) = 2n + 1 + 2i$$
$$w(u_i) = 2n + 2i$$

Thus $\chi_{ld}(F_n) \leq 2n + 1$. Suppose $f(c) = a$, $f(v_1) = 1$, $f(u_1) = 2$. Then $w(v_1) = a + 2$, $w(u_1) = a + 1$ and $w(c) = n(2n + 1)$. Thus vertex $c$ receives the first color. Here $v_i$ and $u_i$ are adjacent vertices, if we need the minimum proper colors for the graph $F_n$, there is a only one possible that $v_i$ and $u_i$, for $1 \leq i \leq n$ receive two distinct colors. Therefore $\chi_{ld}(F_n) \geq 2n + 1$. Hence $\chi_{ld}(F_n) = 2n + 1$. ⬜

Note that the join of two graphs $K_2$ and $K_1$ represents a friendship graph. Hence we proved the following Corollary.

**Corollary 2.16.** Let $G = nK_2 + K_1$, where $n \geq 3$, then $\chi_{ld}(G) = 2n + 1$.

The cartesian product of two graphs $G_1 \times G_2$ is the graph with vertex set $V_1 \times V_2$ with two vertices $u = (u_1, u_2) \in V_1$ and $v = (v_1, v_2) \in V_2$ are adjacent if $u_1 = v_1$ and $u_2$ is adjacent to $v_2$ or $u_2 = v_2$ and $u_1$ is adjacent to $v_1$. A Ladder graph $L_n$ is obtained by the cartesian product of two graphs $P_2$ and $P_n$.

**Theorem 2.17.** $\chi_{ld}(L_n) \leq 2(n - 1)$, for $n \geq 3$.

*Proof.* Let $V(L_n) = \{v_i, u_i, \text{for } 1 \leq i \leq 2n\}$ with $|V(L_n)| = 2n$. Define a bijective function $f : V \to \{1, 2, 3, ..., 2n\}$ by

$$f(v_i) = i, \text{for } 1 \leq i \leq n$$
$$f(u_i) = 2n + 1 - i, \text{for } 1 \leq i \leq n$$
Then the vertex weights are

\[
    w(v_i) = \begin{cases} 
        2(n+1), & \text{for } i = 1 \\
        2n + 1 + i, & \text{for } 2 \leq i \leq n - 1 \\
        2n, & \text{for } i = n 
    \end{cases}
\]

\[
    w(u_i) = \begin{cases} 
        2n, & \text{for } i = 1 \\
        4n + 2 - i, & \text{for } 2 \leq i \leq n - 1 \\
        2(n+1), & \text{for } i = n 
    \end{cases}
\]

Hence \(\chi_{ld}(L_n) \leq 2(n-1)\).

\[\square\]

### 2.4 Corona Product of Graph \(G\) with Null Graph \(\overline{K}_m\)

In this section we compute the local distance antimagic vertex coloring for corona product of graph \(G\) with the null graph \(\overline{K}_m\) on \(m \geq 1\), where \(G\) is a star \(S_n\), a complete \(K_n\), and a friendship graph \(F_n\).

**Theorem 2.18.** \(\chi_{ld}(S_n \circ \overline{K}_m) = n + 2\), for \(n \geq 3\), \(m = 1\).

**Proof.** Let \(V(S_n \circ \overline{K}_1) = \{c \cup v_i \cup v_i^1 \cup c^1, \text{ for } 1 \leq i \leq n\}\) and \(E(S_n \circ \overline{K}_1) = \{cv_i \cup v_i v_i^1 \cup cc^1, \text{ for } 1 \leq i \leq n\}\). Then \(|V(S_n \circ \overline{K}_1)| = 2n + 2\) and \(|E(S_n \circ \overline{K}_1)| = 2n + 1\). Define a bijective function \(f : V \rightarrow \{1, 2, \ldots, 2n\}\) by

\[
    f(v_i) = \begin{cases} 
        2n + 1, & \text{for } i = 1 \\
        n + i, & \text{for } 2 \leq i \leq n 
    \end{cases}
\]

\[
    f(v_i^1) = i, \text{ for } 1 \leq i \leq n
\]

\[
    f(c) = n + 1
\]

\[
    f(c^1) = 2n + 2
\]

Then the weight of the vertices are

\[
    w(v_i) = n + 1 + i, \text{ for } 1 \leq i \leq n
\]

\[
    w(v_i^1) = f(v_i)
\]

\[
    w(c) = \frac{3n^2 + 7n + 4}{2}
\]

\[
    w(c^1) = f(c)
\]

Thus \(\chi_{ld}(S_n \circ \overline{K}_1) \leq n + 2\). \[\square\]
Theorem 2.19. \( \chi_{td}(S_n \circ K_m) = n + 3 \), for every \( n \geq 3 \) and \( m \geq 2 \).

Proof. Let \( V(S_n \circ K_m) = \{c \cup v_i \cup v_i^j \cup c^j, 1 \leq i \leq n, 1 \leq j \leq m\} \) and \( E(S_n \circ K_m) = \{cv_i \cup v_i^j \cup ce^j, 1 \leq i \leq n, 1 \leq j \leq m\} \). Then \( |V(S_n \circ K_m)| = (n+1)(m+1) \) and \( |E(S_n \circ K_m)| = n(m+1) + m \). Now, define the function \( f : V \rightarrow \{1, 2, \ldots, (n+1)(m+1)\} \) by

Case 1: when \( m \geq 2 \), is even

\[
\begin{align*}
    f(c) &= (m+1)n + 1 \\
    f(c^j) &= (m+1)n + 1 + j, \text{ for } 1 \leq j \leq m \\
    f(v_i) &= mn + i, \text{ for } 1 \leq i \leq n \\
    f(v_i^j) &= \begin{cases} 
    n(j-1) + i, & j \text{ is odd, } 1 \leq i \leq n, 1 \leq j \leq m \\
    nj + 1 - i, & j \text{ is even, } 1 \leq i \leq n, 1 \leq j \leq m 
    \end{cases}
\end{align*}
\]

Then the vertex weights are

\[
\begin{align*}
    w(c) &= 2mn + n(n+1) + 2m(mn + n + 1) + m(m+1) \\
    w(v_i) &= m(mn + 1) + 2(mn + n + 1), \text{ for } 1 \leq i \leq n \\
    w(v_i^j) &= f(v_i), \text{ for } 1 \leq i \leq n \\
    w(c^j) &= f(c), \text{ for } 1 \leq j \leq m
\end{align*}
\]

Case 2: when \( m \geq 3 \), is odd

\[
\begin{align*}
    f(c) &= (m+1)n + 1 \\
    f(c^j) &= (m+1)n + 1 + j, \text{ for } 1 \leq j \leq m \\
    f(v_i) &= 4n + 2 - 2i, \text{ for } 1 \leq i \leq n \\
    f(v_i^j) &= \begin{cases} 
    n(j-1) + i, & j \text{ is even, } 4 \leq j \leq m - 1, 1 \leq i \leq n \\
    4n + (j-4)n + i, & j \text{ is odd, } 5 \leq j \leq m, 1 \leq i \leq n \\
    4n + (j-3)n + 1 - i, & j \text{ is even, } 4 \leq j \leq m - 1, 1 \leq i \leq n \\
    \end{cases}
\end{align*}
\]

Then the vertex weights are

\[
\begin{align*}
    w(c) &= 2mn(m+1) + m(m+3) + 2n(3n+1) \\
    w(v_i) &= 2n(m+6) + (m-3)(n+1) + [m(m+1) - 12]n + 4, \text{ for } 1 \leq i \leq n \\
    w(v_i^j) &= f(v_i) \\
    w(c^j) &= f(c)
\end{align*}
\]
Thus $\chi_{ld}(S_n \circ \overline{K_m}) \leq n + 3$, where $m \geq 2$. We know that chromatic number $\chi(S_n \circ \overline{K_m}) = 2$. In the local distance antimagic labeling, the weight of the pendant vertex is the label of its adjacent vertex. The graph $S_n \circ \overline{K_m}$ has $n$-different sets of the pendant vertices. Therefore the graph receives $n + 1$ colors. Then the minimum possible weight of the central vertex is $\frac{n(n + 1)}{2}$. Therefore $w(c) > |V(G)|$, the central vertex receives $n + 2$ color. Also the minimum vertex weight of $v_i$, $w(v_i) = f(v_i) + f(c) = \frac{(nm + 1)(m + 1)}{2n}$. $w(v_i) > |V(G)|$ and $w(v_i)$ is adjacent to $w(c)$. Therefore $w(v_i)$ need a new color. Hence the graph $\chi_{ld}(S_n \circ \overline{K_m}) \geq n + 3$. Therefore $\chi_{ld}(S_n \circ \overline{K_m}) = n + 3$. □

**Theorem 2.20.** $\chi_{ld}(K_n \circ \overline{K_m}) = 2n$, for $n \geq 3$, $m \geq 1$.

**Proof.** Consider a graph $K_n \circ \overline{K_m}$ denotes the corona product of complete graph $K_n$, for $n \geq 3$ vertices and the complement of $K_m$, for $m \geq 1$. Let $V(K_n \circ \overline{K_m}) = \{v_i \cup v_i^j, \text{for } 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(K_n \circ \overline{K_m}) = \{v_iv_{i+1} \cup v_iv_n \cup v_i^jv_i^j, \text{for } 1 \leq i \leq n, 1 \leq j \leq m\}$. Then $|V(K_n \circ \overline{K_m})| = n(m + 1)$ and $|E(K_n \circ \overline{K_m})| = \frac{n(n + 2m - 1)}{2}$. Define a bijective function $f : V \rightarrow \{1, 2, \ldots, 2n\}$ by

$$f(v_i^j) = n(j + 1) - i + 1, \text{for } 1 \leq i \leq n, 1 \leq j \leq m$$

$$f(v_i) = i, \text{for } 1 \leq i \leq n, 1 \leq j \leq m.$$ 

The weight of the vertices are

$$w(v_i^j) = i, \text{for } 1 \leq i \leq n, 1 \leq j \leq m$$

$$w(v_i) = \frac{n(n + 1 - 2i)}{2} + \sum_{j=1}^{m} [n(j + 1) - i + 1], \text{for } 1 \leq i \leq n, 1 \leq j \leq m.$$ 

Thus $\chi_{ld}(K_n \circ \overline{K_m}) \leq 2n$. Suppose $m \geq 1$ and $n = 3$, the minimum possible weight of the vertex $w(v_i) = \frac{(m + n)(m + n - 1)}{2} = |V(G)|$. Then the minimum possible weight of the vertex $v_1$, $w(v_1) = f(v_2) + f(v_3) + f(v_1^1) = 6$. In this case $v_2$ and $v_3$ receives the label 1 and 2 respectively. If $w(v_1) = w(v_3^1)$ or $w(v_3^1)$, then the minimum possibility of the vertex weights are $w(v_1^1)$ and $w(v_3^1)$ is not equal to 6. Similar argument is applied for $v_1^2$ and $w(v_3^1)$. Thus $\chi_{ld}(K_3 \circ \overline{K_1}) = 2n$. For $n \geq 3$, $m \geq 2$, the minimum possible weight $w(v_i) > |V(K_n \circ \overline{K_m})|$. Thus $v_i$, for $1 \leq i \leq n$ receives $n$ distinct colors. It is obvious that weight of all the pendant vertex receives the label of its adjacent vertex. Then the graph has $n$ different set of colors. Therefore $\chi_{ld}(K_n \circ \overline{K_m}) = 2n$. □

**Theorem 2.21.** $\chi_{ld}(F_n \circ \overline{K_m}) = 2n + 4$, for $n \geq 2$, $m \geq 1$.

**Proof.** Let $V(F_n \circ \overline{K_m}) = \{c \cup v_i \cup u_i \cup v_i^j \cup v_i^j \cup v_i^j, 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(F_n \circ \overline{K_m}) = \{cv_i \cup cu_i \cup v_iv_i^j \cup w_iw_i^j \cup c \cup v_i^j, 1 \leq i \leq n, 1 \leq j \leq m\}$. Then
\[ |V(F_n \circ \overline{K_m})| = (m + 1)(2n + 1) \text{ and } |E(F_n \circ \overline{K_m})| = m(2n + 1) + 3n. \]

Define a bijective function \( f : V \rightarrow \{1, 2, ..., (m + 1)(2n + 1)\} \) by

\[
\begin{align*}
  f(v_i) &= 2i - 1, \text{ for } 1 \leq i \leq n \\
  f(u_i) &= 2i, \text{ for } 1 \leq i \leq n \\
  f(c) &= 2n + 1
\end{align*}
\]

**Case 1:** \( m = 1 \)

\[
\begin{align*}
  f(v_i^1) &= 4n + 3 - 2i, \text{ for } 1 \leq i \leq n \\
  f(u_i^1) &= 4n + 2 - 2i, \text{ for } 1 \leq i \leq n \\
  f(c^1) &= 4n + 2
\end{align*}
\]

The vertex weights are

\[
\begin{align*}
  w(v_i) &= 6n + 4 \\
  w(u_i) &= 6n + 2 \\
  w(c) &= 2n^2 + 5n + 2
\end{align*}
\]

**Case 2:** When \( m = 2 \)

\[
\begin{align*}
  f(v_i^j) &= \begin{cases} 
    5n + 2 - i, & \text{for } 1 \leq i \leq n, j = 1 \\
    6n + 2 - i, & \text{for } 1 \leq i \leq n, j = 2 
  \end{cases} \\
  f(u_i^j) &= \begin{cases} 
    3n + 2 - i, & \text{for } 1 \leq i \leq n, j = 1 \\
    4n + 2 - i, & \text{for } 1 \leq i \leq n, j = 2 
  \end{cases}
\end{align*}
\]

The vertex weights are

\[
\begin{align*}
  w(v_i) &= 13n + 5 \\
  w(u_i) &= 9n + 4 \\
  w(c) &= 2n^2 + 13n + 5
\end{align*}
\]

**Case 3:** When \( m \geq 3, \text{ is odd} \)

\[
\begin{align*}
  f(v_i^j) &= \begin{cases} 
    n(j + m + 1) + i + 1, & \text{if } j \text{ is even, } 2 \leq j \leq m - 1 \\
    n(j + m + 2) - i + 2, & \text{if } j \text{ is odd, } 3 \leq j \leq m 
  \end{cases} \\
  f(u_i^j) &= \begin{cases} 
    n(j + 2) + i + 1, & \text{if } j \text{ is even, } 2 \leq j \leq m - 1 \\
    4n + 2 - i, & \text{if } j \text{ is odd, } 3 \leq j \leq m 
  \end{cases} \\
  f(c^j) &= 2n(m + 1) + j + 1, \quad 1 \leq j \leq m
\end{align*}
\]
Then the vertex weights are

\[ w(v_i) = 6n + 4 + \frac{n[m(m + 1) - 2] + (m - 1)[n(2m + 3) + 3]}{2} \]

\[ w(u_i) = 6n + 2 + \frac{n[m(m + 1) - 2] + (m - 1)(5n + 3)}{2} \]

\[ w(c) = \frac{2n(2n + 1) + 2m[2n(m + 1) + 1] + m(m + 1)}{2} \]

**Case 4:** When \( m \geq 4 \), is even

\[ f(v_j) = \begin{cases} 
  n(j + m + 1) + i + 1, & \text{j is odd, } 3 \leq j \leq m - 1 \\
  n(j + m + 2) - i + 2, & \text{j is even, } 4 \leq j \leq m 
\end{cases} \]

\[ f(u_j) = \begin{cases} 
  n(j + 3) + i + 1, & \text{j is odd, } 3 \leq j \leq m - 1 \\
  n(j + 4) - i + 2, & \text{j is even, } 4 \leq j \leq m 
\end{cases} \]

\[ f(c_j) = 2n(m + 1) + j + 1, \text{ for } 1 \leq j \leq m \]

Then the vertex weights are

\[ w(v_i) = 13n + 5 + \frac{n[m(m + 1) - 6] + (m - 2)[n(2m + 3) + 3]}{2} \]

\[ w(u_i) = 9n + 4 + \frac{n[m(m + 1) - 6] + (m - 2)(7n + 3)}{2} \]

\[ w(c) = \frac{2n(2n + 1) + 2m[2n(m + 1) + 1] + m(m + 1)}{2} \]

Thus, \( \chi_{ld}(F_n \circ \overline{K_m}) \leq 2n + 4 \). We know that chromatic number \( \chi(F_n \circ \overline{K_m}) = 3 \).

In the local distance antimagic labeling, the weight of the pendant vertex is the label of its adjacent vertex. The graph \( F_n \circ \overline{K_m} \) has \( 2n \)-different sets of the pendant vertices. Therefore the graph receives \( 2n + 1 \) colors. Then the minimum possible weight of the central vertex is \( \frac{(2n + m)(2n + m + 1)}{2} \). Therefore \( w(c) > |V(G)| \), the central vertex receives \( 2n + 2 \) color. Also the minimum vertex weight of \( v_i \), \( w(v_i) = f(u_i) + f(v_j) + f(c) = \frac{(nm + 2)(nm + 3)}{2n} \), so \( w(v_i) > |V(G)| \). Therefore \( w(v_i) \) receives an another new color \( 2n + 3 \). Similar argument occurs for vertex \( u_i \). Thus \( u_i \) receives a new color \( 2n + 4 \). Hence \( \chi_{ld}(F_n \circ \overline{K_m}) \geq 2n + 4 \). Therefore \( \chi_{ld}(F_n \circ \overline{K_m}) = 2n + 4 \). \( \square \)

### 3 Conclusion and Scope

In this paper, we proved that some star-related graphs, complete graphs, complete \( k \)-partite graphs, friendship graphs, ladder graphs and some corona product
of graphs are local distance antimagic and hence computed $\chi_{ld}(G)$. In general, we observe that $\chi_{ld}(G) = \chi(G)$ for star graphs, line graph of star graphs, complete graphs, complete $k$-partite graphs, complete multipartite graphs. Further, $\chi_{ld}(G) > \chi(G)$ for subdivision of star graphs, bistar graphs, subdivision of bistars, shadow graph of bistars, doublestar graphs, friendship graphs, ladder graphs and corona product of graphs. It is clear that proved that, $\chi_{ld}(G) \geq \chi(G)$. Thus, we obtained $\chi_{ld}(G) \geq \chi(G)$ and obviously $\chi_{ld}(G) = n$ for complete graph $K_n$. Hence the following problems arise naturally.

**Problem 3.1.** Characterise the class of graphs for which $\chi_{ld}(G) = n$.

There are several classes of graphs such as $K_n$, $S_n$, $L(S_n)$, $K_{m,n}$, $K_{x,y,z}$ and $K_{p_1,p_2,...,p_k}$ for which $\chi_{ld}(G) = \chi(G)$. Hence we arrive the following problem.

**Problem 3.2.** Characterise the class of graphs for which $\chi_{ld}(G) = \chi(G)$.

**Problem 3.3.** Characterise graphs with $\chi_{ld}(G \circ K_m) = 2n$.

In Theorem 2.17, we have determined the local distance chromatic number of the ladder graph $\chi_{ld}(L_n) \leq 2(n - 1)$. The problem is open for the remaining case.

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