A forward–backward splitting algorithm for the minimization of non-smooth convex functionals in Banach space

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Abstract

We consider the task of computing an approximate minimizer of the sum of a smooth and a non-smooth convex functional, respectively, in Banach space. Motivated by the classical forward–backward splitting method for the subgradients in Hilbert space, we propose a generalization which involves the iterative solution of simpler subproblems. Descent and convergence properties of this new algorithm are studied. Furthermore, the results are applied to the minimization of Tikhonov-functionals associated with linear inverse problems and semi-norm penalization in Banach spaces. With the help of Bregman–Taylor-distance estimates, rates of convergence for the forward–backward splitting procedure are obtained. Examples which demonstrate the applicability are given, in particular, a generalization of the iterative soft-thresholding method by Daubechies, Defrise and De Mol to Banach spaces as well as total-variation-based image restoration in higher dimensions are presented.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The purpose of this paper is, on the one hand, to introduce an iterative forward–backward splitting procedure for the minimization of functionals of type

$$\min_{u \in X} F(u) + \Phi(u)$$

in Banach spaces and to analyze its convergence properties. Here, $F$ represents a convex smooth functional while the convex $\Phi$ is allowed to be non-smooth. On the other hand, the application of this algorithm to Tikhonov functionals associated with linear inverse problems
in Banach space is studied. We consider, in particular, general regularization terms which are only powers of semi-norms instead of norms. Moreover, examples which show the range of applicability of the algorithm as a computational method are presented.

The forward–backward splitting algorithm for minimization in Banach space suggested in this work tries to establish a bridge between the well-known forward–backward splitting in Hilbert space [11, 13, 24] and minimization algorithms in Banach space, which require in general more analysis. For example, in the situation of Banach spaces, gradient-based (or subdifferential-based) methods always have to deal with the problem that the gradient is an element of the dual space and can therefore not directly used as a descent direction. One common approach of circumventing this difficulty is performing the step in the gradient direction in the dual space and use appropriate duality mappings to link this procedure to the primal space [2, 28]. Such a procedure applied to (1) can be seen as a full explicit step for $\partial (F + \Phi)$ and convergence can often be achieved with the help of a priori step-size assumptions. In contrast to this, forward–backward splitting algorithms also involve an implicit step by applying a resolvent mapping. The notion of resolvents can also be generalized to Banach spaces by introducing duality mappings [21], combining both explicit and implicit steps to a forward–backward splitting algorithm, however, has not been considered so far.

The paper can be outlined as follows. We present, in section 2, a generalization of the forward–backward splitting algorithm which operates in Banach spaces and coincides with the usual method in case of Hilbert spaces. The central step for the proposed method is the successive solution of problems of type

$$\min_{v \in X} \frac{\|v - u^n\|_X^p}{p} + \tau_n (\langle F'(u^n), v \rangle + \Phi(v))$$

which are in general easier to solve than the original problem. In particular, we will show in section 3 that the algorithm stops or the functional values converge as $n \to \infty$ with an asymptotic rate of $n^{1-p}$ if $X$ is reflexive and $F'$ is locally $(p-1)$-Hölder continuous. There, we only have to assume that the step-sizes obey some upper and lower bounds. Moreover, under certain conditions, strong convergence with convergence rates will be proven, in particular that the $q$-convexity of $\Phi$ implies the rate $n^{(1-p)/q}$.

In section 4, we will apply these results to the linear inverse problem of solving $Ku = f$ with $K : X \to Y$ which is regularized with a semi-norm of some potentially smaller space as penalty term and leading to the problem of minimizing a non-smooth Tikhonov functional:

$$\min_{u \in X} \frac{\|Ku - f\|_Y^p}{p} + \alpha \frac{\|u\|_q^q}{q}.$$ 

The forward–backward splitting algorithm applied to these type of functionals will be discussed. It turns out that we have convergence with rate $n^{(1-p)/q}$ whenever the data space $Y$ is $p$-smooth and $q$-convex and the regularizing semi-norm is $q$-convex in a certain sense.

In section 5, examples are given on how to compute the algorithm. Also, basic numerical calculations are shown. We consider linear inverse problems with sparsity constraints in Banach spaces which leads to a generalization of the results for the iterative soft-thresholding procedure in [14] to Banach spaces. Moreover, it is also discussed how to apply the algorithm to linear inverse problems with total-variation regularization in higher dimensions where an embedding into the Hilbert space $L^2(\Omega)$ is impossible. The paper finally concludes with some remarks in section 6.
2. A forward–backward splitting algorithm in Banach space

Let $X$ be a reflexive Banach space in which the functional (1) has to be minimized. Assume that both $F$ and $\Phi$ are proper, convex and lower semi-continuous functionals such that $F + \Phi$ is coercive. In this setting, we require that $F$ represents the 'smooth part' of the functional where $\Phi$ is allowed to be non-smooth. Specifically, it is assumed that $F$ is differentiable with derivative which is, on each bounded set, $(p - 1)$ Hölder continuous for some $1 < p \leq 2$, i.e.

$$
\|F'(u) - F'(v)\|_{X^*} \leq \|F\|_{p-1} \|u - v\|_X^{p-1}
$$

for $\|u\|_X, \|v\|_X \leq C$.

We propose the following iterative procedure in order to find a solution of (1).

(1) Start with a $u^0 \in X$ with $\Phi(u^0) < \infty$ and $n = 0$. Estimate, using the coercivity of $F + \Phi$, a norm bound $\|u\|_X \leq C$ for all $(F + \Phi)(u) \leq (F + \Phi)(u^0)$ and choose $\|F\|_{p-1}$ accordingly.

(2) Compute the next iterate $u^{n+1}$ as follows. Compute $w^n = F'(u^n)$ and determine, for an $\tau_n > 0$ satisfying $0 < \tau_n \leq \frac{p(1 - \delta)}{\|F\|_{p-1}}$, the solutions of the auxiliary minimization problem

$$
\min_{v \in X} \frac{\|v - u^n\|_X^p}{p} + \tau_n((u^n, v) + \Phi(v)).
$$

Choose, if necessary, $u^{n+1}$ as a solution of (3) which minimizes $\|v - u^n\|_X$ among all solutions $v$ of (3) to ensure that the fixed points of $u^n \mapsto u^{n+1}$ are exactly the minimizers of (1).

(3) If $u^{n+1}$ is not optimal, continue with $n := n + 1$ and repeat with step (2), after optionally adjusting $\|F\|_{p-1}$ analogously to step (1).

Remark 1. This iterative procedure can be seen as a generalization of the forward–backward splitting procedure for the subgradients (which is in turn some kind of generalized gradient projection method [5]) in case of $X$ and $Y$ being a Hilbert spaces and $p = 2$. This reads as, in terms of resolvents (see [6, 17, 29], for example, for introductions to these notions),

$$
u^{n+1} = (I + \tau_n \partial \Phi)^{-1}(I - \tau_n F')(u^n)
$$

or, equivalently,

$$
u^{n+1} = \arg\min_{v \in X} \frac{\|v - (u^n - \tau_n F'(u^n))\|_Y^2}{2} + \tau_n \Phi(v).
$$

Such an operation does not make sense in Banach spaces, since $F'(u^n) \in X^*$ cannot be subtracted from $u^n \in X$. However, (4) is equivalent to

$$
u^{n+1} = \arg\min_{v \in X} \frac{\|v - u^n\|_Y^2}{2} + \tau_n ((F'(u^n), v) + \Phi(v)),
$$

which only involves a duality pairing for $F'(u^n)$. Hence, one can replace $\|v - u^n\|_Y^2/2$ by $\|v - u^n\|_X^p/p$ and $X, Y$ with Banach spaces and ends up with (3), which defines a sensible operation (see also proposition 1). Denoting $J_p = \partial \|\cdot\|_p^p$ and $P_{\tau_n}(u, w) = (J_p(-u) + \tau \partial \Phi)^{-1}(w)$, one can interpret $u^n \mapsto (u^n, -\tau_n F'(u^n))$ as being a generalized forward step, while $(u^n, -\tau_n F'(u^n)) \mapsto P_{\tau_n}(u^n, -\tau_n F'(u^n))$ represents the corresponding generalized backward step.
Proposition 1. The problem
\[ \min_{v \in X} \frac{\|v - u\|_X^p}{p} + \tau\langle (\langle w, v \rangle + \Phi(v) \rangle \] has a solution for each \( u \in X \), \( w \in X^* \) and \( \tau \geq 0 \). Moreover, there always exists a solution \( u^{n+1} \) which minimizes \( \|v - u^n\|_X \) among all solutions \( v \).

Proof. The case \( \tau = 0 \) is trivial, so let \( \tau > 0 \) in the following. We first show the coercivity of the objective functional. For this purpose, note that \( N_p(v) = \|v - u\|_X^p \) grows faster than \( \|v\|_X \), i.e. \( N_p(v)/\|v\|_X \to \infty \) whenever \( \|v\|_X \to \infty \). Moreover, for an \( u^* \) minimizing (1) we have \( \Phi(v) \geq \Phi(u^*) + \langle w^*, v - u^* \rangle \) where \( -w^* = -F'(u^*) \in \partial \Phi(u^*) \) since \( F \) is continuous in \( X \) implying that \( \tau(F + \Phi) = F' + \partial \Phi \), see [29]. Hence, we can estimate
\[ \frac{\|v - u\|_X^p}{p} + \tau(\langle w, v \rangle + \Phi(v)) \geq \|v\|_X \left( \frac{N_p(v)}{\|v\|_X} - \tau \|w + w^*\|_{X^*} + \tau \frac{\Phi(u^*) - \langle w^*, u^* \rangle}{\|v\|_X} \right) \]
for some \( L > 0 \) and all \( v \) whose norm is large enough, showing that the functional in (5) is coercive.

It follows that the functional in (5) is proper, convex, lower semi-continuous and coercive in a reflexive Banach space, consequently, at least one solution exists. Finally, denote by \( M \) the set of solutions of (5), which is non-empty. Also, \( M \) is convex and closed since the functional in (5) is convex and lower semi-continuous, respectively, in the reflexive Banach space \( X \). Consequently, by standard arguments from calculus of variations,
\[ \min_{v \in X} \|v - u^n\|_X + I_M(v) \]
admits a solution. Thus, \( u^{n+1} \) is well defined.

\[ \square \]

Proposition 2. The solutions \( u^* \) of the problem (1) are exactly the fixed points of the iteration (for each \( \tau_n > 0 \)).

Proof. Suppose that \( u^* = u^n \) is optimal. Then, since \( \partial(F + \Phi) = F' + \partial \Phi \) we know that \( w^* = F'(u^*) \) also satisfies \( -w^* \in \partial \Phi(u^n) \). Now, the solutions of (3) can be characterized by each \( v \in X \) solving the inclusion relation
\[ -\tau_n w^* \in J_p(v - u^n) + \tau_n \partial \Phi(v) \]
with \( J_p \) being the \( p \)-duality relation in \( X \), i.e. \( J_p = \partial \frac{1}{p} \|\cdot\|_p \). Obviously, \( v = u^* \) is a solution and from the requirement that \( u^{n+1} \) is the solution which minimizes \( \|v - u^n\|_X \) among all solutions follows \( u^{n+1} = u^n \), hence \( u^* \) is a fixed point.

On the other hand, if \( u^n = u^{n+1} \), then, for the corresponding \( u^n \) holds
\[ -\tau_n w^* \in J_p(u^{n+1} - u^n) + \tau_n \partial \Phi(u^{n+1}) \Rightarrow -\tau_n w^* \in \tau_n \partial \Phi(u^n) \]
since \( J_p(0) = \{0\} \) (by Asplund’s theorem, see [12]), meaning that \( u^n \) is optimal by \( w^* = F'(u^n) \).

These results justify in a way that the proposed iteration, which can be interpreted as a fixed-point iteration, makes sense. It is, however, not clear whether we can achieve convergence to a minimizer if the optimality conditions are not satisfied. For this purpose, the algorithm has to be examined more deeply. We will use the descent of the objective functional in (1) in order to obtain conditions under which convergence holds.
3. Descent properties and convergence

This section deals with descent properties for the proximal forward–backward splitting algorithm. It is shown that each iteration leads to a sufficient descent of the functional whenever (2) is satisfied. Since the minimization problem (1) is convex, we can also obtain convergence rates for the functional, which will be done in the following. The proof is essentially based on four steps which are in part inspired by the argumentation in [4, 5, 15].

After introducing a functional \( D(u^n) \) which measures the descent of the objective functional, we prove a descent property which, subsequently, leads to convergence rates for the distance of the functional values to the minimum. Finally, under assumptions on the Bregman (and Bregman–Taylor) distances, convergence rates for the distance to the minimizer follow.

**Lemma 1.** For each iterate and each \( v \in X \), we have the inequality

\[
\frac{\langle J_p(u^n - u^{n+1}), v - u^{n+1} \rangle}{\tau_n} \leq \Phi(v) - \Phi(u^{n+1}) + \langle w^n, v - u^{n+1} \rangle.
\]  

Furthermore, with

\[
D(u^n) = \Phi(u^n) - \Phi(u^{n+1}) + \langle w^n, u^n - u^{n+1} \rangle
\]

it holds that

\[
\frac{\|u^n - u^{n+1}\|_X^p}{\tau_n} \leq D(u^n).
\]

**Proof.** Since \( u^{n+1} \) solves (3) with data \( w^n \) and \( u^n \), the subgradient relation

\[
-J_p(u^{n+1} - u^n) \in \tau_n (\partial \Phi(u^{n+1}) + w^n)
\]

holds and, consequently, by the subgradient inequality,

\[
\Phi(u^{n+1}) + \langle w^n, u^{n+1} \rangle - \tau_n^{-1} \langle J_p(u^{n+1} - u^n), v - u^{n+1} \rangle \leq \Phi(v) + \langle w^n, v \rangle
\]

for each \( v \in X \). Rearranging terms and noting that \( J_p(-u) = -J_p(u) \) for each \( u \in X \) yields (6). The inequality (8) then follows by letting \( v = u^n \) and noting that \( \langle J_p(u^n - u^{n+1}), u^n - u^{n+1} \rangle = \|u^n - u^{n+1}\|_X^p \) by definition of the duality relation. \( \square \)

Next, we prove a descent property which will be crucial for the convergence analysis. It will make use of the ‘descent measure’ \( D(u^n) \) introduced in (7).

**Proposition 3.** The iteration satisfies

\[
(F + \Phi)(u^{n+1}) \leq (F + \Phi)(u^n) - \left(1 - \tau_n \frac{\|F\|_{p-1}}{p} \right) D(u^n)
\]

with \( D(u^n) \geq 0 \) defined by (7).

**Proof.** First note that from (8) follows that \( D(u^n) \geq 0 \) and, together with proposition 2, that \( D(u^n) = 0 \) if and only if \( u^n \) is optimal. Using the definition of \( D(u^n) \) gives

\[
(F + \Phi)(u^n) - (F + \Phi)(u^{n+1}) = D(u^n) + (F(u^n) - F(u^{n+1}) - \langle w^n, u^n - u^{n+1} \rangle). 
\]

Note that we can write

\[
F(u^{n+1}) - F(u^n) - \langle w^n, u^{n+1} - u^n \rangle = \int_0^1 \langle w(t) - w^n, u^{n+1} - u^n \rangle \, dt
\]
with \( w(t) = F'(u^n + t(u^{n+1} - u^n)) \). We now want to estimate the absolute value of (11) in terms of \( D(u^n) \):

\[
\left| \int_0^1 (w(t) - w^n, u^{n+1} - u^n) \, dt \right| \leq \int_0^1 \|w(t) - w^n\|_X \|u^{n+1} - u^n\|_X \, dt \\
\leq \int_0^1 \|F'\|_{p-1} \|t(u^{n+1} - u^n)\|_{X}^{p-1} \|u^{n+1} - u^n\|_X \, dt \\
= \|F'\|_{p-1} \|u^{n+1} - u^n\|_X^p \leq \frac{\tau_n \|F'\|_{p-1}}{p} D(u^n)
\]

by employing the Hölder-continuity assumption as well as the estimate (8). The claimed statement finally follows from the combination of this with (10).

Note that proposition 3 together with (2) yields a guaranteed descent of the functional \( F + \Phi \). Since in this case the boundedness \( 0 < \tau \leq \tau_n \) is also given, the convergence of the functional values to the minimum is immediate as we will see in the following lemma. But first, introduce the functional distance

\[
r_n = (F + \Phi)(u^n) - \left( \min_{u \in X} (F + \Phi)(u) \right)
\]

which allows us to write (9) as

\[
r_n - r_{n+1} \geq \left( 1 - \tau_n \frac{\|F'\|_{p-1}}{p} \right) D(u^n).
\]

**Lemma 2.** Assume that \( F + \Phi \) is coercive or the sequence \( \{u^n\} \) is bounded. Then, the sequence \( \{r_n\} \) according to (12) satisfies

\[
r_n - r_{n+1} \geq c_0 r_n^{p'}
\]

for each \( n \), with \( p' \) the dual exponent \( 1/p + 1/p' = 1 \) and some \( c_0 > 0 \).

**Proof.** First note that, according to proposition 3, \( \{r_n\} \) is a non-increasing sequence. If \( F + \Phi \) is coercive, this immediately means \( \|u^n - u^*\| \leq C_1 \) for some minimizer \( u^* \in X \) and all \( n \). The same holds true if \( \{u^n\} \) is already bounded.

Observe that the convexity of \( F \) as well as (6) gives

\[
r_n \leq \Phi(u^n) - \Phi(u^*) + \langle w^n, u^n - u^* \rangle
\]

\[
= \langle u^n, u^n - u^{n+1} \rangle + \Phi(u^n) - \Phi(u^*) + \langle w^n, u^{n+1} - u^* \rangle
\]

\[
= D(u^n) + \Phi(u^{n+1}) - \Phi(u^*) + \langle w^n, u^{n+1} - u^* \rangle
\]

\[
\leq D(u^n) + \tau_n^{-1} \|J_p(u^n - u^{n+1}), u^{n+1} - u^*\|_X
\]

\[
\leq D(u^n) + \tau_n^{-1} \|J_p(u^n - u^{n+1})\|_X \|u^{n+1} - u^n\|_X.
\]

Now, since \( p > 1 \),

\[
\tau_n^{-1} \|J_p(u^n - u^{n+1})\|_X = \tau_n^{-1} \|u^n - u^{n+1}\|_X^{p-1}
\]

\[
= \left( \tau_n^{-1} \|u^n - u^{n+1}\|_X^{1/p} \right)^{1/p'} \tau_n^{-1/p'}
\]

where \( p' \) is the dual exponent, i.e. \( \frac{1}{p} + \frac{1}{p'} = 1 \). Further, applying (8), (13) and taking the step-size constraint (2) into account yields

\[
\delta r_n \leq r_n - r_{n+1} + C_1 (r_n - r_{n+1})^{1/p} (\delta \Sigma)^{1/p}. 
\]

Note that \( r_n - r_{n+1} \leq r_0 \) since \( \{r_n\} \) is non-increasing. This finally gives

\[
\delta r_n \leq \left( C_1 (\delta \Sigma)^{-1/p} \right) (r_n - r_{n+1})^{1/p'} \Rightarrow \left( \frac{\delta^{1/p'} \Sigma^{1/p}}{(\delta^{-1/p} r_0)^{1/p} + C_1} \right)^{p'} r_n^{p'} \leq r_n - r_{n+1}.
\]
Proposition 4. Under the prerequisites of lemma 2, the functional values for \( \{u^n\} \) converge to the minimum with rate

\[ r_n \leq Cn^{1-p} \]

for some \( C > 0 \).

Proof. Apply the mean value theorem to get the identity

\[
\frac{1}{r_n^{p'-1}} - \frac{1}{r_n^{p'-1}} = \frac{t_n^{p'-1} - r_n^{p'-1}}{(r_n r_{n+1})^{p'-1}} = \frac{(p'-1)\varphi^{p-2}(r_n - r_{n+1})}{(r_n r_{n+1})^{p'-1}}
\]

with \( r_{n+1} \leq \vartheta \leq r_n \). Thus, \( \varphi^{p-2} \geq r_{n+1}^{p'-1} r_n^{-1} \) and, by lemma 2,

\[
\frac{1}{r_{n+1}^{p'-1}} - \frac{1}{r_n^{p'-1}} \geq \frac{(p'-1)c_0 r_n^{p'-1} r_{n+1}^{p'-1}}{(r_n r_{n+1})^{p'-1}} = (p'-1)c_0.
\]

Summing up then yields

\[
\frac{1}{r_n^{p'-1}} - \frac{1}{r_0^{p'-1}} \geq \sum_{i=0}^{n-1} \frac{1}{r_{n+1}^{p'-1}} - \frac{1}{r_i^{p'-1}} \geq n(p'-1)c_0
\]

and consequently,

\[
r_n^{p'-1} \leq \left( r_0^{1-p'} + c_0(p'-1)n \right)^{-1} \Rightarrow r_n \leq Cn^{1-p}
\]

since \( 1/(1-p') = 1 - p \).

This result immediately gives us weak subsequential convergence to some minimizer.

Proposition 5. In the situation of lemma 2, the sequence \( \{u^n\} \) possesses at least one weak accumulation point. Each weak accumulation point is a solution of (1). In particular, if the minimizer \( u^* \) is unique, then \( u^n \rightharpoonup u^* \).

Proof. Since \( r_n \leq n^{1-p} \), the sequence is a minimizing sequence, thus, due to the weak lower semi-continuity of \( F + \Phi \), each weakly convergent subsequence is a minimizer. Moreover, it also follows that \( \{u^n\} \) is a bounded sequence in the reflexive Banach space \( X \), meaning that there is a weakly-convergent subsequence. The statement \( u^n \rightharpoonup u^* \) in case of uniqueness follows by the usual subsequence argument.

To establish strong convergence, one has to examine the functionals \( F \) and \( \Phi \) more closely. One approach is to consider the following Bregman-like distance of \( \Phi \) in solution \( u^* \) of (1):

\[
R(u) = \Phi(u) - \Phi(u^*) + \langle F(u^*), u - u^* \rangle
\]

which is non-negative since \( -F(u^*) \in \partial \Phi(u^*) \). Note that if \( \partial \Phi(u^*) \) is consisting of one point, \( R \) is indeed the Bregman distance. By optimality of \( u^* \) and the subgradient inequality, we have for the iterates \( u^n \) that

\[
r_n \geq \Phi(u^n) - \Phi(u^*) + \langle F(u^*), u^n - u^* \rangle = R(u^n)
\]

hence \( R(u^n) \leq Cn^{1-p} \) by proposition 4. Also note that \( R(u) = 0 \) for optimal \( u \). The usual way to achieve convergence is to postulate decay behavior for \( R \) as the argument approaches \( u^* \).

Definition 1. Let \( \Phi : X \to \mathbb{R} \cup \{\infty\} \) be proper, convex and lower semi-continuous. The functional \( \Phi \) is called totally convex in \( u^* \in X \), if, for each \( w \in \partial \Phi(u^*) \) and \( \{u^n\} \) it holds that

\[
\Phi(u^n) - \Phi(u^*) - \langle w, u^n - u^* \rangle \to 0 \quad \Rightarrow \quad \|u^n - u^*\|_X \to 0.
\]
Likewise, $\Phi$ is convex of power-type $q$ (or $q$-convex) in $u^* \in X$ with $q \in [2, \infty]$, if for all $M > 0$ and $w \in \partial \Phi(u^*)$ there exists a $c > 0$ such that for all $\|u - u^*\|_X \leq M$ we have
\[
\Phi(u) - \Phi(u^*) - \langle w, u - u^* \rangle \geq c \|u - u^*\|_X^q.
\]

The notion of total convexity of functionals is well known in the literature [7, 8], convexity of power type $q$ is also referred to as $q$-uniform convexity [3]. The former term is, however, often used in conjunction with norms of Banach spaces for which an equivalent definition in terms of the modulus of convexity (or rotundity) is used [32].

Now, if $\Phi$ is totally convex in $u^*$, then the sequence $\{u^n\}$ also converges strongly to the minimizer since $R(u^n) \to 0$. Additionally, the minimizer has to be unique since $\|u^{**} - u^*\| > 0$ and $R(u^{**}) = 0$ would violate the total convexity property. The latter considerations prove:

**Theorem 1.** If $\Phi$ is totally convex, then $\{u^n\}$ converges to the unique minimizer $u^*$ in the strong sense.

**Remark 2.** The notion of total convexity is well known in the study of convergence of numerical algorithms and can be established for a variety of functionals of interest, for instance, for $\Phi(u) = \|u\|_Y^p$ if $Y$ is (locally) uniformly convex and $r > 1$ [9]. If $Y$ is continuously embedded in $X$, then $\|u^n - u^*\|_X \leq C \|u^n - u^*\|_Y$, thus $R(u^n) \to 0$ implies $\|u^n - u^*\|_Y \to 0$ and consequently $\|u^n - u^*\|_X \to 0$.

As one can easily see, the notion of $q$-convexity of $\Phi$ in $u^*$ is useful to obtain bounds for the speed of convergence: additionally to strong convergence ($q$-convex implies totally convex), since $\{u^n\}$ is bounded, one can choose a bounded neighborhood in which the sequence is contained and obtains
\[
\|u^n - u^*\|_X \leq (c^{-1} R(u^n))^{1/q} \leq (c^{-1} r_n)^{1/q} \leq c^{-1/q} C^{1/q} n^{1-p/q}
\]
for all $n$ meaning that $u^n \to u^*$ with asymptotic rate $n^{1-p/q}$. So, we can note:

**Theorem 2.** If $\Phi$ is $q$-convex, then $\{u^n\}$ converges to the unique minimizer $u^*$ with asymptotic rate $n^{1-p/q}$.

**Remark 3.** Again, a variety of functionals is $q$-convex, typically for $q \geq 2$. The notion translates to norms as follows: if $Y$ is a Banach space which is convex of power-type $q$ for $q > 1$ (see [23] for an introduction to this notion) which is continuously embedded in $X$, then $\Phi(u) = \|u\|_Y^p$ is $q$-convex. Consequently, one obtains convergence with rate $n^{1-p/q}$.

Note that many known spaces are $q$-convex. For instance, if $r > 1$, each $L^r(\Omega)$ is convex of power-type $q = \max\{2, r\}$ for arbitrary measure spaces and the respective constants are known, see [19, 25]. The analog applies to Sobolev spaces $H^{m,r}(\Omega)$ associated with arbitrary domains $\Omega$: They are also convex of power-type $\max\{2, r\}$. In fact, as a consequence of a theorem of Dvoretzky [16], any Banach space can be at most convex of power-type $2$ [23].

While the Bregman distance (or the Bregman-like distance $R$) turns out to be successful in proving convergence for the minimization algorithm for many functionals, there are some cases, in which $\Phi$ fails to be $q$-convex or totally convex. In these situations, one can also take the Taylor distance of $F$ into account, which is the remainder of the Taylor expansion of $F$ up to order 1:
\[
T(v) = F(v) - F(u^*) - \langle F'(u^*), v - u^* \rangle.
\]
Since $F$ is convex, $T(v) \geq 0$, hence one can indeed speak of some distance (which is in general not symmetric or satisfying the triangle inequality). In fact, the Taylor distance is also a Bregman distance, but here we make the distinction in order to emphasize that the Taylor and Bregman distances are the respective smooth and non-smooth concepts for measuring distances by means of functionals.

With the introduction of $R$ and $T$, the distance of $(F + \Phi)(v)$ to its minimizer can be expressed as

$$ (F + \Phi)(v) - (F + \Phi)(u^*) = T(v) + R(v), \quad (16) $$

which means that it can be split into a Bregman part (with respect to $\Phi$) and a Taylor part (with respect to $F$). In some situations, this splitting can be useful, especially for minimizing functionals of the Tikhonov-type, as the following section shows.

4. Application to Tikhonov functionals

Consider the general problem of minimizing a typical Tikhonov-type functional for a linear inverse problem,

$$ \min_{u \in X} \| Ku - f \|_Y^r + \alpha |u|_Z^s \quad (17) $$

where $r > 1, s \geq 1, \alpha > 0, X$ is a reflexive Banach space, $Y$ is a Banach space with $K : X \to Y$ continuously and some given data $f \in Y$. Finally, let $|\cdot|_Z$ be a semi-norm of the (not necessarily reflexive) Banach space $Z$ which is continuously embedded in $X$. It becomes a minimization problem of type (1) with

$$ F(u) = \| Ku - f \|_Y^r, \quad \Phi(u) = \frac{\alpha |u|_Z^s}{s} \quad (18) $$

with the usual extension $\Phi(u) = \infty$ whenever $u \in X \setminus Z$. The aim of this section is to analyze (17) with respect to the convergence of the forward–backward splitting algorithm applied to (18).

First, we focus on the minimization problem and specify which semi-norms we want to allow as regularization functionals. Also, we need a condition on the linear operator $K$.

**Condition 1.**

1. Let $|\cdot|_Z$ be a semi-norm on the Banach space $Z$ such that $Z = Z_0 \oplus Z_1$ with $Z_0 = \{ z \in Z : |z|_Z = 0 \}$ closed in $X$ and $|\cdot|_Z$ being equivalent to $\|\cdot\|_Z$ on $Z_1$.  
2. Let $K : X \to Y$ be a linear and continuous operator such that $K$, restricted to $Z_0 \subset X$ is continuously invertible.

**Remark 4.** The first condition in condition 1 is satisfied for many semi-norms used in practice. For example, consider $X = L^1(\Omega)$ and $Z = H^{1,1}(\Omega)$ with $|z|_Z = \|\nabla z\|_r$. Assume that $\Omega$ is a bounded domain, then $Z_0 = \text{span}\{\chi_\Omega\}$ is closed in $X$ and $Z_1 = \{ z \in H^{1,1}(\Omega) : \int_\Omega z \, dx = 0 \}$, for instance, gives $Z = Z_0 \oplus Z_1$. Provided that the Poincaré–Wirtinger inequality holds (with constant $C$), we have for $z \in Z_1$

$$ \|z\|_r + \|\nabla z\|_r \leq (C + 1) \|\nabla z\|_r \leq (C + 1)(\|z\|_r + \|\nabla z\|_r) $$

meaning that $|\cdot|_Z$ and $\|\cdot\|_Z$ are indeed equivalent on $Z_1$.

Moreover, note that by considering $X/(\ker(K) \cap Z_0)$ and $Z/(\ker(K) \cap Z_0)$ instead of $X$ and $Z$, respectively, the second condition in condition 1 is satisfied whenever the range of $K$ restricted to $Z_0$ is closed in $Y$ (by the open mapping theorem). This is in particular the case when $Z_0$ is finite dimensional.
Let us briefly obtain the existence of minimizers under the above conditions.

**Proposition 6.** Under the assumption that condition 1 is satisfied, the minimization problem (17) possesses at least one solution in $X$.

**Proof.** First, note that since $Z = Z_0 \oplus Z_1$ with $Z_0$ and $Z_1$ closed, there is, due to the closed-graph theorem, a continuous projection $P : Z \to Z$ such that $\text{rg}(P) = Z_1$ and $\text{ker}(P) = Z_0$. With this projection, $\|P\|_Z$ is equivalent to $|\cdot|_Z$ (see also [22] for a similar situation).

Verify in the following that $F + \Phi$ is coercive in $X$. Suppose that $\Phi(u^n) \leq C_1$ for some sequence $\{u^n\} \subset X$. In particular, $\{u^n\} \subset Z$ by the definition of $\Phi$ and $\|Pu^n\|_X \leq C_2\|Pu^n\|_Z \leq C_3|u^n|_Z \leq C_4$ meaning that $Pu^n$ makes sense and is bounded in $X$ whenever $\Phi(u^n)$ is bounded. Likewise, examining $F$, we get with $Q = I - P : Z \to Z_0$ that

$$\frac{(rF(u^n))^2}{2} = \frac{\|Ku - f\|^2_Y}{2} \geq \frac{1}{2}\|KPu - f\|_Y^2 - \|KQu\|_Y^2$$

Now suppose there is a sequence $\{u^n\}$ where $F + \Phi$ is bounded. The claim is that $\|KQu^n\|_Y$ is also bounded. Assume the opposite, i.e. that $\|KQu^n\|_Y \to \infty$. From the above argumentation we have $\|Pu^n\|_X \leq C_4$, hence $\|KPu^n - f\|_Y \leq C_3$. The estimate on $F$ then reads as

$$\frac{1}{2}(rF(u^n))^2/2 \geq -C_5\|KQu^n\|_Y + \frac{1}{2}\|KQu^n\|_Y^2$$

implying that $F(u^n) \to \infty$, which is a contradiction.

Consequently, there holds $\|KQu^n\|_Y \leq C_5$. By condition 1, $K$ is continuously invertible on $Z_0 \subset X$, so finally

$$\|a^n\|_X \leq \|KQu^n\|_Y + \|Pu^n\|_X \leq C_7\|KQu^n\|_Y + C_4 \leq C_8$$

showing that $F + \Phi$ is coercive.

Consequently, $F + \Phi$ is proper, convex, lower semi-continuous and coercive on the reflexive Banach space $X$, so at least one minimizer exists. \qed

Next, we turn to examining the problem more closely and some properties of the functionals in (17). Both $F$ and $\Phi$ are convex, proper and lower semi-continuous. The differentiability of $F$, however, is strongly connected with the differentiability of the norm in $Y$, since $K$ is arbitrarily smooth. Since the forward–backward splitting algorithm also demands some local Hölder continuity for $F'$, we have to impose conditions. The following notion is known to be directly connected with the differentiability of the norm and the continuity of the derivatives.

**Definition 2.** A Banach space $Y$ is called smooth of power-type $p$ (or $p$-smooth) with $p \in [1, 2]$, if there exists a constant $C > 0$ such that for all $u, v \in Y$ it holds that

$$\frac{\|v\|_Y^p}{p} - \frac{\|u\|_Y^p}{p} - \langle j_p(u), v - u \rangle \leq C\|v - u\|_Y^p.$$

Here, $j_p = \partial\|\cdot\|_Y^p / p$ denotes the $p$-duality mapping between $Y$ and $Y^*$. Now the central result which connects $p$-smoothness with differentiability of the norm is the following (see [33]):
derivative is given by

\[ A \text{Proposition 7. If } Y \text{ is smooth of power-type } p, \text{ then } \| \cdot \|_Y^p \text{ is continuously differentiable with derivative } j_Y \text{ which is moreover } (p - 1) \text{ Hölder continuous.} \]

Furthermore, for \( r \geq p \), the functional \( \| \cdot \|_Y^p/r \) is continuously differentiable. Its derivative is given by \( j_r \), which is still \((p-1)\) Hölder continuous on each bounded subset of \( Y \).

Assuming \( r \geq p \) and that \( Y \) is \( p \)-smooth gives us, together with the usual differentiation rules, that \( F \) is differentiable with derivative

\[ F'(u) = K^* j_r(Ku - f). \]

Since \( K \) is continuous, \( j_r \) is Hölder continuous of order \( p - 1 \) on bounded sets by proposition 7 and keeping in mind that \( \| j_r \|_{p-1} \) might vary on bounded sets, we can estimate the norm \( \| F' \|_{p-1} \) on bounded sets as follows:

\[ \| F' \|_{p-1} \leq \| K^* \| \| j_r \|_{p-1} \| K \|^{p-1} \leq \| j_r \|_{p-1} \| K \|^p. \]

Hence, \( F' \) possesses the Hölder continuity required by the convergence results for the forward–backward splitting algorithm.

As already noted in theorems 1 and 2, convexity of the penalty can lead to convergence with some rate. Here, we also want to extend the results to certain convexity of semi-norms which we define as follows:

\textbf{Definition 3.} Let \( |\cdot|_Z \) be a semi-norm according to condition 1. Then, the functional \( \Phi = \alpha|\cdot|_Z^q/s \) is called totally convex, if, for each fixed \( z^* \in Z \), each sequence \( \{z^n\} \) in \( Z \) and \( \xi \in \partial \Phi(z^*) \) there holds

\[ \alpha \frac{|z^n|_Z^q}{s} - \alpha \frac{|z^*_n|^q}{s} - (\xi, z^n - z^*) \to 0 \quad \Rightarrow \quad |z^n - z^*|_Z \to 0. \]

Likewise, \( \Phi = \alpha|\cdot|_Z^q/q \) is called convex of power-type \( q \) in \( z^* \) if, for each \( M > 0 \) and \( \xi \in \partial \Phi(z^*) \) there exists a \( c > 0 \) such that for all \( |z - z^*|_Z \leq M \) the estimate

\[ \alpha \frac{|z|_Z^q}{q} - \alpha \frac{|z^*_n|^q}{q} - (\xi, z - z^*) \geq c |z - z^*|^q_1 \]

is satisfied.

\textbf{Remark 5.} Analogously to proposition 7, one knows that if a \( |\cdot|_Z^q/q \) is convex of power-type \( q \) on bounded sets for some \( q \geq 2 \), then the functional \( \Phi = |\cdot|_Z^q/s \) is also convex of power-type \( q \) for all \( 1 < s \leq q \) on bounded sets. The same holds true for the convexity of power-type for norms according to definition 1.

With this notion, one is able to give convergence statements and rates for the algorithm applied to the minimization problem (17).

\textbf{Theorem 3.} If, in the situation of proposition 6, the space \( Y \) is smooth of power-type \( p \) for \( p \leq r \) and the functional \( \Phi = \alpha|\cdot|_Z^q/s \) as well as the norm in \( Y \) is totally convex, then \( \{u^n\} \) converges to the unique minimizer \( u^* \). If, moreover, \( |\cdot|_Y^p/q \) as well as \( Y \) is convex of power-type \( q \) for \( q \geq \max(r, s, 2) \) in a minimizer \( u^* \), then \( u^n \to u^* \) in \( X \) with rate \( O(\alpha(1-p)/q) \).

\textbf{Proof.} First verify that the sequence \( \{u^n\} \) is a minimizing sequence for which the associated \( r_n \) vanish like \( n^{1-p} \) which means verifying the prerequisites of proposition 4. Both \( F \) and \( \Phi \) are proper, convex and lower semi-continuous in the reflexive Banach space \( X \), \( F' \) is \((p-1)\) Hölder continuous on each bounded set by proposition 7 and we already saw in proposition 6 that \( F + \Phi \) is coercive. Hence, proposition 4 is applicable. It remains to show the asserted strong convergence. As already mentioned at the end of section 3, the
Bregman distance is no longer sufficient in order to show convergence, so we have to use Taylor–Bregman-distance estimates.

For that purpose, consider the Bregman-like distance \( R(u^n) \) according to (14) and Taylor distance \( T \) introduced in (15) (both with respect to \( F \) and \( \Phi \) as chosen in (18) and in the minimizer \( u^* \)). Remember that we can split the functional distance \( r_n \) according to (12) into Bregman and Taylor parts (16), i.e. \( r_n = R(u^n) + T(u^n) \).

First suppose that \( |·|_Z \) is totally convex, meaning that from \( r_n \geq R(u^n) = \alpha \frac{|u^n|_Z^2}{s} - \alpha \frac{|u^*|_Z^2}{s} + \langle F'(u^*), u^n - u^* \rangle \) and \( r_n \leq C_1 n^{1-p} \) (see proposition 4) follows \( |u^n - u^*|_Z \to 0 \). On the other hand, observe analogously that

\[
\begin{align*}
\|P(u^n)\|_Y &\geq \frac{\|Ku^n - f\|_Y}{r} - \frac{\|Ku^* - f\|_Y}{r} - \langle f, (Ku^n - f) - (Ku^* - f) \rangle \\
\|Q(u^n)\|_Y &\geq \frac{\|Ku^n - f\|_Y}{2r} - \frac{\|Ku^* - f\|_Y}{2r} - \langle f, (Ku^n - f) - (Ku^* - f) \rangle
\end{align*}
\]

so the total convexity of \( \|·\|_Y \) implies \( \|K(u^n - u^*)\|_Y \to 0 \). Note that since \( |u^n| \) is a minimizing sequence, we can reuse the arguments from proposition 6 as well as the projections \( P \) and \( Q \) to obtain that \( |u^n| \subset Z \) and \( \|P(u^n - u^*)\|_Z \leq C_3 \). Thus, we consider \( u \in Z \) in the following and are eventually able to set \( u = u^n - u^* \). It holds that \( \text{rg}(Q) = Z_0 \), hence exploiting again the second part of condition 1 gives

\[
\|KQu\|_Y \geq c_1 \|Qu\|_Y.
\]

Using the well-known convexity estimate for \( \frac{1}{q} |·|^q \) yields

\[
\frac{|Ku^n|_Y^q}{q} \geq \frac{1}{q} \|KQu\|_Y - \frac{\|KQu\|_Y^q}{q} \geq \frac{\|KQu\|_Y^q}{q} - \frac{\|KQu\|_Y^{q-1} \|KPu\|_Y + \frac{2^{2-q}}{q} \|KPu\|_Y^q}{q} \geq \frac{\|KQu\|_Y^q}{2q} - \frac{(2(q - 1))^{q-1} - (2^{-q})}{q} \|KPu\|_Y^q
\]

where, for the latter, the inequality \( a^{q-1} b \leq (q')^{-1} \epsilon^q a^q + q^{-1} \epsilon^{-q} b^q \) for \( a, b \geq 0 \) has been utilized.

Together with the estimate on \( \|KQu\|_Y \) and the continuity of \( K \), one gets

\[
\|Qu\|_X \leq c_1^{-1} \|KQu\|_Y \leq c_3 (\|KPu\|_Y + \|Ku\|_Y) \leq c_3 (C_1 \|Qu\|_Y + \|Ku\|_Y)
\]

and consequently

\[
|u^n - u^*|_X \leq 2^{q-1} (\|P(u^n - u^*)\|_Y + \|Q(u^n - u^*)\|_Y) \leq C_3 (|u^n - u^*|_Y + \|K(u^n - u^*)\|_Y).
\]

(19) Since \( R(u^n) \to 0 \) and \( T(u^n) \to 0 \) imply \( |u^n - u^*|_Z \to 0 \) as well as \( \|K(u^n - u^*)\|_Y \to 0 \), respectively, we have convergence \( u^n \to u^* \) in \( X \).

Regarding the uniqueness, assume that \( u^* \) is also a minimizer, hence \( R(u^*) = T(u^*) = 0 \) and \( u^* \in Z \). The total convexity then yields \( |u^* - u^*|_Z = 0 \Rightarrow \|P(u^* - u^*)\|_Z = 0 \) as well as \( \|K(u^* - u^*)\|_Y = 0 \). From the latter follows \( \|Q(u^* - u^*)\|_Z = 0 \) and consequently the uniqueness statement \( \|u^* - u^*\|_Z = 0 \).

In case \( |·|_Y^q/q \) is \( q \)-convex, we can write, having remark 5 in mind,

\[
|u^n - u^*|_Y^q \leq c_2^{-1} \left( \alpha \frac{|u^n|_Y^2}{s} - \alpha \frac{|u^*|_Y^2}{s} + \langle F'(u^*), u^n - u^* \rangle \right) = C_9 R(u^n).
\]

(20)
where $c_2$ depends on the $X$-norm bound of the sequence $\{u^n\}$. On the other hand, if $Y$ is $q$-convex, the Taylor distance can be estimated as follows:

$$
T(u^n) = \frac{\| Ku^n - f \|_Y^q}{r} \leq \frac{\| Ku^n - f \|_Y^q}{r} - (j_r(Ku^n - f), (Ku^n - f) - (Ku^* - f)) \\
\geq c_3 \| Ku^n - u^* \|_Y^q.
$$

(21)

Hence, (19) together with (20) and (21) becomes, because of (16),

$$
\| u^n - u^* \|_X^q \leq C_0 (C_0 T(u^n) + c_3^{-1} T(u^n)) \leq C_{10} r_n.
$$

But $r_n \leq C_1 n^{-1-p}$, hence $u^n \to u^*$ with rate $O(n^{1-p/q})$. \hfill \Box

5. Examples and applications

This section demonstrates some applications for the iterative minimization procedure discussed in this paper.

We start with an example which is of rather general nature. It shows that the forward–backward splitting procedure amounts to an iterative thresholding-like algorithm when applied to linear inverse problems with sparsity constraints. Afterward, numerical computations showing the performance of this algorithm in the discrete case are presented.

Example 1. Consider a linear inverse problem with sparsity constraints in Banach spaces. The setting can be summarized as follows: we are given a linear and continuous operator $K : \ell^p \to Y$ where $r > 1$, $Y$ is a $p$-smooth and $q$-convex Banach space with $p \leq r \leq q$, $q \geq 2$ and some data $f \in Y$ and want to solve the problem of minimizing the Tikhonov functional

$$
\min_{u \in \ell^p} \frac{\| Ku - f \|_Y^q}{r} + \sum_{k=1}^{\infty} \alpha_k |u_k|^s.
$$

(22)

where $\{\alpha_k\}$ is a sequence bounded by $0 < \underline{a} \leq \alpha_k \leq \overline{a} < \infty$ and $1 \leq s \leq r$.

Such situations occur, for example, when one tries to solve $Au = f$ for some linear and continuous $A : X \to Y$, $X$ being an $L^r(\mathbb{R}^d)$ or a Besov space $B_{\sigma,r}^s(\mathbb{R}^d)$ which is equivalently described using a properly rescaled basis synthesis operator $B : \ell^p \to X$ with respect to appropriate scaling functions/wavelets [26]. Utilizing the basis-expansion coefficients for regularization (giving the $s$th power of a norm which is equivalent to some $B_{\sigma, \partial}^p, s(\mathbb{R}^d)$) and denoting $K = AB$ then leads to Tikhonov functionals of the type (22), see [14] for details.

How does the associated forward–backward splitting algorithm look like? First, we assume that the duality map $j_r$ in $Y$ can be computed. This is for example the case for $Y = L^p(\mathbb{R}^d)$:

$$
j_r(u) = \text{sgn}(u) |u|^{p-1} \|u\|_{L^p}^{-p-1}.
$$

Note that $L^p(\mathbb{R}^d)$ is also smooth of power-type $p \leq \min(2, p^*)$ and one can, without greater effort, compute estimates for the H"older-constants $\|j_r\|_{p-1}$ on bounded sets provided that $p \leq r$. Consequently, it is reasonable to assume that $F(u) = K^* j_r (Ku - f)$ is computationally accessible as well as the constants needed for (2).

The main difficulty is therefore computing solutions of (3) which reads as, denoting $w^n = K^* j_r (Ku^n - f)$,

$$
\min_{v \in \ell^p} \sum_{k=1}^{\infty} r \| v - u^n \|_{\ell^p}^{p-r} \|v_k - u^n_k\|^r_\ell + \tau_n \left( w^n_k v_k + \alpha_k \frac{|v_k|^s}{s} \right).
$$

(23)
It is easy to see that the only coupling between different \( k \) is via the factor \( z = \frac{p}{r} \| v - u^p \|_r^{−p} \) (if \( p \neq r \)), so one can take the minimizers of

\[
\Psi_{\sigma,t}(x) = \frac{|x - y|^r}{r} + \sigma x + t \frac{|x|^p}{s}
\]
as ansatz with \( \sigma = z\alpha u^p \) and \( t = z\tau\alpha u^p \). Since \( \sigma \) enters only linearly in \( \Psi_{\sigma,t}(x) \), it is convenient to write \( \operatorname{argmin}_{x \in \mathbb{R}} \Psi_{\sigma,t}(x) = (\partial\Psi_{\sigma,t})^{-1}(-\sigma) \) with \( \Psi_{\sigma,t} = \frac{1}{l}|x - y|^r + t \frac{|x|^p}{s} \). The latter can be expressed as \((\partial\Psi_{\sigma,t})^{-1} = S_{\sigma,t}\). \n
\[
S_{\sigma,t}(x) = \begin{cases} (\text{sgn}(-y)|\cdot y|^{r-1} + t \text{sgn}(\cdot)|\cdot^{r-1})^{-1}(x) & \text{for } s > 1 \\ y + \text{sgn}(\partial\Psi_{\sigma,t}(x)/\partial x)|\partial\Psi_{\sigma,t}(x)|^{1/(r-1)} & \text{for } s = 1 \end{cases}
\]

and

\[
T_{\sigma,t}(x) = \begin{cases} x + t & \text{for } x \leq -\text{sgn}(y)|y|^{r-1} - t \\ x - t & \text{for } x \geq -\text{sgn}(y)|y|^{r-1} + t \\ -\text{sgn}(y)|y|^{r-1} & \text{else.} \end{cases}
\]

It is notable that for \( s = 1 \), each \( S_{\sigma,t} \) is a thresholding-like function (see figure 1). For \( r = 2 \), one can easily verify the identity \( S_{\sigma,t}(x) = \text{sgn}(x+y)(|x+y| - t) \), meaning that \( S_{\sigma,t} \) is the well-known soft-thresholding. In order to get a solution for (23), one introduces the operator which applies \( S_{\sigma,t} \) pointwise, i.e. \( S_{\alpha,s}(w) = S_{\alpha,s}(w) \) such that optimality is achieved if and only if \( \| v - u^p \|_r^{−p} \).

For \( r = p \) follows \( z = 1 \), so one can express the solution explicitly, for the other cases, it is necessary to compute the value for \( z \) numerically.

To introduce some simplification, note that for \( u^0 = 0 \) follows \( \| u \|_r \leq \| u \|_s \leq \alpha^{-1}(\Phi(u) \leq (F + \Phi)(u^0)) \) yielding \( \| u \|_r \leq \left( \| f \|_s \sqrt{\alpha/(\Phi r)} \right)^{1/\alpha} \). The forward–backward splitting algorithm for the solution of (22) then reads as follows.

1. Initialize with \( n = 0, u^0 = 0 \) and choose \( \| F \|_{p^{-1}} \) locally for \( \| u \|_r \leq \left( \| f \|_s \sqrt{\alpha/(\Phi r)} \right)^{1/\alpha} \).
2. Compute the value \( u^h = K^* j_r(Ku^h - f) \) and solve, for a \( \tau \), satisfying (2), the scalar equation \( u^{n+1} = S_{\alpha,s}(\tau u^n) \), \( \tau = \frac{p}{r} \| u^{n+1} - u^p \|_r^{−p} \).
3. Take, if optimality is not reached, \( u^{n+1} \) as the next iterate and continue with step (2) and \( n := n + 1 \).
It is notable that for $s = 1$, the iterates are always sparse, as a consequence of the optimality condition for the auxiliary problem (23) and the fact that sequences in $\ell^q$ are always null sequences. Regarding the convergence of the algorithm, one easily verifies that theorem 3 is applicable for $s > 1$ since $\Phi$ is the $s$th power of an equivalent norm on $\ell^q$ which is moreover $q$-convex and $Y$ is, by assumption, $q$-convex (although this is actually not needed). It follows that in that case we have the convergence rate $O(n^{(1-p)/q})$.

For the case $s = 1$, $\Phi$ is not even strictly convex and things are a little bit more complicated. But basically, one only has to apply the tools already introduced in a slightly different manner. Also, the following condition turns out to be crucial for the argumentation: let us assume that $K$ possesses the finite basis injectivity property, that is for every finite index set $J \subset \mathbb{N}$ the restriction $K|_J$ is injective (meaning that from $Ku = 0$ and $u_k = 0$ for each $k \notin J$ follows $u = 0$). Under such an assumption, it is also possible to obtain a convergence rate of $O(n^{(1-p)/q})$ where $q$ is the convexity of the space $Y$.

This can be seen analogously to the proof of theorem 3 and also follows a similar line of argument in [5]. Observe that for an optimal $u^*$ it holds that $-F'(u^*) = -K^* j_r(Ku^* - f) \in \partial \Phi(u^*)$ or, somehow weaker,

$$u^* \text{ optimal } \Rightarrow \begin{cases} |K^* j_r(Ku^* - f)|_k \leq \alpha_k & \text{for } u_k^* = 0 \\ |K^* j_r(Ku^* - f)|_k = \alpha_k & \text{for } u_k^* \neq 0. \end{cases}$$

Let $u^*$ be a minimizer and denote by $J = \{k \in \mathbb{N} : |K^* j_r(Ku^* - f)|_k = \alpha_k\}$ which has to be a finite set since otherwise $K^* j_r(Ku^* - f) \notin \ell^q$ (remember the assumption $\alpha_k \geq \alpha > 0$). Likewise, there exists a $\rho < 1$ such that $|K^* j_r(Ku^* - f)|_k \leq \alpha_k \rho$ for all $k \notin J$. For convenience, denote by $P$ the continuous projection $(Pu)_k = u_k(\chi_{\{n,j\}})_k$, by $Q = I - P$ and define the semi-norms $|z|_1 = \|Pz\|_1$ in the space $Z = \ell^1$ as well as $|z|_q = \|Pz\|_q$ in $\ell^q$.

We derive an estimate which somehow states the $q$-convexity of $\Phi$ with respect to the semi-norm $|.|_q$. Observe that for $k \notin J$, there has to be $u_k^* = 0$, hence one can estimate, for all $\|v\|_1 \leq M$,

$$R(v) \geq \sum_{k \notin J} \alpha_k (|v_k| - |u^*_k|) + F'(u^*_k)(v_k - u_k^*) \geq \sum_{k \notin J} \alpha_k (|v_k| - |F'(u^*_k)||v_k|) \geq (1 - \rho) \sum_{k \notin J} \alpha_k |v_k| \geq (1 - \rho) c |v - u^*|_1.$$ 

Then one estimates $|v - u^*|_1 \geq |v - u^*|_q \geq M^{1-q}|v - u^*|^q_2$ which leads to

$$R(v) \geq c |v - u^*|^q_2$$  \hspace{1cm} (25)

for a $c > 0$ which only depends on $M$. On the other hand, observe that a variant of condition 1 is satisfied: $\ell^1 = Z = Z_0 \oplus Z_1$ with $Z_0 = \text{rg}(Q)$ finite dimensional and hence closed in $\ell^q$ and $|.|_q$ being exactly the norm on $Z_1 = \text{rg}(P)$. By the finite basis injectivity property, $K$ is injective on $\text{rg}(Q)$ and since the latter is finite dimensional, also continuously invertible. But, (25) and the latter are exactly what is utilized in the proof of theorem 3 to show the desired convergence rate. Hence, by repeating the arguments there, one obtains $\|u^* - u^*\|_q = O(n^{(1-p)/q})$.

**Example 2.** We like to present numerical computations for a variant of the algorithm developed in example 1. The problem we consider is inverting the integration operator on $[0, 1]$ in $L^p([0, 1]) \rightarrow L^q([0, 1])$ which is, for simplicity, discretized (with a delta-peak basis) and penalized with the discrete $L^1([0, 1])$-norm:

$$\min_{u \in L^1([0, 1])} \frac{\|Ku - f\|_p^p}{p} + \alpha \|u\|_1, \quad Ku(t) = \int_0^t u(s) \, ds.$$  \hspace{1cm} (26)

The forward–backward splitting algorithm applied to the discretized problem then is just the iterative thresholding-like procedure computed in example 1 restricted to finitely many
dimensions. Computations for $p = 1.5$ and $p = 2$ for noisy data have been performed, see figure 2. The regularization parameter has been tuned in order to yield approximately the same discrepancy in the respective norms. As one can see, choosing $p$ less than 2 may favor more sparsity: compared to $p = 1.5$, the solution for $p = 2$ has approximately 50% more non-zero elements.

Furthermore, as predicted by the theory, the numerical algorithm indeed converged with some rate, in practice, however, it turns out that the convergence is somewhat stable on the one hand but very slow on the other hand and many iterations are needed to achieve accurate results.

The following example focuses on presenting an application in which it is natural to consider the Banach-space setting and on showing that the forward–backward splitting procedure leads to a convergent algorithm.

**Example 3.** Consider the problem of restoring an image in higher dimensions with a total-variation penalty term:

$$
\min_{u \in L^p(\Omega)} \frac{\| Ku - f \|_p^p}{p} + \alpha \text{TV}(u),
$$

(27)
Figure 3. Numerical illustration of three-dimensional total-variation regularization for image deblurring. On the left-hand side, you can see, respectively, the 2D slices of the 3D dataset, while on the right-hand side, some isosurfaces of a cut of the data are depicted. The rows show, from top to bottom, the original artificially created data $u$, which have been blurred and disturbed with noise to form the data $f$ and the outcome of the iterative forward–backward splitting algorithm $u^*$ for the deblurring problem with total-variation penalization.

with $\Omega \subset \mathbb{R}^d$ being a bounded domain such that $\text{BV}(\Omega)$ is compactly embedded in appropriate Lebesgue spaces. Here, $K$ denotes a linear and continuous operator mapping $L^p(\Omega) \to L^p(\Omega)$ with $1 < p \leq d/(d - 1)$. This covers in particular the convolution with kernels which are only Radon measures, i.e. $Ku = u * k$ with $k \in \mathcal{M}(\mathbb{R}^d)$, such that the usually assumed continuity $L^p(\Omega) \to L^2(\Omega)$ does not necessarily hold [31], especially for $d \geq 3$. Moreover, in general, coercivity in $L^2(\Omega)$ fails and it is necessary to consider the Banach space setting. From [1] we know that (27) indeed admits a solution in $L^p(\Omega)$ under general conditions.

In the following, we focus on the problem of restoring a blurred three-dimensional image, i.e. $Ku = u * k$ for some non-negative point-spread function $k$ with $\|k\|_1 = 1$ and $d = 3$. Such a task arises, for example, in confocal microscopy [30]. In order to apply the algorithm, we have to solve auxiliary problems of the type

$$
\min_{v \in L^p(\Omega)} \frac{\|v - u\|_p^p}{p} + s(\langle w, v \rangle + \alpha \text{TV}(v))
$$

(28)
Figure 4. Reconstruction of noisy blurred three-dimensional microscopy data showing cortical
neurons in transgenic mice. Again, slices and isosurface representations of the true image \( u \),
the noisy data \( f \) and the solution of the TV-regularization problem \( u^* \) (from top to bottom) are
depicted. Here, the minimization algorithm is also able to remove the noise artifacts from the
data. However, a reduction of the contrast and some loss of detail can be observed as is typical for
total-variation based regularization. (Dataset from http://152.19.37.82/pages/datasample.php, see
also [18].)

with \( u \in L^p(\Omega) \) and \( w \in L^{p'}(\Omega) \) which resembles the total-variation denoising functional.
Following [10, 20], one approach to solve (28) is to consider the Fenchel-predual problem
which equivalently reads as

\[
\min_{\|z\|_\infty \leq \alpha, \|\text{div } z - s w\|_{p'} = 0 \text{ on } \partial \Omega} \frac{\|\text{div } z - s w\|_{p'}}{p'} + \langle \text{div } z, u \rangle.
\]

In the discrete setting, with an appropriate linear discrete divergence operator, the above
becomes a smooth minimization problem with convex constraints which can, for example, be
solved approximately by a gradient projection method [15] or a semi-smooth Newton method
[27]. Finally, once a solution \( z^* \) of the predual problem is known, a solution of (28) can be
obtained by the corresponding Fenchel optimality conditions which, in this case, lead to the identity \( v^* = u + j_p(\text{div} z^* - sw) \).

Hence, one can actually perform the forward–backward splitting procedure in practice. Regarding its convergence, we are only able to apply proposition 5 and get weak convergence of \( \{u^n\} \) to the minimizer \( u^* \) in \( L^p(\Omega) \) (note that the minimizer has to be unique since \( K \) is injective). But additionally, one easily deduces that \( \{BV(u^n)\} \) is also bounded which gives, by compactness, the strong convergence in case \( 1 < p < 3/2 \). Consequently, considering the inverse problem in Banach space yields a convergent algorithm for regularization with the total-variation semi-norm. The convergence, however, comes without an estimate for its speed.

Based on the arguments presented above, numerical computations have been carried out. You can see the outcome of the algorithm for some sample data in figures 3 and 4.

6. Summary and conclusions

The aim of this paper was to show that there is a meaningful generalization of the forward–backward splitting algorithm to general Banach spaces. The main idea was to write the forward–backward step as a minimization problem (4) and to generalize this problem to (3) which defines the iteration. Convergence of this procedure was achieved by proving a descent rate of \( n^{1-p} \) of the functional distance on the one hand and utilizing notions of convexity of the non-smooth functional \( \Phi \) to establish norm-convergence of rate \( n^{(1-p)/q} \). This rate is, however, rather slow in comparison to, e.g. linear convergence. But, we have convergence nevertheless, and in order to prove that the general procedure converges, it suffices to look at the functional for which the backward step is performed.

These abstract results were applied to the concrete setting of Tikhonov functionals in Banach space. The forward–backward splitting algorithm was applied to the computational minimization of functionals of Tikhonov-type with semi-norm regularization and, using Bregman–Taylor-distance estimates, convergence was proven provided the linear operator has a continuous inverse on the space where the semi-norm vanishes. In particular, convergence rates translated to the convexity of the regularizing semi-norm as well as to the smoothness and convexity of the underlying data space, the latter originating from the situation that semi-norms are in general invariant on whole subspaces.

As the examples showed, the results are applicable for Tikhonov functionals considered in practice. In particular, the algorithm can be used to deal with sparsity constraints in Banach space and to derive a convergent generalization of the popular iterative soft-thresholding procedure by Daubechies, Defrise and De Mol to Banach spaces. The resulting algorithm shares many properties: it is easy to implement, produces sparse iterates, but also converges very slowly, what was to expect since its prototype in Hilbert space also admits very slow convergence. Thus, there is the need to accelerate the procedure by, for example, utilizing better step-size rules or higher-order methods. Finally, the method also works for image restoration problems of dimension 3 which can, in general, not be solved in \( L^2(\Omega) \) anymore. Although the theory does not yield estimates for the convergence rate, we are still able to obtain strong convergence in some \( L^p(\Omega) \).

References

[1] Acar R and Vogel C 1994 Analysis of bounded variation penalty method for ill-posed problems Inverse Problems 10 1217–29
