Coherent state approach to quantum clock in a model where the Hamiltonian is a difference between two harmonic oscillators

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Abstract

In order to study the "problem of time", Rovelli proposed a model of a two harmonic oscillator system where one of the oscillators can be thought of as a 'clock' for the other oscillator. In this paper we examine a model where the Hamiltonian is a difference between two harmonic oscillators, and we consider one of them which has the minus sign as a 'clock'. Klauder's projection operator approach to generalized coherent states is used to define physical states and operators. The resolution of unity is derived in terms of a gauge invariant coordinate. We investigate the 'quantum clock' and show that the evolution described by it is identical to the classical motion when the energy becomes large.

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I. INTRODUCTION

One of the measure conceptual problems in quantum gravity is the "problem of time" [1]. In order to study it Rovelli proposed an interesting model of a two harmonic oscillator system where one of the oscillators can be thought of as a 'clock' for the other oscillator [2]. He showed that the 'clock' can describe a natural time evolution, even though the system has a time reparametrization invariance. In a similar model Lawrie and Epp studied an evolution which is governed by an exact Heisenberg equation, and they considered coherent states and introduced a window function to investigate an approximate analytical time dependence of the system [3]. Recently, Ashworth utilized Klauder’s projection operator approach to generalized coherent states [4] for the double harmonic oscillator system [5]. Using Marolf’s gauge invariant statement [6], he introduced 'time' by the phase of an oscillator, 'clock', and he showed that the time evolution described by the 'clock' agrees with the classical equation of motion when the energy becomes large.

On the other hand it is well known that the gravitational degree of freedom has a minus sign in the Hamiltonian of quantum cosmology [7]. The Hamiltonian can be written as a difference between two harmonic oscillators in some cases, for example, the five-dimensional Kaluza-Klein cosmology by Wudka [8] and the minisuperspace model by Hartle-Hawking [9] if time variable is redefined and the cosmological constant is assumed to be zero.

In this paper we examine a model where the Hamiltonian is a difference between two harmonic oscillators, and we consider one of them which has the minus sign in the Hamiltonian as a 'clock'. The projection operator approach to generalized coherent states is used to define physical states. We deduce a resolution
of unity with respect to gauge invariant states by virtue of a coordinate transformation. In the same way physical operators are expressed in terms of gauge invariant states and physical symbols. We investigate the ’quantum clock’ and show that the evolution described by it is identical to the classical motion when the energy becomes large.

In §2 we will consider a model where the Hamiltonian is a difference between two harmonic oscillators, and we will use the projection operator approach to generalized coherent states in order to obtain physical states. In §3 the resolution of unity will be derived in terms of a gauge invariant coordinate. In §4 we will project operators to the physical space, and we will define a ’quantum clock’ and show that the evolution described by it is same with the classical motion when the energy becomes large. We summarize in §5. Appendix A is devoted to derive the gauge transformation of our system. In Appendix B it will be shown that our result of the resolution of unity agrees with that in Ref. [10].

II. A MODEL OF TWO HARMONIC OSCILLATORS

Let us consider the following action which is a difference between two harmonic oscillators:

\[
S = \int dt L ,
\]

\[
L = \frac{1}{2\lambda} \left[ \left( \frac{dq_1}{dt} \right)^2 - \left( \frac{dq_2}{dt} \right)^2 \right] - \frac{\lambda}{2} \left[ \omega^2 (q_1^2 - q_2^2) - 2E \right] .
\]

The action (1) has the time reparametrization invariance, and the Hamiltonian reads

\[
H = \lambda (H_1 - H_2 - E) ,
\]
where \( H_i = \frac{1}{2}(p_i^2 + \omega^2 q_i^2) \), \((i = 1, 2)\). If we define the proper time, \( \tau = \int_0^t dt' \lambda(t') \), the classical equations of motion for \( q_1, q_2 \) are \( \ddot{q}_i = -\omega^2 q_i \) \((i = 1, 2)\), \( \ddot{q}_i = d^2q_i/d\tau^2 \). Therefore, \( q_1 \) and \( q_2 \) are ordinary harmonic oscillators with only one exceptional point that \( q_2 \) has a minus sign in the Hamiltonian (2).

We write the classical solution of this system as

\[
q_{cl}^1 = A \cos(\omega \tau + \phi_1), \quad q_{cl}^2 = B \cos(-\omega \tau + \phi_2),
\]

where we have assumed that the two harmonic oscillators have opposite dependence on the proper time. The reason of this assumption is because under the gauge transformation, that is the time translation, the phases of the two harmonic oscillators are transformed into opposite direction, which is discussed in Appendix A. Then the classical motion of each harmonic oscillator can be also written by another harmonic oscillator as

\[
q_{cl}^1 = A \cos\left(-\cos^{-1}\frac{q_{cl}^2}{B} + \phi_1 + \phi_2\right), \quad q_{cl}^2 = B \cos\left(-\cos^{-1}\frac{q_{cl}^1}{A} + \phi_1 + \phi_2\right).
\]

This expression shows that either \( q_{cl}^1 \) or \( q_{cl}^2 \) can be used for a classical clock for \( q_{cl}^1 \) or \( q_{cl}^2 \), respectively.

Since the action (1) has no time derivative of \( \lambda \), we get the constraint,

\[
H = 0, \quad \text{namely} \quad H_1 - H_2 = E,
\]

which corresponds to the time reparametrization invariance. The equations (3), (5) mean that the classical amplitudes of the oscillators must satisfy

\[
(A\omega)^2 - (B\omega)^2 = 2E.
\]

\(^1\)It was pointed out by Prof. T. Kubota that the phases of two harmonic oscillators are transformed into opposite direction under the gauge transformation.
The consideration in Appendix A suggests that the gauge transformation can be written as
\[ \lambda \rightarrow \lambda + \epsilon, \quad \phi_1 \rightarrow \phi_1 - \epsilon \omega t, \quad \phi_2 \rightarrow \phi_2 + \epsilon \omega t. \]

Hence in our case the summation of initial phases of the two harmonic oscillators \( \phi_1 + \phi_2 \) is gauge invariant.

To quantize this model, we impose the canonical commutation relations for Heisenberg operators, \( \hat{Q}_j, \hat{P}_k \),
\[
\begin{align*}
[ \hat{Q}_j, \hat{P}_k ] &= i \hbar \delta_{jk}, \\
[ \hat{Q}_j, \hat{Q}_k ] &= 0,
\end{align*}
\]
and\[
[ \hat{P}_j, \hat{P}_k ] = 0.
\]

Provided we define annihilation operators by
\[
a = \sqrt{\frac{\omega}{2\hbar}} \hat{Q}_1 + \frac{i}{\sqrt{2\hbar} \omega} \hat{P}_1, \quad b = \sqrt{\frac{\omega}{2\hbar}} \hat{Q}_2 + \frac{i}{\sqrt{2\hbar} \omega} \hat{P}_2,
\]
then we obtain \([a, a^\dagger] = [b, b^\dagger] = 1, \ [a, b] = [a^\dagger, b^\dagger] = 0\). From Eqs. (5) and (8) the constraint operator can be written as
\[
\hat{\Phi} = a^\dagger a - b^\dagger b - E',
\]
with \( E' = E/\hbar \omega \).

We start from the the coherent states for the two harmonic oscillators,
\[
| \alpha, \beta \rangle = e^{-(|\alpha|^2+|\beta|^2)/2} \sum_{n,m=0}^{\infty} \frac{\alpha^n \beta^m}{\sqrt{n!} \sqrt{m!}} |n, m\rangle,
\]
where \( |n, m\rangle = \frac{1}{\sqrt{n!} \sqrt{m!}} (a^\dagger)^n (b^\dagger)^m |0, 0\rangle \) and \( \alpha, \beta \) are arbitrary complex numbers [11]. These coherent states satisfy the properties:
\[
\begin{align*}
a | \alpha, \beta \rangle &= \alpha | \alpha, \beta \rangle, \quad b | \alpha, \beta \rangle = \beta | \alpha, \beta \rangle, \\
\langle \alpha, \beta | | \alpha, \beta \rangle &= 1, \quad \mathcal{I} = \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} | \alpha, \beta \rangle \langle \alpha, \beta |,\end{align*}
\]
with \( d^2 \alpha = d(\text{Re} \alpha) d(\text{Im} \alpha) \). Using Eqs. (8), (11), we obtain the diagonal element of \( \hat{Q}_1 \) and \( \hat{Q}_2 \) as

\[
q_1(\alpha, \beta) = \langle \alpha, \beta | \hat{Q}_1 | \alpha, \beta \rangle = \sqrt{\frac{\hbar}{2\omega}} \left( \alpha + \bar{\alpha} \right),
\]

\[
q_2(\alpha, \beta) = \langle \alpha, \beta | \hat{Q}_2 | \alpha, \beta \rangle = \sqrt{\frac{\hbar}{2\omega}} \left( \beta + \bar{\beta} \right). \tag{12}
\]

In the same way as Ref. [5], we utilize Klauder’s projection operator approach to generalized coherent states [4]. Projecting \( | \alpha, \beta \rangle \) on the physical states as

\[
| \alpha, \beta \rangle_{\text{phys}} = \mathcal{P} | \alpha, \beta \rangle, \quad \mathcal{P} = \int d\mu(\lambda) \ e^{-i\lambda \Phi}, \tag{13}
\]

we have

\[
| \alpha, \beta \rangle_{\text{phys}} = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)} \sum_{n,m=0}^{\infty} \frac{\alpha^n \beta^m}{\sqrt{n!} \sqrt{m!}} \int d\mu(\lambda) \ e^{-i\lambda(n-m-E')} |n,m\rangle,
\]

\[
= e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)} \sum_{m=0}^{\infty} \frac{\alpha^{m+m'} \beta^m}{\sqrt{(m+m')!} \sqrt{m!}} |m+m', m\rangle. \tag{14}
\]

Here we have set \( E' = m' = n - m \). The norms of these states are

\[
\langle \alpha, \beta | \alpha, \beta \rangle_{\text{phys}} = \langle \alpha, \beta | \mathcal{P} | \alpha, \beta \rangle,
\]

\[
= e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)} \sum_{m=0}^{\infty} \frac{|\alpha|^{2(m+m')} |\beta|^{2m}}{m!(m+m')!},
\]

\[
= e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)} \left| \frac{\alpha^{m'}}{\beta} \right| I_{m'}(2|\alpha|\beta), \tag{15}
\]

where we have used the formula [12],

\[
\sum_{n=0}^{\infty} \frac{\left( \frac{x}{2} \right)^{2n}}{n! (k+n)!} = \left( \frac{2}{x} \right)^k I_k(x), \tag{16}
\]

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and $I_k(x)$ is a modified Bessel function. Normalized physical states, $|\alpha, \beta\rangle_{n.phys}$ can be written as

$$
|\alpha, \beta\rangle_{n.phys} = |\alpha, \beta\rangle_{phys} / \sqrt{\langle \alpha, \beta | \alpha, \beta \rangle_{phys}},
$$

$$
= \frac{|\beta|^{m'/2} \alpha^{m'}}{\sqrt{I_{m'}(2|\alpha\beta|)}} \sum_{m=0}^{\infty} \frac{(\alpha\beta)^m}{\sqrt{(m + m')!m!}} |m + m', m\rangle. \quad (17)
$$

### III. RESOLUTION OF UNITY

As indicated in Appendix A the gauge transformation generated by the constraint transforms the complex coordinates as $\alpha \rightarrow \alpha e^{i\varphi}$ and $\beta \rightarrow \beta e^{-i\varphi}$. Whence, if we define a complex coordinate, $\xi = \alpha\beta$, then $\xi$ is gauge independent. Now $|\xi|$ is a product of the amplitudes of the two harmonic oscillators, and $\arg \xi$ is the sum of their phases. We will see that $\xi$ is sufficient to describe the resolution of unity in the physical space. Let us define the minus of the phase of the second harmonic oscillator $q_2$ that has the minus sign in the Hamiltonian (2) as $\theta$, that is $\beta = |\beta| e^{-i\theta} \quad (0 \leq \theta < 2\pi)$. In principle, any of the two oscillators could be used as a 'clock'. However, we will consider the classical limit as when the energy becomes large, namely the first oscillator $q_1$ becomes large. So we will later regard $q_2$ as a 'clock' and $\theta$ as 'time' in this system. We can factor out the dependence on $\theta$ from $|\alpha, \beta\rangle_{n.phys}$,

$$
|\alpha, \beta\rangle_{n.phys} = e^{im'\theta} |\xi\rangle, \quad (18)
$$

$$
|\xi\rangle = \frac{\xi^{m'}}{|\xi|^{m'/2} \sqrt{I_{m'}(|2\xi|)}} \sum_{m=0}^{\infty} \frac{\xi^m}{\sqrt{(m + m')!m!}} |m + m', m\rangle.
$$

The unity operator in the full phase space can be projected on the physical
\[ I' = \mathcal{P} \mathcal{I} \mathcal{P} = \int \frac{d^2 \alpha}{\pi} \frac{d^2 \beta}{\pi} \mathcal{P} | \alpha, \beta \rangle \langle \alpha, \beta | \mathcal{P} . \]  

Suppose we change the coordinates,

\[ \tilde{r} = |\alpha|^2 - |eta|^2, \]

\[ e^{-i\theta} = \frac{\beta}{|\beta|}, \]

\[ \xi = \alpha \beta, \]

we have

\[ \alpha = \sqrt{\frac{\tilde{r}_+}{2}} \frac{\xi}{|\xi|} e^{i\theta}, \quad \beta = \sqrt{-\frac{\tilde{r}_-}{2}} e^{-i\theta}, \]

where \( \tilde{r}_\pm = \tilde{r} \pm \sqrt{\tilde{r}^2 + 4|\xi|^2} \), \( \tilde{r}_+ \tilde{r}_- = -4|\xi|^2 \). The absolute value of the Jacobian \(|J|\) associated with this change of coordinates is calculated as

\[ |J| = \frac{1}{2\sqrt{\tilde{r}^2 + 4|\xi|^2}} = \frac{1}{2r}, \]

where we have defined \( r = |\alpha|^2 + |eta|^2 = \sqrt{\tilde{r}^2 + 4|\xi|^2}, \quad (r \geq 2|\xi|) \). Using Eqs. (15), (18)-(21), we deduce the resolution of unity,

\[ I' = \int \frac{d^2 \alpha}{\pi} \frac{d^2 \beta}{\pi} | \langle \alpha, \beta | \mathcal{P} | \alpha, \beta \rangle | \xi \rangle \langle \xi | , \]

\[ = \int d\tilde{r} d\theta d^2 \xi \frac{1}{2\pi^2 r} e^{-r} \left( \frac{\tilde{r}_+}{2|\xi|} \right)^{m'} I_{m'}(2|\xi|) | \xi \rangle \langle \xi | . \]

Because \( \tilde{r} = \pm \sqrt{r^2 - 4|\xi|^2} \quad (+ \text{for } |\alpha| \geq |\beta|), \quad - \text{for } |\alpha| < |\beta|) \), we are led to

\[ I' = \int \frac{d^2 \xi}{\pi} \frac{I_{m'}(2|\xi|)}{(2|\xi|)^{m'} f_{m'}(|\xi|)} | \xi \rangle \langle \xi | . \]

\[ ^2 \text{It is easier to consider the inverse change of coordinates and to derive } |J^{-1}| = 2r \text{ than to calculate } |J| \text{ directly, which was suggested by Prof. T. Kubota.} \]
\[ f_{m'}(|\xi|) = \int_{-\infty}^{\infty} d\tilde{r} \frac{e^{-r}}{r} (\tilde{r}_+)^{m'} = \int_{2|\xi|}^{\infty} dr \frac{e^{-r}}{\sqrt{r^2 - 4|\xi|^2}} \left[ r_{m'}^m + r_{m'}^{-m'} \right], \]
with \( r_{\pm} = r \pm \sqrt{r^2 - 4|\xi|^2} \). Owing to the following formula [13]:
\[
\int_a^{\infty} dx \frac{(x + \sqrt{x^2 - a^2})^\nu + (x - \sqrt{x^2 - a^2})^\nu}{\sqrt{x^2 - a^2}} e^{-px} = 2a^\nu K_\nu(ap) \quad (a > 0, \ Re \ p > 0),
\]
where \( K_\nu \) is a modified Bessel function, we can obtain the explicit expression of \( f_{m'}(|\xi|) \) as
\[
f_{m'}(|\xi|) = 2 (2|\xi|)^{m'} K_m(2|\xi|). \quad (23)
\]
Finally we can derive the resolution of unity,
\[
\mathcal{T}' = \frac{2}{\pi} \int d^2x I_{m'}(2|\xi|) K_{m'}(2|\xi|) |\xi \rangle \langle \xi |, \quad (24)
\]
from Eqs. (22), (23).

Now the constraint equations (6), (9) suggest that the underlying symmetry of our model is \( SU(1,1) \). As shown in Appendix B, it is possible to prove that our result (24) agrees with the equation (3.22) in Ref. [10] which is the resolution of unity for generalized coherent states associated with the Lie algebra of \( SU(1,1) \).

IV. PROJECTION OF OPERATORS AND QUANTUM CLOCK

According to Ref. [5], let us define a symbol for an arbitrary operator \( \tilde{O}(\hat{Q}, \hat{P}) \) on the physical space as
\[
o(q, p)|_{\text{phys}} = \frac{\langle q, p | \mathcal{P} \tilde{O}(\hat{Q}, \hat{P}) \mathcal{P} | q, p \rangle}{|\langle q, p | \mathcal{P} | q, p \rangle|}, \quad (25)
\]

and let us project $\hat{O}(\hat{Q}, \hat{P})$ to a well-defined operator on the physical states as

$$\hat{O}(\hat{Q}, \hat{P})|_{\text{phys}} = \int d\mu(q, p) \, o(q, p) \, \mathcal{P}| \, q, \, p \rangle \langle q, \, p | \mathcal{P} \, .$$  \hfill (26)

In the same way as the resolution of unity, we can rewrite this equation into the form,

$$\hat{O}(\hat{Q}, \hat{P})|_{\text{phys}} = \frac{2}{\pi} \int d^2 \xi \, K_{m'}(2|\xi|) \, o'(\xi) \, |\xi \rangle \langle \xi| \, ,$$  \hfill (27)

where $r$ and $\tilde{r}_+$ were defined in Eqs. (20), (21). Note that $o'(\xi)$ is the projected symbol and $o'(\xi) = 1$ when $o(\alpha, \beta) = 1$ .

Unless the symbol $o(\xi, \tilde{r}, \theta)$ changes very much with respect to $\tilde{r}$, the integrand of $o'(\xi)$ ($X$ below) approaches a Gaussian function around $\tilde{r} \approx m'$ , when the energy of the system $E' = m'$ becomes large:

$$X = \frac{e^{-\sqrt{\tilde{r}^2 + 4|\xi|^2}}}{2K_{m'}(2|\xi|) \sqrt{\tilde{r}^2 + 4|\xi|^2}} \left( \sqrt{\tilde{r}^2 + 4|\xi|^2} / 2|\xi| \right)^{m'} \, ,$$

$$\to \frac{1}{\sqrt{2\pi m'}} \exp \left[ -\frac{(\tilde{r} - m')^2}{2m'} \right] \, .$$  \hfill (28)

Here we have used the asymptotic form of $K_\nu$ in Ref. [14],

$$K_\nu(\nu z) \sim \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu \eta}}{(1 + z^2)^{1/4}} \, (\nu \to \infty) \, ,$$  \hfill (29)

$$\eta = \sqrt{1 + z^2} + \log \frac{z}{1 + \sqrt{1 + z^2}} \, ,$$

and we have assumed $\tilde{r} \gg |\xi| \, , m' \gg 1$ . Fig. 1 demonstrates the relation between $X$ and $\tilde{r}$ , when $m' = 10, 100, 1000$ and $|\xi| = 1$ . The limit (28) of
$X$ means that $\lim_{m' \to \infty} \int_{-\infty}^{\infty} d\tilde{r} X = 1$. Thus $X$ becomes a delta function $\delta(\tilde{r} - m')$ in the classical limit. This means that, when $m'$ is large, the projection of the symbol satisfies $o'(\xi) \approx \int_{0}^{2\pi} \frac{d\theta}{2\pi} o(\xi, m', \theta)$, and, if the symbol $o$ is gauge independent, namely $o$ does not depend on $\theta$, we have $o'(\xi) \approx o(\xi, m', \theta_0)$, where $\theta_0$ is an arbitrary constant ($0 \leq \theta_0 < 2\pi$).

Fig. 1

For example, let us take $\hat{Q}_1$ and $\hat{Q}_2$ for $\hat{O}(\hat{Q}, \hat{P})$, then we have

\[
q'_1(\xi) \propto \int_{0}^{2\pi} d\theta \cos(\phi_+ + \theta) = 0,
q'_2(\xi) \propto \int_{0}^{2\pi} d\theta \cos \theta = 0,
\]

where we have used Eqs. (12), (27) and have defined $\xi = |\xi| e^{i\phi_+}$. This result is rather natural, since the average positions of operators over one period of the oscillator are zero [5]. Note that the gauge transformation is 'time translation' in this system, we must choose a specific time to avoid this result.

Following Ashworth, we use Marolf's gauge invariant statement [6],

\[
o|_{q=s} = \int dt \frac{dq}{dt} \delta[q(t) - s] \circ(t).
\]

Let us consider the second oscillator $q_2$ which has the minus sign in the Hamiltonian as a 'clock', and let us regard the minus of its phase $\theta$ as 'time' in our system. So we take $q = q_2(\theta)$, $s = B \cos (\omega \tau - \phi_2) = q_2^{cl}$, and we obtain

\[
o|_{q_2=s} = \int d\theta \frac{dq_2}{d\theta} \delta[q_2(\theta) - B \cos (\omega \tau - \phi_2)] o(\theta),
= \int d\theta \delta[\theta - (\omega \tau - \phi_2)] o(\theta) = o(\omega \tau - \phi_2).
\]
This means that we can replace \( o(\xi, \tilde{r}, \theta) \) by \( o(\xi, \tilde{r}, \omega \tau - \phi_2) \), so Eq. (27) gives

\[
do'(\xi, s) = o'(\xi)|_{q_2=s},
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} d\tilde{r} \int_0^{2\pi} d\theta \frac{e^{-r}}{r} m'_{m' + 1/2} K_{m' + 1/2}(2|\xi|) o(\xi, \tilde{r}, \omega \tau - \phi_2) .
\]  

(31)

Choosing \( q_1 \) as \( o \), we obtain

\[
q'_1(\xi)|_{q_2=s} = \frac{1}{2} \int_{-\infty}^{\infty} d\tilde{r} \frac{e^{-r}}{r} m'_{m' + 1/2} K_{m' + 1/2}(2|\xi|) o(\xi, \tilde{r}, \omega \tau - \phi_2),
\]

where \( \phi_+ \) is the phase of \( \xi \), and \( m'_{m'+1/2}(|\xi|) \) is defined in Eqs. (22) with the replacement, \( m' \rightarrow m' + \frac{1}{2} \). Since Eq. (23) means that \( m'_{m'+1/2}(|\xi|) = 2 (2|\xi|)^{m'+1/2} K_{m'+1/2}(2|\xi|) \), we arrive at

\[
q'_1(\xi)|_{q_2=s} = \sqrt{\frac{\hbar}{\omega}} \sqrt{2|\xi|} \frac{K_{m'+1/2}(2|\xi|)}{K_{m'+1/2}(2|\xi|)} \cos (\omega \tau - \phi_2 + \phi_+) .
\]

(32)

In the classical limit \( E' = m' \rightarrow \infty \), the asymptotic form of the modified Bessel function (29) gives \( K_{m'+1/2}(2|\xi|)/K_{m'+1/2}(2|\xi|) \approx \sqrt{m'/|\xi|} \), and

\[
q'_1(\xi)|_{q_2=s} \approx \sqrt{\frac{2\hbar}{\omega}} \sqrt{m'} \cos (\omega \tau - \phi_2 + \phi_+) ,
\]

\[
\approx A \cos (\omega \tau - \phi_2 + \phi_+) .
\]

(33)

Here \( A \) is the amplitude of the first oscillator, and we have used \( \tilde{r} \approx m' \), \( \tilde{r} \gg |\xi| \).

Note that \( \xi \) is gauge invariant and its phase \( \phi_+ \) is the same as the initial phase sum \( \phi_1 + \phi_2 \). Hence the right-hand side of Eq. (33) is identical to the classical

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solution $q_i$ in Eqs. (3). Namely, the evolution of the first operator $q_1$ described by the ‘quantum clock’ $q_2$ is identical to the classical motion when the energy becomes large.

V. SUMMARY

We examined a model where the Hamiltonian is a difference between two harmonic oscillators, and we considered one of them which has the minus sign in the Hamiltonian as a ‘clock’. The projection operator approach to generalized coherent states was used to define physical states. We deduced a resolution of unity with respect to gauge invariant states. In the same way, physical operators were expressed in terms of gauge invariant states and physical symbols. We investigated the ‘quantum clock’ and showed that the evolution described by it is identical to the classical motion when the energy becomes large.

As a future work, it will be interesting to apply the projection operator approach to coherent states in order to study the time evolution of the five-dimensional Kaluza-Klein cosmology by Wudka [8] and the minisuperspace model by Hartle-Hawking [9] when the cosmological constant is zero.

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APPENDIX A

First we begin by a pair of creation and annihilation operators of a harmonic oscillator \( a^\dagger, a \) which satisfy \([a, a^\dagger] = 1\). Let us formally define the polar decomposition of \( a \) as

\[
a = e^{i\theta_a} |a| ,
\]

where \( |a| \) and \( \theta_a \) are the absolute value operator and the phase operator of \( a \), respectively [15]. Then the number operator \( N_a = a^\dagger a = |a|^2 \) satisfies

\[
[N_a, e^{i\theta_a}] = -e^{i\theta_a} .
\]

In the following expansion,

\[
e^{i\theta_a} N_a e^{-i\theta_a} = N_a + i[\theta_a, N_a] + \frac{i^2}{2!}[[\theta_a, [\theta_a, N_a]] + \cdots ,
\]

the left-hand side is equal to \( N_a + 1 \) by Eq. (A2), and the right-hand side becomes \( N_a + i[\theta_a, N_a] \), because \([\theta_a, N_a] \) is a c-number. Therefore we obtain

\[
[N_a, \theta_a] = i .
\]

Next let us consider another pair of creation and annihilation operators \( b^\dagger, b \) with \([b, b^\dagger] = 1\), then similar equations as Eqs. (A1)-(A3) hold with respect to \( b \). We examine two cases where the Hamiltonian is the sum or the difference of two harmonic oscillators.

Case 1: \( \hat{H}_+ = \hbar \omega (N_a + N_b - E') \)

Since Eq. (A3) means \([\hat{H}_+, \theta_a] = i\hbar \omega \) and \([\hat{H}_+, \theta_b] = i\hbar \omega \), we have

\[
[\hat{H}_+, \theta_a + \theta_b] = 2i\hbar \omega , \quad [\hat{H}_+, \theta_a - \theta_b] = 0 .
\]

Therefore the phase difference \( \theta_a - \theta_b \) is gauge invariant, and the phase sum \( \theta_a + \theta_b \) is not invariant in this case. This case was investigated in Ref. [5].
Case 2: $\hat{H}_- = \hbar \omega (N_a - N_b - E')$

Since Eq. (A3) means $[\hat{H}_-, \theta_a] = i\hbar \omega$ and $[\hat{H}_-, \theta_b] = -i\hbar \omega$, we have

$$[\hat{H}_-, \theta_a + \theta_b] = 0, \quad [\hat{H}_-, \theta_a - \theta_b] = 2i\hbar \omega . \quad (A5)$$

Hence the phase sum $\theta_a + \theta_b$ is gauge invariant, and the phase difference $\theta_a - \theta_b$ is not invariant in this case.

From Eq. (9), $\hat{H}_- = \hbar \omega \hat{\Phi}$, and $\hat{H}_-$ is the generator of the gauge transformation (time translation) of our system. So Eqs. (A5) suggest that the phase of each harmonic oscillator transforms into opposite direction under the gauge transformation.

**APPENDIX B**

Barut and Girardello [10] derived the resolution of unity for generalized coherent states associated with the Lie algebra of $SU(1,1)$ as

$$\mathcal{T}' = \int d\sigma(z) |z\rangle \langle z| , \quad (B1)$$

$$\sigma(z) = \sigma(\rho) \rho d\rho d\varphi , \quad \sigma(\rho) = \frac{4}{\pi \Gamma(-2\Phi)} \left(\sqrt{2\rho}\right)^{-2\Phi-1} K_{1+2\Phi}(2\sqrt{2\rho}) ,$$

where $z = \rho e^{i\varphi}$, $|z| = \rho$ and $-2\Phi - 1 = 0, 1, 2, \cdots$. Here we have corrected an erratum of $\sigma(\rho)$ in Ref. [10], namely $K_{1/2+\Phi} \rightarrow K_{1+2\Phi}$. The reason of this is because the formula in Ref. [16] was wrong, and it should be replaced by

$$\int_0^\infty dx \ 2x^{\alpha+\beta} K_{2(\alpha+\beta)}(2\sqrt{x}) x^{s-1} = \Gamma(2\alpha+s) \Gamma(2\beta+s) . \quad (B2)$$

This formula can be proved by the integral expression of $K_\nu(z)$ [17], $K_\nu(z) = \int_0^\infty dt \exp(-z \cosh t) \cosh(\nu t)$, a change of the order of integration and the following formula [18]:

$$\int_0^\infty dx \ \frac{\cosh bx}{\cosh cx} = \frac{2^{\nu-2}}{e\Gamma(\nu)} \Gamma\left(\frac{\nu c + b}{2c}\right) \Gamma\left(\frac{\nu c - b}{2c}\right) .$$
We can also easily assure Eq. (B2) in a special case that \( \alpha = 1/4, \beta = 0, s = 1 \), using \( K_{1/2}(z) = \sqrt{\pi/2z} \exp(-z) \) \[19\].

Let us write the normalized state of \(|z\rangle\) as \(|z\rangle_n\), then we have

\[
|z\rangle\langle z| = \langle z|z \rangle |z\rangle_n \langle z|_n,
\]

(B3)

where we have used Eq. (16) and have identified \( m' = -2\Phi - 1 \). If we write \( \xi = \sqrt{2}z \) and \(|z\rangle_n \langle z| = |\xi \rangle \langle \xi|\), then Eqs. (B1) is identical to Eq. (24). Therefore, we have established that our resolution of unity agrees with that in Ref. [10].
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Figure Captions

FIG. 1. The relation between $X$ and $\tilde{r}$, when $m' = 10, 100, 1000$ and $|\xi| = 1$. 
