Moser-Trudinger inequalities for singular Liouville systems

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Abstract

In this paper we prove a Moser-Trudinger inequality for the Euler-Lagrange functional of a general singular Liouville system. We characterize the values of the parameters which yield coercivity for the functional and we give necessary conditions for boundedness from below. We also provide a sharp inequality under some assumptions on the coefficients of the system.

1 Introduction

An essential tool in the study of the embeddings of Sobolev spaces is the Moser-Trudinger inequality, which gives compact embedding in any $L^p$ space for finite $p \geq 1$ and also exponential integrability.

If we consider a 2-dimensional compact Riemannian manifold $(\Sigma, g)$, due to well-known works from Moser [20] and Fontana [15] we get

$$\log \int_{\Sigma} e^u dV_g - \int_{\Sigma} u dV_g \leq \frac{1}{16\pi} \int_{\Sigma} |\nabla u|^2 dV_g + C \quad \forall u \in H^1(\Sigma),$$

where $\nabla = \nabla_g$ is the gradient given by the metric $g$ and $C = C_{\Sigma, g}$ is a constant depending only on $\Sigma$ and $g$.

This inequality has fundamental importance in the study of the Liouville equations of the kind

$$-\Delta u = \rho \left( \frac{he^u}{\int_{\Sigma} he^u dV_g} - 1 \right),$$

where $\Delta = \Delta_g$ is the Laplace-Beltrami operator, $\rho$ a positive real parameter, $h$ a positive smooth function and $\Sigma$ is supposed, without loss of generality, to have area equal to $|\Sigma| = 1$.

In fact, the solutions of (2) are critical points of the functional

$$I_\rho(u) = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 dV_g + \rho \left( \int_{\Sigma} u dV_g - \log \int_{\Sigma} he^u dV_g \right);$$

using the inequality (1) we can control the last term by the Dirichlet energy, thus showing that $I_\rho$ is bounded from below on $H^1(\Sigma)$ if and only if $\rho$ is smaller or equal to $8\pi$.

Equations like (2) have great importance in different contexts like the Gaussian curvature prescription problem (see for instance [8, 9]) and abelian Chern-Simons models in theoretical

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physics ([21, 24]).

An extension of the inequality (1), which takes into consideration power-type weights, was given by Chen [10] and Trojanov [22]. For a given \( p \in \Sigma \) and \( \alpha \in (-1, 0] \), they showed that

\[
(1 + \alpha) \left( \log \int_{\Sigma} d(\cdot,p)^{2\alpha} e^u dV_g - \int_{\Sigma} u dV_g \right) \leq \frac{1}{16 \pi} \int_{\Sigma} |\nabla u|^2 dV_g + \frac{C}{\alpha} \quad \forall u \in H^1(\Sigma). \tag{3}
\]

This inequality allows to treat singularities in the equation (2), that is to consider equations like

\[
-\Delta u = \rho \left( \frac{h e^u}{\int_{\Sigma} he^u dV_g} - 1 \right) - 4\pi \sum_{m=1}^{M} \alpha_m (\delta_{p_m} - 1), \tag{4}
\]

where we take arbitrary \( p_1, \ldots, p_M \in \Sigma \) and \( \alpha_m > -1 \) for any \( m \in \{1, \ldots, M\} \). This is a natural extension of (2), which allows to consider the same problems in a more general context. For instance, it arises in the Gaussian curvature prescription problem on surfaces with conical singularities and in Chern-Simons vortices theory.

Defining \( G_x \) as the Green function of \(-\Delta\) on \( \Sigma \) centered at a point \( x \), through the change of variables

\[
u \to u + 4\pi \sum_{m=1}^{M} \alpha_m G_{p_m},
\]

equation (4) becomes

\[
-\Delta u = \rho \left( \frac{\tilde{h} e^u}{\int_{\Sigma} \tilde{h} e^u dV_g} - 1 \right)
\]

with \( \tilde{h} = he^{-4\pi \sum_{m=1}^{M} \alpha_m G_{p_m}} \).

Since \( G_x \) has the same behavior as \( \frac{1}{2\pi} \log \frac{1}{d(\cdot, x)} \) around \( x \), then \( \tilde{h} \) behaves like \( d(\cdot, p_m)^{2\alpha_m} \) around each singular point \( p_m \), hence the energy functional

\[
\tilde{I}_\rho(u) = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 dV_g + \rho \left( \int_{\Sigma} u dV_g - \log \int_{\Sigma} \tilde{h} e^u dV_g \right)
\]

can be estimated, as in the regular case, using (3).

The purpose of this paper is to extend inequality (3) to singular Liouville systems of the type

\[
-\Delta u_i = \sum_{j=1}^{N} a_{ij} \rho_j \left( \frac{h_j e^{u_j}}{\int_{\Sigma} h_j e^{u_j} dV_g} - 1 \right) - 4\pi \sum_{m=1}^{M} \alpha_{im} (\delta_{p_m} - 1), \quad i = 1, \ldots, N,
\]

where \( A = (a_{ij}) \) is a \( N \times N \) symmetric positive definite matrix and \( \rho_i, h_i, \alpha_{im} \) are as before.

Applying, similarly to (5), the change of variables

\[
u_i \to u_i + 4\pi \sum_{m=1}^{M} \alpha_{im} G_{p_m},
\]

the system becomes

\[
-\Delta u_i = \sum_{j=1}^{N} a_{ij} \rho_j \left( \frac{\tilde{h}_j e^{u_j}}{\int_{\Sigma} \tilde{h}_j e^{u_j} dV_g} - 1 \right), \quad i = 1, \ldots, N, \tag{6}
\]

with \( \tilde{h}_j \) having the same behavior around the singular points.

The system has a variational formulation with the energy functional

\[
J_\rho(u) := \frac{1}{2} \sum_{i,j=1}^{N} a^{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j dV_g + \sum_{i=1}^{N} \rho_i \left( \int_{\Sigma} u_idV_g - \log \int_{\Sigma} \tilde{h}_i e^{u_i} dV_g \right), \tag{7}
\]
with $a_{ij}$ indicating the entries of the inverse matrix $A^{-1}$ of $A$.

A recent paper by the author and Malchiodi ([4]) gives an answer for the particular case of the $SU(3)$ Toda system, that is $N = 2$ and $A$ is the Cartan matrix

$$
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}.
$$

This is a particularly interesting case, due to its application in the description of holomorphic curves in $\mathbb{CP}^N$ in geometry ([5, 7, 11]) and in the non-abelian Chern-Simons theory in physics ([14, 21, 24]).

The authors prove a sharp inequality, that is they show that the functional $J_\rho$ is bounded from below if and only if both the parameters $\rho_i$ are less or equal than $4\pi \min \left\{ 1, 1 + \min_m \alpha_{im} \right\}$, thus extending the result in the regular case from [17].

Concerning general regular Liouville systems, Wang [23] gave necessary and sufficient conditions for the boundedness from below of $J_\rho$, following previous results in [12, 13] for the problem on Euclidean domains with Dirichlet boundary conditions.

In these papers, the authors introduce, for any $I \subset \{1, \ldots, N\}$, the following function of the parameter $\rho$:

$$
\Lambda_I(\rho) = 8\pi \sum_{i \in I} \rho_i - \sum_{i,j \in I} a_{ij} \rho_i \rho_j. 
$$

What they prove is boundedness from below for $J_\rho$ for any $\rho \in \mathbb{R}^N_+$ which satisfies $\Lambda_I(\rho) > 0$ for all the subsets $I$ of $\{1, \ldots, N\}$, whereas $\inf_{H^1(\Sigma)^N} J_\rho = -\infty$ whenever $\Lambda_I(\rho) < 0$ for some $I \subset \{1, \ldots, N\}$.

The first main result of this paper is an extension of the results from [12, 13, 23] to the case of singularities.

Similarly to Liouville equation, we will have to multiply some quantities by $1 + \alpha_{im}$ and, as easily follows by the boundedness of $\tilde{h}_i$’s, if all the $\alpha_{im}$’s are positive, then things go as in the case of no singularities.

Precisely, we have:

**Theorem 1.1.**

Let $J_\rho$ be the functional defined by (7) and set, for $\rho \in \mathbb{R}^N_+$, $x \in \Sigma$ and $i \in I \subset \{1, \ldots, N\}$:

$$
\alpha_i(x) = \begin{cases} 
\alpha_{im} & \text{if } x = p_m \\
0 & \text{otherwise}
\end{cases} \quad \Lambda_{I,x}(\rho) := 8\pi \sum_{i \in I} (1 + \alpha_i(x)) \rho_i - \sum_{i,j \in I} a_{ij} \rho_i \rho_j. 
$$

$$
\Lambda(\rho) := \min_{I \subset \{1, \ldots, N\}, x \in \Sigma} \Lambda_{I,x}(\rho).
$$

Then, $J_\rho$ is bounded from below if $\Lambda(\rho) > 0$, whereas $J_\rho$ is unbounded from below if $\Lambda(\rho) < 0$.

Notice that, in the definition of $A$, the minimum makes sense because it is taken in a finite set, since $\alpha_i(x) = 0$ for all points of $\Sigma$ but a finite number, and for all the former points $\Lambda_{I,x}$ is defined in the same way.

As a consequence of this result, we obtain information about the existence of solutions for the system (6).

**Corollary 1.2.**

The functional $J_\rho$ is coercive in $H^1(\Sigma)$ if and only if $\Lambda(\rho) > 0$. Therefore, if this occurs, then $J_\rho$ admits a minimizer $u$ which solves (6).
Theorem 1.1 leaves an open question about what happens when $\Lambda(\rho) = 0$. In this case, as we will see in the following Sections, one encounters blow-up phenomena which are not yet fully known for general systems.

Anyway, we can say something more if we assume in addition $a_{ij} \leq 0$ for any $i,j \in \{1, \ldots, N\}$ with $i \neq j$. First of all, we notice that in this case

$$\Lambda(\rho) = \min_{i \in \{1, \ldots, N\}} \left(8\pi(1 + \tilde{\alpha}_i)\rho_i - \rho_i^2 \right)$$

where $\tilde{\alpha}_i = \min \left\{ 0, \min_{m \in \{1, \ldots, M\}} \alpha_{im} \right\}$, hence the sufficient condition in Theorem 1.1 is equivalent to assuming $\rho_i < \frac{8\pi(1 + \tilde{\alpha}_i)}{a_{ii}}$ for any $i$.

With this assumption, studying what happens when $\Lambda_I(\rho) = 0$ is reduced to a single-component local blow-up, which can be treated by using an inequality from [1]. Therefore, we get the following sharp result:

**Theorem 1.3.**

Let $J_\rho$ be defined by (7), $\tilde{\alpha}_i$ as in Theorem 1.1, and suppose $a_{ij} \leq 0$ for any $i,j \in \{1, \ldots, N\}$ with $i \neq j$. Then, $J_\rho$ is bounded from below on $H^1(\Sigma)^N$ if and only if $\rho_i \leq \frac{8\pi(1 + \tilde{\alpha}_i)}{a_{ii}}$ for any $i \in \{1, \ldots, N\}$.

The plan of this paper is the following: in Section 2 we will introduce some notations and some preliminary results which will be needed to prove the two main theorems. In Section 3 we will show a sort of Concentration-compactness theorem, showing the possible non-compactness phenomena for solutions of the system (6). Finally, in Sections 4 and 5 we will give the proof of the two main theorems.

## 2 Notations and preliminaries

In this section, we will give some useful notation and some known preliminary results which will be needed to prove the two main theorems.

Given two points $x, y \in \Sigma$, we will indicate the metric distance on $\Sigma$ between them as $d(x, y)$. We will indicate the open metric ball centered in $p$ having radius $r$ as

$$B_r(x) := \{ y \in \Sigma : d(x, y) < r \}.$$

For any subset of a topological space $A \subset X$ we indicate its closure as $\overline{A}$ and its interior part as $A$.

Given a function $u \in L^1(\Sigma)$, the symbol $\overline{u}$ will indicate the average of $u$ on $\Sigma$. Since we assume $|\Sigma| = 1$, we can write:

$$\overline{u} = \int_\Sigma u \, dV_g = \int_\Sigma u \, dV_g.$$

We will indicate the subset of $H^1(\Sigma)$ which contains the functions with zero average as

$$\overline{H}^1(\Sigma) := \{ u \in H^1(\Sigma); \overline{u} = 0 \}.$$

Since the functional $J_\rho$ defined by (7) is invariant by addition of constants, it will not be restrictive to study it on $\overline{H}^1(\Sigma)^N$ rather than on $H^1(\Sigma)^N$. 

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On the other hand, for a planar Euclidean domain $\Omega \subset \mathbb{R}^2$ with smooth boundary and a function $u \in H^1(\Omega)$ we will indicate with the symbol $u|_{\partial \Omega}$ the trace of $u$ on the boundary of $\Omega$. The space of functions with zero trace will be denoted by

$$H_0^1(\Omega) := \{ u \in H^1(\Omega); \ u|_{\partial \Omega} = 0 \}. \quad (9)$$

We will indicate with the letter $C$ large constants which can vary among different lines and formulas. To underline the dependence of $C$ on some parameter $\alpha$, we indicate with $C_\alpha$ and so on.

We will denote as $o_\alpha(1)$ quantities which tend to 0 as $\alpha$ tends to 0 or to $+\infty$ and we will similarly indicate bounded quantities as $O_\alpha(1)$, omitting in both cases the subscript(s) when it is evident from the context.

First of all, we need two results from Brezis and Merle [6]. The first is a classical estimate about exponential integrability of solutions of some elliptic PDEs.

**Lemma 2.1.** ([6], Theorem 1)

Take $r > 0$, $\Omega := B_r(0) \subset \mathbb{R}^2$, $f \in L^1(\Omega)$ with $\|f\|_{L^1(\Omega)} < 4\pi$ and $u$ solving

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega \end{cases}.$$  

Then, for any $q \in \left[1, \frac{4\pi}{\|f\|_{L^1(\Omega)}}\right]$ there exists a constant $C = C_{q,r}$ such that $\int_{\Omega} e^{qu(x)} dx \leq C$.

The second result we need, which has been extended in [2, 3], is a concentration-compactness theorem for scalar Liouville-type equations, which will be somehow extended to systems in Section 3:

**Lemma 2.2.** ([2], Theorem 2.1; [3], Theorem 5; [6], Theorem 3)

Take $r > 0$, $\alpha > -1$, $\Omega := B_r(0) \subset \mathbb{R}^2$, $\{V^k\}_{k \in \mathbb{N}} \subset L^\infty(\Omega)$ such that $\frac{1}{C} \leq V^k \leq C$ for some $C > 0$ and $u^k$ solving

$$\begin{cases} -\Delta u^k = V^k |x|^{2\alpha}e^{u^k} & \\ \int_{\Omega} |x|^{2\alpha}e^{u^k(x)} dx \leq C \end{cases}.$$  

Then, up to subsequences, one of the following occurs:

1. $u^k$ is bounded in $L^\infty_{\text{loc}}(\Omega)$.
2. $u^k \xrightarrow{k \to +\infty} -\infty$ in $L^\infty_{\text{loc}}(\Omega)$.
3. The set $S := \{ x \in \Omega : \exists x^k \xrightarrow{k \to +\infty} x \text{ such that } u^k(x^k) \xrightarrow{k \to +\infty} +\infty \}$ is nonempty and $u^k \xrightarrow{k \to +\infty} -\infty$ in $L^\infty_{\text{loc}}(\Omega \setminus S)$.

A crucial role in the proof of both Theorem 1.1 and 1.3 will be played by the concentration values of the sequences of solutions of (6).

For a sequence $u^k = \{u^k_1, \ldots, u^k_N\}_{k \in \mathbb{N}}$ of solutions of (6) with $\rho = \rho^k = \{\rho^k_1, \ldots, \rho^k_N\}$, we define (up to subsequences), for $i \in \{1, \ldots, N\}$, the concentration value of its $i^{th}$ component around a point $x \in \Sigma$ as

$$\sigma_i(x) := \lim_{r \to 0} \lim_{k \to +\infty} \rho^k_i \frac{\int_{B_r(x)} h^k e^{u^k} dV_g}{\int_{\Sigma} h_i e^{u_i^k} dV_g}. \quad (10)$$
In a recent paper ([18], see also [16] for the regular case) it was proved, by a Pohožaev identity, that the concentration values satisfy the following algebraic relation, which involves the same quantities which are in Theorem 1.1:

**Proposition 2.3. ([16], Lemma 2.2; [18], Proposition 3.1)**

Let \( \{u^k\}_{k \in \mathbb{N}} \) be a sequence of solutions of (6), \( \alpha_i(x) \) and \( \Lambda_{x,k} \) as in (8) and \( \sigma(x) = (\sigma_1(x), \ldots, \sigma_N(x)) \) as in (10). Then,

\[
\Lambda_{1, \ldots , N,x} \sigma(x) = 8\pi \sum_{i=1}^{N} (1 + \alpha_i(x)) \sigma_i(x) - \sum_{i,j=1}^{N} a_{ij} \sigma_i(x) \sigma_j(x) = 0.
\]

To study the concentration phenomena of solutions of (6) we will use the following simple but useful calculus Lemma:

**Lemma 2.4. ([17], Lemma 4.4)**

Let \( \{a^k\}_{k \in \mathbb{N}} \) and \( \{b^k\}_{k \in \mathbb{N}} \) two sequences of real numbers satisfying

\[
a^k \rightarrow k \rightarrow +\infty + \infty \quad \lim_{k \rightarrow +\infty} \frac{b^k}{a^k} \leq 0.
\]

Then, there exists a smooth function \( F : [0, +\infty) \rightarrow \mathbb{R} \) which satisfies, up to subsequences,

\[
0 < F'(t) < 1 \quad \forall t > 0 \quad F'(t) \rightarrow 0 \quad F(a^k) - b^k \rightarrow k \rightarrow +\infty +\infty.
\]

Finally, as anticipated in the introduction, we will need a singular Moser-Trudinger inequality for Euclidean domains by Adimurthi and Sandeep [1], and its straightforward corollary.

**Theorem 2.5. ([1], Theorem 2.1)**

For any \( r > 0, \alpha \in (-1, 0) \) there exists a constant \( C = C_{\alpha,r} \) such that if \( \Omega := B_r(0) \subset \mathbb{R}^2 \) and \( u \in H^1_0(\Omega) \), then

\[
\int_{\Omega} |\nabla u(x)|^2 \, dx \leq 1 \quad \Rightarrow \quad \int_{\Omega} |x|^{2\alpha} e^{4\pi(1+\alpha)u(x)^2} \, dx \leq C
\]

**Corollary 2.6.**

For any \( r > 0, \alpha \in (-1, 0) \) there exists a constant \( C = C_{\alpha,r} \) such that if \( \Omega := B_r(0) \subset \mathbb{R}^2 \) and \( u \in H^1_0(\Omega) \), then

\[
(1 + \alpha) \log \int_{\Omega} |x|^{2\alpha} e^{u(x)} \, dx \leq \frac{1}{16\pi} \int_{\Omega} |\nabla u(x)|^2 \, dx + C
\]

**Proof.**

By the elementary inequality \( u \leq au^2 + \frac{1}{4a} \) with \( a = \frac{4\pi(1+\alpha)}{\int_{\Omega} |\nabla u(y)|^2 \, dy} \) we get

\[
(1 + \alpha) \log \int_{\Omega} |x|^{2\alpha} e^{u(x)} \, dx \leq (1 + \alpha) \log \int_{\Omega} |x|^{2\alpha} e^{au(x)^2 + \frac{1}{4a}} \, dx
\]

\[
= \frac{1}{16\pi} \int_{\Omega} |\nabla u(y)|^2 \, dy + (1 + \alpha) \log \int_{\Omega} |x|^{2\alpha} e^{4\pi(1+\alpha)\left(\frac{u(x)}{\sqrt{\int_{\Omega} |\nabla u(x)|^2 \, dx}}\right)^2} \, dx
\]

\[
\leq \frac{1}{16\pi} \int_{\Omega} |\nabla u(y)|^2 \, dy + C.
\]

\[\square\]
3 A Concentration-compactness theorem

The aim of this section is to prove a result which describes the concentration phenomena for the solutions of (6), extending what was done for the two-dimensional Toda system in [4, 19].

We actually have to normalize such solutions to bypass the issues of invariance by translation by constants and to have the parameter \( \rho \) multiplying only the constant term. In fact, for any solution \( u \) of (6) the functions

\[
v_i := u_i - \log \int_\Omega \tilde{h}_i e^{u_i} dV_g + \log \rho_i
\]

solve

\[
\begin{aligned}
-\Delta v_i &= \sum_{j=1}^{N} a_{ij} \left( \tilde{h}_j e^{v_j} - \rho_j \right) & i &= 1, \ldots, N.
\end{aligned}
\]

Moreover, we can rewrite in a shorter way (10) as

\[
\sigma_i(x) = \lim_{r \to 0} \lim_{k \to +\infty} \int_{B_r(x)} \tilde{h}_i e^{v_i} dV_g.
\]

For such functions, we get the following concentration-compactness alternative:

**Theorem 3.1.**

Let \( \{ u^k \}_{k \in \mathbb{N}} \) be a sequence of solutions of (6) with \( \rho^k \to \rho \in \mathbb{R}_+^N \) and \( h^k_i = V_i^k \tilde{h}_i \) where \( V_i^k \to 1 \) in \( C^1(\Sigma)^N \), let \( \{ v^k \}_{k \in \mathbb{N}} \) be defined as in (11) and let \( S_i \) be defined, for \( i \in \{ 1, \ldots, N \} \), by

\[
S_i := \left\{ x \in \Sigma : \exists x^k \to k \to +\infty, x \text{ such that } v^k_i(x) \to +\infty \right\}.
\]

Then, up to subsequences, one of the following occurs:

1. If \( S_i = \emptyset \) for any \( i \in \{ 1, \ldots, N \} \), then \( v^k \to v \) in \( W^{2,q}(\Sigma)^N \) for some \( q > 1 \) and some \( v \) which solves (12).

2. If \( S_i \neq \emptyset \) for some \( i \), then it is a finite set for all such \( i \)’s. If this occurs, then there is a subset \( J \subset \{ 1, \ldots, N \} \) such that \( v^k_j \to -\infty \) in \( L^\infty_{\text{loc}} \left( \Sigma \setminus \bigcup_{l=1}^{N} S_l \right) \) for any \( j \in J \)

and \( v^k_j \to v_j \) in \( W^{2,q}_{\text{loc}} \left( \Sigma \setminus \bigcup_{l=1}^{N} S_l \right) \) for some \( q > 1 \) and some suitable \( v_j \), for any \( j \in \{ 1, \ldots, N \} \setminus J \).

Moreover, if \( S_j = \emptyset \), then the second alternative occurs for \( v^k_j \); on the other hand, if \( S_j \setminus \bigcup_{i \neq j} S_i \neq \emptyset \), then the first alternative occurs.

We need two preliminary lemmas.

The first is a Harnack-type alternative for sequences of solutions of PDEs. It is inspired by [6, 19].

**Lemma 3.2.**

Let \( \Omega \subset \Sigma \) be a connected open subset, \( \{ f^k \}_{k \in \mathbb{N}} \) a bounded sequence in \( L^q_{\text{loc}}(\Omega) \cap L^1(\Omega) \) for some \( q > 1 \) and \( \{ w^k \}_{k \in \mathbb{N}} \) bounded from above and solving \( -\Delta w^k = f^k \) in \( \Omega \). Then, up to subsequences, one of the following alternatives holds:

1. \( w^k \) is uniformly bounded in \( L^\infty_{\text{loc}}(\Omega) \).
2. \( w^k \to -\infty \) uniformly in \( L^\infty_{\text{loc}}(\Omega) \).

**Proof.**

Take a compact set \( K \subset \Omega \) and cover it with balls of radius \( \frac{r}{2} \), with \( r \) smaller than the injectivity radius of \( \Sigma \). By compactness, we can write \( K \subset \bigcup_{h=1}^{H} B_{\frac{r}{2}}(x_h) \). If the second alternative does not occur, then up to relabeling we get \( \sup_{B_r(x_1)} w^k \geq -C \).

Then, we consider the solution \( z^k \) of

\[
\begin{cases}
-\Delta z^k = f_k & \text{in } B_r(x_1) \\
z^k = 0 & \text{on } \partial B_r(x_1)
\end{cases}
\]

which is bounded in \( L^\infty(B_r(x_1)) \) by elliptic estimates. This means that, for a large constant \( C \), the function \( C - w^k + z^k \) is positive, harmonic and bounded from below on \( B_r(x_1) \), and moreover its infimum is bounded from above; therefore, applying the Harnack inequality (which is allowed since \( r \) is small enough) we get that \( C - w^k + z^k \) is uniformly bounded in \( L^\infty(B_{\frac{r}{2}}(x_1)) \); hence \( w^k \) is.

At this point, by connectedness, we can relabel the index \( h \) in such a way that \( B_{\frac{r}{2}}(x_h) \cap B_{\frac{r}{2}}(x_{h+1}) \neq \emptyset \) for any \( h \in \{1, \ldots, H-1\} \) and we repeat the argument for \( B_{\frac{r}{2}}(x_2) \): since it has nonempty intersection with \( B_{\frac{r}{2}}(x_1) \), we have \( \sup_{B_{\frac{r}{2}}(x_2)} w^k \geq -C \), hence we get boundedness in \( L^\infty(B_{\frac{r}{2}}(x_2)) \).

In the same way, we obtain the same result in all the balls \( B_{\frac{r}{2}}(x_h) \), whose union contains \( K \), therefore \( w^k \) must be uniformly bounded on \( K \) and we get the conclusion. \( \square \)

The second Lemma basically says that if all the concentration values in a point are under a certain threshold, and in particular if all of them equal zero, then compactness occurs around that point.

On the other hand, if a point belongs to some set \( S_i \), then at least a fixed amount of mass has to accumulate around it; hence, being the total mass uniformly bounded from above, this can occur only for a finite number of points, so we deduce the finiteness of the \( S_i \)'s. Precisely, we have the following, inspired again by [19], Lemma 4.4:

**Lemma 3.3.**

Let \( \{ v^k \}_{k \in \mathbb{N}} \) and \( S_i \) be as in Theorem 3.1 and \( \sigma_i \) as in (10), and suppose \( \sigma_i(x) < \sigma^0_i := \frac{4\pi \min\{1, 1 + \alpha_i(x)\}}{\sum_{j=1}^{N} a_{ij}^2} \) for any \( i \in \{1, \ldots, N\} \). Then, \( x \not\in S_i \) for any \( i \in \{1, \ldots, N\} \).

**Proof.**

First of all we notice that \( \sigma^0_i \) is well-defined for any \( i \) because \( \alpha_i > 0 \), hence \( \sum_{j=1}^{N} a_{ij}^2 > 0 \).

Under the hypotheses of the Lemma, for large \( k \) and small \( r \) we have

\[
\int_{B_r(x)} \frac{\beta e^{\gamma v^k}}{r^2} dV < \sigma^0_i.
\]

(14)

Let us consider the functions \( w^k_i \) and \( z^k_i \) defined by

\[
\begin{cases}
-\Delta w^k_i = -\sum_{j=1}^{N} a_{ij} \rho^k_j & \text{in } B_r(x) \\
w^k_i = 0 & \text{on } \partial B_r(x)
\end{cases}
\]

\[
\begin{cases}
-\Delta z^k_i = f^k_i & \text{in } B_r(x) \\
z^k_i = 0 & \text{on } \partial B_r(x)
\end{cases}
\]
\begin{equation}
\begin{cases}
-\Delta z^k_i = \sum_{i,j=1}^{N} a^i_{ij} \tilde{h}^k e^{z^k_j} & \text{in } B_r(x) \\
z^k_i = 0 & \text{on } \partial B_r(x)
\end{cases}
\end{equation}

Is it evident that the \(w^k_i\)'s are uniformly bounded in \(L^\infty(B_r(x))\). As for the \(z^k_i\)'s, we can suppose to be working on a Euclidean disc, up to applying a perturbation to \(h^k\) which is smaller as \(r\) is smaller, hence for \(r\) small enough we still have the strict estimate (14). Therefore, we get \(\| -\Delta z^k_i \|_{L^1(B_r(x))} < 4\pi \min\{1, 1 + \alpha_i(x)\}\) and we can apply Lemma 2.1 to obtain
\[
\int_{B_r(x)} e^{q|z^k_i|} dV_g \leq C \quad \text{for some } q > \frac{1}{\min\{1, 1 + \alpha_i(x)\}}.
\]

If \(\alpha_i(x) \geq 0\), then taking \(q \in \left(1, \frac{4\pi}{\| -\Delta z^k_i \|_{L^1(B_r(x))}}\right)\) we have
\[
\int_{B_r(x)} (\tilde{h}^k e^{z^k_i})^q dV_g \leq C_r \int_{B_r(x)} e^{q|z^k_i|} dV_g \leq C.
\]

On the other hand, if \(\alpha_i(x) < 0\), we choose
\[
q \in \left(1, \frac{4\pi}{\| -\Delta z^k_i \|_{L^1(B_r(x))} - 4\pi \alpha_i(x)}\right) \quad \text{and, applying H"older's inequality,}
\]
\[
\int_{B_r(x)} (\tilde{h}^k e^{z^k_i})^q dV_g \leq C \int_{B_r(x)} d(\cdot, x)^{2q\alpha_i(x)} e^{q|z^k_i|} dV_g
\]
\[
\leq C \left( \int_{B_r(x)} d(\cdot, x)^{2q\alpha_i(x)} dV_g \right)^{\frac{1}{q}} \left( \int_{B_r(x)} e^{q|z^k_i|} dV_g \right)^{1 - \frac{1}{q}}
\]
\[
\leq C,
\]

because \(qq'\alpha_i(x) > -1\) and \(q' < 1\); hence \(\tilde{h}^k e^{z^k_i}\) is uniformly bounded in \(L^q(B_r(x))\) for some \(q > 1\).

Now, let us consider \(v^k_i - z^k_i - w^k_i\); it is a subharmonic sequence by construction, so for any \(y \in B_{\tilde{z}}(x)\) we get
\[
v^k_i(y) - z^k_i(y) - w^k_i(y) \leq \int_{B_{\tilde{z}}(y)} (v^k_i - z^k_i - w^k_i) dV_g
\]
\[
\leq C \int_{B_{\tilde{z}}(y)} (v^k_i - z^k_i - w^k_i)^+ dV_g
\]
\[
\leq C \int_{B_r(x)} (v^k_i - z^k_i)^+ + w^k_i dV_g
\]
\[
\leq C \int_{B_r(x)} (v^k_i - z^k_i)^+ dV_g.
\]

Moreover, since the maximum principle yields \(z^k_i \geq 0\), taking \(\theta = \begin{cases} 1 & \text{if } \alpha_i(x) \leq 0 \\ \in \left(0, \frac{1}{1 + \alpha_i(x)}\right) & \text{if } \alpha_i(x) > 0 \end{cases}\),
we get
\[ \int_{B_r(x)} (v_i^k - z_i^k)^+ \, dV_g \leq \int_{B_r(x)} v_i^k \, dV_g \]
\[ \leq C_0 \int_{B_r(x)} e^{\theta v_i^k} \, dV_g \]
\[ \leq C \left\| \tilde{h}_i \right\|_{L^\infty(B_r(x))} \left( \int_{B_r(x)} \tilde{h}_i e^{\theta v_i^k} \, dV_g \right)^\theta \]
\[ \leq C. \]

Therefore, we showed that \( v_i^k - z_i^k - w_i^k \) is bounded from above in \( B_\varepsilon(x) \), that is \( e^{v_i^k - z_i^k - w_i^k} \) is uniformly bounded in \( L^\infty(B_\varepsilon(x)) \). Since the same holds for \( e^{v_i^k} \) and \( \tilde{h}_i e^{v_i^k} \) is uniformly bounded in \( L^q(B_\varepsilon(x)) \) for some \( q > 1 \), we deduce that also
\[ \tilde{h}_i e^{v_i^k} = \tilde{h}_i e^{z_i^k} e^{v_i^k - z_i^k - w_i^k} e^{w_i^k} \]
is bounded in the same \( L^q(B_\varepsilon(x)) \).

We then have an estimate on \( = -\Delta z_i^k \|_{L^q(B_\varepsilon(x))} \) for any \( i \in \{1, \ldots, N\} \), hence by standard elliptic estimates we deduce that \( z_i^k \) is uniformly bounded in \( L^\infty(B_\varepsilon(x)) \). Therefore, we also deduce that
\[ u_i^k = (v_i^k - z_i^k - w_i^k) + z_i^k + w_i^k \]
is bounded from above on \( B_\varepsilon(x) \), which is equivalent to saying \( x \not\in \bigcup_{i=1}^N S_i. \]

From this proof, we notice that, under the assumptions of Theorem 1.3, the same result holds for any single index \( i \in \{1, \ldots, N\} \). In other words, the upper bound on one \( \sigma_i \) implies that \( x \not\in S_i \).

**Corollary 3.4.**

Suppose \( a_{ij} \leq 0 \) for any \( i \neq j \). Then, for any given \( i \in \{1, \ldots, N\} \) the following conditions are equivalent:

1. \( x \in S_i \).
2. \( \sigma_i(x) \neq 0 \).
3. \( \sigma_i(x) \geq \sigma_i^0 = \frac{4\pi \min \{1, 1 + a_i(x)\}}{a_{ii}}. \)

**Proof.**

The third statement trivially implies the second and the second implies the first, since if \( v_i^k \) is bounded from above in \( B_r(x) \) then \( h_i e^{v_i^k} \) is bounded in \( L^q(B_r(x)) \). Finally, if \( \sigma_i(x) < \sigma_i^0 \) then the sequence \( z_i^k \) defined by (15) is bounded in \( L^q \) so one can argue as in Lemma 3.3 to get boundedness from above of \( v_i^k \) around \( x \), that is \( x \not\in S_i \).

We can now prove the main theorem of this section.

**Proof of Theorem 3.1.**

If \( S_i = \emptyset \) for any \( i \), then \( e^{v_i^k} \) is bounded in \( L^\infty(\Sigma) \), so \( -\Delta v_i^k \) is bounded in \( L^q(\Sigma) \) for any \( q \in \left[ 1, \frac{1}{\max_{j \in \{1, \ldots, N\}} (-\alpha_j)} \right] \). Therefore, we can apply Lemma 3.2 to \( v_i^k \) on \( \Sigma \), where we
must have the first alternative for every $i$, since otherwise the dominated convergence would give
\[
\int_{\Sigma} \hbar_i^k e^{v_i^k} \, dV_g \to_{{k \to +\infty}} 0 \quad \text{which is absurd; standard elliptic estimates allow to conclude compactness in } W^{2,q}(\Sigma).
\]
Suppose now $S_i \neq \emptyset$ for some $i$; from Lemma 3.3 we deduce
\[
|S_i| \sigma_i^0 \leq \sum_{x \in S_i} \max_j \sigma_j(x) \leq \sum_{j=1}^N \sum_{x \in S_i} \sigma_j(x) \leq \sum_{j=1}^N \rho_j,
\]
hence $S_i$ is finite.

For any $j \in \{1, \ldots, N\}$, we can apply Lemma 3.2 on $\Sigma \setminus \bigcup_{i=1}^N S_i$ with $f^k = \sum_{i=1}^N a_{ij} \left( \hbar_i^k e^{v_i^k} - \rho_i^k \right)$,
since the last function is bounded in $L^2_{\text{loc}}(\Sigma \setminus \bigcup_{i=1}^N S_i)$. Therefore, either $v_i^k$ goes to $-\infty$ or it is bounded in $L^\infty_{\text{loc}}$, and in the last case we get compactness in $W^{2,q}$ by applying again standard elliptic regularity.

In the case $S_i = \emptyset$, $\hbar_i^k e^{v_i^k}$ is bounded in $L^2(\Sigma)$, so the alternative $v_i^k \to_{k \to +\infty} -\infty$ almost everywhere in $\Sigma$ has to be excluded because, as before, through dominated convergence it would give\[
\int_{\Sigma} \hbar_i^k e^{v_i^k} \, dV_g \to_{k \to +\infty} 0.
\]
Finally, suppose without loss of generality to have $S_1 \setminus \bigcup_{j=2}^N S_j = \emptyset$. Take a point $x$ belonging to this set, $r > 0$ such that
\[
B_r(x) \cap \bigcup_{j=1}^N S_j = \{x\} \quad \text{and} \quad \left( B_r(x) \setminus \{x\} \right) \cap \bigcup_{m=1}^M \{p_m\} = \emptyset,
\]
and $w^k$ defined by
\[
\left\{ \begin{array}{ll}
-\Delta w^k &= - \sum_{j=2}^N a_{ij} \hbar_i^k e^{v_i^k} + \sum_{j=1}^N a_{1j} \rho_j^k & \text{in } B_r(x) \\
& 0 & \text{on } \partial B_r(x) .
\end{array} \right.
\]
By our assumptions, $-\Delta w^k$ is bounded in $L^q(B_r(x))$ for some $q > 1$, so $w^k$ is bounded in $L^\infty(B_r(x))$. Therefore, $v_i^k + w^k$ solves
\[
\left\{ \begin{array}{ll}
-\Delta (v_i^k + w^k) &= V^k d(\cdot, x)^{2\alpha_1(x)} e^{v_i^k + w^k} & \text{with } V^k = a_{11} d(\cdot, x)^{-2\alpha_1(x)} \hbar_i^k e^{-w^k} . \\
\int_{B_r(x)} d(\cdot, x)^{2\alpha_1(x)} e^{v_i^k + w^k} \, dV_g & \leq C \int_{B_r(x)} a_{11} \hbar_i^k e^{v_i^k} \, dV_g \leq C .
\end{array} \right.
\]
So, we can apply Lemma 2.2 to $v_i^k + w^k$ on $B_r(x)$ (since for small $r$ we can suppose to deal with Euclidean discs). By how we choose $x$ and $r$, there must occur the third alternative in the Lemma with $S = \{x\}$; therefore, $v_i^k$ cannot be uniformly bounded in $L^\infty_{\text{loc}}(\Sigma \setminus \bigcup_{i=1}^N S_i)$, hence it must tend to $-\infty$ outside the $S_i$. The proof is now complete.

4 Proof of Theorem 1.1.
Here we will prove the theorem which gives sufficient and necessary conditions for the functional $J_\rho$ to be bounded from below. In other words, setting

$$L := \{\rho \in \mathbb{R}^N_+: J_\rho \text{ is bounded from below on } H^1(\Sigma)\}, \quad (16)$$

we will prove that $\{\Lambda > 0\} \subset L \subset \{\Lambda \geq 0\}$.

As a first thing, we notice that the set $L$ is not empty and it verifies a simple monotonicity condition.

**Lemma 4.1.**
The set $L$ defined by (16) is nonempty. Moreover, for any $\rho \in L$ then $\rho' \in L$ provided $\rho'_i \leq \rho_i$ for any $i \in \{1, \ldots, N\}$.

**Proof.**
Let $\lambda > 0$ be the biggest eigenvalue of the matrix $(a_{ij})$. Then,

$$J_\rho(u) \geq \sum_{i=1}^N \left( \frac{1}{2\lambda} \int_{\Sigma} |\nabla u_i|^2 dV_g - \rho_i \log \int_{\Sigma} \tilde{h}_i e^{u_i} dV_g \right).$$

Therefore, from scalar Moser-Trudinger inequality (3), we deduce that $J_\rho$ is bounded from below if $\rho_i \leq \frac{8\pi(1 + \tilde{\alpha}_i)}{\lambda}$, hence $L \neq \emptyset$.

Suppose now $\rho \in L$ and $\rho'_i \leq \rho_i$ for any $i$. Then, through Jensen’s inequality, we get

$$J_{\rho'}(u) = J_\rho(u) + \sum_{i=1}^N (\rho_i - \rho'_i) \int_{\Sigma} e^{u_i} \log \tilde{h}_i dV_g$$

$$\geq -C + \sum_{i=1}^N (\rho_i - \rho'_i) \int_{\Sigma} \log \tilde{h}_i dV_g$$

$$\geq -C$$

for any $u \in H^1(\Sigma)^N$, hence the claim. \(\square\)

It is interesting to observe that a similar monotonicity condition is also satisfied by the set $\{\Lambda > 0\}$ (although one can easily see that it is not true if we replace $\Lambda$ with $\Lambda_{I,x}$).

**Lemma 4.2.**
Let $\rho, \rho' \in \mathbb{R}^N_+$ be such that $\Lambda(\rho) > 0$ and $\rho'_i \leq \rho_i$ for any $i \in \{1, \ldots, N\}$. Then, $\Lambda(\rho') > 0$.

**Proof.**
Suppose by contradiction $\Lambda(\rho') \leq 0$, that is $\Lambda_{I,x}(\rho') \leq 0$ for some $I, x$.

This cannot occur for $I = \{i\}$ because it would mean $\rho'_i \geq \frac{8\pi(1 + \alpha_i(x))}{a_{ii}}$, so the same inequality would for $\rho_i$, hence $\Lambda(\rho) \leq \Lambda_{I,x}(\rho) \leq 0$. 

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Therefore, there must be some $I, x$ such that $\Lambda_{I, x}(\rho) \leq 0$ and $\Lambda_{I\setminus\{i\}, x}(\rho) > 0$ for any $i \in I$; this implies
\[
0 < \Lambda_{I\setminus\{i\}, x}(\rho) - \Lambda_{I, x}(\rho) = 2 \sum_{j \in I} a_{ij} \rho'_j - a_{ii} \rho'_i - 8\pi (1 + \alpha_i(x)) \rho'_i - 2 \sum_{j \in I\setminus\{i\}} a_{ij} \rho_j.
\]
It will be not restrictive to suppose, from now on, $\rho'_1 \leq \rho_1$ and $\rho'_i = \rho_i$ for any $i \geq 2$, since the general case can be treated by exchanging the indexes and iterating. Assuming this, we must have $1 \in I$, therefore we obtain:
\[
0 < \Lambda_{I, x}(\rho) - \Lambda_{I, x}(\rho) = 8\pi (1 + \alpha_1(x))(\rho_1 - \rho'_1) - a_{11} (\rho'_1 - \rho_1) - 2 \sum_{j \in I\setminus\{1\}} a_{1j} \rho_j.
\]
which is negative by (17). We found a contradiction. 

Lemma 4.3.
Suppose $\rho \in L$. Then, there exists a constant $C = C_\rho$ such that
\[
J_\rho(u) \geq \frac{1}{C} \sum_{i=1}^N \hat{\Sigma} |\nabla u_i|^2 dV_g - C.
\]
Moreover, $J_\rho$ admits a minimizer which solves (6).

Proof.
Choosing $\delta \in \left(0, \frac{d(\rho, \partial L)}{\sqrt{N} |\rho|}\right)$ one has $(1 + \delta)\rho \in L$, so
\[
J_\rho(u) = \frac{\delta}{2(1 + \delta)} \sum_{i,j=1}^N a_{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j dV_g + \frac{1}{1 + \delta} J_{(1+\delta)\rho}(u)
\]
\[
\geq \frac{\delta}{2(1 + \delta)} \sum_{i=1}^N \int_{\Sigma} |\nabla u_i|^2 dV_g - C,
\]
hence we get the former claim.

To get the latter, we notice that, due to invariance by translation, any minimizer can be
supposed to be in $H^1(\Sigma)^N$; therefore, we can restrict $J_\rho$ to this subspace. Here, the above inequality implies coercivity, and it is immediate to see that $J_\rho$ is also lower semi-continuous, hence the existence of minimizers follows from direct methods of calculus of variations.

Lemma 4.4.
Suppose $\rho \in \partial L$. Then, there exists a sequence $\{u^k\}_{k \in \mathbb{N}} \subset H^1(\Sigma)^N$ such that

$$\sum_{i=1}^{N} \int_{\Sigma} |\nabla u^k_i|^2 \, dV_g \xrightarrow{k \to +\infty} +\infty, \quad \lim_{k \to +\infty} \frac{J_\rho(u^k)}{\sum_{i=1}^{N} \int_{\Sigma} |\nabla u^k_i|^2 \, dV_g} \leq 0$$

Proof.
We first notice that $(1 - \delta)\rho \in L$ for any $\delta \in (0, 1)$. In fact, otherwise, from Lemma 4.1 we would get $\rho' \notin L$ as soon as $\rho'_i \geq (1 - \delta)\rho_i$ for some $i$, hence $\rho \notin \partial L$.

Now, suppose by contradiction that for any sequence $u^k$ one gets

$$\sum_{i=1}^{N} \int_{\Sigma} |\nabla u^k_i|^2 \, dV_g \xrightarrow{k \to +\infty} +\infty \quad \Rightarrow \quad \lim_{k \to +\infty} \frac{J_\rho(u^k)}{\sum_{i=1}^{N} \int_{\Sigma} |\nabla u^k_i|^2 \, dV_g} \geq \varepsilon > 0.$$

Therefore, we would have

$$J_\rho(u) \geq \frac{\varepsilon}{2} \sum_{i=1}^{N} \int_{\Sigma} |\nabla u_i|^2 \, dV_g - C;$$

hence, indicating as $\lambda'$ the smallest eigenvalue of the matrix $A$, for small $\delta$ we would get

$$J_\rho(u) = (1 + \delta)J_{(1+\delta)\rho}(u) - \frac{\delta}{2} \sum_{i,j=1}^{N} a^{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j \, dV_g$$

$$\geq (1 + \delta) \left( \frac{\varepsilon}{2} - \frac{\delta}{2N} \right) \sum_{i=1}^{N} \int_{\Sigma} \nabla u_i|^2 \, dV_g - C$$

$$\geq -C.$$

So, we obtain $(1 + \delta)\rho \in L$; being also $(1 - \delta)\rho \in L$ (by Lemma 4.1), we get a contradiction with $\rho \notin \partial L$.

To see what happens when $\rho \in \partial L$, we build an auxiliary functional using Lemma 2.4.

Lemma 4.5.
Fix $\tilde{\rho} \in \partial L$ and define:

$$a^k_\tilde{\rho} := \frac{1}{2} \sum_{i,j=1}^{N} a^{ij} \int_{\Sigma} \nabla u^k_i \cdot \nabla u^k_j \, dV_g \quad b^k_\tilde{\rho} := J_{\tilde{\rho}}(u^k)$$

$$\tilde{J}_{\tilde{\rho},\rho}(u) = J_\rho(u) - F_{\tilde{\rho}} \left( \frac{1}{2} \sum_{i,j=1}^{N} a^{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j \, dV_g \right).$$
where $u^k$ is given by Lemma 4.4 and $F_{\tilde{\rho}}$ by Lemma 2.4.

If $\rho \in L$, then $\tilde{J}_{\tilde{\rho}, \rho}$ is bounded from below on $H^1(\Sigma)^N$ and its infimum is achieved by a function which satisfies

$$-\Delta \sum_{i,j=1}^{N} (\delta_{ij} - a^{ij} f) u_j = \sum_{j=1}^{N} a_{ij} \rho_j \left( \frac{\tilde{h}_j e^{u_j}}{\int_{\Sigma} \tilde{h}_j e^{u_j} dV_g} - 1 \right), \quad i = 1, \ldots, N,$$

with $f = (F_{\tilde{\rho}})' \left( \frac{1}{2} \sum_{i,j=1}^{N} a^{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j dV_g \right)$.

On the other hand, $\tilde{J}_{\tilde{\rho}, \rho}$ is unbounded from below.

**Proof.**

For $\rho \in L$, we can argue as in Lemma 4.3, since the continuity follows from the regularity of $F$ and the coercivity from the behavior of $F'$ at the infinity.

For $\rho = \tilde{\rho}$, if we take $u^k$ as in Lemma 4.4 we get

$$\tilde{J}_{\tilde{\rho}, \rho} (u^k) = b^k_\rho - F_{\tilde{\rho}} (u^k_\rho) \to_{k \to +\infty} -\infty.$$

Now we can prove the first half of Theorem 1.1, that is $L$ is bounded from below if $\Lambda < 0$.

**Proof of $\{ \Lambda > 0 \} \subset L$.**

Suppose by contradiction there is some $\tilde{\rho} \in \partial L$ with $\Lambda(\rho) > 0$ and take a sequence $\rho^k \in L$ with $\rho^k \to_{k \to +\infty} \tilde{\rho}$.

Then, by Lemma 4.5, the auxiliary functional $J_{\tilde{\rho}, \rho^k}$ admits a minimizer $u^k$, so, the functions $v^k$ defined as in (11) solve

$$\begin{cases}
-\Delta v^k_i = \sum_{j,l=1}^{N} a_{ij} b^{i,k} \left( \tilde{h}_j e^{v^k_j} - \tilde{\rho}_j^k \right) \\
\int_{\Sigma} \tilde{h}_i e^{v^k_i} dV_g = \rho_i^k
\end{cases}$$

where $b^{i,k}$ is the inverse matrix of $b^k_{ij} := \delta_{ij} - a^{ij} f^k$, hence $b^{i,k} \to_{k \to +\infty} \delta_{ij}$.

We can then apply Theorem 3.1. The first alternative is excluded, since otherwise we would get, for any $u \in H^1(\Sigma)^N$,

$$J_{\tilde{\rho}, \rho} (u) = \lim_{k \to +\infty} J_{\tilde{\rho}, \rho^k} (u) \geq \lim_{k \to +\infty} J_{\tilde{\rho}, \rho^k} (v^k) \geq J_{\tilde{\rho}, \rho} (v) > -\infty,$$

thus contradicting Lemma 4.5.

Therefore, blow up must occur; this means, by Lemma 3.3, that $\sigma_i (p) \neq 0$ for some $i \in \{1, \ldots, N\}$ and some $p \in \Sigma$.

By Proposition 2.3 follows $\Lambda(\sigma) \leq 0$. On the other hand, since $\sigma_i \leq \tilde{\rho}_i$ for any $i$, Lemma 4.2 yields $\Lambda(\tilde{\rho}) \leq 0$, which contradicts our assumptions.

To prove the unboundedness from below of $J_{\tilde{\rho}}$ in the case $\Lambda(\rho) < 0$ we will use suitable test functions, whose properties are described by the following:

**Lemma 4.6.**

Define, for $x \in \Sigma$ and $\lambda > 0$, $\varphi = \varphi^{\lambda,x}$ as

$$\varphi_i := \log \frac{1}{1 + (\lambda d(\cdot, x))^{2(1+\alpha_i(x))}}.$$
Then, for $\lambda \to +\infty$, one has

$$\int_{\Sigma} \nabla \varphi_i \cdot \nabla \varphi_j dV_g \leq 8\pi(1 + \alpha_i(x))(1 + \alpha_j(x)) \log \lambda + C$$

$$\overline{\psi_i} = -2(1 + \alpha_i(x)) \log \lambda + O(1)$$

$$\int_{\Sigma} \tilde{h}_i e^{\sum_{j=1}^{N} c_j \varphi_j} dV_g \geq \frac{C}{\lambda^{2(1+\alpha_i(x))}}$$

if $\sum_{i=1}^{N} c_i (1 + \alpha_j(x)) > 1 + \alpha_i(x)$.

**Proof.**

Since $|\nabla d(\cdot, x)| = 1$ almost everywhere, we can write

$$|\nabla \varphi_i| = \frac{2(1 + \alpha_i(x))\lambda^{2(1+\alpha_i(x))}d(\cdot, x)^{1+2\alpha_i(x)} \nabla d(\cdot, x)}{1 + (\lambda d(\cdot, x))^{2(1+\alpha_i(x))}}$$

$$= \frac{2(1 + \alpha_i(x))\lambda^{2(1+\alpha_i(x))}d(\cdot, x)^{1+2\alpha_i(x)}}{1 + (\lambda d(\cdot, x))^{2(1+\alpha_i(x))}}$$

$$\leq 2(1 + \alpha_i(x)) \min \left\{ \lambda^{2(1+\alpha_i(x))}d(\cdot, x)^{1+2\alpha_i(x)}, \frac{1}{d(\cdot, x)} \right\}.$$

Therefore, we get

$$\left| \int_{\Sigma} \nabla \varphi_i \cdot \nabla \varphi_j dV_g \right| \leq \int_{\Sigma} |\nabla \varphi_i| |\nabla \varphi_j| dV_g$$

$$\leq 4(1 + \alpha_i(x))(1 + \alpha_j(x)) \left( \lambda^{4+2\alpha_i(x)+2\alpha_j(x)} \int_{B^+_\lambda(x)} d(\cdot, x)^{2+2\alpha_i(x)+2\alpha_j(x)} dV_g \right)$$

$$+ \int_{\Sigma \setminus B^+_\lambda(x)} \frac{dV_g}{d(\cdot, x)^2}$$

$$\leq C + 8\pi(1 + \alpha_i(x))(1 + \alpha_j(x)) \log \lambda.$$

For the other estimates, we use the elementary inequalities

$$\max \left\{ 1, (\lambda d(\cdot, x))^{2(1+\alpha_i(x))} \right\} \leq 1 + (\lambda d(\cdot, x))^{2(1+\alpha_i(x))} \leq 2 \max \left\{ 1, (\lambda d(\cdot, x))^{2(1+\alpha_i(x))} \right\},$$

hence one can replace $\varphi_i$ with

$$\psi_i := \log \min \left\{ 1, \frac{1}{(\lambda d(\cdot, x))^{2(1+\alpha_i(x))}} \right\} \begin{cases} 
0 & \text{on } B^+_\lambda(x) \\
-2(1 + \alpha_i(x))(\log \lambda + \log d(\cdot, x)) & \text{on } \Sigma \setminus B^+_\lambda(x)
\end{cases}.$$

What we obtain is

$$\overline{\psi_i} = \overline{\psi_i} + O(1)$$

$$= -2(1 + \alpha_i(x)) \int_{\Sigma \setminus B^+_\lambda(x)} (\log \lambda + \log d(\cdot, x)) dV_g + O(1)$$

$$= -2(1 + \alpha_i(x)) \log \lambda + O(1)$$

and, choosing $r > 0$ such that $B_r(x)$ does not contain any of the points $\{p_m\}_{m=1}^{M}$ except possibly $x$,

$$\int_{\Sigma} \tilde{h}_i e^{\sum_{j=1}^{N} c_j \varphi_j} dV_g \geq C \int_{B_r(x) \setminus B^+_\lambda(x)} \tilde{h}_i e^{\sum_{j=1}^{N} c_j \psi_j} dV_g$$

$$\geq \frac{C}{\lambda^{2 \sum_{j=1}^{N} c_j (1 + \alpha_j(x))}} \int_{B_r(x) \setminus B^+_\lambda(x)} d(\cdot, x)^{2\alpha_i(x)} - 2 \sum_{i=1}^{N} c_i (1 + \alpha_i(x)) dV_g$$

$$\geq \frac{C}{\lambda^{2(1+\alpha_i(x))}}.$$
hence the claim.

Proof of $L \subset \{ \Lambda \geq 0 \}$.
Take $\rho, I, x$ such that $\Lambda_{I,x}(\rho) < 0$ and $\Lambda_{I\setminus\{i\},x}(\rho) \geq 0$ for any $i \in I$, and consider the family of functions $u^\lambda$ defined by

$$u^\lambda_i = \sum_{j \in I} \frac{a_{ij} \rho_j}{4\pi(1 + \alpha_i(x))} \varphi_j^{\lambda,x}.$$ 

By Jensen’s inequality we get

$$J_{\rho}(u^\lambda) \leq 1.$$ 

At this point, we would like to apply Lemma 4.6 to estimate $J_{\rho}(u^\lambda)$; to be able to do this, we have to verify that

$$\frac{1}{4\pi} \sum_{i,j \in I} a_{ij} \rho_j > 1 + \alpha_i(x) \quad \forall i \in I.$$ 

If $I = \{i\}$, then $\rho_i > \frac{8\pi(1 + \alpha_i(x))}{a_{ii}}$, so it follows immediately; for the other cases, it follows from (17).

So, we can apply Lemma 4.6 and we get from the previous estimates:

$$J_{\rho}(u^\lambda) \leq \left( \frac{1}{4\pi} \sum_{i,j \in I} a_{ij} \rho_j - \frac{1}{2\pi} \sum_{i,j \in I} a_{ij} \rho_i \rho_j + 2 \sum_{i \in I} \rho_i(1 + \alpha_i(x)) \right) \log \lambda + C$$

$$\rightarrow \frac{-\Lambda_{I,x}(\rho)}{4\pi} \log \lambda + C \quad k \rightarrow +\infty.$$ 

Proof of Corollary 1.2.
The coercivity in the case $\Lambda < 0$, hence the existence of minimizing solutions for (6) follows from Theorem 1.1 and Lemma 4.3.

If instead $\Lambda(\rho) \geq 0$, then one can find out the lack of coercivity by arguing as before with the sequence $u^\lambda$, which verifies

$$\sum_{i=1}^N \int_{\Sigma} |\nabla u^\lambda|^2 dV_g \rightarrow +\infty \quad \lambda \rightarrow +\infty \quad J_{\rho}(u^\lambda) \leq \frac{-\Lambda_{I,x}(\rho)}{4\pi} \log \lambda + C \leq C.$$
5 Proof of Theorem 1.3.

Here we will finally prove a sharp inequality in the case when the matrix $a_{ij}$ has non-positive entries outside its main diagonal.

As already pointed out in the introduction, the function $\Lambda(\rho)$ can be written in a much shorter form under these assumptions, so the condition $\Lambda(\rho) \geq 0$ is equivalent to $\rho_i \leq \frac{8\pi(1 + \tilde{\alpha}_i)}{a_{ii}}$ for any $i \in \{1, \ldots, N\}$. Moreover, thanks to Lemma 4.1, in order to prove Theorem 1.3 for all such $\rho$'s it will suffice to consider

$$\rho^0 := \left(\frac{8\pi(1 + \tilde{\alpha}_1)}{a_{11}}, \ldots, \frac{8\pi(1 + \tilde{\alpha}_N)}{a_{NN}}\right).$$

By what we proved in the previous Section, for any sequence $\rho^k \not\to \rho^0$, one has

$$\inf_{H_1^1(\Sigma)} J_{\rho^k}(u^k) = J_{\rho^k}(u^k) \geq -C_{\rho^k},$$

so Theorem 1.3 will follow by showing that, for a given sequence $\{\rho^k\}_{k \in \mathbb{N}}$, the constant $C_k = C_{\rho^k}$ can be chosen independently of $k$.

As a first thing, we provide a Lemma which shows the possible blow-up scenarios for such a sequence $u^k$.

Here, the assumption on $a_{ij}$ is crucial since it reduces largely the possible cases.

Lemma 5.1.

Let $\rho^0$ be as in (18), $\{\rho^k\}_{k \in \mathbb{N}}$ such that $\rho^k \not\to \rho^0$, $u^k$ a minimizer of $J_{\rho^k}$ and $v^k$ as in (11). Then, up to subsequences, there exists a set $I \subset \{1, \ldots, N\}$ such that:

1. If $i \in I$, then $S_i = \{x_i\}$ for some $x_i \in \Sigma$ which satisfy $\tilde{\alpha}_i = \alpha_i(x_i)$ and $\sigma_i(x_i) = \rho^0_i$, and $v_i^k \to -\infty$ in $L^\infty_{\text{loc}} \left(\Sigma \setminus \bigcup_{j \in I} \{x_j\}\right)$.

2. If $i \notin I$, then $S_i = \emptyset$ and $v_i^k \to v_i$ in $W^{2,q}_{\text{loc}} \left(\Sigma \setminus \bigcup_{j \in I} \{x_j\}\right)$ for some $q > 1$ and some suitable $v_i$.

Moreover, if $a_{ij} < 0$ then $x_i \neq x_j$.

Proof.

From Theorem 3.1 we get a $I \subset \{1, \ldots, N\}$ such that $S_i \neq \emptyset$ for $i \in I$.

If $S_i \neq \emptyset$, then by Corollary 3.4 one gets $\sigma_i(x) > 0$ for any $x \in S_i$; moreover,

$$\sigma_j(x) \leq \rho^0_j \leq \frac{8\pi(1 + \alpha_j(x))}{a_{jj}}$$

for all $j$'s, hence

$$0 = \Lambda_{(1, \ldots, N), x}(\sigma(x)) \geq \sum_{j=1}^{N} \left(8\pi(1 + \alpha_j(x))\sigma_j(x) - a_{jj}\sigma_j(x)^2\right) \geq 8\pi(1 + \alpha_i(x))\sigma_i(x) - a_{ii}\sigma_i(x)^2 \geq 0.$$
Therefore, all these inequality must actually be equalities.
From the last, we have \( \sigma_{i}(x) = \rho_{i}^{0} = \frac{8\pi(1 + \alpha_{i}(x))}{a_{ij}} \), hence \( \alpha_{i}(x) = \alpha_{i} \). On the other hand, since \( \sum_{x \in S_{i}} \sigma_{i}(x) \leq \rho_{i}^{0} \), it must be \( \sigma_{i}(x) = 0 \) for all but one \( x_{i} \in S_{i} \), so Corollary 3.4 yields \( S_{i} = \{ x_{i} \} \).
Let us now show that \( v_{i}^{k} \xrightarrow[k \to +\infty]{} -\infty \) in \( L_{loc}^{\infty} \). Otherwise, Theorem 3.1 would imply \( v_{i}^{k} \xrightarrow[k \to +\infty]{} v_{i} \) almost everywhere, therefore by Fatou’s Lemma we would get the following contradiction:
\[
\sigma_{i}(x_{i}) < \int_{\Sigma} \tilde{h}_{i}e^{v_{i}}dV_{g} + \sigma_{i}(x_{i}) \leq \int_{\Sigma} \tilde{h}_{i}e^{v_{i}}dV_{g} = \rho_{i}^{k} \leq \rho_{i} = \sigma_{i}(x_{i}).
\]
Since also inequality (19) has to be an equality, we get \( a_{ij}\sigma_{i}(x_{i})\sigma_{j}(x_{i}) \) for any \( i, j \in I \), so whenever \( a_{ij} < 0 \) there must be \( x_{i} = x_{j} \), so \( x_{i} \neq x_{j} \). Finally, if \( S_{i} = \emptyset \), the convergence in \( W_{loc}^{2,q} \) follows from what we just proved and Theorem 3.1.

Basically, we showed that if a component of the sequence \( v^{k} \) blows up, then all its mass concentrates at a single point which has the lowest singularity coefficient.
The next Lemma gives some more important information about the convergence or the blow-up of the components of \( v^{k} \).

**Lemma 5.2.**
Let \( v_{i}^{k}, v_{i}, \rho_{i}^{0}, I \) and \( x_{i} \) as in Lemma 5.1. Then,

1. If \( i \in I \), then the sequence \( v_{i}^{k} - v_{i} \) converges to some \( G_{i} \) in \( W_{loc}^{2,q}(\Sigma \setminus \bigcup_{j \in I} \{ x_{j} \}) \) for some \( q > 1 \) and weakly in \( W^{1,q'}(\Sigma) \) for any \( q' \in (1,2) \), and \( G_{i} \) solves:

\[
\begin{align*}
-\Delta G_{i} &= \sum_{j \in I} a_{ij}\rho_{i}^{0}(\delta_{x_{j}} - 1) + \sum_{j \notin I} a_{ij}
\left( \tilde{h}_{j}e^{v_{j}} - \rho_{j}^{0} \right). \\
\mathcal{G}_{i} &= 0
\end{align*}
\]

2. If \( i \notin I \), then \( v_{i}^{k} \xrightarrow[k \to +\infty]{} v_{i} \) in the same space, and \( v_{i} \) solves:

\[
\begin{align*}
-\Delta v_{i} &= \sum_{j \notin I} a_{ij}\rho_{i}^{0}(\delta_{x_{j}} - 1) + \sum_{j \notin I} a_{ij}
\left( \tilde{h}_{j}e^{v_{j}} - \rho_{j}^{0} \right). \\
\int_{\Sigma} \tilde{h}_{i}e^{v_{i}}dV_{g} &= \rho_{i}^{0}
\end{align*}
\]  

(20)

**Proof.**
From Lemma 5.1 follows that, for \( i \in I \), \( \tilde{h}_{i}e^{v_{i}} \xrightarrow[k \to +\infty]{} \rho_{i}^{0}\delta_{x_{i}} \) in the sense of measures; in fact, if \( \phi \in C(\Sigma) \), then

\[
\left| \int_{\Sigma} \tilde{h}_{i}e^{v_{i}}\phi dV_{g} - \rho_{i}^{0}\phi(x_{i}) \right| \leq \varepsilon \int_{\Sigma} \tilde{h}_{i}e^{v_{i}}|\phi - \phi(x_{i})|dV_{g} + |\rho_{i}^{k} - \rho_{i}^{0}|\left| \phi(x_{i}) \right|
\]

\[
\leq \varepsilon \int_{B_{\delta}(x_{i})} \tilde{h}_{i}e^{v_{i}}dV_{g} + 2\| \phi \|_{L^{\infty}(\Sigma)} \int_{\Sigma \setminus B_{\delta}(x_{i})} \tilde{h}_{i}e^{v_{i}}dV_{g}
\]

\[
+ |\rho_{i}^{k} - \rho_{i}^{0}|\| \phi \|_{L^{\infty}(\Sigma)}
\]

\[
\leq \varepsilon \rho_{i}^{k} + 2\| \phi \|_{L^{\infty}(\Sigma)} \phi(1) + o(1)\| \phi \|_{L^{\infty}(\Sigma)}.
\]
which is, choosing properly $\varepsilon$, arbitrarily small. Therefore, $v_i$ solves (20).

On the other hand, if $q' \in (1, 2)$, then $\frac{q'}{q' - 1} > 2$, so any function $\phi \in W^{1, \frac{q'}{q' - 1}}(\Sigma)$ is actually continuous, hence

$$\left| \int_\Sigma \nabla \left( v^k_i - \bar{v}^k_i - G_i \right) \cdot \nabla \phi dV \right| = \left| \int_\Sigma (-\Delta v^k_i + \Delta G_i) \phi dV \right|$$

$$\leq \sum_{j \in I} a_{ij} \int_\Sigma \tilde{h}_j e^{v^k_i} \phi dV - \rho_j^0 \phi(p)$$

$$+ \sum_{j \notin I} a_{ij} \left| \int_\Sigma \tilde{h}_j \left( e^{v^k_i} - e^{v_i} \right) \phi dV \right| \quad \xrightarrow{k \to +\infty} 0.$$

Therefore, we get weak convergence in $W^{1, q'}(\Sigma)$ for any $q' \in (1, 2)$; standard elliptic estimates yield convergence in $W^{2,q}_{\text{loc}} \left( \Sigma \setminus \bigcup_{j \in I} \{x_j\} \right)$.

In the same way we prove the same convergence of $v^k_i$ to $v_i$. \qed

From these information about the blow-up profile of $v^k$ we deduce an important fact which will be used to prove the main Theorem:

**Corollary 5.3.**

Let $v^k$ and $x_i$ be as in Lemmas 5.1 and 5.2 and $w^k$ be defined by $w^k_i = \sum_{j=1}^N a_{ij} v^k_j$ for $i \in \{1, \ldots, N\}$.

Then, $w^k_i - \bar{w}^k_i$ is uniformly bounded in $W^{2,q}_{\text{loc}}(\Sigma \setminus \{x_i\})$ for some $q > 1$ if $i \in \mathcal{I}$, whereas if $i \notin \mathcal{I}$ it is bounded in $W^{2,q}(\Sigma)$.

**Proof.**

Since $-\Delta w^k_i = \tilde{h}_i e^{v^k_i} - \rho_i^k$, the claim follows from the boundedness of $e^{v^k_i}$ in $L^q_{\text{loc}}(\Sigma \setminus \{x_i\})$ and from standard elliptic estimates. \qed

The last Lemma we need is a localized scalar Moser-Trudinger inequality for the blowing-up sequence.

**Lemma 5.4.**

Let $w^k_i$ be as in Corollary 5.3 and $x_i$ as in the previous Lemmas. Then, for any $i \in \mathcal{I}$ and any small $r > 0$ one has

$$\frac{a_{ii}}{2} \int_{B_r(x_i)} \left| \nabla w^k_i \right|^2 dV - \rho_i^k \log \int_{B_r(x_i)} \tilde{h}_i e^{a_{ii}(w^k_i - \bar{w}^k_i)} dV \geq -C_r.$$

**Proof.**

Since $\Sigma$ is locally conformally flat, we can choose $r$ small enough so that we can apply Corollary 2.6 up to modifying $\tilde{h}_i$. We also take $r$ so small that $\overline{B_r(x_i)}$ contains neither any $x_j$ for $x_j \neq x_i$ nor any $p_m$ for $m = 1, \ldots, M$ (except possibly $x_i$).

Let $z^k_i$ be the solution of

\[
\begin{align*}
-\Delta z^k_i &= \tilde{h}_i e^{v^k_i} - \rho_i^k & \text{in } B_r(x_i) \\
z^k_i &= 0 & \text{on } \partial B_r(x_i)
\end{align*}
\]

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Then, \( w^k_i - \overline{w^k_i} - z^k_i \) is harmonic and it has the same value as \( w^k_i - \overline{w^k_i} \) on \( \partial B_r(x_i) \), so from standard estimates and the maximum principle

\[
\| w^k_i - \overline{w^k_i} - z^k_i \|_{C^1(B_r(x_i))} \leq C \| w^k_i - \overline{w^k_i} - z^k_i \|_{L^\infty(B_r(x_i))} \leq C \| w^k_i - \overline{w^k_i} \|_{L^\infty(\partial B_r(x_i))} \leq C.
\]

From Lemma 5.2 we get

\[
\left| \int_{B_r(x_i)} |\nabla w^k_i|^2 \, dV_g - \int_{B_r(x_i)} |\nabla z^k_i|^2 \, dV_g \right| = \left| \int_{B_r(x_i)} |\nabla (w^k_i - z^k_i)|^2 \, dV_g + 2 \int_{B_r(x_i)} \nabla w^k_i \cdot \nabla (w^k_i - z^k_i) \, dV_g \right|
\]

\[
\leq \int_{B_r(x_i)} |\nabla (w^k_i - z^k_i)|^2 \, dV_g + 2 \| \nabla w^k_i \|_{L^1(\Sigma)} \| \nabla (w^k_i - z^k_i) \|_{L^\infty(B_r(x_i))} \leq C_r.
\]

Moreover,

\[
\int_{B_r(x_i)} \tilde{h}_i e^{a_{ii}(w^k_i - \overline{w^k_i})} \, dV_g \leq e^{a_{ii}\| w^k_i - \overline{w^k_i} - z^k_i \|_{L^\infty(B_r(x_i))}} \int_{B_r(x_i)} \tilde{h}_i e^{a_{ii}z^k_i} \, dV_g \leq C_r \int_{B_r(x_i)} d(\cdot, x_i)^{2\tilde{\alpha}_i} e^{a_{ii}z^k_i} \, dV_g.
\]

Therefore, since \( \tilde{\alpha}_i \leq 0 \) and \( a_{ii}\rho^k_i \leq 8\pi(1 + \tilde{\alpha}_i) \), we can apply Corollary 2.6 to get the claim:

\[
\frac{a_{ii}}{2} \int_{B_r(x_i)} |\nabla w^k_i|^2 \, dV_g - \rho^k_i \log \int_{B_r(x_i)} \tilde{h}_i e^{a_{ii}(w^k_i - \overline{w^k_i})} \, dV_g \geq \frac{1}{2a_{ii}} \int_{B_r(x_i)} |\nabla (a_{ii}z^k_i)|^2 \, dV_g - \rho^k_i \log \int_{B_r(x_i)} d(\cdot, x_i)^{2\tilde{\alpha}_i} e^{a_{ii}z^k_i} \, dV_g - C_r
\]

\[
\geq -C_r.
\]

**Proof of Theorem 1.3.**

As noticed before, it suffices to prove the boundedness from below of \( J_{\rho^k} (u^k) \) for a sequence \( \rho^k \not\to \rho^0 \) and a sequence of minimizers \( u^k \) for \( J_{\rho^k} \); moreover, due to invariance by addition of constants, one can consider \( e^k \) in place of \( u^k \).

Let us start by estimating the term involving the gradients.

From Corollary 5.3 we deduce that the integral of \( |\nabla w^k_i|^2 \) outside a neighborhood of \( x_i \) is uniformly bounded for any \( i \in I \), and the integral on the whole \( \Sigma \) is bounded if \( i \not\in I \). For the same reason, the integral of \( a_{ij} \nabla w^k_i \cdot \nabla w^k_j \) on the whole surface is uniformly bounded. In fact, if \( a_{ij} \neq 0 \), then \( x_i \neq x_j \)

\[
\left| \int_{\Sigma} \nabla w^k_i \cdot \nabla w^k_j \, dV_g \right| \leq \int_{\Sigma \setminus B_r(x_j)} |\nabla w^k_i \cdot \nabla w^k_j| \, dV_g + \int_{\Sigma \setminus B_r(x_i)} |\nabla w^k_i \cdot \nabla w^k_j| \, dV_g \leq \| \nabla w^k_i \|_{L^2(\Sigma)} \| \nabla w^k_j \|_{L^2(\Sigma \setminus B_r(x_j))} + \| \nabla w^k_i \|_{L^2(\Sigma \setminus B_r(x_i))} \| \nabla w^k_j \|_{L^2(\Sigma)} \leq C_r,
\]

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where \( q' = \frac{2q}{3q - 2} < 2 \) and \( q'' = \frac{2q}{2 - q} \), with \( q \) as in Corollary 5.3 (or \( (q', q'') = (1, \infty) \) if \( q \geq 2 \)). Therefore, we can write
\[
\sum_{i,j=1}^{N} a_{ij} \int_{\Sigma} \nabla v_i^k \cdot \nabla v_j^k dV_g = \sum_{i,j=1}^{N} a_{ij} \int_{\Sigma} \nabla w_i^k \cdot \nabla w_j^k dV_g \\
\geq \sum_{i \in I} a_{ii} \int_{B_r(x_i)} |\nabla w_i^k|^2 dV_g - C_r.
\]

To deal with the other term in the functional, we use the boundedness of \( w_i^k \) away from \( x_i \): choosing \( r \) as in Lemma 5.4, we get
\[
\int_{\Sigma} \tilde{h}_i e^{v_i^k - \overline{v}^k} dV_g \leq 2 \int_{B_r(x_i)} \tilde{h}_i e^{v_i^k - \overline{v}^k} dV_g \\
= 2 \int_{B_r(x_i)} \tilde{h}_i e^{\sum_{j=1}^{N} a_{ij} \left( w_j^k - \overline{w}^k \right)} dV_g \\
\leq C_r \int_{B_r(x_i)} \tilde{h}_i e^{a_{ii} \left( w_i^k - \overline{w}^k \right)} dV_g.
\]

Therefore, using Lemma 5.4 we obtain
\[
J_{\rho_k}(u^k) = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij} \int_{\Sigma} \nabla v_i^k \cdot \nabla v_j^k dV_g - \sum_{i=1}^{N} \rho_i \int_{\Sigma} \tilde{h}_i e^{v_i^k - \overline{v}^k} dV_g \\
\geq \sum_{i \in I} \left( \frac{a_{ii}}{2} \int_{B_r(x_i)} |\nabla w_i^k|^2 dV_g - \rho_i \int_{B_r(x_i)} \tilde{h}_i e^{a_{ii} \left( w_i^k - \overline{w}^k \right)} dV_g \right) - C_r \\
\geq -C_r.
\]

Since the choice of \( r \) does not depend on \( k \), the proof is complete. \( \square \)

**Remark 5.5.**

The same arguments used to prove Theorems 1.1 and 1.3 can be applied to get the same results in the case of a compact surface with boundary \( \Sigma \) for the functional
\[
J_{\rho}(u) = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij} \int_{\Sigma} \nabla u_i \cdot \nabla u_j dV_g - \sum_{i=1}^{N} \rho_i \left( \log \int_{\Sigma} \tilde{h}_i e^{u_i} dV_g \right)
\]
on the space \( H^1_{0}(\Sigma) \) defined by (9). Its critical points solve
\[
\begin{align*}
-\Delta u_i &= \sum_{j=1}^{N} a_{ij} \rho_j \frac{\tilde{h}_j e^{u_j}}{\int_{\Sigma} \tilde{h}_j e^{u_j} dV_g} & i = 1, \ldots, N. \\
u_i|_{\partial\Sigma} &= 0
\end{align*}
\]

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