Maximally Chaotic Dynamical Systems
and
Fundamental Interactions

George Savvidy

Institute of Nuclear and Particle Physics, NCSR Demokritos, GR-15310 Athens, Greece
A.I. Alikhanyan National Science Laboratory, Yerevan, 0036, Armenia
Institut für Theoretische Physik, Universität Leipzig, D-04109 Leipzig, Germany

Abstract

We give a general review on the application of Ergodic theory to the investigation of dynamics of the Yang-Mills gauge fields and of the gravitational systems, as well as its application in the Monte Carlo method and fluid dynamics. In ergodic theory the maximally chaotic dynamical systems (MCDS) can be defined as dynamical systems that have nonzero Kolmogorov entropy. The hyperbolic dynamical systems that fulfil the Anosov C-condition belong to the MCDS insofar as they have exponential instability of their phase trajectories and positive Kolmogorov entropy. It follows that the C-condition defines a rich class of MCDS that span over an open set in the space of all dynamical systems. The large class of Anosov-Kolmogorov MCDS is realised on Riemannian manifolds of negative sectional curvatures and on high-dimensional tori. The interest in MCDS is rooted in the attempts to understand the relaxation phenomena, the foundations of the statistical mechanics, the appearance of turbulence in fluid dynamics, the non-linear dynamics of Yang-Mills field and gravitating N-body systems as well as black hole thermodynamics. Our aim is to investigate classical- and quantum-mechanical properties of MCDS and their role in the theory of fundamental interactions.

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1 Introduction

It seems natural to define a maximally chaotic dynamical system (MCDS) as a system that has a nonzero Kolmogorov entropy. A large class of hyperbolic MCDS was constructed by Anosov. These are systems that fulfil the C-condition, a condition that is sufficient for a system to be a MCDS. The Anosov hyperbolicity C-condition is about the local and uniform exponential instability of phase trajectories that leads to the mixing of all orders and to a positive Kolmogorov entropy. The uniqueness of the Anosov C-condition lies in the fact that it defines a rich class of MCDS that span over an open set in the space of all dynamical systems, meaning that in an arbitrary small neighbourhood of a given MCDS the dynamical systems are homeomorphic.

Examples of MCDS were discovered and discussed in the earlier investigations by Lobachevsky, Artin, Hadamard, Hedlund, Hopf, Birkhoff and others, as well as in more recent investigations. Here we shall introduce and discuss the classical- and quantum-mechanical properties of MCDS, the application of the ergodic theory to the investigation of non-Abelian gauge fields and gravitational systems, fluid dynamics and stability of the atmosphere as well as application in Monte Carlo method.

In recent years the quantum-mechanical concept of maximally chaotic systems was developed in a series of publications. The thermodynamics of black holes exhibits the extraordinary property of fast relaxation, in the sense that an arbitrary perturbation to a black hole "scrambles" as fast as possible over the horizon, making it indistinguishable from a thermal distribution. It has been conjectured that black holes are the fastest scramblers in nature. The influence of chaos on time-dependent double commutator of two observables can develop no faster than exponentially with the Lyapunov exponent $2\pi Tt$ that grows linearly in temperature and time. The linear growth is saturated in gravitational and dual to the gravity systems.

We are interested in analysing the behaviour of the out-of-time-order correlation functions and the double commutators in the case of well defined MCDS investigating the "influence and remnants" of the classical chaos on the quantum-mechanical behaviour of the quantised MCDS. Considering MCDS in their quantum-mechanical regime would help to identify the traces of classical chaos in quantum-mechanical regime and clarify the natural meaning of the quantum chaos. In particular, we shall consider the non-Abelian...
gauge field theory and the Artin system that is defined on a surface of constant negative curvature, a finite-area patch on $AdS_2$ \cite{3,129,152}.

The review is organised as follows.

Elements of Ergodic Theory. In the second section we shall define the Kolmogorov entropy and discuss the classification of the dynamical systems (DS) with respect to their statistical/chaotic properties \cite{20,22,23}. These are ergodic, mixing, n-fold mixing, and finally, the K-systems, which have mixing of all orders and the nonzero Kolmogorov entropy. This consideration defines the hierarchy of DS by their increasing chaotic/stochastic properties, with MCDS on the "top" of this hierarchy list.

Maximally Chaotic Dynamical Systems. The basic question is: Do MCDS exist? The Anosov hyperbolicity C-condition defines a rich class of MCDS that span over an open set in the space of all dynamical systems and is about the local and uniform exponential instability of the phase trajectories. It appears that a large class of MCDS can be realised on the Riemannian manifolds of negative sectional curvatures and on high-dimensional tori. We shall consider the general properties of the MCDS in the third section. The MCDS have very strong instability of their phase trajectories and, in fact, the instability is as strong as it can be in principle \cite{115}. The distance between infinitesimally close trajectories increases exponentially and on a closed phase space of the dynamical system this leads to the uniform distribution of phase trajectories over the whole phase space and exposes extended statistical properties \cite{4}. The MCDS are structurally stable \cite{4} spanning an open set in the space of all dynamical systems, which means that in an arbitrary small neighbourhood of a given MCDS the dynamical systems are homeomorphic \cite{3}. The other important property of the MCDS is that they have a countable set of periodic trajectories. The set of points on the periodic trajectories is everywhere dense in the phase space of MCDS. The periodic trajectories and non-periodic trajectories are filling out the phase space of an MCDS in a way very similar to the rational and irrational numbers on a Euclidean space \cite{4,32,34,99,100}.

Anosov C-condition and Geodesic Flows. The hyperbolic geodesic flow on Riemannian manifolds of negative sectional curvatures will be considered in the fourth section \cite{4,10,12,13,40}. It was proven by Anosov that the geodesic flow on a closed Riemannian manifold of negative sectional curvatures fulfils the C-condition and therefore defines a large class of MCDS. This result provides a powerful tool for the investigation of the Hamiltonian systems \cite{10}. If the time evolution of a classical Hamiltonian system under investigation can be reformulated as a geodesic flow on a Riemannian manifold and if all its sectional curvatures are negative, then it represents a
MCDS system. The MCDS approach the equilibrium state with exponential rate that depends on the Kolmogorov-Sinai entropy. The larger the entropy is, the faster a physical system moves toward its equilibrium.\textsuperscript{19,22,24,25,75,83,101,151,152}

Non-Abelian Gauge Field Theory Dynamics. In the fifth section we shall consider the classical and quantum mechanics of the Yang-Mills fields.\textsuperscript{11,17,18,53,55,62,120,121,123–126} In the case of space homogeneous gauge fields the Yang-Mills equations reduce to a classical mechanical system, so called Yang-Mills classical mechanics (YMCM), and represent the bosonic part of the matrix models.\textsuperscript{47,56–62} It has finite degrees of freedom,\textsuperscript{41–44} isotropic and homogeneous energy momentum tensor (40) and relativistic equation of state (41).\textsuperscript{65} By using the energy and momentum conservation integrals the system can be reduced to a system of a lower dimension, and the fundamental question is if the residual system is integrable and has additional hidden conserved integrals\textsuperscript{23} or is non-integrable and chaotic. The evolution of the YMCM can be formulated as a geodesic flow on a Riemannian manifold equipped with the Maupertuis’s metric. The investigation of sectional curvatures demonstrates that in the vicinity of the equipotential surface it is negative and causes the exponential instability of phase trajectories. The numerical integration also confirms this conclusion. The natural question that arises here is to what extent the classical chaos influences the quantum-mechanical properties of the non-Abelian gauge fields. The corresponding quantum-mechanical system defines an important class of matrix models.\textsuperscript{45,46,56–60} We shall discuss their spectral properties and the traces of the classical chaos in their quantum-mechanical regime.

Gravitational Systems, N-body Problem. An interesting application of the Anosov C-systems theory was found in the investigation of the relaxation phenomena in stellar systems like globular clusters and galaxies.\textsuperscript{73,75} Here again, one can use the Maupertuis’s metric in order to reformulate the evolution of the N-body system in Newtonian gravity as a geodesic flow on a Riemannian manifold. Investigation of sectional curvatures allows to estimate the exponential divergency of the phase trajectories and the relaxation time toward the stationary distribution of the star velocities in elliptic galaxies and globular clusters.\textsuperscript{73} This relaxation time is by few orders of magnitude shorter than the Chandrasekhar binary relaxation time.\textsuperscript{74,76–78} The difference is rooted in the fact that in this approach one can take into account the long-range interaction between stars through their collective contribution into the sectional curvature. The Hubble Deep Field and Hubble eXtreme Deep Field images revealed a large number of distant stars.

\textsuperscript{3}The apparent inhomogeneity of the energy momentum tensor (39) in Electrodynamics due to the term \(-E_iE_j - H_iH_j\) is a critical barrier for a successful vector field driven inflation.\textsuperscript{63,64} In Yang Mills theory the energy momentum tensor is perfectly homogeneous (40) and opens a room of possibilities for a vector field driven inflation.\textsuperscript{65} I would like to thank Prof. Viatcheslav Mukhanov for the discussion of this point.
young galaxies seemingly in a non-equilibrium state, while the stars in the nearby older galaxies show a more regular distribution of velocities and shapes, the result of the collective relaxation phenomenon.

**Artin Hyperbolic System.** Of special interest are the MCDS that are defined on a patch of the hyperbolic Lobachevsky plane of constant negative curvature. An example of such a system was defined in a brilliant article published in 1924 by the mathematician Emil Artin. The dynamical system is defined on the fundamental region of the Lobachevsky plane that is obtained by the identification of points congruent with respect to the modular group $SL(2, Z)$, a discrete subgroup of the Lobachevsky plane isometries $SL(2, R)$. The fundamental region in this case is a hyperbolic triangle, a non-compact region of a finite area. The geodesic trajectories are bounded to propagate on the fundamental hyperbolic triangle and are exponentially unstable. In the classical regime the exponential divergency of the geodesic trajectories resulted into the universal exponential decay of the classical correlation functions. The Artin symbolic dynamics, the differential geometry and group-theoretical methods of Gelfand and Fomin are used to investigate the exponential decay rate of the classical correlation functions in the seventh section.

**Quantum Mechanics of Artin System.** In the eighth section we shall describe the quantisation of the Artin system and review the derivation of the Maass wave functions corresponding to a continuous spectrum. There is a great interest in considering quantisation of the hyperbolic dynamical systems and investigation of their quantum-mechanical properties. This subject is very closely related to the investigation of quantum mechanics of classically chaotic systems in gravity. In the eighth section we shall study the behaviour of the correlation functions of the Artin hyperbolic system in its quantum-mechanical regime. In order to investigate the behaviour of the correlation functions in the quantum-mechanical regime it is necessary to know the spectrum of the system and the corresponding wave functions. In the case of the modular group the energy spectrum has a continuous part originating from asymptotically free motion inside an infinitely long channel extended in the vertical direction of the fundamental region. It also has infinitely many discrete energy states corresponding to the motion at the "bottom" of the fundamental triangle. The spectral problem has a deep number-theoretical origin and was partially solved in a series of pioneering articles. It was solved partially because the discrete spectrum and the corresponding wave functions are still not known analytically. The general properties of the discrete spectrum were derived by using Selberg trace formula. Numerical calculations of the discrete energy levels were performed for many energy states.
Out-of-time-order Correlation Functions. Having in hand the explicit expression of the wave functions one can analyse the quantum-mechanical behaviour of the correlation functions in order to investigate the traces of the classical chaos in the quantum-mechanical regime. In the ninth section we shall consider the correlation functions of the Liouville-like operators and shall demonstrate that all two- and four-point correlation functions decay exponentially with time, with the exponents that depend on temperature. Alternatively, the double commutator of the Liouville-like operators separated in time grows exponentially. This growth is reminiscent of the local exponential divergency of trajectories when it was considered in the classical regime. The results are presented in the Fig. The double commutator increases exponentially in time with the exponent $1/\chi(\beta)$. The ratio with respect to the maximal growth exponent is presented in Fig. The temperature dependence of Artin exponent $1/\chi(\beta)$ relative to the maximum growth $1/\beta$ is shown by blue dots in the far right figure. At high temperatures the Artin-Lyapunov exponent $1/\chi(\beta)$ is less than the maximal exponent $1/\beta$, but at low temperatures this is not any more true and we observe a breaking of the saturation regime. In order to confirm this result it seems important to investigate the behaviour of correlation functions and double commutators for alternative observables by using more powerful computer code and hardware than it was available to us.

Artin-Maass Resonances and Riemann Zeta Function Zeros. In the tenth section we shall demonstrate that the Riemann zeta-function zeros define the position and the widths of the resonances of the quantised Artin hyperbolic system. A possible relation of the zeta-function zeros and quantum-mechanical spectrum was discussed in the past, the Pólya-Hilbert conjecture proposed the idea of finding an eigenvalue problem with the spectrum containing the zeros of the Riemann zeta-function. The quantum-mechanical resonances have more complicated pole structure compared to a pure discrete spectrum and can be adequately described in terms of the scattering S-matrix theory. We shall use the S-matrix approach to analyse the scattering phenomenon in a quantised Artin system. As it was discussed above, the Artin dynamical system is defined on the fundamental region of the modular group on the Lobachevsky plane. It has a finite area and an infinite extension in the vertical direction that corresponds to a cusp. In the quantum-mechanical regime the system can be associated with the narrow infinitely long waveguide stretched out to infinity along the vertical axis and a cavity resonator attached to it at the bottom. That suggests a physical interpretation of the Maass automorphic wave function in the form of an incoming plane wave of a given energy entering the resonator and bouncing back to infinity. As the energy of the incoming wave comes close to the eigenmodes of the cavity a pronounced resonance behaviour shows up in the scattering amplitude. The
condition on the absence of incoming waves allows to find the position of these singularities. The poles of the S-matrix are located in the complex energy plane \( E_n - \frac{i\Gamma_n}{2} \), where \( E_n \) is the energy and \( \Gamma_n \) is the width of the \( n \)'th resonance and can be expressed in terms of zeros \( u_n \) of the Riemann zeta function \( \zeta(\frac{1}{2} - iu_n) = 0, \ n = 1, 2, ... \) as

\[
E_n = \frac{u_n^2}{4} + \frac{3}{16}, \quad \Gamma_n = u_n.
\]

It is an intriguing relation between the quantum-mechanical spectrum of MCDS and zeros of the Riemann zeta-function that can help to ”translate and dualise” the problem of distribution of the Riemann zeta-function zeros into its physical context.

C-cascades. Entropy and Periodic Trajectories. In the eleventh section we shall turn our attention to the investigation of the second class of the MCDS systems that is defined on high-dimensional tori. In order that the automorphisms of a torus fulfil the C-condition it is necessary and sufficient that the evolution operator \( T \) has no eigenvalues on a unit circle and has the determinant equal to one. Therefore \( T \) is an automorphism of the torus onto itself. All trajectories with rational coordinates, and only they, are periodic trajectories of the automorphisms of a torus. The entropy of the C-system on a torus is equal to the logarithmic sum of all eigenvalues that lie outside of the unit circle.

MIXMAX Random Number Generator. It was suggested in 1986 to use the MCDS defined on a torus to generate high-quality pseudorandom numbers for the Monte-Carlo method. Usually pseudorandom numbers are generated by deterministic recursive rules. Such rules produce pseudorandom numbers, and it is a great challenge to design pseudorandom number generators that produce high-quality sequences. Although numerous RNGs introduced in the last decades fulfil most of the requirements and are frequently used in simulations, each of them has some weak properties that influence the results and they are less suitable for demanding MC simulations. We shall describe the details of the computer implementation of the torus automorphisms, the computation of the periods of generated random sequences used in Monte Carlo simulations. In a typical computer implementation of the torus automorphisms the initial vector will have rational components. If the denominator is taken to be a prime number, then the recursion is realised on extended Galois field \( GF[p^N] \) and allows to find the period of the trajectories in terms of \( p \) and the properties of the characteristic polynomial of the evolution operator \( T \). We present the derivation of the explicit formulas for the Kolmogorov entropy in case of torus automorphisms and the properties of the periodic trajectories and their density distribution as a function of Kolmogorov entropy.

The efficient implementation of the MIXMAX MCDS generators for Monte Carlo simulations was developed. The MIXMAX generators demonstrated excellent statistical
properties, high performance and superior high-quality output and became a multidisciplinary usable product. The main characteristics of the generators are: a) MIXMAX is an original and genuine 64-bit generator, is one of the fastest generators producing a 64-bit pseudorandom number in approximately 4 nanoseconds, b) has very large Kolmogorov entropy of 0.9 per/bit, c) long periods of order of $10^{120} - 10^{5000}$, d) a new skipping algorithm generates seeds and guarantees that streams are not overlapping. The MIXMAX generators were integrated into the concurrent and distributed MC toolkit Geant4, the foundation library CLHEP and data analysis framework ROOT. These software tools have wide applications in High Energy Physics at CERN, in CMS experiment at SLAC, FNAL and KEK National Laboratories and are part of the CERN’s active Technology Transfer policy. The generator is available in the PYTHIA event generator. The MIXMAX code can be downloaded from the GSL-GNU Scientific Library and boost C++ Libraries.

Turning C-cascade into C-flow. In the thirteenth section we shall consider continuous in time dynamical systems constructed so that the discrete time slices will coincide with a given discrete authomorphism. In Anosov demonstrated how any C-cascade on a torus can be embedded into a continuous C-flow. Here we shall describe the Anosov construction that allows to embed a discrete time evolution on a torus into an evolution that is continuous in time. The embedding was defined by a specific identification of the phase space coordinates and by construction of the corresponding C-flow on a smooth Riemannian manifold of higher dimension. In Anosov construction the C-flow was not defined as a geodesic flow. We were interested in formulating and analysing the alternative system that is defined as a geodesic flow on the same Riemannian manifold. The calculation of the corresponding sectional curvatures demonstrated that the geodesic flow has different dynamics and hyperbolic components.

Infinite Dimensional Limit of C-cascade. In the fourteenth section we define the infinite-dimensional limit of the MCDS that are defined on N-dimensional tori. As we have just discussed, these hyperbolic systems found successful application in computer algorithms that generate high-quality pseudorandom numbers for advanced Monte Carlo simulations. The chaotic properties of these systems are increasing with $N$ because the corresponding Kolmogorov-Sinai entropy grows linearly with $N$. We are interested in considering the systems in the limit $N \to \infty$. The limiting system defines the hyperbolic evolution of the continuous functions that is very similar to the evolution of a velocity function describing the hydrodynamic flow of fluids. We compare the chaotic properties of the limiting system with those of the hydrodynamic flow of incompressible ideal fluid on a torus investigated by Arnold. This maximally chaotic system can find application in the Monte Carlo method, statistical physics and digital signal
processing.

**Fluid Dynamics and Stability of the Atmosphere.** It is interesting to know if similar systems were investigated in the past. The solutions of the partial differential equation describing the evolution $t \to g_t(x)$ of the hydrodynamical flow of incompressible ideal fluid filled in a two-dimensional torus $x \in T^2$ can be considered as a continuous area preserving diffeomorphims $SDiff(T^2)$ of a torus $T^2 \to T^2$. In the Arnold approach\cite{arnold1983} the ideal fluid flow is described by the geodesics $g_t(x) \in G$ on the diffeomorphism group $G = SDiff(T^2)$ with the velocity $v_t(x) = \dot{g}_t(x)g_t^{-1}(x)$ belonging to the corresponding algebra $\mathfrak{g} = \text{sdiff}(T^2)$ of divergence-free vector fields. The Riemannian metric on the group $G$ is induced from the metric on a torus,\cite{arnold1983} and the stability of the geodesic flows on the group $G$ can be analysed by investigating the behaviour of the corresponding sectional curvatures $K(v, \delta v)$.\cite{arnold1983}

It was found that the flow that is defined by a parallel velocity field on $T^2$ is unstable because the sectional curvatures are negative and the flow is exponentially unstable. This shows that it is not possible to reasonably predict the weather beyond a certain period if one assumes that the Earth has a torus topology and its atmosphere is a two-dimensional incompressible fluid. A similar stability analysis was performed for the hydrodynamical flow on a two-dimensional sphere $S^2$\cite{arnold1983,arnold1987,arnold1989} as it is important to use the more realistic assumption that the surface of the Earth is a sphere. In all these cases the flow is exponentially unstable in some directions and is stable in some other directions, resulting in the limitation of predictability of the hydrodynamical flow and leading to the principal difficulties of a long-term weather forecasting.

## 2 Hierarchy of DS and Kolmogorov Entropy

In the ergodic theory the dynamical systems are classified by the increase of their statistical-chaotic properties. *Ergodic systems* are defined as follows.\cite{leibenzon1924,leibenzon1926} Let $x = (q, p)$ be a point of the phase space $x \in M$ of the Hamiltonian systems. The canonical coordinates are denoted as $q = (q_1, ..., q_d)$ and $p = (p_1, ..., p_d)$ are the conjugate momenta. The phase space $M$ is equipped with a positive Liouville measure $d\mu(x) = \rho(q, p) dq_1 ... dq_d dp_1 ... dp_d$, which is invariant under the Hamiltonian flow. The operator $T^t x = x_t$ defines the time evolution of the trajectory starting from the initial point $x$ of the phase space. The ergodicity of the DS takes place if for almost all $x \in M$\cite{leibenzon1924,leibenzon1926}:

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t dt f(T^t x) = \int_M f(x) dx, \quad (1)$$

\footnote{In the following we shall be writing $dx$ instead of $d\mu(x)$ in order to simplify the expressions.}
where \( f(x) \) is a function/observable defined on the phase space \( M \). The time average is equal to the space average almost everywhere, that is, for all \( x \) except for a set of measure zero. It follows then that

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t dt f(T^t x) g(x) dx = \int_M f(x) dx \int_M g(x) dx.
\] (2)

Considering the function \( f = \chi_A \) on the phase space that is equal to one on a set \( A \subset M \) and to zero outside, and similarly the function \( g = \chi_B \) on a set \( B \subset M \), one can get

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t dt \mu[T^t A \cap B] = \mu[A] \mu[B],
\] (3)

The measures of the points of the set \( A \) that fall into the set \( B \) are on average proportional to their measures. The systems with stronger chaotic properties have been defined by Gibbs\textsuperscript{[16,22]} The mixing takes place if for any two sets:

\[
\lim_{t \to \infty} \mu[T^t A \cap B] = \mu[A] \mu[B],
\] (4)

that is, a part of the set \( A \) that falls into the set \( B \) is proportional to their measures. Alternatively,

\[
\lim_{t \to \infty} \int f(T^t x) g(x) dx = \int_M f(x) dx \int_M g(x) dx,
\] (5)

which means that the two-point correlation function tends to zero:

\[
D_t(f, g) = \lim_{t \to \infty} \langle f(T^t x) g(x) \rangle - \langle f(x) \rangle \langle g(x) \rangle = 0,
\] (6)

and is known in field theory language as the factorisation property of the two-point correlation functions. The n-fold mixing takes place if for any \( n \) sets

\[
\lim_{t_n, \ldots, t_1 \to \infty} \mu[T^{t_n} A_n \cap \ldots T^{t_2} A_2 \cap T^{t_1} A_1 \cap B] = \mu[A_n] \ldots \mu[A_2] \mu[A_1] \mu[B]
\] (7)

or, alternatively,

\[
D_t(f_n, \ldots, f_1, g) = \lim_{t_n, \ldots, t_1 \to \infty} \langle f_n(T^{t_n} x) \ldots f_1(T^{t_1} x) g(x) \rangle - \langle f_n(x) \rangle \ldots \langle f_1(x) \rangle \langle g(x) \rangle = 0.
\] (8)

A class of dynamical systems that have even stronger chaotic properties was introduced by Kolmogorov in\textsuperscript{[12]} These are the DS’s which have a non-zero entropy, so called quasi-regular DS’s, or simply K-systems. In order to define the Kolmogorov entropy let us consider a discrete time evolution operator \( T^n x = x_n, n = 0, 1, 2, \ldots \) Let \( \alpha = \{A_i\}_{i \in I} \) ( \( I \) is finite or countable) be a measurable partition of the phase space \( M \) into the nonintersecting subsets \( A_i \) that cover the whole phase space \( M \), that is,

\[
\mu(M \setminus \bigcup_{i \in I} A_i) = 0, \quad \mu(A_i \cap A_j) = 0, i \neq j.
\] (9)
and define the entropy of the partition $\alpha$ as

$$h(\alpha) = -\sum_{i \in I} \mu(A_i) \ln \mu(A_i).$$

(10)

If two partitions $\alpha_1$ and $\alpha_2$ differ by a set of measure zero, then their entropies are equal. The refinement partition $\alpha$

$$\alpha = \alpha_1 \lor \alpha_2 \lor ... \lor \alpha_k$$

(11)

of the collection of partitions $\alpha_1, ..., \alpha_k$ is defined as the intersection of all their composing sets $A_i$:

$$\alpha = \{ \bigcap_{i \in I} A_i \mid A_i \in \alpha_i \text{ for all } i \}.$$  

(12)

The entropy of the partition $\alpha$ with respect to the automorphisms $T$ is defined as a limit:

$$h(\alpha, T) = \lim_{n \to \infty} \frac{h(\alpha \lor T \alpha \lor ... \lor T^{n-1} \alpha)}{n}, \quad n = 1, 2, ...$$

(13)

This number is equal to the entropy of the refinement $\beta = \alpha \lor T \alpha \lor ... \lor T^{n-1} \alpha$ that was generated during the iteration of the partition $\alpha$ by the automorphism $T$. Finally the entropy of the automorphism $T$ is defined as a supremum:

$$h(T) = \sup_{\{\alpha\}} h(\alpha, T),$$

(14)

where the supremum is taken over all partitions $\{\alpha\}$ of $M$. It was proven that the K-systems have mixing of all orders: K-mixing $\supset$ infinite mixing, $\supset$ n-fold mixing, $\supset$ mixing $\supset$ ergodicity. The calculation of the entropy for a given dynamical system seems extremely difficult. The theorem proven by Kolmogorov tells that if one finds the so called "generating partition" $\beta$, then

$$h(T) = h(\beta, T),$$

(15)

meaning that the supremum in (14) is reached on a generating partition $\beta$. In some cases the construction of the generating partition $\beta$ allows an explicit calculation of the entropy of a given dynamical system.

In summary, the above consideration allows to define the hierarchy of DS’s with their increasing chaotic properties and with the maximally chaotic K-systems on the “top” of this hierarchy list. The main question now is: Do the maximally chaotic systems exist and how to decide to which ergodicity class belongs a DS under consideration? The Anosov C-condition is a powerful tool defining the criteria under which a DS belongs to a maximally chaotic class. The hyperbolic C-systems introduced by Anosov represent a large class of K-systems and are structurally stable under small perturbations. We shall consider the C-systems in the next section.
3 Anosov Hyperbolic C-systems

In the fundamental work on geodesic flows on closed Riemannian manifolds of negative curvature Anosov pointed out that the basic property of the geodesic flow on such manifolds is the uniform instability of the phase trajectories, which in physical terms means that in the neighbourhood of almost every fixed trajectory the trajectories behave similarly to the trajectories in the neighbourhood of a saddle point (see Fig. 1). In other words, the hyperbolic instability of the dynamical system $T^t$ which is defined by the equations

$$\dot{x} = f(x), \quad x = (x_1, ..., x_d)$$ (16)

takes place for almost all solutions $\delta x \equiv \omega$ of the deviation equation

$$\dot{\omega} = \frac{\partial f}{\partial x} \bigg|_{x(t) = T^t x} \omega$$ (17)

in the neighbourhood of the phase trajectory $x(t) = T^t x$, where $x \in M$.

The exponential instability of geodesics on Riemannian manifolds of constant negative curvature has been studied by many authors, beginning with Lobachevsky and Hadamard and especially by Artin, Hedlund and Hopf. The concept of exponential instability of a dynamical system trajectories appears to be extremely rich and Anosov suggested to elevate it into a fundamental property of a new class of dynamical systems which he called C-systems.

The brilliant idea to consider dynamical systems which have local and homogeneous hyperbolic instability of the phase trajectories is appealing to the intuition and has deep physical content. The richness of the concept is expressed by the fact that the C-systems occupy a nonzero volume in the space of all dynamical systems and have a non-zero Kolmogorov entropy. Anosov provided an extended list of MCDS. The important examples of the MCDS are: i) the geodesic flow on the Riemannian manifolds of variable negative sectional curvatures and ii) C-cascades - the iterations of the hyperbolic automorphisms of tori.

In the forthcoming sections we shall consider these maximally chaotic systems in details and the application of the MCDS theory to the investigation of the Yang-Mills dynamics, the N-body problem in gravity, fluid dynamics and in the Monte Carlo method. We shall consider
Figure 1: The integral curves in the case of saddle point $\dot{x} = -x$, $\dot{y} = y$ are exponentially contracting and expanding near the solution $x = y = 0$. A similar behaviour takes place in the neighbourhood of almost all trajectories of the C-system as one can get convinced inspecting the solutions (29) and (30) of the Jacobi variation equation (21) with negative sectional curvatures.

the quantum-mechanical properties of the maximally chaotic dynamical systems as well and in particular the DS's which are defined on the closed surfaces of constant negative curvature imbedded into the Lobachevsky hyperbolic plane.

4 Geodesic Flow on Manifolds of Negative Sectional Curvatures

Let us consider the stability of the geodesic flow on a Riemannian manifold $Q$ with local coordinates $q^\alpha \in Q$ where $\alpha = 1, 2, ..., 3N$. The functions $q^\alpha(s) \in Q$ define a one-parameter integral curve $\gamma(s)$ on a Riemannian manifold $Q$ and the corresponding velocity vector

$$u^\alpha = \frac{dq^\alpha}{ds}, \quad \alpha = 1, 2, ..., 3N.$$  \hspace{1cm} (18)

The proper time parameter $s$ along the $\gamma(s)$ is equal to the length, while the Riemannian metric on $Q$ is defined as

$$ds^2 = g_{\alpha\beta}dq^\alpha dq^\beta,$$

and therefore

$$g_{\alpha\beta}u^\alpha u^\beta = 1.$$  \hspace{1cm} (19)

A one-parameter family of deformations assumed to form a congruence of world lines $q^\alpha(s) \rightarrow q^\alpha(s, \nu)$. In order to characterise the infinitesimal deformation of the curve $\gamma(s)$ it is convenient to define a separation vector $\delta q^\alpha = \frac{\partial q^\alpha}{\partial \nu} d\nu$, where $\delta q^\alpha$ is a separation of points having equal distance from some arbitrary initial points along two neighbouring curves.

The resulting phase space manifold $(q(s), u(s)) \in M$ has a bundle structure with the base $q \in Q$ and the spheres $S^{3N-1}$ of unit tangent vectors $u^\alpha$ (19) as fibers. The integral curve $\gamma(s)$ fulfills the geodesic equation

$$\frac{d^2q^\alpha}{ds^2} + \Gamma_{\beta\gamma}^\alpha \frac{dq^\beta}{ds} \frac{dq^\gamma}{ds} = 0$$  \hspace{1cm} (20)
and the relative acceleration depends only on the Riemann curvature:

\[ \frac{D^2 \delta q^\alpha}{ds^2} = -R^\alpha_{\beta\gamma\sigma} u^\beta \delta q^\gamma u^\sigma. \] (21)

The above form of the Jacobi equation is difficult to analyse, first of all because it is written in terms of covariant derivatives \( Du^\alpha = du^\alpha + \Gamma^\alpha_{\beta\gamma} u^\beta d\xi^\gamma \). And secondly because it is written in terms of separation of points on trajectories instead of distance between trajectories. Following Anosov it is convenient to represent the Jacobi equation in terms of simple derivatives. The norm of the deviation \( \delta q \) has the form

\[ |\delta q|^2 \equiv g_{\alpha\beta} \delta q^\alpha \delta q^\beta \]

and its second derivative is

\[ \frac{d^2}{ds^2} |\delta q|^2 = 2g_{\alpha\beta} \frac{D^2 \delta q^\alpha}{ds^2} + 2g_{\alpha\beta} \frac{D \delta q^\alpha}{ds} \frac{D \delta q^\beta}{ds}. \] (22)

Using (21) we shall get the Anosov form of the Jacobi equation

\[ \frac{d^2}{ds^2} |\delta q|^2 = -2R_{\alpha\beta\gamma\lambda} u^\beta \delta q^\gamma u^\lambda + 2|\frac{D \delta q}{ds}|^2 = -2K(q,u,\delta q) |u \wedge \delta q|^2 + 2|\delta u|^2 \] (23)

where \( K(q,u,\delta q) \) is the sectional curvature in the two-dimensional directions defined by the velocity vector \( u^\alpha \) and the deviation vector \( \delta q^\beta \):

\[ K(q,u,\delta q) = \frac{R_{\alpha\beta\gamma\sigma} u^\beta \delta q^\gamma u^\sigma}{|u \wedge \delta q|^2}. \] (24)

One can decompose the deviation vector \( \delta q \) into longitudinal and transversal components \( \delta q^\alpha = \delta q^\alpha_{\parallel} + \delta q^\alpha_{\perp} \), where \( \delta q^\alpha_{\parallel} \) describes a translation along the geodesic trajectories and has no physical interest, the transversal component \( \delta q^\alpha_{\perp} \) describes a physically relevant distance between original and infinitesimally close trajectories \( |u \delta q^\alpha_{\perp}| = 0 \). Such decomposition allows to derive the Jacobi equation only in terms of transversal deviation

\[ \frac{d^2}{ds^2} |\delta q^\alpha_{\perp}|^2 = -2K(q,u,\delta q^\alpha_{\perp}) |\delta q^\alpha_{\perp}|^2 + 2|\delta u^\alpha_{\perp}|^2. \] (25)

Because the last term is positive function the following inequality takes place for relative acceleration:

\[ \frac{d^2}{ds^2} |\delta q^\alpha_{\perp}|^2 \geq -2K(q,u,\delta q^\alpha_{\perp}) |\delta q^\alpha_{\perp}|^2. \] (26)

If the sectional curvature is negative and uniformly bound from above by a constant \( \kappa \):

\[ K(q,u,\delta q^\alpha_{\perp}) \leq -\kappa < 0, \quad \text{where} \quad \kappa = \min_{(q,u,\delta q^\alpha_{\perp})} |K(q,u,\delta q^\alpha_{\perp})| \] (27)

9The area spanned by the bivector is simplifies \( |u \wedge \delta q^\alpha_{\perp}|^2 = |u|^2 |\delta q^\alpha_{\perp}|^2 - |u \delta q^\alpha_{\perp}|^2 = |\delta q^\alpha_{\perp}|^2 \), because \( |u|^2 = 1 \) and \( |u \delta q^\alpha_{\perp}| = 0 \).
then
\[ \frac{d^2}{ds^2} |\delta q_\perp|^2 \geq 2\kappa |\delta q_\perp|^2. \]  
(28)
The phase space of solutions of the second-order differential equation is divided into two separate sets \( X_q \) and \( Y_q \). The set \( Y_q \) consists of the solutions with positive first derivative
\[ \frac{d}{ds} |\delta q_\perp(0)|^2 > 0 \]
and exponentially grows with \( s \to +\infty \)
\[ |\delta q_\perp(s)| \geq \frac{1}{2} |\delta q_\perp(0)| e^{\sqrt{2\kappa} s}, \]  
(29)
while the set \( X_q \) consists of the solutions with negative first derivative
\[ \frac{d}{ds} |\delta q_\perp(0)|^2 < 0 \]
and decay exponentially with \( s \to +\infty \)
\[ |\delta q_\perp(s)| \leq \frac{1}{2} |\delta q_\perp(0)| e^{-\sqrt{2\kappa} s}. \]  
(30)
This proves that the geodesic flow on closed Riemannian manifold of negative curvature fulfills the C-conditions, is therefore maximally chaotic and tends to equilibrium with exponential rate. We shall define a relaxation time as
\[ \tau = \frac{1}{\sqrt{2\kappa}}, \]  
(31)
which is inversely proportional to the Kolmogorov entropy.

The above consideration demonstrate that the geodesic flow on closed Riemannian manifold of negative sectional curvature fulfills the C-condition and defines a large class of maximally chaotic systems with nonzero Kolmogorov entropy. This result provides a powerful tool for the investigation of the Hamiltonian systems. If the time evolution of a classical Hamiltonian system under investigation can be reformulated as the geodesic flow on the Riemannian manifold and if all sectional curvatures are negative, then it represents a MCDS. In physical terms this means that the phase space of a DS does not have invariant tori of an integrable system and

\[ \text{It follows from the variation equation (28) and the boundary condition imposed on the deviation } \delta q_\perp \text{ and its first derivative } \frac{d}{ds} |\delta q_\perp|^2 \text{ that for all } s \text{ the } \frac{d^2}{ds^2} |\delta q_\perp|^2 > 0, \text{ therefore } |\delta q_\perp|^2 \text{ is a convex function and its graph is convex downward. Thus the variation equation has no conjugate points because if } \delta q(s_1) = 0, \delta q(s_2) = 0 \text{ and } s_1 \neq s_2 \text{ then } \delta q(s) \equiv 0 \text{ for } s_1 \leq s \leq s_2. \text{ The sets } X_q \text{ and } Y_q \text{ are defined as follows. The set } X_q \text{ consists of the vectors } (\delta q(s), \frac{d\delta q(s)}{ds}) \to 0 \text{ when } s \to +\infty \text{ and the set } Y_q \text{ of the vectors } (\delta q(t), \frac{d\delta q(t)}{ds}) \to 0 \text{ when } s \to -\infty. \text{ If } (\delta q(s), \frac{d\delta q(s)}{ds})_{s=0} \in X_q \text{ then the first derivative is negative } \frac{d}{ds} |\delta q_\perp|^2 < 0 \text{ for all } s. \text{ As well if } (\delta q(s), \frac{d\delta q(s)}{ds})_{s=0} \in Y_q \text{ then } \frac{d^2}{ds^2} |\delta q_\perp|^2 > 0 \text{ for all } s. \]
its trajectories cover the whole phase space. In the next sections we shall apply this approach to the investigation of Yang-Mills Classical Mechanics to the N-body problem in Newtonian gravity and Monte Carlo method. The MCDS have a tendency to approach the equilibrium state with exponential rate depending on the entropy. That will allow to calculate the relaxation time of stars in galaxies and the quality of Monte Carlo generators. The larger the entropy is, the faster a physical system tends to its equilibrium.

5 Classical and Quantum Mechanics of Yang-Mills field

It is crucially important to find and analyse the classical solutions of the Yang-Mills equations without the external sources in Minkowski space, which may prove to be useful for the construction and study of the Yang-Mills theory vacuum and asymptotic states. Searching the solutions of the classical Yang-Mills equations for which the Poynting vector vanishes, that is there is no energy flux, it was found that space-homogeneous gauge fields satisfy the above condition.

For space-homogeneous gauge fields $\partial_i A_i^a = 0$, $i, k = 1, 2, 3$ the Yang-Mills equation reduces to a classical mechanical system with the Hamiltonian of the form

$$H = \sum_i \frac{1}{2} Tr A_i \dot{A}_i + \frac{g^2}{4} \sum_{i,j} Tr [A_i, A_j]^2,$$

where the gauge field $A_i^a(t)$ depends only on time, $i = 1, 2, 3$, the index $a = 1, \ldots, N^2 - 1$ for $SU(N)$ group and in the Hamiltonian gauge $A_0 = 0$ the Gaussian constraint has the form

$$G = [\dot{A}_i, A_i] = 0.$$  

It is natural to call this system the Yang-Mills Classical Mechanics (YMCM). It is a mechanical system with $3 \cdot (N^2 - 1)$ degrees of freedom. It is important to investigate classical equations of motion of this class of non-Abelian gauge fields, the properties of the separate solutions and of the system as a whole. In particular its integrability versus chaotic properties of the system. The YMCM has a number of conserved integrals: the space and isospin angular momenta

$$m_i = \epsilon_{ijk} A_j^a \dot{A}_k^a, \quad n_i^a = f^{abc} A_i^a \dot{A}_i^b, \quad i = 1, 2, 3, \quad a = 1, \ldots, N^2 - 1$$

in total $3 + (N^2 - 1)$ integrals, plus the energy integral (32) (the $n_i^a = 0$ due to the constraint (33)). The original "white colour" solution found in has the form

$$A_i^a = \delta_i^a f(t),$$
and the corresponding chromoelectric and chromomagnetic fields are:

\[ E_i^a = \delta_i^a \dot{f}(t), \quad H_i^a = g\delta_i^a f^2(t). \] (36)

The energy density is:

\[ \epsilon = T_{00} = \frac{3}{2} (\dot{f}^2 + g^2 f^4) = \mu^4, \] (37)

where \( \mu^4 \) is a constant of dimension \( \text{mass}^4 \). Unusual property of this solution is that the chromoelectric and chromomagnetic fields are parallel to each other and therefore the energy flux, the Pyonting vector, vanishes

\[ T_{0i} = S_i = \epsilon_{ijk} E_j^a H_k^a = 0. \] (38)

The other important property is that the space components of \( T_{\mu\nu} \) are diagonal:

\[ T_{ij} = \frac{1}{2} \delta_{ij} (E_k^a E_k^a + H_k^a H_k^a) - E_i^a E_j^a - H_i^a H_j^a = \frac{1}{2} \delta_{ij} \left( \dot{f}^2 + g^2 f^4 \right) = \delta_{ij} p \] (39)

and the full energy momentum tensor has the form of a relativistic matter:

\[ T_{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \] (40)

Indeed it follows from relations (37), (38) and (39) that the Yang Mills equation of state is equivalent to a homogeneous relativistic matter\(^\text{11}\):

\[ p = \frac{1}{3} \epsilon. \] (41)

The space homogeneous time-dependent vacuum solutions of the Yang Mills equations were considered in the context of the cosmological models and inflation in\(^\text{65,66,68–70,72,72}\). The question is if the YMCM system is exactly integrable and has additional integrals of motions or it is a chaotic system. If the number of conserved integrals coincide with the number of degrees of freedom, then the system is exactly integrable and the phase trajectories lie on high-dimensional tori, if there are less integrals, then the trajectories lie on a manifold of a larger dimension, and if there is no conserved integrals at all, then the trajectories are distributed over the full phase-space\(^3\).

\(^{11}\)The apparent inhomogeneity of the energy momentum tensor (39) in Electrodynamics due to the term \(-E_i E_j - H_i H_j\) is a critical barrier for a successful vector field driven inflation\(^\text{63,64}\). In Yang Mills theory the energy momentum tensor is perfectly homogeneous (40) and opens a room of possibilities for a vector field driven inflation\(^\text{65}\). I would like to thank Prof. Viatcheslav Mukhanov for the discussion of this point.
Figure 2: A single trajectory of the YMCM system integrated over a large time interval. The trajectory scatters on the equipotential surface $x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2 = 1$, densely filling the interior region and making visible the six channels of the equipotential surface on which a trajectory scatters.

Let us consider in details the case of the $SU(2)$ gauge group by introducing the angular variables which allow to separate the angular motion from the oscillations by using the substitution

$$A = O_1 E O_2^T,$$

where $E = (x(t), y(t), z(t))$ is a diagonal matrix and $O_1, O_2$ are orthogonal matrices parametrised by Euler angular variables. In this variables the Hamiltonian will take the form

$$H_{FS} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{g^2}{2}(x^2y^2 + y^2z^2 + z^2x^2) + T_{YM},$$

where $T_{YM}$ is the rotational kinetic energy of the the Yang-Mills “top” spinning in space and iso-space. The question is whether the YMCM system is an integrable system or not. The general behaviour of the colour amplitudes $(x(t), y(t), z(t))$ in time is characterised by rapid oscillations, decrease in some colour amplitudes and growth in others, colour “beats” Fig.2. The strong instability of the trajectories with respect to small variations of the initial conditions in the phase space led to the conclusion that the system is stochastic and non-integrable. The search of the conserved integrals of the form $F(p_x, p_y, p_z, x, y, z)$ fulfilling the equation $\{F, H_{FS}\} = 0$ also confirms their absence, except the Hamiltonian itself. The evolution of the YMCM can be formulated as the geodesic flow on a Riemannian manifold with the Maupertuis’s metric (see the details in the next section). The investigation of the sectional curvature demonstrated that it is negative on the equipotential surface and generates exponential instability of the trajectories Fig.2. The solutions of an YMCM system in an arbitrary coordinate system (after a Lorentz boost) are nonlinear plane waves $A_\mu^a(k \cdot x)$ with a nonzero square of the wave vector $k^2 = \mu^2$ chaotically oscillating in space-time.

The natural question which arrises here is to what extent the classical chaos influences the quantum-mechanical properties of the gauge fields. The significance of the answer to this
question consists in the following. In field theory, e.g. in QED, the electromagnetic field is represented in the form of a set of harmonic oscillators whose quantum-mechanical properties (as of an integrable system) are well known, and the interaction between them is taken into account by perturbation theory. Such an approach excellently describes the experimental situation. In QCD the state of things is quite different. The properties of the YMCM as of a MCDS, cannot be established to any finite order of the perturbation theory. Therefore to understand QCD it seems important to investigate the quantum-mechanical properties of the systems which in the classical limit are maximally chaotic.

The natural question arising now is what quantum-mechanical properties does the system with the Hamiltonian (32) possess if in the classical limit \( \hbar \to 0 \) it is maximally chaotic. What is the structure of the energy spectrum and of the wave functions of quantised gauge system, if in the classical limit it is chaotic. The Schrödinger equation for the gauge field theory in the \( A_0 = 0 \) gauge has the following form:

\[
\frac{1}{2} \int d^3x \left[ -\frac{\delta^2}{\delta A_i^a \delta A_i^a} + H_i^a H_i^a \right] \Psi[A] = E \Psi[A],
\]

where \( H_i^a = \frac{1}{2} \epsilon_{ijk} G_{jk}^a (A) \) and the constraint equations are:

\[
\left[ \delta^{ab} \partial_i + g f^{acb} A_i^c \right] \frac{\delta}{\delta A_i^a} \Psi[A] = 0.
\]

In case of space-homogeneous fields the equations will reduce to the Yang-Mills quantum-mechanical system (YMQM) with finite degrees of freedom which defines a special class of quantum-mechanical matrix models. At zero angular momentum \( \hat{m}_i = 0 \) the YMQM Schrödinger equation takes the form (equations (20) and (21) in 46)

\[
\left\{ -\frac{1}{2} D^{-1} \partial_i D \partial_i + \frac{g^2}{2} \left( x_i^2 x_j^2 + x_2 x_3 x_1 \right) \right\} \Psi = E \Psi,
\]

where \( D(x) = |(x_1^2 + x_2^2)|(x_2^2 + x_3^2)|x_3^2 - x_1^2| \). Using the substitution

\[
\Psi(x) = \frac{1}{\sqrt{D(x)}} \Phi(x)
\]

and the fact that the \( D(x) \) is a harmonic function \( \partial^2_i D(x) = 0 \) the equation can be reduced to the form

\[
-\frac{1}{2} \partial_i^2 \Phi + \frac{1}{2} \sum_{i<j} \left( \frac{1}{(x_i - x_j)^2} + \frac{1}{(x_i + x_j)^2} + g^2 x_i^2 x_j^2 \right) \Phi = E \Phi.
\]

The analytical investigation of this Schrödinger equation is a challenging problem because the equation cannot be solved by separation of variables as far as all canonical symmetries
are already extracted and the residual system possess no continuous symmetries. Nevertheless some important properties of the energy spectrum can be established by calculating the volume of the classical phase space defined by the condition $H(p, q) \leq E$. It follows that classical phase space volume is finite and the energy spectrum of YMQM system is discrete. Typically the classically chaotic systems have no degeneracy of the energy spectrum, the energy levels “repulse” from each other similar to the distribution of the eigenvalues of the matrices with randomly distributed elements. The resent investigations revile that the YMQM successfully describes the QCD glueball spectrum.

In the next section we shall consider the $N$-body problem in classical Newtonian gravity analysing the geodesic flow on a Riemannian manifold equipped with the Maupertuis metric.

6 N-body Problem. Collective Relaxation of Stellar Systems

The important application of the theory of MCDS and geodesic flow of manifolds of negative sectional curvatures have found in astrophysics and cosmology. The $N$-body problem in Newtonian gravity can be formulated as a geodesic flow on Riemannian manifold with the conformal Maupertuis metric:

$$ds^2 = (E - U)dp^2 = W \sum_{\alpha=1}^{3N} (dq^\alpha)^2, \quad U = -G \sum_{a<b} \frac{M_a M_b}{|r_a - r_b|},$$

(49)

where $W = E - U$ and $\{q^\alpha\}$ are the coordinates of the $N$ stars:

$$\{q^\alpha\} = \{M_1^{1/2} r_1, \ldots, M_N^{1/2} r_N\}, \quad \alpha = 1, \ldots, 3N.$$

(50)

The equation of the geodesics on Riemannian manifold with the metric $g_{\alpha\beta} = W\delta_{\alpha\beta}$ in (49) has the form

$$\frac{d^2 q^\alpha}{ds^2} + \frac{1}{2W} \left( 2 \frac{\partial W}{\partial q^\gamma} \frac{dq^\alpha}{ds} \frac{dq^\alpha}{ds} - g^{\alpha\gamma} \frac{\partial W}{\partial q^\gamma} g_{\beta\delta} \frac{dq^\beta}{ds} \frac{dq^\delta}{ds} \right) = 0$$

(51)

and coincides with the classical $N$-body equations when the proper time interval $ds$ is replaced by the time interval $dt$ of the form $ds = \sqrt{2W} dt$. The Riemann curvature in (21) for the Maupertuis metric has the form

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2W} \left[ W_{\beta\gamma} g_{\alpha\delta} - W_{\alpha\gamma} g_{\beta\delta} - W_{\beta\delta} g_{\alpha\gamma} + W_{\alpha\delta} g_{\beta\gamma} \right] - \frac{3}{4W^2} \left[ W_{\beta\gamma} W_{\alpha\delta} - W_{\alpha\gamma} W_{\beta\delta} - W_{\beta\delta} W_{\alpha\gamma} + W_{\alpha\delta} W_{\beta\gamma} \right] + \frac{1}{4W^2} \left[ g_{\beta\gamma} g_{\alpha\delta} - g_{\alpha\gamma} g_{\beta\delta} \right] W_\sigma W^\sigma,$$

(52)
where \( W_\alpha = \partial W/\partial q^\alpha \), \( W_{\alpha\beta} = \partial W/\partial q^\alpha \partial q^\beta \) and the scalar curvature is

\[
R = 3N(3N-1) \left[ - \frac{\Delta W}{3NW^2} - \frac{1}{4} \frac{(\nabla W)^2}{W^3} \right]
\]

and \( \Delta W = \partial^2 W/\partial q^\alpha \partial q^\alpha \), \( \nabla W = (\partial W/\partial q^\alpha)(\partial W/\partial q^\alpha) \). Let us now calculate the sectional curvature \(^{[24]}\):

\[
R_{\alpha\beta\gamma\sigma} \delta q^\alpha u^\beta \delta q^\gamma u^\sigma = \frac{1}{2W} \left[ 2|uW''| \delta q||u\delta q| - |\delta qW''| \delta q||uu| - |uW''| \delta q||u\delta q| \right] -
- \frac{3}{4W^2} \left[ 2|uW'| |W'\delta q||u\delta q| - |\delta qW'| |W'\delta q||uu| - |uW'| |W'\delta q||u\delta q| \right] +
+ \frac{1}{4W^2} \left[ |u\delta q|^2 - |uu||\delta q\delta q| \right] |W'W'|.
\]

(54)

For the normal deviation \( q_\perp \) we have \(|u\delta q_\perp| = 0\) and taking into account that the velocity is normalised to unity \(|uu| = 1\) we have

\[
R_{\alpha\beta\gamma\sigma} \delta q^\alpha_\perp u^\beta \delta q^\gamma_\perp u^\sigma = \left( \frac{3}{4W^2} |uW'|^2 - \frac{1}{2W^2} |W'W'| - \frac{1}{2W} |uW''| u \right) |\delta q_\perp|^2

- \frac{1}{2W} |\delta q_\perp W''| |\delta q_\perp| + \frac{3}{2W^2} |\delta q_\perp W'|^2.
\]

(55)

A remarkable regularity in the light distribution in elliptical galaxies suggests that they have reached some form of natural equilibrium and therefore one can conjecture that the average value of the velocities and deviations can be taken in the form:

\[
\overline{u^\alpha u^\beta} = \frac{1}{3N} g^{\alpha\beta} |uu|, \quad \overline{\delta q^\alpha_\perp \delta q^\beta_\perp} = \frac{1}{3N} g^{\alpha\beta} |\delta q_\perp|^2,
\]

(56)

That reduces the equation \((55)\) to the form which contains only the scalar curvature \((53)\):

\[
R_{\alpha\beta\gamma\sigma} \delta q^\alpha_\perp u^\beta \delta q^\gamma_\perp u^\sigma = \left[ - \frac{1}{3N} \frac{\Delta W}{W^2} - \frac{1}{4} \frac{(\nabla W)^2}{W^3} \right] |\delta q_\perp|^2.
\]

(57)

Thus the sectional curvature is proportional to the scalar curvature in this case. As the number of stars in galaxies is very large, \( N \gg 1 \), we shall get that the dominant term in sectional curvature \((57)\) is negative:

\[
K(q, u, \delta q) = \frac{R_{\alpha\beta\gamma\sigma} \delta q^\alpha \delta q^\beta \delta q^\gamma \delta q^\sigma}{|\delta q_\perp|^2} = -\frac{1}{4} \frac{(\nabla W)^2}{W^3} < 0.
\]

(58)

Finally the deviation equation \((26)\) will take the form

\[
\frac{d^2}{dt^2} |\delta q_\perp|^2 \geq \frac{(\nabla W)^2}{W} |\delta q_\perp|^2,
\]

(59)

where we used the relation \( ds = \sqrt{2W}dt \). The solution of this equation was considered in section four and has the form \((28)\) with \( 2\kappa = (\nabla W)^2/W \). Thus the relaxation time \((61)\) can be defined as

\[
\tau = \sqrt{\frac{W}{(\nabla W)^2}}.
\]

(60)
where \( W = T = N \frac{M \langle v^2 \rangle}{2} \) is total kinetic energy of the stars and \( \nabla W \) is a sum of the forces acting on the stars. Each term in the sum can be approximated by a force \( \frac{GM}{d^2} \) acting on a star by a nearby star at a distance \( d \), where \( d \) is the mean distance between stars. Thus we can get

\[
\tau \propto \left( \frac{NM \langle v^2 \rangle}{2} \frac{d^4}{NM(GM)^2} \right)^{1/2} = \frac{v}{2GMn^{2/3}}
\]  

Comparing collective relaxation time with the Chandrasekhar relaxation time \( \tau_b \) which is due to the binary encounters of stars\(^{[72]}\) one can get

\[
\frac{\tau_b}{\tau} = \frac{v^3}{G^2M^2n \log N} = \frac{v^2}{GMn^{1/3} \log N} \propto \frac{d}{r_*}
\]  

where \( r_* = \frac{2GM}{v} \) is a radius of effective binary scattering of stars. As far as the astrophysical observations revile that \( d \gg r_* \) we will get that the collective relaxation time \( \tau \) is much shorter than the binary relaxation time \( \tau \ll \tau_b \). These time scales together with the dynamical time scale \( \tau_d = \frac{D}{v} = D^{3/2}(GNM)^{-1/2} \), the scale corresponding to a time interval for a star to move around a gravitating system of a characteristic size \( D \), are in the following relation

\[
\tau \approx \frac{D}{d} \tau_d, \quad \tau_b \approx \frac{D}{r_*} \tau_d
\]  

reflecting the appearance and the correspondence of time \( (\tau_d, \tau, \tau_b) \) and length \( (D, d, r_*) \) scales in the gravitational systems\(^{[12]}\). One can estimate the relaxation time \( \tau \) for the elliptic galaxies and globular clusters\(^{[13]}\)

\[
\tau \approx 10^8 \text{yr} \left( \frac{v}{10 \text{km/s}} \right) \left( \frac{n}{1 \text{pc}^{-3}} \right)^{-2/3} \left( \frac{M}{M_\odot} \right)^{-1}
\]  

\(^{[12]}\)The above consideration was instigated during a private presentation of the collective relaxation mechanism to Prof. Viktor Ambartsumian. At the end of the presentation he made a remark that there should be some sort of correspondence between time and length scales in the extended gravitational systems. After returning back to the office I calculated the ratios \( \tau_d, \tau, \tau_b \) and found that indeed there is a direct correspondence between time scales \( (\tau_d, \tau, \tau_b) \) and length scales \( (D, d, r_*) \)!

\(^{[13]}\)In 1986 a seminar was organised at the ITEP in Moscow to present the results on the collective relaxation mechanism. After the seminar Prof. Lev Okun suggested that one can arrange a meeting with Prof. Vladimir Arnold for further discussions of the collective relaxation. The meeting was organised at the Moscow State University and then at his home. Instead of discussing the N-body problem - it seemed that he had already been acquainted with the results on the collective relaxation mechanism - Arnold in a very clear physical terms explained the direct and inverse two-dimensional Radon transformation and presented his book "Catastrophe Theory" where he discussed caustics, a wave front propagation and classification of bifurcations.\(^{[29]}\) At the end of the discussion Arnold suggested that the results should be presented also to Prof. Yakov Zeldovich. The meeting was scheduled at the Moscow State University where he had a lecture on that day. After the lecture he felt uncomfortable to proceed with the discussion at the University and drove his Volga car to the Sternberg Astronomical Institute. During the drive he told that in the last lecture he had presented to the students the Pauli exclusion principle and then added that together with George Gamow they had attempted to "explain" it by repulsive force, but it came out to be impossible. (In Pauli's " General Principles of Quantum Mechanics" the author discussed the attempts to explain the exclusion principle by a singular interaction force between two particles and remarked that in such attempts the difficulty lies in keeping the antisymmetric functions still regular, a constraint that is difficult to fulfill. A mathematically flawless realisation of the program was found by Jaffe.\(^{[80]}\) Pauli stressed that the singularities were such that they barely could be realised in reality.) In Sternberg Institute Zeldovich walked around but then suggested to drive to the Kapitza Institute of Physical Problems
where $v$ is the mean velocity, $n$ is the density and $M$ is the mean mass of stars. This time is by few orders of magnitude shorter than the binary relaxation time. It is interesting to observe that the Hubble Deep Field and the Hubble eXtreme Deep Field images revealed a large number of distant young galaxies seemingly in a non-equilibrium state, while the stars in the nearby older galaxies show a more regular distribution of velocities and shapes, reflecting the collective relaxation mechanism of stars.

7 Artin Hyperbolic System

In order to understand better the interrelation between classically chaotic systems and their quantum mechanical counterparts it seems natural to consider DS that are defined on closed surfaces of constant negative curvature. These DS systems have been studied for a long time by mathematicians and have deep roots in number theory, differential geometry, group theory, the theory of Fuchsian groups and automorphic forms.

Let us consider the DS’s which are defined on closed surfaces on the Lobachevsky plane of constant negative curvature. An example of such system has been defined in a brilliant article published in 1924 by the mathematician Emil Artin (see also 36–38). The dynamical system is defined on the fundamental region of the Lobachevsky plane which is obtained by the identification of points congruent with respect to the modular group $SL(2, \mathbb{Z})$, a discrete subgroup of the Lobachevsky plane isometries $SL(2, \mathbb{R})$. The fundamental region in this case
is a hyperbolic triangle Fig.3. The geodesic trajectories are bounded to propagate on the fundamental hyperbolic triangle. The area of the fundamental region is finite and gets a topology of sphere by "gluing" the opposite sides of the triangle as it is shown in Fig.3 and Fig.8. The Artin symbolic dynamics, the differential geometry and group-theoretical methods of Gelfand and Fomin can be used to investigate the decay rate of the classical and quantum mechanical correlation functions. The following three sections are devoted to the Artin system and are based on the results published in the articles [151, 152, 156].

Let us start with the Poincare model of the Lobachevsky plane, i.e. the upper half of the complex plane: \( H = \{ z \in \mathbb{C}, \Im z > 0 \} \) supplied with the metric (we set \( z = x + iy \))

\[
d s^2 = \frac{d x^2 + d y^2}{y^2} \tag{65}
\]

with the Ricci scalar \( R = -2 \). Isometries of this space are given by \( SL(2, \mathbb{R}) \) transformations. The \( SL(2, \mathbb{R}) \) matrix \( (a,b,c,d \text{ are real and } ad - bc = 1) \)

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

acts on a point \( z \) by linear fractional substitutions \( z \rightarrow \frac{az + b}{cz + d} \). Note also that \( g \) and \( -g \) give the same transformation, hence the effective group is \( SL(2, \mathbb{R})/\mathbb{Z}_2 \). We’ll be interested in the space of orbits of a discrete subgroup \( \Gamma \subset SL(2, \mathbb{R}) \) in \( H \). Our main example will be the modular group \( \Gamma = SL(2, \mathbb{Z}) \). A standard choice of the fundamental region \( \mathcal{F} \) of \( SL(2, \mathbb{Z}) \) is displayed in Fig.3. The fundamental region \( \mathcal{F} \) of the modular group consisting of those points between the lines \( x = -\frac{1}{2} \) and \( x = +\frac{1}{2} \) which lie outside the unit circle in Fig.3. The modular triangle \( \mathcal{F} \) has two equal angles \( \alpha = \beta = \frac{\pi}{3} \) and the third one is equal to zero, \( \gamma = 0 \), thus \( \alpha + \beta + \gamma = 2\pi/3 < \pi \). The area of the fundamental region is finite and equals to \( \frac{\pi}{3} \) and gets a topology of sphere by “gluing” the opposite sides of the triangle. The invariant area element on the Lobachevsky plane is proportional to the square root of the determinant of the metric (65):

\[
d \mu(z) = \frac{d x d y}{y^2} \tag{66}
\]

Thus \( \text{area}(\mathcal{F}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} d x \int_{0}^{\infty} d y \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d y}{y^2} = \frac{\pi}{3} \). Following the Artin construction let us consider the model of the Lobachevsky plane realised in the upper half-plane \( y > 0 \) of the complex plane \( z = x + iy \in \mathcal{C} \) with the Poincaré metric which is given by the line element (65).

The Lobachevsky plane is a surface of a constant negative curvature, because its curvature is equal to \( R = g^{ik}R_{ik} = -2 \) and it is twice the Gaussian curvature \( K = -1 \). This metric has two well known properties: 1) it is invariant with respect to all linear substitutions which form
the group \( g \in G \) of isometries of the Lobachevsky plane\(^{15}\)

\[
w = g \cdot z \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \equiv \frac{az + b}{cz + d},
\]

where \( a, b, c, d \) are real coefficients of the matrix \( g \) and the determinant of \( g \) is positive, \( ad - bc > 0 \). The geodesic lines are either semi-circles orthogonal to the real axis or rays perpendicular to the real axis. The equation for the geodesic lines on a curved surface has the form \((20)\), where the Christoffel symbols are evaluated for the metric \((65)\). The geodesic equations take the form

\[
\frac{d^2x}{dt^2} - \frac{2}{y} \frac{dx}{dt} \frac{dy}{dt} = 0, \quad \frac{d^2y}{dt^2} + \frac{1}{y} \left( \frac{dx}{dt} \right)^2 - \frac{1}{y} \left( \frac{dy}{dt} \right)^2 = 0,
\]

and they have two solutions:

\[
x(t) - x_0 = r \tanh(t), \quad y(t) = \frac{r}{\cosh(t)} \quad \leftarrow \text{orthogonal semi-circles}
\]

\[
x(t) = x_0, \quad y(t) = e^t \quad \leftarrow \text{perpendicular rays}.
\]

Here \( x_0 \in (-\infty, +\infty), t \in (-\infty, +\infty) \) and \( r \in (0, \infty) \). By substituting each of the above solutions into the metric \((65)\) one can get convinced that a point on the geodesics curve moves with a unit velocity \((19)\)

\[
\frac{ds}{dt} = 1.
\]

In order to construct a closed surface \( \mathcal{F} \) on the Lobachevsky plane, one can identify all points in the upper half of the plane which are related to each other by the substitution \((67)\) with the integer coefficients and a unit determinant. These transformations form a modular group \( SL(2, \mathbb{Z}) \). The two points \( z \) and \( w \) are identified if:

\[
w = \frac{mz + n}{pz + q}, \quad d = \begin{pmatrix} m & n \\ p & q \end{pmatrix}, \quad d \in SL(2, \mathbb{Z})
\]

with integers \( m, n, p, q \) constrained by the condition \( mq - pn = 1 \). The \( SL(2, \mathbb{Z}) \) is the discrete subgroup of the isometry transformations \( SL(2, \mathbb{R}) \) of \((67)\). The identification creates a regular tessellation of the Lobachevsky plane by congruent hyperbolic triangles in Fig. 3. The Lobachevsky plane is covered by the infinite-order triangular tiling. One of these triangles can be chosen as a fundamental region. That fundamental region \( \mathcal{F} \) of the above modular group \((70)\) is the well known “modular triangle”, consisting of those points between the lines \( x = -\frac{1}{2} \)

\(^{15}\text{G is a subgroup of all Möbius transformations.}\)

\(^{16}\text{The modular group } SL(2, \mathbb{Z}) \text{ serves as an example of the Fuchsian group.} \)
Figure 3: The fundamental region $\mathcal{F}$ is the hyperbolic triangle $ABD$, the vertex $D$ is at infinity of the $y$ axis. The edges of the triangle are the arc $AB$, the rays $AD$ and $BD$. The points on the edges $AD$ and $BD$ and the points of the arcs $AC$ with $CB$ should be identified by the transformations $w = z + 1$ and $w = -1/z$ in order to form the Artin surface $\bar{\mathcal{F}}$ by "gluing" the opposite sides of the modular triangle together Fig.4. The modular transformations (70) of the fundamental region $\mathcal{F}$ create a regular tessellation of the whole Lobachevsky plane by congruent hyperbolic triangles. $K$ is the geodesic trajectory passing through the point $(x, y)$ of $\mathcal{F}$ in the $\vec{v}$ direction.

and $x = +\frac{1}{2}$ which lie outside the unit circle in Fig.3. The area of the fundamental region is finite and equals to $\frac{\pi}{3}$. Inside the modular triangle $\mathcal{F}$ there is exactly one representative among all equivalent points of the Lobachevsky plane with the exception of the points on the triangle edges which are opposite to each other. These points can be identified in order to form a closed surface $\bar{\mathcal{F}}$ by "gluing" the opposite edges of the modular triangle together. In Fig.3 one can see the pairs of points on the sides of the triangle which are identified. Now one can consider the behaviour of the geodesic trajectories defined on the surface $\bar{\mathcal{F}}$ of constant negative curvature.

Let us describe the time evolution of the physical observables $\{f(x,y,\theta)\}$ which are defined on the phase space $(x, y, \theta) \in \mathcal{M}$, where $z = x + iy \in \bar{\mathcal{F}}$ and $\theta \in S^1$ is a direction of a unit velocity vector. For that one should know a time evolution of geodesics on the phase space $\mathcal{M}$. The simplest motion on the ray $CD$ in Fig.3 is given by the solution (68) $x(t) = 0, \ y(t) = e^t$
and can be represented as a group transformation (67):

\[ z_1(t) = g_1(t) \cdot i = \left( \begin{array}{cc} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{array} \right) \cdot i = ie^t, \quad t \in (-\infty, +\infty). \]  

(71)

The other motion on the circle of a unit radius, the arc ACB in Fig. 3, is given by the transformation

\[ z_2(t) = g_2(t) \cdot i = \left( \begin{array}{cc} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{array} \right) \cdot i = \frac{i \cosh(t/2) + \sinh(t/2)}{i \sinh(t/2) + \cosh(t/2)}. \]  

(72)

Because the isometry group \( SL(2, \mathbb{R}) \) acts transitively on the Lobachevsky plane, any geodesic can be mapped into any other geodesic through the action of the group element \( g \in SL(2, \mathbb{R}) \) (67), thus the generic trajectory can be represented in the following form:

\[ z(t) = gg_1(t) \cdot i = \left( \begin{array}{cc} ae^{t/2} & be^{-t/2} \\ ce^{t/2} & de^{-t/2} \end{array} \right) \cdot i, \quad z(t) = iae^t + b \frac{ce^t}{dd}. \]  

(73)

This provides a convenient description of the time evolution of the geodesic flow on the whole Lobachevsky plane with a unit velocity vector (69). In order to project this motion into the closed surface \( \bar{F} \) one should identify the group elements \( g \in SL(2, \mathbb{R}) \) which are connected by the modular transformations \( SL(2, \mathbb{Z}) \). For that one can use a parametrisation of the group elements \( g \in SL(2, \mathbb{R}) \) defined in.\(^{134}\) Any element \( g \) can be defined by the parameters \((\tau, \omega_2)\)

\[ (\tau, \omega_2), \quad \tau \in \mathcal{F}, \quad \omega_2 = \frac{e^{i\theta}}{\sqrt{y}}, \quad 0 \leq \theta \leq 2\pi, \]  

(74)

where \( \tau = x + iy \) are the coordinates in the fundamental region \( \mathcal{F} \) and the angle \( \theta \) defines the direction of the unit velocity vector \( \vec{v} = (\cos \theta, \sin \theta) \) at the point \((x, y)\) (see Fig. 3). The functions \( \{f(x, y, \theta)\} \) on the phase space \( \mathcal{M} \) can be written as depending on \((\tau, \omega_2)\) and the invariance of the functions with respect to the modular transformations (70) takes the form\(^{17}\)

\[ f(\tau', \omega_2') = f \left( \frac{m\tau + n}{pr + q}, (pr + q)\omega_2 \right) = f(\tau, \omega_2). \]  

(75)

The evolution of the function \( \{f(\tau, u)\} \), where \( u = e^{-2i\theta} \), under the geodesic flow \( g_1(t) \) (71) is defined by the mapping

\[ \tau' = \frac{\tau \cosh(t/2) + u \tau \sinh(t/2)}{\cosh(t/2) + u \sinh(t/2)}, \quad u' = \frac{u \cosh(t/2) + \sinh(t/2)}{\cosh(t/2) + u \sinh(t/2)}. \]  

(76)

The evolution of the observables under the geodesic flow \( g_2(t) \) (72) has a similar form, except of an additional factor \( i \) in front of the variable \( u \). These expressions allow to define the

\(^{17}\)This defines the automorphic functions, the generalisation of the trigonometric, hyperbolic, elliptic and other periodic functions.\(^{129, 133}\)
transformation of the functions \( \{f(x, y, u)\} \) under the time evolution as \( f(x, y, u) \rightarrow f'(x', y', u') \) where
\[
x' = x'(x, y, u, t), \quad y' = y'(x, y, u, t), \quad u' = u'(u, t).
\] (77)

By using the Koopman\cite{Koopman1931} theorem this transformation of functions can be expressed as an action of a one-parameter group of unitary operators \( U_t \):
\[
U_t f(g) = f(gg_t).
\] (78)

Let us calculate transformations which are induced by \( g_1(t) \) and \( g_2(t) \) in (71)-(72). The time evolution is given by the equations (76) and (77):
\[
U_1(t) f(\tau, u) = f\left(\frac{\tau \cosh(t/2) + u \text{ sinh}(t/2)}{\cosh(t/2) + u \text{ sinh}(t/2)}, \frac{\text{cosh}(t/2) + u \text{ sinh}(t/2)}{\text{cosh}(t/2) + u \text{ sinh}(t/2)}\right),
\]
\[
U_2(t) f(\tau, u) = f\left(\frac{\tau \cosh(t/2) + i u \text{ sinh}(t/2)}{\cosh(t/2) + i u \text{ sinh}(t/2)}, \frac{\text{cosh}(t/2) - i u \text{ sinh}(t/2)}{\cosh(t/2) + i u \text{ sinh}(t/2)}\right). 
\] (79)

A one-parameter family of unitary operators \( U_t \) can be represented as an exponent of the self-adjoint operator \( U_t = \exp(iHt) \), thus we have
\[
U_t f(g) = e^{iHt} f(g) = f(gg_t)
\] (80)
and by differentiating it over the time \( t \) at \( t = 0 \) we shall get \( Hf = -i \frac{d}{dt} U_t f|_{t=0} \), that allows to calculate the operators \( H \) corresponding to the \( U_1(t) \) and \( U_2(t) \). Differentiating over time in (79) we shall get for \( H_1 \) and \( H_2 \):
\[
2H_1 = y \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) - i \frac{\partial}{\partial u} - uy \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) + iu^2 \frac{\partial}{\partial u},
\]
\[
2iH_2 = y \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) - i \frac{\partial}{\partial u} + uy \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) - iu^2 \frac{\partial}{\partial u}.
\] (81)

Introducing annihilation and creation operators \( H_- = H_1 - iH_2 \) and \( H_+ = H_1 + iH_2 \) yields
\[
H_+ = y \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) - i \frac{\partial}{\partial u}, \quad H_- = -uy \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) + iu^2 \frac{\partial}{\partial u}
\] (82)
and by calculating the commutator \([H_+, H_-]\) we shall get \( H_0 = u \frac{\partial}{\partial u} \) and their \( sl(2, R) \) algebra is:
\[
[H_+, H_-] = 2H_0, \quad [H_0, H_+] = -H_+, \quad [H_0, H_-] = H_-.
\] (83)

We can also calculate the expression for the invariant Casimir operator:
\[
H = \frac{1}{2}(H_+H_- + H_-H_+) - H_0^2 = -y^2(\partial_x^2 + \partial_y^2) + 2iy\partial_x u\partial_u = -y^2(\partial_x^2 + \partial_y^2) - y\partial_x\partial_y.
\] (84)
Consider a class of functions which fulfil the following two equations:

\[ H_0 f(x, y, u) = -\frac{N}{2} f(x, y, u), \quad H_- f(x, y, u) = 0, \quad (85) \]

where \( N \) is an integer number. The first equation has the solution \( f_N(x, y, u) = (\frac{1}{uy})^{N/2} \psi(x, y) = \omega_N^2 \psi(x, y) \) and by substituting it into the second one we shall get

\[ N \psi(\tau, \bar{\tau}) + (\tau - \bar{\tau}) \frac{\partial \psi(\tau, \bar{\tau})}{\partial \tau} = 0, \quad (86) \]

By taking \( \psi(\tau, \bar{\tau}) = (\tau - \bar{\tau})^N \Phi(\tau, \bar{\tau}) \) we shall get the equation

\[ \frac{\partial \Phi}{\partial \tau} = 0, \quad \text{that is,} \quad \Phi(\tau, \bar{\tau}) \text{ is an anti-holomorphic function} \]

\[ f(\omega_2, \tau, \bar{\tau}) = \frac{1}{\omega_N^2} \Phi(\tau), \quad (87) \]

The invariance under the action of the modular transformation \((75)\) will take the form

\[ \Phi(\frac{m \tau + n \bar{\tau}}{p \tau + q}) = \Phi(\tau), \quad (88) \]

and \( \Phi(\tau) \) is a theta function of weight \( N \). The invariant integration measure on the group \( G \) is given by the formula

\[ d\mu = \frac{dx dy}{y^2} d\theta, \quad (89) \]

and the invariant product of functions on the phase space \((x, y, \theta) \in M\) will be given by the integral

\[ (f_1, f_2) = \int_0^{2\pi} d\theta \int F f_1(\theta, x, y) f_2(\theta, x, y) dxdy y^2. \quad (90) \]

It was demonstrated that the functions on the phase space are of the form \((87)\), where \( \tau = x + iy \) and \( \frac{(\tau - \bar{\tau})}{2i} = y \), thus the expression for the scalar product will takes the following form:

\[ (f_1, f_2) = \int_0^{2\pi} d\theta \int F \Phi_1(\tau) \Phi_2(\bar{\tau}) \frac{1}{|\omega_2|^{2N}} \frac{dxdy}{y^2} = 2\pi \int F \Phi_1(\tau) \Phi_2(\bar{\tau}) y^{N-2} dxdy, \quad (91) \]

where \( N \geq 2 \). This expression for the scalar product allows to calculate the two-point correlation functions under the geodesic flow \((71)-(73)\).

A correlation function can be defined as an integral over a pair of functions in which the first one is stationary and the second one evolves with the geodesic flow:

\[ D_t(f_1, f_2) = \int_M f_1(g) f_2(g_x) d\mu. \quad (92) \]

By using \((76)\) and \((77)\) one can represent the integral in the following form:

\[ D_t(f_1, f_2) = \int_0^{2\pi} \int F f_1[x, y, \theta] f_2[x'(x, y, \theta, t), y'(x, y, \theta, t), \theta'(\theta, t)] \frac{dxdy}{y^2} d\theta. \quad (93) \]

\(^{18}\)The factors \((2i)^N\) have been absorbed by the redefinition of \( \Phi \).
From (78), (79) and (76), (79) it follows that

\[ f_1(\omega_2, \tau, \tau) = \frac{1}{\omega_2^N} \Phi_1(\tau), \]  

\[ f_2(\omega'_2, \tau', \tau') = \frac{1}{\omega_2^N} \Phi_2 \left( \frac{\tau \cosh(t/2) + e^{-2i\theta} \tau \sinh(t/2)}{\cosh(t/2) + e^{-2i\theta} \sinh(t/2)} \right). \]

Therefore the correlation function takes the following form:

\[ D_t(f_1, f_2) = \int_0^{2\pi} d\theta \int F \Phi_1(\tau) \Phi_2(\tau') \frac{y^{N-2} dx dy}{(\cosh(t/2) + e^{-2i\theta} \sinh(t/2))^N}. \]

The upper bound on the correlations functions is

\[ |D_t(f_1, f_2)| \leq \int_0^{2\pi} d\theta \int F |\Phi_1(\tau)\Phi_2(\tau')| \frac{y^{N-2} dx dy}{(\cosh(t/2) + e^{-2i\theta} \sinh(t/2))^N} \]

and in the limit \( t \to +\infty \) the correlation function exponentially decays:

\[ |D_t(f_1, f_2)| \leq M_{\phi_1\phi_2}(\epsilon) e^{-\frac{K}{2} |t|}. \]

If the surface curvature is \( K \) and the metric has the form \( ds^2 = \frac{dx^2 + dy^2}{Ky^2} \), then in the last formula the exponential factor will be

\[ |D_t(f_1, f_2)| \leq M_{\phi_1\phi_2}(\epsilon) e^{-\frac{K}{2} |t|}. \]

and the characteristic time decay \( |D_t(f_1, f_2)| \) will take the form

\[ \tau = \frac{2}{NK}. \]

The decay time of the correlation functions is shorter when the surface has a larger negative curvature or, in other words, when the divergency of the trajectories is stronger.

The behaviour of the phase trajectories and of the correlations functions discussed in the previous sections emphasise the fact that a local exponential divergency of the phase trajectories is translated into the exponential decay of the correlation functions at a universal rate expressible in terms of the entropy \( h(T) \). This observation also provides a qualitative understanding of why the correlation functions decay exponentially: under the action of the hyperbolic evolution \( T^t \) on the observable \( f(T^t x) \) the initial oscillating frequencies of \( f(x) \) are stretched apart toward the high frequency modes, while the frequencies of the observable \( g(x) \) remain stationary. As a result the overlapping integral of the function \( f(T^t x) \) with the \( g(x) \) falls exponentially.

The earlier investigation of the correlation functions of Anosov geodesic flows was performed in [146,150] by using alternative approaches. In our analyses we have used the time evolution equations and the properties of the automorphic functions [151].
8 Quantum Mechanics of Artin System

In the previous section we analysed the behaviour of the classical correlation functions defined on the phase space of the Artin system, which represent a finite-area patch on $AdS_2$ and demonstrated their exponential decay. In this section we shall investigate the quantum-mechanical behaviour of the correlation functions of quantised Artin system. There is a great interest in considering quantisation of the hyperbolic dynamical systems and investigation of their quantum-mechanical properties. This subject is very closely related with the investigation of quantum mechanical properties of classically non-integrable systems.

In classical regime the exponential divergency of geodesic trajectories instigate the universal exponential decay of its classical correlation functions. In order to investigate the behaviour of the correlation functions in quantum-mechanical regime it is necessary to know the spectrum of the system and the corresponding wave functions. The spectral problem has deep number-theoretical origin and was partially solved in a series of pioneering publications. It was solved partially because the discrete spectrum and the corresponding wave functions are not known analytically. The energy spectrum has a continuous part corresponding to the free motion along the infinite "y-channel" extended in the vertical direction of the fundamental region $F$ as well as infinitely many discrete energy states corresponding to a bounded motion at the "bottom" of the fundamental triangle Fig. 4. The general properties of the discrete spectrum have been investigated by using Selberg trace formula. Numerical calculation of the discrete energy levels were performed for many energy states.

Figure 4: The incoming and outgoing plane waves. The plane wave $e^{-ip\tilde{y}}$ incoming from infinity of the $y$ axis, the vertex $D$, elastically scatters on the boundary $ACB$ of the fundamental triangle $F$ in Fig. 3, the bottom of the Artin surface $\bar{F}$. The reflection amplitude $\theta(\frac{1}{2} + ip)/\theta(\frac{1}{2} - ip)$ is a pure phase and is given by the expression in front of the outgoing plane wave $e^{ip\tilde{y}}$. The rest of the wave function describes the standing waves in the $x$ direction between boundaries $x = \pm 1/2$ with the amplitudes, which are exponentially decreasing. On the right figure is an artistic image of the Artin surface, as far as it cannot be smoothly embedded into $R^3$. 
Here we shall review the quantisation of the Artin system defined on a finite-area patch on $AdS_2$ and the derivation of the Maass wave function corresponding to the continuous spectrum. We shall use the Poincaré representation of the Maass non-holomorphic automorphic wave function. By introducing a natural physical variable $\tilde{y}$ for the distance in the vertical direction $\int dy/y = \ln y = \tilde{y}$ and the corresponding momentum $p_y$ we shall represent the wave functions in the form appealing to the physical intuition:

$$\psi_p(x, \tilde{y}) = e^{-ip\tilde{y}} + \frac{\theta(\frac{1}{2}+ip)}{\theta(\frac{1}{2}-ip)} e^{ip\tilde{y}} + \frac{4}{\theta(\frac{1}{2}-ip)} \sum_{l=1}^{\infty} \tau_{ip}(l) K_{ip}(2\pi le^{\tilde{y}}) \cos(2\pi lx).$$

(98)

The first two terms describe the incoming and outgoing plane waves. The plane wave $e^{-ip\tilde{y}}$ incoming from infinity (the vertex $D$) along the $y$ axis of Fig. 4 elastically scatters on the boundary $ACB$ of the fundamental triangle $\mathcal{F}$ and reflects backwards $e^{ip\tilde{y}}$. The reflection amplitude is a pure phase and is given by the expression in front of the outgoing plane wave:

$$\frac{\theta(\frac{1}{2}+ip)}{\theta(\frac{1}{2}-ip)} = \exp [i\varphi(p)].$$

(99)

The rest of the wave function describes the standing waves $\cos(2\pi lx)$ between boundaries $x = \pm 1/2$ and exponentially decreasing amplitude $K_{ip}(2\pi ly)$ in $y$ and index $l$ \cite{124}. The continuous energy spectrum is given by the formula

$$E = p^2 + \frac{1}{4}.$$

(100)

It was conjectured that the wave functions of the discrete spectrum have the following form \cite{132,133,135,136,141,142,144}

$$\psi_n(z) = \sum_{l=1}^{\infty} c_l(n) K_{iu_n}(2\pi le^{\tilde{y}}) \left\{ \frac{\cos(2\pi lx)}{\sin(2\pi lx)} \right\},$$

(101)

where the spectrum $E_n = \frac{1}{4} + u_n^2$ and the coefficients $c_l(n)$ are not known analytically but were computed numerically for many values of $n$ \cite{141,132,144}.

Having in hand the explicit expression of the wave function one can analyse a quantum-mechanical behaviour of the correlation functions and double commutators defined in \cite{120}

$$\mathcal{D}_2(\beta, t) = \langle A(t) B(0) e^{-\beta H} \rangle, \quad \mathcal{D}_4(\beta, t) = \langle A(t) B(0) A(t) B(0) e^{-\beta H} \rangle$$

(102)

$$C(\beta, t) = -\langle [A(t), B(0)]^2 e^{-\beta H} \rangle,$$

(103)

and the operators $A$ and $B$ which we chose to be of the Louiville type \cite{152}

$$A(N) = e^{-2N\tilde{y}}, \quad N = 1, 2, \ldots,$$

(104)

Analysing the basic matrix elements of the Louiville-like operators \cite{104} we shall demonstrate that all two- and four-point correlation functions \cite{102} decay exponentially in time, with the
exponents which depend on temperature Fig.5 and Fig.6. These exponents define the quantum-mechanical decay time $t_d(\beta)$. The square of the commutator of the Louiville-like operators separated in time grows exponentially Fig.7. This growth is reminiscent of the local exponential divergency of classical trajectories. The exponential growth Fig.7 of the commutator was approximated by the expression

$$C(\beta, t) \sim K(\beta) e^{\frac{2\pi}{\chi(\beta)} t}. \quad (105)$$

It was conjectured that a maximal growth of double commutator is linear in temperature

$$C(\beta, t)_{max} \sim K(\beta) e^{\frac{2\pi}{\beta} t}. \quad (106)$$

It is therefore important to investigate the ratio with respect to the maximal growth exponent

$$\frac{C(\beta, t)_{max}}{C(\beta, t)_{\text{Artin-AdS}}} \sim R(\beta) e^{\left(\frac{1}{\beta} - \frac{1}{\chi(\beta)}\right)2\pi t}. \quad (107)$$

The result is presented in Fig.7. The temperature dependence of $1/\chi(\beta)$ relative to the maximum growth $1/\beta$ is shown by blue dots in the far right figure. At high temperatures the Artin Lyapunov exponent $1/\chi(\beta)$ is less than the maximal exponent $1/\beta$, but at low temperatures this is not any more true and we observe a breaking of the saturation regime.[120][122]

1. **Quantisation**

Let us consider the geodesic flow on $\mathcal{F}$ described by the action

$$S = \int ds = \int \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y} \, dt \quad (108)$$

and the equations of motion

$$\frac{d}{dt} \frac{\dot{x}}{y\sqrt{\dot{x}^2 + \dot{y}^2}} = 0, \quad \frac{d}{dt} \frac{\dot{y}}{y\sqrt{\dot{x}^2 + \dot{y}^2}} + \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y^2} = 0. \quad (109)$$

Notice the invariance of the action and of the equations under time reparametrisations $t \to t(\tau)$. The presence of this local gauge symmetry indicates that we have a constrained dynamical system.[154] A convenient gauge fixing which specifies the time parameter $t$ to be proportional to the proper time, is archived by imposing the condition

$$\frac{\dot{x}^2 + \dot{y}^2}{y^2} = 2H, \quad (110)$$

where $H$ is a constant. In this gauge the equations (109) will take the form

$$\frac{d}{dt} \left( \frac{\dot{x}}{y^2} \right) = 0, \quad \frac{d}{dt} \left( \frac{\dot{y}}{y^2} \right) + \frac{2H}{y} = 0. \quad (111)$$
Defining the canonical momenta as \( p_x = \frac{\dot{x}}{\sqrt{y}} \), \( p_y = \frac{\dot{y}}{\sqrt{y}} \), conjugate to the coordinates \((x, y)\), one can get the geodesic equations (111) in the Hamiltonian form:

\[
\dot{p}_x = 0, \quad \dot{p}_y = -\frac{2H}{y}.
\] (112)

The Hamiltonian will take the form

\[
H = \frac{1}{2} y^2 (p_x^2 + p_y^2)
\] (113)

and the corresponding equations will take the following form:

\[
\dot{x} = \frac{\partial H}{\partial p_x} = y^2 p_x, \quad \dot{y} = \frac{\partial H}{\partial p_y} = y^2 p_y
\] (114)

\[
\dot{p}_x = -\frac{\partial H}{\partial x} = 0, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = -y(p_x^2 + p_y^2) = -\frac{2H}{y},
\]

and they coincide with (112). The advantage of the gauge (110) is that the Hamiltonian (113) coincides with the constraint.

Now it is fairly standard to quantize this Hamiltonian system by replacing in (113) \( p_x = -i \frac{\partial}{\partial x}, p_y = -i \frac{\partial}{\partial y} \) and considering time independent Schrödinger equation \( H\psi = E\psi \). The resulting equation explicitly reads:

\[
-y^2(\partial_x^2 + \partial_y^2)\psi = E\psi.
\] (115)

On the LHS one easily recognises the Laplace operator in Poincare metric (65). It is easy to see that the Hamiltonian is positive semi-definite Hermitian operator:

\[
-\int \psi^*(x, y) y^2 (\partial_x^2 + \partial_y^2) \psi(x, y) \frac{dxdy}{y^2} = \int (|\partial_x \psi(x, y)|^2 + |\partial_y \psi(x, y)|^2) dxdy \geq 0.
\] (116)

It is convenient to parametrise the energy \( E = s(1 - s) \) and to rewrite the Schrödinger equation as

\[
-y^2(\partial_x^2 + \partial_y^2) \psi(x, y) = s(1 - s) \psi(x, y).
\] (117)

As far as \( E \) is real and semi-positive and parametrisation is symmetric with respect to \( s \leftrightarrow 1 - s \) it follows that the parameter \( s \) should be chosen within the ranges

\[
s \in [1/2, 1] \text{ or } s = 1/2 + iu, \quad u \in [0, \infty].
\] (118)

One should impose the ”periodic” boundary condition on the wave function with respect to the modular group

\[
\psi(\frac{az + b}{cz + d}) = \psi(z), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z})
\] (119)
in order to have the wave function which is properly defined on the fundamental region \( \bar{F} \) shown in Fig. 3. Taking into account that the transformation \( T : z \rightarrow z + 1 \) is the element of \( SL(2, Z) \) and imposing the periodicity condition \( (119) \) \( \psi(z) = \psi(z + 1) \) one can get the following Fourier expansion \( \psi(x, y) = \sum_{n=-\infty}^{\infty} f_n(y) \exp(2\pi inx) \). Inserting this into Eq. (117), for the Fourier component \( f_n(y) \) one can get the equation:

\[
\frac{d^2 f_n(y)}{dy^2} + (s(1-s) - 4\pi^2 n^2) f_n(y) = 0.
\]

In the case \( n \neq 0 \) the solution which exponentially decays at large \( y \) reads \( f_n(y) = \sqrt{y} K_{s-\frac{1}{2}}(2\pi n|y|) \) and for \( n = 0 \) one can get

\[
f_0(y) = c_0 y^s + c_0' y^{1-s}.
\]

Thus the solution can be represented in the form

\[
\psi(x, y) = c_0 y^s + c_0' y^{1-s} + \sqrt{y} \sum_{n=-\infty}^{\infty} c_n K_{s-\frac{1}{2}}(2\pi n|y|) \exp(2\pi inx),
\]

(120)

where the coefficients \( c_0, c_0', c_n \) should be defined via periodic boundary condition \( (119) \), that is, with respect to the second transformation \( S : z \rightarrow -1/z : \psi(z) = \psi(-1/z) \). The corresponding functional equation defines the coefficients \( c_0, c_0', c_n \). It is difficult to solve this equation and we will consider an alternative solution in the next section.

\[\textit{2. Maass Wave Functions of Continuous Spectrum}\]

Another option is to take a particular solution and perform summation over all nonequivalent transformations of the \( SL(2, Z) \) group. Let us demonstrate this strategy by using the solution (120) when \( c_0 = 1, c_0' = c_n = 0 \):

\[
\psi(z) = y^s = (\Im z)^s.
\]

The \( \Gamma_\infty \) is the subgroup of \( \Gamma = SL(2, Z) \) generating shifts \( z \rightarrow z + n, n \in Z \). Since \( y^s \) is already invariant with respect to \( \Gamma_\infty \), one should perform summation over the conjugacy classes \( \Gamma_\infty \setminus \Gamma \). There is a bijection between the set of mutually prime pairs \((c, d)\) with \((c, d) \neq (0, 0)\) and the set of conjugacy classes \( \Gamma_\infty \setminus \Gamma \). The fact that the integers \((c, d)\) are mutually prime integers means that their greatest common divisor (gcd) is equal to one: \( \gcd(c, d) = 1 \). As a result, it is defined by the classical Poincaré series representation \( \text{and for the sum of our interest we get} \)

\[
\psi_s(z) \equiv \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} (\Im (\gamma z))^s = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2 \atop gcd(c,d)=1} \frac{y^s}{((cx + d)^2 + c^2y^2)^s},
\]

(121)
where, as explained above, the sum on r.h.s. is taken over all mutually prime pairs \((c, d)\). To evaluate the sum one should multiply both sides of the eq. \([121]\) by 
\[\sum_{n=1}^{\infty} \frac{1}{n^{2s}} \equiv \zeta(2s)\]
so that the wave function will be expressed in terms of the Eisenstein series:

\[
\zeta(2s) \psi_s(z) = \frac{1}{2} \sum_{(m,k) \in \mathbb{Z}^2, (m,k) \neq (0,0)} \frac{y^s}{((mx + k)^2 + m^2y^2)^s}.
\] (122)

The evaluation of the sum can be performed explicitly and allows to represent the \([122]\) in the following form:

\[
\zeta(2s) \psi_s(x, y) = \zeta(2s)y^s + \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2}) \zeta(2s - 1)}{\Gamma(s)} y^{1-s} + \sqrt{y} \frac{4\pi^s}{\Gamma(s)} \sum_{l=1}^{\infty} \tau_{s-\frac{1}{2}}(l) K_{s-\frac{1}{2}}(2\pi ly) \cos(2\pi lx),
\] (123)

where the modified Bessel’s \(K\) function is given by the expression

\[
K_{iu}(y) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-y \cosh t} e^{iu t} dt
\] (124)

and \(\tau_{ip}(n) = \sum_{a \cdot b = n} \left(\frac{n}{b}\right)^{ip}\). By using Riemann’s reflection relation

\[
\zeta(s) = \frac{\pi^{s-\frac{i}{2}} \Gamma \left(\frac{1-s}{2}\right)}{\Gamma \left(\frac{s}{2}\right)} \zeta(1-s)
\] (125)

and introducing the function

\[
\theta(s) = \pi^{-s} \zeta(2s) \Gamma(s)
\] (126)

we get an elegant expression of the eigenfunctions obtained by Maass:

\[
\theta(s) \psi_s(z) = \theta(s)y^s + \theta(1-s) y^{1-s} + 4 \sqrt{y} \sum_{l=1}^{\infty} \tau_{s-\frac{1}{2}}(l) K_{s-\frac{1}{2}}(2\pi ly) \cos(2\pi lx).
\] (127)

This wave function is well defined in the complex \(s\) plane and has a simple pole at \(s = 1\). The physical continuous spectrum was defined in \([118]\), where \(s = \frac{1}{2} + iu, u \in [0, \infty]\), therefore

\[
E = s(1-s) = \frac{1}{4} + u^2.
\] (128)

The continuous spectrum wave functions \(\psi_s(z)\) are delta function normalisable. The wave function \([127]\) can be conveniently represented also in the form

\[
\psi_{\frac{1}{2}+iu}(z) = y^{\frac{1}{2}+iu} + \frac{\theta(\frac{1}{2} - iu)}{\theta(\frac{1}{2} + iu)} y^{\frac{1}{2}-iu} + \frac{4 \sqrt{y}}{\theta(\frac{1}{2} + iu)} \sum_{l=1}^{\infty} \tau_{iu}(l) K_{iu}(2\pi ly) \cos(2\pi lx),
\] (129)
where $K_{-iu}(y) = K_{iu}(y)$, $\tau_{-iu}(l) = \tau_{iu}(l)$. The physical interpretation of the wave function becomes more transparent if one introduce the new variables

$$
\tilde{y} = \ln y, \quad p = -u, \quad E = p^2 + \frac{1}{4},
$$

(130)
as well as the alternative normalisation of the wave function $\psi_p(x, \tilde{y}) \equiv y^{-\frac{1}{2}} \psi_{\frac{1}{2}+iu}(z)$

$$
\psi_p(x, \tilde{y}) = e^{-ip\tilde{y}} + \frac{\theta(\frac{1}{2} + ip)}{\theta(\frac{1}{2} - ip)} e^{ip\tilde{y}} + \frac{4}{\theta(\frac{1}{2} - ip)} \sum_{l=1}^{\infty} \tau_{ip}(2\pi l e^{\tilde{y}}) \cos(2\pi lx).
$$

(131)
The first two terms describe the incoming and outgoing plane waves. The plane wave $e^{-ip\tilde{y}}$ incoming along the $y$ axis elastically scatters on the boundary $ACB$ of the fundamental region $F$. The reflection amplitude is a pure phase and is given by the expression in front of the outgoing plane wave $e^{ip\tilde{y}}$

$$
\frac{\theta(\frac{1}{2} + ip)}{\theta(\frac{1}{2} - ip)} = \exp [i \varphi(p)].
$$

(132)

The rest of the wave function describes the standing waves between boundaries $x = \pm 1/2$ with exponentially decreasing amplitudes $K_{ip}(2\pi ly)$.

In addition to the continuous spectrum the system (117) may have a discrete spectrum. The number of discrete states is infinite: $E_0 = 0 < E_1 < E_2 < \ldots \to \infty$. The spectrum is extended to infinity - unbounded from above - and lacks any accumulation point except infinity. The wave functions of the discrete spectrum have the form

$$
D_2(\beta, t) = \langle A(t) B(0) e^{-\beta H} \rangle = \sum_n \langle n | e^{iHt} A(0) e^{-iHt} B(0) e^{-\beta H} | n \rangle = \sum_{n,m} e^{i(E_n - E_m)t - \beta E_n} \langle n | A(0) | m \rangle \langle m | B(0) | n \rangle.
$$

(133)

The energy eigenvalues (222) are parametrised by $n = \frac{1}{2} + iu$, $E_n = \frac{1}{4} + u^2$ and $m = \frac{1}{2} + iv$, $E_m = \frac{1}{4} + v^2$, thus

$$
D_2(\beta, t) = \int_0^{+\infty} \int_0^{+\infty} du \, dv \, e^{i(u^2 - v^2)t - \beta(\frac{1}{2} + u^2)} \int_F \psi_{\frac{1}{2} - iu}(z) A \psi_{\frac{1}{2} + iv}(z) d\mu(z) \int_F \psi_{\frac{1}{2} - iu}(w) B \psi_{\frac{1}{2} + iv}(w) d\mu(w).
$$

(134)

9 Out-of-Time-Order Correlation Functions of Artin System

Here we are interested to analyse the behaviour of the out-of-time-order correlation functions (143), (148), (149) and the double commutators (144) in the case of well defined Artin MCDS investigating the ”influence and remnants” of the classical chaos on the quantum mechanical behaviour of the quantised system. Considering Artin system in its quantum mechanical regime would help to identify the traces of classical chaos and clarify a natural meaning of quantum chaos on a finite-area patch on $AdS_2$. The two-point correlation function is defined as:

$$
D_2(\beta, t) = \langle A(t) B(0) e^{-\beta H} \rangle = \sum_n \langle n | e^{iHt} A(0) e^{-iHt} B(0) e^{-\beta H} | n \rangle = \sum_{n,m} e^{i(E_n - E_m)t - \beta E_n} \langle n | A(0) | m \rangle \langle m | B(0) | n \rangle.
$$

(133)
where the complex conjugate function is $\psi^*_{\frac{1}{2}+iu}(z) = \psi_{\frac{1}{2}-iu}(z)$. Defining the basic matrix element as

$$A_{uv} = \int_{\mathcal{F}} \psi_{\frac{1}{2}-iu}(z) A \psi_{\frac{1}{2}+iv}(z) \, d\mu(z) = \int_{-1/2}^{1/2} dx \int_{\frac{i}{2}\log(1-x^2)}^{\infty} \frac{dy}{\sqrt{1-x^2}} \psi_{\frac{1}{2}-iu}(z) A \psi_{\frac{1}{2}+iv}(z) \tag{135}$$

for the two-point correlation function one can get

$$\mathcal{D}_2(\beta, t) = \int_{-\infty}^{+\infty} e^{i(n^2-v^2)t-\beta(\frac{1}{4}+n^2)} A_{uv} B_{vu} \, dudv. \tag{136}$$

In terms of the new variables $\mathcal{F}$ the basic matrix element (135) will take the form

$$A_{pq} = \int_{-1/2}^{1/2} dx \int_{\frac{i}{2}\log(1-x^2)}^{\infty} dy \psi^*_p(x, y) \left( e^{-\frac{i}{2}y} A e^{\frac{i}{2}y} \right) \psi_q(x, y). \tag{137}$$

The matrix element (135), (137) plays a fundamental role in the investigation of the correlation functions because all correlations can be expressed through it. One should choose also appropriate observables $A$ and $B$. The operator $y^{-2}$ seems very appropriate for two reasons. Firstly, the convergence of the integrals over the fundamental region $\mathcal{F}$ will be well defined. Secondly, this operator is reminiscent of the exponentiated Louiville field since $y^{-2} = e^{-2\tilde{y}}$. Thus the interest is in calculating the matrix element (137) for the observables in the form of the Louiville-like operators $\mathcal{L}^2$

$$A(N) = e^{-2Ny} \tag{138}$$

with matrix element

$$A_{pq}(N) = \int_{-1/2}^{1/2} dx \int_{\frac{i}{2}\log(1-x^2)}^{\infty} dy \, \psi^*_p(x, y) e^{-2Ny} \psi_q(x, y), \quad N = 1, 2, ..., \tag{139}$$

The other interesting observable is $A = \cos(2\pi Nx), N = 1, 2, ...$. The evaluation of the above matrix elements is convenient to perform using a perturbation expansion in which the part of the wave function (131) containing the Bessel’s functions and the contribution of the discrete spectrum is considered as a perturbation. These terms of the perturbative expansion are small and don’t change the physical behaviour of the correlation functions. The reason behind this fact is that in the integration region $\Im z \gg 1, \Im w \gg 1$ of the matrix element (135) the Bessel’s functions decay exponentially. Therefore the contribution of these high modes is small (analogues to the so called mini-superspace approximation in the Liouville theory). In the first approximation of the wave function (131) for the matrix element one can get

$$A_{pq}(N) = \frac{2F_1\left(\frac{1}{2}, N+\frac{i}{2}; \frac{3}{2}; \frac{1}{2}\right)}{2N+i(p-q)} + \frac{2F_1\left(\frac{1}{2}, N+i\frac{p+q}{2}; \frac{3}{2}; \frac{1}{2}\right)}{2N+i(p+q)} e^{-i\phi(q)} + \frac{2F_1\left(\frac{1}{2}, N-i\frac{p+q}{2}; \frac{3}{2}; \frac{1}{2}\right)}{2N-i(p+q)} e^{i\phi(p)} + \frac{2F_1\left(\frac{1}{2}, N-i\frac{p-q}{2}; \frac{3}{2}; \frac{1}{2}\right)}{2N+i(p-q)} e^{i(\phi(p)-\phi(q))}. \tag{140}$$

37
where the scattering phase $\varphi(p)$ was defined in (132). Thus the correlation function between Louville-like fields taken in the power $N$ and $M$ respectively is:

$$D_2(\beta, t) = \int_{-\infty}^{+\infty} e^{i(p^2-q^2)t-\beta(\frac{1}{4}+p^2)}A_{pq}(N)A_{qp}(M)\,dpdq.$$  \hspace{1cm} (141)

This expression is very convenient for the analytical and numerical analyses. It is expected that the two-point correlation function decay exponentially\textsuperscript{120}

$$D_2(\beta, t) \sim K_2(\beta) e^{-\frac{t}{t_2(\beta)}},$$ \hspace{1cm} (142)

where $t_2(\beta)$ is a characteristic time scale of the quantum-mechanical system. The exponential decay of the two-point correlation function with time at different temperatures is shown in Fig.5. The dependence of the exponent $t_2(\beta)$ and of the prefactor $K_2(\beta)$ as a function of temperature are presented in Fig.5. At high and low temperatures $t_2(\beta)$ tends to the fixed values shown in the Fig.5 in dimensionless units. The out-of-time-order four-point correlation function of interest was defined in\textsuperscript{120} as follows:

$$D_4(\beta, t) = \langle A(t)B(0)A(t)B(0)e^{-\beta H} \rangle = \sum_{n,m,l,r} e^{i(E_n-E_m+E_l-E_r)t-\beta E_n}$$

$$\langle n\mid A(0)\mid m\rangle \langle m\mid B(0)\mid l\rangle \langle l\mid A(0)\mid r\rangle \langle r\mid B(0)\mid n\rangle.$$ \hspace{1cm} (143)

The other important observable is the double commutator of the Louville-like operators separated in time\textsuperscript{120}

$$C(\beta, t) = -\langle [A(t), B(0)]^2 e^{-\beta H} \rangle.$$ \hspace{1cm} (144)

The energy eigenvalues we shall parametrise as $n = \frac{1}{2} + iu$, $m = \frac{1}{2} + iv,l = \frac{1}{2} + il$ and $r = \frac{1}{2} + ir$, thus from (143) we shall get\textsuperscript{120}

$$D_4(\beta, t) = \int_{-\infty}^{+\infty} e^{i(u^2-v^2+l^2-r^2)t-\beta(\frac{1}{4}+u^2)}A_{uv}B_{vl}A_{il}B_{ru}dudvdlldr.$$ \hspace{1cm} (145)
Figure 6: The exponential decay of the correlation function $D_4(\beta, t)$ as a function of time at $\beta = 1$. The functions $D'_4(\beta, t), D''_4(\beta, t), D'''_4(\beta, t)$ demonstrate a similar exponential decay $\exp\left(-\frac{t}{t_4(\beta)}\right)$. The temperature dependence of the exponent $t_4(\beta)$ has a well defined high and low temperature limits and is shown on the r.h.s. graph. The corresponding limiting values of the function $t_4(\beta)$ in dimensionless units are $t_4(0) = 0.112$ and $t_4(\infty) = 0.163$. The exponent $t_2(\beta)$ of the two-point correlation function is shown in the Fig.5.

In terms of the variables (130) the four-point correlation function (145) will take the following form:

$$D_4(\beta, t) = \int_{-\infty}^{+\infty} e^{i(p^2 - q^2 + l^2 - r^2) - \beta(\frac{1}{4} + \nu^2)} A_{pq}(N) A_{ql}(M) A_{lr}(N) A_{rp}(M) dpdqdl dr. \quad (146)$$

As it was suggested in (120) the most important correlation function indicating the traces of the classical chaotic dynamics in quantum regime is (144)

$$C(\beta, t) = \langle [A(t), B(0)]^2 e^{-\beta H} \rangle = -D_4(\beta, t) + D'_4(\beta, t) + D''_4(\beta, t) - D'''_4(\beta, t). \quad (147)$$

For the Artin system one can get that (102)

$$D'_4(\beta, t) + D''_4(\beta, t) = 2 \int_{-\infty}^{+\infty} e^{-\beta(\frac{1}{4} + \nu^2)} \cos (q^2 - r^2)t \cos (p^2 - q^2 + l^2 - r^2)t A_{pq}(N) A_{ql}(M) A_{lr}(N) A_{rp}(M) dpdqdl dr. \quad (148)$$

and

$$D_4(\beta, t) + D'''_4(\beta, t) = 2 \int_{-\infty}^{+\infty} e^{-\beta(\frac{1}{4} + \nu^2)} \cos (p^2 - q^2 + l^2 - r^2)t \cos (q^2 - r^2)t A_{pq}(N) A_{ql}(M) A_{lr}(N) A_{rp}(M) dpdqdl dr. \quad (149)$$

The Fig.6 shows the behaviour of the four-point correlation $D_4(\beta, t)$ as the function of the temperature and time. All four correlation functions decay exponentially as it is shown in Fig.6

$$D_4(\beta, t) \sim K(\beta) e^{-\frac{t}{t_4(\beta)}}. \quad (150)$$

Turning to the investigation of the double commutator (147) it is convenient to represent it in the following form (102)

$$C(\beta, t) = 2 \int_{-\infty}^{+\infty} e^{-\beta(\frac{1}{4} + \nu^2)} \{ \cos (q^2 - r^2)t - \cos (p^2 - q^2 + l^2 - r^2)t \} A_{pq}(N) A_{ql}(M) A_{lr}(N) A_{rp}(M) dpdqdl dr, \quad (151)$$
Figure 7: Time evolution of the correlation function \( C(\beta, t) \) \(^{(151)}\) at temperature \( \beta = 1 \). For the short time intervals the function \( C(\beta, t) \) exponentially increases with time. The exponent \( \chi(\beta) \) \(^{(152)}\) slowly decreases with temperature \( \beta \). The temperature dependence of Artin exponent \( 1/\chi(\beta) \) relative to the maximum growth \( 1/\beta \) is shown by blue dots in the far right figure. At high temperatures the Lyapunov exponent \( 1/\chi(\beta) \) in Artin system is smaller than the maximal one \( 1/\beta \), but at low temperatures this is not any more true, we observe a breaking of the saturation regime.

where \(^{(148)}\) and \(^{(149)}\) have been used. The results of the integration are presented in the Fig.7. For a short time intervals the \( C(\beta, t) \) increases exponentially in time

\[
C(\beta, t)_{\text{Artin–AdS}} \sim K(\beta) e^{\frac{4\pi}{\chi(\beta)} t},
\]

with the exponent \( 1/\chi(\beta) \). The important ratio with respect to the maximal growth exponent

\[
\frac{C(\beta, t)_{\text{max}}}{C(\beta, t)_{\text{Artin–AdS}}} \sim R(\beta) e^{\frac{1}{\beta – \frac{1}{\chi(\beta)}^{2\pi t}}}
\]

is presented in Fig.7. The temperature dependence of Artin exponent \( 1/\chi(\beta) \) relative to the maximum growth \( 1/\beta \) is shown by blue dots in the far right figure. At high temperatures the Artin Lyapunov exponent \( 1/\chi(\beta) \) is less than the maximal exponent \( 1/\beta \), but at low temperatures this is not any more true and we observe a breaking of the saturation regime\(^{120,122}\).

In order to confirm this result it seems important to investigate the behaviour of correlation functions and double commutators \(^{(133)-(151)}\) for alternative observables, as well as using more powerful computer codes and hardware as it was available to us.

10 Artin-Maass Resonances and Riemann Zeta Function Zeros

Here we shall demonstrate that the Riemann zeta function zeros define the position and the widths of the resonances of the quantised Artin system\(^{156}\). As it was discussed in previous sections the Artin system is defined on the fundamental region of the modular group on the Lobachevsky plane. It has a finite area and an infinite extension in the vertical direction that correspond to a cusp. In classical regime the geodesic flow on this non-compact surface of constant negative curvature represents one of the most chaotic dynamical systems, has mixing of all orders, continuous classical Koopman spectrum \(^{78}, (81)\) and non-zero Kolmogorov entropy.
The Arin system is defined on a non-compact surface $\bar{F}$ of constant negative curvature which has a topology of sphere with a cusp on the north pole which is stretched to infinity. The deficit angles on the vertices of the Artin surface can be computed using the formula $2\pi - \alpha$, thus $\int K \sqrt{g} d^2\xi = (\frac{\pi}{3} + (2\pi - \frac{2\pi}{3}) + (2\pi - \frac{2\pi}{2}) + (2\pi - 0) = 4\pi$.

In quantum-mechanical regime the system can be associated with the narrow infinitely long waveguide stretched out to infinity along the vertical axis and a cavity resonator attached to it at the bottom Fig.8. That suggests a physical interpretation of the Maass automorphic wave function in the form of an incoming plane wave of a given energy which is entering the resonator, bouncing and scatters back to infinity. As the energy of the incoming wave comes close to the eigenmodes of the cavity a pronounced resonance behaviour shows up in the scattering amplitude.

We already presented above the Maass wave function in terms of the natural physical variable $\tilde{\gamma}$, which is the distance in the vertical direction on the Lobachevsky plane $\ln y = \tilde{\gamma}$, and of the corresponding momentum $\tilde{p}$. The plane wave $e^{-i\tilde{p}\tilde{\gamma}}$ incoming from infinity $D$ along the $y$ axis in Fig.3, Fig.4 and Fig.8 elastically scatters on the boundary $ACB$ of the fundamental triangle $\mathcal{F}$. The reflection amplitude is a pure phase and is given by the expression in front of the outgoing plane wave $e^{i\tilde{p}\tilde{\gamma}}$:

$$S = \frac{\theta(\frac{1}{2} + ip)}{\theta(\frac{1}{2} - ip)} = \exp[2i\delta(p)].$$  \hspace{1cm} (154)

The continuous energy spectrum is given by the formula.

$$E = p^2 + \frac{1}{4}.$$  \hspace{1cm} (155)

As we suggested the system can be described in terms of an infinitely long narrow waveguide with a cavity resonator attached to it at the bottom Fig.4 and Fig.8. In order to support this interpretation we will calculate the area of the Artin surface which is below the fixed coordinate $y_0 = e^{\tilde{y}_0}$:

$$\text{Area}(\mathcal{F}_0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{\sqrt{1-x^2}}^{y_0} dy = \frac{\pi}{3} - e^{-\tilde{y}_0} = \text{Area}(\mathcal{F}) - e^{-\tilde{y}_0}.$$  \hspace{1cm} (156)
Figure 9: The system can be described as a narrow infinitely long waveguide stretched to infinity along the vertical direction and a cavity resonator attached to it at the bottom ACB.

The area $\text{Area}(\mathcal{F}) - \text{Area}(\mathcal{F}_0)$ above the ordinate $\tilde{y}_0$ is therefore exponentially small $e^{-\tilde{y}_0}$. The horizontal ($dy = 0$) size of the Artin surface is also decreases exponentially in the vertical direction:

$$L_0 = 2 \int ds = 2 \int \frac{\sqrt{dx^2 + dy^2}}{y} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{y_0} = e^{-\tilde{y}_0}. \quad (157)$$

The cavity has its eigenmodes and as the energy of the incoming wave $E = p^2 + \frac{1}{4}$ became close to the eigenmodes of the cavity one should expect a pronounced resonance behaviour of the scattering amplitude.

To trace such behaviour let us consider the analytical continuation of the Maass wave function (131) to the complex energy plane $E$. The analytical continuation of the scattering amplitudes as a function of the energy $E$ considered as a complex variable allows to establish important spectral properties of the quantum-mechanical system. In particular, the method of analytic continuation allows to determine the real and complex S-matrix poles. The real poles on the physical sheet correspond to the discrete energy levels and the complex poles on the second sheet below the cut correspond to the resonances in the quantum-mechanical system.

The asymptotic form of the wave function can be represented in the following form:

$$\psi = A(E) e^{ip\tilde{y}} + B(E) e^{-ip\tilde{y}}, \quad p = \sqrt{E - 1/4}. \quad (158)$$

In order to make the functions $A(E)$ and $B(E)$ single-valued one should cut the complex plane along the real axis at the $E = 1/4$. The complex plane with this cut defines a physical sheet. To the left from the cut, at energies $E < 1/4$, the wave function takes the following form:

$$\psi = A(E) e^{-\sqrt{|E-1/4|} \tilde{y}} + B(E) e^{\sqrt{|E-1/4|} \tilde{y}}, \quad (159)$$
The resonances $E_n - i \frac{\Gamma_n}{2}$ are located under the cut on the right hand side of the real axis.

where the exponential factors are real and one of them decreases and the other one increases at $\tilde{y} \to \infty$. The bound states are characterised by the fact that the corresponding wave function tends to zero at $\tilde{y} \to \infty$, thus for the bound states the second term in (159) vanishes $B(E_n) = 0$.158 If a system is unbounded then its energy spectrum may be continuous and the energy spectrum can be quasi-discrete, consisting of smeared levels of a width $\Gamma$.158 These states are also described by the absence incoming waves now in (158):

$$B(E_n - i \frac{\Gamma_n}{2}) = 0$$

and define the complex eigenvalues of the form158

$$\varepsilon_n = E_n - i \frac{\Gamma_n}{2},$$

where $E + n$ and $\Gamma_n$ are both real and positive. The complex poles $E_n - i \frac{\Gamma_n}{2}$ are located under the cut on the right hand side of the real axis Fig.10. To find a position of a resonance one can expand $B(E)$ near a resonance (161) as $B(E) = (E - E_n + i \frac{\Gamma_n}{2}) b_n + ...$ and represent the wave function (158) in the form

$$\psi \approx b_n^*(E - E_n - i \frac{\Gamma_n}{2}) e^{ip\tilde{y}} + b_n (E - E_n + i \frac{\Gamma_n}{2}) e^{-ip\tilde{y}}.$$ (162)

The S-matrix takes the following form158

$$S = e^{2i\delta} = \frac{E - E_n - i \frac{\Gamma_n}{2}}{E - E_n + i \frac{\Gamma_n}{2}} e^{2i\delta_n},$$

where $e^{2i\delta_n} = b_n^*/b_n$. We can find now the positions of the poles in quantum Artin system. Let us consider the asymptotic behaviour of the Maass wave function (131) at large $\tilde{y}$. The conditions (160) of the absence of incoming wave due to (126) will take the form156

$$\theta(\frac{1}{2} - ip) = \frac{\zeta(1 - 2ip)\Gamma(\frac{1}{2} - ip)}{\pi \frac{1}{2} - ip} = 0.$$ (164)
The solution can be expressed in terms of the zeros of the Riemann zeta function:

\[ \zeta\left(\frac{1}{2} - iu_n\right) = 0, \quad n = 1, 2, \ldots, \quad u_n > 0. \]  

(165)

Thus one should solve the equation

\[ 1 - 2ip_n = \frac{1}{2} - iu_n. \]  

(166)

The location of poles is therefore at the following values of the complex momenta

\[ p_n = \frac{u_n}{2} - i \frac{1}{4}, \quad n = 1, 2, \ldots. \]  

(167)

and at the complex energies (155) :

\[ E = p_n^2 + \frac{1}{4} = (\frac{u_n}{2} - i \frac{1}{4})^2 + \frac{1}{4} = \frac{u_n^2}{4} + \frac{3}{16} - i \frac{u_n}{4}. \]  

(168)

These poles correspond to the resonances (161):

\[ E_n = \frac{u_n^2}{4} + \frac{3}{16}, \quad \Gamma_n = \frac{u_n}{2}. \]  

(169)

One can conjecture the following representation of the S-matrix (154):

\[ S = e^{2i\delta} = \frac{\theta(\frac{1}{2} + ip) - \theta(\frac{1}{2} - ip)}{\theta(\frac{1}{2} + ip) - \theta(\frac{1}{2} - ip)} = \sum_{n=1}^{\infty} \frac{E - E_n - i\Gamma_n/2}{E - E_n + i\Gamma_n/2} e^{2i\delta_n} \]  

(170)

with yet unknown phases \( \delta_n \). In order to justify the above representation of the S-matrix one can find the location of the poles on the second sheet by using expansion of the S-matrix (154) at the ”bumps” which occur along the real axis at energies

\[ E_n = \frac{u_n^2}{4} + \frac{3}{16}. \]  

(171)

The expansion will take the following form:

\[ S|_{E\approx E_n} = \frac{\theta(\frac{1}{2} + i\sqrt{E_n - \frac{1}{4}})}{\theta(\frac{1}{2} - i\sqrt{E_n - \frac{1}{4}})} = \frac{\theta(\frac{1}{2} + i\sqrt{E_n - \frac{1}{4}}) + \theta'(\frac{1}{2} + i\sqrt{E_n - \frac{1}{4}})}{\theta(\frac{1}{2} - i\sqrt{E_n - \frac{1}{4}}) + \theta'(\frac{1}{2} - i\sqrt{E_n - \frac{1}{4}})} \frac{E - E_n}{E - E_n + i\Gamma_n/2} e^{2i\delta_n}. \]  

(172)

where

\[ E_n' - i\Gamma_n' / 2 = E_n - \frac{\theta(\frac{1}{2} - i\sqrt{E_n - \frac{1}{4}})}{\theta'(\frac{1}{2} - i\sqrt{E_n - \frac{1}{4}})}, \quad e^{2i\delta_n'} = \frac{\theta'(\frac{1}{2} + i\sqrt{E_n - \frac{1}{4}})}{\theta'(\frac{1}{2} - i\sqrt{E_n - \frac{1}{4}})}. \]  

(173)
Figure 11: At each point $w$ of the C-system the tangent space $R^m_w$ is decomposable into a direct sum of two linear spaces $Y^l_w$ and $X^k_w$. The expanding and contracting geodesic flows are $\gamma^+$ and $\gamma^-$. The expanding and contracting invariant foliations $\Sigma^l_w$ and $\Sigma^k_w$ are transversal to the geodesic flows and their corresponding tangent spaces are $Y^l_w$ and $X^k_w$.

Thus

$$E'_n - E_n = -\frac{\Re \theta\left(\frac{1}{2} - i \sqrt{E_n - \frac{1}{4}}\right)}{\theta'\left(\frac{1}{2} - i \sqrt{E_n - \frac{1}{4}}\right)}, \quad -i\Gamma'_n/2 = -\frac{\theta\left(\frac{1}{2} - i \sqrt{E_n - \frac{1}{4}}\right)}{\theta'\left(\frac{1}{2} - i \sqrt{E_n - \frac{1}{4}}\right)}$$

(174)

and all quantities $E'_n$, $\Gamma'_n/2$ and $\delta'_n$ are real. Considering the first ten zeros of the zeta function which are known numerically\(^{[159],[160]}\) one can calculate the position of the resonances and their widths using the approximation formulas (174) and get convinced that the energies and the widths of the resonances given by the exact formula (169) and the one given by the approximation formulas (174) are consistent within the two percent deviation.

It has been observed that numerical calculation of the discrete energy eigenstates (101) of the Artin system is unstable and shows slow convergence\(^{[141],[142],[144]}\). It seems that this can be caused by the presence of the resonances. If the eigenvalue lies within the continuous energy band of a resonance it should be difficult to distinguish and separate it from the continuous resonance spectrum.

### 11 C-cascade. Entropy and Periodic Trajectories

In this section we shall turn our attention to the investigation of the second class of the MCDS defined on high dimensional tori with a discrete in time evolution\(^{[4]}\). The systems with discrete time\(^{[4]}\) is defined as a cascade on the $d$-dimensional compact phase space $M^d$ induced by the diffeomorphisms $T : M^d \to M^d$. The iterations are defined by a repeated action of the operator $\{T^n, -\infty < n < +\infty\}$, where $n$ is an integer number. The tangent space at the point $x \in M^d$ is denoted by $R^d_x$ and the tangent vector bundle by $\mathcal{R}(M^d)$. The diffeomorphism $\{T^n\}$ induces the mapping of the tangent spaces $\tilde{T}^n : R^d_x \to R^d_{T^n x}$. The C-condition requires that the tangent space $R^d_x$ at each point $x$ of the $d$-dimensional phase space $M^d$ of the dynamical system $\{T^n\}$ should be decomposable into a direct sum of the two linear spaces $X^k_x$ and $Y^l_x$ with the following...
standard deviation

\[ \sigma \]

properties:

\[ f \]

of the function \[ \langle \] that the fluctuations of the time averages from the phase space integral statistically weakly dependent. Therefore for the C-systems on a torus it was demonstrated embedded into the foliation \( \Sigma \)

\[ \log \] of the volume expansion rate

convenient way to calculate the entropy of MCDS is to integrate over the phase space the

Let us turn now to the calculation of the corresponding Kolmogorov-Sinai entropy. The most

multiplied by \( M \)

and if the rate of expansion of the volume of the \( l \)-dimensional cube is \( \lambda(x) \), then

\[ h(T) = \int_{M^d} \ln \lambda(x) dx. \]
Here the volume of the $M^d$ is normalised to 1.

Let us consider the automorphisms of a torus generated by the linear transformation

$$x_i \rightarrow \sum_{j=1}^{n} T_{ij} x_j, \quad (\text{mod} \ 1),$$

(179)

where the integer matrix $T$ has a determinant equal to one $\det T = 1$. In order for the automorphisms of the torus (179) to fulfil the C-condition (175) it is necessary and sufficient that the matrix $T$ has no eigenvalues on the unit circle. Thus the spectrum $\{\lambda = \lambda_1, ..., \lambda_n\}$ of the matrix $T$ should fulfil the following two conditions:

1) $\det T = \lambda_1 \lambda_2 ... \lambda_n = 1$

2) $|\lambda_i| \neq 1, \quad \forall i.$  

(180)

Because the determinant of the matrix $T$ is equal to one, the Liouville’s measure $d\mu = dx_1 ... dx_n$ is invariant under the action of $T$. The inverse matrix $T^{-1}$ is also an integer matrix because $\det T = 1$. Therefore $T$ is an automorphism of the torus onto itself. All trajectories with rational coordinates $(x_1, ..., x_n)$, and only they, are periodic trajectories of the automorphisms of the torus (179). The above conditions (180) on the eigenvalues of the matrix $T$ are sufficient to prove that the system belongs to the class of Anosov C-systems and therefore has mixing properties defined above (4)- (8). Because the C-systems have mixing of all orders it follows that the C-systems are exhibiting the decay of the correlation functions of any order.

For the automorphisms on a torus (179), the coefficient $\lambda(x)$ does not depend on the phase space coordinates $x$ and is equal to the product of eigenvalues $\{\lambda_\beta\}$ with modulus larger than one (222):

$$\lambda(w) = \prod_{\beta=1}^{l} \lambda_\beta.$$  

(181)

Thus the entropy of the Anosov automorphisms on a torus (179), (180) can be calculated and is equal to the sum (4,28,29,31,34)

$$h(T) = \sum_{|\lambda_\beta| > 1} \ln |\lambda_\beta|.$$  

(182)

Thus the entropy $h(T)$ directly depends on the spectrum of the operator $T$. This fact allows to characterise and compare the chaotic properties of dynamical C-systems quantitatively computing and comparing their entropies.

The entropy defines the variety and richness of the periodic trajectories of the C-systems (4,32,34,40). The C-systems have a countable set of everywhere dense periodic trajectories (4). The $E^m$ cover of the torus $W^m$ allows to translate every set of points on the torus into a set of points on Euclidean space $E^m$ and the space of functions on the torus into the periodic functions on $E^m$. To every
Figure 12: The automorphisms on a two-torus. The a) depicts the parallel lines along the eigenvectors and b) depicts their positions after the action of the automorphism.

closed curve $\gamma$ on a torus corresponds a curve $\phi : [0, 1] \to E^m$ for which $\phi(0) = \phi(1) \mod 1$ and if $\phi(1) - \phi(0) = (p_1, ..., p_m)$, then the corresponding winding numbers on a torus are $p_i \in \mathbb{Z}$.

Let us fix the integer number $N$, then the points on a torus with the coordinates having a denominator $N$ form a finite set $\{p_1/N, ..., p_m/N\}$. The automorphism (180) with integer entries transform this set of points into itself, therefore all these points belong to periodic trajectories. Let $w = (w_1, ..., w_m)$ be a point of a trajectory with the period $n > 1$. Then

$$T^m w = w + p,$$

(183)

where $p$ is an integer vector. The above equation with respect to $w$ has nonzero determinant, therefore the components of $w$ are rational.

Thus the periodic trajectories of the period $n$ of the automorphism $T$ are given by the solution of the equation (183), where $p \in \mathbb{Z}^m$ is an integer vector and $w = (w_1, ..., w_m) \in W^m$. As $p$ varies in $\mathbb{Z}^m$ the solutions of the equation (183) determine a fundamental domain $D_n$ in the covering Euclidian space $E^m$ of the volume $\mu(D_n) = 1/|Det(T^m - 1)|$. Therefore the number of all points $N_n$ on the periodic trajectories of the period $n$ is given by the corresponding inverse volume:

$$N_n = |Det(T^m - 1)| = |\prod_{i=1}^m (\lambda^n_i - 1)|.$$

(184)

Using the theorem of Bowen which states that the entropy of the automorphism $T$ can be represented in terms of $N_n$ defined in (184):

$$h(T) = \lim_{n \to \infty} \frac{1}{n} \ln N_n,$$

(185)

one can derive the formula for the entropy (182) for the automorphism $T$ in terms of its eigenvalues:

$$h(T) = \lim_{n \to \infty} \frac{1}{n} \ln(|\prod_{i=1}^m (\lambda^n_i - 1)|) = \sum_{|\lambda_\beta| > 1} \ln |\lambda_\beta|.$$

(186)

Let us now define the number of periodic trajectories of the period $n$ by $\pi(n)$. Then the number of all points $N_n$ on the periodic trajectories of the period $n$ can be written in the following form:

$$N_n = \sum_{l \text{ div } n} l \cdot \pi(l),$$

(187)
where $l$ divides $n$. Using again the Bowen result (185) one can get

$$N_n = \sum_{l \mid n} l \pi(l) \sim e^{nh(T)}.$$  \hspace{1cm} (188)

This result can be rephrased as a statement that the number of points on the periodic trajectories of the period $n$ exponentially grows with the entropy.

Excluding the periodic trajectories which divide $n$ (for example $T^n w = T^{l_1 l_2} T^{l_1} w$, where $n = l_1 l_2$ and $T^{l_1} w = w$) one can get the number of periodic trajectories of period $n$ which are not divisible. For that one should represent the $\pi(n)$ in the following form:

$$\pi(n) = \frac{1}{n} \left( \sum_{l \mid n} l \pi(l) - \sum_{l \mid n, l < n} l \pi(l) \right)$$  \hspace{1cm} (189)

and from (189) and (188) it follows that

$$\pi(n) \sim \frac{e^{nh(T)}}{n} \left( 1 - \frac{\sum_{l \mid n, l < n} l \pi(l)}{\sum_{l \mid n} l \pi(l)} \right) \sim \frac{e^{nh(T)}}{n},$$  \hspace{1cm} (190)

because the ratio in the bracket is strictly smaller than one. This result tells that a system with larger entropy $\Delta h = h(T_1) - h(T_2) > 0$ is more densely populated by the periodic trajectories of the same period $n$:

$$\frac{\pi_1(n)}{\pi_2(n)} \sim e^{n \Delta h}.$$  \hspace{1cm} (191)

The next important result of the Bowen theorem\textsuperscript{33,34} states that

$$\int_{W^m} f(w) d\mu(w) = \lim_{n \to \infty} \frac{1}{N_n} \sum_{w \in \Gamma_n} f(w),$$  \hspace{1cm} (192)

where $\Gamma_n$ is a set of all points on the trajectories of period $n$. The total number of points in the set $\Gamma_n$ we defined earlier as $N_n$.

This result has important consequences for the calculation of the integrals on the manifold $W^m$, because, as it follows from (192), the integration reduces to the summation over all points of periodic trajectories. It is appealing to consider periodic trajectories of the period $n$ which is a prime number. Because every infinite subsequence of convergent sequence converges to the same limit we can consider in (192) only terms with the prime periods. In that case $N_n = n \pi(n)$ and the above formula becomes:

$$\int_{W^m} f(w) d\mu(w) = \lim_{n \to \infty} \frac{1}{n \pi(n)} \sum_{j=1}^{\pi(n) n-1} \sum_{i=0}^{n-1} f(T^i w_j),$$  \hspace{1cm} (193)

where the summation is over all points of the trajectory $T^i w_j$ and over all distinct trajectories of period $n$ which are enumerated by index $j$. The $w_j$ is the initial point of the trajectory $j$.\textsuperscript{19}

\textsuperscript{19}It appears to be a difficult mathematical problem to decide whether two vectors $w_1$ and $w_2$ belong to the same or to distinct trajectories.
From the above consideration it follows that the convergence is guaranteed if one sums over all trajectories of the same period $n$. One can conjecture that all $\pi(n)$ trajectories at the very large period $n$ contribute equally into the sum (193), therefore the integral (193) can be reduced to a sum over fixed trajectory

$$\frac{1}{n} \sum_{i=0}^{n-1} f(T^i w).$$

Thus the knowledge of the spectrum allows to calculate the entropy (182) and the number of periodic trajectories of a period less than $n$ grows exponentially (190).

## 12 MIXMAX Random Number Generator

It was suggested in 1986 in\cite{96} to use the MCDS defined on a torus to generate high quality pseudorandom numbers for Monte-Carlo method. The modern powerful computers open a new era for the application of the Monte-Carlo Method\cite{90,93,96,114,115} for the simulation of physical systems with many degrees of freedom and of higher complexity. The Monte-Carlo simulation is an important computational technique in many areas of natural sciences, and it has significant application in particle and nuclear physics, quantum physics, statistical physics, quantum chemistry, material science, among many other multidisciplinary applications. At the heart of the Monte-Carlo (MC) simulations are pseudorandom number generators (RNG).

Usually pseudo random numbers are generated by deterministic recursive rules\cite{90,93,96}. Such rules produce pseudorandom numbers, and it is a great challenge to design pseudo random number generators that produce high quality sequences. Although numerous RNGs introduced in the last decades fulfil most of the requirements and are frequently used in simulations, each of them has some weak properties which influence the results\cite{113} and are less suitable for demanding MC simulations which are performed for the high energy experiments at CERN and other research centres. The RNGs are essentially used in high energy experiments at CERN for the design of the efficient particle detectors and for the statistical analysis of the experimental data.

In order to fulfil these demanding requirements it is necessary to have a solid theoretical and mathematical background on which the RNG’s are based. RNG should have a long period, be statistically robust, efficient, portable and have a possibility to change and adjust the internal characteristics in order to make RNG suitable for concrete problems of high complexity. In\cite{96} it was suggested that Anosov C-systems,\cite{4} defined on a high dimensional torus, are excellent candidates for the pseudo-random number generators. The C-system chosen in\cite{96} was the one which realises linear automorphism $T$ defined in (179). For convenience in this section the
Figure 13: The tangent vector $\omega \in R_x$ at $x \in M^2$ is decomposable into the sum $R_x = X_x \oplus Y_x$ where the spaces $X_x$ and $Y_x$ are defined by the eigenvectors of the $2 \times 2$ matrix $T(2,0)$ \[^{195}\]. It is exponentially contracting the distances on $X_x$ and expanding the distances on $Y_x$ (details are given in Appendix A).

Figure 14: The distribution of the eigenvalue of operator $A = T(N,s)$ and of its inverse $T^{-1}(N,s)$ in the complex $\lambda$ plane. All eigenvalues are lying outside of the unit circle. The points of the curve that are inside the unit circle represent the eigenvalues with modulus less than one $\{\lambda_{a}\}$ and outside the circle the eigenvalues $\{\lambda_{b}\}$ with modulus larger than one.

dimension $n$ of the phase space $M$ is denoted by $N$. A particular matrix chosen in \[^{111}\] was defined for all $N \geq 2$. The operators $T(N,s)$ are parametrised by the integers $N$ and $s$

$$T(N,s) = \begin{pmatrix}
1 & 1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 2 & 1 & 1 & \ldots & 1 & 1 \\
1 & 3 & 2 & 1 & \ldots & 1 & 1 \\
1 & 4 & 3 & 2 & \ldots & 1 & 1 \\
& & & & & & \\
1 & N & N - 1 & N - 2 & \ldots & 3 & 2
\end{pmatrix} \quad (195)$$

Its entries are all integers $T_{ij} \in \mathbb{Z}$ and $Det T = 1$. The spectrum and the value of the Kolmogorov entropy can be calculated. It is defined recursively, since the matrix of size $N + 1$ contains in it the matrix of the size $N$. In order to generate pseudo-random vectors $x_n = T^n x$, one should choose the initial vector $x = (x_1, \ldots, x_N)$, called the “seed”, with at least one non-zero component to avoid fixed point of $T$, which is at the origin. The eigenvalues of the $T$ matrix \[^{195}\] are widely dispersed for all $N$, see Fig. 14 from reference \[^{107}\] The spectrum is “multi-scale”, with trajectories exhibiting exponential instabilities at different scales \[^{14}\] The spectrum of the operator $T(N,s)$ has two real eigenvalues for even $N$ and three for odd $N$, all the rest of the eigenvalues are complex and lying on leaf-shaped curves. It is seen that the spectrum tends to a universal limiting form as $N$ tends to infinity, and the complex eigenvalues $1/\lambda$ (of the
inverse operator) lie asymptotically on the cardioid curve Fig.14 which has the representation

\[ r(\phi) = 4 \cos^2(\phi/2) \]  

(196)
in the polar coordinates \( \lambda = r \exp(i\phi) \). From the above analytical expression for eigenvalues it follows that the eigenvalues satisfying the condition \( 0 < |\lambda_\phi| < 1 \) are in the range \(-2\pi/3 < \phi < 2\pi/3\) and the ones satisfying the condition \( 1 < |\lambda_\phi| < 4\pi/3 \) are in the interval \( 2\pi/3 < \phi < 4\pi/3 \). One can conjecture that there exists a limiting infinite-dimensional dynamical system with continuous space coordinate and discrete time with the above spectrum. The entropy of the C-K system \( T(N,s) \) can now be calculated for large values of \( N \) as an integral over eigenvalues (196):

\[
h(T) = \sum_\alpha \ln|\frac{1}{\lambda_\alpha}| = \sum_{-2\pi/3 < \phi_i < 2\pi/3} \ln(4 \cos^2(\phi_i/2) \to N \int_{-2\pi/3}^{2\pi/3} \ln(4 \cos^2(\phi/2) \frac{d\phi}{2\pi} = \frac{2}{\pi} N \]  

(197)
and to confirm that the entropy increases linearly with the dimension \( N \) of the operator \( T(N,s) \).

The period of the trajectories of the system \( T(N,s) \) was found in [195] and is characterised by a prime number \( p \). In [97] the necessary and sufficient criterion were formulated for the sequence to be of the maximal possible period:

\[
\tau = \frac{p^N - 1}{p - 1} \sim e^{N \ln p}, \quad N \gg 1.
\]  

(198)
It follows then that the period of the trajectories exponentially increases with the size of the operator \( T(N,s) \).

**Computer Implementation.** In a typical computer implementation of the automorphism (195) the initial vector will have rational components \( u_i = a_i/p \), where \( a_i \) and \( p \) are natural numbers. Therefore it is convenient to represent \( u_i \) by its numerator \( a_i \) in computer memory and define the iteration in terms of \( a_i \):

\[
a_i \to \sum_{j=1}^{N} T_{ij} a_j \mod p.
\]  

(199)
If the denominator \( p \) is taken to be a prime number, then the recursion is realised on extended Galois field \( GF[p^N] \) and allows to find the period of the trajectories in terms of \( p \) and the properties of the characteristic polynomial \( P(x) \) of the matrix \( T \). If the characteristic polynomial \( P(x) \) of matrix \( T \) is primitive in the extended Galois field \( GF[p^N] \), then

\[
T^q = p_0 \mathbb{I} \quad \text{where} \quad q = \frac{p^N - 1}{p - 1},
\]  

(200)

\[^{20}\text{The general theory of Galois field and the periods of its elements can be found in.}\]
where \( p_0 \) is a free term of the polynomial \( P(x) \) and is a *primitive element* of \( GF[p] \). Since our matrix \( T \) has \( p_0 = \text{Det}T = 1 \), the polynomial \( P(x) \) of \( T \) cannot be primitive. The solution suggested in\(^{97}\) is to define the necessary and sufficient conditions for the period \( q \) to attain its maximum are the following:

1. \( T^q = \mathbb{I} \ (\text{mod } p) \), where \( q = \frac{p^N - 1}{p - 1} \)

2. \( T^{q/r} \neq \mathbb{I} \ (\text{mod } p) \), for any \( r \) which is a prime divisor of \( q \).

The first condition is equivalent to the requirement that the characteristic polynomial is irreducible. The second condition can be checked if the integer factorisation of \( q \) is available;\(^{97}\) then the period of the sequence is equal to \(^{221}\) and is independent of the seed. There are precisely \( p - 1 \) distinct trajectories which together fill up all states of the \( GF[p^N] \) lattice:

\[
q \ (p - 1) = p^N - 1.
\] (201)

In\(^{97}\) the actual value of \( p \) was taken as \( p = 2^{61} - 1 \), the largest Mersenne number that fits into an unsigned integer on current 64-bit computer architectures. For the matrix of the size \( N = 256 \) the period in that case is \( q \approx 10^{4600} \). The algorithm which allows the efficient implementation of the generator in actual computer hardware, reducing the matrix multiplication to the \( O(N) \) operations was found in\(^{97}\). The other advantage of this implementation is that it allows to make "jumps" into any point on a periodic trajectory without calculating all previous coordinates on a trajectory, which typically has a very large period \( q \approx 10^{4600} \). The MIXMAX generators were integrated into the concurrent and distributed MC toolkit Geant4\(^{104}\) the foundation library CLHEP\(^{103}\) and data analysis framework ROOT\(^{105}\). These software tools have wide applications in High Energy Physics at CERN, in CMS experiment\(^{106,107}\) at SLAC, FNAL and KEK National Laboratories and are part of the CERN’s active Technology Transfer policy. The generator is available in the PYTHIA event generator\(^{108}\). The MIXMAX code can be downloaded from the GSL-GNU Scientific Library\(^{109}\).

13 **Turning C-cascade into C-flow**

In\(^{13}\) Anosov demonstrated how any C-cascade on a torus can be embedded into a certain C-flow. The embedding was defined by the identification \(^{202}\) and the corresponding C-flow on a smooth Riemannian manifold \( W^{m+1} \) with the metric \(^{209}\) was defined by the equations \(^{204}\). We are interested here to analyse the *geodesic flow* on the same Riemannian manifold \( W^{m+1} \). The geodesic flow has different dynamics \(^{210}\) and as we shall demonstrate below has very interesting hyperbolic components different from \(^{204}\).
Figure 15: The identification of the $W^2 \times \{0\}$ with $W^2 \times \{1\}$ by the formula $(w, 1) \equiv (Tw, 0)$ of a cylinder $W^2 \times [0, 1]$, where $[0, 1] = \{u \mid 0 \leq u \leq 1\}$. The resulting compact manifold $W^3$ has a bundle structure with the base $S^1$ and fibres of the type $W^2$. The manifold $W^3$ has the local coordinates $\tilde{w} = (w^1, w^2, u)$.

Let us consider a $C$-cascade on a torus $W^m$ and increase its dimension $m$ by one unit constructing a cylinder $W^m \times [0, 1]$, where $[0, 1] = \{u \mid 0 \leq u \leq 1\}$, and identifying $W^m \times \{0\}$ with $W^m \times \{1\}$ by the formula:

$$(w, 1) \equiv (Tw, 0).$$  \hspace{1cm} (202)

Here $T$ is diffeomorphism:

$$w^i \rightarrow \sum T_{i,j}w^j, \hspace{1cm} (mod \ 1).$$  \hspace{1cm} (203)

The resulting compact Riemannian manifold $W^{m+1}$ has a bundle structure with the base $S^1$ and fibres of the type $W^m$. The manifold $W^{m+1}$ has the local coordinates $\tilde{w} = (w^1, ..., w^m, u)$ shown in Fig.15. The C-flow $T^t$ on the manifold $W^{m+1}$ is defined by the equations\footnote{204}

$$\frac{dw^1}{dt} = 0, ..., \frac{dw^m}{dt} = 0, \frac{du}{dt} = 1.$$  \hspace{1cm} (204)

For this flow the tangent space $R^{m+1}_{\tilde{w}}$ can be represented as a direct sum of three subspaces: contracting and expanding linear spaces $X^k_{\tilde{w}}$, $Y^l_{\tilde{w}}$ and $Z_{\tilde{w}}$:

$$R^{m+1}_{\tilde{w}} = X^k_{\tilde{w}} \oplus Y^l_{\tilde{w}} \oplus Z_{\tilde{w}}.$$  \hspace{1cm} (205)

The linear space $X^k_{\tilde{w}}$ is tangent to the fibre $W^m \times u$ and is parallel to the eigenvectors corresponding to the eigenvalues which are lying inside the unit circle $0 < |\lambda_\alpha| < 1$ and $Y^l_{\tilde{w}}$ is tangent to the fibre $W^m \times u$ and is parallel to the eigenvectors corresponding to the eigenvalues which are lying outside of the unit circle $1 < |\lambda_\beta|$. $Z_{\tilde{w}}$ is collinear to the phase space velocity \footnote{204}. Under the derivative mapping of the \footnote{204} the vectors \footnote{211} from $X^k_{\tilde{w}}$ and $Y^l_{\tilde{w}}$ are contracting and expanding:

$$|\tilde{T}^t v_1| = \lambda^2_2 |v_1|, \hspace{1cm} |\tilde{T}^t v_2| = \lambda^1_1 |v_2|.$$  \hspace{1cm} (206)

This identification of contracting and expanding spaces proves that \footnote{204} indeed defines a C-flow.\footnote{4}
It is also interesting to analyse the \textit{geodesic flow} on a Riemannian manifold $W^{m+1}$. The equations for the geodesic flow on $W^{m+1}$

$$\frac{d^2\tilde{u}^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{d\tilde{u}^\nu}{dt} \frac{d\tilde{u}^\rho}{dt} = 0$$

are different from the flow equations defined by the equations \textcolor{red}{(204)} and our goal is to learn if the geodesic flow has also the properties of the C-flow. The answer to this question is not obvious and requires investigation of the curvature structure of the manifold $W^{m+1}$. If all sectional curvatures are negative then geodesic flow defines a C-flow.\textcolor{red}{[1]} For simplicity let us consider the automorphisms of a two-dimensional torus which is defined by the $2 \times 2$ matrix $T(2,0)$ \textcolor{red}{(195)}. The metric on the corresponding manifold $W^3$ can be defined as \textcolor{red}{(23)}

$$ds^2 = e^{2u}[\lambda_1 dw^1 + (1 - \lambda_1)dw^2]^2 + e^{2u}[\lambda_2 dw^1 + (1 - \lambda_2)dw^2]^2 + du^2 = g_{\mu\nu}d\tilde{u}^\mu d\tilde{u}^\nu,$$

where $0 < \lambda_2 < 1 < \lambda_1$ are eigenvalues of the matrix $T(2,0)$ and fulfil the relations $\lambda_1\lambda_2 = 1$, $\lambda_1 + \lambda_2 = 3$. The metric is invariant under the transformation

$$w^1 = 2w^1 - w^2, \quad w^2 = -w^1 + w^2, \quad u = u' - 1$$

and is therefore consistent with the identification \textcolor{red}{(202)}. The metric tensor has the form

$$g_{\mu\nu}(u) = \begin{pmatrix}
\lambda_1^{2+2u} + \lambda_2^{2+2u} & (1 - \lambda_1)\lambda_1^{1+2u} + (1 - \lambda_2)\lambda_2^{1+2u} & 0 \\
(1 - \lambda_1)\lambda_1^{1+2u} + (1 - \lambda_2)\lambda_2^{1+2u} & (1 - \lambda_1)^2\lambda_1^{2u} + (1 - \lambda_2)^2\lambda_2^{2u} & 0 \\
0 & 0 & 1
\end{pmatrix}$$

and the corresponding geodesic equations take the following form:

$$\ddot{w}^1 + 2(\lambda_1 - 1)ln\lambda_1 \lambda_1^{1+2u} w^1 \dot{u} - 4(\lambda_1 - 1)ln\lambda_1 \lambda_1^{1+2u} w^2 \dot{u} = 0$$

$$\ddot{w}^2 - 2(\lambda_1 - 1)ln\lambda_1 \lambda_1^{1+2u} w^1 \dot{u} - 4(\lambda_1 - 1)ln\lambda_1 \lambda_1^{1+2u} w^1 \dot{u} = 0$$

$$\ddot{u} + \frac{(\lambda_1^{4u+4})ln\lambda_1}{\lambda_1^{u+2}} w^1 \dot{w}^1 + 2\frac{(1+\lambda_1^{4u+3}(\lambda_1 - 1)ln\lambda_1}{\lambda_1^{u+2}} w^1 \dot{w}^2 +$$

$$\frac{(\lambda_1^{4u+2})(\lambda_1 - 1)^2ln\lambda_1}{\lambda_1^{u+2}} w^2 \dot{w}^2 = 0.$$ 

One can get convinced that these equations are invariant under the transformation \textcolor{red}{(208)}. In order to study a stability of the geodesic flow one has to compute the sectional curvatures. We shall choose the orthogonal frame in the directions of the linear spaces $X^1_{\tilde{w}}, Y^1_{\tilde{w}}$ and $Z_{\tilde{w}}$. The corresponding vectors are:

$$v_1 = (\lambda_1 - 1, \lambda_1, 0), \quad v_2 = (\lambda_2 - 1, \lambda_2, 0), \quad v_3 = (0, 0, 1)$$

and in the metric \textcolor{red}{(209)} they have the lengths:

$$|v_1|^2 = (\lambda_1 - \lambda_2)^2\lambda_2^{2u}, \quad |v_2|^2 = (\lambda_1 - \lambda_2)^2\lambda_1^{2u}, \quad |v_3|^2 = 1.$$
The corresponding sectional curvatures can be computed and the following values:

\[
\begin{align*}
K_{12} &= \frac{R_{\mu\nu\lambda\rho}v_\mu v_\nu v_\lambda v_\rho}{|v_1 \wedge v_2|^2} = \ln^2 \lambda_1 \\
K_{13} &= \frac{R_{\mu\nu\lambda\rho}v_\mu v_3 v_\lambda v_\rho}{|v_1 \wedge v_3|^2} = -\ln^2 \lambda_2 \\
K_{23} &= \frac{R_{\mu\nu\lambda\rho}v_2 v_\nu v_\lambda v_\rho}{|v_2 \wedge v_3|^2} = -\ln^2 \lambda_1.
\end{align*}
\]

It follows from the above equations that the geodesic flow is exponentially unstable on the planes (1,3) and (2,3) and is stable in the plane (1,2). This behaviour is dual to the flow \([204]\) which is unstable in (1,2) plane and is stable in (1,3) and (2,3) planes. The scalar curvature is

\[
R = R_{\mu\nu\lambda\rho}g^{\mu\lambda}g^{\nu\rho} = 2(K_{12} + K_{13} + K_{23}) = -2 \ln^2 \lambda_1 = -2h(T)^2, \tag{214}
\]

where \(h(T)\) is the entropy of the automorphism \(T(2,0)\).

14 Infinite Dimensional Limit of the C-cascade

We are interested to consider the infinite-dimensional limit \(N \to \infty\) of the system \([179]\) when the operator \(T(N)\) is given by the \(N \times N\) matrix with all integer entries \(T_{ij} \in \mathbb{Z}\) and has the form given in \([195,196]\). It has the determinant equal to one and its spectrum has the form \([186]\) (see also \([196]\))

\[
\begin{align*}
\lambda_j &= 1 - 2 \exp(i \pi j/N) + \exp(2i \pi j/N) = -4 \sin^2\left(\frac{\pi j}{2N}\right) \exp(i \pi j/N) \\
&= -4 \sin^2\left(\frac{\pi j}{2N}\right) \exp(i \pi j/N) \\
j &= -N, -N + 2, \ldots, N - 2, N,
\end{align*}
\]

shown in Fig.14. The kernel \(T(N)u_0 = 0\) of the operator \(T(N)\) consists of only one vector with zero components \(u_0 = (0, ..., 0)\). The eigenvalues fulfil the C-condition \([180]\) and the entropy of the system can be calculated for the large values of \(N\) as a sum over eigenvalues:

\[
h(A) = \sum_{\beta} \ln |\lambda_\beta| = \sum_{-2\pi/3 < \phi_j < 2\pi/3} \ln(4 \cos^2(\phi_j/2)) \to N \int_{-2\pi/3}^{2\pi/3} \ln(4 \cos^2(\phi/2)) \frac{d\phi}{2\pi} = \frac{2}{\pi} N, \tag{216}
\]

where the eigenvalues are given in \([196]\) and \(\phi_j = \frac{\pi j}{N}\). The entropy increases linearly with the dimension \(N\) of the operator \(T(N)\). We note that the special form of the matrix \(T(N)\) in \([195]\) has highly desirable property of having a widely spread, nearly continuum spectrum of eigenvalues \([196]\) shown in Fig.14 and indicating that the exponential mixing takes place in many scales.\(^{[96]}\)

We are interested in defining and investigating the system that appears in the infinite-dimensional limit of \(T(N)\) when \(N \to \infty\). The size of state vector \(u = (u_1, ..., u_N)\) tends to
infinity and it seems natural to expect that it can be represented by a continuous function $\psi$ defined in an infinite-dimensional space $\mathcal{H} = \{ \psi(u) \}$, possibly a Hilbert space, and the operator $T(N)$ will reduce to a differential operator $T$ acting in $\mathcal{H}$. The existence of such a limit would mean that $T$ defines a "measure preserving" transformation of continuous functions $T^n \psi = \psi^{(n)}$ in $\mathcal{H}$ that will have maximally strong chaotic properties. Our intension is to define this limiting system, to explore its properties and possible applications in Monte Carlo method. It seems that the investigation of infinite-dimensional "fully chaotic" transformation of continuous functions may also help to understand better a chaotic/turbulent motion of fluids. As it was demonstrated by Arnold\cite{arnold1983,arnold1984} the solutions of the partial differential equation describing the evolution of the hydrodynamical flow of incompressible ideal fluid can be considered as a continuous measure preserving transformation of fluid velocity and the evolution is partially chaotic, that is, the flow is exponentially unstable in some directions and is stable in other directions (the details will be discussed in the last paragraph).

In many areas of mathematics the consideration of infinite-dimensional limits is an ambiguous procedure, and a priory there is no guarantee that a sensible limit of a finite-dimensional structure exists. In our case the dimension of the N-dimensional torus $S^1 \otimes \ldots \otimes S^1$ tends to infinity and it is unclear what type of phase space should be taken in the limit. The hint that a sensible limit may exist comes from the fact that as $N \to \infty$ the eigenvalues (196) fill out the cardioid curve more and more dense without producing any deformation of the cardioid curve that can be seen in figure Fig.16\cite{figure16}. The other important hint is that the inverse matrix $T(N)^{-1}$ is reminiscent to the matrix that represents the discrete version of the second-order differential operator.\cite{inverse_matrix} If one supposes that the state vector becomes a function $\psi(x)$ with its argument on a real line $x \in \mathbb{R}$, then it seems natural to look for a differential operator of the second order and the one that will reproduce the eigenvalue spectrum distributed on cardioid
curve. Having in mind the above consideration let us consider the differential operator

\[ T = 1 - 2 \exp \left( \frac{d}{dx} \right) + \exp \left( 2 \frac{d}{dx} \right) = \frac{d^2}{dx^2} + \frac{d^3}{dx^3} + \frac{7}{12} \frac{d^4}{dx^4} + \ldots \]  

(217)

acting in the Hilbert space of functions \( \mathcal{H} = \{ \psi(x) \} \) defined on the interval \( x \in [-\infty, +\infty] \). The series expansion of the operator has indeed a second-order differential operator and also infinitely many high derivative terms

\[ T = \frac{d^2}{dx^2} + \frac{d^3}{dx^3} + \frac{7}{12} \frac{d^4}{dx^4} + \ldots \]

Its spectral characteristics are defined by the eigenvalue equation

\[ T \psi(x) = \lambda \psi(x). \]  

(218)

Searching the eigenfunctions in the form of plane waves

\[ \psi_a(x) = e^{iax} \]  

(219)

one can find that the spectrum represents a continuous cardioid curve on the complex plane

\[ \lambda(a) = 1 - 2e^{ia} + e^{2ia} = 4e^{i(a+\pi)} \sin^2 \left( \frac{a}{2} \right). \]  

(220)

It is similar to the discrete spectrum (196) and has the periodic structure

\[ \lambda(a + 2\pi k) = \lambda(a), \quad k = 0, \pm 1, \pm 2, \ldots \]  

(221)

The spectrum is continuous and the eigenvalues are distributed in the complex plane representing the cardioid curve shown in Fig.12. As the real momentum parameter \( a \) varies on the real line interval the eigenvalues run around the cardioid infinitely many time (221). The eigenvalues of the operator \( A \) can be divided into two sets \( \{ \lambda_\alpha \} \) and \( \{ \lambda_\beta \} \) with modulus smaller and larger than one:

\[ 0 < |\lambda_\alpha| < 1, \quad 1 < |\lambda_\beta|. \]  

(222)

The eigenvalues \( \lambda_\alpha \) and \( \lambda_\beta \) can be found using (220):

\[ \lambda_\alpha = 4e^{i(a+\pi)} \sin^2 \left( \frac{a}{2} \right), \quad \text{when} \quad -\frac{\pi}{3} + 2\pi k < a < +\frac{\pi}{3} + 2\pi k; \]

\[ \lambda_\beta = 4e^{i(a+\pi)} \sin^2 \left( \frac{a}{2} \right), \quad \text{when} \quad +\frac{\pi}{3} + 2\pi k < a < +\frac{5\pi}{3} + 2\pi k, \]  

(223)

where \( k = 0, \pm 1, \pm 2, \ldots \). This structure of the spectrum repeats itself with the period \( 2\pi \). There are two eigenvalues \( \lambda = 1 \) corresponding to \( a = \pm \pi/3 \) where the cardioid intersects a unit circle shown in Fig.12.
In order to establish the fact that the operator $T$ is defining a measure-preserving transformation one should calculate the determinant of the operator $T$. The measure-preserving transformations of the phase space is a characteristic property of Hamiltonian systems that is expressed in terms of the Liouville’s theorem and represent a large class of dynamical systems that are considered in ergodic theory.\[11,16,19,22,23\] Using the fact that $\ln \det T = \text{Tr} \ln T$ we will have

$$\ln \det A_x = \sum_{k=-\infty}^{\infty} \int_{-\pi+2\pi k}^{\pi+2\pi k} \ln[e^{i(a+\pi)}4\sin^2\left(\frac{a}{2}\right)] \frac{da}{2\pi} = 0,$$

that is, the determinant is equal to one $\det T = 1$ and the operator $T$ is defining a measure-preserving transformation. One can define now the homomorphism of the Hilbert space $\mathcal{H} = \{\psi(x)\}$ in terms of the operator $T$ as

$$\psi^{(1)}(x) = T\psi(x)$$

and the dynamical system on the infinite dimensional phase space $\mathcal{H}$ as:

$$\psi^{(n)}(x) = T^n\psi(x) \mod 1, \quad n = 0, 1, 2, \ldots$$

The homomorphism (226) is defined by mod 1 operation meaning that the functions are wrapping around an infinitely long cylinder $R^1 \otimes S^1$. The function $\psi(x)$ maps $R^1$ to $S^1$ ($\psi : R^1 \rightarrow S^1$).

The determinant of the operator $T$ is equal to one and the eigenvalues are distributed inside and outside of the unit circle, and we have an example of infinite-dimensional hyperbolic C-K system of the type (180). To get convinced that the operator $T$ represents a hyperbolic C-K system one should establish the existence of exponentially expanding and contracting foliations.\[4\] Let us consider the evolution of the infinitesimal perturbation $\psi \rightarrow \psi + \delta \psi$ under the action of $T$ operator in analogy with the geodesic deviation equation:

$$\delta \psi^{(n)}(x) = T^n \delta \psi(x).$$

The deviation $\delta L$ can be evaluated by using the standard inner product in Hilbert space and the mean value theorem\[22\]:

$$\delta L_n = \langle \delta \psi^+ | T^n \delta \psi \rangle = \int_{-\infty}^{+\infty} \delta \psi^+(x) T^n \delta \psi(x) dx = \int_{-\infty}^{+\infty} \int_{\Delta a} \frac{da'}{2\pi} \int_{\Delta a} \frac{da}{2\pi} e^{-i a' x} \delta \phi^+(a') \left[4 \sin^2\left(\frac{a}{2}\right)\right]^n e^{in(a+\pi)} e^{iax} \delta \phi(a) dx = \frac{1}{2\pi} \int_{\Delta a} da e^{in(a+\pi)} \left[4 \sin^2\left(\frac{a}{2}\right)\right]^n |\delta \phi(a)|^2 = e^{in(\bar{a}+\pi)} e^{n \ln [4 \sin^2\left(\frac{a}{2}\right)]} |\delta \phi|^2, \quad a \in (a, b)$$

\[\bar{a}\]One can imagine this cylinder as appearing in the "decompactification" of a 2-torus $S^1 \otimes S^1$, when cutting a 2-torus with a non-contractible circle $S^1$ and stretching the open boundary circles to the infinities.

\[22\]If $f(x)$ is a continuous function and $g(x)$ is a nonnegative integrable function on $[a, b]$, then there exists some number $\bar{x} \in (a, b)$, such that $\int_a^b f(x)g(x)dx = f(\bar{x}) \int_a^b g(x)dx$. 59
where $\bar{a}$ is a number in the interval $\Delta a \subset (\pi/3, 5\pi/3)$ and $|\delta \phi^2| = \frac{1}{2\pi} \int_{\Delta a} |\delta \phi|^2 da$. The absolute value of the deviation is growing exponentially with the iteration time $n$ as

$$|\delta L_n| \sim |\delta \phi^2| e^{n \ln [4 \sin^2(\frac{1}{2})]}.$$  \hfill (229)

In the above perturbation the Fourier spectrum of the function $\delta \psi(x)$ is localised in the region of the spectrum $\Delta a$, where the eigenvalues $\lambda_\beta$ are larger than one, $\Delta a \subset (\pi/3, 5\pi/3)$, and are defined in (223). The integration over $a$ and $a'$ was specified to be in that region $\Delta a$ and the perturbation function had the following form:

$$\delta \psi(x) = \int_{\Delta a} \delta \phi(a) e^{iax} \frac{da}{2\pi}. \hfill (230)$$

In a similar way one can get convinced that the exponential contraction takes place when the Fourier spectrum of the perturbation function is localised in the part of the spectrum $\lambda_\alpha$, where the eigenvalues are less than one $\Delta a \subset (-\pi/3, \pi/3)$ in (223). The existence of exponentially expanding and contracting foliations is a sufficient condition for a dynamical system to expose strong statistical/chaotic properties and, in particular, to have nonzero Kolmogorov-Sinai entropy and to be classified as a hyperbolic C-K system. In physical terms this means that the Fourier amplitudes $\phi(a)$ of the initial state vector $\psi(x)$ are stretched and compressed depending on whether the value of $a$ is in the interval $a \in (\frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k)$ or in the interval $a \in (-\frac{\pi}{3} + 2\pi k, \frac{\pi}{3} + 2\pi k)$, where $k = 0, \pm 1, \pm 2, \ldots$.  

The above consideration allows to calculate the Kolmogorov entropy of the system per unit of the iteration time $n$ as it is was defined by Kolmogorov. The new aspect that appears in this infinite-dimensional system case is that the spectrum (220) is continuous and repeats itself infinitely many times. For that reason the standard Kolmogorov definition of the entropy per unit iteration time is equal to infinity. That can be observed also from the equation (216) when $N \to \infty$. In this circumstances one can propose to calculate the entropy per unit period (221) of the spectrum (220):

$$h(T) = \sum_\alpha \ln \frac{1}{|\lambda_\alpha|} = - \int^{-\pi/3}_{\pi/3} \ln[4 \sin^2(\frac{a}{2})] \frac{da}{2\pi} = 2i[Li_2(e^{i\frac{5\pi}{3}}) - Li_2(e^{i\frac{\pi}{3}})] \sim \frac{2}{\pi}, \hfill (231)$$

where we used the fact that $\prod_\alpha \lambda_\alpha \prod_\beta \lambda_\beta = 1$ and $Li_n(z)$ is the polylogarithm function. This result is understandable in the sense that the finite-dimensional matrix system (195) considered above had the entropy $\sim \frac{2}{\pi} N$, where $N$ is the dimension of the matrix operator. As far as the operator $T$ can be considered as the infinite dimensional limit $N \to \infty$ of (195), the standard Kolmogorov entropy of the system (226) tends to infinity but its entropy ”per spectral period” is finite.
Figure 17: The figure demonstrates the result of a triple iteration \((226)\) \(\psi^{(3)}(x) = T^3\psi(x)\) of the smooth function \(\psi(x) = \sin(x) + \sin(2x) - \cos(3x) + \sin(4x)\).

The inverse operator \(G\) is defined by the equation

\[ T \ G(x-y) = \sum_{k=-\infty}^{\infty} \delta(x-k) \]  

(232)

and has the following solution:

\[ G(x-y) = \frac{1}{2}(x-y)^2 \sum_{k=-\infty}^{\infty} e^{2\pi k(x-y)}. \]  

(233)

The general solution of (232) can be expressed as a sum the fixed solution (233) and an arbitrary element of the kernel \(K\). The kernel subspace \(K\) of the operator \(T\) is defined by the equation

\[ T\psi_0(x) = 0 \]  

(234)

where \(c_1, c_2, a_k\) are arbitrary constants. The inverse transformation \(\psi^{(-n)} = G^n\psi, \ n = 0, 1, 2, \ldots\) is therefore defined modulo kernel (234). We will defined it in its most simple form (233):

\[ \psi^{(-1)} = \int G(x-y)\psi(y)dy = \sum_{k=-\infty}^{\infty} k^2 \psi(x-k) \mod 1. \]  

(235)

Thus the evolution of the system is given in both “time directions” by (235) and (225). Because the determinant of the operator \(T\) is equal to one on a quotient space \(\mathcal{H}/\mathcal{K}\) of functions on a cylinder \(R^1 \otimes S^1\), where \(\mathcal{K}\) is a kernel (234), it is natural to think that the operator \(T\) defines a measure preserving transformation. In this circumstances one can try to define a measure that is invariant with respect to the transformations generated by \(T\). The volume element \(\mathcal{V}\) in the quotient space \(\mathcal{H}/\mathcal{K}\) can be defined by using the Wiener-Feynman functional integral:

\[ \mathcal{V}_A = \int_{\mathcal{C}} F[\psi]D\psi(x), \]  

(236)

where \(A\) is a subset in \(\mathcal{H}/\mathcal{K}\) and the functional \(F[T\psi] = F[\psi]\) should be invariant under the action of the transformations generated by the operator \(T\). The measure \(D\psi(x)\) is invariant because the determinant of the corresponding Jacobian operator is equal to one (224). The invariant functional \(F[\psi]\) can be appropriately chosen.
It is interesting to know if the similar systems were investigated in the past? The solutions of the partial differential equation describing the evolution $t \rightarrow g_t(x)$ of the hydrodynamical flow of incompressible ideal fluid filled in a two dimensional torus $x \in \mathcal{T}^2$ can be considered as a continuous area preserving diffeomorphims $SDiff(\mathcal{T}^2)$ of a torus $\mathcal{T}^2 \rightarrow \mathcal{T}^2$. In Arnold approach the ideal fluid flow is described by the geodesics $g_t(x) \in G$ on the diffeomorphism group $G = SDiff(\mathcal{T}^2)$ with the velocity $v_t(x) = \dot{g}_t(x)g_t^{-1}(x)$ belonging to the corresponding algebra $\mathfrak{g}=sdiff(\mathcal{T}^2)$ of divergence free vector fields. The Riemannian metric on the group $G$ is induced from the metric on a torus and the stability of the geodesic flows on the group $G$ can be analysed by investigating the behaviour of the corresponding sectional curvatures $K(v,\delta v)$. It was found that the flows that are defined by a parallel velocity field on $\mathcal{T}^2$ are unstable because the sectional curvatures are negative and the flow is exponentially unstable. In other directions the sectional curvatures are positive and the flows are stable. The extension to high-dimensional torus $\mathcal{T}^N$ can be found in.

This shows that it is not possible to reasonably predict the weather beyond a certain period if one assume that the Earth has torus topology and its atmosphere is a two-dimensional incompressible fluid. A similar stability analysis was performed for the hydrodynamical flow on a two-dimensional sphere $S^2$ as it is important to use the more realistic assumption that the surface of the Earth is a sphere. In all these cases the flow is exponentially unstable in some directions and is stable in some other directions, resulting in the limitation of predictability of the hydrodynamical flow and leading to the principal difficulties of a long-term weather forecasting.

Comparing these systems with the system considered above one can observe that here we have discrete in time transformations of the phase space and, secondly, the system shows up exponential instability of its geodesics in the full quotient phase space $\mathcal{H}/\mathcal{K}$. This full phase space chaotic behaviour can find application in Monte Carlo method, statistical physics and most probably in digital signal processing in communication systems.

The main idea behind of this approach is that the motion of an abstract "rigid body" rotating in high-dimensional Euclidean space $\vec{r} \in E$ that is invariant under the isometry group $g(\xi) \in G$ can be described in terms of geodesic flow on a corresponding group manifold. The stationary frame coordinates of the "rigid body" are defined as $\vec{r} = g_t \vec{Q}$, where $g_t = g(\xi(t))$ is a time-dependent element of the matrix group $G$ and $\vec{Q}$ are the frame coordinates rigidly
fixed to the rotating “body” $\dot{Q} = 0$. Thus
\[ \ddot{r} = \dot{g}_t \ddot{Q} = \dot{g}_t g_t^{-1} g_t \ddot{Q} = \hat{\omega}_s \dddot{r}. \] (237)

The matrix of angular velocity in stationary frame
\[ \hat{\omega}_s = \dot{g}_t g_t^{-1} \] (238)
is a right-invariant one-form ($d(gg_0)(gg_0)^{-1} = dgg^{-1}$, where $g_0$ is a fixed element of the group $G$). The matrix of angular velocity in rotating frame
\[ \hat{\Omega}_c = g_t^{-1} \dot{g}_t \] (239)
is left invariant because $(g_0g)^{-1}d(g_0g) = g^{-1}dg$. It follows that
\[ \hat{\omega}_s = \dot{g}_t g_t^{-1} = g_t g_t^{-1} \dot{g}_t^{-1} = g_t \hat{\Omega}_c g_t^{-1}. \] (240)

The kinetic energy is defined as a sum of the kinetic energies of all ”parts” of the rotating ”body” through the velocities (237):
\[ T = \frac{1}{2} \sum_a m_a \dddot{r}_a \dddot{r}_a = \frac{1}{2} \text{Tr}(\hat{I}_s \hat{\omega}_s \hat{\omega}_s^+) = -\frac{1}{2} \text{Tr}(\hat{I}_s \hat{\omega}_s \hat{\omega}_s), \] (241)
where one should use the relation $\dddot{r} = g_t \dddot{Q}$, and therefore
\[ \hat{I}_s = \sum_a m_a \dddot{r}_a \dddot{r}_a^+ = g_t \hat{I} g_t^{-1}, \quad \hat{I} = \sum_a m_a Q_a Q_a^+. \] (242)

The matrix $\hat{I}$ is a symmetric positive definite constant matrix that determines the ”moment of inertia” in the frame rigidly fixed to the rotating ”body”. The matrix of angular momentum in stationary frame is
\[ \hat{m}_s = \hat{I}_s \hat{\omega}_s \] (243)
and the corresponding angular momentum in rotating frame can be defined by projection of $\hat{m}_s$ into the rotating frame:
\[ \hat{m}_s = g_t \hat{M}_c g_t^{-1}; \] (244)
thus
\[ \hat{M}_c = \hat{I} \hat{\Omega}_c, \] (245)
where one should use the relations (243), (242) and (238), (239). In terms of rotating frame coordinates the kinetic energy (241) will take the form
\[ T = -\frac{1}{2} \text{Tr}(\hat{I}_s \hat{\omega}_s \hat{\omega}_s) = -\frac{1}{2} \text{Tr}(\hat{m}_s \hat{\omega}_s) = -\frac{1}{2} \text{Tr}(\hat{M}_c \hat{\Omega}_c) = -\frac{1}{2} \text{Tr}(\hat{I} \hat{\Omega}_c \hat{\Omega}_c), \] (246)

\[ ^{23}\text{The general relation between operators in stationary and rotating frames is } A_s = g_t A_c g_t^{-1}. \]
\[ ^{24}\text{The square of angular momentum is conserved: } \text{Tr}(\hat{m}_s^2) = \text{Tr}([g_t \hat{M}_c g_t^{-1}]^2) = \text{Tr}(\hat{M}_c^2). \]
where we used the relations (244) and (240). As it follows from (246) and (251), in mathematical terms the matrix \( \hat{I} \) defines the alternative Euclidean structure on the group algebra \( \langle a, b \rangle = Tr(T_a \hat{I} T_b) \), where \( T^a \) are the generators of the algebra \( g \) and \( \hat{\Omega}_c = \sum_a T_a \Omega^a \).

Using the relation (244) and the conservation of the angular momentum in stationary frame \( \dot{m}_s = 0 \) one can get the generalised Euler equation

\[
\dot{g}_t \hat{M}_c g_t^{-1} + g_t \dot{\hat{M}}_c g_t^{-1} - g_t \dot{\hat{M}}_c g_t^{-1} \dot{g}_t g_t^{-1} = 0,
\]

which can be represented in the following standard form:

\[
\dot{\hat{M}}_c + \hat{\Omega}_c \hat{M}_c - \hat{M}_c \hat{\Omega}_c = 0,
\]

or equivalently as

\[
\frac{d\hat{M}_c}{dt} = [\hat{M}_c, \hat{\Omega}_c]. \tag{247}
\]

In terms of angular velocity it takes the following form:

\[
\hat{I} \frac{d\hat{\Omega}_c}{dt} = [\hat{\Omega}_c, \hat{\Omega}_c], \quad \frac{d\hat{\Omega}_c}{dt} = \hat{I}^{-1} [\hat{\Omega}_c, \hat{\Omega}_c] = \Gamma(\hat{\Omega}_c, \hat{\Omega}_c). \tag{248}
\]

The last Euler equation can be represented in the form of geodesic equation

\[
\frac{d\Omega^a}{dt} + \Gamma^a_{bd} \Omega^b \Omega^d = 0, \tag{249}
\]

where \( \Gamma^a_{bd} \) are the Christopher symbols of the metric (251). The kinetic energy defines the left invariant metric on the group:

\[
ds^2 = Tr(\hat{I} \hat{\Omega}_c \hat{\Omega}_c) dt^2 = Tr(\hat{I} g^{-1} dg g^{-1} dg) = Tr(\hat{I} g^{-1} \frac{\partial g}{\partial \xi^a} g^{-1} \frac{\partial g}{\partial \xi^b}) d\xi^a d\xi^b, \tag{250}
\]

where the \( \xi^a \) are parameters of the Lie group \( G \) and

\[
ds^2 = g_{ab} d\xi^a d\xi^b, \quad g_{ab} = Tr(\hat{I} g^{-1} \frac{\partial g}{\partial \xi^a} \hat{I} g^{-1} \frac{\partial g}{\partial \xi^b}). \tag{251}
\]

The calculation of the components of the Riemann tensor and of the sectional curvatures

\[
K(\xi, \eta) = \frac{R_{abcd} \xi^a \eta^b \xi^c \eta^d}{[\xi \wedge \eta]^2} \tag{252}
\]

allows to investigate the stability of the geodesic flows even in the cases when \( G \) is the infinite-dimensional group of diffeomorphisms \( SDiff(M) \) that describes the flow of incompressible ideal fluid filled in a manifold \( M \).
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References

[1] A.N. Kolmogorov, New metrical invariant of transitive dynamical systems and automorphisms of Lebesgue spaces, Dokl. Acad. Nauk SSSR, 119 (1958) 861-865

[2] A.N. Kolmogorov, On the entropy per unit time as a metrical invariant of automorphism, Dokl. Acad. Nauk SSSR, 124 (1959) 754-755

[3] A.N. Kolmogorov, General Theory of Dynamical Systems and Classical Mechanics, Proceedings of the International Congress of Mathematicians, Amsterdam, 1 (1954) 315-333

[4] D. V. Anosov, Geodesic flows on closed Riemannian manifolds with negative curvature, Trudy Mat. Inst. Steklov., Vol. 90 (1967) 3 - 210

[5] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, Mathematische Zeitschrift 32 (1930) 702-728

[6] G. A. Leonov and N. V. Kuznetsov, Time-Varying Linearization and the Perron Effects, Int.J. of Bifurcation and Chaos, Vol. 17 (2007) 1079-1107; https://doi.org/10.1142/S0218127407017732; https://www.math.spbu.ru/user/nk/PDF/Lyapunov-exponent-Sign-inversion-Perron-effect.pdf

[7] N. I. Lobachevsky, Complete Collected Works, Vol. I-IV (Russian), Moscow-Leningrad (Publisher GITTL) 1946-1951,
[8] Emil Artin, *Ein mechanisches system mit quasiergodischen bahnen*, E. Abh. Math. Semin. Univ. Hambg. 3 (1924) 170.

[9] J. Hadamard, *Sur le billiard non Euclidean*, Soc. Sci. Bordeaux, Proc. Verbaux 1898, 147 (1898); J. Math Pure Appl. 4 (1898) 27.

[10] G. Hedlund, *The dynamics of geodesic flow*, Bull. Am. Math. Soc. 45 (1939) 241-246

[11] E. Hopf, *Proof of Gibbs Hypothesis on the Tendency Toward Statistical Equilibrium*, Mathematics 18 (1932) 333.

[12] E. Hopf, *Statistik der Lösungen geodätischer Probleme vom unstabilen Typus. II*, Math. Ann. 117 (1940) 590-608

[13] E. Hopf, *Statistik der Lösungen geodätischer Probleme vom unstabilen Typus. II*, Math. Ann. 117 (1940) 590-608

[14] E. Hopf, *Ergodic theory and the geodesic flow on surfaces of constant negative curvature*, Bull. Amer. Math. Soc., 77 (1971) 863-877.

[15] D. V. Anosov and Ya. G. Sinai, *Certain smooth ergodic systems*, Uspehi Mat. Nauk 22 (1967), no. 5 (137), 107-172; Russian Math. Surveys 22 (1967), 103-167.

[16] J.W. Gibbs, *Elementary principles in statistical mechanics*, Charles Scribner’s Sons, New York 1902

[17] G. D. Birkhoff, *Proof of the ergodic theorem*, Proc. Natl. Acad. Sci. USA, 17 (1931) 656D660, doi:10.1073/pnas.17.12.656,

[18] B. O. Koopman, *Hamiltonian Systems and Transformations in Hilbert Space*, Proc. Nat. Acad. Sci. 17 (1931) 315

[19] N.S. Krylov, *Works on the foundation of statistical physics*, M.- L. Izdatelstvo Acad. Nauk. SSSR, 1950; (Princeton University Press, 1979)

[20] P. R. Halmos, *Lectures on Ergodic Theory*, Dover Publishing, Inc. Mineola, New York (2017)

[21] Ya.G. Sinai, *On the Notion of Entropy of a Dynamical System*, Doklady of Russian Academy of Sciences, 124 (1959) 768-771.

[22] I. P. Kornfeld, S. V. Fomin, Y. G. Sinai, *Ergodic Theory*, Springer, 1982
[23] V. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics*, (The Mathematical physics monograph series) Benjamin (July 5, 1968), 286pp.

[24] V. A. Rokhlin, *Metric properties of endomorphisms of compact commutative groups*, Izv. Akad. Nauk SSSR Ser. Mat., Volume 28, Issue 4 (1964) 867- 874

[25] V. P. Leonov, *On the central limit theorem for ergodic endomorphisms of the compact commutative groups*, Dokl. Acad. Nauk SSSR, 124 No: 5 (1969) 980-983

[26] V. A. Rokhlin, *On the endomorphisms of compact commutative groups*, Izv. Akad. Nauk, vol. 13 (1949), p.329

[27] V. A. Rokhlin, *On the entropy of automorphisms of compact commutative groups*, Teor. Ver. i Pril., vol. 3, issue 3 (1961) p. 351

[28] S. Smale, *Differentiable dynamical systems*. Bull. Am. Math. Soc. 73 (1967) 747-817

[29] Ya. G. Sinai, *Markov partitions and C-diffeomorphisms*, Funkcional. Anal, i Prilozen. 2 (1968),64-89; Functional Anal. Appl. 2 (1968) 61-82.

[30] Ya. G. Sinai, *Proceedings of the International Congress of Mathematicians*, Uppsala (1963) 540-559.

[31] G. A. Margulis, *Certain measures that are connected with C-flows on compact manifolds*, Funkcional. Anal, i Prilozen. 4 (1970) 62-76; Functional Anal. Appli. 4 (1970) 55-67.

[32] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*. (Lecture Notes in Mathematics, no. 470: A. Dold and B. Eckmann, editors). Springer-Verlag (Heidelberg, 1975), 108 pp.

[33] R. Bowen, *Periodic orbits for hyperbolic flows*, Amer. J. Math., 94 (1972), 1-30.

[34] R. Bowen, *Periodic points and measures for axiom A diffeomorphisms*, Trans. Am. Math. Soc. 154 (1971) 377-397

[35] A. L. Gines, *Metrical properties of the endomorphisms on m-dimensional torus*, Dokl. Acad. Nauk SSSR, 138 (1961) 991-993

[36] M. C. Gutzwiller, *Stochastic Behaviour in Quantum Scattering*, Physica 7D (1983) 341-355.
[37] M. C. Gutzwiller, *Periodic orbits and classical quantization conditions*, J. Math. Phys. **12** (1971) 343-358 doi:10.1063/1.1665596

[38] M. C. Gutzwiller, *Classical Quantisation of a Hamiltonian with Ergodic Behaviour*, Phys. Rev. Lett. **45** (1980) 150-153

[39] G. K. Savvidy, *Infrared Instability of the Vacuum State of Gauge Theories and Asymptotic Freedom*, Phys. Lett. B **71** (1977), 133-134 doi:10.1016/0370-2693(77)90759-6

[40] G. Savvidy, *Maximally chaotic dynamical systems*, Annals Phys. **421** (2020), 168274 doi:10.1016/j.aop.2020.168274

[41] G. Baseyan, S. Matinyan and G. Savvidy, *Nonlinear plane waves in the massless Yang-Mills theory*, Pisma Zh. Eksp. Teor. Fiz. **29** (1979) 641-644

[42] S. Matinyan, G. Savvidy and N. Ter-Arutyunyan-Savvidi, *Classical Yang-Mills mechanics. Nonlinear colour oscillations*, Zh. Eksp. Teor. Fiz. **80** (1980) 830-838

[43] G. M. Asatrian and G. K. Savvidy, *Configuration Manifold of Yang-Mills Classical Mechanics*, Phys. Lett. A **99** (1983) 290. doi:10.1016/0375-9601(83)90887-3

[44] G. Savvidy, *The Yang-Mills classical mechanics as a Kolmogorov system*, Phys. Lett. **130B** (1983) 303-307

[45] G. K. Savvidy, *Classical and Quantum Mechanics of Non-Abelian Gauge Fields*, Nucl. Phys. B **246** (1984) 302. doi:10.1016/0550-3213(84)90298-0

[46] G. K. Savvidy, *Yang-Mills Quantum Mechanics*, Phys. Lett. **159B** (1985) 325. doi:10.1016/0370-2693(85)90260-6

[47] J. Hoppe, MIT Ph.D. Thesis, (1982).

[48] B. Eckhardt, G. Hose and E. Pollak, *Quantum mechanics of a classically chaotic system: Observations on scars, periodic orbits, and vibrational adiabaticity*, Phys. Rev. A **39** (1989) 198

[49] T. Akutagawa, K. Hashimoto, T. Sasaki and R. Watanabe,*Out-of-time-order correlator in coupled harmonic oscillators*, [arXiv:2004.04381 [hep-th]].

[50] E. P. Wigner, *Characteristic Vectors of Bordered Matrices With Infinite Dimensions*, **62** (1955) 548; **65** (1957) 203; *On the Distribution of the Roots of Certain Symmetric Matrices*, **67** (1958) 325.
[51] M. L. Mehta and M. Gaudin, *On the Density of Eigenvalues of Random Matrices*, Nucl. Phys. **18** (1960) 420.

[52] F. J. Dyson, *Statistical theory of the energy levels of complex system*, J Math. Phys. **3** (1962) 140, 157, 166.

[53] B.V. Chirikov, D.L. Shepelyansky, *Stochastic oscillation of classical Yang-Mills fields*, JETP Lett. **34** (1981) 163.

[54] L. Ermann and D. L. Shepelyansky, *Deconfinement of classical Yang-Mills color fields in a disorder potential*, Chaos **31** (2021), 093106 doi:10.1063/5.0057969 [arXiv:2103.16621 [cond-mat.str-el]].

[55] E.S. Nikolaevsky, L.N. Shchur, *Nonintegrability of the classical Yang-Mills fields*, JETP Lett. **36** (1982) 218.

[56] B. de Wit, M. Luscher, H. Nicolai, *The supermembrane is unstable*, Nucl. Phys. B **320** (1989) 135.

[57] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M theory as a matrix model: A Conjecture*, Phys. Rev. D **55** (1997), 5112-5128 doi:10.1103/PhysRevD.55.5112 [arXiv:hep-th/9610043 [hep-th]].

[58] N. Acharyya, A. P. Balachandran, M. Pandey, S. Sanyal and S. Vaidya, *Glueball spectra from a matrix model of pure Yang–Mills theory*, Int. J. Mod. Phys. A **33** (2018) no.13, 1850073 doi:10.1142/S0217751X18500732 [arXiv:1606.08711 [hep-th]].

[59] A. P. Balachandran, S. Vaidya and A. R. de Queiroz, *A Matrix Model for QCD*, Mod. Phys. Lett. A **30** (2015) no.16, 1550080 doi:10.1142/S0217732315500807 [arXiv:1412.7900 [hep-th]].

[60] H. P. Pavel, *Low-energy spectrum of SU(3) Yang-Mills Quantum Mechanics*, [arXiv:2112.06248 [hep-th]].

[61] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Monte Carlo studies of the IIB matrix model at large N*, JHEP **07** (2000), 011 doi:10.1088/1126-6708/2000/07/011 [arXiv:hep-th/0005147 [hep-th]].

[62] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Large N dynamics of dimensionally reduced 4-D SU(N) superYang-Mills theory*, JHEP **07** (2000), 013 doi:10.1088/1126-6708/2000/07/013 [arXiv:hep-th/0003208 [hep-th]].
[63] A. Golovnev, V. Mukhanov and V. Vanchurin, *Vector Inflation*, JCAP **06** (2008), 009 doi:10.1088/1475-7516/2008/06/009 [arXiv:0802.2068 [astro-ph]].

[64] A. Golovnev, V. Mukhanov and V. Vanchurin, *Gravitational waves in vector inflation*, JCAP **11** (2008), 018 doi:10.1088/1475-7516/2008/11/018 [arXiv:0810.4304 [astro-ph]].

[65] G. Savvidy, *Gauge field theory vacuum and cosmological inflation without scalar field*, Annals Phys. **436** (2022), 168681 doi:10.1016/j.aop.2021.168681 [arXiv:2109.02162 [hep-th]].

[66] A. Maleknejad and M. M. Sheikh-Jabbari, *Non-Abelian Gauge Field Inflation*, Phys. Rev. D **84** (2011), 043515 doi:10.1103/PhysRevD.84.043515 [arXiv:1102.1932 [hep-ph]].

[67] E. Elizalde, A. J. Lopez-Revelles, S. D. Odintsov and S. Y. Vernov, *Cosmological models with Yang-Mills fields*, Phys. Atom. Nucl. **76** (2013), 996-1003 doi:10.1134/S1063778813080097

[68] E. Elizalde, A. J. Lopez-Revelles, S. D. Odintsov and S. Y. Vernov, *Cosmological models with Yang-Mills fields*, Phys. Atom. Nucl. **76** (2013), 996-1003 doi:10.1134/S1063778813080097

[69] P. Adshead and M. Wyman, *Gauge-flation trajectories in Chromo-Natural Inflation*, Phys. Rev. D **86** (2012), 043530 doi:10.1103/PhysRevD.86.043530 [arXiv:1203.2264 [hep-th]].

[70] R. Pasechnik, V. Beylin and G. Vereshkov, *Possible compensation of the QCD vacuum contribution to the dark energy*, Phys. Rev. D **88** (2013) no.2, 023509 doi:10.1103/PhysRevD.88.023509 [arXiv:1302.5934 [gr-qc]].

[71] R. Pasechnik, *Quantum Yang–Mills Dark Energy*, Universe **2** (2016) no.1, 4 doi:10.3390/universe2010004 [arXiv:1605.07610 [gr-qc]].

[72] R. Pasechnik, *Quantum Yang–Mills Dark Energy*, Universe **2** (2016) no.1, 4 doi:10.3390/universe2010004 [arXiv:1605.07610 [gr-qc]].

[73] V. Gurzadyan and G. Savvidy, *Collective relaxation of stellar systems*, Astron. Astrophys. **160** (1986) 203

[74] S. Chandrasekhar, *Principles of Stellar Dynamics*, Chicago: University of Chicago Press; London: Cambridge University Press, 1942.
[75] G. W. Gibbons, *The Jacobi-metric for timelike geodesics in static spacetimes*, Class. Quant. Grav. 33 (2016) no.2, 025004 doi:10.1088/0264-9381/33/2/025004 [arXiv:1508.06755 [gr-qc]].

[76] K. R. Lang, *Astrophysical Formulae: Space, Time, Matter and Cosmology*, Springer-Verlag, Berlin, Heidelberg, New York 2006.

[77] J. Binney and S. Tremaine, *Galactic Dynamics*. Princeton University Press (2008).

[78] D. Heggie and P. Hut, *The Gravitational Million-Body Problem: A Multidisciplinary Approach to Star Cluster Dynamics*. Cambridge University Press (2003).

[79] V. I. Arnold, V. S. Afrajmovich, Y.S. Il’yashenko, L. P. Shil’nikov, *Dynamical Systems V: Bifurcation Theory and Catastrophe Theory* Springer (2013) https://doi.org/10.1007/978-3-642-57884-7

[80] G. Jaffé, *Über die Lösungen der Schrödingergleichung bei singulären Wechselwirkungspotentialen*, Zs.f.Phys. 66 (1930) 748.

[81] D. Lynden-Bell, *Statistical Mechanics of Violent Relaxation In Stellar Systems*, Mon. Not. R. astr. Soc. 136 (1967) 101-121.

[82] J. Milnor, *Curvatures of Left Invariant Metrics on Lie Groups*, Adv.Math. 21 (1976) 293-329

[83] V. Arnold, *Sur la géométrie des groupes de Lie de dimension infinie et ses applications en hydrodynamique des fluides parfaits*, Ann.Inst. Fourier (Grenoble) 16, No 1 (1966) 319-361

[84] A. Arakelian and G. K. Savvidy, *Geometry of a Group of Area Preserving Diffeomorphisms*, Phys. Lett. B 223 (1989), 41-46 doi:10.1016/0370-2693(89)90916-7

[85] T. A. Arakelian and G. K. Savvidy, *Cocycles of Area Preserving Diffeomorphisms and Anomalies in Theory of Relativistic Surfaces*, Phys. Lett. B 214 (1988), 350 doi:10.1016/0370-2693(88)91375-5

[86] A. M. Lukatzki, *On curvature of the group of measure preserving diffeomorphisms of n-dimensional torus*, Uspekhi Mat. Nauk 36 (1981) 187

[87] N. K. Smolentsev, *Diffeomorphism Groups of Compact Manifolds*, J. Math. Sciences 146 (2007) 6213
[88] K. Yoshida, *Riemannian curvature on the group of area-preserving diffeomorphisms (motions of fluid) of 2-sphere*, Physica D **100** (1997) 3, https://doi.org/10.1016/S0167-2789(96)00192-3

[89] J. S. Dowker and M. Wei, *Area Preserving Diffeomorphisms and The Stability of the Atmosphere*, Class. Quantum Grav. **7** (1990) 2361-2365.

[90] N. C. Metropolis and S. Ulam, *The Monte Carlo method*, J. Amer. Statistical Assoc. **44** (1949) 335-341

[91] N. C. Metropolis, G. Reitwiesner and J. von Neuman, *Statistical Treatment of Values of First 2000 Decimal Digits of e and of π Calculated on the ENIAC*, Math. Tables and Other Aids to Comp. **4** (1950) 109-111

[92] J. Von Neuman, *Various Techniques Used in Connection with Random Digits. Chapter 13 of Proceedings of Symposium on "Monte Carlo Method" held June-July 1949 in Los Angeles*, J. Res. Nat. Bur. Stand. Appl. Math. Ser. **12** (1951) 36-38

[93] I. M. Sobol, *The Monte Carlo Method*, Univ. of Chicago Press, Chicago, 1974

[94] R. Lidl and H. Niederreiter, *Finite Fields*, Addison-Wesley, Reading, MA, 1983, see also *Finite fields, pseudorandom numbers, and quasirandom points*, in : Finite fields, Coding theory, and Advance in Communications and Computing. (G.L.Mullen and P.J.S.Shine, eds) pp. 375-394, Marcel Dekker, N.Y. 1993.

[95] N. Niki, *Finite field arithmetic and multidimensional uniform pseudorandom numbers (in Japanese)*, Proc. Inst. Statist. Math. **32** (1984) 231.

[96] G. Savvidy and N. Ter-Arutyunyan-Savvidy, *On the Monte Carlo simulation of physical systems*, J.Comput.Phys. **97** (1991) 566; Preprint EFI-865-16-86-YEREVAN, Jan. 1986.

[97] K. Savvidy, *The MIXMAX random number generator*, Comput.Phys.Commun. **196** (2015) 161-165. [http://dx.doi.org/10.1016/j.cpc.2015.06.003](http://dx.doi.org/10.1016/j.cpc.2015.06.003); arXiv:1404.5355

[98] K. Savvidy and G. Savvidy, *Spectrum and Entropy of C-systems. MIXMAX random number generator*, Chaos Solitons Fractals **91** (2016) 33 doi:10.1016/j.chaos.2016.05.003 arXiv:1510.06274 [math.DS]]

[99] G. Savvidy, *Anosov C-systems and random number generators*, Theor. Math. Phys. **188** (2016) 1155; doi:10.1134/S004057791608002X arXiv:1507.06348 [hep-th]].
A. Görlich, M. Kalomenopoulos, K. Savvidy and G. Savvidy, Distribution of periodic trajectories of C-K systems MIXMAX pseudorandom number generator, Int. J. Mod. Phys. C 28 (2016) no.03, 1750032 doi:10.1142/S0129183117500322 [arXiv:1608.03496 [nlin.CD]].

G. Savvidy and K. Savvidy, Exponential decay of correlations functions in MIXMAX generator of pseudorandom numbers, Chaos Solitons Fractals 107 (2018) 244. doi:10.1016/j.chaos.2018.01.007

K. Savvidy, MIXMAX code C/C++
HEPFORGE.ORG, http://mixmax.hepforge.org

The foundation library CLHEP

http://proj-clhep.web.cern.ch/proj-clhep/
https://gitlab.cern.ch/CLHEP/CLHEP/-/blob/develop/Random/Random/
MixMaxRng.h

Geant4. Concurrent and Distributed MC toolkit,
http://geant4.web.cern.ch

ROOT. Data analysis framework,
https://root.cern.ch/doc/master/classROOT_1_1Math_1_1MixMaxEngine.html
https://root.cern.ch/doc/master/classTRandom.html
https://root.cern.ch/doc/master/mixmax_8h_source.html

V. Ivanchenko and S. Banerjee, Upgrade of CMS Full Simulation for Run 2, EPJ Web of Conferences 214 (2019) 02012, https://doi.org/10.1051/epjconf/201921402012

V. Ivanchenko, https://indico.cern.ch/event/731433/contributions/3015654/attachments/1680131/2698971/CMSsim.pdf
https://indico.cern.ch/event/587955/contributions/2937635/attachments/1679273/2706817/PosterCMS_SIM_v4.pdf

T. Sjöstrand et al, An Introduction to PYTHIA 8.2 Comput. Phys.Commun. 191 (2015) 159 [arXiv:1410.3012 [hep-ph]], http://home.thep.lu.se/~bierlich/misc/keyword-manual/Welcome.html

GSL-GNU Scientific Library, GNU Operating System, https://www.gnu.org/software/gsl/
[110] Boost C++ Libraries, https://www.boost.org/doc/libs/1_76_0/boost/random/mixmax.hpp

[111] N. Akopov, G. Savvidy and N. Ter-Arutyunyan-Savvidy, Matrix generator of pseudorandom numbers, J.Comput.Phys. 97 (1991) 573; EFI-867-18-86-YEREVAN, Jan. 1986.

[112] G. G. Athanasiu, E. G. Floratos, G. K. Savvidy K-system generator of pseudorandom numbers on Galois field, Int. J. Mod. Phys. C 8 (1997) 555-565.

[113] P. L’Ecuyer and R. Simard, TestU01: A C Library for Empirical Testing of Random Number Generators, ACM Transactions on Mathematical Software, 33 (2007) 1-40.

[114] V. Demchik, Pseudo-random number generators for Monte Carlo simulations on Graphics Processing Units, Comput. Phys. Commun. 182 (2011) 692 arXiv:1003.1898 [hep-lat]]

[115] M. Falcioni, L. Palatella, S. Pigolotti and A. Vulpiani, Properties making a chaotic system a good Pseudo Random Number Generator, Phys.Rev. E 72 (2005) 016220.

[116] A. I. Larkin and Y. N. Ovchinnikov, Quasiclassical Method in the Theory of Superconductivity, Soviet JETP 28 (1969) 1200.

[117] J. A. Wheeler, Information, Physics, Quantum: The Search for Links, Proc. 3rd Int. Symp. Foundations of Quantum Mechanics, Tokyo, 1989, pp.354-368.

[118] Y. Sekino and L. Susskind, ”Fast Scramblers,” JHEP 10, 065 (2008) arXiv:0808.2096 [hep-th]].

[119] S. H. Shenker and D. Stanford, Black holes and the butterfly effect, JHEP 1403 (2014) 067 doi:10.1007/JHEP03(2014)067 arXiv:1306.0622 [hep-th]].

[120] J. Maldacena, S. H. Shenker and D. Stanford, A bound on chaos, JHEP 1608 (2016) 106 doi:10.1007/JHEP08(2016)106 arXiv:1503.01409 [hep-th]].

[121] G. Gur-Ari, M. Hanada and S. H. Shenker, Chaos in Classical D0-Brane Mechanics, JHEP 1602 (2016) 091 doi:10.1007/JHEP02(2016)091 arXiv:1512.00019 [hep-th]]

[122] J. S. Cotler et al., Black Holes and Random Matrices, JHEP 1705 (2017) 118 Erratum: [JHEP 1809 (2018) 002] doi:10.1007/JHEP09(2018)002, 10.1007/JHEP05(2017)118 arXiv:1611.04650 [hep-th]].
[123] I. Y. Aref’eva, A. S. Koshelev and P. B. Medvedev, *Chaos order transition in Matrix theory*, Mod. Phys. Lett. A **13** (1998) 2481, [hep-th/9804021].

[124] I. Y. Aref’eva, P. B. Medvedev, O. A. Rytchkov and I. V. Volovich, *Chaos in M(atrix) theory*, Chaos Solitons Fractals **10** (1999) 213, [hep-th/9710032].

[125] I. Y. Aref’eva, A. S. Koshelev and P. B. Medvedev, *On stable sector in supermembrane matrix model*, Nucl. Phys. B **579** (2000) 411 doi:10.1016/S0550-3213(00)00205-4 [hep-th/9911149].

[126] I. Y. Aref’eva and I. V. Volovich, *Holographic thermalization*, Theor. Math. Phys. **174** (2013) 186 [Teor. Mat. Fiz. **174** (2013) 216]. doi:10.1007/s11232-013-0016-2

[127] M. Hanada, H. Shimada and M. Tezuka, *Universality in Chaos: Lyapunov Spectrum and Random Matrix Theory*, Phys. Rev. E **97** (2018) no.2, 022224 doi:10.1103/PhysRevE.97.022224 [arXiv:1702.06935 [hep-th]].

[128] T. Anous and C. Cogburn, *Mini-BFSS in Silico*, [arXiv:1701.07511 [hep-th]].

[129] Henri Poincaré *Théorie des Groupes Fuchsiens*, Acta Mathematica, **1** (1882) 1.

[130] Henri Poincaré *Mémoire sur les Fonctions Fuchsiennes*, Acta Mathematica, **1** (1882) 193-294.

[131] Lazarus Fuchs, *Ueber eine Klasse von Funktionen mehrerer Variablen, welche durch Umkehrung der Integrale von Lösungen der linearen Differentialgleichungen mit rationalen Coefficienten entstehen*, J. Reine Angew. Math., **89** (1880) 151-169

[132] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen*, Math. Ann. **121**, No 2 (1949), 141-183.

[133] W. Roelcke, *Über die Wellengleichung bei Grenzkreisgruppen erster Art*, Sitzungsber. Heidelberg. Acad. Wiss. **4** Abh. (1953/1956), 161-267.

[134] I.M.Gelfand and S.V. Fomin , *Geodesic flows on manifolds of constant negative curvature*, Uspekhi Mat. Nauk, **7** (1952) 118-137. Amer.Math.Soc. Translation **1** (1965) 49-65.

[135] A.Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, Indian Journ. Math. Soc. **20** (1956) 47-87.
[136] A. Selberg, *Discontinuous groups and harmonic analysis*, Proceedings of Stockholm Mathematical Congress (1962).

[137] L. D. Faddeev, *Expansion in eigenfunctions of the Laplace operator on the fundamental domain of a discrete group on the Lobachevsky plane*, Trans. Moscow Math. Soc., 17 (1967) 357-386.

[138] L. D. Faddeev, A. B. Venkov and V. L. Kalinin *A non-arithmetic derivation of the Selberg trace formula*, J. Soviet Math., 8 2 (1977) 171-199.

[139] L. A. Takhtajan, *Etudes of the resolvent*, Russ. Math. Surveys 75 (2020) no.1, 147-186 doi:10.1070/RM9917 [arXiv:2004.11950 [math.SP]].

[140] D. A. Hejhal, *The Selberg Trace Formula for PSL(2, R)*, Lecture Notes in Mathematics 548, Springer-Verlag Vol. 1 1976.

[141] D. A. Hejhal, *Eigenvalues of the Laplacian for PSL(2, Z) : some new results and computational techniques*, in International Symposium in Memory of Hua Loo-Keng (ed. by Gong, Lu, Wang, Yang), Science Press and Springer-Verlag 1 (1991) 59-102.

[142] D. A. Hejhal and B. Berg, *Some new results concerning eigenvalues of the non-Euclidean Laplacian for PSL(2, Z)*, Univ. of Minn. Math. Report No. 82-172 (1982) 7pp.

[143] L. R. Ford, *An Introduction to the Theory of Automorphic Functions*, London, 1915, Publisher G. Bell.

[144] A. Winkler, *Cusp forms and Hecke groups*, J. Reine Angew. Math. 386 (1988) 187

[145] D. Bump, *Automorphic Forms and Representations*, Cambridge Studies in Advance Mathematics: 55, Cambridge University Press, 1998.

[146] P. Collet, H. Epstein and G. Gallavotti, *Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and Their Mixing Properties*, Commun. Math. Phys. 95 (1984) 61-112

[147] M. Pollicott, *On the rate of mixing of Axiom A flows*, Invent. math. 81 (1985) 413-426.

[148] C. C. Moore, *Exponential decay of correlation coefficients for geodesic flows*, Group representations, ergodic theory, operator algebras, and mathematical physics (Berkeley, Calif., 1984), Math. Sci. Res. Inst. Publ., vol. 6, Springer, New York, 1987, 163 - 181.
[149] D. Dolgopyat, *On Decay of Correlations in Anosov Flows*, Annals of Mathematics Second Series, **147** (1998) 357-390

[150] N. I. Chernov, *Markov Approximations and Decay of Correlations for Anosov Flows*, Annals of Mathematics Second Series. **147** (1998) 269-324

[151] H. R. Poghosyan, H. M. Babujian and G. K. Savvidy, *Artin Billiard: Exponential Decay of Correlation Functions*, Theor. Math. Phys. **197** (2018) no.2, 1592 doi:10.1134/S004057791811003X [arXiv:1802.04543 [nlin.CD]].

[152] H. Babujian, R. Poghossian and G. Savvidy, *Correlation Functions of Classical and Quantum Artin System defined on Lobachevsky Plane and Scrambling Time*, arXiv:1808.02132 [hep-th].

[153] Y. A. Kordyukov and I. A. Taimanov, *Trace Formula For The Magnetic Laplacian On A Compact Hyperbolic Surface*, arXiv:2202.06055 [math.DG].

[154] L. D. Faddeev, *Feynman integral for singular Lagrangians*, Theor. Math. Phys. **1** (1969) 1 [Teor. Mat. Fiz. **1** (1969) 3]. doi:10.1007/BF01028566

[155] B. Riemann *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, November 1859.

[156] G. Savvidy and K. Savvidy, *Quantum-Mechanical interpretation of Riemann zeta function zeros*, arXiv:1809.09491 [math-ph].

[157] D. Schumayer and D. A. W. Hutchinson, *Physics of the Riemann Hypothesis*, Rev. Mod. Phys. **83** (2011) 307 doi:10.1103/RevModPhys.83.307 [arXiv:1101.3116 [math-ph]].

[158] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, 3rd Edition, eBook ISBN: 9781483149127, Imprint: Pergamon, Published Date: 23rd May 1977 (Chapter 17: ELASTIC COLLISIONS).

[159] A. M. Turing, *Some calculations of the Riemann zeta-function*, Proceedings of the London Mathematical Society, Third Series, **3** (1953) 99-117; doi:10.1112/plms/s3-3.1.99

[160] X. Gourdon, *The 10^{13} first zeros of the Riemann Zeta function, and zeros computation at very large height*, October 24-th 2004.

[161] R. Loren and D. B. Benson, *J. Comput. System Sci. 27*, 400 (1983).
[162] M. Lee, *Int. J. Phys.* 18, 255 (2010) [Erratum: *ibid.* 18, 440 (2010)].

[163] OPAL Collab. (G. Abbiendi et al.), *Eur. J. Phys. C* 11, 217 (1999).

[164] R. Loren and D. B. Benson, *Introduction to String Field Theory*, 2nd edn. (Springer-Verlag, New York, 1999).

[165] R. Loren and D. B. Benson (eds.), *Introduction to String Field Theory*, 2nd edn. (Springer-Verlag, New York, 1999).

[166] C. M. Wang, J. N. Reddy and K. H. Lee, New set of buckling parameters, in *Shear Deformable Beams*, ed. T. Rex (Elsevier, Oxford, 2000), p. 201.

[167] R. Loren, J. Li and D. B. Benson, Deterministic flow-chart interpretations, in *Introduction to String Field Theory*, Ad. Series in Math. Phys., Vol. 3 (Springer-Verlag, New York, 1999), p. 401.

[168] R. Loren, J. Li and D. B. Benson, Deterministic flow-chart interpretations, in *Proc. 3rd Int. Conf. Entity-Relationship Approach*, eds. C. G. Davis and R. T. Yeh (North-Holland, Amsterdam, 1983), p. 421.

[169] R. Loren, J. Li and D. B. Benson, Deterministic flow-chart interpretations, to appear in *J. Comput. System Sci.*