Evolution behaviour of kink breathers and lump-$M$-solitons ($M \to \infty$) for the (3+1)-dimensional Hirota–Satsuma–Ito-like equation

Long-Xing Li

Abstract In this paper, some novel lump solutions and interaction phenomenon between lump and kink $M$-soliton are investigated. Firstly, we study the evolution and degeneration behaviour of kink breather wave solution with different forms for the (3+1)-dimensional Hirota–Satsuma–Ito-like equation by symbolic computation and Hirota bilinear form. In the process of degeneration of breather waves, some novel lump solutions are derived by the limit method. In addition, $M$-fissionable soliton and the interaction phenomenon between lump solutions and kink $M$-solitons (lump-$M$-solitons) are investigated, and the theorem and corollary about the conditions for the existence of the interaction phenomenon are given and proved further. The lump-$M$-solitons with different types are studied to illustrate the correctness and availability of the given theorem and corollary, such as lump-cos type, lump-cosh-exponential type and lump-cosh-cos-cosh type. Several three-dimensional figures are drawn to better depict the nonlinear dynamic behaviours including the oscillation of breather wave, the emergence of lump, the evolution behaviour of fission and fusion of lump-$M$-solitons and so on.

Keywords Hirota bilinear form · Breather wave · Lump-$M$-soliton · Interaction phenomenon

1 Introduction

It is well-known that nonlinear partial different equations (PDEs), involved both space and time variables, are fundamental models in nonlinear science, especially in nonlinear dynamics. The wave phenomenon of nonlinear PDEs are playing an essential role in studying various areas of physics and mathematics, such as nonlinear optics, plasmas, oceanography and Bose–Einstein condensates [1–6]. Solitons, breathers, lumps and rogue waves are typical wave models to study nonlinear scientific issues [7–12]. Lump wave is a special kind of rational function wave with the characteristics of energy concentration and localization property in the space [13–18]. Therefore, a series of methods has been quickly developed to obtain lump wave including the first mathematical description of lump wave [19], the long wave limit method [20,21], the direct method [22], the parameter limit method [23], etc. An effective method was proposed to study the $M$-lump solutions of the integrable systems in [21]; this positive method has attracted a lot of researchers’ attention. Many valuable and interesting results have sprung up lump waves of the (2+1)-dimensional nonlinear equations [24–29]. Lump solutions of the (3+1)-dimensional nonlinear systems, such as potential Yu–Toda–Sasa–Fukuyama equation [30], Sharma–Tasso–Olver-like equation [31], B-type Kadomtsev–Petviashvili–Boussinesq equation [32,33]. Lump waves of the (4+1)-dimensional nonlinear equa-
The (3+1)-dimensional Hirota–Satsuma–Ito-like equation is a new nonlinear wave model which is firstly proposed in [36], and it is generated from (2+1)-dimensional Hirota–Satsuma–Ito equation [37] and Hirota–Satsuma equation [38], which reflect abundant physical meaning in nonlinear shallow waves. The equation reads
\[
\alpha u_{yt} + u_{xxx} + 3u_xu_{xt} + 3u_xu_t + \beta u_{xt} = 0, \quad (1)
\]
where \(\alpha\) and \(\beta\) are real nonzero constants, and \(u(x, y, z, t)\) describes the unidirectional propagation of shallow water waves. The lump solutions are obtained and some interaction phenomena are discussed in [36]. Based on the previous research, we mainly focus on investigating evolution and deformation of kink breather waves, the emergence of lump solutions, interaction phenomenon of lump-\(M\)-soliton and analysing dynamical behaviour of each kinds of solutions.

However, to the best of our knowledge, Eq.(1) has many properties and exact solutions that have not been studied, which deserves further study and discussion. The evolution and degradation of breather wave solutions, and the superposition between lump and kink \(M\)-soliton \((M \to \infty)\) with different kinds have not been studied thoroughly. In this manuscript, we aim to construct breather wave solutions firstly, as degeneration of breather waves, lump waves emerge. Then, the interaction between lump and \(M\)-soliton solutions of Eq.(1) is given. The structure is as follows: In Sect. 2, based on Hirota bilinear form, we construct kink breather waves with different forms, and lump waves will be derived from breather waves by the limit method. In Sect. 3, \(M\)-fissionable soliton and sufficient conditions of the existence for interaction between lump and kink \(M\)-solitons will be obtained, and theoretical proof of superposition behaviour and examples will be given. Finally, some conclusions are concluded in the last section.

### 2 Evolution and degeneration from breather to Lump solution

In this section, based on the Hirota bilinear method, the breather wave solutions of Eq.(1) with different structural forms are constructed. Under the action of oscillation of breather wave, some lump or lump-type solutions are found out by using parameter limit method [9, 24]. Through Painlevé analysis, we assume
\[
u(x, y, z, t) = (lnf)_x. \quad (2)
\]
where \(f\) is an unknown real-valued function about \(x, y, z, t\). Equation(1) can be converted to the bilinear operator \(D\) as follows:
\[
(D_x^3 D_t + \alpha D_y D_t + \beta D_z D_t) f \cdot f = 0. \quad (3)
\]
The Hirota bilinear operator \(D_x^n D_y^m\) are defined by [31] \((n, m \geq 0)\)
\[
D_x^n D_y^m D_z^l f(x, t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^l f(x, y, z, t) |_{x' = x, y' = y, z' = z, t' = t}.
\]

On the basis of [8, 9], we choose the following test function:
\[
f(x, y, z, t) = \delta_1 cosh \xi_1 + \delta_2 cos \xi_2 + \delta_3 cosh \xi_3, \quad (4)
\]
where \(\xi_i = k_i (a_i x + b_i y + c_i z + d_i t + \gamma_i)\), \(k_i, a_i, b_i, c_i, d_i, \gamma_i (i = 1, 2, 3)\) are free real parameters to be determined. Substituting Eq.(4) into Eq.(3) with computer algebra software Mathematica 12, we get two sets of parameter relations.
Evolution behaviour of kink breathers

Case 1:

\[
\begin{align*}
\xi_i &= \frac{a_i k_i}{k_i}, \\
\theta_i &= \frac{a_i k_i x - \beta c_1 y + c_1 z - s d t + \gamma_1}{\gamma_2 + \gamma_3}.
\end{align*}
\]

where \(a_i k_i d_1 \neq 0 (i = 1, 2, 3)\). A multi-breather solitary solution is constructed by inserting Eq.(4) and Eq.(5) into Eq.(2) as follows:

\[
u (x, y, z, t) = 2 a_i k_i \delta_i \sinh \delta_i (x + a_i k_i \delta_i \sinh \delta_i + a_i k_i \sinh \delta_i),
\]

where

\[
\begin{align*}
\delta_1 &= k_1 (\frac{a_i k_i x}{a_i k_i} + \frac{\beta c_1 y + c_1 z}{\gamma_2 + \gamma_3}), \\
\delta_2 &= k_2 (\frac{a_i k_i x}{a_i k_i} + \frac{\beta c_1 y + c_1 z}{\gamma_2 + \gamma_3}), \\
\delta_3 &= k_3 (\frac{a_i k_i x}{a_i k_i} + \frac{\beta c_1 y + c_1 z}{\gamma_2 + \gamma_3}).
\end{align*}
\]

The solution \(\nu\) expressed in Eq.(6) is a kind of double breather wave solution, which takes on the characteristics of double solitary waves with the variable \(\delta_1\) and \(\delta_2\), and the characteristics of periodic solitary waves with the variable \(\delta_3\). Besides, under the action of the function \(\cos \delta_2\), \(\nu\) also shows the characteristics of local oscillation. Therefore, the breather wave solutions with these characteristics are named double breathers [39, 40]. The oscillation behaviour of double breathers is shown in Fig. 1a–c, when we choose the parameters as \(\alpha = 20, \beta = 1, \delta_1 = 2, \delta_2 = 4, \delta_3 = \frac{3}{4}, a_1 = \frac{1}{4}, a_2 = \frac{2}{4}, a_3 = \frac{3}{4}, c_1 = c_2 = c_3 = 1, d_1 = 4, d_2 = 4, d_3 = \frac{1}{2}, z = 0\).

In order to construct the lump solutions from Eq.(6), the parameters \(\delta_i (i = 1, 2, 3)\) must satisfy:

\[
\lim_{k_i \to 0} (\delta_1 + \delta_2 + \delta_3) = 0,
\]

so we take \(k_2 = r k_1, k_3 = s k_1, \delta_1 = \mu^2 \cosh (l k_1), \delta_2 = -(\mu^2 + v^2) \cosh (m k_1), \delta_3 = \nu^2 \cosh (n k_1)\), where \(\mu, v\) are real numbers, and \(m, n, l, r, s\) are real numbers or pure imaginary numbers. Letting \(k_1 \to 0\), lump solution is given by

\[
\begin{align*}
u (x, y, z, t) &= \frac{\mu^2 \mu^2 + 2 \mu^2 + v^2 + r^2 \mu^2 + s^2 \mu^2}{\mu^2 + v^2 + r^2 + s^2 \mu^2} \\
&= \frac{\mu^2 \mu^2 + 2 \mu^2 + v^2 + r^2 \mu^2 + s^2 \mu^2 + (m^2 + n^2) \mu^2 + (m^2 + n^2) v^2}{\mu^2 + v^2 + r^2 + s^2 \mu^2}.
\end{align*}
\]

where \(\mu, v, r, s\) tend to \(\infty\) in any direction, and shows the characteristic of polynomial attenuation on all spatial variables. Meanwhile, we know that the solution is non-singular as the parameters satisfy \((m^2 + l^2) \mu^2 + (m^2 + n^2) v^2 > 0\) from the expression of Eq.(9). From Fig. 2, the lump solution \(\nu\) has a upward peak and a downward valley, and the lump solution with this structures is called lump or lump-type solution [15, 17].

Case 2:

\[
\begin{align*}
\delta_2 &= 0, \\
b_1 &= \frac{-\beta c_1 - a_1^2 k_1^2 + 3 a_1 a_2^2 k_1^2}{\alpha}, \\
b_2 &= \frac{-\beta c_2 + a_2^2 k_1^2 - 3 a_1^2 a_2^2 k_1^2}{\alpha}, \\
d_2 &= \frac{-a_1 d_1 k_1^2 k_1^2}{a_2^2 k_1^2 k_1^2}.
\end{align*}
\]

So we get

\[
\begin{align*}
f (x, y, z, t) &= \mu^2 \cosh (k_1 (a_1 x + b_1 y + c_1 z + d_1 t + \gamma_1)) \\
&+ \delta_2 \cosh (k_2 (a_2 x + b_2 y + c_2 z + d_2 t + \gamma_2)).
\end{align*}
\]

An exact one breather wave solution of HSII equation is obtained by combining Eqs. (10), (11) and (2), and the spatial structure is shown in Fig. 3, and the oscillation behaviour of solitary wave and periodic wave is displayed vividly.
Fig. 1  Spatial oscillation of double breathers in Eq. (6) as a $\gamma_i = -2$, b $\gamma_i = 0$, c $\gamma_i = 2(i = 1, 2, 3)$, at $t = 0$

Fig. 2  Spatial structure of lump solution in Eq. (9) as $\alpha = \beta = 1$, $l = n = 0$, $s = 1$, $r = 2$, $m = 1$, $\mu = \frac{1}{4}$, $\nu = 1$, $a_3 = 1$, $c_1 = 0$, $c_2 = -3$, $c_3 = -1$, $d_2 = 0$, $d_3 = 1$ and a $\gamma_i = -2$, b $\gamma_i = 0$, c $\gamma_i = 2(i = 1, 2, 3)$, at $z = 0$, $t = 0$
Similarly, we take $k_2 = r k_1$, $\delta_1 = \cosh(k_1)$, $b_2 = -\cos(m k_1)$, and let $k_1 \to 0$. New type lump solution is obtained (Fig. 4).

$$u(x, y, z, t) = \frac{4(a_1 \zeta_1 + r a_2 \zeta_2)}{\zeta_1^2 + \zeta_2^2 + l^2 + m^2}, \quad (12)$$

where $\zeta_1 = a_1 x - \frac{\beta c_1}{a} y + c_1 z + d_1 t + \gamma_1$, $\zeta_2 = r a_2 x - \frac{\beta r c_2}{a} y + r c_2 z - \frac{a_1 d_1}{r a_2} t + \gamma_2$, $r a_2 \neq 0$ and $l^2 + m^2 > 0$.

When $a_2 = 0$, lump solution is as follows:

$$u(x, y, z, t) = \frac{4(a_1 \zeta_1 + r a_2 \zeta_2)}{\zeta_1^2 + \zeta_2^2}, \quad (13)$$

where $\zeta_1 = a_1 x - \frac{\beta c_1}{a} y + c_1 z + \gamma_1$, $\zeta_2 = -\frac{\beta r c_2}{a} y + r c_2 z + r d_2 t + \gamma_2$.

3 Interaction between lump solutions and kink $M$-solitons

3.1 Kink $M$-soliton solutions

We choose the multi-soliton function as follows:

$$f(x, y, z, t) = 1 + \sum_{j=1}^{M} \delta_j e^{(p_j x + q_j y + r_j z + s_j t + \sigma_j)}. \quad (13)$$

Based on the transformation Eq.(2) and bilinear form Eq.(3), then $s_j = 0$, $q_j = -\frac{p_j + \beta r_j}{a}$ for $j = 1, 2, \ldots, M$. To better observe the evolution behaviour of kink $M$-soliton, Fig. 5 provides the spatial structure of superposition of the kink $M$-soliton as the number of solitons increases. This constructed $M$-soliton shown in Fig. 5 is also named $M$-fissionable wave solution, and [41] described the dynamic properties and asymptotic behaviour particularly.

3.2 Theoretical proof of superposition behaviour

In this section, the lump $M$-solitons will be investigated, which are also called interaction solutions by some researches [37,42–44]. And a theorem is given and proved it completely. Now, we assume a test function consisting of quadratic and exponential functions of sum type as follows:

$$F(x, y, z, t) = a_0 + \sum_{i=1}^{N} (a_i x + b_i y + c_i z + d_i z + \gamma_i)^2 + \sum_{j=1}^{M} \delta_j e^{(p_j x + q_j y + r_j z + s_j t + \sigma_j)} \quad (14)$$

def \Phi(x, y, z, t) + \Psi(x, y, z, t). \quad (14)$$

where $a_0, a_i, b_i, c_i, d_i (i = 1, 2, 3, \ldots, N)$ and $\delta_j, p_j, q_j, r_j, s_j, \sigma_j (j = 1, 2, 3, \ldots, M)$ are some free real parameters, $\Phi(x, y, z, t)$ represents a quadratic
polynomial function, and $\Psi(x, y, z, t)$ is a multisoliton function. Substituting Eq. (14) into Eq. (2), some new interaction solutions of (3+1)-dimensional HSII equation can be obtained.

**Theorem** Assume the function $\Phi$ is a solution of Eq. (3), and the parameters satisfy the relations $\sum_{i=1}^{N} a_i d_i = 0$, $\sum_{i=1}^{N} (\alpha b_i + \beta c_i) d_i = 0$, $s_j = 0$ and $q_j = -\frac{p_j^{3}}{\alpha} + \beta r_j$, then $F(x, y, z, t)$ is also a solution of Eq. (3).

**Proof** By the Hirota bilinear operator $D_-$, we have

$$P(D_x, D_y, D_z, D_t)F \cdot F = ((D_x^3 + \alpha D_y + \beta D_z) D_t)$$

$$(\Phi \cdot \Phi \cdot \Phi + \Psi \cdot \Phi \cdot \Psi + \Phi \cdot \Psi \cdot \Psi).$$

(15)

Thanks to $\Phi(x, y, z, t)$ is a solution of Eq. (3) and $s_j = 0, \Psi_t = 0$, we have

$$P(D_x, D_y, D_z, D_t)(\Phi \cdot \Phi) = 0,$$

$$P(D_x, D_y, D_z, D_t)(\Psi \cdot \Psi) = 0,$$

(16)
so that

\[ P(D_x, D_y, D_z, D_t)F \cdot F = \left( (D_x^2 + \alpha D_y + \beta D_z) D_t \right) \]

\[ (\Phi \cdot \Phi + \Psi + \Psi + \Phi + \Phi \cdot \Psi) \]

\[ = 2(D_x^2 + \alpha D_y + \beta D_z)(\Phi \cdot \Psi) \]

\[ = 2(D_x^2 + \alpha D_y + \beta D_z)(\Phi_1, \Psi - \Phi_1, \Psi) \]

\[ = 4((\Phi_1, \Psi - \Phi_1, \Psi_x + 3\Phi_1, \Psi_{xx} - \Psi_1, \Psi_{xxx}) \]

\[ + \alpha(\Phi_1, \Psi - \Phi_1, \Psi_t) + \beta(\Phi_2, \Psi - \Phi_2, \Psi_t)) \]

\[ = 4((\Phi_1, \Psi - \Phi_1, \Psi_x + 3\Phi_1, \Psi_{xx} - \Psi_1, \Psi_{xxx}) \]

\[ + \alpha\Psi_1 + \beta(\Phi_2, \Psi - \Phi_2, \Psi_t) \]

According to the form of \( \Phi \) and \( \Psi \), some relationship of parameters is as follows:

\[ \sum_{i=1}^{N} a_i d_i = 0, \quad \sum_{i=1}^{N} (\alpha b_i + \beta c_i) d_i = 0, \quad s_j = 0, \]

\[ q_j = -\frac{p_j^3 + \beta r_j}{\alpha}, \quad (18) \]

and

\[ \Psi_{xxx} + \alpha\Psi_y + \beta\Phi_z = \sum_{j=1}^{M} \delta_j p_j e^{\eta_j} \]

\[ + \alpha \sum_{j=1}^{M} \delta_j q_j e^{\eta_j} \]

\[ + \beta \sum_{j=1}^{M} \delta_j r_j e^{\eta_j} = 0, \quad (19) \]

where \( \eta_j = p_j x + q_j y + r_j z + s_j t + \sigma_j \). Substituting Eq.(18) and (19) into Eq.(17), the theorem is proved completely.

**Corollary** Assume

\[ g(x, y, z, t) = a_0 + \sum_{i=1}^{N} (a_i x + b_i y + c_i z + d_i t) \]

\[ + \gamma_j^2 + \Phi(x, y, z, t). \quad (20) \]

If the function \( \Phi(x, y, z, t) \) satisfies \( \Phi_t = 0 \) and \( \Phi_{xxx} + \alpha\Phi_y + \beta\Phi_z = 0 \), and the parameters satisfy the relations \( \sum_{i=1}^{N} a_i d_i = 0, \sum_{i=1}^{N} (\alpha b_i + \beta c_i) d_i = 0 \). Then, \( g(x, y, z, t) \) is also a solution of Eq.(3).

### 3.3 Superposition behaviour of Lump-\( M \)-soliton

In this subsection, we will give some examples to illustrate the effectiveness of above theorem and corollary, the evolution behaviour of spatiotemporal structure of lump-\( M \)-solitons with the change of soliton number \( M \) and time \( t \) will be studied.

When \( N = 2, M = 0 \) in Eq. (14), substituting Eq. (14) into Eq. (3), we obtain the following two solutions with the aid of Mathematica:

\[ a_1 = -\frac{a_2 d_2}{d_1}, \quad c_1 = -\frac{a b_1}{\beta}, \quad c_2 = -\frac{a b_2}{\beta}. \quad (21) \]

\[ a_2 = -\frac{a_1 d_1}{d_2}, \quad b_1 = 0, \quad b_2 = -\frac{a c_2}{\beta}, \quad c_1 = 0. \quad (22) \]

\[ a_1 = 0, \quad b_1 = -\frac{a c_1}{\beta}, \quad b_2 = -\frac{a c_2}{\beta}, \quad d_1 = 0. \quad (23) \]

\[ a_2 = 0, \quad b_1 = -\frac{a c_1}{\beta}, \quad b_2 = -\frac{a c_2}{\beta}, \quad d_1 = 0. \quad (24) \]

Firstly, we give the application of the theorem, the spatiotemporal evolution behaviour of lump-\( M \)-solitons with the change of soliton number \( M \) by drawing three-dimensional plots (Figs. 6, 7, 8). Combining Eqs. (2), (14) and Eq. (23), the interaction solutions are given by

\[ u(x, y, z, t) = \frac{4a_2(x_2 x_0 - \frac{a_2 c_2}{\alpha} y + c_2 z + \gamma_2) + 2 \sum_{j=1}^{M} \delta_j p_j e^{\eta_j}}{a_0 + \left( \frac{-p_1 c_1}{\alpha} y + c_1 z + d_1 t + \gamma_1 \right)^2 + \left( \alpha x_2 - \frac{\alpha c_2}{\alpha} y + c_2 z + \gamma_2 \right)^2 + \Psi} \quad (25) \]

where \( \eta_j = p_j x + q_j y + r_j z + s_j t + \sigma_j \).

From Figs. 6, 7, 8, the fission behaviours of the spatiotemporal structure of lump-\( M \)-soliton with the increase in the number \( M \) are studied, and we take lump-1-soliton, lump-2-soliton and lump-3-soliton as the main research and presentation objects. In fact, the lump-\( M \)-soliton (\( M \rightarrow +\infty \)) solutions have similar evolution structures. The lump generates from the kink wave and then separates from it with the increase in the time. By the change process of \( u \), we can see that the lump always exists as \( t \rightarrow +\infty \), and their amplitude including lump and kink wave remains constant. The parameters in Figs. 6, 7, 8 are taken as \( \alpha = -3, \beta = 2, \alpha_0 = 4, a_2 = 2, c_1 = 1, c_2 = 2, d_1 = -1, \gamma_1 = \gamma_2 = 0, \delta_1 = \delta_2 = 3, p_1 = 1, p_2 = -1, p_3 = -1, r_1 = -1, r_2 = -2, r_3 = -3, \sigma_1 = \sigma_2 = \sigma_3 = 0, z = 1 \).

Secondly, we investigate the application of the corollary. Actually, there are so many functions \( \Phi(x, y, z, t) \) that satisfy \( \Phi_t = 0 \) and \( \Phi_{xxx} + \alpha\Phi_y + \beta\Phi_z = 0 \). We will give two examples as follows.

**Example 1** We study a new interaction phenomenon by combining Eq. (23) and the corollary. Here, the test function is written as:

\[ f(x, y, z, t) = a_0 + \left( \frac{-\beta c_1}{\alpha} y + c_1 z + d_1 t + \gamma_1 \right)^2 \]

\[ + \left( a_2 x - \frac{\alpha c_2}{\alpha} y + c_2 z + \gamma_2 \right)^2 \]

\[ + \sum_{i=1}^{L} q_i \cosh(p_i x - \frac{p_i^3 + \beta_1}{\alpha} y + r_i z + s_i). \quad (26) \]
Substituting Eq. (26) into Eq. (2), interaction solutions of lump-$L$-$\cosh (L \to +\infty)$ type are obtained. In order to better demonstrate the behaviour of spatial structure with the increase in the number of cosh-type $L$. In Fig. 9, the lump generates from the kink wave, and it is gradually drowned or swallowed by the kink wave. When $t = 0$, the amplitude gets to the extreme point, and the collision between a lump and 1-cosh kink wave is completely nonelastic. The fusion process is shown in Fig. 10, and the lump generates from the of the intersection of 2-cosh kink wave; then, it vanishes ad the amplitude changes rapidly. This phenomenon is similar to the rogue wave; it means a kind of rogue wave will appear based on the interaction between a lump and 2-cosh kink wave. The lump generates from the intersection of 3-cosh kink wave in Fig. 11. As the time goes by, and the lump wave is gradually drowned or swallowed by the kink wave. Figures are drawn with the parameters taken as $\alpha = -3, \beta = 2, a_0 = 4, a_2 = 8, c_1 = 1, c_2 = 2, d_1 = -1, \gamma_1 = \gamma_2 = 0, \delta_1 = \delta_2 = \delta_3 = 1, p_1 = 1, p_2 = \frac{1}{2}, p_3 = \frac{1}{4}, r_1 = -1, r_2 = 1, r_3 = -2, \sigma_1 = \sigma_2 = \sigma_3 = 0, z = 1$.

**Example 2** We study the interaction phenomenon between lump and $M$-solitons with different types, combining Eq. (23) and the corollary. Assume the test function is written as

\[\begin{align*}
\end{align*}\]
Evolution behaviour of kink breathers

Fig. 9  Spatial superposition behaviour of lump-$L$-soliton as $L = 1$, $a t = -10$, $b t = 0$, $c t = 15$. $d t = 35$

Fig. 10  Spatial superposition behaviour of lump-$L$-soliton as $L = 2$, $a t = -10$, $b t = 0$, $c t = 15$. $d t = 35$

Fig. 11  Spatial superposition behaviour of lump-$L$-soliton as $L = 3$, $a t = -10$, $b t = 0$, $c t = 15$. $d t = 35$

\[ f(x, y, z, t) = a_0 + \sum_{i=1}^{N} (a_i x + b_i y + c_i z + d_i z + \gamma_i)^2 + \sum_{j=1}^{M} \delta_j e^{p_j x - \frac{p_j^2 + p_j^j}{2 a} y + r_j z + s_j} + \sum_{k=1}^{K} \lambda_k \cos(q_k x - \frac{q_k^2 + q_k h_k}{2 a} y + h_k z + \omega_k) + \sum_{l=1}^{L} \varphi_l \cosh(q_l x - \frac{q_l^2 + q_l \psi_l}{a} y + \psi_l z + \nu_l). \]  

(27)

Some new interaction solutions are obtained, and the spatial structures of the lump and $M$-solitons with different types are given in Fig. 12. In Fig. 12, different types of interaction are as follows:

(a) Interaction of lump-cos type as $\alpha = -1, \beta = 1, a_0 = 4, a_2 = 1, c_1 = 1, c_2 = 0, d_1 = -1, \gamma_1 = \gamma_2 = 0, \lambda_1 = 1, q_1 = 1, h_1 = -1, \omega_1 = 0, z = 1$ at time $t = 0$.

(b) Interaction of lump-cos-exponential type as $\alpha = -1, \beta = 1, a_0 = 4, a_2 = 1, c_1 = 1, c_2 = 0, d_1 = -1, \gamma_1 = \gamma_2 = 0, \lambda_1 = 1, q_1 = 1, h_1 = -1, \omega_1 = 0, \delta_1 = 1, p_1 = 1, r_1 = -1, s_1 = 0, z = 1$ at time $t = 0$.

(c) Interaction of lump-cosh-cos-cosh type as $\alpha = -1, \beta = 1, a_0 = 4, a_2 = 1, c_1 = 1, c_2 = 0, d_1 = -1, \gamma_1 = \gamma_2 = 0, \lambda_1 = 1, q_1 = 1, h_1 = 1, \omega_1 = 0, \omega_1 = 0, \varphi_1 = 1, \varphi_2 = 0, \psi_1 = 0, \psi_2 = 0, \varphi_3 = 0, \psi_2 = 0, z = 1$ at time $t = 0$. 
4 Conclusion

Based on symbolic computation and Hirota bilinear method, the evolution and degradation behaviours of breather wave solutions with different kinds are investigated; some new lump and lump-type solutions are obtained from the breather wave solutions via using the parameter limit method. Besides, inspired by the previous researches on the study of interaction phenomenon, we have studied the superposition behaviour between lump solution and different types of $M$-solitons ($M \to \infty$) for (3+1)-dimensional HSII equation, and the theorem proof has been proposed firstly in this paper. Meanwhile, several three-dimensional spatiotemporal structure figures of breather waves and the interaction solutions are drawn to better reflect the oscillation of solitary wave and the evolution of interaction behaviour, including the oscillation and fusion–fission phenomenon, superposition and evolution behaviour of lump-$M$-solitons and so on. Figures 1 and 3 show the oscillation of kink breather wave including double breather and breather solitary wave. In Figs. 2 and 4, lump solutions are generated from kink breather wave, it is also kink lump waves. $M$-soliton solutions are known as $M$-fissionable wave which are shown in Fig. 5. Some diverse interaction phenomenon shown in the following figures have great significance to the nonlinear waves in fluid mechanics. In this paper, the methods used to obtain new exact solutions also can be extended to solve other nonlinear partial different equations. For some important classical mathematical and physical models, the methods of studying solutions are very significant.

Acknowledgements The authors would like to express their sincere thanks to referees for their enthusiastic guidance and help. This work was supported by Scientific and Technological Innovation Team of Nonlinear Analysis and Algebra with Their Applications in Universities of Yunnan Province, China, Grant No. 2020CXTD25.

Data availability The authors declare that data supporting the findings of this study are available within the article, and the figures are concrete expression.

Declaration

Conflict of interest The authors declare that there are no conflicts of interests with publication of this work.

Ethical approval The authors ensure the compliance with ethical standards for this work.
References

1. Zabusky, N.J., Kruskal, M.D.: Interaction of solitons in a collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. 15(6), 240–243 (1965)
2. Kivshar, Y.S., Malomed, B.A.: Dynamics of solitons in nearly integrable system. Rev. Mod. Phys. 61(4), 763–915 (1989)
3. Mihalache, D.: Multidimensional localized structures in optics and Bose-Einstein condensates: a selection of recent studies. J. Roman. Phys. 59(3), 295–312 (2014)
4. Forte, S.: Quantum mechanics and field theory with fractional spin and statistics. Appl. Phys. Lett. 64(1), 193–236 (1992)
5. Kibler, B., Fatome, J., Finot, C., Millot, G., Dias, F., Genty, G., Akhmediev, N., Dudley, J.M.: The peregrine soliton in nonlinear fibre optics. Nat. Phys. 6(10), 790–795 (2010)
6. Malomed, B., Torner, L., Wise, F., Mihalache, D.: On multidimensional solitons and their legacy in contemporary atomic, molecular and optical physics. J. Phys. B At. Mol. Opt. Phys. 49(17), 170502 (2016)
7. Dai, Z.D., Wang, C.J., Liu, J.: Inclined periodic homoclinic breather and rogue waves for the (1+1)-dimensional Boussinesq equation. Pramana J. Phys. 83(4), 473–480 (2014)
8. Zhang, R.F., Bilige, S.: Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation. Nonlinear Dyn. 95, 3041–3048 (2019)
9. Liu, J., Mu, G., Dai, Z.D., Lou, H.Y.: Spatiotemporal deformation of multi-soliton to (2+1)-dimensional KdV equation. Nonlinear Dyn. 86, 355–360 (2016)
10. Zhang, R.F., Bilige, S., Chaolu, T.: Fractal solitons, arbitrary function solutions, exact periodic wave and breathers for a nonlinear partial differential equation by using bilinear neural network method. J. Syst. Sci. Complex. 34(1), 122–139 (2021)
11. Zhang, R.F., Bilige, S., Liu, J.G., Li, M.C.: Bright-dark solitons and interaction phenomenon for p-gBKP equation by using bilinear neural network method. Phys. Scr. 96, 025224 (2021)
12. Zhang, R.F., Li, M.C., Yin, H.M.: Rogue wave solutions and the bright and dark solitons of the (3+1)-dimensional Jimbo-Miwa equation. Nonlinear Dyn. 103, 1071–1079 (2021)
13. Ma, W.X., Yong, X.L., Zhang, H.Q.: Diversity of interaction solutions to the (2+1)-dimensional Ito equation. Comput. Math. Appl. 75(1), 289–295 (2018)
14. Ma, C.H., Deng, A.P.: Lump solution of (2+1)-dimensional Boussinesq equation. Commun. Theor. Phys. 65(05), 546–552 (2016)
15. Ma, W.X., Zhou, Y.: Lump solutions to nonlinear partial different equations via Horita bilinear forms. J. Differ. Equ. 264(4), 2633–2659 (2018)
16. Zhao, Z.L., Chen, Y., Han, B.: Lump soliton, mixed lump stripe and periodic lump solutions of a (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation. Mod. Phys. Lett. B 31(14), 1750157 (2017)
17. Wang, C.J.: Spatiotemporal deformation of lump solution to (2+1)-dimensional KdV equation. Nonlinear Dynam. 84(2), 697–702 (2016)
18. Peng, W.Q., Tian, S.F., Zhang, T.T.: Analysis on lump, lump off and rogue waves with predictability to the (2+1)-dimensional B-type Kadomtsev-Petviashvili equation. Phys. Lett. A 382, 2701–2708 (2018)
19. Manakov, M.Q., Zakharov, V.E., Bordag, L.A.: Analysis on lump, Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction. Phys. Lett. A 63(3), 205–206 (1977)
20. Ablowitz, M.J., Satsuma, J.: Solitons and rational solutions of nonlinear evolution equations. J. Math. Phys. 19(10), 2180–2186 (1978)
21. Satsuma, J., Ablowitz, M.J.: Two-dimensional lumps in nonlinear dispersive system. J. Math. Phys. 20(7), 1496–1503 (1979)
22. Ma, W.X., Zhou, Y., Dougherty, R.: Lump-type solutions to nonlinear differential equations derived from generalized bilinear equations. Int. J. Mod. Phys. B 30, 1640018 (2016)
23. Tan, W., Dai, Z.D., Xie, J.L., Qiu, D.Q.: Parameter limit method and its application in the (4+1)-dimensional Fokas equation. Comput. Math. App. 75(12), 4214–4220 (2018)
24. Tan, W., Dai, Z.D., Xie, J.L., Yin, Z.Y.: Dynamics of multi-breathers, N-solitons and M-lump solutions in the (2+1)-dimensional KdV equation. Nonlinear Dyn. 96, 1605–1614 (2019)
25. Wang, C.J., Fang, H., Tang, X.X.: State transition of lump-type waves for the (2+1)-dimensional generalized KdV equation. Nonlinear Dyn. 95, 2943–2961 (2019)
26. Tan, W.: Some new dynamical behaviour of double breathers and lump-N-solitons for the Ito equation. Int. J. Comput. Math. 98(5), 961–974 (2021)
27. Ma, W.X., Zhang, H.Q.: Lump solutions to the (2+1)-dimensional Sawada–Kotera equation. Nonlinear Dyn. 87(4), 2305–2310 (2017)
28. Ren, B., Lin, J., Lou, Z.M.: Lump and their interaction solutions of a (2+1)-dimensional generalized potential Kadomtsev-Petviashvili equation. J. App. Anal. Comput. 10(3), 935–945 (2020)
29. Zhang, R.F., Li, M.C., Albishari, M., Zheng, F.C., Lan, Z.Z.: Generalized lump solutions, classical lump solutions and rogue waves of the (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada-like equation. Appl. Math. Comput. 403, 126201 (2021)
30. Tan, W., Dai, Z.D.: Dynamics of kinky wave for (3+1)-dimensional potential Yu-Tada-Sasa-Fukuyama equation. Nonlinear Dyn. 85(2), 817–823 (2016)
31. Sun, Y., Tian, B., Xie, X.Y., Yin, H.M.: Rogue waves and lump solitons for a (3+1)-dimensional B-type Kadomtsev-Petviashvili equation in fluid dynamics. Wave Random Complex. 28(3), 544–552 (2018)
32. Wang, H., Tian, S.F., Chen, Y., Zhang, T.T.: Dynamics of kink solitary waves and lump waves with interaction phenomena in a generalized (3+1)-dimensional Kadomtsev-Petviashvili-Boussinesq equation. Int. J. Comput. Math. 97(11), 2178–2190 (2020)
33. Verma, P., Kaur, L.: Integrability, bilinearization and analytic study of new form of (3+1)-dimensional B-type Kadomtsev-Petviashvili(BPK)-Boussinesq equation. Appl. Math. Comput. 346, 879–886 (2019)
34. Hirota, R.: Direct Methods in Soliton Theory. Springer, Berlin (1980)
35. Ma, W.X.: N-soliton solution of a combined pKP-BKP equation. J. Geom. Phys. 165, 104191 (2021)
36. Chen, S.J., Lu, X., Ma, W.X.: Bäcklund transformation, exact solution and interaction behaviour of the (3+1)-dimensional Hirota-Satsuma-Ito-like equation. Commun. Nonlinear Sci. Num. Simul. 83, 105135 (2020)
37. Zhou, Y., Manukure, S., Ma, W.X.: Lump and lump-soliton solutions to the Hirota-Satsuma-Ito equation. Commun. Nonlinear Sci. Num. Simul. 68, 56–62 (2019)
38. Hirota, R., Satsuma, J.: N-soliton solutions of model equation for shallow water waves. J. Phys. Soc. Japan 40(2), 611–612 (1976)
39. Tan, W., Zhang, W., Zhang, J.: Evolutionary behaviour of breathers and interaction solutions with $M$-solitons for (2+1)-dimensional KdV system. Appl. Math. Lett. 101((C)), 106063 (2020)
40. Tian, Y., Dai, Z.D.: Rogue waves and new multi-wave solutions of the (2+1)-dimensional Ito equation. Z. Naturforsch. A 70(6), 437–443 (2015)
41. Chen, A.H., Wang, F.F.: Fissionable wave solutions, lump solutions and interactional solutions for the (2+1)-dimensional Sawada-Kotera equation. Phys. Scr. 94(5), 055206 (2019)
42. Hossen, M.B., Roshid, H.O., Ali, M.Z.: Characteristics of the solitary waves and rogue waves with interaction phenomena in a (2+1)-dimensional breaking soliton equation. Phys. Lett. A 382(19), 1268–1274 (2018)
43. Wang, X.B., Tian, S.F., Qin, C.Y., Zhang, T.T.: Dynamics of the breathers, rogue waves and solitary waves in the (2+1)-dimensional Ito equation. Appl. Math. Lett. 68, 40–47 (2017)
44. Yang, J.Y., Ma, W.X.: Abundant interaction solutions of the KP equation. Nonlinear Dynam. 89(1), 1539–1544 (2012)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.