ENTROPY GENERATION IN QUANTUM GRAVITY
AND BLACK HOLE REMNANTS*

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Abstract

The area entropy $A/4$ and the related Hawking temperature in the presence of event horizons are rederived, for de Sitter and black hole topologies, as a consequence of a tunneling of the wave functional associated to the classical coupled matter and gravitational fields. The extension of the wave functional outside the barrier provides a reservoir of quantum states which allows for an additive constant to $A/4$. While, in a semi-classical analysis, this gives no new information in the de Sitter case, it yields an infinite constant in the black hole case. Evaporating black holes would then leave residual “planckons” - Planckian remnants with infinite degeneracy. Generic planckons can neither decay into, nor be directly formed from,
ordinary matter in a finite time. Such opening at the Planck scale of an infinite Hilbert space is expected to provide the ultraviolet cutoff required to render the theory finite in the sector of large scale physics.
1. Introduction

Tunneling in quantum gravity can generate entropy\[1,2\]. To understand how such an apparent violation of unitarity may arise, let us first consider a classical spacetime background geometry with compact Cauchy hypersurfaces. If quantum fluctuations of the background are taken into account, quantum gravity leaves no “external” time parameter to describe the evolution of matter configurations in this background. Indeed, the solutions of the Wheeler-De Witt equation\[3\]

\[
\mathcal{H}\ket{\Psi} = 0
\]

where $\mathcal{H}$ is the Hamiltonian density of the interacting gravity-matter system can contain no reference to such time when there are no contribution to the energy from surface terms at spatial infinity\[4\]. This is due to the vanishing of the time displacement generator and even though the theory can be unambiguously formulated only at the semi-classical level, such a consequence of reparametrization invariance should have a more general range of validity.

To parametrize evolution, one then needs a “clock” which would define time through correlations\[5\], namely correlations of matter configurations with ordered sequences of spatial geometries. If quantum fluctuations of the metric field can be neglected, the field components $g_{ij}$ at every point of space can always be parametrized by a classical time parameter, in accordance with the classical equations of motion. This classical time, which is in fact a function of the $g_{ij}$, can be used to describe the evolution of matter and constitutes thus such a dynamical “clock” correlating matter to the gravitational field\[6\]. This description is available in the Hamilton-Jacoby limit of (1) where the classical background evolving in time is represented by a coherent superposition of “forward” waves formed from eigenstates of (1). When quantum metric field fluctuations are taken into account, “backward” waves, which can be interpreted as flowing backwards in time, are unavoidably generated from (1) and the operational significance of the metric clock
gets lost outside the domain of validity of the Hamilton-Jacoby limit. Nevertheless, in domains of metric field configurations where both forward and backward waves are present but where quantum fluctuations are sufficiently small, interferences with such “time reversed” semi-classical solutions will in general be negligible\(^\star\).

Projecting then out the backward waves restores the operational significance of the metric clock but the evolution marked by the correlation time is no more unitary: information has been lost in projecting these backward waves stemming from regions where quantum fluctuations of the clock are significant. This is only an apparent violation of unitarity which would be disposed of if the full content of the theory would be kept, perhaps eventually by reinterpreting backward waves in terms of the creation of “universe” quanta through a further quantization of the wavefunction (1).

This apparent violation of unitarity is particularly marked if the gravitational clock experiences the strong quantum fluctuations arising from a tunneling process. This can be illustrated from the simple analogy, represented in Fig.1, offered by a nonrelativistic closed system of total fixed energy where a particle in one space dimension \(x\) moving in a potential \(U(x)\) plays the role of a clock for surrounding matter and tunnels through a large potential barrier. Outside the barrier, the clock is well approximated by semi-classical waves, but if on the left of the turning point \(a\) one would take only forward waves, one would inevitably have on the right of the other turning point \(b\) both forward and backward waves with large amplitudes compared with the original ones. The ratio between the squares of the forward amplitudes on the right and on the left of the barrier for a component of the clock wave with given clock energy \(E_c\) is the inverse transmission coefficient \(N_0(E_c)\) through the barrier and provides a measure of the apparent violation of unitarity. In the Hamilton-Jacoby limit of quantum gravity, the characterisation of tunneling amplitudes by inverse transmission coefficients \(N_0\) will appear as the natural one to compute the entropy transferable reversibly between the metric clock

\(^\star\) For a recent discussion of related problems see reference [7].
and matter. More precisely, we shall see that, in this limit, spherically symmetric spacetimes bounded by event horizons are in general connected by tunneling to another manifold and that the entropy gained by tunneling from the latter to the former is, for large barriers, $\log N_0$. Explicit evaluation of this tunneling entropy yields $\ln N_0 = A/4$ where $A$ is the area of the event horizon. In this way the horizon thermodynamics of Gibbons and Hawking\cite{8} is recovered. But the present approach has potentially additional information.

The tunneling entropy $\ln N_0$ is in last analysis an effect of quantum fluctuations in quantum gravity. Therefore, despite the fact that no violation of unitarity would appear in a complete description including backward waves, this entropy should be expressible in terms of density of states of matter and gravity. Tunneling offers an interesting perspective in this direction because it enlarges the semi-classical wave function of spacetime to include in its description the other side of the barrier. This can yield a reservoir of quantum states which may provide, in addition to the $\exp(A/4)$ states building the entropy, residual states which would be expressed as an “integration constant” in the total entropy $S$ of spacetime. Thus we shall write

$$S = A/4 + C$$

and try to get some information about the constant $C$ by analysing both sides of the barrier.

The knowledge of $C$ is crucial, in particular for the understanding of the black hole behaviour at the final stage of evaporation.

A infinitely large value of $C$ would indeed indicate that the evaporation can only radiate a finite number of “surface” states of order $\exp(A/4)$ out of an infinite set of available internal states. This mismatch would entirely modify the black hole evaporation process at its last stage and bring the decay to a halt. Indeed when the black hole evaporates to the Planck scale, it becomes, if $C$ is infinite, a “planckon”\cite{9}, that is a remnant with infinite degeneracy. Causality and unitarity prevent the decay and the production in a finite time of planckons directly
out of ordinary matter for nearly all such states\textsuperscript{[9]}. Namely, if a planckon state $|A_i\rangle$ of finite size and mass $m$ decays (or is produced) within a finite time $\tau$ in an approximately flat space-time background, the total number of possible final (or initial) states is limited through causality by the number $\mathcal{N}(\tau)$ of orthogonal states with total mass $m$ in a volume $\tau^3$. Assuming the number of quantum fields which describe physics at scales large compared to the Planck scale to be finite, $\mathcal{N}(\tau)$ is a finite number. Unitarity then implies that if the dimension $\nu(m)$ of the Hilbert space spanned by the degenerate states $|A_i\rangle$ becomes greater than $\mathcal{N}(\tau)$, a subspace of planckon states whose dimension is $\nu(m) - \mathcal{N}(\tau)$ will be, for times smaller than $\tau$, orthogonal to the Hilbert space of states formed by these quantum fields. Thus, when $\nu(m) \to \infty$, generic planckons cannot decay (nor be formed) in a finite time. Of course the above argument does not preclude the very formation of the finite number of distinct planckons which can be generated in a finite time as remnants of macroscopic decaying black holes. This time is however unrelated to the time for their decay (creation) directly at the Planck size into (from) ordinary matter quanta; the latter time is generically infinite.

On the other hand, a zero or finite value of $C$ would lead to the disappearance of the hole at the end point of evaporation and hence probably imply a genuine violation of unitarity within our universe\textsuperscript{*}.

The main content of our work is that, in an asymptotically flat background, the constant $C$ for black holes, as deduced from a WKB analysis of tunneling, is in fact infinite. Hence the present approach indicates that the solution of the unitarity problem posed by the black hole decay is provided by planckon remnants. This conclusion is however contingent upon the limitation of the semi-classical approach to quantum gravity used here and remains therefore a tentative one.

We shall first review the computation of tunneling amplitudes in quantum gravity through static barriers\textsuperscript{[2]} and compute the tunneling entropy for the case

\textsuperscript{*} For a comprehensive review on recent attempts to solve the black hole unitarity puzzle, see reference [10]. See also reference [11].
of de Sitter spacetime topologies. However the estimation of $C$ appears in this case intractable within the semi-classical approximation. We then shall analyse in similar terms the black hole geometries\textsuperscript{[12]}. The above-mentioned results will be derived and discussed.

2. Tunneling amplitudes in quantum gravity

Our basic action in four dimensional Minkowski space-time will be

$$S = S_{grav} + S_{matter}$$

where $S_{grav}$ has the conventional form ($G = 1$):

$$S_{grav} = -\frac{1}{16\pi} \int \sqrt{-g} R d^4x$$

and $S_{matter}$ contains sufficiently many free parameters to allow for the stress tensors considered below. A possible cosmological constant term can be included in the matter action.

Consider (Fig.2) in general two spacelike hypersurfaces $\Sigma_1$ and $\Sigma_2$ which are turning points in superspace (or turning hypersurfaces) along which solutions of the Minkowskian classical equations of motion for gravity and matter meet a classical solution of their Euclidean extension. $\Sigma_1$ and $\Sigma_2$ are thus the boundaries of a region $\mathcal{E}$ of Euclidean space-time defined by the Euclidean solution. If $\mathcal{E}$ can be continuously shrunk to zero one can span $\mathcal{E}$ by a continuous set of hypersurfaces $\tau_e = \text{constant}$ such that $\tau_e \equiv \tau_{e,1}$ on $\Sigma_1$ and $\tau_e \equiv \tau_{e,2}$ on $\Sigma_2$. These $\tau_e = \text{constant}$ surfaces define a Euclidean coordinate system which we shall call synchronous; the Euclidean metric in $\mathcal{E}$ can be written in the form

$$ds^2 = N^2(\tau_e, x_k) d\tau_e^2 + g_{ij}(\tau, x_k) dx^i dx^j$$

where $N(\tau_e, x_k)$ is a lapse function. The Euclidean action $S_e$ over $\mathcal{E}$, from $\Sigma_1$ to $\Sigma_2$, is obtained by analytic continuation from the Minkowskian action (3) and can
be written as

\[ S_e(\Sigma_2, \Sigma_1) = \int_{\mathcal{E}} \Pi^{ij} g'_{ij} d^4x + \int_{\mathcal{E}} \Pi^a \Phi'_a d^4x - \int_{\mathcal{E}} (g_{ij} \Pi^{ij})' d^4x - \frac{1}{8\pi} \int_{\mathcal{E}} \partial_k[(\partial_j N) g^{kj} \sqrt{g^{(3)}}] d^4x. \quad (6) \]

Here \( \Pi^{ij} \) and \( \Pi^a \) are the Euclidean momenta conjugate to the gravitational fields \( g_{ij} \) and to the matter fields \( \phi_a \); \( g^{(3)} \) is the three dimensional determinant and the \( \prime \) symbol indicates a derivative with respect to \( \tau_e \). In the gauges (5), \( \Pi^{ij} \) is expressed as

\[ \Pi^{ij} = \frac{\sqrt{g^{(3)}}}{32\pi N} [g^{im} g^{jn} - g^{ij} g^{mn}] g'_{mn}. \quad (7) \]

On the turning hypersurfaces \( \Sigma_1 \) and \( \Sigma_2 \), all field momenta \( (\Pi^{ij}, \Pi^a) \) are zero in the synchronous system and the third term in (6) vanishes. The last term in (6) also vanishes if the hypersurfaces \( \Sigma_1 \) and \( \Sigma_2 \) are compact but may receive contributions from infinity otherwise. In this case, we shall assume that turning hypersurfaces merge at infinity sufficiently fast so that the Euclidean action \( S_e(\Sigma_2, \Sigma_1) \) does not get contributions from the last term in (6). The classical Minkowskian solution in the space-time \( \mathcal{M}_1 \) containing \( \Sigma_1 \) can be represented quantum mechanically by a “forward wave” solution \( \Psi(g_{ij}, \phi_a) \) of the Wheeler-De Witt equation (1) in the Hamilton-Jacoby limit. At \( \Sigma_1 \), this wave function enters, in the WKB limit, the Euclidean region \( \mathcal{E} \) and leaves it at \( \Sigma_2 \) to penetrate a new Minkowskian space-time \( \mathcal{M}_2 \). The tunneling of \( \Psi(g_{ij}, \phi_a) \) through \( \mathcal{E} \) engenders in addition to the “forward wave” solution a time reversed “backward wave”. The inverse transmission coefficient \( N_0 \) through the barrier measures the ratio of the norms of the forward waves at \( \Sigma_2 \) and \( \Sigma_1 \). For large \( N_0 \) one may write in the synchronous system

\[ N_0 = \exp \left[ -2 \left( \int_{\mathcal{E}} \Pi^{ij} g'_{ij} d^4x + \int_{\mathcal{E}} \Pi^a \Phi'_a d^4x \right) \right]. \quad (8) \]

As all surface terms in (6) vanish in this system, (8) can be rewritten in the
coordinate invariant form

\[ N_0 = \exp[2S_e(\Sigma_1, \Sigma_2)]. \] (9)

Let us examine the case where the Euclidean manifold \( \mathcal{E} \) is static in the sense that it admits a Killing symmetry. We can take advantage of the covariance of the action \( S_e \) and express it in terms of a new “static” coordinate system, possibly singular, with momenta everywhere vanishing in \( \mathcal{E} \). In this way, momenta in (8) get squeezed into the last surface term of (6) and one gets

\[ N_0 = \Delta t_e \frac{1}{4\pi} \int_{\mathcal{E}} \partial_k[(\partial_j N)g^{kj}\sqrt{g(3)}]d^3x \] (10)

where \( \Delta t_e \) is the Euclidean time needed to span \( \mathcal{E} \) in the static system. The tunneling amplitude will then be computable from this surface term only, even when the static parametrisation is singular.

3. The tunneling entropy in de Sitter spacetime topologies

Let us first illustrate the equivalence implied by (9) of the expressions (8) and (10) for the de Sitter spacetime which is the classical solution of pure gravity in the presence of a cosmological constant \( \Lambda \). In the present formalism, \( \Lambda \) should be viewed as the Lagrangian density of the matter action in (2); it plays the role of a matter distributed with rest energy density \( \sigma = \Lambda \) and obeying the equation of state \( \sigma = -p \) where \( p \) is a (negative) pressure. The full Minkowskian solution is the 4-hyperboloid (Fig.3) which can be parametrized by the minisuperspace metric

\[ ds^2 = d\tau^2 - a^2d\sigma^2; \quad a = r_h \cosh \frac{\tau}{r_h} \] (11)

where \( r_h = (3/8\pi \Lambda) \). The hypersurface \( \tau = 0 \) is a turning hypersurface connecting the hyperboloid to the Euclidean solution consisting of the 4-sphere which can be
described in a synchronous system by replacing in (11) $\tau$ by $-i\tau_e$. This yields the Euclidean scale factor

$$a_e = r_h \cos \frac{\tau_e}{r_h},$$

(12)

The half-sphere delimited by $-\pi r_h/2 \leq \tau_e \leq 0$ has another turning point at, say, the south pole $\tau_e = -\pi r_h/2$ where $a_e = 0$**: in the above synchronous system the space integral of the momenta in a $\tau_e = $ constant hypersurface vanishes in the vicinity of this point and so does the third term in (6). The half-sphere considered constitute the domain $E$ through which a “wormhole” at, say, $a_e = 0$ is connected by tunneling to the de Sitter spacetime. The inverse transmission coefficient $N_0$ can be straightforwardly computed from (8) using (7) and one gets

$$N_0 = \exp \left[ \frac{3\pi}{2} \left( \int_{-\pi r_h/2}^{+\pi r_h/2} a_e \left( \frac{da_e}{d\tau_e} \right)^2 d\tau_e \right) \right] = \exp(\pi r_h^2) = \exp(A/4)$$

(13)

where $A$ is the area of the event horizon.

The significance of this result is best appreciated when the 4-sphere is described in static coordinates:

$$ds^2 = \left( 1 - \frac{r^2}{r_h^2} \right) dt_e^2 + \left( 1 - \frac{r^2}{r_h^2} \right)^{-1} dr^2 + r^2 d\Omega^2.$$  

(14)

In this static frame, all momenta vanish everywhere on the sphere and the tunneling is expressible by the surface term (10) only where the radial integration is carried from $r = 0$ to $r = r_h$. The Euclidean time is periodic with period $T^{-1} = 2\pi r_h$. Using (10) with $\Delta t_e = (1/2)T^{-1} = \pi r_h$, one recovers the result (13).

It is now easy to verify that the equality between the inverse transmission coefficient and $\exp(A/4)$ is maintained when the de Sitter spacetime is perturbed by spherically symmetric static matter distributions[2]. This establishes the validity of (13) for these generalized de Sitter spacetimes.

* In fact, any point on the half-sphere can be taken as a turning hypersurface.
Let us tentatively take boundary conditions in field space by assigning pure forward waves at the wormhole turning point. The probability of finding an expanding generalized de Sitter spacetime for a corresponding wormhole state is then $N_0$, since in the classical limit interferences between spaces evolving forward or backward in time must be negligible. Assuming that all wormhole states are equally probable, we get from (13) that the relative probability of finding two matter configurations in the generalized de Sitter spacetimes is

$$\frac{N_0^{(1)}}{N_0^{(2)}} = \exp \left[ \frac{A^{(1)}}{4} - \frac{A^{(2)}}{4} \right].$$  \hspace{1cm} (15)$$

Integrating the constraint equation $\mathcal{H} = 0$ over a static domain of the Minkowskian spacetime one gets

$$\frac{1}{16\pi} \int \sqrt{-g} R d^3 x + H_{\text{matter}} + \frac{\mathcal{T} A}{4} = 0$$  \hspace{1cm} (16)$$

where $H_{\text{matter}}$ is the total matter energy. The variation of (16) yields

$$-\frac{\delta A}{4} = \mathcal{T}^{-1} \delta \lambda H_{\text{matter}}$$  \hspace{1cm} (17)$$

where $\lambda$ labels the explicit dependence of $H_{\text{matter}}$ on all other (non gravitational) “external” parameters. Equation (17) is the differential Killing identity of reference [13].

It now follows from (15) and (17) that matter configurations with neighbouring energies in a static patch of a generalized de Sitter spacetime would be Boltzmann distributed at the global temperature $\mathcal{T}$ provided our ignorance about wormhole states allows to take them to be equally probable. Thus the temperature of the static patch is $\mathcal{T}$ and therefore (17) also implies that $A/4$ is (up to an integration constant $C_{\text{deSitter}}$) the entropy of spacetime and that the latter is in thermal equilibrium with the surrounding matter. As the entropy must be an intrinsic property
of spacetime, not only is equilibrium a consequence of the chosen boundary conditions in field space but the converse is also true: the temperature obtained directly from (17) with the spacetime entropy identified as \( A/4 + C_{\text{deSitter}} \) must agree at equilibrium with the thermal distribution generated from the field boundary conditions. This justifies a posteriori the above choice of boundary conditions \(^\dagger\).

The tunneling approach to the horizon entropy and temperature\([1],[2]\) used here differs from the analysis based on the Euclidean periodicity of Green’s functions\([8]\) in two respects. On the one hand, the present approach yields the thermal spectrum, and then the entropy, from the backreaction of the thermal matter on the gravitational field, in contradistinction to the Green’s function approach. On the other hand however, the thermal matter considered here is taken in the classical limit while the Green’s function method describes genuine quantum radiation. Both methods fall short of a fully consistent quantum treatment of the backreaction. But as stated in the introduction the present approach may uncover from the hidden side of the barrier a density of state building the full entropy. Unfortunately, for the de Sitter spacetimes considered above, the hidden side is a wormhole whose description cannot be achieved in our semi-classical approach. Hence, for de Sitter spacetimes, we do not gain at this stage any information on the integration constant \( C_{\text{deSitter}} \) which measures the density of states left when the full spacetime reduces to the (planckian) wormhole. As we shall now see the situation appears quite different in the case of black hole geometries.

4. The tunneling entropy of black holes

A Schwarzschild static patch of an eternal black hole of mass \( m_0 \) is described by the metric

\[ ds^2 = (1 - \frac{2m_0}{r}) dt^2 - (1 - \frac{2m_0}{r})^{-1} dr^2 - r^2 d\Omega^2. \tag{18} \]

\(^\dagger\) up to changes which would not alter the probability ratios in the large \( N_0 \) limit.
Surrounding the black hole by static matter generalizes (18) to
\[ ds^2 = g_{00}(r) \, dt^2 - g_{11}(r) \, dr^2 - r^2 \, d\Omega^2 \] (19)
where in absence of outer horizon one has
\[ r \to \infty : g_{00}(r) = g_{11}^{-1}(r) \to 1 - \frac{2MJ}{r}. \] (20)
Here \( M = M(\infty) \) is the total mass and
\[
M(r) = m_0 + \int_{2m_0}^{r} 4\pi z^2 \sigma(z) J \, dz
\]
\[
g_{11}^{-1}(r) = 1 - \frac{2M(r)J}{r}.
\] (21)
\[
g_{00}(r) = \left( 1 - \frac{2M(r)J}{r} \right) \exp \left[ - \int_{r}^{\infty} (\sigma + p_1) 8\pi z g_{11}(z) \, dz \right]
\]
The metric (19) can be extended to the four quadrants of a Kruskal space and we choose identical matter distributions in the two Schwarzschild patches to keep a twofold symmetry around the Kruskal time axis. The Kruskal diagram is depicted in Fig.4 where we have also indicated its Euclidean extension \( T_e = iT \) resulting from the analytic continuation of the static metric (19) to the periodic time \( t_e = it \). The Euclidean periodicity is
\[
\mathcal{T} = \frac{1}{4\pi} \left[ g_{00}(2m_0) g_{11}(2m_0) \right]^{-1/2} \left. \frac{dg_{00}(r)}{dr} \right|_{r=2m_0}
\] (22)
or from (21)
\[
\mathcal{T} = \frac{1}{8\pi m_0} \exp \left[ - \int_{2m_0}^{\infty} (\sigma + p_1) 4\pi z g_{11} \, dz \right].
\] (23)
The Euclidean extension of the black hole surrounded by static matter is represented in Fig.5.
In contradistinction to the de Sitter case, there is clearly no WKB tunneling from a wormhole to a black hole because of the mismatch in topologies. One is therefore lead to investigate possible tunnelings between two black holes geometries \((B.H.)_1\) and \((B.H.)_2\) constituted respectively by black holes of mass \(m_0\) and \(m\) \((m > m_0)\) surrounded by matter. The Euclidean sections of \((B.H.)_1\) and \((B.H.)_2\), depicted in Fig.6, are engendered by a rotation of half a Euclidean period of the hypersurfaces \(a_1\) and \(a_2\) labeled by \(T = 0\) in their Kruskal diagrams. These are turning hypersurfaces along which Minkowskian and Euclidean black holes meet. We now search for two black holes such that \(a_1\) and \(a_2\) are also the boundaries of an Euclidean solution \(E\) of the Euclidean equations of motion through which tunneling can take place from one Minkowskian black hole to the other. A necessary condition for this to happen is that the total mass \(M\) of the two black hole-matter systems and their Euclidean period \(T^{-1}\) be the same, so that the turning hypersurfaces \(a_1\) and \(a_2\) of the two geometries merge at spatial infinity.

Let us choose identical matter distributions outside a radius \(r_c = 2m + \eta\). \(\eta\) is positive and such that the mass of the matter between the horizon and \(r_c\) is 0 for \((B.H.)_2\) and thus \(m - m_0\) for \((B.H.)_1\). Keeping \(m\) and the matter distribution outside \(r_c\) fixed, we now decrease \(m_0\) towards 0. From (23), in order to keep the Euclidean period \(T^{-1}\) constant for \((B.H.)_1\), we have also to decrease \(\eta\) towards 0. As \(\eta \to 0\) the mass \(m - m_0\) surrounding the infinitesimal mass \(m_0\) in \((B.H.)_1\) approaches its own Schwartzschild radius. It then follows from (21) that \(g_{00}(r)\) tends to zero for the whole interval \(2m_0 < r < 2m\). In other words any frequency stemming from the neighbourhood of the small mass hole is infinitely redshifted by the matter in that interval, as encoded in the damping exponentials in equations (21) and (23).

It is clear, from the static coordinate description (19) extended to Euclidean times \(t_e = it\), that the Euclidean sections of \((B.H.)_1\) and \((B.H.)_2\) coincide for \(r > 2m + \eta\) but, while for \((B.H.)_2\) the Euclidean section terminates at \(r = 2m\), \((B.H.)_1\) presents an extra “needle” in the region \(2m_0 < r < 2m\) whose 4-volume is vanishingly small when \(\eta \to 0\). As we now show, this is where tunneling between
$(B.H.)_1$ and $(B.H.)_2$ occurs.

To this effect, following the notations of section 2, we identify at finite $\eta$ the first turning hypersurface $\Sigma_1$ through which tunneling takes place with $a_1$ and consider instead of a second turning hypersurface $\Sigma_2$ a hypersurface $a'_2$ which lies in the Euclidean section of both $(B.H.)_1$ and $(B.H.)_2$; thus $r$ is greater than $2m + \eta$ everywhere on $a'_2$. When $\eta \to 0$, we can choose $a'_2$ arbitrarily close to $a_2$. One can then prove$^{[12]}$ that all gravitational momenta vanish in this limit on $a'_2$ in a synchronous system. We may then identify $a'_2$ with $\Sigma_2$. The region $\mathcal{E}$ is thus contained in the needle $2m_0 < r < 2m + \eta$. Because of the Kruskal twofold symmetry $a_1$ is mapped onto itself by a Euclidean time rotation of half a period and thus $\mathcal{E}$ spans only half the needle 4-volume. From (9), we learn that the inverse transmission coefficient $N_0$ is simply the exponential of the total Euclidean action of the needle. Although the limiting 4-volume of the needle vanishes, the action will turn out to be finite. It is in fact computable as the difference between the Euclidean actions of the two black holes cut off at the arbitrary radius $r_c$ greater than $2m$ because the two geometries and the two actions coincide for all $r > r_c$.

We thus write

$$N_0 = \exp[ S_e^{(B.H.)_2} - S_e^{(B.H.)_1} ].$$

To evaluate these actions we take advantage of the covariance to express them in terms of the static coordinate system (19) with $t = -it_e$. Using equations (10) and (22) and the fact that the integrand in (10) is the same at $r_c$ for $S_e^{(B.H.)_1}$ and for $S_e^{(B.H.)_2}$, we get

$$N_0 = \exp[ 4\pi m^2 - 4\pi m_0^2 (\eta \to 0) ]$$

or, as $m_0$ vanishes in the limit,

$$N_0 = \exp A/4$$

where $A = 16\pi m^2$ is the area of the event horizon of the black hole.
We have thus learned that black holes are related by quantum tunneling to another classical solution for gravity and matter, namely to a “germ black hole” of infinitesimal mass determining the spacetime topology surrounded by a static distribution of matter characterized by a vanishing $g_{00}(r)$. This domain of space-time is characterized by a limiting light-like Killing vector. When the space-time geometry presents a 4-domain endowed with such a Killing vector, we shall call the domain an “achronon”. All spherically symmetric achronon configurations will exhibit an infinite time dilation in the Schwarzschild time $t$, or equivalently massless modes emitted by the achronon are infinitely redshifted. Classically, the achronon has the “frozen” appearance of a collapse at infinite Schwarzschild time. The difference is that it is also frozen in space-time.

To see that achronons can indeed be constructed, at least in a phenomenological fluid model, we shall build a shell model with the required properties.

Let us consider a static spherically symmetric distribution of matter surrounded by an extended shell comprised between two radii $r_a$ and $r_b$. We define

$$\hat{\sigma} \equiv \int_{r_a}^{r_b} \sigma g_{11}^{1/2} dr, \quad \hat{p}_\theta \equiv \int_{r_a}^{r_b} p_\theta g_{11}^{1/2} dr, \quad \hat{p}_1 \equiv \int_{r_a}^{r_b} p_1 g_{11}^{1/2} dr$$

(27)

where $p_\theta = -T_\theta^\theta$ and $p_\phi = -T_\phi^\phi$. Assuming $p_1 = 0$, one may perform the thin shell limit $r_b \to r_a = R$ in these integrals using (21) and the Bianchi identity

$$p_\theta = p_\phi = \frac{1}{4} (\sigma + p_1) \frac{8\pi r^2 p_1 + 2M(r)/r}{1 - 2M(r)/r} + \frac{1}{2} r p_1' + p_1.$$  

(28)

One then gets

$$4\pi R \hat{\sigma} = (1 - 2m^-/R)^{3/2} - (1 - 2m/R)^{3/2}$$

(29)

$$8\pi R \hat{p}_\theta = \frac{1 - m/R}{(1 - 2m/R)^{3/2}} - \frac{1 - m^-/R}{(1 - 2m^-/R)^{3/2}}; \quad \hat{p}_1 = 0$$

(30)

where $m$ and $m^-$ are the values of $M(r)$ respectively at $r_b$ and $r_a$ and $m_s = m - m^-$ is thus the mass of the shell. Equations (29) and (30) are the standard result.[14]
As the radius $R$ approaches $2m$, these solutions become physically meaningless when $\hat{p}_\theta$ becomes greater than $\hat{\sigma}$: this violates the “dominant energy condition”\cite{15}, implying the existence of observers for which the momentum flow of the classical matter becomes spacelike; in fact, the shell is mechanically unstable even before this condition is violated\cite{16}.

The divergence of $\hat{p}_\theta$ when $R \to 2m$ appears in (30) because of the vanishing denominator in (28). Equation (30) depends however crucially on the radial pressure being zero inside the shell. Relaxing this condition it is possible to avoid all singularities of the stress tensor as $R \to 2m$ by requiring $p_1$ inside the shell to satisfy, before performing the thin shell limit,

$$4\pi r^2 p_1 + \frac{M(r)}{r} = 0.$$ \hspace{1cm} (31)

This solution is unsatisfactory if the (extended) shell sits in an arbitrary background because of the finite discontinuity of the radial pressure across the shell boundaries which would lead to singularities in $p_\theta$. We may ensure continuity of the radial pressure by immersing the shell in suitable left and right backgrounds. To avoid reintroducing stress divergences when $r^b$ approaches $2M(r^b)$ these should satisfy $(\sigma + p_1) = 0$ at the shell boundaries. One can now perform the thin shell limit. The finite discontinuity of $p_1(r)$ at $r = R$ leads now to

$$\hat{\sigma} = -\frac{\hat{\sigma}}{2}, \quad \hat{p}_1 = 0$$ \hspace{1cm} (32)

instead of (30), while $\hat{\sigma}$ is still given by (29). The dominant energy condition is now satisfied everywhere and provided the background is smooth enough in the neighbourhood of the shell, no stress divergences will appear when it approaches the Schwarzschild radius. Performing the explicit integration over the shell in the exponential term in (21), one gets for $g_{00}(r)$ in the region $0 \leq r < 2m$,

$$g_{00}(r) = (1 - \frac{2M(r)}{r}) \left[ \frac{R - 2m}{R - 2m} \right] \exp \left[ -\frac{R}{r} \int_{r}^{\infty} (\sigma + p_1) 8\pi z g_{11} \, dz \right].$$ \hspace{1cm} (34)

Here the radius $R$ of the shell is taken at $R = 2m + \eta$ where $\eta$ is a positive infinites-
imal and the symbol $\mathcal{R}$ means that the integral is carried over the regular matter contribution only. Clearly, $g_{00}(r) = O(\eta)$ for $0 \leq r < 2m$, $t$ arbitrary and the above matter distribution constitutes indeed an achronon. These phenomenological shell models are designed only to illustrate achronon properties but are hardly physically relevant as such and we shall tentatively assume that realistic field theoretic achronons do exist. This is of course a crucial issue which deserves further study.

We now relate in general the tunneling as encoded by (26) to the black hole entropy. This can be done following the analysis of the de Sitter case. Assume all states formed by achronons of mass $m$ surrounded by matter configurations of mass $M - m$, $M$ fixed, to be equally probable. This amounts here to assume the validity of the microcanonical ensemble as achronons can be viewed just as lumps of ordinary matter taken out from the surroundings. The relative probability of finding two black hole geometries for a given total mass $M$ is then given by (15) with $A^{(1)}$ and $A^{(2)}$ identified here with the black hole areas. The differential Killing identity (17) follows as before from the integrated constraint equation, the only difference being in general an additional term $\delta M$ on the right hand side arising from a surface term at spatial infinity. As $M$ is kept fixed, this term plays no role and (17) remains valid as such. Therefore, in analogy with the de Sitter case, matter configurations with neighbouring energies in a static Schwarzschild patch of an eternal black hole surrounded by matter have a Boltzmann distribution at a global temperature $T$. The latter now coincides with the local temperature at spatial infinity. $\delta A/4$ is the differential entropy of the hole and $A/4$ is the amount of entropy transferable to matter reversibly. The total black hole entropy is

$$S = A/4 + C_{B,H}. \quad (34)$$

where $\exp C_{B,H}$ measures the number of quantum states of a residual planckian black hole. The boundary condition in field space at equilibrium are such that the wave functional of an eternal black hole has a small amplitude describing an achronon configuration whose relative weight with respect to the classical black hole configuration is of order $\exp -(A/8)$.
5. From achronon to planckons

We now discuss the nature and the significance of $C_{B.H.}$. The entropy $A/4$ which can be exchanged reversibly from a black hole to ordinary matter was derived in the preceding section from the existence of a “potential barrier” between a black hole of mass $m$ and an achronon of the same mass. This was done in the context of eternal black holes admitting a Kruskal twofold symmetry, so that there are in fact two achronons imbedded in two causally disconnected static spaces. Within each space black hole-achronon states are in thermal equilibrium with their surroundings. We are therefore led to picture in such a space a quantum black hole state, in the semi-classical limit, as a quantum superposition of two coherent (normalized) states, $|B.H.\rangle$ and $|A\rangle$ representing respectively a classical black hole and a classical achronon. The relative weight of the two states is approximately, up to a phase, $\exp(-A/8)$. It follows from detailed balance at equilibrium between radiated matter and the black hole that the same superposition should hold for a the black hole who would only emit (and not receive) thermal radiation at the equilibrium temperature. As a black hole formed from collapse would asymptotically emit such a thermal flux, we infer that the collapsing black hole approaches a state $|C\rangle$ containing an achronon component with the same weight as in thermal equilibrium. We thus write

$$|C\rangle \simeq |B.H.\rangle + \exp(-A/8)|A\rangle.$$  (35)

To a classical single black hole configuration one may associate many distinct classical achronon configurations. In the shell model, for instance, there are infinitely many distinct classical matter configurations of the same total mass $m$. The argument is however much more general and infinite quantum degeneracy of the achronon is a direct consequence of the infinite time dilation. Indeed, the Hamiltonian $H_{\text{matter}}$ is of the form

$$H_{\text{matter}} = \int \sqrt{g_{00}} K(\phi_a, g_{ij}, \Pi_a) \, d^3x.$$  (36)
and all its eigenvalues are squashed towards zero by the Schwarzschild time dilation factor $\sqrt{g_{00}}$, thus generating an infinite number of orthogonal zero energy modes on top of the original achronon.

The infinity of zero energy modes around any background implies an infinite degeneracy of achronons of given mass and thus an infinity of distinct quantum black hole states of the same mass differing by the achronon component of their wave function. This infinite degeneracy of the quantum black hole provides the reservoir from which are taken the finite number of “surface” quantum states $\exp A/4$ counted by the area entropy $A/4$ transferable reversibly to outside matter.

Except for providing a rational for the large but finite testable entropy of the black hole, achronons do not modify the behaviour of large macroscopic black holes. However when their mass is reduced by evaporation and approaches the Planck mass the barrier disappears and quantum superposition completely mixes the two components. Of course, this means that both the description in terms of semiclassical configurations and of tunneling disappears. What remains however as a consequence of unitarity, is the infinity of distinct orthogonal quantum states available which have no counterpart in the finite number of decayed states. The quantum black hole has become a planckon$^{[12]}$, that is a Planckian mass object with infinite degeneracy. In terms of equation (34), this means that in a asymptotically flat background, the integration constant of the black hole entropy $C_{B.H.}$ is infinite. As discussed in the introduction, this means that in such a background a generic planckon cannot decay nor be formed in a finite time.

This conclusion however is contingent upon the validity at the qualitative level of our semi-classical approach and upon the possibility of building achronons from genuine field theory. The main question is whether or not the quantum backreaction of the matter on the metric removes the infinite degeneracy. Answering it requires further analysis.

Finally, it is of interest to note that the planckon solution to the unitarity problem posed by the evaporating black hole would have, at a fundamental level,
far reaching implications on the spectrum of quantum gravity. The opening at the Planck size of an infinite number of states, an unavoidable consequence of the existence of planckons, may appear as a horrendous complication which could make quantum gravity definitely unmanageable but hopefully the converse may be true. Indeed planckons should make quantum gravity ultraviolet finite. The Hilbert space of physical states available to macroscopic observer must be orthogonal to the infinite set of states describing Planckian bound states. Their wave function at Planckian scales where planckon configurations are concentrated are therefore expected to be vanishingly small. In this way, planckons would provide the required short distance cut-off for a consistent field theoretic description of quantum gravity within our universe while leaving the largest part of its information content hidden at the Planck scale.

An operational formulation of quantum gravity applicable within our universe and based on conventional four dimensional gravity might thus well be within reach. Nevertheless, the sudden widening of the spectrum of physical states at the Planck scale and the relative scarcity of states which describe large distance physics suggest that a fully consistent theory cannot be formulated in terms of only long range quantum fields (including the metric), and a larger scheme may be required to cope with the infinite amount of information relegated to the Planck scale.

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Figure Captions

Figure 1. Tunneling of a nonrelativistic “clock”.

The energy of the clock $E_c$ is represented by the dashed line. On the left of the turning point $a$ the clock is well represented by a forward wave depicted here by a single arrow. On the right of the turning point $b$ the amplification of the forward wave and the large concomitant backward wave are indicated.

Figure 2. Tunneling in quantum gravity.

The two solid curves represent turning hypersurfaces $\Sigma_1$ and $\Sigma_2$ separating the dark gray Euclidean region $E$ from two Minkowskian spacetimes depicted in light gray.

Figure 3. Tunneling in de Sitter topology

The heavy solid line delineates a 4-hyperboloid and the thin one a wormhole. The Euclidean domain $E$ constituted by a half 4-sphere is delineated by a dashed line. The dotted circle is the turning hypersurface $\tau = 0$.

Figure 4. The Kruskal representation of a black hole, eventually surrounded by static matter.

The dashed straight line is the Euclidean axis $T_e$ and the dashed circle is the analytic continuation in Euclidean time of the solid hyperbolae representing trajectories $r =$constant in the static patches I and III. These are separated from the dynamical regions II and IV by the horizons $r = 2m_0$ where lay the past and future singularities $r = 0$ depicted by the dashed hyperbolae. The Schwarzschild time $t$ run on opposite directions on the two hyperbolae $r =$constant and the Euclidean time $t_e$ spans the period $T^{-\infty}$ on the analytically continued circle.

Figure 5. Euclidean black hole surrounded by static matter.
Each point is a 2-sphere and the circles span the Euclidean time $t_e$. The heavy solid line is the turning hypersurface described in Kruskal time by $T = 0$.

Figure 6. Black hole tunneling.

The figure represents the Euclidean sections of the two black hole geometries $(B.H.)_1$ and $(B.H.)_2$. The $(B.H.)_1$ geometry is depicted by thick lines and the $(B.H.)_2$ geometry by thin lines in the region where it differs from the first.

The curve $a_1$ represents a turning hypersurface of $(B.H.)_1$, to be identified with $\Sigma_1$. The curve $a_2$ represents a turning hypersurface of $(B.H.)_2$. The curve $a'_2$ represents a hypersurface which lays in the intersection of the Euclidean sections of $(B.H.)_1$ and $(B.H.)_2$ and tends to $\Sigma_2$ in the limit $m_0 \to 0$. 
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