Mathematical model of the porous-capillary body in construction materials

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Abstract. Mathematical model of the process of liquid absorption by a porous-capillary body is proposed in this article. Such model is required in construction materials for theoretical description of the kinetics of liquid absorption by the building materials with porous-capillary structure. It is customary to use ordinary second order differential equation for a one-dimensional description of the motion liquid in a cylindrical capillary (Sheikin A.E., Chekhovsky Yu.V., Brusser M.I., 1979). The replacement of length by volume, it means the transition to the three-dimensional case in this equation, is wrong (Korolev E.V., Grishina A.N., Vdovin M.I., Albakasov A.I., 2016). Attempt to take into account all the physical factors of the medium absorption process by the porous-capillary body is doomed to failure in advance. In such cases it is useful at the same time apply as a probabilistic-statistical approach for analysing the results laboratory experiments, and the apparatus of differential equations for mathematical description of the physical process itself. Differential equation for description the process of liquid absorption by a porous capillary body in the three-dimensional case is taken into account the basic physical laws of the liquid penetration process into a building material with a porous capillary structure and known solutions of the required differential equation. The article presents a mathematical model of the process of liquid absorption by a capillary-porous body; it also studies the time dependence of volume mass absorption by a capillary-porous body, and proposes an analytical expression to account the deceleration process caused by appearance of resistance forces during medium absorption by a capillary-porous body.

1. Introduction

The mathematical model of the process of liquid absorption by a capillary-porous body is required for theoretical description of the kinetics of liquid absorption by building materials with a capillary-porous structure. Kinetic models are generally reduced to differential equations, describing the process under analysis.

In terms of kinetic dependences for absorption of various media, different characteristics can be calculated (particularly, using procedures from [1]), which are normally used in forecasting of material properties [2] or structural-load capacity [3] without proper justification.

The work [4] proposes a differential equation of second order, describing the motion of a liquid in a cylindrical capillary:

\[
\frac{d^2x}{dt^2} + \frac{1}{x} \left( \frac{dx}{dt} \right)^2 + \frac{8\eta}{\rho_c r^2} \frac{dx}{dt} - \frac{2}{x} \frac{\sigma}{\rho_c} + g \sin(\beta) = 0
\]  

(1)
(where $x$ – liquid column length in the capillary; $t$ – time of liquid motion in the capillary; $r$ – capillary radius; $\eta$ – liquid dynamic viscosity; $\rho_c$ – liquid density; $g$ – gravity acceleration; $r_m$ – liquid meniscus radius in the capillary; $\sigma$ – liquid surface tension; $\beta$ – capillary axis tilt angle). As one may expect, an equation of the form (1) can be applied not only to the relation of the liquid column length in the capillary, but also to the absorbed liquid volume. The work [5] proved that substitution of the length for the absorbed medium volume would lead to wrong conclusions, so an equation of such type cannot be used as a mathematical model of the process of liquid absorption by a capillary-porous body.

The work [5] puts forth an assumption that at the moment $t$ the mass of a medium, absorbed by a capillary-porous body, the porous space of which is presented by capillaries of a length divisible by the layer thickness $h$, is equal to:

$$W(t) = \frac{\rho_n \cdot \rho_m}{a_0^4} \sum_{i=1}^{i=k} \left( [a_0 - 2h(i - 1)]^3 - [a_0 - 2hi]^3 \right)$$

(2)

where $a_o$ – size of the cubic body edge; $\rho_n$ – density of air cavities (capillaries) or a volume fraction of the porous space of a capillary-porous body capable of medium absorption (effective porosity); $\rho_m$ – average density of a capillary-porous body; $h$ – selected layer thickness, where $h \ll a_o/2$; $i$ – number of the layer; $k$ – proportionality factor; $t$ – time.

2. The results of the model study

Upon completion of identity transformation of the equation (2), its simplified form can be obtained:

$$W(t) = \rho \left[1 - \left(1 - \frac{2kh}{a_0} \cdot t\right)^3\right]$$

(3)

where $\rho = \rho_m \cdot \rho_n$.

To take into account the deceleration process during medium absorption by a capillary-porous body due to appearance of resistance forces, this mode can be modified on the assumption that the rate of filling of capillary-porous body layers with the medium is non-linear, that is:

$$W(t) = \rho \left[1 - \left(1 - (K_M \cdot t)^\alpha\right)^3\right]$$

(4)

Here $K_M$ stands for a generalized parameter, characterizing certain properties of a building material sample under analysis:

$$K_M = \left(2 \cdot \frac{k \cdot h}{a_0}\right)^\frac{1}{\alpha}.$$

To determine the physical meaning of the coefficient $K_M$ please note that the volume (mass) of medium absorption by a capillary-porous body is a non-decreasing function, so the function is maximum at $t = T_{max}$. This argument (time) value corresponds to the maximum value of the function $W(t)$:

$$W(T_{max}) = \rho.$$

It follows that

$$(K_M \cdot T_{max})^\alpha = 1,$$

so the generalized parameter $K_M$ is inversely proportional to the time of maximum medium absorption by a capillary-porous body:

$$K_M = \frac{1}{T_{max}}$$

(5)

Thus,
\[ W(t) = \begin{cases} 
0, & t < 0 \\
\rho \left( 1 - \left( 1 - \left( \frac{t}{T_{\text{max}}} \right)^{a} \right)^{3} \right), & 0 \leq t \leq T_{\text{max}} \\
\rho, & t > T_{\text{max}} 
\end{cases} \]  \hspace{1cm} (6)

\( \rho \) – product of effective porosity by average density of a capillary-porous body,

\( T_{\text{max}} \) – time of maximum medium absorption by a capillary-porous body,

\( \alpha \) – non-linearity parameter, characterizing geometrical properties of the material.

Figure 1 shows diagrams of functions of type (6) for different parameters \( \alpha \) at the same values of \( \rho \) (= 25\%) and \( T_{\text{max}} \) (= 15 time units).

Figure 1. Volume mass medium absorption by a capillary-porous body at the constant value of the parameter \( \alpha \)

The proposed model modification is three-parameter, so at least three laboratory tests are required for determination of the parameters used:

\[ \begin{align*} 
W(t_1) &= W_1 \\
W(t_2) &= W_2 \\
W(t_3) &= W_3 
\end{align*} \hspace{1cm} (7) \]

If time values satisfy the inequality system:

\[ 0 < t_1 < t_2 < t_3 \leq T_{\text{max}} \hspace{1cm} (8) \]

we will have a system consisting of three non-linear equations with three unknown variables:
\[
\begin{aligned}
\rho \left( 1 - \left( \frac{t_1}{T_{\text{max}}} \right)^\alpha \right)^3 &= W_1 \\
\rho \left( 1 - \left( \frac{t_2}{T_{\text{max}}} \right)^\alpha \right)^3 &= W_2 \\
\rho \left( 1 - \left( \frac{t_3}{T_{\text{max}}} \right)^\alpha \right)^3 &= W_3
\end{aligned}
\]  

(9)

In the general case, the system (9) cannot be unambiguously solved in terms of the included unknown variables \( \alpha, \rho \) and \( T_{\text{max}} \).

From (9) it follows:

\[
\alpha = \frac{\ln t_2 - \ln t_1}{\ln t_3 - \ln t_1}, \\
\rho = \frac{W_2 - W_1}{t_2 - t_1}, \\
\left( 1 - \left( \frac{t_1}{T_{\text{max}}} \right)^\alpha \right)^3 - \left( 1 - \left( \frac{t_2}{T_{\text{max}}} \right)^\alpha \right)^3 - \left( 1 - \left( \frac{t_3}{T_{\text{max}}} \right)^\alpha \right)^3 = 0.
\]

(10)

Hence, we have two implicit equations for determination of one parameter \( \rho \) (which actually can produce two different solutions); then the parameter \( \alpha \) can be uniquely determined, and afterwards the parameter \( T_{\text{max}} \). Moreover, during determination of the parameter \( T_{\text{max}} \), there are two equations that can produce different solutions again.

To eliminate such ambiguity in the solution, it is only required to fulfill the following condition during the test:

\[ 0 < t_1 < t_2 < T_{\text{max}} < t_3 \]  

(11)

In this case, the system (11) will look as follows:

\[
\alpha = \frac{\ln \left( 1 - \left( \frac{W_2}{W_3} \right)^{1/3} \right) - \ln \left( 1 - \left( \frac{W_1}{W_3} \right)^{1/3} \right)}{\ln t_2 - \ln t_1}, \\
\rho = W_3, \\
\left( 1 - \left( \frac{t_1}{T_{\text{max}}} \right)^\alpha \right)^3 - \left( 1 - \left( \frac{t_2}{T_{\text{max}}} \right)^\alpha \right)^3 - \left( 1 - \left( \frac{t_3}{T_{\text{max}}} \right)^\alpha \right)^3 = \frac{W_2 - W_1}{W_3}.
\]

(12)

Thus, if conditions (11) from the system (7) are satisfied, the parameters \( \alpha \) and \( \rho \) can be uniquely determined, and the parameter \( T_{\text{max}} \) can be determined from the third equation reduced to a cubic one.

Figure 2 shows two diagrams plotted for solution of the third equation of the system (12), presented implicitly as \( F(T_{\text{max}}) = 0 \), where
\[ t_1 = 1; \ t_2 = 5; \ \frac{W_2 - W_1}{W_3} = 0.27; \alpha_1 = 0.45; \alpha_2 = 0.55. \]

**Figure 2.** Example of graphical solution of the third equation of the system (12)

It is interesting to note that the function \( W(t) \) satisfies the following non-homogeneous differential equation of second order:

\[
\frac{d^2W}{dt^2} + \frac{1-5\cdot\alpha}{t} \frac{dW}{dt} + 6 \left( \frac{\alpha}{t} \right)^2 \left( W(t) - \rho \left( \frac{t}{T_{\text{max}}} \right)^\alpha \right) = 0 \tag{13}
\]

with the boundary conditions

\[ W(0) = 0, \ W(T_{\text{max}}) = \rho \tag{14} \]

If \( \alpha = 1 \) the equation (13) will be as follows:

\[
\frac{d^2W}{dt^2} - \frac{4}{t} \frac{dW}{dt} + 6 \cdot W(t) - \frac{6 \cdot \rho}{T_{\text{max}} \cdot t} = 0. \tag{15}
\]

Now let’s pass over to the second modification of the model (3). The parameter \( \alpha \) should not be a constant value. In general case, \( \alpha \) is a time function specified for the interval of 0 to \( T_{\text{max}} \), and this function is non-increasing. Let us assume that this function is as follows:

\[ \alpha(t) = 1 - \left( \frac{t}{T_{\text{max}}} \right)^\beta, \ t \in [0, T_{\text{max}}] \tag{16} \]

**Figure 3** shows three functions of the analyzed type.
In this case, the model remains three-parameter:

\[
W(t) = \begin{cases} 
0, & t < 0 \\
\rho \left[1 - \left(1 - \frac{t}{T_{max}} \right)^{\frac{t}{T_{max}}} \right], & 0 \leq t \leq T_{max} \\
\rho, & t > T_{max}
\end{cases}
\]  

(17)

Figure 4 shows diagrams of functions of type (17) for different parameters $\beta$ at equal values of $\rho (= 25\%)$ and $T_{max} (= 15$ time units).

The model of type (17), when conditions (8) are satisfied for three laboratory tests, results in the following system:
\[
\begin{align*}
1-\left(\frac{t_1}{T_{\text{max}}}\right)^\beta \ln \left(\frac{t_1}{T_{\text{max}}}\right) & = \ln \left(1 - \frac{W_1}{\rho}\right)^{1/3} \\
1-\left(\frac{t_2}{T_{\text{max}}}\right)^\beta \ln \left(\frac{t_2}{T_{\text{max}}}\right) & = \ln \left(1 - \frac{W_2}{\rho}\right)^{1/3} \\
1-\left(\frac{t_3}{T_{\text{max}}}\right)^\beta \ln \left(\frac{t_3}{T_{\text{max}}}\right) & = \ln \left(1 - \frac{W_3}{\rho}\right)^{1/3}
\end{align*}
\] (18)

The system (18), like the system (9), cannot be solved uniquely in relation to the included unknown variables \(\beta, \rho\) and \(T_{\text{max}}\).

3. Conclusion
For first-approximation modeling of the process of liquid absorption by a capillary-porous body, it is preferable to use a simple non-linear, three-parameter model of the type (6):

For arrangement of a laboratory test for determination of parameters of the model (6) the conditions (11) shall be satisfied.

For selection of a refined model, it is necessary to find its parameters from the system (18).

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