Three Methods to Solve Two Classes of Integral Equations of the Second Kind

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Abstract: Three methods to solve two classes of integral equations of the second kind are introduced in this paper. Firstly, two Kantorovich methods are proposed and examined to numerically solving an integral equation appearing from mathematical modeling in biology. We use a sequence of orthogonal finite rank projections. The first method is based on general grid projections. The second one is established by using the shifted Legendre polynomials. We present a new convergence analysis results and we prove the associated theorems. Secondly, a new Nyström method is introduced for solving Fredholm integral equation of the second kind.

Key Words: Nyström method, Kantorovich method, Integral equations, Orthogonal finite rank projections.

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1. Introduction

Over the last decades, lots of important problems in applied mathematics, science, physics, engineering, biology, electrodynamics, mechanics, economics and other different fields of computer, science and engineering are written and modeled in the form of integral equations. However, there are many obstacles to directly solve these equations. Thus, we should solve these equations by using numerical methods. Recently, several numerical results have been developed for solving integral equations. Among the most approximation schemes, Kantorovich method is the most efficient.

More recently, Mennouni established an improved convergence analysis via Kulkarni method (cf. [7]) to approximate the solution of integro-differential equation in $L^2([-1, 1], \mathbb{C})$ by using the Legendre polynomials. In [8], the author introduced an efficient Galerkin method for a class of Cauchy singular integral equations of the second kind with constant coefficients in $L^2([0, 1], \mathbb{C})$, the author used a sequence of orthogonal finite rank projections. The aim of [6] is to applied the Kulkarni method and a Galerkin method for solving second kind noncompact bounded operator equations. Moreover, the author used a sequence of orthogonal finite rank projections to approximate the solution of singular integral equations of the second kind with Cauchy kernel. The main idea of [10] is to propose a collocation method for solving singular integro-differential equations with logarithmic kernel using airfoil polynomials. The goal of [9] is to numerically solve the Cauchy integro-differential equations using the projection method based on the Legendre polynomials.
In this paper, we introduce three methods to solve two classes of integral equations of the second kind. The first main idea of this work is to extend and improve the results of previous works via two Kantorovich methods for solving an integral equation arising from a problem in mathematical biology. The second one is to develop new Nyström method to solve a Fredholm integral equation of the second kind. In the first Kantorovich method we use a general grid projections. In the second one we exploit the shifted Legendre polynomials in our approach. The convergence analysis and new results are presented in this work.

2. Two Kantorovich methods for solving an integral equation appearing from mathematical modeling in biology

2.1. Integral equation appearing from mathematical modeling in biology

Let us consider the following integral equation of the second kind

\[ x(s) \int_0^1 k(s-t)dt = \int_0^1 x(\tau)k(\tau-s)d\tau + g(s), \quad 0 \leq s \leq 1, \]  

(2.1)

where \( k(\cdot, \cdot) \) is a Fredholm kernel, and \( g \) is a known function.

Equation (2.1) reads as

\[ x(s) - \int_0^1 x(\tau)k(\tau-s)d\tau = f(s), \quad 0 \leq s \leq 1, \]  

(2.2)

where

\[ f(s) := \frac{g(s)}{\int_0^1 k(s-t)dt}. \]

Define the integral operator \( T \):

\[ Tx(s) := \int_0^1 x(\tau)k(\tau-s)d\tau / \int_0^1 k(s-t)dt, \quad 0 \leq s \leq 1. \]

Set \( \mathcal{H} := L^2[0,1] \). Suppose that \( k \in L^1[0,1] \), \( k > 0 \) almost everywhere. We recall that for each \( f \in \mathcal{H} \), \( T \) is compact from \( \mathcal{H} \) into itself, (see [4]). Hence, the integral equation (2.1) has a unique solution \( x \in \mathcal{H} \).

Let \( I \) denote the identity operator on \( \mathcal{H} \). Eq. (2.2) can be rewritten in operator form as follows:

\[ (I - T) x = f. \]

The purpose of this work is to approximate \( x \) through the solution \( x_n \) of the Kantorovich equation

\[ (I - \pi_n T) x_n = f. \]  

(2.3)

2.2. Projection approximations using general grids

Let \( (s_{n,j})_{j=0}^n \) be a grid on \([0,1]\) such that

\[ 0 < s_{n,0} < s_{n,1} < \ldots < s_{n,n} < 1. \]

Set

\[ h_{n,i} := s_{n,i} - s_{n,i-1}, \quad i \in \llbracket 1, n \rrbracket, \quad h_n := (h_{n,1}, h_{n,2}, \ldots, h_{n,n}). \]

Let us consider \((\pi_n)_{n \geq 1}\), a sequence of bounded projections each one of finite rank, such that

\[ \pi_n x := \sum_{j=1}^n \langle x, e_{n,j} \rangle e_{n,j}, \]
where
\[ e_{n,j} := \frac{\phi_{n,j}}{\sqrt{h_{n,j}}}, \quad \phi_{n,j}(s) := \begin{cases} 1 & \text{for } s \in (s_{n,j-1}, s_{n,j}], \\ 0 & \text{otherwise}. \end{cases} \]

Let
\[ J_n := \{s_{n,j}, \quad j \in [0, n]\}. \]

Define the modulus of continuity of the function \( \psi \in \mathcal{H} \) relative to \( h_n \) as follows:
\[ \omega_2(\psi, J_n) := \sup_{0 \leq \delta \leq h_n} \left( \int_0^1 |\psi(\tau + \delta) - \psi(\tau)|^2 d\tau \right)^{1/2}. \]

All functions are extended by 0 outside \([0, 1]\). We recall that
\[ \omega_2(\psi, J_n) \to 0 \text{ as } n \to \infty \text{ for all } \psi \in \mathcal{H}, \]
and that, for all \( \psi \in \mathcal{H} \) (cf. [2]),
\[ \|(I - \pi_n)\psi\| \leq \omega_2(\psi, J_n). \quad (2.4) \]

### 2.3. First Kantorovich method via general grids

We have
\[ \pi_n T x := \sum_{j=1}^{n} \langle Tx, e_{n,j} \rangle e_{n,j}. \]

Applying \( T \) to both sides of equation (2.3), and performing the inner product with \( e_{n,i} \) to both sides of this equation, we get
\[ \langle Tx_n, e_{n,i} \rangle - \sum_{j=1}^{n} \langle Tx_n, e_{n,j} \rangle \langle Te_{n,j}, e_{n,i} \rangle = \langle Tf, e_{n,i} \rangle, \]

or, equivalently,
\[ (I_n - A_n)X_n = b_n, \quad (2.5) \]

where
\[ X_n(j) := \langle Tx_n, e_{n,j} \rangle, \]

and
\[ A_n(i, j) := \langle Te_{n,j}, e_{n,i} \rangle, \]
\[ b_n(i) := \langle Tf, e_{n,i} \rangle. \]

Hence
\[ A_n(i, j) := \frac{1}{\sqrt{h_{n,j} h_{n,i}}} \int_{s_{i-1}}^{s_i} \int_{s_{j-1}}^{s_j} k(\tau - s) \, d\tau \, ds, \]
\[ b_n(i) := \frac{1}{\sqrt{h_{n,i}}} \int_{s_{i-1}}^{s_i} \int_0^1 f(\tau) k(\tau - s) \, d\tau \, ds. \]
2.4. Convergence analysis of the first Kantorovich method

For all \( x \in \mathcal{H} \),

\[
\lim_{n \to \infty} \|\pi_n T x - T x\| = 0,
\]

and since \( T \) is compact,

\[
\lim_{n \to \infty} \| (\pi_n T - T) T \| = 0, \quad \lim_{n \to \infty} \| (\pi_n T - T) \pi_n T \| = 0.
\]

**Theorem 2.1.** There exists a positive constant \( M \), such that

\[
\| x_n - x \| \leq M [\omega_2(x, J_n) + \omega_2(f, J_n)].
\]

**Proof.** In fact

\[
\pi_n x = \pi_n T x + \pi_n f.
\]

Since

\[
x - \pi_n x = x - x_n + x_n - \pi_n x
= x - x_n + (\pi_n T x_n + f) - (\pi_n T x + \pi_n f)
= x - x_n + \pi_n T (x_n - x) + (I - \pi_n) f
= (I - \pi_n T) (x - x_n) + (I - \pi_n) f.
\]

Hence

\[
x - x_n = (I - \pi_n T)^{-1} [(I - \pi_n) x - (I - \pi_n) f],
\]

and since \( T \) is compact, the

\[
M := \sup_{n \geq N} \| (I - \pi_n T)^{-1} \|,
\]

is finite. Using (2.4), we get the desired result. \( \Box \)

2.5. Second Kantorovich method via shifted Legendre polynomials

The aim of this section is to use Kantorovich method for solving (2.1) via shifted Legendre polynomials. For this purpose, let \((L_n)_{n \geq 0}\) denote the sequence of Legendre polynomials. The Shifted Legendre Polynomial \(\tilde{L}_n(s)\) is defined as

\[
\tilde{L}_n(s) := L_n(2s - 1).
\]

Let us consider

\[
\tilde{e}_{n,j} := \sqrt{2j + 1} \tilde{L}_j,
\]

the corresponding normalized sequence. Let \((\pi_n)_{n \geq 0}\) be the sequence of bounded finite rank orthogonal projections defined by

\[
\pi_n x := \sum_{j=0}^{n-1} \langle x, \tilde{e}_{n,j} \rangle \tilde{e}_{n,j}.
\]

Hence, for \( y \in \mathcal{H} \),

\[
\lim_{n \to \infty} \|\pi_n y - y\| = 0.
\]

We recall that (cf. [3]) there exists \( C > 0 \) such that, for all \( y \in H^r([0, 1], \mathbb{C}) \),

\[
\| (I - \pi_n) y \| \leq C n^{-r} \| y \|_r. \tag{2.6}
\]

It follows from (2.3) that

\[
(I_n - \tilde{A}_n) \tilde{X}_n = \tilde{b}_n,
\]
where

\[ A_n(i, j) := \sqrt{2j + 1}\sqrt{2i + 1} \int_0^1 \int_0^1 L_j(\tau)k(\tau - s) \frac{d\tau}{\int_0^1 k(s - t) dt} \tilde{L}_i(s) ds dt, \]

\[ b_n(i) := \sqrt{2i + 1} \int_0^1 \int_0^1 f(\tau)k(\tau - s) \frac{d\tau}{\int_0^1 k(s - t) dt} \tilde{L}_i(s) ds dt. \]

Once the above system is solved, \( x_n \) is recovered as

\[ x_n = \sum_{j=0}^{n-1} X_n(j) \sqrt{2j + 1} \tilde{L}_j + f. \]

2.6. Convergence analysis of second Kantorovich method

**Theorem 2.2.** Assume that \( f \in H^r([0, 1], \mathbb{C}) \) for some \( r > 0 \). Then, there exists \( \alpha > 0 \) such that

\[ \frac{\|x_n - x\|}{\|x\|_r} \leq \alpha \|T\|n^{-r}. \]

**Proof.** We have

\[ x_n - x = \left[ (I - \pi_n T)^{-1} f - (I - T)^{-1} f \right] \]

\[ = (I - \pi_n T)^{-1} [ (\pi_n - I) Tx ], \]

and hence

\[ \|x_n - x\| \leq M C_0 n^{-r} \|Tx\|_r, \]

for some positive constant \( C_0 \), so that

\[ \frac{\|x_n - x\|}{\|x\|_r} \leq \alpha \|T\|n^{-r}, \quad \alpha := MC_0. \]

\[ \square \]

**Theorem 2.3.** Assume that \( f \in H^r([0, 1], \mathbb{C}) \) for some \( r > 0 \). Then, there exists \( \beta > 0 \) such that

\[ \|x_n - x\| \leq \beta n^{-r} [\|x\|_r + \|f\|_r] \]

**Proof.** Recall that

\[ \pi_n x = \pi_n Tx + \pi_n f. \]

As above and as in [1], we have

\[ x - \pi_n x = (I - \pi_n T)(x_n - x) + (I - \pi_n) f. \]

Hence

\[ x_n - x = (I - \pi_n T)^{-1} [ (I - \pi_n) x + (\pi_n - I) f ], \]

using (2.6), we get

\[ \|x_n - x\| \leq M n^{-r} [C_1 \|x\|_r + C_2 \|f\|_r], \]

for some positive constants \( C_1, C_2 \), as we wanted to prove.

\[ \square \]
3. A new Nyström method for solving Fredholm integral equation

3.1. Nyström method

Let us consider the following Fredholm integral equation of the second kind

\[ u(s) - \int_{-1}^{1} k(s, t) u(t) dt = f(s), \quad s \in I := [-1, 1], \tag{3.1} \]

We recall the quadrature formula introduced in [11] as follows

\[ \int_{-1}^{1} h(s) ds \approx \sum_{\nu=1}^{n} \omega_{\nu} h(\tau_{\nu}) + \omega_{e}(+) h(\tau_{e}) + \omega_{e}(-) h(-\tau_{e}), \]

where

\[ \tau_{\nu} = \cos \theta_{\nu}, \quad \theta_{\nu} = \frac{\nu}{n+1} \pi, \quad \nu = 1, 2, \ldots, n \]

\[ \pm \tau_{e} = \pm \cos \theta_{e}, \quad \theta_{e} = \frac{\pi}{2(n+1)}. \]

In this section, we introduce a new Nyström method for solving the integral equation (3.1)

\[ \dot{u}(s) - \left[ \omega_{e}^{(+)} k(s, \tau_{e}) \dot{u}(\tau_{e}) + \sum_{\nu=1}^{n} \omega_{\nu} k(s, \tau_{\nu}) \dot{u}(\tau_{\nu}) + \omega_{e}(-) k(s, -\tau_{e}) \dot{u}(-\tau_{e}) \right] = f(s). \tag{3.2} \]

Collocating (3.2) at \( \tau_{i} \) for \( i = 1, \ldots, n \), we get the following linear system

\[ \dot{u}(\tau_{i}) - \left[ \omega_{e}^{(+)} k(\tau_{i}, \tau_{e}) \dot{u}(\tau_{e}) + \sum_{\nu=1}^{n} \omega_{\nu} k(\tau_{i}, \tau_{\nu}) \dot{u}(\tau_{\nu}) + \omega_{e}(-) k(\tau_{i}, -\tau_{e}) \dot{u}(-\tau_{e}) \right] = f(\tau_{i}), \]

that is

\[ \dot{u}(\tau_{i}) - \sum_{\nu=0}^{n+1} \omega_{\nu} k(\tau_{i}, \tau_{\nu}) \dot{u}(\tau_{\nu}) = f(\tau_{i}), \tag{3.3} \]

with

\[ \omega_{0} = \omega_{e}^{(+)}, \quad \tau_{0} = \tau_{e}, \]

and

\[ \omega_{n+1} = \omega_{e}(-), \quad \tau_{n+1} = -\tau_{e}, \]

which is a system of \( n + 2 \) linear equations with the unknown

\[ \dot{u}_{n} := [\dot{u}(\tau_{0}), \dot{u}(\tau_{1}), \ldots, \dot{u}(\tau_{n+1})], \]

\[ \dot{f}_{n} := [f(\tau_{0}), f(\tau_{1}), \ldots, f(\tau_{n+1})], \]

and

\[ \dot{A}_{n} := \dot{A}_{n}(i, \nu) = \omega_{\nu} k(\tau_{i}, \tau_{\nu}). \]

Hence the following linear system

\[ (I - \dot{A}_{n}) \dot{u}_{n} = \dot{f}_{n}. \]
3.2. Explicit linear system

The corresponding explicit formulae for the weights of formula is presented in [11] as follows

\[
\omega_\nu = \frac{2}{n+1} \left\{ 1 - 2 \sum_{k=1}^{[(n+1)/2]} \frac{\cos 2k\theta_\nu}{4k^2-1} - \frac{\cos 2[(n+1)/2]\theta_\nu}{2[(n+1)/2] + 1} \right\},
\]

hence

\[
\omega_\nu = \frac{2}{n+1} \left\{ 1 - 2 \sum_{k=1}^{[(n+1)/2]} \cos 2k\theta_\nu \cos 2[(n+1)/2]\theta_\nu \right\}, \quad \nu = 1, 2, \ldots, n,
\]

where

\[
\omega_\nu = \frac{4\sin \theta_\nu}{n+1} \sum_{k=1}^{[(n+1)/2]} \frac{\sin(2k-1)\theta_\nu}{2k-1}, \quad \nu = 1, 2, \ldots, n.
\]

Moreover, the author of [11] proved that the weights \( \overline{\omega}_\nu, \overline{\omega}_\nu^+ \) and \( \overline{\omega}_\nu^- \) are given by

\[
\overline{\omega}_\nu = \omega_\nu + \frac{2\sin^2 \theta_\nu \cos 2[(n+1)/2]\theta_\nu}{(2[n/2] + 1)(2[(n+1)/2] + 1) \sin(\theta_\nu + \theta_c) \sin(\theta_\nu - \theta_c)},
\]

with \( \nu = 1, 2, \ldots, n, \)

\[
\overline{\omega}_\nu^+ = \overline{\omega}_\nu^- = \begin{cases} 
\sin \theta_c \\ \frac{(n+1)\tan \theta_c}{n(n+2)} 
\end{cases}, \quad \text{if } n \text{ is even},
\]

Equation (3.3) reads as

\[
\hat{u}(\tau_i) - \sum_{\nu=1}^{n} \left[ \omega_\nu + \frac{2\sin^2 \theta_\nu \cos 2[(n+1)/2]\theta_\nu}{(2[n/2] + 1)(2[(n+1)/2] + 1) \sin(\theta_\nu + \theta_c) \sin(\theta_\nu - \theta_c)} \right] k(\tau_i, \tau_\nu) \hat{u}(\tau_\nu) \\
- \frac{\sin \theta_c}{n+1} [k(\tau_i, \tau_c)\hat{u}(\tau_c) + k(\tau_i, -\tau_c)\hat{u}(-\tau_c)] \\
= f(\tau_i) \quad \text{for } n \text{ even,}
\]

and

\[
\hat{u}(\tau_i) - \sum_{\nu=1}^{n} \left[ \omega_\nu + \frac{2\sin^2 \theta_\nu \cos 2[(n+1)/2]\theta_\nu}{(2[n/2] + 1)(2[(n+1)/2] + 1) \sin(\theta_\nu + \theta_c) \sin(\theta_\nu - \theta_c)} \right] k(\tau_i, \tau_\nu) \hat{u}(\tau_\nu) \\
- \frac{(n+1)\tan \theta_c}{n(n+2)} [k(\tau_i, \tau_c)\hat{u}(\tau_c) + k(\tau_i, -\tau_c)\hat{u}(-\tau_c)] \\
= f(\tau_i) \quad \text{for } n \text{ odd.}
\]

References

1. M. Ahues, F.D. d’Almeida, R.R. Fernandes, Piecewise Constant Galerkin Approximations of Weakly Singular Integral Equations, International Journal of Pure and Applied Mathematics, 55, (2009), 569–580.
2. A. Amosov, M. Ahues, A. Largillier, Superconvergence of Some Projection Approximations for Weakly Singular Integral Equations Using General Grids, SIAM Journal on Numerical Analysis, 47, (2008), 646–674.
3. K.E. Atkinson, Han, Theoretical Numerical Analysis: A Functional Analysis Framework, 3rd edition Springer-Verlag, 2009.
4. S. Eveson, An integral equation arising from a problem in mathematical biology, Bull Lon Math Soc. 23 (1991) 293–299.
5. R. Kulkarni, A Superconvergence Result For Solutions of Compact Operator Equations, Bull. Austral. Math. Soc. 68 (2003) pp. 517-528.
6. A. Mennouni, Two Projection Methods for Skew-Hermitian Operator Equations, Mathematical and Computer Modelling, 55 (2012) pp. 1649-1654.

7. A. Mennouni, Improvement by projection for integro-differential equations. submitted to J. Computational Applied Mathematics.

8. A. Mennouni, Piecewise constant Galerkin method for a class of Cauchy singular integral equations of the second kind in $L^2$. J. Computational Applied Mathematics, 326 (2017), 268–272.

9. A. Mennouni, A projection method for solving Cauchy singular integro-differential equations, Applied Mathematics Letters 25, (2012) 986-989.

10. A. Mennouni, Airfoil polynomials for solving integro-differential equations with logarithmic kernel, Applied Mathematics and Computation 218 (2012) 11947-11951.

11. S. E. Notaris, On a corrected Fejér quadrature formula of the second kind, Numerische Mathematik 133(2): 279-302 (2016).

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