Efficient Coalgebraic Partition Refinement

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Abstract

We present a generic partition refinement algorithm that quotients coalgebraic systems by behavioural equivalence, an important task in reactive verification; coalgebraic generality implies in particular that we cover not only classical relational systems but also various forms of weighted systems. Under assumptions on the type functor that allow representing its finite coalgebras in terms of nodes and edges, our algorithm runs in time $O(m \cdot \log n)$ where $n$ and $m$ are the numbers of nodes and edges, respectively. Instances of our generic algorithm thus match the runtime of the best known algorithms for unlabelled transition systems, Markov chains, and deterministic automata (with fixed alphabets), and improve the best known algorithms for Segala systems.

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1 Introduction

Minimization under bisimilarity is the task of identifying all states in a reactive system that exhibit the same behaviour. Minimization appears as a subtask in state space reduction (e.g. [7]) or non-interference checking [37]. The notion of bisimulation was first defined for relational systems [36, 27, 29]; it was later extended to other system types including probabilistic systems [26, 12] and weighted automata [8]. In fact, the importance of minimization under bisimilarity appears to increase with the complexity of the underlying system type. E.g., while in LTL model checking, minimization drastically reduces the state space but, depending on the application, does not necessarily lead to a speedup in the overall balance [14], in probabilistic model checking, minimization under strong bisimilarity does lead to substantial efficiency gains [22].

The algorithms of minimization, often referred to as partition refinement or lumping, has received a fair amount of attention. Since bisimilarity is a greatest fixpoint, it is more or less immediate that it can be calculated in polynomial time by approximating this fixpoint from above following Kleene’s fixpoint theorem. In the relational setting, Kanellakis and Smolka [21] introduced an algorithm that in fact runs in time $O(nm)$ where $n$ is the number of nodes and $m$ is the number of transitions. An even more efficient algorithm running in time $O(m \log n)$ was later described by Paige and Tarjan [28]; this bound holds even if the number of action labels is not fixed [34]. Current algorithms typically apply further optimizations to the Paige-Tarjan algorithm, thus achieving better average-case behaviour but the same worst-case behaviour [13]. Probabilistic minimization has undergone a similarly dynamic development [5, 10, 39], and the best algorithms for minimization of Markov chains now have the same $O(m \log n)$ run time as the relational Paige-Tarjan algorithm [18, 11, 35].

Using ideas from abstract interpretation, Ranzato and Tapparo [30] have developed a
relational partition refinement algorithm that is generic over notions of process equivalence. As instances, they recover the classical Paige-Tarjan algorithm for strong bisimilarity and an algorithm for stuttering equivalence, and obtain new algorithms for simulation equivalence and for a new process equivalence.

In this paper we follow an orthogonal approach and provide a generic partition refinement algorithm that can be instantiated for many different types of systems (e.g., nondeterministic, probabilistic, weighted). We achieve this by methods of universal coalgebra [31]. That is, we encapsulate transition types of systems as endofunctors on sets (or a more general category), and model systems as coalgebras for a given type functor.

Our work proceeds on several levels of abstraction. On the most abstract level (Section 3) we work with coalgebras for a monomorphism-preserving endofunctor on a category with image factorizations. Here we present a quite general category-theoretic partition refinement algorithm, and we prove its correctness. The algorithm is parametrized over a select routine that determines which observations are used to split blocks of states; the corner case where all available observations are used yields known coalgebraic final chain algorithms, e.g. [25].

Next, we present an optimized version of our algorithm (Section 4) that needs more restrictive conditions to ensure correctness; specifically, we need to assume that the type endofunctor satisfies a condition we call zippability in order to allow for incremental computation of partitions. This property holds, e.g., for all polynomial endofunctors on sets and for the type functors of labelled and weighted transition systems, but not for all endofunctors of interest. In particular, zippable functors fail to be closed under composition, as exemplified by the double covariant powerset functor \(PP\) on sets, for which the optimized algorithm is in fact incorrect. However, it turns out that obstacles of this type can be removed by moving to multi-sorted coalgebras [32], so we do eventually obtain an efficient partition refinement algorithm for coalgebras of composite functors, including \(PP\)-coalgebras as well as (probabilistic) Segala systems [33].

Finally, we analyse the run time of our algorithm (Section 5). To this end, we make our algorithm parametric in an abstract refinement interface to the type functor, which encapsulates the incremental calculation of partitions in the optimized version of the algorithm. We show that if the interface operations can be implemented in linear time, then the algorithm runs in time \(O(m \log n)\), where \(n\) is the number of states and \(m\) the number of ‘edges’ in a syntactic encoding of the input coalgebra. We thus recover the most efficient known algorithms for transition systems (Paige and Tarjan [28]) and for weighted systems (Valmari and Franceschinis [35]). Using the mentioned modularity results, we also obtain an \(O((m + n) \log(m + n))\) algorithm for Segala systems, to our knowledge a new result (more precisely, we improve an earlier bound established by Baier, Engelen, and Majster-Cederbaum [5], roughly speaking by letting only non-zero probabilistic edges enter into the time bound). The algorithm and its analysis apply also to generalized polynomial functors on sets; in particular, for the functor \(2 \times (-)^A\), which models deterministic finite automata, we obtain the same complexity \(O(n \log n)\) as for Hopcroft’s classical minimization algorithm for a fixed alphabet \(A\) [17, 24, 15].

2 Preliminaries

We assume that readers are familiar with basic category theory [3]. For the convenience of the reader we recall some concepts that are central for the categorical version of the algorithm.

**Notation 2.1.** The terminal object is denoted by 1, with unique arrows \(!: A \to 1\), and the product of objects \(A, B\) by \(A \to B \to \text{A} \times \text{B}\). Given \(f : D \to A\) and \(g : D \to B\), the morphism induced by the universal property of the product \(A \times B\) is denoted by
\( \{f, g\} : D \to A \times B \). The kernel \( \ker f \) of a morphism \( f \) is the pullback of \( f \) along itself. We write \( \rightarrow \) for regular epimorphisms (i.e. coequalizers), and \( \Rightarrow \) for monomorphisms.

Kernels allow us to talk about equivalence relations in a category. In particular in \( \text{Set} \), there is a bijection between kernels and equivalence relations in the usual sense: For a map \( f : D \to A \), \( \ker f = \{(x, y) \mid fx = fy\} \) is the equivalence relation induced by \( f \). Generally, relations (i.e. jointly monic spans of morphisms) in a category are ordered by inclusion in the obvious way. We say that a kernel \( K \) is finer than a kernel \( K' \) if \( K \) is included in \( K' \).

We use intersection \( \cap \) and union \( \cup \) of kernels for meets and joins in the inclusion ordering on relations (not equivalence relations or kernels); in this notation, \( \ker \{f, g\} = \ker f \cap \ker g \). In \( \text{Set} \), a map \( f : D \to A \) factors through its kernel, via the map \( [-]_f : D \to D/\ker f \) taking equivalence classes

\[ [x]_f := \{x' \in D \mid fx = fx'\} = \{x' \in D \mid (x, x') \in \ker f\} \]

Well-definedness of functions on \( \text{image} \) is determined precisely by the universal property of \( [-]_f \) as a coequalizer of \( \ker f \Rightarrow D \). In particular, \( f \) induces an injection \( D/\ker f \to A \); together with \( [-]_f \), this is the factorization of \( f \) into a regular epimorphism and a monomorphism. Categorically, this is captured by the following assumptions.

**Assumption 2.2.** We assume throughout that \( \mathcal{C} \) is a finitely complete category that has coequalizers and image factorizations, i.e. every morphism \( f \) has a factorization \( f = m \cdot e \) as a regular epimorphism \( e \) followed by a monomorphism \( m \). We call the codomain of \( e \) the image of \( f \), and denote it by \( D/\ker f \). Regular epis in \( \mathcal{C} \) are closed under composition and right cancellation [3, Prop. 14.14].

**Examples 2.3.** Examples of categories satisfying Assumption 2.2 abound. In particular, every regular category with coequalizers satisfies our assumptions. The category \( \text{Set} \) of sets and maps is, of course, regular. Every topos is regular, and so is every finitary variety (e.g. monoids, groups, vector spaces etc.). Posets and topological spaces fail to be regular but still satisfy our assumptions. The category of algebras for a finitary signature satisfying given equational axioms (e.g. monoids, \( \mathcal{C} \) is a regular category with coequalizers and \( \mathcal{C} \) satisfies our assumptions. The category \( \mathcal{C} \) of algebras for a finitary signature satisfying given equational axioms (e.g. monoids, groups, vector spaces etc.). Posets and topological spaces fail to be regular but still satisfy our assumptions. For a set \( S \) of sorts, the category \( \text{Set}^S \) of \( S \)-sorted sets has \( S \)-tuples of sets as objects. We write \( \chi_S : X \to 2 \) for the characteristic function of a subset \( S \subseteq X \), i.e. for \( x \in X \) we have \( \chi_S(x) = 1 \) if \( x \in S \) and \( \chi_S(x) = 0 \) otherwise. We will also use a three-valued version:

**Definition 2.4.** For \( S \subseteq C \subseteq X \), define \( \chi_S^C : X \to 3 \) by \( \chi_S^C(x) = 0 \) if \( x \in C \) but \( \chi_S^C(x) = 1 \) if \( x \in S \setminus C \) and \( \chi_S^C(x) = 2 \). (This is essentially \( \langle \chi_S, \chi_C \rangle : X \to 4 \) without the impossible case \( x \in S \setminus C \)).

**Coalgebras.** We briefly recall basic notions from coalgebra. For introductory texts, see [31, 20, 1, 19]. Given an endofunctor \( H : \mathcal{C} \to \mathcal{C} \), a coalgebra is pair \((C, c)\) where \( C \) is an object of \( \mathcal{C} \) called the carrier and thought of as an object of states, and \( c : C \to HC \) is a morphism called the structure of the coalgebra. Our leading examples are the following.

**Example 2.5.** 1. Labelled transition systems with labels from a set \( A \) are coalgebras for the functor \( HX = \mathcal{P}(A \times X) \) (and unlabelled transition systems are simply coalgebras for \( \mathcal{P} \)). Explicitly, a coalgebra \( c : C \to HC \) assigns to each state \( x \) a set \( c(x) \subseteq \mathcal{P}(A \times X) \), and this represents the transition structure at \( x \): it has an \( a \)-transition to \( y \) if \( (a, y) \in c(x) \).

2. Weighted transition systems with weights drawn from a commutative monoid are modelled as coalgebras as follows. For the given commutative monoid \((M, +, 0)\), we consider the monoid-valued functor \( M^{(-)} \) on \( \text{Set} \) given for any map \( h : X \to Y \) by

\[
M^{(X)} = \{f : X \to M \mid f(x) \neq 0 \text{ for finitely many } x\}, \quad M^{(h)}(f)(y) = \sum_{h(x)=y} f(x).
\]
$M$-weighted transition systems are in bijective correspondence with coalgebras for $M^{(-)}$ [16] (and for $M$-weighted labelled transition systems one takes $(M^{(-)})^k$).

3. Probabilistic transition systems are modelled coalgebraically using the distribution functor $D$. This is the subfunctor $DX \subseteq \mathbb{R}_{\geq 0}^X$, where $\mathbb{R}_{\geq 0}$ is the monoid of addition on the non-negative reals, given by $DX = \{ f \in \mathbb{R}_{\geq 0}^X \mid \sum_{x \in X} f(x) = 1 \}$.

4. The finite powerset functor $P_l$ is a monoid-valued functor for the Boolean monoid $\mathbb{B} = (2, v, 0)$. The bag functor $B_l$, which assigns to a set $X$ the set of bags (i.e., finite multisets) on $X$, is the monoid-valued functor for the additive monoid of natural numbers.

5. Simple (resp. general) Segala systems [33] strictly alternate between non-deterministic and probabilistic transitions; they can be modeled as coalgebras for the set functor $P_l(A \times D(-))$ (resp. $P_lD(A \times -)$).

A coalgebra morphism from a coalgebra $(C, c)$ to a coalgebra $(D, d)$ is a morphism $h : C \to D$ such that $d \cdot h = Hh \cdot c$; intuitively, coalgebra morphisms preserve observable behaviour. Coalgebras and their morphisms form a category $\text{Coalg}(H)$. The forgetful functor $\text{Coalg}(H) \to C$ creates all colimits, so $\text{Coalg}(H)$ has all colimits that $C$ has.

A subcoalgebra of a coalgebra $(C, c)$ is represented by a coalgebra morphism $m : (D, d) \to (C, c)$ such that $m$ is a monomorphism in $C$. Likewise, a quotient of a coalgebra $(C, c)$ is represented by a coalgebra morphism $q : (C, c) \to (D, d)$ carried by a regular epimorphism $q$ of $C$. If $H$ preserves monomorphisms, then the image factorization structure on $C$ lifts to coalgebras.

**Definition 2.6.** A coalgebra is *simple* if it does not have any non-trivial quotients.

Equivalently, a coalgebra $(C, c)$ is simple if every coalgebra morphism with domain $(C, c)$ is carried by a monomorphism. Intuitively, in a simple coalgebra all states exhibiting the same observable behaviour are already identified. This paper is concerned with the design of algorithms for computing the simple quotient of a given coalgebra:

**Lemma 2.7.** The simple quotient of a coalgebra is unique (up to isomorphism).

Intuitively speaking, two elements (possibly in different coalgebras) are called behaviourally equivalent if they can be identified by coalgebra morphisms. Hence, the simple quotient of a coalgebra is its quotient modulo behavioural equivalence. In our main examples, this means that we minimize w.r.t. standard bisimilarity-type equivalences.

**Example 2.8.** Behavioural equivalence instantiates to various notions of bisimilarity:

1. Park-Milner bisimilarity on labelled transition systems;
2. weighted bisimilarity on weighted transition systems [23, Proposition 2];
3. stochastic bisimilarity on probabilistic transition systems [23];
4. Segala bisimilarity on simple and general Segala systems [6, Theorem 4.2].

## 3 A Categorical Algorithm for Behavioural Equivalence

We proceed to describe a categorical partition refinement algorithm that computes the simple quotient of a given coalgebra under fairly general assumptions.

**Assumption 3.1.** Assume that $H$ is an endofunctor on $C$ that preserves monomorphisms.

Note that mono preservation is w.l.o.g. for $C = \text{Set}$. Roughly, for a given coalgebra $\xi : X \to HX$ in $\text{Set}$, a partition refinement algorithm maintains a quotient $q : X \to X/Q$ that distinguishes some (but possibly not all) states with different behaviour, and in fact, initially $q$ typically identifies everything. The algorithm repeats the following steps:
1. Gather new information on which states should become separated by using $X \xrightarrow{\xi} HX \xrightarrow{Hq} HX/Q$, i.e., by identifying equivalence classes under $q$ that contain states whose behaviour is observed to differ under one more step of the transition structure $\xi$.

2. Use parts of this information to refine $q$ and repeat until $q$ does not change any more.

One of the core ideas of the Paige-Tarjan partition refinement algorithm [28] is to not use all information immediately in the second step. Recall that the algorithm maintains two partitions $Y$ and $Z$ of the state set $X$ of the given transition system; the elements of $Y$ are called subblocks and the elements of $Z$ are called compound blocks. The partition $Y$ is a refinement of the partition $Z$. The key to the time efficiency of the algorithm is to select in each iteration a subblock that is at most half of the size of the compound block it belongs to. At the present high level of generality (which in particular does not know about sizes of objects), we encapsulate the subblock selection in a routine select, assumed as a parameter to our algorithm:

**Definition 3.2.** A select routine is an operation that receives a chain of two regular epis $X \xrightarrow{v} Y \xrightarrow{z} Z$ and returns some morphism $k : Y \to K$ into some object $K$. We call $Y$ the subblocks and $Z$ the compound blocks.

The idea is that the morphism $k$ throws away some of the information provided by the refinement $Y$. For example, in the Paige-Tarjan algorithm it models the selection of one compound block to be split in two parts, which then induce the further refinement of $Y$.

**Example 3.3.** 1. In the classical Paige-Tarjan algorithm [28], i.e., for $C = \text{Set}$, one wants to find a proper subblock that is at most half of the size of the compound block it sits in. So let $S \in Y$ such that $2 \cdot |y^{-1}([S])| \leq |(zy)^{-1}([z(S)])|$. Here, $z(S)$ is the compound block containing $S$. Then we let select$(z,y)$ be $k : Y \to 3$ given by $k(x) = 2$ if $x = S$, else $k(x) = 1$ if $z(x) = z(S)$, and $k(x) = 0$ otherwise; i.e. $k = \chi_{S}$ (Definition 2.4). If $Y$ and $Z$ are encoded as partitions of $X$, then $S$ and $C := z(S)$ are subsets of $X$ and $k \cdot y = \chi_{S}$. If there is no such $S \in Y$, then $z$ is bijective, i.e., there is no compound block from $Z$ that needs to be refined. In this case, $k$ does not matter and we simply put $k = 1 : Y \to 1$.

2. One obvious choice for $k$ is to take the identity on $Y$, so that all of the information present in $Y$ is used for further refinement. We will discuss this in Remark 3.12.

3. Two other, trivial, choices are $k = 1 : Y \to 1$ and $k = z$. Since both of these choices provide no extra information, this will leave the partitions unchanged, see Lemma 3.14.

Given a select routine, the most general form of our partition refinement works as follows.

**Algorithm 3.4.** Given a coalgebra $\xi : X \to HX$, we successively refine equivalence relations $Q$ and $P$ on $X$, maintaining the invariant that $P$ is finer than $Q$. In each step, we take into account new information on the behaviour of states, represented by a map $q : X \to K$, and accumulate this information in a map $\bar{q} : X \to \bar{K}$. To facilitate the analysis, these variables are indexed over loop iterations in the description. Initial values are

$$Q_{0} = X \times X \quad q_{0} = 1 : X \to 1 = K_{0} \quad P_{0} = \ker(X \xrightarrow{\xi} HX \xrightarrow{H1} H1).$$

We then iterate the following steps while $P_{i} \neq Q_{i}$, for $i \geq 0$:

1. $X/P_{i} \xrightarrow{k_{i+1}} K_{i+1} := \text{select}(X \to X/P_{i} \to X/Q_{i})$, using that $X/P_{i}$ is finer than $X/Q_{i}$

2. $q_{i+1} := X \xrightarrow{q_{i}} X/P_{i} \xrightarrow{k_{i+1}} K_{i+1} \quad \bar{q}_{i+1} := (\bar{q}_{i},q_{i+1}) : X \to \bar{K}_{i} \times K_{i+1}$
3. $Q_{i+1} := \ker \bar{q}_{i+1}$ (or $= \ker \langle \bar{q}_i, q_{i+1} \rangle = \ker \bar{q}_i \cap \ker q_{i+1}$)

4. $P_{i+1} := \ker \langle X \xrightarrow{\xi} HX \xrightarrow{H\bar{q}_{i+1}} \prod_{j \leq i+1} K_j \rangle$

Upon termination, the algorithm returns $X/P_i = X/Q_i$ as the simple quotient of $(X, \xi)$.

**Proposition 3.7.** For spans $R \rightrightarrows X$, we will denote the canonical quotient by $\kappa_R : X \rightarrow X/R$.

We proceed to prove correctness, i.e. that the algorithm really does return the simple quotient of $(X, \xi)$. We fix the notation in Algorithm 3.4 throughout. Since $\bar{q}$ accumulates more information in every step, it is clear that $P$ and $Q$ are really being successively refined:

**Lemma 3.6.** For every $i$, $P_{i+1}$ is finer than $P_i$, $Q_{i+1}$ is finer than $Q_i$, and $P_i$ is finer than $Q_{i+1}$.

If we suppress the termination on $P_i = Q_i$ for a moment, then the algorithm thus computes equivalence relations refining each other. At each step, select decides which part of the information present in $P_i$ but not in $Q_i$ should be used to refine $Q_i$ to $Q_{i+1}$.

**Proposition 3.7.** There exist morphisms $\xi/Q_i : X/P_i \rightarrow H(X/Q_i)$ for $i \geq 0$ (necessarily unique) such that (3.2) commutes.

Upon termination the morphism $\xi/Q_i$ yields the structure of a quotient coalgebra of $\xi$.

**Corollary 3.8.** If $P_i = Q_i$ then $X/Q_i$ carries a unique coalgebra structure forming a quotient of $\xi : X \rightarrow HX$. This means intuitively that all states that are merged by the algorithm are actually behaviourally equivalent. The following property captures the converse:

**Lemma 3.9.** Let $h : (X, \xi) \rightarrow (D, d)$ be a quotient of $(X, \xi)$. Then $\ker h$ is finer than both $P_i$ and $Q_i$, for all $i \geq 0$.

**Theorem 3.10** (Correctness). If $P_i = Q_i$, then $\xi/Q_i : X/Q_i \rightarrow HX/Q_i$ is a simple coalgebra.

**Remark 3.11.** Most classical partition refinement algorithms are parametrized by an initial partition $\kappa_I : X \rightarrow X/I$. We start with the trivial partition $!: X \rightarrow 1$ because a non-trivial initial partition might split equivalent behaviours and then would invalidate Lemma 3.9. To accomodate an initial partition $X/I$ coalgebraically, replace $(X, \xi)$ with the coalgebra $(\xi, \kappa_I)$ for the functor $H(-) \times X/I$ – indeed, already $P_0$ will then be finer than $I$.

We look in more detail at two corner cases of the algorithm, where the select routine retains all available information, respectively none:

**Remark 3.12.** Recall that $H$ induces the final sequence:

$$1 \leftarrow H1 \xrightarrow{H} H2 \xrightarrow{H^2} \cdots \xrightarrow{H^{i+1}} H^i \xrightarrow{H^i} H^{i+1} \xrightarrow{H^{i+1}} \cdots$$

Every coalgebra $\xi : X \rightarrow HX$ then induces a canonical cone $\xi^{(i)} : X \rightarrow H^i1$ on the final sequence, defined inductively by $\xi^{(0)} = !$, $\xi^{(i+1)} = H\xi^{(i)} \cdot \xi$. The objects $H^n1$ may be thought of as domains of $n$-step behaviour for $H$-coalgebras. If $C = \text{Set}$ and $X$ is finite, then states $x$ and $y$ are behaviourally equivalent iff $\xi(x) = \xi(y)$ for all $i < \omega$ [38].

The vertical inclusions in (3.1) reflect that only some and not necessarily all of the information present in the relation $P_i$ (resp. the quotient $X/P_i$) is used for further refinement.
If indeed everything is used, i.e., we have $k_{i+1} := \text{id}_{X/P_i}$, then these inclusions become isomorphisms and then our algorithm simply computes the kernels of the morphisms in the canonical cone, i.e., $Q_i = \ker \xi^{(i)}$.

That is, when select retains all available information, then Algorithm 3.4 just becomes a standard final chain algorithm (e.g. [25]). The other extreme is the following:

- **Definition 3.13.** We say that select discards all new information at $i + 1$ if $k_{i+1}$ factors through the morphism $X/P_i \to X/Q_i$, witnessing that $P_i$ is finer than $Q_i$, see Lemma 3.6.

- **Lemma 3.14.** The algorithm fails to progress in the $i + 1$-th iteration, i.e. $Q_{i+1} = Q_i$, iff select discards all new information at $i + 1$.

- **Corollary 3.15.** If $\mathcal{C}$ is (concrete over) $\text{Set}^S$ and select never discards all new information, then Algorithm 3.4 terminates and computes the simple quotient of a given finite coalgebra.

Indeed, Proposition 3.7 shows that we obtain a chain of successively finer quotients of $X$, and by Lemma 3.14 this chain must finally converge (i.e. $P_i = Q_i$ will hold).

## 4 Incremental Partition Refinement

In the most generic version of the partition refinement algorithm (Algorithm 3.4), the partitions are recomputed from scratch in every step: In Step 4 of the algorithm, $P_{i+1} = \ker(H(q_i, q_{i+1}) \cdot \xi)$ is computed from the information $\tilde{q}_i$, accumulated so far and the new information $q_{i+1}$, but in general one cannot exploit that the kernel of $\tilde{q}_i$ has already been computed. We now present a refinement of the algorithm in which the partitions are computed incrementally, i.e. $P_{i+1}$ is computed from $P_i$ and $q_{i+1}$. This requires the type functor $H$ to be zippable (Definition 4.1). The algorithm will be further refined in the next section.

Note that in Step 3, Algorithm 3.4 computes a kernel $Q_{i+1} = \ker \tilde{q}_{i+1} = \ker \langle \tilde{q}_i, q_{i+1} \rangle$. In general, the kernel of a pair $(a, b) : D \to A \times B$ is an intersection $\ker a \cap \ker b$. Hence, the partition for such a kernel can be computed in two steps: 1. Compute $D/\ker a$. 2. Refine every block in $D/\ker a$ with respect to $b : D \to B$. Algorithm 3.4 can thus be implemented to keep track of the partition $X/Q_i$ and then refine this partition by $q_{i+1}$ in each iteration.

However, the same trick cannot be applied immediately to the computation of $X/P_i$, because of the functor $H$ inside the computation of the kernel: $P_{i+1} = \ker(H(\tilde{q}_i, q_{i+1}) \cdot \xi)$. In the following, we will provide sufficient conditions for $H$, $a : D \to A$, $b : D \to B$ to satisfy

$$\ker H(a, b) = \ker(Ha, Hb).$$

As soon as this holds for $a = \tilde{q}_i, b = q_{i+1}$, we can optimize the algorithm by changing Step 4 to

$$P_{i+1} := \ker \langle H\tilde{q}_i, \xi, Hq_{i+1} \cdot \xi \rangle. \quad (4.1)$$

- **Definition 4.1.** A functor $H$ is zippable if the following morphism is a monomorphism:

$$\text{unzip}_{H,A,B} : H(A + B) \xrightarrow{\langle H(\text{id}_A), H(\text{id}_B) \rangle} H(A + 1) \times H(1 + B)$$

Intuitively, if $H$ is a functor on $\text{Set}$, we think of elements $t$ of $H(A + B)$ as shallow terms with variables from $A + B$. Then zippability means that each $t$ is uniquely determined by the two terms obtained by replacing $A$- and $B$-variables, respectively, by some placeholder $\_\_$, viz. the element of 1, as in the examples in Figure 1.

In the following, we work in the category $\mathcal{C} = \text{Set}^S$ of $\mathcal{S}$-sorted sets. However, most proofs are category-theoretic to clarify where sets are really needed and where the arguments are more generic.
Additionally, we will need to enforce constraints on the subblock $S$. While this approach is sufficient for systems with real-valued weights [35], it may in general let the optimization be incorrect for $P_lP_l$ for $P_l$, and hence is zippable by property 1 and Lemma 4.3.

Example 4.6. The finitary functor $P_lP_l$ fails to be zippable, as shown in Figure 1. First, this shows that zippable functors are not closed under quotients, since any finitary functor is a quotient of a polynomial, hence zippable, functor (recall that a Fin-functor $F$ is finitary if $FX = \bigcup \{F_i[Y] \mid i : Y \to X \text{ and } Y \text{ finite}\}$). Secondly, this shows that zippable functors are not closed under composition. One can extend the counterexample to a coalgebra to show that the optimization is incorrect for $P_lP_l$ and select $= \chi_S$. We will remedy this later by making use of a second sort, i.e. by working in Set² (Remark 4.13).

Additionally, we will need to enforce constraints on the select routine to arrive at the desired optimization (4.1). This is because in general, $\ker H(a, b)$ differs from $\ker(Ha, Hb)$ even for $H$ zippable; e.g. for $H = P_l$ and for $\pi_1, \pi_2$ denoting binary product projections, $(P_l\pi_1, P_l\pi_2)$ in general fails to be injective although $P_l(\pi_1, \pi_2) = P_l\text{id} = \text{id}$.

The next example illustrates this issue, and a related one: One might be tempted to implement splitting by a subblock $S$ by $q_i = \chi_S$. While this approach is sufficient for systems with real-valued weights [35], it may in general let $\ker(Hq_i, q_{i+1})\cdot \xi$ and $\ker(Hq_i\cdot \xi, Hq_{i+1}\cdot \xi)$ differ even for zippable $H$, thus rendering the algorithm incomplete:

Example 4.7. Consider the coalgebra $\xi : X \to HY$ for the zippable functor $H = \{\nabla, \bullet\} \times P_l(-)$ illustrated in Figure 2 (essentially a Kripke model). The initial partition $X/P_0$ splits by shape and by $P_l$!, i.e. states with and without successors are split (Figure 2a). Now, suppose that select returns $k_i := \text{id}_{X/P_0} i.e., retains all information (cf. Remark 3.12), so that $Q_1 = P_0$ and $P_1$ puts $c_1$ and $c_2$ into different blocks (Figure 2b). We now analyse the next partition that arises when we split w.r.t. the subblock $S = \{c_1\}$ but not w.r.t. the rest.
C \ S of the compound block C = \{c_1, c_2\}; in other words, we take k_2 := \chi_c(\{c_1\}) : X/P_1 \to 2,
making q_2 = \chi_c(\{c_1\}) : X \to 2. Then, H(q_1, q_2) \cdot \xi \text{ splits } t_1 \text{ from } t_2,
because }t_1 \text{ has a successor } c_2 \text{ with } q_1(c_2) = \{c_1, c_2\} \text{ and } q_2(c_2) = 0 \text{ whereas } t_2 \text{ has no such successor. However, } t_1, t_2 \text{ fail to be split by } H(q_1, H(q_2) \cdot \xi \text{ because their successors do not differ when looking at successor blocks in } X/Q_1 \text{ and } X/\ker \chi_S \text{ separately: both have } \{c_1, c_2\} \text{ and } \{c_3\} \text{ as successor blocks in } X/Q_1 \text{ and } \{c_1\} \text{ and } X \setminus \{c_1\} \text{ as successors in } X/\ker \chi_S. \text{ Formally:}

\begin{align*}
Hq_1 \cdot \xi(t_1) &= (\text{id } \times P_1 \kappa_{P_0}) \cdot \xi(t_1) = (\bullet, \{\{c_1, c_2\}, \{c_3\}\}) = Hq_1 \cdot \xi(t_2) \\
Hq_2 \cdot \xi(t_1) &= (\text{id } \times P_1 \chi_{c_1}) \cdot \xi(t_1) = (\bullet, \{0, 1\}) = Hq_2 \cdot \xi(t_2)
\end{align*}

So if we computed P_2 iteratively as in (4.1) for q_2 = \chi_S, then t_1 and t_2 would not be split, and we would reach the termination condition P_2 = Q_2 before all behaviourally inequivalent states have been separated.

Already Paige and Tarjan [28, Step 6 of the Algorithm] note that one additionally needs to split by C \ S = \{c_3\}, which is accomplished by splitting by q_i = \chi^c_S. This is formally captured by the condition we introduce next.

\section*{Definition 4.8. A select routine respects compound blocks if whenever }k = \text{select}(X \twoheadrightarrow Y \to Z)\text{ then }\ker k \cup \ker z \text{ is a kernel.}

In Set^S, \cup \text{ denotes the usual union of multi-sorted relations; and since reflexive and symmetric relations are closed under unions, the definition boils down to }\ker k \cup \ker z \text{ being transitive. We can rephrase the condition more explicitly, restricting to the single-sorted case for readability:}

\begin{lemma}
For a : D \to A, b : D \to B \text{ in Set, the following are equivalent:}
\begin{enumerate}
\item \ker a \cup \ker b \rightrightarrows D \text{ is a kernel.}
\item \ker a \cup \ker b \rightrightarrows D \text{ is the kernel of the pushout of } a \text{ and } b.
\item For all x, y, z \in D, ax = ay \text{ and } by = bz \text{ implies } ax = az \text{ or } bx = by = bz.
\item For all x \in D, [x]_a \subseteq [x]_b \text{ or } [x]_b \subseteq [x]_a.
\end{enumerate}
\end{lemma}

The last item states that when going from a-equivalence classes to b-equivalence classes, the classes either merge or split, but do not merge with other classes and split at the same time. Note that in Figure 3, Q_1 \cup \ker \chi_S fails to be transitive, while Q_1 \cup \ker \chi^c_S \text{ is transitive.}

\section*{Example 4.10. All select}(X \twoheadrightarrow Y \to Z)\text{ routines in Example 3.3 respect compound blocks.}

\section*{Proposition 4.11. Let a : D \to A, b : D \to B \text{ be a span such that }\ker a \cup \ker b \text{ is a kernel, and let } H : \text{Set} \to D \text{ be a zippable functor preserving monos. Then we have}

\begin{equation}
\ker(Ha, Hb) = \ker H(a, b).
\end{equation}
We thus obtain soundness of optimization (4.1); summing up:

**Corollary 4.12.** Suppose that $H$ is a zippable endofunctor on $\text{Set}$ and that select respects compound blocks and never discards all new information. Then Algorithm 3.4 with optimization (4.1) terminates and computes the simple quotient of a given finite $H$-coalgebra.

**Remark 4.13.** Like most results on set coalgebras, the above extends to multisorted sets by componentwise arguments, and this allows dealing with complex composite functors [32].

**Example 4.14.** Applying this to the functors $F = \mathcal{P}_1, G = A \times (-)$, and $H = \mathcal{D}$, we obtain simple (resp. general) Segala systems as coalgebras for $FGH$ (resp. $FHG$). For simple Segala systems, the $\text{Set}^3$ functor is defined by $(X,Y,Z) \mapsto (FY,GZ,HX)$.

### 5 Efficient Calculation of Kernels

In Algorithm 3.4, it is left unspecified how the kernels are computed concretely. We proceed to define a more concrete algorithm based on a *refinement interface* of the functor. This interface is aimed at efficient implementation of the *refinement step* in the algorithm. Specifically from now on, we split along $\xi : X \to HY$ w.r.t. a subblock $S \subseteq C \in Y/Q$, and need to compute how the splitting of $C$ into $S$ and $C \setminus S$ within $Y/Q$ affects the partition $X/P$.

The low complexity of Paige-Tarjan-style algorithms hinges on this refinement step running in time $O(|\text{pred}(S)|)$, where $\text{pred}(y)$ denotes the set of predecessors of some $y \in Y$ in the given transition system. In order to speak about “predecessors” w.r.t. more general $\xi : X \to HY$, the refinement interface will provide an encoding of $H$-coalgebras as sets of states with successor states encoded as bags (implemented as lists up to ordering; recall that $B_W$ denotes the set of bags over $Z$) of $A$-labelled edges, where $A$ is an appropriate label alphabet. Moreover, the interface will allow us to talk about the behaviour of elements of $X$ w.r.t. the splitting of $C$ into $S$ and $C \setminus S$, looking only at points in $S$.

**Definition 5.1.** A *refinement interface* for a $\text{Set}$-functor $H$ is formed by a set $A$ of *labels*, a set $W$ of *weights* and functions

$$
\begin{align*}
\text{b} : HY & \to B_W(A \times Y), & \text{init} : H1 \times B_W A & \to W, \\
w : \mathcal{P}_1 Y & \to HY \to W, & \text{update} : B_W A \times W & \to W \times H3 \times W
\end{align*}
$$

such that for every $S \subseteq C \subseteq Y$, the diagrams

\begin{equation}
\begin{array}{c}
H1 \times B_W A \xrightarrow{(\text{H1}, b, \text{init})} W \\
\xrightarrow{w(Y)} \xrightarrow{\text{init}} \xrightarrow{(w(S), H_X \otimes \text{w}(C \setminus S))} W \times H3 \times W
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
HY \xrightarrow{(H, \text{b}, \text{init})} W \\
\xrightarrow{w(C)} \xrightarrow{(\text{init}, w(C))} \xrightarrow{\text{update}} W \times H3 \times W
\end{array}
\end{equation}
commute, where \( \text{fil}_S : B_i(A \times Y) \to B_i(A) \) is the filter function \( \text{fil}_S(f)(a) = \sum_{y \in S} f(a, y) \) for \( S \subseteq Y \). The significance of the set \( H3 \) is that when using a set \( S \subseteq C \subseteq X \) as a splitter, we want to split every block \( B \) in such a way that it becomes compatible with \( S \) and \( C \setminus S \), i.e. we group the elements \( s \in B \) by the value of \( H \chi_C \cdot \xi(s) \in H3 \). The set \( W \) depends on the functor. But in most cases \( W = H2 \) and \( w(C) = H \chi_C : HY \to H2 \) are sufficient.

In an implementation, we do not require a refinement interface to provide \( w \) explicitly, because the algorithm will compute the values of \( w \) incrementally using (5.1), and \( \mathcal{B} \) need not be implemented because we assume the input coalgebra to be already encoded via \( \mathcal{B} \):

\[
\begin{align*}
\text{Definition 5.2.} & \quad \text{Given an interface of } H \text{ (Definition 5.1), an } \textit{encoding} \text{ of a morphism } \xi : X \to HY \text{ is given by a set } E \text{ and maps} \\
& \quad \text{graph} : E \to X \times A \times Y & \text{type} : X \to H1
\end{align*}
\]

such that \( (\mathcal{B} \cdot \xi(x))(a, y) = \{ e \in E \mid \text{graph}(e) = (x, a, y) \} \), and with type \( = H! \cdot \xi \).

Intuitively, an encoding presents the morphism \( \xi \) as a graph with edge labels from \( A \).

\[
\text{Lemma 5.3.} \quad \text{Every morphism } \xi : X \to HY \text{ has a canonical encoding where } E \text{ is the obvious set of edges of } \mathcal{B} \cdot \xi : X \to B_i(A \times Y). \text{ If } X \text{ is finite, then so is } E.
\]

\[
\begin{align*}
\text{Example 5.4.} & \quad \text{In the following examples, we take } W = H2 \text{ and } w(C) = H \chi_C : HY \to H2. \text{ We use the helper function } \text{val} := \langle H(= 2), \text{id}, H(= 1) \rangle : H3 \to H2 \times H3 \times H2, \text{ where} \\
& \quad (= x) : 3 \to 2 \text{ is the equality check for } x \in \{1, 2\}, \text{ and in each case define } \text{update} = \text{val} \cdot \text{up for some function } \text{up} : B_iA \times H2 \to H3. \text{ We implicitly convert sets into bags.}
\end{align*}
\]

1. For the monoid-valued functor \( G^{(-)} \), for an Abelian group \( (G, +, 0) \), we take labels \( A = G \) and define \( \mathcal{B}(f) = \{ (f(y), y) \mid y \in Y, f(y) \neq 0 \} \) (which is finite because \( f \) is finitely supported). With \( W = H2 = G \times G \), the weight \( w(C) = H \chi_C : HY \to G \times G \) is the accumulated weight of \( Y \setminus C \) and \( C \). Then the remaining functions are

\[
\text{init}(b_1, e) = (0, \Sigma e) \quad \text{and} \quad \text{up}(e, (r, c)) = (r, c - \Sigma e, \Sigma e),
\]

where \( \Sigma : B_iG \to G \) is the obvious summation map.

2. Similarly to the case \( \mathbb{N}^{(-)} \), one has the following \text{init} and \text{up} functions for the distribution functor \( \mathcal{D} \): put \( \text{init}(h_1, e) = (0, 1) \in \mathcal{D}2 \subseteq [0, 1]^2 \) and \( \text{up}(e, (r, c)) = (r, c - \Sigma e, \Sigma e) \) if the latter lies in \( \mathcal{D}3 \), and \( (0, 0, 1) \) otherwise.

3. Similarly, one obtains a refinement interface for \( B_i = \mathbb{N}^{(-)} \) adjusting the one for \( \mathbb{Z}^{(-)} \); in fact, \text{init} remains unchanged and \( \text{up}(e, (r, c)) = (r, c - \Sigma e, \Sigma e) \) if the middle component is a natural number and \( (0, 0, 0) \) otherwise.

4. Given a polynomial functor \( H_{\Sigma} \) for a signature \( \Sigma \) with bounded arity (i.e. there exists \( k \) such that every arity is at most \( k \)), the labels \( A = \mathbb{N} \) encode the indices of the parameters:

\[
\begin{align*}
b(\sigma(y_1, \ldots, y_n)) & = \{ (1, y_1), \ldots, (n, y_n) \} & \text{init}(\sigma(0, \ldots, 0), f) & = \sigma(1, \ldots, 1) \\
\text{up}(I, \sigma(b_1, \ldots, b_n)) & = \sigma(b_1 + (1 \in I), \ldots, b_i + (i \in I), \ldots, b_n + (n \in I))
\end{align*}
\]

Here \( b_i + (i \in I) \) means \( b_i + 1 \) if \( i \in I \) and \( b_i \) otherwise. Since \( I \) are the indices of the parameters in the subblock, \( i \in I \) happens only if \( b_i = 1 \).

One example where \( W = H2 \) does not suffice is the powerset functor \( \mathcal{P} \): Even if we know for a \( t \in \mathcal{P}Y \) that it contains elements in \( C \subseteq Y \), in \( S \subseteq C \), and outside \( C \) (i.e. we know \( \mathcal{P} \chi_S(t), \mathcal{P} \chi_C \in \mathcal{P}2 \)), we cannot determine whether there are any elements in \( C \setminus S \) – but as seen in Example 4.7, we need to include this information.
Example 5.5. The interface for the powerset functor needs to count the edges into blocks $C \ni S$ in order to know whether there are edges into $C \setminus S$, as described by Paige and Tarjan [28]. What happens formally is that first the interface for $\mathbb{N}^{(2)}$ is implemented for edge weights at most 1, and then the middle component of the result of update is adjusted. So $W = \mathbb{N}^{(2)}$, and the encoding $b : \mathcal{P}_1 Y \hookrightarrow B_1(1 \times Y)$ is the obvious inclusion. Then

$$\text{init}(h_1, e) = (0, |e|) \quad \text{and} \quad w(C(M)) = B_1\chi_C(M) = (|M \setminus C|, |C \cap M|)$$

$$\text{update}(n, (e, r)) = \langle B_1(= 2), (\succ 0)^3, B_1(= 1) \rangle(r, c - |n|, |n|)$$

$$= ((r + c - |n|, |n|), (r \succ 0, c - |n|, r \succ 0, c - |n|, c - |n|), (r + |n|, c - |n|)),$$

where $x \succ 0$ is 0 if $x = 0$ and 1 otherwise.

Assumption 5.6. From now on, assume a Set-functor $H$ with a refinement interface such that init and update run in linear time and elements of $H3$ can be compared in constant time.

Example 5.7. The refinement interfaces in Examples 5.4 and 5.5 satisfy Assumption 5.6.

Remark 5.8. In the implementation, we encode the partitions $X/P$, $Y/Q$ as doubly linked lists of the blocks they contain, and each block is in turn encoded as a doubly linked list of its elements. The elements $x \in X$ and $y \in Y$ each hold a pointer to the corresponding list entry in the blocks containing them. This allows removing elements from a block in $O(1)$.

The algorithm maintains the following mutable data structures:

- An array toSub : $X \to B_1 E$, mapping $x \in X$ to its outgoing edges ending in the currently processed subblock.
- A pointer mapping edges to memory addresses: lastW : $E \to \mathbb{N}$.
- A store of last values deref : $\mathbb{N} \to W$.
- For each block $B$ a list of markings markB $\subseteq B \times \mathbb{N}$.

Notation 5.9. In the following we write $e = x \xrightarrow{a} y$ in lieu of graph(e) = $(x, a, y)$.

Definition 5.10 (Invariants). Our correctness proof below establishes the following properties that we call the invariants:

1. For all $x \in X$, toSub(x) = $\emptyset$, i.e. toSub is empty everywhere.
2. For $e_i = x_i \xrightarrow{a_i} y_i, i \in \{1, 2\}$, lastW(e_1) = lastW(e_2) $\iff$ $x_1 = x_2$ and $[y_1]_{\Re_Q} = [y_2]_{\Re_Q}$.
3. For each $e = x \xrightarrow{a} y$, $C := [y]_{\Re_Q} \in Y/Q$, $w(C, \xi(x)) = \text{deref} \cdot \text{lastW}(e)$.
4. For $x_1, x_2 \in B \in X/P$, $C \in Y/Q$, $(x_1, x_2) \in \ker(H\chi_C \cdot \xi)$.

In the following code listings, we use square brackets for array lookups and updates in order to emphasize they run in constant time. We assume that the functions graph : $E \to X \times A \times Y$ and type : $X \to H1$ are implemented as arrays. In the initialization step the predecessor array pred : $Y \to \mathcal{P}_1 E$, pred(y) = \{e \in E \mid e = x \xrightarrow{a} y\} is computed. Sets and bags are implemented as lists. We only insert elements into sets not yet containing them.

We say that we group a finite set $Z$ by $f : Z \to Z'$ to indicate that we compute $[\cdot]_f$. This is done by sorting the elements of $z \in Z$ by a binary encoding of $f(z)$ using any $O(|Z| \cdot \log |Z|)$ sorting algorithm, and then grouping elements with the same $f(z)$ into blocks. In order to keep the overall complexity for the grouping operations low enough, one needs to use a possible majority candidate during sorting, following Valmari and Franceschini [35]. The algorithm computing the initial partition is listed in Figure 4.
1: for $e \in E$, $e = x \xrightarrow{a} y$ do
2: add $e$ to toSub[$x$] and pred[$y$].
3: for $x \in X$ do
4: $p_X :=$ new cell in deref containing init(type[$x$], $B_i(\pi_2 \cdot$ graph)(toSub[$x$]))
5: for $e \in$ toSub[$x$] do lastW[$e$] = $p_X$
6: toSub[$x$] := $\emptyset$
7: $X/P :=$ group $X$ by type $: X \rightarrow H1$, $Y/Q := \{Y\}$.
8: Figure 4 The initialization procedure

\textbf{Lemma 5.11.} The initialization procedure runs in time $O(|E| + |X| \cdot \log |X|)$ and makes the invariants true.

The algorithm for one refinement step along a morphism $\xi : X \rightarrow HY$ is listed in Figure 5.

In the first part, all blocks $B \in X/P$ are collected that have an edge into $S$, together with $v_\emptyset \in H3$ which represents $H_{\chi_S^C} \cdot \xi(x)$ for any $x \in B$ that has no edge into $S$. For each $x \in X$, toSub[$x$] collects the edges from $x$ into $S$. The markings mark$_B$ list those elements $x \in B$ that have an edge into $S$, together with a pointer to $w(C, x)$.

In the second part, each block $B$ with an edge into $S$ is refined w.r.t. $H_{\chi_S^C} \cdot \xi$. First, for any $(x, p_C) \in$ mark$_B$, we compute $w(S, x)$. $v^x = H_{\chi_S^C} \cdot \xi(x)$, and $w(C \setminus S, x)$ using update. Then, the weight of all edges $x \rightarrow C \setminus S$ is updated to $w(C \setminus S, x)$ and the weight of all edges $x \rightarrow S$ needs to be stored in a new cell containing $w(S, x)$. For all unmarked $x \in B$, we know that $H_{\chi_S^C} \cdot \xi(x) = v_\emptyset$, so all $x$ with $v^x = v_\emptyset$ stay in $B$. All other $x \in B$ are removed and distributed to new blocks w.r.t. $v^x$.

\textbf{Theorem 5.12.} Split($X/P, Y/Q, S \subseteq C \in Y/Q$) refines $X/P$ by $H_{\chi_S^C} \cdot \xi : X \rightarrow H3$.

\textbf{Lemma 5.13.} After running Split, the invariants hold.

\textbf{Lemma 5.14.} Lines 1–23 in Split run in time $O(\sum_{y \in S} |\text{pred}(y)|)$.

\textbf{Lemma 5.15.} For $S_i \subseteq C_i \in Y/Q$, $0 \leq i < k$, with $2 \cdot |S_i| \leq |C_i|$ and $Q_{i+1} = \ker(\kappa_Q, \chi_S^{C_i})$:

\begin{align*}
\text{Split}(X/P, Y/Q, S \subseteq C \in Y/Q) & : M := \emptyset \subseteq X/P \times H3 \\
1: & \text{for } y \in S, e \in \text{pred}[y] \text{ do} \\
2: & x \xrightarrow{a} y := e \\
3: & B := \text{block with } x \in B \in X/P \\
4: & \text{if mark}_B \text{ is empty then} \\
5: & w_C := \text{deref} \cdot \text{lastW}[e] \\
6: & v_\emptyset := \pi_2 \cdot \text{update}(\emptyset, w_C) \\
7: & \text{add } (B, v_\emptyset) \text{ to } M \\
8: & \text{if toSub}[x] = \emptyset \text{ then} \\
9: & \text{add } (x, \text{lastW}[e]) \text{ to mark}_B \\
10: & \text{add } e \text{ to toSub}[x] \\
11: & \text{(a) Collecting predecessor blocks} \\
12: & \text{for } (B, v_\emptyset) \in M \text{ do} \\
13: & B_{\emptyset} := \emptyset \subseteq X \times H3 \\
14: & \text{for } (x, p_C) \in \text{mark}_B \text{ do} \\
15: & \ell := B_i(\pi_2 \cdot \text{graph})(\text{toSub}[x]) \\
16: & (w_S^x, v^x, w_C^x) := \text{update}(\ell, \text{deref}[p_C]) \\
17: & \text{deref}[p_C] := w_C^x \\
18: & p_S := \text{new cell containing } w_S^x \\
19: & \text{for } e \in \text{toSub}[x] \text{ do } \text{lastW}[e] := p_S \\
20: & \text{toSub}[x] := \emptyset \\
21: & \text{if } v^x \neq v_\emptyset \text{ then} \\
22: & \text{remove } x \text{ from } B \\
23: & \text{insert } (x, v^x) \text{ into } B_{\emptyset} \\
24: & B_1 \times \{v_1\}, \ldots, B_k \times \{v_k\} := \\
25: & \text{group } B_{\emptyset} \text{ by } \pi_2 : X \times H3 \rightarrow H3 \\
26: & \text{insert } B_1, \ldots, B_k := \text{into } X/P \\
\end{align*}

\textbf{Figure 5} Refining $X/P$ w.r.t. $\chi_S^C : Y \rightarrow 3$ and $Y/Q$ along $\xi : X \rightarrow HY$.
1. For each \( y \in Y \), \( \{|i < k \mid y \in S_i\} \leq \log_2 |Y| \).
2. \( \text{SPLIT}(S_i \subseteq C_i \in Y/Q_i) \) for all \( 0 \leq i < k \) takes at most \( O(|E| \cdot \log |Y|) \) time in total.

Bringing Sections 3, 4, and 5 together, take a coalgebra \( \xi : X \to HX \) for a zippable Set-functor \( H \) with a given refinement interface where \( \text{init} \) and \( \text{update} \) run in linear and comparison in constant time. Instantiate Algorithm 3.4 with the \( \text{select} \) routine from Example 3.3.1, making \( q_{i+1} = k_{i+1} \cdot \kappa_P = \chi_{S_i}^{C_i} \) with \( 2 \cdot |S| \leq |C|, S_i, C_i \subseteq X \). Replace line 4 by

\[
X/P_{i+1} = \text{SPLIT}(X/P_i, X/Q_i, S_i \subseteq C_i).
\] (4.1')

By Theorem 5.12 this is equivalent to (4.1), and hence, by Corollary 4.12, to the original line 4, since \( \chi_{S_i}^{C_i} \) respects compound blocks. By Lemmas 5.11 and 5.15.2, we have

\[\textbf{Theorem 5.16.} \] The above instance of Algorithm 3.4 computes the quotient modulo behavioural equivalence of a given coalgebra in time \( O((m + n) \cdot \log n) \), for \( m = |E|, n = |X| \).

\[\textbf{Example 5.17.} \]
1. For \( H = \mathcal{I} \times P_1 \), we obtain the classical Paige-Tarjan algorithm \[28\] (with initial partition \( \mathcal{I} \)), with the same complexity \( O((m + n) \cdot \log n) \).
2. For \( HX = \mathcal{I} \times \mathbb{R}^{(X)} \), we solve Markov chain lumping with an initial partition \( \mathcal{I} \) in time \( O((m + n) \cdot \log n) \), like the best known algorithm (Valmari and Franceschinis [35]).
3. For infinite \( A \), we need to decompose the functor \( P_2(\mathcal{A} \times (-)) \) for labelled transition systems into \( P_1 \) and \( A \times (-) \), and thus obtain run time \( O(m \cdot \log m) \), like in \[13\] but slower than Valmari’s \( O(m \cdot \log n) \) [34]. For fixed finite \( A \), we run in time \( O(m \log n) \).
4. Hopcroft’s classical automata minimization \[17\] is obtained by \( HX = 2 \times X^A \), with running time \( O(n \cdot \log n) \) for fixed alphabet \( A \). For non-fixed \( A \) the best known complexity is \( O(|A| \cdot n \cdot \log n) \) [15, 24]. By using decomposition into \( 2 \times P_1 \) and \( N \times (-) \) we obtain \( O(|A| \cdot n \cdot \log n + |A| \cdot n \cdot \log |A|) \).
5. We quotient simple (resp. general) Segala systems [33] by bisimilarity after decomposition into three sorts (cf. Example 4.14). The time bound \( O((n + m) \log(n + m)) \) slightly improves on the previous bound \[5\].

\section{Conclusions and Future Work}

We have presented a generic algorithm that quotients coalgebras by behavioural equivalence. We have started from a category-theoretic procedure that works for every mono-preserving functor on a category with image factorizations, and have then developed an improved algorithm for zippable endofunctors on \( \text{Set} \). Provided the given type functor can be equipped with an efficient implementation of a refinement interface, we have finally arrived at a concrete procedure that runs in time \( O((m + n) \log n) \) where \( m \) is the number of edges and \( n \) the number of nodes in a graph-based representation of the input coalgebra. We have shown that this instantiates to (minor variants of) several known efficient partition refinement algorithms: the classical Hopcroft algorithm \[17\] for minimization of DFAs, the Paige-Tarjan algorithm for unlabelled transition systems \[28\], and Valmari and Franceschinis’s lumping algorithm for weighted transition systems [35]. Moreover, we obtain a new algorithm for simple Segala systems that is asymptotically faster than previous algorithms \[5\]. Coverage of Segala systems is based on modularity results in multi-sorted coalgebra \[32\].

It remains open whether our approach can be extended to, e.g., the monotone neighbourhood functor, which is not itself zippable and also does not have an obvious factorization into zippable functors. We do expect that our algorithm applies beyond weighted systems.
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\section{Omitted Details and Proofs}

\subsection*{Details for Section 2}

\begin{remark}
\begin{enumerate}
\item The following \textit{diagonalization property} holds for image factorizations:
Given a commutative square $m \cdot f = g \cdot e$ where $m$ is a monomorphism and $e$ a regular epimorphism, there exists a (necessarily unique) \textit{diagonal} $d$ such that $m \cdot d = g$ and $d \cdot e = f$. In particular, image factorizations are unique up to isomorphism, so $\text{Im}(f)$ is well-defined up to isomorphism.

\item For any object $X$ there is a bijective correspondence between kernels of morphisms $f$ with domain $X$ and regular quotients of $X$. Indeed, in one direction take the coequalizer of a given kernel pair $\ker f$. In the reverse direction, take the kernel of a given regular epimorphism $e : X \to Y$ (see \cite[Prop. 11.22(2)]{[3]}).

\item It follows that two morphisms $f : X \to Y$ and $g : X \to Y'$ have the same kernel iff they have the same image:
\[ \text{Im } f = \text{Im } g \iff \ker f = \ker g. \]

To see this, take the image factorizations of $f = m \cdot e$ and $g = m' \cdot e'$, respectively, and use that $\ker f = \ker e$ and $\ker g = \ker e'$.

\item A \textit{relation} is a jointly monic parallel pair of morphisms $f, g : E \rightrightarrows X$ (not necessarily a kernel pair). We write $\kappa_E : X \rightrightarrows X/E$ for their coequalizer; we refer to the object $X/E$ as the \textit{quotient} of $X$ modulo $E$, and to $\kappa_E$ as the \textit{quotient map}. Indeed, in $\text{Set}$, $X/E$ is the usual quotient of $X$ modulo the equivalence relation generated by $\{(fx, gx) \mid x \in E\}$. When $f$ and $g$ and $X$ are clear from the context we just write the object $E$ for the relation.

\item We say that a morphism $f : X \to Y$ is \textit{well-defined} on (the equivalence classes of) a relation $\pi_1, \pi_2 : E \rightrightarrows X$ if
\[
\begin{array}{ccc}
E & \xrightarrow{\pi_1} & X \\
\downarrow{\pi_2} & \mapright{f} & \downarrow{f} \\
X & \xrightarrow{f} & Y
\end{array}
\]
commutes. Then by the universal property of $\kappa_E : X \to X/E$ we obtain a unique morphism $f' : X/E \to Y$ such that $f = f' \cdot \kappa_E$. In $\text{Set}$, this is the usual well-definedness of the map $f$ on the equivalence classes in $X/E$ witnessed by the map $f'$.

\item Following the standard terminology in $\text{Set}$, we say that a quotient $X/E_1$ is \textit{finer} than (or a \textit{refinement} of) another quotient $X/E_2$ if the quotient map $\kappa_{E_2}$ is well-defined on $E_1$. This induces a refinement relation on kernels, described as follows: $\pi_1, \pi_2 : E \rightrightarrows X$ is \textit{finer} than $\pi'_1, \pi'_2 : E' \rightrightarrows X$ if there exists a morphism $m : E \to E'$ (necessarily monic) such that $\pi'_i \cdot m = \pi_i$, $i = 1, 2$.

\end{enumerate}
\end{remark}

\begin{lemma}
We have $\ker(m \cdot f) = \ker f$ for every $f : X \to Y$ and every monic $m : Y \rightrightarrows Z$.
\end{lemma}

\begin{proof}
This can be shown by checking directly that $\ker f$ is the kernel of $m \cdot f$ and using that $m$ is monic.
\end{proof}

\begin{lemma}
$C$ has pushouts of regular epimorphisms (i.e. of spans containing at least one regular epimorphism).
\end{lemma}
Proof. Let \( X \xleftarrow{e} Y \xrightarrow{h} W \) be a span, with \( e \) a regular epimorphism. Let \((\pi_1, \pi_2)\) be the kernel pair of \( e \), and let \( q : W \to Z \) be the coequalizer of \( h\pi_1 \) and \( h\pi_2 \) (both exist by our running assumptions). Then \( e \) is the coequalizer of \( \pi_1, \pi_2 \), so that there exists \( r : X \to Z \) such that \( re = qh \). We claim that

\[
\begin{array}{ccc}
Y & \xrightarrow{e} & X \\
\downarrow & & \downarrow r \\
W & \xrightarrow{q} & Z
\end{array}
\]

is a pushout. Uniqueness of mediating morphisms is clear since \( q \) is epic; we show existence. So let \( W \twoheadrightarrow U \xleftarrow{f} X \) be a competitor, i.e. \( fe = gh \). Then \( gh\pi_1 = fe\pi_1 = fe\pi_2 = gh\pi_2 \), so by the coequalizer property of \( q \) we obtain \( k : Z \to U \) such that \( kq = g \). It remains to check that \( kr = f \). Now \( kre = kqh = gh = fe \), which implies the claim because \( e \) is epic.

Lemma A.4. \( \text{Coalg}(H) \) has all coequalizers and pushouts of regular epimorphisms.

Proof. Since the forgetful functor \( \text{Coalg}(H) \to \mathcal{C} \) creates all colimits, the statement follows directly by our running assumptions and Lemma A.3.

Proof of Lemma 2.7

Uniqueness up to isomorphism means:

Lemma A.5. Let \((C,c)\) be a coalgebra, and let \( e_i : (C,c) \to (D_i,d_i) \), \( i = 1, 2 \), be quotients with \((D_i,d_i)\) simple. Then \((D_1,d_1)\) and \((D_2,d_2)\) (more precisely the quotients \( e_1 \) and \( e_2 \)) are isomorphic.

Proof. By Lemma A.4, there is a pushout \( D_1 \xrightarrow{f_1} E \xleftarrow{f_2} D_2 \) of \( D_1 \xleftarrow{e_1} C \xrightarrow{e_2} D_2 \) in \( \text{Coalg}(H) \). Since regular epimorphisms are generally stable under pushouts, \( f_1 \) and \( f_2 \) are regular epimorphisms, hence isomorphisms because \( D_1 \) and \( D_2 \) are simple; this proves the claim.

Behavioural equivalence between coalgebras

Remark A.6. Using elementwise notation for intuition, ‘elements’ \( x \in C \) and \( y \in D \) of coalgebras \((C,c)\) and \((D,d)\) are behaviourally equivalent (written \( x \sim y \)) if they can be merged by coalgebra morphisms: \( x \sim y \) iff there exists a coalgebra \((E,e)\) and coalgebra morphisms \( f : (C,c) \to (E,e) \), \( g : (D,d) \to (E,e) \) such that \( f(x) = g(y) \). Under our running assumptions, any two behaviourally equivalent elements can be identified under a regular quotient, so that a simple quotient of a coalgebra already identifies all behaviourally equivalent elements: Reformulated in proper categorical terms, we claim that every pullback of two coalgebra morphisms \( f, g : (C,d) \to (D,d) \) is contained in the kernel pair of some morphism \( e : (C,d) \to (E,e) \). Indeed, by Lemma A.4 we can take \( e = qf = qg \) where \( q \) is the coequalizer of \( f \) and \( g \) in \( \text{Coalg}(H) \).

A final coalgebra is a terminal object in the category of coalgebras, i.e. a coalgebra \((C,c)\) such that every coalgebra \((D,d)\) has a unique coalgebra morphism into \((C,c)\). There are reasonable conditions under which a final coalgebra is guaranteed to exist, e.g. when \( \mathcal{C} \) is a locally presentable category (in particular, when \( \mathcal{C} = \text{Set} \)) and \( H \) is accessible. If \((C,c)\) is a final coalgebra and \( H \) preserves monos, then we can describe the simple quotient of a coalgebra \((D,d)\) as the image of \((D,d)\) under the unique morphism into \((C,c)\); in particular, in this case every coalgebra has a simple quotient.
Details for Section 3

Notes on Assumption 3.1

For \( C = \text{Set} \), the assumption that \( H \) preserves monos is w.l.o.g. First note that every endofunctor on sets preserves non-empty monos. Moreover, for any set functor \( H \) there exists a set functor \( H' \) that is naturally isomorphic to \( H \) on the full subcategory of all non-empty sets [2, Theorem 3.4.5], and hence has essentially the same coalgebras as \( H \) since there is only one coalgebra structure on \( \emptyset \).

Proof of Lemma 3.6

1. \( P_{i+1} \) finer than \( P_i \) and \( Q_{i+1} \) finer than \( Q_i \): Let \( p : \prod_{j \leq i+1} K_j \rightarrow \prod_{j \leq i} K_j \) be the product projection. Clearly we have \( \bar{q}_i = p \cdot \bar{q}_{i+1} \) and therefore, for the kernel pair \( \pi_1, \pi_2 : Q_{i+1} \rightrightarrows X \) we clearly have

\[
\bar{q}_i \cdot \pi_1 = p \cdot \bar{q}_{i+1} \cdot \pi_1 = p \cdot \bar{q}_{i+1} \cdot \pi_2 = \bar{q}_i \cdot \pi_2.
\]

Hence, we obtain a unique \( Q_{i+1} \rightarrow Q_i \) commuting with the projections of the kernel pairs.

Similarly, for the kernel pair \( \pi_1, \pi_2 : P_{i+1} \rightrightarrows X \) we have

\[
(H \bar{q}_i \cdot \xi) \cdot \pi_1 = Hp \cdot H \bar{q}_{i+1} \cdot \xi \cdot \pi_1 = Hp \cdot H \bar{q}_{i+1} \cdot \xi \cdot \pi_2 = (H \bar{q}_i \cdot \xi) \cdot \pi_2.
\]

Thus, there exists a unique morphism \( P_{i+1} \rightarrow P_i \) commuting with the kernel pair projections.

2. \( P_i \) finer than \( Q_{i+1} \): Induction on \( i \). Since \( Q_{i+1} = \ker \bar{q}_{i+1} \) and \( \bar{q}_{i+1} = \langle q_0, \ldots, q_{i+1} \rangle \), it suffices to show that \( P_i \) is finer than \( \ker q_j \) for \( j = 0, \ldots, i+1 \). For \( j \leq i \), we have by Lemma 3.6 that \( P_i \) is finer than \( P_j \), which is finer than \( \ker q_j \) by induction. Moreover, \( P_i \) is finer than \( \kernel q_{i+1} \) because \( q_{i+1} \) factors through \( X \rightarrow X/P_i \) by construction.

Proof of Proposition 3.7

Since \( Q_i = \ker \bar{q}_i \), the image factorization of \( \bar{q}_i \) has the form \( \bar{q}_i = m \cdot \kappa_{Q_i} \). By definition of \( P_i \) and since \( H \) preserves monos, we thus have \( P_i = \ker (H \bar{q}_i \cdot \xi) = \ker (H \kappa_{Q_i} \cdot \xi) \), and hence obtain \( Q/Q_i \) as in (3.2) by the coequalizer property of \( \kappa_{P_i} \).

Proof of Lemma 3.9

We claim that

\[
\text{if } \ker h \text{ is finer than } Q_i, \text{ then } \ker h \text{ is finer than } P_i.
\]

This is seen as follows: If \( \ker h \) is finer than \( Q_i \), then \( \kappa_{Q_i} : X \rightarrow X/Q_i \) factors through \( h : X \rightarrow D \), so that \( H \kappa_{Q_i} \cdot \xi \) factors through \( Hh \cdot \xi \) and hence through \( h \), since \( Hh \cdot \xi = d \cdot h \).

Since \( P_i = \ker (H \kappa_{Q_i} \cdot \xi) \), this implies that \( \ker h \) is finer than \( P_i \). The claim of the lemma is then proved by induction: for \( i = 0 \), the claim for \( Q_0 = X \times X \) is trivial, and the one for \( P_0 \) follows by (A.1). The inductive step is by Lemma 3.6 and (A.1).

Proof of Theorem 3.10

Let \( h : (X/Q_i, \xi/Q_i) \rightarrow (D, d) \) be a quotient. Then \( h \cdot \kappa_{Q_i} : (X, \xi) \rightarrow (D, d) \) is a quotient of \( (X, \xi) \), so by Lemma 3.9, \( \ker (h \cdot \kappa_{Q_i}) \) is finer than \( Q_i \). Of course, \( Q_i \) is also finer than \( \ker (h \cdot \kappa_{Q_i}) \), so \( h \) is an isomorphism.
The horizontal morphism is monic since monos are closed under coproducts in \( \ker \) such that the following diagram shows that the morphism is monic for any sets \( q_i \) through \( \kappa P_i \). Similarly, the inductive hypothesis implies that we have a mono \( n \) such that \( \zeta^{(i)} = nq_i \). Since \( H \) preserves monomorphisms, this implies that

\[
Q_{i+1} = P_i \cdot \ker(Hq_i) = \ker(Hq_i) = \ker(H\zeta^{(i)}(\xi)) = \ker(\zeta^{(i+1)}).
\]

**Proof of Lemma 3.14**

First note that select does not retain any new information in \( k_{i+1} = k_{i+1}\kappa P_i \) factors through \( f_i\kappa P_i = \kappa Q_{i+1} \). Now we reason as follows: \( Q_i = \ker q_i \) is finer than \( Q_{i+1} = \ker(q_i, q_{i+1}) \) iff \( Q_i \) is finer than \( \ker q_{i+1} \) iff \( q_{i+1} \) factors through \( \kappa Q_i \).

**Details for Section 4**

**Proof of Lemma 4.3**

Let \( F, G \) be endofunctors.

1. Suppose that both \( F \) and \( G \) are zippable. To see that \( F \times G \) is zippable one uses that monos are closed under products:

\[
\begin{align*}
F(A + B) \times G(A + B) \xrightarrow{\text{unzip}_{F,A,B} \times \text{unzip}_{G,A,B}} F(A + 1) \times F(1 + B) \times G(A + 1) \times G(1 + B) \\
\xrightarrow{\text{iff}} F(A + 1) \times F(1 + B) \times G(A + 1) \times G(1 + B)
\end{align*}
\]

2. Suppose again that \( F \) and \( G \) are zippable. To see that \( F + G \) is zippable consider the diagram below:

\[
\begin{align*}
F(A + B) + G(A + B) \xrightarrow{\text{unzip}_{F,A,B} + \text{unzip}_{G,A,B}} (F(A + 1) \times F(1 + B)) + (G(A + 1) \times G(1 + B)) \\
\xrightarrow{\text{iff}} (F(A + 1) + G(A + 1)) \times (F(1 + B) + G(1 + B))
\end{align*}
\]

The horizontal morphism is monic since monos are closed under coproducts in \( C \). The vertical morphism is monic since for any sets \( A_i \) and \( B_i \), \( i = 1, 2 \), the following morphism clearly is a monomorphism:

\[
(A_1 \times A_2) + (A_2 \times A_1) \xrightarrow{[ \langle \pi_1 + \pi_1, \langle \pi_2 + \pi_2 \rangle \rangle]} (A_1 + A_2) \times (B_1 + B_2)
\]

3. Suppose now that \( F \) is a subfunctor of \( G \) via \( s : F \rightarrow G \), where \( G \) is zippable. Then the following diagram shows that \( F \) is zippable, too:

\[
\begin{align*}
F(A + B) \xrightarrow{s_{A+1}} F(A + 1) \times F(1 + B) \\
\xrightarrow{\text{unzip}_{F,A,B}} G(A + B) \xrightarrow{\text{unzip}_{G,A,B}} G(A + 1) \times G(1 + B)
\end{align*}
\]
Proof of Lemma 4.4

Let \( \alpha : H(X + Y) \rightarrow HX \times HY \) be componentwise monic. Then the square

\[
\begin{array}{c}
H(A + B) \\ \downarrow^{\alpha_{A,B}} \\
HA \times HB \\
\end{array} \xrightarrow{\text{unzip}} \begin{array}{c}
H(A + 1) \times H(1 + B) \\ \downarrow^{\alpha_{A,1} \times \alpha_{1,B}} \\
(HA \times H1) \times (H1 \times HB) \\
\end{array}
\]

commutes, by naturality of \( \alpha \) in each of the components. The bottom morphism is monic because it has a left inverse, \( \pi_1 \times \pi_2 \). Therefore, unzip is monic as well.

\[\blacktriangleright\text{Remark A.7.}\] Out of the above results, only zippability of the identity and coproducts of zippable functors depend on our assumptions on \( C \) (see Assumption 2.2). Indeed, zippable functors are closed under coproducts as soon as monomorphism are closed under coproducts, which is satisfied in most categories of interest. Zippability of the identity holds whenever \( C \) is extensive, i.e. it has well-behaved set-like coproducts. Formally, a category is extensive if it has finite coproducts and pullbacks along coproduct injections such that coproducts are

1. disjoint, i.e., coproduct injections are monomorphic and the pullback of distinct coproduct injections is 0 (the initial object),
2. universal, i.e., the pullbacks of a morphism \( h : Z \rightarrow A + B \) along the coproduct injections, yields a coproduct \( Z = X + Y \) and \( h = f + g \):

\[
\begin{array}{ccc}
X & \xrightarrow{x} & Z \\
\downarrow^{f} & & \downarrow^{h} \\
A & \xrightarrow{\text{inl}} & A + B \\
\end{array}
\]

In an extensive category coproducts commute with pullbacks, and therefore monomorphisms are closed under coproducts.

Examples of extensive categories are the categories of sets, posets and graphs as well as any presheaf category. In addition, the categories of unary algebras and of Jónsson-Tarski algebras (i.e. algebras \( A \) with one binary operation \( A \times A \rightarrow A \) that is an isomorphism) are extensive. More generally, any topos is extensive.

The category of monoids is not extensive.

Details for Example 4.6

The following example shows that the optimized algorithm is not correct for the non-zippable functor \( P \). The \texttt{select} routine here \( \chi^C_S \) even fulfills the latter assumption that \texttt{select} respects compounds blocks (Definition 4.8).

\[\blacktriangleright\text{Example A.8.}\] Consider the following coalgebra \( \xi : X \rightarrow HX \) for \( HX = 2 \times P \) : \( X \):

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\downarrow^{a_1} \quad \downarrow^{a_2} \quad \downarrow^{a_3} \quad \downarrow^{a_4} \\
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\downarrow^{a_5} \quad \downarrow^{a_6} \quad \downarrow^{a_7} \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\downarrow^{b_1} \quad \downarrow^{b_2} \quad \downarrow^{b_3} \quad \downarrow^{b_4} \\
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\downarrow^{b_5} \quad \downarrow^{b_6} \quad \downarrow^{b_7} \\
\end{array}
\]

States \( x \) with \( \pi_1(\xi(x)) = 1 \) are indicated by the circle. When computing only

\[
P_{i+1} = \ker(Hq_i \cdot \xi, Hq_{i+1} \cdot \xi) = \ker P_i \cap \ker(Hq_{i+1} \cdot \xi)
\]

instead of \( P_i \), then \( a_1 \) and \( b_1 \) are not distinguished, although they are behaviourally different.
In order to simplify the partitions, we define abbreviations for the circle and non-circle states without successors, and the rest:

\[ F := \{a_2, a_7, b_2, b_6\} \quad N := \{a_4, a_6, b_4, b_7\} \quad C := \{a_1, a_3, a_5, b_1, b_3\} \]

Running the optimized algorithm, i.e. computing \( Q_i \) and \( P'_i \), one obtains the following sequence of partitions.

| \( i \) | \( q_i \) | \( X/Q_i \) | \( X/P'_i \) |
|------|-------|-------|-------|
| 0    | ! \( X \) | \{X\}  | \{F, N, C\} |
| 1    | \( \kappa P_0 : X \to X/P'_0 \) | \{F, N, C\}  | \{F, N, \{a_1, b_1\}, \{a_4, b_4\}, \{a_5, b_5\}\} |
| 2    | \( \chi_S^C : X \to 3 \), for \( S = \{a_3, b_3\} \) \{F, N, \{a_3, b_3\}, \{a_5, b_5\}\} \{F, N, \{a_1, b_1\}, \{a_3, b_3\}, \{a_5, b_5\}\} |

Note that in the step \( i = 2 \) one obtains the same result for \( S' := \{a_5, b_3\} \) or \( S'' := \{a_1, b_1\} \). For \( S \) as in the table, \( a_1 \) and \( b_1 \) are not split in \( X/P'_2 \) because:

\[
H\chi_S^C \cdot \xi(a_1) = H\chi_S^C \{(a_2, a_3), \{a_4, a_5\}\} = \{(0, 2), \{0, 1\}\} = \{(0, 1), \{0, 2\}\} = H\chi_S^C \{(b_2, b_3), \{b_4, b_5\}\} = H\chi_S^C \cdot \xi(b_1)
\]

Now the algorithm terminates because \( X/Q_2 = X/P_2 \), but without distinguishing \( a_1 \) from \( b_1 \).

**Proof of Lemma 4.9**

**Proof.** 4. \( \Rightarrow \) 1. In Set, kernels are equivalence relations. Obviously, \( \ker a \cup \ker b \) is both reflexive and symmetric. For transitivity, take \((x, y), (y, z) \in \ker a \cup \ker b \). Then \( x, z \in [y]_a \cup [y]_b \). If 
\[ \bar{y} \in [y]_b, \text{ then } x, z \in [y]_b \text{ and } x, z \in \ker b; \text{ otherwise } (x, z) \in \ker a. \]

1. \( \Rightarrow \) 2. In Set, monomorphisms are stable under pushouts, so it is sufficient to show that \( \ker a \cup \ker b \) is the kernel of the pushout of the epi-parts of \( a \) and \( b \). In other words, w.l.o.g. we may assume that \( a \) and \( b \) are epic, and we need to check that \( \ker a \cup \ker b \) is the kernel of \( p := p_A \cdot a = p_B \cdot b \), with

\[
\begin{array}{ccc}
D & \xrightarrow{a} & A \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
B & \xrightarrow{p_B} & P, \\
\end{array}
\]

Let \( \ker a \cup \ker b \) be the kernel of some \( y : D \to Y \). Then, \( y \) makes the projections of \( \ker a \) (resp. \( \ker b \)) equal and hence the coequalizer \( a \) (resp. \( b \)) induces a unique \( y_A \) (resp. \( y_B \)):

\[
\begin{array}{ccc}
\ker a & \xrightarrow{\pi_1} & D \\
\downarrow \pi_2 & & \downarrow y \\
\ker a \cup \ker b & \xrightarrow{\pi_1} & Y \\
\downarrow \pi_2 & & \downarrow y \\
A & \xrightarrow{y} & B \\
\end{array}
\]

Because of \( y_B \cdot b = y = y_A \cdot a, (y_A, y_B) \) is a competing cocone for the pushout. This induces a cocone morphism \( y_P : (P, p_A, p_B) \to (Y, y_A, y_B) \), and we have

\[ y_P \cdot p = y_P \cdot p_A \cdot a = y_A \cdot a = y. \]
With this, we are ready to show that \( \ker a \cup \ker b \) is a kernel for \( p \). Consider two morphisms \( c_1 : C \to D, c_2 : C \to D \) with \( p \cdot c_1 = p \cdot c_2 \), then we have
\[
y \cdot c_1 = y_p \cdot p \cdot c_1 = y_p \cdot p \cdot c_2 = y \cdot c_2.
\]
This induces a unique cone morphism \( C \to \ker a \cup \ker b \) as desired.

2. \( \Rightarrow \) 3. Take \( x, y, z \in D \) with \( a(x) = a(y) \) and \( b(y) = b(z) \). Then \( a(x) \) and \( b(z) \) are identified in the pushout \( P \):
\[
p(x) = p_A \cdot a(x) = p_A \cdot a(y) = p_B \cdot b(y) = p_B \cdot b(z) = p(z).
\]
This shows that \( (x, z) \) lies in \( \ker a \cup \ker b \), hence we have that \( a(x) = a(z) \) or \( b(x) = b(z) \).

3. \( \Rightarrow \) 4. For a given \( y \in D \), there is nothing to show in the case where \( [y]_a \subseteq [y]_b \). Otherwise if \( [y]_a \not\subseteq [y]_b \), then there is some \( x \in [y]_a \), i.e. such that \( a(x) = a(y) \), with \( b(x) \neq b(y) \). Now let \( z \in [y]_a \), i.e. \( b(y) = b(z) \). Then, by assumption, \( a(x) = a(y) = a(z) \) or \( b(x) = b(y) = b(z) \). Since the latter does not hold, we have \( a(y) = a(z) \), i.e. \( z \in [y]_a \).

**Proof of Example 4.10**

1. For \( S \in Y \) and \( C := [S]_r \), \( \chi(C) \) respects compound blocks by Lemma 4.9.4:
   - For \( p \in Y \setminus C \), \( z(p) \neq z(S) \) and so \( [p]_a \subseteq Y \setminus C = [p]_b \).
   - For \( p \in C \), \( z(p) = z(S) \) and so \( [p]_a \subseteq C = [p]_b \).

2. The \select \( \) routine returning the identity respects compound blocks, because for any morphism \( a : D \to A \), \( \ker a \cup \ker \text{id}_D = \ker a \) is a kernel.

3. The constant \( k = 1 \) respects compound blocks, because for all \( p \in Y : [p]_a \subseteq Y = [p]_b \).

**Proof of Proposition 4.11**

\[ \xymatrix@C=1.5cm{ H(A + B) \ar[r]^{(H(A + g), H(f + B))} & H(A + D) \times H(C + B) } \]

is monic.

**Proof.** By finality of \( 1 \), the diagram
\[
\xymatrix{ H(A + B) \ar[r]^{\text{univ}_{H(A + B), H(f + B)}} & H(A + D) \times H(C + B) \\
\ar[u]_{(H(A + g), H(f + B))} \ar[r]_{H(A + 1) \times H(1 + B)} & H(A + 1) \times H(1 + B) }
\]

commutes. Since the diagonal arrow is monic, so is \( (H(A + g), H(f + B)) \).

**Proof of Proposition 4.11.** Define
\[
D_A = \{ x \in D \mid [x]_a \subseteq [x]_b \} \xrightarrow{d_A} D \quad \quad D_B = \{ x \in D \mid [x]_b \not\subseteq [x]_a \} \xrightarrow{d_B} D
\]

By construction and Lemma 4.9, we have the isomorphism \( \phi = [d_A, d_B]^{-1} : D \xrightarrow{\simeq} D_A + D_B \).

We denote the images of the restrictions of \( a \) and \( b \) to \( D_A \) and \( D_B \), respectively, by \( a' : D_A \to A', b' : D_B \to B' \). We claim that we can define maps \( c_A \) and \( c_B \) such that
\[
\xymatrix{ D \ar[r]^{d_A} & D_A \ar[r]^{d_A} & D \ar[r]^{d_B} & D_B \ar[r]^{d_B} & D \ar[r]^{d_A} & D_A \ar[r]^{d_A} & D \ar[r]^{d_B} & D_B \ar[r]^{d_B} & D \ar@{.>}[lu]_{a} \ar@{.>}[lu]_{a} }
\]

\[
\xymatrix{ A \ar[r]^{i_A} & A' \ar[r]^{c_B} & B \ar@{.>}[lu]_{a} \ar@{.>}[lu]_{a} } \quad B \ar[r]^{j_B} & B' \ar@{.>}[lu]_{a} \ar@{.>}[lu]_{a} }\]
1. Put \( c_B(a(x)) = b(x) \) for \( x \in D_A \). Firstly, \( c_B \) is well-defined, because \( a(x_1) = a(x_2) \) implies \( x_2 \in [x_1]_a \subseteq [x_1]_b \) and so \( bx_1 = bx_2 \). Secondly, to see that \( c_B(a) \) is in \( B \setminus B' \), assume \( b(x) \in B' \) for \( x \in D_A \). Then there is \( y \in D_B \) such that \( b(y) = b(x) \). Hence \( a(x) = a(y) \) because \( y \in D_B \) and \( x \in [y]_a \). This leads to the following contradiction:

\[
[x]_b = [y]_b \not\subseteq [y]_a = [x]_a \subseteq [x]_b.
\]

2. Analogously, put \( c_A(b(x)) = a(x) \) for \( x \in D_B \). Well-definedness is proved similarly as for \( c_B \); the image restricts to \( A \setminus A' \) by the same argument as before but with \( \subseteq \) and \( \not\subseteq \) swapped in the last line.

Next we consider the diagram below, which commutes by construction of \( c_A, c_B \):

\[
\begin{array}{ccc}
H D & \xrightarrow{H(a,b)} & H(A \times B) \\
\downarrow H\phi & & \downarrow H\phi \\
H(D_A + D_B) & \xrightarrow{c_A + c_B} & H((A' + A') \times (B \setminus B' + B')) \\
\downarrow H(a+b') & & \downarrow H(a+b') \\
H(A' + B') & \xrightarrow{c_A + c_B} & H((A' + A') \times (B \setminus B' + B')) \\
\end{array}
\]

The left hand morphism at the bottom is monic because \( H \) is zippable and by Lemma A.9. The second morphism is an isomorphism. Thus, the diagonal from \( H(A' + B') \) to \( H(A \times B) \) is also monic and we can conclude using Lemma A.2 for each of the colored monomorphisms:

\[
\ker(\langle H_a, H_b \rangle) = \ker(\langle H_{\pi_1}, H_{\pi_2} \rangle \cdot H(a,b)) = \ker( H(a' + b') \cdot H\phi ) = \ker H(a, b).
\]

**Proof of Corollary 4.12**

**Proposition A.10.** Whenever \( \ker(a : D \to A) = \ker(b : D \to B) \) then \( \ker(a \cdot g) = \ker(b \cdot g) \), for \( g : W \to D \).

**Proof.** The kernel \( \ker(a \cdot g) \) can be obtained uniquely from \( \ker a \) by pasting pullback squares as shown below:

\[
\begin{array}{ccc}
\ker(a \cdot g) & \xrightarrow{\bullet} & W \\
\downarrow & & \downarrow g \\
\bullet & \xrightarrow{\ker a} & D \\
\downarrow a & & \downarrow a \\
W & \xrightarrow{g} & D & \xrightarrow{a}
\end{array}
\]

So if \( \ker a = \ker b \), then \( \ker(a \cdot g) = \ker(b \cdot g) \).

Corollary 4.12 is immediate from

**Lemma A.11.** If \( H : \text{Set} \to \text{Set} \) is zippable and select respects compound blocks, then \( P_{i+1} = \ker( H(\bar{q}_i, q_{i+1}) \cdot \xi ) = P_i \cap \ker( Hq_{i+1} : \xi ) \).

**Proof (Lemma A.11).** For \( H \) zippable and \( f_i, k_i \) as in Algorithm 3.4 we have proved:
select respects compound blocks ⇔ ker $f_i \cup \ker k_{i+1}$ is a kernel
can be trivially decomposed into $(\xi, \text{id}) : (X, GX) \rightarrow (Fx, GX)$. (More generally, trivially decomposing a coalgebra via $\xi \mapsto (\xi, \text{id})$ and composing a multi-sorted coalgebra via $(x, y) \mapsto F(y \cdot x)$, respectively, are the object mappings of an adjoint pair of functors, see [32].) Furthermore, if $q : (X, \xi) \rightarrow (Y, \zeta)$ is a quotient coalgebra, one sees that $(\xi, \text{id})$ is a quotient coalgebra of $(x, y)$ via $(q, Gq \cdot y)$. Conversely, if $(x', y')$ is any quotient coalgebra of $(x, y)$ via some $(q_1, q_2)$, say, than $Fy' \cdot x'$ is a quotient coalgebra of $\xi = Fy \cdot x$. Consequently, $\xi$ is a simple coalgebra iff $(x, y)$ is, and therefore the optimized algorithm in the multi-sorted setting computes the correct partition for a composition of Set-functors.

Now suppose that, $(x', y') : (X', Y') \rightarrow (FY', GX')$ is the simple quotient of $(x, y)$ via $(q_1, q_2) : (X, Y) \rightarrow (X', Y')$, say. Then, $(X', FY' \cdot x')$ is clearly a quotient of $(X, \xi)$ via $q_1$. In order to show simplicity suppose that we have an $FG$-coalgebra morphism $h$ from $(X', FY' \cdot x')$ to some $(Z, \zeta)$. Then $(h, Gh \cdot y')$ is clearly an $H$-coalgebra morphism from $(x', y')$ to $(\xi, \text{id})$ (we consider the two sorts separately below – the right-hand component is trivial and the left-hand component states that $q$ is an $FG$-coalgebra morphism from $(X', FY' \cdot x')$ to $(Z, \zeta)$:

\[
\begin{array}{ccc}
X' & \xrightarrow{x'} & FY' \\
\downarrow^h & & \downarrow^{F(Gh \cdot y')} \\
Z & \xrightarrow{\zeta} & FGZ
\end{array}
\quad
\begin{array}{ccc}
Y' & \xrightarrow{y'} & GX' \\
\downarrow^{Gh} & & \downarrow^{Gh} \\
GZ & \xrightarrow{\text{id}} & GZ
\end{array}
\]

Since $(x', y')$ is simple in Set we know that $(h, Gh \cdot y')$ is monic, whence $h$ is injective and we are done.

**Details for Section 5**

**Proof of Lemma 5.3**

Define $E$ as follows. Compose $\xi$ with $b$ and the inclusion into the set of all maps $A \times Y \rightarrow \mathbb{N}$:

\[
X \xrightarrow{\xi} HY \xrightarrow{b} B(A \times Y) = \mathbb{N}^{(A \times Y)} \hookrightarrow \mathbb{N}^{A \times Y}.
\]

Its uncurrying is a map $\text{cnt} : X \times A \times Y \rightarrow \mathbb{N}$, and we let

\[
E := \bigsqcup_{e \in X \times A \times Y} \text{cnt}(e),
\]

where each $\text{cnt}(e) \in \mathbb{N}$ is considered as a finite ordinal number. By copairing we then obtain a unique morphism $\text{graph} : E \rightarrow X \times A \times Y$ defined on the coproduct components as

\[
\text{cnt}(e) \xrightarrow{i} 1 \xrightarrow{1} X \times A \times Y,
\]

and we put type $= H! \cdot \xi$. Note that if $X$ is finite, then so is $E$, since all $b \cdot \xi(x)$ are finitely supported.
Details for Example 5.4

Proof. We verify (5.1) for each of the three examples.

In general, for $S \subseteq C \subseteq Y$, we have $\text{val} \cdot H_{\chi_S} = \langle H_{\chi_S}, H_{\chi_S}^C, H_{\chi_C \setminus S} \rangle$. Hence to verify the axiom for update = $\text{val} \cdot \text{up}$ it suffices to verify that $\text{up} \cdot \langle \text{fil}_S \cdot \mathbf{b}, H_{\chi_C} \rangle = H_{\chi_S}^C$; in fact, using $w(C) = H_{\chi_C}$ we have:

$$\text{update} \cdot \langle \text{fil}_S \cdot \mathbf{b}, w(C) \rangle = \text{val} \cdot \text{up} \cdot \langle \text{fil}_S \cdot \mathbf{b}, H_{\chi_C} \rangle$$
$$= \text{val} \cdot H_{\chi_S}^C$$
$$= \langle H_{\chi_S}, H_{\chi_S}^C, H_{\chi_C \setminus S} \rangle$$
$$= \langle w(S), H_{\chi_S}^C, w(C \setminus S) \rangle.$$

1. For any $f \in HY = G(Y)$, we have:

$$\text{init}(H!(f), B_{\uparrow} \chi_1 \cdot \mathbf{b}(f)) = (0, \sum_{y \in Y} B_{\uparrow} \chi_1 \cdot \mathbf{b}(f))$$
$$= (0, \sum_{y \in Y} f(y)) = (0, \sum_{y \in Y} f(y)) = G(\chi) Y(f) = w(Y)(f),$$

$$\text{up}(\text{fil}_S(b(f)), H_{\chi_C}(f)) = \text{up}(\{f(y) \mid y \in S\}, \{\sum_{y \in C} f(y), \sum_{y \notin C} f(y)\})$$
$$= (\sum_{y \in Y \setminus C} f(y), \sum_{y \in C} f(y) - \sum_{y \in S} f(y), \sum_{y \in S} f(y))$$
$$= (\sum_{y \in Y \setminus C} f(y), \sum_{y \in C \setminus S} f(y), \sum_{y \in S} f(y)) = H_{\chi_S}^C(f).$$

2. The axiom for init clearly holds since for any $f \in DY$, we have $\sum B_{\uparrow} \chi_1 \cdot \mathbf{b}(f) = \sum_{y \in Y} f(y) = 1$.

For the axiom for up the proof is identical as in the previous point; in fact, note that for an $f \in DY$ all components of the triple $(\sum_{y \in Y \setminus C} f(y), \sum_{y \in C \setminus S} f(y), \sum_{y \in S} f(y))$ are in $[0, 1]$ and their sum is $\sum_{y \in Y} f(y) = 1$. Thus, this triple lies in $D_Y$ and is equal to $D_{\chi_S}^C(f)$.

3. For the refinement interface for $\mathbb{N}(\cdot)$ we argue similarly: if $f$ lies in $\mathbb{N}(Y)$ then $\sum_{y \in Y} f(y)$ lies in $\mathbb{N}$ and so do the components of the triple in the proof of the axiom of up, whence we obtain $\mathbb{N}(\chi_S)(f)$.

4. Let $t = \sigma(y_1, \ldots, y_n) \in H_Y$ with $\sigma$ of arity $n$, let $c_i = \chi_C(y_i)$, and $I = \{1 \leq i \leq n \mid y_i \in S\}$.

$$\text{init}(H^!(t), B_{\uparrow} \chi_1 \cdot \mathbf{b}(t)) = \text{init}(\sigma(0, \ldots, 0), B_{\uparrow} \chi_1 \cdot \{((1, y_1), \ldots, (n, y_n))\})$$
$$= \text{init}(\sigma(0, \ldots, 0), \{1, \ldots, 1\}) = \sigma(1, \ldots, 1)$$
$$= \sigma(\chi_C(y_1), \ldots, \chi_C(y_n)) = H_{\chi_C}^!(t).$$

$$\text{up}(\text{fil}_S \cdot \mathbf{b}(t), H_{\chi_C}(t)) = \text{up}(\text{fil}_S \cdot \{((1, y_1), \ldots, (n, y_n))\}, \sigma(c_1, \ldots, c_n))$$
$$= \text{up}(\{1, \sigma(c_1, \ldots, c_n)\})$$
$$= \sigma(c_1 + (1 \in I), \ldots, c_i + (i \in I), \ldots, c_n + (n \in I))$$
$$= \sigma(\chi_C(y_1), \ldots, \chi_C(y_i), \ldots, \chi_C(y_n))$$
$$= H_{\chi_C}^!(t).$$
In the penultimate step it is used that:

\[
\begin{align*}
y_i \in Y \setminus C & \Rightarrow c_1 + (i \in I) = 0 + 0 = \chi_{\Sigma}^C(y_i) \\
y_i \in C \setminus S & \Rightarrow c_1 + (i \in I) = 1 + 0 = \chi_{\Sigma}^C(y_i) \\
y_i \in S & \Rightarrow c_1 + (i \in I) = 1 + 1 = \chi_{\Sigma}^S(y_i)
\end{align*}
\]

\[\blacktriangleleft\]

**Information in \(w(C)\)**

The functor-specific \(w(C)\) is not always \(H\chi_C\), but always has at least this information:

\[\blacktriangleright\textbf{Proposition A.12.} \text{ For any refinement interface, } H\chi_C = H(1) \cdot \pi_2 \cdot \text{update}(\emptyset) \cdot w(C).\]

**Proof.** The axiom for \text{update} and definition of \text{fil}_\omega makes the following diagram commute:

\[
\begin{array}{ccc}
\{\emptyset!,w(C)\} & \xrightarrow{(b,\text{fil}_\omega,w(C))} & B(Y) \times W \\
\downarrow \text{H}_Y & & \downarrow \text{update} \\
B_i(N \times Y) \times B(N(2)) & \xrightarrow{\text{fil}_\omega \times \text{id}} & B(N(3)) \times B(N(3))
\end{array}
\]

\[\blacktriangleleft\]

**Details for Example 5.5**

**Proof.** We prove (5.1) for the refinement interface of the finite powerset functor.

The axiom for \text{init} is proved analogously as for \(N^{(-)}\) in the proof for Example 5.4.1.

Note that we have \text{update} = \(N^{(0)} \cdot (\triangleright_0) \cdot \text{id} \cdot \text{fil}_B\), where \text{up} is as for \(N^{(-)}\).

Now, we need to show the commutativity of the diagram below:

\[
\begin{array}{ccc}
P_i Y \cong B(Y) & \xrightarrow{\text{in}_B \cdot \text{init}_B} & N(2) \times B(N(3)) \\
\downarrow \text{id} \times \text{fil}_\omega \times \text{id} & & \downarrow \text{id} \times \text{fil}_\omega \times \text{id} \\
B_i(N \times Y) \times B(N(2)) & \xrightarrow{\text{fil}_\omega \times \text{id}} & B(N(3)) \times B(N(3))
\end{array}
\]

The inner left-hand triangle clearly commutes. The square below it involving \text{up} and the middle lower triangle commute as shown in Example 5.4.1. The first and the third component of the remaining right-hand part clearly commute, and for the second component let \(f \in B(Y)\) and compute as follows:

\[
B_{\chi_{\Sigma}}(f) = \left( \bigvee_{y \in Y \setminus C} f(y), \bigvee_{y \in C \setminus S} f(y), \bigvee_{y \in S} f(y) \right)
\]

\[
= \left( 0 < \sum_{y \in Y \setminus C} \text{in}_B f(y), 0 < \sum_{y \in C \setminus S} \text{in}_B f(y), 0 < \sum_{y \in S} \text{in}_B f(y) \right)
\]

\[
= (\triangleright_0)^3 \cdot (g \mapsto \left( \sum_{y \in Y \setminus C} g(y), \sum_{y \in C \setminus S} g(y), \sum_{y \in S} g(y) \right)) \cdot \text{fil}_B(f)
\]

\[
= (\triangleright_0)^3 \cdot N(\chi_{\Sigma}) \cdot \text{fil}_B(f).
\]

\[\blacktriangleleft\]
Details for Example 5.7

For all those examples using the val-function from Example 5.4, first note that val runs in linear time (with a constant factor of 3, because val basically returns three copies of its input).

For all the monoid-valued functors $G(\cdot)$ for an abelian group, for $\mathbb{N}$ and for $\mathcal{D}$, all the operations, including the summation $\Sigma e$, run linearly in the size of the input. If the elements $g \in G$ have a finite representation, then so have the elements of $G^{(2)}$ and thus comparing elements of $g_1, g_2 \in G^{(2)}$ is running in constant time.

For a polynomial functor $H_{\Sigma}$ with bounded arities, we assume that the name of the operation symbol $\sigma \in \Sigma$ is encoded by a constant-size integer. So we can also assume that comparison of these integers run in constant time. Since the signature has bounded arities, the maximum arity available in $\Sigma$ is independent from the concrete morphism $X \rightarrow H_{\Sigma}Y$, so the comparison of the arguments of two flat $\Sigma$-terms $t_1, t_2 \in H_{\Sigma}3$ also runs in constant time.

1. The first parameter of type $H_{\Sigma}1$ can be encoded as simply a operation symbol $\sigma$. Let $t \in H_{\Sigma}2$ be fixed. Then one explicitly implements

$$\text{init}(\sigma, f) = \begin{cases} \sigma(1, \ldots, 1) & \text{if } \text{arity}(\sigma) = |f| \\ t & \text{otherwise.} \end{cases}$$

Both the check and the construction of $\sigma(1, \ldots, 1)$ are bounded linearly by the size of $f$. The second case runs in constant time, since we fixed $t$ beforehand.

2. In $\text{up}(I, \sigma(b_1, \ldots, b_n))$, we can not naively check all the $1 \in I, \ldots, n \in I$ queries, since this would lead to a quadratic run-time. Instead we precompute all the queries’ results together.

1: define the array elem with indices $1 \ldots n$, and each cell storing a value in 2.
2: init elem to 0 everywhere.
3: for $i \in I$ with $i \leq n$ do elem[$i$] := 1.
4: return $\sigma(b_1 + \text{elem}[1], \ldots, b_n + \text{elem}[n])$.

The running time of every line is bound by $|I| + n$.

Proof of Lemma 5.11

Proof. The grouping line 7 takes $O(|X| \cdot \log |X|)$ time. The first loop takes $O(|E|)$ steps, and the second one takes $O(|X| + |E|)$ time in total over all $x \in X$ since init is assumed to run in linear time. For the invariants:

1. By line 6.
2. After the procedure, for $e_i = x_i \overset{a_i}{\rightarrow} y_i, i \in \{1, 2\}$, lastW($e_1$) = lastW($e_2$) iff $x_1 = x_2$.
3. This is just the axiom for init in (5.1), because $Y/Q = \{Y\}$ and $\text{deref} \cdot \text{lastW}(e) = \text{init}(\text{type}(x), B; \pi_1 \cdot b \cdot \xi(x))$.
4. Since $\ker(H_{XY} \cdot \xi) = \ker(H! \cdot \xi)$, this is just the way $X/P$ is constructed. \(\blacksquare\)

Proof of Theorem 5.12

\(\blacksquare\) Lemma A.13. Assume that the invariants hold. Then after part (a) of Figure 5, for the given $S \subseteq C \subseteq Y/Q$ we have:

1. For all $x \in X$: $\text{toSub}(x) = \{e \in E \mid e = x \overset{a}{\rightarrow} y, y \in S\}$
2. For all $x \in X$: $\text{fil}_S \cdot b \cdot \xi(x) = B_t(\pi_2 \cdot \text{graph})(\text{toSub}(x))$. 

3. M : X/P \rightarrow H3 is a partial map with M(B) = H^C_{X_B} \cdot \xi(x), if there exists an e = x \xrightarrow{a} y, x \in B, y \in S, and M(B) is undefined otherwise.

4. For each B \in X/P, we have a partial map mark_B : B \rightarrow \mathbb{N} defined by mark_B(x) = \text{lastW}(x) if there exists some e = x \xrightarrow{a} y, y \in S and mark_B(x) is undefined otherwise.

5. If defined on x, deref \cdot mark_B(x) = w(C,\xi(x)).

6. If mark_B is undefined on x, then fil_S(b \cdot \xi(x)) = \emptyset and then H^C_{X_B} \cdot \xi(x) = H^C_{\emptyset} \cdot \xi(x).

Proof. 1. By lines 2 and 11,

\text{toSub}(x) = \{ e \in \text{pred}(y) \mid y \in S, e = x \xrightarrow{a} y \}

= \{ e \in E \mid y \in S, e = x \xrightarrow{a} y \}.

2. \text{fil}_S(b \cdot \xi(x))(a) = \sum_{y \in S} (b \cdot \xi(x))(a, y) = \sum_{y \in S} | \{ e \in E \mid e = x \xrightarrow{a} y \} |

= | \{ e \in E \mid e = x \xrightarrow{a} y, y \in S \} |.

3. By construction M is defined precisely for those blocks B which have at least one element x with an edge e = x \xrightarrow{a} y to S. Let C = [y]_Q \in Y/Q. Then by invariant 3 we know that M(B) is

\pi_2 \cdot \text{update}(\emptyset, \text{deref} \cdot \text{lastW}(e)) = \pi_2 \cdot \text{update}(\emptyset, w(C,\xi(x)))

= \pi_2 \cdot \text{update}(\text{fil}_S(b \cdot \xi(x)), w(C,\xi(x))) \overset{\text{(5.1)}}{=} H^C_{X_B} \cdot \xi(x)

for some e = x \xrightarrow{a} y, x \in B, y \in S. Since ker(H^C_{X_B} \cdot \xi) = ker(H^C_{X_B} \cdot \xi), invariant 4 proves the well-definedness.

4. This is precisely, how mark_B has been constructed. The well-definedness follows from invariant 2. Note that for any B on which M is undefined, the list mark_B is empty.

5. If mark_B(x) = p_C is defined, then p_C = lastW(e) for some e \in toSub(x), and so deref(p_C) = deref(lastW(e) = w(C,\xi(x)) by invariant 3.

6. If x \in B is not marked in B, then x was never contained in line 3. Hence, toSub(x) = \emptyset, and we have

\text{fil}_S(b \cdot \xi(x))(a) = | \{ e \in E \mid e = x \xrightarrow{a} y, y \in S \} | = |\text{toSub}(x)| = 0.

Furthermore we have

H^C_{X_B} \cdot \xi(x) = \pi_2 \cdot \text{update}(\text{fil}_S \cdot b \cdot \xi(x), w(C,\xi(x))) \overset{\text{(5.1)}}{=} \pi_2 \cdot \text{update}(\emptyset, w(C,\xi(x)))

= \pi_2 \cdot \text{update}(\text{fil}_S \cdot b \cdot \xi(x), w(C,\xi(x))) \overset{\text{as just shown}}{=}

= \pi_2 \cdot \text{update}(\text{fil}_S \cdot b \cdot \xi(x), w(C,\xi(x))) \overset{\text{by definition}}{=}

H^C_{X_B} \cdot \xi(x) \overset{\text{(5.1)}}{=} \pi_2 \cdot \text{update}(\text{fil}_S \cdot b \cdot \xi(x), w(C,\xi(x)))

\textbf{Lemma A.14.} Given M(B) = \nu_B in line 12. Then after line 23, B_{\pi_2} is a partial map B \rightarrow H3 defined by B_{\pi_2}(x) = H^C_{X_B} \cdot \xi(x) if H^C_{X_B} \cdot \xi(x) \neq \nu_B and undefined otherwise.

Proof. Suppose first that x \in B is marked, i.e. we have p_C = mark_B(x) and lines 14–23 are executed. Then we have

\begin{align*}
(w^x, v^x, w^x_{C|S}) &= \text{update}(B,(\pi_2 \cdot \text{graph})(\text{toSub}(x)), \text{deref}(p_C)) \overset{\text{by line 16}}{=}

&= \text{update}(\text{fil}_S \cdot b \cdot \xi(x), w(C,\xi(x))) \overset{\text{by Lemma A.13, 2 and 5}}{=}

&= (w(S,\xi(x)), H^C_{X_B} \cdot \xi(x), w(C \setminus S,\xi(x))) \overset{\text{(5.1)}}{=}
\end{align*}
Thus, if \( H_X^S \cdot \xi(x) = v_\varnothing \), \( B_\varnothing \) remains undefined because of line 21, and otherwise gets correctly defined in line 23.

Now suppose that \( x \in B \) is not marked. Then by Lemma A.13 item 6 we know that 
\( \text{files}(B \cdot \xi(x)) = \varnothing \). Since \((B, v_\varnothing) \in M\), we know by Lemma A.13.3 that 
\( v_\varnothing = H_X^C \cdot \xi(x') \) for some \( x' \in B \). Invariant 4 then implies that 
\( H_X^C \cdot \xi(x) = H_X^C \cdot \xi(x') \) since \( \ker(H_X^C \cdot \xi) = \ker(H_X^C \cdot \xi) \). Then, by Lemma A.13.6, we have 
\[
H_X^C \cdot \xi(x) = H_X^C \cdot \xi(x) = H_X^C \cdot \xi(x') = v_\varnothing.
\]

**Proof of Theorem 5.12.** After Lemma A.14, we know that all \( B \) in \( M \) are refined by \( H_X^S \cdot \xi \).

Now let \( B \) be not in \( M \). Then \( \text{mark}_B \) is undefined everywhere, so for all \( x \in B \), we have by Lemma A.13 item 6 that 
\( H_X^C \cdot \xi(x) = H_X^C \cdot \xi(x) \), which is the same for all \( x \in B \) by invariant 4. Hence any \( B \) not in \( M \) is not split by \( H_X^C \cdot \xi \).

**Proof for Lemma 5.13.**

**Proof.** We denote the former values of \( P, Q, \text{deref}, \text{lastW} \) using the subscript \( \text{old} \).

1. It is easy to see that \( \text{toSub}(x) \) becomes non-empty in line 11 only for marked \( x \), and for those \( x \) it is emptied again in line 20.

2. Take \( e_1 = x_1 \xrightarrow{e_1} y_1, e_2 = x_2 \xrightarrow{e_2} y_2 \).

\[ \Rightarrow \text{Assume } \text{lastW}(e_1) = \text{lastW}(e_2). \] If \( \text{lastW}(e_1) = p_S \) is assigned in line 19 for some marked \( x \), then \( x_1 = x_2 = x \) and \( y_1, y_2 \in S \in Y/Q \). Otherwise, \( \text{lastW}(e_1) = \text{lastW}_\text{old}(e_1) \) and so \( \text{lastW}_\text{old}(e_1) = \text{lastW}_\text{old}(e_2) \) and the desired property follows from the invariant for \( \text{lastW}_\text{old} \).

\[ \Rightarrow \text{If } x_1 = x_2 \text{ and } y_1, y_2 \in D \in Y/Q, \text{ then we perform a case distinction on } D. \] If \( D = S \), then \( \text{lastW}(e_1) = \text{lastW}(e_1) = p_S \). If \( D = C \setminus S \), then \( \text{lastW}(e_1) = \text{lastW}_\text{old}(e_1) = \text{lastW}_\text{old}(e_2) = \text{lastW}(e_2) \). Otherwise, \( D \in Y/Q_\text{old} \setminus \{C\} \) and again \( \text{lastW}(e_1) = \text{lastW}_\text{old}(e_1), i \in \{1, 2\} \).

3. Let \( e = x \xrightarrow{a} y, D := [y]_{c\xi} \in Y/Q \) and do a case distinction on \( D \):

\[
\begin{align*}
D = S \quad &\Rightarrow \text{deref} \cdot \text{lastW}(e) = \text{deref}(p_S) = w_S^x = w(S, \xi(x)), \\
D = C \setminus S \quad &\Rightarrow \text{deref} \cdot \text{lastW}(e) = \text{deref}(p_C) = w_C^x = w(C \setminus S, \xi(x)), \\
D \in Y/Q_\text{old} \setminus \{C\} \quad &\Rightarrow \text{deref} \cdot \text{lastW}(e) = \text{deref}_\text{old} \cdot \text{lastW}(e) = w(D, \xi(x));
\end{align*}
\]

note that the first equation in the second case holds due to lines 10 and 14. For the first two cases note that 
\( w_S^x = w(S, \xi(x)) \) and \( w_C^x = w(C \setminus S, \xi(x)) \) by line 16, Lemma A.13, items 2 and 5, and by the axiom for update in (5.1) (see the computation in the proof of Lemma A.14).

4. Take \( x_1, x_2 \in B' \in X/P \) and \( D \in Y/Q \) and let \( B := [x_1]_{P_\text{old}} = [x_2]_{P_\text{old}} \in X/P_\text{old} \).

a. If \( M(B) \) is defined, then \( H_X^C \cdot \xi(x_1) = H_X^C \cdot \xi(x_2) \) otherwise they would have been put into different blocks in line 24. So \( (x_1, x_2) \in \ker(H_X^D \cdot \xi(x_1)) \) is obvious for \( D = S \) and \( D = C \setminus S \). For any other \( D \in Y/Q_\text{old} \), \( (x_1, x_2) \in \ker(H_X^D \cdot \xi) \) by the invariant of the previous partition.

b. If \( M(B) \) is undefined, then \( \text{mark}_B \) is undefined everywhere, in particular for \( x_1 \) and \( x_2 \). Then, by Lemma A.13 item 6 we have \( H_X^C \cdot \xi(x_i) = H_X^C \cdot \xi(x_i) \) for \( i = 1, 2 \).
Since $C \in Y/Q_{\text{old}}$ and $\ker(H^C_{\phi_0} \cdot \xi) = \ker(H^C_{\chi} \cdot \xi)$, we have $H^C_{\chi} \cdot \xi(x_1) = H^C_{\phi_0} \cdot \xi(x_2)$ by invariant 4, and so $(x_1, x_2) \in \ker(H^C_{\chi} \cdot \xi)$. By case distinction on $D$, we conclude:

\[
D = S \quad \Rightarrow \quad (x_1, x_2) \in \ker(H^C_{\chi} \cdot \xi) \subseteq \ker(H(= 2) \cdot H^C_{\chi} \cdot \xi)
\]

\[
D = C \setminus S \quad \Rightarrow \quad (x_1, x_2) \in \ker(H^C_{\chi} \cdot \xi) \subseteq \ker(H(= 1) \cdot H^C_{\chi} \cdot \xi)
\]

\[
D \in Y/Q_{\text{old}} \setminus \{C\} \quad \Rightarrow \quad (x_1, x_2) \in \ker(H^D_{\chi} \cdot \xi), \quad H^C_{\chi,y} = H^D_{\chi, x}
\]

where the last statement holds by invariant 4 for $Y/Q_{\text{old}}$.

\section*{Proof of Lemma 5.14}

\textbf{Proof.} The for-loop in Figure 5a has $\sum_{y \in S} |\text{pred}(y)|$ iterations, each consisting of constantly many operations taking constant time. Since each loop appends one element to some initially empty $\text{toSub}(x)$, we have

\[
\sum_{y \in S} |\text{pred}(y)| = \sum_{x \in X} |\text{toSub}(x)|
\]

In the body of the for-loop in line 14, the only statements not running in constant time are $\ell := B(\pi_2 \cdot \text{graph}) \cdot \text{toSub}(x)$ (line 15), $\text{update}(\ell, \text{deref}(p_C))$ (line 16), and the loop in line 19. The time for each of them is linear in the length of $\text{toSub}(x)$. The loop in line 14 has at most one iteration per $x \in X$. Hence, since each $x$ is contained in at most one block $B$ from line 12, the overall complexity of line 12 to 23 is at most $\sum_{x \in X} |\text{toSub}(x)| = \sum_{y \in S} |\text{pred}(y)|$, as desired.

\section*{Details on the overall complexity for sortings in line 24}

\textbf{Remark A.15.} Recall that when grouping $Z$ by $f : Z \to Z'$ one calls an element $p \in Z'$ a \textit{possible majority candidate} (PMC) if either

\[
|\{z \in Z \mid f(z) = p\}| \geq |\{z \in Z \mid f(z) \neq p\}|
\]

or if no element fulfilling (A.2) exists. A PMC can be computed in linear time [4, Sect. 4.3.3].

When grouping $Z$ by $f$ using a PMC, one first determines a PMC $p \in Z'$, and then only sorts and groups $\{z \mid f(z) \neq p\}$ by $f$ using an $n \cdot \log n$ sorting algorithm.

\textbf{Lemma A.16.} \textit{Summing over all iterations, the total time spent on grouping $B_{s\phi}$ using a PMC is in $O(|E| \cdot \log |X|)$}.

The proof is the same as in the weighted setting of Valmari and Franceschinis [35, Lemma 5]. For the convenience of the reader, an adaptation to our setting is provided:

\textbf{Proof.} Formally we need to prove that for a family $S_i \subseteq Y/Q_{i}$, $1 \leq i \leq k$, the overall time spend on grouping the $B_{s\phi}$ in all the runs of $\text{SPLIT}$ is in total $O(|E| \cdot \log |X|)$.

First, we characterize the subset $M_B \subseteq B_{s\phi}$ of elements that have edges into both $S_i$ and $C_i \setminus S_i$. In the second part, we show that if we assume each sorting step of $B_{s\phi}$ is bound by $2 \cdot |M_B| \cdot \log(2 \cdot |M_B|)$, then the overall complexity is as desired. In the third part, we use a PMC to argue that sorting each $B_{s\phi}$ is indeed bounded as assumed. Since we assume that comparing two elements of $H3$ runs in constant time, the time needed for sorting amounts to the number of comparisons needed while sorting, i.e. $O(n \cdot \log n)$ many.

1. For a $(B, v_{s\phi}) \in M$ in the $i$th iteration consider $B_{s\phi}$. We define...
Let the number of blocks to which \( H\chi_{C_i} \) provides that we have an invariant \( 4 \) provides that we have outgoing edges of \( x \), not marked, and so \( H\chi_{C_i} \cdot \xi(x) = H\chi_{C_i} \cdot \xi(x) \) holds by Lemma A.13 item 3, the domain of \( B_{\neq 0} \) is \( \mathcal{L}_B \cup M_B^i \). Any \( x \in B \) with no edge to \( S_i \) is not marked, and so \( H\chi_{C_i} \cdot \xi(x) \) holds by Lemma A.13 item 6; by contraposition, any \( x \in \mathcal{L}_B \cup M_B^i \) has some edge into \( S_i \). We can make a similar observation for \( H\chi_{C_i} \cdot \xi \).

If \( x \in B \) has no edge to \( C_i \setminus S_i \), then \( \text{fil}_S(b \cdot \xi(x)) = B_i \pi_1 \cdot b \cdot \xi(x) \) by the definition of \( \text{fil}_S \), and therefore we have:

\[
H\chi_{C_i} \cdot \xi(x) = \pi_2 \cdot \text{update}(\text{fil}_S(b \cdot \xi(x)), w(C_i)) = \pi_2 \cdot \text{update}(B_i \pi_1 \cdot b \cdot \xi(x), w(C_i)) = \pi_2 \cdot \text{update}(\text{fil}_S(b \cdot \xi(x)), w(C_i)) \] \hspace{1cm} (5.1)

By contraposition, all \( x \in M_B^i \) have an edge to \( C_i \setminus S_i \).

Note that \( \ker(H\chi_{C_i} \cdot \xi) = \ker(H\chi_{C_i} \cdot \xi) \).

Since \( \mathcal{L}_B \subseteq B \subseteq X/P_i \) and \( C_i \subseteq Y/Q_i \), invariant 4 provides that we have an \( \ell_B^i \in H \) such that \( \ell_B^i = H\chi_{C_i} \cdot \xi(x) \) for all \( x \in \mathcal{L}_B^i \). Hence, we have obtained \( \ell_B^i \in H \) such that

\[
\mathcal{L}_B^i = \{ x \in B_{\neq 0} \mid H\chi_{C_i} \cdot \xi(x) = \ell_B^i \} \quad \text{and} \quad M_B^i = \{ x \in B_{\neq 0} \mid H\chi_{C_i} \cdot \xi(x) \neq \ell_B^i \}.
\]

2. Let the number of blocks to which \( x \) has an edge be denoted by

\[
\#_{Q}^i(x) = |\{ D \in Y/Q_i \mid e = x \xrightarrow{a} y, y \in D \}|.
\]

Clearly, this number is bounded by the number of outgoing edges of \( x \), i.e. \( \#_{Q}^i(x) \leq |b \cdot \xi(x)| \), and so

\[
\sum_{x \in X} \#_{Q}^i(x) \leq \sum_{x \in X} |b \cdot \xi(x)| = |E|.
\]

Define

\[
\#_{M}^i(x) = |\{0 \leq j \leq i \mid x \in \text{ is in some } M_B^j\}|.
\]

If in the \( i \)th iteration \( x \) is in the middle block \( M_B^i \), then \( \#_{Q}^{i+1}(x) = \#_{Q}^i(x) + 1 \), since \( x \) has both an edge to \( S_i \) and \( C_i \setminus S_i \). It follows that for all \( i \), \( \#_{M}^i(x) \leq \#_{Q}^i(x) \), and

\[
\sum_{x \in X} \#_{M}^i(x) \leq \sum_{x \in X} \#_{Q}^i(x) \leq |E|.
\]

Let \( T \) denote the total number of middle blocks \( M_B \), \( 1 \leq i \leq k \), \( B \) in the \( i \)th \( M \), and let \( M_t \), \( 1 \leq t \leq T \), be the \( t \)th middle block. The sum of the sizes of all middle blocks is the same as summing up, how often each \( x \in X \) was contained in a middle block, i.e.

\[
\sum_{t=1}^{T} |M_t| = \sum_{x \in X} \#_{M}^i(x) \leq |E|.
\]

Using the previous bounds and the obvious \( |M_t| \leq |X| \), we now obtain

\[
\sum_{t=1}^{T} 2 \cdot |M_t| \cdot \log(2 \cdot |M_t|) \leq \sum_{t=1}^{T} 2 \cdot |M_t| \cdot \log(2 \cdot |X|) = 2 \cdot (\sum_{t=1}^{T} |M_t|) \cdot \log(2 \cdot |X|) \leq 2 \cdot |E| \cdot \log(2 \cdot |X|) = 2 \cdot |E| \cdot \log(|X|) + 2 \cdot |E| \cdot \log(2) \in O(|E \cdot \log(|X|)|).
\]
3. We prove that sorting \( B_{\Delta} \) in the \( i \)th iteration is bound by \( 2 \cdot |M_B| \cdot \log(2 \cdot |M_B|) \) by case distinction on the possible majority candidate:

   If \( \ell_B \) is the possible majority candidate, then the sorting of \( B_{\Delta} \) sorts precisely \( M_B \) which indeed amounts to
   \[ |M_B| \cdot \log(|M_B|) \leq 2 \cdot |M_B| \cdot \log(2 \cdot |M_B|). \]

   If \( \ell_B \) is not the possible majority candidate, then \( |L_B| \leq |M_B| \). In this case sorting \( B_{\Delta} \) is bounded by
   \[ (|L_B| + |M_B|) \cdot \log(|L_B| + |M_B|) \leq 2 \cdot |M_B| \cdot \log(2 \cdot |M_B|). \]

Proof of Lemma 5.15

Proof. 1. Clearly \( Q_{i+1} \) is finer than \( Q_i \). Moreover, since \( \chi_{S_i}^C \) merges all elements of \( S_i \) we have \( S_i \in Y/Q_{i+1} \). For any \( i < j \) with \( y \in S_i \) and \( y \in S_j \), we know that \( C_j \subseteq S_i \) since \( C_i \) is the block containing \( y \) in the refinement \( Y/Q_j \) of \( Y/Q_{i+1} \) in which \( S_i \) contains \( y \). Hence, we have \( 2 \cdot |S_i| \leq |C_j| \leq |S_i| \). Now let \( i_1 < \ldots < i_n \) be all the elements in \( \{ i < k \mid y \in S_i \} \). Since \( y \in S_{i_1}, \ldots, y \in S_{i_n} \), we have \( 2^{\ell} \cdot |S_{i_n}| \leq |S_{i_1}| \). Thus \( |\{ i < k \mid y \in S_i \}| = n = \log_2(2^{\ell}) = \log_2(2^{\ell} \cdot |S_{i_n}|) \leq \log_2 |S_{i_1}| \leq \log_2 |Y| \), where the last inequality holds since \( S_{i_1} \subseteq Y \).

2. In the \( O \) calculus we have as the total time complexity:

\[
\sum_{0 \leq i < k} \sum_{y \in S_i} \mid \text{pred}(y) \mid = \sum_{y \in Y} \sum_{0 \leq i < k, S_i \ni y} \mid \text{pred}(y) \mid = \sum_{y \in Y} (\mid \text{pred}(y) \mid \cdot \sum_{0 \leq i < k, S_i \ni y} 1) \\
\leq \sum_{y \in Y} (\mid \text{pred}(y) \mid \cdot \log |Y|) = (\sum_{y \in Y} \mid \text{pred}(y) \mid) \cdot \log |Y| \\
= |E| \cdot \log |Y|,
\]

where the inequality in the second line holds by the first part of our lemma.

Details for Example 5.17

There are two ways to handle the functors of type \( H = I \times G \). The first way is to modify the functor interface as follows:

\[
G1 \leftrightarrow I \times G1 \quad b \rightarrow I \times GY \xrightarrow{\text{pred}} GY \rightarrow B_i(A \times Y) \\
W \leftrightarrow I \times W \quad \text{init} \leftrightarrow \text{id}_I \times \text{init}
\]

update is replaced by the following

\[
\text{update}^I \circ (\text{update}^I) \circ (B_i(A \times W) \times I) \rightarrow (W \times G3 \times W) \times I \rightarrow ((I \times W) \times (I \times G3) \times (I \times W))
\]

where the first and the last morphism are obvious. The second way is to decompose the functor into \( I \times (-) \) and \( G \); this only introduces one new edge per state holding the state’s value in \( I \). Both ways do not affect the complexity in the \( O \) notation. This applies to the examples of unlabeled transition systems and weighted systems.

3. When decomposing a coalgebra \( X \rightarrow \mathcal{P}_I(A \times X) \) (with \( |E| \) edges) into a multisorted coalgebra for \( \mathcal{P}_I \) and \( A \times (-) \), the new sort \( Y \) contains one element per edge. So the multisorted coalgebra has \( |X| + |E| \) states and still \( |E| \) edges, leading to a complexity of \( O((|X| + |E|) \cdot \log(|X| + |E|)) \).
4. To obtain the alphabet size as part of the input to our algorithm we consider DFAs as labelled transition systems encoding the letters of the input alphabet as natural numbers, i.e. coalgebras for $X \rightarrow 2 \times \mathcal{P}_f(\mathbb{N} \times X)$. Decomposing the type functor into $F = 2 \times \mathcal{P}_f$ and $G = \mathbb{N} \times (-)$ we equivalently get a coalgebra $(X,Y) \rightarrow (FY, GX)$ over $\text{Set}^2$.

5. If you have a segala system as a coalgebra $\xi : X \rightarrow \mathcal{P}_f(A \times DX)$, then Baier, Engelen, and Majster-Cederbaum [5] define the number of states and edges respectively as $n = |X|, m_p = \sum_{x \in X} |\xi(x)|$.

The decomposition of this coalgebra results in maps $p : X \rightarrow \mathcal{P}_f Y, a : Y \rightarrow A \times Z, d : Z \rightarrow DX$.

The system $\xi$ mentions at most $m_p$ distributions, so $|Y|, |Z| \leq m_p$. The maps $p$ and $a$ need at most $m_p$ edges, and let $m_d$ denote the number of edges needed to encode $d$. Then we have $n + 2 \cdot m_p$ states and $2 \cdot m_p + m_d$ edges, and thus we have a complexity of $O((m_p + m_d + n) \cdot \log(m_p + n))$

whereas the complexity in [5] is $O((m_p \cdot n) \cdot (\log m_p + \log n)))$.

Details for Section 6

Example A.17. The monotone neighbourhood functor mapping a set $X$ to

$$\mathcal{M}(X) = \{ N \subseteq \mathcal{P}X \mid A \in N \land B \supseteq A \implies B \in N\}$$

is not zippable. There are neighbourhoods which are identified by unzip; indeed let $A = \{a_1, a_2\}, B = \{b_1, b_2\}$ and denote by $(-)^\uparrow$ the upwards-closure:

$$\text{unzip}\left(\{{\{a_1, b_1\}, \{a_2, b_2\}}\right)^\uparrow = \left(\{{\{a_1, *\}, \{a_2, *\}}\right)^\uparrow, \{{*, b_1\}, \{*, b_2\}}\right)^\uparrow$$

$$= \text{unzip}\left(\{{a_1, b_2\}, \{a_2, b_1\}\right)^\uparrow.$$.}