Constraints and Period Relations in Bosonic Strings at Genus-\(g\)

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Abstract

We examine some of the implications of implementing the usual boundary conditions on the closed bosonic string in the hamiltonian framework. Using the KN formalism, it is shown that at the quantum level, the resulting constraints lead to relations among the periods of the basis 1-forms. These are compared with those of Riemanns’ which arise from a different consideration.

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1. Introduction.

Formulation of conformal field theories on arbitrary genus Riemann surfaces has been the focus of intense research in recent years. Primarily fuelled by a need for a systematic study of multiloop diagrams in string theories, it has evolved along two major lines. One approach uses path integrals [1] and the other relies on operator techniques [2]. Recently, in a series of papers [3,4,5], Krichever and Novikov (KN) have introduced a new operator formalism which lends itself to a conceptually simpler and more elegant treatment of such theories. Its appealing feature is that it naturally extends the canonical operator formalism, which has been widely used in studies of conformal field theories on the sphere, to surfaces of arbitrary genus. Basically, they introduced global $\lambda$-differential bases that generalizes the usual Laurent bases which is global only for the sphere. Here, globalization of a theory is obtained by replacing the Laurent basis, in which fields are usually expanded, to these new bases. The process of quantization is then carried out in the traditional way in which the coefficients of the expansions are regarded as operators acting on some Fock space.

Another important aspect of the KN formalism is that it admits a global definition of time which in turn allows for a hamiltonian description of a theory. Lugo and Russo [6] have given such a formulation of the closed bosonic string. By using the KN basis, they have defined a hamiltonian in much the same way as one would do on a sphere and furnished the corresponding equations of motion. In this letter, we address some issues that were not taken up in their paper. Here we examine the boundary conditions and the resulting implications when the theory is quantized. In particular, we show that in implementing these conditions at the quantum level, certain relations among the periods of the basis 1-forms are required. Apart from these physical requirements, it is further shown that the same conditions also have a purely topological origin. Indeed, it turns out that they can be obtained directly from the basis 1-forms. Closer examination of these relations, further reveals strong relationships among the periods which are reminiscent of the period relations of Riemann. In fact, for a restricted subset of the periods, these relations appear to be the higher-order analogs of Riemanns’ relations. For $g = 1$, in particular, it is shown that the period relations obtained above reduce to those of Riemanns’.

2. Some Background and Conventions.

We begin by summarizing some results of Refs.[3,4] pertaining to the KN-bases. Let $\Sigma$ be a Riemann surface of genus $g$ with two distinguished points $P_+$ and $P_-$. One can introduce local coordinates $z_+$ and $z_-$ around these points such that $z_{\pm}(P_{\pm}) = 0$. Now,
in accordance with the Riemann-Roch theorem, the basis of meromorphic functions holomorphic outside \( P_+ \) and \( P_- \) is constructed by specifying the order of the poles or zeros at these points. Explicitly in the neighbourhood of \( P_\pm \) they are given by

\[
A_n(z_\pm) = \sum_{m=0}^{\infty} a_{n,m}^\pm n^{-g/2+m}, \quad |n| > g/2, \quad (2.1a)
\]

\[
A_n(z_\pm) = \sum_{m=0}^{\infty} a_{n,m}^\pm n^{-g/2+1/2-1/2+m}, \quad |n| \leq g/2, \quad n \neq g/2, \quad (2.1b)
\]

\[
A_n(z_\pm) = 1, \quad n = g/2, \quad (2.1c)
\]

where \( a_{n,0}^+ = 1 \) and \( n \) takes integral values if \( g \) is even and half integral values for odd \( g \). Similarly a basis for the space of differentials in the neighbourhood of \( P_\pm \) takes the form

\[
\omega_n(z_\pm) = \sum_{m=0}^{\infty} \omega_{n,m}^\pm n^{-g/2-1+m} d z_\pm, \quad |n| > g/2, \quad (2.2a)
\]

\[
\omega_n(z_\pm) = \sum_{m=0}^{\infty} \omega_{n,m}^\pm n^{-g/2+1/2-1/2+m} d z_\pm, \quad |n| \leq g/2, \quad n \neq g/2, \quad (2.2b)
\]

\[
\omega_n(z_\pm) = \sum_{m=0}^{\infty} \omega_{n,m}^\pm n^{-1+m} d z_\pm, \quad n = g/2, \quad (2.2c)
\]

with the coefficients \( \omega_{n,0}^+ = 1, \omega_{g/2,0}^- = -1 \). It is worth noting that the differentials are dual to the basis of meromorphic functions in the following sense,

\[
\frac{1}{2\pi i} \oint_{C_+} A_m \omega_n = -\frac{1}{2\pi i} \oint_{C_-} A_m \omega_n = \delta_{mn} \quad (2.3)
\]

where \( C_+(C_-) \) denotes any contour around \( P_+(P_-) \) but not including \( P_-(P_+) \). The contours \( C_+ \) and \( C_- \) can be chosen as the level lines of a single-valued function

\[
\tau(P) = 1/2 \int_{P_0}^{P} (\omega_{g/2} + \overline{\omega}_{g/2}) \quad (2.4)
\]

such that

\[
\tau = \{ Q \in \Sigma | \tau(Q) = \tau, \quad \tau \in \mathbb{R} \} \quad (2.5)
\]

for a fixed \( Q_0 \in \Sigma \), so that as \( \tau \rightarrow \pm \infty \) they become circles around \( P_\pm \).

Thus in these bases, any continuously differential function \( F(Q) \) and smooth differential \( \Omega(Q) \) on \( C_\tau \) are given by the following expansions respectively:

\[
F(Q) = \sum_m A_m(Q) F_m \quad \text{where} \quad F_m = \frac{1}{2\pi i} \oint_{C_\tau} F(Q') \omega_m(Q') \quad (2.6)
\]
and
\[ \Omega(Q) = \sum_m \omega_m(Q) \Omega_m \quad \text{where} \quad \Omega_m = \frac{1}{2\pi i} \oint_{C_\tau} \Omega(Q') A_m(Q'). \quad (2.7) \]

Next let us briefly recall some results of Ref. [6] that will be useful in the ensuing discussion. To describe the closed bosonic string it is convenient to introduce the following parameterization:
\[ w(P) = \tau(P) + i\sigma(P). \quad (2.6) \]

Here \( \tau(P) \) is defined in (2.4) and \( \sigma(P) \) is given by
\[ \sigma(P) = \frac{1}{(2i)} \int_{P_0}^P (\omega_{g/2} - \overline{\omega}_{g/2}). \quad (2.7) \]

The phase space for the classical closed bosonic string is defined as the space of functions \( X^\mu \) and the differentials \( P^\mu, \mu = 0, 1, 2, \ldots D-1 \). For a fixed \( \tau \) the phase space is characterized by the poisson bracket
\[ \{P^\mu(Q), X^\nu(Q')\} = -\eta^{\mu\nu} \Delta_\tau(Q,Q'); \quad Q,Q' \in C_\tau \quad (2.8) \]
where \( \eta^{\mu\nu} \) is the \( D \) dimensional Minkowski metric with signature (-1,1,...,1) and \( \Delta_\tau \) is the delta function defined over \( C_\tau \),
\[ \Delta_\tau(Q,Q') = \frac{1}{(2\pi i)} \sum_n \omega_n(Q) A_n(Q'). \quad (2.9) \]

The dynamical content of the theory is embodied in the energy momentum tensor \( t \) which is given by the sum of the holomorphic (\( T \)) and the antiholomorphic (\( \overline{T} \)) parts:
\[ T = -\frac{1}{4}(dX + 2\pi P)^2, \quad \overline{T} = -\frac{1}{4}(dX - 2\pi P)^2. \quad (2.10) \]

The hamiltonian, in turn, is obtained from \( t \) and is given by
\[ H(\tau) = -\frac{1}{(2\pi)} \oint_{C_\tau} (t|e_\sigma) \]
\[ = \frac{1}{(4\pi)} \oint_{C_\tau} (dX^2 + 4\pi^2 P^2|e_\sigma) \quad (2.11) \]
where \( e_\sigma \) is a meromorphic vector field defined in terms of vector fields \( e_w \) and \( e_{\overline{w}} \) that are dual to \( \omega_{g/2} \) and \( \overline{\omega}_{g/2} \):
\[ e_\sigma = i(e_w - e_{\overline{w}}); \quad (2.12) \]
From this Hamiltonian, one is led to the equations of motion which, for $X^\mu$, read

$$\partial \overline{\partial} X^\mu(Q) = 0.$$  \hfill (2.14)

For the conjugate momentum, one has

$$P^\mu = 1/(2\pi)(\partial - \overline{\partial})X^\mu.$$  \hfill (2.15)

It should be noted that the equation holds for all points on $\Sigma$ except for $P_\pm$ and eqn.(2.14), in particular, implies that $\partial X^\mu(\overline{\partial} X^\mu)$ has to be holomorphic (antiholomorphic) everywhere except $P_\pm$. This in turn suggests the following expansions:

$$\partial X^\mu(Q) = (i/\sqrt{2}) \sum_n \alpha^\mu_n \omega_n(Q)$$  \hfill (2.16a)

$$\overline{\partial} X^\mu(Q) = (i/\sqrt{2}) \sum_n \overline{\alpha}^\mu_n \overline{\omega}_n(Q).$$  \hfill (2.16b)

By imposing the boundary conditions

$$\oint_{C_r} dX^\mu = \oint_{\alpha_i} dX^\mu = \oint_{\beta_i} dX^\mu = 0$$  \hfill (2.17)

where $(\alpha_i, \beta_i), \ i = 1, 2, ..., g$ is the standard homology basis, one can integrate $dX^\mu$ to give

$$X^\mu(Q) = \int_{Q_0}^Q dX^\mu = x^\mu - ip^\mu \tau(Q) + (i/\sqrt{2}) \sum_{n \neq g/2} \alpha^\mu_n B_n(Q) + \overline{\alpha}^\mu_n \overline{B}_n(Q)$$  \hfill (2.18)

where $B_n(Q) = \int_Q^Q \omega_n$ and $x^\mu = X^\mu(Q_0)$. It is worth noting that relations (2.17) essentially ensure the single-valuedness of $X^\mu$. When expansions (2.16a) and (2.16b) are introduced, they also imply the following relations among the coefficients $\alpha^\mu_n$ and $\overline{\alpha}^\mu_n$:

$$\alpha^\mu_{g/2} = \overline{\alpha}^\mu_{g/2} = -p^\mu/\sqrt{2}$$  \hfill (2.19)

$$\sum_n \alpha^\mu_n a^i_n + \overline{\alpha}^\mu_n \overline{a}^i_n = 0$$  \hfill (2.20a)

$$\sum_n \alpha^\mu_n b^i_n + \overline{\alpha}^\mu_n \overline{b}^i_n = 0$$  \hfill (2.20b)
where
\[ a^i_n = \oint_{\alpha_i} \omega_n; \quad b^i_n = \oint_{\beta_i} \omega_n. \tag{2.21} \]

Incorporating these into (2.18) one has
\[ X^\mu(Q) = x^\mu - i p^\mu \tau(Q) + (i/\sqrt{2}) \sum_{|n| > g/2} \alpha^\mu_n \phi_n(Q) + \overline{\alpha}^\mu_n \overline{\phi}_n(Q), \tag{2.22} \]

where \( \phi_n(Q) \) are harmonic functions given by
\[ \phi_n(Q) = \int_{Q_0}^Q (\omega_n - \sum_{j=1}^g (F_{nj} \eta_j + G_{nj} \overline{\eta}_j)) \tag{2.23} \]
with \( \{ \eta_i \} \) constituting a basis of holomorphic differentials with normalization
\[ \oint_{\alpha_i} \eta_j = \delta_{ij}, \quad \oint_{\beta_i} \eta_j = \Omega_{ij}, \tag{2.24} \]
and
\[ F_{nj} = (i/2) \sum_{i=1}^g ((\Omega_2^{-1} \Omega)_{ji} a^i_n - (\Omega_2^{-1})_{ji} b^i_n) \tag{2.25a} \]
\[ G_{nj} = (i/2) \sum_{i=1}^g ((\Omega_2^{-1} \overline{\Omega})_{ji} \overline{a}^i_n - (\Omega_2^{-1})_{ji} \overline{b}^i_n) \tag{2.25b} \]
\[ \Omega_2 \equiv \text{Im } \Omega. \]

In quantizing the theory, the usual prescription is to regard the coefficients \( \alpha^\mu_n \) and \( \overline{\alpha}^\mu_n \) in the above expansions as operators acting on some Fock space and by replacing the poisson brackets with the corresponding quantum ones \( (\{ , \} \rightarrow 1/i[ , ]). \) For (2.8), we have
\[ [P^\mu(Q), X^\nu(Q')] = -i \eta^\mu\nu \Delta_\tau(Q, Q'), \quad Q, Q' \in C_\tau \tag{2.26} \]
which leads to the following commutation relations for the coefficients:
\[ [\alpha^\mu_n, \alpha^\nu_m] = \gamma_{nm} \eta^\mu\nu, \quad [\overline{\alpha}^\mu_n, \overline{\alpha}^\nu_m] = \overline{\gamma}_{nm} \eta^\mu\nu, \tag{2.27a} \]
\[ [\alpha^\mu_n, \overline{\alpha}^\nu_m] = 0, \quad [x^\mu, p^\nu] = -i \eta^\mu\nu, \tag{2.27b} \]
where
\[ \gamma_{nm} = (1/2\pi i) \oint_{C_\tau} dA_n A_m. \tag{2.28} \]
The corresponding Fock space is generated by $\alpha_n^\mu, \bar{\alpha}_n^\mu$ with the vacuum defined by

$$\alpha_n^\mu|0 > = \bar{\alpha}_n^\mu|0 > = 0, \quad n \geq g/2 \quad (2.29a)$$

$$< 0|\alpha_n^\mu =< 0|\bar{\alpha}_n^\mu = 0, \quad n < -g/2 \text{ or } n = g/2. \quad (2.29b)$$

Note that it is inconsistent to set $\alpha_n^\mu|0 > = \bar{\alpha}_n^\mu|0 > = 0$ for $n \in I \equiv [-g/2, g/2)$ as this will be incompatible with the commutation relations (since $\gamma_{nm} \neq 0$ for $n, m \in I$). In a quantum theory it is also necessary to introduce normal ordering in products of operators. For the above oscillators it is defined as

$$: \alpha_n^\mu : = \begin{cases} \alpha_n^\mu \alpha_m^\nu & \text{if } n < -g/2 \text{ or } m > g/2 \\ \alpha_m^\nu \alpha_n^\mu & \text{if } n > g/2 \text{ or } m < -g/2 \end{cases} \quad (2.30)$$

As for $n, m \in I$, conditions (2.20a) and (2.20b) induce the following expression:

$$: \alpha_n^\mu \alpha_m^\nu : = : \alpha_n^\mu : \alpha_m^\nu : - \eta^{\mu\nu} \sum_{k < -g/2}^g \sum_{l > g/2}^g (a^{-1})_{im}(a^{-1})_{jm}(\gamma_{kl}F_{il}F_{jk} + \bar{\gamma}_{kl}G_{il}G_{jk}). \quad (2.31)$$

With these definitions the hamiltonian is given by

$$H(\tau) = 1/2 \sum_{n,m} (l_{nm}(\tau) : \alpha_n^\mu : \alpha_m^\nu : + \bar{l}_{nm}(\tau) : \bar{\alpha}_n^\mu \bar{\alpha}_m^\nu : ) \eta^{\mu\nu} \quad (2.32)$$

where

$$l_{nm}(\tau) = (1/2\pi i) \oint_{C_r} (\omega_n|e_w)\omega_m = l_{mn}(\tau). \quad (2.33)$$

3. Constraints.

Having briefly reviewed the formulation of the bosonic string, we will now study some of the implications of the underlying assumptions. Note that conditions (2.17), which ensure the single-valuedness of $X^\mu$, are inherently topological as they arise as a result of $\Sigma$ being non-simply connected. When the appropriate KN expansions are used, they lead to constraints among the coefficients $\alpha_n^\mu$ and $\bar{\alpha}_n^\mu$. With these coefficients being regarded as operators in a quantized theory, it becomes necessary to ask whether these constraints hold as operator equations. For this to be implemented consistently, one must require that the constraints commute among themselves. Moreover, these constraints should not depend on time and must therefore also commute with the hamiltonian.
Before proceeding further, let us denote the constraints as follows:

\[
\psi^\mu \equiv \alpha^\mu_{g/2} + p^\mu / \sqrt{2} = 0, \quad \overline{\psi}^\mu \equiv \overline{\alpha}^\mu_{g/2} + p^\mu / \sqrt{2} = 0
\] (3.1a)

\[
\Phi^\mu_i \equiv \sum_n \alpha^\mu_n a^i_n + \overline{\pi}^\mu_i = 0, \quad \Psi^\mu_i \equiv \sum_n \alpha^\mu_n b^i_n + \overline{\pi}^\mu_i = 0.
\] (3.1b)

Now, with the constraints being regarded as operator equations, we must require that they commute with all \(\alpha^\mu_n, \overline{\alpha}^\mu_n\) for consistency. For \(\psi^\mu\) and \(\overline{\psi}^\mu\), the commutators are trivially satisfied as \(\gamma_m g/2 = \gamma g/2 m = 0\) for every \(m\) while for \(\Phi^\mu_i\) and \(\Psi^\mu_i\) we obtain

\[
[\alpha^\mu_n, \Phi^\nu_j] = \eta^{\mu\nu} \sum_m \gamma_{nm} a^i_m, \quad [\pi^\mu_i, \Phi^\nu_j] = \eta^{\mu\nu} \sum_m \gamma_{nm} \pi^i_m,
\] (3.2a)

\[
[\alpha^\mu_n, \Psi^\nu_j] = \eta^{\mu\nu} \sum_m \gamma_{nm} b^i_m, \quad [\pi^\mu_i, \Psi^\nu_j] = \eta^{\mu\nu} \sum_m \gamma_{nm} \overline{b}^i_m,
\] (3.2b)

which for consistency imply

\[
\sum_m \gamma_{nm} a^i_m = \sum_m \gamma_{nm} \overline{a}^i_m = 0 \quad (3.3a)
\]

\[
\sum_m \gamma_{nm} b^i_m = \sum_m \gamma_{nm} \overline{b}^i_m = 0 \quad (3.3b)
\]

These conditions are sufficient in ensuring that the commutators between the constraints vanish:

\[
[\Phi^\mu_i, \Phi^\nu_j] = \eta^{\mu\nu} \sum_{m,n} (\gamma_{nm} a^i_m a^j_n + \gamma_{nm} \overline{a}^i_n \overline{a}^j_m),
\] (3.4a)

\[
[\Phi^\mu_i, \Psi^\nu_j] = \eta^{\mu\nu} \sum_{m,n} (\gamma_{nm} a^i_m b^j_n + \gamma_{nm} \overline{a}^i_n \overline{b}^j_m),
\] (3.4b)

\[
[\Psi^\mu_i, \Psi^\nu_j] = \eta^{\mu\nu} \sum_{m,n} (\gamma_{nm} b^i_m b^j_n + \gamma_{nm} \overline{b}^i_n \overline{b}^j_m).
\] (3.4c)

Note that \(\psi^\mu\) and \(\overline{\psi}^\mu\) commute trivially with the rest. Likewise, the commutators with the Hamiltonian:

\[
[H(\tau), \psi^\mu] = [H(\tau), \overline{\psi}^\mu] = 0,
\] (3.5a)

\[
[H(\tau), \Phi^\mu_i] = \sum_{n,m,k} (\gamma_{nk} a^i_k l_{nm} a^\mu_m + \gamma_{nk} \overline{a}^i_k \overline{l}_{nm} \overline{a}^\mu_m),
\] (3.5b)

\[
[H(\tau), \Psi^\mu_i] = \sum_{n,m,k} (\gamma_{nk} b^i_k l_{nm} a^\mu_m + \gamma_{nk} \overline{b}^i_k \overline{l}_{nm} \overline{a}^\mu_m).
\] (3.5c)
Note that the elements \( \{ a^i_n \} \) \( \{ b^i_n \} \) \( \{ a^i_{g/2} \} \) \( \{ b^i_{g/2} \} \) vanish under these conditions.

It is important to note that while eqns.(3.3a) and (3.3b) have emerged as consistency requirement in the process of quantization, they are also implied by the KN bases. Indeed, by setting \( \Omega(Q) = dA_n(Q) \) in (2.7) and using (2.28), we have

\[
dA_n(Q) = \sum_m \gamma_{nm} \omega_m(Q)
\]

which essentially expresses an exact form in terms of the basis 1-forms. If we now integrate this relation over a cycle, say \( \alpha_i \), then we find that the integral over \( dA_n(Q) \) is zero since all the periods of an exact 1-form are necessarily zero. The right hand side, on the other hand, reduces to \( \sum_m \gamma_{nm} a_i^m \) which immediately leads us to (3.3a). Similarly (3.3b) is obtained by integrating (3.6) over the cycle \( \beta_i \).

Another point to be noted is that the number of terms in (3.3a) and (3.3b) for each \( n \) and \( i \) are not infinite as it appears but finite because of the following ‘locality’ conditions:

\[
\gamma_{nm} = 0 \quad \text{for} \quad \begin{cases} 
-g > n + m > g, & |n| > g/2, \ |m| > g/2 \\
-g - 1 > n + m > g, & \text{if either} |n| \leq g/2 \ \text{or} \ |m| \leq g/2.
\end{cases} \tag{3.7}
\]

As a result of these restrictions, we find that for each \( n \in I \equiv [-g/2, g/2) \)

\[
\sum_{m=-g-n-1}^{g-n} \gamma_{nm} a^i_m = \sum_{m=-g-n-1}^{g-n} \gamma_{nm} b^i_m = 0 \quad (i = 1, 2, \ldots g). \tag{3.8a}
\]

By noting that \( \gamma_{n,g/2} = 0 = \gamma_{n,n} \), the above equations constitute a system of \( 2g \) equations in \( 2g \) terms which we write explicitly as

\[
\begin{pmatrix}
  a^1_{-g-n-1} & a^1_{-g-n} & \cdots & a^1_{g-n} \\
  \vdots & \vdots & & \vdots \\
  a^g_{-g-n-1} & a^g_{-g-n} & \cdots & a^g_{g-n} \\
  b^1_{-g-n-1} & b^1_{-g-n} & \cdots & b^1_{g-n} \\
  \vdots & \vdots & & \vdots \\
  b^g_{-g-n-1} & b^g_{-g-n} & \cdots & b^g_{g-n}
\end{pmatrix}
\begin{pmatrix}
  \gamma^1_{-g-n-1} \\
  \vdots \\
  \gamma^g_{-g-n-1} \\
  \gamma^1_{n-g-n} \\
  \vdots \\
  \gamma^g_{n-g-n}
\end{pmatrix} = 0. \tag{3.8b}
\]

Note that the elements \( \{ a^i_n, b^i_n, a^i_{g/2}, b^i_{g/2} \} \) have been removed since the \( \gamma \)'s multiplied to these terms are zero. For \( n \not\in I \) the summation over \( m \) differs for different values of \( n \). When \( -3g/2 - 1 \leq n \leq -g/2 - 1 \), \( m \) runs over the interval \([-g-n-1, g-n] \) while
for $g/2 + 1 \leq n < -3g/2 - 1$ it extends over $[-g-n, g-n]$. In both of these cases we have 2$g + 1$ terms in 2$g$ equations. †

Clearly, for the set of equations (3.8) to be consistent, one must require that

$$\det \begin{pmatrix} a_{g-n-1} & a_{-g-n} & \cdots & a_{g-n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{g-n-1} & a_{-g-n} & \cdots & a_{g-n} \\ b_{g-n-1} & b_{-g-n} & \cdots & b_{g-n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{g-n-1} & b_{-g-n} & \cdots & b_{g-n} \end{pmatrix} = 0,$$

(3.9)
since, otherwise, the above equations would imply that all the $\gamma$’s are zero. This is required for all $n \in I$ which means that there are $g$ determinants that must vanish and these lead to relations among the the periods which are independent of the $\gamma$’s.

At this juncture it is worth noting that there are other relations among the periods, namely the period relations of Riemann [7], which have a different origin from the relations above. Indeed, these relations are obtained by considering the normal form $\mathcal{M}$ of $\Sigma$ which is obtained by ‘cutting’ $\Sigma$ along the cycles $\alpha_i$ and $\beta_i$. For a surface of genus $g$, this yields a $4g$-sided polygon which is simply-connected. If $\theta$ is a differential of the first or second kind ‡ then we can write $\theta = df$ on $\mathcal{M}$. With $f$ being single-valued on $\mathcal{M}$ we have for an arbitrary differential $\tau$ [7],

$$\oint_{\delta\mathcal{M}} f \tau = \sum_{i=1}^{g} \left[ \int_{\alpha_i} \theta \int_{\beta_i} \tau - \int_{\alpha_i} \tau \int_{\beta_i} \theta \right] = 2\pi i \left( \text{sum of residues of } (f\tau) \right),$$

(3.10)

which is the period relation between the differentials $\theta$ and $\tau$.

For the basis 1-forms $\{\omega_n\}$, with poles at $P_\pm$ we have

$$C_{nm} \equiv \sum_{i=1}^{g} (a_i^m b_i^n - b_i^m a_i^n) = 2\pi i \left( \text{res}_{P_+} (f_n \omega_m) + \text{res}_{P_-} (f_n \omega_m) \right) \quad n \neq g/2,$$

(3.11)

where $f_n = \int \omega_n$. In the above expression, we have excluded $n = g/2$ since the corresponding differential $\omega_{g/2}$ is a differential of the third kind and cannot therefore be expressed

† Note that for $-3g/2 - 1 \leq n \leq -g/2 - 1$, $m = g/2$ lies in the interval $[-g-n-1, g-n]$ so that the corresponding $a$ and $b$ terms are excluded.

‡ A differential is of the first kind if it is analytic everwhere. It is of the second kind if all the residues of its poles are zero and is of the third kind if it has non-zero residues.
as an exact form. This is the only differential of the third kind in the basis. The 1-forms corresponding to \( n \in I \) are of the first kind while the rest are all of the second kind.

4. Further Analysis.

It is important to note that while the period relations are quadratic in the periods, relations (3.3a) are linear. This naturally raises the question as to whether the above relations imply anything more than those of Riemann’s or, for that matter, whether the two are compatible. To this end, let us see what these relations imply insofar as (3.3a) are concerned. To make the analysis more transparent, it is instructive to consider the \( g = 1 \) case. Besides simplifying the analysis, one also has the advantage of working with an explicit form for the KN bases. Indeed, the basis for \( \{ A_n \} \) is given in terms of the well studied elliptic functions [8,9]:

\[
A_n(z) = \frac{\sigma^{n-1/2}(z - z_0) \sigma(z + 2n z_0)}{\sigma^{n+1/2}(z + z_0)} \frac{\sigma^{n+1/2}(2z_0)}{\sigma((2n + 1)z_0)}, \quad n \neq -1/2, \quad (4.1a)
\]

\[
A_{-1/2}(z) = \frac{\sigma^2(z)}{\sigma(z + z_0) \sigma(z - z_0)} \frac{\sigma(2z_0)}{\sigma^2(z)} \quad (4.1b)
\]

where \( \sigma(z) \) is the Wierstrass sigma-function. It should be noted that the poles are at the points \( z = \pm z_0 \). Its dual basis of one-forms \( \omega_n \) satisfying (2.3) can be written as

\[
\omega_n = A_n(z)dz, \quad n \neq 1/2 \quad (4.2a)
\]

\[
\omega_{1/2} = (A_{1/2}(z) - 4\zeta(z_0) + 2\zeta(2z_0))dz \quad (4.2b)
\]

where \( \zeta(z) \) is the Wierstrass zeta-function. In the above bases, one can compute the \( \gamma \)'s explicitly using (2.28) and these are given by [8]

\[
\gamma_{nm}(z_0) = 0, \quad |n + m| > 1, \quad (n, m) \neq (-\frac{1}{2}, -\frac{3}{2}), \quad (4.3a)
\]

\[
\gamma_{n,1-n}(z_0) = n - \frac{1}{2}, \quad (4.3b)
\]

\[
\gamma_{n,-n}(z_0) = (n + \frac{1}{2})D(2z_0(n + \frac{1}{2})) + (n - \frac{1}{2})D(2z_0(\frac{1}{2} - n)), \quad (4.3c)
\]

\[
\gamma_{n,-1-n}(z_0) = (n + \frac{1}{2})[\zeta'(2z_0(n + \frac{1}{2})) - \zeta'(2z_0)], \quad (4.3d)
\]

\[
\gamma_{-1/2,-3/2}(z_0) = -\frac{\sigma(4z_0)}{\sigma^4(2z_0)} \quad (4.3e)
\]

where \( D(2z_0m) \equiv \zeta(2z_0m) - m\zeta(2z_0) \).
Now the equations (3.3a) corresponding to different values of $n$ read as:

\[
\begin{align*}
\gamma_{-1/2,-3/2} a_{-3/2} + \gamma_{-1/2,3/2} a_{3/2} &= 0, \\
n &= -1/2, \\
n &= 3/2, \\
n &= -3/2, \\
|n| &\geq 5/2, \\
\gamma_{n,-1-n} a_{-1-n} + \gamma_n a_n + \gamma_{n,1-n} a_{1-n} &= 0.
\end{align*}
\]

(4.4a)

For $g = 1$ eqn.(3.9) reduces to

\[
\begin{pmatrix}
a_{3/2} \\
b_{3/2}
\end{pmatrix}
\begin{pmatrix}
a_{-3/2} \\
b_{-3/2}
\end{pmatrix}
= 0
\]

(4.5)

which is precisely the condition that $\mathcal{C}_{3/2,-3/2}$ vanishes. Before evaluating this, it is interesting to note that eqns. (4.4b) and (4.4c) also lead to a similar vanishing determinant. To see this, we eliminate the $a_{-1/2}$ term from (4.4b) and (4.4c) and by using (4.4a) we obtain

\[
\begin{align*}
\gamma_{-3/2,-1/2} \gamma_{-5/2,3/2} a_{-5/2} - \gamma_{3/2,-1/2} \gamma_{5/2,-3/2} a_{5/2} &= 0
\end{align*}
\]

(4.6)

which together with its $\beta$-cycle counterpart requires that

\[
\begin{pmatrix}
a_{5/2} \\
b_{5/2}
\end{pmatrix}
\begin{pmatrix}
a_{-5/2} \\
b_{-5/2}
\end{pmatrix}
= 0.
\]

(4.7)

In fact by pairing equations for $n$ and $-n$ in (4.4d), it can be shown inductively that

\[
\gamma_{-3/2,-1/2} \left( \prod_{m=5/2}^{n} \gamma_{-m,m-1} \right) a_n + (-1)^{n-3/2} \gamma_{3/2,-1/2} \left( \prod_{m=5/2}^{n} \gamma_{m,1-m} \right) a_n = 0
\]

(4.8)

for $n = 5/2, 7/2, \ldots$. This means that one must, in general, have

\[
\begin{pmatrix}
a_n \\
b_n
\end{pmatrix}
\begin{pmatrix}
a_{-n} \\
b_{-n}
\end{pmatrix}
= 0
\]

(4.9)

which in turn demands that $\mathcal{C}_{n,-n}$ vanishes.

Now let us calculate $\mathcal{C}_{n,-n}$ for $|n| \geq 3/2$. To do this, we need an explicit representation of $f_n$. Since $f_n$ is only required in the vicinity of $P_{\pm}$, we have

\[
f_n(z_{\pm}) = \int A_{-n}(z_{\pm}) dz_{\pm} = \sum_{k=0}^{\infty} \frac{a_{-n,k}^{\pm}}{k + n + 1/2} z_{\pm}^{k + n + 1/2},
\]

(4.10)

§ Here we list only the equations with the $a$-cycles. It should be noted that these equations are paired with a similar equations in which the $a$-terms are replaced by the $b$-terms.
where the coefficients $a_{n,k}^{\pm}$ can be obtained from the following integral:

$$a_{n,k}^{\pm} = \oint_{\pm z_0} \frac{dz}{2\pi i} (z \mp z_0)^{\mp n-k-1/2} A_n(z). \quad (4.11)$$

Then by using the above form for $f_n$, the residues in (3.11) can be computed and this gives

$$C_{n,m} = 2\pi i \left\{ \sum_{l=0}^{n+m-1} \frac{a_{n,n+m-1-l}^{+} a_{n,m,l}^{+}}{m-l - 1/2} - \sum_{l=0}^{n-m-1} \frac{a_{n,-n-m-1-l}^{-} a_{n,m,l}^{-}}{-m-l - 1/2} \right\}. \quad (4.12)$$

Note that the first sum is zero for $n+m < 1$ while the second is zero for $n+m > 1$. From this we can surmise that $C_{n,m} = 0$ for $m = -n$ which means that both (4.5) and (4.9) are compatible with the period relations. This, however, does not conclusively prove that all the equations are consistent. Indeed, relation (4.8) only replaces one of the two paired equations (corresponding to $n$ and $-n$ in (4.4d)). For full consistency, one must also show compatibility for the remaining equations. To this end consider (4.4d) with $n$ replaced by $-n$:

$$\gamma_{-n,1+n} a_{1+n} + \gamma_{-n,n} a_n + \gamma_{-n,n-1} a_{n-1} = 0 \quad (4.13)$$

with the corresponding $\beta$-cycle equation:

$$\gamma_{-n,1+n} b_{1+n} + \gamma_{-n,n} b_n + \gamma_{-n,n-1} b_{n-1} = 0. \quad (4.14)$$

Then by multiplying (4.13) by $b_m$ and (4.14) by $a_m$ and subtracting, we have

$$\gamma_{-n,1+n} C_{1+n,m} + \gamma_{-n,n} C_{n,m} + \gamma_{-n,n-1} C_{n-1,m} = 0 \quad (4.15)$$

which should hold for all $n$ and $m$. It is not very easy to verify this for generic values of $n$ and $m$, particularly when $|n+m| \gg 1$, since this entails computing a large number of coefficients $a_{n,k}^{\pm}$ in the sum of eqn.(4.12). However, when $|n+m|$ is small, the task becomes tractable. For instance when $n+m = 0$, we have

$$C_{1+n,-n} = 2\pi i \frac{a_{-n,0,n,0}^{+}}{n - 1/2} = -\frac{2\pi i}{n + 1/2}, \quad (4.16a)$$

$$C_{n-1,-n} = 2\pi i \frac{a_{-n,0,n,0}^{-}}{n - 1/2} = \frac{2\pi i}{n - 1/2} \frac{\sigma^2(2z_0)\sigma^2((2n-1)z_0)}{\sigma((2n+1)z_0)\sigma((2n-3)z_0)} \quad (4.16b)$$

Then by using the identity (see Ref.[9])

$$\varphi(u) - \varphi(v) = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)} \quad (4.17)$$
where \( \varphi(u) = -\zeta'(u) \) is the Wierstrass \( \varphi \) function, one can show that (4.15) with \( m = -n \) holds for all \( |n| \geq 3/2 \). Similarly we have also verified the equation for \( n + m = \pm 1 \).

5. Concluding Remarks.

In the analysis for \( g = 1 \) above, we have shown that for (3.9) at least, there is no inconsistency with the corresponding period relation. In fact the condition coincides exactly with the period relation. The situation for \( g > 1 \) is not quite the same. Here we have \( g \) vanishing determinants, each resulting in an equation of order \( 2g \) in the periods. The period relations of Riemann on the other hand are always quadratic in the periods irrespective of \( g \). For the \( 2g \) periods appearing in each determinant, we have \( g(2g - 1) \) period relations that must be compatible. For example, in the \( g = 2 \) case the two vanishing determinants are

\[
\begin{vmatrix}
a^{1}_{-2} & a^{1}_{0} & a^{1}_{2} & a^{1}_{3} \\
a^{2}_{-2} & a^{2}_{0} & a^{2}_{2} & a^{2}_{3} \\
b^{1}_{-2} & b^{1}_{0} & b^{1}_{2} & b^{1}_{3} \\
b^{2}_{-2} & b^{2}_{0} & b^{2}_{2} & b^{2}_{3}
\end{vmatrix}
\quad \text{and} \quad
\begin{vmatrix}
a^{1}_{-3} & a^{1}_{-2} & a^{1}_{-1} & a^{1}_{2} \\
a^{2}_{-3} & a^{2}_{-2} & a^{2}_{-1} & a^{2}_{2} \\
b^{1}_{-3} & b^{1}_{-2} & b^{1}_{-1} & b^{1}_{2} \\
b^{2}_{-3} & b^{2}_{-2} & b^{2}_{-1} & b^{2}_{2}
\end{vmatrix}
\]

which are quartic in the periods. The first determinant should be compared with six period relations associated with the 1-forms \( \{\omega_{-2}, \omega_{0}, \omega_{2}, \omega_{3}\} \) i.e. \( \{C_{-2,0}, C_{-2,2}, C_{-2,3}, C_{0,2}, C_{0,3}, C_{-1,2}\} \) while the second determinant should be checked with those among \( \{\omega_{-3}, \omega_{-2}, \omega_{-1}, \omega_{2}\} \). We would also like to remark that conditions (3.3a) and (3.3b) appear to be very strong in the following sense. Apart from being linear, they also relate the \( a \) and the \( b \) periods separately. In the \( g = 1 \) case for instance, all the \( a \)-periods can be expressed in terms of the \( a_{-1/2} \) and \( a_{3/2} \) (or \( a_{3/2} \)) through the equations (4.4a-4.4d) and (4.8). The \( b \)-periods are also similarly related. The period relations of Riemann on the other hand, relate the \( a \) and \( b \) periods of two 1-forms quadratically.

It is also interesting to note that the conditions (3.3a) and (3.3b) also arise under more general boundary conditions. If we allow \( X^\mu \) to be multi-valued then conditions (2.17) are replaced by [10]

\[
\oint_{C_r} dX^\mu = 2\pi l^\mu, \quad \oint_{\alpha_i} dX^\mu = 2\pi n_i^\mu, \quad \oint_{\beta_i} dX^\mu = 2\pi m_i^\mu
\]

where \( l^\mu, m_i^\mu, n_i^\mu \in \mathbb{Z} \). It is easy to see that the commutators as in (3.2a) and (3.2b) will also reduce the corresponding constraints to (3.3a) and (3.3b).

Finally to summarize briefly, we have shown that in fulfilling the boundary conditions in the closed bosonic string at genus \( g \), certain constraints on the coefficients of the fields and the periods are implied. At the quantum level, the implementation of these constraints
as operator equations lead to linear relations among the periods of the basis 1-forms. We have further shown that these relations are also intrinsic to the 1-forms. Some of these relations imply relationships among the periods that appear to be the higher-order analogs of Riemann’s period relations. In the $g = 1$ case, the relations turn out to be precisely those of Riemann's.
REFERENCES
[1.] Polyakov A.M. (1981) Phys. Lett. B103 207, 211.
[2.] Ishibashi N., Matsuo Y. and Ooguri H. (1987) Mod. Phys. Lett. A2 119; Vafa C. (1987) Phys. Lett. B190 47.
[3.] Krichever I.M. and Novikov S.P. (1987) Funct. Anal. Pril. 21 No.2 46.
[4.] Krichever I.M. and Novikov S.P. (1987) Funct. Anal. Pril. 21 No.4 47.
[5.] Krichever I.M. and Novikov S.P. (1989) Funct. Anal. Pril. 23 No.1 24.
[6.] Lugo A. and Russo J. (1989) Nucl. Phys. B322 210.
[7.] Cohn H. (1967) Conformal Mapping on Riemann Surfaces, (McGraw-Hill).
[8.] Mezincescu L., Nepomechie R. I. and Zachos C.K. (1989) Nucl. Phys. B315 43.
[9.] Chandrasekharan K. (1985) Elliptic Functions, (Springer Verlag).
[10.] Russo J. (1989) Mod. Phys. Lett. A4 2349.