Extended Complex Yang-Mills Instanton Sheaves

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(Dated: September 26, 2018)

Abstract

In the search of YM instanton sheaves with topological charge two, the rank of β matrix in the monad construction can be dropped from the bundle case with rank β= 2 to either rank β= 1 [4] or 0 on some points of CP³ of the sheaf cases. In this paper, we first show that the sheaf case with rank β= 0 does not exist for the previous construction of SU(2) complex YM instantons [3]. We then show that in the new ”extended complex YM instantons” discovered in this paper, rank β can be either 2 on the whole CP³ (bundle) with some given ADHM data or 1, 0 on some points of CP³ with other ADHM data (sheaves). These extended SU(2) complex YM instantons have no real instanton counterparts.

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I. INTRODUCTION

Recent developments of complex Yang-Mills (YM) instantons [1–6] have revealed many new mathematical structures of YM instantons which were not realized in the previous real YM instantons [7–17]. In particular, for the case of $SU(2)$ complex YM instantons (or non-compact $SL(2,\mathbb{C})$ [18], real YM instantons [19, 20]), it was shown that for some ADHM data at some points on $CP^3$ the vector bundle description of 2-instanton breaks down, and one is led to use a description in terms of torsion free sheaves for these ”instanton sheaves” on $CP^3$ [4]. The existence of these 2-instanton sheaves was soon extended to the complex $k$-instanton sheaves with higher topological charges $k = 3, 4$ [5].
On the other hand, one unexpected result obtained in \cite{6} was the great simplification or "solvability" of calculation of the field strength $F$ associated with the sheaf ADHM data, and the explicit form of a class of $SU(2)$ complex YM 2-instanton field strength without removable singularities can be exactly calculated! This "solvability" was not available for the previous real YM $k$-instanton calculation with $k \geq 2$, and was presumably related to the imposing of the sheaf condition $[1, 2]$ on YM instanton sheaves.

Moreover, one sheaf line which is the real line supporting sheaf points on $CP^3$ of $SU(2)$ YM 2-instanton sheaf was identified and found to be a special jumping line over $S^4$ spacetime $[6]$. In addition, the order of the singularity structure of the connection $A$ and the field strength $F$ at the corresponding singular point on $S^4$ associated with this sheaf line in $CP^3$ was found to be higher than those of other singular points associated with the normal jumping lines $[6]$. These results suggest that the existence of YM instanton sheaves on $CP^3$ is closely related to the new singular structure of the corresponding $A$ and $F$ on $S^4$ spacetime.

In the search of YM instanton sheaves with topological charge two $[4]$, the rank of $\beta$ matrix in the monad construction can be dropped from 2 of the bundle case to either 1 presented in $[4]$ or 0 on some points of $CP^3$ of the sheaf cases. In this paper, we will first show that the sheaf case of rank $\beta=0$ does not exist for the previous construction of $SU(2)$ complex YM instantons ($\det \neq 0$). We then show that in the new "extended complex YM instantons" ($\det = 0$) discovered in this paper, the rank of $\beta$ can be either 2 on the whole $CP^3$ with some given ADHM data (bundle) or 1, 0 on some points of $CP^3$ with other ADHM data (sheaves). We will see that these new extended $SU(2)$ complex YM instantons have no real instanton counterparts. See Eq.\,(3.48) and Eq.\,(3.49) in section III for details.

One interesting result we obtained in the search of extended complex YM instantons was the discovery of the existence of instanton sheaf structure with diagonal $y$ ADHM data. For the $\det \neq 0$ complex YM instantons constructed previously $[3]$, it was shown $[4]$ that there was no instanton sheaf solutions with diagonal $y$ ADHM data. Recent results of new real YM instantons on some special backgrounds can be found at $[21, 22]$.

The new complex YM 2-instanton solutions discovered in this paper do not increase the number of moduli parameters $16k - 6$ ($k = 2$) of the previous construction $[3]$. They are "limiting solutions" living on the boundary of the moduli space. We expect the existence
of these new extended complex YM instanton solutions for general topological charge $k$. Recently, some of irreducible components of the boundary of the moduli space of instanton sheaves of charge $k$ on $CP^3$ have been constructed in [23, 24] by mathematicians. In this paper, we will give some explicit constructions for $k = 2$ case which will be more accessible to physicists. The relation of the construction in this paper to the results in [23, 24] remains to be studied.

This paper is organized as following. In section II, we review the first complex YM 2-instanton sheaves we discovered in [4]. These complex YM 2-instanton sheaves are of rank $\beta = 1$ and $\det \neq 0$. In section III, we show that there is no complex YM 2-instanton sheaves with rank $\beta = 0$ and $\det \neq 0$. The results of section III led us to consider the complex YM 2-instanton sheaves with $\det = 0$ which were calculated in section IV. In section IV, we discover many complex YM instanton solutions with rank $\beta = 0, 1$ (sheaves) and 2 (bundle) respectively. These solutions are all of $\det = 0$, and thus are new and not considered previously. For illustration, we give examples of explicit complex YM 2-instanton solutions for each case. In addition, an example of new extended complex YM 2-instanton field strength for the sheaf case with rank $\beta = 0$ and $\det = 0$ will be given in Eq.(4.163). Finally, the conclusion will be given in section V.

II. REVIEW OF YM 2-INSTANTON SHEAVES WITH RANK $\beta = 1$ AND $\det \neq 0$

In this section, we briefly review the biquaternion construction of $SU(2)$ complex ADHM instantons [3, 4]. We will use three approaches to construct the complex YM instantons. We will pay attention to the existence of sheaf structures of YM 2-instanton sheaves [3, 5] in the monad construction.

A. The Biquaternion construction of $SU(2)$ complex ADHM instantons

In this approach, the first step was to introduce the $(k + 1) \times k$ biquaternion matrix $\Delta(x) = a + bx$

$$\Delta(x)_{ab} = a_{ab} + b_{ab}x, \quad a_{ab} = a^\mu_{ab}e_\mu, \quad b_{ab} = b^\mu_{ab}e_\mu \quad (2.1)$$
where $a_{ab}^\mu$ and $b_{ab}^\mu$ are complex numbers, and $a_{ab}$ and $b_{ab}$ are biquaternions. In the biquaternion construction of $SU(2)$ complex ADHM instantons, the quadratic condition on the biquaternion matrix $\Delta(x)$ of $SU(2)$ complex instantons reads

$$\Delta(x)^\circ \Delta(x) = f^{-1} = \text{symmetric, non-singular } k \times k \text{ matrix for } x \notin J \quad (2.2)$$

where for $x \in J$,

$$\det \Delta(x)^\circ \Delta(x) = 0. \quad (2.3)$$

The set $J$ is called jumping lines. An explicit example of jumping lines will be given in Eq. (4.171). The biconjugation in Eq. (2.3) is defined in Eq. (2.5) below. There are no jumping lines for $SU(2)$ real instantons on $S^4$.

The biconjugation $[25]$ of a biquaternion

$$z = z_\mu e_\mu, \quad z_\mu \in C, \quad (4.4)$$

is defined to be

$$z^\circ = z_\mu e_\mu^\dagger = z_0 e_0 - z_1 e_1 - z_2 e_2 - z_3 e_3 = x^\dagger + y^\dagger i. \quad (2.5)$$

In some occasion, the unit quarternions can be expressed as Pauli matrices

$$e_0 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_i \rightarrow -i \sigma_i; \quad i = 1, 2, 3. \quad (2.6)$$

We will use the norm square of a biquaternion, and it is defined to be

$$|z|^2_c = z^\circ z = (z_0)^2 + (z_1)^2 + (z_2)^2 + (z_3)^2, \quad (2.7)$$

which is a complex number in general. This property will turn out to be important for the construction of the new extended complex YM instantons in this paper. See Eq. (3.48) in section III.

It was shown that $[4]$ there are no sheaf structures for the $SU(2)$ complex YM 1-instanton neither complex diagonal $k$-instantons (for $k = 2$ case with det $\neq 0$). So, for simplicity, we will choose $k = 2$ to calculate and identify complex YM instanton sheaf structures of the $SU(2)$ (diagonal and non-diagonal) complex YM 2-instantons with det $= 0$ in this paper.
B. The $SU(2)$ complex ADHM equations

The second method to construct $SU(2)$ complex ADHM data is to solve the complex ADHM equations \([26, 27]\)

\[
[B_{11}, B_{12}] + I_1 J_1 = 0, \tag{2.8a}
\]

\[
[B_{21}, B_{22}] + I_2 J_2 = 0, \tag{2.8b}
\]

\[
[B_{11}, B_{22}] + [B_{21}, B_{12}] + I_1 J_2 + I_2 J_1 = 0. \tag{2.8c}
\]

In this approach, one key step is to use the explicit matrix representation (EMR) \([4]\) of the biquaternion and do the rearrangement rule \([4]\) to explicitly identify the complex ADHM data \((B_{lm}, J_m, J_m)\) with \(l, m = 1, 2\) from the \(\Delta(x)\) matrix in Eq.\((2.1)\).

For illustration, we calculate the $SU(2)$ diagonal 2-instanton case. In the EMR, a biquaternion can be written as a $2 \times 2$ complex matrix

\[
z = z^0 e_0 + z^1 e_1 + z^2 e_2 + z^3 e_3
\]

\[
= \begin{bmatrix}
(a^0 + b^3) + i (b^0 - a^3) & (-a^2 + b^1) + i (-b^2 - a^1) \\
(a^2 + b^1) + i (b^2 - a^1) & (a^0 - b^3) + i (b^0 + a^3)
\end{bmatrix}
\tag{2.9}
\]

where \(a^\mu\) and \(b^\mu\) are real and imaginary parts of \(z^\mu\) respectively. For the diagonal 2-instanton

\[
a = \begin{bmatrix}
\lambda_1 & \lambda_2 \\
y_{11} & 0 \\
0 & y_{22}
\end{bmatrix}
\]

\[
\begin{bmatrix}
p_1 + iq_1 & 0 & p_2 + iq_2 & 0 \\
0 & p_1 + iq_1 & 0 & p_2 + iq_2 \\
y_{11}^0 - iy_{11}^3 & (y_{11}^2 + iy_{11}^1) & 0 & 0 \\
y_{11}^2 - iy_{11}^1 & y_{11}^0 + iy_{11}^3 & 0 & 0 \\
0 & 0 & y_{22}^0 - iy_{22}^3 & -(y_{22}^2 + iy_{22}^1) \\
0 & 0 & y_{22}^2 - iy_{22}^1 & y_{22}^0 + iy_{22}^3
\end{bmatrix}
\tag{2.10}
\]

\[
\begin{bmatrix}
 p_1 + iq_1 \\
0 \\
y_{11}^0 - iy_{11}^3 \\
y_{11}^2 - iy_{11}^1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
p_2 + iq_2 \\
p_1 + iq_1 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
y_{11}^0 - iy_{11}^3 \\
y_{11}^2 - iy_{11}^1 \\
y_{22}^0 - iy_{22}^3 \\
y_{22}^2 - iy_{22}^1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
y_{11}^0 + iy_{11}^3 \\
y_{11}^2 + iy_{11}^1 \\
y_{22}^0 + iy_{22}^3 \\
y_{22}^2 + iy_{22}^1
\end{bmatrix}
\tag{2.11}
\]

\[
\begin{bmatrix}
 J_1 & J_2 \\
B_{11} & B_{21} \\
B_{12} & B_{22}
\end{bmatrix}
\]
where in Eq.(2.11) we have performed the rearrangement rule for an element $z_{ij}$ in a

\[ z_{2n-1,2m-1} \rightarrow z_{n,m}, \]
\[ z_{2n-1,2m} \rightarrow z_{n,k+m}, \]
\[ z_{2n,2m-1} \rightarrow z_{k+n,m}, \]
\[ z_{2n,2m} \rightarrow z_{k+n,k+m}. \] \hspace{1cm} (2.12)

The EMR and the rearrangement rule for $a^\circ$ can be similarly performed. With the identification in Eq.(2.11) (and a similar one for $a^\circ$), one can show that the quadratic condition in Eq.(2.2) leads to the complex ADHM equations in Eq.(2.8a) to Eq.(2.8c).

For the $SU(2)$ real ADHM instantons, one imposes the conditions

\[ I_1 = J^\dagger, J_2 = -I, J_1 = I^\dagger, J_2 = J, \]
\[ B_{11} = B_2^\dagger, B_{12} = B_1^\dagger, B_{21} = -B_1, B_{22} = B_2 \] \hspace{1cm} (2.13a)

to recover the real ADHM equations

\[ [B_1, B_2] + IJ = 0, \] \hspace{1cm} (2.14a)
\[ [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0. \] \hspace{1cm} (2.14b)

C. The monad construction and YM 2-instanton sheaves

The third method to construct $SU(2)$ complex ADHM instantons is the monad construction. This method is particular suitable for constructing instanton sheaves. One introduces the $\alpha$ and $\beta$ matrices as functions of homogeneous coordinates $z, w, x, y$ of $CP^3$ and defines

\[ \alpha = \begin{bmatrix} zB_{11} + wB_{21} + x \\ zB_{12} + wB_{22} + y \\ zJ_1 + wJ_2 \end{bmatrix}, \] \hspace{1cm} (2.15a)
\[ \beta = \begin{bmatrix} -zB_{12} - wB_{22} - y \\ zB_{11} + wB_{21} + x \\ zI_1 + wI_2 \end{bmatrix}. \] \hspace{1cm} (2.15b)

It can be shown that the condition

\[ \beta\alpha = 0 \] \hspace{1cm} (2.16)

is satisfied if and only if the complex ADHM equations in Eq.(2.8a) to Eq.(2.8c) holds.
In the monad construction of the holomorphic vector bundles, either $\beta$ is not surjective or $\alpha$ is not injective at some points of $CP^3$ for some ADHM data, the dimension of $(\text{Ker } \beta / \text{Im } \alpha)$ varies from point to point on $CP^3$, and one encounters "instanton sheaves" on $CP^3$. In our previous publication [1], we discovered that for some ADHM data at some sheaf points on $CP^3$, there exists a common eigenvector $u$ in the costable condition $\alpha u = 0$ or 

\[(zB_{11} + wB_{21}) u = -xu, \quad (2.17a)\] 
\[(zB_{12} + wB_{22}) u = -yu, \quad (2.17b)\] 
\[(zJ_{1} + wJ_{2}) u = 0. \quad (2.17c)\]

So $\alpha$ is not injective there and the dimension of $(\text{Ker } \beta / \text{Im } \alpha)$ is not a constant over $CP^3$. Similar discussion can be done for cases with $\beta$ not surjective [1, 6]. That is, for some ADHM data at some sheaf points on $CP^3$, there exists a common eigenvector $v$ in the stable condition [1]

\[v\beta = 0. \quad (2.18)\]

We will choose to work on rank $\beta$ in this paper.

The first example of YM instanton sheaf discovered in [4] was the 2-instanton sheaf. For points $[x : y : z : w] = [0 : 0 : 1 : 1]$ on $CP^3$ with the ADHM data

\[
\begin{bmatrix}
\lambda_1 & \lambda_2 \\
y_{11} & y_{12} \\
y_{12} & y_{22}
\end{bmatrix} =
\begin{bmatrix}
a & 0 & 0 & ia \\
0 & a & ia & 0 \\
\frac{1}{\sqrt{2}}a & 0 & 0 & \frac{a}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}}a & \frac{a}{\sqrt{2}} & 0 \\
0 & a & \frac{1}{\sqrt{2}}a & 0 \\
\frac{a}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}}a
\end{bmatrix}, \quad a \in C, a \neq 0, \quad (2.19)
\]

it was shown that $\alpha$ is not injective.

Moreover, to understand the change of dimensionality of vector bundles at the sheaf points, we can calculate $\alpha$ and $\beta$ at the sheaf point $[x : y : z : w] = [0 : 0 : 1 : 1]$ to be
\[(a \neq 0)\]

\[
\alpha_{[0:0:1:1]} = \begin{pmatrix}
-\frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} \\
\frac{a}{\sqrt{2}} & i\frac{a}{\sqrt{2}} \\
\frac{a}{\sqrt{2}} & \frac{i a}{\sqrt{2}} \\
a & ia \\
a & ia
\end{pmatrix}, \beta_{[0:0:1:1]} = \begin{pmatrix}
\frac{ia}{\sqrt{2}} & -\frac{a}{\sqrt{2}} & \frac{i a}{\sqrt{2}} & \frac{a}{\sqrt{2}} \\
-\frac{a}{\sqrt{2}} & \frac{ia}{\sqrt{2}} & \frac{a}{\sqrt{2}} & -ia \\
\frac{i a}{\sqrt{2}} & \frac{a}{\sqrt{2}} & -ia & ia \\
a & i a & a & \end{pmatrix}, \tag{2.20}
\]

which are both of rank 1. So the dimensions of \(\text{Im} \alpha_{[0:0:1:1]}\) and \(\text{Ker} \beta_{[0:0:1:1]}\) are 1 and 5 (or \(\text{rank} \beta = 1\)) respectively, which imply the dimension of the quotient space

\[\dim(\text{Ker} \beta_{[0:0:1:1]} / \text{Im} \alpha_{[0:0:1:1]}) = 5 - 1 = 4. \tag{2.21}\]

Note that for points other than sheaf points, \(\dim(\text{Ker} \beta / \text{Im} \alpha) = 4 - 2 = 2\).

**III. NON-EXISTENCE OF YM 2-INSTANTON SHEAVES WITH RANK \(\beta = 0\) AND \(\det \neq 0\)**

Following the discovery of YM 2-instanton sheaves with rank \(\beta = 1\) on some points of \(CP^3\) (see Eq.(2.20)), it is very natural to look for the rest case of YM 2-instanton sheaves with rank \(\beta = 0\) on some points of \(CP^3\) with given ADHM data. We first define the ADHM data as

\[y_{11} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}, y_{22} = -\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \tag{3.22}\]

\[\lambda_1 = \begin{pmatrix} \lambda^0_1 - i \lambda^3_1 - (\lambda^2_1 + i \lambda^1_1) \\ \lambda^2_1 - i \lambda^1_1 \end{pmatrix}, \lambda_1^\oplus = \begin{pmatrix} \lambda^0_1 + i \lambda^3_1 & \lambda^2_1 + i \lambda^1_1 \\ - (\lambda^2_1 - i \lambda^1_1) & \lambda^0_1 - i \lambda^3_1 \end{pmatrix}, \tag{3.23}\]

\[\lambda_2 = \begin{pmatrix} \lambda^0_2 - i \lambda^3_2 - (\lambda^2_2 + i \lambda^1_2) \\ \lambda^2_2 - i \lambda^1_2 \end{pmatrix}, \lambda_2^\oplus = \begin{pmatrix} \lambda^0_2 + i \lambda^3_2 & \lambda^2_2 + i \lambda^1_2 \\ - (\lambda^2_2 - i \lambda^1_2) & \lambda^0_2 - i \lambda^3_2 \end{pmatrix}. \tag{3.24}\]

Eq.(2.2) then implies \[3, 16\]

\[y_{12} = y_{21} = \frac{1}{2} \frac{(y_{11} - y_{22})}{|y_{11} - y_{22}|^2} (\lambda^\oplus_2 \lambda_1 - \lambda^\prime_1 \lambda_2). \tag{3.25}\]
The denominator of $y_{12}$ can be calculated to be

$$y_{11} - y_{22} = \begin{pmatrix} d_1 + a_1 & d_2 + a_2 \\ d_3 + a_3 & d_4 + a_4 \end{pmatrix},$$

$$|y_{11} - y_{22}|_c^2 = (d_1 + a_1) (d_4 + a_4) - (d_2 + a_2) (d_3 + a_3) = \det.$$  \hspace{1cm} (3.26)

The vanishing of the det defined above will be used to search for the new YM 2-instanton solutions in section IV.

It is important to note that since in general $|y_{11} - y_{22}|_c^2$ is a complex number, the vanishing of det defined above does not mean $y_{11} = y_{22}$. While in the real instanton case \cite{[16]}, $|y_{11} - y_{22}|^2$ is a non-negative real number and $|y_{11} - y_{22}|^2 = 0$ implies $y_{11} = y_{22}$ which is not allowed for the 2-instanton solutions. So the extended $SU(2)$ complex YM instantons discovered in this paper have no real instanton counterparts.

To simplify the calculation, one introduces

$$l = \begin{vmatrix} \lambda_1^0 \lambda_2^2 \\ \lambda_1^1 \lambda_2^1 \\ \lambda_2^0 \lambda_2^1 \\ \lambda_2^1 \lambda_2^1 \end{vmatrix}, \quad m = \begin{vmatrix} \lambda_1^0 \lambda_1^1 \\ \lambda_1^0 \lambda_1^1 \\ \lambda_2^1 \lambda_2^1 \\ \lambda_2^1 \lambda_2^1 \end{vmatrix}, \quad n = \begin{vmatrix} \lambda_1^0 \lambda_2^2 \\ \lambda_1^0 \lambda_2^2 \\ \lambda_2^1 \lambda_2^1 \\ \lambda_2^1 \lambda_2^1 \end{vmatrix},$$

which imply

$$(\lambda_2^2 \lambda_1^1 - \lambda_1^0 \lambda_2^0) = 2i \begin{pmatrix} l \\ m + in \end{pmatrix}.$$  \hspace{1cm} (3.27)

Finally we end up with the expression

$$y_{12} = \frac{1}{2} \frac{(y_{11} - y_{22})}{|y_{11} - y_{22}|_c^2} (\lambda_2^2 \lambda_1^1 - \lambda_1^0 \lambda_2^0)$$

$$= \frac{i}{\det} \left[ \frac{(d_1 + a_1) l + (d_2 + a_2) (m + in)}{(d_3 + a_3) l + (d_4 + a_4) (m + in)} \right] \left[ (d_1 + a_1) (m - in) - (d_2 + a_2) l \right] \left[ (d_3 + a_3) (m - in) - (d_4 + a_4) l \right].$$  \hspace{1cm} (3.28)
We are now ready to identify the ADHM data. The EMR of the $y$ data is

$$
\begin{pmatrix}
  (d_1 + a_1) l & (d_1 + a_1) (m - in) \\
  + (d_2 + a_2) (m + in) & - (d_2 + a_2) l \\
  (d_3 + a_3) l & (d_3 + a_3) (m - in) \\
  + (d_4 + a_4) (m + in) & - (d_4 + a_4) l
\end{pmatrix}
\frac{i}{\det} =
\begin{pmatrix}
  (d_1 + a_1) l \\
  + (d_2 + a_2) (m + in) \\
  (d_3 + a_3) l \\
  + (d_4 + a_4) (m + in)
\end{pmatrix}
\begin{pmatrix}
  (d_1 + a_1) (m - in) \\
  - (d_2 + a_2) l \\
  (d_3 + a_3) (m - in) \\
  - (d_4 + a_4) l
\end{pmatrix}.
\tag{3.30}
$$

After imposing the rearrangement rule, we can identify the following ADHM data

\begin{align*}
B_{11} &= \begin{pmatrix}
  d_1 & \\
  \frac{i}{\det} & \\
  + (d_2 + a_2) (m + in)
\end{pmatrix}
\begin{pmatrix}
  (d_1 + a_1) l \\
  + (d_2 + a_2) (m + in)
\end{pmatrix}, \tag{3.31}

B_{21} &= \begin{pmatrix}
  d_2 & \\
  \frac{i}{\det} & \\
  - (d_2 + a_2) l
\end{pmatrix}
\begin{pmatrix}
  (d_1 + a_1) (m - in) \\
  - (d_2 + a_2) l
\end{pmatrix}, \tag{3.32}

B_{12} &= \begin{pmatrix}
  d_3 & \\
  \frac{i}{\det} & \\
  + (d_4 + a_4) (m + in)
\end{pmatrix}
\begin{pmatrix}
  (d_3 + a_3) l \\
  + (d_4 + a_4) (m + in)
\end{pmatrix}, \tag{3.33}

B_{22} &= \begin{pmatrix}
  d_4 & \\
  \frac{i}{\det} & \\
  - (d_4 + a_4) l
\end{pmatrix}
\begin{pmatrix}
  (d_3 + a_3) (m - in) \\
  - (d_4 + a_4) l
\end{pmatrix}. \tag{3.34}
\end{align*}
\[
I_1 = \begin{pmatrix}
-\lambda_1^2 + i\lambda_1^1 & \lambda_0^0 - i\lambda_1^3 \\
-\lambda_2^2 + i\lambda_2^1 & \lambda_0^0 - i\lambda_2^3
\end{pmatrix},
\]
\[
I_2 = \begin{pmatrix}
-(\lambda_0^0 + i\lambda_1^3) - (\lambda_1^2 + i\lambda_1^1) \\
-(\lambda_2^0 + i\lambda_2^3) - (\lambda_2^2 + i\lambda_2^1)
\end{pmatrix}.
\]

We can now consider the rank \(\beta = 0\) or \(\beta = 0\) case.

(1) For those points on \(CP^3\) with \(z = 1\), \(\beta = 0\) means

\[
B_{12} + wB_{22} + y = 0,
\]
\[
B_{11} + wB_{21} + x = 0,
\]
\[
I_1 + wI_2 = 0.
\]

By using Eq.(3.37) and Eq.(3.38), we obtain

\[
\begin{pmatrix}
\frac{d_3 + wd_4 + y}{\det} \\
\frac{\det}{[(d_3 + a_3)l + (d_4 + a_4)(m + in)]} \\
\frac{\det}{w[(d_3 + a_3)(m - in) - (d_4 + a_4)l]}
\end{pmatrix}
\begin{pmatrix}
[(d_3 + a_3)l + (d_4 + a_4)(m + in)] \\
+w[(d_3 + a_3)(m - in) - (d_4 + a_4)l]
\end{pmatrix}
\begin{pmatrix}
a_3 - wa_4 + y
\end{pmatrix} = 0.
\]

and

\[
\begin{pmatrix}
\frac{d_1 + wd_2 + x}{\det} \\
\frac{\det}{[(d_1 + a_3)l + (d_2 + a_2)(m + in)]} \\
\frac{\det}{w[(d_1 + a_3)(m - in) - (d_2 + a_2)l]}
\end{pmatrix}
\begin{pmatrix}
[(d_1 + a_3)l + (d_2 + a_2)(m + in)] \\
+w[(d_1 + a_3)(m - in) - (d_2 + a_2)l]
\end{pmatrix}
\begin{pmatrix}
a_1 - wa_2 + x
\end{pmatrix} = 0.
\]

respectively. We note that Eq.(3.40) gives

\[
d_3 + wd_4 + y = -a_3 - wa_4 + y = 0,
\]

or

\[
w = \frac{d_3 + a_3}{a_4 - d_4}.
\]

Eq.(3.41) gives

\[
d_1 + wd_2 + x = -a_3 - wa_4 + x = 0,
\]
or
\[ w = -\frac{(a_1 + d_1)}{d_2 + a_2}. \] (3.45)

We see from Eq. (3.43) and Eq. (3.45) that
\[ \frac{d_3 + a_3}{-a_4 - d_4} = -\frac{(a_1 + d_1)}{d_2 + a_2}, \] (3.46)

which means
\[ \text{det} = 0. \] (3.47)

We will call instanton solutions with \( \text{det} = 0 \) the "extended complex YM instanton solutions" which were not considered in the previous construction [3]. Moreover, as already mentioned in the paragraph after Eq. (3.26), since in general \( |y_{11} - y_{22}|^2 \) is a complex number, the vanishing of \( \text{det} \) does not mean \( y_{11} = y_{22} \), or
\[ \text{det} = |y_{11} - y_{22}|^2 = 0 \Rightarrow y_{11} = y_{22}. \] (3.48)

So it is possible to have complex YM 2-instanton solutions with \( \text{det} = 0 \). While in the previous \( SU(2) \) real instanton case [16], \( |y_{11} - y_{22}|^2 \) is a non-negative real number and
\[ |y_{11} - y_{22}|^2 = 0 \Rightarrow y_{11} = y_{22} \] (3.49)

which is not allowed for the real YM 2-instanton solutions. So the extended \( SU(2) \) complex YM instantons discovered in this paper have no real instanton counterparts.

(2) For those points on \( CP^3 \) with \( z = 0, w = 1, \beta = 0 \) means

\[ -B_{22} - y = 0, \] (3.50)
\[ B_{21} + x = 0, \] (3.51)
\[ I_2 = 0. \] (3.52)

We see that Eq. (3.50) gives
\[
\begin{bmatrix}
  \frac{i}{\text{det}} \left[ (d_3 + a_3) (m - in) - (d_4 + a_4) l \right] & d_4 + y \\
  \frac{i}{\text{det}} \left[ (d_3 + a_3) (m - in) - (d_4 + a_4) l \right] & -\beta + y
\end{bmatrix}
\] = 0, (3.53)

which implies
\[ d_4 = -a_4 = -y. \] (3.54)
On the other hand, Eq. (3.51) gives

$$\begin{pmatrix}
d_2 + x \\
\frac{i}{\det} [(d_1 + a_3) (m - in) - (d_2 + a_2) l]
\end{pmatrix}
= 0, \quad (3.55)$$

which implies

$$d_2 = -a_2 = -x. \quad (3.56)$$

Eq. (3.54) and Eq. (3.56) imply

$$\det = (d_1 + a_1) (d_4 + a_4) - (d_2 + a_2) (d_3 + a_3) = 0, \quad (3.57)$$

which again corresponds to the extended case.

(3) Finally for \( z = 0 \) and \( w = 0 \), \( \beta = 0 \) in Eq. (2.15b) gives \( x = 0 = y \). Thus for this case there is no point on \( CP^3 \) which supports \( \text{rank} \beta = 0 \).

We thus have completed the proof that \( \text{rank} \beta = 0 \) or \( \beta = 0 \) implies \( \det = 0 \). So there is no \( \text{rank} \beta = 0 \) with \( \det \neq 0 \) complex YM instanton sheaf solutions. In all our previous construction of complex YM instantons (sheaves) \[3, 4\], we have assumed \( \det \neq 0 \) as in the real YM instanton case. In the next section, we are looking for \( \det = 0 \) (see Eq. (3.48)) complex YM instanton solutions both for the sheaf (\( \text{rank} \beta = 0, 1 \) on some points of \( CP^3 \)) and bundle (\( \text{rank} \beta = 2 \) on the whole \( CP^3 \)) cases.

**IV. EXTENDED YM 2-INSTANTON SOLUTIONS WITH \( \det = 0 \)**

In this section, we are looking for complex YM 2-instanton solutions with

$$\det = |y_{11} - y_{22}|_c^2 = (d_1 + a_1) (d_4 + a_4) - (d_2 + a_2) (d_3 + a_3) = 0. \quad (4.58)$$

The idea is that whether one can factor out a \( \det \) factor from the numerator of the rhs of Eq. (3.25). These solutions were not considered in \[3\], and if exist, they correspond to "limiting solutions" living on the boundary of the moduli space. We will discuss the solutions with three different cases with \( \text{rank} \beta = 0, 1 \) on some points of \( CP^3 \) with some given ADHM data (sheaves) and \( \text{rank} \beta = 2 \) on the whole \( CP^3 \) for some other given ADHM data (bundle) respectively.
A. Sheaf case with rank $\beta = 0$

For the first case, we consider $\beta = 0$

$$zB_{12} + wB_{22} + y = 0,$$  \hspace{1cm} (4.59)

$$zB_{11} + wB_{21} + x = 0,$$  \hspace{1cm} (4.60)

$$zI_1 + wI_2 = 0.$$  \hspace{1cm} (4.61)

and assuming $z \neq 0$, $w \neq 0$ on $CP^3$. We first note that Eq.(4.61) or

$$I_1 + \frac{w}{z}I_2 = 0$$  \hspace{1cm} (4.62)

can be used to solve half of the number of $\lambda$ parameters. Indeed, the vanishing of the upper two components of

$$\begin{bmatrix}
-\lambda_1^2 + i\lambda_1^1 - \frac{w}{z} (\lambda_1^0 + i\lambda_1^3) \lambda_1^0 - i\lambda_1^3 - \frac{w}{z} (\lambda_1^2 + i\lambda_1^1) \\
-\lambda_2^2 + i\lambda_2^1 - \frac{w}{z} (\lambda_2^0 + i\lambda_2^3) \lambda_2^0 - i\lambda_2^3 - \frac{w}{z} (\lambda_2^2 + i\lambda_2^1)
\end{bmatrix} = 0$$  \hspace{1cm} (4.63)

give

$$-\lambda_1^2 + i\lambda_1^1 = \frac{w}{z} \lambda_1^0 + i\frac{w}{z} \lambda_1^3,$$  \hspace{1cm} (4.64)

$$\lambda_1^2 + i\lambda_1^1 = (\frac{w}{z})^{-1} \lambda_1^0 - i(\frac{w}{z})^{-1} \lambda_1^3,$$  \hspace{1cm} (4.65)

which can be used to get

$$\lambda_1^2 = \frac{1}{2} \left[ \left( (\frac{w}{z})^{-1} - \frac{w}{z} \right) \lambda_1^0 - i \left( (\frac{w}{z})^{-1} + \frac{w}{z} \right) \lambda_1^3 \right]$$
$$= \frac{1}{2} (\delta \lambda_1^0 - i\rho \lambda_1^3)$$  \hspace{1cm} (4.66)

and

$$\lambda_1^1 = \frac{1}{2 i} \left[ \left( (\frac{w}{z})^{-1} + \frac{w}{z} \right) \lambda_1^0 - i \left( (\frac{w}{z})^{-1} - \frac{w}{z} \right) \lambda_1^3 \right]$$
$$= \frac{1}{2 i} (\rho \lambda_1^0 - i\delta \lambda_1^3)$$  \hspace{1cm} (4.67)

where we have defined

$$\delta = (\frac{w}{z})^{-1} - \frac{w}{z}, \rho = (\frac{w}{z})^{-1} + \frac{w}{z}.$$  \hspace{1cm} (4.68)
Similarly the vanishing of the lower two components of Eq. (3.39) solves another two \( \lambda \), and we end up with

\[ \lambda_1^2 = \frac{1}{2} (\delta \lambda_1^0 - i \rho \lambda_1^3) , \]  
\[ \lambda_1^1 = \frac{1}{2i} (\rho \lambda_1^0 - i \delta \lambda_1^1) , \]  
\[ \lambda_2^2 = \frac{1}{2} (\delta \lambda_2^0 - i \rho \lambda_2^3) , \]  
\[ \lambda_2^1 = \frac{1}{2i} (\rho \lambda_2^0 - i \delta \lambda_2^2) . \]  

With the expressions in Eq. (4.69) to Eq. (4.72), one can simplify \( l, m, n \) to obtain

\[ l = 2 \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix} , \]  
\[ m = -\delta \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix} , \]  
\[ n = -i\rho \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix} . \]  

One can now easily calculate the expression

\[ \lambda_2^0 \lambda_1 - \lambda_1^0 \lambda_2 = 4i \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix} \begin{bmatrix} 1 & -\left(\frac{w}{z}\right)^{-1} \\ \frac{w}{z} & -1 \end{bmatrix} \]  

\[ \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix} \]  

to obtain

\[ y_{12} = \frac{1}{2} \left( \frac{y_{11} - y_{22}}{|y_{11} - y_{22}|^2} \right) \left( \lambda_2^0 \lambda_1 - \lambda_1^0 \lambda_2 \right) \]  
\[ = \frac{2i \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix}}{|y_{11} - y_{22}|^2} \begin{pmatrix} (d_1 + a_2) \\ (d_2 + a_2) \end{pmatrix} \begin{pmatrix} 1 & -\left(\frac{w}{z}\right)^{-1} \\ \frac{w}{z} & -1 \end{pmatrix} \]  
\[ = \frac{2i \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix}}{|y_{11} - y_{22}|^2} \begin{pmatrix} (d_1 + a_2 + \frac{w}{z} (d_2 + a_2) - \left(\frac{w}{z}\right)^{-1} (d_1 + a_1) - (d_2 + a_2)) \\ (d_3 + a_3 + \frac{w}{z} (d_4 + a_4) - \left(\frac{w}{z}\right)^{-1} (d_3 + a_3) - (d_4 + a_4)) \end{pmatrix} . \]

Note that the first column of the matrix in Eq. (4.79) is proportional to the second column with proportional constant \(-\left(\frac{w}{z}\right)^{-1}\) due to the structure of the second matrix in Eq. (4.78). 

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To continue the calculation, we begin with general value of det, and after factoring out a det factor from the numerator of the rhs of Eq. (4.78), we will take the $\text{det} \to 0$ limit. There are two channels to achieve this factorization.

1. **Factorization I**

We first note that the det can be written as

$$\text{det} = (d_4 + a_4) \left[ (d_1 + a_1) - \frac{(d_2 + a_2)(d_3 + a_3)}{(d_4 + a_4)} \right]$$  \hspace{1cm} (4.80)

or

$$\left( \frac{1}{d_4 + a_4} \right) \text{det} = \left[ (d_1 + a_1) - (d_2 + a_2) \frac{(d_3 + a_3)}{(d_4 + a_4)} \right].$$  \hspace{1cm} (4.81)

On the other hand, we remember that

$$zB_{12} + wB_{22} + y = 0,$$
$$zB_{11} + wB_{21} + x = 0$$

imply

$$\frac{w}{z} = \frac{d_3 + a_3}{-a_4 - d_4} = \frac{- (a_1 + d_1)}{d_2 + a_2},$$  \hspace{1cm} (4.82)

which means $\text{det} = 0$. Let’s now consider nonvanishing det (with $\beta \neq 0$) and rewrite Eq. (4.81) as

$$\left( \frac{1}{d_4 + a_4} \right) \text{det} = \left[ (d_1 + a_1) + (d_2 + a_2) \left( \frac{w}{z} + \varepsilon' \right) \right].$$  \hspace{1cm} (4.83)

Note that as $\beta \to 0$, we have $\text{det} \to 0$, which gives

$$\frac{w}{z} = \frac{(d_3 + a_3)}{(d_4 + a_4)} = - \frac{(a_1 + d_1)}{d_2 + a_2}$$  \hspace{1cm} (4.84)

and

$$\varepsilon' \to 0.$$  \hspace{1cm} (4.85)

We can rewrite $(d_2 + a_2) \varepsilon' = \varepsilon \text{det}$ with $\varepsilon$ a finite number as $\text{det} \to 0$, and obtain

$$\left( \frac{1}{d_4 + a_4} + \varepsilon \right) \text{det} = \left[ (d_1 + a_1) + \frac{w}{z} (d_2 + a_2) \right].$$  \hspace{1cm} (4.86)

We can now use Eq. (4.86) to express $\frac{w}{z}$ in terms of det as

$$\frac{w}{z} = - \frac{(d_3 + a_3)}{d_4 + a_4} + \left( \varepsilon \cdot \frac{\text{det}}{d_2 + a_2} \right),$$  \hspace{1cm} (4.87)
and calculate
\[(d_3 + a_3) + \frac{w}{z} (d_4 + a_4) = \left( \frac{\varepsilon \det (d_4 + a_4)}{d_2 + a_2} \right). \tag{4.88} \]

We conclude that Eq.(4.79) can be written as
\[
y_{12} = \frac{2i}{\det} \left( \left( \varepsilon + \frac{1}{d_4 + a_4} \right) \det - \frac{w}{z} \right)^{-1} \left( \varepsilon + \frac{1}{d_4 + a_4} \right) \det \lambda_1^0 \lambda_1^3 \begin{vmatrix} \lambda_2^0 & \lambda_2^3 \end{vmatrix} \tag{4.89} \]
\[
\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{pmatrix} \tag{4.90} \]

with
\[
c_1 = 2i \left( \varepsilon + \frac{1}{d_4 + a_4} \right) \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix}, \tag{4.91} \]
\[
c_3 = 2i \left( \varepsilon \frac{d_4 + a_4}{d_2 + a_2} \right) \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix}, \tag{4.92} \]
\[
\begin{pmatrix} c_2 \\ c_4 \end{pmatrix} = -\frac{w}{z}^{-1} \begin{pmatrix} c_1 \\ c_3 \end{pmatrix}. \tag{4.93} \]

It is important to note that to achieve the factorization, \(\frac{w}{z}\) is an arbitrary number (except \(z \neq 0, w \neq 0\) on \(CP^3\)) and \(\varepsilon\) is finite as \(\det \to 0\) in Eq.(4.89).

To obtain Eq.(4.90), we have cancelled out the \(\det\) factor and take the \(\det \to 0\) limit. We thus have explicitly shown the existence of the extended complex YM 2-instanton sheaf solutions with \(\det = 0\) and rank\(\beta = 0\). These new complex YM instanton solutions were not considered in [3], and live on the boundary of the moduli space. Their existences strongly depend on the structure of the matrix in Eq.(4.78).

Finally we need to check the validities of Eq.(4.59) and Eq.(4.60). The \(B\) matrix of the ADHM data can be written as
\[
B_{11} = \begin{pmatrix} d_1 & c_1 \\ c_1 & -a_1 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} d_2 & c_2 \\ c_2 & -a_2 \end{pmatrix}, \tag{4.94} \]
\[
B_{12} = \begin{pmatrix} d_3 & c_3 \\ c_3 & -a_3 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} d_4 & c_4 \\ c_4 & -a_4 \end{pmatrix}. \tag{4.95} \]

The vanishing of the diagonal terms of
\[
zB_{11} + wB_{21} + x = \begin{pmatrix} d_1 z + d_2 w + x & c_1 z + c_2 w \\ c_1 z + c_2 w & -a_1 z - a_2 w + x \end{pmatrix} = 0 \tag{4.96} \]
and
\[ zb_{12} + wb_{22} + y = \begin{pmatrix} d_3z + d_4w + y & c_3z + c_4w \\ c_3z + c_4w & -a_3z - a_4w + y \end{pmatrix} = 0 \] (4.97)

mean Eq. (4.84), which implies \( \det = 0 \), and
\[ x = -d_1z - wd_2, \] (4.98)
\[ y = -d_3z - wd_4. \] (4.99)

The vanishing of the off-diagonal terms of Eq. (4.96) and Eq. (4.97) mean
\[ c_1 + \frac{w}{z}c_2 = 0, \] (4.100)
\[ c_3 + \frac{w}{z}c_4 = 0, \] (4.101)

which are results of Eq. (4.93). Thus the solution set for this case contain 11 parameters.

7 = 4 + 4 − 1 from \( a_j \) and \( d_j \) subject to \( \det = 0 \), and 4 more parameters from \( \lambda_1^0, \lambda_1^3, \lambda_2^0 \) and \( \lambda_2^3 \).

For completeness and to include the cases of \( z = 0 \) or \( w = 0 \), we have explicitly checked the validity of the complex ADHM equations of the above ADHM data.

In the end of this section, we give one explicit example of the extended YM 2-instanton sheaf for this case. Let’s begin with the following ADHM data
\[ y_{11} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y_{22} = -\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & -p \\ -p & 0 \end{pmatrix}, \] (4.102)

which gives
\[ \det = 1 - p^2. \] (4.103)

For half of the \( \lambda \) ADHM data, we assume
\[ \lambda_1^0 = 1, \lambda_1^3 = 1, \lambda_2^0 = i, \lambda_2^3 = -i. \] (4.104)

With the expressions in Eq. (4.69) to Eq. (4.72) and the following information
\[ \frac{w}{z} = -\left( \frac{d_3 + a_3}{d_4 + a_4} \right) \rightarrow 1, \quad \frac{z}{w} \rightarrow 1 \text{ as } p \rightarrow -1, \] (4.105)

the other half of \( \lambda \) ADHM data can be calculated to be
\[ \lambda_1^1 = -i, \lambda_1^2 = -i, \lambda_2^1 = 1, \lambda_2^2 = -1. \] (4.106)
We can now use the following data

\[ \lambda_1 = \begin{pmatrix} \lambda_0^1 - i\lambda_0^3 - (\lambda_1^2 + i\lambda_1^1) \\ \lambda_1^0 - i\lambda_1^1 \end{pmatrix} = \begin{pmatrix} 1 - i & -1 + i \\ 1 - i & 1 + i \end{pmatrix}, \]

\[ \lambda_1^\oplus = \begin{pmatrix} \lambda_0^0 + i\lambda_0^3 & \lambda_2^2 + i\lambda_1^2 \\ -(\lambda_1^0 - i\lambda_1^1) & \lambda_1^0 - i\lambda_1^3 \end{pmatrix} = \begin{pmatrix} 1 + i & 1 - i \\ 1 + i & 1 - i \end{pmatrix}, \]

\[ \lambda_2 = \begin{pmatrix} \lambda_2^0 - i\lambda_2^3 - (\lambda_2^2 + i\lambda_2^1) \\ \lambda_2^0 - i\lambda_2^2 \end{pmatrix} = \begin{pmatrix} i - 1 & 1 - i \\ -1 + i & 1 + i \end{pmatrix}, \]

\[ \lambda_2^\oplus = \begin{pmatrix} \lambda_2^0 + i\lambda_2^3 & \lambda_2^2 + i\lambda_2^1 \\ -(\lambda_2^2 - i\lambda_2^1) & \lambda_2^0 - i\lambda_2^3 \end{pmatrix} = \begin{pmatrix} 1 + i & -1 + i \\ 1 + i & 1 - i \end{pmatrix} \]

(4.107)

to calculate \( y_{12} \) and obtain

\[ y_{12} = \frac{1}{2} \left( \frac{y_{11} - y_{22}}{y_{11} - y_{22}} \right)^2 (\lambda_2^\oplus \lambda_1 - \lambda_1^\oplus \lambda_2) \]

\[ = 4 \begin{pmatrix} 1 & p \\ p & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{4}{1 - p^2} \left( \frac{1-p^2}{1-p} - \frac{1-p^2}{1-p} \right) \]

\[ = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \]

(4.108)

where we have seen the factorization of the \( \det = 1 - p^2 \) factor as expected.

In the \( p \to -1 \) limit, we can calculate the following \( y \) matrices

\[ y_{11} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y_{22} = -\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y_{12} = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}. \]

(4.109)

One can now use Eq.(4.104), Eq.(4.106) and Eq.(4.109) to calculate the following ADHM
We can now check the three complex ADHM equations

\[
J_1 = \begin{pmatrix}
\lambda_1^0 - i\lambda_1^3 & \lambda_2^0 - i\lambda_2^3 \\
\lambda_1^2 - i\lambda_1^1 & \lambda_2^2 - i\lambda_2^1
\end{pmatrix} = \begin{pmatrix} 1 - i & i - 1 \\ -1 - i & -1 - i \end{pmatrix},
\]

\[
J_2 = \begin{pmatrix}
- (\lambda_1^2 + i\lambda_1^1) & - (\lambda_2^2 + i\lambda_2^1) \\
\lambda_1^0 + i\lambda_1^3 & \lambda_2^0 + i\lambda_2^3
\end{pmatrix} = \begin{pmatrix} -1 + i & 1 - i \\ 1 + i & i + 1 \end{pmatrix},
\]

\[
I_1 = \begin{pmatrix}
- (\lambda_1^2 - i\lambda_1^1) & \lambda_1^0 + i\lambda_1^3 \\
\lambda_2^2 - i\lambda_2^1 & \lambda_1^0 + i\lambda_1^3
\end{pmatrix} = \begin{pmatrix} 1 + i & 1 - i \\ 1 + i & -1 + i \end{pmatrix},
\]

\[
I_2 = \begin{pmatrix}
- (\lambda_1^0 + i\lambda_1^3) - (\lambda_1^0 + i\lambda_1^3) \\
- (\lambda_2^0 + i\lambda_2^3) - (\lambda_2^1 + i\lambda_2^3)
\end{pmatrix} = \begin{pmatrix} -1 - 1 + i \\ -1 - 1 - i \end{pmatrix}, \tag{4.110}
\]

and

\[
B_{11} = \begin{pmatrix} d_1 & c_1 \\ c_1 & -a_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} d_2 & c_2 \\ c_2 & -a_2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix},
\]

\[
B_{12} = \begin{pmatrix} d_3 & c_3 \\ c_3 & -a_3 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} d_4 & c_4 \\ c_4 & -a_4 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix}. \tag{4.111}
\]

We can now check the three complex ADHM equations

\[
[B_{11}, B_{12}] + I_1 J_1
= \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 + i & 1 - i \\ 1 + i & -1 + i \end{pmatrix} \begin{pmatrix} 1 - i & i - 1 \\ -1 - i & -1 - i \end{pmatrix}
= \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.112}
\]

\[
[B_{21}, B_{22}] + I_2 J_2
= \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} -1 - i & -1 + i \\ -1 - i & 1 - i \end{pmatrix} \begin{pmatrix} -1 + i & 1 - i \\ 1 + i & i + 1 \end{pmatrix}
= \begin{pmatrix} 4 & 0 \\ -4 & 4 \end{pmatrix} - \begin{pmatrix} 4 & -4 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.113}
\]
and

\[[B_{11}, B_{22}] + [B_{21}, B_{12}] + I_1 J_2 + I_2 J_1\]

\[
= \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}
\]

\[
+ \begin{pmatrix} 1 + i & 1 - i \\ 1 + i & -1 + i \end{pmatrix} \begin{pmatrix} -1 + i & 1 - i \\ 1 - i & 1 - i \end{pmatrix} + \begin{pmatrix} -1 - i & -1 + i \\ -1 - i & 1 - i \end{pmatrix} \begin{pmatrix} 1 - i & i - 1 \\ -1 - i & -1 - i \end{pmatrix}
\]

\[
= \begin{pmatrix} -3 & -2 \\ 2 & -4 \end{pmatrix} - \begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix} + \begin{pmatrix} -4 & -2 \\ 2 & -3 \end{pmatrix} - \begin{pmatrix} -4 & 2 \\ 2 & -3 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} .
\] (4.114)

It remains to check

\[
z B_{11} + w B_{21} + x = \begin{pmatrix} z + x & 2z - 2w \\ 2z - 2w & w + x \end{pmatrix} = 0,
\]

\[
z B_{12} + w B_{22} + y = \begin{pmatrix} w + y & 2z - 2w \\ 2z - 2w & z + y \end{pmatrix} = 0.
\] (4.115)

The solutions of Eq. (4.115) are

\[
w = z, x = -z, y = -z,
\] (4.116)

which represent a point on \(CP^3\)

\[
[x : y : z : w] = [-1 : -1 : 1 : 1].
\] (4.117)

This is the point where jumping occurs.

2. Factorization II

For the second case, we first assume \(\det \neq 0\) and set

\[
\begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix} = \det \cdot \varepsilon.
\] (4.118)
After cancelling out the det factor and take the det $\to 0$ limit, Eq.(4.79) can then be written as

$$y_{12} = \frac{2i}{|y_{11} - y_{22}|^2} \begin{vmatrix} \lambda_1^0 & \lambda_3^0 \\ \lambda_1^3 & \lambda_3^3 \end{vmatrix} \begin{pmatrix} d_1 + a_1 & d_2 + a_2 \\ d_3 + a_3 & d_4 + a_4 \end{pmatrix} \begin{pmatrix} 1 & -\left(\frac{w}{z}\right)^{-1} \\ \frac{w}{z} & -1 \end{pmatrix}$$

$$= 2i\tilde{\varepsilon} \begin{pmatrix} (d_1 + a_1) + \frac{w}{z} (d_2 + a_2) - (\frac{w}{z})^{-1} (d_1 + a_1) - (d_2 + a_2) \\ (d_3 + a_3) + \frac{w}{z} (d_4 + a_4) - (\frac{w}{z})^{-1} (d_3 + a_3) - (d_4 + a_4) \end{pmatrix} \quad (4.119)$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (4.120)$$

where we have used Eq.(4.84) in the det $\to 0$ limit. The result of Eq.(4.120) is not surprising since in the calculation of case I, we already knew that in the det $\to 0$ limit, the matrix in Eq.(4.119) factored out a det factor. However, it is surprising to see that for the extended complex YM 2-instantons with diagonal $y$ ADHM data, there exist YM instanton sheaf structure! For the det $\neq 0$ complex YM instantons constructed previously [3], it was shown [4] that there is no instanton sheaf solutions with diagonal $y$ ADHM data.

As in the case I, we need to check the validities of Eq.(4.59) and Eq.(4.60). The vanishing of the diagonal terms of

$$zB_{11} + wB_{21} + x = 0, \quad (4.121)$$
$$zB_{12} + wB_{22} + y = 0 \quad (4.122)$$

mean Eq.(4.84) which implies det = 0, and

$$x = -d_1z - wd_2, \quad (4.123)$$
$$y = -d_3z - wd_4. \quad (4.124)$$

On the other hand, the vanishing of the off-diagonal terms of Eq.(4.121) and Eq.(4.122) are results of Eq.(4.93). There is another constraint from the following equation

$$\begin{vmatrix} \lambda_1^0 & \lambda_3^0 \\ \lambda_1^3 & \lambda_3^3 \end{vmatrix} = 0. \quad (4.125)$$
In the following we give one explicit example of YM 2-instanton sheaf for this case. Similar to the example for the case I, let’s take the following ADHM data

\[
y_{11} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
y_{22} = - \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} 0 & -p \\ -p & 0 \end{pmatrix},
\]

which gives

\[
\det = 1 - p^2.
\]

We assume (for simplicity, take \(\tilde{\epsilon} = 1\))

\[
\begin{vmatrix}
\lambda_0 & \lambda_3^2 \\
\lambda_0 & \lambda_3^2
\end{vmatrix} = \begin{vmatrix} 1 & p \\ p & 1 \end{vmatrix} = 1 - p^2 = \det \cdot \tilde{\epsilon}.
\]

Since

\[
\frac{w}{z} = - \left( \frac{d_3 + a_3}{d_4 + a_4} \right) = -p \to 1, \frac{z}{w} \to 1 \text{ as } p \to -1,
\]

we have

\[
\delta = \left( \frac{w}{z} \right)^{-1} - \frac{w}{z} \to 0, \rho = \left( \frac{w}{z} \right)^{-1} + \frac{w}{z} \to 2.
\]

So all other parameters can be calculated to be

\[
\begin{align*}
\lambda_1^2 &= \frac{1}{2} \left( \delta \lambda_1^0 - i \rho \lambda_1^3 \right) = i, \\
\lambda_1^1 &= \frac{1}{2i} \left( \rho \lambda_1^0 - i \delta \lambda_1^3 \right) = -i, \\
\lambda_2^2 &= \frac{1}{2} \left( \delta \lambda_2^0 - i \rho \lambda_2^3 \right) = -i, \\
\lambda_2^1 &= \frac{1}{2i} \left( \rho \lambda_2^0 - i \delta \lambda_2^3 \right) = i
\end{align*}
\]

as \(p \to -1\). Finally \(y_{12}\) can be calculated to be

\[
y_{12} = \frac{1}{2} \frac{(y_{11} - y_{22})}{|y_{11} - y_{22}|^2} (\lambda_2^\otimes \lambda_1 - \lambda_1^\otimes \lambda_2)
\]

\[
= \frac{1}{2} \frac{1}{1 - a^2} \begin{vmatrix} 1 & p \\ p & 1 \end{vmatrix} \begin{vmatrix} \lambda_0^0 & \lambda_3^0 \\ \lambda_0^2 & \lambda_3^2 \end{vmatrix} = \frac{1}{2} \frac{1}{1 - p^2} \begin{vmatrix} 1 & p \\ p & 1 \end{vmatrix} \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} (1 - p^2) \tilde{\epsilon}
\]

\[
= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \tilde{\epsilon} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

as expected.
The next step is to calculate ADHM data. One easily obtains

\[ J_1 = \begin{pmatrix} \lambda_1^0 - i\lambda_1^3 & \lambda_2^0 - i\lambda_2^3 \\ \lambda_1^2 - i\lambda_1^1 & \lambda_2^2 - i\lambda_2^1 \end{pmatrix} = \begin{pmatrix} 1 + i & -i - 1 \\ -1 + i & 1 - i \end{pmatrix}, \]

\[ J_2 = \begin{pmatrix} -(\lambda_1^2 + i\lambda_1^1) - (\lambda_2^2 + i\lambda_2^1) \\ \lambda_1^0 + i\lambda_1^3 & \lambda_2^0 + i\lambda_2^3 \end{pmatrix} = \begin{pmatrix} -1 - i & 1 + i \\ 1 - i & i - 1 \end{pmatrix} = -J_1, \]

\[ I_1 = \begin{pmatrix} - (\lambda_1^0 - i\lambda_1^1) & \lambda_2^0 - i\lambda_2^1 \\ - (\lambda_2^0 - i\lambda_2^1) & \lambda_2^0 - i\lambda_2^1 \end{pmatrix} = \begin{pmatrix} 1 - i & 1 + i \\ -1 + i & -1 - i \end{pmatrix}, \]

\[ I_2 = \begin{pmatrix} -(\lambda_1^0 + i\lambda_1^1) - (\lambda_2^0 + i\lambda_2^1) \\ - (\lambda_2^0 + i\lambda_2^1) - (\lambda_2^0 + i\lambda_2^1) \end{pmatrix} = \begin{pmatrix} 1 + i & -1 - i \\ 1 - i & 1 + i \end{pmatrix} = -I_1, \quad (4.133) \]

and the following \( B_{ij} \) matrices

\[ B_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ B_{12} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.134) \]

The complex ADHM equations can be verified to be

\[
[B_{11}, B_{12}] + I_1J_1 \\
= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 - i & 1 + i \\ -1 + i & -1 - i \end{pmatrix} \begin{pmatrix} 1 + i & -i - 1 \\ -1 + i & 1 - i \end{pmatrix} \\
= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.135)
\]

\[
[B_{21}, B_{22}] + I_2J_2 = [B_{21}, B_{22}] + I_1J_1 \\
= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.136)
\]
and

\[
[B_{11}, B_{22}] + [B_{21}, B_{12}] + I_1 J_2 + I_2 J_1 = [B_{11}, B_{22}] + [B_{21}, B_{12}] - I_1 J_1 - I_1 J_1
= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

(4.137)

Finally we need to check the following equations

\[
z B_{11} + w B_{21} + x = \begin{pmatrix} z + x & 0 \\ 0 & w + x \end{pmatrix} = 0,
\]

(4.138)

\[
z B_{12} + w B_{22} + y = \begin{pmatrix} w + y & 0 \\ 0 & z + y \end{pmatrix} = 0.
\]

(4.139)

Since we want \( w/z = 1 \), we obtain

\[
w = z, x = -z, y = -z,
\]

(4.140)

which represents a point on \( CP^3 \)

\[
[x : y : z : w] = [-1 : -1 : 1 : 1].
\]

(4.141)

This is the sheaf point where jumping occurs.

In the following, as an example, we will calculate explicitly the complex YM 2-instanton field strength for the ADHM data given in Eq.(4.126), Eq.(4.128), Eq.(4.131) and Eq.(4.132).

The \( \Delta \) matrix in Eq.(2.1) is \( [16] \)

\[
\Delta = \begin{pmatrix} \lambda_1 & \lambda_2 \\ x + y_{11} & 0 \\ 0 & x + y_{22} \end{pmatrix},
\]

(4.142)

and its biquaternion conjugation is \( [3] \)

\[
\Delta^\circ = \begin{pmatrix} \lambda_1^\circ & x^\circ + y_{11}^\circ & 0 \\ \lambda_2^\circ & 0 & x^\circ + y_{22}^\circ \end{pmatrix}.
\]

(4.143)
Note that $x^\circ = x^\dagger$ as $x_\mu$ is real. The next step is to introduce

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

(4.144)

which satisfies

$$\Delta^\circ v = 0,$$

(4.145)

and the normalization condition

$$v^\circ v = 1.$$  

(4.146)

Eq.(4.145) can be written as

$$\begin{pmatrix} \lambda_1^\circ & x^\circ + y_{11}^\circ \\ \lambda_2^\circ & x^\circ + y_{22}^\circ \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

(4.147)

or

$$\lambda_1^\circ v_1 + (x^\circ + y_{11}^\circ) v_2 = 0,$$

(4.148)

$$\lambda_2^\circ v_1 + (x^\circ + y_{22}^\circ) v_3 = 0,$$

(4.149)

which can be solved to be

$$v_2 = \frac{-(x + y_{11}) \lambda_1^\circ}{|x + y_{11}|_c^2} v_1, \quad \quad \quad \quad (4.150)$$

$$v_3 = \frac{-(x + y_{22}) \lambda_2^\circ}{|x + y_{22}|_c^2} v_1.$$  

(4.151)

It is important to note that since $y_{12} = 0$, the solvability of $v$ is greatly simplified.

The normalization condition in Eq.(4.146) can now be written as

$$v_1^\circ v_1 + \frac{v_1^\circ \lambda_1 (x + y_{11})^\circ (x + y_{11}) \lambda_1^\circ v_1}{|x + y_{11}|_c^2} + \frac{v_1^\circ \lambda_2 (x + y_{22})^\circ (x + y_{22}) \lambda_2^\circ v_1}{|x + y_{22}|_c^2} = 1$$

(4.152)

or

$$v_1^\circ v_1 \left[ 1 + \frac{|\lambda_1|^2}{|x + y_{11}|_c^2} + \frac{|\lambda_2|^2}{|x + y_{22}|_c^2} \right] = 1.$$

(4.153)

Note that

$$|\lambda_1|^2 = 1^2 + (-i)^2 + (i)^2 + (-1)^2 = 0,$$

(4.154)

$$|\lambda_2|^2 = (-1)^2 + (i)^2 + (-i)^2 + (1)^2 = 0,$$

(4.155)
so we have

\[ v_1^* v_1 = 1 \]  \hspace{1cm} (4.156)

For simplicity, we can choose the \( v_1 \) biquaternion to be a pure number

\[ v_1 = 1 \]  \hspace{1cm} (4.157)

to get

\[
v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -(x+y_{11}) \lambda_1^e \\ -(x+y_{22}) \lambda_2^e \end{pmatrix} .
\]  \hspace{1cm} (4.158)

Let’s now calculate \( f^{-1} \) to be

\[
\begin{align*}
f^{-1} &= \Delta^e \Delta = \begin{pmatrix} \lambda_1^e & x^e + y_{11}^e \\ \lambda_2^e & x^e + y_{22}^e \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ x + y_{11} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1^e \lambda_1 + |x+y_{11}|_c^2 & \lambda_1^e \lambda_2 \\ \lambda_2^e \lambda_1 & \lambda_2^e \lambda_2 + |x+y_{22}|_c^2 \end{pmatrix} \\
&= \begin{pmatrix} |x+y_{11}|_c^2 & 0 \\ 0 & |x+y_{22}|_c^2 \end{pmatrix} .
\end{align*}
\]  \hspace{1cm} (4.159)

where we have used the results of Eq.(4.154) and Eq.(4.155) and the following amazing simple results

\[
\begin{align*}
\lambda_1^e \lambda_2 &= (e_0 + ie_1 - ie_2 + e_3)(-e_0 + ie_1 - ie_2 + e_3) = 0, \\
\lambda_2^e \lambda_1 &= (-e_0 - ie_1 + ie_2 - e_3)(e_0 - ie_1 + ie_2 - e_3) = 0.
\end{align*}
\]  \hspace{1cm} (4.160) (4.161)

So we have

\[
f = \begin{pmatrix} \frac{1}{|x+y_{11}|_c} & 0 \\ 0 & \frac{1}{|x+y_{22}|_c} \end{pmatrix} .
\]  \hspace{1cm} (4.162)
Finally one can explicitly calculate the complex YM 2-instanton field strength to be

\[ F_{\mu\nu} = \varphi \cdot b \left( e_{\mu} e_{\nu}^\dagger - e_{\nu} e_{\mu}^\dagger \right) f \varphi \cdot \varphi \]

\[
= \left( 1 - \frac{\lambda_1 (x + y_{11})^\varphi}{|x + y_{11}|_c^\varphi} \right) - \frac{\lambda_2 (x + y_{22})^\varphi}{|x + y_{22}|_c^\varphi} \right) \left( e_{\mu} e_{\nu}^\dagger - e_{\nu} e_{\mu}^\dagger \right) \left( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} \frac{1}{|x + y_{11}|_c^\varphi} & \frac{0}{|x + y_{22}|_c^\varphi} & \frac{0}{|x + y_{22}|_c^\varphi} \\ \frac{0}{|x + y_{11}|_c^\varphi} & \frac{1}{|x + y_{22}|_c^\varphi} & \frac{0}{|x + y_{22}|_c^\varphi} \\ \frac{0}{|x + y_{11}|_c^\varphi} & \frac{0}{|x + y_{22}|_c^\varphi} & \frac{1}{|x + y_{22}|_c^\varphi} \end{pmatrix} \right)
\]

\[
= \lambda_1 \left( x + y_{11} \right)^\varphi \left( e_{\mu} e_{\nu}^\dagger - e_{\nu} e_{\mu}^\dagger \right) \left( x + y_{11} \right) \lambda_1^\varphi + \lambda_2 \left( x + y_{22} \right)^\varphi \left( e_{\mu} e_{\nu}^\dagger - e_{\nu} e_{\mu}^\dagger \right) \left( x + y_{22} \right) \lambda_2^\varphi.
\]

(4.163)

The amazingly simple result of Eq. (4.163) is to be compared to the CFTW [10] real 2-instanton solution [28] which is lengthy and quite complicated. Presumably, there are many simplified mechanisms [6] for the calculation of complex YM instanton sheaves which make the solvability available. Moreover, the two terms in Eq. (4.163) give the singular structure of the field strength at the sheaf point \( [x : y : z : w] = [-1 : -1 : 1 : 1] \) on \( CP^3 \) as expected [6].

Let’s check this singular structure in the following. We will follow the calculation in [6]. First of all, we note that the Plücker coordinate of the real line in \( CP^3 \) corresponding to the sheaf point \([-1 : -1 : 1 : 1]\) or in short sheaf line [6] can be calculated to be

\[ [-1 : -1 : 1 : 1] \cap \sigma[-1 : -1 : 1 : 1] = [-1 : -1 : 1 : 1] \cap [-1 : 1 : 1 : -1] = [-2 : 0 : 2 : -2 : 0 : -2] \cong [1 : 0 : -1 : 1 : 0 : 1] = [z_{12} \ z_{13} \ z_{14} \ z_{23} \ z_{24} \ z_{34}] \]

(4.164)

where the \( \sigma \) map was defined to be [6]

\[ \sigma : [z_1 : z_2 : z_3 : z_4] \rightarrow [\overline{z}_2 : -\overline{z}_1 : \overline{z}_4 : -\overline{z}_3]. \]

(4.165)

On the other hand, the projection of the sheaf point \([-1 : -1 : 1 : 1]\) on \( CP^3 \) down to \( S^4 \)
is

\[ x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 \]
\[ = \begin{pmatrix} x_0 - i x_3 & (x_2 + i x_1) \\ x_2 - i x_1 & x_0 + i x_3 \end{pmatrix} \]
\[ = \begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix} \]

with the following identification [6]

\[-x_{21} = z_{13}, \]
\[x_{11} = z_{14}, \]
\[-x_{22} = z_{23}, \]
\[x_{12} = z_{24}, \]
\[1 = z_{34}. \]

(4.167)

With the above inputs, one easily calculates the sheaf point on \( CP^3 \) down to \( S^4 \) to be

\[(x_0, x_1, x_2, x_3) = (-1, 0, 0, 0). \]

(4.168)

Finally by using the data \( y_{11} = e_0 \) in Eq.(4.126), we get

\[x + y_{11} = -e_0 + e_0 = 0, \]

(4.169)

which gives the singular structure of the first term in Eq.(4.163). One notes that the sheaf point in Eq.(4.168) is also a singular point of the second term in Eq.(4.163)

\[|x + y_{22}|^2_c = (-1)^2 + (i)^2 + 0^2 + 0^2 = 0. \]

(4.170)

Indeed, the set of jumping lines \( J \) defined in Eq.(2.3) for the ADHM data in Eq.(4.142) can be calculated to be

\[ \det \Delta(x)^\circ \Delta(x) = \det f^{-1} = |x + y_{11}|^2_c \cdot |x + y_{11}|^2_c \]
\[= \left[(x_0 + 1)^2 + x_1^2 + x_2^2 + x_3^2\right]\left[x_0^2 + (x_1 + i)^2 + x_2^2 + x_3^2\right] = 0 \]

(4.171)

where we have used the result of Eq.(4.159). In section IV.C of our previous publication [6], the order of singularity of a jumping line was defined to be the singularity in \( f \) or structure of zero in Eq.(4.171). For the vanishing of the first factor in Eq.(4.171)

\[(x_0 + 1)^2 + x_1^2 + x_2^2 + x_3^2 = 0, \]

(4.172)
we get the sheaf point in Eq. (4.168). On the other hand, the real solutions of
\[ x_0^2 + (x_1 + i)^2 + x_2^2 + x_3^2 = 0 \]  
(4.173)
are
\[ x_1 = 0, \quad x_0^2 + x_2^2 + x_3^2 = 1, \]  
(4.174)
which is a unit 2-sphere on the \( x_0 - x_2 - x_3 \) hyperplane. Note that Eq. (4.170) implies the sheaf point in Eq. (4.168) sits on the unit 2-sphere in Eq. (4.174). This is consistent with our previous result that a sheaf line is always a jumping line [6]. Moreover, it is obvious that the order of singularity of \( f \) at the sheaf point \( x = (x_0, x_1, x_2, x_3) = (-1, 0, 0, 0) \) is higher than those of other points on the 2-sphere associated with normal jumping lines. This is again consistent with the conjecture made in [6].

B. Sheaf case with rank \( \beta = 1 \)

In this section, we consider another sheaf case with rank \( \beta = 1 \) on some points of \( CP^3 \) for some given ADHM data. As in the rank \( \beta = 0 \) case, we are looking for the factorization of a det factor in the \( y_{12} \) matrix
\[ y_{12} = \frac{i}{|y_{11} - y_{22}|^2} \begin{pmatrix} d_1 + a_1 & d_2 + a_2 \\ d_3 + a_3 & d_4 + a_4 \end{pmatrix} \begin{pmatrix} l & m - in \\ m + in & -l \end{pmatrix} \]
\[ \begin{pmatrix} A & B \\ \tilde{A} & \tilde{B} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]  
(4.175)
We will see that under some general assumption on the moduli parameters, the matrix on the rhs of the following equation
\[ \begin{pmatrix} A & B \\ \tilde{A} & \tilde{B} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Aa + Bc & Ab + Bd \\ \tilde{A}a + \tilde{B}c & \tilde{A}b + \tilde{B}d \end{pmatrix} \]  
(4.176)
factors out a common det factor
\[ \text{det} = |y_{11} - y_{22}|^2 = A\tilde{B} - B\tilde{A}. \]  
(4.177)
Let’s first define
\[ \frac{Aa + Bc}{Ab + Bd} = -w' \]  
(4.178)
where $w'$ will be identified to be the $w$ coordinate in $CP^3$ in the later calculation. We begin the calculation by assuming

$$Ab + Bd = \varepsilon \det \tag{4.179}$$

where $\varepsilon$ is a finite number as $\det \to 0$. One can then solve $d$

$$d = \frac{\varepsilon \det - Ab}{B} \tag{4.180}$$

to obtain

$$\tilde{A}b + \tilde{B}d$$

$$= (Ab + Bd) + b \left( \tilde{A} - A \right) + d \left( \tilde{B} - B \right)$$

$$= \varepsilon \det + b \left( \tilde{A} - A \right) + \left( \frac{\varepsilon \det - Ab}{B} \right) \left( \tilde{B} - B \right)$$

$$= \varepsilon \det \left( 1 + \frac{\tilde{B} - B}{B} \right) + b \left[ \left( \tilde{A} - A \right) B - A \left( \tilde{B} - B \right) \right]$$

$$= \det \left[ \varepsilon \frac{\tilde{B}}{B} - \frac{b}{B} \right] = \det \tilde{\varepsilon} \tag{4.181}$$

where we have defined

$$\tilde{\varepsilon} = \varepsilon \frac{\tilde{B}}{B} - \frac{b}{B} \tag{4.182}$$

which was assumed to be finite as $\det \to 0$.

We can factor out a det factor in $Aa + Bc$ by using Eq.(4.178) and Eq.(4.179) to obtain

$$Aa + Bc = -w' \varepsilon \det \tag{4.183}$$

Finally, let’s calculate

$$\tilde{A}a + \tilde{B}c$$

$$= \left( \det \tilde{\varepsilon} \right) \left( \frac{\tilde{A}a + \tilde{B}c}{Ab + Bd} \right)$$

$$= \left( \det \tilde{\varepsilon} \right) \left( \frac{Aa + Bc}{Ab + Bd} \right) \frac{(ad - bc)}{\left( \tilde{A}b + \tilde{B}d \right) \left( Ab + Bd \right)} \left( \frac{A\tilde{B} - B\tilde{A}}{\tilde{A}b + \tilde{B}d} \right)$$

$$= (-w') \tilde{\varepsilon} \det - \left( \frac{ad - bc}{\tilde{\varepsilon} \det} \frac{\det}{\tilde{\varepsilon} \det} \right) \tilde{\varepsilon} \det \tag{4.184}$$
The factor $ad - bc$ in Eq. (4.184) can be calculated to be

$$ad - bc$$

$$= a \left( \varepsilon \text{det} - Ab \right) - bc$$

$$= \frac{a\varepsilon}{B} \text{det} - \frac{b}{B} (Aa + Bc)$$

(4.185)

where we have used Eq. (4.180). Finally

$$\tilde{A}a + \tilde{B}c = (-w') \tilde{\varepsilon} \text{det} - \left( \frac{ad - bc}{\tilde{\varepsilon} \text{det}} \varepsilon \text{det} \right) \tilde{\varepsilon} \text{det}$$

$$= -w' \tilde{\varepsilon} \text{det} - \frac{a}{B} \text{det} + \frac{b}{\varepsilon B} (Aa + Bc)$$

$$= \text{det} \left( -w' \tilde{\varepsilon} - \frac{a}{B} \right) + \frac{b}{\varepsilon B} (-w' \tilde{\varepsilon} \text{det})$$

$$= \left( \delta - \frac{w'b}{B} \right) \text{det}$$

(4.186)

where $\left( \delta - \frac{w'b}{B} \right)$ is assumed to be finite as $\text{det} \to 0$ and $\delta$ is defined to be

$$\delta = -w' \tilde{\varepsilon} - \frac{a}{B}.$$  

(4.187)

In sum, we have achieved the factorization

$$\begin{bmatrix} A & B \\ \tilde{A} & \tilde{B} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -w' \tilde{\varepsilon} \text{det} & \varepsilon \text{det} \\ (\delta - \frac{w'b}{B}) \text{det} & \tilde{\varepsilon} \text{det} \end{bmatrix}. $$

(4.188)

We can now calculate the finite $y_{12}$ in the $\text{det} \to 0$ limit to be

$$y_{12} = \frac{i}{\text{det}} \begin{bmatrix} A & B \\ \tilde{A} & \tilde{B} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{i}{\text{det}} \begin{bmatrix} -w' \varepsilon \tilde{\varepsilon} \\ (\delta - \frac{w'b}{B}) \tilde{\varepsilon} \text{det} \end{bmatrix}.$$ 

(4.189)

On the other hand, if we assume

$$\left( \delta - \frac{w'b}{B} \right) = -w' \tilde{\varepsilon}$$

(4.190)

we get

$$\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} + w' \begin{bmatrix} c_2 \\ c_4 \end{bmatrix} = 0.$$ 

(4.191)
After identifying $\delta$ in Eq. (4.187) and Eq. (4.190), we obtain

$$w' = \frac{-a}{b}. \quad (4.192)$$

We are now ready to check the rank $\beta = 1$ condition. For simplicity, we take $z = 1$ to obtain

$$I_1 + wI_2 = \begin{pmatrix} -\lambda_1^2 + i\lambda_1^1 - w (\lambda_0^0 + i\lambda_3^3) & \lambda_0^0 - i\lambda_1^3 - w (\lambda_1^2 + i\lambda_1^1) \\ -\lambda_2^2 + i\lambda_2^1 - w (\lambda_2^0 + i\lambda_2^3) & \lambda_2^0 - i\lambda_2^3 - w (\lambda_2^2 + i\lambda_2^1) \end{pmatrix}, \quad (4.193)$$

$$B_{11} + wB_{21} + x = \begin{pmatrix} d_1 + d_2w + x & c_1 + c_2w \\ c_1 + c_2w & -a_1 - a_2w + x \end{pmatrix}, \quad (4.194)$$

$$B_{12} + wB_{22} + y = \begin{pmatrix} d_3 + d_4w + y & c_3 + c_4w \\ c_3 + c_4w & -a_3 - a_4w + y \end{pmatrix}. \quad (4.195)$$

We assume a common eigenvector to be

$$v = (1, 0) \quad (4.196)$$

in Eq. (2.18), and take the following particular solutions (vanishing of the first rows of matrices in Eq. (4.193) to Eq. (4.195))

$$-\lambda_1^2 + i\lambda_1^1 - w (\lambda_0^0 + i\lambda_3^3) = 0, \quad (4.197)$$

$$\lambda_0^0 - i\lambda_1^3 - w (\lambda_1^2 + i\lambda_1^1) = 0 \quad (4.198)$$

$$d_1 + d_2w + x = 0, \quad (4.199)$$

$$c_1 + c_2w = 0, \quad (4.200)$$

$$d_3 + d_4w + y = 0, \quad (4.201)$$

$$c_3 + c_4w = 0, \quad (4.202)$$

To solve Eq. (4.200) and Eq. (4.202), we choose $w = w'$ or

$$w = w' = -\frac{Aa + Bc}{Ab + Bd} = \frac{-a}{b}, \quad (4.203)$$

and make use of Eq. (4.191). On the other hand, Eq. (4.199) and Eq. (4.201) give

$$x = -d_1 - d_2w, \; y = -d_3 - d_4w. \quad (4.204)$$
Finally Eq. (4.197) and Eq. (4.198) put constraints on the $\lambda$ parameters. In addition, one needs to take into account the condition of finite $c_j$ in Eq. (4.189).

In the following, let’s give one explicit example of rank $\beta = 1$ and $\det = 0$ case. We begin with the following ADHM data

$$
y_{11} = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
y_{22} = - \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} -p & 0 \\ 0 & -p \end{pmatrix}.
$$

which give

$$
det = p^2 - 1.
$$

Now Eq. (4.176) can be written as

$$
\begin{pmatrix} p & 1 \\ 1 & p \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} pa + c & pb - a \\ a - pc & b - pa \end{pmatrix}.
$$

Then Eq. (4.178) and Eq. (4.192) imply

$$
\frac{pa + c}{pb - a} = \frac{a}{b},
$$

which gives $c = 1 - p$. On the other hand, Eq. (4.179) gives (take $\varepsilon = 1$)

$$
pb - a = p^2 - 1.
$$

For simplicity, we choose $b = p - 1$, then $a = 1 - p$. We can now calculate $y_{12}$ to be

$$
y_{12} = \frac{i}{p^2 - 1} \begin{pmatrix} p & 1 \\ 1 & p \end{pmatrix} \begin{pmatrix} 1 - p & p - 1 \\ 1 - p & p - 1 \end{pmatrix}
= \frac{i}{p^2 - 1} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} (p^2 - 1) = i \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.
$$

We thus have achieved the factorization and have cancelled out the $\det = p^2 - 1$ factor in Eq. (4.210). Moreover the parameters we introduced during the calculation of factorization can be calculated to be

$$
\varepsilon = 1, \tilde{\varepsilon} = 1, \delta = -3,
$$

which are finite as $\det \to 0$.

We will take the $p \to -1$ limit after the factorization. In this limit, we get

$$
\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} l & m - in \\ m + in & -l \end{pmatrix}.
$$
which give

\[
\begin{align*}
l &= \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix} - \begin{vmatrix} \lambda_1^1 & \lambda_1^2 \\ \lambda_2^1 & \lambda_2^2 \end{vmatrix} = 2, \\
m &= \begin{vmatrix} \lambda_1^0 & \lambda_1^1 \\ \lambda_2^0 & \lambda_2^1 \end{vmatrix} - \begin{vmatrix} \lambda_1^2 & \lambda_1^3 \\ \lambda_2^2 & \lambda_2^3 \end{vmatrix} = 0, \\
n &= \begin{vmatrix} \lambda_1^0 & \lambda_1^2 \\ \lambda_2^0 & \lambda_2^2 \end{vmatrix} - \begin{vmatrix} \lambda_1^3 & \lambda_1^1 \\ \lambda_2^3 & \lambda_2^1 \end{vmatrix} = -2i.
\end{align*}
\]

On the other hand, with \( w = 1 \), Eq.(4.197) and Eq.(4.198) become

\[
\begin{align*}
-\lambda_1^2 + i\lambda_1^1 - (\lambda_1^0 + i\lambda_1^3) &= 0, \\
\lambda_1^0 - i\lambda_1^3 - (\lambda_1^2 + i\lambda_1^1) &= 0.
\end{align*}
\]

For illustration, we will find a particular solution of Eq.(4.213) to Eq.(4.217). First of all, we note that Eq.(4.216) and Eq.(4.217) give

\[
\begin{align*}
\lambda_1^0 - i\lambda_1^3 &= 0, \\
i\lambda_1^3 + \lambda_1^2 &= 0.
\end{align*}
\]

We choose the particular solution of Eq.(4.218) and Eq.(4.219) to be

\[
\lambda_1^0 = 1, \lambda_1^1 = -i.
\]

To further simplify the calculation, we choose

\[
\lambda_1^2 = 0 = \lambda_1^3 \text{ and } \lambda_2^0 = 0 = \lambda_2^1.
\]

With these choices, Eq.(4.214) is trivially satisfied and Eq.(4.213) and Eq.(4.215) give

\[
\begin{align*}
\lambda_2^0 - i\lambda_2^3 &= -2i, \\
\lambda_2^3 + i\lambda_2^2 &= 2.
\end{align*}
\]

We see that Eq.(4.222) and Eq.(4.223) are similar, and for simplicity we choose one particular solution to be

\[
\lambda_2^2 = -i = \lambda_2^3 = 1.
\]

Finally

\[
x = -d_1 - d_2 w = -1, \quad y = -d_3 - d_4 w = -1.
\]
We can now check that rank $\beta = 1$. Indeed the $\beta$ matrix can be explicitly calculated to be

$$
\beta = \begin{bmatrix}
-zB_{12} - wB_{22} - y & zB_{11} + wB_{21} + x & zI_1 + wI_2
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0
0 & -2 & 0 & -2 & 0 & 0
\end{bmatrix}.
$$

(4.226)

We conclude that for the ADHM data given in Eq.(4.205), Eq.(4.210), Eq.(4.220), Eq.(4.221) and Eq.(4.224), rank $\beta = 1$ at point $[x : y : z : w] = [-1 : -1 : 1 : 1]$ on $CP^3$. So this is a solution of complex YM instanton sheaf. Note that this extended complex YM instanton sheaf solution was not considered in [4] since det = 0.

C. Bundle case with rank $\beta = 2$

In this section, we consider the bundle solutions with rank $\beta = 2$ on the whole $CP^3$ for some given ADHM data. We are again looking for the factorization of a det factor in the $y_{12}$ matrix. We will demonstrate the existence of a class of extended complex YM 2-instanton solutions with rank $\beta = 2$ and det $= |y_{11} - y_{22}|_c^2 = 0$. We begin with the following ADHM data

$$
y_{11} \equiv \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 1 & -i \end{pmatrix}, y_{22} = -\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = -\begin{pmatrix} a_1 & a_2 \\ a_3 - ia_3 \end{pmatrix},
$$

(4.227)

with $a_4 = -ia_3$. The det can then be calculated to be

$$
\det = |y_{11} - y_{22}|^2_c = -(a_3 + 1) (ia_1 + a_2).
$$

(4.228)

We will take $(ia_1 + a_2) \to 0$ limit after factoring out the det factor in $y_{12}$. The $\lambda$ parameters are chosen to be

$$
\lambda_0^1 = 1 + \varepsilon, \lambda_1^1 = 0, \lambda_2^3 = i, \lambda_3^3 = \delta,
\lambda_0^0 = 0, \lambda_1^2 = 1, \lambda_2^2 = 0, \lambda_3^3 = 0
$$

(4.229)

where

$$
\varepsilon = \frac{-a_1 + ia_2}{2 (1 + a_1)} \to 0,
$$

(4.230)

$$
\delta = \frac{-a_1 + ia_2}{2i (1 + a_1)} \to 0
$$

(4.231)
in the \((ia_1 + a_2) \to 0\) limit. The \(l, m, n\) parameters defined in Eq. (3.27) can be calculated to be

\[
l = i, m = 1 + \varepsilon, n = -\delta. \tag{4.232}
\]

We are now ready to calculate \(y_{12}\)

\[
y_{12} = \frac{i}{|y_{11} - y_{22}|^2} \begin{pmatrix}
1 + a_1 & -i + a_2 \\
1 + a_3 & -i - ia_3
\end{pmatrix}
\begin{pmatrix}
l & m - in \\
m + in & -l
\end{pmatrix}
\begin{pmatrix}
i a_1 + a_2 & 0 \\
0 & \frac{1 \pm ia_2}{1 + a_1}
\end{pmatrix}
\begin{pmatrix}
i + \varepsilon + i\delta & 1 + \varepsilon + i\delta \\
1 + \varepsilon - i\delta & -i
\end{pmatrix}
\begin{pmatrix}
1 + a_1 & -i + a_2 \\
1 + a_3 & -i + a_4
\end{pmatrix}
\begin{pmatrix}
i & \frac{1 + ia_2}{1 + a_1} \\
1 + a_1 & -i
\end{pmatrix}
\tag{4.233}
\]

where we have used

\[
1 + \varepsilon + i\delta = \frac{2(1 + a_1) - a_1 + ia_2 + (-a_1 + ia_2)}{2(1 + a_1)} = \frac{1 + ia_2}{1 + a_1}, \tag{4.234}
\]

\[
1 + \varepsilon - i\delta = \frac{2(1 + a_1) - a_1 + ia_2 - (-a_1 + ia_2)}{2(1 + a_1)} = 1. \tag{4.235}
\]

The parameter \(y_{12}\) can be further calculated to be

\[
y_{12} = \frac{i}{|y_{11} - y_{22}|^2} \begin{pmatrix}
1 + a_3 & 0 \\
0 & -i|y_{11} - y_{22}|^2
\end{pmatrix}
\begin{pmatrix}
|y_{11} - y_{22}|^2 & 0 \\
0 & 1 + a_3
\end{pmatrix}
\begin{pmatrix}
1 + a_1 & 0 \\
0 & \frac{1}{1 + a_1}
\end{pmatrix}
\tag{4.236}
\]

where Eq. (4.228) has been used. One can now factor out the det factor in Eq. (4.236) and take the \(\text{det} \to 0\) limit to get

\[
y_{12} = \begin{pmatrix}
-\frac{i}{1 + a_3} & 0 \\
0 & \frac{1}{1 + a_1}
\end{pmatrix}. \tag{4.237}
\]
So the $y$ data is given by

$$
\begin{pmatrix}
  y_{11} & y_{12} \\
y_{12} & y_{22}
\end{pmatrix} = \begin{pmatrix}
  1 - i & -i \\
  1 - i & 0 \\
-i & 0 \\
0 & 1 + a_3
\end{pmatrix} \begin{pmatrix}
  0 & 0 \\
a_1 & a_2 \\
a_3 & -ia_3
\end{pmatrix} = \begin{pmatrix}
  1 & -i \\
  1 - i & 0 \\
-i & 0 \\
0 & 1 + a_3
\end{pmatrix} \begin{pmatrix}
  0 & 0 \\
a_1 & -a_2 \\
0 & -ia_3
\end{pmatrix}.
$$

(4.238)

After the rearrangement, we obtain the $B_{ij}$ matrix

$$B_{11} = \begin{pmatrix}
  1 & -i \\
-i & 0 \\
0 & 1 + a_3
\end{pmatrix}, \quad B_{21} = \begin{pmatrix}
  0 & 0 \\
-i & 0 \\
0 & 1 + a_3
\end{pmatrix}.
$$

(4.239)

$$B_{12} = \begin{pmatrix}
  1 & 0 \\
0 & -a_3
\end{pmatrix}, \quad B_{22} = \begin{pmatrix}
  0 & 0 \\
-i & 1 + a_3
\end{pmatrix}.
$$

(4.240)

The next step is to check the rank $\beta = 2$ condition. For this we will divide the whole $CP^3$ into three parts.

(1) For the first case, we set $z = 1$. The first two components of the $\beta$ matrix can be calculated to be

$$B_{11} + wB_{21} + x
\begin{pmatrix}
  1 - iw + x & -i \\
-i & -a_1 - a_2w + x
\end{pmatrix},
$$

(4.241)

$$B_{12} + wB_{22} + y
\begin{pmatrix}
  1 - iw + y & 1 \\
1 & -a_3 + ia_3w + y
\end{pmatrix},
$$

(4.242)

and the last component

$$I_1 + wI_2 = \begin{pmatrix}
  -\lambda_1^2 + i\lambda_1^1 - w(\lambda_1^0 + i\lambda_1^1) & \lambda_1^0 - i\lambda_1^3 - w(\lambda_1^2 + i\lambda_1^1) \\
-\lambda_2^2 + i\lambda_2^1 - w(\lambda_2^0 + i\lambda_2^1) & \lambda_2^0 - i\lambda_2^3 - w(\lambda_2^2 + i\lambda_2^1)
\end{pmatrix} = \begin{pmatrix}
  -i - w & 1 - wi \\
i & -iw
\end{pmatrix}.
$$

(4.243)
Let’s assume there exists sheaf solution for this case. We will see soon that a contradiction results. We first note that one necessary condition to have sheaf solution is

$$\det (I_1 + wI_2) = 0, \quad (4.244)$$

which can be written as

$$(n + im)w^2 + 2ilw + (n - im) = 0. \quad (4.245)$$

The solution of Eq.(4.245) in the $\det \to 0$ limit is

$$w = \frac{1}{n + im} \left[ -il \pm \sqrt{- (l^2 + m^2 + n^2)} \right] = \frac{-i \pm (-i) \sqrt{- (2\varepsilon + \varepsilon^2 + \delta^2)}}{1 + \varepsilon + i\delta} \to -i. \quad (4.246)$$

For this case, Eq.(4.241) and Eq.(4.242) reduce to

$$B_{11} + wB_{21} + x = \begin{pmatrix} x & \frac{-i}{1+\alpha_3} \\ \frac{-i}{1+\alpha_3} & x \end{pmatrix}, \quad (4.247)$$

$$B_{12} + wB_{22} + y = \begin{pmatrix} y & \frac{-i}{1+\alpha_1} \\ \frac{-i}{1+\alpha_1} & y \end{pmatrix}, \quad (4.248)$$

and Eq.(4.243) reduces to

$$I_1 + wI_2 = \begin{pmatrix} 0 & 0 \\ i & -1 \end{pmatrix}. \quad (4.249)$$

In the end, the $\beta$ matrix can be written as

$$\beta = \begin{pmatrix} -zB_{12} - wB_{22} - y & zB_{11} + wB_{21} + x & zI_1 + wI_2 \\ -y & \frac{i}{1+\alpha_1} & x \\ \frac{i}{1+\alpha_1} & -y & \frac{i}{1+\alpha_3} \end{pmatrix} \begin{pmatrix} -i + x & 0 \\ 0 & -a_2 + x \end{pmatrix}, \quad (4.250)$$

which gives $\text{rank} \beta = 2$ for any $x$ and $y$. We conclude that there exists no sheaf solution for this case. So this is an extended solution in bundle case.

(2) For the second case, we set $z = 0, w = 1$. The $\beta$ matrix can be calculated to be

$$zB_{11} + wB_{21} + x = \begin{pmatrix} -i + x & 0 \\ 0 & -a_2 + x \end{pmatrix}, \quad (4.251)$$
\( zB_{12} + wB_{22} + y = \begin{pmatrix} -i + y & \frac{1}{1+a_1} \\ \frac{1}{1+a_1} & ia_3 + y \end{pmatrix} \) (4.252)

and

\( zI_1 + wI_2 = \begin{pmatrix} -1 & -i \\ 0 & -i \end{pmatrix} \). (4.253)

So we get

\[
\beta = \begin{bmatrix} -zB_{12} - wB_{22} - y & zB_{11} + wB_{21} + x & zI_1 + wI_2 \\
(i - y & -\frac{1}{1+a_1} & -i + x & 0 & -1 & -i \\
-\frac{1}{1+a_1} & -ia_3 - y & 0 & -a_2 + x & 0 & -i \end{bmatrix},
\]

which gives \( \text{rank} \beta = 2 \), again a bundle case.

(3) For the third case, we set \( z = 0, w = 0 \). The \( \beta \) matrix can be calculated to be

\[
\begin{align*}
zB_{11} + wB_{21} + x &= \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \\
zB_{12} + wB_{22} + y &= \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, \\
zI_1 + wI_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

So we get

\[
\beta = \begin{bmatrix} -zB_{12} - wB_{22} - y & zB_{11} + wB_{21} + x & zI_1 + wI_2 \\
-y & 0 & x & 0 & 0 & 0 \\
0 & -y & 0 & x & 0 & 0 \end{bmatrix},
\]

which gives \( \text{rank} \beta = 2 \) (\( x, y \) can’t be both zeros), again a bundle case.

In this subsection, we thus have explicitly demonstrated a class of extended complex YM 2-instanton solution with \( \text{rank} \beta = 2 \) on the whole \( CP^3 \).

V. CONCLUSION

In this paper, we first show that the complex YM 2-instanton sheaf with \( \text{rank} \beta = 0 \) does not exist for the previous construction of \( SU(2) \) complex YM instantons with
\[ \text{det} = |y_{11} - y_{22}|_c^2 \neq 0 \] The reason has been that complex YM 2-instanton with rank \( \beta = 0 \) implies \( \text{det} = |y_{11} - y_{22}|_c^2 = 0 \) which, although is allowed and possibly exist in the construction of complex YM 2-instanton solutions, was not considered previously \([3]\). We then proceed to show the existence of the new extended (or \( \text{det} = 0 \)) complex YM 2-instantons in this paper.

Moreover, we discovered that the rank \( \beta \) of these new extended complex YM instantons can be either 2 on the whole \( CP^3 \) for some given ADHM data (bundle) or 1, 0 on some points of \( CP^3 \) with some other given ADHM data (sheaves). We have also calculated explicit examples of these new instanton solutions with various rank \( \beta \). These extended \( SU(2) \) complex YM instantons have no real instanton counterparts.

One unexpected result we obtained in the search of extended complex YM instantons was the discovery of the existence of instanton sheaf structure with diagonal \( y \) ADHM data. For the \( \text{det} \neq 0 \) complex YM instantons constructed previously \([3]\), it was shown \([4]\) that there is no instanton sheaf solutions with diagonal \( y \) ADHM data.

It will be a challenge to generalize the calculation of 2-instanton solutions presented in this paper to \( k \)-instanton solutions with higher topological charges. Another interesting issue is to check the solvability of the field strength \( F \) of the new complex YM 2-instanton sheaves discovered in this paper. The first explicit example of complex YM 2-instanton field strength corresponding to sheaf case with rank \( \beta = 1 \) and \( \text{det} \neq 0 \) was given in \([6]\). In this paper, we give the second example of complex YM 2-instanton field strength in Eq.\((4.163)\). This second example corresponds to the sheaf case with rank \( \beta = 0 \) and \( \text{det} = 0 \). The "simple" explicit form of 2-instanton field strength seems not available in the literature for the real instanton case \([28]\).

Acknowledgments

The work of J.C. Lee is supported in part by the Ministry of Science and Technology and S.T. Yau center of NCTU, Taiwan. The work of I-H. Tsai is supported by the Ministry of
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