ON THE PROBLEM OF UNIQUE CONTINUATION FOR THE $p$-LAPLACE EQUATION

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Abstract. We consider the problem of unique continuation for the solutions to the $p$-Laplace equation

$$\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0,$$

where $1 < p < \infty$. Namely, can two different solutions coincide in an open subset of their common domain of definition? We obtain some partial results on this interesting problem.

1. Introduction

Consider the $p$-Laplace equation in an open connected set $G \subset \mathbb{R}^n$, $n \geq 2$,

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty. \quad (1.1)$$

For $p = 2$ we recover the Laplace equation $\Delta u = 0$. The classical unique continuation principle for the $p$-Laplace equation can be formulated as follows

(i) Let $u_1$ and $u_2$ be two solutions to (1.1) such that $u_1 = u_2$ in an open subset of $G$. Then $u_1 \equiv u_2$ in $G$.

(ii) Let $u$ be a solution to (1.1) such that $u = 0$ in an open subset of $G$. Then $u \equiv 0$ in $G$.

The latter formulation is equivalent to the following: (ii') Let $u$ be a solution to (1.1) and consider two open concentric balls $B_r \subset B_R \subset G$ such that $u = 0$ on $B_r$, then $u \equiv 0$ in $B_R$. The problem of unique continuation, both (i) and (ii), is still, to the best of our knowledge, an open problem, except for the linear case $p = 2$. The planar case for (ii) has been solved by Manfredi in [12], see also Bojarski and Iwaniec [1], as they have observed that the complex gradient of a solution to (1.1) is quasiregular.

We refrain from giving a detailed bibliographical account on the literature on unique continuation results for elliptic equations. We refer to the papers [4] and [5] by Garofalo and Lin, and suggest the reader to consult also their bibliographies for more detailed information on the subject.

2000 Mathematics Subject Classification. Primary: 35J92; Secondary: 35B60, 35J70.
In the present paper, we deal with the problem of unique continuation by studying a certain generalization of Almgren’s frequency function for the $p$-Laplacian. By this approach we have obtained some partial results on the unique continuation problem in both cases (i) and (ii). These results, along with the notation and the preliminary results, are stated in §2. The proofs can be found in §§3–5.

2. Results

Let $G$ be an open connected subset of $\mathbb{R}^n$. We consider the $p$-Laplace equation (1.1) in the weak form

$$
\int_G |\nabla u|^{p-2}\nabla u \cdot \nabla \eta \, dx = 0,
$$

where $\eta \in C_0^\infty(G)$ and $1 < p < \infty$. We refer the reader to, e.g., Heinonen et al. [7] and Lindqvist [11] for a detailed study of the $p$-Laplace equation and various properties of its solutions. We mention in passing, however, that the weak solutions of (1.1) are $C^{1,\alpha}_{\text{loc}}(G)$, where $\alpha$ depends on $n$ and $p$. We refer to DiBenedetto [2], Lewis [10], and Tolksdorf [15] for this regularity result.

Hence, without loss of generality, we may redefine $u$ so that $u \in W^{1,p}_{\text{loc}}(G) \cap C^1(G)$.

Let us introduce the frequency function

$$
F_p(r) = \frac{r \int_{B(x,r)} |\nabla u|^p \, dx}{\int_{\partial B(x,r)} |u|^p \, dS},
$$

where $\bar{B}(x,r) \subset G$; we denote

$$
D(r) = \int_{B(x,r)} |\nabla u|^p \, dx \quad \text{and} \quad I(r) = \int_{\partial B(x,r)} |u|^p \, dS.
$$

Observe that $F_p(r)$ is not defined for such radii $r$ for which $I(r) = 0$. We remark that $F_p(r)$ is a generalization of the well known Almgren frequency function

$$
F_2(r) = \frac{r \int_{B(x,r)} |\nabla u|^2 \, dx}{\int_{\partial B(x,r)} |u|^2 \, dS}
$$

for harmonic functions in $\mathbb{R}^n$. To the best of our knowledge, $F_p(r)$, $p \neq 2$, has not been previously studied in the literature. It might be interesting to study other generalizations, for instance, the case in which $r$ is replaced with $r^{p-1}$ in (2.2). We have, however, omitted such considerations here.

The main results of the present paper are the following theorems.

**Theorem 2.4.** Suppose $u \in W^{1,p}_{\text{loc}}(G) \cap C^2(G)$ is a solution to the $p$-Laplace equation in $G$. Consider an affine function

$$
L(x) = l(x) + l_0,
$$

where ...
where \( l_0 \in \mathbb{R} \) and
\[
l(x) = \sum_{i=1}^{n} \alpha_i x_i \neq 0.
\]

Then if \( u(x) = L(x) \) in \( B_r \subset G \), \( u(x) = L(x) \) for every \( x \in G \).

Observe that the affine function \( L(x) \) satisfies the \( p \)-Laplace equation.

The proof of Theorem 2.4 can be generalized to justify the following, see Remark 3.5 for a more detailed discussion. Suppose \( u, v \in W^{1,p}_{\text{loc}}(G) \cap C^2(G) \) are two solutions to the \( p \)-Laplace equation in \( G \). Assume further that \( \nabla v \neq 0 \) in \( G \). Then if \( u(x) = v(x) \) in \( B_r \subset G \), \( u(x) = v(x) \) for every \( x \in G \).

Toward more general results we state the following theorems.

**Theorem 2.5.** Suppose \( u \in C^1(G) \). Assume further that there exist two concentric balls \( B_{r_b} \subset \overline{B}_{R_b} \subset G \) such that the frequency function \( F_p(r) \) is defined, i.e., \( I(r) > 0 \) for every \( r \in (r_b, R_b] \), and moreover, \( \|F_p\|_{L^\infty((r_b, R_b])} < \infty \). Then there exists some \( r^* \in (r_b, R_b] \) such that
\[
\int_{\partial B_{r_1}} |u|^p dS \leq 4 \int_{\partial B_{r_2}} |u|^p dS, \tag{2.6}
\]
for every \( r_1, r_2 \in (r_b, r^*] \). In particular, the following weak doubling property is valid
\[
\int_{\partial B_r} |u|^p dS \leq 4 \int_{\partial B_{r^*}} |u|^p dS, \tag{2.7}
\]
for every \( r \in (r_b, r^*] \).

In the following we formulate a partial result on the unique continuation problem for the \( p \)-Laplace equation. It says that the local boundedness of the frequency function implies the unique continuation principle. In this respect the situation is similar to the linear case \( p = 2 \), and we thus generalize this phenomenon to every \( 1 < p < \infty \).

**Theorem 2.8.** Suppose \( u \) is a solution to the \( p \)-Laplace equation in \( G \). Consider arbitrary concentric balls \( B_{r_b} \subset \overline{B}_{R_b} \subset G \). Assume the following: whenever \( I(r) > 0 \) for every \( r \in (r_b, R_b] \), then \( \|F_p\|_{L^\infty((r_b, R_b])} < \infty \). Then the following unique continuation principle follows: If \( u \) vanishes on some open ball in \( G \), then \( u \) is identically zero in \( G \).

It remains open problem whether the frequency function \( F_p(r) \) is locally bounded for the solutions to the \( p \)-Laplace equation. Local boundedness combined with the method of the present paper would solve the unique continuation problem for equation (1.1). In § 4 we discuss some observations which might be of interest for further studies.
2.1. Preliminaries. Throughout the paper $G$ is an open connected subset of $\mathbb{R}^n$, $n \geq 2$, and $1 < p < \infty$. We use the notation $B_r = B(x, r)$ for concentric open balls of radii $r$ centered at $x \in G$. Unless otherwise stated, the letter $C$ denotes various positive and finite constants whose exact values are unimportant and may vary from line to line. Moreover, $dx = dx_1 \ldots dx_n$ denotes the Lebesgue volume element in $\mathbb{R}^n$, whereas $dS$ denotes the surface element. We denote by $|E|$ the $n$-dimensional Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^n$. The characteristic function of $E$ is denoted by $\chi_E$. Along $\partial B_r$ is defined the outward pointing unit normal vector field at $x \in \partial B_r$ and is denoted by $\nu(x) = (\nu_1, \ldots, \nu_n)(x)$. We will also write $u_\nu = \nabla u \cdot \nu$ or $\partial u / \partial \nu$ for the directional derivative of $u$. Define sets $P_r$ and $N_r$ as follows

$$P_r = \{ x \in \partial B_r : u(x) > 0 \} \quad \text{and} \quad N_r = \{ x \in \partial B_r : u(x) \leq 0 \}.$$ 

We obtain the following formula for the derivative of $I(r)$ in (2.2).

$$I'(r) = \frac{\partial}{\partial r} \left( r^{n-1} \int_{P_r \cap \partial B_1} u^p(r \omega) \, d\omega + r^{n-1} \int_{N_r \cap \partial B_1} (-u)^p(r \omega) \, d\omega \right)$$

$$= \frac{n-1}{r} r^{n-1} \left( \int_{P_r \cap \partial B_1} u^p(r \omega) \, d\omega + \int_{N_r \cap \partial B_1} (-u)^p(r \omega) \, d\omega \right)$$

$$+ pr^{n-1} \left( \int_{P_r \cap \partial B_1} u^{p-1}(r \omega) u_r(r \omega) \, d\omega - \int_{N_r \cap \partial B_1} (-u)^{p-1}(r \omega) u_r(r \omega) \, d\omega \right)$$

$$= \frac{n-1}{r} \int_{\partial B_r} \left( u \chi_{P_r} \right)^p + (-u \chi_{N_r})^p \, dS$$

$$+ p \int_{\partial B_r} \left( u \chi_{P_r} \right)^{p-1} - (-u \chi_{N_r})^{p-1} \frac{\partial u}{\partial \nu} \, dS \quad (2.9)$$

Formula (2.9) gives us the inequality

$$I'(r) \leq \frac{n-1}{r} I(r) + p \int_{\partial B_r} |u|^{p-1} |u_\nu| \, dS. \quad (2.10)$$

We shall also need the following formula for the solutions to the $p$-Laplace equation. It is probably earlier known in the literature, but we provide it here due to the lack of references.

**Lemma 2.11.** Suppose $u$ is a solution to the $p$-Laplace equation in $G$. Then the following identity holds for the $p$-Dirichlet integral

$$\int_{B_r} |
abla u|^p \, dx = \int_{\partial B_r} |
abla u|^{p-2} u u_\nu \, dS \quad (2.12)$$

for every $\overline{B}_r \subset G$. 4
Proof. As in the classical case \( p = 2 \) the proof is based on the Gauss–Green theorem. In the general case \( 1 < p < \infty \), however, \( u \) is not necessarily in \( C^2(G) \). Hence we have to use an approximation argument; we will use the approximation method presented by Lewis in [10].

Consider a ball \( B_r \) and a bounded open set \( D \) such that \( \overline{B_r} \subset \overline{D} \subset G \). Let \( 0 < \varepsilon < 1 \). Following [10] we construct a sequence of functions \( \hat{u}_\varepsilon \in W^{1,p}(D) \cap C^\infty(D) \) such that they minimize the variational integral

\[
\mathcal{I}_\varepsilon(\psi) = \int_D \left( |\nabla \psi|^2 + \varepsilon \right)^{p/2} \, dx,
\]

over all admissible functions in \( \mathcal{F}_u(D) = \{ v \in W^{1,p}(D) : v - u \in W^{1,p}_0(D) \} \). It is well known that the minimizing function \( \hat{u}_\varepsilon \) is unique. The function \( \hat{u}_\varepsilon \) is a solution to uniformly elliptic equation in the weak form

\[
\int_D \left( |\nabla \hat{u}_\varepsilon|^2 + \varepsilon \right)^{(p-1)/2} \nabla \hat{u}_\varepsilon \cdot \nabla \eta \, dx = 0
\]

for all \( \eta \in C^\infty(D) \), which is equivalent to

\[
\nabla \cdot \left( \left( |\nabla \hat{u}_\varepsilon|^2 + \varepsilon \right)^{(p-1)/2} \nabla \hat{u}_\varepsilon \right) = 0
\]

by the Gauss–Green theorem and the fact, cf. Lewis [10], that \( \hat{u}_\varepsilon \in C^\infty(D) \). Then we consider the vector field

\[
U_\varepsilon = \hat{u}_\varepsilon \left( |\nabla \hat{u}_\varepsilon|^2 + \varepsilon \right)^{(p-1)/2} \nabla \hat{u}_\varepsilon,
\]

It is clear that \( U_\varepsilon \in C^1(D) \). We may apply the Gauss–Green theorem to \( U_\varepsilon \) and obtain the following formula

\[
\int_{B_r} |\nabla \hat{u}_\varepsilon|^2 \left( |\nabla \hat{u}_\varepsilon|^2 + \varepsilon \right)^{(p-1)/2} \, dx = \int_{\partial B_r} \left( |\nabla \hat{u}_\varepsilon|^2 + \varepsilon \right)^{(p-1)/2} \hat{u}_\varepsilon \frac{\partial \hat{u}_\varepsilon}{\partial \nu} \, dS.
\]

(2.13)

Above we used the fact that

\[
\int_{B_r} \hat{u}_\varepsilon \nabla \cdot \left( \left( |\nabla \hat{u}_\varepsilon|^2 + \varepsilon \right)^{(p-1)/2} \nabla \hat{u}_\varepsilon \right) \, dx = 0.
\]

Due to Lewis [10] Theorem 1], there exists \( \alpha > 0 \), depending only on \( p \) and \( n \), and positive \( A < \infty \), depending only on \( p, n, \) and \( D \), such that

\[
\max_{x \in \overline{D}} |\nabla \hat{u}_\varepsilon(x)| \leq A,
\]

(2.14)

and for each \( x, y \in D \)

\[
|\nabla \hat{u}_\varepsilon(x) - \nabla \hat{u}_\varepsilon(y)| \leq A |x - y|^\alpha.
\]

(2.15)

In particular, constants \( A \) and \( \alpha \) are independent of \( \varepsilon \). From (2.14), (2.15), the Poincaré inequality (see, e.g., [7]) and from the weak compactness of \( W^{1,p} \), it follows that a subsequence of \( \{ \hat{u}_\varepsilon \} \) converges weakly to a function \( v \) in \( W^{1,p} \), and \( v \in \mathcal{F}_u(D) \). To prove that \( v \) minimizes the
$p$-Dirichlet integral $I = \int_D |\nabla \psi|^p \, dx$ over $\mathcal{F}_a(D)$, suppose $\psi \in \mathcal{F}_a(D)$ is arbitrary. Since $\hat{u}_\varepsilon$ is the minimizing function we obtain

$$I(\psi) = \lim_{\varepsilon \to 0} I_\varepsilon(\psi) \geq \liminf_{\varepsilon \to 0} I_\varepsilon(\hat{u}_\varepsilon) \geq I(v),$$

where in the last inequality we used Reshetnyak’s lower semicontinuity theorem, see [13, Theorem 1.1]. Hence $v$ minimizes the $p$-Dirichlet integral in $\mathcal{F}_a(D)$, and so $v = u$.

To apply the Ascoli–Arzela principle we need to verify that the sequences $\{\hat{u}_\varepsilon\}$ and $\{\nabla \hat{u}_\varepsilon\}$ are uniformly bounded and equicontinuous. These two properties for the latter sequence follow from (2.14) and (2.15). In addition, equicontinuity of $\{\hat{u}_\varepsilon\}$ follows from (2.14) and Morrey’s lemma, see, e.g., [13, § 2.3, Lemma 4.1]. That the sequence is uniformly bounded follows from the weak maximum principle of the $p$-Laplace equation.

The Ascoli–Arzela theorem implies that there exists a subsequence of $\{\hat{u}_\varepsilon\}$ and of $\{\nabla \hat{u}_\varepsilon\}$, both still denoted by $\{\hat{u}_\varepsilon\}$ and $\{\nabla \hat{u}_\varepsilon\}$, such that $\hat{u}_\varepsilon$ and $\nabla \hat{u}_\varepsilon$ converge uniformly to $u$ and $\nabla u$ in $\overline{D}$, respectively. We then obtain the identity (2.12) by passing to the limit in (2.13).

**Remark 2.16.** Equation (2.12) is a generalization of the corresponding equation for harmonic functions in $\mathbb{R}^n$

$$\int_{B_r} |\nabla u|^2 \, dx = \int_{\partial B_r} uu_\nu \, dS. $$

From this identity one deduces that the denominator of $F_2(r)$ in (2.3) is non-decreasing. For general $1 < p < \infty$ we do not know whether $I(r)$ is monotone.

It is noteworthy that one can even provide a characterization for harmonic functions by way of the above identity by applying the well known Radó type theorem.

We can readily deduce the following from Lemma 2.11

**Lemma 2.17.** Suppose $u$ is a solution to the $p$-Laplace equation. Then the following inequality is valid

$$p \int_{B_r} |\nabla u|^p \, dx \leq (p - 1) \int_{\partial B_r} |\nabla u|^p \, dS + \int_{\partial B_r} |u|^p \, dS. \quad (2.18) $$

**Proof.** From (2.12) using Young’s inequality we simply obtain

$$\int_{B_r} |\nabla u|^p \, dx \leq \int_{\partial B_r} |\nabla u|^{p-1} |u| \, dS$$

$$\leq \frac{p-1}{p} \int_{\partial B_r} |\nabla u|^p \, dS + \frac{1}{p} \int_{\partial B_r} |u|^p \, dS.$$
Remark 2.19. Suppose $u$ is a solution to the $p$-Laplace equation and let $Z = \{x \in B_r : u(x) = 0\}$. If there exists a constant $0 < \gamma < 1$ such that $|Z| \geq \gamma |B_r|$, then there exists a constant $C$, depending on $n$, $p$, and $\gamma$, such that, cf. Giusti [6],

$$\int_{B_r} |u|^p \, dx \leq C r^p \int_{B_r} |\nabla u|^p \, dx.$$

Plugging this into (2.18) we have

$$\int_{B_r} |u|^p \, dx \leq C r^p \left( \int_{\partial B_r} |\nabla u|^p \, dS + \int_{\partial B_r} |u|^p \, dS \right),$$

where $C$ depends on $n$, $p$, and $\gamma$.

3. Proof of Theorem 2.4

Note that the $p$-Laplace equation, (1.1), can be written in a different form as follows

$$\Delta_p u = |\nabla u|^{p-4} \left( |\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i} u_{x_j} \right) = 0,$$

and we may study the equation

$$|\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i} u_{x_j} = 0. \quad (3.1)$$

Equation (3.1) characterizes the weak solutions $u \in C^2(G)$ of the $p$-Laplace equation. We invoke Juutinen et al. [8] and Lindqvist [11] for this nontrivial fact. Consider affine function $L(x) = l(x) + l_0$, $l_0 \in \mathbb{R}$, $l(x) \neq 0$. We shall show that the difference $u - L$, where $u$ is a solution to (3.1), satisfies a modified uniformly elliptic equation of the form

$$\sum_{i,j=1}^{n} a_{ij} v_{x_i} v_{x_j} + \sum_{i=1}^{n} b_i(x) v_{x_i} = 0$$

with constant coefficients $(a_{ij})$ and the drift term $b_i(x)$ is continuous in $G$. Clearly, $\Delta_p L = 0$. Let $\alpha := (\alpha_1, \ldots, \alpha_n) = \nabla L$, and we denote the
difference \( u - L \) by \( h \). We proceed by manipulating (3.1) as follows

\[
0 = |\nabla u|^2 \Delta u + (p - 2) \sum_{i, j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} \\
= |(\nabla u - \alpha) + \alpha|^2 \Delta u \\
+ (p - 2) \sum_{i, j=1}^{n} ((u_{x_i} - \alpha_i) + \alpha_i) ((u_{x_j} - \alpha_j) + \alpha_j) u_{x_i x_j} \\
= (|\nabla u - \alpha|^2 + 2(\nabla u - \alpha) \cdot \alpha + |\alpha|^2) \Delta u \\
+ (p - 2) \left( \sum_{i, j=1}^{n} (u_{x_i} - \alpha_i)(u_{x_j} - \alpha_j)u_{x_i x_j} \\
+ \sum_{i, j=1}^{n} \alpha_j (u_{x_i} - \alpha_i)u_{x_i x_j} + \sum_{i, j=1}^{n} \alpha_i (u_{x_j} - \alpha_j)u_{x_i x_j} \\
+ \sum_{i, j=1}^{n} \alpha_i \alpha_j u_{x_i x_j} \right) .
\]

After rearranging the terms we obtain

\[
|\alpha|^2 \Delta u + (p - 2) \sum_{i, j=1}^{n} \alpha_i \alpha_j u_{x_i x_j} + |\nabla u - \alpha|^2 \Delta u + 2(\nabla u - \alpha) \cdot \alpha \Delta u \\
+ (p - 2) \left( \sum_{i, j=1}^{n} (u_{x_i} - \alpha_i)(u_{x_j} - \alpha_j)u_{x_i x_j} \\
+ \sum_{i, j=1}^{n} \alpha_j (u_{x_i} - \alpha_i)u_{x_i x_j} + \sum_{i, j=1}^{n} \alpha_i (u_{x_j} - \alpha_j)u_{x_i x_j} \right) = 0.
\]

Clearly \( \Delta u = \Delta h \) and \( \nabla h = \nabla u - \alpha \), thus we get the following equation

\[
|\alpha|^2 \Delta h + (p - 2) \sum_{i, j=1}^{n} \alpha_i \alpha_j h_{x_i x_j} + |\nabla h|^2 \Delta h + 2(\nabla h \cdot \alpha) \Delta h \\
+ (p - 2) \left( \sum_{i, j=1}^{n} h_{x_i} h_{x_j} h_{x_i x_j} + \sum_{i, j=1}^{n} \alpha_j h_{x_i} h_{x_i x_j} + \sum_{i, j=1}^{n} \alpha_i h_{x_j} h_{x_i x_j} \right) = 0.
\]

By inspecting this last equation we observe that it can be written in the following form

\[
|\alpha|^2 \Delta h + (p - 2) \sum_{i, j=1}^{n} \alpha_i \alpha_j h_{x_i x_j} + \sum_{i=1}^{n} b_i(x) h_{x_i} = 0. \tag{3.2}
\]
We study the quadratic form in (3.2). Note that by the Schwarz inequality
\[ \sum_{i,j=1}^{n} \alpha_i \alpha_j \xi_i \xi_j = \left( \sum_{i=1}^{n} \alpha_i \xi_i \right)^2 \leq |\alpha|^2 |\xi|^2. \]
Thus, in the case in which \( p \geq 2 \) we obtain the following
\[ (p-1)|\alpha|^2 |\xi|^2 \geq |\alpha|^2 |\xi|^2 + (p-2) \sum_{i,j=1}^{n} \alpha_i \alpha_j \xi_i \xi_j \geq |\alpha|^2 |\xi|^2 > 0, \]
whereas if \( 1 < p < 2 \) we deduce the following
\[ |\alpha|^2 |\xi|^2 \geq |\alpha|^2 |\xi|^2 + (p-2) \sum_{i,j=1}^{n} \alpha_i \alpha_j \xi_i \xi_j \geq (p-1)|\alpha|^2 |\xi|^2 > 0. \]
Hence, the quadratic form is positive definite, and equation (3.2) is uniformly elliptic for all \( 1 < p < \infty \). Moreover, since the principal part coefficients are constants (3.2) can be written in the divergence form.

Due to results by Garofalo and Lin in [5] and [4], the unique continuation principle is valid for the equation
\[ - \nabla \cdot (A(x) \nabla u) + b(x) \cdot \nabla u + V(x) u = 0, \quad (3.3) \]
where \( A(x) = (a_{ij}(x))_{i,j=1}^{n} \) is a real symmetric matrix-valued function satisfying the uniform ellipticity condition and it is Lipschitz continuous. The lower order terms, the drift coefficient \( b(x) \) and the potential \( V(x) \), are even allowed to have singularities. The reader should consult (1.4)–(1.6) in [5] for the exact structure conditions of \( b \) and \( V \).

To be more precise, one of the main results in [5] is that if \( v \) is a solution to (3.3) in \( G \), then \( v \) satisfies the following doubling property
\[ \int_{B_{2r}} v^2 \, dx \leq C \int_{B_r} v^2 \, dx, \quad (3.4) \]
where \( B_{2r} \subset B_{\bar{r}} \subset G \), and the constant \( C \) depends on \( n, v \), the ellipticity and the Lipschitz constant of \( A(x) \), and the local properties of \( b(x) \) and \( V \), and \( \bar{r} \) depends on the aforementioned parameters but not on the function \( v \). See [5] for more details. Then if \( v \) vanishes of infinite order at \( x_0 \in G \), i.e.,
\[ \int_{B(x_0,r)} v^2 \, dx = O(r^k), \]
for every \( k \in \mathbb{N} \), as \( r \to 0 \), \( v \) must vanish identically in \( G \). This is a consequence of (3.4), we consult the proof of Theorem 1.2 in [4] for this fact. Also the following follows by such reasoning: if \( v \) vanishes identically on a subdomain of \( G \), then it vanishes on the whole \( G \); see Tao–Zhang [14, Corollary 2.6].
To conclude, since our equation (3.2) is of the type (3.3) with $V \equiv 0$ and the drift term, $b(x)$, is continuous, we obtain the claim from the results in [5] as explained above.

**Remark 3.5.** An argument many ways analogous to the preceding proof justifies the following more general claim: Suppose $u, v \in W^{1,p}_{\text{loc}}(G) \cap C^{2}(G)$ are two solutions to the $p$-Laplace equation in $G$. Assume further that $\nabla v \neq 0$ in $G$. Then if $u(x) = v(x)$ in $B_{r} \subset G$, $u(x) = v(x)$ for every $x \in G$. This observation is obtained by considering (3.1) which is satisfied by both $u$ and $v$. By subtracting and denoting $h = u - v$ we end up having the following equation in nondivergence form

$$|\nabla v|^2 \Delta h + (p - 2) \sum_{i,j=1}^{n} v_{x_i} v_{x_j} h_{x_i x_j} + ((\nabla v + \nabla u) \cdot \nabla h) \Delta u$$

$$+ (p - 2) \sum_{i,j=1}^{n} u_{x_i} x_{j} \left( v_{x_i} h_{x_j} + u_{x_j} h_{x_i} \right) = 0. \quad (3.6)$$

Since $u$ and $v$ are in $C^{2}(G)$ it is well known that equation (3.6) can be rewritten into divergence form, see, e.g., [3, §6]. In addition, equation (3.6) in divergence form is uniformly elliptic for all $1 < p < \infty$ since $\nabla v \neq 0$ in $G$. A reasoning similar to the one in the preceding proof gives the claim.

### 4. Proof of Theorem 2.5

Let $u$ be an arbitrary function in $C^{1}(G)$ and consider two balls $B_{r} \subset \overline{B}_{s} \subset G$ such that $0 < I(r) \leq I(s)$ for $r \leq s$, $r, s \in (r_{b}, R_{b}]$. Integrate both sides of inequality (2.10) over $(r, s)$ to get the following estimate

$$I(s) - I(r) \leq (n - 1) \int_{r}^{s} \frac{I(t)}{t} dt + p \int_{r}^{s} \left( \int_{\partial B_{t}} |u|^{p-1} |u_{\nu}| dS \right) dt$$

$$\leq (n - 1) I(s) \log \frac{s}{r} + \int_{r}^{s} \left( (\varepsilon_{0}p)^{-1/(p-1)} \frac{p - 1}{t^{1/(p-1)}} \right) \int_{\partial B_{t}} |u|^{p} dS$$

$$+ \varepsilon_{0} pt \int_{\partial B_{t}} |\nabla u|^{p} dS \right) dt$$

$$\leq (n - 1) I(s) \log \frac{s}{r} + (\varepsilon_{0}p)^{-1/(p-1)} (p - 1) I(s) \int_{r}^{s} t^{-1/(p-1)} dt$$

$$+ \varepsilon_{0} ps \int_{B_{s}} |\nabla u|^{p} dx. \quad (4.1)$$

We applied above Young’s inequality

$$ab \leq \varepsilon_{0} a^{p} + (\varepsilon_{0}p)^{-q/p} b^{q} \quad 10$$
\( \varepsilon_0 > 0 \), in the case in which \( a = |u_r|^{t/\rho} \) and \( b = |u|^{p-1}t^{-1/p} \). We shall fix \( \varepsilon_0 \) later. We divide inequality (4.1) by \( I(s) \) and obtain
\[
\frac{I(s) - I(r)}{I(s)} \leq (n - 1) \log \frac{s}{r} + (\varepsilon_0 p)^{-1/(p-1)}(p - 1) \int_r^s t^{-1/(p-1)} \, dt \\
+ \varepsilon_0 p F_p(s)
\]
for every \( r, s \in (r_b, r_0], \ r \leq s \) such that \( I(r) \leq I(s) \). Since the frequency function \( F_p(r) \) is locally bounded by the hypothesis of the theorem, we denote \( M = \|F_p\|_{L^\infty((r_b, r_0])} < \infty \). In addition, we note that functions \( \log \frac{s}{r} \) and \( \int_r^s t^{-1/(p-1)} \, dt \) in (4.2) tend to zero when \( r \) goes to \( s \). In order to get each term on the right-hand side in (4.2) smaller than, say, 1/4 we first set \( \varepsilon_0 = 1/(4pM) \) and then choose a radius \( r_0 \in (r_b, r_0] \) so close to \( r_b \) such that
\[
(n - 1) \log \frac{s}{r} \leq \frac{1}{4} \quad \text{and} \quad (\varepsilon_0 p)^{-1/(p-1)}(p - 1) \int_r^s t^{-1/(p-1)} \, dt \leq \frac{1}{4}
\]
for each pair \( r, s \in (r_b, r_0], \ r \leq s \). Since \( r \mapsto I(r) \) is continuous on \([r_b, r_0]\), there exists a radius \( r^* \in (r_b, r_0] \) such that
\[
I(r^*) = \max_{r \in [r_b, r_0]} I(r).
\]
Then we clearly have \( 0 < I(r) \leq I(r^*) \) for each \( r \in (r_b, r_0] \). Therefore, by the above reasoning, we obtain for \( r \in (r_b, r^*) \subset (r_b, r_0] \) the following
\[
\frac{I(r^*) - I(r)}{I(r^*)} \leq \frac{3}{4},
\]
and hence a weak doubling property for all radii \( r \in (r_b, r^*) \)
\[
I(r^*) \leq 4I(r).
\]
We stress here that although the constant in the preceding weak doubling property is uniform, the radius \( r^* \) depends on the function \( u \). Inequality (2.6) follows from the fact that \( r^* \) provides the maximum value of \( I(r) \) on \([r_b, r_0]\).

5. Proof of Theorem 2.8

Suppose on the contrary that the function \( u \), a non-trivial solution to the \( p \)-Laplace equation (1.1), vanishes in a ball \( \overline{B}_{r_1} \), but \( u \) is not identically zero in a concentric open ball \( B_{r_2} \), where \( \overline{B}_{r_2} \subset G \). We remark that the frequency function, \( F_p(r) \), is not defined on \([0, r_1]\).

Let \( t > 0 \) and consider an open ball \( B_t \) which is concentric with \( B_{r_1} \) and \( B_{r_2} \). Define
\[
s = \sup\{ t > 0 : u|_{\partial B_t} \equiv 0 \}.
\]
The aforementioned assumptions imply that \( s \in (r_1, r_2] \). We note, in addition, that due to Lemma 2.11 we may conclude \( u|_{\partial B_{r_2}} \) does not vanish identically for any radii \( \rho \in (s, r_2] \), hence \( I(\rho) \neq 0 \). Then the frequency function \( F_p(r) \) is defined on \((s, r_2]\), moreover \( r \mapsto F_p(r) \) is
absolutely continuous on this half open interval, and by the hypothesis of the theorem $F_p(r)$ is bounded on $(s, r_2]$. Theorem 2.5 implies the existence of a radius $r^* \in (s, r_2]$ such that the following weak doubling property holds

$$I(r^*) \leq 4I(r),$$

for every $r \in (s, r^*]$. Since $I(r) \to 0$ as $r \to s$ we have reached a contradiction.

6. Further remarks

We close the paper by giving a few remarks which might be of interest for further studies.

Suppose $u$ is a non-trivial solution to the $p$-Laplace equation. Assume further that there exists a positive constant $A < \infty$ such that for any $B_r \subset G$

$$\int_{\partial B_r} |\nabla u|^p dS \leq A \int_{\partial B_r} |u|^p dS. \quad (6.1)$$

Combining (6.1) with (2.18) we obtain

$$\int_{B_r} |\nabla u|^p dx \leq C \int_{\partial B_r} |u|^p dS,$$

for some $C$ depending only on $p$ and $A$, and hence that $\|F_p\|_{L^\infty((r_b, R_b])} < \infty$. Theorem 2.8 implies that $u$ satisfies the unique continuation principle.

Theorem 2.5 tells that the boundedness of the frequency function implies (2.6) and, more importantly, the weak doubling property (2.7). In the following, we shall show that also the converse is true in a situation in which a certain additional assumption, which is valid in the case $p = 2$, is satisfied. Suppose inequality (2.6) holds for every $r_1, r_2 \in (r_b, r^*]$. Assume further that there exists a positive constant $A < \infty$ such that

$$\int_{B_r} |u|^p dx \leq Ar \int_{\partial B_r} |u|^p dS, \quad (6.2)$$

where $B_r \subset G$. Let $u$ be a solution to the $p$-Laplace equation. It therefore satisfies a Caccioppoli type estimate. More precisely, there exists a positive constant $C < \infty$, depending on $p$, such that for all $B_r \subset B_{r'} \subset G$ we have

$$\int_{B_r} |\nabla u|^p dx \leq \frac{C}{(p - r)^p} \int_{B_{r'}} |u|^p dx.$$ 

Let $r \in (r_b, r^*]$. The weak doubling property (2.7), the Caccioppoli estimate, and (6.2) altogether imply the following estimate

$$F_p(r) = \frac{r \int_{B_r} |\nabla u|^p dx}{\int_{\partial B_r} |u|^p dS} \leq \frac{C_r}{(r^* - r)^p} \frac{\int_{B_{r^*}} |u|^p dx}{\int_{\partial B_{r^*}} |u|^p dS} \leq C \frac{rr^*}{(r^* - r)^p}.$$
where the constant $C$ depends on $n$, $p$, and $A$. To conclude, the frequency function remains bounded as $r$ tends to $r_b$.

We close the paper by remarking that convexity of $\int_{B_r} |u|^p \, dx$ implies (6.2) with $A = 1$. For harmonic functions, moreover, it is easy to prove that both

$$\int_{B_r} u^2 \, dx \quad \text{and} \quad \int_{\partial B_r} u^2 \, dS$$

are indeed convex in $\mathbb{R}^n$, $n \geq 2$. Convexity of the latter follows by showing that

$$\frac{d}{dr} \left( \frac{1}{r} \int_{\partial B_r} u^2 \, dS \right) \geq 0.$$

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