Existence and Stability of Periodic Orbits in $N$-Dimensional Piecewise Linear Continuous Maps

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Piecewise smooth maps are known to exhibit a wide range of dynamical features including numerous types of periodic orbits. Predicting regions in parameter space where such periodic orbits might occur and determining their stability is crucial to characterize the dynamics of the system. However, obtaining the conditions of existence and stability of these periodic orbits generally use brute force methods which require successive application of the iterative map on a phase point. In this article, we propose a faster and more elegant way of obtaining those conditions without iterating the complete map. The method revolves around direct computation of higher powers of matrices without computing the lower ones and is applicable on any dimension of the phase space. In the later part of the article, we also illustrate the use of this method in computing the regions of existence and stability of a particular class of periodic orbits in three dimensions.

Keywords: Border Collision Bifurcation, Piecewise Smooth Maps, Periodic Orbit

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I. INTRODUCTION

Presence of stable and unstable periodic orbits play a vital role in determining the dynamical properties of a system. While presence of stable periodic orbits form the basins of attraction, the unstable periodic orbits play a crucial role in forming the basin boundaries of the attractors. In fact appearance, disappearance or change in stability of these periodic orbits are responsible for numerous bifurcation phenomena. Hence an important component in characterising a given system is determining the parameters for which a particular periodic orbit might exist.

Piecewise smooth maps are useful in describing systems whose evolution is given by different smooth functions in different regions of space. Examples of piecewise smooth systems include switching electrical circuits [1, 2], impacting mechanical systems [3], walking robots [4, 5], cardiac dynamics [6] and neural spiking [7]. Apart from their applicability, piecewise smooth maps have also attracted attention due to their rich dynamical properties. Even maps which are piecewise linear and have nonlinearities only across the switching manifold, exhibit features like robust chaos [8] and existence of infinitely many co-existing attractors [9]. Study of piecewise linear maps attains even more importance as they can describe the behaviour of the piecewise smooth systems near the border separating the partitions of the phase space, in which case the matrices take a specific form. This article aims to study the conditions of existence and stability of periodic orbits in such piecewise linear maps in any dimension.

Periodic orbits in piecewise linear maps have been seen in many studies involving one, two and three dimensional maps [10–12]. Some of the works have also computed the regions in parameter space where specific types of orbits exist and are stable [12–14]. However most of the studies used brute force application of the iterative map to obtain them. Another iterative technique, invented by the Russian mathematician Leonov [16, 17] has also been used to obtain the existence condition of periodic orbits [13]. Moreover, the methods used there were specific to the type of orbit being analysed.

In this work, we seek to obtain a generic method of finding the existence and stability criteria for a more general class of periodic orbits. The technique developed might be used for systems of arbitrary dimension. Moreover, as the technique is essentially an algorithm to obtain higher iterates of the map without going through the intermediate ones, it can be used to simplify extraction of other relevant information about the map. Additionally, the maps we work on are in their normal forms. Hence the ambit of the results obtained spreads across all piecewise smooth maps as appropriate coordinate transformation near the border of non-smoothness yields the normal form of the maps.

The paper is organized as follows. After defining the notations and conventions used in the article in the Section-2, we derive the generic conditions for existence and stability of periodic orbits in a dimension independent form. We then develop a technique which exploits the form of the map to obtain the conditions of existence and stability in $N$ dimensions in an elegant way. Thereafter, we take the specific case of two dimensional systems, where the simplicity of the map allows us to obtain stronger analytical results. Finally we showcase the utility of the technique developed by computing the the parameter regions in which certain classes of stable periodic orbits exist in a three dimensional piecewise linear maps.

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map.

II. THE PIECEWISE LINEAR MAP IN THE NORMAL FORM

In earlier literature it has been shown that, on proper choice of axes, any \( N \) dimensional piecewise smooth continuous map can be linearised near the border to have the following form [10]

\[
G_\mu(X) = \begin{cases}
    M_L X + \zeta : x_1 \leq 0 \\
    M_R X + \zeta : x_1 > 0
\end{cases}
\]

where \( M_L, M_R \in \mathbb{R}^{N \times N} \) are matrices in their normal form given by [15]

\[
M_J = \begin{pmatrix}
    -d_1^{(J)} & 1 & 0 & \cdots & 0 \\
    -d_2^{(J)} & 0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \cdots & \vdots \\
    -d_{N-1}^{(J)} & 0 & 0 & \cdots & 1 \\
    -d_N^{(J)} & 0 & 0 & \cdots & 0
\end{pmatrix} : J \in \{L, R\}
\]

with \( d_i^{(J)} \) being the coefficient of \( \lambda^i \) in the characteristic polynomial of \( M_J \); \( X = (x_1, \ldots, x_N)^T \in \mathbb{R}^N \) being a generic point in the phase space and \( \zeta = (\mu, \ldots, 0)^T \).

Since the coefficient of \( \lambda^i \) in the characteristic polynomial of a matrix is the sum of eigenvalues of the matrix taken \( i \) at a time (apart from an alternating sign); we recast (2) as

\[
M_J = \begin{pmatrix}
    \rho_1^{(J)} & 1 & 0 & \cdots & 0 \\
    -\rho_2^{(J)} & 0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \cdots & \vdots \\
    -(1)^{N-2}\rho_{N-1}^{(J)} & 0 & 0 & \cdots & 1 \\
    -(1)^{N-2}\rho_N^{(J)} & 0 & 0 & \cdots & 0
\end{pmatrix} : J \in \{L, R\}
\]

for further analysis. Here \( \rho_i^{(J)} \) is the sum of the eigenvalues of \( M_J \) taken \( i \) at a time, i.e., \( \rho_i^{(J)} = (-1)^{i-1}d_i^{(J)} \). For instance, in two and three dimensions, the matrix in (3) takes the form,

\[
M_J = \begin{pmatrix}
    \tau_J & 1 \\
    -\delta_J & 0
\end{pmatrix} : J \in \{L, R\}
\]

and

\[
M_J = \begin{pmatrix}
    \tau_J & 1 & 0 \\
    -\sigma_J & 0 & 1 \\
    \delta_J & 0 & 0
\end{pmatrix} : J \in \{L, R\}
\]

respectively. Here \( \tau_J \) and \( \delta_J \) are the trace and determinant the \( M_J \); and \( \sigma_J \) is the sum of the eigenvalues of \( M_J \) taken two at a time.

Any \( p \)-periodic orbit in such a system can be represented by a sequence of points \( \{X_0, \ldots, X_{p-1}\} \) such that \( X_{i+1} = G_\mu(X_i) \) where \( i \) is a natural number or zero, and \( X_p = X_0 \). We can also associate a symbol to each point on the periodic orbit depending on the partition in which it lies. If a point of the periodic orbit has \( x_1 \leq 0 \), it is assigned the symbol \( L \) (meaning left). Otherwise it is assigned the symbol \( R \) (meaning right). This associates a sequence of \( R \) and \( L \) to each periodic orbit. For example, an \( LLLL \) (also written as \( L^3R \)) orbit consists of three points with \( x_1 \leq 0 \) and a single point with \( x_1 > 0 \). To remove the ambiguity arising from possible cyclic permutation of the points, in this article we adopt the following convention: We number the points \( X_i \) such that \( X_0 \) is assigned the symbol \( R \) and \( X_p \) is assigned the symbol \( L \). However while referring of the orbits, we collect the symbols in the reverse order such that the symbol for the orbit starts with \( L \) and ends with \( R \). Furthermore, for sake of simplicity, we would restrict ourselves to orbits of the form \( L^mR^n \) where \( m \) and \( n \) are natural numbers. Note that since the map in each of the partitions is linear, a periodic orbit lying completely within any one of the partitions does not exist.

III. EXISTENCE AND STABILITY OF PERIODIC ORBITS

In 1959, Leonov [16, 17] did a detailed study of nested period adding bifurcation structure occurring in piecewise-linear discontinuous 1D maps. The algorithmic way proposed in his work was recently used[13] to analyse border collision bifurcations in one dimensional maps. We extend the analysis to obtain the existence criteria for period orbits in \( N \) dimensions.

Consider an \( L^mR^n \) orbit formed by the points \( \{X_0, \ldots, X_{m+n-1}\} \). According to the convention described in the last section, we assume the points \( X_0, \ldots, X_{m-1} \) have \( x_1 \leq 0 \) and the points \( X_m, \ldots, X_{m+n-1} \) have \( x_1 > 0 \). Hence the evolution of \( X_0 \) under the map \( G_\mu \) is given as,

\[
X_1 = M_R X_0 + \zeta
\]

\[
X_2 = M_R^2 X_0 + (I + M_R) \zeta
\]

\[
X_3 = M_R^3 X_0 + (I + M_R + M_R^2) \zeta
\]

\[
\vdots
\]

\[
X_{m-1} = M_R^{m-1} X_0 + \phi_{R,n-1} \zeta
\]

\[
X_n = M_R^n X_0 + \phi_{R,n} \zeta
\]

\[
X_{n+1} = M_L M_R^n X_0 + M_L \phi_{R,n} \zeta + \zeta
\]

\[
\vdots
\]

\[
X_{m+n-1} = M_L^{m-1} M_R^n X_0 + (M_L^{m-1} \phi_{R,n} + \phi_{L,m-1}) \zeta
\]

\[
X_{m+n} = M_L^m M_R^n X_0 + (M_L^m \phi_{R,n} + \phi_{L,m}) \zeta.
\]

Here \( I \) is the \( N \times N \) identity matrix and \( \phi_{J,k} = I + M_J + M_J^2 + \ldots + M_J^{k-1} \). On using the formula for GP of
matrices, it can be written as,
\[ \phi_{J,k} = (I - M_J)^{-1} (I - M_J^k) \quad : J \in \{L, R\} \]  
(6)
if \( I - M_J \) is invertible.

On substituting \( X_{m+n} = X_0 \), we get the expression for \( X_0 \) in terms of known quantities,
\[ X_0 = (I - M_L^m M_R^n)^{-1} (M_L^m \phi_{R,n} + \phi_{L,m}) \zeta \]  
(7)
when \( (I - M_L^m M_R^n) \) is invertible. Once \( X_0 \) is determined, all other points of the orbit can be calculated by evolving \( X_0 \) under the map \( G_\mu \). For existence of the \( L^mR^n \) orbit, we need to ensure that all points of the periodic orbit are in their correct partitions. However the linearity of the maps on either side of the barrier ensures that period exists if this condition is satisfied by the first and the last points in each partition. Hence ensuring that the \( x_1 \) coordinates of \( X_0 \) and \( X_{n-1} \) are positive; and those of \( X_n \) and \( X_{n+m-1} \) are negative, would yield the conditions for the existence of the \( L^mR^n \) orbit.

For periodic orbits to be stable, we require the trace \( T \) and determinant \( \Delta \) of the ordered product of the Jacobian matrices at each point of the periodic orbit to follow
\[ (1 - \Delta) < T < (1 + \Delta). \]  
(8)
Since the Jacobian at any point with \( x_1 \leq 0 \) is \( M_L \) and that at any point with \( x_1 > 0 \) is \( M_R \), the matrix whose trace and determinant are in question is \( M_L^m M_R^n \).

A close look at equations (7) and (8) reveal that, the primary requirement in evaluating those expressions is to compute \( M_L^m \) and \( M_R^n \). Typically this is done algorithmically by brute force matrix multiplication. The following sections aims to provide an algebraic way to compute the powers of the matrices and hence analytically compute the regions of existence and stability of \( L^mR^n \) orbits.

IV. COMPUTING \( M^n \) FOR \( N \) DIMENSIONS

In this section, we present a technique for computing \( n^{th} \) powers of the matrix \( M \), given by
\[ M = \begin{pmatrix} \rho_1 & 1 & 0 & \cdots & 0 \\ -\rho_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{N-2} \rho_{N-1} & 0 & 0 & \cdots & 1 \\ (-1)^{N-1} \rho_N & 0 & 0 & \cdots & 0 \end{pmatrix}. \]  
(9)

Note that this matrix is the same as the matrices \( M_L \) or \( M_R \) defined in (3) except that the subscripts \( L \) and \( R \) are ignored. This is done for notational simplicity. The results derived here are applicable to both \( M_L \) and \( M_R \).

The \( n^{th} \) power of any general \( N \times N \) matrix
\[ A = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,N} \\ \vdots & \ddots & \vdots \\ \theta_{N,1} & \cdots & \theta_{N,N} \end{pmatrix}, \]  
(10)
can be written in the form
\[ A^n = \begin{pmatrix} \theta_{1,1}^{(n)} & \cdots & \theta_{1,N}^{(n)} \\ \vdots & \ddots & \vdots \\ \theta_{N,1}^{(n)} & \cdots & \theta_{N,N}^{(n)} \end{pmatrix}. \]  
(11)
The sequence of matrices \( A, A^2, A^3, \ldots A^n \) is composed of sequences of its individual elements, and therefore \( N^2 \) such independent sequences are needed to construct the matrix \( A^n \). However, we show that due to the structure of the matrix \( M \) in (9), only \( N \) sequences are required to construct the matrix \( M^n \).

Let each of the \( N \) sequences be denoted by \( \Gamma_i \). Let \( \Gamma_{i,j} \) denote the \( j^{th} \) term of the sequence \( \Gamma_i \). Also note that for notational simplicity in the further analysis, we start numbering the terms of the sequence \( \Gamma_i \) from \( 2 - N \) instead of 1. For example, in order to compute \( M^n \) in three dimensions, we would require three sequences: \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \); and the terms of the sequences will be numbered as \( \Gamma_{1,1}, \Gamma_{1,0}, \Gamma_{1,1}, \Gamma_{1,2}, \ldots \) for the first sequence, \( \Gamma_{2,1}, \Gamma_{2,0}, \Gamma_{2,1}, \Gamma_{2,2}, \ldots \) for the second sequence and \( \Gamma_{3,1}, \Gamma_{3,0}, \Gamma_{3,1}, \Gamma_{3,2}, \ldots \) for the third sequence.

The terms of this sequence shall be defined iteratively. However, before giving the iterative relation, we first give the first \( N \) terms of the sequence. This initialisation is done such that the \( N \) elements of the \( i^{th} \) row of \( M \) give the first \( N \) terms of \( \Gamma_i \) in the reverse order, i.e.,
\[ M = \begin{pmatrix} \Gamma_{1,1} & \Gamma_{1,0} & \cdots & \Gamma_{1,2-N} \\ \Gamma_{2,1} & \Gamma_{2,0} & \cdots & \Gamma_{2,2-N} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{N,1} & \Gamma_{N,0} & \cdots & \Gamma_{N,2-N} \end{pmatrix}. \]  
(12)

Again taking the three dimensional case as an example, we would have the initialisation as,
\[ \Gamma_{1,1} = 0, \quad \Gamma_{1,0} = 1, \quad \Gamma_{1,1} = \pi \]
\[ \Gamma_{2,1} = 1, \quad \Gamma_{2,0} = 0, \quad \Gamma_{2,1} = -\sigma \]
\[ \Gamma_{3,1} = 0, \quad \Gamma_{3,0} = 0, \quad \Gamma_{3,1} = \delta. \]  
(13)

Note the due to the initialisation given above, the terms from \( \Gamma_{i,2-N} \) to \( \Gamma_{i,1} \) are defined. Now, the further terms of the sequence are defined iteratively as
\[ \Gamma_{i,j} = \sum_{k=1}^{N} (-1)^{k-1} \rho_k \Gamma_{i,j-k} \quad : j \in [2, \infty). \]  
(14)

With this definition of \( \Gamma_{i,j} \), the matrix \( M^n \) can be written as
\[ M^n = \begin{pmatrix} \Gamma_{1,n} & \Gamma_{1,n-1} & \cdots & \Gamma_{1,n-(N-1)} \\ \Gamma_{2,n} & \Gamma_{2,n-(N-1)} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{N,n} & \Gamma_{N,n-1} & \cdots & \Gamma_{N,n-(N-1)} \end{pmatrix}. \]  
(15)
or more explicitly as
\[ [M^n]_{i,j} = \Gamma_{i,n-(j-1)} \quad \forall n \in \mathbb{N}. \]  
(16)
The proof of the result hinges on the structure of the matrix \( M \). Note that (16) is bound to be satisfied for \( n = 1 \) due to the way the series \( \Gamma_i \) is initialised in (12). Also note that, apart from the first column, the only non-zero elements of the matrix \( M \) are in the superdiagonal and are all equal to 1. Hence, while obtaining \( M^{k+1} \) from \( M^k \), when \( M \) is multiplied from the right, the columns of \( M^k \) are simply shifted to right, except the rightmost column which is lost. Therefore, only the first column needs to be computed, which is given by (14). The detailed proof of the result is given in the Appendix A.

Hence in order to compute \( M^{k+1} \) from \( M^k \), one needs to compute only \( N \) new elements corresponding to the first column of the matrix; as compared to computing \( N^2 \) new elements for a generic \( N \times N \) matrix. Now let us consider the special case of 2 dimensions where further simplification can be done and the \( n^{th} \) power of \( M \) can be computed non-iteratively.

**V. COMPUTING \( M^n \) FOR 2 DIMENSIONS**

In two dimensions the matrix appearing in the normal form map is given by (4). Dropping the subscripts for notational simplicity as in the previous section gives us

\[
M = \begin{pmatrix} \tau & 1 \\ -\delta & 0 \end{pmatrix}. \tag{17}
\]

From the results obtained in the previous section, it can be said that \( M^n \) would be determined by two independent sequences \( a \) and \( b \) and would be of the form

\[
M^n = \begin{pmatrix} a_n & a_{n-1} \\ b_n & b_{n-1} \end{pmatrix}. \tag{18}
\]

However on explicit calculation (as done in Appendix C) it can be shown that the terms of the two sequences are related as

\[
b_i = -\delta a_{i-1} \quad \forall i \geq 0. \tag{19}
\]

Substituting (19) in (18) gives the final form of \( M^n \) as

\[
M^n = \begin{pmatrix} a_n & a_{n-1} \\ -\delta a_{n-1} & -\delta a_{n-2} \end{pmatrix} \quad \forall n \in \mathbb{N} \tag{20}
\]

where

\[
a_n = \tau a_{n-1} - \delta a_{n-2} \quad \forall i \in \mathbb{N} \tag{21}
\]

with initial conditions \( a_0 = 1 \) and \( a_{-1} = 0 \). Apart from the iterative definition, it is also possible to explicitly determine \( a_n \) in terms of known quantities.

In terms of \( \tau \) and \( \delta \), \( a_n \) is given as

\[
a_n = \sum_{m=0}^{[\frac{n}{2}]} (-1)^m n-m \cdot C_{n-m} \cdot \frac{\tau^m \cdot \delta^{n-2m}}{m!} \quad \forall n \geq 0 \tag{22}
\]

and \( a_{-1} = 0 \). Here \([ \cdot ]\) is the greatest integer function and \( C_{n} \) is the coefficient of \( x^n \) in the binomial expansion of \((1 + x)^n \). Using the properties of \( C_{n} \), it can be shown the \( a_n \) as expressed in (22) satisfies (21). However as the proof is lengthy, it is given in Appendix B.

We can also obtain another representation of \( a_n \) if the results are expressed in terms of the eigenvalues of \( M \),

\[
\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}. \tag{23}
\]

To do so, we put \( \tau = \lambda_1 + \lambda_2 \) and \( \delta = \lambda_1 \lambda_2 \) in (17).

We then decompose \( M \) as \( M = U D U^{-1} \) where \( D \) is the diagonal matrix with \( \lambda_1 \) and \( \lambda_2 \) on its diagonals and \( U \) is the matrix with the eigenvectors of \( M \) as the columns.

Then \( M^n = U D^n U^{-1} \), which when computed explicitly gives

\[
M^n = \frac{1}{\lambda_2 - \lambda_1} \left( \lambda_1^{n+1} - \lambda_2^{n+1} \lambda_1^n - \lambda_2^n \right)
\]

which can be recast into the form of (20) with the definition of \( a_n \) as

\[
a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}. \tag{24}
\]

The detailed proof of this result is given in the appendix E. It can also be directly seen that (24) satisfies (21).

Hence in order to compute \( M^n \) for a \( 2 \times 2 \) matrix in the form (17), we use (22) or (24) to get \( a_n, a_{n-1}, a_{n-2} \) and form the matrix given in (20).

The form of the matrix \( M^n \) in (20) may be substituted directly into (6) to obtain the sum of the GP as

\[
\phi_n = \left( \begin{array}{c} f_n \\ -\delta f_{n-1} \end{array} \right) \left( \begin{array}{c} f_{n-1} \\ 1 - \delta f_{n-2} \end{array} \right)
\]

where

\[
f_n = \frac{1 - a_n + \delta a_{n-1}}{1 - \tau + \delta}. \tag{26}
\]

The detailed proof of this result is given in appendix D. Note that the proof is independent of the explicit form of \( a_n \) and uses only the propagation rule (21).

In the next section, we use the results presented in the previous section to obtain parameter regions where certain classes of stable periodic orbits exist in 3 dimensions.

**VI. APPLICATION: STABLE \( L^n R \) ORBITS IN 3 DIMENSIONS**

In 2012, parameter regions where stable \( L^n R \) orbits exist in two dimensions were computed to find the parameter regions of multiple attractor bifurcations [12]. In this section, we extend the result to demonstrate the use of the technique developed in this article to find the regions of existence of stable \( L^n R \) periodic orbits in 3
where
\[ \Gamma_{i,n} = \tau \Gamma_{i,n-1} - \sigma \Gamma_{i,n-2} + \delta \Gamma_{i,n-3} \]  
(29)
for \( i \in \{1, 2, 3\} \) and \( n > 1 \). For \(-1 \leq n \leq 1\), the terms are taken from \( M \) according to (13). Using these expressions, we compute the required powers of \( M_L \) and \( M_R \), and substitute them in (7) and (8) to obtain the required conditions for existence and stability of \( L^n R \) orbits for various values of \( n \). The parameter values for which the stable orbits exist are shown in Fig. 1 where the plausible regions are shown in the two dimensional projection of the six dimensional space.

VII. CONCLUSION

In this article we developed a faster and a more elegant technique to compute the existence and stability conditions for periodic orbits of the form \( L^n R \) in an arbitrary dimensional piecewise linear continuous map. The technique is based on easier computation of powers of \( N \times N \) matrices in their normal form. Due to the structure of the matrices involved, it was found that the elements of the resulting matrix were interrelated; and in order to compute the \( n^{th} \) power of the matrix, only \( N \) out of the \( N^2 \) elements need to be computed. These \( N \) elements in turn can be obtained as simple sequences defined iteratively. Moreover, in the special case of 2 dimensional matrices, further simplifications were made. Notably, explicit expressions for the terms of the sequence in terms of the given parameters and eigenvalues of the matrices were obtained. This allows for a direct evaluation of any power of the \( 2 \times 2 \) normal form matrix without computing the intermediate powers.

Once the powers of the matrices are computed, they can be substituted in the generic expressions of existence and stability of orbits to obtain the regions in parameter space where they exist. We also apply the technique developed to 3 dimensional systems and obtain the regions where \( L^n R \) orbits exist.

The technique developed here can also simplify the computation of stability and existence criteria for other more complex periodic orbits of the form \( L^{m_1} R \) \( L^{m_2} R \) \( L^{m_3} R \), etc. The conditions would also involve computing powers of normal forms of the matrix; although the precise conditions might involve more complicated expressions. Hence, in principle, one can obtain the existence and stability conditions for any periodic orbit in a piecewise linear map of any dimension.

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Appendix: Derivations and Motivations

Appendix A: Finding $M^n$ for a Matrix $M$ in the Normal Form

**Theorem A.1** Let $M$ a matrix of the form (9). Then the elements of $M^n$ are given as

$$[M^n]_{i,j} = \Gamma_{i,n-(j-1)} \quad \forall n \in \mathbb{N}$$

(A1)

where $[A]_{i,j}$ is the element of $A$ corresponding to $i^{th}$ row and $j^{th}$ column and $\Gamma_{i,j}$ is defined in (12) and (14).

**Proof** Let us assume the most general form of $M^n$

$$M^n = \begin{pmatrix} \theta^{(n)}_{1,1} & \ldots & \theta^{(n)}_{1,N} \\ \vdots & \ddots & \vdots \\ \theta^{(n)}_{N,1} & \ldots & \theta^{(n)}_{N,N} \end{pmatrix}$$

(A2)

for some $n \geq 1$. Then

$$M^{n+1} = \begin{pmatrix} \theta^{(n+1)}_{1,1} & \theta^{(n+1)}_{1,2} & \ldots & \theta^{(n+1)}_{1,N} \\ \theta^{(n+1)}_{2,1} & \theta^{(n+1)}_{2,2} & \ldots & \theta^{(n+1)}_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \theta^{(n+1)}_{N,1} & \theta^{(n+1)}_{N,2} & \ldots & \theta^{(n+1)}_{N,N} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} (-1)^{k-1} \rho_k \theta^{(n)}_{1,k} \\ \sum_{k=1}^{N} (-1)^{k-1} \rho_k \theta^{(n)}_{2,k} \\ \vdots \\ \sum_{k=1}^{N} (-1)^{k-1} \rho_k \theta^{(n)}_{N,k} \end{pmatrix} \begin{pmatrix} \theta^{(n)}_{1,1} & \ldots & \theta^{(n)}_{1,N-1} \\ \theta^{(n)}_{2,1} & \ldots & \theta^{(n)}_{2,N-1} \\ \vdots \\ \theta^{(n)}_{N,1} & \ldots & \theta^{(n)}_{N,N-1} \end{pmatrix}$$

Hence

$$\theta^{(n+1)}_{i,1} = \sum_{k=1}^{N} (-1)^{k-1} \rho_k \theta^{(n)}_{i,k} \quad : 1 \leq i \leq N, n \geq 1$$

(A3)

and

$$\theta^{(n+1)}_{i,j} = \theta^{(n)}_{i,j-1} \quad : 1 \leq i \leq N, 1 < j \leq N, n \geq 1.$$  

(A4)

We now use (A4) iteratively to obtain

$$\theta^{(n+1)}_{i,j} = \theta^{(n)}_{i,j-1} = \theta^{(n-1)}_{i,j-2} = \ldots = \theta^{(n-(j-2))}_{i,1} \quad : 1 \leq i \leq N, 1 < j \leq N, n \geq 1.$$  

(A5)

Now if we define

$$\theta^{(n+1)}_{i,1} = \Gamma_{i,n+1} \quad : 1 \leq i \leq N$$

(A6)

then

$$\theta^{(n+1)}_{i,j} = \theta^{(n-(j-2))}_{i,1} = \Gamma_{i,n-(j-2)} \quad : 1 \leq i \leq N, 1 < j \leq N, n \geq 1.$$  

(A7)

Combining (A6) and (A7); and replacing $n+1$ by $n$ we get

$$\theta^{(n)}_{i,j} = \Gamma_{i,n-(j-1)} \quad : 1 \leq i,j \leq N, n \geq 2.$$  

(A8)

Now, by definition

$$\Gamma_{i,n+1} = \theta^{(n+1)}_{i,1} = \sum_{k=1}^{N} (-1)^{k-1} \rho_k \theta^{(n)}_{i,k} \quad : 1 \leq i \leq N, n \geq 1.$$  

(A9)
Using (A8) in the right hand side gives us

\[ \Gamma_{i,n+1} = \theta_{i,1}^{(n+1)} = \sum_{k=1}^{N} (-1)^{k-1} \rho_k \Gamma_{i,n-(k-1)} : 1 \leq i \leq N, n \geq 1. \quad (A10) \]

Finally replacing \( n + 1 \) by \( j \), we have an iterative relation for \( \Gamma_{i,j} \) as

\[ \Gamma_{i,j} = \sum_{k=1}^{N} (-1)^{k-1} \rho_k \Gamma_{i,j-k} : 1 \leq i \leq N, j \geq 2. \quad (A11) \]

Now note that substituting \( n = 1 \) in (A8) gives

\[ \theta_{i,j}^{(1)} = \Gamma_{1,j} : 1 \leq i, j \leq N. \quad (A12) \]

However by definition of \( \theta_{i,j}^{(n)} \), we have

\[ \theta_{i,j}^{(1)} = [M^1]_{i,j} = M_{i,j} : 1 \leq i, j \leq N. \quad (A13) \]

Hence,

\[ \Gamma_{i,2-j} = M_{i,j} : 1 \leq i, j \leq N \quad (A14) \]

which on replacing \( 2 - j \) by \( j \) yields

\[ \Gamma_{i,j} = M_{i,2-j} : 1 \leq i \leq N, -N + 2 \leq j \leq 1 \quad (A15) \]

Appendix B: The Progression Rule of Fundamental Sequence

Lemma B.1 The \( n^{th} \) term of the sequence defined in (22) is related to its two preceding terms by the relation

\[ a_n = \tau a_{n-1} - \delta a_{n-2}. \quad (B1) \]

Proof For \( n = 0 \),

\[ a_{n+1} = a_1 = \tau \]
\[ a_n = a_0 = 1 \]
\[ a_{n-1} = a_{-1} = 0 \]

Hence (B1) is true for \( n = 0 \).

For the other \( n \), we prove it separately for even and odd \( n \).

If \( n \) is even, then

\[ \left[ \frac{n}{2} \right] = \frac{n}{2}, \quad \left[ \frac{n-1}{2} \right] = \frac{n}{2} - 1, \quad \left[ \frac{n+1}{2} \right] = \frac{n}{2}. \quad (B2) \]
Now,

\[
\tau a_n - \delta a_{n-1} = \tau \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^m n - m C_m \delta^m \tau^{n-2m} - \delta \sum_{m=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^m n - 1 - m C_m \delta^m \tau^{n-1-2m}
\]

\[
= \sum_{m=0}^{\frac{n}{2}} (-1)^m n - m C_m \delta^m \tau^{n-2m} - \sum_{m=0}^{\frac{n-1}{2}} (-1)^m n - 1 - m C_m \delta^m \tau^{n-1-2m}
\]

\[
= \tau^{n+1} + \sum_{m=1}^{\frac{n}{2}} (-1)^m n - m C_m \delta^m \tau^{n-1-2m} - \sum_{m=0}^{\frac{n-1}{2}} (-1)^m n - 1 - m C_m \delta^m \tau^{n-1-2m}
\]

\[
= \tau^{n+1} + \sum_{m=1}^{\frac{n}{2}} (-1)^m n - m C_m \delta^m \tau^{n-1-2m} + \sum_{m=1}^{\frac{n-1}{2}} (-1)^m n - 1 - m C_{m-1} \delta^m \tau^{n-1-2m}
\]

\[
= \tau^{n+1} + \sum_{m=1}^{\frac{n}{2}} (-1)^m \left( n - m C_m + n - m C_{m-1} \right) \delta^m \tau^{n-1-2m}
\]

\[
= \tau^{n+1} + \sum_{m=1}^{\frac{n}{2}} (-1)^m n + 1 - m C_m \delta^m \tau^{n-1-2m}
\]

\[
= \sum_{m=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} (-1)^m n + 1 - m C_m \delta^m \tau^{n-1-2m}
\]

\[
= a_{n+1}.
\]

If \( n \) is odd then

\[
\left\lfloor \frac{n}{2} \right\rfloor = \frac{n - 1}{2}, \quad \left\lfloor \frac{n - 1}{2} \right\rfloor = \frac{n - 1}{2}, \quad \left\lfloor \frac{n + 1}{2} \right\rfloor = \frac{n + 1}{2}.
\]

(B3)
Now,

\[ \tau a_n - \delta a_{n-1} = \tau \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m n^{-m} C_m \delta^m \tau^{n-2m} - \delta \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^m n^{-1-m} C_m \delta^m \tau^{n-1-2m} \]

\[ = \sum_{m=0}^{n-1} \left( (-1)^m n^{-m} C_m \delta^m \tau^{n-2m} \right) - \sum_{m=0}^{n-1} \left( (-1)^m n^{-1-m} C_m \delta^m \tau^{n-1-2m} \right) \]

\[ = \tau^{n+1} + \sum_{m=1}^{n-1} \left( (-1)^m n^{-m} C_m \delta^m \tau^{n+1-2m} \right) - \sum_{m=0}^{n-1} \left( (-1)^m n^{-1-m} C_m \delta^m \tau^{n+1-2m} \right) - (-1) \frac{n+1}{2} \delta^{\frac{n+1}{2}} \]

\[ = \tau^{n+1} + \sum_{m=1}^{n-1} \left( (-1)^m \left( n^{-m} C_m + n^{-m} C_{m-1} \right) \delta^m \tau^{n+1-2m} \right) + (-1) \frac{n+1}{2} \delta^{\frac{n+1}{2}} \]

\[ = \tau^{n+1} + \sum_{m=1}^{n-1} \left( (-1)^m n^{-1-m} C_m \delta^m \tau^{n+1-2m} \right) + (-1) \frac{n+1}{2} \delta^{\frac{n+1}{2}} \]

\[ = \sum_{m=0}^{n} \left( (-1)^m n^{-1-m} C_m \delta^m \tau^{n+1-2m} \right) \]

\[ = a_{n+1}. \]

Hence the result is true for all \( n \in \mathbb{N} \).

**Appendix C: Motivating the Form of \( a_n \)**

In this appendix, we give the motivation for obtaining the fundamental sequence in the form

\[ a_n = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m n^{-m} C_m \delta^m \tau^{n-2m} \quad \forall n \geq 0 \quad (C1) \]

and \( a_{-1} = 0 \). In other terms, we try to understand, how the matrix

\[ M = \begin{pmatrix} \tau & 1 \\ -\delta & 0 \end{pmatrix}. \quad (C2) \]

yields

\[ M^n = \begin{pmatrix} a_n & a_{n-1} \\ -\delta a_{n-1} & -\delta a_{n-2} \end{pmatrix} \quad \forall n \in \mathbb{N} \quad (C3) \]

with \( a_n \) defined in \( (C1) \).

For an matrix general \( 2 \times 2 \) matrix \( M \), if we assume

\[ M^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad (C4) \]

then the sequence

\[ M, M^2, M^3, \ldots, M^n \quad (C5) \]

or

\[ \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad (C6) \]
is a set of four sequences: \( \{a_n\}, \{b_n\}, \{c_n\} \) and \( \{d_n\} \).

The aim of the section is to show that these four sequences are restricted by constraints that allow the matrix to be expressed in terms of a single sequence.

To show this, we assume that (C4) holds for the matrix defined in (C2). Then

\[
M^{n+1} = M^n M
\]

\[
\begin{pmatrix}
  a_{n+1} & b_{n+1} \\
  c_{n+1} & d_{n+1}
\end{pmatrix} = \begin{pmatrix}
  a_n & b_n \\
  c_n & d_n
\end{pmatrix} \begin{pmatrix}
  \tau & 1 \\
 -\delta & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
  a_{n+1} & b_{n+1} \\
  c_{n+1} & d_{n+1}
\end{pmatrix} = \begin{pmatrix}
  \tau a_n - \delta b_n & a_n \\
  \tau c_n - \delta d_n & c_n
\end{pmatrix}
\]

Hence

\[
b_{n+1} = a_n
\]
\[
d_{n+1} = c_n
\]

or

\[
b_n = a_{n-1} \quad \text{(C7)}
\]
\[
d_n = c_{n-1} \quad \text{(C8)}
\]

and

\[
a_{n+1} = \tau a_n - \delta b_n \quad \text{(C9)}
\]
\[
c_{n+1} = \tau c_n - \delta d_n \quad \text{(C10)}
\]

Substituting (C7) in (C9) and (C8) in (C10), we get

\[
a_{n+1} = \tau a_n - \delta a_{n-1} \quad \text{(C11)}
\]
\[
c_{n+1} = \tau c_n - \delta c_{n-1} \quad \text{(C12)}
\]

Hence, at the current stage

\[
M^n = \begin{pmatrix}
  a_n & a_{n-1} \\
  c_n & c_{n-1}
\end{pmatrix} \quad \text{(C13)}
\]

with \( a_n \) and \( c_n \) satisfying (C11) and (C12) respectively.

Moreover, as we know

\[
M^1 = \begin{pmatrix}
  a_1 & a_0 \\
  c_1 & c_0
\end{pmatrix} = M = \begin{pmatrix}
  \tau & 1 \\
 -\delta & 0
\end{pmatrix}
\]

therefore

\[
a_1 = \tau, \ a_0 = 1, \ c_1 = -\delta, \ c_0 = 0 \quad \text{(C14)}
\]

Using (C11) and (C12) in conjunction with the initial conditions in (C14), can write the complete sequence of \( a_n \) and \( c_n \). The first few terms are shown below.
TABLE I. List of all $\eta_{m,n}$ values

| $\eta_{m,n}$ | 0 | 1 | 2 | 3 | 4 |
|--------------|---|---|---|---|---|
| 1            | 1 |
| 2            | 1 | 1 |
| 3            | 1 | 2 |
| 4            | 1 | 3 | 1 |
| 5            | 1 | 4 | 3 |
| 6            | 1 | 5 | 6 | 1 |
| 7            | 1 | 6 | 10| 4 |
| 8            | 1 | 7 | 15| 10| 1 |
| 9            | 1 | 8 | 21| 20| 5 |

$a_1 = \tau$
$a_2 = \tau^2 - \delta$
$a_3 = \tau^3 - 2\tau\delta$
$a_4 = \tau^4 - 3\tau^2\delta + \delta^2$
$a_5 = \tau^5 - 4\tau^3\delta + 3\tau\delta^2$
$a_6 = \tau^6 - 5\tau^4\delta + 6\tau^2\delta^2 - \delta^3$
$a_7 = \tau^7 - 6\tau^5\delta + 10\tau^3\delta^2 - 4\tau\delta^3$
$a_8 = \tau^8 - 7\tau^6\delta + 15\tau^4\delta^2 - 10\tau^2\delta^3 + \delta^4$
$a_9 = \tau^9 - 8\tau^7\delta + 21\tau^5\delta^2 - 20\tau^3\delta^3 + 5\tau\delta^4$

It may be noted that if written in the appropriate form, it becomes clear that

$$c_n = -\delta a_n \quad \forall n \in \mathbb{N}.$$  \hspace{1cm} (C15)

In order to extend the result to $c_0$, we define $a_{-1} = 0$ which allows us to write $M^n$ as

$$M^n = \begin{pmatrix} a_n & a_{n-1} \\ -\delta a_{n-1} & -\delta a_{n-2} \end{pmatrix}$$  \hspace{1cm} (C16)

where

$$a_{n+1} = \tau a_n - \delta a_{n-1} \quad \forall n \geq 0$$  \hspace{1cm} (C17)

and $a_{-1} = 0$.

To obtain the analytic expression for $a_n$, we note the following in expansions given earlier. when terms under summation for a particular $n$ value are arranged in the ascending order of powers of $\delta$,

- There are $\left[\frac{n}{2}\right] + 1$ terms in the series.
- Powers of $\delta$ start from 0 and increase in steps of 1.
- Powers of $\tau$ start from $n$ and decrease in steps of 2.
- Sign of each term alternates between plus and minus starting from a plus
- A numerical coefficient precedes each term.

Hence $a_n$ can be written as

$$a_n = \sum_{m=0}^{\left[\frac{n}{2}\right]} (-1)^m \eta_{m,n} \delta^m \tau^{n-2m}$$  \hspace{1cm} (C18)
A look at the \( \eta_{m,n} \) values in Table I reveal that

\[
\eta_{m,n} = \eta_{m,n-1} + \eta_{m-1,n-2}
\]  
(C19)

which seems similar to the properties of the binomial coefficients. In fact,

\[
\eta_{m,n} = \binom{n-m}{m}
\]  
(C20)

and the structure of the Pascal’s triangle might be evidently seen in the table.

Appendix D: The Form of \( \phi_n \)

**Corollary D.1** The sum of the geometric progression

\[
\phi_n = I + M + M^2 + \ldots + M^n
\]

is given as

\[
\phi_n = \begin{pmatrix}
    f_n & f_{n-1} \\
    -\delta f_{n-1} & 1 - \delta f_{n-2}
\end{pmatrix}
\]  
(D1)

where

\[
f_n = \frac{1 - a_n + \delta a_{n-1}}{1 - \tau + \delta}.
\]  
(D2)

**Proof** Using the formula for sum of GP, we can write \( \phi_n \) as

\[
\phi_n = \frac{I - M^n}{I - M}
\]  
(D3)

Hence

\[
\phi_n = \begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}
- \begin{pmatrix}
    \tau & 1 \\
    -\delta & 0
\end{pmatrix}
- \begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}
- \begin{pmatrix}
    a_n & a_{n-1} \\
    -\delta a_{n-1} & -\delta a_{n-2}
\end{pmatrix}
\]  

\[
= \begin{pmatrix}
    1 - \tau & -1 \\
    \delta & 1
\end{pmatrix}
- \begin{pmatrix}
    1 - a_n & -a_{n-1} \\
    \delta a_{n-1} & 1 + \delta a_{n-2}
\end{pmatrix}
\]  

\[
= \frac{1}{1 - \tau + \delta}
\begin{pmatrix}
    1 - a_n + \delta a_{n-1} & 1 - a_{n-1} + \delta a_{n-2} \\
    -\delta (1 - a_n) + \delta a_{n-1} (1 - \tau) & \delta a_{n-1} + (1 - \tau) (1 + \delta a_{n-2})
\end{pmatrix}
\]  

\[
= \begin{pmatrix}
    \frac{1 - a_n + \delta a_{n-1}}{1 - \tau + \delta} & \frac{1 - a_{n-1} + \delta a_{n-2}}{1 - \tau + \delta} \\
    -\frac{\delta (1 - a_n) + \delta a_{n-1} (1 - \tau)}{1 - \tau + \delta} & \frac{\delta a_{n-1} + (1 - \tau) (1 + \delta a_{n-2})}{1 - \tau + \delta}
\end{pmatrix}
\]

Currently, \( \phi_n \) is of the form

\[
\phi_n = \begin{pmatrix}
    f_n & f_{n-1} \\
    \sigma_1 & \sigma_2
\end{pmatrix}
\]  
(D4)
with
\[ \sigma_1 = \frac{-\delta (1 - a_n) + \delta a_{n-1} (1 - \tau)}{1 - \tau + \delta} \]  
and
\[ \sigma_2 = \frac{\delta a_{n-1} + (1 - \tau) (1 + \delta a_{n-2})}{1 - \tau + \delta}. \]

Using lemma B.1, we simplify \( \sigma_1 \) and \( \sigma_2 \) as

\[
\sigma_1 = \frac{-\delta (1 - a_n) + \delta a_{n-1} (1 - \tau)}{1 - \tau + \delta} = \frac{-\delta + \delta a_n + \delta a_{n-1} - \delta \tau a_{n-1}}{1 - \tau + \delta} = \frac{-\delta (1 - a_n - a_{n-1} + \tau a_{n-1})}{1 - \tau + \delta} = \frac{-\delta (1 - a_n - a_{n-1} + a_n + \delta a_{n-2})}{1 - \tau + \delta} = \frac{-\delta (1 - a_{n-1} + \delta a_{n-2})}{1 - \tau + \delta} = -\delta f_{n-1}.
\]

And

\[
\sigma_2 = \frac{\delta a_{n-1} + (1 - \tau) (1 + \delta a_{n-2})}{1 - \tau + \delta} = \frac{\delta a_{n-1} + 1 + \delta a_{n-2} - \tau - \delta \tau a_{n-2}}{1 - \tau + \delta} = \frac{1 - \tau + \delta (a_{n-1} + a_{n-2} - \tau a_{n-2})}{1 - \tau + \delta} = \frac{1 - \tau + \delta (a_{n-1} - \delta a_{n-3})}{1 - \tau + \delta} = \frac{1 - \tau + \delta (-1 + 1 + a_{n-1} - \delta a_{n-3})}{1 - \tau + \delta} = \frac{1 - \tau + \delta (-1 + a_{n-1} - \delta a_{n-3})}{1 - \tau + \delta} = \frac{1 - \tau + \delta - \delta (1 - a_{n-1} + \delta a_{n-3})}{1 - \tau + \delta} = 1 - \delta f_{n-2}.
\]

Therefore,
\[ \phi_n = \begin{pmatrix} f_n & f_{n-1} \\ -\delta f_{n-1} & 1 - \delta f_{n-2} \end{pmatrix}. \]  

(D7)

Appendix E: \( M^n \) in Terms of Eigenvalues

Theorem E.1 Let
\[ M = \begin{pmatrix} \tau & 1 \\ -\delta & 0 \end{pmatrix} \]  

(E1)
with the eigenvalues

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$$  \hspace{1cm} (E2)

then

$$M^n = \begin{pmatrix} a_n & a_{n-1} \\ -\delta a_{n-1} & -\delta a_{n-2} \end{pmatrix} \quad \forall n \in \mathbb{N} \hspace{1cm} (E3)$$

where

$$a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}$$  \hspace{1cm} (E4)

**Proof** Let $\lambda_1$ and $\lambda_2$ be the eigenvectors of $M$. Substituting the trace $\tau = \lambda_1 + \lambda_2$ and determinant $\delta = \lambda_1\lambda_2$, we have

$$M = \begin{pmatrix} \lambda_1 + \lambda_2 & 1 \\ -\lambda_1\lambda_2 & 0 \end{pmatrix}. \hspace{1cm} (E5)$$

Simple calculation of shows that the eigenvector corresponding to $\lambda_1$ is $\begin{pmatrix} 1 \\ -\lambda_2 \end{pmatrix}$ and that corresponding to $\lambda_2$ is $\begin{pmatrix} 1 \\ -\lambda_1 \end{pmatrix}$. Hence, we may construct a matrix $U$ with eigenvectors as columns,

$$U = \begin{pmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{pmatrix}. \hspace{1cm} (E6)$$

and a diagonal matrix $D$ with the eigenvalues

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \hspace{1cm} (E7)$$

such that

$$M = UDU^{-1}. \hspace{1cm} (E8)$$

and hence

$$M^n = U D^n U^{-1}. \hspace{1cm} (E9)$$

Substituting the values, we get

$$M^n = \begin{pmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{pmatrix}^{-1}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} -\lambda_1 & -1 \\ \lambda_2 & 1 \end{pmatrix}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{pmatrix} \begin{pmatrix} -\lambda_1^{n+1} & -\lambda_2^n \\ \lambda_2^{n+1} & \lambda_1^n \end{pmatrix}$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2^{n+1} - \lambda_1^{n+1} & \lambda_2^n - \lambda_1^n \\ \lambda_1^{n+1} - \lambda_2^{n+1} \lambda_2 - \lambda_2^{n+1} \lambda_1 \end{pmatrix}$$
which can be recast as

\[ M^n = \begin{pmatrix} a_n & a_{n-1} \\ -\delta a_{n-1} & -\delta a_{n-2} \end{pmatrix} \]  

(E10)

with

\[ a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}. \]  

(E11)