Separability Criteria for Arbitrary Quantum Systems

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The purpose of this paper is to obtain a sufficient and necessary condition as a criteria to test whether an arbitrary multipartite state is entangled or not. Based on the tensor expression of a multipartite pure state, the paper show that a state is separable iff $|C(\rho)|=0$ for pure states and iff $C(\rho)$ vanishes for mixed states.

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INTRODUCTION

Entanglement is an essential ingredient in quantum information and the central feature of quantum mechanics which distinguishes a quantum system from its classical counterpart. As an important physical resource, it is also widely applied to a lot of quantum information processing (QIP): quantum computation [1], quantum cryptography [2], quantum teleportation [3], quantum dense coding [4] and so on.

Entanglement arises only if there has been interactions between the subsystems of a multipartite system from physics or only if the quantum state is nonseparable or nonfactorized from mathematics. Even though a lot of efforts have been made on how to tell whether a given quantum state is entangled (separable) or not, only bipartite entanglement measures [6,7,8,9] as separability criteria have been for the most part well understood. Even though the separability criteria for pure states [10,11,12] are versatile and complex, there does not exist a unified one: A general formulation of multipartite mixed states is relatively lacking and remains an open problem.

Recently, Reference [5] has presented a sufficient and necessary condition for separability of tripartite qubit systems by arranging $a_{ijk}s$ of a pure state $|\psi\rangle_{ABC} = \sum a_{ijk} |i\rangle_A |j\rangle_B |k\rangle_C$ as a three-order tensor in $2 \times 2 \times 2$ dimension. The introduction of the new skill has provided an effective way to generalize the criteria to tripartite states in arbitrary dimension and to multipartite quantum systems.

In this Letter, we continue Ref.[5] to give out a general formulation of separability criterion for arbitrary quantum systems. The paper is organized as follows: Firstly, we present a sufficient and necessary condition of separability for tripartite pure states. Secondly, we generalize the condition to the case of mixed states. Lastly, we generalize our result to multipartite quantum systems.

SEPARABILITY CRITERION FOR TRIPARTITE PURE STATES

We begin with the definition of our tensors. Unlike the previous definition of tensors, for convenience, all the quantities with indices, such as $T_{ij\cdots k}$ and so on, are called tensors here. The number of the indices is called the order of the tensor. Therefore, the set of all one-order tensors is the set of vectors, and the set of all two-order tensors is the one of matrices. Three-order tensors $T_{ijk}$ are matrices (vectors) if any one (two) of their three indices is (are) fixed.

The elements of a three-order tensor can be arranged at the node of the grid in three-dimensional Hilbert space, such as the tensor $T_{ijk}$ with $i, j, k = 0, 1, 2$ shown in figure 1. From geometry, every fixed index corresponds to a group of parallel planes which are perpendicular to the vector which the fixed index corresponds to. The planes corresponding to different fixed indices are mutually perpendicular.

Definition.-Let $T_{ijk}$ is a three-order tensor in $n_1 \times n_2 \times n_3$ dimension with $i = 0, 1, \cdots, n_1 - 1$, $j = 0, 1, \cdots, n_2 - 1$ and $k = 0, 1, \cdots, n_3 - 1$. $T'_{i'j'k'}$ is any sub-tensor in $m \times m \times m$ dimension by selecting any $m$ planes from every group of parallel ones corresponding to three different fixed indices.

FIG. 1: Three-order tensor of the coefficients of a tripartite pure state.
f(T_{i'j'k'}^f) = |C| = \frac{1}{\sqrt{3}} \sqrt{\sum_\alpha \langle C^{\alpha}_\gamma \rangle \langle C^{\alpha}_\gamma \rangle},

which is introduced in Ref.[5], where \( C^{\alpha}_\gamma \) = \( (T_{i'j'k'})^{s1}T_{i'j'k'} \) with \( s1 = -\sigma_y \otimes \sigma_y \otimes I \), \( s2 = -\sigma_y \otimes I \otimes \sigma_y \), \( s3 = -I \otimes \sigma_y \otimes \sigma_y \), \( s4 = -I \otimes \sigma_y \otimes I \), \( s5 = -\sigma_y \otimes I \otimes \sigma_y \), \( s6 = -\sigma_y \otimes \sigma_y \otimes I \), (here \( \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), and \( I_v = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \).

Analogous to Ref.[5], considering any a tripartite pure state |\( \psi \rangle \rangle_{ABC} = \sum a_{ijk} |i \rangle_A |j \rangle_B |k \rangle_C \) with \( i = 0, 1, \ldots, n_1 - 1, j = 0, 1, \ldots, n_2 - 1 \) and \( k = 0, 1, \ldots, n_3 - 1 \), if arranging the coefficients of the state as a tensor denoted by \( A_{ijk} \) (every coefficient corresponds to a node of the grid), every line in the grid corresponds to a vector and every plane corresponds to a matrix. One can easily find that, the tripartite state is fully separable iff all the parallel vectors are linear relevant. A necessary and sufficient condition for this which is easily proved is that the second compound tensor \( C_2(A_{ijk}) \) must be zero. Namely, every element of the tensor must be zero. Consider the state |\( \psi \rangle \rangle_{ABC} \) written in vector notation |\( \psi \rangle \rangle_{ABC} = |a_{000}, a_{010}, \ldots, a_{n_3n_2n_1}, a_{010}, \ldots, a_{n_3n_2n_1-1} \rangle \rangle_{ABC} \), one can obtain an equivalent expression of above relation, i.e.

\[
|C^{\alpha}_\beta_\gamma(\psi)| = \frac{1}{\sqrt{3}} \sqrt{\sum_\gamma (C^{p}_{\alpha_\beta_\gamma}(\psi))^2} = 0
\]

holds for any \( \alpha \), \( \beta \), and \( \gamma \), where \( C^{p}_{\alpha_\beta_\gamma} = \langle \psi_{ABC} \rangle s^{p}_{\alpha_\beta_\gamma} |\psi_{ABC} \rangle \rangle_{ABC} \) with \( s^{1}_{\alpha_\beta_\gamma} = -L_\alpha \otimes \beta \otimes \gamma \), \( s^{2}_{\alpha_\beta_\gamma} = -L_\alpha \otimes \beta \otimes \gamma \), \( s^{3}_{\alpha_\beta_\gamma} = -L_\beta \otimes \gamma \otimes \alpha \), \( s^{4}_{\alpha_\beta_\gamma} = -L_\beta \otimes \gamma \otimes \alpha \), \( s^{5}_{\alpha_\beta_\gamma} = -L_\alpha \otimes \gamma \otimes \beta \), \( s^{6}_{\alpha_\beta_\gamma} = -L_\alpha \otimes \gamma \otimes \beta \), \( L_\alpha \), \( L_\beta \), and \( L_\gamma \) are the generators of \( SO(n_1) \), \( SO(n_2) \) and \( SO(n_3) \), respectively; \( I_{\alpha} \), \( I_{\beta} \), and \( I_{\gamma} \) are the unit matrices in \( n_1 \), \( n_2 \), and \( n_3 \) dimension, respectively; with \( \alpha = 1, 2, \ldots, n_1(n_1-1)/2 \), \( \beta = 1, 2, \ldots, n_2(n_2-1)/2 \), and \( \gamma = 1, 2, \ldots, n_3(n_3-1)/2 \). \( |M| \) denotes the modulus of the elements of the matrix \( M \).

Then we can construct a new vector \( C = \bigoplus C_{\alpha_\beta_\gamma} \) and employ the length of the vector

\[
|C(\psi)| = \sqrt{\sum_\alpha \langle C^{p}_{\alpha_\beta_\gamma}(\psi) \rangle^2} = \frac{1}{\sqrt{3}} \sqrt{\sum_\gamma \sum_\alpha \langle C^{p}_{\alpha_\beta_\gamma}(\psi) \rangle^2}
\]

as the criterion of separability.

**Separability Criterion for Tripartite Mixed States**

The tripartite mixed states \( \rho = \sum_\gamma \omega_k |\psi^k \rangle \langle \psi^k | \) can be written in matrix notation as \( \rho = \Psi W \Psi^\dagger \), where \( W \) is a diagonal matrix with \( W_{kk} = \omega_k \), the columns of the matrix \( \Psi \) correspond to the vectors \( \psi^k \). Consider the eigenvalue decomposition, \( \rho = \Phi M \Phi^\dagger \), where \( M \) is a diagonal matrix whose diagonal elements are the eigenvalues of \( \rho \), and \( \Phi \) is a unitary matrix whose columns are the eigenvectors of \( \rho \). From Ref.\[5\], one can get \( \Psi W^{1/2} = \Phi M^{1/2} T \), where \( T \) is a Right-unitary matrix. The tripartite mixed states are fully separable iff there exist a decomposition such that \( \psi^k \) for every \( k \) is fully separable. The entanglement measure of formation can be defined as the infimum of the average \( C(\psi^k) \).

Namely, \( C(\rho) = \inf \sum_\gamma \omega_k |C(\psi^k)| \), if \( C(\rho) \) is assigned as the entanglement measure for tripartite mixed states. Therefore, for any a decomposition

\[
\rho = \sum_\gamma \omega_k |\psi^k \rangle \langle \psi^k |
\]

one can get

\[
C(\rho) = \inf \sum_\gamma \omega_k |C(\psi^k)| = \inf \sum_\gamma \omega_k \frac{1}{\sqrt{3}} \sqrt{\sum_\gamma \sum_\alpha (|\psi^k \rangle \langle \psi^k | s^{p}_{\alpha_\beta_\gamma} |\psi^k \rangle \langle \psi^k |)^2}
\]

According to the Mincowski inequality

\[
\left( \sum_{i=1}^k \left( \sum_{j=1}^n x_{ij} \right)^p \right)^{1/p} \leq \sum_{k=1}^n \left( \sum_{i=1}^n (x_{ij})^p \right)^{1/p}, \quad p > 1,
\]

(1)
one can easily obtained

\[
C(\rho) \geq \inf_{\mathbf{T}} \left\{ \frac{1}{\sqrt{3}} \sum_{p} \sum_{\alpha \beta \gamma} \left( \sum_{k} \omega_{k} \left| \left\langle \psi^{k} \right| s_{\alpha \beta \gamma}^{p} \left| \psi^{k} \right\rangle \right| \right)^{2} \right\} \]

\[
= \inf_{\mathbf{T}} \frac{1}{\sqrt{3}} \left\{ \sum_{p} \sum_{\alpha \beta \gamma} \left( \sum_{k} \left| \psi^{k} \right| W^{1/2} s_{\alpha \beta \gamma}^{p} W^{1/2} \psi \left| \psi^{k} \right\rangle \right) \right\}^{2} \]

\[
= \inf_{\mathbf{T}} \frac{1}{\sqrt{3}} \sum_{k} \left| \mathbf{T} \left( \sum_{p} \sum_{\alpha \beta \gamma} \left( z_{\alpha \beta \gamma}^{p} A_{\alpha \beta \gamma}^{p} \right) \right) \right|_{kk}^{2} \]

where \( A_{\alpha \beta \gamma}^{p} = M^{1/2} \Phi^{k} s_{\alpha \beta \gamma}^{p} \Phi M^{1/2} \) for any \( z_{\alpha \beta \gamma}^{p} = y_{\alpha \beta \gamma}^{p} e^{i \phi} \) with \( y_{\alpha \beta \gamma}^{p} > 0 \), \( \sum_{\alpha} y_{\alpha \beta \gamma}^{p} = 1 \), and Cauchy-Schwarz inequality

\[
\left( \sum_{i} x_{i}^{2} \right)^{1/2} \left( \sum_{i} y_{i}^{2} \right)^{1/2} \geq \sum_{i} x_{i} y_{i},
\]

are applied at the last step. The infimum of equation (3) is given by \( \max_{z \in C} \lambda_{1}(z) - \sum_{i=1}^{n} \lambda_{i}(z) \) with \( \lambda_{i}(z) \)s are the singular values, in decreasing order, of the matrix \( \frac{1}{\sqrt{3}} \sum_{p} \sum_{\alpha \beta \gamma} z_{\alpha \beta \gamma}^{p} A_{\alpha \beta \gamma}^{p} \). Therefore, we can express \( C(\rho) \) as

\[
C(\rho) = \max \{ 0, \max_{z \in C} \lambda_{1}(z) - \sum_{i=1}^{n} \lambda_{i}(z) \}. \tag{4}
\]

It is not difficult to find that \( C(\rho) = 0 \) is a sufficient and necessary condition of separability for mixed states according to the whole procedure of derivation.

**SEPARABILITY CRITERION FOR MULTIPARTITE SYSTEMS**

A general \( N \)-partite pure states

\[
|\psi\rangle_{AB\ldots N} = \sum_{i_{j},\ldots,k} a_{i_{j},\ldots,k} |i_{j},\ldots,k\rangle,
\]

\[i \in [0, n_{1} - 1], j \in [0, n_{2} - 1], \ldots, k \in [0, n_{N} - (\bar{k})]\]

is separable iff \( |\psi\rangle_{AB\ldots N} = \sum_{i_{j},\ldots,k} a_{i_{j},\ldots,k} |i_{A}\rangle \otimes |j\rangle_{B} \otimes \cdots \otimes |k\rangle_{N} \). Analogously, if the coefficients \( a_{i_{j},\ldots,k} \)s are arranged as an \( N \)-order tensor, one can easily find that, \( |\psi\rangle_{AB\ldots N} \) is separable iff all the vectors which are mutually parallel are linear relevant. In order to obtain a mathematical rigorous criterion, we have to redefine matrices \( s_{ij}^{\alpha_{\beta} \ldots \lambda} \)

\[
s_{ij}^{\alpha_{\beta} \ldots \lambda} = L_{\alpha} \otimes L_{\beta} \otimes |L_{\gamma}| \otimes \cdots \otimes |L_{\bar{k}}| \otimes I_{\rho} \otimes \cdots \otimes I_{\lambda}, \tag{6}
\]

\[
i = 0, 1, \ldots, N - 2, \ j = 1, 2, \ldots, \left( \frac{N}{2} \right) \]

where \( L_{x} \) denotes the generators of \( SO(n_{p}), x = 1, 2, \ldots, n_{p}(n_{p}-1) \), with \( p \) standing for the \( p \)th subsystem. \( i \) in above equation (6) states that there are \( i \) absolute values of generators. Note that the order of \( L_{\alpha}, L_{\beta}, |L_{\gamma}|, \ldots, |L_{\bar{k}}|, I_{\rho}, \ldots, I_{\lambda} \) in equation (6) must cover all the permutations with \( j \) as a index showing the \( j \)th permutation.

(2) Hence, the criterion for pure states can be expressed as following, if an \( N \)-partite pure state is separable iff

\[
|C(\psi)| = \sqrt{\sum_{\alpha_{\beta} \ldots \lambda} (\psi)^{\alpha_{\beta} \ldots \lambda}_{N}^{ij} (\psi)^{ij}_{N}^{\alpha_{\beta} \ldots \lambda}},
\]

where the sum is over all the indices and \( 1/\sqrt{\left( \frac{N}{2} \right)} \) is a normalized factor.

According to the same procedure of the derivation to Section II, one can easily obtain the criterion of separability for an \( N \)-partite mixed state by testing whether \( C(\rho) \) vanishes with

\[
C(\rho) = \max \left\{ 0, \max_{z \in C} \lambda_{1}(z) - \sum_{i=1}^{n} \lambda_{i}(z) \right\},
\]

where \( \lambda_{i}(z) \)s are the singular values, in decreasing order, of the matrix \( \sum_{i_{j},\ldots,k} z_{i_{j},\ldots,k}^{\alpha_{\beta} \ldots \lambda} A_{i_{j},\ldots,k}^{\alpha_{\beta} \ldots \lambda} / \sqrt{\left( \frac{N}{2} \right)} \).

Let us finally note that, our result can also be reduced to Wootters, concurrence when \( N = 2 \) in 2\times2 dimension or the result in Ref.[5] owing to the only generator \( \sigma_{y} \) of \( SO(2) \). Therefore, the final result presented in this Letter is a general one suitable for arbitrary quantum systems.

**CONCLUSION**

As a summary, in this paper, we generalize the result presented in Ref.[5] to arbitrary quantum systems. The result in this paper provides a sufficient and necessary condition as a general criterion to test whether a given multipartite system is separable or not. The criterion for pure states is convenient and analytic, but it has to turn to a numerical optimization for mixed states, however, a simple treatment similar to Ref.[5] is often enough. The fruitful conclusion similar to [8] is expectable and commendable.

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