CONFORMAL INVARIANCE IN RANDOM CLUSTER MODELS. 
II. FULL SCALING LIMIT AS A BRANCHING SLE.

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Abstract. In the second article of this series, we establish the convergence of the loop ensemble of interfaces in the random cluster Ising model to a conformal loop ensemble (CLE) — thus completely describing the scaling limit of the model in terms of the random geometry of interfaces. The central tool of the present article is the convergence of an exploration tree of the discrete loop ensemble to a branching SLE($\frac{16}{3}, -\frac{2}{3}$). Such branching version of the Schramm’s SLE not only enjoys the locality property, but also arises logically from the Ising model observables.

1. Introduction

Starting with the introduction of the Lenz-Ising model of ferromagnetism, lattice models of natural phenomena played important part in modern mathematics and physics. While overly simplified — continuous phenomena with infinite number of states are restricted to a discrete lattice with finite number of spin configurations, and complicated interactions are simplified to simple next-neighbor ones — they often give a very accurate qualitative description of what we observe in nature. In particular, they exhibit phase transitions when temperature passes through the critical or Curie point, and critical system is expected to enjoy (in the scaling limit) universality and (at least in 2D) conformal invariance.

While this is well understood on the physical and computational level, mathematical proofs (and understanding) are often lacking. In the first paper of this series [32], one of us established conformal invariance of some observables in the FK Ising model at criticality, from which description of a single domain wall as a universal, conformally invariant, fractal curve was — the so-called Schramm’s SLE(16/3) — was deduced [5]. The mathematical theory of such curves was started by Oded Schramm in his seminal paper [23]. The SLE curves are obtained by running a Loewner evolution with a Brownian driving term, and form a one-parameter family of fractals, interesting in themselves [22, 18]. Schramm has shown, that all scaling limits of interfaces or domain walls, if they exist and are conformally invariant, are always described by SLEs; for the exact formulation of the principle, see [23, 30, 14, 13]. A generalization of SLE is the conformal loop ensemble (CLE), which describes the joint law of all the interfaces in a model.

So far, convergence of a single discrete interface to SLE($\kappa$)’s has been established for but a few models: $\kappa = 2$ and $\kappa = 8$ [19], $\kappa = 3$ and $\kappa = \frac{16}{3}$ [5], $\kappa = 4$ [24, 25] and $\kappa = 6$ [29, 31]. However, the framework for the full scaling limit, including all interfaces, is less developed: $\kappa = 3$ [3], $\kappa = \frac{16}{3}$ [16] and $\kappa = 6$ [4].
The present article extends the convergence showed in [16] to include all the interfaces, not just those (infinitely many in the limit) that touch the boundary. Effectively, we give a geometric description of the full scaling limit of the FK Ising model, which is universal, conformally invariant, and can be obtained by a canonical coupling of branching SLE curves.

1.1. Fortuin–Kasteleyn representation of the Ising model. For general background on the Ising model, the random cluster model and other models of statistical physics, see the books [2, 10, 11, 20]. See also the first article [32], Section 2.

1.1.1. Notation and definitions for graphs. In this article, the lattice $L^\bullet$ is the square lattice $\mathbb{Z}^2$ rotated by $\pi/4$ and scaled by $\sqrt{2}$, $L^\circ$ is its dual lattice, which itself is also a square lattice, and $L^\diamond$ is their (common) medial lattice. More specifically, we define three lattices $G = (V(G), E(G))$, where $G = L^\bullet, L^\circ, L^\diamond$, as

$$V(L^\bullet) = \{(i,j) \in \mathbb{Z}^2 : i + j \text{ even}\}, \quad E(L^\bullet) = \left\{ \{v, w\} \subset V(L^\bullet) : |v - w| = \sqrt{2}\right\},$$

$$V(L^\circ) = \{(i,j) \in \mathbb{Z}^2 : i + j \text{ odd}\}, \quad E(L^\circ) = \left\{ \{v, w\} \subset V(L^\circ) : |v - w| = \sqrt{2}\right\},$$

$$V(L^\diamond) = (1/2 + \mathbb{Z})^2, \quad E(L^\diamond) = \left\{ \{v, w\} \subset V(L^\diamond) : |v - w| = 1\right\}.$$

Notice that sites of $L^\diamond$ are the midpoints of the edges of $L^\bullet$ and $L^\circ$. Denote the set of midpoints of the edges of $L^\circ$ as

$$V_{\text{mid}} = \left\{ (i,j) \in \left( \frac{1}{2} \mathbb{Z} \right)^2 : i + j \in \mathbb{Z} + \frac{1}{2} \right\}. \quad (1)$$

It is natural to identify midpoints $V_{\text{mid}}$ with their corresponding edges $E(L^\circ)$.

We call the vertices and edges of $V(L^\bullet)$ black and the vertices and edges of $V(L^\circ)$ white. Correspondingly the faces of $L^\diamond$ are colored black and white depending whether the center of that face belongs to $V(L^\bullet)$ or $V(L^\circ)$.

The directed version $L^\diamond\rightarrow$ is defined by setting $V(L^\diamond\rightarrow) = V(L^\circ)$ and orienting the edges around any black face in the counter-clockwise direction.

![Figure 1.](image.png)

**Figure 1.** The square lattices we are considering are $L^\bullet$ formed by the centers of the black squares, $L^\circ$ formed by the centers of the white squares and $L^\diamond$ formed by the corners of the black and white squares. We will also consider the square–octagon lattice $L^\blacklozenge$ which we see as a modification of $L^\diamond$.

The *modified medial lattice* $L^\blacklozenge$, which is a square–octagon lattice, is obtained from $L^\diamond$ by replacing each site by a small square. See Figure 1. The faces of $L^\blacklozenge$ are referred to as octagons (black or white) and small squares. The oriented lattice $L_{\blacklozenge\rightarrow}$ is obtained from $L^\blacklozenge$ by orienting the edges around black and white octagonal faces in counter-clockwise and clockwise directions, respectively.
Definition 1.1. A simply connected, non-empty, bounded domain \( \Omega \) is said to be a \textit{wired} \( L^\leftrightarrow \)-domain (or \textit{admissible domain}) if \( \partial \Omega \) oriented in counter-clockwise direction is a path in \( L^\leftrightarrow \).

\begin{figure}[h]
\centering
(a) Oriented lattice \( L^\leftrightarrow \).
\hfill
(b) Wired \( L^\leftrightarrow \)-domain.
\caption{The oriented lattice and a discrete admissible domain on it.}
\end{figure}

See Figure 2 for an example of such a domain. The wired \( L^\leftrightarrow \)-domains are in one to one correspondence with non-empty finite subgraphs of \( L^\bullet \) which are simply connected, i.e., they are graphs who have an unique unbounded face and the rest of the faces are squares.

1.1.2. FK Ising model. Let \( G \) be a simply connected subgraph of the square lattice \( L^\bullet \) corresponding to a wired \( L^\leftrightarrow \)-domain. Consider the random cluster measure \( \mu = \mu_{p,q}^1 \) of \( G \) with all \textit{wired boundary conditions} in the special case of the critical FK Ising model, that is, when \( q = 2 \) and \( p = \sqrt{2}/(1 + \sqrt{2}) \). For the concrete definition, see the references given above and the equation (2) below. Its dual model is also a critical FK Ising model, now with free boundary conditions on the dual graph \( G^\circ \) of \( G \) which is a (simply connected) subgraph of \( L^\circ \). They have common loop representation on the corresponding subgraph \( G^\bullet \) of the modified medial lattice \( L^\bullet \).

We call a collection of loops \( \mathcal{L} = (L_j)_{j=1,...,N} \) on \( G^\bullet \) dense collection of non-intersecting loops (DCNIL) if

- each \( L_j \subset G^\bullet \) is a simple loop
- \( L_j \) and \( L_k \) are vertex-disjoint when \( j \neq k \)
- for every edge \( e \in E^\circ \) there is a loop \( L_j \) that visits \( e \). Here we use the fact that \( E^\circ \) is naturally a subset of \( E^\bullet \).

We consider loop collections only modulo permutations, that is, two objects are equal if they are permutations of each other. Let the collection of all the loops in the loop representation be \( \Theta = (\theta_j)_{j=1,...,N} \). Then DCNIL is exactly the support of \( \Theta \) and for any DCNIL collection \( C \) of loops

\[ \mu(\Theta = \mathcal{L}) = \frac{1}{Z} (\sqrt{2})^{\# \text{ of loops in } \mathcal{L}} \]  

(2)

where \( Z \) is a normalizing constant.

We denote the “external” boundary of the domain by \( \partial G^\bullet \) and the “internal” boundary, which is the outermost (simple) loop can be drawn in \( G^\bullet \), as \( \partial_t G^\bullet \), that
is, $\partial G^*$ and $\partial_1 G^*$ are as close as possible and the layer of wired edges of $G^*$ lies between them.

1.2. Exploration tree of a loop ensemble. Suppose that we are given a wired $\mathbb{L}_\infty$-domain $\Omega$ and a DCNIL loop collection $\mathcal{L} = (L_j)_{j=1}^{\ldots,N}$. We wish to define a spanning tree which corresponds to $\mathcal{L}$ in a one-to-one manner with an easy rule to recover $\mathcal{L}$ from the spanning tree. We follow here the ideas of [27].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The branch of an exploration tree from $a$ (the root which is the inwards pointing red arrow) to $e$.}
\end{figure}

Select a small square $S_1$ next to the boundary. We can assume that it shares exactly one edge with the boundary (if it shares two, it is a “bottle neck” — a case we exclude and which doesn’t play any role in the continuum limit). Let the edge in $S_1$ incoming to the domain be $a$ and the outgoing edge be $b$, see also Figure 3 or Figure 4. Let $e \in E(\mathbb{L}_\infty\ominus L_j)$ (that is, as an edge in $E(\mathbb{L}_\infty\ominus L_j)$ it lies between white and black octagon). Then define in the following way the branch $T_e$ from the root $a$ to the target $e$.

- Cut open the loop $L_{j_1}$ that goes through the edge passing from the tail of $b$ to the head of $a$ by removing that edge. Follow from $a$ the $L_{j_k}$ until the disconnection of $e$ and $b$ on the lattice. Suppose that it happens on the small square $S_2$.
- Let $n \geq 1$ and suppose that we have constructed the branch following the loops $L_{j_k}$, $k = 1, 2, \ldots, n$ until we are at the square $S_{n+1}$ and on the loop $L_{j_n}$. Instead of following the loop $L_{j_n}$ by an edge which would take the branch to a component disconnected $e$ we use the other possible edge on $S_{n+1}$ (which is not on any loop) and we arrive to a new (unexplored) loop $L_{j_{n+1}}$. Then we follow that loop until disconnection of $b$ and $e$. Suppose that it happens at the small square $S_{n+2}$. We continue this construction recursively.
Figure 4. The target edges, when loops are recovered from the tree, are the thick colored arrows in the picture.

- The process ends when we reach $e$. Suppose that it happens on a loop $L_{j'}$.
  Rename the loops in the sequence as $L'_{j',k}$, $k = 1, 2, \ldots, N' = N'(e)$, and the small square sequences as $S'_{e,k}$.

This defines the simple lattice path $T_e$ from the root $a$ to the target $e$ which we call the branch of the exploration tree. The collection $\mathcal{T} = (T_e)_e$ where $e$ runs over all edges $e \in E(L_a)$, is called the exploration tree of the loop collection $\mathcal{L} = (L_j)_{j=1,\ldots,N}$. This construction is illustrated in Figure 3.

When we consider $\mathcal{T} = (T_e)_e$ as a collection of edges of $L_a$ it forms a rooted spanning tree of the graph with vertices $V(L_a) \cap (\{a\} \cup \Omega)$ and all edges connecting pairs of them.

We say that the branches $T_e$ (or rather their coupling) are target independent or local, in the sense that the initial segments of $T_e$ and $T_{e'}$ are equal until they disconnect $e$ and $e'$ on the graph. Even the sequences $L'_{j',k}$ and $L'_{j',k}$ and on the other hand the sequences $S'_{e,k}$ and $S'_{e',k}$ agree until the disconnection.

The “tree-to-loops” construction is illustrated in Figure 4 and it is the inverse of the above “loops-to-tree” construction. Each loop corresponds to exactly one small square where branching of the tree occurs. Suppose that $e_1 \in E(L_a)$ is the incoming edge used by the branch to arrive to the small square for the first time and $e_2 \in E(L_a)$ is the other incoming edge (opposite to $e_1$ in the square). Then the loop is reconstructed when we follow the branch to $e_2$ and keep the part after the first exit from the small square and then closing the loop by adding the side of the small square that goes from the head of $e_2$ to that exit point.

Finally let us emphasize the geometric characteristic of the branching point. As it is illustrated in Figure 5, any typical branching point $S_n$ in the scaling limit is uniquely characterized as been a “5-arm point” of a branch in the tree. That is,
in the figure, the branch goes through or close to the square \( S_n \) so that the branch forms a “5-arm figure” — two orange, one gray and two green arms.

(a) A schematic illustration of the loops–tree correspondence. See the figure (b), for an enlargement of the neighborhood of \( S_n \).

(b) The 5-arms formed by 2 orange, 1 gray and 2 green arms emanating from the corners of the square \( O_n \).

**Figure 5.** The correspondence between a loop ensemble and a tree and the geometric “5-arm property” of a branching point.

1.2.1. **Notation for the scaling limit.** Let \( \delta > 0 \). Suppose that \( \Omega_\delta \) is a wired \( \delta \mathbb{H} \)-domain and \( 0 \in \Omega_\delta \). Take a small square that share exactly one edge with the boundary. One of the edges of the square start from the boundary and one ends at the boundary. Call them \( a_\delta \) and \( b_\delta \), respectively.

We shall consider the random loop collection (loop ensemble) \( \Theta_\delta \) on each \( \Omega_\delta \), \( \delta > 0 \), which are distributed as the loop representation of FK Ising model (on the corresponding graph). Define also \( T_\delta \) to be the exploration tree of \( \Theta_\delta \) with the root \( a_\delta \).

When \( \Omega_\delta \) is a sequence converging in the Carathéodory sense with respect to 0 as \( \delta \to 0 \), the **scaling limit** is the limit \( \lim_{\delta \to 0} (\Theta_\delta, T_\delta) \) with respect to a suitable topology. See Section 1.3.3 below on the discussion on the topology.

1.3. **SLEs, CLEs and conformal invariant scaling limits.**

1.3.1. **Schramm–Loewner evolution.** Let \( \gamma : [0, \infty) \to \mathbb{H} \setminus \{0\} \) be a curve such that \( \gamma(0) \in \partial \mathbb{D} \). Denote the connected component containing 0 in \( \mathbb{H} \setminus \gamma([0, t]) \) by \( D_t \).

Then \( D_t \) is simply connected and there exists a unique conformal and onto map \( g_t : D_t \to \mathbb{H} \) such that \( g_t(0) = 0 \) and \( g'_t(0) > 0 \), by the Riemann mapping theorem. By moving to so called capacity parametrization, we may assume that \( \gamma \) is parametrized
such that \( g_t(z) = e^t z + \mathcal{O}(|z|^2) \) as \( z \to 0 \).

This map satisfies the Loewner equation in \( \mathbb{D} \)

\[
g_t(z) = -g_t(z) \frac{g_t(z) + U_t}{g_t(z) - U_t}, \quad g_0(z) = z
\]

for each \( t \in [0, \infty) \) and \( z \in \mathbb{D} \).

**Definition 1.2.** A random curve \( \gamma \) in \( \mathbb{D} \) is a radial SLE(\( \kappa \)), if \( U_t = \exp(i\sqrt{\kappa}B_t) \) for some Brownian motion \((B_t)_{t \in (0, \infty)}\).

**Definition 1.3.** A random curve \( \gamma \) in \( \mathbb{D} \) is a radial SLE(\( \kappa, \kappa - 6 \)) (we assume that \( \kappa \in (4, 8) \)), if \( U_t \) is the first coordinate of an adapted, continuous semimartingale \((U_t, V_t)\) such that \( U_0 = V_0 \),

\[
V_t = V_0 - \int_0^t V_s \frac{V_s + U_s}{V_s - U_s} \, ds
\]

for all \( t \),

\[
dU_t = i\sqrt{\kappa}U_t \, dB_t + \left( -\frac{\kappa}{2} U_t - \frac{\kappa - 6}{2} \frac{U_t + V_t}{U_t - V_t} \right) \, dt
\]

for all \( t \) such that \( U_t \neq V_t \) (for some Brownian motion \((B_t)_{t \in (0, \infty)}\)) and \( \arg(V_t/U_t) \in [0, 2\pi) \) is instantaneously reflecting at 0 and \( 2\pi \), meaning in particular that

\[
\mathbb{P} \left[ \int_0^\infty 1_{U_t = V_t} \, dt = 0 \right] = 1.
\]

**Remark 1.4.** The chordal and radial SLE(\( \kappa, \kappa - 6 \)) only differ by the fact that the target point for the former process is on the boundary while the one for the latter process is in the bulk. Their laws until the disconnection of the two alternative target points are the same, and after the disconnection the processes turn towards their own target points. This follows since the sum of “\( \rho \)'s” is equal to hence \( \kappa - 6 \) and there is no force applied by the marked points \( \infty \) or 0 (in \( \mathbb{H} \) and \( \mathbb{D} \), respectively). This is the target independence or locality property of SLE(\( \kappa, \kappa - 6 \)). This property is not valid for the chordal and radial SLE(\( \kappa \)) whose laws are different (though absolutely continuous with respect to each other on appropriately chosen time intervals); see [26] for the transformation rule between the upper half-plane and the unit disc.

1.3.2. **Conformal loop ensembles.** Suppose that we are given a family of probability measures \( (\mu^\Theta)_{\Omega} \) where \( \Omega \) runs over simply connected domains and \( \mu^\Omega \) is the law of a random loop collection on \( \Omega \). If \( \Theta = (\theta_j)_j \) is distributed according to \( \mu^\Omega \), we suppose that almost surely each loop \( \theta_j \) is simple, \( \theta_i \cap \theta_j = \emptyset \) when \( i \neq j \) and they satisfy the following properties:

- **(Conformal invariance (CI))** If \( \psi : \Omega \to \mathbb{C} \) is conformal and \( \psi^* \) is its push-forward map, then \( \psi^* \mu^\Omega = \mu^\psi(\Omega) \).

- **(Domain Markov property (DMP))** If \( \Omega' \subset \Omega \) is a simply connected domain, \( J' \) is the set indices \( j \) such that \( \theta_j \cap (\Omega \setminus \Omega') = \emptyset \) and \( \hat{\Omega} \) is equal to \( \Omega' \setminus \bigcup_{j \in J'} \text{int}(\gamma_j) \), then the law of \( (\gamma_j)_{j \in J'} \) is equal to \( \mu^{\hat{\Omega}} \).

If the collection \( \Theta = (\theta_j)_j \) satisfy these properties, we call it conformal loop ensemble (CLE).

It turns out that loops in CLE’s are SLE-type curves [28]. See Section 1 of [28] for several formulations of this kind of a result. A given CLE corresponds to

\footnote{In fact, we make an assumption here that the capacity increases on any time interval and that the capacity tends to \( \infty \). The latter statement is equivalent to \( \lim \inf_{t \to \infty} |\gamma(t)| = 0 \).}
SLE(κ) with a unique κ ∈ (8/3, 4]. We use the notation CLE(κ) for the CLE that corresponds to SLE(κ). See [28] for uniqueness statement on CLE’s.

A third view that we adopt to CLE is the branching SLE(κ, κ − 6) construction of CLE(κ), κ ∈ (8/3, 8), which allows the extension of the definition to values κ ∈ (4, 8) which is highly relevant for this article. This process is a collection of curves γz from the root a ∈ ∂D to the target z, where z runs over all points in D.

**Definition 1.5.** The random collection of curves (γz)z is a branching SLE(κ, κ − 6), if the law of γz is the (radial) SLE(κ, κ − 6) from a to z and moreover the curves are coupled so that for each z ≠ z′ it holds that γz and γz′ are equal until the disconnection of z and z′ by γz (or γz′).

A tree in graph theory is a connected graph without any cycles, or equivalently a graph such that any pair of points is connected by a unique simple path. In the same spirit, it is natural to say that the branching SLE(κ, κ − 6) forms a tree: from the root a to any (generic) point z there is a unique path γz and between any (generic) points z ≠ z′ the unique path follows the reversal of γz to the branching point of γz and γz′ and then γz′ from that point to z′.

### 1.3.3. Metrics for curve collections

In this subsection, we present first the topology for the convergence for branches and trees and then for loops and loop ensembles.

**(Metrics for branches and trees)** Consider a triplet (Ω, Ψ; T) where

- Ω is a simply connected domain
- Ψ : Ω → D is a conformal and onto map
- T : [0, 1] → Ω is a curve such that there exists a curve T_D in D parametrized by the d-capacity such that Ψ ∘ T and T_D are equal up to a non-decreasing reparametrization.

Define using the supremum norm a metric for the d-capacity parametrized curves

\[ d_{\text{curve}}(T_1, T_2) = d_{\text{curve}}((Ω_1, Ψ_1; T_1), (Ω_2, Ψ_2; T_2)) := \|\Psi_1 ∘ T_1 - Ψ_2 ∘ T_2\|_{∞} :\]

**Definition 1.6.** A rooted tree T = (x_0; (T_x)_{x ∈ S}) is pair such that x_0 is a point called root and (T_x)_{x ∈ S} a set of curves starting at x_0 indexed by a set of points S so that x ∈ S is the other endpoint of T_x.

Define a metric for trees as

\[ d_{\text{tree}}(T_1, T_2) = d_{\text{tree}}((Ω_1, Ψ_1; T_1), (Ω_2, Ψ_2; T_2)) := \max \left\{ \sup_{T_1} \inf_{T_2} d_{\text{curve}}(T_1, T_2), \sup_{T_2} \inf_{T_1} d_{\text{curve}}(T_1, T_2) \right\} \]

where T_k runs over all the branches of T_k, for k = 1, 2. This is the familiar Hausdorff metric for bounded closed sets.

**(Metrics for loops and loop ensembles)** Similarly we define metrics for loops and loop ensembles. The difference is that there are no marked points for loops and thus there is no natural starting or ending point and we cannot describe it in a canonical way with Loewner evolutions. Thus it makes sense to define in the following way. Let

\[ d_{\text{loop}}(L_1, L_2) = d_{\text{loop}}((Ω_1, Ψ_1; L_1), (Ω_2, Ψ_2; L_2)) = \inf_{f_1, f_2} \|f_1 - f_2\|_{∞} \]

where f_k runs over all parametrizations of Ψ_k ∘ L_k. The metric d_{L,F} for loop ensembles is defined to be the Hausdorff metric for bounded, closed sets of loops.
1.4. **The statement of the main theorem.** The following theorem is the main theorem of this article establishing the convergence of FK Ising loop ensemble to CLE(16/3).

**Theorem 1.7.** The joint law of the FK Ising loop ensemble in a discrete domain $\Omega_\delta$ and its exploration tree (rooted at $a_\delta$) converges in distribution to the joint law of CLE($\kappa$) and its SLE($\kappa, \kappa - 6$) exploration tree with $\kappa = 16/3$ in the topology described above.

We will develop the tools for its proof in the subsequent sections and present the proof in Section 6.

2. **The discrete holomorphic observable and its scaling limit**

![Figure 6](image.png)

**Figure 6.** A Dobrushin domain has two boundary arcs, one with wired (black) boundary and the other with free (white) boundary.

We call the edges where the boundary conditions change $e_i$ and $e_o$; later $a$ and $b$, respectively. We call a fixed target edge $f$ and its halves $f_o$ and $f_i$; later we will also notation $w$ for the target.

2.1. **The discrete observable.** Let us consider FK Ising on a square lattice with a lattice mesh parameter $\delta > 0$. In that setup, suppose that we are given a Dobrushin domain $\Omega = \Omega_\delta$ with an incoming edge $e_i$ and an outgoing edge $e_o$, see Figure 6 for the definition. Denote the set of directed edges of the medial lattice by $E(\mathbb{L}_\delta \to \mathbb{L}_\delta)$. As usual, a directed edge $e \in E(\mathbb{L}_\delta \to \mathbb{L}_\delta)$ is given by an ordered pair $(e_-, e_+) \in (V(\mathbb{L}_\delta \to \mathbb{L}_\delta))^2$.

We fix an interior edge $f \in E(\mathbb{L}_\delta \to \mathbb{L}_\delta)$ which we split into two halves $f_o$ and $f_i$ which are outgoing and incoming edges, respectively, in the new graph $G$. At first, $(f_o)_+$ and $(f_i)_-$ are not connected by an edge.

**Definition 2.1.** We define two enhanced graphs $G_{e_o \sim e_i}$ and $G_{f_o \sim f_i}$ by adding an edge between $(e_o)_+$ and $(e_i)_-$ or between $(f_o)_+$ and $(f_i)_-$, respectively.

Here $e_1 \sim e_2$ denotes an “external arc”, that is, an edge outside of the graph $G$ that are added, which is a kind of a “boundary condition.” In contrast, $(e_1 \sim e_2, e_3 \sim e_4)$ would be an internal arc configuration, which is a connection pattern in the loop configuration (of the FK loop representation) and which can be interpreted as an event.
Define a function
\[ F = F_{e_0 \rightarrow e_1} : E(\mathbb{L}^d) \ni e \mapsto -\mathbb{E}_{e_0 \rightarrow e_1} \left( 1_{e \in \gamma} e^{-\frac{i}{2} W(e_0,e)} \right) \] (9)
where \( W(e_0,e) \) is the winding along \( \gamma^r \) from \( e_0 \) to \( e \). Here the expected value is taken with respect to the critical FK Ising loop measure on the planar graph \( G_{e_0 \rightarrow e_1} \). The measure is supported on loop configurations with an path \( \gamma \) from \( f_i \) to \( f_0 \) and in the formula (9), \( \gamma^r \) denotes the reversal of \( \gamma \). There are two natural ways to define the winding along the arc from \((e_0)_+\) to \((e_1)_-\) but both choices lead to the same value for \( F \): namely, the difference in \( W(f_0,e) \) is \( \pm 4\pi \) hence \( e^{-\frac{i}{2} W(e_0,e)} \) is well-defined.

Set
\[ \lambda = e^{-\frac{i}{2}} \] (10)
then the observable (9) is given by calculating the number of left and right turns from \( f_0 \) to \( e \) along \( \gamma^r \) and weighting the partition function by \( -\lambda \) signed number of turns.

The “fermionic observable” introduced in the first paper \([32]\) is given by
\[ \tilde{F} = \tilde{F}_{f_0 \rightarrow f_1} : E \ni e \mapsto \mathbb{E}_{f_0 \rightarrow f_1} \left( 1_{e \in \gamma} e^{-\frac{i}{2} W(e_0,e)} \right) \] (11)
where \( W(e_0,e) \) is the winding along \( \gamma^r \) from \( e_0 \) to \( e \), \( \gamma \) is the path from \( e_1 \) to \( e_0 \) on \( G_{f_0 \rightarrow f_1} = G \) and \( \gamma^r \) is the reversal of \( \gamma \). Notice that the difference of (9) and (11) is only in the graph being used.

In addition to the notation \( \Omega_\delta \), let us introduce \( a_\delta \), \( b_\delta \) and \( w_\delta \) for the heads of the edges \( e_1, e_0 \) and \( f_0 \), respectively. Then \((\Omega_\delta, a_\delta, b_\delta, w_\delta)\) is a domain in the complex plane with two marked boundary points and a marked interior point, in that order.

For each pair \((\Omega, w)\) where \( \Omega \) is a simply connected domain (\( \neq \mathbb{C} \)) and \( w \in \Omega \), let \( \Psi_{(\Omega, w)} : \Omega \to \mathbb{D} \) be the unique conformal and onto map satisfying \( \Psi_{(\Omega, w)}(w) = 0 \) and \( \Psi_{(\Omega, w)}'(w) > 0 \).

**Definition 2.2.** We say that \((\Omega_\delta, a_\delta, b_\delta, w_\delta)\) converges to \((\Omega, a, b, w)\), where \( a, b \) can be prime ends (generalized boundary points), in the Carathéodory sense if the sequence of conformal maps \( \Psi_{(\Omega, w)}^{-1} \) converges uniformly on compact subsets of \( \mathbb{D} \) to the conformal map \( \Psi_{(\Omega, w)}^{-1} \) as \( \delta \) tends to zero and in addition \( \lim_{\delta \to 0} \Psi_{(\Omega_\delta, w_\delta)}(a_\delta) = \Psi_{(\Omega, w)}(a) \) and \( \lim_{\delta \to 0} \Psi_{(\Omega_\delta, w_\delta)}(b_\delta) = \Psi_{(\Omega, w)}(b) \). In the last two equations, the values of the right-hand sides exist as boundary points of \( \mathbb{D} \).

For a fixed sequence of domains \((\Omega_\delta, a_\delta, b_\delta, w_\delta)\), denote the observables in \((\Omega_\delta, a_\delta, b_\delta, w_\delta)\) as \( F_\delta \) and \( \tilde{F}_\delta \).

### 2.2. The scaling limit of the observable.
For \( \alpha, \beta \in \mathbb{R}, z \in \mathbb{D} \) and \( u, v \in \partial \mathbb{D} \), define
\[ F_\mathbb{D}(z; u, v) = \sqrt{\frac{1}{z^2} - 1 + i \alpha \frac{1}{z} + \beta \left( \frac{1}{z - u} - \frac{1}{z - v} \right)} \] (12)
where \( \alpha \in \mathbb{R} \) and \( \beta > 0 \). Define also
\[ \tilde{F}_\mathbb{D}(z; u, v) = \pi^{-1} \sqrt{\frac{1}{z - u} - \frac{1}{z - v}} \] (13)

**Remark 2.3.** Consider a function
\[ H_\mathbb{D} = \text{Im} \int F_\mathbb{D}^2 \, dz \] (14)
Figure 7. Boundary values and the pole of $H_D$. 

Then $H_D$ is constant on both arcs $uv$ and $vu$. Let those constant be equal to $\zeta$ and $\xi$, respectively. Here $uv$ is the counterclockwise arc on $\partial D$ from $u$ to $v$ and $vu$ from $v$ to $u$. Then $\xi - \zeta = \beta \pi > 0$. Moreover, from (12) it follows that 

$$H_D(z) = \text{Im} \left( -\frac{1}{z} \right) + O \left( \log \frac{1}{|z|} \right)$$

(15) as $z \to 0$. See also Figure 7.

Without any loss of generality for the scaling limit of the exploration tree or the loop ensemble, we assume that $f$ is a horizontal edge pointing to the east.

**Theorem 2.4.** Suppose that $(\Omega_\delta, a_\delta, b_\delta, w_\delta)$ converges to $(\Omega, a, b, w)$ and let $\Psi : \Omega \to \mathbb{D}$ be the conformal and onto map such that $\Psi(w) = 0$ and $\Psi'(w) > 0$. As $\delta$ tends to zero, $\delta^{-1} F_\delta$ and $\delta^{-1/2} \tilde{F}_\delta$ converge (up to absolute positive multiplicative constants) to the scaling limits $F$ and $\tilde{F}$, respectively, which are uniquely determined by 

$$F(z; a, b, w) = \sqrt{\Psi'(w) \Psi'(z)} F_D(\Psi(z); \Psi(a), \Psi(b))$$

(16)

$$\tilde{F}(z; a, b, w) = \sqrt{\Psi'(z)} \tilde{F}_D(\Psi(z); \Psi(a), \Psi(b))$$

(17)

with $\alpha$ and $\beta$ given in terms of $\arg u < \arg v < \arg u + 2\pi$ where $u = \Psi(a)$ and $u = \Psi(b)$ as 

$$\alpha = 2 \cos \left( \frac{\arg v - \arg u}{2} \right)$$

(18)

$$\beta = 2 \sin \left( \frac{\arg v - \arg u}{2} \right) \cos^2 \left( \frac{\arg u + \arg v + \pi}{4} \right).$$

(19)

The degenerate cases $a = b$ are obtained as limit of the formulas (18) and (19) as $\arg v - \arg u$ tends to $0$ or $2\pi$.

The proof is given in Section 3.2, except the “algebraic part” which is presented next.

**2.3. Determination of the coefficients $\alpha$ and $\beta$.** In this section, we determine the coefficients $\alpha$ and $\beta$ in (12) under a hypothesis (called ($\ast$) or ($\ast'$) below) which we verify in the proof of Theorem 2.4.
2.3.1. Zeros of $Q$ and the coefficients $\alpha$ and $\beta$ in the case $u \neq v$. Write

$$F_D(z) = \sqrt{\frac{Q(z)}{z^2(z-u)(z-v)}}$$

where $F_D$ is as in (12).

In this section we expand $Q$ as

$$Q(z) = (-z^2 + i\alpha z + 1)(z-u)(z-v) + \beta(u-v)z^2$$

$$= -z^4 + (i\alpha + u + v)z^3 + (1 - i\alpha(u+v) - uv + \beta(u-v))z^2$$

$$+ (-u - v + i\alpha u v)z + u v$$

which we compare to another expression later.

Now we claim that

(*) $Q$ has to have two zeros $n$ and $m$, both of multiplicity two, such that one of them lies on the arc $uv$ and the other one on $vu$.

We will verify the claim (*) in the proof of Theorem 2.4 in Section 3.2. Basically it results from the fact that the singularities of $F$ at $a$ and $b$ are of the same type as the singularities of $\tilde{F}$ at $a$ and $b$. Thus we define the coefficient in front of that singularity, say, at $a$ by comparing $F$ to $\tilde{F}$.

The observation (*) makes it possible to determine $\alpha$ and $\beta$ in terms of $u$ and $v$. Let’s write

$$u = e^{i\upsilon}, \quad v = e^{i\phi}, \quad \upsilon < \phi < \upsilon + 2\pi.$$ 

Expand $Q$ as

$$Q(z) = -(z-m)^2(z-n)^2$$

$$= -z^4 + 2(m+n)z^3 - (m^2 + n^2 + 4mn)z^2 + 2mn(m+n)z - m^2n^2 \quad (21)$$

and compare (20) and (21). If we ignore for the time being the coefficient of $z^2$, we have to solve the equation system

$$i\alpha + u + v = 2(m+n) \quad (22)$$

$$i\alpha uv - u - v = 2mn(m+n) \quad (23)$$

$$uv = -m^2n^2 \quad (24)$$

for $\alpha$, $m$ and $n$. Let $\rho \in \mathbb{C}$ be such that $\rho^2 = -uv$ and let’s suppose that

$$mn = \rho \quad (25)$$

Then (24) is satisfied. We resolve the choice of $\rho$ later.

Now by (22) and (23)

$$i\alpha uv - u - v = \rho(i\alpha + u + v)$$

When $w \neq -1$, this gives

$$\alpha = i\rho^{-1}(u + v).$$

Plugging this back in (22) gives

$$m + n = \frac{1}{2}(1 - \rho^{-1})(u + v) =: \mu \quad (26)$$

Then

$$m = \frac{\mu + \sqrt{\mu^2 - 4\rho}}{2}, \quad n = \frac{\mu - \sqrt{\mu^2 - 4\rho}}{2}.$$
It’s not necessary to solve these equations explicitly. It suffices to verify later that

\[ \mu \in \mathbb{R} \sqrt{\rho} \text{ and } \mu^2 - 4\rho \in \mathbb{R} - \rho. \]

Then \(|m| = |n| = 1\) and both arcs \(uv\) and \(vu\) contain one of the points \(m, n\).

Solve next \(\beta\) from the coefficient of \(Q\) and using (25) and (26)

\[
-(u-v)\beta = m^2 + n^2 + 4mn + 1 - i\alpha (u + v) - uw
\]

\[
= \frac{1}{4}(1 - \rho^{-1})(u + v)^2 + 2\rho + 1 + \rho^{-1}(u + v)^2 + \rho^2
\]

\[
= \frac{1}{4}(1 + \rho^{-1})(u + v)^2 + (1 + \rho)^2
\]

\[
= \frac{1}{4}(4 + \rho^{-2}(u + v)^2)(1 + \rho)^2 = -\frac{1}{4}(u-v)^2(1 + \rho)^2
\]

Use the explicit formula \(\rho = \exp(i\upsilon + \phi \pm \pi/2)\) to write that

\[
\beta = \frac{1}{4}\frac{u-v}{uw}(1 + \rho)^2
\]

\[
= \frac{1}{4}e^{-i\frac{\upsilon+\phi}{2}} \cdot [-2i \sin\left(\frac{\phi - v}{2}\right)] \cdot e^{i\frac{\upsilon+\phi+\pi}{2}} \cdot \left[2\cos\left(\frac{v + \phi \pm \pi}{4}\right)\right]^2
\]

Hence \(\beta \geq 0\) only when \(\rho = \exp(i\frac{\upsilon+\phi+\pi}{2})\). Thus we have show the following result.

**Proposition 2.5.** A function of the form (12) satisfies (\(\ast\)) if and only if

\[
\alpha = 2\cos\left(\frac{\phi - v}{2}\right)
\]

\[
\beta = 2\sin\left(\frac{\phi - v}{2}\right) \cos^2\left(\frac{v + \phi + \pi}{4}\right)
\]

where \(v = \arg u\), \(\phi = \arg v\) and they satisfy \(v < \phi < v + 2\pi\).

2.3.2. The special case \(u = v\). If \(u = v\), then \(\beta = 0\). Write

\[
F_D(z) = \sqrt{Q(z) \frac{1}{z^2}}
\]

where

\[
\hat{Q}(z) = -z^2 + i\alpha z + 1
\]  

(27)

A similar claim as (\(\ast\)) states that

(\(\ast'\)) \(Q\) has a zero \(n\) of multiplicity two. When \(\arg u = \arg v\), then

\[
\partial_t H_D > 0 \text{ piecewise everywhere and when } \arg v = \arg u + 2\pi, \text{ then }
\]

\[
\partial_t H_D < 0 \text{ piecewise everywhere.}
\]

We will verify the claim (\(\ast'\)) also in the proof of Theorem 2.4 in Section 3.2. Here \(\partial_t\) is the derivative to the direction of the outer normal at the boundary of \(D\).

Write

\[
\hat{Q}(z) = -(z - n)^2 = -z^2 + 2nz - n^2.
\]  

(28)

Then \(n = \pm i\) and \(\alpha = \pm 2\) by comparing (27) and (28).

Write

\[
H_D(z) = \text{Im} \left(-z - \frac{1}{z} + i\alpha \log z + \text{const.}\right).
\]

Since \(\partial_t = r\partial_r = x\partial_x + y\partial_y = z\partial + \bar{z}\partial\),

\[
\partial_t H_D(z) = \text{Im} \left(-z + \frac{1}{z} + i\alpha\right) = \alpha - 2\sin \arg z
\]
for any \( z \in \partial \mathbb{D} \). When \( \alpha = 2 \), then \( n = i \) and \( \partial_n H_D > 0 \) except at \( z = n \), where as when \( \alpha = -2 \), then \( n = -i \) and \( \partial_n H_D < 0 \) except at \( z = n \). Notice that these are consistent with taking the corresponding limits of the formulas in Proposition 2.5.

3. The scaling limit of the observable

3.1. The operators \( \Delta \) and \( \partial \), Green’s function etc. On a square lattice \( \Gamma \) define the linear operator \( \Delta_1^\Gamma : \mathbb{C}^{V(\Gamma)} \rightarrow \mathbb{C}^{V(\Gamma)} \) called unnormalized discrete Laplace operator as

\[
\Delta_1^\Gamma H(z) = \sum_{w \sim z} H(w) - H(z).
\]

Similarly for define the unnormalized discrete versions of the complex derivatives \( \partial \) and \( \bar{\partial} \) as operators \( \partial_1^\Gamma : \mathbb{C}^{V(\Gamma)} \rightarrow \mathbb{C}^{V(\Gamma^*)} \) and \( \bar{\partial}_1^\Gamma : \mathbb{C}^{V(\Gamma)} \rightarrow \mathbb{C}^{V(\Gamma^*)} \) defined by

\[
\partial_1^\Gamma H(z) = \frac{1}{2} \sum_{w \sim z} \frac{\bar{w} - z}{|w - z|} H(w)
\]

and

\[
\bar{\partial}_1^\Gamma H(z) = \frac{1}{2} \sum_{w \sim z} \frac{w - z}{|w - z|} H(w)
\]

where \( w \sim_{\Gamma^*} z \) means that \( z \) and \( w \) are neighbors on the square lattice with vertices \( V(\Gamma) \cup V(\Gamma^*) \). Notice that \( \partial_1^\Gamma \) and \( \bar{\partial}_1^\Gamma \) map \( \mathbb{C}^{V(\Gamma^*)} \) to \( \mathbb{C}^{V(\Gamma)} \). Usually we drop the upper index \( \Gamma \) and furthermore for instance, denote both \( \partial_1^\Gamma \) and \( \bar{\partial}_1^\Gamma \) by \( \partial_1 \). There is no possibility for confusion since they operate on different spaces.

Let’s apply the above operators in a slightly more concrete situation. Suppose now that \( \Gamma_\bullet \) is a square lattice which is \( \sqrt{2}\mathbb{Z} \) rotated by the angle \( \pi/4 \) and \( \Gamma_\circ \) its dual lattice. Set \( \Gamma \) to be the square lattice with vertices \( V(\Gamma_\bullet) \cup V(\Gamma_\circ) \). Let \( \Delta_{1,\bullet} \) be the Laplace operator of \( \Gamma_\bullet \). It standard to verify the following result.

**Lemma 3.1.** \( \bar{\partial}_1 \partial_1 |_{V(\Gamma_\bullet)} = \partial_1 \bar{\partial}_1 |_{V(\Gamma_\circ)} = \frac{1}{4} \Delta_{1,\bullet} \).

To get operators that correspond to the continuum operators, we define

\[
\Delta_\bullet = \frac{1}{2\delta^2} \Delta_{1,\bullet}, \quad \partial = \frac{1}{\sqrt{2}\delta} \partial_1, \quad \bar{\partial} = \frac{1}{\sqrt{2}\delta} \bar{\partial}_1.
\]

The next result gives the existence of discrete Green’s function. The proof can be found in [17].

**Proposition 3.2.** For each \( z_0 \in V(\Gamma_\bullet) \), there exists function a unique function \( G_{z_0} : \mathbb{C}^{V(\Gamma^*)} \rightarrow \mathbb{C} \) such that \( G_{z_0}(z_0) = 0 \), \( \Delta_{1,\bullet} G_{z_0}(\cdot) = \delta_{\cdot,z_0} \) and \( G_{z_0} \) grows sublinearly at infinity. It satisfies the asymptotic equality

\[
G_{z_0}(z) = \frac{1}{2\pi} \log \left( \frac{|z - z_0|}{\sqrt{2}\delta} \right) + C + \mathcal{O}\left( \frac{\delta^2}{|z - z_0|^2} \right) \quad \text{as } z \rightarrow \infty.
\]

Extend \( G_{z_0} \) holomorphically to \( \Gamma_\circ \), that is, suppose that it satisfies \( \bar{\partial}_1 G_{z_0} = 0 \).

The extension is defined up to an additive constant and well-defined locally, but globally it might be multivalued. However \( C := \partial_1 G_{z_0} \) is well-defined and single-valued globally. It satisfies for \( z \in \Gamma^* \),

\[
\mathcal{C}(z) = \sigma(z) D_\bullet G_{z_0}(z)
\]

where \( D_\bullet \) is the (unnormalized) difference operator along the edge of \( \Gamma_\bullet \) going through the site \( z \) (which is the midpoint of the edge) and \( \sigma(z) \) takes values \( e^{i(\frac{k}{2} + \frac{k}{2})} \),
Next take a square lattice $\Gamma_o$ so that the vertices $V(\Gamma)$ are the midpoints of the horizontal edges of $\Gamma_o$. Then the vertices $V(\Gamma^*)$ are the midpoints of the vertical edges of $\Gamma_o$ and we can define $C$ on the vertical edges to be the value of $C$ on $\Gamma^*$ constructed above. Define $C(z)$ for $z \in V(\Gamma_o)$ to be the sum $C$ on the two vertical edges ending to $z$.

It is straightforward to verify the following result. For the definition spin (strongly) preholomorphic, see [32, 7, 8, 16].

**Proposition 3.3.** The discrete Cauchy kernel $C : V(\Gamma_o) \to \mathbb{C}$ satisfies the following properties.

1. $C$ is spin preholomorphic everywhere except on the edge corresponding to $z_0$ — in the sense that the projections of the values of $C$ at the two endpoints of any edge to the complex line ($\mathbb{R}, i\mathbb{R}, \lambda \mathbb{R}$ or $\lambda^* \mathbb{R}$) corresponding to edge are equal except at $z_0$.

2. At $z_0$,
   \[
   \partial_{\lambda} C(z_0) = \frac{1}{2} \left[ \lambda (C(e_{NE}) - C(e_{SW})) - \lambda (C(e_{NW}) - C(e_{SE})) \right] = \frac{1}{4}
   \]
   where $e_\alpha$ are the vertical edges starting from the endpoints of the horizontal edge whose midpoint $z_0$ is, and $\alpha = NE, NW, SW, SE$ are the directions from $z_0$ to the midpoints of those edges.

3. The asymptotic equality
   \[
   C(z) = \frac{\sqrt{2}}{2\pi} \frac{\delta}{z - z_0} + O\left(\frac{\delta^2}{|z - z_0|^2}\right)
   \]  
   holds as $z \to \infty$.

**Proof.** The first and second claim are straightforward to verify from the definitions.

The third claim follows from the definition of $C$ at a vertex as the sum of the values of the two neighboring vertical edges. At the vertical edges, use (29) and (30).

3.2. Convergence of the observable.

**Proof of Theorem 2.4.** The convergence of $\delta^{-1/2} F_\delta$ to $\hat{F}$ was shown in [32]. Thus we need to only show the convergence of $\delta^{-1} F_\delta$ to $F$.

The key element of the proof the convergence of $\delta^{-1/2} F_\delta$ in [32] was that $\hat{F}_\delta$ is spin preholomorhic or discrete holomorphic. This means $\hat{F}_\delta(z)$, which is defined for $z \in V(\delta \mathbb{L}) \cap \Omega_\delta$ as the sum of the values of $\hat{F}_\delta$ on the two neighboring horizontal edges or equivalently on the two neighboring vertical edges, satisfies a relation on each edge that the projection of the values at the two endpoints of the edge to one of the lines (corresponding to the edge, see Figure 10 in [16]) $\mathbb{R}, i\mathbb{R}, \lambda \mathbb{R}$ or $\lambda^* \mathbb{R}$ are equal. The same argument goes through for $F_\delta$ showing that it is spin holomorphic everywhere except at the edge $f$ whose center $w$ is. At $f$, it fails to be holomorphic by amount $\partial_{\lambda} F_\delta = \frac{1}{2} \left[ \lambda (F_\delta(e_{NE}) - F_\delta(e_{SW})) - \lambda (F_\delta(e_{NW}) - F_\delta(e_{SE})) \right] = \frac{1}{2} [\sqrt{2} - (-\sqrt{2})] = \sqrt{2}$.

Next we make the following assumption

\textbf{(***)} On any compact subset of $\Omega \setminus \{w\}$ the sequence of functions $\{\frac{1}{\delta} F_\delta\}_{\delta > 0}$ is uniformly bounded.
We will later show that the assumption (***) holds.

Take any sequence of compact sets $K_n$ increasing to $\Omega \setminus \{w\}$. Using standard arguments of [8] for spin preholomorphic functions, we can show that $(\frac{1}{\delta} F_\delta)_{\delta>0}$ are equicontinuous on any $K_n$ and hence by a diagonal argument any subsequence of $(\frac{1}{\delta} F_\delta)_{\delta>0}$ contains a subsequence which converges uniformly on any compact subset of $\Omega \setminus \{w\}$. Let $F = \lim_{n \to \infty} \frac{1}{\delta} F_{\delta_n}$.

Let $C_\delta$ be the discrete Cauchy kernel introduced above with the “singularity” at $w$ scaled by $4\sqrt{2}$. Write $\hat{F}_\delta = F_\delta - C_\delta$. Then $\hat{F}_\delta$ can be extended holomorphically to the whole $\Omega_\delta$ including $w$, since $\partial_1 F_\delta$ and $\partial_\delta C$ cancel exactly at $w$. We can assume that $\partial B(w, r) \subset K_n$ for some $n$. If $\frac{1}{\delta} |F_\delta| \leq M$ in $K_n$ and $\frac{1}{\delta} |C_\delta| \leq C$ near $\partial B(w, r)$, then $\frac{1}{\delta} |\hat{F}_\delta| \leq M + C$ near $\partial B(w, r)$. By summing the function $\frac{1}{\delta} |\hat{F}_\delta|$ against the Cauchy kernel, we can extend this estimate to the interior of $B(w, r)$. Therefore $\frac{1}{\delta} \hat{F}_\delta$ remains bounded on any compact subset of $\Omega$ and we can extract a subsequence $\frac{1}{\delta} \hat{F}_{\delta_n}$ that converges on any compact subset of $\Omega$. Thus the subsequence $\frac{1}{\delta} F_{\delta_n}$ converges to a function of the form

$$F(z) = \frac{4}{\pi} \frac{1}{z - w} + \hat{F}(z)$$

where $\hat{F}$ is a holomorphic function on $\Omega$.

By the assumption (***) $\frac{1}{\delta} H_\delta$ is uniformly bounded on compact subsets of $\Omega \setminus \{w\}$ where $H_\delta$ is the discrete version of $\text{Im} \int F_\delta^2 dz$ defined as

$$H_\delta(B) - H_\delta(W) = |F_\delta(e)|^2$$

where $B$ and $W$ are neighboring black and white squares and $e$ is the common edge of the squares. Then $H_\delta$ is approximately discrete harmonic, see [32] Section 3. Hence $\frac{1}{\delta} H_\delta$ is equicontinuous on compact subset by arguments of [8] and we can extract a subsequence that converges uniformly on any of the set $K_n$. We can assume that this sequence is $\delta_n$ chosen above. By (32) and by the fact that there is no monodromy around $w$, the limit of $\frac{1}{\delta_n} H_{\delta_n}$ has to be of the form ($C > 0$ is an absolute constant which we get from (32))

$$H = \lim_{n \to \infty} \frac{1}{\delta_n} H_{\delta_n} = C \left[ \frac{1}{\pi} \text{Im} \left( -\frac{1}{z - w} + i\alpha \log (z - w) \right) + \hat{H}(z) \right]$$

where $\alpha \in \mathbb{R}$ is a constant and $\hat{H}$ is a harmonic function on $\Omega$. Since the boundary conditions of $H_\delta$ where Dirichlet on both boundary arcs say $\zeta_\delta$ and $\xi_\delta$ with $\beta_\delta := \xi_\delta - \zeta_\delta > 0$, the quantities $\frac{1}{\delta_n} \zeta_{\delta_n}$ and $\frac{1}{\delta_n} \xi_{\delta_n}$ must remain bounded. Otherwise $\frac{1}{\delta_n} H_{\delta_n}$ wouldn’t converge. Hence $H$ has piecewise constant boundary values, i.e. it satisfies Dirichlet boundary conditions which can be described in the following way: if $\Psi : \Omega \to \mathbb{D}$ is conformal and onto and such that $\Psi(w) = 0$ and $\Psi'(w) > 0$, then $H = H_D \circ \Psi$ (up to the constant $C$) where

$$H_D = \frac{1}{\pi} \text{Im} \left( -\frac{1}{z} - z + i\alpha \log z - \beta \log \frac{z - u}{z - v} \right) + \text{const.}$$

We have to show that $\alpha \in \mathbb{R}$ and $\beta \geq 0$ are uniquely determined. We do this by showing that the assumptions (*) and (**) of Section 2.3 hold.

First we will observe that $F$ is single valued. This follows directly from the properties $F_\delta$.

---

2Essentially, near that part of the boundary, where the values go to $\pm \infty$ with the quickest rate, the uniform boundedness in compact subsets and the large boundary values are in contradiction.
Next we write \( \beta_\delta = |\tilde{F}_\delta(f)|^2 \). Counting the changes in the number of loops when \( f_\alpha \sim f_i \) is removed from the loop configuration of \( G_{f_\alpha \sim f_i} \) and \( e_\alpha \sim e_i \) is added, shows that \( \beta_\delta = |\tilde{F}_\delta(e_\alpha)|^2 = |F_\delta(e_i)|^2 \). The convergence of \( \tilde{F}_\delta \) shows that \( \delta^{-1/2} \beta_\delta \to \beta \) for some \( \beta > 0 \).

Let \( \varepsilon_\delta = F_\delta(e_\alpha)/\tilde{F}_\delta(e_\alpha) \). Notice that \( \varepsilon_\delta \) is real-valued. Then \( |\varepsilon_\delta|^2 = \beta_\delta \) by the above argument and \( \varepsilon_\delta = -F_\delta(e_i)/\tilde{F}_\delta(e_i) \) by the fact that the two open paths in the loop configuration concatenated with \( f_\alpha \sim f_i \) and \( e_\alpha \sim e_i \) form together a closed simple loop, which thus makes one full \( \pm 2\pi \) turn.

Next we will notice that by using \( \tilde{F}_\delta \), we can define

\[
F_\delta^{(a)} = F_\delta + \varepsilon_\delta \tilde{F}_\delta, \quad F_\delta^{(b)} = F_\delta - \varepsilon_\delta \tilde{F}_\delta
\]

which are discrete holomorphic in \( \Omega_\delta \setminus \{f\} \). The corresponding functions \( H_\delta^{(a)} \) and \( H_\delta^{(b)} \) are approximately discrete harmonic in the same sense as \( H \). The function \( H_\delta^{(a)} \) doesn’t have jump at \( a \) and similarly, \( H_\delta^{(b)} \) doesn’t have jump at \( b \). Therefore after transforming conformally to the unit disc, it follows that the scaling limits \( F_D^{(a)} \) and \( F_D^{(b)} \) can be continued holomorphically to neighborhoods of \( a \) and \( b \), respectively.

Consequently, for \( \varepsilon = \lim_{\delta \to 0} \delta^{-1/2} \varepsilon_\delta \)

\[
F_D(z) = -\varepsilon F_D(z) + \text{holomorphic}, \quad \text{as } z \to u \quad (35)
\]

\[
F_D(z) = +\varepsilon F_D(z) + \text{holomorphic}, \quad \text{as } z \to v \quad (36)
\]

where \( u = \Psi(a) \) and \( v = \Psi(b) \).

Now (12) follows from (34). Write \( F_D = \sqrt{Q(z)/P(z)} \) as we did in Section 2.3. If \( F_D(e^{\theta i}) = \phi(\theta) \tau(\theta)^{-1/2} \), then the only way that the properties (35) and (36) can be satisfied is that \( \phi(\theta) \) is zero for some \( \theta = \theta_1 \in (v, \phi) \) as well as for some \( \theta = \theta_2 \in (\phi, v + 2\pi) \). Here \( v = \arg u \) and \( \phi = \arg v \).

Consequently, it follows that \( Q(z) \) has root of order 2 + 4\( k_1 \) or 2 + 4\( k_2 \) for some integer \( k_j \), \( j = 1, 2 \), at \( m = e^{\theta_1} \) and at \( n = e^{\theta_2} \), respectively. Since \( Q \) is of order 4, \( k_1 = k_2 = 0 \). Thus \( \text{Q}(z) = -(z - m)^2(z - n)^2 \). Thus \( (\ast) \) follows. A similar argument gives \( (\ast') \).

By the calculation of Section 2.3 the values of \( \alpha \) and \( \beta \) are uniquely determined and given by Proposition 2.5. This shows that the limit of \( \frac{1}{\delta_n} H_\delta_n \) is unique along any subsequence such that \( \frac{1}{\delta_n} F_\delta_n \) and \( \frac{1}{\delta_n} H_\delta_n \) converge to \( F \) and \( H \). Since \( F = \sqrt{\psi} \) where \( \psi \) is any holomorphic function with \( H = \text{Im} \psi \), also \( F \) is uniquely determined. Since it holds that a subsequence of any subsequence of \( \frac{1}{\delta_n} F_\delta_n \) converges to this unique \( F_\delta \), the whole sequence converges to \( F \). We have arrived to the claim of the theorem.

It remains to be shown that the assumption \( (\ast) \) holds. Assume on contrary that in a compact subset \( K \) of \( \Omega \setminus \{w\} \) the sequence \( M_n = \sup_K |\frac{1}{\delta_n} F_\delta_n| \) goes to infinity. Since increasing \( K \) only increases \( M_n \) we can assume that \( A(w, r/2, 2r) \subset K \) for some \( r > 0 \). The sequence \( \frac{1}{\delta_n M_n} F_\delta_n \) is uniformly bounded on \( K \) and hence equicontinuous.

Define \( \tilde{F}_\delta = F_\delta - C_\delta \) as we did above. The functions \( \frac{1}{\delta_n M_n} \tilde{F}_\delta_n \) extend holomorphically to \( K := K \cup B(w, r) \) and is uniformly bounded on \( K \) and hence equicontinuous. Take a subsequence, still denoted by \( \delta_n \), such that \( \frac{1}{\delta_n M_n} F_\delta_n \) and \( \frac{1}{\delta_n M_n} \tilde{F}_\delta_n \) converge to some \( F \) and \( \tilde{F} \). The functions \( F \) and \( \tilde{F} \) are holomorphic and \( F \) is not identically zero.

Define functions \( H_\delta \) and \( \tilde{H}_\delta \) similarly as in (33) using the spin preholomorphic functions \( F_\delta \) and \( \tilde{F}_\delta \), respectively. Then \( \frac{1}{\delta_n M_n^2} H_\delta_n \) and \( \frac{1}{\delta_n M_n^2} \tilde{H}_\delta_n \) are uniformly bounded in \( K \) and \( \tilde{K} \) respectively. It follows from the fact that the boundary values...
of $H$ are piecewise constant and that $\frac{1}{\delta_n M_n^2} H_{\delta_n}$ uniformly bounded on $\Omega \setminus B(w, r)$. Hence we can take a subsequence (still denoted by $\delta_n$) such that $\frac{1}{\delta_n M_n^2} H_{\delta_n}$ and $\frac{1}{\delta_n M_n^2} \hat{H}_{\delta_n}$ converge to some function $H$ and $\hat{H}$, respectively. We can assume that the former converges uniformly on any compact subset of $\Omega \setminus \{w, a, b\}$ and the latter on $\hat{K}$, which includes a neighborhood $w$. Now

$$H = \text{Im} \int \lim_{n \to \infty} \frac{1}{\delta_n M_n^2} \left(C + \hat{F}\right)^2 dz = \text{Im} \lim_{n \to \infty} \int \frac{1}{\delta_n M_n^2} \hat{F}^2 dz = H_0$$

on $\hat{K} \setminus \{w\}$. It follows that $H$ extends harmonically to $w$ and hence $H = H_D \circ \Psi$ where $\Psi : \Omega \to \mathbb{D}$ is conformal and

$$H_D(z) = -\beta \text{Im} \log \frac{z-u}{z-v} + \text{const.}$$

To reach a contradiction, we will show that $H_D$ has a critical point somewhere in $\overline{\mathbb{D}} \setminus \{u, v\}$.

Notice that the boundary values of $H_\delta$ are $\zeta, \xi, \eta$ and $\eta + 1$ on the arcs $a_\delta b_\delta$ and $b_\delta a_\delta$ and at the points $w_\delta - i\delta/2$ and $w_\delta + i\delta/2$, respectively. We know that $0 < \xi - \zeta < 1$. Therefore either $\xi < \eta + 1$ or $\eta < \zeta$. Suppose that the former happens. The other case can be dealt with in a similar manner.

By maximum principle for $H_\delta^*$ from any point $z$ there exists a path to the boundary of the domain or to $w_\delta$ such that $H_\delta^*$ is strictly increasing along the path. By the values of the normal derivative on the boundary the path can only hit $b_\delta a_\delta$ or $w_\delta$. Furthermore the points that can be connected to $b_\delta a_\delta$ form a connected set and likewise the points that can be connected to $w_\delta$ and those sets exhaust the whole set of vertices. Define the boundary $I$ between those sets as being the set of edges whose one end is in one of the sets and the other is in the other set. Let $x^*$ be the vertex in $I$ that has the maximal value of $H_\delta^*$ in that set. Since $\eta + 1 > \xi$, the point $x^*$ can’t be close to $w_\delta$ for small $\delta$. By compactness of $\overline{\Omega}$ we can suppose that $x_n^*$ converges as $n \to \infty$ to some point $x^* \in \overline{\Omega} \setminus \{w\}$.

Suppose that $x^* \in \Omega \setminus \{w\}$. Then for any $r > 0$ such that $B(x^*, r) \subset \Omega \setminus \{w\}$ it holds that $H_\delta^* - H_\delta^*(x_n^*)$ changes sign at least 4 times along $\partial B(x^*, r)$. Furthermore the angles at which the peaks and valleys appear are bounded away from each other. By convergence of $\frac{1}{\delta_n M_n^2} H_{\delta_n}$ to $H$ to this continuous to hold for $x^*$. Since $H = \text{Im} \Psi$ for some holomorphic $\Psi$ in the neighborhood, we find that $\Psi'(x^*) = 0$ and $x^*$ is a saddle point for $H$. This is a contradiction.

A similar conclusion on a contradiction can be made for $x^* \in \partial \Omega$. Thus (***) holds.

Finally, we need still the following result on the convergence of the observables which can be extracted from the above proof.

**Corollary 3.4** (Uniform convergence of observables over a class of domains). The convergence in Theorem 2.4 is uniform with respect to $(\Omega; a, b, w)$ whenever $B(w, r) \subset \Omega \subset B(w, R)$ and rate of convergence in the Carathéodory convergence is uniform.

4. **A priori bounds for exploration trees and loop ensembles**

In this section, we present some results which we classify as a priori results. They describe properties that ensure the regularity of branches, trees and loops in a manner that is needed for their convergence.
4.1. One-to-one correspondence of the tree and the loop-ensemble in the limit. In the discrete setting we are given a tree–loop ensemble pair. The tree and the loop ensemble are in one-to-one correspondence as explained earlier. Recall that,

- given the loop ensemble, the tree is constructed by the exploration process which follows the loops and jumps to the next loop at points where the followed loop turns away from the target point.
- the loops are recovered from the tree by noticing that each loop corresponds to a small square where the branching to that loop occurs (this correspondence is 1-to-1, when we also count the root as one of the branching points).

We take the incoming edge, which is opposite to the other incoming edge that we used to arrive to the small square for the first time, and select the branch corresponding to that target edge. The loop is constructed from the branch by taking the part between the first exit and last arrival and then adding to the path the edge of the small square that closes it to a loop.

We will show that the probability laws that we are considering form a precompact set in the topology of weak convergence of probability measures. Take a subsequence of the sequence of the tree–loop ensemble pairs that converges weakly. We can choose a probability space so that they converge almost surely. Next theorem summarizes the convergence of the tree–loop ensemble pair. The second assertion basically means that there is a way to reconstruct the loops from the tree also in the limit.

**Theorem 4.1.** Let \( (\Theta^D, T^D) \) be the almost sure limit of \( (\Theta^D_n, T^D_n) \) as \( n \to \infty \). Write \( \Theta^D = (\theta_j)_{j \in J} \) and \( \Theta^D_n = (\theta_{n,j})_{j \in J} \) (with possible repetitions) such that almost surely for all \( j \in J \), \( \theta_{n,j} \) converges to \( \theta_j \) as \( n \to \infty \), and then set \( x_{n,j} \) to be the target point of the branch of \( T^D_{n,j} \) that corresponds to \( \theta_{n,j} \) in the above bijection (described in the beginning of Section 4.1). Then

- Almost surely, the point \( x_{n,j} \) converge to some point \( x_j \) as \( n \to \infty \) and the branch \( T^{x_{n,j}}_+ \) converge to some branch denoted by \( T^{x_j}_+ \) as \( n \to \infty \) for all \( j \).
- Furthermore, the points \( x_j \) that correspond to non-trivial loops (\( \theta_j \) is not a point) are distinct and they form a dense subset of \( \overline{D} \) and \( \overline{T} \) is the closure of \( (T^{x_j}_+)_{j \in J} \).

- On the other hand, \( (T^{x_j}_+)_{j \in J} \) is characterized as being the subset of \( T \) that contains all the branches of \( T \) that have a triplepoint in the bulk or a double-point on the boundary. Furthermore, that double or triple point is unique and it is the target point (that is, endpoint) of that branch. Any loop \( \theta_j \) can be reconstructed from \( (T^{x_j}_+)_{j \in J} \) so that the loop \( \theta_j \) is the part between the second last and last visit to \( x_j^+ \) by \( T^{x_j}_+ \).

The proof is postponed to Section 6.

4.2. \( n \)-arms bounds for the tree.

4.2.1. The \( n \)-arms bound and consequences. Let’s start this subsection by stating a result which follows from the results of [1, 15, 16]. The assumptions, Condition G1 and G2, are presented in the next subsection.

Recall the general setup: we are given a collection \( (\psi, \mathbb{P}) \in \Sigma \) where the conformal map \( \psi \) contains also the information about its domain of definition \( (\Omega, v_{\text{root}}, z_0) = (\Omega(\psi), v_{\text{root}}(\psi), z_0(\psi)) \) through the requirements

\[
\psi^{-1}(\mathbb{D}) = \Omega, \quad \psi(v_{\text{root}}) = -1 \quad \text{and} \quad \psi(z_0) = 0
\]
and $\mathbb{P}$ is a probability measure on the space of trees on the domain $\Omega$ with the fixed root $v_{\text{root}}$. Furthermore, we assume that, if $\mathcal{T}$ is distributed according to $\mathbb{P}$, each $T \in \mathcal{T}$ has some suitable parametrization.

Given a collection $\Sigma$ of pairs $(\psi, \mathbb{P})$ we define the collection $\Sigma_D = \{ \psi \mathbb{P} : (\psi, \mathbb{P}) \in \Sigma \}$ where $\psi \mathbb{P}$ is the pushforward measure defined by $(\psi \mathbb{P})(E) = \mathbb{P}(\psi^{-1}(E))$.

The following result is the prototype of a priori result which we consider and thus stated here early on. We call a random variable $X$ tight over a collection of probability measures $\mathbb{P}$ on the probability space, if for each $\varepsilon > 0$ there exists a constant $M > 0$ such that $\mathbb{P}(|X| < M) > 1 - \varepsilon$ for all $\mathbb{P}$. Remember that a crossing of an annulus $A(z_0, r, R) = \{ z \in \mathbb{C} : r < |z - z_0| < R \}$ is a segment of a curve that intersects both connected components of $\mathbb{C} \setminus A(z_0, r, R)$.

**Theorem 4.2.** Suppose that a collection $\Sigma$ of pairs $(\psi, \mathbb{P})$ satisfies Condition G1 or G2 (stated below explicitly). Then the following claim holds

- for any $\Delta > 0$, there exists $n \in \mathbb{N}$ and $K > 0$ such that the following holds

$$
\mathbb{P}(\text{at least } n \text{ disjoint segments of } \mathcal{T} \text{ cross } A(z_0, r, R)) \leq K \left( \frac{r}{R} \right)^\Delta
$$

for all $\mathbb{P} \in \Sigma_D$.

and there exists a positive number $\alpha, \alpha' > 0$ such that the following claims hold

- if for each $r > 0$, $M_r$ is the minimum of all $m$ such that each $T \in \mathcal{T}$ can be split into $m$ segments of diameter less or equal to $r$, then there exists a random variable $K(T)$ such that $K$ is a tight random variable for the family $\Sigma_D$ and $M_r \leq K(T) r^{-\frac{2}{\alpha}}$ for all $r > 0$.

- All branches of $\mathcal{T}$ can be jointly parametrized so that they are all $\alpha'$-H"older continuous and the H"older norm can be bounded by a random variable $K'(T)$ such that $K'$ is a tight random variable for the family $\Sigma_D$.

In particular, these conclusions hold for the FK Ising exploration tree.

The theorem highlights the importance of probability bounds on multiple crossings of annuli. Each of the claims have their own applications below although they are closely related, see [1].

4.2.2. The annulus crossing property. In this subsection, we state the assumptions (Condition G1 and G2) used in the previous subsection. They are elementary probability bounds on multiple crossings. We essentially repeat here the definition and results from [16].

To present the condition in a slightly more abstract setting, we replace the random tree $\mathcal{T} = (T_x)$ by a more abstract collection of curves which we assume to satisfy the essential requirements of $(T_x)$. Suppose $\gamma_k, k = 0, \ldots, N - 1$, is a finite collection of simple (random) curves and that they are each parametrized by $[0, 1]$. The chosen permutation of these objects specifies the order of exploration of these curves. More specifically, set $\gamma(t) = \gamma_k(t - k)$ when $t \in [k, k + 1)$.

For a given domain $\Omega$ and for a given simple (random) curve $\gamma$ on $\Omega$, we always set $\Omega_\tau = \Omega \setminus \gamma[0, \tau]$ for each (random) time $\tau$. Similarly, for a given domain $\Omega$ and for a given finite collection of simple (random) curves $\gamma_k$ on $\Omega$, we always set $\Omega_{\tau_k} = \Omega \setminus \gamma[0, \tau]$ for each (random) time $\tau$. We call $\Omega_\tau$ or $\Omega_{\tau_k}$ the domain at time $\tau$.

The following definition generalizes Definition 2.3 from [15].
Definition 4.3. For a fixed domain $(\Omega, v_{\text{root}})$ and for fixed explored collection $\gamma$ of curves $\gamma_x, x \in V_{\text{target}}$, where each curve $\gamma_x$ is contained in $\Omega$, starting from $v_{\text{root}}$ and ending at a point $x$ in the set $V_{\text{target}}$, define for any annulus $A = A(z_0, r, R)$, for every (random) time $\tau \in [0, N]$ and $x \in V_{\text{target}}$, $A_{\tau}^{u,x} = \emptyset$ if $\partial B(z_0, r) \cap \partial A_{\tau} = \emptyset$ and
\[
A_{\tau}^{u,x} = \left\{ z \in \Omega_{\tau} \cap A : \text{the connected component of } z \text{ in } \Omega_{\tau} \cap A \text{ doesn't disconnect } \gamma(\tau) \text{ from } x \text{ in } \Omega_{\tau} \right\}
\]
otherwise. Define also
\[
A_{\tau}^{f,x} = \left\{ z \in \Omega_{\tau} \cap A : \text{the connected component of } z \text{ in } \Omega_{\tau} \cap A \text{ is crossed by any path connecting } \gamma(\tau) \text{ to } x \text{ in } \Omega_{\tau} \right\}
\]
and set $A_{\tau}^{u} = \bigcap_{x \in V_{\text{target}}} A_{\tau}^{u,x}$ and $A_{\tau}^{f} = \bigcup_{x \in V_{\text{target}}} A_{\tau}^{f,x}$. We say that $A_{\tau}^{u,x}$ is avoidable for $\gamma_x$ and $A_{\tau}^{u}$ is avoidable for all (branches). We say that $A_{\tau}^{f,x}$ is unavoidable for $\gamma_x$ and $A_{\tau}^{f}$ is unavoidable for at least one (branch). Here and in what follows we only consider allowed lattice paths when we talk about connectedness.

Condition G1. Let $\Sigma$ be as above. If there exists $C > 1$ such that for any $(\psi, P) \in \Sigma$, for any stopping time $0 \leq \tau \leq N$ and for any annulus $A = A(z_0, r, R)$ where $0 < Cr \leq R$, it holds that
\[
P \left( \frac{\gamma[\tau, N] \text{ makes a crossing of } A \text{ which is contained in } A_{\tau}^{u} \text{ or which is contained in } A_{\tau}^{f} \text{ and the first minimal crossing doesn't have branching points on both sides}}{F_\tau} \right) < \frac{1}{2} \tag{37}
\]
then the family $\Sigma$ is said to satisfy a geometric joint unforced–forced crossing bound Call the event above $E^{u,f}$.

See Figure 8 for more information about different types of branching points.

Condition G2. The family $\Sigma$ is said to satisfy a geometric joint unforced–forced crossing power-law bound if there exist $K > 0$ and $\Delta > 0$ such that for any $(\psi, P) \in \Sigma$, for any stopping time $0 \leq \tau \leq N$ and for any annulus $A = A(z_0, r, R)$ where $0 < r \leq R$,
\[
\text{LHS} \leq K \left( \frac{r}{R} \right)^\Delta.
\]
Here LHS is the left-hand side of (37).

Condition G1 and G2 explicitly verified in [16] in generality covering the setup of the current article. And thus we have the following, see [16].

Theorem 4.4. If $\Sigma$ is the collection of pairs $(\phi, P)$ where $\phi$ satisfies the above properties and also that $U(\phi)$ is a discrete domain with some lattice mesh, and $P$ is the law of the critical FK Ising model exploration tree $T$ on $U(\phi)$, then $\Sigma$ satisfies Conditions G1 and G2.

As shown in [15], this type of bounds behave well under conformal maps. We have uniform control on how the constants in Conditions G1 and G2 change if we transform the random objects conformally from one domain to another.

Theorem 4.5. If $\Sigma$ is as in Theorem 4.4, then $\Sigma_D$ satisfies Conditions G1 and G2.

4.3. A priori bounds for a branch. In this subsection, we study precompactness of family of laws of a single branch. The topology is given by the uniform convergence of capacity-parametrized curves.
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\[ \gamma(\tau) \]

\[ \gamma(\tau) \]

(a) Crossing with branching points on both sides

(b) Crossing without any branching points

(c) Crossing with branching points only on its left

(d) Crossing with branching points only on its right

Figure 8. Condition G1 or G2 imply that the crossing events of any of the types illustrated in the figures (b), (c) and (d) has small probability. The longer black arrow is the crossing event considered in (37) and the shorter grey arrows are the crossings of the annulus that are still possible afterwards.

4.3.1. The main lemma for a radial curve. For similar results as presented in this subsection, see Proposition 6.4 in [13] or Appendix A.2 in [15].

Let \( \gamma : [0, \infty) \to \mathbb{C} \) be a simple curve such that \( |\gamma(0)| = 1, 0 < |\gamma(t)| < 1 \) for all \( t \in (0, \infty) \) and \( \lim_{t \to \infty} \gamma(t) = 0 \). Assume also that \( \gamma \) is parametrized by the d-capacity, which can always be done in this case. Denote the driving process of \( \gamma \) by \( U \). To emphasize the driving process, let’s use the notation \( \gamma_U = \gamma \) as well as \( f_U(t, z) = g_t^{-1}(z) \).

Define for \( \varepsilon \in (0, 1) \)
\[
F_U(t, \varepsilon) = f_U(t, (1 - \varepsilon)U_t).
\] (38)

When the Loewner chain is generated by a curve \( \gamma_U \), \( F_U \) extends to a continuous function on \( [0, \infty) \times [0, 1) \). Consequently, \( \lim_{\varepsilon \to 0} \sup_{t \in [0,T]} |F_U(t, \varepsilon) - \gamma_U(t)| = 0 \) for all \( T > 0 \). To get an uniform property of this type, we look at curves \( \gamma_U \) that satisfy
\[
\sup_{t \in [0,T]} |F_U(t, \varepsilon) - \gamma_U(t)| \leq \lambda(\varepsilon)
\] (39)

where \( \lambda : (0, 1) \to (0, \infty) \) is a function such that \( \lim_{\varepsilon \to 0} \lambda(\varepsilon) = 0 \). It is natural to define for any such function \( \lambda \) and any \( T > 0 \),
\[ \mathcal{E}_{\lambda,T} = \{ U : \exists \gamma_U \text{ as above and } (39) \text{ holds } \forall \varepsilon \in (0, 1) \} \).

Notice that \( \varepsilon \mapsto F_U(t, \varepsilon) \) is the so called hyperbolic geodesic between 0 and \( \gamma_U(t) \) in the domain \( \mathbb{D} \setminus \gamma_U(0, t] \).
In next subsections, we apply the following lemma to a branch in the FK Ising exploration tree.

**Lemma 4.6.** Let \( \lambda : (0, 1) \to (0, \infty) \) be a function such that \( \lim_{\varepsilon \to 0} \lambda(\varepsilon) = 0 \). The map from \( U \in \mathcal{E}_{\lambda,T} \) to \( \gamma_U \) is uniformly continuous. More specifically, for each \( T > 0 \) and \( \varepsilon \in (0, 1) \), there exists a constant \( C = C(T, \varepsilon) \) such that if \( \lambda \) is as above and \( U_k \in \mathcal{E}_{\lambda,T} \) for \( k = 1, 2 \), then

\[
\| \gamma_{U_1} - \gamma_{U_2} \|_{\infty, [0, T]} \leq C(T, \varepsilon) \| U_1 - U_2 \|_{\infty, [0, T]} + 2\lambda(\varepsilon). \tag{40}
\]

**Proof.** Fix \( T > 0 \). It is fairly straightforward to show that there is a constant \( C(T, \varepsilon) \) such that for any \( z_k \in \mathbb{C} \) such that \( |z_k| \leq 1 - \varepsilon \) and for any continuous \( U_k : [0, \infty) \to \partial \mathbb{D} \) it holds that

\[
\sup_{t \in [0,T]} |f_{U_1}(t, z_1) - f_{U_2}(t, z_2)| \leq \frac{C(T, \varepsilon)}{2} (\| U_1 - U_2 \|_{\infty, [0,T]} + |z_1 - z_2|). \tag{41}
\]

For instance, this can be derived using the same route as Proposition 6.1 of [13]. Namely, first use a version of Lemma 5.6 of [13] to relate \( f_k(t, z) \) to a time-reversal of the (direct) Loewner flow \( g(t, z) \) for a specific driving term. Then use an argument similar to Lemma 6.1 of [13] to estimate the difference of the solutions of the time-reversed Loewner equation for the two driving terms and for two initial values. We leave the details to the reader.

Let \( z_k = (1 - \varepsilon)U_1(t), k = 1, 2 \). Then \( |z_1 - z_2| \leq \| U_1 - U_2 \|_{\infty, [0, T]} \) and it follows from (41) that

\[
|F_{U_1}(t, \varepsilon) - F_{U_2}(t, \varepsilon)| \leq C(T, \varepsilon) \| U_1 - U_2 \|_{\infty, [0,T]} \tag{42}
\]

for all \( t \in [0, T] \).

Next use the assumption that \( U_1, U_2 \in \mathcal{E}_{\lambda,T} \). Then by the triangle inequality, the inequality (40) follows.

Let’s finalize the proof by showing that the uniform continuity of the map \( U \mapsto \gamma_U \). For any \( \varepsilon > 0 \), choose \( \varepsilon > 0 \) such that \( \lambda(\varepsilon) \leq \varepsilon / 3 \). Then choose \( \delta > 0 \) such that \( C(T, \varepsilon) \delta \leq \varepsilon / 3 \). It follows from (40) that for any \( U_1, U_2 \in \mathcal{E}_{\lambda,T} \) such that \( \| U_1 - U_2 \|_{\infty, [0, T]} < \delta \), it holds that \( \| \gamma_{U_1} - \gamma_{U_2} \|_{\infty, [0, T]} < C(T, \varepsilon) \delta + 2\lambda(\varepsilon) < \varepsilon \). \[ \square \]

**4.3.2. Uniform probability bound on the modulus of continuity of the driving process.**

As demonstrated in [15], a probability bound on the modulus of continuity of the Loewner driving term follows from the probability bound on annulus crossing by considering crossings of thin conformal rectangles along the boundary of the domain. The argument in [15] is written for the Loewner equation in the upper half-plane, but the argument adapts directly to the Loewner equation in \( \mathbb{D} \). For instance, from the Loewner equation in \( \mathbb{D} \), we can deduce that there is a constant \( c > 0 \) such that \( |\gamma(t)| \geq 1 - ct \) and consequently, \( \max_{s \in [0,t]} |\arg U_s - \arg U_0| \geq L > 0 \) where \( L \) and \( t \) are small and \( L \) is much greater than \( t \), only if \( \gamma \) exits \( \{ z : |z| > 1 - ct, \arg z - \arg U_0 \geq L/2 \} \) from the “sides”. Consequently the following theorem holds. For the original result, see [15], Section 3.3.

**Theorem 4.7.** Let \( \beta \in (0, \frac{1}{2}) \). For any branch of \( T \) with the target point in a compact subset of \( \mathbb{D} \) and with the \( d \)-capacity parametrization, the driving process is \( \beta \)-Hölder continuous and the Hölder norm is a tight random variable for the family \( \Sigma_{\mathbb{D}} \).
4.3.3. Uniform probability bound on the modulus of continuity of the hyperbolic geodesic. Similarly as in the previous subsection, we can adapt the theory in [15] to the case of the Loewner equation of $\mathbb{D}$ to deduce the following result.

**Theorem 4.8.** For any $T > 0$, there exists a function $\lambda : (0, 1) \to (0, \infty)$ such that $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = 0$ and the following holds. Any branch of $\mathcal{T}$ with the target point in a compact subset of $\mathbb{D}$ (39) holds for $T > 0$ and $0 < \varepsilon < \varepsilon_0$ where $\varepsilon_0$ is a tight random variable for the family $\Sigma_{\mathbb{D}}$.

Let’s stress here that this result is proven using the general $n$-arms bounds — and the implied bound for the tortuosity (Theorem 4.2 and, in particular, its second assertion) — and the more specific 6-arms bound. See [15], Sections 3.2 and 3.4.

The next result is the main theorem among the a priori bounds for a branch.

**Theorem 4.9** (Tightness of a single branch in the uniform convergence in the $d$-capacity parametrization). For each $\varepsilon > 0$, there exists a compact subset $K$ of the space $C([0, \infty))$ such that the following holds. Any branch of $\mathcal{T}$ with the target point in a compact subset of $\mathbb{D}$ with the $d$-capacity parametrization is contained in $K$ with probability at least $1 - \varepsilon$ uniformly for the family $\Sigma_{\mathbb{D}}$.

**Proof.** Let $D_1 \subset \mathbb{D}$ be compact and $\eta > 0$ such that $\text{dist}(D_1, \partial \mathbb{D}) \geq 2\eta$. For any $x \in D_1$, let $\psi_x$ be the conformal and onto selfmap of the unit disc such that $\psi_x(x) = 0$. Then $\psi_x$ and $\psi_x^{-1}$ are Lipschitz continuous on $\overline{\mathbb{D}}$ with a uniform Lipschitz norm over all $x$ by a direct calculation. Thus we can in fact assume that $x = 0$ and consider only the branches from the root (which can be set to be $-1$) to the target $x = 0$.

For each $n$, let $K_n$ be the set of simple curves $\gamma$ going from the root to the target and parametrized with the $d$-capacity such that for some $\alpha_0 > 0$, $\beta \in (0, \frac{1}{2})$, $C_n$, $\lambda_n : (0, 1) \to (0, \infty)$ such that $\lim_{n \to 0} \lambda_n(u) = 0$ and $v_n$, the following holds

- $\gamma([n, \infty)) \subset B(0, e^{-\alpha_0n})$
- $\gamma(t), t \in [0, n]$, satisfies that its driving term is $\beta$-Hölder continuous and the Hölder norm is at most $C_n$
- the bound (39), where $\lambda$ is replaced by $\lambda_0$ and $\varepsilon$ is replaced by $u$, holds for $\lambda, t \in [0, n]$ and $u \in (v_n)$

and it holds that $\mathbb{P}(K_n) \geq 1 - 2^{-n\varepsilon}$. Such $\alpha_0 > 0$, $\beta$, $C_n$, $\lambda_n$ and $v_n$ exist for $n \geq n_0$ for some $n_0 \in \mathbb{N}$ by using Condition G2, Theorem 4.7 and Theorem 4.8 for the three claims, respectively, and them bounding the probability of the intersection of three events from below by 1 minus the sum of the probabilities of their complementary events.

Let $K_\infty = \bigcap_{n \geq n_0} K_n$. Then $\mathbb{P}(K_n) \geq 1 - \varepsilon$. Let $K = \overline{K_\infty}$ where the closure is on the uniform convergence of the $d$-capacity-parametrized curves on compact subintervals of $[0, \infty)$. Using Lemma 4.6 it is straightforward to check that $K$ is sequentially compact and thus compact. \(\square\)

4.4. **A priori bounds for trees.** It is straightforward to extend the convergence of a single branch to that of many, but fixed number of, branches. Below we present an argument which shows that a tree with fixed, large number of branches is close to the full tree with uniformly high probability.

Let $I_\eta = (\eta\mathbb{Z}^2) \cap \mathbb{D}$. For each $x \in I_\eta$, choose a point $z_x \in \Omega_\delta \cap \delta V_{\text{mid}}$ such that $\psi(z_x) \in B(x, \eta)$. This can be done using the following lemma.

**Lemma 4.10.** For each $\eta, r, R$, there exists $\delta_0 > 0$ such that the following holds. The set $\psi(\Omega_\delta \cap \delta V_{\text{mid}}) \cap B(x, \eta)$ is non-empty when $\delta \in (0, \delta_0)$ and $B(0, r) \subset \Omega_\delta \subset B(0, R)$. 

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For the finite tree approximation of the full tree, we use the following scheme to select the set of target points:

- include all the set of points \( z_x \) where \( x \) runs over all points of \( \tilde{I}_\eta \)
- if some components with diameter greater than \( \eta \) still exists in the complement of the tree in \( \mathbb{D} \) (notice the conformal image of the tree), include a target point in all those components (more accurately, cut that component into four equal quarters in the direction of the diameter. Then select the target point in one of the non-neighboring quarters to the quarter of the branching point to that component.) Repeat this second step until the maximal diameter of the domains to be explored is less than \( \eta \).

Denote the chosen set of target points as \( I_{\delta,\eta} \subset \Omega_\delta \). Let \( T_\delta \) be the (full) exploration tree on \( \Omega_\delta \) and let \( T_\delta^D = \psi^{-1}(T_\delta) \). Define \( \tilde{T}_\delta(I_{\delta,\eta}) \) and \( \tilde{T}_\delta^D(I_{\delta,\eta}) \) be the restrictions of \( T_\delta \) and \( T_\delta^D \) to the branches of \( I_{\delta,\eta} \).

It follows directly that

\[
d_{\text{tree}} \left( \tilde{T}_\delta^D(I_{\delta,\eta}), T_\delta^D \right) < \eta.
\]

It remains to show the following result.

**Theorem 4.11.** For each \( \eta > 0 \) and \( \varepsilon > 0 \), there exists a constant \( M > 0 \) such that

\[
\sup_{\mathbb{P}_\delta} \mathbb{P}_\delta [ \#(I_{\delta,\eta}) > M ] < \varepsilon.
\]

Notice that way the sequence of points was constructed implies that all the conclusions of we made in this section for fixed target points hold also for these random target points.

**Proof of Lemma 4.10.** Suppose first that \( |x| < 1 - \eta/4 \). Then by Koebe distortion theorem, there exists a uniform constant \( \varepsilon > 0 \) such that \( B(\psi(x), \varepsilon) \subset \psi(B(x, \eta/8)) \). Thus the claim holds for those \( x \)'s when \( \delta_0 < \varepsilon/2 \).

Suppose then that \( |x| \geq 1 - \eta/4 \). Let \( J = B(x, \eta) \cap \partial \mathbb{D} \). If the length \( \psi^{-1}(J) \) is greater than \( \delta \), then \( \psi^{-1}(J) \) contains at least one lattice point (which is on the boundary) and the claim follows. If \( \psi(J) \) shorter than \( \delta \) and it doesn’t contain any lattice points, then the endpoints are on the same edge of the lattice. Take one of its endpoints. It follows easily that the diameter of the image of the line segment connecting that endpoint to the closest end of the edge under the map \( \psi \) is at most \( 2\pi/\sqrt{\log \delta^{-1}} \) using an estimate on the length distortion of conformal maps such as Proposition 2.2 in [21].

**Proof of Theorem 4.11.** Use an argument similar to the proof of Theorem 5 of [4]. Namely notice that each time we select a new target point in the above scheme we create a segment of the exploration tree with diameter greater than \( \eta/4 \). Furthermore, they are all disjoint. Since there is \( n \) such that there is no \( n \)-arms event of the tree between scales \( \tilde{\varepsilon} \) and \( \eta/16 \) by the results of Section 4.2 where \( \tilde{\varepsilon}^{-1} \) is a tight random variable, it follows that we need only at most \( n\tilde{\varepsilon}^{-2} \) points in the above construction. This quantity is tight and thus with uniform high probability less than a given large number.

**4.5. A priori bounds for loop ensembles.** Any loop in \( \Theta \) can be constructed from the tree by the reverse construction presented in Section 4.1. By construction, any loop is a subpath of the corresponding branch of the exploration tree. Thus Theorem 4.2 implies the following result.
Theorem 4.12. The family of probability laws of $\Theta$ is tight in the metric space of loop collections. More specifically, all loops in $\Theta$ can be jointly parametrized so that they form a Hölder continuous family with a uniform Hölder norm which is a tight random variable over the family of probability laws.

5. Determining the law of a branch from the observable

5.1. Simple martingales from $F$. Recall that the value of $F_\delta(z)$ is a discrete-time martingale as a process in time variable $n$ when $\Omega_\delta$ is replaced by $\Omega_\delta \setminus \gamma(0,n]$ where $\gamma$ is the branch of $w$ with the lattice step parametrization. By the uniform convergence of Corollary 3.4, it follows that $F(z)$ is a a continuous-time martingale as a process in time variable $t$ when $\Omega$ is replaced by $\Omega \setminus \gamma(0,t]$ where $\gamma$ is the (subsequent) scaling limit of the branch to $w$. For the proof of this type of statement, see the proof of Theorem 1 in [5].

To be able to benefit from the martingale property, we search for simple expressions that we can extract from $F$ which are martingales. See also Proposition 5.1 in [16].

5.1.1. Expansion of the observable of the branch. Set $\alpha(t)$ and $\beta(t)$ to be the coefficients of the observable defined as in the Proposition 2.5 when $v = \Upsilon_t$ and $\phi = \Phi_t$.

Let’s use the expansion of the Loewner map of $\mathbb{D}$
\[ g_t(z) = e^t(z + c(t) z^2 + \ldots) \]
\[ g'_t(z) = e^t(1 + 2c(t) z + \ldots). \]

Notice that there is a big simplification in the expression
\[ \frac{g'_t(0) g'_t(z)}{g_t(z)^2} = \frac{1}{z^2} \frac{1 + 2c(t)z + \ldots}{1 + 2c(t)z + \ldots} = \frac{1}{z^2} (1 + O(z^2)) \]
which doesn’t contain any $z^{-1}$ term.

The expansion of the observable around the origin is
\[ \sqrt{g'_t(0) g'_t(z)} F_\mathbb{D} (g_t(z); e^{i \Upsilon_t}, e^{i \Phi_t}) = \frac{1}{z} (1 + O(z^2)). \]
\[ \cdot \sqrt{1 + i \alpha(t) g_t(z) - g_t(z)^2 - \beta(t) g_t(z)^2 \left( \frac{1}{g_t(z) - e^{i \Upsilon_t}} - \frac{1}{g_t(z) - e^{i \Phi_t}} \right)} \]
\[ = \frac{1}{z} (1 + e^t \alpha(t) z + O(z^2)). \]

The first non-trivial coefficient is
\[ M(t) := e^t \cos \left( \frac{\Phi_t - \Upsilon_t}{2} \right). \]

By the martingale property of the observable, the process $(M(t))_{t \geq 0}$ is a martingale.

5.1.2. Value of the “chordal” observable at $w$. The leading coefficient of
\[ \text{Re} \left( \sqrt{g'_t(z)} F_\mathbb{D} (g_t(z); e^{i \Upsilon_t}, e^{i \Phi_t}) \right) = e^t \sqrt{\beta(t)} + O(|z|) \]
is
\[ N(t) = \pm e^t \sqrt{\sin \left( \frac{\Phi_t - \Upsilon_t}{2} \right)} \cos \left( \frac{\Upsilon_t + \Phi_t + \pi}{4} \right). \]

By the martingale property of the observable, the process $(N(t))_{t \geq 0}$, when $\pm$ are symmetrically distributed random signs independently sampled on each excursion of $\Phi_t - \Upsilon_t \geq 0$, is a martingale.
5.2. Solution of the martingale problem. Next we will show that the fact that $M(t)$ and $N(t)$ are martingales implies that the law of the scaling limit of a single branch is SLE($\frac{16}{3}, -\frac{2}{3}$).

Remember that for SLE($\kappa, \rho$)
\[
d\upsilon_t = \sqrt{\kappa} dB_t - \frac{\rho}{2} \cot \left( \frac{\phi_t - \upsilon_t}{2} \right) dt
\]

(43)
\[
\dot{\phi}_t = \cot \left( \frac{\phi_t - \upsilon_t}{2} \right).
\]

(44)

The process $U_t = e^{i\upsilon_t}$ is the driving process of Loewner equation of $D$ and $V_t = e^{i\phi_t}$ is the other marked point. Notice that then
\[
d(\phi_t - \upsilon_t) = -\sqrt{\kappa} dB_t + \frac{\rho + 2}{2} \cot \left( \frac{\phi_t - \upsilon_t}{2} \right) dt
\]

which we call the stochastic differential equation of a (unnormalized) radial Bessel process with parameters $\kappa$ and $\rho$. Notice that by a time change we can get rid of one of the parameters.

5.2.1. Solution of the martingale problem. Start from the processes
\[
M(t) = e^t \cos \left( \frac{\Phi_t - \Upsilon_t}{2} \right)
\]
\[
N(t) = \pm e^{\frac{\omega}{2}} \sqrt{\sin \left( \frac{\Phi_t - \Upsilon_t}{2} \right)} \cos \left( \frac{\Upsilon_t + \Phi_t + \pi}{4} \right).
\]

which are martingales as shown above, where $\pm$-signs are i.i.d. symmetric coin flips for each excursion of $\Phi_t - \Upsilon_t$ away from 0 or $2\pi$. Since the processes are continuous martingales, we can do stochastic analysis with them, see for instance [9] for background. For example, Itô’s lemma holds for these processes.

Define auxiliary processes
\[
X_t = \frac{\Phi_t - \Upsilon_t}{2}
\]
\[
Z_t = e^{-t} M(t) = \cos X_t.
\]

It holds that
\[
dZ_t = -Z_t dt + e^{-t} dM_t
\]

(45)

where
\[
F(z, \phi) = \frac{1}{2} (1 - z^2)^{\frac{1}{4}} \left( \sqrt{1 + z} \left( \cos \frac{\phi}{2} - \sin \frac{\phi}{2} \right) + \sqrt{1 - z} \left( \cos \frac{\phi}{2} + \sin \frac{\phi}{2} \right) \right)
\]

(46)

Write the Loewner equation (44) as
\[
\dot{\phi}_t = \frac{Z_t}{\sqrt{1 - Z^2_t}}
\]

By Itô’s lemma, when $Z_t \neq \pm 1$,
\[
dN_t = \left[ \frac{1}{2} F_{zz}(Z_t, \Phi_t) e^{-2t} d(M)_t - F_z(Z_t, \Phi_t) Z_t dt + F_b(Z_t, \Phi_t) \frac{Z_t}{\sqrt{1 - Z^2_t}} dt + \frac{1}{2} F_z(Z_t, \Phi_t) dt \right] e^{\frac{\omega}{2}} + F_z(Z_t, \Phi_t) e^{-t} dM_t
\]

(47)
Since $N_t$ is a martingale, the quantity inside the brackets vanishes identically. We will prove the following result below.

**Lemma 5.1.** $\mathbb{P}[\int_0^\infty 1_{\gamma_t = \phi_t} dt = 0] = 1$

By this lemma and Lemma 5.3 of [16], it follows that $d\langle M \rangle_t$ is absolutely continuous with respect to $dt$ and that if we write $d\langle M \rangle_t = a_t^2 e^{2t} dt$, then there exists a Brownian motion $(B_t)_{t \in [0, \infty)}$ s.t. $dM_t = a_t e^t dB_t$. By (47)

$$a_t^2 = \frac{2F_x(Z_t, \Phi_t)Z_t - 2F_y(Z_t, \Phi_t)\frac{Z_t}{\sqrt{1-Z_t^2}} - F(Z_t, \Phi_t)}{F_{xx}(Z_t, \Phi_t)}$$

$$= \frac{4}{3}(1 - Z_t^2).$$

(48)

Notice the extremely simple expression on the right-hand side.

Notice next that by (45) and Itô’s lemma

$$dZ_t = -Z_t dt + \frac{2}{\sqrt{3}} \sqrt{1-Z_t^2} dB_t$$

(49)

$$\implies dX_t = d\arccos Z_t = \frac{2}{\sqrt{3}} \frac{Z_t}{\sqrt{1-Z_t^2}} dt - \frac{4}{\sqrt{3}} dB_t$$

$$= \frac{2}{3} \cot(X_t) dt - \frac{4}{\sqrt{3}} dB_t$$

(50)

That is, $X_t$ follows the law of radial Bessel process of $\kappa = 16/3$ and $\rho = \kappa - 6 = -2/3$.

By comparing to the usual Bessel process, it follows that $\int_0^t \cot \left( \frac{\Phi_s - \Upsilon_t}{2} \right) ds$ is finite and continuous. Thus it follows that

$$\Phi_t = \Phi_0 + \int_0^t \cot \left( \frac{\Phi_s - \Upsilon_s}{2} \right) ds + \Lambda_t^+ - \Lambda_t^-$$

where $\Lambda_t^+$ and $\Lambda_t^-$ are non-decreasing in $t$ and are constant on each excursion of $\Phi_t - \Upsilon_t$ away from 0 or $2\pi$ such that $\Lambda_t^+$ and $\Lambda_t^-$ can increase only when $\Phi_t - \Upsilon_t$ hits 0 or $2\pi$, respectively. The argument similar to the one in Section 5.5.3 in [16], which is based on the regularity result Theorem 4.7 above in the present case, shows that $\Lambda_t^+$ and $\Lambda_t^-$ are identically zero.

### 5.2.2. Instantaneous reflection at $\Phi_t = \Upsilon_t$.

It remains to prove Lemma 5.1.

**Lemma 5.2.** Let $A \subset \mathbb{R}$ be a countable set. Then $\int_0^\infty 1_{\Phi_t \in A} dt = 0$.

**Proof.** The claim follows from the fact that $t \mapsto \Phi_t$ is strictly increasing. Thus $\{t \in [0, \infty) : \Phi_t \in A\}$ is a countable set and has zero Lebesgue measure. $\square$

**Proof of Lemma 5.1.** Let $c(\phi) = \cos \frac{\phi}{2} - \sin \frac{\phi}{2}$ and write $A = \{\phi \in \mathbb{R} : c(\phi) = 0\}$. Then by Lemma 5.2, the Lebesgue measure $\int_0^\infty 1_{\Phi_t \in A} dt = 0$. Therefore it is sufficient to show that almost surely $\int_0^\infty 1_{\gamma_t = \phi, \Phi_t \notin A} dt = 0$.

Let $f_\phi(z) = F(z, \phi)$ where $F$ is as in (46). Then for any $\phi \notin A$, $f^{\phi^{-1}}(n) = 1 - 2 \left( \frac{n}{c(\phi)} \right)^4 + \mathcal{O}(n^6)$ near $n = 0$ and $(n, \phi) \mapsto f^{\phi^{-1}}(n)$ is twice differentiable function of $\phi$ and $n$ with similar bounds on the derivatives.

Now when $\Phi_t \notin A$ and $Z_t$ is near 1, it holds that

$$M_t = f^{\phi_1^{-1}}(N_t e^{-\frac{t}{2}}) e^t = e^t - \frac{N_t^4 e^{-t}}{c(\Phi_t)^4} + \mathcal{O}(N_t^6).$$
Write
\[ 1_{|N_t| \leq \varepsilon} 1_{|c(\Phi_t)| > \delta} \, dM_t \]
\[ = 1_{|N_t| \leq \varepsilon} 1_{|c(\Phi_t)| > \delta} \left( (e^t + O(\varepsilon^4)) \, dt + O(\varepsilon^2) \, dN_t + O(\varepsilon^2) \, d\langle N \rangle_t + O(\varepsilon^4) \, d\Phi_t \right). \]

By the fact that stochastic integrals of bounded integrands with respect to a martingale are martingales, it follows that
\[ E \left[ \int_0^T 1_{|N_t| \leq \varepsilon} 1_{|c(\Phi_t)| > \delta} \, e^t \, dt \right] = E \left[ \int_0^T 1_{|N_t| \leq \varepsilon} 1_{|c(\Phi_t)| > \delta} \, dM_t \right] + O(\varepsilon^2) = O(\varepsilon^2). \]

This shows that indeed \( P \left[ \int_0^T 1_{N_t=0} 1_{|c(\Phi_t)| > \delta} \, dt = 0 \right] = 1 \) for any \( \delta > 0 \). The claim follows from this and Lemma 5.2.

6. The proof of the main theorem on convergence of FK Ising loop ensemble to CLE(16/3)

The proof of Theorem 1.7. We can choose convergent subsequences by the results of Section 4. The convergence holds in the topology specified in Section 1.3.3.

The sequence of observables converges by the results of Section 3. The convergence is uniform over the class of domains and the scaling limit of the observable which is a solution of a boundary value problem depends continuously on the initial segment of the curve. Consequently, the martingale property extends to the scaling limit of the observable, as we told in Section 5.

The law of the driving process of a branch of the exploration tree is uniquely characterized by the results of Section 5. Finally, it is then possible to extend the convergence to the full tree by the results of Section 4. The law of the scaling limit of the loop ensemble is uniquely determined by Theorem 4.1 which is proven below.

\[ \text{The proof of Theorem 4.1.} \]

We can assume that \((\Theta^D, T^D)\) is the almost sure limit of \((\Theta_{D_n}^D, T_{D_n}^D)\) as \( n \to \infty \) by Theorem 1.7 and the discussion in Section 4.1.

The first claim follows from the observation that for any point in \( \mathbb{D} \) and for ball around it, the branch to that point will make a non-trivial loop around the point in the ball. Any of the starting point of an “excursions” after that is one of the points \( x_j \). By an excursion, we mean a segment \( T([s, t]) \) such that if we denote the component of that point in \( \mathbb{D} \setminus T([0, s]) \) by \( D_s \), then \( T((s, t)) \subset D_s \) and \( T(u) \in \partial D_s \) for both \( u = s, t \).

Let us use the following shortcut to prove the last claim of Theorem 4.1. We assume that we know that any finite subtree as in Section 4.4 converges to a tree of SLE\((16/3, -\frac{2}{3})\) curves. It follows from the discussion of Section 1.2 that the target points corresponding to loops are triple points of corresponding branches in the bulk and double points on the boundary. If there would be other other triple points in the bulk or double points on the boundary, there is, with high probability (depending on the parameter \( \eta \) in Section 4.4), a branch in the finite tree such that that point is on that branch. This is in contradiction with the properties of SLE\((16/3, -\frac{2}{3})\) curves.

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